Limits on quantum deletion from no signaling principle

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One of the fundamental restrictions that quantum mechanics imposes is the “No deletion Theorem” which tells us that given two identical unknown quantum states, it is impossible to delete one of them. But nevertheless if not perfect, people have tried to delete it approximately. In these approximate deleting processes our basic target is to delete one of the two identical copies as much as possible while preserving the other copy. In this brief report, by using the No communication theorem (NCT) (impossibility of sending signal faster than light using a quantum resource) as a guiding principle, we obtain a bound on the sum of the fidelity of deletion and the fidelity of preservation. Our result not only brings out the complementary relation between these two fidelities but also predicts the optimal value of the fidelity of deletion achievable for a given fidelity of preservation under no signaling constraint. This work eventually saturates the quest for finding out the optimal value of deletion within the NCT framework.

I. INTRODUCTION

In the last two decades quantum information processing has emerged as a powerful tool for implementing several tasks that cannot be done using classical means. These task include super dense coding [1], teleportation [2], remote state preparation [3], broadcasting [4] and key generation [5]. These are no longer only theoretical possibilities but also have been experimentally demonstrated. On one hand, quantum mechanics has given us a significant advantage in the information processing tasks in terms of things that are doable while on the other hand it has also forbidden us from doing operations which are otherwise possible in the classical world. These impossible operations in quantum information processing also known as No go Theorems [6] are responsible for making the information private and secure. One of the impossibilities which is otherwise classically possible is deleting an arbitrary quantum state. This is also known as No Deletion Theorem [7]. More specifically, it states that the linearity of quantum theory prohibits us to delete an unknown quantum state from two identical copies in either a reversible or an irreversible manner. It is interesting to note that quantum deletion and erasure of a quantum state are not the same process. In quantum theory, the erasure of a single unknown state is considered as swapping it with some standard state and then trashing it into the environment. In contrast, quantum deletion [3] is more of reversible uncopying of an unknown quantum state. In principle we can perfectly erase quantum state as long we are not concerned in preserving the other copy. However it is impossible to delete an arbitrary quantum state keeping the other state as it is. It has been shown that in addition to the linear structure of quantum mechanics, other principles like unitarity, no signaling, incomparability and conservation of entanglement are not congruous to the concept of perfect deletion [9]. Interestingly it was shown that not only deletion other impossible operations [6] are consistent with these physical principles [10]. However, if one tries to delete an unknown quantum state probabilistically, then it is possible with a success probability of less than unity [11]. It has also been shown that using these probabilistic deletion machines one cannot send super-luminal signals probabilistically [12]. Since perfect deletion is not possible, it is interesting to see whether one can delete an unknown state imperfectly. Researchers have devised various approximate deletion machines. These deletion machines are either state dependent or state independent [13]. Recent explorations have revealed that one can construct a universal quantum deletion machine [14], and its fidelity can be further enhanced by the application of suitable unitary transformation [15]. It was shown that cloning and deleting a quantum state exhibits complementary relationship with each other in terms of the correlation generated in each of these processes [16]. These deletion machines can have various applications in quantum information theory [17]. However, the search for optimal quantum deletion machine is not over as no approximate deletion operation was proven to be optimal.

In physics, the No-communication theorem (NCT) is a no-go theorem which states that, during measurement of an entangled quantum state, it is not possible for one observer, by making a measurement of a subsystem of the total state, to communicate information to another observer. The theorem is important because, there is a speculation that, whether by using quantum entanglement there exists a possibility of instantaneous communication between widely separated parties.
As we have mentioned earlier that in principle quantum deletion is different from the quantum erasure. Quantum mechanics allows perfectly to erase/delete quantum state as long we are not constrained to keep the preservation fidelity of the other state equal to 1. In this work we also find that indeed it is the case, as the fidelity of deletion of the second qubit can ideally go to 1 when we are not bothered about keeping the preservation fidelity 1. No deletion theorem already tells us that it is impossible to delete one qubit entirely keeping the other qubit as it is. In other words it is impossible to achieve the sum of these two fidelities equal to 2. So naturally the question arises if not 2 what is the optimal value that this sum can reach. In this work we find out that under the NCT framework, this sum can go at max to the value of 1.5. This is the optimal value that an imperfect universal state independent deletion machine can achieve in principle at the same time being consistent with the no signaling principle. This brings out a unique complementary aspect in the fidelity of deletion with the fidelity of preservation. Not only that, we also give the optimal value of the fidelity of deletion for a given value of the fidelity of preservation and vice versa. This is independent of the sum of the two fidelities. The sum obtained in both the cases is again bounded by the value we obtain overall.

II. BOUNDS ON QUANTUM DELETION; NO COMMUNICATION THEOREM (NCT)

In this section we provide the bounds on the sum of the fidelity of deletion and fidelity of preservation under the no signaling condition. This bound gives us the optimal value of the joint fidelity that can be achieved. We also give the optimal value of the fidelity of deletion for a given value of the fidelity of preservation.

Quantum Deletion and No-deletion theorem

A perfect deletion machine performs a unitary transformation which takes as input $|\psi\rangle \otimes |\psi\rangle$ and transforms it into $|\psi\rangle \otimes |\Sigma\rangle$ where $|\Sigma\rangle$ is referred to as a blank state. The famous No-deletion theorem states that there exists no unitary transformation which can perform perfect deletion. However, this does not rule out the possibility of carrying out this task approximately. An approximate deletion machine transforms the input into $|\psi_1\rangle \otimes |\psi_2\rangle$ where $|\psi_1\rangle$ is a state close to the input state $|\psi\rangle$ and $|\psi_2\rangle$ is a state close to the blank state $|\Sigma\rangle$. Correspondingly, we define two different notions of measuring the performance of the deletion machine.

Fidelity of preservation:

It is defined as the fidelity with which the first copy of the input state is preserved, given by the overlap between the input state $|\psi\rangle$ and the partial output state $\rho_1$

$$F_p = \langle \psi | \rho_1 | \psi \rangle$$ (1)

Fidelity of deletion:

It is defined as the fidelity with which the second copy of the input state is deleted, given by the overlap between the blank state $|\Sigma\rangle$ and the partial output $\rho_2$

$$F_d = \langle \Sigma | \rho_2 | \Sigma \rangle$$ (2)

where $\rho_{12}$ is the overall output state, $\rho_1 = Tr_2(\rho_{12})$ and $\rho_2 = Tr_1(\rho_{12})$

No Communication Theorem (NCT)

The no-communication theorem is a no-go theorem which tells us that, during measurement of an entangled quantum state, it is not possible for any one of the observer, by making a measurement of a subsystem of the total state, to communicate information to another observer. In other words when two parties are spatially separated and if one party carries out the measurement in either of the two basis, the output states corresponding to measurements in two different basis $[|\uparrow\rangle, |\downarrow\rangle], [\rightarrow, \leftarrow]$ will remain identical. The party will not able to distinguish the act of measurement done by the other party as this will lead to the violation of causality.

Let us start with two identical states 1 and 2 parametrized by the Bloch parameter $\vec{m}$ as an input to the imperfect quantum deletion machine (QDM). The combined product state acting as an input is of the form,

$$\rho_{12}^{in}(\vec{m}) = \rho_1(\vec{m}) \otimes \rho_1(\vec{m})$$, (3)

where $\rho(\vec{m}) = \frac{1+\vec{m} \cdot \vec{\sigma}}{2}$. If, for example, perfect quantum deleting machines would exists, then, by deleting the mixtures corresponding to different directions $\vec{m}$ could be distinguished. But perfect deleting machine do not exist. However, imperfect QDM exist. In full generality, the combined output state obtained after the application of imperfect quantum deleting machine (QDM) can be written as:

$$\rho_{12}^{out}(\vec{m}) = \frac{1}{4}(I+(\eta_1\vec{m} \cdot \vec{\sigma} I)\otimes I) + (I \otimes (\eta_2 \vec{b} \cdot \vec{\sigma})) + \sum_{i,j}(t_{ij}\sigma_i \otimes \sigma_j))$$

(4)

The blank state paramatrized by the vector $\vec{b} = \{b_x, b_y, b_z\}$ is given by,

$$\rho(\vec{b}) = \frac{1 + \vec{b} \cdot \vec{\sigma}}{2}$$. (5)

After tracing out the qubit we have the reduced density operators for the states 1 and 2 are given by,

$$\rho_{1}^{out} = Tr_2(\rho_{12}^{out}) = \frac{1 + \eta_1 \vec{m} \cdot \vec{\sigma}}{2}$$,

$$\rho_{2}^{out} = Tr_1(\rho_{12}^{out}) = \frac{1 + \eta_2 \vec{b} \cdot \vec{\sigma}}{2}$$. (6)
The fidelity of preservation as well as the fidelity of deletion based on the output states are given by,

$$ F_p = \text{Tr}(\rho(\vec{m}) \cdot \rho^\text{out}_1) = 1 + \eta_1, \quad F_d = \text{Tr}(\rho(\vec{b}) \cdot \rho^\text{out}_2) = 1 + \eta_2. \quad (7) $$

Here we consider a universal deletion machine (QDM) which should act similarly on all states thereby satisfying,

$$ \rho^\text{out}(U\vec{m}) = U \otimes U \rho^\text{out}(\vec{m})U^\dagger \otimes U^\dagger, \quad (8) $$

for all unitary operator $U$ acting on 2-dimensional spin $\frac{1}{2}$ Hilbert space. Due to the covariance property, $\rho^\text{out}(\vec{m})$ is invariant under rotation around the direction $\vec{m}$ leading to the commutator $[e^{i\alpha \vec{n} \cdot \vec{d}}, \rho^\text{out}(\vec{m})] = 0$ for all $\alpha$. This imposes restrictions on the $t_{ij}$ parameters as well as on the blank state Bloch vector. If we take $\vec{m}$ to be in the $x$-direction, the commutator leads to the conditions:

$$ t_{yz} = -t_{zy}, t_{yy} = t_{zz}, \text{and } t_{xy} = t_{yx} = b_z = b_y = 0 $$

and the output state takes form

$$ \rho^\text{out}_{1,2}(-) = \frac{1}{4} (I + (\eta_1 \sigma_x \otimes I) + (I \otimes (\eta_2 b_z \sigma_z)) $$

$$ + t_{xx}(\sigma_x \otimes \sigma_x) + t_{zz}(\sigma_z \otimes \sigma_z) + (\eta_2 \sigma_y \otimes \sigma_y) $$

$$ + t_{xy}(\sigma_z \otimes \sigma_x) + \sigma_y \otimes \sigma_y) $$

$$= \frac{1}{4} \left[ \begin{array}{ccc} 1 + t_{zz} & b_z \eta_2 - it_{zy} & \eta_1 + it_{ty} & t_{xx} - t_{zz} \\ b_z \eta_2 + it_{zy} & 1 - t_{zz} & t_{xz} + t_{zz} & \eta_1 - it_{zy} \\ \eta_1 - it_{zy} & t_{xx} + t_{zz} & 1 - t_{zz} & b_z \eta_2 + it_{zy} \\ t_{xx} - t_{zz} & \eta_1 + it_{zy} & b_z \eta_2 - it_{zy} & 1 + t_{zz} \end{array} \right]. \quad (9) $$

Now, we apply the no signaling condition, which imposes the constraint that the mixtures of output states corresponding to indistinguishable mixtures of input states are indistinguishable, similar to the approach taken in [15]. Here, we have two-qubit inputs and it is easy to verify that the following holds

$$ \rho(\uparrow \otimes \rho(\uparrow) + \rho(\uparrow) \otimes \rho(\downarrow) + \rho(\downarrow \otimes \rho(\downarrow) + \rho(\downarrow) \otimes \rho(\downarrow) $$

$$ = \rho(- \otimes \rho(-) + \rho(- \otimes \rho(-) $$

$$ + \rho(- \otimes \rho(-) + \rho(- \otimes \rho(-). \quad (10) $$

Out of these, only the ones that are symmetrical are valid inputs to the deletion machine i.e $\rho(\uparrow \otimes \rho(\uparrow), \rho(\downarrow \otimes \rho(\downarrow) \rho(- \otimes \rho(-) \rho(- \otimes \rho(-) \rho(\downarrow) \otimes \rho(-). \quad (11) $$

We assume general pure state outputs for each of the invalid pure input states

$$ U_d \otimes U_d(\rho(\uparrow) \otimes \rho(\downarrow))U_d^\dagger \otimes U_d^\dagger = |\phi\rangle \langle \phi|, \quad U_d \otimes U_d(\rho(\downarrow) \otimes \rho(\uparrow))U_d^\dagger \otimes U_d^\dagger = |\gamma\rangle \langle \gamma|, \quad U_d \otimes U_d(\rho(-) \otimes \rho(-))U_d^\dagger \otimes U_d^\dagger = |\phi'\rangle \langle \phi'|, \quad U_d \otimes U_d(\rho(-) \otimes \rho(-))U_d^\dagger \otimes U_d^\dagger = |\gamma'\rangle \langle \gamma'|, \quad (11) $$

where $U_d$ refers to the unitary corresponding to universal deletion machine and $|\phi\rangle = \sum_{i=1}^{4} p_i |i\rangle, |\phi'\rangle = \sum_{i=5}^{8} p_i |i\rangle, |\gamma\rangle = \sum_{i=1}^{4} q_i |i\rangle$ and $|\gamma'\rangle = \sum_{i=5}^{8} q_i |i\rangle$ such that

$$ \sum_{i=1}^{4} p_i^2 = \sum_{i=1}^{4} q_i^2 = \sum_{i=5}^{8} q_i^2 = 1. $$

The outputs corresponding to the valid inputs are obtained as in Eq. 14 followed by the application of the covariance condition. Finally we apply the condition for no signaling. This imposes the following restrictions on the output matrices as well as the output states for the invalid parameters,

$$ t_{xy} = t_{yx} = 0, p_2 p_4 - p_6 p_8 + q_2 q_4 - q_6 q_8 = 0 $$

$$ p_4 p_6 - p_2 p_8 + q_4 q_6 - q_2 q_8 = 0 $$

$$ b_z \eta_2 = 2(p_2^2 - p_6^2 + q_2^2 - q_6^2) = 2(p_2^2 - p_6^2 + q_2^2 - q_6^2) $$

$$ b_z \eta_2 = 2(p_2^2 - p_6^2 + q_2^2 - q_6^2) = 2(p_2^2 - p_6^2 + q_2^2 - q_6^2) $$

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$$ b_z \eta_2 = 2(p_2 p_4 - p_6 p_8 + q_2 q_4 - q_6 q_8) $$

$$ t_{xx} + t_{zz} = 2(p_2 p_4 - p_6 p_8 + q_2 q_4 - q_6 q_8 + t_{yy}) $$

$$ t_{xx} - t_{zz} = 2(p_2 p_4 - p_6 p_8 + q_2 q_4 - q_6 q_8). \quad (12) $$

Finally, we enforce the non-negativity of the eigenvalues of the output matrices. For example, the eigenvalues of $\rho(-)$ are:

$$ \frac{1}{4} (1 + \eta_1 \pm b_z \eta_2 + t_{xx}) $$

$$ \frac{1}{4} (1 - t_{xx} \pm \sqrt{(\eta_1 - b_z \eta_2)^2 + 4(t_{xy}^2 + t_{zz}^2)}) \quad (13) $$

The maximum value of $F_d + F_p$ subject to non-negativity of the eigenvalues of the output matrices and the condition in [12] is $1.5$ achieved at the following configuration of the parameters:

$$ \eta_1 = b_z = 1, \eta_2 = t_{xx} = t_{yy} = t_{zz} = 0, p_1 = p_5 = 1 $$

$$ q_1 = q_5 = 1, p_i = q_i = 0, i \in \{2, 3, 4, 6, 7, 8\} $$

This shows that the maximum fidelity sum that can be obtained under no signaling constraint is $\frac{3}{2}$. Any quantum deleting machine (QDM) obtaining this fidelity will be called as optimal deleting machine. Thus we see the no signaling constraint is a powerful guide to find the limits of quantum mechanics.

Now, we investigate the tradeoff between fidelity of deletion $F_d$ and fidelity of preservation $F_p$ by numerically fixing one of them and maximizing the other. In Fig 1a we fix $F_d$ and maximize $F_p$ whereas in Fig 1c the situation is reversed. In both the figures we observe that the bound is nearly saturated throughout.

**III. CONCLUSION**

There have been several attempts to design optimal approximate deletion machines in the past. In this paper, we achieve a bound on the performance of a universal
(a) The blue line shows the variation of the maximum value of $F_p$ at a fixed value of $F_d$. The equation for the red line is $F_d + \max(F_p) = 1.5$

(b) The blue line shows the variation of the maximum value of $F_d$ at a fixed value of $F_p$. The equation for the red line is $F_p + \max(F_d) = 1.5$

FIG. 1: Complementary nature of the two kinds of fidelity

state-independent deletion machine using no-signaling as a guiding principle. The sum of the two kinds of fidelity namely the fidelity of deletion and the fidelity of preservation is upper bounded by $\frac{3}{2}$. It should be noted that this is less than the sum obtained by existing machines because those are state-dependent machines. It would be a challenging and important task in future to design a universal state independent deletion machine which can match the performance given by this bound.

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