SIMPLE CURVES ON SURFACES

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To Larry Siebenmann on the occasion of his sixtieth birthday

Abstract. We study simple closed geodesics on a hyperbolic surface of genus $g$ with $b$ geodesic boundary components and $c$ cusps. We show that the number of such geodesics of length at most $L$ is of order $L^{6g+2b+2c-6}$. This answers a long-standing open question.

Let $\mathcal{S}$ be a hyperbolic surface of genus $g$ with $c$ cusps and $b$ boundary components. In this paper we study the set of simple (that is, without self-intersections) closed geodesics on $\mathcal{S}$. More precisely we study the counting function $N(L, \mathcal{S})$ – the number of simple geodesics of length no greater than $L$ on the surface $\mathcal{S}$. We show that there are constants $c_1$ and $c_2$ (depending on only on $\mathcal{S}$), such that

$$c_1 L^{6g-6+2b+2c} \leq N(L, \mathcal{S}) \leq c_2 L^{6g-6+2b+2c}.$$

The estimate (1) should be put into proper perspective, and it is with this end that we give the following historical summary on the study of simple closed curves on surfaces. This study goes back all the way to the beginning of the subject of geometry and topology of surfaces (that is, the work of Henri Poincaré and Max Dehn), and some of the subsequent work we will mention is a development (if not actually a repetition) of this work. Being as it may, one line of inquiry has been group theoretic: suppose $\gamma$ is an element in the fundamental group of $\mathcal{S}$, how do we decide whether or not $\gamma$ is represented by a simple loop? This question was probably known to Nielsen for the case of a punctured torus, however, the earliest reference known to me is the paper of Osborne and Zieschang [14]. In the general case, the first reasonable algorithm for determining whether an element of $\pi_1(\mathcal{S})$ can be represented by a simple curve was given by H. Zieschang [20, 21], and D. Chillingworth [5, 6], following earlier work of Reinhart [16] – Zieschang’s algorithm is primarily group-theoretic, whilst Reinhart–Chillingworth is more geometric. This work has been rendered more explicit by Birman and Series [2, 3], and roughly at the same time Cohen and Lustig [7, 10] had extended the Birman–Series algorithm to determine the minimum number of intersections.

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between curves representing two homotopy classes (which includes the self-intersection number of a single curve as a special case).

Unfortunately, while this algorithmic work is very interesting (the words in the fundamental group represented by simple closed curves are a direct generalisation of Sturm sequences – these are precisely the words in the free group on two generators represented by simple curves on the punctured torus, as remarked by Birman and Series in their papers), they do not seem to be usable for estimating the number of such words as a function of the length of the word (which is, up to constant factors, the same as the length of the corresponding geodesic).

This brings us to the the counting question. It seems that the problem of counting all geodesics of bounded length has been resolved almost entirely, due to the work of Delsarte, Huber, and Selberg in the constant curvature case, and Margulis, Bowen, Ruelle, and others in the variable curvature case. In all cases the estimate is that the number of geodesics of length bounded above by \( L \) is asymptotic to \( \exp(hL)/L \), where \( h \) is the topological entropy of the geodesic flow. In particular, \( h = 1 \) for every finite area hyperbolic surface. Thus, the growth of the number of all geodesics on all such surface depends (up to first order) neither on the topology, nor on the actual hyperbolic metric. This would seem to indicate that the set of all closed geodesics is not a very geometric object. Now, for simple geodesics, things are a lot more subtle. For example, for the simplest hyperbolic surface – the thrice–punctured sphere – there are none. After that, things become more complicated. For the four times punctured sphere, Beardon, Lehner, and Sheingorn [1] had shown that the number of simple geodesics grew at least linearly and at most quadratically, as a function of length. Since the four-punctured sphere and the once-punctured torus are essentially the same, this implies the same estimate for the torus. On the other hand, Thurston’s theories of measured foliations (see [8]) and (dually) projective laminations imply that:

\[
\begin{align*}
    c_1 L^{6g-6+2b+2c} & \leq M(L, S) \leq c_2 L^{6g-6+2b+2c},
\end{align*}
\]

where \( M(L, S) \) is the number of collections of pairwise non-intersecting simple closed geodesics of total length no greater than \( L \) on \( S \).

It is allowed to take multiple copies of any given curve; its contribution to the total length is then multiplied by the multiplicity.

**Notation.** Such a collection of curves will henceforth be called a multicurve.

Since on a once-punctured torus every multicurve is connected (though possibly covers itself multiple times), the estimate (1) for the torus follows, in essence, from the estimate (2) when \( g = 1 \), and \( b + c = 1 \). This also implies the estimate (1) for the 4-times punctured sphere. This estimate only appeared in print in my paper with Greg McShane [11]. In that paper (see also [2]) we actually show a much stronger result: \( N(L, S) \) for \( S \) a
punctured torus is asymptotic to $c_S L^2$, where the coefficient $c_S$ depends on the hyperbolic structure, varies real-analytically over moduli space of tori, and goes to infinity at infinity of moduli space (hence is not constant).

For $S$ a surface of genus 2, the estimate (1) follows from the work of Haas and Susskind [9].

In general, Birman and Series [3] have shown that for any genus, the number of simple curves (actually the number of curves with a bounded number of self-intersections) grows at most polynomially, with the exponent depending on the topological type of the surface. For simple curves their result follows immediately from the estimate (2), which provides an upper bound for the number of simple closed curves of bounded length. The harder part (and the subject of this paper) is proving the lower bound, It should be noted that this is claimed (indirectly) in the paper [15]. However, the argument there is extremely incomplete, and has never been generally accepted.

More recently, Geoff Mess (private communication) has claimed to have improved the estimate (2) for $M(S, L)$ (the number of multicurves to an asymptotic result (crudely speaking, showing that one can choose $c_1$ and $c_2$ arbitrarily close to each other). Furthermore, he has claimed to be able to show analytic variation of the resulting constant over moduli space.

Of course, a really interesting question is whether there is an asymptotic form of (1). The argument proving the estimate (1) (and occupying the rest of this paper) seems to indicate that such a result should exist, but it seems difficult.

Here is an outline of the rest of the paper:

Section 1 contains some background facts. In Sections 2 and 3 the basic method is developed and used to prove estimate (1) for $g = 0, 1$. The case of arbitrary genus requires a couple of other refinements, and is addressed in Section 4.

**Notation.** In the sequel, whenever constants are used (denoted by $c$, $c_1$, etc), it is to be understood that these depend solely on the hyperbolic metric on the surface in question. The same letter can (and does) denote different numbers in different places in the paper.

1. **Background**

   In this section we assemble some necessary background facts.

   **Theorem 1.1.** Let $S$ be a hyperbolic surface, and $C$ a cusp of $S$. Then there is an embedded horodisk neighborhood of $C$ of area 2 which contains meets no simple closed geodesic of $S$.

   **Proof.** This theorem goes back to Poincaré, for a proof see [11].

   **Theorem 1.2.** Let $\gamma, \beta$ be simple closed geodesics on a hyperbolic surface $S$, and let $T_\beta(\gamma)$ be the Dehn twist of $\gamma$ around $\beta$. Then $\ell(T_\beta(\gamma)) \leq \ell(\gamma) + i(\gamma, \beta) \ell(\beta)$, where $i(\gamma, \beta)$ is the geometric intersection number of $\gamma$ and $\beta$. 
Figure 1. first kind

**Proof.** This follows (among other things) from the variational formula of Wolpert [19] on the change of length of curves under earthquake deformations. See also [18].

**Theorem 1.3.** Given two closed curves $\beta_0, \gamma_0$, the smallest geometric intersection number between curves $\beta, \gamma$, freely homotopic to $\beta_0, \gamma_0$, respectively, is realized for $\beta, \gamma$ geodesic.

**Proof.** This was also known to Poincaré, for a proof see [17].

From now on, $\Gamma_{A_1 \ldots A_n}$ shall denote the sphere $S^2$ missing $n$ disks (usually equipped with a hyperbolic metric with $n$ geodesic boundary components).

**Theorem 1.4.** There is only one homotopy class of non-boundary-parallel simple curves on $\Gamma_{ABC}$ beginning and ending on the same boundary component $A$.

**Proof.** See diagram 1.

**Theorem 1.5.** There is only one homotopy class of curves on $\Gamma_{ABC}$ joining boundary component $A$ to boundary component $B$.

**Proof.** See diagram 2.

**Theorem 1.4** is relevant in view of:

**Theorem 1.6.** No geodesic segment in $\Gamma_{ABC}$ can be boundary parallel.

**Proof.** There are no geodesic bigons in $\mathbb{H}^2$.

These observations are sufficient for us to begin counting simple curves.
2. LOW GENUS

To be systematic, we start with the easiest case:

**Theorem 2.1.** Any simple closed geodesic on $\Gamma_{ABC}$ is a boundary component.

*Proof.* Exercise. \qed

**Theorem 2.2.** For any hyperbolic structure $S$ on the 4-punctured sphere, $\exists c_S, L_0 > 0,$ such that for any $L > L_0$, the number of simple closed geodesics on $S$ for length not exceeding $L_0$ is not less than $c_SL^2$.

**Remark 2.3.** Of course, a stronger statement follows from \cite{11, 12}, see the Introduction, but we use the proof of this result to introduce the techniques and notation for the rest of the paper.

*Proof.* Let $\Gamma_{ABCD}$ be the 4-punctured sphere in question. Pick a simple loop $E$, separating $\Gamma_{ABCD}$ into two thrice-punctured spheres, $\Gamma_{ABE}$, and $\Gamma_{CDE}$ (see diagram 3). Let $\gamma$ be a simple geodesic. There are a finite number of such which do not intersect the separating curve $E$ (either 4 or 0, depending on whether or not one counts boundary curves). The curve $\gamma$ could be homotopic to $E$, but if not, it must intersect $E$ transversely in $2k$ points. Let $\gamma$ be such a curve. Note that $\gamma \cap \Gamma_{ABE}$ consists of $k$ geodesic segments, having all of their endpoints on $E$. Up to homotopy, there is only one way to thus place $k$ segments in $\Gamma_{ABE}$, by Theorem 1.4. See Figure 4. The intersection of $\gamma$ with $\Gamma_{CDE}$ looks similar. Note that the length of $\gamma \cap \Gamma_{CDE} \asymp k$.

Consider the inverse operation: Given two diagrams which look like figure 4 we can glue them together with a rational twist $\frac{p}{2k}$. The integer part $t = \lfloor \frac{p}{2k} \rfloor$ corresponds to twisting $\gamma t$ times around $E$. The fractional part correspond to the change of the identification map: A diagram (with an orientation) has a canonical labelling (shown in Figure 4). Gluing with no
twist corresponds to attaching the strand labelled 1 to one labelled 1, and so on. Twisting by \(q\) corresponds to gluing the strand labelled \(l\) to the strand labelled \(l + q\), if \(l + q < k\), to \(2k - (l + q)\) if \(l < k\), \(l + q > k\) and so on.

It is not hard to see that the twist \(\frac{p}{2k}\) leads to a connected curve if and only if \(p\) and \(2k\) are relatively prime. By observation 1.2, the length of \(\gamma\) twisted by \(\frac{p}{2k}\) is bounded above by \(p\ell(E)/2\), while the number of twists not exceeding \(N\) leading to connected curves is

\[
\frac{\phi(k)}{k} 2kN = 2\phi(k)N,
\]

where \(\phi\) denotes the Euler totient function.

By the previous observations, the length of curves obtained thereby is bounded above by \(kN\ell(E)\), so to obtain curves of length not exceeding \(L\), we must take \(N \leq \frac{L}{k\ell(E)}\), thus, for a fixed \(k\) we have

\[
\frac{\phi(k)L}{k\ell(E)}
\]
curves. Since $k$ could be anything up to $cL$ (the constant $c$ depending on the metric on $\Gamma_{ABCD}$), we see that

$$
\mathcal{N}(L, \Gamma_{ABCD}) \geq \sum_{k=1}^{cL} \frac{L}{\ell(E)} \frac{\phi(k)}{k} \geq c'L^2,
$$

where we use $\mathcal{N}(L, S)$ to denote the number of simple geodesics on a surface $S$ of length not exceeding $L$. The last inequality in Eq. (4) is the consequence of Lemma 2.4 below (the argument is standard in number theory; we give it here for completeness).

Lemma 2.4.

$$
\sum_{m=1}^{n} \frac{\phi(m)}{m} = \frac{6}{\pi^2} n + O(\log n),
$$

where $\phi$ denotes the Euler totient function.

Proof. First, note that

$$
\sum_{d|n} \phi(d) = n.
$$

This follows, for example, from the observation that the number of elements of order $d|n$ in the cyclic group of order $n$ is equal to $\phi(d)$. From equation (5), we have, by Möbius inversion, that

$$
\frac{\phi(n)}{n} = \sum_{d|n} \frac{\mu(d)}{d}.
$$

Using equation (6), we have

$$
S(n) = \sum_{m \leq n} \frac{\phi(m)}{m} = \sum_{m \leq n} \sum_{d|m} \frac{\mu(d)}{d}.
$$

Changing the order of summation, we see that

$$
S(n) = \sum_{d \leq n} \sum_{j \leq \frac{n}{d}} \frac{\mu(d)}{d} = \sum_{d \leq n} \left\lfloor \frac{n}{d} \right\rfloor \frac{\mu(d)}{d},
$$

where $\lfloor x \rfloor$ denotes the integer part of $x$. Since $|x - \lfloor x \rfloor| < 1$, we have the estimate

$$
|S(n) - S_0(n)| < \sum_{d \leq n} \left| \frac{\mu(d)}{d} \right| < \sum_{d \leq n} \frac{1}{d} = \log(n) + O(1),
$$

where

$$
S_0(n) = \sum_{d \leq n} \frac{n \mu(d)}{d} = n \sum_{d \leq n} \frac{\mu(d)}{d^2}.
$$
Note that
\[ \sum_{j=1}^{\infty} \frac{\mu(j)}{j^s} = \frac{1}{\zeta(s)}, \]
where \( \zeta \) is the Riemann \( \zeta \) function, while
\[ |\sum_{j=n+1}^{\infty} \frac{\mu(j)}{j^2}| \leq \sum_{j=n+1}^{\infty} \frac{1}{j^2} = O(n^{-1}). \]

Putting all these estimates together, we get the conclusion of the lemma. \( \Box \)

This analysis will now be extended to deal with a sphere with \( k \) boundary components. We will prove

**Theorem 2.5.** The number of simple geodesics of length bounded by \( L \) on a sphere with \( c \) boundary components grows like \( L^{2c-6} \).

**Proof.** As before, take the sphere \( \Gamma_{A_1\ldots A_k} \) and cut it into a sphere \( \Gamma_{A_1\ldots A_k-2E} \) with \( k-1 \) boundary components and a pair of pants \( \Gamma_{E A_{k-1} A_k} \). We will, again, count the simple geodesics \( \gamma \) which intersect \( E \) \( 2k \) times, then sum over the possible values of \( k \). The intersection of such a curve with \( \Gamma_{E A_{k-1} A_k} \) was already studied in the proof of Theorem 2.2. It remains to analyse \( \gamma \cap \Gamma_{A_1\ldots A_k-2E} \). This is a collection of \( k \) disjoint segments having all of their endpoints on \( E \), and we will analyse this inductively. We will prove the following

**Lemma 2.6.** Let \( G = \Gamma_{A_1\ldots A_c} \) be a sphere with \( c \) boundary components. The number of \( k \) component multicurves on \( G \) of length bounded above by \( L \) is bounded below by a constant times \( k^{c-3}L^{c-3} \).

**Proof of lemma 2.6.** Consider a sphere with \( c \) boundary components, \( \Gamma_{A_1\ldots A_{c-1}E} \). We cut off a 3-punctured sphere \( \Gamma_{F A_{c-1} E} \). The intersection of \( \gamma \) with this sphere is a collection of segments, \( 2m \) of which go from \( F \) to \( E \), while \( k-m \) go from \( E \) to itself (see diagram 5).
Remark 2.7. There is another combinatorial possibility, shown in Figure 6, but as we are only interested in a lower bound, we ignore it.

The intersection of $\gamma$ with $\Gamma_{A_1 \ldots A_{c-1} F}$ has $m < k$ connected components. Thus, we have the inequality

$$N(c, k, L) \geq c \sum_{m=0}^{k} \int_{0}^{L} (L - x) dN(c - 1, m, x),$$

where the integral is in the sense of Stieltjes, and $N(c, l, L)$ is the number of $l$-component multicurves of length bounded above by $L$ on a sphere with $c$ boundary components beginning and ending on a fixed component.

Remark 2.8. $f(x) \asymp x^k, k \geq 0$, then

$$\int_{0}^{L} (L - x) df(x) = \int_{0}^{L} f(x) dx \asymp x^{k+1},$$

integrating by parts.

Proof of inequality (7). Note that each term in the sum comes from the intersection of $\gamma$ with $\Gamma_{A_1 \ldots A_{c-1} F}$ having $m$ components. In each case, if that intersection has length $x$, we can twist the length of that intersection is $x$, and we can twist a number of times around $F$ to bring the length up to $L$. This number is proportional to $L - x$, (up to a constant, of order of length of $F$).

For example, for $c = 3$,

$$N(3, l, L) = \begin{cases} 0 & L < L_0, \\ 1 & \text{otherwise}. \end{cases}$$

For $c = 4$,

$$N(4, l, L) \geq cL,$$
by virtue of Remark 2.8. An easy inductive argument shows that
\[ N(c, l, L) \geq c^{l-3}L^{c-3}. \]

To complete the proof of Theorem 2.5, use Lemma 2.6, and essentially repeat the counting argument in the proof of that Lemma verbatim.

3. SURFACES OF GENUS ONE

To extend our estimates to surfaces of genus 1, we follow the same basic strategy: given a surface of genus 1 with \( c \) boundary components (we will denote such a surface by \( T_{A_1\ldots A_c} \), we cut along a nonseparating simple curve \( E \) to obtain a surface of genus 0 with \( c + 2 \) boundary components: \( \Gamma_{E_1 A_1 \ldots A_c E_2} \). Given a simple curve \( \gamma \) on \( T_{A_1\ldots A_c} \), \( \gamma \cap \partial \Gamma_{E_1 A_1 \ldots A_c E_2} \) is a collection of segments, each of whose endpoints lies either on \( E_1 \) or \( E_2 \). Conversely, given such a collection of segments, of total length \( x \), we know that by fractional twisting we can produce several curves \( \gamma_1, \ldots, \gamma_N \), by identifying the boundary components \( E_1 \) and \( E_2 \) with a fractional twist. The number \( N \) of such curves is of order \( L - x \), just as in the proof of Theorem 2.2. We now need to count collections of segments as above, having \( k \) intersections with \( E_1 \) and \( l \) intersections with \( E_2 \). In fact, for the purposes of induction we will count collections of segments having \( k \) intersections with \( E_1 \) and \( l \) intersections with \( E_2 \). For \( c = 1 \), the integers \( k \) and \( l \) determine the curve completely, so
\[ N(3, k, L) = \begin{cases} 0 & \text{if } L < L_0, \\ 1 & \text{otherwise.} \end{cases} \]

For bigger \( c \), we cut the \( \Gamma_{E_1 A_1 \ldots A_c E_2} \) into two pieces: \( \Gamma_{E_1 A_1 F} \) and \( \Gamma_{F A_2 \ldots A_c E_2} \). We now classify the possibilities with respect to the number \( m \) of intersections with the separating curve \( F \), and to simplify matters, we will assume that \( m \leq l \) – this will give a sufficiently good lower bound. This restricts the combinatorial possibilities of the intersection of \( \gamma \) with \( \Gamma_{E_1 A_1 F} \) to one: see figure 7. The same argument as in the proof of Theorem 2.5 gives
\[ N(c, l, k, L) \geq \sum_{m \leq l} N(c - 1, m, k, x)(L - x), \]

which gives the following estimate by the same argument as in the proof of Theorem 2.5.

**Theorem 3.1.** The number of simple geodesics not longer than \( L \) on a surface of genus 1 with \( c \) punctures is of order \( L^{2c} \).
The main difference between the lower genus situation covered in the last two sections and the higher genus case considered now is in the analysis of which fractional twists give connected curves. In the low genus case, the cyclic order of the intersections of \( k \) segments with a closed loop is always the same: \( 1, 2, \ldots, k, k', \ldots, 2', 1' \), which simplifies the analysis considerably. By contrast, in higher genus, many more permutations are possible, and it is not \textit{a priori} obvious how to deal with them. It turns out that we can avoid dealing with the problem entirely: we only count those curves which behave in a planar fashion, and these suffice for the lower bound that we seek. It is at first surprising that we will not have thrown out the baby with the bathwater, but there is a simple heuristic explanation: since any collection of simple, pair-wise non-intersecting, and pair-wise non-isotopic curves contains at most \( 3g - 3 \) elements, any multicurve falls naturally into (at most) \( 3g - 3 \) subsets, the curves in which are pair-wise parallel, thus the permutation group action is closer to that of \( S_g \) than of \( S_k \) (for a \( k \) component multicurve), and thus, if we assume that every permutation is equally likely, we only lose a constant factor (in fixed genus \( g \)). The argument in the rest of this section is a direct counting argument, which appears to bear out this heuristic reasoning (which seems difficult to push through directly).

The actual argument proceeds, as before, by cutting up the surface into simpler pieces.

Consider a closed surface of genus \( g \) (the case of arbitrary signature will follow by combining the analysis below with the analysis in Section 2), and cut it along a curve \( E \) to get two pieces: one, \( T_E \), a torus with one boundary component, the other \( T_E^{g-1} \) a surface of genus \( g - 1 \) with one boundary component. The intersection of the simple curve \( \gamma \) with \( E \) will be a collection of \( k \) segments having all of their endpoints on \( E \).
The piece more amenable to analysis is $T_E$. We analyse it by cutting it further into a thrice-punctured sphere $\Gamma_{F_1EF_2}$. There are two combinatorial possibilities for a collection of $k$ mutually non-intersecting segments on this surface: one is shown in figure 9, the other in figure 8.

We forbid the configuration shown in figure 9, since this has the wrong cycle type vis-à-vis $E$.

We have the constraints that $l_1 + l_3 = k$, while $l_1 + l_2 = l_3 + l_2$, implying that $l_1 = l_3$.

We are allowed to glue $F_1$ to $F_2$ with a twist, with the proviso that we get no closed connected components. This seems like a different sort of problem from the one we encountered in Sections 2 and 3, but luckily there is a simple trick which allows us to reduce it to that case. To wit, we glue in a disk with boundary $E$, and use it to connect each endpoint $i$ to its counterpart $i'$. If, after further identifying $F_1$ to $F_2$ with a twist, the resulting curve is connected, obviously no circle components were created. However, this new problem is exactly the one analysed in Section 3. In particular, this tells
us that a positive proportion of the fractional twists are allowed. Now, the total length of a collection as a function of $l_1$, $l_3$, and the number $\tau$ of twists is bounded above by $c_1 k + c_2 l_3 + c_3 \tau$, and thus the total number of systems of length bounded above by $L$ is at least $c(L - c_1 k)$.

To analyze the other piece, $T^g_E$, we use an inductive decomposition, much as before. The case of $g = 1$ was done above. If $g > 1$, we cut the surface into a torus $T_{EF}$ with two boundary components, and a $T^g_{E^{-1}}$. We are now reduced to analysing the torus $T_{EF}$. We want information about the collections of segments which intersect $E$ $k$ times, intersect $F$ $l$ times, and have a total of $k + l$ connected components. In order to do this we (yet again) cut $T_{EF}$ into two thrice-punctured spheres $\Gamma_{EC_1C_2}$ and $\Gamma_{FC_1C_2}$. Let the number of intersections with $C_1$ be $m$, while the number of intersections with $C_2$ be $n$. By our requirement on the permutation type of intersection with $E$, the intersection of our system with $\Gamma_{EC_1C_2}$ must look like Figure 10, and similarly for $\Gamma_{FC_1C_2}$ (Figure 11). We know, furthermore, that $k_2 + k_3 = k$, $k_1 + k_2 = m$, $k_1 + k_3 = n$, which implies that

$$k_1 = \frac{m + n - k}{2}, \quad k_2 = \frac{k + m - n}{2}, \quad k_3 = \frac{k + n - m}{2}.$$  

We now have to worry about two problems. One is that certain “horizontal” strands (those between $C_1$ and $C_2$) might close up into loops. The second is that certain strands might enter and leave through the boundary component $F$. Either way, we would get a non-connected multicurve. We deal with both of these by, first, imposing two additional inequalities. $2k_1 \leq l_1$, and $2l_1 \leq \min(m, n)$. We then allow only those twists which connect one of the $k_1$ strands to one of the $l_1$ strands (there are at least $l_1 - k_1$ such), and in addition imposing the condition that the horizontal loops do not close up – this is not extremely restrictive, since there we can twist by at least $k_1/2$ around $C_1$, and also by $k_1/2$ around $C_2$, (see figures 10 and for notation).

It is not hard to see that, subject to these restrictions, any strand coming into $C_1$ from below will either leave straightaway upwards, or will cycle.
around between $C_1$ and $C_2$ for a while before leaving in that same manner. Likewise for strands coming into $C_2$ from below. The same sort of counting as before will show that the set of permissible multicurves of length bounded above by $x$ is at least $cx^5$, since these corresponds to points in a cone in 5 dimensions (corresponding to twisting around $c_1$, twisting around $c_2$, and the parameters $l, m, n$). Our constraints are all inequality constraints, and thus will cut out a non-degenerate cone. The rest of the inductive argument is as in Sections 2 and 3, and so for compact surfaces we obtain the claimed result

**Theorem 4.1.** The number of simple geodesics of length bounded by $L$ on a compact surface of genus $g$ grows like $L^{6g-6}$

The estimate for arbitrary signature follows as indicated in the beginning of this section.

5. Conclusions and musings

The reader will have noted that the estimates on the density of connected multicurves among all multicurves become worse and worse as the genus of our surfaces increases. This is, to an extent, a reflection of reality (in fact, it is easy to see that this density decreases exponentially as a function of $g+c$. However, it is clear that the estimates one might obtain by our methods are far from optimal.

Another observation one might make is that the methods of this paper are obviously insufficient for deriving asymptotic results (extending those for the punctured torus as in [1], [2]). While there is some possibility that the method used in the low genus case might be pushed to get results of this type, for general genus this seems hopeless. However, the very existence of an asymptotic formula is in some doubt. A problem which might be tractable by the current methods is one of finding order of growth results for curves with a bounded number of self-intersections (the estimates of Birman and Series are acknowledged by the authors not to be sharp).
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REFERENCES

1. A. Beardon, J. Lehner, M. Sheingorn. Closed geodesics on a Riemann surface with applications to the Markov spectrum, Transactions of the American Mathematical Society, 295 (1986), no. 2, pp. 635–647.
2. Joan Birman and Caroline Series. An algorithm for simple curves on surfaces, Journal of the London Mathematical Society (2), 29 (1984), no. 2, 331–342.
3. Joan Birman and Caroline Series. Geodesics with bounded intersection number on surfaces are sparsely distributed, Topology, 24 (1985), no. 2, pp. 217–225.
4. Joan Birman and Caroline Series. Dehn’s algorithm revisited, with applications to simple curves on surfaces, Combinatorial group theory and topology (Alta, Utah, 1984), pp. 451–478, Annals of Mathematics Studies 111, Princeton University Press, Princeton, NJ, 1987.
5. David Chillingworth. Winding numbers on surfaces. I, Math. Ann. 196 (1972), pp. 218–249.
6. David Chillingworth. Winding numbers on surfaces. II, Math. Ann. 199 (1972), pp. 131–153.
7. Marshall Cohen and Martin Lustig. Paths of geodesics and geometric intersection numbers. I, Combinatorial group theory and topology (Alta, Utah, 1984), pp. 479–500, Annals of Mathematics Studies 111, Princeton University Press, Princeton, NJ, 1987.
8. Albert Fathi, François Laudenbach, Valentin Poénaru. Travaux de Thurston sur les surfaces, With an English Astrisque, 66-67. Société Mathématique de France, Paris, 1979.
9. Andrew Haas and Perry Susskind. The connectivity of multicurves determined by integral weight train tracks, Transactions of the American Mathematical Society, 329 (1992), no. 2, pp. 637–652.
10. Martin Lustig. Paths of geodesics and geometric intersection numbers. II, Combinatorial group theory and topology (Alta, Utah, 1984), pp. 501–543, Annals of Mathematics Studies 111, Princeton University Press, Princeton, NJ, 1987.
11. Greg McShane and Igor Rivin. Simple curves on hyperbolic tori, C. R. Acad. Sci. Paris Sér. I. Math., 320, no. 12, June 1995.
12. Greg McShane and Igor Rivin. Geometry of geodesics and a norm on homology, International Mathematics Research Notices, February 1995.
13. Geoffrey Mess. Private communication.
14. R. Osborne and H. Zieschang. Primitives in the free group on two generators, Inventiones Mathematicae, 63 (1981), no. 1, pp. 17–24.
15. Mary Rees. An alternative approach to the ergodic theory of measured foliations on surfaces, Ergodic Theory Dynamical Systems, 1 (1981), no. 4, pp. 461–488.
16. Bruce Reinhart. The winding number on two manifolds, Ann. Inst. Fourier. Grenoble, 10 (1960), pp. 271–283.
17. Igor Rivin. Intrinsic geometry of convex ideal polyhedra in hyperbolic 3-space, Analysis, algebra, and computers in mathematical research (Luleå, 1992), Lecture Notes in Pure and Appl. Math., 156, Dekker, New York, 1994, pp. 275–291.
18. Caroline Series. Wolpert’s formula, Warwick University Preprint, 1996.
19. Scott Wolpert. On the symplectic geometry of deformations of a hyperbolic surface, Annals of Mathematics (2) 117 (1983), no. 2, pp. 207–234.
20. Heiner Zieschang. *Algorithmen für einfache Kurven auf Flächen*, *Math. Scand.* 17(1965), pp. 17–40.
21. Heiner Zieschang. *Algorithmen für einfache Kurven auf Flächen. II*, *Math. Scand.* 25(1969), pp. 49–58.

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