Resolvents and Yosida Approximations of Displacement Mappings of Isometries

Salihah Alwadani1 · Heinz H. Bauschke1 · Julian P. Revalski2 · Xianfu Wang1

Received: 14 October 2020 / Accepted: 23 March 2021 / Published online: 12 April 2021
© The Author(s), under exclusive licence to Springer Nature B.V. 2021

Abstract
Maximally monotone operators are fundamental objects in modern optimization. The main classes of monotone operators are subdifferential operators and matrices with a positive semidefinite symmetric part. In this paper, we study a nice class of monotone operators: displacement mappings of isometries of finite order. We derive explicit formulas for resolvents, Yosida approximations, and (set-valued and Moore-Penrose) inverses. We illustrate our results by considering certain rational rotators and circular shift operators.

Keywords Circular shift · Displacement mapping · Isometry of finite order · Maximally monotone operator · Moore-Penrose inverse · Nonexpansive mapping · Resolvent · Set-valued inverse · Yosida approximation

Mathematics Subject Classification (2020) Primary 47H05 · 47H09; Secondary 47A06 · 90C25.

Dedicated to Terry Rockafellar on the occasion of his 85th birthday

© Heinz H. Bauschke
heinz.bauschke@ubc.ca

Salihah Alwadani
saliha01@mail.ubc.ca

Julian P. Revalski
revalski@math.bas.bg

Xianfu Wang
shawn.wang@ubc.ca

1 Mathematics, University of British Columbia, Kelowna, B.C. V1V 1V7, Canada
2 Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Acad. G. Bonchev str., Block 8, 1113 Sofia, Bulgaria
1 Introduction

Throughout this paper, we assume that $X$ is a real Hilbert space with inner product $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{R}$, and induced norm $\| \cdot \|$, that $X \neq \{0\}$, and that $R : X \to X$ is a linear isometry of finite order $m \in \{2, 3, \ldots\}$:

$$R^m = \text{Id}.$$ 

Here Id denotes the identity operator on $X$. It follows that $R$ is surjective and that $\|R\| = 1$ and hence $R$ is nonexpansive, i.e., Lipschitz continuous with constant 1. Therefore, by, e.g., [9, Theorem VI.5.1],

$$R^* = R^{-1} = R^{m-1}.$$ 

We also define throughout the paper

$$M := \text{Id} - R.$$ 

Following [17, Exercise 12.16], we shall refer to $M$ as the displacement mapping of $R$. Indeed, [17, Exercise 12.16] states that $M$ is maximally monotone when $X = \mathbb{R}^n$; this result remains true in general as well [4, Example 20.29]. Monotone operators play a major role in modern optimization due to the fact that their zeros are often solutions to inclusion or optimization problems. For more on monotone operator theory, we refer the reader to [4, 10, 11, 17–19, 21, 22]. The main examples of monotone operators are subdifferential operators of convex functions and positive semidefinite matrices.

Displacement mappings of nonexpansive mappings form a nice class of monotone operators. These have turned out to be very useful in optimization. Here are some examples. The papers [3] and [6] on asymptotic regularity of projection mappings and firmly nonexpansive mappings rely critically on the displacement mapping $M$. The analysis of the range of the Douglas–Rachford operator in [5] employed displacement mappings to obtain duality results. Asymptotic regularity results for nonexpansive mappings were generalized in [7] to displacement mappings. In turn, a new application of the Brezis–Haraux theorem led to a completion of this study in the recent paper [8] along with sharp and limiting examples.

The purpose of this paper is to present a comprehensive analysis of $M$ from the point of view of monotone operator theory. We provide elegant and explicit formulas for resolvents and Yosida approximations for $M$ and its inverse.

The paper is organized as follows. In Section 2, we derive a formula for the resolvent of $\gamma M$ (see Corollary 2.2) and discuss asymptotic behaviour as $\gamma \to 0^+$ or $\gamma \to +\infty$. The set-valued and Moore-Penrose inverses of $M$ are provided in Section 3. In Section 4, we obtain formulae for the resolvents and Yosida approximations. Section 5 presents concrete examples based on rational rotators and circular shift operators. The final Section 6 offers some concluding remarks. Notation is standard and follows largely [4].

2 The Resolvent of $\gamma M$

Associated with any maximally monotone operator $A$ is the so-called resolvent $J_A := (A + \text{Id})^{-1}$ which turns out to be a nice firmly nonexpansive operator with full domain. Resolvents not only provide an alternative view on monotone operators because one can recover the underlying maximally monotone operator via $J_A^{-1} - \text{Id}$ but they also are crucial
for the formulation of algorithms for finding zeros of $A$ (e.g., the celebrated proximal point algorithm [16]).

The main purpose of this section is to derive a formula for the resolvent $J_{\gamma M} := (\text{Id} + \gamma M)^{-1}$, where

$$\gamma > 0.$$ 

Because $\gamma M$ is still maximally monotone, the resolvent $J_{\gamma M}$ has full domain. Let us start with a result that holds true for displacement mappings of linear nonexpansive mappings.

**Theorem 2.1** Let $S : X \rightarrow X$ be nonexpansive and linear. Then

$$J_{\gamma (\text{Id} - S)} = \sum_{k=0}^{\infty} \frac{\gamma^k}{(1 + \gamma)^{k+1}} S^k.$$ 

**Proof** We have, using [13, Theorem 7.3-1] in (1),

$$J_{\gamma (\text{Id} - S)} = (\text{Id} + \gamma (\text{Id} - S))^{-1} = \left((1 + \gamma) \left(\text{Id} - \frac{\gamma}{1 + \gamma} S\right)\right)^{-1}$$

$$= (\text{Id} - \frac{\gamma}{1 + \gamma} S)^{-1} \circ \frac{1}{1 + \gamma} \text{Id} = \frac{1}{1 + \gamma} \left((\text{Id} - \frac{\gamma}{1 + \gamma} S)^{-1}\right)$$

$$= \frac{1}{1 + \gamma} \sum_{k=0}^{\infty} \left(\frac{\gamma}{1 + \gamma}\right)^k S^k$$

because $\|\gamma/(1 + \gamma)S\| \leq \gamma/(1 + \gamma) < 1$. \hfill $\square$

**Corollary 2.2** (resolvent of $\gamma M$) We have

$$J_{\gamma M} = \frac{1}{(1 + \gamma)^m - \gamma^m} \sum_{k=0}^{m-1} (1 + \gamma)^{m-1-k} \gamma^k R^k.$$ 

**Proof** Using Theorem 2.1 and the assumption that $R$ is of finite order $m$, we have

$$J_{\gamma M} = \frac{1}{1 + \gamma} \sum_{k=0}^{\infty} \left(\frac{\gamma}{1 + \gamma}\right)^k R^k$$

$$= \frac{1}{1 + \gamma} \left(\sum_{k=0}^{m-1} \left(\frac{\gamma}{1 + \gamma}\right)^k R^k + \left(\frac{\gamma}{1 + \gamma}\right)^m \sum_{k=0}^{m-1} \left(\frac{\gamma}{1 + \gamma}\right)^k R^k + \cdots\right)$$

$$= \frac{1}{1 + \gamma} \left(1 + \left(\frac{\gamma}{1 + \gamma}\right)^m + \left(\frac{\gamma}{1 + \gamma}\right)^{2m} + \cdots\right) \sum_{k=0}^{m-1} \left(\frac{\gamma}{1 + \gamma}\right)^k R^k$$

$$= \frac{1}{1 + \gamma} \frac{1}{1 - \left(\frac{\gamma}{1 + \gamma}\right)^m} \sum_{k=0}^{m-1} \left(\frac{\gamma}{1 + \gamma}\right)^k R^k$$

$$= \frac{(1 + \gamma)^{m-1}}{(1 + \gamma)^m - \gamma^m} \sum_{k=0}^{m-1} \left(\frac{\gamma}{1 + \gamma}\right)^k R^k$$

and the result follows. \hfill $\square$
Remark 2.3 Consider the formula for $J_{\gamma M}$ from Corollary 2.2. The coefficients for $R^k$ are positive and sum up to 1; hence,

$$J_{\gamma M} \in \text{conv}\{\text{Id}, R, \ldots, R^{m-1}\}$$

and $J_{\gamma M}|_{\ker M} = \text{Id}|_{\text{Fix} R}$. In particular, if $\text{Fix} R = \ker M \supsetneq \{0\}$, then $J_{\gamma M}$ cannot be a Banach contraction.

Next, we set

$$D := \ker(M) = \text{Fix} R,$$

which is a closed linear subspace of $X$. This allows us to describe the asymptotic behaviour of $J_{\gamma M}$ as $\gamma$ tends either to $0^+$ or to $+\infty$.

**Proposition 2.4** We have

$$\lim_{\gamma \to 0^+} J_{\gamma M} = \text{Id} \quad (3)$$

and

$$\lim_{\gamma \to +\infty} J_{\gamma M} = P_D = \frac{1}{m} \sum_{k=0}^{m-1} R^k, \quad (4)$$

where the limits are understood in the pointwise sense and $P_D$ denotes the orthogonal projector onto $D$.

**Proof** The main tool is Corollary 2.2.

The limit (3) follows from (2), and it follows also from [4, Theorem 23.48] by noting that $\text{dom} M = \text{dom} M = X$ and hence $P\text{dom} M = P_X = \text{Id}$.

Let’s turn to (4). The left equation follows directly from [4, Theorem 23.48(i)] because $D = M^{-1}(0)$. Finally, as $\gamma \to +\infty$, we have

$$J_{\gamma M} = \frac{1}{(1 + \gamma)^m - \gamma^m} \sum_{k=0}^{m-1} \frac{(1 + \gamma)^{m-1-k} \gamma^k}{\gamma^m} R^k$$

$$= \frac{1}{(1 + \frac{1}{\gamma})^m - 1^m} \frac{1}{\gamma} \sum_{k=0}^{m-1} (1 + \frac{1}{\gamma})^{m-1-k} R^k$$

$$= \frac{1}{(1 + \frac{1}{\gamma})^m - 1^m} \frac{1}{\gamma} \sum_{k=0}^{m-1} (1 + \frac{1}{\gamma})^{m-1-k} R^k$$

$$\to \frac{1}{m} \sum_{k=0}^{m-1} R^k$$

because the derivative of $\xi \mapsto \xi^m$ at 1 is $m$. \hfill \Box

Remark 2.5 If both $T_1$ and $T_2$ are operators that are polynomials in $R$, then clearly $T_1$ and $T_2$ commute:

$$T_1 T_2 = T_2 T_1.$$ 

In particular, by Proposition 2.4, both $P_D$ and $P_{D^\perp} = \text{Id} - P_D$, the projector onto $D$ and its orthogonal complement respectively, commute with any operator that is a polynomial in $R$.  

 Springer
We conclude this section with a connection to the mean ergodic theorem.

Remark 2.6 The linear mean ergodic theorem (see, e.g., [14, Theorem II.11] and [15, Chapter X, Section 144]) states that

\[ P_{\text{Fix } S} = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} S^k \quad (5) \]

pointwise for any surjective isometric (or even just nonexpansive) linear operator \( S : X \to X \). Because \( R^m = \text{Id} \) and \( D = \text{Fix } R \), (5) yields in particular as \( N \to +\infty \) that

\[ P_D \leftarrow \frac{1}{Nm-1} \sum_{k=0}^{Nm-1} R^k = \frac{N}{Nm-1} \sum_{k=0}^{m-1} R^k \to \frac{1}{m} \sum_{k=0}^{m-1} R^k, \]

i.e., an alternative proof of the right identity in (4).

3 The Inverse \( M^{-1} \) and Moore-Penrose Inverse \( M^\dagger \)

The operator \( M \) is a continuous linear operator on \( X \). Unfortunately, \( M \) is neither injective nor surjective. This raises some very natural questions: What is the set-valued inverse \( M^{-1} \)? What is the Moore-Penrose inverse of \( M \)? It is very satisfying that complete answers to these questions are possible, and we provide these in this section. We start by considering the kernel and the range of \( M \).

**Proposition 3.1** We have

\[ \ker M = D = \ker M^* \quad (6) \]

and

\[ \text{ran } M = D^\perp = \text{ran } M^*; \quad (7) \]

in particular, \( \text{ran } M \) is closed.

**Proof** By [4, Proposition 20.17],

\[ \ker M = \ker M^* \text{ and } \text{ran } M = \text{ran } M^*. \quad (8) \]

This and the definition of \( D \) yield (6).

Now let \( y \in X \). Assume first that \( y \in \text{ran } M \). Then \( y \in \text{ran } M = \text{ran } M^* = (\ker M)^\perp = D^\perp \). Conversely, we assume that \( y \in D^\perp \) and we set

\[ x := \frac{1}{m} \sum_{k=0}^{m-2} (m - 1 - k) R^k y. \]
Using (4) in (9), we obtain
\[Mx = (\text{Id} - R)x\]
\[= \frac{1}{m} \sum_{k=0}^{m-2} (m - 1 - k)R^ky - \frac{1}{m} \sum_{k=0}^{m-2} (m - 1 - k)R^{k+1}y\]
\[= \frac{1}{m} \sum_{k=0}^{m-2} (m - 1 - k)R^ky - \frac{1}{m} \sum_{k=1}^{m-1} (m - k)R^ky\]
\[= \frac{m - 1}{m}y - \frac{1}{m} R^{m-1}y - \frac{1}{m} \sum_{k=1}^{m-2} R^ky\]
\[= \left( \text{Id} - \frac{1}{m} \sum_{k=0}^{m-1} R^k \right)y\]
\[= \left( \text{Id} - PD \right)y\]
\[= y.\]  
Hence \(y = Mx \in \text{ran } M\) and thus \(D^\perp \subseteq \text{ran } M\). Altogether, we see that
\[\text{ran } M = D^\perp \text{ is closed}.\]  
The remaining conclusion follows by combining (10), (8), and the fact that \(\text{ran } M^*\) is closed (because \(\text{ran } M\) is and \([4, \text{Corollary 15.34}]\) applies).

We now define the continuous linear operator
\[T := \frac{1}{2m} \sum_{k=1}^{m-1} (m - 2k)R^k\]  
which will turn out to be key to the study of \(M^{-1}\).

**Proposition 3.2** The operator \(T\) satisfies the following:

(i)
\[T = \frac{1}{2m} \sum_{k=1}^{[m/2]} (m - 2k)(R^k - R^{m-k}).\]  
(ii) \(T^* = -T\), i.e., \(T\) is a skew linear operator.
(iii) \(\text{ran } T \subseteq D^\perp\).

**Proof** (i): This follows easily by considering two cases (\(m\) is odd and \(m\) is even).

(ii): The skew part of \(R^k\) is \(\frac{1}{2}(R^k - (R^k)^*) = \frac{1}{2}(R^k - (R^*)^k) = \frac{1}{2}(R^k - (R^{-1})^k) = \frac{1}{2}(R^k - R^{m-k})\). Hence each term in the sum (12) is skew, and therefore so is \(T\).

(iii): The formula for \(PD\) in (4) yields
\[P_{D^\perp} = \frac{m - 1}{m} \text{Id} - \frac{1}{m} \sum_{i=1}^{m-1} R^i.\]
Using this and (11), we obtain (using the empty-sum convention)

\[
2m^2 P_{D\perp} T = \left( (m - 1) \text{Id} - \sum_{i=1}^{m-1} R^i \right) \left( \sum_{j=1}^{m-1} (m - 2j) R^j \right)
\]

\[
= \left( - \sum_{i=1}^{m-1} (m - 2(m - i)) \right) \text{Id}
\]

\[
+ \sum_{k=1}^{m-1} \left( (m - 1)(m - 2k) - \sum_{i=1}^{k-1} (m - 2(k - i)) - \sum_{i=k+1}^{m-1} (m - 2(m + k - i)) \right) R^k
\]

\[
= (0) \text{Id} + \sum_{k=1}^{m-1} \left( (m - 1)(m - 2k) - (k - 1)(m - k) + k(m - 1 - k) \right) R^k
\]

\[
= \sum_{k=1}^{m-1} m(m - 2k) R^k = 2m^2 T.
\]

Hence \( P_{D\perp} T = T \); equivalently, \( \text{ran} \ T \subseteq D\perp \).

We are now able to provide a formula for the inverse of \( M \).

\textbf{Theorem 3.3} We have

\[ M^{-1} = \frac{1}{2} \text{Id} + T + N_{D\perp}, \]

where \( N_{D\perp} = \partial \iota_{D\perp} \) denotes the normal cone operator of \( D\perp \).

\textbf{Proof} Using (7), we observe that \( \text{dom}(T + N_{D\perp}) = D\perp = \text{ran} \ M = \text{dom} \ M^{-1} = \text{dom}(M^{-1} - \frac{1}{2} \text{Id}) \). So pick an arbitrary

\[
y \in D\perp.
\]

In view of the definition of \( N_{D\perp} \), it suffices to show that

\[
M^{-1} y - \frac{1}{2} y \approx Ty + D.
\]

Let \( x \in M^{-1} y \). Then \( Mx = y \) and \( M^{-1} y = x + \ker M = x + D \) by (6). Hence we must show that

\[
x + D - \frac{1}{2} y \approx Ty + D,
\]

which is equivalent to \( x + D - \frac{1}{2}(x - Rx) \approx T(x - Rx) + D \) and to

\[
(x + Rx) + D \approx 2T(x - Rx) + D.
\]

Note that \( P_{D\perp} (x + Rx) = P_{D\perp} (2T(x - Rx)) \Leftrightarrow 2T(x - Rx) - (x + Rx) \in D \Leftrightarrow \ker M \Leftrightarrow M(2T(x - Rx)) = M(x + Rx) = (\text{Id} - R)(x + Rx) \). Hence we must prove that

\[
M(2T(x - Rx)) \approx x - R^2 x. \quad (13)
\]
We now work toward the proof of (13). First, note that
\[
2mT(x - Rx) = \sum_{k=1}^{m-1} (m - 2k)R^k(x - Rx)
\]
\[
= \sum_{k=1}^{m-1} (m - 2k)R^k x - \sum_{k=1}^{m-1} (m - 2k)R^{k+1} x
\]
\[
= \sum_{k=1}^{m-1} (m - 2k)R^k x - \sum_{k=2}^{m} (m + 2 - 2k)R^k x
\]
\[
= (m - 2)x + (m - 2)Rx - \sum_{k=2}^{m-1} R^k x,
\]
which implies
\[
2T(x - Rx) = \frac{m - 2}{m}x + \frac{m - 2}{m}Rx - \frac{2}{m} \sum_{k=2}^{m-1} R^k x.
\] (14)

Using (14), we see that
\[
M(2T(x - Rx)) = (\text{Id} - R)(2T(x - Rx))
\]
\[
= (2T(x - Rx)) - R(2T(x - Rx))
\]
\[
= \frac{m - 2}{m}x + \frac{m - 2}{m}Rx - \frac{2}{m} \sum_{k=2}^{m-1} R^k x
\]
\[
- \frac{m - 2}{m}x - \frac{2}{m} \sum_{k=2}^{m-1} R^k x - \frac{m - 2}{m}R^2 x + \frac{2}{m} \sum_{k=2}^{m-1} R^{k+1} x
\]
\[
= \frac{m - 2}{m}x - \frac{2}{m} \sum_{k=2}^{m-1} R^k x - \frac{m - 2}{m}R^2 x + \frac{2}{m} \sum_{k=2}^{m-1} R^k x
\]
\[
= x - R^2 x,
\]
i.e., (13) does hold, as desired! \qed

Remark 3.4 Consider Theorem 3.3. Then \(M^{-1} - \frac{1}{2} \text{Id}\) is monotone; in other words, \(M^{-1}\) is \(\frac{1}{2}\)-strongly monotone. If \(R \neq \text{Id}\), then \(D \neq X\); hence \(D^{\perp} \supseteq \{0\}\) and \(T|_{D^{\perp}} - \varepsilon \text{Id}\) cannot be monotone because \(T\) is skew. It follows that the constant \(\frac{1}{2}\) is sharp:

\(M^{-1}\) is not \(\sigma\)-strongly monotone if \(R \neq \text{Id}\) and \(\sigma > \frac{1}{2}\).

Recall that given \(y \in X\), the vector \(M^\dagger y\) is the (unique) minimum norm vector among all the solution vectors \(x\) to the least squares problem

\[
\|Mx - y\| = \min_{z \in X} \|Mz - y\|.
\]

(We refer the reader to [12] for further information on generalized inverses.) We now present a pleasant formula for the Moore-Penrose inverse \(M^\dagger\) of \(M\).
Theorem 3.5  The Moore-Penrose inverse of $M$ is

$$M^\dagger = \frac{1}{2} P_D^\perp (\text{Id} + 2T) = \sum_{k=0}^{m-1} \frac{m-1-2k}{2m} R^k.$$  \hfill (15)

Proof  Recall that by (4)

$$\text{Id} - P_D = P_D^\perp = \text{Id} - \frac{1}{m} \sum_{k=0}^{m-1} R^k,$$

which is a polynomial in $R$. Using also [6, Proposition 2.1] and the fact that $T$ is also a polynomial in $R$ (see Remark 2.5), we have

$$M^\dagger = P_{\text{ran} M^*} \circ M^{-1} \circ P_{\text{ran} M} = P_D^\perp \circ (\frac{1}{2} \text{Id} + T + N_D^\perp) \circ P_D^\perp$$

$$= \frac{1}{2} P_D^\perp + P_D^\perp T P_D^\perp$$

$$= \frac{1}{2} P_D^\perp (\text{Id} + 2T)$$

$$= \frac{1}{2m} \left( (m - 1) \text{Id} - \sum_{i=1}^{m-1} R^i \right) \left( \text{Id} + 2T \right)$$

$$= \frac{1}{2m} \left( (m - 1) \text{Id} - \sum_{i=1}^{m-1} R^i \right) \left( \text{Id} + \frac{1}{m} \sum_{j=1}^{m-1} (m - 2j) R^j \right)$$

$$= \frac{1}{2m^2} \left( (m - 1) \text{Id} - \sum_{i=1}^{m-1} R^i \right) \left( m \text{Id} + \sum_{j=1}^{m-1} (m - 2j) R^j \right)$$

$$= \frac{1}{2m^2} \left( (m - 1) \text{Id} - \sum_{j=0}^{m-1} (m - 2j) R^j \right).$$  \hfill (16)

We thus established the left identity in (15). To obtain the right identity in (15), we use the very last expression (16) for $M^\dagger$ and compute the coefficients of $\text{Id}$, $R$, $R^2$, ..., $R^{m-1}$.

The coefficient for $\text{Id}$ is

$$\frac{1}{2m^2} \left( (m - 1)m - \sum_{i=1}^{m-1} (m - 2(m - i)) \right) = \frac{m - 1}{2m}$$

as needed. The coefficient for $R$ is

$$\frac{1}{2m^2} \left( (m - 1)(m - 2) - (m) - \sum_{i=2}^{m-1} (m - 2(m + 1 - i)) \right) = \frac{m - 3}{2m}$$

as needed. The coefficient for the general $R^k$ is

$$\frac{(m - 1)(m - 2i) - \sum_{i=1}^{k} (m - 2(k - i)) - \sum_{i=k+1}^{m-1} (m - 2(m + k - i))}{2m^2} = \frac{m - 1 - 2k}{2m}$$

as needed. We thus verified the right equality in (15). \hfill \Box
Corollary 3.6 For all \( y \in \text{ran } M = D^\perp \), we have
\[
M^{-1}y = M^\dagger y + D = D + \sum_{k=0}^{m-1} \frac{m - 1 - 2k}{2m} R^k y.
\]

Proof It is well known (see, e.g., [4, Proposition 3.31]) that \( MM^\dagger = P_{\text{ran } M} \) which readily implies the conclusion. \(\square\)

Remark 3.7 We mention in passing that the results in this section combined with work on decompositions of monotone linear relations lead to a Borwein-Wiersma decomposition
\[
M^{-1} = \partial \left( \frac{1}{4} \|\cdot\|^2 + \iota_{D^\perp} \right) + T
\]
of \( M^{-1} \). The required background is nicely detailed in Liangjin Yao’s PhD thesis [20].

4 Resolvents and Yosida Approximations

Given a maximally monotone operator \( A \) on \( X \), recall that its resolvent is defined by \( J_A = (\text{Id} + A)^{-1} \). We have computed the resolvent \( J_{\gamma M} \) already in Corollary 2.2. In this section, we present a formula for \( J_{\gamma M^{-1}} \). Moreover, we provide formulas for the Yosida approximations \( \gamma M \) and \( \gamma M^{-1} \), and we also discuss Lipschitz properties of \( J_{\gamma M} \). Recall that the Yosida approximation of \( A \) of index \( \gamma > 0 \) is defined by \( \gamma A := \frac{1}{\gamma} (\text{Id} - J_{\gamma A}) \). Yosida approximations are powerful tools to study monotone operators. They can be viewed as regularizations and approximations of \( A \) because \( \gamma A \) is a single-valued Lipschitz continuous operator on \( X \) and \( \gamma A \) approximates \( A \) in the sense that \( \gamma Ax \to P_{Ax}(0) \in Ax \) as \( \gamma \to 0^+ \). (For this and more, see, e.g., [4, Chapter 23].)

Recall that we proved in Corollary 2.2 that
\[
J_{\gamma M} = \frac{1}{(1 + \gamma)^m - \gamma^m} \sum_{k=0}^{m-1} (1 + \gamma)^{m-1-k} \gamma^k R^k; \quad (17)
\]
indeed, this will give us the following result quickly.

Theorem 4.1 We have
\[
J_{\gamma M^{-1}} = \text{Id} - J_{(1/\gamma)M}, \quad (18)
\]
\[
\gamma M = \frac{1}{\gamma} (\text{Id} - J_{\gamma M}), \quad (19)
\]
and
\[
\gamma(M^{-1}) = \frac{1}{\gamma} J_{(1/\gamma)M} = \frac{1}{(1 + \gamma)^m - 1} \sum_{k=0}^{m-1} (1 + \gamma)^{m-1-k} R^k. \quad (20)
\]

Proof By the linearity of \( M \) and [4, Proposition 23.20], we have
\[
J_{\gamma M^{-1}} = (1/\gamma) J_{(1/\gamma)^{-1}M^{-1}} \circ (1/\gamma)^{-1} \text{Id} = \text{Id} - J_{(1/\gamma)M},
\]
and (18) holds. (19) is just the definition, while (20) is clear from (18). The remaining identity follows by employing (17). \(\square\)
We already observed in $M^{-1}$ is $\frac{1}{2}$-strongly monotone and that the constant $\frac{1}{2}$ is sharp (unless $R = \text{Id}$). This implies that the resolvent $J_{\gamma M}^{-1}$ is a Banach contraction — in general, this observation is already in Rockafellar’s seminal paper [16]:

**Proposition 4.2** $J_{\gamma M}^{-1}$ is a Banach contraction with Lipschitz constant $\frac{2}{2 + \gamma} < 1$. If $D \neq \{0\}$, then $J_{\gamma M}$ cannot be a Banach contraction.

**Proof** Using Remark 3.4 and [4, Proposition 23.13], we have the implications $M^{-1} - \frac{1}{2} \text{Id}$ is skew (and hence monotone) $\Leftrightarrow \gamma M^{-1} - (\gamma/2) \text{Id}$ is monotone $\Leftrightarrow \gamma M^{-1}$ is $\beta$-strongly monotone with $\beta := \gamma/2$ $\Leftrightarrow J_{\gamma M}^{-1}$ is $(1 + \beta)$ cocoercive $\Rightarrow J_{\gamma M}^{-1}$ is a Banach contraction with constant $1/(1 + \beta) = 2/(2 + \gamma) < 1$.

Finally, if $D \neq \{0\}$, then $D = \ker(\gamma M) = \text{Fix} J_{\gamma M}$ contains infinitely many points and thus $J_{\gamma M}$ cannot be a Banach contraction. 

We will see in Section 5.1 below that the contraction constant cannot be improved in general.

## 5 Examples

In this section, we turn to concrete examples to illustrate our results. It is highly satisfying that complete formulas are available. For reasons of space, we restrict our attention to $m \in \{2, 3, 4\}$. The underlying linear isometry of finite order will be either a rational rotator or a circular right-shift operator.

### 5.1 Rational Rotators

Let us start with rational rotators. In this subsection, we assume that $X = \mathbb{R}^2$ and $R = \begin{pmatrix} \cos(2\pi/m) & -\sin(2\pi/m) \\ \sin(2\pi/m) & \cos(2\pi/m) \end{pmatrix}$.

Then $R^m = \text{Id}$ and $D = \text{Fix} R = \{(0, 0)\} \subseteq X$. Using (17) and (18), we record the following formulas for $J_{\gamma M}$, $J_{\gamma M}^{-1}$, $\gamma M$, and $\gamma(M^{-1})$:

If $m = 2$, then

$$J_{\gamma M} = \frac{1}{1 + 2\gamma} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad J_{\gamma M}^{-1} = \frac{2}{2 + \gamma} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\gamma M = \frac{2}{1 + 2\gamma} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \gamma(M^{-1}) = \frac{1}{2 + \gamma} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$  

If $m = 3$, then

$$J_{\gamma M} = \frac{1}{2 + 6\gamma + 6\gamma^2} \begin{pmatrix} 2 + 3\gamma & -\sqrt{3}\gamma \\ \sqrt{3}\gamma & 2 + 3\gamma \end{pmatrix}, \quad J_{\gamma M}^{-1} = \frac{1}{6 + 6\gamma + 2\gamma^2} \begin{pmatrix} 6 + 3\gamma & \sqrt{3}\gamma \\ -\sqrt{3}\gamma & 6 + 3\gamma \end{pmatrix},$$

$$\gamma M = \frac{1}{2 + 6\gamma + 6\gamma^2} \begin{pmatrix} 3 + 6\gamma & \sqrt{3} \\ -\sqrt{3} & 3 + 6\gamma \end{pmatrix}, \quad \gamma(M^{-1}) = \frac{1}{6 + 6\gamma + 2\gamma^2} \begin{pmatrix} 3 + 2\gamma & -\sqrt{3} \\ \sqrt{3} & 3 + 2\gamma \end{pmatrix}.$$  

If $m = 4$, then

$$J_{\gamma M} = \frac{1}{1 + 2\gamma + 2\gamma^2} \begin{pmatrix} 1 + \gamma & -\gamma \\ \gamma & 1 + \gamma \end{pmatrix}, \quad J_{\gamma M}^{-1} = \frac{1}{2 + 2\gamma + \gamma^2} \begin{pmatrix} 2 + \gamma & \gamma \\ -\gamma & 2 + \gamma \end{pmatrix},$$

$$\gamma M = \frac{1}{1 + 2\gamma + 2\gamma^2} \begin{pmatrix} 1 + \gamma & -\gamma \\ \gamma & 1 + \gamma \end{pmatrix}, \quad \gamma(M^{-1}) = \frac{1}{2 + 2\gamma + \gamma^2} \begin{pmatrix} 2 + \gamma & \gamma \\ -\gamma & 2 + \gamma \end{pmatrix}.$$
\[
\gamma M = \frac{1}{1 + 2\gamma + 2\gamma^2} \begin{pmatrix} 1 + 2\gamma & 1 \\ -1 & 1 + 2\gamma \end{pmatrix}, \quad \text{and} \quad \gamma(M^{-1}) = \frac{1}{2 + 2\gamma + \gamma^2} \begin{pmatrix} 1 + \gamma & -1 \\ 1 & 1 + \gamma \end{pmatrix}.
\]

Higher values of \( m \) lead to unwieldy matrices. We do note that for \( m = 2 \), we have
\[
J_{\gamma M^{-1}} = 2/(2 + \gamma) \text{ Id}; \quad \text{consequently, the constant } 2/(2 + \gamma) \text{ in Proposition 4.2 is sharp.}
\]

5.2 Circular Shift Operators

In this subsection, we assume that \( H \) is another real Hilbert space,

\[
X = H^m \text{ and } R: X \to X: (x_1, x_2, \ldots, x_m) \mapsto (x_m, x_1, \ldots, x_{m-1})
\]
is the circular right-shift operator. Then

\[
R^m = \text{Id} \quad \text{and} \quad D = \text{Fix } R = \{(x, x, \ldots, x) \in X \mid x \in H\},
\]
is the “diagonal” subspace of \( X \) with orthogonal complement

\[
D^\perp = \{(x_1, x_2, \ldots, x_m) \in X \mid x_1 + x_2 + \cdots + x_m = 0\}.
\]

Some of our results in this paper were derived in [6]; however, with less elegant proofs (sometimes relying on the specific form of \( R \)). Using (17) and (18) once again, we calculate

\[
J_{\gamma M}, \quad J_{\gamma M^{-1}}, \quad \gamma M, \quad \text{and} \quad \gamma(M^{-1}) \quad \text{for } m \in \{2, 3\}.
\]

These formulas are all new. The matrices in the following are to be interpreted as block matrices where a numerical entry \( \alpha \) stands for \( \alpha \text{ Id} | H \).

If \( m = 2 \), then

\[
J_{\gamma M} = \frac{1}{1 + 2\gamma} \begin{pmatrix} 1 + \gamma & \gamma \\ \gamma & 1 + \gamma \end{pmatrix} \quad \text{and} \quad J_{\gamma M^{-1}} = \frac{1}{2 + \gamma} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.
\]

and

\[
\gamma M = \frac{1}{1 + 2\gamma} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \quad \text{and} \quad \gamma(M^{-1}) = \frac{1}{(2 + \gamma)\gamma} \begin{pmatrix} 1 + \gamma & 1 \\ 1 & 1 + \gamma \end{pmatrix}.
\]

If \( m = 3 \), then

\[
J_{\gamma M} = \frac{1}{1 + 3\gamma + 3\gamma^2} \begin{pmatrix} (1 + \gamma)^2 & \gamma^2 \\ (1 + \gamma)\gamma & (1 + \gamma)^2 \gamma \end{pmatrix} \begin{pmatrix} (1 + \gamma)^2 \gamma & (1 + \gamma)\gamma \\ (1 + \gamma)^2 \gamma & (1 + \gamma)^2 \end{pmatrix};
\]

\[
J_{\gamma M^{-1}} = \frac{1}{3 + 3\gamma + \gamma^2} \begin{pmatrix} 2 + \gamma & -1 & -(1 + \gamma) \\ -(1 + \gamma) & 2 + \gamma & -1 \\ -1 & -(1 + \gamma) & 2 + \gamma \end{pmatrix};
\]

\[
\gamma M = \frac{1}{1 + 3\gamma + 3\gamma^2} \begin{pmatrix} 1 + 2\gamma & -\gamma & -1 - \gamma \\ -1 - \gamma & 1 + 2\gamma & -\gamma \\ -\gamma & -1 - \gamma & 1 + 2\gamma \end{pmatrix};
\]

\[
\gamma(M^{-1}) = \frac{1}{(3 + 3\gamma + \gamma^2)\gamma} \begin{pmatrix} (1 + \gamma)^2 & 1 \\ 1 + \gamma & (1 + \gamma)^2 \end{pmatrix} \begin{pmatrix} 1 + \gamma & 1 \\ 1 + \gamma & (1 + \gamma)^2 \end{pmatrix}.
\]

We refrain from listing formulas for \( m \geq 4 \) due to their complexity.

6 Concluding Remarks

Monotone operator theory is a fascinating and useful area of set-valued and variational analysis. In this paper, we carefully analyzed the displacement mapping of an isometry of finite order. We provided new and explicit formulas for inverses (both in the set-valued and
Moore-Penrose sense) as well as for resolvents and Yosida approximants. This is a valuable contribution because explicit formulas are rather uncommon in this area. We believe these formulas will turn out to be useful not only for discovering new results but also for providing examples or counterexamples. (See also [1] and [2] for very recent applications of our work.)

**Acknowledgments** HHB and XW are supported by the Natural Sciences and Engineering Research Council of Canada.

**References**

1. Alwadani, S., Bauschke, H.H., Revalski, J.P., Wang, X.: The difference vectors for convex sets and a resolution of the geometry conjecture. arXiv:2012.04784 (2020)
2. Alwadani, S., Bauschke, H.H., Wang, X.: The Attouch-Théra duality, generalized cycles and gap vectors. arXiv. arXiv:2101.05857 (2021)
3. Bauschke, H.H.: The composition of finitely many projections onto closed convex sets in Hilbert space is asymptotically regular. Proceedings of the American Mathematical Society **131**, 141–146 (2003)
4. Bauschke, H.H., Combettes, P.L.: Convex Analysis and Monotone Operator Theory in Hilbert spaces, second edition Springer (2017)
5. Bauschke, H.H., Hare, W.L., Moursi, W.M.: On The range of the Douglas-Rachford operator. Math. Oper. Res. **41**, 884–897 (2016)
6. Bauschke, H.H., Martin-Márquez, V., Moffat, S.M., Wang, X.: Compositions and convex combinations of asymptotically regular firmly nonexpansive mappings are also asymptotically regular. Fixed Point Theory and Applications **2012**(53), 1–11 (2012). [https://fixedpointtheoryandapplications.springeropen.com/articles/10.1186/1687-1812-2012-53](https://fixedpointtheoryandapplications.springeropen.com/articles/10.1186/1687-1812-2012-53)
7. Bauschke, H.H., Moursi, W.M.: The magnitude of the minimal displacement vector for compositions and convex combinations of firmly nonexpansive mappings. Optim. Lett. **12**, 1465–1474 (2018)
8. Bauschke, H.H., Moursi, W.M.: On the minimal displacement vector of compositions and convex combinations of nonexpansive mappings. Found. Comput. Math. **20**, 1653–1666 (2020)
9. Berberian, S.K.: Introduction to Hilbert Space. Oxford University Press, Oxford (1961)
10. Brézis, H.: Operateurs Maximaux Monotones et Semi-Groupes de Contractions dans les Espaces de Hilbert. Elsevier, North-Holland (1973)
11. Burachik, R.S., Iusem, A.N.: Set-Valued Mappings and Enlargements of Monotone Operators. Springer-Verlag, Berlin (2008)
12. Groetsch, C.W.: Generalized Inverses of Linear Operators, Marcel Dekker (1977)
13. Kreyszig, E.: Introductory Functional Analysis with Applications. Wiley, Hoboken (1989)
14. Reed, M., Simon, B.: Methods of Modern Mathematical Physics I. Functional Analysis, revised and enlarged edition. Academic Press, Cambridge (1980)
15. Riesz, F., Sz.-Nagy, B.: Functional Analysis. Dover, United States (1990)
16. Rockafellar, R.T.: Monotone operators and the proximal point algorithm. SIAM J. Control. Optim. **14**, 877–898 (1976)
17. Rockafellar, R.T., Wets, R.J.-B.: Variational Analysis, Springer-Verlag corrected 3rd printing (2009)
18. Simons, S.: Minimax and Monotonicity. Springer-Verlag, Berlin (1998)
19. Simons, S.: From Hahn-Banach to Monotonicity. Springer-Verlag, Berlin (2008)
20. Yao, L.: On monotone linear relations and the sum problem in Banach spaces, PhD thesis, UBC Okanagan. [http://hdl.handle.net/2429/39970](http://hdl.handle.net/2429/39970) (2011)
21. Zeidler, E.: Nonlinear Functional Analysis and Its Applications II/A, Linear Monotone Operators. Springer-Verlag, Berlin (1990)
22. Zeidler, E.: Nonlinear Functional Analysis and Its Applications II/B, Nonlinear Monotone Operators. Springer-Verlag, Berlin (1990)

**Publisher’s Note** Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.