Remark on inverse scattering for the Schrödinger equation with cubic convolution nonlinearity

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Abstract. We discuss inverse scattering for the nonlinear Schrödinger equation. The nonlinearity of the equation is an approximate expression of the nonlocal nonlinearity. We give a reconstruction formula for determining the nonlinearity by the knowledge of given scattering states.

1. Introduction
This paper is concerned with the inverse scattering problem for the nonlinear Schrödinger equation

$$i\partial_t u + \Delta u = F(u), \quad (t, x) \in \mathbb{R} \times \mathbb{R}^n.$$  \hspace{1cm} (1)

Here, $u = u(t, x)$ is a complex-valued unknown function, $\partial_t = \partial/\partial t$, $\Delta$ is the Laplacian in $\mathbb{R}^n$. The term $F$ is a nonlocal term

$$F(u) = (V * |u|^2)u,$$  \hspace{1cm} (2)

where $V$ is a real-valued measurable function and $*$ denotes the convolution in the space variables. The nonlinearity $F(u)$, which is called a cubic convolution, is an approximate expression of the nonlocal interaction with the interaction potential $V$. The equation (1) is initially studied by Chadam–Glassey [1].

To formulate the inverse scattering problem, we introduce the definition of the scattering operator for nonlinear evolution equation

$$i\partial_t v(t) + J(v(t)) = f(v(t)),$$  \hspace{1cm} (3)

where $v$ is a complex-valued function on the Hilbert space $X$, $J$ is a self-adjoint operator on $X$ and $f$ is a nonlinear mapping on some subspace of $X$. Let $B(\delta; X)$ be the set of all $\phi \in X$ with $\|\phi\|_X \leq \delta$. The scattering operator $S$ is defined by the mapping

$$S : B(\delta; X) \ni \phi_- \mapsto \phi_+ \in X$$

if the following condition holds for some $\delta > 0$ and some $Z \subset C(\mathbb{R}; X)$:

**For any $\phi_- \in B(\delta; X)$, there uniquely exist $v \in Z$ and $\phi_+ \in X$ satisfying**

$$v(t) = e^{itJ} \phi_- + \frac{1}{i} \int_{\mathbb{R}} e^{i(t-\tau)J} f(v(\tau))d\tau \quad \text{for any } t \in \mathbb{R},$$  \hspace{1cm} (4)
Therefore, we can define the scattering operator

\[ \lim_{t \to \pm \infty} \| v(t) - e^{itJ} \phi \|_X = 0, \]

where \( e^{itJ} \phi \) is a solution to the Cauchy problem for

\[
\begin{cases}
  i \partial_t v(t) + J(v(t)) = 0, \\
v(0) = \phi.
\end{cases}
\]

There is a substantial literature on the scattering theory for (1) (see for instance Nakanishi-Ozawa [5] and references therein). In particular, we can see from the method of Mochizuki [4] that if \( n \geq 3 \),

\[
|V(x)| \leq C(|x|^{-\sigma_1} + |x|^{-\sigma_2}), \quad 2 \leq \sigma_1, \sigma_2 \begin{cases} < n & \text{if } n \leq 4, \\
 \leq 4 & \text{if } n \geq 5,
\end{cases}
\]

then the scattering operator \( S \) is well-defined on some 0-neighborhood of \( H^1(\mathbb{R}^n) = (1 - \Delta)^{-1/2} L^2(\mathbb{R}^n) \). More precisely, we have the following:

**Theorem 1.1** Assume that \( F \) satisfies (2) and that \( V \) fulfills (5). Let \( X_1 = H^1(\mathbb{R}^n), \ Y_1 = L^3(\mathbb{R}; L^3(\mathbb{R}^n)), \ 1/r = 1/2 - 2/3n \) and \( Z_1 = C(\mathbb{R}; X_1) \cap Y_1 \). Then we can find \( \delta > 0 \) such that if \( \phi_- \in B(\delta; X_1) \), there is a unique \( (u, \phi_+) \in Z \times X_1 \) such that \( u \) is a time-global solution to (1) and we have

\[
\lim_{t \to \pm \infty} \| u(t) - e^{it\Delta} \phi_+ \|_{X_1} = 0.
\]

Therefore, we can define the scattering operator

\[ S : B(\delta; X) \ni \phi_- \mapsto \phi_+ \in X. \]

Furthermore, we have

\[
\begin{align*}
\| u \|_{Z_1} & \leq C \| \phi_- \|_{X_1}, \\
\| u - e^{it\Delta} \phi_- \|_{Z_1} & \leq C \| \phi_- \|_{X_1}^3, \\
\phi_+ & = \phi_- + \frac{1}{i} \int_{-\infty}^{+\infty} e^{-it\Delta} F(u(t))dt.
\end{align*}
\]

The inverse scattering problem for the nonlinear equation is to recover the nonlinearity from the knowledge of the scattering states \( (\phi_-, S(\phi_-)) \). Before we treat (1), we review the inverse scattering problem for the Schrödinger equation with power nonlinearity briefly. Strauss [8] considered the inverse scattering problem for the nonlinear Schrödinger equation

\[
i \partial_t u + \Delta u = V(x)|u|^{p-1}u, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^n.
\]

Suppose that \( p \) is an integer satisfying

\[
\begin{cases}
p > 4 & \text{if } n = 1, \\
p > 3 & \text{if } n = 2, \\
p \geq 3 & \text{if } n \geq 3
\end{cases}
\]

and \( V(x) \) is real-valued continuous and bounded, together with to order \( l > 3n/4 \). Then the scattering operator \( S \) is well-defined. It was shown that \( V(x) \) is recovered from the scattering operator the following: For any \( \phi \in H^1 \cap L^{1+1/p}, \)

\[
V(x_0) = \frac{\lim_{\alpha \to 0} \alpha^{-\frac{(n+2)}{2}} I[\phi_{\alpha, x_0}]}{\int_{\mathbb{R}} \| e^{it\Delta} \phi \|_{p+1} dt},
\]

where \( I[\phi_{\alpha, x_0}] \) is an integral.
where $\phi_{\alpha,x_0}(x) = \phi(\alpha^{-1}(x - x_0)), x, x_0 \in \mathbb{R}^n$ and

$$I[\phi] = \lim_{\varepsilon \to 0} \frac{i}{\varepsilon^p} \langle (S - i d)(\varepsilon \phi), \phi \rangle_{L^2(\mathbb{R}^n)}.$$ 

The above limit is called the small amplitude limit. Later, by Weder [12]–[16], the above result was extended to more general cases.

Unfortunately, the above method to obtain the reconstruction formula (9) is not applicable to the case (1). The essential point to prove the formula (9) is the change of variables in the following integral:

$$I[\phi] = \int_{\mathbb{R}} \int_{\mathbb{R}^n} V(x) |e^{it\Delta} \phi|^{p+1} dx dt.$$

By changing variable $x$ by $\alpha^{-1}(x - x_0)$, we have

$$I[\phi_{\alpha,x_0}] = \alpha^{n+2} \int_{\mathbb{R}} \int_{\mathbb{R}^n} V(x_0 + \alpha x) |e^{it\Delta} \phi|^{p+1} dx dt.$$

Therefore, as $\alpha \to 0$, we can take out the value $V(x_0)$ from the inside integral. Applying the same method to (1), we obtain

$$I[\phi_{\alpha,x_0}] = \alpha^{2n+2} \int_{\mathbb{R}} \int_{\mathbb{R}^n} (V(\alpha \cdot) * |e^{it\Delta} \phi|^2) |e^{it\Delta} \phi|^2 dx dt.$$

However, we can not let $\alpha \to 0$, because

$$\int_{\mathbb{R}} \int_{\mathbb{R}^n} (V(0) * |e^{it\Delta} \phi|^2) |e^{it\Delta} \phi|^2 dx dt$$

does not converge.

The inverse scattering problem for the nonlinear Schrödinger equation with a cubic convolution was initially studied by Watanabe [9]. Later, Watanabe [11] and Sasaki [7] consider the inverse scattering problem for

$$i\partial_t u + \Delta u = F^\sigma_j(u), \quad (t, x) \in \mathbb{R} \times \mathbb{R}^n, \quad j = 1, 2,$$

where $\lambda_j \in C^1(\mathbb{R}^n) \cap W^1_\infty(\mathbb{R}^n), \lambda_j(0) \neq 0, j = 1, 2.$

Let us assume that the interaction potential $V$ fulfills either of the following two conditions:

(A) There exist $N \in \{1, 2, \cdots\}$ and $\{\lambda_j\}_{j=1}^N, \{\sigma_j\}_{j=1}^N \subset \mathbb{R}$ such that

$$V(x) = \sum_{j=1}^N \lambda_j |x|^{-\sigma_j},$$

where $\lambda_j \neq 0$ for any $j \geq 1$

and

$$2 \leq \sigma_1 \begin{cases} < n & \text{if } n \leq 4, \\ \leq 4 & \text{if } n \geq 5, \end{cases} \quad \sigma_1 > \sigma_j > \sigma_k \geq 2 \text{ if } 1 < j < k \leq N.$$

(B) $V$ is a real-valued radial function belonging to the Schwartz class $S(\mathbb{R}^n)$.
We write $V \in (A)$ (resp. $V \in (B)$) if and only if $V$ fulfills the condition (A) (resp. (B)). We remark that there exist some $V_A \in (A)$ and $V_B \in (B)$ which do not have the form $F_j^\sigma$. Since $V \in (A) \cup (B)$ satisfies (5), we have the scattering operator $S$ and have properties (6)–(8).

In this paper, we shall prove that if $V \in (A)$, then we can reconstruct $V$, and that if $V \in (B)$, then we can determine the exact value of $(d^3V)(0)$. Here, $d = d/d\rho$, 

$$V_0 : \mathbb{R} \ni \rho \mapsto (\mathcal{F}V)(\rho, 0, \cdots, 0) \in \mathbb{R}^n$$

and $\mathcal{F}$ denotes the Fourier transform on $S(\mathbb{R}^n)$. For any $V \in (B)$, we immediately see that $V_0$ is a real-valued function belonging to $S(\mathbb{R})$ and 

$$V_0(\rho) = V_0(-\rho) = (\mathcal{F}V)(\xi)$$

if $\rho = |\xi|$.

Now let $s \in \mathbb{R}$, $1 \leq r \leq \infty$, $H^s_r(\mathbb{R}^n)$ be the Sobolev space $(1 - \Delta)^{-s/2}L^r(\mathbb{R}^n)$, and $H^s(\mathbb{R}^n) = H^s_2(\mathbb{R}^n)$. For $\alpha > 0$ and $\phi : \mathbb{R}^n \to \mathbb{R}$, we denote $\phi(\alpha^{-1}x)$ by $\phi_\alpha(x)$. A functional $T[\phi]$ is given by 

$$T[\phi] = \lim_{\varepsilon \to 0} \frac{i}{\varepsilon^3} \int (S - id)(\varepsilon \phi), \phi \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the inner product on $L^2(\mathbb{R}^n)$. For $j = 1, 2, \cdots, N$, we put 

$$I_j = \int_\mathbb{R} \langle \cdot, \cdot \rangle^{-\alpha_j} \ast |e^{it\Delta}\phi|^2, |e^{it\Delta}\phi|^2 \rangle dt.$$ 

We define $\psi(\alpha)$ and $\varphi(\alpha)$ by 

$$\psi(\alpha) = \alpha^{-2n-2}T[\phi_\alpha],$$

and 

$$\varphi(\alpha) = \alpha^{n+2}T[\phi_{\alpha^{-1}}]$$

respectively. The Fourier transform on $S(\mathbb{R}^n)$ is given by 

$$(\mathcal{F}\phi)(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} \phi(x) dx$$

for all $\phi \in S(\mathbb{R}^n)$. It is well-known that the Fourier transform on $S'(\mathbb{R}^n)$ is defined uniquely. We also define the inverse Fourier transform on $S'(\mathbb{R}^n)$ by 

$$\mathcal{F}^{-1}\phi(\xi) = (\mathcal{F}\tilde{\phi})(\xi),$$

where we set $\tilde{\phi}(x) = \phi(-x)$. We are ready to state our results.

**Theorem 1.2** Let $n \geq 3$. Assume that the nonlinearity $F$ has the form (2), and that $V \in (A)$. Put $\phi \in X_1 \setminus \{0\}$. Then we can determine $N$, $\{\lambda_j\}_{j=1}^N$ and $\{\sigma_j\}_{j=1}^N$ by the following steps:

**(Step I.)** We have 

$$\sigma_1 = -\lim_{\alpha \to 0} \frac{\ln |\psi(e\alpha)|}{|\psi(\alpha)| + 1}. \tag{11}$$

Furthermore, using the given $\sigma_1$, we obtain 

$$\lambda_1 = I_1^{-1}\lim_{\alpha \to 0}(\alpha^{\sigma_1}\psi(\alpha)). \tag{12}$$
(Step II-1.) Suppose that we have already determined \( \{ \lambda_j \}_{j=1}^{k} \) and \( \{ \sigma_j \}_{j=1}^{k} \). Put
\[
\psi^k(\alpha) = \psi(\alpha) - \sum_{j=1}^{k} \lambda_j I_j.
\]
If \( \lim_{\alpha \to 0} |\psi^k(\alpha)| = 0 \), then \( N = k \).

(Step II-2.) Suppose that we have already determined \( \{ \lambda_j \}_{j=1}^{k} \) and \( \{ \sigma_j \}_{j=1}^{k} \). If \( \lim_{\alpha \to 0} |\psi^k(\alpha)| = \infty \), then we have \( N > k \) and
\[
\sigma_{k+1} = - \lim_{\alpha \to 0} \frac{|\psi^k(e\alpha)|}{|\psi^k(\alpha)| + 1}.
\]
Furthermore, using the given \( \sigma_{k+1} \), we obtain
\[
\lambda_{k+1} = I_{k+1}^{-1} \lim_{\alpha \to 0} (\alpha \sigma^k \psi^k(\alpha)).
\]

**Remark 1.** The value of \( \lim_{\alpha \to 0} |\psi^k(\alpha)| \) must be either 0 or \( \infty \).

**Theorem 1.3** Let \( n \geq 3 \). Assume that the nonlinearity \( F \) has the form (2) and that \( V \in (B) \). Then we have for any \( M = 0, 1, \cdots \) and any \( \phi \in H^{M/2+n/4-1/2}(\mathbb{R}^n) \setminus \{0\} \),
\[
(d^M V_0)(0) = \lim_{\alpha \to 0} \frac{(d^M \phi)(\alpha)}{\| \mathcal{F} V \|_2}.
\]

**Remark 2.** Suppose that \( A \in (B) \) and that we have already seen that \( \mathcal{F} V \) is analytic. Let \( \phi \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\} \). If we obtain \( \lim\sup_{M \to \infty} \| (d^M V_0)(0) \|_2^{1/2} = 0 \), then it follows from the Maclaurin theorem that
\[
V_0(\rho) = \sum_{M=0}^{\infty} \frac{(d^M V_0)(0)}{M!} \rho^M.
\]
Therefore, for any \( x \in \mathbb{R}^n \), we can reconstruct \( V \) by
\[
V(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \sum_{M=0}^{\infty} \frac{(d^M V_0)(0)}{M!} e^{ix \cdot \xi} |\xi|^M d\xi.
\]

2. **Proof of theorems**

In this section, we give the proof of Theorems 1.2 and 1.3. Throughout this section, we assume that \( n \geq 3 \), the nonlinearity \( F \) has the form (2) and the interaction potential \( V \) belongs to \( (A) \cup (B) \). Furthermore, let \( S \) be the scattering operator for (1) given by Theorem 1.1. Using the method of Watanabe [10], we see that
\[
T[\phi] = \int_{\mathbb{R}} \langle V \ast |e^{it\Delta} \phi|^2, |e^{it\Delta} \phi|^2 \rangle dt.
\]
Since
\[
e^{it\Delta} \phi_\alpha = (e^{it\alpha^{-2}\Delta} \phi)_\alpha,
\]
we see that
\[
T[\phi_\alpha] = \alpha^{2n+2} \int_{\mathbb{R}} \langle V_{\alpha^{-1}} \ast |e^{it\Delta} \phi|^2, |e^{it\Delta} \phi|^2 \rangle dt.
\]
We first prove Theorem 1.2.

**Proof of Theorem 1.2.** Suppose that \( V \in (A) \) and \( 0 < \alpha < 1 \). Since
\[
V_{\alpha^{-1}}(x) = \sum_{j=1}^{N} \alpha^{-\sigma_j} \lambda_j |x|^{-\sigma_j},
\]
we obtain

$$\psi(\alpha) = \alpha^{-2n-2} T[\phi_0] = \int_R \left( \langle \sum_{j=1}^N \alpha^{-\sigma_j} \lambda_j |x|^{-\sigma_j} \rangle * |e^{it\Delta} \phi|^2, |e^{it\Delta} \phi|^2 \rangle \right) dt.$$

Hence we see that

$$\psi(\alpha) = \sum_{j=1}^N \alpha^{-\sigma_j} \lambda_j \int_R \left( \langle |x|^{-\sigma_j} \rangle * |e^{it\Delta} \phi|^2, |e^{it\Delta} \phi|^2 \rangle \right) dt = \sum_{j=1}^N \alpha^{-\sigma_j} \lambda_j I_j.$$ (16)

Therefore, we obtain

$$\frac{|\psi(e\alpha)|}{|\psi(\alpha)|+1} = \frac{\sum_{j=1}^N e^{-\sigma_j} \alpha^{-\sigma_j} \lambda_j I_j}{\sum_{j=1}^N \alpha^{-\sigma_j} \lambda_j I_j + 1} = \frac{\sum_{j=1}^N e^{-\sigma_j} \alpha^{\sigma_1-\sigma_j} \lambda_j I_j}{\sum_{j=1}^N \alpha^{\sigma_1-\sigma_j} \lambda_j I_j + \alpha^{\sigma_1}}.$$

By \(\sigma_1 > \sigma_j\) for any \(j > 1\), it follows from the boundedness of \(I_j\) that

$$\lim_{\alpha \to 0} \frac{|\psi(e\alpha)|}{|\psi(\alpha)| + 1} = e^{-\sigma_1}.$$

Thus, (11) holds. From (16), we immediately obtain (12).

Since

$$\psi^k(\alpha) = \sum_{j=k+1}^N \alpha^{-\sigma_j} \lambda_j I_j,$$

we can determine \(N, \{\lambda_j\}_{j=1}^N\) and \(\{\sigma_j\}_{j=1}^N\) by using the steps (Step II-1) and (Step II-2). This completes the proof. \(\square\)

For the proof of Theorem 1.3, we prepare the following lemma:

**Lemma 2.1** Let \(m \geq 2\). For any \(s > 0\) and any \(\phi \in H^{s+m/4-1/2}(\mathbb{R}^m)\), we have

$$\int_R \left\| |s\mathcal{F}(|e^{it\Delta} \phi|^2)| \right\|^2_{L^2(\mathbb{R}^m)} dt \leq C \|\phi\|^4_{H^{s+m/4-1/2}(\mathbb{R}^m)}.$$

**Proof.** Put \(1/\beta = 1/2 - 1/2m\). Then we see from Ginibre-Velo [2] that

$$\int_R \|e^{it\Delta} \phi\|^4_{H^s(\mathbb{R}^m)} dt \leq C \|\phi\|^4_{H^s(\mathbb{R}^m)}.$$

From the embedding \(H^{s+m/4-1/2}(\mathbb{R}^m) \hookrightarrow H^s(\mathbb{R}^m)\), we obtain

$$\int_R \|e^{it\Delta} \phi\|^4_{H^s(\mathbb{R}^m)} dt \leq C \|\phi\|^4_{H^{s+m/4-1/2}(\mathbb{R}^m)}.$$ (18)

Since \(|\xi|^s \leq (1 + |\xi|^2)^{s/2}\) and \((1-\Delta)^{s/2} f = \mathcal{F}^{-1}\left( (1+|\xi|^2)^{s/2} \mathcal{F} f \right)\), the Plancherel theorem implies that

$$\int_R \left\| |s\mathcal{F}(|e^{it\Delta} \phi|^2)| \right\|^2_{L^2(\mathbb{R}^m)} dt \leq C \int_R \|e^{it\Delta} \phi\|^2_{H^t(\mathbb{R}^m)} dt.$$ (19)

By Ponce [6], we have

$$\|e^{it\Delta} \phi\|^2_{H^t(\mathbb{R}^m)} \leq C \|e^{it\Delta} \phi\|^4_{H^s(\mathbb{R}^m)}.$$
Thus, we see from (18) and (19) that (17) holds. □

Proof of Theorem 1.3. Suppose that $V \in (B)$ and $0 < \alpha < 1$. By
\[ \mathcal{F}V_\alpha = \alpha^{-n}(\mathcal{F}V)_{\alpha^{-1}}, \]
it follows from (15) and the Plancherel theorem that
\[ \varphi(\alpha) = \alpha^{n+2}T[\varphi_{\alpha^{-1}}] = \int_R \langle (\mathcal{F}V)_{\alpha^{-1}}\mathcal{F}(|e^{it \Delta} \phi|^2), \mathcal{F}(|e^{it \Delta} \phi|^2) \rangle dt. \]
By (17) and $(\mathcal{F}V)_{\alpha^{-1}}(\xi) = V_0(\alpha|\xi|)$, we obtain
\[ \frac{d^M}{d\alpha^M} \varphi(\alpha) = \int_R \langle |\cdot|^M (d^M V_0)(\alpha\cdot)\mathcal{F}(|e^{it \Delta} \phi|^2), \mathcal{F}(|e^{it \Delta} \phi|^2) \rangle dt, \]
for any $M = 0, 1, \cdots$. The Lebesgue dominated theorem implies that
\[ \lim_{\alpha \to 0} (d^M \varphi)(\alpha) = (d^M V_0)(0) \int_R \langle |\cdot|^M \mathcal{F}(|e^{it \Delta} \phi|^2), \mathcal{F}(|e^{it \Delta} \phi|^2) \rangle dt. \]
Thus, we have (13). This completes the proof. □

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