Stability of holonomic quantum computations

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Abstract

We study the stability of holonomic quantum computations with respect to errors in assignment of control parameters. The general expression for fidelity is obtained. In the small errors limit the simple formulae for the fidelity decrease rate is derived.

\textbf{Key words:} Holonomic quantum computer, fidelity, non-abelian Stokes theorem

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Holonomic approach to quantum computations was primarily proposed in the Ref. [1]. Optical holonomic quantum computer was proposed to realize this idea in a non-linear Kerr medium with the degenerate states of laser beams to be interpreted as qubits and logical gates to be realized by employing the existing devices of quantum optics [2]. Another implementation of holonomic quantum computer based on trapped ions in optical cavity was discussed recently from the viewpoint of its resilience to control parameter errors (undesirable fluctuations of the laser beams amplitudes) [3]. Fidelity primarily proposed in the Ref. [4] is widely used as a measure of the stability of quantum systems.
Fidelity close to unity means stability of quantum system with respect to small perturbations of the Hamiltonian (or more generally evolution operator) \[5\]. Particularly it defines the resilience of quantum computations \[6\]. Another approaches to determine the stability of quantum systems were also proposed (for instance see the Ref. \[7\]). It was demonstrated in particular for cavity model of holonomic quantum computer that this realization of logical gates provides fidelity and a success rate close to unity \[8\]. In this Letter we derive the general expression for the fidelity of any holonomic quantum computation. Taylor expansion for fidelity as function of the control errors magnitude for the general case of holonomic quantum computer is obtained in the most relevant limit of small errors. For this purpose the non-abelian Stokes theorem primarily utilized in connection with the confinement problem of quarks and gluons \[9\] is used. The simple formulae for the fidelity decrease rate when control errors grow is extracted. It is worth to note that fidelity in particular for holonomic quantum computer is mathematically similar to the Wilson loop defined in the theory of strong interactions of elementary particles (quantum chromodynamics), see the Ref. \[10\]. Namely, the "area law" for Wilson loop means confinement of quarks and gluons, as well the similar behavior of fidelity demonstrates the instability of quantum system. This corresponds to the connection between quantum chaos and confinement in the sense of the Ref. \[7\].

In holonomic quantum computer non-abelian geometric phases (holonomies) are used for implementation of unitary transformations (quantum gates) in the subspace \(\mathbb{C}^N\) spanned on eigenvectors corresponding the degenerate eigenvalue of parametric isospectral family of Hamiltonians \[F = \{H(\lambda) = U(\lambda)H_0U(\lambda)^+\}_{\lambda \in \Lambda} \\[1\]. The \(\lambda\)'s are the control parameters and
\( M \) represents the space of the control parameters. The subspace \( C^N \) is called quantum code (\( N \) is the dimension of the degenerate computational subspace). Quantum gates are realized when control parameters are adiabatically driven along the loops in the control manifold \( M \). The unitary operator mapping the initial state vector into the final one has the form \( \bigoplus_{l=1}^{R} e^{i\phi_l \Gamma_\gamma(A^l_\mu)} \), where \( l \) enumerates the energy levels of the system, \( \phi_l \) is the dynamical phase, \( R \) is the number of different energy levels of the system under consideration and the holonomy associated with the loop \( \gamma \in M \) is given by [11]:

\[
\Gamma_\gamma(A_\mu) = \hat{P} \exp i \int_\gamma A_\mu d\lambda_\mu.
\]

Here \( \hat{P} \) denotes the path ordering operator, \( A_\mu \) is the matrix valued adiabatic connection given by the expression:

\[
(A_\mu)_{mn} = \int d^3x \psi_m^\ast(x, \lambda) \left( -i \frac{\partial}{\partial \lambda_\mu} \right) \psi_n(x, \lambda),
\]

where integration goes over the spatial coordinates \( x \) and \( \psi_n(x, \lambda), \ n = \overline{1, N} \) are basis functions of the corresponding eigenspace \( C^N \). Dynamical phase will be omitted bellow due to the suitable choice of the zero energy level [1]. We shall consider the single subspace (no energy level crossings are assumed).

Fidelity for holonomic quantum computer in the general case can be written as follows:

\[
f = tr \left( \rho \Gamma_{\gamma'}^{-1} \Gamma_{\gamma_0} \right) = tr \left( \rho \hat{P} \exp \left\{ i \int_{\delta\gamma} A_\mu d\lambda_\mu \right\} \right)
\]

where \( \gamma_0 \) is the adiabatic loop implementing the desirable quantum gate, \( \gamma' \) is the actual loop with some deviations from \( \gamma_0 \) due to practically unavoidable errors in assignment of control parameters, \( \rho \) denotes the density matrix of
the quantum state which stability is studied. The loop $\delta \gamma$ is defined by the relation $\delta \gamma = \gamma' - 1 \cdot \gamma_0$. The --operation means that to obtain the contour $\delta \gamma$ we primarily go over the loop $\gamma_0$ in the straight direction and after that travel along the loop $\gamma'$ in the opposite direction. For the definition and properties of this operation see Ref. [1]. The contour $\delta \gamma$ is assumed to be closed i.e. to form a loop.

Using the non-abelian Stokes theorem [9] is it easy to obtain the following expression for the fidelity (3):

$$ f = tr \rho \hat{P} \exp \left( i \int_{\delta S} [\hat{P} e^{i \int_{x_0}^{z} A_\mu d\lambda_\mu} ] F_{\chi \varrho} [\hat{P} e^{i \int_{x_0}^{z} A_\nu d\lambda_\nu} ] d\sigma_{\chi \varrho}(z) \right). \tag{4} $$

Here $F_{\chi \varrho} = \partial_\chi A_\varrho - \partial_\varrho A_\chi - i[A_\chi, A_\varrho]_-$ is the curvature tensor in the space of the control parameters, $\partial_\chi \equiv \partial/\partial \lambda_\chi$, $[,]_-$ denotes the commutator, $\delta S$ is the surface spanned on the loop $\delta \gamma$ and $d\sigma_{\chi \varrho}$ is the projection of the infinitesimally small surface element of $\delta S$ on the coordinate plane $\chi \varrho$ of the control parameter space. In the formulae (4) and everywhere bellow the summation over the indices of the curvature tensor $F_{\chi \varrho}$ is performed under the condition $\chi > \varrho$.

Current point $z$ is located on the loop $\delta \gamma$ and point $x_0$ is the arbitrary one. In the Ref. [9] it was demonstrated that when the non-abelian Stokes theorem is applied the result does not depend on the particular form of $\delta S$ and the point $x_0$ is arbitrary if there are no monopole-like and string-like topological structures. This is the case assumed here. The general case deserves further investigation.

The general expression (4) is not convenient for the practical use. To simplify it we consider more relevant limit of small errors. Namely, we assume that the errors in the assignment of control parameters $\delta \lambda_\mu$ are small and sat-
isfy the restrictions $|\delta \lambda_\mu| < ||A_\mu||^{-1}$, $|\delta \lambda_\chi \delta \lambda_\rho| < ||F_{\chi\rho}||^{-1}$ ($\chi > \rho$). The connection and curvature tensor are calculated in some point $\lambda_0$ defining the position of the small loop $\delta \gamma$. The norm is defined as follows $||B|| = sup \{ \sqrt{\langle \psi | B^+ B | \psi \rangle} | |\psi \rangle \in C^N, \langle \psi | \psi \rangle = 1 \}$. Here $B$ denotes both the connection matrix components $A_\mu$ and the curvature tensor matrix components $F_{\chi\rho}$ ($\chi > \rho$). Then the following Taylor expansion in the point $\lambda_0 \in M$ can be obtained for the fidelity (4):

$$f = 1 + itr \rho F_{\chi\rho} \delta \lambda_\chi \delta \lambda_\rho - tr \rho [A_\mu, F_{\chi\rho}] \delta \lambda_\mu \delta \lambda_\chi \delta \lambda_\rho + (itr \rho A_\mu F_{\chi\rho} A_\nu -$$

$$- \frac{i}{2} tr \rho [F_{\chi\rho}, A_\mu A_\nu]_+ - \frac{1}{2} tr \rho F_{\chi\rho} F_{\mu\nu}) \delta \lambda_\chi \delta \lambda_\rho \delta \lambda_\mu \delta \lambda_\nu + O(\delta \lambda^5). \quad (5)$$

Here the point $\lambda_0$ defines the position of the small loop $\delta \gamma$ in the space of parameters $M$ and $[.,.]_+$ denotes the anti-commutator. Under the stated restriction on the control parameter errors higher order terms in Eq.(5) give less significant contribution to the fidelity then the terms having the lower order on $\delta \lambda$. It is seen that the Taylor expansion for fidelity does not have the linear on $\delta \lambda$ term. The cancellation of the first order terms was noticed for the particular implementation of holonomic quantum computer on trapped ions [3].

We have demonstrated that this conclusion does not depend on particular realization of holonomic quantum computation. The reason is that the difference between two vector parallel transported along two different infinitesimally small paths from one point to another is proportional to the area enclosed by these contours and it does not depend on their lengths.

Equality of the fidelity to the unity means that the computations are stable. Namely, the result of computations is not changed, when the quantum gate is perturbed. Expression (5) shows that fidelity slightly deviates from unity for
small errors in assignment of control parameters. For zero errors fidelity equals unity as it should be. The rate of fidelity deviation from unity extracted from the second item of the expansion (5) is determined by the curvature tensor:

\[
\frac{\delta f}{\delta S_{\mu \nu}} = |tr\rho F_{\mu \nu}| \tag{6}
\]

where \(\delta S_{\mu \nu} = \delta \lambda_{\mu} \delta \lambda_{\nu}\) is the "error area" in the plane \(\mu \nu\). We see that the less the curvature tensor leads the less the deviation of the fidelity from unity and therefore calculations are more stable. This result agrees with one of the Ref. [3], where for the particular realization of holonomic quantum computer on trapped ions it was demonstrated the one-qubit holonomic gate to be resilient to control errors for large values of the squeezing parameter (exponentially small values of the curvature tensor components). From the physical point of view this result can be understood as follows. The curvature tensor (as well as the area enclosed) determines the difference between two vector parallel transported along two different infinitesimally small paths from one point to another. Therefore the less the curvature means the less the difference between state vectors parallel transported along loops \(\gamma_0\) and \(\gamma'\). Thus the computation is more stable. The condition for robust holonomic quantum computation is \(F_{\mu \nu} = 0\) for points belonging to the loop \(\gamma_0\).

In conclusion, exploring the similarity between the mathematical apparatus used in non-abelian gauge field theories and one used for description of holonomic quantum computation we obtained the general expression for fidelity of holonomic quantum computer. Its Taylor expansion in the small control errors limit is derived and the simple formulae for the rate of the fidelity deviation from the unity is obtained. Our general results are in agreement with ones obtained for the particular realization of holonomic quantum computer.
on trapped ions. The formal mathematical analogy between the fidelity and Wilson loop was noted. The question about the relationship between the confinement of the strongly interacting quarks and gluons and the stability of the corresponding systems is under the investigation.

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