FRACTIONAL ORDER PSEUDOPARABOLIC PARTIAL DIFFERENTIAL EQUATION: ULAM-HYERS STABILITY

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Abstract. Using Gronwall inequality we will investigate the Ulam-Hyers and generalized Ulam-Hyers-Rassias stabilities for the solution of a fractional order pseudoparabolic partial differential equation.

Keywords: Pseudoparabolic fractional partial differential equation, ψ-Hilfer fractional partial derivative, Ulam-Hyers stability, generalized Ulam-Hyers-Rassias stability.

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1. Introduction

In 1695, Leibniz formulated a question, addressed to l’Hospital, involving a possible generalization of the derivative of whole order to a derivative of order, in principle, arbitrary, and may even be complex. l’Hospital returned the question to Leibniz, questioning him in the case where the order of the derivative was middle and what a possible interpretation might be. In an audacious and prophetic response, Leibniz presents the result and states this is apparently a paradox that one day will generate several important consequences. Thus, the fractional calculus (FC) begins, and since then numerous derivative and integral formulations have been introduced [18, 21, 23, 24, 29, 30, 32, 33]. Among the various formulations, recently Sousa and Oliveira [32], introduced the so-called ψ-Hilfer fractional derivative of a single variable that generalizes a wide class of other formulations of fractional derivatives such as: Riemann-Liouville, Caputo, Hilfer, Riesz and other more recent, for example, generalized Caputo derivative [28]. In order to study the stability of solutions of partial differential equations by means of a fractional derivative, especially the ψ-Hilfer type, there is a need to extend the definition to N variables [34].

We mention a history, similar to FC, how the first idea of stability of functional equations came about. In 1940, Ulam and Hyers exchanged correspondences on the stability study of solutions of differential equations. Since then, this theme, has been a motivator for many researchers, especially mathematicians [22, 39]. Subsequently, this type of stability has come to be called Ulam-Hyers stability. Study the various types of stability, be they of the type Ulam-Hyers, Ulam-Hyers-Rassias, semi-Ulam-Hyers-Rassias, Ulam-Hyers-Mittag-Leffler, δ-Ulam-Hyers-Rassias [6, 7, 11, 12, 13, 14, 15, 19, 26, 35, 36, 37, 38, 40] of solutions of partial and/or ordinary differential equations, by means of fractional derivatives, has been
Several researchers started to focus on the study of Ulam-Hyers and generalized Ulam-Hyers-Rassias stabilities, among others. However, we have mentioned that Abbas and Benchohra [1, 5, 13] are researchers whose study of stabilities is directed to solutions of fractional partial differential equations. Among some works that these authors have performed, we highlight: the study involving partial differential equations of the hyperbolic type and partial differential equations with delay in time, [1, 2, 3, 4, 5, 8, 9, 13, 16]. In this sense, numerous studies were carried out and FC becomes a new area of application and consequently gains more space within the mathematical analysis. Other researchers such as Long et al. [25], Ahmad et al. [17], Choung et al. [20] and Zhang [41] have published works related to the stability of solutions of partial differential equations, in some cases involving fractional neutral stochastic partial integro-differential equations.

In this paper we will consider the following fractional order pseudoparabolic partial differential equation

\[
(1.1) \quad \frac{\partial^{3\alpha}_{\beta,\psi} u}{\partial \beta^{\alpha}_{\psi} x^{2\alpha} \partial \beta^{\alpha}_{\psi} y^{\alpha}} (x, y) = f \left( x, y, u(x, y), \frac{\partial^{\alpha}_{\beta,\psi} u}{\partial \beta^{\alpha}_{\psi} y^{\alpha}} (x, y), \frac{\partial^{2\alpha}_{\beta,\psi} u}{\partial \beta^{\alpha}_{\psi} x^{2\alpha}} (x, y) \right),
\]

where \( \frac{\partial^{3\alpha}_{\beta,\psi} u}{\partial \beta^{\alpha}_{\psi} x^{2\alpha} \partial \beta^{\alpha}_{\psi} y^{\alpha}} (\cdot, \cdot) \) is the \( \psi \)-Hilfer fractional partial derivative [34] with the parameters \( \frac{2}{3} < \alpha \leq 1, \ 0 \leq \beta \leq 1 \) and \( 0 \leq x < a, \ 0 \leq y < b \), being \( f \in C \left( [0, a) \times [0, b) \times \mathbb{B}^3, \mathbb{B} \right) \) and \((\mathbb{B}, |\cdot|)\) a real or complex Banach space.

The main motivation of this paper is to present a study on the Ulam-Hyers and generalized Ulam-Hyers-Rassias stabilities of the solution of a fractional order pseudoparabolic partial differential equation. For this purpose, we use the \( \psi \)-Hilfer fractional derivative of \( N \) variables and the Gronwall inequality, in order to contribute to the study of stabilities and provide a new and interesting result for future research.

The paper is organized as follows: In section 2, we will present the definition of the \( \psi \)-Riemann-Liouville fractional integral of a function relative to another function of \( N \) variables and the \( \psi \)-Hilfer fractional partial derivative. Moreover, through the \( \psi \)-Hilfer fractional partial derivative, we will present a new version for the definition of Ulam-Hyers and Ulam-Hyers-Rassias stabilities, the Gronwall inequality and some remarks. In section 3, our main result, we will study the Ulam-Hyers and generalized Ulam-Hyers-Rassias stabilities of the solution of a fractional order pseudoparabolic partial differential equation. Concluding remarks close the paper.
2. Preliminaries

In this section we will present the definition of \( \psi \)-Riemann-Liouville fractional integral and \( \psi \)-Hilfer fractional derivative of \( N \) variables, as well as Gronwall’s lemma, fundamental in the study of solutions of differential equations. In this sense, the definitions of Ulam-Hyers and generalized Ulam-Hyers-Rassias stabilities are introduced in an adapted version associated with the type of partial differential equation to be studied.

First, we present the definition of \( \psi \)-Riemann-Liouville fractional integral, fundamental to the \( \psi \)-Hilfer fractional derivative approach.

**Definition 1.** [34] Let \( \theta = (\theta_1, \theta_2, ..., \theta_N) \) and \( \alpha = (\alpha_1, \alpha_2, ..., \alpha_N) \), where \( 0 < \alpha_1, \alpha_2, ..., \alpha_N < 1 \), \( N \in \mathbb{N} \). Also put \( \widetilde{I} = I_1 \times I_2 \times \cdots \times I_N = [\theta_1, a_1] \times [\theta_2, a_2] \times \cdots \times [\theta_N, a_N] \), where \( a_1, a_2, ..., a_N \) and \( \theta_1, \theta_2, ..., \theta_N \) are positive constants. Also let \( \psi (\cdot) \) be an increasing and positive monotone function on \((\theta_1, a_1], (\theta_2, a_2], ..., (\theta_N, a_N] \), having a continuous derivative \( \psi' (\cdot) \) on \((\theta_1, a_1], (\theta_2, a_2], ..., (\theta_N, a_N] \). The \( \psi \)-Riemann-Liouville partial integral of \( N \) variables \( u = (u_1, u_2, ..., u_N) \in L^1 (\tilde{I}) \) is defined by

\[
I_{\theta, x}^{\alpha, \psi} u(x) = \frac{1}{\Gamma (\alpha_j)} \int_a \int \cdots \int_{a_j} \psi' (s_j) (\psi (x_j) - \psi (s_j))^{\alpha_j-1} u(s_j) ds_j,
\]

with \( \psi' (s_j) (\psi (x_j) - \psi (s_j))^{\alpha_j-1} = \psi' (s_1) (\psi (x_1) - \psi (s_1))^{\alpha_1-1} \psi' (s_2) (\psi (x_2) - \psi (s_2))^{\alpha_2-1} \cdot \cdot \cdot \psi' (s_N) (\psi (x_N) - \psi (s_N))^{\alpha_N-1} \), and using the notation \( \Gamma (\alpha_j) = \Gamma (\alpha_1) \Gamma (\alpha_2) \cdots \Gamma (\alpha_N) \), \( u(s_j) = u(s_1) u(s_2) \cdots u(s_N) \) and \( ds_j = ds_1 ds_2 \cdots ds_N \), \( j \in \{1, 2, ..., N\} \) with \( N \in \mathbb{N} \).

From the fractional partial integral Eq.\( (2.1) \), it is possible to obtain other fractional partial integrals, that is Erdelyi-Kober fractional partial integral, Katugampola fractional partial integral, Weyl fractional partial integral, among others. In addition, each fractional partial integral obtained here is an extension of its respective fractional integral [21, 24, 30, 32].

In particular, taking \( N = 2 \) and \( \theta_1 = \theta_2 = 0 \) in Eq.\( (2.1) \) we have the fractional partial integral that will be used in what follows,

\[
I_{\theta}^{\alpha, \psi} u(x_1, x_2) = \frac{1}{\Gamma (\alpha_1) \Gamma (\alpha_2)} \int_0^{x_1} \int_0^{x_2} \psi' (s_1) \psi' (s_2) (\psi (x_1) - \psi (s_1))^{\alpha_1-1} (\psi (x_2) - \psi (s_2))^{\alpha_2-1} u(s_1, s_2) ds_1 ds_2,
\]

(2.2)

with \( 0 < \alpha_1, \alpha_2 < 1 \).

Also, we have

\[
I_{0+}^{\alpha_1, \psi} u(x_1, x_2) = \frac{1}{\Gamma (\alpha_1)} \int_0^{x_1} \psi' (s_1) (\psi (x_1) - \psi (s_1))^{\alpha_1-1} u(s_1, s_2) ds_1
\]

(2.3)

and

\[
I_{0+}^{\alpha_2, \psi} u(x_1, x_2) = \frac{1}{\Gamma (\alpha_2)} \int_0^{x_2} \psi' (s_2) (\psi (x_2) - \psi (s_2))^{\alpha_2-1} u(s_1, s_2) ds_2,
\]

(2.4)
with $0 < \alpha_1, \alpha_2 \leq 1$.

Using the $\psi$-Riemann-Liouville fractional partial integral, we present the $\psi$-Hilfer fractional partial derivative. The following definition is an extension of the recent fractional derivative of a variable recently introduced by Sousa and Oliveira [32].

**Definition 2.** [34] Let $\theta = (\theta_1, \theta_2, \ldots, \theta_N)$ and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$, where $0 < \alpha_1, \alpha_2, \ldots, \alpha_N < 1$, $N \in \mathbb{N}$. Also put $\bar{I} = I_{a_1} \times I_{a_2} \times \cdots \times I_{a_N} = [\theta_1, a_1] \times [\theta_2, a_2] \times \cdots \times [\theta_N, a_N]$, where $a_1, a_2, \ldots, a_N$ and $\theta_1, \theta_2, \ldots, \theta_N$ are positive constants. Also let $u, \psi \in C^\alpha(\bar{I}, \mathbb{R})$ two functions such that $\psi$ is increasing and $\psi'(x_i) \neq 0$, $i \in \{1, 2, \ldots, N\}$, $x_i \in \bar{I}$, $N \in \mathbb{N}$. The $\psi$-Hilfer fractional partial derivative of $N$ variables denoted by $H_{\alpha,\beta;\psi}^\theta(\cdot)$ of a function, of order $\alpha$ and type $0 \leq \beta_1, \beta_2, \ldots, \beta_N \leq 1$, is defined by

\[
H_{\alpha,\beta;\psi}^\theta u(x) = I_{\alpha,\beta;\psi}^\theta u(x) = \int_{\theta}^{\theta + \psi} \frac{1}{\psi'(x_j)} \frac{\partial^N}{\partial x_j} I_{\alpha,\beta;\psi}^{(1-\beta)(1-\alpha)} u(x_j),
\]

with $\partial x_j = \partial x_1 \partial x_2 \cdots \partial x_N$ and $\psi'(x_j) = \psi'(x_1) \psi'(x_2) \cdots \psi'(x_N)$, $j \in \{1, 2, \ldots, N\}$, $N \in \mathbb{N}$.

In the same way that a large class of fractional partial integrals can be obtained, as particular cases, it is also possible for the $\psi$-Hilfer fractional partial derivative. This vast class will be omitted here, however we suggest the following papers [32, 34].

Taking $N = 2$ in Eq. (2.5), we present the partial fractional derivative that will be used in this paper,

\[
H_{\alpha,\beta;\psi}^\theta u(x_1, x_2) = I_{\alpha,\beta;\psi}^\theta u(x_1, x_2) = I_{\alpha,\beta;\psi}^{(1-\alpha)} \left(\frac{1}{\psi'(x_1)} \frac{\partial^2}{\partial x_1 \partial x_2} I_{\alpha,\beta;\psi}^{(1-\beta)(1-\alpha)} u(x_1, x_2)\right).
\]

Also, we use the following notation

\[
H_{\alpha,\beta;\psi}^\theta u(x_1, x_2) = \frac{\partial^\alpha_{\beta;\psi} u}{\partial \beta;\psi x_2 \partial x_1 \partial \beta;\psi y_\alpha}(x_1, x_2).
\]

We remark that, the following Gronwall lemma is an important tool in proving the main results of this paper.

**Lemma 1.** [31] One assumes that

1. $u, v, h \in C([a, b], \mathbb{R}_+)$;
2. For any $t \geq a$ and $\psi(t)$ is increasing and $\psi'(t)$ for all $t \in [a, b]$ one has

\[
u(t) \leq v(t) + h(t) \int_a^t \psi'(s) (\psi(t) - \psi(s))^{\alpha-1} u(s) ds,
\]

3. $h(t)$ is nonnegative and nondecreasing.
Then, we have
\[ u (t) \leq v (t) \mathbb{E}_\alpha [h (t) \Gamma (\alpha) (\psi (t) - \psi (0))^\alpha], \]
for any \( t \geq a \) and being \( \mathbb{E}_\alpha (\cdot) \) the one-parameter Mittag-Leffler function.

To facilitate the development of the calculations, we introduce the following notation:

1. \( \mathcal{W}^p_{v_1,v_2} f(x,y) := \int_0^p f(x,y, v(x,y), v_1(x,y), v_2(x,y)) \, dp; \)
2. \( \mathcal{W}^p_{u_1,u_2} f(x,y) := \int_0^p f(x,y, u(x,y), u_1(x,y), u_2(x,y)) \, dp; \)
3. \( \mathcal{W}^p_{v,u} f(x,y) := \int_0^p \left| f(x,y, v(x,y), v_1(x,y), v_2(x,y)) \right| \, dp \)
4. \( \mathcal{W}^p \varphi (x,y) := \int_0^p \varphi (x,y) \, dp; \)
5. \( \Psi^\gamma (x,0) := \frac{(\psi (x) - \psi (0))^{\gamma - 1}}{\Gamma (\gamma)}; \)
6. \( \Psi^\gamma (0,y) := \frac{(\psi (y) - \psi (0))^{\gamma - 1}}{\Gamma (\gamma)}; \)

All of the above items exist and are well defined.

Let \( a, b \in (0, \infty), \varepsilon > 0, \varphi \in C ([0,a] \times [0, b], \mathbb{R}_+) \) and \( (\mathbb{B}, | \cdot |) \) be a real or complex Banach space.

We consider the following inequalities
\[
(2.8) \quad \left| \frac{\partial^{2\alpha}_{x,y}}{\partial y^\alpha \partial x^{2\alpha}} \frac{\partial^2 v}{\partial x^2} (x,y) \right| \leq \varepsilon, \quad x \in [0,a), y \in [0,b); \\
(2.9) \quad \left| \frac{\partial^{2\alpha}_{x,y}}{\partial y^\alpha \partial x^{2\alpha}} \frac{\partial^2 v}{\partial x^2} (x,y) \right| \leq \varphi (x,y),
\]
where \( 2/3 < \alpha \leq 1 \) and \( 0 \leq \beta \leq 1 \).

Note that, a function \( u : [0,a] \times [0,b) \rightarrow \mathbb{B} \) is a solution of Eq. (1.1) if \( u \in C ([0,a] \times [0,b)) \cap C^1 ([0,a] \times [0,b]), \frac{\partial^{2\alpha}_{x,y}}{\partial y^\alpha \partial x^{2\alpha}} \in C ([0,a] \times [0,b]), \frac{\partial^{2\alpha}_{x,y}}{\partial y^\alpha \partial x^{2\alpha}} \in C ([0,a] \times [0,b]) \) and \( u \) satisfies the Eq. (1.1).

The following definitions were adapted for the \( \psi \)-Hilfer fractional derivative for two variables and we used in [27, 34].
Definition 3. The solution of Eq. (1.1) is Ulam-Hyers stability if there exist real numbers $C_1^{f}, C_2^{f}$ and $C_3^{f} > 0$ such that for any $\varepsilon > 0$ and for any solution $v$ to the inequality Eq. (2.8) with

$$|v(x, y) - u(x, y)| \leq C_1^f \varepsilon,$$

$$\left| \frac{\partial^\alpha_{\beta, \psi} v}{\partial_{\beta, \psi} y^\alpha}(x, y) - \frac{\partial^\alpha_{\beta, \psi} u}{\partial_{\beta, \psi} y^\alpha}(x, y) \right| \leq C_2^f \varepsilon,$$

$$\left| \frac{\partial^{2\alpha}_{\beta, \psi} v}{\partial_{\beta, \psi} x^{2\alpha}}(x, y) - \frac{\partial^{2\alpha}_{\beta, \psi} u}{\partial_{\beta, \psi} x^{2\alpha}}(x, y) \right| \leq C_3^f \varepsilon,$$

$x \in [0, a), y \in [0, b)$ with $\frac{2}{3} < \alpha \leq 1$ and $0 \leq \beta \leq 1.$

Definition 4. The solution of Eq. (1.1) admits generalized Ulam-Hyers-Rassias stability if there exist real numbers $C_1^{f, \varphi}, C_2^{f, \varphi}$ and $C_3^{f, \varphi} > 0$ such that for any $\varepsilon > 0$ and for any solution $v$ to the inequality Eq. (2.9) with

$$|v(x, y) - u(x, y)| \leq C_1^{f, \varphi} \varphi(x, y),$$

$$\left| \frac{\partial^\alpha_{\beta, \psi} v}{\partial_{\beta, \psi} y^\alpha}(x, y) - \frac{\partial^\alpha_{\beta, \psi} u}{\partial_{\beta, \psi} y^\alpha}(x, y) \right| \leq C_2^{f, \varphi} \varphi(x, y),$$

$$\left| \frac{\partial^{2\alpha}_{\beta, \psi} v}{\partial_{\beta, \psi} x^{2\alpha}}(x, y) - \frac{\partial^{2\alpha}_{\beta, \psi} u}{\partial_{\beta, \psi} x^{2\alpha}}(x, y) \right| \leq C_3^{f, \varphi} \varphi(x, y),$$

$x \in [0, a), y \in [0, b)$ with $\frac{2}{3} < \alpha \leq 1$ and $0 \leq \beta \leq 1.$

Remark 1. A function $v$ is a solution to the inequality Eq. (2.8) if, and only if, there exists a function $g \in C ([0, a) \times [0, b), \mathbb{B})$, which depends on $v$, such that

1. For all $\varepsilon > 0$, $|g(x, y)| \leq \varepsilon$, $\forall x \in [0, a), \forall y \in [0, b);$ (2) $\forall x \in [0, a), \forall y \in [0, b),$

$$\frac{\partial^{3\alpha}_{\beta, \psi} v}{\partial_{\beta, \psi} x^{2\alpha} \partial_{\beta, \psi} y^\alpha}(x, y) = f \left( x, y, v(x, y), \frac{\partial^{\alpha}_{\beta, \psi} v}{\partial_{\beta, \psi} y^\alpha}(x, y), \frac{\partial^{2\alpha}_{\beta, \psi} v}{\partial_{\beta, \psi} x^{2\alpha}}(x, y) \right) + g(x, y)$$

with $\frac{2}{3} < \alpha \leq 1$ and $0 \leq \beta \leq 1.$

Remark 2. A function $v$ is a solution to the inequality Eq. (2.9) if, and only if, there exists a function $g \in C ([0, a) \times [0, b), \mathbb{B})$, which depends on $v$, such that

1. $|g(x, y)| \leq \varphi(x, y), \forall x \in [0, a), \forall y \in [0, b);$
Then, we can write
\[
\frac{\partial^{2a}_{\beta;\psi}v}{\partial^{2a}_{\beta;\psi}x^2} (x, y) = f \left( x, y, v(x, y), \frac{\partial^{a}_{\beta;\psi}v}{\partial^{a}_{\beta;\psi}x} (x, y), \frac{\partial^{2a}_{\beta;\psi}v}{\partial^{2a}_{\beta;\psi}x^2} (x, y) \right) + g(x, y)
\]
with \( \frac{2}{3} < \alpha \leq 1 \) and \( 0 \leq \beta \leq 1 \).

**Remark 3.** If \( v \) is a solution to the inequality Eq. (2.8), then \((v, v_1, v_2)\), is a solution of the following system of integral inequalities:

\[
\left| v(x, y) - \Psi^\gamma (0, y) v(x, 0) - \Psi^\gamma (x, 0) v(0, y) - \Psi^\gamma (x, 0) xv_x (0, y) \right| \leq \frac{\varepsilon x}{\Gamma (\alpha_1 + 1) \Gamma (\alpha_2 + 1)} \left( \psi (x) - \psi (0) \right)^{\alpha_1} \left( \psi (y) - \psi (0) \right)^{\alpha_2};
\]

\[
\left| v_1(x, y) - \Psi^\gamma (x, 0) v_1(0, y) - \Psi^\gamma (x, 0) x v_1x (0, y) - I_{0+}^{\alpha_1;\psi} \left( \mathcal{W}^{p,v}_{v_1,v_2} f(x, y) \right) \right| \leq \frac{\varepsilon x}{\Gamma (\alpha_1 + 1)} \left( \psi (x) - \psi (0) \right)^{\alpha_1};
\]

\[
\left| v_2(x, y) - \Psi^\gamma (x, 0) v_2(0, y) - I_{0+}^{\alpha_2;\psi} \left( \mathcal{W}^{p,v}_{v_1,v_2} f(x, y) \right) \right| \leq \frac{\varepsilon x}{\Gamma (\alpha_1 + 1) \Gamma (\alpha_2 + 1)} \left( \psi (y) - \psi (0) \right)^{\alpha_2};
\]

where \( x \in [0, a) \), \( y \in [0, b) \), \( \frac{2}{3} < \alpha \leq 1 \), \( 0 \leq \gamma \leq 1 \), \( v_1 = \frac{\partial^{a}_{\beta;\psi}v}{\partial^{a}_{\beta;\psi}x} \) and \( v_2 = \frac{\partial^{2a}_{\beta;\psi}v}{\partial^{2a}_{\beta;\psi}x^2} \).

**Proof.** From Eq. (2.9) we have,

\[
v(x, y) - \Psi^\gamma (0, y) v(x, 0) + \Psi^\gamma (x, 0) v(0, y) + \Psi^\gamma (x, 0) x v_x (0, y) - I_{0+}^{\alpha_1;\psi} \left( \mathcal{W}^{p,v}_{v_1,v_2} f(x, y) \right) + I_{0+}^{\alpha_2;\psi} \left( \mathcal{W}^{p,v} \varphi (x, y) \right).
\]

Then, we can write

\[
\left| v(x, y) - \Psi^\gamma (0, y) v(x, 0) - \Psi^\gamma (x, 0) v(0, y) - \Psi^\gamma (x, 0) x v_x (0, y) \right| \leq I_{0+}^{\alpha_1;\psi} \left( \mathcal{W}^{p,v}_{v_1,v_2} f(x, y) \right);
\]

\[
\left| I_{0+}^{\alpha_1;\psi} \left( \mathcal{W}^{p,v} \varphi (x, y) \right) \right| \leq I_{0+}^{\alpha_1;\psi} \left( \int_0^p \varepsilon dp \right) = \frac{\varepsilon x}{\Gamma (\alpha_1 + 1) \Gamma (\alpha_2 + 1)} \left( \psi (x) - \psi (0) \right)^{\alpha_1} \left( \psi (y) - \psi (0) \right)^{\alpha_2};
\]

In this sense, we have the inequalities

\[
\left| v_1(x, y) - \Psi^\gamma (x, 0) v_1(0, y) - \Psi^\gamma (x, 0) x v_1x (0, y) - I_{0+}^{\alpha_1;\psi} \left( \mathcal{W}^{p,v}_{v_1,v_2} f(x, y) \right) \right| \leq \frac{\varepsilon x}{\Gamma (\alpha_1 + 1) \Gamma (\alpha_2 + 1)} \left( \psi (x) - \psi (0) \right)^{\alpha_1} \left( \psi (y) - \psi (0) \right)^{\alpha_2};
\]
and
\[
\left| v_2(x, y) - \Psi(0, y) v_2(x, 0) - I_{0^+}^{\alpha_2} f(x, y, v(x, y), v_1(x, y), v_2(x, y)) \right| \\
\leq \frac{\varepsilon (\psi(y) - \psi(0))^{\alpha_2}}{\Gamma(\alpha_2 + 1)}.
\]

Remark 4. If \( v \) is a solution of the Eq. (2.9), then \((v, v_1, v_2)\) is a solution of the following system of integral inequalities:
\[
\left| v(x, y) - \Psi(0, y) v(x, 0) - \Psi(0, x) v_0(x, 0) - I_{0^+}^{\alpha_1} \left( \mathcal{W}_{v_1, v_2} f(x, y) \right) \right| \\
\leq I_{0^+}^{\alpha_2} (\mathcal{W}^{p} \varphi(x, y));
\]
\[
\left| v_1(x, y) - \Psi(0, x) v_1(0, y) - \Psi(0, x) v_1(x, 0) - I_{0^+}^{\alpha_1} \left( \mathcal{W}_{v_1, v_2} f(x, y) \right) \right| \\
\leq I_{0^+}^{\alpha_1} (\mathcal{W}^{p} \varphi(x, y));
\]
\[
\left| v_2(x, y) - \Psi(0, y) v_2(x, 0) - I_{0^+}^{\alpha_2} f(x, y, v(x, y), v_1(x, y), v_2(x, y)) \right| \\
\leq I_{0^+}^{\alpha_2} (\mathcal{W}^{p} \varphi(x, y));
\]
\[
x \in [0, a), y \in [0, b), \frac{2}{3} < \alpha \leq 1, 0 \leq \gamma \leq 1, v_1 = \frac{\partial_{\beta; \psi, y} v}{\partial_{\beta; \psi, x^{2\alpha}}} \text{ and } v_2 = \frac{\partial_{\beta; \psi, x}^{2\alpha} v}{\partial_{\beta; \psi, x^{2\alpha}}}.\]

3. Main results

In this section we present the main results obtained in this paper, the Ulam-Hyers and generalized Ulam-Hyers-Rassias stabilities for the solution of the fractional partial differential equation of the pseudoparabolic type Eq. (1.1), as well as the uniqueness of solutions. This section will be divided into two sub-sections, for better development and understanding of results.

3.1. Ulam-Hyers Stability. Now, we consider Ulam-Hyers stability of the Eq. (1.1). For this problem we have the following result.

Theorem 1. We suppose that
\[
(1) \ a < \infty, \ \ b < \infty \\
(2) \ f \in C([0, a] \times [0, b] \times \mathbb{B}^3, \mathbb{B}) \\
(3) \ \exists L_f > 0 \ \text{such that} \\
(3.1) \ |f(x, y, u_1, u_2, u_3) - f(x, y, v_1, v_2, v_3)| \leq L_f \max_{i \in \{1, 2, 3\}} |u_i - v_i|; \\
\text{for all } x \in [0, a], y \in [0, b] \ \text{and} \ u_1, u_2, u_3, v_1, v_2, v_3 \in \mathbb{B}.
\]
Then, we have (a) For \( h \in C^2 ([0, a], \mathbb{B}) \), \( g_1, g_2 \in C^1 ([0, b], \mathbb{B}) \) the Eq. (1.1) has a unique solution with

\[
\begin{align*}
I^\alpha_0 u (x, 0) &= h (x), \ x \in [0, a] \\
I^\beta_0 u (y, 0) &= g_1 (y), \ y \in [0, b] \\
&= g_2 (y), \ y \in [0, b]
\end{align*}
\]

(b) The Eq. (1.1) is Ulam-Hyers stable.

Proof. (a) If \( u (x, y) \) is a solution to the problem Eq. (1.1) and Eq. (3.2), then

\[
\left( u, \frac{\partial^\alpha_{\beta_1} u}{\partial^\beta_2}, \frac{\partial^\alpha_{\beta_1} u}{\partial^\beta_2} \right)
\]

is a solution to the system:

\begin{align*}
\begin{cases}
\frac{\partial^\alpha_{\beta_1} u (x, y)}{\partial^\beta_2} = \Psi^\alpha (x, y), & x, y \in [0, a], b \\
\frac{\partial^\beta_{\gamma} u (x, y)}{\partial^\alpha} = \Psi^\beta (x, y), & x, y \in [0, a], b \\
\frac{\partial^\beta_{\gamma} u (x, y)}{\partial^\alpha} = \Psi^\beta (x, y), & x, y \in [0, a], b \\
\frac{\partial^\beta_{\gamma} u (x, y)}{\partial^\alpha} = \Psi^\beta (x, y), & x, y \in [0, a], b \\
\end{cases}
\end{align*}

where \( u_1 (x, y) = \frac{\partial^\alpha_{\beta_1} u (x, y)}{\partial^\beta_2} \), \( u_2 (x, y) = \frac{\partial^\beta_{\gamma} u (x, y)}{\partial^\alpha} \), \( g_2 (y) = \frac{\partial^\beta_{\gamma} u (x, y)}{\partial^\alpha} \), \( g_1 (y) = \frac{\partial^\beta_{\gamma} u (x, y)}{\partial^\alpha} \). We denote the right hand side of the equation in Eq. (3.3) by \( A_1, A_2, A_3 \), respectively. The system Eq. (3.3), then becomes

\[
\begin{align*}
\begin{cases}
u (x, y) = A_1 (u, u_1, u_2) (x, y) \\
u_1 (x, y) = A_2 (u, u_1, u_2) (x, y) \\
u_2 (x, y) = A_3 (u, u_1, u_2) (x, y)
\end{cases}
\end{align*}
\]

Let \( X := C ([0, a] \times [0, b]) \times C ([0, a] \times [0, b]) \times C ([0, a] \times [0, b]) \) and for any \( \delta > 0 \).

Consider the Bielecki norm on \( X \):

\[
\| (u, u_1, u_2) \|_X := \max \{ M_1, M_2, M_3 \}
\]

with

\[
\begin{align*}
M_1 := & \max_{(x, y) \in [0, a] \times [0, b]} | u (x, y) | \exp (- \delta (x + y)) \\
M_2 := & \max_{(x, y) \in [0, a] \times [0, b]} | u_1 (x, y) | \exp (- \delta (x + y)) \\
M_3 := & \max_{(x, y) \in [0, a] \times [0, b]} | u_2 (x, y) | \exp (- \delta (x + y))
\end{align*}
\]

Then \( (X, \| \cdot \|_X) \) is an ordered \( L \)-space. We will define the operator \( A : X \to X \) by

\[
(u, u_1, u_2) \to (A_1 (u, u_1, u_2), A_2 (u, u_1, u_2), A_3 (u, u_1, u_2)).
\]
Using the hypotheses (1)-(3), we have

\[(3.4) \quad \| A(\bar{u}, \bar{u}_1, \bar{u}_2) - A(\bar{u}, \bar{u}_1, \bar{u}_2) \|_B \leq \frac{L_f}{\delta} \| (\bar{u}, \bar{u}_1, \bar{u}_2) - (\bar{u}, \bar{u}_1, \bar{u}_2) \|_B.\]

Taking \(\delta > 0\) such that \(\frac{L_f}{\delta} < 1\) in relation Eq. (3.4), the operator \(A\) is a contraction and by the contraction principle the conclusion follows.

(b) Let \(v\) be a solution to the inequality Eq. (2.9). Let \(v\) be the unique solution of the Eq. (1.1) satisfying the conditions:

\[(3.5) \quad \begin{cases}
I_{\theta}^{1-\gamma,\psi} u(x, 0) = v(x, 0), \quad x \in [0, a] \\
I_{\theta}^{1-\gamma,\psi} u(x, 0) = v(0, y), \quad y \in [0, b] \\
u_x(x, y) = v_x(0, y), \quad y \in [0, b]
\end{cases}
\]

with \(\gamma = \alpha + \beta(1 - \alpha)\).

From Remark 3, the condition (3) and Lemma 1 (Gronwall Lemma), we have that

\[
|v(x, y) - u(x, y)| \leq |v(x, y) - \Psi^\gamma (0, y) v(x, 0) - \Psi^\gamma (x, 0) v(0, y) - \Psi^\gamma (x, 0) xv_x(0, y)| + I_{\theta}^{\alpha,\psi} \left( \frac{\mathcal{W}_{\psi, \psi} f(x, y)}{\Gamma(\alpha_1 + 1)} \right) \leq \frac{\alpha^\alpha (\psi(b) - \psi(0))^\alpha_2 (\psi(a) - \psi(0))^\alpha_1}{\Gamma(\alpha_1 + 1) \Gamma(\alpha_2 + 1)} E_a [L_f \alpha \Gamma(\alpha_1) \Gamma(\alpha_2) (\psi(b) - \psi(0))^\alpha_2 (\psi(a) - \psi(0))^\alpha_1] = C_f^1 \varepsilon
\]

where

\[C_f^1 := \frac{\alpha^\alpha (\psi(b) - \psi(0))^\alpha_2 (\psi(a) - \psi(0))^\alpha_1}{\Gamma(\alpha_1 + 1) \Gamma(\alpha_2 + 1)} E_a [L_f \alpha \Gamma(\alpha_1) \Gamma(\alpha_2) (\psi(b) - \psi(0))^\alpha_2 (\psi(a) - \psi(0))^\alpha_1] \]

and \(E_a(\cdot)\) is the one-parameter Mittag-Leffler function.

By performing the same process as above, we obtain the following inequalities

\[
|v_1(x, y) - u_1(x, y)| \leq \frac{\left( \frac{\alpha^\alpha (\psi(a) - \psi(0))^\alpha_1}{\Gamma(\alpha_1 + 1)} \right) E_a [L_f \alpha \psi(\alpha) - \psi(0))^\alpha_1 \Gamma(\alpha_1)] = C_f^2 \varepsilon
\]
where \( C_f^2 := \frac{a (\psi (a) - \psi (0))^\alpha_1}{\Gamma (\alpha_1 + 1)} \mathbb{E}_\alpha [L_f a (\psi (a) - \psi (0))^\alpha_1 \Gamma (\alpha_1)] \) and

\[
|v_2 (x, y) - u_2 (x, y)| \leq \epsilon \frac{a (\psi (b) - \psi (0))^\alpha_2}{\Gamma (\alpha_2 + 1)} \mathbb{E}_\alpha [L_f (\psi (b) - \psi (0))^\alpha_2] = C_f^3 \epsilon
\]

where \( C_f^3 := \frac{(\psi (b) - \psi (0))^\alpha_2}{\Gamma (\alpha_2 + 1)} \mathbb{E}_\alpha [L_f (\psi (b) - \psi (0))^\alpha_2] \).

So, the Eq. (1.1) is Ulam-Hyers stable.

3.2. Generalized Ulam-Hyers-Rassias Stability.

**Theorem 2.** We suppose that

1. \( f \in C ([0, \infty) \times [0, \infty) \times \mathbb{B}^3, \mathbb{B}) \);
2. There exists \( L_f \in C^1 ([0, \infty) \times [0, \infty), \mathbb{R}_+) \), such that

\[
|f (x, y, u_1, u_2, u_3) - f (x, y, v_1, v_2, v_3)| \leq L_f (x, y) \max_{i \in \{1, 2, 3\}} |u_i - v_i|
\]

for all \( x, y \in [0, \infty) \), \( u_1, u_2, u_3, v_1, v_2, v_3 \in \mathbb{B} \);
3. There exists \( \lambda_1, \lambda_2, \lambda_3 > 0 \) such that

\[
\begin{align*}
I_0^{\alpha_1 \psi} (\mathcal{W} f_1^1 (x, y)) & \leq \lambda_1^1 \psi (x, y) \\
I_0^{\alpha_2 \psi} (\mathcal{W} f_2^2 (x, y)) & \leq \lambda_2^2 \psi (x, y) \\
I_0^{\alpha_3 \psi} (\mathcal{W} f_3^3 (x, y)) & \leq \lambda_3^3 \psi (x, y)
\end{align*}
\]

for all \( x, y \in [0, \infty) \).
4. \( \varphi : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) is increasing.

Then the Eq. (1.1) \( (a = \infty, b = \infty) \) is generalized Ulam-Hyers-Rassias stable.

**Proof.** Let \( v \) be a solution of the inequality Eq. (2.9) and \( u (x, y) \) the unique solution of the problem

\[
\begin{align*}
\frac{\partial^{\alpha_1 \psi} u}{\partial \beta_1 \psi x^{2 \alpha_1} \partial \beta_1 \psi y^{\alpha_1}} (x, y) = f (x, y, u (x, y), \frac{\partial^{\alpha_1 \psi} u}{\partial \beta_1 \psi y^{\alpha_1}} (x, y), \frac{\partial^{\alpha_1 \psi} u}{\partial \beta_1 \psi x^{2 \alpha_1}} (x, y)) \\
I_0^{1 - \gamma \psi} u (x, 0) = v (x, 0) \\
I_0^{1 - \gamma \psi} u (0, y) = v (0, y) \\
v_x (0, y) = v_x (0, y)
\end{align*}
\]

for all \( x, y \in [0, \infty) \) and with \( \gamma = \alpha + \beta (1 - \alpha) \), then \( (u, u_1, u_2) \) is a solution of the system:

\[
\begin{align*}
u (x, y) = & \Psi^\gamma (0, y) v (x, 0) + \Psi^\gamma (x, 0) v (0, y) + \Psi^\gamma (x, 0) x v_x (0, y) + I_0^{\alpha_1 \psi} (\mathcal{W}^{p,u}_{u_1,u_2} f (x, y)) \\
u_1 (x, y) = & \Psi^\gamma (x, 0) v_1 (0, y) + \Psi^\gamma (x, 0) x v_{1x} (0, y) + I_0^{\alpha_1 \psi} (\mathcal{W}^{p,u}_{u_1,u_2} f (x, y)) \\
u_2 (x, y) = & \Psi^\gamma (0, y) v_2 (0, x) + I_0^{\alpha_1 \psi} (f (x, y, u (x, y), u_1 (x, y), u_2 (x, y)))
\end{align*}
\]
On the other hand, by Remark 4 and using the hypotheses (3), we get

\[
\begin{align*}
|v(x, y) - \Psi^\gamma(0, y)v(x, 0) - \Psi^\gamma(x, 0)v(0, y) - \Psi^\gamma(x, 0)xv_x(0, y)| & \leq I_\theta^{\alpha;\psi}(\mathcal{W}^{p, \varphi}(x, y)) \\
& \leq \lambda_\varphi^1\varphi(x, y), \ x, y \in [0, \infty); \\
|v_1(x, y) - \Psi^\gamma(x, 0)v_1(0, y) - \Psi^\gamma(x, 0)xv_{1x}(0, y) - I_{0+;x}^{\alpha_1;\psi}(\mathcal{W}^{p, u}_{u_1, u_2}f(x, y))| & \leq I_{0+;x}^{\alpha_1;\psi}(\mathcal{W}^{p, \varphi}(x, y)) \\
|v_2(x, y) - \Psi^\gamma(0, y)v_2(0, y) - I_{0+;y}^{\alpha_2;\psi}(f(x, y, u(x, y), u_1(x, y), u_2(x, y)))| & \leq \lambda_\varphi^2\varphi(x, y), \ x, y \in [0, \infty); \\
\end{align*}
\]

(3.7) \quad \leq \quad (3.8) \quad \leq \quad (3.9)

In this sense, from Eq. (3.7), Eq. (3.8), Eq. (3.9) and using the Gronwall Lemma, we obtain

\[
\begin{align*}
|v(x, y) - u(x, y)| & \leq \lambda_\varphi^1\varphi(x, y) + I_\theta^{\alpha;\psi}(\int_0^p L_f(x, y) \max_{i \in \{1, 2, 3\}} |v_i(x, y) - u_i(x, y)| dp) \\
& \leq \lambda_\varphi^1\varphi(x, y) E_\alpha \left[ \int_0^p L_f(x, y) \Gamma(\alpha_1) \Gamma(\alpha_2) (\psi(b) - \psi(0))^{\alpha_2} (\psi(a) - \psi(0))^{\alpha_1} dp \right] \\
& \leq \lambda_\varphi^1\varphi(x, y) E_\alpha \left[ \int_0^p L_f(x, y) \Gamma(\alpha_1) \Gamma(\alpha_2) (\psi(\infty) - \psi(0))^{\alpha_2} (\psi(\infty) - \psi(0))^{\alpha_1} dp \right] \\
& \leq C_{1, f, \varphi}^{\alpha;\psi}(x, y)
\end{align*}
\]

(3.10)

where \(C_{1, f, \varphi}^{\alpha;\psi} := \lambda_\varphi^1 E_\alpha \left[ \int_0^p L_f(x, y) \Gamma(\alpha_1) \Gamma(\alpha_2) (\psi(\infty) - \psi(0))^{\alpha} (\psi(\infty) - \psi(0))^{\alpha} dp \right] \), \(\frac{2}{3} < \alpha \leq 1\) and \(\psi(\infty) < \infty\), \(x, y \in [0, \infty)\) and \(E_\alpha(\cdot)\) is the one-parameter Mittag-Leffler function.

Performing the same steps as above to obtain the inequality Eq. (3.10), we can write

\[
|v_1(x, y) - u_1(x, y)| \leq C_{1, f, \varphi}^{\alpha;\psi}(x, y)
\]

where \(C_{1, f, \varphi}^{\alpha;\psi} := \lambda_\varphi^1 E_\alpha \left[ \int_0^p L_f(x, y) \Gamma(\alpha_1) (\psi(\infty) - \psi(0))^{\alpha_1} dp \right] \), \(\frac{2}{3} < \alpha \leq 1\) and \(\psi(\infty) < \infty\), \(x, y \in [0, \infty)\).

Also, we get

\[
|v_2(x, y) - u_2(x, y)| \leq C_{1, f, \varphi}^{\alpha;\psi}(x, y)
\]
where $C_{f,\varphi}^3 := \lambda_\varphi^3 K_\alpha \left[ \int_0^p L_f(x, y) \Gamma(\alpha_2) (\psi(\infty) - \psi(0))^{\alpha_2} \, dp \right], \quad \frac{2}{3} < \alpha \leq 1$ and $\psi(\infty) < \infty$, $x, y \in [0, \infty)$.

Then, the solution of the Eq.(1.1) is generalized Ulam-Hyers-Rassias stable.

\[ \square \]

4. Concluding remarks

In this paper, we have presented and proposed new stability results of the Ulam-Hyers type of solutions of fractional partial differential equations, contributing to the diffusion of results in the fractional calculus theme, particularly, in the fractional analysis, significantly enriching this field of study.

In this sense, we have been successful in presenting stability results of Ulam-Hyers and generalized Ulam-Hyers-Rassias solution of a fractional order pseudoparabolic partial differential equation, using the $\psi$-Hilfer fractional partial derivative of $N$ variables and the Gronwall inequality. It is also possible to note that there are authors who use in their works, the Banach fixed-point theorem to discuss stability [11, 35].

As mentioned in the introduction, the study of the stability of solutions of fractional differential equations is in a growing, and numerous researchers have contributed to such advancement. However, it is still possible to obtain other new Ulam-Hyers stability results from other types of partial differential equations: hyperbolic, parabolic and elliptical. On the other hand, a theme that gains special attention is the study of stability of the solution of the linear heat equation, by means of integral transforms, that is, the Fourier transform and the Laplace transform. Studies in this direction must be presented and published in the near future.

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