Abstract

We prove the following characterizations of nonstandard models of ZFC (Zermelo-Fraenkel set theory with the axiom of choice) that have an expansion to a model of GB (Gödel-Bernays class theory) plus \( \Delta^1_1\)-CA (the scheme of \( \Delta^1_1\)-Comprehension). In what follows, \( M(\alpha) := (V(\alpha), \in)^M, \) \( L_M \) is the set of formulae of the infinitary logic \( L_{\infty, \omega} \) that appear in the well-founded part of \( M \), and \( \Sigma^1_1\)-AC is the scheme of \( \Sigma^1_1\)-Choice.

**Theorem A.** The following are equivalent for a nonstandard model \( M \) of ZFC of any cardinality:

(a) \( M(\alpha) \prec_{L_M} M \) for an unbounded collection of \( \alpha \in \text{Ord}^M \).

(b) \( (M, \mathcal{X}) \models GB + \Delta^1_1\)-CA, where \( \mathcal{X} \) is the family of \( L_M \)-definable subsets of \( M \).

(c) There is \( \mathcal{X} \) such that \( (M, \mathcal{X}) \models GB + \Delta^1_1\)-CA.

**Theorem B.** The following are equivalent for a countable nonstandard model of ZFC:

(a) \( M(\alpha) \prec_{L_M} M \) for an unbounded collection of \( \alpha \in \text{Ord}^M \).

(b) There is \( \mathcal{X} \) such that \( (M, \mathcal{X}) \models GB + \Delta^1_1\)-CA + \( \Sigma^1_1\)-AC.

1. INTRODUCTION

The point of departure of this paper is Theorem 1.1 below, which characterizes recursively saturated models of PA (Peano Arithmetic) as precisely those nonstandard models of PA that are expandable to models of certain subsystems of second order arithmetic. In what follows, ACA\(_0\) is the well-known subsystem of second order arithmetic whose first order part is PA, \( \Delta^1_1\)-CA (respectively \( \Sigma^1_1\)-AC) is the scheme of \( \Delta^1_1\)-Comprehension (respectively \( \Sigma^1_1\)-Choice), and \( \text{Def}(M) \) is the family of first order definable (parameters allowed) subsets of \( M \).

**1.1. Theorem.** (Barwise-Schlipf [BS]) The following are equivalent for a nonstandard model \( M \) of PA of any cardinality:

(a) \( M \) is recursively saturated.
(b) There is $\mathcal{X}$ such that $(\mathcal{M}, \mathcal{X}) \models \text{ACA}_0 + \Delta^1_1\text{-CA}$.

(c) $(\mathcal{M}, \text{Def}(\mathcal{M})) \models \text{ACA}_0 + \Delta^1_1\text{-CA} + \Sigma^1_1\text{-AC}$.

The argument for $(a) \Rightarrow (c)$ given by Barwise and Schlipf used the machinery of admissible set theory. Not long after, an elementary argument was found by Feferman and Stavi (independently), as reported in Smoryński [Sm]. However, the proof presented for $(b) \Rightarrow (a)$ by Barwise and Schlipf was shown in [ES] to be impaired by a significant gap, and additionally, a correct proof (using a technique not available to Barwise and Schlipf) was presented. Now, prospering from a 45 year hindsight, we can say that the hard part of Theorem 1.1 is $(b) \Rightarrow (a)$, and the straightforward part is $(a) \Rightarrow (c)$ ($(c) \Rightarrow (b)$ is trivial, of course).

An analogue of Theorem 1.1 in the realm of set theory was presented by Schlipf, as in Theorem 1.2 below, in which $o(\mathcal{M})$ is the ordinal height of the well-founded part of $\mathcal{M}$, and $o(\text{HYP}(\mathcal{M}))$ is the ordinal height of $\text{HYP}(\mathcal{M})$, where $\text{HYP}(\mathcal{M})$ is the least admissible structure over $\mathcal{M}$, as defined in Barwise’s definitive text [B] on admissible set theory. Theorem 1.2 implies the analogue of Theorem 1.1 for models of ZF (in which PA is replaced by ZF, and ACA$_0$ is replaced by GB), using Schlipf’s characterization of recursive saturation in terms of $o(\text{HYP}(\mathcal{M})) = \omega$.

1.2. Theorem. (Schlipf [Sch]) The following two conditions are equivalent for a nonstandard model $\mathcal{M}$ of ZF of any cardinality:

(a) There is $\mathcal{X}$ such that $(\mathcal{M}, \mathcal{X}) \models \text{GB} + \Delta^1_1\text{-CA}$.

(b) $o(\mathcal{M}) = o(\text{HYP}(\mathcal{M}))$, and $\mathcal{M}$ satisfies ZF with replacement and separation for formulae involving predicates for all relations on $\mathcal{M}$ that appear in $\text{HYP}(\mathcal{M})$.

Moreover, if $\mathcal{M}$ is a countable nonstandard model of ZFC, then $(a)$ and $(b)$ are equivalent to:

(c) There is $\mathcal{X}$ such that $(\mathcal{M}, \mathcal{X}) \models \text{GB} + \Delta^1_1\text{-CA} + \Sigma^1_1\text{-AC}$.

In a different direction, the paper [En] studies the family of so-called condensable models of ZF, a family that includes all resplendent models of ZF (and in particular, all countable recursively saturated models of ZF). In the terminology of [En], a model $\mathcal{M} \models \text{ZF}$ is condensable if $\mathcal{M} \equiv \mathcal{M}(\alpha) \preceq_{\text{LM}} \mathcal{M}$ for some “ordinal” $\alpha \in \text{Ord}^\mathcal{M}$, where $\mathcal{M}(\alpha) := (V(\alpha),\in)^\mathcal{M}$ and $\text{LM}$ is the set of formulae of the infinitary logic $\text{L}_{\text{\infty,\omega}}$ that appear in the well-founded part of $\mathcal{M}$. The following theorem gives various characterizations of the notion of condensability (see Section 2 for the definitions of the technical notions used in the statement of Theorem 1.3).

1.3. Theorem. [En] The following are equivalent for a countable nonstandard model $\mathcal{M}$ of ZF:

(a) $\mathcal{M}$ is cofinally condensable, i.e., $\mathcal{M} \cong \mathcal{M}(\alpha) \preceq_{\text{LM}} \mathcal{M}$ for an unbounded collection of $\alpha \in \text{Ord}^\mathcal{M}$.

(b) $\mathcal{M}$ is condensable.

(c) For some nonstandard $\gamma \in \text{Ord}^\mathcal{M}$ and some $S \subseteq M$, $S$ is an amenable $\gamma$-satisfaction class on $\mathcal{M}$.

(d) $\mathcal{M}(\alpha) \preceq_{\text{LM}} \mathcal{M}$ for an unbounded collection of $\alpha \in \text{Ord}^\mathcal{M}$.

(e) $\mathcal{M}$ is $\omega$-saturated and $\mathcal{M} \models \text{ZF}(\text{LM})$.

Moreover, without the assumption of countability of $\mathcal{M}$, the following implications hold:

\[ (a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (e) \]

\footnote{We suspect that the implication $(b) \Rightarrow (a)$ fails for some uncountable model of ZF; the implications $(c) \Rightarrow (b)$ and $(d) \Rightarrow (c)$ were shown to be irreversible in [En].}
The main results of this paper are Theorems A and B below that tie Theorems 1.2 and 1.3 together. The proofs of these results do not rely on machinery from admissible set theory, in particular we obtain a new proof, from first principles, of the equivalence of (a) and (c) of Theorem 1.2 for a countable nonstandard model \( \mathcal{M} \) of ZFC. Note that if \( \mathcal{M} \) is \( \omega \)-nonstandard, then condition (a) in Theorems A and B below is equivalent to recursive saturation of \( \mathcal{M} \).

**Theorem A.** The following are equivalent for a nonstandard model \( \mathcal{M} \) of ZFC of any cardinality: 

(a) \( \mathcal{M}(\alpha) \prec_{e, \mathcal{M}} \mathcal{M} \) for an unbounded collection of \( \alpha \in \text{Ord}^\mathcal{M} \).

(b) \( (\mathcal{M}, \mathcal{X}) \models \text{GB} + \Delta_1^1\text{-CA} \), where \( \mathcal{X} \) is the family of \( \mathbb{L}_\mathcal{M} \)-definable subsets of \( \mathcal{M} \).

(c) There is \( \mathcal{X} \) such that \( (\mathcal{M}, \mathcal{X}) \models \text{GB} + \Delta_1^1\text{-CA} \).

**Theorem B.** The following are equivalent for a countable nonstandard model of ZFC:

(a) \( \mathcal{M}(\alpha) \prec_{e, \mathcal{M}} \mathcal{M} \) for an unbounded collection of \( \alpha \in \text{Ord}^\mathcal{M} \).

(b) There is \( \mathcal{X} \) such that \( (\mathcal{M}, \mathcal{X}) \models \text{GB} + \Delta_1^1\text{-CA} + \Sigma_1^1\text{-AC} \).

We suspect that Theorem A can be strengthened by weakening ZFC to ZF. As explained in Remark 4.4, in Theorem B, ZFC cannot be weakened to ZF, and the assumption of countability of \( \mathcal{M} \) is essential. The proof of Theorem A is presented in Section 3, and Theorem B is established in Section 4.

### 2. PRELIMINARIES

In this section we collect the basic definitions, notations, conventions, and results that will be used in the statements and proofs of our main results in Sections 3 and 4.

**2.1. Definition.** (Models, languages, and theories) Models will be represented using calligraphic fonts (\( \mathcal{M}, \mathcal{N}, \text{etc.} \)) and their universes will be represented using the corresponding roman fonts (\( M, N, \text{etc.} \)). In the definitions below, \( \mathcal{M} \) is a model of ZF and \( e^\mathcal{M} \) is the membership relation of \( \mathcal{M} \).

(a) \( \text{Ord}^\mathcal{M} \) is the class of “ordinals” of \( \mathcal{M}, \) i.e., \( \text{Ord}^\mathcal{M} := \{ m \in M : M \models \text{Ord}(m) \} \), where \( \text{Ord}(x) \) expresses “\( x \) is transitive and is well-ordered by \( \in \)”.

(b) \( \mathcal{M} \) is nonstandard if \( e^\mathcal{M} \) is ill-founded (equivalently: if \( (\text{Ord}, e)^\mathcal{M} \) is ill-founded). \( \mathcal{M} \) is \( \omega \)-nonstandard if \( (\omega, e)^\mathcal{M} \) is ill-founded.

(c) For \( c \in M \), \( \text{Ext}_M(c) \) is the \( \mathcal{M} \)-extension of \( c \), i.e., \( \text{Ext}_M(c) := \{ m \in M : m^\mathcal{M} = c \} \). We say that a subset \( X \) of \( M \) is coded in \( \mathcal{M} \) if there is some \( c \in M \) such that \( \text{Ext}_M(c) = X \). For \( A \subseteq M \), \( \text{Cod}_A(\mathcal{M}) \) is the collection of sets of the form \( \bigwedge \text{Ext}_M(c) \), where \( c \in M \).

(d) The well-founded part of \( \mathcal{M} \), denoted \( \text{WF}(\mathcal{M}) \), consists of all elements \( m \) of \( \mathcal{M} \) such that there is no infinite sequence \( \langle a_n : n < \omega \rangle \) with \( m = a_0 \) and \( a_{n+1} \in^\mathcal{M} a_n \) for all \( n \in \omega \). Given \( m \in M \), we say that \( m \) is a nonstandard element of \( \mathcal{M} \) if \( m \notin \text{WF}(\mathcal{M}) \). We denote the submodel of \( \mathcal{M} \) whose universe is \( \text{WF}(\mathcal{M}) \) by \( \text{WF}(\mathcal{M}) \). It is well-known that if \( \mathcal{M} \) is a model of ZF, then \( \text{WF}(\mathcal{M}) \) satisfies KP (Kripke-Platek set theory) [\text{H} Chapter II, Theorem 8.4].

- It is important to bear in mind that we will identify \( \text{WF}(\mathcal{M}) \) with its transitive collapse.

(e) \( o(\mathcal{M}) \) (read as: the ordinal of \( \mathcal{M} \)) is the supremum of all ordinals in \( \text{WF}(\mathcal{M}) \).

(f) Let \( \mathcal{L}_\text{set} \) be the usual vocabulary \( \{ =, \in \} \) of set theory. In this paper we use \( \mathbb{L}_{\infty, \omega} \) to denote the infinitary language based on the vocabulary \( \mathcal{L}_\text{set} \). Thus \( \mathbb{L}_{\infty, \omega} \) is a set-theoretic language that allows conjunctions and disjunctions of sets (but not proper classes) of formulae, subject to the restriction that such infinitary formulae have at most finitely many free variables. Given a set \( \Psi \) of formulae, we denote such conjunctions and disjunctions respectively as \( \bigwedge \Psi \) and \( \bigvee \Psi \).
In the interest of efficiency, we will treat disjunction and universal quantification as defined notions.

(g) \( L_{\delta,\omega} \) is the sublanguage of \( L_{\infty,\omega} \) that allows conjunctions and disjunctions of sets of formulae of cardinality less than \( \delta \). Note that \( L_{\omega,\omega} \) is none other than the usual first order language of set theory, and that in general the language \( L_{\delta,\omega} \) only uses finite strings of quantifiers (as indicated by the \( \omega \) in the subscript).

(h) We say that \( F \) is a fragment of \( L_{\infty,\omega} \) if \( F \) is a set of formulae of \( L_{\infty,\omega} \) that is closed under subformulae, renaming of free variables, existential quantification, negation, and conjunction.

- A fragment of \( L_{\infty,\omega} \) that plays a central role in this paper is \( L_{\mathcal{M}} := L_{\infty,\omega} \cap WF(M) \). Note that if \( M \) is countable, \( L_{\mathcal{M}} = L_{\omega_1,\omega} \cap WF(M) \).

(i) Given a fragment \( F \) of \( L_{\infty,\omega} \), and \( L_{\text{set}} \)-structures \( N_1 \) and \( N_2 \), we write \( N_1 \preceq_F N_2 \) to indicate that \( N_1 \) is a submodel of \( N_2 \), and for all \( \varphi(x_1, \ldots, x_n) \in F \) and all tuples \( (a_1, \ldots, a_n) \) from \( N_1 \), we have:

\[ N_1 \models \varphi(a_1, \ldots, a_n) \text{ iff } N_2 \models \varphi(a_1, \ldots, a_n). \]

(j) Given a fragment \( F \) of \( L_{\infty,\omega} \), \( \text{Th}_F(M) \) is the set of sentences (closed formulae) of \( F \) that hold in \( M \), and \( \text{ZF}(F) \) is the natural extension of \( \text{ZF} \) in which the usual schemes of separation and collection are extended to the schemes \( \text{Sep}(F) \) and \( \text{Coll}(F) \) so as to allow formulae in \( F \) to be used for “separating” and “collecting” (respectively).

(k) For \( \varphi \in L_{\infty,\omega} \), the depth of \( \varphi \), denoted \( \text{Depth}(\varphi) \), is the ordinal defined recursively by the following clauses:

1. \( \text{Depth}(\varphi) = 0 \) if \( \varphi \) is an atomic formula.
2. \( \text{Depth}(\varphi) = \text{Depth}(\psi) + 1 \) if \( \varphi = \neg \psi \).
3. \( \text{Depth}(\varphi) = \text{Depth}(\psi) + 1 \) if \( \varphi = \exists x \psi \).
4. \( \text{Depth}(\varphi) = \sup\{\text{Depth}(\psi) + 1 : \psi \in \Psi \} \), if \( \varphi = \land \Psi \).

(l) For an ordinal \( \alpha \) we use \( D(\alpha) \) to denote \( \{ \varphi \in L_{\infty,\omega} : \text{Depth}(\varphi) \leq \alpha \} \). Within KP, one can code each formula \( \varphi \in L_{\infty,\omega} \) with a set \( \langle \varphi \rangle \) as in Chapter 3 of [12], but in the interest of better readability we will often identify a formula with its code. This coding allows us to construe statements such as \( \varphi \in L_{\infty,\omega} \) and \( \text{Depth}(\varphi) = \alpha \) as statements in the first order language of set theory. It is easy to see that for a sufficiently large \( k \in \omega \), \( D(\alpha) \subseteq V(\omega + k\alpha) \) for each ordinal \( \alpha \). This makes it clear that \( L_{\mathcal{M}} = \bigcup_{\alpha \in \text{dom}(\mathcal{M})} D^\mathcal{M}(\alpha) \).

(m) Suppose \( \mathcal{M} \) is nonstandard and \( W := WF(\mathcal{M}) \). \( \mathcal{M} \) is \( W \)-saturated if for every \( k \in \omega \) and every type \( p(x, y_1, \ldots, y_k) \), and for every \( k \)-tuple \( \overline{a} \) of parameters from \( \mathcal{M} \), \( p(x, \overline{a}) \) is realized in \( \mathcal{M} \) provided the following three conditions are satisfied:

1. \( p(x, \overline{a}) \subseteq L_\mathcal{M} \).
2. \( p(x, \overline{a}) \in \text{Cod}_W(\mathcal{M}) \).
3. \( \forall \varphi \in W \mathcal{M} \models \exists x \left( \bigwedge_{\varphi \in \varphi(x, \overline{a})} \varphi(x, \overline{a}) \right) \).

(n) Every model of GB can be put in the form \( (\mathcal{N}, \mathcal{X}) \), where \( \mathcal{N} \models \text{ZF} \) and \( \mathcal{X} \subseteq \mathcal{P}(\mathcal{N}) \).

2.2. Definition. Suppose \( \mathcal{M} \) is a model of ZF, and \( S \subseteq M \).

(a) \( S \) is separative (over \( \mathcal{M} \)) if \( (\mathcal{M}, S) \) satisfies the separation scheme \( \text{Sep}(S) \) in the extended language that includes a fresh predicate \( S \) (interpreted by \( S \)).

(b) \( S \) is collective (over \( \mathcal{M} \)) if \( (\mathcal{M}, S) \) satisfies the collection scheme \( \text{Coll}(S) \) in the extended language that includes a fresh predicate \( S \) (interpreted by \( S \)).
(c) $S$ is amenable (over $\mathcal{M}$) if $S$ is both separative and collective. In other words, $S$ is amenable if $(\mathcal{M}, S)$ satisfies the replacement scheme $\text{Repl}(S)$ in the extended language that includes a fresh predicate $S$ (interpreted by $S$). Note that if $(\mathcal{M}, \mathfrak{X})$ is a model of GB, then each element of $\mathfrak{X}$ is amenable over $\mathcal{M}$.

(d) For $\alpha \in \text{Ord}^\mathcal{M}$, $S$ is an $\alpha$-satisfaction class (over $\mathcal{M}$) if $S$ correctly decides the truth of atomic sentences, and $S$ satisfies Tarski’s compositional clauses of a truth predicate for $D^{\mathcal{M}}_\alpha(\alpha)$-sentences (see below for the precise definition). $S$ is an $\infty$-satisfaction class over $\mathcal{M}$ if $S$ is an $\alpha$-satisfaction class over $\mathcal{M}$ for every $\alpha \in \text{Ord}^\mathcal{M}$.

We elaborate the meaning of (d) above. Reasoning within ZF, for each object $a$ in the universe of sets, let $c_a$ be a constant symbol denoting $a$ (where the map $\alpha \mapsto c_a$ is $\Delta_1$), and let $\text{Sent}^+(\alpha, x)$ be the set-theoretic formula (with an ordinal parameter $\alpha$ and the free variable $x$) that defines the proper class of sentences of the form $\varphi(c_{a_1}, \cdots, c_{a_n})$, where $\varphi(x_1, \cdots, x_n) \in D(\alpha)$ (the superscript $+$ on $\text{Sent}^+(\alpha, x)$ indicates that $x$ is a sentence in the language augmented with the indicated proper class of constant symbols). Then $S$ is an $\alpha$-satisfaction class over $\mathcal{M}$ if $(\mathcal{M}, S) \models \text{Sat}(\alpha, S)$, where $\text{Sat}(\alpha, S)$ is the conjunction of the universal generalizations of axioms (I) through (IV) below:

\begin{enumerate}
  \item[(I)] $(S(\bar{c}_a = c_b) \leftrightarrow a = b) \land (S(\bar{c}_a \in c_b) \leftrightarrow a \in b)$.
  \item[(II)] $(\text{Sent}^+(\alpha, \varphi) \land (\varphi = \neg \psi)) \rightarrow (S(\varphi) \leftrightarrow \neg S(\psi))$.
  \item[(III)] $(\text{Sent}^+(\alpha, \varphi) \land (\varphi = \bigwedge \Psi)) \rightarrow (S(\varphi) \leftrightarrow \forall \psi \in \Psi S(\psi))$.
  \item[(IV)] $(\text{Sent}^+(\alpha, \varphi) \land (\varphi = \exists x \psi(x))) \rightarrow (S(\varphi) \leftrightarrow \exists x S(\psi(c_x)))$.
\end{enumerate}

(e) For $\alpha < o(\mathcal{M})$, $S$ is the $\alpha$-satisfaction class over $\mathcal{M}$, if $S$ is the usual Tarskian satisfaction class for formulae in $L_\mathcal{M}$ of depth less than $\alpha$, i.e., the unique $\alpha$-satisfaction class $S$ over $\mathcal{M}$ such that $S$ satisfies: (V) $\forall x (S(x) \rightarrow \text{Sent}^+(\alpha, x))$.

Finally, the $o(\mathcal{M})$-satisfaction class over $\mathcal{M}$ is the usual Tarskian satisfaction class for formulae in $L_\mathcal{M}$ of depth less than $o(\mathcal{M})$, i.e., the union of all Tarskian $\alpha$-satisfaction classes over $\mathcal{M}$ as $\alpha$ ranges in $o(\mathcal{M})$.

- In the interest of a lighter notation, if $S$ is an $\alpha$-satisfaction class over $\mathcal{M}$ (for a possibly nonstandard $\alpha \in \text{Ord}^\mathcal{M}$), $\varphi(x_1, \cdots, x_n)$ is an $n$-ary formula of $D^\mathcal{M}(\alpha)$, and $a_1, \cdots, a_n$ are in $\mathcal{M}$, we will often write $\varphi(a_1, \cdots, a_n) \in S$ instead of $\varphi(c_{a_1}, \cdots, c_{a_n}) \in S$.

The following proposition is immediately derivable from the relevant definitions.

2.3. Proposition. If $S$ is an $\alpha$-satisfaction class over $\mathcal{M}$ for some nonstandard ordinal $\alpha$ of $\mathcal{M}$, then for all $n$-ary formula $\varphi(x_1, \cdots, x_n)$ of $L_\mathcal{M}$ and all $n$-tuples $(a_1, \cdots, a_n)$ from $\mathcal{M}$, we have:

$$\mathcal{M} \models \varphi(a_1, \cdots, a_n) \iff \varphi(a_1, \cdots, a_n) \in S.$$  

In particular, for all sentences $\varphi$ of $L_\mathcal{M}$, $\varphi \in S$ iff $\varphi \in \text{Th}_{L_\mathcal{M}}(\mathcal{M})$.

2.4. Remark. Reasoning within ZF, given any limit ordinal $\gamma$, $(V(\gamma), \epsilon)$ carries a separative $\gamma$-satisfaction class $S$ since we can take $S$ to be the Tarskian satisfaction class on $(V(\gamma), \epsilon)$ for formulae of depth less than $\gamma$. More specifically, the Tarski recursive construction/definition of truth works equally well in this more general context of infinitary languages since $(V(\gamma), \epsilon)$ forms a set. Observe that $(V(\gamma), \epsilon, S) \models \text{Sep}(S)$ comes “for free” since for any $X \subseteq V(\gamma)$ the expansion $(V(\gamma), \epsilon, X)$ satisfies the scheme of separation in the extended language.

2.5. Proposition. [En] Proposition 2.5] If $\mathcal{M} \models \text{KP}$, then for each nonzero $\alpha \in o(\mathcal{M})$ there is a formula $\text{Sat}_\alpha(x) \in L_\mathcal{M}$ such that $\text{Sat}_\alpha^\mathcal{M}(x)$ is the $\alpha$-satisfaction class over $\mathcal{M}$.

The following general version of the elementary chains theorem of model theory can verified by a routine adaptation of the usual proof of the $L_{\omega,\omega}$-version (e.g., as in [CK] Theorem 3.1.9).
2.6. Proposition. (Elementary Chains) Suppose $\mathbb{L} \subseteq \mathbb{L}_{\infty, \omega}$ where $\mathbb{L}$ is closed under subformulae; $(I, \prec)$ is a linear order; $(\mathcal{M}_i : i \in I)$ is an $\mathbb{L}$-elementary chain (i.e., $\mathcal{M}_i \prec \mathcal{M}_j$ whenever $i \prec j$); and $\mathcal{M} = \bigcup_{i \in I} \mathcal{M}_i$. Then for any $\varphi$ over $D$, below can be readily obtained by putting Proposition 2.6 above together with Proposition 2.8 of $[En]$. The closed unboundedness of the class of $\varphi$-reflective ordinals is needed in the proof of Lemma 4.3 of this paper, where it is important to arrange arbitrarily large $\varphi$-reflective ordinals of countable cofinality.

2.7. Proposition. (Reflection) Suppose $\mathcal{M} \models \text{ZF}(\mathbb{L}_\mathcal{M})$, and for each $\varphi \in \mathbb{L}_\mathcal{M}$ where $\varphi$ is $n$-ary, let $\text{Ref}_\varphi(\gamma)$ be the $\mathbb{L}_\mathcal{M}$-formula:

$$\forall x_1 \in V(\gamma) \cdots \forall x_n \in V(\gamma) [\varphi(x_1, \cdots, x_n) \iff \varphi^{V(\gamma)}(x_1, \cdots, x_n)].$$

Then for any $\alpha \in o(\mathcal{M})$ there is a closed unbounded collection of ordinals $\gamma \in \text{Ord}^{\mathcal{M}}$ such that $\mathcal{M}(\gamma)$ reflects all formulae in $D^{\mathcal{M}(\alpha)}$, i.e., $\mathcal{M} \models \text{Ref}_\varphi(\gamma)$ for all $\mathbb{L}_\mathcal{M}$-formulae $\varphi$ of depth less than $\alpha$.

The notion “$f$ is $\lambda$-onto $X$” introduced in Definition 2.8 below, and the corresponding existence result (Proposition 2.9), are adaptations of Definition 2.1 and Lemma 2.2 of Kaufmann and Schmerl’s work $[KS]$ on models of arithmetic to the setting of set theory. Lemma 2.2 of $[KS]$ played a key role in the proof presented in $[ES]$ of the direction $(b) \Rightarrow (a)$ of Theorem 1.1. Proposition 2.9 plays an analogous role in the proof of the direction $(b) \Rightarrow (a)$ of Theorem A.

2.8. Definition. A set $I$ is an ordinal interval if $I = \{\gamma : \alpha < \gamma < \beta\}$ for some ordinals $\alpha$ and $\beta$. Suppose $f : I \to X$, where $I$ is an ordinal interval, $X$ is some set, and $\lambda$ is an ordinal. The notion $f$ is $\lambda$-onto $X$ is defined by recursion on $\lambda$ as follows:

- $f$ is 0-onto $X$ means: $f$ is onto $X$.

- For $\lambda = \gamma + 1$, $f$ is $\lambda$-onto $X$ means: for each $Y \subseteq X$ there is an ordinal interval $J \subseteq I$ such that $f \restriction J$ is $\gamma$-onto $Y$.

- For a limit ordinal $\lambda$, $f$ is $\lambda$-onto $X$ means: $\forall \gamma < \lambda f$ is $\gamma$-onto $X$.

2.9. Proposition. (ZFC) Given any set $X$ and any ordinal $\lambda$ there is some ordinal interval $I$ and a function $f : I \to X$ such that $f$ is $\lambda$-onto $X$.

Proof. We use induction on $\lambda$. The case $\lambda = 0$ is clear since we are working in ZFC. If $\lambda = \gamma + 1$, then for each subset $Y$ of $X$ there is an ordinal interval $I_Y$ and some function $h_Y : I_Y \to Y$ such that $h_Y$ is $\gamma$-onto $Y$. Use AC to enumerate $\mathcal{P}(X)$ as $\{Y_\alpha : \alpha < \kappa\}$, where $\kappa = |\mathcal{P}(X)|$. Let $I_\alpha := I_{Y_\alpha}$ and $h_\alpha := h_{Y_\alpha}$ for each $\alpha < \kappa$ and choose an ordinal interval $I$ that is order isomorphic to the well-ordering $\sum_{\alpha < \kappa} I_\alpha$. More explicitly, let $Z = \{(\alpha) \times I_\alpha : \alpha < \kappa\}$ and let $\prec$ be the lexicographic order on $Z$. Then since $\prec$ is a well-ordering, there is an ordinal interval $I$ and an isomorphism $F$ between $(Z, \prec)$ and $(I, \in)$. Note that $F((\alpha) \times I_\alpha) \cap F((\beta) \times I_\beta) = \emptyset$ when $\alpha$ and $\beta$ are distinct elements of $\kappa$. Since for each $\alpha < \kappa$ the function $h_\alpha : I_\alpha \to Y_\alpha$ has the property of being $\lambda$-onto $Y_\alpha$, the isomorphism $F$ allows us to construct functions $f_\alpha : F((\alpha) \times I_\alpha) \to Y_\alpha$ such that each $f_\alpha$ is $\lambda$-onto $Y_\alpha$. This will ensure that $\bigcup_{\alpha < \kappa} f_\alpha$ is a function from $I$ to $X$ that is $\gamma + 1$-onto $X$.

For limit $\lambda$ we use a strategy similar to the successor case. By inductive assumption, for each $\gamma < \lambda$ there is some ordinal interval $I_\gamma$ and a function $f_\gamma : I_\gamma \to X$ such that $f_\gamma$ is $\gamma$-onto $X$. We can therefore
find an ordinal interval \( I \) and an isomorphism \( F \) between the well-ordering \( \sum_{\gamma<\lambda} I_\gamma \) and \( I \). For each \( \gamma < \lambda \) we can then construct functions \( f_\gamma : F(\{\gamma\} \times I_\gamma) \to X \) such that \( f_\gamma \) is \( \gamma \)-onto \( X \). It is evident that \( \bigcup_{a<\lambda} f_\alpha \) is a function from \( I \) to \( X \) that is \( \lambda \)-onto \( X \). \( \square \)

2.10. Remark. Recall that \( \Sigma_k^1 \)-AC (AC for the axiom of choice) is the scheme consisting of formulae of the form

\[
\forall x \exists X \psi(x, X) \rightarrow \exists Y \forall x \psi(x, (Y)_x),
\]

where \( \psi(x, X) \) is a \( \Sigma_k^1 \)-formula (parameters allowed), and \( \Sigma_k^1 \)-Coll (Coll for Collection) is the scheme consisting of formulae of the form

\[
\forall x \exists X \psi(x, X) \rightarrow \exists Y \forall x \exists y \psi(x, (Y)_y),
\]

where \( \psi(x, X) \) is a \( \Sigma_k^1 \)-formula (again, with parameters allowed). In the above

\[
(Y)_x := \{ y : \langle x, y \rangle \in Y \},
\]

where \( \langle x, y \rangle \) is a canonical pairing function.

(a) It is well-known that \( \Sigma_k^1 \)-AC implies \( \Delta_k^1 \)-CA for all \( k < \omega \); an easy proof in the arithmetical setting can be found in [Si, Lemma VII.6.6(1)]; the same proof readily works in the set-theoretic context.

(b) Let GBC be the result of augmenting GB with the global axiom of choice. It is well-known that in the presence of GBC, (1) \( \Sigma_k^1 \)-AC is equivalent to \( \Sigma_k^1 \)-Coll, and (2) global choice is provable in GB + \( \Sigma_k^1 \)-AC. For more detail, see, e.g., [Fu, Section 3.1].

Let \( \mathcal{L}_{\infty, \omega} \) be the extension of \( \mathcal{L}_{\infty, \omega} \) based on the extended vocabulary \( \mathcal{L}_{\text{set}} = \{ =, \in, f \} \), where \( f \) is a function symbol (for a global choice function), and let \( \mathcal{L} = \mathcal{L}_{\infty, \omega} \cap \text{WF}(\mathcal{N}) \), where \( \mathcal{N} \models \text{ZF} \) (\( \mathcal{N} \) need not be nonstandard, so \( \text{WF}(\mathcal{N}) \) might be the whole of \( \mathcal{N} \)). The following result is the infinitary generalization of the well-known theorem that global choice can be generically added to models of ZFC of countable cofinality [Fe] (and its proof is similar to the proof of the finitary case). A proof of part (b) of Proposition 2.11 can be found in [Sch, Theorem 11].

2.11. Proposition. (Forcing Global Choice) Let \( \mathcal{N} \models \text{ZFC}(\mathcal{L}_\mathcal{N}) \), and \( \mathbb{P} \) be the class notion of forcing consisting of set choice functions in \( \mathcal{N} \), ordered by set inclusion.

(a) If \( \text{Ord}^\mathcal{N} \) has countable cofinality, then there is an \( \mathcal{L}_\mathcal{N} \)-generic filter \( G \subseteq \mathbb{P} \), in the sense that \( G \) is a filter that intersects every dense subset of \( \mathbb{P} \) that is definable in \( \mathcal{N} \) by a formula in \( \mathcal{L}_\mathcal{N} \) (parameters allowed).

(b) If \( \mathcal{G} \) is an \( \mathcal{L}_\mathcal{N} \)-generic filter over \( \mathbb{P} \), and \( f = \cup \mathcal{G} \), then \( f \) is a global choice function over \( \mathcal{N} \), and \( (\mathcal{N}, f) \models \text{ZF}(\mathcal{L}_\mathcal{N}) \).
3. PROOF OF THEOREM A

In this section we establish the first main result of our paper. In part (b) of the following theorem, Def$L_M$ is the family of $L_M$-definable subsets of $M$ (parameters allowed).

3.1. Theorem. The following are equivalent for a nonstandard model $M$ of ZF of any cardinality:

(a) $M(\alpha) \preceq_{L_M} M$ for an unbounded collection of $\alpha \in \text{Ord}^M$.

(b) $(M, \mathfrak{X}) \models GB + \Delta^1_1$-$\text{CA}$, for $\mathfrak{X} = \text{Def}_{L_M}$.

(c) There is $\mathfrak{X}$ such that $(M, \mathfrak{X}) \models GB + \Delta^1_1$-$\text{CA}$.

Proof. Since (b) $\Rightarrow$ (c) is trivial, it suffices to establish (a) $\Rightarrow$ (b) and (c) $\Rightarrow$ (a).

(a) $\Rightarrow$ (b). Assume (a). Then by (d) $\iff$ (e) of Theorem 1.3, we have:

(1) $M$ satisfies ZF($L_M$), and

(2) $M$ is $W$-saturated.

(1) makes it clear that GB holds in $(M, \mathfrak{X})$. We will use (2) to show that $\Delta^1_1$-$\text{CA}$ holds in $(M, \mathfrak{X})$. To this end, let $U \subseteq M$ such that $U$ is defined in $(M, \mathfrak{X})$ by a $\Sigma^1_1$-formula $\exists X \psi^+(X, x, A)$, and $M \setminus U$ is defined in $(M, \mathfrak{X})$ by a $\Sigma^1_1$-formula $\exists X \psi^-(X, A, x)$, where $A \in \mathfrak{X}$ is a class parameter definable by the $L_M$-formula $\alpha(m, v)$. Here $m \in M$ is a set parameter; note that we may assume without loss of generality that the only parameter in $\psi^+$ and in $\psi^-$ is a class parameter $A$. Consider the infinitary formulae $\theta^+(x)$ and $\theta^-(x)$ defined as follows:

\[
\theta^+(x) := \bigvee_{\varphi(y,v) \in L_M} \exists y \psi^+(X/\varphi(y,v), A/\alpha(m,v), x); \text{ and } \\
\theta^-(x) := \bigvee_{\varphi(y,v) \in L_M} \exists y \psi^-(X/\varphi(y,v), A/\alpha(m,v), x).
\]

In the above $\psi^+(X/\varphi(y,v), A/\alpha(m,v), x)$ (respectively $\psi^-(X/\varphi(y,v), A/\alpha(m,v), x)$) is the result of replacing all occurrences of subformulae of the form $w \in X$ (where $w$ is a variable) in $\psi^+$ (respectively in $\psi^-$) by $\varphi(y,w)$, and replacing all occurrences of subformulae of the form $w \in A$ in $\psi^+$ (respectively in $\psi^-$) by $\alpha(m,w)$, and re-naming variables to avoid unintended clashes. Since each $X \in \mathfrak{X}$ can be written in the form $\{v \in M : M \models \varphi(m_1, v)\}$ (where $m_1 \in M$ is a parameter), $U$ is definable in $M$ by $\theta^+(x)$ and $M \setminus U$ is definable in $M$ by $\theta^-(x)$. Therefore we have:

(3) $M \models \forall x (\theta^+(x) \lor \theta^-(x))$.

Next, we aim to verify (4) below. In what follows D$^M(\alpha)$ is as in part (k) of Definition 2.1.

(4) There is some $\alpha \in o(M)$ such that $M \models \forall x (\theta^+_\alpha(x) \lor \theta^-\alpha(x))$, where

\[
\theta^+_\alpha(x) := \bigvee_{\varphi(y,v) \in D^M(\alpha)} \exists y \psi^+(X/\varphi(y,v), A/\alpha(m,v), x); \text{ and } \\
\theta^-\alpha(x) := \bigvee_{\varphi(y,v) \in D^M(\alpha)} \exists y \psi^-(X/\varphi(y,v), A/\alpha(m,v), x).
\]

Notice that (4) implies that $U$ is definable in $M$ by $\theta^+_\alpha(x)$, so the verification of $\Delta^1_1$-$\text{CA}$ will be complete once we establish (4) since $\theta^+_\alpha(x) \in L_M$ and $\{a \in M : M \models \theta^+_\alpha(a)\} \subseteq \mathfrak{X}$. To establish (4) we argue by contradiction. Suppose

(5) $M \models \exists x \neg (\theta^+_\alpha(x) \lor \theta^-\alpha(x))$ for each $\alpha \in o(M)$.

Consider the $L_M$-type $p(x)$, where
\[ p(x) := \{ - (\theta_\alpha^+(x) \lor \theta_\alpha^-(x)) : \alpha \in \text{o}(M) \}. \]

It is easy to see that \( p(x) \in \text{Cod}_W(M) \). By (5), for each \( \alpha \in \text{o}(M) \), \( p(x) \cap D^M(\alpha) \) is realized in \( M \), so by \( W \)-saturation of \( M \), \( p(x) \) is realized in \( M \), i.e., \( M \models \exists x \neg (\theta_\alpha^+(x) \lor \theta_\alpha^-(x)) \), which contradicts (3) and finishes the proof of (4).

(c) \( \Rightarrow \) (a). This is the hard direction of Theorem 3.1 and will require a good deal of preliminary lemmata. It will be proved as Lemma 3.6. In part (a) of Lemma 3.2, \( \text{Sat}_\alpha^M \) is as in Proposition 2.5.

3.2. Lemma. If \( (M, \mathfrak{X}) \models \text{GB} + \Delta^1_1 \text{-CA} \), then the following hold:

(a) \( \text{Sat}_\alpha^M \in \mathfrak{X} \) for each nonzero \( \alpha \in \text{o}(M) \).

(b) \( M \models \text{ZF}(L_M) \).

Proof. To see that (a) holds we will use induction on \( \alpha \) to verify that \( \text{Sat}_\alpha^M \) is \( \Delta^1_1 \)-definable in \( (M, \mathfrak{X}) \) for each \( \alpha \in \text{o}(M) \). Recall that \( \text{Sat}(\alpha, S) \) is the first order formula that expresses “\( S \) is an \( \alpha \)-satisfaction class” (as in Definition 2.2). Suppose \( \text{Sat}_\alpha^M \in \mathfrak{X} \) for some \( \alpha \in \text{o}(M) \). Then for each \( m \in M \) we have:

\[
\begin{align*}
m \in \text{Sat}_{\alpha+1}^M & \text{ iff } \\
(M, \mathfrak{X}) \models \exists S [\text{Sat}(\alpha, S) \land (\text{Depth}(m) < \alpha) \land (\text{Neg}(m) \lor \text{Exist}(m) \lor \text{Conj}(m))] \text{, where}
\end{align*}
\]

\[
\begin{align*}
\text{Neg}(x) := \exists y [[x = \neg y] \land \neg S(y)]; \\
\text{Exist}(x) := \exists y \exists v [[x = \exists v y(v)'] \land \exists v S(y(c_v))]; \text{ and} \\
\text{Conj}(x) := \exists y [(x = \neg y') \land \forall z \in y S(z)].
\end{align*}
\]

Similarly, for each \( m \in M \) we have:

\[
\begin{align*}
m \in \text{Sat}_{\alpha+1}^M & \text{ iff } \\
(M, \mathfrak{X}) \models \forall S [\text{Sat}(\alpha, S) \land \text{Depth}(m) = \alpha) \rightarrow (\text{Neg}(m) \lor \text{Exist}(m) \lor \text{Conj}(m))].
\end{align*}
\]

Thus \( \text{Sat}_{\alpha+1}^M \) has both a \( \Sigma^1_1 \) and a \( \Pi^1_1 \) definition in \( (M, \mathfrak{X}) \). The limit case is more straightforward since for limit \( \alpha \) the following hold for each \( m \in M \):

\[
\begin{align*}
m \in \text{Sat}_\alpha^M & \text{ iff } (M, \mathfrak{X}) \models \exists \beta < \alpha [(\text{Depth}(m) = \beta) \land \exists S (\text{Sat}(\beta, S) \land S(m))], \text{ and} \\
m \in \text{Sat}_\alpha^M & \text{ iff } (M, \mathfrak{X}) \models \exists \beta < \alpha [(\text{Depth}(m) = \beta) \land \forall S (\text{Sat}(\beta, S) \rightarrow S(m))] \text{.}
\end{align*}
\]

This concludes the proof of (a). Note that (b) is an immediate consequence of (a) since the veracity of any \( L_M \)-instance of replacement in \( M \) follows from the amenability of \( \text{Sat}_\alpha^M \) over \( M \) for a sufficiently large \( \alpha \in \text{o}(M) \). \( \square \) (Lemma 3.2)

The notion of paradefinability introduced in Definition 3.3 below is the set-theoretical analogue of the notion of recursive \( \sigma \)-definability in \( \text{ES} \).

3.3. Definition. Suppose \( M \models \text{KP} \) and \( A \subseteq M \). Then, \( A \) is \text{paradefinable} in \( M \), if there is a sequence \( \langle \varphi_\alpha(x, \bar{y}) : \alpha < \text{o}(M) \rangle \in \text{Cod}_W(M) \) of \( L_M \)-formulae (where \( \bar{y} \) is a finite tuple whose length is independent of \( \alpha \)) such that for some fixed tuple of parameters \( \bar{m} \) in \( M \) of the same length as \( \bar{y} \), each \( \varphi_\alpha(x, \bar{m}) \) defines a subset \( A_\alpha \subseteq M \) (in \( M \)), with \( A = \bigcup_{\alpha < \text{o}(M)} \quad A_\alpha \). Under these conditions, we say that \( A \) is \text{paradefinable} by \( \langle \varphi_\alpha(x, \bar{m}) : \alpha < \text{o}(M) \rangle \).

3.4. Lemma. If \( M \models \text{KP} \), then the following are paradefinable in \( M \):

(a) \( \text{o}(M) \).

(b) \( \text{WF}(M) \).
(e) The $o(M)$-satisfaction class on $M$.

**Proof.**

(a) $o(M)$ is pardefinable in $M$ by $\langle E_\alpha(x) : \alpha < o(M) \rangle$, where $E_\alpha(x)$ (which defines $\{\alpha\}$) is constructed by recursion via:

$$E_0(x) := \forall y (y \notin x), \text{ and for } \alpha > 0, E_\alpha(x) := \forall y \left( y \in x \leftrightarrow \bigvee_{\lambda < \alpha} E_\lambda(y) \right).$$

(b) WF($M$) is pardefinable in $M$ by $\langle \exists y (E_\alpha(y) \land x \in V(y)) : \alpha < o(M) \rangle$.

(c) The $o(M)$-satisfaction class on $M$ is pardefinable in $M$ by $\langle Sat_\alpha(x) : \alpha < o(M) \rangle$. \square (Lemma 3.4)

Part (b) of the next lemma is the set-theoretical analogue of [ES Lemma 1 (b)].

3.5. **Lemma.** Suppose $(M, \bar{X}) \models GB + \Delta^1_1$-CA, for some nonstandard $M$ that is not $W$-saturated. Then:

(a) For all $\delta \in \text{Ord}^M (\delta \in o(M) \iff \exists S \in \bar{X} (M, S) \models \text{Sat}(\delta, S))$.

(b) If $A \subseteq M$ is pardefinable in $M$, then $A$ is $\Sigma^1_1$-definable in $(M, \bar{X})$.

**Proof.** To establish (a), first note that the assumption of the failure of $W$-saturation in $M$ by Theorem 1.3 implies that there is a $S \in \bar{X}$ such that $S$ is a $\gamma$-satisfaction class over $M$ for any nonstandard $\gamma \in \text{Ord}^M$. Combined with part (a) of Lemma 3.2, this makes it clear that (a) holds. To verify (b), let $A$ be pardefinable by $\langle \varphi_\alpha(x, \bar{m}) : \alpha < o(M) \rangle$. By replacing $\varphi_\alpha(x, \bar{y})$ with $\bigvee_{\beta \leq \alpha} \varphi_\beta(x, \bar{y})$, we can assume that $\text{Depth}(\varphi_\alpha(x, \bar{y})) < \text{Depth}(\varphi_\beta(x, \bar{y}))$ for all $\alpha < \beta < o(M)$. Let $\delta$ be a nonstandard element of $\text{Ord}^M$ such that $\langle \varphi_\alpha(x, \bar{y}) : \alpha < \delta \rangle$ is in $M$ and extends $\langle \varphi_\alpha(x, \bar{y}) : \alpha < o(M) \rangle$ and $\text{Depth}(\varphi_\alpha(x, \bar{y})) < o(M)$ for all $\alpha < o(M)$. Then $A$ is $\Sigma^1_1$-definable in $(M, \bar{X})$ by the formula $\exists X \theta(x, X)$, where

$$\theta(x, X) = \exists \gamma [\text{Sat}(\gamma, X) \land \exists \alpha < \delta \text{ Depth}(\varphi_\alpha(x, \bar{y})) < \gamma \land \varphi_\alpha(c_x, \bar{m}) \in X].$$

By part (a) of the lemma, this makes it evident that $A$ is $\Sigma^1_1$-definable in $(M, \bar{X})$. \square (Lemma 3.5)

- The proof of Theorem 3.1 will be complete once we verify Lemma 3.6 below, which takes care of the direction (c) $\Rightarrow$ (a) of Theorem 3.1. The proof of Lemma 3.6 is rather complicated and we therefore beg the reader’s indulgence.

3.6. **Lemma.** If $(M, \bar{X}) \models GB + \Delta^1_1$-CA and $M$ is nonstandard, then $M(\alpha) \prec L_M M$ for an unbounded collection of $\alpha \in \text{Ord}^M$.

**Proof.** Suppose not, then by Theorem 1.3, $M$ is not $W$-saturated, so by part (a) of Lemma 3.5 we conclude:

1. For all $\delta \in \text{Ord}^M (\delta \in o(M) \iff \exists S \in \bar{X} (M, S) \models \text{Sat}(\delta, S))$.

By our supposition there is some $\beta \in \text{Ord}^M$ such that:

2. There is no $\alpha \in \text{Ord}^M$ above $\beta$ with $M(\alpha) \prec L_M M$.

Since $M \models ZF(L_M)$, by Theorem 2.7 (Reflection), in the real world there is a sequence $\langle \gamma_\alpha : \alpha \in o(M) \rangle$ of ordinals of $M$ such that for each $\alpha \in o(M)$ the following holds:

$$M \models "\gamma_\alpha \text{ is the first ordinal } \gamma > \beta \text{ such that } V(\gamma) \prec_{D(\alpha)} V",$$

where $V(x) \prec_{D(\alpha)} V$ abbreviates the $L_M$-formula $\bigwedge_{\varphi \in D(\alpha)} \text{Ref}_x(\varphi)$ (Ref$_x$ is as in Proposition 2.7).

Let $\Gamma = \{\gamma_\alpha : \alpha \in o(M)\}$. Clearly $\gamma_\alpha \leq \gamma_\xi$ for $\alpha < \xi < o(M)$.
We now distinguish between the following two cases, and will show that each leads to a contradiction, thus proving Lemma 3.6 (recall that the proof of Lemma 3.6 starts with “Suppose not”). Our proof was inspired by the proof of [ES, Theorem 4], which the reader is highly advised to review before reading the proof below, especially since the argument for Case B below is a more complex version of the argument for the “tall case” in the proof of [ES, Theorem 4]. One of the reasons for this increase in complexity has to do with the fact that in nonstandard models of arithmetic (and in ω-nonstandard models of set theory) it is easy to find an ill-founded subset of the nonstandard ordinals of M that is paradefinable since we can choose A to be \{c - n : n ∈ ω\}, where c is any nonstandard finite ordinal. The existence of such an ill-founded A plays a key role in the proof of [KS, Lemma 2.4] since in conjunction with the arithmetical analogue of Proposition 2.9, it allows one to deduce that if some recursive type is omitted, then a recursive type consisting of formulae describing an ordinal interval is omitted. However, in a nonstandard model M of set theory that is ω-standard, the existence of an ill-founded paradefinable subset of the nonstandard ordinals of M takes far more effort to establish (with the help of additional assumptions, as indicated in the proof of Case B below).

**Case A:** Γ is cofinal in OrdM. We wish to show that Γ is Δ1-definable in (M, X). By part (b) of Lemma 3.5, it is sufficient to show that both Γ and its complement are paradefinable in M. The definition of Γ makes it clear that Γ is paradefinable in M by \( \langle \varphi^+_\alpha(x, \beta) : \alpha \in o(M) \rangle \), where \( \varphi^+_\alpha(x, \beta) \) is the following formula:

\[
(x ∈ Ord ∧ β ∈ x) ∧ (V(x) ≤_{D(α)} V) ∧ ∀y ∈ x \neg (V(y) <_{D(α)} V).
\]

To see that the complement of Γ is also paradefinable in M, observe that \( γ_α ≤ γ_β \) whenever \( α ≤ β < o(M) \), and by Proposition 2.6 (Elementary Chains) Γ is a closed subset of OrdM, i.e., for limit \( ξ ∈ o(M) \), \( γ_ξ = sup\{γ_α : α < ξ\} \). Thus for each \( ν ∈ OrdM \setminus Γ \), there is some \( α ∈ o(M) \) such that \( γ_α < ν < γ_{α+1} \). So the complement of Γ is paradefinable in M by \( \langle θ(x) ∨ ψ^-_\alpha(x, \beta) : \alpha ∈ o(M) \rangle \), where:

\[
θ(x) := x \notin Ord,
\]

\[
ψ^-_\alpha(x, \beta) := [∃ y (ψ^+_0(y, β) ∧ x ∈ y)], \quad \text{and}
\]

\[
ψ^-_\alpha(x, \beta) := [∃ y ∃ z (ψ^+_\alpha(y, β) ∧ ψ^+_{α+1}(z, β) ∧ (y ∈ x ∧ z))],
\]

Therefore Γ ∈ X, which implies that Γ is amenable over M, so coupled with the fact that Γ is cofinal in OrdM we conclude that there is some \( f ∈ X \) such that \( f \) is an isomorphism between Γ and OrdM (both ordered by ∈M). This contradicts the fact that Γ is well-founded and OrdM is ill-founded, and thus shows that Case A is impossible.

**Case B:** Γ is bounded in OrdM. In this case, by (2) the supremum of Γ does not exist in OrdM. Fix an upper bound \( δ ∈ OrdM \) for Γ, and for each \( α ∈ o(M) \) let

\[
ψ_α(x, β, δ) := (β ∈ x ∈ δ) ∧ (V(x) ≤_{D(α)} V).
\]

Note that if \( α \) and \( ξ \) are in \( o(M) \) with \( α ≤ ξ \), then \( M ⊨ ∀x (ψ_β(x, ξ, δ) → ψ_α(x, ξ, δ)) \). Consider the \( L_{M'} \)-type:

\[
p(x, β, δ) := \{ψ_α(x, β, δ) : α ∈ o(M)\}.
\]

Clearly:

(3) \( p(x, δ) ∈ Cod_Y(M), M ⊨ ∃ x ψ_α(x, β, δ) \) for each \( α ∈ o(M) \).

Moreover, (2) implies:

(4) \( M ⊨ ∀x \bigvee_{α ∈ o(M)} ¬ψ_α(x, β, δ) \).

In the real world define \( \langle δ_α : α ∈ o(M) \rangle \) with:
\[ \delta_0 = \delta, \text{ and } \delta_\alpha = \max \{ \xi \in \delta : V(\xi) \not\prec_{D(\alpha)} V \}. \]

It is easy to see that \(\delta_\alpha\)s are well-defined for each \(\alpha \in o(\mathcal{M})\). More specifically, let \(X_\alpha = \{ \xi \in \delta : V(\xi) \not\prec_{D(\alpha)} V \} \). Then by the choice of \(\delta\), \(X_\alpha\) is nonempty for each \(\alpha \in o(\mathcal{M})\); and by part (b) of Lemma 3.2, \(X_\alpha\) is coded in \(\mathcal{M}\), so \(\sup (X_\alpha)\) is well-defined, and by Proposition 2.6 (Elementary Chains) \(\sup (X_\alpha) \in X_\alpha\), so \(X_\alpha\) is well-defined. It should also be clear that:

5. \(\{ \delta_\alpha : \alpha \in o(\mathcal{M}) \}\) is parodefined in \(\mathcal{M}\),
6. \(\delta_\alpha \geq \delta_\beta\) if \(\alpha \leq \beta \in o(\mathcal{M})\), and
7. \(\delta_\alpha > \gamma_\nu > \beta\) if \(\nu \in o(\mathcal{M})\) and \(\alpha \in o(\mathcal{M})\).

Next, we observe:

8. For each \(\alpha \in o(\mathcal{M})\) \(\exists \beta \in o(\mathcal{M})\) such that \(\beta > \alpha \) and \(\delta_\alpha > \delta_\beta\).

To see that (8) is true, note that \(\delta_\alpha\) is parodefined by (7), so if (8) is false, then \(V(\delta_\alpha) \not\prec_{D(\beta)} V\) for all \(\beta \in o(\mathcal{M})\), which contradicts (2). Thus (8) implies that \(\{ \delta_\alpha : \alpha \in o(\mathcal{M}) \}\) is ill-founded when viewed as a subset of \(Ord^\mathcal{M}\). Moreover, for any \(\alpha \in o(\mathcal{M})\), \(\{ \delta_\beta : \beta < \alpha \}\) is finite. To verify this, first note that there is a fixed natural number \(k\) such that the depth of the \(\mathbb{L}_\mathcal{M}\)-formula that defines \(\delta_\beta\) for any \(\beta \in o(\mathcal{M})\) is at most \(\beta + k\). Therefore in light of (5) and (6) and the fact that \(Sat^\mathcal{M}_\alpha\) (which is present in \(\mathcal{X}\) by part (a) of Lemma 3.2. and is therefore amenable over \(\mathcal{M}\)) can evaluate the defining formulae of \(\{ \delta_\beta : \beta < \alpha \}\), the well-foundedness of \(Ord^\mathcal{M}\) as viewed in \(\mathcal{M}\) implies that \(\{ \delta_\beta : \beta < \alpha \}\) is finite from the point of view of \(\mathcal{M}\). Therefore \(\{ \delta_\beta : \beta < \alpha \}\) is finite in the real world (this is clear if \(\mathcal{M}\) is \(\omega\)-standard; if \(\mathcal{M}\) is \(\omega\)-nonstandard, then it is trivial since \(\alpha\) would have to be a finite ordinal since \(\alpha \in o(\mathcal{M})\)). Putting all this together, we conclude:

9. The order type of \(\{ \delta_\alpha : \alpha \in o(\mathcal{M}) \}\) under \(\in^\mathcal{M}\) is \(\omega^*\) (i.e., the reversal of \(\omega\)).

We will now use the sequence \(\langle \delta_\alpha : \alpha \in o(\mathcal{M}) \rangle\) to describe a type \(\overline{p}(x, \beta, \delta)\) such that \(\overline{p}(x, \overline{y}) \in \text{Cod}_W(\mathcal{M})\), with \(\overline{p}(x, \overline{y}) = \{ \overline{\psi}_\alpha(x, \overline{y}) : \alpha \in o(\mathcal{M}) \}\), where each \(\overline{\psi}_\alpha(x, \beta, \delta)\) describes an ordinal interval, i.e.,

\[ \overline{\psi}_\alpha(x, \beta, \delta) := s_\alpha(\beta, \delta) < x < t_\alpha(\beta, \delta), \]

for an appropriate choice of \(\mathbb{L}_\mathcal{M}\)-definable terms \(\langle s_\alpha(\beta, \delta) : \alpha \in o(\mathcal{M}) \rangle\) and \(\langle t_\alpha(\beta, \delta) : \alpha \in o(\mathcal{M}) \rangle\), where each \(s_\alpha(\beta, \delta)\) and \(t_\alpha(\beta, \delta)\) is in \(Ord^\mathcal{M}\), and \(s_\alpha\) and \(t_\alpha\) are defined below.

Let \(I(x, y) = \{ z : x \in z \in y \}\). In \(\mathcal{M}\), define \(X\) as the ordinal interval \(I(\beta, \delta)\), and apply Proposition 2.9 to get hold of a function \(f\) and some ordinal interval \(I\) such that \(f : I \to X\) and \(f\) is \(\langle\rangle\)-onto \(X\). Let \(I_0\) be the \(\langle\rangle\)-first ordinal interval \(I\) that supports such a function, where \(\langle\rangle\\) is a canonical well-ordering of all ordinal subintervals. Then define \(s_0\) and \(t_0\) so that \(I_0 = I(s_0(\beta, \delta), t_0(\beta, \delta))\). For \(\alpha > 0\) we define \(s_\alpha\) and \(t_\alpha\) by recursion on \(\alpha\):

- If \(\alpha\) is a successor ordinal \(\lambda + 1\), then \(s_\alpha(\beta, \delta)\) and \(t_\alpha(\beta, \delta)\) are respectively the left and right end points of the \(\langle\rangle\)-first ordinal subinterval \(I\) of the ordinal interval \(I(s_\lambda(\beta, \delta), t_\lambda(\beta, \delta))\) such that \(f \upharpoonright I\) is \(\delta_{\lambda}\)-onto \(\{ x \in I(\beta, \delta) : \psi_\lambda(x, \beta, \delta) \}\).

- If \(\alpha\) is a limit ordinal, then \(s_\alpha(\beta, \delta)\) and \(t_\alpha(\beta, \delta)\) are respectively the left and right end points of the \(\langle\rangle\)-first (ordinal) subinterval \(I\) of \(I(s_{\lambda_0}(\beta, \delta), t_{\lambda_0}(\beta, \delta))\) such that \(f \upharpoonright I\) is \(\delta_{\lambda_0}\)-onto \(\{ x : \psi_\alpha(x, \beta, \delta) \}\), where \(\lambda_0\) is the first ordinal below \(\alpha\) for which the tail \(\langle \delta_\lambda : \lambda_0 \leq \lambda < \alpha \rangle\) is a constant sequence.

Next we will show:

10. \(\mathcal{M} \models \exists x \overline{\psi}_\alpha(x, \beta, \delta)\) for each \(\alpha \in o(\mathcal{M})\).
Naturally, we use induction on $\alpha$ to verify (10). Proposition 2.9 and part (b) of Lemma 3.2 make it clear that the induction smoothly goes through for the base case and the successor case. The limit case requires the additional fact that if $\alpha$ is a limit ordinal, then by (9) there is some $\lambda_0 < \alpha$ such that the tail $\langle \delta_\lambda : \lambda_0 \leq \lambda < \alpha \rangle$ is a constant sequence.

Finally we will establish:

(11) $\hat{p}(x, \beta, \delta)$ is not realized in $M$.

To verify (11) recall that within $M$, $f$ maps each interval $I(s_\alpha(\beta, \delta), t_\alpha(\beta, \delta))$ into $\{x : \psi_\alpha(x, \beta, \delta)\}$. Therefore if some element $m$ of $M$ realizes $\hat{p}(x, \beta, \delta)$, then $f(m)$ realizes $p(x, \beta, \delta)$, which contradicts (4). We are now finally ready to wrap up the proof. Let

$I := \{x \in \text{Ord}^M : \exists \alpha \in o(M) (x < s_\alpha(\beta, \delta))\}$.

It is evident that $I$ is paradefinable in $M$. The complement of $I$ can written as:

$M \setminus I = \{x : x \notin \text{Ord}^M \lor \exists \alpha \in o(M) (x > t_\alpha(\beta, \delta))\}$,

which makes it clear that $M \setminus I$ is also paradefinable in $M$. Therefore by part (b) of Lemma 3.5 both $I$ and its complement are $\Sigma_1^1$-definable in $(M, \mathcal{X})$ and thus $I \in \mathcal{X}$, which implies that the supremum of $I$ exists in $M$ (since each element of $\mathcal{X}$ is separative over $M$). This contradicts (11) and concludes the demonstration that Case B is impossible.

4. PROOF OF THEOREM B

In this section we establish the second main result of this paper.

4.1. Theorem. The following are equivalent for a countable nonstandard model $M$ of ZFC:

(a) $M(\alpha) \prec_{L^M} M$ for an unbounded collection of $\alpha \in \text{Ord}^M$.

(b) There is $\mathcal{X}$ such that $(M, \mathcal{X}) \models \text{GB} + \Delta^1_1\text{-CA} + \Sigma^1_1\text{-AC}$.

Proof. Suppose $M$ is a nonstandard model of ZFC. By Theorem 3.1 $(b) \Rightarrow (a)$ holds, so we will focus on establishing $(a) \Rightarrow (b)$. This will be done in two stages.

Stage 1. We use forcing with set choice functions (as in Proposition 2.11) to expand $M$ to a model $(M, f)$ that satisfies the following properties:

1. $f$ is a global choice function over $M$, and $(M, f) \models \text{ZF}(\mathcal{L}_M)$.

2. $(M, f)$ is $W$-saturated.

Part (b) of Proposition 2.11 assures us that (1) holds. The verification of (2) involves a careful choice of the generic choice function. For this purpose we first verify Lemmas 4.2 and 4.3 below. In Lemma 4.2 the expression “$\alpha$ is a Beth-fixed point” means that $\alpha = \beth(\alpha)$, where $\beth$ is the Beth function. It is well-known that $\alpha$ is a Beth-fixed point iff $V(\alpha)$ is a $\Sigma_1$-elementary submodel of the universe $V$ of sets.

4.2. Lemma. (ZFC) If $\alpha$ is a Beth-fixed point and $\alpha$ has countable cofinality, and $\mathcal{N} := (V(\alpha), \in)$, then there is an $\mathcal{L}_N$-generic global choice function $f$ over $\mathcal{N}$.

Proof. This is a minor variant of part(a) of Proposition 2.11 (Forcing Global Choice).

4.3. Lemma. If $M$ is a $W$-saturated model of $\text{ZF}(\mathcal{L}_M)$, then $M(\alpha) \prec_{L^M} M$ for an unbounded collection of $\alpha \in \text{Ord}^M$ such that $M \models \text{cf}(\alpha) = \omega$.

Proof. Fix a nonstandard $\delta \in \text{Ord}^M$ and consider the type $p(x, \delta)$ (where $\delta$ is treated as a parameter) consisting of the formula
(δ ∈ x) ∧ (x ∈ Ord) ∧ (cf(x) = ω),

together with formulae of the form Refφ(x) as in Theorem 2.7 (Reflection), where φ ranges in L_M. It is easy to see that \( p(x, y) \) satisfies conditions (m1) and (m2) of part (m) of Definition 2.1. Moreover, by Proposition 2.7 (Reflection), \( p(x, δ) \) also satisfies condition (m3) of the same definition (since each closed and unbounded subset of ordinals has unboundedly many members of countable cofinality). Therefore by the assumption of \( W \)-saturation of \( M \), \( p(x, δ) \) is realized in \( M \) by some \( γ \), which makes it clear that \( γ \) is nonstandard and \( M(γ) ≪_{L_M} M \). □ (Lemma 4.3)

By Proposition 2.7 (Reflection) we can fix a sequence \( ⟨α_n : n < ω⟩ \) that is cofinal in \( Ord^M \) such that \( M(α_n) ≪_{L_M} M \) and \( M |= cf(α_n) = ω \). Then we build an \( L_M \)-generic choice function \( f \) over \( M \) by recursively building a sequence of conditions \( ⟨p_n : n < ω⟩ \), as we shall explain. Thanks to Lemma 4.2 (applied within \( M \)) we can get hold of a condition \( p_1 \) whose domain is \( M(α_1) \) such that \( p_1 \) is \( L_{M(α_1)} \)-generic over \( M(α_1) \). Generally, given a condition \( p_n \) in \( M \) whose domain is \( M(α_n) \) and which is \( L_{M(α_n)} \)-generic over \( M(α_n) \), we can use Lemma 4.2 to extend \( p_n \) to a condition \( p_{n+1} \) whose domain is \( M(α_{n+1}) \), and which is \( L_{M(α_{n+1})} \)-generic over \( M(α_{n+1}) \). Then by the choice of \( ⟨α_n : n < ω⟩ \), the union \( f \) of these conditions \( ⟨p_n : n < ω⟩ \) will be \( L_{M} \)-generic over \( M \). Moreover, \( f \) will have the key property that \( f \upharpoonright M(α_n) \) is \( L_{M(α_n)} \)-generic over \( M(α_n) \) for every \( n < ω \) (and thus truth-and-forcing holds for each of these approximations). Then thanks again to truth-and-forcing, together with the fact that \( M(α_n) ≪_{L_M} M \) for each \( n < ω \), we can conclude:

\[
(∗) \quad (M(α_n), f \upharpoonright M(α_n)) ≪_{L_M} (M, f) \quad \text{for each } n < ω.
\]

More explicitly, suppose \( (M(α_n), f \upharpoonright M(α_n)) \models φ(a) \) for some \( L_M \)-formula \( φ(x) \) and some \( a ∈ M(α_n) \). Then for some condition \( p ∈ f \upharpoonright M(α_n) \), we have \( M(α_n) \models [p \models φ(a)] \), and thus by elementarity \( M \models [p \models φ(a)] \), which by genericity of \( f \) assures us that \( (M, f) \models φ(a) \). Note that \( (∗) \) guarantees that the adjunction of the global choice function \( f \) to \( M \) preserves \( W \)-saturation and concludes Stage 1 of the proof.

**Stage 2.** Let \( f \) be the global choice function constructed in Stage 1, and let \( X = Def_{L_M}(M, f) \), i.e., the family of subsets of \( M \) that are definable in \( (M, f) \) by some \( L_M \)-formula (parameters allowed).

We will treat \( f \) as a binary predicate so that variables are the only terms in \( \overline{L}_M \) (this will slightly simplify matters in the argument below). By part (b) of Proposition 2.11, \( (M, f) \models ZF(\overline{L}_M) \), which makes it clear that GBC holds in \( (M, X) \). Recall from part (a) of Remark 2.10 that \( Δ^1_1 \)-CA is provable in GB + \( Σ^1_1 \)-AC, and that in the presence of GBC, \( Σ^1_1 \)-AC is equivalent to \( Σ^1_1 \)-Coll. Hence in light of the fact that GBC holds in \( (M, X) \) the proof of (b) will be complete once we verify that \( Σ^1_1 \)-Coll holds in \( (M, X) \). For this purpose, suppose for some parameter \( A ∈ X \) we have

\[
(1) \quad (M, X) \models ∀x \exists X ψ(x, X, A).
\]

Let \( α(m, v) \) be the \( L_M \)-formula that defines \( A \), where \( m ∈ M \) is a set parameter. Then

\[
(2) \quad (M, X) \models ∀x \ θ(x), \text{ where}
θ(x) := \bigvee_{φ(w,v) ∈ L_M} \exists y \ ψ(x, X/φ(y,v), A/α(m,v)),
\]

and \( ψ(X/φ(y,v), A/α(m,v), x) \) is the result of replacing all occurrences of subformulae of the form \( w ∈ X \) (where \( w \) is a variable) in \( ψ \) by \( φ(w,v) \), and replacing all occurrences of subformulae \( w ∈ A \) in \( ψ \) by \( α(w,v) \). In these replacements, we will assume that some variables will be renamed to avoid unintended clashes.

Let \( D^M(α) \) consist of all formulae of \( \overline{L}_{∞,ω} \) of depth less than \( α \) that appear in \( M \). We claim that (3) below holds.

\[
(3) \quad \text{There is some } α ∈ o(M) \text{ such that } M \models ∀x \ θ_α(x), \text{ where}
\]

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Suppose (3) is false, then we have:

4. \( M \models \exists x \neg \theta_\alpha(x) \) for each \( \alpha \in o(M) \).

Consider the \( L_M \)-type \( p(x) := \{ \neg \theta_\alpha(x) : \alpha \in o(M) \} \). It is easy to see that \( p(x) \in \text{Cod}_W(M) \). By the assumption that (3) is false, for each \( \alpha \in o(M) \), \( p(x) \cap M(\alpha) \) is realized in \( M \), by \( W \)-saturation of \( M \), \( p(x) \) is realized in \( M \), i.e., \( M \models \exists x \neg \theta(x) \), which contradicts (2) and completes the verification of (3).

Let \( B = \text{Def}_{\mathcal{M}}(\alpha)(M, f) \), i.e., the subfamily of \( X \) consisting of subsets of \( M \) that are definable in \( (M, f) \) by some \( L_M \)-formula of depth less than \( \alpha \). Note that \( B \in X \) since there is some \( \beta \in o(M) \) with \( \beta > \text{Depth}(\sigma) \) for each \( \sigma \in \mathcal{D}_\alpha(M) \), and \( \text{Sat}_\beta(M, f) \in X \) by (a minor variant of) Proposition 2.5. Therefore, by (3) we have

\[
(5) \quad (M, X) \models \forall x \exists y \psi(x, X, (B)_y).
\]

By quantifying out \( B \), (5) readily yields

\[
(6) \quad (M, X) \models \exists Y \forall x \exists y \psi(x, Y, (Y)_y).
\]

This concludes the verification of \( \Sigma^1_1 \)-Collection in \( (M, X) \). \( \square \) (Theorem 4.1)

4.4. Remark. The proof of \((b) \Rightarrow (a)\) of Theorem 4.1 does not invoke the countability of \( M \), but the direction \((a) \Rightarrow (b)\) does, and indeed this direction of the theorem can fail for an uncountable model \( M \), e.g., if \( M \) is a recursively saturated rather classless model of \( \text{ZFC} + \forall x (V \neq \text{HOD}(x)) \), where \( \text{HOD}(x) \) is the class of sets that are hereditarily ordinal definable from the parameter \( x \). More explicitly, it is well-known that \( \text{ZFC} + \forall x (V \neq \text{HOD}(x)) \) is consistent, assuming that \( \text{ZF} \) is consistent.\(^2\) On the other hand, Kaufmann \([\text{K}]\) showed, using the combinatorial principle \( \Diamond_{\omega_1} \), that every countable model \( M_0 \) of \( \text{ZF} \) has an elementary end extension \( M \) that is recursively saturated and rather classless, and later Shelah \([\text{Sh}]\) used an absoluteness argument to eliminate \( \Diamond_{\omega_1} \). Here the rather classlessness of \( M \) means that if \( X \) is a subset of \( M \) that is piecewise coded in \( M \), then \( X \) is parametrically definable in \( M \) (\( X \) is piecewise coded in \( M \) means that for every \( \alpha \in \text{Ord}^M \), \( V^M(\alpha) \cap X \) is coded by an element of \( M \)), then \( X \) is parametrically definable in \( M \). Therefore if \( M \) is a recursively saturated rather classless model of \( \text{ZFC} + \forall x (V \neq \text{HOD}(x)) \), then by recursive saturation of \( M \), \( M \) satisfies condition \((a)\) of Theorem 4.1, but it does not satisfy condition \((b)\) of Theorem 4.1 since if \( M \) expands to a model \( (M, X) \) of \( \text{GB} + \Sigma^1_1 \)-\( \text{AC} \), then as pointed out in part \((b)\) of Remark 2.10, there is a global choice function \( F \) coded in \( X \). But the veracity of \( \text{GB} \) in \( (M, X) \) implies that \( F \) is piecewise coded in \( M \) and therefore \( F \) is parametrically definable in \( M \), which contradicts the fact that \( \forall x (V \neq \text{HOD}(x)) \) holds in \( M \).

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\(^2\)Easton proved (in his unpublished dissertation \([\text{Ea}]\)) that, assuming \( \text{Con}(\text{ZF}) \), there is a model \( M \) of \( \text{ZFC} \) which carries no \( M \)-definable global choice function for the class of pairs in \( M \); and in particular \( \exists x (V = \text{HOD}(x)) \) fails in \( M \). Easton’s theorem was exposited by Felgner \([\text{Fe}]\) p.231; for a more recent and streamlined account, see Hamkins’ MathOverflow answer \([\text{H}]\).
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