The Bergman kernel for the Vekua equation

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1 | INTRODUCTION

The Bergman kernel is a useful tool in complex analysis with important applications in differential geometry, in theory of analytic and harmonic functions and in the study of conformal mappings¹ (see also Krantz² for new results and extensions to several complex variables). The idea behind its construction is rather simple. The space of square integrable analytic functions on a planar domain (called Bergman space) is associated with a kernel that allows the reproduction of any function in the Bergman space.

This kind of reproducing property was the base of important generalizations. On the one hand, S. Bergman and his collaborators developed a method for solving boundary value problems for elliptic PDEs with variable coefficients, based on the construction of some appropriate reproducing kernels.³ On the other hand, analogous reproducing properties appeared in the work of other mathematicians of that time (and even earlier) in the study of other functional spaces. In Aronszajn,⁴ the abstract theory of reproducing kernel Hilbert spaces (RKHS) was developed. The Bergman-type kernels are concrete examples of RKHS. For the main results, applications, and historical facts about the RKHS theory, see Aronszajn, Bell, Saitoh, and Sawano.⁴-⁷ Despite the fact that the RKHS theory is well developed, when a concrete reproducing kernel is found, special properties of the particular kernel and new connections with different branches of analysis arise.

The goal of this paper is to establish the existence and some basic properties of the Bergman kernel for the Vekua equation. The paper is organized as follows. In Section 2, we present some well-known properties of solutions of the Vekua equation (1) needed throughout the paper, and in Section 3, we define and study the corresponding Bergman space $H^2(Ω)$. Contrary to the case of the Bergman space of analytic functions, $H^2(Ω)$ is a real linear space (in general, not complex). By introducing a suitable scalar product, we realize that $H^2(Ω)$ is a Hilbert space and establish useful convergence properties. Using this, we show in Section 4 that the real and the imaginary parts of the pointwise evaluation are continuous linear functionals on $H^2(Ω)$. This leads first to the existence of the Bergman kernel in $H^2(Ω)$ and second to a realization of a reproducing property in the space $H^2(Ω)$. Finally, in Section 5, we provide a method for constructing the Bergman kernel.
kernel in terms of any countable orthonormal basis in $H^2(\Omega)$. Moreover, we prove that such countable orthonormal basis always exists and provide some suggestions for its construction. Additionally, we obtain a representation for the Bergman projection from $L^2(\Omega)$ onto $H^2(\Omega)$.

## 2 PROPERTIES OF SOLUTIONS OF THE VEKUA EQUATION

Let $\Omega \subset \mathbb{C}$ be a domain (that is, open and connected) and $K \subset \Omega$ be a compact subset. A function $W : K \to \mathbb{C}$ is called Hölder continuous on $K$ with exponent $0 < \alpha < 1$ if there exists $M > 0$ such that

$$|W(z_1) - W(z_2)| \leq M|z_1 - z_2|^{\alpha}$$

for all $z_1, z_2 \in K$. The set of such functions is denoted by $C^\alpha(K)$. We define $C^\alpha(\Omega)$ to be the space of functions $W : \Omega \to \mathbb{C}$ such that $W \in C^\alpha(K)$ for any compact $K \subset \Omega$. By $L^p(\Omega)$, $1 \leq p < \infty$, we denote the classical Lebesgue space of complex valued measurable functions $\varphi$ enjoying the property $\int_{\Omega} |\varphi(z)|^p \, dx\,dy < \infty$. $L^{\infty}(\Omega)$ denotes the space of essentially bounded measurable functions $\varphi$ with the norm $\|\varphi\|_{\infty} := \text{ess sup}_{z \in \Omega} |\varphi(z)|$. The space $L^p_{\text{loc}}(\Omega)$, $1 \leq p \leq \infty$ consists of all functions $\varphi$ such that $\varphi \in L^p(K)$ for any compact $K \subset \Omega$.

We shall deal with weak solutions of the Vekua equation

$$\partial_{\overline{z}} W = aW + b\overline{W}, \text{ in } \Omega, \quad (1)$$

where $\partial_{\overline{z}} = \frac{1}{2} (\partial_x - i\partial_y)$ and the coefficients $a, b \in L^\infty(\Omega)$. A function $W \in L^1_{\text{loc}}(\Omega)$ is a weak solution of (1) if $W$ has weak partial derivatives $\partial_\xi W, \partial_\eta W \in L^1_{\text{loc}}(\Omega)$, and (1) is satisfied a.e. in $\Omega$.

**Remark 1.** Weak solutions of $\partial_{\overline{z}} W = 0$ in $\Omega$ are analytic functions in $\Omega$.\(^8\, p\, 32\)

**Proposition 1** (Vekua\(^8\, p\, 140\)). If $W$ is a weak solution of (1) in $\Omega$, then $W \in C^\alpha(\Omega)$ for any $0 < \alpha < 1$.

**Proposition 2.** Consider the integral operator

$$(T_\Omega \varphi)(z) := \frac{1}{\pi} \int_{\Omega} \frac{\varphi(\zeta)}{\zeta - \overline{z}} \, d\Omega_\zeta.$$  

where $\Omega \subset \mathbb{C}$ is a bounded domain. If $\varphi \in L^\infty(\Omega)$, then

(i) $|(T_\Omega \varphi)(z)| \leq k_1 \|\varphi\|_{\infty}$ for all $z \in \Omega$, $k_1$ depending only on the area of $\Omega$ (see Bers\(^9\, p\, 7\) or Vekua\(^8\, p\, 40\));

(ii) $T_\Omega \varphi$ has weak first order partial derivatives in $\Omega$ and $(\partial_{\overline{z}} T_\Omega \varphi)(z) = \varphi(\overline{z})$ in $\Omega$ (see Vekua\(^8\, pp\, 29 and 72\)).

**Proposition 3.** Let $\Omega$ be a bounded domain and $W \in L^\infty(\Omega)$. Set

$$\Phi = W - T_\Omega (aW + b\overline{W}).$$  

Then, $W$ is a solution of (1) in $\Omega$ iff $\Phi$ is analytic in $\Omega$.

**Proof.** First note that as $W \in L^\infty(\Omega)$, then $aW + b\overline{W} \in L^\infty(\Omega)$ and from Proposition 2, part (ii), $T_\Omega (aW + b\overline{W})$ has weak first-order partial derivatives, and $\partial_{\overline{z}} T_\Omega (aW + b\overline{W}) = aW + b\overline{W}$ in $\Omega$. From here and by (2), we see that $\Phi$ has weak first-order partial derivatives in $\Omega$ if $W$ does. Moreover, application of $\partial_{\overline{z}}$ to $\Phi$ gives $\partial_{\overline{z}} \Phi = \partial_{\overline{z}} W - (aW + b\overline{W})$ from which we conclude that $W$ is a solution of (1) iff $\Phi$ is analytic.

**Proposition 4.** The limit of a uniformly convergent sequence of solutions of (1) is a solution of (1) as well.

**Proof.** Let $W_n(z)$ be a sequence of solutions of (1) in $\Omega$ such that $W_n(z) \to W(z)$ uniformly in $\Omega$. Let $K \subset \Omega$ be a compact subdomain and denote by $\text{Int} \, K$ the interior of $K$. It follows from Proposition 1 that $W_n(z) \in C^\alpha(K)$ and from Proposition 3 that the functions $\Phi_n$ defined by $\Phi_n = W_n - T_\Omega (aW_n + b\overline{W}_n)$ are analytic in $\text{Int} \, K$. Then, the uniform limit $\Phi = W - T_\Omega (aW + b\overline{W})$ is also an analytic function in $\text{Int} \, K$. The last equality together with Proposition 3 allows one to conclude that $W$ is a solution of (1) in $\text{Int} \, K$. As $K \subset \Omega$ is any compact subdomain the final result follows. \(\square\)
Theorem 1 (Basic lemma\(^8\). \(p. 144\)). Let \( \Omega \) be a bounded domain and \( W \) a solution of (1) (allowed to be unbounded) in \( \Omega \) and set

\[
g(z) := \begin{cases} 
a(z) + \frac{W(z)}{\overline{W(z)}} b(z), & \text{if } W(z) \neq 0 \text{ and } z \in \Omega, 
\end{cases}
\]

\[a(z) + b(z), \quad \text{if } W(z) = 0 \text{ and } z \in \Omega,
\]

\[S := T_{\overline{g}}. \text{ Then, there exists an analytic function } \Psi \text{ in } \Omega \text{ such that}
\]

\[W = \Psi e^S \text{ in } \Omega. \quad (3)
\]

Remark 2. We emphasize that the function \( S \) from the previous theorem is bounded in \( \Omega \) by a constant that does not depend on \( W \). Indeed, from the inequality \(|g(z)| \leq |a(z)| + |b(z)| \leq \|a\|_{\infty} + \|b\|_{\infty}, z \in \Omega, \) and by part (i) of Proposition 2, we see that \(|S(z)| \leq k_1(\|a\|_{\infty} + \|b\|_{\infty}) \) where \( k_1 \) depends only on the area of \( \Omega \).

In the theory of generalized analytic functions, formulas (3) and (2) are often called the representations of first and second kind, respectively.\(^8,10\)

3 | THE BERGMAN SPACE FOR THE VEKUA EQUATION

We define the Bergman space of the Vekua equation (1) as

\[H^2(\Omega) = \{ W \in L^2(\Omega) : \partial_\Omega W = aW + bW \text{ in } \Omega \},\]

where \( \Omega \) is a bounded domain and \( a, b \in L^\infty(\Omega) \).

It would be convenient to introduce an inner product structure on \( H^2(\Omega) \), regarding it as a subspace of \( L^2(\Omega) \). However, since \( H^2(\Omega) \) is a real linear space, it is necessary to consider \( L^2(\Omega) \) as a real linear space as well. The following remark is useful for what follows.

Remark 3. Let \( L_\mathbb{C} \) be a complex linear space and \( \langle \cdot, \cdot \rangle_\mathbb{C} \) a (complex) inner product defined there. Denote by \( L_\mathbb{R} \) the same set (as \( L_\mathbb{C} \)) but understood as a real linear space. Then, it is straightforward to check that \( \langle u, v \rangle_\mathbb{R} := \text{Re} \langle u, v \rangle_\mathbb{C} \) is a (real) inner product in \( L_\mathbb{R} \). Moreover, both scalar products induce the same norm, that is,

\[\|u\|_\mathbb{C} = \sqrt{\langle u, u \rangle_\mathbb{C}} = \sqrt{\langle u, u \rangle_\mathbb{R}} = \|u\|_\mathbb{R} \text{ for all } u \in L_\mathbb{C} \equiv L_\mathbb{R}.
\]

Also, if \( L_\mathbb{C} \) is a complex Hilbert space, then \( L_\mathbb{R} \) is a real Hilbert space.

From now on, and according to the previous remark, we consider the real Hilbert space \( L^2(\Omega) \) with the inner product given by

\[\langle W, V \rangle = \text{Re} \int_\Omega W \overline{V} \, dx \, dy
\]

\[= \int_\Omega \text{Re} W \text{Re} V \, dx \, dy + \int_\Omega \text{Im} W \text{Im} V \, dx \, dy.
\]

The norm and the inner product in \( H^2(\Omega) \) are those induced by \( L^2(\Omega) \),

\[\langle W, V \rangle_{H^2(\Omega)} = \text{Re} \int_\Omega W \overline{V} \, dx \, dy, \quad \|W\|_{H^2(\Omega)} := \|W\|_{L^2(\Omega)}.
\]

If \( a = b \equiv 0 \), then \( H^2(\Omega) \) coincides with the classical Bergman space of analytic functions which we shall denote by \( A^2(\Omega) \) (however, and contrary to the classical case, in the present paper, \( A^2(\Omega) \) is understood as a real space).

The next two propositions generalize corresponding well-known results for \( A^2(\Omega) \).\(^5,11\)

Proposition 5. Let \( K \subset \Omega \) be a compact. There is a constant \( C_K > 0 \) such that for all \( W \in H^2(\Omega) \), the following inequality is valid

\[\sup_{z \in K} |W(z)| \leq C_K \|W\|_{H^2(\Omega)}.
\]
Proof. Let $K \subset \Omega$ be a compact and let $W \in H^2(\Omega)$. From Proposition 1, there exist $\Psi$ an analytic function in $\Omega$ and $S$ a bounded function in $\Omega$ such that $W = \Psi e^S$. It is clear that $\Psi \in A^1(\Omega)$ (see also Klimentov\textsuperscript{10}, theorem 2.1.1). According to Remark 2, there exists a constant $M > 0$ independent of $W$ such that $|e^{S(z)}|, |e^{-S(z)}| \leq M$, for all $z \in \mathbb{C}$. Hence,

$$\sup_{z \in K} |W(z)| \leq M \sup_{z \in K} |\Psi(z)|.$$  

Moreover, according to Krantz\textsuperscript{11}, lemma 1.2.1 and Vekua,\textsuperscript{8} there exists a constant $D_K > 0$, depending on $K$, such that

$$\sup_{z \in K} |\Psi(z)| \leq D_K ord_2 \Psi, \quad \forall \Psi \in A^2(\Omega).$$

Combining the last inequalities, we obtain the desired result

$$\sup_{z \in K} |W(z)| \leq MD_K ord_2 \Psi = MD_K \left\| W e^{-S} \right\|_{L^2(\Omega)} \leq M^2 D_K \|W\|_{L^2(\Omega)}.$$  

$\square$

**Proposition 6.** $H^2(\Omega)$ is a Hilbert space. Moreover, the convergence in $H^2(\Omega)$ implies the uniform convergence on all compact subsets of $\Omega$.

Proof. Let $W_n$ be a Cauchy sequence in $H^2(\Omega)$, and let $K$ be a compact subset of $\Omega$. As $W_n$ is a Cauchy sequence in the Hilbert space $L^2(\Omega)$, there exists $W \in L^2(\Omega)$ such that $W_n \to W$, $n \to \infty$ with respect to the $L^2$ norm. Using this together with Proposition 5, we see that $W_n$ is a Cauchy sequence in the Banach space $C(K)$. Thus, $W_n$ converges uniformly on $K$ to some function $\tilde{W} \in C(K)$. It follows from Proposition 4 that $\tilde{W}$ is a solution of the Vekua equation. In order to finish the proof, let us show that $W = \tilde{W}$ almost everywhere in $K$. It is easy to see from the above reasoning that $W_n \to W$ in $L^2(K)$ as well as $W_n \to \tilde{W}$ in $L^2(K)$. Thus, $W = \tilde{W}$ as elements of $L^2(K)$, and the proposition is proved.  

$\square$

4 | THE BERGMAN KERNEL FOR THE VEKUA EQUATION

Let $\zeta \in \Omega$. From Proposition 5, it follows that both functionals $R_\zeta$ and $I_\zeta$ acting from $H^2(\Omega)$ to $\mathbb{R}$ as $R_\zeta W := \text{Re} W(\zeta)$, $I_\zeta W := \text{Im} W(\zeta)$ are continuous. Thus, by Riesz' representation theorem, there are functions $K(\zeta, z) \equiv k_\zeta(z) \in H^2(\Omega)$ and $L(\zeta, z) \equiv l_\zeta(z) \in H^2(\Omega)$ such that

$$\text{Re} W(\zeta) = \langle W(z), K(\zeta, z) \rangle_{H^2(\Omega)}$$

(4)

and

$$\text{Im} W(\zeta) = \langle W(z), L(\zeta, z) \rangle_{H^2(\Omega)}$$

(5)

for all $W \in H^2(\Omega)$. The kernels $K(\zeta, z)$ and $L(\zeta, z)$ enjoy the following interesting relations.

**Proposition 7.** For any $\zeta, z \in \Omega$, the equalities hold

$$\text{Re} K(\zeta, z) = \text{Re} K(z, \zeta), \quad \text{Im} L(\zeta, z) = \text{Im} L(z, \zeta),$$

and

$$\text{Re} L(\zeta, z) = \text{Im} K(z, \zeta).$$

Proof. Let $\zeta_1 \in \Omega$. Taking $W(z) = K(\zeta_1, z)$ in (4), we obtain $\text{Re} K(\zeta_1, z) = \langle K(\zeta_1, z), K(\zeta_1, z) \rangle_{H^2(\Omega)}$. Moreover, since $H^2(\Omega)$ is a real space,

$$\text{Re} K(\zeta_1, \zeta) = \langle K(\zeta_1, z), K(\zeta, z) \rangle_{H^2(\Omega)} = \langle K(z, z), K(\zeta_1, z) \rangle_{H^2(\Omega)} = \text{Re} K(\zeta, \zeta_1).$$

The other equalities are proved similarly.  

$\square$
Gathering equalities (4) and (5), we obtain
\[
W(\zeta) = (W(z), K(\zeta, z))_{H^2(\Omega)} + i(W(z), L(\zeta, z))_{H^2(\Omega)}
\]
\[
= \int_{\Omega} \{ (\text{Re} K(\zeta, z) + i \text{Re} L(\zeta, z)) \text{Re} W(z) + (\text{Im} K(\zeta, z) + i \text{Im} L(\zeta, z)) \text{Im} W(z) \} \, dx \, dy.
\]

With the aid of Proposition 7, this equality can be written in the form
\[
W(\zeta) = \int_{\Omega} \{ K(z, \zeta) \text{Re} W(z) + L(z, \zeta) \text{Im} W(z) \} \, dx \, dy.
\]

**Definition 1.** We define the Bergman kernel of the Vekua equation with coefficient \(\alpha \in \mathbb{C}\) and center \(\zeta \in \Omega\) as
\[
B(\alpha, \zeta, z) := (\text{Re} \, \alpha) K(z, \zeta) + (\text{Im} \, \alpha) L(z, \zeta).
\]

The previous construction leads to the following statement.

**Proposition 8.** The Bergman kernel \(B(\alpha, \zeta, z)\) belongs to \(H^2(\Omega)\) in the variable \(z\) and enjoys the reproducing property
\[
W(\zeta) = \int_{\Omega} B(W(z), z, \zeta) \, dx \, dy, \quad \zeta \in \Omega
\]
(7)

for all \(W \in H^2(\Omega)\).

**Remark 4.** Consider the case \(a = b = 0\), meaning that the Vekua equation defines analytic functions. Hence, if \(W \in A^2(\Omega)\), then \(iW \in A^2(\Omega)\) and \(\langle W, iV \rangle_{A^2(\Omega)} = -\langle iW, V \rangle_{A^2(\Omega)}\). Using this together with (4) and (5), it is easy to see that
\[
\text{Re} \, L(z, \zeta) = -\text{Re} \, L(z, \zeta), \quad \text{Im} \, L(z, \zeta) = \text{Re} \, K(z, \zeta).
\]

On the other hand, from the last equalities and Proposition 7, it follows that \(K(z, \zeta) = -iL(z, \zeta)\). Substituting this into (6) gives us the classical Bergman reproducing formula
\[
W(\zeta) = \int_{\Omega} K(z, \zeta) W(z) \, dx \, dy.
\]

5 | **CONSTRUCTION OF THE BERGMAN KERNEL BY MEANS OF A COUNTABLE ORTHONORMAL SYSTEM, THE BERGMAN PROJECTION**

Since \(H^2(\Omega)\) is a closed subspace of \(L^2(\Omega)\), then
\[
L^2(\Omega) = H^2(\Omega) \oplus (H^2(\Omega))^\perp,
\]
(8)
and there exists an orthogonal projection \(P_{\Omega}\) from \(L^2(\Omega)\) onto \(H^2(\Omega)\). \(P_{\Omega}\) is named the Bergman projection.

**Proposition 9.** Let \(\varphi \in L^2(\Omega)\). Then,
\[
(P_{\Omega} \varphi)(\zeta) = \int_{\Omega} B(\varphi(z), z, \zeta) \, dx \, dy, \quad \zeta \in \Omega.
\]

**Proof.** From (4), (5) and by Proposition 7, it is straightforward to see that
\[
\int_{\Omega} B(\varphi(z), z, \zeta) \, dx \, dy = \langle \varphi(z), K(\zeta, z) \rangle_{H^2(\Omega)} + i \langle \varphi(z), L(\zeta, z) \rangle_{H^2(\Omega)}.
\]
Let $Q_\Omega$ be the operator defined on $L^2(\Omega)$ by the right-hand side of this equality. According to (8), if $\varphi \in L^2(\Omega)$, there exist $W \in H^2(\Omega)$ and $\psi \in (H^2(\Omega))^\perp$ such that $\varphi = W + \psi$. Since $\psi$ is orthogonal to both $K(\zeta, z)$ and $L(\zeta, z)$ (both kernels belong to $H^2(\Omega)$), then $Q_\Omega \psi = 0$, and hence,

$$Q_\Omega \varphi = Q_\Omega (W + \psi) = Q_\Omega W + Q_\Omega \psi = Q_\Omega W.$$ 

This shows that $Q_\Omega^2 = Q_\Omega$ and that the range of $Q_\Omega$ is $H^2(\Omega)$, or equivalently, $Q_\Omega$ is the orthogonal projection from $L^2(\Omega)$ onto $H^2(\Omega)$. Hence, $Q_\Omega = P_\Omega$, and the proof is finished. \hfill $\Box$

A subset $M$ of a normed linear space $X$ is called complete if span $M$ is dense in $X$. Obviously, if $H$ is a Hilbert space, then from a countable and complete subset of $H$, it is possible to construct a countable orthonormal basis of $H$, first removing the linearly dependent elements and then applying the Gram–Schmidt orthonormalization process. Moreover, under the notation of Remark 3, it is clear that if $\{\varphi_n\}_{n \in \mathbb{N}}$ is a complete subset of $H_\mathbb{C}$, then $\{\varphi_n, i\varphi_n\}_{n \in \mathbb{N}}$ is a complete subset of $H_\mathbb{R}$.

**Remark 5.** If $\Omega$ is a bounded simply connected domain whose boundary is also a boundary of an infinite region, then the system of the usual nonnegative powers $\{z^n, iz^n\}_{n \in \mathbb{N}_0}$ is complete in $A^2(\Omega)$.\textsuperscript{12,13}

**Proposition 10.** $H^2(\Omega)$ has a countable complete subset.

**Proof.** It is well-know that $L^2(\Omega)$ has a countable orthonormal basis $\{\varphi_n\}_{n \in \mathbb{N}}$ (see\textsuperscript{14}). Thus, given $W \in H^2(\Omega) \subset L^2(\Omega)$, we can write the corresponding Fourier series $W = \sum_{n=1}^{\infty} c_n \varphi_n$ with $c_n = \langle W, \varphi_n \rangle_{L^2(\Omega)}$. Since $P_\Omega$ is a bounded operator and a projection onto $H^2(\Omega)$, after applying $P_\Omega$ to both sides of the above series, we get $W = P_\Omega W = \sum_{n=1}^{\infty} c_n P_\Omega \varphi_n$. This means that span $\{P_\Omega \varphi_n\}_{n \in \mathbb{N}}$ is dense in $H^2(\Omega)$. The statement is proved. \hfill $\Box$

**Proposition 11.** Let $\{\varphi_n\}_{n \in \mathbb{N}}$ be an orthonormal basis of $H^2(\Omega)$ and $W \in H^2(\Omega)$. Then,

$$W(z) = \sum_{n=1}^{\infty} \langle W, \varphi_n \rangle_{H^2(\Omega)} \varphi_n(z). \quad (9)$$

The series converges in $H^2(\Omega)$ with respect to the variable $z$ and uniformly on compact subsets of $\Omega$. In particular,

$$B(\alpha, \zeta, z) = \sum_{n=1}^{\infty} [\text{Re}(\bar{\alpha} \varphi_n(\zeta))] \varphi_n(z).$$

**Proof.** Formula (9) is the representation of $W$ as a Fourier series corresponding to the orthonormal basis $\{\varphi_n\}$; thus, it is convergent in $H^2(\Omega)$. The uniform convergence then follows due to Proposition 6. The last part of the statement follows from (9) taking $W(z) = B(\alpha, \zeta, z)$. \hfill $\Box$

**Remark 6.** An analogue of the Runge theorem from complex analysis is available for the Vekua equation, replacing the usual powers of complex analysis by a special countable system of solutions of the Vekua equation called formal powers.\textsuperscript{9} There are general conditions under which the system of formal powers can be constructed by a simple algorithm.\textsuperscript{9,15-17} The previous proposition is obviously related to the Runge theorem because of uniform convergence on compact sets. We conjecture that if $\Omega$ is a Jordan domain, then the system of nonnegative formal powers is complete in $H^2(\Omega)$.

### 6. CONCLUSIONS

The Bergman space for the Vekua equation with essentially bounded measurable coefficients in bounded domains is considered. The existence and several properties of the corresponding Bergman reproducing kernel are proved. Its series representation in terms of any orthonormal basis of the Bergman space is obtained.
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