Prescribed Webster Scalar Curvatures on Compact Pseudo-Hermitian Manifolds

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Abstract
In this paper, we investigate the problem of prescribing Webster scalar curvatures on compact pseudo-Hermitian manifolds. In terms of the method of upper and lower solutions and the perturbation theory of self-adjoint operators, we can describe some sets of Webster scalar curvature functions which can be realized through pointwise CR conformal deformations and CR conformally equivalent deformations respectively from a given pseudo-Hermitian structure.

Keywords Compact pseudo-Hermitian manifold · Webster scalar curvature · CR conformal deformation

Mathematics Subject Classification 32V20 · 53C21 · 53C25

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1 Introduction

In Riemannian geometry, the problem of finding a conformal metric on a compact Riemannian manifold with a prescribed scalar curvature has been investigated extensively (cf. [8, 18, 29–31, 35, 36, 38, 42] and the references therein). Its special case that the candidate scalar curvature function is constant is the well-known Yamabe problem, which was settled down by a series of works due to Yamabe, Trudinger, Aubin, and Schoen (cf. [1, 40, 43, 48]).

The following problem is a CR analog of prescribing scalar curvature problem: given any smooth function $\hat{\rho}$ on a compact strictly pseudoconvex CR manifold $M$ of real dimension $2n + 1$ with contact form $\theta$, does there exist a contact form $\hat{\theta}$ CR conformal to $\theta$, that is, $\hat{\theta} = u^{2/n} \theta$ for some positive function $u$, such that its Webster scalar curvature $\text{Scal}_{\hat{\theta}} = \hat{\rho}$? It is equivalent to solving the following partial differential equation

$$-\left(2 + \frac{2}{n}\right) \Delta_{\theta} u + \text{Scal}_{\theta} u = \hat{\rho} u^{1 + \frac{2}{n}} \text{ on } M \quad (1.1)$$

for $u > 0$, where $\text{Scal}_{\theta}$ is the Webster scalar curvature of $(M, \theta)$. When $\hat{\rho}$ is constant, the above problem is referred to as CR Yamabe problem, which was solved by Jerison and Lee (cf. [25, 26]), Gamara and Yacoub (cf. [13, 17]). Another interesting special case for the prescribed Webster scalar curvature problem is to consider the domain manifold to be a CR sphere $S^{2n+1}$. Similar to the Riemannian case, this problem is not always solvable. Indeed, Cheng [6] gave a Kazdan–Warner type necessary condition for the solution $u$ and the prescribed function $\hat{\rho}$. Besides, in [7, 12, 20–23, 33, 37, 41], if $\hat{\rho}$ satisfies suitable conditions, some existence results were established for the prescribed Webster scalar curvature problem on $S^{2n+1}$ by means of variational, topological, perturbation methods, Webster scalar curvature flow, or the theory of critical points, etc. In [3–5, 14, 15, 24, 48], the authors investigated the problem on strictly pseudoconvex spherical CR manifolds. In [19], using geometric flow, Ho proved that any negative smooth function $\hat{\rho}$ can be prescribed as the Webster scalar curvature in the CR conformal class, provided that $\text{dim } M = 3$ and the CR Yamabe invariant of $M$ is negative. In [34], the authors studied the prescribed Webster scalar curvature problem on a pseudo-Hermitian manifold in arbitrary CR dimension with negative CR Yamabe invariant. Using variational techniques, they established several non-existence, existence, and multiplicity results when the function $\hat{\rho}$ is sign-changing.

In this paper, we investigate the prescribed Webster scalar curvature problem on a compact strictly pseudoconvex CR manifold $M$, following the original approaches in [29, 30], but adjusting their related arguments to subelliptic version. In this way, we are able to establish several existence results for the problem on a compact strictly pseudoconvex CR manifold in arbitrary CR dimension. To state our main results, let us introduce some notations. Given a compact strictly pseudoconvex CR manifold $(M^{2n+1}, H, J, \theta)$ (also called the pseudo-Hermitian manifold, see Sect. 2), where $(H, J)$ is a CR structure of type $(n, 1)$ and $\theta$ is a pseudo-Hermitian structure with positive Levi form, let $\text{PC}(\theta)$ denote the set of smooth functions on $M$ that are the Webster scalar curvatures of pseudo-Hermitian structures $\hat{\theta}$ in the CR conformal...
class $\left[ \theta \right] = \left\{ u\theta : 0 < u \in C^\infty (M) \right\}$. In other words, $PC(\theta)$ is the set of smooth functions $\hat{\rho}$ for which one can find a positive solution of (1.1). Let $Y_M(\theta)$ be the CR Yamabe constant (see (2.21) or (2.22)) and $\lambda_1$ be the first eigenvalue of operator $L = -\left( 2 + \frac{2}{n} \right) \Delta_\theta + \text{Scal}_\theta$ (also see (2.19)).

Using the method of upper and lower solutions on CR manifolds, we obtain the following conclusions.

**Theorem 1.1** Let $(M^{2n+1}, H, J, \theta)$ be a compact pseudo-Hermitian manifold. Then

(A) The following statements are equivalent:

(a1) $\lambda_1 < 0$.

(a2) $\{\hat{\rho} \in C^\infty (M) : \hat{\rho} < 0\} \subset PC(\theta)$.

(a3) $\{\hat{\rho} \in C^\infty (M) : \hat{\rho} < 0\} \cap PC(\theta) \neq \emptyset$.

(a4) $Y_M(\theta) < 0$.

(B) The following statements are equivalent:

(b1) $\lambda_1 = 0$

(b2) $0 \in PC(\theta)$.

(b3) $Y_M(\theta) = 0$.

(c) The following statements are equivalent:

(c1) $\lambda_1 > 0$.

(c2) $\{\hat{\rho} \in C^\infty (M) : \hat{\rho} > 0\} \cap PC(\theta) \neq \emptyset$.

(c3) $Y_M(\theta) > 0$.

Note that the results in (B) of Theorem 1.1 essentially belong to [25] as a special case of CR Yamabe problem, and we state these results here for relative completeness, while the prescribed Webster scalar curvature problem for more general nonvanishing function $\hat{\rho}$ in the case of $\lambda_1 = 0$ is still an unsolved problem.

Since in general not all smooth functions $\hat{\rho}$ can be realized as the Webster scalar curvature of some pseudo-Hermitian structure $\hat{\theta}$ pointwise CR conformal to $\theta$, i.e., $\hat{\theta} \in [\theta]$ (cf. [6], [34]), we will try to enhance the possibility of realizing $\hat{\rho}$ as the prescribed Webster scalar curvature by relaxing the desired pseudo-Hermitian structure $(\hat{H}, \hat{J}, \hat{\theta})$ to be CR conformally equivalent to $(H, J, \theta)$, i.e., there is a map $\Phi \in \text{Diff}(M)$ such that $\Phi^* \hat{\theta} \in [\theta]$, $\hat{H} = d\Phi(H)$, $\hat{J} = d\Phi \circ J \circ (d\Phi)^{-1}$. For simplicity, let $CE(\theta)$ denote the set of smooth functions on $M$ which are the Webster scalar curvatures of $(\hat{H}, \hat{J}, \hat{\theta})$ CR conformally equivalent to $(H, J, \theta)$. In other words, $CE(\theta)$ is the set of smooth functions $\hat{\rho}$ for which one can find a map $\Phi \in \text{Diff}(M)$ such that

$$-\left( 2 + \frac{2}{n} \right) \Delta_\theta u + \text{Scal}_\theta u = (\hat{\rho} \circ \Phi) u^{1+\frac{2}{n}} \quad \text{on } M$$

admits a positive solution. By the inverse function theorem and perturbation methods in our cases, we obtain

**Theorem 1.2** Let $(M^{2n+1}, H, J, \theta)$ be a compact pseudo-Hermitian manifold.

(1) If $\lambda_1 < 0$, then $CE(\theta) = \{\hat{\rho} \in C^\infty (M) : \hat{\rho} < 0 \text{ somewhere} \}$.

(2) If $\lambda_1 = 0$, then $CE(\theta) = \{\hat{\rho} \in C^\infty (M) : \hat{\rho} \text{ changes sign on } M \} \cup \{0\}$. 
(3) If $\lambda_1 > 0$, then $C E(\theta) = \{ \hat{\rho} \in C^\infty(M) : \hat{\rho} > 0 \text{ somewhere} \}$.

In particular, if $\hat{\rho}$ is a smooth function on $M$ with changing sign, then it belongs to $C E(\theta)$ regardless of the sign of $\lambda_1$. In other words, any smooth function with changing sign can be realized as some Webster scalar curvature.

**Corollary 1.3** Let $(M^{2n+1}, H, J, \theta)$ be a compact pseudo-Hermitian manifold. If $\hat{\rho} \in C^\infty(M)$ and it changes sign on $M$, then there exists a structure $(\hat{H}, \hat{J}, \hat{\theta})$ on $M$ such that $(M, \hat{H}, \hat{J}, \hat{\theta})$ is a pseudo-Hermitian manifold with the Webster scalar curvature $\hat{\rho}$ and is CR conformally equivalent to $(M, H, J, \theta)$.

## 2 Preliminaries

In this section, we will introduce the notions and notations of pseudo-Hermitian geometry (cf. [10]).

Let $M$ be an orientable real smooth manifold with $\dim \mathbb{R} M = 2n + 1$. A CR structure on $M$ is a complex subbundle $T_{1,0} M$ of complex rank $n$ of the complexified tangent bundle $T M \otimes \mathbb{C}$ satisfying

$$T_{1,0} M \cap T_{0,1} M = \{ 0 \}, \quad [\Gamma(T_{1,0} M), \Gamma(T_{1,0} M)] \subseteq \Gamma(T_{1,0} M)$$

where $T_{0,1} M = \overline{T_{1,0} M}$. The complex subbundle $T_{1,0} M$ corresponds to a real rank $2n$ subbundle of $T M$:

$$H = \text{Re} \{ T_{1,0} M \oplus T_{0,1} M \},$$

which is called the Levi distribution. Clearly, it carries a natural complex structure $J$ defined as

$$J(X + \bar{X}) = \sqrt{-1}(X - \bar{X})$$

for any $X \in T_{1,0} M$. Equivalently, the CR structure may be described by the pair $(H, J)$. Let $(M, H, J)$ and $(\tilde{M}, \tilde{H}, \tilde{J})$ be two CR manifolds. A smooth map $f : (M, H, J) \to (\tilde{M}, \tilde{H}, \tilde{J})$ is called a CR map if it satisfies

$$df(H) \subset \tilde{H}, \quad df \circ J = \tilde{J} \circ df \text{ on } H.$$ 

Furthermore, $f$ is said to be a CR isomorphism if it is a $C^\infty$ diffeomorphism and a CR map.

Since both $M$ and $H$ are orientable, there is a global nowhere vanishing 1-form $\theta$ with $H = \ker \theta$, which is called a pseudo-Hermitian structure on $M$. The corresponding Levi form is defined as

$$L_\theta(X, Y) = d\theta(X, JY)$$

(2.5)
for any $X, Y \in H$. The integrability assumption of $T_{1,0}M$ implies $L_\theta$ is $J$-invariant and symmetric. If the CR manifold $M$ admits a pseudo-Hermitian structure $\theta$ such that $L_\theta$ is positive definite, then $(M, H, J)$ is said to be strictly pseudoconvex. Henceforth we will assume that $(M, H, J)$ is a strictly pseudoconvex CR manifold and $\theta$ is a pseudo-Hermitian structure with positive Levi form. The quadruple $(M^{2n+1}, H, J, \theta)$ is referred to as a pseudo-Hermitian manifold.

Let $\theta, \hat{\theta}$ be two pseudo-Hermitian structures on the CR manifold $(M, H, J)$, whose Levi forms are positive definite. Since $\dim \mathbb{R} TM/H = 1$, $\theta, \hat{\theta}$ are related by

$$\hat{\theta} = u \theta$$

(2.6)

for some nowhere vanishing function $u \in C^\infty(M)$. Applying the exterior differentiation operator $d$ to (2.6), we get

$$L_{\hat{\theta}} = u L_\theta.$$  

(2.7)

Since both $L_\theta$ and $L_{\hat{\theta}}$ are positive definite, we see that $u$ is positive everywhere. Given a CR structure $(H, J)$, then the set of all its pseudo-Hermitian structures with positive Levi form is exactly

$$\theta = \{ u \theta : 0 < u \in C^\infty(M) \},$$

(2.8)

where $\theta$ is one pseudo-Hermitian structure of $(H, J)$ with positive Levi form. A property on a CR manifold $(M, \theta)$ is said to be CR invariant if it is invariant for all pseudo-Hermitian structures in $[\theta]$.

On a pseudo-Hermitian manifold $(M^{2n+1}, H, J, \theta)$, there is a unique globally defined nowhere vanishing tangent vector field $\xi$ on $M$ such that

$$\theta(\xi) = 1, \quad d\theta(\xi, \cdot) = 0,$$

(2.9)

which is usually called the Reeb vector field. Hence, we have a splitting of the tangent bundle

$$TM = H \oplus \mathbb{R} \xi,$$

(2.10)

which leads to a natural projection $\pi_H : TM \to H$ and a Riemannian metric on $M$ (the Webster metric)

$$g_\theta = \pi_H^* L_\theta + \theta \otimes \theta,$$

(2.11)

where $(\pi_H^* L_\theta)(X, Y) = L_\theta(\pi_H X, \pi_H Y)$ for $X, Y \in TM$. On a pseudo-Hermitian manifold, there is a unique linear connection $\nabla$ called Tanaka–Webster connection preserving the CR structure and Webster metric (cf. Theorem 1.3 of [10]). For a function $u \in C^2(M)$, one can define the sub-Laplacian of $u$ as the divergence of
horizontal gradient:

\[ \Delta_\theta u = \text{div}(\nabla^H u), \tag{2.12} \]

where \( \nabla^H u = \pi_H \nabla u \). Then the integration by parts yields

\[ \int_M (\Delta_\theta u)v\Psi^\theta = -\int_M L_\theta (\nabla^H u, \nabla^H v)\Psi^\theta, \tag{2.13} \]

for \( u, v \in C^2(M) \) with compact support, where \( \Psi^\theta = \theta \wedge (d\theta)^n \) is a volume form of \((M^{2n+1}, H, J, \theta)\). If we choose a local \( g_\theta \)-orthonormal real frame \( \{X_\alpha\}_{\alpha=1}^{2n} \) of \( H \) defined on an open set \( U \subset M \), then the sub-Laplacian of \( u \) can be expressed by

\[ \Delta_\theta u = \sum_{\alpha=1}^{2n} X_\alpha^2 u + X_0 u \tag{2.14} \]

where \( X_0 = -\sum_{\alpha=1}^{2n} \nabla_{X_\alpha} X_\alpha \in H \). From the positive definiteness of the Levi form \( L_\theta \) on \( H \), i.e.,

\[ 0 < L_\theta(X, X) = d\theta(X, JX) = -\theta([X, JX]), \quad \forall X \in H \setminus \{0\}, \tag{2.15} \]

it follows that \( X_0, X_1, \ldots, X_{2n} \) together with their commutators span the tangent spaces at any point of \( U \). Moreover, let \((V, x_1, x_2, \ldots, x_{2n+1})\) be a local coordinate system on \( M \), and set \( X_\alpha = \sum_{i=1}^{2n+1} b_\alpha^i \frac{\partial}{\partial x^i} \), where \( b_\alpha^i : V \to \mathbb{R} \) are smooth functions. Then

\[ \Delta_\theta u = \sum_{i,j=1}^{2n} a^{ij} \frac{\partial^2 u}{\partial x^i \partial x^j} - \sum_{i,j,k=1}^{2n+1} a^{ij} \Gamma_{ij}^k \frac{\partial u}{\partial x^k}, \tag{2.16} \]

where \( a^{ij} = \sum_{\alpha=1}^{2n} b_\alpha^i b_\alpha^j \), \( \Gamma_{ij}^k \) are the Christoffel symbols of \( \nabla \) with respect to the local frame \( \{\frac{\partial}{\partial x^i}\}_{i=1}^{2n+1} \).

The curvature theory of Tanaka–Webster connection was developed in [45] (cf. also [10]). In particular, Webster defined a scalar curvature associated with a pseudo-Hermitian structure \( \theta \), which is referred to as Webster scalar curvature in literature. For a pseudo-Hermitian manifold \((M^{2n+1}, H, J, \theta)\), we say that a pseudo-Hermitian structure \( \hat\theta \) on \( M \) is pointwise CR conformal to \( \theta \) if \( \hat\theta = u^2 \theta \) for some positive function \( u \in C^\infty(M) \). The pseudo-Hermitian manifold \((M^{2n+1}, H, J, \hat\theta)\) is said to be a pointwise CR conformal deformation of \((M^{2n+1}, H, J, \theta)\). Furthermore, from [32] and [25], the Webster scalar curvatures of \( \theta \) and \( \hat\theta \) have the following relationship:

\[ -b_n \Delta_\theta u + \rho u = \hat\rho u^a \tag{2.17} \]

where \( \hat\rho \) is the Webster scalar curvature of \((M, H, J, \hat\theta)\), and \( a = 1 + \frac{2}{n}, \ b_n = 2 + \frac{2}{n} \). For convenience, let \( PC(\theta) \) denote the set of \( C^\infty(M) \) functions which are Webster
scalar curvatures of all $\hat{\theta} \in [\theta]$. In other words, $PC(\theta)$ is the set of $C^\infty(M)$ functions for which one can find a positive solution of (2.17).

Now we assume that $(M^{2n+1}, H, J, \theta)$ is a compact pseudo-Hermitian manifold. Set

$$L = -b_n \Delta_\theta + \rho$$

(2.18)

and let $\lambda_1$ be the first eigenvalue of the operator $L$, that is

$$\lambda_1 = \inf_{0 < u \in C^\infty(M)} \frac{\int_M (b_n |\nabla H u|^2_\theta + \rho u^2)^\theta}{\int_M u^2 \Psi_\theta},$$

(2.19)

where $S^2_1(M)$ is the Folland-Stein space (cf. [11], [39]), the norm $| \cdot |_\theta$ is induced by $g_\theta$. If $\psi$ is an eigenfunction corresponding to $\lambda_1$, then $L \psi = \lambda_1 \psi$. Note that $\psi$ is $C^\infty$ and nowhere vanishing (cf. [44]), so we may assume that $\psi > 0$ and thus

$$\lambda_1 = \inf_{0 < u \in C^\infty(M)} \frac{\int_M (b_n |\nabla H u|^2_\theta + \rho u^2)^\theta}{\int_M u^2 \Psi_\theta}.$$ (2.20)

Recall that CR Yamabe constant is given by

$$Y_M(\theta) = \inf_{0 < u \in C^\infty(M)} \frac{\int_M (b_n |\nabla H u|^2_\theta + \rho u^2)^\theta}{(\int_M u^{2 + \frac{2}{n}} \Psi_\theta)^{\frac{n}{n+1}}},$$ (2.21)

$$= \inf_{\hat{\theta} \in [\theta]} \frac{\int_M \hat{\rho} \Psi_\hat{\theta}}{(\int_M \Psi_\hat{\theta})^{\frac{n}{n+1}}},$$ (2.22)

which is a CR invariant.

Given a pseudo-Hermitian manifold $(M^{2n+1}, H, J, \theta)$, we say that the structures $(\hat{H}, \hat{J}, \hat{\theta})$ is CR conformally equivalent to $(H, J, \theta)$ if there is a map $\Phi \in \text{Diff}(M)$ and $0 < u \in C^\infty(M)$ such that

$$\Phi^* \hat{\theta} = u^\frac{2}{n} \theta, \quad \hat{H} = d\Phi(H), \quad \hat{J} = d\Phi \circ J \circ (d\Phi)^{-1}.$$ (2.23)

Clearly, $\hat{J}$ is a complex structure on $\hat{H}$ and $\Phi : (M, H, J, \theta) \to (M, \hat{H}, \hat{J}, \hat{\theta})$ is a CR isomorphism, where the pseudo-Hermitian manifold $(M, \hat{H}, \hat{J}, \hat{\theta})$ is called a CR conformally equivalent deformation of $(M, H, J, \theta)$. Furthermore, Webster scalar curvatures have the following relationship:

$$-b_n \Delta_\theta u + \rho u = (\hat{\rho} \circ \Phi) u^a,$$ (2.24)

where $\hat{\rho}$ is the Webster scalar curvature of $(M, \hat{H}, \hat{J}, \hat{\theta})$, and $a = 1 + \frac{2}{n}$, $b_n = 2 + \frac{2}{n}$. Similarly, let $CE(\theta)$ denote the set of $C^\infty(M)$ functions which are the Webster scalar curvatures of $(M, H, J, \theta)$. In other words, $CE(\theta)$ is the set of $C^\infty(M)$ functions $\hat{\rho}$,
for which one can find a map $\Phi \in \text{Diff}(M)$ such that (2.24) admits a positive solution on $M$. Clearly, $PC(\theta)$ is a subset of $CE(\theta)$.

At the end of this section, we recall the Folland-Stein spaces on the pseudo-Hermitian manifold $(M^{2n+1}, H, J, \theta)$ briefly (cf. [11], [39]), which are the generalized Sobolev spaces compatible to the CR structure $(H, J)$. Let $\{X_\alpha\}_{\alpha=1}^{2n}$ be a local $G_\theta$-orthonormal real frame of $H$ defined on an open subset $U \subset M$. For any $k \in \mathbb{N}_+$ and $1 < p < +\infty$, the Folland-Stein spaces on $U$ is defined by

$$S^p_k(U) = \{ f \in L^p(U) : X_i f \in L^p(U), s \leq k, X_{ij} f \in L^p(U) \} \quad (2.25)$$

with the norms

$$\| f \|_{S^p_k(U)} = \| f \|_{L^p(U)} + \sum_{1 \leq s \leq k} \| X_i \cdots X_s f \|_{L^p(U)} \quad (2.26)$$

where the $L^p$-norm of $f$ is defined by $\| f \|_{L^p(U)} = \left( \int_U |f|^p \Psi_\theta \right)^{1/p}$. By the partition of unity, we can also define $S^p_k(\Omega)$ and $S^p_k(M)$, where $\Omega$ is any open subset of $M$. Since the sub-Laplacian operator $\Delta_\theta$ can be written as (2.14), then by the theory of Rothschild and Stein (cf. [39]), we have the following theorem.

**Theorem 2.1** Suppose that $u \in L^p_{\text{loc}}(\Omega)$ and satisfies

$$\Delta_\theta u = v \quad \text{on } \Omega. \quad (2.27)$$

If $v \in S^p_k(\Omega)$, then $\chi u \in S^p_{k+2}(\Omega)$ for any $\chi \in C^\infty_0(\Omega), 1 < p < +\infty, k = 0, 1, 2, \ldots$. Moreover, there exists a constant $C_\chi > 0$ independent of $u$ and $v$ such that

$$\| \chi u \|_{S^p_{k+2}(\Omega)} \leq C_\chi \left( \| v \|_{S^p_k(\Omega)} + \| u \|_{L^p(\Omega)} \right). \quad (2.28)$$

### 3 Pointwise CR Conformal Deformations with Prescribed Webster Scalar Curvature

In this section, we will investigate the set $PC(\theta)$ on a compact pseudo-Hermitian manifold $(M^{2n+1}, H, J, \theta)$ when $\lambda_1 < 0, \lambda_1 = 0$ and $\lambda_1 > 0$ respectively. Firstly, we consider the case $\lambda_1 < 0$. For this case, we will use the method of upper and lower solutions on pseudo-Hermitian manifolds. For this purpose, we need the following existence and comparison results.

**Lemma 3.1** Let $(M^{2n+1}, H, J, \theta)$ be a compact pseudo-Hermitian manifold. Let $L_1 = -\Delta_\theta + f$, where $f \in C^\infty(M)$. Then

1. If $f > 0$, then $L_1 : S^2_2(M) \rightarrow L^2(M)$ is invertible.
2. The equation $L_1 w = g$ has a weak solution if and only if $\langle g, w \rangle_{L^2} = 0$ for any solution $w$ of $L_1 w = 0$. 

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(3) If \( f > 0 \) and \( v \) is a \( S^2_1(M) \) function satisfying

\[
L_1 v \geq 0,
\]

which means

\[
\int_M \left( \nabla^H v \cdot \nabla^H \varphi + f v \varphi \right) \Psi \theta \geq 0
\]

for any nonnegative function \( \varphi \in S^2_1(M) \), then \( v \geq 0 \).

**Proof** (1) Let us show that for any \( g \in L^2(M) \), there is a unique solution \( u \in S^2_2(M) \) such that \( L_1 u = g \). Here we can relax the requirement \( u \in S^2_2(M) \) to \( u \in S^2_1(M) \).

Indeed, if \( u \in S^2_1(M) \) and \( L_1 u = -\Delta \theta u + f u = g \), then \( -\Delta \theta u = g - f u \in L^2(M) \). By Theorem 2.1 and the partition of unity on \( M \), we obtain \( u \in S^2_2(M) \).

Set

\[
(u, v) = \langle L_1 u, v \rangle_{L^2} = \int_M \left( \nabla^H u \cdot \nabla^H v + f u v \right) \Psi \theta,
\]

where \( \cdot \) is the inner product induced by the Webster metric \( g_\theta \) and \( u, v \in S^2_1(M) \). By a simple computation, we obtain

\[
(u, u) \geq c_1 \| u \|^2_{S^2_1(M)}
\]

and

\[
(u, v) \leq c_2 \| u \|_{S^2_1(M)} \| v \|_{S^2_1(M)}
\]

where \( c_1, c_2 \) are two positive constants. Therefore, the space \( S^2_1(M) \) with the inner product \( \langle \cdot, \cdot \rangle \) is a Hilbert space. By Cauchy–Schwarz inequality and (3.4), we have

\[
\| \langle g, v \rangle_{L^2} \| \leq \| g \|_{L^2} \| v \|_{L^2} \leq c_1^{-\frac{1}{2}} \| g \|_{L^2} (v, v)^{\frac{1}{2}}
\]

for any \( v \in S^2_1(M) \), which implies \( \langle g, v \rangle_{L^2} \) is a bounded linear functional of \( v \in S^2_1(M) \). Applying the Riesz representation theorem, there is a unique \( u \in S^2_1(M) \) such that

\[
\langle g, v \rangle_{L^2} = (u, v) = \langle L_1 u, v \rangle_{L^2}
\]

for any \( v \in S^2_1(M) \).

(2) Since \( M \) is compact and \( f \in C^\infty(M) \), there is a positive constant \( \lambda > 0 \) such that \( f + \lambda > 0 \) on \( M \). In view of the part (1) of this lemma, the inverse operator \( L_2 = (L_1 + \lambda)^{-1} : L^2(M) \to S^2_1(M) \) exists. Using \( S^2_1(M) \subset \subset L^2(M) \) (cf. Theorem 3.15 of [10], [9]) yields that \( L_2 : L^2(M) \to L^2(M) \) is a completely
continuous. The equation $L_1v = g$ is equivalent to $v - \lambda L_2v = L_2g$. Applying the Fredholm-Riesz-Schauder theory (cf. [2]) and the facts $L_1^* = L_1$, $L_2^* = L_2$, we get that $v - \lambda L_2v = L_2g$ has a weak solution if and only if $\langle L_2g, w \rangle_{L^2} = 0$ where $w$ satisfies $w - \lambda L_2w = 0$ which is equivalent to $L_1w = 0$. From

$$\langle g, w \rangle_{L^2} = \langle L_2^{-1}L_2g, w \rangle_{L^2} = \langle L_2g, L_2^{-1}w \rangle_{L^2} = \lambda \langle L_2g, w \rangle_{L^2}, \quad (3.8)$$

it follows that $L_1v = g$ has a weak solution if and only if $\langle g, w \rangle_{L^2} = 0$ for any solution $w$ of $L_1w = 0$.

(3) Since $v \in S^2_1(M)$, then $v_- = \min\{v, 0\} \in S^2_1(M)$ which can be proved by a small modification of the argument in [16, Lemma 5.2, Example 5.3, pp. 124-125]. Taking $\varphi = -v_-$ in (3.2), we get

$$\int_M |\nabla^H v_-|^2 \Psi^\theta \leq -\int_M f(v_-)^2 \Psi^\theta, \quad (3.9)$$

which implies $v_- = 0$ since $f > 0$. Hence, $v \geq 0$. 

Using the above lemma, we obtain the following result, which is a pseudo-Hermitian version of Lemma 2.6 of [29].

**Lemma 3.2** Let $(M^{2n+1}, H, J, \theta)$ be a compact pseudo-Hermitian manifold. Assume that $f(x, u) \in C^\infty(M \times \mathbb{R})$. If there are two functions $u_+, u_- \in C^0(M) \cap S^2_1(M)$ satisfying

$$-\Delta_{\theta} u_+ + f(x, u_+) \geq 0 \quad \text{in } M, \quad (3.10)$$

$$-\Delta_{\theta} u_- + f(x, u_-) \leq 0 \quad \text{in } M \quad (3.11)$$

$$u_+ \geq u_- \quad \text{in } M, \quad (3.12)$$

then there exists a function $u \in C^\infty(M)$ such that

$$-\Delta_{\theta} u + f(x, u) = 0 \quad \text{in } M, \quad (3.13)$$

$$u_- \leq u \leq u_+ \quad \text{in } M. \quad (3.14)$$

Here $u_+$ and $u_-$ are called the upper and lower solutions of (3.13) respectively.

**Proof** Set $A_1 = \min_M u_-, A_2 = \max_M u_+ + I = [A_1, A_2]$. Since $f(x, u) \in C^\infty(M \times \mathbb{R})$, there exists a constant $\lambda > 0$ such that $f(x, u) = -f(x, u) + \lambda u$ is increasing with respect to $u \in I$ for any fixed $x \in M$. In order to find a solution of (3.13) and (3.14), we consider the sequence $\{u_k\}$ defined for $k \geq 1$ by

$$\begin{cases}
(-\Delta_{\theta} + \lambda)u_k = \tilde{f}(x, u_{k-1}) \\
u_0 = u_-.
\end{cases} \quad (3.15)$$
By (3.10), (3.11), (3.12) and (3.15), we have

\[
(-\Delta_\theta + \lambda)(u_+ - u_1) \geq 0, \tag{3.16}
\]
\[
(-\Delta_\theta + \lambda)(u_1 - u_-) \geq 0. \tag{3.17}
\]

According to Lemma 3.1,

\[
u_- \leq u_1 \leq u_+ \tag{3.18}
\]

Iterating the above procedure yields

\[
u_- \leq u_1 \leq u_2 \leq \ldots \leq u_+. \tag{3.19}
\]

Set \( u = \lim_{k \to \infty} u_k \), then \( u \) is a weak solution of (3.13) and (3.14). Since \( \text{Im} u \subset I \) and \( f(\cdot, \cdot) \in C^\infty(M \times \mathbb{R}) \), we conclude that \( f(x,u) \in L^p(M) \) with \( p > 2n + 1 \).

By the regularity result for \( \Delta_\theta \) (cf. Theorem 2.1), we have \( u \in S^p_2(M) \), and thus \( f(x,u) \in S^p_2(M) \). Repeating the above argument, we obtain that \( u \in S^p_2(M) \) for any \( k \in \mathbb{N}_+ \). Therefore, \( u \in C^\infty(M) \) because of \( S^p_2(M) \subset W^{k,p}(M) \subset C^{k-1}(M) \) for any \( k \in \mathbb{N}_+ \) (cf. Theorem 19.1 of [11]), where \( W^{k,p}(M) \) is the classical Sobolev space.

Remark 3.3 The authors of [34] proved that when \( f(x,u) = -b_n^{-1}(\rho u - \hat{\rho} u^a) \), the equation (3.13) admits a weak solution satisfying (3.14) if it has a weak lower solution \( u_- \) and a weak upper solution \( u_+ \).

In terms of Lemma 3.2, and by a similar argument of [29], we obtain

Theorem 3.4 Let \((M^{2n+1}, H, J, \theta)\) be a compact pseudo-Hermitian manifold and \( \hat{\rho} \) be a smooth negative function on \( M \). Then \( \hat{\rho} \in PC(\theta) \) if and only if \( \lambda_1 < 0 \).

Proof If \( \hat{\rho} \in PC(\theta) \), then there is a positive function \( u \in C^\infty(M) \) such that \( u \) satisfies the prescribed Webster scalar curvature equation (2.17), that is, \( Lu = \hat{\rho} u^a \), where \( L \) is given by (2.18) and \( a = 1 + \frac{2}{n} \). Let \( \psi \) be the positive eigenfunction associated with \( \lambda_1 \) of \( L \). Then

\[
\lambda_1 \langle \psi, u \rangle_{L^2} = \langle L\psi, u \rangle_{L^2} = \langle \psi, Lu \rangle_{L^2} = \langle \psi, \hat{\rho} u^a \rangle_{L^2} < 0 \tag{3.20}
\]

from which it follows that \( \lambda_1 < 0 \). Conversely, if \( \lambda_1 < 0 \), then there is a positive solution \( u \in C^\infty(M) \) such that \( Lu = \hat{\rho} u^a \) by the existence of upper and lower solutions, hence \( \hat{\rho} \in PC(\theta) \). Indeed, Let \( u_+ \equiv \alpha > 0 \) where \( \alpha \) is a constant large enough so that

\[
Lu_+ - \hat{\rho} u_+^a = \alpha(\rho - \hat{\rho} \alpha^{a-1}) \geq 0. \tag{3.21}
\]
On the other hand, let \( u_\pm = \beta \psi \) where \( \beta > 0 \) is so small that \( u_- \leq \alpha \equiv u_+ \) and \( u_- \leq \left( \frac{\lambda_1}{\inf M \hat{\rho}} \right)^{\frac{1}{a-1}} \). Then
\[
Lu_- = \beta L \psi = \lambda_1 \beta \psi = \lambda_1 u_- \leq \hat{\rho} u_+^\alpha.
\] (3.22)

Therefore, by Lemma 3.2, there exists a smooth solution \( u \) satisfying \( Lu = \hat{\rho} u^\alpha \) and \( 0 < u_- \leq u \leq u_+ \), i.e., \( \hat{\rho} \in PC(\theta) \).

**Remark 3.5** From the above proof, if \( \lambda_1 < 0 \), then there always exists a lower solution \( u_- \in C^\infty(M) \) of (2.17) with \( 0 < u_- < u \), where \( u \) is a given \( C^\infty(M) \) function.

**Remark 3.6** By a flow method, Ho [19] proved that every negative function \( \hat{\rho} \in PC(\theta) \) if the CR Yamabe constant \( Y_M(\theta) < 0 \) and \( \dim \mathbb{R} M = 3 \).

**Remark 3.7** In [34], authors gave the following results.

1. When \( \hat{\rho} \) is a smooth nonpositive function on \((M^{2n+1}, H, J, \theta)\) with \( Y_M(\theta) < 0 \) such that the set \( \{ x \in M : \hat{\rho}(x) = 0 \} \) has positive measure, authors gave a necessary and sufficient condition for \( \hat{\rho} \in PC(\theta) \).
2. If \( \hat{\rho} \) is a smooth nonpositive function on \((M^{2n+1}, H, J, \theta)\), then \( \hat{\rho} \) is the Webster scalar curvature of at most one pseudo-Hermitian structure \( \hat{\theta} \in [\theta] \).

Making use of Lemma 3.2 again, we can establish the following property of the set \( PC(\theta) \).

**Proposition 3.8** Let \((M^{2n+1}, H, J, \theta)\) be a compact pseudo-Hermitian manifold with \( \lambda_1 < 0 \). If \( \hat{\rho} \in PC(\theta) \) and \( \hat{\rho}_1 \leq \hat{\rho} \), then \( \hat{\rho}_1 \in PC(\theta) \).

**Proof** To prove \( \hat{\rho}_1 \in PC(\theta) \), we just need to find a positive solution of
\[
-b_n \Delta_\theta u + \rho u = \hat{\rho}_1 u^\alpha.
\] (3.23)

where \( \rho \) is the Webster scalar curvature of \((M^{2n+1}, H, J, \theta)\). We will use the method of upper and lower solutions again. From Remark 3.5, we know that there exists a lower solution of the above equation. Hence, it suffices to find an upper solution of (3.23). Since \( \hat{\rho} \in PC(\theta) \), there is a positive solution \( u \in C^\infty(M) \) of
\[
-b_n \Delta_\theta u + \rho u = \hat{\rho} u^\alpha.
\]
If \( \hat{\rho}_1 \leq \hat{\rho} \), then \( u \) is an upper solution of (3.23). Indeed,
\[
-b_n \Delta_\theta u + \rho u - \hat{\rho}_1 u^\alpha = (-b_n \Delta_\theta u + \rho u - \hat{\rho} u^\alpha) + (\hat{\rho} - \hat{\rho}_1) u^\alpha \geq 0.
\] (3.24)

Hence we may get a positive solution of (3.23).

**Remark 3.9** If \( \hat{\rho} \in PC(\theta) \) and \( \hat{\rho}_1 = \alpha \hat{\rho} \) for some constant \( \alpha > 0 \), then \( \hat{\rho}_1 \in PC(\theta) \) regardless of the sign of \( \lambda_1 \). Indeed, since \( \hat{\rho} \in PC(\theta) \), there is a positive solution \( u \in C^\infty(M) \) of (2.17), then \( \alpha \frac{1}{a-1} u \) is a solution of (3.23), so \( \hat{\rho}_1 \in PC(\theta) \).

Now we turn to the case \( \lambda_1 = 0 \).
Proposition 3.10 Let \((M^{2n+1}, H, J, \theta)\) be a compact pseudo-Hermitian manifold. Then \(0 \in PC(\theta)\) if and only if \(\lambda_1 = 0\).

Proof Assume that \(0 \in PC(\theta)\), that is, there is a positive solution \(u \in C^\infty(M)\) of
\[
Lu = -b_n \Delta_\theta u + \rho u = 0.
\]
From \(\lambda_1 \langle \psi, u \rangle_{L^2} = \langle L\psi, u \rangle_{L^2} = \langle \psi, Lu \rangle_{L^2} = 0\) (3.25) where \(\psi\) is the positive eigenfunction of \(\lambda_1\) of \(L\), we deduce that \(\lambda_1 = 0\). Conversely, if \(\lambda_1 = 0\), then the associated eigenfunction \(\psi\) realizes the zero Webster curvature, i.e., \(0 \in PC(\theta)\). \(\Box\)

Since by Proposition 3.10 if \(\lambda_1 = 0\) then one can always find a pseudo-Hermitian structure \(\hat{\theta} \in [\theta]\) of zero Webster scalar curvature, we can without loss of generality restrict our attention to the case that \((M^{2n+1}, H, J, \theta)\) already has a zero Webster scalar curvature \(\rho \equiv 0\).

Proposition 3.11 Let \((M^{2n+1}, H, J, \theta)\) be a compact pseudo-Hermitian manifold with Webster scalar curvature \(\rho \equiv 0\). If \(0 \not\equiv \hat{\rho} \in PC(\theta)\), then \(\hat{\rho}\) must change sign on \(M\) and
\[
\int_M \hat{\rho} \Psi^\theta < 0.
\]

Proof Since \(\hat{\rho} \in PC(\theta)\), there is a positive solution \(u \in C^\infty(M)\) such that
\[
-b_n \Delta_\theta u = \hat{\rho} u^a,
\]
where \(b_n = 2 + \frac{2}{n}\), \(a = 1 + \frac{2}{n}\). Hence,
\[
\int_M \hat{\rho} u^a \Psi^\theta = -b_n \int_M \Delta_\theta u \Psi^\theta = 0.
\]
Therefore, \(\hat{\rho}\) must change sign on \(M\) since \(u > 0\) and \(\hat{\rho} \not\equiv 0\). Furthermore, multiplying (3.26) by \(u^{-a}\) and integrating by parts yield
\[
\int_M \hat{\rho} \Psi^\theta = -b_n \int_M u^{-a} \Delta_\theta u \Psi^\theta = -ab_n \int_M u^{-a-1} |\nabla H u|^2 \Psi^\theta < 0.
\]
\(\Box\)

In the case \(\lambda_1 > 0\), there is a substitute for Theorem 3.4 as follows.

Proposition 3.12 Let \((M^{2n+1}, H, J, \theta)\) be a compact pseudo-Hermitian manifold. Then \(\lambda_1 > 0\) if and only if there is a positive function \(\hat{\rho} \in C^\infty(M)\) such that \(\hat{\rho} \in PC(\theta)\).

Proof Let \(\psi\) be the positive eigenfunction associated with \(\lambda_1\) of \(L\). If there is a positive function \(\hat{\rho} \in C^\infty(M)\) such that \(\hat{\rho} \in PC(\theta)\), then \(Lu = \hat{\rho} u^a\) has a positive solution \(u\), and
\[
\lambda_1 \langle \psi, u \rangle_{L^2} = \langle L\psi, u \rangle_{L^2} = \langle \psi, Lu \rangle_{L^2} = \langle \psi, \hat{\rho} u^a \rangle_{L^2} > 0
\]
which implies that \( \lambda_1 > 0 \). Conversely, if \( \lambda_1 > 0 \), then
\[
L\psi = \lambda_1 \psi = (\lambda_1 \psi^{1-a}) \psi^a. \tag{3.30}
\]

Pick \( \hat{\rho} = \lambda_1 \psi^{1-a} > 0 \), then \( L\psi = \hat{\rho} \psi^a \), so \( \hat{\rho} \in PC(\theta) \).

Combining Theorem 3.4, Proposition 3.10, Proposition 3.12 with the definition of the CR Yamabe constant, we can give the proof of Theorem 1.1.

**Proof of Theorem 1.1** First, we show the assertion (A). Clearly, \( (a1) \iff (a2) \) has already been proved in Theorem 3.4. The statement \( (a3) \iff (a4) \) can be deduced from (2.22) and Theorem 3.4 of [25]. It remains to prove the statement \( (a2) \iff (a3) \). The case \( (a2) \Rightarrow (a3) \) is trivial. Let us consider \( (a3) \Rightarrow (a2) \). Assume that \( \hat{\rho}_0 \in \{ \hat{\rho} \in C^\infty(M) : \hat{\rho} < 0 \} \cap PC(\theta) \). It follows from Theorem 3.4 that \( \lambda_1 < 0 \). In terms of the equivalence \( (a1) \iff (a2) \), we see that \( (a2) \) holds.

Next, we consider the assertion (B). Obviously, \( (b1) \iff (b2) \) has been proved in Proposition 3.10. \( (b3) \Rightarrow (b2) \) follows from Theorem 3.4 of [25]. Now let us consider the case \( (b2) \Rightarrow (b3) \). By (2.22), we have \( Y_M(\theta) \leq 0 \). If \( Y_M(\theta) < 0 \), then \( \lambda_1 < 0 \) by the results in (A) of this theorem. However, according to \( (b1) \iff (b2) \), \( 0 \in PC(\theta) \) implies \( \lambda_1 = 0 \), which leads to a contradiction. Therefore, \( Y_M(\theta) = 0 \).

Finally, we treat the assertion (C). It is obvious that \( (c1) \iff (c2) \) can be obtained by Proposition 3.12. \( (c3) \Rightarrow (c2) \) may be deduced by the solvability of CR Yamabe problem (cf. [13], [17], [25]). For the case \( (c2) \Rightarrow (c3) \), using the equivalence \( (c1) \iff (c2) \), we have \( \lambda_1 > 0 \). Combining \( (a1) \iff (a4) \) and \( (b1) \iff (b3) \) yields \( Y_M(\theta) > 0 \).

From Theorem 1.1, we can easily see that

**Corollary 3.13** Let \( (M^{2n+1}, H, J, \theta) \) be a compact pseudo-Hermitian manifold. Then \( \lambda_1 \) and \( Y_M(\theta) \) have the same sign or are both zero, and thus the sign of \( \lambda_1 \) is a CR invariant.

**Remark 3.14** In [44], it is easy to see that Corollary 3.13 is implicit in his argument, although it is not clearly pointed out. Indeed, it was proved directly by using the transformation law (3.1) in [25] under the CR pointwise conformal deformations and the solvability of CR Yamabe problem.

### 4 CR Conformally Equivalent Deformations with Prescribed Webster Scalar Curvature

In this section, we will determine the set \( CE(\theta) \) on a compact pseudo-Hermitian manifold \( (M^{2n+1}, H, J, \theta) \) with \( \lambda_1 < 0, \lambda_1 = 0 \) and \( \lambda_1 > 0 \) respectively.

Let us consider a second-order quasilinear differential operator \( T \) on the compact pseudo-Hermitian manifold \( (M^{2n+1}, H, J, \theta) \):
\[
Tu = u^{-a} (-b_n \Delta_{\theta} u + \rho u), \tag{4.1}
\]
where $a = 1 + \frac{2}{n}$, $b_n = 2 + \frac{2}{n}$ and $\rho \in C^\infty(M)$. Note that under the local $g_\theta$-orthonormal real frame \( \{X_\alpha\}_{\alpha=1}^{2n} \) of $H$ defined on $U \subset M$, by (2.14) we may rewrite $T$ as

$$
Tu = -\sum_{\alpha=1}^{2n} b_n u^{-a} X_\alpha^2 u - b_n u^{-a} X_0 u + \rho u^{1-a},
$$

(4.2)

where $X_0 = -\sum_{\alpha=1}^{2n} \nabla X_\alpha X_\alpha \in H$, and $X_0, X_1, \ldots, X_{2n}$ together with their commutators span the tangent spaces at any point of $U$. The linearization of $T$ at a given positive function $u_0 \in C^\infty(M)$ is

$$
T'(u_0)v = \left. \frac{d}{dt} \right|_{t=0} T(u_0 + tv)
= b_n u_0^{-a} \left\{ -\Delta_\theta v + \left( a \frac{\Delta_\theta u_0}{u_0} + \frac{1-a}{b_n} \rho \right) v \right\}
= b_n u_0^{-a} \left\{ -\sum_{\alpha=1}^{2n} X_\alpha^2 v - X_0 v + \left( a \frac{\Delta_\theta u_0}{u_0} + \frac{1-a}{b_n} \rho \right) v \right\}
$$

(4.3)

where $v \in S_2^p(M)$. Set

$$
A(u_0)v = -\Delta_\theta v + \left( a \frac{\Delta_\theta u_0}{u_0} + \frac{1-a}{b_n} \rho \right) v,
$$

(4.4)

which is a linear self-adjoint operator with ker $T'(u_0) = \ker A(u_0)$.

**Lemma 4.1** Let $(M^{2n+1}, H, J, \theta)$ be a compact pseudo-Hermitian manifold. Let $L_3 : S_2^p(M) \to L^p(M)$ be the operator defined as in (4.3) with $0 < u_0 \in C^\infty(M)$ and $\rho \in C^\infty(M)$. Assume that $p > 2n + 1$. If ker $L_3 = 0$, then

$$
\|v\|_{S_2^p(M)} \leq C \|L_3 v\|_{L^p(M)}
$$

(4.5)

for any $v \in S_2^p(M)$, where $C$ is a positive constant independent of $v$. Therefore, the operator $L_3 : S_2^p(M) \to L^p(M)$ is bijective with a continuous inverse.

**Proof** Picking a partition of unity on the compact manifold $M$ and using Theorem 2.1, we get that there exists a constant $C$ independent of $v$ such that

$$
\|v\|_{S_2^p(M)} \leq C \left( \| -\Delta_\theta v\|_{L^p(M)} + \|v\|_{L^p(M)} \right)
$$

(4.6)
for any \( v \in S_2^p(M) \). Since \( 0 < u_0 \in C^\infty(M) \) and \( \rho \in C^\infty(M) \), then from (4.6) it follows that

\[
\|v\|_{S_2^p(M)} \leq C (\| - \Delta_\Omega v + hv\|_{L^p(M)} + \|h\|_{L^p(M)} + \|v\|_{L^p(M)}) \\
\leq C (C'\|b_nu_0^{-a}(-\Delta_\Omega v + hv)\|_{L^p(M)} + C''\|v\|_{L^p(M)}) \\
\leq C_1 (\|L_3v\|_{L^p(M)} + \|v\|_{L^p(M)}),
\]

(4.7)

where \( h = a\Delta_\Omega u_0 + \frac{1-a}{bn}\rho \), \( C' = b_n^{-1}(\sup_M u_0)^a \), \( C'' = \sup_M |h| + 1 \), \( C_1 = \max\{CC', CC''\} \). In order to get (4.5), it is sufficient to prove that

\[
\|v\|_{L^p(M)} \leq C_2 \|L_3v\|_{L^p(M)}
\]

(4.8)

for any \( v \in S_2^p(M) \), where \( C_2 > 0 \) is a constant independent of \( v \). If not, there is a sequence \( \{v_n\} \subset S_2^p(M) \) such that \( \|v_n\|_{L^p(M)} = 1 \) but \( \|L_3v_n\|_{L^p(M)} \to 0 \) as \( n \to +\infty \). Then by (4.7), we have

\[
\|v_n\|_{S_2^p(M)} \leq C_3,
\]

(4.9)

where the constant \( C_3 > 0 \) is independent of \( n \). Using the compactly embedding theorem \( S_2^p(M) \subset W^{1,p}(M) \subset \subset C^0(M) \) (cf. Theorem 19.1 of [11]) yields that there exists a subsequence \( v_{n_k} \) of \( v_n \) and a function \( v \in C^0(M) \) such that \( \lim_{k \to +\infty} \|v_{n_k} - v\|_{C^0(M)} = 0 \), and thus \( \lim_{k \to +\infty} \|v_{n_k} - v\|_{L^p(M)} = 0 \), where \( W^{1,p}(M) \) is the classical Sobolev space with \( p > 2n + 1 \). According to (4.7) and the triangle inequality, we have

\[
\|v_{n_i} - v_{n_j}\|_{S_2^p(M)} \leq C_1 (\|L_3v_{n_i}\|_{L^p(M)} + \|L_3v_{n_j}\|_{L^p(M)} \\
+ \|v_{n_i} - v\|_{L^p(M)} + \|v - v_{n_j}\|_{L^p(M)}) \to 0
\]

(4.10)

as \( i, j \to \infty \), i.e., \( \{v_{n_k}\} \) is a Cauchy sequence in \( S_2^p(M) \), so \( \lim_{k \to +\infty} \|v_{n_k} - v\|_{S_2^p(M)} = 0 \). By the continuity of the operator \( L_3 : S_2^p(M) \to L^p(M) \), we obtain that

\[
\|L_3v\|_{L^p(M)} = \lim_{k \to +\infty} \|L_3v_{n_k}\|_{L^p(M)} = 0.
\]

(4.11)

Hence, \( L_3v = 0 \) in \( L^p(M) \). By \( \ker L_3 = 0 \), we get \( v = 0 \) in \( S_2^p(M) \). However, from \( \|v_{n_k}\|_{L^p(M)} = 1 \) for any \( k \), we deduce that \( \|v\|_{L^p(M)} = 1 \), which leads to a contradiction.

Now we go to prove the last conclusion of this lemma, namely, \( L_3 : S_2^p(M) \to L^p(M) \) is bijective with a continuous inverse. In fact, the condition \( \ker L_3 = 0 \) implies the injectivity. So it is sufficient to show the existence of \( L_3v = f \) for any \( f \in L^p(M) \). According to the fact that \( C^\infty(M) \) is dense in \( L^p(M) \), there is a sequence \( \{f_j\} \subset C^\infty(M) \) such that \( \lim_{j \to \infty} \|f_j - f\|_{L^p(M)} = 0 \). In terms of Lemma 3.1, (4.3) and
regularity results in [46], there exists \( v_j \in C^\infty(M) \) such that \( L_3 v_j = f_j \). Using (4.5) and \( \lim_{j \to \infty} \| f_j - f \|_{L^p(M)} = 0 \) yields that
\[
\| v_i - v_j \|_{S^2_p(M)} \leq C \| L_3 v_i - L_3 v_j \|_{L^p(M)} = C \| f_i - f_j \|_{L^p(M)} \to 0 \quad (4.12)
\]
as \( i, j \to +\infty \), so \( \{v_i\} \) is a Cauchy sequence in \( S^2_p(M) \). Hence, \( v_j \to v \) in \( S^2_p(M) \) due to the completeness of \( (S^2_p(M), \| \cdot \|_{S^2_p(M)}) \). Since \( L_3 : S^2_p(M) \to L^p(M) \) is a continuous map,
\[
\| L_3 v - f \|_{L^p(M)} \leq \| L_3 v - L_3 v_j \|_{L^p(M)} + \| L_3 v_j - f \|_{L^p(M)} = \| L_3 v - L_3 v_j \|_{L^p(M)} + \| f_j - f \|_{L^p(M)} \to 0 \quad (4.13)
\]
as \( j \to +\infty \). Consequently, \( L_3 v = f \). Hence, \( L_3 : S^2_p(M) \to L^p(M) \) is a bijective continuous linear map, and thus \( L_3 \) has a continuous inverse map due to the open mapping theorem of Banach.

Using the inverse function theorem for Banach spaces and the regularity results in [46], we have the following theorem.

**Theorem 4.2** Let \( (M^{2n+1}, H, J, \theta) \) be a compact pseudo-Hermitian manifold, and let \( T \) be the operator defined as in (4.1) with \( \rho \in C^\infty(M) \). Assume that \( 0 < u_0 \in C^\infty(M), \rho > 2n + 1 \). If the linearization \( T'(u_0) : S^2_p(M) \to L^p(M) \) is injective (and thus is invertible), then there exists a constant \( \eta > 0 \) such that for any function \( f \in C^\infty(M) \) with \( \| f - T(u_0) \|_{L^p(M)} < \eta \), there is a positive function \( u \in C^\infty(M) \) satisfying \( T(u) = f \).

According to [27, Chapter 5, Theorem 4.10, p.291], the spectrum of the self-adjoint operator \( A(u) \) depends continuously on \( u \in \{u \in C^\infty(M) : u > 0\} \). Furthermore, from the proof of Lemma 3.1 (2), we know that the resolvent of the self-adjoint operator \( A(u) \) is compact for any positive function \( u \in C^\infty(M) \). Therefore, the spectrum of \( A(u) \) is discrete, and every eigenvalue is of finite multiplicity. In addition, given a function \( z \in C^\infty(M) \), the self-adjoint operator \( A(u + tz) \) depends analytically on \( t \) for \( |t| \) small enough, hence so do the eigenvalues and eigenfunctions of \( A(u + tz) \) (cf. [28]). After a process similar to Theorem 4.5 and Lemma 4.6 in [30], we have the following perturbation theorem for \( T' \).

**Theorem 4.3** The second-order linear degenerate elliptic operator \( T'(u) : S^2_p(M) \to L^p(M) \) is bijective on an open dense subset of the set \( \{u \in C^\infty(M) : u > 0\} \).

For our purpose in this section, we need the following approximation theorem (cf. Theorem 2.1 of [30]).

**Theorem 4.4** Let \( N \) be a connected manifold with dimension \( n \geq 2 \) and let \( f \in C(N) \cap L^p(N) \). Then a function \( g \in L^p(N) \) is in the \( L^p \)-closure of \( O_f \) if and only if \( \inf_N f \leq g(x) \leq \sup_N f \) for almost all \( x \in N \). Here \( O_f \) is the orbit of \( f \) under the group of diffeomorphism of \( N \).

Making use of the above three theorems yields the key lemma as follows.
Lemma 4.5 For a given smooth function $\hat{\rho}$ on a compact pseudo-Hermitian manifold $(M^{2n+1}, H, J, \theta)$ with the Webster scalar curvature $\rho$, if $\min_M \hat{\rho} < C \rho < \max_M \hat{\rho}$ for some constant $C > 0$, then $\hat{\rho} \in CE(\theta)$.

Proof Let $u_0 \equiv 1$. According to Theorem 4.3, for any $\epsilon > 0$, there exists a smooth function $u_1$ so close to $u_0$ that

$$\|T(u_1) - \rho\|_\infty = \|T(u_1) - T(u_0)\|_\infty < \epsilon, \quad (4.14)$$

and $T'(u_1)$ is invertible. Picking $\epsilon$ sufficiently small and using the assumption $C^{-1} \min_M \hat{\rho} < \rho < C^{-1} \max_M \hat{\rho}$ yield

$$C^{-1} \min_M \hat{\rho} < T(u_1) < C^{-1} \max_M \hat{\rho}. \quad (4.15)$$

By Theorem 4.4, we obtain that for any $\eta > 0$, there exists a diffeomorphism $\Phi_1$ of $M$ such that

$$\|C^{-1} \hat{\rho} \circ \Phi - T(u_1)\|_{L^p(M)} < \eta. \quad (4.16)$$

Making use of Theorem 4.2, we get that there exists a positive solution $u \in C^\infty(M)$ of $T(u) = C^{-1} \hat{\rho} \circ \Phi$. Set $v = C^{-\frac{1}{n-1}} u$, then $v$ is a positive solution of $T(v) = \hat{\rho} \circ \Phi$. Therefore, $\hat{\rho} \in CE(\theta)$. \qed

In terms of the above key lemma, we can give the proof of Theorem 1.2.

Proof of Theorem 1.2 (1) Since $\lambda_1 < 0$ and Theorem 3.4, there is a pseudo-Hermitian structure $\theta_1 \in [\theta]$ such that the corresponding Webster scalar curvature equals -1. If $\hat{\rho}$ is negative somewhere, then $\min_M \hat{\rho} < -C < \max_M \hat{\rho}$ for some constant $C > 0$. According to Lemma 4.5, we have $\hat{\rho} \in CE(\theta_1) = CE(\theta)$. Conversely, if $\hat{\rho} \in CE(\theta)$, then there exists a diffeomorphism $\Phi$ of $M$ and a positive function $u \in C^\infty(M)$ such that $Lu = (\hat{\rho} \circ \Phi) u^a$. Let $\psi$ be the positive eigenfunction associated with the eigenvalue $\lambda_1$ of $L$, then

$$0 > \lambda_1 \langle \psi, u \rangle_{L^2} = \langle L \psi, u \rangle_{L^2} = \langle \psi, Lu \rangle_{L^2} = \langle \psi, (\hat{\rho} \circ \Phi) u^a \rangle_{L^2}. \quad (4.17)$$

Consequently, $\hat{\rho}$ must be negative somewhere on $M$.

(2) From Proposition 3.10, it follows that $0 \in PC(\theta) \subset CE(\theta)$. Similar to part (1) of this theorem, we can get the conclusion easily.

(3) According to Theorem 1.1 and the results about CR Yamabe problem (cf. [13], [17], [25]), $\lambda_1 > 0$ implies that there is a positive constant $\rho_1 \in PC(\theta)$. By an argument similar to part (1) of this theorem, we can obtain that $\hat{\rho} \in CE(\theta)$ if and only if $\hat{\rho}$ is positive somewhere. \qed

Before the end of this section, we point out that the sign of $\lambda_1$ is invariant under CR conformally equivalent deformations.

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Theorem 4.6 Let \((M^{2n+1}, H, J, \theta)\) be a compact pseudo-Hermitian manifold. If the structures \((\hat{H}, \hat{J}, \hat{\theta})\) and \((H, J, \theta)\) are CR conformally equivalent, then \(\lambda_1(\theta)\) and \(\lambda_1(\hat{\theta})\) have the same sign or are both zero.

**Proof** Since \((\hat{H}, \hat{J}, \hat{\theta})\) and \((H, J, \theta)\) are CR conformally equivalent, then there is a map \(\Phi \in \text{Diff}(M)\) and \(0 < u \in C^\infty(M)\) such that

\[
\Phi^* \hat{\theta} = u^2 \hat{\theta}, \quad \hat{H} = d\Phi(H), \quad \hat{J} = d\Phi \circ J \circ (d\Phi)^{-1}.
\]  

(4.18)

Set \(\tilde{\theta} = u^2 \theta\), hence \(\Phi : (M, H, J, \theta) \rightarrow (M, \hat{H}, \hat{J}, \hat{\theta})\) is a CR isomorphism with \(\Phi^* \tilde{\theta} = \tilde{\theta}\). Therefore, \(\lambda_1(\tilde{\theta}) = \lambda_1(\theta)\). From Corollary 3.13, it follows that \(\lambda_1(\tilde{\theta})\) and \(\lambda_1(\theta)\) have the same sign or are both zero. \(\Box\)

**Appendix**

In this section, we give an alternative proof of Theorem 3.4 by following the spirit of [30]. Although the expressions are different, the following theorem and Theorem 3.4 are completely equivalent.

Theorem 4.7 Let \((M^{2n+1}, H, J, \theta)\) be a compact pseudo-Hermitian manifold. Then \(S = \{ f \in C^\infty(M) : f < 0 \} \subset \text{Im} T\) if and only if \(\lambda_1 < 0\), where \(T\) is as in (4.1) with \(\rho \in C^\infty(M)\), and \(\text{Im} T = \{ Tu : 0 < u \in S^2_\rho(M) \}\).

**Proof** If \(S \subset \text{Im} T\), then \(-1 \in \text{Im} T\), i.e., \(T(u) = -1\) for some \(C^\infty(M)\) function \(u > 0\). Thus, \(Lu = -u^a\). Let \(\psi\) be the positive eigenfunction of \(L\) with respect to \(\lambda_1\), namely, \(L\psi = \lambda_1 \psi\). So we have the following

\[
\lambda_1(\psi, u)_{L^2} = \langle L\psi, u \rangle_{L^2} = \langle \psi, Lu \rangle_{L^2} = -\langle \psi, u^a \rangle_{L^2} < 0
\]

(4.19)

which implies \(\lambda_1 < 0\).

For the converse, assume that \(\lambda_1 < 0\). Set \(K = S \cap \text{Im} T\). From \(L\psi = \lambda_1 \psi\), we have \(T(\psi) = \lambda_1 \psi^{1-a} < 0\), thus \(\lambda_1 \psi^{1-a} \in K\). Consequently, \(K\) is nonempty. Clearly, \(S\) is connected. In order to prove \(K = S\), it is sufficient to prove \(K\) is a both open and closed subset in \(S\). For openness, we will show that for any \(u > 0\), \(T(u) \in K\) implies \(\ker T'(u) = \ker A(u) = 0\), where \(A(u)\) is defined by (4.4), and thus \(K\) is open subset of \(S\) in terms of Theorem 4.2. Let \(\mu_1\) be the first eigenvalue of \(A(u)\) and \(\phi\) be the corresponding positive eigenfunction. By (4.4), we have

\[
\mu_1(\phi, u)_{L^2} = \langle A(u)\phi, u \rangle_{L^2} = \langle \phi, A(u)u \rangle_{L^2} = \frac{1-a}{b_n} \langle \phi, T(u)u^a \rangle_{L^2} > 0
\]

(4.20)

which gives \(\mu_1 > 0\), and so \(\ker A(u) = 0\). For closeness, we assume that \(f_j \in K\) and \(f_j \xrightarrow{C^0} f \in S\), we need prove \(f \in K\), i.e., there is \(u \in S^2_\rho(M)\) such that \(T(u) = f\). Since \(f_j \in K\), there exists a function \(0 < u_j \in C^\infty(M)\) satisfying \(T(u_j) = f_j\). Let
\[
    w_j = \log \frac{u_j}{\psi},
\]
where \( \psi \) is the positive eigenfunction associated with the first eigenvalue \( \lambda_1 \) of \( L \). Then \( w_j \) satisfies
\[
    -b_n \Delta_\theta w_j - b_n \nabla^H w_j \cdot \left( \nabla^H w_j + 2 \frac{\nabla^H \psi}{\psi} \right) = -\lambda_1 + f_j \psi^{a-1} e^{(a-1)w_j} \quad (4.21)
\]
where \( \cdot \) is the inner product induced by the Webster metric \( g_\theta \). Considering the maximum and minimum of \( w_j \) and using the classical maximum principle, it is easy to show that there are two constants \( m_1, m_2 > 0 \) independent of \( j \) such that
\[
    0 < m_1 \leq u_j \leq m_2.
\]
Hence, applying Lemma 4.1 to the operator \( L_4 = -\Delta_\theta + id \), we have
\[
    \| u_j \|_{S_2^p(M)} \leq C \| L_4 u_j \|_{L^p(M)} = C \left\| \frac{1}{b_n} (f_j u_j^a - \rho u_j) + u_j \right\|_{L^p(M)} \leq \hat{C} \quad (4.22)
\]
where \( C, \hat{C} \) are constants independent of \( j \). Using the compactly embedding theorem \( S_2^p(M) \subset W^{1,p}(M) \subset C^0(M) \) (\( p > 2n + 1 \), cf. Theorem 19.1 of [11]), there exists a subsequence \( \{u_{jk}\} \) such that \( u_{jk} \overset{C^0}{\to} u \) as \( k \to +\infty \), where \( u > 0 \) in \( M \) since \( 0 < m_1 \leq u_{jk} \leq m_2 \). Moreover, the subsequence \( \{u_{jk}\} \) is a Cauchy sequence in \( S_2^p(M) \), because
\[
    \| u_{jk} - u_{jl} \|_{S_2^p(M)} \leq C \| L_4(u_{jk} - u_{jl}) \|_{L^p(M)} = C \left\| (u_{jk} - u_{jl}) + \frac{1}{b_n} (f_{jk} u_{jk}^a - f_{jl} u_{jl}^a + \rho u_{jl} - \rho u_{jk}) \right\|_{L^p(M)} \to 0, \quad (4.23)
\]
as \( k, l \to +\infty \). Therefore, \( u_{jk} \to u \) in \( S_2^p(M) \) as \( k \to +\infty \). Let \( k \to \infty \) in \( T(u_{jk}) = f_{jk} \), by the continuity of \( T : S_2^p(M) \to L^p(M) \), we obtain \( T(u) = f \), so \( f \in K \). \( \square \)

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