Abstract

We propose $N = 4$ twisted superspace formalism in four dimensions by introducing Dirac-Kähler twist. In addition to the BRST charge as a scalar counterpart of twisted supercharge we find vector and tensor twisted supercharges. By introducing twisted chiral superfield we explicitly construct off-shell twisted $N = 4$ SUSY invariant action. We can propose variety of supergauge invariant actions by introducing twisted vector superfield. We may, however, need to find further constraints to identify twisted $N = 4$ super Yang-Mills action. We propose a superconnection formalism of twisted superspace where constraints play a crucial role. It turns out that $N = 4$ superalgebra of Dirac-Kähler twist can be decomposed into $N = 2$ sectors. We can then construct twisted $N = 2$ super Yang-Mills actions by the superconnection formalism of twisted superspace in two and four dimensions.
1 Introduction

One of the most fundamental questions in modern particle physics is to understand the origin of the supersymmetry (SUSY) between boson and fermion. It was pointed out that quantized version of four-dimensional topological Yang-Mills action with instanton gauge fixing led to $N = 2$ twisted super Yang-Mills action \([1–5]\). The twisting procedure generates matter fermions from the ghost-related fermions by relating the spin of matter fermion and internal $R$ symmetry of $N = 2$ SUSY algebra. Even though this is a very special example, we may recognize that SUSY appears from the quantization procedure of bosonic topological theory. Many related works appeared to find better understandings of the connection between the SUSY and the topological field theory \([6–11]\) \([12, 13]\).

One of the important characteristics of the Witten’s twisting procedure is that the BRST charge of the quantization of topological field theories is responsible for the twisted SUSY as a supercharge. In the investigations of the quantization of topological field theories of Schwarz type; Chern-Simons action and BF actions, a new type of vector SUSY was discovered \([14–17]\). It was recognized that this vector SUSY belongs to a twisted version of an extended SUSY of $N = 2$ or $N = 4$. The origin of the vector SUSY was recognized in some particular examples to be related to the fact that the energy-momentum tensor can be expressed as a pure BRST variation \([18–20]\). It was later stressed that a (pseudo) scalar SUSY was also accompanied together with BRST and vector SUSY \([21]\). Then there appeared many related works \([22–30]\) \([31–34]\). The connection of the extended SUSY and the quantization procedure of anti-field formalism by Batalin and Vilkovisky was also investigated \([35, 36]\).

One of the authors (N.K.) and Tsukioka pointed out that the two-dimensional version of topological Yang-Mills action obtained from generalized gauge theory \([37–40]\) with instanton gauge fixing led to two-dimensional version of $N = 2$ twisted super Yang-Mills action \([41]\). It was found in this investigation that the twisting procedure to relate between the ghost-related fermions and matter fermions is essentially Dirac-Kähler fermion mechanism \([42–49]\) and the ”flavor” degrees of freedom of the Dirac-Kähler fermion can be interpreted as that of the extended SUSY \([41]\). It turned out that this Dirac-Kähler twisting mechanism works universally in the quantization of topological field theory and an extended SUSY is generated. Here the BRST charge in the quantization is equivalent to the scalar component of twisted supercharge \([50]\). It was then recognized that the twisted superspace formulation is hidden behind the formulation. In the previous paper two of the authors (J.K. and N.K.) with Uchida proposed twisted superspace formalism for $N = 2$ twisted SUSY in two dimensions and derived off-shell SUSY invariant BF, Wess-Zumino and super Yang-Mills actions \([50]\). The quantized action of Yang-Mills type in two dimensions has the close connection with $N = 2$ super Yang-Mills action obtained from the different context \([51, 52]\). In this paper we propose four-dimensional $N = 4$ twisted superspace formalism as a natural extension of the $N = D = 2$ twisted superspace formalism. Related works with the similar contexts with our formulation was given.
by Labastida and collaborators for $N = 2$ twisted SUSY by spinor formulation in two and four dimensions [53,54], while our formulation is based on the scalar-vector-tensor formulation due to the Dirac-Kähler twisting procedure.

Concerning to the twisting procedure of $N = 4$ SUSY there are several twisting procedure [55–59]. The twisting procedure we propose in this paper has close connection with that of Marcus [56]. Off-shell $N = 4$ SUSY invariant actions were proposed and the corresponding superspace formulation was investigated by several authors but so far it is not successful [60–62].

One of the other important motivations of current investigation comes from the recent lattice SUSY investigation [63]. It is well known that the Dirac-Kähler fermion mechanism is fundamentally related to the lattice formulation [64–66] [67,68]. In fact recently $N = 2$ twisted superspace in two dimensions has been successfully formulated on a lattice with an introduction of mild non-commutability [69–74] for lattice difference operator and twisted supercharges [63]. It is strongly suggested that $N = 4$ twisted superspace formalism in four dimensions is important to formulate four-dimensional SUSY on a lattice. In particular $N = 4$ twisted super Yang-Mills action is needed to formulate supergauge invariant action on a four-dimensional lattice.

The recent AdS/CFT correspondence [75, 76] from superstring formulation also suggests that superspace formulation of $N = 4$ super Yang-Mills action will help to understand the fundamental structure of Yang-Mills theory based on the brane dynamics.

This paper is organized as follows: We first give the general formulation of $N = 4$ twisted SUSY algebra based on Dirac-Kähler twist in section 2. Then we formulate twisted superspace and superfield for $N = D = 2$ and $N = D = 4$ in section 3. We introduce twisted chiral and vector superfields and propose off-shell twisted SUSY invariant actions. In section 4 we propose superconnection formalism to formulate twisted $N = 2$ SUSY invariant super Yang-Mills actions in two and four dimensions. We summarize the results in section 5. We provide several appendices to give the details of tensor kinematics of twisted algebra and full expression of twisted $N = 4$ SUSY invariant action.

2 Twisted SUSY from Dirac-Kähler twist

The twisting procedure was first proposed by Witten in the derivation of the twisted version of $N = 2$ super Yang-Mills action [1]. It was soon recognized that the super Yang-Mills action can be derived by the quantization of topological Yang-Mills action with instanton gauge fixing, where the ghost-related fields turned into matter fermions via twisting mechanism [2–5]. It was also shown that $N = D = 2$ twisted super Yang-Mills action can be derived from instanton gauge fixing of the two-dimensional version of generalized topological Yang-Mills theory in the similar way as the four-dimensional case [41]. It was then found in the $N = D = 2$ twisted SUSY formulation that the twisting mechanism is essentially related to the Dirac-
Kähler fermion mechanism [41]. Then it has led to the proposal of the $N = 2$ twisted superspace formalism in two dimensions [50]. Here we propose twisting mechanism of $N = 4$ twisted super symmetry in four dimensions. We may call this twisting procedure as Dirac-Kähler twist. In order to show that a general formulation of the Dirac-Kähler twist of twisted SUSY for $N = 2$ in two dimensions and $N = 4$ in four dimensions has an intimate similarity, we construct the four-dimensional formulation parallel to the two-dimensional case.

The supercharges of extended supersymmetry algebra satisfy the following relations:

$$\{Q_{\alpha i}, Q_{j\beta}\} = 2\delta_{ij} P_\mu \gamma^\mu, \quad (2.1)$$

where the indices $\{\alpha, \beta\}$ and the indices $\{i, j\}$ are Lorentz spinor and internal $R$-symmetry suffix of an extended SUSY, respectively. For $N = 2$ extended SUSY in two dimensions we take $i, j = 1, 2$, while we take $i, j = 1, 2, 3, 4$ for $N = 4$ extended SUSY in four dimensions. Since we introduce $\gamma$-matrix in the right hand side of (2.1) the conjugate supercharge $Q^*_{i\alpha}$ should be related to $Q_{\alpha i}$. $\gamma^\mu$ and $P_\mu$ are the corresponding $\gamma$-matrix and momentum generator in two and four dimensions, respectively, and the explicit representation of four-dimensional $\gamma^\mu$ is given in Appendix A. Throughout this paper we consider Euclidean spacetime.

### 2.1 $N = 2$ twisted SUSY in two dimensions

In defining Dirac-Kähler twist we identify the right index of the supercharge $Q_{\alpha i}$ as spinor suffix, then we can decompose the charge into the following scalar, vector and pseudo-scalar components which we call twisted supercharges:

$$Q_{\alpha i} = (1s + \gamma^\mu s_\mu + \gamma^5 \tilde{s})_{\alpha i}. \quad (2.2)$$

We introduce the conjugate supercharge as $Q^*_{i\alpha} = (C^{-1} Q^T C)_{i\alpha}$ where we can take $C = 1$ in two-dimensional Euclidean spacetime and thus $Q^*_{i\alpha} = Q_{\alpha i}$. The details of the notation in two dimensions can be found in [50].

The relations (2.1) can now be rewritten by the twisted generators as:

$$\{s, s_\mu\} = P_\mu, \quad \{\tilde{s}, s_\mu\} = -\epsilon_{\mu\nu} P_\nu, \quad s^2 = \tilde{s}^2 = \{s, \tilde{s}\} = \{s_\mu, s_\nu\} = 0. \quad (2.3)$$

This is the twisted $N = D = 2$ SUSY algebra obtained from Dirac-Kähler twist.

Similar to the supercharges we can introduce twisted superparameters as

$$\theta_{\alpha i} = \frac{1}{2} \left(1\theta + \gamma^\mu \theta_\mu + \gamma^5 \tilde{\theta}\right)_{\alpha i}, \quad (2.4)$$

then we can define $N = 2$ supersymmetry transformation as

$$\delta \theta = \theta_{\alpha i} Q_{\alpha i} = \theta s + \theta^\mu s_\mu + \tilde{\theta} \tilde{s}. \quad (2.5)$$
In general $N = 2$ SUSY in two dimensions includes super-Poincaré symmetry together with $R$-symmetry as follows:
\[
\begin{align*}
[P_\mu, Q_{\alpha i}] &= 0, \\
[J, Q_{\alpha i}] &= \frac{i}{2} (\gamma^5)_{\alpha}^{\beta} Q_{\beta i}, \\
[R, Q_{\alpha i}] &= \frac{i}{2} (\gamma^5)_{i}^{j} Q_{\alpha j}, \\
[J, P_\mu] &= i\epsilon_\mu^\nu P_\nu, \\
[P_\mu, P_\nu] &= [P_\mu, R] = [J, R] = 0.
\end{align*}
\] (2.6)

$J$ and $R$ are generators of $SO(2)$ Lorentz and $SO(2)_I$ internal rotation of extended SUSY called $R$ symmetry, respectively, while $\gamma^5$ can be identified as the rotation generator of spinor suffix for those rotations.

The essential meaning of the Dirac-Kähler twist is to identify the extended SUSY indices as the spinor ones. Then the internal extended SUSY should transform as spinor under the Lorentz transformation. This will lead to a redefinition of the energy-momentum tensor and the Lorentz rotation generator.

We can redefine the energy-momentum tensor $T_{\mu\nu}$ as the following relation without breaking the conservation law:
\[
T'_{\mu\nu} = T_{\mu\nu} + \epsilon_{\mu\rho} \partial^\rho R_\nu + \epsilon_{\nu\rho} \partial^\rho R_\mu,
\] (2.7)

where $R_\mu$ is the conserved current associated with $R$ symmetry [1, 10, 53]. This modification leads to a redefinition of the Lorentz rotation generator,
\[
J' = J + R.
\] (2.8)

This new rotation group is interpreted as the diagonal subgroup of $SO(2) \times SO(2)_I$.

The twisted version of the super-Poincaré and $R$-symmetry algebra can be obtained as follows:
\[
\begin{align*}
[J', s] &= [J', \tilde{s}] = 0, \quad [J', s_\mu] = i\epsilon_{\mu\nu} s^\nu, \\
[R, s] &= \frac{i}{2} \tilde{s}, \quad [R, s_\mu] = \frac{i}{2} \epsilon_{\mu\nu} s^\nu, \quad [R, \tilde{s}] = -\frac{i}{2} s, \\
[J', P_\mu] &= i\epsilon_{\mu\nu} P^\nu, \\
[P_\mu, P_\nu] &= [P_\mu, R] = [J', R] = 0.
\end{align*}
\] (2.9)

This is the twisted $D = N = 2$ SUSY algebra.

### 2.2 $N = 4$ twisted SUSY in four dimensions

In this subsection we consider an extension of the twisting procedure using Dirac-Kähler twist into four dimensions. The Dirac-Kähler fermion mechanism is formulated in any dimensions [42–49]. It is particularly convenient to formulate in even dimensions since the corresponding Clifford algebra is unambiguously defined with a chiral generator. Similar to the two-dimensional case, we introduce the Dirac-Kähler twisting procedure in four dimensions. In this case the situation is more involved
since the charge conjugation matrix $C$ cannot be taken as unit matrix any more. Throughout this paper we consider the four-dimensional Euclidean spacetime.

As in the two-dimensional case we identify the extended SUSY suffix $\{i\}$ as spinor suffix of a diagonal subgroup $SO(4) \times SO(4)$. Analogous to the Dirac-Kähler mechanism, we can expand $Q_{\alpha i}$ as:

$$Q_{\alpha i} = \frac{1}{\sqrt{2}} \left( 1 s + \gamma^\mu s_\mu + \frac{1}{2} \gamma^{\mu\nu} s_{\mu\nu} + \frac{1}{3!} \gamma^{\mu\nu\rho} s_{\mu\nu\rho} + \frac{1}{4!} \gamma^{\mu\nu\rho\sigma} s_{\mu\nu\rho\sigma} \right)_{\alpha i}$$

where $\tilde{s}_\mu \equiv \frac{1}{3!} \epsilon_{\mu\nu\rho\sigma} s^{\nu\rho\sigma}$ and $\tilde{s} \equiv \frac{1}{4!} \epsilon^{\mu\nu\rho\sigma} s_{\mu\nu\rho\sigma}$. We define the following conjugate supercharge:

$$\overline{Q}_{i\alpha} = (C^{-1} Q^T C)_{i\alpha},$$

where $C$ is the charge conjugation matrix in four-dimensional Euclidean spacetime and satisfies

$$\gamma_\mu^T = C \gamma_\mu C^{-1}, \quad C^T = -C.$$ 

(2.12)

We give explicit representation of the $\gamma$-matrix and the charge conjugation matrix in four dimensions and useful relations in Appendix A.

The algebra (2.1) may be equivalently rewritten as

$$\{Q_{\alpha i}, Q_{\beta j}\} = 2 C_{ji} (\gamma^\nu C^{-1})_{\alpha\beta} P_\nu.$$ 

(2.13)

The twisted supercharges $\{s_I\}$ are related to the supercharges $Q_{\alpha i}$ of $N = 4$ extended SUSY as:

$$s = \frac{1}{2\sqrt{2}} \text{Tr} Q,$$

$$s_\mu = \frac{1}{2\sqrt{2}} \text{Tr} (Q \gamma_\mu),$$

$$s_{\mu\nu} = -\frac{1}{2\sqrt{2}} \text{Tr} (Q \gamma_{\mu\nu}),$$

$$\tilde{s}_\mu = -\frac{1}{2\sqrt{2}} \text{Tr} (Q \tilde{\gamma}_\mu),$$

$$\tilde{s} = \frac{1}{2\sqrt{2}} \text{Tr} (Q \gamma_5).$$

(2.14)

We can then explicitly calculate the commutation relations of supercharges $\{s_I\}$

$$\{s, s_\mu\} = P_\mu, \quad \{s_\mu, s_\rho\} = -\left( \delta_{\mu\rho} P_\sigma - \delta_{\mu\sigma} P_\rho \right),$$

$$\{\tilde{s}, s_\mu\} = P_\mu, \quad \{\tilde{s}_\mu, s_\rho\} = \epsilon_{\mu\rho\sigma\nu} P^\nu,$$

$$\{s, s_{\mu\nu}\} = \{s, \tilde{s}_\mu\} = \{s, \tilde{s}\} = \{s_\mu, \tilde{s}_\nu\} = \{s_\mu, \tilde{s}\} = \{s_{\mu\nu}, \tilde{s}\} = 0,$$

$$\{s, s\} = \{s_\mu, s_\nu\} = \{s_{\mu\nu}, s_{\rho\sigma}\} = \{\tilde{s}_\mu, \tilde{s}_\nu\} = \{\tilde{s}, \tilde{s}\} = 0.$$ 

(2.15)
Here we introduce Grassmann odd twisted SUSY parameters corresponding to the supercharges (2.10)
\[
\theta_{\alpha i} = \frac{1}{2\sqrt{2}} \left( \theta + \gamma^\mu \theta_\mu + \frac{1}{2} \gamma^{\mu\nu} \theta_{\mu\nu} + \tilde{\gamma}^\mu \tilde{\theta}_\mu + \gamma_5 \tilde{\theta} \right)_{\alpha i}.
\] (2.16)

Then the explicit form of $N = 4$ twisted SUSY generator can be given by
\[
\delta \theta = \overline{\theta}_{\alpha i} Q_{\alpha i} = \theta s + \theta^\mu s_\mu + \frac{1}{2} \theta^{\mu\nu} s_{\mu\nu} + \tilde{\theta}^\mu \tilde{s}_\mu + \tilde{\theta} \tilde{s},
\] (2.17)

where $\theta$ has the same relation as (2.10) with $\theta$:
\[
\theta_{\alpha i} = (C^{-1} \theta^T C)_{\alpha i}.
\]

Next we consider super-Poincaré and $R$-rotation of $N = 4$ SUSY algebra,
\[
\begin{align*}
\{P_\mu, P_\nu\} &= 0, \\
\{Q_{\alpha i}, P_\mu\} &= 0, \\
\{M_{\mu\nu}, M_{\rho\sigma}\} &= -i(\delta_{\mu\rho} M_{\nu\sigma} - \delta_{\nu\rho} M_{\mu\sigma} - \delta_{\mu\sigma} M_{\nu\rho} + \delta_{\nu\sigma} M_{\mu\rho}), \\
\{P_\mu, M_{\rho\sigma}\} &= i(\delta_{\mu\rho} P_\sigma - \delta_{\mu\sigma} P_\rho), \\
\{Q_{\alpha i}, M_{\mu\nu}\} &= \frac{i}{2}(\gamma_{\mu\nu})_{\alpha\beta} Q_{\beta i}, \tag{2.18} \\
\{R_{\mu\nu}, R_{\rho\sigma}\} &= -i(\delta_{\mu\rho} R_{\nu\sigma} - \delta_{\nu\rho} R_{\mu\sigma} - \delta_{\mu\sigma} R_{\nu\rho} + \delta_{\nu\sigma} R_{\mu\rho}), \\
\{Q_{\alpha i}, R_{\mu\nu}\} &= \frac{i}{2}(\Gamma_{\mu\nu})_{ij} Q_{\alpha j}, \\
\{R_{\mu\nu}, P_\rho\} &= 0, \\
\{R_{\mu\nu}, M_{\rho\sigma}\} &= 0,
\end{align*}
\]

where $M_{\mu\nu}$ is $SO(4)$ Lorentz generator and $R_{\mu\nu}$ is $SO(4)_I$ internal space rotation generator called $R$-symmetry. $\Gamma_\mu$ satisfies the same Clifford algebra as $\gamma_\mu$ and can be identified as the rotation generators of spinor suffix of $SO(4)_I$.

We introduce the Dirac-Kähler twisting procedure where we identify the internal SUSY suffix $\{i\}$ as spinor suffix of $SO(4)_I$. We can then identify $\Gamma_\mu^\mu (\gamma_\mu)_{\alpha\beta}$ in particular $\Gamma_\mu^\mu = (\gamma_\mu)^T$. In this case the twisted algebras do not have the Lorentz invariance of original Lorentz group any more. For the $R$-rotation we obtain
\[
\{Q_{\alpha i}, R_{\mu\nu}\} = \frac{i}{2}(\Gamma_{\mu\nu})_{ij} Q_{\alpha j} \\
\rightarrow [Q_{\alpha i}, R_{\mu\nu}] = -\frac{i}{2}(\gamma_{\mu\nu})_{ij} Q_{\alpha j} \\
= -\frac{i}{2} Q_{\alpha j}(\gamma_{\mu\nu})_{ji}. \tag{2.19}
\]

Substituting $Q_{\alpha i}$ into (2.10), we obtain the following algebraic relations:
\[
\begin{align*}
[s, R_{\mu\nu}] &= \frac{i}{2} s_{\mu\nu}, \\
[s_\rho, R_{\mu\nu}] &= \frac{i}{2}(\delta_{\mu\rho} s_\nu - \delta_{\nu\rho} s_\mu) - \frac{i}{2} \epsilon_{\mu\rho\sigma} \tilde{s}_\sigma,
\end{align*}
\]
\[ [s_{\rho\sigma}, R_{\mu\nu}] = \frac{-i}{2} (\delta_{\mu\rho} \delta_{\nu\sigma} - \delta_{\nu\rho} \delta_{\mu\sigma}) s - \frac{i}{2} (\delta_{\mu\rho} s_{\sigma\nu} - \delta_{\mu\sigma} s_{\rho\nu} - \delta_{\nu\rho} s_{\mu\sigma} + \delta_{\nu\sigma} s_{\rho\mu}) + \frac{i}{2} \epsilon_{\mu\nu\rho\sigma} \tilde{s}, \]
\[ [\tilde{s}_\rho, R_{\mu\nu}] = -\frac{i}{2} \epsilon_{\mu\nu\rho\sigma} s^\sigma + \frac{i}{2} (\delta_{\mu\rho} \tilde{s}_\nu - \delta_{\nu\rho} \tilde{s}_\mu), \]
\[ [\tilde{s}, R_{\mu\nu}] = -\frac{i}{4} \epsilon_{\mu\nu\rho\sigma} s^{\rho\sigma}. \] (2.20)

Other algebraic relations can be obtained in the similar way. It should be noted that \( \{ s_I \} \) are not tensors under the original Lorentz transformation. They are, however, tensors of newly defined Lorentz generators,

\[ M'_{\mu\nu} = M_{\mu\nu} + R_{\mu\nu}. \] (2.21)

Using the new Lorentz generator \( M'_{\mu\nu} \), we can represent the super-Poincaré and \( R \)-symmetry part of \( N = 4 \) twisted SUSY algebras as follows:

\[ [P_\mu, P_\nu] = 0, \]
\[ [s, P_\rho] = [s_\mu, P_\rho] = [s_{\mu\nu}, P_\rho] = [\tilde{s}_\mu, P_\rho] = [\tilde{s}, P_\rho] = 0, \]
\[ [M'_{\mu\nu}, M'_{\rho\sigma}] = -i (\delta_{\mu\rho} M'_{\nu\sigma} - \delta_{\nu\rho} M'_{\mu\sigma} - \delta_{\mu\sigma} M'_{\rho\nu} + \delta_{\nu\sigma} M'_{\rho\mu}), \]
\[ [P_\mu, M'_{\rho\sigma}] = i (\delta_{\mu\rho} P_\sigma - \delta_{\mu\sigma} P_\rho), \]
\[ [s, M'_{\rho\sigma}] = [\tilde{s}_\mu, M'_{\rho\sigma}] = 0, \]
\[ [s_{\rho\sigma}, M'_{\mu\nu}] = -i (\delta_{\mu\rho} s_{\sigma\nu} - \delta_{\mu\sigma} s_{\rho\nu} - \delta_{\nu\rho} s_{\sigma\mu} + \delta_{\nu\sigma} s_{\rho\mu}), \]
\[ [R_{\mu\nu}, R_{\rho\sigma}] = -i (\delta_{\mu\rho} R_{\nu\sigma} - \delta_{\nu\rho} R_{\mu\sigma} - \delta_{\mu\sigma} R_{\nu\rho} + \delta_{\nu\sigma} R_{\mu\rho}), \]
\[ [s, R_{\mu\nu}] = \frac{i}{2} \tilde{s}_{\mu\nu}, \]
\[ [s_\rho, R_{\mu\nu}] = \frac{i}{2} (\delta_{\rho\nu} \tilde{s}_\mu - \delta_{\rho\mu} \tilde{s}_\nu) - \frac{i}{2} \epsilon_{\mu\nu\rho\sigma} \tilde{s}^\sigma, \]
\[ [s_{\rho\sigma}, R_{\mu\nu}] = -\frac{i}{2} (\delta_{\mu\rho} s_{\sigma\nu} - \delta_{\mu\sigma} s_{\rho\nu} - \delta_{\nu\rho} s_{\sigma\mu} + \delta_{\nu\sigma} s_{\rho\mu}) + \frac{i}{2} \epsilon_{\mu\nu\rho\sigma} \tilde{s}, \]
\[ [\tilde{s}_\rho, R_{\mu\nu}] = -\frac{i}{2} \epsilon_{\mu\nu\rho\sigma} s^\sigma + \frac{i}{2} (\delta_{\mu\rho} \tilde{s}_\nu - \delta_{\nu\rho} \tilde{s}_\mu), \]
\[ [\tilde{s}, R_{\mu\nu}] = -\frac{i}{4} \epsilon_{\mu\nu\rho\sigma} s^{\rho\sigma}, \]
\[ [R_{\mu\nu}, P_\rho] = 0, \]
\[ [R_{\mu\nu}, M'_{\rho\sigma}] = -i (\delta_{\mu\rho} R_{\nu\sigma} - \delta_{\nu\rho} R_{\mu\sigma} - \delta_{\mu\sigma} R_{\nu\rho} + \delta_{\nu\sigma} R_{\mu\rho}). \] (2.22)

As we can see in this algebra, \( s, s_\mu, s_{\mu\nu}, \tilde{s}_\nu, \tilde{s} \) transform as scalar, vector, tensor, pseudo-vector and pseudo-scalar, respectively under the new Lorentz generator \( M'_{\mu\nu} \).

Next we consider how the relation (2.21) has an effect on the transformation law.
of component fields. We define the Dirac-Kähler field as follows \[42–49\],

$$\Psi_{\alpha i} = \frac{1}{\sqrt{2}} \left( \psi + \gamma^\mu \psi_\mu + \frac{1}{2} \gamma^{\mu\nu} \psi_{\mu\nu} + \tilde{\gamma}^\mu \tilde{\psi}_\mu + \gamma^5 \tilde{\psi} \right)_{\alpha i}, \quad (2.23)$$

where \(\Psi_{\alpha i}\) appears in the \(N = 4\) extended supersymmetric theory while \(\{\psi, \psi_\mu, \psi_{\mu\nu}, \tilde{\psi}_\mu, \tilde{\psi}\}\) appear in the twisted \(N = 4\) supersymmetric theory. As \(M'\) is the Lorentz generator in the twisted theory, we can define the transformation laws of \(\{\psi, \psi_\mu, \psi_{\mu\nu}, \tilde{\psi}_\mu, \tilde{\psi}\}\) as

$$\delta_{M'} \psi = 0, \quad \delta_{M'} \psi_\mu = 2ik_{\mu\nu} \psi^\nu, \quad \delta_{M'} \psi_{\mu\nu} = -2i(k_{\mu}^\rho \psi_{\nu\rho} - k_{\nu}^\rho \psi_{\mu\rho}), \quad \delta_{M'} \tilde{\psi}_\mu = 2ik_{\mu\nu} \tilde{\psi}^\nu, \quad \delta_{M'} \tilde{\psi} = 0, \quad (2.24)$$

where \(k_{\mu\nu}\) is bosonic anti-symmetric tensor. Therefore the Dirac-Kähler field transforms in the following form:

$$\delta_{M'} \Psi = \frac{i}{2} [k_{\mu\nu} \gamma^{\mu\nu}, \Psi]. \quad (2.25)$$

We can also define the R-symmetry transformation law of these fields, \(\delta_R\), as

$$\delta_R \psi = i 2k_{\mu\nu} \psi_{\mu\nu}, \quad \delta_R \psi_\mu = ik_{\mu\nu} \psi^\nu - \frac{i}{2} k^{\rho\sigma} \epsilon_{\rho\sigma\mu\nu} \tilde{\psi}^\nu, \quad \delta_R \psi_{\mu\nu} = -ik_{\mu\nu} \psi - i(k_{\mu}^\rho \psi_{\nu\rho} - k_{\nu}^\rho \psi_{\mu\rho}) + \frac{i}{2} k^{\rho\sigma} \epsilon_{\rho\sigma\mu\nu} \tilde{\psi}, \quad \delta_R \tilde{\psi}_\mu = -\frac{i}{2} k^{\rho\sigma} \epsilon_{\rho\sigma\mu\nu} \psi^\nu + ik_{\mu\nu} \tilde{\psi}^\nu, \quad \delta_R \tilde{\psi} = -\frac{i}{4} k^{\rho\sigma} \epsilon_{\mu\sigma\rho\nu} \psi^{\mu\nu}. \quad (2.26)$$

The Dirac-Kähler field transforms under the R-symmetry transformation in the following:

$$\delta_R \Psi = -\frac{i}{2} \Psi k_{\rho\sigma} \gamma^{\rho\sigma}. \quad (2.27)$$

Here we may replace \(\gamma_{\mu\nu}\) by \(\Gamma_{\mu\nu}\) and then

$$\delta_R \Psi_{\alpha i} = \frac{i}{2} k_{\rho\sigma} (\Gamma^{\rho\sigma})_{ij} \Psi_{\alpha j}. \quad (2.28)$$
This transformation is the R-symmetry transformation of spinor field with $SO(4)_I$ internal symmetry. On the other hand the Lorentz transformation induced by $M = M' - R$ is

$$
\delta_M \Psi = \delta_{M'} \Psi - \delta_R \Psi
= \frac{i}{2} [k_{\mu \nu} \gamma^{\mu \nu}, \Psi] + \frac{i}{2} \Psi k_{\rho \sigma} \gamma^{\rho \sigma}
= \frac{i}{2} k_{\mu \nu} \gamma^{\mu \nu} \Psi,
$$

(2.29)

which precisely coincides with the Lorentz transformation of a spinor field. Thus the R-symmetry generator $R$ plays the role of shifting the integer spin ghost-related fermions into the half integer spin matter fermions. This shows that Dirac-Kähler fermion mechanism is essentially related to the twisting procedure of $N = 4$ extended SUSY in four dimensions just like the two-dimensional case [41, 50].

3 Twisted superspace and superfield

In the Dirac-Kähler twisting procedure we have introduced the twisted supercharges in the following form:

$$
Q_{\alpha i} = a_D \left( \gamma^I s_I \right)_{\alpha i},
$$

(3.30)

where the normalization constant $a_D$ has a dimension dependence in our notation:

$$
\begin{align*}
&\begin{cases}
a_2 = 1 & (N = 2 \text{ in two dimensions}), \\
al_4 = \frac{1}{\sqrt{2}} & (N = 4 \text{ in four dimensions}),
\end{cases} \\
&\left\{ 
\right.
\end{align*}
$$

(3.31)

We then introduce the corresponding twisted superparameters as:

$$
\theta_{\alpha i} = \frac{a_D}{2} \left( \gamma^I \theta_I \right)_{\alpha i}.
$$

(3.32)

Then we can define a twisted SUSY transformation as

$$
\delta_{\theta} = \bar{\theta}_{\alpha i} Q_{\alpha i} = \bar{\theta}_I s_I.
$$

(3.33)

We can then define the following supergroup element acting on a twisted superspace:

$$
G(x^\mu, \theta_A) = e^{i(-x^\mu P_\mu + \delta_\theta)}.
$$

(3.34)

$N = D = 2$ and $N = D = 4$ twisted superspace of extended SUSY is defined in the parameter space of $(x^\mu, \theta_I)$.

By using the relations (2.3) or (2.15) and Baker-Hausdorff formula, we can obtain the following relation:

$$
G(0, \xi_I) \ G(x^\mu, \theta_I) = G(x^\mu + b^\mu, \theta_I + \xi_I),
$$

(3.35)
where

\[ b^\mu = \frac{i}{2} \xi \theta^\mu + \frac{i}{2} \xi^\mu \theta + \frac{i}{2} e^{\mu \nu} \xi^\nu \tilde{\theta} + \frac{i}{2} e^{\mu \nu} \xi \tilde{\theta}^\nu \]

\[(N = 2 \text{ in two dimensions}),\]

\[ b^\mu = \frac{i}{2} \xi \theta^\mu + \frac{i}{2} \xi^\mu \theta - \frac{i}{2} \xi^\nu \theta^\mu - \frac{i}{4} e^{\mu \nu \rho \sigma} \xi_\nu \theta_{\rho \sigma} - \frac{i}{4} e^{\mu \nu \rho \sigma} \xi_{\rho \sigma} \tilde{\theta}_\nu + \frac{i}{2} \xi^\mu + \frac{i}{2} \xi \tilde{\theta}^\mu \]

\[(N = 4 \text{ in four dimensions}).\]

(3.36)

This multiplication induces a shift transformation in superspace \((x^\mu, \{\theta_I\})\):

\[(x^\mu, \{\theta_I\}) \rightarrow (x^\mu + b^\mu, \{\theta_I + \xi_I\}),\]

which is in general generated by supercharge differential operators \(Q_A\). Accordingly supercharge differential operators are introduced to satisfy the following parameter shifts:

\[ \delta \xi (x^\mu, \{\theta_I\}) = (\xi_I Q_I) x^\mu, \]

\[ \delta \xi \phi (x^\mu, \{\theta_I\}) = (\xi_I Q_I) \phi. \]

We introduce a general superfield \(\Upsilon(x, \{\theta_I\})\),

\[ \Upsilon(x, \{\theta_I\}) = \phi + \theta_I \phi_I + \frac{1}{2} \theta_I \theta_J \phi_{IJ} + \cdots. \]

(3.39)

We then define twisted SUSY transformation of component fields as follows:

\[ \delta \xi \Upsilon(x, \{\theta_I\}) = \delta \xi \phi + \theta_I \delta \xi \phi_I + \frac{1}{2} \theta_I \theta_J \delta \xi \phi_{IJ} + \cdots \]

\[ \equiv (\xi_I Q_I) \Upsilon(x, \{\theta_I\}), \]

where \(\delta \xi = \xi_I s_I\). It should be noted that this operator applies not on the superfield but on component fields as well. We thus obtain the full SUSY transformation law of component fields in principle. In general superfield is, however, highly reducible and thus we need to introduce a constraint on the general superfield. Once we know the SUSY transformation of the component fields by any means, we can construct the superfield itself as follows:

\[ \Upsilon(x, \{\theta_I\}) = e^{\delta \phi} \]

\[ = \phi + \delta \phi + \frac{1}{2} (\delta \phi)^2 + \cdots, \]

(3.41)

where \(\phi\) is the parent field to generate other component fields in the same twisted supermultiplet.
### 3.1 $N = 2$ twisted superspace in two dimensions

The twisted $N = 2$ supercharge differential operators generating the parameter shift (3.37) and (3.38) are given by

\[
\begin{align*}
Q &= \frac{\partial}{\partial \theta} + \frac{i}{2} \theta \mu \partial_\mu, \\
Q_\mu &= \frac{\partial}{\partial \theta^\mu} + \frac{i}{2} \theta \partial_\mu - \frac{i}{2} \bar{\theta} \epsilon_{\mu\nu} \partial^\nu, \\
\bar{Q} &= \frac{\partial}{\partial \bar{\theta}} - \frac{i}{2} \theta \mu \epsilon_{\mu\nu} \partial^\nu,
\end{align*}
\]  

which satisfy the following $N = 2$ twisted superalgebra:

\[
\begin{align*}
\{Q, Q_\mu\} &= i \partial_\mu, \\
\{\bar{Q}, Q_\mu\} &= -i \epsilon_{\mu\nu} \partial^\nu, \\
\{Q_\mu, Q_\nu\} &= \{Q, \bar{Q}\} = \bar{Q}^2 = Q^2 = 0.
\end{align*}
\]  

(3.43)

It should be note that the sign of spacetime derivative is reversed with respect to the original algebra (2.3) due to the reverse order in operation. We introduce superderivative differential operators \{\mathcal{D}_I\} which anticommute with \{Q_I\},

\[
\{Q^I, \mathcal{D}^J\} = 0.
\]  

(3.44)

We find

\[
\begin{align*}
\mathcal{D} &= \frac{\partial}{\partial \theta} - \frac{i}{2} \theta \mu \partial_\mu, \\
\mathcal{D}_\mu &= \frac{\partial}{\partial \theta^\mu} - \frac{i}{2} \theta \partial_\mu + \frac{i}{2} \bar{\theta} \epsilon_{\mu\nu} \partial^\nu, \\
\tilde{\mathcal{D}} &= \frac{\partial}{\partial \bar{\theta}} + \frac{i}{2} \theta \mu \epsilon_{\mu\nu} \partial^\nu.
\end{align*}
\]  

(3.45)

We can see that these operators satisfy the following algebra:

\[
\begin{align*}
\{\mathcal{D}, \mathcal{D}_\mu\} &= -i \partial_\mu, \\
\{\tilde{\mathcal{D}}, \mathcal{D}_\mu\} &= i \epsilon_{\mu\nu} \partial^\nu, \\
\{\mathcal{D}_\mu, \mathcal{D}_\nu\} &= \{\mathcal{D}, \tilde{\mathcal{D}}\} = \mathcal{D}^2 = \tilde{\mathcal{D}}^2 = 0.
\end{align*}
\]  

(3.46)

which has the same algebraic structure as (2.3). Similarly we can derive the twisted angular momentum and $R$-symmetry differential operators which satisfy the $N = 2$ twisted SUSY algebra (2.9):

\[
\begin{align*}
J' &= i \epsilon^{\mu\nu} x_\mu \partial_\nu + i \epsilon^{\mu\nu} \theta_\mu \frac{\partial}{\partial \theta^\nu}, \\
R &= -\frac{i}{2} \frac{\partial}{\partial \theta} + \frac{i}{2} \bar{\theta} \frac{\partial}{\partial \theta} + \frac{i}{2} \epsilon^{\mu\nu} \theta_\mu \frac{\partial}{\partial \theta^\nu},
\end{align*}
\]  

(3.47)

where

\[
J' = J + R.
\]  

(3.48)
3.2 $N=4$ twisted superspace in four dimensions

In the previous section we have derived $N=4$ twisted SUSY algebra based on the Dirac-Kähler twisting procedure. In this section we will construct a $N=4$ twisted superspace formalism. The $N=4$ twisted supercharges satisfy the algebra (2.15):

\[
\{s, s_\mu\} = -i \partial_\mu, \quad \{s_\mu, s_\rho\sigma\} = i (\delta_\mu_\rho \partial_\sigma - \delta_\mu_\sigma \partial_\rho) \equiv i \mathcal{D}_\mu, \rho \sigma, \\
\{\tilde{s}, \tilde{s}_\mu\} = -i \partial_\mu, \quad \{\tilde{s}_\mu, s_\rho\sigma\} = -i \epsilon_{\rho\sigma\mu\nu} \partial_\nu, \\
\{\text{others}\} = 0, \tag{3.49}
\]

where we have introduced an explicit representation of momentum $P_\mu = -i \partial_\mu$. We construct superspace formalism based on this $N=4$ twisted SUSY algebra.

The twisted supercharge differential operators generating the parameter shifts (3.37) and (3.38) for the $N=4$ twisted superspace are given by

\[
\begin{align*}
Q & = \frac{\partial}{\partial \theta} + i \frac{\theta_\mu}{2} \partial_\mu, \\
Q_\mu & = \frac{\partial}{\partial \theta^{\mu}} + i \frac{\theta_\sigma}{2} \partial_\sigma - i \frac{1}{2} \theta_\rho \epsilon_{\rho\sigma\mu\nu} \partial_\nu, \\
Q_\rho\sigma & = \frac{\partial}{\partial \theta^{\rho\sigma}} - \frac{i}{2} (\theta_\rho \partial_\sigma - \theta_\sigma \partial_\rho) + i \frac{1}{2} \theta_\nu \epsilon_{\rho\sigma\nu\mu} \partial_\mu, \\
\tilde{Q}_\mu & = \frac{\partial}{\partial \theta^{\mu}} + i \frac{\theta_\rho \epsilon_{\rho\sigma\nu\mu} \partial_\nu + i \frac{1}{2} \theta_\mu \partial_\mu, \\
\tilde{Q} & = \frac{\partial}{\partial \tilde{\theta}} + i \frac{\tilde{\theta}^\mu}{2} \partial_\mu. \tag{3.50}
\end{align*}
\]

These differential operators satisfy the following relations:

\[
\begin{align*}
\{Q, Q_\mu\} & = i \partial_\mu, \quad \{Q_\mu, Q_\rho\sigma\} = -i \mathcal{D}_\mu, \rho \sigma, \\
\{\tilde{Q}, \tilde{Q}_\mu\} & = i \partial_\mu, \quad \{\tilde{Q}_\mu, Q_\rho\sigma\} = i \epsilon_{\rho\sigma\mu\nu} \partial_\nu, \\
\{\text{others}\} & = 0, \tag{3.51}
\end{align*}
\]

where we have introduced the following notations:

\[
\begin{align*}
\mathcal{D}_\mu, \rho \sigma & \equiv \delta_\mu_\rho \partial_\sigma - \delta_\mu_\sigma \partial_\rho = \delta_\mu_\nu, \rho \sigma \partial_\nu, \\
\delta_\mu_\nu, \rho \sigma & \equiv \delta_\mu_\rho \delta_\nu_\sigma - \delta_\mu_\sigma \delta_\nu_\rho, \\
\frac{\partial \theta^{\mu\nu}}{\partial \theta^{\rho\sigma}} & \equiv \delta_\mu^\rho \delta_\nu_\sigma - \delta_\mu^\nu \delta_\rho_\sigma = \delta^{\mu\nu}_{\rho\sigma}. \tag{3.52}
\end{align*}
\]

Next we introduce superderivative differential operators $\{\mathcal{D}_I\}$ which anticommute with all differential operators $\{Q_I\}$:

\[
\begin{align*}
\mathcal{D} & = \frac{\partial}{\partial \theta} - i \frac{\theta_\mu}{2} \partial_\mu, \\
\mathcal{D}_\mu & = \frac{\partial}{\partial \theta^{\mu}} - i \frac{\theta_\rho}{2} \partial_\rho + i \frac{1}{2} \theta_\mu \partial_\mu, 
\end{align*}
\]
We use these operators \( \{ \mathcal{D}_I \} \) to impose chiral and anti-chiral conditions and to reduce the unnecessary degrees of freedom in a twisted superfield. The differential operator of the twisted Lorentz generator \( M'_\rho\sigma \) and the R-symmetry generator \( R_\rho\sigma \) are given by

\[
M'_\rho\sigma = -i\delta_{\rho\sigma,\alpha\beta}x^\alpha \partial^\beta - i\delta_{\rho\sigma,\alpha\beta}\theta^\alpha \frac{\partial}{\partial \theta^\beta} - i\delta_{\rho\sigma,\alpha\beta}\tilde{\theta}^\alpha \frac{\partial}{\partial \tilde{\theta}^\beta} - i\delta_{\rho\sigma,\alpha\beta}\theta^\alpha \partial^\beta - i\delta_{\rho\sigma,\alpha\beta}\tilde{\theta}^\alpha \partial^\beta - i\delta_{\rho\sigma,\alpha\beta}\theta^\alpha \partial^\beta,
\]

\[
R_\rho\sigma = -i\frac{1}{2}\epsilon_{\rho\sigma,\alpha\beta}\theta^\alpha \frac{\partial}{\partial \theta^\beta} + i\frac{1}{2}\epsilon_{\rho\sigma,\alpha\beta}\tilde{\theta}^\alpha \frac{\partial}{\partial \tilde{\theta}^\beta} + i\frac{1}{2}\epsilon_{\rho\sigma,\alpha\beta}\tilde{\theta}^\alpha \frac{\partial}{\partial \theta^\beta} + i\frac{1}{2}\theta^\alpha \frac{\partial}{\partial \theta},
\]

where the newly defined Lorentz generators \( M'_\rho\sigma \) and the original Lorentz generators \( M_\rho\sigma \) have the same relation as (2.21),

\[
M'_\rho\sigma = M_\rho\sigma + R_\rho\sigma.
\]

### 3.3 Chiral and anti-chiral superfields

We consider chiral superfield characterized by the following condition:

\[
\mathcal{D}_\rho\sigma \Psi(x^\mu, \{ \theta_I \}) = \mathcal{D}_\rho\sigma \Psi(x^\mu, \{ \theta_I \}) = \tilde{\mathcal{D}}_\rho\sigma \Psi(x^\mu, \{ \theta_I \}) = 0,
\]

where \( \Psi \) has all the \( \{ \theta_I \} = \{ \theta, \theta^\mu, \theta^{\mu\nu}, \tilde{\theta}, \tilde{\theta}^\mu \} \) dependence at this stage. The details of the twisted chiral superfield formulation for \( N = D = 2 \) can be found in [50, 63]. The reason why we call these condition as the chiral condition stems from the Dirac-Kähler fermion formulation on a lattice. The chiral transformation on a lattice is interpreted as an interchange between the even sites and odd sites which in turn can be translated into the interchange between even and odd differential forms. We may then identify the conditions in (3.30) as the chiral sector of even forms.

There are several possible treatments to solve this type of differential equations. It is convenient to rewrite the chiral and anti-chiral conditions by using the operator which satisfy the following relations for the differential operators:

\[
U \mathcal{D} U^{-1} = \frac{\partial}{\partial \theta}, \quad U \mathcal{D}_\rho\sigma U^{-1} = \frac{\partial}{\partial \theta^\rho\sigma}, \quad U \tilde{\mathcal{D}} U^{-1} = \frac{\partial}{\partial \tilde{\theta}},
\]

\[
(3.57)
\]
\[ U^{-1} \mathfrak{D}_\mu U = \frac{\partial}{\partial \theta^\mu}, \quad U^{-1} \tilde{\mathfrak{D}}_\mu U = \frac{\partial}{\partial \tilde{\theta}^\mu}, \] (3.58)

where

\[ U = e^{-a^\mu \partial_\mu}, \quad a^\mu = \frac{i}{2} \partial_\mu \theta^\mu + \frac{i}{2} \partial_\nu \theta^\nu + \frac{i}{4} \epsilon^{\mu \nu \rho \sigma} \tilde{\theta}_\nu \theta_\rho \sigma - \frac{i}{2} \tilde{\theta}^\mu \tilde{\theta}. \] (3.59)

Then the chiral conditions (3.56) can be transformed into

\[ \frac{\partial}{\partial \theta} \Psi(x^\mu, \{\theta_1\}) = 0, \quad \frac{\partial}{\partial \theta^\mu \rho} \Psi(x^\mu, \{\theta_1\}) = 0, \quad \frac{\partial}{\partial \theta} \Psi(x^\mu, \{\theta_1\}) = 0, \] (3.60)

which leads

\[ \Psi(x^\mu, \{\theta_1\}) = \Psi'(x^\mu, \theta^\mu, \tilde{\theta}^\mu). \] (3.61)

Then the solution for the original chiral condition (3.56) is obtained as

\[ \Psi(x^\mu, \{\theta_1\}) = U^{-1} \Psi'(x^\mu, \theta^\mu, \tilde{\theta}^\mu) = \Psi'(y^\mu, \theta^\mu, \tilde{\theta}^\mu), \] (3.62)

where

\[ y^\mu = x^\mu + a^\mu. \] (3.63)

This solution can be expanded as follows:

\[ \Psi'(y^\mu, \theta^\mu, \tilde{\theta}^\mu) = \Psi^0(y^\mu, \tilde{\theta}^\mu) + \theta^m \Psi_1^m(y^\mu, \tilde{\theta}^\mu) + \frac{1}{2} \theta^m \theta^n \Psi_2^{mn}(y^\mu, \tilde{\theta}^\mu) + (\tilde{\theta}^3)^m \Psi_3^m(y^\mu, \tilde{\theta}^\mu) + \theta^4 \Psi_4^4(y^\mu, \tilde{\theta}^\mu), \] (3.64)

where \( (\theta^3)^m = \frac{1}{3!} \theta^\mu \theta^\nu \theta^\rho \epsilon_{\mu \nu \rho.}^\ m \) and \( \theta^4 = \frac{1}{4!} \epsilon_{\mu \nu \rho \sigma} \theta^\mu \theta^\nu \theta^\rho \theta^\sigma \) with \( \{\Psi^0, \Psi_1^m, \Psi_2^{mn}, \Psi_3^m, \Psi_4^4\} \) defined by

\[
\begin{align*}
\Psi^0(y^\mu, \tilde{\theta}^\mu) & = \psi^0 + \tilde{\theta}^\mu \psi^0_{\mu}, \\
\Psi_1^m(y^\mu, \tilde{\theta}^\mu) & = \psi^1_{m,\mu} + \tilde{\theta}^\mu \psi^1_{\mu,\mu} + \frac{1}{2} \tilde{\theta}^\mu \tilde{\theta}^\nu \psi^1_{\mu,\nu} + (\tilde{\theta}^3)^\sigma \tilde{\psi}^0_{\sigma} + \tilde{\psi}^4 \tilde{\psi}^0, \\
\Psi_2^{mn}(y^\mu, \tilde{\theta}^\mu) & = \psi^2_{mn,\mu} + \tilde{\theta}^\mu \psi^2_{\mu,\mu} + \frac{1}{2} \tilde{\theta}^\mu \tilde{\theta}^\nu \psi^2_{\mu,\nu} + (\tilde{\theta}^3)^\sigma \tilde{\psi}^2_{mn,\sigma} + \tilde{\psi}^4 \tilde{\psi}^2_{mn,\sigma}, \\
\Psi_3^m(y^\mu, \tilde{\theta}^\mu) & = \psi^3_{m,\mu} + \tilde{\theta}^\mu \psi^3_{\mu,\mu} + \frac{1}{2} \tilde{\theta}^\mu \tilde{\theta}^\nu \psi^3_{\mu,\nu} + (\tilde{\theta}^3)^\sigma \tilde{\psi}^3_{m,\sigma} + \tilde{\psi}^4 \tilde{\psi}^3_{m,\sigma}, \\
\Psi_4^4(y^\mu, \tilde{\theta}^\mu) & = \psi^4 + \tilde{\theta}^\mu \psi^4_{\mu} + \frac{1}{2} \tilde{\theta}^\mu \tilde{\theta}^\nu \psi^4_{\mu,\nu} + (\tilde{\theta}^3)^\sigma \tilde{\psi}^4_{\sigma} + \tilde{\psi}^4 \tilde{\psi}^4,
\end{align*}
\]
The relation given in (3.62) can be understood also from the following relation:

\[
\Psi(x^\mu, \{\theta\}) = e^{\delta_\theta \psi_0(x)} = e^{(\theta^\mu s_\mu + \tilde{\theta}^\mu \tilde{s}_\mu)} e^{(\theta s + \frac{1}{2} \theta^{\mu\nu} s_{\mu\nu})} e^{a^\mu \partial_\mu \psi_0(x)}
\]

\[
= e^{(\theta^\mu s_\mu + \tilde{\theta}^\mu \tilde{s}_\mu)} e^{(\theta s + \frac{1}{2} \theta^{\mu\nu} s_{\mu\nu})} \psi_0(y)
\]

\[
= \Psi'(y^\mu, \theta^\mu, \tilde{\theta}^\mu),
\]

(3.65)

where we can recognize that the all the component fields can be identified as the fields obtained by operating the twisted supercharge \( s_\mu \) and \( \tilde{s}_\mu \) to the parent field \( \psi_0(y) \).

Anti-chiral conditions for a superfield are given by

\[
D^\mu \Phi(x^\mu, \{\theta\}) = \tilde{D}^\mu \Phi(x^\mu, \{\theta\}) = 0.
\]

(3.66)

Similar to the chiral condition (3.60), we can transform the original anti-chiral condition (3.66) into the following form:

\[
U^{-1} D^\mu U U^{-1} \Phi(x^\mu, \{\theta\}) = \frac{\partial}{\partial \theta^\mu} U^{-1} \Phi(x^\mu, \{\theta\}) = 0,
\]

\[
D^\mu U U^{-1} \Phi(x^\mu, \{\theta\}) = \frac{\partial}{\partial \tilde{\theta}^\mu} U^{-1} \Phi(x^\mu, \{\theta\}) = 0,
\]

(3.67)

which leads

\[
U^{-1} \Phi(x^\mu, \{\theta\}) = \Phi'(x^\mu, \theta, \theta^\rho, \tilde{\theta}).
\]

(3.68)

Then the solution for the original anti-chiral condition (3.66) is obtained as

\[
\Phi(x^\mu, \{\theta\}) = U \Phi'(x^\mu, \theta, \theta^\rho, \tilde{\theta}) = \Phi'(w^\mu, \theta, \theta^\rho, \tilde{\theta}),
\]

(3.69)

where

\[
w^\mu = x^\mu - a^\mu.
\]

(3.70)

This solution can be expanded as follows:

\[
\Phi'(w^\mu, \theta, \tilde{\theta}, \theta^{\mu\nu}) = \Phi^0(w^\mu, \theta^{\mu\nu}) + \theta \Phi^1(w^\mu, \theta^{\mu\nu}) + \tilde{\theta} \Phi^2(w^\mu, \theta^{\mu\nu}) + \theta \tilde{\theta} \Phi^3(w^\mu, \theta^{\mu\nu}),
\]

(3.71)

with

\[
\Phi^m( w^\mu, \theta^{\mu\nu} ) = \phi^m + \frac{1}{2} \theta^{ab} \phi^m_{ab} + \frac{1}{4! 2^2} \theta^{ab} \theta^{cd} \phi^m_{abcd} + \frac{1}{6! 2^4} \theta^{ab} \theta^{cd} \theta^{ef} \phi^m_{abcd,ef} + \frac{1}{4! 2^7} \theta^{ab} \theta^{cd} \theta^{ef} \theta^{gh} \Gamma_{abcd,ef,gh,ijkl} \phi^m_{ijkl} + \frac{1}{5! 2^6} \theta^{ab} \theta^{cd} \theta^{ef} \theta^{gh} \theta^{ij} \Gamma_{abcd,ef,gh,ijkl} \phi^m_{ijkl} + \frac{1}{6! 2^6} \theta^{ab} \theta^{cd} \theta^{ef} \theta^{gh} \theta^{ij} \theta^{kl} \Gamma_{abcd,ef,gh,ijkl} \phi^m_{ijkl},
\]

(3.72)
where \((m = 0, 1, 2, 3)\), \(\Gamma_{ab,cd,ef,gh,ij,kl}\) is like a six-dimensional totally anti-symmetric tensor. Independent degrees of freedom of the tensor is six because of the anti-symmetric nature of the suffix \(\{ab\}\). We give the definition of \(\Gamma_{ab,cd,ef,gh,ij,kl}\) and useful relations in the Appendix B.

Similar to the relation \(3.65\), the corresponding relation for anti-chiral field \(3.69\) can be understood from the following relation as well:

\[
\Phi(x^\mu, \{\theta\}) = e^{\delta \theta} \phi^0(x) = e^{(\theta s + \frac{1}{2} \theta \nu s_{\mu \nu} + \theta \delta)} e^{(\theta s \mu + \tilde{\theta} s_{\mu})} e^{\alpha \nu \partial_{\mu} \phi^0(x)}
\]

\[
= e^{(\theta s + \frac{1}{2} \theta \nu s_{\mu \nu} + \theta \delta)} e^{(\theta s \mu + \tilde{\theta} s_{\mu})} \phi^0(w)
\]

\[
= \Phi'(w^\mu, \theta, \theta^{\rho \sigma}, \tilde{\theta}),
\]

where we can again recognize that the all the component fields can be identified as the fields obtained by operating the twisted supercharge \(s, \tilde{s}\) and \(s_{\mu \nu}\) to the parent field \(\phi^0(w)\).

The SUSY transformations of the chiral and anti-chiral superfields are given by

\[
s_I \Psi(x^\mu, \{\theta\}) = Q_I \Psi(x^\mu, \{\theta\}),
\]

\[
s_I \Phi(x^\mu, \{\theta\}) = Q_I \Phi(x^\mu, \{\theta\}),
\]

where \(\{s_I\} = \{s, s_{\mu \nu}, \tilde{s}_{\mu}, \tilde{s}\}\) and \(\{Q_I\} = \{Q, Q_{\mu}, Q_{\mu \nu}, Q_{\mu}, Q\}\).

These SUSY transformation can be transformed into the following form by using the operator \(U\):

\[
s_I U \Psi(x^\mu, \{\theta\}) = U Q_I U^{-1} U \Psi(x^\mu, \{\theta\}),
\]

\[
s_I U^{-1} \Phi(x^\mu, \{\theta\}) = U^{-1} Q_I U U^{-1} \Phi(x^\mu, \{\theta\}),
\]

which can be equivalently written as

\[
s_I \Psi'(x^\mu, \theta^\mu, \tilde{\theta}^\mu) = Q_I' \Psi'(x^\mu, \theta^\mu, \tilde{\theta}^\mu),
\]

\[
s_I \Phi'(x^\mu, \theta, \theta^{\rho \sigma}, \tilde{\theta}) = Q''_I \Phi'(x^\mu, \theta, \theta^{\rho \sigma}, \tilde{\theta}),
\]

where \(Q'_I = U Q_I U^{-1}\) and \(Q''_I = U^{-1} Q_I U\) are given by

\[
Q' = \frac{\partial}{\partial \theta} + i \theta^\mu \partial_\mu, \quad \tilde{Q'} = \frac{\partial}{\partial \tilde{\theta}} + i \tilde{\theta}^\mu \partial_\mu,
\]

\[
Q''_\mu = \frac{\partial}{\partial \theta_\mu}, \quad \tilde{Q''}_\mu = \frac{\partial}{\partial \tilde{\theta}_\mu},
\]

\[
Q''_{\rho \sigma} = \frac{\partial}{\partial \theta_{\rho \sigma}} - i (\theta_\rho \partial_{\sigma} - \theta_{\sigma} \partial_\rho) + i \tilde{\theta}^\mu \epsilon_{\mu \rho \sigma} \partial'_{\nu},
\]

\[
Q'' = \frac{\partial}{\partial \theta}, \quad \tilde{Q}'' = \frac{\partial}{\partial \tilde{\theta}}, \quad Q''_{\rho \sigma} = \frac{\partial}{\partial \theta_{\rho \sigma}},
\]

\[
Q'_{\mu} = \frac{\partial}{\partial \theta_\mu} + i \theta \partial_\mu - i \theta_{\mu \nu} \partial'_{\nu},
\]

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\[ \tilde{Q}''_{\mu} = \frac{\partial}{\partial \theta} + i \theta^{\mu} \partial_{\mu}, \quad \tilde{Q}'_{\mu} = \frac{\partial}{\partial \tilde{\theta}} - i \theta^{\mu} \epsilon_{\mu \nu \rho} \partial^{\nu} + i \tilde{\theta} \partial^{\mu}. \] (3.78)

The explicit form of the \( N = 4 \) twisted SUSY transformation of the component fields for chiral and anti-chiral superfield are given in the Appendix C.

4 Twisted SUSY invariant actions

Based on the twisted superspace formalism given in the previous section and further introduction of twisted vector superfield we derive variety of SUSY invariant actions.

4.1 Off-shell \( N = 4 \) twisted SUSY invariant action

We first show that a naive extension of the \( N = D = 2 \) twisted SUSY invariant action to \( N = 4 \) in four dimensions leads to an action with a lengthy expression. This is, however, a first nontrivial example of an action which has off-shell \( N = 4 \) twisted SUSY invariance.

The chiral and anti-chiral conditions for \( N = 2 \) twisted superfield in two dimensions can, respectively, be given by the following conditions:

\[ \mathcal{D} \Psi(x^{\mu}, \theta, \tilde{\theta}, \theta^{\mu}) = 0, \quad \tilde{\mathcal{D}} \Psi(x^{\mu}, \theta, \tilde{\theta}, \theta^{\mu}) = 0 \] (4.1)

and

\[ \mathcal{D}_{\mu} \overline{\Psi}(x^{\mu}, \theta, \tilde{\theta}, \theta^{\mu}) = 0. \] (4.2)

Parallel to the formulation given in the previous subsections, we can obtain fermionic chiral and anti-chiral superfields as

\[ \Psi(x^{\mu}, \theta, \tilde{\theta}, \theta^{\mu}) = \Psi'(y^{\mu}, \theta^{\mu}) = i e^{\theta^{\mu} s_{\mu}} c(y) = i c(y) + \theta^{\mu} \omega_{\mu}(y) + i \theta^{2} \lambda(y), \]
\[ \overline{\Psi}(x^{\mu}, \theta, \tilde{\theta}, \theta^{\mu}) = \overline{\Psi}(w^{\mu}, \theta, \tilde{\theta}) = i e^{\theta^{s} + \tilde{s}_{\mu}} \tau(w) = i \tau(w) + \theta b(w) + \tilde{\theta} \phi(w) - i \tilde{\theta} \rho(w), \] (4.3)

where

\[ y^{\mu} = x^{\mu} + \frac{i}{2} \theta^{\mu} \theta^{\nu} - \frac{i}{2} \epsilon^{\mu \nu \rho} \theta^{\rho} \tilde{\theta}, \]
\[ w^{\mu} = x^{\mu} - \frac{i}{2} \theta^{\mu} \theta^{\nu} + \frac{i}{2} \epsilon^{\mu \nu \rho} \theta^{\rho} \tilde{\theta}. \] (4.4)

Here the parent fields \( c(y) \) and \( \tau(w) \) of chiral and anti-chiral superfields are Grassmann odd fields. The two-dimensional counterparts of the supercharge operators \( Q' \) in (3.77) and \( Q'' \) in (3.78) are given by

\[ Q' = \frac{\partial}{\partial \theta} + i \theta^{\mu} \partial_{\mu}, \quad \tilde{Q}' = \frac{\partial}{\partial \tilde{\theta}} - i \theta^{\mu} \epsilon_{\mu \nu \rho} \partial^{\nu}, \quad Q''_{\mu} = \frac{\partial}{\partial \theta^{\mu}}, \] (4.5)
\[ Q'' = \frac{\partial}{\partial \tilde{\theta}}, \quad \tilde{Q}'' = \frac{\partial}{\partial \tilde{\theta}}, \quad Q'_{\mu} = \frac{\partial}{\partial \theta^{\mu}} + i \theta \partial_{\mu} - i \tilde{\theta} \epsilon_{\mu \nu} \partial^{\nu}. \] (4.6)
We can then obtain $N = 2$ twisted SUSY transformation of the component fields of the (anti-)chiral field which we show in Table 1. The details of the twisted chiral superfield formulation for $N = D = 2$ can be found in [50, 63].

Table 1: $N = 2$ twisted SUSY transformation.

|   | $s$   | $s_\mu$ | $\tilde{s}$ |
|---|-------|---------|-------------|
| $c$ | 0     | $-i\omega_\mu$ | 0           |
| $\omega_\nu$ | $\partial_\nu c$ | $-i\epsilon_{\mu\nu}\lambda$ | $-\epsilon_{\mu\nu}\partial^\nu c$ |
| $\lambda$ | $e^{\mu\nu}\partial_\mu\omega_\nu$ | 0 | $-\partial^\mu\omega_\mu$ |
| $\bar{c}$ | $-ib$ | 0 | $-i\phi$ |
| $b$ | 0     | $\partial_\mu \bar{c}$ | $-i\rho$ |
| $\phi$ | $i\rho$ | $-\epsilon_{\mu\nu}\partial^\nu \bar{c}$ | 0 |
| $\rho$ | 0     | $-\partial_\mu \phi - \epsilon_{\mu\nu}\partial^\nu b$ | 0 |

As we have seen from the formulation of Dirac-Kähler twist, the two-dimensional and four-dimensional formulation of the twisted superspace have close similarity. It is then natural to expect that we can obtain the four-dimensional counterpart of the action (4.7) which has off-shell twisted $N = 4$ SYM invariance. The bi-linear product of the chiral superfield $\Psi$ in (3.64) and the anti-chiral superfield $\Phi$ in (3.71) leads to the following action:

$$S = \int d^4x d^4\theta \Phi(x) \Psi(x)$$<br>$$= \int d^4x \left\{ -\bar{\psi}^3 \gamma^\mu \psi^3_{\mu} + i\bar{\psi}^3 \gamma^\mu \psi_{\mu} + \frac{i}{2} \mathcal{D}_{\sigma,ab} \bar{\psi}^3_{\sigma,ab} \gamma^\mu \psi^3_{ab,\gamma} - \frac{i}{2} \epsilon_{abcd} \gamma^\mu \psi^3_{ab,cd} - \frac{1}{4} \mathcal{D}_{\mu,ab} \mathcal{D}_{\nu,cd} \bar{\psi}^3_{\mu,ab,cd} \gamma^\sigma \psi^3_{\sigma,\rho} - \frac{1}{2} \epsilon_{abcd} \gamma^\mu \psi^3_{ab,cd} \gamma^\delta \psi^3_{\delta,\alpha} \right\},$$

where $\delta_\theta$ is defined in (3.33). This action can be identified as the quantized BF action with auxiliary fields and has off-shell $N = 2$ twisted SUSY invariance up to the surface terms by construction.

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where $\delta_\theta$ is defined in (3.33). This action can be identified as the quantized BF action with auxiliary fields and has off-shell $N = 2$ twisted SUSY invariance up to the surface terms by construction.
where $\Phi$ and $\Psi$ are Grassmann odd superfields. Here we display the action up to the second order of derivative. We show the complete expression of the action in the Appendix D. As we can see, the full expression of the action is very lengthy and includes up to the 8-th order of derivatives. In two-dimensional case the action has an interpretation that it is the quantized version of two-dimensional BF theory \cite{50}. This four-dimensional action does not have an obvious correspondence with a known action. It is, however, interesting to recognize that this is the first example of the action which is derived from $N = 4$ twisted superfields and has the exact off-shell $N = 4$ twisted SUSY in four dimensions. To keep the exact off-shell $N = 4$ twisted SUSY we need 8-th order derivative terms, which is very non-trivial and can never be found unless we have twisted superspace formulation.

\[ \text{(4.9)} \]

**4.2 $N = 2$ decomposition of $N = 4$ twisted SUSY algebra in four dimensions**

In this subsection we introduce a (anti-)self-dual decomposition of the $N = 4$ twisted SUSY into two $N = 2$ twisted SUSY sectors. We decompose twisted supercharges as follows:

\[
\left\{ \begin{array}{l}
 s^\pm \equiv \frac{1}{\sqrt{2}} (s \pm \tilde{s}), \\
 s^\pm_\mu \equiv \frac{1}{\sqrt{2}} (s_\mu \pm \tilde{s}_\mu), \\
 s^\pm_{\mu\nu} \equiv \frac{1}{\sqrt{2}} (s_{\mu\nu} \pm \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} s^{\rho\sigma}),
\end{array} \right.
\]

where the second rank tensor twisted supercharge satisfies (anti-)self-dual condition

\[
\frac{1}{2} \epsilon_{\mu\nu\rho\sigma} s^{\pm\rho\sigma} = \mp s^\pm_{\mu\nu}. \quad \text{(4.10)}
\]

We may identify this decomposition as self-dual and anti-self-dual decomposition. These supercharges satisfy the following two $N = 2$ factorized algebras:

\[
\begin{align*}
\{ s^\pm, s^\pm_\mu \} & = -i \partial_\mu, \\
\{ s^\pm_{\mu\nu}, s^\pm_\rho \} & = i (\delta_{\mu\nu,\rho\sigma} \partial^\sigma \mp \epsilon_{\mu\nu\rho\sigma} \partial^\sigma) \\
& \equiv i D^\pm_{\rho,\mu\nu}, \\
\{ \text{others} \} & = 0.
\end{align*}
\]

(4.11)

The supercharges $\{ s^+_I \} \equiv \{ s^+, s^+_\mu, s^+_\mu \}$ are the dual partner of $\{ s^-_I \} \equiv \{ s^-, s^-_\mu, s^-_\mu \}$ and anticommute with each other. It is interesting to note that the algebras of self-dual and anti-self-dual supercharges close by themselves and construct twisted $N = 2$ superalgebra in four dimensions. In other words the $N = 4$ superalgebra generated by Dirac-Kähler twist can be decomposed into two dual pairs of $N = 2$ twisted superalgebras which we use to formulate $N = 2$ twisted superspace formalism in the
later subsection 5.2. Corresponding to the factorization of the supercharges, we can decompose the superparameters in the similar way by the following relation:

$$\delta_\theta \equiv \frac{\partial}{\partial \theta^\pm} + \frac{i}{2} \theta^\pm \partial_\mu,$$

$$\equiv \delta_{\theta+} + \delta_{\theta-}, \quad (4.12)$$

where

$$\delta_{\theta+} \equiv \theta^+ s^+ + \frac{1}{4} \theta^\pm s^\pm + \frac{1}{4} \theta^\mu s^\mu,$$

$$\delta_{\theta-} \equiv \theta^- s^- + \frac{1}{4} \theta^\pm s^\pm + \frac{1}{4} \theta^\mu s^\mu, \quad (4.13)$$

with

$$\begin{cases} 
\theta^\pm \equiv \frac{1}{\sqrt{2}} (\theta \pm \tilde{\theta}), \\
\theta^\pm_\mu \equiv \frac{1}{\sqrt{2}} (\theta^\pm_\mu \pm \tilde{\theta}^\pm_\mu), \\
\theta^\pm_\mu_\nu \equiv \frac{1}{\sqrt{2}} (\theta^-_\mu \pm \theta^\mu_\nu \pm \frac{1}{2} \epsilon_\mu \nu \rho \sigma \theta^- \rho \sigma). 
\end{cases}$$

By using the above dual decomposed superparameters we can find supercharge differential operators:

$$Q^\pm = \frac{\partial}{\partial \theta^\pm} + \frac{i}{2} \theta^\pm \partial_\mu,$$

$$Q^\pm_\mu = \frac{\partial}{\partial \theta^\pm_\mu} + \frac{i}{2} \theta^\pm_\mu \partial_\nu - \frac{i}{2} \theta^\pm_\mu \partial_\nu ,$$

$$Q^\pm_\mu_\nu = \frac{\partial}{\partial \theta^\pm_\mu_\nu} - \frac{i}{2} \theta^\pm_\rho \partial_\rho,$$  \quad (4.14)

where we introduce the following notations:

$$D^\pm_\rho_\mu_\nu = \delta^\pm_\rho_\mu_\nu \partial^\rho ,$$  \quad (4.15)

$$\delta^\pm_\rho_\mu_\nu = \frac{\partial}{\partial \theta^\pm_\rho_\mu_\nu} \theta^\pm_\rho_\nu = \delta_\rho_\mu \delta_\nu_\sigma - \delta_\rho_\sigma \delta_\nu_\mu + \epsilon_\mu_\nu_\rho_\sigma \partial^\rho .$$

These $N = 2$ supercharge differential operators satisfy the following relations:

$$\{Q^\pm, Q^\pm_\mu\} = i \partial_\mu ,$$

$$\{Q^\pm_\rho_\mu_\nu, Q^\pm_\rho\} = -i D^\pm_\rho_\mu_\nu ,$$

$$\{\text{others}\} = 0 , \quad (4.16)$$

where the sign of the spacetime derivative is reversed with respect to the algebra \[4.11\]. Similar to the $N = 4$ full twisted algebra, we introduce superdifferential operators which anticommute with all the supercharge differential operators $\{Q^\pm_\rho\}$,

$$D^\pm = \frac{\partial}{\partial \theta^\pm} - \frac{i}{2} \theta^\pm_\mu \partial_\mu ,$$

$$D^\pm_\mu = \frac{\partial}{\partial \theta^\pm_\mu} - \frac{i}{2} \theta^\pm_\mu \partial_\mu + \frac{i}{2} \theta^\pm_\mu \partial_\nu ,$$

\[4.17\]}
\[
\mathcal{D}_{\mu \nu}^+ = \frac{\partial}{\partial \theta^{\pm \mu}} + \frac{i}{2} \theta_{\mu \nu}^+ \mathcal{D}_{\mu \nu}^+,
\]  

(4.17)

which satisfy the same algebra as (4.11) with replacements: \( s_I^+ \to \mathcal{D}_I^+ \).

We now consider only the positive part of supercharges \( \{s_I^+\} \), the corresponding supercharge differential operators \( \{\mathcal{Q}_I^+\} \) and superdifferential operators \( \{\mathcal{D}_I^+\} \), which fulfill \( N = 2 \) twisted SUSY algebra in four dimensions. We can impose chiral and anti-chiral conditions by using these four-dimensional \( N = 2 \) positive superdifferential operators:

\[
\mathcal{D}^+ \Phi = \mathcal{D}_{\mu \nu}^+ \Phi = 0,
\]

(4.18)

\[
\mathcal{D}_\mu^+ \Psi = 0.
\]

(4.19)

The following combination of the coordinates satisfy the chiral condition (4.18):

\[
\begin{cases}
y^\mu = x^\mu + \frac{i}{2} \theta^+ \theta^{+ \mu} - \frac{i}{2} \theta^+ \theta^{+ \mu}, \\
\theta^{+ \mu}.
\end{cases}
\]

Thus the general solution of chiral superfield \( \Phi \) can be given by

\[
\Phi(y^\mu, \theta^{+ \mu}) = e^{\theta^{+ \mu} s_I^+} \phi = \phi + \theta^{+ \mu} C_\mu + \frac{1}{2} \theta^{+ \mu} \theta^{+ \nu} \phi_{\mu \nu} + \theta^{+ \mu} \tilde{\psi}_\mu + \theta^{+ \mu} \phi.
\]

(4.20)

where \( \theta^{+ \mu} \equiv \frac{1}{2} \epsilon_{\mu \nu \rho \sigma} \theta^{+ \nu} \theta^{+ \rho} \theta^{+ \sigma} \), \( \theta^{+ 4} \equiv \frac{1}{4} \epsilon_{\mu \nu \rho \sigma} \theta^{+ \nu} \theta^{+ \rho} \theta^{+ \sigma} \). We show the twisted \( N = 2 \) SUSY transformation of the Abelian chiral multiplets of the super field in Table 2. We can now define the following action:

| \( s^+ \) | \( s_\mu^+ \) | \( s_A^+ \) |
|-------|---------|---------|
| \( \phi \) | 0 | \( C_\mu \) | 0 |
| \( C_\nu \) | \( -i \partial_\nu \phi \) | \( -\phi_{\mu \nu} \) | \( iD_{\nu A} \phi \) |
| \( \phi_{\rho \sigma} \) | \( i(\partial_\rho C_\sigma - \partial_\sigma C_\rho) \) | \( -\epsilon_{\mu \rho \sigma} \tilde{\psi}_\mu \) | \(-i(D_{\rho A} C_\sigma - D_{\sigma A} C_\rho) \) |
| \( \tilde{\psi}_\nu \) | \( -i \epsilon_{\rho \sigma \nu} \partial^\sigma \phi^{+ \rho} \) | \( -\delta_{\mu \nu} \phi \) | \( \frac{i}{2} D_{\mu A} \epsilon_{\mu \rho \sigma} \phi_{\rho \sigma} \) |
| \( \tilde{\phi} \) | \( i \partial^\mu \tilde{\psi}_\mu \) | 0 | \(-iD_{\mu A} \tilde{\psi}_\mu \) |

Table 2: Twisted \( N = 2 \) SUSY transformation of Abelian chiral multiplets.

\[
S = \int d^4 y d^4 \theta \left( \Phi(y, \theta^{+ \mu}) \right)^2
\]

\[
= \int d^4 y \frac{1}{4!} \epsilon^{\mu \nu \rho \sigma} s_\mu^+ s_\nu^+ s_\rho^+ s_\sigma^+ (\phi^2)
\]

\[
= 2 \int d^4 x \{ \phi \tilde{\phi} - C_\mu \tilde{\psi}_\mu + \frac{1}{8} \epsilon^{\mu \nu \rho \sigma} \phi_{\mu \nu} \phi_{\rho \sigma} \},
\]

(4.21)

where the coordinate shift \( y^\mu \to x^\mu \) is allowed since this action includes only chiral superfield which has only \( y^\mu \) dependence. Due to this possible coordinate shift the
action does not include any derivative terms. It should be compared with the action of previous subsection which has a bilinear form of chiral and anti-chiral superfields and each of the superfield has different chiral coordinate dependence, which is the origin of generating derivatives.

Next we consider the anti-chiral condition \((4.19)\). We find that the following combination of coordinate satisfies the condition:

\[
\left\{ \begin{array}{l}
 w^\mu = x^\mu - \frac{i}{2} \theta^+ \theta^{+\mu} + \frac{1}{2} \theta^{+\mu} \theta^{+\nu}, \\
 \theta^+, \quad \theta^{+A}_A.
\end{array} \right.
\]

Then the general solution for anti-chiral superfield can be given by

\[
\Psi(w, \theta^+, \theta^{+A}) = e^{\theta^+ s^+ + \frac{1}{2} \theta^{+\nu} s^{+\mu}_\nu} \bar{\phi} = -\phi + \frac{1}{4} \theta^{+A} \chi_+^+ + \frac{1}{4} (\theta^{+2})^A A M_+^A + \frac{1}{4} (\theta^{+2})^{A_+} A \chi_+ + (\theta^{+3}) L, \tag{4.22}
\]

where \((\theta^{+2})_A \equiv \frac{1}{24} \Gamma^{+}_{ABC} \theta^{+B} \theta^{+C}\) and \((\theta^{+3}) \equiv \frac{1}{3!} \Gamma^{+}_{ABC} \theta^{+A} \theta^{+B} \theta^{+C}\). Here the suffixes \(A, B, C\) denote tensor suffix, for example \(\phi_A^{+\mu} \psi_{\mu}^{+A}\) stands for \(\phi_{\mu}^{+\nu} \psi_{\mu}^{+\nu}\). \(\Gamma^{\pm}_{ABC}\) is defined as follows:

\[
\Gamma^{\pm\mu\alpha, \nu\beta, \rho\gamma} = \delta^{\alpha\nu} \delta^{\beta\rho} \delta^{\gamma\mu} + \delta^{\mu\nu} \delta^{\beta\gamma} \delta^{\rho\alpha} + \delta^{\alpha\beta} \delta^{\nu\gamma} \delta^{\rho\mu} + \delta^{\mu\beta} \delta^{\nu\gamma} \delta^{\rho\alpha} - (\delta^{\alpha\nu} \delta^{\beta\rho} \delta^{\gamma\mu} + \delta^{\mu\nu} \delta^{\beta\gamma} \delta^{\rho\alpha} + \delta^{\alpha\beta} \delta^{\nu\gamma} \delta^{\rho\mu} + \delta^{\mu\beta} \delta^{\nu\gamma} \delta^{\rho\alpha})
\]

\[
\mp \varepsilon^{\mu\alpha\beta\gamma} \delta^{\nu\rho} \equiv \varepsilon^{\mu\alpha\nu\rho} \delta^{\beta\gamma} \pm \varepsilon^{\mu\alpha\rho\nu} \delta^{\beta\gamma} \pm \varepsilon^{\mu\rho\alpha\nu} \delta^{\beta\gamma} \pm \varepsilon^{\mu\rho\gamma\alpha} \delta^{\nu\beta}. \tag{4.23}
\]

We summarize details of tensor notations and useful formulae in Appendix E. We show twisted \(N = 2\) SUSY transformation of Abelian anti-chiral multiplets in Table 3.

| \(s^+\) | \(s^{+\mu}_\mu\) | \(s^{+\mu}_\mu\) |
|---|---|---|
| \(\phi\) | \(\chi\) | 0 |
| \(\chi^+\) | \(-N^+_B\) | \(-iD_{\mu,B} \phi\) |
| \(M^+_B\) | \(-\tilde{\chi}_B\) | \(-\frac{i}{4} \Gamma^{+}_{B\mu\nu} \delta^{\nu} \chi^+_C\) |
| \(\bar{\chi}\) | \(-L\) | \(i\partial^{\nu} M^{+\nu}_B\) |
| \(\chi\) | 0 | \(-i \partial_{\mu} \phi\) |
| \(N^+_B\) | 0 | \(i (\partial_{\mu} \chi^+_B + D^{+\mu}_B \chi)\) |
| \(\tilde{\chi}_B\) | 0 | \(-i (\partial_{\mu} M^+_B + \frac{i}{4} \Gamma^{+}_{B\mu\nu} \delta^{\nu} N^+_C)\) |
| \(L\) | 0 | \(i (\partial_{\mu} \bar{\chi} + \partial^{\nu} \bar{\chi}^{+\mu}_B)\) |

Table 3: Twisted \(N = 2\) SUSY transformation of Abelian anti-chiral multiplets

A twisted \(N = 2\) SUSY invariant action can be also obtained by anti-chiral superfield as

\[
S = \int d^4 w d^4 \theta (\Psi(w, \theta^+, \theta^{+A}))^2
\]
\[
\int d^4 w \, s^+ \frac{1}{3!^4} \Gamma^{+A,B,C} s^+_A s^+_B s^+_C (\bar{\phi}^2)
\]
\[
= \int d^4 x \{ \bar{\phi} L + \frac{1}{4} \chi^+ A \chi^+ + \frac{1}{4} M^+ A N^+ + \bar{\chi} \chi \}, \tag{4.24}
\]

where we make the coordinate shift \( w^\mu \rightarrow x^\mu \) again. By the same reason as the previous action there is no derivative in the action. The action has \( N = 2 \) twisted SUSY invariance where the supertransformation of the anti-chiral superfield is given in Table 3. In the following subsection we introduce superconnection formalism where several constraints will be introduced. The constraints relate the chiral and anti-chiral superfields and generate derivatives and eventually super Yang-Mills actions can be generated.

4.3 Vector superfield and gauge symmetry for \( N = 4 \) twisted SUSY invariant actions

In this subsection we propose to formulate a twisted version of vector superfield to derive supergauge invariant actions. Using the operators \( \{ \mathcal{D}^\pm \} \) defined in (4.17), we may impose two types of chiral conditions. The first type is given as follows:

\[
\mathcal{D}^+ \Psi = \mathcal{D}^- \Psi = \mathcal{D}^+_{\mu \nu} \Psi = \mathcal{D}^-_{\mu \nu} \Psi = 0, \tag{4.25}
\]
\[
\mathcal{D}^+ \Phi = \mathcal{D}^- \Phi = 0, \tag{4.26}
\]

which are essentially equivalent to the previous chiral conditions (3.56) and (3.66), respectively. Here we propose to impose other type of chiral conditions

\[
\mathcal{D}^+ \Phi^+ = \mathcal{D}^+_{\mu \nu} \Phi^+ = \mathcal{D}^-_{\mu \nu} \Phi^+ = 0, \tag{4.27}
\]
\[
\mathcal{D}^- \Phi^- = \mathcal{D}^-_{\mu \nu} \Phi^- = \mathcal{D}^+_{\mu \nu} \Phi^- = 0, \tag{4.28}
\]

where we find that these conditions are mutually transformed into each other by the replacement: \( + \leftrightarrow - \). We may loosely abuse the words, chiral and anti-chiral in this section, instead of self-dual and anti-self-dual of the previous subsection, and call (4.27) as a chiral condition for a chiral superfield \( \Phi^+ \) and (4.28) as an anti-chiral condition for an anti-chiral superfield \( \Phi^- \).

The simplest way to find a general solution of the chiral condition (4.27) is to recognize that the following parameters are solution of the chiral condition:

\[
\begin{cases}
  y^{+ - \mu} = x^\mu + a^{+ \mu} - a^{- \mu}, \\
  \theta^-, \quad \theta^-_{\mu \nu}, \quad \theta^+_{\mu},
\end{cases}
\]

where

\[
a^{\pm \mu} = \frac{i}{2} \theta^{\pm} \theta^{\pm \mu} + \frac{i}{2} \theta^{\pm} \theta^{\pm \mu}. \tag{4.29}
\]

Therefore a general solution of the chiral condition (4.27) is given by an arbitrary function of these parameters:

\[
\Phi^+ = \Phi^+(y^{+ - \mu}, \theta^-, \theta^-_{\mu \nu}, \theta^+_{\mu}). \tag{4.30}
\]
Similarly we find that the following parameters satisfy the anti-chiral condition (4.28),
\[
\begin{align*}
  y^{-+} = x^\mu - a^+ - a^-,
  \theta^+, \quad \theta^{++}_{\mu\nu}, \quad \theta^-.
\end{align*}
\]
Thus a general solution can be given by
\[
\Phi^- = \Phi^- (y^{-+}, \theta^+, \theta^{++}_{\mu\nu}, \theta^-).
\]
Supertransformation of the component fields of these chiral superfields can be obtained by the similar procedure as in the subsection 3.3. We, however, don’t bother to write them explicitly here.

Let us now introduce an operator \( P \) which interchanges the chirality of the superparameters and coordinate, \(+ \leftrightarrow -\):
\[
P \{ y^{\pm \pm \mu}, \theta^\pm, \theta^{\pm \pm}_{\mu\nu}, \theta^\pm \} = \{ y^{\mp \pm \mu}, \theta^\mp, \theta^{\mp \pm}_{\mu\nu}, \theta^\mp \}.
\]

We define a chiral and anti-chiral superfields; \( \Lambda^+ \) and \( \Lambda^- \), which satisfy the following conditions:
\[
\begin{align*}
  \mathcal{D}^+ \Lambda^+ &= \mathcal{D}^+_{\mu\nu} \Lambda^+ = \mathcal{D}^- \Lambda^+ = 0, \\
  \mathcal{D}^- \Lambda^- &= \mathcal{D}^-_{\mu\nu} \Lambda^- = \mathcal{D}^+ \Lambda^- = 0.
\end{align*}
\]
A chiral pair of superfields \( \Lambda^+ \) and \( \Lambda^- \) which satisfy these chirally conjugate conditions can be related by the following conjugate relation:
\[
P \Lambda^\pm (y^{\pm \pm \mu}, \theta^\mp, \theta^{\mp \pm}_{\mu\nu}, \theta^\mp) = \Lambda^\mp (y^{\mp \mp \mu}, \theta^\mp, \theta^{\mp \mp}_{\mu\nu}, \theta^\mp).
\]

These chiral and anti-chiral superfields can be expanded into component fields as follows:
\[
\begin{align*}
  \Lambda^\pm (y^{\pm \pm \mu}, \theta^\mp, \theta^{\mp \pm}_{\mu\nu}, \theta^\pm) &= \lambda^\pm_0 (x) + \theta^{\pm \mu} \lambda^{\pm \mu}_1 (x) + \frac{1}{4} \theta^{\mp A} \lambda^{\pm \mu}_{1A} (x) + \theta^{\mp} \lambda^{\pm}_1 (x) \\
  &\quad + \frac{1}{2} \theta^{\pm \mu} \theta^{\pm \nu} \lambda^{\pm \mu\nu}_2 (x) + \frac{1}{4} \theta^{\mp A} \theta^{\pm \mu} \lambda^{\pm \mu}_{2A} (x) + \frac{1}{4} \theta^{\pm A} \theta^{\mp B} \lambda^{\pm \mu \nu}_{2AB} (x) \\
  &\quad + \theta^{\mp \mu} \theta^{\pm \nu} \lambda^{\pm \mu\nu}_2 (x) + \frac{1}{4} \theta^{\mp A} \theta^{\pm \mu} \lambda^{\pm \mu}_{2A} (x) \\
  &\quad + \frac{i}{2} \theta^{\mp \mu} \partial_{\rho} \lambda^{\pm \mu}_0 (x) + \frac{i}{2} \theta^{\pm \rho} \partial_{\mu} \lambda^{\pm \mu}_0 (x) - \frac{i}{2} \theta^{\mp} \partial_{\mu} \lambda^{\pm \mu}_0 (x) \\
  &\quad - \frac{i}{2} \theta^{\pm \rho} \partial_{\mu} \lambda^{\mp \mu}_0 (x) + \cdots,
\end{align*}
\]
where we have expanded up to the second order of twisted superparameters.

We define the vector superfield which satisfies the following condition:
\[
P V (x^\mu, \theta^\pm, \theta^{\pm \mu}_{\mu\nu}, \theta^\pm) = V (x^\mu, \theta^\pm, \theta^{\pm \mu}_{\mu\nu}, \theta^\pm).
\]

Here we can introduce the following chiral supergauge transformation for the vector superfield:
\[
e^V \rightarrow e^{-\Lambda^-} e^V e^{-\Lambda^+}
\]
where we may consider non-Abelian vector superfield and thus the two chiral gauge superparameters \( \Lambda^+, \Lambda^- \) may carry non-Abelian nature and should satisfy the chiral and anti-chiral conditions \((4.33)\) and \((4.34)\), respectively. Then the small supergauge transformation can be given by

\[
\delta V = -\Lambda^+-\Lambda^- - \frac{1}{2} [V, \Lambda^+ - \Lambda^-] + \cdots.
\]

The vector superfield can be expanded into component fields as follows:

\[
V = A_0(x) + \theta^{+\mu} A^+_\mu + \frac{1}{4} \theta^{-A} A^-_{1A} + \theta^- A^-_1 \\
+ \theta^{-\mu} A^-_{1\mu} + \frac{1}{4} \theta^{+A} A^+_1 + \theta^+ A^+_1 \\
+ \theta^+ \theta^{+\mu} A^+_{2\mu} + \theta^- \theta^{-\mu} A^-_{2\mu} \\
+ \theta^+ \theta^{+\rho\mu} B^+_{2\mu} + \theta^- \theta^{-\rho\mu} B^-_{2\mu} \cdots,
\]

where we keep only some of the component fields in the second order superparameter terms.

As we can see from the small supergauge transformation \((4.38)\) we can gauge away the component fields \((A_0, A^+_\mu, A^-_{1A}, A^+_1, \cdots)\) by the corresponding gauge parameters \((-\lambda^+_0 - \lambda^-_0, \lambda^+_1, \lambda^-_{1A}, \lambda^+_1, \cdots)\), respectively. In fact we can recognize that all the component fields of the vector superfield, which are coefficients of any combination of the product \((\theta^-, \theta^{-\mu\rho}, \theta^+_\rho)\) or \((\theta^+, \theta^{+\mu\rho}, \theta^-_\rho)\), can be gauged away. We identify this gauging away procedure as Wess-Zumino gauge choice. Then we can find gauge fields \(A^+_\mu\) and \(B^+_{2\mu}\) which have the following gauge transformation:

\[
\delta A^+_\mu = -\frac{i}{2} \partial_\mu (\lambda^+_0 - \lambda^-_0) - \frac{i}{2} [A^+_{2\mu}, \lambda^+_0 - \lambda^-_0]
\]

\[
\delta B^+_{2\mu} = -\frac{i}{2} \partial_\mu (\lambda^+_0 - \lambda^-_0) - \frac{i}{2} [B^+_{2\mu}, \lambda^+_0 - \lambda^-_0].
\]

It is interesting to note that there is only one chiral pair of gauge parameters \(-\frac{i}{2} (\lambda^+_0 - \lambda^-_0)\) for two chiral pairs of gauge fields \(A^+\) and \(B^+\).

Next we consider the gauge invariant quantity which satisfies the chiral constraint \((4.27)\). We first define the following differential operator:

\[
\mathcal{D}^{8+} = \mathcal{D}^+ \Gamma^{+ABC} \mathcal{D}^+_A \mathcal{D}^+_B \mathcal{D}^+_C \frac{1}{3!4!} \epsilon^{\mu\nu\rho\sigma} \mathcal{D}_\mu \mathcal{D}_\nu \mathcal{D}_\rho \mathcal{D}_\sigma,
\]

where the suffixes \(A, B, C\) denote tensor suffixes similar as in \((4.22)\) and \(\Gamma^{+ABC}\) is defined in \((4.23)\). Since the chiral and anti-chiral supergauge parameters \(\Lambda^+\) and \(\Lambda^-\) satisfy the chiral and anti-chiral conditions, they satisfy the following relations:

\[
e^{\mp \Lambda^\pm} (\mathcal{D}^\pm, \mathcal{D}^\pm_{\mu\nu\rho\sigma}) = (\mathcal{D}^\pm, \mathcal{D}^\pm_{\mu\nu\rho\sigma}) e^{\mp \Lambda^\pm}.
\]
When the differential operator $\mathcal{D}^{8+}$ is applied to a general superfield $\Phi$, $\mathcal{D}^{8+}\Phi$ satisfies the chiral condition \((4.27)\) up to possible total divergence terms. It is then possible to find all the possible chiral invariant combinations of the differential operators \{\(\mathcal{D}^{-}, \mathcal{D}^{A}, \mathcal{D}^{\mu}\)\} acting on a vector superfield $e^V$:

\[
W^{1-} = \mathcal{D}^{8+}e^{-}V\mathcal{D}^{-}e^{V}, \quad W^{1+}_{A} = \mathcal{D}^{8+}e^{-}V\mathcal{D}^{A}e^{V}, \quad W^{1+}_{\mu} = \mathcal{D}^{8+}e^{-}V\mathcal{D}^{\mu}e^{V}, \\
W^{2-}_{A} = \mathcal{D}^{8+}e^{-}V(\mathcal{D}^{2-})_A e^{V}, \quad W^{2+}_{\mu\nu} = \mathcal{D}^{8+}e^{-}V(\mathcal{D}^{2+})_{\mu\nu} e^{V}, \quad W^{2+}_{\mu} = \mathcal{D}^{8+}e^{-}V\mathcal{D}^{-}\mathcal{D}^{A}e^{V}, \\
W^{3-} = \mathcal{D}^{8+}e^{-}V(\mathcal{D}^{3-})e^{V}, \quad W^{3+}_{\mu} = \mathcal{D}^{8+}e^{-}V(\mathcal{D}^{3+})_{\mu} e^{V}, \quad W^{3+}_{A} = \mathcal{D}^{8+}e^{-}V\mathcal{D}^{-}(\mathcal{D}^{2-})_A e^{V}, \\
W^{4-} = \mathcal{D}^{8+}e^{-}V\mathcal{D}^{-}(\mathcal{D}^{3-})e^{V}, \quad W^{4+} = \mathcal{D}^{8+}e^{-}V(\mathcal{D}^{4+})e^{V},
\]

where

\[
(\mathcal{D}^{2-})_A = \frac{1}{2! \cdot 4^2} \Gamma_{ABC}^{-\mu} \mathcal{D}^{-B} \mathcal{D}^{-C}, \quad (\mathcal{D}^{2+})_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \mathcal{D}^{+\rho} \mathcal{D}^{+\sigma}, \\
(\mathcal{D}^{3-}) = \frac{1}{3! \cdot 4^2} \Gamma_{ABC}^{-A} \mathcal{D}^{-B} \mathcal{D}^{-C}, \quad (\mathcal{D}^{3+})_{\mu} = \frac{1}{3} \epsilon_{\mu\nu\rho\sigma} \mathcal{D}^{+\nu} \mathcal{D}^{+\rho} \mathcal{D}^{+\sigma}, \\
(\mathcal{D}^{4+}) = \frac{1}{4!} \epsilon_{\mu\nu\rho\sigma} \mathcal{D}^{+\mu} \mathcal{D}^{+\nu} \mathcal{D}^{+\rho} \mathcal{D}^{+\sigma}.
\]

These “supercurvature” terms \(\{W_I\}\) transform adjointly under a supergauge transformation and satisfy the chiral condition \((4.27)\). Opposite chiral combinations of the “supercurvature” terms can be obtained simply by the interchange of the chirality for the above terms: \(+ \leftrightarrow -\).

We can then formally construct a variety of supergauge invariant actions which have off-shell \(N=4\) twisted SUSY and non-Abelian gauge symmetry:

\[
S_1 = \int d^4x \ d^8+\theta \ Tr(W^{1-}W^{1-}),
\]
\[
S_2 = \int d^4x \ d^8+\theta \ Tr(W^{4-}\tilde{W}^{4+}),
\]
\[
\ldots.
\]

where

\[
d^8+\theta = d\theta - \frac{1}{3! \cdot 4^3} \Gamma_{ABC}^{-A} d\theta^{-B} d\theta^{-C} \frac{1}{4!} \epsilon_{\mu\nu\rho\sigma} d\theta^{+\mu} d\theta^{+\nu} d\theta^{+\rho} d\theta^{+\sigma}.
\]

It should be noted that the chiral superfields \(\{W_I\}\) given above are function of \((y^{+\mu}, \theta^-\theta^-A, \theta^+_\mu)\) and thus the coordinate change \(y^{+\mu} \to x^\mu\) is allowed since the action includes only the chiral superfields and thus \(d^8+\theta\) include only chiral super-parameters.
These actions are highly reducible in the sense that they may include too many superfluous component fields even if we take the Wess-Zumino gauge. As we have shown in the previous subsection 4.1 and in the Appendix D, the $N = 4$ twisted supersymmetric action constructed from chiral and anti-chiral pair of superfields have included too many terms. We expect that these gauge invariant actions given above include too many fields as well when they are expressed by the component fields.

It is, however, not unreasonable to expect that these actions may include twisted $N = 4$ super Yang-Mills terms in the action since they include gauge fields as we have shown in (4.40). If we try to find how the suspected gauge fields, $A_{2\mu}^\pm$ and $B_{2\mu}^\pm$ are included in the action, we find too many spacetime derivatives. For example in the Abelian gauge we can consider how the gauge field $A_{2\mu}^+$ survives after the operation of $D_8^+ D_7^-$ and before the integration $d\theta^{8+}$ in

$$W^{1-} = D_8^+ D_7^- V = D_8^+ D_7^- (\cdots + \theta^+ \theta^{+\mu} A_{2\mu}^+ + \cdots)$$

$$= \frac{1}{3! \cdot 4^3 4!} \frac{i}{2} \Gamma^{ABC} \theta^{+\alpha} \theta^{+\beta} \theta^{+\gamma} D_{\alpha,A}^+ D_{\beta,B}^+ D_{\gamma,C}^+$$

$$\epsilon^{\mu\nu\rho\sigma} (\theta^- \partial_{\nu} - \theta^- \partial_{\nu} \partial_{\nu}) (\theta^- \partial_{\rho} - \theta^- \partial_{\rho} \partial_{\rho}) (\theta^- \partial_{\sigma} - \theta^- \partial_{\sigma} \partial_{\sigma}) \theta^{+\tau} \partial_\mu A_{2\tau}^+ + \cdots .$$

As we can see that this term includes 7-th power of spacetime derivative. Other component gauge fields include higher derivatives as well. Thus the action (4.45) and (4.46) include the gauge terms but higher derivatives are accompanied and then Yang-Mills term is missing. We may further need to impose constraints in addition to the chiral conditions to find super Yang-Mills actions in the vector superfield formulation. In the next section we introduce superconnection to the corresponding superderivatives and impose constraints to find super Yang-Mills action.

5 Superconnection formalism

5.1 $N = 2$ superconnection formalism in two dimensions

In the previous subsection we have given a general formulation to obtain off-shell $N = 4$ twisted SUSY invariant and gauge invariant actions by introducing vector superfield. The supergauge invariance was introduced by the transformation of the vector superfield. It turned out that the proposed supergauge invariant actions include too higher derivatives and thus we expect that it is necessary to impose further constraints to find twisted $N = 4$ SUSY invariant Yang-Mills action. Here we propose an alternative approach, twisted superconnection formalism, to find supergauge invariant actions. A similar type of formulation was proposed by Labastida and his collaborators by the spinor notations [53, 54]. Here the formulation is based on the Dirac-Kähler twist to generate twisted superspace. Since the four-dimensional twisted $N = 4$ SUSY invariant formulation is expected to be obtained quite parallel to the two-dimensional $N = 2$ version of the formulation, we first show the simpler twisted $N = D = 2$ formulation of superconnection formalism.
We first introduce fermionic superconnection \( \{ \Gamma_I \} \) which allows us to define the following gauge superspace covariant derivatives:

\[
\begin{align*}
\nabla & \equiv \mathcal{D} - i \Gamma, \\
\tilde{\nabla} & \equiv \tilde{\mathcal{D}} - i \tilde{\Gamma}, \\
\nabla_\mu & \equiv \mathcal{D}_\mu - i \Gamma_\mu,
\end{align*}
\]

(5.1)

where \( \mathcal{D}, \tilde{\mathcal{D}} \) and \( \mathcal{D}_\mu \) are two-dimensional superderivatives defined in (3.45) and satisfy the \( N = 2 \) twisted SUSY algebra (3.46). We further introduce the following bosonic covariant derivative:

\[
\nabla_\mu \equiv \partial_\mu - i \Gamma_\mu,
\]

(5.2)

where the lowest \( \theta \) independent component of \( \Gamma_\mu \) is identified as the usual gauge field: \( \Gamma_\mu|_{\theta=0} = \omega_\mu \). We call them as supercovariant derivatives. It should be noted that these superconnections \( \{ \Gamma_I \} \) are superfields.

Corresponding to the superconnections \( \{ \Gamma_I \} \), we can introduce the following supergauge transformation:

\[
\delta_{\text{gauge}} \Gamma_I = \nabla_I K \equiv \mathcal{D}_I K - i[\Gamma_I, K],
\]

(5.3)

where \( K \) is an arbitrary superfield. Since the superconnections include many superfluous component fields, we gauge away them by the supergauge freedom of the superfield \( K \), by taking Wess-Zumino gauge, and keep the usual gauge degrees of freedom.

Even after the Wess-Zumino gauge fixing, we still have many component fields. In the superconnection formalism we impose reasonable constraints to reduce the unnecessary component fields in the superfields. In general the anticommutators of those supercovariant derivatives \( \{ \nabla_I \} = \{ \nabla, \tilde{\nabla}, \nabla_\mu \} \) have the following form:

\[
\{ \nabla_I, \nabla_J \} = T_{IJ}^\mu \nabla_\mu - i \mathcal{F}_{IJ},
\]

(5.4)

where we identify \( \mathcal{F}_{IJ} \) as supercurvatures and \( T_{IJ}^\mu \) as superspace torsions with a possible bosonic covariant derivative \( \nabla_\mu \). We also introduce fermionic supercurvature

\[
[\nabla_I, \nabla_J] = -i \mathcal{F}_{IJ},
\]

(5.5)

and a bosonic curvature which includes usual curvature term:

\[
[\nabla_\mu, \nabla_\nu] = -i \mathcal{F}_{\mu\nu},
\]

(5.6)

where torsion terms are suppressed. The lowest (\( \theta \)-independent) component of \( \mathcal{F}_{\mu\nu} \) coincides with the usual curvature:

\[
\mathcal{F}_{\mu\nu}|_{\theta=0} \equiv F_{\mu\nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu - i[\omega_\mu, \omega_\nu].
\]

(5.7)
We summarize the commutators and anticommutators of these supercovariant derivatives in Table 4 where torsion terms are specified as in the Table.

In order to suppress superfluous component fields we propose to impose the following constraints on the supercurvatures:

\[
\mathcal{W}_1 = \mathcal{W}_2 \equiv \mathcal{W}, \quad \mathcal{W}_3 = 0, \quad \mathcal{F}_{\mu\nu} \equiv \delta_{\mu\nu} F, \quad \mathcal{F}_\mu = \tilde{\mathcal{F}}_\mu = 0. \quad (5.8)
\]

We eventually find out that these constraints are necessary conditions to derive twisted \( N = 2 \) SUSY invariant Yang-Mills actions and can be interpreted as suppression conditions of higher spin fields from the superspace. In the Appendix F we explicitly show in the details how the component fields of the superconnection can be gauged away, by taking Wess-Zumino gauge, to be consistent with these constraints.

We can then obtain series of nontrivial relations via Jacobi identity of these supercovariant derivatives. For example,

\[
[\nabla, \{\tilde{\nabla}, \nabla_\mu\}] + [\tilde{\nabla}, \{\nabla, \nabla_\mu\}] + [\nabla_\mu, \{\nabla, \tilde{\nabla}\}] = 0,
\]

\[
\to [\nabla, i\epsilon_{\mu\nu} \nabla_\nu] + [\tilde{\nabla}, -i\nabla_\mu] + [\nabla_\mu, 0] = 0,
\]

\[
\to \epsilon_{\mu\nu} \mathcal{F}_\nu - \tilde{\mathcal{F}}_\mu = 0. \quad (5.9)
\]

We can similarly obtain other identities,

\[
\nabla \mathcal{W} \equiv \mathcal{D} \mathcal{W} - i[\Gamma, \mathcal{W}] = 0, \quad (5.10)
\]

\[
\tilde{\nabla} \mathcal{W} \equiv \tilde{\mathcal{D}} \mathcal{W} - i[\tilde{\Gamma}, \mathcal{W}] = 0, \quad (5.11)
\]

\[
\nabla_\mu \mathcal{F} \equiv \mathcal{D}_\mu \mathcal{F} - i[\Gamma_\mu, \mathcal{F}] = 0, \quad (5.12)
\]

\[
\mathcal{F}_\mu = -\frac{i}{2} \nabla_\mu \mathcal{W}, \quad (5.13)
\]

\[
\tilde{\mathcal{F}}_\mu = -\frac{i}{2} \epsilon_{\mu\nu} \nabla^\nu \mathcal{W}, \quad (5.14)
\]

\[
\mathcal{F}_{\mu\nu} = -\frac{i}{2} \delta_{\mu\nu} \nabla \mathcal{F} + \frac{i}{2} \epsilon_{\mu\nu} \tilde{\nabla} \mathcal{F}, \quad (5.15)
\]

\[
\mathcal{F}_{\mu\nu} = -\frac{1}{2} \epsilon_{\mu\nu} \nabla \nabla \mathcal{W} + \frac{i}{4} \epsilon_{\mu\rho} \epsilon_{\nu\sigma} \nabla_\rho \nabla_\sigma \mathcal{W}, \quad (5.16)
\]

where we assume that the gauge algebra is non-Abelian.
If the gauge algebra is Abelian the first three conditions (5.10), (5.11) and (5.12) are simplified and lead to chiral and anti-chiral conditions,

\[ \mathcal{D}W = \tilde{\mathcal{D}}W = 0, \]
\[ \mathcal{D}_\mu \mathcal{F} = 0. \]  

Following to the same procedure given in subsection 4.1, we find the following component expansion of chiral and anti-chiral superfields:

\[ W(y) = A + \theta^\mu \lambda_\mu + \theta^2 D, \]
\[ \mathcal{F}(w) = B + \theta \rho + \tilde{\theta} \tilde{\rho} + \theta \tilde{\theta} E. \]  

The lowest (\( \theta = 0 \)) component of the constraint (5.16) in the Abelian case leads to the following relation:

\[ -D + E = \epsilon^{\mu \nu} F_{\mu \nu}, \]  

where \( F_{\mu \nu} \equiv \partial_\mu \omega_\nu - \partial_\nu \omega_\mu \). Since the explicit form of the chiral and anti-chiral superfields is given, the \( N = 2 \) twisted SUSY transformation of the component fields can be obtained by the same procedure of subsection 4.1.

In the case of non-Abelian gauge group the constraints (5.10) (5.11) and (5.12) are not (anti-)chiral condition anymore, we cannot obtain the component field expansion of (anti-)chiral superfield easily like in the Abelian case. We thus proceed differently to obtain the component field expansion of (anti-)chiral superfield. We first derive twisted SUSY transformation of the all component fields of the (anti-)chiral superfield by using the constraints obtained from the Jacobi identities.

We first identify the lowest component of the superfields as follows:

\[ W| = A, \quad \nabla_\mu W| = \lambda_\mu, \quad \frac{1}{2} \epsilon^{\mu \nu} \nabla_\mu \nabla_\nu W| = -D, \]
\[ \mathcal{F}| = B, \quad \nabla \mathcal{F}| = \rho, \quad \tilde{\nabla} \mathcal{F}| = \tilde{\rho}, \quad \nabla \tilde{\nabla} \mathcal{F}| = -E, \]

where \( | = |_{\theta = 0} \).

For example the SUSY transformation of the field \( A \) can be derived by the constraint (5.10),

\[ sA = sW| = \nabla W| = [\nabla, i\{\nabla, \nabla\}]| = 0, \]  

where we can replace \( s \) with \( \nabla \) since the \( \theta \)-independent terms coincide due to the choice of Wess-Zumino gauge. We show another example in the following:

\[ s\rho = s\nabla \mathcal{F}| = \nabla \nabla \mathcal{F}| \]
\[ = \nabla (\mathcal{D} \mathcal{F} - i[\Gamma, \mathcal{F}]|) \]
\[ = \mathcal{D}(\mathcal{D} \mathcal{F} - i[\Gamma, \mathcal{F}]|) - i[\Gamma, \mathcal{D} \mathcal{F} - i[\Gamma, \mathcal{F}]|] \]
\[ = -i[\mathcal{D} \Gamma, \mathcal{F}]| - \{\Gamma, [\Gamma, \mathcal{F}]\}| \]
\[ -i\mathcal{W} = \{\nabla, \nabla\} = \{\mathcal{D} - i\Gamma, \mathcal{D} - i\Gamma\} = -2i\mathcal{D} - 2\Gamma^2. \]

We then find
\[ s\rho = -\frac{i}{2}[\mathcal{W}, \mathcal{F}] = -\frac{i}{2}[A, B]. \] (5.26)

Similarly we can find twisted SUSY transformation of all component fields, which we show in Table 5, where we introduce the notation of usual covariant derivative: \( D_{\mu} = \partial_{\mu} - i[\mathcal{W}, \cdots] \).

Table 5: Twisted \( N = 2 \) SUSY transformation of the component fields of non-Abelian super Yang-Mills theory with \( D_{\mu} = \partial_{\mu} - i[\mathcal{W}, \cdots] \).

| \( s \) | \( s_{\mu} \) | \( \tilde{s} \) |
|---|---|---|
| \( A \) | 0 | \( \lambda_{\mu} \) | 0 |
| \( \lambda_{\nu} \) | \( -iD_{\nu} A \) | \( -\epsilon_{\mu\nu} D + \frac{i}{2}\delta_{\mu\nu}[A, B] \) | \( i\epsilon_{\nu\rho} D^\rho A \) |
| \( D \) | \( i\epsilon^{\mu\nu} D_{\mu}\lambda_{\nu} + \frac{i}{2}[A, \tilde{\rho}] \) | \( \frac{1}{2}[B, \epsilon_{\mu\nu}\lambda^\nu] \) | \( -iD^\mu \lambda_{\mu} - \frac{i}{2}[A, \rho] \) |
| \( B \) | \( \frac{1}{2}[A, B] \) | \( 0 \) | \( \tilde{\rho} \) |
| \( \rho \) | \( -E \) | \( -iD_{\mu} B \) | \( E \) |
| \( \tilde{\rho} \) | \( -i\frac{1}{2}[A, \tilde{\rho}] \) | \( i\epsilon_{\mu\lambda} D^\rho B \) | \( -\frac{i}{2}[A, \rho] \) |
| \( E \) | \( iD_{\mu}\tilde{\rho} + i\epsilon_{\mu\nu} D^\nu \rho + \frac{i}{2}[B, \epsilon_{\mu\nu}\lambda^\nu] \) | \( iD^\mu \lambda_{\mu} - i[A, \rho] \) |
| \( \omega_{\nu} \) | \( -i\frac{1}{2}\lambda_{\nu} \) | \( \frac{1}{2}(\epsilon_{\mu\nu}\tilde{\rho} - \delta_{\mu\nu}\rho) \) | \( -\frac{1}{2}\epsilon_{\nu\rho}\lambda^\rho \) |

The SUSY transformation of the gauge field can be derived as follows. The lowest components of the constraint (5.16),
\[ \mathcal{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu} \nabla \nabla \mathcal{F} + \frac{i}{4} \epsilon_{\mu\nu\rho\sigma} \nabla_{\rho} \nabla_{\sigma} \mathcal{W}, \] (5.27)
leads
\[ F_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu}(E - D), \] (5.28)
where \( F_{\mu\nu} \equiv \partial_{\mu}\omega_{\nu} - \partial_{\nu}\omega_{\mu} - i[\omega_{\mu}, \omega_{\nu}] \). Here we define a new field
\[ G = E + D, \] (5.29)
and then the degrees of freedom of \( E \) and \( D \) can be replaced by \( \epsilon^{\mu\nu} F_{\mu\nu} \) and \( G \).

Operating \( s \) charge on (5.28), we obtain the following relation:
\[ sE - sD = -i\epsilon^{\mu\nu} D_{\mu}\lambda_{\nu}. \]
where we have used the $s$ transformation of $E$ and $D$ from the Table 5. Thus the $s$ transformation law of the gauge filed can be identified up to the gauge transformation as follows:

$$s\omega_\mu = -\frac{i}{2}\lambda_\mu.$$  \hfill (5.31)

We can derive other twisted supercharge transformations of the gauge field in the similar way. It should be understood in the Table 5 that the transformation of the fields $E$ and $D$ can be replaced by that of $G$ and the gauge field $\omega_\mu$ due to the relations (5.28) and (5.29).

As we can see in the identification of SUSY transformation in (5.30), the twisted SUSY algebra is expected to be valid up to some gauge transformations. We can confirm that the twisted SUSY algebra is in fact valid up to the gauge transformation with the following specific choice of gauge parameters:

$$\begin{align*}
\{s, s\} &\varphi = \delta_{\text{gauge}} (-A) \varphi, \\
\{s_\mu, s_\nu\} &\varphi = \delta_{\mu\nu} \delta_{\text{gauge}} (-B) \varphi, \\
\{\tilde{s}, \tilde{s}\} &\varphi = \delta_{\text{gauge}} G \varphi, \\
\{\tilde{s}, s_\mu\} &\varphi = i\epsilon_{\mu\nu}\partial^\nu \varphi + \delta_{\text{gauge}} (-i\epsilon_{\mu\nu} \omega_\nu) \varphi, \\
\{s, \tilde{s}\} &\varphi = 0,
\end{align*}$$

\hfill (5.32)

where $\delta_{\text{gauge}} (\epsilon)$ denotes gauge transformation of parameter $\epsilon$ for any component fields $\{\varphi\} = \{A, \lambda_\mu, D, B, \rho, \tilde{\rho}, E, \omega_\mu\}$.

Since we now know all the twisted $N = 2$ SUSY transformation of the component fields of the superfields $\mathcal{F}$ and $\mathcal{W}$, we can reconstruct the component fields expansion of these superfields. Since $A$ and $B$ are parent fields of $\mathcal{F}$ and $\mathcal{W}$, respectively, we may define as

$$\begin{align*}
\mathcal{W} &= e^{\delta_\theta} A, \\
\mathcal{F} &= e^{\delta_\theta} B.
\end{align*}$$

Explicit forms of $\mathcal{W}$ and $\mathcal{F}$ are

$$\begin{align*} 
\mathcal{W}(x) &= A + \theta\theta^i \frac{i}{2} D_\mu A + \theta^\mu \theta^i \frac{i}{2} \epsilon_{\mu\nu} D^\nu A + \theta^4 \frac{1}{4} D^\mu D_\mu A \\
&+ \theta^\mu \lambda_\mu + \theta^2 \frac{i}{2} \epsilon_{\mu\nu} D_\mu \lambda_\nu - \theta^2 \bar{\theta}^i \frac{i}{2} D^\mu \lambda_\mu \\
&+ \theta^2 D \\
&+ \theta^2 \frac{i}{3} [A, \tilde{\rho}] - \theta^2 \bar{\theta}^i \frac{i}{3} [A, \rho] + \theta^2 \bar{\theta}^i \frac{i}{6} [A, \epsilon_{\mu\nu} \lambda_\nu] \\
&- \theta^4 \frac{i}{6} \{\lambda^\mu, \lambda_\mu\} + \theta^4 \frac{1}{8} [A, [A, B]],
\end{align*}$$

\hfill (5.35)
\[ \mathcal{F}(x) = B - \theta \theta^\mu \frac{i}{2} D_\mu B - \theta^\mu \tilde{\theta} \frac{i}{2} \epsilon_{\mu \nu} D^\nu B + \theta^4 \frac{1}{4} D^\mu D_\mu B + \theta^\rho - \theta \theta^\mu \tilde{\theta} \frac{i}{2} \epsilon_{\mu \nu} D^\nu \rho + \tilde{\theta} \tilde{\rho} - \theta^\mu \tilde{\theta} \frac{i}{2} D_\mu \tilde{\rho} + \theta \tilde{\theta}(F + D) \]
\[ + \theta^2 \frac{i}{6} \{ \tilde{\rho}, B \} - \theta^2 \tilde{\theta} \frac{i}{6} \{ \rho, B \} + \theta \theta^\mu \tilde{\theta} \frac{i}{3} \{ \epsilon_{\mu \nu} \lambda^\nu, B \} \]
\[ + \theta^4 \{ \frac{1}{8} [B, [B, A]] - \frac{i}{6} \{ \rho, \rho \} - \frac{i}{6} \{ \tilde{\rho}, \tilde{\rho} \} \}. \] (5.36)

Using the superfields \( F \) and \( W \), we can construct off-shell \( N = 2 \) twisted SUSY invariant action with non-Abelian gauge symmetry:

\[
S = \int d^2 x d^4 \theta \text{Tr} W(x) \mathcal{F}(x) = \int d^2 x \frac{1}{2} s^\mu s_\mu s^{\dot{\nu}} s_{\dot{\nu}} \text{Tr}(AB) = \int d^2 x \text{Tr} \left( -\frac{1}{2} F^\mu F_{\mu} + \frac{G^2}{4} + D^\mu D_\mu A B - iD^\mu \lambda_\mu \rho - i\epsilon^{\mu \nu} D_\mu \lambda_\nu \tilde{\rho} \right. \]
\[ - \frac{1}{4} [A, B]^2 - \frac{i}{2} \{ \rho, \rho \} A - \frac{i}{2} \{ \tilde{\rho}, \tilde{\rho} \} A - \frac{i}{2} \{ \lambda^\mu, \lambda_\mu \} B \right) \]. \] (5.37)

We identify this action as off-shell twisted \( N = 2 \) SUSY invariant super Yang-Mills action. This action is equivalent to the kinetic terms of quantized topological Yang-Mills action which was derived previously [41, 50].

### 5.2 \( N = 2 \) superconnection formalism in four dimensions

As we show in the previous subsection twisted \( N = 2 \) SUSY invariant super Yang-Mills action has been derived by the superconnection formalism in two dimensions. Based on the philosophy of Dirac-Kähler twisting procedure we naively expect that twisted \( N = 4 \) SUSY invariant super Yang-Mills action may be derived by the superconnection formalism parallel to the \( N = D = 2 \) superconnection formalism. It turns out that it is highly nontrivial to find out twisted \( N = 4 \) counterparts of the constraints (5.8) in a systematic way. Instead here we consider superconnection formalism of \( N = 2 \) truncated version of twisted \( N = 4 \) SUSY algebra, which is given in subsection 4.2. This formulation is closely related to the \( N = 2 \) formulation given by Alvarez and Labastida who used spinor formulation of usual four-dimensional extended \( N = 2 \) SUSY algebra [54]. Hereafter we omit the \( + \) suffix from \( N = 2 \) twisted SUSY counterpart decomposed from \( N = 4 \) twisted SUSY for supercharges and differential operators: \( \{ s^I_\pm \} = \{ s^+, s^{\dot{+}}, s^{\dot{+}}_\mu \} \), \( \{ \mathcal{D}^+_I \} = \{ \mathcal{D}^+, \mathcal{D}^+_\mu, \mathcal{D}^+_\mu \} \) and \( \{ Q^I_\pm \} = \{ Q^+, Q^{\dot{+}}, Q^{\dot{+}}_\mu \} \).

As in the two-dimensional formulation we first introduce supercovariant derivatives by introducing fermionic superconnections \( \Gamma_0, \Gamma_\mu, \Gamma_\dot{\mu} \) and a bosonic supercon-
\begin{align}
\n\nabla_0 &= \mathcal{D} - i\Gamma_0, \\
\nabla_\mu &= \mathcal{D}_\mu - i\Gamma_\mu, \\
\nabla_{\mu\nu} &= \mathcal{D}_{\mu\nu} - i\Gamma_{\mu\nu}, \\
\nabla_\mu &= \partial_\mu - i\Gamma_\mu,
\end{align}

(5.38)

where the superderivative differential operators $\mathcal{D}^+ = \mathcal{D}, \mathcal{D}_{\mu}^+ = \mathcal{D}_{\mu}$ and $\mathcal{D}_{\mu\nu}^+ = \mathcal{D}_{\mu\nu}$ are given in (4.17) and satisfy the four-dimensional $N = 2$ twisted SUSY algebra (4.11) with replacements: $s^+_I \rightarrow \mathcal{D}^+_I$. The lowest ($\theta = 0$) component of the connection $\Gamma_\mu$ is identified as the usual gauge field $\Gamma_\mu|_{\theta=0} = \omega_\mu$.

We can introduce the following supergauge transformation:

$$
\delta_{\text{gauge}} \Gamma_I = \nabla_I K \equiv \mathcal{D}_I K - i[\Gamma_I, K],
$$

(5.39)

where $K$ is an arbitrary superfield. We gauge away superfluous component fields by the supergauge freedom of the superfield $K$ by taking Wess-Zumino gauge and keep the usual gauge degrees of freedom.

Parallel to the two-dimensional notations we summarize the commutators and anticommutators of these supercovariant derivatives in Table 6. Here the lowest

| $\nabla_0$ | $\nabla_B$ | $\nabla_\nu$ | $\nabla_\mu$ |
| --- | --- | --- | --- |
| $-i\mathcal{F}_{0,0}$ | $-i\mathcal{F}_{0,B}$ | $-i\mathcal{F}_{\nu} - i\mathcal{F}_{0,\nu}$ | $-i\mathcal{F}_{0,\mu}$ |
| $-i\mathcal{F}_{A,B}$ | $i\delta_{A,\mu\nu\rho}\mathcal{D}^\rho - i\mathcal{F}_{B,\nu}$ | $-i\mathcal{F}_{\mu,\nu}$ | $-i\mathcal{F}_{\mu,\nu}$ |
| $\nabla_\mu$ | $\nabla_\nu$ | $\nabla_\mu$ | $\nabla_\mu$ |

Table 6: Definition of $N = 2$ Supercurvatures

component of $\mathcal{F}_{\mu,\nu}$ is usual curvature:

$$
\mathcal{F}_{\mu,\nu}| = F_{\mu,\nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu - i[\omega_\mu, \omega_\nu].
$$

(5.40)

We impose the following constraints:

$$
\mathcal{F}_{A,B} = \delta_{A,B}^+ F_{0,0}, \\
\mathcal{F}_{\mu,\nu} = \delta_{\mu,\nu}^+ \mathcal{W}, \\
\mathcal{F}_{0,\mu} = \mathcal{F}_{0,A} = \mathcal{F}_{\nu,A} = 0,
$$

(5.41)

which can be interpreted as the suppression conditions of higher spin fields to construct twisted $N = 2$ super Yang-Mills multiplets. Then the Table 6 changes into Table 7 after taking into account the constraints.

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\[ \begin{array}{|c|c|c|c|}
\hline
\nabla_0 & \nabla_B & \nabla_\nu & \nabla_\mu \\
\hline
-iF_{0,0} & 0 & -i\nabla_\nu & -iF_{0,\mu} \\
-i\delta_{A,B}^+F_{0,0} & i\delta_{A,\nu\rho}^+\nabla_\rho & -i\delta_{\mu\nu}^+ & -iF_{A,\mu} \\
\hline
\end{array} \]

Table 7: \( N = 2 \) Supercurvatures after the constraints

Since we have imposed the constraints (5.41), we can obtain series of nontrivial relations by Jacobi identities:

\[ \begin{align*}
\nabla_0 F_{0,0} & \equiv \mathcal{D} F_{0,0} - i[\Gamma_0, F_{0,0}] = 0, \\
\nabla_A F_{0,0} & \equiv \mathcal{D} A F_{0,0} - i[\Gamma_A, F_{0,0}] = 0, \\
\nabla_\mu W & \equiv \mathcal{D} \mu W - i[\Gamma_\mu, W] = 0,
\end{align*} \]

where we assume that the gauge algebra is non-Abelian.

5.2.1 Abelian case

If the gauge algebra is Abelian the first three constraints (5.42), (5.43) and (5.44) turn into the chiral and anti-chiral constraints:

\[ \begin{align*}
\mathcal{D} F_{0,0} & = 0, \\
\mathcal{D} A F_{0,0} & = 0, \\
\mathcal{D} \mu W & = 0.
\end{align*} \]

We can solve these chiral and anti-chiral constraints (5.50) and (5.51), respectively, and expand these superfields as follows:

\[ \begin{align*}
F_{0,0}(y, \theta^+\mu) & = \phi + \theta^+\mu C_\mu + \frac{1}{2} \theta^+\mu \theta^+\nu \phi_{\mu\nu} + \theta^+\nu \bar{\phi} + \theta^+\bar{\phi}, \\
W(w, \theta^+, \theta^+A) & = -\bar{\phi} + \frac{1}{4} \theta^A \bar{\chi}_A^+ + \frac{1}{4} (\theta^2^+)^A M_A^+ + (\theta^3^+) \bar{\chi}.
\end{align*} \]
\[ + \theta^+(\chi + \frac{1}{4} \theta^A N^+_A + \frac{1}{4} (\theta^{2+})^A \chi^+_A + (\theta^3+) L). \] (5.52)

In two-dimensional \( N = 2 \) case we have obtained only one relation (5.21) between the usual curvature and the component fields of chiral and anti-chiral superfields. In four-dimensional twisted \( N = 2 \) formulation the constraints (5.48) and (5.49) lead to the following relations:

\[
\tilde{\phi} = -\partial^2 \phi, \\
\tilde{\psi}^\mu = i(\partial^\mu \chi - \partial_\nu \chi^{+\mu\nu}), \\
\phi_A \equiv \phi_A^+ + \phi_A^- = \phi_A - 2F_A^-, \\
N_A^+ = \phi_A^+ + 2F_A^+, \\
M_A^+ = \phi_A^+ - 2F_A^+, \\
\tilde{\psi} = -i\partial^\mu C_\mu, \\
L = \partial^2 \phi, \\
\tilde{\chi}_A^+ = i\delta^+_{\lambda, \rho \sigma} \partial^{\rho} \phi^{\sigma} = i(\partial \phi)^+_A, \\
\text{where we define a (anti-)self-dual part of the curvature:}
\]

\[ F^\pm_{\mu\nu} \equiv \frac{1}{4} \delta^\pm_{\mu\nu, \rho\sigma} F^{\rho\sigma}. \] (5.55)

Taking into account these relations, we obtain the explicit form of the superfields \( F_{0,0} \) and \( W \) as

\[
\begin{align*}
F_{0,0} &= \phi + \theta^{+\mu} C_\mu + \frac{1}{2} \theta^{+\mu} \theta^{+\nu}(\phi_{\mu\nu}^+ - 2F_{\mu\nu}^-) \\
&\quad + i\theta^{+\mu}(\partial^\mu \chi - \partial_\nu \chi^{+\mu\nu}) - \theta^{4+} \partial^2 \phi, \\
W &= \phi^{+\lambda} A^{\lambda} + \frac{1}{4} (\theta^{2+})^A (\phi_A^+ - 2F_A^+) - i(\theta^3^+) \partial^\mu C_\mu \\
&\quad + \theta^+(\chi + \frac{1}{4} \theta^A \chi^+_A + \frac{1}{4} (\theta^{2+})^A (\partial \phi)^+_A + (\theta^3^+) \partial^2 \phi). 
\end{align*}
\] (5.56)

Using these superfields, we can obtain off-shell twisted \( N = 2 \) SUSY invariant actions:

\[
S_1 = \frac{1}{2} \int d^4 y d^4 \theta (F_{0,0})^2 \\
= \frac{1}{2} \int d^4 y \frac{1}{4!} \epsilon^{\mu\nu\rho\sigma} s^+_\mu s^-_\rho s^+_\nu s^-_\sigma (\phi^2) \\
= \int d^4 x \{ -\phi \partial^2 \phi - iC_\mu (\partial^\mu \chi - \partial_\nu \chi^{+\mu\nu}) - \frac{1}{4} (\phi_{\mu\nu}^+)^2 + (F_{\mu\nu}^-)^2\}, \quad (5.58)
\]

\[
S_2 = \frac{1}{2} \int d^4 w d^4 \theta W^2
\]

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\[ \frac{1}{2} \int d^4w \, s^+ \frac{1}{3!} \Gamma^{+A,B,C} s^+ A_B s^+ C (\vec{\phi}) \]
\[ = - \int d^4x \{ (-\phi \partial^\mu \vec{\phi} - i C_\mu (\partial^\mu \chi - \partial^\nu \chi^{+\mu \nu}) - \frac{1}{4} (\phi^{+\mu})^2 + (F^{+\mu})^2) \}. \] (5.59)

We can see that these actions (5.58) and (5.59) are Abelian version of \( N = 2 \) twisted SUSY invariant super Yang-Mills action a la Donaldson-Witten. The difference of these actions are topologically invariant surface term.

Here it is interesting to recognize that these actions \( S_1 \) and \( S_2 \), respectively, have the same forms as (4.21) and (4.24) which don’t have derivatives in the Lagrangian. It turns out that the derivatives in the actions \( S_1 \) and \( S_2 \) are generated by the constraint (5.41) which in turn relate the chiral and anti-chiral superfields. Eventually the corresponding actions (5.58) and (5.59) are related and the difference is only the topological surface term.

Twisted \( N = 2 \) SUSY transformation can be obtained by operating superdifferential operators to the superfields and is given in Table 8. As in the two-dimensional case the twisted \( N = 2 \) SUSY algebra (4.11) is valid modulo gauge transformation.

### 5.2.2 Non-Abelian case

If the gauge algebra is non-Abelian the constraints (5.42), (5.43) and (5.44) are no longer chiral and anti-chiral constraints and thus \( F_{0,0} \) and \( W \) are not (anti-)chiral superfields. We cannot then obtain the explicit component fields expansion of the superfields from simple arguments. We then proceed to derive the twisted SUSY transformation of component fields from the constraint relations obtained from Jacobi identity as in the two-dimensional non-Abelian case. We first identify the following component fields:

\[ F_{0,0} = \phi, \quad \nabla_\mu F_{0,0} = C_\mu, \quad \frac{1}{2} (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) F_{0,0} = -\phi_{\mu\nu}, \]
\[ \frac{1}{3!} \epsilon^{\mu\nu\rho\sigma} \nabla_\nu \nabla_\rho \nabla_\sigma F_{0,0} = -\bar{\phi}^\mu, \quad \frac{1}{4!} \epsilon^{\mu\nu\rho\sigma} \nabla_\mu \nabla_\nu \nabla_\rho \nabla_\sigma F_{0,0} = \bar{\phi} \quad \text{(5.60)} \]
\[ \mathcal{W} = \bar{\phi}, \quad \nabla_A \mathcal{W} = \chi^+_A, \quad \nabla_0 \mathcal{W} = \chi, \quad \nabla_A \nabla_0 \mathcal{W} = N_A^+, \quad \nabla_0 \nabla_A \mathcal{W} = -L. \] (5.61)

We can then obtain the twisted \( N = 2 \) SUSY transformation of these component fields by Jacobi identities with Wess-Zumino gauge as in the two-dimensional case. For example we find the transformation law of \( \phi \) as follows:

\[ s\phi = QF_0,0 = -\nabla_0, i{\nabla_0, \nabla_0} = 0. \] (5.62)

We can similarly find the transformation laws of all the other component fields. We show the list of twisted \( N = 2 \) SUSY transformation of component fields in Table 9.

| Field | \( s^+ \) | \( s^+_\mu \) |
|-------|---------|---------|
| \( \phi \) | 0 | \( C_\mu \) |
| \( C_\nu \) | \(-iD_\nu \phi \) | \(-i\delta^+_{B,\mu\nu}(D^\sigma \chi - D_\sigma \chi^{+\mu\nu}) + \frac{i}{2}\delta^+_{B,\mu\nu}[\phi, C_\nu] \) |
| \( \phi^+_B \) | \( \frac{i}{2}\delta^+_{B,\rho\sigma}(D^\rho C^\sigma - \frac{2}{3}[\chi, \phi]) \) | \(-\phi^+_{\mu\nu} + 2F^-_{\mu\nu} + \frac{i}{2}\delta_{\mu\nu}[\phi, \bar{\phi}] \) |
| \( \phi \) | \( \chi \) | \( i\delta^+_{B,\mu\nu} D^\nu \bar{\phi} \) |
| \( \chi^+_B \) | \(-\phi^+_B - 2F^+_B \) | \(-iD_\nu \bar{\phi} \) |
| \( \chi \) | \(-\frac{i}{2}[\phi, \bar{\phi}] \) | \(-\frac{i}{2}(\delta_{\mu\nu} \chi - \chi^+_{\mu\nu}) \) |
| \( \omega_{\nu} \) | \(-\frac{i}{2}C_\nu \) | |

| Field | \( \mathcal{W}^A \) |
|-------|-----|
| \( \phi \) | 0 |
| \( C_\nu \) | \( i\delta^+_{A,\nu\rho} D^\sigma \phi \) |
| \( \phi^+_B \) | \(-\frac{i}{2}\delta^+_{B,\alpha\beta}\delta^+_{\alpha\gamma}(D^\gamma C^\beta - \frac{1}{3}[\chi, \phi]) + \frac{i}{2}\delta^+_{A,B}[\chi, \phi] \) |
| \( \phi \) | \(-\phi^+_B + 2F^+_B - \frac{i}{2}\delta_{A,B}[\phi, \bar{\phi}] \) |
| \( \chi^+_A \) | \( \chi^+_A \) |
| \( \chi \) | \(-\frac{1}{4}\Gamma^{BC}(\phi^+ + 2F^+ C) - \frac{i}{2}\delta_{A,B}[\phi, \bar{\phi}] \) |
| \( \omega_{\nu} \) | \(-\frac{i}{2}\delta^+_{A,\nu\rho} C^\rho \) |

Table 9: \( N = 2 \) twisted SUSY transformation of component fields for non-Abelian Donaldson-Witten theory

The \( N = 2 \) twisted SUSY algebra is closed for the fields up to the gauge transformation of the following specific gauge parameter choice:

\[ \{s^+, s^+\} \phi = \delta_{gauge}(-\phi) \phi \]
\[ \{s^+_\mu, s^+_\nu\} \varphi = \delta_{\mu\nu} \delta_{\text{gauge}}(-\psi) \varphi, \]
\[ \{s^+_A, s^+_B\} \varphi = \delta^+_{AB} \delta_{\text{gauge}} (-\phi) \varphi; \]
\[ \{s^+, s^+_A\} \varphi = 0, \]
\[ \{s^+, s^+_\mu\} \varphi = -i(\partial_\mu \varphi + \delta_{\text{gauge}} (-A_\mu) \varphi), \]
\[ \{s^+_A, s^+_\mu\} \varphi = i\delta^+_{A,\mu}(\partial^\nu \varphi + \delta_{\text{gauge}} (-A^\nu) \varphi), \] (5.63)

where \( \{\varphi\} = \{\phi, C_\mu, \phi^+_A, \bar{\phi}, \chi^+_A, \chi, \omega_\mu\} \) and \( \delta_{\text{gauge}} \phi \omega_\mu = \partial_\mu \phi - \frac{i}{2} [\omega_\mu, \phi] \) for the gauge field and \( \delta_{\text{gauge}} \phi \cdots = -i[\cdots, \phi] \) for the other fields. We can see that \( N = 2 \) twisted SUSY algebra in four dimensions is closed modulo the gauge transformation.

We can now define the superfield \( \mathcal{F}_{0,0} \) and \( \mathcal{W} \) from the parent fields \( \phi \) and \( \bar{\phi} \) as follows:

\[ \mathcal{F}_{0,0}(y^\mu, \theta^{+\mu}) = e^{\theta^{+\mu} s^+_\mu} e^{\theta^+ s^+_+ \frac{1}{2} \theta^{+\mu \nu} s^+_{\mu \nu}} \phi \]
\[ = e^{\theta^{+\mu} s^+_\mu} \phi, \] (5.64)
\[ \mathcal{W}(w^\mu, \theta^{+\mu}) = e^{\theta^{+\mu} + \frac{1}{2} \theta^{+\mu \nu} s^+_{\mu \nu} e^{\theta^+ s^+_+ \frac{1}{2} \theta^{+\mu \nu} s^+_{\mu \nu}}} \]
\[ = e^{\theta^{+\mu} + \frac{1}{2} \theta^{+\mu \nu} s^+_{\mu \nu} \phi}, \] (5.65)

The superfield \( \mathcal{F}_{0,0} \) and \( \mathcal{W} \) are then given by

\[ \mathcal{F}_{0,0}(y^\mu, \theta^{+\mu}) = \phi(y^\mu) + \theta^{+\mu} C_\mu + \frac{1}{2} \theta^{+\mu \nu}(\phi^+_{\mu \nu} - 2F^+_{\mu \nu}) \]
\[ + i(\theta^{3+})_{\mu}(D^\mu \chi - D_\nu \chi^{+ \mu \nu} + \frac{1}{2} [\phi, C_\mu]) \]
\[ + \theta^{+\mu} (-D^\mu D_\mu \phi + \frac{i}{8} \{\chi^{+ \mu \nu}, \chi^{+ \mu \nu}\}) \]
\[ + \frac{i}{2} \{\chi, \chi\} - \frac{1}{4} [\phi, [\phi, \phi]]), \] (5.66)

\[ \mathcal{W}(w^\mu, \theta, \theta^{+\mu}) = \bar{\phi}(w^\mu) + \theta \chi + \frac{1}{4} \theta^+ A^+ \chi^+_A + \frac{1}{4} \theta^+ A^+(\phi^+_A + 2F^+_A) \]
\[ + \frac{1}{4} (\theta^{2+})^4 (\phi^+_A - 2F^+_A) \]
\[ + \theta^+ (\theta^{2+})^4 \left( \frac{i}{4} \delta^+_{A, \mu \nu} D^\mu C^\nu - \frac{i}{8} [\chi^+_A, \phi] \right) \]
\[ + \theta^{3+} (-iD^\mu C_\mu + \frac{i}{2} [\chi, \phi]) \]
\[ + \theta^+ \theta^{3+} (D^\mu D_\mu \phi - \frac{i}{2} \{C^\mu, C_\mu\} - \frac{1}{4} [\phi, [\phi, \phi]]). \] (5.67)

We can now define the following off-shell \( N = 2 \) twisted SUSY invariant actions:

\[ S_1 = \frac{1}{2} \int d^4y d^4\theta \text{ Tr} \mathcal{F}_{0,0}^2 \]
\[ = \frac{1}{2} \int d^4y \frac{1}{4!} e^{\mu \nu \rho \sigma} s^+_{\mu} s^+_{\nu} s^+_{\rho} s^+_{\sigma} \text{ Tr}(\phi^2), \]
\[ S_2 = \frac{1}{2} \int d^4wd^4\theta \text{ Tr} \mathcal{W}^2 \]
\[ = \frac{1}{2} \int d^4w \frac{1}{3!} \delta^{+ A, B, C} s^+_{A} s^+_{B} s^+_{C} \text{ Tr}(\bar{\phi}^2), \] (5.68)
where \( \int d^4 \theta (\theta^{4+}) = 1 \), \( \int d^4 \theta (\theta^{3+}) = 1 \). We can obtain the explicit form of these actions,

\[
S_1 = \int d^4 x \text{Tr} \left( - \phi D\mu D\nu \phi - iC\mu (D\mu \chi - D\nu \chi_{\mu\nu}^+) - \frac{1}{4} (\phi^{+})^2 + (F_{\mu\nu})^2 \\
+ \frac{i}{2} \phi \{ \chi, \chi \} + \frac{i}{8} \phi \{ \chi^{+A}, \chi^{+}_{A} \} + \frac{i}{2} \bar{\phi} \{ C^\mu, C_{\mu} \} + \frac{1}{4} [\phi, \bar{\phi}]^2 \right),
\]

\[
S_2 = - \int d^4 x \text{Tr} \left( \phi D\mu D\nu \phi - iC\mu (D\mu \chi - D\nu \chi_{\mu\nu}^+) - \frac{1}{4} (\phi^{+})^2 + (F_{\mu\nu})^2 \\
+ \frac{i}{2} \phi \{ \chi, \chi \} + \frac{i}{8} \phi \{ \chi^{+A}, \chi^{+}_{A} \} + \frac{i}{2} \bar{\phi} \{ C^\mu, C_{\mu} \} + \frac{1}{4} [\phi, \bar{\phi}]^2 \right).
\]

The difference of these actions is only topological surface term. These actions are equivalent to non-Abelian version of Donaldson-Witten action which has off-shell twisted \( N = 2 \) SUSY invariance.

We have derived the four-dimensional quantized topological Yang-Mills action by using superconnection formulation. Thus our twisted \( N = 4 \) superspace formalism naturally includes our twisted \( N = 2 \) superspace formalism as a subalgebra.

### 5.2.3 Dirac-Kähler matter fermion

As we have already shown that Dirac-Kähler(D-K) twisting procedure of supercharges generated \( N = 4 \) twisted superalgebra which is decomposed into two sets of \( N = 2 \) twisted superalgebra by (anti-)self-dual decomposition. On the other hand the D-K fermion mechanism transforms fermionic anti-symmetric tensor fields into matter fermions, we expect that this type of decomposition will also be generated in the matter fermion sector of D-K fermion. Here we explicitly construct twisted \( N = 2 \) sector of D-K fermion.

We claim that the following two ”flavor” Dirac fermion action is essentially equivalent to \( N = 2 \) sector of D-K fermion constructed from the D-K fermion mechanism:

\[
\int d^4 x \sum_{i=1}^{2} \Psi^{i \mu N=2} \gamma^\mu \partial \mu \Psi^{i \mu N=2},
\]

where \( \Psi^{N=2} = (\Psi^{i N=2})^\dagger \) and \( \Psi^{i N=2} \) is \( SU(2) \) Majorana fermion which satisfies the following condition [77]:

\[
(\Psi^{i N=2})^* = \epsilon^{ij} B \Psi^{j N=2},
\]

where \( B = -\gamma^{1} \gamma^{3} \) and \( \gamma^\mu = B^{-1} \gamma^\mu B \).

The \( N = 2 \) sector of the D-K fermion \( \Psi^{i N=2} \) is the chirally projected sector of the original D-K fermion on the ”flavor” suffix or \( R \)-symmetry related suffix as follows:

\[
(\Psi^{N=2})_{\alpha i} = (\Psi)_{\alpha j}(P_+)_{ji},
\]
where $P_\pm = \frac{1}{2}(1 \pm \gamma_5)$ and the D-K fermion $\Psi$ is defined in (2.23) as
\[
\Psi_{\alpha i} = \frac{1}{2\sqrt{2}}(1\psi + \gamma^\mu \psi_\mu + \frac{1}{2}\gamma^{\mu\nu}\psi_{\mu\nu} + \gamma^\mu \tilde{\psi}_\mu + \gamma_5 \tilde{\psi})_{\alpha i},
\]
(5.74)

where $i = 1, 2, 3, 4$.

The action (5.71) can be transformed as follows:
\[
\int d^4x \sum_{i=1}^{2} \bar{\Psi}_{iN=2}^{iN=2} \gamma^\mu \partial_\mu \Psi_{iN=2}
\]
\[
= \int d^4x \text{Tr} \bar{\Psi} \gamma^\mu \partial_\mu \Psi P_+
\]
\[
= \int d^4x \frac{1}{2} \left\{ (\psi + \tilde{\psi}) \partial_\mu (\psi^\mu + \tilde{\psi}^\mu) - (\psi_\mu + \tilde{\psi}_\mu) (\psi^\mu - \frac{1}{2} \epsilon_{\mu\rho\sigma} \psi_\rho \psi_\sigma) \right\}
\]
\[
= \int d^4x \left\{ \chi \partial_\mu C_\mu - C_\mu \partial_\rho \chi_\rho^+ \right\},
\]
(5.75)

where $\chi = \frac{1}{\sqrt{2}}(\psi + \tilde{\psi})$, $C_\mu = \frac{1}{\sqrt{2}}(\psi^\mu + \tilde{\psi}^\mu)$, $\chi_\rho^+ = \frac{1}{\sqrt{2}}(\psi^\rho - \frac{1}{2} \epsilon^{\rho\mu\sigma} \psi_\mu \psi_\sigma)$. These terms are included in the action (5.69). Then the action (5.69) eventually turns into the following form:
\[
S_1 = \int d^4x \text{Tr} \left( -\phi D^\mu D^\mu \phi - i\bar{\Psi}_{iN=2}^{iN=2} \gamma^\mu D_\mu \Psi_{iN=2} - \frac{1}{4} \phi^+ \phi \right) \right)
\]
\[
+ i\phi \bar{\Psi}_{iN=2}^{iN=2} (1 + \gamma_5) \Psi_{iN=2} + i\phi \bar{\Psi}_{iN=2}^{iN=2} (1 - \gamma_5) \Psi_{iN=2} + \frac{1}{4} \phi \phi^+ \phi \phi^+ \right)
\]
(5.76)

where fermions are now transformed into $SU(2)$ Majorana fermions via Dirac-Kähler fermion mechanism.

6 Conclusions and Discussions

We have proposed four-dimensional twisted $N = 4$ superspace formalism based on the Dirac-Kähler twisting procedure. In the Dirac-Kähler twist, $N = 2$ twisted superspace formalism in two dimensions and that of $N = 4$ in four dimensions have close similarity in the formulation. We have examined the formulation in various cases to derive off-shell invariant twisted SUSY invariant actions. To see the basic structure in simpler examples we have given $D = N = 2$ formulation as well. In order to find $N = 4$ invariant action with gauge symmetry in four dimensions, we have introduced twisted vector superfield and found formal expressions of actions which have off-shell $N = 4$ twisted SUSY invariance. It turns out, however, the actions have still many superfluous component fields and too higher derivatives for the gauge fields and we may still need to find further constraints to obtain $N = 4$ twisted super Yang-Mills action.
We have then proposed twisted superconnection formalism to find supergauge
invariant Yang-Mills actions. In two dimensions we have successfully derived $N = 2$
twisted SUSY invariant super Yang-Mills actions. In four dimensions we have not
yet found out constraints to derive $N = 4$ twisted SUSY invariant super Yang-Mills
action. By decomposing the twisted $N = 4$ algebra into two sectors of $N = 2$ algebra
and thus truncating the full algebra into $N = 2$, we have proposed twisted $N = 2$
superspace formalism in four dimensions. We have then examined the superconnec-
tion formalism on the $N = 2$ sector of twisted SUSY algebra. We have successfully
derived $N = 2$ super Yang-Mills action from the $N = 2$ twisted superspace and
thus the super Yang-Mills action is off-shell $N = 2$ twisted SUSY invariant. In the
final form of the twisted $N = 2$ SUSY invariant Yang-Mills action, the ghost-related
fermions in the twisted sector turn into matter fermions by the Dirac-Kähler fermion
mechanism.

One of the main aims of establishing this twisted superspace formalism in four
dimensions is to derive $N = 4$ twisted SUSY invariant Yang-Mills action. As we
have seen already from some examples of $N = 4$ twisted SUSY invariant actions,
there are either too many superfluous component fields in the action or too higher
derivatives in the action. The strategy to find further constraints in vector superfield
and superconnection formulations are not obvious. The constraints we imposed for
the superconnection formalism can be interpreted as the suppression condition of
higher spin fields in the supermultiplets. We may hope that this type of condition
may lead a clue to find consistent constraints for superconnections of $N = 4$ twisted
SUSY algebra. It is known that there are some nontrivial examples of off-shell usual
$N = 4$ SUSY invariant Yang-Mills action [60–62] which may have twisted superspace
counterpart of the formulation.

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Appendix A  Four-dimensional $\gamma$-matrix and related formulae

We use the following Euclidean four-dimensional $\gamma$-matrix throughout this paper. The $\gamma$-matrixes satisfy the following Clifford algebra:

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu \nu}.$$  \hspace{1cm} (A.1)

We use the following representation of $\gamma$-matrix:

$$\gamma^\mu = \begin{pmatrix} 0 & i\sigma^\mu \\ i\sigma^\mu & 0 \end{pmatrix},$$

where $\sigma^\mu = (\sigma^1, \sigma^2, \sigma^3, \sigma^4), \sigma^\mu = (-\sigma^1, -\sigma^2, -\sigma^3, \sigma^4)$ with $\sigma^4 = i1_{2\times2}$ and $\{\sigma^i\}(i = 1, 2, 3)$ are Pauli matrixes. We introduce the following charge conjugation matrix $C$ and $B$-matrix:

$$\gamma^\mu T = C\gamma_\mu C^{-1}, \quad C \equiv -\gamma_1\gamma_3, \quad C^T = -C, \quad \gamma_5 \equiv \gamma_1\gamma_2\gamma_3\gamma_4,$$

$$\gamma^\mu \equiv B\gamma^\mu B^{-1}, \quad B^*B = -1, \quad B \equiv -\gamma^1\gamma^3.$$  \hspace{1cm} (A.2)

We then introduce the following notations:

$$\gamma_{\mu\nu} \equiv \frac{1}{2}[\gamma_\mu, \gamma_\nu], \quad \tilde{\gamma}_\mu \equiv \gamma_\mu\gamma_5.$$  \hspace{1cm} (A.3)

We list useful formulae which are used in the algebraic manipulation of $N = 4$ twisted SUSY algebra:

$$\gamma^\mu \gamma^\nu = \gamma^{\mu\nu} + \delta^{\mu\nu},$$  \hspace{1cm} (A.4)

$$\gamma^\mu \gamma^\nu \gamma^\rho = \gamma^{\mu\nu\rho} + \gamma^\mu \delta^{\nu\rho} - \gamma^\nu \delta^{\mu\rho} + \gamma^\rho \delta^{\mu\nu},$$

$$\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma = \epsilon^{\mu\nu\rho\sigma}\gamma_5 + \gamma^\mu \delta^{\nu\rho}\delta^{\sigma} - \gamma^\rho \delta^{\nu\sigma}\delta^{\mu} + \gamma^\sigma \delta^{\nu\rho}\delta^{\mu} - \gamma^\nu \delta^{\rho\sigma}\delta^{\mu} + \gamma^\nu \delta^{\rho\mu}\delta^{\sigma} + \gamma^\rho \delta^{\nu\sigma}\delta^{\mu} + \gamma^\nu \delta^{\rho\sigma}\delta^{\mu},$$  \hspace{1cm} (A.5)

$$\gamma^{\mu\nu} \gamma^\rho = -\epsilon^{\mu\nu\rho\sigma}\gamma_\sigma + \gamma^\mu \delta^{\nu\rho} - \gamma^\nu \delta^{\mu\rho},$$  \hspace{1cm} (A.7)

$$\gamma^{\mu\nu} \gamma^{\rho\sigma} = \epsilon^{\mu\nu\rho\sigma}\gamma_5 - \gamma^{\mu\rho} \delta^{\nu\sigma} + \gamma^{\mu\sigma} \delta^{\nu\rho} + \gamma^{\nu\rho} \delta^{\mu\sigma} - \gamma^{\nu\sigma} \delta^{\mu\rho} - \delta^{\mu\rho} \delta^{\nu\sigma} + \delta^{\mu\sigma} \delta^{\nu\rho},$$  \hspace{1cm} (A.8)

$$\gamma^\mu \tilde{\gamma}^\rho = -\epsilon^{\mu\rho\sigma}\gamma_\sigma + \tilde{\gamma}^\mu \delta^{\rho\nu} - \tilde{\gamma}^\nu \delta^{\rho\mu},$$  \hspace{1cm} (A.9)

$$\gamma^{\mu\nu} \gamma_5 = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\gamma_{\rho\sigma},$$  \hspace{1cm} (A.10)

$$\gamma^\rho \gamma^{\mu\nu} = -\epsilon^{\mu\nu\rho\sigma}\gamma_\sigma - \gamma^\rho \delta^{\mu\nu} + \gamma^\mu \delta^{\rho\nu},$$  \hspace{1cm} (A.11)

$$\tilde{\gamma}^\rho \gamma^{\mu\nu} = -\epsilon^{\mu\nu\rho\sigma}\gamma_\sigma - \tilde{\gamma}^\rho \delta^{\mu\nu} + \tilde{\gamma}^\nu \delta^{\rho\mu},$$  \hspace{1cm} (A.12)

$$\gamma_5 \gamma^{\mu\nu} = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\gamma_{\rho\sigma}.$$  \hspace{1cm} (A.13)
Appendix B  Definition of $\Gamma^{ab,cd,ef,gh,ij,kl}$

We introduce the following symbol:

$$\theta^{ab}\theta^{cd}\theta^{ef}\theta^{gh}\theta^{ij}\theta^{kl} \equiv \Gamma^{ab,cd,ef,gh,ij,kl}(\theta^{12}\theta^{13}\theta^{14}\theta^{23}\theta^{24}\theta^{34}),$$  \hspace{1cm} (B.1)$$

where $\theta^{ab}$ are twisted superparameters with anti-symmetric tensor suffix. These tensor suffixes have six independent degrees of freedom and thus $\Gamma$ behaves like a six-dimensional totally anti-symmetric tensor. Here we denote the tensor suffixes $\{ab\}$ by the upper case $A$. We can derive the following formulae:

$$\Gamma^{ABCDEF}\Gamma_{ABCDEF} = 6!^2 6^2,$$  \hspace{1cm} (B.2)$$

$$\Gamma^{ABCDEF}\Gamma_{ABCDHG} = 5!^2 5^2 \delta^F_G,$$  \hspace{1cm} (B.3)$$

$$\Gamma^{ABCDEF}\Gamma_{ABCDEF} = 4!^2 4^2 (\delta^E_G \delta^F_H - \delta^E_H \delta^F_G),$$  \hspace{1cm} (B.4)$$

$$\Gamma^{ABCDEF}\Gamma_{ABCDEF} = 3!^2 3^2 \{\delta^D_G \delta^E_H \delta^F_I - \delta^D_I \delta^E_H \delta^F_G + \text{cyclic in GHI} \},$$  \hspace{1cm} (B.5)$$

$$\Gamma^{ABCDEF}\Gamma_{ABGHIJ} = | \delta^C_G \delta^D_G \delta^E_G \delta^F_G |
| \delta^C_H \delta^D_H \delta^E_H \delta^F_H |
| \delta^C_I \delta^D_I \delta^E_I \delta^F_I |
| \delta^C_J \delta^D_J \delta^E_J \delta^F_J |,$$  \hspace{1cm} (B.6)$$

$$\Gamma^{ABCDEF}\Gamma_{AGHIJK} = | \delta^B_G \delta^C_G \delta^D_G \delta^E_G \delta^F_G |
| \delta^B_H \delta^C_H \delta^D_H \delta^E_H \delta^F_H |
| \delta^B_I \delta^C_I \delta^D_I \delta^E_I \delta^F_I |
| \delta^B_J \delta^C_J \delta^D_J \delta^E_J \delta^F_J |
| \delta^B_K \delta^C_K \delta^D_K \delta^E_K \delta^F_K |,$$  \hspace{1cm} (B.7)$$

$$\Gamma^{ABCDEF}\Gamma_{GHIJKL} = | \delta^A_G \delta^B_G \delta^C_G \delta^D_G \delta^E_G \delta^F_G |
| \delta^A_H \delta^B_H \delta^C_H \delta^D_H \delta^E_H \delta^F_H |
| \delta^A_I \delta^B_I \delta^C_I \delta^D_I \delta^E_I \delta^F_I |
| \delta^A_J \delta^B_J \delta^C_J \delta^D_J \delta^E_J \delta^F_J |
| \delta^A_K \delta^B_K \delta^C_K \delta^D_K \delta^E_K \delta^F_K |
| \delta^A_L \delta^B_L \delta^C_L \delta^D_L \delta^E_L \delta^F_L |.$$  \hspace{1cm} (B.8)
Appendix C  $N = 4$ twisted SUSY transformation of chiral multiplets

We show the full list of the $N = 4$ twisted SUSY transformation of chiral multiplets given in subsection 3.3.

1 Chiral Multiplets

| $\psi$ | $S$ | $S_\mu$ | $S_\mu$ | $S$ |
|--------|-----|---------|---------|-----|
| $\psi^0$ | 0 | $\psi^0_\mu$ | $\psi^0_\mu$ | 0 |
| $\psi^1_a$ | 0 | $-\psi^1_{\mu,a}$ | $-\psi^1_{\mu,a}$ | $-i\partial_\mu \psi^1$ |
| $\psi^0_{ab}$ | 0 | $\psi^0_{\mu,ab}$ | $\epsilon_{ab\mu\nu} \psi^0_{\mu,\nu}$ | $i(\partial_\mu \psi^0_{\nu,b} - \partial_\nu \psi^0_{\mu,a})$ |
| $\psi^0_a$ | 0 | $-\psi^0_{\mu,\alpha}$ | $\delta_{\mu,\alpha} \psi^0_a$ | $\frac{i}{2} \epsilon_\alpha \epsilon_{bcd} \partial_b \psi^0_{\mu,cd}$ |
| $\psi^0$ | 0 | $\psi^1_{\mu}$ | 0 | $-i\partial^\mu \psi^0$ |

| $\psi$ | $S$ | $S_\mu$ | $S_\mu$ | $S$ |
|--------|-----|---------|---------|-----|
| $\psi^1_m$ | $-i\partial_m \psi^0$ | $-\psi^2_{jm,m}$ | $\psi^1_{m,\mu}$ | 0 |
| $\psi^2_m$ | $i\partial_m \psi^1_m$ | $\psi^2_{jm,a}$ | $i(\partial_a \psi^1_m - \partial_m \psi^1_a)$ | $\frac{i}{2} \epsilon_\alpha \epsilon_{bcd} \partial_b \psi^1_{m,cd}$ |
| $\psi^1_{mn,a}$ | $-i\partial_m \psi^0_{n,a}$ | $-\psi^2_{jm,m}$ | $\delta_{m,a} \psi^1_{m,\alpha}$ | $0$ |
| $\psi^2_{mn,ab}$ | $i(\partial_m \psi^1_{n,a} - \partial_n \psi^1_{m,a})$ | $\epsilon_{mn,\mu} \psi^3_{\mu,a}$ | $\epsilon^{\nu}_{mn,\mu} \psi^3_{\nu,a}$ | $-i\partial^\mu \psi^2_{mn,a}$ |
| $\psi^3_{mn,a}$ | $-i(\partial_m \psi^1_{n,a} - \partial_n \psi^1_{m,a})$ | $\epsilon_{mn,\mu} \psi^3_{\mu,a}$ | $\epsilon^{\nu}_{mn,\mu} \psi^3_{\nu,a}$ | $-i\partial^\mu \psi^2_{mn,a}$ |
| $\psi^3_{mn}$ | $i(\partial_m \psi^1_{n,a} - \partial_n \psi^1_{m,a})$ | $\epsilon_{mn,\mu} \psi^3_{\mu,a}$ | $\epsilon^{\nu}_{mn,\mu} \psi^3_{\nu,a}$ | $-i\partial^\mu \psi^2_{mn,a}$ |

| $\psi$ | $S$ | $S_\mu$ | $S_\mu$ | $S$ |
|--------|-----|---------|---------|-----|
| $\psi^3_m$ | $\frac{i}{2} \epsilon_m \epsilon_{\mu\nu} \partial_\mu \psi^2_{\nu,m}$ | $\delta_{m,\mu} \psi^1_{m,\mu}$ | $\psi^3_{m,\mu}$ | 0 |
| $\psi^4_m$ | $-\frac{i}{2} \epsilon_m \epsilon_{\mu\nu} \partial_\mu \psi^2_{\nu,m}$ | $-\delta_{m,\mu} \psi^3_{m,\mu}$ | $-i\partial_\mu \psi^4_m$ |
| $\psi^4_{mn,ab}$ | $i(\partial_m \psi^1_{n,a} - \partial_n \psi^1_{m,a})$ | $\epsilon_{mn,\mu} \psi^3_{\mu,a}$ | $\epsilon^{\nu}_{mn,\mu} \psi^3_{\nu,a}$ | $-i\partial^\mu \psi^4_{mn,a}$ |
| $\psi^4_{mn}$ | $i(\partial_m \psi^1_{n,a} - \partial_n \psi^1_{m,a})$ | $\epsilon_{mn,\mu} \psi^3_{\mu,a}$ | $\epsilon^{\nu}_{mn,\mu} \psi^3_{\nu,a}$ | $-i\partial^\mu \psi^4_{mn,a}$ |

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\[
\begin{array}{|c|c|c|c|c|}
\hline
\psi_{\mu} & S & S_{\mu} & S_{\mu} & S \\
\hline
\psi^4 & -i\partial_{\mu}\psi^3_{\mu} & 0 & \psi^4_{\mu} & 0 \\
\psi^4_{\mu} & i\partial_{\mu}\psi^3_{\mu} & 0 & -\psi^4_{\mu,a} & -i\partial_{\mu}\psi^4_{a} \\
\psi^4_{\mu ab} & -i\partial_{\mu}\psi^3_{\mu,ab} & 0 & \epsilon_{ab\mu}\psi^4_{\nu} & i(\partial_{\alpha}\psi^4_{\beta,b} - \partial_{b}\psi^4_{\alpha,a}) \\
\psi^4_{\mu a} & i\partial_{\mu}\psi^3_{\mu,a} & 0 & \delta_{\mu,a}\psi^4_{\nu} & \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\partial_{\rho}\psi^4_{\sigma,cd} \\
\psi^4 & -i\partial_{\mu}\psi^3_{\mu} & 0 & 0 & -i\partial_{\mu}\psi^4_{a} \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|}
\hline
\psi & S_{\mu
u} \\
\hline
\psi^4 & 0 \\
\psi^4_{\mu} & -i\epsilon_{\mu
u\rho\sigma}\partial_{\rho}\psi^4_{\sigma} \\
\psi^4_{\mu ab} & i(\epsilon_{\mu
u\rho\sigma}\partial_{\rho}\psi^4 b_{\sigma} - \epsilon_{\mu
u\rho\sigma}\partial_{\rho}\psi^4_{\sigma,a}) \\
\psi^4_{\mu a} & \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\epsilon_{\mu
u\rho\sigma}\partial_{\rho}\psi_{\sigma,cd} \\
\psi^4 & i\epsilon_{\mu\nu\rho\sigma}\partial_{\rho}\psi^4_{\sigma} \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|}
\hline
\psi & S_{\mu
u} \\
\hline
\psi^4_{mn} & -i(\delta_{m,\mu}\partial_{\nu} - \delta_{m,\nu}\partial_{\mu})\psi^4_{m} + i(\delta_{n,\mu}\partial_{\nu} - \delta_{n,\nu}\partial_{\mu})\psi^4_{n} \\
\psi^4_{mn,a} & i(\delta_{m,\mu}\partial_{\nu} - \delta_{m,\nu}\partial_{\mu})\psi^4_{n,a} + i(\delta_{n,\mu}\partial_{\nu} - \delta_{n,\nu}\partial_{\mu})\psi^4_{m,a} - i\epsilon_{\mu\nu\rho\sigma}\partial_{\rho}\psi_{mn}^4 \\
\psi^4_{mn,ab} & -i\delta_{mn,\mu}\partial_{\nu}\psi^4_{1,n,a} + i\delta_{mn,\nu}\partial_{\mu}\psi^4_{1,m,a} + i\epsilon_{\mu\nu\rho\sigma}\partial_{\rho}\psi_{mn}^4,\beta \\
\psi^4_{mn,a} & i\delta_{mn,\mu}\partial_{\nu}\psi^4_{1,n,a} - i\delta_{mn,\nu}\partial_{\mu}\psi^4_{1,m,a} + \frac{1}{2}\epsilon_{\alpha\beta}\epsilon_{\mu\nu\rho\sigma}\partial_{\rho}\psi_{mn}^4,\beta \\
\psi^4_{mn,\beta} & -i(\delta_{m,\mu}\partial_{\nu} - \delta_{m,\nu}\partial_{\mu})\psi^4_{1,n,\beta} + i(\delta_{n,\mu}\partial_{\nu} - \delta_{n,\nu}\partial_{\mu})\psi^4_{1,m,\beta} + i\epsilon_{\mu\nu\rho\sigma}\partial_{\rho}\psi_{mn}^4_{\beta} \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|}
\hline
\psi & S_{\mu
u} \\
\hline
\psi^4_{m} & -\frac{1}{2}(\epsilon_{m,\alpha}^{\mu\beta}\partial_{\nu} - \epsilon_{m,\mu}^{\alpha\beta}\partial_{\nu})\psi^4_{\alpha\beta} \\
\psi^4_{m,a} & \frac{1}{2}(\epsilon_{m,\alpha}^{\mu\beta}\partial_{\nu} - \epsilon_{m,\mu}^{\alpha\beta}\partial_{\nu})\psi^4_{\alpha\beta,a} - i\epsilon_{\mu\nu\rho\sigma}\partial_{\rho}\psi^4_{m,\beta} \\
\psi^4_{m,ab} & -\frac{1}{2}(\epsilon_{m,\alpha}^{\mu\beta}\partial_{\nu} - \epsilon_{m,\mu}^{\alpha\beta}\partial_{\nu})\psi^4_{\alpha\beta,ab} + i\epsilon_{\mu\nu\rho\sigma}\partial_{\rho}\psi^4_{m,\beta} - i\epsilon_{\mu\nu\rho\sigma}\partial_{\rho}\psi^4_{m,\beta} \\
\psi^4_{m,a} & \frac{1}{2}(\epsilon_{m,\alpha}^{\mu\beta}\partial_{\nu} - \epsilon_{m,\mu}^{\alpha\beta}\partial_{\nu})\psi^4_{\alpha\beta,a} + \frac{1}{2}\epsilon_{\alpha\beta}\epsilon_{\mu\nu\rho\sigma}\partial_{\rho}\psi^4_{m,\beta} \\
\psi^4_{m,\beta} & -\frac{1}{2}(\epsilon_{m,\alpha}^{\mu\beta}\partial_{\nu} - \epsilon_{m,\mu}^{\alpha\beta}\partial_{\nu})\psi^4_{\alpha\beta,\beta} + i\epsilon_{\mu\nu\rho\sigma}\partial_{\rho}\psi^4_{m,\beta} \\
\hline
\end{array}
\]

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2. Anti-chiral Multiplet

| $\phi$          | $S$ | $S_\mu$         |
|-----------------|-----|-----------------|
| $\phi^0_{ab}$   | $-\phi^0_{ab}$ | $iD_{\mu,ab}\phi^0_{ab}$ |
| $\phi^0_{ab,cd}$ | $\phi^0_{ab,cd}$ | $-i(D_{\mu,ab}\phi^0_{ab,cd} - D_{\mu,cd}\phi^0_{ab})$ |
| $\phi^0_{ab,cd,ef}$ | $\phi^0_{ab,cd,ef}$ | $i(D_{\mu,ab}\phi^0_{ab,cd,ef} + D_{\mu,ef}\phi^0_{ab,cd} + D_{\mu,cd}\phi^0_{ab})$ |
| $\phi^0_{ab,cd,ef}$ | $\phi^0_{ab,cd,ef}$ | $-i\frac{1}{3!\epsilon}\Gamma_{ab,cd,ef,gh,ij,kl}D_{\mu,ef}\phi^0_{gh,ij,kl}$ |
| $\phi^0_{ab}$   | $-\phi^0_{ab}$ | $-\frac{i}{2}D_{\mu,cd}\phi^0_{ab}$ |
| $\phi^0$       | $\phi^1$ | $\frac{i}{2}D_{\mu,ab}\phi^1_{ab}$ |

| $\phi$          | $S$ | $S_\mu$         |
|-----------------|-----|-----------------|
| $\phi^2_{ab}$   | $-\phi^3_{ab}$ | $iD_{\mu,ab}\phi^2_{ab}$ |
| $\phi^2_{ab,cd}$ | $-\phi^3_{ab,cd}$ | $-i(D_{\mu,ab}\phi^2_{ab,cd} - D_{\mu,cd}\phi^2_{ab})$ |
| $\phi^2_{ab,cd,ef}$ | $\phi^3_{ab,cd,ef}$ | $i(D_{\mu,ab}\phi^3_{ab,cd,ef} + D_{\mu,ef}\phi^3_{ab,cd} + D_{\mu,cd}\phi^3_{ab})$ |
| $\phi^2_{ab,cd,ef}$ | $-\phi^3_{ab,cd,ef}$ | $-i\frac{1}{3!\epsilon}\Gamma_{ab,cd,ef,gh,ij,kl}D_{\mu,ef}\phi^2_{gh,ij,kl}$ |
| $\phi^3_{ab}$   | $\phi^3_{ab}$ | $-\frac{i}{2}D_{\mu,cd}\phi^2_{ab}$ |
| $\phi^2$       | $-\phi^3$ | $\frac{i}{2}D_{\mu,ab}\phi^2_{ab}$ |
\[
\begin{array}{|c|c|c|}
\hline
\phi & S & S_\mu \\
\hline
\phi^2 & 0 & i\partial_\mu \phi^2 \\
\phi^{ab} & 0 & -i\partial_\mu \phi^{2}_{ab} + iD_{\mu,ab}\phi^3 \\
\phi^{ab,cd} & 0 & i\partial_\mu \phi^{2}_{ab,cd} - i(D_{\mu,ab}\phi^{3}_{cd} - D_{\mu,cd}\phi^{3}_{ab}) \\
\phi^{ab,cd,ef} & 0 & -i\partial_\mu \phi^{2}_{ab,cd,ef} + i(D_{\mu,ab}\phi^{3}_{cd,ef} + D_{\mu,ef}\phi^{3}_{ab,cd} + D_{\mu,cd}\phi^{3}_{ef,ab}) \\
\phi^3_{ab,cd} & 0 & i\partial_\mu \phi^{2}_{ab,cd} - i\frac{1}{2\nu^2} \Gamma_{ab,cd,ef} \phi^{4}_{gh,ij,kl} \phi^{3}_{gh,ij,kl} \\
\phi^3_{ab} & 0 & -i\partial_\mu \phi^{2}_{ab} - \frac{1}{2} D_{\mu,cd}\phi^{3}_{ab} \\
\phi^3 & 0 & i\partial_\mu \phi^2 + \frac{1}{2} D_{\mu,ab}\phi^3_{ab} \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|}
\hline
\phi \phi^0 & S_{\mu\nu} & \phi \phi^1 \\
\hline
\phi^0 & \phi^0_{\mu\nu} & \phi^1 \phi^{\mu} \\
\phi^{ab} & -\phi^{\mu}_{ab} & -\phi^{\mu,ab} \\
\phi^{ab,cd} & \phi^{\mu,cd}_{ab,cd} & \phi^{\mu,cd}_{ab,cd} \\
\phi^{ab,cd,ef} & -\phi^{\mu,cd}_{ab,cd,ef} \phi^{gh,ij} \phi^{3}_{gh,ij} \\
\phi^{2}_{ab,cd} & \phi^{\mu,cd}_{ab,cd} - \delta_{\mu,cd}\phi^{2}_{ab} \\
\phi^{2}_{ab} & \delta_{\mu,\phi^{\mu}}^{ab} & \delta_{\mu,\phi^{\mu}}^{ab} \\
\phi^{2} & 0 & 0 \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|}
\hline
\phi \phi^0 & S_{\mu\nu} & \phi \phi^1 \\
\hline
\phi^0 & \phi^0_{\mu\nu} & \phi^1 \phi^{\mu} \\
\phi^{ab} & -\phi^{\mu}_{ab} & -\phi^{\mu,ab} \\
\phi^{ab,cd} & \phi^{\mu,cd}_{ab,cd} & \phi^{\mu,cd}_{ab,cd} \\
\phi^{ab,cd,ef} & -\phi^{\mu,cd}_{ab,cd,ef} \phi^{gh,ij} \phi^{3}_{gh,ij} \\
\phi^{2}_{ab,cd} & \phi^{\mu,cd}_{ab,cd} - \delta_{\mu,cd}\phi^{2}_{ab} \\
\phi^{2}_{ab} & \delta_{\mu,\phi^{\mu}}^{ab} & \delta_{\mu,\phi^{\mu}}^{ab} \\
\phi^{2} & 0 & 0 \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|}
\hline
\phi \phi^0 & S_\mu & S \\
\hline
\phi^0 & 0 & \phi^2 \\
\phi^{ab} & -i\epsilon_{ab\mu\nu}\phi^{0}_{\mu\nu} & -\phi^{ab} \\
\phi^{ab,cd} & i(\epsilon_{ab\mu\nu}\phi^{0}_{\mu\nu} - \epsilon_{cd\mu\nu}\phi^{0}_{\ab}) & \phi^{ab,cd} \\
\phi^{ab,cd,ef} & -i(\epsilon_{ab\mu\nu}\phi^{0}_{\mu\nu,ef} + \epsilon_{ef\mu\nu}\phi^{0}_{\ab,cd} + \epsilon_{cd\mu\nu}\phi^{0}_{\ef,ab}) & -\phi^{ab,cd,ef} \\
\phi^{2}_{ab,cd} & \frac{i}{2\nu^2} \Gamma_{ab,cd,ef} \epsilon_{ef\mu\nu}\phi^{0}_{gh,ij,kl} \epsilon_{gh,ij,kl} & \phi^{2}_{ab,cd} \\
\phi^{2}_{ab} & \frac{1}{2} \epsilon_{cd\mu\nu}\phi^{0}_{\mu\nu,ab} & -\phi^{2}_{ab} \\
\phi^{2} & \frac{1}{2} \epsilon_{ab\mu\nu}\phi^{0}_{\mu\nu} & \phi^{2} \\
\hline
\end{array}
\]
### Appendix D  \( N = 4 \) twisted supersymmetric action

The full action (4.9) has the following form:

\[
\mathcal{L} = s s \epsilon_{\mu \nu \rho} s_\mu s_\nu s_\rho s_\sigma \epsilon_{\mu \rho a} s_\mu s_\nu s_\rho s_\sigma \Gamma^{A B C D E F} S_{A B} S_{C D} S_{E F} \Phi^{0} \bar{\psi}^0
\]

\[
= -\bar{\psi}^3 \psi^3 - \bar{\psi}^2 \psi^2 + \bar{\psi}^1 \psi^1 + \bar{\psi}^0 \psi^0
\]

\[
+ \frac{i}{2} D_{\sigma \alpha} \bar{\psi}^1 \psi^1 \sigma + \frac{1}{4} D_{\sigma \alpha} \bar{\psi}^2 \psi^2 \sigma^2 + \frac{1}{4} D_{\sigma \alpha} \bar{\psi}^3 \psi^3 \sigma^2
\]
\[
- \frac{1}{4^2} \epsilon_{\mu \nu \rho \sigma} D_{\mu,ab} D_{\nu,cd} \phi^{3,ab,cd,\psi_0,2,\rho,\sigma} + \frac{i}{8} \epsilon_{\mu \nu \rho \sigma} D_{\mu,ab} D_{\nu,cd} \phi^{2,ab,cd,\partial_\rho \psi_1,\sigma} \\
+ \frac{i}{4^2} \epsilon_{\mu \nu \rho \sigma} D_{\mu,ab} D_{\nu,cd} \phi^{1,ab,cd,\partial_\rho \psi_2,\rho,\sigma} - \frac{1}{8} \epsilon_{\mu \nu \rho \sigma} D_{\mu,ab} D_{\nu,cd} \phi^{0,ab,cd,\partial_\rho \psi_1,\sigma,\rho} \\
+ \frac{i}{4 \cdot 43!24} \epsilon_{\mu \nu \rho \sigma} D_{\mu,ab} D_{\nu,cd} D_{\rho,ef} \Gamma^{ab,cd,ef,gh,ij,kl} \phi^{3,gh,ij,kl,\psi_1,\sigma} \\
- \frac{1}{4 \cdot 43!24} \epsilon_{\mu \nu \rho \sigma} D_{\mu,ab} D_{\nu,cd} D_{\rho,ef} \Gamma^{ab,cd,ef,gh,ij,kl} \phi^{2,gh,ij,kl,\partial_\sigma \psi_0,\rho} \\
+ \frac{i}{4 \cdot 4 \cdot 43!24} \epsilon_{\mu \nu \rho \sigma} D_{\mu,ab} D_{\nu,cd} D_{\rho,ef} \Gamma^{ab,cd,ef,gh,ij,kl} \phi^{1,gh,ij,kl,\partial_\sigma \psi_1,\rho,\sigma} - \frac{i}{8 \cdot 4 \cdot 4 \cdot 43!24} \epsilon_{\mu \nu \rho \sigma} D_{\mu,ab} D_{\nu,cd} D_{\rho,ef} \Gamma^{ab,cd,ef,gh,ij,kl} \phi^{0,gh,ij,kl,\partial_\sigma \psi_0,\rho,\sigma,\rho} \\
+ \frac{3}{4 \cdot 4 \cdot 43!24} \epsilon_{\mu \nu \rho \sigma} D_{\mu,ab} D_{\nu,cd} D_{\rho,ef} \Gamma^{ab,cd,ef,gh,ij,kl} \phi^{3,gh,ij,kl,\psi_0,\sigma} - \frac{1}{2} \epsilon_{abcd} \phi^{2,be,cf,\psi_0,3,\delta} \\
+ \frac{1}{4} \epsilon_{abcd} \phi^{1,be,cf,\epsilon_{\alpha \beta \gamma} \partial_\alpha \psi_1,\beta,\gamma} + \frac{i}{4} \epsilon_{abcd} \phi^{0,be,cf,\epsilon_{\alpha \beta \gamma} \partial_\alpha \partial_\gamma \psi_0,\beta,\gamma} \\
+ \frac{i}{8} \epsilon_{abcd} \phi^{2,be,cf,\epsilon_{\alpha \beta \gamma} \partial_\alpha \psi_2,\beta,\gamma} + \frac{i}{8} \epsilon_{abcd} \phi^{3,be,cf,\epsilon_{\alpha \beta \gamma} \partial_\alpha \psi_3,\beta,\gamma} \\
+ \frac{1}{16} \epsilon_{abcd} \phi^{4,be,cf,\epsilon_{\alpha \beta \gamma} \partial_\alpha \psi_4,\beta,\gamma} \\
+ \frac{3}{2 \cdot 4 \cdot 43!24} \epsilon_{abcd} \phi^{1,be,cf,\epsilon_{\alpha \beta \gamma} \partial_\alpha \partial_\beta \partial_\gamma \psi_1,\beta,\gamma} \\
- \frac{3}{2 \cdot 4 \cdot 43!24} \epsilon_{abcd} \phi^{0,be,cf,\epsilon_{\alpha \beta \gamma} \partial_\alpha \partial_\beta \partial_\gamma \psi_0,\beta,\gamma} \\
- \frac{3}{2 \cdot 4 \cdot 4 \cdot 43!24} \epsilon_{abcd} \phi^{1,be,cf,\epsilon_{\alpha \beta \gamma} \partial_\alpha \partial_\beta \partial_\gamma \psi_1,\beta,\gamma} \\
+ \frac{3}{2 \cdot 4 \cdot 4 \cdot 43!24} \epsilon_{abcd} \phi^{2,be,cf,\epsilon_{\alpha \beta \gamma} \partial_\alpha \partial_\beta \partial_\gamma \psi_2,\beta,\gamma} \\
+ \frac{3}{2 \cdot 4 \cdot 4 \cdot 43!24} \epsilon_{abcd} \phi^{3,be,cf,\epsilon_{\alpha \beta \gamma} \partial_\alpha \partial_\beta \partial_\gamma \psi_3,\beta,\gamma} \\
+ \frac{3}{4 \cdot 4 \cdot 4 \cdot 43!24} \epsilon_{abcd} \phi^{4,be,cf,\epsilon_{\alpha \beta \gamma} \partial_\alpha \partial_\beta \partial_\gamma \psi_4,\beta,\gamma}
\]
\[ + \frac{3}{2} \frac{1}{4!3^{12}4^{1}} \epsilon^{\alpha \beta \gamma \delta} \epsilon_{a b c d e f g h} \partial \Gamma^{\alpha \beta \gamma \delta} \phi^{0} \Gamma_{\mu \nu \rho \sigma} \partial^{\mu \nu \rho \sigma} \partial_{\psi}^{2} \]
\[ - i \frac{3}{4!4!3^{12}4^{1}} \epsilon^{\alpha \beta \gamma \delta} \epsilon_{a b c d e f g h} \partial \Gamma^{\alpha \beta \gamma \delta} \phi^{0} \Gamma_{\mu \nu \rho \sigma} \partial^{\mu \nu \rho \sigma} \partial_{\psi}^{2} \]
\[ + \frac{3}{4!} \frac{1}{4!3^{12}4^{1}} \epsilon^{\alpha \beta \gamma \delta} \epsilon_{a b c d e f g h} \partial \Gamma^{\alpha \beta \gamma \delta} \phi^{0} \Gamma_{\mu \nu \rho \sigma} \partial^{\mu \nu \rho \sigma} \partial_{\psi}^{2} \]

\[
\times \epsilon^{\mu \nu \rho \sigma} D_{\mu \nu} \partial_{\psi}^{2} \partial_{\rho} \partial_{\sigma} \partial_{\phi}^{1} \psi \]
\[ + \frac{3}{4!} \frac{1}{4!3^{12}4^{1}} \epsilon^{\alpha \beta \gamma \delta} \epsilon_{a b c d e f g h} \partial \Gamma^{\alpha \beta \gamma \delta} \phi^{0} \Gamma_{\mu \nu \rho \sigma} \partial^{\mu \nu \rho \sigma} \partial_{\psi}^{2} \]
\[ + \frac{1}{4!3^{12}4^{1}} \epsilon^{\alpha \beta \gamma \delta} \epsilon_{a b c d e f g h} \partial \Gamma^{\alpha \beta \gamma \delta} \phi^{0} \Gamma_{\mu \nu \rho \sigma} \partial^{\mu \nu \rho \sigma} \partial_{\psi}^{2} \]
\[ - \frac{1}{4!} \frac{1}{4!3^{12}4^{1}} \epsilon^{\alpha \beta \gamma \delta} \epsilon_{a b c d e f g h} \partial \Gamma^{\alpha \beta \gamma \delta} \phi^{0} \Gamma_{\mu \nu \rho \sigma} \partial^{\mu \nu \rho \sigma} \partial_{\psi}^{2} \]
\[ + \frac{1}{4!} \frac{1}{4!3^{12}4^{1}} \epsilon^{\alpha \beta \gamma \delta} \epsilon_{a b c d e f g h} \partial \Gamma^{\alpha \beta \gamma \delta} \phi^{0} \Gamma_{\mu \nu \rho \sigma} \partial^{\mu \nu \rho \sigma} \partial_{\psi}^{2} \]
\[ = \frac{3}{4!3^{12}4^{1}} \epsilon^{\alpha \beta \gamma \delta} \epsilon_{a b c d e f g h} \partial \Gamma^{\alpha \beta \gamma \delta} \phi^{0} \Gamma_{\mu \nu \rho \sigma} \partial^{\mu \nu \rho \sigma} \partial_{\psi}^{2} \]

**Appendix E Useful formulae in four dimensions**

Definition of \( \delta^{+}_{\mu \nu, \rho \sigma} \) is
\[
\delta^{+}_{\mu \nu, \rho \sigma} = \delta_{\mu \rho} \delta_{\nu \sigma} - \delta_{\mu \sigma} \delta_{\nu \rho} - \varepsilon_{\mu \nu \rho \sigma},
\] (E.1)
\[
\delta^{+ A}_{\mu \nu} = 4 \cdot 3,
\] (E.2)

where the suffix \( A \) denotes tensor suffix \( \mu \nu \). Thus \( \delta^{+ A}_{\mu \nu} \) stands for \( \delta^{+ \mu \nu} \). We introduce self-dual field \( \phi^{+ \mu \nu} \) (\( \phi^{+ \mu \nu} = \frac{1}{2} \epsilon^{\mu \nu \rho \sigma} \phi^{+ \rho \sigma} \))

\[
\delta^{+}_{\mu \nu, \rho \sigma} \phi^{+ \rho \sigma} = (\delta_{\mu \rho} \delta_{\nu \sigma} - \delta_{\mu \sigma} \delta_{\nu \rho} - \varepsilon_{\mu \nu \rho \sigma}) \phi^{+ \rho \sigma}
\]
Variants of the definition of $\Gamma^{+ABC}$ are

$$
\Gamma^{+\mu\alpha,\nu\beta,\rho\gamma} = 
\begin{align*}
\delta^{\alpha\nu}\delta^{\beta\nu}\delta^{\gamma\nu} &+ \delta^{\mu\nu}\delta^{\beta\rho}\delta^{\gamma\alpha} + \delta^{\alpha\beta}\delta^{\nu\gamma}\delta^{\rho\mu} + \delta^{\mu\beta}\delta^{\nu\rho}\delta^{\gamma\alpha} \\
- (\delta^{\alpha\nu}\delta^{\beta\gamma}\delta^{\rho\mu} + \delta^{\mu\nu}\delta^{\beta\rho}\delta^{\gamma\alpha} + \delta^{\alpha\beta}\delta^{\nu\rho}\delta^{\gamma\mu} + \delta^{\mu\beta}\delta^{\nu\gamma}\delta^{\rho\alpha}) \\
- \varepsilon^{\mu\alpha\beta\gamma} \delta^{\nu\rho} - \varepsilon^{\mu\alpha\nu\beta} \delta^{\beta\gamma} + \varepsilon^{\mu\alpha\nu\gamma} \delta^{\beta\rho} + \varepsilon^{\mu\alpha\beta\rho} \delta^{\nu\gamma},
\end{align*}
$$

(E.3)

$$
\Gamma^{+AB\rho\sigma} = \frac{1}{2}(\delta^{+A,\nu\rho} \delta^{+B},\nu_{\sigma} - \delta^{+A,\nu\sigma} \delta^{+B},\nu_{\rho}),
$$

(E.4)

$$
\Gamma^{+ABC} = \frac{1}{4}\delta^{+A,\nu\rho} \delta^{+B},\nu_{\delta+C,\rho\sigma}.
$$

(E.5)

Useful formulae are in order.

$$
\varepsilon^{\mu\alpha\gamma\nu} \delta^{\beta\rho} = \delta^{\mu\beta} \varepsilon^{\rho\alpha\gamma\nu} + \delta^{\nu\beta} \varepsilon^{\rho\mu\alpha\gamma} - \delta^{\rho\beta} \varepsilon^{\gamma\mu\alpha\nu} - \delta^{\gamma\alpha} \varepsilon^{\rho\mu\beta\nu}
$$

$$
= \delta^{\mu\rho} \varepsilon^{\beta\alpha\gamma\nu} + \delta^{\nu\rho} \varepsilon^{\beta\mu\alpha\gamma} - \delta^{\rho\beta} \varepsilon^{\gamma\mu\alpha\nu} - \delta^{\gamma\alpha} \varepsilon^{\rho\mu\beta\nu}
$$

$$
= \frac{1}{2}(\delta^{\beta\rho} \varepsilon^{\rho\alpha\gamma\nu} + \delta^{\gamma\rho} \varepsilon^{\beta\mu\alpha\gamma} - \delta^{\alpha\beta} \varepsilon^{\rho\mu\gamma\nu} - \delta^{\rho\beta} \varepsilon^{\gamma\mu\alpha\nu}
$$

$$
+ \delta^{\mu\rho} \varepsilon^{\beta\alpha\gamma\nu} + \delta^{\nu\rho} \varepsilon^{\beta\mu\alpha\gamma} - \delta^{\rho\beta} \varepsilon^{\gamma\mu\alpha\nu} - \delta^{\gamma\alpha} \varepsilon^{\rho\mu\beta\nu}),
$$

(E.6)

$$
- \varepsilon^{\mu\alpha\gamma\nu} \delta^{\beta\rho} - \varepsilon^{\mu\alpha\rho\beta} \delta^{\nu\gamma} + \varepsilon^{\mu\alpha\nu\beta} \delta^{\rho\gamma} + \varepsilon^{\mu\alpha\gamma\beta} \delta^{\nu\rho}
$$

$$
= \frac{1}{2}(-\delta^{\mu\beta} \varepsilon^{\rho\alpha\gamma\nu} - \delta^{\nu\beta} \varepsilon^{\rho\alpha\gamma\nu} + \delta^{\mu\gamma} \varepsilon^{\beta\rho\alpha\nu} + \delta^{\nu\gamma} \varepsilon^{\beta\rho\alpha\nu}
$$

$$
+ \delta^{\alpha\beta} \varepsilon^{\rho\mu\gamma\nu} + \delta^{\nu\mu} \varepsilon^{\gamma\rho\beta\nu} - \delta^{\alpha\gamma} \varepsilon^{\beta\mu\rho\nu} - \delta^{\rho\mu} \varepsilon^{\gamma\alpha\beta\nu}),
$$

(E.7)

$$
\Gamma^{+ABC}, \Gamma^{+}_{ABC} = 3! \cdot 4^{3},
$$

(E.8)

$$
\Gamma^{+ABC}, \Gamma^{+}_{ABD} = 2! \cdot 4^{2} \delta^{+C}_{D},
$$

(E.9)

$$
\Gamma^{+ABC}, \Gamma^{+}_{ADE} = 4(\delta^{+B}_{D}\delta^{+C}_{E} - \delta^{+B}_{E}\delta^{+C}_{D}),
$$

(E.10)

$$
\Gamma^{+ABC}, \Gamma^{+}_{DEF} = \left(\delta^{+A}_{D}\delta^{+B}_{E}\delta^{+C}_{F} + \text{ (cyclic in DEF)}\right) - \left(\delta^{+A}_{F}\delta^{+B}_{E}\delta^{+C}_{D} + \text{ (cyclic in DEF)}\right),
$$

(E.11)

$$
\mathcal{D}^{+}_{\mu,\rho\sigma} = \delta_{\mu\rho} \partial_{\sigma} - \delta_{\mu\sigma} \partial_{\rho} - \varepsilon_{\mu\rho\sigma} \partial^{\nu},
$$

(E.12)

$$
\Gamma^{+\mu,\rho\sigma} = 2\delta^{+\mu,\rho\sigma},
$$

(E.13)
\[ \delta^{+\mu\nu}_\rho = 3\delta^{\mu\nu}, \quad (E.14) \]

\[ \Gamma^{+ABC}D_{\mu,A}D_{\nu,B}D_{\rho,C} = -4^3\epsilon_{\sigma\mu\rho}\partial^2\partial^\sigma, \quad (E.15) \]

\[ \Gamma^{+ABC}D_{\mu,A}D_{\nu,B} = 4^2\left(\partial_\mu D^{+C}_{\nu} - \partial_\nu D^{+C}_{\mu} - \partial^2\delta^{+}_{\mu\nu}\right), \quad (E.16) \]

\[ \delta^{+\mu\nu,\alpha\gamma}_{\rho\sigma,\beta} = \delta_{\alpha\beta}\delta^{+\mu\nu}_{\rho\sigma} + \delta_{\beta\rho}\delta^{+\mu\nu}_{\alpha\sigma} - \delta_{\beta\sigma}\delta^{+\mu\nu}_{\alpha\rho} - \delta_{\alpha\rho}\delta^{+\mu\nu}_{\beta\sigma} + \delta_{\alpha\sigma}\delta^{+\mu\nu}_{\beta\rho}, \quad (E.17) \]

\[ \delta^{+\mu\nu,\alpha\gamma}_{\rho\sigma,\beta} + \delta^{+\mu\nu}_{\beta\gamma}\delta^{+\mu\nu}_{\rho\sigma,\alpha} = 2\delta_{\alpha\beta}\delta^{+\mu\nu}_{\rho\sigma}, \quad (E.18) \]

\[ \delta^{+\mu\nu,\alpha\gamma}_{\rho\sigma,\beta} - \delta^{+\mu\nu}_{\beta\gamma}\delta^{+\mu\nu}_{\rho\sigma,\alpha} = 2(\delta_{\beta\rho}\delta^{+\mu\nu}_{\alpha\sigma} - \delta_{\beta\sigma}\delta^{+\mu\nu}_{\alpha\rho} - \delta_{\alpha\rho}\delta^{+\mu\nu}_{\beta\sigma} + \delta_{\alpha\sigma}\delta^{+\mu\nu}_{\beta\rho}). \quad (E.19) \]

**Appendix F  Wess-Zumino gauge in two dimensions**

In this Appendix we explicitly show how the superfluous component fields in the superconnection can be gauged away consistently with the constraints by Wess-Zumino gauge. We consider twisted \(N = D = 2\) Abelian case for simplicity. We have introduced the fermionic superconnection \(\{\Gamma_I\}\) as in (5.1). Here we first show the explicit component expansion of the superconnection as:

\[ \Gamma = \Gamma^0 \]
\[ + \theta\Gamma^1 + \theta^\mu\Gamma^1_\mu + \tilde{\theta}\tilde{\Gamma}^1 \]
\[ + \theta\tilde{\theta}\tilde{\Gamma}^2 + \theta^2\tilde{\Gamma}^2 + \theta^\mu\Gamma^2_\mu + \theta^\mu\tilde{\theta}\tilde{\Gamma}^2_\mu + \cdots, \quad (F.1) \]

\[ \tilde{\Gamma} = H^0 \]
\[ + \theta H^1 + \theta^\mu H^1_\mu + \tilde{\theta}\tilde{H}^1 \]
\[ + \theta\tilde{\theta}H^2 + \theta^2\tilde{H}^2 + \theta^\mu H^2_\mu + \theta^\mu\tilde{\theta}\tilde{H}^2_\mu + \cdots, \quad (F.2) \]

\[ \Gamma_\mu = M^0_\mu \]
\[ + \theta M^1_\mu + \theta^\rho M^1_{\mu\rho} + \tilde{\theta}\tilde{M}^1_\mu \]
\[ + \theta\tilde{\theta}M^2_\mu + \theta^2\tilde{M}^2_\mu + \theta^\rho M^2_{\mu\rho} + \theta^\rho\tilde{\theta}\tilde{M}^2_{\mu\rho} + \cdots, \quad (F.3) \]

\[ \Gamma_\mu = \omega_\mu \]
According to the introduction of the superconnection we introduce the following
gauge transformation:

\[
\delta \Gamma = D K, \quad (F.5)
\]
\[
\delta \tilde{\Gamma} = \tilde{D} K, \quad (F.6)
\]
\[
\delta \Gamma_\mu = D_\mu K, \quad (F.7)
\]
\[
\delta \tilde{\Gamma}_\mu = \tilde{D}_\mu K, \quad (F.8)
\]

where \( K \) is the supergauge parameter defined by

\[
K = K^0 + \theta K^1 + \theta^\mu K^1_\mu + \tilde{\theta} \tilde{K}^1
+ \theta \tilde{\theta} K^2 + \theta^2 \tilde{K}^2 + \theta^\mu \tilde{\theta} K^2_\mu
+ \theta \theta^2 K^3 + \theta^2 \tilde{\theta} K^3 + \theta^\mu \tilde{\theta} K^3_\mu
+ \theta^4 K^4. \quad (F.9)
\]

All the component fields in the supergauge parameter \( K \) except for the \( K^0 \) which is
identified as an ordinary gauge parameter can be used to gauge away the component
fields of the superconnection \( \{ \Gamma_I \} \).

We show some of gauge transformations in (F.5) and (F.6)

\[
\delta \Gamma^0 = K^1, \quad \delta \Gamma^1 = 0, \quad (F.10)
\]
\[
\delta \Gamma^1_\mu = K^2_\mu - \frac{i}{2} \partial_\mu K^0, \quad \delta \tilde{\Gamma}^1 = K^2, \quad (F.11)
\]
\[
\delta \tilde{H}^0 = \tilde{K}^1, \quad \delta \tilde{H}^1 = -K^2,
\]
\[
\delta \tilde{H}^1_\mu = -\tilde{K}^2_\mu + \frac{i}{2} \epsilon_{\mu \nu} \partial_\nu K^0, \quad \delta \tilde{H}^1 = 0.
\]

For example we can gauge away \( \Gamma^0 \) by using the gauge parameter \( K^1 \), similarly \( H^0 \) can be gauged away by \( \tilde{K}^1 \) and so on. In order to gauge away \( \tilde{\Gamma}^1 \) and \( H^1 \), we
have to use the same gauge parameter \( K^2 \) and thus it is \textit{a priori} not obvious if this
gauging away procedure is consistent or not. It turns out that the consistency comes
out from the constraints in (5.8). Here we explicitly show curvature constraints (5.8),

\[
\mathcal{W}_1 = 2 D \Gamma = \mathcal{W}, \quad (F.12)
\]
\[
\mathcal{W}_2 = 2 \tilde{D} \tilde{\Gamma} = \mathcal{W}, \quad (F.13)
\]
\[
\mathcal{W}_3 = \mathcal{D} \tilde{\Gamma} + \tilde{D} \Gamma = 0, \quad (F.14)
\]
\[
\mathcal{F}_{\mu \nu} = \mathcal{D}_\mu \Gamma_\nu - \mathcal{D}_\nu \Gamma_\mu = \delta_{\mu \nu} \mathcal{F}, \quad (F.15)
\]

55
The constraint (F.14) leads to the following relation:

\[ 0 = H^1 + \bar{\Gamma}^1 + \theta(-\Gamma^2) + \theta^2(H_\mu^2 - \bar{\Gamma}_\mu^2) + \bar{\theta}(H^2) \]
\[ + \theta\bar{\theta}(0) + \theta^2(H^3 - \frac{i}{2}\epsilon^{\mu\nu}\partial_\mu H^1 + \bar{\Gamma}^3 + \frac{i}{2}\partial^\mu \Gamma^1) \]
\[ + \theta\theta^\mu(\frac{i}{2}\partial_\mu H^1 + \Gamma^3 - \frac{i}{2}\epsilon^{\mu\nu}\partial^\mu \Gamma^1) \]
\[ + \theta^\mu\bar{\theta}(H_\mu^3 - \frac{i}{2}\partial_\mu \bar{H}^1 + \frac{i}{2}\epsilon_\mu \partial^\nu \bar{\Gamma}^1) + \cdots, \] (F.18)

This constraint thus requires, \( H^1 + \bar{\Gamma}^1 = 0, \Gamma^2 = 0, \quad H_\mu^2 - \bar{\Gamma}_\mu^2 = 0, \) and so on. Thus \( \bar{\Gamma}^1 \) and \( H^1 \) can be gauge away by the same gauge parameter \( K^2 \).

We can then gauge away most of the component fields in the superconnection except for the following fields: \( \Gamma^1 = \frac{1}{2}A, \Gamma^2_\mu = \frac{1}{2}\lambda_\mu, \Gamma^3 = \frac{1}{2}D + \frac{1}{8}\epsilon^{\mu\nu}\Gamma^1, M^1_{\mu\nu} = \frac{1}{2}\delta_{\mu\nu}B, M^2_{\mu\nu} = -\frac{1}{2}(\delta_{\mu\nu}\rho + \epsilon_{\mu\nu}\bar{\rho}) \) and \( \omega_\mu \). As we can see these are the basic component fields of the chiral supermultiplets.

We show the most explicit form of the superconnection with non-Abelian gauge algebra in the following:

\[ \Gamma = \Gamma^0(0) \]
\[ + \theta\Gamma^1(\frac{1}{2}A) + \theta^\mu\Gamma^1_{\mu}(-\frac{i}{2}\omega_\mu) + \bar{\theta}\bar{\Gamma}^1(0) \]
\[ + \theta\bar{\theta}\Gamma^2(0) + \theta^2\bar{\Gamma}^2(-\frac{1}{3}\bar{\rho}) + \theta\theta^\mu\Gamma^2_{\mu}(\frac{1}{2}\lambda_\mu) + \theta^\mu\bar{\theta}\bar{\Gamma}^2_{\mu}(-\frac{i}{6}\epsilon_{\mu\nu}\lambda^\nu) \]
\[ + \theta\theta^2\bar{\Gamma}^3(\frac{1}{2}D + \frac{1}{4}\epsilon^{\rho\sigma}F_{\rho\sigma})) + \theta^2\bar{\theta}\bar{\Gamma}^3_{\mu}(\frac{i}{8}[A, B]) + \theta\theta^\mu\bar{\theta}\bar{\Gamma}^3_{\mu}(\frac{i}{4}\epsilon_{\mu\nu}D^\nu A) \]
\[ + \theta^4\Gamma^4(\frac{i}{6}D^{\mu}\lambda_{\mu} - \frac{i}{6}[A, \rho]), \] (F.19)

\[ \Gamma_{\mu} = M^0_{\mu}(0) \]
\[ + \theta M^1_{\mu}(-\frac{i}{2}\omega_\mu) + \theta^\rho M^2_{\mu\rho}(\frac{1}{2}\delta_{\mu\rho}B) + \bar{\theta}M^1_{\mu}(\frac{i}{2}\epsilon_{\mu\nu}\omega^\nu) \]
\[ + \theta\bar{\theta}M^2_{\rho}(\frac{1}{3}\epsilon_{\mu\nu}\lambda^\nu) + \theta^\rho M^3_{\mu\rho}(0) + \theta\theta^\rho M^3_{\mu\rho}(\frac{1}{2}\delta_{\mu\rho}\rho - \frac{1}{6}\epsilon_{\mu\rho}\bar{\rho}) \]
\[ + \theta^\rho\bar{\theta}M^2_{\mu\rho}(\frac{1}{2}\delta_{\mu\rho}\bar{\rho} - \frac{1}{6}\epsilon_{\mu\rho}\rho) \]
\begin{align*}
+ \theta^2 M^3_\mu \left( \frac{i}{4} \epsilon_{\mu\rho} D^\rho B \right) + \theta^2 \bar{\theta} M^3_\mu \left( \frac{i}{4} D_\mu B \right) + \theta \theta^\rho \bar{\theta} \tilde{M}^3_{\mu\rho} \left( -\frac{1}{2} \delta_{\mu\rho} (D + \frac{3}{4} \epsilon^{\nu\sigma} F_{\nu\sigma}) + \frac{i}{8} \epsilon_{\mu\rho} [A, B] \right) \\
+ \theta^4 M^4_\mu \left( -\frac{i}{6} D_\mu \rho - \frac{i}{6} \epsilon_{\mu\nu} D^\nu \bar{\rho} - \frac{i}{6} [B, \lambda_{\mu}] \right). 
\end{align*}

Using these superconnections, we can explicitly construct supercurvatures.
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