Stream cipher based on quasigroup string transformations in $\mathbb{Z}_p^*$

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Abstract. In this paper we design a stream cipher that uses the algebraic structure of the multiplicative group $\mathbb{Z}_p^*$ (where $p$ is a big prime number used in ElGamal algorithm), by defining a quasigroup of order $p - 1$ and by doing quasigroup string transformations. The cryptographical strength of the proposed stream cipher is based on the fact that breaking it would be at least as hard as solving systems of multivariate polynomial equations modulo big prime number $p$ which is NP-hard problem and there are no known fast randomized or deterministic algorithms for solving it. Unlikely the speed of known ciphers that work in $\mathbb{Z}_p^*$ for big prime numbers $p$, the speed of this stream cipher both in encryption and decryption phase is comparable with the fastest symmetric-key stream ciphers.

Key words: quasigroups, quasigroup string transformations, stream cipher, public-key, ElGamal

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1 Introduction

From the point of view how encryption algorithms encrypt information that is repeated several times during the phase of communication, they are divided on stream ciphers and block ciphers. While block ciphers always give the same output of cipher texts for the same input blocks of plain text, the stream ciphers give different outputs for the same sequences of plain text. On the other side, depending on the type of the keys used in cryptographic algorithm, and the way they are used, there is another classification of encryption algorithms: symmetric-key and public-key algorithms. Symmetric-key algorithms need the correspondents in the communication to share a same key that is previously
exchanged through some secure channel that is out of the scope of the definition of the algorithm, while in public-key algorithms the problem of exchanging the communication key is a part of the algorithm and no secure channel is necessary for that purpose.

Stream cipher algorithms can be either symmetric-key or public-key. Regarding the speed of encryption and decryption procedures, symmetric-key stream ciphers are much more faster than public-key ones. That is because the symmetric-key stream ciphers usually use fast register operations such as shifting, rotation, and bit by bit logical operations, while the most popular and known public-key algorithms usually use modular exponentiation. Thus, the public-key algorithms are around 1000 times slower than symmetric-key algorithms.

A well-known public-key stream cipher is Blum-Goldwasser probabilistic public-key encryption scheme [1]. Even though the speed of that algorithm in encryption phase is much faster than RSA encryption, the speed of that algorithm in decryption phase is similar or in some cases even slower than the speed of RSA algorithm, ([2] p. 310-311). In fact, the lack of the speed of public-key stream ciphers is one of the main reasons why they are not widely used in stream communication.

Diffie-Hellman algorithm was proposed in 1976 [3] and introduced the concept of public-key cryptography. That algorithm usually is used for establishing a key exchange between two correspondents, and then, the communication is usually continued by some symmetric fast algorithm (either block or stream cipher). In 1985 ElGamal proposed a public-key cryptosystem based on Diffie-Hellman algorithm [4]. One of the disadvantages of ElGamal algorithm is that cipher text is two times longer than corresponding plain text, which makes it unsuitable for using it as a stream cipher.

In this paper beside the theory of finite fields we use also the theory of quasigroups and Latin Squares. Although quasigroups (or Latin squares) are used in design of many modern symmetric cryptographic algorithms [5], [6] they are not in the main stream of cryptographic paradigms. During the last 10 years several cryptographic algorithms were developed based on quasigroups [7], [8], [9]. Those algorithms base their security on assumptions that other problems such as factoring of natural numbers or discrete logarithm problems can not be solved in polynomial time - and thus have solid theoretical ground for their security. However, for all of those algorithms, because they usually use sets of Latin squares (quasigroups), their implementation is several orders of magnitude slower than other cryptographic algorithms in their category, based usually on bit manipulation and shifting registers.

Excellent introductory materials about theory of quasigroups the reader can find in [10] and [11] and some applications of quasigroups and Latin squares in [12], [13], [14], [15].

In cryptographic algorithms introduced in [16] and [17], and the following papers [18], [19], [20], [21] and [22] the authors use quasigroups to define so-called “quasigroup string transformations”. By those algorithms they define a stream cipher whose principles are used in this paper. For effective encryption
and decryption, the quasigroup stream cipher uses a set of leaders that are in fact the secret and symmetric key. However, quasigroups used in those algorithms are of the order from 16 to 256, and complete multiplicative table have to be known, before encryption/decryption starts.

In the paper [23] in order to solve the problem of fast generation of a quasigroups of order $p − 1$ where $p$ is a prime number, authors propose a fast way for generating a quasigroups by knowing only the first row in the multiplicative table of the quasigroup. That first row is in fact a permutation of the elements $\mathbb{Z}_p^* = \mathbb{Z}_p \setminus \{0\} = \{1, 2, ..., p − 1\}$ and by knowing only that permutation it is possible to define the product of any two elements such that a quasigroup will be formed.

In this paper we will define a stream cipher that in its initialization phase uses ElGamal algorithm, then the encryption is made by using quasigroup string transformations and the definition of a quasigroup is based by knowing only one permutation in the set of $\mathbb{Z}_p^*$.

The organization of the paper is following: In Section 2 we will give basic definitions of the ElGamal algorithm, quasigroup stream cipher and fast quasigroup definition from a known permutation. In Section 3 we will define the new stream cipher and we will give an example with a small value of $p$, in Section 4 we will examine the cryptographical strength of the proposed stream cipher, and in Sections 5 we will give the conclusions.

2 Basic definitions

In our description of cryptographic algorithms we will use the usual notification that the correspondents in the communication are Alice and Bob.

2.1 Basic ElGamal encryption algorithm

The ElGamal encryption algorithm uses a big prime number $p$, and uses the operations of modular exponentiation and modular multiplication. There are three phases of the algorithm: Key generation, Encryption and Decryption. The algorithm is the following:

**Key generation** Alice generates her public and private keys as follows:
1. Generate a large random prime number $p$ and a generator $\alpha$ of the multiplicative group $\mathbb{Z}_p^*$ of the integers $\{1, 2, ..., p − 1\}$.
2. Select a random integer $a$, $1 ≤ a ≤ p − 2$ and compute $\alpha^a \text{mod} p$.
3. Alice’s public key is the triplet $(p, \alpha, \alpha^a)$; Alice’s private key is $a$.

**Encryption** Bob encrypts a message $m$ for Alice by doing the following:
1. Obtain Alice’s authentic public key $(p, \alpha, \alpha^a)$.
2. Represent the message as an integer $m$ in the range $\{0, 1, ..., p − 1\}$.
3. Select a random integer $e$, $1 ≤ e ≤ p − 2$.
4. Compute $\gamma = \alpha^e \text{mod} p$ and $\delta = m \cdot (\alpha^a)^e \text{mod} p$.
5. Send the ciphertext $c = (\gamma, \delta)$. 
Decryption  To recover the message $m$ Alice should do the following:
1. Use the private key $a$ to compute $\alpha^{-ae} = \gamma^a$
2. Recover $m$ by computing $m = \delta \cdot \alpha^{-ae} \mod p$

It is obvious that message expansion in ElGamal algorithm is by factor 2, because Bob sends the cipher text $c = (\gamma, \delta)$ that has twice the length of the message $m$. That fact is considered as a serious disadvantage of the algorithm. Simple analysis of the algorithms show that in a phase of encryption it uses two modular exponentiations and one modular multiplication, while in phase of Decryption it uses one modular exponentiation, one calculation of an inverse element in multiplicative group $\mathbb{Z}_p^*$ (calculation of the element $\gamma^{-1} \mod p$) and one modular multiplication. For the security analysis, and security issues about used prime numbers in ElGamal algorithm the reader can see [2].

2.2 Definition of basic quasigroup string transformations

In this subsection we will give some definitions from the theory of quasigroups and define a basic quasigroup string transformations. We say “basic” string transformations, because in [17] much more complex quasigroup string transformations are defined, but we will not use them in our definition of the stream cipher.

Definition 1. Let $Q = \{a_1, a_2, \ldots, a_n\}$ be a finite set of $n$ elements. A quasigroup $(Q, \cdot)$ is a groupoid satisfying the law

$$\left( \forall u, v \in Q \right) \left( \exists x, y \in Q \right) \quad u \cdot x = v \quad \& \quad y \cdot u = v. \quad (1)$$

Given a quasigroup $(Q, \cdot)$ a new operation $\cdot^{-1}$ on the set $Q$ can be derived by:

$$\cdot^{-1}(x, y) = z \iff x \cdot z = y \quad (2)$$

It easy to prove the following

Lemma 1. The groupoid $(Q, \cdot^{-1})$ is a quasigroup. $\square$

Instead of the symbol $\cdot^{-1}$ we will use the symbol $\\backslash$ and we will say that the quasigroup $(Q, \\backslash)$ is the left parastrophe (or conjugate in some literature) adjoint to the quasigroup $(Q, \cdot)$.

Then from the definition of $\\backslash$ it follows that

$$x \cdot y = z \iff y = x \\backslash z. \quad (3)$$

and

$$x \\backslash (x \cdot y) = y, \quad x \cdot (x \\backslash y) = y. \quad (4)$$

In what follow we will give basic definitions for quasigroup string transformations and address several theorems and properties which are proved in [17].

Consider an alphabet (i.e. a finite set) $Q$, and denote by $Q^+$ the set of all nonempty words (i.e. finite strings) formed by the elements of $Q$. The elements
of $Q^+$ will be rather denoted by $a_1a_2\ldots a_n$ than $(a_1, a_2, \ldots, a_n)$, where $a_i \in Q$. Let $*$ be a quasigroup operation on the set $Q$, i.e. consider a quasigroup $(Q, *)$. For each $a \in Q$ we define two functions $e_a, d_a : Q^+ \rightarrow Q^+$ as follows.

Let $a_i \in Q$, $a = a_1a_2\ldots a_n$. Then

$$e_a(a) = b_1b_2\ldots b_n \iff b_1 = a * a_1, \ b_2 = b_1 * a_2, \ldots, \ b_n = b_{n-1} * a_n$$

i.e. $b_{i+1} = b_i * a_{i+1}$ for each $i = 0, 1, \ldots, n - 1$, where $b_0 = a$.

and

$$d_a(a) = c_1c_2\ldots c_n \iff c_1 = a * a_1, \ c_2 = a_1 * a_2, \ldots, \ c_n = a_{n-1} * a_n$$

i.e. $c_{i+1} = a_i * a_{i+1}$ for each $i = 0, 1, \ldots, n - 1$, where $c_0 = a$.

**Definition 2.** The functions $e_a, d_a$ are called $e$- and $d$- string transformation of $Q^+$ based on the operation $*$ with leader $a$.

Note that $e_a \circ d_a = d_a \circ e_a = 1_a$ i.e. $e_a$ and $d_a$ are mutually inverse string transformations. A graphical representation of $e_a$ and $d_a$ is shown on Fig. 1 and Fig. 2. Next we will extend the definition of $e$- and $d$- string transformations with the following

**Definition 3.** If we choose $k$ leaders $a_1, a_2, \ldots, a_k \in Q$ (not necessarily distinct), then the compositions of mappings

$$E_k = E_{a_1\ldots a_k} = e_{a_1} \circ e_{a_2} \circ \cdots \circ e_{a_k}$$

and

$$D_k = D_{a_1\ldots a_k} = d_{a_1} \circ d_{a_2} \circ \cdots \circ d_{a_k}$$

are called $E$- and $D$- quasigroup string transformations of $Q^+$ respectively.
In [17] the following two lemmas are proved:

Lemma 2. The functions $E_k$ and $D_k$ are permutations on $Q^+$.

Lemma 3. In a quasigroup $(Q, *)$, with a given set of $k$ leaders $\{a_1, a_2, \ldots, a_k\}$ the inverse of $E_k = E_{a_1 \cdots a_k} = e_{a_1} \circ \cdots \circ e_{a_k}$, is $E_k^{-1} = D_{a_k \cdots a_1} = d_{a_k} \circ \cdots \circ d_{a_1}$.

Now it is clear that for any quasigroup string transformation $E$ the pair of functions $(E, E^{-1})$ can be considered as a pair of an encryption and decryption function for the strings on an alphabet $Q$. More formally we give the following definition of a quasigroup stream cipher:

Definition 4. For a given quasigroup $(Q, *)$, and a given $k$-tuple $(a_1, a_2, \ldots, a_k)$ of leaders $a_i \in Q$, the system $((Q, *), (a_1, a_2, \ldots, a_k), E_{a_1 \cdots a_k}, D_{a_k \cdots a_1})$ defines a quasigroup stream cipher on the strings in $Q^+$.

2.3 Definition of a quasigroup of big order $p - 1$

The construction of a Latin squares is discussed in [9], [12] and [14]. However, the construction of such Latin squares is not suitable for our purposes in this paper, because we want to define a quasigroup of order $p - 1$ where $p$ is big prime number with more then 1024 bits. That problem can be solved by the approach that is described in [23]. Namely, if we have a permutation

$P = \begin{pmatrix} 1 & 2 & \cdots & j & \cdots & p - 1 \\ a_{1j} & a_{12} & \cdots & a_{1j} & \cdots & a_{1p - 1} \end{pmatrix}$,

where $(a_{11}, a_{12}, \cdots, a_{1j}, \cdots, a_{1p - 1})$ is the first row of the quasigroup that we want to define, then by defining $i * j = i \times a_{1j} \mod p$ we will define a quasigroup $(Q, *)$ of order $p - 1$.

We will define a permutation of the elements in $\mathbb{Z}_p^*$ by the following lemma:

Lemma 4. For a given prime number $p$, and a given number $K$, $1 \leq K \leq p - 2$, the function $f_K(j) = \frac{1}{1 + (K + j) \mod (p - 1)} \mod p$ is a permutation of the elements in $\mathbb{Z}_p^*$.

Now, we can prove the following

Lemma 5. The multiplication operation $*$ defined in the set $Q = \{1, 2, \ldots, p - 1\}$ as:

$$i * j = i \times f_K(j) \mod p$$

(5)

defines a quasigroup $(Q, *)$.

From the last Lemma, we have the following

Corollary 1. If we define the following function

$$g(i, j, K) = (\{i \times j^{-1} \mod p\} - 1 - K) \mod (p - 1)$$

(6)
A quasigroup stream cipher in $\mathbb{Z}_p^*$

that takes the arguments $i, j$ and $K$ from the set $\{1, 2, \ldots, p-1\}$, i.e. maps the set $\{1, 2, \ldots, p-1\}^3$ into the set $\{0, 1, 2, \ldots, p-2\}$ then the left parastrophe $(Q, \backslash)$ of a quasigroup $(Q, *)$ defined by (5) is defined as:

$$i \backslash j = \begin{cases} \frac{g(i,j,K)}{\alpha} & \text{if } g(i,j,K) \neq 0 \\ p-1 & \text{if } g(i,j,K) = 0 \end{cases}$$

(7)

To be consistent with the notation of $f_K(j)$, we will use the notation $g_K(i,j)$ instead the notation $g(i,j,K)$. Additional reason for doing that will be offered in the next section, where once the value of $K$ is chosen, it will remain fixed for different values of $i$ and $j$.

3 A quasigroup stream cipher in $\mathbb{Z}_p^*$

In this section we will define a quasigroup stream cipher that combines algorithms described in previous section. The algorithm is as follows:

A quasigroup stream cipher

**Key generation.** Alice generates her public and private keys as follows:

1. Generate a large random prime number $p$ and a generator $\alpha$ of the multiplicative group $\mathbb{Z}_p^*$ of the integers $\{1, 2, \ldots, p-1\}$.
2. Select a random integer $a$, $1 \leq a \leq p-2$ and compute $\alpha^a \mod p$
3. Alice’s public key is the triplet $(p, \alpha, \alpha^a)$; Alice’s private key is $a$.

**Session key generation.** Bob wants to establish secure stream channel with Alice by doing the following:

1. Obtain Alice’s authentic public key $(p, \alpha, \alpha^a)$.
2. Select a random integer $K$, $1 \leq K \leq p-1$ by which a quasigroup $(Q, *)$ will be defined for the elements $\{1, 2, \ldots, p-1\}$ with equation (5).
3. Encrypt $K$ by ElGamal algorithm, obtaining $C = (\Gamma, \Delta)$.
4. Select $k \geq 3$ random integers $a_i$, $i = 1, 2, \ldots, k$, $1 \leq a_i \leq p-2$ to be leaders for quasigroup stream cipher and encrypt them by ElGamal algorithm, obtaining $C_i = (\Gamma_i, \Delta_i)$, $i = 1, 2, \ldots, k$.
5. Send $C_i$.

**Establishment of a secure stream cipher.** Alice will establish secure stream channel with Bob by doing the following:

1. Decrypt $C$ by ElGamal decryption procedure, obtaining $K$ by which a left parastrophe $(Q, \backslash)$ will be defined with equation (7).
2. Decrypt $C_i$ by ElGamal decryption procedure, obtaining the integers $a_i$, $i = 1, 2, \ldots, k$, $1 \leq a_i \leq p-1$ to be leaders for quasigroup stream cipher.

**Stream Encryption.** Bob encrypts messages $m_\mu$ from the message stream $m_1, m_2, \ldots$ by doing the following:

1. Represent every message part $m_\mu$ as an integer in the range $\{0, 1, \ldots, p-1\}$. 
Session key generation. Bob wants to establish secure stream channel with Alice. Alice generates her public and private keys as follows:

Key generation.

1. Obtain Alice’s authentic public key \((p, \alpha, \alpha^a) = (65537, 13, 29656)\); Alice’s private key is \(a = 10307\).

2. Select a random integer \(K = 35469\) by which a quasigroup \((Q, \ast)\) will be defined for the elements \(\{1, 2, \ldots, 65536\}\) with equation

\[
i \ast j = 1 + (i \ast j) \mod 65536.
\]

3. Encrypt \(K\) by ElGamal algorithm, obtaining \(C = (\Gamma, \Delta) = (1845, 57308)\) (by using the random exponent to be \(e = 53882\)).

4. Select \(k = 3\) random integers \(a_1, a_2, a_3 = (41866, 44005, 27025)\) to be initial leaders for quasigroup stream cipher and encrypt them by ElGamal algorithm, obtaining \(C_1 = (\Gamma_1, \Delta_1) = (13023, 32389)\), \(C_2 = (\Gamma_2, \Delta_2) = (39691, 7691)\) and \(C_3 = (\Gamma_3, \Delta_3) = (14791, 21654)\) (by using random exponents to be 19495, 7737 and 4256).

5. Send \(C_1, C_2\) and \(C_3\).

Establishment of a secure stream cipher. Alice will establish secure stream channel with Bob by doing the following:

1. Decrypt \(C = (\Gamma, \Delta) = (1845, 57308)\) by ElGamal decryption procedure, obtaining \(K = 35469\) by which a left parastrophe \((Q, \backslash)\) will be defined with equation \((7)\) i.e. \(i \backslash j = (i \ast j^{-1} \mod p) - 1 \mod (p - 1)\).

2. Decrypt \(C_1 = (\Gamma_1, \Delta_1) = (13023, 32389)\), \(C_2 = (\Gamma_2, \Delta_2) = (39691, 7691)\) and \(C_3 = (\Gamma_3, \Delta_3) = (14791, 21654)\) by ElGamal decryption procedure, obtaining the integers \(a_1, a_2, a_3 = (41866, 44005, 27025)\) to be initial leaders for quasigroup stream cipher.
Stream Encryption. Bob encrypts messages \( m_\mu \) from the message stream \( m_1, m_2, \ldots \) by doing the following:

1. Suppose that Bob wants to send the following three successive messages \( (m_1, m_2, m_3, \ldots) = (64816, 47513, 52916, \ldots) \).
2. He iteratively compute
   \[
   m^{(1)}_1 = a_1 * m_1 = 41866 \ast 64816 = 6851,
   m^{(2)}_1 = a_2 * m^{(1)}_1 = 44005 \ast 6851 = 44908,
   m^{(3)}_1 = a_3 * m^{(2)}_1 = 27025 \ast 44908 = 19753.
   \]
3. Set \( c_1 = m^{(3)}_1 = 19753 \) and update the values of the leaders by \( (a_1, a_2, a_3) = (m^{(1)}_1, m^{(2)}_1, 1 + (m^{(1)}_1 + m^{(2)}_1 + m^{(3)}_1) \mod (p-1)) = (6851, 44908, 5977) \).
4. Send the ciphertext \( c_1 = 19753 \).
5. He then repeats the steps 2.–4. for \( m_2 = 47513 \) and so on.

Stream Decryption. To decrypt the part \( c_\mu \) of the cipher text stream \( c_1, c_2, \ldots \), Alice should do the following:

1. Obtain cipher text part \( c_1 = 19753 \).
2. Iteratively compute
   \[
   c^{(3)}_1 = a_3 \backslash c_1 = 27025 \backslash 19753 = 44908,
   c^{(2)}_1 = a_2 \backslash c^{(3)}_1 = 44005 \backslash 44908 = 6851,
   c^{(1)}_1 = a_1 \backslash c^{(2)}_1 = 41866 \backslash 6851 = 64816.
   \]
3. Recover \( m_1 = c^{(1)}_1 = 64816 \) and update the values of the leaders by \( a_2 = c^{(3)}_1 = 44908, a_1 = c^{(2)}_1 = 6851 \) and \( a_3 = 1 + (c^{(1)}_1 + c^{(2)}_1 + c^{(3)}_1) \mod (p-1) = 1 + (19753 + 44908 + 6851) \mod (p-1) = 5977 \).
4. She then repeats the steps 2. and 3. for \( c_2 \) and so on.

4 Cryptographical strength of the quasigroup stream cipher in \( \mathbb{Z}_p^* \)

The proposed algorithm has two parts. The first part is the part that is ElGamal algorithm, and the cryptographical strength of that part is based on the strength of ElGamal algorithm i.e. on cryptographical strength of Diffie-Helman algorithm which further relies its security on intractability of Discrete Logarithm Problem.

The second part is the part where fast stream cipher transformations are performed using \( k \) leaders that are unknown for an adversary. In what follows we will examine the cryptographical strength of the stream cipher depending on the number of leaders \( k \). We will assume that the quasigroup stream cipher is broken if the adversary find some of the symmetric parts of the stream i.e. if he find somehow the number \( K \) which defines the permutation in \( \mathbb{Z}_p^* \) or any of the initial leaders \( a_1, a_2, \ldots, a_k \).

4.1 The case \( k = 1 \)

Let \( k = 1 \), and let suppose that the adversary have one pair of known plaintext and ciphertext \( (M, C) = (m_1, m_2, m_3, \ldots, c_1, c_2, c_3, \ldots) \). By having that information he will try to obtain some knowledge about the value \( K \) which defines the quasigroup \( (Q, \ast) \) and about the initial leader \( a_1 \). From the definition of the algorithm it follows that \( c_1 = a_1 \ast m_1 \) and \( c_2 = c_1 \ast m_2, \) i.e.
\[
\begin{align*}
  c_1 &= a_1 \mod (p-1) \\
  c_2 &= c_1 + (K + m_2) \mod (p-1) \\
  c_3 &= c_2 + (K + m_3) \mod (p-1)
\end{align*}
\]

where \(a_1\) and \(K\) are not known. The last system can be reduced to a quadratic polynomial equation with one unknown \(K\) in the field \(\mathbb{Z}_p\). Such type of univariate quadratic polynomial equations can be easily solved for any prime number \(p\) [24] p.37. So, if the number of used leaders is \(k = 1\) the stream cipher is easily breakable.

### 4.2 The case \(k = 2\)

For the case when \(k \geq 2\) we will make an analysis of the strength of the algorithm by assuming that adversary can apply the chosen plaintext attack, i.e. we will assume that the adversary knows what is the outcome from encryption of the plaintext stream where all messages \(m_i = p - 2, i = 1, 2, \ldots\) i.e. he knows the following pair of plaintext and ciphertext: \((M, C) = (p - 2, p - 2, p - 2, p - 2, \ldots, c_1, c_2, c_3, c_4 \ldots)\). With that special case, the equations for quasigroup transformations are simplified since for any \(c \in \mathbb{Z}^*_p\):

\[
c \times (p - 2) = c \times f_K(p - 2) \mod p = \frac{c}{1 + (K - p - 2) \mod (p - 1)} \mod p = \frac{c}{K} \mod p
\]

We will make an additional assumption, in order to simplify the equations that have to be solved. Namely, instead of complicated usage of modulo \(p\) and modulo \(p - 1\) in the obtained equations, we will only use operations modulo \(p\). Although the solutions for those equations are not necessary solutions for the real equations involving modulo \(p\) and modulo \(p - 1\), we will show that even those simplified equations are hard to solve if the number of used leaders \(k\) is sufficiently large.

So, by mentioned simplifications and assumptions, for \(k = 2\) the adversary will obtain the following system of equations in \(\mathbb{Z}_p\):

\[
\begin{align*}
  c_1 &= \frac{a_1}{1 + (K + m_1) \mod (p-1)} \\
  c_2 &= \frac{c_1 + (K + m_2) \mod (p-1)}{1 + (K + m_2) \mod (p-1)} \\
  c_3 &= \frac{c_2 + (K + m_3) \mod (p-1)}{1 + (K + m_3) \mod (p-1)}
\end{align*}
\]

The last system can be reduced to the following univariate polynomial equation of degree 3 in \(\mathbb{Z}_p\) with unknown variable \(K\):

\[
c_3 K^3 + (-2 c_2 + c_3 - c_2 c_3) K^2 + (c_1 - c_2 + c_2^2) K + c_2 c_3 - c_1 c_3 = 0
\]

For those type of polynomials there are efficient (running in polynomial time) randomized algorithms for solving them in \(\mathbb{Z}_p^*\) (see for example [24] p.37, p.123-p.132).
So, we could say that the case with two leaders i.e. when \( k = 2 \) when the equations are simplified and we only work modulo \( p \), can be successfully attacked by the chosen plaintext attack.

We are not aware if there are some known randomized or deterministic algorithms for solving equations that involve both modulo \( p \) and modulo \( p - 1 \) which is much complicated and harder to solve case, but taking conservative approach, we will consider that the case \( k = 2 \) is not safe.

### 4.3 The case \( k = 3 \)

For the case when \( k = 3 \), and by supposing that a possible adversary have one pair of known chosen plaintext and ciphertext \((M, C) = (p - 2, p - 2, p - 2, \ldots, c_1, c_2, c_3, c_4 \ldots)\), he can obtain the following system of simplified equations:

\[
\begin{align*}
    c_1 &= \frac{a_1}{1 + K + \frac{a_2}{1 + \frac{a_1}{K} + K}} \\
    c_2 &= \frac{c_1 + \frac{a_1}{K} + \frac{a_2}{1 + \frac{a_1}{K} + K}}{1 + \frac{a_1}{K} + \frac{a_2}{1 + \frac{a_1}{K} + K}} \\
    c_3 &= \frac{c_2 + \frac{a_1}{K^2} + \left(1 + \frac{a_1}{K} + K\right) \left(1 + \frac{a_2}{K} + K\right) \left(1 + \frac{a_3}{K} + K\right)}{1 + \frac{a_1}{K} + \frac{a_2}{1 + \frac{a_1}{K} + K} \left(1 + \frac{a_3}{K} + K\right) \left(1 + \frac{a_4}{K} + K\right)} \\
    c_4 &= \frac{c_3 + \frac{a_1}{K^2} + \left(1 + \frac{a_1}{K} + K\right) \left(1 + \frac{a_2}{K} + K\right) \left(1 + \frac{a_3}{K} + K\right) \left(1 + \frac{a_4}{K} + K\right)}{1 + \frac{a_1}{K} + \frac{a_2}{1 + \frac{a_1}{K} + K} \left(1 + \frac{a_3}{K} + K\right) \left(1 + \frac{a_4}{K} + K\right) \left(1 + \frac{a_5}{K} + K\right)}
\end{align*}
\]

If we introduce two new variables \( A_1 = \frac{a_1}{K} \) and \( A_2 = \frac{a_2}{1 + \frac{a_1}{K} + K} \) we can reduce the above system to the system of two bivariate polynomial equations:

\[
\begin{align*}
    P_1(K, A_1) &= 0 \\
    P_2(K, A_1) &= 0
\end{align*}
\]

where in the first polynomial \( P_1 \), the degree of \( K \) is 7, and the degree of \( A_1 \) is 3, and in the second polynomial \( P_2 \), the degree of \( K \) is 12 and the degree of \( A_1 \) is 5.

It is clear that if we continue several more steps, with a usage of several more leaders, the complexity of the system to be solved would increase even more. Although the obtained equations have specific structure we can ask the following question: Are there any fast (in polynomial time, deterministic or randomized) algorithms for solving systems of multivariate polynomials modulo big prime number \( p \). We can try to find the answer in the results of modern Number Theory. Namely, two areas of research are connected with posted question: 1. Factorization of multivariate polynomials modulo prime number and 2. Solving systems of multivariate polynomials modulo prime number. Although in the last
decades we see dramatic breakthrough in factorization of multivariate polynomials modulo prime numbers (see for example [25], [26], [27], [28] and [29]), and in some cases the results of that breakthrough increase our knowledge how to solve systems of multivariate polynomials modulo prime number, it was shown that finding the roots of systems of multivariate polynomials modulo big prime number \( p \) is equivalent to solving an NP-hard problem (see [30] and [31]).

From above discussion, we can say that we have strong evidence that breaking the proposed stream cipher would be as hard as solving in polynomial time some NP-hard problem.

5 Conclusions and further directions

In these conclusions, we would like to say something about the speed of the proposed stream cipher. For every block \( m_\mu \) and \( c_\mu \) both in encryption and decryption phase \( k \) modular multiplications and \( k \) modular calculations of inverse element are needed, but doesn’t need operations of modular exponentiation. If we have in mind that modular multiplication and modular division operations modulo \( p \) can be implemented in \( O(\log^2 p) \), that means that total number of operations have complexity of \( O(k \log^2 p) \). In other words, calculated as operations per byte, the proposed stream cipher is much faster then cryptographic algorithms that work over \( \mathbb{Z}_p^* \), and can approach the speed of fast symmetric-key stream ciphers. However, the benefits for using the proposed stream cipher are that its cryptographic strength is equivalent as solving in polynomial time (with deterministic or with randomized algorithm) NP-hard problems.

From other point of view, the proposed stream cipher tries to bridge the gap between fast symmetric-key algorithms and slow public-key algorithms, using the flexibility of key-exchange possibilities of the public-key algorithms, and the speed of symmetric-key algorithms. The mathematical structure of the domain of encoded messages is the same in both parts, i.e. the transformations are done in the set of \( \mathbb{Z}_p^* = \mathbb{Z}_p \setminus \{0\} = \{1, 2, ..., p - 1\} \).

In practical implementation, in order to avoid the common disadvantage of all public-key algorithms that is the expansion of the original message first in the process of transformation of a message \( m \) into an integer from the set \( \{1, 2, \ldots, p - 1\} \) and then in the process of encryption, we should implement the proposed algorithm with a prime number \( p \) which has the form of a Fermat prime number \( F_n = 2^{2^n} + 1 \). However, for \( n = 3 \), and \( n = 4 \) the prime numbers \( F_3 = 257 \) and \( F_4 = 65537 \) are too small for cryptographic purposes, and there are no prime Fermat numbers for \( n > 4 \). To overcome that disadvantage we propose the use of a prime numbers of the form \( p_l = 2^{8l} + 3 \). For example \( p_{98}, p_{213}, \) and \( p_{251} \) are prime numbers with 784, 1704 and 2008 bits respectfully. For example let suppose that we use a prime number \( p_{251} \) with 2008 bits. Then, in the process of message transformation, we could simply treat every consecutive 2008 bits i.e. 251 bytes as an input message and to add one extra byte that will be in fact the total message expansion.
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