ALL GRAPHS
WITH PAIRED-DOMINATION NUMBER TWO
LESS THAN THEIR ORDER

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Abstract. Let \( G = (V, E) \) be a graph with no isolated vertices. A set \( S \subseteq V \) is a paired-dominating set of \( G \) if every vertex not in \( S \) is adjacent with some vertex in \( S \) and the subgraph induced by \( S \) contains a perfect matching. The paired-domination number \( \gamma_p(G) \) of \( G \) is defined to be the minimum cardinality of a paired-dominating set of \( G \). Let \( G \) be a graph of order \( n \). In [Paired-domination in graphs, Networks 32 (1998), 199–206] Haynes and Slater described graphs \( G \) with \( \gamma_p(G) = n \) and also graphs with \( \gamma_p(G) = n - 1 \). In this paper we show all graphs for which \( \gamma_p(G) = n - 2 \).

Keywords: paired-domination, paired-domination number.

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1. INTRODUCTION

All graphs considered in this paper are finite, undirected, without loops, multiple edges and isolated vertices. Let \( G = (V, E) \) be a graph with the vertex set \( V \) and the edge set \( E \). Then we use the convention \( V = V(G) \) and \( E = E(G) \). Let \( G \) and \( G' \) be two graphs. If \( V(G) \subseteq V(G') \) and \( E(G) \subseteq E(G') \) then \( G \) is a subgraph of \( G' \) (and \( G' \) is a supergraph of \( G \)), written as \( G \subseteq G' \). The number of vertices of \( G \) is called the order of \( G \) and is denoted by \( n(G) \). When there is no confusion we use the abbreviation \( n(G) = n \). Let \( C_n \) and \( P_n \) denote the cycle and the path of order \( n \), respectively. The open neighborhood of a vertex \( v \in V \) in \( G \) is denoted \( N_G(v) = N(v) \) and defined by \( N(v) = \{ u \in V : vu \in E \} \) and the closed neighborhood \( N[v] \) of \( v \) is \( N[v] = N(v) \cup \{ v \} \). For a set \( S \) of vertices the open neighborhood \( N(S) \) is defined as the union of open neighborhoods \( N(v) \) of vertices \( v \in S \), the closed neighborhood \( N[S] \) of \( S \) is \( N[S] = N(S) \cup S \). The degree \( d_G(v) = d(v) \) of a vertex \( v \) in \( G \) is the number of edges incident to \( v \) in \( G \); by our definition of a graph, this is equal to \( |N(v)| \).
A leaf in a graph is a vertex of degree one. A subdivided star $K^*_1,t$ is a star $K_{1,t}$, where each edge is subdivided exactly once.

In the present paper we continue the study of paired-domination. Problems related to paired-domination in graphs appear in [1–5]. A set $M$ of independent edges in a graph $G$ is called a matching in $G$. A perfect matching $M$ in $G$ is a matching in $G$ such that every vertex of $G$ is incident to an edge of $M$. A set $S \subseteq V$ is a paired-dominating set, denoted PDS, of a graph $G$ if every vertex in $V - S$ is adjacent to a vertex in $S$ and the subgraph $G[S]$ induced by $S$ contains a perfect matching $M$. Therefore, a paired-dominating set $S$ is a dominating set $S = \{u_1,v_1,u_2,v_2,\ldots,u_k,v_k\}$ with matching $M = \{e_1,e_2,\ldots,e_k\}$, where $e_i = u_iv_i$, $i = 1,\ldots,k$. Then we say that $u_i$ and $v_i$ are paired in $S$. Observe that in every graph without isolated vertices the end-vertices of any maximal matching form a PDS. The paired-domination number of $G$, denoted $\gamma_p(G)$, is the minimum cardinality of a PDS of $G$. We will call a set $S$ a $\gamma_p(G)$-set if $S$ is a paired-dominating set of cardinality $\gamma_p(G)$. The following statement is an immediate consequence of the definition of PDS.

**Observation 1.1** ([4]). If $u$ is adjacent to a leaf of $G$, then $u$ is in every PDS.

Haynes and Slater [4] show that for a connected graph $G$ of order at least six and with minimum degree $\delta(G) \geq 2$, two-thirds of its order is the bound for $\gamma_p(G)$.

**Theorem 1.2** ([4]). If a connected graph $G$ has $n \geq 6$ and $\delta(G) \geq 2$, then

$$\gamma_p(G) \leq 2n/3.$$ 

Henning in [5] characterizes the graphs that achieve equality in the bound of Theorem 1.2.

In [4] the authors give the solutions of the graph-equations $\gamma_p(G) = n$ and $\gamma_p(G) = n - 1$, where $G$ is a graph of order $n$.

**Theorem 1.3** ([4]). A graph $G$ with no isolated vertices has $\gamma_p(G) = n$ if and only if $G$ is $mK_2$.

Let $\mathcal{F}$ be the collection of graphs $C_3$, $C_5$, and the subdivided stars $K^*_1,t$. Now, we can formulate the following statements.

**Theorem 1.4** ([4]). For a connected graph $G$ with $n \geq 3$, $\gamma_p(G) \leq n - 1$ with equality if and only if $G \in \mathcal{F}$.

**Corollary 1.5** ([4]). If $G$ is a graph with $\gamma_p(G) = n - 1$, then $G = H \cup rK_2$ for $H \in \mathcal{F}$ and $r \geq 0$.

In the present paper we consider the graph-equation

$$\gamma_p(G) = n - 2, \quad (1.1)$$

where $n \geq 4$ is the order of a graph $G$.

Our aim in this paper is to find all graphs $G$ satisfying (1.1). For this purpose we need the following definition and statements.
Definition 1.6. A subgraph $G$ of a graph $G'$ is called a special subgraph of $G'$, and $G'$ is a special supergraph of $G$, if either $V(G) = V(G')$ or the subgraph $G'[V(G') - V(G)]$ has a perfect matching.

It is clear that if $V(G) = V(G')$ then the concepts “subgraph” and “special subgraph” are equivalent. Now we can formulate the following fact.

Fact 1.7. Let $G$ be a special subgraph of $G'$.

(a) If $S$ is a PDS in $G$ then $S' = S \cup (V(G') - V(G))$ is a PDS in $G'$.

(b) If $\gamma_p(G) = n - r$ then $\gamma_p(G') \leq n' - r$, where $n = |V(G)|$, $n' = |V(G')|$ and $0 \leq r \leq n - 2$.

Proof. 

(a) Assume that

$$S = \{u_1, v_1, u_2, v_2, \ldots, u_t, v_t\} \quad \text{and} \quad V(G') - V(G) = \{u_{t+1}, v_{t+1}, \ldots, u_k, v_k\},$$

where $u_i$ and $v_i$ are paired in $S$ (for $i = 1, \ldots, t$) and in $V(G') - V(G)$ (for $i = t+1, \ldots, k$). Hence $M = \{e_1, e_2, \ldots, e_k\}$, where $e_i = u_i v_i$, for $i = 1, \ldots, k$, is a perfect matching in $G'[S']$. By definition of a PDS and by $V(G) - S = V(G') - S'$ we obtain the statement of a).

(b) Let $S$ be a $\gamma_p$-set in $G$, thus $|V(G) - S| = r$. It follows from a) that $S' = S \cup (V(G') - V(G))$ is a PDS in $G'$. Moreover, we have the equality

$$|S'| = n' - |V(G') - S'| = n' - |V(G) - S| = n' - r.$$

Therefore we obtain $\gamma_p(G') \leq |S'| = n' - r$. 

Now assume that $G$ is a connected graph of order $n \geq 4$ satisfying (1.1). Let $S = \{u_1, v_1, u_2, v_2, \ldots, u_k, v_k\}$ be a $\gamma_p(G)$-set with a perfect matching $M = \{e_1, e_2, \ldots, e_k\}$, where $e_i = u_i v_i$ for $i = 1, 2, \ldots, k$, and $V - S = \{x, y\}$. By letting $\alpha(S)$ denote the minimum cardinality of a subset of $S$ which dominates $V - S$, i.e.

$$\alpha(S) = \min\{|S'| : S' \subseteq S, V - S \subseteq N(S')\}.$$

Let $S_i$ be any set of size $\alpha(S)$ such that $S_i \subseteq S$ and $V - S \subseteq N(S_i)$. For $S$, $M$ and $S_i$, we define a graph $H$ as follows:

$$V(H) = V(G) \quad \text{and} \quad E(H) = M \cup \{uv : u \in S_i, v \in \{x, y\}\}.$$ 

It is clear that $H$ is a spanning forest of $G$; we denote it as $G_{sf}(S, M, S_i)$.

2. THE MAIN RESULT

The main purpose of this paper is to construct all graphs $G$ of order $n$ for which $\gamma_p(G) = n - 2$. At first consider the family $G$ of graphs in Fig. 1. We shall show that only the graphs in family $G$ are connected and satisfy condition (1.1).
All graphs with paired-domination number two less than their order

Fig. 1. Graphs in family \( \mathcal{G} \)

**Theorem 2.1.** Let \( G \) be a connected graph of order \( n \geq 4 \). Then \( \gamma_p(G) = n - 2 \) if and only if \( G \in \mathcal{G} \).

**Proof.** Our aim is to construct all connected graphs \( G \) for which (1.1) holds. Let \( G \) be a connected graph of order \( n \geq 4 \) satisfying (1.1). We shall prove that \( G \in \mathcal{G} \).

Let us consider the following cases.

Case 1. There exists a \( \gamma_p(G) \)-set \( S \) such that \( \alpha(S) = 1 \).

Case 1.1. \( k = 1 \). Then we have the graphs shown in Fig. 2. It is clear that the graphs \( H_i \) satisfy (1.1) and \( H_i = G_i \) for \( i = 1, 2, 3, 4 \).

Figure 2 illustrates the graphs \( H_i \), where the shaded vertices form a \( \gamma_p \)-set. We shall continue to use this convention in our proof.

At present for \( k \geq 2 \) we shall find all connected graphs \( G \) satisfying (1.1) and having a \( \gamma_p(G) \)-set \( S \) with \( \alpha(S) = 1 \). It is clear that in Case 1 any graph \( G_{sf}(S, M, S_i) \) is independent of the choice of \( S, M \) and \( S_i \), so we can write \( G_{sf}(S, M, S_i) = G_{sf} \). The spanning forest \( G_{sf} \) consists of \( k \) components \( G^{(1)}, G^{(2)}, \ldots, G^{(k)} \), where \( G^{(1)} = K_{1,3} \).
with $V(K_{1,3}) = \{x, y, u_1, v_1\}$, where $u_1$ is the central vertex, while $G^{(i)} = K_2$ for $i = 2, \ldots, k$ (see Fig. 3). Now by adding suitable edges to $G_{sf}$ we are able to reconstruct $G$.

![Fig. 3. The spanning forest of $G$](image)

**Case 1.2.** $k = 2$. Now we start with the graph $H_5$ (Fig. 4). In our construction of the desired connected graphs we add one or more edges to $H_5$. Thus, let us consider the following cases regarding the number of these edges.

**Case 1.2.1.** One edge (Fig. 5). One can see that $H_6 = G_5$ satisfies (1.1) but $H_7$ does not.

![Fig. 4. The spanning forest $H_5$](image)

![Fig. 5. The graphs obtained by adding one edge to $H_5$](image)

**Case 1.2.2.** Two edges. For $H_7$ we have $\gamma_p(H_7) = 6 - 4 = |V(H_7)| - 4$. Thus, by Fact 1.7 b) for any special supergraph $G'$ of $H_7$ we obtain $\gamma_p(G') \leq |V(G')| - 4$. Hence, we deduce that it suffices to add one edge to $H_6$. Since adding the edges $u_1u_2$ or $u_1v_2$
leads to $H_7$, we shall omit these edges in our construction. Now consider the graphs
of Fig. 6.

Certainly, $\gamma_p(H_8) = n - 4$, $\gamma_p(H_9) = n - 2$ and $H_i = G_{i-3}$ for $i = 9, \ldots, 12$. Using the
above argument for $H_8$ we do not take $v_1u_2$. Let us consider the following cases.

Case 1.2.3. Three edges. It follows from Fact 1.7 b) that it suffices to add one edge
to $H_i$ for $i = 9, \ldots, 12$.

Case 1.2.3.1. $H_9$. Observe that $H_i = G_{i-3}$, $i = 13, 14, 15$, satisfy (1.1). Moreover, the
graphs depicted in Fig. 7 are the unique graphs for which (1.1) holds in this case.
Indeed, the edge $v_2y$ leads to a supergraph of $H_8$, and joining $u_2$ to $x$ we have $H_{15}$.

Case 1.2.3.2. $H_{10}$. Then we obtain a supergraph of $H_7$ by means of edge $v_2y$, a
supergraph of $H_8$ by means of $xy$, $u_2x$, instead by adding $u_2y$ we return to $H_{15}$.

Therefore, it remains to research the graph of Fig. 8. It obvious that (1.1) holds for
$H_{16} = G_{13}$. 

Fig. 6. Adding a new edge to $H_6$

Fig. 7. $H_9 + e$

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Case 1.2.3.3. $H_{11}$. Then it suffices to consider the graph of Fig. 9. Really, edges $v_2x$, $v_2y$ lead to a supergraph of $H_8$ and $u_2x$, $u_2y$ lead to $H_{13}$. Observe that for $H_{17} = G_{14}$ equality (1.1) is true.

Case 1.2.3.4. $H_{12}$. Here we do not obtain any new graph satisfying (1.1). Indeed, we obtain: a supergraph of $H_7$ (by adding $v_1x$), a supergraph of $H_8$ (by $v_2x$), $H_{13}$ (by $u_2x$), $H_{14}$ (by $u_2y$) and $H_{16}$ (by $v_2y$).

Case 1.2.4. Four edges.

Case 1.2.4.1. $H_{13}$. Let $G$ be a graph obtained by adding a new edge $e$ to $H_{13}$. If $e = v_1y$ then $H_7 \subseteq G$; if $e = v_2y, v_2x$, then $H_8 \subseteq G$ and for $e = v_1x, u_2x$ we have the graph $G_{15} \in \mathcal{G}$ (Fig. 10).

Case 1.2.4.2. $H_{14}$. Keeping the above convention we note: if $e = xy$ then $H_7 \subseteq G$; if $e = v_2y, v_2x, u_2x$ then $H_8 \subseteq G$. 

Fig. 8. The graph obtained from $H_{10}$ by adding an edge

Fig. 9. $H_{11} + e$

Fig. 10. $H_{15} + e$
Case 1.2.4.3. $H_{15}$. If $e = v_2y$ then $H_7 \subseteq G$; if $e = xy, v_1y, u_2x$ then $H_8 \subseteq G$; if $e = v_1x$ then $G = G_{15}$. It is easy to see that (1.1) is true for $G_{15}$.

Case 1.2.4.4. $H_{16}$. In this case we conclude: if $e = xy$ then $H_7 \subseteq G$; if $e = v_2y, u_2x$ then $H_8 \subseteq G$; if $e = u_2y$ then $G = G_{15}$.

Case 1.2.4.5. $H_{17}$. Then we obtain the following results: if $e = v_1x, v_2y, u_2y$ then $H_7 \subseteq G$; if $e = v_2x$ then $H_8 \subseteq G$; if $e = u_2x$ then we have the graph $H_{18}$ depicted in Fig. 11. It is clear that $H_{18} = G_{15}$.

Case 1.2.5. Five edges.

Case 1.2.5.1. $G_{15}$. Then it suffices to consider the following: if $e = v_1y$ then $H_7 \subseteq G$; if $e = v_1x$ then $H_8 \subseteq G$. Therefore, Case 1.2 is complete.

For case $k \geq 3$ we only consider graphs satisfying the condition $G[S'] = G_{sf}[S'] = K_{1,3}$ for $S' = \{x, y, u_1, v_1\}$. In other words, $G$ contains the induced star $K_{1,3}$, where $V(K_{1,3}) = \{x, y, u_1, v_1\}$ and $u_1$ is the central vertex.

Case 1.3. $k = 3$. Then we start with the basic graph of Fig. 12. To obtain connected graphs we add two or more edges to $H_{19}$ and investigate whether (1.1) holds for the resulting graphs. At first we find a forbidden subgraph $H \subseteq G$ i.e. such that $\gamma_p(H) = n - 4$. We have already shown two forbidden special subgraphs $H_7, H_8$, and we now present the other one in Fig. 13. For a while we return to the general case $k \geq 3$. The forbidden special subgraphs $H_7$ and $H_{20}$ determine a means of construction of graphs $G$ from $G_{sf}$.

Case 1.2.5. Five edges.
**Claim 1.** Let $G$ be a connected graph satisfying (1.1) and obtained from $G_{sf} = H_{19}$. Then vertex $u_i$ or $v_i$, $i = 2, \ldots, k$, can be adjacent to the vertices $v_1, x, y$, only.

Now we add at least two edges to $H_{19}$. We consider the following cases.

**Case 1.3.1. Two edges.** Then we obtain the graphs $H_{21}$ and $H_{22}$ for which (1.1) holds (Fig. 14).

**Case 1.3.2. Three edges.** At present it suffices to add one edge in $H_{21}, H_{22}$. This way we obtain the graphs depicted in Figure 15. Observe that (1.1) fails for $H_{24}$ since $H_{20} \subseteq H_{24}$. Thus, $H_{23}$ satisfies (1.1) but $H_i$, $i = 24, 25, 26$, do not.

**Case 1.3.3. Four edges.** By adding one edge to $H_{23}$ we obtain the unique graph for which (1.1) holds (see Fig. 16). One can verify that in the remaining options we have special supergraphs of $H_7$, $H_8$, $H_{20}$, $H_{25}$ or $H_{26}$.

**Case 1.3.4. Five edges.** Each new edge in $H_{27}$ leads to a special supergraph of $H_7$, $H_8$, $H_{20}$, $H_{25}$ or $H_{26}$. But the following statement is obvious.

**Claim 2.** The graphs $H_7$, $H_8$, $H_{20}$, $H_{25}$ and $H_{26}$ are forbidden special subgraphs for (1.1).
We now study a generalization of the case $k = 3$. We keep our earlier assumption regarding the induced star $K_{1,3}$ with vertex set $\{u_1, v_1, x, y\}$.

Case 1.4. $k \geq 3$. Then we give one property of graphs satisfying (1.1).

Claim 3. Let $G$ be a connected graph for which (1.1) holds and $k \geq 3$. If $G$ contains the induced star $K_{1,3}$ with $V(K_{1,3}) = \{x, y, u_1, v_1\}$ then at least one vertex of $K_{1,3}$ is a leaf in $G$.

Proof. Consider some cases.

Case A. $k = 3$. It follows from our earlier investigations that $H_{21}$, $H_{22}$, $H_{23}$ and $H_{27}$ are the unique connected graphs satisfying (1.1) in this case. Thus, we have the desired result.

Case B. $k \geq 4$. Claim 1 and Fact 1.7 b) imply that a special subgraph $G[S]$ induced by $S = \{x, y, u_1, v_1, u_2, v_2, u_3, v_3\}$ is connected and satisfies (1.1), i.e. it must be one of the graphs $H_{21}$, $H_{22}$, $H_{23}$, $H_{27}$.

Case B.1. $G[S] = H_{21}$. We show that $x$ is a leaf in $G$. Suppose not and let $x$ be adjacent to $v_i$, where $i \geq 4$. Then we obtain the graph $H_{28}$ in Fig. 17, for which (1.1) does not hold.
Case B.2. $G[S] = H_{22}$.

Case B.2.1. Suppose that in $G$ vertex $v_i$, $i \geq 4$, is adjacent to $x$ and $y$. Then for graph $H_{29}$ depicted in Fig. 18 equality (1.1) is false since $H_{20} \subseteq H_{29}$.

Case B.2.2. Assume that in $G$ vertices $v_i$ and $u_i$, $i \geq 4$ are adjacent to $x$ and $y$, respectively (see Fig. 19). In this way we obtain graph $H_{30}$ which does not satisfy (1.1) since $H_{26} \subseteq H_{30}$.

Case B.2.3. Now, in $G$ let vertices $v_i$ and $u_j$, $4 \leq i < j$, be adjacent to $x$ and $y$, respectively (Fig. 20). As can be seen, (1.1) fails for $H_{31}$, furthermore $u_j$ is paired with $y$, $u_i$ with $v_i$, $u_3$ with $v_3$ and $v_1$ with $v_2$. It follows from the above consideration that we omit the cases: $G[S] = H_{23}$ and $G[S] = H_{27}$, since $H_{21}, H_{22}$ are subgraphs of $H_{23}, H_{27}$. In all cases we obtain special subgraphs of $G$ for which (1.1) fails, therefore $G$ does not satisfy (1.1), a contradiction.
All graphs with paired-domination number two less than their order

We are now in a position to construct the desired graphs for \( k \geq 3 \). Let \( G \) be a connected graph satisfying the following conditions:

a) (1.1) holds,

b) \( k \geq 3 \),

c) \( G \) contains the induced \( K_{1,3} \) with \( V(K_{1,3}) = \{x, y, u_1, v_1\} \).

According to Claims 1–3 we can reconstruct \( G \) based on \( G_{sf} \). By Claim 3, at least one vertex of \( K_{1,3} \), say \( x \), is a leaf in \( G \). Hence, by Claim 1, a vertex \( u_i \) or \( v_i \), \( i = 2, \ldots, k \), can be adjacent to \( v_1 \), \( y \), only. Observe that one vertex among \( u_i, v_i \), for \( i = 2, \ldots, k \), is a leaf. Indeed, if \( v_1y \) and \( u_iy \) (\( v_1v_1 \) and \( u_iv_1 \)) are edges of \( G \) then \( H_8 \) is a special subgraph of \( G \), but if \( v_1y, u_iv_1 \in E(G) \) then \( H_{25} \) is a special subgraph of \( G \) (Fig. 21). From the above investigations we obtain the desired graph in Fig. 22. One can see that (1.1) holds for \( H_{32} = G_{16} \). We emphasize that the numbers of edges \( yu_i \) or \( v_1w_i, yp_j, v_1z_m \) can be zero here.
Note that the graphs $H_{21}$, $H_{22}$, $H_{23}$, and $H_{27}$ are particular instances of $H_{32}$. We next describe desired graphs $G$ based on $H_{32}$. We now discard the assumption concerning the induced star $K_{1,3}$ i.e. edges joining $x$, $y$, $v_1$ are allowable. At first we add the edge $yv_1$ to $H_{32}$ and obtain graph $H_{33} = G_{17}$ which satisfies (1.1) (Fig. 23).

We now consider the following exhaustive cases (Fig. 24). It easy to see that (1.1) is true for $H_{34} = G_{18}$ and $H_{35} = G_{19}$ but is false for $H_i$, $i = 36, \ldots, 39$.

Case 2. Each $\gamma_p(G)$-set $S$ satisfies $\alpha(S) = 2$.

Case 2.1. There exists a set $S$ containing vertices $u,v$ that dominate $\{x,y\}$ such that $u$ is paired with $v$ in some perfect matching $M$ of $S$. Without loss of generality we may assume that $u = u_1$, $v = v_1$.

Case 2.1.1. $k = 1$. Then the unique graphs $H_{40} = G_{20}$ and $H_{41} = G_{21}$ satisfying (1.1) are depicted in Fig. 25.

Now for a connected graph $G$ with $k \geq 2$ the spanning forest $G_{sf}(S,M,S_i) = G_{sf}$ for $S_i = \{u,v\}$ is the sum of components $G^{(1)}, G^{(2)}, \ldots, G^{(k)}$, where $G^{(1)} = P_4$ and $G^{(i)} = K_2$ for $i = 2, \ldots, k$ (Fig. 26).

Case 2.1.2. $k = 2$. Now we start with the spanning forest of Fig. 27. In our construction of the desired connected graphs we add at least one edge to the graph $H_{42}$. Therefore, consider the following cases.
All graphs with paired-domination number two less than their order

Fig. 23. The family $G_{17}$

Fig. 24. The exhaustive cases

Fig. 25. The case for $k = 1$
Case 2.1.2.1. One edge (Fig. 28). Then we have $H_{43} = G_{22}$ and $H_{44} = G_{23}$ satisfy (1.1).

Case 2.1.2.2. Two edges. Now by adding one edge to $H_{43}$ and $H_{44}$ we obtain some graphs by exhaustion (Fig. 29). Observe (1.1) fails for $H_{45}$, $H_{46}$ and holds for $H_{47} = G_{24}$, $H_{48} = G_{25}$ and $H_{49} = G_{26}$. Moreover graphs $H_i$ for $i = 50, 51, 52$ are discussed in Case 1.

Case 2.1.2.3. Three edges. Then it suffices to add one edge to $H_i$, $i = 47, 48, 49$. One resulting graph is the graph $H_{53}$ depicted in Fig. 30, which does not satisfy (1.1). One can verify that the remaining graphs in this case are supergraphs of $H_{45}$, $H_{46}$ or are graphs discussed in Case 1.
All graphs with paired-domination number two less than their order

Case 2.1.3. \( k \geq 3 \). At first we show some graphs for which (1.1) does not hold (Fig. 31). For \( H_i, i = 54, \ldots, 57 \), (1.1) is false; in \( H_{54} \) the vertex \( u_1 \) is paired with \( u_2 \) and \( v_1 \) with \( u_3 \).

Now we start with the spanning forest depicted in Fig. 32.

Taking account of the forbidden special subgraphs \( H_i, i = 54, \ldots, 57 \), we can reconstruct \( G \) based on \( G_{sf} \). By the connectedness of \( G \) it is necessary to join vertices of both the edges \( u_i v_1, u_j v_j \) with at least one vertex among \( u_1, u_1, x, y \). Thus we consider the following cases (without loss of generality we take the vertices \( u_i \) and \( u_j \) of the above edges). If \( u_i u_1 \in E(G) \) then we have two options: \( u_j u_1 \in E(G) \) or \( u_j x \in E(G) \). Instead, if \( u_i x \in E(G) \) then we have the following options: \( u_j x \in E(G) \) or \( u_i u_1 \in E(G) \). Replace \( u_1 \) by \( v_1 \) and \( x \) by \( y \) we obtain analogous results. This way we construct the desired graph \( G = H_{58} \) for which (1.1) holds (Fig. 33). Note that \( H_{58} = G_{27} \). We end this case with adding new edges in \( H_{58} \). At first, if \( u_i z \in E(G) \) and \( v_i z \in E(G) \), where \( 2 \leq i \leq k \), \( z = u_1, v_1, x, y \), then we return to Case 1. Therefore,
let us consider all possible cases, which are depicted in Fig. 34. Then we obtain that (1.1) is true for \( H_{60} = G_{28} \) but is false for \( H_{59} \) and \( H_{61} \).

Fig. 31. The forbidden graphs

Fig. 32. The spanning forest for \( k \geq 3 \), where \( 2 \leq i < j \leq k \)

Fig. 33. \( H_{58} = G_{27} \)

Case 2.2. For each \( S \) and for all vertices \( u, v \in S \) that dominate \( \{x, y\} \) the vertex \( u \) is not paired with \( v \) in any perfect matching of \( S \). In this case the spanning forest \( G_{sf}(S, M, S_i) = G_{sf} \), for each \( M \) and \( S_i = \{u, v\} \), is depicted in Fig. 35.
All graphs with paired-domination number two less than their order

Now we search for connected graphs based on $G_{sf}$ and consider the following cases.

Case 2.2.1. $k = 2$. Then by adding one edge we obtain the three options of Fig. 36: $H_{62}$ does not satisfy (1.1) while $H_{63} = G_5$ and $H_{64} = G_{23}$.

Case 2.2.2. $k = 3$. Now consider the spanning forest depicted in Fig. 37. By joining the vertices $u_1, v_1, x$ to $u_2, v_2, y$ we could obtain $H_i, i = 62, 63, 64$, or their supergraphs. Hence the obtained graphs do not satisfy (1.1) or belong to Case 1 or Case 2.1. Therefore, it suffices to consider edges joining the above vertices to $u_3$ or $v_3$ (Fig. 38). Then $H_i, i = 65, \ldots, 69$, do not satisfy (1.1) but $H_{70}$ belongs to the family $G_{10}$. 

Fig. 34. $H_{58} + e$

Fig. 35. The spanning forest $G_{sf}$ of a connected graph $G$

Fig. 36. The case $k = 2$

Fig. 38. $H_{65} + e$, $H_{66} + e$, $H_{67} + e$, $H_{68} + e$, $H_{69} + e$.
Case 2.2.3. \( k > 3 \). Then we obtain graphs for which (1.1) fails or graphs belonging to Case 1.

Conversely, let \( G \) be any graph of the family \( \mathcal{G} \). It follows from the former investigations that (1.1) holds for \( G \).  
\[ \square \]
We end this paper with the following statement obtained by Theorems 1.3, 1.4, 2.1 and Corollary 1.5.

**Corollary 2.2.** If $G$ is a graph of order $n \geq 4$, then $\gamma_p(G) = n - 2$ if and only if
1) exactly two of the components of $G$ are isomorphic to graphs of the family $F$ given in Theorem 1.4 and every other component is $K_2$ or
2) exactly one of the components of $G$ is isomorphic to a graph of the family $G$ given in Theorem 2.1 and every other component is $K_2$.

**REFERENCES**

[1] M. Chellali, T.W. Haynes, *Trees with unique minimum paired-dominating set*, Ars Combin. 73 (2004), 3–12.

[2] S. Fitzpatrick, B. Hartnell, *Paired-domination*, Discuss. Math. Graph Theory 18 (1998), 63–72.

[3] T.W. Haynes, S.T. Hedetniemi, P.J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, New York, 1998.

[4] T.W. Haynes, P.J. Slater, *Paired-domination in graphs*, Networks 32 (1998), 199–206.

[5] M.A. Henning, *Graphs with large paired-domination number*, J. Comb. Optim. 13 (2007), 61–78.

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