Golden ratio prediction for solar neutrino mixing

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New Journal of Physics 11 (2009) 063026 (13pp)
Received 8 March 2009
Published 16 June 2009
Online at http://www.njp.org/
doi:10.1088/1367-2630/11/6/063026

Abstract. It has recently been speculated that the solar neutrino mixing angle is connected to the golden ratio $\varphi$. Two such proposals have been made, $\cot \theta_{12} = \varphi$ and $\cos \theta_{12} = \varphi/2$. We compare these ansätze and discuss a model leading to $\cos \theta_{12} = \varphi/2$ based on the dihedral group $D_{10}$. This symmetry is a natural candidate because the angle in the expression $\cos \theta_{12} = \varphi/2$ is simply $\pi/5$, or $36^\circ$. This is the exterior angle of a decagon and $D_{10}$ is its rotational symmetry group. We also estimate radiative corrections to the golden ratio predictions.

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1. Introduction

The question of what kind of flavor model underlies the peculiar features of lepton mixing is one of the dominating ones in contemporary theoretical neutrino physics. One hopes that precision measurements of the flavor parameters will provide hints toward the symmetry principle behind the apparent regularities. We will in this paper discuss one intriguing example of this line of thought.

All these issues are linked to the structure of the neutrino (and the charged lepton) mass matrix. Typically, the smallness of $|U_{e3}|$ and the close-to-maximality of $\theta_{23}$ are—in the charged lepton basis—attributed to the presence of an approximate $\mu-\tau$ symmetry:

$$m_\nu = \begin{pmatrix} A & B & B \\ \cdot & D & E \\ \cdot & \cdot & D \end{pmatrix}.$$  (1)

The eigenvector to the eigenvalue $D - E$ indeed is $(0, -1, 1)^T$, but solar neutrino mixing is unconstrained by the matrix given above. If in addition to $\mu-\tau$ symmetry the condition $A + B = D + E$ holds, then the value $\sin^2 \theta_{12} = \frac{1}{3}$ is obtained: the infamous tri-bimaximal mixing (TBM) [1], which dominates the current theoretical literature on lepton flavor model building. However, comparing the TBM parameters ($\sin^2 \theta_{12} = \frac{1}{3}$, $\sin^2 \theta_{13} = 0$ and $\sin^2 \theta_{23} = \frac{1}{2}$) with the current best-fit, 1, 2 and 3$\sigma$ ranges [2] (very similar results are found in [3])

$$\begin{array}{ccc}
\sin^2 \theta_{12} & \sin^2 \theta_{13} & \sin^2 \theta_{23} \\
0.304 & 0.01 & 0.50 \\
0.288-0.326 & \lesssim 0.026 & 0.44-0.57 \\
0.27-0.35 & \lesssim 0.040 & 0.39-0.63 \\
0.25-0.37 & \lesssim 0.056 & 0.36-0.67 \\
\end{array}$$  (2)

one notes that there is ample room for mixing scenarios other than TBM. In this short note, we consider alternatives to TBM and focus on the fascinating possibility of linking solar neutrino mixing with the golden ratio $\phi = \varphi^2 - 1 = \frac{1}{2} (1 + \sqrt{5})$. Two such proposals have recently been made. The first one is [4]–[6]

(A): $\cot \theta_{12} = \varphi \Rightarrow \sin^2 \theta_{12} = \frac{1}{1 + \varphi^2} = \frac{2}{5 + \sqrt{5}} \simeq 0.276.$  (3)

The second possibility is [7]

(B): $\cos \frac{\theta_{12}}{2} = \varphi \Rightarrow \sin^2 \frac{\theta_{12}}{2} = \frac{1}{4} (3 - \varphi) = \frac{5 - \sqrt{5}}{8} \simeq 0.345.$  (4)

It can be seen that both predictions lie within the current 2$\sigma$ range$^2$. The possibility (A) was first noted in [4], and discussed in more detail in [5], where it was also mentioned that $A_5$ might be a candidate for the underlying flavor symmetry group. In this spirit a model based on $A_5$, which can lead to $\cot \theta_{12} = \varphi$, has been outlined in [6]. The reason why $A_5$ is the candidate symmetry is because this group is isomorphic to the rotational group of the icosahedron and its geometrical features can be linked to the golden ratio. For instance, the 12 vertices of an icosahedron with edge length 2 have Cartesian coordinates $(0, \pm 1, \pm \varphi)$, $(\pm 1, \pm \varphi, 0)$ and $(\pm \varphi, 0, \pm 1)$. A peculiar feature of $\cot \theta_{12} = \varphi$ is that the angle also gives $\tan 2\theta_{12} = 2$, and this can be obtained from

$^2$ Actually, prediction (A) would lie very slightly outside the 2$\sigma$ range of [3], which is $\sin^2 \theta_{12} = 0.278$–0.352.

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a simple matrix proportional to [5]

\[ m_\nu \propto \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}. \]  

(5)

This matrix is invariant under a \( Z_2 \) symmetry generated by [5]

\[ S = \frac{1}{\sqrt{5}} \begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix}. \]  

(6)

where invariance is fulfilled when \( S^T m_\nu S = m_\nu \).

Now consider the second golden ratio prediction \( \cos \theta_{12} = \frac{\varphi}{2} \), which corresponds simply to \( \theta_{12} = \frac{\pi}{5} \). A mixing scenario based on this value was proposed with a purely phenomenological purpose in [7]. A unified parametrization of both the Cabibbo–Kobayashi–Maskawa (CKM) and the Pontecorvo–Maki–Nakagawa–Sakata (PMNS) matrix was constructed by choosing in addition to the lepton mixing angle \( \theta_{12} = \frac{\pi}{5} \) a similar expression for the quark sector, namely \( \theta_{12}^{Q} = \frac{\pi}{12} \). The resulting value for the sine of the Cabibbo angle, \( \sin \theta_{12}^{Q} = \left( \sqrt{3} - 1 \right) / \sqrt{8} \), is also a simple algebraic and irrational number. The point made in [7] was that at zeroth order the CKM matrix is a 12-rotation with angle \( \frac{\pi}{12} \), whereas the PMNS matrix is a 12-rotation with angle \( \frac{\pi}{5} \) multiplied with an additional maximal (atmospheric) 23-rotation. To correct the 12-angles of the quark and lepton sectors to their respective best-fit values, one needs to multiply both zeroth order mixing matrices with a small 12-rotation. It turns out that one can achieve this with a universal (i.e. the same for quarks and leptons) angle \( \epsilon_{12} \simeq -0.03 \) [7].

In the present paper we concentrate on the possible theoretical origin of the golden ratio prediction (B). We stress that flavor models based on the symmetry group \( D_{10} \) are natural candidates to generate \( \theta_{12} = \frac{\pi}{5} \). The dihedral group \( D_{10} \) is the rotational symmetry group of a decagon and the exterior angle in a decagon is nothing but \( \frac{\pi}{5} \), or 36°. Indeed, we will present a model based on \( D_{10} \) in the following section 2. We remark that also \( D_5 \), the rotational symmetry group of a regular pentagon, could be possible. In a pentagon the length of a diagonal is \( \varphi \) times the length of a side. The triangle formed by the diagonal and two sides has one angle of 108° (the internal angle) and two angles with 36° each. However, here we focus on \( D_{10} \) because it turns out that the vacuum alignment we need in our model is simplified due to the larger number of representations in \( D_{10} \). Note that just as considering \( A_5 \) for the golden ratio prediction (A) was motivated by geometrical considerations, the use of the (mathematically simpler) pentagon or decagon symmetry group is here motivated by prediction (B). These are examples for the hope mentioned in the beginning, namely that precision measurements may give us hints toward the underlying symmetry behind flavor physics\(^3\).

The present paper is structured as follows: after discussing general symmetry properties of mass matrices with \( \cos \theta_{12} = \frac{\varphi}{2} \) and an explicit \( D_{10} \) model in section 2 we will in section 3 deal with renormalization group (RG) corrections to both golden ratio predictions (A) and (B), before we conclude in section 4.

2. Golden ratio prediction \( \theta_{12} = \frac{\pi}{5} \) and dihedral groups

We have seen in the introduction that there is a simple \( Z_2 \) under which a mass matrix generating \( \cot \theta_{12} = \varphi \) is invariant, see equations (5) and (6). The second golden ratio proposal (B) in

\(^3\) TBM is usually obtained with models based on \( A_4 \), the symmetry group of a tetrahedron ([8] and references therein). Here the angle between two faces (the dihedral angle) is \( 2\theta_{\text{TBM}} \), where \( \sin^2 \theta_{\text{TBM}} = 1/3 \).
equation (4) corresponds to $\tan 2\theta_{12} = \sqrt{1 + \phi^2}/(\phi - 1)$, and therefore it diagonalizes a less straightforward matrix. Nevertheless, in this case one can make use of $Z_2$ invariance as well, however, the charged lepton sector has also to be taken into account. We will first discuss this for the simplified two-flavor case with symmetric mass matrices, before making the transition to dihedral groups and then to the explicit model based on $D_{10}$ that we will construct.

The generators of the $Z_2$ under which the neutrino mass matrix $m_\nu$ and the charged lepton mass matrix $m_\ell$ have to be invariant are

$$S_{\nu,\ell} = \left( \begin{array}{cc} 0 & e^{-i\Phi_{\nu,\ell}} \\ e^{i\Phi_{\nu,\ell}} & 0 \end{array} \right),$$

respectively. The matrices $m_\nu$ and $m_\ell$ are invariant when they have the following structure:

$$m_{\nu,\ell} = \begin{pmatrix} A_{\nu,\ell} e^{i\Phi_{\nu,\ell}} & B_{\nu,\ell} \\ B_{\nu,\ell} & A_{\nu,\ell} e^{-i\Phi_{\nu,\ell}} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A_{\nu,\ell} & B_{\nu,\ell} \\ B_{\nu,\ell} & A_{\nu,\ell} \end{pmatrix} \begin{pmatrix} 0 & e^{-i\Phi_{\nu,\ell}} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\equiv P_{\nu,\ell} \begin{pmatrix} A_{\nu,\ell} & B_{\nu,\ell} \\ B_{\nu,\ell} & A_{\nu,\ell} \end{pmatrix} Q_{\nu,\ell}.$$

The inner matrix can be written as

$$\begin{pmatrix} A_{\nu,\ell} & B_{\nu,\ell} \\ B_{\nu,\ell} & A_{\nu,\ell} \end{pmatrix} = \tilde{U}_{\nu,\ell}^T \text{diag}(A_{\nu,\ell} - B_{\nu,\ell}, A_{\nu,\ell} + B_{\nu,\ell}) \tilde{U}_{\nu,\ell}, \quad \text{where} \quad \tilde{U}_{\nu,\ell} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}. \quad (9)$$

The total diagonalization matrices of $m_\nu$ and $m_\ell$ are $U_{\nu,\ell} = P_{\nu,\ell} \tilde{U}_{\nu,\ell}$ and the physical mixing matrix is their product $U = U_{\nu}^\dagger U_{\nu} = \tilde{U}_{\nu}^\dagger P_{\nu} \tilde{U}_{\nu}$. The 11-element is found to be

$$|U_{11}|^2 = \left| \cos \frac{1}{2} (\Phi_{\nu} - \Phi_{\ell}) \right|^2. \quad (10)$$

The fact that a nontrivial phase matrix lies in between the two maximal rotations $\tilde{U}_{\nu}^\dagger$ and $\tilde{U}_{\nu}$ is crucial. Obviously, at this stage any mixing angle can be generated. However, the observation made in [9] was that the phase factors in equation (7) can be linked to group theoretical flavor model building with dihedral groups $D_n$. To make the connection from equation (10) to dihedral groups, we note that the flavor symmetry $D_n$ has two-dimensional representations $2_j$, with $j = 1, \ldots, \frac{n}{2} - 1$ ($j = 1, \ldots, \frac{n-1}{2}$) for integer (odd) $n$, generated by

$$A = \begin{pmatrix} e^{2\pi i j \frac{n}{2}} & 0 \\ 0 & e^{-2\pi i j \frac{n}{2}} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$ \quad (11)

$Z_2$ subgroups are generated by

$$BA^k = \begin{pmatrix} 0 & e^{-2\pi i j \frac{n}{2} k} \\ e^{2\pi i j \frac{n}{2} k} & 0 \end{pmatrix}$$

with integer $k$. This is just the required form of a $Z_2$ generator in equation (7). It is now possible to construct models in which the two fermions transform under the representation $2_j$ of $D_n$, and $D_n$ is broken such that $m_\nu$ is left invariant under $BA^k$ and $m_\ell$ is left invariant under $BA^k$ [9]. Consequently, the relation in equation (10) is obtained and we can identify

$$|U_{11}|^2 = \left| \cos \frac{\pi j}{n} (k_\nu - k_\ell) \right|^2. \quad (13)$$
are thus generated by dimension 5 operators, the neutrino masses by dimension 6 operators flavons, which acquire VEVs and thereby break the flavor group. The charged lepton masses neutrinos also vanishes. Mass for the leptons is generated by coupling them to gauge singlet couplings are allowed for the charged leptons and the dimension 5 operator giving mass to the $Z$ prediction of the solar mixing angle, while the auxiliary Abelian symmetry 
We augment the MSSM by a flavor symmetry $D$ generated by an effective operator coupling to two Higgs vacuum expectation values (VEVs).

Hence, a natural candidate to implement the requested value of $\pi/5$ is e.g. $D_{10}$. This is no surprise given the observation that we made in the introduction, namely that $\pi/5$ is the exterior angle of a decagon and that $D_{10}$ is its rotational symmetry group.

We continue with an explicit model: we work in the framework of the MSSM without explicitly introducing right-handed neutrinos. Majorana masses for the light neutrinos are thus generated by an effective operator coupling to two Higgs vacuum expectation values (VEVs).

We augment the MSSM by a flavor symmetry $D_{10} \times Z_5$. The symmetry $D_{10}$ is used for our prediction of the solar mixing angle, while the auxiliary Abelian symmetry $Z_5$ separates the charged lepton and neutrino sectors. Due to the flavor symmetry, no renormalizable Yukawa couplings are allowed for the charged leptons and the dimension 5 operator giving mass to the neutrinos also vanishes. Mass for the leptons is generated by coupling them to gauge singlet flavons, which acquire VEVs and thereby break the flavor group. The charged lepton masses are thus generated by dimension 5 operators, the neutrino masses by dimension 6 operators$^4$.

The transformation properties of the MSSM leptons and Higgs fields, as well as the representations under which the flavons transform, are given in table 1. The multiplication table and the Clebsch–Gordan coefficients of $D_{10}$ are relegated to appendix A. Note that the fermions and the flavons that couple to them are all in unfaithful representations of $D_{10}$ (i.e. in $2_2$ and $2_4$), so that here a $D_5$ structure would have sufficed. However, the full $D_{10}$ structure is needed to achieve the desired vacuum alignment. We can continue by constructing the Yukawa superpotential, giving the leading order terms for both charged lepton and neutrino masses:

$$ w_Y = y_1^e (l_1 e^c l_2 e^c + l_2 e^c l_1 e^c) \frac{h_u}{\Lambda} + y_2^e (l_1 e^c \rho^c + l_2 e^c \rho^c) \frac{h_d}{\Lambda} + y_3^e (l_1 e^c \chi^c + l_2 e^c \chi^c) \frac{h_d}{\Lambda} $$

$$ + y_4^e (l_1 e^c \rho^c + l_2 e^c \rho^c) \frac{h_d}{\Lambda} + y_5^e l_3 e^c \sigma^c \frac{h_d}{\Lambda} $$

$$ + y_6^e l_1 l_2 \sigma^e \frac{h_u}{A^2} + y_7^e l_2 l_1 \sigma^e \frac{h_u}{A^2} + y_8^e (l_1 l_1 \chi^c + l_2 l_2 \chi^c) \frac{h_2^u}{\Lambda^2} + y_9^e l_3 l_3 \sigma^c \frac{h_u^2}{A^2} . $$

As we will show below in appendix B, introducing appropriate ‘driving fields’ and minimizing the flavon superpotential leads to the following VEVs for the flavons:

$$ \langle \chi_1^c \rangle = v_e \left( \frac{1}{e^{2\pi i k/5}} \right) , \quad \langle \chi_2^c \rangle = v_e \left( \frac{1}{e^{3\pi i k/5}} \right) , \quad \langle \rho_1^c \rangle = z_e \left( \frac{1}{e^{4\pi i k/5}} \right) , \quad \langle \rho_2^c \rangle = \left( \frac{1}{e^{5\pi i k/5}} \right) . $$

$^4$ In a model including quarks, this may explain $m_\tau \ll m_1$ without invoking a large $\tan \beta$. 

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**Table 1.** Particle content of the $D_{10}$ model: $l_i$ are the three left-handed lepton doublets, $e_i^c$ are the right-handed charged lepton singlets and $h_{u,d}$ are the MSSM Higgs doublets. Furthermore we have flavons $\sigma^e, \chi_{1,2}^c, \xi_{1,2}^e, \rho_{1,2}^c, \sigma^\nu, \phi_{1,2}^\nu, \chi_{1,2}^e$ and $\xi_{1,2}^e$, which only transform under $D_{10} \times Z_5$. The phase $\omega = e^{2\pi i/5}$ is the fifth root of unity.

| Field | $l_{1,2}$ | $l_3$ | $e_{1,2}^c$ | $e_3^c$ | $h_{u,d}$ | $\sigma^e$ | $\chi_{1,2}^e$ | $\xi_{1,2}^e$ | $\rho_{1,2}^c$ | $\sigma^\nu$ | $\phi_{1,2}^\nu$ | $\chi_{1,2}^e$ | $\xi_{1,2}^e$ |
|-------|-----------|-------|-------------|---------|-----------|-------------|----------------|----------------|----------------|-------------|----------------|----------------|-----------|
| $D_{10}$ | $2_4$ | $\omega$ | $1_1$ | $\omega^2$ | $1_1$ | $\omega^2$ | $2_2$ | $\omega^3$ | $2_4$ | $\omega^3$ | $2_4$ | $\omega^3$ | $2_4$ |
| $Z_5$ | $\omega$ | $\omega^2$ | $\omega^3$ | $\omega^4$ | $\omega^5$ | $\omega^5$ | $\omega^5$ | $\omega^5$ | $\omega^5$ | $\omega^5$ | $\omega^5$ | $\omega^5$ | $\omega^5$ |
where \( k \) is an odd integer between 1 and 9, and
\[
\begin{align*}
\begin{pmatrix} \langle \varphi^e_1 \rangle \\ \langle \varphi^e_2 \rangle \end{pmatrix} &= v_e \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} \langle \chi^\mu_1 \rangle \\ \langle \chi^\mu_2 \rangle \end{pmatrix} = w_\mu \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} \langle \xi^\tau_1 \rangle \\ \langle \xi^\tau_2 \rangle \end{pmatrix} = z_\tau \begin{pmatrix} 1 \\ 1 \end{pmatrix}.
\end{align*}
\]
(16)

The VEVs of the singlet flavons \( \langle \sigma^e \rangle = x_e \) and \( \langle \sigma^\nu \rangle = x_\nu \) are assumed to be also non-vanishing. The VEV structure leads to the following mass matrices:
\[
m_\ell = \frac{\langle h_d \rangle}{\Lambda} \begin{pmatrix}
y^e_2 z_e & y^e_1 v_e e^{2\pi i k/5} & y^e_4 v_e e^{4\pi i k/5} \\
y^e_1 v_e e^{2\pi i k/5} & y^e_2 z_e & y^e_4 z_e \\
y^e_3 v_e e^{2\pi i k/5} & y^e_5 v_e & y^e_6 x_e
\end{pmatrix},
\]
\[
m_\nu = \frac{\langle h_u \rangle^2}{\Lambda^2} \begin{pmatrix}y_2 w_\nu & y_1 x_\nu & 0 \\
y_2 w_\nu & y_3 x_\nu & 0 \\
0 & 0 & y_3 x_\nu
\end{pmatrix}.
\]
(17)

To see that indeed the golden ratio prediction is obtained from the above two matrices, note that for the choice \( k = 3 \) the relevant matrix \( m_\ell m_\ell^\dagger \) takes the form
\[
m_\ell m_\ell^\dagger = \begin{pmatrix}A & B e^{-2i\Phi} & D e^{i(\delta-\Phi)} \\
B e^{2i\Phi} & A & D e^{i(\delta+\Phi)} \\
D e^{-i(\delta-\Phi)} & D e^{-i(\delta+\Phi)} & G \end{pmatrix}, \quad \text{where} \quad \Phi = \frac{4\pi}{5}.
\]
(18)

The quantities \( A, B, D \) and \( G \) are real and positive, \( \delta \) is a phase. To obtain the golden ratio prediction for the solar mixing angle, we have to set in equations (15) and (17) \( k = 3 \) or 7. From the other possibilities \( k = 1 \) or 9 would give a solar mixing angle of \( 2\pi/5 \), whereas \( k = 5 \) would give a vanishing solar mixing angle. This small number of degeneracies can not be resolved by the flavon potential. Looking at the last matrix \( m_\ell m_\ell^\dagger \) in equation (18), one immediately recognizes the \( Z_2 \)-invariance of the upper left 12-block, which is just the invariance we were looking for, see equation (8). To be precise, the \( D_{10} \) was broken in a way that \( m_\ell m_\ell^\dagger \) is left invariant under \( B A^3 \), whereas the neutrino mass matrix \( m_\nu \) is left invariant under \( B A^0 = B \). Inserting this in equation (13), where we have to set \( j = 4 \) because the first and second left-handed lepton doublets transform as \( 24 \), we expect \( |U_{e1}|^2 = |\cos \frac{2\pi}{5}|^2 \), which is indeed equivalent to an angle of \( \pi/5 \). We will explicitly check this in the following. Diagonalizing \( m_\ell m_\ell^\dagger \) with the relation
\[
U_\ell = \text{diag}(m^2_e, m^2_\mu, m^2_\tau) \quad \text{achieved with the matrix}
\]
\[
\begin{pmatrix}
\sqrt{-\frac{1}{2}} & \sqrt{\frac{1}{2}} & 0 \\
\sqrt{-\frac{1}{2}} & -\sqrt{\frac{1}{2}} & 0 \\
0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \theta_{23} & 0 \\
0 & \sin \theta_{23} & \cos \theta_{23}
\end{pmatrix}.
\]
(19)

The diagonal phase matrix on the left is crucial. The rotation angle in the 23-axis is given by
\[
\tan 2\theta_{23} = \frac{2\sqrt{2} D}{G - A - B},
\]
(20)
and the charged lepton masses are given by
\[
m^2_e = A - B, \quad m^2_\mu, \tau = \frac{1}{2} \left[(A + B + G) \pm w (A + B - G)\right],
\]
(21)
where
\[
w = \sqrt{1 + 8 D^2 / (A + B - G)^2}.
\]
The neutrino mass matrix is diagonalized via \( U^\dagger \nu m \nu U = m^{\text{diag}} \), with

\[
U_\nu = \begin{pmatrix}
-\sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} & 0 \\
\sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} & 0 \\
0 & 0 & 1
\end{pmatrix} P.
\]  

(22)

The eigenvalues have in general nontrivial phases, which are taken into account in the diagonal matrix \( P \), and their absolute values are

\[
m_1 = \frac{\langle h_u \rangle^2}{\Lambda^2} |y_2 w_\nu - y_1 x_\nu|,
\]

\[
m_2 = \frac{\langle h_u \rangle^2}{\Lambda^2} |y_2 w_\nu + y_1 x_\nu|,
\]

\[
m_3 = \frac{\langle h_u \rangle^2}{\Lambda^2} |y_3 x_\nu|.
\]  

(23)

We note that the model makes no predictions about the neutrino masses or their ordering. Nevertheless, one can easily convince oneself that the number of free parameters in the model is enough to fit the neutrino and charged lepton masses, as well as the large atmospheric neutrino mixing angle \( \theta_{23} \). The model in general does not predict \( \theta_{23} \) to be maximal, which is not an issue given the fact that it is the lepton mixing parameter with the largest allowed range. However, maximal mixing is compatible with the model. We have \( \theta_{23} = \pi/4 \) when \( G = A + B \), in which case \( m^2_{\mu,\tau} = A + B \mp \sqrt{2} \Delta D \) and \( m^2_\ell \) as in equation (21). The fact that there is not more predictivity can be traced to the fact that there is a comparably large number of flavon fields required in order to make the model work. This is the price one unfortunately has to pay if one insists on the rather peculiar value of \( \theta_{12} \). Given the fact that current data allows for this very interesting possibility, one should nevertheless pursue the task of constructing models leading to it.

The final PMNS matrix is

\[
U = U_\ell^T U_\nu.
\]  

(24)

One finds that \( U_{e3} \) is vanishing and that atmospheric neutrino mixing is governed by \( \tan 2\theta_{23} \) given by equation (20). As mentioned above, the PMNS matrix has a nontrivial phase matrix including \( \Phi \) in between the two maximal 12-rotations, one of which stems from \( U_\ell \), the other from \( U_\nu \). As discussed above, this is the origin of the required result. Indeed, the 12-element of \( U \) is

\[
|U_{e2}|^2 = \sin^2 \Phi = \sin^2 \pi/5,
\]  

(25)

and due to \( U_{e3} = 0 \) this is just \( \sin^2 \theta_{12} \). We have thus achieved our goal of predicting \( \theta_{12} = \pi/5 \).

As discussed in appendix B, higher order corrections to the scenario, as well as flavor changing neutral currents, can be estimated to give only very small contributions.

3. Renormalization corrections to the golden ratio predictions

It is worth discussing RG effects on the golden ratio predictions, because any symmetry leading to the predictions discussed in this paper could presumably be operating at a high-energy scale \( \Lambda \), and the observables have to be evolved down to the low energy scale \( \lambda \). Note that RG corrections to \( |U_{e3}| \) and \( \theta_{33} \) are typically suppressed with respect to the running of \( \theta_{12} \) by a factor of \( \Delta m^2_{\ell\nu} / \Delta m^2_{\ell\nu} \). As the initial values of both \( |U_{e3}| \) and \( \theta_{33} \) need not be specified here (other than being small or close to maximal, respectively) we do not comment on their RG shift.

We will stay model independent here and estimate the corrections as a function of the unknown neutrino mass values and ordering. An expression for \( \dot{\theta}_{12} \), where the dot denotes the derivative
with respect to $t = \ln \mu/\mu_0$ with $\mu$ the renormalization scale, is given e.g. in [10]. One can therefrom estimate the shift for the solar neutrino mixing angle:

$$\theta_{12} \simeq \theta_{12}^0 + k_{12} \epsilon_{RG} ,$$

where $\theta_{12}^0$ is the initial value of $\theta_{12}$ (here given by equation (3) or (4)) and

$$\epsilon_{RG} \equiv c \frac{m_\tau^2}{16\pi^2 v_0^2} \ln \frac{\Lambda}{\lambda}$$

with $v_0 = 246$ GeV, $c = -3/2$ in the SM and $(1 + \tan^2 \beta)$ in the MSSM. Neutrino physics is included in

$$k_{12} = \sin 2\theta_{12}^0 \sin^2 \theta_{23}^0 \frac{|m_1 + m_2 e^{2i\alpha}|^2}{\Delta m^2_\odot}.$$  

Consequently, from equation (26) one finds

$$\sin^2 \theta_{12} \simeq \sin^2 \theta_{12}^0 + k_{12} \epsilon_{RG} \sin 2\theta_{12}^0 .$$

Note that the Majorana phase $\alpha$ can suppress the running. As is well known, $\theta_{12}$ decreases in the SM and increases in the MSSM, independent of the sign of $\Delta m^2_{\lambda}$. The following numerical estimates are done with $\sin^2 \theta_{23}^0 = 1/2$, $\Lambda/\lambda = 10^{10}$ and with $\Delta m^2_\odot$, $\Delta m^2_{\lambda}$ fixed for simplicity at their current best-fit values [2]. In the normal hierarchy (NH, $m_3 \simeq \sqrt{\Delta m^2_{\lambda}}$, $m_2 \simeq \sqrt{\Delta m^2_\odot} \gg m_1$) the running in the SM is completely negligible. In the case of the MSSM,
even for $\tan \beta = 40$ the shift in $\sin^2 \theta_{12}$ is not more than 1.5%. This changes in the inverted hierarchy (IH, $m_2 \simeq m_1 \simeq \sqrt{\Delta m^2_{\odot}} \gg m_3$), where for the MSSM and $\tan \beta = 10$ the value of $\sin^2 \theta_{12}$ can increase by around 10%. In the SM, again, the shift is at less than half a percent not measurable. For quasi-degenerate (QD) neutrinos with a common mass scale of 0.2 eV the SM allows shifts of around 3%, whereas in the MSSM the shift can be as large as the value of $\sin^2 \theta_{12}$, even for small values of $\tan \beta = 5$. We illustrate this in figure 1, where we used equation (29) to show the RG-induced shifts of $\sin^2 \theta_{12}$ for a normal mass hierarchy (SM and MSSM with $\tan \beta = 40$), an IH (SM and MSSM with $\tan \beta = 10$), as well as QD neutrinos (smallest mass 0.2 eV for the SM and MSSM with $\tan \beta = 5$). In the cases of a normal and inverted hierarchy we have chosen (at high scale) 0.001 eV for the smallest neutrino mass. To a good approximation and unless in the MSSM $\tan \beta$ is very large, the running of the neutrino masses can be described by a rescaling, with basically no dependence on the other neutrino parameters [10]. Because $k_{12}$ from equation (28) has the masses appearing in the denominator and numerator, their running cancels in our approximation as long as $|k_{12} \epsilon_{RG}| \ll 1$. The range of the corrections in figure 1 is due to the unknown Majorana phases. For illustration, we also include the shifts for TBM.

To bring $\theta_{12}$ very close to the best-fit value, the prediction (A) requires the MSSM and IH or QD, whereas prediction (B) (and TMB) requires the SM with rather large neutrino masses. If future data lead to more precise determinations of $\sin^2 \theta_{12}$ and other neutrino parameters, one will be able to rule out some of the existing possibilities.

4. Summary

Precision flavor data may give hints toward the underlying physics. We have stressed in this paper that current data imply that the golden ratio $\varphi$ can be connected to solar neutrino mixing. With $\cot \theta_{12} = \varphi$ and $\cos \theta_{12} = \varphi/2$ there are two appealing possibilities, not too far away from current best-fit values and compatible with current 2$\sigma$ ranges. We have compared these values, estimated radiative corrections and in particular proposed a model based on the dihedral group $D_{10}$ leading to the relation $\cos \theta_{12} = \varphi/2$. The angle leading to $\cos \theta_{12} = \varphi/2$ is $\theta_{12} = \pi/5$, closely linked to the symmetry of a decagon, which naturally leads one to consider its rotational symmetry group $D_{10}$.

Acknowledgments

This work was supported by the ERC under the Starting Grant MANITOP and by the Deutsche Forschungsgemeinschaft in the Transregio 27 ‘Neutrinos and beyond—weakly interacting particles in physics, astrophysics and cosmology’ (WR). AB acknowledges support from the Studienstiftung des Deutschen Volkes.

Appendix A. Multiplication rules and Clebsch–Gordan coefficients of $D_{10}$

We present here the Clebsch–Gordan coefficients for $D_{10}$. The multiplication rules for the Kronecker products are given in table A.1. For $s_i \sim 1_i$ and $(a_1, a_2)^T \sim 2_j$, we find

$$\begin{align*}
(s_1 a_1) \sim 2_j, \quad & (s_2 a_1) \sim 2_j, \quad (s_3 a_2) \sim 2_j, \quad \text{and} \quad (s_4 a_2) \sim 2_{5-j},
\end{align*}$$

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If depending on whether $i$ (The Clebsch–Gordan coefficients for the product of supersymmetric Lagrangian, the superpotential must have a $U$ Again, the first case is relevant for $k$ or driving fields \[= 5 \text{ holds the covariants read} \]

\[
\begin{array}{cccc}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
\end{array}
\]

\[
\times \begin{array}{cccc}
2 & 1 & 2 & 3 \\
2 & 1 & 2 & 3 \\
2 & 1 & 2 & 3 \\
2 & 1 & 2 & 3 \\
\end{array}
\]

\[
\begin{array}{cccc}
1_1 + 1_3 + 1_3 & 2_1 + 1_3 & 2_3 + 1_3 & 1_1 + 1_3 + 1_3 \\
2_1 + 2_3 & 1_1 + 1_3 + 2_3 & 1_3 + 1_3 + 2_3 & 2_3 + 2_3 \\
1_1 + 1_3 + 1_3 & 1_3 + 1_3 + 1_3 & 2_1 + 1_3 & 2_3 + 1_3 \\
1_1 + 1_3 + 1_3 & 2_1 + 1_3 & 1_3 + 1_3 + 1_3 & 2_3 + 1_3 \\
\end{array}
\]

The Clebsch–Gordan coefficients for the product of $(a_1, a_2)^T$ with $(b_1, b_2)^T$, both in $\sim 2_i$, read

\[
a_1 b_2 + a_2 b_1 \sim 1_1, \quad a_1 b_2 - a_2 b_1 \sim 1_2,
\]

\[
\begin{array}{c}
(a_1 b_1) \\
(a_2 b_2)
\end{array} \sim 2_j \quad \text{or} \quad \begin{array}{c}
(a_2 b_1) \\
(a_1 b_2)
\end{array} \sim 2_j
\]

depending on whether $i = 1, 2$ or $i = 3, 4$. For the two doublets $(a_1, a_2)^T \sim 2_i$ and $(b_1, b_2)^T \sim 2_j$ we find for $i + j \neq 5$

\[
\begin{array}{c}
(a_1 b_2) \\
(a_2 b_1)
\end{array} \sim 2_k \quad (k = i - j) \quad \text{or} \quad \begin{array}{c}
(a_2 b_1) \\
(a_1 b_2)
\end{array} \sim 2_k \quad (k = j - i),
\]

\[
\begin{array}{c}
(a_1 b_1) \\
(a_2 b_2)
\end{array} \sim 2_l \quad (l = i + j) \quad \text{or} \quad \begin{array}{c}
(a_2 b_2) \\
(a_1 b_1)
\end{array} \sim 2_l \quad (l = 10 - (i + j)).
\]

If $i + j = 5$ holds the covariants read

\[
a_1 b_1 + a_2 b_2 \sim 1_3, \quad a_1 b_1 - a_2 b_2 \sim 1_4,
\]

\[
\begin{array}{c}
(a_1 b_2) \\
(a_2 b_1)
\end{array} \sim 2_k \quad \text{or} \quad \begin{array}{c}
(a_2 b_2) \\
(a_1 b_1)
\end{array} \sim 2_k.
\]

Again, the first case is relevant for $k = i - j$, whereas the second one is valid for $k = j - i$.

**Appendix B. VEV alignment of the $D_{10} \times Z_5$ model**

To obtain the necessary vacuum alignment in the flavon potential, we need to introduce $U(1)_R$ and driving fields \[12\]. Regular $R$-parity is a subgroup of the $U(1)_R$. To ensure a supersymmetric Lagrangian, the superpotential must have a $U(1)_R$ charge of 2. The superfields

\[
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containing the SM fermions have an $R$-charge of 1, whereas the Higgs fields have an $R$-charge of zero. Hence, for the Yukawa superpotential given in equation (14) to be viable, the flavons also need to have a vanishing $R$-charge. Consequently, for the flavon superpotential one needs to introduce additional flavor-charged fields, having an $R$-charge of 2. The transformation properties of these driving fields are given in table B.1. The flavon superpotential can then be divided into two parts

$$w_f = w_{f,e} + w_{f,v},$$

(B.1)

where $w_{f,e}$ and $w_{f,v}$ are responsible for the vacuum alignment of the flavons contributing to the charged lepton and neutrino masses, respectively. We begin by considering the charged lepton part:

$$w_{f,e} = a_e (\chi_1^e \xi_1^e + \chi_2^e \xi_2^e) \psi_0^e + b_e (\chi_1^e \xi_1^e \varphi^e_0 + \chi_2^e \xi_1^e \varphi_2^e) + c_e (\xi_1^e \rho_1^e \varphi_1^e + \xi_2^e \rho_2^e \varphi_2^e)$$

$$+ d_e (\xi_2^e \xi_1^e + \xi_1^e \rho_1^e) \sigma^e + f_e (\xi_1^e \rho_1^e \xi_1^e + \xi_2^e \rho_2^e \xi_2^e).$$

(B.2)

As the flavor symmetry is broken at a high scale, the scalar potential can be minimized in the supersymmetric limit. The flavons and driving fields are not charged under any gauge group, so the scalar potential is given by the F-terms alone. Hence, we can determine the supersymmetric minimum of the potential by setting the F-terms of the driving fields to zero:

$$\frac{\partial w_{f,e}}{\partial \psi_0^e} = a_e (\chi_1^e \xi_1^e + \chi_2^e \xi_2^e) = 0,$$

$$\frac{\partial w_{f,e}}{\partial \varphi_1^e} = b_e \chi_1^e \xi_2^e + c_e \xi_1^e \rho_2^e = 0,$$

$$\frac{\partial w_{f,e}}{\partial \varphi_2^e} = b_e \chi_2^e \xi_1^e + c_e \xi_2^e \rho_1^e = 0,$$

$$\frac{\partial w_{f,e}}{\partial \xi_1^e} = d_e \xi_2^e \sigma^e + f_e \xi_1^e \rho_1^e = 0,$$

$$\frac{\partial w_{f,e}}{\partial \xi_2^e} = d_e \xi_1^e \sigma^e + f_e \xi_2^e \rho_2^e = 0.$$

Similarly, from the neutrino part

$$w_{f,v} = a_\nu (\chi_1^\nu \xi_1^\nu - \chi_2^\nu \xi_2^\nu) \psi_0^v + b_\nu (\varphi_1^\nu \chi_1^\nu \xi_2^0 + \varphi_2^\nu \chi_2^\nu \xi_1^0) + c_\nu (\xi_1^\nu \varphi_1^0 + \xi_2^\nu \xi_1^0) \sigma^\nu$$

$$+ d_\nu (\varphi_1^\nu \varphi_2^\nu \chi_1^0 + \varphi_2^\nu \chi_2^\nu \xi_1^0) = 0,$$

(B.3)

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we obtain a minimum of the potential by setting the F-terms of the driving fields to zero:

\[
\frac{\partial w_{f,\nu}}{\partial \psi_{0\nu}} = a_\nu (\chi_{1\nu} \xi_{1\nu} - \chi_{2\nu} \xi_{2\nu}) = 0,
\]

\[
\frac{\partial w_{f,\nu}}{\partial \chi_{0\nu}^1} = d_\nu \phi_{1\nu}^2 + f_\nu (\phi_{2\nu}^2 + g_\nu \chi_{2\nu} \sigma_{2\nu}) = 0,
\]

\[
\frac{\partial w_{f,\nu}}{\partial \chi_{0\nu}^2} = d_\nu \phi_{2\nu}^2 + f_\nu (\phi_{1\nu}^2 + g_\nu \chi_{1\nu} \sigma_{1\nu}) = 0,
\]

\[
\frac{\partial w_{f,\nu}}{\partial \xi_{0\nu}^1} = b_\nu \phi_{2\nu} \chi_{1\nu} + c_\nu \xi_{2\nu} \sigma_{2\nu} = 0,
\]

\[
\frac{\partial w_{f,\nu}}{\partial \xi_{0\nu}^2} = b_\nu \phi_{1\nu} \chi_{1\nu} + c_\nu \xi_{1\nu} \sigma_{1\nu} = 0.
\]

As advocated above, these two sets of equations are uniquely solved by the VEV configurations given in equations (15) and (16), where we have set a possible relative phase in the doublet of VEVs of the flavons in the charged lepton sector to zero. This can be done without loss of generality, as only the phase difference between the two sectors is phenomenologically relevant. We have also assumed that none of the parameters in the superpotential vanish. For the charged lepton sector, the flavon VEVs \(w_e\) and \(x_e\) are free parameters (which we take to be nonzero), while

\[
v_e = e^{4\pi i k/5} \frac{c_e d_e x_e}{b_e f_e}, \quad z_e = e^{8\pi i k/5} \frac{d_e x_e}{f_e}.
\]

Similarly, \(v_\nu\) and \(x_\nu\) are free parameters (again taken to be non-vanishing) and

\[
w_\nu = -\frac{c_\nu f_\nu x_\nu v_\nu^2}{c_\nu g_\nu x_\nu^2 - b_\nu d_\nu v_\nu^2}, \quad z_\nu = \frac{b_\nu f_\nu v_\nu^3}{c_\nu g_\nu x_\nu^2 - b_\nu d_\nu v_\nu^2}.
\]

The driving fields themselves are only allowed vanishing VEVs, as can be inferred from considering the F-terms of the flavons. Note that since we cannot make the cutoff scale \(\Lambda\) arbitrarily large, we need to take into account NLO corrections to both the Yukawa and flavon superpotentials. We also should be careful regarding potentially dangerous flavor changing neutral currents induced by the flavons. All this could be taken into account by carefully studying the mass spectrum of the scalars. Given the sizable number of fields this is a formidable task, but fortunately it suffices to make some general estimates, which agree well quantitatively with a lengthy explicit calculation in a similar model [13]: the \(\tau\) lepton mass, see equation (21), is of the order of \(\langle f \rangle v/\Lambda\), where \(\langle f \rangle\) is a flavon VEV, \(v\) the Higgs VEV (\(\simeq 10^2\) GeV) and \(\Lambda\) the cutoff scale. The neutrino mass, see equation (23), is of the order of \(\langle f \rangle v^2/\Lambda^2\). With the charged lepton \(\tau\) mass \(\simeq 2\) GeV and the neutrino mass \(\simeq 0.1\) eV it follows that \(\Lambda \simeq 10^{12}\) GeV and \(\langle f \rangle \simeq 10^{10}\) GeV. Now we can estimate that the flavon mass is also of the order of \(\langle f \rangle\). NLO corrections to the potential, and therefore to the neutrino and charged lepton mass matrices, are of the order of \(\langle f \rangle /\Lambda \simeq 10^{-2}\) and therefore under control. Any potentially dangerous flavor changing neutral currents are also suppressed by the heavy mass scale \(\langle f \rangle\).
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