STABILITY OF VISCOUS SHOCK PROFILES FOR DISSIPATIVE
SYMMETRIC HYPERBOLIC-PARABOLIC SYSTEMS

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Abstract. Combining pointwise Green’s function bounds obtained in a companion paper [MZ.2] with earlier, spectral stability results obtained in [HuZ], we establish nonlinear orbital stability of small amplitude viscous shock profiles for the class of dissipative symmetric hyperbolic-parabolic systems identified by Kawashima [Kaw], notably including compressible Navier–Stokes equations and the equations of magnetohydrodynamics, obtaining sharp rates of decay in $L^p$ with respect to small $L^1 \cap H^3$ perturbations, $2 \leq p \leq \infty$. Our analysis follows the approach introduced in [MZ.1] to treat stability of relaxation profiles.

Section 1. Introduction

Consider the class of degenerate parabolic conservation laws of dissipative, symmetric hyperbolic–parabolic type in the sense of Kawashima [Kaw], i.e., systems
\begin{equation}
G(\bar{U})_t + F(\bar{U})_x = (B(\bar{U})\bar{U}_x)_x,
\end{equation}
satisfying
\begin{enumerate}
\item[(A1)] $dF, dG, B$ symmetric, $dG > 0$, $B \geq 0$ (symmetric hyperbolic–parabolicity),
\item[(A2)] No eigenvector of $dFdG^{-1}$ lies in the kernel of $BdG^{-1}$ (dissipativity),
\item[(A3)] $\bar{U} = \left( \begin{array}{c} \bar{u} \\ \bar{v} \end{array} \right)$, \quad $B = \left( \begin{array}{cc} 0 & 0 \\ 0 & b \end{array} \right)$, \quad $b > 0$ (block structure),
\end{enumerate}
in some neighborhood $\mathcal{U}$ of a particular base point $\bar{U}_*$, where $\bar{U} \in \mathbb{R}^n$, $\bar{u} \in \mathbb{R}^{n-r}$, $\bar{v} \in \mathbb{R}^r$, and $b \in \mathbb{R}^{r \times r}$. As discussed in [Kaw], this class of equations includes many physical models arising in continuum mechanics: in particular, compressible Navier–Stokes equations and the equations of compressible magnetohydrodynamics (MHD). We make also the regularity assumption
\begin{enumerate}
\item[(H0)] $F, G, B \in C^3$,
\end{enumerate}
needed for our later analysis.

An interesting class of solutions of (1.1) are viscous shock profiles, or asymptotically constant traveling wave solutions
\begin{equation}
\bar{U} = \bar{U}(x - st); \quad \lim_{z \to \pm \infty} \bar{U}(z) = \bar{U}_\pm
\end{equation}

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connecting endstates $\tilde{U}_\pm$ corresponding to (discontinuous) shock waves of the associated hyperbolic system

$$G(\bar{U})_t + F(\bar{U})_x = 0.$$  

Existence of viscous profiles has been treated in the large for compressible Navier–Stokes by, e.g., Gilbarg [Gi] and for MHD by Conley and Smoller [CS] (see [Fre.1] for a treatment in the small amplitude case). Most recently, Freistühler [Fre.2] has carried out a general treatment of existence in the small, analogous to that of Majda and Pego in the strictly parabolic case [MP], for arbitrary models (1.1) satisfying (A1)–(A3).

Let

$$a_1(u) \leq \cdots \leq a_n(u)$$

denote the eigenvalues of $A_G := dF dG^{-1}(\tilde{U})$, $r_j(\tilde{U})$ and $l_j(\tilde{U})$ a smooth choice of associated right and left eigenvectors, $l_j \cdot r_k = \delta^j_k$, and assume within neighborhood $\mathcal{U}$ of base point $\tilde{U}_*$ that:

(H1) The $p$th characteristic field is of multiplicity one, i.e. $a_p(\tilde{U})$ is a simple eigenvalue of $A_G(\tilde{U})$.

(H2) The $p$th characteristic field is genuinely nonlinear, i.e. $\nabla a_p \cdot r_p(\tilde{U}) \neq 0$.

(H3) The eigenvalues of $dF_{11}dG_{11}^{-1}$ (necessarily real and semisimple) are: (i) of constant multiplicity; and, (ii) different from $a_p$.

Then, we have:

**Proposition 1.1 ([Fre.2])**. Let (A1)–(A3), and (H1)–(H3) hold, and $F, G, B \in C^2$ in (1.1) (implied in particular by (H0)). Then, for left and right states $\tilde{U}_\pm$ lying within a sufficiently small neighborhood $V \subset \mathcal{U}$ of $\tilde{U}_*$, and speeds $s$ lying within a sufficiently small neighborhood of $a_p(\tilde{U}_*)$, there exists a viscous profile (1.2) that is “local” in the sense that the image of $\bar{u}(\cdot)$ lies entirely within $V$ if and only if the triple $(\tilde{U}_-, \tilde{U}_+, s)$ satisfies both the Rankine–Hugoniot relations:

(RH) $$s[G] = [F],$$

and the Lax characteristic conditions for a $p$-shock:

(L) $$a_p(\bar{U}_-) > s > a_p(\bar{U}_+); \quad \text{sgn} \ (a_j(\bar{U}_-) - s) = \text{sgn} \ (a_j(\bar{U}_+) - s) \neq 0 \text{ for } j \neq p.$$  

(Note: The structure theorem of Lax [La,Sm] implies that (RH), always a necessary condition for existence of profiles, holds for $\bar{U}_\pm \in V$ only if $s$ lies near some $a_j(\tilde{U}_*)$; thus, the restriction on speed $s$ is only the assumption that the triple $(\tilde{U}_-, \tilde{U}_+, s)$ be associated with the $p$th and not some other characteristic field).

**Remark 1.2.** The generically satisfied conditions (H1) and (H2) are implied by strict hyperbolicity and genuine nonlinearity, respectively, of the associated hyperbolic system (1.3). In particular, they hold always for the equations of compressible
gas dynamics, under the assumption of an ideal gas [Sm]. Condition (H3)(ii) corresponds to the requirement that “hyperbolic,” or unsmoothed modes in the solution be noncharacteristic, at least as regards the principal characteristic speed $a_p$. This technical condition is needed in order that the traveling wave ODE be of nondegenerate type, and is a standard assumption in the theory. See assumption (4) of [Fre.2], assumption (H1)(ii) of [Z.3], Appendix A.2, or assumption (H5) of [SZ] for restatements in various different contexts. Condition (H3)(i) is an additional technical assumption that was used in the detailed Green’s function analysis carried out in [MZ.2] for slightly more general systems; we suspect that it can be dropped in the present, symmetrizable case. Conditions (H3)(i)–(ii) are likewise satisfied for compressible ideal gas dynamics, for which $dF_{11}dG_{11}^{-1}$ is $1 \times 1$ and identically equal to particle velocity ($u$ in Eulerian coordinates, 0 in Lagrangian coordinates).

Stability of viscous profiles has been examined for compressible Navier–Stokes equations in [MN, KMN, L.2], with various partial results concerning special (mainly zero-mass) initial data; the results of [KMN, L.2] are restricted to small amplitude profiles, while the results of [MN] in the case of an isentropic $\gamma$-law gas apply to profiles of amplitude $\alpha(\gamma)$, with $\alpha \to \infty$ as $\gamma \to 1$. Yet, as pointed out recently in the fluid-dynamical survey [Te], the basic problem of stability of gas-dynamical shocks in its full generality remains open even for small amplitude waves, a significant gap in the theory of compressible gas dynamics. More recently, Humpherys and Zumbrun [HuZ] have established strong spectral stability of general, small amplitude shock waves of Kawashima class systems, of the type constructed by Freistühler, generalizing a corresponding result of Goodman [Go.1–2] in the strictly parabolic case (see Section 2, below). As discussed in [ZH, HuZ], strong spectral stability is roughly equivalent to, but slightly weaker than stability with respect to zero-mass perturbations.

In [MZ.2], applying the general machinery developed in [ZH, MZ.1], we have shown, for a slightly more general class of systems and for profiles of arbitrary amplitude and type, that strong spectral stability, plus hyperbolic stability of the corresponding ideal shock (always satisfied for weak Lax shocks satisfying (H1)–(H3)), are necessary and sufficient conditions for linearized orbital stability, and, moreover, yield extremely detailed pointwise bounds on the Green’s function of the linearized evolution equations, analogous to those obtained for relaxation profiles in [MZ.1]. In combination with the result of [HuZ], these results imply $L^1 \cap L^p \to L^p$ linearized orbital stability of small-amplitude shock profiles of Kawashima class systems, with sharp rates of decay for all $1 \leq p \leq \infty$.

The purpose of the present paper, extending the work of [HuZ, MZ.2], is to establish $L^1 \cap H^3 \to L^p$ nonlinear orbital stability as solutions of (1.1) of small-amplitude profiles of general Kawashima class systems, with sharp rates of decay for all $1 \leq p \leq \infty$. More precisely, we shall establish:

**Theorem 1.3.** Let there hold (A1)–(A3) and (H0)–(H3) for a general relaxation model (1.1), with $\bar{U}_*$ and $U$ as above. Then, for $V \subset U$ sufficiently small, the viscous
profiles $\bar{U}$ described in Proposition 1.1 are nonlinearly orbitally stable from $L^1 \cap H^3$ to $L^p$, for all $p \geq 2$. More precisely, for initial perturbations $U_0 := \bar{U}_0 - \bar{U}$ such that $|U_0|_{L^1 \cap H^3} \leq \zeta_0$, $\zeta_0$ sufficiently small, the solution $\bar{U} = (\bar{u}, \bar{v})(x, t)$ of (1.1) with initial data $\bar{U}_0$ satisfies

\begin{equation}
|\bar{U}(x, t) - \bar{U}(x - \delta(t))|_{L^p} \leq C\zeta_0 (1 + t)^{-\frac{1}{2}(1-1/p)}
\end{equation}

and

\begin{equation}
|\bar{v}(x, t) - \bar{v}(x - \delta(t))|_{L^p} \leq C\zeta_0 (1 + t)^{-\frac{1}{2}(1-1/p)},
\end{equation}

for all $2 \leq p \leq \infty$, for some $\delta(t)$ satisfying

\begin{equation}
|\dot{\delta}(t)| \leq C\zeta_0 (1 + t)^{-\frac{1}{2}}
\end{equation}

and

\begin{equation}
|\delta(t)| \leq C\zeta_0.
\end{equation}

In particular, this implies nonlinear stability of all small amplitude gas-dynamical profiles under the assumptions of an ideal gas; see Remark 1.2 above.

Our analysis in this paper, and in the companion paper [MZ.2], follows closely to that used in [MZ.1] to treat stability of relaxation profiles, making use of structural similarities between the two problems pointed out in [Ze.2,Z.3]; in turn, the argument of [MZ.1] makes substantial use of the pointwise semigroup machinery introduced in [ZH,Z.2] to treat the parabolic case. As are general features of the pointwise semigroup approach, we obtain through this program extremely detailed pointwise bounds on the Green’s function (more properly, distribution) of the linearized operator about the wave, sufficiently strong that the nonlinear stability analysis becomes comparatively simple. Moreover, these bounds (though not the nonlinear stability argument in which they are used, see Remark 1.5 just below) depend only on a generalized spectral stability (i.e., Evans function) condition and not on the amplitude or type (Lax, under-, or overcompressive) of the wave.

Remark 1.4. The assumption of genuine nonlinearity (H2) is not needed either for the existence or the stability result, but is made only to simplify the discussion. Though we stated above only the restriction to the genuinely nonlinear case, existence was in fact treated for the general (nongenuinely nonlinear) case in [Fre.2], substituting for the Lax entropy condition (L) the strict Liu entropy inequality (E) of [L.3]. Likewise, as described in [HuZ], the result of spectral stability may be extended to the general case by substituting for the “Goodman-type” weighted energy estimates of [HuZ], Section 5, the variation introduced by Fries [Fri.1–2] to treat the nongenuinely nonlinear case for strictly parabolic viscosities. Accordingly, the result of Theorem 1.3 holds also in this case.
Remark 1.5. The restriction to small-amplitude shocks arises only through the energy estimates used to close the nonlinear iteration argument. In particular, it should be possible for the isentropic, $\gamma$-law gas case to treat amplitudes of the same order $\alpha(\gamma)$ treated in [MN]. A fundamental open problem is to remove the amplitude restriction altogether, as done in the strictly parabolic case [ZH,Z.2] and in the case of discrete kinetic relaxation models [MZ.1], replacing it with the generalized spectral condition of the linearized theory.

Remark 1.6. Useful necessary conditions for viscous stability have been obtained in [Z.3] for arbitrary amplitude profiles of the more general class of models considered in [MZ.2], using the stability index of [GZ,BSZ]. Strengthened versions of the one-dimensional inviscid stability criteria of Erpenbeck–Majda [Er,M.1–3], these readily yield examples of unstable large-amplitude profiles, similarly as in the strictly parabolic case (see, e.g., [GZ,FreZ,ZS,Z.3]). This shows that the spectral stability requirement is not vacuous in the large amplitude case.

Plan of the paper. In Section 2 we cite the spectral stability result of [HuZ], and in Section 3 the pointwise Green’s function bounds of [MZ.2]. As an immediate corollary, we establish in Section 4 the linearized orbital stability of general shock profiles satisfying the necessary conditions of spectral and hyperbolic stability: in particular, of small amplitude profiles of Kawashima class systems. Finally, we establish in Section 5 the main result of nonlinear stability of weak profiles of Kawashima class systems, by a modified version of the argument of [MZ.1].

Note: Liu and Zeng have informed us [LZe.2] that they also have obtained nonlinear stability of weak Navier–Stokes profiles, by a different argument based on the approximate Green’s function approach of [L.2]. This method is inherently limited to weak shocks of classical, Lax type, both at the linear and nonlinear level.

Section 2. Spectral stability.

Take without loss of generality $s = 0$, so that $\tilde{U} = \tilde{U}(x)$ becomes a stationary solution. Then, the linearized equations of (1.1) about $\tilde{U}$ take the form

$$U_t = LU := -(A^0)^{-1}(AU)_x + (A^0)^{-1}(BU)_x,$$

where

$$B := B(\tilde{U}), \quad A^0 := dG(\tilde{U}), \quad Av := dF(\tilde{U})v - (dBv)\tilde{U}_x.$$

Definition 2.1. We call the profile $\tilde{U}(\cdot)$ strongly spectrally stable if the linearized operator $L$ about the wave has no spectrum in the closed unstable complex half-plane $\{\lambda : \text{Re } \lambda \geq 0\}$ except at the origin, $\lambda = 0$. (Recall, [Sat], that $\lambda = 0$ is always in the spectrum of $L$, since $L\tilde{U}_x = 0$ by direct calculation/differentiation of the traveling wave ODE).
The spectral stability of small-amplitude viscous profiles of the slightly more general class of dissipative, symmetrizable degenerate parabolic systems has been investigated in [HuZ]. To apply these results, we have only to note that (1.1), expressed with respect to variable $\tilde{G} := G(\tilde{U})$, becomes

$$(2.3) \quad \tilde{G}_t + F(\tilde{U}(\tilde{G}))_x = (B(\tilde{U}(\tilde{G}))\tilde{U}(\tilde{G})_x)_x,$$

or, in quasilinear form,

$$(2.4) \quad \tilde{G}_t + F_G G_U^{-1} \tilde{G}_x = (BG_U^{-1} \tilde{G}_x)_x.$$

System (2.4) is evidently symmetrizable (by the symmetric positive definite $G_U^{-1}$), and dissipative (since the dissipativity condition is independent of coordinate system), with the additional block structure property that the left kernel of the new viscosity matrix $BG_U^{-1}$ is constant. This is almost the class of equations considered in [HuZ], the difference being that there the block structure assumption was that the viscosity matrix have constant right instead of left kernel. This discrepancy is unimportant in the analysis, since, at the linearized level, one case may be converted to the other by the (linear) change of coordinates $A^0 V := U$, corresponding to similarity transform $M \rightarrow (A^0)^{-1} MA^0$. Alternatively, in the particular case of our interest, we may simply carry out energy estimates in the natural coordinates of (2.1); for related calculations, see Section 4.1, below.

Thus, the results of [HuZ] apply to the somewhat larger class of dissipative symmetrizable systems of form (2.3)–(2.3), with the block structure condition that either the left or right kernel of the viscosity matrix be constant, and we may conclude, in particular:

**Theorem 2.2 [HuZ].** Let (A1)–(A3) and (H0)–(H3) hold, and let $\tilde{U}(x - st)$ be a viscous shock solution of (1.1) such that the profile $\{\tilde{u}(z)\}$ lies entirely within a sufficiently small neighborhood $V \subset U$ of $\tilde{U}_*$, and the speed $s$ lies within a sufficiently small neighborhood of $a_p(\tilde{U}_*)$. Then, $\tilde{U}$ is strongly spectrally stable, in the sense of Definition 2.1 above.

**Section 3. Pointwise Green’s function bounds.**

In a companion paper to this one [MZ.1], we have investigated linearized stability and behavior of viscous profiles of systems of the general form (2.4), not necessarily symmetrizable, satisfying the standard set of conditions identified in [Z.3]. As remarked in [Z.3,MZ.1], these hold always for symmetric systems (1.1) satisfying (A1)–(A3), (H0)–(H3), and the small-amplitude profiles described in Proposition 1.1; however, they may also hold in much greater generality, in particular for shocks of large amplitude, or nonclassical type. For simplicity of exposition, we shall restrict our discussion here to the present case of interest; for the general case, see [MZ.1].
The first main result of [MZ.2], generalizing the corresponding results established for viscous, strictly parabolic, shocks in [ZH], and for relaxation shocks in [MZ.1], is:

**Theorem 3.1 [MZ.2].** Under assumptions (A1)–(A3), (H0)–(H3), small amplitude shock profiles are \( L^1 \cap L^p \rightarrow L^p \) linearly orbitally stable for \( p > 1 \) if and only if they are strongly spectrally stable, with sharp decay bounds

\[
|U(\cdot, t) + \delta(t)\bar{U}'(\cdot)|_{L^p} \leq C(1 + t)^{-\frac{1}{2}(1-1/p)}(|U_0|_{L^1} + |U_0|_{L^p})
\]

for initial data \( U_0 \in L^1 \cap L^p \), some choice of \( \delta \).

Theorem 3.1 is obtained (see, e.g., Corollary 4.2, below for one direction) as a consequence of detailed, pointwise bounds on the Green’s function (more properly speaking, distribution) \( G(x, t; y) \) of the linearized evolution equations

\[
G_t = L_G G := -(A_G G)_x + (B_G G)_x,
\]

\[
B_G := B(A^0)^{-1}(\bar{U}(x)), \quad A_G v := dF(A^0)^{-1}v - dB(\bar{U}(x))v\bar{U}'(x),
\]

corresponding to (2.4), i.e., expressed with respect to the conservative variable \( G \). We now describe these bounds, for use in the following sections.

Let \( a_j^\pm, j = 1, \ldots (n) \) denote the eigenvalues of \( A_G(\pm \infty) \), and \( l_j^\pm \) and \( r_j^\pm \) associated left and right eigenvectors, respectively, normalized so that \( l_j^\pm r_k^\pm = \delta_{jk}^j \). In case \( A_G^\pm \) is strictly hyperbolic, these are uniquely defined. In the general case, we require further that \( l_j^\pm, r_j^\pm \) be left and right eigenvectors also of \( P_j^\pm B_G^\pm P_j^\pm \), \( P_j^\pm := R_j^\pm L_j^\pm t \), where \( L_j^\pm \) and \( R_j^\pm \) denote \( m_j^\pm \times m_j^\pm \) left and right eigenblocks associated with the \( m_j^\pm \)-fold eigenvalue \( a_j^\pm \), normalized so that \( L_j^\pm R_j^\pm = I_{m_j^\pm} \). (Note: The matrix \( P_j^\pm B_G^\pm P_j^\pm \sim L_j^\pm B_G^\pm R_j^\pm \) is necessarily diagonalizable, by simultaneous symmetrizability of \( A_G, B_G \).)

Eigenvalues \( a_j(x) \), and eigenvectors \( l_j, r_j \) correspond to large-time convection rates and modes of propagation of the degenerate model (2.4). Likewise, let \( a_j^*(x), j = 1, \ldots, (n - r) \) denote the eigenvalues of

\[
A_G^*: = A_G,11 - A_G,12 B_G^{-1}_{12} B_G,21 = A_{11}(A_{11}^0)^{-1},
\]

and \( l_j^*(x), r_j^*(x) \in \mathbb{R}^{n-r} \) associated left and right eigenvectors, normalized so that \( l_j^* r_j^* \equiv \delta^j_k \). More generally, for an \( m_j^* \)-fold eigenvalue, we choose \( (n-r) \times m_j^* \) blocks \( \hat{L}_j^* \) and \( \hat{R}_j^* \) of eigenvectors satisfying the dynamical normalization

\[
L_j^* \partial_x R_j^* \equiv 0,
\]
along with the usual static normalization $L^*_j R_j \equiv \delta^*_k I_{m^*_j}$; as shown in Lemma 4.9, [MZ.1], this may always be achieved with bounded $L^*_j, R^*_j$. Associated with $L^*_j, R^*_j$, define extended, $n \times m^*_j$ blocks

$$L^*_j := \begin{pmatrix} L^*_j & 0 \\ -B^{-1}_{G,22} B_{G,21} R^*_j \end{pmatrix}, \quad \mathcal{R}^*_j := \begin{pmatrix} R^*_j \\ -B^{-1}_{G,22} B_{G,21} R^*_j \end{pmatrix}. \quad (3.5)$$

Eigenvalues $a^*_j$ and eigenmodes $L^*_j, \mathcal{R}^*_j$ correspond, respectively, to short-time hyperbolic characteristic speeds and modes of propagation for the reduced, hyperbolic part of degenerate system (2.4). Note that our discussion in the introduction of condition (H3) is validated by the second equality in (3.4), a nontrivial consequence of symmetry/block structure in the original system (1.1).

Define time-asymptotic, scalar diffusion rates

$$\beta^\pm_j := \left( l^t_j B_{Gj} \right) \pm, \quad j = 1, \ldots, n, \quad (3.6)$$

and local, $m_j \times m_j$ dissipation coefficients

$$\eta^*_j(x) := -l^t_j D^* r^*_j(x), \quad j = 1, \ldots, J \leq n - r, \quad (3.7)$$

where

$$D^* := A_{G,12} \left[ A_{G,21} - A_{G,22} B^{-1}_{G,22} B_{G,21} + A^*_j B^{-1}_{G,22} B_{G,21} \right], \quad (3.8)$$

is an effective dissipation precisely analogous to the effective diffusion predicted by formal, Chapman–Enskog expansion in the (dual) relaxation case. As described in Appendix A2 of [MZ.2], these quantities arise in a natural way, through Taylor expansion of the (frozen-coefficient) Fourier symbol

$$-i\xi A_G(x) - \xi^2 B_G(x) \quad (3.9)$$

of the linearized operator $L_G$ about $\xi = 0$ and $\xi = \infty$, respectively.

The important quantities $\eta^*_j, \beta_j$ were identified by Zeng [Ze.1, LZe.1] in her study by Fourier transform techniques of decay to constant (necessarily equilibrium) solutions $(\bar{u}, \bar{v}) \equiv (u\pm, v\pm)$ of relaxation systems, corresponding at the linearized level to the study of the limiting equations

$$U_t = L^\pm_G U := -A^\pm_G U_x + B^\pm_G U_{xx} \quad (3.10)$$

as $x \to \pm \infty$ of the linearized evolution equations (3.2). As a consequence of dissipativity, (A2), we have (see, e.g., [Kaw, LZe.1]) that

$$\beta^\pm_j > 0, \quad j = 1, \ldots, n \quad (3.11)$$

and

$$\text{Re} \sigma(\eta^*_j(x)) > 0, \quad j = 1, \ldots, J \leq n - r. \quad (3.12)$$
Proposition 3.2 [MZ.2]. For weak shock profiles, under assumptions (A1)–(A3), (H0)–(H3), the Green’s function $G(x, t; y)$ associated with the linearized evolution equations (3.2) may in the Lax or overcompressive case be decomposed as

$$G(x, t; y) = H + E + S + R,$$

where, for $y \leq 0$:

$$H(x, t; y) := \sum_{j=1}^{J} R^*_j(x) \zeta^*_j(y) \delta_{x-\bar{a}_j t}(-y) \mathcal{L}^*_j(y)$$

(3.14)

and

$$E(x, t; y) := \sum_{a_k^- > 0} \left[ e_k^0 \right] G'(x)_{k}^{-t} \left( \text{erf} \left( \frac{y + a^-_k t}{\sqrt{4 \beta^-_k t}} \right) - \text{erf} \left( \frac{y - a^-_k t}{\sqrt{4 \beta^-_k t}} \right) \right),$$

(3.15)

$$S(x, t; y) := \chi_{\{t \geq 1\}} \sum_{a_k^- < 0} r_k^- l_k^{-t} (4 \pi \beta^-_k t)^{-1/2} e^{-(x-y-a^-_k t)^2/4 \beta^-_k t}$$

$$+ \chi_{\{t \geq 1\}} \sum_{a_k^- > 0, a_j^- < 0} e_k^j \left[ r_j^- l_j^{-t} (4 \pi \beta^-_{jk} t)^{-1/2} e^{-(x-z^-_{jk} t)^2/4 \beta^-_{jk} t} \left( \frac{e^x}{e^x + e^{-x}} \right) \right]$$

$$+ \chi_{\{t \geq 1\}} \sum_{a_k^- > 0, a_j^+ > 0} e_k^j \left[ r_j^+ l_j^{-t} (4 \pi \beta^+_{jk} t)^{-1/2} e^{-(x-z^+_{jk} t)^2/4 \beta^+_{jk} t} \left( \frac{e^x}{e^x + e^{-x}} \right) \right],$$

(3.16)

denote hyperbolic, excited, and scattering terms, respectively, and $R$ denotes a faster decaying residual, satisfying:

$$R(x, t; y) = O(e^{-\eta(|x-y|+t)})$$

$$+ \sum_{k=1}^{n} \left( (t+1)^{-1/2} e^{-\eta x^+} + e^{-\eta |x|} \right) t^{-1/2} e^{-(x-y-a^-_k t)^2/M t}$$

$$+ \sum_{a_k^- > 0, a_j^- < 0} \chi_{\{a_k^- t \geq |y|\}} O((t+1)^{-1/2} t^{-1/2}) e^{-(x-a_j^- (t-|y/a^-_k|))^2/M t} e^{-\eta x^+},$$

$$+ \sum_{a_k^- > 0, a_j^+ > 0} \chi_{\{a_k^- t \geq |y|\}} O((t+1)^{-1/2} t^{-1/2}) e^{-(x-a_j^+ (t-|y/a^-_k|))^2/M t} e^{-\eta x^+},$$

$$+ \sum_{a_k^- > 0, a_j^- < 0} \chi_{\{a_k^- t \geq |y|\}} O((t+1)^{-1/2} t^{-1/2}) e^{-(x-a_j^- (t-|y/a^-_k|))^2/M t} e^{-\eta x^-},$$

$$+ \sum_{a_k^- > 0, a_j^+ > 0} \chi_{\{a_k^- t \geq |y|\}} O((t+1)^{-1/2} t^{-1/2}) e^{-(x-a_j^+ (t-|y/a^-_k|))^2/M t} e^{-\eta x^-},$$

$$+ \sum_{a_k^- > 0, a_j^- < 0} \chi_{\{a_k^- t \geq |y|\}} O((t+1)^{-1/2} t^{-1/2}) e^{-(x-a_j^- (t-|y/a^-_k|))^2/M t} e^{-\eta x^-},$$

$$+ \sum_{a_k^- > 0, a_j^+ > 0} \chi_{\{a_k^- t \geq |y|\}} O((t+1)^{-1/2} t^{-1/2}) e^{-(x-a_j^+ (t-|y/a^-_k|))^2/M t} e^{-\eta x^-}.$$
\[ R_y(x, t; y) = \sum_{j=1}^{J} \mathcal{O}(e^{-\eta t}) \delta_{x-a_j^+ t}(-y) + \mathcal{O}(e^{-\eta(|x-y|+t)}) \]
\[ + \sum_{k=1}^{n} \mathcal{O} \left( (t+1)^{-1/2} e^{-\eta x^+} + e^{-\eta|x|} \right) t^{-1} e^{-(x-y-a_k^- t)^2/Mt} \]
\[ + \sum_{a_k^- > 0, a_j^- < 0} \chi_{\{|a_k^- t| \geq |y|\}} \mathcal{O}((t+1)^{-1/2} t^{-1}) e^{-(x-a_j^- (t-|y/a_k^-|))^2/Mt} e^{-\eta x^+} \]
\[ + \sum_{a_k^- > 0, a_j^+ > 0} \chi_{\{|a_k^- t| \geq |y|\}} \mathcal{O}((t+1)^{-1/2} t^{-1}) e^{-(x-a_j^+ (t-|y/a_k^-|))^2/Mt} e^{-\eta x^-} , \]

\[ R_x(x, t; y) = \sum_{j=1}^{J} \mathcal{O}(e^{-\eta t}) \delta_{x-a_j^+ t}(-y) + \mathcal{O}(e^{-\eta(|x-y|+t)}) \]
\[ + \sum_{k=1}^{n} \mathcal{O} \left( (t+1)^{-1} e^{-\eta x^+} + e^{-\eta|x|} \right) t^{-1} (t+1)^{1/2} e^{-(x-y-a_k^- t)^2/Mt} \]
\[ + \sum_{a_k^- > 0, a_j^- < 0} \chi_{\{|a_k^- t| \geq |y|\}} \mathcal{O}(t+1)^{-1/2} t^{-1} e^{-(x-a_j^- (t-|y/a_k^-|))^2/Mt} e^{-\eta x^+} \]
\[ + \sum_{a_k^- > 0, a_j^+ > 0} \chi_{\{|a_k^- t| \geq |y|\}} \mathcal{O}(t+1)^{-1/2} t^{-1} e^{-(x-a_j^+ (t-|y/a_k^-|))^2/Mt} e^{-\eta x^-} , \]

and
\[ R_{xy}(x, t; y) = (\partial / \partial y) \left( \sum_{j=1}^{J} \mathcal{O}(e^{-\eta t}) \delta_{x-a_j^+ t}(-y) \right) \]
\[ + \sum_{j=1}^{J} \mathcal{O}(e^{-\eta t}) \delta_{x-a_j^+ t}(-y) + \mathcal{O}(e^{-\eta(|x-y|+t)}) \]
\[ + \sum_{k=1}^{n} \mathcal{O} \left( (t+1)^{-3/2} e^{-\eta x^+} + e^{-\eta|x|} \right) t^{-3/2} (t+1) e^{-(x-y-a_k^- t)^2/Mt} \]
\[ + \sum_{a_k^- > 0, a_j^- < 0} \chi_{\{|a_k^- t| \geq |y|\}} \mathcal{O}(t+1)^{-1/2} t^{-3/2} e^{-(x-a_j^- (t-|y/a_k^-|))^2/Mt} e^{-\eta x^+} \]
\[ + \sum_{a_k^- > 0, a_j^+ > 0} \chi_{\{|a_k^- t| \geq |y|\}} \mathcal{O}(t+1)^{-1/2} t^{-3/2} e^{-(x-a_j^+ (t-|y/a_k^-|))^2/Mt} e^{-\eta x^-} , \]

for some \( \eta, M > 0 \), where \( x^\pm \) denotes the positive/negative part of \( x \), and indicator
function $\chi_{t[a_k^{-}]t[|y|]}$ is one for $|a_k^{-}t| \geq |y|$ and zero otherwise. Symmetric bounds hold for $y \geq 0$.

Here, the averaged convection rates $\bar{a}_j^{+}(y,t)$ in (3.14) and (3.19)–(3.20) denote the time-averages over $[0,t]$ of $a_j^{+}(x)$ along characteristic paths $z_j^{+} = z_j^{+}(y,t)$ defined by

$$(3.21) \quad \frac{dz_j^{+}}{dt} = a_j^{+}(z_j^{+}), \quad z_j^{+}(0) = y,$$

and the dissipation matrix $\zeta_j^{+} = \zeta_j^{+}(y,t) \in \mathbb{R}^{m_j \times m_j}$ is defined by the dissipative flow

$$(3.22) \quad \frac{d\zeta_j^{+}}{dt} = -\eta_j^{+}(z_j^{+})\zeta_j^{+}, \quad \zeta_j^{+}(y) = I_{m_j}.$$

Similarly, in (3.16),

$$(3.23) \quad z_{jk}^{\pm}(y,t) := a_{jk}^{\pm} \left( t - \frac{|y|}{|a_k^{-}|} \right)$$

and

$$(3.24) \quad \bar{\beta}_{jk}^{\pm}(x,t;y) := \frac{|x^{\pm}|}{|a_{jk}^{\pm}t|} \beta_{jk}^{\pm} + \frac{|y|}{|a_k^{-}t|} \left( \frac{a_{jk}^{\pm}}{a_k^{-}} \right)^2 \beta_{k}^{-},$$

represent, respectively, approximate scattered characteristic paths and the time-averaged diffusion rates along those paths. In all equations, $a_j$, $a_j^{\pm}$, $l_j$, $L_j^{\pm}$, $r_j$, $R_j^{\pm}$, $\beta_j^{\pm}$ and $\eta_j^{+}$ are as defined just above, and scattering coefficients $[c_{j,i}^{\pm}]$, $i = -,0,+$, are constants, uniquely determined by

$$(3.25) \quad \sum_{a_j^{+}>0} [c_{j,-}^{\pm}]r_{jk}^{-} + \sum_{a_j^{-}<0} [c_{j,+}^{\pm}]r_{jk}^{+} + [c_{k,-}^{0}](G(+\infty) - G(-\infty)) = r_{k}^{-}$$

for each $k = 1, \ldots, n$, and satisfying

$$(3.26) \quad \sum_{a_k^{-}>0} [c_{k,-}^{0}]l_{k}^{-} = \sum_{a_k^{+}<0} [c_{k,+}^{0}]l_{k}^{+} = \pi,$$

where the constant vector $\pi$ is the left zero effective eigenfunction of $L_G$ associated with the right eigenfunction $G'$.

Proposition 3.2, the variable-coefficient generalization of the constant-coefficient results of [Ze.1,LZe.1], was established in [MZ.2] by Laplace transform (i.e., semigroup) techniques generalizing the Fourier transform approach of [Ze.1–2,LZe]; for discussion/geometric interpretation, see [Z.2,MZ.1–2]. In our stability analysis, we will use only a small part of the detailed information given in the proposition, namely $L^p \rightarrow L^q$ estimates on the time-decaying portion $H + S + R$ of the Green’s function $G$ (see Lemma 4.1, below). However, the stationary portion $E$ of the Green’s function must be estimated accurately for an efficient stability analysis.
Lemma 3.3. Under the assumptions of Proposition 3.2, there holds, additionally,
\[ \Pi_2(A^0(x))^{-1} H(x, t; y) \equiv 0, \]
where \( \Pi_2 := \begin{pmatrix} 0 & 0 \\ 0 & I_r \end{pmatrix} \) denotes projection onto the final \( r \) coordinates of \( G \).

Proof. Equivalently, we must show that \( \Pi_2(A^0)^{-1} R_j^* \equiv 0 \) for all \( 1 \leq j \leq J \).
This is most easily verified by the intrinsic property of \( L_j^* \) and \( R_j^* \) (readily seen from our formulae) that they lie, respectively, in the left and right kernel of \( B_G \).
For, this gives
\[ 0 \equiv B_G R_j^* := \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} (A^0)^{-1} R_j = b \Pi_2(A^0)^{-1} R_j^*, \]
yielding the result by invertibility of \( b \), (see condition (A3)). \( \blacksquare \)

Lemma 3.3 quantifies the observation that, in \( U = (u, v)^t \) coordinates, the “parabolic” variable \( v \) experiences smoothing under the evolution of (2.1), whereas the “hyperbolic” coordinate \( v \) does not. (Recall that \( U = (A^0)^{-1} G \).

Section 4. Linearized stability.

We now show, for the Lax case under consideration, that linearized orbital stability follows immediately from the pointwise bounds of Proposition 3.2, thus partially recovering the result of [MZ.2] that was stated in Proposition 3.1. This analysis motivates the nonlinear argument to follow in Section 5.

We carry out our analysis with respect to the conservative variable \( G = A^0(x)U \), i.e. with respect to linearized equations (3.2). Similarly as in [Z.2, MZ.1–2], define the linear instantaneous projection:
\[ \varphi(x, t) := \int_{-\infty}^{+\infty} E(x, t; y) G_0(y) \, dy =: -\delta(t) \tilde{G}'(x), \]
where \( G_0 \) denotes the initial data for (2.1), and \( \tilde{G} = G(\tilde{U}(x)) \) as usual. The amplitude \( \delta \) may be expressed, alternatively, as
\[ \delta(t) = -\int_{-\infty}^{+\infty} e(y, t) G_0(y) \, dy, \]
where
\[ E(x, t; y) =: \tilde{G}'(x) e(y, t), \]
i.e.,

\[ e(y, t) := \sum_{a_k} \left( \text{erf}(y + a_k t / \sqrt{4\beta_k t}) - \text{erf}(y - a_k t / \sqrt{4\beta_k t}) \right) l_k \]

for \( y \leq 0 \), and symmetrically for \( y \geq 0 \).

Then, the solution \( G \) of (2.1) satisfies

\[ G(x, t) - \varphi(x, t) = \int_{-\infty}^{+\infty} (H + \tilde{G})(x, t; 0)G_0(y) \, dy, \]

where

\[ \tilde{G} := S + R \]

is the regular part and \( H \) the singular part of the time-decaying portion of the Green’s function \( \mathcal{G} \).

**Lemma 4.1.** For weak shock profiles of (1.1), under assumptions (A1)–(A3), (H0)–(H3), there hold:

\[ | \int_{-\infty}^{+\infty} \tilde{G}(y, t; y) f(y) dy |_{L^p} \leq C(1 + t)^{-\frac{1}{2}(1-1/r)} |f|_{L^q}, \]

\[ | \int_{-\infty}^{+\infty} \tilde{G}_y(y, t; y) f(y) dy |_{L^p} \leq C(1 + t)^{-\frac{1}{2}(1-1/r)-1/2} |f|_{L^q} + C e^{-\eta t} |f|_{L^p}, \]

\[ | \int_{-\infty}^{+\infty} \tilde{G}_x(y, t; y) f(y) dy |_{L^p} \leq C(1 + t)^{-\frac{1}{2}(1-1/r)-1/2} |f|_{L^q} + C e^{-\eta t} |f|_{L^p}, \]

\[ | \int_{-\infty}^{+\infty} \tilde{G}_{xy}(y, t; y) f(y) dy |_{L^p} \leq C(1 + t)^{-\frac{1}{2}(1-1/r)-1/2} |f|_{L^q} + C e^{-\eta t} |f|_{W^{1,p}}, \]

and

\[ | \int_{-\infty}^{+\infty} H(y, t; y) f(y) dy |_{L^p} \leq C e^{-\eta t} |f|_{L^p}, \]

for all \( t \geq 0 \), some \( C, \eta > 0 \), for any \( 1 \leq q \leq p \) (equivalently, \( 1 \leq r \leq p \)) and \( f \in L^q \cap W^{1,p} \), where \( 1/r + 1/q = 1 + 1/p \).

**Proof.** Bounds (4.6)–(4.9) follow by the Hausdorff-Young inequality together with bounds (3.16) and (3.17)–(3.20), precisely as in [Z.2,MZ.1–2]. Bound (4.10) follows by direct computation and the fact that particle paths \( z_j(y, t) \) satisfy uniform bounds

\[ 1/C \leq |(\partial/\partial y) z_j| < C, \]

for all \( y, t \), by the fact that characteristic speeds \( a_j(x) \) converge exponentially as \( x \to \pm \infty \) to constant states.
Corollary 4.2. Let $\bar{U}$ be a weak shock profile of (1.1), under assumptions (A1)–(A3), (H0)–(H3). Then, strong spectral stability implies $L^1 \cap L^p \to L^p$ linearized orbital stability, for any $p > 1$. More precisely, for initial data $U_0 \in L^1 \cap L^p$, the solution $U = (u, v)^t(x, t)$ of (2.1) satisfies the linear decay bounds

\begin{equation}
|U(\cdot, t) + \delta(t)\bar{U}(\cdot)|_{L^p} \leq C(1 + t)^{-\frac{1}{2}(1-1/p)}(|U_0|_{L^1} + |U_0|_{L^p}).
\end{equation}

Moreover, provided $U_0 \in W^{1,p}$, there hold also the derivative bounds

\begin{equation}
|(v(\cdot, t) + \delta(t)\bar{v}(\cdot))_x|_{L^p} \leq C(1 + t)^{-\frac{1}{2}(1-1/p)}(|U_0|_{L^1} + |U_0|_{W^{1,p}}).
\end{equation}

Proof. For (4.11), it is equivalent to show that, for initial data $G_0 \in L^1 \cap L^p$, the solution $G(x, t)$ of (3.2) satisfies

\begin{equation}
|G(\cdot, t) - \varphi(\cdot, t)|_{L^p} \leq C(1 + t)^{-\frac{1}{2}(1-1/p)}(|G_0|_{L^1} + |G_0|_{W^{1,p}}).
\end{equation}

But, this follows immediately from (4.4) and bounds (4.6) and (4.10), with $q = p$. Likewise, for (4.12), it is equivalent to show that, for initial data $G_0 \in L^1 \cap W^{1,p}$, the solution $G(x, t)$ of (3.2) satisfies

\begin{equation}
|(\Pi_2(A^0)^{-1}(G(\cdot, t) - \varphi(\cdot, t)))_x|_{L^p} \leq C(1 + t)^{-\frac{1}{2}(1-1/p)}(|G_0|_{L^1} + |G_0|_{W^{1,p}}).
\end{equation}

This follows, similarly, from the derivative analog

\begin{equation}
(\Pi_2(A^0)^{-1}(G(x, t) - \varphi(x, t)))_x = \int_{-\infty}^{+\infty} (\Pi_2(A^0)^{-1}(H + \delta y))_x (x, t; 0)G_0(y) dy
\end{equation}
of (4.4), together with (3.27) and bounds (4.6), (4.8), with $q = p$. \]

Section 5. Nonlinear stability.

Finally, we establish our main result of nonlinear orbital stability with respect to perturbations $U_0 \in L^1 \cap H^3$ of weak viscous profiles (necessarily Lax type, by Proposition 1.1) of dissipative, symmetric hyperbolic–parabolic systems of type (1.1), of strength $\varepsilon > 0$ sufficiently small with respect to the parameters of the system in question. We follow the basic iteration scheme of [MZ.1,Z.2]; for precursors of this scheme, see [Go.2,K.1–2,LZ.1–2,ZH,ZH.1–2].

Define the nonlinear perturbation

\begin{equation}
U(x, t) := \bar{U}(x + \delta(t), t) - \bar{U}(x),
\end{equation}

where $\delta(t)$ (estimating shock location) is to be determined later; for definiteness, fix $\delta(0) = 0$. Substituting (5.1) into (1.1), we obtain

\begin{equation}
G(\bar{U})_t + F(\bar{U})_x - (B(\bar{U})\bar{U}_x)_x = \delta(t)G(\bar{U})_x,
\end{equation}

and thereby the basic perturbation equation

\begin{equation}
\left(G(\bar{U}) - G(\bar{U})_t \right) + \left(F(\bar{U}) - F(\bar{U})_t\right) + \left(B(\bar{U})\bar{U}_x - B(\bar{U})\bar{U}_x\right)_x = \delta(t)\dot{G}(\bar{U})_x,
\end{equation}

where $\bar{U}$ now denotes $\bar{U}(x + \delta(t), t)$ and $\bar{U}$ denotes $\bar{U}(x)$.

5.1. Energy estimates. We begin by establishing the following basic energy estimate, by which we will eventually close our nonlinear iteration argument:
Proposition 5.1. Let $U_0 \in H^3$, and suppose that, for $0 \leq t \leq T$, both the supremum of $|\dot{\delta}|$ and the $H^2 \cap W^{2,\infty}$ norm of the solution $U = (u,v)^t$ of (5.1)–(5.3) remain bounded by a sufficiently small constant $\zeta > 0$, for a small-amplitude shock profile as described in Proposition 1.1 of a system (1.1) satisfying (A1)–(A3), (H0)–(H3), with $\varepsilon := |\tilde{U}_+ - \tilde{U}_-|$ sufficiently small. Then, there hold the bounds:

$$
|U|_{H^3}(t), \int_0^t (|U_x|_{H^2} + |v_x|_{H^3})^2(s)ds \leq C \int_0^t (|U|_{L^\infty} + |U|_{L^2}^2 + \dot{\delta}^2)(s)ds,
$$

for all $0 \leq t \leq T$.

Remark 5.2. Note, with the expected decay rates $|U|_{L^\infty}, |\dot{\delta}(t)| \sim C(1 + t)^{-1/2}$, and $|U|_{L^2} \sim C(1 + t^{-1/4})$, that the righthand side of (5.4) becomes order $\log(1 + t)$, very nearly recovering the order one bound available in the constant-coefficient case [Kaw].

Lemma 5.3 ([SK]). Assuming (A1), condition (A2) is equivalent to either of:

(K1) There exists a smooth skew-symmetric matrix $K(u)$ such that

$$
\text{Re} \left( K(A^0)^{-1}A + B \right)(u) \geq \theta > 0.
$$

$A^0$, $A$, $B$ as in (2.2).

(K2) For some $\theta > 0$, there holds

$$
\text{Re} \sigma(-i\xi(A^0)^{-1}A(u) - |\xi|^2(A^0)^{-1}B(u)) \leq -\theta|\xi|^2/(1 + |\xi|^2),
$$

for all $\xi \in \mathbb{R}$.

Proof. These and other useful equivalent formulations are established in [SK].

Lemma 5.4. Let (A1)–(A3) and (H1)–(H3) hold, and let $\bar{U}(x - st)$ be a viscous shock solution such that the profile $\{\bar{U}(z)\}$ lies entirely within a sufficiently small neighborhood $V \subset U$ of $\tilde{U}_*$, and the speed $s$ lies within a sufficiently small neighborhood of $a_p(\tilde{U}_*)$: i.e., a profile as described in Proposition 1.1. Then, letting $\varepsilon := |\tilde{U}_+ - \tilde{U}_-|$ denote shock strength, we have for $q = 0, \ldots, 4$ the uniform bounds:

$$
|\partial^q_x \bar{U}(x)| \leq C \varepsilon^{q+1}e^{-\theta\varepsilon|x|},
$$

$$
|\partial^q_t \bar{U}|_{L^p} \leq C \varepsilon^{q+1 - 1/p},
$$

for some $C$, $\theta > 0$.

Proof. Though the bounds (5.7) are not explicitly stated in [Fre.2], they follow immediately from the detailed description of center manifold dynamics obtained in the proof, exactly as in the strictly parabolic case [MP].
Proof of Proposition 5.1. Writing (5.1) in quasilinear form
\[(\bar{A}^0 \bar{U}_t - \bar{A}^0 \bar{U}_t) + (\bar{A} \bar{U}_x - \bar{A} \bar{U}_x) - (\bar{B} \bar{U}_x - \bar{B} \bar{U}_x)_x = \dot{\delta}(t) \bar{A}^0 \bar{U}_x,\]
where
\[(\bar{A}^0 := A^0(\bar{U}), \bar{A}^0 := A^0(\bar{U}); \bar{A} := A(\bar{U}), \bar{A} := A(\bar{U}); \bar{B} := B(\bar{U}), \bar{B} := B(\bar{U}),\]
using the quadratic Leibnitz relation
\[A_2 U_2 - A_1 U_1 = A_2(U_2 - U_1) + (A_2 - A_1)U_1,\]
and recalling the block structure assumption (A3), we obtain the alternative perturbation equation:
\[(\bar{A}^0 \bar{U}_t + \bar{A} \bar{U}_x - (\bar{B} \bar{U}_x)_x = M_1(U) \bar{U}_x + (M_2(U) \bar{U}_x)_x + \dot{\delta}(t) \bar{A}^0 \bar{U}_x + \dot{\delta}(t) \bar{A}^0 \bar{U}_x,\]
where
\[M_1(U) := \bar{A} - \bar{A} = \left( \int_0^1 dA(\bar{U} + \theta U) \, d\theta \right) U,\]
and
\[M_2(U) := \bar{B} - \bar{B} = \begin{pmatrix} 0 & 0 \\ 0 & \left( \int_0^1 db(\bar{U} + \theta U) \, d\theta \right) U \end{pmatrix},\]
with
\[\bar{b} := b(\bar{U}), \bar{b} := b(\bar{U}).\]

We now carry out a series of successively higher order energy estimates of the type formalized by Kawashima [Ka.1]. The origin of this approach goes back to [K,MN] in the context of gas dynamics; see, e.g., [HoZ] for further discussion/references.

Let \( \bar{K} \) denote the skew-symmetric matrix described in Lemma 5.3 associated with \( \bar{A}^0, \bar{A}, \bar{B} \). Then, regarding \( \bar{A}^0, \bar{K}, \) we have the bounds
\[(\bar{A}^0_x = dA^0(\bar{U}) \bar{U}_x, \quad \bar{K}_x = dK(\bar{U}) \bar{U}_x, \quad \bar{A}_x = dA(\bar{U}) \bar{U}_x, \quad \bar{B}_x = dB(\bar{U}) \bar{U}_x, \quad \bar{A}^0_t = dA^0(\bar{U}) \bar{U}_t, \quad \bar{K}_t = dK(\bar{U}) \bar{U}_t, \quad \bar{A}_t = dA(\bar{U}) \bar{U}_t, \quad \bar{B}_t = dB(\bar{U}) \bar{U}_t, \]
and (from defining equations (5.1)–(5.2)):
\[|\bar{U}_x| = |U_x + \bar{U}_x| \leq |U_x| + |\bar{U}_x|\]
\(|\tilde{U}_t| \leq C(|\tilde{U}_x| + |\tilde{\nu}_{xx}| + |\tilde{\delta}|)|\tilde{U}_x|\)

\begin{equation}
(5.17)
\leq C(|U_x| + |\tilde{U}_x| + |v_{xx}| + |\tilde{\nu}_{xx}| + |\tilde{\delta}|)|U_x| + |\tilde{\delta}|)|\tilde{U}_x|\)
\end{equation}

\leq C(|U_x| + |\tilde{U}_x| + |v_{xx}| + |\tilde{\nu}_{xx}|).

Thus, in particular

\begin{equation}
(5.18)\quad |\tilde{\delta}|, |\tilde{A}_x^0|, |\tilde{K}_x|, |\tilde{\nu}_x|, |\tilde{B}_x|, |\tilde{A}_t|, |\tilde{K}_t|, |\tilde{\nu}_t|, |\tilde{B}_t| \leq C(\zeta + \varepsilon).
\end{equation}

\textbf{H}^1 \textit{estimate.} We first perform a standard, “Friedrichs-type” estimate for symmetric hyperbolic systems. Taking the \(L^2\) inner product of \(U\) against (5.11), we obtain after rearrangement/integration by parts, and several applications of Young’s inequality, the energy estimate

\begin{equation}
(5.19)\quad \frac{1}{2} \langle U, \tilde{A}^0 U \rangle_t = \langle U, \tilde{A}^0 U_t \rangle + \frac{1}{2} \langle U, \tilde{A}_x^0 U \rangle
\end{equation}

\begin{equation}
= -\langle U, \tilde{A} U_x \rangle + \langle U, (\tilde{B} U)_x \rangle_x + \langle U, M_1(U) \tilde{U}_x \rangle + \langle U, (M_2(U) \tilde{U}_x)_{xx} \rangle
\end{equation}

\begin{equation}
+ \tilde{\delta}(t) \langle U, \tilde{A}^0 U_x \rangle + \tilde{\delta}(t) \langle U, \tilde{A}_x^0 U \rangle + \frac{1}{2} \langle U, \tilde{A}_t^0 U \rangle
\end{equation}

\begin{equation}
= \frac{1}{2} \langle U, \tilde{A}_x U \rangle - \langle U_x, \tilde{B} U \rangle + \langle U, M_1(U) \tilde{U}_x \rangle - \langle U, M_2(U) \tilde{U}_x \rangle
\end{equation}

\begin{equation}
- \frac{1}{2} \tilde{\delta}(t) \langle U, \tilde{A}^0 U \rangle + \tilde{\delta}(t) \langle U, \tilde{A}_x^0 U \rangle + \frac{1}{2} \langle U, \tilde{A}_t^0 U \rangle
\end{equation}

\begin{equation}
\leq -\langle U_x, \tilde{B} U \rangle
\end{equation}

\begin{equation}
+ C \int \left( |U_x| |\tilde{U}_x| + |v_{xx}|) |U|^2 + |v_x||U||U_x| + |\tilde{\delta}| |U||\tilde{U}_x| \right)
\end{equation}

\begin{equation}
\leq -\langle U_x, \tilde{B} U \rangle
\end{equation}

\begin{equation}
+ C \int \left( |U_x|^2 + |U|^2 + |v_x|^2 + |v_{xx}|^2)(|U| + |\tilde{U}_x|) + |\tilde{\delta}|^2 |\tilde{U}_x| \right)
\end{equation}

\begin{equation}
\leq -\langle U_x, \tilde{B} U \rangle
\end{equation}

\begin{equation}
\quad + C \left( |U|_{L^\infty} (|U|_{L^\infty} + |U|^2_{L^2}) + |\tilde{\delta}|^2 \right) + C(\varepsilon + \zeta) \left( |U_x|^2_{L^2} + |v_{xx}|^2_{L^2} \right),
\end{equation}

where \(\varepsilon, \zeta > 0\) is as in the statement of the Proposition. Here, we have freely used the weak shock assumption and consequent bounds (5.7), as well as (5.15). (Note: we have also used in a crucial way the block-diagonal form of \(M_2\) in estimating \(|\langle U_x, M_2(U) \tilde{U}_x \rangle| \leq C \int |v_x||U||\tilde{U}_x|\) in the first inequality).

Likewise, differentiating (5.11), taking the \(L^2\) inner product of \(U_x\) against the resulting equation, and substituting the result into

\begin{equation}
(5.20)\quad \frac{1}{2} \langle U_x, \tilde{A}^0 U_x \rangle_t = \langle U_x, (\tilde{A}^0 U_t)_{xx} \rangle - \langle U_x, \tilde{A}_x^0 U_t \rangle + \frac{1}{2} \langle U_x, \tilde{A}_t^0 U_x \rangle,
\end{equation}
we obtain after an integration by parts:
\[ (5.21) \]
\[
\frac{1}{2} \langle U_x, \tilde{A}^0 U_x \rangle_t = -\langle U_x, (A U_x) \rangle + \langle U, (\tilde{B} U_x)_x \rangle + \langle U, M_1(U) \tilde{U}_x \rangle + \langle U, (M_2(U) \tilde{U}_x)_x \rangle \\
+ \delta(t) \langle U, \tilde{A}^0 U_x \rangle + \dot{\delta}(t) \langle U, \tilde{A}^0 \tilde{U}_x \rangle + \frac{1}{2} \langle U, \tilde{A}_t^0 U \rangle \\
= -\frac{1}{2} \langle U_x, \tilde{A} U_x \rangle - \langle U_{xx}, \tilde{B} U_{xx} \rangle - \langle U_{xx}, \tilde{B}_x U_x \rangle + \langle U_x, (M_1(U) \tilde{U}_x)_x \rangle \\
- \langle U_{xx}, (M_2(U) \tilde{U}_x)_x \rangle + \frac{1}{2} \delta(t) \langle U_x, \tilde{A}^0 U_x \rangle + \dot{\delta}(t) \langle U_x, \tilde{A}^0 \tilde{U}_x \rangle \\
+ \delta(t) \langle U_x, \tilde{A}_t^0 U_x \rangle - \langle U_x, \tilde{A}_x^0 U_t \rangle + \frac{1}{2} \langle U_x, \tilde{A}_t^0 U_x \rangle,
\]
which by (5.18), plus various applications of Young’s inequality yields the next-order energy estimate:
\[ (5.22) \]
\[
\frac{1}{2} \langle U_x, \tilde{A}^0 U_x \rangle_t \leq -\langle U_{xx}, \tilde{B} U_{xx} \rangle \\
+ C \int \left( |U|^2 + |\dot{\delta}|^2 \right) (|\tilde{U}_{xx}| + |\tilde{U}_x|) + (\varepsilon + \zeta) (|U_x|^2 + |U_x| v_{xx}) \right), \\
\leq -\langle U_{xx}, \tilde{B} U_{xx} \rangle \\
+ C \left( |U|^2_{L^\infty} + |\dot{\delta}|^2 \right) + C(\varepsilon + \zeta) (|U_x|^2_{L^2} + |v_{xx}|^2_{L^2}) \\
\leq -\frac{1}{2} \langle U_{xx}, \tilde{B} U_{xx} \rangle \\
+ C \left( |U|^2_{L^\infty} + |\dot{\delta}|^2 \right) + C(\varepsilon + \zeta)|U_x|^2_{L^2}.
\]

Next, we perform a nonstandard, “Kawashima-type” derivative estimate. Taking the $L^2$ inner product of $U_x$ against $\tilde{K}(\tilde{A}^0)^{-1}$ times (5.11), and noting that (integrating by parts, and using skew-symmetry of $\tilde{K}$)
\[ (5.23) \]
\[
\frac{1}{2} \langle U_x, \tilde{K} U \rangle_t = \frac{1}{2} \langle U_x, \tilde{K} U_t \rangle + \frac{1}{2} \langle U_{xt}, \tilde{K} U \rangle + \frac{1}{2} \langle U_x, \tilde{K}_t U \rangle \\
= \frac{1}{2} \langle U_x, \tilde{K} U_t \rangle - \frac{1}{2} \langle U_t, \tilde{K} U_x \rangle \\
- \frac{1}{2} \langle U_t, \tilde{K}_x U_x \rangle + \frac{1}{2} \langle U_x, \tilde{K}_t U \rangle \\
= \langle U_x, \tilde{K} U_t \rangle + \frac{1}{2} \langle U, \tilde{K}_x U_t \rangle + \frac{1}{2} \langle U_x, \tilde{K}_t U \rangle,
\]
we obtain by calculations similar to the above the auxiliary energy estimate:
\[ (5.24) \]
\[
\frac{1}{2} \langle U_x, \tilde{K} U \rangle_t \leq -\langle U_x, \tilde{K}(\tilde{A}^0)^{-1} \tilde{A} U_x \rangle \\
+ C(\zeta + \varepsilon)|U_x|^2_{L^2} + C(\zeta^{-1}|v_{xx}|^2_{L^2} + C(|U|^2_{L^\infty} + |\dot{\delta}(t)|^2).
Adding (5.19), (5.24), and (5.22) times a suitably large constant \( M > 0 \), and recalling (5.5), we obtain, finally:

\[
(5.25) \quad \frac{1}{2} \left( \langle U, \tilde{A}^0 U \rangle + \langle U_x, \tilde{K} U \rangle + M \langle U_x, \tilde{A}^0 U_x \rangle \right)_t \\
\leq -\theta(|U_x|^2_{L^2} + |v_x|^2_{L^2}) + C \left( |U|_{L^\infty}(|U|_{L^\infty} + |U|^2_{L^2}) + |\delta|^2 \right),
\]

\( \theta > 0 \), for any \( \zeta, \varepsilon \) sufficiently small, and \( M, C > 0 \) sufficiently large.

**Higher order estimates.** Performing the same procedure on the once- and twice-differentiated versions of equation (5.11), we obtain, likewise, the \( H^q \) estimates, \( q = 2, 3, \) of:

\[
(5.26) \quad \frac{1}{2} \left( \langle \partial_x^{q-1} U, \tilde{A}^0 \partial_x^{q-1} U \rangle + \langle \partial_x^{q-1} U_x, \tilde{K} \partial_x^{q-1} U \rangle + M \langle \partial_x^q U, \tilde{A}^0 \partial_x^q U \rangle \right)_t \\
\leq -\theta(|\partial_x^q U|^2_{L^2} + |\partial_x^{q+1} v|^2_{L^2}) \\
+ (\varepsilon + \zeta)|U_x|^2_{H^{q-2}} + C \left( |U|_{L^\infty}(|U|_{L^\infty} + |U|^2_{L^2}) + |\delta|^2 \right).
\]

We omit the calculations, which are entirely similar to those carried out already.

**Final estimate.** Summing our \( H^q \) estimates from \( q = 1 \) to \( 3 \), and telescoping the sum of the righthand sides, we thus obtain, for \( \varepsilon, \zeta \) sufficiently small:

\[
(5.27) \quad \sum_{q=1}^{3} \left( \langle \tilde{A}^0 \partial_x^{q-1} U, \partial_x^{q-1} U \rangle + \frac{1}{2} \langle \partial_x^q U, \tilde{K} \partial_x^{q-1} U \rangle + M \langle \tilde{A}^0 \partial_x^q U, \partial_x^q U \rangle \right)_t \\
\leq -\theta(|U_x|_{H^2} + |v_x|^2_{H^3}) + C \left( |U|_{L^\infty}(|U|_{L^\infty} + |U|^2_{L^2}) + |\delta|^2 \right),
\]

or, integrating from 0 to \( t \):

\[
(5.28) \quad \sum_{q=1}^{3} \left( \langle \tilde{A}^0 \partial_x^{q-1} U, \partial_x^{q-1} U \rangle + \frac{1}{2} \langle \partial_x^q U, \tilde{K} \partial_x^{q-1} U \rangle + M \langle \tilde{A}^0 \partial_x^q U, \partial_x^q U \rangle \right)_t \bigg|_0^t \\
\leq -\int_0^t \theta(|U_x|_{H^2} + |v_x|^2_{H^3})(s)ds \\
+ C \int_0^t \left( |U|_{L^\infty}(|U|_{L^\infty} + |U|^2_{L^2}) + |\delta|^2 \right)(s)ds.
\]

Noting that, for \( M \) sufficiently large, we have for each \( q \), by Young’s inequality, and positive definiteness of \( \tilde{A}^0 \):

\[
(5.29) \quad \langle \partial_x^{q-1} U, \tilde{A}^0 \partial_x^{q-1} U \rangle + \frac{1}{2} \langle \partial_x^q U, \tilde{K} \partial_x^{q-1} U \rangle + M \langle \partial_x^q U, \tilde{A}^0 \partial_x^q U \rangle (t) \geq \theta(|\partial_x^{q-1} U|_{L^2}^2 + |\partial_x^q U|^2_{L^2}),
\]
for some $\theta > 0$, we may rearrange (5.28) to obtain our ultimate goal:

$$\begin{align*}
|U|_{H^3}^2(t) + \int_0^t \theta(|U_x|_{H^2} + |v_x|_{H^2})(s)ds \\
\leq C|U|_{H^3}^2(0) + C \int_0^t \left( |U|_{L^\infty} + |U|_{L^2}^2 + |\dot{\delta}|^2 \right)(s)ds,
\end{align*}$$

(5.30)

from which the result immediately follows. \hfill \blacksquare

5.2. **Nonlinear iteration.** We now carry out the nonlinear iteration, following [Z.2, MZ.1]. For this stage of the argument, it will be convenient to work again with the conservative variable

$$G := G(\bar{U}) - G(\bar{U}),$$

(5.31)

writing (5.1) in the more standard form:

$$G_t - LG = N(G, G_x) + \dot{\delta}(t)(\bar{G}_x + G_x),$$

(5.32)

$$\bar{G} := G(\bar{U}),$$

where

$$N(G, G_x) = O(|G|^2 + |G||v_x|),$$

$$N(G, G_x)_x = O(|G|^2 + |G||v_x| + |G_x||v_x| + |G||v_{xx}|),$$

(5.33)

so long as $|G|, |G_x|$ remain bounded. Here, $\nu$ denotes the second coordinate of the alternative perturbation variable $U = (u, v)^t$ defined in (5.1).

By Duhamel’s principle, and the fact that

$$\int_{-\infty}^\infty G(x, t; y)\bar{G}_x(y)dy = e^{Lt}\bar{G}_x(x) = \bar{G}_x(x),$$

(5.34)

we have, formally,

$$G(x, t) = \int_{-\infty}^\infty G(x, t; y)G_0(y)dy$$

$$+ \int_0^t \int_{-\infty}^\infty G_y(x, t - s; y)(N(G, G_x) + \dot{\delta}G)(y, s)dyds$$

$$+ \delta(t)\bar{G}_x.$$  

(5.35)

Defining, by analogy with the linear case, the nonlinear instantaneous projection:

$$\varphi(x, t) := -\delta(t)\bar{G}_x$$

$$:= \int_{-\infty}^\infty E(x, t; y)G_0(y)dy$$

$$- \int_0^t \int_{-\infty}^\infty E_y(x, t - s; y)(N(G, G_x) + \dot{\delta}G)(y, s)dy,$$

(5.36)
or equivalently, the instantaneous shock location:

\[
\delta(t) = -\int_{-\infty}^{\infty} e(y, t) G_0(y) dy \\
+ \int_0^t \int_{-\infty}^{+\infty} e_y(y, t-s)(N(G, G_x) + \dot{\delta}G)(y, s) dy ds,
\]

(5.37)

where \(E, e\) are defined as in (3.15), (4.3), and recalling (4.5), we thus obtain the reduced equations:

\[
G(x, t) = \int_{-\infty}^{\infty} (H + \tilde{G})(x, t; y) G_0(y) dy \\
+ \int_0^t \int_{-\infty}^{+\infty} H(x, t-s; y)(N(G, G_x)(y, s) dy ds \\
- \int_0^t \int_{-\infty}^{\infty} \tilde{G}_y(x, t-\dot{s}; y)(N(G, G_x) + \dot{\delta}G)(y, s) dy ds,
\]

(5.38)

and, differentiating (5.37) with respect to \(t\),

\[
\dot{\delta}(t) = -\int_{-\infty}^{\infty} e(t, y) G_0(y) dy \\
+ \int_0^t \int_{-\infty}^{+\infty} e_{yt}(y, t-s)(N(G, G_x) + \dot{\delta}G)(y, s) dy ds.
\]

(5.39)

Note: In deriving (5.39), we have used the fact that \(e_y(y, s) \to 0\) as \(s \to 0\), as the difference of approaching heat kernels, in evaluating the boundary term

\[
\int_{-\infty}^{+\infty} e_y(y, 0)(N(G, G_x) + \dot{\delta}G)(y, t) dy = 0.
\]

(5.40)

(Indeed, \(|e_y(\cdot, s)|_{L^1} \to 0\), see Remark 2.6, below).

The defining relation \(\delta(t)\dot{u}_x := -\varphi\) in (5.36) can be motivated heuristically by

\[
\tilde{G}(x, t) - \varphi(x, t) \sim G = \begin{pmatrix} u \\ v \end{pmatrix} (x + \delta(t), t) - \begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} (x) \\
\sim \tilde{G}(x, t) + \delta(t) \tilde{G}_x(x),
\]

where \(\tilde{G}\) denotes the solution of the linearized perturbation equations, and \(G\) the background profile. Alternatively, it can be thought of as the requirement that the instantaneous projection of the shifted (nonlinear) perturbation variable \(G\) be zero, [HZ.1–2].
Lemma 5.5 [Z.2]. The kernel $e$ satisfies

\begin{align}
|e_y(\cdot, t)|_{L^p}, |e_t(\cdot, t)|_{L^p} &\leq C t^{-\frac{1}{2}(1-1/p)}, \\
|e_{ty}(\cdot, t)|_{L^p} &\leq C t^{-\frac{1}{2}(1-1/p)-1/2},
\end{align}

for all $t > 0$. Moreover, for $y \leq 0$ we have the pointwise bounds

\begin{align}
|e_y(y, t)|, |e_t(y, t)| &\leq C t^{-\frac{1}{2}} e^{-\frac{(y+a-\cdot)^2}{4Mt}}, \\
|e_{ty}(y, t)| &\leq C t^{-1} e^{-\frac{(y+a-\cdot)^2}{4Mt}},
\end{align}

for $M > 0$ sufficiently large (i.e. $> 4b_\pm$), and symmetrically for $y \geq 0$.

**Proof.** For definiteness, take $y \leq 0$. Then, (4.3) gives

\begin{align}
e_y(y, t) &= \left(\frac{1}{u_+ - u_-}\right) (K(y + a_-t, t) - K(y - a_-t, t)), \\
e_t(y, t) &= \left(\frac{1}{u_+ - u_-}\right) ((K + K_y)(y + a_-t, t) - (K + K_y)(y - a_-t, t)), \\
e_{ty}(y, t) &= \left(\frac{1}{u_+ - u_-}\right) ((K_y + K_{yy})(y + a_-t, t) - (K_y + K_{yy})(y - a_-t, t)),
\end{align}

where

\begin{equation}
K(y, t) := \frac{e^{-y^2/4b_-t}}{\sqrt{4\pi b_-t}}
\end{equation}

denotes an appropriate heat kernel. The pointwise bounds (5.43)–(5.44) follow immediately for $t \geq 1$ by properties of the heat kernel, in turn yielding (5.41)–(5.42) in this case. The bounds for small time $t \leq 1$ follow from estimates

\begin{align}
|K_y(y + a_-t, t) - K_y(y - a_-t, t)| &= \left| \int_{y+a_-t}^{y-a_-t} K_{yy}(z, t) dz \right| \\
&\leq C t^{-3/2} \int_{y+a_-t}^{y-a_-t} e^{-\frac{z^2}{4Mt}} dz \\
&\leq C t^{-1/2} e^{-\frac{(y+a-\cdot)^2}{4Mt}},
\end{align}
and, similarly,

\[ |K_{yy}(y + a_+, t) - K_{yy}(y - a_-, t)| = | \int_{-a_-}^{a_+} K_{yy}(z, t) \, dz | \]

(5.50)

\[ \leq Ct^{-2} \int_{y-a_-}^{y+a_+} e^{-\frac{(y-a_-t)^2}{4Mt}} \, dz, \]

\[ \leq Ct^{-1} e^{-\frac{(y+a_-t)^2}{4Mt}}. \]

The bounds for \( |e_y| \) are again immediate. Note that we have taken crucial account of cancellation in the small time estimates of \( e_t, e_{ty} \). □

**Remark 5.6:** For \( t \leq 1 \), a calculation analogous to that of (5.49) yields

\[ |e_y(y, t)| \leq C e^{-\frac{(y+a_-t)^2}{4Mt}}, \]

and thus \( |e(\cdot, s)|_{L^1} \to 0 \) as \( s \to 0 \).

With these preparations, we are ready to carry out our analysis:

**Proof of Theorem 1.3.** Define

\[ \zeta(t) := \sup_{0 \leq s \leq t, 2 \leq p \leq \infty} \left[ (|U(\cdot, s)|_{L^p} + |v_x(\cdot, s)|_{L^p})(1 + s)\frac{1}{p}(1 - \frac{1}{p}) + |\dot{\delta}(s)|(1 + s)^{1/2} + (|\delta(s)| + |U(\cdot, s)|_{H^2} + |U(\cdot, s)|_{W^2,\infty}) \right]. \]

(5.51)

We shall establish:

**Claim.** For all \( t \geq 0 \) for which a solution exists with \( \zeta \) uniformly bounded by some fixed, sufficiently small constant, there holds

(5.52)

\[ \zeta(t) \leq C_2(\zeta_0 + \zeta(t)^2). \]

From this result, it follows by continuous induction that, provided \( \zeta_0 < 1/4C_2 \), there holds

(5.53)

\[ \zeta(t) \leq 2C_2\zeta_0 \]

for all \( t \geq 0 \) such that \( \zeta \) remains small. By standard short-time theory/continuation, we find that the solution (unique, in this regularity class) in fact remains in \( H^2 \) for all \( t \geq 0 \), with bound (5.53), at once yielding existence and the claimed sharp \( L^p \) bounds, \( 2 \leq p \leq \infty \). Thus, it remains only to establish the claim above.

**Proof of Claim.** We must show that each of the quantities \( |U|_{L^p}(1 + s)^{\frac{1}{2}(1 - \frac{1}{p})}, |v_x|_{L^p}(1 + s)^{\frac{1}{2}(1 - \frac{1}{p})}, |\dot{\delta}|(1 + s)^{1/2}, |\delta|, |U|_{H^2}, \) and \( |U|_{W^2,\infty} \) are separately bounded by

(5.54)

\[ C(\zeta_0 + \zeta(t)^2), \]
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for some $C > 0$, all $0 \leq s \leq t$, so long as $\zeta$ remains sufficiently small.

Provided that we can establish the others, the final two bounds follow easily from
the energy estimates of Proposition 5.1. For, by bounds (5.54) on $|\dot{\delta}|$ and $|U|_{L^\infty}$, and
the assumption that $|U|_{H^2 \cap W^{2,\infty}} \leq \zeta$ remains small, we obtain from (5.4) that

\begin{equation}
|U|_{H^3}(s) \leq C_3(\zeta_0 + \zeta(t)^2) \log(1 + S),
\end{equation}

for $0 \leq s \leq t$. Interpolating with assumed bound

$|U|_{L^2}(s) \leq C_3(\zeta_0 + \zeta(t)^2)(1 + s)^{-1/4}$,

we obtain

\begin{equation}
|U|_{H^2}(s) \leq C_3(\zeta_0 + \zeta(t)^2)|U|_{H^2}^{2/6}|U|_{L^2}^{1/6}
\leq C_3(\zeta_0 + \zeta(t)^2)(1 + t)^{-1/25},
\end{equation}

and, by Sobolev estimate,

\begin{equation}
|U|_{W^{2,\infty}} \leq |U|_{H^2}^{1/2}|U|_{H^3}^{1/2} \leq C_3(\zeta_0 + \zeta(t)^2)(1 + t)^{-1/51},
\end{equation}

both better than claimed.

Thus, in order to establish the result, we have only to establish the remaining
bounds, on $|U|_{L^p}$, $|v_x|_{L^p}$, $|\dot{\delta}|$, and $|\delta|$. These will be carried out using the Green’s
function estimates of Lemma 4.1. Accordingly, we first convert the problem to
conservative, $G$ coordinates, via:

\textbf{Observation 5.7.} It is sufficient to establish corresponding bounds on $|G|_{L^p}$, $|v_{G,x}|_{L^p}$, $|\dot{\delta}|$, and $|\delta|$, where $v_G(x, t) := A^0(x)^{-1}G(x, t)$.

\textit{Proof.} We have

\begin{equation}
U = G^{-1}G^\prime - G^{-1}{\bar{G}} = (A^0)^{-1}_{\text{ave}}(x, t)G
\end{equation}

\begin{equation}
:= \left( \int_0^1 (A^0)^{-1}(\bar{U}(x) + \theta U(x, t))d\theta \right) G,
\end{equation}

where

\begin{equation}
|(A^0)^{-1}_{\text{ave}}(x, t) - A^0(x)^{-1}| \leq C|U| \leq C|G|
\end{equation}

and

\begin{equation}
|((A^0)^{-1}_{\text{ave}}(x, t) - A^0(x)^{-1})_x| \leq C(|\bar{U}_x| + |U_x|) \leq \zeta,
\end{equation}

whence

$|U|_{L^p} \leq C|G|_{L^p}$.
and
\[|U_x - ((A^0)^{-1}G)_x|_{L^p} \leq C(|G|_{L^p} |G_x|_{L^\infty} + \zeta |G|_{L^p}) \leq C\zeta |G|_{L^p},\]
from which the result easily follows.

By (5.38)–(5.39), we have
\[
|G|_{L^p}(t) \leq |\int_{-\infty}^{\infty} (H + \tilde{G})(x, t; y)G_0(y)dy|_{L^p} \\
+ |\int_{0}^{t} \int_{-\infty}^{\infty} H(x, t - s; y)(N(G, G_x) + \dot{\delta}G_x)(y, s)dy ds|_{L^p} \\
+ |\int_{0}^{t} \int_{-\infty}^{\infty} \tilde{G}_y(x, t - s; y)(N(G, G_x) + \dot{G})(y, s)dy ds|_{L^p} \\
=: I_a + I_b + I_c,
\]
\[
|v_{G,x}|_{L^p}(t) := |((A^0)^{-1}G)_x|_{L^p}(t) \leq |\int_{-\infty}^{\infty} \left((A^0)^{-1}\tilde{G}\right)_x(x, t; y)G_0(y)dy|_{L^p} \\
+ |\int_{0}^{t} \int_{-\infty}^{\infty} \left((A^0)^{-1}\tilde{G}_y\right)_x(x, t - s; y)(N(G, G_x) + \dot{G})(y, s)dy ds|_{L^p} \\
=: II_a + II_b,
\]
\[
|\dot{\delta}|(t) \leq |\int_{-\infty}^{\infty} e_{t}(y, t)G_0(y)dy| \\
+ |\int_{0}^{t} \int_{-\infty}^{+\infty} e_{yt}(y, t - s)(N(G, G_x) + \dot{\delta}G)(y, s)dy ds| \\
=: III_a + III_b,
\]
and
\[
|\dot{\delta}|(t) \leq |\int_{-\infty}^{\infty} e_{y}(y, t)G_0(y)dy| \\
+ |\int_{0}^{t} \int_{-\infty}^{+\infty} e_{y}(y, t - s)(N(G, G_x) + \dot{G})(y, s)dy ds| \\
=: IV_a + IV_b.
\]

We estimate each term in turn, following the approach of [Z.2,MZ.1]. The linear term \( I_a \) satisfies bound
\[
I_a \leq C\zeta_0 (1 + t)^{-\frac{1}{2}(1-1/p)},
\]
as already shown in the proof of Corollary 4.2. Likewise, applying the bounds of Lemma 4.1 together with (5.33) and definition (5.51), we have

\[
I_b = \left| \int_0^t \int_{-\infty}^{\infty} H(x, t - s; y)(N(G, G_x)x + \hat{\delta}G_x)(y, s)dy ds \right|_{L^p}
\]

\[
\leq C \int_0^t e^{-\eta(t-s)}(|G|_{L^\infty} + |v_{G,x}|_{L^\infty} + |\hat{\delta}|)|G|_{W^{2,p}}(s)ds
\]

\[
\leq C\zeta(t)^2 \int_0^t e^{-\eta(t-s)}(1 + s)^{-1/2}ds
\]

\[
\leq C\zeta(t)^2(1 + t)^{-1/2},
\]

and (taking \(q = 2\) in (4.7))

\[
I_c = \left| \int_0^t \int_{-\infty}^{\infty} \tilde{G}_y(x, t - s; y)(N(G, G_x) + \hat{\delta}G)(y, s)dy ds \right|_{L^p}
\]

\[
\leq C \int_0^t e^{-\eta(t-s)}(|G|_{L^\infty} + |v_{G,x}|_{L^\infty} + |\hat{\delta}|)|G|_{L^p}(s)ds
\]

\[
+ C \int_0^t (t - s)^{-3/4 + 1/2p}(|G|_{L^\infty} + |v_{G,x}|_{L^\infty} + |\hat{\delta}|)|G|_{L^2}(s)ds
\]

\[
\leq C\zeta(t)^2 \int_0^t e^{-\eta(t-s)}(1 + s)^{-\frac{1}{2}(1-1/p)-1/2}ds
\]

\[
+ C\zeta(t)^2 \int_0^t (t - s)^{-3/4 + 1/2p}(1 + s)^{-3/4}ds
\]

\[
\leq C\zeta(t)^2(1 + t)^{-\frac{1}{2}(1-1/p)}.
\]

Summing bounds (5.65)–(5.67), we obtain (5.54), as claimed, for \(2 \leq p \leq \infty\). The desired bounds on \(II_a\) and \(II_b\) follow by an identical calculation, once we notice that, in Lemma 4.1, \(((A^0)^{-1}\tilde{G})\) and \(((A^0)^{-1}\tilde{G})_y\) satisfy the same \(L^q \to L^p\) bounds as do \((H + \tilde{G})\) and \((H + \tilde{G})_y\), respectively.

Similarly, applying the bounds of Lemma 5.5 together with definition (5.51), we find that

\[
III_a = \left| \int_{-\infty}^{\infty} e_t(y, t)G_0(y)dy \right|
\]

\[
\leq \left| e_t(y, t) \right|_{L^\infty}(t)|G_0|_{L^1}
\]

\[
\leq C\zeta_0(1 + t)^{-1/2}
\]
and

\[ III_b = \left| \int_0^t \int_{-\infty}^{+\infty} e_{yt}(y, t - s) \left( N(G, G_x) + \hat{\delta} G \right)(y, s) dy ds \right| \]

\[ \leq \int_0^t |e_{yt}|_{L^2}(t - s)(|G|_{L^\infty} + |v_{G,x}|_{L^\infty} + |\hat{\delta}|)|G|_{L^2}(s) ds \]

\[ \leq C \zeta(t)^2 \int_0^t (t - s)^{-3/4}(1 + s)^{-3/4} ds \]

\[ \leq C \zeta(t)^2 (1 + t)^{-1/2}, \]

while

\[ IV_a = \left| \int_{-\infty}^{\infty} e(y, t) G_0(y) dy \right| \]

\[ \leq |e(y, t)|_{L^\infty}(t)|G_0|_{L^1} \]

\[ \leq C \zeta_0 \]

and

\[ IV_b = \left| \int_0^t \int_{-\infty}^{+\infty} e_y(y, t - s) \hat{\delta} G(y, s) dy ds \right| \]

\[ \leq \int_0^t |e_y|_{L^2}(t - s)(|G|_{L^\infty} + |v_{G,x}|_{L^\infty} + |\hat{\delta}|)|G|_{L^2}(s) ds \]

\[ \leq C \zeta(t)^2 \int_0^t (t - s)^{-1/4}(1 + s)^{-3/4} ds \]

\[ \leq C \zeta(t)^2. \]

Summing (5.68)–(5.69) and (5.70)–(5.71), we obtain (5.54) as claimed.

This completes the proof of the claim, and the result.

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