WAVE PROPAGATION IN RANDOM WAVEGUIDES

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Dedicated to Professor Roger Temam with admiration

Abstract. We study uncertainty bounds and statistics of wave solutions through a random waveguide which possesses certain random inhomogeneities. The waveguide is composed of several homogeneous media with random interfaces. The main focus is on two homogeneous media which are layered randomly and periodically in space. Solutions of stochastic and deterministic problems are compared. The waveguide media parameters pertaining to the latter are the averaged values of the random parameters of the former. We investigate the eigenmodes coupling due to random inhomogeneities in media, i.e. random changes of the media parameters. We present an efficient numerical method via Legendre Polynomial Chaos expansion for obtaining output statistics including mean, variance and probability distribution of the wave solutions. Based on the statistical studies, we present uncertainty bounds and quantify the robustness of the solutions with respect to random changes of interfaces.

1. Introduction. Many classes of problems in computational science and engineering are characterized by uncertainties. Even if the mathematical equations are a perfect representation of physical reality, its solutions will not describe reality unless we have perfect knowledge about the initial and boundary conditions which are often difficult to prescribe with high accuracy. Some uncertainties are inherited from the data, or from a lack of knowledge of the underlying physics or from the impossibility of fully describing the solutions of a system possessing a large number of degrees of freedom (see e.g. [1], [15], [18]). The goal of uncertainty quantification is to investigate the impact of errors, or uncertainty, on the solutions. These uncertainties come from incomplete knowledge of physical parameters, initial and boundary conditions, location of media interfaces in electromagnetic and acoustical waveguides, errors in measurements and other sources. To predict or control solutions from governing equations or to quantify uncertainty in the solutions, we

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should take uncertainty quantification into account in order to fully understand simulation results and subsequently the true physics. Problems with uncertainties include Stochastic Partial Differential Equations (SPDE) (see e.g. [11], [6], [3], [4]) and PDE supplemented with uncertain data (initial/boundary conditions, parameter values) etc. (see e.g. [29], [26], [30], [2], [14], [16]). In this paper we investigate statistical properties of the solutions of hyperbolic equations in randomly layered media characterized by the presence of random interfaces. In different contexts, i.e. different type of equations, you may refer to [7], [23], [24], [25], [27], [22] and [32].

We aim to construct a method for obtaining output statistics of solutions of one-dimensional wave equation in a periodic waveguide pertaining to the random media parameter $c$:

$$
\begin{align*}
   u_{tt} &= c^2 u_{xx}, \quad 0 < x < 2\pi, \quad t > 0, \\
   u(x, t) &= u(x + 2\pi, t), \\
   u(x, 0) &= g(x), \\
   u_t(x, 0) &= h(x), 
\end{align*}
$$

(1)

where the initial conditions $g, h$ are deterministic and $2\pi$-periodic smooth functions, and the parameter $c$, which depends on the media properties, is defined with uncertainties. More precisely, $c = c^{\xi_1,\xi_2}(x) = c_0$ for $x \in (0, \xi_1) \cup (\xi_2, 2\pi)$ and $c_1$ for $x \in (\xi_1, \xi_2)$ with $c_0, c_1 \geq d > 0$, $c_0 \neq c_1$ (see Fig. 1), and $\xi_1, \xi_2$ are random parameters which are defined in (3) below. The media, in particular, waveguide, can be polluted or damaged for some reasons and then the properties of some parts of the media might change. However, in general, such changes or location of imperfections and impurities are uncertain. We thus would like to compute the output solutions taking into account such uncertainties. The equation (1) is not an energy conservative form. However, the work presented here can be extended to a conservative form, i.e. $u_{tt} = (c^2 u_x)_x$.

A waveguide, e.g. optical fiber, is a material or structure which guides electromagnetic or sound waves and they vary from slab waveguides to fiber or channel ones. Although our model problem is posed on one-dimensional periodic space, the numerical techniques which take into account randomness of waveguide media properties can be extended to 2 or 3 dimensional spaces in which real waveguides can be constructed.

Let us first consider two kinds of homogeneous media characterized by the parameters $c_0$, $c_1$, resp., which interface randomly and periodically in space. We assume that the random parameters $\xi_1, \xi_2$ are independent and satisfy

$$
0 \leq l_1 < \xi_1 < l_2 \leq l_3 < \xi_2 < l_4 \leq 2\pi,
$$

(2)

and we also assume that $\xi_1, \xi_2$ are uniformly distributed over $(l_1, l_2), (l_3, l_4)$, respectively. We can then write probability density functions $f_1(\xi_1), f_2(\xi_2)$ as

$$
f_1(\xi_1) = \frac{1}{l_2 - l_1} \chi_{(l_1,l_2)}(\xi_1), \quad f_2(\xi_2) = \frac{1}{l_4 - l_3} \chi_{(l_3,l_4)}(\xi_2),
$$

(3)

where $\chi_A(\xi)$ is the characteristic function of $A$. Random parameters with non-uniform distributions will be discussed in future studies.

We approximate $u$ as follows

$$
u = u^{\xi_1,\xi_2} = \sum_{k=0}^{N_1} \sum_{l=0}^{N_2} u_{kl}(x, t) P_{k_1,l_2}^{1,2}(\xi_1) P_{l_3,l_4}^{3,4}(\xi_2)
$$

(4)
\[ c = c_0 \quad \shortmid \quad c = c_1 \quad \shortmid \quad c = c_0 \]

\[ l_1 \quad \xi_1 \quad l_2 \quad \xi_2 \quad l_3 \quad \xi_2 \quad l_4 \]

\[ c_0^2 \quad c_1^2 \quad c_0^2 \]

**Figure 1.** Two random interfaces between two kinds of homogeneous media (top); The plot of \( \mu^2(x) = \mathbb{E}((\xi_1, \xi_2)^2) \) (bottom).

where \( P_k^{l_1, l_2}, P_k^{l_3, l_4} \) are shifted Legendre polynomials such that

\[ P_k^{a,b}(\xi) = P_k \left( \frac{2\xi - a - b}{b - a} \right) \]

and \( P_k \) are the standard Legendre polynomials with span \((-1, 1)\). We call (4) Legendre polynomial chaos (PC) expansions and the coefficients \( u_{ij}(x, t) \) PC mode (see [9], [10], [31]).

Thanks to the orthogonality of Legendre polynomials, we can explicitly obtain the expectation and variance. That is,

\[ E(u) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u(\xi_1, \xi_2) f_1(\xi_1) f_2(\xi_2) d\xi_1 d\xi_2 = u_{00}(x, t), \]

\[ \text{Var}(u) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (u(\xi_1, \xi_2))^2 f_1(\xi_1) f_2(\xi_2) d\xi_1 d\xi_2 - (E(u))^2 \]

\[ = \left( u_{01}(x, t) \right)^2 + \left( u_{10}(x, t) \right)^2 + \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} \frac{(u_{ij}(x, t))^2}{(2i+1)(2j+1)}. \]

Substituting (4) in (1), multiplying by \( P_k^{l_1, l_2}(\xi_1), P_k^{l_3, l_4}(\xi_2) \) and integrating over \( \omega = (l_1, l_2) \times (l_3, l_4) \) we can obtain that, for \( u = (u_{00}, \ldots, u_{ij}, \ldots, u_{N_1, N_2}) \),

\[
    \begin{cases}
    u_{tt} = \Lambda(x)u_{xx}, & 0 < x < 2\pi, \ t > 0, \\
    u(x, t) = u(x + 2\pi, t), \\
    u(x, 0) = (g(x), 0, \ldots, 0), \\
    u_t(x, 0) = (h(x), 0, \ldots, 0),
    \end{cases}
\]

where

\[ \Lambda(x) = \]

\[ \left( \frac{2i + 1}{l_2 - l_1} \right) \left( \frac{2j + 1}{l_4 - l_3} \right) \int_{\omega} (\xi_1, \xi_2(x))^2 P_k^{l_1, l_2}(\xi_1) P_k^{l_3, l_4}(\xi_2) d\xi_1 d\xi_2. \]

**Remark 1.** We note that in the system (8) the matrix \( \Lambda(x) \) is continuous over \( x \in (0, 2\pi) \) and thus the modal solutions \( u_{ij}(x, t) \) are expected to be more regular than the solutions of the original stochastic problems (1) with discontinuous parameter \( c \).
Remark 2. Although the data \(g(x), h(x)\) are deterministic, their stochastic versions can be also taken into account by writing (4) for \(u = g, h\) and then the vector of their PC modes is replaced for the right-hand sides of (8)\textsubscript{3,4}.

So far we have considered two random interfaces. In the next section we demonstrate how to reduce the computational size by decomposing the parameter \(c = c^{\xi_1,\xi_2}(x)\) and we will also consider \(n\)-random interfaces below.

1.1. 2- interfaces. We now decompose \(c^{\xi_1,\xi_2}\) as

\[
c^{\xi_1,\xi_2}(x) = c^{\xi_1}(x) c^{\xi_2}(x)
\]

where \(c^{\xi_1}(x) = c_0\) for \(x \in (0, \xi_1), \ c_1\) for \(x \in (\xi_1, 2\pi)\) and \(c^{\xi_2}(x) = 1\) for \(x \in (0, \xi_2),\ c_1^{-1}c_0\) for \(x \in (\xi_2, 2\pi)\). Note that \(c^{\xi_1}(x)c^{\xi_2}(x) = c_0\) for \(x \in [0, \xi_1) \cup (\xi_2, 2\pi]\), and \(c_1\) for \(x \in (\xi_1, \xi_2)\) as required.

Using the Kronecker product we can write

\[
\Lambda(x) = L_1(x) \otimes L_2(x) = L_2(x) \otimes L_1(x),
\]

where

\[
L_1(x) = \left( \frac{2i + 1}{l_2 - l_1} \int_{l_1}^{l_2} (c^{\xi_1}(x))^2 P^{l_1,l_2}_i(\xi_1) P^{l_1,l_2}_i(\xi_1) d\xi_1 \right),
\]

\[
L_2(x) = \left( \frac{2j + 1}{l_4 - l_3} \int_{l_3}^{l_4} (c^{\xi_2}(x))^2 P^{l_3,l_4}_j(\xi_2) P^{l_3,l_4}_j(\xi_2) d\xi_2 \right).
\]

Noting that

\[
u = U^S
\]

where \(U = U(x, t) = (u_{ij})\) and \(U^S\) is the stacking of \(U\) and we can then rewrite (8):

\[
U^S_{it} = (L_2(x) \otimes L_1(x)) U^S_{xx}.
\]

Using the fact \((B^T \otimes A)X^S = (AXB)^S\) and noting that \(L_1(x), L_2(x)\) are symmetric we can write

\[
U^S_{it} = (L_1(x)U_{xx}L_2(x))^S,
\]

and thus

\[
U_{it} = L_1(x)U_{xx}L_2(x).
\]

The computational size can then be substantially reduced by writing that, for \(0 \leq l_1 < l_2 \leq l_3 < l_4 \leq 2\pi,

\[
U_{it} = \begin{cases} 
(c_0)^2 U_{xx} & \text{for } x \in [0, l_1) \cup (l_4, 2\pi], \\
L_1(x)U_{xx} & \text{for } x \in [l_1, l_2], \\
(c_1)^2 U_{xx} & \text{for } x \in [l_2, l_3), \\
(c_1)^2 U_{xx}L_2(x) & \text{for } x \in [l_3, l_4).
\end{cases}
\]

Comparing with (8) the matrix size involved is reduced from \((N_1N_2)^2\) (for \(\Lambda(x)\)) to either \(N_1^2\) or \(N_2^2\) (for \(L_1(x)\), or \(L_2(x)\)).

The numerical plots in Figs. 2-3 show respectively mean, variance, PC modes and the decay of PC modes. We observe that the PC modes \(u_{ij}, i, j = 0, 1, \cdots, 10\), decay exponentially and they decay faster with small variance (or fluctuation) than with large variance. The PC modes in the Figures are computed from the matrix equation (17) or (18) supplemented with the parameter values \(c_0 = 1, c_1 = 2,\) and
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Figure 2. Mean, variance and PC modes at $t = 2$ with $N_1 = N_2 = 10$; (a) with a relatively small fluctuation ($l_1 = 1.0708, l_2 = 2.0708, l_3 = 4.2124, l_4 = 5.2124$); (b) with a relatively large fluctuation ($l_1 = 0.5708, l_2 = 2.5708, l_3 = 3.7124, l_4 = 5.7124$).

Figure 3. Exponential decay of PC modes in log scale with $N_1 = N_2 = 10$, i.e. $\log_{10}(\max_x |u_{ij}(x,t)|)$ at $t = 2$, $i, j = 0, 1, \cdots, 10$; (a) with a relatively small fluctuation ($l_1 = 1.0708, l_2 = 2.0708, l_3 = 4.2124, l_4 = 5.2124$); (b) with a relatively large fluctuation ($l_1 = 0.5708, l_2 = 2.5708, l_3 = 3.7124, l_4 = 5.7124$).

the initial conditions $u = v$, $u_0 = 0$ at $t = 0$ where $v$ is an eigenvector given by (24) below with the eigenvalue $\omega^2 = 1.7444405$ for (a), $2.0806212$ for (b).

Remark 3. It is noteworthy to investigate the entries of $\Lambda(x)$ as either $l_1$ or $l_2$ (similarly, either $l_3$ or $l_4$) is close to $x$. We first observe that for $x \in (l_1, l_2)$, setting
1.1. 

\[
\delta_{ik} = 1 \text{ if } i = k, \text{ and } 0 \text{ if } i \neq k, \\
L_1(x)_{ik} = \frac{2i + 1}{l_2 - l_1} \int_{l_1}^{l_2} (\xi_i(x))^2 P_i^{l_1,l_2}(\xi_1) P_k^{l_1,l_2}(\xi_1) d\xi_1 \\
= \frac{2i + 1}{l_2 - l_1} \left( c_i^2 \int_{l_1}^{x} + c_0^2 \int_{x}^{l_2} \right) P_i^{l_1,l_2}(\xi_1) P_k^{l_1,l_2}(\xi_1) d\xi_1. 
\] (19)

Hence we can see that \( L_1(x)_{ik} \rightarrow c_i^2 \delta_{ik} \) as \( l_2 \rightarrow x \) and \( L_1(x)_{ik} \rightarrow c_0^2 \delta_{ik} \) as \( l_1 \rightarrow x \).

Similarly, for \( x \in (l_3, l_4) \), we can also infer that \( L_2(x)_{ik} \rightarrow c_i^{-2} c_0^2 \delta_{ik} \) as \( l_4 \rightarrow x \) and \( L_2(x)_{ik} \rightarrow \delta_{ik} \) as \( l_3 \rightarrow x \). For \( x \in (l_1, l_2)^c \), \( L_1(x) \) is a diagonal matrix \( (L_2(x) \) is too for \( x \in (l_3, l_4)^c \). This implies that \( \Lambda(x) \) tends to be diagonalized as the lengths \( \bar{l}_1, \bar{l}_2, \bar{l}_3, \bar{l}_4 \) tend to be small, i.e. the variances of \( \xi_1, \xi_2 \) approach zero.

Here we infer that as the variances of \( \xi_1, \xi_2 \) are getting larger, the matrix \( \Lambda(x) \), \( x \in (l_1, l_2) \cup (l_3, l_4) \), tends to be dense and the PC modes broadly interact each other in the \( x \)-space.

1.2. 

n- interfaces. We can extend to \( n \)- random interfaces at \( x = \xi_1, \cdots, \xi_n \). We write

\[
c = c^{\xi_1, \cdots, \xi_n}(x) = c_i \text{ for } x \in (\xi_i, \xi_{i+1}) 
\] (20)

where \( i = 0, 1, \cdots, n \) with \( \xi_0 = 0, \xi_{n+1} = 2\pi, c_0 = c_{n+1} \) by periodicity. As before we can decompose the parameter \( c \):

\[
c = c_1^{\xi_1}(x) \cdots c_n^{\xi_n}(x) 
\] (21)

where \( c_i^{\xi_i}(x) = c_0 \) for \( x \in (0, \xi_1) \), \( c_1 \) for \( x \in (\xi_1, 2\pi) \) and \( c_i^{\xi_i}(x) = 1 \) for \( x \in (0, \xi_i) \), \( c_{i+1}^{-1} \) for \( x \in (\xi_i, 2\pi) \), for \( i = 1, \cdots, n \).

Considering uniform distributions over \( (l_{2i-1}, l_{2i}) \) for the random parameters \( \xi_i \) we can follow a similar decomposition as in Sec. 1.1. We thus find that

\[
\Lambda(x) = \left( \prod_{k=1}^{n} \frac{2l_k + 1}{l_{2k} - l_{2k-1}} \right) \\
\cdot \left( \int_{\omega} (\xi_1^{\xi_n}(x))^2 P_1^{l_1,l_2}(\xi_1) \cdots P_n^{l_{n-1},l_n}(\xi_n) P_{l_1,l_2}(\xi_1) \cdots P_{l_{n-1},l_n}(\xi_n) \right) 
\] (22)

where \( \omega = (l_1, l_2) \times \cdots \times (l_{2n-1}, l_{2n}) \) and we can then write

\[
\Lambda(x) = L_1(x) \otimes L_2(x) \otimes \cdots \otimes L_n(x). 
\] (23)

Here we note that the matrices \( L_k(x) \)'s are symmetric, i.e. \( L_k(x)^T = L_k(x) \). Simulations on \( n \)- multiple random interfaces will be discussed in subsequent articles.

2. Robustness of wave solutions with respect to random changes of waveguides interfaces. In this section we investigate some statistical properties of the wave solutions of (1) and the robustness of the solutions with respect to random changes of interfaces. For this, we compute the stochastic solutions of (1), and as in Figure 4 we compare their averages \( u_0 \) with the deterministic solutions of (1) with \( c^2 = \mu^2(x) = \mathbb{E}(c^{\xi_1, \xi_2}(x)^2) \) (see Fig. 1), i.e. (35) below. In the deterministic problem we consider a special solution, i.e. an eigenmode \( v^* = e^{i\omega t} v(x) \), where \( v \) is an eigenvector (or carrier) of the (deterministic) Helmholtz equation (see [17], [5]):

\[
v_{xx} + \frac{\omega^2}{\mu^2(x)} v = 0. 
\] (24)
Figure 4. Robustness of stochastic solutions of (1) with $c_0 = 1, c_1 = 2$ at $t = 2$ with $N_1 = N_2 = 10$; (first row) with a relatively small fluctuation ($l_1 = 1.4708, l_2 = 1.6708, l_3 = 4.61239, l_4 = 4.81239$); (second row) with a relatively large fluctuation ($l_1 = 1.0708, l_2 = 2.0708, l_3 = 4.2124, l_4 = 5.2124$); eigenvectors $v$ (a)-(c) in the order of frequency (left), the error in log scale between the mean $u_00$ as in (8) and the solution $u^*$ as in (35), i.e. $\log_{10}|(u_00 - u^*)(x, 2)|$ (right) where the initial conditions are $u = v$ (a)-(c), $u_t = 0$ at $t = 0$.

Note that $u^*$ is the solution of (35) below with initial conditions $u^* = v(x)$ and $u^*_t = 0$ at $t = 0$. Comparing with the stochastic solutions of (1) with the same initial conditions we investigate the robustness of the eigenmode $u^*$ through the waveguides. The eigenvalue $\omega^2$ can be computed from a discrete problem described in (39) below.

We first obtain the expectation and variance of $\xi_1 \xi_2(x)^2$ in the lemma below.
Lemma 2.1. The expectation \( \mu^2(x) = \mathbb{E}(c^{\xi_1, \xi_2}(x)^2) \) and variance \( \text{Var}(c^{\xi_1, \xi_2}(x)^2) \) are as follows:

\[
\mathbb{E}(c^{\xi_1, \xi_2}(x)^2) = c_0^2 + (c_1^2 - c_0^2)I(x),
\]

\[
\text{Var}(c^{\xi_1, \xi_2}(x)^2) = (c_1^2 - c_0^2)^2(I(x) - I^2(x)),
\]

where

\[
I(x) = I_{l_1, l_2, l_3, l_4}(x) = \frac{x - l_1}{l_2 - l_1} \chi(x_{l_1}) + \chi(x_{l_2}, l_3)(x) + \frac{l_4 - x}{l_4 - l_3} \chi(x_{l_3, l_4})(x).
\]

Proof. We write

\[
\begin{align*}
\mathbb{E}(c^{\xi_1, \xi_2}(x)^2) &= \int_0^{2\pi} \int_0^{2\pi} (c_0^2 \chi(0, \xi_1)(x) + c_1^2 \chi(\xi_1, \xi_2)(x) + c_0^2 \chi(\xi_2, 2\pi)(x) f_1(\xi_1) f_2(\xi_2) d\xi_1 d\xi_2) \\
&= c_0^2 + (c_1^2 - c_0^2) \int_0^{2\pi} \chi(0, \xi_1)(x) f_1(\xi_1) d\xi_1 + (c_1^2 - c_0^2) \int_0^{2\pi} \chi(\xi_1, \xi_2)(x) f_2(\xi_2) d\xi_2 \\
&= c_0^2 + (c_1^2 - c_0^2) \left\{ \int_0^{2\pi} f_2(\xi_2) d\xi_2 - \int_x f_1(\xi_1) d\xi_1 \right\},
\end{align*}
\]

and thus by elementary calculations (25) follows. We first find that \( \mathbb{E}((c^{\xi_1, \xi_2}(x))^4) \) is exactly (28) with \( c_0^2 \) being replaced by \( c_1^4 \) and \( c_1^2 \) by \( c_1^4 \) and then, since \( \text{Var}(c^{\xi_1, \xi_2}(x)^2) = \mathbb{E}((c^{\xi_1, \xi_2}(x))^2) - (\mathbb{E}(c^{\xi_1, \xi_2}(x))^2)^2 \), (26) easily follows.

Remark 4. Since \( \xi_1, \xi_2 \) are uniformly distributed over \( (l_1, l_2), (l_3, l_4) \), respectively, we easily compute the probabilities \( P(c(x)) \):

\[
P(c(x) = c_0) = \chi_{[0, l_1]}(x) + \frac{l_2 - x}{l_2 - l_1} \chi_{(l_1, l_2)}(x) + \frac{x - l_3}{l_4 - l_3} \chi_{(l_3, l_4)}(x) + \chi_{(l_4, 2\pi]}(x),
\]

\[
P(c(x) = c_1) = \frac{x - l_1}{l_2 - l_1} \chi_{(l_1, l_2)}(x) + \chi_{(l_3, l_4)}(x) + \frac{l_4 - x}{l_4 - l_3} \chi_{(l_3, l_4)}(x).
\]

Without a difficulty, since \( P(c = c_0) + P(c = c_1) = 1 \), we can find that

\[
\mu^2(x) = c_0^2 P(c = c_0) + c_1^2 P(c = c_1) = c_0^2 + (c_1^2 - c_0^2) P(c = c_1),
\]

which leads to (25).

2.1. Homogeneous media. It is interesting to analyze the behaviors of solutions in homogeneous media, i.e. the media parameters are constant. In particular, we consider the average \( \mu \) as in (32) below which does not depend on \( x \) and the stochastic parameters \( \xi_1, \xi_2 \). By elementary calculations, we can evaluate the average \( \mu^2 \) over a space period:

\[
\mu^2 = \frac{1}{2\pi} \int_0^{2\pi} \mathbb{E}(c^{\xi_1, \xi_2}(x)^2) dx = c_0^2 + \frac{(c_1^2 - c_0^2)(l_1 + l_2 - l_4 - l_3)}{4\pi}.
\]

We first compare the solutions of two deterministic problems (1) with \( c = \mu(x) \) and \( c = \bar{\mu} \) respectively and obtain the error between the two solutions. More explanations follow in Remark 5 below.
Lemma 2.2. Let $u^*$ be a solution of the deterministic problem (1) with $c^2 = \mu(x)^2 = \mathbb{E}(c^2\xi_1^2(x)^2)$ and $\bar{u}$ of the deterministic problem (1) with $c^2 = \bar{\mu}^2 = \mathbb{E}(c^2\xi_2^2(x)^2)$. Then we have the following estimates for $u^* - \bar{u}$:

$$
|u^* - \bar{u}|_{L^\infty(0,T;H^1(\Omega))} + |(u^* - \bar{u})|_{L^\infty(0,T;L^2(\Omega))} \\
\leq \kappa|\sigma_0^2 - \sigma_1^2|[(l_4 - l_3)^{-1} + (l_4 - l_3)^{-1}]||\bar{u}|_{L^2(0,T;H^2(\Omega))}.
$$

(33)

Proof. Setting $w = u^* - \bar{u}$ we can write the equation for $w$: $w_{tt} = \mu(x)^2w_{xx} + (\mu(x)^2 - \bar{\mu}^2)\bar{u}_{xx}$. Multiplying by $w_t$ and integrating over $\Omega$, thanks to Poincaré inequality, see e.g. [5], we find that

$$
\frac{1}{2}\int |w_t|^2 + \frac{1}{2}\int \mu(x)^2|w_x|^2 = -\int (\mu(x)^2)_{x}w_xw_t + \int (\mu(x)^2 - \bar{\mu}^2)\bar{u}_{xx}w_t
$$

(34)

$$
\leq \kappa((\mu(x)^2)_{x}|_{L^\infty}(|w_x|_{L^2} + |\bar{u}_{xx}|_{L^2})|w_t|_{L^2}
$$

Noting $\mu^2(x) \geq d^2 > 0$ and from (25) we find that

$$(\mu^2(x))_{x} = (\sigma_0^2 - \sigma_1^2)\left\{\frac{1}{l_2 - l_1}\chi_{(l_1,l_2)}(x) + \frac{1}{l_4 - l_3}\chi_{(l_3,l_4)}(x)\right\}
$$

and applying the Gronwall inequality the estimate (33) follows.

Remark 5. The deterministic problem (1) with $c = \mu(x)$ (inhomogeneous media) (see the plot of $\mu^2(x)$ in Fig. 1) corresponds to the problem (8) with $N_1 = N_2 = 0$, “mean dynamics”. From (8) we can write

$$
\left\{
\begin{array}{ll}
u^*_{tt} = \mu^2(x)u_{xx}, & 0 < x < 2\pi, t > 0, \\
u^*(x,t) = u^*(x + 2\pi, t), \\
u^*(x,0) = g(x), \\
u^*_t(x,0) = h(x).
\end{array}
\right.
$$

(35)

The robustness of the stochastic solutions with respect to randomness can be measured by comparing the two mean solutions $u_{00}$ of (8) respectively with $N_1 = N_2 = 0$ (i.e. (35)) and $N_1, N_2$ sufficiently large. The numerical results are shown in Figure 4 for various eigenvectors and variance (fluctuation) of the random interfaces. The two solutions are averaged outputs: the first one is a deterministic solution with averaged stochastic parameters $\xi_1, \xi_2$ and the second one is an average of the stochastic output solution $u$ of (1).

For homogeneous media, i.e. $c = \bar{\mu}$, thanks to d’Alembert’s formula, the explicit solutions are given by

$$
\bar{u}(x,t) = \frac{1}{2}[g(x - \bar{\mu}t) + g(x + \bar{\mu}t)] + \frac{1}{2\bar{\mu}}\int_{x-\bar{\mu}t}^{x+\bar{\mu}t} h(s)ds.
$$

(36)

The errors between $u^*$ and $\bar{u}$ as in (33) imply that the error is only up the inhomogeneities which introduce coupling of eigenmodes. The eigenmode coupling due to random inhomogeneities is described in the next section.

2.2. Inhomogeneous media due to random inhomogeneities. Due to inhomogeneities in media (note $c = c\xi_1\xi_2(x)$) we observe eigenmodes coupling of the ones from the wave equations in homogeneous media (see e.g. [8], [6]). Moreover, the randomness of the media interfaces complicates the coupling; due to the randomness, we have already observed the PC modes coupling as indicated in (8). Here the randomness makes the matrix $\Lambda(x)$ dense, i.e. the PC modes all interact each other for $x$ in the ranges of random interfaces. The eigenmodes of the wave solutions of (1) in a homogeneous media, e.g. $c = \bar{\mu}$, are easily found. Setting
c = c₀ (homogeneous media) in (1) the eigenmodes are of the form \( u = e^{i\omega (t + \frac{x}{c_0})} \).

Taking into account the 2\pi-periodicity in \( x \) we find that \( \omega/c_0 = j, j = 0, 1, \cdots \). Hence the wave solution \( u \) can be written \( u = \sum_{j=0}^{\infty} v_j e^{ij(c_0t+\frac{x}{c_0})} \). However, in an inhomogeneous media, the eigenmodes are coupled. Writing \( u = \sum_{j=0}^{\infty} v_j e^{ijx} \) and substituting for \( u \) in (1) we obtain that, for \( k = 0, 1, \cdots, \)

\[
v''_k(t) + c_0^2 k^2 v_k(t) + \frac{c_1^2 - c_0^2}{2\pi} \sum_{j=0}^{\infty} v_j(t) \left[ j^2 \int_{\xi_1}^{\xi_2} \cos((j - k)x) \, dx \right] = 0, \tag{37}
\]

which is supplemented with the initial conditions derived from (1)\textsubscript{3,4}. This equation is a system of ordinary differential equations and well-posed. The random inhomogeneities make all eigenmodes coupled with each other.

2.3. Stochastic eigenvalue problems. From the wave equations (1) setting \( u = e^{i\omega t} v(x) \) we can obtain a non-deterministic Helmholtz equation:

\[
v_{xx} + \frac{\omega^2}{c^2(x)} v = 0. \tag{38}
\]

If the variance of noise is small, we expect to find a good frequency \( \omega \) which might be close to the one from the deterministic problem (24).

Although the eigenvalues and eigenvectors for the Helmholtz equation (24) may not be found analytically, discretizing the problem by Fourier spectral method we obtain a linear system (see [21])

\[
A \mathbf{v} = \omega^2 B \mathbf{v}, \tag{39}
\]

where \( A \) is the spectral differentiation matrix for \(-\partial^2/\partial x^2\), \( B = \text{diag}(1/\mu^2(x_j)) \) and \( \mathbf{v} = (v(x_j)) \). The eigenvalues \( \omega^2 \) and corresponding eigenvectors \( \mathbf{v} \) can be found using e.g. \textit{eig} function in MATLAB. For the stochastic problem (38) we obtain a random matrix \( B = B^{\xi_1,\xi_2} = \text{diag}(1/c^2(x_j)) \) instead. We can now numerically compare the eigenvalues and eigenvectors for (39) and for (39) with \( B \) being replaced by \( B^{\xi_1,\xi_2} \). We expect that the eigenvector corresponding to the smallest eigenvalue has a low frequency which makes robust with the change of the lengths \( l_1, l_2, l_3, l_4 \) (i.e. the variances of \( \xi_1, \xi_2 \)), whereas the eigenvector with a large eigenvalue tends to be sensitive to the change of the lengths. For more discussion on stochastic eigenvalue problems (38) and (39), the readers are referred to [19],[28] respectively.

2.4. Numerical examples. Figure 4 shows the effect of random fluctuation of interfaces to the wave solutions of (1). The mean solution \( \bar{u}_{00} \) of (8) is very close to the solution \( u^* \) of the mean dynamics (35) when the variance or random fluctuation is small (first row) and the eigenvector \( v \) has a low frequency or small eigenvalue (a). More precisely, the mean \( \bar{u}_{00} \) of solution with high frequency (c) is much sensitive to even small random fluctuation whereas the one with low frequency is robust to it. From this we can infer that the uncertainty caused by the random interfaces cannot be ignored only except in the case when the random fluctuation is small and the solutions have low frequency. We have to take into account the uncertainty in the output solutions where we use the PC method (4). It satisfactory describes the output stochastic solutions by solving the modal system (8). Furthermore, the exponential decay of the PC modes as in (3) supports using the PC method.

In Figure 5 via Monte Carlo simulations applied to the PC expansions (4) we plot the cumulative distribution function (CDF) for the solutions \( u(x,t) \) of Eq. (1) as in [13],[20]. If the variance of the random interfaces pertaining to the parameter
$c$ is small as in (a), we also observe small variance in the output $u(x,t)$ whereas the large variance of them results in large variance of the output as in (b). If the input variance of random interfaces is small, we may ignore the output variance and compute the mean dynamics. However, if it is not small, solutions from the mean dynamics are very different from the mean output as indicated in Figures 4, 5.

3. Conclusions. We have constructed a numerical tool to investigate statistical quantities, e.g. CDF, mean and variance, of the wave solutions of Eq. (1) with randomly layered media, more precisely, media with random interfaces. The basic idea of the numerical tool is projecting the stochastic equations (1) on a random field via the so-called Legendre polynomial chaos and computing the PC modes by standard numerical tools. We compute and compare the mean solutions $u = u_{00}(x,t)$ respectively with $N_1 = N_2 = 0$, i.e. the mean dynamics (35), and with $N_1, N_2$ sufficiently large. The difference between those two solutions indicates the robustness with respect to random interfaces. It turns out that the eigenmodes with a small/large eigenvalue is robust/sensitive with the random changes of media interfaces. This implies that a slow/fast part of solutions is less/more likely to be subjected to randomness. 

We have implemented an efficient numerical tool taking into account two random interfaces due to different media properties. More interfaces will be discussed in a future study. We can also consider inverse problems like finding an interface position based on the desired probability distributions of wave solutions (see e.g. [12]) and these problems will be studied in subsequent articles.

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