On parameters of the Levi-Civita solution

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Abstract

The Levi-Civita (LC) solution is matched to a cylindrical shell of an anisotropic fluid. The fluid satisfies the energy conditions when the mass parameter $\sigma$ is in the range $0 \leq \sigma \leq 1$. The mass per unit length of the shell is given explicitly in terms of $\sigma$, which has a finite maximum. The relevance of the results to the non-existence of horizons in the LC solution and to gauge cosmic strings is pointed out.

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1 Introduction

The number of exact solutions of the Einstein field equations has been increasing dramatically, especially after the discovery of the so-called soliton techniques in the late seventies [1]. However, there are very few of them with physical interpretations [2]. The understanding of their physics is closely related to the understanding of their sources. In fact, it is exactly in this vein that the work of finding a source for the Kerr vacuum solution still continues, and so far no success is claimed, although the Kerr solution was found more than thirty years ago.

Another rather embarrassing case is the solution found by Levi-Civita in 1919 [3]. The Levi-Civita (LC) solution is usually written in the form

\[ds^2 = R^{4\sigma}dT^2 - R^{4\sigma(2\sigma-1)}(dR^2 + dZ^2) - C^{-2}R^{2-4\sigma}d\varphi^2,\]  

(1)

where \(\sigma\) and \(C\) are two arbitrary constants, and \(\{x^\mu\} = \{T, R, Z, \varphi\}\) are the usual cylindrical coordinates with \(-\infty < T, Z < +\infty,\) \(R \geq 0,\) \(0 \leq \varphi \leq 2\pi,\) and the hypersurfaces \(\varphi = 0, 2\pi\) being identified. Except for the cases \(\sigma = -\frac{1}{2}, 0, \frac{1}{4}, \frac{1}{2}\), the metric is Petrov type I [4, 5], and has three Killing vectors, \(\xi_{(0)}^\mu = \delta_T^\mu, \xi_{(2)}^\mu = \delta_Z^\mu,\) and \(\xi_{(3)}^\mu = \delta_\varphi^\mu.\) For \(\sigma = -\frac{1}{2}, \frac{1}{4}\), the metric is Petrov type D, and there is a fourth Killing vector, \(\xi_{(-1/2)}^\mu = \varphi \delta_Z^\mu - Z \delta_\varphi^\mu,\) and \(\xi_{(1/4)}^\mu = \varphi \delta_T^\mu - T \delta_\varphi^\mu,\) respectively. The Killing vector \(\xi_{(-1/2)}^\mu\) corresponds to a
rotation in the \((Z, \varphi)\) plane, and the corresponding solution is locally isometric to Taub’s plane solution \(\footnote{[6]}\). This property suggests to some authors \(\footnote{[7]}\) that Taub’s solution has in fact cylindrical symmetry. However, we intend to take the opposite view, since the intrinsic curvature of the \((Z, \varphi)\) plane is identically zero, and we consider this as the main criterion for plane symmetry. The Killing vector \(\xi^\mu_{(1/4)}\) corresponds to a Lorentz boost. For \(\sigma = 0, \frac{1}{2}\), the metric is locally flat, and in the latter case it is written in a uniformly accelerated system of coordinates. Except for the last two cases, the space-time is asymptotically flat in the radial direction and singular on the axis. This can be seen, for example, from the Kretschmann scalar

\[
\mathcal{R} \equiv R_{\alpha \beta \gamma \delta} R^{\alpha \beta \gamma \delta} = \frac{64 A \sigma^2 (2 \sigma - 1)^2}{R^4 A},
\]

where \(A \equiv 4 \sigma^2 - 2 \sigma + 1\). The singularity is usually believed to represent a cylindrical source, and in realistic models it should be replaced by a regular region filled with matter. However, this kind of explanation is unsatisfactory, because it leaves open the question of whether or not the matter can be realistic. Considering the Newtonian limit and time-like geodesics, Gautreau and Hoffman \(\footnote{[4]}\) found that the above interpretation holds only for \(0 \leq \sigma < \frac{1}{4}\), and the parameter \(\sigma\) represents the mass per unit length. This conclusion was later confirmed by constructing explicit sources for the LC solution \(\footnote{[8], [9], [10]}\).
and was recently extended to $0 \leq \sigma < \frac{1}{2}$ \cite{11, 12}. It is believed that $\sigma = \frac{1}{2}$ is the maximal value allowed to LC solution such that it can be interpreted as representing the gravitational field produced by a cylindrical source\footnote{In \cite{12}, a particular source for the LC solution with $\sigma = 1$ was found, and the source is composed of “rather bizarre relativistic material.” Yet, in \cite{13} it was remarked that when imposing the strong energy condition, the parameter $\sigma$ has to be in the range $0 \leq \sigma \leq 1$, but no detail was given.}. In any case, when $\sigma$ takes the latter range, a fundamental question arises: in what form is the parameter $\sigma$ related to the mass per unit length of the source, since $\sigma = 0, \frac{1}{2}$ all correspond to a spacetime that is locally flat? Bonnor and Martins \cite{5} proposed a particular relation for the range $0 \leq \sigma \leq \frac{1}{4}$, which is clearly not applicable beyond it.

On the other hand, the LC solution does not possess any horizons. If our present understanding about the formation of black holes is correct, this seems to indicate that there is an upper limit to the allowed linear mass densities of these cylinders, and that this limit is always below the critical linear mass, above which horizons are expected to be formed \cite{9}.

In this paper, we shall address the above mentioned problems. In particular, by considering more general sources to the LC vacuum solution, we shall show that all the gravitational fields of the LC solution with $0 \leq \sigma \leq 1$ are produced by physically reasonable cylindrical sources, and that the mass per unit length has a finite maximum at $\sigma = \frac{1}{2}$. Specifically, the paper is
organized as follows: in section 2, the matching of the LC solutions to that of Minkowski is given and the energy conditions of the resulting shell on the matching hypersurface are considered; in section 3 the various definitions of the mass per unit length of the shell are considered, while in section 4 our main conclusions are presented.

2 Matching the LC solutions to that of Minkowski

In the previous investigations, specific equations of state of matter fields were usually assumed. Thus all the results obtained so far seem model-dependent. In order to avoid this problem, we model the cylinder source as an infinitely thin cylindrical shell with a finite radius, and inside the shell the spacetime is assumed to be Minkowski. Although here we do not have a theorem similar to that of Birkhoff in the spherically symmetric case [1], it is quite reasonable to assume that the source is static, since the LC solution is static. Therefore, our assumption that inside the static shell the spacetime is Minkowski has no loss of generality, since it is the only static spacetime that has zero mass density distribution [14]. By matching Minkowski geometry to that of LC, in general, we obtain a massive shell on the matching hypersurface. To obtain the most general shell, we require that the metric coefficients be only continuous across the shell, the minimal requirement in order to have the
Einstein field equations meaningful [13]. Then, using the formulae given by Taub [14], we calculate the surface energy-momentum tensor \( T \). Of course, matter shells such obtained are not always physically realistic, unless we further impose the energy conditions [19]. Thin shells of matter might be formed in the early stages of the Universe, and may provide the necessary perturbations to the formations of galaxies and the large-scale structure of the Universe [20].

Before going into the details, we ought to mention the other physical parameter \( C \). Marder [21] first realized that the LC solution has two physically independent constants, \( \sigma \) and \( C \), while Bonnor [8] was the one who first related the constant \( C \) to an angular defect. Recently, this angular defect was further interpreted as representing gauge cosmic strings \([22,10]\), being objects formed in the early Universe [20].

To get the most general matching, we first make the following coordinate transformations

\[
t = \alpha T, \quad r = A^{-1} R^4 + a, \quad z = \beta Z, \quad (2)
\]

where \( \alpha, \beta \) and \( a \) are arbitrary constants. Clearly, the above transformations leave the ranges of the coordinates unchanged. Then, in terms of the new
coordinates \( \{x^\mu\} \equiv \{t, r, z, \varphi\} \), the metric (1) takes the Gaussian form

\[
ds^2 = \alpha^{-2} B^{4\sigma/A} dt^2 - dr^2 - \beta^{-2} B^{4\sigma(2\sigma-1)/A} dz^2 - C^{-2} B^{2(1-2\sigma)/A} d\varphi^2, \tag{3}
\]

where \( B \equiv A(r - a) \). In this new system of coordinates, the spacetime singularity at \( R = 0 \) is mapped to \( r = a \). In the following, we shall assume that the matter shell is located on the hypersurface \( r = r_0 \), where \( r_0 \) is a constant and that the LC solution (3) describes the region \( r \geq r_0 \). To avoid spacetime singularities outside the shell we require \( r_0 > a \). Inside the shell \( (0 \leq r \leq r_0) \), the spacetime is Minkowski and the metric takes the form

\[
ds^2 = dt^2 - dr^2 - dz^2 - r^2 d\varphi^2. \tag{4}
\]

To have the Einstein field equations meaningful on \( r = r_0 \), we impose the first junction conditions \( g_{\mu\nu}|_{r=r_0^+} = g_{\mu\nu}|_{r=r_0^-} \). Clearly, for \( \mu, \nu = 0, 2 \), these conditions can be always satisfied by properly choosing the two arbitrary constants \( \alpha \) and \( \beta \), while for \( \mu, \nu = 3 \) they become

\[
r_0^2 = C^{-2} [A(r_0 - a)]^{2(1-2\sigma)/A}. \tag{5}
\]

With the above requirement, we can see that the first derivatives of the metric coefficients with respect to \( r \) are, in general, discontinuous across the hypersurface \( r = r_0 \), which gives rise to thin shells [16].
Following Taub (see also [14]), we first introduce a tensor $b_{\mu\nu}$ via the relations

\[ [g_{\mu\nu,\lambda}]^+ = g_{\mu\nu,\lambda}^+ \bigg|_{r=r_0} - [g_{\mu\nu,\lambda}]^- \bigg|_{r=r_0} = n_\lambda b_{\mu\nu}, \tag{6} \]

where $n_\lambda$ is the normal vector to the hypersurface $r = r_0$, and is now given by $n_\lambda = \delta^r_\lambda$. Once $b_{\mu\nu}$ is known, the surface energy-momentum tensor $\tau_{\mu\nu}$ can be read off from the expression

\[ \tau_{\mu\nu} = \frac{1}{2} \left\{ b(n g_{\mu\nu} - n_{\mu} n_{\nu}) + n_\lambda (n_{\mu} b_{\nu}^\lambda + n_{\nu} b_{\mu}^\lambda) - (n b_{\mu\nu} + n_\lambda n_\delta b^{\delta\lambda} g_{\mu\nu}) \right\}, \tag{7} \]

where $n \equiv n_\lambda n^\lambda$ and $b \equiv b_\lambda^\lambda$. In terms of $\tau_{\mu\nu}$ the energy-momentum tensor $T_{\mu\nu}$ of the four-dimensional spacetime takes the form $T_{\mu\nu} = \tau_{\mu\nu} \delta(r - r_0)$, where $\delta(r - r_0)$ denotes the Dirac delta function.

From Eqs.(3), (4) and (6), it can be shown that the non-vanishing components of $b_{\mu\nu}$ now are given by

\[ b_{00} = 4\sigma [A(r_0 - a)]^{(4\sigma-A)/A}, \quad b_{22} = 4\sigma(2\sigma - 1) [A(r_0 - a)]^{[4\sigma(2\sigma-1)-A]/A}, \]

\[ b_{33} = 2r_0 - \frac{2(1-2\sigma)}{C} [A(r_0 - a)]^{[2(1-2\sigma)-A]/A}. \tag{8} \]

Substituting the above expressions into Eq.(7), we find that the surface energy-momentum tensor can be written as

\[ \tau_{\mu\nu} = \rho t_{\mu} t_{\nu} + p_z z_\mu z_{\nu} + p_\phi \phi_\mu \phi_{\nu}, \tag{9} \]
where \( t_\mu = \delta^0_\mu \), \( z_\mu = \delta^2_\mu \), \( \varphi_\mu = r_0 \delta^3_\mu \), and

\[
\rho = \frac{2\sigma r_0 - aA}{A r_0 (r_0 - a)}, \quad p_z = \frac{2\sigma r_0 (1 - 2\sigma r_0) - aA}{A r_0 (r_0 - a)}, \quad p_\varphi = \frac{4\sigma^2}{A r_0 (r_0 - a)}.
\]  

\( (10) \)

In writing the above equations we have chosen units such that \( G = 1 = c \), where \( G \) denotes the gravitational constant and \( c \) the speed of light.

The above expressions show that the shell consists of an anisotropic fluid with surface energy density \( \rho \) and principal pressures \( p_z \) and \( p_\varphi \) in the \( z \)- and \( \varphi \)-directions, respectively. Clearly this interpretation is valid only provided that the fluid satisfies certain energy conditions \[19\]. Using the above expressions it is easy to show that all the three energy conditions, weak, strong and dominant, are satisfied for \( 0 \leq \sigma \leq 1 \), by appropriately choosing the constant \( a \). In particular, when \( \sigma = 0 \), the solution gives rise to a cosmic string with a finite radius \[22\],

\[
\rho = -p_z = - \frac{a}{r_0 (r_0 - a)} = \frac{1}{r_0} \left( 1 - \frac{1}{C} \right), \quad p_\varphi = 0.
\]  

\( (11) \)

When \( a = 0 \), Eq.(10) reduces to

\[
\rho = \frac{\sigma}{4\pi A r_0}, \quad p_z = \frac{\sigma (1 - 2\sigma)}{4\pi A r_0}, \quad p_\varphi = \frac{\sigma^2}{2\pi A r_0}.
\]  

\( (12) \)

It can be shown too that in this case the weak and strong energy conditions are satisfied for \( 0 \leq \sigma \leq 1 \), while the dominant energy condition is satisfied for \( 0 \leq \sigma \leq 1/2 \). If we further set \( \sigma = 0 \), we shall find that no cosmic
string now exists. This is because in this case the first junction condition (5) requires \( C = 1 \), which means no angular defect. Thus, to obtain a cosmic string solution for the case \( \sigma = 0 \), it is necessary to have \( a \neq 0 \).

However, since in this paper we are mainly interested in the physical meaning of \( \sigma \), in the following section we shall, without loss of generality, restrict ourselves only to this case. It can be shown that the main conclusions obtained there also hold in the general case.

3 The mass per unit length of the shell

There exist various definitions of the mass per unit length of a cylinder \[21, 23, 24\]. Here we are mainly interested in the one given by Vishveshwar and Winicour (VW) \[24\], since it always gives the correct Newtonian limit and seems more suitable for the present case. The VW definition is based on the time translation and rotational Killing vectors, which now are \( \xi^\mu_{(0)} = \delta^\mu_t \), and \( \xi^\mu_{(3)} = \delta^\mu_\varphi \), respectively. In terms of these vectors, the mass per unit length \( \mu \) of a cylinder is defined by

\[
\mu = -\frac{1}{2\tau}(\lambda_{33}\lambda_{00,r} - \lambda_{03}\lambda_{03,r}),
\]

(13)

where

\[
\lambda_{00} = \xi^\nu_\nu(0), \quad \lambda_{03} = \xi^\nu_\nu(3),
\]
\[
\lambda_{33} = \zeta'_{(3)}\zeta_{(\nu)}, \quad \tau^2 = -2(\lambda_{00}\lambda_{33} - \lambda_{03}^2). \quad (14)
\]

Applying the above definition to the present case, we find
\[
\mu = \frac{\sigma}{4\sigma^2 - 2\sigma + 1}, \quad (15)
\]
which shows that when \(0 < \sigma \ll 1\), we have \(\mu \approx \sigma\). This is consistent with its Newtonian limit \([4]\). However as \(\sigma\) increases, \(\mu\) monotonically increases until \(\sigma = \frac{1}{2}\) where it reaches its maximum \(\mu = \mu_{\text{max}} = \frac{1}{2}\) in the chosen units. This maximal value is twice that obtained in \([9]\) for a particular dust source. Thus the results obtained here and those in \([9]\) strongly suggest that the linear mass of static cylinders has an upper limit, although with particular solutions it is difficult to find the exact limit. After the point \(\sigma = \frac{1}{2}\), \(\mu\) starts to decrease as \(\sigma\) continuously increases. Therefore, \(\sigma = \frac{1}{2}\) is a turning point.

This point is also the critical point that separates “normal” gauge cosmic strings from the supermassive ones. In \([25]\) it is shown that the spacetime of a gauge cosmic string asymptotically approaches the one of Eq.(3) with \(\sigma = 0\) and \(C > 1\), when the mass per unit length of the string is very low, while in the opposite limit, the spacetime asymptotically approaches the one of Eq.(3) with \(\sigma = 1\). In between these two states there exists a critical state that separates the normal strings from the supermassive ones, and this critical state is exactly given by the LC solution (3) with \(\sigma = \frac{1}{2}\).
On the other hand, using the definition of Israel [23], we find that

\[
\mu_{\text{Israel}} = \int (\rho + p_z + p_\phi)\delta(r - r_0)\sqrt{g}drd\phi = \sigma,
\]

while the “inertial” mass per unit length of Marder [21] yields

\[
\mu_{\text{Marder}} = \int \rho\delta(r - r_0)\sqrt{g_{(2)}}drd\phi = \frac{\sigma}{2(4\sigma^2 - 2\sigma + 1)},
\]

where \(g_{(2)}\) is the determinant of the induced metric on the 2-surface \(S\) defined by \(t, z = \text{Const.}\). Clearly, in the present case Marder’s definition does not give the correct Newtonian limit, but Israel’s does. Moreover, Eq.(17) has a maximum at \(\sigma = 1/2\), while Eq.(16) does not.

4 Conclusions

In this paper, we have shown that the LC solution with \(0 \leq \sigma \leq 1\) can be produced by physically realistic cylindrical sources, and that the mass per unit length of the cylinder, \(\mu\), depends only on the parameter \(\sigma\), and is given explicitly by Eq.(15). When \(0 < \sigma \ll 1\), we have \(\mu \approx \sigma\), which is consistent with its Newtonian limit. As \(\sigma\) increases, \(\mu\) is monotonically increasing until \(\sigma = 1/2\), where it reaches its maximum \(\mu = \mu_{\text{max}} = 1/2\). This explains why no horizons exist in the LC solution. When \(\sigma < 0\), Eq.(12) shows that the shell does not satisfy any of the energy conditions, and the mass of the shell is negative. Nonetheless this is also consistent with its Newtonian limit [4, 5].
But when $\sigma \ll 0$ one has to consider the above results with great caution since once the energy conditions are violated we could, in principle, have any kind of sources. In particular, when $\sigma = -\frac{1}{2}$, Eq. (7) shows that $\rho, p_z$ and $p_\varphi$ are all negative. Since the solution with $\sigma = -\frac{1}{2}$ is locally isometric to Taub’s plane solution [3], can we conclude that the source for the Taub solution also has negative mass? This is a question that is still under our investigation.

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