COUNTEREXAMPLES TO FUJITA’S CONJECTURE ON SURFACES IN POSITIVE CHARACTERISTIC

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Abstract. We present counterexamples to Fujita’s conjecture in positive characteristic. More precisely, given any algebraically closed field \( k \) of characteristic \( p > 0 \) and any positive integer \( m \), we show there exists a smooth projective surface \( S \) over \( k \) admitting an ample Cartier divisor \( A \) such that the adjoint linear system \( |K_S + mA| \) is not free of base points.

1. Introduction

The study of adjoint linear systems plays an important role in the classification of varieties. In this direction, T. Fujita \[Fuj87\] proposed the following famous conjecture:

**Conjecture 1.1** (Fujita’s conjecture). Let \( X \) be a smooth projective variety of dimension \( n \) over an algebraically closed field of characteristic zero and \( A \) an ample Cartier divisor on \( X \).

i) (Freeness) For \( m \geq n + 1 \), the adjoint linear system \( |K_X + mA| \) is free of base points.

ii) (Very ampleness) For \( m \geq n + 2 \), the adjoint linear system \( |K_X + mA| \) is very ample.

By taking \( X = \mathbb{P}^n \) and \( A \) a hyperplane, it can be seen that the bounds of \( m \) in both the freeness and the very ampleness parts of this conjecture are optimal.

On curves Fujita’s conjecture follows from Riemann-Roch formula immediately and on surfaces it has been proved by Reider’s elegant method \[Rei88\]. The freeness part of this conjecture has now been proved up to dimension five (see \[EL93, Hel97, Kaw97, YZ20\]) while the very ampleness part remains widely open when \( \dim X \geq 3 \). On the other hand, for varieties of arbitrary dimension, there have been many other important “Fujita’s conjecture type” results (see \[Dem93, Kol93, AS95, Hel97, Smi97\] and so on). One of these remarkable results was due to Angehrn and Siu \[AS95\]: \( |K_X + mA| \) is base point free for \( m \geq \left( \frac{n + 1}{2} \right) + 1 \). We refer the readers to \[Laz04, § 10.4A\] for a brief review about both the related results and techniques.

Although Fujita’s conjecture was originally formulated in characteristic zero, many people have studied the positive characteristic version. For convenience, let us simply say Conjecture 1.1 to mean ‘the positive characteristic version of Conjecture 1.1’ when there is no risk of confusion.

Along this direction, in dimension two, by adapting Reider’s method to characteristic \( p \), Shepherd-Barron \[SB91, Corollary 8\] was able to prove Conjecture 1.1 in dimension two except for surfaces of general type and quasi-elliptic surfaces, and
more recently it is claimed that Conjecture 1.1 also holds for the latter case by Chen [Che]. On the other hand, following from the celebrated works of Deligne and Illusie [DI87] and of Langer [Lan15], Reider’s method applies to surfaces admitting a $W_2$-lifting and therefore Conjecture 1.1 is also true in the $W_2$-lifting case. Besides, some other ‘Fujita’s conjecture type’ results are also obtained for surfaces (see [Nak93a, Nak93b, Ter99, DCF15] and so on). For example Di Cerbo and Fanelli [DCF15] proved $|2K_X + 38A|$ is very ample for a surface.

For varieties of arbitrary dimension, by results of Smith and Keeler ([Kee08, Kee19, Smi97, Smi00]), it is known that Conjecture 1.1 is also true under an additional assumption that $\mathcal{O}_X(A)$ is globally generated. Furthermore, there are many other attempts to prove Conjecture 1.1 in positive characteristic (see [Sch14, MS14, Mur19] and so on). However, we find examples disproving Fujita’s conjecture in positive characteristic, and there does not even exist a Fujita type bound. In particular, we have a negative answer to Schwede’s question [Sch14, pp. 71] concerning the stable adjoint linear subsystems in dimension at least two. Our main result is the following theorem.

**Theorem 1.2.** Let $k$ be an arbitrary algebraically closed field of positive characteristic and $m \in \mathbb{N}_+$ an arbitrary positive integer. Then there exists a smooth projective surface $S$ over $k$ admitting an ample Cartier divisor $A$ such that the complete linear system $|K_S + mA|$ is not free of base point.

The surface $S$ in Theorem 1.2 is a generalization of the so-called Raynaud’s surface ([Ray78, Muk13]). As is known to experts, comparing with the case in characteristic zero, one disadvantage in dealing with Conjecture 1.1 is the failure of Kodaira type vanishing in positive characteristic [Che]. In history, Raynaud’s surface is the first counterexample to Kodaira’s vanishing ([Ray78]). So we firstly checked Conjecture 1.1 for Raynaud’s surfaces, and with a non-trivial observation we actually discovered that Conjecture 1.1 already fails for some special Raynaud’s surfaces provided the characteristic $p$ is large. To obtain examples for all primes, we made a ‘passage from $p$ to $q = p^n$’ generalization of Raynaud’s surfaces to obtain the final example (cf. §2) as required in Theorem 1.2.

Finally, concerning with the Seshadri constant, we have the following comments on the pair $(S, A)$ in Theorem 1.2. Fixing the base field $k$ and $3 \leq m \in \mathbb{N}_+$, there is a divisor $\Gamma_2$ on $S$ (cf. §2.5) and a positive constant $b(m)$ such that the Seshadri constant $\varepsilon(A, x) < b(m)$ for any $x \in \Gamma_2$ and $\varepsilon(A, x) \geq 1$ when $x \in S \setminus \Gamma_2$. The constants $b(m)$ are furthermore such that $\lim_{m \to \infty} b(m) = 0$. In particular, although the adjoint system $|K_S + mA|$ is not free of base points, it does define a birational map whenever $m > 4$. Indeed, as $\varepsilon(A, x) \geq 1$ when $x \in S \setminus \Gamma_2$, we have the Frobenius-Seshadri constant $\varepsilon_F(mA, x) \geq \frac{m}{2} > 2$ for such $x$ (cf. [MS14, Prop. 2.12]). It then follows from [MS14, Thm. 1.1(2)] that $|K_S + mA|$ defines a birational map. So in

*However, Murayama’s opinion that ‘the failure of Kodaira-type vanishing is not the main obstacle to Fujita’s conjecture’ seems right [Mur19, Principal 1.6]. For instance, as Chen [Che] claimed, Fujita’s conjecture holds for quasi-elliptic surfaces while Kodaira’s vanishing fails on these surfaces [Ray78, Zhe17].
spite that effectivity on the freeness and very ampleness of adjoint linear systems fails due to Theorem 1.2 the effectivity on birationality still makes sense.

**Question 1.3.** Is there a constant $B(n,p)$ such that for any $n$-dimensional smooth projective variety $X$ defined over a field of characteristic $p$ and an ample Cartier divisor $A$ on $X$, the adjoint linear system $|K_X + mA|$ defines a birational map if $m \geq B(n,p)$? Furthermore, can we expect $B(n,p)$ is independent on $p$?

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2. A generalization of Raynaud’s surface

In this section, we are going to present a certain generalization of Raynaud’s surface given in [Ray78]. In the next section, we will show our generalization does give the example as required in Theorem 1.2. Let us fix an algebraically closed field $k$ of characteristic $p > 0$.

2.1. Outline of the generalization. Let us briefly recall Raynaud’s surface and then outline how we generalize his construction.

Raynaud’s surface $S_1$ is, roughly speaking, a cyclic cover $\pi_1 : S_1 \rightarrow \mathbb{P}$ of degree prime to $p$ over a ruled surface $\rho : \mathbb{P} \rightarrow C$ over a curve $C$ such that $\pi_1$ is branched along two smooth irreducible divisors $\Sigma_i \subseteq \mathbb{P}$, $i = 1, 2$ satisfying

(i) $\rho|_{\Sigma_1} : \Sigma_1 \rightarrow C$ is an isomorphism, namely, $\Sigma_1$ is a section of $\rho$;
(ii) $\rho|_{\Sigma_2} : \Sigma_2 \rightarrow C$ is a purely inseparable morphism of degree $p$; and
(iii) $\Sigma_1 \cdot \Sigma_2 = 0$.

Our generalization of Raynaud’s construction is to replace (ii) with

(ii*) the restriction map $\rho|_{\Sigma_2} : \Sigma_2 \rightarrow C$ is a purely inseparable morphism of degree $q = p^n$ for any prescribed $n \in \mathbb{N}_+$.

2.2. The base curve. Fix an arbitrary positive integer $n$ and set $q = p^n$. We firstly give the base curve $C$, which is obtained by a slight modification of the example in [Muk13, Exm. 1.3].

Let $C \subseteq \mathbb{P}^2_k = \text{Proj}(k[X, Y, Z])$ be the plane curve defined by the equation:

$$Y^{qe} - X^{qe-1}Y = XZ^{qe-1},$$

where $e \in \mathbb{N}_+$ is a free variable. It is easy to check that $C$ is a smooth curve and

$$2g(C) - 2 = qe(qe - 3).$$

Take $\infty := [0, 0, 1]$ on $C$. Then $U_1 := C \setminus \infty = C \cap \{X \neq 0\}$ is an affine open subset defined by $y_1^{qe} - y_1 = z^{qe-1}$ with $y_1 = Y/X$ and $z = Z/X$. As a result, $dz$ is a generator of $\Omega^1_C|_{U_1}$ since $dy_1 = z^{qe-2}dz$. In particular, we have

$$K_C = \text{div}(dz) = (2g(C) - 2)\infty = qe(qe - 3) \cdot \infty.$$
Next set \( V := C \cap \{ Z \neq 0 \} \). Then \( V \subset C \) is an open affine subset defined by the equation \( y^{qe} - x^{qe-1}y = x \) with \( y = Y/Z \) and \( x = X/Z \). The special point \( \infty \) is now given by \( x = y = 0 \). After some easy local calculations, we have:

- \( y \) is a local parameter at \( \infty = [0, 0, 1] \) and invertible on \( V \setminus \infty \), and
- \( v_\infty(x) = qe \). Indeed, by \( x = y^{ue} - x^{ue-1}y \), it follows that
  \[ v_\infty(x) \geq \min \{ qe \cdot v_\infty(y) = qe, (qe - 1) \cdot v_\infty(x) + 1 \} > 0 \]
  and thus \( (qe - 1) \cdot v_\infty(x) + 1 > qe \), which gives the inequality \( v_\infty(x) = qe \).

As a result, the function

\[ \gamma := \left( \frac{x}{y^{ue}} \right)^{qe-2} \cdot y = \left( 1 - \left( \frac{x}{y} \right)^{qe-1} \right)^{qe-2} \cdot y = \left( 1 - \left( y^{ue-1} - x^{ue-1} \right)^{qe-1} \right)^{qe-2} \cdot y \in \mathcal{O}(V) \]

is also a local parameter at \( \infty \). In particular, \( d\gamma \) is a generator of \( \Omega^1_{C, \infty} \) and hence there is an open neighbourhood \( \infty \in U_2 \subset V \) such that \( d\gamma \) is a generator of \( \Omega^1_{C|U_2} \).

Finally, we have

\[ z - y^{qe} = \frac{1}{x} - \frac{1}{y^{ue}} = \frac{x^{ue-1}y}{xy^{ue}} = y^{qe(qe-3)} \gamma, \]

or equivalently,

\[ z = (y^{qe(qe-3)})q \gamma + (y^{-e})^q. \]

In summation, we have an affine covering \( C = U_1 \cup U_2 \) along with two rational functions \( z_1 = z, z_2 = \gamma \) such that

- \( z_1 \) is regular on \( U_i \) and \( dz_1 \) is a generator of \( \Omega^1_{C/k|U_i} \), and
- the translation relation of \( z_1 \) is given by:

\[ (3) \quad z_1 = \alpha z_2 + \beta^q, \]

where \( \alpha = y^{e(qe-3)} \in \mathcal{O}_C(U_1 \cap U_2)^* \) and \( \beta = y^{-e} \in \mathcal{O}_C(U_1 \cap U_2) \).

**Remark 2.1.** Note that as \( y \) and \( \alpha \) are invertible on \( V \setminus \infty \), it turns out that \( d\gamma = dz_2 \) is a generator of \( \Omega^1_{C|V} \) and we can in fact take \( U_2 = V \).

2.3. A rank two locally free sheaf \( \mathcal{E} \) on \( C \). By the previous constructions, \( z_i \in \mathcal{O}_{U_i} \) and \( dz_i \) is a generator of \( \Omega^1_{C/k|U_i} \). As a consequence, in the following finite purely inseparable cover \( V_i := \text{Spec}(\mathcal{O}_{U_i}[t]/(t^q - z_i)) \rightarrow U_i \), the source \( V_i \) is regular. In fact, by Jacobian criterion,

\[ \Omega_{V_i/k} = \frac{(\mathcal{O}_{U_i} \otimes \mathcal{O}_{U_i} \mathcal{O}_{V_i}) \oplus \mathcal{O}_{V_i} \cdot dt}{\mathcal{O}_{V_i} \cdot dt (t^q - z_i)} = \frac{\mathcal{O}_{V_i} \cdot dz_i \oplus \mathcal{O}_{V_i} \cdot dt}{\mathcal{O}_{V_i} \cdot dz_i} \simeq \mathcal{O}_{V_i} \cdot dt \simeq \mathcal{O}_{V_i}. \]

Therefore, the covering map \( V_i \rightarrow U_i \) coincides with the \( n \)-th Frobenius map, and we have

\[ F^n_{*} \mathcal{O}_C|U_i = \mathcal{O}_{U_i}[t]/(t^q - z_i) = \mathcal{O}_{U_i}[[\sqrt{z_i}]] = \bigoplus_{j=0}^{q-1} \mathcal{O}_C \cdot \sqrt[ q ]{z_i}. \]

Then we construct a rank two locally free subsheaf \( \mathcal{E} \subseteq F^n_{*} \mathcal{O}_C \) as follows. On \( U_i \), \( \mathcal{E} \) is given by:

\[ (4) \quad \mathcal{E}|_{U_i} = \mathcal{O}_{U_i} \cdot X_i(:= 1) \oplus \mathcal{O}_{U_i} \cdot Y_i(:= \sqrt{z_i}) \subseteq F^n_{*} \mathcal{O}_C|U_i = \bigoplus_{j=0}^{q-1} \mathcal{O}_C \cdot \sqrt[ q ]{z_i}. \]
Here note that by equation (3), we have the following translation relation on $U_1 \cap U_2$:

$$
\begin{align*}
X_1 &= X_2, \\
Y_1 &= \alpha \cdot Y_2 + \beta \cdot X_2.
\end{align*}
$$

where $\alpha = y^{(q-3)} \in O_C(U_1 \cap U_2)^*$ and $\beta = y^{-e} \in O_C(U_1 \cap U_2)$. So the above construction of $\mathcal{E}$ makes sense.

By construction, as a subsheaf of $F^*_nO_C$, $\mathcal{E}$ contains the natural saturated subsheaf $O_C \subseteq F^*_nO_C$ and hence it induces an exact sequence:

$$0 \to O_C \to \mathcal{E} \to \mathcal{L} \to 0.$$  

(5)

As $O_C$ is locally generated by 1 on $U_i$, $\mathcal{L}$ is an invertible sheaf with generators $\eta_i := \theta(Y_i)$ on $U_i$ and translation relation $\eta_i = \alpha \cdot \eta_2$. As $v_\infty(\alpha) = e(q-3)$ and $\text{div}(\eta_i) = e(q-3) \cdot \infty$, we have

$$\mathcal{L} \cong O_C(e(q-3) \cdot \infty) \quad \text{and} \quad \mathcal{L}^q \cong \omega_C.$$  

(6)

2.4. The ruled surface $\mathbb{P}(\mathcal{E})$. In this subsection we will study the ruled surface

$$\rho : \mathbb{P}(\mathcal{E}) := \text{Proj}(\mathcal{E}) \to C.$$  

As mentioned in §2.3, this ruled surface should admit a special configuration of two smooth irreducible divisors $\Sigma_1, \Sigma_2$. We now give their constructions in the following.

Firstly, the exact sequence (5) gives a section divisor $\Sigma_1 \subseteq |O_C(1)|$, which is locally defined by $X_i = 0$ on $U_i(\mathcal{E}) = \text{Proj}(\mathcal{O}_{U_i}[X_i, Y_i])$. Then

- $\Sigma_1$ is a section of $\rho$; and
- $O_{\Sigma_1}(1) = \rho^*\mathcal{L}|_{\Sigma_1}$, $\Sigma_1^2 = \text{deg } \mathcal{L} = (2g(C) - 2)/q > 0$. In particular, $\Sigma_1$ is nef and big.

Secondly, the restriction of the multiplicative map to $O_C \otimes O_C F^*_nO_C$

$$\iota : \mathcal{E} \otimes O_C F^*_nO_C \to F^*_nO_C \otimes O_C F^*_nO_C \xrightarrow{\text{multiplication}} F^*_nO_C$$

gives a splitting of (5) after tensoring with $F^*_nO_C$ as the left diagram below. This splitting then gives a map $\nu$ as the right commutative diagram below such that $\nu^*O_{\mathbb{P}(\mathcal{E})}(1) \cong O_C$.

In fact, this $\iota$ is nothing but a surjective map $(F^n)^*\mathcal{E} \to O_C$ (here $C$ is the source of the $n$-th Frobenius map). Thus $\iota$ induces a morphism $\nu$ as above so that $\nu^*O_{\mathbb{P}(\mathcal{E})}(1) \cong O_C$. We simply take $\Sigma_2 := \nu(C)$ and it follows from

$$0 = \text{deg}_C(\nu^*O_{\mathbb{P}(\mathcal{E})}(1)) = \text{deg}(\nu) \cdot (\Sigma_1 \cdot \Sigma_2)$$

that $\Sigma_1 \cap \Sigma_2 = \emptyset$. More concretely, on $U_i$, $\iota$ is given by $X_i \mapsto 1$ and $Y_i \mapsto \sqrt{z_i}$. Correspondingly, on $\mathbb{P}_{U_i}(\mathcal{E})(= \text{Proj}(\mathcal{O}_{U_i}[X_i, Y_i]))$, $\Sigma_2$ is defined by $Y_i^q - z_iX_i^q = 0$ while $\Sigma_1$ is defined by $X_i = 0$. Moreover, from these equations we see that $\Sigma_1 \cap \Sigma_2 = \emptyset$, and the projective $\Sigma_2 \to C$ coincides with $F^n : C \to C$ (cf. §2.3).

In summation, we have
• $\Sigma_2$ is smooth and $\rho_{|\Sigma_2}: \Sigma_2 \to C$ is the $n$-th iterated Frobenius map of $C$;
• $\Sigma_1 \cap \Sigma_2 = \emptyset$; and
• $\Sigma_2 \in |O_{\mathbb{P}(E)}(q) \otimes \rho^*\omega_C^{-1}|$. In fact, we have $\Sigma_2 \in |O_{\mathbb{P}(E)}(q) \otimes \rho^*H|$ for some invertible sheaf $H$ on $C$. As $\Sigma_2 \cap \Sigma_1 = \emptyset$, we have $(O_{\mathbb{P}(E)}(q) \otimes \rho^*H)|_{\Sigma_1} = O_{\Sigma_1}$ and hence $\rho^*H \cong O_{\Sigma_1}(-q) \cong \rho^*L^{-q} \cong \rho^*\omega_C^{-1}$.

So far, we have given the configuration as promised in §2.1.

2.5. A generalized Raynaud’s surface. Given the triple pair $(\mathbb{P}(E), \Sigma_1, \Sigma_2)$ as above, we shall construct a generalization of Raynaud’s surface.

In the following we assume in the defining equation (1) of $C$, the free integral variable $e$ is such that

\[(q + 1) \mid qe(qe - 3) = 2g(C) - 2.\]

This assumption can be easily fulfilled, e.g., by taking $(q + 1) \mid e$. Then denote the invertible sheaf $N := O_C(\frac{qe(qe - 3)}{q + 1} \cdot \infty)$, and we have

$N^{q+1} \cong O_C(qe(qe - 3) \cdot \infty) \cong \omega_C$

by (2). As a result, the invertible sheaf $M := O_{\mathbb{P}(E)}(1) \otimes \rho^*N^{-1}$ satisfies

$M^{q+1} = O_{\mathbb{P}(E)}(q + 1) \otimes \rho^*\omega_C^{-1} \cong O_{\mathbb{P}(E)}(\Sigma_1 + \Sigma_2)$.

It is well known the data $(M, \Sigma_1 + \Sigma_2)$ gives a finite flat $(q + 1)$-cyclic cover

$\pi: S = \text{Spec}_{O_{\mathbb{P}(E)}}(O_{\mathbb{P}(E)} \otimes M^{-1} \oplus \cdots \oplus M^{-q}) \to \mathbb{P}(E)$

branched along $\Sigma := \Sigma_1 + \Sigma_2$. Here the $O_{\mathbb{P}(E)}$-algebra structure of

$O_{\mathbb{P}(E)} \oplus M^{-1} \oplus \cdots \oplus M^{-q}$

is defined by the embedding $M^{-q+1} \cong O_X(-\Sigma_1 - \Sigma_2) \hookrightarrow O_X$.

Since $\Sigma$ is smooth, the surface $S$ is also smooth.

Remark 2.2. In the above construction, we can also replace the $(q+1)$-cyclic cover by an $s$-cyclic cover with $s \mid (q+1)$ such that the branched divisor remains $\Sigma$. Raynaud’s surfaces given in [Ray78] are exactly the above surfaces with $n = 1$ and $s = 2$ if $p \neq 2$ and $s = 3$ if $p = 2$.

Since $\pi$ is branched along $\Sigma_i$, $i = 1, 2$, we have divisors $\Gamma_i \subseteq S$ such that $\pi^*\Sigma_i = (q+1)\Gamma_i$. In particular, the divisor $\Gamma_1$ is again nef and big (cf. §2.4) and hence $\Gamma_1 + f^*R$ is ample for any ample divisor $R$ on $C$.

Finally, we have

\begin{align*}
\omega_S &\cong \pi^*(\omega_{\mathbb{P}(E)} \otimes M^q) \cong \pi^*(O_{\mathbb{P}(E)}(q-2) \otimes \rho^*(L \otimes N)) \\
&\cong \pi^*(O_{\mathbb{P}(E)}(q-2) \otimes \rho^*(L^2(-l\infty)))
\end{align*}

(7)
where
\[ l := \frac{e(qe - 3)}{q + 1} \]
and the third “≃” is due to \( L \simeq \mathcal{O}_C(e(qe - 3) \cdot \infty) \) and
\[ L^{-1} \otimes \mathcal{N} \simeq \mathcal{O}_C((-\frac{qe(qe - 3)}{q} + \frac{qe(qe - 3)}{q + 1} + l \cdot \infty)) = \mathcal{O}_C(-l \cdot \infty). \]

3. Adjoint linear systems on generalized Raynaud’s surfaces

We keep the notations in the previous section. For any ample divisor \( R \) on \( C \), we set \( A_R = \Gamma_1 + f^*R \). As aforementioned, this divisor is ample. The main result is

**Theorem 3.1.** Assume in the defining equation (7) of \( C \), the free integral variable \( e \) satisfies
\[ (\star) \quad l - (q - 1) = \frac{e(qe - 3)}{q + 1} - q + 1 \geq e(q - 2). \]
Then for any \( 1 \leq m \leq q \) there is an open dense subset \( U_m \subseteq C \) such that for any \( Q \in U_m \) and any Cartier divisor \( R \) on \( C \) of degree one satisfying \( mR \sim (m - 1) \cdot \infty + Q \), the adjoint systems \( |K_S + mAR| \) has a base point \( \Gamma_2 \cap f^{-1}(Q) \).

The condition (\( \star \)) is easily satisfied by taking \( e \gg q \), and the existence of \( R \) for every \( Q \) is guaranteed by the divisibility of \( \text{Pic}^0(C) \). We will see that the condition (\( \star \)) is just needed to guarantee a certain element to be zero in \( H^1(U, \omega_C) \) when we prove the non-surjectivity of \( \phi \) in Lemma 3.3. The rest part of this section is devoted to proving Theorem 3.1.

3.1. A non-free criterion. From the construction of \( S \), we have
\[ \pi_* \mathcal{O}_S = \mathcal{O}_{\mathbb{P}(\mathcal{E})} \bigoplus \mathcal{M}^{-1} \bigoplus \cdots \bigoplus \mathcal{M}^{-q}. \]
Moreover, we have the following proposition on the decomposition of certain ideal sheaves.

**Proposition 3.2** (cf. [Zhe17, Prop. 3.3]). For \( 1 \leq r < q + 1 \) the ideal sheaf \( \mathcal{O}_S(-r\Gamma_1) \) has a decomposition
\[ \pi_* \mathcal{O}_S(-r\Gamma_1) = (\bigoplus_{i=0}^{r-1} \mathcal{M}^{-i}(-\Sigma_1)) \bigoplus (\bigoplus_{i=r}^{q} \mathcal{M}^{-i}) \subseteq \pi_* \mathcal{O}_S = \bigoplus_{i=0}^{q} \mathcal{M}^{-i} \]
which is compatible with the decomposition of \( \pi_* \mathcal{O}_S \).

**Corollary 3.3.** Let \( \mathcal{F} \) be an invertible sheaf on \( \mathbb{P}(\mathcal{E}) \). Then for \( 1 \leq r < q + 1 \) we have a decomposition
\[ \pi_* (\pi^* \mathcal{F}(-r\Gamma_1)) = (\bigoplus_{i=0}^{r-1} \mathcal{M}^{-i} \otimes \mathcal{F}(-\Sigma_1)) \bigoplus (\bigoplus_{i=r}^{q} \mathcal{M}^{-i} \otimes \mathcal{F}) \subseteq \bigoplus_{i=0}^{q} \mathcal{M}^{-i} \otimes \mathcal{F}, \]
and the global sections of \( \pi^* \mathcal{F}(-r\Gamma_1) \) from all but the first direct summand \( \mathcal{M}^0 \otimes \mathcal{F}(-\Sigma_1) \simeq \mathcal{F}(-\Sigma_1) \) vanish along \( \Gamma_2 \).

In particular, if the first summand \( \mathcal{F}(-\Sigma_1) \) as \( \mathcal{O}_{\mathbb{P}(\mathcal{E})} \)-module has a base point \( Q_0 \in \Sigma_2 \), then \( \pi^* \mathcal{F}(-r\Gamma_1) \) has a base point \( \pi^{-1}(Q_0) \in \Gamma_2 \).
Proof. The decomposition follows immediately from Proposition 3.2 by the projection formula. For simplicity we use $\tilde{\mathcal{F}}_i$, $i = 0, 1, \cdots, q$ to denote the corresponding direct summand of $\pi_*(\pi^*\mathcal{F}(-r\Gamma_1))$ in the above decomposition. Then we have

$$H^0(S, \pi^*\mathcal{F}(-r\Gamma_1)) \simeq H^0(\mathbb{P}(\mathcal{E}), \tilde{\mathcal{F}}_0) \bigoplus \bigoplus_{i > 0} H^0(\mathbb{P}(\mathcal{E}), \bigoplus_{i > 0} \tilde{\mathcal{F}}_i).$$

For the remaining assertions, it suffices to show the sections from $H^0(\mathbb{P}(\mathcal{E}), \bigoplus_{i > 0} \tilde{\mathcal{F}}_i)$ do not generate $\pi^*\mathcal{F}(-r\Gamma_1)$ at each point of $\Sigma_2$, or equivalently the natural homomorphism

$$H^0(\mathbb{P}(\mathcal{E}), \bigoplus_{i > 0} \tilde{\mathcal{F}}_i) \otimes \pi_*\mathcal{O}_S \rightarrow \pi_*(\pi^*\mathcal{F}(-r\Gamma_1))$$

is not surjective at each point of $\Sigma_2 = \pi(\Gamma_2)$. In practice, we shall verify that the composition homomorphism

$$\left(\bigoplus_{i=0}^q \mathcal{M}^{-i}\right) \otimes \left(\bigoplus_{i=1}^q \tilde{\mathcal{F}}_i\right) \rightarrow \pi_*(\pi^*\mathcal{F}(-r\Gamma_1)) \rightarrow \tilde{\mathcal{F}}_0 = \mathcal{F}(-\Sigma_1)$$

is not surjective at each point of $\Sigma_2$, where the second map denotes the projection to the first direct summand. Keep in mind the compatibility of this decomposition of $\pi_*(\pi^*\mathcal{F}(-r\Gamma_1))$ as a $\pi_*\mathcal{O}_S \cong \bigoplus \mathcal{M}^{-i}$-module. Locally we can get sections of $\tilde{\mathcal{F}}_0 = \mathcal{F}(-\Sigma_1)$ from the direct summands $\tilde{\mathcal{F}}_i$, $i > 0$ as follows:

- when $1 \leq i \leq r - 1$, $\mathcal{M}^{-i-q-1} \otimes (\mathcal{M}^{-i} \otimes \mathcal{F}(-\Sigma_1)) \simeq \mathcal{F}(-2\Sigma_1 - \Sigma_2) \subset \mathcal{F}(-\Sigma_1)$ as a sub-sheaf determined by tensor with the ideal sheaf $\mathcal{O}(-\Sigma_1 - \Sigma_2)$;
- when $r \leq i \leq q$, $\mathcal{M}^{-i-q-1} \otimes (\mathcal{M}^{-i} \otimes \mathcal{F}) \simeq \mathcal{F}(-\Sigma_1 - \Sigma_2) \subset \mathcal{F}(-\Sigma_1)$ as a sub-sheaf determined by tensor with the ideal sheaf $\mathcal{O}(-\Sigma_2)$.

We see that the local sections of $\tilde{\mathcal{F}}_0$ from $(\bigoplus_{i=0}^q \mathcal{M}^{-i}) \otimes (\bigoplus_{i=1}^q \tilde{\mathcal{F}}_i)$ vanish along $\Sigma_2$ and conclude the result. $\square$

Note that by (7), we have

$$\mathcal{O}_S(K_S + mA_R) = \pi^*(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(q - 2) \otimes \rho^*(\mathcal{L} \otimes \mathcal{N}(mR)))(m\Gamma_1)$$

$$= \pi^*(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(q - 1) \otimes \rho^*(\mathcal{L} \otimes \mathcal{N}(mR)))(-(q + 1 - m)\Gamma_1).$$

Therefore, by Corollary 3.3 Theorem 3.1 follows from the next proposition.

**Proposition 3.4.** Let $\mathcal{P} := \mathcal{O}_{\mathbb{P}(\mathcal{E})}(q - 2) \otimes \rho^*(\mathcal{L} \otimes \mathcal{N})$. When $(\star)$ holds, there is an open dense subset $\mathcal{U}_m \subseteq C$ such that for every closed point $Q \in \mathcal{U}_m$, the natural map

$$H^0(\mathbb{P}(\mathcal{E}), \mathcal{P} \otimes \rho^*\mathcal{O}_C((m - 1) \cdot \infty)) \otimes_{s_Q} \otimes H^0(\mathbb{P}(\mathcal{E}), \mathcal{P} \otimes \rho^*\mathcal{O}_C((m - 1) \cdot \infty + Q))$$

is an isomorphism, where $s_Q \in H^0(C, \mathcal{O}_C(Q))$ is a section corresponding to $Q$.

In fact, this proposition says that the fixed part of the linear system

$$|\mathcal{O}_{\mathbb{P}(\mathcal{E})}(q - 1) \otimes \rho^*(\mathcal{L} \otimes \mathcal{N}(mR))|(-\Sigma_1)|$$

$$= |\mathcal{O}_{\mathbb{P}(\mathcal{E})}(q - 2) \otimes \rho^*(\mathcal{L} \otimes \mathcal{N}(mR))|$$

$$= |\mathcal{P} \otimes \rho^*\mathcal{O}_C((m - 1) \cdot \infty + Q)|$$
contains \( F_Q := \rho^{-1}(Q) \) as a component (and hence admits the base point \( F_Q \cap \Sigma_2 \)).

3.2. Proof of Proposition 3.4. For simplicity, we set 
\[
N := \left( \frac{e(qe - 3)}{q + 1} - (m - 1) \right) = (l + 1 - m)
\]
and \( \Delta := N \cdot \infty \), and hence \( L \otimes N((m - 1) \cdot \infty) = L^2(-\Delta) \) by (8). An immediate consequence of the assumption (\( \star \)) is that
\[
\deg \Delta = N \geq l + 1 - q \geq e(q - 2).
\]

For Proposition 3.4, it is equivalent to the following:

**Proposition 3.5.** When (\( \star \)) holds, there is an open dense subset \( U_m \subseteq C \) such that for every closed point \( Q \in U_m \), the natural map
\[
H^0(C, S^{q-2}(\mathcal{E}) \otimes L^2(-\Delta)) \cong H^0(C, S^{q-2}(\mathcal{E}) \otimes L^2(-\Delta + Q))
\]
is an isomorphism.

Note that we have the following filtration due to the exact sequence (8),
\[
S^{-1}(\mathcal{E}) := 0 \subseteq S^0(\mathcal{E}) \subseteq \cdots \subseteq S^{q-1}(\mathcal{E})
\]
In fact, the sheaf \( \mathcal{E} \) is an extension of \( \mathcal{L} \) by \( \mathcal{O}_C \). So we have the natural ascending filtrations \( S^r(\mathcal{E}) \simeq S^r(\mathcal{E}) \otimes \mathcal{O}_C^{q-1} \subseteq S^{q-1}(\mathcal{E}), r = -1, \ldots, q - 1 \) (see [Har77, II, Ex. 5.16]). Concretely, with the notation from (12) the inclusion \( S^{-1}(\mathcal{E}) \subseteq S^0(\mathcal{E}) \) is induced by
\[
X_i^{r-1-j} Y_i^j \mapsto X_i^{r-1-j} Y_i^j, j = 0, \ldots, r - 1, \text{ on } U_i.
\]
One can also realize the above filtration equivalently in another way as follows. The inclusion \( \mathcal{E} \subseteq F^n \mathcal{O}_C \) along with the \( \mathcal{O}_C \)-algebra structure of \( F^n \mathcal{O}_C \) induces natural embeddings \( S^r(\mathcal{E}) \subseteq F^n \mathcal{O}_C \) for \( r \leq q - 1 \). Concretely, by the construction of \( \mathcal{E} \) the embedding can be described as follows:
\[
S^r(\mathcal{E})|_{U_i} = \bigoplus_{j=0}^{r} \mathcal{O}_C \cdot X_i^{r-1-j} Y_i^j (= 1^{r-j} \cdot q^{j} z_i^j) \subseteq F^n \mathcal{O}_C|_{U_i} = \bigoplus_{j=0}^{q-1} \mathcal{O}_C \cdot q^j z_i^j,
\]
hence the filtration (13) coincides with the natural filtration of \( F^n \mathcal{O}_C \).

In either viewpoint, we have canonical isomorphisms
\[
S^r(\mathcal{E})/S^{r-1}(\mathcal{E}) \simeq \mathcal{L}^r, r = 0, \ldots, q - 1.
\]
These isomorphisms, in terms of the second viewpoint, are locally given as \( Y_i^r(= \sqrt[3]{z_i}) \mapsto \eta_i^r \) on \( U_i, i = 1, 2 \). We also consider the quotient sheaf \( S^{q-2}(\mathcal{E})/S^{r-1}(\mathcal{E}) \), which on \( U_i \) is generated by \( X_i^{q-2-j} Y_i^j = q^j z_i^j, j = r, \ldots, q - 2 \). Then from the filtration (13), we get the exact sequence:
\[
0 \rightarrow \mathcal{L}^r(\simeq S^r(\mathcal{E})/S^{r-1}(\mathcal{E})) \rightarrow S^{q-2}(\mathcal{E})/S^{r-1}(\mathcal{E}) \rightarrow S^{q-2}(\mathcal{E})/S^r(\mathcal{E}) \rightarrow 0.
\]
Lemma 3.6. Put $S^r := (S^{r-2}(\mathcal{E})/S^{r-1}(\mathcal{E})) \otimes \mathcal{L}^{2}$, $r = 0, \ldots, q - 2$. Then for any $r = 0, \ldots, q - 2$, there is an open dense subset $V_r \subseteq U_1 = C \setminus \infty$ such that the natural maps

$$H^0(C, S^r(-\Delta)) \overset{\otimes_q}{\to} H^0(C, S^r(-\Delta + Q))$$

are isomorphisms for all $Q \in V_r$.

Note that the assertion in this lemma when $r = 0$ is all what we need (see Proposition 3.5). We are going to prove this lemma by a descending induction on $r$. And before running the induction process, we give the following lemma.

Lemma 3.7. For any $0 \leq i \leq q$, the natural map

$$H^0(C, L^i(-\Delta)) \overset{\otimes_q}{\to} H^0(C, L^i(-\Delta + Q))$$

is an isomorphism for any $Q \neq \infty$.

Proof. Recall that $L^q \sim K_C$. By Riemann-Roch formula and Serre duality, it suffices to prove the natural map

$$H^1(C, L^i(-\Delta + Q))^\vee \cong H^0(C, L^{q-i}(\Delta - Q)) \overset{\otimes_q}{\to} H^0(C, L^{q-i}(\Delta)) \cong H^1(C, L^i(-\Delta))^\vee$$

is not an isomorphism for all $Q \neq \infty$. This is true because $Q$ is not a base point of $|L^{q-i}(\Delta)| = |((q-i)e(ge-3)+N)\cdot \infty|$. \qed

Proof of Lemma 3.6. First set $r = q - 2$, then we have $S^{q-2} = L^q$. So Lemma 3.7 with $i = q$ gives Lemma 3.6 for $r = q - 2$ and $V_{q-2}$ can be taken as $U_1 = C \setminus \infty$.

We argue by a descending induction on $r$. Now assume the statement holds for $r + 1 \leq q - 2$. Let’s consider the case $r$. By tensoring (15) with $L^2$ we obtain the exact sequence

$$0 \to L^{r+2} \to S^r \to S^{r+1} \to 0.$$  

Then tensoring the above exact sequence with $O_C(-\Delta)$ and $O_C(-\Delta + Q)$ for $Q \in V_{r+1}$, we obtain the following commutative diagram of exact sequences:

$$
\begin{array}{ccc}
0 & \to & L^{r+2}(-\Delta) \\
\downarrow & & \downarrow \\
0 & \to & L^{r+2}(-\Delta + Q) \\
\end{array}
\begin{array}{ccc}
S^r(-\Delta) & \to & S^{r+1}(-\Delta) \\
\downarrow & & \downarrow \\
S^r(-\Delta + Q) & \to & S^{r+1}(-\Delta + Q) \\
\end{array}
\to 0
$$

Taking the cohomology of the above diagram we obtain

$$
H^0(L^{r+2}(-\Delta)) \overset{\otimes_q}{\to} H^0(S^r(-\Delta)) \overset{\alpha_r}{\to} H^0(S^{r+1}(-\Delta)) \overset{\phi}{\to} H^1(L^{r+2}(-\Delta))
\cong \alpha_r
\quad \cong \alpha_{r+1}
\quad \cong \phi
\quad \cong \otimes_q
$$

where the horizontal sequences are exact, the leftmost vertical isomorphism follows from Lemma 3.7 and $\alpha_{r+1}$ is an isomorphism by the inductive assumption on $r + 1$.

Now we want to find some open dense subset $V_r \subseteq V_{r+1}$ such that $\alpha_r$ is an isomorphism for any $Q \in V_r$. In fact, we only need to verify that $\alpha_r$ is surjective. Applying the Five Lemma, it suffices to verify that the map $\text{Im}(\phi) \overset{\otimes_q}{\to} H^1(C, L^{r+2}(-\Delta + Q))$ is injective, which is equivalent to the following condition
Recall also \( \beta \), we take a nonzero section

By Serre’s duality, we only need to find \( Z \) such that for any \( s \in Z \), \( s \) is a proper subspace. This is done by the next lemma.

Lemma 3.8. The associated map

\[ \phi : H^0(C, S^{r+1}(-\Delta)) \rightarrow H^1(C, L^{r+2}(-\Delta)) \]

is not surjective for \( r = 0, \ldots, q - 3 \).

Proof. By Serre’s duality, we only need to find

\[ 0 \neq \omega \in H^0(C, \omega_C \otimes L^{-r-2}(\Delta)) \cong H^1(C, L^{r+2}(-\Delta))^\vee \]

such that for any \( s \in H^0(C, S^{r+1}(-\Delta)) \) the pairing \( \langle \omega, \phi(s) \rangle = 0 \), which means

\[ \omega \otimes \phi(s) = 0 \in H^1(C, \omega_C) \cong H^1(U = \{U_1, U_2\}, \omega_C). \]

We will actually take

\[ \omega = dz_1 \otimes \eta_1^{-r-2} = \alpha^{q-r-2}dz_2 \otimes \eta_2^{-r-2} \in H^0(C, \omega_C \otimes L^{-r-2}) \subseteq H^0(C, \omega_C \otimes L^{-r-2}(\Delta)). \]

Recall that \( \eta_i \) are the local generators of \( L \) on \( U_i \) with \( \eta_1 = \alpha \cdot \eta_2 \) and \( \alpha = y^{q(q-3)}. \)

Recall also \( \beta = y^{-e} \) and \( y_N \) is a generator for \( O_C(-\Delta) \) on \( U_2 \). Now take an arbitrary
Formulae and text from the image.
3.3. Conclusion. In summary, for any fixed algebraically closed field $k$ of characteristic $p > 0$ and any positive integer $m \geq 1$, we can take $q = p^n \geq m$ and $e = (q + 1)e_0$ for some $e_0 \in \mathbb{N}_+$ such that (■) and (之星) hold. Then Theorem 3.1 shows that on the associated smooth projective surface $S$ constructed in §2, there is an ample divisor $A$ such that $|K_S + mA|$ is not free of base points. Namely, we have proven Theorem 1.2.

References

[AS95] Urban Angehrn and Yum Tong Siu, Effective freeness and point separation for adjoint bundles, Invent. Math. 122 (1995), no. 2, 291–308, DOI 10.1007/BF01231446. MR1358978
[Che] Yen-An Chen, Fujita’s Conjecture for Quasi-Elliptic Surfaces, arXiv preprint arXiv:1907.09046
[DI87] Pierre Deligne and Luc Illusie, Relèvements modulo $p^2$ et décomposition du complexe de de Rham, Invent. Math. 89 (1987), no. 2, 247–270, DOI 10.1007/BF01389078.
[Dem93] Jean-Pierre Demailly, A numerical criterion for very ample line bundles, J. Differential Geom. 37 (1993), no. 2, 323–374. MR1205448
[DCF15] Gabriele Di Cerbo and Andrea Fanelli, Effective Matsusaka’s theorem for surfaces in characteristic $p$, Algebra Number Theory 9 (2015), no. 6, 1453–1475, DOI 10.2140/ant.2015.9.1453. MR3397408
[EL93] Lawrence Ein and Robert Lazarsfeld, Global generation of pluricanonical and adjoint linear series on smooth projective threefolds, J. Amer. Math. Soc. 6 (1993), no. 4, 875–903. MR1207013
[Fuj87] Takao Fujita, On polarized manifolds whose adjoint bundles are not semipositive, Algebraic geometry, Sendai, 1985, Adv. Stud. Pure Math., vol. 10, North-Holland, Amsterdam, 1987, pp. 167–178, DOI 10.2969/aspm/01010167. MR946238
[Har77] Robin Hartshorne, Algebraic geometry, Graduate Texts in Mathematics, No. 52, Springer-Verlag, New York-Heidelberg, 1977. MR0463157
[Hel97] Stefan Helmke, On Fujita’s conjecture, Duke Math. J. 88 (1997), no. 2, 201–216, DOI 10.1215/S0012-7094-97-08807-4. MR1455517
[Kaw97] Yujiro Kawamata, On Fujita’s freeness conjecture for 3-folds and 4-folds, Math. Ann. 308 (1997), no. 3, 491–505. MR1457742
[Kee08] Dennis S. Keeler, Fujita’s conjecture and Frobenius amplitude, Amer. J. Math. 130 (2008), no. 5, 1327–1336, DOI 10.1353/ajm.0.0015. MR2450210
[Kee19] ______, Erratum to: Fujita’s conjecture and Frobenius amplitude, Amer. J. Math. 141 (2019), no. 5, 1477–1478, DOI 10.1353/ajm.2019.0039. MR4011807
[Kol93] János Kollár, Effective base point freeness, Math. Ann. 296 (1993), no. 4, 595–605, DOI 10.1007/BF01445123.
[Lau15] Adrian Langer, Bogomolov’s inequality for Higgs sheaves in positive characteristic, Invent. Math. 199 (2015), no. 3, 889–920, DOI 10.1007/s00222-014-0534-z.
[Laz04] Robert Lazarsfeld, Positivity in algebraic geometry. II, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 49, Springer-Verlag, Berlin, 2004. Positivity for vector bundles, and multiplier ideals. MR2095472
[Muk13] Shigeru Mukai, Counterexamples to Kodaira’s vanishing and Yau’s inequality in positive characteristics, Kyoto J. Math. 53 (2013), no. 2, 515–532, DOI 10.1215/21562261-2081279. MR3079312
[Mur19] Takumi Murayama, Seshadri Constants and Fujita’s Conjecture via Positive Characteristic Methods, ProQuest LLC, Ann Arbor, MI, 2019. Thesis (Ph.D.)–University of Michigan. MR4024420
[MS14] Mircea Mustaţă and Karl Schwede, A Frobenius variant of Seshadri constants, Math. Ann. 358 (2014), no. 3-4, 861–878, DOI 10.1007/s00208-013-0976-4. MR3175143
[Nak93a] Tohru Nakashima, *Adjoint linear systems on a surface of general type in positive characteristic*, Pacific J. Math. 160 (1993), no. 1, 129–132. MR1227507

[Nak93b] , *On Reider’s method for surfaces in positive characteristic*, J. Reine Angew. Math. 438 (1993), 175–185, DOI 10.1515/crll.1993.438.175.

[Ray78] Michel. Raynaud, *Contre-exemple au “vanishing theorem” en caractéristique p > 0*, C. P. Ramanujam—a tribute, Tata Inst. Fund. Res. Studies in Math., vol. 8, Springer, Berlin-New York, 1978, pp. 273–278 (French). MR541027

[Rei88] Igor Reider, *Vector bundles of rank 2 and linear systems on algebraic surfaces*, Ann. of Math. (2) 127 (1988), no. 2, 309–316. MR932299

[SB91] Nicolas. I. Shepherd-Barron, *Unstable vector bundles and linear systems on surfaces in characteristic p*, Invent. Math. 106 (1991), no. 2, 243–262, DOI 10.1007/BF01243912. MR1128214

[Sch14] Karl Schwede, *A canonical linear system associated to adjoint divisors in characteristic p > 0*, J. Reine Angew. Math. 696 (2014), 69–87, DOI 10.1515/crelle-2012-0087. MR3276163

[Smi97] Karen E. Smith, *Fujita’s freeness conjecture in terms of local cohomology*, J. Algebraic Geom. 6 (1997), no. 3, 417–429. MR1487221

[Smi00] , *A tight closure proof of Fujita’s freeness conjecture for very ample line bundles*, Math. Ann. 317 (2000), no. 2, 285–293, DOI 10.1007/s002080000094. MR1764238

[Ter99] Hiroyuki Terakawa, *The d-very ampleness on a projective surface in positive characteristic*, Pacific J. Math. 187 (1999), no. 1, 187–199, DOI 10.2140/pjm.1999.187.187. MR1674325

[YZ20] Fei Ye and Zhixian Zhu, *On Fujita’s freeness conjecture in dimension 5*, Adv. Math. 371 (2020), 107210, 56, DOI 10.1016/j.aim.2020.107210. With an appendix by Jun Lu.

[Zhe17] Xudong Zheng, *Counterexamples of Kodaira vanishing for smooth surfaces of general type in positive characteristic*, J. Pure Appl. Algebra 221 (2017), no. 10, 2431–2444, DOI 10.1016/j.jpaa.2016.12.030. MR3646309

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