Kraśkiewicz-Pragacz modules and Pieri and dual Pieri rules for Schubert polynomials

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November 29, 2017

Abstract. In their 1987 paper Kraśkiewicz and Pragacz defined certain modules, which we call KP modules, over the upper triangular Lie algebra whose characters are Schubert polynomials. In a previous work the author showed that the tensor product of KP modules always has a KP filtration, i.e. a filtration whose each successive quotients are isomorphic to KP modules. In this paper we explicitly construct such filtrations for certain special cases of these tensor product modules, namely $S^w \otimes S^d(K^i)$ and $S^w \otimes \wedge^d(K^i)$, corresponding to Pieri and dual Pieri rules for Schubert polynomials.

1 Introduction

Schubert polynomials are one of the main subjects in algebraic combinatorics. One of the tools for studying Schubert polynomials is the modules introduced by Kraśkiewicz and Pragacz. These modules, which here we call KP modules, are modules over the upper triangular Lie algebra and have the property that their characters with respect to the diagonal matrices are Schubert polynomials.

It is known that a product of Schubert polynomials is always a positive sum of Schubert polynomials. The previously known proof for this positivity property uses the geometry of the flag variety. In [8] the author showed that the tensor product of two KP modules always has a filtration by KP modules and thus gave a representation-theoretic proof for this positivity. Although the proof there does not give explicit constructions for the KP filtrations, it may provide a new viewpoint for the notorious Schubert-LR problem asking for a combinatorial positive rule for the coefficient in the expansion of products of Schubert polynomials into a sum of Schubert polynomials.

*This work was supported by Grant-in-Aid for JSPS Fellows No. 15J05373.
There are some cases where the expansions of products of Schubert polynomials are explicitly known. Examples of such cases include the Pieri and the dual Pieri rules for Schubert polynomials ([1], [6], [7], [10]). These are the cases where one of the Schubert polynomials is a complete symmetric function $h_d(x_1, \ldots, x_i)$ or an elementary symmetric function $e_d(x_1, \ldots, x_i)$. The purpose of this paper is to investigate the structure of tensor product modules corresponding to these products and to give explicit constructions of KP filtrations for these modules.

The structure of this paper is as follows. In Section 2 we prepare some definitions and results on Schubert polynomials and KP modules. In Section 3 we review the Pieri and the dual Pieri rules for Schubert polynomials. In Section 4 we give explicit constructions for KP filtrations of the corresponding tensor product modules $S_w \otimes S^d(K^i)$ and $S_w \otimes \bigwedge^d(K^i)$. In Section 5 we give a proof of the main result.

2 Preliminaries

Let $\mathbb{N}$ be the set of all positive integers. By a permutation we mean a bijection from $\mathbb{N}$ to itself which fixes all but finitely many points. The graph of a permutation $w$ is the set $\{(i, w(i)) : i \in \mathbb{N}\} \subset \mathbb{N}^2$. For $i < j$, let $t_{ij}$ denote the permutation which exchanges $i$ and $j$ and fixes all other points. Let $s_i = t_{i,i+1}$. For a permutation $w$, let $\ell(w) = \#\{i < j : w(i) > w(j)\}$. For a permutation $w$ and positive integers $p < q$, if $\ell(wt_{pq}) = \ell(w) + 1$ we write $wt_{pq} > w$. It is well known that this condition is equivalent to saying that $w(p) < w(q)$ and there exists no $p < r < q$ satisfying $w(p) < w(r) < w(q)$. For a permutation $w$ let $I(w) = \{(i, j) : i < j, w(i) > w(j)\}$.

For a polynomial $f = f(x_1, x_2, \ldots)$ and $i \in \mathbb{N}$ define $\partial_i f = \frac{f - s_i f}{x_i - x_{i+1}}$. For a permutation $w$ we can assign its Schubert polynomial $\mathfrak{S}_w \in \mathbb{Z}[x_1, x_2, \ldots]$ which is recursively defined by

- $\mathfrak{S}_w = x_1^{n-1} x_2^{n-2} \cdots x_{n-1}$ if $w(1) = n, w(2) = n-1, \ldots, w(n) = 1$ and $w(i) = i (i > n)$ for some $n$, and
- $\mathfrak{S}_{ws_i} = \partial_i \mathfrak{S}_w$ if $\ell(ws_i) < \ell(w)$.

Hereafter let us fix a positive integer $n$. Let

$$ S^{(n)} = \{ w : \text{permutation, } w(n+1) < w(n+2) < \cdots \}. $$

Note that if $w \in S^{(n)}$ then $I(w) \subset \{1, \ldots, n\} \times \mathbb{N}$. Let $K$ be a field of characteristic zero. Let $\mathfrak{b} = \mathfrak{h}_n$ denote the Lie algebra of all $n \times n$ upper triangular matrices over $K$. For a $\mathfrak{b}$-module $M$ and $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$, let $M_\lambda = \{ m \in M : h m = \langle \lambda, h \rangle m \ (\forall h = \text{diag}(h_1, \ldots, h_n)) \}$ where $\langle \lambda, h \rangle = \sum_i \lambda_i h_i$. If $M$ is a direct sum of these $M_\lambda$ and these $M_\lambda$ are finite dimensional then we say that $M$ is a weight module and we define $\text{ch}(M) = \sum_\lambda \dim M_\lambda x^\lambda$ where $x^\lambda = x_1^{\lambda_1} \cdots x_n^{\lambda_n}$. For $1 \leq i < j \leq n$ let $e_{ij} \in \mathfrak{b}$ be the matrix with 1 at the $(i, j)$-th position and all other coordinates 0.
Let $U$ be a vector space spanned by a basis $\{u_{ij} : 1 \leq i \leq n, j \in \mathbb{N}\}$. Let $T = \bigoplus_{d=0}^{\infty} \bigwedge^d U$. The Lie algebra $\mathfrak{b}$ acts on $U$ by $e_{pq}u_{ij} = \delta_{iq}e_{pj}$ and thus on $T$.

For $w \in S(n)$ let $u_w = \bigwedge_{(i,j) \in I(w)} u_{ij} \in \bigwedge^{I(w)} U \subset T$. The Kraskiewicz-Pragacz module $S_w$ (or the KP module for short) associated to $w$ is the $\mathfrak{b}$-submodule of $\bigwedge^{I(w)} U \subset T$ generated by $u_w$. In [5] Kraskiewicz and Pragacz showed the following:

**Theorem 2.1** ([5], Remark 1.6 and Theorem 4.1]). $S_w$ is a weight module and $\text{ch}(S_w) = S_w$.

**Example 2.2.** If $w = s_i$, then $u_w = u_{i,i+1} \in U$ and thus $S_w = \bigoplus_{1 \leq j \leq i} Ku_{j,i+1} =: K^i$ on which $\mathfrak{b}$ acts by $e_{pq}u_j = \delta_{qj}u_p$.

A KP filtration of a $\mathfrak{b}$-module $M$ is a filtration $0 = M_0 \subset \cdots \subset M_r = M$ such that each $M_i/M_{i-1}$ is isomorphic to some KP module.

### 3 Pieri and dual Pieri rules for Schubert polynomials

**Definition 3.1.** For $w \in S_{\infty}$, $i \geq 1$ and $d \geq 0$, let

$$X_{i,d}(w) = \{t_{p_1q_1}t_{p_2q_2}\cdots t_{p_dq_d} : p_j \leq i, q_j > i, w_1 \prec w_2 \prec \cdots, w_1(p_1) < w_2(p_2) < \cdots\}$$

and

$$Y_{i,d}(w) = \{t_{p_1q_1}t_{p_2q_2}\cdots t_{p_dq_d} : p_j \leq i, q_j > i, w_1 \prec w_2 \prec \cdots, w_1(q_1) > w_2(q_1) > \cdots\}$$

where $w_1 = w, w_2 = wt_{p_1q_1}, w_3 = wt_{p_1q_1}t_{p_2q_2}, \cdots$.

Note that the condition for $X_{i,d}(w)$ (resp. $Y_{i,d}(w)$) implies that $q_1, \ldots, q_d$ (resp. $p_1, \ldots, p_d$) are all different.

**Theorem 3.2** (Conjectured in [1] and proved in [10], also appears with different formulations in [6] and [7]). We have

$$\mathfrak{S}_w \cdot h_d(x_1, \ldots, x_i) = \sum_{\zeta \in X_{i,d}(w)} \mathfrak{S}_{w\zeta}$$

and

$$\mathfrak{S}_w \cdot e_d(x_1, \ldots, x_i) = \sum_{\zeta \in Y_{i,d}(w)} \mathfrak{S}_{w\zeta}$$

where $h_d$ and $e_d$ denote the complete and elementary symmetric functions respectively. 

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1 The formulation of dual Pieri rule here is slightly different from the one in [1] but they are easily shown to be equivalent.
Note here that the permutation \( \zeta \in X_{i,d}(w) \) (or \( Y_{i,d}(w) \)) in fact uniquely determines its decomposition into transpositions satisfying the conditions in Definition 3.1. So we can write, without ambiguity, for example “for \( \zeta = t_{p_1,q_1} \cdots t_{p_{q_2},q_2} \in X_{i,d}(w) \) define (something) as (some formula involving \( p_j \) and \( q_j \))”. Hereafter if we write such we will always assume the conditions in Definition 3.1.

4 Explicit Pieri and dual Pieri rules for KP modules

The author showed in [8] that the tensor product of KP modules always has a KP filtration. Since \( S^d(K^i) \) and \( \bigwedge^d(K^i) \) \((1 \leq i \leq n, d \geq 1)\) are special cases of KP modules, \( S_w \otimes S^d(K^i) \) and \( S_w \otimes \bigwedge^d(K^i) \) \((w \in S^{(n)})\) have KP filtrations. In this section we give explicit constructions for these filtrations.

For positive integers \( p \leq q \) we define an operator \( e'_q \) acting on \( T \) as \( e'_q(\sum u_{a,b} \cdot \cdots \cdot v_{a,b} \cdot \cdots \cdot u \cdot \cdots \cdot v) = \sum u_{a,b} \cdots \cdot v_{a,b} \cdots \cdot u \cdot \cdots \cdot v \). Let these operators act on \( T \otimes S^d(K^i) \) and \( T \otimes \bigwedge^d(K^i) \) by applying them on the left-hand side tensor component. Also for \( j \geq 1 \) define an operator \( \mu_j : T \otimes \bigwedge^a(K^i) \to T \otimes \bigwedge^{a-1}(K^i) \) \((a \geq 1)\) by \( u \otimes (v_1 \otimes v_2 \otimes \cdots) \mapsto (\iota_j(v_1) \wedge u) \otimes (v_2 \otimes v_3 \otimes \cdots) \) where \( \iota_j(u_p) = u_{p,j} \) \((1 \leq p \leq i)\). We denote the restrictions of \( \mu_j \) to \( T \otimes S^a(K^i) \) and \( T \otimes \bigwedge^a(K^i) \) (seen as submodules of \( T \otimes \bigwedge^a(K^i) \)) by the same symbol. Note that \( e'_q \) and \( \mu_j \) give \( \mathfrak{g} \)-endomorphisms on \( T \otimes S^*(K^i) \) and \( T \otimes \bigwedge^*(K^i) \).

For a permutation \( z \) and positive integers \( p < q \) let \( m_{pq}(z) = \# \{ r < q : z(p) < z(r) < z(q) \} \) and \( m'_{pq}(z) = \# \{ r < p : z(p) < z(r) < z(q) \} \). For \( \zeta = t_{p_1,q_1} \cdots t_{p_{q_2},q_2} \in X_{i,d}(w) \) (resp. \( Y_{i,d}(w) \)) define

\[
v_{\zeta} = \left( \prod_j m_{p_j,q_j}(w_j) \right) u_w \otimes \left( \prod_j u_{p_j} \right)
\]

\[
= \left( \prod_j m_{p_j,q_j}(w_j) \right) u_w \otimes \left( \sum_{\sigma \in S_d} u_{p_{\sigma(1)}} \otimes \cdots \otimes u_{p_{\sigma(d)}} \right) \in S_w \otimes S^d(K^i)
\]

(resp.

\[
v_{\zeta} = \left( \prod_j m'_{p_j,q_j}(w_j) \right) u_w \otimes \left( \prod_j u_{p_j} \right)
\]

\[
= \left( \prod_j m'_{p_j,q_j}(w_j) \right) u_w \otimes \left( \sum_{\sigma \in S_d} \text{sgn} \sigma \cdot u_{p_{\sigma(1)}} \otimes \cdots \otimes u_{p_{\sigma(d)}} \right) \in S_w \otimes \bigwedge^d(K^i)
\]

) where \( w_j = wt_{p_1,q_1} \cdots t_{p_{q_2},q_2} \) as in Definition 3.1. Note that these are also well-defined even if some \( q_j \) are greater than \( n \), since in such a case \( m_{p_j,q_j}(w_j) = 0 \). Note also that the products of the operators \( e_{p_j,q_j} \) above are well-defined since
the operators $e'_{q, p_j}$ ($p_j \leq i$, $q_j > i$) commute with each others. Also, for such $\zeta$, define a $\mathfrak{b}$-homo morphism $\phi_\zeta : T \otimes \bigotimes_{\zeta}^d (K^i) \to T$ by

$$
\phi_\zeta = \mu_q \cdots \mu_{q_1} \prod_{j} (e'_{q, p_j})^m_{q, p_j}(w_j).
$$

Note that the order in the product symbol does not matter since the operators $e'_{q, p_j}$ commute.

Let $\leq \text{ and } \prec$ denote the lexicographic and reverse lexicographic orderings on permutations respectively, i.e. for permutations $u$ and $v$, $u \prec v$ (resp. $u \prec v$) if there exists a $k$ such that $u(j) = v(j)$ for all $j < k$ (resp. $j > k$) and $u(k) < v(k)$.

**Proposition 4.1.** For $\zeta, \zeta' \in X_{i,d}(w)$ (resp. $Y_{i,d}(w)$),

- $\phi_\zeta(\zeta)$ is a nonzero multiple of $u_{w\zeta} \in T$, and
- $\phi_\zeta(\zeta') = 0$ if $(w\zeta)^{-1} < (w\zeta')^{-1}$ (resp. $(w\zeta)^{-1} < (w\zeta')^{-1}$).

The proof for this proposition is given in the next section. Here we first see that Proposition 4.1 gives desired filtrations.

For a $\mathfrak{b}$-module $M$ and elements $x, y, z \in M$ let $(x, y, z)$ denote the submodule generated by these elements. Consider the sequence of submodules

$$
0 \subset (v_{\zeta_1}) \subset (v_{\zeta_2}) \subset \cdots \subset (v_{\zeta} : \zeta \in X_{i,d}(w) \text{ (resp. } Y_{i,d}(w)))
$$

inside $S_w \otimes S^d(K^i)$ (resp. $S_w \otimes \bigwedge^d(K^i)$), where $\zeta_1, \zeta_2, \ldots \in X_{i,d}(w)$ (resp. $Y_{i,d}(w)$) are all the elements ordered increasingly by the lexicographic (resp. reverse lexicographic) ordering of $(w\zeta)^{-1}$. From the proposition we see that there are surjections $(v_{\zeta_1}, \cdots, v_{\zeta})/(v_{\zeta_1}, \cdots, v_{\zeta-1}) \to S_{w\zeta_i}$ induced from $\phi_\zeta$.

Thus we have

$$
\dim(S_w \otimes S^d(K^i)) \geq \dim(v_{\zeta} : \zeta \in X_{i,d}(w)) \geq \sum_{\zeta \in X_{i,d}(w)} \dim S_{w\zeta} = \dim(S_w \otimes S^d(K^i))
$$

and

$$
\dim(S_w \otimes \bigwedge^d(K^i)) \geq \dim(v_{\zeta} : \zeta \in Y_{i,d}(w)) \geq \sum_{\zeta \in Y_{i,d}(w)} \dim S_{w\zeta} = \dim(S_w \otimes \bigwedge^d(K^i))
$$

respectively, where the last equalities are by Proposition 3.2. So the equalities must hold everywhere. Thus $(v_{\zeta} : \zeta \in X_{i,d}(w) \text{ (resp. } Y_{i,d}(w))) = S_w \otimes S^d(K^i)$ (resp. $S_w \otimes \bigwedge^d(K^i)$) and the surjections above are in fact isomorphisms. So, in conclusion, we get from Proposition 4.1 the following:
Lemma 5.1. The proof of Theorem 4.2 which will be done below).

Proof. This is essentially the same as [9, Lemma 5.8].

Lemma 5.2. Let \( w \) be a permutation, \( i \geq 1 \) and \( d \geq 0 \). Let \( \zeta = t_{p_1,q_1} \cdots t_{p_d,q_d} \in X_{i,d}(w) \) (resp. \( Y_{i,d}(w) \)) and \( 1 \leq a \leq d \). Suppose that there exists no \( b < a \) satisfying \( p_b = p_a \) (resp. \( q_b = q_a \)). Then \( m_{p_a q_a}(w) = m_{p_a q_a}(w) \) and \( m'_{q_a p_a}(w) = m'_{q_a p_a}(w) \). Let \( w' = w_{t_{p_1,q_1} \cdots t_{p_{a-1},q_{a-1}}} \) as in Definition 5.4.

Proof. We show the case \( \zeta \in X_{i,d}(w) \): the other case can be treated similarly. First note that \( p_1, \ldots, p_{a-1} \neq p_a \) by the hypothesis. Also, as we have remarked before, \( q_1, \ldots, q_d \) are all different. Thus the proof is now reduced to the following lemma.

Lemma 5.3. Let \( p < q, p' < q' \) and suppose

- \( \{p, q\} \cap \{p', q'\} = \emptyset \), and

5 Proof of Proposition 4.1
Proof. Let us begin with a simple observation: suppose there exist two rectangles \( R_1 \) and \( R_2 \) with edges parallel to coordinate axes. Suppose that no two edges of these rectangles lie on the same line. Then, checking all the possibilities we see that

\[
\#(\text{NW and SE corners of } R_1 \text{ lying inside } R_2) - \#(\text{NE and SW corners of } R_1 \text{ lying inside } R_2)
\]

\[
= \#(\text{NW and SE corners of } R_2 \text{ lying inside } R_1) - \#(\text{NE and SW corners of } R_2 \text{ lying inside } R_1).
\]

First consider the case \( R_1 = [p, q] \times [w(p), w(q)] \) and \( R_2 = [p', q'] \times [w(p'), w(q')] \) in the observation above. \( wt_{p'q'}t_{pq} \gg wt_{p'q'} \gg w \) implies that the first term in the left-hand side and the second term in the right-hand side vanish (here the coordinate system is taken so that points with smaller coordinates go NW). Thus all the terms must vanish. In particular the first term on the right-hand side vanishes and thus \( wt_{pq} \gg w \).

We have shown that none of the points \((p, w(p)), (p, w(q)), (q, w(p)) \) and \((q, w(q)) \) lie in \([p', q'] \times [w(p'), w(q')]\). Since \( m_{pq}(w) \) (resp. \( m_{p'q'}(w) \)) is the number of points of the graph of \( w \) lying inside the rectangle \( R_1 = [q, M] \times [w(p), w(q)] \) (resp. \( R_1' = [-M, p] \times [w(p), w(q)] \) for \( M \gg 0 \) and the graphs of \( w \) and \( wt_{p'q'} \) differ only at the vertices of the rectangle \( R_2 = [p', q'] \times [w(p'), w(q')] \), applying the observation to these rectangles shows the remaining claims.

\[ \square \]

Proof of Proposition 4.1

Proof for \( X_{i,d}(w) \): We assume \((w_\zeta)^{-1} \preceq (w_\zeta')^{-1} \) and show that \( \phi_\zeta'(v_\zeta) = 0 \) unless \( \zeta' = \zeta \) and \( \phi_\zeta(v_\zeta) \) is a nonzero multiple of \( u_{w_\zeta} \). Let \( \zeta = t_{p_1q_1} \cdots t_{p_aq_a} \) and \( \zeta' = t_{p'_1q'_1} \cdots t_{p'_aq'_a} \) as in Definition 3.1. We write \( w_\alpha = wt_{p_1q_1} \cdots t_{p_{a-1}q_{a-1}} \) and \( w_\alpha' = wt_{p'_1q'_1} \cdots t_{p'_{a-1}q'_{a-1}} \).
For $\zeta = \prod_j t_{p_j q_j}$ and $\zeta' = \prod_j t_{p'_j q'_j}$ in $X_{i, d}(w)$ we have

$$\phi(\zeta)(u_\zeta) = \sum_{\sigma \in S_d} \left( u_{\sigma(d)} q_d \land \cdots \land u_{\sigma(1)} q_1 \land \left( \prod_{j=1}^d E_j \prod_{j=1}^d E'_j \cdot u_w \right) \right) \cdots \left( * \right)$$

where $E_j = e_{p_j q_j}^w(w_j)$ and $E'_j = (e_{q'_j p'_j}^w)^{m_{q'_j} m_{p'_j}}(w_w)$.

If $w(p_1) < w(p'_1)$, then $(w\zeta)^{-1}(w(p_1)) = q_1 > p_1 = (w\zeta')^{-1}(w(p_1))$ and $(w\zeta)^{-1}(j) = w^{-1}(j) = (w\zeta')^{-1}(j)$ for all $j < w(p_1)$, and this contradicts the assumption $(w\zeta)^{-1} \leq (w\zeta')^{-1}$. Thus $w(p_1) \geq w(p'_1)$. Also, by a similar argument, if $p_1 = p'_1$ then $q_1 \leq q'_1$.

Fix $\sigma \in S_d$. Let $1 \leq a \leq d$ be minimal such that $p_a = p_{\sigma(a)}$. Note that this in particular implies $u_a(p_a) = w(p_a)$. We have

$$u_{\sigma(d)} q_d \land \cdots \land u_{\sigma(1)} q_1 \land \left( \prod_{j=1}^d E_j \prod_{j=1}^d E'_j \cdot u_w \right)$$

$$= u_{\sigma(d)} q_d \land \cdots \land u_{\sigma(2)} q_2 \land \prod_{j \neq a} E_j \prod_{j \neq a} E'_j \cdot \left( u_{\sigma(1)} q_1 \land E_a E'_a \right) u_w$$

$$= u_{\sigma(d)} q_d \land \cdots \land u_{\sigma(2)} q_2 \land \prod_{j \neq a} E_j \prod_{j \neq a} E'_j \cdot \left( u_{p_a q_a} (w) (e_{q'_j p'_j}^w)^{m_{q'_j} m_{p'_j}} (w) u_w \right)$$

where the last equality is by Lemma 5.2 (note that $w'_1 = w$ by definition).

First consider the case $w(p_1) > w(p'_1)$. We show that the summand in $(*)$ vanishes for all $\sigma$. It suffices to show $u_{\sigma(q'_1)} \land e_{p_a q_a} (w) (e_{q'_j p'_j}^w)^{m_{q'_j} m_{p'_j}} (w) u_w = 0$. We have $w(p_a) = w_a(p_a) \geq w(p_1) > w(p'_1)$. Thus by Lemma 5.1 we see $u_{\sigma(q'_1)} \land e_{p_a q_a} (w) (e_{q'_j p'_j}^w)^{m_{q'_j} m_{p'_j}} (w) u_w = 0$ (note that $w_{t_{p_a q_a}} > w$ by Lemma 5.2).

Next consider the case $w(p_1) = w(p'_1)$ and $a > 1$. In this case we see $u_{\sigma(q'_1)} \land e_{p_a q_a} (w) (e_{q'_j p'_j}^w)^{m_{q'_j} m_{p'_j}} (w) u_w = 0$ since $w(p_a) = w_a(p_a) > w(p_1) = w(p'_1)$.

Next consider the case $w(p_1) = w(p'_1)$, $a = 1$ and $q_1 < q'_1$. Then since $w_{t_{p_1 q_1}}, w_{t_{p'_1 q'_1}} > w$ it follows that $w(q'_1) < w(q_1)$. So again by Lemma 5.1 we see $u_{\sigma(q'_1)} \land e_{p_a q_a} (w) (e_{q'_j p'_j}^w)^{m_{q'_j} m_{p'_j}} (w) u_w = 0$.

So the only remaining summands in $(*)$ are the ones with $(p_1, q_1) = (p'_1, q'_1)$ and $a = 1$, i.e. $p_{\sigma(1)} = p_1$. It is easy to see that the sum of such summands is a nonzero multiple of the sum of terms with $\sigma(1) = 1$. If $\sigma(1) = 1$ we have, by
the latter part of Lemma 5.1

\[ u_{p_{(d)}} q_d' \land \cdots \land u_{p_{(1)}} q_1' \land \left( \prod_{j=1}^{d} E_j \prod_{j=1}^{d} E_j' \cdot u_w \right) \]

\[ = u_{p_{(d)}} q_d' \land \cdots \land u_{p_{(2)}} q_2' \land \left( \prod_{j=2}^{d} E_j \prod_{j=2}^{d} E_j' \cdot \left( u_{p_1 q_1} \land e_{p_1 q_1}^{m_{p_1 q_1}(w)} (e_{q_1 p_1}^{m_{q_1 p_1}(w)}) u_w \right) \right) \]

\[ = (\text{nonzero const.}) \cdot u_{p_{(d)}} q_d' \land \cdots \land u_{p_{(2)}} q_2' \land \left( \prod_{j=2}^{d} E_j \prod_{j=2}^{d} E_j' \cdot u_{wt_{p_1 q_1}} \right). \]

So, working inductively on \( d \) (using \( wt_{p_1 q_1}, t_{p_2 q_2} \cdots t_{p_d q_d} \) and \( t_{p_2 q_2} \cdots t_{p_d q_d}' \) in place of \( w, \zeta \) and \( \zeta' \) respectively, noting that if \((p_1, q_1) = (p_1', q_1')\) then \((w\zeta)^{-1} \leq \text{lex}_{\leq}(w\zeta')^{-1}\) implies \((wt_{p_1 q_1}) t_{p_2 q_2} \cdots t_{p_d q_d})^{-1} = (w\zeta)^{-1} \leq (w\zeta')^{-1} = ((wt_{p_1 q_1}) t_{p_2 q_2}' \cdots t_{p_d q_d})^{-1}\)

we see that:

- \( u_{p_{(d)}} q_d' \land \cdots \land u_{p_{(1)}} q_1' \land \left( \prod_{j=1}^{d} E_j \prod_{j=1}^{d} E_j' \cdot u_w \right) \) vanishes if \((w\zeta)^{-1} < (w\zeta')^{-1}\),
  - or if \( \zeta' = \zeta \) and \( \sigma \neq \text{id} \), and
  - if \( \zeta' = \zeta \) and \( \sigma = \text{id} \) then it is a nonzero multiple of \( u_w \zeta \).

This finishes the proof for \( X_{i,d}(w) \).

Proof for \( Y_{i,d}(w) \): This proceeds much similarly to the previous case. Here instead of \((*)\) we use

\[ \phi_{r_{\text{lex}}} (v_{\zeta}) = \sum_{\sigma \in S_d} \left( \text{sgn}(\sigma) \cdot u_{p_{(d)}} q_d' \land \cdots \land u_{p_{(1)}} q_1' \land \left( \prod_{j=1}^{d} E_j \prod_{j=1}^{d} E_j' \cdot u_w \right) \right) \]

\[ = \sum_{\sigma \in S_d} \left( u_{p_d q_{d-1}} \land \cdots \land u_{p_1 q_{1-1}} \land \left( \prod_{j=1}^{d} E_j \prod_{j=1}^{d} E_j' \cdot u_w \right) \right) \]

where \( E_j = e_{p_{j+1} q_j}^{m_{p_{j+1} q_j}(w_j)} \) and \( E_j' = (e_{q_{j+1} p_j})^{m_{q_{j+1} p_j}(w_j)} \) as before.

We assume \((w\zeta)^{-1} \leq (w\zeta')^{-1}\). Fix \( \sigma \) and take \( 1 \leq a \leq d \) minimal with \( q_a' = q_{\sigma^{-1}(a)}' \). By an argument similar to the above, it suffices to show that

\[ u_{p_1 q_1} \land e_{p_1 q_1}^{m_{p_1 q_1}(w)} (e_{q_1 p_1}^{m_{q_1 p_1}(w)}) u_w \text{ is zero unless } a = 1 \text{ and } (p_1, q_1) = (p'_1, q'_1), \]

and in such a case it is a nonzero multiple of \( u_{wt_{p_1 q_1}} \).

Since \((w\zeta)^{-1} \leq (w\zeta')^{-1}\) by the hypothesis, we see that \( w(q_1) \geq w(q'_1) \), and that if \( w(q_1) = w(q'_1) \) then \( p_1 \leq p'_1 \).

If \( w(q_1) > w(q'_1) \) then the claim follows from Lemma 5.1 since \( w(q_1) > w(q'_1) \geq w'_a(q'_a) = w'_a(q''_a) \). If \( w(q_1) = w(q'_1) \) and \( a > 1 \) then the claim follows from Lemma 5.1 since in this case \( w(q_1) = w(q'_1) > w(q''_a) \) by \( wt_{p_1 q_1}, wt_{p'_1 q'_1} > w \).

If \( q_1 = q'_1, a = 1 \) and \( p_1 < p'_1 \) the claim follows from Lemma 5.1 since \( w(p_1) > 

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Finally if \((p_1, q_1) = (p_1', q_1')\) and \(a = 1\) then
\[
 u_{p_1 q_1} \wedge e_{p_1 q_1}^{m_{p_1 q_1}(w)} (e_{q_1 p_1}^{e_p} m_{a p_1}(w) u_w =
 u_{p_1 q_1} \wedge e_{p_1 q_1}^{m_{p_1 q_1}(w)} (e_{q_1 p_1}^{e_p} m_{a p_1}(w) u_w
\]
is a constant multiple of \(u_{w p_1 q_1}\) by Lemma 5.1.

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