EULER SYSTEMS FOR $\text{GSp}_4 \times \text{GL}_2$

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Abstract. For a non-endoscopic cohomological cuspidal automorphic representation of $\text{GSp}_4 \times \text{GL}_2$, assumed to be $p$-ordinary, we construct an Euler system for the Galois representation associated to it. Both the construction and the verification of tame norm relations are based on Novodvorsky’s integral formula for the $L$-function of $\text{GSp}_4 \times \text{GL}_2$.

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1. Introduction

1.1. Recent progress in Euler systems. The theory of Euler systems is one of the most powerful tools available for studying the arithmetic of Galois representations, and especially for controlling Selmer groups. It was first introduced by Kolyvagin [Kol90], inspired by works of Thaine [Tha88] and his own [Kol88], and later formalized by Rubin [Rub00]. However, the construction of Euler systems is a difficult problem, and there are relatively few known examples of Euler systems.

A common strategy for constructing Euler systems is pushing forward cohomology classes on certain special cycles of Shimura varieties of varying levels. The prototypical example is Kolyvagin’s Euler system of Heegner points [Kol88], which uses codimension one cycles, the Heegner points, on modular curves, although one should note that this is not exactly an Euler system in the sense of [Rub00], but rather a so-called “anticyclotomic” Euler system. Another related construction is Kato’s Euler system [Kat04], which utilizes the cup product of Siegel units on modular curves.

Recently a series of examples have been constructed using this technique, including the work of [LLZ14] (embedding of Shimura varieties corresponding to $GL_2 \hookrightarrow GL_2 \times GL_2$, $LLZ18$ ($GL_2 \hookrightarrow \text{Res}_{F/\mathbb{Q}} GL_2$ with $F$ a real quadratic field), and $LSZ19$ ($GL_2 \times GL_2 \hookrightarrow GSp_4$). The common starting point for all these recent constructions is Siegel units on modular curves. Pushing forward these cohomology classes via the corresponding closed embedding of Shimura varieties, one gets motivic cohomology classes in the target Shimura variety. The embeddings are suitably “twisted” in order to get the desired levels in the Shimura variety, which can be then realized as one base Shimura variety base-changed to various cyclotomic extensions of $\mathbb{Q}$. One then considers the étale realization of such motivic cohomology classes to find elements in étale cohomology. Finally, an application of the Hochschild–Serre spectral sequence yields Galois cohomology classes satisfying the desired norm relations.

In [LLZ14] and [LLZ18], the proof of the norm relations involves explicit yet laborious double coset computations. In [LSZ19], a novel approach using a multiplicity-one argument in local representation theory is adopted, bypassing the complicated calculations on $GSp_4$ that would have been practically intractable. More specifically, Loeffler, Skinner and Zerbes use methods of smooth representation theory to reduce the tame norm-compatibility statement to a far easier, purely local statement involving Bessel models of unramified representations of $GSp_4(\mathbb{Q}_\ell)$. This reduction is possible thanks to a case of the local Gan–Gross–Prasad conjecture due to Kato, Murase and Sugano [KMS03], showing that the space of $SO_4(\mathbb{Q}_\ell)$-invariant linear functionals on an irreducible spherical representation of $SO_4(\mathbb{Q}_\ell) \times SO_5(\mathbb{Q}_\ell)$ has dimension as most one. Note that $SO_4$ and $SO_5$ are the projectivization of $GL_2 \times GL_2$, $GL_2$ and $GSp_4$, respectively. This technique promises to be applicable in other settings where local multiplicity one results of this type are known. In [Gro20], the norm relations for the Euler system of Hilbert modular surfaces as first constructed in [LLZ18] is improved and reproved using this type of multiplicity one arguments.

Exactly the same multiplicity one result, between $SO_4$ and $SO_5$, can also be applied in our setting of the natural embedding $GL_2 \times GL_2 \hookrightarrow GSp_4 \times GL_2$, as suggested in [LSZ19] and [LZ18].

1.2. Outline of main results and the proofs. Let $G = GSp_4 \times GL_2$. Let $\Pi = \Pi_f \otimes \Pi_{\infty}$ be a cuspidal automorphic representation of $G(\mathbb{A}_f)$ with $\Pi_{\infty}$ a unitary discrete series of weight $(k_1, k_2, k)$ with $k_1 \geq k_2 \geq 3$, $k \geq 2$. Let $K \subset G(\mathbb{A}_f)$ be the level of $\Pi$, and $\Sigma$ be the set of primes at which $K$ is not hyperspecial. Let $W_{\Pi}$ be the $p$-adic Galois representation associated to $\Pi$, namely for each $\ell \notin \Sigma \cup \{p\}$, $P_{\ell}(\ell^{-s}) = L(\Pi_{\ell}, s - \frac{k_1 + k_2 + k - 4}{2})^{-1}$,
where $P_t(X) = \det(1 - \text{Frob}_t^{-1} \cdot X | W_{\Pi})$ is the characteristic polynomial of the geometric Frobenius $\text{Frob}_t^{-1}$ on $W_{\Pi}$.

We state our main theorem, which is a combination of Definition 8.15 and Proposition 8.16, 8.17.

**Main Theorem.** Assume that $\Pi$ is non-endoscopic. Assume that $\Pi$ is unramified and (Borel) ordinary at $p$. Fix an integer $e > 1$ prime to $\Sigma \cup \{2, 3, p\}$. Let $r$ be an integer such that $\max(0, 1 - k_2 + k) \leq r \leq \min(k_1 - k_2, k - 2)$. Then there exists a Galois-stable lattice $T_{\Pi}^* \subset W_{\Pi}^*$ so that an Euler system for $T_{\Pi}^*(2 - k_2 - r)$ exists. More precisely, for each square-free integer $M \geq 1$ prime to $\Sigma \cup \{p\}$ and an integer $m \geq 0$, there exists $e_{\ell, m}^{[\Pi, r]} \in H^1(\mathbb{Q}(\mu_{\ell M^m}), T_{\Pi}^*(2 - k_2 - r))$ satisfying

\[
\text{cores}_{e_{\ell, m}^{[\Pi, r]}} e_{\ell M^m}^{[\Pi, r]} = e_{\ell M^m}^{[\Pi, r]},
\]

(tame norm relation) cores $e_{\ell, m}^{[\Pi, r]} e_{\ell M^m}^{[\Pi, r]} = P_t(1 - k_2 - r e^{-1}) \cdot e_{\ell M^m}^{[\Pi, r]}.$

where $\sigma_e$ is the arithmetic Frobenius of $\ell$ in $\text{Gal}(\mathbb{Q}(\mu_{\ell M^m}/\mathbb{Q}))$.

We give a brief overview of how the Euler system classes $e_{\ell, m}^{[\Pi, r]}$ are constructed, and how to show the norm relations. The method is similar to [LSZ19].

**Construction of Euler system classes.** Let $H = \text{GL}_2 \times_{\text{GL}_1} \text{GL}_2$ and $G = \text{GSp}_4 \times_{\text{GL}_1} \text{GL}_2$. For each open compact $U \subset G(\mathbb{A}_f)$, let $Y_G(U)$ denote the Shimura variety for $G$ of level $U$. For each of certain parameters $(a, b, c, r) \in \mathbb{Z}^4$, we will construct a $G(\mathbb{A}_f)$-equivariant map, the symbol map (Section 5.4)

\[\text{Symbl}: S(\mathbb{A}_f^2) \otimes_{\mathbb{H}(H(\mathbb{A}_f)))} \mathbb{H}(G(\mathbb{A}_f)) \rightarrow \varinjlim U H^5_{\text{mot}}(Y_G(U), \mathbb{Q}).\]

Here $S(\mathbb{A}_f^2)$ is the space of Schwartz functions on $\mathbb{A}_f^2$, regarded as an $H(\mathbb{A}_f)$-representation via the projection to its first factor $H \rightarrow \text{GL}_2$, $\mathbb{H}(\mathbb{Q})$ denotes the Hecke algebra, $\mathbb{D}$ is some relative Chow motive over $Y_G$ arising from algebraic representations of $G$. The construction of Symbl records how to construct Euler system classes by pushing forward Eisenstein classes as mentioned above: The space of Schwartz functions $S(\mathbb{A}_f^2)$ parametrizes Eisenstein classes, and the Hecke algebra $\mathbb{H}(G(\mathbb{A}_f))$ gives the freedom to twist the pushforward of Eisenstein classes.

Let $K \subset G(\mathbb{A}_f)$ be a level subgroup, unramified outside $p$ and a finite set $\Sigma \not= \{p\}$ of primes. For any integer $n$ coprime to $\Sigma$, the base-extension $Y_G(K) \times_{\text{Spec} \mathbb{Q}} \text{Spec} \mathbb{Q}(\mu_n)$ is again a Shimura variety for $G$ of some level smaller than $K$. This identification, along with explicit choices of $K_p \subset G(\mathbb{Q}_p)$ and test data $\phi_{\ell^M^m, n} \in S(\mathbb{A}_f^2), \xi_{\ell^M^m, n} \in \mathbb{H}(G(\mathbb{A}_f))$ as input to Symbl, we can define the motivic Euler system classes (Definition 8.1)

\[z_{\ell^M^m} \in H^5_{\text{mot}}(Y_G(K) \times_{\text{Spec} \mathbb{Q}} \text{Spec} \mathbb{Q}(\mu_{\ell^M^m}), \mathbb{D}),\]

for $M \geq 1$ a square-free integer coprime to $(p, \Sigma)$ and an integer $m \geq 0$.

To get an Euler system associated to the Galois representation $W_{\Pi}^*$, note that $\Pi_f^* \otimes W_{\Pi}^*$ appears with multiplicity 1 as a direct summand of $\varinjlim_U H^4_{\text{et}}(Y_G(U), \mathbb{Q}),$ and our conditions on $\Pi$ would ensure it does not contribute to cohomology outside degree 4. Choosing a vector $\varphi \in \Pi_f$ thus results in a homomorphism of Galois representations $\Pi_f^* \otimes W_{\Pi}^* \rightarrow W_{\Pi}^*$, which factors through $(\Pi_f^*)^K \otimes W_{\Pi}^*$ if $\varphi$ is $K$-invariant. This, combined with the Hochschild–Serre spectral sequence, gives a map

\[H^4_{\text{et}}(Y_G(K) \times_{\text{Spec} \mathbb{Q}} \text{Spec} \mathbb{Q}(\mu_{\ell^M^m}), \mathbb{D}) \rightarrow H^1(\mathbb{Q}(\mu_{\ell^M^m}), W_{\Pi}^*). \]

The images of the étale realizations of $z_{\ell^M^m}$ give a collection of cohomology classes $z_{\ell^M^m}^{[\Pi, r]} \in H^1(\mathbb{Q}(\mu_{\ell^M^m}), W_{\Pi}^*)$, and we will prove that they satisfy the norm relations of an Euler system. If we carefully trace the parameters $(a, b, c, r) \in \mathbb{Z}^4$ and take care of integrality of the Euler system classes (which depends on an auxiliary choice of integer $e$), we will construct $e_{\ell, m}^{[\Pi, r]}$ as in the Main Theorem.

**Norm relations.** We will focus on the tame norm relations. The wild norm relations can be dealt with in a similar fashion and the lack of Euler factor makes it easier. We have seen that the symbol map induces a $G(\mathbb{A}_f)$-equivariant bilinear pairing

\[3: S(\mathbb{A}_f^2) \otimes_{\mathbb{H}(H(\mathbb{A}_f)))} \mathbb{H}(G(\mathbb{A}_f)) \otimes \Pi_f \rightarrow H^1(\mathbb{Q}(\mu_{\ell^M^m}), W_{\Pi}^*), \]
Via Frobenius reciprocity, \( \mathfrak{f} \) corresponds to an \( H(\mathbb{A}_f) \)-equivariant pairing

\[
\mathfrak{f}: S(\mathbb{A}_{f}^2) \otimes \Pi_f \to H^1(\mathbb{Q}(\mu_{MP^{\infty}}), W^*_1).
\]

As the symbol map is constructed out of the Eisenstein class map, it can be shown that it suffices to study the \( \mathfrak{f} \) and \( \mathfrak{f} \) replacing \( S(\mathbb{A}_{f}^2) \) by principal series \( \tau \) of \( \text{GL}_2 \). Note that the classes \( z^H_{MP^{\infty}} \) and \( \text{cores}^{Q}_{\mathbb{Q}(\mu_{MP^{\infty}})}(z^H_{MP^{\infty}}) \) are given by images of the \( \mathfrak{f} \), with different choices of test data only differing at \( \ell \). Hence it suffices to study \( \mathfrak{f}_\ell \), or equivalently \( \mathfrak{f}_\ell \). Therefore we are in the situation of comparing the values of different local test data under the \( H(\mathbb{Q}_\ell) \)-equivariant pairing

\[
\mathfrak{f}_\ell: \tau_\ell \otimes \Pi_\ell \to \mathbb{C}.
\]

By a known case of the Gan–Gross–Prasad conjecture due to [KMS03], the space of such \( H(\mathbb{Q}_\ell) \)-equivariant maps is 1-dimensional, and for each principal series \( \tau_\ell = I(\chi, \psi) \) we may construct an explicit basis element \( \mathfrak{f}_{\ell, \psi} \) using zeta integrals (Proposition 7.18). Thus the desired tame norm relations are reduced to explicit computations of local zeta integrals (Section 7.2).

**Choice of local test data.** For the tame norm relation to actually hold, we need to choose two suitable local test data in \( \tau_\ell \otimes \Pi_\ell \) so that the their image under \( \mathfrak{f}_{\ell, \psi} \) differ by a ratio of \( L(0, \Pi_\ell)^{-1} \). The test vector in \( \tau_\ell \) will be some Siegel sections, which is fairly standard. It is the test vectors in \( \Pi_\ell \) that will play an important role. Our \( \Pi_\ell \) will be assumed to be generic. Let \( \varphi_0 \) be the normalized spherical vector of \( \Pi_\ell \). The local zeta integral at \( \varphi_0 \) can be rewritten as a double sum of Whittaker functions at certain values, and the Casselman–Shalika formula says the double sum would give the local \( L \)-factor \( L(0, \Pi_\ell) \). We can similarly write down the local zeta integral at Hecke translates of \( \varphi_0 \) as (different) double sums of Whittaker functions. Playing with the double sums, we can find a correct Hecke translate of \( \varphi_0 \) at which the local zeta integral is 1.

1.3. **Organization.** Finally we give a summary of the the contents of the paper. Section 3 to 6 are largely background knowledge. In Section 3 we recall algebraic representation theory; in particular we investigate the branching law: how irreducible representations of \( G \) decomposes as irreducible representations of \( H \). In Section 4 we fix notations for Shimura varieties, construct coefficient sheaves from the algebraic representations, and recall Eisenstein classes on modular curves. In Section 5, we construct the symbol map according to the integral formula for \( L \)-functions of \( G \). In Section 6, we recall local representation theory, in preparation to study the local symbol maps. Section 7 is the core of the paper: Section 7.2, 7.4 and 7.5 contains the key computation needed for tame norm relation, integrality, and wild norm relation for Euler system classes. In Section 8, we assemble the results to construct Euler system elements and prove their norm relations.

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2. **General Notations**

- Let \( \widehat{\mathbb{Z}} := \lim_{\rightarrow} \mathbb{Z}/n\mathbb{Z} \) be the profinite completion of \( \mathbb{Z} \) and \( \mathbb{A}_f := \widehat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q} \) be the ring of the finite adèles of \( \mathbb{Q} \). We fix an embedding \( \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_\ell \) for any prime \( \ell \).

- Let \( J \) be the skew-symmetric \( 4 \times 4 \) matrix over \( \mathbb{Z} \) given by

\[
\begin{pmatrix}
1 & 1 \\
0 & -1 \\
-1 & 0 \\
0 & 1
\end{pmatrix}.
\]

Let \( \text{GSp}_4 \) be the group scheme over \( \mathbb{Z} \) defined by

\[
\text{GSp}_4(R) := \{(g, \mu) \in \text{GL}_4(R) \times \text{GL}_4(R) : g^t \cdot J \cdot g = \mu J\}
\]

for any commutative unital ring \( R \). Write \( \mu : \text{GSp}_4 \to \text{GL}_4 \) for the symplectic multiplier map.
3. Algebraic Representations and Branching Laws

3.1. Algebraic representations. We first recall the representations of $G$: GL$_2$. Write $t_1$ and $t_2$ for the characters of the diagonal torus of GL$_2$, given by projection onto the two diagonal entries. Note that $t_1 + t_2$ is the determinant $\det$ restricting to the torus and that $\{t_1, t_2\}$ is a basis of the character group of the torus. For an integer $k \geq 0$, we denote by $\text{Sym}^k$ the $k$-th symmetric power of the standard representation of GL$_2$. Note that $\text{Sym}^k$ is the unique (up to isomorphism) algebraic representation of GL$_2$ with highest weight $kt_1$. Here the positivity of roots is defined with respect to the Borel subgroup of upper triangular matrices of GL$_2$. It has dimension $k + 1$ and the central character $x \mapsto x^k$. Also, the dual representation satisfies

$$(\text{Sym}^k) \cong \text{Sym}^k \otimes \det^{-k}.$$ 

Now we turn to the representations of GSp$_4$. Write $\chi_1, \ldots, \chi_4$ for the characters of the diagonal torus of GSp$_4$ given by projection onto the four diagonal entries. Note that $\chi_1 + \chi_4 = \chi_2 + \chi_3$ is the symplectic multiplier $\mu$ restricting to the torus and that $\{\chi_1, \chi_2, \mu\}$ is a basis of the character group of the torus. For integers $a, b \geq 0$, we denote by $V^{a,b}$ the unique (up to isomorphism) irreducible algebraic representation of GSp$_4$ whose highest weight, with respect to the Borel subgroup of upper triangular matrices of GSp$_4$, is the character $(a + b)\chi_1 + a\chi_2$. The representation $V^{a,b}$ has dimension $\frac{1}{6}(a + 1)(b + 1)(a + b + 2)(2a + b + 3)$ and the central character $x \mapsto x^{2a+b}$. The dual representation satisfies

$$(V^{a,b})^* \cong V^{a,b} \otimes \mu^{-2a-b}.$$ 

We may thus deduce the representation theory of $G = \text{GSp}_4 \times_{\text{GL}_1} \text{GL}_2$ from the above. For any integers $a, b, c \geq 0$, set

$$W^{a,b,c} := V^{a,b} \boxtimes \text{Sym}^c.$$ 

This is an irreducible algebraic representations of $G$ with highest weight $(a + b)\chi_1 + b\chi_2 + ct_1$.

3.2. Branching laws. We are interested in $\iota^*(W^{a,b,c})$, the restriction of $W^{a,b,c}$ to $H$ via the embedding $\iota : H \hookrightarrow G$. We have the following branching law.

Proposition 3.1. For any integers $a, b, c \geq 0$, we have

$$\iota^*(W^{a,b,c}) = \bigoplus_{q=0}^a \bigoplus_{r=0}^b \bigoplus_{j=0}^{\min(a-q+r,c)} \text{Sym}^{a+b-q-r} \boxtimes \text{Sym}^{a-q+r+c-2j} \otimes \mu^j q.$$ 

Proof. We first observe that the embedding $\iota : H \hookrightarrow G$ factors as

$$H = \text{GL}_2 \times_{\text{GL}_1} \text{GL}_2 \overset{(\text{id}_{\text{GL}_2} \circ \iota_1)}\longrightarrow \text{GL}_2 \times_{\text{GL}_1} \text{GL}_2 \times_{\text{GL}_1} \text{GL}_2 \overset{(\iota_2 \circ \text{id}_{\text{GL}_2})}{\longrightarrow} \text{GSp}_4 \times_{\text{GL}_1} \text{GL}_2 = G,$$

where $\iota_1 : \text{GL}_2 \hookrightarrow \text{GL}_2 \times_{\text{GL}_1} \text{GL}_2$ is the diagonal embedding and $\iota_2 = \text{pr}_1 \circ \iota : \text{GL}_2 \times_{\text{GL}_1} \text{GL}_2 \hookrightarrow \text{GSp}_4$. The branching laws for the embeddings $\iota_1$ and $\iota_2$ are known; the former is the Clebsch–Gordan formula

$$\iota_1^*(\text{Sym}^k \boxtimes \text{Sym}^l) = \bigoplus_{j=0}^{\min(k, l)} \text{Sym}^{k+l-2j} \otimes \mu^j,$$

and the latter is

$$\iota_2^*(V^{a,b}) = \bigoplus_{q=0}^a \bigoplus_{r=0}^b \text{Sym}^{a+b-q-r} \boxtimes \text{Sym}^{a-q+r} \otimes \mu^q.$$ 

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we can define the inequality $c$ for the irreducible $\pi$ which is the pullback from the first factor of a $GL_1$ generated by $e$. Moreover, if $r \leq a + b$, then such $H$-subrepresentations are exactly

$$\left( Sym^{b+c-2r} \otimes \mu^{a+r} \right) \boxtimes 1,$$

where $r$ is an integer such that $\max(0, -a + c) \leq r \leq \min(b, c)$.

**Proof.** According to Proposition 3.1, the representation $\nu^*(W^{a, b, c})$ has an irreducible $H$-subrepresentation which is the pullback from the first factor of a $GL_2$-irreducible representation equivalent to that there exists non-negative integers $q \leq a$, $r \leq b$, and $j \leq \min(a - q + r, c)$ with $a - q + r + c - 2j = 0$. By the condition on $j$, the equality $a - q + r + c - 2j = 0$ implies that $a - q + r = c = j$. Since $0 \leq q \leq a$ and $r \leq b$, the inequality $c \leq a + b$ follows from the equality $a - q + r = c$.

If $c \leq a + b$, $0 \leq q \leq a$, $0 \leq r \leq b$, and $a - q + r = c$, then Proposition 3.1 says that the representation $\nu^*(W^{a, b, c})$ has the irreducible $H$-subrepresentation

$$Sym^{a+b-q-r} \boxtimes 1 \otimes \mu^{a+r} = (Sym^{b+c-2r} \otimes \mu^{a+r}) \boxtimes 1.$$

Moreover, since $r = -a + c + q$ and $0 \leq q \leq a$, we have $-a + c \leq r \leq c$. Combining the condition $0 \leq r \leq b$, we obtain the desired range of $r$.

Following Corollary 3.2, we will now show that there is actually a canonical injective homomorphism of $H$-representations

$$br^{[a, b, c, r]} : (Sym^{b+c-2r} \otimes \det^{a+r}) \boxtimes 1 \rightarrow \nu^*(W^{a, b, c}),$$

which will be referred to as the *branching maps*.

We fix the following choices
- $u \in V^{1,0}$ with weight $\chi_1 + \chi_2$, a highest weight vector,
- $\nu' \in V^{1,0}$ with weight $\mu$,
- $v \in V^{0,1}$ with weight $\chi_1$, a highest weight vector,
- $\nu' \in V^{0,1}$ with weight $\chi_2$,
- $w \in Sym^1$ with weight $t_1$, a highest weight vector,
- $w' \in Sym^1$ with weight $t_2$.

In fact, $V^{0,1}$ is isomorphic to the standard representation of $GSp_4 \subseteq GL_4$, so we fix an isomorphism and choose $v$ to be the vector corresponding to $e_1$, where $e_1, \ldots, e_4$ is the standard basis for the standard representation. Under this isomorphism, we choose $\nu'$ to be $e_2$. Next, we identify $V^{1,0}$ with the 5-dimensional irreducible subrepresentation of $\bigwedge^2 V^{0,1}$, and choose $u$ to be with $v \wedge v'$, or $e_1 \wedge e_2$. Then $w'$ is $e_1 \wedge e_4 - e_2 \wedge e_3$.

Finally, $Sym^1$ is the standard representation of $GL_2$. We take the representation space to be the subspace generated by $e_2, e_3$, and choose $w$ to be $e_2$. Then $w'$ is $e_3$. Note that the choice of $v$ determines all the other five vectors.

Now we can define

$$v^{a, b, c, r} := u^{c-r} \cdot v^{b-r} \cdot (u')^{a-c+r} \cdot (v')^r \cdot (w')^c.$$  

It is easily seen that $v^{a, b, c, r}$ lies in $W^{a, b, c}$. In addition, it has $G$-highest weight $(b+c-2r)\chi_1 + c\chi_2 - c\chi_1 + (a+r)\mu$ and hence $H$-highest weight $(b+c-2r)t_1 \boxtimes 1 + (a+r)\mu$. This means that $v^{a, b, c, r}$ is a highest weight vector for the irreducible $H$-subrepresentation of $\nu^*(W^{a, b, c})$ which is isomorphic to $(Sym^{b+c-2r} \otimes \mu^{a+r}) \Box 1$. Hence we can define

$$br^{[a, b, c, r]} : (Sym^{b+c-2r} \otimes \mu^{a+r}) \Box 1 \rightarrow \nu^*(W^{a, b, c}).$$

(see [LSZ19, Proposition 4.3.1]). Hence $\nu^*(W^{a, b, c})$ is equal to

\[
(i_1, \text{id}_{GL_2})^* \circ (\text{id}_{GL_2}, t_2)^* (W^{a, b, c}) = \bigoplus_{q=0}^{a} \bigoplus_{r=0}^{b} Sym^{a+b-q-r} \boxtimes \nu_1^* (Sym^{a-r} \boxtimes Sym^c) \otimes \mu^q
\]

\[
= \bigoplus_{q=0}^{a} \bigoplus_{r=0}^{\min(a-q+r,c)} Sym^{a+b-q-r} \boxtimes Sym^{a-q+r+c-2j} \otimes \mu^{q+j}.
\]
by sending the highest weight vector \((w^b+c−2r \otimes (w \wedge w')^{a+r}) \boxtimes 1\) to \(v^{a,b,c,r}\). Note that changing our fixed choice of highest weight vector \(v\) to a scalar multiple \(c \cdot v\) only changes the the two highest weight vectors by the same scalar \(c^{a+b+c}\), and thus the branching map \(br^{[a,b,c,r]}\) just defined is independent of the choice of \(u, u', v, v', w,\) and \(w'\).

3.3. Integrality. In this subsection, we discuss the integrality of branching maps. Let \(\lambda\) be a dominant integral weight of \(G\), \(W_\lambda\) the corresponding representation, and \(w_\lambda \in W_\lambda\) a highest weight vector of \(W_\lambda\). The pair \((W_\lambda, w_\lambda)\) is unique up to a unique isomorphism.

An **admissible lattice** in \(W_\lambda\) is a \(\mathbb{Z}\)-lattice \(L\) such that

- the homomorphism \(G \to GL(W_\lambda)\) extends to a homomorphism \(G \to GL(L)\) of group schemes over \(\mathbb{Z}\),
- the intersection of \(L\) with the highest weight space of \(W_\lambda\) is \(\mathbb{Z}w_\lambda\).

It is known that there are finitely many admissible lattices. Let \(W_{\lambda,Z}\) be the maximal admissible lattice. For two dominant integral weights \(\lambda\) and \(\lambda'\) of \(G\), the image of \(W_{\lambda,Z} \otimes W_{\lambda',Z}\) under the natural \(G\)-homomorphism

\[W_\lambda \otimes W_{\lambda'} \to W_{\lambda+\lambda'}\]

is an admissible lattice, so the map exist integrally

\[W_{\lambda,Z} \otimes W_{\lambda',Z} \to W_{\lambda+\lambda',Z} \]

Now we prove that the branching maps also exist integrally.

**Proposition 3.3.** Let \(a, b,\) and \(c\) be non-negative integers with \(c \leq a+b\). For any integer \(r\) with \(\max(0, -a+c) \leq r \leq \min(b, c)\), the branching map \(br^{[a,b,c,r]}\) induces a morphism of integral representations

\[br^{[a,b,c,r]}: (TSym^{b+c−2r} \otimes \mu^{a+r}) \boxtimes 1 \to \iota^*(W_{Z}^{a,b,c}),\]

where \(TSym^{b+c−2r}\), the symmetric tensors, is the minimal admissible lattice in \(Sym^{b+c−2r}\).

**Proof.** First of all, \(L := (br^{[a,b,c,r]})^{-1}(W_{Z}^{a,b,c})\) is a lattice of \(Sym^{b+c−2r} \otimes \mu^{a+r}\) stable under the \(H\)-action. Also by our construction of branching maps, \(L\) contains the highest weight vector \(w^{b+c−2r} \otimes (w \wedge w')^{a+r} \boxtimes 1\). Hence \(L\) is an admissible lattice of \(Sym^{b+c−2r} \otimes \mu^{a+r}\) and contains the minimal admissible lattice. \(\square\)

4. Cohomology and Eisenstein Classes

4.1. Modular varieties. We introduce some notations for modular varieties. For \(K^{GL_2} \subset GL_2(\mathbb{A}_f)\) a open compact subgroup, We let \(Y_{GL_2}(K^{GL_2})\) denote the modular curve over \(\mathbb{Q}\) of level \(K\). It has \(\mathbb{C}\)-points

\[Y_{GL_2}(K^{GL_2})(\mathbb{C}) = GL_2(\mathbb{Q}) \backslash H^\pm \times GL_2(\mathbb{A}_f)/K^{GL_2}\]

where \(H^\pm\) denotes the union of the upper and lower half planes in \(\mathbb{C}\). Also let \(Y_{GL_2} := \lim_{\longrightarrow}^{KL_2} Y_{GL_2}(K^{GL_2})\). Similarly, we denote by \(Y_H\) and \(Y_G\) the canonical model over \(\mathbb{Q}\) for the Shimura varieties of \(H\) and \(G\), respectively.

For any open compact subgroup \(K \subset G(\mathbb{A}_f)\), the embedding \(\iota: H \hookrightarrow G\) induces a morphism of varieties over \(\mathbb{Q}\)

\[\iota: Y_H(K \cap H(\mathbb{A}_f)) \to Y_G(K)\]

4.2. **Coefficient sheaves**. In this section, let \(a, b, c, r \geq 0\) be integers with \(c \leq a + b\) and \(\max(0, -a+c) \leq r \leq \min(b, c)\).

By [Anc15, Théorème 8.6] (see [LSZ19, Section 6]), the algebraic \(G\)-representation \(W^{a,b,c}_{\mathbb{Q}}\) over \(\mathbb{Q}\) induces a \(G(\mathbb{A}_f)\)-equivariant relative Chow motive \(\mathcal{M}^{a,b,c}_{\mathbb{Q}}\) over \(Y_G\). We also have a \(GL_2(\mathbb{A}_f)\)-equivariant relative Chow motive \(\mathcal{K}_Q^k\) over \(Y_{GL_2}\) associated to the representation \((\mu^{k})^* = Sym^k \otimes \det^{-k}\) of \(GL_2\) (Note the dual.) Then, by the work of Torzewski [Tor20], the branching map \(br^{[a,b,c,r]}\) in (1) gives a morphism of \(H(\mathbb{A}_f)\)-equivariant relative Chow motives

\[br^{[a,b,c,r]}: \mathcal{K}^{b+c−2r}_{Q} \boxtimes 1 \to \iota^*(\mathcal{M}^{a,b,c+(-a−r)(−a−r)})\]

where for any integer \(m\), the symbol \([m]\) means twisting by the character \(|\mu(-)|^m\) of \(G(\mathbb{A}_f)\). Note that the \(G(\mathbb{A}_f)\)-equivariant relative Chow motive associated to \(\mu\) is \(\mathbb{Q}(-1)[-1]\) (see [LSZ19, Lemma 6.2.2]).
4.3. Eisenstein classes for GL₂. For any space $X$ and field $F$, let $S(X, F)$ denote the space of Schwartz functions on $X$ with values in $F$. Let $S₀(\mathbb{A}_f^2, F) \subset S(\mathbb{A}_f^2, F)$ be the subspace of functions $\phi$ such that $\phi(0, 0) = 0$. For $e > 1$ an integer, let $eS₀(\mathbb{A}_f^2, \mathbb{Z}) \subset S₀(\mathbb{A}_f^2, \mathbb{Q})$ denote the subgroup of functions of the form $\phi = \phi(e) \cdot \prod_{\ell | e} \chi(\mathbb{Z}^2_\ell)$, where $\phi(e) \in S(\mathbb{A}_f^2, \mathbb{Z})$.

Recall that we have the Siegel units on modular curves $Y_{GL₂}$.

**Theorem 4.1.** There is a $GL₂(\mathbb{A}_f)$-equivariant map

$$S₀(\mathbb{A}_f^2, \mathbb{Q}) \to H₁^{mot}(Y_{GL₂}(\mathbb{Q}(1))) = O^×(Y_{GL₂}) \otimes \mathbb{Q}, \phi \mapsto g_\phi$$

so that $g_\phi = \chi((a, b) + n\mathbb{Z}^2)$ for an integer $N ≥ 1$ and $(a, b) \in \mathbb{Q}^2 - N\mathbb{Z}^2$, then $g_\phi$ is the Siegel unit $g_{a/N, b/N}$.

**Proof.** See [LSZ19, Proposition 7.1.1].

There is also an integral version of Siegel units.

**Theorem 4.2.** If $e$ is a positive integer prime to 6, then there is a map

$$eS₀(\mathbb{A}_f, \mathbb{Z}) \to H₁^{mot}(Y_{GL₂}(\mathbb{Z}(1))) = O^×(Y_{GL₂}), \phi \mapsto e g_\phi$$

so that $e g_\phi = \left(e^{-2} - \begin{pmatrix} e & 1 \\ e & 0 \end{pmatrix}\right) g_\phi$ as elements of $H₁^{mot}(Y_{GL₂}(\mathbb{Q}(1)))$.

**Proof.** See [LSZ19, Proposition 7.1.2].

For higher weights, there is the map of Eisenstein symbol, being an analogue of Siegel units.

**Theorem 4.3.** For an integer $k ≥ 1$, there is a $GL₂(\mathbb{A}_f)$-equivariant map

$$S(\mathbb{A}_f^2, \mathbb{Q}) \to H₁^{mot}(Y_{GL₂}, \mathcal{H}_Q^k(1)), \phi \mapsto \text{Eis}^k_{mot, \phi}$$

the motivic Eisenstein symbol, satisfying the following property: the pullback of the de Rham realization $r_{dR}(\text{Eis}^k_{mot, \phi})$ to the upper half plane is the $\mathcal{H}_Q^k$-valued differential 1-form

$$-F_\phi^{(k+2)}(\tau)(2\pi i dz)^k(2\pi i d\tau)$$

where $F_\phi^{(k+2)}$ is the Eisenstein series defined by

$$F_\phi^{(k+2)}(\tau) = \frac{(k + 1)!}{(-2\pi i)^k} \sum_{x, y \in \mathbb{Q}, (x, y) ≠ (0, 0)} \frac{\phi(x, y)}{(x \tau + y)^{k+2}}.$$

**Proof.** See [LSZ19, Theorem 7.2.2].

To have an integral version of Eisenstein symbols, we need to go the étale realization. Let $p$ be a prime number. Let $\mathcal{H}_p^k$, denote the $p$-adic realization of the Chow motive $\mathcal{H}_Q^k$. Also let $\mathcal{H}^k_{Z_p}$, the étale $\mathbb{Z}_p$-sheaf associated to $\text{TSym}^k\mathbb{Z}^2 \otimes \text{det}^{-k}$, the minimal admissible lattice in $(\text{Sym}^k)^* = \text{Sym}^k \otimes \text{det}^{-k}$. We have $\mathcal{H}_p^k = \mathcal{H}^k_{Z_p} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. For any open compact subgroup $K_{GL₂} \subset G(\mathbb{A}_f)$, we have an étale realization map

$$r_{\text{ét}} : H₁^{mot}(Y_{GL₂}(K_{GL₂}), \mathcal{H}_Q^k(1)) \to H₁^{\text{ét}}(Y_{GL₂}(K_{GL₂}), \mathcal{H}_p^k(1)).$$

**Theorem 4.4.** If $e$ is prime to 6$p$, then for each neat compact open subgroup $K \subset GL₂(\mathbb{A}_f^{(p)}) \times \prod_{\ell | e(p) \mathbb{Z}_\ell}$, there is a map

$$e_pS(\mathbb{A}_f^2, \mathbb{Z}_p)^K \to H₁^{\text{ét}}(Y_{GL₂}(K), \mathcal{H}_p^k(1)), \phi \mapsto e \text{Eis}^k_{\text{ét}, \phi}$$

so that $e \text{Eis}^k_{\text{ét}, \phi} = \left(e^{-2} - \begin{pmatrix} e & 1 \\ e & 0 \end{pmatrix}\right) r_{\text{ét}}(\text{Eis}^k_{mot, \phi})$ as elements in $H₁^{\text{ét}}(Y_{GL₂}(K), \mathcal{H}_p^k(1))$.

**Proof.** See [LSZ19, Proposition 7.2.4].

The next few theorems tell us that to understand the map of Eisenstein symbol in terms of $GL₂(\mathbb{A}_f)$-representations, it suffices to consider the principal series of $GL₂$.
Definition 4.5. Let $k \geq 0$ be an integer. Given a (finite-order) character $\nu: \mathbb{Q}_{\ell_0}^\times \to \mathbb{C}^\times$ such that $\nu(-1) = (-1)^k$, let $I(| \cdot |^{1/2+1/k} \nu, | \cdot |^{-1/2})$ be the space of smooth functions $f: \text{GL}_2(\mathbb{A}_F) \to \mathbb{C}$ such that

$$f\left(\begin{array}{ll} a & b \\ c & d \end{array}\right) g = |a|^{k+1} |d|^{-1} f(g).$$

The space $I(| \cdot |^{1+2/k} \nu, | \cdot |^{-1/2})$ is equipped with a $\text{GL}_2(\mathbb{A}_F)$-action of right translation. cf. Definition 6.1.

Theorem 4.6. There is a $\text{GL}_2(\mathbb{A}_F)$-equivariant isomorphism

$$\partial_0: \mathcal{O}_2^\times (\mathcal{Q}_{\mathfrak{GL}_2})/(\mathcal{Q}_{\mathfrak{GL}_2})^\times \otimes \mathbb{C} \to I^0(| \cdot |^{1/2}, | \cdot |^{-1/2}) \oplus \bigoplus_{\nu \neq 1} I(| \cdot |^{1/2+1/k} \nu, | \cdot |^{-1/2}).$$

Here $I^0(| \cdot |^{1/2}, | \cdot |^{-1/2})$ is the kernel of $I(| \cdot |^{1/2}, | \cdot |^{-1/2}) \to \mathbb{C}$, $f \mapsto \int_{\text{GL}_2(\mathbb{A}_F)/B(\mathbb{A}_F)} f \, dg$, a codimension one sub-representation of $I(| \cdot |^{1/2}, | \cdot |^{-1/2})$.

Proof. See [LSZ19, Theorem 7.3.2].

Theorem 4.7. For an integer $k \geq 1$, there is a $\text{GL}_2(\mathbb{A}_F)$-equivariant surjective map

$$\partial_k: H^1_{\text{mot}}(\mathcal{Q}_{\mathfrak{GL}_2}, \mathcal{H}_k^k(1)) \otimes \mathbb{C} \to \bigoplus_{\nu} I(| \cdot |^{1+1/k} \nu, | \cdot |^{-1/2}).$$

Moreover, $\partial_k$ is an isomorphism on the image of the Eisenstein symbol map $\phi \mapsto \text{Eis}_{\text{mot}, \phi}^k$.

Proof. See [LSZ19, Theorem 7.3.3].

Theorem 4.8. Let $k \geq 0$ be an integer. Let $\nu = \prod \nu_i$ be a finite-order character of $\mathbb{Q}_{\ell_0}^\times \backslash \mathbb{A}_F^\times$ so that $\nu(-1) = (-1)^k$. Write $S(\mathbb{A}_F^2, \mathbb{C})^\nu \subset S(\mathbb{A}_F^2, \mathbb{C})$ for the subspace of functions on which $\mathbb{Z}^k$ acts via $\nu$. Then for $\phi = \prod \phi_i \in S(\mathbb{A}_F^2, \mathbb{C})^\nu$ (when $k = 0, \nu = 1$ further assume that $\phi(0, 0) = 0$),

$$\partial_k(\text{Eis}_{\text{mot}, \phi}^k) = \frac{2(k+1)!L(k+2, \nu, \mathbb{C})}{(-2\pi i)^{k+2}} \prod_{\ell} f_{\phi_i, \chi_i, \psi_i},$$

where $\chi_i = | \cdot |^{1+1/k} \nu_i, \psi_i = | \cdot |^{-1/2}$, and $f_{\phi_i, \chi_i, \psi_i}$ is the Siegel section defined in Proposition 6.7.

Proof. See [LSZ19, Proposition 7.3.4].

5. The symbol map

5.1. Motivation: Novodvorský’s integral formula for $L$-functions. Let $\Pi = \prod \Pi_\ell$ (resp. $\pi = \prod \pi_\ell$) be a generic irreducible admissible automorphic representation of $\text{GSp}_4(\mathbb{A})$ (resp. $\text{GL}_2(\mathbb{A})$), and assume $\Pi$ is cuspidal. In [Nov79] (see also [Sou84]), Novodvorský constructs the automorphic $L$-function $L(s, \Pi \otimes \pi) := L(s, \Pi \otimes \pi, r)$ where $r$ is the natural 8-dimensional representation of the $L$-group of $\text{GSp}_4 \times \text{GL}_2$, which is isomorphic to $\text{GSp}_4(\mathbb{C}) \times \text{GL}_2(\mathbb{C})$. The automorphic $L$-function $L(s, \Pi \otimes \pi)$ is constructed as the product of local $L$-functions $L(s, \Pi_\ell \otimes \pi_\ell)$, and the local $L$-function $L(s, \Pi_\ell \otimes \pi_\ell)$ is defined as the common denominator of minimal degree of the integrals $\mathcal{I}(\psi_{\pi_\ell} \otimes \varphi_\ell, \Phi, s)$ defined below.

The generic assumption we made on $\Pi$ and $\pi$ means that they admit Whittaker models. The definition will be recalled in Section 7.2. Let $\Psi$ be an unramified character of $\mathbb{Q}_p$. Let $W_{\varphi_\Pi}$ denote a Whittaker function for $\varphi_{\Pi_\ell} \in \Pi_\ell$ with respect to $\Psi$, and $W_{\varphi_{\pi_\ell}}$ denote a Whittaker function for $\varphi_{\pi_\ell} \in \pi_\ell$ with respect to $W_{\Psi}^{-1}$.

Recall that we have an embedding $H := \text{GL}_2 \times_{\text{GL}_1} \text{GL}_2 \hookrightarrow G := \text{GSp}_4 \times_{\text{GL}_1} \text{GL}_2$.

Definition 5.1 (Local zeta integral). For $\varphi_{\Pi_\ell} \in \Pi_\ell, \varphi_{\pi_\ell} \in \pi_\ell$. Let $\Phi$ be a locally constant complex function on $\mathbb{Q}_\ell \otimes \mathbb{Q}_l$. Define

$$\mathcal{I}(\varphi_{\Pi_\ell} \otimes \varphi_{\pi_\ell}, \Phi, s) := \int_{(\mathcal{C} \cap H)(\mathbb{Q}_s)} W_{\varphi_{\Pi_\ell}}(h_1, h_2) W_{\varphi_{\pi_\ell}}(h_2) f_{\varphi_{\ell}, \varphi_{\ell}}(h_1) \, |\det h_1|_s^s \, dh.$$
In fact, let \( \varphi_\Pi = \prod_\ell \varphi_{\Pi_\ell} \in \Pi \) and \( \varphi_\pi = \prod_\ell \varphi_{\pi_\ell} \in \pi \), then the global integral
\[
\int_{C(A_0)H(Q)\setminus H(A_0)} \varphi_\Pi(h_1, h_2) \varphi_\pi(h_2) E^\Phi(h_1; \omega, s) |\det h_1|^{s} \, dh
\]
\[
= \int_{\phi \in \mathcal{F}(A_0 \setminus \phi, \Phi, s)} W_{\varphi_\Pi}(h_1, h_2) W_{\varphi_\pi}(h_2) f^\Phi_{\omega, s}(h_1) |\det h_1|^{s} \, dh
\]
decomposes into the product of local integrals \( \mathcal{F}(\varphi_\Pi \otimes \varphi_\pi, \Phi, s) \) over all \( \ell \). Here \( E^\Phi(h_1; \omega, s) = \sum_{\gamma} f^\Phi_{\omega, s}(\gamma h_1) \) is the Eisenstein series for \( GL_2 \) used by Jacquet in [Jac72], and \( \omega = \omega_\Pi \omega_\pi \) is the product of the central characters of \( \Pi \) and \( \pi \).

5.2. **The symbol map at finite level.** Motivated by Novodvorsky’s integral formula, we ought to construct an Euler system making use of Eisenstein symbols coming from the first \( GL_2 \)-factor of \( H \).

**Lemma 5.2.** Let \( U' \subset U \subset G(\mathbb{A}_f) \) be open compact subgroups, and define \( V' := U' \cap H(\mathbb{A}_f) \) and \( V := U \cap H(\mathbb{A}_f) \). We have the commutative diagram
\[
\begin{array}{ccc}
Y_H(V') & \xrightarrow{(\iota_U')^\ast} & Y_G(U') \\
\downarrow_{\text{pr}_V'} & & \downarrow_{\text{pr}_G'} \\
Y_H(V) & \xrightarrow{\iota_U} & Y_G(U)
\end{array}
\]

This induces maps on the cohomologies
\[
H^1_{\text{mot}}(Y_H(V'), \iota^\ast \mathcal{M}^a,b,c,*) \xrightarrow{(\iota_U')^\ast} H^5_{\text{mot}}(Y_G(U'), \mathcal{M}_Q^{a,b,c,*(2)})
\]
\[
H^1_{\text{mot}}(Y_H(V), \iota^\ast \mathcal{M}^a,b,c,*) \xrightarrow{\iota_U^\ast} H^5_{\text{mot}}(Y_G(U), \mathcal{M}_Q^{a,b,c,*(2)})
\]

Let \( x \in H^1_{\text{mot}}(Y_H(V), \iota^\ast \mathcal{M}^a,b,c,*) \). Then
\[
\text{Vol}(V') \cdot (\text{pr}_U')_\ast \circ \iota_U',* (x) = \text{Vol}(V) \cdot \iota_U,*(x)
\]
in \( H^5_{\text{mot}}(Y_G(U), \mathcal{M}_Q^{a,b,c,*(2)}) \).

**Proof.** The functoriality of pushforward maps gives
\[
(\text{pr}_U')_\ast \circ \iota_U',* = \iota_U, \ast \circ (\text{pr}_V^\ast)_\ast.
\]

Now we apply maps on both sides of the above identity to \( x \). Since \( x \) is fixed by \( V \), the right hand side is
\[
[V : V'] \cdot \iota_U,*(x) = \frac{\text{Vol} V}{\text{Vol} V'} \cdot \iota_U,*(x),
\]
which proves the claim. \( \square \)

Let \( U \subset G(\mathbb{A}_f) \) be an open compact subgroup. For any \( U' \subset U \subset G(\mathbb{A}_f) \), let \( V' := U' \cap H(\mathbb{A}_f) \) and \( V'^{(1)} := \text{pr}_1(V') \subset GL_2(\mathbb{A}_f) \). Let \( a, b, c \geq 0 \) be integers and \( c \leq a + b \). Let \( r \) be an integer such that \( \max(0, -a + c) \leq r \leq \min(b, c) \). We have the following composition of maps
\[
S(A^2_f, \mathbb{Q})^{V'^{(1)}} \xrightarrow{\text{Eis}_{\text{mot}}^{b+a+c+2r}} H^1_{\text{mot}}(Y_{GL_2}(V'^{(1)}), \mathcal{M}_Q^{b+a+c+2r}(1))
\]
\[
\xrightarrow{(\text{pr}_1)^\ast} H^1_{\text{mot}}(Y_H(V'), \mathcal{M}_Q^{b+a+c+2r}(1) \boxtimes 1)
\]
\[
\xrightarrow{\text{br}_{[a,b,c,r]}^\ast} H^1_{\text{mot}}(Y_H(V'), \iota^\ast \mathcal{M}_Q^{a,b,c,*(1 - a + r)[a - r]})
\]
\[
\xrightarrow{\text{Vol}(V')^{-1} \ast} H^5_{\text{mot}}(Y_G(U'), \mathcal{M}_Q^{a,b,c,*(3 - a - r)[a - r]})
\]
\[
\xrightarrow{(\text{pr}_U')_\ast} H^5_{\text{mot}}(Y_G(U), \mathcal{M}_Q^{a,b,c,*(3 - a - r)[a - r]}),
\]
Lemma 5.4. A smooth representation of $\Phi(\mathbb{Q})$ which equals $\lim_{V''} S(2, \mathbb{Q})^{\mathbb{Q}(1)} \to H_{mot}^2 \mathcal{Y}(U), \mathcal{H}_Q^{a,b,c,*} (3-a-r)|-a-r].$

By Lemma 5.3, these maps are compatible under $S(2, \mathbb{Q})^{\mathbb{Q}(1)} \to S(2, \mathbb{Q})^{\mathbb{Q}(1)}$ for $U'' \subset U'$ and hence gives rise to

$$\text{Symbl}_{(a,b,c,r)}^{U'}: S(2, \mathbb{Q}) = \lim_{V''} S(2, \mathbb{Q})^{\mathbb{Q}(1)} \to H_{mot}^2 \mathcal{Y}(U), \mathcal{H}_Q^{a,b,c,*} (3-a-r)|-a-r].$$

Moreover, $\text{Symbl}_{(a,b,c,r)}^{U'} = (\text{pr}_{U'})_* \circ \text{Symbl}_{(a,b,c,r)}^{U'}$ by construction.

5.3. Hecke Algebra. Let $\mathcal{H}_\ell(G)$ be the Hecke algebra of locally constant, compactly supported $\mathbb{C}$-valued functions on $G(\mathbb{Q}_\ell)$. Fix a left-invariant Haar measure $dg$ on normalized so that $Vol G(\mathbb{Z}_\ell) = 1$. The product structure on $\mathcal{H}_\ell(G)$ is given by convolution: given $\xi_1, \xi_2 \in \mathcal{H}_\ell(G)$,

$$(\xi_1 \cdot \xi_2)(-) := \int_{G(\mathbb{Q}_\ell)} \xi_1(g)\xi_2(g^{-1} - ) dg.$$

It is well-known that if $\sigma$ is a smooth representation of $G(\mathbb{Q}_\ell)$, then $\sigma$ can also be viewed as a left $\mathcal{H}_\ell(G)$-module: given $\varphi \in \sigma$ and $\xi \in \mathcal{H}_\ell(G)$,

$$\xi \cdot \varphi := \int_{G(\mathbb{Q}_\ell)} \xi(g) (g^{-1} \cdot \varphi) dg.$$

We view $\mathcal{H}_\ell(G)$ as a left $G(\mathbb{Q}_\ell)$-representation and a right $H(\mathbb{Q}_\ell)$-representation by

$$(g \cdot \xi \cdot h)(-) = \xi(h - g),$$

where $g \in G(\mathbb{Q}_\ell), h \in H(\mathbb{Q}_\ell)$ and $\xi \in \mathcal{H}_\ell(G)$. This makes $\mathcal{H}_\ell(G)$ into a left $\mathcal{H}_\ell(G)$-module and a right $\mathcal{H}_\ell(H)$-module, but note that the left $\mathcal{H}_\ell(G)$-module structure is given by $(\xi_1 \cdot \xi_2)(-) = \int_{G(\mathbb{Q}_\ell)} \xi_1(g)\xi_2(-g) dg$ rather than the one given by convolution on the left.

The reason for this choice of left $G(\mathbb{Q}_\ell)$-action (rather than $(g \cdot \xi)(-) = \xi(g^{-1} - )$) will be clear in Lemma 5.4.

**Lemma 5.3.** Let $\sigma$ be a smooth representation of $G(\mathbb{Q}_\ell)$, and $\sigma^\vee$ its contragredient. For every $\xi \in \mathcal{H}_\ell(G)$, let $\xi' \in \mathcal{H}_\ell(G)$ be the function defined by $\xi'(g) = \xi(g^{-1})$. Then for any $\Phi \in \sigma^\vee$ and $\varphi \in \sigma$, we have

$$\Phi(\xi \cdot \varphi) = (\xi' \cdot \Phi)(\varphi).$$

**Proof.** By linearity of $\Phi$ we have

$$\Phi(\xi \cdot \varphi) = \Phi(\int_{G(\mathbb{Q}_\ell)} \xi(g)(g \cdot \varphi) dg) = \int_{G(\mathbb{Q}_\ell)} \xi(g)\Phi(g \cdot \varphi) dg.$$

On the other hand,

$$(\xi' \cdot \Phi)(\varphi) = \left(\int_{G(\mathbb{Q}_\ell)} \xi(g^{-1})(g \cdot \Phi) dg\right)(\varphi) = \int_{G(\mathbb{Q}_\ell)} \xi(g^{-1})\Phi(g^{-1} \cdot \varphi) dg,$$

which equals $\Phi(\xi \cdot \varphi)$ as required. \qed

Let $\tau$ be a smooth representation of $GL_2(\mathbb{Q}_\ell)$, and $\sigma$ be a smooth representation of $G(\mathbb{Q}_\ell)$. Regard $\tau$ as a smooth representation of $H(\mathbb{Q}_\ell)$ through the first projection $pr_1 : H(\mathbb{Q}_\ell) \to GL_2(\mathbb{Q}_\ell)$. Let

$$\mathcal{X}(\tau, \sigma^\vee) = \text{Hom}_{\mathcal{H}_\ell(G)}(\mathcal{H}_\ell(G) \otimes_{\mathcal{H}_\ell(H)} \tau, \sigma^\vee).$$

**Lemma 5.4.** Let $\mathfrak{z} \in \mathcal{X}(\tau, \sigma^\vee)$ and $\xi_1, \xi_2 \in \mathcal{H}_\ell(G)$. Then

$$\xi_2 \cdot \mathfrak{z}((\xi_1 \otimes \phi)) = \mathfrak{z}((\xi_1 \xi_2) \otimes \phi).$$
Proof. By linearity and $G(\mathbb{Q}_\ell)$-equivariance of $3$,

$$
\xi_2 \cdot 3(\xi_1 \otimes \phi) = \int_{G(\mathbb{Q}_\ell)} \xi_2(g^{-1}) (g \cdot 3(\xi_1 \otimes \phi)) \, dg
$$

$$
= 3 \left( \int_{G(\mathbb{Q}_\ell)} \xi_2(g^{-1})(g \cdot \xi_1) \, dg \right) \otimes \phi
$$

$$
= 3 \left( \int_{G(\mathbb{Q}_\ell)} \xi_2(g^{-1})\xi_1(-g) \, dg \right) \otimes \phi
$$

$$
= 3((\xi_1 \xi_2) \otimes \phi).
$$

The formula in the above lemma will appear a lot in our computation with Hecke actions. From now on, we will change notation to write $3: \tau \otimes \mathcal{H}_\ell \mathcal{H}_\ell(G) \rightarrow \sigma^\vee$ instead of $3: \mathcal{H}_\ell(G) \otimes \mathcal{H}_\ell(H) \tau \rightarrow \sigma^\vee$ to signify our focus on the right $\mathcal{H}_\ell(G)$-action on $\mathcal{H}_\ell(G)$ given by right convolution (and to align our notation with that used by [LSZ19]). Hence the formula becomes

$$
\xi_2 \cdot 3(\phi \otimes \xi_1) = 3(\phi \otimes \xi_1 \xi_2).
$$

Despite the notation, the tensor is still with respect to the left $\mathcal{H}_\ell(H)$-module $\tau$ and the right $\mathcal{H}_\ell(H)$-module $\mathcal{H}_\ell(G)$.

5.4. The symbol map at infinite level. Let $\mathcal{H}(G(\mathbb{A}_f)) = \bigotimes_f \mathcal{H}_\ell(G)$ be the Hecke algebra of locally constant, compactly supported $\mathbb{C}$-valued functions on $G(\mathbb{A}_f)$. We view $S(G(\mathbb{A}_f), \mathbb{Q})$ as a left $\mathcal{H}(H(\mathbb{A}_f))$-module through the first projection $pr_1: H \rightarrow GL_2$ and extend $\mathcal{H}(G(\mathbb{A}_f))$-linearly the symbol maps $\text{Symbl}_{U}^{a,b,c,r}$ at different level to define

$$
\text{Symbl}_{U}^{a,b,c,r}: S(G(\mathbb{A}_f), \mathbb{Q}) \otimes \mathcal{H}(H(\mathbb{A}_f)) \mathcal{H}(G(\mathbb{A}_f)) \rightarrow H^S_{\text{mot}}(Y_G, \mathcal{W}_{\mathbb{Q}}^{a,b,c,r}(3-a-r)][-a-r].
$$

We want to have an explicit description of $\text{Symbl}_{U}^{a,b,c,r}$ as what we had for the finite level $\text{Symbl}_{U}^{a,b,c,r}$. First, because of the relation $\text{Symbl}_{U}^{a,b,c,r} = (pr_U^f)^* \circ \text{Symbl}_{U}^{a,b,c,r}$, for any open compact subgroup $U \subset G(\mathbb{A}_f)$, we have

$$
\text{Symbl}_{U}^{a,b,c,r}(ch(U) \otimes \phi) = \text{Symbl}_{U}^{a,b,c,r}(\phi).
$$

Then because of the $\mathcal{H}(G(\mathbb{A}_f))$-equivariance of $\text{Symbl}_{U}^{a,b,c,r}$, for generators $ch(gU) \in \mathcal{H}(G(\mathbb{A}_f))U$, we have

$$
\text{Symbl}_{U}^{a,b,c,r}(\phi \otimes ch(gU)) = \text{Symbl}_{U}^{a,b,c,r}(\phi \otimes \frac{1}{\text{Vol}U} \text{ch}(gUg^{-1}) \text{ch}(gU))
$$

$$
= \frac{1}{\text{Vol}U} \text{ch}(gU^{-1}) \cdot \text{Symbl}_{U}^{a,b,c,r}(\phi \otimes \text{ch}(gUg^{-1}))
$$

$$
= \frac{1}{\text{Vol}U} \text{ch}(g^{-1} \cdot gUg^{-1}) \cdot \text{Symbl}_{U}^{a,b,c,r}(\phi \otimes \text{ch}(gUg^{-1}))
$$

$$
= g^{-1} \cdot \text{Symbl}_{U}^{a,b,c,r}(\phi \otimes \text{ch}(gUg^{-1}))
$$

$$
= g^{-1} \cdot \text{Symbl}_{gUg^{-1}}^{a,b,c,r}(\phi).
$$

Here the second equality uses the formula (2).

5.5. Integrality of the symbol map. We examine more carefully the integrality of the symbol map. First we need to define a variant of the $\text{Symbl}_{U}^{a,b,c,r}$. Let $e$ be an integer prime to $6p$. Let $U \subset G(\mathbb{A}_f^{(e)}) \times \prod_{f|p} \mathbb{Z}_f$ be an open compact subgroup. For any $U' \subset U \subset G(\mathbb{A}_f)$, let $V':= U' \cap H(\mathbb{A}_f)$ and $V'^{(1)} := \text{pr}_1 V' \subset$...
GL_2(\mathbb{A}_f). For \( b + c - 2r \geq 1 \), again we can consider the composition of maps

\[
ep S(\mathbb{A}_f^2, \mathbb{Z}_p)_{V(1)} \xrightarrow{\text{Ein}} H^1_{et}(YGL_2(V(1)), \mathbb{H}^{b+c-2r}(1)) \xrightarrow{(pr,1)^*} H^1_{et}(YH, \mathbb{H}^{b+c-2r}(1) \boxtimes 1) \xrightarrow{b_{[a,b,c,r]}} H^1_{et}(YH(V'), \epsilon^*((\mathcal{W}_p)^a_{b,c,r}(1-a-r)[-a-r])) \xrightarrow{\text{Vol}(V')^{-1}} \text{Vol}(V') : H^5_{et}(Y_G(U'), \mathcal{W}_p^{a,b,c,r}(3-a-r))[-a-r] \xrightarrow{(pr,1')^*} \text{Vol}(V') : H^5_{et}(Y_G(U), \mathcal{W}_p^{a,b,c,r}(3-a-r))[-a-r],
\]

which gives rise to

\[
ep \text{Sym}^{a,b,c,r}_U : \ep S(\mathbb{A}_f^2, \mathbb{Z}_p) \rightarrow H^5_{et}(Y_G(U), \mathcal{W}_p^{a,b,c,r}(3-a-r))[-a-r]
\]

Note that the integrality of \( \ep \text{Sym}^{a,b,c,r}_U(\phi) \) is controlled by \( \text{Vol} V' \), where \( V' = U' \cap H(\mathbb{A}_f) \) for some \( U' \subset U \) and \( V' \) fixes \( \phi \).

Let \( \mathcal{H}_p(G(\mathbb{A}_f^p) \times \mathbb{Z}_p) \) be the algebra under convolution of locally constant, compactly supported \( \mathbb{Z}_p \)-valued functions on \( G(\mathbb{A}_f^p) \times \mathbb{Z}_p \). Again we can extend \( \mathcal{H}_p(G(\mathbb{A}_f^p) \times \mathbb{Z}_p) \)-linearly to define

\[
\ep \text{Sym}^{a,b,c,r}_U : \ep S(\mathbb{A}_f^2, \mathbb{Z}_p) \otimes \mathcal{H}_p(G(\mathbb{A}_f^p) \times \mathbb{Z}_p) \rightarrow H^5_{et}(Y_G, \mathcal{W}_p^{a,b,c,r}(3-a-r))[-a-r].
\]

Namely, \( \ep \text{Sym}^{a,b,c,r}_U(\phi \otimes \text{ch}(U)) := \ep \text{Sym}^{a,b,c,r}_U(\phi) \), and \( \mathcal{H}_p(G(\mathbb{A}_f^p) \times \mathbb{Z}_p) \)-equivariance gives

\[
\ep \text{Sym}^{a,b,c,r}_U(\phi \otimes \text{ch}(gU)) = g^{-1} \cdot \ep \text{Sym}^{a,b,c,r}_U(\phi).
\]

In particular, if \( \phi \) is fixed by \( gUg^{-1} \cap H(\mathbb{A}_f) \), then the integrality of \( \ep \text{Sym}^{a,b,c,r}_U(\phi \otimes \text{ch}(gU)) \) is controlled by \( \text{Vol}(gUg^{-1} \cap H(\mathbb{A}_f)) \). We summarize this in the following Proposition:

**Proposition 5.5.** Let \( \xi = \sum_m m_\eta \text{ch}(\eta U) \in \mathcal{H}_p(G(\mathbb{A}_f^p) \times \mathbb{Z}_p) \) and \( \phi \in \ep S(\mathbb{A}_f^2, \mathbb{Z}_p) \). If \( m_\eta \cdot \text{Vol}(\eta U \eta^{-1} \cap H(\mathbb{A}_f)) \in \mathbb{Z}_p \) for all \( \eta \), then

\[
\ep \text{Sym}^{a,b,c,r}_U(\phi \otimes \xi) \in H^5_{et}(Y_G(U), \mathcal{W}_p^{a,b,c,r}(3-a-r))[-a-r]
\]

lies in the image of the cohomology \( H^5_{et}(Y_G(U), \mathcal{W}_p^{a,b,c,r}(3-a-r))[-a-r] \) with integral coefficient.

By Theorem 4.4, \( \ep \text{Sym}^{a,b,c,r}_U \) and \( \text{Sym}^{a,b,c,r}_U \) are related by

\[
\ep \text{Sym}^{a,b,c,r}_U(\phi \otimes \xi) = r_{et} \circ \text{Sym}^{a,b,c,r}_U \left( (e^2 - e^{-(b+c-2r)})(\begin{pmatrix} e & 0 \\ 1 & 1 \end{pmatrix})^{-1} \right) \phi \otimes \xi
\]

as elements in \( H^5_{et}(Y_G(U), \mathcal{W}_p^{a,b,c,r}(3-a-r))[-a-r] \), for any \( \xi \otimes \phi \in \mathcal{H}_p(G(\mathbb{A}_f^p) \times \mathbb{Z}_p) \otimes \ep S(\mathbb{A}_f^2, \mathbb{Z}_p) \).

We will be choosing suitable \( \phi \otimes \xi \)'s, so that their image under the symbol map give the Euler system elements. In the next two sections, we do some local representation theory, in order to know what \( \phi \otimes \xi \) to choose to make norm relations of Euler system elements satisfied.

### 6. Local Representation Theory

We recall some results of local representation theory that will be needed to verify the norm relations of an Euler system. The main references are [Ren10] and Chapter 4 of [Bum97], Section 3 of [LSZ19] and Section 1 of [Gro20] also have excellent summaries of the materials, and we follow their accounts closely.
6.1. **Principal series of** $GL_2(\mathbb{Q}_\ell)$. Let $| \cdot |$ denote the standard $\ell$-adic norm on $\mathbb{Q}_\ell$ normalized such that $|\ell| = \ell^{-1}$. For $\chi$ a smooth character of $\mathbb{Q}_\ell^\times$, we write $L(\chi, s)$ for the local $L$-factor defined as

$$L(\chi, s) = L(\chi|\cdot|, 0) = \begin{cases} (1 - \chi(\ell)\ell^{-s})^{-1} & \text{if } \chi|_{\mathbb{Z}_\ell^\times} = 1, \\ 1 & \text{otherwise.} \end{cases}$$

**Definition 6.1.** Given two smooth characters $\chi$ and $\psi$ of $\mathbb{Q}_\ell^\times$, let $I(\chi, \psi)$ be the space of smooth functions $f : GL_2(\mathbb{Q}_\ell) \to \mathbb{C}$ such that

$$f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}g\right) = \chi(a)\psi(d)|a/d|^{1/2}f(g),$$

equipped with a $GL_2(\mathbb{Q}_\ell)$-action of right translation.

This is the normalized induction from the standard Borel group. It is well-known that if $\chi/\psi \neq |\cdot|^\pm 1$, then $I(\chi, \psi)$ is an irreducible representation. If $\chi/\psi = |\cdot|^1$, then $I(\chi, \psi)$ has an irreducible codimension one invariant subspace. If $\chi/\psi = |\cdot|^{-1}$, then $I(\chi, \psi)$ has a one-dimensional invariant subspace and the quotient representation is irreducible.

Write $dx$ (respectively, $d^\times x$, $dg$) for the Haar measure on $\mathbb{Q}_\ell$ (respectively, $\mathbb{Q}_\ell^\times$, $GL_2(\mathbb{Q}_\ell)$) normalized so that $\mathbb{Z}_\ell$ (respectively, $\mathbb{Z}_\ell^\times$, $GL_2(\mathbb{Z}_\ell)$) has unit volume. There is a natural pairing $I(\chi, \psi) \times I(\chi^{-1}, \psi^{-1}) \to \mathbb{C}$ defined by

$$\langle f_1, f_2 \rangle = \int_{GL_2(\mathbb{Z}_\ell)} f_1(g)f_2(g)dg,$$

under which $I(\chi^{-1}, \psi^{-1})$ is identified with the dual of $I(\chi, \psi)$.

**Definition 6.2.** Let $\chi$ and $\psi$ be smooth characters of $\mathbb{Q}_\ell$. A flat section of the family of representations $I(\chi|\cdot|^s, \psi|\cdot|^s)$ indexed by $s_1, s_2 \in \mathbb{C}$ is a function $GL_2(\mathbb{Q}_\ell) \times \mathbb{C}^2 \to \mathbb{C}$, $(g, s_1, s_2) \mapsto f_{s_1, s_2}(g)$ such that for all fixed $s_1, s_2$, the function $g \mapsto f_{s_1, s_2}(g)$ is in $I(\chi|\cdot|^s_1, \psi|\cdot|^s_2)$, and the restriction of $f_{s_1, s_2}$ to $GL_2(\mathbb{Z}_\ell)$ is independent of $s_1$ and $s_2$.

**Remark 6.3.** From the Iwasawa decomposition (Proposition 4.5.2 of [Bum97]), one sees that every $f \in I(\chi, \psi)$ extends to a unique flat section.

Next we recall the definition of the intertwining operator. Fix characters $\chi, \psi$ of $\mathbb{Q}_\ell^\times$ and write them as

$$\chi = \xi_1|\cdot|^s_1, \quad \psi = \xi_2|\cdot|^s_2,$$

where $\xi_1, \xi_2$ are unitary characters of $s_1, s_2 \in \mathbb{C}$. Let $f \in I(\chi, \psi)$. For $g \in GL_2(\mathbb{Q}_\ell)$, consider

$$Mf(g) = \int_{\mathbb{Q}_\ell} f(w\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}g)dx,$$

where $w = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$. This integral is well-defined if $\text{Re}(s_1 - s_2) > 0$.

**Proposition 6.4** ([Bum97], Proposition 4.5.6). If $\text{Re}(s_1 - s_2) > 0$, then the integral defining $Mf(g)$ is absolutely convergent and defines a nonzero intertwining map

$$M : I(\chi, \psi) \to I(\psi, \chi), \quad f \mapsto Mf.$$  

The next proposition allows us to extend the intertwining operator $M$ to all $s_1, s_2 \in \mathbb{C}$ such that $\chi \neq \psi$.

**Proposition 6.5** ([Bum97], Proposition 4.5.7). Fix $f \in I(\chi, \psi)$, and let $f_{s_1, s_2}$ denote the flat section passing through $f$. Then for fixed $g \in GL_2(\mathbb{Q}_\ell)$, the integral $Mf_{s_1, s_2}(g)$, originally defined for $\text{Re}(s_1 - s_2) > 0$, has analytic continuation to all $s_1, s_2 \in \mathbb{C}$ where $\chi \neq \psi$, and defines a nonzero intertwining operator

$$M : I(\chi, \psi) \to I(\psi, \chi).$$

In particular, by Schur’s lemma, when $I(\chi, \psi)$ is irreducible (and hence so is $I(\psi, \chi)$), the intertwining operator $M$ is an isomorphism. On the other hand, when $\chi/\psi = |\cdot|$ (resp. $\chi/\psi = |\cdot|^{-1}$), the kernel of $M$ is precisely the unique codimension one (resp. one-dimensional) irreducible sub-representation of $I(\chi, \psi)$, and $M$ maps the quotient isomorphically onto the one-dimensional (resp. codimension one) irreducible sub-representation of $I(\psi, \chi)$.
6.2. Siegel sections. We follow the account of Section 3.2 of [LSZ19], See also Section 1.5 of [Gro20].

Definition 6.6. Let \( S(Q^2_\ell, \mathbb{C}) \) be the space of Schwartz functions on \( Q^2_\ell \), and \( \text{GL}_2(Q_\ell) \) acts on \( S(Q^2_\ell, \mathbb{C}) \) by right translation. For \( \phi \in S(Q^2_\ell, \mathbb{C}) \), we use \( \hat{\phi} \) to denote its Fourier transform

\[
\hat{\phi}(x, y) = \int_{Q^2_\ell} e_{\ell}(xv - yu)\phi(u, v)du dv,
\]

where \( e_{\ell}(x) \) is the standard additive character of \( \mathbb{Q}_\ell \) taking \( 1/\ell^n \) to \( \exp(2\pi i / \ell^n) \).

Proposition 6.7 ([LSZ19], Proposition 3.2.2). Let \( \phi \in S(Q^2_\ell, \mathbb{C}) \) and \( \chi, \psi \) be characters of \( \mathbb{Q}_\ell^\times \). There is a well-defined element \( f_{\phi, \chi, \psi} \in I(\chi, \psi) \) defined by integrals satisfying

\[
\begin{align*}
  f_{\phi, \chi, \psi}(h) &= \chi(\det g)^{-1}\det g|^{-1/2}f_{\phi, \chi, \psi}(hg), \\
  f_{\phi, \chi, \psi}(h) &= \psi(\det g)^{-1}\det g|^{-1/2}f_{\phi, \chi, \psi}(hg).
\end{align*}
\]

In particular, if \( \psi = | \cdot |^{-1/2} \), the map

\[
S(Q^2_\ell, \mathbb{C}) \to I(\chi, \psi), \quad \phi \mapsto f_{\phi, \chi, \psi}
\]

is \( \text{GL}_2(\mathbb{Q}_\ell) \)-equivariant.

Proposition 6.8 ([LSZ19], Proposition 3.2.3). We have

\[
M(f_{\phi, \chi, \psi}) = \frac{\varepsilon(\psi/\chi)}{L(\chi/\psi, 1)} f_{\hat{\phi}, \psi, \chi},
\]

where \( \varepsilon(\psi/\chi) \) is the local \( \varepsilon \)-factor (a non-zero scalar, equal to 1 if \( \chi/\psi \) is unramified).

Definition 6.9.

- For integers \( s \geq 0, t \geq 0 \), define functions \( \phi_{s, t} \in S(Q^2_\ell, \mathbb{C}) \) by

\[
\phi_{s, t} := \begin{cases} 
  \text{ch}(\ell^s \mathbb{Z}_\ell \times (1 + \ell^t \mathbb{Z}_\ell)) & \text{if } t > 0, \\
  \text{ch}(\ell^s \mathbb{Z}_\ell \times \mathbb{Z}_\ell^\times) & \text{if } s > 0 \text{ and } t = 0, \\
  \text{ch}(\mathbb{Z}_\ell \times \mathbb{Z}_\ell) & \text{if } s = t = 0.
\end{cases}
\]

- For integers \( s \geq t \geq 0 \), define open compact subgroups of \( \text{GL}_2(\mathbb{Q}_\ell) \)

\[
K^{\text{GL}_2}(\ell^s, \ell^t) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\mathbb{Z}_\ell) : c \equiv 0 \mod \ell^s, d \equiv 1 \mod \ell^t \right\}.
\]

Note that \( K^{\text{GL}_2}(\ell^s, \ell^t) \) is the stabilizer in \( \text{GL}_2(\mathbb{Q}_\ell) \) of \( \phi_{s, t} \) (when \( s \geq t \)) and that

\[
\text{Vol}(K^{\text{GL}_2}(\ell^s, \ell^t)) = \begin{cases}
  \ell^{2-s-t}(\ell^2 - 1)^{-1} & \text{if } t > 0, \\
  \ell^{1-t}(\ell + 1)^{-1} & \text{if } s > t = 0, \\
  1 & \text{if } s = t = 0.
\end{cases}
\]

- For integers \( s \geq t \geq 0 \), define open compact subgroups of \( H(\mathbb{Q}_\ell) \)

\[
K^H(\ell^s, \ell^t) := (K^{\text{GL}_2}(\ell^s, \ell^t) \times \text{GL}_2(\mathbb{Z}_\ell)) \cap H(\mathbb{Z}_\ell).
\]

Note that \( K^H(\ell^s, \ell^t) \) is the stabilizer of \( \phi_{s, t} \) in \( H(\mathbb{Q}_\ell) \) of \( \phi_{s, t} \) (when \( s \geq t \)).

Lemma 6.10 ([LSZ19], Lemma 3.2.5). Let \( \chi, \psi \) be unramified characters. Denote by \( \text{B}^{\text{GL}_2} \subset \text{GL}_2 \) the standard Borel subgroup of upper triangular matrices. Then the function \( f_{\phi_{1, 0}, \chi, \psi} \) is supported on \( \text{B}^{\text{GL}_2}(\mathbb{Q}_\ell)K^{\text{GL}_2}(\ell^t, 1) \), and

\[
f_{\phi_{1, 0}, \chi, \psi}(1) = \begin{cases}
  L(\chi/\psi, 1)^{-1} & \text{if } t > 0, \\
  1 & \text{if } t = 0.
\end{cases}
\]
6.3. Principal series of $\text{GSp}_4(\mathbb{Q}_\ell)$. Next we collect some basic results on principal series representations of $\text{GSp}_4(\mathbb{Q}_\ell)$. We follow closely the account of [LSZ19], Section 3.5. The standard reference is [RS07].

**Definition 6.11.** Let $\chi_1, \chi_2, \rho$ be smooth characters of $\mathbb{Q}_\ell^\times$ such that

$$| \cdot |^{\pm 1} \not\in \{ \chi_1, \chi_2, \chi_1\chi_2, \chi_1/\chi_2 \}. $$

We define $\chi_1 \times \chi_2 \rtimes \rho$ to be the representation of $\text{GSp}_4(\mathbb{Q}_\ell)$ given by the space of smooth functions $f : \text{GSp}_4(\mathbb{Q}_\ell) \to \mathbb{C}$ satisfying

$$f \left( \begin{array}{cccc} a & * & * & * \\ b & * & * & cb^{-1} \\ * & * & ca^{-1} & * \\ \end{array} \right) g = \frac{|a^2b|}{|c|^{3/2}} \chi_1(a)\chi_2(b)c \rho(c)f(g),$$

with $\text{GSp}_4(\mathbb{Q}_\ell)$ acting by right translation. The representation $\chi_1 \times \chi_2 \rtimes \rho$ is called an irreducible principal series.

**Remark 6.12.** This representation has central character $\chi_1\chi_2 \rho^2$. The condition on $\chi_1, \chi_2$ implies it is irreducible and generic. In fact, this is the only type among the 11 groups of irreducible, admissible, non-supercuspidal representations of $\text{GSp}_4(\mathbb{Q}_\ell)$ which is both generic and spherical (see table A.1 of [RS07] and table 3 of [Sch05]).

If $\eta$ is a smooth character of $\mathbb{Q}_\ell^\times$, we may regard it as a character of $\text{GSp}_4(\mathbb{Q}_\ell)$ via the multiplier map. Then twisting $\chi_1 \times \chi_2 \rtimes \rho$ by $\eta$ results in the representation $\chi_1 \times \chi_2 \rtimes \rho\eta$.

**Lemma 6.13.** Let $\sigma = \chi_1 \times \chi_2 \rtimes \rho$ be an irreducible principal series as above, and $\eta$ a smooth character of $\mathbb{Q}_\ell^\times$. Then the twist $\sigma \otimes \eta$ of $\sigma$ by $\eta$ is equivalent to $\eta$ if and only if at least one of the following conditions is satisfied:

- $\eta = 1$;
- $\eta = \chi_1$ and $\chi_1^2 = 1$;
- $\eta = \chi_2$ and $\chi_2^2 = 1$;
- $\eta = \chi_1\chi_2$ and $\chi_1^2 = \chi_2^2 = 1$.

**Proof.** By Theorem 4.2 of [ST93], $\sigma \otimes \eta = \chi_1 \times \chi_2 \rtimes \rho\eta$ is isomorphic to $\sigma$ if and only if $(\chi_1, \chi_2, \rho)$ and $(\chi_1\chi_2, \rho\eta)$ are in the same orbit of the Weyl group acting on characters of the diagonal torus. Using the explicit representatives of the 8-element Weyl group as given in Section 2.1 of [RS07], we see $(\chi_1, \chi_2, \rho\eta)$ must be one of the following: $(\chi_1, \chi_2, \rho), (\chi_2, \chi_1, \rho), (\chi_1^{-1}, \chi_2, \rho\chi_1), (\chi_2, \chi_1^{-1}, \rho\chi_1), (\chi_1, \chi_2^{-1}, \rho\chi_2), (\chi_2^{-1}, \chi_1, \rho\chi_2), (\chi_1^{-1}, \chi_2^{-1}, \rho\chi_1\chi_2), (\chi_2^{-1}, \chi_1^{-1}, \rho\chi_1\chi_2)$. The desired claim then follows immediately.

**Definition 6.14.** Let $\sigma = \chi_1 \times \chi_2 \rtimes \rho$ be an irreducible principal series as defined above. The local spin $L$-factor of $\sigma$ is defined as

$$L(s, \sigma) = L(\sigma \otimes | \cdot |^{s}, 0) = L(\rho, s)L(\rho\chi_1, s)L(\rho\chi_2, s)L(\rho\chi_1\chi_2, s).$$

We record the following characterization of irreducible generic unramified representations.

**Proposition 6.15** ([LSZ19], Proposition 3.5.3; see also [RS07], Section 2.2)). Let $\sigma = \chi_1 \otimes \chi_2 \rtimes \rho$ be an irreducible principal series. Then $\sigma$ is unramified if and only if all three characters $\chi_1, \chi_2, \rho$ are all unramified. Moreover, every irreducible, generic, unramified representation of $\text{GSp}_4(\mathbb{Q}_\ell)$ is isomorphic to $\chi_1 \otimes \chi_2 \rtimes \rho$ for a unique Weyl-group orbit of unramified characters $(\chi_1, \chi_2, \rho)$ satisfying (3).

**Proposition 6.16.** Let $\sigma = \chi_1 \times \chi_2 \rtimes \rho$ and $\tau' = I(\chi; \psi)$ be irreducible principal series representations, of $\text{GSp}_4(\mathbb{Q}_\ell)$ and $\text{GL}_2(\mathbb{Q}_\ell)$, respectively. Suppose neither of the following cases happens:

- $\chi_i = \chi/\psi$ is quadratic for either $i = 1$ or $i = 2$;
- $\chi_1\chi_2 = \chi/\psi$ and both $\chi_1, \chi_2$ are quadratic.

Then $\sigma \otimes \tau'$ remains irreducible as a representation of $G(\mathbb{Q}_\ell) \subset \text{GSp}_4(\mathbb{Q}_\ell) \times \text{GL}_2(\mathbb{Q}_\ell)$. 


Proof. Let \( Z = \{ (aI_4, bI_2) \in \text{GSp}_4(\mathbb{Q}_\ell) \times \text{GL}_2(\mathbb{Q}_\ell) : a, b \in \mathbb{Q}_\ell^\times \} \) be the center of \( \text{GSp}_4(\mathbb{Q}_\ell) \times \text{GL}_2(\mathbb{Q}_\ell) \). As \( \sigma' \otimes \tau' \) is irreducible as a \( \text{GSp}_4(\mathbb{Q}_\ell) \times \text{GL}_2(\mathbb{Q}_\ell) \)-representation, \( Z \) acts as scalars. Thus \( \sigma' \otimes \tau' \) is irreducible as a \( G(\mathbb{Q}_\ell) \)-representation if and only if it is also irreducible as an \( Z \cdot G(\mathbb{Q}_\ell) \)-representation.

Now we are in the situation of [GK82], Lemma 2.1. By part (d) of the cited lemma, it suffices to prove there is no character \( \nu = \nu_1 \nu_2 \) of \( \text{GSp}_4(\mathbb{Q}_\ell) \times \text{GL}_2(\mathbb{Q}_\ell) \) trivial on \( Z \cdot G(\mathbb{Q}_\ell) \), such that \( \sigma' \otimes \tau' \simeq (\sigma' \otimes \nu_1) \otimes (\tau' \otimes \nu_2) \). Here \( \nu_1 \) (respectively, \( \nu_2 \)) is a character of \( \mathbb{Q}_\ell^\times \), regarded as a character of \( \text{GSp}_4(\mathbb{Q}_\ell) \) via the multiplier map (respectively, a character of \( \text{GL}_2(\mathbb{Q}_\ell) \) via determinant).

The condition that \( \nu \) is trivial on \( G(\mathbb{Q}_\ell) \) implies that \( \nu_1 \nu_2 = 1 \). Thus neither of \( \nu_1 \) and \( \nu_2 \) can be trivial. Moreover, \( \tau' \simeq \tau' \otimes \nu_2 \) implies \( \nu_2 = \chi / \psi \) and is quadratic. On the other hand, Lemma 6.13 implies either \( \nu_1 \) is trivial, or equals to one of \( \chi_1 \) and \( \chi_2 \) and is quadratic, or equals to \( \chi_1 \chi_2 \) and both \( \chi_1 \) and \( \chi_2 \) is quadratic. It is clear none of the cases is possible given our assumption.

From now on, we will assume \( \sigma = \sigma' \otimes \tau' \) satisfies the assumption of Proposition 6.16.

6.4. Multiplicity one. Let \( \tau \) be a smooth representation of \( \text{GL}_2(\mathbb{Q}_\ell) \), and \( \sigma \) be a smooth representation of \( G(\mathbb{Q}_\ell) \). Regard \( \tau \) as a smooth representation of \( H(\mathbb{Q}_\ell) \) through the first projection \( pr_1 : H(\mathbb{Q}_\ell) \to \text{GL}_2(\mathbb{Q}_\ell) \).

We defined in Section 5.3

\[
\mathcal{X}(\tau, \sigma^\vee) = \text{Hom}_{H_t(G)}(\tau \otimes H_t(H) \cdot \mathcal{H}_t(G), \sigma^\vee).
\]

As before, despite the notation, the tensor is with respect to the left \( \mathcal{H}_t(H) \)-module \( \tau \) and the right \( \mathcal{H}_t(H) \)-module \( \mathcal{H}_t(G) \).

Proposition 6.17. There is a canonical bijection between \( \mathcal{X}(\tau, \sigma^\vee) \) and \( \text{Hom}_H(\tau \otimes \sigma|_H, \mathbb{C}) \). More precisely, if \( \mathfrak{z} \in \mathcal{X}(\tau, \sigma^\vee) \) corresponds to \( z \in \text{Hom}_H(\tau \otimes \sigma|_H, \mathbb{C}) \), then

\[
\mathfrak{z}(\xi \otimes \phi)(\varphi) = z(\phi \otimes (\xi \cdot \varphi)).
\]

Proof. Note that \( \tau \otimes H_t(H) \mathcal{H}_t(G) \) is the left \( \mathcal{H}_t(G) \)-module corresponding to the smooth \( G(\mathbb{Q}_\ell) \)-representation \( \text{ind}_H^G \tau \), the compact induction of \( \tau \) from \( H \) to \( G \) ([Ren10, III.2.6 Théorème]). Hence

\[
\mathcal{X}(\tau, \sigma^\vee) = \text{Hom}_G \left( \text{ind}_H^G \tau, \sigma^\vee \right) = \text{Hom}_G \left( \left( \text{ind}_H^G \tau \right) \otimes \sigma, \mathbb{C} \right) = \text{Hom}_G \left( \sigma, \left( \text{ind}_H^G \tau \right)^\vee \right) = \text{Hom}_G \left( \sigma, \text{ind}_H^G (\tau^\vee) \right) = \text{Hom}_H (\sigma|_H, \tau^\vee) = \text{Hom}_H (\tau \otimes \sigma|_H, \mathbb{C} ).
\]

Here the fourth equality is [Ren10, III.2.7 Théorème] and the fifth equality is the Frobenius reciprocity ([Ren10, III.2.5 Théorème]).

Theorem 6.18 (Kato–Murase–Sugano, [LSZ19, Theorem 3.7.5]). Let \( \sigma' \) be an irreducible unramified principal series of \( \text{GSp}_4(\mathbb{Q}_\ell) \) with central character \( \omega_{\sigma'} \). Let \( (\chi_1, \chi_2) \), \( (\psi_1, \psi_2) \) be pairs of unramified characters of \( \mathbb{Q}_\ell^\times \) satisfying \( \chi_1 / \chi_2 \psi_1 / \psi_2 \omega_{\sigma'} = 1 \). Suppose neither \( \chi_1 / \psi_1 \) nor \( \chi_2 / \psi_2 \) is quadratic or \( | \cdot |^{-1} \). Then

\[
\dim \text{Hom}_H (I_H (\chi_1 \psi_1) \otimes \sigma'|_H, \mathbb{C}) \leq 1,
\]

where \( I_H (\chi, \psi) \) is \( I(\chi_1, \psi_1) \otimes I(\chi_2, \psi_2) \) regarded as an \( H \)-representation under restriction \( H \hookrightarrow \text{GL}_2 \times \text{GL}_2 \).

Corollary 6.19. Let \( \sigma = \sigma' \otimes \tau' \) be a representation of \( G(\mathbb{Q}_\ell) \) satisfying the assumption of Proposition 6.16, which we further assumed to be unramified. In particular, \( \sigma' \) and \( \tau' = I(\chi_2, \psi_2) \) are irreducible unramified principal series of \( \text{GSp}_4(\mathbb{Q}_\ell) \) and \( \text{GL}_2(\mathbb{Q}_\ell) \), respectively. Let \( \chi_1, \psi_1 \) be unramified characters of \( \mathbb{Q}_\ell^\times \) satisfying \( \chi_1 \chi_2 \psi_1 / \psi_2 \omega_{\sigma'} = 1 \), where \( \omega_{\sigma'} \) is the central character of \( \sigma' \). Suppose neither \( \chi_1 / \psi_1 \) nor \( \chi_2 / \psi_2 \) is quadratic or \( | \cdot |^{-1} \). Then \( \dim \text{Hom}_H (I(\chi_1, \psi_1) \otimes \sigma|_H, \mathbb{C}) \leq 1 \).

Proof. Denote \( I(\chi_1, \psi_1) \) by \( \tau \). Observe that \( \tau \otimes \sigma|_H \) is equal to \( (\tau \otimes \tau') \otimes (\sigma'|_H) \), where \( \tau \otimes \tau' \) is a representation of \( H \) with \( H \) acting on \( \tau \) through the first \( \text{GL}_2 \)-factor, and on \( \tau' \) through the second \( \text{GL}_2 \)-factor. Then the multiplicity one result follows from Theorem 6.18.
7. LOCAL FORMULA FOR NORM RELATIONS

In this section, we derive formula in local representation theory that will be useful later for proving norm relations of Euler system.

7.1. Double coset operators.

Definition 7.1. For integers \( m \geq 0 \) and \( n \geq 0 \), define open compact subgroups of \( G(\mathbb{Q}_\ell) \)

\[
K_{m,n} := \left\{ g \in G(\mathbb{Q}_\ell) : g \equiv \left( \begin{array}{cccc} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ \end{array} \right) \pmod{\ell^n}, \mu(g) \equiv 1 \pmod{\ell^m} \right\}
\]

\[
B_{m,n} := \left\{ g \in G(\mathbb{Q}_\ell) : g \equiv \left( \begin{array}{cccc} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ \end{array} \right) \pmod{\ell^n}, \mu(g) \equiv 1 \pmod{\ell^m} \right\}
\]

We denote by \( K_{m,n}^{(1)} \) (resp. \( K_{m,n}^{(2)} \)) the \( \text{GSp}_4 \) (resp. \( \text{GL}_2 \)) -component of \( K_{m,n} \). Similarly for \( B_{m,n}^{(1)} \) and \( B_{m,n}^{(2)} \).

Consider the following matrices in \( G(\mathbb{Q}_\ell) \). Note that except for \( u_3, u_4 \), the other \( u_i \)'s are in \( H(\mathbb{Q}_\ell) \).

\[
\begin{align*}
&u_0 := \left( \begin{array}{ccc} 1 & & \\
& 1 & \\
& & 1 \\
\end{array} \right), & & u_1 := \left( \begin{array}{c} \ell \\
& 1 \\
& & 1 \\
\end{array} \right), \\
&u_2 := \left( \begin{array}{cc} \ell^2 & \\
& \ell \\
\end{array} \right), & & u_3 := \left( \begin{array}{c} \ell^2 \\
& \ell \\
\end{array} \right), \\
&u_4 := \left( \begin{array}{cc} \ell^2 & \\
& 1 \\
\end{array} \right), & & u_5 := \left( \begin{array}{c} \ell^2 \\
& \ell \\
\end{array} \right), \\
&u_6 := \left( \begin{array}{cc} \ell^4 & \\
& \ell^2 \\
\end{array} \right).
\end{align*}
\]

Recall from Section 5.3 that \( \mathcal{H}_\ell(G) \) is the Hecke algebra of locally constant, compactly supported \( \mathbb{C} \)-valued functions on \( G(\mathbb{Q}_\ell) \).

Definition 7.2. Let \( m \geq 0 \) and \( n \geq 1 \) be integers. (Here we assume the stronger assumption \( n \geq 1 \) than in Definition 7.1).

- For \( i \in \{0, \ldots, 6\} \), define Hecke operators

\[
U_i(\ell) := \text{ch}(K_{m,n}u_iG(\mathbb{Z}_\ell)) \in \mathcal{H}_\ell(G),
\]

\[
U_i^{K_{1,0}}(\ell) := \text{ch}(K_{m,n}u_iK_{1,0}) \in \mathcal{H}_\ell(G) \text{ for } m \geq 1.
\]

- For \( i = 1, 2, 5 \), also define Hecke operators

\[
U_i^{B_{m,n}}(\ell) := \frac{1}{\text{Vol}B_{m,n}} \text{ch}(B_{m,n}u_iB_{m,n}) \in \mathcal{H}_\ell(G).
\]
Remark 7.3. Let \( u = \left( \begin{array}{ccc} \ell^{a_1} & \rho_{a_2} & \rho_{a_3} \\ \rho_{a_4} & \ell^{a_5} & \rho_{a_6} \end{array} \right), \left( \begin{array}{cc} \phi_1 & \phi_2 \\ \phi_3 & \phi_4 \end{array} \right) \) with \( a_1 \geq a_2 \geq a_3 \geq a_4 \geq 0 \). For any integers \( m \geq 0, n \geq 0 \), we have

\[
u^{-1}K_{m,n}u \cap G(\mathbb{Z}_\ell) \subset K_{m,n}.
\]

In particular, the canonical map

\[
K_{m,n}uK_{m,n}/K_{m,n} \rightarrow K_{m,n}uG(\mathbb{Z}_\ell)/G(\mathbb{Z}_\ell)
\]

is a bijection. Hence if \( n \geq 1 \), the double coset \( K_{m,n}uG(\mathbb{Z}_\ell) \) dose not depend on the choice of the integers \( m, n \). Furthermore if \( m \geq 1 \) and \( n \geq 1 \), then \( K_{m,n}uK_{1,0} \) is also independent of \( m, n \). So the definition of \( U_i(\ell) \) (resp. \( U_i^{K_{1,0}}(\ell) \)) above is independent of \( m \geq 0, n \geq 1 \) (resp. \( m \geq 1, n \geq 1 \)).

**Definition 7.4.** For \( i \in \{1, \ldots, 6\} \), let \( J_i \) be the set of left \( K_{1,1}\)-coset representatives of \( K_{1,1}u_iK_{1,1} \) to be given by Lemma 7.5 below. By Remark 7.3, \( J_i \) is also the set of left \( G(\mathbb{Z}_\ell)\)-coset representatives of \( K_{1,1}u_iG(\mathbb{Z}_\ell) \), and the set of left \( K_{1,0}\)-coset representatives of \( K_{1,1}u_iK_{1,0} \).

In order to carry out explicit calculations using these Hecke operators, we will need the following double coset decomposition.

**Lemma 7.5.** Let \( m \geq 0 \) and \( n \geq 1 \) be integers.

- For the \( \text{GSp}_4 \)-component and \( K_{m,n}^{(1)} \)-level, we have the double coset decomposition

\[
K_{m,n}^{(1)}u_1K_{m,n}^{(1)} = \bigcup_{x, y, z \in \mathbb{Z}/\ell} \left( \begin{array}{ccc} \ell & x & y \\ \ell & z & x \\ 1 & 1 & 1 \end{array} \right) K_{m,n}^{(1)} \bigcup_{x, y, z \in \mathbb{Z}/\ell} \left( \begin{array}{ccc} \ell & x & y \\ \ell & z & x \\ 1 & 1 & 1 \end{array} \right) K_{m,n}^{(1)}
\]

\[
K_{m,n}^{(1)}u_2K_{m,n}^{(1)} = K_{m,n}^{(1)}u_3K_{m,n}^{(1)} = \bigcup_{x, y, z \in \mathbb{Z}/\ell} \left( \begin{array}{ccc} \ell^2 & x & y \\ \ell & z & x \\ 1 & 1 & 1 \end{array} \right) K_{m,n}^{(1)}
\]

\[
K_{m,n}^{(1)}u_4K_{m,n}^{(1)} = \bigcup_{x, y, z \in \mathbb{Z}/\ell} \left( \begin{array}{ccc} \ell^2 & x & y \\ \ell^2 & z & x \\ 1 & 1 & 1 \end{array} \right) K_{m,n}^{(1)} \bigcup_{x, y, z \in \mathbb{Z}/\ell} \left( \begin{array}{ccc} \ell^2 & x & y \\ \ell^2 & z & x \\ 1 & 1 & 1 \end{array} \right) K_{m,n}^{(1)}
\]

\[
K_{m,n}^{(1)}u_5K_{m,n}^{(1)} = \bigcup_{x, y, z \in \mathbb{Z}/\ell} \left( \begin{array}{ccc} \ell^3 & x & y \\ \ell^2 & x & y \\ 1 & 1 & 1 \end{array} \right) K_{m,n}^{(1)} \bigcup_{x, y, z \in \mathbb{Z}/\ell} \left( \begin{array}{ccc} \ell^3 & x & y \\ \ell^2 & x & y \\ 1 & 1 & 1 \end{array} \right) K_{m,n}^{(1)}
\]

\[
K_{m,n}^{(1)}u_6K_{m,n}^{(1)} = \bigcup_{x, y, z \in \mathbb{Z}/\ell} \left( \begin{array}{ccc} \ell^4 & x & y \\ \ell^2 & x & y \\ 1 & 1 & 1 \end{array} \right) K_{m,n}^{(1)}
\]

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Theorem 7.7

A Whittaker functional of $\sigma$ over a local field. Fix a non-degenerate character of a maximal unipotent subgroup (This comes from 6.1.1 of [RS07] for the $W_Q$ in our case of $1'$). Let $\tau$ be an irreducible admissible representation of a quasi-split reductive group $G(\mathbb{Q}_\ell)$ (resp. $\tau'$). Let $\Psi$ be an unramified character of $\mathbb{Q}_\ell$ in the sense that

$$\Psi(\mathbb{Z}_\ell) = 1 \quad \text{and} \quad \Psi(\ell \cdot \mathbb{Z}_\ell) \neq 1.$$

Definition 7.6. A Whittaker functional of $\sigma'$ (resp. $\tau'$) with respect to $\Psi$ (resp. $\Psi^{-1}$) is a linear function

$$W_{\sigma'} : \sigma' \to \mathbb{C} \quad \text{(resp.} \ W_{\tau'} : \tau' \to \mathbb{C} \text{)}$$

satisfying the Whittaker relation

$$W_{\sigma'} \left( \begin{pmatrix} 1 & y & * & * \\ x & 1 & * & -y \\ & & & 1 \end{pmatrix} \right) g) = \Psi(x + y) W_{\sigma'}(g) \quad \text{(resp.} \ W_{\tau'} \left( \begin{pmatrix} 1 & x & 0 & 0 \\ 0 & 1 & 0 & 0 \\ & & & 1 \end{pmatrix} \right) g) = \Psi^{-1}(x) W_{\tau'}(g)).$$

Theorem 7.7 ([Sha74]). Let $\pi$ be an irreducible admissible representation of a quasi-split reductive group over a local field. Fix a non-degenerate character of a maximal unipotent subgroup (This comes from $\Psi$ and $\Psi^{-1}$ in our case of $\text{GSp}_4$ and $\text{GL}_2$, respectively.) Then the vector space of Whittaker functionals of $\pi$ with respect to the non-degenerate character has dimension $\leq 1$.

One says $\pi$ is generic if it admits a nonzero Whittaker functional.

We assume throughout that $\sigma'$ and $\tau'$ are generic. Let $W_{\sigma'}$ (resp. $W_{\tau'}$) be a non-zero Whittaker functional of $\sigma'$ (resp. $\tau'$). Then for each $\varphi_{\sigma'} \in \sigma'$ (resp. $\varphi_{\tau'} \in \tau'$), one can define a function

$$W_{\sigma', \varphi_{\sigma'}} : \text{GSp}_4(\mathbb{Q}_\ell) \to \mathbb{C} \quad \text{g} \mapsto W_{\sigma'}(g \cdot \varphi_{\sigma'})$$

(resp. $W_{\tau', \varphi_{\tau'}} : \text{GL}_2(\mathbb{Q}_\ell) \to \mathbb{C} \quad \text{g} \mapsto W_{\tau'}(g \cdot \varphi_{\tau'}))$

which by construction satisfies the Whittaker relation. There is a left $\text{GSp}_4(\mathbb{Q}_\ell)$ (resp. $\text{GL}_2(\mathbb{Q}_\ell)$)-action on $\{W_{\sigma', \varphi_{\sigma'}}\}$ (resp. $\{W_{\tau', \varphi_{\tau'}}\}$):

$$g \cdot W_{\sigma', \varphi_{\sigma'}}(\cdot) = W_{\sigma', \varphi_{\sigma'}}(\cdot - g) \quad \text{(resp.} \ g \cdot W_{\tau', \varphi_{\tau'}}(\cdot) = W_{\tau', \varphi_{\tau'}}(\cdot - g) \text{)}$$

and the canonical map $\sigma' \to \{W_{\sigma', \varphi_{\sigma'}}\}$ (resp. $\tau' \to \{W_{\tau', \varphi_{\tau'}}\}$) is an isomorphism of $\text{GSp}_4(\mathbb{Q}_\ell)$ (resp. $\text{GL}_2$)-representations. The space $\{W_{\sigma', \varphi_{\sigma'}}\}$ (resp. $\{W_{\tau', \varphi_{\tau'}}\}$) is called the Whittaker model of $\sigma'$ (resp. $\tau'$).
Recall from Definition 5.1 that we have the local zeta integral
\[ \mathcal{Z}(\varphi_{\sigma'}, \omega, \eta) = \int_{CN(H(\mathbb{Q}))} W_{\sigma', \varphi_{\sigma'}}(h_1, h_2) W_{\tau', \varphi_{\tau'}}(h_2) \chi_{GL}(h_1) |\det h_1|^s dh. \]
Here \( h = (h_1, h_2) \in H, C \) is the center of \( GSp_4(\mathbb{Q}_l) \), \( N \subset H(\mathbb{Q}_l) \) is the standard maximal unipotent subgroup, \( \Phi \) is a locally constant complex function on \( \mathbb{Q}_l \otimes \mathbb{Q}_l, \omega = \omega_{\varphi_{\tau'}} \) is the product of central characters of \( \sigma' \) and \( \tau' \), and \( f_{\omega, \sigma'}^H(h_1) \) := \( \int_{\mathbb{Q}_l^X} \Phi((0, x)x h_1) \omega(x) |x|^{2s} dx \).

**Definition 7.8** ([Sou84, Equation (3.8)]). Let \( \eta \) be an unramified character of \( \mathbb{Q}_l^* \), used for twisting. Define
\[ l(\varphi_{\sigma'} \otimes \varphi_{\tau'}, \eta, s) := \int_{\mathbb{Q}_l^* \times \mathbb{Q}_l^*} W_{\sigma', \varphi_{\sigma'}}(xy, x) W_{\tau', \varphi_{\tau'}}(x, y) |x|^{s-2} |y|^s \eta(xy) dx dy \]
and a normalized version
\[ Z(\varphi, \eta, s) := L(s, \sigma \otimes \eta)^{-1} l(\varphi, \eta, s). \]

**Definition 7.9.** Denote by \( \varphi_0 = \varphi_{\sigma', 0} \otimes \varphi_{\tau', 0} \) a spherical vector of \( \sigma = \sigma' \otimes \tau' \), normalized in the sense that
\[ W_{\sigma', \varphi_{\sigma'}, 0}(1) W_{\tau', \varphi_{\tau'}, 0}(1) = 1. \]

**Proposition 7.10.** \( Z(\varphi_0, \eta, s) = L(2s, \eta^2 \omega)^{-1}. \)

*Proof.* Since \( \varphi_0 \) is a spherical vector, we can rewrite the integral \( l(\varphi_0, \eta, s) \) as
\[ l(\varphi_0, \eta, s) = \sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} W_{\sigma', \varphi_{\sigma'}}(\ell_i j, \ell_i j) W_{\tau', \varphi_{\tau'}}(\ell_j, \ell_j) \chi_{\Lambda_i, j}(t_{\sigma'}) \chi_{\Lambda_j, j}(t_{\tau'}). \]

The Casselman–Shalika formula says that
\[ W_{\sigma', \varphi_{\sigma'}, 0}(\ell_i j, \ell_i j) = \delta_{GSp_4}^{1/2}(\ell_i j, \ell_i j) \cdot \chi_{\Lambda_i, j}(t_{\sigma'}), \quad \text{for } i \geq 0, j \geq 0, \]
\[ W_{\tau', \varphi_{\tau'}, 0}(\ell_j, \ell_j) = \delta_{GL_2}^{1/2}(\ell_j, \ell_j) \cdot \chi_{\Lambda_j, j}(t_{\tau'}), \quad \text{for } i \geq 0, \]
and 0 for \( i, j \) otherwise. Here \( \delta_{GSp_4} \) (resp. \( \delta_{GL_2} \)) is the modulus function of the Borel subgroup of \( GSp_4 \) (resp. \( GL_2 \)); in particular, \( \delta_{GSp_4}(\ell_1, \ell_2, \ell_3, \ell_4) = \epsilon^{-3\lambda_1 + 2\lambda_2 + \lambda_3} \) and \( \delta_{GL_2}(\ell_1, \ell_2) = \epsilon^{-\lambda_1 + \lambda_2} \).

\( \chi_{\Lambda_i, j}(t_{\sigma'}) \) (resp. \( \chi_{\Lambda_j, j}(t_{\tau'}) \)) is the character of the irreducible representation of \( GSp_4 \) (resp. \( GL_2 \)) with highest weight \((i + j)\lambda_1 + j\lambda_2 \) (resp. \((i + j)t_1 + jt_2 \)) defined at the beginning of Section 3. And \( t_{\sigma'} \) (resp. \( t_{\tau'} \)) is the semisimple conjugacy class in the \( L \)-group of \( GSp_4 \) (resp. \( GL_2 \)) associated to \( \sigma' \) (resp. \( \tau' \)). Note that in the above formula, we have implicitly used the fact that \( k \) \( GSp_4 \), the \( L \)-group of \( GSp_4 \), is isomorphic to \( GSp_4(\mathbb{C}) \), and similarly \( k \) \( GL_2 \cong GL_2(\mathbb{C}) \).

Hence if we denote \( \eta(\ell) \epsilon^{-s} \) by \( \chi \), then
\[ l(\varphi_0, \eta, s) = \sum_{i \geq 0, j \geq 0} \chi_{\Lambda_i, j}(t_{\sigma'}) \chi_{\Lambda_j, j}(t_{\tau'}) \chi^{i + 2j}. \]

A similar calculation to [GPSR87, p.139-p.140], except that we have a twist by \( \eta \) and we use \( GSp_4 \) instead of \( Sp_4 \), then implies that
\[ l(\varphi_0, \eta, s) = L(2s, \wedge^2(\tau' \otimes \eta) \otimes \omega_{\sigma'})^{-1} L(s, \sigma' \otimes \tau' \otimes \eta). \]
Note that $\wedge^2 (\tau' \otimes \eta) \otimes \omega_{\sigma'} = \eta^2 \det(\tau') \omega_{\sigma'} = \eta^2 \omega$. Hence

$$Z(\varphi_0, \eta, s) = L(s, \sigma \otimes \eta)^{-1} l(\varphi_0, \eta, s) = L(2s, \eta^2, \omega)^{-1}.$$ 

□

To simplify notation, we write

$$W_{\sigma', \varphi_{\sigma'}}(i, j) := W_{\sigma', \varphi_{\sigma'}}(\begin{pmatrix} \ell_{i+2j} & \ell_{i+j} & \ell_j & 1 \\ \ell_{i+j} & \ell_j & 1 \\ \ell_j & 1 \\ 1 \end{pmatrix})$$

$$W_{\tau', \varphi_{\tau'}}(i, j) := W_{\tau', \varphi_{\tau'}}(\begin{pmatrix} \ell_{i+j} & \ell_j \\ \ell_j & 1 \end{pmatrix})$$

and when $\varphi_{\sigma'}$ (resp. $\varphi_{\tau'}$) is the normalized spherical vector $\varphi_{\sigma', 0}$ (resp. $\varphi_{\tau', 0}$), we further omit the subscript of $\varphi_{\sigma'}$ (resp. $\varphi_{\tau'}$).

Breaking up the local zeta integral into double sums, we see

$$l(\varphi_0, \eta, s) = \sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} W_{\sigma'}(i, j) W_{\tau'}(i, j) \ell^{2(i+j)} X^{i+2j}$$

$$= \sum_{i \geq 0} \sum_{j \geq 0} W_{\sigma'}(i, j) W_{\tau'}(i, j) \ell^{2(i+j)} X^{i+2j}.$$ 

More generally, we have the following:
Proposition 7.11. We have the following identities:

\[ l(U_1(\ell)\varphi_0, \eta, s) = \ell^2 X^{-1} \left( \sum_{i \geq 0} \sum_{j \geq 1} \sum_{i \geq 1} \sum_{j \geq 0} W_{\sigma}(i, j) W_{\tau}(i, j) \ell^{2(i+j)} X^{i+j} \right) \]

\[ + \ell X^{-1} \sum_{i \geq 2} \sum_{j \geq 0} W_{\sigma}(i, j) W_{\tau}(i - 2, j + 1) \ell^{2(i+j)} X^{i+j} \]

\[ + \ell^3 X^{-1} \sum_{i \geq 0} \sum_{j \geq 1} W_{\sigma}(i, j) W_{\tau}(i + 2, j - 1) \ell^{2(i+j)} X^{i+j}, \]

\[ l(U_2(\ell)\varphi_0, \eta, s) = \ell^2 X^{-2} \sum_{i \geq 0} \sum_{j \geq 1} W_{\sigma}(i, j) W_{\tau}(i, j) \ell^{2(i+j)} X^{i+j}, \]

\[ l((U_3(\ell) + U_2(\ell)) \varphi_0, \eta, s) = (\ell - 1) \ell^2 X^{-2} \sum_{i \geq 1} \sum_{j \geq 1} W_{\sigma}(i, j) W_{\tau}(i, j) \ell^{2(i+j)} X^{i+j} \]

\[ + \ell^2 X^{-2} \sum_{i \geq 2} \sum_{j \geq 1} W_{\sigma}(i, j) W_{\tau}(i - 2, j + 1) \ell^{2(i+j)} X^{i+j} \]

\[ + \ell^4 X^{-2} \sum_{i \geq 0} \sum_{j \geq 1} W_{\sigma}(i, j) W_{\tau}(i + 2, j - 1) \ell^{2(i+j)} X^{i+j}, \]

\[ l((U_4(\ell) + U_2(\ell)) \varphi_0, \eta, s) = (\ell - 1) \ell^2 X^{-2} \sum_{i \geq 1} \sum_{j \geq 1} W_{\sigma}(i, j) W_{\tau}(i, j) \ell^{2(i+j)} X^{i+j} \]

\[ + \ell^2 X^{-2} \sum_{i \geq 2} \sum_{j \geq 0} W_{\sigma}(i, j) W_{\tau}(i - 2, j + 1) \ell^{2(i+j)} X^{i+j} \]

\[ + \ell^4 X^{-2} \sum_{i \geq 0} \sum_{j \geq 1} W_{\sigma}(i, j) W_{\tau}(i + 2, j - 1) \ell^{2(i+j)} X^{i+j}, \]

\[ l(U_5(\ell)\varphi_0, \eta, s) = \ell^4 X^{-3} \left( \sum_{i \geq 1} \sum_{j \geq 1} \sum_{i \geq 2} \sum_{j \geq 2} W_{\sigma}(i, j) W_{\tau}(i, j) \ell^{2(i+j)} X^{i+j} \right) \]

\[ + \ell^3 X^{-3} \sum_{i \geq 2} \sum_{j \geq 1} W_{\sigma}(i, j) W_{\tau}(i - 2, j + 1) \ell^{2(i+j)} X^{i+j} \]

\[ + \ell^5 X^{-3} \sum_{i \geq 0} \sum_{j \geq 1} W_{\sigma}(i, j) W_{\tau}(i + 2, j - 1) \ell^{2(i+j)} X^{i+j}, \]

\[ l(U_6(\ell)\varphi_0, \eta, s) = \ell^4 X^{-4} \sum_{i \geq 0} \sum_{j \geq 2} W_{\sigma}(i, j) W_{\tau}(i, j) \ell^{2(i+j)} X^{i+j}, \]

Proof. The Proposition follows from explicit calculations of local zeta integrals. We will only show the proof for \( U_4(\ell) + U_2(\ell) \). The others are completely analogous.

Following the explicit double coset decomposition in Lemma 7.5, we see

\[ (U_4(\ell) + U_2(\ell)) \varphi_0 = \sum_{\gamma \in J_1 \cup J_2} \gamma \cdot \varphi_0. \]

Here \( \gamma \) runs through the disjoint union \( J_4 \sqcup J_2 = J_4^1 \sqcup (J_2^1 \sqcup J_2^2) \sqcup J_2^3 \) with

\[ J_4^1 = \{(A_{t,y}^{x,z}, \ell \cdot I_2) : x, y, z \in \{0, \ldots, \ell^2 - 1\}\}, \]

\[ J_2^1 \sqcup J_2 = \{(A_{t,y}^{x,z,w}, \ell \cdot I_2) : x, y, w \in \{0, \ldots, \ell - 1\}, z \in \{0, \ldots, \ell - 1\}\}, \]

\[ J_2^3 = \{(A_{t,y}^{x,z,w}, \ell \cdot I_2) : x, y \in \{0, \ldots, \ell^2 - 1\}\}, \]

where to simplify notation (we only use the following notation in the proof) we write

\[
A_{t,y}^{x,z} := \begin{pmatrix} \ell^2 & x & y \\ \ell^2 & z & x \\ 1 & 1 \end{pmatrix}, \quad A_{t,y}^{x,z,w} := \begin{pmatrix} \ell^2 & \ell x & \ell y + \ell z \\ \ell & w & y \\ \ell & -x & 1 \end{pmatrix}, \quad A_{t,y}^{x,y} := \begin{pmatrix} \ell^2 & x & y \\ 1 & \ell^2 & -x \\ 1 & 1 \end{pmatrix}
\]
and $I_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Because the GL$_2$-component $\gamma^{(2)}$ of any $\gamma \in J_1 \cup J_2$ is $\ell \cdot I_2$, the zeta integral $l((U_4(\ell) + U_2(\ell)) \varphi_0, \eta, s)$ becomes

$$\sum_{\gamma \in J_1 \cup J_2} \sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} W_\sigma(\begin{pmatrix} \ell^{i+j} & \ell^{i+j} \\ \ell^i & 1 \end{pmatrix}) A_{\gamma} W_\tau(\begin{pmatrix} \ell^{i+j+1} & \ell^{j+1} \\ \ell^i & 1 \end{pmatrix}) X^{i+j}.$$  \hfill (5)

- For $\gamma \in J_1^1$, we have
  $$A_{\gamma} = \begin{pmatrix} \ell^{i+j} & \ell^{i+j} \\ \ell^i & 1 \end{pmatrix} A_{1, \gamma} = \begin{pmatrix} 1 & \ell^{i+j} & \ell^{i+j} \ell^z \\ \ell^i & 1 & 1 \end{pmatrix}$$

so by definition of Whittaker functions

$$W_\sigma(\begin{pmatrix} \ell^{i+j} \\ \ell^i \end{pmatrix}) A_{1, \gamma} W_\tau(\begin{pmatrix} \ell^{i+j+1} \\ \ell^i \end{pmatrix}) = \Psi(\ell^z) W_\sigma(i + 2, j).$$

Note that

$$\sum_{0 \leq z < \ell^2} \Psi(\ell^z) = \begin{cases} \ell^2 & \text{if } i \geq 0; \\ 0 & \text{if } i = -1 \text{ or } -2, \end{cases}$$

by the unramified condition on $\Phi$. Thus the contribution to equation (5) is

$$\sum_{x, y, z \in \mathbb{Z}/\ell^2} \sum_{i \geq 0} \sum_{j \geq 0} W_\sigma(\begin{pmatrix} \ell^{i+j} & \ell^{i+j} \\ \ell^i & 1 \end{pmatrix}) A_{\gamma} W_\tau(\begin{pmatrix} \ell^{i+j+1} & \ell^{j+1} \\ \ell^i & 1 \end{pmatrix}) X^{i+j}$$

$$= \ell^6 \sum_{i, j \geq 0} W_\sigma(i + 2, j) W_\tau(i, j + 1) X^{i+j}$$

$$= \ell^2 X^{-2} \sum_{i \geq 2} \sum_{j \geq 0} W_\sigma(i, j) W_\tau(i - 2, j + 1) X^{i+j}.$$  \hfill (6)

Here note that the terms with $i < -2$ in the sum on the left vanishes because $W_\sigma(i + 2, j) = 0$, and similarly for terms with $j < 0$.

- For $\gamma \in J_2^1 \cup J_2$, we have
  $$A_{\gamma} = \begin{pmatrix} \ell^{i+j} & \ell^{i+j} \\ \ell^i & 1 \end{pmatrix} A_{2, \gamma} = \begin{pmatrix} 1 & \ell x & \ell^{i+j} y + \frac{\ell^z}{\ell} \\ \ell^{-1} w & 1 & \ell^{i+j} y \\ \ell & 1 & -\ell^i x \end{pmatrix}$$

so we are summing

$$\Psi(\ell^{i-1} w + \ell^i x) W_\sigma(i, j + 1) W_\tau(i, j + 1) X^{i+j}.$$ 

Note that

$$\sum_{0 \leq x, w < \ell} \Psi(\ell^{i-1} w + \ell^i x) = \sum_{0 \leq w < \ell} \Psi(\ell^{i-1} w) \sum_{0 \leq x < \ell} \Psi(\ell^i x)$$

$$= \begin{cases} \ell^2 - \ell & \text{if } i \geq 1 \text{ and } j \geq 0; \\ 0 & \text{if } i = 0 \text{ or } j = -1. \end{cases}$$
Thus the contribution to equation (5) is
\[
\sum_{\ell, j, i \in \mathbb{Z}/\mathbb{Z}, \chi, \psi} \sum_{i \geq 0} W_{\sigma}(\ell \chi) A_{\ell} W_{\tau}(\ell \psi) \ell^{2(i+j)} X^{i+j+1}
\]
(7)
\[
= \ell^4 (\ell - 1) \sum_{i \geq 0} W_{\sigma}(i, j) W_{\tau}(i, j + 1) \ell^{2(i+j)} X^{i+j+1}
\]
\[
= \ell^2 (\ell - 1) X^{-2} \sum_{i \geq 0} W_{\sigma}(i, j) W_{\tau}(i, j) \ell^{2(i+j)} X^{i+j+1}.
\]

**For** \( \gamma \in J^1 \), we have
\[
\begin{pmatrix}
\ell^2 + j \\
\ell^2 + j \\
\ell^2 + j
\end{pmatrix}
\begin{pmatrix}
\ell^2 + j \\
\ell^2 + j \\
\ell^2 + j
\end{pmatrix}
A_{\gamma} = \begin{pmatrix}
1 & \ell^2 + j & 1 \\
1 & 1 & -\ell^2 + j \\
1 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
\ell^2 + j \\
\ell^2 + j \\
\ell^2 + j
\end{pmatrix}
\]
so we are summing
\[
\Psi(\ell \chi) W_{\sigma}(i - 2, j + 2) W_{\tau}(i, j + 1) \ell^{2(i+j)} X^{i+j+1}.
\]

Note that
\[
\sum_{0 \leq j < \ell^2} \Psi(\ell \chi) = \begin{cases}
\ell^2 & \text{if } j \geq 0; \\
0 & \text{if } j = -1 \text{ or } -2.
\end{cases}
\]

Thus the contribution to equation (5) is
\[
\sum_{\ell, j, i \in \mathbb{Z}/\mathbb{Z}, \gamma, \delta} \sum_{i \geq 0} W_{\sigma}(\ell \chi) A_{\ell} W_{\tau}(\ell \psi) \ell^{2(i+j)} X^{i+j+1}
\]
(8)
\[
= \ell^4 (\ell - 1) \sum_{i \geq 0} W_{\sigma}(i - 2, j + 2) W_{\tau}(i, j + 1) \ell^{2(i+j)} X^{i+j+1}
\]
\[
= \ell^4 X^{-2} \sum_{i \geq 0} W_{\sigma}(i, j) W_{\tau}(i + 2, j - 1) \ell^{2(i+j)} X^{i+j+1}.
\]

Combining equations (6), (7) and (8) in the above three cases immediately yields the desired identity. \( \square \)

**Definition 7.12.** For \( s \in \mathbb{C}, \) define
\[
\xi_s(\ell) := \frac{1}{\ell^{s+2}} U_1(\ell) + \frac{2\eta^2(\ell)}{\ell^{s+3}} U_2(\ell) + \frac{\eta^2(\ell)}{\ell^{s+3}} U_3(\ell) + \frac{2\eta^2(\ell)}{\ell^{s+3}} U_4(\ell) - \frac{\eta^3(\ell)}{\ell^{s+4}} U_5(\ell) + \frac{2\eta^4(\ell)}{\ell^{s+4}} U_6(\ell).
\]

**Corollary 7.13.** \( Z(\xi_s(\ell) \phi_0, \eta, s) = L(s, \sigma \otimes \eta)^{-1}. \)

**Proof.** A direct calculation using Proposition 7.11 shows \( L(\xi_s(\ell) \phi_0, \eta, s) = 1, \) and hence
\[
Z(\xi_s(\ell) \phi_0, \eta, s) = L(s, \sigma \otimes \eta)^{-1} l(\xi_s(\ell) \phi_0, \eta, s) = L(s, \sigma \otimes \eta)^{-1}.
\]
\( \square \)

Let \( \chi, \psi \) be two characters of \( \mathbb{Q}_1^X \) such that \( \chi/\psi \neq |.|^{1/2} \) and \( \chi \psi = \omega^{-1} \). From now on we take \( \eta = \psi|.|^{1/2}. \)

**Definition 7.14.** For \( \varphi \in \sigma \) and \( s \in \mathbb{C}, \) define a function \( z_{\varphi,s} : H(\mathbb{Q}_1) \to \mathbb{C} \) by
\[
z_{\varphi,s}(h) = Z(h \cdot \varphi, \eta, s).
\]

**Lemma 7.15.** \( z_{\varphi,s} \in I(|.|^{-s}\psi^{-1}, |.|^{-s}\chi^{-1}) \otimes I(|.|^{s}, |.|^{-s}). \)

**Proof.** This follows from a straightforward calculation of the \( H \)-action on \( z_{\varphi,s}. \) \( \square \)
In particular, we have an $H(\mathbb{Q}_\ell)$-equivariant map
\[ \sigma \to I([|\cdot|^{-s}y^{-1}, |\cdot|^s\chi^{-1}]) \otimes I([|\cdot|^s, |\cdot|^{-s}]), \quad \varphi \mapsto z_{\varphi,s}. \]

**Proposition 7.16.**
\[ z_{\varphi_0,s}(1) = L(\psi/\chi, 2s + 1)^{-1}; \]
\[ z_{\xi,\ell \varphi_0,s}(1) = L(s, \sigma \otimes \eta)^{-1}. \]

**Proof.** The first part follows directly from Proposition 7.10 and the fact $\eta^2 \omega = |\cdot|\psi/\chi$. The second part is a direct consequence of Corollary 7.13. \ \qed

**Definition 7.17.** Let $\phi \in \text{Hom}_{H(\mathbb{Q}_\ell)}(I(\chi, \psi) \otimes \sigma, \mathbb{C})$ be nonzero.

**Proposition 7.18.** $\hat{\lambda}_{\chi,\psi} \in \text{Hom}_{H(\mathbb{Q}_\ell)}(I(\chi, \psi) \otimes \sigma, \mathbb{C})$ is nonzero.

**Proof.** As $\chi/\psi \neq |\cdot|^{\pm 1}$, according to equation (10) below, $\hat{\lambda}_{\chi,\psi}(F_{\varphi_0,0}) \neq 0$, which in particular implies $\hat{\lambda}_{\chi,\psi}$ is nonzero. \ \qed

**Remark 7.19.** The assumption $\chi/\psi \neq |\cdot|^{\pm 1}$ can be lessened just as in [LSZ19]. In fact, as we are only using the constant function $1 \in I([|\cdot|^{-1/2}, |\cdot|^1/2])$ in the pairing defining $\hat{\lambda}_{\chi,\psi}$, it factors through
\[ I(\psi^{-1} |\cdot|, \chi^{-1} |\cdot|) \otimes \mathbb{C} \xrightarrow{\lambda_{\chi,\psi}} \mathbb{C} \]
where the first $\mathbb{C}$ is the trivial sub-representation of $I([|\cdot|^{-1/2}, |\cdot|^1/2])$, and the second $\mathbb{C}$ is the trivial quotient representation of $I([|\cdot|^1/2, |\cdot|^{-1/2})$. Now write $\tau' = I(\chi', \psi')$, and assume the functions $L(\sigma' \otimes \psi', s)$ and $L(\psi/\chi, s + 1/2) L(\psi/\chi', s + 1/2)$ do not both have a pole at $s = 1/2$. Then the $H(\mathbb{Q}_\ell)$-equivariant map
\[ \sigma \mapsto I(\psi^{-1}, \chi^{-1}) \otimes I([|\cdot|^{1/2}, |\cdot|^{-1/2}) \to I(\psi^{-1}, \chi^{-1}) \otimes \mathbb{C} \]
factors through the unique irreducible quotient of $\sigma$, and has its image landing in the unique irreducible sub-representation of $I(\psi^{-1}, \chi^{-1})$. This follows directly from Lemma 3.7.2 of [LSZ19], which in turn builds on an explicit computation using Mackey theory. Then we may define $\hat{\lambda}_{\chi,\psi}(f \otimes \phi) = \lim_{s \to 0} L(\psi/\chi, 2s + 1)(M(f_s) \otimes 1, z_{\varphi,s}).$ However, for ease of exposition, we will keep assuming $\chi/\psi \neq |\cdot|$ and $\tau$ is irreducible.

To further simplify notation, fixing the characters $\chi, \psi$, for every $\phi \in \mathcal{S}(\mathbb{Q}_\ell^2, \mathbb{C})$, we write
\[ F_\phi := f_{\phi,\chi,\psi} \in I(\chi, \psi), \]
for the Siegel section defined in Proposition 6.7.

We choose $\chi = |\cdot|^{1/2+k}\nu$ and $\psi = |\cdot|^{-1/2}$, where $k \geq 1$ is an integer and $\nu$ is a finite-order unramified character. Then $\eta = \psi^1 |\cdot|^{1/2} = 1$, and by Proposition 6.7, the map $\phi \mapsto F_\phi$ is $GL_2(\mathbb{Q}_\ell)$-equivariant.

**Proposition 7.20.** Let $\sigma = \sigma' \otimes \tau'$ be an unramified representation of $G(\mathbb{Q}_\ell)$ satisfying the assumption of Proposition 6.16. As above let $\chi = |\cdot|^{1/2+k}\nu, \psi = |\cdot|^{-1/2}$ and $\chi/\psi, \omega = 1$, where $\omega$ is the central character of $\sigma$. Let $\hat{\lambda} \in \text{Hom}_{H(\mathbb{Q}_\ell)}(I(\chi, \psi) \otimes \sigma, \mathbb{C})$. Then for any integer $t \geq 1$,
\[ \hat{\lambda}(F_{\phi_0,t}, \varphi_0) = \frac{1}{t!^{-1}(t + 1)}(1 - \frac{e^k}{\nu(t)})\hat{\lambda}(F_{\phi_0,0}, \varphi_0), \]
\[ \hat{\lambda}(F_{\phi_0,1}, \xi_{\ell}(t)\varphi_0) = \frac{1}{t!^{-1}(t + 1)}L(0, \sigma)^{-1}\hat{\lambda}(F_{\phi_0,0}, \varphi_0). \]
Proof. We know \( \dim \text{Hom}_{H(Q)}(I(\chi, \psi) \otimes \sigma, \mathbb{C}) \leq 1 \) from Corollary 6.19 and \( \mathfrak{z}_{X, \psi} \) is nonzero by Proposition 7.18, so it suffices to prove the proposition for \( \mathfrak{z} = \mathfrak{z}_{X, \psi} \).

By definition, \( F_{\phi, \omega} \) is the value at \( s = 0 \) of the Siegel section \( f_{\phi, \omega, \chi, -\infty} \), and by Proposition 6.8, we have

\[
M(f_{\phi, \omega, \chi, -\infty}) = L(\chi/\psi, 1 - 2s) f_{\phi, \omega, \psi, +\infty}.
\]

On the other hand, by Lemma 6.10, \( f_{\phi, \omega, \psi, +\infty} \) restricted to \( \text{GL}_2(\mathbb{Z}) \) is supported on \( K_1^{GR}(\ell^t, 1) \). Since \( \phi_{\omega, \chi} \) is invariant under \( K_1^{GR}(\ell^t, 1) \), it follows that \( f_{\phi, \omega, \psi, +\infty} \otimes 1 \) restricted to \( H(\mathbb{Z}_t) \) is a scalar multiple of \( \text{ch}(K_1^{GR}(\ell^t, 1)) \). Moreover, since \( \varphi_0 \) is the spherical vector, it is fixed by \( H(\mathbb{Z}_t) \), which means \( z_{\varphi_0, s} \) is constant on \( H(\mathbb{Z}_t) \). Therefore

\[
\langle M(f_{\phi, \omega, \chi, -\infty}), z_{\varphi_0, s} \rangle = L(\chi/\psi, 1 - 2s) \int_{H(\mathbb{Z}_t)} f_{\phi, \omega, \psi, +\infty}(h) z_{\varphi_0, s}(h) dh.
\]

Here in the last equality we have used Lemma 6.10 and Proposition 7.16. It follows that

\[
\mathfrak{z}_{X, \psi}(F_{\phi, \omega, \varphi_0}) = \lim_{s \to 0} L(\psi/\chi, 1 + 2s)^{-1} M(f_{\phi, \omega, \chi, -\infty}) \otimes 1, \quad z_{\varphi_0, s}
\]

(10)

As \( L(\psi/\chi, 1)^{-1} = 1 - \ell^{-1} \psi/\chi(\ell) = 1 - \chi^s(\chi^s) \), this proves the first part of the proposition.

For the second part, by Remark 7.3 each \( U_1(\ell) \) is invariant under left-translation of \( K_{0,1} \), so \( z_{\xi_0(\ell), \varphi_0, s} \) is constant on \( K_1^{GR}(\ell, 1) = K_{0,1} \cap H(\mathbb{Z}_t) \). Then a similar argument as above shows that if \( t > 0 \),

\[
\mathfrak{z}_{X, \psi}(F_{\phi, \omega, \xi_0(\ell), \varphi_0}) = \lim_{s \to 0} L(\psi/\chi, 1 + 2s)^{-1} \cdot \text{Vol} K_1^{GR}(\ell^t, 1) z_{\xi_0(\ell), \varphi_0, s}(1)
\]

using Proposition 7.16 for the value of \( z_{\xi_0(\ell), \varphi_0, s}(1) \). The desired conclusion follows immediately by comparing the formula for \( \mathfrak{z}_{X, \psi}(F_{\phi, \omega, \xi_0(\ell), \varphi_0}) \) and \( \mathfrak{z}_{X, \psi}(F_{\phi, \omega, \varphi_0}) \). \( \square \)

Recall that for \( \tau \) a smooth representation of \( GL_2(Q) \) and \( \sigma \) a smooth representation of \( G(Q) \), we defined \( \mathfrak{X}(\tau, \sigma^\vee) = \text{Hom}_{H(\mathbb{C})}(\tau \otimes H(\mathbb{C}), \mathfrak{U}(G), \sigma^\vee) \) at the beginning of Section 6.4.

**Definition 7.21.** Given \( \mathfrak{z} \in \text{Hom}_{H}(I(\chi, \psi) \otimes \sigma|_H, \mathbb{C}) \), define \( \mathfrak{z} \in \mathfrak{X}(S(Q^\ell_{\mathbb{C}}, \mathbb{C}), \sigma^\vee) \) as the image of \( \mathfrak{z} \) under the map

\[\text{Hom}_{H}(I(\chi, \psi) \otimes \sigma|_H, \mathbb{C}) \xrightarrow{6.17} \mathfrak{X}(I(\chi, \psi), \sigma^\vee) \rightarrow \mathfrak{X}(S(Q^\ell_{\mathbb{C}}, \mathbb{C}), \sigma^\vee)\]

where the second map is induced by \( S(Q^\ell_{\mathbb{C}}, \mathbb{C}) \rightarrow I(\chi, \psi), \phi \mapsto F_\phi \). More precisely, for \( \varphi \in \sigma \),

\[
\mathfrak{z}(\phi \otimes \xi)(\varphi) = \mathfrak{z}(F_\phi \otimes \xi \varphi).
\]

Recall for integers \( s \geq t \geq 0 \) we defined in Definition 6.9 the locally constant function \( \phi_{s,t} \in S(Q^\ell_{\mathbb{C}}, \mathbb{C}) \) and its stabilizer \( K_1^{GR}(\ell^t, \ell^t) \subset GL_2(\mathbb{Z}_t) \).

**Lemma 7.22.** Let \( \xi \in H(\mathbb{C}) \) be invariant under left-translation by the principal congruence subgroup of level \( \ell^T \) in \( H(\mathbb{Z}_d) \) for some \( T \geq 0 \). Then

\[
\frac{1}{\text{Vol} K_1^{GR}(\ell^t, \ell^t)} \mathfrak{z}(\phi_{s,t} \otimes \xi)
\]

is independent of \( s \geq t \geq T \).
Proof. First observe that the coset representatives of $\mathbb{K}_1^{\text{GL}_2}(\ell^t,\ell^t)/\mathbb{K}_1^{\text{GL}_2}(\ell^t,\ell^t)$ can be chosen to be inside the principle congruence subgroup of level $\ell^t$ in $\text{GL}_2(\mathbb{Q}_\ell)$. Hence we have
\[
\mathcal{Z}(\phi_{t,t} \otimes \xi) = \sum_{\gamma \in \mathbb{K}_1^{\text{GL}_2}(\ell^t,\ell^t)/\mathbb{K}_1^{\text{GL}_2}(\ell^t,\ell^t)} \mathcal{Z}(\gamma \cdot \phi_{s,t} \otimes \xi) \\
= \sum_{\gamma \in \mathbb{K}_1^{\text{GL}_2}(\ell^t,\ell^t)/\mathbb{K}_1^{\text{GL}_2}(\ell^t,\ell^t)} \mathcal{Z}(\phi_{s,t} \otimes \xi \cdot \gamma) \\
= \frac{\text{Vol} \mathbb{K}_1^{\text{GL}_2}(\ell^t,\ell^t)}{\text{Vol} \mathbb{K}_1^{\text{GL}_2}(\ell^t,\ell^t)} \mathcal{Z}(\phi_{s,t} \otimes \xi).
\]
Here the third equality follows from our choice of $\gamma$ inside the principle congruence subgroup of level $\ell^t$ in $\text{GL}_2(\mathbb{Q}_\ell)$, and by assumption $\xi$ is invariant under the left-translation it. \hfill \Box

Definition 7.23. We write $\mathcal{Z}(\phi_\infty \otimes \xi) := \lim_{s,t \to \infty} \frac{1}{\text{Vol} \mathbb{K}_1^{\text{GL}_2}(\ell^s,\ell^t)} \mathcal{Z}(\phi_{s,t} \otimes \xi)$ for this limit value.

Remark 7.24. Let $U$ be an open compact subgroup of $G(\mathbb{Q}_\ell)$ and $\eta \in G(\mathbb{Q}_\ell)$. The group $\eta U \eta^{-1} \cap H(\mathbb{Z}_\ell)$ contains the principal congruence subgroup of level $\ell^T$ in $H(\mathbb{Z}_\ell)$ for some $T \geq 0$ since $\eta U \eta^{-1} \cap H(\mathbb{Z}_\ell)$ is an open compact subgroup of $H(\mathbb{Z}_\ell)$. Hence $\text{ch}(\eta U)$ is invariant under left-translation by the principal congruence subgroup of level $\ell^T$ in $H(\mathbb{Z}_\ell)$ for some $T \geq 0$. In particular, one can define $\mathcal{Z}(\phi_\infty \otimes \xi)$ for any element $\xi \in H(\ell)(G)$.

Theorem 7.25.
\[
\mathcal{Z}(\phi_\infty \otimes \xi_0(\ell)) = L(0,\sigma)^{-1} \mathcal{Z}(\phi_{0,0} \otimes \text{ch}(G(\mathbb{Z}_\ell))).
\]

Proof. Translating the second identity of Theorem 7.20 in terms of $\mathcal{Z}$, together with the fact that the representation $\sigma$ is irreducible and thus generated by $\varphi_0$ gives
\[
\mathcal{Z}(\phi_{t,0} \otimes \xi_0(\ell)) = \frac{1}{\ell^{t-1}(\ell + 1)} L(0,\sigma)^{-1} \mathcal{Z}(\phi_{0,0} \otimes \text{ch}(G(\mathbb{Z}_\ell))).
\]
Since $\xi_0(\ell)$ is invariant under left-translation by $K_{0,1}$, Lemma 7.22 implies that
\[
\mathcal{Z}(\phi_\infty \otimes \xi_0(\ell)) = \frac{1}{\text{Vol} \mathbb{K}_1^{\text{GL}_2}(\ell^t,\ell^t)} \mathcal{Z}(\phi_{t,t} \otimes \xi_0(\ell)) = \ell^{2t-2}(\ell^2 - 1) \mathcal{Z}(\phi_{t,t} \otimes \xi_0(\ell))
\]
for any integer $t \geq 1$.

Note that $\phi_{t,0} = \text{ch}(\ell^t \mathbb{Z}_\ell \times \mathbb{Z}_\ell^\times)$ and $\phi_{t,t} = \text{ch}(\ell^t \mathbb{Z}_\ell \times (1 + \ell^t \mathbb{Z}_\ell))$ are related via
\[
\phi_{t,0} = \sum_{a \in (\mathbb{Z}/\ell^t \mathbb{Z})^\times} \left( \begin{array}{c} 1 \\ a \end{array} \right) \cdot \phi_{t,t}.
\]
Hence by $H(\mathbb{Q}_\ell)$-equivariance of $\mathcal{Z},$
\[
\mathcal{Z}(\phi_{t,0} \otimes \xi_0(\ell)) = \sum_{a \in (\mathbb{Z}/\ell^t \mathbb{Z})^\times} \mathcal{Z}(\phi_{t,t} \otimes \xi_0(\ell)) \cdot \left( \begin{array}{c} 1 \\ a \end{array} \right) = \ell^{t-1}(\ell - 1) \mathcal{Z}(\phi_{t,t} \otimes \xi_0(\ell)),
\]
where in the last identity we used the fact that $\xi_0(\ell)$ is invariant under left-translation by $K_{0,1}$, and
\[
\left( \begin{array}{c} 1 \\ a \end{array} \right) \in K_{0,1} \cap H(\mathbb{Z}_\ell). \] The desired result follows by combining all these equations:
\[
\mathcal{Z}(\phi_\infty \otimes \xi_0(\ell)) = \ell^{2t-2}(\ell^2 - 1) \mathcal{Z}(\phi_{t,t} \otimes \xi_0(\ell)) \\
= \ell^{t-1}(\ell + 1) \mathcal{Z}(\phi_{t,0} \otimes \xi_0(\ell)) \\
= L(0,\sigma)^{-1} \mathcal{Z}(\phi_{0,0} \otimes \text{ch}(G(\mathbb{Z}_\ell))).
\]
\hfill \Box
Lemma 7.26. Let $U \subset G(\mathbb{Q}_\ell)$ be an open compact subgroup and $v$ an integer. Take an element $h \in H(\mathbb{Q}_\ell)$ satisfying $\text{pr}_1(h) \in \left(\ell^v \cdot \mathbb{Z}_\ell^\times, \mathbb{Q}_\ell, 1\right)$. Then for any $g \in G(\mathbb{Q}_\ell)$, we have
\[
3(\phi_\infty \otimes \text{ch}(hgU)) = \ell^{-v} \cdot 3(\phi_\infty \otimes \text{ch}(gU)).
\]
In particular, if $h \in U$ and $v = 0$, we have
\[
3(\phi_\infty \otimes \text{ch}(hg^{-1}U)) = 3(\phi_\infty \otimes \text{ch}(hgU)) = 3(\phi_\infty \otimes \text{ch}(gU)).
\]
Proof. Take a sufficiently large integer $T > 0$ such that $\text{ch}(gU)$ and $\text{ch}(hgU)$ are invariant under left-translation by the principal congruence subgroup of level $\ell T$ in $H(\mathbb{Z}_\ell)$. Since $\text{pr}_1(h) \in \left(\ell^v \cdot \mathbb{Z}_\ell^\times, \mathbb{Q}_\ell, 1\right)$, one can take integers $s$ and $t$ satisfying $\min\{s + v, s\} \geq t \geq T$ and
\[
h^{-1} \cdot \phi_{s,t} = \phi_{s+v,t}.
\]
Then we have
\[
3(\phi_\infty \otimes \text{ch}(hgU)) = \frac{1}{\text{Vol}(K_1^{\text{GL}_2}(\ell^s, \ell^t))} 3(\phi_{s,t} \otimes \text{ch}(hgU))
\]
\[
= \frac{1}{\text{Vol}(K_1^{\text{GL}_2}(\ell^s, \ell^t))} 3(\phi_{s+v,t} \otimes \text{ch}(gU))
\]
\[
= \ell^{-v} \cdot 3(\phi_\infty \otimes \text{ch}(gU)),
\]
where the second equality follows from the $H(\mathbb{Q}_\ell)$-equivariance of $3$ and the third equality follows from $\ell^v \cdot \text{Vol}(K_1^{\text{GL}_2}(\ell^{s+v}, \ell^t)) = \text{Vol}(K_1^{\text{GL}_2}(\ell^s, \ell^t))$.

Lemma 7.27. Let $i, j, m, n \geq 0$ be integers, and $U \subset G(\mathbb{Q}_\ell)$ be either $K_{m,n}$ or $B_{m,n}$.

1. For any $\alpha, \beta \in \mathbb{Z}_\ell^\times$ with $\alpha \beta \in 1 + \ell^m \mathbb{Z}_\ell$ and $a, b, c \in \mathbb{Q}_\ell$, we have
\[
3(\phi_\infty \otimes \text{ch}(n_{i,j}^{a,b,c}U)) = 3(\phi_\infty \otimes \text{ch}(n_{i,j}^{\beta a,ab,\alpha \beta^{-1}c}U)).
\]

2. For any $a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbb{Q}_\ell$, we have
\[
n_{i,j}^{a_1,b_1,c_1} \circ n_{i,j}^{a_2,b_2,c_2} = n_{i,j}^{a_1+a_2, b_1+b_2, c_1+c_2}.
\]

In particular, if $a_1 \equiv a_2$ (mod $\ell^t$), $b_1 \equiv b_2$ (mod $\ell^t$), and $c_1 \equiv c_2$ (mod $\ell^t$), then we have
\[
\text{ch}(n_{i,j}^{a_1,b_1,c_1} U) = \text{ch}(n_{i,j}^{a_2,b_2,c_2} U).
\]
Proof. The element $n_{i,j}^{a,b,c}$ is conjugate to $n_{i,j}^{\beta a,ab,\alpha \beta^{-1}c}$ via \((\begin{pmatrix} \alpha \\ & \beta \end{pmatrix}, \begin{pmatrix} \alpha \\ & \beta \end{pmatrix})\), which fixes $\phi_{s,t}$ for any $s, t \geq 0$ and belongs to $K$. Hence claim (1) follows from Lemma 7.26. Claim (2) is by direct computation. □

7.4. Divisibility of $m_\eta$.

Definition 7.28. Define
\[
\xi_0^{K_1,0}(\ell) := \text{ch}(K_{1,0}) - \ell^{-2}U_1^{K_1,0}(\ell) + 2\ell^{-3}U_2^{K_1,0}(\ell) + \ell^{-3}U_3^{K_1,0}(\ell)
\]
\[
+ \ell^{-3}U_4^{K_1,0}(\ell) - \ell^{-4}U_5^{K_1,0}(\ell) + \ell^{-4}U_6^{K_1,0}(\ell),
\]
In particular, we have $\text{pr}_1^{K_1,0} \xi_0^{K_1,0}(\ell) = \xi_0(\ell)$, where $\xi_0(\ell)$ was defined in Definition 7.12.
For $3 \in \mathcal{X}(S(Q^2_\ell, \mathbb{C}), \sigma^\vee)$, put
\[ 3(\eta) := 3(\phi_\infty \otimes \text{ch}(\eta K_{1,0})) \]
for notational simplicity. Then we can write
\[ 3(\phi_\infty \otimes \xi_0^{K_{1,0}}(\ell)) = \sum_{\eta} m_\eta 3(\eta) \]
for some $m_\eta \in \mathbb{Q}$. In this section, we record a divisibility result of the multiplicities $m_\eta$ that will be useful later when proving the constructed Euler system is integral.

Recall from Definition 7.4 that $J_i$ is the set of left $K_{1,1}$-coset representatives of $K_{1,1}u_iK_{1,1}$ as given by Lemma 7.5, and by Remark 7.3, $J_i$ is also the set of left $K_{1,0}$-coset representatives of $K_{1,1}u_iK_{1,0}$. In the next six lemmas, we simplify the sum $\sum_{\eta \in J_i} 3(\eta)$ for each of $i = 1, 2, \ldots, 6$, and combine the results to conclude divisibility of $m_\eta$. The computations are rather involved and repetitive, so we put them in the appendix.

**Lemma 7.29.**
\[ \sum_{\eta \in J_1} 3(\eta) = (\ell + 1) \cdot 3(\eta_{0,0}^{0,0,0}) + (\ell^2 - 1) \cdot 3(\eta_{1,0}^{1,0,0}) + \ell(\ell + 1) \cdot 3(\eta_{0,1}^{0,0,1}) + 3(\eta_{1,1}^{0,1,1}). \]

*Proof.* See Lemma A.12. □

**Lemma 7.30.**
\[ \sum_{\eta \in J_2} 3(\eta) = 3(\eta_{0,0}^{0,0,0}) + (\ell^2 - 1) \cdot 3(\eta_{1,0}^{1,0,0}). \]

*Proof.* See Lemma A.13. □

**Lemma 7.31.**
\[ \sum_{\eta \in J_3} 3(\eta) = \ell(\ell + 1) \cdot 3(\eta_{0,1}^{0,0,1}) + \ell^2(\ell^2 - 1) \cdot 3(\eta_{1,1}^{1,0,1}) + \ell(\ell^2 - 1) \cdot 3(\eta_{1,1}^{0,1,1}) \]

*Proof.* See Lemma A.14. □

**Lemma 7.32.**
\[ \sum_{\eta \in J_4} 3(\eta) = \ell(\ell + 1) \cdot 3(\eta_{0,1}^{0,0,1}) + \ell(\ell^2 - 1) \cdot 3(\eta_{1,1}^{0,1,1}) + \ell^2(\ell^2 - 1) \cdot 3(\eta_{2,1}^{-1,1,1}). \]

*Proof.* See Lemma A.15. □

**Lemma 7.33.**
\[ \sum_{\eta \in J_5} 3(\eta) = (\ell + 1) \cdot 3(\eta_{0,0}^{0,0,0}) + (\ell + 1)(\ell^2 - 1) \cdot 3(\eta_{1,0}^{1,0,0}) + \ell^2(\ell^2 - 1) \cdot 3(\eta_{2,0}^{1,0,0}) \]
\[ + \ell(\ell + 1) \cdot 3(\eta_{0,1}^{0,0,1}) + \ell(\ell^2 - 1) \cdot 3(\eta_{1,1}^{0,1,1}) + \ell^2(\ell^2 - 1) \cdot 3(\eta_{1,1}^{1,0,1}) \]
\[ + \ell^2(\ell^2 - 1) \cdot 3(\eta_{2,1}^{-1,1,1}) + \ell^2(\ell^2 - 1) \cdot 3(\phi_\infty \otimes \text{ch}(\eta_{2,1}^{0,1,1} \cdot K_{0,0})). \]

*Proof.* See Lemma A.16. □

**Lemma 7.34.**
\[ \sum_{\eta \in J_6} 3(\eta) = 3(\eta_{0,0}^{0,0,0}) + (\ell^2 - 1) \cdot 3(\eta_{1,0}^{1,0,0}) + \ell^2(\ell^2 - 1) \cdot 3(\eta_{2,0}^{1,0,0}). \]

*Proof.* See Lemma A.17. □

**Definition 7.35.** Define
\[ \xi_{\text{Simple}} := \ell(\ell - 1)^3(\ell + 1)^2 \cdot \left( \text{ch}(K_{1,0}) - \text{ch}(\eta_{1,0}^{1,0,0} K_{1,0}) - \text{ch}(\eta_{0,0}^{0,0,1} K_{1,0}) + \ell \cdot \text{ch}(\eta_{1,1}^{1,0,1} K_{1,0}) \right. \]
\[ - (\ell - 1) \cdot \text{ch}(\eta_{1,1}^{1,0,1} K_{1,0}) + \ell \cdot \text{ch}(\eta_{2,1}^{-1,1,1} K_{1,0}) \]
\[ + \ell^3(\ell - 1)^2(\ell + 1)^2 \cdot \text{ch}(\eta_{2,1}^{0,1,1} K_{0,0}). \]
Combining the above six lemmas, we obtain the following:

**Theorem 7.36.** We have the following identity
\[
\mathcal{X}(\phi_\infty \otimes \xi_0^{K_{1,0}}(\ell)) = \frac{1}{\ell^2(\ell^2 - 1)} \cdot \mathcal{X}(\phi_\infty \otimes \xi_{\text{Simple}}).
\]

**Proof.** This theorem follows from Lemmas 7.29 through 7.34 and the definition that
\[
\xi_0^{K_{1,0}}(\ell) = \text{ch}(K_{1,0}) - \ell^{-2} \sum_{\eta \in J_1} \text{ch}(\eta K_{1,0}) + 2\ell^{-3} \sum_{\eta \in J_2} \text{ch}(\eta K_{1,0})
\]
\[
+ \ell^{-3} \sum_{\eta \in J_3 \cup J_4} \text{ch}(\eta K_{1,0}) - \ell^{-4} \sum_{\eta \in J_5} \text{ch}(\eta K_{1,0}) + \ell^{-4} \sum_{\eta \in J_6} \text{ch}(\eta K_{1,0}).
\]

\[\square\]

**Remark 7.37.** Let \( p \neq \ell \) be a prime. Then for any open compact subgroup \( V \) of \( H(\mathbb{Q}_\ell) \), we have
\[
\text{Vol}(V) \in \frac{1}{(\ell - 1)^3(\ell + 1)^2} \mathbb{Z}_p.
\]

Furthermore, since \( G(\mathbb{Z}_\ell) \) contains diagonal matrices with integral coefficients,
\[
\text{Vol}(gG(\mathbb{Z}_\ell)g^{-1} \cap H(\mathbb{Z}_\ell)) \in \frac{1}{(\ell - 1)^2(\ell + 1)^2} \mathbb{Z}_p
\]
for any \( g \in G(\mathbb{Q}_\ell) \). Hence from the definition of \( \xi_{\text{Simple}} = \sum_\eta m_\eta \text{ch}(\eta K_{1,0}) + \sum_{\eta'} m_{\eta'} \text{ch}(\eta' K_{0,0}) \), we see that
\[
m_\eta \cdot \text{Vol}(\eta K_{1,0})^{-1} \cap H(\mathbb{A}_f) \in \mathbb{Z}_p \quad \text{and} \quad m_{\eta'} \cdot \text{Vol}(\eta K_{0,0})^{-1} \cap H(\mathbb{A}_f) \in \mathbb{Z}_p
\]
for all \( \eta, \eta' \). This is the condition in Proposition 5.5, which will be used later to prove integrality of Euler system classes.

**Theorem 7.38.**
\[
\sum_{g \in G(\mathbb{Z}_\ell)/K_{1,0}} g \cdot \mathcal{X}(\phi_{1,1} \otimes \xi_0^{K_{1,0}}(\ell)) = \frac{L(0, \sigma)^{-1}}{\ell^2 - 1} \cdot \mathcal{X}(\phi_0 \otimes \text{ch}(G(\mathbb{Z}_\ell))).
\]

**Proof.** Since \( \xi_0^{K_{1,0}}(\ell) \) is invariant under left-translation by \( K_{1,1} \), Lemma 7.22 implies that
\[
\sum_{g \in G(\mathbb{Z}_\ell)/K_{1,0}} g \cdot \mathcal{X}(\phi_{1,1} \otimes \xi_0^{K_{1,0}}(\ell)) = \frac{1}{\ell^2 - 1} \sum_{g \in G(\mathbb{Z}_\ell)/K_{1,0}} g \cdot \mathcal{X}(\phi_\infty \otimes \xi_0^{K_{1,0}}(\ell))
\]
\[
= \frac{1}{\ell^2 - 1} \mathcal{X}(\phi_\infty \otimes \xi_0(\ell)).
\]

The theorem then follows from Theorem 7.25.

\[\square\]

### 7.5. Formula for wild norm relations.

**Proposition 7.39.** Let \( i, j, m, n \) be non-negative integers, and \( n \geq 1 \). When \( i > 0, j > 0 \) and \( 2i + j \geq m \), we have
\[
\mathcal{X} \left( \phi_\infty \otimes \text{ch}(\eta_{i,j+1}^{0,1,1} B_{m,n}) \right) = \frac{U_{1}^{B_{m,n}}(\ell)^{\prime}}{\ell^3} \cdot \mathcal{X}(\phi_\infty \otimes \text{ch}(\eta_{i,j}^{0,1,1} B_{m,n})).
\]

**Proof.** Since \( n \geq 1 \), we have
\[
U_{1}^{B_{m,n}}(\ell)^{\prime} \cdot \mathcal{X}(\phi_\infty \otimes \text{ch}(\eta_{i,j}^{0,1,1} B_{m,n})) = \mathcal{X}(\phi_\infty \otimes \text{ch}(\eta_{i,j}^{0,1,1} B_{m,n}) \cdot U_{1}^{B_{m,n}}(\ell))
\]
\[
= \sum_{x, y, u, v \in \mathbb{Z}/\ell} \mathcal{X} \left( \phi_\infty \otimes \text{ch}(\eta_{i,j}^{0,1,1}) \left( \begin{pmatrix} \ell & x \\ z & 1 \end{pmatrix}, \begin{pmatrix} \ell & u \\ 1 & 1 \end{pmatrix} B_{m,n} \right) \right).
\]

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Furthermore, we have
\[
\eta_{i,j}^{0,1,1}(\ell, x, y | z, \ell) = (\ell, x, y | \ell, u) \eta_{i+1,j+1}^{0,1,1}
\]
and hence Lemma 7.26 implies that
\[
U_1^{B_{m,n}}(\ell)^{'} 3(\phi_\infty \otimes \text{ch}(\eta_{i,j}^{0,1,1} B_{m,n})) = \ell \cdot \sum_{x, u \in \mathbb{Z}/\ell} 3(\phi_\infty \otimes \text{ch}(\eta_{i+1,j+1}^{0,1,1} + \ell^t u B_{m,n})).
\]

Since \(i > 0\) and \(j > 0\), the elements \(1 + \ell^t x\) and \(1 + \ell^t u\) are units. Lemma 7.27(1) along with \(2i + j \geq m\) shows that
\[
\sum_{x, u \in \mathbb{Z}/\ell} 3(\phi_\infty \otimes \text{ch}(\eta_{i+1,j+1}^{0,1,1} B_{m,n})) = \ell^2 \cdot 3(\phi_\infty \otimes \text{ch}(\eta_{i,j}^{0,1,1} B_{m,n})).
\]

**Proposition 7.40.** Let \(i, j, m, n\) be non-negative integers and \(n \geq 1\). When \(i \geq 2j \geq n\) and \(i \geq m\), we have
\[
3(\phi_\infty \otimes \text{ch}(\eta_{i,j}^{0,1,1} B_{m,n})) = \frac{U_2^{B_{m,n}}(\ell)^{'} 3(\phi_\infty \otimes \text{ch}(\eta_{i,j}^{0,1,1} B_{m,n}))}{\ell^2}.
\]

**Proof.** Since \(n \geq 1\), we have
\[
U_2^{B_{m,n}}(\ell)^{'} 3(\phi_\infty \otimes \text{ch}(\eta_{i,j}^{0,1,1} B_{m,n})) = 3(\phi_\infty \otimes \text{ch}(\eta_{i,j}^{0,1,1} B_{m,n}) \cdot U_2^{B_{m,n}}(\ell))
\]
\[
= \sum_{x, y, z, u \in \mathbb{Z}/\ell} 3(\phi_\infty \otimes \text{ch}(\eta_{i,j}^{0,1,1}(\ell^2, x, y, z, \ell, y, -x, \ell)^B_{m,n})).
\]

Since
\[
\eta_{i,j}^{0,1,1}(\ell, x, y, z, \ell, y, -x, \ell) = (\ell^2, x, y, z - 2x \ell^{-i}, \ell, y, -x, \ell) \eta_{i+1,j}^{0,1,1}
\]
\[
= (\ell^2, z - 2x \ell^{-i}, \ell) \eta_{i+1,j}^{0,1,1} y_{i,j}^{1,1}.
\]

Lemma 7.26 implies that
\[
U_2^{B_{m,n}}(\ell)^{'} 3(\phi_\infty \otimes \text{ch}(\eta_{i,j}^{0,1,1} B_{m,n})) = \sum_{x, y, z, u \in \mathbb{Z}/\ell} 3(\phi_\infty \otimes \text{ch}(\eta_{i+1,j}^{0,1,1} + \ell^t u B_{m,n})).
\]

We have the assumption that \(i \geq 2j \geq n\) and \(i \geq m\). Write \(a = \ell^t x\) and \(b = 1 + \ell^t y\) to simplify notation. Since we assume \(i \geq n\), we have
\[
h := (\begin{pmatrix} b \\ \ell \end{pmatrix}, \begin{pmatrix} b \\ -a \end{pmatrix}) \in H(\mathbb{Q} / \ell) \cap B_{m,n}.
\]

Since we also assume \(i \geq 2j\), \(i - j \geq n\) and \(i \geq m\), we have
\[
h^{-1} \eta_{i+1,j}^{0,1,1} h B_{m,n} = \eta_{i+1,j}^{0,1,1} (\begin{pmatrix} b^{-1} - ab^{-1} \ell^{-j} \\ ab^{-1} \\ ab^{-1} \ell^{-j} - ab^{-1} \ell^{-2j} \\ ab^{-1} \ell^{-j} + 1 \end{pmatrix} (\begin{pmatrix} b \\ -a \end{pmatrix})) B_{m,n}
\]
\[
= \eta_{i+1,j}^{0,1,1} B_{m,n}.
\]
Hence Lemma 7.26 shows that
\[
\sum_{x,y \in \mathbb{Z}/\ell} 3 \left( \phi_\infty \otimes \text{ch}(\eta_{i+1,j}^{x,1+y,1}B_{m,n}) \right) = \ell^2 \cdot 3 \left( \phi_\infty \otimes \text{ch}(\eta_{i+1,j}^{0,1}B_{m,n}) \right).
\]

\[\square\]

**Lemma 7.41.** Let \( n \geq 1 \) be an integer. For \( i \in \{1, 2\} \), let
\[
\gamma_i := \left( \begin{array}{c}
\gamma_i^1 \\
\gamma_i^2 \\
\gamma_i^3
\end{array} \right)
\]
be a diagonal matrix satisfying \( b_i \geq a_i \geq 0 \) and \( d_i \geq c_i \geq 0 \). Then we have
\[
\text{ch}(B_{m,n}u_1u_2B_{m,n}) = \text{Vol}(B_{m,n})^{-1} \cdot \text{ch}(B_{m,n}u_1B_{m,n}) \cdot \text{ch}(B_{m,n}u_2B_{m,n}).
\]

**Proof.** Write
\[
B_{m,n}u_1B_{m,n} = \bigcup_{i \in I} \gamma_1, i u_1 B_{m,n}
\]
and
\[
B_{m,n}u_1B_{m,n} = \bigcup_{j \in J} B_{m,n}u_2 \gamma_2, j = \bigcup_{j \in J} u_2 \gamma_2, j B_{m,n}.
\]
Then
\[
\text{Vol}(B_{m,n})^{-1} \cdot \text{ch}(B_{m,n}u_1B_{m,n}) \cdot \text{ch}(B_{m,n}u_2B_{m,n}) = \sum_{i,j} \text{ch}(\gamma_{1,i} u_1 B_{m,n} u_2 \gamma_{2,j})
\]
\[
= \sum_i \text{ch}(\gamma_{1,i} u_1 B_{m,n} u_2 B_{m,n}).
\]

Let us show that \( \bigcup_i \gamma_{1,i} u_1 B_{m,n} u_2 B_{m,n} = B_{m,n} u_1 u_2 B_{m,n} \). For any lower triangular matrix \( g \in B_{m,n} \), we have \( gu_2 B_{m,n} = u_2 B_{m,n} \) by the choice of \( u_2 \). Hence we have
\[
B_{m,n}u_2B_{m,n} = Bu_2B_{m,n},
\]
where \( B \) denotes the subgroup of upper-triangular matrices. Furthermore, \( B_{m,n}u_1B = B_{m,n}u_1 \), and thus
\[
\bigcup_i \gamma_{1,i} u_1 B_{m,n} u_2 B_{m,n} = B_{m,n} u_1 B_{m,n} u_2 B_{m,n} = B_{m,n} u_1 Bu_2 B_{m,n} = B_{m,n} u_1 u_2 B_{m,n}.
\]

Next, let us show that the union \( \bigcup_i \gamma_{1,i} u_1 B_{m,n} u_2 B_{m,n} \) is disjoint. It suffices to show the union
\[
\bigcup_{i,j} \gamma_{1,i} u_1 \gamma_{2,j} u_2 B_{m,n}
\]
is disjoint. This follows from the facts that \( \bigcup_{i,j} \gamma_{1,i} u_1 \gamma_{2,j} u_2 B_{m,n} = B_{m,n} u_1 u_2 B_{m,n} \) and
\[
\# B_{m,n} u_1 u_2 B_{m,n} / B_{m,n} = \# I + \# J.
\]

Therefore, we see that
\[
\sum_i \text{ch}(\gamma_{1,i} u_1 B_{m,n} u_2 B_{m,n}) = \text{ch}(B_{m,n} u_1 u_2 B_{m,n}).
\]

\[\square\]

**Corollary 7.42.** Let \( 3 \in \mathcal{X}(\tau, \sigma^\vee) \) with \( \tau = S(\mathbb{Q}_\ell^2, \mathbb{C}) \). Let \( m \geq n \geq 1 \) be integers. Then
\[
3 \left( \phi_\infty \otimes \text{ch}(\eta_{2m+2,m+1}^{0,1}B_{m,n}) \right) = \frac{U_{B_{m,n}}^m(\ell)^{\prime}}{33} \cdot 3 \left( \phi_\infty \otimes \text{ch}(\eta_{2m,m}^{0,1}B_{m,n}) \right).
\]
Proof. This follows directly combining Propositions 7.39 and 7.40 and Lemma 7.41. When \(m \geq n \geq 1\), we have

\[
3 \left( \phi_{\infty} \otimes \text{ch}(\eta_{2m+2,m+1}^{0,1}) \right) \equiv \frac{7}{3} \left( \frac{U_{B_{m,n}}}{\ell^3} \right) \cdot 3 \left( \phi_{\infty} \otimes \text{ch}(\eta_{2m+1,m}^{0,1} B_{m,n}) \right)
\]

\[
= \frac{7}{15} \left( \frac{U_{B_{m,n}}}{\ell^3} \right) \cdot 3 \left( \phi_{\infty} \otimes \text{ch}(\eta_{2m,m}^{0,1} B_{m,n}) \right).
\]

\[\square\]

7.6. Heuristic of the choice of \(U(p)'\). The heuristic why \(U_5(p)'\) is the correct Hecke operator involved in wild norm relation is explained more generally in [LZ17]. Let \(V\) be a \(p\)-adic representation of \(\text{Gal}_\mathbb{Q}\).

Definition 7.43 (LZ17, Definition 7.1). A Panchishkin sub-representation of \(V\) at \(p\) is a subspace \(V^+ \subset V\) such that

- \(V^+\) is stable under \(\text{Gal}_\mathbb{Q}^p\)
- \(V^+\) has Hodge–Tate weights \(\geq 1\)
- \(V/V^+\) has Hodge–Tate weights \(\leq 0\)

Here our convention is that cyclotomic characters have Hodge–Tate weight +1.

Conjecture 7.44 (LZ17, Conjecture 7.4). Let \(c \in \text{Gal}_\mathbb{Q}\) be complex conjugation. Let

\[r(V) := \max\{0, \dim V^e - 1 - \#\text{non-negative Hodge–Tate weights of } V\}.\]

Suppose that \(r(V) = 1, r(V^* (1)) = 0\), and \(V\) has a Panchishkin sub-representation. Then there is an Euler system \(c_m \in H^1(Q(\mu_m), V)\) for \(V\).

Let \(\Pi = \Pi_f \otimes \Pi_\infty\) be a cuspidal automorphic representation of \(G(A_f)\) with \(\Pi_\infty\) a unitary discrete series of weight \((k_1, k_2, k)\), with \(k_1 \geq k_2 \geq 3, k \geq 2\). Then we have an associated Galois representation \(V = W_{\Pi_f}\), which is 8-dimensional. Write \(V = V_1 \otimes V_2\), where \(V_1\) comes from the \(\text{GSp}_4\) component and \(V_2\) comes from the \text{GL}_2\)-component. Then \(V_1 = 1\) is 2-dimensional and \(V_2 = 1\) is 1-dimensional, and so \(V_{c} = 1\) is 4-dimensional. Also \(V_1\) has Hodge–Tate weights \(0, k_2 - 2, k_1 - 1, k_1 + k_2 - 3\), and \(V_2\) has Hodge–Tate weights \(0, k - 1\). According to the conjecture, we need a 3-dimensional Panchishkin sub-representation \(V^+\) in the 8-dimensional \(V\) to construct an Euler system. The 3-dimensional sub-representation comes from the (non-direct) sum of the tensor product of a 2-dimensional sub-representation of \(V_1\) with the 1-dimensional sub-representation of \(V_2\), and the tensor product of a 1-dimensional sub-representation of \(V_1\) with \(V_2\). Hence we need to assume that \(\Pi\) is (Borel) ordinary at \(p\) for the Panchishkin sub-representation to exist. This is why \(U_5(p)'\) shows up in the formula for wild norm relations.

8. Euler system elements and their norm relations

Fix a prime \(p\), and a finite set of primes \(\Sigma\) not containing \(p\), and an open compact subgroup \(K_{\Sigma} \subset G(Q_\Sigma) := \prod_{\ell \in \Sigma} G(Q_\ell)\). Choose a \(\phi_{\Sigma} \in \mathcal{S}(Q_{\Sigma}^2, \mathbb{Z})\). Then choose an open compact subgroup \(K_{\Sigma}^H \subset H(Q_\Sigma) \subset K_{\Sigma}\) acting trivially on \(\phi_{\Sigma}\).

Recall that we defined the symbol map in Section 5.4

\[\text{Sym}_a^{[a,b,c,r]}: \mathcal{S}(A_f^2, Q) \otimes _{H(H(A_f))} H(G(A_f)) \to H_{\text{mot}}^b(Y_G, \mathfrak{W}_{Q}^{a,b,c,r}(3-a-r))[a-r],\]

and the integral version in Section 5.5

\[e\text{Sym}_a^{[a,b,c,r]}: \text{cpSy}(A_f^2, \mathbb{Z}_p) \otimes _{H(H(A_f))} H_{\mathbb{Z}_p}(G(A_f^p) \times \mathbb{Z}_p)) \to H_{\text{mot}}^a(Y_G, \mathfrak{W}_{Q_p}^{a,b,c,r}(3-a-r))[a-r].\]

which are related by

\[e\text{Sym}_a^{[a,b,c,r]}(\phi \otimes \xi) = r_{\text{et}} \circ \text{Sym}_a^{[a,b,c,r]} \left( e^{a} - e^{-(b+c-2r)} \left( \begin{array}{cc} c & 1 \\ 1 & 1 \end{array} \right)^{-1} \right) \phi \otimes \xi \]
8.1. Definition of Euler system elements.

**Definition 8.1.** Let $\Sigma$ be a finite set of primes. For each $\ell \in \Sigma$, fix a choice of $\phi_\ell \in S(\mathbb{Q}_\ell^2, \mathbb{Z})$ and an open compact subgroup $K_\ell \subset G(\mathbb{Q}_\ell)$.

For any square-free integer $M \geq 1$ coprime to $\Sigma \cup \{p\}$, and integers $m, n \geq 1$, define

$$z^{[a,b,c,r]}_{\text{mot},Mp^m,p^n} := \text{Symbl}^{[a,b,c,r]}(\phi_{Mp^m,p^n} \otimes \xi_{Mp^m,p^n}).$$

Here

- $\phi_{Mp^m,p^n} = \bigotimes_\ell \phi_\ell \in S(\mathbb{A}_f^2, \mathbb{Z})$, where for $\ell \in \Sigma$, $\phi_\ell$ is our fixed choice, and for $\ell \notin \Sigma$,

$$\phi_\ell = \begin{cases} 
\phi_{0,0} = \text{ch}(\mathbb{Z}_\ell \times \mathbb{Z}_\ell) & \ell \nmid Mp \\
\phi_{1,1} = \text{ch}(\ell \mathbb{Z}_\ell \times (1 + \ell \mathbb{Z}_\ell)) & \ell \mid M
\end{cases}$$

- $\xi_{Mp^m,p^n} = \bigotimes_\ell \xi_\ell \in \mathcal{H}(G(\mathbb{A}_f))$, where

$$\xi_\ell = \begin{cases} 
\frac{1}{\text{Vol}(K_\ell \cap H(\mathbb{Z}_\ell))} \text{ch}(K_\ell) & \ell \in \Sigma \\
\frac{1}{\text{Vol}(H(\mathbb{Z}_\ell))} \xi^{K_\ell}_{0,0}(\ell) & \ell \nmid Mp, \ell \notin \Sigma
\end{cases}$$

- $K_\ell = \begin{cases} 
G(\mathbb{Z}_\ell) & \ell \nmid Mp \\
K_{1,0} & \ell \mid M \\
B_{m,n} & \ell = p
\end{cases}$

Also let $K_{Mp^m,p^n} := \bigotimes_\ell K_\ell$, where for $\ell \in \Sigma$, $K_\ell$ is our fixed choice, and for $\ell \notin \Sigma$,

Since $\phi_{Mp^m,p^n} \otimes \xi_{Mp^m,p^n}$ is invariant under $K_{Mp^m,p^n}$, so is $z^{[a,b,c,r]}_{\text{mot},Mp^m,p^n}$. Hence $z^{[a,b,c,r]}_{\text{mot},Mp^m,p^n}$ is an element in $H^5_{\text{mot}}(Y_{G}(K_{Mp^m,p^n}), \mathbb{Q}^{a,b,c,*}(3 - a - r)).$ Note that we get rid of the twist $[-a - r]$.

**Remark 8.2.** The appearance of $4m+n$ in the definition of $\phi_p$ and $\xi_p$ can in fact be replaced by any sufficiently large integer $t$ so that

- $\xi_p$ is invariant under left-translation of the principal congruence subgroup of level $p^t$ in $H(\mathbb{Z}_p)$; equivalently, the principal congruence subgroup of level $p^t$ in $H(\mathbb{Z}_p)$ is contained in $\eta^{0,1,1}_{2m,m}B_{m,n}(\eta^{0,1,1}_{2m,m}B_{m,n})^{-1} \cap H(\mathbb{Z}_p)$. This condition is used to apply Lemma 7.22 to rewrite formula involving $\phi_{\infty}$ in terms of $\phi_{t,t}$.

- $\frac{p^m}{\text{Vol}(K_{p^t}(\mathbb{Z}_p^2))} \text{Vol}(\eta^{0,1,1}_{0,0} \times \eta^{0,1,1}_{0,0} \times (\eta^{0,1,1}_{0,0} \times \eta^{0,1,1}_{0,0}^{-1})) \in \mathbb{Z}_p$, where $s_m$ is to be defined below. This condition is used to apply Proposition 5.5 to check integrality in Theorem 8.7 below.

We need an integral version of $z^{[a,b,c,r]}_{\text{mot},Mp^m,p^n}$. Note that the naive idea of simply replacing $\text{Symbl}^{[a,b,c,r]}$ by $e\text{Symbl}^{[a,b,c,r]}$ to define $e^{[a,b,c,r]}_{\text{mot},Mp^m,p^n}(\phi_{Mp^m,p^n} \otimes \xi_{Mp^m,p^n})$ does not work. This is because

1. When $M$ is such that there exists $\ell \mid \gcd(M, e)$, then $\phi_\ell \neq \phi_{0,0}$, so our chosen $\phi_{Mp^m,p^n}$ does not lie in $eG(K_\ell^2, \mathbb{Z}_p)$.
2. The support of $\xi_p = \text{ch}(\eta_{m}B_{m,n})$ is not contained in $G(\mathbb{Z}_p)$, so our chosen $\xi_{Mp^m,p^n}$ does not lie in $H_{\text{mot}}(G(\mathbb{A}_f \times \mathbb{Z}_p)).$ Hence it does not make sense to consider $z\text{Symbl}^{[a,b,c,r]}(\phi_{Mp^m,p^n} \otimes \xi_{Mp^m,p^n})$. The remedy is to use the following Lemma 8.3 to reinterpret $\text{Symbl}^{[a,b,c,r]}(\phi_{Mp^m,p^n} \otimes \xi_{Mp^m,p^n})$.

Let $s_m = \prod_\ell s_\ell \in H(\mathbb{A}_f) \subset G(\mathbb{A}_f)$, where

$$s_\ell = \begin{cases} 
(\text{id}_2, \text{id}_2) \in H(\mathbb{Q}_\ell) & \ell \neq p \\
\left(\begin{array}{c} p^{3m} \\ 1 \\
p^{2m} \\
p^{n}
\end{array}\right) \in H(\mathbb{Q}_p) & \ell = p
\end{cases}.$$
We abuse notation to let \( s_m \) also denote the induced map on \( Y_G \) of right translation by \( s_m \)

\[
Y_G(s_m K s_m^{-1}) \xrightarrow{s_m} Y_G(K).
\]

for arbitrary level \( K \subset G(\mathbb{A}_f) \). We also have a natural morphism of (resp. integral) sheaves on \( Y_G(s_m K s_m^{-1}) \)

\[
s_{m,*} : \mathcal{W}_G(a,b,c) \xrightarrow{s_m} \mathcal{W}_G(a,b,c) \quad \text{(resp. } s_{m,*} : \mathcal{W}_G(a,b,c) \xrightarrow{s_m} \mathcal{W}_G(a,b,c))
\]

given by the action of \( s_m^{-1} \) on the representation \( W_n^{a,b,c,*} \) of \( G \). We again denote the map on cohomology induced by \( s_m \) and \( s_{m,*} \),

\[
s_{m,*} : H^\text{mot}_G(Y_G(s_m K s_m^{-1}), \mathcal{W}_G^{a,b,c,*}) \to H^\text{mot}_G(Y_G(K), \mathcal{W}_G^{a,b,c,*})
\]

(resp. \( s_{m,*} : H^\text{ét}_G(Y_G(s_m K s_m^{-1}), \mathcal{W}_G^{a,b,c,*}) \to H^\text{ét}_G(Y_G(K), \mathcal{W}_G^{a,b,c,*}) \). 

**Lemma 8.3.**

\[
\text{Symbl}^{[a,b,c,r]}(\phi_{Mp^{m,p^n} \otimes \xi_{Mp^{m,p^n}}}) = s_{m,*} \left( \text{Symbl}^{[a,b,c,r]}(s_m : \phi_{Mp^{m,p^n} \otimes s_m : \xi_{Mp^{m,p^n}}}) \right).
\]

**Proof.** This is by the definition of \( s_{m,*} \), the \( G \)-equivariance of \( \text{Symbl}^{[a,b,c,r]} \), and the \( H \)-equivariance between \( S(\mathbb{A}_f^2, \mathbb{Q}) \) and \( \mathcal{H}(G(\mathbb{A}_f)) \):

\[
s_{m,*} \left( \text{Symbl}^{[a,b,c,r]}(s_m : \phi_{Mp^{m,p^n} \otimes s_m : \xi_{Mp^{m,p^n}}}) \right)
= \text{Symbl}^{[a,b,c,r]}(s_m : \phi_{Mp^{m,p^n} \otimes s^{-1}m : \xi_{Mp^{m,p^n}}})
= \text{Symbl}^{[a,b,c,r]}(s^{-1}m : \phi_{Mp^{m,p^n} \otimes s^{-1}m : \xi_{Mp^{m,p^n}}})
= \text{Symbl}^{[a,b,c,r]}(\phi_{Mp^{m,p^n} \otimes s^{-1}m : \xi_{Mp^{m,p^n}}})
\]

We choose and fix an integer \( e > 1 \) which is prime to \( \Sigma \cup \{2, 3, p\} \).

**Lemma 8.4.**

1. \( s_m : \phi_{Mp^{m,p^n}} \in \mathcal{S}((\mathbb{A}_f^p) \times \mathbb{Z}_p)^2, \mathbb{Z}_p) \) if \( M \) is prime to \( e \).
2. \( s_m : \xi_{Mp^{m,p^n}} : s_m^{-1} \in \mathcal{H}_{\text{Zp}}(G(\mathbb{A}_f^p) \times \mathbb{Z}_p) \) if \( n \geq 3m \).

**Proof.**

1. We need to check that for \( \ell | e \), \( s_\ell : \phi_\ell = \text{ch}(\mathbb{Z}_p^2) \), and that \( s_p : \phi_p \) has support contained in \( \mathbb{Z}_p^2 \).
   - For \( \ell \not| e \), we also have \( \ell \not| M \) by our choice of \( e \). Then \( s_\ell : \phi_\ell = (1_d, 1_d) : \text{ch}(\mathbb{Z}_p^2) = \text{ch}(\mathbb{Z}_p^2) \).
   - \( s_p : \phi_p = \text{ch}(\mathbb{Z}_p^{m+n} \times (1 + p^{m+n} \mathbb{Z}_p)) \) has support in \( \mathbb{Z}_p^2 \).

2. By construction \( \xi \) is a \( \mathbb{Z}_p \)-linear combination of characteristic functions, so it takes value in \( \mathbb{Z}_p \). It remains to check that \( s_m : \xi_p \cdot s_m^{-1} \) has support contained in \( G(\mathbb{Z}_p) \).

Indeed

\[
s_m : \xi_p \cdot s_m^{-1} = s_m : \text{ch}(\eta_{2m,m} B_{m,n} \cdot s_m^{-1})
= \text{ch}(s_m \eta_{2m,m} B_{m,n} s_m^{-1})
= \text{ch}(\eta_{0,0} B_{m,n} s_m^{-1})
\]

Now \( \eta_{0,0} \in \mathbb{Z}_p \), and \( s_m B_{m,n} s_m^{-1} \subset \mathbb{Z}_p \) because we assumed \( n \geq 3m \).

We are now ready to define \( \varepsilon_{[a,b,q,r]}^{[a,b,q,r]} \).

**Definition 8.5.** For \( M \geq 1 \) a square-free integer coprime to \( \Sigma \) and \( ep \), and integers \( m, n \geq 1 \) define

\[
\varepsilon_{[a,b,q,r]}^{[a,b,q,r]} := \left\{ \begin{array}{ll}
s_{m,*} \left( \text{Symbl}^{[a,b,c,r]}(s_m : \phi_{Mp^{m,p^n} \otimes s_m : \xi_{Mp^{m,p^n}}}) \right) & n \geq 3m \\
\left( \frac{K_{Mp^{m,p^n}+3m}}{\text{Pr}K_{Mp^{m,p^n}}^{m+n+3m}} \right)_{\text{et}} \varepsilon_{[a,b,q,r]}^{[a,b,q,r]}(3 - a - r)) & n < 3m \\
\end{array} \right.
\]

The definition makes sense by Lemma 8.4, and \( \varepsilon_{[a,b,q,r]}^{[a,b,q,r]} \) is an element in \( H^2_{\text{et}}(Y_G(K_{Mp^{m,p^n}}), \mathcal{W}_G^{[a,b,c,r]}(3 - a - r)) \). Again we get rid of the twist \([-a - r] \).
Proposition 8.6. We have

\[ e_{\ell, M_{p^n}}^{[a, b, q, r]} = (e^2 - e^{-(b+c-2r)}) \begin{pmatrix} e & e \\ e & e \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot r_{\ell, M_{p^n}}(z_{\text{mot}, M_{p^n}, p^n}) \]

in \( H^5_{et}(Y_G(K_{M_{p^n}}), \mathcal{W}_{Q_p}^{a, b, c, r}(3 - a - r)) \).

Proof. To simplify notation, we let

\[ e_{\ell, M_{p^n}}^{[a, b, q, r]} \]

Assume first that \( n \geq 3m \). Then

\[ e_{\ell, M_{p^n}}^{[a, b, q, r]} = s_{m, e} \begin{pmatrix} \text{Symbl}^{[a, b, q, r]}(s_{m} \cdot \phi_{M_{p^n}} \otimes s_{m} \cdot \xi_{M_{p^n}} \cdot s_{m}^{-1}) \\ s_{m, e} \cdot r_{\ell, M_{p^n}}(e_{m, e} \cdot \phi_{M_{p^n}} \otimes e_{m, e} \cdot \xi_{M_{p^n}} \cdot e_{m, e}^{-1}) \end{pmatrix} \]

where the third to last equality uses the fact that \( e_{\ell} \) is in the center of \( \mathbb{Z} \langle \mathbb{A}_f \rangle \), and the second to last equality uses the \( G \)-equivariance of \( \text{Symbl}^{[a, b, q, r]} \).

The case of \( n < 3m \) follows by further noting that the motivic classes are compatible with levels:

\[ \begin{pmatrix} \text{pr}_{K_{M_{p^{n+3m}}}}^{K_{M_{p^n}}} \end{pmatrix} _* \begin{pmatrix} [a, b, q, r]_{\text{mot}, M_{p^{n+3m}}} \\ [a, b, q, r]_{\text{mot}, M_{p^n}} \end{pmatrix} = [a, b, q, r]_{\text{mot}, M_{p^n}}. \]

This is because it can be easily checked that \( \phi_{M_{p^n}} \otimes \xi_{M_{p^n}} \) appearing in the definition of \( z_{\text{mot}, M_{p^n}} \) are compatible with levels, i.e.,

\[ \sum_{g \in K_{M_{p^n}} \backslash K_{M_{p^{n+1}}}} \phi_{M_{p^n}} \otimes g \cdot \xi_{M_{p^n}} = \phi_{M_{p^n}} \otimes \xi_{M_{p^n}}. \]

\[ \square \]

Theorem 8.7 (Integrality). Assume that \( 5m \geq n \). Then \( e_{\ell, M_{p^n}}^{[a, b, c, r]} \in H^5_{et}(Y_G(K_{M_{p^n}}), \mathcal{W}_{\mathbb{Q}_p}^{a, b, c, r}(3 - a - r)) \) is in the image of \( H^5_{et}(Y_G(K_{M_{p^n}}), \mathcal{W}_{\mathbb{Z}_p}^{a, b, c, r}(3 - a - r)) \) for all \( \eta \).

Proof. We use Proposition 5.5 to show the integrality of \( e_{\ell, M_{p^n}}^{[a, b, c, r]} \). It suffices to check the assumption of Proposition 5.5 for each prime \( \ell \); namely if we write \( \xi_{\ell} = \sum_{n} m_{n} \text{ch}(\eta K_{\ell}) \), then \( m_{n \eta} \text{Vol}(\eta K_{\eta^{-1}} \cap H(\mathbb{Z}_\ell)) = \mathbb{Z}_p \) for all \( \eta \).

For \( \ell \nmid M_{p} \), this is clear as \( \xi_{\ell} = \frac{1}{\text{Vol}(K_{\ell} \cap H(\mathbb{Z}_\ell))} \text{ch}(K_{\ell}) \).

For \( \ell | M_{p} \), this follows from Theorem 7.36 and Remark 7.37.
For \( \ell = p \), we have seen in the proof of Lemma 8.4 that
\[
\eta_{0,1}^{0,1} s_m \cdot \xi_p \cdot s_m^{-1} = \chi(\eta_{0,0}^{0,1} s_m B_{m,n} s_m^{-1}).
\]
Observe that
\[
\eta_{0,0}^{0,1} s_m B_{m,n} s_m^{-1} \cap H(\mathbb{Z}_p) \text{ contains }
\]
\[
\{ h = (h_1, h_2) \in H(\mathbb{Z}_p) : h_1 \equiv \left( \begin{array}{cc} * & * \\ 1 & 1 \end{array} \right) \mod p^{3m+n}, h_2 \equiv \left( \begin{array}{cc} * & * \\ 1 & 1 \end{array} \right) \mod p^{m+n}, \mu(h) \equiv 1 \mod p^m \}
\]
which has index in \( H(\mathbb{Z}_p) \) being \((p + 1)^2 (p - 1) p^{3m+3n-3}\). Hence
\[
(p^2 - 1) p^{3m+2n-2} \cdot \text{Vol} \left( \eta_{0,0}^{0,1} s_m B_{m,n} s_m^{-1} \cap H(\mathbb{Z}_p) \right) \in \frac{p^{5m-n+1}}{p+1} \mathbb{Z}_p \subset \mathbb{Z}_p,
\]
if \( 5m \geq n \).

8.2. Midway results on wild norm relations: two methods.

**Proposition 8.8.** Let \( m \geq 1 \) be an integer. Then
\[
\left( \left[ K_{M^{m+1}, p^m} \right]_{\text{mot}} \right)_{\mathbb{Z}} \left[ \mathbb{Z}_{\text{mot}, M^{m+1}, p^m} \right] = \frac{U_5^{B_{m,n}}(p)}{p^{3(a+r)}} \cdot \left[ \mathbb{Z}_{\text{mot}, M^{m+1}, p^m} \right].
\]

**Proof.** By construction (Definition 8.1), \( \phi_{M^{m+1}, p^n} \otimes \xi_{M^{m+1}, p^n} \) and \( \phi_{M^{m+1}, p^n} \otimes \xi_{M^{m+1}, p^{m+1}} \) only differ at \( p \), so we only need to check the equality at \( p \). Also because we defined \( \xi_{m,n}^{[a,b,q,r]} \) omitting the twist \([-a-r] = \|\mu(-)\|^{-a-r} \), we want to show is
\[
3 \left( \phi_{\infty} \otimes p^{5(m+1)} \chi(\eta_{0,1}^{0,1} B_{m,n}) \right) = \frac{U_5^{B_{m,n}}(p)'}{p^{3(a+r)}} 3 \left( \phi_{\infty} \otimes p^{5m} \chi(\eta_{0,1}^{0,1} B_{m,n}) \right).
\]
where \( 3 \) is the \( p \)-component of \( \text{Symm}^{[a,b,c,r]} \). This then is what we proved in Corollary 7.42.

There is an alternative way to prove the wild norm relations as in Proposition 8.8. Instead of using local representation theory, there is a geometric method to establish the wild norm relations. The method has the advantage that the wild norm relations hold in étale cohomology with integral coefficients, while Proposition 8.8 is a result in cohomology with rational coefficients, and hence forgets about torsion. The geometric method is formalized by Loeffler [Loe19]. We recall his notations and show how to modify the results to apply to our case.

Let \( T_G \) be a maximal torus of \( G \) and \( B_G \supset T_G \) a Borel subgroup of \( G \) so that \( T_G = T_G \cap H \) and \( B_H = B_G \cap H \) is a maximal torus and a Borel subgroup of \( G \), respectively. Let \( Q_H^0 \) be a mirabolic in \( H \), and \( Q_G = L_G \cdot N_G \) a parabolic with its Levi decomposition in \( G \).

In our case, we take \( T_G \) to be the diagonal matrices, \( B_G \) to be the upper triangular matrices, \( Q_H^0 = \left( \left( \begin{array}{cc} * & * \\ 1 & 1 \end{array} \right), \right) \) and \( Q_G = B_G \). Consider the left action of \( G \) on the flag variety \( F = G/Q_G \), where \( Q_G \) is the opposite of \( Q_G \), i.e., the lower triangular matrices in our case.

We verify the assumptions in [Loe19, Section 4.3].

**Lemma 8.9.** Let \( u = (\eta_{0,1}^{0,1})^{-1} \in G(\mathbb{Z}_p) \). Then

(A) The \( Q_H^0 \)-orbit of \( u \) is open in \( F \).

(B) We have \( u^{-1} Q_H^0 u \cap Q_G^0 \subset Q_G^0 \), where \( Q_G^0 = N_G \cdot L_G^0 \) for some normal reductive subgroup \( L_G^0 \subset L_G \).

**Proof.** Assumption (A) is equivalent to saying \( u^{-1} Q_H^0 u \cdot Q_G^0 \) is open in \( G \). Then both assumptions are verified by a direct computation: For \( \left( \begin{array}{cc} a & b \\ c & d \end{array} \right), \left( \begin{array}{cc} a' & b' \\ c' & d' \end{array} \right) \in Q_H^0 \), we have
\[
u^{-1} Q_H^0 u = \left( \begin{array}{cccc} a & c' & d' - a & b - c' \\ a' & b' & 1 - a' & c' \\ c' & d' - c' & -c' & d' \\ 1 & 1 & 1 & 1 \end{array} \right), \left( \begin{array}{cccc} a' + c' & -a' - c' + b' + d' \\ a' & b' & 1 - a' & c' \\ c' & d' - c' & -c' & d' \\ 1 & 1 & 1 & 1 \end{array} \right).
\]

Hence we may pick \( L_G^0 = \{\left( \begin{array}{cc} 1 & a \\ 1 & 1 \end{array} \right), \left( \begin{array}{cc} 1 & a \\ 1 & 1 \end{array} \right)\} \).

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Let $\eta \in X_*(T_G)$ be a cocharacter which factors through the center $Z(L_G)$ and strictly dominant. Set $\tau = \eta(p)$. For $m \geq 0$ an integer, define open compact subgroups of $G(\mathbb{Z}_p)$:

$$U_m := \{ g \in G(\mathbb{Z}_p) : \tau \equiv g \mod p^m \in \mathbb{Q}_G^0 \}$$

$$U'_m := \{ g \in G(\mathbb{Z}_p) : \tau \equiv g \mod p^{m+1} \in \mathbb{Q}_G^0 \} = U_m \cap \tau U_m \tau^{-1}$$

$$V_m := \tau^{-m}U_m \tau^m,$$

so $U_m \supset U'_m \supset U_{m+1}$. Let $\mathcal{T}$ be the Hecke operator defined by the double coset $U_m \tau^{-1}U_m$.

Let $M_H \to M_G$ be a map of Cartesian cohomology functors (defined in [Loe19, Section 2]). In our situation, we take $M_H(-) = H^*_c(\mathcal{H}_H(-), \tau^* \mathcal{H}^0_{\mathcal{M}, \eta, c, s}^a)$ and $M_G(-) = H^*_c(Y_G(-), \mathcal{H}^0_{\mathcal{M}, \eta, c, s}(2))$. Given a cohomology functor $M$, one can define the Iwasawa completion: for any (not necessarily open) compact subgroups $K$

$$M_{Iw}(K) = \varprojlim_{U \supseteq K} M(U)$$

where $U$ runs through open compact subgroups containing $K$, and the transition maps are given by push-forwards. If $z_H$ is an element in $M_{Iw}(\mathbb{Q}_H^0)$, we can define $z_{G,m}$ to be the image of $z_H$ under the map

$$M_{Iw}(\mathbb{Q}_H^0) \xrightarrow{m^*} M_{Iw}(\mathbb{Q}_H^0 \cap u^{-1}U_mu) \xrightarrow{u \mapsto z_{G,m}} M_{Iw}(\mathbb{Q}_H).$$

Let $\xi_m = \tau_\eta^m \cdot z_{G,m}$.

**Theorem 8.10** ([Loe19, Proposition 4.5.2]). $(\Pr_{V_m^{n+1}}^{m+1}) \xi_{m+1} = \mathcal{T} \cdot \xi_m$.

The proof of this theorem is a formal argument in cohomology, as long as the following key lemma on the interplay between the level subgroups $U_m$ and $U'_m$ holds.

**Lemma 8.11** ([Loe19, Lemma 4.4.1]). Suppose $m \geq 1$. Then

(i) We have $u^{-1}Q_H^0 \cap U'_m = u^{-1}Q_H^0 \cap U_m + 1$

(ii) We have $[u^{-1}Q_H^0 \cap U_m : u^{-1}Q_H^0 \cap U'_m] = [U_m : U'_m]$

We cannot directly apply Theorem 8.10 and Lemma 8.11 because the level subgroups we use are different from the $U_m$ defined here. In our situation, we take $\tau = u_5$ and hence $\mathcal{T} = U_5(p')$. We also take $z_H$ to be

$$b^{[a,b,c,r]} \circ \Pr_1 \circ e \Eis_{\mathcal{T}}^{b+c-2r}(\phi_\infty) = \left( \frac{1}{\Vol K_{M_p', p^n}} b^{[a,b,c,r]} \circ \Pr_1 \circ e \Eis_{\mathcal{T}}^{b+c-2r}(\phi_{M_{p', p^n}}) \right).$$

Then $z_{\mathcal{T}, M_{p', p^n}}$ (Definition 8.5 is constructed from $z_H$ similar to how $z_{G,m}$ is constructed from $z_H$), except that $U_m$ is replaced by $s_m B_{m,n+3m}^{-1}$ and a Tate twist by $1 - a - r$.

Define $U_m' = s_m B_{m,n+3m}^{-1}, U_m' = U_m \cap \tau U_m \tau^{-1}$, and $V_m = \tau^{-m}U_m \tau^m = B_{m,n+3m}$, so $U_m \supset U'_m \supset U_{m+1}$. We have an analogue of Lemma 8.11.

**Lemma 8.12.** Suppose $m \geq 1$. Then

(i) We have $u^{-1}Q_H^0 \cap U'_m = u^{-1}Q_H^0 \cap U_m + 1$

(ii) We have $[u^{-1}Q_H^0 \cap U_m : u^{-1}Q_H^0 \cap U'_m] = [U_m : U'_m]$

**Proof.** The proof of (i) is very similar to that of Lemma 8.11. By a direct computation,

$$q \in U'_m \iff q \in \begin{pmatrix} Z_p & p^{m+1} & p^{2m+2} & p^{3m+3} \\ p^n & p^{n+1} & p^{n+2m} & p^{n+3m} \\ p^{n+2m} & p^{n+2m+2} & p^{n+2m+2} & p^{n+3m+3} \end{pmatrix}, \begin{pmatrix} Z_p & p^n \\ p^n & Z_p \end{pmatrix}$$

and

$$q \in U_{m+1} \iff q \in \begin{pmatrix} Z_p & p^{m+1} & p^{2m+2} & p^{3m+3} \\ p^n & p^{n+1} & p^{n+2m+2} & p^{n+3m+3} \end{pmatrix}, \begin{pmatrix} Z_p & p^n \\ p^n & Z_p \end{pmatrix}. $$

It can then be computed that their intersections with $u^{-1}Q_H^0$ are the same. Here we implicitly used a stronger version of assumption (B) in the computation.
To prove (ii), we need to show that there is a set of representatives for \( \mathcal{U}_m \setminus \mathcal{U}_m \) contained \( u^{-1}Q_H^0u \). We have an isomorphism \( \mathcal{U}_m \setminus \mathcal{U}_m = N_{m+1} \setminus N_m \), where \( N_m = \tau^*N_G(Z_p)\tau^{-m} \). Assumption (A) says that the orbit of identity under \( u^{-1}Q_H^0u \) is open as a \( Z_p \)-subscheme of \( F = G/Q_G \), and hence contains the kernel of reduction modulo \( p \). As a result, \( u^{-1}Q_H^0u \cdot Q_G \supset N_mQ_G \) for any integer \( m \geq 0 \). Indeed, if we write \( Q_G(p^s) \subset Q_G(Z_p) \) for the kernel of reduction modulo \( p^s \), then we still have \( u^{-1}Q_H^0u \cdot Q_G(p^s) \supset N_mQ_G(p^s) \) for any integer \( m \geq 0 \). Let \( x \in N_m \). Then there exists some \( q \in Q_G^0 \) such that \( u^{-1}qu \in N_{m+1}Q_G(p)^{3m} \). So \( u^{-1}qu \in \mathcal{U}_m \) and maps to \( x \) under the isomorphism \( \mathcal{U}_m \setminus \mathcal{U}_m = N_{m+1} \setminus N_m \).

Having this Lemma, the proof of Theorem 8.10 then gives us

**Theorem 8.13.** Let \( m \geq 1 \) be an integer. Then

\[
\left( p^{K^{[a,b,q,r]}}_{K^{[a,b,q,r]}} \right) \left( e^{\pi_{et,M^{[a,b,q,r]}}} \right) = U_{5}^{B^{[a,b,q,r]}}(p) \cdot e^{\pi_{et,M^{[a,b,q,r]}}} \cdot p^{[a,b,q,r]}
\]

8.3. **Projection to \( \Pi \)-component and norm relations.** Let \( \Pi = \Pi_f \otimes \Pi_{\infty} \) be a cuspidal automorphic representation of \( G(\mathbb{A}_f) \) with \( \Pi_{\infty} \) a unitary discrete series. Assume that \( \Pi \) is non-endoscopic. Let \( K \subset G(\mathbb{A}_f) \) be the level of \( \Pi \).

Let \( k_1 \geq k_2 \geq 3 \) and \( k \geq 2 \) be integers. Let \( a = k_2 - 3, b = k_1 - k_2, c = k - 2 \), so \( a, b, c \geq 0 \). Let \( \Sigma(k_1, k_2, k) \) denote the set of isomorphism classes of representations \( \Pi_f \) of \( G(\mathbb{A}_f) \) which are the finite part of a cuspidal automorphic representation \( \Pi = \Pi_f \otimes \Pi_{\infty} \) where \( \Pi_{\infty} \) is a unitary discrete series of weight \((k_1, k_2, k)\).

**Theorem 8.14** (Taylor, Weissauer [Wei05]). There is a \( G(\mathbb{A}_f) \times \text{Gal}_{\mathbb{Q}} \)-equivariant decomposition

\[
H^4_{et}(Y_G(Q), \mathcal{W}^{a,b,c}) \otimes \mathbb{Q}_p \cong \bigoplus_{\Pi_f \in \Sigma(k_1,k_2,k)} \Pi_f \left[ \frac{2a+b+c}{2} \right] \otimes W_{\Pi_f},
\]

where \( W_{\Pi_f} \) is a finite dimensional \( p \)-adic representation of \( \text{Gal}_{\mathbb{Q}} \). Here \( \Pi_f[r] := \Pi_f \otimes \|\mu(\cdot)\|^r \).

If \( \Pi \) is non-endoscopic, then the semi-regularization of \( W_{\Pi_f} \) is isomorphic to \( \rho_{\Pi_f} \), the unique \((\text{up to isomorphism})\) semisimple Galois representation so that \( \det(1 - \rho_{\Pi_f}(\text{Frob}_q^{-1}) \cdot \epsilon^{-s}) = L(\Pi_f, s - 2a+b+c)^{-1} \).

Let \( \Pi \) be non-endoscopic of level \( K \), and write \( m_{\Pi_f} \) for the associated maximal ideal of the spherical Hecke algebra away from \( K \). (As the notation suggests, \( m_{\Pi_f} \) only depends on the finite part \( \Pi_f \) of \( \Pi \).) Then the localization of \( H^4_{et}(Y_G(Q), \mathcal{W}^{a,b,c}) \) at \( m_{\Pi_f} \) is zero unless \( i = 4 \), in which case the localization is equal to that of \( H^4_{et} \). We rewrite the above decomposition as

\[
H^4_{et}(Y_G(Q), \mathcal{W}^{a,b,c}) \otimes \mathbb{Q}_p \cong \bigoplus_{\Pi_f \in \Sigma(k_1,k_2,k)} \Pi_f^* \left[ \frac{2a+b+c}{2} \right] \otimes W_{\Pi_f}^*.
\]

The Poincaré duality says there is a perfect pairing

\[
H^4_{et}(Y_G(Q), \mathcal{W}^{a,b,c}) \times H^4_{et}(Y_G(Q), \mathcal{W}^{a,b,c}) \rightarrow \mathbb{Q}_p(-4).
\]

Combining this with the above decomposition, and recall that \( \mathcal{W}^{a,b,c} = \mathcal{W}^{a,b,c}[2a+b+c](2a+b+c) \), we obtain

\[
W_{\Pi_f}^* \cong W_{\Pi_f}(-2a-b-c-4).
\]

Hence we can again rewrite the decomposition as

\[
H^4_{et}(Y_G(Q), \mathcal{W}^{a,b,c}) \otimes \mathbb{Q}_p \cong \bigoplus_{\Pi_f \in \Sigma(k_1,k_2,k)} \Pi_f^* \left[ \frac{2a+b+c}{2} \right] \otimes W_{\Pi_f}^*.
\]

There is a Hochschild–Serre spectral sequence

\[
E_2^{r,s} = H^r(Q, H^s_{et}(Y_G(Q), \mathcal{W}^{a,b,c}(n))) \Rightarrow H^{r+s}(Y_G(Q), \mathcal{W}^{a,b,c}(n)),
\]

for any integer \( n \), compatible with the Hecke action. Localizing at \( m_{\Pi_f} \), we obtain

\[
H^4_{et}(Y_G(Q), \mathcal{W}^{a,b,c}(n))_{m_{\Pi_f}} \otimes \mathbb{Q}_p = \bigoplus_{\Pi_f \sim \Pi_f} \Pi_f^* \left[ \frac{2a+b+c}{2} \right] \otimes H^4(Q, W_{\Pi_f}(-n-4)).
We write 
\[ \text{pr}_{\Pi^*} : H^0_{\et}(Y_G(K), \mathcal{W}_{\mathbb{Q}_p}^{a,b,c,*}(3-a-r)) \to \Pi_f \left[ \frac{2a + b + c}{2} \right] \otimes H^1(\mathbb{Q}, W^{*}_{\Pi_f}(-1 - a - r)) \]
for the map given by localization at \( \mathfrak{m}_{\Pi_f} \) and then projection onto the isotypical component.

From now on, we assume that \( \Pi \) is unramified and ordinary at \( p \), i.e., \( \Pi_p \) is unramified and an eigenvalue of \( U_{B_{0,1}}^p(p) \) on \( \Pi_p \left[ \frac{-2a + b + c}{2} \right] \) is a \( p \)-adic unit. Let such an eigenvalue be \( \alpha \), and the corresponding eigenvector be \( v_{\alpha} \in \Pi_f \left[ \frac{-2a + b + c}{2} \right] \). Then \( v_{\alpha} \) induces 
\[ \text{pr}_\alpha : \Pi_f \left[ \frac{2a + b + c}{2} \right] \to \mathbb{Q}_p. \]

Let \( \Sigma \neq p \) be the set of primes \( \ell \) for which \( K_{\ell} \neq \mathbb{G}(\mathbb{Z}_\ell) \). For any square-free integer \( M \geq 1 \) coprime to \( \Sigma \cup \{ p \} \) and any integer \( m \geq 0 \), we consider the open compact subgroups \( K_{M_{p^m}, p} \) as defined in Definition 8.1. We have isomorphisms 
\[ Y_G(K_{M_{p^m}, p}) \cong Y_G(K_{p^0 B_{0,1}}) \times Y_{GL_1}(1 + M_{p^m} \hat{\mathbb{Z}}) \cong Y_G(K_{p^0 B_{0,1}}) \times \text{Spec} \mathbb{Q}[X]/\Phi_{M_{p^m}}(X) \]
where \( \Phi_{M_{p^m}}(X) \) is the cyclotomic polynomial of degree \( M_{p^m} \). Hence 
\[ H^1_{\et}(Y_G(K_{M_{p^m}, p}), \mathcal{W}_{\mathbb{Q}_p}^{a,b,c}) \cong H^1_{\et}(Y_G(Q(\mu_{M_{p^m}})(K_{p^0 B_{0,1}}), \mathcal{W}_{\mathbb{Q}_p}^{a,b,c}) \cong Q_p[\text{Gal}(\mathbb{Q}(\mu_{M_{p^m}})/\mathbb{Q})] \otimes Q_p H^1(\mathbb{Q}, W^{*}_{\Pi_f}(-1 - a - r)) \]
Then composing \( \text{pr}_\alpha \) and \( \text{pr}_{\Pi^*} \) for the level \( K_{M_{p^m}, p} \) gives 
\[ H^1_{\et}(Y_G(Q(\mu_{M_{p^m}})(K_{M_{p^m}, p}), \mathcal{W}_{\mathbb{Q}_p}^{a,b,c,*}(3-a-r)) \]
\[ \text{pr}_\alpha \circ \text{pr}_{\Pi^*} \]
\[ \mathbb{Q}_p[\text{Gal}(\mathbb{Q}(\mu_{M_{p^m}})/\mathbb{Q})] \otimes Q_p H^1(\mathbb{Q}, W^{*}_{\Pi_f}(-1 - a - r)) \]

**Definition 8.15 (Euler system classes).** Fix an integer \( e > 1 \) which is prime to \( \Sigma \cup \{ 2, 3, p \} \). Assume that \( c \leq a + b \), and let \( r \) be an integer such that \( \max(0, -a + c) \leq r \leq \min(b, c) \). For any square-free integer \( M \geq 1 \) coprime to \( \Sigma \cup \{ p \} \), and any integer \( m \geq 0 \), define 
\[ e^{[\Pi, r]}_{\text{cores} Q(\mu_{M_{p^m}})} := \left( \frac{\left( \frac{a}{a} + r \right)}{\alpha}_{[a,b,c]}^{\text{cores} Q(\mu_{M_{p^m}})} \right)_{\Pi_f} \text{pr}_\alpha \circ \text{pr}_{\Pi^*} \left( e^{[a,b,c]}_{\text{cores} Q(\mu_{M_{p^m}})} \right)_{\Pi_f} \]
where \( \sigma_p \) is the image of \( p^{-1} \) under the Artin reciprocity map \( Q_p^* \to K_{\ell}^* \to \text{Gal}(\mathbb{Q}(\mu_{M_{p^m}})/\mathbb{Q}) \). By Shapiro’s Lemma, this is an element in \( H^1(\mathbb{Q}(\mu_{M_{p^m}}), W^{*}_{\Pi_f}(-1 - a - r)) \).

In fact, note that \( \text{pr}_\alpha \) induces 
\[ H^1_{\et}(Y_G(K_{p^0 B_{0,1}}), \mathcal{W}_{\mathbb{Q}_p}^{a,b,c,*}) \otimes Q_p \mathbb{Q}_p \to W^{*}_{\Pi_f} \]
and we denote by \( T^{\Pi}_{\text{II}} \) the image of \( H^1_{\et}(Y_G(K_{p^0 B_{0,1}}), \mathcal{W}_{\mathbb{Q}_p}^{a,b,c,*}) \otimes \mathbb{Z}_p \mathbb{Q}_p \) under this map. Then \( T^{\Pi}_{\text{II}} \) is a Galois-stable lattice in \( W^{*}_{\Pi_f} \), and by Theorem 8.7, \( e^{[\Pi, r]}_{\text{cores} Q(\mu_{M_{p^m}})} \) lies in \( H^1(\mathbb{Q}(\mu_{M_{p^m}}), T^{\Pi}_{\text{II}}(-1 - a - r)) \).

**Proposition 8.16 (Wild norm relation).** Let \( m \geq 0 \) be an integer. Then 
\[ \text{cores} Q(\mu_{M_{p^m+1}})_{\text{cores} Q(\mu_{M_{p^m}})} e^{[\Pi, r]}_{\text{cores} Q(\mu_{M_{p^m+1}})} = e^{[\Pi, r]}_{\text{cores} Q(\mu_{M_{p^m+1}})} \]

**Proof.** The \( m = 0 \) case is by definition. For \( m \geq 1 \), this follows from Proposition 8.8, noting that under \( \text{pr}_\alpha \), \( U_{5_{B_{0,1}}}^m(p)^r \) acts as \( \alpha \), and the identification of \( Y_G(K_{M_{p^m}, p}) \cong Y_G(K_{p^0 B_{0,1}}) \times \text{Spec} \mathbb{Q}(\mu_{M_{p^m}}) \) intertwines \( U_{5_{B_{0,1}}}^m(p)^r \) with \( U_{5_{B_{0,1}}}^m(p)^{\sigma_p^{-3}} \) (see [LSZ19, Sec 5.4]).
Proposition 8.17 (Tame norm relation). Let $m \geq 0$ be an integer. Then
\[ \text{core}^{Q_{(\mu_{Mp^m})}}_{Q_{(\mu_{Mp^m})}} \cdot \varepsilon^r_{\ell} = P_\ell(\ell^{-2-a-r} \sigma^{-1}_\ell) \cdot \varepsilon^r_{\ell} \]
where $P_\ell(X) = \det(1 - X \text{Frob}_q^{-1} | W_{P_\ell})$, and $\sigma$ is the arithmetic Frobenius at $\ell$ in $\text{Gal}(Q_{(\mu_{Mp^m})}/Q)$.

Proof. Using Shapiro’s Lemma, we have the following commutative diagram
\[
\begin{array}{c}
H^1(Q, Q_p[\text{Gal}(Q_{(\mu_{Mp^m})}/Q)]) \otimes W_{P_\ell}(-1 - a - r) \xrightarrow{\sim} H^1(Q_{(\mu_{Mp^m})}, W_{P_\ell}(-1 - a - r)) \\
\downarrow^\text{pr}_{\text{Gal}(Q_{(\mu_{Mp^m})}/Q)} \quad \downarrow^\text{core}^{Q_{(\mu_{Mp^m})}}_{Q_{(\mu_{Mp^m})}}
\end{array}
\]

The upper (resp. lower) isomorphism respects the $\text{Gal}(Q_{(\mu_{Mp^m})}/Q)$ (resp. $\text{Gal}(Q_{(\mu_{Mp^m})}/Q)$)-action on both sides. On the left hand side, the action is induced from the $\text{Gal}(Q_{(\mu_{Mp^m})}/Q)$-action on $Q_p[\text{Gal}(Q_{(\mu_{Mp^m})}/Q)] \otimes W_{P_\ell}$ given by $g \cdot (g' \otimes w) = gg' \otimes g^{-1}w$. On the right hand side, the action is the usual one: given $g \in \text{Gal}(Q_{(\mu_{Mp^m})}/Q)$, there is an automorphism on $H^1(Q_{(\mu_{Mp^m})}, W_{P_\ell}(-1 - a - r))$ coming from $\text{Gal}(Q_{(\mu_{Mp^m})} \to \text{Gal}(Q_{(\mu_{Mp^m})}), h \mapsto ghg^{-1}$ and $W_{P_\ell}^\ast(-1 - a - r) \to W_{P_\ell}^\ast(-1 - a - r), w \mapsto g^{-1}w$. So we need to prove
\[
\text{pr}_{\text{Gal}(Q_{(\mu_{Mp^m})}/Q)} \cdot \varepsilon^r_{\ell} = P_\ell(\ell^{-2-a-r} \sigma^{-1}_\ell) \cdot \varepsilon^r_{\ell}.
\]

By construction (Definition 8.1, 8.5 and 8.15), the two sides of the equality only differ at $\ell$, so using the relation between integral classes and rational classes (Proposition 8.6) we only need to show
\[
\sum_{g \in K_{0,0}/K_{1,0}} g \cdot 3 \left( \phi_{1,1} \otimes (\ell^2 - 1) \cdot \xi_{0,0}^0(\ell) \right) = P_\ell(\ell^{-2-a-r} \sigma^{-1}_\ell) \cdot 3 \left( \phi_{0,0} \otimes \text{ch}(G(Z_\ell)) \right)
\]
where
\[
3 : S(Q_\ell^2) \otimes H(G(Z_\ell)) \to \Pi_\ell^\ast \left[ \frac{b + c - 2r}{2} \right]_{G(\mathbb{Q}_\ell)} \otimes H^1(Q, Q_p[\text{Gal}(Q_{(\mu_{Mp^m})}/Q)] \otimes W_{P_\ell}^\ast(-1 - a - r))
\]
is the $\ell$-component of $\text{Symbl}_{[a,b,c,r]}$ composed with $\text{pr}_{\Pi_\ell}$.

Note that by construction $\text{Symbl}_{[a,b,c,r]}$ factors through $\text{Eis}_{b+c-2r}$. We discuss the case $b + c - 2r > 0$ and $b + c - 2r = 0$ separately.

- $b + c - 2r > 0$:
  By Theorem 4.7 the image of $\text{Eis}_{b+c-2r}$ is isomorphic via $\partial_{b+c-2r}$ to $\bigoplus_\eta I((\cdot - b + c - 2r + 1/2\eta, \cdot - 1/2)$. Also $\Pi_\ell$ is an unramified principal series of $G(\mathbb{Q}_\ell)$. Hence 3 satisfies the assumptions of Theorem 7.38.
  When $M = 1$ and $m = 0$, the $\text{Gal}(Q_{(\mu_{Mp^m})}/Q)$-action is trivial, so
  \[ P_\ell(\ell^{-2-a-r} \sigma^{-1}_\ell) = P_\ell(\ell^{-2-a-r}) = L(1-b+c+2r, \Pi_\ell^{-1})^{-1} = 1 \frac{-b+c+2r}{2}, \Pi_\ell^{-1} \]
  Hence (11) follows from Theorem 7.38. For general $M$ and $m$, we apply the $M = 1, m = 0$ case replacing $\Pi$ by $\Pi \otimes \chi^{-1}$, where
  \[ \chi : (\mathbb{Z}/Mp^m)^\times \to \text{Gal}(Q_{(\mu_{Mp^m})}/Q) \to \mathbb{C}^\times \]
is a Dirichlet character. Since $W_{P_\ell}^\ast \otimes \chi^{-1} = W_{\Pi_\ell} \otimes \chi$, and
  \[ L(s, \Pi_\ell \otimes \chi^{-1}) = P_\ell(\ell^{-s-a-b+c-2r} \chi^{-1}(\ell)) = P_\ell(\ell^{-s-2b+c+2r} \chi(\sigma^{-1}_\ell)), \]
we conclude that (11) is also true in this case.

- $b + c - 2r = 0$:
  By Theorem 4.6, $\mathcal{O}^\times(Y_{GL_2})$ surjects via $\partial_0$ onto $I^0(\cdot - 1/2, \cdot | - 1/2)$ and $\bigoplus_\eta I((\cdot - 1/2, \cdot | - 1/2)$ with the kernel being a sum of 1-dimensional (and hence non-generic) representations of $GL_2$. Any $3 : I(\chi, \psi) \otimes \mathcal{H}(H) \to \text{Symbl}^\ast$ must factor through the unique irreducible (generic) quotient of $I(\chi, \psi)$. We can then apply Theorem 7.38 and argue as in the case $b + c - 2r > 0$ above.
\[ \square \]
Appendix A. Proofs of Lemmas 7.29 to 7.34

In the appendix we present the proofs of Lemmas 7.29 through 7.34. Let $i, j$ be integers and $a, b, c \in \mathbb{Z}_\ell$. Recall the notation

$$\eta_{i,j}^{a,b,c} = \begin{pmatrix} 1 & a\ell^{-i} & b\ell^{-i} \\ 1 & 1 & -a\ell^{-i} \\ 1 & 1 & -a\ell^{-i} \end{pmatrix} \in G(\mathbb{Q}_\ell).$$

In addition, we will also write

$$\theta_{i,j}^{a,b,c} = \begin{pmatrix} 1 & a\ell^{-i} & b\ell^{-i} \\ 1 & 1 & -a\ell^{-i} \\ 1 & 1 & -a\ell^{-i} \end{pmatrix} \in G(\mathbb{Q}_\ell)$$

and

$$\lambda_{i,j}^{a,b,c} = \begin{pmatrix} 1 & a\ell^{-i} & b\ell^{-i} \\ 1 & 1 & -a\ell^{-i} \\ 1 & 1 & -a\ell^{-i} \end{pmatrix} \in G(\mathbb{Q}_\ell).$$

A.1. More conjugation lemmas. From now on let $m, n \geq 0$ be integers. We recall Lemma 7.26 in the following lemma, then generalize Lemma 7.27, and later prove more technical lemmas which will be used in the computation.

**Lemma A.1.** Let $v$ be an integer. Take an element $h \in H(\mathbb{Q}_\ell)$ satisfying $\text{pr}_1(h) \in \left(\ell^v \cdot \mathbb{Z}_\ell^\times, \mathbb{Q}_\ell, 0, 1\right)$. Then for any $g \in G(\mathbb{Q}_\ell)$, we have

$$3(\phi_\infty \otimes \text{ch}(h g K_{m,n})) = \ell^{-v} \cdot 3(\phi_\infty \otimes \text{ch}(g K_{m,n})).$$

In particular, if $h \in K_{m,n}$ and $v = 0$, we have

$$3(\phi_\infty \otimes \text{ch}(h g K_{m,n})) = 3(\phi_\infty \otimes \text{ch}(h K_{m,n})) = 3(\phi_\infty \otimes \text{ch}(g K_{m,n})).$$

**Corollary A.2.**

1. For any $\alpha, \beta \in \mathbb{Z}_\ell^\times$ with $\alpha \beta \in 1 + \ell^m \mathbb{Z}_\ell$ and $a, b, c \in \mathbb{Q}_\ell$, we have

$$3(\phi_\infty \otimes \text{ch}(\eta_{i,j}^{a,b,c} K_{m,n})) = 3(\phi_\infty \otimes \text{ch}(\eta_{i,j}^{\alpha \beta a, \alpha \beta b, \alpha \beta c} K_{m,n})).$$

$$3(\phi_\infty \otimes \text{ch}(\theta_{i,j}^{a,b,c} K_{m,n})) = 3(\phi_\infty \otimes \text{ch}(\theta_{i,j}^{\alpha \beta a, \alpha \beta b, \alpha \beta c} K_{m,n})).$$

$$3(\phi_\infty \otimes \text{ch}(\lambda_{i,j}^{a,b,c} K_{m,n})) = 3(\phi_\infty \otimes \text{ch}(\lambda_{i,j}^{\alpha \beta a, \alpha \beta b, \alpha \beta c} K_{m,n})).$$

2. For any $a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbb{Z}_\ell$, we have

$$\eta_{i,j}^{a_1, b_1, c_1, a_2, b_2, c_2} = \eta_{i,j}^{a_1 + a_2, b_1 + b_2, c_1 + c_2},$$

$$\theta_{i,j}^{a_1, b_1, c_1, a_2, b_2, c_2} = \theta_{i,j}^{a_1 + a_2, b_1 + b_2, c_1 + c_2},$$

$$\lambda_{i,j}^{a_1, b_1, c_1, a_2, b_2, c_2} = \lambda_{i,j}^{a_1 + a_2, b_1 + b_2, c_1 + c_2}.$$

In particular, if $a_1 \equiv a_2$ (mod $\ell^t$), $b_1 \equiv b_2$ (mod $\ell^t$) and $c_1 \equiv c_2$ (mod $\ell^t$), then we have

$$\text{ch}(\eta_{i,j}^{a_1, b_1, c_1} K_{m,n}) = \text{ch}(\eta_{i,j}^{a_2, b_2, c_2} K_{m,n}),$$

$$\text{ch}(\theta_{i,j}^{a_1, b_1, c_1} K_{m,n}) = \text{ch}(\theta_{i,j}^{a_2, b_2, c_2} K_{m,n}),$$

$$\text{ch}(\lambda_{i,j}^{a_1, b_1, c_1} K_{m,n}) = \text{ch}(\lambda_{i,j}^{a_2, b_2, c_2} K_{m,n}).$$

**Proof.** The element $\eta_{i,j}^{a,b,c}$ (resp. $\theta_{i,j}^{a,b,c}$, $\lambda_{i,j}^{a,b,c}$) is conjugate to $\eta_{i,j}^{\alpha \beta a, \alpha \beta b, \alpha \beta c}$ (resp. $\theta_{i,j}^{\alpha \beta a, \alpha \beta b, \alpha \beta c}$, $\lambda_{i,j}^{\alpha \beta a, \alpha \beta b, \alpha \beta c}$) via $\left(\frac{\alpha \beta}{1}, \frac{\alpha}{\beta}\right)$, which fixes $\phi_{s,t}$ for any $s, t \geq 0$ and belongs to $K_{m,n}$. Hence claim (1) follows from Lemma A.1. Claim (2) is by direct computation.

□
Corollary A.3.
(1) For any $\alpha, \beta, \gamma, \delta \in \mathbb{Z}_\ell$ with $\alpha \delta - \beta \gamma = 1$, we have
\[ 3(\phi_\infty \otimes \text{ch}(\eta_{i,j}^{a,b,c} K_{m,n})) = 3(\phi_\infty \otimes \text{ch}(\eta_{i,0}^{\delta a - \gamma b, -\beta a + ab, 0} K_{m,n})) \]
Hence if $a, b \in \mathbb{Z}_\ell$ so that $(a, b) \neq (0, 0)$, then
\[ 3(\phi_\infty \otimes \text{ch}(\eta_{i,0}^{a,b,0} K_{m,n})) = 3(\phi_\infty \otimes \text{ch}(\eta_{i-v,0}^{1,0,0} K_{m,n})), \]
where $v := \min\{\text{ord}_\ell(a), \text{ord}_\ell(b)\} < \infty$.
(2) For any $\beta \in \mathbb{Z}_\ell$, we have
\[ 3(\phi_\infty \otimes \text{ch}(\eta_{i,j}^{a,b,c} K_{m,n})) = 3(\phi_\infty \otimes \text{ch}(\eta_{i,j}^{a,-\beta a+b,c} K_{m,n})). \]
Hence if $\text{ord}_\ell(a) \leq \text{ord}_\ell(b)$, then
\[ 3(\phi_\infty \otimes \text{ch}(\eta_{i,j}^{a,b,c} K_{m,n})) = 3(\phi_\infty \otimes \text{ch}(\eta_{i,j}^{a,0,c} K_{m,n})). \]

Proof.
(1) Apply Lemma A.1 to $h = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in H(\mathbb{Q}_\ell) \cap K_{m,n}$.
(2) Apply Lemma A.1 to $h = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 & \beta \\ 1 & 1 \end{pmatrix} \in H(\mathbb{Q}_\ell) \cap K_{m,n}$.

Corollary A.4. Let $a, b, c \in \mathbb{Z}_\ell$. If $i \leq 2$ and $j \leq 2$, we have
\[ 3(\phi_\infty \otimes \text{ch}(\theta_{i,j}^{a,b,c} K_{m,n})) = 3(\phi_\infty \otimes \text{ch}(\theta_{i,0}^{a,b,-ac\ell^2-cj,0} K_{m,n})). \]
In particular,
\[ 3(\phi_\infty \otimes \text{ch}(\theta_{1,1}^{a,b,c} K_{m,n})) = 3(\phi_\infty \otimes \text{ch}(\theta_{1,0}^{a,b,0} K_{m,n})). \]

Proof. Put
\[ h := \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} (\ell^2 - 1)^{-1}c\ell^2 & -1 \\ 1 & 1 \end{pmatrix} \in H(\mathbb{Q}_\ell) \cap K_{m,n}. \]
Since $j \leq 2$ and $c \in \mathbb{Z}_\ell$, we have $h \in H(\mathbb{Z}_\ell)$. Note that $(\ell^2 - 1)^{-1}ac\ell^2 - cj \equiv -ac\ell^2 - cj \pmod{\ell^i}$ since we assume $i \leq 2$. Since
\[ h \cdot \theta_{i,j}^{a,b,c} \cdot h^{-1} = \theta_{i,0}^{a,b,-(\ell^2 - 1)^{-1}ac\ell^2-cj,0}, \]
this corollary follows from Lemmas A.1 and A.2.

Remark A.5. When $i \leq 2$ and $j \leq 2$, Corollary A.4 shows that
\[ \sum_{b=0}^{\ell^i-1} 3(\phi_\infty \otimes \text{ch}(\theta_{i,j}^{a,b,c} K_{m,n})) = \sum_{b=0}^{\ell^i-1} 3(\phi_\infty \otimes \text{ch}(\theta_{i,0}^{a,b,0} K_{m,n})). \]
and
\[ \sum_{b=0}^{\ell-1} 3(\phi_\infty \otimes \text{ch}(\theta_{i,1}^{a,0,b,c} K_{m,n})) = \sum_{b=0}^{\ell-1} 3(\phi_\infty \otimes \text{ch}(\theta_{i,0}^{a,0,b,0} K_{m,n})). \]

Corollary A.6. For $j \leq 0$ and $c \in \mathbb{Z}_\ell$, we have
\[ \text{ch}(i_{i,j}^{a,b,c} K_{m,n}) = \text{ch}(i_{i,0}^{a,b,0} K_{m,n}). \]

Proof. This follows immediately from Corollary A.2 (2) and the fact that $c \in \ell^i \mathbb{Z}_\ell$.

Corollary A.7.
\[ 3(\phi_\infty \otimes \text{ch}(\theta_{i,0}^{a,b,0} K_{m,n})) = 3(\phi_\infty \otimes \text{ch}(i_{i,0}^{b,-a,0} K_{m,n})). \]

Proof. Apply Lemma A.1 to $w := \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix} \in H(\mathbb{Q}_\ell) \cap K_{m,n}$. 

□
Corollary A.8. Let \( a, b \in \mathbb{Z}_\ell \) and \( c \in \mathbb{Z}_\ell^\times \) be a unit. For any \( x \in \mathbb{Z}_\ell^\times \) with \( xc \equiv 1 \pmod{\ell} \), we have
\[
3(\phi_\infty \otimes \text{ch}(\eta_{i_1,1}^{a,b,c} K_{m,n})) = 3(\phi_\infty \otimes \text{ch}(\iota_{i_0}^{a+bx\ell,b,0} K_{m,n})).
\]
Proof. Put \( h := \begin{pmatrix} 1 & \ell \\ 1 & -x\ell \\ -1 & 0 \\ 1 & 1 \end{pmatrix} \in H(Q_\ell) \cap K_{m,n} \). Then we have
\[
h \eta_{i_1,1}^{a,b,c} h^{-1} = \iota_{i_0}^{a+bx\ell,b,0} \left( I_4 \left( \begin{array}{cc} \ell & c \\ -x & -1(1-xc) \end{array} \right) \begin{pmatrix} 1 & 0 \\ x\ell & 1 \end{pmatrix} \right).
\]
Since \( xc \equiv 1 \pmod{\ell} \), we see that
\[
h \eta_{i_1,1}^{a,b,c} h^{-1} K_{m,n} = \iota_{i_0}^{a+bx\ell,b,0} K_{m,n},
\]
and Lemma A.1 implies that
\[
3(\phi_\infty \otimes \text{ch}(\eta_{i_1,1}^{a,b,c} K_{m,n})) = 3(\phi_\infty \otimes \text{ch}(\iota_{i_0}^{a+bx\ell,b,0} K_{m,n})).
\]
□

Corollary A.9. Let \( a, b \in \mathbb{Z}_\ell \) and \( c \in \mathbb{Z}_\ell^\times \) be a unit.

(1) For any \( \alpha \in \mathbb{Z}_\ell^\times \), we have
\[
3(\phi_\infty \otimes \text{ch}(\eta_{i_1,1}^{a,b,c} K_{m,n})) = 3(\phi_\infty \otimes \text{ch}(\eta_{i_1,1}^{\alpha,\alpha^{-1}b,1} K_{m,n})).
\]

(2) For any \( \alpha \in \mathbb{Z}_\ell^\times \), we have
\[
\sum_{a=0}^{\ell-1} 3(\phi_\infty \otimes \text{ch}(\eta_{i_2,1}^{a,b,c} K_{m,n})) = \sum_{a=0}^{\ell-1} 3(\phi_\infty \otimes \text{ch}(\eta_{i_2,1}^{a,ab,b,1} K_{m,n})).
\]
Proof. Use Corollaries A.2 and A.8.

(1)
\[
3(\phi_\infty \otimes \text{ch}(\eta_{i_1,1}^{a,b,c} K_{m,n})) = 3(\phi_\infty \otimes \text{ch}(\iota_{i_0}^{a,b,0} K_{m,n})) = 3(\phi_\infty \otimes \text{ch}(\iota_{i_0}^{\alpha,\alpha^{-1}b,0} K_{m,n})) = 3(\phi_\infty \otimes \text{ch}(\eta_{i_1,1}^{\alpha,\alpha^{-1}b,1} K_{m,n})).
\]

(2)
\[
\sum_{a=0}^{\ell-1} 3(\phi_\infty \otimes \text{ch}(\eta_{i_2,1}^{a,b,c} K_{m,n})) = \sum_{a=0}^{\ell-1} 3(\phi_\infty \otimes \text{ch}(\iota_{i_0}^{a+bc^{-1},b,0} K_{m,n})) = \sum_{a=0}^{\ell-1} 3(\phi_\infty \otimes \text{ch}(\iota_{i_0}^{a,b,0} K_{m,n})) = \sum_{a=0}^{\ell-1} 3(\phi_\infty \otimes \text{ch}(\iota_{i_0}^{a^{-1}a,ab,0} K_{m,n})) = \sum_{a=0}^{\ell-1} 3(\phi_\infty \otimes \text{ch}(\eta_{i_2,1}^{a,ab,b,1} K_{m,n})) = \sum_{a=0}^{\ell-1} 3(\phi_\infty \otimes \text{ch}(\eta_{i_2,1}^{a,ab,b,1} K_{m,n})).
\]
□
Remark A.10. When \( a \in \mathbb{Z}_\ell^* \), \( b \in \mathbb{Z}_\ell \), Corollaries A.3(2), A.8, and A.9 show that
\[
3(\phi_\infty \otimes \text{ch}(\ell_{1,0} a K_{m,n})) = 3(\phi_\infty \otimes \text{ch}(\eta_{1,1} a K_{m,n})) = 3(\phi_\infty \otimes \text{ch}(h_{1,1} a K_{m,n})) = 3(\phi_\infty \otimes \text{ch}(1,0,1 K_{m,n})).
\]

By Corollary A.7, we also have
\[
3(\phi_\infty \otimes \text{ch}(\theta_{1,0} a K_{m,n})) = 3(\phi_\infty \otimes \text{ch}(0,1,0 K_{m,n})).
\]

Lemma A.11. We have
\[
\sum_{a=1}^{\ell-1} 3(\phi_\infty \otimes \text{ch}(\ell_{2,0} a K_{1,0})) = 3(\phi_\infty \otimes \text{ch}(\ell_{2,0} K_{0,0})) = 3(\phi_\infty \otimes \text{ch}(\eta_{2,1} K_{0,0})).
\]

Proof. Put \( h_a := \begin{pmatrix} a \\ 1 \end{pmatrix} \). Then \( h_\ell h_a = h_\ell = \ell \). Since \( K_{0,0} = \bigcup_{a=1}^{\ell-1} h_a K_{1,0} \), Lemma A.1 shows that
\[
3(\phi_\infty \otimes \text{ch}(\ell_{1,0} a K_{0,0})) = \sum_{a=0}^{\ell-1} 3(\phi_\infty \otimes \text{ch}(h_a^{-1} \ell_{2,0} K_{1,0})) = \sum_{a=0}^{\ell-1} 3(\phi_\infty \otimes \text{ch}(h_a^{-1} \ell_{2,0} K_{1,0})).
\]
The second equality follows from Corollary A.8.

A.2. Computations. Recall the notation
\[
3(g) := 3(\phi_\infty \otimes \text{ch}(g K_{1,0})),
\]
for any element \( g \in G(\mathbb{Q}_\ell) \).

A.2.1. \( J_1 \). The finite set \( J_1 \) is the disjoint union of the following four subsets:
\[
J_1^1 = \{(A_1, B_1^u): x, y, z, u \in \{0, \ldots, \ell - 1\}\},
J_1^2 = \{(A_1^{xy}, B_2): x, y, z \in \{0, \ldots, \ell - 1\}\},
J_1^3 = \{(A_x, B_1^u): x, y, u \in \{0, \ldots, \ell - 1\}\},
J_1^4 = \{(A_2^{xy}, B_2^u): x, y \in \{0, \ldots, \ell - 1\}\},
\]
where to simplify notation (we only use the following notation in this subsubsection) we write
\[
A_1^{xy} = \begin{pmatrix} \ell & x & y \\ \ell & z & x \\ 1 & 1 & 1 \end{pmatrix}, \quad A_2^{xy} = \begin{pmatrix} \ell & x & y \\ 1 & \ell & -x \\ 1 & 1 \end{pmatrix},
\]
and
\[
B_1^u = \begin{pmatrix} \ell & u \\ 1 & 1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 \\ \ell \end{pmatrix}.
\]

Lemma A.12.
(1) \[
\sum_{\eta \in J_1^1} 3(\eta) = \ell \cdot 3(\eta_{0,0,0}) + \ell(\ell - 1) \cdot 3(\eta_{1,0,0}) + \ell(\ell - 1) \cdot 3(\eta_{0,1,0}) + \ell(\ell - 1)^2 \cdot 3(\eta_{1,1,1}).
\]

(2) \[
\sum_{\eta \in J_1^2} 3(\eta) = 3(\eta_{0,0,0}) + (\ell - 1) \cdot 3(\eta_{1,0,1}).
\]

(3) \[
\sum_{\eta \in J_1^3} 3(\eta) = \sum_{\eta \in J_1^2} 3(\eta).
\]

(4) \[
\sum_{\eta \in J_1^4} 3(\eta) = 3(\eta_{0,0,0}) + (\ell - 1) \cdot 3(\eta_{1,0,0}).
\]

In particular, since \(J_1 = J_1^1 \sqcup J_1^2 \sqcup J_1^3 \sqcup J_1^4\),
\[
\sum_{\eta \in J_1} 3(\eta) = (\ell + 1) \cdot 3(\eta_{0,0,0}) + (\ell^2 - 1) \cdot 3(\eta_{1,0,0}) + \ell(\ell + 1) \cdot 3(\eta_{0,1,1}) + \ell(\ell^2 - 1) \cdot 3(\eta_{1,1,1}).
\]

Proof.

(1) For representatives in \(J_1^1\), we have
\[
(A_{1,x,y,z}^z, B_1^u) = \left(\begin{array}{c}
\ell \\
y \\
z \\
1
\end{array}\right), \eta_{1,1}^{0,x,u-z}.
\]

Hence Lemma A.1 implies that
\[
\sum_{\eta \in J_1^2} 3(\eta) = \sum_{x,z,u=0}^{\ell-1} 3(\eta_{1,1}^{0,x,u-z}).
\]

Corollaries A.3 and A.9(1) shows that
\[
\sum_{x,z,u=0}^{\ell-1} 3(\eta_{1,1}^{0,x,u-z}) = \ell \cdot \sum_{x=0}^{\ell-1} 3(\eta_{1,0}^{0,x,0}) + \ell(\ell - 1) \cdot \sum_{x=0}^{\ell-1} 3(\eta_{1,1}^{0,x,1})
\]
\[
= \ell \cdot 3(\eta_{0,0,0}) + \ell(\ell - 1) \cdot 3(\eta_{1,0,0}) + \ell(\ell - 1) \cdot 3(\eta_{0,1,0}) + \ell(\ell - 1)^2 \cdot 3(\eta_{1,1,1}).
\]

(2) For representatives in \(J_1^2\), we have
\[
(A_{1,x,y,z}^y, B_2^u) = \left(\begin{array}{c}
\ell \\
y \\
z \\
1
\end{array}\right), \eta_{1,-1}^{0,x,-z}.
\]

Lemma A.1 and Corollary A.6 imply that
\[
\sum_{\eta \in J_1^3} 3(\eta) = \sum_{x,z=0}^{\ell-1} 3(\eta_{1,1,-1}^{0,x,-z}) = \ell \cdot \sum_{x=0}^{\ell-1} 3(\eta_{1,0}^{0,x,0}),
\]

and Corollary A.2 and Corollary A.8 show that
\[
\sum_{x=0}^{\ell-1} 3(\eta_{1,0}^{0,x,0}) = 3(\eta_{0,0,0}) + (\ell - 1) \cdot 3(\eta_{1,0}^{0,1,0})
\]
\[
= 3(\eta_{0,0,1}) + (\ell - 1) \cdot 3(\eta_{1,1}^{-,1,1})
\]
\[
= 3(\eta_{0,0,1}) + (\ell - 1) \cdot 3(\eta_{1,1}^{0,1,1}).
\]
(3) For representatives in $J_3^1$, we have
\[(A_2^{x,y}, B_1^u) = \left( \left( \begin{array}{c} \ell \\ y \\ 1 \end{array} \right), \left( \begin{array}{c} 1 \\ \ell \end{array} \right) \right) \cdot \theta_{1,1}^{x,0,u}.\]

Lemma A.1 implies that
\[\sum_{\eta \in J_3^1} 3(\eta) = 3(\theta_{1,1}^{x,0,u})\]

Furthermore, Corollaries A.4 and A.7 imply that
\[\sum_{x,u=0}^{\ell-1} 3(\theta_{1,1}^{x,0,u}) = \ell \cdot \sum_{x=0}^{\ell-1} 3(\eta_{1,0}^{0,0,0}) = \eta_{1,0}^{0,0,0} \cdot \sum_{\eta \in J_3^1} 3(\eta).

(4) For representatives in $J_4^1$, we have
\[(A_2^{x,y}, B_2) = \left( \left( \begin{array}{c} \ell \\ y \\ 1 \end{array} \right), \left( \begin{array}{c} 1 \\ \ell \end{array} \right) \right) \cdot \eta_{1,0}^{x,0,0}.\]

Hence Lemma A.1 and Corollary A.2 imply that
\[\sum_{\eta \in J_4^1} 3(\eta) = 3(\eta_{1,0}^{x,0,0}) = 3(\eta_{0,0,0}^{0,0,0}) + (\ell - 1) \cdot 3(\eta_{1,0}^{1,0,0}).\]

\[\square\]

A.2.2. $J_2$. We have
\[J_2 = \{(A_1^{x,y,z}, \ell \cdot I_2) : x, y \in \{0, \ldots, \ell - 1\}, z \in \{0, \ldots, \ell^2 - 1\}\},\]
where to simplify notation (we only use the following notation in this subsection) we write
\[A_1^{x,y,z} = \left( \begin{array}{cccc} \ell^2 & \ell x & \ell y & z \\ \ell x & \ell & y & -x \\ \ell y & & \ell & -1 \\ -x & & 1 & \ell \end{array} \right)\quad \text{and} \quad I_2 = \left( \begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \end{array} \right).\]

Lemma A.13.
\[\sum_{\eta \in J_2} 3(\eta) = 3(\eta_{0,0,0}^{0,0,0}) + (\ell^2 - 1) \cdot 3(\eta_{1,0}^{1,0,0}).\]

Proof. For representatives in $J_2$, we have
\[(A_1^{x,y,z}, \ell \cdot I_2) = \left( \left( \begin{array}{c} \ell^2 \\ z \\ 1 \end{array} \right), \left( \begin{array}{c} 1 \\ \ell \\ \ell \end{array} \right) \right) \cdot \eta_{1,0}^{x,y,0}.\]

Lemma A.1 implies that
\[\sum_{\eta \in J_2} 3(\eta) = \sum_{x,y=0}^{\ell-1} 3(\eta_{1,0}^{x,y,0}),\]
and Corollary A.3 shows that
\[\sum_{x,y=0}^{\ell-1} 3(\eta_{1,0}^{x,y,0}) = 3(\eta_{0,0,0}^{0,0,0}) + (\ell^2 - 1) \cdot 3(\eta_{1,0}^{1,0,0}).\]

\[\square\]
A.2.3. $J_3$. The finite set $J_3$ is the disjoint union of the following three subsets:

$J_3^1 = \{(x, y, z, B_1^u) : x, y \in \{0, \ldots, \ell - 1\}, z, u \in \{1, \ldots, \ell^2 - 1\}\},$

$J_3^2 = \{(x, y, z, B_2^u) : x, y \in \{0, \ldots, \ell - 1\}, z \in \{1, \ldots, \ell^2 - 1\}, u \in \{1, \ldots, \ell - 1\}\},$

$J_3^3 = \{(x, y, z, B_3) : x, y \in \{0, \ldots, \ell - 1\}, z \in \{1, \ldots, \ell^2 - 1\}\},$

where to simplify notation (we only use the following notation in this subsubsection) we write

$$A_1^{\sigma, \tau} = \begin{pmatrix} \ell^2 & \ell x & \ell y & z \\ \ell y & y & \ell & -x \\ \ell & -x & 1 \end{pmatrix}$$

and

$$B_1^u = \begin{pmatrix} \ell^2 & u \\ \ell & 1 \end{pmatrix}, \quad B_2^u = \begin{pmatrix} \ell & u \\ \ell & 1 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 1 \\ \ell^2 \end{pmatrix}.$$ Put

$$S_3 := 3(\eta_{0,0,1}) + \ell(\ell - 1) \cdot 3(\eta_{1,0,1}) + (\ell - 1) \cdot 3(\eta_{1,1,1}).$$

Note that we have

$$\sum_{x,y=0}^{\ell-1} 3(\theta_{1,0,0}^{x,y,0}) A_{1,7}^7 \sum_{x,y=0}^{\ell-1} 3(\theta_{1,0,0}^{x,y,0}) A_{1,8}^8 \sum_{x,y=0}^{\ell-1} 3(\eta_{1,1,1}) A_{1,9}^9 S_3$$

**Lemma A.14.**

1. $$\sum_{\eta \in J_3^1} 3(\eta) = \ell^2 S_3.$$  

2. $$\sum_{\eta \in J_3^2} 3(\eta) = (\ell - 1) S_3.$$  

3. $$\sum_{\eta \in J_3^3} 3(\eta) = S_3.$$  

In particular, since $J_3 = J_3^1 \sqcup J_3^2 \sqcup J_3^3$,

$$\sum_{\eta \in J_3} 3(\eta) = \ell(\ell + 1) \cdot 3(\eta_{0,0,1}) + \ell^2(\ell - 1) \cdot 3(\eta_{1,0,1}) + \ell(\ell - 1) \cdot 3(\eta_{1,1,1}).$$

**Proof.** Put

$$h_z := \left( \begin{pmatrix} \ell^2 & z \\ 1 & \ell \end{pmatrix}, \begin{pmatrix} \ell \end{pmatrix} \right).$$

1. For representatives in $J_3^1$, we have

$$\langle A_1^{\sigma, \tau}, B_1^u \rangle = h_z \cdot \theta_{1,1,1}^{x,y,u}.$$  

Lemma A.1 implies that

$$\sum_{\eta \in J_3^1} 3(\eta) = \sum_{u=0}^{\ell^2-1} \sum_{x,y=0}^{\ell-1} 3(\theta_{1,1,1}^{x,y,u}),$$

and Corollary A.4 (see Remark A.5) implies that

$$\sum_{u=0}^{\ell^2-1} \sum_{x,y=0}^{\ell-1} 3(\theta_{1,1,1}^{x,y,u}) = \ell^2 S_3.$$
(2) For representatives in \( J_3^2 \), we have
\[
(A_{1}^{x,y,z}, B_{2}^{u}) = h_{z} \cdot \eta_{1,1}^{x,y,u}.
\]

Lemma A.1 implies that
\[
\sum_{\eta \in J_3^2} \mathcal{J}(\eta) = \sum_{u=1}^{\ell-1} \sum_{x,y=0}^{\ell-1} \mathcal{J}(\eta_{1,1}^{x,y,u}).
\]

Since \( \ell \nmid u \), Corollary A.9(1) implies that
\[
\ell - 1 \sum_{u=1}^{\ell-1} \mathcal{J}(\eta_{1,1}^{x,y,u}) = \ell - 1 \mathcal{J}(\eta_{1,1}^{x,y,1}) = (\ell - 1)S_3.
\]

(3) For representatives in \( J_3^3 \), we have
\[
(A_{1}^{x,y,z}, B_{3}) = h_{z} \cdot \iota_{1,0}^{x,y,0}.
\]

Lemma A.1 implies that
\[
\sum_{\eta \in J_3^3} \mathcal{J}(\eta) = \sum_{x,y=0}^{\ell-1} \mathcal{J}(\iota_{1,0}^{x,y,0}) = S_3.
\]

\[\Box\]

A.2.4. \( J_4 \). The finite set \( J_4 \) is the disjoint union of the following three subsets:
\[
J_4^1 = \{ (A_{1}^{x,y,z}, \ell \cdot I_2) : x, y, z \in \{0, \ldots, \ell^{2} - 1\} \},
J_4^2 = \{ (A_{2}^{x,y,z,w}, \ell \cdot I_2) : x, y \in \{0, \ldots, \ell - 1\}, z \in \{0, \ldots, \ell^{2} - 1\}, w \in \{1, \ldots, \ell - 1\} \},
J_4^3 = \{ (A_{3}^{x,y}, \ell \cdot I_2) : x, y \in \{0, \ldots, \ell^{2} - 1\} \},
\]
where to simplify notation (we only use the following notation in this subsubsection) we write
\[
A_{1}^{x,y,z} = \begin{pmatrix} \ell^2 & x & y \\ \ell^2 & z & x \\ 1 & 1 \end{pmatrix},
A_{2}^{x,y,z,w} = \begin{pmatrix} \ell^2 & \ell x & wx + \ell y & z \\ \ell & w & y & -x \\ y & \ell & -x & 1 \end{pmatrix},
A_{3}^{x,y} = \begin{pmatrix} \ell^2 & x & y \\ 1 & 1 \end{pmatrix}
\]
and \( I_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \). Put
\[
S_4 := \mathcal{J}(\eta_{0,0}^{0,1}) + (\ell - 1) \cdot \mathcal{J}(\eta_{1,1}^{0,1,1}) + \ell(\ell - 1) \cdot \mathcal{J}(\eta_{2,1}^{\ell,1,1}).
\]
Note that we have
\[
\sum_{x=0}^{\ell^2 - 1} \mathcal{J}(\iota_{2,0}^{x,0}) = S_4
\]
by Corollaries A.2 and A.8.

Lemma A.15.

(1)
\[
\sum_{\eta \in J_4^1} \mathcal{J}(\eta) = \ell^2 S_4.
\]

(2)
\[
\sum_{\eta \in J_4^2} \mathcal{J}(\eta) = (\ell - 1)S_4.
\]
\[
\sum_{\eta \in J_4} \mathfrak{Z}(\eta) = S_4.
\]

In particular, since \(J_4 = J_4^1 \sqcup J_4^2 \sqcup J_4^3\),

\[
\sum_{\eta \in J_4} \mathfrak{Z}(\eta) = \ell(\ell + 1) \cdot 3(\eta_{0,1}^{0,0,1}) + \ell(\ell^2 - 1) \cdot 3(\eta_{1,1}^{0,1,1}) + \ell^2(\ell^2 - 1) \cdot 3(\eta_{2,1}^{-\ell,1,1}).
\]

**Proof.**

(1) Put
\[
h_{z,y} := \left(\begin{array}{c}
\ell^2 \\
y \\
1
\end{array}\right), \left(\begin{array}{c}
\ell^2 \\
z \\
1
\end{array}\right).
\]

For representatives in \(J_4^1\), we have

\[
(A_1^{x,y,z}, \ell \cdot I_2) = h_{z,y} \cdot (0, x, -z).
\]

Lemma A.1 and Corollary A.6 show that

\[
\sum_{\eta \in J_4^1} \mathfrak{Z}(\eta) = \ell^2 - 1 \sum_{x,z=0} \mathfrak{Z}(\eta_{0,1}^{0,0,1}) = \ell^2 - 1 \sum_{x=0} \mathfrak{Z}(\eta_{2,0}^{0,0,0}) = \ell^2 S_4.
\]

(2) Put
\[
h_{z,w} := \left(\begin{array}{c}
\ell^2 \\
z \\
1
\end{array}\right), \left(\begin{array}{c}
w \\
\ell
\end{array}\right).
\]

For representatives in \(J_4^2\), we have

\[
(A_2^{x,y,z,w}, \ell \cdot I_2) = h_{z,w} \cdot (x, w + \ell x + \ell y, -w).
\]

Lemma A.1 implies that

\[
\sum_{\eta \in J_4^2} \mathfrak{Z}(\eta) = \sum_{x,y=0}^{\ell - 1} \sum_{w=1}^{\ell - 1} \mathfrak{Z}(\eta_{2,1}^{0,0,1}) = (\ell - 1) \sum_{x=0} \mathfrak{Z}(\eta_{2,0}^{0,0,0}) = (\ell - 1)S_4.
\]

(3) Put
\[
h_y := \left(\begin{array}{c}
\ell^2 \\
y \\
1
\end{array}\right), \left(\begin{array}{c}
1 \\
\ell^2
\end{array}\right).
\]

For representatives in \(J_4^3\), we have

\[
(A_3^{x,y}, \ell \cdot I_2) = h_y \cdot (x, 0, 0).
\]

Lemma A.1 and Corollary A.7 imply that

\[
\sum_{\eta \in J_4^3} \mathfrak{Z}(\eta) = \sum_{x=0}^{\ell^2 - 1} \mathfrak{Z}(\eta_{2,0}^{0,0,0}) = \sum_{x=0}^{\ell^2 - 1} \mathfrak{Z}(\eta_{2,0}^{0,0,0}) = S_4.
\]
A.2.5. \( J_5 \). The finite \( J_5 \) is the disjoint union of the following four subsets:

\[
J_5^1 = \{(A_1^{x,y,z,w}, B_1^u) : x, w, u \in \{0, \ldots, \ell - 1\}, y \in \{0, \ldots, \ell^2 - 1\}, z \in \{0, \ldots, \ell^3 - 1\}\},
\]

\[
J_5^2 = \{(A_2^{x,y,z,w}, B_2) : x, w, u \in \{0, \ldots, \ell - 1\}, y \in \{0, \ldots, \ell^2 - 1\}, z \in \{0, \ldots, \ell^3 - 1\}\},
\]

\[
J_5^3 = \{(A_2^{x,y,z}, B_1^u) : x, u \in \{0, \ldots, \ell - 1\}, y \in \{0, \ldots, \ell^2 - 1\}, z \in \{0, \ldots, \ell^3 - 1\}\},
\]

\[
J_5^4 = \{(A_2^{x,y,z}, B_2) : x \in \{0, \ldots, \ell - 1\}, y \in \{0, \ldots, \ell^2 - 1\}, z \in \{0, \ldots, \ell^3 - 1\}\},
\]

where to simplify notation (we only use the following notation in this subsection) we write

\[
A_1^{x,y,z,w} = \begin{pmatrix} \ell^3 & \ell^2 & \ell y & z \\ \ell^2 & \ell & y - wx & -x \\ \ell & 1 & 1 & 1 \end{pmatrix}, \quad A_2^{x,y,z} = \begin{pmatrix} \ell^3 & \ell y & \ell^2 & z \\ \ell & \ell^2 & -y & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix},
\]

and

\[
B_1^u = \begin{pmatrix} \ell^2 \\ \ell u \end{pmatrix}, \quad B_2 = \begin{pmatrix} \ell \\ \ell^2 \end{pmatrix}.
\]

Put

\[
S_5 := \sum_{x=0}^{\ell-1} \sum_{y=0}^{\ell^2-1} \mathfrak{A}(\ell x, y, 0).
\]

Corollary A.2 and Remark A.10 show that

\[
S_5 = \mathfrak{A}(0, 0, 0) + (\ell - 1) \cdot \mathfrak{A}(0, 1, 0) + \ell(\ell - 1) \cdot \mathfrak{A}(1, 0, 0) + \ell(\ell - 1) \cdot \mathfrak{A}(1, 0, 0) + \ell(\ell - 1) \cdot \sum_{x=0}^{\ell-1} \mathfrak{A}(\ell x, 1, 0),
\]

which by Corollary A.8 and Lemma A.11 is equal to

\[
\mathfrak{A}(0, 0, 1) + (\ell - 1) \cdot \mathfrak{A}(1, 1, 1) + \ell(\ell - 1) \cdot \mathfrak{A}(1, 0, 1) + \ell(\ell - 1) \cdot \mathfrak{A}(2, 1, 1) + \ell(\ell - 1) \cdot \mathfrak{A}(\phi_\infty \odot \text{ch}(0, 1, K_0, 0)),
\]

where the second term happens when \( \ell \mid y \) and \((x, y) \neq (0, 0)\) and the third term happens when \( \ell \nmid y \).

Lemma A.16.

1. \[
\sum_{\eta \in J_5^1} \mathfrak{A}(\eta) = \ell \cdot \mathfrak{A}(0, 0, 0) + \ell(\ell^2 - 1) \cdot \mathfrak{A}(1, 0, 0) + \ell^3(\ell - 1) \cdot \mathfrak{A}(2, 0, 0) + \ell(\ell - 1) S_5
\]
2. \[
\sum_{\eta \in J_5^2} \mathfrak{A}(\eta) = \ell S_5.
\]
3. \[
\sum_{\eta \in J_5^3} \mathfrak{A}(\eta) = \sum_{\eta \in J_5^4} \mathfrak{A}(\eta) = \ell S_5.
\]
4. \[
\sum_{\eta \in J_5^4} \mathfrak{A}(\eta) = \mathfrak{A}(0, 0, 0) + (\ell^2 - 1) \cdot \mathfrak{A}(1, 0, 0) + \ell^2(\ell - 1) \cdot \mathfrak{A}(2, 0, 0).
\]

In particular, since \( J_5 = J_5^1 \cup J_5^2 \cup J_5^3 \cup J_5^4 \),

\[
\sum_{\eta \in J_5} \mathfrak{A}(\eta) = \ell(1) \cdot \mathfrak{A}(0, 0, 0) + (\ell + 1)(\ell^2 - 1) \cdot \mathfrak{A}(1, 0, 0) + \ell^2(\ell - 1) \cdot \mathfrak{A}(2, 0, 0)
\]

\[
+ \ell(\ell + 1) \cdot \mathfrak{A}(0, 1, 0) + \ell(\ell^2 - 1) \cdot \mathfrak{A}(1, 1, 0) + \ell^2(\ell - 1) \cdot \mathfrak{A}(1, 0, 1)
\]

\[
+ \ell^2(\ell - 1) \cdot \mathfrak{A}(2, 1, 0) + \ell^2(\ell - 1) \cdot \mathfrak{A}(2, 1, 1) + \ell^2(\ell - 1) \cdot \mathfrak{A}(\phi_\infty \odot \text{ch}(0, 1, K_0, 0)).
\]

Proof.
(1) Put
\[ h_{z,w} := \left( \binom{\ell^3}{z}, \binom{\ell^2}{w} \ell \right). \]
For representatives in \( J^1_3 \), we have
\[ (A^1_{x,y,z,w}, B^1_1) = h_{z,w} \cdot \eta_{2,1}. \]
Lemma A.1 implies that
\[ \sum_{\eta \in J^1_3} 3(\eta) = \sum_{x,w,u=0}^{\ell-1} \sum_{y=0}^{\ell^2-1} 3(\eta_{2,1, \ell^3,y,u,w}), \]
and Corollary A.9(2) shows that
\[ \sum_{x,w,u=0 \atop y=0}^{\ell-1} \sum_{y=0}^{\ell^2-1} 3(\eta_{2,1, \ell^3,y,u,w}) = \ell \cdot \sum_{x=0 \atop y=0}^{\ell-1} \sum_{y=0}^{\ell^2-1} 3(\eta_{2,0, \ell^3,y,u,w}) + \ell(\ell - 1) \cdot \sum_{x=0 \atop y=0}^{\ell-1} \sum_{y=0}^{\ell^2-1} 3(\eta_{2,1, \ell^3,y,u,w}). \]
Corollary A.3(1) shows that the sum \( \sum_{x=0 \atop y=0}^{\ell-1} \sum_{y=0}^{\ell^2-1} 3(\eta_{2,0, \ell^3,y,u,w}) \) is equal to the sum
\[ 3(\eta_{2,0, \ell^3,y,u,w}) + (\ell^2 - 1) \cdot 3(\eta_{2,0, \ell^3,y,u,w}) + \ell^2(\ell - 1) \cdot 3(\eta_{2,0, \ell^3,y,u,w}), \]
where the second term happens when \( \ell \mid y \) and \( (x,y) \neq (0,0) \), and the third term happens when \( \ell \nmid y \). Corollary A.8 shows that
\[ \sum_{x=0 \atop y=0}^{\ell-1} \sum_{y=0}^{\ell^2-1} 3(\eta_{2,1, \ell^3,y,u,w}) = \sum_{x=0 \atop y=0}^{\ell-1} \sum_{y=0}^{\ell^2-1} 3(\eta_{2,0, \ell^3,y,u,w}) = \sum_{x=0 \atop y=0}^{\ell-1} \sum_{y=0}^{\ell^2-1} 3(\eta_{2,0, \ell^3,y,u,w}). \]

(2) Put
\[ h_{z,w} := \left( \binom{\ell^3}{z}, \binom{\ell^2}{w} \ell \right). \]
For representatives in \( J^2_3 \), we have
\[ (A^1_{x,y,z,w}, B^2_2) = h_{z,w} \cdot \ell_{2,-1}^{\ell y,x,w}. \]
Lemma A.1 implies that
\[ \sum_{\eta \in J^2_3} 3(\eta) = \sum_{x,w,u=0}^{\ell-1} \sum_{y=0}^{\ell^2-1} 3(\ell_{2,-1}^{\ell y,x,w}), \]
and Corollary A.6 implies that
\[ \sum_{x,w,u=0 \atop y=0}^{\ell-1} \sum_{y=0}^{\ell^2-1} 3(\ell_{2,-1}^{\ell y,x,w}) = \ell \cdot \sum_{x=0 \atop y=0}^{\ell-1} \sum_{y=0}^{\ell^2-1} 3(\ell_{2,0}^{\ell y,x,w}). \]

(3) Put
\[ h_{z} := \left( \binom{\ell^3}{z}, \binom{\ell^2}{w} \ell \right). \]
For representatives in \( J^3_3 \), we have
\[ (A^1_{x,y,z,w}, B^3_3) = h_{z} \cdot \theta_{2,1}^{\ell y,x,u}. \]
Lemma A.1 implies that
\[ \sum_{\eta \in J^3_3} 3(\eta) = \sum_{x,w,u=0}^{\ell-1} \sum_{y=0}^{\ell^2-1} 3(\theta_{2,1}^{\ell y,x,u}), \]
and Corollaries A.4 and A.7 (see Remark A.5) show that

\[
\sum_{x,u=0}^{\ell-1} \sum_{y=0}^{\ell^2-1} 3(\theta_{2,1}^{y,\ell,x,u}) = \ell \cdot \sum_{x=0}^{\ell-1} \sum_{y=0}^{\ell^2-1} 3(\theta_{2,0}^{y,\ell,0}) = \ell \cdot \sum_{x=0}^{\ell-1} \sum_{y=0}^{\ell^2-1} 3(\ell,x,y,0) = \sum_{\eta \in J_2^0} 3(\eta).
\]

(4) Put

\[
h_z := \left( \begin{pmatrix} \ell^4 & z \\ 1 & \ell^2 \end{pmatrix}, \left( \begin{pmatrix} \ell^2 \\ y \end{pmatrix} \right) \right).
\]

For representatives in \( J_5^1 \), we have

\[
(A_1^{x,y,z}, B_2) = h_z \cdot \eta_{2,0}^{y,\ell,x,0}.
\]

Lemma A.1 implies that

\[
\sum_{\eta \in J_5^1} 3(\eta) = \sum_{x=0}^{\ell-1} \sum_{y=0}^{\ell^2-1} 3(\eta_{2,0}^{y,\ell,x,0}),
\]

and Corollary A.3 implies that

\[
\sum_{x=0}^{\ell-1} \sum_{y=0}^{\ell^2-1} 3(\eta_{2,0}^{y,\ell,x,0}) = 3(\eta_{0,0,0}^{0,0,0}) + (\ell^2 - 1) \cdot 3(\eta_{1,0,0}^{1,0,0}) + \ell^2(\ell - 1) \cdot 3(\eta_{2,0,0}^{1,0,0}).
\]

\( \square \)

A.2.6. \( J_6 \). By definition, we have

\( J_6 = \{(A_1^{x,y,z}, \ell^2 \cdot I_2): x, y \in \{0, \ldots, \ell^2 - 1\}, z \in \{0, \ldots, \ell^4 - 1\}\} \),

where to simplify notation (we only use the following notation in this subsubsection) we write

\[
A_1^{x,y,z} = \begin{pmatrix} \ell^4 & \ell^2 x & \ell^2 y & z \\ \ell^2 & y & \ell^2 & -x \\ \ell^2 & \ell^2 & 1 \\ 1 & 1 \end{pmatrix}
\]

and \( I_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \).

Lemma A.17.

\[
\sum_{\eta \in J_6} 3(\eta) = 3(\eta_{0,0,0}^{0,0,0}) + (\ell^2 - 1) \cdot 3(\eta_{1,0,0}^{1,0,0}) + \ell^2(\ell^2 - 1) \cdot 3(\eta_{2,0,0}^{1,0,0}).
\]

Proof. Put

\[
h_z := \left( \begin{pmatrix} \ell^4 & z \\ 1 & \ell^2 \end{pmatrix}, \left( \begin{pmatrix} \ell^2 \\ y \end{pmatrix} \right) \right).
\]

For representatives in \( J_6 \), we have

\[
(A_1^{x,y,z}, B_2) = h_z \cdot \eta_{2,0}^{x,y,0}.
\]

Lemma A.1 implies that

\[
\sum_{\eta \in J_6} 3(\eta) = \sum_{x,y=0}^{\ell^2-1} 3(\eta_{2,0}^{x,y,0}),
\]

and Corollary A.3 shows that

\[
\sum_{x,y=0}^{\ell^2-1} 3(\eta_{2,0}^{x,y,0}) = 3(\eta_{0,0,0}^{0,0,0}) + (\ell^2 - 1) \cdot 3(\eta_{1,0,0}^{1,0,0}) + \ell^2(\ell^2 - 1) \cdot 3(\eta_{2,0,0}^{1,0,0}).
\]

where the second term happens when \( \ell \mid x, \ell \mid y, (x, y) \neq (0, 0) \), and the third term happens when \( \ell \nmid x \) or \( \ell \nmid y \). \( \square \)
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