Tracking the Empirical Distribution of a Markov-modulated Duplication-Deletion Random Graph

Maziyar Hamdi, Vikram Krishnamurthy, and George Yin

Abstract

This paper considers a Markov-modulated duplication-deletion random graph where at each time instant, one node can either join or leave the network; the probabilities of joining or leaving evolve according to the realization of a finite state Markov chain. The paper comprises of 2 results. First, motivated by social network applications, we analyze the asymptotic behavior of the degree distribution of the Markov-modulated random graph. Using the asymptotic degree distribution, an expression is obtained for the delay in searching such graphs. Second, a stochastic approximation algorithm is presented to track empirical degree distribution as it evolves over time. The tracking performance of the algorithm is analyzed in terms of mean square error and a functional central limit theorem is presented for the asymptotic tracking error.

Index Terms

Complex networks, empirical degree distribution, giant component, Markov-modulated random graphs, power law, searchability, stochastic approximation.

I. INTRODUCTION

Dynamic random graphs have been widely used to model social networks, biological networks [1] and Internet graphs [2]. Motivated by analyzing social networks, this paper considers Markov-modulated dynamic random graphs of the duplication-deletion type which we now describe:

Let $n = 0, 1, 2, \ldots$ denote discrete time. Let $\theta$ denote a discrete time Markov chain with state space \{1, 2, \ldots, $M$\}, evolving according to the $M \times M$ transition probability matrix $A^\rho$ and initial probability

Maziyar Hamdi and Vikram Krishnamurthy are with the Department of Electrical and Computer Engineering, University of British Columbia, Vancouver, Canada. Email: \{maziyarh,vikramk\}@ece.ubc.ca.

George Yin is with the Department of Mathematics, Wayne State University, Detroit, US. Email: gyin@math.wayne.edu.

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distribution $\pi_0$. A Markov-modulated duplication-deletion random graph is parameterized by the 7-tuple $(M, A^\rho, \pi_0, r, p, q, G_0)$. Here $p$ and $q$ are $M$-dimensional vectors with elements $p(i)$ and $q(i) \in [0,1]$, $i = 1, \ldots, M$. $p(i)$ denote the connection probabilities and $q(i)$ denote the deletion probabilities. Also, $r \in [0,1]$ denotes the probability of duplication step and $G_0$ denotes the initial graph at time 0. $G_0$ can be any finite simple connected graph. For simplicity we assume that $G_0$ is a simple connected graph with size $N_0$. The duplication-deletion random graph is constructed as follows:

**Algorithm 1** Markov-modulated Duplication-deletion Graph parameterized by $(M, A^\rho, \pi_0, r, p, q, G_0)$

At time $n$, given the graph $G_n$ and Markov chain state $\theta_n$, simulate the following events:

**Step 1: Duplication step:** With probability $r$ implement the following steps:

- Choose node $u$ from graph $G_n$ randomly with uniform distribution.
- **Vertex-duplication**: Generate a new node $v$.
- **Edge-duplication**:
  - Connect node $u$ to node $v$. (A new edge between $u$ and $v$ is added to the graph.)
  - Connect each neighbor of node $u$ with probability $p(\theta_n)$ to node $v$. These connection events are statistically independent.

**Step 2: Deletion Step:** With probability $q(\theta_n)$ implement the following step:

- **Edge-deletion**: Choose node $w$ randomly from $G_n$ with uniform distribution. Delete node $w$ and all edges connected to node $w$ in graph $G_n$.
- **Duplication Step**: Implement Step 1.

**Step 3:** Denote the resulting graph as $G_{n+1}$.

Generate Markov state $\theta_{n+1}$ using transition matrix $A^\rho$.

**Step 4: Network Manager’s Diagnostics:** The network manager computes the estimates of the expected degree distribution. Denote the resulting graph as $G_{n+1}$.

Set $n \rightarrow n + 1$ and go to Step 1.

For convenience in our analysis, assume that a node generated in the duplication step cannot be eliminated in the deletion step immediately after its generation. Also to prevent the isolated nodes, assume that the neighbor of a node with degree one cannot be eliminated in the deletion step. The duplication step (Step 2) is purely for convenience — it ensures that the graph size does not decrease. The Markov-modulated random graph generated by Algorithm I mimics social networks where the interaction between nodes evolves over time due to underlying dynamics such as seasonal variations (e.g., the high school
friendship social network evolving over time with different winter/summer dynamics). In such cases, the connection/deletion probabilities $p, q$ evolve with time. Algorithm 1 models these time variations as a finite state Markov chain $\theta_n$ with transition matrix $A^\rho$.

**Context: Why is the degree distribution important?**

The expected degree distribution yields useful information about the connectivity of the random graph. For example, if a majority of nodes in the random graph have relatively high degrees, the graph is highly connected and a message can be transferred between two arbitrary nodes with shorter paths. However, if a majority of nodes have smaller degrees then for transmitting a message throughout the network, longer paths are needed, see [3]. Also, the degree distribution can be used to determine the existence of “giant component” 1. The existence of a giant component has important implications in social networks in terms of modeling information propagation in a social network and in human disease modeling [4], [5], [6]. If the average degree of a random graph is strictly greater than one then with probability one there exists a unique giant component [2] and the size of this component can be computed from the expected degree sequence. The average degree and the size of giant component is computed at each time as a measure of connectivity by the monitoring node. Another application of tracking the expected degree distribution is to estimate adaptively the “searchability” of the network. The searchability of a social network [7] is the average number of nodes that need to be accessed to reach another node. In this paper, we track the searchability of the network by means of tracking the expected degree distribution at each time.

**Main Results and Paper Organization:**

**Notation:** At each time $n$, let $N_n$ denote the number of nodes of graph $G_n$. Also, let $f_n(i)$ denote the number of vertices of graph $G_n$ with degree $i$. Clearly $\sum_{i \geq 1} f_n(i) = N_n$. Define the “empirical vertex degree distribution” as

$$g_n(i) = \frac{f_n(i)}{N_n}, \quad \text{for } 1 \leq i \leq N_n. \quad (1)$$

Note that $g_n(i)$ can be viewed as a probability mass function since $g_n(i) \geq 0$ and $\sum_i g_n(i) = 1$. Let $\overline{g}_n = \text{E}\{g_n\}$ denoted the “expected vertex degree distribution” where $g_n$ is the empirical degree distribution defined in [1].

1 A giant component is a connected component with size $O(n)$ where $n$ is the total number of vertices in the graph.
Given the above Markov-modulated random graph, this paper presents three main results.

**Result 1:** *Asymptotic Degree Distribution Analysis of fixed size Markov-modulated duplication-deletion random graph*

Consider the sequence of finite duplication-deletion random graphs \( \{G_n\} \), generated by Algorithm 1 with \( r = 0 \). Clearly the number of vertices in the graph generated by Algorithm 1 with \( r = 0 \) satisfies \( N_n = N_0 \) for \( n = 1, 2, \ldots \) (The size of random graph is fixed.). Assume that the Markov chain \( \theta_n \) evolves according to a slow transition matrix \( A^\rho = I + \rho Q \), where \( Q \) is a generator matrix and \( \rho \) is a small positive constant. A novel degree distribution analysis is provided for the fixed size Markov-modulated duplication-deletion random graph in Sec II. Theorem 2.1 shows that for each \( \theta_n = \theta \), the expected degree distribution of the finite random, \( \overline{g}(\theta) \), can be computed from (12).

The asymptotic degree distribution analysis allows us to investigate the searchability and connectivity of the random graph generated according to Algorithm 1 as described in Sec II. Also, using the asymptotic degree distribution, the existence and size of the giant component in the random graph can be explored.

**Result 2:** *Tracking the Empirical Degree Distribution*

In Sec III we address the following two questions:

- *How can a network manager estimate (track) the empirical degree distribution using a stochastic approximation algorithm without knowledge of Markovian dynamics?*
- *How good is the estimate \( \tilde{g}_n \) generated by the stochastic approximation algorithm (2) when the random graph evolves according to Algorithm 1?*

In Sec III we propose a stochastic approximation algorithm to estimate the degree of each node in random graph which can be modeled by Algorithm 1. Consider the finite Markov-modulated duplication-deletion random graph generated by Algorithm 1 with 7-tuple \( (M, A^\rho, \pi_0, p, q, r, G_0) \) where \( r = 0 \). Suppose at each time \( n \), noisy measurements, \( y_n \) the empirical distribution of \( g_n \) are obtained by the administrator of the social network. The network manager does not have information about the Markovian dynamics and deploys a non-parametric stochastic approximation algorithm to estimate the expected vertex degree distribution. More precisely, given these measurements \( y_n, n = 1, 2, \ldots \), the network administrator aims to estimate the time varying expected vertex distribution \( \overline{g}(\theta_n) \). It deploys the following constant step size stochastic approximation algorithm:

\[
\tilde{g}_{n+1} = \tilde{g}_n + \varepsilon [y_{n+1} - \tilde{g}_n] \tag{2}
\]

Here \( \varepsilon > 0 \) denotes a small positive step size. Eq. (2) is merely an exponentially discounted empirical
distribution of the noisy node degree. Let \( \tilde{g}_n = \hat{g}_n - E\{\bar{g}(\theta_n)\} \) denote the tracking error of the estimate of the empirical distribution of node degree. We present three results regarding the tracking performance of the degree distribution of the random graph:

- **2-a. Mean square error analysis:** Theorem 3.1 in Sec. III-A shows that the mean squared of tracking error (the distance between \( E\{\bar{g}(\theta_n)\} \) and the estimated probability mass function (PMF) \( \hat{g}_n \)) is of order of \( O \left( \varepsilon + \rho + \frac{\rho^2}{\varepsilon} \right) \). (Recall \( \varepsilon \) is the step size of the stochastic approximation algorithm and \( \rho \) parameterizes the speed of the underlying un-observed Markovian dynamics). Derivation of this result uses error bounds on two-time scale Markov chains and perturbed Liapunov function methods.

- **2-b. Weak convergence analysis:** Theorem 3.2 in Sec. III-B shows that the asymptotic behavior of the stochastic approximation algorithm (2) converges weakly to the solution of a switched Markovian ordinary differential equation
  \[ \frac{d\hat{g}(t)}{dt} = -\hat{g}(t) + \bar{g}(\theta(t)), \quad \hat{g}(0) = \hat{g}_0. \] (3)

- **2-c. Functional central limit theorem for scaled tracking error:** How can the tracking error in the empirical distribution estimate be quantified? Sec. III-C investigates the asymptotic behavior of the scaled tracking error. Similar to [8], it is shown that the interpolated scaled tracking error (between the expected and the estimated PMF) converges weakly to the solution of a switching diffusion. Let \( \nu_k = \frac{\tilde{g}_k - E\{\bar{g}(\theta_k)\}}{\sqrt{\varepsilon}} \) denote the scaled tracking error. Theorem 3.3 in Sec. III-C proves that under reasonable conditions, the interpolated sequence of iterates, \( \nu_k(t) = \nu_k \) for \( k \in [k\varepsilon, (k+1)\varepsilon) \) converges weakly to the solution of the following Markovian switched diffusion process
  \[ d\nu(t) = -\nu(t)dt + \left( \Sigma(\theta(t)) \right) d\omega, \] (4)
  where \( \omega(\cdot) \) is an \( \mathbb{R}^{N_0} \)-dimensional standard Brownian motion and \( \Sigma(\theta) \in \mathbb{R}^{N_0 \times N_0} \) is the covariance matrix. Eq. (4) (and Theorem 3.3) are functional central limit theorems. The dynamics of the error in (4) follow a Markov-modulated diffusion process. The covariance \( \Sigma(\theta(t)) \) for large \( t \) is used as a measure for the asymptotic convergence rate of the tracking algorithm.

Note that the Markovian assumption only appear in our analysis, the stochastic approximation algorithm (2) does not assume knowledge of the underlying Markov chain. (In [9], [10] this analysis falls under the class of analysis of a stochastic approximation algorithm with a Markovian hyperparameter.)

**Result 3: Power law component for infinite duplication-deletion random graph without Markovian dynamics**

Sec. IV extends the results of Sec. III and investigates the dynamics of the graph generated according to Algorithm I with \( r = 1 \) and when there are no Markovian dynamics, that is, \( M = 1 \). Since \( r = 1 \)
for \( n \geq 0 \), \( G_{n+1} \) has one more vertex compared to \( G_n \). In particular, since \( G_0 \) is an empty set, \( G_n \) has \( n \) nodes, that is, \( N_n = n \). Theorem 4.1 proves that the expected node degree distribution \( \mathbb{E}_n \) satisfies a power law as \( n \to \infty \). That is,

\[
\log \mathbb{E}_n(i) = \alpha - \beta \log i \quad \text{as} \quad n \to \infty
\]

where \( \alpha \) and \( \beta \) are non-negative real numbers. The power law component, \( \beta \), satisfies

\[
(1 + q)(p^{\beta-1} + p\beta - p) = 1 + q \beta.
\]  

(5)

where \( p \) and \( q \) are the probabilities defined in Algorithm 1. The above result slightly extends [1], [11] where only a duplication model is considered. Theorem 4.1 parametrizes the degree distribution of the infinite duplication-deletion random graph without Markovian dynamics generated by Algorithm 1 by the power law component. Theorem 4.1 allows us to explore the searchability of the network and also the existence and size of the giant component of the infinite duplication-deletion random graph without Markovian dynamics.

Related Works:

We refer to [12], [13] for a comprehensive development of stochastic approximation algorithms. Here, the related literature on dynamic social networks is reviewed briefly. The evolution of random graphs is investigated in several papers,[14], [15]. The book [16] provides a detailed expositions of random graphs. The model of Pastor-Satorras et al.[11] makes the basis for the model which is studied and generalized in this paper. In this model, at each time step, a new node joins the network. In the literature, it has been shown that the degree distribution of such network satisfies power law[17], [18]. In random graphs which satisfy the power law, the number of nodes with an specific degree depends on a parameter called power law component. A general complex graph generated by any arbitrary pure duplication, may not satisfy the power law. The power law distribution is satisfied in many other networks such as WWW-graphs, peer-to-peer networks, phone call graphs and various massive social networks (e.g. Yahoo, MSN, Facebook)[19], [20], [21], [22], [23], [24], [25]. The power law component describes asymptotic behavior of an online social network e.g. maximum degree, existence of giant component, diameter of the graph, and etc. [26] provides condition on the evolution of the graph to satisfy power law and shows that as a result of having an edge between nodes \( u \) and \( v \), the resulting graph satisfies power law.
II. ASYMPTOTIC DEGREE DISTRIBUTION ANALYSIS OF THE FIXED SIZE MARKOV-MODULATED RANDOM GRAPH

This section presents degree distribution analysis of the fixed size Markov-modulated duplication-deletion random graph. Consider the fixed size Markov-modulated duplication-deletion random graph generated according to Algorithm 1 with 7-tuple \((M, A^\rho, \pi_0, p, q, r, G_0)\) where \(r = 0\). The number of vertices in the graph generated by Algorithm 1 with \(r = 0\) is always \(N_0\) and the size of the graphs is fixed. Recall from Sec I, the state space of \(\{\theta_n\}\) is denoted as \(\mathcal{M} = \{1, 2, ..., M\}\),

and the transition probability matrix of \(\theta_n\) is

\[
A^\rho = I + \rho Q.
\]

Here \(\rho\) is a small positive real number and so \(\theta_n\) is a “slow” Markov chain. \(I\) is an \(M \times M\) identity matrix, and \(Q\) is an irreducible generator of a continues-time Markov chain. Let \(q_{ij}\) denote the elements of the generator matrix \(Q\) such that

- (A) \(q_{ij} \geq 0\) if \(i \neq j\) and \(\forall i, \sum_{j=1}^{M} q_{ij} = 0\). For simplicity, we assume that the initial distribution \(\pi_0\) is independent of \(\rho\). \(Q\) is irreducible\(^2\).

Theorem 2.1 below proves that the expected degree distribution of the fixed size markov-modulated duplication-deletion random graph satisfies a recursive equation from which the expected degree distribution can be found.

**Theorem 2.1:** Consider the fixed size Markov-modulated duplication-deletion random graph generated according to Algorithm 1 with 7-tuple \((M, A^\rho, \pi_0, p, q, r, G_0)\) where \(A^\rho = I + \rho Q\) and \(r = 0\). Let \(\mathcal{g}_n^\theta = E\{g_n|\theta_n = \theta\}\). The expected degree distribution of nodes in the fixed size Markov-modulated duplication-deletion random graph, \(\mathcal{g}_n^\theta\), satisfies the following recursion

\[
\mathcal{g}_{n+1} = (I + \frac{1}{N_0} L'(\theta))\mathcal{g}_{n},
\]

\(^2\)The assumption of irreducibility implies that there exists a unique stationary distribution for this Markov chain, \(\pi \in \mathbb{R}^{M \times 1}\) such that

\[
\pi' = \pi' A^\rho.
\]
where $'$ denotes transpose of a matrix and $L(\theta_n)$, with elements defined in (10), is a generator matrix (that is, each row adds to zero and each diagonal element of $L(\theta_n)$ is negative):

$$l_{ji} = \begin{cases} 
0 & j < i - 1 \\
q(\theta_n)p(\theta_n)^{i-1} + q(\theta_n)(1 + p(\theta_n)(i - 1)) & j = i - 1 \\
q(\theta_n)p(\theta_n)^{i-1}(1 - p(\theta_n)) - q(\theta_n)(i + 2 + p(\theta_n)i) & j = i \\
q(\theta_n)(\frac{i+1}{i-1})p(\theta_n)^{i-1}(1 - p(\theta_n))^2 + q(\theta_n)(i + 1) & j = i + 1 \\
q(\theta_n)(\frac{j}{i-1})p(\theta_n)^{j-1}(1 - p(\theta_n))^{j-i+1} & j > i + 1 
\end{cases} \quad (10)$$

for $1 \leq i, j \leq N_0$

The proof is presented in Appendix A.

Theorem 2.1 shows that the evolution of the expected degree distribution in a fixed size Markov-modulated duplication-deletion random graph satisfies (9). Eq. (9) can be re-written as

$$\overline{g}^0_{n+1} = B'(\theta)\overline{g}^0_n, \quad (11)$$

where $B(\theta_n) = I + \frac{1}{N_0}L(\theta_n)$. Since $L(\theta_n)$ is a generator, for sufficiently large $N_0$, $B(\theta_n)$ can be considered as the transition matrix of a Markov chain. Hence, for each state of the Markov chain $\theta_n = \theta \in \{1,2,\ldots,M\}$, there exists a unique stationary distribution $\overline{g}(\theta)$ such that

$$\overline{g}(\theta) = B'(\theta)\overline{g}(\theta). \quad (12)$$

Therefore from (12), the expected degree distribution of the fixed size Markov-modulated duplication-deletion random graph can be computed for each state of the underlying Markov chain $\theta_n = \theta$. Note that the underlying markov chain $\theta_n$ depends on the small parameter $\rho$. The main idea is that although $\theta_n$ is time-varying but it is piecewise constant and since $\rho$ is small parameter, it changes slowly over time. Also from (12), the evolution of $\overline{g}^0_n$ depends on $\frac{1}{N_0}$. Our assumption throughout this paper is that $\rho \ll \frac{1}{N_0}$. This means that the evolution of $\overline{g}^0_n$ is faster than the evolution of $\theta_n$ or equivalently it can be said that $\overline{g}^0_n$ reaches its stationary distribution ($\overline{g}(\theta)$) before the state of $\theta_n$ changes.

**Example: Searchability of a Network**

So far in this section, an asymptotic analysis of the degree distribution was presented for a random graph generated according to Algorithm I. We now comment briefly on how the degree distribution can be used to investigate the searchability of the network. This also motivates the stochastic approximation algorithm presented in Sec.III as will be described below. The search problem arises in a network when a specific node faces a problem (request) whose solution is at other node (e.g., delivering a letter to a
specific person or finding a web page with specific information). Assume [7] that on receiving a search request, each node follows the following protocol: (a) It address the request if it or its neighbors have the solution; otherwise (b) it relays the request to one of its neighbors chosen uniformly. The objective is to find the expected search delay, that is, the expected number of steps until the request is addressed.

**Lemma 2.1:** Consider the sequence of fixed size Markov-modulated duplication-deletion random graph obtained by Algorithm 1 \{G_n\}, with \((M, A^\rho, \pi_0, p, q, r, G_0)\) where \(A^\rho = I + \rho Q\) and \(r = 0\) and expected degree distribution \(\bar{g}_n\). The expected search delay is

\[
\lambda(N_0) = O\left(\frac{N_0 \overline{d_1}}{\overline{d_2} - \overline{d_1}}\right),
\]

as \(n \to \infty\) where \(\overline{d_1} = \sum_{i=1}^{N_0} \bar{g}_n(i)\) and \(\overline{d_2} = \sum_{i=1}^{N_0} i^2 \bar{g}_n(i)\).

**Proof:** See Chapter 5 of [7] and recall that size of the considered random graph is \(N_0\). □

Lemma 2.1 implies that, if the empirical degree distribution of the possibly time-varying network can tracked accurately, then such an estimate can be used to track the searchability of the network. Also, using the estimated degree distribution and Lemma 2.1 we can address the following design problem as: How can \(p\) and \(q\) in Algorithm 1 be chosen so that the average delay does not exceed a threshold? Using the stochastic approximation algorithm in (2) (see Sec.III below for the convergence proof), we can estimate the expected degree distribution, \(\bar{g}_n\), and from that, we can compute \(\overline{d_1}\) and \(\overline{d_2}\). Then, from Lemma 2.1 we can find the measure of searchability and compare it with the maximum acceptable average delay and modify the parameters of Algorithm 1 accordingly. We illustrate searchability in numerical examples given in Sec.V.

**III. ESTIMATING (TRACKING) THE DEGREE DISTRIBUTION OF THE FIXED SIZE MARKOV-MODULATED DUPLICATION-DELETION RANDOM GRAPH**

In Sec.II a degree distribution analysis is provided for the fixed size Markov-modulated duplication-deletion random graph generated by Algorithm 1 with 7-tuple \((M, A^\rho, \pi_0, r, p, q, G_0)\), where \(r = 0\), \(G_0\) is a simple connected graphe of size \(N_0\) and \(A^\rho\) is defined in (7). In this section we assume that the empirical degree distribution of this graph, \(g_n\), is observed in noise by a network administrator. How can the network administrator track the expected degree distribution of the fixed size Markov-modulated duplication deletion random graph without knowing the dynamics of the graph? Suppose that the vertex distribution \(f_n\) generated according to Algorithm 1 is measured in noise by the administrator of the social network. That is, the measurement is

\[
\hat{f}_n = f_n + \omega_n.
\]
Here, at each time $n$, the elements $\omega_n(i)$ of the noise vector are integer-valued zero mean random variables and $\sum_{i\geq 1} \omega_n(i) = 0$. The zero sum assumption ensures that $\hat{f}_n$ is a valid empirical distribution. In terms of the empirical vertex distribution, we can rewrite this measurement process as

$$y_n(i) = \frac{\hat{f}_n(i)}{\sum_{i\geq 0} \hat{f}_n(i)} = \frac{\hat{f}_n(i)}{N_0} = g_n(i) + \frac{1}{N_0} \omega_n(i)$$

that the vertex distribution $g_n$ of the graph $G_n$ generated according to Algorithm 1 is measured in noise by the administrator of the social network. That is, the measurement is

$$y_n = g_n + \epsilon_n$$  \hspace{1cm} (15)

where $\epsilon_n = \frac{\omega_n}{N_0}$. Recall that $N_n = N_0$ when $r = 0$. The normalized noisy observations from the monitoring node, $y_n$, are used to estimate the empirical probability mass function of degree of each node. To estimate a time varying PMF, the following stochastic approximation algorithm with constant step size, $\epsilon$ (where $\epsilon$ denotes a small positive constant), is used to estimate the empirical probability mass function:

$$\hat{g}_{n+1} = \hat{g}_n + \epsilon (y_n - \hat{g}_n).$$  \hspace{1cm} (16)

Note that the stochastic approximation algorithm (16) does not assume any knowledge of the Markov-modulated dynamics of the graph. The Markov chain assumption for the random graph dynamics is only used in our convergence and tracking analysis. Our goal is to analyze how well the algorithm tracks the empirical node degree of the graph. This section studies the asymptotic behavior of the estimated degree distribution. Let $E\{\theta_n\}$ denote the expectation of $\theta_n$ with respect to $\sigma$-algebra, $G$, generated by $\{y_k, \quad k \leq n\}$. First, we show that the difference between $E\{\theta_n\}$ and $\hat{g}_n$, obtained by stochastic approximation, is bounded and the upper bound depends on $\epsilon$ and $\rho$.

A. Tracking Error of the Stochastic Approximation Algorithm

Recall that the tracking error is $\tilde{g}_n = g_n - E\{\theta_n\}$. Theorem 3.1 below shows that the difference between sample path and the expected probability mass function is small - -implying that the stochastic approximation algorithm can successfully track the Markov-modulated node distribution given noisy measurements (We again emphasize that not knowledge of the Markov chain parameters are required in the algorithm). It also finds the order of this difference in terms of $\epsilon$ and $\rho$.

**Theorem 3.1:** Consider the random graph $(M, A^\rho, \pi_0, p, q, r, G_0)$. Suppose that $\rho^2 = o(\epsilon^3)$. Then for

\[Note that in this paper, we assume that $\rho = O(\epsilon)$, therefore $\rho^2 = o(\epsilon)$ is a consequence.\]
sufficiently large $n$ the tracking error of the stochastic approximation (2) is
\[
\mathbb{E}|\tilde{g}_n|^2 = O\left(\varepsilon + \rho + \frac{\rho^2}{\varepsilon}\right).
\] (17)

The proof of Theorem 3.1 is presented in Appendix E. In the proof, the perturbed Lyapunov function methods are used. As a corollary of Theorem 3.1, we obtain the following mean squares convergence result.

Corollary 3.1: Under the conditions of Theorem 3.1 if $\rho = O(\varepsilon)$ we have
\[
\mathbb{E}|\tilde{g}_n|^2 = O(\varepsilon).
\]
and therefore,
\[
\limsup_{\varepsilon \to 0} \mathbb{E}|\tilde{g}_n|^2 = 0.
\]

B. Limit System of Regime-Switching Ordinary Differential Equations

Theorem 3.2 shows that the sequence of estimates generated by the stochastic approximation algorithm (16) converges weakly to the dynamics of a Markov-modulated ordinary differential equation.

Theorem 3.2: Consider the Markov-modulated random graph generated by Algorithm 1 and the sequence of estimates $\{\hat{g}_n\}$ generated by stochastic approximation algorithm (16). Assume condition (A) holds, and $\rho = O(\varepsilon)$. Define the continuous-time interpolated process
\[
\hat{g}_\varepsilon(t) = \hat{g}_n, \quad \theta_\varepsilon(t) = \theta_n, \quad t \in [n\varepsilon, (n+1)\varepsilon).
\] (18)
Then as $\varepsilon \to 0$, $(\hat{g}_\varepsilon(\cdot), \theta_\varepsilon(\cdot))$ converges weakly to $(\hat{g}(\cdot), \theta(\cdot))$ such that $\theta$ is continuous-time Markov chain with generator $Q$ and $\hat{g}(\cdot)$ satisfies the Markov-modulated ordinary differential equation (ODE)
\[
\frac{d\hat{g}(t)}{dt} = -\hat{g}(t) + \overline{f}(\theta(t)), \quad \hat{g}(0) = \hat{g}_0,
\] (19)
where $\overline{g}(\theta)$ is defined in (12).

Note that (19) is a Markov-modulated ordinary differential equation. The above theorem asserts that the empirical measure obtained by stochastic approximation algorithm (16) converges weakly to Markovian switched ODE (19). As mentioned in Sec. I this is unusual since typically in averaging of stochastic approximation algorithms, convergence occurs to a deterministic differential equation. The intuition behind that the estimates obtained by (16) converges to a Markov-modulated ODE (rather than a deterministic ODE) is that the Markov chain (with transition matrix $I + \rho Q$ ) evolves on the same time scale as the stochastic approximation algorithm with step size $\varepsilon$ (when $\rho = O(\varepsilon)$). If the Markov chain evolved on
a faster time scale, then the limiting dynamics would indeed be a deterministic ODE weighed by the stationary distribution for the Markov chain. If the Markov chain evolved slower than the dynamics of the stochastic approximation algorithm, then the asymptotic behavior would also be a deterministic ODE with the Markov chain being a constant.

### C. Scaled Tracking Error

The following theorem studies the behavior of the scaled tracking error between the estimates generated by the stochastic approximation algorithm (16) and the expected degree distribution and proves that this error should also satisfy a switching diffusion equation. Theorem 3.3 gives a functional central limit theorem for this scaled tracking error. Let \( \nu_k = \frac{\bar{g}_k - E\{g(\theta_k)\}}{\sqrt{\varepsilon}} \) denote the scaled tracking error.

**Theorem 3.3:** Assume condition (A) holds. Define \( \nu^{\varepsilon}(t) = \nu_k \) for \( t \in [k \varepsilon, (k + 1) \varepsilon) \). Then \( (\nu^{\varepsilon}(\cdot), \theta^{\varepsilon}(\cdot)) \) converges weakly \( (\nu(\cdot), \theta(\cdot)) \) such that \( \nu(\cdot) \) is the solution of the following Markovian switched diffusion process

\[
\nu(t) = -\int_0^t \nu(s) ds + \int_0^t \Sigma(\theta) d\omega(\tau),
\]

where \( \omega(\cdot) \) is an \( \mathbb{R}^{N_0} \)-dimensional standard Brownian motion. The covariance matrix, \( \Sigma(\theta) \), in (20) can be explicitly computed as

\[
\Sigma(\theta) = Z(\theta)'D(\theta) + D(\theta)Z(\theta) - D(\theta) - \bar{g}(\theta)g'(\theta).
\]

Here, \( D(\theta) = \text{diag}(g(\theta, 1), \ldots, g(\theta, N_0)) \) and \( Z(\theta) = (I - B(\theta) + 1\bar{g}(\theta))^{-1} \) where \( B(\theta_n) \) and \( \bar{g}(\theta) \) are defined in (II) and (12), respectively.

For general switching processes, we refer to [27]. In fact, more complex continuous-state dependent switching rather than Markovian switching was considered there. Eq. (21) reveals that the covariance matrix of the tracking error depends on \( B(\theta) \) and \( \bar{g}(\theta) \) and consequently on the parameters of \( p \) and \( q \) of the random graph. Recall from Sec.II that \( B(\theta) \) is the transition matrix of the Markov chain which models the evolution of the expected degree distribution in Markov modulated random graph and can be computed from Theorem2.1. We can interpret the covariance matrix in terms of searchability of the graph defined in Sec.II. Sec.V provides numerical examples that show that the trace of the covariance matrix \( \Sigma(\theta) \) is proportional to the searchability of the graph generated by Algorithm 1. Numerical examples in Sec.V also show that the trace of covariance of the tracking error is proportional to the average degree of nodes.
IV. DISCUSSION AND EXTENSION: POWER LAW COMPONENT FOR INFINITE DUALPLICATION-DELETION RANDOM GRAPH WITHOUT MARKOVIAN DYNAMICS

In Sec. II, a degree distribution analysis is provided for the fixed size Markov-modulated random graph generated according to Algorithm 1 with $r = 0$. This section extends the results of Sec. II to the infinite duplication-deletion random graph without Markovian dynamics. Here, we investigate the random graph generated according to Algorithm 1 with $r = 1$ and when there are no Markovian dynamics, that is, $M = 1$. Since $r = 1$ for $n \geq 0$, $G_{n+1}$ has one more vertex compared to $G_n$. In particular, since $G_0$ is an empty set, $G_n$ has $n$ nodes, that is, $N_n = n$. In this section, employing the same approach used in the proof of Theorem 2.1 it is shown that the infinite duplication-deletion random graph without Markovian dynamics generated by Algorithm 1 with $r = 1$ satisfies a power law and an expression is derived for the power law component. Let us first define the power law:

**Definition 4.1 (Power Law):** Consider the infinite duplication-deletion random graph without Markovian dynamics generated according to Algorithm 1 with 7-tuple $(M, A^\rho, \pi_0, p, q, r, G_0)$. Let $n_k$ denote the number of nodes of degree $k$ in a random graph $G_n$. Then $G_n$ satisfies a power law distribution if $n_k$ is proportional to $k^{-\beta}$ for a fixed $\beta > 1$: $\log n_k = \alpha - \beta \log k$, where $\alpha$ is a constant. $\beta$ is called power law component.

**Theorem 4.1:** Consider the infinite random graph with Markovian dynamics $G_n$ obtained by Algorithm 1 with 7-tuple $(1, 1, 1, 1, p, q, G_0)$ with the expected degree distribution $\bar{g}_n$. As $n \to \infty$, $G_n$ satisfies a power law. That is

$$\log \bar{g}_n(i) = \alpha - \beta \log i,$$

(22)

where the power law component, $\beta$, can be computed from following equation.

$$(1 + q)(p^{\beta-1} + p\beta - p) = 1 + \beta q,$$

(23)

where $p$ and $q$ are the probabilities defined in duplication and deletion steps.

**Remark 1. Outline of Proof:** The proof of Theorem 4.1 which is presented in Appendix B consists of two steps: (i) finding the power law component and (ii) showing that the degree distribution converges to a power law as $n \to \infty$. To find the power law component, we derive a recursive equation for the number of nodes with degree $i + 1$ at time $n + 1$, $f_{n+1}(i + 1)$, in terms of degree of nodes in graph $G_n$. Then, this recursive equation is rearranged to equation for the power law component. To prove that the degree distribution satisfies a power law, we define a new parameter $h_n(i) = \frac{1}{n} \sum_{k=1}^{i} E\{f_n(k)\}$ and we show that $\lim_{n \to \infty} h_n(i) = \sum_{k=1}^{i} Ck^{-\beta}$ where $\beta$ is the power law component computed by the solving
The recursive equation. Theorem 4.1 asserts that the infinite duplication-deletion random graph without Markovian dynamics generated by Algorithm 1 satisfies a power law and provides an expression for the power law component. The significance of this theorem is that it ensures that with use of one single parameter (the power law component), we can describe the degree distribution of large numbers of nodes in graphs that model social networks.

Fig. 1. The power law component for the non-Markovian random graph generated according to Algorithm 1 obtained by (23) for different values of $p$ and $q$ in Algorithm 1.

Remark 2. Power Law Component: Let $\beta^*$ denote the solution of (23). Then the power law component is defined as $\beta = \max\{1, \beta^*\}$. Fig. 1 shows the power law component and $\beta^*$ versus $p$ for different values of probability of deletion, $q$. As can be seen in Fig. 1 the power law component is increasing in $q$ and decreasing in $p$.

V. NUMERICAL EXAMPLES

In this section, numerical examples are given to illustrate the results from Sec. II, Sec. III, and Sec. IV. The main conclusions are:

(i) The infinite duplication-deletion random graph without Markovian dynamics generated by Algorithm 1 satisfies a power law as stated in Theorem 4.1. This is illustrated in Example 1 below.
(ii) The degree distribution of the fixed size duplication-deletion random graph generated by Algorithm 1 can be computed from Theorem 2.1. When $N_0$ (the size of the random graph) is sufficiently large, numerical results show that the degree distribution satisfies a power law as well. This is shown in Example 2 below.

(iii) The estimates obtained by stochastic approximation algorithm (16) follow the expected probability distribution precisely without information about the Markovian dynamics. This is illustrated in Example 3 below.

(iv) The larger the trace of the asymptotic covariance of the scaled tracking error, the greater the average degree of nodes and the searchability of the graph. This is illustrated in Example 4 below.

**Example 1**: Consider an infinite duplication-deletion random graph without Markovian dynamics generated by Algorithm 1 with $p = 0.5$ and $q = 0.1$. Theorem 4.1 implies that the degree sequence of the resulting graph satisfies a power law with exponent computed using (40). Fig. 2 shows the number of nodes with specific degree on a logarithmic scale for both horizontal and vertical axes. It can be inferred from the linearity in Fig 2 (excluding the nodes with very small degree), that the resulting graph from duplication-deletion process satisfies a power law. As can be seen in Fig 2, the power law is a better approximation for the middle points compared to both ends.

![Fig. 2. Illustration of Theorem 4.1. The degree distribution of the duplication-deletion random graph satisfies a power law.](image)

The parameters are specified in Example 1 of Sec V.
Example 2: Consider the fixed size duplication-deletion random graph obtained by Algorithm 1 with $r = 0$, $N_0 = 10$, $p = 0.4$, and $q = 0.1$. (We consider no Markovian dynamics here to illustrate Theorem 2.1.) Fig. 4 depicts the degree distribution of the fixed size duplication-deletion random graph obtained by Theorem 2.1. As can be seen in Fig. 4, the computed degree distribution is close to that obtained by simulation. The numerical results show that the degree distribution of the fixed size random graph also satisfies a power law for some values of $p$ when the size of random graph is sufficiently large. Fig. 3 shows the number of nodes with specific degree for the fixed size random graph obtained by Algorithm 1 with $r = 0$, $N_0 = 1000$, $p = 0.4$, and $q = 0.1$ on a logarithmic scale for both horizontal and vertical axes.

Fig. 3. The degree distribution of the fixed size duplication-deletion random graph satisfies a power law when $N_0$ is sufficiently large. The parameters are specified in Example 2 of Sec. V.

Fig. 4. Illustration of Theorem 2.1: The degree distribution of the fixed size duplication-deletion random graph. The parameters are specified in Example 2 of Sec. V.

Example 3: Consider the fixed size Markov-modulated duplication-deletion random graph generated by Algorithm 1 with $r = 0$ and $N_0 = 500$. Assume that the underlying Markov chain has three states, $M = 3$. We choose the following values for probabilities of connection and deletion: state (1): $p = q = 0.05$, state (2): $p = 0.2$ and $q = 0.1$, and state (3): $p = 0.4$, $q = 0.15$. The sample path of the Markov chain jumps at times $n = 3000$ from state (1) to state (2) and $n = 6000$ from state (2) to state (3). As the state of the Markov chain changes, the expected degree distribution, $\bar{g}(\theta)$, obtained by (12) evolves over time. The corresponding values for the expected degree distribution (for $i = 3$) are shown in Fig. 5 by a dotted line. The estimated probability mass function, $\hat{g}_n$, obtained by the stochastic approximation algorithm
is plotted in Fig. 5 using a solid line. The figure shows that the estimates obtained by the stochastic approximation algorithm (16) follow the expected degree distribution obtained by (12) precisely without any information about the Markovian dynamics.

**Example 4**: Consider the fixed size Markov-modulated duplication-deletion random graph obtained by Algorithm 1 with $M = 91$ and $r = 0$ and $N_0 = 1000$. For each value of $p(\theta) = 0.04 + \theta \times 0.01, \theta \in \{1, 2, \ldots, 91\}$ and $q \in \{0.05, 0.1, 0.15, 0.2\}$, we compute $L(\theta)$ from (10) and consequently the stationary distribution, $\overline{\gamma}(\theta)$, from (12). As expected, the stationary distribution does not depend on $q$ because only the deletion step in Algorithm 1 occurs with probability $q$. From $\overline{\gamma}(\theta)$, we compute the average degree of nodes, $\overline{d}_1$. Fig. 6 shows the average degree of nodes versus the probability of the connection in Algorithm 1. As can be seen in Fig. 5, with increasing the probability of connection in Algorithm 1, the average degree of nodes in the graph (which is a measure for the connectivity of the graph, see [2]) increases.

![Fig. 5. Illustration of Theorem 3.1](image1)

![Fig. 6. The average degree of nodes (as a measure of connectivity) of the fixed size Markov-modulated duplication-deletion random graph obtained by Algorithm 1 for different values of the probability of connection, $p$, in Algorithm 1. The parameters are specified in Example 4 of Sec. V](image2)

Then for each value of $p(\theta) = 0.04 + \theta \times 0.01, \theta \in \{1, 2, \ldots, 91\}$ and $q \in \{0.05, 0.1, 0.15, 0.2\}$, the covariance matrix is computed using (6). Fig. 7 depicts the trace of the covariance matrix, $\text{trace}(\Sigma(\theta))$, for each value of $p$ and $q$ versus the corresponding average degree of nodes (for each value of $p$). As can be seen in Fig. 7, the trace of the covariance matrix is larger when the average degree of nodes is
higher (the graph is highly connected).

Recall from Lemma 2.1, the order of delay in the searching problem can be computed by \( \lambda(N_0) = O\left(\frac{N_0d_1}{d_2-d_1}\right)\). Knowing the degree distribution \( g(\theta) \), \( d_1 \) and \( d_2 \) can be computed for each value of \( p \in \{0.05, 0.06, \ldots, 0.95\} \). Fig.8 shows the trace of the covariance matrix versus \( \left(\frac{d_1}{d_2-d_1}\right) \) as a measure of the searchability for each value of \( q \in \{0.05, 0.1, 0.15, 0.2\} \). As can be seen in Fig.8, the trace of covariance matrix is larger when the order of delay in the search problem in (13) is smaller.\(^4\)

![Fig. 7](image7.png)  
**Fig. 7.** The trace of the covariance matrix of the scaled tracking error, \( \text{trace}(\Sigma(\theta)) \), versus the average degree of nodes as a measure of connectivity of the network. The parameters are specified in Example 3 of Sec. V.

![Fig. 8](image8.png)  
**Fig. 8.** The trace of the covariance matrix of the scaled tracking error, \( \text{trace}(\Sigma(\theta)) \), versus the order of delay in the searching problem as a measure of searchability of the network. The parameters are specified in Example 3 of Sec. V.

### VI. CONCLUSION

This paper analyzed the dynamics of a duplication-deletion graph where at each time instant, one node can either join or leave the graph (An extension to the duplication model of [1], [11]). The power law component for such graph was computed using the result of Theorem 4.1. Also a Markov-modulated random graph was proposed to model the social networks whose evolution changes over time. Using the stochastic approximation algorithms, the probability mass function of degree of each node is estimated. Then, an upper bound was derived for the distance between the estimated and the expected PMF. As a result of this bound, we showed that the scaled tracking error between the expected PMF and the

\(^4\)This means that the target node can be found in the search problem with smaller number of steps.
estimated one weakly converges to a diffusion process. From that, the covariance of this error can be computed. Finally, we presented a discussion on application of this work in controlling a social network using the degree distribution obtained by stochastic approximation. In this case it is assumed that the network manager observes the degree of active users and this observation is noisy due to the activity profile of users. Using the estimated degree distribution, the network manager can track the level of connectivity (by computing the orders of size of giant component) and the searchability of the network (by computing the order of delay).

APPENDIX

A. Proof of Theorem 2.1

Proof: To find the degree distribution of nodes, we find a relation between the number of nodes with specific degree at time \( n \) and the degree distribution of the graph at time \( n - 1 \). Given the resulting graph at time \( n \), we are trying to find the expected number of nodes with degree \( i + 1 \) at time \( n + 1 \). The following events can occur that result in a node with degree \( i + 1 \) at time \( n + 1 \):

- A node with degree \( i \) is chosen at the duplication step as a parent node. In this case, there will be another edge connecting the new node to the parent node in the edge-duplication step. Probability of choosing a node with degree \( i \) is \( \frac{f_n(i)}{N_n} \). If a node with degree \( i \) is not chosen itself but one of its neighbors is selected as parent node, there is also a chance for this node to have another edge (with probability of \( p \)). This node has \( i \) neighbors therefore, the corresponding probability is \( p_i \).

- The degree of the most recently generated node (in the vertex-duplication) increases to \( i + 1 \) in the edge-duplication step. This means that, this node is connected to \( "i" \) neighbors of the parent node.
and remains unchanged in the deletion step. The probability of this scenario is

\[
\left(1 - \frac{q(i+2) + q(1+p(i+1))}{N_n}\right) \sum_{j \geq i} \frac{1}{N_n} f_n(j)(j)_i p^j (1 - p)^{j-i}.
\]

- A node with degree \(i+2\) remains unchanged in the duplication step and one of its neighbors is eliminated in the deletion step. The probability of this event is \(q \left(\frac{i+2}{N_n}\right) \left(1 - \frac{p(i+2)+1}{N_n}\right)\).

- The degree of the node generated in the deletion-step increases to \(i+1\) (As described in Sec. I, in deletion-step to maintain the total number of nodes, a new node is generated and connected to the graph). The probability of this scenario is \(q \sum_{j \geq i} \frac{1}{N_n} f_n(j)(j)_i p^j (1 - p)^{j-i}\).

- A node with degree \(i\) remains unchanged in the duplication step and the same node or one of its neighbors selected in the duplication part of the deletion step. The corresponding probability is \(\frac{q(1+pi)}{N_n} \left(1 - \frac{1+pi}{N_n}\right)\).

- The degree of a node with \(i+1\) neighbors increases in the duplication step and one of its neighbors is eliminated in the deletion step. The corresponding probability is \(q \left(\frac{i+2}{N_n}\right) \left(1 - \frac{p(i+2)+1}{N_n}\right)\).

Let \(\Omega\) denote the set of all arbitrary graphs and \(\mathcal{F}_n\) denote the sigma algebra generated by graphs \(G_\tau, \tau \leq n\). Considering the above events that result in a node with degree \(i+1\) at time \(n+1\), the following recurrence formula can be derived for the conditional expectation of \(f_{n+1}(i+1)\):

\[
\mathbb{E}\{f_{n+1}(i+1)|\mathcal{F}_n\} = \left(1 - \frac{q(i+2) + (1+p(i+1))}{N_n}\right) \left(1 - \frac{p(i+1)+1}{N_n}\right) f_n(i+1)
\]

\[
+ \left(1 - q \left(\frac{i+1}{N_n}\right) \left(1 + \frac{pi}{N_n}\right)\right) \left(1 + \frac{pi}{N_n}\right) f_n(i)
\]

\[
+ \left(1 - q \left(\frac{i+2}{N_n}\right) \left(1 + p(i+1)\right)\right) \sum_{j \geq i} \frac{1}{N_n} f_n(j)(j)_i p^j (1 - p)^{j-i}
\]

\[
+ q \sum_{j \geq i} \frac{1}{N_n} f_n(j)(j)_i p^j (1 - p)^{j-i}
\]

\[
+ q \left(\frac{i+2}{N_n}\right) \left(1 - \frac{p(i+2)+1}{N_n}\right) f_n(i+2)
\]

\[
+ q \left(\frac{1+pi}{N_n}\right) \left(1 - \frac{1+pi}{N_n}\right) f_n(i)
\]

\[
+ q \left(\frac{i+2}{N_n}\right) \left(p(i+1)+1\right) f_n(i+1).
\]

(24)
Let $f_\theta^n(i) = \mathbb{E}\{f_n(i) | \theta_n = \theta\}$. By taking expectation of both sides of (24) with respect to trivial sigma algebra $\{\Omega, \emptyset\}$, the smoothing property of conditional expectations yields.

$$f_\theta^{n+1}(i+1) = \left(1 - q \left(\frac{i + 2}{N_n} + \frac{1 + p(i + 1)}{N_n}\right)\right) \left(1 - \frac{p(i + 1) + 1}{N_n}\right) f_\theta^n(i + 1)$$

$$+ \left(1 - q \left(\frac{i + 1}{N_n} + \frac{1 + p(i)}{N_n}\right)\right) \left(\frac{1 + pi}{N_n}\right) f_\theta^n(i)$$

$$+ \left(1 - q \left(\frac{i + 2}{N_n} + \frac{1 + p(i + 1)}{N_n}\right)\right) \sum_{j \geq i} \frac{1}{N_n} f_\theta^n(j) \binom{j}{i} p^i (1 - p)^{j-i}$$

$$+ q \sum_{j \geq i} \frac{1}{N_n} f_\theta^n(j) \binom{j}{i} p^i (1 - p)^{j-i}$$

$$+ q \left(\frac{i + 2}{N_n}\right) \left(1 - \frac{p(i + 2) + 1}{N_n}\right) f_\theta^n(i + 2)$$

$$+ q \left(\frac{1 + pi}{N_n}\right) \left(1 - \frac{1 + pi}{N_n}\right) f_\theta^n(i)$$

$$+ q \left(\frac{i + 2}{N_n}\right) \left(\frac{p(i + 1) + 1}{N_n}\right) f_\theta^n(i + 1). \quad (25)$$

Assuming that size of the graph is sufficiently large, each term like $f_n(i') N_n^2$ can be neglected for large $N_n$. So (25) can be re-written as

$$f_\theta^{n+1}(i + 1) = \left(1 - \frac{q(\theta)(i + 2) + q(\theta)(p(\theta)(i + 1) + 1)}{N_n}\right) f_\theta^n(i + 1)$$

$$+ \left(\frac{1 + p(\theta)i}{N_n}\right) f_\theta^n(i) + q(\theta) \left(\frac{i + 2}{N_n}\right) f_\theta^n(i + 2)$$

$$+ q(\theta) \sum_{j \geq i} \frac{1}{N_n} f_\theta^n(\theta, j) \binom{j}{i} p(\theta(i + 1))^i (1 - p(\theta(i + 1)))^{j-i}. \quad (26)$$

Using (25), we can write the following recursion for the $(i + 1)$-th element of $\hat{g}_\theta^n(n + 1)$.

$$\hat{g}_\theta^{n+1}(i + 1) = \left(\frac{N_n - (q(\theta)(i + 2) + q(\theta)(p(\theta)(i + 1) + 1))}{N_{n+1}}\right) \hat{g}_\theta^n(i + 1)$$

$$+ \left(\frac{1 + p(\theta)i}{N_{n+1}}\right) \hat{g}_\theta^n(i) + q(\theta) \left(\frac{i + 2}{N_{n+1}}\right) \hat{g}_\theta^n(i + 2)$$

$$+ q(\theta) \sum_{j \geq i} \frac{1}{N_{n+1}} \hat{g}_\theta^n(j) \binom{j}{i} p(\theta)^i (1 - p(\theta))^{j-i}. \quad (27)$$
Since the probability of duplication step \( r = 0 \), the number of vertices does not increase. Thus, \( N_n = N_0 \) and (27) can be written as

\[
\theta_n(i + 1) = \left(1 - \frac{1}{N_0} (q(\theta)(i + 2) + q(\theta)(p(\theta)(i + 1) + 1)) + \frac{1}{N_0}(1 + p(\theta)i)q(\theta)\theta_n(i) + \frac{1}{N_0}q(\theta)(i + 2)\theta_n(i + 2)\right) + \frac{1}{N_0}q(\theta)\sum_{j \geq i} \theta_n(j) \left(\frac{i}{j} p(\theta)^j (1 - p(\theta))^{j-i}\right).
\]

(28)

From (28), it is clear that the vector \( \theta_n(i + 1) \) depends on elements of \( \theta_n(i) \). In a matrix notation, (28) can be re-arranged as

\[
\theta_n(i + 1) = (I + \frac{1}{N_0} L(\theta)) \theta_n(i),
\]

(29)

where \( L(\theta_n) \) is defined as (10).

To prove that \( L(\theta_n) \) is a generator, we need to show that \( l_{ii} < 0 \) and \( \sum_{i=1}^{N_0} l_{ki} = 0 \).

\[
\sum_{i=1}^{N_0} l_{ki} = -(q(\theta_n)(k + 1) + q(\theta_n)(1 + p(\theta_n)k) + (1 + p(\theta_n)k)q(\theta_n)
+ q(\theta_n)k + q(\theta_n) \sum_{k \leq i - 1} \left(\frac{k}{i - 1}\right) p(\theta_n)^{i-1} (1 - p(\theta_n))^{k-i+1}
\]

\[
= -q(\theta_n) + q(\theta_n) \sum_{k \leq i - 1} \left(\frac{k}{i - 1}\right) p(\theta_n)^{i-1} (1 - p(\theta_n))^{k-i+1}.
\]

(30)

Let \( m = i - 1 \). (30) can be rewritten as

\[
\sum_{i=1}^{N_0} l_{ik} = -q(\theta_n) + q(\theta_n) \sum_{m=0}^{k} \left(\frac{k}{m}\right) p(\theta_n)^{m} (1 - p(\theta_n))^{k-m}
\]

\[
= -q(\theta_n) + q(\theta_n)(1 - p(\theta_n)) \sum_{m=0}^{k} \left(\frac{k}{m}\right) \left(\frac{p(\theta_n)}{1 - p(\theta_n)}\right)^{m}
\]

(31)

We know that \( \sum_{m=0}^{k} \left(\frac{k}{m}\right) a^m = (1 + a)^k \), so (31) can be written as

\[
\sum_{i=1}^{N_0} l_{ik} = -q(\theta_n) + q(\theta_n)(1 - p(\theta_n)) \left(\frac{1}{1 - p(\theta_n)}\right)^k
\]

\[
= 0.
\]

(32)

Also it can be shown that if \( q(\theta_n) < \frac{p(\theta_n)(1-p(\theta_n))}{2+p(\theta_n)} \), then \( l_{ii} < 0 \).
B. Proof of Theorem 4.1

Proof: To prove Theorem 4.1, we first compute the power law component, $\beta$, and then we prove
that the expected degree distribution converges to the power law distribution with component $\beta$. Let
$\overline{f}_n(i) = E\{f_n(i)\}$. Similar to (24), $\overline{f}_n(\theta_n, i)$ can be written as

$$
\overline{f}_{n+1}(i + 1) = \left(1 - \frac{q(i + 2) + (1 + p(i + 1))}{N_n}\right) \left(1 - \frac{p(i + 1) + 1}{N_n}\right) \overline{f}_n(i + 1)
+ \left(1 - \frac{q(i + 1) + (1 + p i)}{N_n}\right) \frac{1}{N_n} \overline{f}_n(i)
+ \left(1 - \frac{q(i + 2) + (1 + p(i + 1))}{N_n}\right) \sum_{j \geq i} \frac{1}{N_n} \overline{f}_n(j) \left(\frac{j}{i}\right) p^j (1 - p)^{j-i}
+ q \sum_{j \geq i} \frac{1}{N_n} \overline{f}_n(j) \left(\frac{j}{i}\right) p^j (1 - p)^{j-i}
+ q \left(\frac{i + 2}{N_n}\right) \left(1 - \frac{p(i + 2) + 1}{N_n}\right) \overline{f}_n(i + 2)
+ q \left(\frac{1 + p i}{N_n}\right) \left(1 - \frac{1 + p i}{N_n}\right) \overline{f}_n(i)
+ q \left(\frac{i + 2}{N_n}\right) \left(\frac{p(i + 1) + 1}{N_n}\right) \overline{f}_n(i + 1).
$$

(33)

To compute the power law component, we can heuristically assume that $\overline{f}_n(i) = a_i n$ as $N_n = n$ goes
to infinity (we will prove this precisely later on this section). Therefore, each term like $\frac{\overline{f}_n(i)}{N_n}$
can be neglected as $n$ approaches infinity. So (33) can be re-written as

$$
\overline{f}_{n+1}(i + 1) = \left(1 - \frac{q(i + 2) + (1 + q)(p(i + 1) + 1)}{N_n}\right) \overline{f}_n(i + 1) + \left(1 + q\right) \frac{(1 + q)(1 + q) a_i + q(i + 2) a_{i+2}}{N_n} \overline{f}_n(i)
+ q \left(\frac{i + 2}{N_n}\right) \overline{f}_n(i + 2) + (1 + q) \sum_{j \geq i} \frac{1}{N_n} \overline{f}_n(j) \left(\frac{j}{i}\right) p^j (1 - p)^{j-i}.
$$

(34)

Substituting $\overline{f}_n(j) = a_j n$ and $N_n = n$ in (34) yields

$$
a_{i+1}(n + 1) = a_{i+1} n - a_{i+1} \left(1 + p(i + 1) + q(i + 2)\right) + (1 + q)(1 + p i) a_i + q(i + 2) a_{i+2}
+ (1 + q) \sum_{j \geq i} a_j \left(\frac{j}{i}\right) p^j (1 - p)^{j-i}.
$$

(35)

Taking all terms with $a_{i+1}$ to the left hand side, we have

$$
a_{i+1} \left(1 + (1 + q)(1 + p(i + 1)) + q(i + 2)\right) = (1 + q) \left(1 + p i) a_i + \sum_{j \geq i} a_j \left(\frac{j}{i}\right) p^j (1 - p)^{j-i}\right)
+ q(i + 2) a_{i+2}.
$$

(36)
Dividing both sides of (36) by \(a_i\) yields
\[
\frac{a_{i+1}}{a_i} \left(1 + (1 + q)(1 + p(i + 1)) + q(i + 2)\right) = (1 + q) \left((1 + pi) + \sum_{j \geq i} \frac{a_j}{a_i} \binom{j}{i} p^j (1 - p)^{j-i}\right)
+ q(i + 2) \frac{a_{i+2}}{a_i}.
\]
(37)

Solving Equation (36) for \(a_i\), we can complete the proof of Theorem 4.1. The following lemma whose proof can be found in [1] is used to solve the recurrence relation for \(a_i\).

Lemma 1.1:
\[
\sum_{j \geq i} \frac{a_j}{a_i} \binom{j}{i} p^j (1 - p)^{j-i} = p^{\beta-1} + O \left(\frac{1}{i}\right).
\]
(38)

Proof: The proof is presented in Appendix C. □

To solve (36) for \(a_i\), we can further assume that \(a_i = C i^{-\beta}\) [2]. Therefore, \(\frac{a_{i+1}}{a_i} = \left(\frac{i+1}{i}\right)^{-\beta}\)
\[
\left(1 - \frac{\beta}{i}\right) \left(1 + (1 + q)(1 + p(i + 1)) + q(i + 2)\right) = (1 + q)(1 + pi + p^{\beta-1})
+ O \left(\frac{1}{i}\right) + q(i + 2) \left(1 - \frac{2\beta}{i}\right).
\]
(39)

Neglecting the \(O \left(\frac{1}{i}\right)\) terms, yields
\[
(1 + q)(p^{\beta-1} + p\beta - p) = 1 + \beta q.
\]
(40)

Note that the proof presented above depends on few assumptions. To give a rigorous proof, the succeeding steps should be followed as described in [2]:

• First, we need to show that the limit \(\lim_{n \to \infty} \frac{1}{n} \mathbb{E}\{f_n(i)\}\) exists.

• Let \(a_i\) be the solution of (36) such that \(\sum_{i=1}^{\infty} a_i = 1\) and \(a_0 = 0\), then it is needed to show that
\[
\lim_{n \to \infty} \frac{1}{n} \mathbb{E}\{f_n(i)\} = a_i.
\]
(41)

• Finally, we should show that \(a_i\) is proportional to \(i^{-\beta}\), where \(\beta\) is the root of (40).

To complete the proof we define new function as follows \(h_n(i) = \frac{1}{n} \sum_{k=1}^{i} \mathbb{E}\{f_n(k)\}\) which can be described as CDF of degree of each node in random graph. It is sufficient to show that for all \(i > 0\),
\[
\lim_{n \to \infty} h_n(i) = \sum_{k=1}^{i} a_k
\]
(42)

where \(a_i\) is the solution of (36). It is obvious if (42) holds, \(h_n(i) - h_n(i - 1) = a_i\) and thus
\[
\lim_{n \to \infty} \frac{1}{n} \mathbb{E}\{f_n(i)\} = a_i
\]
The following lemma gives a recurrence formula to compute the value of \( h(n+1, i) \).

**Lemma 1.2:**

\[
h_{n+1}(i) = D_{n+1}(i) h_n(i) + B_{n+1}(i) h_n(i-1) + C_{n+1}(i) h_n(i+1) + \frac{1 + q}{n+1} \sum_{j \geq i-1} h_n(j) F(j, i-1, p), \tag{43}
\]

where

\[
D_{n+1}(i) = \left( \frac{n - (q(i + 2) + (1 + q)(pi + 1))}{n+1} \right),
\]

\[
B_{n+1}(i) = \frac{(1 + q)(1 + pi)}{n+1},
\]

\[
C_{n+1}(i) = \frac{q(i+1)}{n+1},
\]

\[
F(j, i, p) = \sum_{k=0}^{i} \binom{j}{k} p^k (1-p)^{j-k} - \sum_{k=0}^{i} \binom{j+1}{k} p^k (1-p)^{j+1-k}.
\]

This lemma can be proved by induction. The complete proof can be found in Appendix D. The recursive equation presented in Lemma 1.2 is used later to prove that the degree distribution converges to a power law.

**Lemma 1.3:** Let \( s_i = \sum_{k=1}^{i} a_i \) and

\[
\omega(n) = \sup_{i \geq 1} \frac{h_n(i)}{s_i},
\]

where \( h_n(i) \) satisfies (43). Then the limit \( \lim_{n \to \infty} \omega(n) \) exists and we have \( \lim_{n \to \infty} \omega(n) = 1 \).

**Sketch of the proof:** Knowing that \( h_n(i) \) satisfies the recurrence formula (43), the proof is similar to [2]. Plugging \( i = n \) in (44) yields \( \omega(n) \geq \frac{h_n(n)}{s_n} \geq \frac{1}{s_n} \geq 1 \). Using the Lemma 1.2 and similar to [2], it can be shown that \( \omega(n+1) \leq \omega(n) \). \( \omega(n) \) is bounded and decreasing, so the limit of \( \lim_{n \to \infty} \omega(n) \) exists. To show \( \lim_{n \to \infty} \omega(n) = 1 \), we assume that \( \lim_{n \to \infty} \omega(n) = c \). It can be shown that if \( c \neq 1 \), \( \omega(n) \leq 1 \) is violated. Thus \( c = 1 \) and the proof is complete.
C. Proof of Lemma 1.1

Proof:

\[
\sum_{j \geq i} \frac{a_j}{a_i} \binom{j}{i} p^i (1 - p)^{j-i} = \sum_{j \geq i} \frac{i}{j} \binom{j}{i} p^i (1 - p)^{j-i} = \sum_{j \geq i} \binom{j}{j-i} p^i (1 - p)^{j-i} = \left(1 + O\left(\frac{1}{i}\right)\right) \sum_{j \geq i} \binom{j}{j-i} p^i (1 - p)^{j-i} = \left(1 + O\left(\frac{1}{i}\right)\right) \sum_{k=0}^{\beta-i} \binom{\beta-i}{k} p^i (1 - p)^k = \left(1 + O\left(\frac{1}{i}\right)\right) p^i p^{\beta-i-1} = \left(1 + O\left(\frac{1}{i}\right)\right) p^{\beta-1}.
\]

(45)

D. Proof of Lemma 1.2

We prove the lemma by induction on \(i\):

For \(i = 1\): It is sufficient to show that:

\[h(n+1, 1) = D_{n+1}(1)h(n, 1) + C_{n+1}(1)h(n, 2) + \frac{1}{n+1} \sum_{j \geq 1} h(n, j)F(j, 0, p).\]

Also using the definition of \(F(j, i, p)\), we can rewrite \(F(j, 0, p)\) as \((1 - p)^j - (1 - p)^{j+1}\). The number of nodes with degree one at time \(n+1\) can be written as following

\[
E\{f(n+1, 1)\} = \left(1 - \frac{(1+q)(1+p)+q}{n}\right) E\{f_n(1)\} + \frac{2q}{n} E\{f_n(2)\} + (1 + q) \sum_{j \geq 1} \frac{1}{n} E\{f_n(j)\}(1 - p)^j.
\]

(46)

Note that (46) is slightly different from the general equation for each \(i\), (34). Because as described in Sec.2, neighbors of a node with degree one cannot be eliminated from the graph to maintain the connectivity in the graph. Therefore, a node with degree one can change in the deletion step if that node
is selected in the deletion step (with probability $q$). Using (46), $h(n+1,1)$ can be written as

$$h(n+1,1) = \frac{1}{n+1} \mathbb{E}\{f(n+1,1)\}$$

$$= \frac{1}{n+1} \left( \left( 1 - \frac{(1+q)(1+p)+q}{n} \right) \mathbb{E}\{f_n(1)\} + \frac{2q}{n} \mathbb{E}\{f_n(2)\} \right)$$

$$+ \frac{1}{n+1} \sum_{j \geq 1} \frac{1+q}{n} \mathbb{E}\{f_n(j)\}(1-p)^j.$$  \hspace{1cm} (47)

We know that $h(n,0) = 0$ for all $n$. Using the definition of $h(\cdot, \cdot)$ and (46), (47) can be re-arranged as follows

$$h(n+1,1) = \frac{1}{n+1} \left( \left( n - (1+q)(1+p) + q \right) h(n,1) + \frac{2q}{n} \left( h(n,2) - h(n,1) \right) \right)$$

$$+ (1+q) \sum_{j \geq 1} (h(n,j) - h(n,j-1))(1-p)^j$$

$$= \frac{1}{n+1} \left( \left( n - (3q + (1+q)(1+p)) \right) h(n,1) + \frac{2q}{n} h(n,2) \right)$$

$$+ \frac{1+q}{n+1} \sum_{j \geq 1} (h(n,j) - h(n,j-1))(1-p)^j$$  \hspace{1cm} (48)

$\sum_{j \geq 1} (h(n,j) - h(n,j-1))(1-p)^j$ can be written in terms of the $F(j,0,p)$.

$$\sum_{j \geq 1} (h(n,j) - h(n,j-1))(1-p)^j = \sum_{j \geq 1} h(n,j)(1-p)^j - \sum_{j \geq 1} (h(n,j-1)(1-p)^j$$

$$= \sum_{j \geq 1} h(n,j)(1-p)^j - \sum_{j \geq 1} (h(n,j)(1-p)^j + 1$$

$$= \sum_{j \geq 1} h(n,j) ( (1-p)^j - (1-p)^{j+1})$$

$$= \sum_{j \geq 1} h(n,j) F(j,0,p).$$  \hspace{1cm} (49)

Substituting (49) in (48) yields

$$h(n+1,1) = \frac{1}{n+1} \left( \left( n - ((1+q)(1+p) + 3q) \right) h(n,1) + \frac{2q}{n} h(n,2) + (1+q) \sum_{j \geq 1} h(n,j) F(j,0,p) \right)$$

$$= D_{n+1}(1) h(n,1) + C_{n+1}(1) h(n,2) + \frac{1+q}{n+1} \sum_{j \geq 1} h(n,j) F(j,0,p).$$  \hspace{1cm} (50)

Thus (43) holds for $i = 1$. Now it is assumed that (43) holds for $i = k$, we want to show that it also
holds for \( i = k + 1 \).

\[
\mathbb{E}\{f(n + 1, k + 1)\} = \left(1 - \frac{q(k + 2) + (1 + q)(p(k + 1) + 1)}{n}\right) \mathbb{E}\{f(n, k + 1)\} \\
+ \left(\frac{(1 + q)(1 + pk)}{n}\right) \mathbb{E}\{f_n(k)\} + \left(\frac{q(k + 2)}{n}\right) \mathbb{E}\{f_n(k + 2)\} \\
+ (1 + q) \sum_{j \leq k} \frac{f_n(j)}{n} \binom{j}{k} p^k(1 - p)^{j-k}.
\]  

(51)

from definition of \( h(n, k) \), we have : \( \mathbb{E}\{f_n(k)\} = n \left(h(n, k) - h(n, k - 1)\right) \). Eq. (51) can be re-written as follows

\[
\mathbb{E}\{f(n + 1, k + 1)\} = \left(n - \left(q(k + 2) + (1 + q)(p(k + 1) + 1)\right)\right) \left(h(n, k + 1) - h(n, k)\right) \\
+ (1 + q)(1 + pk)(h(n, k) - h(n, k - 1)) + q(k + 2)(h(n, k + 2) - h(n, k + 1)) \\
+ (1 + q) \sum_{j \leq k} (h(n, j) - h(n, j - 1)) \binom{j}{k} p^k(1 - p)^{j-k}.
\]  

(52)

Using the Abel summation identity, and knowing that

\[
F(j, k, p) = \sum_{k=0}^{k} \binom{j}{k} p^k(1 - p)^{j-k} - \sum_{k=0}^{k} \binom{j + 1}{k} p^k(1 - p)^{j+1-k},
\]

the last term can be written as

\[
\sum_{j \leq k} (h(n, j) - h(n, j - 1)) \binom{j}{k} p^k(1 - p)^{j-k}
\]

(53)

\[
= \sum_{j \geq k} \left( \binom{j}{k} p^k(1 - p)^{j-k} - \binom{j + 1}{k} p^k(1 - p)^{j+1-k} \right) - p^k h(n, k - 1)
\]

\[
= -p^k h(n, k - 1) + \sum_{j \geq k} h(n, j)(F(j, k, p) - F(j, k - 1, p)).
\]  

(54)

Substituting (53) in (52) yields

\[
\mathbb{E}\{f(n + 1, k + 1)\} = h(n, k + 2)(q(k + 2)) + h(n, k + 1)\left(n - (2q(k + 2) + (1 + q)(p(k + 1) + 1))\right) \\
+ h(n, k)\left(1 + q\right)(2 + p(2k + 1)) \\
+ q(k + 2) - n) + h(n, k - 1)(1 + q)(-1 - pk - p^k) \\
+ (1 + q) \sum_{j \geq k} h(n, j)(F(j, k, p) - F(j, k - 1, p)).
\]  

(55)

The value of \( h(n + 1, k + 1) \) can be computed using \( h(n, k + 1) \) and \( \mathbb{E}\{f_n(k + 1)\} \) as follows

\[
h(n + 1, k + 1) = h(n + 1, k) + \frac{1}{n + 1} \mathbb{E}\{f(n + 1, k + 1)\}.
\]  

(56)
Eq. (55) gives an expression for $E\{f(n+1,k+1)\}$ in terms of the value of $h(\cdot,\cdot)$ at time $n$. Substituting (55) in (56) gives a recursive equation for computing $h(n+1,k+1)$:

$$h(n+1,k+1) = h(n+1,k) + \frac{1}{n+1} E\{f(n+1,k+1)\}$$

$$= D_{n+1}(k)h(n,k) + B_{n+1}(k)h(n,k-1) + C_{n+1}h(n,k+1)$$

$$+ \frac{1+q}{n+1} \sum_{j=k-1} h(n,j)F(j,k-1,p)$$

$$+ \frac{1}{n+1} \left( h(n,k+2)(q(k+2)) + h(n,k+1) \right)$$

$$\left( n - (2q(k+2) + (1+q)(p(k+1)+1)) \right)$$

$$+ h(n,k) \left( (1+q)(2 + p(2k+1)) + h(n,k-1)(1+q)(-1 - pk - p^k) \right)$$

$$+ (1+q) \sum_{j \geq k} h(n,j)(F(j,k,p) - F(j,k-1,p)) \right).$$

We assume that (34) holds for $i = k$ so substituting the values for $D_{n+1}(k)$, $B_{n+1}(k)$, and $C_{n+1}(k)$ from (34) in (57) yields

$$h(n+1,k+1) = h(n,k+2) \left( \frac{q(k+2)}{n+1} \right) + h(n,k+1) \left( \frac{n - (q(k+3) + (1+q)(p(k+1)+1))}{n+1} \right)$$

$$+ h(n,k) \left( \frac{(1+q)(1+p(k+1))}{n+1} \right) + \frac{1+q}{n+1} \sum_{j \geq k} h(n,j)(F(j,k,p)).$$

(58) can be written as follows

$$h(n+1,k+1) = D_{n+1}(k+1)h(n,k+1) + B_{n+1}(k+1)h(n,k) + C_{n+1}(k+1)h(n,k+2)$$

$$+ \frac{1+q}{n+1} \sum_{j \geq k} h(n,j)F(j,k,p).$$

(59)

Thus, (34) holds for $i = k+1$ and the proof is completed by induction.

E. Proof of Theorem 3.1

Proof: Define the Liapunov function $V(x) = (x'x)/2$ for $x \in \mathbb{R}^{N_0}$. Use $E_n$ to denote the conditional expectation with respect to the $\sigma$-algebra, $\mathcal{H}_n$, generated by $\{y_j, \theta_j, \ j \leq n\}$.

$$E_n\{V(\bar{g}_{n+1}) - V(\bar{g}_n)\} = E_n\left\{ \bar{g}_n'[-\varepsilon \bar{g}_n + \varepsilon (y_{n+1} - E\{\bar{g}(\theta_n)\}) + E\{\bar{g}(\theta_n) - \bar{g}(\theta_{n+1})\}] \right\}$$

$$+ E_n\left\{ | - \varepsilon \bar{g}_n + \varepsilon (y_{n+1} - E\{\bar{g}(\theta_n)\}) + E\{\bar{g}(\theta_n) - \bar{g}(\theta_{n+1})\}|^2 \right\},$$

(60)
where \( y_{n+1} \) and \( \overline{f}(\theta_n) \) are vectors in \( \mathbb{R}^{N_0} \) with elements \( y_n(i) \) and \( \overline{f}(\theta_n, i) \), \( 1 \leq i \leq N_0 \), respectively. It is easily seen that

\[
E_n\{\overline{f}(\theta_n) - \overline{f}(\theta_{n+1})\} = O(\rho), \quad E_n\{|\overline{f}(\theta_n) - \overline{f}(\theta_{n+1})|^2\} = O(\rho). \quad (61)
\]

Using \( K \) to denote a generic positive value (with the notation \( KK = K \) and \( K + K = K \)), a familiar inequality \( ab \leq \frac{a^2 + b^2}{2} \) yields

\[
O(\varepsilon \rho) = O(\varepsilon^2 + \rho^2). \quad (62)
\]

Moreover we have \( |\overline{g}_n| = |\overline{g}_n| \cdot 1 \leq (|\overline{g}_n|^2 + 1)/2. \) Thus

\[
O(\rho)|\overline{g}_n| \leq O(\rho) \left( V(\overline{g}_n) + 1 \right). \quad (63)
\]

Then detailed estimates lead to

\[
E_n \left\{ \left| -\varepsilon \overline{g}_n + \varepsilon (y_{n+1} - E\{\overline{f}(\theta_n)\}) \right| + E\{\overline{f}(\theta_n) - \overline{f}(\theta_{n+1})\} \right|^2 \right\} = O(\varepsilon^2 + \rho^2)(V(\overline{g}_n) + 1) \quad (64)
\]

Furthermore, we can obtain that

\[
E_n\{V(\overline{g}_{n+1}) - V(\overline{g}_n)\} = -2\varepsilon V(\overline{g}_n) + \varepsilon E_n\{\overline{g}_{n+1} - E\overline{f}(\theta_n)\} + E_n\{\overline{g}_n E\{\overline{f}(\theta_{n+1}) - \overline{f}(\theta_n)\}\} + O(\varepsilon^2 + \rho^2)(V(\overline{g}_n) + 1). \quad (65)
\]

Define \( V_1^\rho \) and \( V_2^\rho \) as following

\[
V_1^\rho(\overline{g}, n) = \varepsilon \sum_{j=n}^{\infty} \overline{g}^j E_n\{y_{j+1} - E\overline{f}(\theta_j)\},
\]

\[
V_2^\rho(\overline{g}, n) = \sum_{j=n}^{\infty} \overline{g}^j E_n\{\overline{f}(\theta_j) - \overline{f}(\theta_{j+1})\}. \quad (66)
\]

It can be shown that

\[
|V_1^\rho(\overline{g}, n)| = O(\varepsilon)\left(V(\overline{g}) + 1\right), \quad (67)
\]

\[
|V_2^\rho(\overline{g}, n)| = O(\rho)\left(V(\overline{g}) + 1\right). \quad (68)
\]

Define \( W(\overline{g}, n) \) as

\[
W(\overline{g}, n) = V(\overline{g}) + V_1^\rho(\overline{g}, n) + V_2^\rho(\overline{g}, n). \quad (69)
\]

This leads to

\[
E_n\{W(\overline{g}_{n+1}, n + 1) - W(\overline{g}_n, n)\} = E_n\{V(\overline{g}_{n+1}) - V(\overline{g}_n)\} + E_n\{V_1^\rho(\overline{g}_{n+1}, n + 1) - V_1^\rho(\overline{g}_n, n)\} + E_n\{V_2^\rho(\overline{g}_{n+1}, n + 1) - V_2^\rho(\overline{g}_n, n)\}. \quad (69)
\]
Moreover,
\[ E_n \{ W(\tilde{g}_{n+1}, n+1) - W(\tilde{g}_n, n) \} = -2\varepsilon V(\tilde{g}_n) + O(\varepsilon^2 + \rho^2)(V(\tilde{g}_n) + 1). \] (70)

Eq. (70) can be rewritten as
\[ E_n \{ W(\tilde{g}_{n+1}, n+1) - W(\tilde{g}_n, n) \} \leq -2\varepsilon W(\tilde{g}_n, n) + O(\varepsilon^2 + \rho^2)(W(\tilde{g}_n, n) + 1). \] (71)

If \( \varepsilon \) and \( \rho \) are chosen small enough, then there exists a small \( \lambda \) such that \(-2\varepsilon + O(\rho^2) + O(\varepsilon^2) \leq -\lambda\varepsilon\).
So (71) can be re-arranged to the following,
\[ E_n \{ W(\tilde{g}_{n+1}, n+1) \} \leq (1 - \lambda\varepsilon)W(\tilde{g}_n, n) + O(\varepsilon^2 + \rho^2). \] (72)

Taking expectation of both sides yields
\[ E \{ W(\tilde{g}_{n+1}, n+1) \} \leq (1 - \lambda\varepsilon)E \{ W(\tilde{g}_n, n) \} + O(\varepsilon^2 + \rho^2). \] (73)

Iterating on (73) yields
\[ E \{ W(\tilde{g}_{n+1}, n+1) \} \leq (1 - \lambda\varepsilon)^n E \{ W(\tilde{g}_{N}, N) \} + \sum_{j=N}^{n} O(\varepsilon^2 + \rho^2)(1 - \lambda\varepsilon)^j, \] (74)

so
\[ E \{ W(\tilde{g}_{n+1}, n+1) \} \leq (1 - \lambda\varepsilon)^n E \{ W(\tilde{g}_{N}, N) \} + O \left( \varepsilon + \frac{\rho^2}{\varepsilon} \right). \] (75)

If \( n \) is large enough we can approximate \((1 - \lambda\varepsilon)^n = O(\varepsilon^2 + \rho^2)\)\(= O(\varepsilon)\)
\[ E \{ W(\tilde{g}_{n+1}, n+1) \} \leq O \left( \varepsilon + \frac{\rho^2}{\varepsilon} \right). \] (76)

Finally, using (67) and replacing \( W(\tilde{g}_{n+1}, n+1) \) with \( V(\tilde{g}_{n+1}) \), we obtain
\[ E \{ V(\tilde{g}_{n+1}) \} \leq O \left( \rho + \varepsilon + \frac{\rho^2}{\varepsilon} \right). \] (77)

F. Sketch of the Proof of Theorem 3.2

Since the proof is similar to [13, Theorem 4.5], we only indicate the main steps needed and omit most of the vabatim details.

(1) First we show that the two component process \((\tilde{g}^\varepsilon(\cdot), \theta^\varepsilon(\cdot))\) is tight in \( D([0, T] : \mathbb{R}^{N_0} \times \mathcal{M}) \). Using the techniques as in [28, Theorem 4.3], it can be shown that \( \theta^\varepsilon(\cdot) \) converges weakly to a continuous-time Markov chain generated by \( Q \). Thus, we mainly need to consider \( \tilde{g}^\varepsilon(\cdot) \). We show that
\[ \lim_{\Delta \to 0} \limsup_{\varepsilon \to 0} E \left[ \sup_{0 \leq s \leq \Delta} E_t^\varepsilon [ \tilde{g}^\varepsilon(t + s) - \tilde{g}^\varepsilon(t)]^2 \right] = 0, \]
where $E_t^\varepsilon$ denotes the conditioning on the past information up to $t$. Then the tightness follows from the criterion [29, p. 47].

(2) Since $(\hat{g}^\varepsilon(\cdot), \theta^\varepsilon(\cdot))$ is tight, we can extract weakly convergent subsequence according to the Prohorov theorem (see [12]). To figure out the limit, we show that $(\hat{g}^\varepsilon(\cdot), \theta^\varepsilon(\cdot))$ is a solution of the martingale problem with operator $L_0$. For each $i \in M$ and continuously differential function with compact support $f(\cdot, i)$, the operator is given by

$$L_0f(\hat{g}, i) = \nabla f'(\hat{g}, i)[-\hat{g} + \overline{g}(i)] + \sum_{j \in M} q_{ij} f(\hat{g}, j), \ i \in M. \quad (78)$$

We can further demonstrate the martingale problem with operator $L_0$ has a unique solution in the sense in distribution. Thus the desired convergence property follows.

G. Sketch of the Proof of Theorem \ref{thm:main}

(1) First note

$$\nu_{n+1} = \nu_n - \varepsilon \nu_n + \sqrt{\varepsilon} (y_{n+1} - E \overline{g}(\theta_n)) + \frac{E[\overline{g}(\theta_n) - \overline{g}(\theta_{n+1})]}{\sqrt{\varepsilon}}. \quad (79)$$

Again, the approach is similar to that of [13, Theorem 5.6]. So again, we will be brief.

(2) Define an operator

$$Lf(\nu, i) = -\nabla f'(\nu, i) \nu + \frac{1}{2}(\nabla^2 f(\nu, i) \Sigma(i)) + \sum_{j \in M} q_{ij} f(\nu, j), \ i \in M, \quad (80)$$

for function $f(\cdot, i)$ that has continuous partial derivatives with respect to $\nu$ up to the second order and that has compact support. It can be show that the associated martingale problem has a unique solution in the sense in distribution.

(3) It is natural now to work with a truncated process. For a fixed but otherwise arbitrary $r_1 > 0$, define a truncation function

$$q_{r_1}(x) = \begin{cases} 1, & \text{if } x \in S_{r_1}, \\ 0, & \text{if } x \in \mathbb{R}^N_0 - S_{r_1}, \end{cases}$$

where $S_{r_1} = \{x \in \mathbb{R}^N_0 : |x| \leq r_1\}$. Then we get the truncated iterates

$$\nu_{n+1}^{r_1} = \nu_n^{r_1} - \varepsilon \nu_n^{r_1} q_{r_1}(\nu_n) + \sqrt{\varepsilon} (y_{n+1} - E \overline{g}(\theta_n)) + \frac{E[\overline{g}(\theta_n) - \overline{g}(\theta_{n+1})]}{\sqrt{\varepsilon}} q_{r_1}(\nu_n). \quad (81)$$

Define $\nu^{\varepsilon,r_1}(t) = \nu_n^{r_1}$ for $t \in [\varepsilon n, \varepsilon n + \varepsilon)$. Then $\nu^{\varepsilon,r_1}(\cdot)$ is an $r$-truncation of $\nu^\varepsilon(\cdot)$; see [12, p. 284] for a definition. We then show the truncated process $(\nu^{\varepsilon,r_1}(\cdot), \theta^\varepsilon(\cdot))$ is tight. Moreover, by Prohorov’s theorem,
we can extract a convergent subsequence with limit \((\nu^r(\cdot), \theta(\cdot))\) such that the limit \((\nu^r(\cdot), \theta(\cdot))\) is the solution of the martingale problem with operator \(L^r\) defined by

\[
L^r f^r(\nu, i) = -\nabla f^r(\nu, i)\nu + \frac{1}{2} \text{tr}\left[\nabla^2 f^r(\nu, i)\Sigma(i)\right] + \sum_{j \in \mathcal{M}} q_{ij} f^r(\nu, j), \quad i \in \mathcal{M},
\]

(82)

where \(f^r(\nu, i) = f(\nu, i)q^r(\nu)\).

(4) Letting \(r_1 \to \infty\), we show that the un-truncated process also converges and the limit denoted by \((\nu(\cdot), \theta(\cdot))\) is precisely the martingale problem with operator \(L\) defined in (80). Furthermore, the limit covariance can be evaluated as in [13, Lemma 5.2].

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