ON THE NOTION OF A BASIS OF A FINITE DIMENSIONAL VECTOR SPACE

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Abstract

In this Note, we show that the notion of a basis of a finite-dimensional vector space could be introduced by an argument much weaker than Gauss’ reduction method. Our aim is to give a short proof of a simply formulated lemma, which in fact is equivalent to the theorem on frame extension, using only a simple notion of the kernel of a linear mapping, without any reference to special results, and derive the notions of basis and dimension in a quite intuitive and logically appropriate way, as well as obtain their basic properties, including a lucid proof of Steinitz’s theorem.

1. Introduction

To introduce the notion of a basis of a finite dimensional vector space the following Proposition or its equivalent should be proved:

Given two $n$-frames (sequence of linearly independent vectors) $\|e_j\|_n, \|f_j\|_n$ in a vector space, the vector $f_j$ contained in the linear span of $\|e_j\|_n$, $f_j \in [\|e_j\|_n], j = 1, \ldots, n$, there exists a uniquely defined invertible $n \times n$-matrix of scalars $A$, satisfying the equations

$$\|f_j\|_n = \|e_j\|_n A \iff \|e_j\|_n = \|f_j\|_n A^{-1}.$$ 

Existence and uniqueness of the matrix $A$ is evident, but proving its invertibility is always a delicate pedagogical problem for every author of an introductory text on Linear Algebra, since it should be given right at the beginning of the exposition, after the vector space axioms are listed.

A standard argument of proving the Proposition is based (explicitly or implicitly) on Gauss’ reduction process of transforming a square matrix to a triangular form, often combined with the Steinitz’s theorem on frame extension.

In this Note we suggest, instead of solving the above equations, to solve the system of inclusions

$$e_i \in [\|f_j\|_n], i = 1, \ldots, n,$$

which is equivalent under given conditions to the above equations. The proof is short and based on an argument much weaker than the reduction process - additionally to the list of axioms we only need the notion of the kernel of a linear mapping, without any special constructions performed on the data.

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2. Preliminary remarks

The logical procedure of introducing a basis of a finite dimensional vector space is completely equivalent to that of inverting a nondegenerate square matrix of scalars and is reduced to proving the following proposition or its equivalent, after which the path to the notions of a basis and dimension are almost uniquely determined and short.

Proposition. If two $n$-frames of a vector space (not necessarily finite dimensional), $\|e_j\|_n$ and $\|f_j\|_n$, are given, where the vectors $f_j$, $j = 1, \ldots, n$, are contained in the linear span $\langle \|e_j\|_n \rangle$, then there exists an invertible $n \times n$ matrix of scalars $A$ such that

\[
\|f_j\|_n = \|e_j\|_n A \iff \|e_j\|_n = \|f_j\|_n A^{-1}.
\]

The existence and uniqueness of $A$ are evident since the sequences $\|e_j\|_n$, $\|f_j\|_n$ are linearly independent, but proving its invertibility is always a delicate pedagogical problem for every author of an introductory text on Linear Algebra, since the proof should be given right at the beginning of the exposition, after the vector space axioms are listed, very often even before linear mappings are introduced. A standard way of solving the problem is to apply to $A$ (implicitly or explicitly) Gauss’ method of reduction of a square matrix to a triangular form, often combined with Steinitz’s theorem on a frame extension. In this respect, it is instructive to compare expositions of the corresponding material in four texts [1], [2], [3] and [4], the last three published after [1] with the time lag of more than 50 years. In this note, instead of solving equations (1), we prove the system of inclusions

\[
e_i \in \langle \|f_j\|_n \rangle, \quad i = 1, \ldots, n,
\]

which is equivalent to (1) since $\|e_j\|_n$, $\|f_j\|_n$ are linearly independent. The proof is much shorter and based on an argument much weaker than the reduction process. Additionally to the list of axioms we only need the notion of the kernel of a linear mapping, without any special constructions performed on the data.

In section 3, inclusions (2) are presented and discussed as a short and simply formulated Basic Lemma, from which, in sections 4 - 6, the notions of a basis and dimension of a finite dimensional vector space along with their basic properties (including a lucid proof of the frame extension theorem) are derived in a quite intuitive and logically proper way. Finally, in Section 7, the Basic Lemma is proved. In the next Section 2 we list several necessary initial notions used in this note. To conclude the section, we should remark that in most texts on introductory Linear Algebra, linear mappings are introduced too late, contrary to a well known motto, according to which “morphisms in a category are at least as useful as the objects are”. We think that linear mappings and their basic properties should be exposed in introductory texts right after the vector space axioms are listed and properly discussed, and consider this Note as a proper example supporting our opinion.

3. A short list of necessary initial notions

Together with the standard general set-theoretic notions related to a mapping $L$,

\[
dom L = E, \ codom L = F, \ im L = L(E),
\]
it is useful to have at the disposal from the very beginning specific linear notions of a (linear) subspace \( F \subset E \) and the factor \( E/F \), in particular, the notions of the kernel \( \ker L \) and factor \( E/\ker L \).

For an adequate definition of the basis and its appropriate discussion, we should have a certain freedom in handling finite sequences of vectors in \( E \). An arbitrary sequence of length \( n \) of vectors in \( E \) is presented as an \( n \)-row matrix,

\[
\|x_1, \ldots, x_n\| = \|x_j\|_n = \|x_j\|, \ x_j \in E, \ j = 1, \ldots, n.
\]

Every sequence of vectors could be extended by ascribing to it new vectors from the space. It is useful to remember that a sequence of length \( n \) of vectors in \( E \) is not a subset of \( E \), but rather a function to \( E \) on the ordered set of first \( n \) naturals, where \( \text{im}||x_j|| \subset E \). If \( \text{im}||x_j|| \) is a subset of a subspace \( F \subset E \) we say that the sequence \( ||x_j|| \) belongs to \( F \) and write \( ||x_j|| \prec F \).

The initial list of basic items should also contain the following notions: linear combination of vectors of a sequence, linearly independent and dependent sequences of vectors of \( E \), \( n \) - frames — linearly independent sequences of length \( n \), or rank \( n \),

\[
\|e_j\| = \|e_j\|_n, \ \text{rank} \|e_j\|_n = n.
\]

We also need the following notions: the rank of an arbitrary sequence, \( \text{rank} ||x_j|| \), \( ||x_j|| \prec E \), defined as the maximal rank of frames contained (as subsequences) in \( ||x_j|| \), the linear span \( \|\|x_j||\| \subset E \) of an arbitrary sequence of vectors \( ||x_j|| \prec E \), linear span of an arbitrary subset \( A \subset E \) — minimal subspaces in \( E \) (with respect to the set-theoretic inclusion) containing, respectively, subsets \( \text{im}||x_j|| \) and \( A \).

4. **The Basic Lemma. Formulation and a Preliminary Discussion**

A frame \( ||e_j|| \prec F \subset E \) is *maximal in a subspace* \( F \) if the rank of an arbitrary sequence \( ||x_j|| \prec F \) is bounded by the rank of \( ||e_j|| \),

\[
||x_j|| \prec F \implies \text{rank} \|e_j\| \geq \text{rank} \|x_j\|.
\]

A frame \( \|e_j\|_n \prec F \) could be extended (is extendable) in \( F \), if there exists a vector \( f \in F \) such that the extended sequence \( e_1, \ldots, e_n, f \) is again a frame (of rank \( n+1 \)). Intuitively, maximality of a frame could be considered as a “generalized version” of the “individual property” of a frame to be non-extendable. Every maximal frame in \( F \) is evidently not extendable. The inversion of the assertion is also true — every non-extendable frame in \( F \) is maximal in \( F \) (Steinitz’s extension theorem), but the proof is not trivial and in fact belongs right to the core of the problem under discussion — of giving proper definitions of a basis and dimension of a finite dimensional vector space.

Now we shall formulate a simple lemma, which easily clears up all interrelations between the notions of maximality of a frame and its ability “to be extendable”, and suggests natural intuitive definitions of a basis and dimension.

**The Basic Lemma.** Every frame \( ||e_j|| \prec E \) is maximal in its linear span \( \|\|e_j||\| \) and extendable in every subspace \( F \supset \|\|e_j||\| \), if the inclusion is strong.

The ability of being “extendable” under the given conditions is evident — it is achieved by ascribing to \( \|e_j|| \) an arbitrary vector \( f \in F \), \( f \notin \|\|e_j||\| \), the maximality of \( \|e_j|| \) in the linear span \( \|\|e_j||\| \) is equivalent to each of the following two assertions.
1. Every frame $\|f_j\|_n$ of rank $n$ in the linear span $[\|e_j\|_n]$ is not extendable there, or, equivalently, the following system of $n$ inclusions is valid,

$$e_i \in [\|f_j\|_n], \; i = 1, \ldots, n.$$  \hspace{1cm} (3)

2. The rank of an arbitrary sequence in $E$ consisting of linear combinations of a fixed sequence of $n$ vectors from $E$ does not exceed $n$.

A simple inductive proof of the system of inclusions (3), based on the notion of the kernel of a linear mapping, is given (as already stated) in the final Section 6. Before, in next two Sections, we shall derive simple and intuitive definitions of a basis and dimension of a finite dimensional vector space from the formulated lemma, as well as give a lucid proof of Steinitz’s theorem on the frame extension.

5. Definition of a basis and dimension of a finite dimensional vector space

A vector space $E$ is finite dimensional if it contains a finite set of generators — a finite subset $G \subset E$ with the linear span coinciding with $E$. Since $G$ is finite, it contains a frame $\|e_j\|_n \prec G$ (of maximal rank in $G$), which spans the whole space $E = [\|e_j\|_n]$, hence, according to the lemma, the frame is maximal in $E$. Hence every finite dimensional vector space contains maximal frames — the bases of the space. Their common rank $n$ is the dimension of the space, and every vector $x \in E$ is uniquely represented as

$$x = \sum_{i=1}^{n} \lambda^\alpha e_\alpha, \; \lambda^j \in \Lambda, \; j = 1, \ldots, n.$$  

Thus, every basis of a finite dimensional vector space is an irreducible system of generators of the space.

Conversely, if a sequence $\|x_j\|_n$ in a vector space $E$ is given such that every vector $x \in E$ is uniquely represented as

$$x = \sum_{i=1}^{n} \lambda^\alpha x_\alpha, \; \lambda^j \in \Lambda,$$

then, according to the lemma, $\|x_j\|_n$ is a maximal frame of rank $n$, or a basis in $E$.

6. The Steinitz’ theorem on the frame extension

In $k + l$-dimensional vector space $E^{k+l}$ a basis $B = \|e_j\|_{k+l}$ and an arbitrary $k$-frame $\|f_j\|_k$ are given. From general considerations it easily follows by induction that $B$ contains a subsequence $e_{i_1}, \ldots, e_{i_r}$ of a certain length $r \leq k + l$ such that the frame $\|f_j\|_k$ extended by the subsequence is a frame of length $k + r$,

$$B' = \|f_1, \ldots, f_k; e_{i_1}, \ldots, e_{i_r}\| \prec E^{k+l},$$

containing the linear span of $B$, which is the whole space $E^{k+l}$, hence

$$[B'] = E^{k+l}.$$  

According to the lemma, the obtained frame $B'$ is maximal in $E^{k+l}$, i.e. is a basis of the $k + l$-dimensional vector space $E^{k+l}$, hence $r = l$, and we have extended the initial $k$-frame $\|f_j\|_k$ by a subsequence of length $r = l$ of a preassigned basis $B$ to a new basis $B'$ of the space $E^{k+l}$. 

7. The basic lemma. Proof

We shall prove the system of inclusions (2) by induction performed on the rank \( n \) of the frame \( \|e_j\|_n \). The assertion is evident for \( n = 1 \) since in this case the linear span coincides with the family of vectors
\[
\|e_j\|_1 = \{ \lambda e_1 | \lambda \in \Lambda \}.
\]
Assuming that the assertion is proved for all natural \( k \leq n - 1 \), consider the case \( k = n \). Introduce \( n \) linear mappings
\[
L_i \in Hom \left( \|e_j\|_n, \|f_1, \ldots, \hat{f}_i, \ldots, f_n\| \right), \quad i = 1, \ldots, n,
\]
by defining \( L_i \) on vectors \( e_1, \ldots, e_n \) according to the equations
\[
L_i e_j = f_j, \quad i \neq j, \quad L_i e_i = 0, \quad i, j = 1, \ldots, n.
\]
The kernel of \( L_i \) coincides with the subspace
\[
ker L_i = \{ \lambda e_i | \lambda \in \Lambda \},
\]
and the image — with the linear span
\[
im L_i = [\|f_1, \ldots, \hat{f}_i, \ldots, f_n\|].
\]
The restriction of \( L_i \) on the subspace \( [\|f_j\|_n] \subset [\|e_j\|_n] \) has a nonzero kernel since otherwise the \( n \)-frame \( [L_i f_j], j = 1, \ldots, n \) would be embedded in the linear span of an \( (n - 1) \)-frame \( [f_1, \ldots, \hat{f}_i, \ldots, f_n] \), which contradicts the inductive assumption. Hence,
\[
\lambda e_i \in ker L_i \big|_{[\|f_j\|_n]} \quad \forall \lambda \in \Lambda, \quad i = 1, \ldots, n,
\]
or
\[
e_i \in [\|f_j\|_n] \quad \forall i = 1, \ldots, n.
\]
This proves the Lemma.

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