Achievable Information Rates for Probabilistic Amplitude Shaping: A Minimum-Randomness Approach via Random Sign-Coding Arguments

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Abstract—Probabilistic amplitude shaping (PAS) is a coded modulation strategy in which constellation shaping and channel coding are combined. PAS has attracted considerable attention in both wireless and optical communications. Achievable information rates (AIRs) of PAS have been investigated in the literature using Gallager’s error exponent approach. In particular, it has been shown that PAS achieves the capacity of a memoryless channel. In this work, we revisit the capacity-achieving property of PAS, and derive AIRs using weak typicality. We provide alternative proofs based on random sign-coding arguments. Our objective is to minimize the randomness in the random coding experiment. Accordingly, in our proofs, only some signs of the channel inputs are drawn from a random code, while the remaining signs and the amplitudes are produced constructively. We consider both symbol-metric and bit-metric decoding.

Index Terms—Probabilistic amplitude shaping, achievable information rate, symbol-metric decoding, bit-metric decoding.

I. INTRODUCTION

Coded modulation (CM) refers to the design of forward error correction (FEC) codes and high-order modulation formats which are combined to reliably transmit more than one bit per channel use. Examples of CM strategies include multilevel coding (MLC) [1], [2] in which each address bit of the signal point is protected by an individual binary FEC code, and trellis CM [3] which combines the functions of a trellis-based channel code and a modulator. Among many CM strategies, bit-interleaved CM (BICM) [4], [5] which combines a high-order modulation format with a binary FEC code using a binary labeling strategy, and uses bit-metric decoding (BMD) at the receiver, is the de-facto standard for CM. BICM is included in multiple wireless communication standards such as the IEEE 802.11 [6] and the DVB-S2 [7]. BICM is also nowadays the de-facto CM alternative for fiber optical communications.

Proposed in [8], probabilistic amplitude shaping (PAS) integrates constellation shaping into existing BICM systems. The shaping gap that exists for channels with non-uniform capacity-achieving distributions, e.g., the additive white Gaussian noise (AWGN) channel [9, Ch. 9], can be closed with PAS. To this end, an amplitude shaping block converts binary information strings into shaped amplitude sequences in an invertible manner. Then a systematic FEC code produces parity bits encoding the binary labels of these amplitudes. These parity bits are used to select the signs, and the combination of the amplitudes and the signs, i.e., probabilistically shaped channel inputs, are transmitted over the channel. PAS has attracted considerable attention in fiber optical communications due to its availability of providing rate adaptivity.

Achievable information rates (AIRs) of PAS have been investigated in the literature [10], [11], [12]. It has been shown that the capacity of any memoryless channel can be achieved with PAS, e.g., in [11, Example 10.4]. The achievability proofs in the literature are based on Gallager’s error exponent approach [13, Ch. 5] and strong typicality [14, Ch. 1].

In this work, we provide alternative proofs of achievability for PAS, and revisit its capacity-achieving property. We use random sign-coding arguments based on weak typicality [9, Secs. 3.1 and 15.2], and consider both symbol-metric decoding (SMD) and BMD. As explained in Sec. II-D, the main contributions of this paper are (1) to provide alternative proofs of achievability for PAS, and (2) to unify results somewhat scattered in the literature.

This work is organized as follows. In Sec. II, we briefly summarize the related literature on CM, AIRs, and PAS, and state our contribution. In Sec. III, we provide some background information on typical sequences, and establish the system model. Finally, in Sec. IV, we provide random sign-coding arguments to derive AIRs for PAS, and consequently, show that it achieves capacity. Conclusions are drawn in Sec. V.

II. RELATED WORK AND OUR CONTRIBUTION

A. Achievable Information Rates

For a memoryless channel which is characterized by an input alphabet $\mathcal{X}$, input distribution $p(x)$, and channel law $p(y|x)$, the maximum AIR is the mutual information (MI)
$I(X; Y)$ of the channel input $X$ and output $Y$. Consequently, the capacity of this channel is defined as $I(X; Y)$ maximized over all possible input distributions $p(x)$, typically under an average power constraint, e.g., in [9, Sec. 9.1]. The MI can be achieved, e.g., with MLC and multi-stage decoding [1], [2].

In BICM systems, channel inputs are uniquely labeled with $\log_2 |\mathcal{X}| = m$-bit binary strings. At the transmitter, the output of a binary FEC code is mapped to channel inputs using this labeling strategy. At the receiver, BMD is employed, i.e., binary labels $C_1, C_2, \cdots, C_m$ are assumed to be independent, and consequently, the symbol-wise decoding metric is written as the product of bit-metrics

$$q_i(x, y) = \prod_{i=1}^{m} q_i(c_i, y).$$

Since the metric in (1) is in general not proportional to $p(y|x)$, i.e., there is a mismatch between the actual channel law and the one assumed at the receiver, this setup is called mismatched decoding.

Different AIRs have been derived for this so-called mismatched decoding setup. One of these is the generalized MI (GMI) [15], [16]

$$\text{GMI} (p(x)) = \max_{s \geq 0} E \left[ \log \frac{[q(X, Y)]^s}{\sum_{x \in \mathcal{X}} p(x) [q(x, Y)]^s} \right],$$

which reduces to [18, Thm. 4.11, Coroll. 4.12], [19]

$$\text{GMI} (p(c_1)p(c_2)\cdots p(c_m)) = \sum_{i=1}^{m} I(C_i; Y),$$

when the bit-levels are independent at the transmitter, i.e.,

$$p(x) = p(c_1, c_2, \cdots, c_m) = p(c_1)p(c_2)\cdots p(c_m),$$

and

$$q_i(c_i, y) = p(y|c_i).$$

The rate (3) is achievable for both uniform and shaped bit-levels [5], [20]. The parameter that maximizes (2) to obtain (3) is $s = 1$.

Another AIR for mismatched decoding is the LM rate [16], [21]

$$\text{LM} (p(x)) = \max_{s \geq 0, r(\cdot)} E \left[ \log \frac{[q(X, Y)]^s r(X)}{\sum_{x \in \mathcal{X}} p(x) [q(x, Y)]^s r(x)} \right],$$

where $r(\cdot)$ is a real-valued cost function defined on $\mathcal{X}$. The expectations in (2) and (5) are taken with respect to $p(x, y)$.

1**Notation:** Capital letters $X$ are used to denote random variables, lower case letters $x$ to specify their realizations. Random vectors are indicated by $X$ while their realizations are by $x$. Element-wise multiplication of $x$ and $y$ is denoted by $x \odot y$. Calligraphic letters $\mathcal{X}$ represent sets while the Cartesian product of $\mathcal{X}$ and $\mathcal{Y}$ is shown as $\mathcal{X} \times \mathcal{Y}$. We denote by $\mathcal{X}^n$ the $n$-fold Cartesian product of $\mathcal{X}$ with itself. Probability density and mass functions over $X$ are denoted by $p(x)$. We use $1[\cdot]$ to indicate the indicator function which is 1 when its argument is true and 0 otherwise. The entropy (in bits) of $X$ is denoted by $H(X)$, the expected value of $X$ by $E[X]$.

2Here we assume that $|\mathcal{X}|$ is an integer power of two.

3The problem of computing the bit-level distributions that maximize the GMI in (3) is shown to be non-convex in [17].

When there is dependence among bit-levels, i.e., $p(c_1, c_2, \cdots, c_m)$ is not a product distribution, the rate [22], [23]

$$R_{BMD} (p(x)) = \left[ H(C_1C_2\cdots C_m) - \sum_{i=1}^{m} H(C_i|Y) \right]^+.\hspace{1cm}(6)$$

has been shown to be achievable by BMD for any joint input distribution $p(c_1, c_2, \cdots, c_m)$. Here $\lceil \cdot \rceil$ denotes $\max\{0, \cdot \}$. In [22], [23], the achievability of (6) is derived using random coding arguments based on strong typicality [14, Ch. 1]. Later in [24, Lemma 1], it is shown that (6) is an instance of the so-called LM rate (5) for $s = 1$, the symbol decoding metric (1), bit decoding metrics (4), and the cost function

$$r(c_1, c_2, \cdots, c_m) = \frac{\prod_{i=1}^{m} p(c_i)}{p(c_1, c_2, \cdots, c_m)}.$$

**B. Probabilistic Amplitude Shaping: Model**

PAS [8] is a capacity-achieving CM strategy in which constellation shaping and FEC coding are combined as shown in Fig. 1. In PAS, first an amplitude shaping block maps $k$-bit information strings to $n$-amplitude shaped sequences $a = (a_1, a_2, \cdots, a_n)$ in an invertible manner. These amplitudes are drawn from a $2^{\gamma n}$-ary alphabet $A$. The amplitude shaping block can be realized using constant composition distribution matching [25], multiset-partition distribution matching [26], shell mapping [27], enumerative sphere shaping [28], etc.

After $n$ amplitudes are generated, binary labels $c_1, c_2, \cdots, c_{m-1}$ of the amplitudes $a$, and an additional $\gamma n$-bit information string $s = (s_1, s_2, \cdots, s_{\gamma n})$ are fed to a $(m-1 + \gamma)/m$ systematic FEC encoder. The encoder produces $(1-\gamma)n$ parity bits $s_2 = (s_{\gamma n+1}, s_{\gamma n+2}, \cdots, s_n)$. The additional data bits $s_1$ and the parity bits $s_2$ are used as the signs $\mathfrak{s} = (s_1, s_2, \cdots, s_n)$ for the amplitudes $a$. Finally, probabilistically shaped channel inputs $x = \mathfrak{s} \otimes a$ are transmitted through the channel. Here $\gamma$ is the rate of the additional information in bits per symbol (bit/1-D), or equivalently, the fraction of signs that are selected directly by data bits. The transmission rate of PAS is $R = k/n + \gamma$ in bit/1-D.

**C. Probabilistic Amplitude Shaping: Achievable Rates**

Based on Gallager's error exponent approach [13, Ch. 5], AIRs of PAS have been investigated in [10], [11], [12]. In [10], a random code ensemble is considered from which the channel inputs $x$ are drawn. Then the AIR in [10, Eqs. (32)-(34)] is derived for a general memoryless decoding metric $q(x, y)$. It is shown that by properly selecting $q(x, y), I(X; Y)$ and the rate (6) can be recovered from the derived AIR, and consequently, they can be achieved with PAS.

Computing error exponents for PAS is also the main concern of the work presented in [11, Ch. 10]. The difference from [10] is in the random coding setup. In [11, Ch. 10], a random code ensemble is considered from which only the signs $\mathfrak{s}$ of the channel inputs are drawn at random. We call this the random sign-coding setup. The error exponent [11, Eq. (10.42)] is then derived again for a general memoryless decoding metric. Error
exponents of PAS have also been examined based on the joint source-channel coding (JSCC) setup in [12], [29]. Random sign-coding is considered in [12], [29], but only with SMD, and only for the specific case where $\gamma = 0$.

### D. Our Contribution

Here, we revisit the capacity-achieving property of PAS, and derive AIRs in the typicality framework. We use weak typicality [9, Secs. 3.1 and 15.2], and consider SMD with an arbitrary value of $0 \leq \gamma \leq 1$. We then derive AIRs for BMD, and unify some scattered results in the literature. Zooming in on the random sign-coding structure, we provide alternative proofs of achievability based on random sign-coding arguments. Our focus is to keep the amount of randomness which is in the random coding procedure as small as possible, or equivalently, to mimic the actual situation in which channel inputs are produced constructively as close as possible. To this end, in our random sign-coding experiment, both the amplitude sequences ($a$) and the sign sequence parts ($s_1$) which are information bits are constructively produced, and only the remaining signs ($s_2$) are randomly generated as illustrated in Fig. 2.

![Fig. 2: The scope of the random coding experiments considered in this work and in [10], [11], [12].](image)

We note that our approach is different than the random sign-coding setup considered in [11] and [12] where all signs ($s_1$ and $s_2$) are generated randomly which is called partially systematic encoding in [11, Ch. 10]. Furthermore, we define a special type of typicality (B-typicality, see Definition 1 below) that allows us to avoid the mismatched JSSC approach of [12].

### III. Preliminaries

#### A. Typical Sequences

We will provide achievability proofs based on weak typicality. For the properties of the corresponding typical sets and sequences, we refer the reader to [9, Secs. 3.1 and 15.2].

Let $0 < \varepsilon < 1$ and $n$ be a positive integer. Consider joint probability distribution $p(u,v) = p(u)p(v|u)$ where $u \in U$ and $v \in V$. Let $u$ and $v$ denote $n$-sequences over the alphabets $U$ and $V$, respectively. Let $A^n_v(U)$ and $A^n_u(UV)$ denote typical and jointly typical sets, respectively. The notation $(U)$ and $(UV)$ is used to emphasize that $A^n_v(U)$ is a set of vectors $u = (u_1, u_2, \ldots, u_n)$ while $A^n_u(UV)$ is a set of vectors $(u, v) = ((u_1, u_2, \ldots, u_n), (v_1, v_2, \ldots, v_n))$.

**Definition 1 (B-typicality).** Let the input probability distribution $p(u)$ together with the transition probability distribution $p(v|u)$ determine the joint probability distribution $p(u, v) = p(u)p(v|u)$. Now we define

$$B^n_v(U) \triangleq \{ u : u \in A^n_v(U) \}$$

and

$$\Pr \{ (u, v) \in A^n_v(UV) \mid U = u \} \geq 1 - \varepsilon,$$

where $V$ is the output sequence of a “channel” $p(v|u)$ when sequence $u$ is input.

The set $B^n_v(U)$ in (8) guarantees that, a sequence $u$ in this typical set, will with high probability generate a sequence $v$ that is jointly typical with $u$. We note that $U$ and/or $V$ can be composite. The set $B^n_v(U)$ has three properties, as stated in the following Lemma. The proof of Lemma 1 is given in Appendix VI-A.

**Lemma 1 (B-typicality properties).** The set $B^n_v(U)$ in Definition 1 has the following properties:

- **$P_1$**: For $u \in B^n_v(U)$,
  $$2^{-n(H(U) + \varepsilon)} \leq p(u) \leq 2^{-n(H(U) - \varepsilon)}.$$  
- **$P_2$**: For $n$ large enough,
  $$\sum_{u \in B^n_v(U)} p(u) \leq \varepsilon.$$
- **$P_3$**: $|B^n_v(U)| \leq 2^{n(H(U) + \varepsilon)}$, while for $n$ large enough
  $$|B^n_v(U)| \geq (1 - \varepsilon)2^{n(H(U) - \varepsilon)}.$$  

#### B. Channel Model

We consider communication over a real memoryless channel with input $X$. We assume that the inputs are chosen from a discrete set $\mathcal{X}$, i.e., the signal constellation. The output $Y$ is a quantized version of the continuous channel output. This leads
to a discrete-input discrete-output channel \(\{X, p(y|x), Y\}\).

Finally, we assume that the channel output is quantized with a resolution high enough that the discrete-output channel is an accurate model for the underlying continuous-output channel.

C. Binary Labeling

We use \(2^m\)-ary amplitude shift keying (M-ASK) alphabets \(X = \{-M + 1, -M + 3, \ldots, M - 1\}\) where \(M = 2^m\). The integer \(m\) is the number of bits needed to label \(x \in X\). We note that \(X\) is symmetric around the origin and can be factorized as \(X = S \times A\). Here \(S = \{-1, +1\}\) and \(A = \{+1, +3, \ldots, M - 1\}\) are the sign and amplitude alphabets, respectively. Accordingly, any channel input \(x \in X\) can be written as the multiplication of a sign and an amplitude, i.e., \(x = s \otimes a\). The amplitude is addressed by \(m-1\) amplitude bits \(B_1 B_2 \cdots B_{m-1}\), while the sign is addressed by a sign bit \(S\). We emphasize that unlike the case in Sec. II-A, we use \(SB_1 B_2 \cdots B_{m-1}\) to denote a channel input instead of \(C_1 C_2 \cdots C_m\). Amplitudes and signs of \(x \in X\) are tabulated for 8-ASK in Example 1.

Example 1. Table 1 shows the corresponding amplitude-sign pair for each \(x \in X\) for 8-ASK.

| Table 1 |
|---------|

| \(a\) | 7 | 5 | 3 | 1 | 1 | 3 | 5 | 7 |
|-------|---|---|---|---|---|---|---|---|
| \(s\) | -1 | -1 | 1 | -1 | 1 | +1 | 1 | +1 |
| \(x\) | -7 | -5 | -3 | -1 | +1 | +3 | +5 | +7 |

D. Amplitude Shaping and Sign-Coding

We cast the PAS structure shown in Fig. 1 as a sign-coding structure as in Fig. 3. The sign-coding setup consists of two layers: a shaping layer and a coding layer.

Definition 2 (Sign-coding). Let \(A\) be distributed with \(p(a)\) over \(a \in A\), and \(S\) be uniform over \{-1, +1\} and independent of \(A\). Together with the channel law \(p(y|x)\), this leads to a joint distribution \(p(a, s, y) = p(a)p(s)p(y|x)\) where \(x = s \otimes a\). Next, fix \(\varepsilon > 0\) and \(n > 0\), and consider the set \(B_{SY,\varepsilon}^n(A)\) of \(B\)-typical amplitude sequences \(\alpha\), and note that \(|B_{SY,\varepsilon}^n(A)| \approx 2^{nH(A)}\). Also note that in the context of Definition 1, in \(B_{SY,\varepsilon}^n(A)\) amplitude \(A\) is in the role of \(U\) and pair \(SY\) replaces \(V\).

Now for every message index pair \((m_n, m_s)\), with \(m_n \in \{1, 2, \cdots, M_n\}\) and \(m_s \in \{1, 2, \cdots, M_s\}\), a sign-coding structure consists of (see Fig. 3):

- **A shaping layer** that produces for every message index \(m_n\), a length-\(n\) shaped amplitude sequence \(\alpha(m_n) = (a_1(m_n), a_2(m_n), \cdots, a_n(m_n)) \in B_{SY,\varepsilon}^n(A)\) where the mapping is one-to-one. The set \(B\) of amplitude sequences is assumed to be shaped but uncoded.
- An additional \(n_1\)-bit information string in the form of a sign sequence part \(\tilde{x}(m_s) = (s_1(m_s), s_2(m_s), \cdots, s_{n_1}(m_s))\) for every message index \(m_s\).

- **A coding layer** that extends the sign sequence \(\tilde{x}(m_s)\) by adding a second length-\(n_2\) sign sequence part \(\tilde{x}'(m_a, m_s) = (s_{n_1+1}(m_a, m_s), s_{n_2+1}(m_a, m_s), \cdots, s_n(m_a, m_s))\) of length-\(n_2\) to \(\tilde{a}(m_a)\) and \(\tilde{x}(m_s)\), for all \(m_a\) and \(m_s\). This is obtained by using an encoder that produces redundant signs in the set \(S\) from \(\tilde{a}(m_a)\) and \(\tilde{x}(m_s)\). Here \(n_1 + n_2 = n\).

Finally, the transmitted sequence is \(z(m_a, m_s) = \tilde{a}(m_a) \otimes \tilde{x}(m_s)\), where \(z(m_a, m_s) = (\tilde{x}(m_s), \tilde{x}'(m_a, m_s))\). The sign-coding setup with \(n_1 = 0\) \((\gamma = 0)\) is called basic sign-coding, while the setup with \(n_1 > 0\) \((\gamma > 0)\) is called modified sign-coding.

In the following, the concept of AIR is formally defined in the sign-coding context.

Definition 3 (Achievable information rate). A rate \(R\) is said to be achievable if for every \(\delta > 0\) and \(n\) large enough, there exists a sign-coding encoder and a decoder such that \((1/n) \log_2 (M_n M_s) \geq R - \delta\) and error probability \(P_e \leq \delta\).

IV. RANDOM SIGN-CODING ALGORITHMS

Here we investigate AIRs of the sign-coding architecture in Fig. 3. We consider both SMD and BMD at the receiver. In what follows, three AIRs are presented. The proofs are based on \(B\)-typicality and random sign-coding arguments, and are given in Appendices. As indicated in Definition 2, signs \(S\) are assumed to be uniform in the proofs.

A. Sign-Coding with Symbol-Metric Decoding

Theorem 1 (Basic sign-coding with SMD). For any memoryless channel \(\{X, p(y|x), Y\}\) with amplitude shaping and basic sign-coding, the rate

\[ R_{\text{SMD}}^{\infty} = \max_{p(a): H(A) \leq I(SA; Y)} H(A), \tag{10} \]

is achievable using SMD.

Theorem 1 implies that for any memoryless channel, the rate \(H(A)\) is achievable with basic sign-coding, as long as \(H(A) \leq I(SA; Y) = I(X; Y)\) is satisfied. For the AWGN channel, this means that a range of rate-SNR pairs are achievable. One of these points, \(H(A) = I(SA; Y)\), is on the capacity-SNR curve, for which \(H(SA|Y) = 1\). Here SNR denotes the signal-to-noise ratio. This will be explained more clearly in Example 2. Here we used \(H(SA) = H(A) + 1\) which is due to the assumption that \(S\) and \(A\) are independent, and \(S\) is uniform.

Theorem 2 (Modified sign-coding with SMD). For any memoryless channel \(\{X, p(y|x), Y\}\) with amplitude shaping and modified sign-coding, the rate

\[ R_{\text{SMD}}^{\infty} = \max_{p(a), \gamma: H(A)+\gamma \leq I(SA; Y)} H(A) + \gamma, \tag{11} \]

is achievable using SMD.

Theorem 2 implies that for any memoryless channel, the rate \(H(A) + \gamma\) is achievable with modified sign-coding, as long as
In this paper, we studied achievable information rates (AIRs) of probabilistic amplitude shaping (PAS) for memoryless channels. In contrast to the existing literature in which Gallager’s error exponent approach was followed, we used weak typicality framework. Random sign-coding arguments were introduced to upper-bound the probability of error of a so-called sign-coding structure. Achievability of the mutual information is demonstrated. Sign-coding combined with amplitude shaping corresponds to PAS, and consequently, PAS achieves capacity.

\[ H(A) + \gamma \leq I(SA;Y) = I(X;Y) \] is satisfied. For the AWGN channel, this means all points on the capacity-SNR curve for which \( H(SA|Y) \leq 1 - \gamma \) are achievable. This follows from

\[ H(SA|Y) = H(A) + 1 - I(SA;Y), \] (12)

and the constraint in the maximization in (11).

**Example 2.** We consider the AWGN channel with average power constraint \( E[X^2] \leq P \). Figure 4 shows the capacity of 4-ASK

\[ C_{4-ASK} = \max_{p(x):|x|^2 \leq P} I(X;Y), \] (13)

together with the amplitude entropy \( H(A) \) of the distribution that achieves this capacity. Here SNR = \( E[X^2]/\sigma^2 \), and \( \sigma^2 \) is the noise variance. Basic sign-coding achieves capacity only for SNR = 0.72 dB, i.e., at the point where \( H(A) = I(X;Y) \) which is \( C_{4-ASK} = 0.562 \) bit/1-D. We see from Fig. 4 that the shaping gap is negligible around this point, i.e., the capacity \( C_{4-ASK} \) of 4-ASK and the MI \( I(X;Y) \) for uniform \( p(x) \) are virtually the same. On the other hand, this gap is significant for larger rates, e.g., it is around 0.42 dB at 1.6 bit/1-D. To achieve rates larger than 0.562 bit/1-D on the capacity-SNR curve, sign-coding with \( \gamma \) > 0 is required. At a given SNR, \( C_{4-ASK} \) can be written as \( C_{4-ASK} = H(A) + \gamma \), i.e., when \( H(A) \) is shifted above by \( \gamma \), the crossing point is again at \( C_{4-ASK} \) for that SNR. We have also plotted the additional rate \( \gamma = C_{4-ASK} - H(A) \) in Fig. 4. As an example, at SNR = 9.74 dB, \( C_{4-ASK} = H(A) + \gamma = 1.6 \) can be achieved with modified sign-coding where \( H(A) = 0.91 \) and \( \gamma = 0.7 \). We observe that sign-coding achieves capacity of 4-ASK for SNR ≥ 0.72 dB.

**B. Sign-Coding with Bit-Metric Decoding**

The following theorem gives an AIR for sign-coding with BMD. For simplicity and due to space limitations, we only consider 8-ASK \( (n = 3) \). However, our result can be generalized to any ASCII alphabet.

\[ R_{BMD}^\gamma = \max_{p(b_1,b_2):H(B_1,B_2) \leq H(S,B_2) - H(S|B_2) - H(B_1|Y) - H(B_2|Y)} H(B_1,B_2). \] (14)

is achievable using BMD.

The result of Theorem 3 can be generalized for any 2m-ASK channel input alphabet such that the rate \( H(A) = H(B_1,B_2,\ldots,B_{m-1}) \) is achievable with basic sign-coding using BMD, as long as \( H(A) \leq R_{BMD} \) is satisfied. Here \( R_{BMD} \) is as given in (6).

**V. CONCLUSIONS**

In this paper, we studied achievable information rates (AIRs) of probabilistic amplitude shaping (PAS) for memoryless channels. In contrast to the existing literature in which Gallager’s error exponent approach was followed, we used weak typicality framework. Random sign-coding arguments were introduced to upper-bound the probability of error of a so-called sign-coding structure. Achievability of the mutual information is demonstrated. Sign-coding combined with amplitude shaping corresponds to PAS, and consequently, PAS achieves capacity.
Our alternative approach is different than the random sign-coding experiments considered in the literature, in the sense that our objective was to minimize the randomness in the random coding procedure. To this end, in our random sign-coding setup, both the amplitudes and the signs of the channel inputs are directly selected by information bits are constructively produced. This resembles the situation in actual coded modulation systems where channel inputs are created constructively.

VI. APPENDIX

A. Proof of Lemma 1

Proof of P1: We see from [9, Eq. (3.6)] that for \( u \in A^n_\varepsilon(U) \),
\[
2^{-n(H(U)+\varepsilon)} \leq p(u) \leq 2^{-n(H(U)-\varepsilon)},
\]
Due to Definition 1, each \( u \in B^0_{V,\varepsilon}(U) \) is also in \( A^n_\varepsilon(U) \), more specifically, \( B^0_{V,\varepsilon}(U) \subset A^n_\varepsilon(U) \). Consequently, (15) also holds for \( u \in B^0_{V,\varepsilon}(U) \) which completes the proof of P1.

Proof of P2: Let \((U,V)\) be i.i.d. with respect to \( p(u,v) \). Then
\[
\Pr\{ (U,V) \notin A^n_\varepsilon(UV) \} = 1 - \Pr\{ (U,V) \in A^n_\varepsilon(UV) \},
\]
\[
\geq 1 - \sum_{u \in A^n_\varepsilon(U)} \Pr\{ (u,V) \in A^n_\varepsilon(UV) | U = u \},
\]
\[
\geq 1 - \left( \sum_{u \in A^n_\varepsilon(U)} p(u) + \sum_{u \notin A^n_\varepsilon(U)} p(u)(1-\varepsilon) \right),
\]
\[
= 1 - \sum_{u \in B^0_{V,\varepsilon}(U)} p(u) - (1-\varepsilon) \left( 1 - \sum_{u \in B^0_{V,\varepsilon}(U)} p(u) \right),
\]
\[
= \varepsilon - \varepsilon \Pr\{ U \in B^0_{V,\varepsilon}(U) \},
\]
from which we obtain
\[
\Pr\{ U \in B^0_{V,\varepsilon}(U) \} \geq 1 - \frac{\Pr\{ (U,V) \notin A^n_\varepsilon(UV) \}}{\varepsilon} \overset{(a)}{\geq} 1 - \varepsilon,
\]
where (a) follows from the weak law of large numbers which indicates that \( \Pr\{ (U,V) \notin A^n_\varepsilon(UV) \} \leq \varepsilon^2 \) for large enough \( n \). Since (a) implies P2, the proof is now complete.

Proof of P3: We see from [9, Thm. 3.1.2] that
\[
|A^n_\varepsilon(U)| \leq 2^{n(H(U)+\varepsilon)},
\]
Since \( B^0_{V,\varepsilon}(U) \subset A^n_\varepsilon(U) \) again by Definition 1, (16) also holds for \( |B^0_{V,\varepsilon}(U)| \). From (a) in the proof of P2, we obtain for \( n \) sufficiently large that
\[
1 - \varepsilon \leq \Pr\{ U \in B^0_{V,\varepsilon}(U) \},
\]
\[
\leq \sum_{u \in B^0_{V,\varepsilon}(U)} 2^{-n(H(U)-\varepsilon)},
\]
\[
= |B^0_{V,\varepsilon}(U)| 2^{-n(H(U)-\varepsilon)},
\]
where (b) follows from (15).

B. Proof of Theorem 1

Code Construction: We fix \( \varepsilon \) and consider the set \( B^0_{V,Y,\varepsilon}(A) \) of \( B \)-typical \( v \)-sequences. These amplitude sequences are indexed \( a(1), a(2), \ldots, a(M_a) \) where \( M_a \Delta \) \( B^0_{V,Y,\varepsilon}(A) \).

Random sign-coding: Now we randomly generate for each sequence \( a \in B^0_{V,Y,\varepsilon}(A) \), a sign sequence \( s \) of length \( n \). Each sign \( s \in S = \{-1,+1\} \) is chosen uniformly and independently of all others. The sign sequences are also indexed \( a(1), a(2), \ldots, a(M_a) \).

Encoding: A message index \( m \) is chosen uniformly from \( \{1, 2, \ldots, M_a\} \). The corresponding amplitude sequence \( a(m) \) is combined with the corresponding randomly generated sign sequence \( s(m) \) into the channel input sequence \( x(m) \). More precisely, we obtain \( x(m) = s(m)a(m) \) for \( i = 1, 2, \ldots, n \), and \( x(m) \) is transmitted through the channel. Note that our encoder is sign-coding since all typical sequences \( a \in B^0_{V,Y,\varepsilon}(A) \) are used.

Decoding: The decoder finds the unique message index \( m \) such that the corresponding amplitude-sign sequence is jointly typical with the received output sequence \( y \), i.e., \( (\hat{a}(m), \hat{s}(m), y) \in A^n_\varepsilon(ASY) \).

Error Analysis: The error probability averaged over the random sign sequences that are generated is \( \overline{P}_e = \overline{P}_e(1) + \overline{P}_e(2) \).

Error of first kind: For the error of the first kind, we can write
\[
\overline{P}_e(1) \leq \sum_{m} \frac{1}{M_a} \sum_{y} \sum_{\hat{a}(m)} p(a(m), \hat{a}(m), y),
\]
where \( \hat{a}(m) \) and \( y \) follow \( p(\hat{a}(m), y|a(m)) = p(\hat{a}(m)|a(m) \in \hat{a}(m)) = 2^{-n}p(y|a(m) \in \hat{a}(m)) \).

Error of second kind: For the error of the second kind, we
obtain

\[ P_e(2) \leq \sum_m \frac{1}{M_a} \sum_{\tilde{y}} 2^{-n} \sum_{y} p(y|g(m), \tilde{y}) \sum_{k \neq m} 2^{-n} \cdot I[(a(k), \tilde{y}, y) \in A_n^\alpha(ASY)] = M_a \sum_{m} 2^{-n} \frac{1}{M_a} \sum_{\tilde{y}} 2^{-n} \sum_{k \neq m} \cdot I[(a(k), \tilde{y}, y) \in A_n^\alpha(ASY)] \]

(a) follows from doing summation over all \(a\) instead of only over the \(B\)-typical ones, and then from the one-to-one correspondence between \((a, \tilde{y})\) and \(\tilde{y}\).

(b) follows from \(M_a = |B_n^\alpha(A)| \leq 2^{n(H(A) + \varepsilon)}\), i.e., \(B\)-typicality property \(P_3\), and (3) holds for all large enough \(n\).

(c) follows from from \(M_a = |B_n^\alpha(A)| \leq 2^{n(H(A) + \varepsilon)}\), i.e., \(B\)-typicality property \(P_3\), and the weak typicality upper bounds for \(|A_n^\alpha(XY)|\), \(p(y)\), and \(p(\tilde{y})\).

The conclusion is that for \(H(A) < I(X;Y) - 10\varepsilon\), the error probability of the second kind

\[ P_e(2) \leq \varepsilon, \quad (20) \]

for \(n\) large enough.

**Result:** Combining (17) and (20), the total error probability \(P_e \leq 2\varepsilon\) for \(n\) large enough. This implies the existence of a basic sign code with total error probability \(P_e = \Pr\{M_a \neq M_a\} \leq 2\varepsilon\). This holds for all \(\varepsilon > 0\), and therefore, the rate

\[ R = H(A) \leq I(X;Y), \quad (21) \]

is achievable with basic sign-coding, which concludes the proof of Theorem 1.

### C. Proof of Theorem 2

**Code Construction:** We fix \(\varepsilon\) and consider the set \(B_n^\alpha(\varepsilon)(A)\) of \(B\)-typical \(a\)-sequences. These amplitude sequences are indexed \(a(1), a(2), \ldots, a(M_a)\) where \(M_a = |B_n^\alpha(\varepsilon)(A)|\). The \(n\gamma\)-length sign-sequence parts are indexed \(\tilde{a}'(1), \tilde{a}'(2), \ldots, \tilde{a}'(M_a)\) where \(M_a = 2^n\gamma\).

**Random sign-coding:** Now we randomly generate for each sequence \(a \in B_n^\alpha(\varepsilon)(A)\) and sign sequence part \(\tilde{a}'\), another sign sequence part \(\tilde{a}''\) of length \(n_2\). Each sign \(s \in S = \{-1, +1\}\) is chosen uniformly and independently of all others. This sign sequence parts are also indexed \(\tilde{a}''(1), \tilde{a}''(2), \ldots, \tilde{a}''(M_a)\).

**Encoding:** A message index pair \((m_a, m_s)\) is chosen such that \(M_a\) is uniform over \(\{1, 2, \ldots, M_a\}\) and \(M_s\) is uniform over \(\{1, 2, \ldots, M_s\}\). The corresponding amplitude sequence \(a(m_a)\) is combined with \(\tilde{a} = (\tilde{a}'(m_a), \tilde{a}''(m_a, m_s))\) into the channel input sequence \(x(m_a, m_s)\). More precisely, we obtain

\[ x_i(m_a, m_s) = \begin{cases} s_i(m_a) a_i(m_a) & \text{for } 1 \leq i \leq n_1, \\ s_i(m_a, m_s) a_i(m_a) & \text{for } n_1 + 1 \leq i \leq n, \end{cases} \]

for \(i = 1, 2, \ldots, n\), and \(x(m_a, m_s)\) is transmitted through the channel.

**Decoding:** The decoder finds the unique message index pair \((\hat{m}_a, \hat{m}_s)\) such that the corresponding amplitude-sign sequence is jointly typical with the received output sequence \(\tilde{y}\), i.e., \((\tilde{a}(\hat{m}_a), \tilde{a}'(\hat{m}_a, \hat{m}_s), \tilde{y}) \in A_n^\alpha(ASY)\).

**Error Analysis:** The error probability averaged over the random sign sequences that are generated is \(P_e = P_e(1) + P_e(2)\).

**Error of first kind:** For the error of the first kind, we can write

\[ P_e(1) \leq \sum_m \frac{1}{M_a} \sum_{\tilde{y}'} 2^{-n_2} \sum_{s \neq \tilde{y}'} p(\tilde{y}'|a(m), \tilde{a}'(m, s'), y) \cdot I[(a(m), \tilde{a}'(m, s'), y) \notin A_n^\alpha(ASY)] \leq \varepsilon. \quad (22) \]

This a direct consequence of Definition 1 since \(a(m) \in B_n^\alpha(\varepsilon)(A)\), and \((\tilde{a}', \tilde{a}'')\) and \(y\) satisfy

\[ p(\tilde{a}', \tilde{a}'', y|a(m)) = p(\tilde{a}', \tilde{a}'') p(y|a(m), \tilde{a}'', \tilde{a}'') = 2^{-n_2} p(y|a(m), \tilde{a}'', \tilde{a}''). \]

**Error of second kind:** For the error of the second kind, we...
where we substituted $n_1 = n\gamma$. Here:

(a) follows from $n = n_1 + n_2$. Moreover we split out $k_a k_s \neq m_a m_s$ into $k_a \neq m_a, k_s \neq m_s$ and $k_a = m_a, k_s \neq m_s$.

(b) follows from $p(s' s'') = 2^{-n}$ and for $a \in \mathcal{B}_{SY,\varepsilon}^0(A)$ from

$$\frac{1}{M_a} \leq 2^{3n\varepsilon} p(a). \quad (23)$$

(c) follows from doing summation over all $a$ instead of only over the $B$-typical ones.

(d) follows from substituting $s$ for $s' s''$.

(e) by working out the summations over $s', s''$ in the first part and $s$ in the second part.

(f) follows from $M_a = | \mathcal{B}_{SY,\varepsilon}^0(A) | \leq 2^{n(H(A)+\varepsilon)}$, i.e., $B$-typicality property $P_3$, and the weak typicality upper bounds for $| \mathcal{A}_c^p(ASY) |$, $p(a)p(s')$, and $p(y)$, more precisely,

- $| \mathcal{A}_c^p(ASY) | \leq 2^{n(H(ASY)+\varepsilon)}$
- $p(a)p(s') \leq 2^{-n(H(ASY)+\varepsilon)}$
- $p(y) \leq 2^{n(H(Y)+\varepsilon)}$

The conclusion is that for $H(A) + \gamma < I(AS;Y) - 10\varepsilon$ and $\gamma < I(S;AY) - 6\varepsilon$, the error probability of the second kind

$$P_e(2) \leq \varepsilon,$$  

(24)

for $n$ large enough. The second constraint is implied by the first one since

$$\gamma + 10\varepsilon \leq I(AS;Y) - H(A),$$

$$\leq I(AS;Y) - I(A;Y),$$

$$= I(S;Y|A),$$

$$\leq I(S;A) + I(S;Y|A),$$

$$= I(S;AY).$$

**Result:** Combining (22) and (24), the total error probability $P_e \leq 2\varepsilon$ for $n$ large enough. This implies the existence of a modified sign code with total error probability $P_e = \Pr\{(M_a, M_s) \neq (M_a, M_s)\} \leq 2\varepsilon$. This holds for all $\varepsilon > 0$, and therefore, the rate

$$R = H(A) + \gamma \leq I(SA; Y) = I(X; Y),$$

(25)

is achievable with modified sign-coding, which concludes the proof of Theorem 2.

**D. Proof of Theorem 3**

**Code Construction:** We fix $\varepsilon$ and consider the set $\mathcal{B}_{SY,\varepsilon}^0(B_1 B_2)$ of $B$-typical $b_1 b_2$-sequences for the joint distribution $\{p(b_1 b_2), (b_1 b_2) \in \{0,1\}^2\}$. These 2-tuple sequences are indexed by $b_1 b_2 \in (B_1 B_2)$. $M = | \mathcal{B}_{SY,\varepsilon}^0(B_1 B_2) |$.

**Random sign-coding:** Now we randomly generate for each sequence $b_1 b_2 \in \mathcal{B}_{SY,\varepsilon}^0(B_1 B_2)$, a sign sequence $s$ of length $n$. Each sign $s \in S = \{+1, -1\}$ is chosen uniformly and independently of all others. The sign sequences are also indexed in $(1), s(2), \cdots, s(M)$. The corresponding 2-tuple sequence $b_1(m)b_2(m)$ is first mapped to an amplitude sequence by
a symbol-wise mapping function \( f(\cdot) \). Then the amplitude sequence is combined with the corresponding randomly generated sign sequence \( \gamma(m) \). More precisely, we obtain \( x_i(m) = s_i(m)/f(b_{1i}(m), b_{2i}(m)) \) for \( i = 1, 2, \ldots, n \), and \( \gamma(m) \) is transmitted through the channel\(^5\).

**Decoding:** The decoder finds the unique message index \( \hat{m} \) such that the corresponding bit and sign sequences are jointly typical\(^b\) with the received output sequence \( y \), i.e., \((\hat{b}_{1}(\hat{m}), \hat{y}) \in A^n_B(1_B Y)\), \((\hat{b}_{2}(\hat{m}), \hat{y}) \in A^n_B(2_B Y)\), and \((\hat{\gamma}(\hat{m}), \hat{y}) \in A^n_\gamma(SY)\).

**Error Analysis:** The error probability averaged over the random sign sequences that are generated is \( \mathcal{P}_e = \mathcal{P}_e(1) + \mathcal{P}_e(2) \).

Error of first kind: For the error of the first kind, we can write
\[
\mathcal{P}_e(1) \leq \sum_{m=1}^{M} \sum_{y} \sum_{\gamma} p(\gamma, y, b_1(m), b_2(m), m) \cdot \mathbb{1}[(b_1(m), y) \in A^n_B(1_B Y), (b_2(m), y) \in A^n_B(2_B Y), (\gamma, y) \in A^n_\gamma(SY)] \leq \epsilon.
\]

This a consequence of Definition 1 since \((b_1 b_2) \in B^n_{SY}(1_B 2_B)\), and \( s \) and \( y \) follow \( p(s, y | b_1(m), b_2(m)) = 2^{-n} p(\gamma | b_1(m), b_2(m), s) \).

Error of second kind: For the error of the second kind, we obtain
\[
\mathcal{P}_e(2) \leq M \sum_{m=1}^{M} \frac{2^{-n}}{M} \sum_{y} \sum_{\gamma} p(\gamma, y, b_1(m), b_2(m), m) \cdot \mathbb{1}[(b_1(k), y) \in A^n_B, (b_2(k), y) \in A^n_B, (\gamma, y) \in A^n_\gamma]\]
\[
= M 2^{d_{\gamma}} \sum_{m=1}^{M} \sum_{y} p(\gamma, y, b_1(m), b_2(m), m) \cdot \mathbb{1}[(b_1(k), y) \in A^n_B, (b_2(k), y) \in A^n_B, (\gamma, y) \in A^n_\gamma] \]
\[
\leq M 2^{d_{\gamma}} \sum_{m=1}^{M} \sum_{\gamma} p(\gamma, y, b_1(m), b_2(m), m) \cdot \mathbb{1}[(\gamma, y) \in A^n_\gamma] \]
\[
\leq M 2^{d_{\gamma}} \sum_{m=1}^{M} \sum_{\gamma} p(\gamma, y, b_1(m), b_2(m), m) \cdot \mathbb{1}[(\gamma, y) \in A^n_\gamma] \]
\[
\leq 2^n(n(H(B_1 B_2) + \epsilon)) \cdot 2^{-n(H(Y) - \epsilon)} \cdot 2^{-n(H(B_1 B_2) S - \epsilon)} \cdot 2^n(n(H(Y) + c + n(H(B_1 Y) + H(S) | Y) + n(H(S) | Y) + 16 \epsilon)) \]
\[
= 2^n(n(H(B_1 B_2) - H(B_1 B_2 S) + H(B_1 Y) + H(B_2 Y) + H(S) | Y) - 16 \epsilon) \leq 2^{-n} \epsilon \leq \epsilon.
\]

where the indexing of \( A^n_B \) is dropped, and assumed to be clear from the context. Here (d) follows from the properties of weak-typicality for \( B_1 \), for \( B_2 \), and for \( S \), all given \( Y \) [9, Eqs. (15.30) & (15.38)].

The conclusion is that for \( H(B_1 B_2) < H(B_1 B_2 S) - H(B_1 Y) - H(B_2 Y) - H(S | Y) - 16 \epsilon \), the error probability of the second kind
\[
\mathcal{P}_e(2) \leq \epsilon,
\]
for \( n \) large enough.

**Result:** Combining (26) and (27), the total error probability \( \mathcal{P}_e \leq 2 \epsilon \) for \( n \) large enough. This implies the existence of a basic sign code with BMD with total error probability \( P_e = \Pr\{M \neq \hat{M} \} \leq 2 \epsilon \). This holds for all \( \epsilon > 0 \), and therefore, the rate
\[
R = H(B_1 B_2) \leq H(B_1 B_2 S) - H(S | Y) - H(B_1 Y) - H(B_2 Y),
\]
is achievable with basic sign-coding using BMD, which concludes the proof of Theorem 3.

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