THE ROTHE METHOD FOR MULTI-TERM TIME FRACTIONAL INTEGRAL DIFFUSION EQUATIONS

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Dedicated to Professor Zhenhai Liu on the occasion of his 60th birthday.
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Abstract. In this paper we study a class of multi-term time fractional integral diffusion equations. Results on existence, uniqueness and regularity of a strong solution are provided through the Rothe method. Several examples are given to illustrate the applicability of main results.

1. Introduction. The fractional calculus as a natural generalization of the classical integer order calculus was introduced firstly in the 17th century by L'Hospital to inquire Leibniz what meaning could be ascribed to \( \frac{d^n u}{dt^n} \) if \( n \) is a fraction. Since then, it attracted attention of many famous mathematicians such as Abel, Euler, Fourier, Laplace, Liouville, Riemann and so forth. However, its development was slow, only a century later, in 1819, Lacroix managed to answer the question concerning the expression \( \frac{d^{0.5} u(t)}{dt^{0.5}} \) for the function \( u(t) = t \). Until the twentieth century, many researchers found that some physical phenomena for viscoelastic materials lead to their description in terms of noninteger order differential equations, for example, fractional Kelvin-Voigt constitutive laws and fractional Maxwell model. After that, more and more researchers have paid their attention to fractional calculus, which is not only a generalization of the classical integer order calculus.

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calculus, but it can accurately describe many phenomena in science and engineering, e.g. in electrodynamics biotechnology, aerodynamics and control of dynamical systems [1, 8, 11, 14, 15, 29, 18, 27, 30].

Fractional order diffusion equations, as an important branch of the fractional calculus theory and applications, are generalizations of the classical diffusion equations. They treat super-diffusive flow processes, which have been studied by Eidelman-Kochubei in [5], Ervin et al. in [6], Li-Xu in [12], Lin-Xu in [13], Yang et al. in [25], Zhang-Xu in [31], and so on. Recently, the existence and uniqueness of a strong solution for the following fractional integral diffusion equation have been established by Raheem-Bahuguna in [19]

\[
\begin{align*}
\frac{\partial u(t)}{\partial t} + Au(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) \, ds + f(t), \quad t \in (0, T], \\
u(0) &= u_0,
\end{align*}
\]

where \(\alpha \in (0, 1)\). Note that the results in [19] are important contributions for the development of fractional diffusion equations. However, to obtain the existence and uniqueness for (1), the condition

\[
\frac{T^{1+\alpha}}{\Gamma(1+\alpha)} < 1,
\]

is required in [19]. This means that [19, Lemma 3.4] provides only a local (in time) unique strong solution of (1). For example, the following fractional integral diffusion equation problem cannot be solved by using main results of [19]

\[
\begin{align*}
\frac{\partial u(t,x)}{\partial t} - \frac{\partial^2 u(t,x)}{\partial x^2} &= \frac{1}{\Gamma(0.5)} \int_0^t (t-s)^{-0.5} u(s,x) \, ds \quad \text{in} \quad (0, 10) \times (0, \pi), \\
u(0,x) &= u_0(x) \quad \text{for all} \quad x \in [0, \pi], \\
u(t,0) = \nu(t,\pi) &= 0 \quad \text{for all} \quad t \in [0, 10],
\end{align*}
\]

since \(\frac{10^{0.5}}{\Gamma(1.5)} > 1\) and the smallness condition (2) is not satisfied. On the other hand, in [19], the order of the derivative \(\alpha\) is constrained to the interval \((0, 1)\). In some situations, the fractional integral diffusion equations can require that the order \(\alpha\) is not constrained to \((0, 1)\) and the integral term is not a single one but it is a multi-term. Being motivated by the above analysis, in this paper, we will remove these drawbacks and investigate the following multi-term time fractional integral diffusion equation, \((\text{FIDE})\) for short, in a Hilbert space \(H\)

\[
\begin{align*}
\frac{\partial u(t)}{\partial t} + Au(t) &= \sum_{i=1}^{k} a_i (\partial^{\alpha_i} u(t)) + f(t), \quad t \in (0, T], \\
u(0) &= u_0,
\end{align*}
\]

where the constants \(a_i, \alpha_i, i = 1, \ldots, k\) are such that \(a_i \geq 0, \alpha_i > 0\), \(-A: D(A) \subseteq H \to H\) is an infinitesimal generator of a \(C_0\)-semigroup of contractions in \(H\), the function \(f: [0, T] \to H\) is Lipschitz continuous, \(\partial^{\alpha_i} u\) denotes the fractional integral of order \(\alpha_i > 0\) of \(u\), and \(u_0 \in D(A)\). It is obvious that if \(a_1 = 1, a_i = 0\) for \(i = 2, \ldots, k\) and \(\alpha_1 \in (0, 1)\), then (3) reduces to (1).

We now give the definition of strong solution for \((\text{FIDE})\).

**Definition 1.1.** A function \(u \in C(0, T; H)\) is called a strong solution of (3) if and only if the following conditions hold

(i) \(u(t) \in D(A)\) for a.e. \(t \in [0, T]\),
(ii) \( u \) is differentiable for a.e. on \([0, T]\),

(iii) \( u \) satisfies the equation (3) for a.e. \( t \in [0, T] \).

In this paper, we will study a class of multi-term time fractional integral diffusion equations (FIDE) in (3). The main results are delivered on existence, uniqueness and regularity of a strong solution to (FIDE). We apply the Rothe method, see \([9, 10, 28]\) for details, combined with a surjectivity result for \(m\)-accretive operator, see \([2, 21, 26]\).

The outline of the paper is as follows. In Section 2, we recall the basic material we use in next sections. In Section 3, we investigate the multi-term time fractional integral diffusion equation (3) and provide our main results. Several examples which illustrate the applicability of main results are given in Section 4.

2. Notation and preliminaries. In this section we recall the basic notation and various results which are needed in the sequel, see \([2, 3, 4, 11, 14, 18, 23, 24, 26]\). We first present definitions from the fractional calculus theory.

**Definition 2.1.** (Riemann-Liouville fractional order integral) Let \( X \) be a Banach space, \( y \in L^1(0, T; X) \) and \( \alpha > 0 \). The fractional integral of order \( \alpha > 0 \) of \( y \) is defined by

\[
0I_t^\alpha y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) \, ds \quad \text{for a.e.} \quad t \in [0, T],
\]

where \( \Gamma \) is the gamma function given by

\[
\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t} \, dt.
\]

In what follows, we denote by \( 0I_t^0 \) the identity operator, i.e., we write \( 0I_t^0 y(t) = y(t) \) for a.e. \( t \in [0, T] \).

Let \( X \) be a Banach space. We denote the space of all absolutely continuous functions from \([0, T]\) to \( X \) by \( AC(0, T; X) \).

**Definition 2.2.** (Caputo fractional order derivative with \( 0 < \alpha \leq 1 \)) Let \( X \) be a Banach space, \( y \in AC(0, T; X) \), and \( \alpha \in (0, 1] \). The Caputo fractional derivative of order \( \alpha \in (0, 1] \) of the function \( y \) is defined by

\[
C_0^\alpha D_t^\alpha y(t) = 0I_t^{1-\alpha} y'(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} y'(s) \, ds \quad \text{for a.e.} \quad t \in [0, T].
\]

It is clear that the Caputo derivative of order \( \alpha = 1 \) reduces to the classical first-order derivative, that is, we have \( C_0^\alpha D_t^\alpha y(t) = y'(t) \) for a.e. \( t \in [0, T] \).

Let \( X \) be a real normed space with its dual \( X^* \), and the norms of \( X \) and \( X^* \) be denoted by \( \| \cdot \| \) and \( \| \cdot \|_{X^*} \), respectively. The value of a functional \( x \in X^* \) at a point \( x \in X \) will be conveniently denoted by \( \langle x^*, x \rangle \). The symbols \( \rightarrow \) and \( \rightharpoonup \) stand for the strong convergence in \( X \) and weak convergence in \( X \). The duality mapping \( J: X \to 2^{X^*} \) is defined by

\[
J(x) = \{ x^* \in X^* \mid \langle x^*, x \rangle = \| x \|^2 = \| x^* \|_{X^*}^2 \} \quad \text{for all} \quad x \in X.
\]

It is well-known, see \([2, \text{Theorem 1.2}]\), that if \( X^* \) is strictly convex, then the duality mapping \( J \) is single valued and demicontinuous. In particular, if \( X \) is a Hilbert space, then the duality mapping \( J \) becomes the identity operator \( I \).
Definition 2.3. Let $X$ be a Banach space and $A: D(A) \subseteq X \to 2^X$ be a multivalued mapping. Then $A$ is called

(i) accretive, if for all $x_1, x_2 \in D(A)$, $y_1 \in Ax_1, y_2 \in Ax_2$ and $z^* \in J(x_1 - x_2)$, we have

$$\langle z^*, y_1 - y_2 \rangle \geq 0,$$

(ii) $m$-accretive, if $A$ is accretive and $I + A$ is surjective, where $I$ denotes the identity operator in $X$.

Remark 1. It is known, by Kato’s lemma, see [2, Lemma 3.1, p.100], that $A$ is accretive if and only if

$$\|x_1 - x_2\| \leq \|x_1 - x_2 + \lambda(y_1 - y_2)\|$$

for all $\lambda > 0, x_i \in D(A)$ and $y_i \in Ax_i, i = 1, 2$.

Let $A: D(A) \subseteq X \to X$ be an accretive operator. Let us define the operators

$$J_\lambda: R(I + \lambda A) \to D(A)$$

and

$$A_\lambda: R(I + \lambda A) \to X$$

by

$$J_\lambda x = (I + \lambda A)^{-1} x \text{ for all } x \in R(I + \lambda A),$$

$$A_\lambda x = \frac{(x - J_\lambda x)}{\lambda} \text{ for all } x \in R(I + \lambda A),$$

where $R(I + \lambda A)$ denotes the range of the operator $I + \lambda A$ and the operator $A_\lambda$ is called the Yosida approximation of $A$ (for details, see [20, p.151] or [2, p.101]). We also recall the following important properties of $m$-accretive operator, which can be found in [2, Proposition 3.4, p.103].

**Proposition 1.** Let $A: X \to 2^X$ be an $m$-accretive operator. Then $A$ is closed and if $\lambda_n \in \mathbb{R}$ and $x_n \in X$ are such that

$$\lambda_n \to 0, \; x_n \to x, \; A_{\lambda_n} x_n \to y, \text{ as } n \to \infty,$$

then $y \in Ax$. If $X^*$ is uniformly convex, then $A$ is demiclosed, and if

$$\lambda_n \to 0, \; x_n \to x, \; A_{\lambda_n} x \to y, \text{ as } n \to \infty,$$

then $y \in Ax$.

Recall that an operator $A: X \to 2^X$ is said to be closed, if $x_n \to x, y_n \to y$ and $y_n \in Ax_n$, then $y \in Ax$. Also, $A$ is said to be demiclosed, if $x_n \to x, y_n \to y$ and $y_n \in Ax_n$ yield $y \in Ax$.

**Remark 2.** It is known, see [17, Theorem 1.4.3(b)] and [2, Proposition 3.3], that if $A: D(A) \subseteq X \to X$ is such that $-A$ is an infinitesimal generator of a $C_0$-semigroup of contractions in $X$, then $A$ is an $m$-accretive operator.

We conclude this section by recalling the Gronwall inequalities which proofs can be found in [7, Lemma 7.25] and [22, Lemma 2.31. p.49], respectively.

**Lemma 2.4.** (Discrete form) Let $T > 0$ be given. For a positive integer $N$, we define $\tau = \frac{T}{N}$. Assume that \(\{y_n\}_{n=1}^{N}\) and \(\{x_n\}_{n=1}^{N}\) are two sequences of nonnegative numbers such that

$$x_n \leq \tau y_n + \tau \sum_{j=1}^{n-1} x_j \text{ for } n = 1, \ldots, N \quad (4)$$
with a positive constant \( \bar{c} \) independent of \( N \) (or \( \tau \)). Then, there exists a positive constant \( c \), independent of \( N \) (or \( \tau \)) such that

\[
x_n \leq c \left( y_n + \tau \sum_{j=1}^{n-1} y_j \right) \quad \text{for} \quad n = 1, \ldots, N.
\]

Therefore, for a constant \( C > 0 \) independent of \( N \) (or \( \tau \)), we have

\[
\max_{1 \leq n \leq N} x_n \leq C \max_{1 \leq n \leq N} y_n.
\]

If, instead of (4), we assume

\[
x_n \leq \bar{c} y_n + \bar{c} \tau \sum_{j=1}^{n} x_j \quad \text{for} \quad n = 1, \ldots, N,
\]

and \( \tau \) is sufficient small, then

\[
x_n \leq c \left( y_n + \tau \sum_{j=1}^{n} y_j \right) \quad \text{for} \quad n = 1, \ldots, N,
\]

and \( \max_{1 \leq n \leq N} x_n \leq C \max_{1 \leq n \leq N} y_n. \)

Lemma 2.5. (Integral form) Assume that \( f, g \in C(a, b; \mathbb{R}) \) satisfy

\[
f(t) \leq g(t) + c \int_{a}^{t} f(s) \, ds \quad \text{for all} \quad t \in [a, b],
\]

where \( c > 0 \) is a constant. Then

\[
f(t) \leq g(t) + c \int_{a}^{t} g(s) e^{c(t-s)} \, ds \quad \text{for all} \quad t \in [a, b].
\]

Moreover, if \( g \) is nondecreasing, then

\[
f(t) \leq g(t) e^{c(t-a)} \quad \text{for all} \quad t \in [a, b].
\]

3. Time fractional integral diffusion equation. In this section, we pass to the formulation of the Rothe method to prove the existence and uniqueness of a strong solution to (FIDE) in (3). It consists in the time discretization in which we define an approximate sequence by using an implicit (backward) Euler formula. Then, at each time step, we will solve a stationary equation. Finally, we construct the piecewise constant and piecewise affine interpolants, and prove a convergence result. In the rest of the paper, \( C > 0 \) denotes a constant whose value may change from line to line.

Let \( N \in \mathbb{N} \) be fixed and denote \( \tau = \frac{T}{N} \). Let \( f_0^\tau = f(0) \) and

\[
f_n^\tau = \frac{1}{\tau} \int_{t_{n-1}^\tau}^{t_n^\tau} f(s) \, ds
\]

for \( n = 1, \ldots, N \), where \( t_n^\tau = n \tau \).

Now, we apply the Rothe method to (FIDE) in (3) and consider the following discretized problem called the Rothe problem for (FIDE).
Problem 1. Find \( \{u^n_\tau\}_{n=0}^{N} \subset H \) such that
\[
\frac{u_1^n - u_0^n}{\tau} + A u_1^n = f^n_\tau
\]
and
\[
\frac{u^n_\tau - u^{n-1}_\tau}{\tau} + A u^n_\tau = \sum_{i=1}^{k} \left( \frac{a_i}{\Gamma(\alpha_i)} \sum_{j=1}^{n-1} \int_{t_{n-1}^j}^{t_n^i} \Gamma(\alpha_i) \left( (t-n-s)^{\alpha_i-1} - (n-j-1)^{\alpha_i} \right) ds \right) + f^{n-1}_\tau
\]
for \( n = 2, \ldots, N \).

The following result provides conditions for the solvability of Problem 1.

Lemma 3.1. Let \( H \) be a Hilbert space and \( a_i \geq 0, \alpha_i > 0 \) for \( i = 1, \ldots, k \). Assume that the operator \(-A: D(A) \subset H \rightarrow H\) is an infinitesimal generator of a \( C_0\)-semigroup of contractions in \( H \) and \( f: [0, T] \rightarrow H \) is a Lipschitz continuous function. Then, for each \( \tau > 0 \), there exists \( \{u^n_\tau\}_{n=0}^{N} \) which solves Problem 1.

Proof. It is enough to prove that the operator \( v + \tau A v \) is surjective for any \( \tau > 0 \). In fact, we get this conclusion readily by the \( m\)-accretiveness of \( A \) and the following property of \( m\)-accretive operator: an accretive operator \( B: X \rightarrow 2^X \) is \( m\)-accretive if and only if \( I + \lambda B \) is surjective for all (equivalently, for some) \( \lambda > 0 \), see [2, Proposition 3.3, p.102].

The next result establishes the estimates for the solution of Problem 1.

Lemma 3.2. Assume that the assumptions of Lemma 3.1 hold. Then, there exist \( \tau_0 > 0 \) and a constant \( C > 0 \) independent of \( \tau \) such that, for all \( \tau \in (0, \tau_0) \), the solution \( \{u^n_\tau\}_{n=0}^{N} \) of Problem 1 satisfies
\[
\max_{n=1,\ldots,N} \| u^n_\tau \| \leq C, \quad (6)
\]
\[
\max_{n=1,\ldots,N} \left\| \frac{u^n_\tau - u^{n-1}_\tau}{\tau} \right\| \leq C. \quad (7)
\]

Proof. We first prove the estimate (6). For \( n = 1 \), we have
\[
u_1^1 + \tau A u_1^1 = u_0 + \tau f_\tau^0.
\]
Multiplying the above equation by \( u_1^0 \) and then applying the \( m\)-accretivity of \( A \), cf. Remark 2, we obtain
\[
\| u_1^1 \|^2 \leq \langle u_1^1, u_1^0 \rangle + \tau \langle A u_1^1, u_1^0 \rangle \leq \| u_0 \| \| u_1^0 \| + \tau \| f_\tau^0 \| \| u_1^0 \|. \quad (8)
\]
Since \( f \) is Lipschitz continuous on \([0, T]\), then there exists a constant \( m_f > 0 \) such that \( \| f_\tau^n \| \leq m_f \) for all \( \tau > 0 \) and \( n \geq 0 \). This combined with (8) implies
\[
\| u_1^1 \| \leq \| u_0 \| + T m_f \leq C. \quad (9)
\]
Next, for \( n = 2, \ldots, N \), we multiply the equation (5) by \( u_t^n \) to obtain
\[
\frac{1}{\tau} (u^n_t - u^{n-1}_t, u^n_t) + \langle Au^n_t, u^n_t \rangle = \langle f^{n-1}_t, u^n_t \rangle
\]
\[
+ \sum_{i=1}^{k} \left( \frac{\alpha_i \tau_{\alpha_i}}{\Gamma(\alpha_i + 1)} \sum_{j=1}^{n-1} \langle u^j_t, u^n_t \rangle [(n - j)^{\alpha_i} - (n - j - 1)^{\alpha_i}] \right).
\]
It follows from the \( m \)-accretivity of the operator \( A \) that
\[
\| u^n_t \| - \| u^{n-1}_t \| \leq \frac{\sum_{i=1}^{k} (\alpha_i \tau_{\alpha_i+1})^{\Gamma(\alpha_i + 1)} \left( \sum_{n=2}^{l} \sum_{j=1}^{n-1} \| u^j_t \| [(n - j)^{\alpha_i} - (n - j - 1)^{\alpha_i}] \right)}{\Gamma(\alpha_i + 1)}
\]
\[
+ \tau \| f^{n-1}_t \|.
\]
Summing up the above inequalities from 2 to \( l \), where \( 2 \leq l \leq N \), we obtain
\[
\| u^l_t \| - \| u^1_t \|
\]
\[
\leq \sum_{i=1}^{k} \left( \frac{\alpha_i \tau_{\alpha_i+1}}{\Gamma(\alpha_i + 1)} \sum_{n=2}^{l} \sum_{j=1}^{n-1} \| u^j_t \| [(n - j)^{\alpha_i} - (n - j - 1)^{\alpha_i}] \right)
\]
\[
+ \tau \sum_{n=2}^{l} \| f^{n-1}_t \|.
\]
Combining this estimate with the fact that
\[
\sum_{n=2}^{l} \sum_{j=1}^{n-1} \| u^j_t \| [(n - j)^{\alpha_i} - (n - j - 1)^{\alpha_i}]
\]
\[
= \sum_{n=2}^{l} \| u^n_t \| \sum_{j=1}^{l-n} [(l - j)^{\alpha_i} - (l - j - 1)^{\alpha_i}]
\]
\[
\leq l^{\alpha_i} \sum_{n=2}^{l} \| u^n_t \|,
\]
and the boundedness of \( u^1_t \), see (9), we deduce
\[
\| u^l_t \| \leq \sum_{i=1}^{k} \left( \frac{\alpha_i \tau_{\alpha_i+1}}{\Gamma(\alpha_i + 1)} \sum_{n=2}^{l} \| u^n_t \| \right) + \tau \sum_{n=2}^{l} \| f^{n-1}_t \| + \| u_0 \| + Tm_f
\]
\[
\leq \sum_{i=1}^{k} \left( \frac{\alpha_i \tau_{\alpha_i}}{\Gamma(\alpha_i + 1)} \sum_{n=2}^{l} \| u^n_t \| \right) + \| u_0 \| + 2Tm_f
\]
\[
= \left( \sum_{i=1}^{k} \frac{\alpha_i \tau_{\alpha_i}}{\Gamma(\alpha_i + 1)} \right) \left( \tau \sum_{n=2}^{l} \| u^n_t \| \right) + \| u_0 \| + 2Tm_f.
\]
Choosing \( \tau_0 = \frac{1}{2} \sum_{i=1}^{k} \frac{\alpha_i \tau_{\alpha_i}}{\Gamma(\alpha_i + 1)} \) and then applying the discrete version of the Gronwall inequality in Lemma 2.4, for all \( \tau \in (0, \tau_0) \), we have
\[
\| u^l_t \| \leq C
\]
for all $2 \leq l \leq N$. The latter combined with (9) implies that the estimate (6) holds.

Now we shall verify the estimate (7). For $n = 1$, it is obvious that

$$\frac{1}{\tau} \| u^n_\tau - u^{n-1}_\tau \|^2 \leq \frac{1}{\tau} \langle u^n_\tau - u^{n-1}_\tau, u^n_\tau - u^{n-1}_\tau \rangle + \langle Au^n_\tau - Au^{n-1}_\tau, u^n_\tau - u^{n-1}_\tau \rangle$$

$$= \langle f^n_\tau - Au^{n-1}_\tau, u^n_\tau - u^{n-1}_\tau \rangle \leq \| f^n_\tau - Au^{n-1}_\tau \| \| u^n_\tau - u^{n-1}_\tau \|$$

and hence

$$\| u^n_\tau - u^{n-1}_\tau \| \leq C.$$  \hfill (10)

Consider the $n$-th equation in (5) and the $(n-1)$-th equation in (5), and then subtract these two equalities to get

$$\frac{u^n_\tau - u^{n-1}_\tau}{\tau} - \frac{u^{n-1}_\tau - u^{n-2}_\tau}{\tau} + Au^n_\tau - Au^{n-1}_\tau$$

$$= f^n_\tau - f^{n-2}_\tau + \left( \sum_{i=1}^{k} \frac{a_i \tau^{\alpha_i}}{\Gamma(\alpha_i + 1)} \right) u^{n-1}_\tau$$

$$+ \sum_{i=1}^{k} \left( \frac{a_i \tau^{\alpha_i}}{\Gamma(\alpha_i + 1)} \sum_{j=1}^{n-2} u^j_\tau \left[ (n-j)^{\alpha_i} - 2(n-j-1)^{\alpha_i} + (n-j-2)^{\alpha_i} \right] \right).$$

Multiplying the above equality by $u^n_\tau - u^{n-1}_\tau$ and then applying the $m$-accretivity of the operator $A$, we obtain

$$\frac{1}{\tau} \| u^n_\tau - u^{n-1}_\tau \|^2 - \frac{1}{\tau} \| u^n_\tau - u^{n-1}_\tau \| \| u^{n-1}_\tau - u^{n-2}_\tau \|$$

$$\leq \| f^n_\tau - f^{n-2}_\tau \| \| u^n_\tau - u^{n-1}_\tau \| + \left( \sum_{i=1}^{k} \frac{a_i \tau^{\alpha_i}}{\Gamma(\alpha_i + 1)} \right) \| u^{n-1}_\tau \| \| u^n_\tau - u^{n-1}_\tau \|$$

$$+ \sum_{i=1}^{k} \left( \frac{a_i \tau^{\alpha_i}}{\Gamma(\alpha_i + 1)} \sum_{j=1}^{n-2} u^j_\tau \left[ (n-j)^{\alpha_i} - 2(n-j-1)^{\alpha_i} + (n-j-2)^{\alpha_i} \right] \right)$$

and

$$\| u^n_\tau - u^{n-1}_\tau \| - \| u^{n-1}_\tau - u^{n-2}_\tau \|$$

$$\leq \sum_{i=1}^{k} \left( \frac{a_i \tau^{\alpha_i}}{\Gamma(\alpha_i + 1)} \sum_{j=1}^{n-2} u^j_\tau \left[ (n-j)^{\alpha_i} - 2(n-j-1)^{\alpha_i} + (n-j-2)^{\alpha_i} \right] \right)$$

$$+ \| f^{n-1}_\tau - f^{n-2}_\tau \| + \left( \sum_{i=1}^{k} \frac{a_i \tau^{\alpha_i}}{\Gamma(\alpha_i + 1)} \right) \| u^{n-1}_\tau \|.$$
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\[ \| f^n_{n-1} - f^n_{n-2} \| + \left( \sum_{i=1}^{k} \frac{a_i \tau^\alpha_i}{\Gamma(\alpha_i + 1)} \right) C. \]  

(11)

Since \( f \) is a Lipschitz continuous function, then there exists a constant \( L_f > 0 \) such that

\[ \| f^n_{n-1} - f^n_{n-2} \| = \frac{1}{\tau} \left\| \int_{t_{n-2}}^{t_{n-1}} (f(s) - f(s - \tau)) \, ds \right\| \leq \tau L_f. \]

We use this inequality and the fact that

\[ \sum_{j=1}^{n-2} \left( (n - j)^{\alpha_i} - 2(n - j - 1)^{\alpha_i} + (n - j - 2)^{\alpha_i} \right) + 1 \]

\[ = (n - 1)^{\alpha_i} - (n - 2)^{\alpha_i} \quad \text{for all} \ i = 1, \ldots, k \]

in the estimate (11) to obtain

\[ \left\| \frac{u^n_{\tau} - u^{n-1}_{\tau}}{\tau} \right\| - \left\| \frac{u^{n-1}_{\tau} - u^{n-2}_{\tau}}{\tau} \right\| \leq \tau L_f \]

\[ + \sum_{i=1}^{k} \left( \frac{C a_i \tau^{\alpha_i}}{\Gamma(\alpha_i + 1)} \right) \left( (n - 1)^{\alpha_i} - (n - 2)^{\alpha_i} \right). \]

Summing up the above inequalities from 2 to \( l \), where \( 2 \leq l \leq N \), and then using (10), we have

\[ \left\| \frac{u^n_{\tau} - u^{n-1}_{\tau}}{\tau} \right\| \leq C + \sum_{n=2}^{l} \sum_{i=1}^{k} \left( \frac{C a_i \tau^{\alpha_i}}{\Gamma(\alpha_i + 1)} \right) \left( (n - 1)^{\alpha_i} - (n - 2)^{\alpha_i} \right) \]

\[ = C + \sum_{i=1}^{k} \left( \frac{C a_i \tau^{\alpha_i}}{\Gamma(\alpha_i + 1)} \right) \sum_{n=2}^{l} \left( (n - 1)^{\alpha_i} - (n - 2)^{\alpha_i} \right) \]

\[ = C + \sum_{i=1}^{k} \left( \frac{C a_i \tau^{\alpha_i}}{\Gamma(\alpha_i + 1)} \right) (l - 1)^{\alpha_i} \]

\[ \leq C \]

for all \( l = 2, \ldots, N \). This means that the estimate (7) holds, which completes the proof of the lemma.

Subsequently, for a given \( \tau > 0 \), we define the piecewise affine function \( u_\tau \), and the piecewise constant interpolant functions \( \overline{u}_\tau \) and \( f_\tau \) as follows

\[ u_\tau(t) = u^n_{\tau} + \frac{t - t_n}{\tau} (u^n_{\tau} - u^{n-1}_{\tau}) \quad \text{for} \quad t \in (t_{n-1}, t_n], \]

\[ \overline{u}_\tau(t) = \begin{cases} u^n_{\tau}, & t \in (t_{n-1}, t_n], \\ u^0_{\tau}, & t = 0, \end{cases} \]

\[ f_\tau(t) = \begin{cases} f^n_{\tau-1}, & t \in (t_{n-1}, t_n], \\ f(0), & t = 0. \end{cases} \]

The main result of this paper is obtained in the following theorem.

**Theorem 3.3.** Assume that hypotheses of Lemma 3.1 hold. Then, there exists a function \( u \in C(0, T; H) \) which is the unique strong solution of (FIDE) in (3) such that \( u_{\tau} \rightarrow u \) in \( C(0, T; H) \), as \( \tau \rightarrow 0 \), and \( u \) is Lipschitz continuous on \( [0, T] \).
Proof. Let $N, M \in \mathbb{N}$ and $\tau = \frac{T}{N}$. Denote by $v_\tau$ the following element

$$v_\tau(t) = \sum_{i=1}^{k} \left( \frac{a_i \tau^{\alpha_i}}{\Gamma(\alpha_i + 1)} \sum_{j=1}^{n-1} u_j^h \left[(n-j)^{\alpha_i} - (n-j-1)^{\alpha_i} \right] \right)$$

for $t \in (t_{n-1}, t_n]$.

$v_\tau(0) = 0$.

According to the estimate (7), we easily calculate that

$$\|u_\tau(t) - \overline{u}_\tau(t)\| = |t-t_n| \left\| \frac{u^h_\tau - u^{h-1}_\tau}{\tau} \right\| \leq \frac{C}{N} \tag{12}$$

for all $t \in (t_{n-1}, t_n]$. On the other hand, by the estimate (6), we can easily obtain, for $t \in (t_{n-1}, t_n]$, that

$$\left\| \frac{t^{\alpha_i}}{\Gamma(\alpha_i + 1)} \sum_{j=1}^{n-1} u_j^h \left[(n-j)^{\alpha_i} - (n-j-1)^{\alpha_i} \right] - a_i \int_{t_{n-1}}^{t} \overline{u}_\tau(s) \, ds \right\|$$

$$= \frac{1}{\Gamma(\alpha_i)} \left| \int_{t_{n-1}}^{t} (t_n-s)^{\alpha_i-1} \overline{u}_\tau(s) \, ds - \int_{0}^{t} (t-s)^{\alpha_i-1} \overline{u}_\tau(s) \, ds \right|$$

$$\leq \frac{1}{\Gamma(\alpha_i)} \int_{t_{n-1}}^{t} \left[ (t_n-s)^{\alpha_i-1} - (t-s)^{\alpha_i-1} \right] |\overline{u}_\tau(s)| \, ds$$

$$+ \frac{1}{\Gamma(\alpha_i)} \int_{t_{n-1}}^{t} (t-s)^{\alpha_i-1} |\overline{u}_\tau(s)| \, ds$$

$$\leq \frac{C}{\Gamma(\alpha_i)} \left( t_n - t_{n-1} \right) |(t-s)^{\alpha_i-1} - (t_{n-1} - s)^{\alpha_i-1}| \, ds$$

$$\leq \frac{C}{\Gamma(\alpha_i + 1)} \left( 2(t-t_{n-1})^{\alpha_i} + t^{\alpha_i} - t_{n-1}^{\alpha_i} \right) \leq \frac{C}{\Gamma(\alpha_i + 1)} \tau^{\alpha_i} \leq \frac{C}{N^{\alpha_i}} \tag{13}$$

for $i = 1, \ldots, k$.

By the definitions of $u_\tau$, $\overline{u}_\tau$, $v_\tau$ and $f_\tau$, it is clear that (5) is equivalent to the following problem

$$\frac{du_\tau(t)}{dt} + A \overline{u}_\tau(t) = v_\tau(t) + f_\tau(t) \quad \text{for all } t \in [0, T]. \tag{14}$$

Analogously, for $h = \frac{T}{N}$, we have

$$\frac{du_h(t)}{dt} + A \overline{u}_h(t) = v_h(t) + f_h(t) \quad \text{for all } t \in [0, T]. \tag{15}$$

Subtracting the equalities (14) and (15), and then multiplying the result by $\overline{u}_\tau(t) - \overline{u}_h(t)$, we have

$$\left\langle \frac{du_\tau(t)}{dt} - \frac{du_h(t)}{dt}, \overline{u}_\tau(t) - \overline{u}_h(t) \right\rangle + \left\langle A \overline{u}_\tau(t) - A \overline{u}_h(t), \overline{u}_\tau(t) - \overline{u}_h(t) \right\rangle$$

$$= \left\langle v_\tau(t) - \sum_{i=1}^{k} a_i (0I_i^n, \overline{u}_\tau(t)), \overline{u}_\tau(t) - \overline{u}_h(t) \right\rangle.$$
Hence, we obtain
\[
\sum_{i=1}^{k} a_i \langle 0 I_t^{\alpha_i} \pi_r(t) - \sum_{i=1}^{k} a_i (0 I_t^{\alpha_i} \pi_h(t)), \pi_r(t) - \pi_h(t) \rangle
\]

From the estimates (6), (7) and inequality (12), we get, for all \( t \in (t_{n-1}, t_n) \)
\[
\frac{d u_r(t)}{d t} - \frac{d u_h(t)}{d t}, u_r(t) - u_h(t) + \langle A \pi_r(t) - A \pi_h(t), \pi_r(t) - \pi_h(t) \rangle
\]

Thus, we deduce the following estimate
\[
\sum_{i=1}^{k} a_i \langle 0 I_t^{\alpha_i} \pi_r(t) - \sum_{i=1}^{k} a_i (0 I_t^{\alpha_i} \pi_h(t)), \pi_r(t) - \pi_h(t) \rangle
\]

Hence, we obtain
\[
\frac{1}{2} \frac{d}{dt} \| u_r(t) - u_h(t) \|^2 \leq C \left( \| u_r(t) - \pi_r(t) \| + \| u_h(t) - \pi_h(t) \| + \| f_r(t) - f_h(t) \| \right)
\]
Now, we consider the last term in the inequality (19). It follows from the Cauchy inequality with \( \varepsilon > 0 \), see [26], that

\[
\frac{1}{2} \frac{d}{dt} \| u_\tau(t) - u_h(t) \|^2 \leq C \left( \frac{1}{N} + \frac{1}{M} + \sum_{i=1}^{k} \left( \frac{1}{N^{\alpha_i}} + \frac{1}{M^{\alpha_i}} \right) \right) \\
+ \sum_{i=1}^{k} a_i \left\| a I_i^{\alpha_i} (u_\tau(t) - u_h(t)) \right\| \| \overline{u}_\tau(t) - u_h(t) \| \\
+ \sum_{i=1}^{k} a_i \left\| a I_i^{\alpha_i} (u_h(t) - u_h(t)) \right\| \| \overline{u}_h(t) - u_h(t) \| \\
+ \sum_{i=1}^{k} a_i \left\| a I_i^{\alpha_i} (u_\tau(t) - u_h(t)) \right\| \| u_\tau(t) - u_h(t) \|. \\
\]

We combine this inequality with (12), (17) and (18) to deduce

\[
\frac{1}{2} \frac{d}{dt} \| u_\tau(t) - u_h(t) \|^2 \leq C \left( \frac{1}{N} + \frac{1}{M} + \sum_{i=1}^{k} \left( \frac{1}{N^{\alpha_i}} + \frac{1}{M^{\alpha_i}} \right) \right) \\
+ \sum_{i=1}^{k} a_i \left( a I_i^{\alpha_i} \left( u_\tau(t) - u_h(t) \right) \right) \| \overline{u}_\tau(t) - u_h(t) \| \\
+ \sum_{i=1}^{k} a_i \left( a I_i^{\alpha_i} \left( u_h(t) - u_h(t) \right) \right) \| \overline{u}_h(t) - u_h(t) \| \\
+ \sum_{i=1}^{k} a_i \left( a I_i^{\alpha_i} \left( u_\tau(t) - u_h(t) \right) \right) \| u_\tau(t) - u_h(t) \| \\
\leq C \left( \frac{1}{N} + \frac{1}{M} + \sum_{i=1}^{k} \left( \frac{1}{N^{\alpha_i}} + \frac{1}{M^{\alpha_i}} \right) \right) \\
+ \sum_{i=1}^{k} \frac{a_i}{\Gamma(\alpha_i)} \int_0^t (t-s)^{\alpha_i-1} \left\| u_\tau(s) - u_h(s) \right\| \left\| u_\tau(t) - u_h(t) \right\| ds. \tag{19}
\]

Now, we consider the last term in the inequality (19). It follows from the Cauchy inequality with \( \varepsilon > 0 \), see [26], that

\[
\sum_{i=1}^{k} \frac{a_i}{\Gamma(\alpha_i)} \int_0^t (t-s)^{\alpha_i-1} \left\| u_\tau(s) - u_h(s) \right\| \left\| u_\tau(t) - u_h(t) \right\| ds \\
\leq \varepsilon \sum_{i=1}^{k} \frac{a_i}{\Gamma(\alpha_i)} \int_0^t (t-s)^{\alpha_i-1} \left\| u_\tau(s) - u_h(s) \right\|^2 ds \\
+ \left( \sum_{i=1}^{k} \frac{a_i T^{\alpha_i}}{4\varepsilon \Gamma(\alpha_i + 1)} \right) \left\| u_\tau(t) - u_h(t) \right\|^2 \\
\leq \left( \sum_{i=1}^{k} \frac{a_i T^{\alpha_i}}{4\varepsilon \Gamma(\alpha_i + 1)} \right) \left\| u_\tau(t) - u_h(t) \right\|^2
\]
uniform convexity of $H$ for all $u, v \in H$ is uniformly Lipschitz continuous for all $t \in [0, T]$, as $\tau \to 0$. Further, by the fact that the function $u_\tau$ is uniformly Lipschitz continuous for all $\tau > 0$, we conclude that $u$ is Lipschitz continuous as well.

Next, we will show that $u$ is a strong solution of (FIDE) in (3). From the convergence $u_\tau \to u$ in $C(0, T; H)$, as $\tau \to 0$, we know that $u_\tau(t) \to u(t)$ in $H$ for all $t \in [0, T]$, as $\tau \to 0$. However, the inequality (12) ensures that $\bar{u}_\tau(t) \to u(t)$ in $H$ for all $t \in [0, T]$, as $\tau \to 0$. According to the $m$-accretivity of the operator $A$, the uniform convexity of $H$, from Proposition 1, we know that $A$ is demiclosed, thus, $A\bar{u}_\tau(t) \to Au(t)$ in $H$ for all $t \in [0, T]$, as $\tau \to 0$.

In addition, it is clear that $\alpha I_{\tau}^{\alpha} u_\tau(t) \to \alpha I_{\tau}^{\alpha} u(t)$ in $H$ for all $t \in [0, T]$, as $\tau \to 0$. Hence, we take into account the inequality (17) to obtain $\alpha I_{\tau}^{\alpha} \bar{u}_\tau(t) \to \alpha I_{\tau}^{\alpha} u(t)$ in $H$ for all $t \in [0, T]$, as $\tau \to 0$. Then, directly from (13), we also get $v_\tau(t) \to \sum_{i=1}^{k} a_i (\alpha I_{\tau}^{\alpha} u(t))$ in $H$ for all $t \in [0, T]$, as $\tau \to 0$. Moreover, we easily obtain $f_\tau \to f$ in $C(0, T; H)$, as $\tau \to 0$.\n
Thus, applying the integral version of the Gronwall inequality, see Lemma 2.5, we conclude

$$
\|u_\tau(t) - u(t)\| \leq C \left( \frac{1}{N} + \frac{1}{M} + \sum_{i=1}^{k} \left( \frac{1}{N\alpha_i} + \frac{1}{M\alpha_i} \right) \right)
$$

for all $t \in [0, T]$, where $C > 0$ is independent of $\tau, h$ and $t$. This means that \{u_\tau\} $\subset C(0, T; H)$ is a Cauchy sequence, and therefore, there exists $u \in C(0, T; H)$ such that $u_\tau \to u$ in $C(0, T; H)$, as $\tau \to 0$. Further, by the fact that the function $u_\tau$ is uniformly Lipschitz continuous for all $\tau > 0$, we conclude that $u$ is Lipschitz continuous as well.
Let \( w \in H \). We multiply (14) by \( w \) and integrate the result on \([0, t]\). We have
\[
\langle u_t(t) - u_0, w \rangle + \int_0^t \langle A \varpi(s), w \rangle \, ds = \int_0^t \langle v_\tau(s), w \rangle \, ds + \int_0^t \langle f_\tau(s), w \rangle \, ds.
\]
Letting \( \tau \to 0 \) and applying the Lebesgue dominated convergence theorem, see [16, Theorem 1.65], we infer
\[
\langle u(t) - u_0, w \rangle + \langle \int_0^t A u(s) \, ds, w \rangle = \sum_{i=1}^k a_i \int_0^t (0 I_i^\alpha u(s)) \, ds + \int_0^t f(s) \, ds, w \rangle \quad \text{for all } w \in H.
\]
Since \( w \in H \) is arbitrary, we have
\[
u(t) - u_0 + \int_0^t A u(s) \, ds = \sum_{i=1}^k a_i \int_0^t (0 I_i^\alpha u(s)) \, ds + \int_0^t f(s) \, ds
\]
for all \( t \in [0, T] \). This implies that
\[
\frac{du(t)}{dt} + A u(t) = \sum_{i=1}^k a_i (0 I_i^\alpha u(t)) + f(t)
\]
for a.e. \( t \in (0, T) \), because \( u \) is a Lipschitz continuous function. From the convergence \( u_0 = u_\tau(0) \to u(0) \) in \( H \), as \( \tau \to 0 \), we have \( u(0) = u_0 \). This means that \( u \) is a strong solution to (FIDE) in (3).

It remains to prove uniqueness of strong solution to (FIDE) in (3). To that end, assume that \( u_1 \) and \( u_2 \) are two solutions to (3). Then, we have
\[
\frac{du_j(t)}{dt} + A u_j(t) = \sum_{i=1}^k a_i (0 I_i^\alpha u_j(t)) + f(t) \quad \text{for a.e. } t \in (0, T),
\]
where \( j = 1, 2 \). Subtracting the above equalities, multiplying the result by \( u_1(t) - u_2(t) \), and using the \( m \)-accretivity of the operator \( A \), we have
\[
\frac{1}{2} \frac{d}{dt} \|u_1(t) - u_2(t)\|^2 \leq \sum_{i=1}^k a_i (0 I_i^\alpha (\|u_1(t) - u_2(t)\|) \|u_1(t) - u_2(t)\|).
\]
We again apply the Cauchy inequality with \( \varepsilon > 0 \) to obtain
\[
\frac{1}{2} \frac{d}{dt} \|u_1(t) - u_2(t)\|^2 \leq \sum_{i=1}^k \frac{T^\alpha_i}{4 \varepsilon \Gamma(\alpha_i + 1)} \|u_1(t) - u_2(t)\|^2
\]
\[
+ \varepsilon \sum_{i=1}^k a_i (0 I_i^\alpha \|u_1(t) - u_2(t)\|^2)
\]
\[
\leq \sum_{i=1}^k \frac{T^\alpha_i}{4 \varepsilon \Gamma(\alpha_i + 1)} \|u_1(t) - u_2(t)\|^2
\]
\[
+ \sum_{i=1}^k \frac{\varepsilon a_i T^\alpha_i}{\Gamma(\alpha_i + 1)} \sup_{s \in [0, t]} \|u_1(s) - u_2(s)\|^2
\]
\[(21)\]
where \( \varepsilon > 0 \) is chosen such that
\[
\sum_{i=1}^{k} \frac{\varepsilon a_i T^{\alpha_i+1}}{\Gamma(\alpha_i + 1)} < \frac{1}{2}.
\]

Integrating the inequality (21) on \([0, t]\), we get
\[
\frac{1}{2} \|u_1(t) - u_2(t)\|^2 \leq \left( \sum_{i=1}^{k} \frac{T^{\alpha_i}}{4\varepsilon \Gamma(\alpha_i + 1)} \right) \int_{0}^{t} \|u_1(s) - u_2(s)\|^2 ds
\]
\[
+ \left( \sum_{i=1}^{k} \frac{\varepsilon a_i T^{\alpha_i+1}}{\Gamma(\alpha_i + 1)} \right) \sup_{s \in [0, t]} \|u_1(s) - u_2(s)\|^2.
\]

Hence
\[
\left( \frac{1}{2} - \sum_{i=1}^{k} \frac{\varepsilon a_i T^{\alpha_i+1}}{\Gamma(\alpha_i + 1)} \right) \|u_1(t) - u_2(t)\|^2
\]
\[
\leq \left( \frac{1}{2} - \sum_{i=1}^{k} \frac{\varepsilon a_i T^{\alpha_i+1}}{\Gamma(\alpha_i + 1)} \right) \sup_{s \in [0, t]} \|u_1(t) - u_2(t)\|^2
\]
\[
\leq \left( \sum_{i=1}^{k} \frac{T^{\alpha_i}}{4\varepsilon \Gamma(\alpha_i + 1)} \right) \int_{0}^{t} \|u_1(s) - u_2(s)\|^2 ds.
\]

Applying the integral Gronwall inequality, see Lemma 2.5, we conclude \( u_1 = u_2 \), which completes the proof of the theorem.

4. Applications. In this section several examples are given to illustrate the main results of Section 3.

Example 1. Consider the following classical integral diffusion equation of the form
\[
\begin{aligned}
&\frac{\partial u(t, x)}{\partial t} - \frac{\partial^2 u(t, x)}{\partial x^2} = \int_{0}^{t} u(s, x) ds \quad \text{in} \quad (0, 10) \times (0, \pi), \\
u(0, x) = u_0(x) \quad \text{for all} \quad x \in [0, \pi], \\
u(t, 0) = u(t, \pi) = 0 \quad \text{for all} \quad t \in [0, 10].
\end{aligned}
\]

For this problem we choose \( k = 1, a_1 = 1, \alpha_1 = 1 \) and \( T = 10 \). We denote by \( u: [0, T] \to L^2(0, \pi; \mathbb{R}) \) the function \( u(t)(x) = u(t, x) \) for all \( t \in [0, T], x \in [0, \pi] \), and put \( H = L^2(0, \pi; \mathbb{R}), f = 0 \) and \( Au = -\frac{\partial^2 u}{\partial x^2} \) with
\[
D(A) = H^2(0, \pi; \mathbb{R}) = \{ u \in L^2(0, \pi; \mathbb{R}) \mid u'' \in L^2(0, \pi; \mathbb{R}) \} \subset H.
\]

It is obvious that all hypotheses of Theorem 3.3 are satisfied. Therefore, by this theorem, we know that the problem has a unique strong solution in \( u \in C(0, T; H) \) and \( u \) is Lipschitz continuous.

Example 2. Consider the following single-term fractional integral diffusion equation of the form
\[
\begin{aligned}
&\frac{\partial u(t, x)}{\partial t} - \frac{\partial^2 u(t, x)}{\partial x^2} = \frac{1}{\Gamma(0.5)} \int_{0}^{t} (t - s)^{-0.5} u(s, x) ds + t^2 x^2 \\
&\quad \text{in} \quad (0, 10) \times (0, \pi), \\
u(0, x) = u_0(x) \quad \text{for all} \quad x \in [0, \pi], \\
u(t, 0) = u(t, \pi) = 0 \quad \text{for all} \quad t \in [0, 10].
\end{aligned}
\]
For this problem, we have \( k = 1, \alpha_1 = 1, \alpha_1 = 0.5 \) and \( T = 10 \). We denote by 
\[ u: [0, T] \to L^2(0, \pi; \mathbb{R}) \]
the function \( u(t)(x) = u(t, x) \) for all \( t \in [0, T], \ x \in [0, \pi] \), 
\( H = L^2(0, \pi; \mathbb{R}) \), \( f(t) = tx^2 \) and the operator \( A \) is defined as in Example 1. It is clear that all assumptions of Theorem 3.3 hold. Therefore, by this theorem, we deduce that the problem has a unique strong solution in \( C(0, T; H) \) and it is a Lipschitz continuous function.

Example 3. Consider the following multi-term fractional integral diffusion equation of the form

\[
\begin{aligned}
\frac{\partial u(t, x)}{\partial t} - \frac{\partial^2 u(t, x)}{\partial x^2} &= \sum_{i=1}^{k} \frac{a_i}{\Gamma(\alpha_i)} \int_{0}^{t} (t - s)^{\alpha_i - 1} u(s, x) \, ds \\
&\quad + f(t, x) \quad \text{in} \quad (0, 10) \times (0, \pi), \\
u(0, x) &= u_0(x) \quad \text{for all} \quad x \in [0, \pi], \\
u(t, 0) &= u(t, \pi) = 0 \quad \text{for all} \quad t \in [0, 10],
\end{aligned}
\]

where \( a_i \geq 0, \alpha_i > 0 \) for all \( i = 1, \ldots, k, \ T = 10 \) and \( f: [0, T] \times [0, \pi] \) is such that \( t \mapsto f(t, x) \) is Lipschitz continuous on \([0, T]\) for all \( x \in [0, \pi] \) and \( x \mapsto f(t, x) \in L^2(0, \pi; \mathbb{R}) \) for all \( t \in [0, T] \).

As before, we denote by \( u: [0, T] \to L^2(0, \pi; \mathbb{R}) \) the function \( u(t)(x) = u(t, x) \) for all \( t \in [0, T], \ x \in [0, \pi] \), \( H = L^2(0, \pi; \mathbb{R}) \) and the operator \( A \) is defined as in Example 1. It is obvious that all hypotheses of Theorem 3.3 are satisfied, and so by this theorem we deduce that the problem has a unique strong solution in \( C(0, T; H) \) and the solution is a Lipschitz continuous function.

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