A COHOMOLOGICAL FRAMEWORK FOR HOMOTOPY MOMENT MAPS

YAËL FRÉGIER, CAMILLE LAURENT-GENGOUX, AND MARCO ZAMBON

Abstract. Given a Lie group acting on a manifold $M$ preserving a closed $n + 1$-form $\omega$, the notion of homotopy moment map for this action was introduced in [10], in terms of $L_\infty$-algebra morphisms. In this note we describe homotopy moment maps as coboundaries of a certain complex. This description simplifies greatly computations, and we use it to study various properties of homotopy moment maps: their relation to equivariant cohomology, their obstruction theory, how they induce new ones on mapping spaces, and their equivalences. The results we obtain extend some of the results of [10].

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Introduction

Recall that a symplectic form is a closed, non-degenerate 2-form. It is natural to consider symmetries of a given symplectic manifold, that is, a Lie group acting on a manifold, preserving the symplectic form. Among such actions, a nice subclass is given by actions that admit a moment map; in that case the infinitesimal generators of the action are hamiltonian vector fields. Actions admitting a moment map enjoy remarkable geometric, algebraic and topological properties, that have been studied extensively in the literature (e.g. symplectic reduction, the relation to equivariant cohomology and localization, convexity theorems,...)

In this note we consider closed $n + 1$-forms for some $n \geq 1$. When they are non-degenerate, they are called multisymplectic form, and are higher analogues of symplectic forms which appear naturally in classical field theory.

Recently Rogers [15] (see also [18]) showed that the algebraic structure underlying a manifold with a closed $n + 1$-form $\omega$ is the one of an $L_\infty$-algebra. This allowed [10] for a natural extension of the notion of moment map to closed forms of arbitrary degree, called homotopy moment map. The latter is phrased in terms of $L_\infty$-algebra morphisms.
The first contribution of this note is to construct, out of the action of a Lie group \( G \) on a manifold \( M \), a chain complex \( C \) with the following property:

- any invariant closed form \( \omega \) gives rise to a cocycle \( \tilde{\omega} \) in \( C \)
- homotopy moment maps are given exactly by the primitives of \( \tilde{\omega} \).

The chain complex \( C \) is simply the product of the Chevalley-Eilenberg complex of the Lie algebra of \( G \), with the de Rham complex of \( M \). The action is encoded by the cocycle \( \tilde{\omega} \). Notice that by the above the set of homotopy moment maps (for a fixed \( \omega \)) has the structure of an affine space, which is unexpected since \( L_\infty \)-algebra morphisms are generally very non-linear objects.

This characterization of homotopy moment maps is very useful: \( L_\infty \)-algebra morphisms are usually quite intricate and cumbersome to work with in an explicit way, while working with coboundaries in a complex is much simpler. In this note we use the above characterization to:

- show that certain extensions of \( \omega \) in the Cartan model give rise to homotopy moment maps (see §4),
- give cohomological obstructions to the existence of homotopy moment maps (see §5),
- show that a homotopy moment map for a \( G \)-action on \((M, \omega)\) induces one on \( \text{Maps}(\Sigma, M) \), the space of maps from any closed and oriented manifold \( \Sigma \) into \( M \), endowed with the closed form obtained from \( \omega \) by transgression (see §6),
- obtain a natural notion of equivalence of homotopy moment maps, both under the requirement that \( \omega \) be kept fixed and allow \( \omega \) to vary (see §7). We show that it is compatible with the geometric notion of equivalence induced by isotopies of the manifold \( M \), and with the notion of equivalence of \( L_\infty \)-morphisms (see Appendix A).

In §4 and §5 we obtain results similar to those of [10], but with much less computational effort. The results obtained in §7 are a significant extension of results obtained in [10], where only closed 3-forms and loop spaces were considered. The equivalences introduced in §7 and their properties extend and justify the work carried out for closed 3-forms in [10, §7.4].

One more application of the characterization of moment maps as coboundaries in \( C \) is the following. Given two manifolds endowed with closed forms, their cartesian product \((M_1 \times M_2, \omega_1 \wedge \omega_2)\) is again an object of the same kind. This construction restricts to the multisymplectic category, but not to the symplectic one. The above characterization of moment maps is used in [17] to construct homotopy moment maps for cartesian products.

**Remark:** Recall that if \( X \) is a Lie algebra, a \( X \)-differential algebra [13] §3 is a graded commutative algebra \( \Omega = \oplus_{i \geq 0} \Omega^i \) with graded derivations \( \iota_v, \mathcal{L}_v \) of degrees \(-1, 0\) (depending linearly on \( v \in X \)) and a derivation \( d \) of degree 1 such that the Cartan relations hold:

\[
[d, d] = 0, \quad [\mathcal{L}_v, d] = 0, \quad [\iota_v, d] = \mathcal{L}_v
\]

\[
[\iota_v, \iota_w] = 0, \quad [\mathcal{L}_v, \mathcal{L}_w] = \mathcal{L}_{[v, w]}_X, \quad [\mathcal{L}_v, \iota_w] = \iota_{[v, w]}_X.
\]

This note is written in terms of geometric objects, but most of it applies also to the algebraic setting obtained replacing the setting we assume in §2 with:

- \( X \) a Lie algebra, \( \Omega \) a \( X \)-differential algebra, \( \omega \in \Omega^{n+1} \) with \( d\omega = 0 \).
- \( g \) a Lie algebra and \( \rho: g \to X \) a Lie algebra morphism, so that \( \mathcal{L}_{\rho(x)}\omega = 0 \) for all \( x \in g \).
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1. Closed forms

We recall briefly how some notions from symplectic geometry apply to closed differential forms of arbitrary degree.

Definition 1.1. Let \((M, \omega)\) be a pre-\(n\)-plectic manifold, i.e., \(M\) is a manifold and \(\omega\) a closed \(n + 1\)-form. An \((n - 1)\)-form \(\alpha\) is Hamiltonian iff there exists a vector field \(v_\alpha \in \mathfrak{X}(M)\) such that

\[
\begin{align*}
\mathit{d}\alpha &= -\iota_{v_\alpha} \omega.
\end{align*}
\]

We say \(v_\alpha\) is a Hamiltonian vector field for \(\alpha\). The set of Hamiltonian \((n - 1)\)-forms is denoted as \(\Omega_{\text{Ham}}^{n-1}(M)\).

In analogy to symplectic geometry, one can endow the set of Hamiltonian \((n - 1)\)-forms with a skew-symmetric bracket, which however is not a Lie bracket. If one passes from \(\Omega_{\text{Ham}}^{n-1}(M)\) to a larger space, one obtains an \(L_\infty\)-algebra \([12]\), which was constructed essentially in \([15, \text{Thm. 5.2}]\), and generalized slightly in \([18, \text{Thm. 6.7}]\).

Definition 1.2. Given a pre-\(n\)-plectic manifold \((M, \omega)\), the observables form an \(L_\infty\) algebra, denoted \(L_\infty(M, \omega) := (L, \{l_k\})\). The underlying graded vector space is given by

\[
L_i = \begin{cases} 
\Omega_{\text{Ham}}^{n-1}(M) & i = 0, \\
\Omega^{n-1+i}(M) & -n + 1 \leq i < 0.
\end{cases}
\]

The maps \(\{l_k : L^\otimes k \to L | 1 \leq k < \infty\}\) are defined as

\[
l_1(\alpha) = \mathit{d}\alpha,
\]

if \(\deg(\alpha) > 0\), and for all \(k > 1\)

\[
l_k(\alpha_1, \ldots, \alpha_k) = \begin{cases} 
0 & \text{if } \deg(\alpha_1 \otimes \cdots \otimes \alpha_k) < 0, \\
\varsigma(k) \iota(v_{\alpha_1} \wedge \cdots \wedge v_{\alpha_k}) \omega & \text{if } \deg(\alpha_1 \otimes \cdots \otimes \alpha_k) = 0,
\end{cases}
\]

where \(v_{\alpha_i}\) is any Hamiltonian vector field associated to \(\alpha_i \in \Omega_{\text{Ham}}^{n-1}(M)\). Here, \(\varsigma(k) = \frac{\varsigma(k - 1) \varsigma(k)}{(-1)^{k+1}} = \frac{1}{(-1)^{k+1}}\). Notice that \(\varsigma(k - 1) \varsigma(k) = (-1)^k\) for all \(k\).

Among the Lie group actions on \(M\) that preserve \(\omega\), it is natural to consider those whose infinitesimal generators are Hamiltonian vector fields. This leads to the following notion \([10, \text{Def. 5.1}]\).

Definition 1.3. A (homotopy) moment map for the action of \(G\) on \((M, \omega)\) is a \(L_\infty\)-morphism \(f : \mathfrak{g} \to L_\infty(M, \omega)\) such that for all \(x \in \mathfrak{g}\)

\[
d(f_1(x)) = -\iota(v_x) \omega.
\]

Saying that \(f\) is a \(L_\infty\)-morphism means that it consists of components \(f_k : \wedge^k \mathfrak{g} \to \Omega^{n-k}(M)\) (for \(k = 1, \ldots, n\)) satisfying

\[
1\text{So } \varsigma(k) = 1,1,-1,-1,1,\ldots \text{ for } k = 1,2,3,4,5,\ldots.
\]
(2) \[ \sum_{1 \leq i < j \leq k} (-1)^{i+j+1} f_{k-1}(\{x_i, x_j\}, x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_k) \]
\[ = df_k(x_1, \ldots, x_k) + \varsigma(k)\iota(v_{x_1} \wedge \cdots \wedge v_{x_k})\omega \]
for \(2 \leq k \leq n\), as well as

(3) \[ \sum_{1 \leq i < j \leq n+1} (-1)^{i+j+1} f_n(\{x_i, x_j\}, x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_{n+1}) = \varsigma(n+1)\iota(v_{x_1} \wedge \cdots \wedge v_{x_{n+1}})\omega. \]

2. A DOUBLE COMPLEX ENCODING MOMENT MAPS

The set-up in the whole of this note is the following:

\((M, \omega)\) is a pre-\(n\)-plectic manifold,

\(G\) is a Lie group acting on \(M\) preserving \(\omega\).

We denote the Lie algebra of \(G\) by \(\mathfrak{g}\), elements of \(\mathfrak{g}\) by \(x\), and the corresponding infinitesimal generators of the action (which are vector fields on \(M\)) by \(v_x\).

In this section we introduce a complex with the property that suitable coboundaries correspond bijectively to moment maps for the action of \(G\) on \((M, \omega)\).

The manifold \(M\) and the Lie algebra \(g\) give rise to a double complex

\( (\wedge^{\geq 1}\mathfrak{g}^* \otimes \Omega(M), d_{\mathfrak{g}}, d) \),

where \(d_{\mathfrak{g}}\) is the Chevalley-Eilenberg differential of \(\mathfrak{g}\) and \(d\) is the De Rham differential of \(M\). We consider the total complex, which we denote by \(C\), with differential

\[ d_{\text{tot}} := d_{\mathfrak{g}} \otimes 1 + 1 \otimes d. \]

We use the Koszul sign convention, hence, on an element of \(\wedge^k \mathfrak{g}^* \otimes \Omega(M)\), \(d_{\text{tot}}\) acts as \(d_{\mathfrak{g}} + (-1)^k d\).

We first need a lemma, which appears (using a slightly different notation) as the Extended Cartan Formula in [13, Lemma 3.4], and which we present without proof.

**Lemma 2.1.** Let \(M\) be a manifold and let \(\Omega\) be an \(N\)-form (not necessarily closed). For all \(k \geq 2\) and all vector fields \(v_1, \ldots, v_k\) we have:

\[ (-1)^k d\iota(v_1 \wedge \cdots \wedge v_k)\Omega = \sum_{1 \leq i < j \leq k} (-1)^{i+j+1} \iota([v_i, v_j] \wedge v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge \hat{v}_j \wedge \cdots \wedge v_k)\Omega \]
\[ + \sum_{1=1}^{k} (-1)^i \iota(v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge v_k)\mathcal{L}_{v_i}\Omega \]
\[ + \iota(v_1 \wedge \cdots \wedge v_k)d\Omega. \]

**Remark 2.2.** The Lie derivative of a form \(\Omega\) along an multivector field \(V = v_1 \wedge \cdots \wedge v_k\) is defined by \(\mathcal{L}_V \Omega := d\iota(V)\Omega - (-1)^k\iota(V)d\Omega\) [9 Def. A2]. From the above we deduce that \((-1)^k \mathcal{L}_V \Omega\) equals the first two terms on the right hand side of the identity in Lemma 2.1.

\(^2\)It can be proven by a direct computation, extending the proof of [10 Lemma 7.2].
Lemma 2.3. For any $G$-invariant $\sigma \in \Omega^N(M)$ define
\begin{equation}
\sigma_k: \wedge^k g \to \Omega^{N-k}(M), \quad (x_1, \ldots, x_k) \mapsto \iota(v_{x_1} \wedge \cdots \wedge v_{x_k})\sigma.
\end{equation}
and $\tilde{\sigma} := \sum_{k=1}^{N} (-1)^{k-1} \sigma_k$. The map
\begin{equation}
\sim : (\Omega(M)^G, d) \to (\wedge^{\geq 1} g^* \otimes \Omega(M), d_{\text{tot}}), \quad \sigma \mapsto \tilde{\sigma}
\end{equation}
intertwines the differentials, that is: $d_{\text{tot}}\tilde{\sigma} = \tilde{d}\sigma$.

Proof. Lemma 2.1 implies that $(-1)^k d\sigma_k = d_g \sigma_{k-1} + (d\sigma)_k$ for all $k \geq 2$. Hence
\begin{equation}
d_{\text{tot}}\tilde{\sigma} = \sum_{k=1}^{N} (-1)^{k-1} (d_g \sigma_k + (-1)^k d\sigma_k)
\end{equation}
\begin{equation}
= \sum_{k=2}^{N+1} (-1)^k d_g \sigma_{k-1} - \sum_{k=1}^{N} d\sigma_k
\end{equation}
\begin{equation}
= \sum_{k=2}^{N+1} ((-1)^k d_g \sigma_{k-1} - d\sigma_k) - d\sigma_1 = \tilde{d}\sigma,
\end{equation}
where in the third equality we used $\sigma_{N+1} = 0$, and in the last one we used the above equation and $-d\sigma_1 = (d\sigma)_1$ (the latter follows from $d(\iota_{v_x} \sigma) + \iota_{v_x} d\sigma = L_{v_x} \sigma = 0$).

Since $\omega$ is a closed differential form, from Lemma 2.3 we obtain:

Corollary 2.4. $\tilde{\omega}$ is $d_{\text{tot}}$-closed.

The next proposition states that moment maps for the action of $G$ on $(M, \omega)$ correspond bijectively to primitives of $\tilde{\omega}$ in $(\mathcal{C}, d_{\text{tot}})$. In particular, moment maps form an affine space, which is somewhat surprising since generally $L_\infty$-morphisms are very non-linear objects.

Proposition 2.5. Let $\varphi = \varphi_1 + \cdots + \varphi_n$, with $\varphi_k \in \wedge^k g^* \otimes \Omega^{n-k}(M)$. Then: $d_{\text{tot}}\varphi = \tilde{\omega}$ iff
\begin{equation}
f_k := \varsigma(k)\varphi_k: \wedge^k g \to \Omega^{n-k}(M),
\end{equation}
for $k = 1, \ldots, n$, are the components of a homotopy moment map for the action of $G$ on $(M, \omega)$.

Proof. $d_{\text{tot}}\varphi = \sum_{k=2}^{n+1} d_g \varphi_{k-1} + \sum_{k=1}^{n} (-1)^k d\varphi_k$ is equal to $\tilde{\omega}$ iff we have
\begin{equation}
-d\varphi_1 = \omega_1
\end{equation}
\begin{equation}
d_g \varphi_{k-1} + (-1)^k d\varphi_k = (-1)^{k-1}\omega_k \quad \text{for all } 2 \leq k \leq n
\end{equation}
\begin{equation}
d_g \varphi_n = (-1)^n \omega_{n+1}.
\end{equation}

Evaluating eq. (6) on $x \in g$ we obtain $d\varphi_1(x) = -\iota_{v_x} \omega$, which is equivalent to eq. (1).

Evaluating eq. (7) on $x_1, \ldots, x_k \in g$ we obtain
\begin{equation}
\sum_{1 \leq i < j \leq k} (-1)^{i+j} \varphi_{k-1}(v_{x_i}, v_{x_j}, v_{x_1}, \ldots, \tilde{v}_{x_1}, \ldots, \tilde{v}_{x_j}, \ldots, v_{x_k})
\end{equation}
\begin{equation}
= -(-1)^k d\varphi_k(v_{x_1}, \ldots, v_{x_k}) + (-1)^{k-1} \iota(v_{x_1} \wedge \cdots \wedge v_{x_k}) \omega.
\end{equation}
Multiplying this equation by $-\varsigma(k-1) = -(-1)^k \varsigma(k)$ we obtain eq. (2). Similarly one sees that eq. (8) is equivalent to eq. (3).

Remark 2.6. The results of this section can be derived also from [8, §3]. See [10] for an explanation of how this derivation goes.
3. Closed forms and moment maps as cocycles

Recall that whenever \( f : (A,d) \rightarrow (A',d') \) is a map of complexes, \((A[1] \oplus A',d_f)\) is a complex with differential \( d_f := \left( \begin{array}{cc} d & 0 \\ -d' & \end{array} \right) \). This is known as the cone construction.

We apply this to the map of complexes \( \tilde{\omega} \) of Lemma 2.3. We obtain:

**Proposition 3.1.** Fix an action of a Lie group \( G \) on a manifold \( M \). Then

\[
\mathcal{B} := \Omega(M)^G[1] \oplus (\wedge^1 \mathfrak{g}^* \otimes \Omega(M)), \quad D := \left( \begin{array}{cc} d & 0 \\ -d_{tot} & \end{array} \right)
\]

is a complex with the property: the \( D \)-closed elements in degree \( n \) are pairs \((\omega[1], \varphi)\) where \( \omega \) is a pre-\( n \)-plectic form and \( \varphi \) corresponds (via Prop. 2.5) to a moment map for \( \omega \).

**Proof.** Compute \( D(\omega[1], \varphi) = (d\omega[1], \tilde{\omega} - d_{tot}\varphi) \) and apply Prop. 2.5 \( \square \)

4. Equivariant cohomology

In this section we recover in a quick way a result of [10], which states that suitable extensions of \( \omega \) in the Cartan model give rise to moment maps (see Prop. 4.4).

The following is a variation of Lemma 2.3.

**Lemma 4.1.** For any \( G \)-equivariant \( F : \mathfrak{g} \rightarrow \Omega^N(M) \), such that \( \iota_{v_x} F(x) = 0 \) for all \( x \in \mathfrak{g} \), define

\[
F_k : \wedge^k \mathfrak{g} \rightarrow \Omega^{N+1-k}(M), \quad F_k(x_1, \ldots, x_k) = \iota(x_1 \wedge \cdots \wedge v_{x_{k-1}})F(x_k)
\]

and \( \tilde{F} := F_1 + \cdots + F_{N+1} \in \wedge^1 \mathfrak{g}^* \otimes \Omega(M) \). Then \( d_{tot}\tilde{F} = -dF \).

**Remark 4.2.** 1) Notice that \( F_1 = F \).

2) If \( \alpha \in \Omega^{N+1}(M) \) is \( G \)-invariant, then \( \alpha : \mathfrak{g} \rightarrow \Omega(M)^N, x \mapsto \iota_{v_x} \alpha \) is \( G \)-equivariant since \( \mathcal{L}_{v_y} (\iota_{v_x} \alpha) = \iota_{[v_y,v_x]} \alpha \). We have \( \tilde{\alpha} = (\alpha_1) \) (where the l.h.s. was defined in Lemma 2.3 and the r.h.s. in Lemma 4.1).

3) \( \tilde{F} \) lies in the \( G \)-invariant part of \( \wedge^1 \mathfrak{g}^* \otimes \Omega(M) \), see [10], §6] for a proof.

**Proof.** Notice first that the condition \( \iota_{v_x} F(x) = 0 \) ensures that \( \tilde{F} \) is well-defined (as a totally skew-symmetric map). It also implies that \( \iota_{v_x} d(F(x)) = \mathcal{L}_{v_x}(F(x)) = F([x,x]) = 0 \), so that \( dF \) is well-defined.

We compute for all \( k \):

\[
(9) \quad (d_g F_{k-1})(x_1, \ldots, x_k) = \sum_{1 \leq i < j \leq k} (-1)^{i+j} F_{k-1}([x_i, x_j], \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_k)
\]

\[
= \sum_{1 \leq i < j \leq k-1} (-1)^{i+j} \iota([v_{x_i}, v_{x_j}] \wedge \cdots \wedge \hat{v}_{x_i} \wedge \cdots \wedge \hat{v}_{x_j} \wedge \cdots \wedge v_{x_{k-1}}) F(x_k)
\]

\[
+ \sum_{i=1}^{k-1} (-1)^{i+k} (-1)^{k-2} \iota(v_{x_1} \wedge \cdots \wedge \hat{v}_{x_i} \wedge \cdots \wedge v_{x_{k-1}}) F([x_i, x_k])
\]

and notice that \( F([x_i, x_k]) = \mathcal{L}_{v_{x_i}} F(x_k) \) by the equivariance of \( F \). Hence

\[
(-1)^{k-1} d(F_k(x_1, \ldots, x_k)) = (-1)^{k-1} d(\iota(v_{x_1} \wedge \cdots \wedge v_{x_{k-1}}) F(x_k))
\]

\[
= (d_g F_{k-1})(x_1, \ldots, x_k) + \iota(v_{x_1} \wedge \cdots \wedge v_{x_{k-1}}) d(F(x_k)),
\]
where in the last equation we used Lemma 2.1 (applied to $\Omega := F(x_k)$) and eq. (9). In other words:

\[(10)\]  
\[(-1)^{k-1}dF_k = d_\varphi F_{k-1} + (dF)_k.\]

We conclude the proof computing

\[d_{tot}\tilde{F} = \sum_{k=1}^{N+1} (d_\varphi F_k + (-1)^k dF_k) = \sum_{k=2}^{N+2} d_\varphi F_{k-1} + \sum_{k=1}^{N+1} (-1)^k dF_k\]

\[= \sum_{k=2}^{N+2} (d_\varphi F_{k-1} + (-1)^k dF_k) - dF_1\]

\[= -\tilde{d}F,
\]

using $F_{N+2} = 0$ in the third equality and eq. (10) in the last one. \[\square\]

Given the action of $G$ on $M$, recall that the Cartan model is the complex $\mathfrak{g}$-equivariant linear maps $\varphi: \mathfrak{g} \to \Omega^{n-1}(M)$, which we can regard as an element of $(\Omega^n(M) \otimes \mathfrak{g}^\ast)^G$, where elements of $\mathfrak{g}^\ast$ are assigned degree two, together with the Cartan differential $d_G$ (see for example [11]). If we choose a basis $x_i$ of $\mathfrak{g}$ and denote by $\xi_i$ the dual basis of $\mathfrak{g}^\ast$, we can write $d_G = d \otimes 1 - \sum_i \iota_{v_{x_i}} \otimes \xi_i$.

**Remark 4.3.** The invariant pre-$n$-plectic form $\omega$ (or, more precisely, $\omega \otimes 1$) is usually not closed in the Cartan model. Given an equivariant linear map $\mu: \mathfrak{g} \to \Omega^{n-1}(M)$, which we can regard as an element of $(\Omega^n(M) \otimes \mathfrak{g}^\ast)^G$, we have [10] §6.1: $\omega - \mu$ is a closed element of the Cartan model if for all $x, y \in \mathfrak{g}$

a) $d\mu(x) = -\iota_{v_x} \omega$ (i.e., $v_x$ is the Hamiltonian vector field of $\mu(x)$),

b) $\mathcal{L}_{v_x} \mu(y) = \mu([x,y])$ (i.e., $\mu: \mathfrak{g} \to \Omega^{n-1}(M)$ is $G$-equivariant),

c) $\iota_{v_x} \mu(x) = 0$.

We recover the main statement of [10] Thm. 6.3:

**Proposition 4.4.** Let $\mu: \mathfrak{g} \to \Omega^{n-1}(M)$ be an equivariant linear map so that $\omega - \mu$ is a cocycle in the Cartan model. Then $d_{tot} \tilde{\mu} = \tilde{\omega}$, and the maps $(1 \leq k \leq n)$

\[f_k: \wedge^k \mathfrak{g} \to \Omega^{n-k}(M),\]

\[(x_1, \ldots, x_k) \mapsto \epsilon(k) \iota(v_{x_1} \wedge \cdots \wedge v_{x_{k-1}}) \mu(x_k)\]

are the components of a homotopy moment map $\mathfrak{g} \to L_\infty(M, \omega)$.

**Proof.** By a) in Remark 4.3 we have the following equality of equivariant maps $\mathfrak{g} \to \Omega^n(M)$:

\[d\mu = -\omega_1,\]

where $\omega_1(x) = \iota_{v_x} \omega$. As $\iota_{v_x} \iota_{v_y} \omega = 0$ for all $x$, we can apply the map $\tilde{\omega}$ (see Lemma 2.1) to obtain $\tilde{d}\mu = -\tilde{\omega}_1$. We have $d_{tot} \mu = -d\tilde{\mu}$ by applying Lemma 2.1 to $\mu$ (we are allowed to do so because of c) in Remark 4.3, and we also have $\tilde{\omega}_1 = \tilde{\omega}$ (see Rem. 4.2). Altogether we obtain

\[d_{tot} \tilde{\mu} = \tilde{\omega}.
\]

We conclude applying Prop. 2.5 to $\varphi := \tilde{\mu}$. \[\square\]

**Remark 4.5.** Prop. 4.4 can be extended [1] [16], as follows: every arbitrary extension of $\omega$ to a cocycle in the Cartan model gives rise to a moment map.

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3 It calculates the equivariant cohomology when $G$ is compact.

4 In [10] Thm. 6.3 it is also shown that the moment map $f$ is equivariant, using b) in Remark 4.3.
5. Obstruction theory

We consider the obstruction theory for the existence of moment maps, obtaining results similar to those contained in [10] §9.1, §9.2.

Fix a point \( p \in M \). It is immediate to check that
\[
r: (\wedge^{\geq 1}g^* \otimes \Omega(M), d_{\text{tot}}) \rightarrow (\wedge g^*, d_g), \quad \eta \otimes \alpha \mapsto \eta \cdot \alpha|_p
\]
is a chain map. Here \( \Omega|_p \in \mathbb{R} \) is declared to vanish if \( \Omega \in \Omega^{\geq 1}(M) \). Since \( \widetilde{\omega} \) is \( d_{\text{tot}} \)-closed by Lemma 2.3, it follows that \( r(\widetilde{\omega}) = (-1)^n \omega_{n+1}|_p \in \wedge^{n+1}g^* \) is \( d_g \)-closed, hence it defines a class in the Chevalley-Eilenberg cohomology \( H_{CE}(g) \).

**Corollary 5.1.** Let \( p \in M \). If a homotopy moment map exists, then \( [\omega_{n+1}|_p] = 0 \).

**Proof.** By Prop. 2.5 a homotopy moment map exists iff \( [\widetilde{\omega}] = 0 \). In this case \( 0 = H(r)([\widetilde{\omega}]) = (-1)^n [\omega_{n+1}|_p] \), where \( H(r) \) denotes the map on cohomology induced by \( r \). \( \square \)

**Corollary 5.2.** If \( [\omega_{n+1}|_p] = 0 \) and \( \]
\[
\text{then there exists a moment map.}
\]

**Proof.** The algebraic Künneth formula ([3 Exerc. 14.23]) and the conditions on the vanishing of cohomology groups imply that \( H(r): H^{n+1}(\wedge^{\geq 1}g^* \otimes \Omega(M), d_{\text{tot}}) \rightarrow H^{n+1}(\wedge g^*, d_g) \) is an isomorphism. Hence \( \widetilde{\omega} \) vanishes iff \( [\omega_{n+1}|_p] \) vanishes. The latter does vanish by assumption, so there exists \( \varphi \) with \( d_{\text{tot}}\varphi = \widetilde{\omega} \), and by Prop. 2.5 the primitive \( \varphi \) gives rise to a homotopy moment map. \( \square \)

6. Actions on mapping spaces

In [10] §11 it is shown that a moment map for a pre-2-plectic manifold \( M \) gives rise to a moment map for the loop space \( LM \) and an induced presymplectic form. Recall that \( LM = M^{S^1} \) consists of all differentiable maps from the circle \( S^1 \) to \( M \). In this section we generalize this, allowing \( M \) to be any pre-\( n \)-plectic manifold and replacing the circle with any compact, orientable manifold.

6.1. Loop spaces. For the sake of exposition, consider first the case of the loop space \( LM \) (an infinite-dimensional Fréchet manifold). The action of \( G \) on \( M \) induces an action on \( LM \), simply given by \((g \cdot \gamma)(t) := g \cdot \gamma(t)\) for all \( \gamma \in LM \) and \( t \in S^1 \). Given an element \( x \) of the Lie algebra \( g \), recall that we denote by \( v_x \) (a vector field on \( M \)) the corresponding infinitesimal generator of the action on \( M \). The corresponding infinitesimal generator of the action on \( LM \), which we denote by \( v_x^L \), is given as follows: \( v_x^L|_{\gamma} = \gamma^*v_x \in \Gamma(\gamma^*TM) = T_{\gamma}LM \), for all \( \gamma \in LM \).

There is a degree preserving map
\[
\ell: \Omega(M) \rightarrow \Omega(LM)[−1]
\]
called transgression, which commutes with the de Rham differential [4 §3.5]. Explicitly, it sends a form \( \alpha \in \Omega^j(M) \) to \( \alpha^L \in \Omega^{j−1}(LM) \) given by
\[
\alpha^L|_{\gamma}(z_1, \ldots, z_{j−1}) = \int_0^{2\pi} \alpha(z_1, \ldots, z_{j−1}, \gamma)|_{\gamma(s)} \, ds \quad \forall \gamma \in LM, \forall z_1, \ldots, z_{j−1} \in T_{\gamma}LM.
\]

5These cohomology classes vanish, for example, if \( H^1(M) = \cdots = H^n(M) = 0 \).

6The notation \( \Omega(LM)[−1] \) refers to the fact that here \( \Omega^{k−1}(LM) \) is assigned degree \( k \).
In particular, the closed form $\omega \in \Omega^{n+1}(M)$ transgresses to a closed form $\omega^\ell \in \Omega^n(LM)$.

Consider the complex $C = (\Lambda^{2,1}\mathfrak{g}^* \otimes \Omega(M), d_{\text{tot}})$ of eq. (4), as well as $C' := (\Lambda^{2,1}\mathfrak{g}^* \otimes \Omega(LM)[-1], d_{\text{tot}})$. The transgression map extends trivially to a degree preserving mapping

$$\text{Id} \otimes \ell : C \to C',$$

which commutes with $d_q$ and the de Rham differential, and hence with $d_{\text{tot}}$. We use the superscript $\ell$ to denote this map too. In particular, given $\varphi = \varphi_1 + \cdots + \varphi_n \in C$ where $\varphi_k \in \Lambda^k \mathfrak{g}^* \otimes \Omega^{n-k}(M)$, we obtain an element $\varphi^\ell \in C'$ with components $\varphi^\ell = (\varphi^\ell)_1 + \cdots + (\varphi^\ell)_{n-1}$ where $(\varphi^\ell)_k := (\varphi_k)^\ell \in \Lambda^k \mathfrak{g}^* \otimes \Omega^{n-k-1}(LM)$.

**Proposition 6.1.** If $\varphi$ corresponds (in the sense of Prop. 6.1) to a homotopy moment map for $(M, \omega)$, then $\varphi^\ell$ corresponds to a homotopy moment map for $(LM, \omega^\ell)$.

**Proof.** If $d_{\text{tot}} \varphi = \tilde{\omega}$ then (11)

$$d_{\text{tot}}(\varphi^\ell) = (d_{\text{tot}} \varphi)^\ell = (\tilde{\omega})^\ell = \tilde{\omega}^\ell.$$

The last equality holds because for all $k$ we have $(\omega_k)^\ell = (\omega^\ell)_k$, as a consequence of

$$(\omega_k)^\ell(x_1, \ldots, x_k)|_\gamma = (\ell(v_{x_1} \land \cdots \land v_{x_k})\omega)|_\gamma = \int_0^{2\pi} \omega(v_{x_1}, \ldots, v_{x_k}, \bullet, \gamma)|_\gamma ds$$

$$= \ell(v_1^\ell \land \cdots \land v_k^\ell)(\omega^\ell)|_\gamma = (\omega^\ell)_k(x_1, \ldots, x_k)|_\gamma,$$

where $x_1, \ldots, x_k \in \mathfrak{g}$, $\gamma \in LM$, and $\bullet$ denotes $n-k$ slots for elements of $T_xLM$. Recall that $(\omega^\ell)_k$ was defined in Lemma 2.3.

Thanks to eq. (11) we can now apply Prop. 2.5 (which holds in the setting of Fréchet manifolds too). \qed

### 6.2. General mapping spaces.

We now generalize Prop. 6.1. Let $\Sigma$ be a compact, oriented manifold of dimension $s$. The $G$ action on $M$ gives rise to $G$ action on $M^\Sigma$, the Fréchet manifold of smooth maps from $\Sigma$ to $M$. (The formulae for this action and the corresponding infinitesimal generators are exactly as in §6.1).

Transgression is the differential-preserving map

$$\ell := \int_{\Sigma} \circ ev^* : \Omega(M) \to \Omega(M^\Sigma)[-s]$$

where $ev: \Sigma \times M^\Sigma \to M$ is the evaluation map and $\int_{\Sigma}$ denotes integration along the fibers [2] Cap. VI.4 of the projection $\Sigma \times M^\Sigma \to M^\Sigma$, which lowers the degree of a differential form by $s$. Notice that the closed form $\omega \in \Omega^{n+1}(M)$ transgresses to a closed form $\omega^\ell \in \Omega^{n+1-s}(M^\Sigma)$.

The transgression map extends trivially to a map of complexes

$$\text{Id} \otimes \ell : C \to C',$$

where now $C' := (\Lambda^{2,1}\mathfrak{g}^* \otimes \Omega(M^\Sigma)[-s], d_{\text{tot}})$. Let $\varphi = \varphi_1 + \cdots + \varphi_n$, where $\varphi_k \in \Lambda^k \mathfrak{g}^* \otimes \Omega^{n-k}(M)$.

**Proposition 6.2.** If $\varphi$ corresponds (in the sense of Prop. 2.5) to a homotopy moment map for $(M, \omega)$, then $\varphi^\ell$ corresponds to a homotopy moment map for $(M^\Sigma, \omega^\ell)$.

**Proof.** The proof is the same as for Prop. 6.1. We just point out that the equation $(\omega_k)^\ell = (\omega^\ell)_k$ for all $k$ is obtained applying the well-known relation $(\ell(v_x)\omega)^\ell = \ell(v_x)\omega^\ell$ for all $x \in \mathfrak{g}$. It can be proven also directly, exactly as in the proof of Prop 6.1, using the explicit description of the integration along the fiber given in [2] Cap. VI.4 and the fact that the derivative
Definition remarking that the choices of Cartan cocycles we allow are also not the most general ones.

Remark 7.2

Corollary 6.3. Let

\[ f: \mathfrak{g} \to L_\infty(M, \omega) \]

be a homotopy moment map for the pre-$n$-plectic manifold $(M, \omega)$ with components $f_k: \wedge^k \mathfrak{g} \to \Omega^{n-k}(M)$, where $k = 1, \ldots, n$.

Then

\[ f^\ell: \mathfrak{g} \to L_\infty(M^\Sigma, \omega^\ell) \]

is a homotopy moment map for the pre-$(n-s)$-plectic manifold $(M^\Sigma, \omega^\ell)$ with components $(f^\ell)_k := (f_k)^\ell: \wedge^k \mathfrak{g} \to \Omega^{n-s-k}(M^\Sigma)$, where $k = 1, \ldots, n-s$.

7. Equivalences

In this section we introduce notions of equivalence for: 1) certain cocycles in the Cartan model, and 2) pairs consisting of closed invariant forms and moment maps. Recall that Prop. 4.4 states that to the former Cartan cocycles one can canonically associate moment maps; we show that the above equivalences are compatible with this association. All along this section we fix an action of a Lie group $G$ on a manifold $M$.

7.1. Equivalences of Cartan cocycles.

Definition 7.1. Two cocycles $C^0 = \omega^0 - \mu^0$ and $C^1 = \omega^1 - \mu^1$ in the Cartan model, with $\omega^0, \omega^1 \in \Omega^{n+1}(M)^G$ and $\mu^0, \mu^1 \in (\Omega^{n-1}(M) \otimes \mathfrak{g}^*)^G$, are equivalent iff they differ by a coboundary of the form $d_G(\alpha + F)$, where

\[ \alpha \in \Omega^n(M)^G \text{ and } F \in (\Omega^{n-2}(M) \otimes \mathfrak{g}^*)^G. \]

Explicitly, $C^1 - C^0 = d_G(\alpha + F)$ means that

- a) $\omega^1 - \omega^0 = d\alpha$
- b) $\mu^1 - \mu^0 = \iota_\alpha - dF$
- c) $\iota(v_x) F(x) = 0$ for all $x \in \mathfrak{g}$,

where we use the short form $\iota_\alpha(x) := \iota_{v_x}\alpha$.

Remark 7.2. In the symplectic case (so $n = 1$), Def. 7.1 reduces to: $\omega^1 - \omega^0 = d\alpha$ and $\mu^1 - \mu^0 = \iota_\alpha$ for some $\alpha \in \Omega^1(M)^G$. In particular, if $\omega^1 = \omega^0$, each function $\iota_{v_x}\alpha$ is constant.

The following proposition states that if two Cartan cocycles are related by a $G$-equivariant diffeomorphism of $M$ isotopic to the identity, then they are equivalent.

Proposition 7.3. Let the Lie group $G$ act on $M$. Let $\omega^0, \omega^1 \in \Omega^{n+1}(M)^G$. Let $\mu^0, \mu^1 \in (\Omega^{n-1}(M) \otimes \mathfrak{g}^*)^G$ so that $C^i := \omega^i - \mu^i$ is a cocycle in the Cartan model. Suppose that there exists a $G$-equivariant diffeomorphism $\psi$, isotopic to $\text{Id}_M$ by $G$-equivariant diffeomorphisms, with

\[ \psi^*\omega^0 = \omega^0, \ \psi^*\mu^1 = \mu^0. \]

(Here $\mu^1$ is viewed as a map $\mathfrak{g} \to \Omega^{n-1}(M)$ and $(\psi^*\mu^1)(x) := (\psi^*\mu^1(x))$ for all $x \in \mathfrak{g}$).

Then $C^1$ and $C^0$ are equivalent in the sense of Def. 7.1.

If $C^0 - C^1$ is exact, in general we may not find a primitive of the form $\alpha + F$ as above. We justify our definition remarking that the choices of Cartan cocycles we allow are also not the most general ones.
Proof. We construct explicitly equivariant Cartan cochains \(\alpha, F\) such that \(d_G(\alpha + F) = C^1 - C^0\).

Let \(\{\psi_s\}_{s \in [0,1]}\) a isotopy from \(\psi^0 = Id_M\) to \(\psi = \psi^1\) by \(G\)-equivariant diffeomorphisms, and denote by \(\{X_s\}_{s \in [0,1]}\) the time-dependent vector field generating \(\{\psi_s\}_{s \in [0,1]}\). Define \(\omega^s\) by \(\psi^*_s(\omega^s) = \omega^0\) and \(\mu^s \in (\Omega^{n-1}(M) \otimes \mathfrak{g}^*)^G\) by \(\psi^*_s(\mu^s) = \mu^0\).

We claim that

\[
\alpha := -\int_0^1 \iota_{X_s}\omega^s \in \Omega^n(M)
\]
satisfies \(\omega^1 - \omega^0 = d\alpha\) (condition a) in Def. \([7.1]\). This follows integrating from \(s = 0\) to \(s = 1\) the equation

\[
\frac{d}{ds}\omega^s = -\mathcal{L}_{X_s}\omega^s = -d\iota_{X_s}\omega^s
\]
where in the first equality we use \([6, \text{Prop. 6.4}]\)

\[
0 = \frac{d}{ds}(\psi^*_s\omega^s) = \psi^*_s(\mathcal{L}_{X_s}\omega^s + \frac{d}{ds}\omega^s).
\]

We claim that

\[
F := \int_0^1 \iota_{X_s}\mu^s \in \Omega^{n-2}(M) \otimes \mathfrak{g}^*
\]
satisfies \(\mu^1 - \mu^0 = \iota_{\nu_0}\alpha - dF\) (condition b) in Def. \([7.1]\). Similarly to the above, this follows integrating from 0 to 1 the following expression, for all \(x \in \mathfrak{g}\):

\[
\frac{d}{ds}\mu^s(x) = -\mathcal{L}_{X_s}(\mu^s(x)) = -\iota_{X_s}(d\mu^s(x)) - d\iota_{X_s}\mu^s(x)
\]
\[
= \iota_{X_s}(\nu_0 \omega^s) - d\iota_{X_s}\mu^s(x)
\]
\[
= \iota_{\nu_0}(\omega^s) - d\iota_{X_s}\mu^s(x).
\]

Here in the first equality we used again \([6, \text{Prop. 6.4}]\), and in the third one \(\iota_{\nu_0}\omega^s = -d(\mu^s(x))\) for all \(x \in \mathfrak{g}\) (see Remark \([4.3] (a))\).

We are left with showing \(\nu_0 F(x) = 0\) for all \(x \in \mathfrak{g}\) (condition c) in Def. \([7.1]\). This holds since \(\nu_0 \mu^s(x) = 0\), a consequence of Remark \([4.3] (c)\) and the fact that \(\omega^s - \mu^s\), being the pullback of a Cartan cocycle by a \(G\)-equivariant map, is itself a Cartan cocycle.

Notice that the \(X_s\) are \(G\)-invariant, as their flow \(\{\psi_s\}\) commutes with the \(G\)-action. Further the \(\omega^s\) are \(G\)-invariant, since \(\omega\) is \(G\)-invariant. Hence \(\alpha\) is \(G\)-invariant. By the same reasoning and the invariance of \(\mu^0\), we see that \(F\) is \(G\)-invariant. \(\square\)

7.2. Equivalences of moment maps. Let \(f : \mathfrak{g} \hookrightarrow L_\infty(M, \omega)\) be a homotopy moment map for \(\omega\), and

\[
(12) \quad \varphi_k := \varsigma(k) f_k : \wedge^k \mathfrak{g} \rightarrow \Omega^{n-k}(M) \quad \text{for } k = 1, \ldots, n.
\]

We know that \(\varphi = \varphi_1 + \cdots + \varphi_n\) satisfies \(d_{\text{tot}} \varphi = \tilde{\omega}\).

Indeed, by Prop. \([2.5]\) this equation characterizes moment maps for \(\omega\). Therefore, if \(\eta \in (\wedge^2 \mathfrak{g}^* \otimes \Omega(M))_{n-1}\), then \(\varphi + d\iota_{\nu_0}\eta\) naturally provides a new moment map for \((M, \omega)\). Further, notice that if \(\alpha \in (\Omega^n)^G\), by Lemma \([2.3]\) we have \(d_{\text{tot}}(\varphi + \tilde{\alpha}) = \omega + d\tilde{\alpha}\), i.e. \(\varphi + \tilde{\alpha}\) provides a moment map for \(\omega + d\alpha\). The following definition, which arises naturally considering the complex \(\mathcal{B}\) introduced in \([3]\) is made so that these two kinds of moment maps are equivalent to the original one.
Definition 7.4. Let ω be an invariant pre-n-plectic form on M and f a moment map for ω, for which we denote by ϕ the corresponding element of \((\wedge^{\geq 1} g^* \otimes \Omega(M))^\ast\) as in Prop. 2.5 and similarly for \((\omega', f')\). The pairs \((\omega, f)\) and \((\omega', f')\) are equivalent if there exist \(\eta \in (\wedge^{\geq 1} g^* \otimes \Omega(M))_{n-1}\) and \(\alpha \in (\Omega^n)^G\) such that
\[
(\omega' - \omega) = d\alpha
\]
\[
(\omega' - \omega) = d\alpha + \bar{\alpha}
\]

Remark 7.5. The equivalence introduced in Def. 7.4 can be phrased as a simple coboundary condition, thereby providing an algebraic justification for Def. 7.4. Indeed, in terms of the complex \(\mathcal{B} = (\Omega(M)^G[1] \oplus (\wedge^{\geq 1} g^* \otimes \Omega(M)), D)\) introduced in §3 the conditions (13)-(14) are simply phrased as
\[
(\omega'[1], \varphi') - (\omega[1], \varphi) = D(\alpha[1], -\eta).
\]

A geometric justification for Def. 7.4 is given in Prop. 7.8.

In Appendix A we compare Def. 7.4 with the natural notion of equivalence for \(L_\infty\)-morphisms. There we show that two homotopy moment maps are equivalent with \(\alpha = 0\) (see Def. 7.4) if they are equivalent as \(L_\infty\)-morphisms.

Remark 7.6. Condition (14), explicitly, is that there exists \(\eta = \eta_1 + \cdots + \eta_{n-1} \in (\wedge^{\geq 1} g^* \otimes \Omega(M))_{n-1}\) and \(\alpha \in (\Omega^n)^G\) with
\[
d\eta_1 + \alpha_1 = (\varphi' - \varphi)_1,
\]
\[
d\eta_k + (1 - 1)^k \eta_k + (-1)^{k-1} \alpha_k = (\varphi' - \varphi)_k \quad \forall k = 2, \ldots, n-1
\]
\[
d\eta_n + (1 - 1)^{n-1} \alpha_n = (\varphi' - \varphi)_n,
\]
where \(\alpha_i\) is defined as in eq. (5).

Remark 7.7. Given a \(G\)-manifold, we can consider \(\omega = 0 \in \Omega_{closed}^{n+1}(M)\) and the zero moment map \(f = 0\). Given \(\alpha \in \Omega^n(M)^G\), applying the operation described in Def. 7.4 provides a moment map for \((M, d\alpha)\), which agrees exactly with the one provided in [10, §8.1] for exact \(n\)-plectic forms admitting an invariant primitive.

The following proposition provides a geometric justification for Def. 7.4. It states that if two moment maps are related by a \(G\)-equivariant diffeomorphism of \(M\) isotopic to the identity, then they are equivalent.

Proposition 7.8. Let the Lie group \(G\) act on \(M\). Let \(\omega^0, \omega^1\) be closed \((n+1)\)-forms preserved by the action. Suppose that there exists a \(G\)-equivariant diffeomorphism \(\psi\), isotopic to \(Id_M\) by \(G\)-equivariant diffeomorphisms, such that \(\psi^\ast \omega^1 = \omega^0\).

Let \(f^i : g \rightarrow L_\infty(M, \omega_i)\) be homotopy moment maps \((i = 0, 1)\) intertwined by \(\psi\) that is, for all their components \((k = 1, \ldots, n)\) we have
\[
\psi^\ast f^1_k = f^0_k.
\]

Then \(f^0\) and \(f^1\) are equivalent in the sense of Def. 7.4.

Proof. Let \(\{\psi_s\}_{s \in [0, 1]}\) a isotopy from \(\psi^0 = Id_M\) to \(\psi = \psi^1\) by \(G\)-equivariant diffeomorphisms, and denote by \(\{X_s\}_{s \in [0, 1]}\) the time-dependent vector field generating \(\{\psi_s\}_{s \in [0, 1]}\).

Define \(\omega^s\) by \(\psi_s^\ast (\omega^s) = \omega^0\) and \(f^s\) by \(\psi_s^\ast (f^s) = f^0\).

The form
\[
\alpha := -\int_0^1 tX_s \omega^s ds \in \Omega^n(M)
\]
satisfies $\omega^1 - \omega^0 = d\alpha$, i.e. eq. (13), as we have already shown at the beginning of the proof of Prop. 7.3.

Now, for all $1 \leq k \leq n$ (and defining $f^0 = 0$) we have

$$\frac{d}{ds} f^s_k = -\mathcal{L}_{x_s} f^s_k = -d(\iota_{x_s} f^s_k) - \iota_{x_s} df^s_k$$

$$= -d(\iota_{x_s} f^s_k) + \iota_{x_s} d\phi f^s_{k-1} + \varsigma(k) \iota_{x_s} \omega^s_k$$

$$= -d(\iota_{x_s} f^s_k) + \iota_{x_s} d\phi f^s_{k-1} + (-1)^k \varsigma(k) (\iota_{x_s} \omega^s)_{k},$$

where in the first equation we used [6] Prop. 6.4, and in the second one $df^s_k = -d\phi f^s_{k-1} - \varsigma(k) \omega^s_k$ (which holds by eq. (7)).

Multiplying by $\varsigma(k)$ the equation $f^1_k - f^0_k = \int_0^1 \frac{d}{ds} f^s_k \, ds$ we hence obtain

$$\varphi^1_k - \varphi^0_k = (-1)^k d\eta_k + d\phi \eta_{k-1} + (-1)^{k-1} \alpha_k,$$

where we define

$$\eta_k: \wedge^k \mathfrak{g} \to \Omega^{n-1-k}(M), \quad \eta_k(x^1, \ldots, x_k) = (-1)^{k-1} \varsigma(k) \int_0^1 \iota(X_s) f^s_k \, ds.$$ 

As this holds for all $1 \leq k \leq n$, we obtain $\varphi_1 - \varphi_0 = d_{\text{tot}} \eta + \alpha$, i.e. eq. (14). \hfill \Box

We finish this subsection discussing equivalences for which the pre-$n$-plectic form is fixed. Fix a pre-$n$-plectic form $\omega$. Restricting Def. 7.3 to the space of moment maps for $\omega$ we obtain: two moment maps $f, f'$ for $\omega$ are equivalent iff there exist $\eta \in (\wedge^1 \mathfrak{g}^* \otimes \Omega(M))_{n-1}$ and a closed form $\alpha \in \Omega^n(M)^G$ such that eq. (14) is satisfied.

The following proposition extends the results of [10] §7.5.

**Proposition 7.9.** There exist a closed 3-form $\omega$ with the following property: There exist an equivariant moment map for $\omega$ which is not equivalent (in the sense of Def. 7.3) to any moment map for $\omega - \mu$ as in Prop. 4.4.

**Proof.** Consider an action of a Lie group $G$ on a connected pre-2-plectic manifold $(M, \omega)$, and let $f$ be an equivariant moment map. Let $f'$ be another equivariant moment map (for the same action on $(M, \omega)$) which is equivalent to $f$. This means exactly that there exist $\eta_1 \in \mathfrak{g}^* \otimes C^\infty(M)$ and a closed form $\alpha \in \Omega^3(M)^G$ satisfying eq. (14). In particular the equation

$$-d\eta_1 + \alpha = (\varphi' - \varphi)_1$$

holds, where we denote $\varphi_1 = f_1, \varphi_2 = f_2$, etc.

For every $x \in \mathfrak{g}$, evaluating the l.h.s. of the eq. (15) on $x$ and applying the interior product $\iota_{v_x}$, we obtain a function. We claim that this function is a constant, which we denote by $C_x$. To see this, first notice that evaluating the l.h.s. of the eq. (15) on $x$ and applying $\iota_{v_x}$ we obtain $\iota_{v_x} (-d\eta_1(x) + \alpha_1(x)) = -\mathcal{L}_{v_x}(\eta_1(x))$. Second, notice that since $\varphi_1, (\varphi')_1, \alpha_1$ are equivariant, it follows that $d\eta_1$ is also equivariant. In particular we have $d\mathcal{L}_{v_x} \eta_1(x) = \mathcal{L}_{v_x} d\eta_1(x) = d\eta_1(x,x) = 0$, i.e., $\mathcal{L}_{v_x} (\eta_1(x)) := -C_x$ is a constant function.

From the above claim we conclude: if there exists $x \in \mathfrak{g}$ such that

i) $\iota_{v_x} \varphi_1(x) \neq 0$

ii) $C_x = 0$

---

Footnote 8: Loosely speaking, the action of $\alpha$ can be interpreted as induced by a gauge transformation on the higher Courant algebroid $TM \oplus \wedge^{n-1}T^*M$ endowed with the $\omega$-twisted Courant bracket.
then necessarily \( i_{v_x}(\varphi')_1(x) \neq 0 \), so \( f' \) can not arise from a Cartan cocycle (compare with Remark 4.3 c)).

Now, following [10] §7.5, we display an example of moment map \( f \) and \( x \in \mathfrak{g} \) satisfying assumption i) and such that for every \( \eta_1 \in \mathfrak{g}^* \otimes C^\infty(M) \) assumption ii) is satisfied. It follows that there exists no moment map which is equivalent to \( f \) and which arises from a Cartan cocycle.

Let \( g \) be the abelian group \( S^1 \times S^1 \), and \((M, \omega) = (S^1 \times S^1 \times \mathbb{R}, d\theta_1 \wedge d\theta_2 \wedge dz)\). We take the infinitesimal action of \( g \) on \( M \) to be given by \((1,0) \in \mathfrak{g} \mapsto \partial_{\theta_1}, (0,1) \mapsto \partial_{\theta_2}\). It is easily checked that

\[
\begin{align*}
\varphi_1 : \mathfrak{g} &\rightarrow \Omega^1_{Ham}(M), & (1,0) &\mapsto zd\theta_2 + d\theta_1, & (0,1) &\mapsto -zd\theta_1 + d\theta_2, \\
\varphi_2 : \wedge^2 \mathfrak{g} &\rightarrow C^\infty(M), & (1,0) \wedge (0,1) &\mapsto -z
\end{align*}
\]

are the components of an equivariant moment map. Let \( x \) be the constant \( \alpha \) which satisfies \( \eta_1 \wedge \alpha = 0 \), hence assumption i) is satisfied. For any \( h \in C^\infty(M) \) such that \( \mathcal{L}_{v_x}(h) = \partial_{\theta_1}(h) \) is a constant, integrating \( \mathcal{L}_{v_x}(h)d\theta_1 \) along the circles \( S^1 \times \{\text{point}\} \times \{\text{point}\} \) of \( M \) one sees by Stokes’ theorem that the constant \( \mathcal{L}_{v_x}(h) \) is necessarily zero. Hence, for any \( \eta_1 \in \mathfrak{g}^* \otimes C^\infty(M) \) we have \( \mathcal{L}_{v_x}(\eta_1(x)) = 0 \), verifying that assumption ii) is satisfied.

\[\square\]

Remark 7.10. The notion of equivalence on the space of moment maps for \( \omega \) mentioned just before Prop. 7.9 should not be confused with the similar but more restrictive one in which \( \alpha = 0 \) is imposed. We refer to the latter notion of equivalence as inner equivalence. Explicitly: two moment maps \( f \) and \( f' \) for \( \omega \) are inner equivalent if there exist \( \eta \in (\wedge^2 \mathfrak{g}^* \otimes \Omega(M))_{n-1} \) such that \( \varphi' - \varphi = d_{\text{tot}}\eta \), where \( \varphi \) denotes the element of \( (\wedge^2 \mathfrak{g}^* \otimes \Omega(M))_n \) corresponding to \( f \) as in Prop. 2.5 and similarly for \( \varphi' \) and \( f' \). The notion of inner equivalence arises naturally if one considers the complex \((C, d_{\text{tot}})\) of §2 (as opposed to the complex \( \mathcal{B} \) introduced in §4).

Notice that when \( \alpha = 0 \), the first equation in Rem. 7.6 says that, for all \( x \in \mathfrak{g} \), the elements \((\varphi')_1(x)\) and \( \varphi_1(x) \) of \( \Omega^{n-1}_{\text{ham}}(M, \omega) \) – which have the same hamiltonian vector field \( v_x \), and hence a priory differ by a closed form – actually differ by an exact form.

7.3. Relation between the two notions of equivalence. We end establishing the relation between the equivalences introduced in Def. 7.1 and Def. 7.4.

Proposition 7.11. Let the Lie group \( G \) act on \( M \). Take two Cartan cocycles \( C^0 = \omega^0 - \mu^0 \) and \( C^1 = \omega^1 - \mu^1 \), with \( \omega^0, \omega^1 \in \Omega^{n+1}(M)^G \) and \( \mu^0, \mu^1 \in (\Omega^{n-1}(M) \otimes \mathfrak{g}^*)^G \). Assume that \( C^0 \) and \( C^1 \) are equivalent in the sense of Def. 7.4.

Then the homotopy moment maps \( f^0 \) and \( f^1 \), induced by the \( \mu^1 \) as in Prop. 4.4, are equivalent in the sense of Def. 7.4.

Proof. Since we assume that \( C^0 \) and \( C^1 \) are equivalent, there is \( \alpha \in \Omega^n(M)^G \) and \( F \in (\Omega^{n-2}(M) \otimes \mathfrak{g}^*)^G \) satisfying the equation appearing below Def. 7.4, that is

\[
\begin{align*}
a) & \quad \omega^1 - \omega^0 = d\alpha \\
b) & \quad \mu^1 - \mu^0 = i_\alpha - dF \\
c) & \quad i_{v_x}F(x) = 0 \quad \text{for all} \quad x \in \mathfrak{g},
\end{align*}
\]

We now check that an equivalence between the homotopy moment maps is given by the form \( \alpha \) and by \( \eta := \tilde{F} \) (notice that \( \tilde{F} \) is well-defined by c)). The relation 13 is just a).

Applying the map \( \sim \) (see Lemma 4.1 to equation b) we obtain

\[
\mu^1 - \mu^0 = i_{\tilde{\alpha}} - d\tilde{F}.
\]
Denoting by $\varphi^i \in (\wedge^i g^* \otimes \Omega(M))_0$ the element corresponding to $f^i$ as in Prop. 2.5 for $i = 0, 1$, we have $\varphi^i = \tilde{\mu}^i$ (to see this, compare the formulæ in Prop. 2.5 and Prop. 1.4). Using $\tilde{\alpha} = \tilde{\alpha}$ (by Rem. 4.2) and $dt \dd F = -d\dd F$ (by Lemma 4.1) we obtain exactly the relation (11).

\section*{Appendix A. Equivalences of moment maps and $L_\infty$-algebra morphisms}

Let $G$ be a Lie group acting on a pre-$n$-plectic manifold $(M, \omega)$. A moment map for this action (Def. 1.3) is in particular an $L_\infty$-morphism $\mathfrak{g} \to L_\infty(M, \omega)$. There is a natural notion of equivalence of $L_\infty$-morphisms, and the aim of this appendix is to show that it coincides with the inner equivalence introduced in Rem. 7.10 (that is, equivalence in the sense of Def. 7.3-imposing $\alpha = 0$).

The notion of equivalence of $L_\infty$-morphisms comes from homotopy theory, and coincides with the one given by equivalences of Maurer-Cartan elements [7]. We express it following [7] §5: let $\tilde{L}, L$ be $L_\infty$-algebras. Consider $\Omega(\mathbb{R}) = \mathbb{R}[t] + \mathbb{R}[t]dt$, the differential graded algebra of polynomial forms on the real line, where $t$ has degree 0 and $dt$ degree 1. Then $L \otimes \Omega(\mathbb{R})$ is again an $L_\infty$-algebra [5] §1.

\begin{definition}
Let $\tilde{L}, L$ be $L_\infty$-algebras. Let $f, f' : \tilde{L} \to L$ be $L_\infty$-morphisms. $f$ and $f'$ are \textbf{equivalent} iff there exists an $L_\infty$-morphism $H : \tilde{L} \to L \otimes \Omega(\mathbb{R})$ such that
\begin{equation}
H|_{t=0,dt=0} = f, \quad H|_{t=1,dt=0} = f'.
\end{equation}
\end{definition}

\begin{proposition}
Two homotopy moment maps $f$ and $f'$ are inner equivalent (see Rem. 7.10) iff they are equivalent in the sense of Def. A.1.
\end{proposition}

\begin{proof}
We first given a characterization of $L_\infty$-morphisms from $\mathfrak{g}$ to $L_\infty(M, \omega) \otimes \Omega(\mathbb{R})$. $L_\infty(M, \omega) \otimes \Omega(\mathbb{R})$ is concentrated in degrees $\leq 1$, and its multibrackets $\{l_k\}$ are as follows [5] §1: for $k \geq 2$ they are given by the multibrackets of $L_\infty(M, \omega)$ extended by $\Omega(\mathbb{R})$-linearity (with no signs involved), and the differential is
\begin{equation}
l_1(\gamma \otimes \Gamma) = D\gamma \otimes \Gamma + (-1)^{\deg(\gamma)} \gamma \otimes \frac{\partial}{\partial t} \Gamma dt,
\end{equation}
where $D$ denotes the differential in $L_\infty(M, \omega)$. All multibrackets, except for the differential, vanish unless all entries are in degree 0 or 1.

We observe that the truncation
\begin{equation}
T := (L_\infty(M, \omega) \otimes \Omega(\mathbb{R}))_{< 0} \oplus \{c \in (L_\infty(M, \omega) \otimes \Omega(\mathbb{R}))_0 : l_1(c) = 0\}
\end{equation}
is closed under the multibrackets. Hence $T$ is a $L_\infty$-algebra for which the multibrackets (except for the differential) vanish unless all entries are in degree zero.

Let
\begin{equation}
H : \wedge^k \mathfrak{g} \to L_\infty(M, \omega) \otimes \Omega(\mathbb{R})
\end{equation}
be a linear map such that $H|_{\wedge^k \mathfrak{g}}$ has degree $1 - k$.

\begin{claim}
$H$ is an $L_\infty$-morphism iff conditions (21) and (22) below are satisfied.
\end{claim}

We divide the proof of the claim in three steps.

\footnotetext{Indeed, $T$ is closed under $l_1$ since $(l_1)^2 = 0$. For the higher brackets, the only non-trivial case to consider is $l_2(\gamma \otimes \Gamma, \gamma' \otimes \Gamma')$ when $\gamma, \Gamma, \gamma', \Gamma'$ all have degree zero. This bracket lies in $T$ since $l_1$ satisfies the Leibniz rule w.r.t. $l_2$.}
A) $H$ is a $L_\infty$-morphism iff the image of the first component $H_1$ is annihilated by $l_1$ and for $2 \leq m \leq n + 1$, for all $x_i \in \mathfrak{g}$

\begin{equation}
\sum_{1 \leq i < j \leq m} (-1)^{i+j+1}H_{m-1}([x_i, x_j], x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_m)
\end{equation}

\begin{equation}
= l_1H_m(x_1, \ldots, x_m) + l_m(H_1(x_1), \ldots, H_1(x_m))
\end{equation}

where $H_{n+1} = 0$. Indeed, if $H$ is a $L_\infty$-morphism, then $H_1$ is a chain map and takes values in $l_1$-closed elements, and therefore $H$ takes values in $T$, so that we can apply [10 3.2]. Conversely, if the image of $H_1$ is annihilated by $l_1$, we can apply [10 3.2], and eq. (17) implies that $H$ is a $L_\infty$-morphism.

B) Write

\begin{equation}
H = h^0(t) + h^1(t)dt
\end{equation}

where $h^0(t)$ and $h^1(t)$ are maps $\wedge \mathfrak{g} \to L_\infty(M, \omega) \otimes \mathbb{R}[t]$. Notice that the component $h^0(t)_k$ has degree $1 - k$, while $h^1(t)_k$ has degree $-k$. The condition that the degree zero component of $H_1$ takes values in $l_1$-closed elements reads

\begin{equation}
\frac{\partial}{\partial t}h^0(t)_1 + dh^1(t)_1 = 0.
\end{equation}

Separating the terms without $dt$ from those containing $dt$, eq. (17) is equivalent to ($2 \leq m \leq n + 1$)

\begin{equation}
\sum_{1 \leq i < j \leq m} (-1)^{i+j+1}h^0(t)_{m-1}([x_i, x_j], x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_m)
\end{equation}

\begin{equation}
= dh^0(t)_m(x_1, \ldots, x_m) + l_m(h^0(t)_1(x_1), \ldots, h^0(t)_1(x_m))
\end{equation}

and

\begin{equation}
\sum_{1 \leq i < j \leq m} (-1)^{i+j+1}h^1(t)_{m-1}([x_i, x_j], x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_m)
\end{equation}

\begin{equation}
= dh^1(t)_m(x_1, \ldots, x_m) + (-1)^{1-m}l_m(h^0(t)_m(x_1, \ldots, x_m),
\end{equation}

where we used $m \geq 2$ and degree counting both to replace $D$ by the de Rham differential $d$, and to conclude that $l_m$ vanishes if one of its arguments is of the form $h^1(t)_1(x_i)$.

C) Eq. (18), (19) and (20) are equivalent to the fact that the following two equations hold for all $t \in \mathbb{R}$:

\begin{equation}
h^0(t): \mathfrak{g} \hookrightarrow L_\infty(M, \omega) \text{ is an } L_\infty \text{ morphism}
\end{equation}

\begin{equation}
d_{tot}h^0(t) = \frac{\partial}{\partial t}h^0(t),
\end{equation}

where the bar denotes the following: if $\xi = \sum_{k=1}^N \xi_k$ with $\xi_k \in \wedge \mathfrak{g}^\ast \otimes \Omega(M)$, then

\begin{equation}
\bar{\xi} := \sum_{k=1}^N \xi(k)\xi_k.
\end{equation}

The equivalence between eq. (19) (for all $2 \leq m \leq n + 1$) and eq. (21) is given again by [10 3.2]. We show the equivalence between eq. (18) and eq. (20) (for all $2 \leq m \leq n + 1$) on one side, and eq. (22) on the other. Notice that the L.H.S. of eq. (22) consists of three kinds of terms, exactly as it happens in eq. (3), (7), (8). The analogue of eq. (6) is equivalent to eq. (18). To take care of the analogues of eq. (7) and (8), write eq. (20) in the form...
\[-d_\theta h^1(t)_{m-1} = dh^1(t)_m + (-1)^{1-m} \frac{\partial}{\partial t} h^0(t)_m \text{ for all } 2 \leq m \leq n + 1, \text{ and use that } h^1(t)_n = 0 \text{ by degree reasons.} \]

Now, given homotopy moment maps \( f, f' \), let \( \varphi := \bar{f}, \varphi' := \bar{f}' \) (where the bar has been defined just above).

Assume first that \( f \) and \( f' \) are inner equivalent, so that there is \( \eta \in \wedge^{\geq 1} \mathfrak{g}^* \otimes \Omega(M)_{n-1} \) with \( \varphi' - \varphi = d_{\text{tot}} \eta \). Define

\[ H = h^0(t) + h^1(t) dt := (\varphi + t d_{\text{tot}} \eta) + \eta dt. \]

Notice that \( H \) satisfies eq. (16). Now we check the two conditions appearing in the above claim. Condition (21) is satisfied, because \( d_{\text{tot}} h^0(t) = d_{\text{tot}} \varphi = \bar{\omega} \) for all \( t \) and because of Prop. \ref{prop:moment_map}. Condition (22) is satisfied as both sides are equal to \( d_{\text{tot}} \eta \). Therefore by the claim \( H \) is an \( L_\infty \)-morphism. We conclude that \( f \) and \( f' \) are equivalent the sense of Def. \ref{def:inner_equivalence}.

Conversely, assume we are given \( H = h^0(t) + h^1(t) dt \) satisfying the conditions of Def. \ref{def:inner_equivalence} that is: \( h^0(1) = f' = \bar{\varphi} \) and \( h^0(0) = f = \bar{\varphi} \), and \( H \) is an \( L_\infty \)-morphism. Then integrating over \( t \) we define

\[ \eta := \int_0^1 \bar{h}^1(t). \]

It satisfies

\[ \varphi' - \varphi = \bar{h}^0(1) - \bar{h}^0(0) = \int_0^1 \frac{\partial}{\partial t} \bar{h}^0(t) = d_{\text{tot}} \eta, \]

where in the last equation condition (22) is used. Hence \( f \) and \( f' \) are inner equivalent. \( \square \)

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Laboratoire de Mathématiques de Lens (Université d’Artois) and Institut für Mathematik (Universität Zürich)

*E-mail address*: yael.fregier@math.uzh.ch, yael.fregier@gmail.com

Laboratoire et Département de Mathématiques UMR 7122 Université de Metz et CNRS

Bat. A, Ile du Saulcy F-57045 Metz Cedex 1, France

*E-mail address*: camille.laurent-gengoux@univ-lorraine.fr

Universidad Autónoma de Madrid (Departamento de Matemáticas) and ICMAT(CSIC-UAM-UC3M-UCM), Campus de Cantoblanco, 28049 - Madrid, Spain

*E-mail address*: marco.zambon@uam.es, marco.zambon@icmat.es, marco.zambon@wis.kuleuven.be