Field Theoretical Approach to Bicritical and Tetracritical Behavior: Static and Dynamics

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Abstract. We discuss the static and dynamic multicritical behavior of three-dimensional systems of $O(n_k) \times O(n_\gamma)$ symmetry as it is explained by the field theoretical renormalization group method. Whereas the static renormalization group functions are currently known within high order expansions, we show that an account of two loop contributions refined by an appropriate resummation technique gives an accurate quantitative description of the multicritical behavior. One of the essential features of the static multicritical behavior obtained already in two loop order for the interesting case of an antiferromagnet in a magnetic field ($n_k = 1$, $n_\gamma = 2$) are the stability of the biconical fixed point and the neighborhood of the stability border lines to the other fixed points leading to very small transient exponents. We further pursue an analysis of dynamical multicritical behavior choosing different forms of critical dynamics and calculating asymptotic and effective dynamical exponents within the minimal subtraction scheme.

Keywords: critical behavior, multicritical points, renormalization group

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INTRODUCTION

Beneath the milestone contributions of N.N. Bogolyubov that shaped modern theoretical physics one definitely should mention his and D.N. Shirkov work on the renormalization group (RG)[1]. Three papers on RG written in the mid-50-ies by three different groups [2] addressed quantum electrodynamics problems, however very soon their importance has been realized in - on the first sight - very different field of phase transitions and critical phenomena. It is generally recognized by now that the success in conceptual understanding and quantitative description of behavior in the vicinity of critical points in different condensed matter systems is due to the effective application of the RG ideas originating from the above papers [3]. It is our pleasure to contribute to these
FIGURE 1. Typical phase diagrams of anisotropic antiferromagnets in a uniform parallel external magnetic field $H$. Types of ordering are schematically shown by arrows. a: the bicritical point. Three phases - an antiferromagnetic phase, a spin flop phase and the paramagnetic phase are in coexistence. The phase transition lines to the paramagnetic phase are second order transition lines, whereas the transition line between the spin flop and the antiferromagnetic phase is of first order. b: the tetracritical point. Four phases - an antiferromagnetic phase, a spin flop phase, an intermediate or mixed phase and the paramagnetic phase - are in coexistence. All transition lines are of the second order in this case. Also indicated is the dynamical universality class of the transition from the paramagnetic to the corresponding ordered phase according to the classification of Hohenberg and Halperin [15] for the three component antiferromagnet.

Multicritical points appear on phase diagrams of various systems that contain several phase transitions lines. In the vicinity of the meeting points of such lines the multicritical behavior is observed, which is characterized by competition of different types of ordering. Prominent examples are given by the antiferromagnets in an external magnetic field like GdALO$_3$, MnF$_2$, MnCl$_2$4D$_2$O, Mn$_2$AS$_4$ (A=Si or Ge) [4]. Other examples are given by the layered cuprate antiferromagnets like (Ca,La)$_{14}$Cu$_{24}$O$_{41}$. Schematic phase diagrams of such systems are shown in Fig.1 in a $H$-$T$ plane. There, multicritical points of two different types are manifested. At a bicritical point (Fig. 1a) three phases are in coexistence, whereas four phases coexist in the tetracritical point (Fig. 1b). On a more general level, the multicritical behavior is inherent to a critical system when some "nonordering" field is applied. Such a field (beside the magnetic field $H$ this may be pressure, stress etc.) may alter non-universal parameters of the system and lead to appearance of the lines of phase transition points. Besides the above example that concern the shift of the Néel point of anisotropic antiferromagnets by a uniform magnetic filed, other examples of multicritical behavior are observed at a shift of the Curie points under applied pressure or depression of the $\lambda$ point in $^4$He at dilution by $^3$He [5].

A field theoretic description of multicritical behavior starts with a static effective Hamiltonian for an $n$-component field $\Phi = (\phi_k, \phi_\xi)$ of $O(n_k)$ $O(n_\xi)$ symmetry ($n_k + n_\xi = n$). An account of

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1 The paper is based on the invited lecture given by one of us (R.F.) at the Conference Statphys’09 dedicated to the 100-th anniversary of N.N.Bogolyubov (23.06-25.06.2009, Lviv, Ukraine)
the interaction between the two order parameters $\phi_k$ and $\phi_\gamma$ leads to different types of multicritical behavior connected with the stable fixed point (FP) found in the RG treatment [7, 8, 9, 10, 11, 14]. In particular, the bicritical point (Fig. 1a) has been connected with the stability of the isotropic Heisenberg fixed point of $O(n_k + n_\gamma)$ symmetry, whereas the tetracritical point (Fig. 1b) corresponds to a FP of $O(n_k)$ $O(n_\gamma)$ symmetry, which might be either the so called biconical FP or the decoupling FP. In the last FP the parallel and the perpendicular components of the order parameter are asymptotically decoupled. If no FP is reached the multicritical point might be of first order, i.e. a triple point.

Quite recently the possible types of phase diagrams in the $H_T$ plane of three dimensional uniaxial anisotropic antiferromagnets have been studied by Monte Carlo simulations [12]. For $n_k = 1$ and $n_\gamma = 2$ a phase diagram with a bicritical point has been found in agreement with earlier simulations [13], but contrary to the results of RG theory in higher loop orders [11].

The dynamics of antiferromagnets in a magnetic field is quite complicated. To account for conservation laws present in such systems, the dynamical equations of motion should contain coupling terms between the two order parameters (the components of the staggered magnetization parallel and perpendicular to the magnetic field, $\phi_k$ and $\phi_\gamma$) and conserved densities (e.g. the parallel component of the magnetization or the energy density). First formulation of the equations of motion at multicritical points has been done in Ref. [17]. The simplest form of dynamics assumes the relaxational behavior for the two order parameters $\phi_k$ and $\phi_\gamma$ (the so-called model A) [15, 18]. Dynamical multicritical behavior within the one-loop approximation has been considered in [17] on the basis of the static one loop results [9]. A further step to the complete model is to include the diffusive dynamics of the slow conserved density leading to a model C like extension. This model has been studied in one loop order in Refs. [17, 19, 20] taking into account only a part of dynamical two loop order terms and one loop statics. In order to get more insight in the dynamics in the vicinity of multicritical points, recently we have reconsidered the above dynamical models within the two loop approximation [16, 21].

In what follows below we briefly summarize an outcome of an RG analysis of multicritical behavior paying special attention to an impact of the non-universal contributions to an asymptotic behavior. In particular, we will show that an account of two loop part of the RG expansions refined by an appropriate resummation technique gives an accurate quantitative description of the static multicritical behavior. Furthermore, we pursue an analysis of dynamical multicritical behavior choosing different forms of critical dynamics and calculating asymptotic and effective dynamical exponents.

**RG FLOWS AND STATIC MULTICRITICAL BEHAVIOR**

The generalized static $O(n_k)$ $O(n_\gamma)$-symmetrical effective Hamiltonian that results from the decomposition of the $n$-component order parameter field into two mutually interacting fields $\phi_k$ and $\phi_\gamma$ of different irreducible representations of dimensions $n_k$ and $n_\gamma$, $n = n_k + n_\gamma$, reads:

\[
\mathcal{H} = \frac{Z}{d^d x} \left( \frac{1}{2} \hat{r}_\gamma \phi_\gamma \phi_\gamma + \frac{1}{2} \sum_{i=1}^{n_k} \nabla_i \phi_k \nabla_i \phi_k + \frac{1}{2} \hat{r}_k \phi_k \phi_k + \frac{1}{2} \sum_{i=1}^{n_\gamma} \nabla_i \phi_\gamma \nabla_i \phi_\gamma + \frac{1}{4!} \left( \phi_\gamma \phi_\gamma + 2 \phi_k \phi_k \phi_k \phi_k + 2 \phi_\gamma \phi_\gamma \phi_\gamma \phi_\gamma \right) \right): 
\]
Here, $\hat{f}_2 \hat{u} \hat{m} \hat{u}_c \hat{g} = \hat{f}_u \hat{g}$ and $\check{r}_2$, $\check{r}_k$ are couplings and masses, correspondingly, index 0 refers to the bare quantities, and central dots stand for scalar products. The decomposition in parallel and perpendicular order parameter components allows to describe the multicritical behavior at the meeting point of two critical lines: (i) the line where $\check{r}_2$ becomes zero and the $n_2$-dimensional components $\phi_{\gamma 0}$ are the order parameter, and (ii) the line where $\check{r}_k$ becomes zero and the order parameter is $\phi_{x 0}$. At the meeting point both quadratic terms become zero and both components of $\phi_0$ have to be taken into account. As has been predicted already by the one-loop RG analysis [8,9], an effective Hamiltonian (11) describes three different types of multicritical behavior that are governed by three different FPs: (i) the isotropic $n$ component Heisenberg FP, called below $\mathcal{H} (n)$, all fourth order couplings are equal in this FP, (ii) the decoupling FP point $\mathscr{D}$, which consists of a combination of the FPs $\mathcal{H} (n_2)$ and $\mathcal{H} (n_k)$ of two decoupled systems and (iii) the biconical FP, $\mathcal{B}$, with nontrivial nonzero couplings. As it was revealed by subsequent calculations [10,11] the FP picture does not change qualitatively with an account of higher orders of the perturbation theory. However, the one-loop results attain essential quantitative changes that lead to drastic modification of the type of a phase diagram. Typical example may be given by the behavior of the $\beta$-functions, that describe flow of the fourth order couplings $\hat{f}_u \hat{g}$ under renormalization. The above functions, calculated in the two-loop approximation with the minimal subtraction RG scheme read [14]:

$$
\beta_{u_2} = \varepsilon u_2 + \frac{(n_2 + 8)}{6} u_2^2 + \frac{n_k}{6} u_2^2 - \frac{(3n_2 + 14)}{12} u_2^2 + \frac{5n_k}{36} u_2 u_2^2 - \frac{n_k}{9} u_2^3; \\
\beta_u = \varepsilon u + \frac{(n_2 + 2)}{6} u_2 u + \frac{(n_k + 2)}{6} u_2 u + \frac{2}{3} u_2^2 + \frac{(n_2 + n_k + 16)}{72} u_2^3 \\
\beta_{u_k} = \varepsilon u_k + \frac{(n_k + 8)}{6} u_k^2 + \frac{n_2}{6} u_k^2 - \frac{(3n_k + 14)}{12} u_k^2 + \frac{5n_k}{36} u_k u_k^2 - \frac{n_2}{9} u_k^3;
$$

Here, $\hat{f}_u \hat{u} \hat{u}_c \hat{g} = \hat{f}_u \hat{g}$ are renormalized couplings and the space dimension $d$ enters the $\beta$-functions via parameter $\varepsilon = 4 - d$. With the $\beta$-functions at hand, one can analyze the flow equations of the fourth-order couplings $\hat{f}_u \hat{g}$:

$$
\frac{d u_a}{d \tau} = \beta_{u_a} (\hat{f}_u \hat{g});
$$

with $a = 2; k$; and the flow parameter $\tau$, and find the FPs $\hat{f}_u \hat{g}$ of these equations as the solutions of the system of equations

$$
\beta_{u_a} (\hat{f}_u \hat{g}) = 0;
$$

Which of these FPs is the stable one depends on the number of components $n_2$ and $n_k$ and the dimension $d$ of space. The scaling properties depend on the symmetry of stable FP.

There are two alternative ways to look for the solutions of the FP equations (6) and, subsequently, for the scaling properties of the system. In one approach, the $\varepsilon$-expansion, the solutions are obtained as series in $\varepsilon$ and then evaluated at the value of interest (at $\varepsilon = 1$ for $d = 3$ theories). Alternatively, one may solve a system of non-linear equations directly at the dimensionality of space of interest (e.g. at $\varepsilon = 1$) [22] and obtain the FP coordinates numerically. The RG expansions being divergent [23], the special resummation techniques are used to get convergent results [24].

As we have discussed already above, depending on the values of $n_k; n_2$; and $d$, the multicritical behavior is governed by one of the three non-trivial FPs: $\mathcal{H}$ $\hat{f}_u \hat{u}_2 = u = u_g \hat{g}$, $\mathcal{B}$ $\hat{f}_u \hat{u}_2 \neq u \neq u_g \hat{g}$, and $\mathcal{D}$ $\hat{f}_u \hat{u}_2 \neq 0 \mu = 0 \mu_k \neq 0 \hat{g}$. In Fig. 2 we show how the stability of these FPs change with
FIGURE 2. Regions of FPs stability in the in the $n_k \cdot n_\gamma$-plane, $d = 3$. The lines separate regions where Heisenberg FP $H$, biconical FP $B$ and decoupling FP $D$ are stable (from left to right). Shown are the $H B$-stability borderlines (dashed lines) and $B D$-stability borderlines (solid lines), in one loop order (thin lines) and resummed two loop order (thick lines). The dots indicate low integer values for order parameter components.

$n_k \cdot n_\gamma$ for $d = 3$. There, we compare the first order $\epsilon$-expansion results \[8, 9\] with the two-loop results \[14\] obtained within the fixed $d = 3$ technique \[22\]. The two-loop results were obtained applying Pad-Borel resummation technique to functions \[23\]–\[25\]. One sees that the borderlines of the FPs stability are drastically shifted to smaller values of OP components. Thus in the case $n_1 = 1$ and $n_\gamma = 2$ FP $B$ (connected with tetracriticality) is stable in two loop order contrary to the one loop calculations where the FP $H$ (connected with bicriticality) is stable. The resummed higher orders of the perturbation theory do not change this result and do not lead to essential changes in the critical exponents either \[11\].

As usually, the asymptotic values of the critical exponents are defined by the stable FP values of the corresponding RG $\zeta$-functions, which we do not expose here. Note that in general, there are distinct exponents $\eta_k$, $\eta_\gamma$ governing spacial decay of the order parameter correlations in directions parallel and perpendicular to the anisotropy axis. As a consequence, there is a pair of $\gamma$-exponents, $\gamma_k$, $\gamma_\gamma$ that govern corresponding isothermal magnetic susceptibilities. However, the above RG procedure assumes that the multicritical system is described by a single diverging length scale and therefore by one correlation length $\xi$ and one corresponding critical exponent $\nu$. This does not hold for decoupled systems where two length scales are present and the usual scaling laws with one length scale break down \[9\]. We give typical numerical values of the exponents in Table I.

Whereas the asymptotic critical exponent values are determined strictly at the FP and correspond to the scaling behavior at the multicritical point, of special interest are the effective critical exponents which are observed in the vicinity of the multicritical point. These are the effective exponents that often are observed experimentally and are measured in MC simulations. In the
TABLE 1. Critical exponents of the $O(1) \times O(2)$ model obtained in different approximations. [14]: resummation of the two-loop RG series at fixed $d = 3$; [6]: first order $\varepsilon$-expansion. [11, 26]: resummed fifth order $\varepsilon$-expansion. Numbers, shown in italic were obtained via familiar scaling relations.

| Reference | FP | $\eta_\zeta$ | $\eta_k$ | $\gamma_\zeta$ | $\gamma_k$ | $\nu$ |
|-----------|----|--------------|----------|----------------|------------|------|
| [14]      | $\mathcal{B}$ | 0.037        | 0.037    | 1.366          | 1.366      | 0.696 |
| [14]      | $\mathcal{H}(3)$ | 0.040        | 0.040    | 1.411          | 1.411      | 0.720 |
| [6]       | $\mathcal{B}$ | 0            | 0        | 1.222          | 1.222      | 0.611 |
| [6]       | $\mathcal{H}(3)$ | 0            | 0        | 1.227          | 1.227      | 0.611 |
| [11]      | $\mathcal{B}$ | 0.037(5)     | 0.037(5) | 1.37(7)        | 1.37(7)    | 0.70(3) |
| [26]      | $\mathcal{H}(3)$ | 0.0375(45)  | 0.0375(45) | 1.382(9) | 1.382(9) | 0.7045(55) |

RG framework, one may estimate the effective exponents from the values of corresponding RG $\zeta$-functions calculated along the RG flow and relate the flow parameter $\gamma$ to the distance to the multicritical point. In Fig. 3 we show the resummed RG flow of Eqs. (5) for different initial conditions [14]. The unstable FPs are shown as filled spheres, the stable biconical FP as filled cube. Let us note that the neighborhood of the stability border lines to the other FPs leads to very small transient exponents. Therefore, the stable FP is not reached for the value of the flow parameter chosen in Fig. 3 (there, the flow parameter has been changed in the interval $40 \ln \gamma 0$).

FIGURE 3. Resummed RG flow of Eqs. (5) for different initial conditions at $d = 3, n_k = 1, n_\zeta = 2$. The unstable FPs are shown as filled spheres, the stable biconical FP as filled cube. The FPs points are connected by separatrices defining the surface which encloses the attraction region.

Defining the effective exponents as explained above, one can evaluate their numerical values along the RG flows of Fig. 3 and in this way predict possible outcome of measuring the scaling properties of different observables at multicritical point. As two typical examples, we show in Fig. 4 the change of the values of isothermal susceptibility effective exponents $\gamma_k, \gamma_\zeta$ and of
FIGURE 4. Effective exponents of different observables in the vicinity of a multicritical point for the the flows of Fig. 3: a: isothermal magnetic susceptibility (solid curves: $\gamma_k$, dashed curves: $\gamma_\gamma$); b: correlation length.

the correlation length critical exponent $\nu$ as the multicritical point is being approached, the limit $T \to T_c$ corresponds to the limit $\nu \to 0$.

Before passing to discussion of peculiarities of dynamic multicritical behavior, let us note a particular feature of the $O(n^k)$--$O(n^\gamma)$ model that becomes evident from the above analysis of the statics. As the stability analysis shows, for the physically interesting case $d = 3, n_k = 1, n_\gamma = 2$ the asymptotic behavior is governed by the biconical FP $B$. Therefore, the tetracritical point is realized (c.f. Fig. 1a). However, depending on the particular microscopic non-universal characteristics of a given system, one may expect a variety of different scenarios for multicritical behavior, including the triple point (that corresponds to the run away solutions of the RG flow equations, c.f. Fig. 3) and bicritical point (when for certain initial condition the Heisenberg FP $H$ is reached).

DYNAMICS IN THE VICINITY OF MULTICRITICAL POINTS

Sketched above particular features of static multicritical behavior are further manifested if the critical dynamics is addressed. Below, we briefly analyze three different from of dynamical behavior in the vicinity of multicritical points.

Relaxational dynamics (model A)

Let us start from the simplest dynamical model, model A, when one assumes relaxational behavior for the two order parameters $\phi_k$ and $\phi_\gamma$. This model has been studied in the one-loop approximation in [17], the two-loop results have been obtained in [14]. The model A type Langevin equations of motion describe two order parameters that relax to equilibrium with the relaxation rates (kinetic coefficients) $\dot{\Gamma}_\gamma$ and $\dot{\Gamma}_k$:

$$\frac{\partial \phi_{\gamma0}}{\partial t} = \dot{\Gamma}_\gamma \frac{\delta \mathcal{H}}{\delta \phi_{\gamma0}} + \Theta_{\phi_{\gamma0}}; \quad \dot{\Gamma}_k; \quad \frac{\partial \phi_k0}{\partial t} = \dot{\Gamma}_k \frac{\delta \mathcal{H}}{\delta \phi_k0} + \Theta_{\phi_k0}; \quad (7)$$
Here, $H$ is the static effective Hamiltonian \textbf{(1)}, index 0 refers to bare (unrenormalized) quantities and the stochastic forces $\Theta_{\phi}, \Theta_{\phi}$ fulfill Einstein relations
\begin{align}
\Gamma \delta \phi & = \Gamma \delta \phi + \Theta_{\phi} ; \\
\Gamma \delta \phi & = \Gamma \delta \phi + \Theta_{\phi} ; \quad (9)
\end{align}
with indices $\alpha = 1; \ldots ; n_k$, corresponding to the two subspaces.

Application of the RG procedure to study dynamical multicritical behavior relies on the Bausch-Janssen-Wagner approach \textbf{(27)}, where the appropriate Lagrangian of the model is studied and dynamic vertex functions are calculated in perturbation theory and renormalized. In such a technique, essential simplification of calculations is achieved due to the possibility to single out a static part of every dynamic vertex function \textbf{(28, 29)}. Renormalization of the kinetic coefficients gives rise to appropriate $\beta$-functions. Here, we reveal the two-loop $\beta$-function for the time-scale ratio $v = \Gamma_k = \Gamma_\gamma$ between the renormalized kinetic coefficients $\Gamma_k$ and $\Gamma_\gamma$. The function reads \textbf{(16)}:
\begin{equation}
\beta_v = \frac{v}{72} \left( n_k + 2 u_k^2 \right) \left( n_\gamma + 2 u_\gamma^2 \right) \left( 6 \ln \frac{4}{3} + 1 \right) n_k u^2 \left[ \frac{2 \ln \frac{1 + v}{2 + v}}{2 + v} + 2 \ln \frac{1 + v}{2 + v} \right] \left( \frac{1 + v}{2 + v} \right) + n_\gamma u^2 \left[ 4 \ln \frac{1 + v}{2 + v} + 2 \ln \frac{1 + v}{2 + v} \right] \left( \frac{1 + v}{2 + v} \right) ; \quad (11)
\end{equation}
As we have noted in the preceding section discussing the static critical behavior, a non universal effective critical behavior may be observed if the values of the static couplings and the time scale ratio are not in a FP but rather are described by the flow equations. For $v$ the flow equation reads
\begin{equation}
\frac{dv}{d\tau} = \beta_v u_k\left(\cdot\right) u_\gamma\left(\cdot\right) u \left(\cdot\right) v \left(\cdot\right) ; \quad (12)
\end{equation}
Below we will show some results about non-universal dynamic multicritical behavior obtained with two-loop accuracy. The numerical results for the static part of the RG function were obtained by means of the resummation technique \textbf{(25)}, whereas no resummation has been applied to the dynamic functions \textbf{(16)}.

One of the quantities of interest that characterize dynamic critical phenomena is the autocorrelation time $\tau$. It is known to diverge as the critical point $T_c$ is approached, the divergency is described by the power law:
\begin{equation}
\tau \sim T^{-\nu} \sim \left( T - T_c \right)^{-\nu} ; \quad (13)
\end{equation}
with the universal correlation length and dynamic critical exponents $\nu$ and $z_\gamma$, correspondingly. In the multicritical phenomena we consider, one distinguishes two dynamical critical exponents, $z_k$ and $z_\gamma$, that govern the power law increase of the autocorrelation time for the order parameters $\phi_k$ and $\phi_\gamma$, correspondingly. In asymptotics they are defined by the stable FP values of the corresponding RG functions. At the strong scaling FP there is only one dynamic time scale and the two exponents are equal whereas at the weak scaling FP they are different and define for each component, parallel and perpendicular, the time scale. As it follows from our calculations \textbf{(16)} and as one may see from the Fig.\textbf{5}, the region of stability of the biconical FP $\beta$ (physically important

\frac{\partial \phi_{\phi_0}}{\partial t} = \Gamma \frac{\partial \mathcal{H}}{\partial \phi_{\phi_0}} + \Theta_{\phi} ;
FIGURE 5. Regions of the different types of dynamic scaling behavior, $\varepsilon = 4$; $d = 1$. The rest of notations are as in Fig. 2.

FIGURE 6. Model A multicritical dynamics. Effective dynamical exponent in for different RG flows in the vicinity of a multicritical point at $d = 3$, $n_k = 1$, $n_\gamma = 2$. The labeling of the flows corresponds to Fig. 3. The exponents for the perpendicular (dashed curves) and parallel (solid curves) components of the OP differ in the non asymptotic region.

The case $d = 3$, $n_k = 1$, $n_\gamma = 2$ belongs to this region) is characterized by the strong scaling dynamics: the time relaxation of both order parameters, $\phi_k$ and $\phi_\gamma$ is governed by the same exponent. In Fig. 6 we show an evolution of this exponent $z_{\text{eff}}$ to its asymptotic value $z = 2$ when the time-scale ratio $\nu$ is set to its FP value and the static couplings $u$ change along the RG flows of Fig. 3. Since the exponents have not reached their (equal) asymptotic values differences between the parallel and perpendicular components of the OP remain.
Conservation of magnetization (model C)

A step towards making the description of dynamic phenomena in the vicinity of a multicritical point more realistic is to take into account possible couplings between the order parameters and conserved densities, that is to consider the model C dynamics \[15, 18\]. In the problem under consideration, there are two types of conserved densities: one is magnetization-like (more precisely, it is the parallel component of the magnetization), another is the energy density. We will not consider this second density here, as far as up to the two-loop order the specific heat critical exponent \( \alpha \) is negative for the case \( d = 3, n_k = 1, n_\gamma = 2 \) which is of most interest here. Therefore, a coupling to the energy density is irrelevant in the RG sense - it vanishes at the FP \[29\]. An account of both the order parameter and the (conserved) scalar density is achieved by an extension of the static functional (1). Now, the corresponding model C static functional reads:

\[
\mathcal{H}^{(C)} = \mathcal{H} + \int d^d x \left( \frac{1}{2} m_0^2 + \frac{1}{2} \gamma_0 \phi_0 \phi' \right) + \frac{1}{2} \hat{\gamma}_k \phi_k \phi_k' + \hat{h} m_0 : \tag{14}
\]

Here, the first term in the right hand side is given by Eq. (1), the density \( m_0 \) \( m_0 \) (x) is a scalar quantity, \( \hat{h} \) is a field conjugated to \( m_0 \), \( \gamma_0 \) and \( \gamma_k \) are asymmetric static couplings between the corresponding order parameters and the conserved density.

In their turn, the relaxational equations of motion (7),(8) are now extended by including a diffusion equation for the scalar density:

\[
\frac{\partial \phi_0}{\partial t} = \frac{\partial \phi_k}{\partial t} = \frac{\partial m_0}{\partial t} = \hat{\lambda} \nabla^2 \delta \mathcal{H}^{(C)} + \theta_m : \tag{17}
\]

Here, the static functional \( \mathcal{H}^{(C)} \) is given by (14), \( \hat{\lambda} \) is a kinetic coefficient of diffusive type for the scalar density, the rest of notations is as in (7),(8). The stochastic forces \( \theta_{\phi_0} \), \( \theta_{\phi_k} \) satisfy the Einstein relations (9), (10), with an additional Einstein relation for the new stochastic force \( \theta_m \):

\[
\theta_m(x,t) \theta_m(x'^0,t'^0) = 2 \hat{\lambda} \nabla^2 \delta(x-x'^0) \delta(t-t'^0) : \tag{18}
\]

The renormalization of the above introduced asymmetric couplings \( \gamma_0 \), \( \gamma_k \) and kinetic coefficient \( \hat{\lambda} \) leads to new RG functions. In particular the RG flow of the time scale ratios

\[
w_\gamma = \frac{\Gamma_\gamma}{\hat{\lambda}} ; \quad w_k = \frac{\Gamma_k}{\hat{\lambda}} \tag{19}
\]

is now governed by the appropriate functions \( \beta_{w_\gamma} \) and \( \beta_{w_k} \), correspondingly. Note that defined for model A time scale ratio \( v \) is equally well defined in terms of (19):

\[
v \frac{\Gamma_k}{\Gamma_\gamma} = \frac{w_k}{w_\gamma} \tag{20}
\]

Therefore, the dynamical FP equations:

\[
\beta_{w_\gamma}(w_\gamma, \eta, v) = \beta_{w_k}(w_k, \eta, v) = \beta_v(w_\gamma, \eta, \nu) = 0 \tag{21}
\]
Equations of motion (15)–(17) describe time evolution of three different observables. Each of them has its own autocorrelation time which, as the multicritical point is reached, may be governed by an independent dynamical critical exponent. In addition to the two exponents defined in the former subsection, $z_k$ and $z_\?$, the dynamical critical exponent $z_m$ for the scalar density is to be considered. Similar, as in the model A case, these three exponents may coincide, in the strong scaling dynamical FP or they may differ, in the weak scaling dynamical FP. Complete stability analysis of the model C RG equations in two-loop approximation is given in Ref. [16]. In particular, it is shown that for the case $d = 3, n_k = 1, n_\? = 2$, where the static FP is the biconical FP $B$, the strong scaling dynamical FP is stable. Physically this means that in asymptotics the multicritical dynamics is characterized by one time scale, and three dynamical exponents coincide. In particular, their asymptotical value was found to be $z_k = z_\? = z_m = 2.18$ [14]. However, as it was revealed in the former sections, the effective multicritical behavior is much richer. In particular, in Fig. 7a we show the RG flows calculated for different dynamical initial conditions when the static couplings are chosen to be fixed at their biconical FP values. The stable dynamical (strong scaling) FP lies outside the region shown. Also shown is the surface $v = w_k = w_?$ to which the flow is restricted by the condition (20). The RG flows of Fig. 7a give rise to difference in the effective dynamical critical exponents, as shown in Fig. 7b. The insert of the figure shows that even for flow parameters as small as $\ln^t = 2000$ the effective exponent $z_?$ has not reached its asymptotic value $2.18$.

The complete dynamic model (model G)

We now restrict ourselves to the case of $n_k = 1, n_\? = 2$ and include mode coupling terms, which correspond to Larmor terms describing the precession of the alternating magnetization and the magnetization around each other. They are well known from the isotropic antiferromagnet without
an external field \[30\]. Then within an external magnetic field the corresponding equations read

\[
\frac{\partial \phi^\alpha_{z,0}}{\partial t} = \Gamma^\alpha_{z,0} \frac{\delta \mathcal{H}^{(C)}}{\delta \phi^\alpha_{z,0}} + \hat{\Gamma}^\alpha_{z,0} \varepsilon^{\alpha\beta\gamma} \frac{\delta \mathcal{H}^{(C)}}{\delta \phi^\beta_{z,0}} + \hat{g} \varepsilon^{\alpha\beta\gamma} \phi^\beta_{z,0} \frac{\delta \mathcal{H}^{(C)}}{\delta m_0} + \theta^\alpha ; \tag{22}
\]

\[
\frac{\partial \phi_{x,0}}{\partial t} = \Gamma^\alpha_{x,0} \frac{\delta \mathcal{H}^{(C)}}{\delta \phi_{x,0}} + \theta_{\phi_x} ; \tag{23}
\]

\[
\frac{\partial m_0}{\partial t} = \hat{\lambda} \nabla^2 \frac{\delta \mathcal{H}^{(C)}}{\delta m_0} + \hat{g} \varepsilon^{\alpha\beta\gamma} \phi^\beta_{z,0} \frac{\delta \mathcal{H}^{(C)}}{\delta \phi^\gamma_{z,0}} + \theta_m ; \tag{24}
\]

Now \(\alpha\) and \(\beta\) indicates the planar components \(xy\) and the Levi-Civita tensor \(\varepsilon^{\alpha\beta\gamma}\) with the third index fixed to \(z\) has been introduced. The parallel component of the OP is its \(z\)-component. This component remains just relaxing, whereas the planar components of the OP are coupled to the \(z\)-component of the magnetization by the precession terms.

A new feature arises because of the simultaneous presence of the mode coupling \(\hat{g}\) and the asymmetric static couplings \(\gamma^\alpha_{\gamma,0}\) and \(\gamma^\alpha_{k,0}\) in \(\mathcal{H}^{(C)}\) \[14\]. The perpendicular relaxation coefficient \(\Gamma^\gamma_{\gamma,0}\) has to be considered a complex quantity where the imaginary part constitute a precession term (second term on the right hand side of \(22\)). Even if in the background such terms are absent they are produced by the renormalization procedure.

The stochastic forces \(\theta_{\phi_z}, \theta_{\phi_k}\) and \(\theta_m\) fulfill Einstein relations

\[
\Gamma \theta^\alpha_{\phi_z} (x^\alpha t) \theta^\beta_{\phi_z} (x^\alpha 0) \delta_{i,0} = 2 \hat{\Gamma}^\alpha_{\phi_z} \delta (x \cdot x^0) \delta (t \cdot t^0) \delta^{\alpha\beta} ; \tag{25}
\]

\[
\Gamma \theta^\alpha_{\phi_k} (x^\alpha t) \theta^\beta_{\phi_k} (x^\alpha 0) \delta_{i,0} = 2 \hat{\Gamma}^\alpha_{\phi_k} \delta (x \cdot x^0) \delta (t \cdot t^0) ; \tag{26}
\]

\[
\Gamma \theta^\alpha_m (x^\alpha t) \theta^\beta_m (x^\alpha 0) \delta_{i,0} = 2 \hat{\lambda} \nabla^2 \delta (x \cdot x^0) \delta (t \cdot t^0) ; \tag{27}
\]

This model has been solved in one loop order in \[17\] using the one loop results of statics. As is has been already seen for the simpler dynamic models models changes are expected in two loop order both by the statics as well as by the dynamic terms especially of model C type. We have calculated the complete field theoretic functions in two loop order \[31\] necessary to calculate the critical (effective) dynamical exponents. Independent whether the Heisenberg or biconical is the stable static FP a first inspection of the flow of the dynamical parameters shows the following: (i) The imaginary part of the perpendicular relaxation rate renormalizes to zero, (ii) the times scale ratios \(v\) \[20\], \(w_k\) \[19\] approach zero and \(w_\gamma\) increases to \(\infty\). Irrespective of the kind of the stable dynamic FP - wether it is a strong scaling FP with very small but finite or a weak scaling FP with zero values for \(v\) and \(w_k\) - the physical observable features of the magnetic transport coefficient are effective ones. The range of effective values for the dynamic exponents corresponding to the relaxation of the perpendicular and parallel alternating magnetization and the magnetization are starting around its Van Hove values \(z_? \quad z_k \quad z_\lambda \quad 2\) in the background and approach for the biconical FP deep in the asymptotic regime

\[
z_? \quad 1 \quad z_k \quad 2 \quad z_\lambda \quad 1 \quad 6 ; \tag{28}
\]

The main prediction according to this result would be that the perpendicular and the parallel component of the OP would scale differently in this region.

The importance of this magnetic system lies in the physical accessibility of the OP, contrary to superfluid \(^4\)He or superfluid mixture of \(^4\)He and \(^3\)He whose dynamics is described by model F
Here all quantities are in principle measurable quantities. Thus the prediction of the different dynamic scaling of the OP components can be tested.

CONCLUSIONS AND OUTLOOK

By this review we wanted to summarize recent progress achieved in theoretical description of the multicritical phenomena. Whereas traditionally RG techniques address critical points in their different realizations, the description of multicritical phenomena is possible both on quantitative and accurate qualitative levels. Moreover, the problem appears to be tractable analytically even if the complicated forms of multicritical dynamics are confronted. As is revealed by the theoretical analysis, a particular feature of static and dynamic behavior inherent to multicritical points is the multitude of fixed points that describe the RG flow. In its turn, this gives rise to rich effective behavior that may be characterized by different types of multicritical points. A natural continuation of performed studies would be to analyze cumulative effects caused on the multicritical behavior by symmetry breaking factors of different forms (single-ion anisotropies, disorder, frustrations) that might be present in a system.

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