TRANSITIVE TRIANGLE TILINGS IN ORIENTED GRAPHS

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Abstract. In this paper, we prove an analogue of Corrádi and Hajnal’s result. There exists \( n_0 \) such that for every \( n \in 3\mathbb{Z} \) when \( n \geq n_0 \) the following holds. If \( G \) is an oriented graph on \( n \) vertices and \( \delta^0(G) \geq 7n/18 \), then \( G \) contains a perfect \( TT_3 \)-tiling, which is a collection of vertex disjoint transitive triangles covering every vertex of \( G \). This result is best possible, as there exists an oriented graph \( G \) on \( n \) vertices without a perfect \( TT_3 \)-tiling and \( \delta^0(G) = \lfloor 7n/18 \rfloor - 1 \).

1. Introduction

Let \( G \) be an oriented graph, that is a directed graph without loops such that between every two vertices there is at most one edge. We write \( xy \) for an edge directed from \( x \) to \( y \). The outdegree \( d^+_G(x) \) of a vertex \( x \) is the number of vertices \( y \) such that \( xy \in E(G) \). Similarly, the indegree \( d^-_G(x) \) of a vertex \( x \) is the number of vertices \( y \) such that \( yx \in E(G) \). Define the minimum outdegree \( \delta^+(G) \) of \( G \) to be the minimal \( d^+_G(x) \) over all vertices \( x \) of \( G \), and define the minimum indegree \( \delta^-(G) \) of \( G \) similarly. Define the minimum semidegree \( \delta^0(G) \) of \( G \) to be \( \min \{ \delta^+(G), \delta^-(G) \} \).

The oriented graph on \( \{v_1, \ldots, v_n\} \) with edge set \( \{v_nv_1\} \cup \{v_iv_{i+1} : i \in \{1, \ldots, n-1\}\} \) is the directed cycle of length \( n \). An oriented graph in which there is an edge between every pair of vertices is called a tournament. A tournament that does not contain a directed cycle is transitive. Up to isomorphism, there are two tournaments on 3 vertices: The directed cycle of length 3, which we refer to as the cyclic triangle, and the transitive tournament on 3 vertices, which we refer to as the transitive triangle or as \( TT_3 \).

A tiling of \( G \) is a collection of vertex disjoint subgraphs called tiles. If every tile is isomorphic to some oriented graph \( H \), then the tiling is an \( H \)-tiling. If every vertex in \( G \) is contained in a tile, then the tiling is perfect. The same definitions are applied to graphs and directed graphs.

In [5], Hajnal and Szemerédi proved that for any \( k, r \in \mathbb{N} \) and for any graph \( G \) on \( kr \) vertices if the minimum degree of \( G \) is at least \( (r-1)k \), then \( G \) has a perfect \( K_r \)-tiling. The case when \( r = 3 \) was proved earlier by Corrádi and Hajnal [1].

The problem of finding cyclic triangle tilings in an oriented graph was considered by Keevash and Sudakov [7], who proved a nearly optimal result: For some \( \varepsilon > 0 \) there exists \( n_0 \) such that if \( G \) is an oriented graph on \( n \geq n_0 \) vertices and \( \delta^0(G) \geq (1/2 - \varepsilon)n \), then \( G \) contains a cyclic triangle tiling that covers all but at most 3 vertices. Furthermore, if \( n \equiv 3 \) (mod 18), then there is a tournament \( T \) such that \( \delta^0(T) \geq (n-1)/2-1 \) which does not have a perfect cyclic triangle tiling. They repeated the following question which was asked by both Cuckler [2] and Yuster [13].

\[ \text{Date: January 3, 2014.} \]
Question 1.1. Does every tournament $T$ on $n \equiv 3 \pmod{6}$ vertices with $\delta^0(T) = (n-1)/2$ have a perfect cyclic triangle tiling?

In this paper, we consider the problem of finding a perfect transitive triangle tiling, proving an analogue to Corrádi and Hajnal’s result for oriented graphs.

Theorem 1.2. There exists $n_0$ such that for every $n \in 3 \mathbb{Z}$ when $n \geq n_0$ the following holds. If $G$ is an oriented graph on $n$ vertices and $\delta^0(G) \geq 7n/18$, then $G$ contains a perfect $TT_3$-tiling.

Treglown [10] conjectured that Theorem 1.2 is true for any $n \in 3 \mathbb{Z}$.

The related problems for directed graphs have been considered (see [12], [4], [3] and [11]).

The following family of examples shows that Theorem 1.2 is tight. Let $n \in 3 \mathbb{Z}$ and let $G$ be the following oriented graph on $n$ vertices. Let $W_1, W_2, W_3, U_1, U_2$ be a partition of $V(G)$ such that

$|W_i| = \left\lfloor \frac{2n/3 + i}{3} \right\rfloor$ for $i \leq 3$, \quad $|U_1| = \left\lceil \frac{n-3}{6} \right\rceil$, \quad $|U_2| = \left\lceil \frac{n-3}{6} \right\rceil$.

The edges of $G$ are all possible directed edges from $W_i$ to $W_{i+1}$, from $U_1$ to $U_2$, from $W_1 \cup W_2$ to $U_1$, from $U_1$ to $W_3$, from $U_2$ to $W_1 \cup W_2$ and from $W_3$ to $U_2$, see Figure 1. Note that $\delta^0(G) = \lceil 2n/9 \rceil + \lceil (n-3)/6 \rceil + 1$, which is achieved by considering a vertex in $W_3$ (the outdegree when $n \equiv 15 \pmod{18}$ and indegree in all other cases). Since every transitive triangle in $G$ contains a vertex in $U_1 \cup U_2$ and $|U_1 \cup U_2| = n/3 - 1$, $G$ does not contain a perfect $TT_3$-tiling.

1.1. Outline of the paper. We prove Theorem 1.2 using a stability approach and the absorption technique. We say that an oriented graph $G$ on $n$ vertices is $\alpha$-extremal if there exists $W \subseteq V(G)$ such that $|W| \geq (2/3 - \alpha)n$ and $G[W]$ does not contain a transitive triangle.

In Section 2, we handle the case when $G$ is not $\alpha$-extremal.

Lemma 1.3. For every $\alpha > 0$ there exists $\varepsilon = \varepsilon(\alpha) > 0$ and $n_0 = n_0(\alpha)$ such that when $G$ is an oriented graph on $n \in 3 \mathbb{Z}$ vertices and $n \geq n_0$ the following holds. If $\delta^0(G) \geq (7/18 - \varepsilon)n$, then $G$ has a perfect $TT_3$-tiling or $G$ is $\alpha$-extremal.

In Section 3 we prove Theorem 1.2 for oriented graphs $G$ which are $\alpha$-extremal.
Lemma 1.4. There exists $\alpha > 0$ and $n_0$ such that when $G$ is an oriented graph on $n \in 3\mathbb{Z}$ vertices and $n \geq n_0$ the following holds. If $\delta^0(G) \geq 7n/18$ and $G$ is $\alpha$-extremal, then there exists a perfect $TT_3$-tiling of $G$.

Lemma 1.3 and Lemma 1.4 together clearly prove Theorem 1.2.

While proving Lemma 1.3 we prove the following result which may be of some interest because it applies for all $n$. Furthermore, it might be possible to extend the proof of this theorem to prove the main theorem for all $n$.

Theorem 1.5. If $G$ is an oriented graph on $n$ vertices and $\delta^0(G) \geq 7n/18$, then there exists a $TT_3$-tiling of $G$ that covers all but at most 11 vertices.

1.2. Notation. Given a graph or digraph $G$, we write $V(G)$ for its vertex set, $E(G)$ for its edge set, and $e(G) = |E(G)|$ for the number of its edges. Given a collection $\mathcal{T}$ of subgraphs, we write $V(\mathcal{T})$ for $\bigcup_{T \in \mathcal{T}} V(T)$. When $\mathcal{W}$ is a collection of vertex subsets we will also use the notation $V(\mathcal{W})$ to denote $\bigcup_{W \in \mathcal{W}} W$.

Suppose that $G$ is an oriented graph. If $x$ is a vertex of $G$, then $N^+_G(x)$ denotes the out-neighborhood of $x$, i.e. the set of all those vertices $y$ for which $xy \in E(G)$. Similarly, $N^-_G(x)$ denotes the in-neighborhood of $x$, i.e. the set of all those vertices $y$ for which $yx \in E(G)$. Note that $d^+_G(x) = |N^+_G(x)|$ and $d^-_G(x) = |N^-_G(x)|$. We write $N_G(x) = N^+_G(x) \cup N^-_G(x)$ and $d_G(x) = d^+_G(x) + d^-_G(x)$. We write $\delta(G)$ and $\Delta(G)$ for the minimum degree and maximum degree of the underlying undirected graph of $G$, respectively. Given a vertex $v$ of $G$ and a set $A \subseteq V(G)$, we define $d^+_G(v,A) = |N^+_G(v) \cap A|$ and define $d^-_G(v,A)$ and $d_G(v,A)$ similarly.

Given $A, B \subseteq V(G)$, let $E_G(A,B)$ be the set of edges in $G$ directed from $A$ to $B$. Similarly, $E_G(A,B)$ denotes the set of edges with one endpoint in $A$ and the other in $B$ and let $e_G(A,B) = |E_G(A,B)|$. For a vertex $v$, we write $E_G(v)$ for $E_G(v,V(G))$. For a vertex set $A \subseteq V(G)$, we write $G[A]$ for the subgraph of $G$ induced by $A$ and let $e_G(A) = e(G[A])$. If $G$ is known from the context, then we may omit the subscript. We let $G[A,B]$ be the bipartite graph in which $a \in A$ is adjacent to $b \in B$ if and only if $ab \in E(G)$ or $ba \in E(G)$. If $x, y, z \in V(G)$ we sometimes refer to $G[\{x, y, z\}]$ as $xyz$ or as $xe$, where $e = yz$ or $e = zy$.

If we refer to a directed path or cyclic triangle as $xyz$, then it must contain the edge set $\{xy, yz\}$ or the edge set $\{xy, yz, zx\}$, respectively.

For $m \in \mathbb{N}$ we write $[m] = \{1, \ldots, m\}$. For any set $V$ we let $\binom{V}{m}$ be the collection of subsets of $V$ that are of order $m$. When it is clear that a variable $i$ must remain in $[m]$ (e.g. when $i$ is the index of $W_1, \ldots, W_m$) we let $i + 1 = 1$ when $i = m$ and $i - 1 = m$ when $i = 1$.

1.3. Preliminary lemmas and propositions. Let $G$ be an oriented graph. Note that if $uw$ is an edge in $G[N^+(v)]$, then $vuw$ is a transitive triangle. We get the following easy proposition.

Proposition 1.6. Let $G$ be a tournament on 4 vertices. Then every vertex of $G$ is contained in a transitive triangle.

Proposition 1.7. Let $G$ be an oriented graph on $n$ vertices. Then

(a) every (directed) edge $uv$ is contained in at least $3\delta^0(G) - n$ transitive triangles $uvw$ such that $w \in N^- (v)$;
(b) every (directed) edge $uv$ is contained in at least $3\delta^0(G) - n$ transitive triangles $uvw$ such that $w \in N^+ (u)$;

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(c) For every directed path $uvw$ on 3 vertices, there are at least $2(3\delta^0(G) - n)$ vertices $x$ such that there exists a transitive triangle in $G[[u, v, w, x]]$ containing $x$ and $v$.

Proof. Let $uw$ be an edge in $G$. Note that every vertex $w$ in $N(u) \cap N^-(v)$ forms a transitive triangle with $uv$. Since $|N(u) \cap N^-(v)| \geq \delta(G) + \delta^-(G) - n \geq 3\delta^0(G) - n$, (a) follows. By a similar argument, (b) also holds.

Let $uvw$ be a directed path on 3 vertices. By (a), there is a set $U \subseteq N^-(v)$ with $|U| \geq 3\delta^0(G) - n$ such that every $u' \in U$ forms a transitive triangle with $uv$. By (b), there is a set $W \subseteq N^+(v)$ with $|W| \geq 3\delta^0(G) - n$ such that every $w' \in W$ forms a transitive triangle with $vw$. Since $U \cap W = \emptyset$, (c) holds. 
\[\] 

2. Non-extremal case

2.1. Absorbing structure. In this section, we prove Lemma 2.2. Roughly speaking, the lemma states that there exists a small vertex set $U \subseteq V(G)$ such that $G[U \cup W]$ contains a perfect $TT_3$-tiling for every small $W \subseteq V(G) \setminus U$. Thus, in order to find a perfect $TT_3$-tiling in $G$, it is sufficient to find a $TT_3$-tiling covering almost all vertices in $G[V(G) \setminus U]$. This technique was introduced by Rödl, Ruciński and Szemerédi [9] to obtain results on matchings in hypergraphs.

For any $r \in \mathbb{N}$ and any collection $H$ of oriented graphs on $[r]$, define $\mathcal{F}(H, G)$ to be the set of functions $f$ from $[r]$ to $V(G)$ such that $f$ is a directed graph homomorphism from some $H \in \mathcal{H}$ to $G$. Let $\mathcal{K}$ be the set of oriented graphs $K$ on $\{1, \ldots, 21\}$ such that both $K$ and $K[\{1, \ldots, 18\}]$ have a perfect $TT_3$-tiling. For any ordered triple $X = (x_1, x_2, x_3)$ of vertices in $G$, let $\mathcal{A}'(X)$ be the set of functions $f \in \mathcal{F}(\mathcal{K}, G)$ such that $f(19) = x_1$, $f(20) = x_2$ and $f(21) = x_3$. Let $\mathcal{A}(X)$ be the set of functions in $\mathcal{A}'(X)$ restricted to $[18]$. Clearly $|\mathcal{A}(X)| = |\mathcal{A}'(X)|$. Note we do not require the function in $\mathcal{F}(H, G)$ to be injective, but at a later stage of the proof non-injective functions will essentially be discarded. We consider non-injective functions only to make the following arguments simpler.

Lemma 2.1. For $\varepsilon_0 = 1/250$ and $0 \leq \varepsilon \leq \varepsilon_0$, there exists $\tau = \tau(\varepsilon) > 0$ and $n_0 = n_0(\varepsilon)$ such that the following holds. If $G$ is an oriented graph on $n \geq n_0$ vertices and $\delta^0(G) \geq (7/18 - \varepsilon)n$, then $|\mathcal{A}(X)| \geq \tau n^{18}$ for every ordered triple $X = (x_1, x_2, x_3)$ of vertices in $G$.

Proof. Let $0 < \beta < (1/249 - \varepsilon)/10$ and $\tau = \beta^{18}$. Let $\mathcal{T}$ be the set of functions from $\{1, 2, 3\}$ to $V(G)$ that are digraph homomorphisms from a transitive triangle on $\{1, 2, 3\}$ to $G$. In other words, $\mathcal{T}$ contains all functions from $\{1, 2, 3\}$ to $V(G)$ whose image induces a transitive triangle. If we let $f(1)$ be any vertex $a \in V(G)$ and let $f(2)$ be any $b \in N_G(a)$, by Proposition 1.7, there are $3\delta^0(G) - n \geq (1/6 - 3\varepsilon)n$ vertices we can assign to $f(3)$ so that $f \in \mathcal{T}$. This gives us that

$$|\mathcal{T}| \geq n \cdot (7/9 - 2\varepsilon)n \cdot (1/6 - 3\varepsilon)n > n^3/9 > (\beta n)^3.$$ 

For any $p \in \mathbb{N}$, let $\mathcal{L}_p$ be the set of oriented graphs $L$ on $[3p + 1]$ such that both $L[\{2, \ldots, 3p + 1\}]$ and $L[\{1, \ldots, 3p\}]$ have perfect $TT_3$-tilings (see Figure 2 for some examples). For any $x, y \in V(G)$, let $\mathcal{C}_p(x, y)$ be the set of $f \in \mathcal{F}(\mathcal{L}_p, G)$ such that $f(1) = x$ and $f(3p + 1) = y$. If $x = y$ we say that $x$ and $y$ are 0-linked. Otherwise, we say that $x$ and $y$ are $p$-linked if $|\mathcal{C}_p(x, y)| \geq (\beta n)^{3p-1}$.
Let $q \geq p$. We have that $|C_q(x, y)| \geq |C_p(x, y)||T|^{q-p}$. Indeed, for any $f \in C_p(x, y)$ and $g_i \in T$ for $i \in [q-p]$, the function

$$h(j) = \begin{cases} f(j) & \text{if } 1 \leq j \leq 3p \\ g_i(k) & \text{if } j = 3p + 3(i-1) + k \text{ for some } i \in [q-p] \text{ and } k \in [3] \\ y & \text{if } j = 3q + 1 \end{cases}$$

is in $C_q(x, y)$. Hence, the inequality holds. Recall that we do not require the functions in $C_q(x, y)$ to be injective. So since $|T| \geq (\beta n)^3$, if $x$ and $y$ are $p$-linked, then $x$ and $y$ are also $q$-linked.

Let $X = (x_1, x_2, x_3)$ be an ordered triple of vertices in $G$. We have that

$$|A(X)| = |A'(X)| \geq \sum_{f \in T} \prod_{i \in 1} |C_2(f(i), x_i)|.$$ 

Indeed, let $f \in T$ and $g_i \in C_2(f(i), x_i)$ for $i \in [3]$, and define

$$h(j) = \begin{cases} g_i(k) & \text{if } j = 6(i-1) + k \text{ for some } i \in [3] \text{ and } k \in [6] \\ x_i & \text{if } j = 18 + i \text{ for some } i \in [3] \end{cases}$$

By the definition of $C_2$ and the fact that the image $h(\{1, 7, 13\}) = f([3])$ induces a transitive triangle in $G$, it is not too hard to see that $h \in A'(X)$ and that the inequality holds. Therefore, because $|T| \geq (\beta n)^3$, we can complete the proof of the lemma by showing that every pair of vertices in $V(G)$ is 2-linked.

For any $U \subseteq V(G)$, let $N(U) = \bigcap_{u \in U} N_G(u)$. If $U = \{x, y\}$ is a 2-set, we often write $N(x, y)$ instead of $N(U)$. We have the following inequality

$$N_G(U) \geq |U|\delta(G) - (|U| - 1)n \geq \left(\frac{9 - 2|U|}{9} - 2|U|\varepsilon\right)n. \quad (1)$$

For any pair $x, y \in V(G)$, let

$$N_1^1(x, y) = N_G^+(x) \cap N_G^+(y), \quad N_2^1(x, y) = N_G^-(x) \cap N_G^+(y),$$

$$N_1^2(x, y) = N_G^+(x) \cap N_G^-(y), \quad N_2^2(x, y) = N_G^-(x) \cap N_G^-(y)$$

Figure 2. A graph in $L_1$ and a graph in $L_2$. 

Figure 3. Orientation of any edges in $G'[N]$. 5
and \( \mathcal{N}(x, y) = \{ N_i^j(x, y) : 1 \leq i, j \leq 2 \} \). Furthermore, let
\[
F(x, y) = \overrightarrow{E}_G(N^2_2(x, y), N(x, y)) \cup \overrightarrow{E}_G(N(x, y), N^1_1(x, y)) \cup \bigcup_{A \in \mathcal{N}(x, y)} E(G[A]),
\]
and note that for every \( e \in F(x, y) \) both \( xe \) and \( ye \) are transitive triangles, so \( e \) corresponds to two distinct homomorphisms in \( \mathcal{C}_1(x, y) \). Therefore, if \( |F(x, y)| \geq (\beta n)^2/2 \), then \( x \) and \( y \) are 1-linked.

**Claim 1.** For any pair \( x, y \in V(G) \), if there exists \( A \in \mathcal{N}(x, y) \) such that \( |A| \geq (2/9 + \beta + 2\varepsilon)n \), then \( x \) and \( y \) are 1-linked.

**Proof.** We have that \( |F(x, y)| \geq e_G(A) \geq |A|((\delta(G) + |A| - n)/2 > (\beta n)^2/2. \)

For a contradiction, assume that there exists a pair \( x, y \in V(G) \) that are not 2-linked and define \( F = F(x, y) \). Therefore, since this implies that \( x \) and \( y \) are not 1-linked, we have that \( |F| < (\beta n)^2/2 \). Let
\[
N^0 = \{ v \in N(x, y) : |E_G(v) \cap F| \geq \beta n \}
\]
and note that \( |N^0| < \beta n \). Let \( G' = G - F, N = N(x, y) \setminus N^0, N^j_i = N_i^j(x, y) \setminus N^0 \) for every \( 1 \leq i, j \leq 2 \), and \( \mathcal{N} = \{ N_i^j : 1 \leq i, j \leq 2 \} \) (see Figure 3).

**Claim 2.** For every \( u \in N \), there exist at least two distinct sets \( A, B \in \mathcal{N} \) such that both \( |N_G(u, A)| \geq 2\beta n \) and \( |N_G(u, B)| \geq 2\beta n \).

**Proof.** If we suppose there exists only one such set \( A \in \mathcal{N} \), then, using (1), we have that
\[
|A| > |N_G(\{u, x, y\})| - |N^0| - |E_G(u) \cap F| = |d_{G'}(u, N \setminus A)| > (1/3 - 6\varepsilon - 8\beta)n > n/4
\]
which, by Claim 1, contradicts the fact that \( x \) and \( y \) are not 1-linked.

Let \( I^+ = \{ v \in N : d_{G'}(v, N) < 2\beta n \} \), \( I^- = \{ v \in N : d_{G'}(v, N) < 2\beta n \} \) and \( I = I^+ \cup I^- \). Note that \( N^1_i \subseteq I^- \) and \( N^2_i \subseteq I^+ \).

**Claim 3.** Every pair \( u, w \) in \( I \) is 1-linked.

**Proof.** Both \( u \) and \( w \) have all but \( 2\beta n \) of one type of neighbors (either in or out) in \( \overline{N} \), so there exists \( A \in \mathcal{N}(u, w) \) such that
\[
|A| \geq 2(\delta^0(G) - 2\beta n) - |\overline{A}| \geq (7/9 - 2\varepsilon - 4\beta)n - (4/9 + 4\varepsilon + \beta)n > n/4.
\]
Applying Claim 1 completes the proof.

**Claim 4.** If \( A \in \mathcal{N} \) and \( |A| \geq (1/18 + 5\varepsilon + 3\beta)n \), then every pair \( u, w \in A \) is 1-linked.

**Proof.** Since \( u \) and \( w \) both have all but \( \beta n \) of their neighbors in \( \overline{N} \),
\[
N_G(u, w) \geq 2(\delta(G) - \beta n) - |\overline{A}| \geq (14/9 - 4\varepsilon - 2\beta)n - (17/18 - 5\varepsilon - 3\beta)n = (11/18 + \beta + \varepsilon)n.
\]
Let \( B = N^1_i(u, w) \cup N^2_i(u, w) \) and \( v \in B \). We have that
\[
|E_G(v) \cap F(u, w)| \geq \delta^0(G) + |N_G(u, w)| - n \geq \beta n.
\]
Therefore, if \( |B| \geq \beta n \), then \( |F(u, w)| \geq (\beta n)^2/2 \) and \( u \) and \( w \) are 1-linked. If \( |B| < \beta n \), then there exists \( i \in \{1, 2\} \) such that
\[
|N^{3-i}_i(u, w)| \geq (|N_G(u, w)| - |B|)/2 > n/4
\]
and the result follows from Claim 1.
Suppose that there are \((\beta n)^2\) pairs \((a, b) \in N_1 \times N_2\). By (1) and the fact that \(a\) and \(b\) are each incident to at most \(\beta n\) edges from \(F\),

\[
|N_{G'}(a, b) \cap (N_1^2 \cup N_2^2)| \geq |N_{G'}(\{a, b, x, y\})| - |N_0| - 2\beta n > \beta n.
\]

Pick \(c \in N_{G'}(a, b) \cap (N_1^2 \cup N_2^2)\) in one of \(\beta n\) ways. If \(c \in N_1^2\), then \(xac\) and \(ybc\) are transitive triangles and if \(c \in N_2^2\), then \(xbc\) and \(yac\) are transitive triangles. By Claim 3, \(a\) and \(b\) are 1-linked which implies that \(x\) and \(y\) are 2-linked, a contradiction.

So we can assume that there exists \(i \in \{1, 2\}\) such that \(|N_{3-i}^-| < \beta n\). For the rest of the argument we will assume \(i = 1\), a similar argument will work for the case when \(i = 2\). By Claim 2 and the fact that \(|N_3^2| < \beta n\), there are at least \((\beta n)^2\) edges in \(E_{G'}(N_1^2, N_2^2)\). Let \(ab\) be one such edge and pick \(j \in \{1, 2\}\) so that \(a \in N_j^{3-j}\) and \(b \in N_{3-j}\).

Assume there are at least \(\beta n\) out-neighbors of \(c\) in \(N_j^{3-j}\). If \(j = 1\), then \(xab\) and \(ybc\) are transitive triangles and if \(j = 2\), then \(xbc\) and \(yab\) are transitive triangles. By Claim 3, \(|N_j^{3-j}| \geq |N(x, y)| - |N_0| - |N_2^2| - 2(2/9 + \beta + 2\varepsilon)n > n/12\).

With Claim 4, this implies that \(a\) and \(c\) are 1-linked. Therefore, \(x\) and \(y\) are 2-linked, a contradiction.

So assume that there are less than \(\beta n\) out-neighbors of \(b\) in \(N_j^{3-j}\). Recall that

\[
\overrightarrow{E}_{G'}(b, N \setminus (N_j^{3-j} \cup N_2^2)) \subseteq F,
\]

so \(b\) has less than \(2\beta n\) out neighbors in \(N\) and hence, \(b \in I^+\). By Claim 3, \(b\) is 1-linked with every vertex in \(N_1^2\), and by Claim 2 we can pick \(c \in N_{G'}(a, 1)\) in \(2\beta n\) ways. If \(j = 1\), then \(xab\) and \(ybc\) are transitive triangles and if \(j = 2\), then \(xbc\) and \(yab\) are transitive triangles. Therefore, since \(b\) and \(c\) are 1-linked, \(x\) and \(y\) are 2-linked, a contradiction. \(\square\)

**Lemma 2.2** (Absorbing Lemma). For every \(0 \leq \varepsilon \leq 1/250\), there exists \(\sigma_0 = \sigma_0(\varepsilon)\) such that for every \(0 < \sigma < \sigma_0\), there exists \(n_0 = n_0(\varepsilon, \sigma)\) such that the following holds. If \(G\) is an oriented graph on \(n \geq n_0\) vertices with \(\delta^0(G) \geq (7/18 - \varepsilon)n\), then \(G\) contains a vertex set \(U \subseteq V(G)\) with \(|U| \leq 3\sigma n\) and \(|U| \in 3\mathbb{Z}\) such that, for every \(W \subseteq V(G)\) \(\setminus U\) with \(|W| \leq 3\sigma^2 n\) and \(|W| \in 3\mathbb{Z}\), \(G[U \cup W] \) contains a perfect \(TT_3\)-tiling.

**Proof.** Let \(\tau = \tau(\varepsilon)\) be the constant given by Lemma 2.1 and let \(\sigma_0 = \tau/(72^2 + 1)\) and let \(0 < \sigma < \sigma_0\). Let \(G\) be sufficiently large oriented graph with \(\delta^0(G) \geq (7/18 - \varepsilon)n\). Let \(\mathcal{F}\) be the set of functions from \([18]\) to \(V(G)\). Call a map \(f \in \mathcal{F}\) absorbing if there exists an ordered triple \(X\) of vertices such that \(f \in A(X)\).

Choose \(\mathcal{U}' \subseteq \mathcal{F}\) by selecting each \(f \in \mathcal{F}\) independently at random with probability \(p = 2\sigma n^{-17}\). Call a pair \(f, g \in \mathcal{F}\) bad if either \(f\) or \(g\) is not injective or the images of \(f\) and \(g\) intersect and note that there are \(n \cdot (36/2) \cdot n^{34}\) bad pairs in \(\mathcal{F}\). Therefore, the expected number of bad pairs in \(\mathcal{U}'\) is less than \(18 \cdot 35 \cdot 4\sigma^2 n\). Thus, using Markov’s inequality, we derive that, with probability more than \(1/2\), \(\mathcal{U}'\) contains at most \((72\sigma)^2 n\) intersecting pairs.

By Chernoff’s bound and Lemma 2.1, with positive probability the set \(\mathcal{U}'\) also satisfies \(|\mathcal{U}'| \leq 3\sigma n|\mathcal{U}|\) and \(|A(X) \cap \mathcal{U}'| \geq \tau \sigma n\) for each ordered triple \(X\) of vertices. By deleting every bad pair from \(\mathcal{U}\) and any \(f \in \mathcal{U}'\) for which \(f\) is not absorbing, we get a \(\mathcal{U} \subset \mathcal{U}'\) consisting of injective homomorphisms with pairwise disjoint images. Moreover, for each ordered triple \(X\) of vertices, there are at least \(\tau \sigma n - (72\sigma)^2 n > \sigma^2 n\) functions in \(A(X)\) \(\cap \mathcal{U}\). Let \(\mathcal{U}\) be the union of the images of every \(f \in \mathcal{U}\). Since \(\mathcal{U}\) consists only of absorbing functions, \(G[\mathcal{U}]\) has a perfect \(TT_3\)-tiling, so \(|\mathcal{U}| \in 3\mathbb{Z}\). For any set \(W \subseteq V\) of size \(|W| \leq 3\sigma^2 n\) and \(|W| \in 3\mathbb{Z}\), \(W\) can be
oriented graph on proof of Theorem 1.5, let

For any $\alpha > 0$ there exists $\varepsilon = \varepsilon(\alpha)$ such that the following holds. If $G$ is an oriented graph on $n$ vertices such that $\delta^0(G) \geq (7/18 - \varepsilon)n$, then either $G$ has a $TT_3$-tiling on all but at most 14 vertices or $G$ is $\alpha$-extremal.

Proof of Theorem 1.5 and Lemma 2.3. For the proof of Lemma 2.3 let $\varepsilon = \alpha/50$. For the proof of Theorem 1.5 let $\varepsilon = 0$. So, in either case, we have that $\delta^0(G) \geq (7/18 - \varepsilon)n$.

Let $\mathcal{M} = \mathcal{T} \cup \mathcal{P} \cup \mathcal{F} \cup \mathcal{I}$ be a collection of vertex disjoint subgraphs of $G$ such that every vertex in $G$ is contained in a subgraph of $\mathcal{M}$, every $T \in \mathcal{T}$ is a transitive triangle, every $P \in \mathcal{P}$ is a directed path on 3 vertices, every $e \in \mathcal{F}$ is an edge and every $v \in \mathcal{I}$ is a single vertex. Clearly such a set $\mathcal{M}$ exists. Assume that $\mathcal{M}$ is selected to maximize $(|\mathcal{T}|, |\mathcal{P}|, |\mathcal{F}|)$ lexicographically. Let $X = V(\mathcal{T})$, $Y = V(\mathcal{P})$ and $Z = V(G) \setminus (X \cup Y)$.

We will show that if $\varepsilon = 0$, then $|\mathcal{P}| \leq 2$ and if $\varepsilon > 0$ and $|\mathcal{P}| \geq 4$, then $G$ is $\alpha$-extremal.

Let $B$ be a $(\mathcal{P}, V(G))$ bipartite graph in which there is an edge between $P \in \mathcal{P}$ and $v \in V(G)$ if and only if $G[P \cup v]$ contains a transitive triangle. By Proposition 1.7 $d_B(P) \geq 2(3\delta^0(G) - n) \geq (1/3 - 6\varepsilon)n$. For every $P \in \mathcal{P}$, by the maximality of $|\mathcal{T}|$, $d_B(P, Y \cup Z) = 0$. Also by the maximality of $|\mathcal{T}|$, for every $T \in \mathcal{T}$ if there exists $x \in V(T)$ such that $d_B(x) \geq 2$, then $d_B(y) = 0$ for every $y \in V(T) - x$. Assume $|\mathcal{P}| \geq 3$ and note that we then have that $e_B(\mathcal{P}, V(T)) \leq |\mathcal{P}|$ for every $T \in \mathcal{T}$. Let

$$
\mathcal{T'} = \{ T \in \mathcal{T} : e_B(\mathcal{P}, V(T)) > 3 \}.
$$

We have that

$$
n + (|\mathcal{P}| - 3)|\mathcal{T}'| > 3|\mathcal{T}| + (|\mathcal{P}| - 3)|\mathcal{T}'| \geq e_B(\mathcal{P}, V(G)) \geq (1/3 - 6\varepsilon)n|\mathcal{P}|.
$$

Which, since $|\mathcal{T}'| < n/3$, is a contradiction when $\varepsilon = 0$, so in this case, we must have that $|\mathcal{P}| \leq 2$. If $\varepsilon > 0$ and $|\mathcal{P}| \geq 4$, then

$$
|\mathcal{T}'| \geq \left( \frac{|\mathcal{P}|}{3(|\mathcal{P}| - 3)} - \frac{1}{|\mathcal{P}| - 3} - \frac{6|\mathcal{P}|\varepsilon}{|\mathcal{P}| - 3} \right) n \geq \left( \frac{1}{3} - 24\varepsilon \right) n.
$$

For every $T \in \mathcal{T}'$, since $e_B(\mathcal{P}, T) \geq 4$, there exists $x_T \in V(T)$ such that $d_B(x_T) \geq 2$. Therefore, by the maximality of $|\mathcal{T}|$, $d_B(x_T) = e_B(\mathcal{P}, V(T)) \geq 4$. Let

$$
W = Y \cup Z \cup \bigcup_{T \in \mathcal{T}'} (V(T) - x_T),
$$

and note that $|W| > (2/3 - 48\varepsilon)n$. The graph $G[W]$ does not contain a transitive triangle. Indeed, if such a triangle $T$ exists and we define $\mathcal{T}'' = \{ T' \in \mathcal{T}' : V(T) \cap V(T') \neq \emptyset \}$ and $B' = B - \{ P \in \mathcal{P} : V(P) \cap V(T) \neq \emptyset \}$, then for every $T' \in \mathcal{T}''$, we have that

$$
d_B(x_T) \geq d_B(x_{T'}) - |Y \cap V(T)| \geq 4 - |Y \cap V(T)| > |X \cap V(T)| \geq |\mathcal{T}'|.
$$
Therefore, there is a matching covering $T''$ in $B'$. The edges in this matching correspond to $|T''|$ disjoint transitive triangles in the graph induced by $(V(T'') \cup Y \cup Z) \setminus V(T)$, contradicting the maximality of $|T|$. Hence, $G$ is $\alpha$-extremal.

Assume that there exist two distinct edges $ab$ and $cd$ in $F$. For any set $U \subseteq V(G)$, define

$$w(U) = d^+_G(a, U) + \sum_{v \in \{b, c, d\}} d_G(v, U).$$

Note that by the maximality of $|P|$, there are no triangles in $G[Z]$, so for every $z \in Z$, $w(z) \leq 2$. For any $P \in \mathcal{P}$, the maximality of $|T|$ implies that there is no transitive triangle in the graph induced by $\{a, b, c, d\} \cup V(P)$. It is not hard to see that, with Proposition \ref{prop:extremal}, this gives us that $d^+_G(a, V(P)) + d_G(b, V(P)) \leq 3$ and $e_G(cd, V(P)) \leq 4$, so $w(P) \leq 7$.

Claim. For every $T \in \mathcal{T}$, $w(T) \leq 8$.

Proof. Assume that there exists $T \in \mathcal{T}$ such that $w(T) \geq 9$. We will show that there exists a disjoint directed path on 3 vertices and a transitive triangle in the graph induced by $\{a, b, c, d\} \cup V(T)$, contradicting the maximality of $|P|$.  

Remove the edges into $a$ from $G$ to form $G'$. Note that this implies that for every $x \in V(T)$, if $e_G(ab, x) = 2$, then $abx$ is a transitive triangle. If, in addition to this, there exists $y \in V(T) - x$ such that $e_G(cd, y) = 2$, then we have the desired $P_3$. This is the case when for one of the edges $e \in \{ab, cd\}$, $d_G(e, V(T)) = 5$. Indeed, if $f \in \{ab, cd\} - e$, then $e_G(f, V(T)) \geq 4$, so we can pick $x \in V(T)$ such that $e_G(f, x) = 2$. Since then $e_G(e, V(T) - x) \geq 3$, we can pick $y \in V(T) - x$ such that $e_G(e, y) \geq 2$. Therefore, we are only left to consider the cases when one of $ab$ or $cd$ and $V(T)$ induce a tournament on 5 vertices in $G'$.

If $e_G(ab, V(T)) = 6$, then $e_G(cd, V(T)) \geq 3$ and for one of $c$ or $d$, say $c$, $e_G(c, V(T)) \geq 2$, so if $x$ and $y$ are the neighbors of $c$ in $V(T)$, $cxy$ is a triangle. If $z \in V(T) - x - y$, then $zab$ is a transitive triangle.

If $e_G(cd, V(T)) = 6$, then $e_G(ab, V(T)) \geq 3$. We can assume that $T$ is the unique transitive triangle in $G'[[a, b] \cup V(T)]$, because, if it was not, then the graph induced by vertices of $V(T)$ not also in this triangle and $\{c, d\}$ would contain a triangle, which contains a directed path on 3 vertices. This implies that $e_G(ab, v) = 1$ for every $v \in V(T)$, and that $e_G(a, V(T)) \leq 1$. This further implies that $b$ has an out neighbor $x \in V(T)$, so $abx$ is a directed path on 3 vertices. Since $G'[[c, d] \cup V(T) - x]$ is a tournament on 4 vertices, we have the desired transitive triangle by Proposition \ref{prop:extremal}.

Therefore,

$$\left(\frac{49}{18} - 7\varepsilon\right)n \leq 7\delta^0(G) \leq w(V(G)) \leq 2|Z| + 7|Y|/3 + 8|X|/3 \leq 8n/3,$$

a contradiction. Hence $|F| \leq 1$.

By the maximality of $|F|$, $I$ is an independent set. Since there are no triangles in $G[Z]$, $e_G(I, e) \leq |I|$ for every $e \in F$. Let $T \in \mathcal{P} \cup \mathcal{T}$. If $e_G(I, V(T)) > 2|I| + 1$, then there exist vertices $v_1, v_2 \in I$ such that $e_G(v_1, V(T)) = e_G(v_2, V(T)) = 3$. Furthermore, in the graph induced by $\{v_1, v_2\} \cup V(T)$, if $T \in \mathcal{P}$, then there is a triangle and a disjoint edge, and if $T \in \mathcal{T}$, then there is a transitive triangle and a disjoint edge. Since both cases violate the
maximality of $|F|$, we have that

$$|I| \left(\frac{7}{9} - 2\varepsilon\right) n \leq |I|\delta(G) \leq e_G(I, V(G) \setminus I)$$

$$\leq |I||F| + (2|I| + 1)|\mathcal{P} \cup \mathcal{T}| \leq \frac{|I||Z|}{2} + \frac{(2|I| + 1)|X \cup Y|}{3} \leq |I|\frac{2n}{3} + \frac{n}{3}.$$

Hence, $|I| \leq 3 + 18\varepsilon$. □

Using Lemmas $\ref{lem:2.2}$ and $\ref{lem:2.3}$ we can prove Lemma $\ref{lem:1.3}$

Proof of Lemma $\ref{lem:1.3}$. Let $\varepsilon = \min\{\varepsilon(\alpha)/2, 1/250\}$ where $\varepsilon(\alpha)$ is as in Lemma $\ref{lem:2.3}$ and let $\sigma_0 = \sigma_0(\varepsilon)$ be as in by Lemma $\ref{lem:2.2}$. Assume that $n$ is sufficiently large. So, by Lemma $\ref{lem:2.2}$ there exists $\sigma < \min\{\sigma_0, \varepsilon/3\}$ for which there exists $U \subseteq V(G)$ such that $|U| \leq 3\sigma n < \varepsilon n$ and the conclusion of Lemma $\ref{lem:2.2}$ holds. Let $G' = G - U$ and note that $\delta^0(G') \geq (7/18 - 2\varepsilon)n$. If we assume $G$ is not $\alpha$-extremal, then by Lemma $\ref{lem:2.3}$ and the fact that $n \in 3\mathbb{Z}$, there exists a $TT_3$-tiling on all but a set $W$ of size at most 12. Since $|W| < 3\sigma^2 n$ there exists a perfect $TT_3$-tiling of $G[U \cup W]$ completing the proof. □

3. The $\alpha$-extremal case

In this section we prove Lemma $\ref{lem:1.3}$. We start with some well-known and simple propositions regarding matchings in graphs.

**Proposition 3.1.** Every graph $G$ on $n$ vertices has a matching of size at least $\min\{|n/2|, \delta(G)\}$.

**Proof.** Let $M$ be a maximum matching in $G$ and assume $|M| < \min\{|n/2|, \delta(G)\}$. Let $U$ be the set of vertices that are incident to an edge in $M$. Because $|M| \leq n/2 - 1$, there exist distinct $x, y \in V(G) \setminus U$. Since $M$ is a maximum matching, $e_G\{x, y\}, V(G) \setminus U) = 0$ which implies

$$e_G\{x, y\}, U) \geq 2\delta(G) > 2|M|,$$

So there exists $e \in M$ such that $e_G\{x, y\}, e) \geq 3$. This contradicts the maximality of $M$. □

**Proposition 3.2.** Let $G$ be an $(X, Y)$-bipartite graph with $d_G(x) \geq a$ for every $x \in X$ and $d_G(y) \geq b$ for every $y \in Y$. If $|X| = |Y|$ and $a + b \geq |X|$, then $G$ contains a perfect matching.

**Proof.** We show that $G$ satisfies Hall’s condition. Let $X' \subseteq X$ be non-empty, let $x \in X'$ and let $Y'$ be the set of vertices in $Y$ that are adjacent to a vertex in $X'$. Clearly, $|Y'| \geq d_G(x) \geq a$, so assume $|X'| > a$. We have that $d_G(y) \geq b > |X' \setminus X|$ for every $y \in Y$. Hence, $y \in Y'$ and $|Y'| = |Y| \geq |X'|$. □

Let $G$ be a $(V_1, V_2)$-bipartite graph. For $X_1 \subseteq V_1$ and $X_2 \subseteq V_2$ both non-empty, define $d_G(X_1, X_2) := \frac{e_G(X_1, X_2)}{|X_1||X_2|}$ to be the density of $G$. For constants $0 < \varepsilon, d < 1$, we say that $G$ is $(d, \varepsilon)$-regular if

$$(1 - \varepsilon)d \leq d_G(X_1, X_2) \leq (1 + \varepsilon)d$$

whenever $|X_i| \geq \varepsilon|V_i|$ for $i = 1, 2$. We say that $G$ is $(d, \varepsilon)$-superregular if $G$ is $(d, \varepsilon)$-regular and $(1 - \varepsilon)d|V_i| \leq d_G(v, V_i) \leq (1 + \varepsilon)d|V_i|$ for every $v \in V_3$, and $i \in \{1, 2\}$. Note that the preceding definitions match the corresponding definitions in $\S$. 3.

**Proposition 3.3.** For any $0 < \varepsilon < 1$, if $G$ is a $(V_1, V_2)$-bipartite graph such that $|V_1| = |V_2| = n$ and $\delta(G) \geq (1 - \varepsilon)n$ then $G$ is $(1, \varepsilon^{1/2})$-superregular.
Proof. It is clear that we only need to show that $G$ is $(1, \varepsilon^{1/2})$-regular. Let $X_i \subseteq V_i$ such that $|X_i| \geq \varepsilon^{1/2} n$ for $i \in \{1, 2\}$. We have that
\[ 1 \geq d(X_1, X_2) \geq \frac{|X_1|(|X_2| - \varepsilon n)}{|X_1||X_2|} = 1 - \frac{\varepsilon n}{|X_2|} \geq 1 - \varepsilon^{1/2}. \] □

The following theorem follows immediately from the Chernoff type bounds on the hypergeometric distribution (see Theorem 2.10 in [6]).

**Theorem 3.4.** For every $0 < \eta < 1$ there exists $k = k(\eta) > 0$ such when $V$ is a set, $X \subseteq V$ and $m$ is a positive integer such that $m \leq |V|$ the following holds. If $|U|$ is selected uniformly at random from \( \binom{V}{m} \), then with probability at least $1 - e^{-km}$
\[ \frac{|X|}{|V|} - \eta \leq \frac{|X \cap U|}{|U|} \leq \frac{|X|}{|V|} + \eta. \]

A partition of a set is *equitable* if the class sizes are differing by at most 1.

**Proposition 3.5.** For every $\eta > 0$ there exist integers $k = k(\eta) > 0$ and $n_0 = n_0(\eta)$ such that when $F$ is an $(A, B)$-bipartite graph with $|A| = |B| = n$ for $n \geq n_0$ the following holds. If an equitable partition $\{A_1, A_2\}$ of $A$ and an equitable partition $\{B_1, B_2\}$ of $B$ are both chosen uniformly at random from all such partitions, then with probability at least $1 - e^{-kn}$ we have
\[ d_F(A, B) - \eta \leq d_F(A_i, B_j) \leq d_F(A, B) + \eta \]
for every $1 \leq i, j \leq 2$.

**Proof.** Choose partitions $\{A_1, A_2\}$ and $\{B_1, B_2\}$ as in the proposition and let $k = 5k(\eta/2)$, where $k(\eta/2)$ is as in in Theorem 3.4 and assume that $n$ is sufficiently large. Let $1 \leq i, j \leq 2$. By Theorem 3.4 for any $v \in A$ with probability at least $1 - e^{-5k|B_j|} \geq 1 - e^{-2kn}$
\[ \frac{d_F(v, B)}{|B|} - \frac{\eta}{2} \leq \frac{d_F(v, B_j)}{|B_j|} \leq \frac{d_F(v, B)}{|B|} + \frac{\eta}{2}, \] (2) and the analogous statement holds for every $v \in B$. So with probability at least
\[ 1 - 2ne^{-2kn} \geq 1 - e^{-kn} \]
(2) holds for every $v \in V(G)$. Therefore,
\[ d_F(A_i, B_j) = \sum_{v \in A_i} d_F(v, B_j) \geq \frac{\sum_{v \in A_i} d_F(v, B)}{|A_i||B_j|} - \frac{\eta}{2} = \frac{\sum_{v \in B} d_F(v, A_i)}{|A_i||B|} - \frac{\eta}{2} \]
\[ \geq \frac{\sum_{v \in B} d_F(v, A)}{|A||B|} - \eta = d_F(A, B) - \eta. \]

By a similar computation, the upper bound also holds. □

**Theorem 3.6** (Kühn and Osthus [8]). For all positive constants $d, \xi_0, \eta \leq 1$ there is a positive $\varepsilon = \varepsilon(d, \xi_0, \eta)$ and an integer $n_0 = n_0(d, \xi_0, \eta)$ such that the following holds for all $n \geq n_0$ and all $\xi \geq \xi_0$. Let $G$ be a $(d, \varepsilon)$-superregular bipartite graph whose vertex classes both have size $n$ and let $F$ be a subgraph of $G$ with $e(F) = \xi e(G)$. Choose a perfect matching $M$ uniformly at random in $G$. Then with probability at least $1 - e^{-\varepsilon n}$ we have
\[ \xi - \eta \leq \frac{|M \cap E(F)|}{|M|} \leq \xi + \eta. \]
Proposition 3.7. Let \( G \) be an oriented graph and let \( x \in V(G) \) and let \( a, b, c \in N_G(x) \). If \( abc \) is a cyclic triangle in \( G \), then \( xe \) is a transitive triangle for at least two edges \( e \in \{ab, bc, ca\} \).

Proof. Let \( i = d^+_G(x, \{a, b, c\}) \). By symmetry, there are four cases depending on the value of \( i \). Furthermore, by reversing the edges of \( G \) it is easy to see that the cases when \( i = j \) are equivalent to the cases when \( i = 3 - j \). It is easy to verify the statement when \( i = 3 \) and when \( i = 2 \), we omit the details.

We will use the following lemma to finish the proof of Lemma 3.4. Lemma 3.8 essentially states that if a graph looks very similar to the graph depicted in Figure 1 except that \( |W_1| + |W_2| + |W_3| = 2(|U_1| + |U_2|) \), then there exists a perfect \( TT_3 \)-tiling of \( G \). As a final step, we show that if \( \delta(G) \geq 7n/18 \) and \( G \) is \( \alpha \)-extremal, then we can remove a small number of vertex disjoint transitive triangles to leave a graph that looks like Figure 1 with the property that \( |W_1| + |W_2| + |W_3| = 2(|U_1| + |U_2|) \).

We say that \( G \) is a blow-up of a cyclic triangle if \( G \) is an oriented graph on \( n \) vertices such that there exists a vertex partition \( W_1, W_2, W_3 \) with \( |W_i| = \lceil (n + i - 1)/3 \rceil \) for \( i \leq 3 \) and every vertex in \( W_i \) contains \( W_{i+1} \) in its out-neighborhood.

For a vertex set \( V \) such that \( |V| = 18m \) for some \( m \in \mathbb{N} \), let \( C(V) \) be the collection of oriented graphs \( C \) on \( V \) for which there is a partition \( \{U, W\} \) of \( V \) such that \( |U| = 6m \), \( |W| = 12m \), \( U \) is an independent set, \( C[W] \) is a blow-up of a cyclic triangle and \( d^+_C(w, U) = d^-_C(w, U) = 3m \) for every \( w \in W \).

Lemma 3.8. There exists a constant \( \beta > 0 \) and an integer \( n_0 \) such that for any \( m \in \mathbb{N} \) where \( n = 18m \geq n_0 \) and any oriented graph \( G \) on \( n \) vertices the following holds. If there exists \( C \in C(V(G)) \) such that \( \Delta(C - G) \leq \beta m \), then \( G \) contains a perfect \( TT_3 \)-tiling.

Proof. Let \( \eta = 1/12 \), \( \varepsilon = \varepsilon(1, \eta/2, \eta/2) \), \( \beta = \min\{\varepsilon^2, 1/24\} \) where \( \varepsilon(1, \eta/2, \eta/2) \) is as in Theorem 3.6. Assume \( m \) is sufficiently large. Let \( \{U, W\} \) be a partition of \( V(G) \) for which \( |W| = 12m \) and \( C[W] \) is a blow-up of a cyclic triangle with parts \( W_1, W_2 \) and \( W_3 \) each of order \( 4m \) such that every vertex in \( W_i \) has \( W_{i+1} \) in its out-neighborhood and \( d^+_C(w, U) = d^-_C(w, U) = 3m \) for every \( w \in W \).\ We may assume \( G \subseteq C \), so there are no transitive triangles in \( G[W] \). Let \( F = E(G[W]) \) and define a bipartite graph \( B \) with classes \( U, F \) as follows. A vertex \( u \in U \) and an edge \( xy \in F \) forms an edge in \( B \) if \( uxy \) is a transitive triangle in \( G \).

Clearly \( |F| \leq 3(4m)^2 = 48m^2 \).

Claim. For every \( u \in U \), \( d_B(u) \geq (2/3 - \beta)48m^2 \).

Proof. Let \( P(u) \) be the set of pairs of the form \( (e, abc) \) where \( a \in W_1 \cap N_G(u) \), \( b \in W_2 \cap N_G(u) \) and \( c \in W_3 \cap N_G(u) \), \( abc \) is a cyclic triangle and \( e \in \{ab, bc, ac\} \cap N_B(u) \). By Proposition 3.7, for every \( (e, abc) \in P(u) \) the cyclic triangle \( abc \) appears at least twice as the second element of a pair in \( P(u) \). Therefore, because \( \Delta(C - G) \leq \beta m \)

\[ |P(u)| \geq 2 \cdot (4 - \beta) m \cdot (4 - 2\beta) m \cdot (4 - 3\beta) m > (1 - 3\beta/2)128m^3. \]

Since any edge can appear as the first element in at most \( 4m \) of the pairs in \( P(u) \),

\[ d_B(u) \geq |P(u)|/(4m) \geq (1 - 3\beta/2)32m^2 = (2/3 - \beta)48m^2. \]
for all $1 \leq k, \ell \leq 2$ and $j \in \{1, 2, 3\} - i$. Let $G_1 = G[W'_1, W'_2]$, $G_2 = G[W'_2, W'_3]$ and $G_3 = G[W'_1, W'_3]$. Note that $\delta(G_i) \geq (2-\beta)m$ for every $i \in [3]$, so $G_i$ is $(1, \beta^{1/2})$-superregular by Proposition 3.3. Therefore, by Theorem 3.6, there exists a perfect matching $M_i$ of $G_i$ such that

$$\frac{|M_i \cap E(F(u))|}{2m} \geq \frac{|E(G_i) \cap E(F(u))|}{e(G_i)} - \frac{\eta}{2} \geq d_{F(u)}(W_i, W_{i+1}) - \eta$$

for every $u \in U$ and $i \in [3]$. Note that Theorem 3.6 does not apply when $|E(G_i) \cap E(F(u))|/e(G_i) \leq \eta/2$, but in that case the inequality is vacuously true. Observe $M = M_1 \cup M_2 \cup M_3$ is a perfect matching of $G[W]$ and for $B' = B[U, M]$, and every $u \in U$

$$\frac{d_{B'}(u)}{|M|} = \frac{|M \cap E(F(u))|}{6m} = \frac{1}{3} \sum_{i=1}^{3} \frac{|M_i \cap E(F(u))|}{2m} \geq \frac{1}{3} \sum_{i=1}^{3} (d_{F(u)}(W_i, W_{i+1}) - \eta) = \frac{48m^2}{d_{B}(u)} - \eta \geq \frac{2}{3} - (\beta + \eta).$$

We also have that for every $e \in M$, $d_{B'}(e) \geq (3 - 2\beta)m > (1/2 - \beta)6m$. Note that since

$$2/3 - (\beta + \eta) + 1/2 - \beta \geq 7/6 - 2\beta - \eta \geq 1,$$

Proposition 3.2 implies that $B'$ has a perfect matching. This perfect matching corresponds to a perfect $TT_{3}$-tiling of $G$. □

Proof of Lemma 3.4. Let $\beta$ be as in Lemma 3.8. Let $\tau = \beta/288$ and let $\alpha = \tau^3$.

Let $\gamma > 0$ and let $\mathcal{W} = \{W_1, W_2, W_3\}$ be a collection of three disjoint vertex subsets. We say that $v \in V(G)$ is $(i, \gamma)$-cyclic for the triple $\mathcal{W}$ if

$$d^{+}_G(v, W_{i-1}) + d_G(v, W_i) + d^-_G(v, W_{i+1}) \leq \gamma n,$$

and that $v$ is $\gamma$-cyclic for $\mathcal{W}$ if $v$ is $(i, \gamma)$-cyclic for some $i$. The triple $\mathcal{W}$ is $\gamma$-cyclic if every vertex in $W_i$ is $(i, \gamma)$-cyclic for every $i \in [3]$. A vertex is $\gamma$-bad for $\mathcal{W}$ if it is not $\gamma$-cyclic. The following claim follows from the preceding definition.

Claim 1. For any $1 > \gamma > \gamma' \geq 0$, if a vertex $v$ is $\gamma$-bad for $\{W_1, W_2, W_3\}$ and $|X| \leq \gamma'n$, then $v$ is $(\gamma - \gamma')$-bad for $\{W_1 \setminus X, W_2 \setminus X, W_3 \setminus X\}$.

For any $\lambda$, we say that $\mathcal{W}$ is $\lambda$-equitable if

$$||W_i| - |W_j|| \leq \lambda n$$

for every $i, j \in [3]$ and $|V(\mathcal{W})| \geq (2/3 - \lambda)n$. Note that this implies that $|W_i| \geq (2/9 - \lambda)n$ for every $i \in [3]$.

Let $\mathcal{W} = \{W_1, W_2, W_3\}$ be a $\lambda$-equitable triple and let $v \in V(G)$ be $(i, \gamma)$-cyclic for $\mathcal{W}$. By the degree condition,

$$d^{+}_G(v, W_{i-1}) + d^{+}_G(v, W_{i+1}) = d_G(v, V(\mathcal{W})) - (d^{+}_G(v, W_{i-1}) + d_G(v, W_i) + d^{+}_G(v, W_{i+1})) \geq |V(\mathcal{W})| - 2n/9 - \gamma n.$$

Therefore, since $|W_1|, |W_2|, |W_3| \geq (2/9 - \lambda)n$, we have the following:

$$d^{+}_G(v, W_{i-1}) \geq |W_{i-1}| + |W_i| - 2n/9 - \gamma n \geq |W_{i-1}| - (\gamma + \lambda)n,$$

$$d^{+}_G(v, W_{i+1}) \geq |W_{i+1}| - (\gamma + \lambda)n$$

and

$$d^{+}_G(v, W_{i+1}) \geq (2/9 - 2\lambda - \gamma)n.$$
Claim 2. Let $0 < \gamma < 1/27$ and let $W = \{W_1, W_2, W_3\}$ be such that $W$ is both $\gamma$-cyclic and $\gamma$-equitable. If $v \in V(G)$ such that there are no transitive triangles in $G[v \cup W]$ that contain $v$, then $v$ is 0-cyclic for $W$.

Proof. Since $|V(W)| \geq (2/3 - \gamma)n > 11n/18$, there exists an $x \in N^+_G(v, W_{i+1})$ for some $i \in [3]$. Let $I_x = N^-_G(x, W_i)$. By \([3]\), $|I_x| \geq (2/9 - 3\gamma)n$. Suppose that $v$ is not $(i, 0)$-cyclic, i.e. there exists $y \in N^+_G(v, W_{i-1} \cup W_i) \cup N^-_G(v, W_i \cup W_{i+1})$.

If $y \in N^+_G(v, W_{i-1} \cup W_i)$, then let $I_y = N^-_G(y, W_{i+1} \cup W_{i-1})$ and if $y \in N^-_G(v, W_i \cup W_{i+1})$, then let $I_y = N^+_G(y, W_{i+1} \cup W_{i-1})$. Again by \([3]\), we have that $|I_y| \geq (2/9 - 3\gamma)n$. Note that $v$ has no neighbors in $I_x \cup I_y$, because any such neighbor would imply a transitive triangle containing $v$ in $G[v \cup W]$. Since $I_x$ and $I_y$ are disjoint,

$$|W| + \delta(G) - n \leq d_G(v, W) \leq |W| - |I_x| - |I_y| \leq |W| - (4/9 - 6\gamma)n < |W| - 2n/9$$

a contradiction. \qed

Recall, since $G$ is $\alpha$-extremal there exists $W \subseteq V(G)$ such that $|W| \geq (2/3 - \alpha)n$ and $G[W]$ does not contain any transitive triangles.

Claim 3. There exists a 0-cyclic partition $W = \{W_1, W_2, W_3\}$ of $W$ such that for every $i \in [3]$

$$(2/9 - \alpha)n \leq |W_i| \leq 2n/9.$$

Proof. Let $G' = G[W]$ and note that $\delta(G') \geq \delta(G) + |W| - n \geq |W| - 2n/9$. \((4)\)

Since $G'$ is $TT_3$-free, for every $v \in W$ the sets $N^+_G(v)$ and $N^-_G(v)$ are independent. This with \(\textit{(4)}\) implies that both sets are of order at most $2n/9$ and hence that $\delta^0(G') \geq \delta(G') - 2n/9 \geq |W| - 4n/9$. \((5)\)

Since $G'$ is $TT_3$-free there exists a cyclic triangle $w_1w_2w_3$ in $G'$. This also implies that, for any $i \in [3]$, the set $\widehat{W}_i = N^+_G(w_{i-1}) \cup N^-_G(w_{i+1})$ is disjoint from $N^+_G(w_i)$. Hence, by \(\textit{(4)}\), $|\widehat{W}_i| \leq 2n/9$. Define $\widehat{W}_i = N^+_G(w_{i-1}) \cap N^-_G(w_{i+1})$. Then we have that $\widehat{W}_i$ is an independent set and, by \(\textit{(4)}\),

$$2n/9 \geq |\widehat{W}_i| \geq d^+_G(x_{i-1}) + d^-_G(x_{i+1}) - |\widehat{W}_i| \geq 2|W| - 10n/9 \geq (2/9 - 2\alpha)n.$$

Note that $\vec{E}_{G'}(\widehat{W}_{i-1}, \widehat{W}_{i+1}) \subseteq \vec{E}_{G'}(N^+_G(w_i), N^-_G(w_i)) = \emptyset$.

This gives us that $\widehat{W} = (\widehat{W}_1, \widehat{W}_2, \widehat{W}_3)$ is 0-cyclic.

Let $X = W \setminus V(W)$. By repeatedly applying Claim\(\textit{2}\) we can iteratively add each $x \in X$ to a set $\widehat{W}_i$ for which $x$ is $(i, 0)$-cyclic for $\widehat{W}$. Let $W = \{W_1, W_2, W_3\}$ be the resulting collection. For every $i \in [3]$, the set $W_i$ is independent, so $|W_i| \leq 2n/9$ by \(\textit{(4)}\) and moreover $|W_i| \geq (2/9 - \alpha)n$. So $W$ is the desired partition of $W$. \qed

Let $U = V(G) \setminus W$. If $v \in V(G)$ is $(i, \gamma)$-cyclic for $W$, then

$$d^+_G(v, U), d^-_G(v, U) \geq \delta^0(G) - \max\{|W_{i+1}|, |W_{i-1}|\} - \gamma n \geq (1/6 - \gamma)n \geq |U|/2 - (\alpha/2 + \gamma)n.$$

\((6)\)
We also have that,
\[ d_G(v, U) \geq \delta(G) - (|W \setminus W_i| + d_G(v, W_i)) \geq 7n/9 - 4n/9 - \gamma n = |U| - (\alpha + \gamma)n. \quad (7) \]

By Claim 3, we can apply (7) with \( \gamma = 0 \) to give us that
\[ e_G(W, U) \geq (|U| - \alpha n)|W| > |U||W| - \alpha n^2. \]

Defining \( Z = \{ u \in U : d_G(u, W) < |W| - \tau n \} \), we have that, since \( \tau^3 = \alpha \),
\[ |Z| < \tau^2 n. \quad (8) \]

Let \( Z(i) \) be the set of vertices in \( Z \) that are \((i, \tau)\)-cyclic for \( W \). Clearly \( Z(1), Z(2) \) and \( Z(3) \)
are disjoint. Let \( Z'' = \bigcup_{i=1}^3 Z(i) \), \( W'_i = W_i \cup Z(i) \), \( \mathcal{W}' = (W'_1, W'_2, W'_3) \), \( W' = V(\mathcal{W}') = W \cup Z'' \), \( U' = U \setminus Z'' \) and \( Z'' = Z \setminus Z'' \). Note that, for every \( i \in [3] \), \((2/9 - \alpha)n \leq |W'_i| \leq 2n/9 + |Z| \) so \( \mathcal{W}' \) is \((2\tau^2)\)-equitable and that every vertex in \( W'_i \) is \((i, \tau)\)-cyclic for \( W \). Since \( |W' \setminus W| \leq |Z| \), this implies that \( \mathcal{W}' \) is \((2\tau)\)-cyclic. We also have that for every \( u \in U' \setminus Z' \),
\[ d_G(u, W') \geq |W| - \tau n \geq |W'| - |Z| - \tau n \geq |W'| - 2\tau n. \quad (9) \]

We will now find three collections \( T_1, T_2, T_3 \) of disjoint transitive triangles. We define \( X_i = V(T_i) \) and \( Y_i = \bigcup_{j=1}^3 X_j \). The collections will be constructed so that the sets \( X_1, X_2, X_3 \)
are disjoint. The collections will also have the following properties:

(P1) \(|W' \setminus X_i| = 2|U' \setminus Y_i|\) for \( i \in \{1, 2, 3\} \),
(P2) \(|Y_3| \leq \tau n\),
(P3) \(|Y_3| \leq \tau n\),
(P4) \(|W'_1 \setminus Y_3| = |W'_2 \setminus Y_3| = |W'_3 \setminus Y_3|\) and
(P5) \(|V(G) \setminus Y_3|\) is divisible by 18.

Assume we have such collections. We claim that \( G - Y_3 \) then satisfies the conditions of Lemma 3.8 with \( \beta = 16 \cdot 18\tau \). To see this, first note that every vertex in \( W'_i \setminus Y_3 \) is \((i, \tau)\)-cyclic for \( W \); the triple \((W'_1 \setminus Y_3, W'_2 \setminus Y_3, W'_3 \setminus Y_3)\) is \((2\tau)\)-cyclic and \((2\tau)\)-equitable; and \((U' \setminus Y_2) \cap Z = \emptyset \). The conclusion then follows from (9), (3), and (4).

We begin the construction by finding a collection \( T_1 \) such that \(|W' \setminus X_1| = 2|U' \setminus X_1|\). Call a transitive triangle \( T \) standard if \(|V(T) \cap W'| = 2|V(T) \cap U'| = 1 \). Every transitive triangle \( T \in T_2 \cup T_3 \) will be standard and this will give us Property (P1).

Let \( c = |W'| - 2n/3 \) and note that \(-\alpha n \leq c \leq |Z''| \leq \tau^2 n \), so \(|c| \leq \tau^2 n \). Simple computations show that the following claim gives the desired collection \( T_1 \). Indeed, if \( c > 0 \), then \(|W'| - 3c = 2(n/3 - c) = 2|U'| \), and if \( c < 0 \), then \(|W'| - |c| = 2(n/3 - |c|) = 2(|U'| - 2|c|) \).

**Claim 4.** There exists a collection \( T_1 \) of \(|c| \) disjoint transitive triangles such that for every \( T \in T_1 \),
- if \( c > 0 \), \( T \subseteq G[W'_i] \); and
- if \( c < 0 \), \(|V(T) \cap W'| = 1 \) and \(|V(T) \cap (U \setminus Z)| = 2 \).

**Proof.** First assume \( c > 0 \) and let \( I = \{ i \in [3] : |W'_i| > 2n/9 \} \) and \( c_i = |W'_i| - 2n/9 \) for \( i \in I \). Note that \( c_i < \tau^2 n \). For every \( i \in I \), by the degree condition, we have that \( \delta(G[W'_i]) \geq c_i \). Therefore, Proposition 3.11 implies that there exists a matching \( M_i \) of size \( c_i \) in \( G[W'_i] \). For every \( xy \in M_i \), \( x \) and \( y \) are \((i, \tau)\)-cyclic and \( W' \) is an \((\tau^2)\)-equitable triple. So, by (3),
\[ |N_{G}(x, W'_{i-1}) \cap N_{G}(y, W'_{i-1})| \geq |W'_{i-1}| - 2(\tau + \tau^2)n, \]

where \( G \) is the graph obtained from \( G \) by removing \( Z \).
and, similarly, $|N^+_G(x, W'_{i+1}) \cap N^+_G(y, W'_{i+1})| \geq |W'_{i+1}| - 2(\tau + \tau^2)n$. Therefore, we can easily match the edges $\bigcup_{i \in I} M_i$ to vertices in $W'$ so that the matching corresponds to disjoint transitive triangles in $G$. Since $\sum_{i \in I} c_i \geq c$ we have the desired collection $T_1$.

Now assume $c < 0$. Let $U'' = U \setminus Z = U' \setminus Z'$. By Claim 1, we have that $|U''| \geq (1/3 - \tau^2)n$ so by the degree condition, $\delta(G[U'']) \geq (1/3 - \tau^2)n$ and there exists a matching $M$ of order $|c| \leq \tau^2n$ in $G[U'']$. By Proposition 1.7 every $e \in E(G)$ has $n/6$ vertices $v$ such that $ev$ is a transitive triangle. Therefore, we can match each edge $e \in M$ to a vertex $v \in V(G) \setminus Z$ so that the $ev$ are disjoint transitive triangles. Let $T'_1$ be this collection of transitive triangles. Suppose that there exists $T \in T'_1$ such that $V(T) \subseteq U''$. By Proposition 1.6 and the fact that, by (9), $\bigcap_{v \in V(T)} N_G(v, W') \geq |W'| - 6\tau n$, it is trivial to replace $T$ with a transitive triangle that has one vertex in $W'$ and an edge from $E(T)$ and is also disjoint from $V(T'_1 - T)$. By replacing every such triangle in this manner, we can create the desired collection $T_1$. □

We now aim to find a collection $T_2$ of standard transitive triangles that satisfies Property [P2]. Note that, by the definition of $Z'$, every vertex in $Z'$ is $\tau$-bad for $W$ and hence is $\tau$-bad for $W'$. The following claim then follows from Claim [1] and Claim [2].

**Claim 5.** There exists a collection $T_2$ of $|Z' \setminus X_1|$ disjoint standard transitive triangles in $G - X_1$ such that $|T \cap Z'| = 1$ for every $T \in T_2$.

**Proof.** Let $T_2$ be a collection of disjoint standard transitive triangles in $G - X_1$ such that for every $T \in T_2$, $|V(T) \cap Z'| = 1$. Let $Y_2 = V(T_2) \cup X_1$. Suppose that $|T_2|$ is maximal among all such collections and that there exists $z \in Z' \setminus Y_2$. Since $z$ is $\tau$-bad for $W'$, by Claim [1] and the fact that $|Y_2| < |X_1| + 3|Z'| < 6\tau^2 < \tau n$, $z$ is $0$-bad for $\{W_1 \setminus Y_2, W_2 \setminus Y_2, W_3 \setminus Y_2\}$. Hence, by Claim [2] there exists a transitive triangle containing $z$ and two vertices in $V(W' \setminus Y_2)$. Adding $T$ to $T_2$ contradicts the maximality of $|T_2|$. □

Let $W''_i = W'_i \setminus Y_2$ for every $i \in [3]$. Since $W'$ is $(2\tau^2)$-equitable and $|Y_2| \leq |X_1| + 3|Z'| \leq 6\tau^2$, the collection $W'' = \{W''_1, W''_2, W''_3\}$ is $(8\tau^2)$-equitable. Because $|T_1 \cup T_2| \leq 2|Z| \leq 2\tau^2 n$, if we can find a collection $T_3$ of at most $17\tau^2 n \leq \tau n/3 - 2\tau^2 n$ disjoint standard transitive triangles in $G - Y_2$ that satisfies (P4) and (P5), we will also satisfy Property (P3). This is quite easy to do, as we now describe.

Let $\pi$ be a permutation of $[3]$ such that $|W''_{\pi(1)}| \leq |W''_{\pi(2)}| \leq |W''_{\pi(3)}|$. Let $M_1, M_2$ and $M_3$ be disjoint edge sets such that their union is a matching and

- $|M_1| = |W''_{\pi(3)}| - |W''_{\pi(2)}|$, $|M_2| = |W''_{\pi(3)}| - |W''_{\pi(1)}|$, $|M_3| = |W''_{\pi(1)}| - |W''_{\pi(2)}|$,

- $M_1 \subseteq E_G(W''_{\pi(3)}, W''_{\pi(1)})$, $M_2 \subseteq E_G(W''_{\pi(3)}, W''_{\pi(2)})$ and $M_3 \subseteq E_G(W''_{\pi(2)}, W''_{\pi(1)})$ for each $i \in [3]$.

Let $M' = M_1 \cup M_2$ and $M = M' \cup M_3$. Note that since $W''$ is $(8\tau^2)$-equitable, $|M'| < |M| \leq 2(8\tau^2 n) + 3 \leq 17\tau^2 n$. Let $vv' \in M$. Since $v$ and $v'$ are both $\tau$-cyclic for $W$, (8) and (7) give us that the number of vertices $x \in U$ such that $xvv'$ is a transitive triangle is at least

$$|N^-_G(x, U) \cap N_G(v', U)| \geq n/6 - \alpha n - 2\tau n.$$  

Therefore, we can find the desired collection $T_3$ by matching edges in either $M$ or $M'$ to unused vertices in $U'$ in the graph $B$. We can clearly satisfy Property (P4) Note that Properties [P1] and [P4] imply that $|V(G) \setminus Y_3| \in 9\mathbb{Z}$. So we can satisfy Property [P5] by picking $M$ or $M'$ appropriately. □
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