THE CLASSIFICATION OF ORTHOGONALLY RIGID
$G_2$-LOCAL SYSTEMS AND RELATED DIFFERENTIAL
OPERATORS

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ABSTRACT. We classify orthogonally rigid local systems of rank 7 on the punctured projective line whose monodromy is dense in the exceptional algebraic group $G_2$. We obtain differential operators corresponding to these local systems under Riemann-Hilbert correspondence.

1. INTRODUCTION

It is well known that the exceptional simple algebraic group $G_2$ can be seen as a subgroup of $GL_7$ and stabilizes the bilinear form

$$x_0^2 + x_1y_1 + x_2y_2 + x_3y_3,$$

where $x_0, x_1, y_1, x_2, y_2, x_3, y_3$ is a suitably chosen basis of the underlying 7-dimensional vector space, cf. [1]. It is the aim of this article to classify the orthogonally rigid local systems $\mathcal{L}$ of rank 7 whose monodromy group is Zariski dense in $G_2(\mathbb{C})$ and hence leaves the above form invariant. Orthogonal rigidity for an irreducible orthogonally self-dual complex local system $\mathcal{L}$ on $\mathbb{P}^1 \setminus \{x_1, \ldots, x_{r+1}\}$ of rank $n$ means that the following dimension formula holds:

\begin{equation}
\sum_{i=1}^{r+1} \text{codim}(C_{O_n}(g_i)) = 2 \dim(O_n),
\end{equation}

where $C_{O_n}(g_i)$ denotes the centralizer of the local monodromy generator $g_i$ in the orthogonal group $O_n$. The dimension formula (1.1) is equivalent to the vanishing of the parabolic cohomology of $\pi_1(\mathbb{P}^1 \setminus \{x_1, \ldots, x_{r+1}\})$ with values in the Lie algebra of $O_n$ (acting adjointly via the monodromy representation of $\mathcal{L}$) and is hence closely related to the dimension of the tangent space of the component of the space of representations of $\pi_1(\mathbb{P}^1 \setminus \{x_1, \ldots, x_{r+1}\})$ with given local monodromies, cf. [18]. The dimension formula is also a necessary condition for the condition that there exist only finitely many equivalence classes of irreducible orthogonally self-dual local systems $\mathcal{L}$ with given local monodromies [17]. Hence, for such local systems, the notion of orthogonal rigidity is weaker as the notion of (physical) rigidity used in [12] (which can be seen as rigidity relative to the larger group $GL_n$) but still strong enough to impose a lot of structure on $\mathcal{L}$.

By the work of N. Katz on the middle convolution functor $MC_{\chi}$, all rigid irreducible local systems $\mathcal{L}$ on the punctured line can be constructed by applying iteratively $MC_{\chi}$ and tensor products with rank-1-sheaves to a rank-1-sheaf. For orthogonally rigid local systems with $G_2$-monodromy we prove that there is a

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similar method of construction: Each such local system can be constructed using
$MC_{\chi}$, tensor products with rank-1-sheaves and exceptional isomorphisms between
small algebraic groups which each have a natural interpretation as tensor opera-
tions like alternating or symmetric products (e.g. $SO_5 = \Lambda^2(Sp_4)$). In some cases,
also rational pullbacks are involved. Our main result is as follows (Thm. 5.1):

**Theorem.** 1.1. Let $\mathcal{L}$ be an orthogonally rigid local system on a punctured
projective line $\mathbb{P}^1 \setminus \{x_1, \ldots, x_r\}$ of rank 7 whose monodromy group is dense in the
exceptional simple group $G_2$. If $\mathcal{L}$ has nontrivial local monodromy at $x_1, \ldots, x_r$,
then $r = 3, 4$ and $\mathcal{L}$ can be constructed by applying iteratively a sequence of the
following operations to a rank-1-system:
- Middle convolutions $MC_{\chi}$, with varying $\chi$.
- Tensor products with rank-1-local systems.
- Tensor operations like symmetric or alternating products.
- Pullbacks along rational functions.

Especially, each such local system which has quasi-unipotent monodromy is mo-
tivic, i.e., arises from the variation of periods of a family of varieties over the
punctured projective line.

A list of the occurring cases together with the local monodromies is given in
Thm. 5.1. Rigid local systems on the punctured line with $G_2$-monodromy were
classified in [8]. Since orthogonal rigidity for irreducible orthogonal local systems
with $G_2$-monodromy is a weaker condition as the usual rigidity condition our clas-
sification contains the rigid local systems from [8] as special cases. We remark
that the verification that the monodromy group is inside the group $G_2$ cannot
be decided looking at the local monodromies alone. To prove this, we make use
of recent results of Bogner and Reiter in [4] on the interpretation of $MC_{\chi}$ at the
level of differential operators, related to the Hadamard product. Miraculously,
the differential operators which belong to the local systems of Thm. 1.1 under
Riemann-Hilbert correspondence can easily be determined and it can be proven
in each case that they have the property that the second alternating square has
rank 14. This implies that the second alternating square of $\mathcal{L}$ decomposes into a
rank-14-factor and a rank-7-factor and hence that the monodromy is contained
in the group $G_2$.

Motivated by the results of Thm. 1.1 one may ask the question, whether any
irreducible orthogonally rigid local system can be obtained by a sequence of
tensor operations, middle convolutions $MC_{\chi}$, and rational pullbacks applied to a
local system of rank one.

2. Preliminaries on convolution operations
Recall the construction of the middle convolution from [12]: Consider the addition
map
$$\pi : \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1, (x, y) \mapsto x + y.$$
Let $\mathcal{L}$ be a complex valued local system on $\mathbb{A}^1 \setminus \{x_1, \ldots, x_r\}$ and let $L = j_\ast \mathcal{L}[1]$, viewed as perverse sheaf on $\mathbb{A}^1$ ($j$ denoting the inclusion of $\mathbb{A}^1 \setminus \{x_1, \ldots, x_r\}$
into $\mathbb{A}^1$). Let $\mathcal{L}_\chi$ be a local system on $\mathbb{G}_m$, defined by a nontrivial character
$\chi : \pi_1(\mathbb{G}_m) \to \mathbb{C}^\times$. We call $\mathcal{L}_\chi$ a Kummer sheaf. Let $L_\chi = (k_\ast \mathcal{L}_\chi)[1]$, where
$k$ denotes the natural inclusion of $\mathbb{G}_m$ to $\mathbb{A}^1$. Sometimes we need the following
variant: using the isomorphism $\mathbb{A}^1 \setminus \{ y \} \to \mathbb{G}_m$, $x \mapsto x - y$, we can view $\mathcal{L}_\chi$ as local system on $\mathbb{A}^1 \setminus \{ y \}$. This local system is then denoted $\mathcal{L}_\chi(x - y)$.

Following Katz [12], one can define the middle convolution of $\mathcal{L}$ with the Kummer sheaf $\mathcal{L}_\chi$ as

\begin{equation}
(2.1) \quad \text{MC}_\chi(\mathcal{L}) := (\text{im}(R\pi_!(\mathcal{L} \boxtimes L_\chi)) \to R\pi_*(\mathcal{L} \boxtimes L_\chi)) [-1]|_{\mathbb{A}^1 \setminus \{x_1, \ldots, x_r\}}.
\end{equation}

**Remark. 2.1.** Since we restrict to $\mathbb{A}^1 \setminus \{x_1, \ldots, x_r\}$, the 0-th and the 2-th higher direct image vanish by the non-triviality of $\mathcal{L}_\chi$, so (2.1) is equivalent to

\begin{equation}
(2.2) \quad \text{MC}_\chi(\mathcal{L}) = (\text{im}(R^1\pi_!(\mathcal{L} \boxtimes k_\ast \mathcal{L}_\chi)) \to R^1\pi_*(\mathcal{L} \boxtimes k_\ast \mathcal{L}_\chi)) |_{\mathbb{A}^1 \setminus \{x_1, \ldots, x_r\}}.
\end{equation}

Hence, the middle convoluted local system $\text{MC}_\chi(\mathcal{L})$ can be seen as variation of the parabolic cohomology groups $H^1(\mathbb{P}^1, i_\ast(\mathcal{L} \boxtimes \mathcal{L}_\chi(x - y)))$ over $\mathbb{A}^1 \setminus \{x_1, \ldots, x_r\}$, where $i$ is the inclusion of $\mathbb{A}^1 \setminus \{x_1, \ldots, x_r, y\}$ into $\mathbb{P}^1$ and the local systems $\mathcal{L}$ and $\mathcal{L}_\chi(x - y)$ are viewed as local systems on $\mathbb{A}^1 \setminus \{x_1, \ldots, x_r, y\}$ via restriction (cf. [12] and [9]).

In the usual way we fix a set of generators $\gamma_1, \ldots, \gamma_{r+1}$ of $\pi_1(\mathbb{A}^1 \setminus \mathfrak{x})$, where $\gamma_i$ ($i = 1, \ldots, r$) is a simple loop which moves counterclockwise around $x_i$, where $\gamma_{r+1}$ moves around $\infty$, such that the product relation $\gamma_1 \cdots \gamma_{r+1} = 1$ holds. Hence, every local system on $\mathbb{A}^1 \setminus \mathfrak{x}$ gives, via its monodromy representation

$$
\rho_{\mathcal{L}} : \pi_1(\mathbb{A}^1 \setminus \mathfrak{x}, x_0) \to \text{GL}(\mathcal{L}_{x_0}) \simeq \text{GL}_n(\mathbb{C}),
$$

rise to its monodromy tuple $(A_1, \ldots, A_{r+1})$, where $A_i = \rho_{\mathcal{L}}(\gamma_i)$. The following result is a consequence of the numerology of the middle convolution (cf. [12] Cor. 3.3.6)):

**Lemma. 2.2.** Let $\mathcal{L}$ be an irreducible local system with monodromy tuple $\mathcal{A} = (A_1, \ldots, A_{r+1}) \in \text{GL}(V)^{r+1}$, s.t. at least two $A_i, A_j, 1 \leq i < j \leq r$ are non trivial. Let $\chi : \pi_1(\mathbb{G}_m) \to \mathbb{C}^\times$ be the character which sends a counterclockwise generator of $\pi_1(\mathbb{G}_m)$ to $\lambda \in \mathbb{C}^\times \setminus \{1\}$. Let $(B_1, \ldots, B_{r+1})$ be the monodromy tuple of $\text{MC}_\chi(\mathcal{L})$. Then the following hold:

(i) The rank $m$ of $\text{MC}_\chi(\mathcal{L})$ is

$$
m = \sum_{i=1}^{r} \text{rk}(A_i - 1) + \text{rk}(\lambda^{-1}A_{r+1} - 1) - \text{rk}(\mathcal{L}).
$$

(ii) Every Jordan block $J(\alpha, l)$ occurring in the Jordan decomposition of $A_i$ contributes a Jordan block $J(\alpha \lambda, l')$ to the Jordan decomposition of $B_i$, where

$$
l' := \begin{cases} 
l, & \text{if } \alpha \neq 1, \lambda^{-1}, \\
l - 1, & \text{if } \alpha = 1, \\
l + 1, & \text{if } \alpha = \lambda^{-1}.
\end{cases}
$$

The only other Jordan blocks which occur in the Jordan decomposition of $B_i$ are blocks of the form $J(1,1)$. 

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(iii) Every Jordan block $J(\alpha^{-1}, l)$ occurring in the Jordan decomposition of $A_{r+1}$ contributes a Jordan block $J(\alpha^{-1}\lambda^{-1}, l')$ to the Jordan decomposition of $B_{r+1}$, where

$$l' := \begin{cases} 
  l, & \text{if } \alpha \neq 1, \lambda^{-1}, \\
  l + 1, & \text{if } \alpha = 1, \\
  l - 1, & \text{if } \alpha = \lambda^{-1}.
\end{cases}$$

The only other Jordan blocks which occur in the Jordan decomposition of $B_{r+1}$ are blocks of the form $J(\lambda^{-1}, 1)$.

By the Riemann-Hilbert correspondence, each local system $\mathcal{L} \in \text{LS}(A^1 \setminus \mathbf{x})$ corresponds to an ordinary differential equation (or, equivalently, an operator $L$ in the Weyl algebra $\mathbb{C}[x, \partial = x \frac{d}{dx}]$) with regular singularities. Let us first describe the tensor operations in the Weyl algebra needed below (cf. [15, Chapter 2] and [4]).

**Definition. 2.3.**

(i) Let $M_1$, $M_2$ be two differential $\mathbb{C}(x)$-modules. The tensor product $(M_1 \otimes \mathbb{C}(x) M_2, \partial_{M_1 \otimes M_2})$ of $M_1$ and $M_2$ over $\mathbb{C}(x)$ is given by the $\mathbb{C}(x)$-vector space $M_1 \otimes \mathbb{C}(x) M_2$ together with the derivation

$$\partial_{M_1 \otimes M_2}(m_1 \otimes m_2) := \partial_{M_1}(m_1) \otimes m_2 + m_1 \otimes \partial_{M_2}(m_2).$$

(ii) Let $L_1, L_2 \in \mathbb{C}(x)[\partial], \partial = \frac{d}{dx}$, be two monic differential operators with corresponding differential modules $((M_1, \partial_{M_1}), \Omega_1)$ and $((M_2, \partial_{M_2}), \Omega_2)$. The tensor product $L_1 \otimes L_2 \in \mathbb{C}(x)[\partial]$ of $L_1$ and $L_2$ over $\mathbb{C}(x)$ is the minimal monic annihilating operator of $\Omega_1 \otimes \Omega_2 \in M_1 \otimes \mathbb{C}(x) M_2$.

**Remark. 2.4.**

(i) By [15 Corollary 2.19], the solution space of $L_1 \otimes L_2$ in the Picard-Vessiot field $K \supset \mathbb{C}(x)$ of the operator is spanned by the set

$$\{y_1 y_2 \mid L_1(y_1) = L_2(y_2) = 0\}.$$

In particular, $L_1 \otimes L_2$ is the monic operator of minimal order, whose solution space is spanned by this set.

(ii) Symmetric and exterior powers of differential modules and differential operators are defined similarly. If $L \in \mathbb{C}(x)[\partial]$ is monic, by [15 Corollary 2.23] and [15 Corollary 2.28] $\text{Sym}^2(L)$ is the monic operator of minimal degree whose solution space is spanned by the set

$$\{y_1 y_2 \mid L(y_i) = 0 \text{ for } i = 1, 2\}$$

and $\Lambda^2(L)$ the monic operator of minimal degree whose solution space is spanned by the set of Wronskians

$$\{\text{Wr}(y_1, y_2) := \det \begin{pmatrix} y_1 & y_2 \\ \partial y_1 & \partial y_2 \end{pmatrix} \mid L(y_i) = 0 \text{ for } i = 1, 2\}.$$

Let $L \in \mathbb{C}[x, \partial]$ be Fuchsian, i.e. $L$ has only regular singularities, and smooth on $A^1 \setminus \{x_1, \ldots, x_r\}$, let $f$ be a solution of $L$, viewed as section of the local system $\mathcal{L}$ of solutions of $L$, and let $a \in \mathbb{Q} \setminus \mathbb{Z}$. For two simple loops $\gamma_p$, $\gamma_q$, based at $x_0 \in A^1 \setminus \{x_1, \ldots, x_r\}$, and moving counterclockwise around $p$, resp. $q$, we define the Pochhammer contour

$$[\gamma_p, \gamma_q] := \gamma_p^{-1} \gamma_q^{-1} \gamma_p \gamma_q.$$
For \( y \in \mathbb{A}^1 \setminus \{x_1, \ldots, x_r\} \), the integral
\[
(2.3) \quad C_a^p(f)(y) := \int_{[\gamma_p, \gamma_y]} f(x)(y-x)^a \frac{dx}{y-x}
\]
is called the convolution of \( f \) and \( x^a \) with respect to the Pochhammer contour \([\gamma_p, \gamma_y]\).

**Remark. 2.5.** If \( x^a \) is a local section of the Kummer sheaf \( \mathcal{L}_x \), then the integral \( \int_{[\gamma_p, \gamma_y]} f(x)(y-x)^a \frac{dx}{y-x} \) represents an element in \( H^1(\mathbb{A}^1 \setminus \{x_1, \ldots, x_r, y\}, \mathcal{L} \otimes \mathcal{L}_{x-y}) \) in the usual way, cf. [3] (where we view \( \mathcal{L} \) and \( \mathcal{L}_x \) as local systems on \( \mathbb{A}^1 \setminus \{x_1, \ldots, x_r, y\} \) by restriction). Under certain conditions (made explicit in [7]), the analytic continuation of the integral (2.3) near the singularities is in the image of the local monodromy and therefore contained in the parabolic cohomology group \( H^1(\mathbb{P}^1, k_*(\mathcal{L} \otimes \mathcal{L}_{x-y})) \leq H^1(\mathbb{A}^1 \setminus \{x_1, \ldots, x_r, y\}, \mathcal{L} \otimes \mathcal{L}_{x-y}) \), cf. [9]. By Remark 2.7, for varying \( y \), the integral \( C_a^p(f)(y) \) can hence be viewed as a section of \( MC_L(x) \).

In a similar way as for \( C_a^p(f)(y) \), define
\[
H_a^p(f)(y) := \int_{[\gamma_p, \gamma_y]} f(x) \left(1 - \frac{y}{x}\right)^{-a} \frac{dx}{x}.
\]
The integral \( H_a^p(f) \) is called the Hadamard product of \( f \) and \((1-x)^{-a}\) with respect to the Pochhammer contour \([\gamma_p, \gamma_y]\). We have the obvious relations
\[
C_a^p(f) = (-1)^{a-1} H_{1-a}^p(x^a f), \quad H_a^p(f) = (-1)^{-a} C_{1-a}^p(x^{a-1} f).
\]

In [4], the following is proved:

**Proposition. 2.6.** Let \( L = \sum_{i=0}^m x^i P_i(\vartheta) \in \mathbb{C}[x, \vartheta] \) be Fuchsian, \( f \) a solution of \( L \) and \( a \in \mathbb{Q} \setminus \mathbb{Z} \). Then \( C_a^p(f) \) is a solution of
\[
\mathcal{C}_a(L) := \sum_{i=0}^m y^i \prod_{j=0}^{i-1} (\vartheta + i - a - j) \prod_{k=0}^{m-i-1} (\vartheta - k) P_i(\vartheta - a) \in \mathbb{C}[y, \vartheta]
\]
for each \( p \in \mathbb{P}^1 \) and \( H_a^p(f) \) is a solution of
\[
\mathcal{H}_a(L) := \sum_{i=0}^m y^i \prod_{j=0}^{i-1} (\vartheta + a + j) \prod_{k=0}^{m-i-1} (\vartheta - k) P_i(\vartheta) \in \mathbb{C}[y, \vartheta].
\]

**Remark. 2.7.** It is shown in [4], Cor. 4.16 that under some mild restrictions the operator \( \mathcal{C}_a(L) \) has a right factor \( L \ast_C (\vartheta - a) \) that coincides with the differential operator associated to the middle convolution \( MC_L(x) \) via the Riemann-Hilbert correspondence (where \( \mathcal{L} \) corresponds to \( L \) under the Riemann-Hilbert correspondence). Similarly we get the statement for the Hadamard product \( \mathcal{H}_a(L) = C_{1-a}(L \otimes (\vartheta - (a - 1))) \) of \( L \) with \( L_a = (\vartheta - x(\vartheta + a)) \). We denote the irreducible right factor by \( L \ast_H L_a \). It will turn out that in our situation we can easily determine the right factor via [4], Prop. 4.17.
3. Jordan forms in $G_2$ and exceptional isomorphisms

Let us collect the information on the conjugacy classes of the simple algebraic group $G_2$. Below, we list the possible Jordan canonical forms of elements of the group $G_2(\mathbb{C}) \leq \text{GL}_7(\mathbb{C})$ together with the dimensions of the centralizers in the groups $G_2(\mathbb{C})$, $\text{SO}_7(\mathbb{C})$ and in the group $\text{GL}_7(\mathbb{C})$. The list exhausts all possible cases, cf. [8, Section 1.3]. We use the following conventions: $1_n \in \mathbb{C}^{n \times n}$ denotes the identity matrix, $J(n)$ denotes a unipotent Jordan block of size $n$, $\omega \in \mathbb{C}^\times$ denotes a primitive 3-rd root of unity, and $i \in \mathbb{C}^\times$ denotes a primitive 4-th root of unity. Moreover, an expression like $(xJ(2), x^{-1}J(2), x^2, x^{-2}, 1)$ denotes a matrix in Jordan canonical form in $\text{GL}_7(\mathbb{C})$ with one Jordan block of size 2 having eigenvalue $x$, one Jordan block of size 2 having eigenvalue $x^{-1}$, and three Jordan blocks of size 1 having eigenvalues $x^2, x^{-2}, 1$ (resp.). We also abbreviate a tuple $(x, \ldots, x)$ of length $n$ by $x1_n$.

The Table of $\text{GL}_7$ conjugacy classes of $G_2$ is as follows:

| Jordan form | Centralizer dimension in $G_2$ | Centralizer dimension in $\text{SO}_7$ | Centralizer dimension in $\text{GL}_7$ | Conditions |
|-------------|-------------------------------|---------------------------------|---------------------------------|------------|
| $(1, 1, 1, 1, 1, 1, 1)$ | 14 | 21 | 49 | |
| $(J(2), J(2), 1, 1, 1)$ | 8 | 13 | 29 | |
| $(J(3), J(2), J(2))$ | 6 | 9 | 19 | |
| $(J(3), J(3), 1)$ | 4 | 7 | 17 | |
| $J(7)$ | 2 | 3 | 7 | |
| $(-1, -1, -1, -1, 1, 1, 1)$ | 6 | 9 | 25 | |
| $(-J(2), -J(2), 1, 1, 1)$ | 4 | 7 | 17 | |
| $(-J(2), -J(2), J(3))$ | 4 | 5 | 11 | |
| $(-J(3), -1, J(3))$ | 2 | 3 | 9 | |
| $(\omega, \omega, 1, \omega^{-1}, \omega^{-1}, \omega^{-1})$ | 8 | 9 | 19 | |
| $(\omega J(2), \omega^{-1} J(2), \omega, \omega^{-1}, 1)$ | 4 | 5 | 11 | |
| $(\omega J(3), \omega^{-1} J(3), 1)$ | 2 | 3 | 7 | |
| $(i, i, -1, 1, i^{-1}, i^{-1}, -1)$ | 4 | 5 | 13 | |
| $(iJ(2), i^{-1} J(2), -1, -1, 1)$ | 2 | 3 | 9 | |
| $(x, x, x^{-1}, x^{-1}, 1, 1, 1)$ | 4 | 7 | 17 | $x^2 \neq 1$ |
| $(x, x, x^2, x^{-1}, x^{-1}, x^{-2})$ | 4 | 5 | 11 | $x^4 \neq 1 \neq x^3$ |
| $(x, -1, -x, 1, -x^{-1}, -1, x^{-1})$ | 2 | 3 | 9 | $x^4 \neq 1$ |
| $(xJ(2), x^{-1} J(2), x^2, x^{-2}, 1)$ | 2 | 3 | 7 | $x^4 \neq 1 \neq x^3$ |
| $(xJ(2), x^{-1} J(2), J(3))$ | 2 | 3 | 7 | $x^2 \neq 1$ |
| $(x, y, xy, 1, (xy)^{-1}, y^{-1}, x^{-1})$ | 2 | 3 | 7 | pairw. diff. eigenvalues |
Later, we will need information on how Jordan forms are transformed under the exceptional isomorphisms $SO_6 = \Lambda^2 SL_4$ and $SO_5 = \Lambda^2 SP_4$. Under the exceptional isomorphism $SO_6 = \Lambda^2 SL_4$ selected Jordan forms are transformed as follows:

| Jordan form in $SO_6$ | $\iff$ | Jordan form $\Lambda^2 SL_4$ |
|-----------------------|--------|-----------------------------|
| $(J(2), J(2), 1, 1)$  | $\iff$ | $(\Lambda^2(J(2), 1, 1)$ |
| $(J(5), 1)$           | $\iff$ | $(\Lambda^2(J(4))$ |
| $(J(3), 1, -1, -1)$   | $\iff$ | $(\Lambda^2(iJ(2), -iJ(2))$ |
| $(\omega J(3), \omega^{-1} J(3))$ | $\iff$ | $(\Lambda^2(\omega J(3), 1)$ |
| $(xJ(2), x^{-1} J(2), x^2, x^{-2})$ | $\iff$ | $(\Lambda^2(xJ(2), 1, x^{-2})$ |
| $(J(3), 1, x^2, x^{-2})$ | $\iff$ | $(\Lambda^2(xJ(2), x^{-1} J(2))$ |
| $(x, y, xy, (xy)^{-1}, y^{-1}, x^{-1})$ | $\iff$ | $(\Lambda^2(x, y, (xy)^{-1}, 1)$ |

Under the exceptional isomorphism $SO_5 = \Lambda^2 SP_4$ the Jordan forms are transformed as follows:

| Jordan form in $SO_5$ | $\iff$ | Jordan form $\Lambda^2 SP_4$ |
|-----------------------|--------|-----------------------------|
| $J(5)$                | $\iff$ | $(\Lambda^2(J(4))$ |
| $(-J(3), -1, 1)$      | $\iff$ | $(\Lambda^2(-J(2), J(2))$ |
| $(J(3), x^2, x^{-2})$ | $\iff$ | $(\Lambda^2(xJ(2), x^{-1} J(2))$ |
| $(xJ(2), x^{-1} J(2), 1)$ | $\iff$ | $(\Lambda^2(J(2), x, x^{-1})$ |
| $(xy, xy^{-1}, x^{-1} y, xy^{-1}, 1)$ | $\iff$ | $(\Lambda^2(x, y, y^{-1}, x^{-1})$ |

The proof of the above statements is a straightforward computation using bases and is hence omitted.

4. THE POSSIBLE CASES

Recall the following result of Scott [10]:

**Lemma. 4.1.** Let $K$ be an algebraically closed field and let $V$ be an $n$-dimensional $K$-vector space. Let $(T_1, \ldots, T_{r+1}) \in \text{GL}(V)^{r+1}$ with $T_1 \cdots T_{r+1} = 1$ such that $\langle T_1, \ldots, T_{r+1} \rangle$ is irreducible. Then the following statements hold:

\[
\sum_{i=1}^{r+1} \text{rk}(T_i - 1) \geq 2n \quad (\text{Scott Formula})
\]

\[
\sum_{i=1}^{r+1} \dim(C_{\text{GL}(V)}(T_i)) \leq (r - 1)n^2 + 2 \quad (\text{Dimension count}),
\]

where $\dim(C_{\text{GL}(V)}(T_i))$ denotes the dimension of the centralizer of $T_i$ in $\text{GL}(V)$.

Let $(J_1, \ldots, J_{r+1})$ be a tuple of matrices in Jordan form which occur in the group $G_2$. We want to answer the question, if there exists a tuple of elements $(T_1, \ldots, T_{r+1}) \in G_2(\mathbb{C})^{r+1}$ having Jordan forms $J_1, \ldots, J_{r+1}$ (resp.) which is the monodromy tuple of an orthogonally rigid local system whose monodromy group
is Zariski dense in the group $G_2(\mathbb{C})$ (especially, this implies that $T_1 \cdots T_{r+1} = 1$ and that the monodromy group $(T_1, \ldots, T_{r+1})$ is irreducible). By definition, orthogonal rigidity implies that

$$\sum_{i=1}^{r+1} \text{codim}(C_{O_7}(T_i)) = \sum_{i=1}^{r+1} (21 - \text{dim}(C_{O_7}(T_i))) = 2 \text{dim}(O_7) = 42. \quad (4.1)$$

Moreover, again by the irreducibility, the tuple $(T_1, \ldots, T_{r+1})$ satisfies the formulae of Lemma 4.1. The condition (4.1) together with the list of possible centralizer dimensions gives the following possibilities for tuples of centralizer dimensions $N^O_7 := \text{dim}(C_{O_7}(T_i))$, resp. $N^{\text{GL}_7}_i := \text{dim}(C_{\text{GL}_7}(T_i))$ (in the case that they contradict the Scott Formula (Lemma 4.1) this is noted in the last column by red.):

| case | $(N^O_7, \ldots, N^O_{r+1})$ | $(N^{\text{GL}_7}_1, \ldots, N^{\text{GL}_7}_{r+1})$ | remarks |
|------|-------------------------------|---------------------------------|---------|
| $P_1$ | (13, 5, 3) | (29, 13, 9) | red. |
| | | (29, 13, 7) | |
| | | (29, 11, 9) | |
| | | (29, 11, 7) | |
| $P_2$ | (9, 9, 3) | (25, 25, 9/7) | red. |
| | | (25, 19, 9) | red. |
| | | (25, 19, 7) | lin. rigid, s. [8] |
| | | (19, 19, 9) | |
| | | (19, 19, 7) | |
| $P_3$ | (9, 7, 5) | (25, 17, 13/11) | red. |
| | | (19, 17, 13) | |
| | | (19, 17, 11) | |
| $P_4$ | (7, 7, 7) | (17, 17, 17) | red., s. [8] |
| $P_5$ | (8, 8, 8, 4) | (13, 13, 9, 7) | |

**Remark.** 4.2. The only irreducible subgroups of $G_2(\mathbb{C})$ are either finite or the group $\text{SL}_2(\mathbb{C})$. Thus the Zariski closure of an irreducible subgroup containing an unipotent element with Jordan form different from $J(7)$ is $G_2(\mathbb{C})$.

5. The main result

**Theorem.** 5.1. Let $\mathcal{L}$ be an orthogonally rigid local system on a punctured projective line $\mathbb{P}^1 \setminus \{x_1, \ldots, x_{r+1}\}$ of rank 7 whose monodromy group is dense in the exceptional simple group $G_2$. If $\mathcal{L}$ has nontrivial local monodromy at $x_1, \ldots, x_{r+1}$, then $r = 2, 3$ and $\mathcal{L}$ can be constructed by applying iteratively a sequence of tensor operations, middle convolutions $\text{MC}_\chi$, and rational pullbacks, applied to a local system of rank one. Moreover the Jordan canonical forms (under the additional assumptions as given in Section [3]) of the local monodromies of $\mathcal{L}$ are as follows:
(i) The case $P_1$:

| $P_1$ | 
|-------|
| 1 | $(\mathbf{J}(2), \mathbf{J}(2), 1, 1, 1)$ $(i, i, -1, 1, -1, -i, -i)$ $\mathbf{J}(7)$ |
| 2 | $(\mathbf{J}(2), \mathbf{J}(2), 1, 1, 1)$ $(i, i, -1, 1, -1, -i, -i)$ $(\omega \mathbf{J}(3), \omega^{-1} \mathbf{J}(3), 1)$ |
| 3 | $(\mathbf{J}(2), \mathbf{J}(2), 1, 1, 1)$ $(i, i, -1, 1, -1, -i, -i)$ $(x \mathbf{J}(2), x^{-1} \mathbf{J}(2), x^2, x^{-2}, 1)$ |
| 4 | $(\mathbf{J}(2), \mathbf{J}(2), 1, 1, 1)$ $(i, i, -1, 1, -1, -i, -i)$ $(x \mathbf{J}(2), x^{-1} \mathbf{J}(2), \mathbf{J}(3))$ |
| 5 | $(\mathbf{J}(2), \mathbf{J}(2), 1, 1, 1)$ $(i, i, -1, 1, -1, -i, -i)$ $(x, y, y, 1, (xy)^{-1}, y^{-1}, x^{-1})$ $\pm i \notin \{x, y, xy\}$ |
| 6 | $(\mathbf{J}(2), \mathbf{J}(2), 1, 1, 1)$ $(-\mathbf{J}(2), -\mathbf{J}(2), \mathbf{J}(3))$ $\mathbf{J}(7)$ |
| 7 | $(\mathbf{J}(2), \mathbf{J}(2), 1, 1, 1)$ $(-\mathbf{J}(2), -\mathbf{J}(2), \mathbf{J}(3))$ $(\omega \mathbf{J}(3), \omega^{-1} \mathbf{J}(3), 1)$ |
| 8 | $(\mathbf{J}(2), \mathbf{J}(2), 1, 1, 1)$ $(-\mathbf{J}(2), -\mathbf{J}(2), \mathbf{J}(3))$ $(x \mathbf{J}(2), x^{-1} \mathbf{J}(2), x^2, x^{-2}, 1)$ |
| 9 | $(\mathbf{J}(2), \mathbf{J}(2), 1, 1, 1)$ $(-\mathbf{J}(2), -\mathbf{J}(2), \mathbf{J}(3))$ $(x \mathbf{J}(2), x^{-1} \mathbf{J}(2), \mathbf{J}(3))$ |
| 10 | $(\mathbf{J}(2), \mathbf{J}(2), 1, 1, 1)$ $(-\mathbf{J}(2), -\mathbf{J}(2), \mathbf{J}(3))$ $(x, y, y, 1, (xy)^{-1}, y^{-1}, x^{-1})$ |
| 11 | $(\mathbf{J}(2), \mathbf{J}(2), 1, 1, 1)$ $(z, z, z^{-1}, z^{-1}, z^2, z^{-2}, 1)$ $(-\mathbf{J}(3), -1, \mathbf{J}(3))$ |
| 12 | $(\mathbf{J}(2), \mathbf{J}(2), 1, 1, 1)$ $(z, z, z^{-1}, z^{-1}, z^2, z^{-2}, 1)$ $(i \mathbf{J}(2), i^{-1} \mathbf{J}(2), -1, -1, 1)$ |
| 13 | $(\mathbf{J}(2), \mathbf{J}(2), 1, 1, 1)$ $(z, z, z^{-1}, z^{-1}, z^2, z^{-2}, 1)$ $(x, -1, -x, 1, -x^{-1}, -1, x^{-1})$ |
| 14 | $(\mathbf{J}(2), \mathbf{J}(2), 1, 1, 1)$ $(z, z, z^{-1}, z^{-1}, z^2, z^{-2}, 1)$ $\mathbf{J}(7)$ |
| 15 | $(\mathbf{J}(2), \mathbf{J}(2), 1, 1, 1)$ $(z, z, z^{-1}, z^{-1}, z^2, z^{-2}, 1)$ $(\omega \mathbf{J}(3), \omega^{-1} \mathbf{J}(3), 1)$ |
| 16 | $(\mathbf{J}(2), \mathbf{J}(2), 1, 1, 1)$ $(z, z, z^{-1}, z^{-1}, z^2, z^{-2}, 1)$ $(x \mathbf{J}(2), x^{-1} \mathbf{J}(2), x^2, x^{-2}, 1)$ $x \neq z^{\pm 1}$ |
| 17 | $(\mathbf{J}(2), \mathbf{J}(2), 1, 1, 1)$ $(z, z, z^{-1}, z^{-1}, z^2, z^{-2}, 1)$ $(x \mathbf{J}(2), x^{-1} \mathbf{J}(2), \mathbf{J}(3))$ $xz^{\pm 1} \neq 1$ |
| 18 | $(\mathbf{J}(2), \mathbf{J}(2), 1, 1, 1)$ $(z, z, z^{-1}, z^{-1}, z^2, z^{-2}, 1)$ $(x, y, y, 1, (xy)^{-1}, y^{-1}, x^{-1})$ $z \notin \{x, y, xy, 1, (xy)^{-1}, y^{-1}, x^{-1}\}$ |
| 19 | $(\mathbf{J}(2), \mathbf{J}(2), 1_3)$ $(\omega \mathbf{J}(2), \omega^{-1} \mathbf{J}(2), \omega, \omega^{-1}, 1)$ $(\omega \mathbf{J}(3), -1, \mathbf{J}(3))$ |
| 20 | $(\mathbf{J}(2), \mathbf{J}(2), 1_3)$ $(\omega \mathbf{J}(2), \omega^{-1} \mathbf{J}(2), \omega, \omega^{-1}, 1)$ $(\omega \mathbf{J}(2), i^{-1} \mathbf{J}(2), -1, -1, 1)$ |
| 21 | $(\mathbf{J}(2), \mathbf{J}(2), 1_3)$ $(\omega \mathbf{J}(2), \omega^{-1} \mathbf{J}(2), \omega, \omega^{-1}, 1)$ $(x, -1, -x, 1, -x^{-1}, -1, x^{-1})$ $x^6 \neq 1$ |
| 22 | $(\mathbf{J}(2), \mathbf{J}(2), 1_3)$ $(\omega \mathbf{J}(2), \omega^{-1} \mathbf{J}(2), \omega, \omega^{-1}, 1)$ $\mathbf{J}(7)$ |
| 23 | $(\mathbf{J}(2), \mathbf{J}(2), 1_3)$ $(\omega \mathbf{J}(2), \omega^{-1} \mathbf{J}(2), \omega, \omega^{-1}, 1)$ $(x \mathbf{J}(2), x^{-1} \mathbf{J}(2), x^2, x^{-2}, 1)$ $x^6 \neq 1$ |
| 24 | $(\mathbf{J}(2), \mathbf{J}(2), 1_3)$ $(\omega \mathbf{J}(2), \omega^{-1} \mathbf{J}(2), \omega, \omega^{-1}, 1)$ $(x \mathbf{J}(2), x^{-1} \mathbf{J}(2), \mathbf{J}(3))$ $x^3 \neq 1$ |
| 25 | $(\mathbf{J}(2), \mathbf{J}(2), 1_3)$ $(\omega \mathbf{J}(2), \omega^{-1} \mathbf{J}(2), \omega, \omega^{-1}, 1)$ $(x, y, y, 1, (xy)^{-1}, y^{-1}, x^{-1})$ $x^3 \neq 1, y^3 \neq 1, (xy)^3 \neq 1$ |
(ii) The case $P_2$: The linearly rigid case $P_2(25, 19, 7)$ is settled in [S] and is therefore omitted.

| $P_2(19, 19, 9)$ | # |
|-----------------|---|
| 2               | (J(2), J(2), J(3)) (J(2), J(2), J(3)) (iJ(2), i^{-1}J(2), 1, -1_2) 2 |
| 3               | (J(2), J(2), J(3)) (J(2), J(2), J(3)) (x, -x, -x^{-1}, x^{-1}, 1, -1_2) 4 |

| $P_2(19, 19, 7)$ | # |
|-----------------|---|
| 1               | (J(2), J(2), J(3)) (J(2), J(2), J(3)) J(7) 1 |
| 2               | (J(2), J(2), J(3)) (J(2), J(2), J(3)) (ω_j(3), ω^{-1}J(3), 1) 1 |
| 3               | (J(2), J(2), J(3)) (J(2), J(2), J(3)) (x_j(2), x^{-1}J(2), x^2, x^{-2}, 1) 2 |
| 4               | (J(2), J(2), J(3)) (J(2), J(2), J(3)) (x_j(2), x^{-1}J(2), J(3)) 2 |
| 5               | (J(2), J(2), J(3)) (J(2), J(2), J(3)) (x, y, xy, 1, (xy)^{-1}, y^{-1}, x^{-1}) 4 |

(iii) The case $P_3$:

| $P_3$ | # |
|-------|---|
| 5     | (J(2), J(2), J(3)) (−J(2), −J(2), 1_3) (i, i, −1, 1, i^{-1}, i^{-1}, −1) 2 |
| 6     | (J(2), J(2), J(3)) (−J(2), −J(2), 1_3) (−J(2), −J(2), J(3)) 1 |
| 7     | (J(2), J(2), J(3)) (−J(2), −J(2), 1_3) (x, x^2, 1, x^{-1}, x^{-1}, x^{-2}) 2 |
| 8     | (J(2), J(2), J(3)) (−J(2), −J(2), 1_3) (ω_j(2), ω^{-1}J(2), ω, ω^{-1}, 1) 2 |
| 9     | (J(2), J(2), J(3)) (z, z, z^{-1}, z^{-1}, 1_3) (i, i, −1, 1, i^{-1}, i^{-1}, −1) 2 |
|       | z^4 ≠ 1 |
| 10    | (J(2), J(2), J(3)) (z, z, z^{-1}, z^{-1}, 1_3) (−J(2), −J(2), J(3)) 2 |
| 11    | (J(2), J(2), J(3)) (z, z, z^{-1}, z^{-1}, 1_3) (x, x, x^2, 1, x^{-1}, x^{-1}, x^{-2}) 2 |
|       | x^{±3}z^{±1} ≠ 1, x^{±1}z^{±1} ≠ 1 |
| 11    | (J(2), J(2), J(3)) (z, z, z^{-1}, z^{-1}, 1_3) (x, x, x^2, 1, x^{-1}, x^{-1}, x^{-2}) 1 |
|       | x^{±3}z^{±1} = 1, x^{±1}z^{±1} = 1 |
| 12    | (J(2), J(2), J(3)) (z, z, z^{-1}, z^{-1}, 1_3) (ω_j(2), ω^{-1}J(2), ω, ω^{-1}, 1) 2 |
|       | z^3 ≠ 1 |
| 12    | (J(2), J(2), J(3)) (z, z, z^{-1}, z^{-1}, 1_3) (ω_j(2), ω^{-1}J(2), ω, ω^{-1}, 1) 1 |
|       | z^3 = 1 |

(iv) The case $P_5$:

| $P_5$ | # |
|-------|---|
| 1     | (J(2), J(2), 1_3) (J(2), J(2), 1_3) (ω_1_3, 1, ω^{-1}1_3) (J(3), J(3), 1) 1 |
| 2     | (J(2), J(2), 1_3) (J(2), J(2), 1_3) (ω_1_3, 1, ω^{-1}1_3) (−J(2), −J(2), 1_3) 1 |
| 3     | (J(2), J(2), 1_3) (J(2), J(2), 1_3) (ω_1_3, 1, ω^{-1}1_3) (x, x, x^{-1}, x^{-1}, 1_3) 1 |
For each tuple of Jordan forms in the above list, there exists a local system $\mathcal{L}$ of rank 7 whose monodromy group is dense in the exceptional simple group $G_2$. The cardinality of equivalence classes with given local monodromies under the diagonal conjugation in $G_2$ is listed under #.

Proof. The proof is divided into the cases $P_1, P_2, P_3, P_5$, where each case is dealt with in one of the following subsections:

5.1. The case $P_1$. We can assume $x_1 = 0, x_2 = 1, x_3 = \infty$. If $\phi, \phi' : \pi_1(\mathbb{G}_m) \to \mathbb{C}^\times$ are characters, there exists a unique local system $\mathcal{L}(\phi, \phi')$ of rank one on $\mathbb{A}^1 \setminus \{0, 1\}$ whose local monodromies at 0, 1 is $\phi, \phi'$, resp. The functor, which sends a local system $\mathcal{L}$ on $\mathbb{A}^1 \setminus \{0, 1\}$ to $\mathcal{L} \otimes \mathcal{L}(\phi, \phi')$ is denoted $\text{MT}_{\mathcal{L}(\phi, \phi')}$. The irreducibility condition and the Scott Formula (Lemma 4.1) imply that only the possibilities listed in the case $P_1$ of Thm. 5.1 occur. At first we prove that in each of the listed cases, there exists an orthogonally rigid local system that can be reduced to a rank 1 system via the middle convolution and tensor products. Applying the functor

$$M_\phi = \text{MC}_\phi \circ \text{MT}_{\mathcal{L}(1, \phi^{-1})} \circ \text{MC}_{\phi^{-1}} \circ \text{MT}_{\mathcal{L}(1, \phi)};$$

where

$$\phi = \begin{cases} 
  i & (1) - 5 \\
  -1 & (6) - 10 \\
  z & (11) - 18 \\
  \omega & (19) - 26 
\end{cases},$$

we obtain an orthogonally rigid local system of rank 5 or 6 by [7 Thm. 2.4.(i)] and [6 Cor. 5.15]. The change of the Jordan form of the local monodromies in each step can be traced via Lemma 2.2. The result is

| nr. | rk | (J(2), J(2), 1) | (J(3), 1, 1, 1) | J(5) |
|-----|----|----------------|-----------------|------|
| 1   | 5  | (J(2), J(2), 1) | (J(3), 1, 1, 1) | J(5) |
| 2   | 6  | (J(2), J(2), 1, 1) | (J(3), 1, 1, 1) | (\omega J(3), J(3), J(3)) |
| 3   | 6  | (J(2), J(2), 1, 1) | (J(3), 1, 1, 1) | (x J(2), x^{-1} J(2), x^2, x^{-2}) |
| 4   | 5  | (J(2), J(2), 1) | (J(3), 1, 1, 1) | (x J(2), x^{-1} J(2), 1) |
| 5   | 6  | (J(2), J(2), 1, 1) | (J(3), 1, 1, 1) | (x, y, xy, (xy)^{-1}, y^{-1}, x^{-1}) |
| 6   | 5  | (J(2), J(2), 1) | J(5) | J(5) |
| 7   | 6  | (J(2), J(2), 1, 1) | (J(5), 1) | (\omega J(3), J(3), J(3)) |
| 8   | 6  | (J(2), J(2), 1, 1) | (J(5), 1) | (x J(2), x^{-1} J(2), x^2, x^{-2}) |
| 9   | 5  | (J(2), J(2), 1) | J(5) | (x J(2), x^{-1} J(2), 1) |
| 10  | 6  | (J(2), J(2), 1, 1) | (J(5), 1) | (x, y, xy, (xy)^{-1}, y^{-1}, x^{-1}) |
Using tensor identities $A^2 \text{SL}_4 = \text{SO}_6$ and $A^2 \text{SP}_4 = \text{SO}_5$ and the effect on the Jordan forms listed in Section 3, we obtain up to two linearly rigid local systems of rank 4. (linear rigidity is synonymously used as physical rigidity in [12] and, in the above terms, it is the same as $\text{GL}_n$-rigidity) In some cases we get only one local system due to the Scott Formula. Their local monodromies in Jordan form is as follows.

| nr. | rk | (J(2), J(2), 1) | (J(3), z^2, z^{-2}) | (−J(3), −1, 1) |
|-----|----|-----------------|---------------------|-----------------|
| 11  | 5  | (J(2), J(2), 1) | (J(3), z^2, z^{-2}) | (−J(3), −1, 1) |
| 12  | 6  | (J(2), J(2), 1) | (J(3), z^2, z^{-2}) | (iJ(2), i^{-1}J(2), −1, −1) |
| 13  | 6  | (J(2), J(2), 1) | (J(3), z^2, z^{-2}) | (x, −1, −x, −x^{-1}, −1, x^{-1}) |
| 14  | 5  | (J(2), J(2), 1) | (J(3), z^2, z^{-2}) | J(5) |
| 15  | 6  | (J(2), J(2), 1) | (J(3), z^2, z^{-2}) | (ωJ(3), ω^{-1}J(3)) |
| 16  | 6  | (J(2), J(2), 1) | (J(3), z^2, z^{-2}) | (xJ(2), x^{-1}J(2), x, x^{-2}) |
| 17  | 5  | (J(2), J(2), 1) | (J(3), z^2, z^{-2}) | (xJ(2), x^{-1}J(2), 1) |
| 18  | 6  | (J(2), J(2), 1) | (J(3), z^2, z^{-2}) | (x, y, xy, (xy)^{-1}, y^{-1}, x^{-1}) |
| 19  | 5  | (J(2), J(2), 1) | (J(3), ω, ω^{-1}) | (−J(3), −1, 1) |
| 20  | 6  | (J(2), J(2), 1) | (J(3), ω, ω^{-1}) | (iJ(2), i^{-1}J(2), −1, −1) |
| 21  | 6  | (J(2), J(2), 1) | (J(3), ω, ω^{-1}) | (xJ(2), x^{-1}J(2), 1, x^2, x^{-2}) |
| 22  | 5  | (J(2), J(2), 1) | (J(3), ω, ω^{-1}) | J(5) |
| 23  | 6  | (J(2), J(2), 1) | (J(3), ω, ω^{-1}) | (xJ(2), x^{-1}J(2), 1, x, x^{-2}) |
| 24  | 5  | (J(2), J(2), 1) | (J(3), ω, ω^{-1}) | (xJ(2), x^{-1}J(2), 1) |
| 25  | 6  | (J(2), J(2), 1) | (J(3), ω, ω^{-1}) | (x, y, xy, (xy)^{-1}, y^{-1}, x^{-1}) |
| \( n \) | \((J(2), 1, 1)\) | \((zJ(2), z^{-1}J(2))\) | \( \pm J(4)\) |
|---|---|---|---|
| 14 | \((J(2), 1, 1)\) | \((zJ(2), z^{-1}J(2))\) | \(\omega J(3), 1\) |
| 15 | \((J(2), 1, 1)\) | \((zJ(2), z^{-1}J(2))\) | \(-(\omega J(3), 1)\) |
| \(z \neq 3\)|
| 16 | \((J(2), 1, 1)\) | \((zJ(2), z^{-1}J(2))\) | \((xJ(2), 1, x^{-2})\) |
| 17 | \((J(2), 1, 1)\) | \((zJ(2), z^{-1}J(2))\) | \(- (xJ(2), 1, x^{-2})\) |
| \(-zx^{\pm 2} \neq 1, -zx^{\pm 1} \neq 1\)|
| 18 | \((J(2), 1, 1)\) | \((zJ(2), z^{-1}J(2))\) | \((x, y, (xy)^{-1}, 1)\) |
| 19 | \((J(2), 1, 1)\) | \((zJ(2), z^{-1}J(2))\) | \(-(x, y, (xy)^{-1}, 1)\) |
| \(-z^{\pm 1} \in \{x, y, xy\}\)|
| 20 | \((J(2), 1, 1)\) | \((zJ(2), z^{-1}J(2))\) | \((J(2), -J(2))\) |
| 21 | \((J(2), 1, 1)\) | \((zJ(2), z^{-1}J(2))\) | \(\pm (iJ(2), 1, -1)\) |
| 22 | \((J(2), 1, 1)\) | \((zJ(2), z^{-1}J(2))\) | \(\pm (x, -x^{-1}, 1, -1)\) |
| 23 | \((J(2), 1, 1)\) | \((zJ(2), z^{-1}J(2))\) | \((xJ(2), 1, x^{-2})\) |
| 24 | \((J(2), 1, 1)\) | \((zJ(2), z^{-1}J(2))\) | \(-(xJ(2), 1, x^{-2})\) |
| \(x^{12} \neq 1\)|
| 25 | \((J(2), 1, 1)\) | \((zJ(2), z^{-1}J(2))\) | \((x, y, (xy)^{-1}, 1)\) |
| 26 | \((J(2), 1, 1)\) | \((zJ(2), z^{-1}J(2))\) | \(-(x, y, (xy)^{-1}, 1)\) |
| \(x^6 \neq 1, y^6 \neq 1, (xy)^6 \neq 1\)|

**Remark. 5.2.** These tuples of local monodromies in Jordan form arise from hypergeometric irreducible local systems. This can be checked by [2] or the Katz Existence algorithm.

From the discussion above we know that there exist at most 2 orthogonally rigid local systems having \(G_2\)-monodromy with the same local monodromies. We reduce the monodromy tuple modulo \(l\) in order to show that there exists at most 1 such local system. Via Prop. A.1 we can compute the normalized structure constant \(n(cl(\sigma_1), \ldots, cl(\sigma_{r+1}))\) (the definition is recalled in the Appendix) corresponding to the reduced monodromy tuple \((\sigma_1, \ldots, \sigma_{r+1})\) via the generic character table of the group \(G_2(q)\). Using CHEVIE, [10], we obtain in the notation of Chang and Ree (cf. Rem. A.5) the following list. (Note that the output of
CHEVIE comes along with a list of possible exceptions depending on the eigenvalues of the conjugacy classes. In most of the cases these exceptions correspond to ours obtained from the Scott Formula. In the remaining cases one can proceed as follows. The characters of $G_2(q)$ fall into finitely many families $\mathcal{F}_j$. Using the character table of $G := G_2(q)$ in CHEVIE or [11] Anhang B, one easily sees that the contribution of most those families

$$\left| \frac{|G|^{r-1}}{\prod_i |C_G(\sigma_i)|} \sum_{\chi \in \mathcal{F}_j} \frac{\prod_i \chi(\sigma_i)}{\chi(1)^{r-1}} \right|$$



to the normalized structure constant is bounded by $c/q$, where $c$ is constant. Thus we finally get that $\lim_k([n(C(q^k))]) < 2$.

| $n(u_2, k_{2,2}, u_6)$ | 1 | $n(u_2, k_{3,1}, u_6)$ | 1 | $n(u_2, h_{1b}, u_6)$ | 1 |
| $n(u_2, k_{2,2}, k_{2,3})$ | 2 - $\frac{3}{2q}$ | $n(u_2, k_{3,1}, k_{2,3})$ | 1 | $n(u_2, h_{1b}, k_{2,3})$ | 1 |
| $n(u_2, k_{2,2}, k_{2,4})$ | 2 - $\frac{1}{2q}$ | $n(u_2, k_{3,1}, k_{2,4})$ | 0 | $n(u_2, h_{1b}, k_{2,4})$ | 0 |
| $n(u_2, k_{2,2}, k_{3,2})$ | 1 | $n(u_2, k_{3,1}, k_{3,2})$ | < 1 | $n(u_2, h_{1b}, k_{3,2})$ | 1 |
| $n(u_2, k_{2,2}, k_{3,3,1})$ | 0 | $n(u_2, k_{3,1}, k_{3,3,1})$ | < 1 | $n(u_2, h_{1b}, k_{3,3,1})$ | 0 |
| $n(u_2, k_{2,2}, k_{3,3,2})$ | 0 | $n(u_2, k_{3,1}, k_{3,3,2})$ | < 1 | $n(u_2, h_{1b}, k_{3,3,2})$ | 0 |
| $n(u_2, k_{2,2}, h_{1a,1})$ | 1 | $n(u_2, k_{3,1}, h_{1a,1})$ | 1 | $n(u_2, h_{1b}, h_{1a,1})$ | 1 |
| $n(u_2, k_{2,2}, h_{1b,1})$ | 1 | $n(u_2, k_{3,1}, h_{1b,1})$ | 1 | $n(u_2, h_{1b}, h_{1b,1})$ | 1 |
| $n(u_2, k_{2,2}, h_1)$ | 1 | $n(u_2, k_{3,1}, h_1)$ | 1 | $n(u_2, h_{1b}, h_1)$ | 1 |

By Thm. [A.3] we have hence at most one orthogonally rigid local system having $G_2$-monodromy with given local monodromies.

To show the existence, we construct a differential operator by translating the middle convolution operations and tensor product operations to the level of differential operators, cf. Remark [2.7]. To simplify the construction we rather work with the middle Hadamard product than with the middle convolution. Let $L = \vartheta(\vartheta - c)(\vartheta - d)(\vartheta + (c + d)) - x(\vartheta + a)^2(\vartheta + 1 - a)^2$ be hypergeometric with Riemann scheme

$$\mathcal{R}(L) = \begin{bmatrix} 0 & 1 & \infty \\ 0 & 0 & a \\ c & 1 & a \\ d & 1 & 1-a \\ -c-d & 2 & 1-a \end{bmatrix},$$

and $L_b = \vartheta - x(\vartheta + b)$, $b \in \{a, 1 - a\}$. Thus we get the formally self adjoint operator

$$P_1 := L_a \ast_H L_{1-a} \ast_H \Lambda^2(L),$$


\[ P_1 = \vartheta (\vartheta - d) (\vartheta + d) (\vartheta - c) (\vartheta + c) (c + d + \vartheta) (-c - d + \vartheta) - \\
x (2\vartheta + 1) (\vartheta + a) (\vartheta + 1 - a) \cdot (\vartheta^4 + 2\vartheta^3 + (2 - c^2 - d^2 - 2a^2 + 2a - cd) \vartheta^2 + \\
(1 - c^2 - d^2 - 2a^2 + 2a - cd) \vartheta - 2a (a - 1)(a^2 - a - cd - d^2 - c^2 + 1)) + \\
x^2 (\vartheta + 1) (\vartheta + 2 - a) (\vartheta + 2 - a) (\vartheta + a) (\vartheta + 2 - a) (\vartheta + 1 - a) (\vartheta + 2 a)
\]

with Riemann scheme

\[
R(P_1) = \begin{cases} 
0 & 1 & \infty \\
0 & 0 & 1 \\
c & 1 & a \\
d & 1 & a + 1 \\
c + d & 2 & 2a \\
-c - d & 3 & 2 - 2a \\
-d & 3 & 1 - a \\
-c & 4 & 2 - a 
\end{cases}
\]

Specializing the parameters \(a, c, d\) suitably we get a differential equation for all the \(P_i\)-cases if we can show that it has \(G_2\)-monodromy. Using MAPLE one gets that \(\Lambda^2(L)\) has degree 14. Thus \(\Lambda^2\) of the monodromy representation decomposes into a rank 7 and a rank 14 representation. Hence, by Table 5 in [14] the monodromy group is contained in \(G_2\). Thus the orthogonally rigid local systems with \(G_2\)-monodromy in the case \(P_1\) are uniquely determined by their local monodromy.

### 5.2. The case \(P_2\).

In the case \(P_2(19,19,9)\) the list of possible Jordan forms of the local monodromies is

| nr. | rk | \(J(2), J(2), J(3)\) | \(J(2), J(2), J(3)\) | \(-J(3), -1, J(3)\) |
|-----|----|-----------------|-----------------|-----------------|
| 1   | 7  | \(J(2), J(2), J(3)\) | \(J(2), J(2), J(3)\) | \(-J(3), -1, J(3)\) |
| 2   | 7  | \(iJ(2), i^{-1}J(2), -1, -1, 1\) |
| 3   | 7  | \(x, -x, -x^{-1}, 1, -1, -1\) |

Applying the functor

\[ M_\varphi = \text{MT}_{L(-1,-1)} \circ \text{MC}_{-1} \circ \text{MT}_{L(-1,-1)} \circ \text{MC}_{-1}, \]

we obtain in each case an orthogonally rigid local system of rank 5 with the following tuple of Jordan forms

| nr. | rk | \(-J(2), -J(2), 1\) | \(-J(2), -J(2), 1\) | \(-J(3), J(3)\) |
|-----|----|-----------------|-----------------|-----------------|
| 1   | 5  | \(-J(2), -J(2), 1\) | \(-J(2), -J(2), 1\) | \(-J(3), J(3)\) |
| 2   | 5  | \(-J(2), -J(2), 1\) | \(-J(2), -J(2), 1\) | \(-J(2), -J(2), 1\) |
| 3   | 5  | \(-J(2), -J(2), 1\) | \(-J(2), -J(2), 1\) | \(-J(2), -J(2), 1\) |

Using the isomorphism

\[ \Lambda^2 SP_4 \cong SO_5 \]
we get

| nr. | rk | J(2), 1, 1 | J(2), 1, 1 | J(2), i, i⁻¹ |
|-----|----|------------|------------|--------------|
| 2   | 4  | −(J(2), 1, 1), (−J(2), 1, 1) | −(J(2), , i⁻¹) |
| 3   | 4  | −(J(2), 1, 1), (−J(2), 1, 1) | ±(ix, i, −i, (ix)⁻¹) |

Applying MC⁻¹ we get an orthogonally rigid local system of rank 3, 4 resp., with the following tuple of Jordan forms of the local monodromies.

| nr. | rk | (J(3), 1) | (J(3), 1) | ±(ix, i, −i, (ix)⁻¹) |
|-----|----|----------|----------|-------------------|
| 2   | 3  | J(3)     | J(3)     | (1, i, i⁻¹)       |
| 3   | 4  | (J(3), 1) | (J(3), 1) | ±(ix, i, −i, (ix)⁻¹) |

Via the isomorphisms

\[ \text{sym}^2 \text{SP}_2 = \text{SO}_3, \quad \text{SL}_2 \otimes \text{SL}_2 = \text{SO}_4 \]

we can decompose it into linearly rigid irreducible local systems \( L_1 \) and \( L_2 \) of rank 2 with the following tuple of Jordan forms.

| nr. | \( L_i \) | J(2) | J(2) | ±(ζ_8, ζ_8⁻¹) |
|-----|---------|------|------|---------------|
| 2   | \( L_1 = L_2 \) | J(2) | J(2) | ±(ζ_8, ζ_8⁻¹) |
| 3   | \( L_1 \) | J(2) | J(2) | (y, y⁻¹)       |
|     | \( L_2 \) | J(2) | J(2) | ±(iy, i⁻¹y⁻¹) |

Remark. 5.3. The cases (2) and (3) are quadratic pullbacks of linearly rigid local systems (cf. [8]) with \( G_2 \)-monodromy \( (A, B, C) \), \( ABC = 1, A^2 = 1 \), via

\[ (A, B, C) \mapsto (A^2, B^A, B, C^2). \]

| nr. | rk | (J(2), J(2), J(3)) | (ζ_8J(2), ζ_8⁻¹J(2), ζ_8^2, ζ_8⁻², 1) |
|-----|----|---------------------|----------------------------------|
| 2   | 7  | (−1, 1, 3)          | (ζ_8J(2), ζ_8⁻¹J(2), ζ_8^2, ζ_8⁻², 1) |
|     |     | (−1, 1, 3)          | (−ζ_8J(2), −ζ_8⁻¹J(2), ζ_8^2, ζ_8⁻², 1) |
| 3   | 7  | (−1, 1, 3)          | (y, iy⁻¹, iy⁻¹, 1, i, −i)       |
|     |     | (−1, 1, 3)          | (y, −iy⁻¹, iy⁻¹, 1, i, −i)       |

This shows the existence of two, four resp., orthogonally rigid local systems with \( G_2 \) monodromy and the same local monodromy in the case (2), (3) resp..

In the case \( P_2(19, 19, 7) \) the list of possible Jordan forms of the local monodromies is as follows:
Otherwise there exists an element \( h \in G_2 \) such that
\[
g_1^h = g_2, \quad g_2^h = g_1^{q_2}, \quad g_3^h = g_3.
\]
Thus
\[
(g_1, g_2, g_3)^{h g_2^{-1}} = (g_2, g_1, g_3^{h g_2^{-1}}).
\]
(h_{g_2}^{-1})^2 \in Z(G_2) = 1.

This gives \( h^2 = g_1 g_2 \). Hence the pullback of \((g_1, h_{g_2}^{-1}, (g_1 h_{g_2}^{-1})^{-1})\) is

\[
(g_1, g_1^{(h_{g_2}^{-1})^{-1}}, (h_{g_2}^{-1})^2, (g_1 h_{g_2}^{-1})^{-2}) = (g_1, g_2, 1, (g_1 h_{g_2}^{-1})^{-2}) =
(g_1, g_2, h^{-2}) = (g_1, g_2, g_3).
\]

Thus in the case (1) there is only one irreducible triple and all irreducible tuples in the case \( P_2(19, 19, 7) \) arise from pullbacks of linearly rigid irreducible triples. For completeness we write down the differential operators with linearly rigid \( G_2 \)-monodromy tuple. For this we translate the construction from [S] to the level of differential operators and obtain formally selfdual operators \( L \) with Riemann-scheme, where

\[
L := 8 (\vartheta - 1)(\vartheta - 2)(\vartheta - 3)(2 \vartheta - 1)(2 \vartheta - 3)(2 \vartheta - 5)(2 \vartheta - 7)
-8x (2 \vartheta - 5)(\vartheta - 1)(\vartheta - 2)(2 \vartheta - 3)(8 \vartheta^2 - 24 \vartheta + 25 - 4(p^2 + q^2 + pq))
+2x^2(\vartheta - 1)(2 \vartheta - 1)(2 \vartheta - 3)(96 \vartheta^4 - 384 \vartheta^3 + (720 - 96(q^2 + p^2 + pq))\vartheta^2 +
(192(q^2 + pq) - 672)\vartheta + 141 + 8(p^2 - 1 + qp + q^2)((2p^2 - 15 + 2qp + 2q^2)) +
x^3(2\vartheta - 1)(-256\vartheta^6 + 768\vartheta^5 + (-1312 + 384(p^2 + q^2 + pq))\vartheta^4 +
(1344 - 768(p^2 + q^2 + pq))\vartheta^3 - (160 + 32(4(p^2 + q^2 + pq) - 21)(p^2 - 1 + qp + q^2))\vartheta^2
+(32(4(p^2 + q^2 + pq) - 9)(p^2 - 1 + qp + q^2))\vartheta +
(64pq^2(q^2 + 2qp + p^2) - 3) - 8(6(p^2 + q^2 + pq) - 5)(p^2 - 1 + qp + q^2)) +
128x^4\vartheta(\vartheta - q)(\vartheta + q)(\vartheta - p)(\vartheta + p)(\vartheta + p + q)(\vartheta - p - q),
\]

\[
\mathcal{R}(L) = \begin{cases} 
0 & 1 & \infty \\
1/2 & 0 & 0 \\
1 & 0 & q \\
3/2 & 1 & p \\
2 & 1 & p + q \\
5/2 & 1 & -p - q \\
3 & 2 & -q \\
7/2 & 2 & -p 
\end{cases}.
\]

5.3. The \( P_3 \) case. In the case \( P_3 \) the list of possible Jordan forms of the local monodromies is as follows.
where having Jordan form of type (J(3),J(3)) we obtain an orthogonally rigid local system of rank 5 with the following tuple of Jordan forms of the local monodromies. The contradiction of rank being 5 and nonexistence (reducibility).

\[
\phi = \begin{cases} 
  i & (1, 5), 9 \\
  -1 & (2, 6), 10 \\
  x & (3, 7), 11 \\
  \omega & (4, 8), 12 
\end{cases}
\]

we obtain an orthogonally rigid local system of rank 5 with the following tuple of Jordan forms of the local monodromies. The contradiction of rank being 5 and having Jordan form of type (J(3),J(3)) in the cases (1)-(4) shows their nonexistence (reducibility).
Using the isomorphism $A^2SP_4 \cong SO_5$ and the Scott Formula we get

| nr. | rk | (i, i, 1, -i) | (-J(2), 1, 1) | (iJ(2), -iJ(2)) |
|-----|----|---------------|---------------|-----------------|
| 5   | 4  | (i, 1, 1, -i) | (-J(2), -J(2), 1) | (J(3), -1, -1) |
| 6   | 4  | (1, -J(2), -J(2)) | (-J(2), -J(2), 1) | J(5) |
| 7   | 4  | (x, x, 1, x^{-1}, 1) | (-J(2), -J(2), 1) | (x^2, J(3), x^{-2}) |
| 8   | 4  | (ω, 1, ω^{-1}, ω^{-1}) | (-J(2), -J(2), 1) | (ω, J(3), ω^{-1}) |
| 9   | 4  | (i, i, 1, -i) | (z, z, z^{-1}, z^{-1}, 1) | (J(3), -1, -1) |
| 10  | 4  | (1, -J(2), -J(2)) | (z, z, z^{-1}, z^{-1}, 1) | J(5) |
| 11  | 4  | (x, x, 1, x^{-1}, 1) | (z, z, z^{-1}, z^{-1}, 1) | (x^2, J(3), x^{-2}) |
| 12  | 4  | (ω, 1, ω^{-1}, ω^{-1}) | (z, z, z^{-1}, z^{-1}, 1) | (ω, J(3), ω^{-1}) |

Applying MC_{-1} we obtain an orthogonally rigid local system of rank 3 or 4 with the following tuple of Jordan forms of the local monodromies.

| nr. | rk | (i, i, 1, -i) | (J(3), 1) | (iJ(2), -iJ(2)) |
|-----|----|---------------|-----------|-----------------|
| 5   | 4  | (i, 1, 1, -i) | (J(3), 1) | (iJ(2), -iJ(2)) |
| 6   | 3  | J(3) | J(3) | J(3) |
| 7   | 4  | (-x, 1, 1, -x^{-1}) | (J(3), 1) | ℤ(xJ(2), x^{-1}J(2)) |
| 8   | 4  | (-ω, 1, 1, -ω^{-1}) | (J(3), 1) | ℤ(ωJ(2), ω^{-1}J(2)) |
| 9   | 4  | (i, 1, 1, -i) | (-z, 1, 1, -z^{-1}) | (iJ(2), -iJ(2)) |
| 10  | 3  | J(3) | (-z, 1, -z^{-1}) | J(3) |
| 11  | 4  | (-x, 1, 1, -x^{-1}) | (-z, 1, 1, -z^{-1}) | ℤ(xJ(2), x^{-1}J(2)) |
| 12  | 4  | (-ω, 1, 1, -ω^{-1}) | (-z, 1, 1, -z^{-1}) | ℤ(ωJ(2), ω^{-1}J(2)) |

Via the isomorphisms sym^2SP_2 ≅ SO_4 and SL_2 ⊕ SL_2 ≅ SO_4 we can decompose it into linearly rigid irreducible local systems $L_1$ and $L_2$ of rank 2 with the following tuple of Jordan forms. The conditions for a product of eigenvalues not being 1 is due to the irreducibility condition from the Scott Formula.
Thus there exist at most 4 orthogonally rigid local systems having $G_2$-monodromy with the same local monodromies. Computing the normalized structure constant of the reduced monodromy tuple we show that there exist at most 2 such local systems.

\[
\begin{array}{|c|c|c|}
\hline
nr. & \mathcal{L}_i & \mathcal{J}(2) \\
\hline
5 & \mathcal{L}_1 & (\zeta_8, \zeta_8^{-1}) \\
6 & \mathcal{L}_2 & (\zeta_8, \zeta_8^{-1}) \\
7 & \mathcal{L}_1 = \mathcal{L}_2 & \mathcal{J}(2) \\
8 & \mathcal{L}_1 & (\bar{x}, \bar{x}^{-1}) \\
9 & \mathcal{L}_2 & (\bar{x}, \bar{x}^{-1}) \\
10 & \mathcal{L}_1 = \mathcal{L}_2 & \mathcal{J}(2) \\
11 & \mathcal{L}_1 & (\bar{x}, \bar{x}^{-1}) \\
12 & \mathcal{L}_2 & (\bar{x}, \bar{x}^{-1}) \\
\hline
\end{array}
\]

\[
\begin{align*}
n(u_2, k_{2,1}, k_{2,2}) &= \frac{3q-1}{q} \\
n(u_2, k_{2,1}, k_{3,1}) &= 2 \\
n(u_2, k_{2,1}, h_{1b}) &= \begin{cases} 2 & o(h_{1b}) | (q-1)/2 \\ 0 & o(h_{1b}) \nmid (q-1)/2 \end{cases} \\
n(u_2, h_{1a}, k_{2,2}) &= \begin{cases} 2 & o(h_{1a}) | (q-1)/2 \\ 0 & o(h_{1a}) \nmid (q-1)/2 \end{cases} \\
n(u_2, h_{1a}, k_{3,1}) &= \begin{cases} 2 & o(h_{1a}) | (q-1)/2 \\ 0 & o(h_{1a}) \nmid (q-1)/2 \end{cases} \\
n(u_2, h_{1a}, h_{1b}) &= \begin{cases} 2 & o(h_{1a}) | (q-1)/2, o(h_{1b}) | (q-1)/2 \\ 0 & o(h_{1b}) \nmid (q-1)/2 \end{cases}
\end{align*}
\]

Replacing $q$ by $q^2$ we see that there are at most 2 such local systems in the case $P_3$ with the same local monodromies. The existence follows from the construction of the corresponding differential operators. Let

\[
P_3 := L_{2c+1/2} *_H L_{-2c+1/2} *_H (L_3 \otimes (\Lambda^2(L_{1/2} *_H (L_0 \otimes (L_1 \otimes L_2))))_{21})
\]
where
\[ \mathcal{R}(L_3) = \begin{pmatrix} 0 & 1 & \infty \\ c & b & 1/2 \\ -c & -b & 1/2 \end{pmatrix}, \quad \mathcal{R}(L_\alpha) = \begin{pmatrix} 0 & 1 & \infty \\ 0 & -\alpha/\alpha & \end{pmatrix}, \alpha \in \{1/2, \pm 2c + 1/2\}, \]

\[ L_0 = \vartheta - 1/2, \quad \mathcal{R}(L_0) = \begin{pmatrix} 0 & 1 & \infty \\ 1/2 & 0 & -1/2 \end{pmatrix} \]

and
\[ L_1 := 4 \left( \vartheta - c \right) \left( \vartheta + c \right) + z \left( -8 \vartheta^2 - 4 \vartheta - 1 - 4b^2 + 4c^2 \right) + z^2 \left( 2\vartheta + 1 \right)^2 \]
\[ L_2 := 4 \left( \vartheta - c \right) \left( \vartheta + c \right) + z \left( -8 \vartheta^2 - 4 \vartheta - 1 - 4b^2 + 20c^2 \right) + z^2 \left( 2\vartheta + 1 + 4c \right) \left( 2\vartheta + 1 - 4c \right) \]
with
\[ \mathcal{R}(L_1) = \begin{pmatrix} 0 & 1 & \infty \\ c & b & 1/2 \\ -c & -b & 1/2 \end{pmatrix}, \quad \mathcal{R}(L_2) = \begin{pmatrix} 0 & 1 & \infty \\ c & b & 2c + 1/2 \\ -c & -b & -2c + 1/2 \end{pmatrix} \]

Then
\[ P_3 = 16 \vartheta^2 \left( \vartheta - 2 \right)^2 \left( \vartheta - 1 \right)^3 - \\
8x\vartheta^2 \left( 2\vartheta - 1 \right) \left( \vartheta - 1 \right)^2 \left( 4 \vartheta^2 - 4 \vartheta + 8b^2 + 5 - 24c^2 \right) + \\
4x^2\vartheta^3 \left( 24\vartheta^4 + \left( 38 - 288c^2 + 64b^2 \right)\vartheta^2 + 16b^2 + 64b^4 - 144c^2 + 7 + 576c^4 - 384c^2b^2 \right) - \\
2x^3 \left( 2 \vartheta + 1 \right) \left( 2\vartheta + 1 + 4c \right) \left( 2\vartheta + 1 - 4c \right) \left( 4\vartheta^4 + 8\vartheta^3 + 11\vartheta^2 + 8\vartheta^2b^2 - 56\vartheta^2c^2 + \\
7\vartheta - 56\vartheta c^2 + 8\vartheta^2b^2 + 64c^4 + 2 + 4b^2 - 36c^2 - 64c^2b^2 \right) + \\
x^4 \left( \vartheta + 1 \right) \left( 2\vartheta + 3 - 4c \right) \left( 2\vartheta + 1 - 4c \right) \left( \vartheta + 1 - 4c \right) \left( \vartheta + 1 + 4c \right) \\
\left( 2\vartheta + 3 + 4c \right) \left( 2\vartheta + 1 + 4c \right) \]
with
\[ \mathcal{R}(P_3) = \begin{pmatrix} 0 & 1 & \infty \\ 0 & 0 & 1 \\ 0 & 1 & 2c + 1/2 \\ 1 & 2 & 2c + 1/2 \\ 1 & 2 & 2c + 3/2 \\ 1 & 3/2 + 2b & 4c + 1 \\ 2 & 3/2 - 2b & -2c + 3/2 \\ 2 & 1/2 - 2b & -2c + 1/2 \end{pmatrix} \]

Thus if we replace \( b \) by \( b + 1/2 \) (or \( c \) by \( c + 1/2 \)) in the construction we get the same local monodromies for \( P_3(b) \) and \( P_3(b + 1/2) \). However if \( L_1 \) is reducible, i.e. if \( \pm b \pm c + 1/2 \in \mathbb{Z} \) or \( L_1(b) \sim L_1(b + 1/2) \), i.e. \( 2b \in 1/2 + \mathbb{Z} \), then there is only one \( P_3 \) with the given local monodromies.
Since \( \Lambda^2 \) yields an operator of degree 14 we get that \( P_3 \) has \( G_2 \)-monodromy.
5.4. The case $P_5$. We start with the possible list of Jordan forms of the local monodromies of orthogonally rigid quadruples with $G_2$-monodromy.

| nr. | rk | \((J(2), J(2), 1_3)\) | \((J(2), J(2), 1_3)\) | \((J(3), 1)\) | \((\omega, \omega^{-1}, 1_2)\) |
|-----|----|----------------|----------------|----------|----------------|
| 1   | 7  | \((J(2), J(2), 1_3)\) | \((J(2), J(2), 1_3)\) | \((J(3), 1)\) | \((\omega, \omega^{-1}, 1_2)\) |
| 2   | 7  | \((J(2), J(2), 1_3)\) | \((J(2), J(2), 1_3)\) | \((\omega, \omega^{-1}, 1_3)\) | \((-J(2), -J(2), 1_3)\) |
| 3   | 7  | \((J(2), J(2), 1_3)\) | \((J(2), J(2), 1_3)\) | \((\omega, \omega^{-1}, 1_3)\) | \((\omega, \omega^{-1}, 1_3)\) |

Applying the functor

\[ M_{\phi} = M_{\mathcal{L}(1,1,\phi^{-1})} \circ M_{\mathcal{L}(1,1,\phi)} \circ M \circ M_{\mathcal{L}(1,1,\phi^{-1})} \circ M, \]

where

\[ \phi = \begin{cases} 
1 & 1 \\
-1 & 2 \\
x & 3 
\end{cases} \]

we obtain an orthogonally rigid local system of rank 4 with the following tuple of Jordan forms of the local monodromies.

| nr. | rk | \((J(2), J(2))\) | \((J(2), J(2))\) | \((J(3), 1)\) | \((\omega, \omega^{-1}, 1_2)\) |
|-----|----|----------------|----------------|----------|----------------|
| 1   | 4  | \((J(2), J(2))\) | \((J(2), J(2))\) | \((J(3), 1)\) | \((\omega, \omega^{-1}, 1_2)\) |
| 2   | 5  | \((J(2), J(2), 1)\) | \((J(2), J(2), 1)\) | \((-J(2), -J(2), 1)\) | \((\omega, \omega^{-1}, 1_3)\) |
| 3   | 5  | \((J(2), J(2), 1)\) | \((J(2), J(2), 1)\) | \((x_1, x^{-1}1_2)\) | \((\omega, \omega^{-1}, 1_3)\) |

Applying the functor

\[ M_{\phi} = M_{\mathcal{L}(1,\phi^{-1}, 1)} \circ M_{\mathcal{L}(1,\phi, 1)} \circ M \circ M_{\mathcal{L}(1,\phi^{-1})} \circ M, \]

where

\[ \phi = \begin{cases} 
1 & 1 \\
-1 & 2 \\
x & 3 
\end{cases} \]

we obtain an orthogonally rigid local system of rank 4 with the following tuple of Jordan forms of the local monodromies.

| nr. | rk | \((J(2), J(2))\) | \((J(2), J(2))\) | \((J(3), 1)\) | \((\omega, \omega^{-1}, 1, 1)\) |
|-----|----|----------------|----------------|----------|----------------|
| 1   | 4  | \((J(2), J(2))\) | \((J(2), J(2))\) | \((J(3), 1)\) | \((\omega, \omega^{-1}, 1, 1)\) |
| 2   | 4  | \((J(2), J(2))\) | \((-J(2), -J(2))\) | \((J(3), 1)\) | \((\omega, \omega^{-1}, 1, 1)\) |
| 3   | 4  | \((J(2), J(2))\) | \((x_1, x^{-1}1_2)\) | \((J(3), 1)\) | \((\omega, \omega^{-1}, 1, 1)\) |

Via the isomorphism

\[ \text{SL}_2 \otimes \text{SL}_2 = \text{SO}_4 \]

we can decompose it into linearly rigid irreducible local systems $\mathcal{L}_1$ and $\mathcal{L}_2$ of rank 2 with the following tuple of Jordan forms of the local monodromies.
From the discussion above we know that there exist at most 2 orthogonally rigid local systems having \(G_2\)-monodromy with the same local monodromies. The generic character table of the group \(G_2(q)\) shows that there exists at most 1 such local system. The normalized structure constant of the reduced monodromy tuple gives

\[
\begin{align*}
n(u_2, u_2, k_3, u_3) &= 1 \\
n(u_2, u_2, k_3, u_4) &= n(u_2, u_2, k_3, u_5) = 0 \\
n(u_2, u_2, k_3, k_{2,1}) &= 1 \\
n(u_2, u_2, k_3, h_{10}) &= 1
\end{align*}
\]

There are infinitely many such quadruples due to the positions \((1, t, 0, \infty)\) of the singularities. Those with singularities at \(1, -1, 0, \infty\) arise from quadratic pullbacks of \(P_1\)-cases with the following tuples of Jordan forms.

\[
\begin{align*}
\text{nr.} & & \mathcal{L}_1 & & \mathcal{L}_2 \\
1 & & \mathbf{J}(2) & & \mathbf{J}(2) & & (\omega, \omega^{-1}) \\
& & \mathbf{J}(2) & & \mathbf{J}(2) & & (\omega, \omega^{-1}) \\
2 & & \mathbf{J}(2) & & \mathbf{J}(2) & & (\omega, \omega^{-1}) \\
& & \mathbf{J}(2) & & \mathbf{J}(2) & & (\omega, \omega^{-1}) \\
3 & & \mathbf{J}(2) & & \mathbf{J}(2) & & (\omega, \omega^{-1}) \\
& & \mathbf{J}(2) & & \mathbf{J}(2) & & (\omega, \omega^{-1})
\end{align*}
\]

This finishes the proof of Thm 5.1. \(\Box\)

**Appendix A. Generic character tables and structure constants**

Let \(\mathcal{C} = (C_1, \ldots, C_{r+1})\) be a tuple of conjugacy classes of a group \(G\) and

\[\Sigma(\mathcal{C}) = \{\sigma \in G^{r+1} \mid \sigma_i \in C_i, \sigma_1 \cdots \sigma_{r+1} = 1\}\]

Then the *normalized structure constant* \(n(\mathcal{C})\) is defined as

\[n(\mathcal{C}) = \frac{\left| \Sigma(\mathcal{C}) \right|}{\left| \text{Inn}(G) \right|}.
\]

The following result is well known (cf. [13, Chap. I, Thm. 5.8]):
Proposition. A.1. Let $G$ be a finite group, let $\text{Irr}(G)$ denote the set of irreducible characters of $G$ and let $\mathcal{C} = (C_1, \ldots, C_{r+1})$ be a tuple of conjugacy classes in $G$ with representatives $\sigma_1, \ldots, \sigma_{r+1}$. Then

$$n(\mathcal{C}) = \frac{|Z(G)| \colon |G|^{|r-1}}{\prod_{i \in \text{Irr}(G)} |C_G(\sigma_i)|} \sum_{\chi \in \text{Irr}(G)} \prod_{i} \chi(\sigma_i) \chi(1)^{r-1}.$$ 

In order to find an upper bound for the number of local systems with the same tuple of local monodromies we use reduction modulo $l$ and derive the bound from the normalized structure constant:

Let $G$ be a reductive algebraic group defined over $\mathbb{Z}$ which is an irreducible subgroup of $\text{GL}_n$ (e.g. $G_2 \leq \text{GL}_7$) and let $\mathcal{C} = (C_1, \ldots, C_{r+1})$ be a tuple of conjugacy classes in $G$. Consider the map

$$\pi : C_1 \times \cdots \times C_{r+1} \to G, (g_1, \ldots, g_{r+1}) \mapsto g_1 \cdots g_{r+1}$$

and let $X := \pi^{-1}(1)$ (with $1 \in G$ the neutral element). The variety $X$ decomposes into irreducible components $X_1, \ldots, X_k$. The following result is the content of [12], Lemma 5.9.3, and will be useful below:

Lemma. A.2. Let $R$ be a subring of $\mathbb{C}$ which is finitely generated as a $\mathbb{Z}$-algebra. Then there exists an $N \in \mathbb{N}_{>0}$ such that for any prime number $\ell$ which does not divide $N$, there exists a finite extension $K_\nu$ of $\mathbb{Q}_\ell$ with valuation ring $O_\nu$ and an isomorphism $\iota : \mathbb{C} \to \mathbb{Q}_\ell$ under which $R$ is mapped into $O_\nu$.

The idea of the proof is as follows: Using Noether normalization, $R$ is an integral extension of $\mathbb{Z} \left[\frac{1}{\nu} \right] \left[ x_1, \ldots, x_{r+1} \right]$, where $x_1, \ldots, x_{r+1}$ are algebraically independent. By the axiom of choice, for any algebraically independent set $\{ y_1, \ldots, y_{r+1} \} \subseteq \mathbb{Q}_\ell$ (where $\ell$ does not divide $N$), there exists an isomorphism $\iota : \mathbb{C} \to \mathbb{Q}_\ell$ which maps $x_i$ to $y_i$, $i = 1, \ldots, r+1$.

Lemma A.2 implies that for any tuple $\mathcal{C}$ of conjugacy classes there exists an $M \in \mathbb{N}_{>0}$ such that for any prime number $\ell$ which does not divide $M$, there exists a finite extension $K_\nu$ of $\mathbb{Q}_\ell$ with valuation ring $O_\nu$ such that $C_1 \times \cdots \times C_{r+1}$ and $X = \pi^{-1}(1)$ is defined over $O_\nu$. Similarly, for any $g = (g_1, \ldots, g_{r+1}) \in X$ there exists an $N \in \mathbb{N}_{>0}$ such that for any prime number $\ell$ which does not divide $N$, there exists a finite extension $K_\nu$ of $\mathbb{Q}_\ell$ with valuation ring $O_\nu$ such that the coefficients of all elements of $g$ are contained in $O_\nu$. Hence, for almost all $\ell$ we find $\nu \mid \ell$ such we can reduce the entries of $g$ modulo the valuation ideal $m_\nu \subseteq O_\nu$. In this way we obtain the reduced tuple $\bar{g} = (\bar{g}_1, \ldots, \bar{g}_{r+1}) \in (C_1, \ldots, C_{r+1})$, where $\bar{C}_i$ is the conjugacy class of $\bar{g}_i$ in $G(\mathbb{F}_q)$, where $\mathbb{F}_q = O_\nu / m_\nu$. For positive natural numbers $k$, let $\mathcal{C}(q^k)$ denote the tuple of conjugacy classes of $\bar{g}_1, \ldots, \bar{g}_{r+1}$ in the group $G(\mathbb{F}_q^k)$.

Theorem. A.3. Suppose that $G$ is an irreducible simple algebraic subgroup of $\text{GL}_n(\mathbb{C})$ defined over $\mathbb{Z}$ and suppose that there exists an $s \in \mathbb{N}_{>0}$ such that

$$\sup \left( \lim_{q \to \infty} n(\mathcal{C}(q^k)) \right) = s,$$

where the supremum is taken over all prime powers $q$ which are cardinalities of the residue fields of $\nu$ as above. Then, up to diagonal $G(\mathbb{C})$-conjugation, there exist at most $s$ tuples

$$g_i := (g_{i,1}, \ldots, g_{i,r+1}) \in C_1 \times \cdots \times C_{r+1} \quad (i = 1, \ldots, s)$$
with \( g_{i,1} \cdots g_{i,r+1} = 1 \) and such that the generated subgroup \( \langle g_{i,1}, \ldots, g_{i,r+1} \rangle \) is irreducible.

**Proof.** Assume that there exist \( t > s \) different equivalence classes (w.r. to diagonal \( G(\mathbb{C}) \)-conjugation) of tuples

\[
g_c = (g_{i,1}, \ldots, g_{i,r+1}) \in C_1 \times \cdots \times C_{r+1} \quad (i = 1, \ldots, t)
\]

with \( g_{i,1} \cdots g_{i,r+1} = 1 \) and such that the generated subgroup \( \langle g_{i,1}, \ldots, g_{i,r+1} \rangle \) is irreducible. We have the following two cases:

**Case 1:** The tuples \( g_c = (g_{i,1}, \ldots, g_{i,r+1}) (i = 1, \ldots, t) \) lie in \( t \) different irreducible components \( X_i \) of \( X \). By Lemma A.2 for almost all \( \ell \) there exists a finite extension \( K_\nu \) of \( \mathbb{Q}_\ell \) such that \( g_c \in X_i(O_\nu) (i = 1, \ldots, t) \). If \( \ell \gg 0 \) and \( k \gg 0 \), then the reductions modulo \( m_\nu \) of the components \( X_i \) remain different. Hence reduction modulo the maximal ideal \( m_\nu \) of \( O_\nu \) leads to \( t \) different equivalence classes (under diagonal conjugation with elements in \( G(\mathbb{F}_q) \)) \( g_c \in (C_1, \ldots, C_{r+1}) \), contrary to \( t > s = \sup_q \lim_k \{ n(C(q^k)) \} \).

**Case 2:** Two of the tuples, say \( g_1 \) and \( g_2 \), lie in the same irreducible component \( X_1 \). Since \( \langle g_1 \rangle \) is irreducible the \( G(\mathbb{C}) \)-stabilizer of \( g_1 \in C_1 \times \cdots \times C_{r+1} \) under diagonal conjugation is equal to the centralizer of \( \langle g_1 \rangle \) and hence coincides with the (finite) centre \( Z(G) \) of \( G \). This implies that the dimension of the component \( X_1 \) of \( X \) with \( g_1 \in X_1 \) is \( \geq \dim G \). Therefore, by the assumption in Case 2, \( \dim X_1 > \dim G \) and, by dimension reasons, there exist infinitely many \( G(\mathbb{C}) \)-orbits \( V_j (j \in J) \) in \( X_1 \). Pick \( u > s \) different orbits \( V_1, \ldots, V_u \) and representatives

\[
v_k \in V_k \quad (k = 1, \ldots, u).
\]

Suppose that the \( V_1, \ldots, V_u \) are defined over \( R \), where we see \( R \) as a subring of \( O_\nu \) (\( \nu(\ell) \)) as above. We claim that for \( \ell \gg 0 \), the reductions modulo-\( \nu \) are different. This can be seen inductively as follows: The orbits are (quasi-)affine varieties inside an ambient affine space \( k^n \). Pick functions \( f_j \) in the vanishing ideals of \( V_j \) with the property that for \( j \neq j' \), there exists \( v_{j'} \in V_{j'} \) with \( f_j(v_{j'}) \neq 0 \). Extending scalars and assuming \( \ell \) large enough, we can assume that the functions \( f_j \) and the \( v_j \) are defined over \( R \) and hence over \( O_\nu \). If \( f_j(v_{j'}) \) is algebraic, then for almost all \( \ell \) the inequality \( f_j(v_{j'}) \neq 0 \) will hold modulo \( \nu \) for all pairs of \( j, j' \) where \( j \neq j' \). If \( f_j(v_{j'}) \) is transcendental, then with the freedom to choose the isomorphism \( \iota : C \to \mathbb{Q}_\ell \) (see the remark following Lemma A.2) in a way that the inequality \( f_j(v_{j'}) \neq 0 \) will hold modulo \( \nu \) for all pairs of \( j, j' \) where \( j \neq j' \). Therefore, the orbits remain different modulo \( \nu \) and hence

\[
\sup_q \lim_k \{ n(C(q^k)) \} \geq u,
\]

a contradiction to \( u > s \). \( \square \)

**Remark. A.4.** Recall that there are character tables for \( G(\mathbb{F}_q) \) which compute the character values of the elements of \( G(\mathbb{F}_q) \) as functions depending on \( q \), the so-called generic character tables. For groups with small Lie rank, these generic character tables are implemented in [10], especially, the case \( G = G_2 \), cf. [5] and [11, Anhang B], can be found there. Using the generic character table of \( G_2(q) \) we can determine \( n(C(q^k)) \) and also \( \sup_q \lim_k \{ n(C(q^k)) \} \) in many cases.

**Remark. A.5.** We give an overview of the class representatives \( c_j \) in \( G_2(q) \) taken from Chang and Ree. In order to determine \( \lim_k \{ n(C(q^k)) \} \) we can assume
that the eigenvalues of all class representatives of $C_1, \ldots, C_{r+1}$ are in $\mathbb{F}_q$. Otherwise we replace $q$ by $q^4$. The generic character table depends on the congruence of $q \mod 12$. Thus we can also assume that $q \equiv 1 \mod 12$. We list in the notation of Chang and Ree [5] for class representative $c_j$ having eigenvalues in $\mathbb{F}_q$ the corresponding Jordan forms. The order of the centralizer of $c_j$ in $G \in \{G_2, O_7, GL_7\}$ is a polynomial in $q$ of degree $d_G := \dim C_G(\mathbb{F}_q)(g_j)$.

| class rep. | Jordan form | $d_{G_2}$ | $d_{O_7}$ | $d_{GL_7}$ | conditions |
|------------|-------------|-----------|-----------|------------|------------|
| 1          | 1           | 14        | 21        | 49         |            |
| $u_1$      | $(J(2), J(2), 1, 1, 1)$ | 8         | 13        | 29         |            |
| $u_2$      | $(J(3), J(2), J(2))$ | 6         | 9         | 19         |            |
| $u_3$      | $(J(3), J(3), J(1))$ | 4         | 7         | 17         |            |
| $u_4$      | $(J(3), J(3), J(1))$ | 4         | 7         | 17         |            |
| $u_5$      | $(J(3), J(3), J(1))$ | 4         | 7         | 17         |            |
| $u_6$      | $J(7)$      |           |           |            |            |
| $k_2$      | $(-1_4, 1_3)$ | 6         | 9         | 25         |            |
| $k_{2,1}$  | $(-J(2), -J(2), 1, 1, 1)$ | 4         | 7         | 17         |            |
| $k_{2,2}$  | $(-J(2), -J(2), J(3))$ | 4         | 5         | 11         |            |
| $k_{2,3}$  | $(-J(3), -J(1), J(3))$ | 2         | 3         | 9          |            |
| $k_{2,4}$  | $(-J(3), -J(1), J(3))$ | 2         | 3         | 9          |            |
| $k_3$      | $(\omega 1_3, 1, \omega^{-1}1_3)$ | 8         | 9         | 19         | $\omega^3 = 1$ |
| $k_{3,1}$  | $(\omega J(2), \omega^{-1} J(2), \omega, \omega^{-1}, 1)$ | 4         | 5         | 11         |            |
| $k_{3,2}$  | $(\omega J(3), \omega^{-1} J(3), 1)$ | 2         | 3         | 7          |            |
| $k_{3,3,1}$ | $(\omega J(3), \omega^{-1} J(3), 1)$ | 2         | 3         | 7          | $i = 1, 2$, $k_{3,3,1}^{-1} \sim k_{3,3,2}$ |
| $h_{1a}$   | $(x, x, x^{-1}, x^{-1}, 1, 1, 1)$ | 4         | 7         | 17         | $x^{q-1} = 1, x^2 \neq 1$ |
| $h_{1a,1}$ | $(x J(2), x^{-1} J(2), J(3))$ | 2         | 3         | 7          |            |
| $h_{1b}$   | $(x, x, x^{2}, 1, x^{-1}, x^{-1}, x^{-2})$ | 4         | 5         | 11         | $x^{q-1} = 1, x^3 \neq 1, x^4 \neq 1$ |
|           | $(i, i, -1, 1, -1, i^{-1}, i^{-1})$ | 4         | 5         | 13         |            |
| $h_{1b,1}$ | $(x J(2), x^{-1} J(2), x^2, x^{-2}, 1)$ | 2         | 3         | 7          |            |
|           | $(i J(2), i^{-1} J(2), -1, -1, 1)$ | 2         | 3         | 9          |            |
| $h_1$      | $(x, y, xy, 1, (xy)^{-1}, y^{-1}, x^{-1})$ | 2         | 3         | 7          | $x^{q-1} = y^{q-1} = 1$ |
|           | $(x, -1, -x, 1, -x^{-1}, -1, x^{-1})$ | 2         | 3         | 9          | pairw. diff. eigenvalues |

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