Time-Varying ALIP Model and Robust Foot-Placement Control for Underactuated Bipedal Robotic Walking on a Swaying Rigid Surface

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Abstract—Controller design for bipedal walking on dynamic rigid surfaces (DRSes), which are rigid surfaces moving in the inertial frame (e.g., ships and airplanes), remains largely underexplored. This paper introduces a hierarchical control approach that achieves stable underactuated bipedal walking on a horizontally oscillating DRS. The highest layer of our approach is a real-time motion planner that generates desired global behaviors (i.e., center of mass trajectories and footstep locations) by stabilizing a reduced-order robot model. One key novelty of this layer is the derivation of the reduced-order model by analytically extending the angular momentum based linear inverted pendulum (ALIP) model from stationary to horizontally moving surfaces. The other novelty is the development of a discrete-time-foot-placement controller that exponentially stabilizes the hybrid, linear, time-varying ALIP. The middle layer translates the desired global behaviors into the robot’s full-body reference trajectories for all directly actuated degrees of freedom, while the lowest layer exponentially tracks those reference trajectories based on the full-order, hybrid, nonlinear robot model. Simulations confirm that the proposed framework ensures stable walking of a planar underactuated biped under different swayng DRS motions and gait types.

I. INTRODUCTION

Bipedal robots can aid in various critical real-world applications on nonstationary man-made platforms, such as firefighting, inspection, and maintenance on ships, trains, and aircraft. Enabling stable legged locomotion on a nonstationary rigid platform, which we call a dynamic rigid surface (DRS) [1], is a fundamentally challenging control problem due to the complexity of the associated robot dynamics that are nonlinear, hybrid, and time varying [2]. The objective of this study is to derive and validate a hierarchical control approach that enables stable bipedal underactuated walking on a swaying DRS (e.g., a vessel’s deck).

Various control approaches have been created to realize provably stable bipedal robotic walking on stationary rigid surfaces. One widely studied method is the hybrid zero dynamics (HZD) framework [3], which stabilizes bipedal walking by explicitly treating the full-order, hybrid, nonlinear robot dynamics. For underactuated robots (e.g., bipeds with point feet), the HZD method exploits input-output linearization to transform the nonlinear robot dynamics associated with the directly actuated degrees of freedom (DOFs) into a linear time-invariant system, which is then stabilized based on the well-studied linear system theory. Due to the use of input-output linearization, internal dynamics exist, and its solutions (e.g., periodic orbits) are typically unstable for walking robots. The HZD method constructs a reduced-order zero dynamics manifold that agrees with the overall hybrid dynamics, and provably stabilizes periodic orbits on that manifold to ensure the stability for the full-order system.

Due to the high dimensionality and strong nonlinearity of a full-order robot model, real-time generation of stable desired trajectories based on the full-order model can be computationally prohibitive for achieving robust bipedal walking on stationary uneven terrains. To that end, researchers have integrated reduced-order model based planning with full-order model based control. A hybrid linear inverted pendulum (LIP) model was developed and used to approximate the hybrid walking dynamics of an underactuated bipedal robot [4]–[6]. Y. Gong et al. proposed a new variant of the LIP that uses the angular momentum about the contact point, instead of the linear velocity of the center of mass (CoM), as a state variable [7], [8], which is called the “ALIP”. V. Paredes et al. introduced a LIP template model to generate a stepping controller with an adaptive learning regulator for ensuring stable walking on a bipedal humanoid robot [9]. Yet, due to the time-varying movement of the surface-foot contact point/region, the dynamic model of bipedal walking on a DRS is explicitly time-varying [2], [10], which is fundamentally different from the typical time-invariant robot dynamics during static-surface locomotion.

Recently, the control problem of stabilizing legged locomotion on a DRS has been initially studied. To provably stabilize quadrupedal walking on a vertically moving DRS, A. Iqbal et al. introduced a nonlinear control approach that explicitly handles the time-varying DRS movement and the hybrid, nonlinear robot dynamics during quadrupedal walking [2]. Also, the classical continuous-time LIP model has been analytically extended from a static surface to a vertically moving DRS, resulting in a homogeneous, time-varying LIP model [11]–[13]. Still, as the walking robot considered is fully actuated, the inherent instability associated with underactuated walking does not exist. Also, these studies assume the surface motion is vertical.
This study introduces a hierarchical control approach that achieves stable underactuated bipedal walking on a periodically swaying rigid surface by explicitly treating the hybrid, time-varying robot dynamics and simultaneously exploiting the complementary advantages of full-order and reduced-order models for walking stabilization. The specific contributions are: (a) analytically extending the ALIP model from stationary surfaces to a horizontally swaying DRS, resulting in a hybrid, nonhomogeneous, time-varying ALIP; (b) synthesizing a discrete-time footstep controller that exponentially stabilizes the hybrid, time-varying ALIP, and formulating an optimization problem to find the desired CoM trajectories and the stabilizing footstep locations; (c) developing a three-layer control approach that ensures the stability for the hybrid, time-varying, nonlinear unactuated robot dynamics by stabilizing the proposed ALIP and by mitigating the model inaccuracy of the ALIP with proper full-order trajectory design and tracking control; and (d) demonstrating the proposed approach enables a full-order robot to stably walk on a swaying DRS under different surface motions and gait types.

II. TIME-VARYING ANGULAR MOMENTUM BASED LINEAR INVERTED PENDULUM MODEL

This section introduces the derivation of the proposed angular momentum based linear inverted pendulum (ALIP) that captures the essential robot dynamics during walking on a horizontally swaying DRS. The ALIP model serves as the basis of the higher-layer planner introduced in Sec. III.

Different from the classical LIP model [14] that takes a robot’s center of mass (CoM) position and velocity as its state, we choose to use the CoM position and the angular momentum about the surface-foot contact point as the state, resulting in an ALIP. Compared with the classical LIP model, such a state choice allows the ALIP to be more accurate in representing the true robot dynamics in the presence of velocities jumps at foot landings, large peak motor torques, or aggressive swing leg motions [7].

A. Angular Momentum about Foot-Surface Contact Point

This subsection introduces the mathematical expression of a biped’s angular momentum about the foot-surface contact point. For simplicity, we consider bipeds with point feet.

1) Continuous swing phase: During a swing phase, one foot of the biped contacts the walking surface, and the other moves in the air. Let $S$ denote the contact point during walking to the surface. For a DRS, the point $S$ moves in the world frame. Let $A$ denote a stationary point in the world frame that instantaneously coincides with $S$ at the given time.

During a single-support phase, the three-dimensional (3-D) vector of angular momentum about the point $S$, denoted as $L_S$, has the following relation with the angular momentum about point $A$, denoted as $L_A$:

$$L_S = L_A + p_{SA} \times (m\dot{v}_{CoM}). \quad (1)$$

Here the vector $p_{SA}$ is the position of point $A$ relative to point $S$, expressed in the world frame. Note that $p_{SA} = \theta$ because point $A$ instantaneously coincides with point $S$ at the given time. The scalar constant $m$ is the total mass of the robot.

The vector $v_{CoM}$ is the absolute CoM velocity with respect to (w.r.t.) the world frame.

We will use the relation in (1) to derive the dynamics of $L_S$ for legged locomotion on a DRS (Sec. II-B).

2) Discrete foot-switching event: At the end of the swing phase, the swing foot touches the walking surface. Without loss of generality, we assume that the support foot begins to swing just after the swing foot touchdown.

Across a foot landing event, the position of the contact point jumps. Let $L_k (k \in \{1, 2, ..., \})$ be the angular momentum about the $k^{th}$ contact point on the walking surface. Let $p_{k-1 \rightarrow k}$ denote the position vector pointing from the $(k-1)^{th}$ to the $k^{th}$ contact point. Let $(\cdot)^-$ and $(\cdot)^+$ respectively represent the values of $(\cdot)$ just before and after the $k^{th}$ foot-landing instant.

Just before the $k^{th}$ foot-landing instant, the angular momentum about the new contact point, $L_{k+1}^-$, can be related to the previous one, $L_k^-$, as follows:

$$L_{k+1}^- = L_k^- + p_{k-1 \rightarrow k} \times (m\dot{v}_{CoM}). \quad (2)$$

Across the $k^{th}$ foot-landing event, the surface-foot impact at the new contact point generates zero impulse torque about the $(k + 1)^{th}$ contact point. Thus, the angular momentum about the $(k + 1)^{th}$ contact point, $L_{k+1}$, does not change across a foot-landing instant; that is, $L_{k+1}^- = L_{k+1}^+$. Accordingly, $L_{k+1}^+$ can be expressed as:

$$L_{k+1}^+ = L_k^- + p_{k-1 \rightarrow k} \times (m\dot{v}_{CoM}). \quad (3)$$

From (3), we know $L_{k+1}^+ = L_k^-$ holds (i.e., the robot’s angular momentum about the contact point is invariant to foot switching) when a planar robot walks on a flat, horizontal surface with zero vertical CoM velocity.

B. 2-D ALIP Dynamics during a Continuous Swing Phase

This subsection introduces the proposed ALIP model that captures the essential continuous-phase robot dynamics associated with 2-D walking on a DRS (see Fig. 1-a).

As explained in subsection A, we choose to use the angular momentum about the contact point as a state variable of the proposed ALIP, in addition to the CoM position. To derive the needed dynamic model with these state variables, we first derive the accurate dynamic model in 3-D and then provide the approximate ALIP model in 2-D.

1) Deriving $L_S$: Recall that point $A$ is static in the world frame and instantaneously coincides with the contact point $S$ (which is attached to the DRS) at the given time. Taking the time derivative of both sides of (1) gives:

$$L_S = L_A + p_{SA} \times (m\dot{v}_{CoM}) + p_{SA} \times (m\dot{v}_{CoM}). \quad (4)$$

Since $p_{SA} = \theta$ and $p_{SA} = -p_S \times (m\dot{v}_{CoM})$, (4) becomes:

$$L_S = L_A - p_S \times (m\dot{v}_{CoM}). \quad (5)$$

As the time derivative of the angular momentum $L_A$ equals

$$\frac{d}{dt} L_A = 2 m \ddot{v}_{CoM}.$$

Fig. 1. Illustration of the proposed hybrid AIP model.

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the sum of the external moments about point A, we have
\[
\dot{L}_A = p_{AC} \times (mg) + \tau \tau \tau_A, (6)
\]
where \(p_{AC}\) is the CoM position relative to point A, \(g\) is the gravitational acceleration, and \(\tau \tau \tau_A\) is the external torque applied at the contact point.

Since \(\tau_A = 0\) for point feet and \(p_{AC} = p_{SC}\) (where \(p_{SC}\) is the CoM position relative to point S), (6) becomes:
\[
\dot{L}_A = p_{SC} \times (mg). (7)
\]
Combining (5) and (7) gives the 3-D dynamics of \(L_S\):
\[
\dot{L}_S = p_{SC} \times (mg) - p_S \times (mv_{CoM}). (8)
\]

2) Deriving \(p_{SC}\). The relative CoM velocity \(p_{SC}\) is:
\[
p_{SC} = v_{CoM} - p_s,\]
where the expression of \(v_{CoM}\) can be obtained through the following relationship between \(L_S\) and the angular momentum about the CoM, denoted as \(L_{CoM}\):
\[
\dot{L}_S = L_{CoM} + p_{SC} \times (mv_{CoM}). (9)
\]

3) 2-D ALIP model: Similar to the classical LIP model [14] and the ALIP model for stationary-surface locomotion [8], we assume that the CoM height above the DRS remains constant. Also, we assume that the DRS only moves horizontally, which is realistic for real-world DRSes such as the deck of a vessel moving in regular sea waves [15]. Thus, the linear velocity of the DRS at the support point \(S\), \(p_S\), and the CoM velocity, \(v_{CoM}\), are parallel for 2-D robots. Accordingly, \(p_s \times (mv_{CoM}) = 0\) holds for 2-D robots, with which (8) becomes:
\[
\dot{L}_S = p_{SC} \times (mg). (10)
\]

By approximating a biped as an inverted pendulum (i.e., a point mass atop a massless leg) [14], the angular momentum about the CoM, \(L_{CoM}\), becomes zero. Thus, (9) becomes:
\[
\dot{L}_S = p_{SC} \times (mv_{CoM}). (11)
\]

Since this study considers 2-D bipeds, we can express the angular momentum \(L_S\) as a scalar variable, which is denoted as \(L_S\). Also, under the assumption that the CoM height is constant, we only need to consider the 1-D horizontal movement along the \(x\)-direction as shown in Fig. 1.

Let \(\Delta x_{SC}, \Delta z_{SC}\) be the coordinates of the position vector \(p_{SC}\). Note that \(\Delta Z_{SC} = 0\) under the assumption of the constant CoM height, and let \(H\) denote the constant CoM height (i.e., \(\Delta Z_{SC} = H\)). Let \(\dot{x}_{SC}\) and \(\dot{z}_{SC}\) be the forward CoM velocity relative to the support point \(S\) (i.e., the \(x\)-component of \(p_{SC}\)) and the horizontal surface velocity (i.e., the \(x\)-component of \(p_S\)), respectively. The 3-D vector equations (9)-(10) can be written in the scalar form as
\[
\dot{L}_S = mgx_{SC} \quad \text{and} \quad \dot{x}_{SC} = \frac{L_S}{mH} - \dot{x}_{SC}. (11)
\]
Accordingly, with the state definition \(x := [x_{SC}, L_S]^T\), the state-space representation of the 2-D ALIP model during a continuous swing phase is given by
\[
x = Ax - f(t) := \begin{bmatrix} 0 & 1 \\ mg & 0 \end{bmatrix} x - \begin{bmatrix} 0 \\ \dot{x}_{SC}(t) \end{bmatrix}. (12)
\]

Note that the proposed continuous-phase ALIP model in (12) is explicitly time-varying due to the time-varying forcing term \(f(t)\) induced by the time-varying velocity of the support point \(S\), \(\dot{x}_{SC}(t)\). Such a time-varying property differs from the previous time-invariant LIP models described in [8], [14]. As any point on a purely translating rigid surface has the same linear velocity, \(\dot{x}_{SC}(t)\) represents the surface velocity. In this study, we assume the surface motion \(\dot{x}_{SC}(t)\) is differentiable.
A. Discrete-Time Foot Placement Control

The continuous-phase ALIP model in (12) is unstable and has no control input. This subsection introduces the derivation of a control strategy that regulates the footstep locations to stabilize the overall hybrid ALIP model. The control command is the footstep position of the ALIP, which is also referred to as swing-foot-landing control [4].

Inspired by [4], we formulate the footstep controller as:

\[ u = u^* + K(x^* - x^*), \]

(14)

where \( u^* \) is the desired footstep location of the ALIP. The vector \( K \) is the feedback gain to be designed with \( K \in \mathbb{R}^{1 \times 2} \). Here, \( x^* \) and \( x^* \) are the actual and the desired pre-impact states of the ALIP, respectively.

From (13) and (14), we obtain the closed-loop discrete-time ALIP model at the \( k^{th} \) \((k = 1, 2, \ldots)\) switching event as

\[ \Delta x = B x^* + G := \begin{bmatrix} -K & 0 \\ 1 \end{bmatrix} x^* + \begin{bmatrix} K x^* - u^* \\ 0 \end{bmatrix}, \quad t = t_k, \]

(15)

where \( 1 \times 2 \) is a \( 1 \times 2 \) zero vector.

B. Stability Condition

The hybrid ALIP system described by (12) and (15) is:

\[ \begin{cases} \dot{x} = Ax - f(t), & t \neq t_k; \\
\Delta x = Bx^* + G, & t = t_k. \end{cases} \]

(16)

The stability condition for the periodic solution of such a linear time-varying nonhomogeneous hybrid system is [17]

Theorem 1: If all eigenvalues of the monodromy matrix associated with the hybrid homogeneous portion of the system in (16) are less than one in modulus, then the hybrid homogeneous system is exponentially stable, and so is the \( T \)-periodic solution \( \psi(t) \) (with \( \psi(t + T) = \psi(t) \)) of the hybrid non-stationary system.

C. Exponential Stabilization of Periodic ALIP Walking

We formulate an offline optimization problem to find a periodic solution \( \psi(t) \) of the hybrid ALIP in (16) along with a stabilizing control gain \( K \).

1) Optimization variables: The optimization variables are \( K, u^* \), and \( x^* \) as they define the footstep controller in (14).

2) Cost function: We choose to minimize the norm of \( K \) to help prevent the controller from overreacting to small values of \( (x^* - x^*) \). Thus, the cost function is \( J = KK^T \).

3) Inequality constraints: The inequality constraints considered include stability and feasibility constraints. The stability constraint ensures that the solution of the ALIP model exponentially converges to the planned trajectory by enforcing the stability condition in Theorem 1. Let \( \mu_i \) be the \( i^{th} \) eigenvalue of the monodromy matrix associated with the homogeneous portion of the hybrid ALIP model in (16). To ensure stability, Theorem 1 requires \( |\mu_i| < 1 \) for any \( i \). The feasibility constraint ensures that the full-order model can achieve the desired trajectory without violating its kinematic limits. Let \( u_{\text{min}} \) and \( u_{\text{max}} \) be the minimum and maximum values of the step length \( u \). Let \( x_{\text{min}} \) and \( x_{\text{max}} \) be the feasible lower and upper bounds of state \( x \). Then, the feasibility constraints are: \( u_{\text{min}} \leq u^* \leq u_{\text{max}} \) and \( x_{\text{min}} \leq x^* \leq x_{\text{max}} \).

4) Equality constraints: We use equality constraints to enforce (a) the desired step length and (b) the periodicity of the solution of the hybrid ALIP system in (16). With \( \psi(t_{k+1}) = x^* \), we express the periodicity constraint as \( \psi(t_{k+1}) = x^* \). Let \( \psi_1(t_{k+1}) \) denote the first element of \( \psi(t_{k+1}) \), i.e., the desired post-impact value of \( x_{\text{sw}} \). Then, the step length constraint is expressed as: \( u^* = x_{\text{sw}} - \psi_1(t_{k+1}) \).

5) User-defined gait parameters: Users typically specify the key features of the desired robot behaviors as input to optimization-based planning. We pre-select the following parameters and variables: CoM height \( H \), gait period \( T \), DRS motion \( x_{\text{sw}}(t) \), and total mass \( m \). Note that pre-specifying a known surface motion \( x_{\text{sw}}(t) \) is realistic for real-world moving platforms (e.g., vessels) equipped with motion sensors [18].

IV. MIDDLE-LAYER WALKING PATTERN PLANNER

This section presents the middle-layer planner of the proposed control approach. This layer generates the full-body reference motions compatible with both the ALIP and the desired global behaviors produced by the higher-layer planner. These full-body trajectories are tracked by the lower-layer controller explained in Sec. V.

A. Full-Body Control Variable Selection

Let \( q \) be the vector of the generalized coordinates of a full-order planar robot. Typically, \( q \) is defined as \( q := [x_b, y_b, \theta_b, q_1, \ldots, q_4]^T \), where the scalar variables \( x_b, y_b, \) and \( \theta_b \) are the pose of the robot’s base in the world frame and \( q_i \) \((i = 1, \ldots, n)\) is the angle of the \( i^{th} \) revolute joint of this robot. We assume the robot has point feet with all revolute joints powered (except ankles). When all joints except the point feet are powered, the planar walking robot is underactuated (with one degree of underactuation) [19].

We consider a planar biped with four revolute joints (right, Fig. 2). We design the control variables \( h_c \) as: \( h_c(q) = [\xi_{\text{CoM}}(q), \theta_b, x_{\text{sw}}, z_{\text{sw}}(q)]^T \), where \( \xi_{\text{CoM}} \) is the CoM height above the DRS, \( \theta_b \) is the trunk orientation, and \( x_{\text{sw}} \) and \( z_{\text{sw}} \) are the forward and vertical swing foot positions relative to the support foot. We choose to directly regulate the CoM height \( \xi_{\text{CoM}} \) to be constant and match that of the ALIP for ensuring the closeness between the ALIP and the true robot dynamics. Also, the trunk angle \( \theta_b \) is controlled for an upright trunk posture and a small angular momentum about the CoM. Finally, by commanding the swing foot position \( (x_{\text{sw}}, z_{\text{sw}}) \), the full-order robot can reliably execute the desired footstep locations supplied by the higher layer.

B. Full-Body Trajectory Generation

We use \( h_d \) to denote the vector of the desired trajectories for the control variables \( h_c \), and define \( h_d \) as: \( h_d = [\phi_1, \phi_2, \phi_3, \phi_4]^T \), where the scalar functions \( \phi_1, \ldots, \phi_4 \) represent the desired trajectories for \( \xi_{\text{CoM}}, \theta_b, x_{\text{sw}}, \) and \( z_{\text{sw}} \).

We parameterize the desired trajectories \( \phi_1, \ldots, \phi_4 \) using Bézier polynomials [3], [20]. To encode the polynomials, we use a time-based phase variable \( s \) that represents how long a step has progressed within a walking step. Let \( T_k \) denote the switching instant at the end of the \( k^{th} \) actual walking step. Then, \( s := \frac{T_k}{T} \) during the actual \((k + 1)^{th}\) step. The higher-level planner constantly updates the Bézier coefficients based on the actual horizontal footstep location.
V. LOWER-LAYER INPUT-OUTPUT LINEARIZING CONTROL BASED ON HYBRID FULL-ORDER MODEL

This section explains the feedback controller that we use to command the directly controlled variables of the actual robot to follow the desired full-body motion.

As inspired by the HZD approach [3], we develop a tracking controller based on the full-order, hybrid, nonlinear robot dynamics model and the exact linearization of the nonlinear map between the control input and the directly commanded tracking error. Also, similar to our recent work on the provable stabilization of DRS walking [2], we explicitly handle the time-varying nature of the robot dynamics through modeling and controller design.

A. Full-Order Dynamic Model

1) Full-order continuous-phase dynamics: A robot with point feet has point contact with the DRS. Let $p_F(q)$ denote the support foot position in the world. Assuming the support foot does not slip on the DRS (i.e., $p_F(q) = p_s(t)$) and defining $J_F(q) := \frac{\partial p}{\partial q}$, the holonomic constraint associated with the point contact is given by

$$J_F \dot{q} + J_0 \dot{q} = \bar{p}_s(t). \quad (17)$$

With Lagrange's method, the continuous-phase robot dynamics during DRS walking can be expressed as:

$$M(q) \ddot{q} + c(q, \dot{q}) = \bar{B} \tau + J_F^T \tau_F,$$  \quad (18)

where $M$ is the inertia matrix, $c$ is the sum of the gravitational, Coriolis, and centrifugal terms, $B$ is the constant input-selection matrix, $\tau_F$ is a vector of joint torques, and $\tau_F$ is the ground reaction force induced by the interaction between the contact foot and the moving surface.

Combining (17) and (18) yields

$$M(q) \ddot{q} + \bar{c}(t, q, \dot{q}) = \bar{B} \tau,$$ \quad (19)

where $\bar{c} := c - J_F^T (J_F M^{-1} J_F^T)^{-1} (J_F M^{-1} c - J_F \bar{p}_s(t))$, and $\bar{B} := B - J_F^T (J_F M^{-1} J_F^T)^{-1} (J_F M^{-1} B) \quad [21]$.

2) Switching surface and impact dynamics: When the swing foot strikes the DRS, an impact occurs, causing jumps in the generalized velocities $\dot{q}$ [3]. The switching surface governing the occurrence of the impact is given by $S_q := \{ (q, \dot{q}) : \dot{z}(q, \dot{q}) = 0, \dot{z}_{sw}(q, \dot{q}) < 0 \}$.

The velocity jump can be described by a reset map: $\dot{q}^+ = R_q(\dot{q}^-)$, where the matrix $R_q$ can be derived as in [3]. Different from the fixed-time switching of the ALIP model, the full-order switching surface is state-triggered [3], which is accurate in capturing the actual robot behavior.

B. Input-output Linearizing Control

The tracking error $y$ is defined as $y = h_c(q) - h_d(s)$. With the following input-output linearizing control law [22]

$$\tau_y = \left[ \frac{\partial h_c}{\partial q} M^{-1} B \right]^{-1} \left[ \frac{\partial h_c}{\partial q} M^{-1} c + v - \frac{\partial h_d}{\partial q} (\frac{\partial h_d}{\partial q} q + \frac{1}{T} \Phi_{\tau}) \right], \quad (20)$$

the output function dynamics becomes $y = v$, which is linear and time-invariant. We can stabilize this linear system using proportional-derivative (PD) control, i.e., $v = -K_p y - K_d \dot{y}$ with PD gains $K_p$ and $K_d$.

VI. SIMULATION VALIDATION

This section reports the simulation validation results of the proposed approach for planar underactuated bipedal walking under different gait types and surface movements.

A. MATLAB Simulation Setup

1) Robot: In MATLAB, a planar robot with point feet, five links, and four motors (right, Fig. 2) is simulated based on the hybrid full-order model given in Sec. V. The detailed mass distribution and geometric properties of this robot are given in Table I. As the robot has one degree of underactuation (Sec. IV-A), the unactuated dynamics is 2-D.

| Physical properties of the planar biped | trunk | shank/thigh |
|----------------------------------------|------|-------------|
| Mass (kg)                              | 38   | 0.63        |
| Length (m)                             | 0.3  | 0.4         |

2) DRS motions: As this study focuses on horizontally and periodically moving surfaces, we simulate the following two swaying DRS motions: (M1) $s_s(t) = 0.03 \sin(\frac{2\pi}{T} t)$ m and (M2) $s_s(t) = 0.03 \sin(\frac{2\pi}{T} t)$ m.

3) Desired gait types: While this study addresses 2-D walking, two types of 2-D gaits are planned and tested, corresponding to the typical walking patterns in the sagittal and lateral planes of 3-D walking. They are: (G1) walking along a single direction, with the two legs periodically passing each other, and (G2) stepping in place w.r.t. the DRS frame, without leg crossing.

4) Simulated cases: Two walking scenarios are simulated: (A) DRS motion (M1) and walking gait (G1) and (B) DRS motion (M2) and walking gait (G2).

5) Implementation of hierarchical controller: To implement the higher-layer planner, we use the MATLAB fmincon command to solve the higher-layer optimization problem. The bounds on the step length and the ALIP state are chosen to represent the kinematic limits of the full-order robot, which are $u_{\text{max}} = -u_{\text{min}} = 0.7m$ and $x_{\text{max}} = -x_{\text{min}} = [0.7m, 40kg m^2/s]^T$.

For case (A), the user-defined parameter is chosen to be $H = 0.81m$, $m = 39.8kg$, and $T = 0.4s$. For case (B), the parameters are the same as case (A) except $T = 0.2s$. The optimization is solved in MATLAB within 10 seconds for both walking gaits. The eigenvalues of cases (A) and (B) are $\mu = -0.023 \pm 0.003i$ and $\mu = -0.340 \pm 0.000i$, respectively.

The middle layer provides the desired full-body trajectories $h_d$. To ensure a close match between the ALIP and full-order robot, we choose $\phi_1$ to match the CoM height $H$ of the ALIP, set $\phi_2$ as constant to minimize trunk motion, and design $\phi_4$ with a relatively small swing foot height. The coefficients of $\phi_3$ determining the desired forward footstep location are updated by the ALIP planner at 100 Hz.

The proposed lower-layer controller is implemented based on (20). The matrix $M$ and vector $c$ are obtained using FROST [23]. The control gains are chosen as $K_p = K_p I$ ($K_p = 2500$) and $K_d = K_d I$ ($K_d = 100$) for both cases (A) and (B), where $I$ is a $4 \times 4$ identity matrix. This choice of PD gains ensures the exponential convergence of the output function $y$ within a continuous phase.

B. Simulation Results

The results of case (A) (Fig 3 a)) and case (B) (Fig 3 b)) indicate that the unactuated state variables of the full-order robot, i.e., the angular momentum about the contact point,
Fig. 3. State trajectories of the ALIP and the full-order robot under a) case (A) and b) case (B). The state trajectories of the full-order model stay close to those of the ALIP model, indicating a small model discrepancy.

Fig. 4. Desired footstep location of the ALIP (i.e., \(x^*\)) and online-planned footstep location of the full-order robot (i.e., \(x\)) for a) case (A) and b) case (B). The small difference between \(x^*\) and \(x\) supports that the full-order robot tracks the desired ALIP footstep locations well. \(L_g\) and the relative CoM position, \(x_{SC}\), remain bounded and close to the planned ALIP trajectories. Figure 3 confirms that our approach enables the underactuated biped to walk stably on a periodically swaying DRS. Figure 4 displays the desired step lengths of the ALIP and full-order robot under cases (A) and (B), indicating that the biped follows the desired ALIP footstep locations well for both cases.

VII. CONCLUSION

This paper has proposed a control approach that achieves provably stable underactuated bipedal walking on a horizontally oscillating DRS. An ALIP model for walking on a DRS was analytically derived with the time-varying surface movement and hybrid robot dynamics explicitly considered. A discrete feedback control strategy was synthesized to stabilize the periodic solution of the hybrid ALIP model whose continuous-phase dynamics is unstable and uncontrollable. A hierarchical control approach was developed to stabilize the unactuated full-order robot dynamics comprising a higher-layer ALIP footstep controller, a middle-layer full-order trajectory generator, and a lower-layer nonlinear feedback controller. Simulations of a planar robot confirmed that the proposed framework stabilizes underactuated walking on a swaying DRS under different surface motions and gait types.

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