BPS Spectrum and Nahm Duality of Matrix Theory Compactifications

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Abstract

In the Matrix theories compactified on the $d$ dimensional torus with $d = 5, 6, 7$ which are described by the theories of Dd-branes in the type IIA or IIB theory on the $d$ dimensional dual torus we analyze the BPS bound states of the background Dd-brane with other objects by taking the matrix theory limit. Through the Nahm duality transformation the flux and the momentum multiplets for these BPS states are shown to be associated with the black holes and the black strings respectively from the viewpoint of the weakly coupled type IIA string theory compactified on the $d - 1$ dimensional torus.

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Many important evidences have been gathered in favor of the Matrix theory description of M-theory [1]. The identification of M-theory with the strong coupling limit of the type IIA string theory is realized by the large \( N \) limit accompanied with the large eleventh radius in the Matrix theory. Explicit constructions of Matrix theory compactifications have been studied in order to obtain full understandings of the compactification of M-theory on compact manifolds. The Matrix theory compactified on \( T^d \) is shown to be equivalent to the \( d + 1 \) dimensional \( U(N) \) supersymmetric Yang-Mills theory (SYM) on a dual torus \( T^d \). \cite{2, 3, 4}. The U duality group of Matrix theory compactified on \( T^3 \) is given by \( SL(3, \mathbb{Z}) \times SL(2, \mathbb{Z}) \) where the \( SL(3, \mathbb{Z}) \) is the geometric symmetry group of \( T^3 \) and the \( SL(2, \mathbb{Z}) \) is the S duality of \( 3 + 1 \) dimensional SYM theory. For \( d > 3 \) the SYM theories are strongly coupled in the ultraviolet region so that extra degrees of freedom need to be added to describe consistent full theories. The SYM theories for \( d > 3 \) are replaced by some more general theories on the world-volumes of extended objects in the string theory in order to control the theories in the ultraviolet region. The Matrix theory on \( T^4 \) is described in terms of the 5-branes of M-theory and the Matrix theory on \( T^5 \) is given by the theory of NS5-branes in the type IIA or IIB string theory \cite{5, 6, 7, 8, 9}. The theory of NS5-branes decouples from the bulk space-time modes to become a well-defined theory as a Matrix theory in the limit that the eleven dimensional Planck length and the string coupling constant are taken to zero with the string length fixed. The compactification on \( T^6 \) is proposed by the formulation of the world-volume theory of Kaluza-Klein (KK) monopoles in eleven dimensions, which become the D6-branes in the type IIA string theory language \cite{10, 11, 12, 13}. Based on the world-volume theory of NS5-branes or D6-branes the U duality group for the Matrix theory on \( T^d \) with \( d = 5 \) or 6 is investigated as a symmetry group for the spectrum of BPS bound states constructed by a NS5-brane or a D6-brane background with some other objects. The structure of U duality group \( E_d \) in the Matrix theory compactified on \( T^d \) with \( d \geq 3 \) is also systematically studied from the effective SYM theory by imposing additional requirements about generalized Montone-Olive duality and permutation \cite{14}, where the BPS spectrum is considered to be reliably determined by this low energy effective theory instead of the consistent full theory.

Recently through an appropriate matrix theory limit the Matrix theory compactified on \( T^d \times S^1 \) with very small radius of the eleventh spatial circle has been constructed as a weakly coupled IIA string theory and shown to be related with the discrete light cone quantization of the Matrix theory with finite radius of the light-like circle \cite{15, 16}. In the Matrix theory on \( T^d \times S^1 \) in the discrete light cone gauge that is described by a \( d + 1 \) dimensional \( U(N) \) SYM theory with finite \( N \), the U duality group \( E_d \) is shown to be enhanced to \( E_{d+1} \) by the Nahm duality symmetry (N duality) \cite{17, 18} which exchanges the rank \( N \) of the gauge group with some of the electric or magnetic fluxes \cite{19}. In the picture of Matrix theory on \( T^d \times S^1 \) the N duality acts by exchanging the longitudinal radius with the radius of a space-like circle so that it is related with the Lorentz invariance of the Matrix theory.

We are so interested in compactifications down to four dimensions that it is a fasci-
nating problem to elucidate the U dualities for the Matrix theory compactifications down to lower dimensions. We aim to elaborate the U duality groups for the Matrix theories compactified on $T^d \times S^1$ with $d = 5, 6, 7$. Making use of the matrix theory limit in the theories of Dd-branes in the type IIA or IIB string theories we will analyze the BPS bound states of a background Dd-brane and some other objects and investigate the structures of U duality groups acting on these BPS states. On the other hand in the viewpoint of the toroidally compactified type IIA string theory at weak coupling the BPS states were originally constructed as extreme black holes or extreme black strings and shown to form the U duality multiplets [20]. We will show that these two kinds of pictures for BPS states are identified through the N duality transformation.

The Matrix theory compactified on $T^d \times S^1$, a rectangular torus with circles of radii $L_i, i = 1, \cdots, d$ and $R$ is described by a $d+1$ dimensional SYM theory on $\tilde{T}^d$, a dual torus with circles of radii $\Sigma_i, i = 1, \cdots, d$ whose gauge coupling is $g_{ym}$. The relationships between them are given by

$$\Sigma_i = \frac{l_s^2}{L_i}, \quad g_{ym}^2 = g_s l_s^{-3} \prod_{i=1}^d \frac{l_s^2}{L_i}, \quad (1)$$

where the IIA string coupling and length $g_s, l_s$ and the radius of the longitudinal dimension $R$ are related with the Planck length $l_p$ as $R = g_s^{2/3} l_p^2 = g_s l_s$. The theory of coincident Dd-branes in the type IIA or IIB string theory on $T^d$ is generated by applying T duality transformations on all the circles of $T^d$ to the Matrix theory on $T^d \times S^1$ that is the theory of D0-branes in the type IIA string theory on $T^d$. At low energies this system reduces to the previous $d+1$ dimensional SYM theory on $\tilde{T}^d$. Since the string coupling of it is written as

$$G_s = g_s \prod_{i=1}^d \frac{l_s}{L_i}, \quad (2)$$

the gauge coupling in the SYM language is specified by $G_s$ as $g_{ym}^2 = G_s l_s^{d-3}$.

We start to devote ourselves to the $d = 5$ case. The Matrix theory on $T^5 \times S^1$ is described by a theory of coincident D5-branes in the type IIB string theory on $\tilde{T}^5$ with coupling $G_s$ and string length $l_s$. In order to study BPS states which will fit into the representations of U duality group we write down the masses of several BPS objects such as Dp-branes with $p = 1, 3, 5$, NS1-brane, NS5-brane and pp-wave

$$M_{D1} = \frac{\Sigma_i}{G_s l_s^2}, \quad M_{D3} = \frac{\Sigma_i \Sigma_j \Sigma_k}{G_s l_s^4}, \quad M_{D5} = \frac{V_5}{G_s l_s^6}, \quad M_{NS1} = \frac{\Sigma_i}{l_s^2}, \quad M_{NS5} = \frac{V_5}{G_s^2 l_s^6}, \quad M_{pp} = \frac{1}{\Sigma_i}, \quad (3)$$

where $V_d = \prod_{i=1}^d \Sigma_i$. Here we take the matrix theory limit $[13, 16]$ where $l_p$ is taken to zero with the ratio $l_s/\sqrt{l_p}$ fixed and the radii $L_i$ are measured in Planck unit;

$$l_p \rightarrow 0,$$
\[ \frac{L_i}{l_p} = \text{constant}, \quad \frac{l_s}{\sqrt{l_p}} = \text{constant}. \] (4)

In this limit \( \Sigma_i \) are kept constant and \( G_s \), which is proportional to \( l_p^{-1} \), becomes strong coupling, while the string coupling constant in the type IIA string theory is small as \( g_s \sim l_p^{3/2} \), which leads to a very small eleventh radius as \( R \sim l_p^2 \). Hence the D5-brane is most heavy as \( M_{D5} \sim l_p^{-2} \) and the others are separated into two groups. One consists of the basic BPS states with masses \( M_{NS1}, M_{D3} \) and \( M_{NS5} \) which are proportional to \( l_p^{-1} \), and the other includes \( M_{D1} \) and \( M_{pp} \) which are kept constant in the matrix theory limit. The former basic BPS objects, whose masses are denoted by \( M_a \), combine with the D5-brane respectively to form non-threshold bound states with masses squared \( M^2 = M^2_{D5} + M_a^2 \).

Subtracting the background we have finite Yang-Mills energies in the matrix theory limit

\[
E_{NS1}^i = \frac{g_{ym}^2 \Sigma_i^2}{2V_5}, \quad E_{D3}^{ij} = \frac{V_5}{2g_{ym}^2 (\Sigma_i \Sigma_j)^2}, \quad E_{NS5} = \frac{V_5}{2g_{ym}^6},
\] (5)

with finite \( g_{ym} \). For any \( d \) it is noted from (3) and (4) that the gauge coupling is kept fixed in the decoupling limit (4), while the string coupling behaves as \( G_s \sim l_p^{3-d}/2 \). Each of the latter basic BPS objects together with the background D5-brane makes a threshold bound state with zero binding energy. The corresponding Yang-Mills energies are given by

\[
E_{D1}^i = \frac{\Sigma_i}{g_{ym}^2}, \quad E_{pp}^i = \frac{1}{\Sigma_i}.
\] (6)

The former bound states in (3) transform under \( SL(5, Z) \) as 5, 10, and 1 respectively so that they give the 16 of \( SO(5, 5, Z) \), the U duality group of the Matrix theory on \( T^5 \times S^1 \), while the latter bound states in (4) transforming as 5, 5 under \( SL(5, Z) \) give the 10 of \( SO(5, 5, Z) \).

Now we will perform the S duality transformation defined by

\[
\hat{\Sigma}_i = \Sigma_i, \quad \hat{G}_s = G_s^{-1}, \quad \hat{l}_s = l_s G_s^{1/2},
\] (7)

which convert the theory of D5-branes at strong coupling into that of NS5-branes at weak coupling in the type IIB string theory. The matrix theory limit characterized by \( G_s \sim l_p^{-1} \), (4) is changed into \( \hat{G}_s \to 0, \hat{\Sigma}_i = \text{constant} \) and \( \hat{l}_s = \text{constant} \). Through (7) the masses of BPS objects in (3) are respectively mapped to

\[
M_{NS1} = \frac{\hat{\Sigma}_i}{\hat{l}_s}, \quad M_{D3} = \frac{\hat{\Sigma}_i \hat{\Sigma}_j \hat{\Sigma}_k}{G_s \hat{l}_s^4}, \quad M_{NS5} = \frac{\hat{V}_5}{G_s \hat{l}_s^6},
\]

\[
M_{D1} = \frac{\hat{\Sigma}_i}{G_s \hat{l}_s^2}, \quad M_{D5} = \frac{\hat{V}_5}{G_s \hat{l}_s^6}, \quad M_{pp} = \frac{1}{\Sigma_i}
\] (8)
as expected where \( \hat{V}_5 = \prod_{i=1}^{5} \hat{\Sigma}_i \) and the NS5-brane is most massive as \( \hat{G}_s \Sigma^2 \), corresponding to the previous D5-brane. It produces a non-threshold bound state with the second massive D1, D3 or D5-brane whose mass is proportional to \( \hat{G}_s^{-1} \). The finite Yang-Mills energies for them are constructed in Ref. [7]. On the other hand the more light NS1-brane or pp-wave combines with the background NS5-brane into a threshold bound state. From this theory of NS5-branes in the type IIB string theory in the limit that the string coupling vanishes with the string length kept fixed the non-critical string theory with (1, 1) supersymmetry is generated and its low energy theory is further described by the six dimensional SYM theory [8]. Moreover let us make the T duality transformation about one of the five directions defined by \( G^A_s = \hat{G}_s \hat{\Sigma}_i / \hat{\Sigma}_i \), \( \Sigma^A_i = \hat{\Sigma}_i / \hat{\Sigma}_i \), \( \Sigma^A_j = \hat{\Sigma}_j (j \neq i) \) and \( l_{sA} = \hat{l}_s \). The masses in (8) are well arranged into

\[
M_{pp} = \frac{1}{\hat{\Sigma}^A_i}, \quad M_{D2} = \frac{\Sigma^A_i \Sigma^2_j}{G^A_s l_{sA}}, \quad M_{NS5} = \frac{V^A_5}{(G^A_s)^2 l^P_{sA}}, \quad M_{D0} = \frac{1}{G^A_s l_{sA}}, \quad M_{D4} = \frac{V^A_5}{G^A_s l^P_{sA} \Sigma^A_i}, \quad M_{NS1} = \frac{\Sigma^A_i}{l^2_{sA}}, \quad (9)
\]

with \( V^A_5 = \prod_{i=1}^{5} \Sigma^A_i \). Indeed if we take the 1-st direction as \( i \), one of the \( M_{D1}, \hat{\Sigma}_1 / \hat{G}_s \hat{l}_s^2 \) yields one \( M_{D0} \) and the other four \( \hat{\Sigma}_j / \hat{G}_s \hat{l}_s^2 \) combine with the six \( M_{D3} = \sum_{j} \hat{\Sigma}_j \hat{\Sigma}_k / \hat{G}_s \hat{l}_s^4 \) into the ten \( M_{D2} \). The remaining four \( M_{D3} = \sum_{j} \hat{\Sigma}_j \hat{\Sigma}_k \hat{\Sigma}_l / \hat{G}_s \hat{l}_s^4 \) and one \( M_{D5} \) become the five \( M_{D4} \). One \( M_{NS5} \) turns to be again the mass of NS5-brane in the type IIA string theory. The four \( \hat{\Sigma}_j / \hat{l}_s^2 \) of the NS1-brane with one \( 1/\hat{\Sigma}_1 \) of the pp-wave and the four \( 1/\hat{\Sigma}_j \) of the pp-wave with one \( \hat{\Sigma}_1 / \hat{l}_s^2 \) of the NS1-brane lead to to the five \( M_{NS1} \) and the five \( M_{pp} \) respectively. In the matrix theory limit which is also specified by \( G^A_s \to 0, l^A_s = constant \) the heaviest NS5-brane background combines with the lightest NS1-brane or pp-wave to form a threshold bound state. This theory of NS5-branes in the type IIA string theory becomes the (2, 0) string theory in the limit that the string coupling vanishes with the string length held fixed. The succeeding zero slope limit \( l^A_s \to 0 \) yields its low energy theory, that is the (2, 0) field theory [1, 8].

Here we return to the theory of D5-branes in the type IIB string theory on \( \hat{T}^5 \). As discussed above the BPS bound states in this theory are characterized by the basic constituent BPS objects. In the list of BPS spectrum in (8) we separate the fifth direction of the dual five-torus from the other directions and classify the masses of the basic BPS states as Table 1. In this table \( i = 1, \cdots, 4 \) and for example with respect to the D3-brane the \( 10 \) of \( SL(5, Z) \) decompose as \( 6 + 4 \) under \( SL(4, Z) \). The Eqs. (9) and (10) make the basic BPS mass spectrum change into that for the theory of D0-branes in the type IIA string theory on \( T^5 \), which is shown in Table 2 with \( V^d_{ai} = \prod_{i=1}^{d} L_i \). In the language of SYM theory on \( \hat{T}^d \) the N duality transformation is defined by

\[
g^2_{ym} \to g^2_{ym} \gamma^d_{N}, \quad \Sigma_i \to \Sigma_i \gamma_N \quad \text{for } i \neq d,
\]
\[ \Sigma_d \rightarrow \Sigma_d, \quad l_s^2 \rightarrow l_s^2 \gamma_N, \]

where \( \gamma_N = R/L_d = g_{ym}^2 l_s^2 \Sigma_d/V_d \). In Table 1 under the N duality the \( \Sigma_5/G_s l_s^2 = \Sigma_5/g_{ym}^2 \) for the D1-brane and the \( 1/\Sigma_i \) for the pp-wave are transformed into \( V_5/g_{ym}^4 l_s^2 = V_5/G_s l_s^6 \) for the NS5-brane and \( \Sigma_i \Sigma_j \Sigma_k/g_{ym}^2 l_s^2 = \Sigma_i \Sigma_j \Sigma_k/G_s l_s^4 \) for the D3-brane. The \( \Sigma_5/l_s^2 \) for the NS1-brane is also transformed into \( V_5/g_{ym}^2 l_s^4 \) which turns out to be \( V_5/G_s l_s^6 \), the mass of D5-brane. The corresponding \( 1/L_5 \) for the pp-wave in Table 2 is accordingly transformed into \( 1/R = 1/g_s l_s \) that is the D0-brane mass in the IIA string theory on \( T^5 \). The others are not changed under the N duality. In the viewpoint of the six dimensional weakly coupled type IIA string theory compactified on \( T^4 \) the BPS states obtained by applying the N duality transformation to the 16 in Table 2 are interpreted to represent the following sixteen electric black holes; 4 black holes arising from pp-waves travelling in each of the 4 toroidal dimensions, one black hole given by the D0-brane, 6 black holes arising from the D2-branes wrapped around the 2-torus, one black hole provided by the D4-brane wrapped around the 4-torus and 4 black holes originating in the NS1-branes wrapped around the 1-torus. The BPS states transformed from the 10 consist of the following ten electric and magnetic black strings; 4 electric strings arising from the D2-branes wrapped around the 1-torus, one magnetic string generated from the NS5-brane wrapped around the 4-torus, 4 magnetic strings arising from the D4-branes wrapped around the 3-torus and one electric string given by the unwrapped NS1-brane.

Now we consider compactifying the Matrix theory on a six-torus down to five dimen-

| NS1  | \( \frac{\Sigma_5}{l_s^2} \) (4), \( \frac{\Sigma_5}{l_s^2} \) (1) | 5   | 16 |
| D3  | \( \frac{\Sigma_5 \Sigma_5 l_s}{G_s l_s^2} \) (6), \( \frac{\Sigma_5 \Sigma_5 l_s}{G_s l_s^2} \) (4) | 10  | 16 |
| NS5 | \( \frac{\Sigma_5}{l_s^2} \) (1) | 1   | 16 |
| D1  | \( \frac{\Sigma_5}{G_s l_s^2} \) (1), \( \frac{\Sigma_5}{G_s l_s^2} \) (4) | 5   | 16 |
| pp  | \( \frac{\Sigma_5}{\Sigma_i} \) (4), \( \frac{\Sigma_5}{\Sigma_i} \) (1) | 5   | 10 |

Table 1: The BPS states for the theory of D5-branes in the type IIB string theory on \( \tilde{T}^5 \).

| PP   | \( \frac{L_5}{L_5} \) | 16 |
| D2   | \( \frac{L_5 L_5}{g_s l_s} \) | 16 |
| NS5  | \( \frac{V_5}{g_{ym} l_s^2} \) | 16 |
| D4   | \( \frac{V_5}{g_{ym} l_s^2} \) | 16 |
| NS1  | \( \frac{V_5}{l_s^2} \) | 16 |

Table 2: The BPS states for the theory of D0-branes in the type IIA string theory of \( T^5 \).
sions. It is described by a theory of coincident D6-branes in the type IIA string theory on $\tilde{T}^6$ with strong string coupling as $G_s \sim l_p^{-3/2}$. The masses of BPS objects are given by

$$M_{D2} = \frac{\Sigma_i \Sigma_j}{G_s l_s^3}, \quad M_{D4} = \frac{\Sigma_i \Sigma_j \Sigma_k \Sigma_l}{G_s l_s^5}, \quad M_{D6} = \frac{V_6}{G_s l_s^7},$$

where $M_{KK}$ is the mass of KK monopole wrapped around the six directions, where Taub-NUT direction is specified by $i \in \{10, 12\}$. The matrix theory limit (4) shows that the D6-brane is most heavy as $M_{D6} \sim l_p^{-2}$ and the $M_{NS1}, M_{D4}, M_{KK}$ behave as $l_p^{-1}$ and the $M_{D2}, M_{pp}, M_{NS5}$ are finite. From these behaviors it is noted that the background D6-brane can make non-threshold bound states with the NS1-brane, D4-brane and KK monopole respectively, whose finite Yang-Mills energies are

$$E_{NS1}^i = \frac{g_{ym}^2 \Sigma_i}{2V_6}, \quad E_{D4}^{ij} = \frac{V_6}{2g_{ym}^2 (\Sigma_i \Sigma_j)^2}, \quad E_{KK}^i = \frac{V_6 \Sigma_i^2}{2g_{ym}^6}.$$

They build up a flux multiplet which is a 27 of $E_6(Z)$, the U duality group for the Matrix theory compactified on $T^6 \times S^1$. The 27 decompose as $6 + 15 + 6$ under $SL(6, Z)$. On the other hand the D2-brane, pp-wave and NS5-brane can respectively combine with the background D6-brane to make threshold bound states whose finite Yang-Mills energies are

$$E_{D2}^{ij} = \frac{\Sigma_i \Sigma_j}{g_{ym}^2}, \quad E_{pp}^i = \frac{1}{\Sigma_i}, \quad E_{NS5}^i = \frac{V_6}{g_{ym}^4 \Sigma_i}.$$

These BPS bound states transform as the 15, 6, 6 under $SL(6, Z)$ and form a momentum multiplet represented by a 27 of $E_6(Z)$.

Let us single out the sixth direction of the dual six-torus from the others in order to classify the masses of the basic BPS states for the theory of D6-branes at strong coupling in the IIA string theory on $\tilde{T}^6$ in Table 3. In this table the BPS states are distributed in a similar way to Table 1. Through (11) and (12) the BPS states in Table 3 are transformed back into those for the theory of D0-branes at weak coupling in the type IIA string theory on $T^6$ as Table 4. In this stage we will make the N duality transformation. The $\Sigma_6/l_s^2$ of NS1 and the $V_6/G_s l_s^6 \Sigma_6$ of NS5 in Table 3 are transformed into the masses of D6-brane and D0-brane, $V_6/G_s l_s^6$ and $1/G_s l_s$ respectively, which further correspond to the masses of D0-brane and D6-brane for the theory of D0-branes in the IIA string theory on $T^6$, $1/g_{ym} l_s$ and $V_6 l_s$ respectively. The $\Sigma_i \Sigma_6/G_s l_s^3$ of D2 and the $1/\Sigma_i$ of pp are interchanged by the $V_6 \Sigma_i / G_s l_s$ of KK and the $V_6 / G_s l_s^5 \Sigma_i \Sigma_6$ of D4 respectively. The others are invariant under the N duality transformation. From Table 4 we deduce that the flux multiplet 27 is transformed under the N duality into the twenty seven electric black holes. Indeed these states are
Table 3: The BPS states for the theory of D6-branes in the type IIA string theory on $\tilde{T}^6$.

| State | Expression | $\Sigma$ | $\Sigma$ | $\Sigma$ | $\Sigma$ |
|-------|------------|--------|--------|--------|--------|
| NS1   | $\frac{1}{\Sigma}$ (5), $\frac{1}{\Sigma}$ (1) | 6      |        |        |        |
| D4    | $\frac{\lambda_6}{\sqrt{\Sigma_6 \Sigma_5 \Sigma_1 \Sigma_3}}$ (10) | $\frac{\lambda_6}{\sqrt{\Sigma_6 \Sigma_5 \Sigma_1 \Sigma_3}}$ (5) | 15     |        | 27     |
| KK    | $\frac{\lambda_6}{\sqrt{\Sigma_6 \Sigma_5 \Sigma_1 \Sigma_3}}$ (1) | $\frac{\lambda_6}{\sqrt{\Sigma_6 \Sigma_5 \Sigma_1 \Sigma_3}}$ (5) | 6      |        |        |
| NS5   | $\frac{\lambda_6}{\sqrt{\Sigma_6 \Sigma_5 \Sigma_1 \Sigma_3}}$ (5), $\frac{\lambda_6}{\sqrt{\Sigma_6 \Sigma_5 \Sigma_1 \Sigma_3}}$ (1) | 6      |        |        |        |
| D2    | $\frac{\lambda_6}{\sqrt{\Sigma_6 \Sigma_5 \Sigma_1 \Sigma_3}}$ (5) | $\frac{\lambda_6}{\sqrt{\Sigma_6 \Sigma_5 \Sigma_1 \Sigma_3}}$ (10) | 15     |        | 27     |
| pp    | $\frac{1}{\Sigma}$ (5) | $\frac{1}{\Sigma}$ (1) | 6      |        |        |

Table 4: The BPS states for the theory of D0-branes in the type IIA string theory on $T^6$.

| State | Expression | $\Sigma_1$ | $\Sigma_1$ |
|-------|------------|------------|------------|
| PP    | $\frac{1}{L_1}$, $\frac{1}{L_0}$ | 6          |            |
| D2    | $\frac{\lambda_6}{q_4 L_1}$ | $\frac{L_1 L_0}{q_4 L_1}$ | 27         |
| NS5   | $\frac{V_6^{\phi}}{g^{(6)} L_0}$ | $\frac{V_6^{\phi}}{g^{(6)} L_0}$ |            |
| KK    | $\frac{V_6^{\phi} L_1}{g^{(6)} L_0}$ | $\frac{V_6^{\phi} L_1}{g^{(6)} L_0}$ | 27         |
| D4    | $\frac{V_6^{\phi}}{g_2 L_1 L_0}$ | $\frac{V_6^{\phi}}{g_2 L_1 L_0}$ |            |
| NS1   | $\frac{L_0}{L_1}$ | $\frac{L_0}{L_1}$ | 27         |


interpreted in the viewpoint of the type IIA string theory compactified on $T^5$ at weak coupling as follows; 6 black holes provided by the pp-waves travelling in each of the 5 toroidal dimensions and the one D0-brane, 15 black holes generated by the 10 D2-branes wrapped around the 2-torus and the 5 NS1-branes wrapped around the 1-torus, and 6 black holes given by the one NS5-brane wrapped around the 5-torus and the 5 D4-branes wrapped around the 4-torus. It is interesting to note that the momentum multiplet $27$ is transformed into the twenty seven magnetic black strings; 6 black strings generated by the 5 KK magnetic monopoles, that are Taub-NUT 5-branes, wrapped around the 5-torus and the one D6-brane wrapped around the 5-torus, 15 black strings produced by the 5 NS5-branes wrapped around the 4-torus and the 10 D4-branes wrapped around the 3-torus, and 6 black strings given by the one NS1-brane wrapped around the 5-torus and the one unwrapped NS1-brane. In this way we observe that these N duality transformed states $27$, $2\bar{7}$ are just extreme electric black holes and extreme magnetic black strings which explicitly appear in the weakly coupled type IIA string theory compactified on $T^5$ as argued in Ref. [20]. The BPS electric black holes $27$ decompose as $16 + 10 + 1$ under $SO(5, 5, Z)$, where the 1 corresponds to the one NS5-brane wrapped around the 5-torus and the 16 are associated with the D0, D2, D4-branes and the 10 correspond to the five directions of pp-wave and the five directions of NS1-brane winding.

Here we extend the previous analysis to consider the Matrix theory compactified on $T^7 \times S^1$ that is described by a theory of coincident D7-branes in the type IIB string theory on $\tilde{T}^7$. The BPS states are specified by the following masses:

$$M_{D3} = \frac{\Sigma_i \Sigma_j \Sigma_k}{G_s l_s^4}, \quad M_{D5} = \frac{\Sigma_i \cdots \Sigma_{i_5}}{G_s l_s^6}, \quad M_{D7} = \frac{V_7}{G_s l_s^8},$$

$$M_{NS1} = \frac{\Sigma_i}{l_s^2}, \quad M_{pp} = \frac{1}{\Sigma_i}, \quad M_{KK} = \frac{V_7 \Sigma_i}{G_s^2 l_s^{10}} \Sigma_j (14)$$

and

$$M_{(NS5)} = \frac{V_7 \Sigma_i \Sigma_j}{G_s^2 l_s^{10}}, \quad M_{(D1)} = \frac{V_7^2}{G_s^{3/6} l_s^{14} \Sigma_i} \quad (15)$$

which are masses for the new type of states that we represent by $(NS5)$ and $(D1)$. There are further basic BPS states whose masses are $V_7^2/G_s^2 l_s^{12} \Sigma_i \Sigma_j \Sigma_k$ and $V_7^2 \Sigma_i/G_s^4 l_s^{16}$, that will not be analyzed. From $M_{NS5} = \Sigma_i \cdots \Sigma_{i_5}/G_s^2 l_s^6$ the mass $M_{(NS5)}$ is produced by making two T duality transformations along two directions transverse to the world-volume of the NS5-brane and the $M_{(D1)}$ is also interpreted to be associated with the rolled-up Taub-NUT 6-brane in Ref. [17]. The $M_{KK}$ is regarded as the mass of Taub-NUT 5-brane with the $i$-th Taub-NUT direction, which is wrapped around the 6-torus transverse to the $j$-th direction. Analyzing the behaviors of masses in the matrix limit we get the structure of the BPS bound states. The most heavy D7-brane with $M_{D7} \sim l_p^{-2}$ can combine with the NS1-brane, D5-brane, (D1)-brane and (NS5)-brane respectively whose masses behave as $l_p^{-1}$, to construct non-threshold bound states that form a flux multiplet, 56 of $E_7(Z)$,
the U duality group for the Matrix theory on $T^7 \times S^1$. The $M_{KK}, M_{D3}$ and $M_{pp}$ show no $l_p$ dependence so that the KK monopole, D3-brane and pp-wave combine with the background D7-brane to build up threshold bound states that belong to a momentum multiplet, while the D1-brane and NS5-brane cannot build up bound states with finite non-zero energies since the $M_{D1} = \Sigma_i / G_{s} l_s^2$ and $M_{NS5}$ are proportional to $l_p$. The finite Yang-Mills energies of the bound states for the flux multiplet are derived as

$$E^{i}_{NS1} = \frac{g_{ym}^2 \Sigma_i^2}{2V_7}, \quad E^{i}_{D5} = \frac{V_7}{2g_{ym}^2 \Sigma_i \Sigma_j}, \quad E^{i}_{(D1)} = \frac{V_7^3}{2g_{ym}^2 \Sigma_i^2}, \quad E^{i}_{(NS5)} = \frac{V_7 \Sigma_i^2 \Sigma_j^2}{2g_{ym}^2}$$

which transform as the 7, 21, 7 and 21 under $SL(7, Z)$.

Separating the seventh direction of the dual seven-torus from the others we carry out the classification of masses of basic BPS states for the theory of D7-branes at strong coupling in the type IIB string theory on $\tilde{T}^7$ in Table 5. The distribution of the BPS states in this table is constructed by regarding those in Tables 1, 3 as suggestive guides, however there are some different structures. Comparing the BPS spectrums of the type IIB theories on $\tilde{T}^5$, $\tilde{T}^6$, $\tilde{T}^7$ in Tables 1, 3, 5 we note that as the dimensions of dual tori increase the NS5-brane in the flux multiplet in Table 1 turns to be in the momentum multiplet as shown in Table 3, which is also the case for the KK monopole in the flux multiplet in Table 3 that goes into the momentum multiplet in Table 5. These basic BPS states are transformed back into those for the theory of D0-branes at weak coupling in the type IIA string theory on $T^7$ as Table 6. Under the N duality transformation in the type IIB string theory on $\tilde{T}^7$ the $\Sigma_i/l_s^2$ of NS1, the $V_7^2 / G_{s}^3 l_{s}^{14} \Sigma_7$ of (D1) and the $V_7 \Sigma_i / G_{s} l_{s}^{2} \Sigma_i$ of KK in Table 5 are mapped to the masses of D7-brane or D1-brane, $V_7 / G_{s} l_{s}^{8} \Sigma_7 / G_{s} l_{s}^{8} \Sigma_i / G_{s} l_{s}^{2}$, which further turn out to be those of D0-brane or D6-brane, $1 / g_s l_s, V_7^2 / g_s l_s^2 L_7, V_7^2 / g_s l_s^2 L_4$ respectively for the theory of D0-branes in the type IIA string theory on $T^7$. The $V_7 \Sigma_i / G_{s} l_{s}^{8} \Sigma_i$ of KK, the $\Sigma_i / \Sigma_j / G_{s} l_{s}^{8} \Sigma_i$ of D3 and the $1 / \Sigma_i$ of pp are mapped to $V_7^2 / G_{s} l_{s}^{14} \Sigma_i$ of (D1), $V_7 \Sigma_i / \Sigma_j / G_{s} l_{s}^{10}$ of (NS5) and $\Sigma_i / \Sigma_j / \Sigma_m / G_{s} l_{s}^{0}$ of D5 to each other. The other states are invariant under the N duality transformation. Gathering these results we find that the

| NS1 | $l_s^7$ (6), $l_s^7$ (1) | 7 |
| D5 | $\frac{l_s^7}{G_{s} l_{s}^{6}}$ (15) | $\frac{l_s^7}{G_{s} l_{s}^{6}}$ (6) | 21 |
| (D1) | $\frac{l_s^7}{G_{s} l_{s}^{10}}$ (1) | $\frac{l_s^7}{G_{s} l_{s}^{10}}$ (6) | 7 |
| (NS5) | $\frac{l_s^7}{G_{s} l_{s}^{10}}$ (6) | $\frac{l_s^7}{G_{s} l_{s}^{10}}$ (15) | 21 |
| KK | $\frac{l_s^7}{G_{s} l_{s}^{10}}$ (6) | $\frac{l_s^7}{G_{s} l_{s}^{10}}$ (6), $\frac{l_s^7}{G_{s} l_{s}^{10}}$ (30) | 42 |
| D3 | $\frac{l_s^7}{G_{s} l_{s}^{10}}$ (15) | $\frac{l_s^7}{G_{s} l_{s}^{10}}$ (20) | 35 |
| pp | $\frac{l_s^7}{G_{s} l_{s}^{10}}$ (6) | $\frac{l_s^7}{G_{s} l_{s}^{10}}$ (1) | 7 |

Table 5: The BPS states for the theory of D7-branes in the type IIB string theory on $\tilde{T}^7$. 

| NS1 | $\frac{l_s^7}{G_{s} l_{s}^{6}}$ (6), $\frac{l_s^7}{G_{s} l_{s}^{6}}$ (1) | 7 |
| D5 | $\frac{l_s^7}{G_{s} l_{s}^{6}}$ (15) | $\frac{l_s^7}{G_{s} l_{s}^{6}}$ (6) | 21 |
| (D1) | $\frac{l_s^7}{G_{s} l_{s}^{10}}$ (1) | $\frac{l_s^7}{G_{s} l_{s}^{10}}$ (6) | 7 |
| (NS5) | $\frac{l_s^7}{G_{s} l_{s}^{10}}$ (6) | $\frac{l_s^7}{G_{s} l_{s}^{10}}$ (15) | 21 |
| KK | $\frac{l_s^7}{G_{s} l_{s}^{10}}$ (6) | $\frac{l_s^7}{G_{s} l_{s}^{10}}$ (6), $\frac{l_s^7}{G_{s} l_{s}^{10}}$ (30) | 42 |
| D3 | $\frac{l_s^7}{G_{s} l_{s}^{10}}$ (15) | $\frac{l_s^7}{G_{s} l_{s}^{10}}$ (20) | 35 |
| pp | $\frac{l_s^7}{G_{s} l_{s}^{10}}$ (6) | $\frac{l_s^7}{G_{s} l_{s}^{10}}$ (1) | 7 |
Table 6: The BPS states for the theory of D0-branes in the type IIA string theory on $T^7$.

N duality transformation maps the flux multiplet 56 in Table 6 to the following electric and magnetic black holes for the weakly coupled type IIA string theory compactified on $T^6$ \[ [20] \]; 7 electric black holes originating in the 6 pp-waves travelling in each of the 6 toroidal dimensions and one D0-brane, 21 electric black holes provided by the 15 D2-branes wrapped around the 2-torus and the 6 NS1-branes wrapped around the 1-torus, 7 magnetic black holes arising from the one D6-brane wrapped around the 6-torus and the 6 KK monopoles wrapped around the 6-torus with a Taub-NUT direction in each of the 6 toroidal dimensions, 21 magnetic black holes given by the 6 NS5-branes wrapped around the 5-torus and the 15 D4-branes wrapped around the 4-torus.

In conclusion we have carried out the N duality transformations concretely for the masses of basic BPS states and found the relations between the BPS spectrums of the Matrix theory compactifications and those of the weakly coupled type IIA string theory compactifications. The flux multiplets in the Matrix theories compactified on $T^d \times S^1$ with $d = 5, 6, 7$ are at first sight interpreted as the black holes as well as the black strings from the viewpoint of the weakly coupled type IIA string theories compactified on $T^{d-1}$. Under the N duality transformation, however they are mapped just only to the extremal BPS black holes. Analogously the momentum multiplets are transformed to the extremal BPS black strings. We have observed that for $d = 5, 6$ the flux multiplets 16 of $SO(5, 5, Z)$, 27 of $E_6(Z)$ correspond to the electric black holes and the momentum multiplets 10, 27 to the 5 magnetic and 5 electric black strings, the twenty seven magnetic black strings, while for $d = 7$ the flux multiplet 56 of $E_7(Z)$ turns out to be identified with the 28 electric and 28 magnetic black holes, thus providing further evidence for the Matrix theory compactifications. The matrix theory limit proves useful when dividing the BPS states into the flux and the momentum multiplets. By interchanging the original role of the eleventh direction with that of one of the other directions we can naturally incorporate the BPS black holes or black strings of the weakly coupled type IIA string theory into the Matrix theory description. It would be interesting to make clear the rich structures
of the BPS spectrums in the compactifications of various kinds of the ten dimensional string theories in terms of the Matrix theory description. The N duality transformation would play an important role for this elucidation and in particular give a clue for relations between the strongly coupled IIA theory compactifications and the weakly coupled IIA theory compactifications.

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