Massless integrable quantum field theories and massless scattering in 1+1 dimensions

P. Fendley† and H. Saleur♠

†Department of Physics, University of Southern California
Los Angeles CA 90089

♠Department of Physics and Department of Mathematics
University of Southern California
Los Angeles CA 90089

These lecture notes provide an elementary introduction to the study of massless integrable quantum field theory in 1+1 dimensions using “massless scattering”. Some previously unpublished results are also presented, including a non-perturbative study of Virasoro conserved quantities.

Lectures presented at Strings 93, May 1993
and at the Trieste Summer School on High Energy Physics and Cosmology, July 1993.

* Packard Fellow
1. Introduction

A massless quantum field theory has no gap in the excitation spectrum. This can be seen, for example, in the Laplace representation of Green’s functions. This of course does not imply scale invariance; in general properties will interpolate between those of the two different conformal field theories describing the UV and IR fixed points. A standard (although a little marginal) example of such theory is the $O(3)$ non-linear sigma model with topological angle $\Theta = \pi$, which has central charges $c_{\text{UV}} = 2$ and $c_{\text{IR}} = 1$.

In 1+1 dimensions, many massless theories are integrable. Such theories include well-known statistical-mechanical models like the continuum limit of the XXZ spin chain and the Kondo problem. Many more are provided by appropriate perturbations of conformal field theories. Their study is interesting for several reasons. A few properties are accessible experimentally; see for example [1,2]. Features of academic interest include Green’s functions with different anomalous dimensions in the UV and IR, the consequences for the topology of the space of relativistic quantum field theories, a better understanding of the second law of thermodynamics associated with renormalization group trajectories, and a way of understanding perturbations of IR fixed points by irrelevant operators.

In these lectures we will discuss the scattering theories associated with integrable massless quantum field theories. In a massless theory the excitations should consist of right-moving and left-moving particles with $p = \pm E$, where we set the speed of light to be 1. $S$-matrices describing the “scattering” of such particles were calculated long ago in [3] for the XXX model. Such objects do not make much sense in traditional $S$-matrix theory where one requires the existence of in and out states; it is difficult for instance to imagine a physical process that would lead to scattering between two particles moving in the same direction at the speed of light. Massless $S$-matrices in 1 + 1 dimensions seem to make sense only in the context of integrable quantum field theories. In this case the scattering is completely elastic: momenta are conserved individually. We build states by acting with creation and annihilation operators on the ground state. These operators have non-trivial commutation properties (the Zamolodchikov-Faddeev algebra [4,5]) encoded in the $S$-matrix (suppressing internal indices describing the particles):

$$R^+(\theta_1)R^+(\theta_2) = S(\theta_1 - \theta_2)R^+(\theta_2)R^+(\theta_1),$$  \hspace{1cm} (1.1)

where by convention $R^+(\theta_i)R^+(\theta_j)$ creates a plane wave with $x_i < x_j$. This formal definition of $S$, directly inspired by the Bethe ansatz equations, makes sense in both
massive and massless cases. Another way of describing this definition is as a matching condition on two-particle wavefunctions.

Some care must be taken when defining massless integrable theories. For instance, the usual proof that an infinite number of conserved quantities implies factorized scattering relies on the possibility of separating wave packets in general [3], which is not possible in the massless case. Analyticity properties are also not completely clear. Since the particles are massless, they are either right- or left-moving. Because S-matrices for pure left-left or right-right scattering can be obtained by a limiting process from physical S-matrices acting on massive particles, one requires from $S_{LL}$ and $S_{RR}$ exactly the same properties as for physical S-matrices. As we will discuss, the case of $S_{LR}$ is subtler. In general these properties can be derived using the definition based on Bethe ansatz wavefunctions, without reference to any in and out states.

After having given the caveats, we would like to explain why finding massless S-matrices is a worthy endeavor. Even if the S-matrix found has no physical meaning in its own right, many quantities calculated from it do. In these notes we will show how to calculate the free energy at non-zero temperature using the thermodynamic Bethe ansatz (TBA). Modular invariance relates this to the Casimir energy on a circle, giving a “c-function”, which for unitary models shows the evolution of the number of degrees of freedom in the flow from ultraviolet to infrared. This calculation can also be modified to give some excited-state energies as well, which give conformal dimensions in the critical limit. We will also show how to obtain the ground-state energy at zero temperature in a background field, a result related to the chiral $U(1)$ anomaly in the critical limit. Finally, we present here some new results on obtaining higher-spin Virasoro conserved charges from the massless scattering.

These results in fact suggest that some aspects of a conformal field theory can be described by a theory of massless particles with no left-right scattering [7]. This can be seen as follows. In these massless but not scale-invariant theories, we have a mass scale $M$; $M = 0$ gives the ultraviolet fixed point while $M \to \infty$ gives the infrared one. The momenta and energy of the particles are parametrized by

$$E = p = \frac{M}{2} e^{\theta} \quad \text{for right movers}$$

$$E = -p = \frac{M}{2} e^{-\theta} \quad \text{for left movers}$$

(1.2)
A Lorentz-invariant $S$-matrix element $S_{LL}$ describing scattering of two left movers depends only on the ratio of the two momenta, so it depends only on $\theta_1 - \theta_2$ and not on $M$. The left-right scattering also depends only on the rapidity difference but does depend on $M$, because the only Lorentz invariant is $s = (p_1 + p_2)^2$. We can always rescale $M \to \infty$ by shifting the rapidities. The $LL$ and $RR$ $S$-matrices are independent of this shift (although $S_{LR}$ is not), so they are characterized solely by properties of the infrared fixed point. In this sense one can think of the $LL$ and $RR$ $S$-matrices as being the $S$-matrices for the conformal field theory. This should not seem bizarre — many properties of four-dimensional field theories (even ones with massless particles like QED) are described by particle theories!

Another reason for studying massless $S$-matrices is that finding them is often an easier task than doing the full Bethe ansatz, a result of the constraints of an integrable theory. In the massive case, this has become a highly-developed art (see [8-10] for reviews), and many of these lessons can be applied to the massless case. The basic method is to guess the particle content based on the knowledge of the symmetries of the problem (and on the Lagrangian, if one is known), and then find the simplest $S$-matrix consistent with these symmetries as well as the criteria of factorizability, unitarity and crossing symmetry. In many cases, such an $S$-matrix is the correct one, as can be checked by a variety of methods. The symmetries, in particular affine quantum group symmetries, must however be analyzed carefully.

In addition to the XXX model, massless $S$-matrices have been found for a number of models. Continuum theories include the flow from the tricritical Ising model to the Ising model [11], the $O(3)$ sigma model at $\Theta = \pi$ and the $SU(2)_1$ principal chiral model [8], the flows between the minimal models [12], the Kondo problem [13], the “sausage” sigma model [14], and the Landau-Ginzburg flows to the $N=2$ minimal models [15]. Lattice models include integrable higher-spin XXX chains [16] and the Hubbard model [17,18].

The purpose of these lectures is to provide a pedagogical introduction, so we will skip many technical details. We have tried to make the sections reasonably independent of one another so that they can be read separately. In Section 2 we give a simple example of a massless field theory, the sine-Gordon model with imaginary potential. Section 3 contains a discussion of the usual Thirring model (for properties we discuss, Thirring and sine-Gordon can be used interchangeably) and its massless limit. Its main purpose is to introduce the physically odd idea of left-left or right-right scattering between massless particles from the Bethe ansatz view point. Section 4 gives a simple introduction to the thermodynamic Bethe ansatz carried out with the example of massless scattering.
Section 5 explores a little more how one can describe certain aspects of conformal field theories by massless particles with no left-right scattering. Previously unpublished results regarding the non-perturbative analysis of conserved quantities in conformal field theories are presented. Section 6 contains some comments on integrable theories with nontrivial left-right scattering. Section 7 is based on [12]. There we carry out explicitly the study of the sine-Gordon model with imaginary coupling and background field, the latter being introduced to get a simpler calculation. We show that even if massless scattering appears a little odd physically, it at least provides the proper analytic continuation of physical quantities in appropriate directions of the parameter space. Section 8 contains conclusions and questions of interest.

2. A simple example of massless field theory

A simple but generic example of integrable massless field theory is provided by the sine-Gordon model with imaginary “coupling” (i.e. prefactor of the cosine term). Recall the hamiltonian

$$H = \int ds \left[ \frac{1}{2} \Pi^2 + \frac{1}{2} (\partial \phi)^2 + \lambda \cos \beta_{SG} \phi \right]$$  \hspace{1cm} (2.1)

where we set $\beta_{UV}^2 = 8\pi \frac{t}{t+1}$ at the UV fixed point. This model is well-studied for $\lambda$ real and known to be a massive field theory with a trivial fixed point in the IR. Taking $\lambda$ imaginary does not look too physical at first. There are however several reasons for doing so. The massless flows between minimal models are obtained as reductions of this model, and are unitary even though (2.1) is not. In addition, (2.1) describes the flow of the $O(n)$ model to its low-temperature phase [13]; this covers interesting physical situations in condensed-matter physics such as self-avoiding polymers.

To see the difference in behavior between $\lambda$ real and imaginary consider the large-$t$ limit, where the cosine term is almost marginal and reliable perturbation theory at order $1/t$ can be carried out following the analysis of the XY model [20]. The RG equations are

$$\frac{d\lambda}{d\beta} = \left( 2 - \frac{\beta^2}{4\pi} \right) \lambda, \hspace{2cm} (2.2)$$

and

$$\frac{1}{\beta^2} \frac{d\beta^2}{db} = -\pi^2 \lambda^2. \hspace{2cm} (2.3)$$
From these one deduces, at first nontrivial order
\[
\frac{d\beta^2}{db} = -\frac{\beta^4}{4\pi} \left[ \left( \frac{8\pi}{\beta^2} - 1 \right)^2 - \frac{1}{t^2} \right].
\] (2.4)
For \(\lambda\) real, the initial derivative of \(\beta\) is negative, and the coupling flows to zero at large distance. For \(\lambda\) imaginary, the initial derivative is positive and the coupling increases monotonically to the IR fixed point with
\[
\beta_{IR}^2 = \frac{8\pi}{t} - \frac{1}{t^2}.
\] (2.5)
Correspondingly both the UV and IR fixed points are Gaussian models with different radii of compactification, and we have a flow “within” \(c = 1\).

The most interesting aspect concerns the evolution of the running central charge. The latter can be defined in several ways away from the fixed points, for instance using the two-point function of the stress energy tensor [21] or finite-size effects [22,23]. Qualitatively, these functions should behave in a similar way and are usually expected to describe the evolution of the number of degrees of freedom. By analogy with the second law of thermodynamics we expect that such a function should decrease when following a RG flow. It has been proven that the first type of \(c\)-function always decreases in unitary theories (the \(c\)-theorem [21]), and all known unitary examples of the second type also decrease. In our non-unitary problem, it is easy to compute them at first non trivial order in \(1/t\) where they coincide. One finds then [12]
\[
c = 1 + \frac{12}{t^3} \frac{e^{4(t-\beta_0)/t}(1 - e^{4(t-\beta_0)/t})}{(1 + e^{4(t-\beta_0)/t})^3},
\] (2.6)
that has a roaming behavior. It indeed has \(c = 1\) in the UV and IR but goes up and down in between, reaching a pair of extrema with values
\[
c_{\pm} = 1 \pm \frac{2}{\sqrt{3}} \frac{1}{t^3}.
\] (2.7)
Of course the fact that \(c\) can increase and does not obey a nonunitary version of the \(c\)-theorem is not surprising and is an obvious consequence of the imaginary coupling. One might hope to slightly modify the perturbation to obtain an exact flow which would stop at the value \(c_+\). This does not seem impossible in view of [24], which describes flows that exactly interpolate between minimal models and flows that go “very near” them.

The fact that \(c\) is not monotonic sheds some doubt on the reliability of the running \(c\)-function as a measure of the number of degrees of freedom. Let us emphasize that (2.6) is indeed related to the absolute ground state of the theory — it is the “\(c_{eff}\)” supposed to qualitatively replace \(c\) in the non-unitary case.
3. Massless scattering at the conformal point

We consider the example of the massive Thirring model on a circle of length $L$ with hamiltonian

$$H = \int dx \left[ -i \left( \psi_1^+ \partial_x \psi_1 - \psi_2^+ \partial_x \psi_2 \right) + m_0 \left( \psi_1^+ \psi_2 + \psi_2^+ \psi_1 \right) + 2g_0 \psi_1^+ \psi_2^+ \psi_2 \psi_1 \right].$$

(3.1)

which is solvable by the Bethe ansatz [25,26]. The analogous calculation is done for the sine-Gordon model in [27]. As a result eigenenergies take the form

$$E = \sum_i m_0 \cosh \xi_i,$$

(3.2)

and momenta

$$P = \sum_i m_0 \sinh \xi_i,$$

(3.3)

where the $\xi_i$ are bare rapidities of pseudoparticles satisfying the Bethe ansatz equations

$$\exp(im_0 L \sinh \xi_i) \prod_j \frac{\sinh(\xi_i - \xi_j + 2i\mu)}{\sinh(\xi_i - \xi_j - 2i\mu)} = 1,$$

(3.4)

and $\cot \mu = -\frac{1}{2}g_0$. One recognizes in (3.4) the conditions for the wave function to be periodic, the phase shift being a combination of a free term and factorized scattering between pairs of pseudoparticles. Thus the elements of the product in (3.4) are the “bare” $S$-matrix elements for the pseudoparticles, which we denote as $S_0(\xi_i - \xi_j) \equiv \exp[i\phi_0(\xi_i - \xi_j)] \equiv \frac{1 + i\Lambda}{1 - i\Lambda}$. Equation (3.4) follows from the form of Bethe wave functions

$$\psi(x_1, \ldots, x_N|\xi_1, \ldots, \xi_N) = \exp(im_0 \sum_i x_i \sinh \xi_i) \prod_{i<j} \left[ 1 + i\Lambda(\xi_i - \xi_j)\text{sign}(\xi_i - \xi_j) \right].$$

(3.5)

This wave function is almost free, with only a phase shift when the coordinates are exchanged. Of course since we are dealing with fermions, the real wave function has still to be antisymmetrized so

$$\Psi(x_1, \ldots, x_N) = \sum_P \text{sign}(P)\psi(x_1, \ldots, x_N|\xi_{P1}, \ldots, \xi_{PN}),$$

(3.6)

(this makes sense because $\Lambda$ is an odd function, or equivalently $S_0(\xi)S_0(-\xi) = 1$) and all rapidities must therefore be different.
The study of the quantum field theory requires building the ground state by filling the appropriate Dirac sea, and then finding the excitations and their scattering. We must mention a difficulty concerning the choice of UV cutoff. In the attractive regime $g_0 > 0$ essentially all cutoffs produce the same results, but the situation is different in the repulsive regime $g_0 < 0$. (This is unfortunately the regime where one truncates the sine-Gordon model to describe massive perturbations of Virasoro minimal models by the $\phi_{13}$ operator \cite{28}.\textsuperscript{\textasteriskcentered}) The rapidity cutoff of \cite{29} leads to many more particles in the spectrum than the cutoff of \cite{30}, which is essentially a lattice cutoff \cite{31}. The latter regularization reproduces the conjectures of \cite{8} and seems in agreement with the expected physics of perturbed minimal models \cite{32}. We avoid entering into technical details and give a “morally” correct discussion in the following.

Another difficulty arises in the eigenfunctions of (3.1) as a result of terms like (sign $\delta$). Depending on the regularization, different relations between the bare coupling $g_0$ and the parameter $\mu$ are found. In \cite{26} the relation $\mu = \frac{\pi + g_0}{2}$ is used instead. We shall carry out the discussion in terms of the variable $\mu$.

The ground state is easily built by filling the sea with antipseudoparticles, i.e. filling the line $\text{Im}(\xi) = \pi$. There are various kind of excitations. We shall discuss only the case of solitons $s$ and antisolitons $a$ (these are the only particles with non-vanishing $U(1)$ charge). For instance a pair of solitons is simply obtained by making two holes in the ground state distribution. Antisolitons, or pairs of solitons and antisolitons are obtained similarly, with the addition of some strings \cite{29} of pseudoparticles around $\text{Im}(\xi) = 0$. Of course the introduction of such holes induces a shift in the distribution of the $\xi$’s, the so-called backflow. As a result the mass and rapidity of the solitons are renormalized, giving

$$E = m \cosh \theta, \quad p = m \sinh \theta, \quad (3.7)$$

with

$$m = m_0 \frac{\gamma e^{A(1-\gamma)}}{\pi(\gamma - 1)} \tan \pi \gamma, \quad \theta = \gamma \lambda, \quad (3.8)$$

with $\gamma \equiv \frac{\pi}{2\mu}$ and $\lambda$ denotes the bare rapidity of the particles ($s, a$) (again there are some slight differences between authors for this formula). As shown in \cite{26} the $S$-matrix of solitons can be extracted from the Bethe ansatz equations. The $S$-matrix elements are defined as follows. Suppose first we have only one soliton, with bare rapidity $\lambda_1$. Then
the total phase shift collected by the wave function when the argument of the soliton goes around the circle is

\[ \phi_1 = m_0 L \sinh \lambda_1 + \sum_j \phi_0(\lambda_1 - \bar{\xi}_j) \]  

(3.9)

where the sum is taken over all pseudoparticles in the sea and \( \bar{\xi} \) indicates shifted (with respect to the ground state) rapidities due to the backflow. Suppose then we have two solitons with bare rapidities \( \lambda_1 \) and \( \lambda_2 \). Then the total phase shift of the wave function when the argument of the first soliton again goes around the circle is

\[ \phi_2 = m_0 L \sinh \lambda_1 + \sum_j \phi_0(\lambda_1 - \tilde{\xi}_j). \]  

(3.10)

where \( \tilde{\xi} \) are shifted rapidities. One then defines the \( S \)-matrix element by

\[ \ln S = i(\phi_2 - \phi_1). \]

Complete computation shows that it depends only on the difference of the rapidities. Moreover one also checks that for more particles, the phase shifts simply add and the scattering can be decomposed as a succession of two-particle ones. The resulting \( S \)-matrix, therefore a solution of the Yang-Baxter equation, has the well-known matrix elements

\[ a = Z(\theta) \sinh \left( \frac{i\pi - \theta}{t} \right), \quad b = Z(\theta) \sinh \left( \frac{\theta}{t} \right), \quad c = Z(\theta) \sinh \left( \frac{i\pi}{t} \right) \]  

(3.11)

where \( a \) corresponds to \( ss \rightarrow ss \) scattering, \( b \) to \( sa \rightarrow sa \) and \( c \) to \( sa \rightarrow as \). We give the the expression for normalization factor \( Z \) in sect. 7. The symmetry under \( s \leftrightarrow a \) gives the remainder of the elements. We have parametrized

\[ \gamma \equiv \frac{\pi}{2\mu} \equiv \frac{\pi}{t+1}. \]  

(3.12)

The \( S \)-matrix \((3.11)\) can be manipulated to exhibit \( \hat{U}_q sl(2) \) symmetry \([33]\) with

\[ q = -\exp \left( -\frac{i\pi}{t} \right). \]  

(3.13)

This is a dynamical symmetry; there is also a kinematical symmetry \( U_{q_0} sl(2) \) with \( q_0 = -\exp \left( -\frac{i\pi}{t+1} \right) \) following from the Bethe ansatz equations \([34]\). Notice the shift of the denominator.

In order to reach the deep UV limit, we let the mass \( m_0 \) go to zero. Particles with non-vanishing energy must have rapidities with very large modulus, of the order \( \xi_0 = \ln(M/m_0) \gg 1 \), where \( M \) is a not-yet-defined parameter with the dimension of mass.
There are thus two regions of interest in which we set respectively \( \xi = \xi_0 + \theta \) and \( \xi = -\xi_0 + \theta \), \( \theta \) remaining finite. The spectrum obviously splits into right and left excitations with

\[
E_R \approx \frac{m_0}{2} e^{\xi_0} e^{\theta} \equiv \frac{M}{2} e^{\theta} = p_R; \quad E_L \approx \frac{m_0}{2} e^{\xi_0} e^{-\theta} \equiv \frac{M}{2} e^{-\theta} = -p_L. \tag{3.14}
\]

and we have a doubling of species \( a_{L,R}, s_{L,R} \) (see 1). For pseudoparticles of the same kind, the phase shifts are unchanged as \( \xi_i - \xi_j = \theta_i - \theta_j \). For particles of different kinds however \( \xi_i - \xi_j \approx \pm 2\xi_0 \to \pm \infty \) so the phase shifts become constants, independent of the rapidities. As a result, in the computation of the \( S \)-matrix, the \( LL \) and \( RR \) scattering are the same as the ones for corresponding massive particles computed above \( S_{LL} = S_{RR} = S^0 \), while the \( LR \) scattering becomes trivial.

The \( S \)-matrices in the massless limit can also be obtained by studying the XXZ spin chain, which is a lattice regularization of the massless Thirring model. This is a simple generalization of the work of [3] on the XXX chain.

It should be clear finally that the properties of a massless theory with trivial \( LR \) scattering are independent of the mass scale \( M \). Indeed, changing \( M \) is equivalent to shifting \( \xi_0 \), and the analysis only depends on rapidity differences.

4. Thermodynamic Bethe ansatz

The technique we now discuss involves computing the free energy of an integrable lattice model (e.g. the XXZ model) or an integrable quantum field theory (e.g. the Thirring model) on an infinite line at finite temperature \( T \). There are two approaches to this calculation. In the traditional “bare” approach, one finds the energy and entropy of the states of the model using the Bethe ansatz. The thermodynamic state is the state which minimizes the free energy. The limit \( T \to 0 \) gives the ground-state energy and the vicinity \( T \approx 0 \) the structure of low-lying excitations. In the massive Thirring model this approach starts with the bare equations (3.4). In the second approach one forgets the bare theory completely and studies instead the thermodynamics of a gas composed of the various “physical” excitations (like the soliton and antisoliton of the sine-Gordon model) scattering with their respective \( S \)-matrices. As in the first approach, one determines their energy and entropy and again minimizes the free energy. The first approach was used in works like

---

1 In some cases like in [6] there is an additional phase in the definition of \( S_{LL} \) and \( S_{RR} \).
The second approach, pioneered in [30], is more recent, and is usually called the thermodynamic Bethe ansatz (TBA).

The second approach allows some convenient short cuts: instead of solving the theory one first establishes that it is integrable, conjectures the excitations and their $S$-matrix using intuition and symmetry arguments (with some care), and uses the TBA to derive various properties. When both approaches can be implemented, they of course give the same results; the quasiparticle excitations used in the second can be found from the Bethe ansatz equations by filling the Fermi or Dirac sea. However, there are examples (at least in the massive case) where a lattice model is not integrable but its continuum limit is an integrable quantum field theory. Usually the only way of defining the continuum model is by a perturbed conformal field theory, so the usual Bethe ansatz methods cannot be applied; the only recourse is to use the second approach. A classic example is the Ising model at $T = T_c$ in a magnetic field [9].

As a simple example we describe the TBA for a single type of massless particle, say right-moving, with energy and momentum parametrized as in (1.2). The scattering is described by a single $S$-matrix element $S_{RR}$. Quantizing a gas of such particles a circle of length $L$ requires the momentum of the $i$th particle to obey

$$\exp\left(i \frac{Me\theta_i}{2} L \right) \prod_{j \neq i} S_{RR}(\theta_i - \theta_j) = 1. \tag{4.1}$$

One can think of this intuitively as bringing the particle around the world through the other particles; one obtains a product of two-particle $S$-matrix elements because the scattering is factorizable. This is the renormalized equivalent of the bare relation (3.4).

Going to the $L \to \infty$ limit, we introduce the density of rapidities indeed occupied by particles $\rho(\theta)$ and the density of holes $\tilde{\rho}$. A hole is a state which is allowed by the quantization condition (4.1) but which is not occupied, so that the density of possible rapidities is $\rho(\theta) + \tilde{\rho}(\theta)$. Taking the derivative of the log of (4.1) yields

$$2\pi[\rho(\theta) + \tilde{\rho}(\theta)] = \frac{ML}{2} e^\theta + \int_{-\infty}^\infty \Phi(\theta - \theta')\rho(\theta'), \tag{4.2}$$

where

$$\Phi(\theta) = \frac{1}{i} \frac{d}{d\theta} \ln S(\theta).$$

---

2 Observe however that there is another integrable lattice model based on $E_8$ that is integrable and has the same scaling limit as Ising in a magnetic field.
To determine which fraction of the levels is occupied we do the thermodynamics \[35\]. The energy is
\[E = \int_{-\infty}^{\infty} \rho(\theta) \frac{M}{2} e^\theta d\theta,\]
and the entropy is
\[S = \int_{-\infty}^{\infty} [(\rho + \tilde{\rho}) \ln(\rho + \tilde{\rho}) - \rho \ln(\rho) - \tilde{\rho} \ln(\tilde{\rho})] d\theta.\]

The free energy per unit length \(F = (E - TS)/L\) is found by minimizing it with respect to \(\rho\). The variations of \(E\) and \(S\) are
\[
\delta E = \int_{-\infty}^{\infty} \delta \rho \frac{M}{2} e^\theta d\theta,
\]
\[
\delta S = \int_{-\infty}^{\infty} [(\delta \rho + \delta \tilde{\rho}) \ln(\rho + \tilde{\rho}) - \delta \rho \ln(\rho) - \delta \tilde{\rho} \ln(\tilde{\rho})] d\theta.
\]

It is convenient to parametrize
\[
\frac{\rho(\theta)}{\tilde{\rho}(\theta)} \equiv \exp \left( -\frac{\epsilon}{T} \right),
\]
(4.3)
giving
\[
\delta S = \int_{-\infty}^{\infty} \left[ \delta \rho \ln \left( 1 + e^{\epsilon/T} \right) + \delta \tilde{\rho} \ln \left( 1 + e^{-\epsilon/T} \right) \right] d\theta.
\]

Using (4.2) allows us to find \(\tilde{\rho}\) in terms of \(\rho\). Denoting convolution by *, this gives
\[
2\pi (\delta \rho + \delta \tilde{\rho}) = \Phi * \delta \rho\]
so
\[
\delta S = \int_{-\infty}^{\infty} \left[ \frac{\epsilon}{T} + \frac{\Phi}{2\pi} * \ln \left( 1 + e^{-\epsilon/T} \right) \right] \delta \rho d\theta.
\]

Hence the extremum of \(F\) occurs for
\[
\frac{M}{2} e^\theta = \epsilon + T \frac{\Phi}{2\pi} * \ln \left( 1 + e^{-\epsilon/T} \right),
\]
(4.4)
and one has then, expressing \(\tilde{\rho}\) from (1.2) and using (4.4)
\[
F = E_0 - T^2 \frac{M}{4\pi T} \int_{-\infty}^{\infty} e^\theta \ln \left( 1 + e^{-\epsilon/T} \right) d\theta.
\]
(4.5)
The ground state energy \(E_0\) cannot be obtained by this method since all the information we use is the structure of excitations above the ground state, so we set \(E_0 = 0\) for the rest of this section.
The limit $T \to 0$ of this system is interesting. We introduce the positive and negative parts of the pseudoenergy satisfying therefore

$$
\frac{M}{2} e^\theta = \epsilon^+ + \epsilon^- - \frac{\Phi}{2\pi} \epsilon^-, 
$$

(4.6)

In this limit the solution is $\epsilon^- = 0$, $\epsilon^+ = \frac{M}{2} e^\theta$. It follows from this and (4.3) that $\rho \to 0$ as $T \to 0$, which is required because our TBA provides the structure of excitations over the ground state. In general $-\epsilon^-$ (resp. $\epsilon^+$) gives the excitation energy for holes (resp. particles).

The knowledge of $\mathcal{F}$ leads to the determination of the central charge of the theory. We have $\mathcal{F} = -\frac{T}{R} \ln Z$, where $Z$ is the partition function of the one dimensional quantum field theory at temperature $T$ (in the following we refer to this point of view as “thermal”). In Euclidean formalism, this corresponds to a theory on a torus with finite size in time direction $R = 1/T$. By modular invariance, identical results should be obtained if one quantizes the theory with $R$ as the space coordinate. For large $L$, $Z = e^{-E(R)L}$, where $E(R)$ is the ground-state (Casimir) energy with space a circle of length $R$. Thus $\mathcal{F} = E(R)/R$. In the following we refer to this as the “finite-size” point of view. Conformal invariance requires that at a fixed point this Casimir energy is $E(R) = -\frac{\pi c}{6R}$, where $c$ is the central charge [37]. Going back to the thermal point of view, $\mathcal{F} = -\frac{\pi c T^2}{6}$ and the specific heat is $\mathcal{C} = -\frac{\pi c T^3}{3}$.

Observe from (4.4) and (4.3) that the free energy does not depend on the mass scale $M$, because it can be rescaled by a shift in rapidities. By dimensional analysis one has therefore $\mathcal{F} = T^2 \times \text{constant}$. This scale invariance is a manifestation of the fact that $S_{LL}$ and $S_{RR}$ are describing only conformal properties. With massive particles or with nontrivial left-right massless scattering, $\mathcal{F}$ does depend on $M/T$, giving a running central charge.

We can analytically find this central charge from (4.4). We take the derivative of (4.4) with respect to $\theta$ and solve for $e^\theta$. Substituting this in (4.5), we have

$$
\mathcal{F} = -\frac{T}{2\pi} \int d\theta \left[ \frac{d\epsilon}{d\theta} \ln(1 + e^{-\epsilon/T}) - \int d\theta' \ln(1 + e^{-\epsilon(\theta)/T}) \Phi(\theta - \theta') \frac{d\epsilon}{d\theta'} \frac{1}{1 + e^{\epsilon(\theta')/T}} \right] 
$$

$$
= -\frac{T}{2\pi} \int d\theta \frac{d\epsilon}{d\theta} \left[ \ln(1 + e^{-\epsilon/T}) + (\epsilon - \frac{M}{2} e^\theta) \frac{1}{1 + e^{\epsilon(T)/T}} \right] 
$$

$$
= -\mathcal{F} - \frac{T}{2\pi} \int d\theta \frac{d\epsilon}{d\theta} \left[ \ln(1 + e^{-\epsilon/T}) + \frac{\epsilon}{1 + e^{\epsilon/T}} \right].
$$
where we use (4.4) again to get to the second line. We can replace the integral over $\theta$ with one over $\epsilon$, giving an ordinary integral

$$
\mathcal{F} = -\frac{T}{4\pi} \int_{\epsilon(-\infty)}^{\infty} d\epsilon \left[ \ln(1 + e^{-\epsilon/T}) + \frac{\epsilon}{1 + e^{\epsilon(\theta)/T}} \right],
$$

A change of variables gives

$$
\mathcal{F} = -\frac{T}{2\pi} L \left( \frac{1}{1 + x_0} \right), \quad (4.7)
$$

where $L(x)$ is the Rogers dilogarithm function

$$
L(x) = -\frac{1}{2} \int_0^x \left( \frac{\ln(1-y)}{y} + \frac{\ln y}{1-y} \right) dy,
$$

and $x_0 \equiv \exp(\epsilon(-\infty)/T)$ is obtained from (4.4) as

$$
\frac{1}{x_0} = \left( 1 + \frac{1}{x_0} \right)^I, \quad (4.8)
$$

with $I = \frac{1}{2\pi} \int \Phi$.

For example, when the $S$ matrix is a constant, $\Phi = 0$, $x_0 = 1$ and

$$
\mathcal{F} = -\frac{T\pi}{24}, \quad (4.9)
$$

where we used $L(1/2) = \frac{\pi^2}{12}$. Here we find $c_L = \frac{1}{4}$. In a left-right-symmetric quantum field theory, the right sector makes the same contribution, giving the total central charge $c = \frac{1}{2}$ required for free fermions.

For the nonunitary Lee-Yang $S$-matrix [38] one has $I = -1$. In that case $x_0 = \frac{1+\sqrt{5}}{2}$. Using $L \left( \frac{3-\sqrt{5}}{2} \right) = \frac{2}{5} \frac{\pi^2}{5}$ one finds $c_R = \frac{3}{10}$ and $c = \frac{2}{5}$ after left and right contributions have been collected. This is indeed the effective central charge for the Lee-Yang problem; it is not equal to the true central charge $c = -\frac{22}{3}$ because of the presence of an operator with negative dimension in the vacuum.

It is possible to do the same computation with the massless pair $(a, s)$ scattering with (3.11). The computation is technically more complicated because the scattering is non-diagonal. We just refer the reader to references [39,32] for details. Simply observe that in the case $\mu = \frac{\pi}{2}$ the scattering becomes diagonal and because of the doubling of the number of species, the above calculation gives rise to $c = 1$ as expected.

Besides the central charge, some conformal dimensions can also be identified using the TBA. To do so one includes an imaginary chemical potential $\mu_b$ (not to be confused with
\( \mu \) in section 3) for each species of particle \( b \). As before, we minimize the corresponding free energy \( G = E - TS - \sum_b \mu_b N_b \), and find similar results with, in most cases, \( \exp\left(-\epsilon_b/T\right) \) replaced by \( \exp\left(-\left(\epsilon_b - \mu_b\right)/T\right) \). In the finite-size point of view, the introduction of a chemical potential amounts to considering the theory on a circle of length \( R \) with twisted boundary conditions. As is well known, the ground state energy in that case gives an effective central charge \( c_{\text{eff}} = c - 24h \) where \( h \) is related to the twist. For \( RR \) scattering given by (3.11) for instance, with \( \mu_s = -\mu_a = i\alpha\pi T/t \) one finds \( h = \frac{\alpha^2}{4(t+1)} \). Thus one recovers the dimensions of vertex operators in a Gaussian model.

The question of reconstructing the whole quantum field theory from a massless scattering theory with no left-right scattering is still open. For Virasoro minimal models two independent such theories are probably necessary, as there are two fundamental quantum groups, or two labels in the Kac table. A bit of progress in this direction is presented in the next section.

### 5. Integrable CFT and massless scattering: Virasoro conserved quantities

We showed in sect. 4 how one obtains the free energy in a massless integrable theory, and compared this result with conformal field theory predictions. In an integrable theory, the energy \( E \equiv \langle E \rangle \) is just the first of an infinite series of conserved charges. These conserved charges can often be expressed as suitably regularized powers of the energy-momentum tensor. Since we derive the particle densities of the thermodynamic state, we can calculate the expectation value of any quantity which can be expressed in terms of the particles. This suggests that the conserved charges should be related to expectation values \( \langle E^n \rangle \). In this section we show that in the sine-Gordon model, this is indeed true, thus verifying non-perturbatively what had been shown classically and perturbatively in the quantum theory [40,41].

We start with free fermions with antiperiodic boundary conditions on a circle of length \( R \) in order to select the ground state. We are thinking about the system in the finite-size point of view discussed in the last section. Consider the quantity

\[
\langle E^n \rangle_{gs} = \frac{1}{2} \left(\frac{2\pi}{R}\right)^n \left\langle \sum_{j=-\infty}^{\infty} (j + 1/2)^n : \psi_{-j-1/2} \psi_{j+1/2} : \right\rangle_{gs}
\]

\[
= (-)^n \frac{1}{2} \left(\frac{2\pi}{R}\right)^n \sum_{j=0}^{\infty} (j + 1/2)^n.
\]

(5.1)
The sum can be evaluated by $\zeta$-function regularization leading to
\[
\langle E^{2k+1} \rangle_{gs} = \frac{1}{2} \left( \frac{2\pi}{R} \right)^{2k+1} \left( 1 - \frac{1}{2^{2k+1}} \right) \zeta(-2k - 1),
\]
and
\[
\langle E^{2k} \rangle_{gs} = 0,
\]
where the last result follows from $\zeta(-2k) = 0$. For $k = 0$ one gets $\mathcal{E} = \langle E \rangle = -\frac{\pi c}{6} R$ with $c = 1/2$ ($c/2$ appears because we concentrate on one chirality).

We can compute the analogous quantity from the thermal point of view by putting the particles on a circle of large length $L$ at temperature $T = 1/R$. The TBA analysis of sect. 4 gives
\[
\langle E^n \rangle_{TBA} = \int_{-\infty}^{\infty} d\theta \left( \frac{Me^\theta}{2} \right)^n \rho(\theta)
= \int_{-\infty}^{\infty} d\theta \left( \frac{Me^\theta}{2} \right)^n \left( \rho(\theta) + \tilde{\rho}(\theta) \right) \frac{1}{1 + \exp(\epsilon/T)}
= nLT \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} \left( \frac{Me^\theta}{2} \right)^n \ln(1 + e^{-\epsilon/T}),
\]
where we used the fact that $2\pi(\rho + \tilde{\rho}) = LT d\epsilon/d\theta$, which is proven by showing that they obey the same integral equation.

For free fermions, $\epsilon = Me^\theta/2$. The expectation value (5.4) is
\[
\langle E^n \rangle_{TBA} = \frac{L}{2\pi R} R^{-n} \int_0^{\infty} x^{n-1} \ln(1 + e^{-x}) dx,
\]
or, restricting to $n = 2k + 1$
\[
\langle E^{2k+1} \rangle_{TBA} = \frac{L}{2\pi R} \frac{(2k+1)!}{R} R^{-2k-1} \left( 1 - \frac{1}{2^{2k+1}} \right) \zeta(2k + 2).
\]
To compare (5.2) and (5.3) recall the identities [12]
\[
\zeta(2k + 2) = \frac{(2\pi)^{2k+2}}{2(2k+2)!} (-1)^k B_{2k+2}; \quad \zeta(-2k - 1) = -\frac{B_{2k+2}}{2k + 2},
\]
where $B_n$ are Bernoulli numbers. Hence we find
\[
\langle E^{2k+1} \rangle_{TBA} = (-1)^{k+1} \frac{L}{R} \langle E^{2k+1} \rangle_{gs}.
\]
For \( k = 0 \) we recover \( \mathcal{F} \) as derived in the previous section, using the fact that \( \mathcal{E} = -\mathcal{F} \) for theories with no left-right scattering. More generally (5.6) follows from the relation of \( \langle E^{2k+1} \rangle_{TBA} \) (resp. \( \langle E^{2k+1} \rangle_{gs} \)) to the energy-momentum tensor component \( T_{xx}^{k+1} \) (resp. \( T_{yy}^{k+1} \)) and \( T_{xx} + T_{yy} = 0 \) at the conformal point.

Notice that when \( n \) is even, the results are quite different, since \( \langle E^{2k} \rangle_{TBA} \neq 0 \) while (5.3) holds. The usual argument is that such even powers do not correspond to any local quantity in the quantum field theory, and therefore the two results cannot be compared as we did in (5.6).

The generalization of this computation to the case of nontrivial scattering is not straightforward, but it is useful. We can compute – at least numerically – the quantities \( \langle E^{2k+1} \rangle_{TBA} \) from (5.4). In general the conformal field theory cannot easily be described by oscillators as in the free theory above, so we do not compute the equivalent of \( \langle E^{2k+1} \rangle_{gs} \). What we can however compute using only the Virasoro algebra are quantities like

\[
\langle \int :T^k: \rangle_h,
\]

where the integral is over the period of the cylinder, and average is taken in a state of conformal weights \((h,h)\), generalizing the ground state. The double dots indicate normal ordering on the cylinder. Recall that this normal ordering leaves room for non-vanishing expectation values. These can be computed by explicitly performing the subtraction of the divergent terms. The result coincides with the simpler zeta regularization. Such a quantity cannot be directly compared to \( \langle E^{2k+1} \rangle_{TBA} \) because of a non-trivial renormalization factor. In the free fermion case for instance \( \langle \int :T^2: \rangle_{gs} = \left( \frac{2\pi}{R} \right)^3 \frac{79}{96} \zeta(-3) \) while \( \langle E^3 \rangle_{gs} = \left( \frac{2\pi}{R} \right)^3 \frac{7}{16} \zeta(-3) \), because \( :T^2:= \frac{3}{8} :\partial^2 \psi \partial \psi : - \frac{5}{24} :\partial^4 \psi \partial \psi : \). However this normalization factor occurs from short-distance singularities and therefore does not depend on boundary conditions. We can therefore compare ratios of the moments of \( T, \partial T, \ldots \) and \( \langle E^n \rangle \)'s for different boundary conditions around the circle, or equivalently the choice of state in which the average (5.7) is taken. As explained in the previous section, in the thermodynamics this means taking imaginary chemical potentials for the particles.

A crucial point is that we must treat the theory as a non-minimal, non-unitary theory with central charge \( c = 1 - \frac{6}{\ell(t+1)} \). We use this value of \( c \), denoted as \( c_{min} \), in the Virasoro algebra computations. Boundary conditions other than these twisted ones give rise to \( c_{eff} = c_{min} - 24h \), where the conformal dimension \( h \) is computed with respect to the \( c_{min} \) ground state. The usual sine-Gordon ground state (with \( c_{eff} = 1 \)) is then interpreted
as arising from an operator of negative dimension. With these definitions one finds for instance, using the Virasoro algebra and ζ-function regularization,

\[ \langle \int : T^2 : \rangle_h = \frac{1}{5760} (10c_{eff}^2 + 40c_{eff} + 4c_{min}), \quad (5.8) \]

\[ \langle \int : T^3 : \rangle_h = -\frac{1}{(24)^3} \left( c_{eff}^3 + 12c_{eff}^2 + \frac{192}{5} c_{eff} + \frac{32}{7} c_{min} + \frac{6}{5} c_{min}c_{eff} \right), \quad (5.9) \]

\[ \langle \int : T\partial^2 T : \rangle_h = -\frac{1}{(24)^3} \left( \frac{48}{5} c_{eff} + \frac{32}{7} c_{min} \right). \quad (5.10) \]

The \( c_{eff} \) result from the non-zero \( \langle L_0 \rangle_h \) on the cylinder.

Our strategy has been simply to compute numerically the values of \( \langle E^{2k+1} \rangle_{TBA} \) for various chemical potentials, denoting these by \( \langle E^{2k+1} \rangle_\alpha \). For sine-Gordon the chemical potentials are \( \mu_s = -\mu_a = i\alpha\pi T/t \); this results in \( c_{eff} = 1 - 24h \), where \( h = \frac{\alpha^2}{4t(t+1)} \).

The general TBA equations with fugacities are written out in [32]. Since the numerics are crucial to obtaining our result, we describe the methods briefly. The multi-function generalization of (4.4) is of the form

\[ \epsilon_a(\theta) = \nu_a(\theta) + \sum_b \int d\theta' \phi_{ab}(\theta - \theta') \ln(1 + \lambda_b e^{-\epsilon_b(\theta')/T}) \]

To find \( \epsilon_a \) numerically, we solve this iteratively. We guess the initial \( \epsilon_a; \epsilon_a = \nu_a \) usually works. Then one evaluates the right-hand side numerically by discretizing the integral; this gives the next guess for \( \epsilon_a \). Usually the iteration converges; occasionally one needs to use take a linear combination of \( x(\text{guess}) + (1-x)(\text{iteration}) \) for the next guess. More elaborate methods to improve convergence are described in [43]. Once this procedure has obtained \( \epsilon_a \) to the desired accuracy, the expression (5.4) for \( \langle E^n \rangle \) can then be numerically evaluated. We note that this numerical procedure for solving non-linear integral equations is generally far simpler to implement that those for solving non-linear differential equations.

The numerical results are rather interesting. As before, we have \( c_{min} = 1 - 6/t(t+1) \) and \( c_{eff} = 1 - 6\alpha^2/t(t+1) \). We find

\[ \langle E^3 \rangle_\alpha = f_1 (10c_{eff}^2 + 40c_{eff} + 4c_{min}) \]

\[ \langle E^5 \rangle_\alpha = g_1 (c_{eff}^3 + 12c_{eff}^2 + 40c_{eff} + 2c_{eff}c_{min} + \frac{16}{3} c_{min} + \frac{8}{21} c_{min}^2). \quad (5.11) \]

Moreover, we find that at least for \( \langle E^3 \rangle \), the prefactor takes reasonably simple values; to excellent numerical accuracy we have

\[ f_2 = \frac{7}{48}, \quad f_3 = \frac{\pi^2}{70}, \quad f_4 = \frac{1001}{7200}, \quad f_5 = \frac{2\pi^2}{143} \]

17
This is a hint that these numbers can be derived analytically from the TBA, but we have tried and failed to do so. We can make the amusing observation that \( \langle E^n \rangle \) for free fermions can be written in terms of polylogarithms, just like the dilogarithms written in the last section for \( \langle E \rangle \). We also note that the free energy of the impurity in the Kondo problem (see [1]) can be written as a sum of these conserved quantities, another hint of interesting hidden structure.

Hence we can fit our numerical results to (5.8) for \( n = 3 \) and to a linear combination of (5.9) and (5.10) for \( n = 5 \). We see that (5.8) is indeed proportional to \( \langle E^3 \rangle_{TBA} \) and that

\[
\langle E^5 \rangle \propto \left\langle \int : T^3 : + \left( \frac{c_{min} + 2}{12} \right) T \partial^2 T : \right\rangle_h.
\] (5.12)

After the numerical computation was completed we checked that (5.12) agrees with the conserved quantity at grade 5 in [44]. This is of course no surprise. Recall that in the classical sine-Gordon theory, the conserved quantities are precisely expressed as the sum of odd powers of the momenta: we simply check here that this result holds in the quantum theory as well. This is expected, but as far as we know, was checked only perturbatively so far [40,41]. Hence by massless scattering we recover not only the central charge and conformal weights of a conformal field theory, but also the conserved quantities which involve the Virasoro algebra itself, making the connection between the two points of view a little closer. One might wonder if there is an action of the Virasoro algebra on the massless particles.

6. Nontrivial left-right scattering

The massless Thirring model has no non-trivial LR scattering because \( L \) and \( R \) excitations are infinitely separated in the rapidity plane. On the other hand, the most interesting situations occur when the LR scattering is non-trivial. In that case, the theory is not scale invariant, and is described by two different conformal field theories in the UV and IR limits. To get such a situation in the Thirring model, we need to arrange for massless \( L \) and \( R \) excitations that both occur around the same region of rapidities. A way to do so is to choose a purely imaginary bare mass in (3.1) \( m_0 = -i|m_0| \). Indeed the result (3.2) still holds, so

\[
E = -i|m_0| \sum_i \cosh \xi_i.
\] (6.1)
If we restrict to the consideration of real energies we need $\text{Im}(\xi) = \pm \pi/2$. Then for $\xi = \pm i \frac{\pi}{2} + \nu$, $e = \pm |m_0| \sinh \nu$. One therefore expects the ground state to resemble figure 2 with the half lines

$$\text{Re}(\xi) < 0, \text{Im}(\xi) = \frac{\pi}{2}; \text{Re}(\xi) > 0, \text{Im}(\xi) = -\frac{\pi}{2}.$$  

(6.2)

filled up. Actually, because the theory is not free, the determination of the ground state is slightly more delicate — the interaction between the various Bethe ansatz roots must be considered. The choice of the cutoff is also important, as well as the sign of $g_0$. One finds typically that the picture (6.2) is almost correct, up to some exponentially decaying density on the other side of the half-lines. This produces therefore the necessary massless excitations around a common region $\xi = 0$. More details of this approach will be presented in [43].

An imaginary mass in the Thirring model is like imaginary prefactor in front of the cosine term in sine-Gordon, so we recover the situation discussed in sect. 2. As explained there the appearance of imaginary numbers is more natural than may appear at first sight. Just as the massive minimal models perturbed by $\phi_{13}$ are related to the ordinary sine-Gordon model [28], the massless flow between minimal models [21,23] is related to the model with imaginary mass. The non-unitarity of the imaginary-mass model does not exclude unitarity for a subsector (the perturbed massless minimal model). For more details see [12].

It is easy to generalize the TBA of section 4 to models with a a non-trivial $S_{LR}$. This time, the running central charge depends nontrivially on $M/T$. Its UV and IR values can be easily found. As discussed in the introduction, we expect that the IR conformal field theory is characterized by only $S_{LL}$ and $S_{RR}$, so its LR scattering should be trivial. One finds as before

$$c_{IR} = c_R + c_L = 2c_R = \frac{6}{\pi^2} L \left( \frac{1}{1 + y_0} \right),$$

(6.3)

where $y_0$ is the solution of $1/y_0 = (1 + 1/y_0)^{I_1}$ with $I_1 = \frac{1}{2\pi} \int \Phi_{LL}$. In the UV coupling between left and right particles has to be considered leading to

$$c_{UV} = \frac{6}{\pi^2} \left[ 2L \left( \frac{1}{1 + x_1} \right) - L \left( \frac{1}{1 + x_0} \right) \right],$$

(6.4)

where $1/x_1 = (1 + 1/x_1)^{I_1 + I_2}$ and $I_2 = \frac{1}{2\pi} \int \Phi_{LR}$.

A simple example of a massless integrable field theory is the flow from tricritical to critical Ising model. As discussed in [11] the spectrum consists of a right mover and a left
mover, the Goldstino resulting from spontaneously-broken supersymmetry. Because the IR conformal field theory is a free fermion, $S_{LL}$ and $S_{RR}$ must be trivial. The left-right scattering is given by

$$S_{RL}(\theta_R - \theta_L) = -\tanh \left( \frac{\theta_R - \theta_L}{2} - i\frac{\pi}{4} \right).$$  \tag{6.5}

The compatibility of left-right and right-left interchange of arguments of the wavefunction requires that

$$S_{LR}(\theta_L - \theta_R)S_{RL}(\theta_R - \theta_L) = 1,$$  \tag{6.6}

so here

$$S_{LR}(\theta_L - \theta_R) = \tanh \left( \frac{\theta_L - \theta_R}{2} - i\frac{\pi}{4} \right).$$  \tag{6.7}

In the IR limit (the Ising model) where $\theta_R - \theta_L \to \infty$ one checks that both matrix elements go to $-1$ as expected. With this $S$-matrix we have $I_1 = 0$ and $I_2 = 1/2$, so $x_0 = 1$ and $x_1 = \frac{\sqrt{5} - 1}{2}$. Using values of dilogarithms given in sect. 4 we find $c_{IR} = 1/2$ and $c_{UV} = 7/10$ as desired.

Following (6.5) and (6.6) notice that

$$S_{RL}(\theta)S_{RL}(-\theta) = -1,$$  \tag{6.8}

a result that must be carefully compared to the usual $S(\theta)S(-\theta) = 1$ for diagonal massive (or left-left or right-right) scattering.

### 7. Sine-Gordon model in a background field

In this section we discuss the sine-Gordon model in a background field coupled to the $U(1)$ soliton-number charge. In the traditional bare approach, this field would modify the Dirac or Fermi sea. In our approach, this makes it energetically favorable for physical particles to appear in the vacuum, even at zero temperature. In the sine-Gordon case with positive background field, only the negatively-charged particles appear in the vacuum. Their mutual scattering is diagonal, so the problem is technically easier than the finite-temperature TBA problem, where both kinds of particles appear in the thermodynamic state. We discuss both the cases $\lambda$ real and $\lambda$ imaginary, hence giving a (partially) non-perturbative treatment of the problem raised in sect. 2. For a more complete study see \cite{12}.
The sine-Gordon Hamiltonian with a constant external $U(1)$ gauge field $A_\mu$ is

$$H = \int dx \left[ \frac{1}{2} \Pi^2 + \frac{1}{2} (\partial \varphi)^2 + \lambda \cos \beta_{SG} \varphi \right] - QA,$$

(7.1)

where

$$Q = \int j_0 dx = \frac{\beta_{SG}}{2\pi} \int_{-\infty}^{\infty} \frac{\partial \varphi}{\partial x} dx$$

(7.2)

In the ordinary case $\lambda$ real, $Q$ is the integer-valued soliton topological charge, normalized so that the soliton (antisoliton) has $Q = 1 (-1)$. $A$ is a constant with the dimension of a length. We then consider the corresponding specific vacuum energy $E(A, \lambda)$ as a function of $A$. As before, we parametrize $\beta_{SG}^2 = \frac{8\pi}{4\pi + 1}$.

Before we turn to the scattering theory, it is worth looking at the action (7.1) from the perturbative (in $\lambda$) point of view. Dimensional arguments as well as explicit perturbative calculations show that the background field works as an infrared cutoff at scales $\sim A$ and therefore if $A \gg \lambda^{(1+t)/2}$ the theory is in the ultraviolet regime. As a leading $A \rightarrow \infty$ approximation we set $\lambda = 0$ in (7.1). This theory is the continuum limit of the XXZ model in a magnetic field, which has been studied in refs. [47-49]. As can easily be inferred from the action, it is a Gaussian model whose radius of compactification depends on $A$ (amusing finite-size corrections occur in the related XXZ model in a field due to commensurability problems between $R$ and the scale $A$). Redefining $\partial \varphi$ by a shift gives for the ground-state energy density

$$E(A, 0) = -\frac{\beta_{SG}^2}{8\pi^2} A^2.$$  

(7.3)

At any critical point, $E(A, 0)$ is proportional to $A^2$, since there is no other scale in the problem. The coefficient is proportional to the chiral anomaly (found from the $J_L J_L$ OPE) [50].

For $\lambda \neq 0$ the scaling argument shows that $E(A, \lambda)$ is a function of the dimensionless variable

$$\xi \equiv \lambda / A^{2/(1+t)}$$

and by parity has a perturbative expansion in $\xi^2$

$$E_{\text{pert}}(\xi) = -\frac{A^2}{\pi} \sum_{l=0}^{\infty} k_{2l} \xi^{2l}.$$  

(7.4)
One can use perturbed conformal field theory to derive

\[ k_0 = \frac{t}{1 + t} \]
\[ k_2 = \frac{\pi^2}{4} \left( \frac{2t}{1 + t} \right)^{2(t-1)/(t+1)} \frac{\Gamma \left( \frac{1 - t}{1 + t} \right)}{\Gamma \left( \frac{2t}{1 + t} \right)} \]

We expect the series (7.4) to have some finite radius of convergence \( \xi_0 \), defining therefore an analytic function \( \mathcal{E}_{\text{pert}}(\xi) \) at \( |\xi| < \xi_0 \). The perturbation theory is the same for \( \lambda \) real or imaginary, so (7.4) holds all around \( \xi = 0 \). However, the scattering theory depends crucially on the nature of \( \lambda \): for real \( \lambda \) the particles are massive, while for \( \lambda \) imaginary they are massless.

Consider first the unitary massive sine-Gordon model (\( \lambda \) real in (7.1)) and let \( m \) be the mass of the corresponding charged particle (soliton). As usual the on-mass-shell momenta \((E, p)\) are parameterized in terms of rapidity \( \theta \)

\[ E = m \cosh \theta \quad ; \quad p = m \sinh \theta \]

In the field (7.1) every soliton (antisoliton) acquires additional energy \(-A \) (\( A \)). It is clear that if \( A \gg m \) the state without particles is no longer the ground state. The true vacuum contains a sea of positively-charged solitons which fill all possible states inside some (\( A \)-dependent) “Fermi interval” \(-B < \theta < B \). The non-trivial scattering of the solitons certainly influences the structure of the ground-state sea. However, only one kind of particle is in the sea (this can be checked more completely \([30,45]\)); for \( A > 0 \) this is the soliton. The solitons scatter diagonally among themselves, the two-particle amplitude being \( a(\theta) \) from (3.11).

As in the finite-temperature case, we define the density of particles \( \rho \) and density of states \( \rho + \tilde{\rho} \). To obtain the ground state energy we minimize

\[ \mathcal{E}^{(Re)}(A) - \mathcal{E}^{(Re)}(0) = \int (m \cosh \theta - A) \rho(\theta) d\theta, \quad (7.5) \]

(the superscript \( Re \) is added to stress that currently we address the ordinary sine-Gordon model with real coupling \( \lambda \)) subject to the quantization

\[ 2\pi[\rho + \tilde{\rho}] = m \cosh \theta + \Phi \star \rho \quad (7.6) \]
where the kernel $\Phi$ follows from the soliton-soliton $S$-matrix and reads explicitly

$$\frac{\Phi(\theta)}{2\pi} = \frac{1}{2\pi i} \frac{d}{d\theta} \log a(\theta) = \int \frac{e^{i\omega \theta} \sinh \frac{\pi (t-1) \omega}{2}}{2 \cosh \frac{\pi \omega}{2} \sinh \frac{\pi \omega}{2}} d\omega. \quad (7.7)$$

The equations (7.5) and (7.6) and the equations for $B$ (given by minimizing the energy with respect to $B$) can be put in a more convenient form by defining the function $f(\theta)$ as

$$f(\theta) = A - m \cosh \theta + \int_{-B}^{B} d\theta' \Phi(\theta - \theta') f(\theta'), \quad (7.8)$$

where this equation is good only for $|\theta| < B$. Replacing $A - m \cosh \theta$ in (7.5) with this and using (7.6) one finds that

$$\mathcal{E}(A) = -\frac{m}{2\pi} \int_{-B}^{B} d\theta \cosh \theta f(\theta). \quad (7.9)$$

The boundary conditions $f(\pm B) = 0$ determine $B$.

We can understand the meaning of the function $f$ as follows. Define $\epsilon^+$ as the energy of particle excitations above the ground state, and $\epsilon^-$ as the energy of holes. By this definition, $\epsilon^+ \geq 0$ and $\epsilon^- \leq 0$. A variation of the energy is thus

$$\delta \mathcal{E}^{(Re)}(A) = \int (m \cosh \theta - A) \delta \rho(\theta) d\theta = \int \epsilon^+ \delta \rho - \epsilon^- \delta \tilde{\rho}. \quad (7.10)$$

Using (7.6) to reexpress $\delta \tilde{\rho}$ as a function of $\delta \rho$ we find

$$\delta \mathcal{E}^{(Re)}(A) = \int \left( \epsilon^+ + \epsilon^- - \frac{\Phi}{2\pi} * \epsilon^- \right) \delta \rho \quad (7.11)$$

so by comparing (7.10) and (7.11) we find

$$m \cosh \theta - A = \epsilon^+ + \epsilon^- - \frac{\Phi}{2\pi} * \epsilon^- . \quad (7.12)$$

Using this in (7.5) gives

$$\mathcal{E}^{(Re)}(A) - \mathcal{E}^{(Re)}(0) = \frac{m}{2\pi} \int d\theta \left[ \cosh \theta \epsilon^- + \epsilon^+ \rho - \epsilon^- \tilde{\rho} \right]. \quad (7.13)$$

In the ground state $\rho(\theta) = 0$ when $\epsilon^+(\theta) > 0$ and $\tilde{\rho}(\theta) = 0$ when $\epsilon^-(\theta) < 0$. This means that the last two terms vanish, and we obtain (7.4), where $f = -\epsilon^-$. From a formal point of view one may interpret the whole scattering approach and the resulting system (7.7)–(7.9) as a way of summing up the perturbative expansion (7.4) at
real $\xi$ and going beyond the radius of convergence along $\lambda$ real. One would like a similar tool to sum up (7.4) at $\lambda$ purely imaginary. To do so we make a guess inspired by the study of the massive Thirring model and section 6. We assume that in the sine-Gordon model with imaginary coupling or Thirring model with imaginary mass, the number of species doubles, so now we have a pair of left and right massless particles. We also assume that the $LL$ (identical to $RR$) and the $LR$ scattering are nontrivial. The energy spectrum (1.2) is gapless. Turn on the background field $A > 0$ as before. The positively charged particles are always excited in the ground state; we assume as in the massive case that they are the only particles contributing to the thermodynamics. Now the right- and left-movers fill respectively the semi-infinite Fermi intervals $-\infty < \theta < B$ and $-B < \theta < \infty$ with some Fermi boundary $B \sim \log A/M$. Again it is a straightforward Bethe ansatz exercise to derive the following system of integral equations

\[ QA - \frac{Me^\theta}{2} = f_R(\theta) - \int_{-\infty}^{B} \Phi_{LL}(\theta - \theta') f_R(\theta') \frac{d\theta'}{2\pi} - \int_{-B}^{\infty} \Phi_{RL}(\theta - \theta') \rho_L(\theta') \frac{d\theta'}{2\pi}; \]

\[ QA - \frac{Me^{-\theta}}{2} = f_L(\theta) - \int_{-B}^{\infty} \Phi_{LL}(\theta - \theta') f_L(\theta') \frac{d\theta'}{2\pi} - \int_{-\infty}^{-B} \Phi_{RL}(\theta - \theta') f_R(\theta') \frac{d\theta'}{2\pi}. \]

The positive functions $f_R(\theta)$ and $f_L(\theta)$ are defined in the Fermi intervals $\infty < \theta < B$ and $-B < \theta < \infty$ respectively, and are restricted by the boundary conditions

\[ f_R(B) = f_L(-B) = 0 \] (7.15)

Finally, the ground state energy, which we now call $\mathcal{E}^{(Im)}(A)$, is evaluated as follows

\[ \mathcal{E}^{(Im)}(A) - \mathcal{E}^{(Im)}(0) = -\frac{M}{2\pi} \int_{-\infty}^{B} e^\theta f_R(\theta) d\theta, \]

where we have taken into account the obvious symmetry $f_R(\theta) = f_L(-\theta)$.

In eqn. (7.14) we introduced the $U(1)$ charges $\pm Q$ of the massless particles, which, with the knowledge assumed in these lectures, we cannot fix in advance. It is possible to carry out the computation with undetermined kernels, and fix them at the end by requiring the result to provide analytic continuation of the perturbative series to $\lambda$ imaginary. To save time, let us give the answer and justify it. One has for $\Phi_{LL}(\theta)$ the same expression as
in the massive case, but with a shift $t \rightarrow t - 1$. For $\Phi_{RL}$ we have the same, with a further shift $\theta \rightarrow \theta + i\frac{\pi}{2}(t - 1)$. This results in

$$
\frac{\Phi_{LL}(\theta)}{2\pi} = \frac{1}{2\pi i} \frac{d}{d\theta} \log a_{LL}(\theta) = \int \frac{e^{i\omega\theta} \sinh \frac{\pi(t-2)\omega}{2}}{2 \cosh \frac{\pi\omega}{2} \sinh \frac{\pi(t-1)\omega}{2}} d\omega,
$$

(7.17)

$$
\frac{\Phi_{RL}(\theta)}{2\pi} = \frac{1}{2\pi i} \frac{d}{d\theta} \log a_{RL}(\theta) = -\int \frac{e^{i\omega\theta} \sinh \frac{\pi\omega}{2}}{2 \cosh \frac{\pi\omega}{2} \sinh \frac{\pi(t-1)\omega}{2}} d\omega.
$$

Although this new BA system (7.14)–(7.16) has a rather different form from that of eqns. (7.7)–(7.9), it is easy to relate the two in the UV region $A \rightarrow \infty$ where in both systems $B \rightarrow \infty$. In this limit the right and left Fermi intervals have a broad overlap at $-B < \theta < B$. Near say the right Fermi boundary $\theta \sim B$ (where the main contribution to (7.16) comes from) we can forget about the left one and solve (7.14) for $f_L(\theta)$ by the Fourier transform with $B \rightarrow \infty$. The resulting equation for $f_R(\theta)$ is

$$
rQA - \frac{Mc^2}{2} = f_R(\theta) - \int_{-\infty}^{B} \Phi(\theta - \theta') f_R(\theta') \frac{d\theta'}{2\pi},
$$

(7.18)

where in terms of the Fourier transforms

$$
\tilde{\Phi}(\omega) = \tilde{\Phi}_{LL}(\omega) + \frac{[\tilde{\Phi}_{RL}(\omega)]^2}{1 - \tilde{\Phi}_{LL}(\omega)} = \frac{\sinh \frac{\pi(t-1)\omega}{2}}{2 \cosh \frac{\pi\omega}{2} \sinh \frac{\pi\omega}{2}},
$$

(7.19)

(compare with eqn. (7.7)) and

$$
r = 1 + \frac{\tilde{\Phi}_{RL}(0)}{1 - \tilde{\Phi}_{LL}(0)} = \frac{t - 1}{t}
$$

It coincides precisely with the corresponding limit $B \rightarrow \infty$ of eq. (7.8) provided

$$
Q = \frac{t}{t - 1},
$$

$$
M = m
$$

(7.20)

We pause to discuss the logic. In the UV limit $\lambda = 0$ and we have a free boson after the shift $\partial \varphi \rightarrow \partial \varphi + A$. We can perturb this fixed point by $\lambda$ real or $\lambda$ purely imaginary. The first case is the usual sine-Gordon in the regime where there are only solitons in the scattering theory. The theory is massive, so the IR fixed point is the trivial one. The second case is like the sine-Gordon model with imaginary $\lambda$, where there are only $L$ and $R$ “solitons”. As discussed in previous sections the theory is massless and the IR fixed point
is now nontrivial. Using known results about flow between minimal models (for example the large-$t$ expansion discussed in sect. 2), we expect the IR fixed point to exhibit the shift $t 	o t - 1$. In the IR limit, the $LR$ scattering becomes negligible, hence the form of $LL$ and $RR$ scattering above. For $LR$ scattering we require that in the opposite UV limit, the two scattering theories look the same, which is natural since they hold in two regimes connected at the UV fixed point.

The expansion (7.4) is obtained by analyzing our two sets of background field energy equations using a generalized Weiner-Hopf technique [16,30]. This has been described in detail in [12], so we do not present the full calculation here. Instead, we will explain how to extract the relevant information from the kernels. The technique relies on the usual Weiner-Hopf trick of dividing the Fourier transforms of the kernels into a product of two pieces, the first of which has no poles or zeroes in the lower half plane and the second none in the upper half plane. Defining $1/K_+(\omega)K_-(\omega) \equiv 1 - \tilde{\Phi}(\omega)$, one finds that expressions of the form

$$
\oint h(\omega)\frac{g(\omega)e^{2i\omega B}}{(\omega - i)^2} d\omega; \quad g(\omega) \equiv \frac{K_+(\omega)}{K_-(\omega)}
$$

(7.21)

occur regularly in the analysis; the contour covers the upper half plane. The function $h(\omega)$ is different depending on where we are in the analysis, but it is analytic in the upper half plane. First we discuss the massive case. The poles in the contour are at $\omega = i$ and at the zeros of $1 - \tilde{\Phi}$, which are at $\omega = 2in/(t + 1)$. The pole at $\omega = i$ results in the bulk contribution $E(0)$. Ignoring the bulk term, (7.21) can be written as a series in $\exp(-4B/(t + 1))$. In particular, the boundary condition results in an equation

$$
\frac{M}{A}e^B = \text{const} + \sum_n h_n g_n e^{-4nB/(t+1)}
$$

where $h_n$ and $g_n$ are the residues of $h(\omega)/(\omega - i)^2$ and $g(\omega)$, respectively. The $h_n$ themselves also obey an equation of this form, so for large $A/M$, we can write $e^B$ and $h_n$ each as a series in $(A/M)^{-4/(t+1)}$. The energy is also given by a term like (7.21), so it too must be a series in $(A/M)^{-4/(t+1)}$ as in (7.4):

$$
\mathcal{E}^{Re}(A, M) = \mathcal{E}^{Re}(0) - \frac{A^2}{\pi} \sum_{n=0}^{\infty} k_n \left( \frac{M}{A} \right)^{4n/(t+1)}.
$$

(7.22)
This gives the result \((7.4)\) for ordinary sine-Gordon. For the imaginary coupling, we must first rewrite the equations \((7.14)\) in Weiner-Hopf form. The general result is that

\[
A \rightarrow Q(1 + \frac{\Phi_{LR}(0)}{1 - \Phi_{LL}(0)})A
\]

\[
\frac{1}{K_+(\omega)K_-(\omega)} = 1 - \Phi_{LL} - \frac{\Phi_{LR}^2}{1 - \Phi_{LL}}
\]

\[
g(\omega) = \frac{K_+(\omega)}{K_-(\omega)} \frac{\Phi_{LR}}{1 - \Phi_{LL}}.
\]

The \(K_+\) and \(K_-\) obtained for the massless case are exactly the same as the ones obtained above for the massive one. The extra piece in the above expression for \(g(\omega)\) is \(- \sinh(\pi \omega/2)/\sinh(\pi t \omega/2)\) here. It results in no other additional poles in the contour because of the zeros in \(g\). Its only effect is to change \(g_n\) to \((-1)^n g_n\) (again ignoring the bulk piece). Since the \(h_n\) above are not changed, we then find that the series for the massless flow is exactly the same as in ordinary sine-Gordon, except that the signs of every other term are different:

\[
\mathcal{E}^{(Im)}(A,M) = \mathcal{E}^{(Im)}(0) - \frac{A^2}{\pi} \sum_{n=0}^{\infty} (-)^n k_n \left( \frac{M}{rQA} \right)^{4n/(1+t)}
\]

with precisely the same coefficients \(k_n\) as in expansion \((7.22)\).

We conclude that up to the known bulk vacuum energy contributions the massless BA system \((7.14)-(7.16)\) gives the correct analytic continuation of the massive one \((7.7)-(7.9)\) to purely imaginary \(\xi\), providing \((7.20)\) holds. In particular, the low-temperature mass scale \(M\) is equal to the high-temperature scale \(m\).

This entire discussion can presumably be put on firmer ground by solving the Thirring model with imaginary mass using the traditional Bethe ansatz approach [45].

8. Conclusions

The examples presented here have mainly been the sine-Gordon and Thirring models, with and without a background field. These calculations can be extended to the truncated (RSOS) cases, the latter situation being completely unitary. See the references [12,31].

It seems that the physics of massless flows is intimately related with the one of symmetry breaking and that the conformally-invariant IR fixed points are generally some sort of Goldstone phase. Two of the simplest cases, the flow from tricritical to critical Ising and
the flow from dilute to dense polymers have to do respectively with $N = 1$ and $N = 2$ spontaneous SUSY breaking (the latter being possible because of the non-unitarity). Moreover, as already mentioned, the sine-Gordon model with imaginary coupling describes the flow from critical to low temperature $O(n)$ model (with $n = 2 \cos \frac{\pi}{T}$). The Mermin-Wagner theorem preventing spontaneous breaking of continuous symmetry does not apply to the cases $n$ non-integer. As a consequence, $O(n)$ models in two dimensions $-2 < n < 2$ have a low-temperature phase which is massless and has properties reminiscent of a Goldstone phase [51] (for instance they qualitatively agree with what can be deduced from the $\epsilon$-expansion in higher dimensions, extended formally to $D = 2$).

Some important questions remain to be addressed. For instance, can one seriously describe conformal field theories using massless scattering (reconstruct Green functions using form factors)? As explained above, massless scattering is a sort of perturbation of the IR fixed point. Can one, using it (i.e. probably using the conserved quantities) give a meaning to the conformal perturbation theory of an IR fixed point by an irrelevant operator?

Acknowledgments: These lectures were presented by H.S. who thanks the organizers and the students at Trieste for many interesting questions. We have benefited a great deal by interacting with our collaborators K. Intriligator, N. Reshetikhin, S. Skorik and Al.B. Zamolodchikov. P.F. and H.S. were supported by the Packard foundation and DOE grant No. DE-FG03-84ER40618.
**Figure Captions**

Figure 1: The structure of the ground state in the massive Thirring model. Left and right massless excitations are observed in the limit $\xi \to \pm \infty$. For instance, a pair of left and right solitons is obtained by making two holes for $|\xi| \gg 1$.

Figure 2: Schematic structure of the ground state for imaginary $m_0$. 

References

[1] N. Andrei, K. Furuya, and J. Lowenstein, Rev. Mod. Phys. 55 (1983) 331; A.M. Tsvelick and P.B. Wiegmann, Adv. Phys. 32 (1983) 453.
[2] I. Affleck, in Les Houches 1988 Fields, Strings, Critical Phenomena, ed. by E. Brezin and J. Zinn-Justin, North Holland.
[3] L.D. Faddeev and L.A. Takhtajan, Phys. Lett. 85A (1981) 375.
[4] F. Smirnov, Th. Math. Phys. 60 (1984) 363.
[5] V.E. Korepin, Comm. Math. Phys. 86 (1982) 391.
[6] S. Parke, Nucl. Phys. B177 (1980) 166.
[7] A.B. Zamolodchikov, Al.B. Zamolodchikov, Nucl. Phys. B379 (1992) 602.
[8] A.B. Zamolodchikov and Al.B. Zamolodchikov, Ann. Phys. 120 (1979) 253.
[9] A.B. Zamolodchikov, Adv. Stud. Pure Math. 19 (1989) 1.
[10] G. Mussardo, Phys. Rep. 218 (1992) 215.
[11] Al.B. Zamolodchikov, Nucl. Phys. B358 (1991) 524.
[12] P. Fendley, H. Saleur and Al.B. Zamolodchikov, “Massless Flows I” and “Massless Flows II”, hepth #9304050 and 9304051, to appear in Int. J. Mod. Phys. A
[13] P. Fendley, Phys. Rev. Lett. 71 (1993) 2485.
[14] V. Fateev, E. Onofri and Al.B. Zamolodchikov, “The sausage model (integrable deformations of O(3) sigma model)”, to appear in Nucl. Phys. B.
[15] P. Fendley and K. Intriligator, “Exact $N=2$ Landau-Ginzburg Flows”, hepth #9307166, to appear in Nucl. Phys. B.
[16] N. Reshetikhin, J. Phys. A24 (1991) 3299.
[17] J. Carmelo, P. Horsch, P.A. Bares and A.A. Ovchinnikov, Phys. Rev. B44 (1991) 9967.
[18] F. Essler and V. Korepin, “Scattering Matrix and Excitation Spectrum of the Hubbard Model”, ITP-SB-93-40.
[19] B. Nienhuis, Phys. Rev. Lett. 49 (1982) 1062.
[20] J.V. Jose, L.P. Kadanoff, S. Kirkpatrick, D.R. Nelson, Phys. Rev. B16 (1977) 217.
[21] A.B. Zamolodchikov, JETP Lett. 43 (1986) 730.
[22] C. Itzykson and H. Saleur, J. Stat. Phys. 48 (1987) 449.
[23] A. Ludwig and J. Cardy, Nucl. Phys. B285 (1987) 687.
[24] Al.B. Zamolodchikov, “Resonance factorized scattering and roaming trajectories”, Ecole Normale preprint ENS-LPS-355
[25] H. Bergknoff and H.B. Thacker, Phys. Rev. D19 (1979) 366.
[26] V.E. Korepin, Th. Math. Phys. 41 (1979) 953.
[27] L. Faddeev, E. Sklyanin and L. Takhtajan, Th. Math. Phys. 40 (1979) 688.
[28] T. Eguchi and S.K. Yang, Phys. Lett. B224 (1989) 373; T. Hollowood and P. Mansfield, Phys. Lett. 226B (1989) 73; M.T. Grisaru, A. Lerda, S. Penati and D. Zanon, Phys.
Lett. B234 (1990) 88; N. Reshetikhin and F. Smirnov, Comm. Math. Phys. 131 (1990) 157.

[29] V. Korepin, Comm. Math. Phys. 76 (1980) 165.
[30] G. Japradize, A. Nersesyan and P. Wiegmann, Nucl. Phys. B230 (1984) 511; P. Wiegmann, Phys. Lett. B152 (1985) 209.
[31] N. Reshetikhin and H. Saleur, “Lattice regularization of massive and massless integrable field theories”, preprint USC-93-020, hepth #9309135.
[32] P. Fendley and H. Saleur, Nucl. Phys. B388 (1992) 609.
[33] D. Bernard and A. Leclair, Nucl. Phys. B340 (1990) 721; G. Felder and A. LeClair, Int. J. Mod. Phys A7 (1992) 239.
[34] V. Pasquier and H. Saleur, Nucl. Phys. B330 (1990) 523.
[35] C.N. Yang and C.P. Yang, J.Math. Phys. 10 (1969) 1115.
[36] Al.B.Zamolodchikov, Nucl. Phys. B342 (1991) 695.
[37] H.W. Blote, J.L. Cardy and M.P. Nightingale, Phys. Rev. Lett. 56 (1986) 742; I. Affleck, Phys. Rev. Lett. 56 (1980) 746.
[38] J. Cardy and G. Mussardo, Phys. Lett. 225B (1989) 275.
[39] P. Fendley and K. Intriligator, Nucl. Phys. B372 (1992) 553.
[40] P.P.Kulish and E.R. Nisimov, Th. Math. Phys. 29 (1976) 161.
[41] V. Korepin and L.D. Faddeev, Th. Math. Phys. 25 (1975) 147.
[42] I. Gradshtein and I. Rishnik, Table of Integrals, Series and Products (Academic Press, 1980).
[43] T. Klassen and E. Melzer, Nucl. Phys. B350 (1990) 635.
[44] R. Sasaki and I. Yamanaka, Adv. Stud. in Pure Math. 16 (1988) 271.
[45] S. Skorik and H. Saleur, work in progress.
[46] M. Ganin, Izv. Vuzov (Math) 33 (1963) 31.
[47] N.M. Bogoliubov, A. Izergin and V. Korepin, Nucl. Phys. B275 (1986) 687.
[48] N.M. Bogoliubov, A. Izergin and N. Reshetikhin, J.Phys. A 20 (1987) 5361.
[49] F. Woynarovich, H.P. Eckle and T.T. Truong, J.Phys. A22 (1989) 4027.
[50] P. Fendley and K. Intriligator, “Central charges without finite-size effects”, to appear in Phys. Lett. B.
[51] H. Saleur, Phys. Rev. B35 (1987) 3657.