Abstract—Massive multiuser multiple-input multiple-output (MU-MIMO) systems are expected to be the core technology in fifth-generation wireless systems because they significantly improve spectral efficiency. However, the requirement for a large number of radio frequency (RF) chains results in high hardware costs and power consumption, which obstruct the commercial deployment of massive MIMO systems. A potential solution is to use low-resolution digital-to-analog converters (DACs)/analog-to-digital converters for each antenna and RF chain. However, using low-resolution DACs at the transmit side directly limits the degree of freedom of output signals and thus poses a challenge to the precoding design. In this study, we develop efficient and universal algorithms for a downlink massive MU-MIMO system with finite-alphabet precodings. Our algorithms are developed based on the alternating direction method of multipliers (ADMM) framework. The original ADMM does not converge in a nonlinear discrete optimization problem. The primary cause of this problem is that the alternating (update) directions in ADMM on one side are biased, and those on the other side are unbiased. By making the two updates consistent in an unbiased manner, we develop two algorithms: one demonstrates excellent performance and the other possesses a significantly low computational complexity. Compared with state-of-the-art techniques, the proposed precoding algorithms present significant advantages in performance and computational complexity.

Index Terms—Massive MIMO, multiuser MIMO, precoding, low-resolution DAC, discrete phase shifter.

I. INTRODUCTION

With the expansion of the Internet of Things and the increase in the data rate demand for mobile devices, the requirement for wireless data rate continues to surge. Massive multiuser multiple-input multiple-output (MU-MIMO) systems are believed to be a key technology for reaching and surpassing the high data rate demand. This technology involves equipping a base station (BS) with a few hundreds of antennas in a centralized [1, 2] or distributed [3] manner to achieve a quasi-orthogonal channel vector between users and a BS. This system demonstrates several advantages, including improvement of network coverage and cell throughput and enhancement of user energy efficiency.

Although the benefits of using massive MU-MIMO systems in BS increase with the number of antennas, the requirement for a large number of radio frequency (RF) chains results in high hardware costs and power consumption, which obstruct the commercial deployment of massive MIMO system. A potential solution is the use of low-resolution digital-to-analog converters (DACs)/analog-to-digital converters (ADCs) for each antenna and RF chain because the hardware complexity and power consumption of DACs/ADCs can be reduced exponentially by decreasing the resolution (in bits) [4]. The problems caused by low-resolution ADCs/DACs have stimulated many discussions [5–32]. Several contributions have been proposed for uplink systems with low-resolution ADCs, and related studies have considered multifold assessments, such as time/frequency synchronization [5], channel estimation [6–12], data detection [8, 9, 11–18], and related performance analyses [12, 14, 19–25]. To date, only a small number of contributions [26–32] consider problems in downlink systems with low-resolution DACs, which piqued our interest.

In downlink massive MU-MIMO systems, BS transmits data to multiple independent user equipment (UE) simultaneously. Maximal ratio transmission, zero-forcing (ZF), and Wiener filter (WF) precoders are commonly used to mitigate inter-user interference (IUI) caused by simultaneous transmission. Using low-resolution DACs at the transmit side directly limits the degree of freedom of output signals. A straightforward approach is to quantize the values of these conventional precoders directly. However, this approach results in a significant performance loss when heavily quantized DACs are applied.

Relevant prior art—The use of 1-bit DACs at the transmitter not only ensures constant-envelope (CE) signals in the input of power amplifiers but also minimizes the energy consumption of a DAC itself. Therefore, massive MU-MIMO systems with 1-bit DACs have elicited much attention [26–32]. For example, a minimum bit error ratio precoder for a 1-bit multiple-input single-output downlink system with 16-QAM signaling was proposed in [27]. This work breaks the myth that the 1-bit precoder is restricted to QPSK signaling. By using biconvex relaxation, [28] proposed 1-bit precoding algorithms for massive MU-MIMO systems that demonstrate better error-rate performance than the ZF precoder directly followed by quantization. Moreover, [28] considered VLSI architectures that enable hundreds of antennas to serve tens of UE. In contrast to [28] where the precoders are designed for 1-bit massive MIMO systems based on the minimum-mean-squared-error (MMSE) criterion to mitigate IUI, [29] changed the design criterion to the minimum bit error-rate (BER). In [30], the proposed precoder design considers the signal distortions caused by 1-bit quantization at the transmitter and receiver. Aside from 1-bit DACs, [26] investigated the problem of downlink precoding with low-resolution DACs (e.g., 1–3 bits) at the BS by using Bussgang’s theorem. [31] extended the work of [26] from cases with frequency-flat channels to those with frequency-selective channels.

Another type of hardware-aware precoding is the use of CE precoding (e.g., [33–38]), which reduces the peak-to-average power ratio in the output signals and thus decreases the linearity requirements at the BS. In CE precoding, the
transmitted signals are strictly limited by a fixed amplitude, and their phases are optimized to minimize IUI. Phase rotation can be implemented by installing an analog or digital phase shifter (PS) in each antenna. The authors in [33–36] assumed that infinite-resolution PSs can generate any required phase. The design of infinite-resolution PSs leads to high hardware complexity and power consumption. Low-resolution PSs are therefore typically used in practice. When finite-resolution PSs are employed, a straightforward approach that utilizes the quantized values of each continuous PS in a finite set also leads to a significant performance loss [37, 38]. Notably, the 1-bit DAC precoding problem can be considered as a special case of CE precoding, in which the phase of the transmitted signal is limited to only four different values.

Thus far, existing finite-alphabet precodings are designed individually. For example, most algorithms are designed specifically for the 1-bit DAC precoding problem, and extending these algorithms from 1-bit DAC precoding to general finite-alphabet precodings remains unjustified. In fact, all of the precoding problems mentioned above are related to a nonlinear least-squares (NLS) problem that attempts to minimize IUI (formed by a minimum Euclidean norm) with a finite-alphabet feasible set. The NLS problem is non-convex and difficult to solve explicitly. A common approach to address this optimization problem is to formulate it into a nonlinear integer (discrete) optimization problem and solve it using the branch-and-cut technique [39] (such as the sphere-decoding method [26]). However, the worst-case complexity of the sphere-decoding method increases significantly with the problem dimensions. Therefore, this approach is unsuitable for massive MIMOs with a large number of antennas. In [40], the authors proposed an algorithm called trellis-based CE precoder (TB_CEP) that searches the precoding by using a trellis structure and only retains a few possible combinations of the trellis states. However, TB_CEP does not provide good trade-off between complexity and performance (which will be shown subsequently in our simulations). Meanwhile, the alternating direction method of multipliers (ADMM) is a common algorithm for nonlinear discrete optimization. The individual steps in ADMM can be implemented exactly. However, ADMM does not converge, and even when it converges, it does not converge to a good suboptimal point.

Contribution—In this study, we develop a universal algorithm for a downlink massive MU-MIMO system with finite-alphabet precodings that minimize IUI. The key contributions of this study are threefold.

- Our algorithms are developed based on the ADMM framework. We reveal that the primary reason for the failure of ADMM to generate a good solution in nonlinear discrete optimization is as follows: the alternating (update) directions in ADMM on one side are biased and those on the other side are unbiased. By making the two updates consistent in an unbiased manner, we develop a universal framework for finite-alphabet precodings called iterative discrete estimation (IDE), which possesses better convergence properties than other local optimization methods. The proposed algorithms possess a unified structure, such that they can be applied to various finite-alphabet problems.
- Although IDE generally achieves excellent error-rate performance, it has a slightly higher complexity than state-of-the-art methods because it requires matrix inversion. Following the same framework as IDE but using an approximation for matrix inversion, we propose a low-complexity version of IDE called IDE2. The simulations show that IDE2 provides good trade-offs between complexity and error-rate performance and is thus highly suitable for massive MU-MIMO systems.
- Most authors (e.g., [26, 28, 31]) only evaluated their precoders under QPSK signaling. In fact, their precoders do not need to perform well under a generally high QAM signaling (e.g., 16-QAM and 64-QAM) because the precoders only need to transform the desired signals into four quadrants for QPSK signaling. In contrast to state-of-the-art precoders, IDE and IDE2 are universal and can work efficiently under any high-level QAM signaling.

The remainder of this paper is organized as follows. Section II introduces the system model and problem formulation. Section III presents ADMM and the reason for its failure to work well in nonlinear discrete optimization problems. The proposed algorithms are introduced, optimized, and simplified. Moreover, the complexity of the proposed algorithms is compared with the complexity of state-of-the-art methods. The simulations are presented in Sections IV. The conclusions are provided in Section V.

Notations—For any matrix $A$, $A^H$ is the conjugate transpose of $A$ and $\text{tr}(A)$ denotes the traces of $A$. $\text{diag}(A)$ returns a diagonal matrix with its diagonal elements containing the diagonal elements of $A$. $E_x\{\cdot\}$ represents the expectation with respect to random variable $x$. When a complex-valued random variable $x$ is the Gaussian distribution with mean $\mu$ and variance $\sigma^2$, we write $x \sim \mathcal{CN}(\mu, \sigma^2)$.

II. System Model and Problem Formulation

A. System Model

As illustrated in Fig. 1, we consider the downlink transmission of a massive MU-MIMO system, in which a BS with $N$ antennas serves $K$ single-antenna users simultaneously in the same time frequency resource. The input output of the downlink channel can be expressed as

$$y = Hx + z,$$  \hspace{1cm} (1)

where $y = [y_1, \ldots, y_K]^T \in \mathbb{C}^K$ contains the received signals of all users, $x = [x_1, \ldots, x_N]^T \in \mathbb{C}^N$ is the transmitted signal from the BS, $H = [H_{k,n}] \in \mathbb{C}^{K \times N}$ denotes the downlink channel with element $H_{k,n}$ being the channel response between transmitting antenna $n$ and user $k$, and $z = [z_1, \ldots, z_K]^T \in \mathbb{C}^K$ is the noise vector. We assume that channel matrix $H$ is perfectly known at the BS and $z_k$’s are i.i.d. circularly symmetric complex Gaussian with mean $0$ and variance $\sigma^2$, that is, $z \sim \mathcal{CN}(0, \sigma^2 I)$.

In the downlink transmission, the BS aims to transmit constellation points $s_k \in \mathcal{O}$ for $k = 1, \ldots, K$ to each of $K$ users, where $\mathcal{O}$ is the set of constellation points (e.g., QPSK, 16-QAM, and 64-QAM). To this end, symbol vector
s = \[s_1, \ldots, s_K\]^T is mapped into \(N\)-dimensional vector \(x\) through a precoder denoted by \(x = \mathcal{P}(s, H)\). The notation \(\mathcal{P}(s, H)\) implies that the precoder not only depends on the transmit constellation points \(s\) but also utilizes the realization of channel matrix \(H\). As an example, if the ZF precoder is used, then we have
\[
x = \mathcal{P}(s, H) = \frac{1}{\beta} H^H (HH^H)^{-1} s, \quad (2)
\]
where \(\beta\) is the precoding factor selected to ensure that a power constraint is satisfied. In this study, we consider the average power constraint as follows:
\[
\frac{1}{N} \mathbb{E}_s \{ \|x\|^2 \} \leq P_{tx}, \quad (3)
\]
where \(P_{tx}\) is the transmit power of each antenna. If \(\mathbb{E}_s \{ss^H\} = \mathbf{I}\), then we obtain \(\beta = \sqrt{\text{tr}((HH^H)^{-1})/(NP_{tx})}\). We define \(NP_{tx}/\sigma^2\) as the signal-to-noise ratio (SNR).

If the ZF precoder is used, then the received signal is \(Hx = \frac{1}{\beta}s\) rather than \(s\). The users should (be able to) rescale the received signal by a factor \(\beta\) to obtain an estimate of the transmit constellation points. Therefore, we define the metric for IUI as
\[
\text{IUI} = \mathbb{E}_x \{ \|s - \beta Hx\|^2 \}. \quad (4)
\]
Although the WF precoder can achieve zero IUI, it presents several challenges to the BS, such as requiring infinite-resolution DACs and high-linearity power amplifiers. In this study, we are interested in a practical setting where each antenna is equipped with a low-cost constrained RF chain. For example, the BS is equipped with low-resolution DACs (e.g., 1-bit DACs), analog PSs, or digital PSs, as illustrated in Figs. 1(a), 1(b), and 1(c), respectively. In these applications, each entry of transmit vector \(x\) is restricted to a finite-alphabet \(\mathcal{X} = \{\chi_0, \ldots, \chi_{M-1}\}\), where \(\chi_m\) represents the possible quantization output. We refer to \(M = |\mathcal{X}|\) and \(B = \log_2 M\) as the number of quantization levels (per dimension) and the number of quantization bits (per dimension), respectively. We call the transmitted signal \(x \in \mathcal{X}^N\) the finite-alphabet precoder.

**B. Problem Formulation**

If the precoder belongs to a finite-alphabet, that is, \(x \in \mathcal{X}^N\), then obtaining zero IUI becomes difficult. The users will experience additional distortion. Our goal is to design a precoder that minimizes IUI under the power constraint (3). Notably, we have rescaled the received signal by the factor \(\beta\). Therefore, given the transmitted symbol \(s\), the mean squared error of the estimated symbols at the receivers can be written as [26]
\[
\|s - \beta Hx\|^2 + \beta^2 K\sigma^2. \quad (5)
\]
Factor \(\beta\) serves as a trade-off between IUI and noise enhancement. By using (5) as the objective function, the precoder design can be formulated as follows:
\[
\min_{x, \beta} \|s - \beta Hx\|^2 + \beta^2 K\sigma^2, \quad \text{s.t.} \quad x \in \mathcal{X}^N, \quad \beta > 0. \quad (6)
\]
If \(x \in \mathcal{X}^N\), then the average power of \(x\) is always determined. Therefore, the power constraint is removed.

The problem formulation in (6) is general and can be used in several applications by setting support \(\mathcal{X}\) to be of several given forms. For example, if the BS is equipped with infinite-resolution DACs, then we can set \(\mathcal{X} = \mathbb{C}\) and introduce the average power constraint (3). In this case, the solution to (6) is the WF precoder [41] expressed as follows:
\[
x = \frac{1}{\beta_{WF}} H^H \left( HH^H + \frac{K\sigma^2 - I}{NP_{tx}} \right)^{-1} s, \quad (7)
\]
where
\[
\beta_{WF} = \sqrt{\frac{\text{tr}(WH^HWF)}{NP_{tx}}}. \quad \text{The 1-bit quantized precoder problem [26, 28] can be obtained by setting } \mathcal{X} = \sqrt{P_{tx}}(\pm 1 \pm j) \text{ with } k = 0, \ldots, M - 1, \text{ the precoder is called a CE precoder [34, 36, 42]. In the CE precoder, a high-efficiency power amplifier can be used because the antenna elements have the same output amplitude. The problem of (6) is also related to an integer programming problem, which is NP-hard in general. Many techniques, such as sphere decoding [26], NOMAD [43], and TB CEP [40], have been proposed to solve these problems. However, the computation complexity of these techniques dramatically increases with the increase in BS antennas } N. \text{ Recent interest has shifted to the design of numerically efficient precoding methods suitable for massive MU-MIMO systems.} \]

1In low-resolution DAC case, we assume the same quantization alphabet for the real and imaginary parts. Therefore, the \(m\)th entry of the transmit vector \(x\) is \(x_n = x_{R,n} + jx_{I,n}\) with \(x_{R,n}, x_{I,n} \in \chi\).
where we denote \( \hat{H} = \beta H \). We develop a numerically efficient algorithm for solving problem (8). We begin by introducing a commonly used ADMM framework for nonconvex problems to explain why it fails to generate a good solution, and we describe in detail our novel algorithms.

A. Why ADMM Fail

To use the ADMM framework, we rewrite problem (8) in a consensus form as follows [44]:

\[
\begin{align*}
\min_{x_1, x} & \quad \|s - \hat{H}x_1\|^2_2 + I_{\mathcal{X}}(x) \\
\text{s.t.} & \quad x_1 - x = 0,
\end{align*}
\]

where \( I_{\mathcal{X}}(\cdot) \) is the indicator function of \( \mathcal{X}^N \), that is,

\[
I_{\mathcal{X}}(x) = \left\{ \begin{array}{ll}
0, & \text{if } x \in \mathcal{X}^N, \\
\infty, & \text{otherwise}.
\end{array} \right.
\]

The augmented Lagrangian of (9) is expressed as

\[
L_r (x_1, x, u) = \|s - \hat{H}x_1\|^2_2 + I_{\mathcal{X}}(x) + u^H (x_1 - x) + \gamma \|x_1 - x\|^2_2,
\]

where \( u \) is the dual vector, and \( \gamma > 0 \) is the penalty parameter (or the augmented Lagrangian parameter). The ADMM for this problem is expressed as

\[
\begin{align*}
x_1^{t+1} &= \arg\min_{x_1} L_r (x_1, x^t, u^t), \\
x^{t+1} &= \arg\min_{x} L_r (x^{t+1}, x, u^t), \\
u^{t+1} &= u^t + \gamma (x_1^{t+1} - x^{t+1} + 1).
\end{align*}
\]

In (12), the \( x_1 \)-update involves solving the IUI minimization problem, the \( x \)-update involves projection onto a finite-alphabet set \( \mathcal{X}^N \), and the \( u \)-update can be interpreted as a consensus adjustment step with step size \( \gamma \). After some algebraic manipulation, (12) can be expressed explicitly as

\[
\begin{align*}
x_1^{t+1} &= (\hat{H}^H \hat{H} + \gamma I)^{-1} (\hat{H}^H s + \gamma x^t - u^t), \\
x^{t+1} &= \Pi_{\mathcal{X}} (x_1^{t+1} + \frac{1}{2\gamma} u^t), \\
u^{t+1} &= u^t + \gamma (x_1^{t+1} - x^{t+1}),
\end{align*}
\]

where \( \Pi_{\mathcal{X}} \) is projected onto \( \{x_n \in \mathcal{X}, n = 1, \ldots, N\} \).

The \( x_1 \)-minimization step (13a) is convex, but the \( x \)-update (13b) is projected onto a nonconvex set \( \mathcal{X}^N \). Although the use of ADMM for nonconvex problems is common, ADMM may not converge. Fig. 2 shows the experimental result of IUI versus iteration, in which \( N = 64, K = 16, \mathcal{X} = \frac{1}{2} \{\pm 1, \pm 2\} \), and \( \gamma = 1 \), to better understand this problem. The IUI of ADMM changes dramatically in each iteration and cannot converge. By checking the program itself, we determine that the main reason for this problem is projection \( \Pi_{\mathcal{X}} \). In each iteration, projection \( \Pi_{\mathcal{X}} \) generates \( x_1^{t+1} \) onto a discrete point, which makes \( x_1^{t+1} \) clearly different from \( x_1^t \), and the variation is too large to make the iteration converge. To solve the rapid change problem, we introduce damping factor \( \alpha \in [0, 1] \) after (13c) as follows:

\[
\begin{align*}
x^{t+1} &\leftarrow \alpha x^t + (1 - \alpha) x^{t+1}, \\
u^{t+1} &\leftarrow \alpha u^t + (1 - \alpha) u^{t+1}.
\end{align*}
\]

We simply refer to ADMM (13) in conjunction with (14) as ADMM2. In Fig. 2, ADMM2 is more stable than ADMM, and its track behaves as a stepped line because as the updates do not surpass a threshold, \( x_1^{t+1} \) does not change in each iteration.

Although ADMM2 behaves like a smooth version of ADMM, the IUI track of ADMM2 is not monotonically decreasing and still cannot converge. By checking the program itself again, we realize that the problem is in the \( u \)-update. This update cannot perform like a consensus adjustment as expected because the alternative update between \( x_1 \) and \( x \) is conflictive. Therefore, we remove dual vector \( u \) in each iteration and obtain the following algorithm

\[
\begin{align*}
x_1^{t+1} &= (\hat{H}^H \hat{H} + \gamma I)^{-1} (\hat{H}^H s + \gamma x^t), \\
x^{t+1} &= \Pi_{\mathcal{X}} (x_1^{t+1}), \\
x^{t+1} &\leftarrow \alpha x^t + (1 - \alpha) x^{t+1}.
\end{align*}
\]

We refer to the algorithm as ADMM3. In ADMM3, \( x_1 \) and \( x \) are updated in an alternating manner without a consensus adjustment. Specifically, the \( x_1 \)-update involves solving a
linearly-constrained minimum Euclidean norm problem, i.e.,
\[
\arg\min_{x_1} \|s - \tilde{H}x_1\|_2^2 + \gamma \|x_1 - x^t\|_2^2.
\] (16)
Then, the x-update projects the resulting point onto a finite-alphabet to obtain the subsequent iteration. This update strategy is straightforward. However, Fig. 2 shows that ADMM3 only updates one time and then falls into a local optimum solution with poor IUI.

B. Proposed Methods

From the previous presented experiments, we realize that all ADMM-based algorithms experience a similar problem: the x1-update intends to minimize IUI, which is not necessarily an alphabet point, whereas the x-update aims to project the result onto an alphabet point. Their updates cannot easily reach a consensus stage. To understand this problem, we rewrite x1-update in (15a) as follows:
\[
x_{1}^{t+1} = x^t + W\left(s - \tilde{H}x^t\right),
\] (17)
where
\[
W = \left(\tilde{H}^H\tilde{H} + \gamma I\right)^{-1}\tilde{H}^H.
\] (18)
From the perspective of estimation theory [45], (17) can be interpreted as the optimal linear MMSE estimate of x given prior knowledge on
\[
E\{x\} = x^t \text{ and } E\{(x - x^t)(x - x^t)^H\} = \frac{1}{\gamma}I.
\] (19)
Such a linear MMSE estimate is biased for each iteration [45] (see Appendix A for this argument). However, the projection step in the x-update always returns an alphabet point, which is an unbiased estimate of x.

To make the two updates consistent, we change (17) into an unbiased version by replacing W with W_u = DW, where D is a diagonal matrix with its entries selected, such that the diagonal elements of W_uH = D(\tilde{H}^H\tilde{H} + \gamma I)^{-1}\tilde{H}^H are 1. If the diagonal elements of W_uH are 1, then the bias in (17) is approximately removed (see Appendix A for this argument). To this end, we let D = [\text{diag}(\tilde{W}\tilde{H})]^{-1}. By substituting the unbiased version of (17) into (15b), we obtain the following algorithm
\[
x^{t+1} = \Pi_X\left(x^t + W_u\left(s - \tilde{H}x^t\right)\right),
\] (20a)
\[
x^{t+1} \leftarrow \alpha x^t + (1 - \alpha) x^{t+1},
\] (20b)
where
\[
W_u = [\text{diag}(\tilde{W}\tilde{H})]^{-1}W.
\] (21)
Notably, we consider the diagonal entries before performing the matrix inversion in (21). Given that each iteration consists of an estimation step and a discrete projection step, we refer to the algorithm as IDE. Fig. 2 shows that the performance of IDE is significantly better than that of ADMM3 and does not have an instability problem similar to that of ADMM2.

The estimation in IDE is based on the problem in (16), in which penalty parameter \(\gamma\) is used to regulate IUI minimization and the previous estimate. Generally, small values of \(\gamma\) tend to produce a small IUI but at the expense of a low convergence rate. Therefore, we use different penalty parameters \(\gamma^t\) for each iteration with the goal of making performance less dependent on the choice of the penalty parameter. Our setting of \(\gamma^t\) is based on a simple observation: from linear estimation theory, \(\gamma\) in (18) should be set as an inverse of covariance (19), such that the estimate (17) can achieve MMSE. Given that the targeted x is unknown, we estimate the error variance as follows:
\[
(\gamma^t)^{-1} = \frac{\|s - \tilde{H}x^t\|^2_2}{\text{tr}(\tilde{H}^H\tilde{H})}.
\] (22)
In (22), \(\text{tr}(\tilde{H}^H\tilde{H})\) serves as a normalization factor to remove the channel effect; thus, \((\gamma^t)^{-1} \approx \frac{1}{\gamma}\|x - x^t\|^2_2\) corresponds to the error variance (19). The algorithm of IDE with adaptive \(\gamma\) is summarized in Algorithm 1. We always use IDE with adaptive \(\gamma\); therefore, we simply refer to Algorithm 1 as IDE. Selection of damping factor \(\alpha\) requires a trade-off between stability and speed of convergence. In Algorithm 1 (as well as Algorithm 2), we set \(\alpha = 0.95\) based on experience.

**Algorithm 1: IDE**

1. **Inputs:** s, \(\tilde{H} = \beta\tilde{H}\)
2. **Initial:** \(t = 0, x^0 = 0, \gamma^0 = 1, \alpha = 0.95\) while \(t < T\) do
3. \(W = (\tilde{H}^H\tilde{H} + \gamma^t I)^{-1}\tilde{H}^H\)
4. \(D = [\text{diag}(\tilde{W}\tilde{H})]^{-1}\)
5. \(x^{t+1} = \Pi_X\left(x^t + DW\left(s - \tilde{H}x^t\right)\right)\)
6. \(\gamma^{t+1} = \frac{\text{tr}(\tilde{H}^H\tilde{H})}{\|s - \tilde{H}x^{t+1}\|^2_2} - 1\)
7. \(\gamma^{t+1} \leftarrow \alpha \gamma^{t+1} + (1 - \alpha) x^{t+1}\)
8. \(\gamma^{t+1} \leftarrow \alpha \gamma^{t+1} + (1 - \alpha) \gamma^{t+1}\)
9. \(t \leftarrow t + 1\)
10. **Output** \(x = x^{t+1}\)

Fig. 2 shows that IDE with adaptive \(\gamma\) converges rapidly and performs significantly better than the other algorithms. Although the performance of IDE is good, its complexity could be high for massive MIMO systems. The complexity of IDE is dominated by the matrix inversion in line 4 of Algorithm 1, which is expressed as \(O(NK^2)\). Notably, matrix inversion has to calculate each iteration because \(\gamma^t\) changes in each iteration.

We address the complexity issue by using an approximation for the matrix inversion. In particular, if \(\gamma\) is large\(^2\), then we can obtain the following approximation
\[
(\tilde{H}^H\tilde{H} + \gamma I)^{-1} \approx \frac{1}{\gamma}I,
\] (23)
which results in \(W \approx \frac{1}{\gamma}\tilde{H}^H\). Consequently, we obtain
\[
W_u \approx [\text{diag}(\tilde{H}^H\tilde{H})]^{-1}\tilde{H}^H.
\] (24)
By substituting \(W_u\) into (20), we obtain a low-complexity version of IDE (summarized in Algorithm 2). We refer to

\(^2\)From (16), a large value of \(\gamma\) implies \(x_1 \approx x^t\); that is, \(x_1\) is close to its previous iteration \(x^t\). Therefore, we observe from Fig. 2 that IDE with this approximation exhibits fast convergence but high IUI.
the algorithm as IDE2. As shown in Fig. 2, the performance of IDE2 is only slightly degenerated because of the low complexity.

Algorithm 2: IDE2

1. Inputs: $s$, $\bar{H} = \beta H$
2. Initial: $t = 0$, $x^0 = 0$, $\alpha = 0.95$
3. while $t < T$ do
   4. $W_u = [\text{diag}(\bar{H}^H\bar{H})]^{-1}\bar{H}^H$
   5. $x^{t+1} = \Pi_X(x^t + W_u(s - \bar{H}x^t))$
   6. $x^{t+1} \leftarrow \alpha x^{t+1} + (1 - \alpha)x^{t+1}$
   7. $t \leftarrow t + 1$
8. Output $x = x^{t+1}$

IDE2 appears nearly similar to classical gradient descent. $F(x) = \|s - \bar{H}x\|^2_2$ denotes the objective function of (8). The negative gradient of $F(x)$ is $-\nabla F(x) = \bar{H}^H(s - \bar{H}x)$. In gradient descent, the update should be in the following form

$$x^{t+1} = \Pi_X \left( x^t + \gamma \bar{H}^H(s - \bar{H}x^t) \right),$$

where step size $\gamma$ should be sufficiently small, such that $F(x^{t+1}) \leq F(x^t)$. The main difference between IDE2 and gradient descent is the step size. The set size in IDE2 is in vector form, i.e., $[\text{diag}(\bar{H}^H\bar{H})]^{-1}$, which attempts to make each update in an unbiased manner. Meanwhile, the set size in gradient descent is a constant $\gamma$, which makes each update in a biased manner.

C. Alternative Update Mechanism for $\beta$

In the previous discussion, we fix precoding factor $\beta$. For any given $x^{t+1}$, precoding factor $\beta$ that minimizes (6) is expressed as follows:

$$\beta^{t+1} = \frac{\text{Re} \{s^H\bar{H}x^{t+1}\}}{|\bar{H}x^{t+1}|^2_2 + K\sigma^2}.$$

Appendix B shows the derivation. Then, we fix the estimate $\beta^{t+1}$ and use the proposed algorithms to derive $x^{t+2}$. The algorithm alternates between the updates of $\beta^t$ and $x^{t+1}$. Specifically, we plug (26) after line 6 of Algorithm 1 and line 5 of Algorithm 2 with initial $\beta^0 = 1$.

D. Complexity Analysis

We analyze the computational complexity of the proposed algorithms and other prior state-of-the-art methods, such as SQUID [26], C1PO [28], and TB_CEP [40], in terms of the number of multiplication operations. We only consider the real-valued model of (8), that is, $\bar{H}$, $s$, and $x$ are real-valued matrices or vectors. Our analytical results can be easily extended to the complex-valued model by using real-valued representation of the complex-valued matrix and vector.\(^3\)

We analyze the complexity of IDE (Algorithm 1). In line 4 of Algorithm 1, given that $\bar{H}$ is fat (i.e., $K < N$), we apply the matrix inversion lemma to $(\bar{H}^H\bar{H} + \gamma^t I)^{-1}$ and only compute the matrix inversion of the small matrix $\gamma^t \bar{H}^H\bar{H} + I$. We cache $(\gamma^t \bar{H}^H\bar{H} + I)^{-1}$ for the subsequent steps in lines 5 and 6. The computation of $(\gamma^t \bar{H}^H\bar{H} + I)^{-1}$ requires $N K^2 + \frac{3}{2} K^3$ multiplications, which involve the cost of forming $\bar{H}^H\bar{H}$ and computing the Cholesky factorization. The subsequent computation of $D$ is implemented as follows:

$$\left[ \frac{1}{\gamma^t} \left( \text{diag}(\bar{H}^H\bar{H}) - \text{diag}(\bar{H}^H(\gamma^t \bar{H}^H\bar{H} + I)^{-1}\bar{H}^H\bar{H}) \right) \right]^{-1},$$

which involves the computation of the diagonal entries of $\bar{H}^H\bar{H}$ (required $NK$ multiplications), the computation of the matrix product of $(\gamma^t \bar{H}^H\bar{H} + I)^{-1}$ and $\bar{H}^H\bar{H}$ (required $K^3$ multiplications), the computation of the diagonal entries of $\bar{H}^H(\gamma^t \bar{H}^H\bar{H} + I)^{-1}\bar{H}^H\bar{H}$ (required $N (K^2 + K)$ multiplications), the computation of a scalar multiplication by $1/\gamma^t$ (required $N$ multiplications), and the computation of the inverse of the diagonal matrix (required $N$ multiplications). The total cost of this step is $2N K^2 + \frac{3}{2} K^3 + 3N K + 2N$. Similarly, the $x$-update in line 6 requires $2N K + N$ multiplications. The cost of projection $\Pi_X$ in the $x$-update is negligible. Line 7 involves $N K + K$ multiplications for computing $\|s - \bar{H}x^{t+1}\|^2_2$. The cost of the damping updates in lines 8 and 9 is negligible because multiplication by constant damping factor $\alpha$ can be implemented using a sequence of shifts and additions or subtractions. Therefore, Algorithm 1 requires a total of $2N K^2 + \frac{3}{2} K^3 + 5N K + 3N + K$ multiplications for $t = 1$. Given that $\bar{H}^H\bar{H}$ does not change in each iteration, we can cache the result to perform the subsequent iterations efficiently. Accordingly, Algorithm 1 requires a total of $2N K^2 + \frac{3}{2} K^3 + 4N K + 3N + K$ multiplications for each iteration when $t = 2$. The total number of multiplications for Algorithm 1 is $T(2N K^2 + \frac{3}{2} K^3 + 4N K + 3N + K) + NK$, where $T$ denotes the number of iterations required to reach a stopping criterion. We summarize the number of multiplications of IDE in Table I.

Then, we analyze the complexity of IDE2 (Algorithm 2). The analysis of IDE2 is nearly similar to that of IDE, except for the fact that IDE2 does not require matrix inversion. In line 4 of Algorithm 2, we only form the reciprocal values of the diagonal entries of $\bar{H}^H\bar{H}$ and cache this result for the subsequent $x$-update, which only costs $NK + N$ multiplications. Line 5 requires $NK + N$ multiplications to compute $W_u s$ and $W_u \bar{H}x^t$, respectively. Similarly, the costs for projection $\Pi_X$ and damping update are negligible. Therefore, Algorithm 2 requires a total of $4NK + 3N$ multiplications for $t = 1$. For $t \geq 2$, lines 4 and 5 only update $W_u \bar{H}x^t$ because the other parts do not change. Accordingly, Algorithm 2 requires a total of $2NK + N$ multiplications for each iteration when $t \geq 2$. The overall complexity required by IDE2 is $T(2NK + N) + 2NK + 2N$ when performing $T$ iterations.

Following the analysis framework mentioned previously, we analyze the complexity of SQUID, C1PO, and TB_CEP and summarize the corresponding results in Table I. In contrast to the other schemes, TB_CEP does not operate in an iterative manner. TB_CEP searches the precoding by using a trellis

\(^3\)A straightforward method is to replace the dimensions in the complexity analysis from $N$ and $K$ to $2M$ and $2K$, respectively.
structure similar to that used by the Viterbi algorithm in decoding or channel equalization. In each step, TB_CEP retains one path from $M^L$ possible combinations of the trellis states. The number of trellis states $M^L$ serves as a trade-off between complexity and performance.

We provide the total number of multiplications in Table II given specific values for system configurations to thoroughly understand the complexity of the mentioned schemes. We fix the number of users to $K = 16$ and show the results for two settings: i) $N = 64$ and ii) $N = 128$. The complexity of SQUID, C1PO, and the proposed algorithms depends on the number of iterations $T$. For SQUID and C1PO, we set $T = 100$ and 24, respectively, following the suggestions of their original proponents [26, 40]. For IDE and IDE2, we set $T = 100$, although good convergence is observed after approximately 50 iterations. Table II indicates that TB_CEP always exhibits relatively higher complexity than the others. IDE2 exhibits the better computational efficiency than the other algorithms for the large MIMO system. IDE requires a slightly higher complexity than SQUID and C1PO but generally achieves excellent error-rate performance (to be shown subsequently in the simulations). In the simulations, we show that even the most efficient IDE2 can achieve better error-rate performance than SQUID, C1PO, and TB_CEP. In addition, the complexity of IDE2 increases linearly with BS antenna size $N$. Consequently, IDE2 exhibits the best trade-offs between complexity and error-rate performance among all the compared algorithms.

### IV. Simulation Results and Discussion

We conduct simulations to evaluate the performance of the proposed methods in terms of average IUI and BER. We consider a setting in which each antenna at the transmitter is equipped with a low-resolution DAC or PS. The 1-bit DAC is the extreme case of this setting. If no specification is provided, then the message symbol is intended for each user is QPSK, and precoding factor $\beta$ is fixed to 1. We fix the number of users to $K = 16$ and scale the total BS antenna number $N = R \times K$ from 32 to 64 and 128 by selecting $R = 2, 4, 8$. We refer to $R$ as the system load factor. The channel responses between the BS and each user follow a circular Gaussian distribution $CN(0, 1)$ in an i.i.d. manner. Performances are evaluated via 30,000 channel realizations. For IDE and IDE2, we set $T = 100$, although good convergence is observed after approximately 50 iterations. Notably, the update of the precoding factor $\beta^t$ depends on $x^t$ as expressed in (26). We begin with a fixed $\beta$ and update $\beta$ every 10 iterations of

| Algorithm | 1-st iteration | Subsequent iteration (each) | $T$ iterations |
|-----------|----------------|-----------------------------|----------------|
| SQUID     | $N^2 K + 2N K^2 + 2N^2 + \frac{1}{2} K^3 + 3N$ | $N^2 + 2N$ | $T(N^2 + 2N) + N^2 K + 2N K^2 + N^2 + \frac{1}{2} K^3 + N$ |
| C1PO      | $NK^2 + \frac{1}{2} K^3 + N^2 + N$ | $N^2 + N$ | $T(N^2 + N) + N K^2 + \frac{1}{2} K^3$ |
| IDE       | $2N K^2 + \frac{1}{2} K^3 + 5N K + 3N$ | $2N K^2 + \frac{1}{2} K^3 + 4N K + 3N$ | $T(2N K^2 + \frac{1}{2} K^3 + 4N K + 3N) + N K$ |
| IDE2      | $4N K + 3N$ | $2N K + N$ | $T(2N K + N) + 2N K + 2N$ |
| TB_CEP    | $N^2 K M^{L+1}$ |                          |                |

### TABLE I

**Computational complexity for different algorithms**

### TABLE II

**The total number of multiplications (MCLs) for different algorithms**

| Algorithm | Max iterations # | MCLs # for $(N, K) = (64, 16)$ | MCLs # for $(N, K) = (128, 16)$ |
|-----------|-----------------|-----------------------------|-----------------------------|
| SQUID     | 100             | 0.52E+6                     | 2.00E+6                     |
| C1PO      | 24              | 0.13E+6                     | 0.46E+6                     |
| IDE       | 100             | 4.21E+6                     | 7.92E+6                     |
| IDE2      | 100             | 0.21E+6                     | 0.42E+6                     |
| Output 4-PSK ($M = 4$) $(N, K, L) = (64, 16, 0.1N)$ | 1000E+6 | 4.80E+24 | 2.00E+40 |

Fig. 3. BER with QPSK signaling as a function of the SNR for different 1-bit DAC precoders with adaptive $\beta$. The number of users is fixed to $K = 16$, and the total BS antenna number is scaled $N = R \times K$ from 32 to 64 and 128 by selecting $R = 2, 4, 8$. 
Table III

| Memory length | TB_CEP | IDE | IDE2 |
|---------------|--------|-----|------|
| L = 3         | 0.0753 |     |      |
| L = 4         | 0.2538 |     |      |
| L = 5         | 0.7546 |     |      |
| L = 6         | 2.0994 |     |      |
| L = 7         | 5.1339 |     |      |
| L = 8         | 7.4925 |     |      |

Fig. 4. IUI versus SNR for different 1-bit DAC precoders with adaptive $\beta$.

Fig. 5. BER with 16-QAM signaling as a function of the SNR for different 1-bit DAC precoders with adaptive $\beta$; $K = 16$, $N = 128$, and $R = 8$.

$x^4$ to ensure that $x^4$ has converged to a good state$^4$.

A. 1-bit Precoding

In Fig. 3, we compare the BER and SNR of our proposed algorithms and state-of-the-art methods, such as SDRr and SQUID [26]. As reported in [26], the performance of SDRr is close to that observed in exhaustive search. Given that exhaustive search is impossible, we regard SDRr as a performance benchmark in this experiment. However, the computational complexity of SDRr is still high with the increase in the number of BS antenna $N$. In [26] and [28], the authors proposed low-complexity versions called SQUID and C1PO, respectively. The performance of SQUID and C1PO is comparable. Thus, we only consider SQUID in the comparison. In this figure, we use the adaptive precoding factor $\beta$ for all of the algorithms (including SDRr and SQUID). For comparison, we also report the performance of the WF precoder for the infinite-resolution case.

We observe that the BER performance of IDE is comparable to that of the benchmark SDRr. IDE has a low computational complexity. The proposed algorithms (i.e., IDE and IDE2) present significant advantages in terms of BER compared with SQUID, particularly when $R$ is small (e.g., $R = 2, 4$). The performance of IDE2 is comparable to that of IDE, and IDE2 has a significantly lower computational complexity than IDE. When $R = 8$, the gap between the performance of the proposed algorithms and that of SQUID becomes negligible. However, this result does not imply that their behaviors are similar. We also show the corresponding IUIs for the three precoding algorithms in Fig. 4 to thoroughly understand their differences. When $R = 8$, the proposed algorithms provide a significantly lower IUI than SQUID. The IUI of SQUID is saturated at -14 dB. The error floor for IUI degenerates the BER performance when a high-modulation symbol (e.g., 16-QAM) is used because such a high modulation has to work in the high-SNR regime. In Fig. 5, we compare the BER with the 16-QAM symbol and 1-bit DACs of our proposed algorithms and other algorithms under $R = 8$ to justify this argument. Notably, the gaps between the performance of SQUID and our proposed algorithms increase with the increase in SNR because the IUI of SQUID saturates in the low-SNR regime (Fig. 4). In addition, the BERs of our proposed algorithms are similar to SDRr.

Another interesting observation from Fig. 4 is that the IUIs of the three algorithms are similar when $R = 2$. However, IDE and IDE2 offer significant advantages in terms of BER compared with SQUID (Fig. 3). The reason is that as $R = 2$, their precoding factors $\beta$ are different and thus lead to different BERs. The precoding factor serves as a trade-off between IUI and noise enhancement. Therefore, this result indicates that IDE and IDE2 provide better trade-off in this regard than SQUID.

B. Low-resolution PSs

We examine our algorithms under a more general setting with finite-resolution PSs, that is, the possible values of the transmitting antenna outputs are from a set of $M$-
achieve the same performance as our proposed algorithms when \( L \) increases to a high level. However, the computational complexity of TB\_CEP is too high to be practical. In addition, Fig. 6 shows that IDE2 can be close to IDE as the number of resolution levels, \( M \), increases.

We consider a small MU-MIMO system with eight BS antennas and two users, that is, \( N = 8 \) and \( K = 2 \), to determine the gaps between the optimal precoder (exhaustive search) and the considered algorithms. The optimal precoding can be obtained by exhaustive search or TB\_CEP with \( L = 8 \).

Fig. 7 compares the BERs of IDE, IDE2, TB\_CEP, and the optimal precoder when the four-phase state of the PSs is employed. We determine that the gap between the performance of the optimal precoder and the performance of IDE (or IDE2) is small, namely, approximately 2 dB for a target BER of \( 10^{-3} \). When \( L = 7 \), TB\_CEP presents a comparable performance to that of the proposed algorithms. However, the computational complexity of TB\_CEP is too high to be practical when the number of BS antennas is large. We summarize the average running times\(^5\) (in seconds) of the algorithms in Table III. IDE and IDE2 provide significant advantages in terms of complexity compared with TB\_CEP.

C. Low-resolution Precoding

In the previous experiments, the algorithms are evaluated by using either 1-bit DACs or PSs at the transmitters. Both of them are CE precoding. Then, we shift our attention to multi-bit precoding for low-resolution DACs, in which the precoding has multiple amplitude levels. When low-resolution DACs are employed, [26] suggests quantizing the values of WF precoding to a finite set directly. In Fig. 8, we compare the BERs of the quantized WF precoding with those of IDE and IDE2 under different DAC resolution precoders, modulation symbols, and load factors.

First, we focus on the case with 4-QAM signaling (Figs. 8(a) and 8(b)). When \( R = 4 \), the 1-bit WF precoder cannot achieve BER below \( 10^{-3} \) and has significant gaps compared with IDE and IDE2. For 2-bit DACs, IDE and IDE2 gain 2 dB compared with the 2-bit WF precoder for a target BER of \( 10^{-3} \). In the case with 3-bit DACs, all algorithms perform similarly, but IDE and IDE2 exhibit the better performance. In this case, all algorithms are close to the limit performance of the infinite-resolution WF precoder. Similar characteristics are observed when \( R = 8 \). Second, we focus on cases with 16-QAM and 64-QAM signaling (Figs. 8(c) to 8(f)). IDE and IDE2 have a significant performance gain over the quantized WF precoder in all quantization levels. For 3-bit DACs, IDE and IDE2 are close to the limit performance of infinite-resolution WF precoding, whereas the 3-bit WF precoder is still far from the limit performance. Moreover, the performance of IDE2 is compared with the performance of IDE while exhibiting significantly lower computational complexity.

The discussions in the previous subsections focus on the 1-bit precoder, low-resolution PSs, and multi-bit precoder.

\(^5\)The simulations are performed with MATLAB v8.6.0 (R2015b) on a 64-bit Windows 7 PC equipped with a 3.4-GHz Intel Core i7-3370 CPU and 4 GB of memory.
In prior state-of-the-art methods, the precoding techniques for different settings are completely different. Notably, our algorithms are not only universal but can also perform well in various finite-alphabet precoders with general QAM signaling.

We propose novel precoding algorithms called IDE and IDE2 for a massive MU-MIMO system where each antenna at the BS is equipped with coarse-resolution DACs or PSs. The algorithms have a unified structure, such that they can

\[ D. \text{ Complexity and Convergence Rate} \]

Fig. 9 compares the average running times versus N for all of the concerned algorithms. We consider PSs with four-phase state and \( T = 100 \). IDE2 exhibits the lowest timing complexity among the algorithms, followed by SQUID and IDE. The timing complexity of SDRr and TB_CEP increases significantly with the number of BS antennas. After integrating all of the previous experiments, we conclude that IDE2 exhibits the best trade-off between performance and complexity among all the algorithms.

Timing complexity heavily depends on the number of iterations \( T \). As previously mentioned, IDE and IDE2 perform well after approximately 50 iterations. Fig. 10 shows the IUI versus iteration under 1-bit DACs with fixed \( \beta \) for \( R = 4, 8 \) and SNR = 0 dB to justify this argument. We observe that IUI only decreases by approximately 1 dB as the iterations increase from 50 to 100. The IUI at \( T = 50 \) is sufficient to obtain good detection for 4-QAM or 16-QAM symbols. In addition, \( \mathbf{x} \) does not obtain a good IUI result after the first few iterations. Therefore, as mentioned at the beginning of this section, we update \( \beta \) after every 10 iterations of \( \mathbf{x}^t \).

V. Conclusion

Fig. 8. BER as a function of the SNR for different DAC resolution precoders with adaptive \( \beta \).

Fig. 9. Average execution time versus the number of BS antennas for all of the concerned precoders. We consider PSs with four-phase state and \( T = 100 \).
be applied to various finite-alphabet input problems that aim to minimize IUI. Compared with state-of-the-art methods, the proposed precoding algorithms show significant advantages in terms of performance and computational complexity. In particular, IDE2 is simple and possesses low per-iteration complexity. Thus, it is highly suitable for massive MU-MIMO systems.

APPENDIX A: APPROXIMATELY UNBIASED ESTIMATOR

This appendix aims to show that (20a) is an approximately unbiased estimator of \( x \). We begin by showing that (17) can be interpreted as the optimal linear MMSE estimate of \( x \) given prior knowledge on (19). To this end, we introduce the following virtual channel model

\[
    s = Hx + \tilde{z},
\]

where \( \tilde{z}_k \)'s are i.i.d circularly-symmetric complex Gaussian with mean 0 and variance 1, that is, \( \tilde{z} \sim CN(0, I) \). Suppose that we wish to construct an estimator for \( x \) from \( s \). We restrict our estimator to be in the form

\[
    \hat{x} = Ws + b,
\]

where matrix \( W \) and vector \( b \) are to be determined to minimize \( E[(x - \hat{x})^H(x - \hat{x})] \). Notably, the expectation is taken over the joint density of \( s \) and \( x \). The optimal \( W \) and \( b \) are expressed as follows [45]:

\[
    W = C_{zs} C_s^{-1}, \quad b = \bar{x} - W\bar{s},
\]

where \( \bar{x} = E\{x\} \), \( \bar{s} = E\{s\} \), \( C_{zs} \) is the cross-covariance matrix between \( x \) and \( s \), and \( C_s \) is the auto-covariance matrix of \( s \). Specifically, by substituting (28), we obtain

\[
    C_{zs} = E\{(x - \bar{x})(s - \bar{s})^H\},
\]

\[
    = E\{(x - \bar{x})(x - \bar{x})^H \tilde{H}^H + (x - \bar{x})\tilde{z}^H\},
\]

\[
    = C_x \tilde{H}^H,
\]

and

\[
    C_s = E\{(s - \bar{s})(s - \bar{s})^H\},
\]

\[
    = E\left\{ \begin{pmatrix} \tilde{H}(x - \bar{x}) + \tilde{z} \\ \tilde{H}(x - \bar{x}) + \tilde{z} \end{pmatrix} \begin{pmatrix} \tilde{H}(x - \bar{x}) + \tilde{z} \\ \tilde{H}(x - \bar{x}) + \tilde{z} \end{pmatrix}^H \right\},
\]

\[
    = \tilde{H} C_x \tilde{H}^H + C_z.
\]

Using prior knowledge on \( x \) in (19) and \( C_z = I \), we obtain

\[
    W = C_{zs} C_s^{-1}
    = \tilde{H}^H(\tilde{H}\tilde{H}^H + \gamma I)^{-1}
    = (\tilde{H}^H\tilde{H} + \gamma I)^{-1}\tilde{H}^H,
\]

\[
    b = x^t - W\tilde{H}x^t,
\]

where (33a) is equal to (33b) using the matrix identity. The optimal linear MMSE estimate is

\[
    \hat{x} = \tilde{x} + W(s - \bar{s}) = x^t + W(s - \tilde{H}x^t),
\]

which is completely identical to (17).

Taking the expected value on both sides of (34) produces

\[
    E\{\hat{x}\} = E\{x^t - WHx^t + Ws\}
    = (I - WH)x^t + WHE\{x\} \neq E\{x\}.
\]

Clearly, \( x \) is a biased estimator of \( x \) as long as \( E\{x\} \neq x^t \). \( x^t \) is obtained from the previous iteration, which could not be equal to \( x^t \) before convergence. We can remove the bias of \( \hat{x} \) by setting \( \gamma = 0 \). In this case, (34) becomes

\[
    \hat{x} = (\tilde{H}^H\tilde{H})^{-1}\tilde{H}^H s.
\]

Moreover, we obtain \( E\{\tilde{H}^H\tilde{H}^{-1}\tilde{H}^H s\} = E\{x\} \). However, this approach degenerates the iteration algorithm because it causes the estimation step to completely ignore knowledge on the previous iteration result \( x^t \).

From the discussion above, we realize that factor \( \gamma \) should be preserved. However, as long as \( \gamma \) appears, the form expressed in (35b) always presents a biased estimate because the main diagonal elements of \( WH \) are smaller than 1. To approximately remove the bias, we normalize the diagonal elements of \( WH \) as follows:

\[
    DW\tilde{H} = [\text{diag}(WH)]^{-1}WH.
\]

By doing so, we obtain

\[
    \hat{x} = x^t + DW(s - \tilde{H}x^t)
\]

and

\[
    E\{\hat{x}\} = (I - F)x^t + FE\{x\},
\]

where \( F = DW\tilde{H} \). Given that the diagonal elements of \( F \) are 1 and the off diagonal elements of \( F \) are smaller than 1, we obtain \( (I - F)x^t \approx 0, FE\{x\} \approx E\{x\}, \) and \( E\{\hat{x}\} \approx E\{x\} \).

According to the central limit theorem, this approximation

\[
    (H^H\tilde{H} + \gamma I)^{-1}(H^H\tilde{H} + \gamma I)H^H = (H^H\tilde{H} + \gamma I)^{-1}H^H.
\]

By pre-multiplying by \( \gamma \), we obtain

\[
    (H^H\tilde{H} + \gamma I)^{-1}(H^H\tilde{H} + \gamma I)H^H = (H^H\tilde{H} + \gamma I)^{-1}H^H.
\]
becomes good when $N$ becomes large. The new estimator is an approximately unbiased estimator.

**APPENDIX B: DERIVATIONS OF $\beta$**

When $x$ is given, the minimum problem (6) can be written in the form:

$$\min_{\alpha} \|s - \beta Hx\|^2 + \beta^2 K \sigma^2,$$

s.t. $\beta > 0$. (40)

We unfold the objective function as follows:

$$(s - \beta Hx)^H (s - \beta Hx) + \beta^2 K \sigma^2$$

becomes large. The new estimator is an approximately unbiased estimator.

By equating the differential of (41) with respect to $\beta$ to zero, we obtain

$$-2{\text{Re}}(s^H Hx) + 2\beta \|Hx\|^2 + 2\beta K \sigma^2 = 0.$$ (42)

If $\|Hx\|^2 + K \sigma^2 \neq 0$, then we obtain

$$\beta = \frac{{\text{Re}}(s^H Hx)}{\|Hx\|^2 + K \sigma^2}.$$ (43)

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