THE EXISTENCE OF FULL DIMENSIONAL INVARIANT TORI FOR 1-DIMENSIONAL NONLINEAR WAVE EQUATION

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Abstract. In this paper we prove the existence and linear stability of full dimensional tori with subexponential decay for 1-dimensional nonlinear wave equation with external parameters, which relies on the method of KAM theory and the idea proposed by Bourgain.

1. Introduction and main result

Consider 1-dimensional nonlinear wave equation (NLW)
\[
\begin{align*}
  u_{tt} &= u_{xx} - V \ast u - u^3 \\
  u(t,0) &= u(t,\pi) = 0, \quad -\infty < t < +\infty,
\end{align*}
\]
(1.1)
on the finite interval \( x \in [0, \pi] \) with Dirichlet boundary conditions
\[
  u(t,0) = u(t,\pi) = 0, \quad -\infty < t < +\infty,
\]
where \( V \ast \) is the Fourier multiplier defined by
\[
\hat{V} \ast u(n) = V_n \hat{u}(n)
\]
and \((V_n)_{n \in \mathbb{N}^*}\) are independently chosen in \([0, 1]\), \( \mathbb{N}^* = \mathbb{N} \setminus \{0\} \).

To state our results, we need some notations and definitions. Let \( z = (z_n)_{n \in \mathbb{N}^*} \) and its complex conjugate \( \bar{z} = (\bar{z}_n)_{n \in \mathbb{N}^*} \). Introduce \( I_n = |z_n|^2 \) and \( J_n = I_n - I_n(0) \), where \( I_n(0) \) will be considered as the initial data. Consider the Hamiltonian \( R \) with the following form
\[
R(z, \bar{z}) = \sum_{a, k, k' \in \mathbb{N}^*} B_{akk'} M_{akk'}
\]
(1.2)
with
\[
M_{akk'} = \prod_{n \in \mathbb{N}^*} I_n(0)^{a_n} z_n^{k_n} \bar{z}_n^{k'_n},
\]
and \( B_{akk'} \) are the coefficients.

Definition 1.1. Fixed any \( a, k, k' \in \mathbb{N}^* \), denote \((n_i)_{i \geq 1}\) the decreasing rearrangement of
\[
\{ n : \text{where } n \text{ is repeated } 2a_n + k_n + k'_n \text{ times} \},
\]
i.e.
\[
(n_i)_{i \geq 1} = (n_1, n_2, \ldots, n_l)
\]
(1.3)
with \( l = \sum_{n \in \mathbb{N}^*} (2a_n + k_n + k'_n) \) and \( n_1 \geq n_2 \geq n_3 \geq \ldots \), and \((n'_i)_{i \geq 1}\) the decreasing rearrangement of
\[
\{ n : \text{where } n \text{ is repeated } |k_n - k'_n| \text{ times} \},
\]

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the following inequalities hold

\[(n_j^*)_{j \geq 1} = (n_1^*, n_2^*, \ldots, n_j^*)\]

with \(j = \sum_{n \in \mathbb{N}} |k_n - k_n'|\) and \(n_1^* \geq n_2^* \geq n_3^* \geq \ldots\).

**Remark 1.2.** Noting that \(2a_n + k_n + k'_n \geq |k_n - k'_n|\), then for any \(1 \leq i \leq \sum_{n \in \mathbb{N}} |k_n - k'_n|\) one has

\[n_i \geq n_i^*,\]

where \((n_i)_{i \geq 1}\) and \((n_i^*)_{i \geq 1}\) are two decreasing rearrangements defined in Definition 1.1

For \(n \in \mathbb{N}^*\), take the \(\omega_n\) to be random in \([0, 1/n]\) and denote \(\|x\| = \text{dist}(x, \mathbb{Z})\).

**Definition 1.3.** (Nonresonant Conditions) For any \(k, k' \in \mathbb{Z}^{N^*}\) with \(k \neq k'\), we say \(\omega = (\omega_n)_{n \in \mathbb{N}^*}\) is nonresonant in the following sense: there exists a real number \(0 < \gamma < 1\) such that the following inequalities hold

\[\left\| \sum_{n \in \mathbb{N}^*} (k_n - k'_n) \omega_n \right\| \geq \gamma \prod_{n \in \mathbb{N}^*} \frac{1}{1 + (k_n - k'_n)^2 n^5},\]

and if \(n_3^* < n_2^*\) and \(\sum_{n \in \mathbb{N}^*} |k_n - k'_n| \geq 3\), then

\[\left\| \sum_{n \in \mathbb{N}^*} (k_n - k'_n) \omega_n \right\| \geq \frac{\gamma^3}{16} \prod_{n \in \mathbb{N}^*} \left( \frac{1}{1 + (k_n - k'_n)^2 n^5} \right)^4,\]

whenever \(0 \neq k - k' = (k_n - k'_n)_{n \in \mathbb{N}^*}\) is a finitely supported sequence of integers.

Given \(\theta \in (0, 1)\) and \(r > 0\), define Banach space \(G^{r, \theta}\) of all complex sequences \(w = (w_1, w_2, \ldots)\) with the finite norm

\[\|w\|_{r, \theta} = \sup_{n \in \mathbb{N}^*} |w_n| e^{\gamma n^\theta}.\]

Now our main result is as follows:

**Theorem 1.4.** Given \(r > 0\), \(0 < \theta < 1\) and a frequency vector \(\omega = (\omega_n)_{n \in \mathbb{N}^*}\) satisfying the nonresonant conditions (1.6) and (1.7), then for sufficiently small \(\epsilon > 0\) there exist \(V = (V_n)_{n \in \mathbb{N}^*}\) with \(V_n \in [0, 1]\), such that (1.1) has a full dimensional invariant torus \(E\) with amplitude in \(G^{r, \theta}\) satisfying:

1. the amplitude \(I = (I_n)_{n \in \mathbb{N}^*}\) of \(E\) restricted as

\[\frac{1}{4} e^{2} e^{-2n^\theta} \leq |I_n| \leq 4e^{2} e^{-2n^\theta};\]

2. the frequency on \(E\) prescribed to be \((n + \omega_n)_{n \in \mathbb{N}^*}\);

3. the invariant tori \(E\) linearly stable.

The existence and linear stability of invariant tori for Hamiltonian PDEs have drawn a lot of concerns during the last decades. There are many related works for 1-dimensional PDEs. See [1, 3, 12, 14, 15, 17, 21, 23, 24, 26, 27] for example. For high dimensional PDEs, Bourgain [7, 8] developed a new method initiated by Craig-Wayne [12] to prove the existence of KAM tori for \(d\)-dimensional nonlinear Schrödinger equations (NLS) and \(d\)-dimensional NLW with \(d \geq 1\), based on the Newton iteration, Fröhlich-Spencer techniques, Harmonic analysis and semi-algebraic set theory. This is so-called C-W-B method. Later, Eliasson-Kuksin [13] proved a classical KAM theorem which can be applied to \(d\)-dimensional NLS. It is obtained
the existence of KAM tori as well as the linear stability of such tori. Also see \[4, 5, 10, 26\] for the related problem.

In the above works, the obtained KAM tori are of low dimension which are the support of the quasi-periodic solutions. It must be noted that the constructed quasi-periodic solutions are not typical in the sense that the low dimensional tori have measure zero for any reasonable measure on the infinite dimensional phase space. It is natural at this point to find the full dimensional tori which are the support of the almost periodic solutions. The first result on the existence of almost periodic solutions for 1-dimensional NLW was given by Bourgain in \[6\] using C-W-B method. Later, Pöschel \[25\] (also see \[16\] by Geng-Xu) constructed the almost periodic solutions for 1-dimensional NLS by the classical KAM method. These almost periodic solutions were obtained by successive small perturbations of quasi-periodic solutions. To avoid the number of the small divisors increasing fast, the action \(I = (I_n)\) must satisfy some very strong compactness properties. In fact, the following super-exponential decay for the action \(I\) is given
\[
I_n \sim C e^{-|n|}, \quad C > 1,
\]
as \(n \to \infty\). It means that these solutions are with very high regularity and looks like the quasi-periodic ones. Hence, Kuksin raised the following open problem (see Problem 7.1 in \[21\]):

*Can the full dimensional KAM tori be expected with a suitable decay, for example,
\[
I_n \sim |n|^{-C}
\]
with some \(C > 0\) as \(|n| \to +\infty\)?*

The first try to obtain the existence of full dimensional tori with slower decay was given by Bourgain \[9\], who proved that 1-dimensional NLS has a full dimensional KAM torus of prescribed frequencies with the actions of the tori obeying the estimates
\[
\frac{1}{2} e^{-r|n|^{1/2}} \leq I_n \leq 2 e^{-r|n|^{1/2}}, \quad r > 0.
\]
Recently, Cong-Liu-Shi-Yuan \[11\] generalized Bourgain’s result from \(\theta = 1/2\) to \(0 < \theta < 1\), i.e. the actions of the tori satisfying
\[
\frac{1}{2} e^{-r|n|^{\theta}} \leq I_n \leq 2 e^{-r|n|^{\theta}}, \quad \theta \in (0, 1) \text{ and } r > 0.
\]
Moreover the authors proved the obtained tori are stable in a sub-exponential long time.

Different from the ideas in \[6\] and \[25\], Bourgain treated all Fourier modes at once under some suitable Diophantine conditions. See the nonresonant conditions \(1.6\) for the details, which is similar as the one given in \[8\]. It is well known that the core of KAM theory is how to deal with small divisor. Note that the conditions \(1.6\) is totally different from the nonresonant conditions used to construct the low dimensional tori, since the factors \(n^5\) appears in the denominator, which causes a much worse small denominator problem. Two key observations are given by Bourgain: one is the inequality \(2.2\) for \(\theta = 1/2\); the other is as follows: let \(n_i\) be a finite set of modes satisfying
\[
|n_1| \geq |n_2| \geq |n_3| \geq \cdots
\]
and
\[
(1.8) \quad n_1 - n_2 + n_3 - n_4 + \cdots = 0.
\]
Note an important fact that in the case of a ‘near’ resonance, there is also a relation
\[
(1.9) \quad n_1^2 - n_2^2 + n_3^2 - n_4^2 + \cdots = o(1).
\]
Unless $n_1 = n_2$, from (1.8) and (1.9) one has

\[ |n_1| + |n_2| \leq C (|n_3| + |n_4| + \cdots), \]

where $C$ is a positive constant. In another word, the first two biggest indices $n_1$ and $n_2$ can be controlled by other indices, which is essential to overcome the small divisor, i.e. giving some good estimate of the solution of homological equation (see Lemma 2.5 for the details).

As everyone knows that NLS and NLW are two typical Hamiltonian PDEs which can be considered as touchstones of KAM theory for infinite dimensional Hamiltonian system (see [18] and [24]). Some properties of these two equations are similar, but the others are not. A main difference is as follows: for NLS the growth of the frequencies are quadric (also called separation property), while the growth of the frequencies is only linear for NLW. The separation property of the frequencies is essential to control the number of the resonant sets. Eliasson-Kuksin [13] proved a classical KAM theorem which can be applied to $d$-dimensional NLS but not for $d$-dimensional NLW.

In this paper, we would like to study the existence of full dimensional tori for NLW (1.1) with subexponential decay. Our approach and its results are parallel to an investigation of 1-dimensional NLS by Bourgain in [9]. Hence some parts of the respective expositions are quite similar. But we decided to repeat them anyway so that the reader need not refer to [9] for the essentials. One main problem is also there is no separation property for the frequencies of NLW. That is to say the conditions (1.9) fail, which causes that the main estimates (1.10) do not hold all the time. To overcome this difficult we will introduce some new nonresonant conditions firstly. Precisely we assume that the frequency $\omega$ satisfies a stronger nonresonant conditions (see (1.6) and (1.7) in Definition 1.3 below), which is helpful to control the solution of homological equation (see Lemma 2.5 for the details). Of course, we have to show such nonresonant conditions hold for most of $\omega$ in the sense of some measure, which is proven in Lemma 4.1. Another problem is that we have to show it is possible to choose some parameters $V = (V_n)$ such that the frequency $\omega$ is fixed during the KAM iterations. Different from the case for NLS, the frequency $\omega$ here belongs to $\ell^2$ instead of $\ell^\infty$. Therefore, the frequency shift should be calculated carefully to guarantee the inverse function theorem works (see (2.76) for the details). To this end, we introduce the modified norm for the Hamiltonian compared to the one defined in [9], which is based on the regularity of the nonlinear terms for NLW (see Definition 2.2 for the details). Also we will give some elementary estimates about this norm. After that, we obtain the existence and linear stability of full dimensional tori with subexponential decay for NLW by a KAM iterative process.

Finally, we also mention a recent work by L. Biasco, J. E. Massetti and M. Procesi [22]. The authors proved the existence of linear stability of almost periodic solution for 1-dimensional NLS with external parameters with a more geometric point of view by constructing a rather abstract counter-term theorem for infinite dimensional Hamiltonian system. Another interesting byproduct is that a construction of elliptic tori independent of their dimension.

2. KAM Theorem

2.1. Some notations and the norm of the Hamiltonian.

\[ \text{Lemma 2.1. Consider the decreasing rearrangement } (n_i)_{i \geq 1} \text{ which is defined by (1.3) in Definition 1.1 and assume that there are } (\mu_i)_{i \geq 1} \text{ with } \mu_i \in \{1, -1\} \text{ such that} \]

\[ \sum_{n \in \mathbb{N}^*} \mu_i n_i = 0. \]
Then for any \( 0 < \theta < 1 \), one has

\[
\sum_{n \in \mathbb{N}^*} (2\alpha_n + k_n + k_n')n^\theta - 2n_1^\theta \geq (2 - 2\theta) \sum_{i \geq 3} n_i^\theta.
\]

**Proof.** The proof of (2.2) is the same as Lemma 2.1 in [11], which generalizes the result given in Lemma 1.1 in [9]. \( \square \)

**Definition 2.2.** For any given \( \rho > 0 \) and \( 0 < \theta < 1 \), define the norm of the Hamiltonian \( R \) (see (1.2)) by

\[
\|R\|_\rho = \sup_{a,k,k' \in \mathbb{N}^*} \left( \prod_{n \in \mathbb{N}^*} n^{\frac{1}{2}(2\alpha_n + k_n + k_n')} \right) |B_{akk'}| e^{\rho \left( \sum_{n \in \mathbb{N}^*} (2\alpha_n + k_n + k_n')n^\theta - 2n_1^\theta \right)}.
\]

(2.3)

For any \( k \in \mathbb{N}^* \), define

\[
\text{supp } k = \{ n : k_n \neq 0 \}.
\]

Rewrite \( R \) as

\[
R = R_0 + R_1 + R_2
\]

where

\[
R_0 = \sum_{\text{supp } k \cap \text{supp } k' = \emptyset} B_{akk'} \mathcal{M}_{akk'},
\]

(2.6)

\[
R_1 = \sum_{m \in \mathbb{N}^*} J_m \left( \sum_{\text{supp } k \cap \text{supp } k' = \emptyset} B_{akk'}^{(m)} \mathcal{M}_{akk'} \right),
\]

(2.7)

\[
R_2 = \sum_{m_1, m_2 \in \mathbb{N}^*} J_{m_1} J_{m_2} \left( \sum_{\text{no assumption}} B_{akk'}^{(m_1, m_2)} \mathcal{M}_{akk'} \right).
\]

(2.8)

Given \( r > 0 \), let

\[
D = \left\{ z = (z_n)_{n \in \mathbb{N}^*} : \frac{1}{2} e^{-rn^\theta} \leq |z_n| \leq e^{-rn^\theta} \right\},
\]

and

\[
\Pi = \left\{ V = (V_n)_{n \in \mathbb{N}^*} : V_n \in [0,1] \right\}.
\]

Then we have the following result:

**Theorem 2.3.** For \( 0 < \theta < 1 \) and \( r > \frac{100\rho}{2-2\theta} > 0 \), suppose the Hamiltonian

\[
H(z, \bar{z}) = N(z, \bar{z}) + \epsilon R(z, \bar{z})
\]

is real analytic on the domain \( D \times \Pi \), where

\[
N(z, \bar{z}) = \sum_{n \in \mathbb{N}^*} \lambda_n(V) |z_n|^2
\]

is a normal form with

\[
\lambda_n(V) = \sqrt{n^2 + V_n},
\]

(2.9)
and $R(z, \bar{z})$ satisfies

$$||R||_\rho \leq 1.$$ 

Then given any $\omega = (\omega_n)_{n \in \mathbb{N}^*}$ satisfying the nonresonant conditions (1.6) and (1.7) and for sufficiently small $\epsilon$ depending on $r, \rho, \theta$ and $\gamma$, there exist $V_\ast \in \Pi$ and a real analytic symplectic coordinate transformation $\Phi : D_\ast \times \{V_\ast\} \to D$, where

$$D_\ast = \left\{ z = (z_n)_{n \in \mathbb{N}^*} : \frac{2}{3} e^{-rn_\theta} \leq |z_n| \leq \frac{5}{6} e^{-rn_\theta} \right\}$$

satisfying

$$\sup_{z \in D_\ast} ||(\Phi - id)(z)||_{r, \theta} \leq \epsilon^{0.4}$$

such that for $H_\ast = H \circ \Phi = N_\ast + R_{2, \ast}$, where

$$N_\ast = \sum_{n \in \mathbb{N}^*} (n + \omega_n)|z_n|^2$$

and $R_{2, \ast}$ has the form of (2.8) and satisfies

$$||R_{2, \ast}||_{10, \rho} \leq \epsilon^{0.4}.$$

2.2. Derivation of homological equations. The proof of Theorem 2.3 employs the rapidly converging iteration scheme of Newton type to deal with small divisor problems introduced by Kolmogorov, involving the infinite sequence of coordinate transformations. At the $s$-th step of the scheme, a Hamiltonian $H_s = N_s + R_s$ is considered, as a small perturbation of some normal form $N_s$ with the form of

$$N_s = \sum_{n \in \mathbb{N}^*} \lambda_{n,s}(V)|z_n|^2,$$

where

$$\lambda_{n,s}(V) = \sqrt{n^2 + \tilde{V}_{n,s}(V)}.$$

A transformation $\Phi_s$ is set up so that

$$H_s \circ \Phi_s = N_{s+1} + R_{s+1}$$

with another normal form $N_{s+1}$ and a much smaller perturbation $R_{s+1}$. We drop the index $s$ of $H_s, N_s, R_s, \Phi_s$ and shorten the index $s + 1$ as +.

We desire to eliminate the terms $R_0, R_1$ in (2.5) by the coordinate transformation $\Phi$, which is obtained as the time-1 map $X^t_F|_{t=1}$ of a Hamiltonian vector field $X_F$ with $F = F_0 + F_1$. Let $F_0$ (resp. $F_1$) has the form of $R_0$ (resp. $R_1$), that is

$$F_0 = \sum_{a, k, k' \in \mathbb{N}^*} \mathcal{F}_{akk'} \mathcal{M}_{akk'},$$

and the homological equations become

$$(2.12) \quad \{N, F\} + R_0 + R_1 = [R_0] + [R_1],$$

and

$$F_1 = \sum_{m \in \mathbb{N}^*} J_m \left( \sum_{a, k, k' \in \mathbb{N}^*} \mathcal{F}_{akk'} \mathcal{M}_{akk'} \right),$$

where $\mathcal{F}$ and $\mathcal{M}$ are given by (2.2) and (2.3), respectively.
where

\[(2.13) \quad [R_0] = \sum_{a \in \mathbb{N}^{n^*}} B_{a00} M_{a00}, \]

and

\[(2.14) \quad [R_1] = \sum_{m \in \mathbb{N}^*} J_m \sum_{a \in \mathbb{N}^{n^*}} B_{a00}(m) M_{a00}. \]

The solutions of the homological equations (2.12) are given by

\[(2.15) \quad F_{akk'} = \frac{B_{akk'}}{\sum_{n \in \mathbb{N}^*}(k_n - k'_n)\lambda_n}, \]

and

\[(2.16) \quad F_{akk'}^{(m)} = \frac{B_{akk'}^{(m)}}{\sum_{n \in \mathbb{N}^*}(k_n - k'_n)\lambda_n}. \]

The new Hamiltonian \(H_+\) has the form

\[H_+ = H \circ \Phi = N + \{N, F\} + R_0 + R_1 + \int_0^1 \left\{ (1 - t)\{N, F\} + R_0 + R_1, F \right\} \circ X_F^t \, dt + R_2 \circ X_F^t \]

\[(2.17) \quad = N_+ + R_+, \]

where

\[(2.18) \quad N_+ = N + [R_0] + [R_1], \]

and

\[(2.19) \quad R_+ = \int_0^1 \left\{ (1 - t)\{N, F\} + R_0 + R_1, F \right\} \circ X_F^t \, dt + R_2 \circ X_F^t. \]

2.3. The solvability of the homological equations \((2.12)\). In this subsection, we will estimate the solutions of the homological equations \((2.12)\). To this end, we define the new norm for the Hamiltonian \(R\) as follows:

\[(2.20) \quad ||R||^+_\rho = \max \{ ||R_0||^+_\rho, ||R_1||^+_\rho, ||R_2||^+_\rho \}, \]

where

\[(2.21) \quad ||R_0||^+_\rho = \sup_{a, k, k' \in \mathbb{N}^{n^*}} \frac{\left( \prod_{n \in \mathbb{N}^*} n^{\frac{1}{2}(2a_n + k_n + k'_n)} \right) |B_{akk'}|}{e^{\rho(\sum_{n \in \mathbb{N}^*}(2a_n + k_n + k'_n)n^* - 2n_1^*)}}, \]

\[(2.22) \quad ||R_1||^+_\rho = \sup_{a, k, k' \in \mathbb{N}^{n^*}} \frac{\left( \prod_{m \in \mathbb{N}^*} n^{\frac{1}{2}(2a_n + k_n + k'_n)} \right) |mB_{akk'}^{(m)}|}{e^{\rho(\sum_{n \in \mathbb{N}^*}(2a_n + k_n + k'_n)n^* + 2m^* - 2n_1^*)}}, \]

\[(2.23) \quad ||R_2||^+_\rho = \sup_{a, k, k' \in \mathbb{N}^{n^*}} \frac{\left( \prod_{m \in \mathbb{N}^*} n^{\frac{1}{2}(2a_n + k_n + k'_n)} \right) |m_1m_2B_{akk'}^{(m_1,m_2)}|}{e^{\rho(\sum_{n \in \mathbb{N}^*}(2a_n + k_n + k'_n)n^* + 2m_1^* + 2m_2^* - 2n_1^*)}}. \]

Moreover, one has the following estimates:
Lemma 2.4. Given any \( \delta, \rho > 0 \), one has

\[
\|R\|_{\rho+\delta}^+ \leq \left( \frac{1}{\delta} \right)^{C(\theta)\delta^{-\frac{1}{\delta}}} \|R\|_{\rho}^+
\]

and

\[
\|R\|_{\rho+\delta}^+ \leq \frac{C(\theta)}{\delta^2} \|R\|_{\rho}^+,
\]

where \( C(\theta) \) is a positive constant depending on \( \theta \) only.

Proof. Firstly, we will prove the inequality (2.24). Write \( M_{akk'} = \prod_{n \in \mathbb{N}^*} I_n(0)^{a_n} I_n^{b_n} I_n^{l_n} I_n^{l_n'} \),

where \( b_n = k_n \wedge k_n' \), \( l_n = k_n - b_n \), \( l_n' = k_n' - b_n \)

and \( l_n l_n' = 0 \) for all \( n \).

Express the term \( \prod_{n \in \mathbb{N}^*} I_n^{b_n} = \prod_{n \in \mathbb{N}^*} (I_n(0) + J_n)^{b_n} \)

by the monomials of the form

\[
\prod_{n \in \mathbb{N}^*} I_n(0)^{b_n},
\]

\[
\sum_{m_n, b_m \geq 1} (I_n(0)^{b_m-1} J_m) \left( \prod_{n \neq m} I_n(0)^{b_n} \right),
\]

\[
\sum_{m_n, b_m \geq 2} (I_n(0)^{b_m}) \left( I_n(0)^r J_m^{b_m-r-2} \right) \left( \prod_{n > m} I_n^{b_n} \right),
\]

and

\[
\sum_{m_1 < m_2, b_{m_1}, b_{m_2} \geq 1} \left( \prod_{n < m_1} I_n(0)^{b_n} \right) \left( I_{m_1}(0)^{b_{m_1}-1} J_{m_1} \right)
\]

\[
\times \left( \prod_{m_1 < n < m_2} I_n(0)^{b_n} \right) \left( I_{m_2}(0)^r J_{m_2}^{b_{m_2}-r-1} \right) \left( \prod_{n > m_2} I_n^{b_n} \right).
\]

Now we will estimate the bounds for the coefficients respectively.

Consider the term \( M_{akk'} = \prod_{n \in \mathbb{N}} I_n(0)^{a_n} z_n^{k_n} z_n^{k_n'} \) with fixed \( a, k, k' \) satisfying \( k_n k_n' = 0 \) for all \( n \). It is easy to see that \( M_{akk'} \) comes from some parts of the terms \( M_{akk'} \) with no assumption for \( \kappa \) and \( \kappa' \). For any given \( n \) one has

\[
I_n(0)^{a_n} z_n^{k_n} z_n^{k_n'} = \sum_{\beta_n = \kappa_n \wedge \kappa_n'} I_n(0)^{\alpha_n + \beta_n} z_n^{\kappa_n - \beta_n} z_n^{\kappa_n' - \beta_n},
\]

where

\[
\alpha_n + \beta_n = a_n,
\]

and

\[
\kappa_n - \beta_n = k_n, \quad \kappa_n' - \beta_n = k_n'.
\]
Hence one has

\[ 2a_n + k_n + k'_n = 2(\alpha_n + \beta_n) + (\kappa_n - \beta_n) + (\kappa'_n - \beta_n) = 2\alpha_n + \kappa_n + \kappa'_n. \]

Moreover, if \( 0 \leq \alpha_n \leq a_n \) is chosen, so \( \beta_n, k_n, k'_n \) are determined. On the other hand,

\[
\left| \left( \prod_{n \in \mathbb{N}'^*} \frac{1}{n^{\frac{1}{2}(2\alpha_n + \kappa_n + \kappa'_n)}} \right) B_{\alpha\kappa\kappa'} \right| 
\leq \| R \| e^{\theta \left( \sum_{n \in \mathbb{N}'^*} (2\alpha_n + \kappa_n + \kappa'_n)n^{\theta} - 2\alpha_n \right)} 
\leq \| R \| e^{\theta \left( \sum_{n \in \mathbb{N}'^*} (2\alpha_n + k_n + k'_n)n^{\theta} - 2\alpha_n \right)},
\]

where the last equality is based on (2.28) and \( \nu_1 = \max \text{supp} \alpha + \kappa + \kappa'. \)

Hence,

\[ (2.29) \quad \left( \prod_{n \in \mathbb{N}'^*} \frac{1}{n^{\frac{1}{2}(2\alpha_n + k_n + k'_n)}} \right) B_{\alpha k k'} \leq \| R \| e^{\theta \left( \sum_{n \in \mathbb{N}'^*} (2\alpha_n + k_n + k'_n)n^{\theta} - 2\alpha_n \right)} \left( \prod_{n \in \mathbb{N}'^*} (1 + a_n) \right). \]

In view of (2.21) and (2.29), we have

\[
\| R_0 \| \leq \| R \| e^{-\theta \left( \sum_{n \in \mathbb{N}'^*} (2\alpha_n + k_n + k'_n)n^{\theta} - 2\alpha_n \right)} \left( \prod_{n \in \mathbb{N}'^*} (1 + a_n) \right)
\leq \| R \| \left( \frac{1}{\theta} \right)^{C(\theta)} \delta^{\frac{\theta}{2}},
\]

where the last inequality is based on (7.37) in [11] and \( C(\theta) \) is a positive constant depending only on \( \theta \).

Next consider the term \( J_m \mathcal{M}_{\alpha k k'} = J_m \prod_{n \in \mathbb{N}'^*} I_n(0)^{a_n} z_n^{k_n} z'_n^{k'_n} \) with fixed \( a, k, k' \) satisfying \( k_n, k'_n = 0 \) for all \( n \). The term \( J_m \mathcal{M}_{\alpha k k'} \) also comes from some parts of the terms \( \mathcal{M}_{\alpha n \kappa \kappa'} \) with no assumption for \( \kappa \) and \( \kappa' \).

For any given \( n \neq m \) one has

\[
I_n(0)^{a_n} z_n^{k_n} z'_n^{k'_n} = \sum_{\beta_n = \kappa_n \wedge \kappa'_n} I_n(0)^{\alpha_n + \beta_n - \kappa_n - \beta_n} z_n^{k_n - \beta_n} z'_n^{k'_n - \beta_n}.
\]

Following (2.26), (2.27) and (2.28), one has

\[
\alpha_n + \beta_n = a_n, \quad \kappa_n - \beta_n = k_n, \quad \kappa'_n - \beta_n = k'_n,
\]

and

\[
2\alpha_n + \kappa_n + \kappa'_n = 2a_n + k_n + k'_n.
\]

Moreover, if \( 0 \leq \alpha_n \leq a_n \) is chosen, so \( \beta_n, k_n, k'_n \) are determined.

For any given \( n = m \) one has

\[
J_m I_m(0)^{a_m} q_m^{k_m} q'_m^{k'_m} = \sum_{\beta_m = \kappa_m \wedge \kappa'_m} \beta_m J_m I_m(0)^{\alpha_m + \beta_m - 1} q_m^{k_m - \beta_m} q'_m^{k'_m - \beta_m}.
\]

Hence,

\[ (2.31) \quad \alpha_m + \beta_m - 1 = a_m, \quad \kappa_m - \beta_m = k_m, \quad \kappa'_m - \beta_m = k'_m. \]
and

\[(2.32) \quad 2\alpha_m + \kappa_m + \kappa'_m = 2a_m + k_m + k'_m + 2.\]

Moreover, if \(0 \leq \alpha_m \leq a_m\) is chosen (noting that \(\alpha_m = a_m + 1 \leftrightarrow \beta_m = 0\)), so \(\beta_m, k_m, k'_m\) are determined. On the other hand,

\[
\left| \left( \prod_{n \in \mathbb{N}^*} n^{(2\alpha_n + \kappa_n + \kappa'_n)} \right) B_{\alpha\kappa\kappa'_m} \right|
\leq \left| R \right|_\rho \epsilon^\theta \left( \sum_{n \in \mathbb{N}^*} (2\alpha_n + \kappa_n + \kappa'_n)n^\theta - 2\nu_1^\theta \right)
\leq \left| R \right|_\rho \epsilon^\theta \left( \sum_{n \in \mathbb{N}^*} (2\alpha_n + \kappa_n + \kappa'_n)n^\theta + 2m^\theta - 2n_1^\theta \right),
\]

where the last equality is based on (2.32) and \(n_1 \leq \nu_1\). Then

\[
\left| \left( \prod_{n \in \mathbb{N}^*} n^{(2\alpha_n + k_n + k'_n)} \right) mB^{(m)}_{\alpha\kappa\kappa'_m} \right|
\leq \left| R \right|_\rho \epsilon^\theta \left( \sum_{n \in \mathbb{N}^*} (2\alpha_n + k_n + k'_n)n^\theta + 2m^\theta - 2n_1^\theta \right)
\leq \left| R \right|_\rho \epsilon^\theta \left( \sum_{n \in \mathbb{N}^*} (2\alpha_n + k_n + k'_n)n^\theta + 2m^\theta - 2n_1^\theta \right) \left( \prod_{n \in \mathbb{N}^*} (1 + a_n) \right) \beta_m
\leq \left| R \right|_\rho \epsilon^\theta \left( \sum_{n \in \mathbb{N}^*} (2\alpha_n + k_n + k'_n)n^\theta + 2m^\theta - 2n_1^\theta \right) \left( \prod_{n \in \mathbb{N}^*} (1 + a_n) \right) (1 + a_m)
\]

(based on (2.31))

\[(2.33) \quad \leq \left| R \right|_\rho \left( \frac{1}{\delta} \right)^{C(\theta)\delta^{-1}}.\]

where the last inequality is based on (7.37) in [11] and \(C(\theta)\) is a positive constant depending only on \(\theta\). In view of (2.22), one has

\[(2.34) \quad \left| R_1 \right|_\rho^+ \leq \left| R \right|_\rho \left( \frac{1}{\delta} \right) \epsilon^\theta \left( \sum_{n \in \mathbb{N}^*} (a_n + b_n) \right)^{C(\theta)\delta^{-1}}.\]

Similarly, one has

\[(2.35) \quad \left| R_2 \right|_\rho^+ \leq \left( \frac{1}{\delta} \right) \epsilon^\theta \left( \sum_{n \in \mathbb{N}^*} (a_n + b_n) \right)^{C(\theta)\delta^{-1}} \left| R \right|_\rho.\]

In view of (2.20), (2.30), (2.34) and (2.35), we finish the proof of (2.24). On the other hand, the coefficient of \(M_{abll'}\) increases by at most a factor

\[
\left( \sum_{n \in \mathbb{N}^*} (a_n + b_n) \right)^2,
\]
then one has
\[
||R||_{\rho+\delta} \leq ||R||_{\rho}^{+} \left( \sum_{n \in \mathbb{N}^{*}} (a_{n} + b_{n}) \right)^{2} e^{-\delta \left( \sum_{n \in \mathbb{N}^{*}} (2a_{n} + k_{n}^{*} + k_{n}') n^{6} - 2n_{1}^{6} \right)}
\]
\[
\leq ||R||_{\rho}^{+} \left( 2 \sum_{i \geq 3} n_{i}^{6} \right)^{2} e^{-\delta (2-2^{\theta}) \sum_{i \geq 3} n_{i}^{6}}
\]
\[
\leq \frac{4}{(2 - 2^{\theta})^{2} \delta^{2}} ||R||_{\rho}^{+},
\]
where the last inequality is based on Lemma 7.5 in [11] with \( p = 2 \), and we finish the proof of (2.25).

Lemma 2.5. Assume \( \omega = (\omega_{n})_{n \in \mathbb{N}^{*}} \), with \( \omega_{n} = \lambda_{n} - n \) satisfies the nonresonant conditions (1.7) and (1.8). Then for any \( \rho > 0, \theta < 1 \) (depending only on \( \theta \)), the solutions of the homological equations (2.13), which are given by (2.15) and (2.16), satisfy
\[
||F||_{\rho+\delta}^{+} \leq \frac{1}{\gamma^{3}} \cdot e^{C(\theta)\delta^{-\frac{5}{2}}} ||R||_{\rho}^{+},
\]
where \( C(\theta) \) is a positive constant depending on \( \theta \) only.

Proof. We distinguish two cases:

Case 1. \( n_{1}^{*} < n_{2}^{*} \).

Since
\[
\sum_{n \in \mathbb{N}^{*}} (k_{n} - k_{n}') n \in \mathbb{Z},
\]
the nonresonant conditions (1.7) implies
\[
(2.37) \quad \left| \sum_{n \in \mathbb{N}^{*}} (k_{n} - k_{n}') \lambda_{n} \right| \geq \frac{\gamma^{3}}{16} \prod_{n \in \mathbb{N}^{*}} \left( \frac{1}{1 + (k_{n} - k_{n}')^{2} n^{6}} \right)^{4}.
\]

Hence,
\[
(\prod_{n \in \mathbb{N}^{*}} n^{\frac{1}{2} (2a_{n} + k_{n} + k_{n}')}} |F_{akk'}| e^{-(\rho+\delta) \left( \sum_{n \in \mathbb{N}^{*}} (2a_{n} + k_{n} + k_{n}') n^{6} - 2n_{1}^{6} \right)}
\]
\[
= \left( \prod_{n \in \mathbb{N}^{*}} n^{\frac{1}{2} (2a_{n} + k_{n} + k_{n}')}} \right) |B_{akk'}| \left| \sum_{n \in \mathbb{N}^{*}} (k_{n} - k_{n}') \lambda_{n} \right| \times e^{-(\rho+\delta) \left( \sum_{n \in \mathbb{N}^{*}} (2a_{n} + k_{n} + k_{n}') n^{6} - 2n_{1}^{6} \right)}
\]
(in view of (2.15))
\[
\leq \frac{16}{\gamma^{3}} ||R_{0}||_{\rho}^{+} \left( \prod_{n \in \mathbb{N}^{*}} \left( 1 + (k_{n} - k_{n}')^{2} n^{6} \right)^{4} \right) \times e^{-\delta \left( \sum_{n \in \mathbb{N}^{*}} (2a_{n} + k_{n} + k_{n}') n^{6} - 2n_{1}^{6} \right)}
\]
(in view of (2.21) and (2.37))
\[
\leq \frac{1}{\gamma^{3}} \cdot e^{C_{1}(\theta)\delta^{-\frac{5}{2}}} ||R_{0}||_{\rho}^{+},
\]
where the last inequality is based on Lemma 4.2 and $C_{1}(\theta)$ is a positive constant depending on $\theta$ only, which finishes the proof of

(2.38) \[ ||F_{0}||_{\rho+\delta}^{+} \leq \frac{1}{\gamma^{3}} \cdot e^{C(\theta)\delta^{-\frac{2}{3}}} ||R_{0}||_{\rho}^{+} \]

in Case. 1.

Case. 2. $n_3^* = n_2^*$.

If

\[ n_1^* \geq 10 \sum_{i \geq 2} n_i^* , \]

then one has

\[ \left| \sum_{n \in \mathbb{N}^{*}} (n_1 - n_1') \lambda_n \right| \geq 1 , \]

where there is no small divisor. Hence we always assume that

\[ n_1^* < 10 \sum_{i \geq 2} n_i^* . \]

In view of $n_3^* = n_2^*$, then one has

\[ n_1^* \leq 11 \sum_{i \geq 3} n_i^* \]

and

\[ (n_1^*)^{\theta} \leq 11^{\theta} \sum_{i \geq 3} (n_i^*)^{\theta} . \]

Moreover,

\[
\sum_{i \geq 1} (n_i^*)^{\theta} \leq (11^{\theta} + 2) \sum_{i \geq 3} (n_i^*)^{\theta} \]

(2.39) \[ \leq (11^{\theta} + 2) \sum_{i \geq 3} n_i^\theta \quad \text{(in view of Remark 1.2)} \]

(2.40) \[ \leq \frac{11^{\theta} + 2}{2 - 2^{\theta}} \left( \sum_{n \in \mathbb{N}^{*}} (2a_n + k_n + k_n') n^{\theta} - 2n_1^{\theta} \right) , \]

where the last inequality is based on (2.2).

Since

\[ \sum_{n \in \mathbb{N}^{*}} (n_1 - n_1') n \in \mathbb{Z} , \]

the nonresonant conditions (1.6) implies

(2.41) \[ \left| \sum_{n \in \mathbb{N}^{*}} (n_1 - n_1') \lambda_n \right| \geq \gamma \prod_{n \in \mathbb{N}^{*}} \frac{1}{1 + (n_1 - n_1')^{2} n^{\delta}} . \]

Following the proof of (2.38) one has

\[
\left( \prod_{n \in \mathbb{N}^{*}} n_1^{\delta}(2a_n + k_n + k_n') \right) |F_{akk'}| e^{-(\rho+\delta)(\sum_{n \in \mathbb{N}^{*}} (2a_n + k_n + k_n') n^{\theta} - 2n_1^{\theta})} \]

\[ \leq \frac{1}{\gamma} \cdot e^{C_{2}(\theta)\delta^{-\frac{2}{3}}} ||R_{0}||_{\rho}^{+} , \]
where $C_2(\theta)$ is a positive constant depending on $\theta$ only, which finishes the proof of

$$(2.42) \quad ||F_0||_{\rho+\delta}^+ \leq \frac{1}{\gamma^2} \cdot e^{C(\theta)\delta_1^+} ||R_0||_{\rho}^+$$

in Case 2.

Similarly, one can prove

$$(2.43) \quad ||F_1||_{\rho+\delta}^+ \leq \frac{1}{\gamma^2} \cdot e^{C(\theta)\delta_1^+} ||R_1||_{\rho}^+.$$

In view of (2.38), (2.42) and (2.43), we finish the proof of (2.36).

2.4. The new perturbation $R_+$ and the new normal form $N_+$. Recall the new term $R_+$ is given by (2.19) and write

$$(2.44) \quad R_+ = R_{0+} + R_{1+} + R_{2+}.$$ 

Now we will estimate $R_{i+}$ for $i = 0, 1, 2$ respectively. To this end, we give the following estimate:

**Lemma 2.6. (Poisson Bracket)** Let $\theta \in (0, 1), \rho > 0$ and $0 < \delta_1, \delta_2 \ll 1$ (depending on $\theta, \rho$). Then one has

$$(2.45) \quad ||\{H_1, H_2\}||_{\rho} \leq \frac{1}{\delta_2} \left( \frac{1}{\delta_1} \right)^{C(\theta)\delta_1^+} ||H_1||_{\rho-\delta_1} ||H_2||_{\rho-\delta_2},$$

where $C(\theta)$ is a positive constant depending on $\theta$ only.

**Proof.** Let

$$H_1 = \sum_{a, k, k' \in \mathbb{N}^*} b_{akk'} M_{akk'}$$

and

$$H_2 = \sum_{A, K, K' \in \mathbb{N}^*} B_{AKK'} M_{AKK'}.$$ 

It follows easily that

$$\{H_1, H_2\} = \sum_{a, k, k, A, K, K' \in \mathbb{N}^*} b_{akk'} B_{AKK'} \{M_{akk'}, M_{AKK'}\},$$

where

$$\{M_{akk'}, M_{AKK'}\} = -i \sum_{j \in \mathbb{N}^*} \left( \prod_{n \neq j} I_n(0)^{a_n + A_n} z_n^{k_n} \right) \times \left( (k_j' K_j' - k_j' K_j) I_j(0)^{a_j + A_j} z_j^{k_j} \right).$$

Then the coefficient of

$$M_{\alpha\kappa'} := \prod_{n \in \mathbb{N}^*} I_n(0)^{\alpha_n} z_n^{\kappa_n}$$

is given by

$$(2.46) \quad B_{\alpha\kappa'} = -i \sum_{j \in \mathbb{N}^*} \sum_{*} \sum_{**} (k_j' K_j' - k_j' K_j) b_{akk'} B_{\alphaKK'},$$

where

$$\sum_{*} = \sum_{a+\Lambda = \alpha},$$
and

\[
\sum_{** \atop n \neq j, k + n = \kappa_n, \kappa'_n + K_n = \kappa'_n, \text{when } n = j, k + n - 1 = \kappa_n, \kappa'_n - 1 = \kappa'_n}
\]

(2.47)

In view of (2.46) in Definition 2.2, one has

\[
\left| \prod_{n \in \mathbb{N}^*} n^\frac{1}{2}(2a_n + k + k') b_{akk'} \right| \\
\leq \left| H_1 \right|_{\rho - \delta_1} e^{(\rho - \delta_1)(\sum_{n \in \mathbb{N}^*} (2a_n + k + k') n^\alpha - 2n^\beta)} \\
= \left| H_1 \right|_{\rho - \delta_1} e^{\rho \left(\sum_{n \in \mathbb{N}^*} (2a_n + k + k') n^\alpha - 2n^\beta\right) - \delta_1 \left(\sum_{n \in \mathbb{N}^*} (2a_n + k + k') n^\alpha - 2n^\beta\right)} \\
\leq \left| H_1 \right|_{\rho - \delta_1} e^{\rho \left(\sum_{n \in \mathbb{N}^*} (2a_n + k + k') n^\alpha - 2n^\beta\right) - (2-2\delta_1) \delta_1 \sum_{i \geq 3} n_i^\beta},
\]

(2.48)

where the last inequality is based on (2.2) in Lemma 2.1.

Similarly,

\[
\prod_{n \in \mathbb{N}^*} n^\frac{1}{2}(2A_n + K_n + K'_n) B_{akk'} \leq \left| H_2 \right|_{\rho - \delta_2} e^{\rho \left(\sum_{n \in \mathbb{N}^*} (2A_n + K_n + K'_n) n^\alpha - 2N^\beta\right) - (2-2\delta_2) \delta_2 \sum_{i \geq 3} N_i^\beta}.
\]

(2.49)

Substituting (2.48) and (2.49) in (2.40) gives

\[
\left| \prod_{n \in \mathbb{N}^*} n^\frac{1}{2}(2a_n + k + k') b_{akk'} \right| \leq \left| H_1 \right|_{\rho - \delta_1} \left| H_2 \right|_{\rho - \delta_2} \sum_{j \in \mathbb{N}^*} \sum_{**} \sum \left| k_{j K_j} - k_{j K_j} \right| \\
\times e^{\rho \left(\sum_{n \in \mathbb{N}^*} (2a_n + k + k') n^\alpha - 2n^\beta + \sum_{n \in \mathbb{N}^*} (2A_n + K_n + K'_n) n^\alpha - 2N^\beta\right)} \\
\times e^{- (2-2\delta_1) \delta_1 \sum_{i \geq 3} n_i^\beta} e^{-(2-2\delta_2) \delta_2 \sum_{i \geq 3} N_i^\beta} \\
= \left| H_1 \right|_{\rho - \delta_1} \left| H_2 \right|_{\rho - \delta_2} \sum_{j \in \mathbb{N}^*} \sum_{**} \sum \left| k_{j K_j} - k_{j K_j} \right| \\
\times e^{\rho \left(\sum_{n \in \mathbb{N}^*} (2a_n + k + k') n^\alpha + 2j\nu_1 - 2n^\beta\right)} \\
\times e^{-(2-2\delta_1) \delta_1 \sum_{i \geq 3} n_i^\beta} e^{-(2-2\delta_2) \delta_2 \sum_{i \geq 3} N_i^\beta} \\
= \left| H_1 \right|_{\rho - \delta_1} \left| H_2 \right|_{\rho - \delta_2} \sum_{j \in \mathbb{N}^*} \sum_{**} \sum \left| k_{j K_j} - k_{j K_j} \right| e^{2\rho (\nu_1^\beta - n_i^\beta - N_i^\beta)} \\
\times e^{-(2-2\delta_1) \delta_1 \sum_{i \geq 3} n_i^\beta} e^{-(2-2\delta_2) \delta_2 \sum_{i \geq 3} N_i^\beta},
\]

(2.50)

where \( \nu_1 = \max\{n : \alpha_n + \kappa_n + \kappa'_n \neq 0\} \).
Following the proof of (4.8) in Lemma 4.1 in [11], one has

\[(2.51) \quad I \leq \frac{1}{\delta_2} \left( \frac{1}{\delta_1} \right)^{C(\theta)} \delta_1^+ \]

where

\[I = \sum_j \sum^*_j \sum^{**}_j |k_j K_j' - k_j' K_j| e^{2\rho (j^g + \nu_j^g - n_j^g - N_j^g)} \times e^{-(2-2\theta)\delta_1 \sum_{i \geq 3} n_i^g e^{-(2-2\theta)\delta_2 \sum_{i \geq 3} N_i^g}}.\]

Hence in view of (2.50) and (2.51), we finish the proof of (2.45).

Based on Lemma 2.6 and following the proof of (4.54)-(4.56) in [11], one has

\[(2.52) \quad || R_0^+ ||_{\rho + 3\delta}^+ \leq \frac{1}{\gamma^3} \cdot e^{3 \cdot \frac{2\rho}{3}} (|| R_0 ||_{\rho}^+ + || R_1 ||_{\rho}^+)^\left(|| R_0 ||_{\rho}^+ + || R_1 ||_{\rho}^+ \right)^2,\]

\[(2.53) \quad || R_1^+ ||_{\rho + 3\delta}^+ \leq \frac{1}{\gamma^3} \cdot e^{3 \cdot \frac{2\rho}{3}} (|| R_0 ||_{\rho}^+ + || R_1 ||_{\rho}^+)^2,\]

\[(2.54) \quad || R_2^+ ||_{\rho + 3\delta}^+ \leq || R_2 ||_{\rho}^+ + \frac{1}{\gamma^3} \cdot e^{3 \cdot \frac{2\rho}{3}} (|| R_0 ||_{\rho}^+ + || R_1 ||_{\rho}^+).\]

The new normal form \( N^+ \) is given in (2.18). Note that \([R_0]\) (in view of (2.13)) is a constant which does not affect the Hamiltonian vector field. Moreover, in view of (2.14), we denote by

\[\omega_{n^+} = \sqrt{n^2 + \tilde{V}_n + \sum_{a \in \mathbb{N}^{\sigma}} B^{(n)}_{a00} M_{a00}},\]

where the terms

\[\sum_{a \in \mathbb{N}^{\sigma}} B^{(n)}_{a00} M_{a00}\]

is the so-called frequency shift. The estimate of \(|\sum_{a \in \mathbb{N}^{\sigma}} B^{(n)}_{a00} M_{a00}|\) will be given in the next section (see (2.76) for the details).

Finally, we give the estimate of the Hamiltonian vector field.

**Lemma 2.7.** Given a Hamiltonian

\[H = \sum_{a,k,k' \in \mathbb{N}^{\sigma}} B_{akk'} M_{akk'},\]

then for any \( r > \left( \frac{1}{2-2\theta} + 3 \right) \rho \) and

\[\sup_{n \in \mathbb{N}^{\sigma}} |I_n(0)| e^{2r n^\theta} < 1,\]

one has

\[(2.55) \quad \sup_{|| z ||_{r,\theta} < 1} ||X_H||_{r,\theta} \leq C(r, \rho, \theta)||H||_{\rho},\]

where \(C(r, \rho, \theta)\) is a positive constant depending on \(r, \rho\) and \(\theta\) only.

**Proof.** Letting

\[\tilde{B}_{akk'} = \left( \prod_{n \in \mathbb{N}^{\sigma}} n^\frac{1}{2} \right) B_{akk'}\]
and noting that
\[ |\overline{B_{akk'}}| \geq |B_{akk'}|, \]
then following the proof of (5.21) in Lemma 5.2 in [11], we finish the proof of (2.55). \(\square\)

2.5. Iteration and Convergence. Now we give the precise set-up of iteration parameters. Let \(s \geq 1\) be the \(s\)-th KAM step.

\[\begin{align*}
r_0 &= \rho, \quad r \geq \frac{1000}{2^{-2}}, \\
\delta_s &= \frac{\rho}{2^s}, \\
\rho_{s+1} &= \rho_s + 3\delta_s, \\
\epsilon_s &= \lambda^{(2)^s}, \quad \text{which dominates the size of the perturbation,} \\
\lambda_s &= e^{-C(\theta)\left(\ln \frac{1}{\epsilon_s}\right)^{\frac{1}{2}}}, \\
\eta_0 &= 1.1 - \sup_{n \in \mathbb{N}^r} \omega_n, \eta_{s+1} = \frac{1}{20}\lambda_s\eta_s, \\
d_0 &= 0, \quad d_{s+1} = d_s + \frac{1}{s+1}d_{s+1}, \\
D_s &= \{(z_n)_{n \in \mathbb{N}^r} : \frac{1}{2} + d_s \leq |z_n|e^{\tau n} \leq 1 - d_s\}. \\
\end{align*}\]

Denote the complex cube of size \(\lambda > 0\):
\[C_\lambda(V^*) = \left\{(V_n)_{n \in \mathbb{N}^r} : |V_n - V_n^*| \leq \lambda \right\}.\]

**Lemma 2.8.** Suppose \(H_s = N_s + R_s\) is real analytic on \(D_s \times C_{\eta_s}(V_n^*)\), where
\[N_s = \sum_{n \in \mathbb{N}^r} \lambda_{n,s}(V)|z_n|^2\]
is a normal form with
\[\lambda_{n,s}(V) = \sqrt{n^2 + \overline{V}_{n,s}(V)}\]
satisfying
\[\sqrt{n^2 + \overline{V}_{n,s}(V^*)} = n + \omega_n,\]
\[\left|\frac{\partial \overline{V}_{n,s}}{\partial V} - I\right|_{l^\infty \rightarrow l^\infty} < d_s e_0^{\frac{1}{2}}.\]
and \(R_s = R_{0,s} + R_{1,s} + R_{2,s}\) satisfying
\[\begin{align*}
|R_{0,s}|_{\rho_s}^+ &\leq \epsilon_s, \\
|R_{1,s}|_{\rho_s} &\leq \epsilon_s^{0.6}, \\
|R_{2,s}|_{\rho_s} &\leq (1 + d_s)\epsilon_0.
\end{align*}\]
Assume that \(\omega = (\omega_n)_{n \in \mathbb{N}^r}\) satisfies the nonresonant conditions \((1.6)\) and \((1.7)\). Then for all \(V \in C_{\eta_s}(V_n^*)\) satisfying \(\overline{V}_n(V) \in C_{\lambda_s}(\omega)\), there exist a real analytic symplectic coordinate transformation \(\Phi_{s+1} : D_{s+1} \rightarrow D_s\) satisfying
\[\begin{align*}
||\Phi_{s+1} - id||_{(r, \theta)}(r, \theta) &\leq \epsilon_s^{0.5}, \\
||D\Phi_{s+1} - I||_{(r, \theta) \rightarrow (r, \theta)} &\leq \epsilon_s^{0.5},
\end{align*}\]
such that for \(H_{s+1} = H_s \circ \Phi_{s+1} = N_{s+1} + R_{s+1}\), the same assumptions as above are satisfied with ‘\(s + 1\)’ in place of ‘\(s\)’, where \(C_{\eta_{s+1}}(V_{n+1}^*) \subset \overline{V}_s^{-1}(C_{\lambda_s}(\omega))\) and
\[||\overline{V}_{s+1} - \overline{V}_s||_\infty \leq \epsilon_s^{0.5},\]
\[ \| V_{s+1}^* - V_s^* \|_\infty \leq 2r_s^{0.5}, \]

where

\[ ||\Phi_{s+1} - id||_{r,\theta} := \sup_{z \in B_{s+1}} \| (\Phi_{s+1} - id)(z) \|_{r,\theta}. \]

**Proof.** In the step \( s \to s + 1 \), there is saving of a factor

\[ e^{-\delta_s \left( \sum_{n \in \mathbb{N}^* \cap (2a_n + k_n + k'_n) n^\theta - 2n_1^s} \right)}. \]

By (2.2), one has

\[ (2.66) \]

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In view of (2.39) and (2.67), one has
\[ \sum_{i \geq 1} (n_i^t)\theta \leq (11^\theta + 2)B_s. \]

Let
\[ \tilde{N}_s = (11^\theta + 2)^{1/2}N_s, \]
then following the proof of (2.68), we have
\[ (2.69) \]
\[ \prod_{n \in N_s} \frac{1}{1 + (k_n - k_n')^2 n^5} > \sigma_s. \]

Assuming \( \tilde{\omega} \in C_{\sigma_s}(\omega) \) and from the lower bound (2.68) and (2.69), the relation (1.6) and (1.7) remain true if we substitute \( \tilde{\omega} \) for \( \omega \). Moreover, there is analyticity on \( C_{\sigma_s}(\omega) \). The transformations \( \Phi_{s+1} \) is obtained as the time-1 map \( X^t_{F_s} \) of the Hamiltonian vector field \( X_{F_s} \) with \( F_s = F_{0,s} + F_{1,s} \). Taking \( \rho = \rho_s, \delta = \delta_s \) in Lemma 2.4, we get
\[ (2.70) \]
\[ \| R_{i,s} \|_{\rho_s + \delta_s}^+ \leq \frac{\rho_s}{\gamma^{2/5} \delta_s} e^{C(\theta)s_0} \| R_{i,s} \|_{\rho_s}^+. \]

where \( i = 0, 1 \). By Lemma 2.4 we get
\[ (2.71) \]
\[ \| F_{i,s} \|_{\rho_s + 2\delta_s} \leq \frac{C(\theta)}{\delta_s^2} \| F_{i,s} \|_{\rho_s + \delta_s}. \]

Combining (2.59), (2.60), (2.70) and (2.71), we get
\[ \| F_s \|_{\rho_s + 2\delta_s} \leq \frac{C(\theta)}{\gamma^{2/5} \delta_s} e^{C(\theta)s_0} (\epsilon_s + \epsilon_s^{0.5}). \]

By Lemma 2.7 we get
\[ \sup_{\| x \|_{r,s} < 1} \| X_{F_s} \|_{r,\theta} \leq C(r, \rho, \theta) \| F_s \|_{\rho_s + 2\delta_s} \leq \frac{C(r, \rho, \theta)}{\gamma^{2/5} \delta_s} e^{C(\theta)s_0} (\epsilon_s + \epsilon_s^{0.5}) \leq \epsilon_s^{0.55}, \]
where noting that \( 0 < \epsilon_0 < 1 \) small enough and depending on \( r, \rho, \theta \) only. Since \( \epsilon_s^{0.55} \ll \frac{1}{\pi s_{s+1}^{1/2}} = d_{s+1} - d_s \), we have \( \Phi_{s+1} : D_{s+1} \to D_s \) with
\[ \| \Phi_{s+1} - id \|_{(r,\theta)} \leq \sup_{x \in D_{s+1}} \| X_{F_s} \|_{r,\theta} \leq \epsilon_s^{0.55} < \epsilon_s^{0.5}, \]
which is the estimate (2.62). Moreover, by Cauchy estimate we get
\[ \| DX_{F_s} - I \|_{(r,\theta) \to (r,\theta)} \leq \frac{1}{d_s} \epsilon_s^{0.55} < \epsilon_s^{0.5}, \]
and thus the estimate (2.63) follows.

Moreover, under the assumptions (2.40) - (2.41) at stage \( s \), we get from (2.42), (2.63) and (2.44) that
\[ \| R_{0,s+1} \|_{\rho_{s+1}}^+ \leq \frac{2d}{e} \left( \epsilon_0^{0.9} + \epsilon_0^{0.3} \right) \leq \epsilon_{s+1}, \]
\[ \| R_{1,s+1} \|_{\rho_{s+1}}^+ \leq \frac{2d}{e} \left( \epsilon_0^{0.3} + \epsilon_0^{0.3} \right) \leq \epsilon_{s+1}. \]
and

$$||R_{2,s+1}||_{p,s+1}^+ \leq ||R_{2,s}||_{p,s}^+ + \frac{2\theta}{\epsilon_0} \left( \epsilon_0^\frac{1}{3} + \epsilon_0 \frac{0.6(\frac{d}{2})}{\epsilon_0} \right)$$

$$\leq (1 + d_s)\epsilon_0 + 2e^ws \epsilon_0$$

$$\leq (1 + d_{s+1})\epsilon_0,$$

which are just the assumptions (2.59)-(2.61) at stage $s + 1$.

Define

$$\Lambda_s(V) = (\Lambda_n,s(V))_{n \in \mathbb{N}^*}$$

with

$$\Lambda_n,s(V) = \lambda_n,s(V) - n.$$

For any $n \in \mathbb{N}^*$, now we would like to prove (2.72)

$$\Lambda_s \left( \mathcal{C}_{\frac{s}{10},s}^\parallel(V_s^*) \right) \subseteq \mathcal{C}_{\sigma_s}(\omega).$$

In view of (2.56), one has

$$\Lambda_n,s(V) = \sqrt{\frac{V_{n,s}(V)}{n^2 + \tilde{V}_{n,s}(V) + n}}.$$

Hence

$$|\Lambda_{n,s}(V)| \leq \frac{1}{n} \left| \sqrt{\tilde{V}_{n,s}(V)} \right|,$$

where noting that

$$\sqrt{n^2 + \tilde{V}_{n,s}(V) + n} \geq n.$$

If $V \in \mathcal{C}_{\frac{s}{10}}^\parallel(V_s^*) \subseteq \mathcal{C}_{\eta_s}(V_s^*)$ and using Cauchy’s estimate, one has

(2.73)

$$\sum_{m \in \mathbb{N}^*} \left| \partial V_{n,s}(V) \right| \leq \frac{2}{\eta_s} \sup_{\mathcal{C}_{\frac{s}{10}}^\parallel(V_s^*)} \left| \tilde{V}_{n,s} \right| < \frac{10}{\eta_s}.$$

Let $V \in \mathcal{C}_{\frac{s}{10},s}^\parallel(V_s^*) \subseteq \mathcal{C}_{\frac{s}{10}}^\parallel(V_s^*)$, then

$$|\Lambda_{n,s}(V) - \omega_n|$$

$$= \left| \sqrt{n^2 + \tilde{V}_{n,s}(V) - \sqrt{n^2 + \tilde{V}_{n,s}(V_s^*)}} \right|$$

$$= \left| \sqrt{\tilde{V}_{n,s}(V) - \tilde{V}_{n,s}(V_s^*)} \right|$$

$$\leq \frac{2}{3n} \cdot \sup_{\mathcal{C}_{\frac{s}{10},s}^\parallel(V_s^*)} \left| \partial \tilde{V}_{n,s} \right|_{l^\infty} \cdot ||V - V_s^*||_{l^\infty}$$

$$< \frac{2}{3n} \cdot 10\eta_s^{-1} \cdot \frac{1}{10} \sigma_s \eta_s \quad \text{(in view of (2.73))}$$

$$= \frac{2\sigma_s}{3n},$$

which finishes the proof of (2.72).
Note that
\[
\left| \left( \prod_{m \in \mathbb{N}^*} m^{2a_m} \right) nE_{a00}^{(n)} \right| \leq \frac{1}{n} \epsilon_0 0.6(\frac{2}{n})^r e^{2\rho_{s+1}(\sum_{m \in \mathbb{N}^*} amm^* + n^* - m^*)}
\]
which implies
\[
\left| E_{a00}^{(n)} \right| \leq 1 \frac{1}{n} \epsilon_0 0.6(\frac{2}{n})^r e^{2\rho_{s+1}(\sum_{m \in \mathbb{N}^*} amm^* + n^* - m^*)}
\]
Assuming further
\[
(2.74)
I_m(0) \leq e^{-2rm^*}
\]
and for any s,
\[
(2.75)
\rho_s < \frac{1}{2} \rho_r
\]
we obtain
\[
\left| \sum_{a \in \mathbb{N}^*} B_{a00}^{(n)}M_{a00} \right| \leq \frac{1}{n} \epsilon_0 0.6(\frac{2}{n})^r \sum_{a \in \mathbb{N}^*} e^{2\rho_{s+1}(\sum_{m \in \mathbb{N}^*} amm^* + n^* - m^*)} \prod_{m \in \mathbb{N}^*} I_m(0)^{a_m}
\]
\[
\leq \frac{1}{n} \epsilon_0 0.6(\frac{2}{n})^r \sum_{a \in \mathbb{N}^*} e^{2\rho_{s+1}(\sum_{m \in \mathbb{N}^*} amm^*)} \prod_{m \in \mathbb{N}^*} I_m(0)^{a_m} (\text{in view of } (2.74))
\]
\[
\leq \frac{1}{n} \epsilon_0 0.6(\frac{2}{n})^r \sum_{a \in \mathbb{N}^*} e^{-r(\sum_{m \in \mathbb{N}^*} amm^*)} \prod_{m \in \mathbb{N}^*} (1 - e^{-rm^*})^{-1} (\text{in view of } (2.75))
\]
\[
\leq \frac{1}{n} \epsilon_0 0.6(\frac{2}{n})^r \left( \frac{1}{r} \right)^{C(\theta)r^{-\frac{1}{m} + \frac{1}{m} \rho_{s+1}}}
\]
i.e.
\[
|\lambda_{n,s+1} - \lambda_{n,s}| < \frac{1}{n} \epsilon_0 0.6(\frac{2}{n})^r \left( \frac{1}{r} \right)^{C(\theta)r^{-\frac{1}{m} + \frac{1}{m} \rho_{s+1}}}
\]
Noting that
\[
\lambda_{n,s+1} - \lambda_{n,s} = \sqrt{n^2 + \tilde{V}_{n,s+1}} - \sqrt{n^2 + \tilde{V}_{n,s}} = \frac{\tilde{V}_{n,s+1} - \tilde{V}_{n,s}}{\sqrt{n^2 + \tilde{V}_{n,s+1}} + \sqrt{n^2 + \tilde{V}_{n,s}}},
\]
then one has
\[
|\tilde{V}_{n,s+1} - \tilde{V}_{n,s}| < \left( \frac{1}{r} \right)^{C(\theta)r^{-\frac{1}{m} + \frac{1}{m} \rho_{s+1}}} \epsilon_0 0.6(\frac{2}{n})^r < \epsilon_s
\]
which verifies (2.64). Further applying Cauchy’s estimate on $C_{\sigma,\eta_s}(V_s^*)$, one gets
\[
\sum_{m \in \mathbb{N}^*} \left| \frac{\partial \tilde{V}_{n+1}}{\partial V_m} - \tilde{V}_{n,s} \right| \leq \frac{10 \epsilon_0^{0.5}}{\sigma_s \eta_s} \\
\leq \left( \frac{10}{\eta_s} \right) e^{C(\rho,\theta) \left( \ln \frac{1}{\epsilon_0} \right) \frac{1}{\epsilon_0^{0.5}}} - 0.5 \ln \frac{1}{\epsilon_0^{0.5}} \\
\leq \left( \frac{1}{\eta_s} \right) e^{-\frac{1}{2} \ln \frac{1}{\epsilon_0^{0.5}}},
\]
(2.77)
\[
= \frac{1}{\eta_s} \epsilon_0 \left( \frac{1}{2} \right)^{n+1}.
\]

Since
\[
\eta_{s+1} = \frac{1}{20} \sigma_s \eta_s,
\]
it follows that
\[
\eta_{s+1} = \eta_s e^{-C(\rho,\theta) \left( \ln \frac{1}{\epsilon_0} \right) \frac{1}{\epsilon_0^{0.5}} \left( \frac{1}{2} \right)^{n+1}} \\
\geq \eta_s e^{-C(\rho,\theta) \left( \ln \frac{1}{\epsilon_0} \right) \left( \frac{1}{2} \right)^n} \quad \text{for } \epsilon_0 \text{ small enough}
\]
(2.78)
\[
= \eta_s e^{C(\rho,\theta) \left( \frac{1}{2} \right)^n} \\
\text{and hence by iterating (2.78) implies}
\]
\[
\eta_s \geq \eta_0 \epsilon_0 \left( \frac{1}{2} \right)^n \\
= \eta_0 \epsilon_0 \left( \frac{1}{2} \right)^n \\
\quad \geq \epsilon_0^{\frac{1}{100}} \left( \frac{1}{2} \right)^n \quad \text{for } \epsilon_0 \text{ small enough}.
\]

On $C_{\sigma,\eta_s}(V_s^*)$ and for any $n$, we deduce from (2.77), (2.79) and the assumption (2.93) that
\[
\sum_{m \in \mathbb{N}^*} \left| \frac{\partial \tilde{V}_{n+1}}{\partial V_m} - \delta_{nm} \right| \leq \sum_{m \in \mathbb{N}^*} \left| \frac{\partial \tilde{V}_{n+1}}{\partial V_m} - \partial \tilde{V}_{n,s} \right| + \sum_{m \in \mathbb{N}^*} \left| \partial \tilde{V}_{n,s} - \delta_{nm} \right| \\
\leq \epsilon_0 \left( \frac{1}{2} \right)^n \left( \frac{1}{2} \right)^n + d_s \epsilon_0 \left( \frac{1}{2} \right)^n \\
\quad < d_{s+1} \epsilon_0 \left( \frac{1}{2} \right)^n,
\]
and consequently
\[
\left\| \frac{\partial \tilde{V}_{n+1}}{\partial V} - I \right\|_{L^\infty \to L^\infty} < d_{s+1} \epsilon_0 \left( \frac{1}{2} \right)^n,
\]
which verifies (2.58) for $s + 1$.

Finally, we will freeze $\omega$ by invoking an inverse function theorem. Consider the following functional equation
\[
\tilde{V}_{n,s+1}(V_s^*) = n \omega_n + \omega_n^2,
\]
and
\[
V_s^* \in C_{\sigma,\eta_s}(V_s^*).
From (2.58) and the standard inverse function theorem implies (2.59) having a solution $V^{s+1}_*$, which verifies (2.57) for $s+1$. Noting that
\[
V^{s+1}_* - V^* = (I - \tilde{V}_{s+1})(V^{s+1}_*) - (I - \tilde{V}_{s+1})(V^*) + (V_s - \tilde{V}_{s+1})(V^*_s),
\]
and using (2.64), (2.58), one has
\[
||V^{s+1}_* - V^*||_\infty \leq (1 + \frac{d_s}{s+1})\epsilon_0 + 2\epsilon^0.5 < 2\epsilon^0.5 \ll \sigma_s \eta_s,
\]
which verifies (2.65) and completes the proof of the iterative lemma.

We are now in a position to prove Theorem 2.3.

**Proof.** To apply iterative lemma with $s = 0$, set
\[
V_{n,0} = n\omega_n + \omega_n^2, \quad \tilde{V}_0 = id, \quad \epsilon_0 = \epsilon,
\]
and consequently (2.57)–(2.61) with $s = 0$ are satisfied. Hence, the iterative lemma applies, and we obtain a decreasing sequence of domains $D_s \times C_\eta(V^*_s)$ and a sequence of transformations
\[
\Phi^s = \Phi_1 \circ \cdots \circ \Phi_s : \quad D_s \times C_\eta(V^*_s) \rightarrow D_0 \times C_\eta(V^*_0),
\]
such that $H \circ \Phi^s = N_s + R_s$ for $s \geq 1$. Moreover, the estimates (2.62)–(2.65) hold. Thus we can show $V^*_s$ converge to a limit $V^*$ with the estimate
\[
||V^*_s - \omega||_\infty \leq \sum_{s=0}^{\infty} 2\epsilon^0.5 < \epsilon^0.4
\]
and $\Phi^s$ converge uniformly on $D_s \times \{V_s\}$, where $D_s = \{(z_n)_{n \in \mathbb{N}^*} : \frac{2}{3} \leq |z_n|e^{rn^\theta} \leq \frac{5}{6}\}$, to $\Phi : D_s \times \{V_s\} \rightarrow D_0$ with the estimates
\[
||\Phi - id||_{(r, \theta)} \leq \epsilon^0.4,
\]
\[
||D\Phi - I||_{(r, \theta) \rightarrow (r, \theta)} \leq \epsilon^0.4.
\]
Hence
\[
H_s = H \circ \Phi = N_s + R_{2,s},
\]
where
\[
N_s = \sum_{n \in \mathbb{N}^*} (n + \omega_n)|z_n|^2
\]
and
\[
||R_{2,s}||_{10,\rho} \leq \epsilon^0.4.
\]

**Remark 2.9.** By (2.55), the Hamiltonian vector field $X_{R_{2,s}}$ is a bounded map from $Gr^{r,\theta}$ into $Gr^{r,\theta}$. Taking
\[
I_n(0) = \frac{3}{4}e^{-2rn^\theta},
\]
we get an invariant torus $\mathcal{T}$ with frequency $(n + \omega_n)_{n \in \mathbb{N}^*}$ for $X_{H_s}$. Moreover, we deduce the torus $\Phi(\mathcal{T})$ is linearly stable from the fact that (2.55) is a normal form of order 2 around the invariant torus.
3. Application to the nonlinear wave equation

We study equation (1.1) as an infinite dimensional hamiltonian system. As the phase space on one may take, for example, the product of the usual Sobolev spaces \( P = H^1_0([0, \pi]) \times L^2([0, \pi]) \) with coordinates \( u \) and \( v = u_t \). Then the hamiltonian of (1.1) is

\[
H = \frac{1}{2} \langle v, v \rangle + \frac{1}{2} \langle Au, u \rangle + \frac{\epsilon}{4} \int_0^\pi u^4 dx,
\]

where \( A = -d^2/dx^2 + V(x) \) and \( \langle \cdot, \cdot \rangle \) denotes the usual scalar product in \( L^2 \). The hamiltonian equations of motions are

\[
u_t = \frac{\partial H}{\partial v} = v, \quad v_t = -\frac{\partial H}{\partial u} = -Au - u^3,
\]

hence they are equal to (1.1).

To rewrite it as a hamiltonian in infinitely many coordinates we make the ansatz

\[
u = S'p = \sum_{n \in \mathbb{N}^*} \sqrt{\lambda_n} p_n \phi_n, \quad v = Sp = \sum_{n \in \mathbb{N}^*} \sqrt{\lambda_n} p_n \phi_n
\]

where

\[
\phi_n = \sqrt{\frac{2}{\pi}} \sin nx \quad \text{for} \quad n = 1, 2, \ldots \text{ are the normalized Dirichlet eigenfunctions of the operator } A \text{ with eigenvalues } \lambda_n = \sqrt{n^2 + V_n}.
\]

We obtain the Hamiltonian

(3.1) \[
H = \Lambda + G = \frac{1}{2} \sum_{n \in \mathbb{N}^*} \lambda_n (p_n^2 + q_n^2) + \frac{\epsilon}{4} \sum_{\pm i, \pm j, \pm k, \pm l = 0} G_{ijkl} q_i q_j q_k q_l,
\]

with

(3.2) \[
G_{ijkl} = \frac{1}{\sqrt{\lambda_i \lambda_j \lambda_k \lambda_l}} \int_0^\pi \phi_i \phi_j \phi_k \phi_l dx.
\]

We introduce the complex coordinates

\[
z_n = \frac{1}{\sqrt{2}} \left( q_n + i p_n \right), \quad \bar{z}_n = \frac{1}{\sqrt{2}} \left( q_n - i p_n \right)
\]

with \( i = \sqrt{-1} \). Then the Hamiltonian (3.1) is turned into

(3.3) \[
H(z, \bar{z}) = \sum_{n \in \mathbb{N}^*} \lambda_n |z_n| \bar{z}_n + \frac{\epsilon}{16} \sum_{\pm i, \pm j, \pm k, \pm l = 0} G_{ijkl} (z_i + \bar{z}_i)(z_j + \bar{z}_j)(z_k + \bar{z}_k)(z_l + \bar{z}_l)
\]

Then the Hamiltonian (3.3) has the form of

\[
H(z, \bar{z}) = N(z, \bar{z}) + R(z, \bar{z}),
\]

where

\[
N(z, \bar{z}) = \sum_{n \in \mathbb{N}^*} \lambda_n |z_n|^2,
\]

and

\[
R(z, \bar{z}) = \frac{\epsilon}{16} \sum_{\pm i, \pm j, \pm k, \pm l = 0} G_{ijkl} (z_i + \bar{z}_i)(z_j + \bar{z}_j)(z_k + \bar{z}_k)(z_l + \bar{z}_l)
\]

In view of (3.2), one has

\[
||R||_p \leq C \epsilon.
\]
Applying Theorem 2.3 and Remark 2.9 we finish the proof of Theorem 1.4.

4. Measure Estimate and Technical Lemma

**Lemma 4.1.** Let the set
\[
\Pi = [0, 1] \times [0, 1/2] \times \cdots [0, 1/n] \times \cdots
\]
with probability measure. Then there exists a subset \( \Pi_\gamma \subset \Pi \) with
\[
\text{meas } \Pi_\gamma \leq C\gamma,
\]
where \( C \) is a positive constant, such that for any \( \omega \in \Pi \setminus \Pi_\gamma \), the inequalities (1.6) and (1.7) holds.

**Proof.** Define the resonant set \( R_l \) by
\[
R_l = \left\{ \omega : \left\| \sum_{n \in \mathbb{N}} l_n \omega_n \right\| < \gamma \prod_{n \in \mathbb{N}} \frac{1}{1 + l_n^2 n^5} \right\},
\]
and
\[
R_1 = \bigcup_{l \in \mathbb{Z}^m} R_l.
\]
Then following the proof of Lemma 4.1 in [9], one has
\[
\text{meas } R_1 \leq C_1 \gamma,
\]
where \( C_1 \) is a positive constant.

Define the resonant set \( \tilde{R}_l \) (where considering \( l = k - k' \)) by
\[
\tilde{R}_l = \left\{ \omega : \left\| \sum_{n \in \mathbb{N}^*} l_n \omega_n \right\| < \frac{\gamma^3}{16} \prod_{n \geq m, n \neq n_1^*, n_2^*} \left( \frac{1}{1 + l_n^2 n^6} \right)^4 \right\},
\]
Then one has
\[
\text{meas } \tilde{R}_l \leq \frac{m \gamma^3}{16} \prod_{n \geq m, n \neq n_1^*, n_2^*} \left( \frac{1}{1 + l_n^2 n^6} \right)^4,
\]
where \( l_j = 0 \) with \( 1 \leq j \leq m - 1 \) and \( l_m \neq 0 \).

Note that
\[
\sum_{n \in \mathbb{N}^*} l_n \omega_n = \sum_{n \in \mathbb{N}^*, n \neq n_1^*, n_2^*} l_n \omega_n + \sigma_{n_1^*} \omega_{n_1^*} + \sigma_{n_2^*} \omega_{n_2^*},
\]
where \( \sigma_{n_1^*}, \sigma_{n_2^*} \in \{-1, 1\} \). Hence, if \( \omega \in \Pi \setminus R_1 \) (where \( R_1 \) is defined in (4.3)) and
\[
n_2^* \geq \frac{4}{\gamma} \prod_{n \geq m, n \neq n_1^*, n_2^*} (1 + l_n^2 n^5),
\]
then
\[ \left\| \sum_{n \in \mathbb{N}^* \setminus \{n_1, n_2\}} l_n \omega_n + \sigma_{n_1} \omega_{n_1} + \sigma_{n_2} \omega_{n_2} \right\| \geq \left\| \sum_{n \in \mathbb{N}^* \setminus \{n_1, n_2\}} l_n \omega_n \right\| - \left\| \omega_{n_1} + \omega_{n_2} \right\| \]
\[ \geq \gamma \prod_{n \in \mathbb{N}^* \setminus \{n_1, n_2\}} \frac{1}{1 + l_n^2 n^5} - \frac{2}{n_2^5} \]
\[ \geq \frac{\gamma}{2} \prod_{n \in \mathbb{N}^* \setminus \{n_1, n_2\}} \frac{1}{1 + l_n^2 n^5} \]
where the last inequality is based on (4.8). Hence, we always assume
\[ (4.9) \quad n_2^* < \frac{4}{\gamma} \prod_{n \in \mathbb{N}^* \setminus \{n_1, n_2\}} (1 + l_n^2 n^5) := A(l). \]

If
\[ (4.10) \quad n_1^* = n_2^*, \]
then one has
\[ (4.11) \quad n_1^* < A(l). \]
If \( n_1^* > n_2^* \), then noting that
\[ (4.12) \quad n_1^* \leq \sum_{i \geq 2} n_i^* , \]
and
\[ (4.13) \quad \sum_{i \geq 2} n_i^* = \sum_{n \in \mathbb{N}^* \setminus \{n_1, n_2\}} |l_n| \leq \prod_{n \in \mathbb{N}^* \setminus \{n_1, n_2\}} (1 + l_n^2 n^5) \]
which implies
\[ (4.14) \quad n_1^* < A(l) \left( \prod_{n \in \mathbb{N}^* \setminus \{n_1, n_2\}} (1 + l_n^2 n^5) \right) = \frac{4}{\gamma} \left( \prod_{n \in \mathbb{N}^* \setminus \{n_1, n_2\}} (1 + l_n^2 n^5) \right)^2 := B(l). \]

Then define the resonant set
\[ (4.15) \quad \mathcal{R}_2 = \bigcup_{l \in \mathbb{Z}^* \setminus \{n_1, n_2\}} \mathcal{R}_l , \]
In view of (4.6), (4.9), (4.14) and following the proof of (4.4), one has
\[ (4.16) \quad \text{meas } \mathcal{R}_2 \leq C_3 \gamma , \]
where \( C_3 \) is a positive constant.

Let
\[ \Pi_g = \Pi \setminus (\mathcal{R}_1 \cup \mathcal{R}_2) , \]
then one has
\begin{equation}
\textup{meas } \Pi_\gamma \leq C_\gamma,
\end{equation}
and for any \( \omega \in \Pi \setminus \Pi_\gamma \), the inequalities (1.6) and (1.7) holds.

Lemma 4.2. The following estimate holds
\[ \left( \prod_{n \in \mathbb{N}^*: n \neq n_1^n, n_2^n} \left( 1 + (k_n - k_n')^2 n^6 \right)^4 \right) \times e^{-\delta (\sum_{n \in \mathbb{N}^*: (2a_n + k_n + k_n') n^\theta - 2n_1^n)} \leq e^{C(\theta) \delta - \frac{\theta}{2}}, \]
where \( C(\theta) \) is a positive constant depending on \( \theta \) only.

Proof. By a direct calculation, one has
\[ \left( \prod_{n \in \mathbb{N}^*: n \neq n_1^n, n_2^n} \left( 1 + (k_n - k_n')^2 n^6 \right)^4 \right) \times e^{-\delta (\sum_{n \in \mathbb{N}^*: (2a_n + k_n + k_n') n^\theta - 2n_1^n)} \leq e^{1 + 48 \left( \sum_{n \in \mathbb{N}^*: n \neq n_1^n} \ln(1 + (k_n - k_n')^2 n^6) \right)} \times e^{-\delta (2 - 2^\theta) (\sum_{n \geq 3} n_1^n)} \leq e^{1 + 48 \left( \sum_{n \in \mathbb{N}^*: n \neq n_1^n, n_2^n} \ln(|k_n - k_n'| n) \right) - \delta (2 - 2^\theta) (\sum_{n \geq 3} n_1^n)} \]
\[ \leq e^{1 + 48 \left( \sum_{n \in \mathbb{N}^*: n \neq n_1^n, n_2^n} \ln(|k_n - k_n'| n) \right) - \delta (2 - 2^\theta) (\sum_{n \geq 3} n_1^n)} \times e^{16e^{||R_0||_p} \frac{\sum_{n > N: n \neq n_1^n} 48 \ln(|k_n - k_n'| n - \delta |k_n - k_n'| n^\theta)}{\Theta}} \]
\[ \leq e^{1 + 48 \left( 2 - 2^\theta \right) \frac{1}{\Theta} \ln \left( \frac{1}{\Theta} \right)} \times e^{16e^{||R_0||_p} \frac{\sum_{n > N: n \neq n_1^n} 48 \ln(|k_n - k_n'| n - \delta |k_n - k_n'| n^\theta)}{\Theta}} \]
\[ \leq e^{C(\theta) \delta - \frac{\theta}{2}}, \]
where \( C(\theta) \) is a positive constant depending on \( \theta \) only.

For \( 0 < \delta \ll 1 \), it is easy to verify the following two facts that:
\begin{enumerate}
\item let \( f(x) = 48 \ln x - \delta x^\theta \), and then
\begin{equation}
\max_{x \geq 1} f(x) = f \left( \left( \frac{48}{\Theta} \right)^{1/\theta} \right) = 48 \ln \left( \left( \frac{48}{\Theta} \right)^{1/\theta} \right) - \frac{48}{\theta} \leq \frac{48}{\theta} \ln \left( \frac{48}{\Theta} \right); \end{equation}
\item for \( k_n \neq k_n' \) and \( n > N = \left( \frac{48}{\Theta} \right)^{4/\theta} \), one has
\begin{equation}
48 \ln \left( |k_n - k_n'| n \right) - \delta \left( |k_n - k_n'| n^\theta \right) < 0. \end{equation}
\end{enumerate}
References

[1] P. Baldi, M. Berti, E. Haus, and R. Montalto. Time quasi-periodic gravity water waves in finite depth. *Invent. Math.*, 214(2):739–911, 2018.

[2] P. Baldi, M. Berti, and R. Montalto. KAM for quasi-linear and fully nonlinear forced perturbations of Airy equation. *Math. Ann.*, 359(1-2):471–536, 2014.

[3] P. Baldi, M. Berti, and R. Montalto. KAM for autonomous quasi-linear perturbations of KdV. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 33(6):1589–1638, 2016.

[4] M. Berti and P. Bolle. Sobolev quasi-periodic solutions of multidimensional wave equations with a multiplicative potential. *Nonlinearity*, 25(9):2579–2613, 2012.

[5] M. Berti and P. Bolle. Quasi-periodic solutions with Sobolev regularity of NLS on $\mathbb{T}^d$ with a multiplicative potential. *J. Eur. Math. Soc.*, 15(1):229–286, 2013.

[6] J. Bourgain. Construction of approximative and almost periodic solutions of perturbed linear Schrödinger and wave equations. *Geom. Funct. Anal.*, 6(2):201–230, 1996.

[7] J. Bourgain. Quasi-periodic solutions of Hamiltonian perturbations of 2D linear Schrödinger equations. *Ann. of Math. (2)*, 148(2):363–439, 1998.

[8] J. Bourgain. *Green’s function estimates for lattice Schrödinger operators and applications*, volume 158 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2005.

[9] J. Bourgain. On invariant tori of full dimension for 1D periodic NLS. *J. Funct. Anal.*, 229(1):62–94, 2005.

[10] J. Bourgain and I. Kachkovskiy. Anderson localization for two interacting quasiperiodic particles. *Geom. Funct. Anal.*, 2019.

[11] H. Cong, J. Liu, Y. Shi, and X. Yuan. The stability of full dimensional KAM tori for nonlinear Schrödinger equation. *J. Differential Equations*, 264(7):4504–4563, 2018.

[12] W. Craig and C. E. Wayne. Newton’s method and periodic solutions of nonlinear wave equations. *Comm. Pure Appl. Math.*, 46(11):1409–1498, 1993.

[13] L. H. Eliasson and S. B. Kuksin. KAM for the nonlinear Schrödinger equation. *Ann. of Math. (2)*, 172(1):371–435, 2010.

[14] R. Feola and M. Procesi. Quasi-periodic solutions for fully nonlinear forced reversible Schrödinger equations. *J. Differential Equations*, 259(7):3389–3447, 2015.

[15] M. Gao and J. Liu. Invariant tori for 1D quintic nonlinear wave equation. *Journal of Differential Equations*, 263:8533–8564, Dec. 2017.

[16] J. Geng and X. Xu. Almost periodic solutions of one dimensional Schrödinger equation with the external parameters. *J. Dynam. Differential Equations*, 25(2):435–450, 2013.

[17] T. Kappeler and J. Pöschel. *KdV & KAM*, volume 45 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge*. Springer-Verlag, Berlin, 2003.

[18] S. Kuksin and J. Pöschel. Invariant Cantor manifolds of quasi-periodic oscillations for a nonlinear Schrödinger equation. *Ann. of Math. (2)*, 143(1):149–179, 1996.

[19] S. B. Kuksin. Hamiltonian perturbations of infinite-dimensional linear systems with imaginary spectrum. *Funktsional. Anal. i Prilozhen.*, 21(3):22–37, 95, 1987.

[20] S. B. Kuksin. *Analysis of Hamiltonian PDEs*, volume 19 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, Oxford, 2000.

[21] S. B. Kuksin. Fifteen years of KAM for PDE. In *Geometry, topology, and mathematical physics*, volume 212 of *Amer. Math. Soc. Transl. Ser. 2*, pages 237–258. Amer. Math. Soc., Providence, RI, 2004.

[22] J. E. M. L. Biasco and M. Procesi. Almost periodic invariant tori for the nls on the circle. *arXiv:1903.07576* 2019.
[23] J. Liu and X. Yuan. A KAM theorem for Hamiltonian partial differential equations with unbounded perturbations. *Comm. Math. Phys.*, 307(3):629–673, 2011.

[24] J. Pöschel. Quasi-periodic solutions for a nonlinear wave equation. *Comment. Math. Helv.*, 71(2):269–296, 1996.

[25] J. Pöschel. On the construction of almost periodic solutions for a nonlinear Schrödinger equation. *Ergodic Theory Dynam. Systems*, 22(5):1537–1549, 2002.

[26] W.-M. Wang. Energy supercritical nonlinear Schrödinger equations: quasiperiodic solutions. *Duke Math. J.*, 165(6):1129–1192, 2016.

[27] C. E. Wayne. Periodic and quasi-periodic solutions of nonlinear wave equations via KAM theory. *Comm. Math. Phys.*, 127(3):479–528, 1990.

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