INTEGRATING TEMPORAL INFORMATION TO SPATIAL INFORMATION IN A NEURAL CIRCUIT

BRABEEBA WANG, NANCY LYNCH

ABSTRACT. In this paper, we consider a network of spiking neurons with a deterministic synchronous firing rule at discrete time. We propose three problems — “first consecutive spikes counting”, “total spikes counting” and “k-spikes temporal to spatial encoding” — to model how brains extract temporal information into spatial information from different neural codings. For a max input length $T$, we design three networks that solve these three problems with matching lower bounds in both time $O(T)$ and number of neurons $O(\log T)$ in all three questions.

1. Introduction

Algorithms in the brain are inherently distributed. Although each neuron has relatively simple dynamics, as a distributed system, a network of neurons shows strong computational power. There have been many attempts to model the brain computationally. At a single-neuron level, theoretical neuroscientists were able to model the dynamics of a single neuron to high accuracy with the Hodgkin-Huxley model [HH52]. At a circuit level, to make the analysis tractable, neuroscientists approximated detailed dynamics of neurons with simplified models such as the nonlinear integrate-and-fire model [FTHvBV03] and the spiking response model [WWvJ97]. Recently, Lynch et al. used stochastic neurons firing at discrete time to solve problems such as winner-take-all and similarity testing [LMP17a, LMP17b]. These models vary in their assumptions about spike/rate code, deterministic/stochastic response, and continuous/discrete time. In this paper, we consider a network of spiking neurons with a deterministic synchronous firing rule in discrete time to simplify the analysis and focus on the computational principles.

One of the most important questions in neuroscience is how humans integrate information over time. Sensory inputs such as visual and auditory stimulus are inherently temporal; however, brains are able to integrate the temporal information to a single concept, such as a moving object in a visual scene, or an entity in a sentence. There are two kinds of neuronal codings: rate coding and temporal coding. Rate coding is a neural coding scheme assuming most of the information is coded in the firing rate of the neurons. It is most commonly seen in muscle in which the higher firing rates of motor neurons correspond to higher intensity in muscle contraction [AZ26]. On the other hand, rate coding cannot be the only neural coding brains employ. A fly is known to react to new stimuli and change its direction of flight within 30-40 ms. There is simply not enough time for neurons to compute averages [RWdRSB96]. Therefore, neuroscientists propose the idea of temporal coding, assuming the information is coded in the specific temporal firing patterns. One of the popular temporal codings is the first-to-spike coding. It has been shown that the timing of the first spike encodes most information of an image in retinal cells [GM08]. We propose three toy problems to model how brains extract information from different coding. “First consecutive spikes counting” (FCSC) counts the first consecutive interval of spikes, which is equivalent to counting the distance between the first two spikes, a prevalent neural coding scheme in sensory cortex. “Total spikes counting” (TSC) counts the number of the spikes over an arbitrary interval, which is an example of rate coding. Lastly, “k-spikes temporal to spatial encoding” (kSTS) is a generalization of “first consecutive spikes counting” and an example of temporal coding. In particular, TSC contains an
interesting difficulty: there are conflicting objectives between maintaining the count when no spike arrives and updating the count when a spike arrives. To overcome this difficulty, we allow the network to enter an unstable intermediate state which carries the information of the count. The intermediate state then converges to a stable state that represents the count after a computation step without inputs. Hitron and Parter, in a newly-submitted paper [HP19], propose a different solution to our TSC problem.

In this paper, we design three networks that solve the above three problems by translating temporal information into spatial information with matching lower bounds in both time $O(T)$ and number of neurons $O(\log T)$ for all three questions. The organization of this paper is as follows. In Section 2, we present the definition of a network of spiking neurons and the problem statements. In Section 3, we present the FCSC network that counts consecutive spikes in binary. In Section 4, we generalize section 3 and present the TSC network that counts spikes over arbitrary intervals. In Section 5, we present the kSTS network that embeds sparse temporal inputs into spatial codings. In Section 6, we discuss our model assumptions and their implication along with possible future directions.

2. Problem Statements/Goals

In this section, we cover the model definition and the following three problems: first consecutive spikes counting (FCSC), total spikes counting (TSC) and $k$-spikes temporal to spatial encoding (kSTS). In particular, we will use FCSC networks as subroutines on a kSTS network.

2.1. Model. In this paper, we consider a network of spiking neurons with deterministic synchronous firing at discrete times. Formally, a neuron $x$ consists of the following data with $t \geq 1$

$$x(t) = \Theta(\sum_{y \in P_x} w_{yx} y(t-1) - b_x)$$

where $x(t)$ is the indicator function of neuron $x$ firing at time $t$, $b_x$ is the threshold (bias) of neuron $i$, $P_x$ is the set of presynaptic neurons of $i$, $w_{yx}$ is the strength of connection from neuron $y$ to neuron $x$ and $\Theta$ is a nonlinear function. Here we take $\Theta$ as the Heaviside function given by $\Theta(z) = 1$ if $z > 0$ and 0 otherwise. At $t = 0$, we let $x(0) = 0$ if $x$ is not one of the input neurons.

2.2. First consecutive spikes counting (FCSC). Given an input neuron $x$ and the max input length $T$, we consider any input firing sequence such that $x(t) = 0$ for all $t \geq T$. Define $L_x$ in terms of this firing sequence as follows: if $x(t) = 1$ for some $t$, then there must exist integers $i, L$ such that $x(t) = 0$ for all $t, t < i$, $x(i+1) = 1$ for all $i, 0 \leq i < L$ and $x(i+L) = 0$. Define $L_x = L$. (i.e. $L$ is the length of the first consecutive spikes interval in the sequence.) Otherwise, that is if $x(t) = 0$ for all $t \geq 0$, then define $L_x = 0$.

Let $\{y_t\}_{0 \leq i < m}$ be $m$ output neurons. Then we say a network of neurons solves FCSC in time $t'$ with $m'$ neurons if there exists an injective function $F : \{0, \cdots , n\} \rightarrow \{0, 1\}^m$ such that $y(t) = F(L_x)$ for all $t, x, t \geq t'$ and the network has $m'$ neurons.

2.3. Total spikes counting (TSC). Given an input neuron $x$ and the max input length $T$, we consider any input firing sequence such that $x(t) = 0$ for all $t \geq T$. Define $L_x = \{t : x(t) = 1, 0 \leq t < T\}$ as the total number of spikes in the sequence. Let $\{y_t\}_{0 \leq i < m}$ be $m$ output neurons. Then we say a network of neurons solves TSC in time $t'$ with $m'$ neurons if there exists an injective function $F : \{0, \cdots , n\} \rightarrow \{0, 1\}^m$ such that $y(t) = F(L_x)$ for all $t, x, t \geq t'$ and the network has $m'$ neurons.
2.4. **k-spikes Temporal to Spatial Encoding.** Given an input neuron \( x \) and the max input length \( T \), we consider any input firing sequence such that \( x(t) = 0 \) for all \( t \geq T \) and \( |\{ t : x(t) = 1, 0 \leq t < T \}| = k \) (i.e., there are spikes at \( k \) distinct time points). We also assume that there is a designated \( x_{end} \) neuron that fires at time \( T \) to notify the network that the input ends. Let \( \{ y_i \}_{0 \leq i < m} \) be \( m \) output neurons. Denote the set of input temporal signals of max input length \( T \) with \( k \) distinct 1 as \( S_{T,k} \). Then we say a network of neurons solves \( k\text{STS} \) in time \( t' \) with \( m' \) neurons if there exists an injective function \( F : S_{T,k} \rightarrow \{0, 1\}^m \) such that \( y_i(t') = F(x) \) for all \( t, x, t \geq t' \) and the network has \( m' \) neurons.

Our contributions in this paper are to design networks that solve these three problems respectively with matching lower bounds.

**Theorem 2.1.** There exists a network with \( O(\log T) \) neurons that solves FCSC problem in \( O(T) \) time.

**Theorem 2.2.** There exists a network with \( O(\log T) \) neurons that solves TSC problem in \( O(T) \) time.

**Theorem 2.3.** There exists a network with \( O(k \log T) \) neurons that solves \( k\text{STS} \) problem in \( O(T) \) time.

It is easy to see that we also have the corresponding information-theoretical lower bound all being \( \Omega(\log T) \) if we treat \( k \) as a constant.

### 3. First Consecutive Spikes Counting

We present the constructions in two stages. At the first stage, we count consecutive spikes in binary transiently. At the second stage, we transform the transient firing into persistent firing. By composing the two stages, we get our desired network.

**First stage:** The network contains neurons \( z_0, \ldots, z_n, \text{in}_1, \ldots, \text{in}_n \) and we build the network inductively. For the base network that counts mod 2, we have

\[
\begin{align*}
w_{xz_0} &= 1, w_{z_0z_0} = -1, b_{z_0} = 0.5.
\end{align*}
\]

By noticing that \( z_0(t) = 1 \) if and only if \( x(t-1) = 1 \) and \( z_0(t-1) = 0 \) for \( t \geq 1 \), we have the following lemma

**Lemma 3.1.** For the mod2 base network, if \( x(t') = 1 \) for all \( t', 0 \leq t' \leq t \) for \( t \geq 0 \), then at time \( t \), \( z_0(t) = t \mod 2 \).

Now we iteratively build the network where \( 1 \leq i \leq n \) on top of the mod2 base network with the following rule:

\[
\begin{align*}
w_{xz_i} &= i + 1, \ w_{z_jz_i} = 1, \forall j, 0 \leq j < i, \ w_{z_k\text{in}_i} = 1, \forall k, 0 < k < i, \ w_{\text{in}_i\text{in}_i} = -i - 1, \ w_{z_i\text{in}_i} = 1, \ w_{z_i\text{z}_i} = i \\
b_{z_i} &= 2i + 0.5, \ b_{\text{in}_i} = i - 0.5
\end{align*}
\]

**Figure 1.** mod2 Base Network
This completes the construction. From the construction, we can deduce the following lemma.

**Lemma 3.2.** For $i > 0$, neurons $z_i, in_i$ fire according to the following rules:

1. $z_i(t) = 1$ if and only if $x(t-1) = 1$, $in_i(t-1) = 0$, and either $z_j(t-1) = 1$ for all $j, 0 \leq j < i$ or $z_i(t-1) = 1$
2. $in_i(t) = 1$ if and only if $z_j(t-1) = 1$ for all $j, 1 \leq j \leq i$

**Proof.**

**Case (1):** The potential of $z_i(t)$ is

$$w_{xzi}x^{(t-1)} + \sum_{j=0}^{i-1} w_{zjzi}z_j^{(t-1)} + w_{ini}in_i^{(t-1)} + w_{zizi}z_i^{(t-1)} = (i+1)x^{(t-1)} + \sum_{j=0}^{i-1} z_j^{(t-1)} - (i+1)in_i^{(t-1)} + iz_i^{(t-1)}$$

**Only if:** Let’s show the only if direction for the firing rule of $z_i(t)$ by proving the contrapositive.

If $x(t-1) = 0$, then the potential of $z_i(t)$ is

$$\sum_{j=0}^{i-1} x_j^{(t-1)} - (i+1)in_i^{(t-1)} + iz_i^{(t-1)} \leq 2i < 2i + 0.5 = b_z$$

If $in_i^{(t-1)} = 1$, then the potential of $z_i(t)$ is

$$(i+1)x^{(t-1)} + \sum_{j=0}^{i-1} z_j^{(t-1)} - (i+1) + iz_i^{(t-1)} \leq 2i < 2i + 0.5 = b_z$$

If there exists $\hat{j}, 0 \leq \hat{j} < i$ such that $z_{\hat{j}}(t-1) = 0$ and $z_i^{(t-1)} = 0$, then the potential of $z_i(t)$ is

$$\sum_{j \neq \hat{j}, 0 \leq j \leq i-1} z_j^{(t-1)} + (i+1)x^{(t-1)} - (i+1)in_i^{(t-1)} \leq 2i < 2i + 0.5 = b_z$$

In all three cases, we have $z_i(t) = 0$.

**If:** For the if direction, if $x(t-1) = 1$, $in_i^{(t-1)} = 0$ and $z_j^{(t-1)} = 1$ for all $j, 0 \leq j < i$, then the
potential of $z_i^{(t)}$ is

$$(i + 1) + \sum_{j=0}^{i-1} 1 + i z_i^{(t-1)} \geq 2i + 1 > 2i + 0.5 = b_{z_i}$$

If $x^{(t-1)} = 1$, $i n_i^{(t-1)} = 0$ and $z_i^{(t-1)} = 1$, then the potential of $z_i^{(t)}$ is

$$(i + 1) + \sum_{j=0}^{i-1} z_j^{(t-1)} + i \geq 2i + 1 > 2i + 0.5 = b_{z_i}$$

In both cases, we have $z_i^{(t)} = 1$.

Case (2): The firing rule of $i n_i^{(t)}$ can be analyzed similarly.

The potential of $i n_i^{(t)}$ is

$$\sum_{j=1}^i w_{z_j i n_i} z_j^{(t-1)} = \sum_{j=1}^i z_j^{(t-1)}$$

Only If: For the only if direction, if there exists $\hat{j}, 1 \leq \hat{j} \leq i$ such that $x^{(t-1)} = 0$, then the potential of $i n_i^{(t)}$ is

$$\sum_{j \neq \hat{j}, 1 \leq j \leq i} z_j^{(t-1)} \leq i - 1 < i - 0.5 = b_{i n_i}$$

We have $i n_i^{(t)} = 0$.

If: For the if direction, if $z_j^{(t-1)} = 1$ for all $j, 1 \leq j \leq i$, then the potential of $i n_i^{(t)}$ is

$$\sum_{j=1}^i 1 = i > i - 0.5 = b_{i n_i}$$

We have $i n_i^{(t)} = 1$ as desired.

Using the above lemma, we can verify that indeed the network at the first stage fires in binary, with $z_i$ encoding the $i$th digit in the binary representation.

**Theorem 3.3.** For $i \geq 1$, $t \geq 0$, if $x^{(t')} = 1$ for all $t', 0 \leq t' \leq t$, then

1. $z_i^{(t)} = a_i$ for $t = \sum_{j=0}^\infty a_j 2^j$ where $a_j \in \{0, 1\}$.
2. $i n_i^{(t)} = 1$ if and only if $t \mod 2^{i+1} = 2^i - 1$ or 0.

**Proof.** First, let’s verify that the claim is true for $z_0$. Since $x^{(t')} = 1$ for all $t', 0 \leq t' \leq t$, $z_0^{(t')} = 1$ if and only if $z_0^{(t-1)} = 0$. This implies exactly $z_0^{(t)} = t \mod 2$ as desired (for all the modular arithmetic at this paper, we choose the smallest nonnegative number from the equivalence class).

Now let’s do the induction on $t$ and we will verify the induction by checking $z_i, i n_i$ fires in according to the induction hypothesis for all $i \geq 1$. When $t = 1$, the induction statement is trivially satisfied for all $i \geq 1$. Fix $i$, we have the following cases:

1. $0 < t \mod 2^{i+1} < 2^i, z_i^{(t-1)} = 0$:
   This implies that $0 \leq t - 1 \mod 2^i < 2^i - 1$. By induction hypothesis, not all $z_j^{(t-1)} = 1$ for $0 \leq j < i$. Now by Lemma 3.2, we have $z_i^{(t)} = 0 = a_i$, $i n_i^{(t)} = 0$ as desired.

2. $t \mod 2^{i+1} = 2^i, z_i^{(t-1)} = 0, i n_i^{(t-1)} = 0$:
   This implies that $t - 1 \mod 2^i = 2^i - 1$. By induction hypothesis, $z_j^{(t-1)} = 1$ for all $j, 0 \leq j < i$. Now by Lemma 3.2, we have $z_i^{(t)} = 1 = a_i$, $i n_i^{(t)} = 0$ as desired.
(3) $2^i < t \mod 2^{i+1} < 2^i + 1, z_i^{(t-1)} = 1, i_{n_i}^{(t-1)} = 0$

This implies that $0 \leq t - 1 \mod 2^i < 2^i - 2$. By induction hypothesis, not $z_j^{(t-1)} = 1$ for all $j, 1 \leq j < i$. Now by Lemma 3.2, we have $z_i^{(t)} = 1 = a_i, i_{n_i}^{(t)} = 0$ as desired.

(4) $t \mod 2^{i+1} = 2^i + 1, z_i^{(t-1)} = 1, i_{n_i}^{(t-1)} = 0$

This implies that $t - 1 \mod 2^i = 2^i - 2$. By induction hypothesis, $z_j^{(t-1)} = 1$ for all $j, 1 \leq j < i$. Now by Lemma 3.2, we have $z_i^{(t)} = 1 = a_i, i_{n_i}^{(t)} = 1$ as desired.

(5) $t \mod 2^{i+1} = 0, z_i^{(t-1)} = 1, i_{n_i}^{(t-1)} = 1$

This implies that $t - 1 \mod 2^i = 2^i - 1$. By induction hypothesis, $z_j^{(t-1)} = 1$, for all $j, 1 \leq j < i$. Now by Lemma 3.2, we have $z_i^{(t)} = 0 = a_i, i_{n_i}^{(t)} = 1$ as desired.

This completes the induction.

Second stage: Now the second stage is a simple “capture network” with input neurons $x_i, z_i$ for all $i, 0 \leq i \leq n$, output neurons $y_i$ for $0 \leq i \leq n$ and an auxiliary neuron $s$. Intuitively, the network persistently captures the state of $z_i$ for all $i, 0 \leq i \leq n$ into $y_i$ for all $i, 0 \leq i \leq n$. We will specify the timing of the states of $y_i$ being captured later. The network is defined as the following:

$$w_{xy} = -2, w_{xs} = -n - 1, w_{ss} = n + 2, w_{yi} = 4, w_{yi} = 1, w_{zi} = w_{zi} = 1, w_{sy} = -1.5, \forall 0 \leq i \leq n$$

and

$$b_{yi} = b_s = 0.5$$

Notice that the above weight ensures the following one step firing rule:

**Lemma 3.4.** For $0 \leq i \leq n$, neurons $y_i^{(t)}$, $s^{(t)}$ fire according to the following rules:

1. $y_i^{(t)} = 1$ if and only if $y_i^{(t-1)} = 1$, or $(y_i^{(t-1)} = 0, s^{(t-1)} = 0$ and $z_i^{(t-1)} = 1$)
2. $s^{(t)} = 1$ if and only if $s^{(t-1)} = 1$, or (there exists $i, i'$ such that $z_i^{(t-1)} = 1 or y_i^{(t-1)} = 1$, and $x^{(t-1)} = 0$)

**Proof.** Case (1): The potential of $y_i^{(t)}$ is

$$w_{xy}x^{(t-1)} + w_{yi}y_i^{(t-1)} + w_{zi}z_i^{(t-1)} + w_{si}, s^{(t-1)} = -2x^{(t-1)} + 4y_i^{(t-1)} + z_i^{(t-1)} - 1.5s^{(t-1)}$$
Only If: Let’s show the only if direction for the firing rule of $y_i^{(t)}$ first. If $y_i^{(t-1)} = 0, x^{(t-1)} = 1$, the potential of $y_i^{(t)}$ is

$$-2 + z_i^{(t-1)} - 1.5s^{(t-1)} \leq -1 < 0.1 = b_y,$$

If $y_i^{(t-1)} = 0, s^{(t-1)} = 1$, the potential of $y_i^{(t)}$ is

$$-2x^{(t-1)} + z_i^{(t-1)} - 1.5 \leq -0.5 < 0.1 = b_y,$$

If $y_i^{(t-1)} = 0, z_i^{(t-1)} = 0$, the potential of $y_i^{(t)}$ is

$$-2x^{(t-1)} - 1.5s^{(t-1)} \leq 0 < 0.1 = b_y,$$

In all three cases, we have $y_i^{(t)} = 0$.

If: For the if direction, if $y_i^{(t-1)} = 1$, then the potential of $y_i^{(t)}$ is

$$-2x^{(t-1)} + 4 + z_i^{(t-1)} - 1.5s^{(t-1)} \geq 0.5 > 0.1 = b_y,$$

If $y_i^{(t-1)} = 0, x^{(t-1)} = 0, s^{(t-1)} = 0, z_i^{(t-1)} = 1$, the potential of $y_i^{(t)}$ is

$$4y_i^{(t-1)} + 1 \geq 1 > 0.1 = b_y,$$

In both cases, we have $y_i^{(t)} = 1$.

**Case (2):** The potential of $s^{(t)}$ is

$$\sum_{j=0}^{n} w_{y_j, s_j} z_j^{(t-1)} + \sum_{j=0}^{n} w_{y_j, y_j} y_j^{(t-1)} + w_{y_j, x_j} x^{(t-1)} + w_{y_j, s_j} s^{(t-1)} = \sum_{j=0}^{n} z_j^{(t-1)} + \sum_{j=0}^{n} y_j^{(t-1)} - (n+1)x^{(t-1)} + (n+2)s^{(t-1)}$$

Only If: For the only if direction, if $s^{(t-1)} = 0, y_j^{(t-1)} = z_j^{(t-1)} = 0$ for all $j, 0 \leq j \leq n$, then the potential of $s^{(t)}$ is

$$-(n+1)x^{(t-1)} \leq 0 < 0.5 = b_s,$$

If $s^{(t-1)} = 0, x^{(t-1)} = 1$, the potential of $s^{(t)}$ is

$$\sum_{j=0}^{n} z_j^{(t-1)} + \sum_{j=0}^{n} z_j^{(t-1)} - (n+1) \leq 0 < 0.5 = b_s,$$

In both cases, we have $s^{(t)} = 0$.

If: For the if direction, if there exists $i, 0 \leq i \leq n$ such that $y_i^{(t-1)} = 1$ and $x^{(t-1)} = 0$, then the potential of $s^{(t)}$ is

$$\sum_{j=0}^{n} z_j^{(t-1)} + \sum_{j \neq i, 0 \leq j \leq n} y_j^{(t-1)} + 1 + (n+2)s^{(t-1)} \geq 1 > 0.5 = b_s,$$

If there exists $i, 0 \leq i \leq n$ such that $z_i^{(t-1)} = 1$ and $x^{(t-1)} = 0$, the potential of $s^{(t)}$ is

$$\sum_{j=0}^{n} y_j^{(t-1)} + \sum_{j \neq i, 0 \leq j \leq n} z_j^{(t-1)} + 1 + (n+2)s^{(t-1)} \geq 1 > 0.5 = b_s,$$

If $s^{(t-1)} = 1$, the potential of $s^{(t)}$ is

$$\sum_{j=0}^{n} z_j^{(t-1)} + \sum_{j=0}^{n} y_j^{(t-1)} - (n+1)x^{(t-1)} + (n+2) \geq 1 > 0.5 = b_s,$$

In all three cases, we have $s^{(t)} = 1$ as desired. □
Now we can describe the behaviors of the capture network in the following theorem. The network persistently captures the state of $z_i$ for all $i, 0 \leq i \leq n$ at the first time point such that $x = 0$ and there exists some $\hat{i}$ such that $z_{\hat{i}} = 1$ into $y_{\hat{i}}$ for all $i, 0 \leq i \leq n$.

**Theorem 3.5.** For the network at the second stage, let $t' \geq 0$ be such that $x(t') = 0$ and there exists $\hat{j}$ such that $z_{\hat{j}}^{(t')} = 1$, and for all $t, 0 \leq t < t'$, either $x(t) = 1$ or $z_i^{(t)} = 0$ for all $i, 0 \leq i \leq n$. Then $y_i^{(t)} = z_i^{(t')}$ for all $i, t, 0 \leq i \leq n, t > t'$.

**Proof.** First by Lemma 3.4 for all $t, 0 < t \leq t'$, $y_i^{(t)} = s_i^{(t)} = 0$ for all $i, 0 \leq i \leq n$. Now at time $t' + 1$, by Lemma 3.4, we see that $y_i^{(t'+1)} = z_i^{(t')}$, $\forall i, 0 \leq i \leq n$ and $s_i^{(t'+1)} = 1$. Now by Lemma 3.4 we know that $s_i^{(t)} = 1$ for all $t, t > t'$. Now by Lemma 3.4 again, if $y_i^{(t'+1)} = 0$, then since $s_i^{(t)} = 1$ for all $t, t > t'$, $y_i^{(t)} = 0$ for all $t > t'$; and if $y_i^{(t'+1)} = 1$, then we also have $y_i^{(t)} = 1$ for all $t, t > t'$ as desired.

Now we are ready to prove the main Theorem 2.1.

**Proof.** We are going to prove the main theorem by composing the networks from stage one and two together. If $x_i^{(t)} = 0$ for all $t, 0 \leq t \leq T$, then the network satisfies the criterion trivially since $y_i^{(t)} = 0$ for all $0 \leq t \leq T$. If not, then there exists $\hat{t} \geq 0, L_x > 0$ such that $x_i^{(t)} = 0$ for all $t, 0 \leq t < \hat{t}, x_i^{(\hat{t}+i)} = 1$ for all $i, 0 \leq i < L_x$ and $x_i^{(\hat{t}+L_x)} = 0$ where $L_x$ is the length of the first consecutive spikes interval. Let $L_x = \sum_{j=0}^{\infty} a_j 2^j$; then by Theorem 3.3 and Lemma 3.1 $z_i^{(\hat{t}+L_x-1)} = a_i$ for all $i, 0 \leq i \leq n$. Now because $L_x > 0$, we know there exists $\hat{j}$ such that $z_{\hat{j}}^{(\hat{t}+L_x)} = 1$ by Theorem 3.3. And by Lemma 3.2, we know $z_i^{(t)} = 0$ for all $i, t, 0 \leq t \leq \hat{t}, 0 \leq i \leq n$. Now the assumption of Theorem 3.5 is satisfied with $t' = \hat{t} + L_x$. By Theorem 3.5, we have $y_i^{(t)} = a_i$ for all $t, i, 0 \leq i \leq n, t \geq \hat{t} + L_x$ as desired. This shows that the above network solves FCSC problem in $O(T)$ times with $O(\log T)$ neurons. □

4. **Total Spikes Counting**

To count the total number of spikes in an arbitrary interval requires persistence of neurons without external spikes. Notice that on FCSC network, each neuron toggles itself according to binary representation without delay. However, persistence of neurons and toggles without delays are conflicting objectives; persistence of neurons stabilizes the network while toggling without delays changes the firing patterns of the network. For example, we use self-inhibition to count mod 2 but if we use self inhibition to count mod 2, the neuron cannot maintain the count during intervals with no inputs. In this section, we circumvent this difficulty by allowing the network to enter an unstable intermediate state that still stores the information of the count when the spikes arrive; however, the network will converge to a clean state that according to binary representation after one step of computation without external signals, and this clean state is stable in an arbitrary interval with no input.

In this section, because the self-inhibition used in Section 3 to count mod 2 cannot induce persistence, we build a network of four neurons to count mod 4 to replace the function of $z_0, z_1$ in Section 3. We then iteratively build the rest of the network that approximately fires in binary on top of the mod 4 counter network.

The construction of the mod 4 counter network is the following:

\[ w_{xf_i} = 1, w_{f_if_i} = 2, 0 \leq i \leq 3, w_{f_j+f_j} = -3, 0 \leq j \leq 2, w_{f_1f_2} = w_{f_2f_3} = w_{f_3f_0} = 1, w_{f_0f_3} = w_{f_3f_1} = -3 \]

and

\[ b_{f_1} = 0.5, b_{f_i} = 1.5, i \neq 1 \]
We have the following lemma to specify the firing rules of $f_i$:

**Lemma 4.1.** Neurons $f_i(t)$ for $0 \leq i < 4$ at time $t, t \geq 1$ fire according to the following rules:

1. $f_1(t) = 1$ if and only if $f_2(t-1) = f_3(t-1) = 0$, and $x(t-1) = 1$ or $f_1(t-1) = 1$

2. $f_i(t) = 1$ for $i \neq 1$ if and only if $f_{(i+1) \mod 4}(t-1) = 0$, and $x(t-1) = 1, f_{(i-1) \mod 4}(t-1) = 1$ or $f_i(t-1) = 1$

**Proof.**

**Case (1):** The potential of $f_1(t)$ is

$$w_{xf_1}x(t-1) + w_{f_1f_1}f_1(t-1) + w_{f_2f_1}f_2(t-1) + w_{f_3f_1}f_3(t-1) = x(t-1) + 2f_1(t-1) - 3f_2(t-1) - 3f_3(t-1)$$

**Only If:** Let's show the only if direction for the firing rule of $f_1(t)$ first. If $f_2(t-1) = 1$, then the potential of $f_1(t)$ is

$$x(t-1) + 2f_1(t-1) - 3f_3(t-1) \leq 0 < 0.5 = b_{f_1}$$

If $f_3(t-1) = 1$, then the potential of $f_1(t)$ is

$$x(t-1) + 2f_1(t-1) - 3f_2(t-1) - 3 \leq 0 < 0.5 = b_{f_1}$$

If $f_1(t-1) = 0, x(t-1) = 0$, then the potential of $f_1(t)$ is

$$-3f_2(t-1) - 3f_3(t-1) \leq 0 < 0.5 = b_{f_1}$$

In all three cases, we have $f_1(t) = 0$.

**If:** For the if direction, if $f_2(t-1) = f_3(t-1) = 0, x(t-1) = 1$, then the potential of $f_1(t)$ is

$$1 + 2f_1(t-1) \geq 1 > 0.5 = b_{f_1}$$

If $f_2(t-1) = f_3(t-1) = 0, x(t-1) = 1, f_1(t-1) = 1$, then the potential of $f_1(t)$ is

$$x(t) + 2 \geq 2 > 0.5 = b_{f_1}$$

In both cases, we have $f_1(t) = 1$. 
Base Case: By Lemma 4.1, we have
\[ w_{x_i}x^{(t-1)} + w_{f_i}f_i^{(t-1)} + w_{f(i-1)}mod 4 f_i^{(t-1)}mod 4 + w_{f(i+1)}mod 4 f_i^{(t-1)}mod 4 = x^{(t-1)} + 2f_i^{(t-1)}mod 4 - 3f_i^{(t-1)}mod 4 \]

Furthermore, the network will be at a clean state again at time \(t\) if:

- \(x^{(t-1)} = 0\), then the potential of \(f_i^{(t)}\) is
  \[ f_i^{(t-1)} = 0 \]
- \(x^{(t-1)} = 1\), then the potential of \(f_i^{(t)}\) is
  \[ f_i^{(t-1)} = 1 \]

Case (2): For \(i \neq 1\), The potential of \(f_i^{(t)}\) is
\[ w_{x_i}x^{(t-1)} + w_{f_i}f_i^{(t-1)} + w_{f(i-1)}mod 4 f_i^{(t-1)}mod 4 + w_{f(i+1)}mod 4 f_i^{(t-1)}mod 4 = x^{(t-1)} + 2f_i^{(t-1)}mod 4 - 3f_i^{(t-1)}mod 4 \]

Only If: For the only if direction, if \(f_{(i+1)}^{(t-1)}mod 4 = 1\), then the potential of \(f_i^{(t)}\) is
\[ x^{(t-1)} + 2f_i^{(t-1)}mod 4 - 3f_i^{(t-1)}mod 4 \leq 1 < 1.5 = b_i \]

Define a clean state with value \(i\), \(0 \leq i < 4\) at time \(t\) of the mod4 counter network to be a state in which \(f_i^{(t)} = 1\) and \(f_j^{(t)} = 0\) for all \(j, j \neq i\). By Lemma 4.1, it is trivial to see that if \(x^{(t)} = 0\) for all \(t, t \geq t'\), then \(f_i^{(t)} = f_i^{(t')}\) for all \(t, t \geq t'\) and all \(i, 0 \leq i < 4\). Using Lemma 4.1, we have the following lemma describing the behaviors of mod4 counter network. Intuitively, when a new input arrives, the network enters an intermediate state in which both neurons represent the old count and the new count fire; when there is no input, the neuron that represents the new count will inhibit the neuron that represents the old count to stabilize the network in a clean state.

Lemma 4.2. Let the mod 4 counter network be at a clean state with value \(i\) at time \(t\). Fix a positive integer \(L\). Let \(x^{(t'+i)} = 1\) for all \(i, 0 \leq i < L\) and \(x^{(t'+L)} = 0\). Then, at time \(t, t' < t < t' + L + 1\), we have the state of the network being
\[ f_{(i+t-\nu)}^{(t)} mod 4 = f_{(i+t-\nu-1)}^{(t)} mod 4 = 1, f_{(i+t-\nu-2)}^{(t)} mod 4 = f_{(i+t-\nu-3)}^{(t)} mod 4 = 0 \]

Furthermore, the network will be at a clean state again at time \(t' + L + 1\) with \(f_{(i+L)}^{(t'+L+1)} mod 4 = 1\).

Proof. First, let’s use induction on \(t\) to prove at time \(t, t' < t < t' + L + 1\), we have the state of the network be
\[ f_{(i+t-\nu)}^{(t)} mod 4 = f_{(i+t-\nu-1)}^{(t)} mod 4 = 1, f_{(i+t-\nu-2)}^{(t)} mod 4 = f_{(i+t-\nu-3)}^{(t)} mod 4 = 0 \]

Base Case: By Lemma 4.1, we have
\[ f_{(i+1)}^{(t'+1)} mod 4 = 1, f_{(i-1)}^{(t'+1)} mod 4 = 1, f_{(i-2)}^{(t'+1)} mod 4 = 0 \]
for the base case.

**Inductive Step:** Now assume the induction hypothesis is true for \( t = k \), since we have \( x^{(k)} = 1 \) by Lemma 4.1, we indeed have

\[
\begin{align*}
    f^{(k+1)}_{(i+k+1-t') \mod 4} &= f^{(k+1)}_{(i+k+1-t'-1) \mod 4} = 1, \\
    f^{(k+1)}_{(i+k+1-t'-2) \mod 4} &= f^{(k+1)}_{(i+k+1-t'-3) \mod 4} = 0
\end{align*}
\]

This completes the induction.

Now since \( x^{(t'+L)} = 0 \), by Lemma 4.1 we can derive the state of the network at time \( t' + L + 1 \):

\[
\begin{align*}
    f^{(t'+L+1)}_{(i+L) \mod 4} &= 1, \\
    f^{(t'+L+1)}_{j} &= 0, \forall j \neq (i + L) \mod 4
\end{align*}
\]

as desired. \( \square \)

Now we iteratively build the network with the following rule on top of the mod 4 counter network,

\[
\begin{align*}
    w_{f3z_i} &= w_{f3i} = 3, \quad w_{f0z_i} = w_{f0i} = -1, \quad w_{xz_i} = w_{xin_i} = 1, \\
    w_{zjz_i} &= w_{zjin_i} = 1, \forall j, 2 \leq j < i, \quad w_{jin_iz_i} = -i - 3, \quad w_{ziz_i} = i + 3
\end{align*}
\]

and

\[
\begin{align*}
    b_{z_i} &= i + 1.5, \quad b_{in_i} = i + 2.5
\end{align*}
\]

**Figure 5.** Total spikes counting (TSC) Network

In the full construction of the TSC network, intuitively, we replace the function of \( z_0, z_1 \) in Section 3 with a mod4 counter network. We design the weights coming from \( f_3, f_0 \) such that they will induce proper carry in an approximate binary representation at \( z_i, i \geq 2 \), and we use a similar idea as the mod 4 counter network to make TSC network converge to an exact binary representation in one computation step without input.

The following lemma specifies the firing rules of \( z_i, in_i \) for \( i \geq 2 \):

**Lemma 4.3.** Neurons \( z_i^{(t)}, in_i^{(t)} \) for \( i \geq 2 \) fire according to the following rules:

1. \( z_i^{(t)} = 1 \) if and only if \( in_i^{(t-1)} = 0 \), and either \( f_3^{(t-1)} = 1, f_0^{(t-1)} = 0, x^{(t-1)} = 1 \) and \( z_j^{(t-1)} = 1 \) for all \( j, 2 \leq j < i \) or \( z_i^{(t-1)} = 1 \).
(2) \( \text{in}_i^{(t)} = 1 \) if and only if \( z_i^{(t-1)} = 1, f_3^{(t-1)} = 1, f_0^{(t-1)} = 0, x^{(t-1)} = 1 \) and \( z_j^{(t-1)} = 1 \) for all \( j, 2 \leq j < i \)

Proof. Case 1: The potential of \( z_i^{(t)} \) is

\[
\begin{align*}
& w_{f_3} f_3^{(t-1)} + w_{f_0} f_0^{(t-1)} + \sum_{j=2}^{i-1} w_{z_j} z_j^{(t-1)} + w_{z_i} z_i^{(t-1)} + w_{\text{in}_i} \text{in}_i^{(t-1)} + w_{x_i} x^{(t-1)} \\
& = 3f_3^{(t-1)} - f_0^{(t-1)} + \sum_{j=2}^{i-1} z_j^{(t-1)} + (i + 3)i - (i + 3) + x^{(t-1)}
\end{align*}
\]

Only If: Let's show the only if direction for the firing rule of \( z_i^{(t)} \) first. If \( \text{in}_i^{(t-1)} = 1 \), the potential of \( z_i^{(t)} \) is

\[
3f_3^{(t-1)} - f_0^{(t-1)} + \sum_{j=2}^{i-1} z_j^{(t-1)} + (i + 3)i - (i + 3) + x^{(t-1)} \leq i + 1 < i + 1.5 = b_{z_i}
\]

If \( f_3^{(t-1)} = 0, z_i^{(t-1)} = 0 \), the potential of \( z_i^{(t)} \) is

\[
- f_0^{(t-1)} + \sum_{j=2}^{i-1} z_j^{(t-1)} - (i + 3)i + x^{(t-1)} \leq i - 1 < i + 1.5 = b_{z_i}
\]

If \( f_0^{(t-1)} = 1, z_i^{(t-1)} = 0 \), the potential of \( z_i^{(t)} \) is

\[
3f_3^{(t-1)} - 1 + \sum_{j=2}^{i-1} z_j^{(t-1)} - (i + 3)i + x^{(t-1)} \leq i + 1 < i + 1.5 = b_{z_i}
\]

If \( x^{(t-1)} = 0, z_i^{(t-1)} = 0 \), the potential of \( z_i^{(t)} \) is

\[
3f_3^{(t-1)} - f_0^{(t-1)} + \sum_{j=2}^{i-1} z_j^{(t-1)} - (i + 3)i + x^{(t-1)} \leq i + 1 < i + 1.5 = b_{z_i}
\]

If \( z_i^{(t-1)} = 0 \) and there exists \( j, 2 \leq j < i \) such that \( z_j^{(t-1)} = 0 \), the potential of \( z_i^{(t)} \) is

\[
3f_3^{(t-1)} - f_0^{(t-1)} + \sum_{j=2}^{i-1} z_j^{(t-1)} - (i + 3)i + x^{(t-1)} \leq i + 1 < i + 1.5 = b_{z_i}
\]

In all cases, we have \( z_i^{(t)} = 0 \).

If: For the if direction, if \( \text{in}_i^{(t-1)} = 0, f_3^{(t-1)} = 1, f_0^{(t-1)} = 0, x^{(t-1)} = 1, z_j^{(t-1)} = 1 \) for all \( j, 2 \leq j < i \), then the potential of \( z_i^{(t)} \) is

\[
3 + \sum_{j=2}^{i-1} 1 + (i + 3)i + 1 \geq i + 2 > i + 1.5 = b_{z_i}
\]

If \( \text{in}_i^{(t-1)} = 0, z_i^{(t-1)} = 1 \), the potential of \( z_i^{(t)} \) is

\[
3f_3^{(t-1)} - f_0^{(t-1)} + \sum_{j=2}^{i-1} z_j^{(t-1)} + (i + 3) + x^{(t-1)} \geq i + 2 > i + 1.5 = b_{z_i}
\]
In both cases, we have $z^{(t)}_i = 1$.

**Case 2:** the potential of $in^{(t)}_i$ is

$$w_{f_3}_{in}, f^{(t-1)}_3 + w_{f_0}_{in}, f^{(t-1)}_0 + \sum_{j=2}^{i-1} w_{z_j}_{in}, z^{(t-1)}_j + w_{z_i}_{in}, z^{(t-1)}_i + w_{X_{in}}, x^{(t-1)}$$

$$= 3f^{(t-1)}_3 - f^{(t-1)}_0 + \sum_{j=2}^{i-1} z^{(t-1)}_j + z^{(t-1)}_i + x^{(t-1)}$$

**Only If:** For the only if direction, if $z^{(t-1)}_i = 0$, then the potential of $in^{(t)}_i$ is

$$3f^{(t-1)}_3 - f^{(t-1)}_0 + \sum_{j=2}^{i-1} z^{(t-1)}_j + z^{(t-1)}_i + x^{(t-1)} \leq i + 2 < i + 2.5 = b_{in_i}$$

If $f^{(t-1)}_3 = 0$, the potential of $in^{(t)}_i$ is

$$-f^{(t-1)}_0 + \sum_{j=2}^{i-1} z^{(t-1)}_j + z^{(t-1)}_i + x^{(t-1)} \leq i < i + 2.5 = b_{in_i}$$

If $f^{(t-1)}_0 = 1$, the potential of $in^{(t)}_i$ is

$$3f^{(t-1)}_3 - 1 + \sum_{j=2}^{i-1} z^{(t-1)}_j + z^{(t-1)}_i + x^{(t-1)} \leq i + 2 < i + 2.5 = b_{in_i}$$

If $x^{(t-1)} = 0$, the potential of $in^{(t)}_i$ is

$$3f^{(t-1)}_3 - f^{(t-1)}_0 + \sum_{j=2}^{i-1} z^{(t-1)}_j + z^{(t-1)}_i \leq i + 2 < i + 2.5 = b_{in_i}$$

If there exists $\hat{j}, 2 \leq \hat{j} < i$ such that $z^{(t)}_{\hat{j}} = 0$, the potential of $in^{(t)}_i$ is

$$3f^{(t-1)}_3 - f^{(t-1)}_0 + \sum_{j \neq \hat{j}, 2 \leq j < i} z^{(t-1)}_j + z^{(t-1)}_i + x^{(t-1)} \leq i + 2 < i + 2.5 = b_{in_i}$$

In all cases, $in^{(t)}_i = 0$.

**If:** For the if direction, if $z^{(t-1)}_i = 1, f^{(t-1)}_3 = 1, f^{(t-1)}_0 = 0, x^{(t-1)} = 1, z^{(t-1)}_j = 1$ for all $j, 2 \leq j < i$, then the potential of $in^{(t)}_i$ is

$$3 + \sum_{j=2}^{i-1} 1 + 1 + 1 \leq i + 3 > i + 2.5 = b_{in_i}$$

We have $in^{(t)}_i = 1$ as desired.

Define a clean state at time $t'$ of TSC network with value $X$ stored be one in which

1. $f^{(t')}_{X \mod 4} = 1, f^{(t')}_{j} = 0, \forall j \neq X \mod 4$ (i.e., the mod4 counter subnetwork is clean with value $X \mod 4$)
2. For $X = \sum_{i=0}^{\infty} a_i, a_i \in \{0, 1\}$, $z^{(t')}_k = a_k, \forall k \geq 2$
3. $in^{(t')}_i = 0$ if $X \mod 2^{i+1} = 2^{i+1} - 1$
So being at a clean state for TSC network with value $X$ stored implies being at a clean state with value $X \mod 4$ for its mod4 counter subnetwork with $z_i$ in binary representation for $i \geq 2$. By Lemma 4.3, it is trivial to see that if $x^{(t)} = 0$ for all $t \geq t'$ and $i \geq 2$, then $f_i^{(t')} = f_i^{(t)}$ for all $t, t' \geq t'$ and all $i$. Using Lemma 4.3, we have the following lemma describing the behaviors of the TSC network.

**Lemma 4.4.** Let TSC network be at a clean state at time $t'$ with value $X$ stored. Fix a positive integer $L$. Let $x^{(t') + i} = 1$ for all $i, 0 \leq i < L$ and $x^{(t' + L)} = 0$. Then, at $t, t' < t < t' + L + 1$, $z_i, in_i$ fire with the following rules for all $i \geq 2$:

1. $1 = X + t - t' \mod 2^{i+1} < 2^i$, $z_i^{(t)} = 0$
   
   This implies that $0 \leq X \mod 2^{i+1} < 2^i - 1$. This shows that not all $z_j^{(t-1)} = 1$ for all $j, j < i$ or $f_j^{(t-1)} = 0$ or $f_j^{(t-1)} = 1$. By Lemma 4.3, we have $z_i^{(t)} = in_i^{(t)} = 0$
2. $1 < X + t - t' \mod 2^{i+1} < 2^i$, $z_i^{(t)} = in_i^{(t)} = 0$
   
   This implies that $2^i - 1 \leq X \mod 2^{i+1} < 2^i - 1$. This shows that either $f_3^{(t-1)} = 1, f_0^{(t-1)} = 0, z_j^{(t-1)} = 1$ for all $j, j < i$ or $z_i^{(t-1)} = 1$ but not both. By Lemma 4.3, we have $z_i^{(t)} = 1, in_i^{(t)} = 0$
3. $X + t - t' \mod 2^{i+1} = 0$
   
   This implies that $X \mod 2^{i+1} = 2^i - 1$. This shows that $f_3^{(t-1)} = 1, f_0^{(t-1)} = 0, z_j^{(t-1)} = 1$ for all $j < i$ and by the definition of a clean state, we have $in_i^{(t-1)} = 0$. Now by Lemma 4.3, we have $z_i^{(t)} = 1, in_i^{(t)} = 1$.

**Inductive Step:** Assume the induction hypothesis is accurate for $t = k$. We have the following cases

1. $1 = X + k + 1 - t' \mod 2^{i+1} < 2^i$
   
   This implies that $X + k - t' \mod 2^{i+1} = 0$. Now by induction hypothesis and Lemma 4.2, we know that $f_3^{(k)} = 1, f_0^{(k)} = 0, z_j^{(k)} = 1, in_j^{(k)} = 1$ for all $j, i \geq j \geq 2$. By Lemma 4.3, we have $z_i^{(k+1)} = 0, in_i^{(k+1)} = 1$
2. $1 < X + k + 1 - t' \mod 2^{i+1} < 2^i$
   
   This implies that $1 \leq X + k - t' \mod 2^{i+1} < 2^i - 1$. By induction hypothesis and Lemma 4.2, this shows that not all $z_j^{(k)} = 1$ for all $j, j < i$ or $f_3^{(k)} = 0$ or $f_0^{(k)} = 1$. By Lemma 4.3, we have $x_i^{(k+1)} = in_i^{(k+1)} = 0$
3. $X + k + 1 - t' \mod 2^{i+1} = 2^i$
   
   This implies that $2^i - 1 \leq X + k - t' \mod 2^{i+1} < 2^i - 1$. By induction hypothesis and Lemma 4.2, this shows that either $f_3^{(k)} = 1, f_0^{(k)} = 0, z_j^{(k)} = 1$ for all $j, j < i$ or $z_i^{(k)} = 1$ but not both. By Lemma 4.3, we have $z_i^{(k+1)} = 1, in_i^{(k+1)} = 0$
4. $X + k + 1 - t' \mod 2^{i+1} = 0$
   
   This implies that $X + k - t' \mod 2^{i+1} = 2^i - 1$. By induction hypothesis and Lemma 4.2.
this shows that all $f_{3}^{(k)} = 1, f_{0}^{(k)} = 0, \in_{i}^{(k)} = 0, z_{j}^{(k)} = 1$ for all $j, j \leq i$. Now by Lemma 4.3, we have $z_{i}^{(t)} = 1, \in_{i}^{(t)} = 1$.

This completes the induction.

Now we just need to show that at time $t' + L + 1$ the network is at a clean state with value $X + L$ stored. We have the following cases:

1. $1 = X + L \mod 2^{i+1} < 2^i$:
   - By above induction, we have $z_{j}^{(t' + L)} = 0$ for $j, j \leq i$. No matter what the value of $\in_{i}^{(t' + L)}$ is, by Lemma 4.3 we have $z_{i}^{(t' + L + 1)} = \in_{i}^{(t' + L + 1)} = 0$.

2. $1 < X + L \mod 2^{i+1} < 2^i, z_{i}^{(t)} = \in_{i}^{(t)} = 0$:
   - By above induction, we have $z_{i}^{(t' + L)} = \in_{i}^{(t' + L)} = 0$. By Lemma 4.3 we have $z_{i}^{(t' + L + 1)} = \in_{i}^{(t' + L + 1)} = 0$

3. $X + L \mod 2^{i+1} \geq 2^i$, we have $z_{i}^{(t' + L)} = 1, \in_{i}^{(t' + L)} = 0$. By Lemma 4.3 we have $z_{i}^{(t' + L + 1)} = \in_{i}^{(t' + L + 1)} = 0$

4. $X + L \mod 2^{i+1} = 0$, we have $z_{i}^{(t' + L)} = 1, \in_{i}^{(t' + L)} = 1$. By Lemma 4.3 we have $z_{i}^{(t' + L + 1)} = 0, \in_{i}^{(t' + L + 1)} = 1$

which is exactly a clean state with value $X + L$ stored combining with Lemma 4.2.

Now we are ready for the main proof of Theorem 2.2.

\textbf{Proof.} Let $f_{i}, z_{j}, 0 \leq i < 4, 2 \leq j \leq n$ be our output neurons. Let there be $X$ spikes in $T$ time steps. Let $[t_{0}, t_{0} + X_{0} - 1], \cdots, [t_{k}, t_{k} + X_{k} - 1]$ be the disjoint maximal intervals of spikes ordered by time (i.e. $x^{(t)} = 1$ if $t \in [t_{i}, t_{i} + X_{i} - 1]$ for some $0 \leq i \leq k$ and $[t_{i}, t_{i} + X_{i}] \cap [t_{j}, t_{j} + X_{j}] = \emptyset$ for all $i \neq j$ and $t_{0} < t_{1} < \cdots < t_{k}, \sum_{i=0}^{k} X_{i} = X$). Now I claim that at time $t_{k} + X_{k} + 1$, the network is at a clean state with value $\sum_{j=0}^{i} X_{j}$ stored. We will prove the claim with induction on $i$. For $i = 0$, apply Lemma 4.4, we get that the network is at a clean state with value $X_{0}$ stored. Assume the network is at a clean state with value $\sum_{j=0}^{i} X_{j}$ stored at time $t_{i} + X_{i} + 1$. Then apply Lemma 4.4 again, we get at time $t_{i+1} + X_{i+1} + 1$, the network is at a clean state with value $\sum_{j=0}^{i+1} X_{j}$ stored at time $t_{i+1} + X_{i+1} + 1$. So at time $t_{k} + X_{k} + 1 = O(T)$, the network is at a clean state with value $\sum_{j=0}^{k} X_{j} = X$ stored as desired. This shows that the above network solves TSC problem in $O(T)$ times with $O(\log T)$ neurons.

\section{5. k-spikes Temporal to Spatial Encoding}

kSTS can be seen as a toy problem to model how brain transforms temporal information to spatial information. With the FCSC network from Section 3, we can derive kSTS network as an easy extension.

First, notice that there is an obvious $O(T)$ solution to kSTS problem which sends the temporal information to $T$ spatial buffers and then captures it as the following:

$$w_{x_{1}z_{1}} = w_{x_{i}x_{i+1}} = 1, \forall i, 1 \leq i \leq T - 1, w_{x_{j}y_{j}} = w_{x_{end}y_{j}} = 1, \forall j, 1 \leq j \leq T$$

and

$$b_{x_{i}} = 0.5, b_{y_{i}} = 1.5, 1 \leq i \leq T$$

Since signals in brain are usually sparse, we can compress the signals better with the information that the signals only have $k$ spikes as a corollary of the FCSC problem. The idea to use a negated FCSC network to count each block and then capture everything at the end. This will gives us a $O(k \log T)$ solution. To be precise, the negated FCSC network is obtained by changing every
connection from \( x \) to its opposite sign and adjust bias accordingly as below:

For the base network that counts mod 2, we have

\[
w_{xz_0} = -1, w_{z_0z_0} = -1, b_{z_0} = -0.5
\]

Now we iteratively build the \( i \)th block of network where \( 1 \leq i \leq n \) with the following rule:

\[
w_{xz_i} = -i-1, \quad w_{z_jz_i} = 1, \quad \forall j, 0 \leq j \leq i, \quad w_{z_ikz_i} = -i-1, \quad w_{z_iz_i} = i
\]

\[
b_{z_i} = i - 0.5, \quad b_{in_i} = i - 0.5
\]

and the capture part

\[
w_{xy_i} = 2, \quad w_{xs} = n + 1, \quad w_{xs} = n + 2, \quad w_{y_iy_i} = 4, \quad w_{z_iz_i} = 1, \quad w_{z_is} = 1, \quad w_{sy_i} = -1.5, \quad \forall 0 \leq i \leq n
\]

and

\[
b_{y_i} = 2.5, \quad b_s = n + 1.5
\]

This network counts the first consecutive non-spikes.

Now to construct the entire network, we proceed in blocks. The 0th block of the network \( N_0 \) contains a complete negated FCSC network. We inductively build the \( i \)th block \( N_i \) of the network as the following, for notation convenience, let \( x \) be \( p_0 \). Then the \( i \)th block of the network contains a negated network of first consecutive counter \( N_i \) with all the biases plus 1 and we have the connection

\[
w_{p_{i-1}p_i} = w_{xp_i} = 1, \quad w_{p_ip_i} = 2, \quad w_{p_iz} = 1, \quad \forall z \in N_i, \quad w_{p_ix_{(i-1)}}, \quad \forall x_{(i-1)j} \in N_{i-1}
\]

and

\[
b_{p_i} = 1.5
\]

It’s easy to see that when the \( i \)th spikes arrive, we begin the counting at the \( i+1 \)th block of network and block any incoming signal in \( i \)th block network because \( i+1 \)th block of network only begins to accepting signals after \( p_{i+1} \) fires.

**Figure 6.** \( k \)-spikes Temporal to Spatial Encoding Network

And at the end, we can capture all the spatial information similarly as the second stage of FCSC network

\[
w_{xeuzy_0} = w_{zyu} = 1, \quad w_{y_izy_z} = 2, \quad b_{zu} = 1.5, \quad \forall z, \quad i, \quad z \in N_i, \quad 0 \leq i \leq k
\]

This proves Theorem 2.3.
6. Discussion and Future Direction

In this paper, we have shown that networks of neurons are capable of integrating temporal information to solve three different tasks with temporal inputs efficiently. Our paper follows similar approaches to Lynch [LMP17a, LMP17b, LM18] by treating neurons as static circuits to explore the computational power of neural circuits. There are three noteworthy points about our model. First, instead of a stochastic model, we use a deterministic one. However, it should be noted that all the results in this paper would still hold under the randomized model of Lynch [LMP17a, LMP17b, LM18] with high probability. Second, we use a model which resets the potential at every round. Therefore, to retain temporal information, many self-excitation connections are employed in our networks. At the other extreme, we can have a model in which the potential does not decay from past rounds. In that model, temporal information can be stored in potentials, but it might require different mechanisms to translate the information from potentials to spikes. The two models thus can lead to different possible computational principles in brains. Third, we used a discrete time model instead of a continuous time model, which would be more biologically plausible. However, this might not be a concern since we can use Maass’s synchronization module [Maa96] to simulate our discrete time model from a continuous time model.

This paper mainly deals with the exact versions of the problems. One possible extension is to consider the approximate versions of the problems. By introducing noise into our models, we might be able to solve the approximate versions of the problems more efficiently. For example, for approximate counting, we aim to output some firing patterns corresponding to a number $\tilde{X}$ such that

$$P(|\tilde{X} - X| > \epsilon X) < \delta$$

is small. The lower bound for this question is $\Omega((\log \log T)$ and finding a matching upper bound can be an interesting future direction. However, approximate versions of the questions are tricky with temporal inputs because the network inevitably reuses random bits if they are stored inside the weights. A possible approach is to use a small number of random bits to generate a large family of $k$-wise independent random functions within neurons.

Another aspect of the temporal input we have not exploited is the time-scale invariance of the problem. In biology, many problems are time-scale invariant. A person who says “apple” fast can be understood as well as a person who says “apple” slowly. If we exploit this invariance, we might be able to reduce the networks’ complexity further.

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