Mass Spectra and Decay Constants of Heavy-Light Axial Quarkonia in the Framework of Bethe-Salpeter Equation

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Abstract
In this work we calculate the mass spectrum and decay constants of ground and excited states of heavy-light P-wave mesons such as $1^{++}$ and $1^{+-}$, with quark composition, $c\bar{u}$, $c\bar{s}$, $b\bar{u}$, $b\bar{s}$, and $b\bar{c}$ in the framework of a QCD motivated Bethe-Salpeter equation (BSE) by making use of the exact treatment of the spin structure $(\gamma_\mu \otimes \gamma_\mu)$ in the interaction kernel. In this $4 \times 4$ BSE framework, the coupled Salpeter equations for $Q\bar{q}$ are first solved for the confining part of interaction, and are shown to decouple under heavy-quark approximation. Then the one-gluon-exchange interaction is perturbatively incorporated, leading to their mass spectral equations. The analytic forms of wave functions obtained from these equations are then used for calculation of leptonic decay constants of ground and excited states of $1^{++}$, and $1^{+-}$ as a test of these wave functions and the overall framework.

Keywords Bethe-Salpeter equation · Salpeter equations · Mass spectral equation · Heavy-Light Quarkonia · Decay constants

1 Introduction
There is a growing interest in the experimental and theoretical studies of heavy-light mesons over the last few years. Studies on heavy-light mesons are important for the determination of Cabibo-Kobayashi-Maskawa (CKM) mass matrix elements. These studies on quarkonia need heavy quark dynamics, which can provide a significant test of Quantum Chromodynamics (QCD). Spectroscopy of heavy quarkonia have been studied through non-perturbative QCD approaches, such as NRQCD [1], QCD sum rule [2], potential models [3–5], lattice QCD [6–8], Bethe-Salpeter equation (BSE) method [9–16], heavy quark effective theory [17], Relativistic Quark Model (RQM) [18, 19], and Chiral perturbation theory [20].

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A lot of investigation has been carried out on S-wave mesons such as pseudoscalar (0\(^+\)) and vector (1\(^-\)) mesons. While comparatively lesser investigation has been carried out on P-wave mesons such as scalar (0\(^+\)), and axial (both 1\(^++\) and 1\(^+-\)) mesons. Further, new states are continuously being discovered at experimental facilities around the world. Some of the recently discovered P-wave conventional quarkonium states are: 

\[ h_b(1P), h_b(2P), \chi_{b1}(3P) \text{ and } \chi_{b2}(3P) \]  

[21, 22]. The discoveries of conventional states 

\[ h_c(1P), h_c(2P), \chi_c(1P), \chi_c(2P), \text{ and } \eta_c(1S), \]  

and the observation of the exotic states like X(3872), X(3915), Y(4260), Z(3930) at Belle, BaBar, LHC, BESIII, CLEO, etc have created a renewed interest in quarkonium physics [21]. These exotic states are the unconventional states that were named (X, Y, Z), that do not fit into conventional quarkonium states. Amongst these states are Z\(_{(c)}(3900)\) and Z\(_{(c)}(4430)\), whose quantum number assignments have been confirmed to be 1\(^+-\) as per the Particle Data Group (PDG) 2020 tables [23]. Further there is a lack of knowledge of decay constants of axial meson states in general.

Thus, this work is devoted to calculation of mass spectrum of ground and excited states of P-wave heavy-light (Q\(\overline{q}\)) mesons, such as axial vector quarkonia, which fall into two classes: 1\(^++\), and 1\(^+-\). Here, 1\(^+-\) mesons have \(L = 1\), but no spin excitation (i.e. \(S = 0\)), while 1\(^++\) have \(L = 1\), and \(S = 1\). This give total angular momentum, \(J = 1\) for both types of mesons. This difference gives these two classes of mesons slightly different masses.

The 1\(^++\) state was first detected in \(p\overline{p}\) collisions by R704 collaboration [24]. \(h_c\) was then found in reaction, \(p\overline{p} \rightarrow h_c \rightarrow \pi^0 J/\psi \rightarrow e^+e^-\) in FNAL E760 experiment, with mass, \(M_{h_c} = 3526.2 \pm 0.2 \pm 0.2\) MeV\(\_\text{c}\), and \(\Gamma_{h_c} < 1.1\) MeV [25]. Very recently, BES III collaboration reported \(h_c\) production in the process, \(e^+e^- \rightarrow \pi^+\pi^-h_c\) [26].

1\(^++\) mesons are seen in \(pp\) collisions. However, not many decays of these mesons are experimentally observed as can be checked from PDG tables [23]. In the present work we thus study mass spectrum, and the leptonic decays of ground and excited states of 1\(^+-\) and 1\(^++\) \(Q\overline{Q}\), and \(Q\overline{q}\) quarkonia. Though the mass spectrum of these quarkonia have been experimentally observed, but so far there is no experimental determination of their leptonic decays [23] so far, though calculations of these quantities have been studied in some models. Our predictions of leptonic decay constants of these mesons may provide a guide to future experiments for determination of these values. Further, the values of leptonic decay constants that we calculate in this work play might play an important role in precise determination of CKM matrix elements, studies of weak decays and meson mixing.

We further wish to study various transitions involving these axial mesons through processes such as, \(A^+ \rightarrow V\gamma\), \(A^- \rightarrow P\gamma\) etc. \((A^+/A^- = 1^{++}/1^{-+}\) (axial vector), \(V = 1^{-}\) (vector), \(P = 0^{-}\) (pseudoscalar), and \(S = 0^{++}\) (scalar) quarkonia\), which have been studied by some models [27], for which experimental data [21, 23] is available, having recently studied the radiative M1 and E1 transitions, such as \(V \rightarrow P\gamma\); \(V \rightarrow S\gamma\), and \(S \rightarrow V\gamma\) [28]. The transitions involving leptonic and radiative decays of axial vector quarkonia would also serve as a test for the analytic forms of wave functions of these mesons calculated analytically in this paper by solving their mass spectral equations.

Here, regarding the mass spectral calculations, we wish to mention that in unequal mass systems such as \(Q\overline{Q}\), the quarks are not very close together. Due to this the confining interaction would dominate over the One-Gluon-Exchange (OGE) interactions for \(Q\overline{Q}\) systems. Thus, the perturbative incorporation of OGE term [29, 30] is a reasonable approximation for heavy-light quarkonia.

The unequal mass kinematics that also gives the partitioning of internal momenta of the hadron rests on the Wightman-Garding definition of internal momenta of individual quarks. We make use of the four effectively 3D Salpeter equations (that are obtained through the 3D...
reduction of the 4D Bethe-Salpeter equation under Covariant Instantaneous Ansatz, which is a Lorentz-invariant generalization of Instantaneous Approximation. We use the full Dirac structure for writing down the wave functions of these hadrons in accordance with [31, 32]. We first solve the Salpeter equations and obtain coupled equations in amplitudes of Dirac structures, which are then decoupled under heavy-quark approximation, and are used to obtain mass spectrum that is explicitly dependent on the principal quantum number, \( N \) in an approximate harmonic oscillator basis. We then incorporate perturbatively the one-gluon-exchange interaction, and obtain not only the mass spectrum, but also the algebraic forms of wave functions in approximate harmonic oscillator basis. These wave functions are then used for analytic calculations of leptonic decay constants of both \( 1^{+−} \) and \( 1^{++} \) mesons. These analytic calculations have advantage of transparency in expressing spectrum in terms of principal quantum number, \( N \), and also obtaining algebraic forms of wave functions that can be employed for calculations of various transitions involving these mesons.

This paper is organized as follows: In Section 2, we introduce the formulation of the \( 4 \times 4 \) Bethe-Salpeter equation under the covariant instantaneous ansatz, and derive the hadron-quark vertex. In Sections 3 and 4, we derive the mass spectral equation of heavy-light \( 1^{+−} \), and \( 1^{++} \) mesons respectively. In Section 4 is devoted to the calculations of their decay constants, Section 5 is devoted to numerical results and discussion.

2 Formulation of the \( 4 \times 4 \) Bethe-Salpeter Equation

Our work is based on QCD motivated Bethe-Salpeter equation in ladder approximation, with an effective four-fermion interaction mediated by a gluonic propagator that serves as the kernel of BSE in the lowest order. The precise form of our kernel includes a confining term along with a one-gluon exchange term. The effective forms of the BS kernel in ladder approximation have recently been used in [16, 33–36]. They can be used to study relativistic bound states (This was shown recently in [33]). As mentioned above, the frame work of Bethe-Salpeter equation is quite general, and provides an effective description of bound quark-antiquark systems through a suitable choice of input kernel for confinement.

We summarize the main points about the \( 4 \times 4 \) Bethe-Salpeter equation under the Covariant Instantaneous Ansatz (CIA). Here, CIA which is a Lorentz-invariant generalization of Instantaneous Approximation (IA), which is used to derive the 3D Salpeter equations [28–30, 37, 38]. We start with a 4D BSE for quark-anti quark system with quarks of constituent masses, \( m_1 \) and \( m_2 \), written in a \( 4 \times 4 \) representation of 4D BS wave function \( \Psi(P,q) \) as:

\[
S_F^{-1}(p_1) \Psi(P,q) S_F^{-1}(-p_2) = i \int \frac{d^4q'}{(2\pi)^4} K(q,q') \Psi(P,q'),
\]

where \( K(q,q') \) is the interaction kernel, and \( S_F^{-1}(\pm p_{1,2}) = \pm i p_{1,2} + m_{1,2} \) are the quark/antiquark propagators, where the momenta of quarks can be expressed as,

\[
p_{1,2} = \hat{m}_{1,2} P \pm q,
\]

where \( \hat{m}_{1,2} = 1/2[1 \pm (m_1^2 - m_2^2)/M^2] \) act as momentum partitioning parameters. We now make use of the Covariant Instantaneous Ansatz, where, \( K(q,q') = K(\tilde{q},\tilde{q}') \) on the BS kernel, where the BS kernel depends entirely on the variable, \( \tilde{q}_\mu = q_\mu - \frac{q_\mu P}{P_z} P_\mu \). Here, \( \tilde{q} \) the transverse component of internal momentum of the hadron, that is orthogonal to the total hadron momentum (\( \tilde{q}.P = 0 \)), and \( \sigma P_\mu = \frac{q_\mu P}{P_z} P_\mu \) is the longitudinal component of
q, that is parallel to \( P \). Here, the 4-dimensional volume element is, \( d^4q = d^3\hat{q}Mds \). Now working on the right side of \( (1) \), and making use of the fact that

\[
\psi(\hat{q}') = \frac{i}{2\pi} \int Mds' \psi(P, q'),
\]

and the fact that the longitudinal component of \( Mds \) of \( q \) does not appear in \( K(\hat{q}, \hat{q}') \), carrying out integration over \( Mds \) on right side of \( (1) \), we obtain,

\[
S_F^{-1}(p_1)\Psi(P, q)S_F^{-1}(-p_2) = \int \frac{d^3\hat{q}'}{(2\pi)^3} K(\hat{q}, \hat{q}')\psi(\hat{q}'),
\]

We can express the 4D BS wave function, \( \Psi(P, \hat{q}) \) as,

\[
\Psi(P, \hat{q}) = S_1(p_1)\Gamma(\hat{q})S_2(-p_2).
\]

Here the 4D hadron-quark vertex, that enters into the definition of the 4D BS wave function in the previous equation, can be identified as,

\[
\Gamma(\hat{q}) = \int \frac{d^3\hat{q}'}{(2\pi)^3} K(\hat{q}, \hat{q}')\psi(\hat{q}').
\]

Thus, it can be observed that \( \Gamma(\hat{q}) \), as seen from the \( (4), (5) \), is directly related to the 4D wave function, \( \Psi(P, q) \), and one can express the 4D BS wave function \( \Psi(P, q) \) in terms of \( \Gamma(\hat{q}) \).

Following a sequence of steps outlined in [38], we get four Salpeter equations (in 4D variable \( \hat{q} \)), which are effective 3D forms of BSE (Salpeter equations) given below:

\[
(M - \omega_1 - \omega_2)\psi^{++}(\hat{q}) = \Lambda_1^{+}(\hat{q})\Gamma(\hat{q})\Lambda_2^{+}(\hat{q})
\]

\[
(M + \omega_1 + \omega_2)\psi^{--}(\hat{q}) = -\Lambda_1^{-}(\hat{q})\Gamma(\hat{q})\Lambda_2^{-}(\hat{q})
\]

\[
\psi^{--}(\hat{q}) = 0.
\]

\[
\psi^{+-}(\hat{q}) = 0.
\]

Regarding the interaction kernel \( K(\hat{q}', \hat{q}) \), as mentioned earlier, we use an effective form of interaction kernel, \( K \) given below in \( (8), \) and \( (10) \), mainly due to the ease with which it is able to keep a link between mass spectroscopy (arising from 3D Salpeter equations given above), and transition amplitudes through an important connection between the 3D wave function \( \psi(\hat{q}) \), that satisfies the first two 3D Salpeter equations (employed for determination of spectra), and the hadron-quark vertex \( \Gamma \) (for calculation of transition amplitudes in 4D basis), as apparent from the structure of the first two Salpeter equations that connect \( \psi(\hat{q}) \) (on LHS), and \( \Gamma(\hat{q}) \) (on RHS).

The kernel can be written as,

\[
K(\hat{q}', \hat{q}) = \left( \frac{1}{2} \hat{\lambda}_1, \frac{1}{2} \hat{\lambda}_2 \right) (\gamma_\mu \otimes \gamma_\mu) V(\hat{q}', \hat{q})
\]

with colour, spin and orbital parts respectively. For a kernel with the above spin dependence, we can rewrite the hadron-quark vertex as \([29]\),

\[
\Gamma(\hat{q}) = \int \frac{d^3\hat{q}'}{(2\pi)^3} V(\hat{q}', \hat{q})\gamma_\mu \psi(\hat{q}')\gamma_\mu,
\]
where, each of the $\gamma_\mu$s sandwich the BS wave function, $\psi(\hat{q})$, with the scalar part of the kernel, $V = V_{OGE} + V_{\text{Confinement}}$ as,

$$V(\hat{q}, \hat{q}') = \frac{4\pi \alpha_s}{(\hat{q} - \hat{q}')^2} + \frac{3}{4} \omega^2_{q\bar{q}} \int d^3r \left( \kappa r^2 - \frac{C_0}{\omega^2_{\bar{q}}} \right) e^{i(\hat{q} - \hat{q}') \cdot \hat{r}},$$

$$\kappa = (1 + 4\hat{m}_1\hat{m}_2A_0M^2r^2)^{-\frac{1}{2}}. \quad (10)$$

The presence of running coupling constant, $\alpha_s$ in $\omega^2_{q\bar{q}}$ provides an explicit QCD motivation to the BSE kernel. It is to be seen that this algebraic form of the potential ensures a smooth transition from nearly harmonic (for $c\bar{u}$) to almost linear (for $b\bar{b}$)\cite{30}.

The structure of the confinement part $V_c(\hat{q}, \hat{q}')$ in terms of $\bar{V}_c$, and its structure is taken from \cite{30, 38}. The framework is quite general so far. To evaluate the mass spectral equations, we have to use the four Salpeter equations in (7).

### 3 Mass Spectral Equation for Heavy-Light $1^{+-}$ Quarkonia

We start with the general form of 4D BS wave function for axial meson ($1^{+-}$) in \cite{31, 32}. Taking its dot product with $\epsilon_{\mu}$, the polarization vector of axial vector meson, we get,

$$\Psi^{-+}(P, q) = \gamma_5 (q \cdot \epsilon)[g_1(q, P) + i \hat{P}g_2(q, P) - i \hat{q}g_3(q, P) + [\hat{P}, \hat{q}]g_4(q, P)]. \quad (11)$$

Then, making use of the 3D reduction and making use of the fact that $\hat{q} \cdot P = 0$, we can write the general decomposition of the instantaneous BS wave function for scalar mesons ($J^{PC} = 1^{+-}$), of dimensionality $M^0$ being composed of various Dirac structures that are multiplied with scalar functions $g_i(\hat{q})$ and various powers of the meson mass $M$ as \cite{29}

$$\Psi^{-+}(\hat{q}) = \gamma_5 \left( \frac{\epsilon \cdot \hat{q}}{M} \right) \left[ g_1(\hat{q}) + i \frac{P}{M} g_2(\hat{q}) - i \frac{\hat{q}}{M} g_3(\hat{q}) + \frac{2}{M^2} \frac{P \hat{q}}{M^2} g_4(\hat{q}) \right]. \quad (12)$$

Till now these amplitudes $g_1, ..., g_4$ in equation above are all independent, and as per the power counting rule \cite{13, 15} proposed by us earlier, the $g_1$, and $g_2$ are the amplitudes associated with the leading Dirac structures, namely $\gamma_5 I$ and $\gamma_5 P/M$, while $f_3$ and $f_4$ will be the amplitudes associated with the sub-leading Dirac structures, namely, $\gamma_5 \hat{q}/M$, and $\gamma_5^{2P\hat{q}/M^2}$.

We now use the last two Salpeter equations $\psi^{+-}(\hat{q}) = \psi^{-+}(\hat{q}) = 0$ in (7), that can be used to obtain the constraint relations between the scalar functions for unequal mass mesons. We wish to mention that due to the two constraint equations, the scalar amplitudes, $g_i(\hat{q})(i = 1, ..., 4)$ are no longer all independent, but are tied together by the relations.

$$g_4 = \frac{M(m_1\omega_2 - m_2\omega_1)}{2(\omega_1 + \omega_2)q^2} g_2,$$

$$g_3 = \frac{(\hat{q}^2 + \omega_1\omega_2 - m_1m_2)M}{q^2(m_1 + m_2)} g_1. \quad (13)$$
Making use of the above relations between amplitudes, we can write the complete 3D Salpeter wave function, $\psi_{A^-}(\hat{q})$ as,

$$
\psi_{A^-}(\hat{q}) = \gamma_S \frac{e^\hat{q}}{M} \left[ g_1 \left( 1 - i \frac{\beta}{\beta^2} \left( q^2 + \omega_1 \omega_2 - m_1 m_2 \right) \right) + g_2 \left( \frac{\gamma}{\gamma^2} + \frac{(m_1 \omega_2 - m_2 \omega_1)}{(\omega_1 + \omega_2) q^2} \right) \right].
$$

(14)

We proceed in the same way as, [29], where on the right side of these equations, we first work with the confining interaction, $V_c(\hat{q})$ alone. The coupled integral equations that result from the first two Salpeter equations are:

$$(M - \omega_1 - \omega_2)[2g_1 + g_2 L] = \int \frac{d^3 \hat{q}'}{(2\pi)^3} V_c(\hat{q}')[g_1 H_1' + g_2 H_2']$$

$$(M + \omega_1 + \omega_2)[2g_1 - g_2 L] = \int \frac{d^3 \hat{q}'}{(2\pi)^3} V_c(\hat{q}')[g_1 H_1' - g_2 H_2']$$

$L = \frac{m_1}{\omega_1} + \frac{m_2}{\omega_2} + \frac{(m_1 \omega_2 - m_2 \omega_1)}{\omega_1 + \omega_2} \left( \frac{1}{\omega_1} - \frac{1}{\omega_2} \right)$

$$H_1' = -4 - \frac{m_1 m_2 + \hat{q}^2}{\omega_1 \omega_2} + 2 \frac{(m_1 - m_2)}{m_1 + m_2} \left( \frac{1}{\omega_1} + \frac{1}{\omega_2} - \frac{1}{\omega_1} \right)$$

$$H_2' = 2 \left( \frac{m_1}{\omega_1} - \frac{m_2}{\omega_2} \right),$$

(15)

where, $\omega_{1,2}^2 = m_{1,2}^2 + \hat{q}^2$, and $H_{1,2}'$ involve $\hat{q}'$. Now, making use of the structure of confining interaction, $V_c(\hat{q}') = V_c(\hat{q} \delta(\hat{q} - \hat{q}'))$ in the model, we can express the above two coupled integral equations into two coupled algebraic equations,

$$(M - \omega_1 - \omega_2)[2g_1 + g_2 L] = V_c(\hat{q}')(g_1 H_1) + g_2 H_2$$

$$(M + \omega_1 + \omega_2)[2g_1 - g_2 L] = V_c(\hat{q}')(g_1 H_1' - g_2 H_2'),$$

(16)

where $H_{1,2}$ now involve $\hat{q}$. To decouple the above algebraic equations, we first add these equations, and substract the second equation from the first. We thus get two algebraic equations that are again coupled in $g_1$, and $g_2$. Eliminating $g_1$ in terms of $g_2$ from the first equation, and putting in the second equation, and similarly, eliminating $g_2$ in terms of $g_1$ from the second equation, and putting in the first equation, we get two identical decoupled equations in $g_1$, and $g_2$ as:

$$\left[ \frac{M^2}{4} - \frac{1}{4}(m_1 + m_2)^2 - \hat{q}^2 \right] g_1(\hat{q}) = -\frac{1}{2}(m_1 + m_2) \overline{V}_c(\hat{q}) g_1(\hat{q})$$

$$\left[ \frac{M^2}{4} - \frac{1}{4}(m_1 + m_2)^2 - \hat{q}^2 \right] g_2(\hat{q}) = -\frac{1}{2}(m_1 + m_2) \overline{V}_c(\hat{q}) g_2(\hat{q}).$$

(17)

Since the two equations are of the same form in scalar functions $g_1(\hat{q})$ and $g_2(\hat{q})$, that are the solutions of identical equations, we can take, $g_1(\hat{q}) \approx g_2(\hat{q})(= \phi_A(\hat{q}))$. Thus, we can write the complete wave function for $1^{++}$ meson as,

$$\psi_{A^-}(\hat{q}) = \gamma_S \frac{e^\hat{q}}{M} \left[ \left( 1 - i \frac{\beta}{\beta^2} \left( q^2 + \omega_1 \omega_2 - m_1 m_2 \right) \right) + \left( i \frac{\gamma}{\gamma^2} + \frac{(m_1 \omega_2 - m_2 \omega_1)}{(\omega_1 + \omega_2) q^2} \right) \right] \phi_A(\hat{q}).$$

(18)
Using the expression for $\hat{V}_c(\hat{q})$ given above, we get the equation,

$$E_A \phi_A^-(\hat{q}) = \left[-\beta_A^+ \hat{\nabla}_q^2 + \hat{q}^2\right] \phi_A^-(\hat{q}),$$  \tag{19}

where the inverse range parameter $\beta_A^-$ can be expressed as,

$$\beta_A^- = \left(\frac{2}{3}(m_1 + m_2)\omega^2_{q\bar{q}}\right)^{\frac{1}{4}},$$

$$\omega_{q\bar{q}} = (4M\hat{m}_1\hat{m}_2\omega^2_0\alpha_s(M))^{1/2},$$

$$\alpha_s = \frac{12\pi}{33 - 2N_f} \log \left(\frac{M^2}{\Lambda^2_{QCD}}\right)^{-1}$$  \tag{20}

The solutions of (19) are calculated by using the power series method. We assume the solutions of (19) are of the form, $\phi(\hat{q}) = \xi(\hat{q}) e^{-\frac{\hat{q}^2}{\beta_A^-}}$. Then (19) can be expressed as,

$$\xi''(\hat{q}) + \left(\frac{2}{\hat{q}} - \frac{2\hat{q}}{\beta_A^-}^2\right)\xi'(\hat{q}) + \left(\frac{E}{\beta_A^-} - \frac{3}{\beta_A^-} - \frac{l(l + 1)}{\hat{q}^2}\right)\xi(\hat{q}) = 0.$$  \tag{21}

The energy eigen values of this equation obtained using the power series method are:

$$E_N = 2\beta_A^- \left( N + \frac{3}{2} \right);$$

$$N = 2n + l$$  \tag{22}

with the quantum number $n$ taking values, $n = 0, 1, 2, \ldots$, and the orbital quantum number $l = 1$ that corresponds to $P$ wave states. This leads to the mass spectral equation for axial vector ($1^-\pi^{-}$) quarkonia as,

$$\frac{1}{4} \left[M^2 - (m_1 + m_2)^2\right] + \frac{C_0\beta^4_A}{\omega^2_0} \sqrt{1 + 8\hat{m}_1\hat{m}_2A_0} \left( N + \frac{3}{2} \right)$$

$$= 2\beta_A^- \left( N + \frac{3}{2} \right), \quad N = 1, 3, 5, \ldots.$$  \tag{23}

It can further be checked that for each value of $n = 0, 1, 2, \ldots$, would thus correspond a polynomial, $\xi(\hat{q})$ of order $2n + 1$ in $\hat{q}$. The normalized odd-parity eigen functions derived as solutions of (19) are:

$$\phi_A^-(1P, \hat{q}) = \sqrt{\frac{7}{3}} \frac{1}{\pi^{3/4}} \frac{1}{\beta^{5/2}_A} \hat{q} e^{-\frac{\hat{q}^2}{2\beta_A^-}},$$

$$\phi_A^-(2P, \hat{q}) = \sqrt{\frac{5}{3}} \frac{1}{\pi^{3/4}} \frac{1}{\beta^{5/2}_A} \hat{q} \left(1 - \frac{2\hat{q}^2}{5\beta^2_A^-}\right) e^{-\frac{\hat{q}^2}{2\beta_A^-}},$$

$$\phi_A^-(3P, \hat{q}) = \sqrt{\frac{35}{12}} \frac{1}{\pi^{3/4}} \frac{1}{\beta^{5/2}_A} \hat{q} \left(1 - \frac{4\hat{q}^2}{5\beta^2_A^-} + \frac{4\hat{q}^4}{35\beta^4_A^-}\right) e^{-\frac{\hat{q}^2}{2\beta_A^-}},$$

$$\phi_A^-(4P, \hat{q}) = \sqrt{\frac{35}{8}} \frac{1}{\pi^{3/4}} \frac{1}{\beta^{5/2}_A} \hat{q} \left(1 - \frac{6\hat{q}^2}{5\beta^2_A^-} + \frac{12\hat{q}^4}{35\beta^4_A^-} - \frac{8\hat{q}^6}{315\beta^6_A^-}\right) e^{-\frac{\hat{q}^2}{2\beta_A^-}}.$$  \tag{24}
Now, we treat the mass spectral equation in (19), which is obtained by taking only the confinement part of the kernel, as an unperturbed spectral equation with the unperturbed wave functions in (24). We then incorporate the one gluon exchange term in the interaction kernel perturbatively (as in [29]) and solve to first order in perturbation theory. The complete mass spectra of ground and excited states of heavy-light axial \((1^+)^-\) quarkonia is

\[
\frac{1}{8\beta^2_{A^-}} \left[ M^2 - (m_1 + m_2)^2 \right] + \frac{C_0\beta^2_{A^-}}{2\omega_0^2} \sqrt{1 + 8\hat{m}_1\hat{m}_2A_0} \left( N + \frac{3}{2} \right) + \gamma (V_{coul}^A) = N + \frac{3}{2}, \quad N = 1, 3, 5, ..., (25)
\]

where \(\langle V_{coul}^A \rangle\) is the expectation value of \(V_{coul}^A\) between the unperturbed states of the axial vector mesons \((1^+)^-\) with \(l = 1\) and \(n = 0, 1, 2, ...\), and \(\gamma\) is introduced as a weighting factor to have the Coulomb term dimensionally consistent with the harmonic term, with \(\gamma\) expressed in units of \(\omega_0^4/(C_0\beta^2_{A^-})\), and it also acts as a measure of the strength of the perturbation. The expectation value of the Coulomb term associated with the OGE term for axial \((1^+)^-\) quarkonia is a single elegant expression for all states, \(|nP >\), (where, \(n = 1, 2, 3, ...\)),

\[
\langle nP | V_{coul}^- | nP \rangle = -\frac{32\pi\alpha_s}{9\beta^2_{A^-}}. \quad (26)
\]

The results of our model for mass spectrum for \((1^+)^-\) \(Q\bar{Q}\) states along with data [23], and other models is given in Table 1. It was observed in our previous works [29, 30] that the mass spectra of mesons of various \(J^{PC}\) \((0^{++}, 0^{-+}, \text{and} 1^{--})\) is somewhat insensitive to a small range of variations of parameter \(\omega_0\), as long as \(\frac{C_0}{\omega_0}\) is a constant. The input parameters of our model obtained by best fit to the spectra of ground states of scalar, pseudoscalar and vector \(Q\bar{Q}\), and \(QQ\) quarkonia are: \(C_0=0.69\), \(\omega_0=0.22\ \text{GeV}\), \(\Lambda_{QCD}=0.250\ \text{GeV}\), and \(A_0=0.01\), with input quark masses \(m_u=0.300\ \text{GeV}, m_s=0.430\ \text{GeV}, m_c=1.490\ \text{GeV},\) and \(m_b=4.690\ \text{GeV}\). Using these set of input parameters, we do the mass spectral calculations of both ground and excited states of heavy-light axial vector \((1^+)^-\) mesons.

The numerical values of \(\gamma\) multiplying \(V_{coul}^A\) that gave reasonable agreement with data and other models are given in Table 1. These can at best be expressed in units of \(\omega_0^4/(C_0\beta^2_{A^-})\).

We also calculated percentage contribution of coulomb term to the mass of each meson state, which are indeed small, as seen in Table 1, justifying the perturbative treatment of the coulomb term for these states. We see that for any \(J^{PC}\), the contribution of coulomb term to meson mass for \(b\bar{u}\), \(b\bar{s}\), and \(c\bar{b}\) mesons is larger than the corresponding contributions from \(c\bar{u}\), \(c\bar{s}\), and \(c\bar{c}\) states. Also as we go to higher radial states of a given meson, the contribution

\[
\begin{array}{c|c}
\text{Mesons} & \gamma \\
\hline
h_c(nP), \chi_{c1}(nP), & 0.085 \\
c\bar{b}(nP) & 0.34 \\
s\bar{b}(nP) & 0.26 \\
u\bar{b}(nP) & 0.26 \\
s\bar{c}(nP), & 0.051 \\
u\bar{c}(nP), & 0.051 \\
\end{array}
\]
Table 2 Mass spectra of ground and excited states of axial $1^{++}$ quarkonia (in GeV) in BSE-CIA (with the percentage contribution of the OGE to meson mass) along with data and results of other models

| Mass (GeV) | BSE-CIA % contribution of OGE | Expt. [23] | BSE | PM | Lattice QCD | RQM |
|-----------|-------------------------------|------------|-----|----|-------------|-----|
| $M_{hc}(1P_1)$ | 3.525 | 10.28% | 3.525±0.00001 | 3.5244 [16] | 3.518 [39] | 3.5059 [41] | 3.525 [40] |
| $M_{hc}(2P_1)$ | 3.743 | 9.25% | 3.888±0.0025 | 3.9358 [16] | 3.934 [39] | 3.927 [40] |
| $M_{hc}(3P_1)$ | 3.963 | 8.12% | 4.478±0.015 | 4.337 [40] |
| $M_{cb}(1P_1)$ | 6.843 | 8.63% | 6.8451 [16] |
| $M_{cb}(2P_1)$ | 7.147 | 8.28% | 7.2755 [16] |
| $M_{cb}(3P_1)$ | 7.478 | 7.78% | |
| $M_{ub}(1P_1)$ | 5.827 | 11.08% | 5.8364 [16] |
| $M_{ub}(2P_1)$ | 6.045 | 11.46% | 6.2803 [16] |
| $M_{ub}(3P_1)$ | 6.299 | 11.27% | |
| $M_{uc}(1P_1)$ | 2.442 | 14.83% | 2.4498 [16] |
| $M_{uc}(2P_1)$ | 2.605 | 13.47% | 2.8304 [16] |
| $M_{uc}(3P_1)$ | 2.748 | 12.03% | |
| $M_{uc}(1P_0)$ | 2.316 | 16.83% | 2.3025 [16] |
| $M_{uc}(2P_0)$ | 2.454 | 15.48% | 2.6511 [16] |
| $M_{uc}(3P_0)$ | 2.601 | 13.88% | |

of coulomb term to mass keeps decreasing from its corresponding contribution for ground states. This means that the radially excited states are loosely bound in comparison to the ground states, which is similar to the case of atoms. The mass spectra of $1^{++}$ states are given in Table 2.

We now give the plots of wave functions for $1^{++}$ states for $hc(nP)$, $uc(nP)$, $ub(nP)$ and $cb(nP)$, as a function of internal momentum, $|\hat{q}|$. We have obtained the general expressions of 3D forms of long distance (nonperturbative) Bethe-Salpeter wave functions for $1^{++}$ heavy-light $Q\bar{q}$ mesons. We have given the plots of these wave functions as a function of the internal momentum, $|\hat{q}|$ in Figs. 1, 2, 3 and 4. For $Q\bar{q}$ systems, the wave functions show a damped oscillatory behavior. For $1^{++}$ mesons, the amplitude of the wave function is 0 at $|\hat{q}| = 0$ (due to the wave functions being odd), then with increase in $\hat{q}$, it reaches a maximum, executes a damped oscillatory behavior, and finally becomes 0. We wish to mention that a very similar behaviour is observed for the plots of $1^{++}$ mesons, due to which we give here only the plots of $1^{++}$ mesons. Further, as regards all the $1^{++}$ mesons, we see that $1P$ states have zero nodes, followed by $2P$ states with one node and $3P$ states with two nodes. Thus, these plots show that the 3D wave functions, $\phi_A^{1-}(nP)$ have $n-1$ nodes. This is a general feature of all quantum mechanical systems forming a bound state. An interesting feature of these plots is that as the mass of the meson,$M$ increases, $\phi(\hat{q}) \to 0$ at a higher value of $|\hat{q}|$. As seen from the plots, the wave functions of heavier mass $Q\bar{q}$ systems (such as $cb, ub$) extend to a much shorter distance than the wave functions of $(hc, uc)$. This implies that the heavier mesons
Fig. 1 Plots of radial wave functions $\phi_{A^c}(\hat{q})$ for $h_c(1P)$, $h_c(2P)$, and $h_c(3P)$ versus $|\hat{q}|$

comprising of $b$ quarks ($u\bar{b}$, $c\bar{b}$ etc.) are more tightly bound than the comparatively lighter mesons comprising of $c$ quark ($h_c$, $u\bar{c}$ etc.).

This feature is also supported by the fact that in general, the percentage contribution of $V_{coul}$ to meson mass, $M$, is larger for $u\bar{b}$, than for $u\bar{c}$.

Fig. 2 Plots of radial wave functions $\phi_{A^c}(\hat{q})$ for $u\bar{c}(1P)$, $u\bar{c}(2P)$, and $u\bar{c}(3P)$ versus $|\hat{q}|$
It is in this sense, the algebraic forms of 3D hadronic BS wave functions can not only provide information about the long range non-perturbative physics, they also tell us the shortest distance to which they can penetrate in a hadron.

We now derive the mass spectral equations of $1^{++}$ quarkonia in the next section.
4 Mass Spectrum of $1^{++}$ Quarkonia

We start with the general form of 4D BS wave function for axial meson ($1^{++}$) in [31, 32]. Taking its dot product with $\epsilon_{\mu}$, the polarization vector of axial vector meson, and making use of the orthogonality relation, $P\epsilon = 0$, we get,

$$\Psi_{A+}(P, q) = \gamma_5 \not{\epsilon} [f_1(q, P) + \frac{P}{M} f_2(q, P) - \frac{\hat{q}}{M} f_3(q, P) + i(\not{P} - \not{q}) f_4(q, P)]$$

$$+ \gamma_5 (\epsilon, q) (f_3(q, P) + 2i \frac{P}{M} f_4(q, P))$$

(27)

Till now these amplitudes $f_1, ..., f_4$ in equation above are all independent. We try to make these amplitudes dimensionless by pulling out various powers of $M$, and write the above expression as,

$$\Psi_{A+}(P, q) = \gamma_5 \not{\epsilon} \left[ i f_1(q, P) + \frac{P}{M} f_2(q, P) - \frac{\hat{q}}{M} f_3(q, P) + i(\not{P} - \not{q}) f_4(q, P) \right]$$

$$+ \gamma_5 (\epsilon, q) \left( f_3(q, P) + 2i \frac{P}{M} f_4(q, P) \right)$$

(28)

With the use of our power-counting rule [13–15], it can be verified that the Dirac structures associated with the amplitudes $f_1$ and $f_2$ are $O(M)$, and are leading, and thus they would contribute the most to any axial-vector meson calculation. Following a similar procedure as in the case of scalar mesons, we can write the Salpeter wave function in terms of only two Dirac amplitudes: $f_1$ and $f_2$. Plugging this wave function together with the projection operators into the first two Salpeter equations, and following the same steps as for the $1^{+-}$ meson case, we get the coupled integral equations in the amplitudes $f_1$ and $f_2$:

$$\begin{align*}
(M + \omega_1 + \omega_2) \left[ \frac{(m_1 - m_2)}{\omega_1 \omega_2} (\hat{q}, \epsilon) f_1(\hat{q}) + \left( \frac{1}{\omega_1} + \frac{1}{\omega_2} \right) \hat{q}, \epsilon f_2(\hat{q}) \right] \\
= -\frac{4}{3} \int \frac{d^3 \hat{q}'}{(2\pi)^3} V(\hat{q}, \hat{q}') \left[ \frac{2(m_1 - m_2)}{\omega_1 \omega_2} \hat{q}', \epsilon f_1(\hat{q}') \right]
\end{align*}$$

$$\begin{align*}
(M - \omega_1 - \omega_2) \left[ -\frac{(m_1 + m_2)}{\omega_1 \omega_2} (\hat{q}, \epsilon) f_1(\hat{q}) + \left( \frac{1}{\omega_1} - \frac{1}{\omega_2} \right) \hat{q}, \epsilon f_2(\hat{q}) \right] \\
= \frac{4}{3} \int \frac{d^3 \hat{q}'}{(2\pi)^3} V(\hat{q}, \hat{q}') \frac{m_1 + m_2}{\omega_1 \omega_2} \hat{q}', \epsilon f_1(\hat{q}')
\end{align*}$$

(29)

Making use of the fact that $V(\hat{q}, \hat{q}') = V(\hat{q}', \hat{q})\delta^3(\hat{q} - \hat{q}')$ on the RHS of the two coupled equations, they convert into two algebraic equations. Following a similar procedure as in case of $1^{+-}$, the two above equations will get decoupled in amplitudes, $f_1$ and $f_2$ that are listed below.

$$\begin{align*}
\left[ \frac{M^2}{4} - \frac{1}{4}(m_1 + m_2)^2 - \hat{q}^2 \right] g_1(\hat{q}) &= -\frac{1}{2} (m_1 + m_2) \overline{V}_c(\hat{q}) g_1(\hat{q}) \\
\left[ \frac{M^2}{4} - \frac{1}{4}(m_1 + m_2)^2 - \hat{q}^2 \right] g_2(\hat{q}) &= -\frac{1}{2} (m_1 + m_2) \overline{V}_c(\hat{q}) g_2(\hat{q}).
\end{align*}$$

(30)

Thus it can be checked that $f_1$ and $f_2$ satisfy the same equation for unequal mass $1^{++}$ meson, and thus, $f_1(\hat{q}) \approx f_2(\hat{q}) \approx \phi_{A+}(\hat{q})$, where $\phi_{A+}(\hat{q})$ can be shown to satisfy the mass
spectral equation,

$$E_A + \phi_A (\hat{q}) = \left[-\beta^2_A + \hat{V}_q^2 + q^2\right] \phi_A (\hat{q}),$$

$$\beta_A^+ = \left(\frac{2}{3} M \omega^2_{q\hat{q}}\right)^{1/2}$$  \hspace{1cm} (31)

The spectrum of $1^{++}$ is again of the $N + \frac{3}{2}$ type, with $N = 2n + l$ with $n = 0, 1, 2, ...$, and $l = 1$ as in the $1^{+-}$ case. The normalized odd parity energy eigen functions of (31) that are obtained by solutions of this equation are similar to $1^{+-}$, with expressions in (24), with the replacement, $\phi_{A-}(\hat{q}) \Rightarrow \phi_{A+}(\hat{q})$, and $\beta_{A-} \Rightarrow \beta_{A+}$, with the inverse range parameter, $\beta_{A+}$ given in (31).

The perturbative incorporation of the coulomb term into the mass spectral equation, leads to the equation,

$$E_A + \phi_A (\hat{q}) = \left[-\beta^2_A + \hat{V}_q^2 + q^2 + V_{coul}\right] \phi_A (\hat{q}),$$

$$\beta_A^+ = \left(\frac{2}{3} M \omega^2_{q\hat{q}}\right)^{1/2}$$  \hspace{1cm} (32)

The solutions of the above spectral equation is,

$$\frac{1}{8\beta^2_A} \left[M^2 - (m_1 + m_2)^2\right] + \frac{C_0 \beta^2_A}{2\omega^2_0} \sqrt{1 + 8\hat{m}_1 \hat{m}_2 A_0 \left(N + \frac{3}{2}\right) + \gamma \langle V_{coul}^A \rangle} = N + \frac{3}{2}, \hspace{0.5cm} N = 1, 3, 5, ...,$$

which leads to the mass spectra. Here, $l = 1$, where $V_{coul}$ is the expectation value of $V_{coul}$ between the unperturbed states of a given quantum number, $n$ (with $l = 1$) for axial mesons, with value,

$$< n P | V_{coul}^A | n P >= -\frac{64}{9} \frac{\pi \alpha_s}{\beta_A^2},$$

for $1P, 2P, 3P, ...$ states. The mass spectra can be calculated numerically by inverting this equation, and is given in Table 3.

We now calculate the leptonic decay constants of $1^{+-}$, and $1^{++}$ mesons.

5 Leptonic Decays of $1^{+-}$ Mesons

The decay constants of $1^{+-}$ states are defined through the relation,

$$< 0|\bar{q}_1 \gamma_{\mu} (1 - \gamma_5) q_2 | A^- > = (f_{A^-}) M \epsilon_{\mu}.$$  \hspace{1cm} (35)

Now, this equation can be expressed as a quark-loop integral,

$$(f_{A^-}) M \epsilon_{\mu} = \sqrt{3} \int \frac{d^3 \hat{q}}{(2\pi)^3} Tr[\Psi_{A-}(\hat{q})(1 - \gamma_5) \gamma_{\mu}],$$

with the wave function, $\Psi_{A-}^A(\hat{q})$ given in (10) as,

$$\psi_{A-}(\hat{q}) = (N_{A-}) \gamma_5 \sigma \hat{q} g_1 \left(1 - i \hat{q} \frac{\hat{q}^2 + \omega_1 \omega_2 - m_1 m_2}{\hat{q}^2 (m_1 + m_2)}\right)$$

$$+ g_2 \left(\frac{P}{M} + \frac{(m_1 \omega_2 - m_2 \omega_1)}{(\omega_1 + \omega_2) \hat{q}^2} \frac{M}{\hat{q}}\right) \phi_{A-}(\hat{q}).$$

$$\beta_A^+ = \left(\frac{2}{3} M \omega^2_{q\hat{q}}\right)^{1/2}$$  \hspace{1cm} (37)
Putting the expression for $\psi^A(\hat{q})$ above, and evaluating trace over the gamma matrices, we obtain,

$$ (f_A^-) M \epsilon_\mu = \sqrt{3} \int \frac{d^3 \hat{q}}{(2\pi)^3} 4(\hat{q},\epsilon) q_\mu \frac{(\hat{q}^2 + \omega_1 \omega_2 - m_1 m_2)}{\hat{q}^2(m_1 + m_2)} \phi_A^- (\hat{q}). \tag{38} $$

Now, multiplying both sides of the above equation by the polarization vector, $\epsilon_\mu$, we get,

$$ (f_A^-) M = \sqrt{3} N_{A^-} \int \frac{d^3 \hat{q}}{(2\pi)^3} 4(\hat{q},\epsilon)^2 \frac{(\hat{q}^2 + \omega_1 \omega_2 - m_1 m_2)}{\hat{q}^2(m_1 + m_2)} \phi_A^- (\hat{q}), \tag{39} $$

where, the 4D BS normalizer of axial $(1^-)$ meson, $N_{A^-}$, can be obtained by solving the current conservation condition,

$$ 2i P_\mu = \int \frac{d^4 q}{(2\pi)^4} Tr \left[ \overline{\psi}(P, q) \left( \frac{\partial}{\partial P_\mu} S_F^{-1}(p_1) \right) \psi(P, q) S_F^{-1}(-p_2) \right] + (1 \leftrightarrow 2) \tag{40} $$

We make use of the fact that $S_F^{-1}(p_{1,2}) = i(\pm i P_{1,2} + m_{1,2})$, where $p_{1,2} = \hat{m}_{1,2} P \pm q$. In the hadron rest frame, where $P = (0, iM)$, and $q = (\hat{q}, i0)$, we can reduce the above equation to the 3D form,

$$ 2i P_\mu = \int \frac{d^3 \hat{q}}{(2\pi)^3} Tr \left[ \overline{\psi}(\hat{q})(-\hat{m}_1 \gamma_\mu) \psi(\hat{q})[-i(\hat{m}_2 P + \hat{q}) + m_2] \right] + (1 \leftrightarrow 2) \tag{41} $$

| Table 3 | Mass spectra of ground and excited states of axial vector $1^{++}$ quarkonia (in GeV) in BSE-CIA (with the percentage contribution of the OGE to meson mass) along with data and results of other models |
|---|---|---|---|---|---|---|
| | BSE-CIA | % contribution of OGE | Expt. [23] | BSE | PM | Lattice QCD | RQM |
| $M_{\chi c}(1P_1)$ | 3.527 | 10.01% | 3.510\(\pm0.00005\) | 3.5244 [16] | 3.581 [39] | 3.4845 [41] | 3.510 [40] |
| $M_{\chi c}(2P_1)$ | 3.811 | 9.23% | 3.871\(\pm0.00017\) | 3.9358 [16] | 3.934 [39] | 3.872 [40] |
| $M_{\chi c}(3P_1)$ | 4.147 | 8.41% | 4.146\(\pm0.0024\) | 4.312 [40] |
| $M_{\bar{b}c}(1P_1)$ | 6.841 | 8.70% | 6.8451 [16] |
| $M_{\bar{b}c}(2P_1)$ | 7.171 | 8.54% | 7.2755 [16] |
| $M_{\bar{b}c}(3P_1)$ | 7.553 | 8.20% | |
| $M_{\bar{b}c}(1P_1)$ | 5.829 | 12.08% | 5.8364 [16] |
| $M_{\bar{b}c}(2P_1)$ | 6.085 | 11.40% | 6.2803 [16] |
| $M_{\bar{b}c}(3P_1)$ | 6.415 | 11.27% | |
| $M_{\bar{b}c}(1P_1)$ | 5.713 | 35.44% | 5.7047 [16] |
| $M_{\bar{b}c}(2P_1)$ | 5.961 | 12.56% | 6.0355 [16] |
| $M_{\bar{b}c}(3P_1)$ | 6.282 | 12.42% | |
| $M_{\bar{b}c}(1P_1)$ | 2.460 | 14.23% | 2.459\(\pm0.0009\) | 2.4498 [16] |
| $M_{\bar{b}c}(2P_1)$ | 2.746 | 13.23% | 2.8304 [16] |
| $M_{\bar{b}c}(3P_1)$ | 3.042 | 12.00% | |
| $M_{\bar{b}c}(1P_0)$ | 2.313 | 17.83% | 2.3025 [16] |
| $M_{\bar{b}c}(2P_0)$ | 2.592 | 16.22% | 2.6511 [16] |
| $M_{\bar{b}c}(3P_0)$ | 2.874 | 13.97% | |
Table 4 Leptonic decay constants, $f_{A^-}$ of ground state (1P) and excited state (2P) and (3P) of heavy-light axial vector ($1^{++}$) mesons (in GeV) in present calculation (BSE-CIA) along with experimental data, and their masses in other models

|                  | BSE-CIA | Expt. | BSE [16] | QCD SR1  | QCD-SR2  |
|------------------|---------|-------|----------|----------|----------|
| $f_{h_1(1P)}$    | 0.161   | -     | 0        | 0.176 [39] | 0.490 [43] |
| $f_{h_2(2P)}$    | 0.094   | -     | 0        | 0.244 [39] |          |
| $f_{h_3(3P)}$    | 0.062   | -     | 0        |          |          |
| $f_{cb_1(1P)}$   | 0.112   | -     | 0.050    |          |          |
| $f_{cb_2(2P)}$   | 0.061   | -     | 0.049    |          |          |
| $f_{cb_3(3P)}$   | 0.038   | -     |          |          |          |
| $f_{sb_1(1P)}$   | 0.235   | -     | 0.076    |          |          |
| $f_{sb_2(2P)}$   | 0.104   | -     | 0.071    |          |          |
| $f_{sb_3(3P)}$   | 0.059   | -     |          |          |          |
| $f_{ub_1(1P)}$   | 0.271   | -     | 0.076    |          |          |
| $f_{ub_2(2P)}$   | 0.111   | -     | 0.070    |          |          |
| $f_{ub_3(3P)}$   | 0.064   | -     |          |          |          |
| $f_{uc_1(1P)}$   | 0.133   | -     | 0.062    |          |          |
| $f_{uc_2(2P)}$   | 0.094   | -     | 0.050    |          |          |
| $f_{uc_3(3P)}$   | 0.073   | -     |          |          |          |
| $f_{us_1(1P)}$   | 0.146   | -     | 0.072    |          |          |
| $f_{us_2(2P)}$   | 0.110   | -     | 0.056    |          |          |
| $f_{us_3(3P)}$   | 0.095   | -     |          |          |          |

Further making use of the fact that $\Psi(\hat{q}) = \gamma_4 \psi(\hat{q}) \gamma_4$ and making use of the expression for the full wave function $\psi(\hat{q})$ expressed above, along with the fact that, $\epsilon_\mu \epsilon_\nu = \frac{1}{3} (\delta_\mu,\nu + P_\mu P_\nu M^2)$, to calculate the 4D BS normalizer, $N_A$.

We can express the decay constants as,

$$f_{A^-} = \frac{4\sqrt{3}}{M} N_A^- \int \frac{d^3\hat{q}}{(2\pi)^3} \frac{(\hat{q}^2 + \omega_1 \omega_2 - m_1 m_2)}{\hat{q}^2 (m_1 + m_2)} \phi_{A^-}(\hat{q}).$$

The values of these decay constants are given in Table 4:

6 Leptonic Decays of $1^{++}$ Mesons

The decay constants of $1^{++}$ states are defined through the relation,

$$<0|\bar{q}_1 \gamma_\mu (1 - \gamma_5) q_2 |A^+> = (f_{A^+}) M \epsilon_\mu.$$

Now, this equation can be expressed as a quark-loop integral,

$$(f_{A^+}) M \epsilon_\mu = \sqrt{3} \int \frac{d^3\hat{q}}{(2\pi)^3} Tr[\Psi_{A^+}(\hat{q})(1 - \gamma_5) \gamma_\mu],$$

where, the wave function, $\Psi_{A^+}(\hat{q})$ is given as,

$$\Psi_{A^+}(\hat{q}) = N_{A^+} \gamma_5 \left[ i \not{\kappa} + \frac{\not{k} + \not{P}}{M} \right] \phi_{A^+}(\hat{q}).$$
Table 5 Leptonic decay constants, \( f_{A^+} \) of ground state (1P) and excited state (2P) and (3P) of heavy-light axial vector (1^{++}) mesons (in GeV.) in present calculation (BSE-CIA) along with experimental data, and their masses in other models.

|                | BSE-CIA | Expt. | BSE [16] | LFQM [42] | QCD-SR2 |
|----------------|---------|-------|----------|-----------|---------|
| \( f_{x\bar{c}}(1P) \) | 0.211   | –     | 0.206    | -0.105    | 0.490   |
| \( f_{x\bar{c}}(2P) \) | 0.248   | –     | -0.207   |           |         |
| \( f_{x\bar{c}}(3P) \) | 0.250   | –     | 0.199    |           |         |
| \( f_{s\bar{b}}(1P) \) | 0.243   |       | 0.157    | -0.166    |         |
| \( f_{s\bar{b}}(2P) \) | 0.173   |       | -0.156   |           |         |
| \( f_{s\bar{b}}(3P) \) | 0.143   |       |          |           |         |
| \( f_{c\bar{u}}(1P) \) | 0.159   |       | 0.219    | -0.159    |         |
| \( f_{c\bar{u}}(2P) \) | 0.137   |       | -0.204   |           |         |
| \( f_{c\bar{u}}(3P) \) | 0.160   |       |          |           |         |
| \( f_{u\bar{c}}(1P) \) | 0.166   |       | 0.211    | -0.177    |         |
| \( f_{u\bar{c}}(2P) \) | 0.129   |       | -0.143   |           |         |
| \( f_{u\bar{c}}(3P) \) | 0.117   |       |          |           |         |

We put the above equation into (35) to calculate the decay constants, \( f_{A^+} \), which is obtained as,

\[
f_{A^+} = \frac{4\sqrt{3}}{M} N_{A^+} \int \frac{d^3 \hat{q}}{(2\pi)^3} \phi_{A^+}(\hat{q}),
\]

where, the 4D BS normalizer of axial meson (1^{++}), \( N_{A^+} \), is again obtained by solving the current conservation conditions.

The leptonic decay constants for 1^{++} quarkonia are given in Table 5.

7 Discussion

We have calculated the mass spectrum and leptonic decay constants of heavy-light (\( Q\bar{q} \)) axial vector mesons, both 1^{+-} and 1^{++} in the framework of 4 \( \times \) 4 Bethe-Salpeter equation. We have employed a 3D reduction of the Bethe-Salpeter equation under Covariant Instantaneous Ansatz (CIA) with an interaction kernel consisting of both the confining and one gluon exchange terms, to derive the algebraic forms of the 3D mass spectral equations, that are explicitly dependent on the principal quantum number, \( N \). Analytical Solutions of these mass spectral equations not only leads to mass spectrum of ground and excited states of heavy-light axial vector (1^{++}) and (1^{+-}) quarkonia, but also the eigen functions of heavy-light quarkonia in an approximate harmonic oscillator basis. These wave functions for heavy-light mesons so derived, are then used to calculate their leptonic decay constants.

Exact treatment of the spin structure \( (\gamma_{\mu} \otimes \gamma_{\mu}) \) is done in the interaction kernel. We first derive analytically the mass spectral equation using only the confining part of the interaction kernel for \( Q\bar{q} \) systems. Then treating this mass spectral equation as the unperturbed equation, we introduce the One-Gluon-Exchange (OGE) perturbatively, and obtain the mass spectra for heavy-light 1^{++}, and 1^{+-} quarkonia, treating the wave functions derived above as the unperturbed wave functions. The parameters used were fit from the mass spectrum of pseudoscalar, vector and scalar heavy-light quarkonia and are given in Section 3 in this paper.
Mass spectral calculation is an important element to study dynamics of hadrons. Further, the analytic solutions of the spectral equations also lead to hadronic wave functions that play an important role in the calculation of various processes involving $Q\overline{Q}$, and $Q\overline{q}$ hadrons. In our calculations, the wave functions were analytically derived from the mass spectral equations in approximate harmonic oscillator basis, and were recently used to calculate the leptonic decays of heavy-light $P$ and $V$ quarkonia [30], two photon decays of $P$ and $S$ quarkonia [29], and single photon radiative $M1$, and $E1$ transitions through the processes, $V \rightarrow P_\gamma$, $V \rightarrow S_\gamma$, and $S \rightarrow V_\gamma$ [28]. In the present work, the wave functions derived from the mass spectral equations are used to calculate the leptonic decay constants of heavy-light $1^{+-}$, and $1^{++}$ quarkonia for which there is no experimental data presently available, and can be used as a guide for experiments.

We studied the plots of hadronic Bethe-Salpeter wave functions calculated analytically in this work. We studied the long distance (nonperturbative) wave functions of $1^{+-}$ mesons as a function of the internal momentum, $|\hat{q}|$ in Figs. 1–4. For $1^{+-}$ mesons, the amplitude is 0 at $|\hat{q}| = 0$ (since wave functions are odd), then with increase in $|\hat{q}|$, it reaches a maximum. After this the amplitude shows a damped oscillatory behavior, and finally becomes 0. We wish to mention that very similar behaviour of plots is observed for $1^{++}$ mesons, due to which we give here only the plots of $1^{+-}$ mesons.

These plots show that the wave functions, $\phi(\hat{q})(nP)$ have $n - 1$ nodes, which is a general feature of quantum mechanical systems forming a bound state. An interesting feature of these plots is that as the mass of the meson, $M$ increases, $\phi(\hat{q}) \rightarrow 0$ at a higher value of $|\hat{q}|$. As further seen from the plots, the wave functions of heavier mass $Q\overline{q}$ systems (such as $c\overline{b}, u\overline{b}$) extend to a much shorter distance than the wave functions of $(h_c, u\overline{c})$, implying thereby that the heavier mesons (comprising of $b$ quarks) are more tightly bound than the comparatively lighter mesons comprising of $c$ quark. This feature is also supported by the fact that in general, the percentage contribution of $V_{\text{coulomb}}$ to meson mass, $M$, is larger for $u\overline{b}$, than for $u\overline{c}$. Thus, the long distance wave functions of axial vector mesons can act as a bridge between the long distance non-perturbative physics, and the short distance perturbative physics. Thus the wave functions calculated analytically by us can lead to studies on a number of processes involving $Q\overline{Q}$, and $Q\overline{q}$ states. These algebraic forms of wave functions are then used to calculate the leptonic decay constants of axial vector (both $1^{+-}$, and $1^{++}$) quarkonia in this work as a test of these wave functions.

We have first obtained the numerical values of masses for ground and excited states of various heavy-light mesons and made comparison of our results with experimental data and other models using the same input parameters that were used for calculation of mass spectra of scalar, vector and pseudoscalar heavy-light quarkonia, and their transitions [28, 30]. We then obtained the numerical values of leptonic decay constants for these heavy-light axial vector quarkonia with the same set of input parameters.

Our results of masses for ground and excited states of heavy-light axial ($1^{+-}$ and $1^{++}$) quarkonia are in reasonable agreements with experimental data and other models. The experimental data [23] for leptonic decay constants of axial vector mesons is not yet currently available, though they have been studied in some models. We will be using the analytical forms of eigen functions for ground and excited states of heavy-light $1^{+-}$, and $1^{++}$ quarkonia to evaluate various transitions involving these quarkonia for further work.

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