Possible Superfluidity  
in thin corrugated Annulus

Zotin K.-H. Chu

WIPM, 30, Xiao-Hong Shan West, Wuhan 430071, PR China and  
Math.-Phys. Centre, 4-601, Building C, Beijingcheng, Baixingkangcheng,  
Road Changsha, Wulumuqi 830013, China

Abstract

We revisit the persistent flow of a superfluid in a thin wavy-rough annulus. The existence of a phase memory around this thin corrugated annulus is shown to be responsible for the energy minima with a periodic dependence on the total momentum which is directly related to the quantization of circulation. We also illustrate the general features using the ideal Bose gas as an example.

1 Introduction

One of the most remarkable properties of macroscopic quantum systems is the phenomenon of persistent flow. In a superconductor, persistent flow is electrical current without resistance: current in a loop of superconducting wire will flow essentially forever [1]. In a superfluid such as liquid helium below the lambda point, the frictionless flow allows persistent circulation in a hollow toroid [2-3]. Although the dynamics of superfluids follows quite directly from the simple assumption that the quantum field of the particles has a mean value which may be treated as a macroscopic variable [4] (Anderson mentioned that it is as legitimate to treat the quantum field amplitude as a macroscopic dynamical variable as it is the position of a solid body; both represent a broken symmetry which, however, cannot be conveniently repaired until one gets to the stage of quantizing and studying the quantum fluctuations of the macroscopic behavior of the system).

To help researchers to easily understand the superfluidity in the ring-type geometry, in this short paper, we only consider the uniform flow around a circular wavy-rough (thin) annulus with the linear dimension of the cross section smaller than the mean radius \( R = (r_i + r_o)/2 \). The amplitude of the wavy-rough corrugations \( \epsilon \) is presumed to be much smaller than the linear dimension of the cross section (say, \( r_o - r_i \) and \( r_i \) or \( r_o \), cf. Fig. 1; i.e. \( \epsilon \ll r_o - r_i \ll R \)).

2 Theoretical Formulations

Let \( x \) be a coordinate along the average ring of the wavy-rough thin annulus \( (x/R \) representing the angle variable). The explicit dependence on the two additional coordinates required to locate
an atom within the cross section is neglected here for simplicity. The system of $N$ atoms can be described by a symmetric wave function $\psi(x_s)$ of the $N$ variables $x_s$ together with the conjugate momenta $p_s$ $(s = 1, \ldots, N)$. The interaction with the walls shall be neglected here so that the total momentum $P = \sum_s p_s$ (or the total angular momentum $PR$) is a constant of the motion. We can obtain a solution

$$\psi(x_s) = \chi(x_s - x_{s'}) \exp[iP(\sum_s x_s)/N\hbar],$$

(1)

of the Schrödinger equation

$$\mathcal{H}\psi = E\psi.$$  

Here,

$$P\chi = 0$$

and $s, s' = 1, \ldots, N$. In fact, we have

$$p_s = \frac{\hbar}{i} \frac{\partial}{\partial x_s}$$

(2)

which only come from the kinetic energy $\sum_s p_s^2/2m$. From above equations and/or mathematical expressions, the substitution for $\psi$ into the Schrödinger equation gives

$$\mathcal{H}\chi = e\chi$$

(3)

together with the total energy

$$E = \frac{P^2}{2M} + e,$$

(4)

where $M = Nm$ is the total mass of the system. Note that it is customary to separate the motion of the center of gravity $(\sum_s x_s)/N$. Thus, $e$ which is due to the relative motion of the atoms, is normally independent of the total momentum $P$.

Nevertheless $\psi$ should be single valued for superfluid systems and must return to the same value when an atom is brought around the ring to its original position. This means whenever $x_s = x_s + 2\pi R$, we have

$$\chi = \chi f$$

with

$$f = \exp[-2\pi i(PR/N\hbar)].$$

Hence, $e$ depends upon $P$ in the superfluid state. However, as $\chi$ depends only on the differences of $x_s$ and then $\chi$ remains the same if all $N$ of them are increased by $2\pi R$. This also requires that $f^N = 1$ and $P = k\hbar/R$ where $k$ is an integer. The latter confirms that the eigenvalue of the total angular momentum $PR$ to be integer multiples of $\hbar$. 
With above reasoning, we also have, for the eigenvalue $e$, $e(P + N\hbar/R) = e(P)$ or $e(P)$ is periodic with $N\hbar/R$. Moreover, as a reverse rotation cannot affect the energy $E$, the same holds for $e$ so that $e$ is an even function of $P$. To be explicit, a stationary state of the system can be characterized by $P$ and an additional set of $N - 1$ quantum numbers, say, $n$. The corresponding eigenvalue of the energy is then of the general form

$$E_n(P) = \frac{P^2}{2M} + e_n(P),$$

(5)

with

$$e_n(P + \frac{N\hbar}{R}) = e_n(P), \quad e_n(-P) = e_n(P).$$

(6)

A dependence of the energy on the total momentum through $e$ indicates a phase memory which, irrespective of its origin, extends around the whole thin wavy-rough annulus.

3 Persistent Flow at T=0

We only consider the conditions at the absolute zero of temperature (T=0) for simplicity. It means $P = 0$ and the ground state of the system will be characterized also by (the set of additional quantum numbers) $n = 0$. Now,

$$E_0(P) = \frac{P^2}{2M} + e_0(P)$$

(7)

represents the lowest value of the energy for a given $P$ and $e_0(P)$ has a minimum at $P = 0$ (and this minimum is periodically repeated : $e_0(P + N\hbar/R) = e_0(P)$). If we presume $e_0(P)$ has a finite slope as $P \to 0$ from either side, then it can be illustrated schematically in Fig. 2. Similarly, as shown in Fig. 3, $E_0(P)$ also exhibits minima when $P$ is an integer multiple of $N\hbar/R$ with an absolute magnitude below a critical value $P_c$. Next, we need to consider the interaction of the system with the (container) walls along (with) the ensuing equilibrium to understand the significance of these minima. After starting from an arbitrary initial state, equilibrium can be reached by transitions involving an exchange of momentum and energy. Once the walls (of the container) are held at zero temperature, the system will be brought to the lowest state and hence to a vanishing momentum (if, as in normal cases, there are no other minima of $E_0$).

If, however, there are other minima the system can reach one of them through a rapid succession of transitions, each involving an energy loss accompanied by a small transfer of momentum to the walls. From this on, only those transitions could occur in which the energy further decreases with a momentum transfer comparable to or larger than the macroscopically large value $N\hbar/R$. Such transitions can thus be safely considered to be so highly rare as to cause a metastability which, in effect, will let the system remain in a state with (momentum) $P_\mu = \mu N\hbar/R$ ($\mu$ is an integer).
In view of the general definition of the drift velocity $u = P/M \ (M = Nm)$, the system therefore exhibits persistent flow with (the drift velocity)

$$u_\mu = \frac{\mu \hbar}{mR}. \quad (8)$$

Meanwhile the circulation defined by the line integral around the ring is $2\pi Ru_\mu = \mu (h/m)$. With this, above just confirms the quantization with the quantum of circulation $h/m$ (valid for $|P_\mu| \leq P_c$).

### 3.1 Critical Velocity

From equation (7) we are concerned, in fact, no longer with a minimum of $E_0$ at such large values of $P_\mu$ that $P^2/2M$ rises more steeply than $e_0$, because it allows a decrease of the energy towards the low-momentum side. Using the derivatives expression and $e'_0(P_\mu) = e'_0(0)$, we have a persistent flow at the drift velocity $u_\mu$ requiring

$$\frac{|P_\mu|}{M} = |u_\mu| < |e'_0(0)|, \quad (9)$$

so that the critical momentum is

$$P_c = M|e'_0(0)| \quad \text{or that} \quad u_c = |e'_0(0)| \quad (10)$$

represents a critical velocity.

The above pedagogical statements or detailed explanations demonstrate the possibility to derive some of the salient features of superfluidity from fundamental principles. We shall give an example, an ideal Bose gas, below to illustrate above mentioned point so that students and researchers can gain more insights into its behavior.

### 3.2 Example: Ideal Bose Gas

We shall consider an ideal Bose gas for a pedagogical illustration. In the interval $0 \leq P \leq N\hbar/R$, the lowest energy for a given value $P = \nu (h/R)$ can be obtained by assigning the momentum $p = \hbar/R$ to $\nu$ atoms and $p = 0$ to the $(N - \nu)$ atoms. We then have

$$E_0(P) = \frac{\nu \hbar^2}{2mR^2} = \frac{P\hbar}{2mR}, \quad (11)$$

and from equation (7) (as $M = Nm$),

$$e_0(P) = \frac{P\hbar}{2mR} \left(1 - \frac{PR}{N\hbar}\right). \quad (12)$$

With other intervals being almost the same (except the period shift), we can determine the corresponding function and demonstrate it by the dashed parabolic curves in Fig. 2 (setting $N\hbar/R \equiv 1$).

Once we add $P^2/2M$ to above expression, we find (for $P_\mu \leq P \leq P_{\mu+1}$)

$$E_0(P) = \frac{P_\mu^2}{2M} + \frac{(P - P_\mu)\hbar}{2mR}(2\mu + 1). \quad (13)$$
The straight segments which connect the points on the parabola \( P^2/2M \) for \( P = P_\mu \) and \( P = P_{\mu+1} \) represent this function by the dashed line in Fig. 3 (setting \( Nh/R = 1 \)). As it has no other minimum than that at \( P = 0 \), we can conclude that the ideal Bose gas does not exhibit persistent flow in a contained at rest. However, we remind the readers that the above property derives from a uniform rotation, rather than a translation, of the container (thin corrugated annulus) so that the invariance of the relative velocity against uniform uniform motion of the system of reference cannot be invoked in concluding on the case of a contained at rest considered here.

4 Conclusion

To conclude in brief, we already demonstrate the possibility to derive some of the characteristic features of superfluidity from fundamental principles in a weakly-corrugated thin annulus and we believe our presentation will be useful to researchers in relevant fields.

References

[1] Fernandez JF 1977 Phys. Rev. B 15 3362
   Bloch F 1968 Phys. Rev. Lett. 21 1241

[2] Leggett AJ 1999 Rev. Mod. Phys. 71 S318
   Lin CC 1959 Phys. Rev. Lett. 2, 245

[3] Morizot O, Colombe Y, Lorent V, Perrin H and Garraway BM 2006 Phys. Rev. A 74 023617
   Javanainen J, Paik SM and Yoo SM 1998 Phys. Rev. A 58 580
   Bloch F 1973 Phys. Rev. A 7 2187
   Bloch F 1974 Phys. Rev. A 10 716

[4] Anderson PW 1966 Rev. Mod. Phys. 38, 298
Fig. 1  Schematic plot of a thin wavy-rough annulus. The amplitude of the corrugation $\epsilon \ll R = (r_i + r_o)/2$. 
Fig. 2  Schematic illustration of $e_0(P)$. It is presumed that finite slopes exist at the periodically repeating minimum. The dashed lines indicate the case of an ideal Bose gas. $P$ is the total momentum. Here, we set $Nh/R \equiv 1$. 
Fig. 3 Schematic illustration of $E_0(P)$. The slope towards lower values of $|P|$ (for heavy solid lines) at successive minima decreases with increasing momentum (preventing the occurrence of further minima above a critical value $P_c$ of $|P|$). The dashed lines indicate the case of an ideal Bose gas. $P$ is the total momentum. Here, we set $N\hbar/R \equiv 1$. 