Spin$^C$ Dirac operators over the flat 3-torus

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Abstract

We determine spectrum and eigenspaces of some families of Spin$^C$ Dirac operators over the flat 3-torus. Our method relies on projections onto appropriate 2-tori on which we use complex geometry.

Furthermore we investigate those families by means of spectral sections (in the sense of Melrose/Piazza). Our aim is to give a hands-on approach to this concept. First we calculate the relevant indices with the help of spectral flows. Then we define the concept of a system of infinitesimal spectral sections which allows us to classify spectral sections for small parameters $R$ up to equivalence in $K$-theory. We undertake these classifications for the families of operators mentioned above.

Our aim is therefore twofold: On the one hand we want to understand the behaviour of Spin$^C$ Dirac operators over a 3-torus, especially for situations which are induced from a 4-manifold with boundary $T^3$. This has prospective applications in generalised Seiberg-Witten theory. On the other hand we want to make the term “spectral section”, for which one normally only knows existence results, more concrete by giving a detailed description in a special situation.

Keywords:
Spin$^C$ Dirac operator; 3-torus; spectral section

Subject classification:
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1 Introduction

In the study of smooth 4-manifolds, especially in the context of (generalised) Seiberg-Witten theory, it would be nice to understand Spin$^C$ Dirac operators
which are induced on the boundary of a compact 4-manifold.

Manifolds with boundary $T^3$ where already studied in this context by [6]. But for generalized Seiberg-Witten theories, also families of operators in non-trivial Spin$^C$ structures become important. Therefore, we undertake a detailed study for some of these families. We now describe the object of investigation:

For every Spin$^C$ structure on $T^3 = \mathbb{R}^3 / \mathbb{Z}^3$ we analyse the family of Dirac operators given by connections $\nabla^K + i\alpha$; here $\nabla^K$ is a fixed background connection (to be constructed below) for an appropriate line bundle $K$ and $\alpha$ comes from the parameter space of closed one-forms.

Our first aim is to determine the spectrum and an orthogonal eigenbasis for these operators. Our strategy is as follows:

1. We write the 3-torus as $S^1$ bundle over a 2-torus (determined by the Spin$^C$ structure).
2. We equip the 2-torus with a complex structure and choose appropriate holomorphic line bundles.
3. We use complex geometry and methods from [1].
4. We combine the calculated terms with exponential functions to get the desired result.

The calculations above will help us to access our second aim: The construction of spectral sections.

For a lattice $\ell \subset H^1(T^3; \mathbb{Z}) \subset H^1(T^3; \mathbb{R})$ look at the family of operators parametrised by $B = (\ell \otimes \mathbb{R})/\ell$. Since we know the concrete spectrum we can calculate all spectral flows in this torus which gives us direct access to the index in $K^1(B)$. By [1] section 2 the vanishing of this index corresponds to the existence of spectral sections.

For small parameters $R$ we give a classification of all spectral sections up to equivalence in $K$-theory.

**Remark 1.** If $\iota : T^3 \hookrightarrow M$ is the boundary of a Spin$^C$ 4-manifold $M$ and $\ell$ is chosen to be a subset of $\iota^*(H^1(M; \mathbb{Z}))$, then one can show that our family of operators is a boundary family in the sense of [4]; this guarantees the existence of spectral sections in this case but does not lead to concrete constructions of them.
2 Definitions

We take $T^3 := \mathbb{R}^3/\mathbb{Z}^3$ to be the flat 3-torus. We identify the first and second cohomology groups with each other by the Hodge star operation. Both of them will be identified with $\mathbb{Z}^3$ or $\mathbb{R}^3$ through the standard (positively oriented) basis $dx_1, dx_2, dx_3$ of $\mathbb{T}^3$.

The trivial Spin structure induces a Spin$^C$ structure with associated bundle $\mathbb{H} = T^3 \times \mathbb{H}$. Here $\mathbb{H} = \text{span}\{e_0, e_1, e_2, e_3\}$ denotes the space of quaternions. It is considered as a complex vector space by left multiplication with $i = e_1$ and as a left-quaternionic vector space by inverse right multiplication.

Now the Spin$^C$ structures can be canonically identified with elements $\hat{k} \in H^2(T^3; \mathbb{Z})$ (for a general explanation of Spin$^C$ structures and their associated bundles see e.g. [5]). For every such element we choose a Hermitian line bundle $K$ with $c_1(K) = \hat{k}$ and a unitary background connection $\nabla^K$; possible choices and constructions will be detailed in the subsequent sections. Then the Spin$^C$ structure $\hat{k}$ has the associated bundle $\mathbb{H} \otimes K$.

For each $K$ and closed one-form $\alpha$ we get a Spin$^C$ Dirac operator

$$D^K_\alpha : \Gamma(\mathbb{H} \otimes K) \to (\mathbb{H} \otimes K)$$

for the connection $\nabla^K + i\alpha$.

These operators will be analysed in the subsequent sections.

3 Spectrum and Eigenbasis

We distinguish two main cases.

3.1 Nontrivial Spin$^C$ structure

We write $\hat{k} = h \cdot k$ with $k \in \mathbb{Z}^3$ and maximal $h \in \mathbb{Z}_+$. Let $W$ be the plane in $\mathbb{R}^3$ orthogonal to $k$ and $\pi_k$ the orthogonal projection. By taking quotients we get a map $\pi^\Lambda_K : T^3 \to T_\Lambda := W/\Lambda$ with $\Lambda = \pi_k(\mathbb{Z}^3)$.

Let $w_1, w_2$ be the basis of a fundamental parallelogram in $\Lambda$. We take $c^i \in [0, 1)$, $i = 1, 2$, with $w_i - c^i \cdot k \in \mathbb{Z}^3$.

Lemma 2. The map $\pi^\Lambda_K : T^3 \to T_\Lambda$ determines a trivial $\mathbb{R}/\mathbb{Z}$-bundle with trivialisation:

$$\begin{align*}
\left[\chi_1 w_1 + \chi_2 w_2 + \chi k\right] &\mapsto \left[\chi_1 w_1 + \chi_2 w_2; \left\langle c^1 \chi_1 + c^2 \chi_2 + \chi \right\rangle\right] \quad (1)
\end{align*}$$
Proof. Direct calculation.

We give $T_\Lambda$ the induced metric and orientation and choose a Hermitian line bundle $L$ over it with $c_1(L) = h$ (in the standard identification of $H^2(T_\Lambda; \mathbb{Z})$ with $\mathbb{Z}$). Furthermore, we equip the bundle with an arbitrary unitary connection $\nabla^L$.

**Definition 3.** We define $K := \pi^{-1}_K(L)$ and $\nabla^K := \pi^{-1}_K(\nabla^L)$. Then we have $c_1(K) = \hat{k}$.

### 3.1.1 Working on $T_\Lambda$

We now look at the corresponding problem on $T_\Lambda$. For each (positive) Chern class $h$, we have an associated bundle $\mathbb{H} \otimes L$ over $T_\Lambda$. Then each closed one-form $\alpha_\Lambda$ over $T_\Lambda$ defines a Dirac operator

$$D^L_{\alpha_\Lambda} : \Gamma(\mathbb{H} \otimes L) \to (\mathbb{H} \otimes L)$$

We give $W$ an arbitrary complex structure and scale everything so that we work on $\mathbb{C}/\{1, \tau\}$ with $\text{im} \tau > 0$. Now we can equip $L$ with a holomorphic structure; we choose it so that $\nabla^L + i\alpha_\Lambda$ becomes the Chern connection of the holomorphic bundle.

This specifies a problem for twisted Dirac operators on a Riemann surface. We use the results of [1, section 5.2], where the eigenspaces of $D^L_{\alpha_\Lambda}$ are described in terms of holomorphic sections.

The eigenspaces can be made explicit using theta functions. A detailed discussion of all calculations and identifications can be found in [3, section 2.c]. The result is the following:

**Lemma 4.** We can explicitly construct a basis of orthogonal eigensections $\sigma_m$, $m \in \mathbb{Z}$, for $D^L_{\alpha_\Lambda}$ with respective eigenvalues

$$\mu_m := \text{sgn} m \sqrt{2\pi h \|k\| \left\lfloor \frac{|m|}{h} \right\rfloor}.$$ 

The eigenvalues are independent of $\alpha_\Lambda$.

### 3.1.2 An eigenbasis for $(D^K_{\alpha})^2$

**Remark 5.** By a standard gauging argument, we can reduce the problem of finding spectrum and eigenspaces from closed one-forms to harmonic one-forms. So from now on we assume $\alpha \in H^1(T^3; \mathbb{R}) \cong \mathbb{R}^3$. 

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We now look at the map $s_l \circ \text{tri}, \ l \in \mathbb{Z}$, where $s_l : \mathbb{R}/\mathbb{Z} \to S^1$ is defined to be $t \mapsto \exp(2\pi it)$ and tri is the map from (1). Its exterior derivative is given by:

$$d(s_l \circ \text{tri}) = 2\pi i (s_l \circ \text{tri}) \ (c^1, c^2, 1).$$

We now want to separate this form into its parallel and orthogonal part with respect to $W$:

$$d(s_l \circ \text{tri}) = 2\pi i (s_l \circ \text{tri}) \cdot (\omega^l \parallel + \omega^l \perp).$$

In the same way we split $\alpha = \alpha^l \parallel + \alpha^l \perp$.

We set $\alpha^l : = \alpha^s + 2\pi \omega^l \parallel$ and use Lemma 4 to determine a basis of sections for $\Gamma(H \otimes L)$ which we call $\sigma^l_m$, $m \in \mathbb{Z}$.

The parameter $\omega^l \parallel$ becomes necessary for our construction since the bundle $T^3 \to T$ is trivial but its metric differs from the orthogonal product $T \times S^1$.

**Definition 6.** Define

$$\hat{\sigma}_{l,m}(v) := (s_l \circ \text{tri})(v) \cdot \pi^* k(\sigma^l_m)(v)$$

This can be interpreted as a combination of a basis of the Dirac operator over $S^1$ with bases over $T$.

**Definition 7.** Let $\lambda_l : = (2\pi l + \langle k, \alpha \rangle)/\|k\|$, where $\langle , \rangle$ means the standard scalar product of $\mathbb{R}^3$ (or, interpreted differently, the evaluation of $k \cup \alpha$ at the orientation class).

**Theorem 8** (Eigenbasis for $(D^K_\alpha)^2$). The set $\{\hat{\sigma}_{l,m} \ | \ l, m \in \mathbb{Z}\}$ forms an orthogonal basis of eigensections for $(D^K_\alpha)^2$ with the respective eigenvalues $\lambda_l^2 + \mu_m^2$.

**Proof.** Applying $D^K_\alpha$ twice and using the definition of $\omega^l$, we see that these sections are indeed eigensections for the given eigenvalues. With a standard calculation (see [3, p.45]), we conclude that the set span $\{\hat{\sigma}_{l,m} \ | \ l, m \in \mathbb{Z}\}$ is dense in the space of $L^2$-sections. The orthogonality can be deduced from the orthogonality of the $\sigma^l_m$ by using the fact that a change of $\alpha_\perp$ changes the spectrum but fixes $\sigma^l_m$. \hfill $\square$

**3.1.3 An eigenbasis for $D^K_\alpha$**

Theorem 8 gives a quadratic equation for $D^K_\alpha$. Furthermore, we know that the Dirac operator on $T$ is graded, so the bases $\sigma^l_m$ split into $\sigma^l_m + \sigma^l_m$. Together this leads us to the following definition:
Definition 9. Let
\[
\sigma_{l,m}^\pm := (s_l \circ \text{tri}) \cdot \left( \left( \lambda_l + \mu_m \pm \sqrt{\lambda_l^2 + \mu_m^2} \right) \pi_\mathcal{K}^\ast (\sigma_{l,m}^+) \right.
\]
\[
+ \left( -\lambda_l + \mu_m \pm \sqrt{\lambda_l^2 + \mu_m^2} \right) \pi_\mathcal{K}^\ast (\sigma_{l,m}^-) \Bigg)\]
\[
\sigma_{l,m}^0 := \hat{\sigma}_{l,m}
\]
and
\[
\nu_{l,m}^\pm := \pm \sqrt{\lambda_l^2 + \mu_m^2}
\]
\[
\nu_{l,m}^0 := \left\{ \begin{array}{ll}
\lambda_l & \text{for } 0 \leq m \leq h - 1 \\
\mu_m & \text{otherwise}
\end{array} \right.
\]

From this set of vectors we have to choose a subset of nonzero vectors whose span is dense.

Theorem 10. We get an orthogonal eigenbasis of $\mathcal{D}_\alpha^K$ by
\[
\left\{ \sigma_{l,m}^\pm \mid (l, m) \in \mathbb{Z}^2 \text{ with } \lambda_l \neq 0 \text{ and } m \geq h \right\}
\]
\[
\cup \left\{ \sigma_{l,m}^0 \mid (l, m) \in \mathbb{Z}^2 \text{ with } \lambda_l = 0 \text{ or } 0 \leq m \leq h - 1 \right\},
\]
which will be written as $M_{l,m}^\pm \cup M_{l,m}^0$. The respective eigenvalues are $\nu_{l,m}^{+/0/-}$.

Proof. We check that all these vectors are nonzero and belong to the defined eigenspaces.

From the construction in [1] we know that $\sigma_{l,m}^- = \sigma_{l,m}^+ + \sigma_{l,m}^-$ implies $\sigma_{h-m-1}^- = \sigma_{h-m}^+ - \sigma_{h-m}^-$. Therefore, we have the $\mathcal{D}_\alpha^K$-invariant subspaces
\[
\text{span} \left\{ \hat{\sigma}_{l,m}, \mathcal{D}_\alpha^K \hat{\sigma}_{l,m} \right\} = \text{span} \left\{ \hat{\sigma}_{l,m}, \hat{\sigma}_{h,m-1} \right\}
\]
They can be used to prove the orthogonality and density of the constructed sections. \hfill \Box

3.2 Trivial Spin$^C$ structure

We look at $\mathcal{D}_\alpha$ on $\Gamma(\mathbb{H}) = \Gamma(\mathbb{C}^2)$ for the standard connection $\nabla^K$. \hfill 6
Let
\[ \sigma_b(x_1, x_2, x_3) := \exp \left( 2\pi i (b_1 x_1 + b_2 x_2 + b_3 x_3) \right) \]

Then we get the basis of sections:
\[ \text{span} \left\{ \sigma_b^+ = (\sigma_b, 0) \mid b \in \mathbb{Z}^3 \right\} \cup \left\{ \sigma_b^- = (0, \sigma_b) \mid b \in \mathbb{Z}^3 \right\} \]

Define \( \beta = \alpha + 2\pi b \).

We use the classical methods of [2] to determine:

**Theorem 11.** We get an orthogonal eigenbasis for \( D_\alpha \) as
\[ \left\{ \|\beta\| \sigma_b^+ - D_\alpha \sigma_b^+ \mid b \in \mathbb{Z}^3 \text{ with } \beta_2 \neq 0 \text{ or } \beta_3 \neq 0 \right\} \]
\[ \cup \left\{ \|\beta\| \sigma_b^+ + D_\alpha \sigma_b^+ \mid b \in \mathbb{Z}^3 \text{ with } \beta_2 \neq 0 \text{ or } \beta_3 \neq 0 \right\} \]
\[ \cup \left\{ \sigma_b^+ \mid \beta_2 = \beta_3 = 0 \right\}. \]

Furthermore, we have for \( \beta_2 \neq 0 \) or \( \beta_3 \neq 0 \):
\[ \text{span} \left\{ \sigma_b^+, \sigma_b^- \right\} = \text{span} \left\{ \|\beta\| \sigma_b^+ - D_\alpha \sigma_b^+, \|\beta\| \sigma_b^+ + D_\alpha \sigma_b^+ \right\}. \]

The spectrum consists of all numbers \( \pm \|\beta(b, \alpha)\| \) for \( b \in \mathbb{Z}^3 \).

**Remark 12.** In the case \( \hat{k} \neq 0 \) the spectrum is determined by \( \alpha_\perp \) while the eigenbasis is determined by \( \alpha_\parallel \). Here every change of \( \alpha \) has influence on both eigenbasis and spectrum.

### 4 Spectral sections

We look at families of Dirac operators over a compact base space \( B \). [4] defined the concept of a spectral section for a constant \( R > 0 \). The most interesting spectral sections are those for small \( R \); they should be classified in the sense of the following definition.

**Definition 13.** Let \( R_{\text{inf}} \) be defined as the infimum of the set
\[ \left\{ R > 0 \mid \text{for } R \text{ exists at least one spectral section} \right\}. \]
Furthermore, choose a (small) positive number $\varepsilon_P$. Then a system of infinitesimal spectral sections is a map
\[
\left[ R_{\text{inf}}, R_{\text{inf}} + \varepsilon_P \right] \times I \to \{ \text{spectral sections for a fixed operator } D \}
\]
where
\[
(R, i) \mapsto P^i_R,
\]
where
1. $I$ is an arbitrary index set,
2. $P^i_R$ is a spectral section for the constant map $R$,
3. every $(P^i_R)_\alpha$, $\alpha \in B$, depends continuously on $R$ (where we consider $(P^i_R)_\alpha$ as operator between $L^2$ spaces), and
4. $\cup_{i \in I} \{ P^i_R \}$ is a representation system for all spectral sections for $R$, i.e. for all possible spectral sections $P_R$ there is a $P^i_R$ with $i \in I$, so that $\text{Im } P_R - \text{Im } P^i_R$ is zero in $K$-theory.

A minimal system of infinitesimal spectral sections is one in which $I$ is chosen minimal (under the inclusion relation).

4.1 Definition of the family

Let $\ell \subset H^1(T^3; \mathbb{Z})$ be a lattice (of non-maximal dimension) and let $B := (\ell \otimes \mathbb{R})/\ell$.

We need the following ingredients for our definition:

- $\ker(d)_{\ell \otimes \mathbb{R}}$: The subset of $\ker(d)$ representing elements in $\ell \otimes \mathbb{R}$.
- $G_\ell$: The subgroup of the gauge group $\text{Map}(T^3, S^1)$ determined by $\ell$.
- The projection $\text{pr}_{T^3} : T^3 \times (\nabla^K + \mathbf{i} \ker(d)_{\ell \otimes \mathbb{R}}) \to T^3$ together with the induced vector bundle $\text{pr}_{T^3}^* (\mathbb{H} \otimes K)$.

If $v$ is an element of the fibre of $\text{pr}_{T^3}^* (\mathbb{H} \otimes K)$ over
\[
(y, \nabla^K + \mathbf{i} \alpha^c) \in T^3 \times (\nabla^K + \mathbf{i} \ker(d)_{\ell \otimes \mathbb{R}}),
\]
we can define the following action of $G_\ell$:
\[
G_\ell \times \text{pr}_{T^3}^* (\mathbb{H} \otimes K) \to \text{pr}_{T^3}^* (\mathbb{H} \otimes K)
\]
\[
\left( u, (v, y, \nabla^K + \mathbf{i} \alpha) \right) \mapsto (u(y) \cdot v, y, \nabla^K + \mathbf{i} \alpha + ud\alpha^{-1}), \quad (2)
\]
The quotient is a bundle over $T^3 \times B$. The connection from the parameter space determines a family of Dirac operators called $\mathcal{D}$.

Depending on $\hat{k}$ and $\ell$ we want to know:

1. Do spectral sections exist?
2. If they exist: What do they look like?

### 4.2 Existence of spectral sections

Following [4] we know that spectral sections for $\mathcal{D}$ exist if and only if the index of $\mathcal{D}$ in $K^1(B)$ vanishes. Let $\mathcal{I}$ be the following composition of isomorphisms (remember that $B$ is a torus of maximal dimension 2):

$$K^1(B) \xrightarrow{\text{Chern}} H^1(B; \mathbb{Z}) \rightarrow (H_1(B; \mathbb{Z}))^* \rightarrow \ell^*$$

**Lemma 14.** Let $a \in H^1(T^3; \mathbb{Z})$ and let $f : (\mathbb{R} \cdot a)/a \rightarrow B$ be the map induced by the inclusion. In this way we get a pullback family $\mathcal{D}^a$ over $(\mathbb{R} \cdot a)/a$. Then the spectral flow of $\mathcal{D}^a$ in positive direction is given by

$$\langle \hat{k}, a \rangle = \langle \hat{k} \cup a, [T^3] \rangle$$

**Proof.** We use our explicit knowledge of the spectrum.

First we assume $\hat{k} \neq 0$: From all eigenvalues $\nu_{l,m}^{+/-0}$ only those of the form $\nu_{l,m}^0$ for $0 \leq m \leq h - 1$ have a chance to cross zero. From the definition we know that $\nu_{l,m}^0 = \lambda_l = (2\pi l + \langle k, \alpha \rangle)/\|k\|$ for which we can count the crossings while running around the circle.

For $\hat{k} = 0$ the spectrum is always symmetric with respect to zero. We see that every spectral flow has to vanish. \hfill \Box

With this Lemma we get a direct access to the following statement:

**Theorem 15.** The isomorphism $\mathcal{I}$ maps the index of $\mathcal{D}$ to the map $x \mapsto \langle \hat{k} \cup x, [T^3] \rangle$ in $\ell^*$.

**Proof.** Take a fundamental basis $a_1, a_2$ of the torus $B$; then an element in $K^1(B)$ is determined by its images in $K^1((\mathbb{R} \cdot a_i)/a_i)$, which we calculate with the formula from the preceding lemma. Since the maps are linear, it is enough to check the theorem for $a_1, a_2$ which is an easy exercise. \hfill \Box

**Corollary 16.** Spectral sections for $\mathcal{D}$ exist if and only if $k \cup \ell = 0$. 

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4.3 Construction of spectral sections for $\hat{k} \neq 0$

**Theorem 17.** If spectral sections exist, the spectrum is constant.

**Proof.** From $k \cup \ell = 0$ we know that for every $\alpha \in (\ell \otimes \mathbb{R})$ we have $\alpha \perp = 0$. From section 3.1.3 we know that this implies a constant spectrum. \hfill $\square$

Therefore, we have $R_{\text{inf}} = 0$. For $\varepsilon_P$ smaller than the smallest eigenvalue of $D$, the spectral sections are fixed everywhere except for the $h$-dimensional kernel of $D$.

Let $I := \{ F \mid F \text{ subbundle of } B \times \mathbb{C}^h \} / \sim \cong \mathbb{Z}^{h-1} \cup \{ 0 \} \cup \{ C^k \}$ and define $P_F|_{\ker D}$ for $R < \varepsilon_P$ as the orthogonal projection onto $F$. This defines a system of infinitesimal spectral sections which is obviously also minimal.

4.4 Construction of spectral sections for $\hat{k} = 0$

We split $\Gamma_{L^2}(\mathbb{H})$ into the 2-dimensional $D_\alpha$-invariant subspaces $\Sigma_b = \text{span}\{ \sigma^+_b, \sigma^-_b \}$. On each of them, we have the two eigenvalues $\pm \| \beta \| = \pm \| \alpha + 2\pi b \|$. For small $R$ we know that for each $\alpha$ there is at most one $b$ with $\| \beta \| \leq R$. So for any spectral section $P$ for $D$ with small $R$ we know that it fixes all $\Sigma_b$. Since $P_\alpha|\Sigma_b : \Sigma_b \to \Sigma_b$ is a one-dimensional orthogonal projection for $\| \beta \| > R$, it has to be a one-dimensional orthogonal projection for all $\beta$ (and, therefore, for all $\alpha$, since $\alpha$ and $\beta$ are in bijective correspondence).

We now assume that $\ell$ is a plane since $\dim \ell \leq 1$ does not lead to interesting conclusions. In addition to the assumptions about $R$ above we assume that $\varepsilon_P$ is smaller than the minimal distance between $\ell \otimes \mathbb{R}$ and any point $b \in \mathbb{Z}^3 \backslash \ell$. This implies that for such $b$ there are no eigenvalues with $\| \beta \| < R$ on $\Sigma_b$.

The space of one-dimensional orthogonal projections on $\mathbb{C}^2$ equals $\mathbb{CP}^1 \cong S^2$. Fix an element $b \in \ell_{\mathbb{Z}} = (\ell \otimes \mathbb{R}) \cap \mathbb{Z}^3$ and look at the corresponding map $P_\beta|\Sigma_b : \ell \otimes \mathbb{R} \to \mathbb{CP}^1$ (written as function of $\beta$). For $\| \beta \| \geq R$ every ray coming from zero will be mapped to one point, producing a circle in $\mathbb{CP}^1$ (this follows from the construction of the eigenbasis).

For $\| \beta \| < R$ we have to continue this map in some way; topologically, the problem is as follows: We have to construct a map from the 2-disc to the 2-sphere which maps the boundary pointwise to the equator. Up to homotopy, there are $\pi_2(S^2) \cong \mathbb{Z}$ many choices for that.

4.4.1 A system of infinitesimal spectral sections

The preceding discussion leads to the following:
Since we had imposed no lower bounds for $R$, we have $R_{\text{inf}} = 0$. Let $\varepsilon_P$ be so small that $\varepsilon_P$ fulfills all conditions mentioned above.

We take $I = \{ g : \ell_Z/\ell \to \pi_2(\mathbb{CP}^1) \}$ and define for each $R < \varepsilon_P$ spectral projections $P^g$. For $b \notin \ell_Z$ these maps are already defined on $\Sigma_b$. For $b \in \ell_Z$, we define $P^g_b$ on $\Sigma_b$ to be a continuation specified by $g(b) \in \pi_2(\mathbb{CP}^1)$ as discussed in the preceding subsection (These continuations can be chosen to depend continuously on the parameters).

Conditions 1 and 2 (from the definition of infinitesimal spectral sections) are clear, 3 can be checked directly (if we specify the continuations explicitly), and 4 follows from the discussion above.

In general this system is not minimal. We can choose a minimal system $J$ by fixing an element $g_0 \in I$ and a point $l_0 \in \ell_Z/\ell$ and defining

$$J = \{ g \in I \mid g(l) = g_0(l) \text{ for } l \neq l_0 \}.$$

This is true because $J$ represents all element of the form $(0, z)$ from $K(B) \cong H^0(B; \mathbb{Z}) \oplus H^2(B; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$.

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