A new continuum limit of matrix models

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Abstract

We define a new scaling limit of matrix models which can be related to the
method of causal dynamical triangulations (CDT) used when investigating two-
dimensional quantum gravity. Surprisingly, the new scaling limit of the matrix
models is also a matrix model, thus explaining why the recently developed CDT
continuum string field theory (arXiv:0802.0719) has a matrix-model representation
(arXiv:0804.0252).
1 Introduction

The great versatility of matrix models or matrix integrals in theoretical physics is well illustrated by their particularly beautiful application in two-dimensional Euclidean quantum gravity (see [1, 2, 3, 4] for reviews). This theory can be defined as a suitable sum over triangulations, so-called “dynamical triangulations” (DT), whose continuum limit is obtained by taking the side lengths $a$ of the triangles to zero. The method of DT was originally introduced as a nonperturbative world-sheet regularization of the Polyakov bosonic string [5, 6, 7]. There it was used with success (or to disappointment, depending on one’s taste) to show rigorously that a tachyon-free version of Polyakov’s bosonic string theory does not exist in target space dimensions $d > 1$ [8]. However, when viewed as a theory of 2d quantum gravity coupled to matter with central charge $c \leq 1$, the theory – noncritical string theory – is perfectly consistent, and matrix models have been used to solve in an elegant way the combinatorial aspects of the DT construction, where one sums over random surfaces glued together from equilateral triangles.

The DT approach possesses a well-defined cut-off, the length $a$ of the lattice links. As has been discussed in many reviews (for instance the ones mentioned above), a continuum limit can be defined when the lattice spacing is taken to zero while simultaneously renormalizing the bare cosmological constant and possibly other matter coupling constants. However, the continuum limit in question has some unconventional properties. In this article we will show that there is another way of taking the scaling limit of the matrix models, which still relates them to a summation over triangulated random surfaces, the so-called causal dynamical triangulations (CDT) [9]. We will show that this limit is in a way more natural and does not lead to the somewhat unconventional renormalization encountered in the standard DT approach.

In the next section, we will review briefly the conventional situation in the simplest case, that of pure Euclidean two-dimensional quantum gravity without matter. We follow the notations in [4]. In Sec. 3 we introduce a new limit of the corresponding matrix model and analyze its interpretation in terms of random surfaces. Sec. 4 discusses the results obtained and outlines possible applications.

2 The old matrix model and its continuum limit

Our starting point is the Hermitian matrix integral

$$Z(\tilde{g}) = \int d\phi \, e^{-N\text{tr} \left( \frac{1}{2} \phi^2 - \frac{2}{3} \phi^3 \right)} = \sum_{k=0}^{\infty} \frac{1}{k!} \int d\phi \, e^{-\frac{1}{2}N\text{tr} \left( \phi^2 \right)} \left( \frac{N\tilde{g}}{3} \text{tr} \phi^3 \right)^k,$$  \hspace{1cm} (1)

where $\phi$ is an $N \times N$ Hermitian matrix. This integral is formal since (assuming $\tilde{g} > 0$) it is not convergent unless an analytic continuation is performed. However,
by power-expanding \( \exp(N \tilde{g} \tr \phi^3) \) as indicated in eq. (1), the matrix integral defines a formal power series in the coupling constant \( \tilde{g} \). Furthermore, this expansion can be given the geometric interpretation of gluing together triangles in all possible ways. The size \( N \) of the matrix acts as a factor, organizing the power series further into a summation over surfaces of fixed topology. It turns out that the power series corresponding to a given fixed topology is convergent with convergence radius \( \tilde{g}_c \) independent of the topology considered, a property that has made the matrix integrals useful in the study of noncritical strings. The particular rearrangement of the power series (1) according to topology, i.e. in powers of \( 1/N^2 \), is called the large-\( N \) expansion.

Let us briefly review how a “continuum limit” of the matrix integrals can be associated with noncritical string theory in the simplest case of \( c = 0 \), “pure” Euclidean two-dimensional quantum gravity, where there is no extended target space (equivalently, no matter fields coupled to 2d gravity). This is best done by calculating an “observable”, the so-called disc amplitude or Hartle-Hawking wave function \( w(z) \) of the 2d universe to leading order in the \( 1/N^2 \) expansion. One has

\[
w(z) \equiv \left\langle \frac{1}{N} \tr \left( \frac{1}{z - \phi} \right) \right\rangle = \frac{1}{N} \sum_{n=0}^{\infty} \frac{\langle \tr \phi^n \rangle}{z^{n+1}},
\]

where

\[
\langle \tr \phi^n \rangle \equiv Z(\tilde{g})^{-1} \int d\phi \tr \phi^n \ e^{-N\tr(\frac{1}{2}\phi^2 - \frac{\tilde{g}}{3}\phi^3)},
\]

again to be viewed as a formal power series in \( \tilde{g} \). To leading order in \( 1/N \) one finds

\[
w(z) = \frac{1}{2} \left( z - \tilde{g}z^2 + \tilde{g}(z - b)\sqrt{(z - c)(z - d)} \right), \quad c > 0, \quad c \geq |d|,
\]

where the constants \( b(\tilde{g}), c(\tilde{g}) \) and \( d(\tilde{g}) \) are functions of \( \tilde{g} \) and uniquely determined by the requirement that \( w(z) \) fall off like \( 1/z \) (with coefficient 1) for \( z \rightarrow \infty \).

The geometric interpretation of \( w(z) \) is as follows: the term in (2) corresponding to \( \langle \tr \phi^n \rangle \) represents the summation over all triangulations with a boundary consisting of \( n \) links and (to leading order in the large-\( N \) expansion) the topology of a disc. Consequently, \( zw(z) \) represents the sum over all triangulations with the topology of a disc and the constant \( \ln z \) has the interpretation of a boundary cosmological constant \( l_b \), such that a boundary of length \( n \) is assigned a weight \( e^{-l_bn} \). In the same way as \( w(z) \) has a power expansion in \( \tilde{g} \), it also has a power expansion in \( 1/z \)

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1By expanding the matrix integral in a power series in \( \tilde{g} \) and subsequently in powers of \( 1/N^2 \), we are of course leaving out any nonperturbative contributions to the matrix integral (defined in some way, e.g. by analytic continuation) which are not captured by such an expansion.

2When performing the Gaussian integral, each boundary link, represented by a factor of \( \phi \) in \( \tr \phi^n \), is glued either to a triangle or to another boundary link. In the latter case we call it a “double link” and it will not be glued to other triangles, see Fig. 1.
Figure 1: Graphical representation of relation (5): The boundary contains one marked link which is part of a triangle or a double link. Associated to each triangle is a weight $\tilde{g}$, and to each double link a weight 1.

and the radius of convergence is $c(\tilde{g})$. The geometric interpretation can be further elaborated upon by noting that $w(z)$ satisfies the combinatorial equation

$$w(z) = z\tilde{g}w(z) + \frac{1}{z}w^2(z) + \frac{1}{z}Q(z, \tilde{g}),$$

(5)

where $Q(z, \tilde{g})$ is a polynomial in $z$ which is uniquely determined by the requirement that the solution to (5) have the form (4) and fall off like $1/z$. The recursion relation leading to (5) is represented graphically in Fig. 1 and will be important in the next section.

The continuum limit of the matrix model, relevant for noncritical string theory, is obtained as follows. Note first that the radius of convergence $\tilde{g}_c$ is determined by the condition

$$b(\tilde{g}) = c(\tilde{g}),$$

(6)

namely, the point where $w(z)$ changes analytic structure from $(z-c)^{1/2}$ to $(z-c)^{3/2}$. Next we fine-tune $\tilde{g}$ to $\tilde{g}_c$ and $z$ to $c(\tilde{g}_c)$ according to

$$\tilde{g} = \tilde{g}_c(1 - \Lambda a^2 + O(a^4)), \quad z = c(\tilde{g}_c) + aZ + O(a^2).$$

(7)

These assignments can be viewed as additive renormalizations of the bare cosmological and boundary cosmological constants such that $\Lambda$ and $Z$ now represent the renormalized coupling constants in the limit where the lattice spacing $a \to 0$. The rationale behind interpreting $\Lambda$ as a continuum cosmological constant comes from considering the term in $w(z)$ with $k$ triangles glued together, for very large $k$. It will appear as

$$s(k) \left( \frac{\tilde{g}}{\tilde{g}_c} \right)^k \approx s(k) e^{-k\Lambda a^2},$$

(8)

\footnote{For a combinatorial interpretation of $Q(z, \tilde{g})$ see, for instance, \cite{4}. $Q(z)$ is present in eq. (5) because Fig. 1 is not correct when the boundary consists of less than two links.}
where $s(k)$ is a subleading term and we associate $A(k) = ka^2$ with a macroscopic area. The coupling constant multiplying this area is by definition proportional to the continuum renormalized cosmological constant. Similarly one is led to the conclusion that $Z$ can be interpreted as a continuum boundary cosmological constant.

Inserting (7) into (4), we obtain in the limit of $a \to 0$ that

$$w(z) = \frac{1}{2} \left( z - \tilde{g} z^2 + \tilde{g} \sqrt{c(g_c) - d(g_c)} \right) a^{3/2} W_E(Z, \Lambda) \left( 1 + O(a) \right),$$

where

$$W_E(Z, \Lambda) = (Z - \sqrt{2\Lambda/3}) \sqrt{Z + 2\sqrt{2\Lambda/3}}$$

is called the continuum Hartle-Hawking wave function or (using string terminology) the continuum disc amplitude.

The above arguments can be generalized from the matrix potential in (1) to a general potential of the form

$$V(\phi) = \frac{1}{2} \phi^2 - g \sum_{i=1}^{n} t_i \phi^i,$$

again leading to a formal power series after expanding the exponential in powers of the coupling constant $g$ and performing the remaining Gaussian integrals. The geometric interpretation is analogous to that for the cubic potential, only that now we glue together $i$-gons and $j$-gons, where $i$ and $j$ run from 1 to $n$, with each $i$-gon assigned a relative weight $t_i$. By allowing such polygons we are clearly considering a generalization of simplicial complexes, but we will for convenience continue to refer to these as “triangulations”. The Hartle-Hawking wave function $w(z)$ now satisfies the combinatorial equation

$$w(z) = g \left( \sum_{i=1}^{n} t_i z^{i-2} \right) w(z) + \frac{1}{z} w^2(z) + \frac{1}{z} Q(z, g),$$

where again the polynomial $Q(z, g)$ is determined by the requirement that $w(z)$ fall off like $1/z$ and have a single cut. The equation has the graphical representation shown in Fig. 2.

As long as the weights $t_i$ are positive (a natural requirement if one wants to assign an area to the $i$-gon), one obtains

$$w(z) = \frac{1}{2} \left( V'(z) + g t_n \sqrt{c(g_c) - d(g_c)} a^{3/2} W_E(Z, \Lambda) \right)$$

in the limit $a \to 0$. In (13), $V'(z)$ denotes the derivative with respect to $z$ of the potential $V(z)$ defined in (11), $g_c$ the critical value of $g$ (for fixed $t_i$’s), i.e. the radius
Figure 2: Graphical representation of relation (12): the marked link of the boundary belongs either to an \(i\)-gon (associated weight \(g_i\)) or a double link (associated weight 1). It is also a graphical representation of eq. (38) if instead of weight 1 we associate a weight \(g_s\) to the marked double link.

of convergence of the formal power expansion in \(g\) for fixed topology (in this case that of the disc), and one has made an ansatz similar to (7), namely,\n\begin{equation}
    g = g_c(1 - \Lambda a^2), \quad z = c(g_c) + aZ
\end{equation}

for the coupling constants. One peculiar aspect of (9) and (13) is the non-scaling part \(V'(z)/2\). This term clearly dominates when \(a \to 0\), and in fact renders the average number of \(i\)-gons present in the ensemble with partition function \(w(z)\) finite, even at the critical point (14). This somewhat embarrassing fact can be circumvented by differentiating \(w(z)\) a sufficient number of times with respect to \(g\) and \(z\), which will make these “non-universal” contributions vanish. For example, differentiating \(w(z)\) twice with respect to \(g\) yields an expression which diverges in the limit \(a \to 0\), allowing one to ignore the finite contribution from non-universal terms,\n\begin{equation}
    \frac{\partial^2 w(z)}{\partial g^2} \propto \frac{1}{a^{5/2}} \frac{1}{\sqrt{\Lambda}} \frac{1}{(Z + \sqrt{2\Lambda}/3)^{3/2}} \propto \frac{1}{a^{5/2}} \frac{\partial^2 W_E(Z, \Lambda)}{\partial \Lambda^2}.
\end{equation}

In this sense one may view \(W_E(Z, \Lambda)\) as the continuum disc function. However, it would be incorrect to say that \(w(z)\) was the regularized Hartle-Hawking wave function from which one obtained the continuum Hartle-Hawking wave function through an additive renormalization of the cosmological and the boundary cosmological constants.

In the following section, we will discuss a new kind of scaling limit for matrix models, where such non-universal term are not present.
3 The new scaling limit of matrix models

Hermitian matrix models are often analyzed in terms of the dynamics of their eigenvalues. Since the action (11) is invariant under the transformation $\phi \to U\phi U^\dagger$, with $U \in U(N)$ a unitary $N \times N$-matrix, one can integrate out the “angular” degrees of freedom. What is left is an integration over the eigenvalues $l_i$ of $\phi$ only,

$$Z(g) \propto \int \prod_{i=1}^N dl_i \, e^{-N \sum V(l_i)} \prod_{k<l} |l_k - l_l|^2,$$

where the last factor, the Vandermonde determinant, comes from integrating over the angular variables, and where

$$\text{tr } V(\phi) = \sum_{i=1}^N V(l_i).$$ (17)

Naively one might expect that the large-$N$ limit is dominated by a saddle-point with $V'(l) = 0$. However, this is not the case since the Vandermonde determinant in (16) contributes in the large-$N$ limit. The cut which appears in $w(z)$ is a direct result of the presence of the Vandermonde determinant. In this way one can say that the dynamics of the eigenvalues is “non-classical”, deviating from $V'(l) = 0$, the size of the cut being a measure of this non-classicality. We will now introduce a new coupling constant $g_s$ in the matrix model by substituting

$$V(\phi) \to \frac{1}{g_s} V(\phi),$$ (18)

and consider the limit $g_s \to 0$. As will become clear in due course, this controls and reduces the size of the cut and thus brings the system closer to a “classical” behaviour.

In the analysis it will be convenient to keep the coupling constant $t_1 > 0$, which can be motivated as follows. Consider a “triangulation” consisting of $T_1$ one-gons, $T_2$ two-gons, $T_3$ triangles, $T_4$ squares etc., up to $T_n n$-gons, and expand the $g$-dependent part of the associated matrix integral

$$Z(g, g_s) = \int d\phi \, e^{-\frac{N}{g} \text{tr} \left( \frac{1}{2} \phi^2 - g \sum_{i=1}^n \frac{1}{i} \phi^i \right)}$$ (19)

in powers of $g$. Each $i$-gon appearing in the triangulation has a factor $g/g_s$ associated with it. At the same time, the Gaussian integration will produce a factor $g_s^L$, where

$$L = \frac{1}{2} T_1 + \frac{3}{2} T_3 + \cdots + \frac{n}{2} T_n$$ (20)
is the number of links in the triangulation, since each Gaussian integration corresponds precisely to a gluing of an $i$-gon and a $j$-gon. The total coupling-constant factor associated with the triangulation is therefore given by

$$g_{T_1 + \cdots + T_n} g_s^{-T_1/2 + T_3/2 + \cdots + (n/2 - 1)T_n}. \quad (21)$$

We observe that in the limit $g_s \to 0$, a necessary condition for obtaining a finite critical value $g_c(g_s)$ is $T_1 > 0$. We should emphasize that the analysis described below can be carried out also if we suppress the appearance of any one-gons (by setting $t_1 = 0$) in our triangulations, but it is slightly more cumbersome since then $g_c(g_s) \to \infty$ as $g_s \to 0$, requiring further rescalings.

For simplicity we will consider the simplest nontrivial model with potential

$$V(\phi) = \frac{1}{g_s} \left(-g\phi + \frac{1}{2}\phi^2 - \frac{g}{3}\phi^3\right) \quad (23)$$

and analyze its behaviour in the limit $g_s \to 0$. The disc amplitude now has the form

$$w(z) = \frac{1}{2g_s} \left(-g + z - gz^2 + g(z - b)\sqrt{(z - c)(z - d)}\right), \quad (24)$$

and the constants $b$, $c$ and $d$ are determined by the requirement that $w(z) \to 1/z$ for $z \to \infty$. Compared with the analysis of the previous section, the algebraic condition fixing the coefficient of $1/z$ to be unity will now enforce a completely different scaling behaviour as $g_s \to 0$.

For the time being, we will think of $g_s$ as small and fixed, and perform the scaling analysis for $g_c(g_s)$. As already mentioned in eq. (6) above, the critical point $g_c$ is determined by the additional requirement that $b(g_c) = c(g_c)$, which presently leads to the equation

$$\frac{(1 - 4g_c^2)^{3/2}}{3} = 12\sqrt{3} g_c^2 g_s. \quad (25)$$

Anticipating that we will be interested in the limit $g_s \to 0$, we write the critical points as

$$g_c(g_s) = \frac{1}{2}(1 - \Delta g_c(g_s)), \quad \Delta g_c(g_s) = \frac{3}{2}g_s^{2/3} + O(g_s^{4/3}), \quad (26)$$

and

$$z_c(g_s) = c(g_c, g_s) = \frac{1}{2g_c(g_s)} \left(1 + \sqrt{\frac{1 - 4g_c(g_s)^2}{3}}\right) = 1 + g_s^{1/3} + O(g_s^{2/3}), \quad (27)$$

In accordance with the remark above this model can be related to the model without one-gons by the field and coupling-constant redefinitions

$$\tilde{\phi} = \sqrt{1 - 4g^2} \left(\phi + \frac{2g}{1 + \sqrt{1 - 4g^2}}\right), \quad \tilde{g} = \frac{g}{\sqrt{1 - 4g^2}}, \quad (22)$$

and

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while the size of the cut in (23), \( c(g_c) - d(g_c) \), behaves as

\[
c(g_c) - d(g_c) = 4g_s^{1/3} + 0(g_s^{2/3}). \tag{28}
\]

Thus the cut shrinks to zero as \( g_s \to 0 \).

Expanding around the critical point given by (26)-(27) a nontrivial limit can be obtained if we insist that in the limit \( a \to 0 \), \( g_s \) scales according to

\[
g_s = G_s a^3, \tag{29}
\]

where \( a \) is the lattice cut-off introduced earlier. With this scaling the size of the cut scales to zero as \( 4a G_s^{1/3} \). In addition \( \sqrt{(z - c)(z - d)} \propto a \) if we introduce the standard identification (14): \( z = c(g_c) + aZ \). This scaling is different from the conventional scaling in Euclidean quantum gravity where \( \sqrt{(z - c)(z - d)} \propto a^{1/2} \) since in that case \( (z - c) \) scales while \( (z - d) \) does not scale.

We can now write

\[
g = g_c(g_s)(1 - a^2\Lambda) = \bar{g}(1 - a^2\Lambda_{cdt} + O(a^4)), \tag{30}
\]

with the identifications

\[
\Lambda_{cdt} \equiv \Lambda + \frac{3}{2}G_s^{2/3}, \quad \bar{g} = \frac{1}{2}, \tag{31}
\]

as well as

\[
z = z_c + aZ = \bar{z} + aZ_{cdt} + O(a^2), \tag{32}
\]

with the identifications

\[
Z_{cdt} \equiv Z + G_s^{1/3}, \quad \bar{z} = 1. \tag{33}
\]

Using these definitions one computes in the limit \( a \to 0 \) that

\[
w(z) = \frac{1}{a} \frac{\Lambda_{cdt} - \frac{1}{2}Z_{cdt}^2 + \frac{1}{2}(Z_{cdt} - H)\sqrt{(Z_{cdt} + H)^2 - 4G_s H}}{2G_s}. \tag{34}
\]

In (34), the constant \( H \) (or rather, its rescaled version \( h = H/\sqrt{2\Lambda_{cdt}} \)) satisfies the third-order equation

\[
h^3 - h + \frac{2G_s}{(2\Lambda_{cdt})^{3/2}} = 0, \tag{35}
\]

which follows from the consistency equations for the constants \( b, c \) and \( d \) in the limit \( a \to 0 \). We thus define

\[
w(z) = \frac{1}{a} W_{cdt}(Z_{cdt}, \Lambda_{cdt}, G_s) \equiv \frac{1}{a} W(Z, \Lambda, G_s) \tag{36}
\]

in terms of the continuum Hartle-Hawking wave functions \( W_{cdt}(Z_{cdt}, \Lambda_{cdt}, G_s) \) and \( W(Z, \Lambda, G_s) \).
Notice that while the cut of $\sqrt{(z - c)(z - d)}$ goes to zero as the lattice spacing $a$, it nevertheless survives in the scaling limit when expressed in terms of renormalized “continuum” variables, as is clear from eq. (34). Only in the limit $G_s \to 0$ it disappears and we have

$$w(z) = \frac{1}{a} W_{\text{cdt}}(Z_{\text{cdt}}, \Lambda_{\text{cdt}}, G_s) \xrightarrow{G_s \to 0} \frac{1}{a} \left( \frac{1}{Z_{\text{cdt}} + \sqrt{2\Lambda_{\text{cdt}}}} \right),$$

which is the original CDT disk amplitude introduced in [9].

A number of comments are in order to put this result into the context of previous work concerning the standard and generalized 2d CDT theory, as well as Euclidean quantum gravity:

1. Eqs. (34) and (35) are precisely the equations that were derived in the generalized CDT model of references [10, 11, 12]. This suggests that $G_s$ should be interpreted as a coupling constant associated with the splitting of a spatial universe into two. This is a meaningful statement in a universe with Lorentzian signature, where (in the simplest case) such a splitting is associated with an isolated point where the metric and its associated light-cone structure are degenerate, which has a diffeomorphism-invariant meaning. This was the motivation for considering such processes in CDT in the first place, leading to a generalization of the original CDT disc amplitude alluded to in (37) to the expression $W_{\text{cdt}}(Z_{\text{cdt}}, \Lambda_{\text{cdt}}, G_s)$ given by eq. (34) when $G_s > 0$. However, we can also formally make such an association in this purely Euclidean matrix model by noting that by introducing the coupling constant $g_s$ (and assuming $t_1 = t_3 = 1$ and $t_i = 0$ otherwise) eq. (12) is changed to

$$w(z) = g \left( \sum_{i=1}^{n} t_i z^{i-2} \right) w(z) + \frac{g_s}{z} w^2(z) + \frac{1}{z} Q(z, g).$$

Going back to Fig. 2, this suggests that one should associate a factor $g_s$ instead of a factor 1 with the graph with the double line. Geometrically this can be viewed as a process where a triangle is removed at a marked link (and a new link is marked at the new boundary), except in the case where the marked link does not belong to a triangle, but is part of a double-link, in which case the double link is removed and the triangulation is separated into two. If one thinks of the recursion process in Fig. 2 as a “peeling away” of the triangulation as proper time advances, the presence of a double link represents the “acausal” splitting point beyond which the triangulation splits into two discs with two separate boundary components (i.e. two separate one-dimensional spatial universes). Associating an explicit weight $g_s$ with this situation and letting $g_s \to 0$ suppresses this process compared to processes where we simply remove an $i$-gon from the triangulation. Nevertheless, in the
limit $a \to 0$ the process survives precisely when we scale $g_s \to 0$ as prescribed by eq. (29). The interpretation of this process, advocated in [13], is that it represents a split of the spatial boundary with respect to (Euclidean) proper time.

(2) Using eq. (35) we can expand $w(z)$ into a power series in $G_s/\Lambda_{\text{cdt}}^{3/2}$ whose radius of convergence is $1/\sqrt[3]{3}$. For fixed values of $\Lambda_{\text{cdt}}$, this value corresponds to the largest value of $G_s$ where (35) has a positive solution for $h$. The existence of such a bound on $G_s$ for fixed $\Lambda_{\text{cdt}}$ was already observed in [10]. This bound can be re-expressed more transparently in the present Euclidean context, where it is more natural to keep the “Euclidean” cosmological constant $\Lambda$ fixed, rather than $\Lambda_{\text{cdt}}$. We have

$$\frac{G_s}{(2\Lambda_{\text{cdt}})^{3/2}} \leq \frac{1}{3\sqrt[3]{3}} \Rightarrow \frac{3G_s^{2/3}}{2\Lambda + 3G_s^{2/3}} \leq 1,$$

which for fixed $\Lambda > 0$ is obviously satisfied for all positive $G_s$. In order to see that the usual Euclidean 2d quantum gravity (characterized by some finite value for $g_s$) can be rederived from the disc amplitude (34), let us expand eq. (34) for large $G_s$. The square root part becomes

$$a^{-1} G_s^{-5/6} \left( Z - \sqrt{2\Lambda/3} \right) \sqrt{Z + 2\sqrt{2\Lambda/3}},$$

which coincides with the generic expression $a^{3/2}W_E(Z, \Lambda)$ in Euclidean 2d quantum gravity (c.f. eqs. (9) and (10)) if we take $G_s$ to infinity as $g_s/a^3$. However, if we reintroduce the same scaling in the $V'(z)$-part of $w(z)$, it does not scale with $a$ but simply goes to a constant. This term would dominate $w(z)$ in the limit $a \to 0$ if one did not remove it by hand, as is usually done in the Euclidean model (see Sec. 2).

(3) Why does the potential $V'(z)$ (and therefore the entire disc amplitude $w(z)$) scale (like $1/a$) in the new continuum limit with $g_s = G_s a^3$, $a \to 0$, contrary to the situation in ordinary Euclidean quantum gravity? This is most clearly seen by looking again at the definitions (30) and (32). Because of the vanishing

$$V'(\bar{z}, \bar{g}) = 0, \quad V''(\bar{z}, \bar{g}) = 0$$

in the point $(\bar{z}, \bar{g}) = (1, 1/2)$, expanding around $(\bar{z}, \bar{g})$ according to (30), (32) leads automatically to a potential which is of order $a^2$ when expressed in terms of the renormalized constants $(Z_{\text{cdt}}, \Lambda_{\text{cdt}})$, precisely like the square-root term when expressed in terms of $(Z_{\text{cdt}}, \Lambda_{\text{cdt}})$.

The point $(\bar{z}, \bar{g})$ differs from the critical point $(z_c(g_s), g_c(g_s))$, as long as $g_s \neq 0$. In fact, both $1/\bar{z}$ and $\bar{g}$ lie beyond the radii of convergence of $1/\bar{z}$ and $g$, which
are precisely $1/z_c(g_s)$ and $g_c(g_s)$. However, since the differences are of order $a$ and $a^2$, respectively, they simply amount to shifts in the renormalized variables, as made explicit in eqs. (31) and (33). Therefore, re-expressing $W(Z, \Lambda, G_s)$ in (36) in terms of the variables $Z_{\text{cdt}}$ and $\Lambda_{\text{cdt}}$ simply leads to the expression $W_{\text{cdt}}(Z_{\text{cdt}}, \Lambda_{\text{cdt}}, G_s)$, first derived in [10]. Similarly, any geometric quantities defined with respect to $Z$ and $\Lambda$ can equally well be expressed in terms of $Z_{\text{cdt}}$ and $\Lambda_{\text{cdt}}$. For instance, the average continuum length of the boundary and the average continuum area of a triangulation are given by

$$\langle L \rangle = \frac{\partial \ln W(Z, \Lambda, G_s)}{\partial Z} = \frac{\partial \ln W_{\text{cdt}}(Z_{\text{cdt}}, \Lambda_{\text{cdt}}, G_s)}{\partial Z_{\text{cdt}}},$$

$$\langle A \rangle = \frac{\partial \ln W(Z, \Lambda, G_s)}{\partial \Lambda} = \frac{\partial \ln W_{\text{cdt}}(Z_{\text{cdt}}, \Lambda_{\text{cdt}}, G_s)}{\partial \Lambda_{\text{cdt}}}. \quad (42)$$

In the limit of $G_s \to 0$, the variables $(Z, \Lambda)$ and $(Z_{\text{cdt}}, \Lambda_{\text{cdt}})$ become identical and the disc amplitude becomes the original CDT amplitude $(Z_{\text{cdt}} + \sqrt{2\Lambda_{\text{cdt}}})^{-1}$ alluded to in (37).

(4) In the matrix potential (23), which formed the starting point of our new scaling analysis, we are still free to perform a change of variables. Inspired by relations (30)–(33), let us transform to new “CDT”-variables

$$\phi \to \bar{z} \hat{I} + a \Phi + O(a^2), \quad (44)$$

at the same time re-expressing $g$ as

$$g = \bar{g}(1 - a^2 \Lambda_{\text{cdt}} + O(a^4)), \quad (45)$$

following eq. (30). Substituting the variable change into the matrix potential, and discarding a $\phi$-independent constant term, one obtains

$$V(\phi) = \bar{V}(\Phi) \equiv \frac{\Lambda_{\text{cdt}} \Phi - \frac{1}{6} \Phi^3}{2G_s} \quad (46)$$

in the limit $a \to 0$, from which it follows that

$$Z(g, g_s) = a^{N^2} Z(\Lambda_{\text{cdt}}, G_s), \quad Z(\Lambda_{\text{cdt}}, G_s) = \int d\Phi \ e^{-N\text{tr} \bar{V}(\Phi)}. \quad (47)$$

The disc amplitude for the potential $\bar{V}(\Phi)$ is precisely $W(Z_{\text{cdt}}, \Lambda_{\text{cdt}}, G_s)$, and since by definition

$$\frac{1}{z - \phi} = \frac{1}{a} \frac{1}{Z_{\text{cdt}} - \Phi}, \quad (48)$$

the first equal sign in eq. (36) follows straightforwardly from the simple algebraic equation (46). We conclude that the continuum generalized CDT-theory
is described by the matrix model with potential $\bar{V}(\Phi)$. While in the present article we have analyzed this only to leading order in $N$, the proof that the generalized CDT-theory is reproduced by the matrix model with potential $\bar{V}(\Phi)$ to all orders in $1/N^2$ was already given in [12]. The important new point made here is that there really is a regularized lattice theory with a geometric interpretation underlying the formal manipulations of eqs. (46)-(47), where $a$ appears merely as a parameter without an obvious geometric interpretation as a cut-off.

(5) We can generalize the discussion above if we drop the restriction that all $t_i$ should be positive. Then there exist choices such that

$$V'(\bar{z}, \bar{g}, \bar{t}_i) = \cdots = V^{(n)}(\bar{z}, \bar{g}, \bar{t}_i) = 0,$$

where the derivatives are with respect to $z$. For a potential satisfying (49) we obtain a nontrivial limit if we introduce the scaling

$$g_s = G_s a^n, \quad g = \bar{g}(1 - a^n \Lambda_{cdt}^{n/2}), \quad z = \bar{z}(1 - aZ_{cdt}).$$

In the limit $a \to 0$ one obtains

$$W_{cdt}(Z_{cdt}, \Lambda_{cdt}) = \frac{V_n(Z_{cdt}) + P_n(Z_{cdt})\sqrt{(Z_{cdt} - C)(Z_{cdt} - D)}}{2G_s},$$

$$V_n(Z_{cdt}) \equiv \Lambda_{cdt}^{n/2} - \frac{(-1)^n}{n}Z_{cdt}^n,$$

where $P_n(Z_{cdt})$ is a polynomial of order $n - 1$. The coefficients of $P_n(Z_{cdt})$ as well as $C$ and $D$ satisfy algebraic equations which are derived from the requirement that $W_{cdt}(Z_{cdt})$ fall off like $1/Z_{cdt}$. In the limit $G_s \to 0$ one finds

$$W_{cdt}(Z_{cdt}, \Lambda_{cdt}, G_s) \to \frac{1}{Z_{cdt} + \sqrt{2}\Lambda_{cdt}},$$

which coincides with the original CDT result, valid when no splitting of the one-dimensional spatial universe is allowed as a function of proper time.

The scaling of $g$ in (50) is identical to the scaling for the conventional multicritical one-matrix model [14, 15] (for reviews see [2, 3]). Precisely as in that case one has a $(n - 1)$-dimensional set of deformations around the solution (51).

4 Discussion

The CDT model has been advocated as a potential candidate for quantum gravity in four dimensions [16, 17], the triangulations in this case constructed by gluing together four-simplices in such a way that the geometry is causal in the sense defined
in GR. If one does not impose any causal constraint these regularized models have so far not led to a continuum theory in four dimensions \[13\]. It is of interest to study to what extent one can lift the causality constraint in the CDT model and study the relation to the pure Euclidean model. This was one of the main motivations for the present investigation in two dimensions. The two-dimensional models are of course only toy models of quantum gravity, but they have the advantage that explicit analytic calculations can be performed. In this spirit it was shown in \[12\] that the loop equations of the matrix model with the cubic potential (46) are identical to the continuum string-field equations of a generalized CDT model. The generalization consisted in the inclusion of topology changes of the spatial universe, which because of their causality-violating nature were excluded from the original 2d CDT model. However, it left open the issue of how to understand this generalized continuum model as the scaling limit of an ordinary matrix model, with the conventional geometric interpretation of implementing a gluing of polygons of side length \(a\), \(a\) being the ultraviolet lattice cut-off, and in addition to understand the precise relation of this model to “ordinary” two-dimensional Euclidean quantum gravity. In this article we have provided such an interpretation and worked out in detail the relation to Euclidean quantum gravity.

This enables us to view the matrix models with a potential of the type \[11\] as representing regularized path integrals over random surfaces, as one would like to do in a theory of two-dimensional quantum gravity. What is rather remarkable is that starting with such a matrix model and taking the continuum limit \(a \to 0\), while scaling \(g_s \to 0\) as \(G_s a^3\), one again ends up with a matrix model, namely, the one given by (46), now describing the continuum CDT string field theory formulated in \[11\].

In the CDT framework, the new coupling constant \(G_s\) describes the splitting of a spatial hypersurface into two. Such a concept is not topological in a two-dimensional sense, but has a diffeomorphism-invariant meaning in spacetimes of Lorentzian signature. Well-behaved causal properties of spacetime were the starting point of the original CDT path-integral formulation, motivated by arguments going back to \[19\]. Each individual, causal configuration in the regularized CDT path integral allowed a rotation to a unique Euclidean spacetime, leading to a modified Euclidean path integral with many fewer geometries than the ordinary Euclidean gravitational path integral. What we have shown here is that also this restricted class of geometries can be captured by starting out with an ordinary matrix model (i.e. one intended to describe Euclidean 2d quantum gravity), and then imposing a “penalty” in the form of a coupling constant \(g_s = G_s a^3\) for the process where the boundary splits into two.

A potentially powerful application of the model described here concerns matter-coupled quantum gravity. It has been shown that in the continuum limit the triangulations appearing in CDT are much more regular than generic triangulations of the disc: a generic triangulation of the disc has fractal dimension four \[13\], while
those appearing in CDT have (fractal) dimension two [10, 20]. As a result, when one couples the Ising model to two-dimensional Euclidean quantum gravity the critical matter exponents will change from the Onsager values to the so-called KPZ values. This change can be traced to the fractal nature of the generic triangulations on which the Ising spins are placed, using DT as the framework for (regularized) two-dimensional Euclidean quantum gravity. By contrast, if one couples the Ising model to Lorentzian quantum gravity in the form of CDT triangulations, one obtains the Onsager values for the critical exponents as has been shown numerically [21] and using a high-$T$ expansion [22]. By generalizing the framework of this article from a one-matrix to a two-matrix model description, one may be able to obtain a two-matrix model description of CDT coupled to the Ising model and in this way prove analytically that the critical exponents of the Ising model on CDT are indeed the Onsager exponents. If this can be done, it might be by far the simplest way to calculate explicitly the Onsager exponents of an Ising lattice model. Work in this direction will be reported elsewhere.

It is also somewhat surprising that the matrix integral (47), with the same dimensions associated to the fields and coupling constants, is encountered in the so-called Dijkgraaf-Vafa correspondence [23]. From the gauge theory side $V(\Phi)$ is then the tree-level superpotential of the adjoint chiral field $\Phi$, which breaks the supersymmetry of the unitary gauge theory from $\mathcal{N} = 2$ to $\mathcal{N} = 1$. If one demands that this tree-level potential corresponds to a renormalizable theory then the form of the potential $V(\Phi)$ is essential unique (and given by (46)). $G_3$ is a dimension three coupling constant coming either from three of the compactified dimensions in topological string theory or alternatively, via the DV-correspondence, from the glueball superfield condensate in the gauge theory. In this formalism a continuum interpretation is associated with the equation defining disk amplitude, rather than with the (discretized) surfaces defined by the matrix integral, the equation defining an algebraic surface. This point of view is also adopted in [24, 25, 26] where the calculation of matrix integrals like (47) has been carried to a new level of perfection. In these applications one encounters eventually much more complicated situations with higher order potentials $V(\Phi)$ which admit multi-cut solutions. It would be interesting if these multi-cut solutions could also find a random surface interpretation.

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