Modularity on vertex operator algebras arising from semisimple primary vectors

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Abstract

In this article, using an idea of the physics superselection principal, we study a modularity on vertex operator algebras arising from semisimple primary vectors. We generalizes the theta functions on vertex operator algebras and prove that the internal automorphisms do not change the genus one twisted conformal blocks.

1 Introduction

In the study of the physics superselection principal and its application in the theory of vertex operator algebras (VOAs), Li introduced the notion of semisimple primary vectors in [Li1]. Let $V$ be a VOA. A vector $u \in V_1$ is called a semisimple primary vector if it satisfies: (i) $L(n)u = 0$ for $n > 0$; (ii) $u(0)u = -\delta_{m,1}\langle u, u \rangle I$ for $m \geq 0$; (iii) $u(0)$ acts on $V$ semisimply with rational eigenvalues. The main feature of a semisimple primary vector $u$ is that it realizes a functor between distinct categories of irreducible $V$-modules. Define the Delta operator associated to $u$ by

$$\Delta(u, z) := z^{u(0)} \exp \left( - \sum_{n=1}^{\infty} \frac{u(n)}{n} (-z)^{-n} \right).$$

(1.1)

Let $g$ be an automorphism on $V$ of finite order and denote by $\sigma(u)$ an internal automorphism $e^{-2\pi\sqrt{-1}u(0)}$ on $V$. Assume that $g$ fixes $u$ so that we have $[g, \sigma(u)] = 1$. It is shown in [Li2] (see also [Li1]) that if $(W, Y_W(\cdot, z))$ is an irreducible $g$-twisted $V$-module, then $(W, Y_W(\Delta(u, z) \cdot, z))$ is an irreducible $g\sigma(u)$-twisted $V$-module. We present a new application of the Delta operators (1.1) to the conformal blocks in the orbifold theory.
Recall the theta functions on VOAs. They were introduced by Miyamoto in \[\text{M1}\] as a generalization of the theta functions on lattices. Let \(u, v\) be vectors in weight one subspace of \(V\) such that \(u_{(1)}v = 0\). On every \(V\)-module \(W\), Miyamoto defined the following formal power series in \([M1]\):

\[
Z_W(u; v; \tau) = e^{\pi \sqrt{-1} \langle u, v \rangle} \text{tr}_W \exp \left( 2\pi \sqrt{-1} u_{(0)} \right) q^{L(0) + u_{(0)} + \frac{1}{2} \langle u, u \rangle - \frac{1}{4} c},
\]

where \(\langle \cdot, \cdot \rangle\) is an invariant bilinear form on \(V\) such that \(u_{(1)}v = \langle u, v \rangle \mathbb{I}\), \(c\) is the central charge of \(V\) and \(q\) denotes \(e^{2\pi \sqrt{-1} \tau}\). Based on Zhu’s theory \([Z]\), Miyamoto proved the modularity of \(Z_W(u; v; \tau)\) in \([M1]\). Surprisingly, the modular transformation law of a theta function on a module is exactly the same as that of character of the module, even though a theta function carries some informations of automorphisms and hence those of twisted modules. On the other hand, using the Delta operators \((1.1)\), we can understand the \(\sigma(u)\) twisted module. Denote \((W, Y_u)\) from a view point of orbifold conformal field theory \([DLM2]\). Combining results in \([M1]\) and \([DLM2]\), in this article we will reveal the property of the internal automorphisms that they do not change the genus one twisted conformal blocks.

Let us explain our results more precisely. Let \(G\) be a finite abelian subgroup of \(\text{Aut}(V)\). Assume that \(V\) is \(C_2\)-cofinite and \(k\)-rational for all \(k \in G\). Let \(u, v\) be mutually commutative rational semisimple primary vectors (see Sec. 4.2) in \(V^G\). For each pair \((g, h)\) in \(G \times G\), denote by \(\{(W^i(g, h), \phi_i(h)) \mid i = 1, \ldots, N\}\) the complete set of inequivalent irreducible \(g\)-twisted \(h\)-stable \(V\)-modules, where \(\phi_i(h)\) are \((\text{fixed}) h\)-stabilizing automorphisms on \(W^i(g, h)\). Define the genus one twisted conformal block \(\mathcal{C}_1(g, h)\) associated to a pair \((g, h)\) in \(G \times G\) as the space of trace functions

\[
T_{W^i(g, h)}(a, \tau) := \text{tr}_{W^i(g, h)} z^{\omega(a)} Y(a, z) \phi_i(h) q^{L(0) - c/24}, \quad 1 \leq i \leq N.
\]

For \(\rho = (\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}) \in \text{SL}_2(\mathbb{Z})\), the following modular transformation is shown in \([DLM2]\):

\[
(\gamma \tau + \delta)^{-\omega[\rho]} T_{W^i(g, h)}(a, \rho \tau) = \sum_{j=1}^{N} A_{ij}(\rho, (g, h)) T_{W^j((g, h)^\rho)}(a, \tau), \quad (1.2)
\]

where \(A_{ij}(\rho, (g, h))\) are the constants independent of \(a\) and \(\tau\), \(\rho \tau\) denotes \((\alpha \tau + \beta)(\gamma \tau + \delta)^{-1}\) and \((g, h)^\rho\) denotes \((g^a h^\gamma, g^\beta h^\delta)\). The matrix \((A_{ij}(\rho, (g, h)))_{ij}\) defines a linear isomorphism from \(\mathcal{C}_1(g, h)\) to \(\mathcal{C}_1((g, h)^\rho)\). Define the Schur polynomial \(p_s(x_1, x_2, \ldots)\) in variables
$$x_1, x_2, \ldots, \text{by the equation:}$$

$$\exp \left( - \sum_{n=1}^{\infty} \frac{x_n}{n} (-z)^n \right) = \sum_{s=0}^{\infty} p_s(x_1, x_2, \ldots) z^{-s}.$$  

Then define a generalized theta function on $W^i(g, h)$ by

$$Z_{W^i(g, h)}(a; (u, v); \tau) := \sum_{s=0}^{\infty} \text{Tr}_{W^i(g, h)} \left\{ (p_s(u(1), u(2), \ldots) a)_{\text{wt}(a) + \lambda_u(a) - s - 1} \right\} \times e^{\pi \sqrt{-1} \text{I}(u, v)} \exp \left( 2\pi \sqrt{-1} \text{I}(0) \right) \phi_i(h)^{-1} q^{L(0)+u(0)+\frac{1}{2}(u, u)-c/24},$$  

where $a \in V, \lambda_u(a)$ is a scalar such that $u(0) a = \lambda_u(a) a$ and $q = e^{2\pi \sqrt{-1} \tau}$. Then our main theorem in this paper is the following:

**Theorem 1.** The generalized theta function $Z_{W^i(g, h)}(a; (u, v); \tau)$ converges on the upper half plane and gives a vector in $C_1(g \sigma(u), h \sigma(v))$ for each $1 \leq i \leq N$. Furthermore, we have the following modular transformation for $\rho = \left( \begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix} \right) \in \text{SL}_2(\mathbb{Z})$:

$$(\gamma \tau + \delta)^{-\text{wt}[a]} Z_{W^i(g, h)} \left( a; (u, v); \frac{\alpha \tau + \beta}{\gamma \tau + \delta} \right) = \sum_{i=1}^{N} A_{ij}(\rho, (g, h)) Z_{W^i((g, h)^\rho)}(a; (\alpha u + \gamma v, \beta u + \delta v); \tau),$$

where $A_{ij}(\rho, (g, h))$ are the constants given by (1.2) and independent of $a, u, v$ and $\tau$.

There is an interesting consequence of Theorem 1. For $\rho \in \text{SL}_2(\mathbb{Z})$, denote by $\Psi_{(g, h)}(\rho)$ the linear isomorphism between $C_1(g, h)$ and $C_1((g, h)^\rho)$ given as (1.2). Define a linear isomorphism $\Omega_{(g, h)}(u, v) : C_1(g, h) \rightarrow C_1(g \sigma(u), h \sigma(v))$ by $T_{W^i(g, h)}(a, \tau) \mapsto Z_{W^i(g, h)}(a; (u, v); \tau)$. Then we obtain the following corollary.

**Corollary 1.** For each $\rho = \left( \begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix} \right) \in \text{SL}_2(\mathbb{Z})$, the following diagram commutes:

$$\begin{array}{ccc}
C_1(g, h) & \xrightarrow{\Psi_{(g, h)}(\rho)} & C_1((g, h)^\rho) \\
\Omega_{(g, h)}(u, v) \downarrow & & \downarrow \Omega_{(g, h)^\rho}(\alpha u + \gamma v, \beta u + \delta v) \\
C_1(g \sigma(u), h \sigma(v)) & \xrightarrow{\Psi_{(g \sigma(u), h \sigma(v))}(\rho)} & C_1((g \sigma(u), h \sigma(v))^\rho).
\end{array}$$  

(1.4)

The corollary above says that the $\text{SL}_2(\mathbb{Z})$-transformation laws for two different genus one twisted conformal blocks $C_1(g, h)$ and $C_1(g \sigma(u), h \sigma(v))$ are the same so that the internal automorphisms do not changes the genus one twisted conformal blocks. This result
seems to suggest that this is true for higher genus conformal blocks, since internal automorphisms \( \sigma(u) \) and \( \sigma(v) \) are generated by the fields in \( V \) itself.

The main idea in the proof of Theorem 1 is to use the Delta operators (1.1) associated to semisimple primary vectors. The Delta operator realizes a functor between the category of weak modules and the category of admissible modules. For an admissible \( V \)-module \((W, Y_W(\cdot, z))\), a \( V \)-module \((W, Y_W(\Delta(u, z) \cdot, z))\) is not admissible in general. Namely, even if \( V \) is rational, we do not know whether \((W, Y_W(\Delta(u, z) \cdot, z))\) is completely reducible or not. Therefore, before we give the proof of Theorem 1, we investigate a relation between \( g \)-rationality and \( g \)-regularity. We extend the result on the spanning sets for weak modules in \([B]\) to the twisted modules. The following is a simple refinement of Lemma 2.4 of \([MB]\).

**Lemma 1.** Let \( V \) be a \( C_2 \)-cofinite VOA of CFT type and \( W \) a weak \( g \)-twisted \( V \)-module generated by one element \( w \). Then \( W \) is linearly spanned by

\[
\alpha_1^{n_1} \cdots \alpha_s^{n_s} w, \quad \alpha^i \in U, \quad n_1 < \cdots < n_s < T,
\]

where \( U \) is a finite dimensional subspace of \( V \) such that \( V = U + C_2(V) \) and \( T \) is a fixed number in \( \frac{1}{|g|} \mathbb{N} \).

It is worth mentioning that the repeat condition in \([B]\) is now removed by Lemma 1. As an application, we also extend the result in \([ABD]\) to the twisted case.

**Corollary 2.** Every \( g \)-rational \( C_2 \)-cofinite VOA of CFT type is actually \( g \)-regular.

This paper is organized as follows. In Section 2 we recall basic definitions. In Section 3 we extend the results in \([B]\) and \([ABD]\) to the twisted case. In Section 4.1 we review the theory on the modular invariance on rational VOAs and in Section 4.2 we review the theory on the physics superselection principal and semisimple primary vectors. Using these theories, we prove Theorem 1 above in Section 4.3. In Section 4.4 we discuss a relation between our theta functions and abelian coset models.

## 2 Preliminaries

In this paper, we mainly treat VOAs of CFT type.

**Definition 2.1.** A VOA \( V \) is called **CFT type** if it has a weight decomposition \( V = \oplus_{n=0}^{\infty} V_n \) without negative weights and its weight zero subspace is spanned by the vacuum, i.e. \( V_0 = \mathbb{C} \mathbf{1} \).
We review the definition of the twisted modules. Let \( g \) be an automorphism on \( V \) of finite order \(|g|\). Then we can decompose \( V \) as a direct sum of eigenspaces for \( g \):
\[
V = V^0 \oplus V^1 \oplus \cdots \oplus V^{|g|-1}, \quad \text{where } V^r := \{a \in V \mid ga = e^{2\pi i r / |g|} a\}.
\]

**Definition 2.2.** A weak \( g \)-twisted \( V \)-module is a vector space \( W \) with a linear map
\[
Y_M(\cdot, z) : a \in V \mapsto Y_M(a, z) = \sum_{n \in \frac{1}{|g|}\mathbb{Z}} a(n)z^{-n-1} \in \text{End}(W)[[z^\pm \frac{1}{|g|}]]
\]
(called the vertex operator on \( M \)) satisfying the following:

(i) \( Y_M(a, z) = \sum_{n \in \mathbb{Z}} a(n+a)z^{-n-1} \frac{1}{|g|} \) for \( a \in V^r \),

(ii) \( a(n)w = 0 \) for \( n \gg 0 \) where \( a \in V \) and \( w \in W \),

(iii) \( Y_M(1, z) = \text{id}_M \),

(iv) The following \( g \)-twisted Jacobi identity holds for \( a \in V^r \) and \( b \in V \):
\[
z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y_M(a, z_1)Y_M(b, z_2) - z_0^{-1}\delta\left(\frac{-z_1 + z_2}{z_0}\right) Y_M(b, z_2)Y_M(a, z_1)
= z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right)\left(\frac{z_1 - z_0}{z_2}\right)^{\frac{1}{|g|}} Y_M(Y_V(a, z_0)b, z_2).
\]

Let us recall two consequences of the twisted Jacobi identity. Let \( W \) be a weak \( g \)-twisted \( V \)-module. For \( a \in V^r \), \( b \in V^s \) and \( w \in W \), there exists \( k \in \mathbb{N} \) such that \( z^{k+\frac{1}{|g|}}Y_W(a, z)w \in W[[z]] \). By (iv), we can derive the following associativity (cf. [Li2]).
\[
(z_2 + z_0)^{k+\frac{1}{|g|}} Y_M(Y_V(a, z_0)b, z_2)w = (z_2 + z_0)^{k+\frac{1}{|g|}} Y_M(a, z_0 + z_2)Y_M(b, z_2)w.
\] (2.1)

Let \( A, B \) be subsets of \( V \) and \( X \) a subset of \( W \). Set \( A \cdot X = \langle a(n)x \mid a \in A, x \in X, n \in \frac{1}{|g|}\mathbb{Z} \rangle \).

Using (2.1), we can show the following associativity-like relation:
\[
A \cdot (B \cdot X) \subseteq (A \cdot B) \cdot X.
\] (2.2)

In particular, \( V \cdot w \) is a submodule of \( W \).

Another consequence of the twisted Jacobi identity is the iterate formula. On \( V \), there exists \( N \gg 0 \) such that \((z_1 - z_2)^NY_V(a, z_1)Y_V(b, z_2) = (z_1 - z_2)^NY_V(b, z_2)Y_V(a, z_1) \). Then the following formula holds on \( W \).
\[
(a_{(m)}b)_{(n+\frac{r}{|g|}+\frac{s}{|g|})} = \sum_{i=0}^{N} \sum_{j=0}^{\infty} (-1)^j \binom{-r/|g|}{i} \binom{m+i}{j} \times \left\{ a_{(m+i-j+\frac{r}{|g|})}b_{(n-i+j+\frac{s}{|g|})} - (-1)^{m+i}b_{(m+n-j+\frac{s}{|g|})}a_{(j+\frac{r}{|g|})} \right\}.
\] (2.3)
Definition 2.3. An admissible $g$-twisted $V$-module is a weak $g$-twisted $V$-module which carries a $\frac{1}{|g|}\mathbb{N}$-grading $M = \oplus_{n \in \mathbb{Z}} M(n)$ such that $a(m) M(n) \subseteq M(n + \text{wt}(a) - m - 1)$ for all $a \in V$.

Definition 2.4. An ordinary $V$-module is a weak $V$-module which carries a $\mathbb{C}$-grading $M = \oplus_{s \in \mathbb{C}} M_s$ such that:

(i) $\dim M_s < \infty$,
(ii) $M_{s+N} = 0$ for any fixed $s$ and sufficiently small integer $N$,
(iii) $L(0)w = sw = \text{wt}(w)w$ for $w \in M_s$.

It follows from definitions that every ordinary $g$-twisted $V$-module is an admissible $g$-twisted $V$-module. Also, it is shown in [DLMI] that an irreducible admissible $g$-twisted $V$-module is an irreducible ordinary $g$-twisted $V$-module.

Definition 2.5. A VOA $V$ is said to be $g$-rational if every admissible $g$-twisted $V$-module is a direct sum of irreducible admissible $g$-twisted $V$-modules. Also, $V$ is said to be $g$-regular if every weak $g$-twisted $V$-module is a direct sum of irreducible ordinary $g$-twisted $V$-modules. A 1-rational (resp. 1-regular) VOA is simply called rational (resp. regular).

Definition 2.6. For $n \geq 2$, set $C_n(W) = \langle a_{(-n)}w \mid a \in V, w \in W \rangle$. An ordinary $V$-module $W$ is said to be $C_n$-cofinite if $W/C_n(W)$ is of finite dimension.

There are many conjectures about rationality. We give some of them below.

(1) Rationality of orbifold VOA [DVVV]: if $V$ is rational then $V^G$ is also rational, where $G$ is a finite automorphism group acting on $V$ and $V^G$ denotes the fixed point subalgebra $\{a \in V \mid ga = a \text{ for all } g \in G\}$.

(2) Rationality and $C_2$-cofiniteness [ABD]: rationality provides $C_2$-cofiniteness.

(3) Rationality and regularity [DLM3]: every rational VOA is regular.

(4) Relation between rationality and $g$-rationality: if $V$ is rational, then $V$ is $g$-rational for any finite automorphism $g$.

Concerning to the conjecture (1) and (4), we prove the following.

Proposition 2.7. Let $V$ be a simple VOA and $g$ an automorphism on $V$ of finite order $|g|$. If the orbifold VOA $V^{(g)}$ is rational, then $V$ is $g$-rational.

Proof: Recall the associative algebras $A_{g,n}(V)$ introduced in [DLMI]. Since $V^{(g)}$ is rational, all $A_{1,n}(V^{(g)})$, $n \in \mathbb{N}$, are semisimple by [DLMI]. Then all $A_{g,n}(V)$, $n \in \frac{1}{|g|}\mathbb{N}$, are also semisimple because they are homomorphic images of semisimple algebras $A_{1,n}(V^{(g)})$. Therefore, by a theorem in [DLMI], $V$ is $g$-rational.
Remark 2.8. By the proposition above, if the conjecture (1) is true for arbitrary finite cyclic group \( G = \langle g \rangle \), then (4) will follow from (1).

Recently, Abe, Buhl and Dong proved the following remarkable theorem.

**Theorem 2.9.** ([ABD, Theorem 4.5]) Every rational \( C_2 \)-cofinite VOA of CFT type is regular.

This result will be generalized to \( g \)-twisted case in the next section.

### 3 Spanning set for twisted VOA-modules

Here we give brief generalizations of the results obtained in [B] and [ABD].

For a VOA \( V \) of CFT type, Gaberdiel and Neitzke showed the following theorem on a spanning set of \( V \).

**Theorem 3.1.** [GN] Let \( V \) be a \( C_2 \)-cofinite VOA of CFT type and write \( V = U + C_2(V) \) with \( \dim U < \infty \). Then \( V \) is spanned by vectors of the form \( \alpha_1(-n_1) \cdots \alpha_k(-n_k) \mathbf{1} \), \( n_1 > \cdots > n_k > 0 \) with each \( \alpha_i \in U \).

We generalize this theorem to weak \( g \)-twisted \( V \)-modules. First, we recall the \( g \)-twisted universal enveloping algebra \( U^g(V) \) of \( V \) in [DLM1]. As a tensor product of two vertex algebras \( \mathbb{C}[t^\pm \gamma] \) and \( V \), \( \hat{V} := \mathbb{C}[t^\pm \gamma] \otimes_C V \) carries a structure of a vertex algebra and \( g_V := \hat{V}/(\frac{d}{dt} \otimes 1 + 1 \otimes L(-1))\hat{V} \) forms a Lie algebra under the 0-th product induced from \( \hat{V} \). Define a linear isomorphism \( \hat{g} \) on \( \hat{V} \) by \( \hat{g}(t^n \otimes a) := e^{-2\pi \sqrt{-1} n t^n} \otimes ga \). Then \( \hat{g} \) defines an automorphism of a vertex algebra \( \hat{V} \) and hence it gives rise to an automorphism of a Lie algebra \( g_V \). Denote by \( g_V^g \) the \( \hat{g} \)-invariants of \( g_V \), which is a Lie subalgebra of \( g_V \). Then the \( g \)-twisted universal enveloping algebra \( U^g(V) \) is defined to be the universal enveloping algebra for \( g_V^g \). The algebra \( U^g(V) \) has a universal property such that for any weak \( g \)-twisted \( V \)-module \( M \), the mapping \( a(n) \in U^g(V) \mapsto a(n) = \text{Res}_{Y_M(a, z)} z^n \in \text{End}(M) \) gives a representation of \( U^g(V) \) on \( M \). It is clear that \( g_V^g \) is spanned by images of elements \( t^{n+\frac{r}{|g|}} \otimes a \) with \( a \in V^r \), \( 0 \leq r \leq |g|-1 \). We denote the image of \( t^{n+\frac{r}{|g|}} \otimes a \) in \( g_V^g \) by \( a(n+\frac{r}{|g|}) \).

By definition, we have the following commutator relation:

\[
[a(m), b(n)] = \sum_{i=0}^{\infty} \binom{m}{i} (a(i)b)(m+n-i).
\]

**Definition 3.2.** For a monomial \( x^1(n_1) \cdots x^k(n_k) \) in \( U^g(V) \), we define its **length** by \( k \), **degree** by \( \text{wt}(x^1) + \cdots + \text{wt}(x^k) \) and **weight** by \( \text{wt}(x^1) - n_1 - 1 + \cdots + (\text{wt}(x^k) - n_k - 1) \).
Let $W$ be a weak $g$-twisted $V$-module generated by one element $w \in W$. In this case, a linear map $\phi_w : x^1(m_1) \cdots x^k(m_k) \in U^g(V) \mapsto x^1_{(m_1)} \cdots x^k_{(m_k)} w \in W = V \cdot w$ gives a surjection.

The idea of the following assertion comes from M. Miyamoto [M3] Lemma 2.4.

**Lemma 3.3.** Let $V$ be a $C_2$-cosemifinite VOA of CFT type and $W$ a weak $g$-twisted $V$-module generated by a non-zero element $w$, i.e. $W = V \cdot w$. Let $U$ be a finite dimensional subspace of $V$ such that both $L(0)$ and $g$ act on $U$ and $V = U + C_2(V)$. Then the image $\phi_w(X) \in W$ of any monomial $X = x^1(m_1) \cdots x^k(m_k)$ in $U^g(V)$ can be expressed as a linear combination of images of monomials $\alpha^1(n_1) \cdots \alpha^k(n_k)$ in $U^g(V)$ such that $\deg \alpha^1(n_1) \cdots \alpha^k(n_k)$ is less or equal to $\deg X$, $\wt \alpha^1(n_1) \cdots \alpha^k(n_k) = \wt X$ and $n_1 < \cdots < n_k$, where $T$ is a fixed element in $\mathbb{Z}$ such that $\phi_w(\beta(m)) = 0$ for all $\beta \in U$ and $m \geq T$.

**Proof:** We divide the proof into several steps.

**Claim 1.** We can express the image $\phi_w(X)$ of any monomial $X = x^1(m_1) \cdots x^k(m_k) \in U^g(V)$ in the following form:

$$\phi_w(X) = \phi_w(A) + \phi_w(B),$$

where $A$ is a linear combination of monomials $\alpha^1(n_1) \cdots \alpha^k(n_k) \in U^g(V)$ with $\alpha^i \in U$ such that $\deg \alpha^1(n_1) \cdots \alpha^k(n_k) = \deg X$ and $\wt \alpha^1(n_1) \cdots \alpha^k(n_k) = \wt X$, and $B$ is a sum of monomials whose degrees are less than $\deg X$ and weights are equal to $\wt X$.

We prove the claim above by induction on $r = \deg X$. The case $r = 0$ is clear. Assume that the claim is true for $r - 1$. Without loss, we may assume that both $L(0)$ and $g$ act on $x^i$, $1 \leq i \leq k$ semisimply and none of them is the vacuum. Then, by inductive assumption, $\phi_w(x^2(m_2) \cdots x^k(m_k))$ can be expressed a linear combination of images of monomials as stated. Therefore, we may assume that $x^2, \cdots, x^k$ are contained in $U$. Since $V = U + C_2(V)$, we can write $x^1 = \alpha^1 + \sum_i \alpha^1_{(-2)} b^i$ with $L(0)$-homogeneous $\alpha^1 \in U$ and $a^i, b^i \in V$ such that $\wt(\alpha^1) = \wt(\alpha^1_{(-2)} b^i) = \wt(x^1)$. Then $X = \alpha^1(m_1)x^2(m_2) \cdots x^k(m_k) + \sum_i (\alpha^1_{(-2)} b^i)(m_1)x^2(m_2) \cdots x^k(m_k)$. Then using (23) we can rewrite the image of second term in the desired form because $\wt(\alpha^1) + \wt(b^i) < \wt(\alpha^1_{(-2)} b^i)$. This completes the proof of Claim 1.

**Claim 2.** Let $A = \alpha^1(m_1) \cdots \alpha^k(m_k) \in U^g(V)$ be a monomial with $\alpha^i \in U$ and $\sigma$ a permutation on the set $\{1, 2, \ldots, k\}$. Then we have the following equality in $W$:

$$\phi_w \left( \alpha^{\sigma(1)}(n_{\sigma(1)}) \cdots \alpha^{\sigma(k)}(n_{\sigma(k)}) \right) = \phi_w(A) + \phi_w(B),$$

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where $B$ is a sum of monomials whose degrees are less than $\deg A$ and weights are equal to $\wt A$.

Again we proceed by induction on $r = \deg A$. The case $r = 0$ is obvious. Assume that the assertion is correct for $\deg A = r - 1$. Then using the commutator formula (3.1) we can rearrange $A$ to be as asserted since $\wt(\alpha_i^j) < \wt(\alpha^i) + \wt(\alpha^j)$ for $p \geq 0$. Thus, Claim 2 holds.

**Claim 3.** Let $A = \alpha^1(m_1) \cdots \alpha^k(m_k) \in \mathcal{U}^p(V)$ be a monomial with $\alpha^i \in U$ and $m_1 \leq \cdots \leq m_k < T$. Then the image $\phi_w(A)$ of $A$ can be expressed in the following form:

$$\phi_w(A) = \phi_w(B) + \phi_w(C),$$

where $B$ is a sum of monomials $\beta^1(n_1) \cdots \beta^s(n_s)$ with $\beta^j \in U$ such that $n_1 < \cdots < n_s$, $s \leq k$, $\deg \beta^1(n_1) \cdots \beta^s(n_s) = \deg A$ and $\wt \beta^1(n_1) \cdots \beta^s(n_s) = \wt A$, and $C$ is a sum of monomials whose degrees are less than $\deg A$ and weights are equal to $\wt A$.

We show that if the assertion is not correct then keeping both degree and weight of $A$ we can make $m_1$ in a monomial $A$ infinitely larger. We define an ordering on $\mathbb{N} \times \mathbb{N}$. For $(r_1, s_1), (r_2, s_2) \in \mathbb{N} \times \mathbb{N}$, we define $(r_1, s_1) > (r_2, s_2)$ if $r_1 > r_2$, or $r_1 = r_2$ and $s_1 > s_2$. By this ordering, $\mathbb{N} \times \mathbb{N}$ becomes a well-ordered set and hence we can perform an induction on $(\deg A, \text{length}A) \in \mathbb{N} \times \mathbb{N}$. Clearly, the assertion is clear for $(\mathbb{N}, 0)$, $(\mathbb{N}, 1)$ and $(0, \mathbb{N})$. So we assume that the assertion is true for all elements in $\mathbb{N} \times \mathbb{N}$ smaller than $(r, s)$ with $r > 0$, $s > 0$. Then, by inductive assumption, we may assume that $m_2 < \cdots < m_k < T$. If $m_1 < m_2$, then we are done. So we have to consider the case $m_1 = m_2$ and the case $m_1 > m_2$. But, the following argument shows that the latter case can be reduced to the former case. Assume that $m_1 > m_2$. Then $A$ can be replaced by a linear combination of $A' = \alpha^1(m_2)\alpha^1(m_1)\alpha^3(m_3) \cdots \alpha^k(m_k)$ and monomials whose degrees are smaller than $\deg A$ and weights are the same as $\wt A$. Then applying Claim 1 and Claim 2 together with inductive assumption to $A'$, we can replace $A$ by a monomial $A'' = (\alpha^1)'(m'_1) \cdots (\alpha^k)'(m'_k)$ such that $(\alpha^1)' \in U$, $m'_1 > m_1$, $m'_1 \leq \cdots \leq m'_k < T$, $\deg A'' = \deg A$ and $\wt A'' = \wt A$. Then, repeating this procedure, we will reach the case $m_1 = m_2 < m_3 < \cdots < m_k$.

Now let us consider the case $m_1 = m_2 < m_3 < \cdots < m_k$. In this case, both $\alpha^1$ and $\alpha^2$ are contained in the same eigenspace, say $V^r$. Write $m_1 = n + \frac{r}{|g|}$. Using the iterate
formula (2.3) on \((\alpha_{(1)}^{1}\alpha^{2}(2m_1+1)\alpha^{3}(m_3)\cdots\alpha^{k}(m_k))\), we get

\[
\phi_w(\alpha^{1}(m_1)\alpha^{2}(m_1)\alpha^{3}(m_3)\cdots\alpha^{k}(m_k)) = \lambda \phi_w((\alpha_{(-1)}^{1}\alpha^{2})(2m_1+1)\alpha^{3}(m_3)\cdots\alpha^{k}(m_k))
\]
\begin{equation}
+ \sum_{i>0} \mu_i \phi_w(\alpha^{1}(m_1+i)\alpha^{2}(m_1-i)\alpha^{3}(m_3)\cdots\alpha^{k}(m_k))
\end{equation}
\begin{equation}
+ \sum_{i>0} \mu'_i \phi_w(\alpha^{2}(m_1+i)\alpha^{2}(m_1-i)\alpha^{3}(m_3)\cdots\alpha^{k}(m_k)) + \phi_w(X),
\end{equation}

where \(X\) is a sum of monomials whose degrees are less than \(\text{deg} A\) and weights are equal to \(\text{wt} A\). Note that in the expansion of \((\alpha_{(-1)}^{1}\alpha^{2})(2m_1+1)\), we can make the coefficient of \(\alpha^{1}(m_1)\alpha^{2}(m_1)\alpha^{3}(m_3)\cdots\alpha^{k}(m_k)\) non-zero by choosing suitable \(N\) in (2.3). The first term in the right-hand side of (2.3) has smaller length than that of \(A\) so that by induction together with Claim 1 and Claim 2 we may omit this term. The second and third terms in the right-hand side of (3.2) shall be reduced to the case \(m_1 > m_2\). Therefore, we obtain a procedure which makes \(m_1\) infinitely larger with keeping \(\text{deg} A\) and \(\text{wt} A\), which must stop in finite steps. Thus, we get Claim 3 and hence we complete the proof of the Lemma 3.3.

Remark 3.4. Even if \(U\) is not finite dimensional, the lemma above still holds when we can take a \(T \in \frac{1}{|g|}\mathbb{Z}\) such that \(\phi_w(\beta(m)) = 0\) for all \(\beta \in U, m \geq T\).

Remark 3.5. By Lemma 3.3 we can remove the repeat condition in [B].

Remark 3.6. There is another proof of Lemma 3.3 in [NT]. See the proof of Theorem 3.2.7 of [NT].

Now we can generalize Theorem 4.5 of [ABD].

**Theorem 3.7.** Every \(g\)-rational \(C_2\)-cofinite VOA of CFT type is \(g\)-regular.

**Proof:** The proof is almost the same as that of Theorem 4.5 of [ABD]. The main idea in the proof of Theorem 4.5 of [ABD] is to show that every weak module has a non-trivial lowest weight vector. By Lemma 3.3 we can find a non-zero lowest weight vector in every weak \(g\)-twisted module. Thus, applying the argument in [ABD] we get the assertion.

There are several corollaries of Lemma 3.3.

**Corollary 3.8.** Let \(V\) be a \(C_2\)-cofinite VOA of CFT type. Then every irreducible weak \(g\)-twisted \(V\)-module \(W\) is an irreducible ordinary \(g\)-twisted \(V\)-module.

**Proof:** By Lemma 3.3 we can introduce a \(\frac{1}{|g|}\mathbb{Z}\)-grading on \(W\). Therefore, every irreducible weak \(g\)-twisted module is exactly an irreducible admissible \(g\)-twisted module. Since every irreducible admissible module is an ordinary module, we get the assertion.
Corollary 3.9. Let $V$ be a $C_2$-cofinite VOA of CFT type. Then every weak $g$-twisted $V$-module is admissible.

Proof: By Proposition 3.6 of [DLM2], the $g$-twisted Zhu algebra $A_g(V)$ (see [DLM1]) is finite dimensional. Then the argument in the proof of Proposition 5.6 of [ABD] with suitable modification leads to the assertion. 

4 Generalized theta functions on VOA-modules

4.1 Modular invariance of trace functions

Let $V$ be a VOA and let $g$ and $h$ be mutually commutative automorphisms on $V$. A $g$-twisted $V$-module $W$ is said to be $h$-stable if there exists a linear isomorphism $\phi_W(h)$ on $W$ such that

$$\phi_W(h)Y_W(a, z) = Y_W(ha, z)\phi_W(h)$$

for all $a \in V$. A linear isomorphism $\phi_W(h)$ is called $h$-stabilizing automorphism or simply stabilizing automorphism on $W$. For an ordinary $g$-twisted $h$-stable $V$-module $W$, we can consider the following $q$-trace.

$$T_W(a, \tau) := \text{tr}_W z^{\text{wt}(a)}Y_W(a, z)\phi_W(h)^{-1}q^{L(0)-c/24}, \quad (4.1)$$

where $q = e^{2\pi \sqrt{-1}\tau}$ and $c$ denotes the central charge of $V$. Zhu [Z] proved that the space spanned by the trace functions above is invariant under the action of the modular group $SL_2(\mathbb{Z})$ in the case $g = h = 1$ and Dong, Li and Mason [DLM2] generalized his result to the case where $g$ and $h$ generate a finite abelian subgroup in Aut($V$). Before we state their results, we have to introduce a structure transformation of vertex operator algebras.

Definition 4.1. ([Z Theorem 4.2.1]) Let $(V, Y(\cdot, z), 1, \omega)$ be a VOA. For each homogeneous $a \in V$, the vertex operator

$$Y[a, z] := e^{z\text{wt}(a)}Y(a, e^z - 1) = \sum_{n \in \mathbb{Z}} a_{[n]} z^{-n-1} \in \text{End}(V)[[z, z^{-1}]]$$

provides another VOA structure on $V$ with the same vacuum vector $1$ and a new Virasoro vector $\tilde{\omega} := \omega - (c/24)1$. We write $Y[\tilde{\omega}, z] = \sum_{n \in \mathbb{Z}} L[n] z^{-n-2}$ and denote $L[0]$-weight subspaces by $V_{[n]} = \{a \in V \mid L[0]a = na\}$. Also, we use $\text{wt}[a]$ to denote the $L[0]$-weight of $a \in V$.

We use the following notation. For $\rho = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(\mathbb{Z})$, $(g, h)^\rho$ denotes $(g^\alpha h^\gamma, g^\beta h^\delta)$ and $\rho \tau$ denotes $(\alpha \tau + \beta)(\gamma \tau + \delta)^{-1}$ for $\tau \in \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}$. 

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Theorem 4.2. ([DLM2]) Let $V$ be a $C_2$-cofinite VOA and let $g, h$ be automorphisms on $V$ generating a finite abelian subgroup in $\text{Aut}(V)$. Then the trace functions (4.1) defined on irreducible $g$-twisted $h$-stable $V$-modules converge to linearly independent holomorphic functions on the upper half plane. Denote by $C_1(g, h)$ the linear space spanned by the trace functions $T_W(a, \tau)$, where $W$ runs over irreducible $g$-twisted $h$-stable $V$-modules.

For $\rho = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$ and $T_W(a, \tau) \in C_1(g, h)$, define

$$(T_W)^\rho(a, \tau) := (\gamma \tau + \delta)^{-\text{wt}[a]} T_W(a, \rho \tau).$$

(4.2)

If $V$ is both $g$-rational and $g^\alpha h^\gamma$-rational, then $\rho$ defines a linear isomorphism from $C_1(g, h)$ to $C_1((g, h)^\rho)$.

The space $C_1(g, h)$ is called a genus one twisted conformal block.

Remark 4.3. This theorem has been generalized to involve intertwining operators in [M2] and [Y].

4.2 Semisimple primary vectors

Here we review the theory of the physics superselection principal and semisimple primary vectors in [Li1].

Definition 4.4. A vector $u \in V$ is called a semisimple primary vector if it satisfies the following.

(i) $L(n)u = \delta_{n,0} h$ for $n \geq 0$.

(ii) $u^{(m)}u = \delta_{m,1} \gamma 1$ for $m \geq 0$ and some $\gamma \in \mathbb{Q}$.

(iii) $u^{(0)}$ acts on $V$ semisimply.

Since $u^{(0)}$ is a derivative operator and keeps each homogeneous subspace of $V$, its exponential operator $\exp(\alpha u^{(0)})$ gives an automorphism of $V$ for any $\alpha \in \mathbb{C}$. In the following, we denote $\exp(-2\pi \sqrt{-1} u^{(0)})$ by $\sigma(u)$. If all eigenvalues of $u^{(0)}$ on $V$ is contained in $\frac{1}{T} \mathbb{Z}$ for some $T \in \mathbb{Z}$, then $\sigma(u)$ have a finite order. We call such a semisimple primary vector rational.

Let $u$ be a rational semisimple primary vector and $g$ an automorphism of $V$ of finite order such that $gu = u$ (so $g\sigma(u) = \sigma(u)g$). Define

$$\Delta(u, z) := z^{u^{(0)}} \exp \left( - \sum_{n=1}^{\infty} \frac{u^{(n)}}{n} (-z)^{-n} \right).$$

(4.3)

Since $u^{(0)}$ acts on $V$ semisimply, $\Delta(u, z)$ is a well-defined operator on $V$. Let $(W, Y_W(\cdot, z))$ be a weak $g$-twisted $V$-module. The following proposition is due to Li [Li2] (see also [Li1]).
Proposition 4.5. ([Li2, Proposition 5.4]) \((W, Y_W(\Delta(u, z) \cdot, z))\) is a weak \(g\sigma(u)\)-twisted \(V\)-module.

Let us denote \((W, Y_W(\Delta(u, z) \cdot, z))\) simply by \(\tilde{W}\). We can write the action of \(a \in V\) on \(\tilde{W}\) in the following way. By the assertion above, there exists a linear isomorphism \(\varphi_W : W \to \tilde{W}\) such that \(Y_W(a, z) \varphi_W = \varphi_W Y_W(\Delta(u, z) a, z)\) for all \(a \in V\). Define the Schur polynomials \(p_s(x_1, x_2, \ldots)\) in variables \(x_1, x_2, \ldots\) by the following equation:

\[
\exp \left( - \sum_{n=1}^{\infty} \frac{x_n}{n} (-z)^{-n} \right) = \sum_{s=0}^{\infty} p_s(x_1, x_2, \ldots) z^{-s}. \tag{4.4}
\]

Assume that \(u(0)a = \lambda a\) for some \(\lambda \in \mathbb{Q}\). Then the vertex operator of \(a\) on \(\tilde{W}\) is given as follows:

\[
Y_{\tilde{W}}(a, z) \varphi_W = \varphi_W Y_W(\Delta(u, z) a, z) = \varphi_W \sum_{s=0}^{\infty} z^{-s+\lambda} Y_W(p_s(u(1), u(2), \ldots) a, z). \tag{4.5}
\]

The Delta operator \(\Delta(u, z)\) has an additive property. A pair of semisimple primary vectors \(u\) and \(v\) such that \(u(0)v = 0\) is called mutually commutative because we have \(\Delta(u, z) \Delta(v, z) = \Delta(v, z) \Delta(u, z) = \Delta(u + v, z)\). In particular, \(u\) is commutative with itself so that \(\Delta(u, z)\) is invertible because \(\Delta(u, z) \Delta(-u, z) = \Delta(0, z) = \text{id}_V\). The following statement is easy.

Proposition 4.6. We have a bijective correspondence between the set of irreducible \(g\)-twisted \(V\)-modules and the set of irreducible \(g\sigma(u)\)-twisted modules through the Delta operator \(\Delta(u, z)\). Furthermore, if an automorphism \(h\) on \(V\) is commutative with \(g\) and acts on \(u\) trivially, then the set of irreducible \(g\)-twisted \(h\)-stable \(V\)-modules and the set of irreducible \(g\sigma(u)\)-twisted \(h\)-stable \(V\)-modules are in one-to-one correspondence.

We will need the following lemma.

Lemma 4.7. Let \(V\) be a \(g\)-rational \(C_2\)-cofinite VOA of CFT type. Then \(V\) is \(g\sigma(u)\)-regular for every rational semisimple primary vector \(u\).

Proof: In this case, \(V\) is \(g\)-regular by Theorem 3.7. Let \((W, Y_W(\cdot, z))\) be a weak \(g\sigma(u)\)-twisted \(V\)-module. Then \((W, Y_W(\Delta(-u, z) \cdot, z))\) is a weak \(g\)-twisted \(V\)-module. Since \(V\) is \(g\)-regular, \((W, Y_W(\Delta(-u, z) \cdot, z))\) is a direct sum of irreducible \(g\)-twisted \(V\)-modules. Then \((W, Y_W(\Delta(u, z) \Delta(-u, z) \cdot, z)) = (W, Y_W(\cdot, z))\) is also a direct sum of irreducible \(g\sigma(u)\)-twisted \(V\)-modules.
4.3 Main Theorem

In the following context, we will work over the following setting.

1. $V$ is a $C_2$-cofinite vertex operator algebra of CFT type.

2. $L(1)V_1 = 0$.

3. $g$ and $h$ are automorphisms on $V$ generating a finite abelian subgroup in $\text{Aut}(V)$.

4. $V$ is $k$-rational for all $k \in \langle g, h \rangle$.

5. $H$ is a set of mutually commutative rational semisimple primary vectors in $V^{(g,h)}$, where $V^{(g,h)}$ denotes the fixed point sub VOA under $\langle g, h \rangle$.

6. For $u \in H$, $\sigma(u)$ denotes $\exp(-2\pi \sqrt{-1}u(0))$ and $\lambda_u(a)$ is a linear function on $V$ defined as $u(0)a = \lambda_u(a)a$ for $a \in V$.

7. $E(H) = \{\sigma(u) \mid u \in H\}$, an abelian subgroup of $\text{Aut}(V)$.

We make some remarks on the assumption above. By (2), $V$ possesses the unique invariant bilinear form $\langle \cdot, \cdot \rangle$ such that $\langle 1, 1 \rangle = -1$ (cf. [Li3]). Note that this bilinear form satisfies $a(1) b = \langle a, b \rangle 1$ for $a, b \in V_1$. The assumption (4) is satisfied if $V^{(k)}$ is rational for all $k \in \langle g, h \rangle$ by Proposition 2.7. All $\sigma(\rho u)$ with $u \in H$ are finite automorphisms on $V$ for any $\rho \in \mathbb{Q}$. Therefore, $H$ forms a $\mathbb{Q}$-vector space in $V_1$. Since $H$ is contained in $V^{(g,h)}$, we have $[\langle g, h \rangle, E(H)] = 1$ in $\text{Aut}(V)$.

Let $k_1, k_2 \in \langle g, h \rangle$. Since $V$ is $k_1$-rational, there are finitely many irreducible inequivalent $k_1$-twisted $k_2$-stable $V$-modules. We denote the complete set of inequivalent irreducible $k_1$-twisted $k_2$-stable $V$-modules by $\{(W^i(g,h), \phi_{i,k_1}(k_2)) \mid i = 1, 2, \ldots, N = N(k_1, k_2)\}$, where $\phi_{i,k_1}(k_2)$ are (fixed) $k_2$-stabilizing automorphisms on $W^i(k_1, k_2)$. Note that the number $N(k_1, k_2)$ of irreducible $k_1$-twisted $k_2$-stable $V$-modules is the same as that of irreducible $k_1^\alpha k_2^\gamma$-twisted $k_1^\beta k_2^\delta$-stable $V$-modules for all $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ by Theorem 4.2. Recall the genus one twisted conformal block $C_1(g,h)$ which is a linear span of $q$-traces

$$T_{W^i(g,h)}(a, \tau) = \text{tr}_{W^i(g,h)} Y(a, z) \phi_{i,g}(h)^{-1} q^{L(0) - c/24}, \ 1 \leq i \leq N.$$ 

For $\rho = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$, we have the following transformation:

$$T_{W^i(g,h)}(a, \rho \tau) = (\gamma \tau + \delta)^{\text{wt}[a]} \sum_{j=1}^{N} A_{ij}(\rho, (g,h)) T_{W^j((g,h)\rho)}(a, \tau), \quad (4.6)$$

where the constants $A_{ij}(\rho, (g,h))$ are given by Theorem 4.2 and independent of $a$ and $\tau$.  

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Definition 4.8. For \( u, v \in H \) and \( a \in V \), define
\[
Z_{W^i(g,h)}(a; (u,v); \tau) := \sum_{s=0}^{\infty} \Tr_{W^i(g,h)} \left\{ (p_s(u(1), u(2), \ldots))a \right\}_{\text{wt}(a) + \lambda_a(a) - s - 1}
\times e^{2\pi \sqrt{-1}(u,v)} \exp \left( 2\pi \sqrt{-1}v(0) \right) \phi_{i,g}(h)^{-1}q^{\text{wt}(0) + u(0) + \frac{1}{2}(u,a) - c/24},
\]
where \( q \) denotes \( e^{2\pi \sqrt{-1} \tau} \), \( p_s(x_1, x_2, \ldots) \) is the Schur polynomial defined by (4.4) and \( \lambda_a(a) \) is a scalar such that \( u(0)a = \lambda_a(a)u \). We call \( Z_{W^i(g,h)}(a; (u,v); \tau) \) a generalized theta function on \( W^i(g,h) \) with respect to \( H \).

Remark 4.9. In [M1], Miyamoto defined the function above in the case when \( g = h = 1 \) and \( a = \mathbb{1} \) and he called \( Z_{W^i(1,1)}(\mathbb{1}; (0, v); \tau)\eta(\tau)^c \) a theta function of \( W^i(1,1) \), where \( \eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \) is the Dedekind eta function.

We consider the modular transformations of \( Z_{W^i(g,h)}(a; (u,v); \tau) \). Our main theorem is the following.

Theorem 4.10. The generalized theta function \( Z_{W^i(g,h)}(a; (u,v); \tau) \) converges to a holomorphic function on the upper half plane and gives a vector in \( \mathcal{C}_1(\sigma(u), \sigma(v)) \) for each \( 1 \leq i \leq N \). Furthermore, we have the following modular transformation for \( \rho = \left( \begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array} \right) \in \text{SL}_2(\mathbb{Z}) \):
\[
(g \tau + \delta)^{-\text{wt}[a]} Z_{W^i(g,h)} \left( a; (u,v); \frac{\alpha \tau + \beta}{\gamma \tau + \delta} \right)
= \sum_{i=1}^{N} A_{ij}(\rho, (g,h)) Z_{W^i(g,h^\rho)}(a; (\alpha u + \gamma v, \beta u + \delta v); \tau),
\]
where \( A_{ij}(\rho, (g,h)) \) are the constants given by [DLM3] as in the equation (4.6).

Remark 4.11. Since \( L[0] = L(0) + \sum_{i \geq 1} c_i L(i) \) for some \( c_i \in \mathbb{C} \), we have \( [L[0], u(0)] = 0 \) for all \( u \in H \). Therefore, \( u(0) \) acts on each \( L[0] \)-homogeneous subspace \( V_{[a]} \) diagonally.

Proof: We divide the proof into two parts. In the first part, we show that the modular transformation \( (g \tau + \delta)^{-\text{wt}[a]} Z_{W^i(g,h)}(a; (u,v); \rho \tau) \) is uniquely expressed as a linear combination of \( Z_{W^j(g,h^\rho)}(a; (\alpha u + \gamma v, \beta u + \delta v); \tau) \), \( 1 \leq j \leq N \). Then we show that the coefficients of the linear combination are exactly given as stated.

Let us consider the meaning of \( Z_{W^i(g,h)}(a; (u,v); \rho \tau) \). By Proposition 4.5, \( \tilde{W}^i(g,h) := (W^i(g,h), Y(\Delta(u,z)) \cdot z) \) is an irreducible \( g \sigma(u) \)-twisted \( V \)-module. Since \( H \) is a subspace of \( V^{(g,h)} \), \( u(0) \) acts on \( \tilde{W}^i(g,h) \) as a derivation
\[
v(0)Y_{\tilde{W}^i(g,h)}(a, z) = Y_{\tilde{W}^i(g,h)}(v(0)a, z) + Y_{\tilde{W}^i(g,h)}(a, z)v(0).
\]
Therefore, we have the following.

\[
\exp\left(-2\pi\sqrt{-1}v(0)\right) Y_{\tilde{W}^i(g,h)}(a, z) = Y_{\tilde{W}^i(g,h)}(\sigma(v)a, z) \exp\left(-2\pi\sqrt{-1}v(0)\right).
\]

(Note that \(\exp\left(-2\pi\sqrt{-1}v(0)\right)\) is well-defined on \(\tilde{W}^i(g,h)\) since it is irreducible.) Namely, \(\tilde{\sigma}(v) := \exp\left(-2\pi\sqrt{-1}v(0)\right)\), here \(v(0) = \text{Res}_{\phi} Y_{\tilde{W}^i(g,h)}(v, z) \in \text{End}(\tilde{W}^i(g,h))\), is a \(\sigma(v)\)-stabilizing automorphism on \(\tilde{W}^i(g,h)\). On the other hand, by definition, there exist linear isomorphisms \(\varphi_{W^i(g,h)} : W^i(g,h) \to \tilde{W}^i(g,h)\), \(1 \leq i \leq N\), such that

\[
Y_{\tilde{W}^i(g,h)}(a, z)\varphi_{W^i(g,h)} = \varphi_{W^i(g,h)}Y_{\tilde{W}^i(g,h)}(\Delta(u, z)a, z).
\]

Then

\[
\varphi_{W^i(g,h)}\phi_{i,g}(h)\varphi_{W^i(g,h)}^{-1} Y_{\tilde{W}^i(g,h)}(a, z) = \varphi_{W^i(g,h)}Y_{\tilde{W}^i(g,h)}(\Delta(u, z)a, z)\varphi_{W^i(g,h)}^{-1}
\]

\[
= \varphi_{W^i(g,h)}Y_{\tilde{W}^i(g,h)}(\Delta(u, z)ha, z)\phi_{i,g}(h)\varphi_{W^i(g,h)}^{-1}
\]

\[
= Y_{\tilde{W}^i(g,h)}(ha, z)\varphi_{W^i(g,h)}\phi_{i,g}(h)\varphi_{W^i(g,h)}^{-1}.
\]

Thus, a composition \(\tilde{\phi}_{i,g}(h) := \varphi_{W^i(g,h)}\phi_{i,g}(h)\varphi_{W^i(g,h)}^{-1}\) provides a \(k_2\)-stabilizing automorphism on \(\tilde{W}^i(g,h)\) for each \(1 \leq i \leq N\). Therefore, we see that all inequivalent irreducible \(g\sigma(u)\)-twisted \(h\sigma(v)\)-stable \(V\)-modules are filled by \(\tilde{W}^i(g,h)\), \(1 \leq i \leq N\) with the stabilizing automorphisms \(\tilde{\phi}_{i,g}(h)\tilde{\sigma}(v)\) by Proposition \ref{prop:irreducible}. Hence, by Theorem \ref{thm:trace} the trace function

\[
T_{\tilde{W}^i(g,h)}(a; (u, v); \tau) := \text{tr}_{\tilde{W}^i(g,h)} z^{\text{wt}(a)} Y_{\tilde{W}^i(g,h)}(a, z)\tilde{\sigma}(v)^{-1}\tilde{\phi}_{i,g}(h)^{-1}q^{L(0)-c/24}
\]

converges on the upper half plane and gives a vector in \(C_1(g\sigma(u), h\sigma(v))\). Since \(V\) is \(g\sigma(u)\)-rational by Lemma \ref{lem:irreducible}, \(C_1(g\sigma(u), h\sigma(v))\) is spanned by \(T_{\tilde{W}^i(g,h)}(a; (u, v); \tau), 1 \leq i \leq N\). Therefore, by Theorem \ref{thm:trace}

\[
\rho = \left(\begin{array}{cc}
a & \beta \\
\gamma & \delta
\end{array}\right) \in SL_2(\mathbb{Z})\]

defines a linear isomorphism between \(C_1((g\sigma(u), h\sigma(v)))\) and \(C_1((g\sigma(u), h\sigma(v)))^\rho\) in the following way:

\[
T_{\tilde{W}^i(g,h)}(a; (u, v); \rho \tau) = (\gamma \tau + \delta)^{\text{wt}(a)} \sum_{j=1}^{N} B_{ij}((g, h), (u, v), \rho) T_{\tilde{W}^i(g,h)^\rho}(a; (\alpha u + \gamma v, \beta u + \delta v); \tau),
\]

where \(B_{ij}((g, h), (u, v), \rho)\) are scalars independent of \(a\) and \(\tau\).
By the way, using $Y_{W^i(g,h)}(a, z) \varphi_{W^i(g,h)} = \varphi_{W^i(g,h)} Y_{W^i(g,h)}(\Delta(u, z) a, z)$, we have

$$T_{W^i(g,h)}(a; (u, v); \tau) = \text{tr}_{W^i(g,h)} z^{\text{wt}(a)} \varphi_{W^i(g,h)} Y(a, z) \tilde{\sigma}(v) z^{-1} \phi_{i,g}(h)^{-1} q^{L(0) - c/24}$$

$$= \text{tr}_{W^i(g,h)} z^{\text{wt}(a)} Y(a, z) \tilde{\sigma}(v) z^{-1} \phi_{i,g}(h)^{-1} q^{L(0) - c/24} \varphi_{W^i(g,h)}$$

$$= \text{tr}_{W^i(g,h)} \varphi_{W^i(g,h)} z^{\text{wt}(a)} Y(\Delta(u, z) a, z) \exp \left(2\pi \sqrt{-1}(v(0) + \langle u, v \rangle)\right) \times \phi_{i,g}(h)^{-1} q^{L(0) + u(0) + \frac{1}{2} \langle u, a \rangle - c/24}$$

$$= e^{\pi \sqrt{-1}(u, v)} Z_{W^i(g,h)}(a; (u, v); \tau),$$

where we have also used that $v(0) \varphi_{W^i(g,h)} = \varphi_{W^i(g,h)}(v(0) + \langle u, v \rangle)$ and $L(0) \varphi_{W^i(g,h)} = \varphi_{W^i(g,h)}(L(0) + u(0) + \frac{1}{2} \langle u, u \rangle)$. Therefore, by (4.8), $(\gamma \tau + \delta)^{-\text{wt}[\alpha]} Z_{W^i(g,h)}(a; (u, v); \rho \tau)$ is a linear combination of $Z_{W^i(g,h)}(a; (\alpha u + \gamma v, \beta u + \delta v); \tau)$.

Next, we show that $B_{i,j}((g, h), (u, v), \rho) = e^{\pi \sqrt{-1}(-\langle \alpha u + \gamma v, \beta u + \delta v \rangle + \langle u, v \rangle)} A_{i,j}(\rho, (g, h))$ for $1 \leq i, j \leq N$, which would complete the proof. Since $B_{i,j}((g, h), (u, v), \rho)$, $1 \leq i, j \leq N$, are independent of $a$, we prove the equality in the case where $a = 1$. We use some results from [Z] and [DLM2].

For $a_1, \ldots, a_n \in V(g, h)$, set

$$S_{W^i(g,h)}((a_1, z_1), \ldots, (a_n, z_n); \tau) := q_{z_1}^{\text{wt}(a_1)} \cdots q_{z_n}^{\text{wt}(a_n)} \text{tr}_{W^i(g,h)} Y(a_1, q_{z_1}) \cdots Y(a_n, q_{z_n}) \phi_{i,g}(h)^{-1} q^{L(0) - c/24},$$

where $q_x$ denotes $e^{2\pi \sqrt{-1}x}$. We deduce a recurrent formula for $S_{W^i(g,h)}$. Before we state it, we introduce the following functions (cf. [Z] and [DLM2]).

The Eisenstein series $G_{2k}(\tau)$ ($k = 2, 3, \ldots$) are series

$$G_{2k}(\tau) = 2\zeta(2k) + \frac{2(2\pi \sqrt{-1})^{2k}}{(2k - 1)!} \sum_{n=1}^{\infty} \frac{n^{2k-1} q^n}{1 - q^n}, \quad (4.9)$$

where $\zeta(2k) = \sum_{n=1}^{\infty} 1/n^{2k}$. We use normalized Eisenstein series

$$E_{2k}(\tau) := \frac{1}{(2\pi \sqrt{-1})^{2k}} G_{2k}(\tau). \quad (4.10)$$

Since $G_{2k}(\tau)$ is a modular form of weight $2k$ for the modular group $SL_2(\mathbb{Z})$, we have

$$E_{2k} \left( \begin{array}{cc} \alpha \tau + \beta \\ \gamma \tau + \delta \end{array} \right) = (\gamma \tau + \delta)^{2k} E_{2k}(\tau) \quad \text{for} \quad \left( \begin{array}{cc} \alpha \\ \gamma \\ \delta \end{array} \right) \in SL_2(\mathbb{Z}). \quad (4.11)$$

We define the functions $\varphi_k(z, \tau)$ ($k \geq 1$) by

$$\varphi_k(z, \tau) := \frac{1}{z^{k}} + (-1)^k \sum_{n=1}^{\infty} \binom{2n+1}{k-1} G_{2n+2}(\tau) z^{2n+2-k}. \quad (4.12)$$
The following modular transformations are well-known:

\[ \wp_k \left( \frac{z}{\gamma \tau + \delta}, \frac{\alpha \tau + \beta}{\gamma \tau + \delta} \right) = (\gamma \tau + \delta)^k \wp_k(z, \tau) \quad \text{for} \quad \left( \begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array} \right) \in SL_2(\mathbb{Z}). \tag{4.13} \]

**Assertion 1.** ([Z, Proposition 4.4.2]) For \( a_1, \ldots, a_n, b \in V^{(g,h)} \), the following recurrent formula holds:

\[
S_{W^i(g,h)} \left( (b, x), (a_1, z_1), \ldots, (a_n, z_n); \tau \right) = S_{W^i(g,h)} \left( (b_{[-1]}a_1, z_1), (a_2, z_2), \ldots, (a_n, z_n); \tau \right) \\
- \sum_{k=2}^{\infty} E_{2k}(\tau) S_{W^i(g,h)} \left( (b_{[2k-1]}a_1, z_1), (a_2, z_2), \ldots, (a_n, z_n); \tau \right) \\
+ \sum_{s=1}^{n} \sum_{m=0}^{\infty} \frac{1}{(2\pi \sqrt{-1})^{m+1}} \{ \wp_{m+1}(x - z_s, \tau) - \wp_{m+1}(z_1 - z_s, \tau) \} \\
\times S_{W^i(g,h)} \left( (a_1, z_1), \ldots, (b_{[m]}a_s, z_s), \ldots, (a_n, z_n); \tau \right).
\]

**Proof:** Since \( W^i(g,h) \) are untwisted ordinary modules for \( V^{(g,h)} \), we can use the same argument as that in [Z] and hence we obtain the same consequence. (Note that our usage of the Eisenstein series differs from that of Zhu in [Z] by scalar multiples.)

Using the recurrent formula above, we can show the following.

**Assertion 2.** For \( a_1, \ldots, a_n \in V^{(g,h)} \) and \( \rho = \left( \begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array} \right) \in SL_2(\mathbb{Z}) \), we have

\[
S_{W^i(g,h)} \left( \left( a_1, \frac{z_1}{\gamma \tau + \delta} \right), \ldots, \left( a_n, \frac{z_n}{\gamma \tau + \delta} \right), \frac{\alpha \tau + \beta}{\gamma \tau + \delta} \right) = (\gamma \tau + \delta)^{\text{wt}[a_1] + \cdots + \text{wt}[a_n]} \sum_{j=1}^{N} A_{ij}(\rho, (g,h)) S_{W^i((g,h)\rho)} \left( (a_1, z_1), \ldots, (a_n, z_n); \tau \right),
\]

where scalars \( A_{ij}(\rho, (g,h)) \) are given by (4.6).

**Proof:** We proceed by induction on \( n \). The case \( n = 1 \) is already known by Theorem 4.2. Since the \( n \)-point trace \( S_{W^i(g,h)} \) is completely determined by the 1-point trace \( T_{W^i(g,h)} \), using (4.11), (4.13) and the recurrent formula obtained in Assertion 1, we get the assertion.
Assertion 3. ([M1 Main Theorem])

Set $Z'_{W^i(g,h)}(u; v; \tau) := Z_{W^i(g,h)}(1; (u, v); \tau)$. For $\rho = (\alpha_\beta_\gamma_\delta) \in SL_2(\mathbb{Z})$, we have

$$Z'_{W^i(g,h)}(u; v; \frac{\alpha\tau + \beta}{\gamma\tau + \delta}) = \sum_{j=1}^{N} A_{ij}(\rho, (g, h)) \ Z'_{W^i(g,h)\rho}(\alpha u + \gamma v; \beta u + \delta v; \tau). \quad (4.14)$$

Proof: The formula above was proved when $g = h = 1$ in [M1]. The proof in [M1] is given by direct calculations on the $n$-point traces. So by tracing calculations in [M1] together with Theorem 4.12 and Assertion 2, one can verify the assertion.

Therefore, we have reached $B((g, h), (u, v), \rho) = e^{\pi\sqrt{-1}((\alpha u + \gamma v, \beta u + \delta v) + (u, v))} A((g, h), \rho)$ by (4.14). This completes the proof of Theorem 4.10.

There is an interesting consequence of Theorem 4.10. For each $\rho \in SL_2(\mathbb{Z})$, let us denote by $\Psi_{(g,h)}(\rho)$ the isomorphism from $C_1(g, h)$ to $C_1((g, h)^\rho)$ given as (4.2). The proof of Theorem 4.10 tells us that the space $C_1(g\sigma(u), h\sigma(v))$ is spanned by functions $Z_{W^i(g,h)}(a; (u, v); \tau)$, $1 \leq i \leq N$ and the matrix representation of $\Psi_{(g,h)}(\rho)$ that of $\Psi_{(g\sigma(u), h\sigma(v))}(\rho)$ are given by the same matrix $(A_{ij}(\rho, (g, h)))_{ij}$. Namely, we have proved that the internal automorphisms do not change the genus one twisted conformal blocks:

**Corollary 4.12.** For a pair $(u, v) \in H \times H$, define a linear isomorphism $\Omega_{(g,h)}(u, v) : C_1(g, h) \to C_1(g\sigma(u), h\sigma(v))$ by $T_{W^i(g,h)}(a; (u, v); \tau) \mapsto Z_{W^i(g,h)}(a; (u, v); \tau)$. Then we have

$$\Psi_{(g\sigma(u), h\sigma(v))}(\rho) \circ \Omega_{(g,h)}(u, v) = \Omega_{(g,h)\rho}(\alpha u + \gamma v, \beta u + \delta v) \circ \Psi_{(g,h)}(\rho)$$

for every $\rho = (\alpha_\beta_\gamma_\delta) \in SL_2(\mathbb{Z})$.

### 4.4 Relation to abelian coset construction

We keep the setup of the previous subsection. In this subsection we consider the case where $g = h = 1$ and $V$ is simple. In addition, we assume that the restriction of the invariant bilinear form $\langle \cdot, \cdot \rangle$ on $H$ is non-degenerate. Set $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Q}} H \subset V_1$ and let \{\$h_1, \ldots, h_{\dim \mathfrak{h}}\}$ be an orthonormal basis of $\mathfrak{h}$. For $h, k \in \mathfrak{h}$, their vertex operators satisfies the commutator relation $[h_{(m)}, k_{(n)}] = \delta_{m+n,0} m \langle h, k \rangle$. Therefore, $\mathfrak{h}$ generates a free bosonic sub VOA $M_{\mathfrak{h}}(1, 0)$ with the Virasoro vector

$$\omega_{\mathfrak{h}} := \frac{1}{2} \sum_{i=1}^{\dim \mathfrak{h}} h_{(-1)}^i h_{(-1)}^i \mathbb{1}.$$ 

For $\alpha \in \mathfrak{h}$, set

$$V^\alpha := \{ x \in V \mid h_{(0)} x = \langle \alpha, h \rangle x \ \text{for} \ h \in \mathfrak{h} \}. $$
and we define $L = \{\alpha \in \mathfrak{h} \mid V^\alpha \neq 0\}$ which is a subgroup of the additive group $\mathfrak{h}$. Then we obtain an $L$-graded structure

$$V = \oplus_{\alpha \in L} V^\alpha$$

with $Y(a, z)b \subset V^{\alpha + \beta}((z))$ for $a \in V^\alpha$, $b \in V^\beta$. Note that $V^0$ is a sub VOA. Since we have assumed that $V$ is $C_2$-cofinite, $L$ is a finitely generated free $\mathbb{Z}$-module. Namely, $L$ equipped with $(\cdot, \cdot)$ is a rational lattice. As $V$ is simple, $L = \{\alpha \in \mathfrak{h} \mid V^\alpha \neq 0\}$. Define the space of highest weight vectors

$$\Omega_V := \{x \in V \mid h_{(n)}x = 0 \quad \text{for} \quad h \in \mathfrak{h}, \quad n \geq 1\}.$$ 

Since $[h_{(0)}, k_{(m)}] = 0$ for $h, k \in \mathfrak{h}$, $m \in \mathbb{Z}$, $h_{(0)}$ preserves $\Omega_V$. Then

$$\Omega_V = \oplus_{\alpha \in L} \Omega^\alpha_V, \quad \text{where} \quad \Omega^\alpha_V = \Omega_V \cap V^\alpha.$$ 

Since $V \cong M_{\mathfrak{h}}(1, 0) \otimes \Omega_V$ as a linear space, we have the following decomposition:

$$V = M_{\mathfrak{h}}(1, 0) \otimes \Omega_V = \oplus_{\alpha \in L} M_{\mathfrak{h}}(1, 0) \otimes \Omega^\alpha_V.$$ 

Note that $\Omega^0_V$ is a sub VOA of $V^0$ with the Virasoro vector $\omega_{\Omega} := \omega - \omega_{\mathfrak{h}}$ and is a commutant subalgebra of $M_{\mathfrak{h}}(1, 0)$ such that $V^0 = M_{\mathfrak{h}}(1, 0) \otimes \Omega^0_V$. For $a \in \Omega^0_V$, define

$$Y_{\Omega_V}(a, z) := E^-(\alpha, z)Y_V(a, z)E^+(\alpha, z)z^{-\alpha(a)},$$

where

$$E^\pm(\alpha, z) := \exp \left( \sum_{n=1}^{\infty} \frac{\Omega(\pm n)}{\pm n} z^n \right).$$

Then it is shown in [DL] and [Li4] that the structure $(\Omega_V, Y_{\Omega_V}(\cdot, z), \mathbb{1}, \omega_{\Omega})$ is a simple $L$-graded generalized vertex operator algebra with central charge $c - \dim \mathfrak{h}$.

Let $W$ be an irreducible $V$-module. Then we also have a decomposition

$$W = M_{\mathfrak{h}}(1, 0) \otimes \Omega_W,$$

where $\Omega_W := \{w \in W \mid h_{(n)}w = 0 \quad \text{for} \quad h \in \mathfrak{h}, \quad n \geq 1\}$ denotes the space of highest weight vectors in $W$. For $\lambda \in \mathfrak{h}$, set $W^\lambda = \{w \in W \mid h_{(0)}w = \langle \lambda, h \rangle w \quad \text{for} \quad h \in \mathfrak{h}\}$. Since $W$ is irreducible, there is an $L$-subset $\lambda_W + L$ of $\mathfrak{h}$ such that $W = \oplus_{\beta \in L + \lambda_W} W^\beta$ and $\Omega_W = \oplus_{\beta \in L + \lambda_W} \Omega^\beta_W$ with $\Omega^\beta_W = W^\beta \cap \Omega_W$. For $a \in \Omega^0_V$, set

$$Y_{\Omega_W}(a, z) := E^-(\alpha, z)Y_W(a, z)E^+(\alpha, z)z^{-\alpha(a)}.$$ 

Then it is shown in [DL] and [Li4] that $(\Omega_W, Y_{\Omega_W}(\cdot, z))$ is an irreducible $(L + \lambda_W)$-graded $\Omega_V$-module. Moreover, we have the following theorem:
Theorem 4.13. ([L4, Theorem 3.16]) For a rational VOA \( V \), the associated \( L \)-graded generalized vertex operator algebra \( (\Omega_V, \omega_1) \) is also rational in the sense that every \( \Omega_V \)-module with an \( L \)-set grading is completely reducible. Moreover, the map which associates to an irreducible \( V \)-module \( W \) an irreducible \( \Omega_V \)-module \( \Omega_W \) defines a bijection between the set of inequivalent irreducible \( V \)-modules and the set of inequivalent irreducible \( \Omega_V \)-modules with \( L \)-set gradings.

Remark 4.14. Even if \( V \) is a simple VOA, the generalized VOA \( \Omega_V \) may contain a nontrivial ideal. However, the theorem above says that there is no \( L \)-graded ideal in \( \Omega_V \).

By this theorem, we can expect that the space of \( q \)-characters \( \text{ch}_{\Omega_W}(\tau) \), where \( W \) runs over irreducible \( V \)-modules, has a modular invariance property. Below we show modular transformation laws of \( \text{ch}_{\Omega_W}(\tau) \).

Let \( a \in V^\alpha \) and \( u, v \in H \). Then \( a_n(W^\beta) \subset W^{\alpha+\beta} \) for \( \beta \in L + \lambda_W, n \in \mathbb{Z} \), whereas \( u_{(0)} \) and \( v_{(0)} \) acts on \( W^\beta \) semisimply. So we have \( Z_W(a; (u, v); \tau) = 0 \) unless \( \alpha = 0 \). That is, our theta function is effective only for elements in \( V^0 = M_h(1, 0) \otimes \Omega_V^0 \).

Proposition 4.15. Let \( W \) be an untwisted \( V \)-module and \( a \in V^\alpha \). If \( \alpha \neq 0 \), then \( Z_W(a; (u, v); \tau) = 0 \).

If \( a \in \Omega_V^0 \subset V^0 \), then \( Y_{\Omega_W}(a, z) = Y_W(a, z) \) so that \( a \) acts on each \( \Omega_W^0 \). Moreover, both \( u_{(0)} \) and \( v_{(0)} \) are also commutative with actions of \( M_h(1, 0) \) on \( W \). Therefore, we have

\[
Z_W(a; (u, v); \tau) = e^{\pi \sqrt{-1}(u,v)} \text{tr}_W o(a) \sigma(v) q^{L(0)+u_{(0)}+\frac{1}{2}(u,u)-\frac{1}{24}c} \\
= e^{\pi \sqrt{-1}(u,v)} \text{tr}_{M_h(1,0)} q^{L_h(0)-\dim h/24} \cdot \text{tr}_{\Omega_W} o(a) \sigma(v) q^{L(0)-L_h(0)+u_{(0)}+\frac{1}{2}(u,u)-\frac{1}{24}(c-\dim h)} \\
= e^{\pi \sqrt{-1}(u,v)} \eta(\tau)^{-\dim h} \cdot \text{tr}_{\Omega_W} o(a) \sigma(v) q^{L_{\Omega}(0)+u_{(0)}+\frac{1}{2}(u,u)-\frac{1}{24}(c-\dim h)},
\]

where \( L_h(0) = (\omega_h)_{(1)} \) and \( L_{\Omega}(0) = (\omega - \omega_1)_{(1)} \) are degree operators on \( M_h(1,0) \) and \( \Omega_W \), respectively. By the above equality, the essential ingredient of a theta function \( Z_W(a; (u, v); \tau) \) comes from the structure of the space \( \Omega_W = \bigoplus_{\beta \in L+\lambda_W} \Omega_W^\beta \) of highest weight vectors. Now for \( a \in \Omega_V \) and \( u, v \in H \), set a trace form on \( \Omega_W \) by

\[
X_{\Omega_W}(a; (u, v); \tau) := e^{\pi \sqrt{-1}(u,v)} \text{tr}_{\Omega_W} o(a) \sigma(v) q^{L_{\Omega}(0)+u_{(0)}+\frac{1}{2}(u,u)-\frac{1}{24}(c-\dim h)}.
\]

Then \( Z_W(a; (u, v); \tau) = \eta(\tau)^{-\dim h} \cdot X_{\Omega_W}(a; (u, v); \tau) \). Let \( \{W^1, \ldots, W^N\} \) be the set of all inequivalent irreducible untwisted \( V \)-modules. Then by Theorem 4.10 we have the following modular transformation laws:
**Theorem 4.16.** For \( a \in \Omega^0_V \) with \( L_0(0)a = wt_\Omega(a) \cdot a, wt_\Omega(a) \in \mathbb{Q} \), we have
\[
X_{\Omega_W}(a; (u, v); \tau + 1) = e^{\pi \sqrt{-1}/12} \sum_{j=1}^{N} T_{ij} X_{W_j}(a; (u, u + v), \tau),
\]
\[
X_{\Omega_W}(a; (u, v); -1/\tau) = (-\sqrt{-1} \tau)^{\dim h/2} \cdot \tau^{wt_\Omega(a)} \sum_{j=1}^{N} S_{ij} X_{W_j}(a; (v, -u); \tau),
\]
where \( T_{ij} = A_{ij}(1/0) \) and \( S_{ij} = A_{ij}(0/1) \) are constants given by Theorem 4.2 as in the equation (4.6).

**Proof:** The following transformation laws are well-known: \( \eta(\tau + 1) = e^{\pi \sqrt{-1}/12} \eta(\tau) \) and \( \eta(-1/\tau) = (-\sqrt{-1} \tau)^{1/2} \eta(\tau) \). Since \( L(0)a = 0 \) by definition of \( \Omega_V \), we have \( L(0)a = (L_0(0) + L_0(0))a = wt_\Omega(a) \cdot a \). Thus by combining Theorem 4.10 with the relation \( Z_{W^i}(a; (u, v); \tau) = \eta(\tau)^{-\dim h} X_{\Omega_W}(a; (u, v); \tau) \), we have the desired equalities.

Note that \( X_{\Omega_W}(I; (0, 0); \tau) = tr_{\Omega_W} q^{\ell(0)(-c - \dim h)/24} = ch_{\Omega_W}(\tau) \) is the \( q \)-character of an \( \Omega_V \)-module \( \Omega_W \). By the theorem above, the space of \( q \)-characters of \( \Omega_V \)-modules is not invariant under \( SL_2(\mathbb{Z}) \) as we have an extra term \( (-\sqrt{-1} \tau)^{-\dim h/2} \). However, we can eliminate the term \( (-\sqrt{-1} \tau)^{-\dim h/2} \) in the following way. Assume that the rational lattice \( L \) contains a positive definite even lattice \( K \) such that \( \text{rank}(L) = \text{rank}(K)(= \dim h) \). Then \( V \) contains a lattice VOA \( V_K \) associated to \( K \), and all \( W^i \) are twisted modules for \( V_K \). We further assume that we can choose a lattice \( K \) such that all \( W^i \) are untwisted \( V_K \)-modules.

How to choose such a lattice \( K \) is shown in [Li4], and such \( K \) always satisfies \( L + \lambda_{W^i} \subset K^0 \) for all \( 1 \leq i \leq N \). Since \( V_K \) is rational, we have the following decomposition for each \( W^i \):
\[
W^i = \bigoplus_{\mu + K \in (L + \lambda_{W^i})/K} V_{K + \mu} \otimes \text{Hom}_{V_K}(V_{K + \mu}, W^i).
\]
Set \( \Omega^\mu_{W^i} := \text{Hom}_{V_K}(V_{K + \mu}, W^i) \) and \( \Omega^K_{W^i} := \bigoplus_{\mu + K \in (L + \lambda_{W^i})/K} \Omega^\mu_{W^i} \). Then it is shown in [DL] and [Li4] that there is an ideal \( I \) of \( \Omega_V \) such that the quotient generalized vertex operator algebra \( \Omega_V/I \) is simple and isomorphic to \( \Omega^K_{W^i} \). So the space \( \Omega^K_{W^i} \) naturally possesses a structure of a simple \( L/K \)-graded generalized vertex operator algebra. Moreover, it is also shown in [Li4] that each \( \Omega^K_{W^i} \) is an irreducible \( (L + \lambda_{W^i})/K \)-graded \( \Omega^K_{W^i} \)-module, and every \( \Omega^K_{W^i} \)-module with an \( L/K \)-set grading is a direct sum of copies of \( \Omega^K_{W^i} \)'s.

On the other hand, we have
\[
Z_{W^i}(I; (0, 0); \tau) = \sum_{\mu + K \in (L + \lambda_{W^i})/K} \text{ch}_{V_{K + \mu}}(\tau) \cdot \text{ch}_{\Omega^\mu_{W^i}}(\tau)
\]
\[
= \sum_{\mu + K \in (L + \lambda_{W^i})/K} \eta(\tau)^{-\text{rank}(K)} \cdot \theta_{K + \mu}(\tau) \cdot \text{ch}_{\Omega^\mu_{W^i}}(\tau)
\]
\[
= \eta(\tau)^{-\dim h} \sum_{\mu + K \in (L + \lambda_{W^i})/K} \theta_{K + \mu}(\tau) \cdot \text{ch}_{\Omega^\mu_{W^i}}(\tau),
\]
22


where $\theta_{K+\mu}(\tau) = \sum_{\beta \in K+\mu} q^{(\beta,\beta)/2}$ are theta functions defined on the lattice $K$. Thus

$$X_{\Omega_{\iota_1}}(1; (0,0); \tau) = \sum_{\mu+K \in (L+\lambda_{\iota_1})/K} \theta_{K+\mu}(\tau) \cdot \text{ch}_{\Omega_{\iota_1}}^{\mu}(\tau).$$

Since the term $(-\sqrt{-1})^{-\dim h/2} = (\tau)^{-\text{rank}(K)/2}$ also appears in the modular transformation of $\theta_{K+\mu}(\tau)$, we have a desired cancellation. By this observation, it is very likely to happen that the space of $q$-characters

$$\text{ch}_{\Omega_{\iota_1}}^{\mu}(\tau) = \sum_{\mu+K \in (L+\lambda_{\iota_1})/K} \text{ch}_{\Omega_{\iota_1}}^{\mu}(\tau)$$

of $\Omega_{\iota_1}^{K}$-modules $\Omega_{\iota_1}^{K}$, $1 \leq i \leq N$, are invariant under the action of $SL_2(\mathbb{Z})$. This question will be discussed in another paper.

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