Asymptotics of the optimum in discrete sequential assignment

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Abstract

We consider the stochastic sequential assignment problem of Derman, Lieberman and Ross (1972) corresponding to a discrete distribution supported on a finite set of points. We use large deviation estimates to compute the asymptotics of the optimal policy as the number of tasks \( N \to \infty \).

Key words: stochastic sequential assignment, optimal policy, large deviation, martingale

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1 Introduction

Let us start with a simple example of the question we are interested in. Consider the following game: you start with a row of \( N \) empty boxes, and a fair dice is rolled \( N \) times. After each roll in turn, you have to assign the rolled value to one of the boxes. After all the rolls have taken place, an \( N \)-digit number is obtained, that you are trying to maximize (see [4] for a popularizing account of this game). Let us be more precise about the sense of maximizing, for which the following are two natural possibilities: (i) maximize the score on average, that is, the expected value of the final \( N \)-digit number; (ii) maximize the probability of obtaining the ‘perfect score’, that is, when the assignment is monotone increasing from right to left.

In the formulation (i), the problem is a special case of stochastic sequential assignment, introduced by Derman, Lieberman and Ross in [2]. They gave a recursive method to compute the optimal policy (when the values ‘rolled’ are i.i.d. samples from any distribution on \( \mathbb{R} \) with finite mean). Formulation (ii) is of interest in the discrete example outlined in the previous paragraph. An especially interesting feature of it is that it undergoes a controllability phase transition. In order to state this, a slightly different initial setup is convenient: assume there are

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$N - N_0$ rounds to go, and the remaining set of empty boxes splits into 5 contiguous intervals of lengths $n_1, \ldots, n_5$, where $n_1 + \cdots + n_5 = N - N_0$, separated by contiguous intervals already filled by the values 2, \ldots, 5; see Figure 1. It follows from the main result of [5] that as $N - N_0 \to \infty$, if the vector $(n_i/(N - N_0))_{i=1}^5$ lies in the interior of a certain parallelepiped $K$, then the probability of obtaining the perfect score under the optimal policy is exponentially close to a constant $c > 0$, whereas, if the same vector is in the exterior of $K$, it goes to 0 exponentially. This critical phenomenon extends to a more general setting, where the $n_i$'s are assigned to edges of a finite graph, and the values 'rolled' are the vertices of the graph; see [5]. Estimates proved in [5] suggest that in the critical region, when $(n_i/(N - N_0))_i$ is near $\partial K$, one can expect the value of the optimal policy to exhibit Gaussian behaviour. Information on the asymptotics of the optimal policy in this region would help establish Gaussian behaviour rigorously.

![Figure 1: Illustration of the setup in which a phase transition occurs. Here $N = 25$, $N_0 = 10$, and there are 15 rounds to go. In the picture $(n_1, \ldots, n_5) = (3, 4, 3, 3, 2)$.

A step towards the above goal is to determine the asymptotic behaviour, as $N \to \infty$, of the optimal policy in both problems (i) and (ii). This turns out to be easier for (i), and is the aim of the present paper. The reason why (i) is easier to analyze is the following. Let $Z_1, \ldots, Z_N$ denote the values assigned to boxes 1, \ldots, $N$, respectively. Then in the case of (i), the objective is to maximize a linear function of the $Z_i$'s (in expectation), whereas in the case of (ii) the objective is to maximize a non-linear function (the indicator of $Z_1 \leq \cdots \leq Z_N$). We expect that in many, but not all discrete valued assignment problems, the formulations (i) and (ii) behave similarly, and we consider the case (i) as the first step in understanding (ii).

Let us from now on restrict to the objective of maximizing the expected score, and ask: when $N$ is large, where are the optimal locations to place the first rolled number, if it is, respectively, 1, \ldots, 6? We will see that these are approximately given by

$$1, \quad N \log_{\frac{2}{9}} \frac{4}{9}, \quad N \log_{\frac{1}{2}} \frac{3}{2}, \quad N \log_{\frac{2}{3}} \frac{2}{3}, \quad N \log_{\frac{1}{2}} \frac{1}{2}, \quad N,$$

respectively, with boxes counted from the right end.

More generally, we consider the problem when the dice has $k \geq 3$ sides, with real values $x_1 < \cdots < x_k$ written on them, and the probability of rolling $x_i$ is $p_i > 0$, $i = 1, \ldots, k$. As mentioned earlier, the optimal policy was found in [2]. When the values to be assigned come from a continuous distribution, it was shown in [1] by a law of large numbers argument, that the asymptotic profile of the optimal policy is given in terms of the scaled quantiles of the underlying distribution. We were surprised not to find in the literature any result on the asymptotics in the
The discrete case stated above. The discrete case is interesting, since the location of the optimum is determined by large deviation events, in contrast with the continuous case.

The structure of the paper is the following. In Section 1.1 we precisely define the model considered, and state our main result. Section 1.2 gives the short large deviation computation that yields the asymptotic optimum values. The proof of the main theorem is given in Section 2.

1.1 The model

Let \( p = (p_i)_{i=1}^k \) be a discrete distribution supported on the points \( x_1 < \cdots < x_k \), such that \( p_i > 0 \) for all \( 1 \leq i \leq k \). Let \( N \geq 1 \), and let \( r_N > \cdots > r_1 \) be given numbers (rewards). Let \( X_1, \ldots, X_N \) be i.i.d. random variables with distribution \( p \). Let \( F_t = \sigma\{X_1, \ldots, X_t\}, 0 \leq t \leq N \).

By a policy we mean a sequence of random indices \( J(1), \ldots, J(N) \in \{1, \ldots, N\} \) such that:

(i) \( J(t) \) is \( F_t \)-measurable; and

(ii) \( J(1), \ldots, J(N) \) are distinct.

By the reward of the policy we mean the random quantity

\[
R(X; J) = \sum_{t=1}^N X_t r_{J(t)}.
\]

The aim is to maximize the expected reward. The theorem below follows directly from a theorem proved by Derman, Lieberman and Ross \[2\]; see also \[6, Section VI.7\]. We denote by \( 1 = \ell_N(1) \leq \ell_N(2) \leq \cdots \leq \ell_N(k-1) \leq \ell_N(k) = N \) the optimal locations for placing the values \( x_i, i = 1, \ldots, k \), among \( N \) remaining empty spaces, \( N \geq 1 \) (making an arbitrary choice in case the optimal policy has ties).

**Theorem 1.1.** \[2\] There exist numbers

\[
-\infty = a_{N,0} < \cdots < a_{N,N} = \infty, \quad N \geq 1,
\]

that only depend on \( p \) and \( (x_i)_{i=1}^k \) (and not on the \( r_i \)'s), such that any optimal policy satisfies

\[
a_{N,\ell_N(i)-1} \leq x_i \leq a_{N,\ell_N(i)}, \quad 1 \leq i \leq k, \quad N \geq 1.
\]

Moreover, we can choose

\[
a_{N,n} = \mathbb{E}[X_{T(n)}], \quad 1 \leq n \leq N - 1, \quad N \geq 2,
\]

where \( T(n) \) is the unique index \( 1 \leq t \leq N - 1 \) such that \( J(t) = n \), and \( J \) is any optimal policy with \( N - 1 \) rounds to go.
We are ready to formulate our main result. Let \( f_i := \sum_{j \leq i} p_j, \) \( i = 0, \ldots, k, \) and let

\[
d_i := \frac{\log \frac{1-f_i}{1-f_{i-1}}}{\log \frac{f_{i-1}}{f_i}}, \quad i = 2, \ldots, k-1.
\]

(1.4)

We extend this definition to \( i = 1 \) and \( i = k \) by setting \( d_1 = 0 \) and \( d_k = 1. \)

**Theorem 1.2.** Let \( k \geq 3, \) and let \( p \) be a positive probability vector. Then any optimal policy satisfies

\[
\ell_N(i) = N(d_i + o(1)), \quad \text{as } N \to \infty, \ i = 1, \ldots, k.
\]

**1.2 Computation of the optimum**

The asymptotic optimum \( d_i \) arises from the equality of two large deviation rates. We give this short computation before delving into the proof of Theorem 1.2. For \( i = 2, \ldots, k-1 \) let us look for the value of \( y \in (f_{i-1}, f_i) \) that yields equality of the large deviation rates:

\[
I_i^-(y) := -\lim_{N \to \infty} \frac{1}{N} \log \mathbb{P} \left[ \sum_{t=1}^{N} 1_{X_t < x_i} \geq yN \right],
\]

\[
I_i^+(y) := -\lim_{N \to \infty} \frac{1}{N} \log \mathbb{P} \left[ \sum_{t=1}^{N} 1_{X_t > x_i} \geq (1-y)N \right].
\]

Using the large deviation rate function for binomial random variables \( \mathbb{R}, \) we have

\[
I_i^-(y) = y \log y + (1-y) \log(1-y) - y \log f_{i-1} - (1-y) \log(1-f_{i-1})
\]

\[
I_i^+(y) = y \log y + (1-y) \log(1-y) - y \log f_i - (1-y) \log(1-f_i).
\]

Therefore, equality occurs for \( y = d_i. \)

**2 Proof of the main theorem**

In this section we give the proof of Theorem 1.2. The idea of the proof is as follows. In order to arrive at a contradiction, assume that \( \limsup_{N \to \infty} \ell_N(i)/N > d_i, \) and \( i \) is smallest with this property. Let \( U_N \) denote the random value that is assigned to the location \( \ell_N(i) - 1, \) that is, \( U_N = X_{\ell_N(i)-1}. \) If we can deduce that \( \mathbb{E}[U_N] > x_i \) for some (sufficiently large) \( N, \) this contradicts Theorem 1.1. Let \( \tau \) be the random time when location \( \ell_N(i) - 1 \) is filled, which is a stopping time:

\[
\tau = \inf\{t \geq 1 : J(t) = \ell_N(i) - 1\}.
\]
Let $I_t$ be the index of location $\ell_N(i) - 1$ among the ‘empty spaces’ at time $t$, that is:

$$I_t = \# \{ 1 \leq n \leq \ell_N(i) - 1 : n \neq J(s) \text{ for any } 1 \leq s \leq t \}, \quad 0 \leq t < \tau.$$ 

Let $\sigma$ be the stopping time

$$\sigma = \inf \{ 0 \leq t < \tau : I_t = \ell_N(i) - t + 2 \} \in \{0, \ldots, \tau - 1\} \cup \{\infty\}.$$ 

Observe that by definition, either $\sigma < \tau$ or $\sigma > \tau$ (when the set of $t$’s considered in the inf is empty).

We prove the following estimates.

**Proposition 2.1.** Let $2 \leq i \leq k - 1$, and assume that:

(i) $\limsup_{N \to \infty} \ell_N(j)/N \leq d_j$ for $1 \leq j \leq i - 1$;

(ii) $\limsup_{N \to \infty} \ell_N(i)/N > b > d_i$ for some $b$.

Then there exist $b'$ satisfying $b > b' > d_i$ such that for all $N$ such that $\ell_N(i) \geq Nb + 1$, we have

$$P[U_N \leq x_{i-1}] \leq \exp \left( -(1 + o(1)) I_i^{-}(b') N \right), \quad \text{as } N \to \infty.$$ 

**Proposition 2.2.** Let $2 \leq i \leq k - 1$, and assume that:

(i) $\limsup_{N \to \infty} \ell_N(i)/N > b > d_i$ for some $b$.

Then there exists $b''$ satisfying $b > b'' > d_i$ and a constant $c_1 = c_1(b, x_i, x_{i+1}) > 0$ such that for an infinite set $G$ of positive integers we have

(a) $\ell_N(i) \geq Nb + 1$ for all $N \in G$;

(b) 

$$E[U_N | \sigma < \tau] \geq x_i + c_1 \exp(- (1 + o(1)) I_i^{+}(b'') N), \quad \text{for all } N \in G. \quad (2.1)$$

(c) Moreover, there exists a constant $c_2 > 0$ such that $P[\sigma < \tau] \geq c_2$ for all $N \in G$.

Let us first show that the two propositions imply the main theorem.

**Proof of Theorem 1.2.** Adding a constant to the random variables $X_i$ changes the reward of any policy by a constant, and hence we may assume without loss of generality that $0 = x_1 < x_2 < \cdots < x_k$.

It is sufficient to show that $\limsup_{N \to \infty} \ell_N(i)/N \leq d_i$, for $1 \leq i \leq k$, since then the analogous inequality for the liminf follows by symmetry. In order to arrive at a contradiction, assume that $i$ is the smallest index such that the inequality fails. Then $2 \leq i \leq k - 1$, and we can fix a number $b$ with $\limsup_{N \to \infty} \ell_N(i)/N > b > d_i$, so that Propositions 2.1 and 2.2 can be applied. Consider the subsequence $G$ provided by Proposition 2.2, and observe that due to part (a) of this proposition, the conclusion of Proposition 2.1 also holds for all $N \in G$.

Since $\sigma \neq \tau$, we have

$$E[U_N] = E[U_N 1_{\sigma < \tau}] + E[U_N 1_{\sigma < \tau}], \quad N \in G.$$
Proposition 2.1 implies that
\[ P[U_N \leq x_{i-1}, \tau < \sigma] \leq P[U_N \leq x_{i-1}] \leq \exp\left(-(1 + o(1)) I_i^- (b') N\right), \ N \in \mathcal{G}. \]

Writing \( \alpha = P[\sigma < \tau] \) this implies
\[ E[U_N 1_{\tau<\sigma}] \geq x_i P[U_N \geq x_i, \tau < \sigma] \geq x_i (1 - \alpha) - x_i \exp\left(-(1 + o(1)) I_i^- (b') N\right), \ N \in \mathcal{G}. \] (2.2)

On the other hand, Proposition 2.2 implies that
\[ E[U_N 1_{\sigma<\tau}] = \alpha E[U_N | \sigma < \tau] \geq \alpha \left(x_i + c_1 \exp\left(-(1 + o(1)) I_i^+ (b'') N\right)\right), \ N \in \mathcal{G}. \] (2.3)

Putting together (2.2) and (2.3), and using that \( I_i^+ (b'') < I_i^+ (d_i) = I_i^- (d_i) < I_i^- (b') \), we get that for a sufficiently large \( N \in \mathcal{G} \) we have
\[ E[U_N] > x_i. \]

This is the desired contradiction, and completes the proof. \( \square \)

Our next task is to prove Proposition 2.1. For what follows, recall the definition of \( I_t \) from the beginning of this section, and observe that we have \( I_0 = \ell_N(i) - 1 \), and the time evolution of \((I_t)_{0 \leq t < \tau}\) is given by:
\[ I_{t+1} = \begin{cases} \ I_t & \text{when } X_{t+1} = x_j \text{ and } I_t < \ell_{N-t}(j), \ 2 \leq j \leq k; \\ \ I_t - 1 & \text{when } X_{t+1} = x_j \text{ and } I_t > \ell_{N-t}(j), \ 1 \leq j \leq k - 1; \end{cases} \] (2.4)

moreover, we have
\[ \tau = t + 1, \ \text{ when } X_{t+1} = x_j \text{ and } I_t = \ell_{N-t}(j), \ 1 \leq j \leq k. \]

**Proof of Proposition 2.1.** We fix \( \delta > 0 \) such that
\[ \delta < \min \{b - d_i, d_i - d_{i-1}, f_{i-1} - d_{i-1}\}. \]

Due to assumption (i), there exists \( N_2 \geq 1 \) such that for all \( N \geq N_2 \) we have \( \ell_N(i - 1)/N \leq d_{i-1} + \delta \). Consider the stopping time
\[ \sigma_1 := \inf\{0 \leq t < \tau: I_t \leq \ell_{N-t}(i - 1)\}. \]

When \( N - t \geq N_2 \), we have
\[ (\ell_N(i) - 1) - \ell_{N-t}(i - 1) \geq Nb - (N - t) (d_{i-1} + \delta). \] (2.5)
Let

\[ N' := N \frac{b - d_{i-1} - \delta}{1 - d_{i-1} - \delta}. \]

A simple computation shows that when \( t < N' \), the right hand side of (2.5) is larger than \( t \). Since \( I_t \) can decrease by at most 1 at each time step, this implies the deterministic inequality \( \sigma_1 \geq N' \).

When \( N - t < N_2 \) and \( N \geq N_3 := 2N_2/(b - d_i) \), we have

\[ (\ell_N(i) - 1) - \ell_{N-t}(i - 1) \geq Nb - N_2 \geq N \frac{b + d_i}{2}. \quad (2.6) \]

Putting \( b' := (b + d_i)/2 \) and

\[ K_t := \begin{cases} N b - (N - t) (d_{i-1} + \delta) & \text{when } N' \leq t \leq N - N_2; \\ N b' & \text{when } N - N_2 < t \leq N, \end{cases} \]

the estimates (2.5) and (2.6) imply that for \( N \geq N_3 \) we have

\[ P[1_{N' \leq t < \tau} \leq N] \leq P[\sigma_1 \leq \tau] \leq P[\exists N' \leq t < \tau \text{ such that } I_t = I_0 - K_t]. \quad (2.7) \]

Taking into account the time-evolution of \((I_t)_{0 \leq t < \tau}\), the right hand side of (2.7) is at most:

\[ P[\exists N' \leq t \leq N \text{ such that } \sum_{s=1}^{t} 1_{X_s \leq x_{i-1}} = K_t] \leq \sum_{t=N'}^{N} P\left[ \sum_{s=1}^{t} 1_{X_s \leq x_{i-1}} = K_t \right]. \quad (2.8) \]

Let us write \( t = xN \), and put \( g(x) = K_t/t \). Observe that we have \((b - d_{i-1} - \delta)/(1 - d_{i-1} - \delta) \leq x \leq 1\), and we have

\[ g(x) = \begin{cases} K_t \frac{1}{x} - \frac{1 - x}{x} (d_{i-1} + \delta) & \text{when } N' \leq t \leq N - N_2; \\ K_t \frac{b'}{x} & \text{when } N - N_2 < t \leq N. \end{cases} \]

Using the large deviation rate function for binomial random variables \[3\], we have

\[ \log P \left[ \sum_{s=1}^{t} 1_{X_s \leq x_{i-1}} = K_t \right] \leq N \left[ o(1) - x g(x) \log g(x) f_{i-1} - x (1 - g(x)) \log \frac{1 - g(x)}{1 - f_{i-1}} \right] \]

\[ =: N \left[ o(1) + x F(x) \right], \quad \text{as } N \to \infty. \quad (2.9) \]

Note that the \( o(1) \) term is uniform in \( x \), since \( t \geq N' \), and \( N' \) grows (linearly) with \( N \).
We first show that the expression $xF(x)$ inside square brackets in the right hand side of (2.9) is increasing when $N' \leq t \leq N - N_2$. We have

$$g'(x) = -\frac{b - d_{i-1} - \delta}{x^2} = -\frac{g(x)}{x} + \frac{d_{i-1} + \delta}{x}.$$ 

We also have

$$d \frac{d}{dx} [xF(x)] = F(x) + xF'(x)$$

$$= F(x) + x \left[ -g'(x) \log \frac{g(x)}{f_{i-1}} + g'(x) \log \frac{1 - g(x)}{1 - f_{i-1}} - g(x) \frac{g'(x)}{g(x)} - (1 - g(x)) \frac{-g'(x)}{1 - g(x)} \right]$$

$$= F(x) + (g(x) - d_{i-1} - \delta) \log \frac{g(x)}{f_{i-1}} + ((1 - g(x)) - (1 - d_{i-1} - \delta)) \log \frac{1 - g(x)}{1 - f_{i-1}}$$

$$= -(d_{i-1} + \delta) \log \frac{g(x)}{f_{i-1}} - (1 - d_{i-1} - \delta) \log \frac{1 - g(x)}{1 - f_{i-1}}. \quad (2.10)$$

Note that $b \leq g(x) \leq 1$. We show that with $c = d_{i-1} + \delta$ and $d = f_{i-1}$, the function

$$h(y) = -c \log \frac{y}{d} - (1 - c) \log \frac{1 - y}{1 - d}$$

is everywhere positive on the interval $y \in [b, 1]$. Indeed, we have $y \geq b > d_{i-1} = d > d_{i-1} + \delta = c$, and

$$h'(y) = -\frac{c}{y} + \frac{1 - c}{1 - y} = \frac{y(1 - c) - c(1 - y)}{y(1 - y)} = \frac{y - c}{y(1 - y)} > 0, \quad \text{for } y \geq d.$$

Therefore, $h(y) > h(d) = 0$. This gives that

$$\sup \left\{ xF(x) : N'/N \leq x \leq 1 - N_2/N \right\} \leq -b \log \frac{b}{f_{i-1}} - (1 - b) \log \frac{1 - b}{1 - f_{i-1}} = -I_i^- (b). \quad (2.11)$$

On the other hand, we have

$$\sup \left\{ xF(x) : 1 - N_2/N \leq x \leq 1 \right\} = -(1 + o(1)) I_i^- (b'), \quad \text{as } N \to \infty. \quad (2.12)$$

The estimates (2.11) and (2.12) ensure that the right hand side of (2.8) is at most

$$\exp(-(1 + o(1)) I_i^- (b') N), \quad \text{as } N \to \infty.$$

This completes the proof. \qed
In order to complement the bound provided by Proposition 2.1, we seek a lower bound on the probability that a value $\geq i + 1$ is assigned to $\ell_N(i) - 1$. To start, the following lemma provides a ‘continuity’ result for the optimal policy.

**Lemma 2.3.** For every $N \geq 1$ we have $\ell_N(i) \in \{\ell_{N-1}(i), \ell_{N-1}(i) - 1\}$.

**Proof.** It is easy to verify from the proof of Theorem 1.1 using induction on $N$, that for all $N \geq 2$ the numbers $\alpha_{N,r}$, $r = 0, \ldots, N$ are strictly increasing.

Fix $2 \leq i \leq k - 1$. Let $r \leq \ell_{N-1}(i) - 2$. Then due to the proof of Theorem 1.1, we have that $\alpha_{N,r}$ is a convex combination of the numbers $\alpha_{N-1,r-1} < \alpha_{N-1,r}$, both of which are at most $x_i$, and there is positive weight on the smaller value. It follows that $\alpha_{N,r} < x_i$, and hence $\ell_N(i) \geq \ell_{N-1}(i) - 1$.

By a similar argument, if $r \geq \ell_{N-1} + 1$, then $\alpha_{N,r}$ is a convex combination of the numbers $\alpha_{N-1,r-1} < \alpha_{N-1,r}$, both of which are at least $x_i$, and there is positive weight on the larger value. Therefore, $\alpha_{N,r} > x_i$ and $\ell_N(i) \leq \ell_{N-1}(i)$.

We will need to restrict $N$ to a set of ‘good’ values. We need the following lemma to define these.

**Lemma 2.4.** Let $2 \leq i \leq k - 1$, let $d_i < v < b < 1$, and assume that $\limsup_{N \to \infty} \ell_N(i)/N > b$. Then there are infinitely many values of $N$ such that the following hold:

(i) $\ell_N(i) \geq bn + 1$;
(ii) we have

\[
\ell_{N-t}(i) \leq \ell_N(i) - vt, \quad 0 \leq t \leq N - 1.
\]  

(2.13)

**Proof.** There are infinitely many numbers $N'$ such that $\ell_{N'}(i) \geq bn' + 1$. We first claim that with $c_1 = (b - v)/(1 - v)$, and all $N'$ sufficiently large, there exists $N$ with $c_1 N' \leq N \leq N'$ such that

\[
\ell_{N-t}(i) \leq \ell_N(i) - vt, \quad 0 \leq t \leq N - 1.
\]  

(2.14)

Should such $N$ not exist, there would be an integer $r \geq 1$ and a sequence of numbers $N' = n_0 > n_1 > \cdots > n_r \geq 1$ with $1 \leq n_r < c_1 N'$, such that

\[
\ell_{n_s}(i) > \ell_{n_{s-1}}(i) - v(n_{s-1} - n_s), \quad s = 1, \ldots, r.
\]

(Indeed, such a sequence can be chosen inductively, using the negation of (2.14) for $N = n_0, n_1, \ldots$ in turn.) This implies that

\[
n_r \geq \ell_{n_r}(i) = \ell_{N'}(i) + \sum_{s=1}^{r} (\ell_{n_s} - \ell_{n_{s-1}}) > \ell_{N'}(i) - v \sum_{s=1}^{r} (n_{s-1} - n_s) = \ell_{N'}(i) - v(N' - n_r)
\]

\[
> bn' - v(N' - n_r) = vn_r + (b - v)N'.
\]
Hence \( n_r \geq N'(b - v)/(1 - v) = c_1 N' \). This is a contradiction, and hence \( N \) exists with the claimed property (2.14).

Let now \( N \) be the largest integer \( \leq N' \) with the property (2.14). Then it also holds that
\[
\ell_{N+t}(i) \leq \ell_N(i) + vt, \quad 0 \leq t \leq N' - N. \tag{2.15}
\]
Indeed, (2.15) holds automatically for \( t = 0 \), and should it be violated for some \( 1 \leq t \leq N' - N \), then with the smallest such \( t \) the value \( N + t \) would also satisfy (2.14), contradicting the maximality of \( N \). Observe that due to (2.15) we have
\[
\ell_N(i) \geq \ell_{N'}(i) - v(N' - N) \geq bN' + 1 - v(N' - N) \geq bN' + 1 - b(N' - N) = bN + 1,
\]
and hence \( N \) satisfies both properties in the Lemma.

We will write \( G = G(b, v, i) \) for the set of values of \( N \geq 1 \) such that both (i)–(ii) in Lemma 2.4 hold.

**Lemma 2.5.** There exists \( c_2 > 0 \) such that for \( N \in G \) we have \( P[\sigma < \tau] \geq c_2 \).

**Proof.** Let \( N \in G \). Due to Lemma 2.3 and (2.13), there exists a deterministic \( t_0 = t_0(N) \) such that on the event \( \{X_1 = \cdots = X_{t_0} = x_k\} \), we have \( I_{t_0} = \ell_{N-t_0}(i) \). Let us choose the smallest such \( t_0 \). Then there exists a deterministic upper bound \( T_0 \), so that \( t_0(N) \leq T_0 \). Due to Lemma 2.3 and (2.13), there exists \( t_1 = t_1(N) > t_0 \) such that on the event \( \{X_1 = \cdots = X_{t_1} = x_k\} \), we have \( \sigma = t_1 < \tau \). Since there is also a deterministic upper bound \( t_1(N) \leq T_1 \), we have \( P[\sigma < \tau] \geq p_{T_1} \).

We introduce the martingale \( Y_t := E[U_N | F_t] \). Observe that \( Y_0 = E[U_N] \) and \( Y_t = U_N \) when \( t \geq \tau \).

**Proof of Proposition 2.2.** On the event \( \sigma < \tau \), consider the stopping time
\[
\sigma'' = \inf\{t > \sigma : I_t = \ell_{N-t}(i) + 1\}.
\]
On the event \( \sigma < \sigma'' < \tau \), we have
\[
E[U_N | F_{\sigma''}] = Y(\sigma'') \geq x_i.
\]
On the event \( \sigma < \tau \leq \sigma'' \), we have \( Y(\tau) \geq x_{i+1} \). We construct an event \( E^+ \subset \{\sigma < \tau \leq \sigma''\} \subset \{\sigma < \tau, U_N > x_i\} \) such that
\[
P[E^+] \geq c_1 \exp(-(1 + o(1)) I_i^+(b'') N). \tag{2.16}
\]

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This will prove the proposition, since using the martingale property we have
\[ E[U_N | \sigma < \tau] = \frac{1}{\mathbb{P}[\sigma < \tau]} (E[U_N 1_{\sigma < \sigma'' < \tau}] + E[U_N 1_{\sigma < \tau < \sigma''}]) \]
\[ = \frac{1}{\mathbb{P}[\sigma < \tau]} (E[Y(\sigma'')] 1_{\sigma < \sigma'' < \tau} + E[Y(\tau) 1_{\sigma < \tau < \sigma''}]) \]
\[ \geq \frac{1}{\mathbb{P}[\sigma < \tau]} (x_i \mathbb{P}[\sigma < \sigma'' < \tau] + x_{i+1} \mathbb{P}[\sigma < \tau < \sigma'']) \]
\[ \geq x_i + (x_{i+1} - x_i) \frac{\mathbb{P}[E^+] \mathbb{P}[\sigma < \tau]}{\mathbb{P}[\sigma < \tau]} \geq x_i + (x_{i+1} - x_i) \mathbb{P}[E^+]. \]

In order to define \( E^+ \) and prove (2.19), we fix a positive integer \( A \) whose value will be determined later. Due to (2.13), there exists an integer \( t_2 \) such that
\[ I_0 = \ell_N - vt_2 + A. \]
Let
\[ E_1^+ = \{ V_1 = \cdots = V_{t_1} = x_k \} \]
\[ E_2^+ = \{ V_{t_1+1} = \cdots = V_{t_2} = x_k \} \]
\[ E^+ = E_1^+ \cap E_2^+ \cap \{ \tau \leq \sigma'' \}. \]
Note that on \( E_1^+ \cap E_2^+ \) we have \( I_{t_2} = I_{t_1} = I_0 = \ell_N - vt_2 + A \). Therefore, due to (2.13) for \( t > t_2 \) we have
\[ \ell_{N-t} \leq \ell_N - vt = \ell_N - vt_2 + A - (t - t_2)v - A = I_{t_2} - v(t - t_2) - A. \]
This implies the inclusion of events:
\[ E^+ \supset E_1^+ \cap E_2^+ \cap \{ \forall t \geq t_2 \text{ we have } \ell_{N-t} + 1 < I_t \} \]
\[ \supset E_1^+ \cap E_2^+ \cap \{ \forall t \geq t_2 \text{ we have } I_{t_2} - v(t - t_2) - (A - 1) < I_t \} \]
\[ \supset E_1^+ \cap E_2^+ \cap \left\{ \forall t \geq t_2 \text{ we have } \sum_{t_2 < s \leq t} 1_{X_s \leq x_i} < A - 1 + v(t - t_2) \right\}. \]
Writing \( Y_s = 1_{X_s \leq x_i} \) and \( G(t) = \{ \sum_{t_2 < s \leq t} Y_s < A - 1 + v(t - t_2) \} \) this gives
\[ \mathbb{P}[E^+] \geq p^{t_2}_{k} \mathbb{P} \left[ \sum_{t_2 < s \leq t} Y_s < A - 1 + v(t - t_2) \text{ for all } t \geq t_2 \right] = p^{t_2}_{k} \mathbb{P} [\cap_{t_2 < t \leq N} G(t)]. \]
Let us fix a number \( b'' \) that satisfies \( d_i < b'' < v \), and put \( M = \{ b''(N - t_2) \} \subset G(N) \). Then due to the definition of \( I_i^+ \), for sufficiently large \( N \) we have
\[ \mathbb{P}[H] = \exp \left( -(1 + o(1)) I_i^+ (b'') (N - t_2) \right) \geq \exp \left( -(1 + o(1)) I_i^+ (b'') N \right). \]
We show that
\[
P[\cup_{t_2 < t < N} G(t)^C \mid H] \leq \sum_{t_2 < t < N} P[G(t)^C \mid H] \leq 1/2,
\] (2.17)
if \( A \) is large enough, which implies the required statement. Under the conditioning on \( H \), \( Y_{t_2 + 1}, \ldots, Y_N \) have the same law as an i.i.d. sequence \( Y_{t_2 + 1}', \ldots, Y_N' \) with
\[
P[Y'_i = 1] = \frac{M}{N - t_2} \quad P[Y'_i = 0] = 1 - \frac{M}{N - t_2}
\]
conditioned on \( \sum_{t_2 < s \leq N} Y'_s = M \). Write \( S'_t = \sum_{t_2 < s \leq t} Y'_s \). Then we need to give an upper bound on
\[
P[S'_t \geq A - 1 + v(t - t_2) \mid S'_N = M],
\]
We distinguish the cases \( N/2 \leq t < N \) and \( t_2 < t < N/2 \). Since the mean of \( Y'_s \) is less than \( v \), when \( N/2 \leq t < N \) we have
\[
P[S'_t \geq A - 1 + v(t - t_2) \mid S'_N = M] \leq \frac{P[S'_t \geq A - 1 + v(t - t_2)]}{P[S'_N = M]} \leq C\sqrt{N} C \exp(-cN),
\]
where we used the local limit theorem to lower bound the denominator. The right hand side sums to less than 1/4, if \( N \) is large enough.

When \( t_2 < t < N/2 \), we can write
\[
P[S'_t \geq A - 1 + v(t - t_2) \mid S'_N = M] = \frac{P[S'_t \geq A - 1 + v(t - t_2), S'_N = M]}{P[S'_N = M]}
\]
\[
= \frac{1}{P[S'_N = M]} \sum_{y \geq A - 1 + v(t - t_2)} P[S'_t = y, S'_N = M]
\]
\[
= \frac{1}{P[S'_N = M]} \sum_{y \geq A - 1 + v(t - t_2)} P[S'_t = y] P[S'_N = M \mid S'_t = y].
\]
For each fixed \( y \), the last conditional probability is a binomial probability with \( N - t \) trials, where the probability of a success is \( P[Y'_i = 1] = \frac{M}{N - t_2} \). This success probability is bounded away from 0 and 1, due to \( M = [b''(N - t_2)] \) and \( 0 < b'' < 1 \). Hence \( P[S'_N = M \mid S'_t = y] \) is bounded above by the maximum of the above binomial distribution, which is \( \leq C/\sqrt{N - t} \leq C/\sqrt{N}/2 \), uniformly in \( y \). This gives that
\[
P[S'_t \geq A - 1 + v(t - t_2) \mid S'_N = M] \leq \frac{1}{P[S'_N = M]} \sum_{y \geq A - 1 + v(t - t_2)} P[S'_t = y] \frac{C}{\sqrt{N}}
\]
\[
\leq P[S'_t \geq A - 1 + v(t - t_2)] \frac{C/\sqrt{N}}{P[S'_N = M]} \leq C P[S'_t \geq A - 1 + v(t - t_2)].
\]
Since again the mean of $Y'_s$ is less than $v$, the right hand side is summable in $t$. Moreover, by choosing $A$ large, we can make it sum to a value less than $1/4$. The two cases together yield the required (2.17), and establish (2.16).

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