THE MODULI SPACES OF PARABOLIC CONNECTIONS WITH A QUADRATIC DIFFERENTIAL AND ISOMONODROMIC DEFORMATIONS

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Abstract. In this paper, we study the moduli spaces of parabolic connections with a quadratic differential. We endow these moduli spaces with symplectic structures by using the fundamental 2-forms on the moduli spaces of parabolic connections (which are phase spaces of isomonodromic deformation systems). Moreover, we see that the moduli spaces of parabolic connections with a quadratic differential are equipped with structures of twisted cotangent bundles.

1. Introduction

We recall the definitions of Lagrangian triples and Hamiltonian data, which are discussed in [5]. Let \( p: X \to S \) be a smooth morphism of smooth varieties. A \( p \)-connection is an \( \mathcal{O}_X \)-linear morphism \( \nabla_S: p^*\Theta_S \to \Theta_X \) such that \( dp \circ \nabla_S = \text{id}_{p^*\Theta_S} \). Here \( \Theta_S \) and \( \Theta_X \) are the tangent sheaves of \( S \) and \( X \), respectively. A \( p \)-connection \( \nabla_S \) is integrable if the corresponding map \( \Theta_S \to p_*\Theta_X \) commutes with brackets. Note that an integrable \( p \)-connection \( \nabla_S \) defines an action of \( \Theta_S \) on relative differential forms \( \Omega^2_{X/S} \) by the Lie derivatives along horizontal vector field \( \nabla_S(\Theta_S) \). A form \( \omega \in \Omega^2_{X/S} \) is \( \nabla_S \)-horizontal if \( \omega \) is fixed by the \( \Theta_S \)-action.

Definition 1.1. Let \( S \) be a smooth variety. An \( S \)-Lagrangian triple consists of a morphism \( \pi: X \to Y \) of \( S \)-varieties \( p_X: X \to S \) and \( p_Y: Y \to S \), a relative 2-form \( \omega \in \Omega^2_{X/S} \) and a \( p_X \)-connection \( \nabla_S \) such that

(i) \( p_X, p_Y \) and \( \pi \) are smooth surjective morphisms,
(ii) a form \( \omega \) is closed and non-degenerate,
(iii) for any \( s \in S \) the morphism \( \pi_s: X_s \to Y_s \) is a twisted cotangent bundle over \( Y_s \), and
(iv) \( \nabla_S \) is integrable and \( \omega \) is \( \nabla_S \)-horizontal.

Definition 1.2. An \( S \)-Hamiltonian datum on an \( S \)-variety \( p_Y: Y \to S \) consists of

(i) a twisted cotangent bundle \( (\tilde{X}, \omega_{\tilde{X}}, \tilde{\pi}): \tilde{X} \to Y \) over \( Y \). Put \( X := \tilde{X} \mod p_Y^*\Omega^1_Y \); this is a \( \Theta^*_{Y/S} \)-torsor over \( Y \); let \( \tilde{\pi}: \tilde{X} \to Y \) be the projections, and
(ii) a section \( h: X \to \tilde{X} \) of \( \tilde{\pi} \) (called Hamiltonian).

Put \( \omega_X := h^*\omega_{\tilde{X}} \), which is a closed 2-form on \( X \). We assume the following integrability axiom holds: For each \( x \in X \) the form \( \omega_x \in \bigwedge^2 \Theta^*_{X,x} \) has rank \( \text{dim} X - \text{dim} Y \).

One has \( S \)-Lagrangian triples from \( S \)-Hamiltonian data \( (\tilde{X}, \omega_{\tilde{X}}, \tilde{\pi}, h) \). In fact, \( \pi: X \to Y \) is as in Definition 1.2. Let \( \omega \) be the relative of \( \omega_X \) and put \( p_X := p_Y \circ \pi \). By the kernels of \( \omega_X: \Theta_X \to \Omega^1_X \), we can determine a \( p_X \)-connection \( \nabla_S \). Then \( (\pi: X \to Y, \omega, \nabla_S) \) is an \( S \)-Lagrangian triple. This correspondence from \( S \)-Hamiltonian data to \( S \)-Lagrangian triples is bijective (see [5]). Moreover, let \( \tilde{X} \) be the twisted cotangent bundle over \( Y \) corresponding to an \( S \)-Lagrangian triple \( (\pi: X \to Y, \omega, \nabla_S) \) and let \( \omega_X \) be the 2-form on \( X \). In [5], it is remarked that the twisted cotangent bundle \( \tilde{X} \) over \( Y \) is isomorphic to the fiber product \( X \times_S T^*S \) as symplectic manifolds. This isomorphism is given by the morphism \( \tilde{r}: \tilde{X} \to X \times_S T^*S, \tilde{r}(\tilde{x}) = (r(\tilde{x}), \tilde{x} - h(r(\tilde{x}))) \). Here the symplectic form on \( X \times_S T^*S \) is equal to the sum of \( \omega_X \) and a standard symplectic form on \( T^*S \). The purpose of this paper is to construct
S-Hamiltonian data \((\tilde{X}, \omega, \pi, h)\) from S-Lagrangian triples \((\pi: X \to Y, \omega_X, \nabla_S)\) by using more concrete argument in the case of isomonodromic deformations.

In our case, \(X\) is a moduli space of pointed smooth projective curves and parabolic connections (see [8] Theorem 2.1 and [9]), \(Y\) is a moduli space of pointed smooth projective curves and quasi-parabolic bundles admitting a parabolic connection, and \(S\) is a moduli space of pointed smooth projective curves. We have projections \(p_X: X \to S, p_Y: Y \to S\) and \(\pi: X \to Y\). The moduli space \(X\) has the relative symplectic form \(\omega\) over \(S\) (see [8] Section 7). The \(p_X\)-connection \(\nabla_S\) is given by the isomonodromic deformations of parabolic connections (see [8] Proposition 8.1). The main result of this paper is to construct the corresponding twisted cotangent bundle \(\tilde{X}\) over \(Y\) by using computation of \(\check{\text{C}}\)ech cohomologies. We construct the twisted cotangent bundle with the remark in mind: The twisted cotangent bundle \(\tilde{X}\) over \(Y\) is isomorphic to the fiber product \(X \times_S T^*S\). Our argument is as follows. First, we consider the fiber product \(X \times_S T^*S\) (which called extended phase space, see [7] Section 7)). The fiber product \(X \times_S T^*S\) is the moduli space of (pointed smooth projective curves and) parabolic connections with a quadratic differential. We describe the tangent sheaf of \(X \times_S T^*S\) and the symplectic form on \(X \times_S T^*S\) by the \(\check{\text{C}}\)ech cohomology (Proposition 3.1 and Proposition 3.6). Second, we describe the cotangent sheaf \(\Omega^1_S\) by the \(\check{\text{C}}\)ech cohomology, and we define an \(\Omega^1_{Y}\)-action on \(X \times_S T^*S\) explicitly (Definition 4.3). We show that by this \(\Omega^1_{Y}\)-action and the symplectic form, \(X \times_S T^*S\) is a twisted cotangent bundle over \(Y\) (Theorem 4.4). The section \(X \to X \times_S T^*S\) given by the zero section of \(T^*S \to S\) is the Hamiltonian of the Hamiltonian datum.

A twisted cotangent bundle over \(Y\) is important for studying quantizations of isomonodromic deformations. In fact, quantizations of isomonodromic deformations may be described by using certain algebras of twisted differential operators, which are quantizations of twisted cotangent bundles (see [8] and [4]). It is expected that the results of this paper are useful to understand quantizations of isomonodromic deformations in the context of a certain algebro-geometric way such as [8] and [9].

The organization of this paper is as follows. In Section 2, we recall basic definitions and basic facts on parabolic connections (in 2.1), Atiyah algebras (in 2.2) and twisted cotangent bundles (in 2.3). In Section 3, we treat moduli spaces of parabolic connections with a quadratic differential. First, we describe the tangent sheaves of these moduli spaces in terms of the hypercohomology of a certain complex. Second, we endow the moduli spaces with symplectic structures. In Section 4, we see that the moduli spaces of parabolic connections with a quadratic differential are equipped with structures of twisted cotangent bundles.

2. Preliminaries

2.1. Moduli space of stable parabolic connections. Following [8], we recall basic definitions and basic facts on parabolic connections. Let \(C\) be a smooth projective curve of genus \(g\). We put

\[ T_n := \{(t_1, \ldots, t_n) \in C \times \cdots \times C \mid t_i \neq t_j \text{ for } i \neq j\} \]

for a positive integer \(n\). For integers \(d, r\) with \(r > 0\), we put

\[ N^{(n)}_r(d) := \left\{ (\nu_j^{(i)})_{0 \leq j \leq r-1} \in \mathbb{C}^{nr} \mid d + \sum_{i,j} \nu_j^{(i)} = 0 \right\}. \]

Take members \(t = (t_1, \ldots, t_n) \in T_n\) and \(\nu = (\nu_j^{(i)})_{1 \leq i \leq n, 0 \leq j \leq r-1} \in N^{(n)}_r(d)\).

**Definition 2.1.** We say \((E, \nabla, \{\nu^{(i)}_j\}_{1 \leq i \leq n})\) is a \((t, \nu)\)-parabolic connection of rank \(r\) and degree \(d\) over \(C\) if

1. \(E\) is a rank \(r\) algebraic vector bundle on \(C\),
2. \(\nabla: E \to E \otimes \Omega^1_C(t_1 + \cdots + t_n)\) is a connection, that is, \(\nabla\) is a homomorphism of sheaves satisfying \(\nabla(fa) = a \otimes df + f \nabla(a)\) for \(f \in \mathcal{O}_C\) and \(a \in E\), and
Remark 2.2. We have
\[
\deg E = \deg(\text{det}(E)) = -\sum_{i=1}^{n} \text{res}_{t_i}(\nabla_{\text{det}(E)}) = -\sum_{i=1}^{n} \sum_{j=0}^{r-1} \nu_{j}^{(i)} = d.
\]

Definition 2.3 (\cite{Definition 2.3}). Take an element \( \nu \in N_r^{(n)}(d) \). We call \( \nu \) special if

1. \( \nu_{j}^{(i)} - \nu_{k}^{(i)} \in \mathbb{Z} \) for some \( i \) and \( j \neq k \), or
2. there exists an integer \( s \) with \( 1 < s < r \) and a subset \( \{ j_1, \ldots, j_s \} \subset \{ 0, \ldots, r-1 \} \) for each \( 1 \leq i \leq n \) such that \( \sum_{i=1}^{n} \sum_{k=1}^{s} \nu_{j_k}^{(i)} \in \mathbb{Z} \).

We call \( \nu \) generic if it is not special.

Take rational numbers \( 0 < \alpha_{1}^{(i)} < \alpha_{2}^{(i)} < \cdots < \alpha_{i}^{(i)} < 1 \) for \( i = 1, \ldots, n \) satisfying \( \alpha_{j}^{(i)} \neq \alpha_{j'}^{(i')} \) for \( (i, j) \neq (i', j') \). We choose a sufficiently generic \( \alpha = (\alpha_{j}^{(i)}) \).

Definition 2.4. A parabolic connection \( (E, \nabla, \{ l_{j}^{(i)} \}_{1 \leq i \leq n}) \) is \( \alpha \)-stable (resp. \( \alpha \)-semistable) if for any proper nonzero subbundle \( F \subset E \) satisfying \( \nabla(F) \subset F \otimes \Omega_{\mathcal{C}/S}^{1}(t_1 + \cdots + t_n) \), the inequality
\[
\frac{\text{deg } F + \sum_{i=1}^{n} \sum_{j=1}^{r} \alpha_{j}^{(i)} \dim((F|_{t_i} \cap l_{j-1}^{(i)})/(F|_{t_i} \cap l_{j}^{(i)}))}{\text{rank } F} < \frac{\text{deg } E + \sum_{i=1}^{n} \sum_{j=1}^{r} \alpha_{j}^{(i)} \dim(l_{j-1}^{(i)}/l_{j}^{(i)})}{\text{rank } E}
\]
holds.

Let \( \tilde{M}_{g,n} \) be a smooth algebraic scheme which is a certain covering of the moduli stack of \( n \)-pointed smooth projective curves of genus \( g \) over \( \mathbb{C} \) and take a universal family \( (\mathcal{C}, t_1, \ldots, t_n) \) over \( \tilde{M}_{g,n} \).

Definition 2.5. We denote the pull-back of \( \mathcal{C} \) and \( \tilde{t} \) by the morphism \( \tilde{M}_{g,n} \times N_{r}^{(n)}(d) \to \tilde{M}_{g,n} \) by the same character \( \mathcal{C} \) and \( \tilde{t} = \{ \tilde{t}_1, \ldots, \tilde{t}_n \} \). Then \( D(\tilde{t}) := \tilde{t}_1 + \cdots + \tilde{t}_n \) becomes an effective Cartier divisor on \( \mathcal{C} \) flat over \( \tilde{M}_{g,n} \times N_{r}^{(n)}(d) \). We also denote by \( \tilde{\nu} \) the pull-back of the universal family on \( N_{r}^{(n)}(d) \) by the morphism \( \tilde{M}_{g,n} \times N_{r}^{(n)}(d) \to N_{r}^{(n)}(d) \). We define a functor \( \mathcal{M}_{\mathcal{C}/\tilde{M}_{g,n}}^{\alpha}(\tilde{t}, r, d) \) from the category of locally noetherian schemes over \( \tilde{M}_{g,n} \times N_{r}^{(n)}(d) \) to the category of sets by
\[
\mathcal{M}_{\mathcal{C}/\tilde{M}_{g,n}}^{\alpha}(\tilde{t}, r, d)(S) := \left\{ (E, \nabla, \{ l_{j}^{(i)} \}) \right\} / \sim
\]
for a locally noetherian scheme \( S \) over \( \tilde{M}_{g,n} \times N_{r}^{(n)}(d) \), where

1. \( E \) is a rank \( r \) algebraic vector bundle on \( \mathcal{C}_S \),
2. \( \nabla: E \to E \otimes \Omega_{\mathcal{C}/S}^{1}(D(\tilde{t})_S) \) is a relative connection,
3. for each \( (\tilde{t}_i)_S, l_{j}^{(i)} \) is a filtration by subbundles \( E|_{(\tilde{t}_i)_S} = l_{j}^{(i)} \subset l_{j+1}^{(i)} \subset \cdots \subset l_{r}^{(i)} = 0 \) such that \( (\text{res}_{(\tilde{t}_i)_S}(\nabla - (\tilde{\nu}_{j}^{(i)})_{S, t_i})|_{l_{j}^{(i)}})(l_{j}^{(i)}) \subset l_{j+1}^{(i)} \) for \( j = 0, \ldots, r-1 \), and
4. for any geometric point \( s \in S \), \( \dim(l_{j}^{(i)}/l_{j+1}^{(i)}) \otimes k(s) = 1 \) for any \( i, j \) and \( (E, \nabla, \{ l_{j}^{(i)} \}) \otimes k(s) \) is \( \alpha \)-stable.

Here \( (E, \nabla, \{ l_{j}^{(i)} \}) \sim (E', \nabla', \{ l_{j}^{(i)} \}) \) if there exist a line bundle \( L \) on \( S \) and an isomorphism \( \sigma: E \simto E' \otimes L \) such that \( \sigma|_{t_i}(l_{j}^{(i)}) = l_{j}^{(i)} \otimes L \) for any \( i, j \) and the diagram
\[
\begin{array}{ccc}
E & \xrightarrow{\nabla} & E \otimes \Omega_{\mathcal{C}/T}^{1}(D(\tilde{t})) \\
\sigma & & \text{res} \otimes \text{id} \\
E' \otimes L & \xrightarrow{\nabla' \otimes \text{id}} & E' \otimes \Omega_{\mathcal{C}/T}^{1}(D(\tilde{t})) \otimes L
\end{array}
\]
Definition 2.7. We define the Atiyah algebra of $E$ as
\[ A_E = \{ \partial \in D_1 \mid \text{symb}_1(\partial) \in \text{id}_E \otimes \Theta_C \subset \mathcal{E}nd(E) \otimes \Theta_C \}. \]
Here, for $v \in D_1$, $\text{symb}_1(v)$ is the symbol of the differential operator $v$.

We have inclusions $D_0 = \mathcal{E}nd(E) \subset A_E \subset D_1$ and the short exact sequence
\[ 0 \longrightarrow \mathcal{E}nd(E) \longrightarrow A_E \xrightarrow{\text{symb}_1} \Theta_C \longrightarrow 0. \]
Fix a positive integer $n$. Let $D = t_1 + \cdots + t_n$ be an effective divisor of $C$ where $t_1, \ldots, t_n$ are distinct points of $C$. We put $A_{E}(D) := \text{symb}_1^{-1}(\Theta_C(-D))$. Then we have the following exact sequence
\[ 0 \longrightarrow \mathcal{E}nd(E) \longrightarrow A_{E}(D) \xrightarrow{\text{symb}_1} \Theta_C(-D) \longrightarrow 0. \]

For a connection $\nabla: E \to E \otimes \Omega^1_U(D)$, we define a splitting
\[ \iota(\nabla): \Theta_C(-D) \longrightarrow A_{E}(D) \]
as follows. Let $U$ be an affine open subset of $C$ where we have a trivialization $E|_U \cong \mathcal{O}_U^r$. We denote by $Af^{-1}df$ a connection matrix of $\nabla$ on $U$ where $f$ is a local defining equation of $t_i$ and $A \in M_r(\mathcal{O}_U)$. For an element $g\frac{\partial}{\partial f} \in \Theta_C(-D)(U)$, we define the element $\iota(\nabla)(g\frac{\partial}{\partial f}) := g \left( \frac{\partial}{\partial f} + Af^{-1} \right) \in A_{E}(D)(U)$, which gives a map $\iota(\nabla)(U): \Theta_C(-D)(U) \to A_{E}(D)(U)$. By this map, we obtain the splitting [3].

2.3. Twisted cotangent bundles. Following [4] Section 2], we recall the definition of twisted cotangent bundles and recall the correspondence between twisted cotangent bundles and $\Omega^1_X$-torsors. Let $X$ be a smooth algebraic variety over $\mathbb{C}$.

Definition 2.8. Let $T^* = T^*(X) \to X$ be the cotangent bundle on $X$. A twisted cotangent bundle on $X$ is a $T^*$-torsor $\pi_\phi: \phi \to X$ (i.e., $\pi_\phi$ is a fibration equipped with a simple transitive action of $T^*$ along the fibers) together with a symplectic form $\omega_\phi$ on $\phi$ such that $\pi_\phi$ is a polarization for $\omega_\phi$ (i.e., $\dim \phi = 2 \dim X$ and the Poisson bracket \{\cdot, \cdot\} vanished on $\pi_\phi^{-1}O_X$) and for any 1-form $\nu$ one has $t^*_\nu(\omega_\phi) = \pi_\phi^*d\nu + \omega$. Here $t_\nu: \phi \to \phi; t_\nu(a) = a + \nu_\pi(a)$ is the translation by $\nu$.

Definition 2.9. Let $d: A^n \to A^{n+1}$ be a morphism of sheaves of abelian groups on $X$, considered as length 2 complex $A^* \subset \text{supp} \text{d}$ supported in degree $n$ and $n + 1$. An $A^*$-torsor is a pair $(F, c)$, where $F$ is an $A^n$-torsor and $c: F \to A^{n+1}$ is a map such that $c(a + \phi) = d(c(a)) + c(\phi)$ for $a \in A^n, \phi \in F$.

Let $\Omega^2_X := (\Omega^2_X \to \Omega^2_{X})$ be the truncated de Rham complex, where $\Omega^2_{X}$ are closed 2-forms on $X$. We recall the correspondence between twisted cotangent bundles and $\Omega^2_X$-torsors. For a twisted cotangent bundle $\phi$, let $\Gamma(\phi)$ be the $\Omega^2_X$-torsor of a section of $\phi$. We define a map $c: \Gamma(\phi) \to \Omega^2_{X}$ by $c(\gamma) := \gamma^*(\omega_\phi)$. Conversely, for an $\Omega^2_X$-torsor $(F, c)$, let $\pi_\phi: \phi \to X$ be the space of the torsor $F$. The symplectic form is defined as the unique form such that for a section $\gamma \in F$ of $\pi_\phi$ the corresponding isomorphism $T^*X \sim \phi; 0 \to \gamma$, identifies $\omega_\phi$ with $\omega + \pi^*c(\gamma)$. Here $\omega$ is the canonical symplectic form on the cotangent bundle $T^*X$.\[ \phantom{2} \]
3. Moduli scheme of parabolic connections with a quadratic differential

In this section, we treat a moduli space of parabolic connections with a quadratic differential. In 3.2 we describe the (algebraic) tangent sheaf of this moduli space in terms of the hypercohomology of a certain complex. Moreover, we describe the analytic tangent sheaf in terms of the hypercohomology of a certain analytic complex. This description is more simple than the algebraic one. In 3.3 we recall a description of the vector fields associated to the isomonodromic deformations in terms of the description of the (algebraic) tangent sheaf. In 3.4 we show that the moduli space of parabolic connections with a quadratic differential is endowed with a symplectic structures. The classical trick of turning a time dependent Hamiltonian flow into an autonomous one by adding variables is well-known. In this trick, the space given by adding the variables to a phase space is called an extended phase space. (Hamiltonians of isomonodromic deformations are time dependent.)

3.1. Moduli space of stable parabolic connections with a quadratic differential. Let $T^*\tilde{M}_{g,n}$ be the total space of the cotangent bundle of $\tilde{M}_{g,n}$. We denote by $\tilde{M}_{C/\tilde{M}_{g,n}}(\tilde{t},r,d)$ the fiber product of $T^*\tilde{M}_{g,n} \times N_{r}^{(n)}(d)$ and $M_{C/\tilde{M}_{g,n}}^{\alpha}(\tilde{t},r,d)$ over $\tilde{M}_{g,n} \times N_{r}^{(n)}(d)$:

\[
\begin{array}{ccc}
\tilde{M}_{C/\tilde{M}_{g,n}}(\tilde{t},r,d) & \longrightarrow & M_{C/\tilde{M}_{g,n}}^{\alpha}(\tilde{t},r,d) \\
\downarrow & & \downarrow \\
T^*\tilde{M}_{g,n} \times N_{r}^{(n)}(d) & \longrightarrow & \tilde{M}_{g,n} \times N_{r}^{(n)}(d).
\end{array}
\]

We call the fiber product $\tilde{M}_{C/\tilde{M}_{g,n}}(\tilde{t},r,d)$ the moduli space of $\alpha$-stable parabolic connections with a quadratic differential. If we take a zero section of $T^*\tilde{M}_{g,n}$, then we have an inclusion

\[
M_{C/\tilde{M}_{g,n}}^{\alpha}(\tilde{t},r,d) \longrightarrow \tilde{M}_{C/\tilde{M}_{g,n}}(\tilde{t},r,d).
\]

Let $(C,\textbf{t}) \in \tilde{M}_{g,n}$. The tangent space of $\tilde{M}_{g,n}$ at $(C,\textbf{t})$ is isomorphic to $H^{1}(C, \Theta_{C}(-D(\textbf{t})))$. By the Serre duality, the cotangent space at $(C,\textbf{t})$ is isomorphic to $H^{0}(C, \Omega_{C}^{2}(D(\textbf{t})))$, which is the space of (global) quadratic differentials on $(C,\textbf{t})$.

3.2. Infinitesimal deformations. For simplicity, we put $\tilde{M} := \tilde{M}_{C/\tilde{M}_{g,n}}^{\alpha}(\tilde{t},r,d)$ and $C_{\tilde{M}} := C \times _{\tilde{M}_{g,n}} \tilde{M}$. Let $(\tilde{E}, \tilde{\nabla}, \{ \tilde{l}_{i,j}^{(i)} \}, \tilde{\psi})$ be a universal family on $C_{\tilde{M}}$. Let $A_{\tilde{E}}(D(\tilde{\textbf{t}}))$ be the relative Atiyah algebra which is the extension

\[
0 \longrightarrow \mathcal{E}nd(\tilde{E}) \longrightarrow A_{\tilde{E}}(D(\tilde{\textbf{t}})) \xrightarrow{\text{symb}} \Theta_{C_{\tilde{M}}/\tilde{M}}(-D(\tilde{\textbf{t}})) \longrightarrow 0,
\]

where $\iota(\tilde{\nabla}) : \Theta_{C_{\tilde{M}}/\tilde{M}}(-D(\tilde{\textbf{t}})) \to A_{\tilde{E}}(D(\tilde{\textbf{t}}))$ is the $\mathcal{O}_{X}$-linear section of $\text{symb}$ associated to the relative connection $\tilde{\nabla}$. We put

\[
\begin{align*}
\tilde{F}^{0} & := \left\{ s \in \mathcal{E}nd(\tilde{E}) \bigg| s|_{l_{i,j} \times \tilde{M}}(\tilde{l}_{i,j}^{(i)}) \subset \tilde{l}_{j}^{(i)} \text{ for any } i,j \right\} \quad \text{and} \\
\mathcal{F}^{0} & := \left\{ s \in A_{\tilde{E}}(D(\tilde{\textbf{t}})) \bigg| (s - \iota(\tilde{\nabla}) \circ \text{symb}(s))|_{l_{i,j} \times \tilde{M}}(\tilde{l}_{i,j}^{(i)}) \subset \tilde{l}_{j}^{(i)} \text{ for any } i,j \right\}.
\end{align*}
\]

Then we have an extension

\[
0 \longrightarrow \tilde{F}^{0} \longrightarrow \mathcal{F}^{0} \xrightarrow{\text{symb}} \Theta_{C_{\tilde{M}}/\tilde{M}}(-D(\tilde{\textbf{t}})) \longrightarrow 0.
\]
We put
\[
\tilde{F}^1 := \{ s \in \mathcal{E}nd(\tilde{E}) \otimes \Omega^1_{\tilde{C}/\tilde{M}}(D(\tilde{t})) \mid \text{res}_{i,M^0 \otimes (t,r,d)}(s)(\tilde{t}_j^{(i)}) \subset \tilde{t}_{j+1}^{(i)} \text{ for any } i,j \} \quad \text{and}
\]
\[
F^1 := \tilde{F}^1 + \Omega^2_{\tilde{C}/\tilde{M}}(D(\tilde{t})).
\]
We define a homomorphism \( \tilde{d}_\phi : \tilde{F}^0 \rightarrow \tilde{F}^1 \) as \( s \mapsto \tilde{\nabla} \circ s - s \circ \tilde{\nabla} \) and we define a homomorphism \( d_{\tilde{\phi}} : \Theta_{\tilde{C}/\tilde{M}}(-D(\tilde{t})) \rightarrow \Omega_{\tilde{C}/\tilde{M}}^2(D(\tilde{t})) \) as follows. Take an affine open covering \( \{ U_\alpha \} \) of \( \tilde{M} \) such that we can take trivializations of \( \Theta_{\tilde{C}/\tilde{M}}(-D(\tilde{t})) \) and \( \Omega_{\tilde{C}/\tilde{M}}^2(D(\tilde{t})) \) on each \( U_\alpha \). For an element \( a\partial / \partial f_\alpha \in \Theta_{\tilde{C}/\tilde{M}}(-D(\tilde{t}))(U_\alpha) \), we define a homomorphism on \( U_\alpha \) by
\[
(7) \quad a \frac{\partial}{\partial f_\alpha} \mapsto \left( \frac{\partial\psi_{U_\alpha}}{\partial f_\alpha} a + 2\psi_{U_\alpha} \frac{\partial a}{\partial f_\alpha} \right) d_\alpha \otimes df_\alpha \in \Omega_{\tilde{C}/\tilde{M}}^2(D(\tilde{t}))(U_\alpha),
\]
where \( \tilde{\psi}_{U_\alpha} = \psi_{U_\alpha} df_\alpha \otimes df_\alpha \). By the homomorphism on each \( U_\alpha \), we can define a homomorphism \( d_{\tilde{\phi}} \). We define a complex \( \mathcal{F}^* \) by the differential \( d_{\mathcal{F}^*} = (d_{\tilde{\phi}},d_\phi) \circ (\text{Id} - \iota(\tilde{\nabla}) \circ \text{symb}_1,\text{symb}_1) \):

\[
\begin{array}{ccc}
\mathcal{F}^0 & & \mathcal{F}^1 \\
\text{(Id} - \iota(\tilde{\nabla})\text{symb}_1,\text{symb}_1) & \downarrow d_{\mathcal{F}^*} & \downarrow (d_{\tilde{\phi}},d_\phi) \text{circ} (\text{Id} - \iota(\tilde{\nabla}) \circ \text{symb}_1,\text{symb}_1) \\
\tilde{F}^0 + \Theta_{\tilde{C}/\tilde{M}}(-D(\tilde{t})) & \rightarrow & \tilde{F}^1 + \Omega_{\tilde{C}/\tilde{M}}^2(D(\tilde{t})).
\end{array}
\]

**Proposition 3.1.** We put \( \tilde{M} = \tilde{M}_{C,\tilde{M}_n}^{\alpha}(\tilde{t},r,d) \) and \( M = M_{C,\tilde{M}_n}^{\alpha}(\tilde{t},r,d) \). Let \( \mathcal{F}^0_{\tilde{M}}, \tilde{F}^0_{\tilde{M}}, \) and \( \tilde{F}^1_{\tilde{M}} \) be the pull-backs of \( \mathcal{F}^0, \tilde{F}^0, \) and \( \tilde{F}^1 \) by the natural immersion \( C_\tilde{M} \rightarrow \tilde{C}_\tilde{M} \), respectively. There exist canonical isomorphisms
\[
\zeta : \Theta_{M/N_r^{(i)}(d)} \tilde{\rightarrow} \mathbb{R}^1(\pi_{\tilde{M}})_* (\mathcal{F}^*),
\]
\[
(8) \quad \zeta : \Theta_{M/N_r^{(i)}(d)} \tilde{\rightarrow} \mathbb{R}^1(\pi_{\tilde{M}})_* (\mathcal{F}^0_{\tilde{M}} \rightarrow \tilde{F}^1_{\tilde{M}}),
\]
where \( \pi_{\tilde{M}} : C_\tilde{M} \rightarrow \tilde{M} \) and \( \pi_{\tilde{M}} : C_M \rightarrow M \) are the natural morphisms.

**Proof.** We show the existence of the isomorphism \( \zeta \). For the existence of the isomorphisms \( \zeta \) and \( \zeta \), see the proof of \cite[Proposition 3.2]{[6]} and the proof of \cite[Theorem 2.1]{[3]}. We take an affine open set \( \tilde{U} \subset \tilde{M} \). Let \( (\tilde{E},\tilde{\nabla},(\tilde{t}_j^{(i)})) \) be the family on \( C \times \tilde{M}_n \tilde{U} \). We take an affine open covering \( C_\tilde{U} = \bigcup \alpha U_\alpha \) such that \( \phi_\alpha : \tilde{E}|_{U_\alpha} \tilde{\rightarrow} \mathcal{O}_{C_\tilde{U}}^r_{\alpha} \) for any \( \alpha \), \( \mathcal{H} \{ i \mid t_i^{(i)}|_{U_\alpha} \cap U_\alpha \neq \emptyset \} \leq 1 \) for any \( \alpha \) and \( \{ \alpha \mid t_i^{(i)}|_{U_\alpha} \cap U_\alpha \neq \emptyset \} \leq 1 \) for any \( \alpha \). Take a relative tangent vector field \( v \in \Theta_{M/N_r^{(i)}(d)}(\tilde{U}) \). The field \( v \) corresponds to a member \( ((C_r,t_r,\psi_r),(E_v,\nabla_v,t_{\{v\}}^{(i)}) \in \tilde{M}(\text{Spec} \mathcal{O}_{C_\tilde{U}}[\epsilon]) \) such that \( ((C_r,t_r,\psi_r),(E_v,\nabla_v,t_{\{v\}}^{(i)})) \otimes \mathcal{O}_{C_\tilde{U}}[\epsilon]/(\epsilon) \cong ((C_r,t_r,\psi_r),(E_v,\nabla_v,t_{\{v\}}^{(i)})) \), where \( \mathcal{O}_{C_\tilde{U}}[\epsilon] = \mathcal{O}_{C_\tilde{U}}[\epsilon]/(t^2) \). Here,

- \( \psi_r \in H^0(C_r,\Omega_{C_\tilde{U}}^2(D(\tilde{t})) \log(D(\tilde{t}))\mathcal{O}_{C_\tilde{U}}[\epsilon]) \), and
- \( \nabla_v : E_v \rightarrow E_v \otimes \Omega_{C_\tilde{U}}^1(D(\tilde{t})) \log(D(\tilde{t}))\mathcal{O}_{C_\tilde{U}}[\epsilon] \) is a connection,

where we define the sheaf \( \Omega_{C_\tilde{U}}^1(D(\tilde{t})) \log(D(\tilde{t}))\mathcal{O}_{C_\tilde{U}}[\epsilon] \) as the coherent subsheaf of \( \Omega_{C_\tilde{U}}^1(D(\tilde{t}))\mathcal{O}_{C_\tilde{U}}[\epsilon] \) locally generated by \( f^{-1}df \) and \( df \) for a local defining equation \( f(D(\tilde{t}))\mathcal{O}_{C_\tilde{U}}[\epsilon] \), which is the pull-back of \( D(\tilde{t}) \) by the morphism \( C_r \rightarrow C_\tilde{U} \rightarrow C_\tilde{U} \rightarrow C \). Set \( U_\alpha := U_\alpha \times \text{Spec} \mathcal{O}_{C_\tilde{U}}[\epsilon] \). Let
\[
\mu_{\alpha\beta}(\epsilon) : U_{\alpha\beta} \times \text{Spec} \mathcal{O}_{C_\tilde{U}}[\epsilon] \tilde{\rightarrow} U_{\alpha\beta} \times \text{Spec} \mathcal{O}_{C_\tilde{U}}[\epsilon]
\]
be an isomorphism associated to the first-order deformation \( C_\epsilon \) of \( C_\tilde{U} \). The isomorphism \( \mu_{\alpha\beta}(\epsilon) \) satisfies
\[
\mu_{\alpha\beta}(\epsilon)^* (\epsilon) = \epsilon, \quad \mu_{\alpha\beta}(\epsilon)^* (f) = f + \epsilon d_{\alpha\beta} f, \quad \text{for } f \in \mathcal{O}_{U_{\alpha\beta}},
\]
for some \( d_{\alpha\beta} \in \Theta_{\mathcal{O}_\varnothing}(-D)(U_{\alpha\beta}) \). We describe \( d_{\alpha\beta} \) as \( d_{\alpha\beta} = \frac{\partial \eta_{\alpha\beta}(\epsilon)}{\partial \epsilon} \mid_{\mathcal{O}_\varnothing} \in \Theta_{\mathcal{O}_\varnothing}(-D)(U_{\alpha\beta}) \). Here, \( f_\alpha \) is a local defining equation of \( \tilde{t}_i|_{\mathcal{O}_\varnothing} \cap U_\alpha \). Set \( \phi_\alpha^*: E_\epsilon|U_\alpha \cong \mathcal{O}^{\mathfrak{m}}_{U_\alpha} \). There is an isomorphism

\[
\varphi_\alpha: E_\epsilon|U_\alpha \xrightarrow{\phi_\alpha^*} \mathcal{O}^{\mathfrak{m}}_{U_\alpha} \xrightarrow{\phi_\alpha^*} \tilde{E}|U_\alpha \cong \mathcal{O}^{\mathfrak{m}}_{\mathcal{O}_\varnothing}[\epsilon]
\]

such that \( \varphi_\alpha \otimes \mathcal{O}_{\mathcal{O}_\varnothing}[\epsilon]/(\epsilon): E_\epsilon \otimes \mathcal{O}_{\mathcal{O}_\varnothing}[\epsilon]/(\epsilon)|U_\alpha \xrightarrow{\sim} \tilde{E}|U_\alpha \otimes \mathcal{O}_{\mathcal{O}_\varnothing}[\epsilon]/(\epsilon) = \tilde{E}|U_\alpha \) is the given isomorphism and that \( \varphi_\alpha|_{\tilde{t}_i \otimes \mathcal{O}_{\mathcal{O}_\varnothing}[\epsilon]}((l_i)^{\varnothing}) = (\tilde{r}_i)|_{U_\alpha \times \text{Spec} \mathcal{O}_{\mathcal{O}_\varnothing}[\epsilon]} \) if \( \tilde{t}_i|_{\mathcal{O}_\varnothing} \cap U_\alpha \neq \emptyset \). Put

\[
\theta_{\alpha\beta}(\epsilon): \mathcal{O}^{\mathfrak{m}}_{U_{\alpha\beta}} \xrightarrow{\phi_\alpha^*} E_\epsilon|U_{\alpha\beta} \xrightarrow{\phi_\alpha^*} \mathcal{O}^{\mathfrak{m}}_{U_{\alpha\beta}},
\]

which is an element of \( \mathcal{E}nd(\mathcal{O}^{\mathfrak{m}}_{U_{\alpha\beta}})(U_{\alpha\beta}) \). We denote \( \theta_{\alpha\beta}(\epsilon) \) by

\[
\theta_{\alpha\beta}(\epsilon) = \tilde{\theta}_{\alpha\beta} + \epsilon \frac{\partial \theta_{\alpha\beta}(\epsilon)}{\partial \epsilon} \quad \text{where} \quad \tilde{\theta}_{\alpha\beta}, \epsilon \frac{\partial \theta_{\alpha\beta}(\epsilon)}{\partial \epsilon} \in \mathcal{E}nd(\mathcal{O}^{\mathfrak{m}}_{U_{\alpha\beta}})(U_{\alpha\beta}).
\]

Set

\[
\eta_{\alpha\beta} := \frac{\partial \theta_{\alpha\beta}(\epsilon)}{\partial \epsilon} (\tilde{\theta}_{\alpha\beta})^{-1} \in \mathcal{E}nd(\mathcal{O}^{\mathfrak{m}}_{U_{\alpha\beta}})(U_{\alpha\beta}).
\]

We define elements \( u_{\alpha\beta} \in \mathcal{F}^0(U_{\alpha\beta}) \) and \( (v_\alpha, w_\alpha) \in \mathcal{F}^1(U_\alpha) \) by

\[
(10) \quad u_{\alpha\beta} := (\phi_\alpha|_{U_{\alpha\beta}})^{-1} \circ (d_{\alpha\beta} + \eta_{\alpha\beta}) \circ \phi_\alpha|_{U_{\alpha\beta}}
\]

\[
(11) \quad v_\alpha := (\varphi_\alpha \otimes \text{id}) \circ \nabla_\epsilon|_{U_\alpha} \circ \varphi_\alpha^{-1} \mid_{U_\alpha} \mod d_\epsilon,
\]

\[
(12) \quad w_\alpha := \psi_\alpha|_{U_\alpha} - \tilde{\psi}|_{U_\alpha} \mod d_\epsilon.
\]

respectively. We can see that

\[
u_{\alpha\gamma} = u_{\alpha\gamma} + u_{\alpha\beta} = 0, \quad \text{and} \quad d_{\mathfrak{F}}(u_{\alpha\beta}) = (v_\beta, w_\beta) - (v_\alpha, w_\alpha).
\]

Then \( \{u_{\alpha\beta}\}, \{(v_\alpha, w_\alpha)\} \) determines an element \( \sigma_\mathfrak{D}(\nu) \) of \( \mathbf{H}^1(\mathfrak{F}_\mathfrak{D}) \). We can check that \( \nu \mapsto \sigma_\mathfrak{D}(\nu) \) determines an isomorphism

\[
\Theta_{\mathfrak{M}/\mathfrak{N}^{(n)}_D}(\tilde{U}) \xrightarrow{\sim} \mathbf{H}^1(\mathfrak{F}_\mathfrak{D}^\bullet): \nu \mapsto \sigma_\mathfrak{D}(\nu).
\]

We denote by \( \xi_\mathfrak{D} \) this isomorphism. The isomorphism \( \xi_\mathfrak{D} \) induces the desired isomorphism \( \xi \). \( \square \)

We describe the analytic tangent sheaf in terms of the hypercohomology of a certain analytic complex. Let \( \nu \) be an element of \( \mathcal{N}_D^{(n)} \). Put \( \mathfrak{M}_\nu = \mathfrak{M}_\nu^\alpha \cap \mathfrak{M}_\nu^\beta \) so that it is the fiber of \( \nu \) under \( \mathfrak{M}_\nu^{\alpha\beta} \). Let \( j: \mathcal{C}_{\mathfrak{M}_\nu} \setminus \{\tilde{t}_1, \ldots, \tilde{t}_n\} \to \mathfrak{M}_\nu \) be the canonical inclusion. Let \( \tilde{V} := \text{Ker} \nabla^{\mathfrak{m}}|_{\mathcal{C}_{\mathfrak{M}_\nu} \setminus \{\tilde{t}_1, \ldots, \tilde{t}_n\}} = \mathfrak{M}_\nu \) be the locally constant sheaf of the locally free \( \pi_{\mathfrak{M}_\nu} \circ j)^{-1}\mathcal{O}_{\mathfrak{M}_\nu} \)-module associated to the relative analytic connection \( \nabla^{\mathfrak{m}} \) on \( \mathcal{C}_{\mathfrak{M}_\nu} \setminus \{\tilde{t}_1, \ldots, \tilde{t}_n\} \). \( \pi_{\mathfrak{M}_\nu}: \mathcal{C}_{\mathfrak{M}_\nu} \to \mathfrak{M}_\nu \) is the natural map.

Assume that \( \nu \) is generic. We define a complex \( \mathcal{F}_\bullet^{\mathfrak{m}} \) by

\[
(13) \quad \mathcal{F}_\bullet^{\mathfrak{m}}: j_* \left( \mathcal{E}nd(\tilde{V}) \right) \otimes \Theta_{\mathcal{C}_{\mathfrak{M}_\nu}/\mathfrak{M}_\nu}(-D(\tilde{t})) \xrightarrow{d_{\xi_\mathfrak{D}} \circ \text{pr}_2} \mathcal{C}_{\mathfrak{M}_\nu}/\mathfrak{M}_\nu \otimes \mathbf{H}^2(\mathfrak{D}_\mathfrak{F}(\tilde{t}))
\]

where \( \text{pr}_2 \) is the second projection. We have the following commutative diagram

\[
\begin{array}{cccc}
\mathcal{F}_0^{\mathfrak{m}} & \xrightarrow{d_{\mathfrak{F}}^{\mathfrak{m}}} & \mathcal{F}_1^{\mathfrak{m}} & \\
\mathcal{F}_0^{\mathfrak{m}} & \downarrow & \downarrow & \\
\end{array}
\]

\[
\begin{array}{cccc}
(j_* \mathcal{E}nd(\tilde{V})) & \otimes \Theta_{\mathcal{C}_{\mathfrak{M}_\nu}/\mathfrak{M}_\nu}(-D(\tilde{t})) & \xrightarrow{d_{\xi_\mathfrak{D}} \circ \text{pr}_2} & \mathcal{C}_{\mathfrak{M}_\nu}/\mathfrak{M}_\nu \otimes \mathbf{H}^2(\mathfrak{D}_\mathfrak{F}(\tilde{t}))
\end{array}
\]

We can show that the homomorphism \( \text{Ker} d_{\mathfrak{F}}|_{\mathcal{C}_{\mathfrak{M}_\nu} \setminus \{\tilde{t}_1, \ldots, \tilde{t}_n\}} \to j_* \left( \mathcal{E}nd(\tilde{V}) \right) \) is an isomorphism and the homomorphism \( d_{\xi_\mathfrak{D}}: \mathcal{F}_0^{\mathfrak{m}} \to \mathcal{F}_1^{\mathfrak{m}} \) is surjective as in the proof of [8, Proposition 7.3]. Then we have the following proposition.
Proposition 3.2. If \( \nu \) is generic, then we have
\[
R^1(\pi_{\tilde{M}_\nu})_*((\mathcal{F}^n)^*) \cong R^1(\pi_{\tilde{M}_\nu})_*((\tilde{\mathcal{F}}^n)^*)
\]
where \( \pi_{\tilde{M}_\nu}: \tilde{M}_\nu \to M_\nu \) is the natural map.

3.3. Isomonodromic deformations. Let \( \nu \) be an element of \( N_\nu(n)(d) \). Put \( M_\nu = M_\nu^\alpha / \tilde{M}_\nu(\tilde{t}, r, d)_\nu \) which is the fiber of \( \nu \) under \( M_\nu^\alpha / \tilde{M}_\nu(\tilde{t}, r, d) \to N_\nu(n)(d) \). Let \( j: C_{M_\nu} \setminus \{ \tilde{t}_1, \ldots, \tilde{t}_n \} M_\nu \to C_{M_\nu} \) be the canonical inclusion. Let \( \text{Ker} \tilde{\nabla}^n \) be the locally constant sheaf of the locally free \( (\pi_{M_\nu} \circ j)^{-1}C_{M_\nu} \)-module associated to the relative analytic connection \( \tilde{\nabla}^n \) on \( C_{M_\nu} \setminus \{ \tilde{t}_1, \ldots, \tilde{t}_n \} M_\nu \), where \( \pi_{M_\nu}: C_{M_\nu} \to M_\nu \) is the natural map.

Definition 3.3. For \( \nu_\mu: M_\nu \to \tilde{M}_{g, n} \), we say a complex foliation \( \mathcal{F} \) is a foliation determined by the isomonodromic deformations if
1. \( \mathcal{F} \) is transverse to each fiber \( (M_\nu)_t = \nu_\mu^{-1}(t), t \in \tilde{M}_{g, n} \), and
2. for each leaf \( l \) on \( M_\nu \), the restriction of the local system \( j_*\text{Ker} \tilde{\nabla}^n|_{C_{M_\nu} \setminus \{ \tilde{t}_1, \ldots, \tilde{t}_n \} M_\nu} \) is constant.

Let \( \mu: \pi_{\mu}^*\Theta_{\tilde{M}_{g, n}} \to R^1(\pi_{M_\nu})_*(\Theta_{C_{M_\nu}/M_\nu}(\tilde{D}(\tilde{t}))) \) be the Kodaira–Spencer map, where \( \pi_{M_\nu}: C_{M_\nu} \to M_\nu \) is the natural morphism. We define a splitting \( D \) of the tangent map \( \Theta_{M_\nu} \to \pi_{\mu}^*(\Theta_{\tilde{M}_{g, n}}) \) as follows:
\[
D: \pi_{\mu}^*(\Theta_{\tilde{M}_{g, n}}) \to \Theta_{M_\nu} \cong R^1(\pi_{M_\nu})_*(\mathcal{F}^0_{M_\nu} \to \tilde{\mathcal{F}}^1_{M_\nu}); \quad \nu \mapsto [\{ t(\tilde{\nabla})(d_{\alpha \beta}) \}, \{ 0 \}],
\]
where \([d_{\alpha \beta}]\) is a description of \( \mu(\nu) \) by the Čech cohomology. Here, we take an affine open covering \( \{ U_\alpha \} \).

Proposition 3.4 ([8] Section 6], [9] Section 8] and [10] Proposition 3.7). The subsheaf \( D(\pi^*(\Theta_{\tilde{M}_{g, n}})) \) determines the foliation determined by the isomonodromic deformations.

We can take a natural lift \( \tilde{\mathcal{D}}: \pi_{\mu}^*(\Theta_{T^*\tilde{M}_{g, n}}) \to \Theta_{\tilde{M}_\nu} \) of \( D: \pi_{\mu}^*(\Theta_{\tilde{M}_{g, n}}) \to \Theta_{M_\nu} \) as follows. We define a complex \( \mathcal{G}^* \) by
\[
\Theta_{C_{M_\nu}/M_\nu}(-\tilde{D}(\tilde{t})) =: \mathcal{G}^0 \xrightarrow{d_{\psi}} \mathcal{G}^1 =: \Omega_{C_{M_\nu}/M_\nu}^{\leq 2}(\tilde{D}(\tilde{t})),
\]
where \( d_{\psi} \) is defined by (i). Then we can show that \( \tilde{\pi}_{\mu}^*(\Theta_{T^*\tilde{M}_{g, n}}) \cong R^1(\pi_{\tilde{M}_\nu})_*(\mathcal{G}^*) \). We define a lift \( \tilde{\mathcal{D}}: \pi_{\mu}^*(\Theta_{T^*\tilde{M}_{g, n}}) \to \Theta_{\tilde{M}_\nu} \) of \( D \) by the following homomorphism
\[
\tilde{\mathcal{D}}: H^1(\mathcal{G}^*_U) \to H^1(\mathcal{F}^*_U): \quad [\{ d_{\alpha \beta} \}, \{ w_\alpha \}] \mapsto [\{ t(\tilde{\nabla})(d_{\alpha \beta}) \}, \{ 0, w_\alpha \}].
\]

3.4. Symplectic structure. First, we recall the canonical symplectic structure \( \omega_{\tilde{M}_{g, n}} \) on \( T^*\tilde{M}_{g, n} \). Let \( U \) be an affine open set of \( T^*\tilde{M}_{g, n} \) and let \( (C_U, \tilde{\psi}) \) be a family of curves and quadratic differentials on \( U \). Let \( \psi_\alpha d\psi_\alpha^\alpha \) be the restriction of \( \tilde{\psi} \) on an affine open set \( U_\alpha \subset C_U \). Let \( \mu_{\alpha \beta} \) be the isomorphism (ii):
\[
f_\alpha = \mu_{\alpha \beta}(f_\beta).
\]
We define a 1-form \( \theta_{\tilde{M}_{g, n}} \) on \( T^*\tilde{M}_{g, n} \) by
\[
\theta_{\tilde{M}_{g, n}}: H^1(G^*_U) \to H^1(\Omega^1_{C_U/U}); \quad [\{ d_{\alpha \beta} \}, \{ w_\alpha \}] \mapsto [\{ d_{\alpha \beta} \psi_\alpha \frac{\partial \mu_{\alpha \beta}}{\partial f_\beta} df_\alpha \}],
\]
where \( G_U \) is the complex \( d_{\psi}: \Theta_{C_U/U}(-D(\tilde{t})) \to \Omega_{C_U/U}^{\leq 2}(D(\tilde{t})) \). The 1-form \( \theta_{\tilde{M}_{g, n}} \) is the canonical 1-form on the cotangent bundle \( T^*\tilde{M}_{g, n} \). Let \( d\theta_{\tilde{M}_{g, n}} \) be the exterior differential of \( \theta_{\tilde{M}_{g, n}} \). The 2-form \( d\theta_{\tilde{M}_{g, n}} \) gives the symplectic form on the cotangent bundle \( T^*\tilde{M}_{g, n} \).
Proposition 3.5. Let $v = \{(d_{\alpha \beta}), \{w_{\alpha}\}\}$ and $v' = \{(d'_{\alpha \beta}), \{w'_{\alpha}\}\}$ be elements of $\mathbf{H}^1(\mathcal{G}^\bullet_{\nu})$. The pairing
\[
\mathbf{H}^1(\mathcal{G}^\bullet_{\nu}) \otimes \mathbf{H}^1(\mathcal{G}^\bullet_{\nu}) \rightarrow \mathbf{H}^2(\Omega^\bullet_{\nu U}/U);
\]
\[
v \otimes w \mapsto \{(2 \cdot d_{\beta \gamma} \circ d'_{\beta \gamma} \circ \psi_\beta), \{-d_{\delta \alpha} \circ w'_\beta - w_\alpha \circ d'_{\alpha \beta}\}\}
\]
coincides with the symplectic form $d\theta_{\tilde{M}_{\nu, \alpha}}$.

Proof. Let $D_v : \mathcal{O}_{U_{\nu}} \rightarrow \mathcal{O}_{U_{\nu}}$ be a derivation corresponding to $v$. We compute the 2-form $d\theta_{\tilde{M}_{\nu, \alpha}}(v, v')$ as follows:
\[
D_v \theta_{\tilde{M}_{\nu, \alpha}}(v') - D_{v'} \theta_{\tilde{M}_{\nu, \alpha}}(v) + \theta_{\tilde{M}_{\nu, \alpha}}([v, v'])
\]
\[
= D_{v'}(\mu_{\beta \gamma})D_v \left( \psi_{\alpha} \frac{\partial \mu_{\alpha \beta}}{\partial f_{\beta}} \right) df_{\alpha} - D_{v'}(\mu_{\beta \gamma})D_v \left( \psi_{\alpha} \frac{\partial \mu_{\alpha \beta}}{\partial f_{\beta}} \right) df_{\alpha}
\]
\[
= D_{v'}(\mu_{\beta \gamma})\psi_{\alpha} \frac{\partial D_v(\mu_{\alpha \beta})}{\partial f_{\beta}} df_{\alpha} - D_{v'}(\mu_{\beta \gamma})\psi_{\alpha} \frac{\partial D_v(\mu_{\alpha \beta})}{\partial f_{\beta}} df_{\alpha}
\]
\[
+ D_{v'}(\mu_{\beta \gamma})\psi_{\alpha} \frac{\partial \mu_{\beta \gamma}}{\partial f_{\beta}} \frac{\partial \mu_{\alpha \beta}}{\partial f_{\beta}} df_{\alpha} - D_{v'}(\mu_{\beta \gamma})\psi_{\alpha} \frac{\partial \mu_{\beta \gamma}}{\partial f_{\beta}} \frac{\partial \mu_{\alpha \beta}}{\partial f_{\beta}} df_{\alpha}
\]
\[
= -d_{\alpha \beta} \psi_{\alpha} \frac{\partial \mu_{\beta \gamma}}{\partial f_{\beta}} df_{\beta} + d_{\alpha \beta} \psi_{\alpha} \frac{\partial \mu_{\beta \gamma}}{\partial f_{\beta}} df_{\beta} - d_{\alpha \beta} \psi_{\alpha} \frac{\partial \mu_{\beta \gamma}}{\partial f_{\beta}} df_{\beta} + d_{\alpha \beta} \psi_{\alpha} \frac{\partial \mu_{\beta \gamma}}{\partial f_{\beta}} df_{\beta} + d_{\alpha \beta} \psi_{\alpha} \frac{\partial \mu_{\beta \gamma}}{\partial f_{\beta}} df_{\beta} + d_{\alpha \beta} \psi_{\alpha} \frac{\partial \mu_{\beta \gamma}}{\partial f_{\beta}} df_{\beta}.
\]
We add the exterior differential of $d_{\alpha \beta} d_{\alpha \beta} \psi_{\alpha}$ to the formula above:
\[
- d_{\alpha \beta} \psi_{\alpha} \frac{\partial \mu_{\beta \gamma}}{\partial f_{\beta}} df_{\beta} + d_{\alpha \beta} \psi_{\alpha} \frac{\partial \mu_{\beta \gamma}}{\partial f_{\beta}} df_{\beta} - d_{\alpha \beta} \psi_{\alpha} \frac{\partial \mu_{\beta \gamma}}{\partial f_{\beta}} df_{\beta} + d_{\alpha \beta} \psi_{\alpha} \frac{\partial \mu_{\beta \gamma}}{\partial f_{\beta}} df_{\beta} + d_{\alpha \beta} \psi_{\alpha} \frac{\partial \mu_{\beta \gamma}}{\partial f_{\beta}} df_{\beta} + d_{\alpha \beta} \psi_{\alpha} \frac{\partial \mu_{\beta \gamma}}{\partial f_{\beta}} df_{\beta}.
\]
By the isomorphism $\mathbf{H}^1(\Omega^1_{\nu U}/U) \cong \mathbf{H}^2(\Omega^\bullet_{\nu U}/U)$, we have this proposition. □

Proposition 3.6. Take a point $v \in N^1_{(n)}(d)$. Let $\tilde{M}_{\nu, \alpha}^\bullet \subseteq (\tilde{t}, r, d)_{\nu}$ be the fiber of $v$ under the composition $\tilde{M}_{\nu, \alpha}^\bullet \subseteq (\tilde{t}, r, d)_{\nu} \rightarrow N^1_{(n)}(d)$. Then the fiber $\tilde{M}_{\nu, \alpha}^\bullet \subseteq (\tilde{t}, r, d)_{\nu}$ has an algebraic symplectic structure.

We can obtain the above proposition by the following two propositions.

Proposition 3.7. There is a non-degenerate relative 2-form $\omega \in H^0(\tilde{M}, \Omega^2_{\tilde{M}/N^1_{(n)}(d)}).$

Proof. We set $\eta(s) := s - \iota(\nabla) \circ \text{symb}_1(s) \in \mathcal{E}(\tilde{E})$, where $s \in \mathcal{F}^0$. For $v = \{(u_{\alpha \beta}), \{(v_{\alpha}, w_{\alpha})\}\} \in \mathbf{H}^1(\tilde{C} \times \tilde{U}, \tilde{F}^\bullet_{\nu})$ and $w = \{(u'_{\alpha \beta}), \{(v'_{\alpha}, w'_{\alpha})\}\} \in \mathbf{H}^1(\tilde{C} \times \tilde{U}, \tilde{F}^\bullet_{\nu})$, we put
\[
\omega_1(v, w) = \{(\text{Tr}(\eta(u_{\alpha \beta}) \circ \eta(u'_{\alpha \beta})), -\text{Tr}(\eta(u_{\alpha \beta}) \circ v'_{\beta}) - \text{Tr}(v_{\alpha} \circ \eta(u'_{\alpha \beta}))\}\}
\]
\[
\omega_2(v, w) = \{(2 \cdot \text{symb}_1(u_{\alpha \beta}) \circ \text{symb}_1(u'_{\alpha \beta}) \circ \psi_\beta), -\text{symb}_1(u_{\alpha \beta}) \circ w'_{\beta} + w_{\alpha} \circ \text{symb}_1(u'_{\alpha \beta})\}\}.
\]
For each affine open subset $\tilde{U} \subset \tilde{M}$, we define a pairing
\[
\mathbf{H}^1(\tilde{C} \times \tilde{U}, \tilde{F}^\bullet_{\nu}) \otimes \mathbf{H}^1(\tilde{C} \times \tilde{U}, \tilde{F}^\bullet_{\nu}) \rightarrow \mathbf{H}^2(\tilde{C} \times \tilde{U}, \Omega^\bullet_{\tilde{C} \times \tilde{U}/\tilde{U}}) \cong H^0(\tilde{O}_{\tilde{U}})
\]
\[
v \otimes w \mapsto \omega_1(v, w) + \omega_2(v, w),
\]
where we consider in Čech cohomology with respect to an affine open covering $\{U_{\alpha}\}$ of $\tilde{C} \times \tilde{T}$, $\{u_{\alpha \beta}\} \in C^1(\mathcal{F}^0)$, $\{(v_{\alpha}, w_{\alpha})\} \in C^0(\mathcal{F}^1)$ and so on. This pairing determines a pairing
\[
\omega : \mathbf{R}^1(\pi_{\tilde{M}})_*(\mathcal{F}^\bullet_{\nu}) \otimes \mathbf{R}^1(\pi_{\tilde{M}})_*(\mathcal{F}^\bullet_{\nu}) \rightarrow \mathcal{O}_{\tilde{M}}.
\]
By the same argument as in the proof of Proposition 3.7, $\omega$ is skew symmetric and non-degenerate. □

Proposition 3.8. For the 2-form constructed in Proposition 3.7, we have $d\omega = 0$. 

Proof. Let $\Theta^\text{initial}_{M_\nu}$ be the subbundle of $\Theta_{M_\nu}$ consisted by the images of the tangent morphism $\Theta_{\tilde{M}_\nu/T^*\tilde{M}_{g,n}} \to \Theta_{M_\nu}$ and let $\Theta^\text{IMD}_{M_\nu}$ be the subbundle of $\Theta_{M_\nu}$ consisted by the images of $\tilde{D}(\hat{\pi}^*_\nu(\Theta_{T^*\tilde{M}_{g,n}})) \to \Theta_{M_\nu}$. We take an affine open set $\tilde{U} \subset \tilde{M}_\nu$. We have a canonical decomposition

$$H^1(F^*_\nu) \to \Theta^\text{initial}_{\tilde{U}} \oplus \Theta^\text{IMD}_{\tilde{U}}, \quad v = \{(u_{\alpha\beta}), \{(v_\alpha), \{w_\alpha\}\} \mapsto v_{\text{initial}} + v_{\text{IMD}},$$

where

$$v_{\text{initial}} = \{(\eta(u_{\alpha\beta})), \{(v_\alpha, 0)\}\} \quad \text{and} \quad v_{\text{IMD}} = \{(\xi(\nabla) \circ \text{symb}_1(u_{\alpha\beta})), \{(0, w_\alpha)\}\}.$$

We may assume that $v$ is generic. Let $\tilde{U}$ be an affine open set of $\tilde{M}_\nu$ and let $(\tilde{E}, \tilde{\nabla}, \{\tilde{\iota}^{(i)}\}, \tilde{\psi})$ be the family on $C \times M_{g,n}$. We take an affine open covering $C_\nu = \bigcup_{\alpha} U_\alpha$ such that $\phi_\alpha : \tilde{E}|_{U_\alpha} \to \mathcal{O}^{|\nu}_U$ for any $\alpha$, $\sharp \{\alpha \mid \tilde{E}|_{C \cap \tilde{M}_\nu} \not\in \emptyset\} \leq 1$ for any $\alpha$ and $\sharp \{\alpha \mid \tilde{E}|_{C \cap \tilde{M}_\nu} \not\in \emptyset\} \leq 1$ for any $i$. If we replace $U_\alpha$ sufficiently smaller, there exists a sheaf $E_\alpha$ on $U_\alpha$ such that $E_\alpha|_{U_\alpha \cap U_\beta} \cong (\pi^{-1}_{M_\nu} \mathcal{O}_{M_\nu}|_{\alpha \cap \beta})_{\nu} \otimes_{\nu} \mathcal{O}_{U_\alpha}$ for any $\beta \neq \alpha$ and an isomorphism $\phi_\alpha : j_*(\tilde{\nabla})|_{U_\alpha} \to E_\alpha$. Here the local system $\tilde{V}$ is defined in [3,2]. For each $\alpha$, $\beta$, we put

$$\varphi_{\alpha\beta} : E_\alpha|_{U_\alpha \cap U_\beta} \to E_\beta|_{U_\alpha \cap U_\beta}.$$

For each $\alpha$, $\beta$, let $\mu_{\alpha\beta} : U_\alpha \to U_\beta$ be an isomorphism such that the gluing scheme of the collection $(U_\alpha, U_\alpha \cap U_\beta, \mu_{\alpha\beta})$ is isomorphic to $C_\nu$. We consider a vector field $v \in H^0(\tilde{U}, \Theta_{M_\nu})$. Then $v$ corresponds to a derivation $D_v : \Theta_{M_\nu} \to \Theta_{M_\nu}$ which naturally induces a morphism

$$D_v : \text{Hom}(E_\beta|_{U_\alpha \cap U_\beta}, E_\alpha|_{U_\alpha \cap U_\beta}) \to \text{Hom}(E_\beta|_{U_\alpha \cap U_\beta}, E_\alpha|_{U_\alpha \cap U_\beta}).$$

The isomorphism $\Theta_{\tilde{M}_\nu} \cong R^1(\pi_{\tilde{M}_\nu})_*((\tilde{F}^*)^\text{an})$ is given by

$$\Theta_{\tilde{M}_\nu} \ni \nu \mapsto \{(\phi_{\alpha\beta}^{-1} \circ D_v(\varphi_{\alpha\beta}) \circ \phi_{\beta\alpha}, \{D_v(\varphi_{\alpha\beta})\})\}. \quad (\tilde{F}^*)^\text{an},$$

and the 2-form $\omega(u,v)$ is given by

$$\omega(u,v) = \omega_1(u,v) + \omega_2(u,v), \quad u, v \in \Theta_{\tilde{M}_\nu}$$

is given by

$$\omega_1(u,v) = \{(\text{Tr}(D\varphi_{\alpha\beta})D\varphi_{\alpha\beta}(v_{\text{initial}}(\varphi_{\alpha\beta})), \varphi_{\alpha\beta})\} \quad \text{and}$$

$$\omega_2(u,v) = \{(D\varphi_{\alpha\beta})D\varphi_{\alpha\beta}(v_{\text{initial}}(\varphi_{\alpha\beta})), \varphi_{\alpha\beta} \} \quad \{D\varphi_{\alpha\beta}(v_{\text{initial}}(\varphi_{\alpha\beta})), \varphi_{\alpha\beta} \}.$$
Let $\mu_1, \ldots, \mu_{3g-3+n}$ be local vector fields on an affine open subset $U \subset \hat{M}_{g,n}$. Let $h_i$ be a linear function on $T^*\hat{M}_{g,n}$ corresponding to the local vector field $\mu_i$ on $U$. Assume that $\{h_i, h_j\}_{\hat{M}_{g,n}} = 0$ for $i, j = 1, \ldots, 3g-3+n$ and $dh_1 \wedge \cdots \wedge dh_{3g-3+n}$ is not identically 0, where $\{\cdot, \cdot\}_{\hat{M}_{g,n}}$ is the Poisson bracket associated to the symplectic structure $\omega_{\hat{M}_{g,n}}$. Put $\hat{U} = (\pi \circ p_{\hat{M}_{g,n}})^{-1}(U)$, where $\hat{\pi}_*: \hat{M}_{C/\hat{M}_{g,n}}^o(t, r, d)_\nu \to T^*\hat{M}_{g,n}$ and $p_{\hat{M}_{g,n}} : T^*\hat{M}_{g,n} \to \hat{M}_{g,n}$. Let $\omega_{\hat{T}^*\hat{M}_{g,n}}$ be the symplectic structure on $T^*\hat{M}_{g,n}$. We define a Hamiltonian $E_i$ on $\hat{U}$ as $\hat{\pi}^*h_i$ for $i = 1, \ldots, 3g-3+n$.

**Proposition 3.10.** Assume that the Hamiltonians $h_i$ and the symplectic structure $\omega_{\hat{T}^*\hat{M}_{g,n}}$ on $U$ give a commuting Hamiltonian system. By the Hamiltonians $E_i$ and the symplectic structure $\omega$ on $\hat{U}$, we have an autonomous commuting Hamiltonian structure on $\hat{U}$. The functions $E_i$ are conserved quantities. On the common level surface $E_1 = 0, \ldots, E_{3g-3+n} = 0$ in $\hat{U}$, one recovers the multi-time dependent dynamics associated to the isomonodromic deformations.

**Proof.** Let $\{\cdot, \cdot\}_{\hat{M}_{g,n}}$ be the Poisson bracket associated to the symplectic structure $\omega_{\hat{T}^*\hat{M}_{g,n}}$ on $T^*\hat{M}_{g,n}$. Let $v_{h_i}$ be the element $\hat{\pi}^{-1}(\Theta_{T^*\hat{M}_{g,n}})(\hat{U})$ defined by the vector field $\{\cdot, h_i\}_{\hat{M}_{g,n}}$ on $\hat{\pi}(U) \subset \hat{M}_{g,n}$. In other words, $\omega_{\hat{T}^*\hat{M}_{g,n}}(v_{h_i}, v) = dh_i(v)$ for any $v \in \Theta_U$. Put $v_{h_i} = \{(d_{\alpha i}^h), \{w_{\alpha i}^h\}\} \in \mathcal{H}^1(G^*)$, where $G^*$ is the complex [16]. Put $v_{h_i}^{\text{IMD}} = \{\{i(\nabla)(d_{\alpha i}^h)), \{0, w_{\alpha i}^h\}\}\} \in \mathcal{H}^1(F^*_U)$. By the diagram (22), we have $\omega(v_{h_i}^{\text{IMD}}, v) = dE_i(v)$ for any $v \in \Theta_{\hat{U}}$, that is, $v_{h_i}^{\text{IMD}} = \{\cdot, E_i\}$, which is the dynamics of the Hamiltonian system associated to $E_i$ and $\omega$.

Since the Hamiltonian system associated to $h_i$ and $\omega_{\hat{T}^*\hat{M}_{g,n}}$ on $U$ is commuting, the Hamiltonian system associated to $E_i$ and $\omega$ is commuting. The common level surface $E_1 = \cdots = E_{3g-3+n} = 0$ is $M^o_{C/\hat{M}_{g,n}}(t, r, d)_\nu$. On this common level surface, the tangent associated to the dynamics is $\{\{i(\nabla)(d_{\alpha i}^h)), \{0\}\}\} \in \mathcal{H}^1(F^*_U \to \tilde{F}^*_U)$, which is a tangent associated to the isomonodromic deformations. \hfill $\Box$

## 4. Moduli stack of stable parabolic connections with a quadratic differential and twisted cotangent bundle

In this section, we see that the moduli spaces of parabolic connections with a quadratic differential are equipped with structures of twisted cotangent bundles. We consider moduli stacks corresponding to the moduli schemes considered in the previous section. We introduce a moduli stack of pointed smooth projective curves and quasi-parabolic bundles. We consider the cotangent bundle of this moduli stack. We describe the tangent sheaf of the total space of this cotangent bundle and the canonical symplectic form on this cotangent bundle. In 4.2, we consider a map from the moduli stack of parabolic connections with a quadratic differential to the moduli stack of pointed smooth projective curves and quasi-parabolic bundles. We endow this map with structure of a twisted cotangent bundle. In 4.3, we consider a relation between parabolic connections with a quadratic differential and extended (parabolic) connections. Extended connections are appeared in [1] and [2].

In this section, we assume that $\nu$ is generic. If $\nu$ is generic, then any $(t, \nu)$-parabolic connection is irreducible. So all $(t, \nu)$-parabolic connections are stable.

### 4.1. Moduli stack of stable parabolic connections with a quadratic differential

Let $\mathcal{M}_{g,n}$ be the moduli stack of $n$-pointed smooth projective curves of genus $g$, where $n$-points consist of distinct points. Let $\mathcal{M}_{g,n}(r, \nu)_{(t, d)}$ be the moduli stack of collections $((C, t, \psi), (E, \nabla, I))$, where $(C, t)$ $(t = (t_1, \ldots, t_n))$ is an $n$-pointed smooth projective curve of genus $g$ over $\mathbb{C}$ where $t_1, \ldots, t_n$ are distinct points, $\psi$ is an element of $H^0(C, \Omega^2(D(t)))$, and $(E, \nabla, I)$ is a $(t, \nu)$-parabolic connection of rank $r$ and of degree $d$ on $C$. Let $\Theta_{\mathcal{M}_{g,n}(r, \nu)_{(t, d)}}$ be the tangent complex of $\mathcal{M}_{g,n}(r, \nu)_{(t, d)}$. Let $\Theta_{\mathcal{M}_{g,n}(r, \nu)_{(t, d)}, x}$ be the fiber of $\Theta_{\mathcal{M}_{g,n}(r, \nu)_{(t, d)}}$ over a point $x : pt \to \mathcal{M}_{g,n}(r, \nu)_{(t, d)}$. Then $H^0(\Theta_{\mathcal{M}_{g,n}(r, \nu)_{(t, d)}, x})$ is isomorphic to $H^1(F^*_x)$. Here, we recall the
complex $F^*_x$: 
\[ F^0 = \{ s \in A_E(D(t)) \mid (s - i(\nabla) \circ \text{sym}_1(s)) \mid l_i, (l^{(i)}_j) \subset l^{(i)}_j \text{ for any } i, j \}; \]
\[ F^1 = \{ s \in \mathcal{E}nd(E) \otimes \Omega^2_{C}(D(t)) \mid \text{res}_i(s)(l^{(i)}_j) \subset l^{(i)}_{j+1} \text{ for any } i, j \}; \]
\[ F^1_x = F^1_x \oplus \Omega^2_{C}(D(t)); \]
and $d_{\Phi^*} := (d_{\Phi^*}, d_{\Phi^*} \circ (\text{Id} - i(\nabla) \circ \text{sym}_1, \text{sym}_1)): F^0 \to F^1_x$,

where $d_{\Phi^*}: \tilde{F}^0 \to \tilde{F}^1$, $s \mapsto \nabla s - s \circ \nabla$ and $d_{\Phi^*}: \Theta_C(-D(t)) \to \Omega^2_{C}(D(t))$ defined by $[\Phi]$. The pairing $H^1(F^*_x) \otimes H^1(F^*_x) \to H^2(\Omega^*_x)$ defined by $[\Phi]$ gives a symplectic structure on $\mathfrak{M}_{g,n}(r, d, \nu)$.

**Definition 4.1.** Let $(C, \mathbf{t})$ be an $n$-pointed smooth projective curve of genus $g$ over $\mathbb{C}$ where $t_1, \ldots, t_n$ are distinct points. We say $(E, \mathbf{l})$ $(l = \{ l^{(i)}_1 \}_{1 \leq i \leq n})$ is a quasi-parabolic bundle of rank $r$ and of degree $d$ on $(C, \mathbf{t})$ if $E$ is a rank $r$ algebraic vector bundle of degree $d$ on $C$, and for each $t_i$, $l^{(i)}_1$ is a filtration $E|_{l_i} = l^{(i)}_0 \supset l^{(i)}_1 \supset \cdots \supset l^{(i)}_r = 0$.

Let $\mathfrak{P}_{g,n}(r, d)$ be the moduli stack of pairs $((C, \mathbf{t}), (E, \mathbf{l}))$, where $(C, \mathbf{t})$ $(\mathbf{t} = (t_1, \ldots, t_n))$ is an $n$-pointed smooth projective curve of genus $g$ over $\mathbb{C}$ where $t_1, \ldots, t_n$ are distinct points, and $(E, \mathbf{l})$ is a quasi-parabolic bundle of rank $r$ and of degree $d$ on $(C, \mathbf{t})$. We have a projection $\mathfrak{P}_{g,n}(r, d) \to \mathfrak{M}_{g,n}$. Let $\mathfrak{P}_{g,n}(r, d, \nu)$ be the substack defined by the condition where a quasi-parabolic bundle admits a $(\mathbf{t}, \nu)$-parabolic connection. Let $\pi_{\mathfrak{P}_{g,n}(r, d, \nu)}$ and $\pi_{\mathfrak{M}_{g,n}}$ be the following morphisms:

\[ \pi_{\mathfrak{P}_{g,n}(r, d, \nu)}: \mathfrak{M}_{g,n}(r, d, \nu) \to \mathfrak{P}_{g,n}(r, d, \nu); \quad ((C, \mathbf{t}, \mathbf{\psi}), (E, \mathbf{\nu}, \mathbf{\lambda})) \mapsto ((C, \mathbf{t}, \mathbf{\psi}), (E, \mathbf{\nu}, \mathbf{\lambda})) \]
\[ \pi_{\mathfrak{M}_{g,n}}: \mathfrak{P}_{g,n}(r, d, \nu) \to \mathfrak{M}_{g,n}; \quad ((C, \mathbf{t}, (E, \mathbf{l}))) \mapsto (C, \mathbf{t}). \]

Let $\Theta_{\mathfrak{P}_{g,n}(r, d, \nu)}$ be the tangent complex of $\mathfrak{P}_{g,n}(r, d, \nu)$. Let $\Theta_{\mathfrak{P}_{g,n}(r, d, \nu), p}$ be the fiber of $\Theta_{\mathfrak{P}_{g,n}(r, d, \nu)}$ over a point $p: pt \to \mathfrak{P}_{g,n}(r, d, \nu)$.

We consider infinitesimal deformations of $p = ((C, \mathbf{t}), (E, \mathbf{l}))$. We put
\[ \widetilde{H}^0_p := \{ s \in \mathcal{E}nd(E) \mid s|_{l_i}(l^{(i)}_j) \subset l^{(i)}_j \text{ for any } i, j \} \quad \text{and} \quad \widetilde{H}^1_p := \{ s \in \mathcal{E}nd(E) \otimes \Omega^2_{C}(D(t)) \mid \text{res}_i(s)(l^{(i)}_j) \subset l^{(i)}_{j+1} \text{ for any } i, j \}.
\]
Note that $(\widetilde{H}^0_p)^* \otimes \Omega^2_{C} \cong \widetilde{H}^1_p$. Put
\[ H^0_p := \{ s \in A_E(D(t)) \mid \mathcal{E}nd(E) \mid s|_{l_i}(l^{(i)}_j) \subset l^{(i)}_j \text{ for any } i, j \} \quad \text{and} \quad H^1_p := (H^0_p)^* \otimes \Omega^2_{C}.
\]
Then we have exact sequences
\[ 0 \to \widetilde{H}^0_p \to H^0_p \xrightarrow{\text{sym}_1} \Theta_C(-D(t)) \to 0 \quad \text{and} \quad 0 \to \Omega^2_{C}(D(t)) \xrightarrow{\nu} H^1_p \xrightarrow{\kappa} \widetilde{H}^1_p \to 0.
\]
We take an affine open covering $\{ U_i \}$ of $C$ so that we can take a trivialization $\phi_i: E|_{U_i} \cong \mathcal{O}^r_{U_i}$ of $E$ on each $U_i$ and the restriction of $H^1_p$ to $U_i$ is $\mathcal{O}_{U_i}$-isomorphic to the direct sum $(\mathcal{H}^1_p|_{U_i} \oplus \Omega^2_{C}(D(t)|_{U_i})$. We fix trivializations of $E$. On $U_i \cap U_j$, the transformation $(\mathcal{H}^1_p|_{U_i} \oplus \Omega^2_{C}(D(t)|_{U_i}))$ is given by
\[ (\Phi_i(f_i)|_{d_i}, \phi_i(f_i)|_{d_i} \otimes d_i) \mapsto (\Phi_j(f_j)|_{d_j}, \phi_j(f_j)|_{d_j} \otimes d_j)
\]
\[ := \left( \theta_{ij}^{-1} \Phi_i(f_i) \theta_{ij} df_i, \phi_i(f_i) df_i \otimes df_i + \text{Tr} \left( \theta_{ij}^{-1} \Phi_i(f_i) \frac{d\theta_{ij}}{df_i} \right) df_i \otimes df_i \right),
\]
where $\theta_{ij} := \phi_i \circ \phi_j^{-1} : \mathcal{O}^r_{U_i \cap U_j} \to \mathcal{O}^r_{U_i \cap U_j}$ is a transition function of $E$. Then $H^0((\mathfrak{P}_{g,n}(r, d, \nu), p)$ is isomorphic to $H^1(\mathcal{H}^1_p)$, and $H^0(\mathcal{H}^1_p)$ is the dual of $H^1(\mathcal{H}^1_p)$. The vector space $H^0(\mathcal{H}^1_p)$ is the space of 1-forms at $p$.

Put $p = ((C, \mathbf{t}), (E, \mathbf{l}))$. Let $\hat{\Phi}_p$ be an element of $H^0(\mathcal{H}^1_p)$, which is described by $(\Phi_p, \phi_p)$ locally, where $\Phi_p df \in \mathcal{H}^1_p$ and $\phi_p d \phi \otimes df \in \Omega^2_{C}(D(t))$. We consider infinitesimal deformations of $(p, \hat{\Phi}_p)$. For $\hat{\Phi}_p$, we
define a complex \( d^0(\hat{H}_p) : H^0_p \to H^1_p \) as follows. For each affine open set \( U \subset C \), we define the image of \( a_U \partial/\partial f_U + \eta_U \in H^0_p(U) \) as
\[
(\Phi_p d_U \circ \eta_U - \eta_U \circ \Phi_p d_U - \frac{\partial(a_U \Phi_p)}{\partial f_U} d_U, \\
\text{Tr} \left( \frac{\partial \eta_U}{\partial f_U} \Phi_p d_U \otimes d_U \right) - a_U \frac{\partial \phi_p}{\partial f_U} d_U \otimes d_U - 2 \frac{\partial a_U}{\partial f_U} \phi_p d_U \otimes d_U).
\]
We can show that this homomorphism on each \( U \) gives a homomorphism \( d^0(\hat{H}_p) : H^0_p \to H^1_p \). We consider the first hypercohomology \( H^1(H_p^*) \) of \( d^0(\hat{H}_p) : H^0_p \to H^1_p \). By the Čech cohomology, an element of \( H^1(H_p^*) \) is described by \([\{(a_{ij} \partial/\partial f_i + \eta_{ij})\}, \{(\hat{v}_i, \hat{w}_i)\}]\), where
\[
a_{jk} \frac{\partial f_i}{\partial f_j} - a_{ik} + a_{ij} = 0,
\]
\[
\theta^{-1}_{ji} \eta_{jk} \theta_{ji} + a_{jk} \theta^{-1}_{ji} \frac{\partial \theta_{ji}}{\partial f_j} - \eta_{ik} + \eta_{ij} = 0, \quad \text{and}
\]
\[
\theta^{-1}_{ji} \hat{v}_j \theta_{ji} \hat{w}_j + \text{Tr} \left( \theta^{-1}_{ji} \hat{v}_j \frac{\partial \theta_{ji}}{\partial f_j} df_i \right) - (\hat{v}_i, \hat{w}_i) = d^0(\hat{H}_p)(\eta_{ij})
\]
for some affine open covering \( \{U_i\}_i \) of \( C \). Infinitesimal deformations of \( (p, \hat{H}_p) \) are parametrized by \( H^1(\hat{H}_p^*) \). Then the fiber of the tangent sheaf of the moduli stack of pairs \(((C, t), (E, \hat{L})), \hat{\Phi}) \) (where \( \hat{\Phi} \in H^0(\hat{H}_p^*) \)) at a point \((p, \hat{H}_p)\) is isomorphic to \( H^1(H_p^*) \). Moreover, we define a paring \( H^1(H_p^*) \otimes H^1(H_p^*) \to H^2(\Omega^*_C) \) by
\[
[\{(a_{ij} \partial/\partial f_i + \eta_{ij})\}, \{(\hat{v}_i, \hat{w}_i)\}] \otimes [\{(a_{ij} \partial/\partial f_i + \eta'_{ij})\}, \{(\hat{v}'_i, \hat{w}'_i)\}] \\
\Rightarrow \left[ \left( \text{Tr}(\eta_{ij} (a_{jk}' \Phi_j)), \text{Tr}(a_{ji} \eta_{ij} \Phi_i) - 2 a_{ij} a_{jk}' \phi_j \right), \right. \\
- \left. \{ - \text{Tr}(\eta_{ij} \hat{v}'_j) + (a_{ji} \hat{w}'_j) - \text{Tr}(\hat{v}_i \eta'_{ij}) + (\hat{w}_i a'_{ij}) \} \right].
\]
(24)

**Proposition 4.2.** This pairing gives a symplectic structure on the moduli stack of pairs \(((C, t), (E, \hat{L})), \hat{\Phi})).

**Proof.** We define a 1-form \( \theta_{\Phi,g,n(r,d,v)} \) by
\[
\theta_{\Phi,g,n(r,d,v)} : H^1(H_p^*) \to H^1(\Omega^*_C);
\]
\[
[(a_{ij} \partial/\partial f_i + \eta_{ij})\}, \{(\hat{v}_i, \hat{w}_i)\}] \mapsto \left[ \text{Tr}(\eta_{ij} \Phi_i df_i) + a_{ij} \phi_i \left( \frac{\partial \phi_i}{\partial f_i} df_i \right) \right]
\]
for each \((\Phi, df, \phi_i, df_i \otimes df_i)\). This 1-form \( \theta_{\Phi,g,n(r,d,v)} \) is the canonical 1-form on the cotangent bundle of \( \Phi_{g,n}(r,d,v) \). Let \( d\theta_{\Phi,g,n(r,d,v)} \) be the exterior differential of \( \theta_{\Phi,g,n(r,d,v)} \). The 2-form \( d\theta_{\Phi,g,n} \) gives the symplectic form on the cotangent bundle of \( \Phi_{g,n}(r,d,v) \). We compute the 2-form \( d\theta_{\Phi,g,n} \) as follows:
\[
\text{Tr} \left( D_v(\theta_{ij}) D_v(\theta^{-1}_{ij} \Phi_i df_i) - D_v(\theta_{ij}) \theta^{-1}_{ij} D_v(\theta_{ij}) \Phi_i df_i \right) + d(a_{ij} \text{Tr}(\eta_{ij} \Phi_i)) \\
= \text{Tr} \left( - D_v(\theta_{ij}) \theta^{-1}_{ij} D_v(\theta_{ij}) \theta^{-1}_{ij} \Phi_i df_i + D_v(\theta_{ij}) \theta^{-1}_{ij} D_v(\theta_{ij}) \theta^{-1}_{ij} \Phi_i df_i \right) \\
D_v(\theta_{ij}) \theta^{-1}_{ij} D_v(\Phi_i df_i) - D_v(\theta_{ij}) \theta^{-1}_{ij} D_v(\Phi_i df_i) + d(a_{ij} \text{Tr}(\eta_{ij} \Phi_i)) \\
= \text{Tr} \left( - \eta_{ij} \left( \Phi_i df_i, \eta_{ij} \right) - d(a_{ij} \Phi_i) \right) - \eta_{ij} \hat{v}'_j + \eta_{ij} \hat{v}_i + a_{ij} \text{Tr}(d(\eta_{ij} \Phi_i)) \\
= - \text{Tr}(\eta_{ij} \hat{v}'_j) + \text{Tr}(\eta_{ij} \hat{v}_i) + a_{ij} \text{Tr}(d(\eta_{ij} \Phi_i)) \\
= \text{Tr}(\eta_{ij} \hat{v}'_j) + \text{Tr}(\eta_{ij} \hat{v}_i) - a_{ij} \text{Tr} \left( \theta^{-1}_{ij} \hat{v}'_j \frac{\partial \theta_{ij}}{\partial f_i} + a_{ij} \text{Tr}(d(\eta_{ij} \Phi_i)) \right).
\]
and
\[ D_{\nu}(\mu_{ji})D_{\nu} \left( \frac{\partial \mu_{ij}}{\partial f_j} \right) df_i - D_{\nu}(\mu_{ji})D_{\nu} \left( \frac{\partial \mu_{ij}}{\partial f_j} \right) df_i + d \left( a_{ij}^i a_{ji} \phi_j \right) \]
\[ = a_{ij} \left( -a_{ij}^i d\phi_i - 2\phi_i \frac{\partial a_{ij}^i}{\partial f_i} df_i \right) - a_{ij}^i \phi_i df_i + a_{ij}^i \phi_i df_i + a_{ij} \Tr \left( \theta_{ji}^{-1} \psi_j \frac{\partial \theta_{ji}}{\partial f_i} \right) - a_{ij} \Tr(d(\eta_{ji})\Phi_i). \]

Then we have this proposition. \( \square \)

Put \( p = ((C, t), (E, l)) \). Let \( \nabla \) be a connection: \( \nabla: E \to E \otimes \Omega^1_C(D(t)) \). For a connection \( \nabla \), we define a decomposition of \( H^0(\mathcal{H}_p) \) as follows:
\[ H^0(\mathcal{H}_p) \to H^0(\mathcal{H}_p) \oplus H^0(\Omega^{\otimes 2}_C(D(t))) \]
\[ \hat{\Phi} \mapsto (\kappa(\hat{\Phi}), \hat{\Phi} - \psi(\nabla, \kappa(\hat{\Phi}))). \]

Here \( \psi(\nabla, \kappa(\hat{\Phi})) \in H^0(\mathcal{H}_p) \) is defined as follows. We take an affine open covering \( \{ U_i \} \) of \( C \) such that on \( U_i \) the connection \( \nabla|_{U_i} \) is described by \( d + A_i df_i \) and the Higgs field \( \kappa(\hat{\Phi})|_{U_i} \) is described by \( \Phi_i df_i \). On each \( U_i \), we define an element \( \psi(\nabla, \kappa(\hat{\Phi}))[U_i] \) as
\[ \psi(\nabla, \kappa(\hat{\Phi}))[U_i] = \left( \Phi_i df_i, \Tr \left( \Phi_i A_i + \frac{1}{2} \Phi_i \Phi_i \right) df_i \otimes df_i \right) \in H^1_p(U_i), \]
which gives an element \( \psi(\nabla, \kappa(\hat{\Phi})) \in H^0(\mathcal{H}_p) \).

4.2. Moduli stack as twisted cotangent bundle. Let \( \Gamma(\pi_{\mathfrak{g}, n(r, d, \nu)}) \) be sections of \( \pi_{\mathfrak{g}, n(r, d, \nu)} \). We take a section \( \sigma \) and put \( \sigma(p) = (\nabla_p, \psi_p) \), where \( \nabla_p: E \to E \otimes \Omega^1_C(D(t)) \) is a connection such that \( (E, l, \nabla) \) is a \((t, \nu)\)-parabolic connection of rank \( r \) and of degree \( d \) on \( C \), and \( \psi_p \in H^0(C, \Omega^{\otimes 2}_C(D(t))) \) for \( p = ((C, t), (E, l)) \).

Definition 4.3. For a 1-form \( \hat{\Phi} \) on \( \mathfrak{g}, n(r, d, \nu) \), we define a translation by
\[ \Gamma(\pi_{\mathfrak{g}, n(r, d, \nu)}) \to \Gamma(\pi_{\mathfrak{g}, n(r, d, \nu)}) \]
\[ \sigma(p) = (\nabla_p, \psi_p) \mapsto \gamma^\nu \sigma(p) := (\nabla_p + \kappa(\hat{\Phi}_p), \psi_p + (\hat{\Phi}_p - \psi(\nabla_p, \kappa(\hat{\Phi}_p))). \]

By this translation \([27]\), we have an \( \Omega^1_{\mathfrak{g}, n(r, d, \nu)} \)-torsor structure on \( \Gamma(\pi_{\mathfrak{g}, n(r, d, \nu)}). \)

Theorem 4.4. Assume that \( \nu \) is generic. Let \( \omega \) be the symplectic form on \( \mathfrak{g}, n(r, d, \nu) \). We define a map \( c: \Gamma(\pi_{\mathfrak{g}, n(r, d, \nu)}) \to \Omega^{2c}_{\mathfrak{g}, n(r, d, \nu)} \) by \( c(\gamma) = \gamma^\nu(\omega) \) for \( \gamma \in \Gamma(\pi_{\mathfrak{g}, n(r, d, \nu)}). \) Then for any \( \hat{\Phi} \in \Omega^1_{\mathfrak{g}, n(r, d, \nu)} \) we have \( c(t_{\hat{\Phi}}(\gamma)) = d(\hat{\Phi}) + c(\gamma). \) That is, \((\Gamma(\pi_{\mathfrak{g}, n(r, d, \nu)}), c)\) is an \( \Omega^{2c}_{\mathfrak{g}, n(r, d, \nu)} \)-torsor.

Proof. Put \( p = ((C, t), (E, l)) \). Let \( v \) be an element of \( H^1(\mathcal{H}_p) \). We take a section \( \sigma \) and put \( \sigma(p) = (\nabla_p, \psi_p) \), where \( \nabla_p: E \to E \otimes \Omega^1_C(D(t)) \) is a connection and \( \psi_p \in H^0(C, \Omega^{\otimes 2}_C(D(t))). \)

We take an affine open covering \( \{ U_i \} \) of \( C \) such that elements of \( H^1(\mathcal{H}_p) \) are described by the Čech cohomology: \( v = \{(a_{ij} \partial f_i - a_{ij} \partial f_j + \eta_{ij}) \}, \) where \( a_{ij} \partial f_i - a_{ij} \partial f_j + \eta_{ij} \in H^1_p(U_i \cap U_j) \). Let \( \nabla_p(v, \epsilon), \hat{\Phi}_p(v, \epsilon), \) and \( \psi_p(v, \epsilon) \) be the infinitesimal deformations of \( \nabla_p, \hat{\Phi}_p \), and \( \psi_p \) associated to \( v \) over Spec \( \mathbb{C}[\epsilon] \), respectively, where \( \epsilon^2 = 0 \). We take local descriptions of the connection, the Higgs field, and the quadratic differentials on \( U_i \) as follows. The connection \( \nabla_p(v, \epsilon) \) and the Higgs field \( \kappa(\hat{\Phi}_p)(v, \epsilon) \) are described as \( d + A_i df_i + \epsilon \nu_i \mod df_i + \Phi_i df_i + \epsilon \phi_i \mod df_i \) on \( U_i \), respectively. Moreover, on \( U_i \) the quadratic differentials \( \psi_p(v, \epsilon) \) and \( (\hat{\Phi}_p - \psi(\nabla_p, \kappa(\hat{\Phi}_p)))(v, \epsilon) \) are described by \( \psi_i df_i \otimes df_i + \epsilon \nu_i df_i \otimes df_i \) and \( \phi_i df_i \otimes df_i + \epsilon \nu_i df_i \otimes df_i \mod df_i \) on \( U_i \), respectively.
We decompose $\Phi_p = \Phi_1 + \Phi_2$, where $\Phi_1 = \psi(\nabla_p, \Phi_p)$ and $\Phi_2 = \Phi_p - \psi(\nabla_p, \Phi_p)$. Then we can compute the exterior differentials of $\Phi_1$ and $\Phi_2$ as follows.

\[
d\Phi_1(v, v') = \left[ \left\{ \text{Tr}(\eta_{ij} a_{i_j}' A_j) + \text{Tr}(2a_{i_j} A_j \nabla_j + a_{i_j} a_{i_j}' A_j \nabla_j) \right\} - \left\{ \sideset{\cdots}{\text{Tr}}(\eta_{ij} a_{i_j}' A_j) + \text{Tr}(a_{i_j} A_j \nabla_j + a_{i_j} a_{i_j}' A_j \nabla_j) \right\} - \text{Tr}(\nabla_j) \right],
\]

\[
d\Phi_2(v, v') = \left[ \left\{ -2a_{i_j} a_{i_j}' \nabla_j - a_{i_j} a_{i_j}' \right\} \right].
\]

On the other hand, the symplectic form is computed as follows. Put $c(\gamma)_1 := \gamma^*(\omega_1)$ and $c(\gamma)_2 := \gamma^*(\omega_2)$, where $\omega_1$ and $\omega_2$ are defined by (19) in the proof of Proposition 3.7. We have

\[
c(t, \nabla)_1(v, v') - c(\gamma)_1(v, v') = d\Phi_1(v, v').
\]

Moreover, we also can show that $c(t, \nabla)_2(v, v') - c(\gamma)_2(v, v') = d\Phi_2(v, v')$. Then we obtain that $(\Gamma(\pi_{\mathcal{P}_n(r,d,u)}), c)$ is an $\Omega^{2\ast}_p, \mathcal{P}_n(r,d,u)$-torsor. □

4.3. Extended parabolic connections. Let $(C, t) \{ t_1, \ldots, t_n \}$ be an $n$-pointed smooth projective curve of genus $g$ over $\mathbb{C}$ where $t_1, \ldots, t_n$ are distinct points. Put $D(t) = t_1 + \cdots + t_n$. We describe a description of $(t, \nabla)$-parabolic connection with a quadratic differential in terms of a “integral kernel” on $C \times C$ as in [1] and [2]. Let $p_1: C \times C \rightarrow C$ and $p_2: C \times C \rightarrow C$ be the first and second projections, respectively. Put $O_C(\ast D(t)) := \lim_{m \rightarrow \infty} O_C(mD(t))$, and $O_C^p(\ast D(t)) := O_C^p \otimes O_C(\ast D(t))$. Let $\mathcal{E}nd^0(E) \subset \mathcal{E}nd(E)$ be the subbundle of traceless endomorphisms of $E$. We define sheaves $K_{D(t)}$ on $C \times C$ as $K_{D(t)}(E) := p_1^*(E \otimes O_C^p(\ast D(t))) \otimes p_2^*(E^* \otimes O_C^p(\ast D(t)))$, where $\Delta \subset C \times C$ is the diagonal. We have a natural injective morphism $C \otimes (\ast D(t)) \otimes \mathcal{E}nd^0(E) \rightarrow K_{D(t)}(E)|_{\Delta}$. We define a sheaf $\mathcal{E}xConn_{D(t)}(E)$ on $3\Delta$ by

\[
\mathcal{E}xConn_{D(t)}(E) = \{ s \in K_{D(t)}(E)|_{\Delta}/(\Omega_C^p(\ast D(t)) \otimes \mathcal{E}nd^0(E)) \mid s|_{\Delta} = \text{Id}_E \}.
\]

Note that we can consider $s|_{\Delta}$ as a connection $s|_{\Delta} : E \rightarrow E \otimes O_C^p(\ast D(t))$. We consider elements of $\mathcal{E}xConn_{D(t)}(E)$ as pairs of connections and quadratic differentials on $C$ locally. Let $U_1$ and $U_2$ be open sets of $C$. Let $(\alpha_i, a_i)$ be an elements of $\mathcal{E}xConn_{D(t)}(E)$ on $U_i$. Here $A_i df_i$ is a connection matrix on $U_i$ and $a_i df_i \otimes df_i \in H^0(U_i, \Omega_C^p(\ast D(t)))$. The transformation of the pair is the following

$\left(A_i df_i, a_i df_i \otimes df_i \right) \mapsto \left( \theta_{ij}^{-1} A_i \theta_{ij} df_i + \theta_{ij}^{-1} d\theta_{ij} \theta_{ij} df_i, \alpha_i + \text{Tr} \left( \theta_{ij}^{-1} A_i \frac{d\theta_{ij}}{df_i} \right) + \frac{1}{2} \text{Tr} \left( \theta_{ij}^{-1} d^2 \theta_{ij} \frac{d\theta_{ij}}{df_i} \right) \right) \text{df}_i \otimes \text{df}_i$ on $U_i \cap U_j$. We may define an $\mathcal{H}_{C,A,E,l}$-action on $\mathcal{E}xConn_{D(t)}(E)$ for any parabolic structures $l$ by [23].

Let $\nabla : E \rightarrow E \otimes \Omega_C^p(\ast D(t))$ be a connection. We can define a global section $\nabla_{Ex} of \mathcal{E}xConn_{D(t)}(E)$ associated to $\nabla$ as follows. Take a trivialization of the locally free sheaf $E$ on an open set $U_i$ of $C$. Let $A_i df_i$ be the connection matrix of $\nabla$ on $U_i$. We define $\nabla_{Ex}|_{U_i} \in \mathcal{E}xConn_{D(t)}(E)(U_i)$ as

\[
\left( A_i df_i, \frac{1}{2} \text{Tr} \left( d(A_i) \otimes df_i \right) + \frac{1}{2} \text{Tr} \left( A_i df_i \otimes A_i df_i \right) \right).
\]
where $d$ is the exterior derivative. Let $((E, \nabla, \{l_{ij}^{(t)}\}), \psi)$ be a $(t, \nu)$-parabolic connection with a quadratic differential. For $(\nabla, \psi)$, we can define a global section $\nabla_{\text{Ex}} + \psi$ of $\mathcal{E}x\text{Conn}_D(t)(E)$ by the above construction. We call this description $((E, \nabla_{\text{Ex}} + \psi, \{l_{ij}^{(t)}\})$ a $(t, \nu)$-extended parabolic connection.

By the identification a parabolic connection with a quadratic differential as an extended parabolic connection, we obtain another $\Omega^1_{g,n}(\mathcal{P}, \nu, \psi)$-torsor structure on $\Gamma(\pi_{\mathcal{P},n}(r,d,\nu))$. We take a section $\theta$ of $\pi_{\mathcal{P},n}(r,d,\nu)$ and put $\sigma(p) = (\nabla_p, \psi_p)$, where $\nabla_p : E \rightarrow E \otimes \Omega^1_{g,n}(D(t))$ is a connection and $\psi_p \in H^0(C, \Omega^1_{\mathcal{C}}(D(t)))$ for $p = ((C, t), (E, \mathcal{I}))$. For a connection $\nabla_p : E \rightarrow E \otimes \Omega^1_{\mathcal{C}}(D(t))$ and $\Phi_p \in H^0(H^1)$, we define $\psi(\nabla_p, \Phi_p) \in H^0(H^1)$ as follows. We take an affine open covering $\{U_i\}$ of $C$ such that on $U_i$ the connection $\nabla_p|_{U_i}$ is described by $d + A_i df_i$ and the Higgs field $\Phi_p|_{U_i}$ is described by $\Phi_i df_i$. On each $U_i$, we define an element $\psi(\nabla_p, \Phi_p)|_{U_i}$ as

$$\psi(\nabla_p, \Phi_p)|_{U_i} = \left(\Phi_i df_i, \text{Tr} \left( \Phi_i A_i df_i \otimes df_i + \frac{1}{2} \Phi_i \Phi_i df_i \otimes df_i + \frac{1}{2} d(\Phi_i) \otimes df_i \right) \right) \in \mathcal{F}_{\mathcal{P}}^1(U_i),$$

which gives an element $\psi(\nabla_p, \Phi_p) \in H^0(H^1)$. For a 1-form $\Phi$ on $\pi_{\mathcal{P},n}(r,d,\nu)$, we define a translation by $t'$, $\Gamma(\pi_{\mathcal{P},n}(r,d,\nu)) \rightarrow \Gamma(\pi_{\mathcal{P},n}(r,d,\nu))$

$$\sigma(p) = (\nabla_p, \psi_p) \mapsto t'(\Phi)(p) := (\nabla_p + \kappa(\Phi_p), \psi_p - \psi(\nabla_p, \kappa(\Phi_p))).$$

By the translation, we have another $\Omega^1_{g,n}(\mathcal{P}, \nu, \psi)$-torsor structure on $\Gamma(\pi_{\mathcal{P},n}(r,d,\nu))$.

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