Single-Peakedness and Total Unimodularity:
Efficiently Solve Voting Problems Without Even Trying

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Abstract
Many NP-hard winner determination problems admit polynomial-time algorithms when restricting inputs to be single-peaked. Commonly, such algorithms employ dynamic programming along the underlying axis. We introduce a new technique: carefully chosen integer linear programming (IP) formulations for certain voting problems admit an LP relaxation which is totally unimodular if preferences are single-peaked, and which thus admits an integral optimal solution. This technique gives fast algorithms for finding optimal committees under the PAV and Chamberlin–Courant voting rules under single-peaked preferences, as well as for certain OWA-based rules. Under single-crossing preferences, Young scores can also be calculated. An advantage of this technique is that no special-purpose algorithm needs to be used to exploit structure in the input preferences: any standard IP solver will terminate in the first iteration if the input is single-peaked, and will continue to work otherwise.

1 Introduction

In a departure from classical voting theory, a growing literature from computational social choice has recently studied multi-winner voting rules: Given diverse preferences of a collection of agents, instead of identifying a single best alternative, we are aiming for a (fixed-size) set of alternatives that jointly are able to represent the preferences of the agents best. Such procedures are useful in a wide variety of circumstances: obvious examples include the election of a parliament, or of a committee representing the interests of members of an organisation. Other applications can be found in group recommendation systems, or for making decisions about which products or services to offer: Which courses should be offered at a university? Which movies should be presented on an airline entertainment system?

Several attractive rules for such tasks have been designed by researchers in political science (e.g., Chamberlin and Courant 1983, Monroe 1995) and more recently by computer scientists (Faliszewski et al. 2016a; Faliszewski et al. 2016b; Skowron, Faliszewski, and Lang 2015). Many of these rules are defined in terms of some objective function: a winning committee is a set of \( k \) candidates that maximises this objective. Unsurprisingly, then, the winner determination problems of such rules are typically NP-hard (Lu and Boutilier 2011). To tackle the complexity of these problems, approximation algorithms (Skowron, Faliszewski, and Slinko 2015) and fixed-parameter tractability approaches (Betzler, Slinko, and Uhlmann 2013; Bredereck et al. 2015) have been developed for these problems, and integer programming formulations have also been designed for them (Potthoff and Brams 1998).

Another approach to efficiently solving these winner determination problems seeks to exploit underlying structure in the preferences reported by the agents (Elkind, Lackner, and Peters 2016). A particularly popular preference restriction in this space is the notion of single-peaked preferences, due to Black (1948) and Arrow (1950). Under this model, the alternative space has a one-dimensional structure: alternatives are ordered on a left-to-right axis; and agents’ preferences are monotonically decreasing as we move further away from their peak (most-preferred alternative). In particular, we can expect preferences to be structured this way when voting over the value of a numerical quantity (such as a tax rate). While single-peaked preferences were first employed to escape impossibility results in social choice theory (Moulin 1991), it also yields positive algorithmic results: Notably, Betzler, Slinko, and Uhlmann (2013) showed that Chamberlin–Courant’s (1983) committee selection rule can be computed efficiently when preferences are single-peaked. These results can be extended to other multi-winner voting rules (Elkind and Ismaili 2015), and to other preference restriction such as single-crossing prefer-
ences, or single-peakedness on trees (Skowron et al. 2013; Elkind, Faliszewski, and Skowron 2014; Peters and Elkind 2016).

The algorithms mentioned all directly exploit the underlying structure of the preferences. For example, the algorithm due to Betzler et al. (2013) proceeds by dynamic programming along the single-peaked axis. Thus, to use this algorithm, we first need a procedure to uncover this axis. Fortunately, efficient recognition algorithms for the mentioned domains are known; in particular, a suitable axis can be found in linear time (Escoffier, Lang, and Özól 2008). Thus, the approach to tractability via preference restrictions would look roughly like shown in Figure 1: Given input preferences, apply recognition algorithms for several preference restrictions, and if any of these tests succeed, use a special-purpose algorithm that performs well when preferences have the uncovered structure. If none of the tests succeed, fall back to some (superpolynomial) general-purpose solver, such as an integer programming solver.

But there is something awkward about linking together this cacophony of algorithms: given the amazing progress in solver technology in recent decades (such as witnessed in SAT competitions or in commercial IP solvers), it might well be faster in practice to skip these intermediate steps and go straight to CPLEX. (This objection becomes stronger once implementation effort is factored in.) Further, modern IP solvers are often able to exploit underlying structure automatically: experimentally, solving times on single-peaked instances are much faster than on random instances.

While such an experimental result is nice, performance guarantees are better. Could it be that, on certain single-peaked instances, an IP solver performs exponentially worse than special-purpose algorithms? The answer is no. This paper shows that integer programming solvers will provably terminate in polynomial time when solving certain voting problems on single-peaked inputs. In more detail, for several voting rules (including Chamberlin–Courant and Proportional Approval Voting), we will design IP formulations which are correct: experimentally, solving times on single-peaked inputs happen to be single-peaked. Since all standard IP solvers first solve the LP relaxation when the preference input happens to be single-peaked. Since all standard IP solvers first solve the LP relaxation, they will terminate with the correct answer in their first iteration. If the instance is not single-peaked, the IP solver might enter further iterations while solving – importantly, our formulations are correct whether or not the input is single-peaked. Moreover, this approach to achieving polynomial time efficiency does not require separately running a recognition algorithm! The IP solver need not know an underlying axis, or even whether the input is single-peaked at all; the LP relaxation will just ‘magically’ have an integral solution. Our proofs rely on establishing that the constraint matrices become totally unimodular in the single-peaked case. Previous applications of this technique include certain tractable cases of the winner determination problems of combinatorial auctions (see Müller 2006).

Our method also allows us to show that a conjecture due to Elkind and Lackner (2015) is false. They consider structure in dichotomous preferences based on approval ballots, where agents only submit a binary yes/no decision for every candidate. In particular, they consider an analogue of single-peakedness in this setting (which they call candidate interval or CI), which requires there to be an ordering (axis) of the candidates such that, for every voter, their set of approved candidates forms an interval of the axis. They then analyse an axiomatically particularly attractive multi-winner rule for the approval setting, known as Proportional Approval Voting (PAV). While Elkind and Lackner (2015) showed that PAV is efficiently computable for a certain subclass of their CI concept, and obtained FPT results for CI preferences, they conjectured that PAV remains NP-hard for CI preferences in general. This conjecture appears to be largely based on the difficulty of solving this problem using dynamic programming approaches that typically work in single-peaked settings. Our method, on the other hand, allows us to find a polynomial-time algorithm for this case, via IP solving.

We then combine the approaches for Chamberlin–Courant and PAV to give a similar polynomial time result for so-called OWA-based multi-winner rules in the case that preferences are single-peaked. We further give an IP formulation that computes Young’s voting rule efficiently for single-crossing preferences, and close by briefly sketching applications of our method to some further rules and settings.

2 Preliminaries

Total Unimodularity A matrix \( A = (a_{ij})_{ij} \in \mathbb{Z}^{m \times n} \) with \( a_{ij} \in \{-1, 0, 1\} \) is called totally unimodular if every square submatrix \( B \) of \( A \) has \( \det B \in \{-1, 0, 1\} \). The following results are well-known. Proofs and much more about their theory can be found in the textbook by Schrijver (1998).

**Theorem 1.** Suppose \( A \in \mathbb{Z}^{m \times n} \) is a totally unimodular matrix, \( b \in \mathbb{Z}^n \) is an integral vector of right-hand sides, and \( c \in \mathbb{Q}^n \) is an objective vector. Then the linear program

\[
\text{max } c^T x \text{ subject to } Ax \leq b
\]

has an integral optimum solution, which is a vertex of the polyhedron \( \{ x : Ax \leq b \} \). Thus, the integer linear program

\[
\text{max } c^T x \text{ subject to } Ax \leq b, x \in \mathbb{Z}^n
\]

IP can be solved using its linear programming relaxation (P).

An optimum solution to (IP) can be found in polynomial time. We will now state some elementary results about totally unimodular matrices.

**Proposition 2.** If \( A \) is totally unimodular, then so is

(1) its transpose \( A^T \),

(2) the matrix \( [A \mid -A] \) obtained from \( A \) by appending the negated columns of \( A \),

(3) the matrix \( [A \mid I] \) where \( I \) is the identity matrix,

(4) any matrix obtained from \( A \) through permuting or deleting rows or columns.

In particular, from (3) and (4) it follows that appending a unit column \((0, \ldots, 1, \ldots, 0)^T\) will not destroy total unimodularity. Further, using these transformations, we can see that Theorem 1 remains true even if we add to (P) constraints giving lower and upper bounds to some variables, if we replace some of the inequality constraints by equality constraints, or change the direction of an inequality.
A binary matrix $A = (a_{ij})_{i,j} \in \{0,1\}^{m \times n}$ has the strong consecutive ones property if the 1-entries of each row form a contiguous block, as in the example on the right. A binary matrix has the consecutive ones property if its columns can be permuted so that the resulting matrix has the strong consecutive ones property.

**Proposition 3.** Every binary matrix with the consecutive ones property is totally unimodular.

By a celebrated result of Seymour (1980), it is possible to decide in polynomial time whether a given matrix is totally unimodular.

**Single-Peaked Preferences.** Let $A$ be a finite set of alternatives, or candidates, and let $m = |A|$. A weak order, or preference relation, is a binary relation $\succ$ over $A$ that is complete and transitive. We write $\succ$ and $\succeq$ for the strict and indifference parts of $\succ$. A linear order is a weak order that, in addition, is antisymmetric, so that $x \sim y$ only if $x = y$. Every preference relation $\succ$ induces a partition of $A$ into indifference classes $A_1, \ldots, A_k$ so that $A_1 \succ A_2 \succ \cdots \succ A_k$.

The profile will be called single-peaked with respect to $\prec$ if for all $x, y \in A_k$. We will say that an alternative $a \in A_i$ has rank $t$ in the ordering $\succ$ and write $\text{rank}(a) = t$; thus the alternatives of rank 1 are the most-preferred alternatives under $\succ$. Finally, we say that any set of the form $\{x \in A : \text{rank}(x) \geq t\}$ is a top-initial segment of $\succ$.

Let $\preceq$ be the (strict part of) a linear order over $A$; we call $\preceq$ an axis. A linear order $\preceq_i$ with most-preferred alternative $p$ (the peak) is single-peaked with respect to $\preceq$ if for every pair of candidates $a, b \in A$ with $p \preceq b \preceq a$ or $a \preceq b \preceq p$, it holds that $b \succ_i a$. For example, if the alternatives in $A$ correspond to different proposed levels of a tax, and the numbers in $A$ are ordered by $\preceq$ in increasing order, then it is sensible to expect voters’ preferences over $A$ to be single-peaked with respect to $\preceq$. Note that $\succ$ is single-peaked with respect to $\preceq$ if and only if all top-initial segments of $\succ$ form an interval of $\preceq$. Accordingly, we will define a weak order $\preceq$ to be single-peaked with respect to $\preceq$ exactly if all top-initial segments of $\succ$ form an interval of $\preceq$. This concept is often known as ‘possibly single-peaked’ (Lackner 2014) because it is equivalent to asking that all the ties in the weak order can be broken in such a way that the resulting linear order is single-peaked.

A profile $P = (\succ_1, \ldots, \succ_n)$ over a set of alternatives $A$ is a list of weak orders over $A$. Each of the orders represents the preferences of a voter; we write $N = [n]$ for the set of voters. The profile will be called single-peaked if there exists some axis $\prec$ over $A$ so that each order $\succ_i$ in $P$ is single-peaked with respect to $\prec$. A profile is single-peaked if and only if the following matrix $M^P_A$ has the consecutive ones property: take one column for each alternative, and one row for each top-initial segment $S$ of each voter’s preference relation; the row is just the incidence vector of $S$. This construction is due to Bartholdi III and Trick (1986).

**Dichotomous Preferences.** A weak order $\succeq$ is dichotomous if it partitions $A$ into at most two indifference classes $A_1 \succ A_2$. The alternatives in $A_1$ are said to be approved by the voter $\succ$. On dichotomous preferences, the notion of single-peakedness essentially coincides with the consecutive ones property (Faliszewski et al. 2011): there needs to be an ordering $\prec$ of the alternative so that each approval set $A_1$ is an interval of $\preceq$. Thus, Elkind and Lackner (2015) use the name Candidate Interval (CI) for single-peakedness in this context.

**Single-Crossing Preferences.** A profile $P = (\succ_1, \ldots, \succ_n)$ of linear orders is called single-crossing if voters can be ordered so that for all $a, b \in A$, the set of voters who prefer $a$ to $b$ form an interval of this ordering. Again, this can be phrased in terms of the consecutive ones property of a matrix $M^P_A$ built from the profile: take one column for each voter, and one row for each of the $m^2$ pairs $a, b \in A$; there is a 1 in this row for each voter with $a \succ b$.

### 3 Proportional Approval Voting

In this section, we will consider Proportional Approval Voting (PAV), a multiwinner voting rule defined for dichotomous (approval) preferences. A naïve way to form a committee would be to select the $k$ alternatives with highest approval score, but this method tends to ignore minority candidates, and so is not representative (Aziz et al. 2015a). PAV attempts to fix this issue. The rule appears to have been first proposed by (Thiele 1895). In the general case, a winning committee under PAV is hard to compute (Aziz et al. 2015b).

Let us define PAV formally. Each voter $i$ submits a set $v_i \subseteq C$ of approved candidates (or, equivalently, a dichotomous weak order with $v_i \succ C \setminus v_i$). We aim to find a good committee $W \subseteq C$ of size $|W| = k$. The intuition behind Proportional Approval Voting (PAV) is that voters are happier with committees that contain more of their approved candidates, but that there are decreasing marginal returns to extra approved candidates in the committee. Concretely, each voter obtains a ‘utility’ of 1 for the first approved candidate in $W$, of $\frac{1}{2}$ for the second, of $\frac{1}{3}$ for the third, and so on. The objective value of a committee $W \subseteq C$ is thus

$$\sum_{i \in N} 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{|W \cap v_i|}.$$  

The choice of harmonic numbers might seem unusual, and one can more generally define a rule $\alpha$-PAV where $\alpha \in \mathbb{R}^k_+$ is a non-increasing scoring vector (so $\alpha_i \geq \alpha_j$ when $i \geq j$). This rule gives $W$ the objective value $\sum_{i \in N} \sum_{t=1}^{|W \cap v_i|} \alpha_t$. Then PAV is just $(1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{k})$-PAV. However, the choice of harmonic numbers is the only vector $\alpha$ that lets $\alpha$-PAV satisfy an axiom called ‘extended justified representation’ (Aziz et al. 2015a), making this a natural choice after all.
Proposition 4. Program (PAV-IP) correctly computes an optimal committee according to α-PAV.

Proof. In any feasible solution of (PAV-IP), the variables \( y_c \) encode a committee of size \( k \). Fix such a committee. Fix such a committee \( W = \{ c \in C : y_c = 1 \} \). We show that in optimum, the objective value of the solution is the α-PAV-value of this committee.

Since \( \alpha \geq 0 \), in optimum, as many \( x_{i,\ell} \) will be set to 1 as constraint (3) allows. Thus, for each \( i \), exactly \( |W \cap u_i| \) many variables \( x_{i,\ell} \) will be set to 1. Since \( \alpha \) is non-increasing wlog, in optimum, these will be variables \( x_{i,1}, \ldots, x_{i,|W \cap u_i|} \). Then the objective value equals the α-PAV-value of \( W \). □

An interesting feature of (PAV-IP) is that the integrality constraints (4) on the variables \( x_{i,\ell} \) can be relaxed to just \( 0 \leq x_{i,\ell} \leq 1 \); this does not change the objective value of the optimum solution. This is because (a) in optimum, the quantity \( \sum_{\ell} x_{i,\ell} \) is integral and (b) it never pays to then have one of the \( x_{i,\ell} \) to be fractional, because this fractional amount can be shifted to \( x_{i,\ell'} \) with (weakly) higher value. This observation might tell us that solving (PAV-IP) is relatively easy as it is “close” to being an LP, and this also seems to be true in practice. On the other hand, it is not necessarily beneficial to relax the integrality constraints (4) when passing (PAV-IP) to an IP solver: the presence of integrality constraints might nudge the solver to keep numerical integrality gaps smaller.

Of course, the point of this paper is to give another reason why (PAV-IP) is “close” to being an LP.

Proposition 5. The constraint matrix of (PAV-IP) is totally unimodular when the input preferences are single-peaked.

Proof. We will use the manipulations allowed by Proposition 2 liberally. In particular, it allows us to ignore the constraints \( 0 \leq x_{i,\ell}, y_c \leq 1 \), and to ignore the difference between equality and inequality constraints. Thus, after permuting columns corresponding to variables \( x_{i,r} \) so that they are sorted by \( i \), the constraint matrix of (PAV-IP) is

\[
A_{\text{PAV-IP}} = \begin{bmatrix}
-I_n & \cdots & -I_n & M_{\text{SP}}^P \\
0_n & \cdots & 0_n & M_{\text{SP}}^P \\
1_m & \cdots & 1_m & \cdots & 1_m
\end{bmatrix}.
\]

If preferences \( P \) are single-peaked, then \( M_{\text{SP}}^P \) has the consecutive ones property, and this is also true after appending a row with all-1s. Thus, \( M_{\text{SP}}^P \) is totally unimodular. Applying Proposition 2 repeatedly to append negations of unit columns, we obtain \( A_{\text{PAV-IP}} \), which is thus totally unimodular. □

Using Theorem 1, we obtain our desired result.

Theorem 6. α-PAV can be computed in polynomial time for single-peaked approval preferences.

4 Chamberlin–Courant’s Rule

Now we leave the domain of dichotomous preferences, and consider the full generality of profiles of weak orders. The definition of Chamberlin–Courant’s rule (1983) is based on the notion of having a representative in the elected committee: each voter is represented by their favourite candidate in the committee, and voters are happier with more preferred representatives. Let \( w \in \mathbb{N}^m \) be a (non-increasing) scoring vector; the standard choice for \( w \) are Borda scores: \( w = (m, m - 1, \ldots, 2, 1) \). Let \( P = \{\succ_1, \ldots, \succ_n\} \) be a profile. Then the objective value of a committee \( W \subseteq C \) according to Chamberlin–Courant’s rule is

\[
\sum_{i \in N} \max \{ w_{\text{rank}(c)} : c \in W \}.
\]

Chamberlin–Courant now returns any committee \( W \subseteq C \) with \( |W| = k \) that maximises this objective.

The rule thus defined can be seen as a (non-metric) facility location problem: each candidate \( c \in C \) is a potential facility location, we are allowed to open exactly \( k \) facilities, and the distance between customers and facilities are determined through \( w \). There is a standard integer programming formulation for this problem using binary variables \( y_c \), denoting whether \( c \) will be opened or not, and variables \( x_{i,c} \), denoting whether facility \( c \) will service voter \( i \).

We will need an alternative formulation based on maximising a number of points. For expositional simplicity, let’s take \( w \) to be Borda scores; other scoring rules can be obtained by weighting the points. Here is another way of thinking about the objective value as defined above: each voter \( i \) can earn a point for each rank \( r \) in \( i \)'s preference order: for every rank \( r \in [m] \), \( i \) earns the point \( x_{i,r} \) if there is a committee member \( c \in W \) with rank \( (c) \geq r \). Then the number of points obtained in total equals the objective values: if \( i \)'s favourite committee member is in rank \( r \), then \( i \) will earn precisely \( w_r = m - r + 1 \) points, namely the points \( x_{i,r}, x_{i,r+1}, \ldots, x_{i,m} \). This view suggests the following integer programming formulation, where we put \( w_r' = w_r - w_{r-1} \) and \( w_0' = w_1' \).

\[
\text{maximise} \sum_{i \in N} \sum_{r \in [m]} w_r' \cdot x_{i,r} \quad \text{(CC-IP)}
\]

subject to

\[
\sum_{c \in C} y_c = k \quad \text{(2)}
\]

\[
x_{i,r} \leq \sum_{c : \text{rank}(c) \geq r} y_c \quad \text{for } i \in N, r \in [m] \quad \text{(3)}
\]

\[
x_{i,r} \in \{0, 1\} \quad \text{for } i \in N, r \in [m]
\]

\[
y_c \in \{0, 1\} \quad \text{for } c \in C
\]
**Proposition 7.** Program (CC-IP) correctly computes an optimal committee according to w-Chamberlin–Courant.

**Proof.** In any feasible solution of (CC-IP), the variables $y_c$ encode a committee of size $k$. Fix such a committee $W = \{ c \in C : y_c = 1 \}$. We show that in optimum, the objective value of the solution is the objective value of this committee according to w-CC.

Since $w^T \geq 0$, in optimum, every $x_{i,r}$ will be set to 1 if constraint (3) allows this. This is the case iff there is a committee member $c \in W$ with rank$_k(c) \geq r$, i.e., iff the ‘point’ $x_{i,r}$ is earned as described above. Then it is clear that the objective of (CC-IP) corresponds to Chamberlin–Courant’s objective.

**Proposition 8.** The constraint matrix of (CC-IP) is totally unimodular when the input preferences are single-peaked.

**Proof.** After similar simplification as in Proposition 5 we see that the constraint matrix of (CC-IP) is

$$A_{\text{CC-IP}} = \begin{bmatrix} -I_{nm} & M_{SP}^P \\ 0 & 1_m \end{bmatrix}.$$

Again, if preferences $P$ are single-peaked, then $M_{SP}^P$ has the consecutive ones property, and this is also true after appending a row with all-1s. Thus, $M_{SP}^P$ is totally unimodular. Applying Proposition 2 repeatedly to append unit columns, we obtain $A_{\text{CC-IP}}$, which is thus totally unimodular.

**Theorem 9.** Chamberlin–Courant with score vector $w$ can be solved in polynomial time for single-peaked preferences.

**5 OWA-based Rules**

In the philosophy behind Chamberlin–Courant, each voter is represented by exactly one committee member, and obtains all ‘utility’ through this representation. In many application scenarios, we may instead seek multirepresentation (Skowron, Faliszewski, and Lang 2015): for example, you might watch several of the movies offered by an inflight-entertainment system. In such scenarios, Chamberlin–Courant might design a suboptimal committee: Skowron et al. (2015) introduce OWA-based multiwinner rules as a more flexible alternative (see also Faliszewski et al. 2016b).

Given a vector $x \in \mathbb{R}^k$, a weight vector $\alpha \in \mathbb{R}^k$ defines an ordered weighted average (OWA) operator as follows: first, sort the entries of $x$ into non-increasing order, so that $x_{\sigma(1)} \geq \ldots \geq x_{\sigma(k)}$; second, apply the weights: the ordered weighted average of $x$ with weights $\alpha$ is given by $\alpha(x) := \sum_{i=1}^k \alpha_i x_{\sigma(i)}$. For example, $\alpha = (1,0,\ldots,0)$ gives the maximum, and $\alpha = (1,1,\ldots,1)$ gives the sum of the numbers in $x$.

Now, a scoring vector $w \in \mathbb{N}^m$ and an OWA $\alpha$ define an OWA-based multi-winner rule as follows: Given a profile $P$, the rule outputs a committee $W = \{ c_1, \ldots, c_k \}$ that maximises the objective value

$$\sum_{i \in N} \alpha(w_{c_1}, \ldots, w_{c_k}).$$

Thus, choosing $\alpha = (1,0,\ldots,0)$ gives us w-Chamberlin–Courant as a special case. Choosing $\alpha = (1,1,0,\ldots,0)$ gives us an analogue of Chamberlin–Courant where voters are represented by their favourite two members of the committee. The OWA-based rules with $\alpha = (1, \frac{1}{2}, \ldots, \frac{1}{k})$ and $w = (1,0,\ldots,0)$ gives us PAV, when given dichotomous preferences as input. Thus, OWA-based rules generalise both Chamberlin–Courant and PAV, and it turns out that we can apply our method to these rules by merging the ideas of (PAV-IP) and (CC-IP). However, our formulation is only valid for non-increasing OWA vectors $\alpha_i \geq \alpha_j$ whenever $i \geq j$.

For the IP, we again put $w_{c}^T = w_{r} - w_{r-1}$ and $w_1^T = w_1$.

**Proposition 10.** If $\alpha$ and $w$ are non-increasing, (OWA-IP) correctly computes an optimal committee according to the OWA-based rule based on $\alpha$ and $w$.

**Proof sketch.** Similarly to previous arguments, in optimum, we will have $x_{i,r} = 1$ if and only if the committee $W = \{ c \in C : y_c = 1 \}$ contains at least $\ell$ candidates that voter $i$ ranks in rank $r$ or above. Thus, the objective value of (OWA-IP) agrees with the defined objective of the OWA-based rule.

The following property is proved very similarly to before.

**Proposition 11.** The constraint matrix of (OWA-IP) is totally unimodular when input preferences are single-peaked.

**Theorem 12.** OWA-based rules with non-increasing OWA operator can be solved in polynomial time for single-peaked preferences.

**6 Young’s Rule**

In contrast to the rules we have considered so far, Young’s (1977) voting rule does not select a committee; it is a single-winner voting rule that is based on extending Condorcet’s rule. Given a profile $P$, an alternative $c \in C$ is a Condorcet winner if $c$ beats every other alternative in a pairwise majority contest, i.e., for every $a \neq c$, the number of voters in $P$ with $c \succ a$ strictly exceeds the number of voters with $a \succ c$.

Still, this case is also efficiently solvable in the single-peaked case: note that a voter’s least-favourite committee members will be either the left-most or the right-most member of the committee; thus it suffices to consider committees of size 2. This idea can be extended to OWA operators $\alpha = (0,\ldots,0,\alpha_k,\ldots,\alpha_1)$ that are zero except for constant many values at the end.
(thus defined, Condorcet winners are unique). As is well-known, not every profile admits a Condorcet winner. How should we choose the outcome in such profiles? Young’s rule chooses the Condorcet winners of the largest subprofiles that do admit a Condorcet winner, where a subprofile can be obtained by deleting voters. In other words, Young’s rule yields all alternatives that can be made Condorcet winners by deleting a minimum number of voters. Young’s rule also assigns to each alternative \( c \) a score, namely the number of voters in a maximum subprofile of \( P \) in which \( c \) is the Condorcet winner; Young winners are the alternatives with maximum score.

Deciding whether a given alternative is a Young winner is \( \Theta^2 \)-complete (Rothe, Spakowski, and Vogel 2003), but is solvable in polynomial time for single-peaked (Brandt et al. 2015) and single-crossing preferences, essentially because such profiles always admit a Condorcet winner (in odd electorates). Since, in general, it is hard to find a Young winner, it is also hard to find the Young score of a given alternative \( a \). This latter problem admits the following simple IP formulation. Here, the variables \( d_i \) indicate whether voter \( i \) is to be deleted from the profile, and \( \text{maj}(b, a) = |\{i \in N : b \succ_i a\}| - |\{i \in N : a \succ_i b\}| \) is the majority margin of \( b \) over \( a \).

\[
\begin{align*}
\text{minimise } & \sum_{i \in N} d_i & \text{ (Young-IP)} \\
\text{subject to } & \sum_{i \in N} d_i \geq \text{maj}(b, a) + 1 & \text{ for } b \in A \setminus \{a\} \\
& d_i \in \{0, 1\} & \text{ for } i \in N
\end{align*}
\]

While previous IPs were easy for single-peaked profiles, this one behaves nicely for single-crossing preferences:

**Proposition 13.** The constraint matrix of (Young-IP) is totally unimodular when input preferences are single-crossing.

**Proof.** After disregarding the constraints \( 0 \leq d_i \leq 1 \), the constraint matrix is a submatrix of \( M_{SC}^P \), consisting of just the rows corresponding to pairs \((b, a)\) for \( b \in A \setminus \{a\}\). If input preferences are single-crossing, then \( M_{SC}^P \) has the consecutive ones property, and hence so do all its submatrices. \( \square \)

It follows that Young scores can be computed in polynomial time when preferences are single-crossing. This in itself is not an impressive result, since there is a very easy direct algorithm for solving this special case, but the advantage of the method via integer programming is that we need not check whether \( P \) is single-crossing or not to obtain the efficiency gains.

7 Some Remarks

**More than Single-Peakedness.** Our polynomial-time results apply to a slightly larger class than just single-peaked profiles: they also apply when \( M_{SP}^P \) (with an all-1s row appended) is totally unimodular but does not necessarily have the consecutive ones property. It can be shown that this is the case whenever \( P \) contains only two distinct voters, or, more generally, when the set of all top-initial segments of \( P \) can also be induced by a two-voter profile. Together with single-peaked profiles, we conjecture that these classes of profiles are precisely the profiles for which the relevant constraint matrices are totally unimodular.

**Egalitarian versions.** We can obtain egalitarian versions of the multi-winner rules that we have discussed by replacing the sum over \( N \) by a minimum in their objective values (Betzler, Slinko, and Uhlmann 2013). For PAV and Chamberlin–Courant, our IP formulations can easily be adapted to answer the question “is there a committee with egalitarian objective value \( \geq L \)?” while preserving total unimodularity in the case of single-peaked preferences. An optimum committee can then be found by a binary search on \( L \). However, it is unclear how this can be achieved for OWA-based rules. It is also unclear how to handle other utility aggregation operators such as lexicin (see Elkind and Ismaili 2015).

**PAV and Voter Intervals.** Elkind and Lackner (2015) define an analogue of single-crossingness for dichotomous preferences called voter interval (VI), which requires the transpose of \( M_{SP}^P \) to have the consecutive ones property. As for CI, they conjectured that PAV remains hard on VI preferences. We could not solve this problem using our method: the constraint \( \sum_{c \in C} y_c = k \) of (PAV-IP) destroys total unimodularity.

**Dodgson’s rule.** An alternative is a Dodgson winner if it can be made a Condorcet winner using a minimum number of swaps of adjacent alternatives. This number of swaps is the Dodgson score of an alternative. Bartholdi III, Tovey, and Trick (1989) give an IP formulation for this problem, which is also used in the treatment of Caragiannis et al. (2009). Sadly, while ‘most’ of the constraint matrix is again identical to \( M_{SP}^P \), some extra constraints (saying that the swaps in each vote should only count once) destroy total unimodularity, so our method cannot be employed for this formulation. Note that while Brandt et al. (2015) give an efficient algorithm for finding a Dodgson winner in the case of single-peaked preferences, the problem of efficiently calculating scores appears to be open and non-trivial. Maybe there is an alternative IP formulation that can be made to work using our approach.

**Kemeny’s rule.** Conitzer, Davenport, and Kalagnanam (2006) present several IP formulations for Kemeny’s rule. The poly-size formulation they give involves constraints enforcing transitivity of the Kemeny ranking; these constraints are not totally unimodular. In any case, most strategies for calculating Kemeny’s rule first calculate all pairwise majority margins; we might as well check for transitivity at this stage — trying to use fancy total unimodularity is unnecessary.

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