WHAT ARE THE SYSTEMS THAT DECOHERE?

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Abstract. The fact that the Environment Induced Decoherence approach offers no general criterion to decide where to place the “cut” between system and environment has been considered as a serious conceptual problem of the proposal. In this letter we argue that this is actually a pseudo-problem, which is dissolved by the fact that decoherence is a phenomenon relative to the relevant observables selected by the measuring arrangement. We also show that, when the spin-bath model is studied from this perspective, certain unexpected results are obtained, as that of a system decohering in interaction with a very small environment.

Introduction. Environment Induced Decoherence (EID), which turns the coherent state of an open system into a decohered mixture, is the clue for the account of the emergence of classicality from quantum mechanics \cite{1,2}. Therefore, the split of the universe into the system $S$ and the environment $E$ is essential for EID. However, since the environment may be “external” or “internal”, the EID approach offers no general criterion to decide where to place the “cut” between system and environment. Zurek considers this fact as a problem for his proposal: “In particular, one issue which has been often taken for granted is looming big, as a foundation of the whole decoherence program. It is the question of what are the ‘systems’ which play such a crucial role in all the discussions of the emergent classicality.” \cite{3}. The aim of this letter is to argue that such a “looming big” problem is actually a pseudo-problem, which is dissolved by the fact that decoherence is a phenomenon relative to the relevant observables selected in each particular case. Precisely, if $O$ is the space of all the observables of a closed system, $O_R \subset O$ is the space of the relevant observables, that is, those that can be experimentally measured. Since decoherence depends on the space $O_R$ considered, and $O_R$ changes with the change of the measuring arrangement, decoherence turns out to be a phenomenon relative to that arrangement.

Let us stress that we use the word ‘relative’ strictly with the same meaning as in special relativity, where it has no subjective content: a reference frame is defined by a set of clocks and rules at rest in an inertial system, and this set is the measuring arrangement. Analogously, a quantum measuring arrangement is a set of devices having experimental access only to the

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observables $O_R \in O_R$; so, it is that arrangement what defines, relatively, the system and its environment. With a certain arrangement, the physicist may observe the decoherence of the system so defined and the emergence of classicality in that system. But a different arrangement defines a different system which may not decohere and, as a consequence, retains its quantum behavior.

We will develop our argument by analyzing the well-known spin-bath model from the general theoretical framework for decoherence presented in a previous work [4].

**The spin-bath model.** The spin-bath model is a very simple model that has been exactly solved in previous papers (see [5]). Let us consider a closed system $U = P + P_i$ where (i) $P$ is a spin-1/2 particle represented in the Hilbert space $\mathcal{H}_P$, and (ii) the $P_i$ are $N$ spin-1/2 particles, each one of which is represented in its own Hilbert space $\mathcal{H}_i$. The complete Hilbert space of the composite system $U$ is, $\mathcal{H} = \mathcal{H}_P \otimes \bigotimes_{i=1}^{N} \mathcal{H}_i$. In the particle $P$, the two eigenstates of the spin operator $S_{s,\vec{v}}$ in direction $\vec{v}$ are $|\uparrow\rangle$ and $|\downarrow\rangle$, such that $S_{s,\vec{v}} |\uparrow\rangle = \frac{1}{2} |\uparrow\rangle$ and $S_{s,\vec{v}} |\downarrow\rangle = -\frac{1}{2} |\downarrow\rangle$.

In each particle $P_i$, the two eigenstates of the corresponding spin operator $S_{i,\vec{v}}$ in direction $\vec{v}$ are $|\uparrow_i\rangle$ and $|\downarrow_i\rangle$, such that $S_{i,\vec{v}} |\uparrow_i\rangle = \frac{1}{2} |\uparrow_i\rangle$ and $S_{i,\vec{v}} |\downarrow_i\rangle = \frac{1}{2} |\downarrow_i\rangle$. Therefore, a pure initial state of $U$ reads

\[
|\psi_0\rangle = (a |\uparrow\rangle + b |\downarrow\rangle) \bigotimes_{i=1}^{N}(\alpha_i |\uparrow_i\rangle + \beta_i |\downarrow_i\rangle)
\]

where the coefficients $a$, $b$, $\alpha_i$, $\beta_i$ are such that satisfy $|a|^2 + |b|^2 = 1$ and $|\alpha_i|^2 + |\beta_i|^2 = 1$. Usually these numbers (and also the $g_i$ below) are taken as aleatory numbers. If $P$ interacts with each one of the $P_i$ but the $P_i$ do not interact with each other, the total Hamiltonian $H$ of the composite system $U$ results (see [5], [6])

\[
H = H_{SE} = S_{s,\vec{v}} \otimes \sum_{i=1}^{N} 2g_i S_{i,\vec{v}} \bigotimes_{j \neq i} I_j
\]

where $I_j$ is the identity operator on the subspace $\mathcal{H}_j$, $S_{s,\vec{v}} = \frac{1}{2} (|\uparrow\rangle \langle \uparrow| - |\downarrow\rangle \langle \downarrow|)$ and $S_{i,\vec{v}} = \frac{1}{2} (|\uparrow_i\rangle \langle \uparrow_i| - |\downarrow_i\rangle \langle \downarrow_i|)$. Under the action of $H$, the state $|\psi_0\rangle$ evolves as $|\psi(t)\rangle = a |\uparrow\rangle |E_{\uparrow}(t)\rangle + b |\downarrow\rangle |E_{\downarrow}(t)\rangle$ where $|E_{\uparrow}(t)\rangle = |E_{\uparrow}(-t)\rangle$ and

\[
|E_{\uparrow}(t)\rangle = \bigotimes_{i=1}^{N} (\alpha_i e^{ig_i t/2} |\uparrow_i\rangle + \beta_i e^{-ig_i t/2} |\downarrow_i\rangle)
\]
If $O$ is the space of observables of the whole system $U$, let us consider a space of relevant observables $O_R \subset O$ such that $O_R \in O_R$ reads

$$O_R = \left( \begin{array}{c}
  s_{\uparrow \uparrow} |\uparrow\rangle \langle \uparrow| \\
  + s_{\uparrow \downarrow} |\uparrow\rangle \langle \downarrow| \\
  + s_{\downarrow \downarrow} |\downarrow\rangle \langle \downarrow| \\
  + s_{\downarrow \uparrow} |\downarrow\rangle \langle \uparrow|
\end{array} \right) \otimes \prod_{i=1}^{N} \left( \begin{array}{c}
  \epsilon^{(i)}_{\uparrow \uparrow} |\uparrow\rangle \langle \uparrow| \\
  + \epsilon^{(i)}_{\uparrow \downarrow} |\uparrow\rangle \langle \downarrow| \\
  + \epsilon^{(i)}_{\downarrow \downarrow} |\downarrow\rangle \langle \downarrow| \\
  + \epsilon^{(i)}_{\downarrow \uparrow} |\downarrow\rangle \langle \uparrow|
\end{array} \right)$$

(4)

Since the operators $O_R$ are Hermitian, the diagonal components $s_{\uparrow \uparrow}$, $s_{\downarrow \downarrow}$, $\epsilon^{(i)}_{\uparrow \uparrow}$, $\epsilon^{(i)}_{\downarrow \downarrow}$ are real numbers and the off-diagonal components are complex numbers satisfying $s_{\uparrow \downarrow} = s^*_{\downarrow \uparrow}$, $\epsilon^{(i)}_{\uparrow \downarrow} = \epsilon^{(i)}_{\downarrow \uparrow}$. Then, the expectation value of the observable $O$ in the state $|\psi(t)\rangle$ can be computed as

$$\langle O_R \rangle_{\psi(t)} = (|a|^2 s_{\uparrow \uparrow} + |b|^2 s_{\downarrow \downarrow}) \Gamma_0(t)
$$

(5)

where (see eqs. (23) and (24) in [3])

$$\Gamma_0(t) = \prod_{i=1}^{N} \left[ |\alpha_i|^2 \epsilon^{(i)}_{\uparrow \uparrow} + |\alpha_i|^2 \beta_i \epsilon^{(i)}_{\downarrow \uparrow} e^{-i\gamma t} \\
  + |\beta_i|^2 \epsilon^{(i)}_{\uparrow \downarrow} + (\alpha_i^* \beta_i \epsilon^{(i)}_{\uparrow \downarrow})^* e^{i\gamma t} \right]$$

(6)

$$\Gamma_1(t) = \prod_{i=1}^{N} \left[ |\alpha_i|^2 \epsilon^{(i)}_{\downarrow \downarrow} e^{i\gamma t} + |\beta_i|^2 \epsilon^{(i)}_{\downarrow \uparrow} e^{-i\gamma t} \\
  + |\alpha_i|^2 \beta_i \epsilon^{(i)}_{\downarrow \uparrow} + (\alpha_i^* \beta_i \epsilon^{(i)}_{\downarrow \uparrow})^* \right]$$

(7)

As a generalization of the usual presentations, we will study two different ways of splitting the whole closed system $U$ into a relevant part and its environment, by considering different choices for the space $O_R$.

**Case 1: Observing the particle $P$.** In the typical situation studied by the EID approach, the system $S$ is simply the particle $P$, and the remaining particles $P_i$ are the environment. Therefore, the relevant observables $O_R \in O_R$ are those corresponding to $P$, and are obtained from eq. [4] by making $\epsilon^{(i)}_{\uparrow \downarrow} = \epsilon^{(i)}_{\downarrow \uparrow} = 1$, $\epsilon^{(i)}_{\downarrow \downarrow} = 0$:

$$O_R = \left( \sum_{s,s'=\uparrow,\downarrow} s_{ss'} |s\rangle \langle s'| \right) \otimes I_i = O_S \otimes \bigotimes_{i=1}^{N} I_i$$

(8)

The expectation value of these observables is given by

$$\langle O_R \rangle_{\psi(t)} = |a|^2 s_{\uparrow \uparrow} + |b|^2 s_{\downarrow \downarrow} + 2 \Re \{ab^* s_{\downarrow \uparrow} r_1(t)\}$$

(9)

where

$$r_1(t) = \prod_{i=1}^{N} \left[ |\alpha_i|^2 e^{i\gamma t} + |\beta_i|^2 e^{-i\gamma t} \right]$$

(10)
By comparing eq. (9) with eq. (5), we see that in this case \( \Gamma_0(t) = 1 \) and \( \Gamma_1(t) = r_1(t) \). Moreover,

\[
|r_1(t)|^2 = \prod_{i=1}^{N}(|\alpha_i|^4 + |\beta_i|^4 + 2|\alpha_i|^2|\beta_i|^2 \cos 2g_i t)
\]

Since \( |\alpha_i|^2 + |\beta_i|^2 = 1 \), then

\[
\max_t (|\alpha_i|^4 + |\beta_i|^4 + 2|\alpha_i|^2|\beta_i|^2 \cos 2g_i t) = \left( (|\alpha_i|^2 + |\beta_i|^2)^2 \right) = 1
\]

and

\[
\min_t (|\alpha_i|^4 + |\beta_i|^4 + 2|\alpha_i|^2|\beta_i|^2 \cos (2g_i t)) = \left( (|\alpha_i|^2 - |\beta_i|^2)^2 \right) = \left( 2|\alpha_i|^2 - 1 \right)^2
\]

If the coefficients \( g_i, \alpha_i \) and \( \beta_i \) are aleatory numbers, then \( (|\alpha_i|^4 + |\beta_i|^4 + 2|\alpha_i|^2|\beta_i|^2 \cos 2g_i t) \) is an aleatory number which, if \( t \neq 0 \), fluctuates between 1 and \( \left( 2|\alpha_i|^2 - 1 \right)^2 \). Let us note that, since the \( |\alpha_i|^2 \) and the \( |\beta_i|^2 \) are aleatory numbers in the closed interval \( [0,1] \), when the environment has many particles (that is, when \( N \to \infty \)), the statistical value of the cases \( |\alpha_i|^2 = 1, |\beta_i|^2 = 1, |\alpha_i|^2 = 0 \) and \( |\beta_i|^2 = 0 \) is zero. In this case, eq. (11) for \( |r_1(t)|^2 \) is an infinite product of numbers belonging to the open interval \( (0,1) \). As a consequence (see [1], [2]),

\[
\lim_{N \to \infty} r_1(t) = 0
\]

In order to know the time-behavior of the expectation value of eq. (12), we have to compute the time-behavior of \( r_1(t) \). If we know that \( r_1(0) = 1 \) for \( N \to \infty \), and that \( \lim_{N \to \infty} r_1(t) = 0 \) for any \( t \neq 0 \), it can be expected that, for \( N \) finite, \( r_1(t) \) will evolve in time from \( r_1(0) = 1 \) to a very small value. Moreover, \( r_1(t) \) is a periodic function because it is a product of periodic functions with periods depending on the coefficients \( g_i \). Nevertheless, since the \( g_i \) are aleatory, the periods of the individual functions are different and, as a consequence, the recurrence time of \( r_1(t) \) will be very large, and strongly increasing with the number \( N \) of particles.

The time-behavior of \( r_1(t) \) was computed by means of a numerical simulation, where the aleatory numbers \( |\alpha_i|^2, |\beta_i|^2 \) and \( g_i \) were obtained from a generator of aleatory numbers: these generator fixed the value of \( |\alpha_i|^2 \), and the \( |\beta_i|^2 \) were computed as \( |\beta_i|^2 = 1 - |\alpha_i|^2 \). The function
Figure 1. Decoherence for $S = P$ with $N = 200$.

$r_1(t)$ for $N = 200$ is plotted in Figure 1 (see also numerical simulations in [6]), which shows that the system $P$ decoheres in interaction with an environment of $N$ particles $P_i$.

Case 2: Observing the particles $P_i$. Although in the usual presentations of the model the system of interest is $P$, as in the previous section, we can conceive different ways of splitting the whole $U$ into an open system and an environment. For instance, it may be the case that the measuring arrangement “observes” a subset of the particles of the environment, e.g., the $p$ first particles $P_j$. In this case, the system of interest is composed by $p$ particles, $S = \sum_{i=1}^{p} P_i$, and the environment is composed by all the remaining particles, $E = P + \sum_{i=p+1}^{N} P_i$. So, in eq. (4), $s_{\uparrow\uparrow} = s_{\downarrow\downarrow} = 1$, $s_{\uparrow\downarrow} = s_{\downarrow\uparrow} = 0$, the coefficients $\epsilon_{j\uparrow\uparrow}$, $\epsilon_{j\downarrow\downarrow}$, $\epsilon_{j\downarrow\uparrow}$ are generic for $j \in \{1...p\}$, and $\epsilon_{i\uparrow\uparrow} = \epsilon_{i\downarrow\downarrow} = 1$, $\epsilon_{i\downarrow\uparrow} = \epsilon_{i\uparrow\downarrow} = 0$ for $i \in \{p+1...N\}$. Then, the relevant observables $O_R \in \mathcal{O}_R \subset \mathcal{O}$ read

\begin{equation}
O_R = I_S \otimes \left( \bigotimes_{j=1}^{p} O_{S_j} \right) \otimes \left( \bigotimes_{i=p+1}^{N} I_i \right)
\end{equation}

where $O_{S_j}$ is given by
\[ O_{S_j} = \epsilon^{(j)}_\uparrow \uparrow | \uparrow_j \rangle \langle \uparrow_j | + \epsilon^{(j)}_\downarrow \downarrow | \downarrow_j \rangle \langle \downarrow_j | + \epsilon^{(j)}_\uparrow \downarrow | \uparrow_j \rangle \langle \downarrow_j | + \epsilon^{(j)}_\downarrow \uparrow | \downarrow_j \rangle \langle \uparrow_j | \]

(16)

Therefore, the expectation value of the relevant observables \( O_R \) is

\[ \langle O_R \rangle_{\psi(t)} = \prod_{i=1}^{p} \left[ |\alpha_i|^2 \epsilon^{(i)}_\uparrow \uparrow + \alpha_i^* \beta_j \epsilon^{(i)}_\downarrow \downarrow e^{-i g_i t} \right. \]

\[ \quad \quad + \left. |\beta_i|^2 \epsilon^{(i)}_\downarrow \downarrow + (\alpha_i^* \beta_j \epsilon^{(i)}_\downarrow \downarrow)^* e^{i g_i t} \right] \]

(17)

If \( p = 1 \), the expectation value of eq. (17) results

\[ \langle O_{R_j} \rangle_{\psi(t)} = |\alpha_j|^2 \epsilon^{(j)}_\uparrow \uparrow + |\beta_j|^2 \epsilon^{(j)}_\downarrow \downarrow + \text{Re} \left( \alpha_j^* \beta_j \epsilon^{(j)}_\downarrow \downarrow e^{i g_j t} \right) \]

(18)

The evolution of \( \langle O_{R_j} \rangle_{\psi(t)} \) depends on the time-behavior of the third term of eq. (18), which can rewritten as

\[ r_2(t) = \text{Re} \left( \alpha_j^* \beta_j \epsilon^{(j)}_\downarrow \downarrow e^{i g_j t} \right) \]

(19)

In this case, numerical simulations are not required to see that \( r_2(t) \) is an oscillating function which, as a consequence, has no limit for \( t \rightarrow \infty \). This means that a single particle \( S = P_j \) with a large environment \( E = P + \sum_{i \neq j} P_i \) of \( N \) particles does not decohere. Nevertheless, this result can be understood by considering that \( P_j \) strongly interacts only with particle \( P \), but does not interact with the rest of the particles \( P_{i \neq j} \); therefore, the interaction of \( S = P_j \) with its environment \( E = P + \sum_{i \neq j} P_i \) is not strong enough to produce decoherence.

In order to obtain the expectation value \( \langle O_{R_j} \rangle_{\psi(t)} \) for \( p > 1 \), we will simplify the computation by considering the particular case for which the relevant observables are

\[ O_R = I_S \otimes \left( \bigotimes_{j=1}^{p} S_x^{(j)} \right) \otimes \left( \bigotimes_{i=p+1}^{N} I_i \right) \]

(20)

where \( S_x^{(j)} \) is the projection of the spin onto the \( x \)-axis of the particle \( P_j \). Then, \( \epsilon^{(j)}_\uparrow \uparrow = \epsilon^{(j)}_\downarrow \downarrow = 0 \), and the expectation value reads

\[ \langle O_R \rangle_{\psi(t)} = r_3(t) = \prod_{i=1}^{p} \left[ 2 \text{Re} \left( \alpha_i^* \beta_j \epsilon^{(j)}_\downarrow \downarrow e^{-i g_j t} \right) \right] \]

(21)

The time-behavior of \( r_3(t) \), with \( p = 4 \), is plotted in Figure 2, where we can see a fast decaying followed by fluctuations around zero. As expected, such fluctuations strongly damp
off with the increase of the number $p$ of particles, as shown in Figure 3 ($p = 8$) and Figure 4 ($p = 10$); with $p = 200$ the plot turns out to be indistinguishable of that obtained for the decoherence of Case 1 with $N = 200$.

The surprising consequence of these results is that the time-behavior is independent of the number $N$ of the particles $P_i$, but only depends on the number $p$ of the particles that constitute the system of interest (see eq. (17)). Therefore, we can consider a limit case of $N = p = 10$, where the system $S$ is composed by the $p = N = 10$ particles and the environment $E$ is a single particle, $E = P$: in this case, as shown in Figure 4, we have to say that a system of 10 particles decoheres as the result of its interaction with a single-particle environment. The situation becomes even more striking as the number $p$ increases: with $N = p = 200$, the system of 200 particles strongly decoheres in interaction with a single-particle environment.

**Conclusions.** The need of selecting a set of relevant observables, in terms of which the time-evolution of the system is described, is explicitly or implicitly admitted by the different approaches to the emergence of classicality: gross observables in van Kampen [7], macroscopic observables of the apparatus in Daneri et al. [8], collective observables in Omnès [9], [10]. It is quite clear that a closed system can be “partitioned” into many different ways and, thus,
there is not a single set of relevant observables essentially privileged (see [11], [12]). Each partition depends on the experimental viewpoint adopted, and represents a decision about which degrees of freedom are to be “observed” and which are disregarded in each case. Since there is no privileged or essential partition, there is no need of an unequivocal criterion to decide where to place the cut between “the” system and “the” environment: the “looming big” problem of defining the systems that decohere vanishes when the relativity of decoherence is recognized.

This conclusion is a natural consequence of the fact that the dynamical postulate of quantum mechanics refers to closed systems: the time-behavior of the parts resulting from different partitions of the closed system has to be inferred from that postulate. Since the total Hamiltonian rules the dynamical evolution of the closed system, then the time-behavior of its parts depends on the form in which the Hamiltonian is decomposed in each particular partition. This means that the occurrence of decoherence cannot be simply inferred from the interaction between a small open system and a large environment: the decomposition of the total Hamiltonian has to be studied in detail in each case, in order to know whether the system of interest resulting from the partition decoheres or not under the action of its self-Hamiltonian and the interaction Hamiltonian. As we have seen, when the phenomenon of decoherence is studied from this
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In this perspective, certain unexpected results are obtained, as the case of a system decohering in interaction with a very small environment. Such a result disagrees with the standard reading of the phenomenon, according to which the dissipation of information and energy from the system to a very large environment is what causes the destruction of the coherence between the states of the system.

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