Anomalous Hall effect in the Abrikosov lattice of type-II superconductors

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We study the temperature and magnetic field dependence of the vortex-core charge in the Abrikosov lattice of an s-wave superconductor based on the augmented quasiclassical equations with the Lorentz force and the pair-potential gradient (PPG) force. It is shown that the charging is dominated by the spatial derivative terms of the pair potential in the PPG force, where the PPG force terms are divided into the two parts, namely, (i) the spatial derivative of the pair potential and (ii) the product of the vector and pair potential. This originates from the fact that the Lorentz force and vector potential terms almost cancel each other out. We also clarify that the vortex-core charging due to the PPG force is caused by the anomalous Hall effect on circulating supercurrents, since the charge is large even in an isolated vortex and in the limit of high-κ, which is also the zero field limit, and originates primarily from the phase of the pair potential, i.e. supercurrents. Moreover, we see that the vortex-core charge peaks near zero temperature and about half the upper critical field, which makes it the most suitable region for a measurement of the charge.

I. INTRODUCTION

It is known that vortices in type-II superconductors have not only a single magnetic flux quantum but also accumulated charge. This vortex-core charging has attracted the interest of many researchers, and this has led to numerous theoretical studies. The earliest studies on the vortex-core charging were carried out based on the phenomenological theory [1,2] and the Bogoliubov-de Gennes (BdG) equations [4,5]. London included the Lorentz force acting on supercurrents in his phenomenological equation, which provides a transparent way to show the existence of the vortex-core charge due to the Lorentz force acting on circulating supercurrents [1,7]. Khomskii and Freimuth also calculated the vortex-core charge due to the chemical potential difference between the vortex-core region in the normal state and its surrounding region in the superconducting state by adopting the pair potential in the form of the step function [2,8]. Matsumoto and Heeb first performed a self-consistent calculation based on the BdG equations, including Maxwell’s equations, in order to calculate the vortex-core charging in a chiral p-wave superconductor [8]. The vortex-core charging in a chiral p-wave superconductor was also calculated by using the Ginzburg–Landau (GL) Lagrangian with the Chern–Simons term [2]. Moreover, the dynamical dipole charge in the vortex core in type-II superconductors under an AC electromagnetic field was calculated by Eschrig et al. [10,11]. It was also shown that the electric charge accumulates even at the vortex core of electrically neutral p-wave superfluids, although it is small compared with the core of superconductors [12]. Experimentally, the vortex-core charge in cuprate superconductors was estimated using the NMR/NQR measurements [13].

Recently, the augmented quasiclassical (AQC) equations of superconductivity with the three force terms were derived by incorporating the next-to-leading-order contributions in the expansion of the Gor’kov equations [14,15] in terms of the quasiclassical parameter $\delta \equiv 1/k_F\xi_0$ to study the vortex-core charging in type-II superconductors [16], where $k_F$ is the amplitude of the Fermi wavenumber and $\xi_0 \equiv \hbar v_F/\Delta_0$ is the coherence length with the energy gap $\Delta_0$ at zero magnetic field and zero temperature, and the amplitude of the Fermi velocity $v_F$. It has been elucidated that the vortex-core charging is caused by the three forces: (i) the Lorentz force that acts on supercurrents $\vec{F}_{\text{L}}$, (ii) pair-potential gradient (PPG) force $\vec{F}_{\text{PPG}}$, and (iii) the pressure difference arising from the slope in the density of states (SDOS) [2,8,16]. In the isolated vortex system near the lower critical field, the vortex-core charge was calculated in s-wave superconductors with a cylindrical Fermi surface [21] and a spherical Fermi surface [16] based on the AQC equations with the three force terms. We found that the vortex-core charging in an s-wave superconductor with a spherical Fermi surface is dominated by the PPG force near zero temperature and by the SDOS pressure near the transition temperature when the magnetic penetration depth is larger than the coherence length, and the PPG force is dominant at low temperatures even if the magnetic penetration depth is almost the same as the coherence length [16]. Thus, the PPG force contributes dominantly to the vortex-core charge in a wide parameter range within the isolated vortex system. Masaki also studied the vortex-core charging in s- and chiral p-wave superconductors with an isolated vortex based on the AQC equations with the Lorentz and PPG forces, and pointed out that the angular parts coming from the phase of the pair potential in the PPG force terms contribute dominantly to the vortex-core charging [22]. The PPG force terms are divided into three parts: the radial parts, which are the radial derivatives of the cylindrical coordinates around the vortex center, the angular parts, which are the angular derivatives, and the vector potential terms coming from the gauge invariance of the AQC equations. Therefore, the vortex-core charge is very small if the chirality and vorticity are an-
tiparallel, i.e. $L_z = 0$, since the phase coming from each cancels out, where $L_z$ is the total angular momentum. If the chirality and vorticity are parallel, i.e. $L_z = 2$, the vortex-core charge is enhanced compared to that in s- and antiparallel chiral $p$-wave superconductors since a vortex has only the vorticity of $L_z = 1$ in s-wave superconductors. These results are consistent with those based on the BdG equations obtained by Matsumoto et al. [3]. While vortex-core charging due to the PPG force is yet to be fully understood, it is well known that the vortex-core charging due to the Lorentz force comes from the magnetic Hall effect due to the Lorentz force acting on circulating supercurrents, and we can understand roughly that the vortex-core charging due to the SDOS pressure is caused by the (effective) chemical potential difference between the core and its surrounding region by assuming a roughly normal metal at the core [16]. On the other hand, one may think about the difference between the charging due to the PPG force and the SDOS pressure, since the chemical potential is exactly equal in the normal and superconducting states of the homogeneous system when considering only the PPG force [16, 21]. We discuss this issue based on the recent results [16, 21, 22] and the calculation in this present paper.

The magnetic field dependence of the vortex-core charge was calculated in s-wave superconductors with a cylindrical Fermi surface [23] and in d-wave superconductors with anisotropic Fermi surfaces used for cuprates [24] based on the AQc equations with only the Lorentz force. It was shown that the charge density at the core has a large peak as a function of the magnetic field. Therefore, the vortex-core charge due to the Lorentz force may be larger than that due to the PPG force in a strong magnetic field region. The main purpose of this present paper is to develop a numerical method for calculating the temperature and magnetic field dependence of the vortex-core charge in the Abrikosov lattice of type-II superconductors microscopically within the AQc equations, to study the forces responsible for the charging in the Abrikosov lattice, and to clarify the temperature and magnetic field dependence of the vortex-core charge in the Abrikosov lattice.

To this end, we calculate the temperature and magnetic field dependence of the vortex-core charge in a two-dimensional s-wave superconductor due to the Lorentz and PPG forces using the AQc equations of superconductivity in the Matsubara formalism. The SDOS pressure terms can be neglected in the case of superconductors with a cylindrical Fermi surface [21]. A recently used method may be more useful, but this formulation still only incorporates the Lorentz force [21]. We here perform the numerical calculation of the vortex-core charging combining the methods in Refs. [21] and [23].

This paper is organized as follows. In Sect. IV we present the formalism based on the AQc equations with the Lorentz and PPG forces in the Matsubara formalism, and show that the PPG force can be neglected in the Meissner state. In Sect. IV we give numerical results for the charging in the vortex lattice of an s-wave superconductor with a cylindrical Fermi surface, and discuss the vortex-core charging due to the PPG force. In Sect. IV we provide a conclusion.

II. FORMALISM

A. Augmented quasiclassical equations

For clean s-wave superconductors in equilibrium, the AQc equations with the PPG and Lorentz forces in the Matsubara formalism are given by [16, 21]

$$
\left[ i \varepsilon_n \hat{\tau}_3 - \hat{\Delta} \hat{\tau}_3, \hat{g} \right] + i \hbar \nu_F \cdot \partial \hat{g} \\
+ \frac{i \hbar}{2} \varepsilon (\nu_F \times \mathbf{B}) \cdot \frac{\partial}{\partial \nu_F} \{ \hat{\tau}_3, \hat{g} \} \\
- \frac{i \hbar}{2} \partial \hat{\Delta} \hat{\tau}_3 \cdot \frac{\partial \hat{g}}{\partial \nu_F} - \frac{i \hbar}{2} \frac{\partial \hat{g}}{\partial \nu_F} \cdot \partial \hat{\Delta} \hat{\tau}_3 = 0,
$$

where $\hat{g} = \hat{g}(\varepsilon_n, \mathbf{p}_F, \mathbf{r})$ and $\hat{\Delta} = \hat{\Delta}(\mathbf{r})$ are the quasiclassical Green’s functions and the pair potential, respectively, $\varepsilon_n = (2n + 1) \pi k_B T$ is the fermion Matsubara energy ($n = 0, \pm 1, \cdots$) with $k_B$ and $T$ denoting the Boltzmann constant and temperature, respectively, $\nu_F$ and $\mathbf{p}_F$ are the Fermi velocity and momentum, respectively, $e < 0$ is the electron charge, $\mathbf{B} = \mathbf{B}(\mathbf{r})$ is the magnetic-flux density, $\partial$ is the gauge-invariant differential operator. The commutators are given by $[\hat{a}, \hat{b}] = \hat{a} \hat{b} - \hat{b} \hat{a}$, and $\{ \hat{a}, \hat{b} \} = \hat{a} \hat{b} + \hat{b} \hat{a}$. The first line in Eq. (1) corresponds to the standard Eilenberger equations [27–30], the second line is the Lorentz force terms [17, 18], and the third line is the PPG force terms [16, 20, 21]. It may not be accurate to call the second line the Lorentz force since the vector potential appears in the other terms [31], but we will do so here for convenience. We also assume spin-singlet pairing without spin paramagnetism. The matrices $\hat{g}$, $\hat{\Delta}$ and $\hat{\tau}_3$ are then expressible as [25]

$$
\hat{g} = \begin{bmatrix}
  g & -if \\
  if & -\bar{g}
\end{bmatrix}, \quad \hat{\Delta} = \begin{bmatrix}
  0 & \Delta \\
  \Delta^* & 0
\end{bmatrix}, \quad \hat{\tau}_3 = \begin{bmatrix}
  1 & 0 \\
  0 & -1
\end{bmatrix},
$$

where the barred functions are defined generally by $\bar{X}(\varepsilon_n, \mathbf{p}_F, \mathbf{r}) = X^*(\varepsilon_n, -\mathbf{p}_F, \mathbf{r})$, and the operator $\partial$ is given by

$$
\partial \equiv \begin{cases}
  \nabla & \text{on } \nu_F \\
  \nabla + i \frac{2eA}{\hbar} & \text{on } \nu_F \text{ or } \Delta^*
\end{cases}, \quad \text{on } \hat{g}
$$

with $\mathbf{A} = \mathbf{A}(\mathbf{r})$ denoting the vector potential.

Following the procedure used in Ref. [5], we expand $g$ and $f$ formally in terms of $\delta$ as $g = g_0 + g_1 + \cdots$ and $f = f_0 + f_1 + \cdots$, where $g_0$ and $f_0$ are the solutions of
shifted Green’s functions in the standard Eilenberger equations. The standard Eilenberger equations are given by [27, 30]

$$\varepsilon_n f_0 + \frac{1}{2} \hbar v_F \cdot \left( \mathbf{\nabla} - i \frac{2eA}{\hbar} \right) f_0 = \Delta g_0, \quad (4a)$$

$$\Delta = \Gamma_0 \pi k_B T \sum_{n=-\infty}^{\infty} \langle f_0 \rangle_F, \quad (4b)$$

$$\mathbf{\nabla} \times \mathbf{\nabla} \times \mathbf{A} = -i 2 \pi e \mu_0 N(0) k_B T \sum_{n=-\infty}^{\infty} \langle \mathbf{v}_F g_0 \rangle_F, \quad (4c)$$

with the normalization condition $g_0 = \text{sgn}(\varepsilon_n) (1 - f_0 f_0)^{1/2}$. Here, $\Gamma_0 \ll 1$ is the dimensionless coupling constant responsible for the Cooper pairing, $\langle \cdot \rangle_F$ is the Fermi surface average normalized as $\langle 1 \rangle_F = 1$, $\mu_0$ is the vacuum permeability, and $N(0)$ is the normal density of states per spin and unit volume at the Fermi energy. Equation (4c) forms a set of self-consistent equations for $g_0$, $\Delta$, and $\mathbf{A}$.

The equation for $g_1$ can be obtained from Eq. (11) as [16, 21]

$$\mathbf{v}_F \cdot \mathbf{\nabla} g_1 = -e \langle \mathbf{v}_F \times \mathbf{B} \rangle \cdot \frac{\partial g_0}{\partial p_F} - \frac{i}{2} \partial \Delta^* \frac{\partial f_0}{\partial p_F} - \frac{i}{2} \partial \Delta \cdot \frac{\partial \bar{f}_0}{\partial p_F}, \quad (5)$$

with $g_1 = -\bar{g}_1$. The electric field $\mathbf{E} = \mathbf{E}(r)$ obeys [16, 21]

$$-\lambda_{TF}^2 \mathbf{\nabla}^2 \mathbf{E} + \mathbf{E} = i \frac{\pi k_B T}{e} \sum_{n=-\infty}^{\infty} \langle \mathbf{\nabla} g_1 \rangle_F, \quad (6)$$

where $\lambda_{TF} \equiv \sqrt{\varepsilon_0 d/2e^2 N(0)}$ is the Thomas–Fermi screening length with $\varepsilon_0$ and $d$ denoting the vacuum permittivity and the thickness, respectively. [22, 33, 34]. This equation enables us to calculate the electric field and charge density microscopically.

### B. Meissner state

Since the angular parts which originate from the phase of the pair potential in the PPG force terms contribute dominantly to the charging in an isolated vortex of type-II superconductors, one may predict that the PPG force acts on supercurrents even in the Meissner state and causes the Hall effect near a surface. However, we show below that the PPG force does not act on supercurrents in the Meissner state. To this end, we derive an expression for the Hall electric field in the Meissner state by solving the AQC equations to study the action of the PPG force on supercurrents. We first express the pair potential and the abnormal Green’s function as $\Delta(r) = |\Delta(r)| e^{i \varphi(r)}$ and $f_0(\varepsilon_n, p_F, r) = \bar{f}_0(\varepsilon_n, p_F, r) e^{i \varphi(r)}$, respectively. Substitute them into Eq. (4a), and neglect the spatial derivative of $\bar{f}_0$. Then we obtain the Doppler-shifted Green’s functions in the standard Eilenberger equations as

$$g_0 = \frac{\tilde{\varepsilon}_n}{\sqrt{\varepsilon_n^2 + |\Delta|^2}}, \quad (7a)$$

$$\bar{f}_0 = \frac{|\Delta|}{\sqrt{\varepsilon_n^2 + |\Delta|^2}}, \quad (7b)$$

where $\tilde{\varepsilon}_n$ is defined by $\tilde{\varepsilon}_n \equiv \varepsilon_n + im v_F \cdot \mathbf{v}_s$ with the superfluid velocity $\mathbf{v}_s \equiv (\hbar/2m)(\nabla \varphi - 2eA/\hbar)$ and the electron mass $m$. We also substitute Eq. (7) into Eq. (4) and then obtain

$$\mathbf{v}_F \cdot \mathbf{\nabla} g_1 = -e \langle \mathbf{v}_F \times \mathbf{B} \rangle \cdot \frac{\partial}{\partial p_F} \tilde{\varepsilon}_n \langle 1 \rangle_F \cdot \langle \mathbf{v}_F \rangle_F, \quad (8)$$

Thus, the PPG force terms all cancel each other out. Since the Doppler shift method is an approximation that neglects the low energy excitations such as the vortex and surface states, we find that the low energy excitations are important for the PPG force to work.

We next assume that the gap amplitude is spatially constant as $|\Delta(r)| = |\Delta|$, and expand $g_0, \bar{f}_0, \mathbf{v}_F \cdot \mathbf{\nabla} g_1$ up to the first-order in $\mathbf{v}_s$ as

$$g_0 = \frac{\varepsilon_n}{\sqrt{\varepsilon_n^2 + |\Delta|^2}} + im v_F \cdot \mathbf{v}_s \frac{|\Delta|^2}{(\varepsilon_n^2 + |\Delta|^2)^{3/2}}, \quad (9a)$$

$$\bar{f}_0 = \frac{|\Delta|}{\sqrt{\varepsilon_n^2 + |\Delta|^2}} - im v_F \cdot \mathbf{v}_s \frac{\varepsilon_n |\Delta|}{(\varepsilon_n^2 + |\Delta|^2)^{3/2}}, \quad (9b)$$

$$\mathbf{v}_F \cdot \mathbf{\nabla} g_1 = -ie \langle \mathbf{v}_F \times \mathbf{B} \rangle \cdot \frac{\partial}{\partial p_F} m \mathbf{v}_F \cdot \mathbf{v}_s \frac{|\Delta|^2}{(\varepsilon_n^2 + |\Delta|^2)^{3/2}}. \quad (9c)$$

Substituting Eqs. (9a) and (9b) into Eqs. (11) and (14c), respectively, and using $\langle \mathbf{v}_F \rangle_F = 0$, we obtain the gap equation and the London equation [28]. Furthermore, considering the region outside the vortex core or a surface without any spatial variation in the gap amplitude, substituting Eq. (9c) into Eq. (10), and using $g_1 = -\bar{g}_1$, the equation for the electric field in the Meissner state is given by

$$-\lambda_{TF}^2 \mathbf{\nabla}^2 \mathbf{E} + \mathbf{E} = \mathbf{B} \times \mathbf{R}_H \mathbf{j}, \quad (10)$$

where $\mathbf{j}$ and $\mathbf{R}_H$ are the current density and the Hall coefficient tensor, respectively, in the Meissner state given by

$$\mathbf{j} = me N(0)(1 - Y) (\mathbf{v}_F \mathbf{F} \mathbf{v}_F) \mathbf{v}_s, \quad (11a)$$

$$\mathbf{R}_H = \frac{1}{2eN(0)} \left( \frac{\partial}{\partial p_F} \mathbf{v}_F \right) (\mathbf{v}_F \mathbf{F} \mathbf{v}_F)^{-1}, \quad (11b)$$

with $Y = Y(T)$ denoting the Yosida function [7, 28, 35] defined by

$$Y = 1 - 2 \pi k_B T \sum_{n=0}^{\infty} \frac{|\Delta|^2}{(\varepsilon_n^2 + |\Delta|^2)^{3/2}}. \quad (12)$$
FIG. 1. Gap amplitude $|\Delta(\mathbf{r})|$ at temperature $T = 0.2T_c$ in units of the zero temperature gap $\Delta_0$ on a square grid with $x$ and $y$ ranging from $[-2\xi_0, +2\xi_0]$ for the average flux densities (a) $\bar{B} = 0.15B_{c2}$, (b) $\bar{B} = 0.42B_{c2}$, and (c) $\bar{B} = 0.88B_{c2}$.

This is the same as the result obtained in Ref. [7], despite taking into account the PPG force. Therefore, the Lorentz force acts on supercurrents in the Meissner state as calculated in the previous work [7], but the PPG force does not. We can also show that the PPG force does not act on the shielding currents in anisotropic [34] and chiral [22] superconductors based on the corresponding AQC equations.

III. NUMERICAL RESULTS

A. Numerical procedures

We solve Eqs. (4), (5), and (6) numerically for a triangular vortex lattice of an $s$-wave type-II superconductor with a cylindrical Fermi surface based on the methods in Refs. [21] and [23]. We take the magnetic field to be along the axial direction of the cylinder. The corresponding vector potential is expressible in terms of the average flux density $\mathbf{B} = (0, 0, \bar{B})$ as $A(r) = (\mathbf{B} \times r) / 2 + \tilde{A}(r)$, where $\tilde{A}$ denotes the spatial variation of the flux density. Functions $\tilde{A}(r)$ and $\Delta(r)$ for the triangular lattice obey the following periodic boundary conditions [36–38]:

$$\tilde{A}(r + \mathbf{R}) = \tilde{A}(r), \quad \Delta(r + \mathbf{R}) = \Delta(r)e^{i\mathbf{B}(r+\mathbf{R}) \cdot i\pi n_1 n_2}, \quad (13a)$$

where $\mathbf{R} = n_1 \mathbf{a}_1 + n_2 \mathbf{a}_2$ with the integers $n_1$ and $n_2$, and $\mathbf{a}_1 = a_2(1/2, \sqrt{3}/2, 0)$ and $\mathbf{a}_2 = a_2(0, 1, 0)$ are the basic vectors of the triangular lattice with the length $a_2$ determined by the flux-quantization condition $(\mathbf{a}_1 \times \mathbf{a}_2) \cdot \mathbf{B} = h/|e|$.

To start with, we solve the standard Eilenberger equations (11) self-consistently for the vortex lattice using the Riccati method [28, 39–41]. The solution is substituted into the right-hand side of Eq. (5), which is solved by using the standard Runge–Kutta method. We next obtain the electric field substituting the solution of Eq. (5) into Eq. (6) and solving Eq. (6), and then calculate the charge density $\rho$ using the Gauss’ law $\rho = \epsilon_0 \nabla \cdot E$ numerically. The results present below are for $\lambda_{TF} = 0.03\xi_0$, $\lambda_0 = 5\xi_0$ and $\delta = 0.03$, where $\lambda_0$ is the mag-
FIG. 3. Charge density \( \rho(\mathbf{r}) \) due to the Lorentz force ((a), (b), and (c)), and the paramagnetic ((d), (e), and (f)) and diamagnetic ((g), (h), and (i)) terms in the PPG force at temperature \( T = 0.2T_c \) in units of \( \rho_0 \equiv \Delta_0 \epsilon_0 d/|e|\xi_0^2 \) on a square grid with \( x \) and \( y \) ranging from \([-2\xi_0, +2\xi_0]\) for the average flux densities \( \bar{B} = 0.15B_{c2}, \bar{B} = 0.42B_{c2}, \) and \( \bar{B} = 0.88B_{c2} \) from right to left, respectively. PM and DM denote paramagnetic and diamagnetic, respectively.

**B. Results**

Figures 1 and 2 plot the spatial variations of the gap amplitude \( |\Delta(\mathbf{r})| \) and the \( z \)-component of the magnetic-flux density \( B(\mathbf{r}) \), respectively, at temperature \( T = 0.2T_c \) for the average flux densities \( \bar{B} = 0.15B_{c2}, \bar{B} = 0.42B_{c2}, \) and \( \bar{B} = 0.88B_{c2} \), respectively. It is shown that the gap amplitude away from the vortex center and its slope at the vortex center become small together, and the \( B/\bar{B} \) at vortex center is also small, i.e. the flux density becomes spatially uniform at strong magnetic fields compared with weak fields. We also find that the distance between the vortices becomes closer and the six-fold symmetrical anisotropy becomes stronger at strong magnetic fields. Thus, we have reproduced the previous work pro-
posed by Ichioka et al. \[38\].

In Fig. 4 we show plots for the spatial dependence of the charge density due to the Lorentz force, and the spatial derivative terms of the pair potential and the terms for the product of the vector and pair potential in the PPG force in units of $\rho_0 \equiv \Delta_{\text{gap}} d/|e|\xi_0^2$ as a function of the magnetic field calculated for temperatures (a) $T = 0.2T_c$ and (b) $T = 0.5T_c$. PM and DM denote paramagnetic and diamagnetic, respectively.

FIG. 4. Charge density at the vortex center $\rho(0)$ due to the Lorentz force (green circular points), and the paramagnetic (blue square points) and diamagnetic (red triangular points) terms in the PPG force in units of $\rho_0 \equiv \Delta_{\text{gap}} d/|e|\xi_0^2$ as a function of the magnetic field calculated for temperatures (a) $T = 0.2T_c$ and (b) $T = 0.5T_c$. PM and DM denote paramagnetic and diamagnetic, respectively.

To compare our result with the vortex-core charge estimated by the NMR/NQR measurements \[13\], we next calculate the order of magnitude for the accumulated charge around a vortex. We here adopt the core region of radius $0.2\xi_0$ and the thickness $d = 10$ Å to roughly estimate the peak value of the accumulated charge $Q$. The vortex-core charge in YBCO is given by $Q \sim 10^{-3}|e|$.
for the following appropriate parameters: \( k_F^{-1} \simeq 1.0 \) Å, \( \Delta_0 \simeq 28 \) meV [14], and \( \xi_0 \simeq 30 \) Å [13]. The amount of charge estimated based on our present calculation is an order of magnitude larger than the charge reported in Ref. [2], owing to the difference in the calculation method and the Thomas–Fermi screening length described above. The order of magnitude of the estimated charge in YBCO is roughly consistent with the experimental results by Kumagai et al. [13].

We finally discuss what the vortex-core charging due to the PPG force is. The vortex-core charge due to the PPG force at weak magnetic fields has the following characteristics: (i) the dominant contribution from the angular parts, i.e. the angular derivatives of the cylindrical coordinates around the vortex center [22], (ii) the PPG force acts on only supercurrents in the vortex state in the core, and (iii) it is relatively large even in the high-\( \kappa \) limit and in the isolated vortex system [17, 21]. The PPG force terms are dominated by the angular parts coming from the phase of the pair potential, and they all cancel each other out away from the core as shown in Subsect. [11]. Hence the presence of supercurrents is essential to PPG force charging in the vortex state. Furthermore, since the vortex-core charge is relatively large even in the high-\( \kappa \) limit and in the isolated vortex system, the large vortex-charging is caused by the PPG force acting on circulating supercurrents in the vortex state even in an area that can be considered zero magnetic field. Thus, we can deduce that the vortex-core charging due to the PPG force is the anomalous Hall effect on supercurrents in the vortex state. Fujimoto showed that the Berry phase effect associated with a spatially modulated superconducting order parameter, i.e. the PPG force, gives rise to a fictitious Lorentz force acting on quasiparticles, based on the augmented quasiclassical theory similar to ours [13]. Therefore, we may also infer that the vortex-charging due to the PPG force is caused by the Berry phase effect on supercurrents in the vortex state.

IV. CONCLUSION

We have developed a numerical method for the study of charging in the vortex lattice state of type-II superconductors based on the AQC equations with the Lorentz and PPG forces. Using it, we have calculated the charge distribution in the vortex lattice of \( s \)-wave superconductors with a cylindrical Fermi surface. We have shown that the vortex-core charge due to the Lorentz force and the terms for the product of the vector and pair potential coming from diamagnetic supercurrents in the PPG force almost cancel each other out, and the total vortex-core charge becomes almost the same as that due to the spatial derivative terms of the pair potential coming from paramagnetic supercurrents in the PPG force. We have also found that low temperatures and magnetic fields about half the upper critical field are suitable for the experimental measurement of electric charge in the vortex core. Moreover, we have shown that the PPG force does not contribute to the charging in the the Meissner state. On the other hand, the PPG force contributes to the vortex-core charging even at zero field such as the isolated vortex system and the high-\( \kappa \) limit, and is dominated by the terms of the pair potential phase, which are related to supercurrents. Therefore, it can be understood that the vortex-core charging due to the PPG force is caused by the anomalous Hall effect on supercurrents in the vortex state. We again emphasize that only the Lorentz force acts on supercurrents away from the vortex core and the Lorentz force may be important for transport phenomena [20, 46].

There still remains many interesting problems in relation to the study of vortex lattice systems using the AQC equations. For example, our present method can be used to study the flux-flow Hall effect in the vortex lattice by combing the AC response theory based on the standard Eilenberger equations [10, 11], and to also calculate the vortex lattice in \( ^3 \)He [47–50].

Kumagai et al. estimated the vortex-core charge in cuprate superconductors by the NMR/NQR measurements. However, they used the local electric field gradient obtained from changes in the nuclear quadrupole resonance frequency to estimate the vortex-core charge experimentally. To the best of our knowledge, direct observation of the vortex-core charge such as the atomic force microscopy measurement has not been achieved yet. We hope that our present study will stimulate more detailed experiments on vortex-core charging.

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