Universal enveloping crossed module of Leibniz crossed modules and representations

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Abstract. The universal enveloping algebra functor $\mathcal{UL}: \mathcal{Lb} \to \mathcal{Alg}$, defined by Loday and Pirashvili [1], is extended to crossed modules. Then we construct an isomorphism between the category of representations of a Leibniz crossed module and the category of left modules over its universal enveloping crossed module of algebras. Note that the procedure followed in the proof for the Lie case cannot be adapted, since the actor in the category of Leibniz crossed modules does not always exist.

1. Introduction

Leibniz algebras, which are a non-antisymmetric generalization of Lie algebras, were introduced in 1965 by Bloh [2], who called them $D$-algebras and referred to the well-known Leibniz identity as differential identity. In 1993 Loday [3] made them popular and studied their (co)homology. From that moment, many authors have studied this structure, obtaining very relevant algebraic results [1, 4] and applications to Geometry [5, 6] and Physics [7].

Crossed modules of groups were described for the first time by Whitehead in the late 1940s [8] as an algebraic model for path-connected CW-spaces whose homotopy groups are trivial in dimensions greater than 2. From that moment, crossed modules of different algebraic objects, not only groups, have been considered, either as tools or as algebraic structures in their own right. For instance, in [1] crossed modules of Leibniz algebras were defined in order to study cohomology.

Observe that in Ellis’s PhD thesis [9] it is proved that, given a category of $\Omega$-groups $\mathcal{C}$ such as the categories of associative and Leibniz algebras, crossed modules, cat$^1$-objects, internal categories and simplicial objects in $\mathcal{C}$ whose normal complexes are of length 1 are equivalent structures. Internal categories can be described in terms of what Baez calls strict 2-dimensional objects (see [10] for groups and [11] for Lie algebras). By analogy to Baez’s terminology, crossed modules of associative algebras (respectively Leibniz algebras) can be viewed as strict associative 2-algebras [12] (respectively strict Leibniz 2-algebras [13, 14]).

In the case of Lie algebras, the universal enveloping algebra plays two important roles: the category of representations of a Lie algebra is isomorphic to the category of left modules...
over its universal enveloping algebra and it is right adjoint to the Liezation functor. For Leibniz algebras, these roles are played by two different functors: Loday and Pirashvili [1] proved that, given a Leibniz algebra \( p \), the category of left \( \mathcal{UL}(p) \)-modules is isomorphic to the category of \( p \)-representations, where \( \mathcal{UL}(p) \) is the universal enveloping algebra of \( p \). On the other hand, if associative algebras are replaced by dialgebras, there exist a universal enveloping dialgebra functor [15], which is right adjoint to the functor that assigns to every dialgebra its corresponding Leibniz algebra.

As Norrie states in [16], it is surprising the ease of the generalization to crossed modules of many properties satisfied by the objects in the base category. In [17], the universal enveloping dialgebra functor is extended to crossed modules. The aim of this article is to extend to crossed modules the functor \( \mathcal{UL} \) and the aforementioned isomorphism between representations of a Leibniz algebra and left modules over its universal enveloping algebra. Observe that the analogous isomorphism in the case of Lie algebras can be easily proved via the actor, but this method cannot be applied in our case, since the actor of a Leibniz crossed module does not necessarily exist [13] (see [18] for the the 1-dimensional case). This makes our approach especially interesting.

In Section 2 we recall some basic definitions and properties, such as the concept of crossed module of associative and Leibniz algebras, along with the notions of the corresponding \( \text{cat}^1 \)-objects. In Section 3 we give proper definitions of left modules over a crossed module of associative algebras and representations of a Leibniz crossed module. In Section 4 we describe the generalization to crossed modules of the functor \( \mathcal{UL} : \mathcal{Lb} \to \mathcal{Alg} \), that is \( \mathcal{XUL} : \mathcal{XLb} \to \mathcal{XAlg} \), which assigns to every Leibniz crossed module its corresponding universal enveloping crossed module of algebras. Finally, we construct an isomorphism between the categories of representations of a Leibniz crossed module and the left modules over its universal enveloping crossed module of algebras.

**Notations and conventions**
Throughout the paper, we fix a commutative ring \( K \) with unit. All algebras are considered over \( K \). The categories of Leibniz and (non-unital) associative algebras will be denoted by \( \mathcal{Lb} \) and \( \mathcal{Alg} \), respectively.

### 2. Preliminaries

**Definition 2.1 ([3]).** A Leibniz algebra \( p \) over \( K \) is a \( K \)-module together with a bilinear operation \([ , ] : p \times p \to p\), called the Leibniz bracket, which satisfies the Leibniz identity:

\[
[[p_1,p_2],p_3] = [p_1,[p_2,p_3]] + [[p_1,p_3],p_2],
\]

for all \( p_1, p_2, p_3 \in p \). A morphism of Leibniz algebras is a \( K \)-linear map that preserves the bracket.

We will denote by \( \mathcal{Lb} \) the category of Leibniz algebras and morphisms of Leibniz algebras. These are in fact right Leibniz algebras. For the opposite structure, that is \([p_1,p_2]' = [p_2,p_1]'\), the left Leibniz identity is

\[
[p_1,[p_2,p_3]]' = [[p_1,p_2]',p_3] + [p_2,[p_1,p_3]]',
\]

for all \( p_1, p_2, p_3 \in p \).
If the bracket of a Leibniz algebra $p$ happens to be antisymmetric, then $p$ is a Lie algebra. Furthermore, every Lie algebra is a Leibniz algebra. For more examples, see [3].

Recall that a Leibniz algebra $p$ acts on another Leibniz algebra $q$ if there are two bilinear maps $p \times q \rightarrow q$, $(p,q) \mapsto [p,q]$ and $q \times p \rightarrow q$, $(q,p) \mapsto [q,p]$, satisfying six relations, which are obtained from the Leibniz identity by taking two elements in $p$ and one in $q$ (three identities) and one element in $p$ and two elements in $q$ (three identities). Given an action of a Leibniz algebra $p$ on another Leibniz algebra $q$, it is possible to consider the semidirect product $q \rtimes p$, whose Leibniz structure is given by:

$$\left[(q_1,p_1),(q_2,p_2)\right] = ([q_1,q_2] + [p_1,q_2] + [q_1,p_2],[p_1,p_2]),$$

for all $(q_1,p_1), (q_2,p_2) \in q \oplus p$.

**Definition 2.2 ([1]).** A representation of a Leibniz algebra $p$ is a $K$-module $M$ equipped with two actions $p \times M \rightarrow M$, $(p,m) \mapsto [p,m]$ and $M \times p \rightarrow M$, $(m,p) \mapsto [m,p]$, satisfying the following three axioms:

$$[m,[p_1,p_2]] = [[m,p_1],p_2] - [[m,p_2],p_1],$$

$$[p_1,[m,p_2]] = [[p_1,m],p_2] - [[p_1,p_2],m],$$

$$[p_1,[p_2,m]] = [[p_1,p_2],m] - [[p_1,m],p_2],$$

for all $m \in M$ and $p_1,p_2 \in p$.

A morphism $f : M \rightarrow N$ of $p$-representations is a $K$-linear map which is compatible with the left and right actions of $p$.

**Remark 2.3.** Given a $p$-representation $M$, we can endow the direct sum of $K$-modules $M \oplus p$ with a Leibniz structure such that $M$ is an abelian ideal and $p$ is a subalgebra. The converse statement is also true. It is evident that the Leibniz structure of $M \oplus p$ is the one of $M \rtimes p$, as described previously.

**Definition 2.4 ([1]).** Let $p'$ and $p''$ be two copies of a Leibniz algebra $p$. We will denote by $l_x$ and $r_x$ the elements of $p'$ and $p''$ corresponding to $x \in p$. Consider the tensor $K$-algebra $T(p' \oplus p'')$, which is associative and unital. Let $I$ be the two-sided ideal corresponding to the relations:

$$[x,y]_r = y_rx_x - x_ry_x,$$

$$[x,y]_l = y_rx_x - x_ry_x,$$

$$x_1(y_r + y_l) = 0.$$

for all $x, y \in p$. The universal enveloping algebra of the Leibniz algebra $p$ is the associative and unital algebra

$$\text{UL}(p) := T(p' \oplus p'')/I.$$  

This construction defines a functor $\text{UL}: \text{Leib} \rightarrow \text{Alg}$.

**Theorem 2.5 ([1]).** The category of representations of the Leibniz algebra $p$ is isomorphic to the category of left modules over $\text{UL}(p)$.
Proof. Let $M$ be a representation of $\mathfrak{p}$. It is possible to define a left action of $\mathcal{UL}(\mathfrak{p})$ on the $K$-module $M$ as follows. Given $x_l \in \mathfrak{p}^l$, $x_r \in \mathfrak{p}^r$ and $m \in M$,

$$x_l \cdot m = [x, m], \quad x_r \cdot m = [m, x].$$

These actions can be extended to an action of $T(\mathfrak{p}^l \oplus \mathfrak{p}^r)$ by composition and linearity. It is not complicated to check that this way $M$ is equipped with a structure of left $\mathcal{UL}(\mathfrak{p})$-module.

Regarding the converse statement, it is immediate that, starting with a left $\mathcal{UL}(\mathfrak{p})$-module, the restrictions of the actions to $\mathfrak{p}^l$ and $\mathfrak{p}^r$ give two actions of $\mathfrak{p}$ which make $M$ into a representation.

Recall that a left module over an associative algebra $A$ can be described as a morphism $\alpha : A \to \text{End}(M)$, where $M$ is a $K$-module.

Both $\text{Lb}$ and $\text{Alg}$ are categories of interest, notion introduced by Orzech in [19]. See [20] for a proper definition and more examples. Categories of interest are a particular case of categories of groups with operations, for which Porter [21] described the notion of crossed module. The following definitions agree with the one given by Porter.

**Definition 2.6.** A crossed module of Leibniz algebras (or Leibniz crossed module) $(q, p, \eta)$ is a morphism of Leibniz algebras $\eta : q \to p$ together with an action of $p$ on $q$ such that

\[
\eta([q, p]) = [p, \eta(q)] \quad \text{and} \quad \eta([q, p]) = [\eta(q), p],
\]

\[
[\eta(q_1), q_2] = [q_1, q_2] = [q_1, \eta(q_2)],
\]

for all $q, q_1, q_2 \in q, p \in p$.

A morphism of Leibniz crossed modules $(\varphi, \psi)$ from $(q, p, \eta)$ to $(q', p', \eta')$ is a pair of Leibniz homomorphisms, $\varphi : q \to q'$ and $\psi : p \to p'$, such that

\[
\psi \eta = \eta' \varphi,
\]

\[
\varphi([q, p]) = [\psi(p), \varphi(q)] \quad \text{and} \quad \varphi([q, p]) = [\varphi(q), \psi(p)],
\]

for all $q \in q, p \in p$.

**Definition 2.7.** A crossed module of algebras $(B, A, \rho)$ is an algebra homomorphism $\rho : B \to A$ together with an action of $A$ on $B$ such that

\[
\rho(ab) = a\rho(b) \quad \text{and} \quad \rho(ba) = \rho(b)a,
\]

\[
\rho(b_1)b_2 = b_1b_2 = b_1\rho(b_2),
\]

for all $a \in A, b_1, b_2 \in B$.

A morphism of crossed modules of algebras $(\varphi, \psi) : (B, A, \rho) \to (B', A', \rho')$ is a pair of algebra homomorphisms, $\varphi : B \to B'$ and $\psi : A \to A'$, such that

\[
\psi \rho = \rho' \varphi,
\]

\[
\varphi(ba) = \varphi(b)\psi(a) \quad \text{and} \quad \varphi(ab) = \psi(a)\varphi(b),
\]

for all $b \in B, a \in A$.

We will denote by $\text{XLb}$ and $\text{XAlg}$ the categories of Leibniz crossed modules and crossed modules of associative algebras, respectively. Crossed modules can be alternatively describe as $\text{cat}^1$-objects, namely:
Definition 2.8. A cat\(^1\)-Leibniz algebra \((p_1, p_0, s, t)\) consists of a Leibniz algebra \(p_1\) together with a Leibniz subalgebra \(p_0\) and the structural morphisms \(s, t : p_1 \to p_0\) such that

\[
\begin{align*}
s|_{p_0} & = t|_{p_0} = \text{id}_{p_0}, \quad \text{(CLb1)} \\
[Ker s, Ker t] & = 0 = [Ker t, Ker s], \quad \text{(CLb2)}
\end{align*}
\]

Definition 2.9. A cat\(^1\)-algebra \((A_1, A_0, \sigma, \tau)\) consists of an algebra \(A_1\) together with a subalgebra \(A_0\) and the structural morphisms \(\sigma, \tau : A_1 \to A_0\) such that

\[
\begin{align*}
\sigma|_{A_0} & = \tau|_{A_0} = \text{id}_{A_0}, \quad \text{(CAs1)} \\
\text{Ker} \sigma \text{ Ker} \tau & = 0 = \text{Ker} \tau \text{ Ker} \sigma. \quad \text{(CAs2)}
\end{align*}
\]

It is a well-known fact (see for instance [9]) that the category of crossed modules of Leibniz algebras (resp. associative algebras) is equivalent to the category of cat\(^1\)-Leibniz algebras (resp. associative algebras).

Given a crossed module of Leibniz algebras \((q, p, \eta)\), the corresponding cat\(^1\)-Leibniz algebra is \((q \rtimes p, p, s, t)\), where \(s(q, p) = p\) and \(t(q, p) = \eta(q) + p\) for all \((q, p) \in q \rtimes p\). Conversely, given a cat\(^1\)-Leibniz algebra \((p_1, p_0, s, t)\), the corresponding Leibniz crossed module is \(t|_{\text{Ker} s} : \text{Ker} s \to p_0\), with the action of \(p_0\) on \(\text{Ker} s\) induced by the bracket in \(p_1\). The equivalence for associative algebras is analogous.

3. Representations of crossed modules

Since our intention is to extend Theorem 2.5 to crossed modules, it is necessary to give a proper definition of representations over Leibniz crossed modules and left modules over crossed modules of algebras.

In the case of crossed modules of associative algebras, by analogy to the 1-dimensional situation, left modules can be described via the endomorphism crossed module:

Definition 3.1. Let \((B, A, \rho)\) be a crossed module of algebras. A left \((B, A, \rho)\)-module is an abelian crossed module of algebras \((V, W, \delta)\), that is \(\delta\) is simply a morphism of \(K\)-modules and the action of \(W\) on \(V\) is trivial, together with a morphism of crossed modules of algebras \((\varphi, \psi) : (B, A, \rho) \to (\text{Hom}_K(W, V), \text{End}(V, W, \delta), \Gamma)\).

Note that \(\text{End}(V, W, \delta)\) is the algebra of all pairs \((\alpha, \beta)\), with \(\alpha \in \text{End}_K(V)\) and \(\beta \in \text{End}_K(W)\), such that \(\beta \delta = \delta \alpha\). Furthermore, \(\Gamma(d) = (d \delta, \delta d)\) for all \(d \in \text{Hom}_K(W, V)\). The action of \(\text{End}(V, W, \delta)\) on \(\text{Hom}_K(W, V)\) is given by

\[
(\alpha, \beta) \cdot d = \alpha d \quad \text{and} \quad d \cdot (\alpha, \beta) = d \beta,
\]

for all \(d \in \text{Hom}_K(W, V)\), \((\alpha, \beta) \in \text{End}_K(V, W, \delta)\). See [22, 23] for further details.

For the categories of crossed modules of groups and Lie algebras, representations can be defined via an object called the actor (see [24, 16]). However this is not the case for Leibniz crossed modules (see [13]). Nevertheless, it is possible to give a definition by equations:

Definition 3.2. A representation of a Leibniz crossed module \((q, p, \eta)\) is an abelian Leibniz crossed module \((N, M, \mu)\) endowed with:
(i) **Actions of the Leibniz algebra** $\mathfrak{p}$ (and so $\mathfrak{q}$ via $\eta$) on $N$ and $M$, such that the homomorphism $\mu$ is $\mathfrak{p}$-equivariant, that is

\[
\mu([p, n]) = [p, \mu(n)],
\]

\[
\mu([n, p]) = [\mu(n), p],
\]

for all $n \in N$ and $p \in \mathfrak{p}$.

(ii) **Two $K$-bilinear maps** $\xi_1: \mathfrak{q} \times M \to N$ and $\xi_2: M \times \mathfrak{q} \to N$ such that

\[
\mu \xi_2(m, q) = [m, q],
\]

\[
\mu \xi_1(q, m) = [q, m],
\]

\[
\xi_2(\mu(n), q) = [n, q],
\]

\[
\xi_1(q, \mu(n)) = [q, n],
\]

\[
\xi_2(m, [p, q]) = \xi_2([m, p], q) - [\xi_2(m, q), p],
\]

\[
\xi_1([p, q], m) = \xi_2([m, p], q) - [p, \xi_2(m, q)],
\]

\[
\xi_2([m, [q, p]]) = [\xi_2([m, q], p) - \xi_2([m, p], q),
\]

\[
\xi_1([q, [p, m]]) = -\xi_1([q, [m, p]],
\]

\[
[p, \xi_1(q, m)] = -[p, \xi_2(m, q)],
\]

for all $q, q' \in \mathfrak{q}$, $p \in \mathfrak{p}$, $n \in N$, $m, m' \in M$.

**Remark 3.3.** As in Remark 2.3, given a $(\mathfrak{q}, \mathfrak{p}, \eta)$-representation $(N, M, \mu)$, we can obtain a Leibniz crossed module structure on $(N \oplus \mathfrak{q}, M \oplus \mathfrak{p}, \mu \oplus \eta)$ where $N$ and $M$ are abelian ideals and $\mathfrak{q}$ and $\mathfrak{p}$ are subalgebras of $N \oplus \mathfrak{q}, M \oplus \mathfrak{p}$ respectively. Conversely it is also true. Moreover, a representation can be seen as an action of $(\mathfrak{q}, \mathfrak{p}, \eta)$ over an abelian Leibniz crossed module $(N, M, \mu)$ in the sense of [13].

### 4. Isomorphism between the categories of representations

In this section, we extend to crossed modules the universal enveloping algebra functor and then we give the construction of an isomorphism between the categories of representations of a Leibniz crossed module and left modules over its corresponding universal enveloping crossed module of algebras. Recall that the method used in the proof of the equivalent result in the case of Lie algebras cannot be applied in our case due to the lack of actor in the category of Leibniz crossed modules.

Let $(\mathfrak{q}, \mathfrak{p}, \eta)$ be a Leibniz crossed module and consider its corresponding cat$^1$-Leibniz algebra

\[
\mathfrak{q} \rtimes \mathfrak{p} \xrightarrow{s} \mathfrak{p},
\]

with $s(q, p) = p$ and $t(q, p) = \eta(q) + p$ for all $(q, p) \in \mathfrak{q} \times \mathfrak{p}$. Now, if we apply $\text{UL}$ to the previous diagram, we get

\[
\text{UL}(\mathfrak{q} \times \mathfrak{p}) \xrightarrow{\text{UL}(s)} \text{UL}(\mathfrak{p}).
\]
Although it is true that $\mathcal{U}(s)|_{\mathcal{U}(p)} = \mathcal{U}(t)|_{\mathcal{U}(p)} = \text{id}_{\mathcal{U}(p)}$, in general, the second condition for cat$^1$-algebras (CAs2) is not satisfied. Nevertheless, we can consider the quotient $\mathcal{U}(q \times p) = \mathcal{U}(q \times p)/X$, where $X = \text{Ker} \mathcal{U}(s)|_{\mathcal{U}(s)} \oplus \text{Ker} \mathcal{U}(t)|_{\mathcal{U}(t)}$. Applying this construction, we observe that

\[
\mathcal{U}(q \times p) \xrightarrow{\mathcal{U}(s)} \mathcal{U}(p)
\]

is clearly a cat$^1$-algebra. Note that $\mathcal{U}(p)$ can be regarded as a subalgebra of $\mathcal{U}(q \times p)$.

We can now define $\mathcal{XUL}(q, p, \eta)$ as the crossed module of associative algebras given by $(\text{Ker} \mathcal{U}(s), \mathcal{U}(p), \mathcal{U}(t)|_{\text{Ker} \mathcal{U}(s)})$. This construction defines a functor $\mathcal{XUL}: \mathcal{Xb} \to \mathcal{XAlg}$.

**Theorem 4.1.** The category of representations of a Leibniz crossed module $(q, p, \eta)$ is isomorphic to the category of left modules over its universal enveloping crossed module of algebras $\mathcal{XUL}(q, p, \eta)$.

**Proof.** Let $(N, M, \mu)$ be a left $(\text{Ker} \mathcal{U}(s), \mathcal{U}(p), \mathcal{U}(t)|_{\text{Ker} \mathcal{U}(s)})$-module. Then we have a homomorphism

\[
(\varphi, \psi): (\text{Ker} \mathcal{U}(s), \mathcal{U}(p), \mathcal{U}(t)|_{\text{Ker} \mathcal{U}(s)}) \to (\text{Hom}_K(M, N), \text{End}(N, M, \mu), \Gamma),
\]

such that $\psi \circ \mathcal{U}(t)|_{\text{Ker} \mathcal{U}(s)} = \Gamma \circ \varphi$, $\varphi(ab) = \varphi(b)\psi(a)$ and $\varphi(ab) = \psi(a)\varphi(b)$. We need to define actions of $p$ on $N$ and $M$ satisfying (LbEQ1) and (LbEQ2) and we need to define $\xi_1: q \times M \to N$ and $\xi_2: M \times q \to N$ satisfying identities (LbM1a)–(LbM5b).

We define the actions of $p$ on $N$ and $M$ as those induced by $\psi: \mathcal{U}(p) \to \text{End}(N, M, \mu)$ as in Theorem 2.5. The identities (LbEQ1) and (LbEQ2) are followed by the properties of $\text{End}(N, M, \mu)$. We define the morphisms $\xi_1$ and $\xi_2$ by $\xi_1(q, m) = \varphi((q, 0)_r)(m)$ and $\xi_2(m, q) = \varphi((q, 0)_r)(m)$. The identities (LbM1a), (LbM2a) and (LbM1b), (LbM2b) are followed by the commutative square $\psi \circ \mathcal{U}(t)|_{\text{Ker} \mathcal{U}(s)} = \Gamma \circ \varphi$ applied to the elements $(q, 0)_r$ and $(q, 0)_l$ respectively. Given the element $([p, q], 0)_r \in \text{Ker} \mathcal{U}(s)$, we have that

\[
([p, q], 0)_r = [(0, p)_r, (q, 0)_r] = (q, 0)_r(0, p)_r - (0, p)_r(q, 0)_r.
\]

Applying $\varphi$ to this relation and using that $\varphi([p, q]) = \varphi(b)\psi(a)$ and $\varphi(ab) = \psi(a)\varphi(b)$ we obtain $\varphi([p, q], 0)_r = \varphi(q, 0)_r\psi_2(0, p)_r - \psi_1(0, p)_r\varphi(q, 0)_r$ which implies (LbM3a). Proceeding in the same way for the elements $([p, q], 0)_l$, $([p, q], 0)_r$ and $([q, 0], 0)_l$ we check that identities (LbM3b), (LbM3c) and (LbM3d) are satisfied. Doing a similar argument on the elements $([q, q'], 0)_r$ and $([q, q'], 0)_l$ we obtain identities (LbM4a) and (LbM4b). Applying $\varphi$ to the relations

\[
(q, 0)_l(0, p)_l = -(q, 0)_l(0, p)_r \quad \text{and} \quad (0, p)_l(q, 0)_l = -(0, p)_l(q, 0)_r,
\]

we have identities (LbM5a) and (LbM5b) respectively.

Conversely, let $(N, M, \mu)$ be a $(q, p, \eta)$-representation. We need to construct a morphism of crossed modules of algebras $(\varphi, \psi)$ from $\mathcal{XUL}(q, p, \eta)$ to $(\text{Hom}_K(M, N), \text{End}(N, M, \mu), \Gamma)$. The homomorphism $\psi = (\psi_1, \psi_2): \mathcal{U}(p) \to \text{End}(N, M, \mu)$ is the homomorphism induced by the actions of $p$ on $N$ and $M$ as in Theorem 2.5. It is well defined by identities (LbEQ1) and
(LbEQ2). Let be the homomorphism \( \Phi: (q \times p)^l \oplus (q \times p)^r \rightarrow \text{Hom}_K(N \otimes M, N \otimes M) \) defined by

\[
\Phi(q, p)_l(n, m) = ([q, n] + [p, n] + \xi_1(q, m), [p, m]),
\]

\[
\Phi(q, p)_r(n, m) = ([n, q] + [n, p] + \xi_2(m, q), [m, p]).
\]

Note that they can also be rewritten as

\[
\Phi(q, p)_l(n, m) = (t(q, p)_l(n) + \xi_1(q, m), s(q, p)_l(m)),
\]

\[
\Phi(q, p)_r(n, m) = (t(q, p)_r(n) + \xi_2(m, q), s(q, p)_l(m)).
\]

By the universal property of the tensor algebra, there is a unique homomorphism

\[
T(\Phi): T((q \times p)^l \oplus (q \times p)^r) \rightarrow \text{Hom}_K(N \otimes M, N \otimes M),
\]

commuting with the inclusion.

We consider the projection \( \pi: \text{Hom}_K(N \otimes M, N \otimes M) \rightarrow \text{Hom}_K(M, N \otimes M) \), where \( \pi(f)(m) = f(0, m) \), and denote \( \varphi' = \pi \circ T(\Phi) \). Given an element of the form \((q, p)_r(q', p')_r - (q', p')_r(q, p)_r + [(q, p)_r, (q', p')_r] \) we obtain that

\[
\varphi'((q, p)_r(q', p')_r - (q', p')_r(q, p)_r + [(q, p)_r, (q', p')_r]) (m)
= [\xi_2(m, q'), q] + [[m, p'], q] + [\xi_2(m, q'), p]
- [\xi_2(m, q), q'] - [[m, p], q'] - [\xi_2(m, q), p']
+ [\xi_2(m, [q, q'])] + [\xi_2(m, [q', p'])] + [\xi_2(m, [q', p])] = 0,
\]

by the properties of \( \xi_2 \).

Analogously, it is possible to prove that \( \varphi' \) vanishes on the other two relations of the universal enveloping algebra. Then \( \varphi' \) factors through \( \mathcal{UL}(q \times p) \). In order to ease notation we will refer to it as \( \varphi' \) as well.

By definition it is clear that \( \varphi'|_{\text{Ker} \mathcal{UL}(s)}(M) \subseteq N \) and \( T(\Phi)((q \times p)^l \oplus (q \times p)^r)(N) = T(t)((q \times p)^l \oplus (q \times p)^r)(N) \). Then \( \varphi' \) factors through \( \text{Ker} \mathcal{UL}(t) \text{Ker} \mathcal{UL}(s) \). Moreover, we have that

\[
\varphi'(q', p')_r(q, p)_r(m) = ([\xi_2(m, q), q'] + [\xi_2(m, q), p'] + [\xi_2(m, p), q])
= (\xi_2(m, [t(q, p)_r, q']) + [\xi_2(m, q'), t(p, q)_r, [\xi_2(m, q), s(q', p')_r, s(q', p), s(q, p), m]),
\]

extending this argument we see that \( \varphi' \) also factors through \( \text{Ker} \mathcal{UL}(s) \text{Ker} \mathcal{UL}(t) \).

\[
\begin{array}{ccc}
(q \times p)^l \oplus (q \times p)^r \xrightarrow{\Phi} \text{Hom}_K(N \otimes M, N \otimes M) & \xrightarrow{\pi} & \text{Hom}_K(M, N \otimes M) \\
T(\Phi) & & \\
\text{Ker} \mathcal{UL}(t) \text{Ker} \mathcal{UL}(s) & & \\
\mathcal{UL}(q \times p) & & \\
\end{array}
\]
Therefore, $\varphi$ will be the restriction of $\varphi'$ to $\text{Ker} \, \mathcal{U}(q \times p)$ and it will take values in $\text{Hom}_K(M,N)$, that is
\[
\varphi: \text{Ker} \, \mathcal{U}(q \times p) \to \text{Hom}_K(M,N).
\]
With these definitions of $\varphi$ and $\psi$, to check that
\[
(\varphi, \psi): (\text{Ker} \, \mathcal{U}(s), \mathcal{U}(p), \mathcal{U}(t)|_{\text{Ker} \, \mathcal{U}(s)}) \to (\text{Hom}_K(M,N), \text{End}(N, M, \mu), \Gamma)
\]
is a morphism of crossed modules of algebras is now a matter of straightforward computations.

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