Dyson’s model in infinite dimensions is irreducible

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Abstract

Dyson’s model in infinite dimensions is a system of Brownian particles interacting via a logarithmic potential with an inverse temperature of $\beta = 2$. The stochastic process is given as a solution to an infinite-dimensional stochastic differential equation. Additionally, a Dirichlet form with the sine$^2$ point process as a reference measure constructs the stochastic process as a functional of the associated configuration-valued diffusion process. In this paper, we prove that Dyson’s model in infinite dimensions is irreducible.

Keywords: Dyson’s model, random matrices, irreducibility, diffusion process, interacting Brownian motion, infinite-dimensional stochastic differential equations, logarithmic potential, Gaussian unitary ensembles

MSC2020: 60B20, 60H10, 60J40, 60J60, 60K35

1 Introduction

This paper considers an infinite-dimensional stochastic differential equation (ISDE) of the form

\[
X_t^i - X_0^i = B_t^i + \frac{\beta}{2} \int_0^t \lim_{r \to \infty} \sum_{|X_u^i - X_u^j| < r, j \neq i} \frac{1}{X_u^i - X_u^j} \, du \quad (i \in \mathbb{Z}).
\]  
(1.1)

For $\beta = 2$, the ISDE was introduced by Spohn [27], who called it Dyson’s model. Spohn derived (1.1) for $\beta = 2$ as an informal limit of Dyson’s Brownian motion in finite dimensions. Here, Dyson’s Brownian motion is a solution of a finite-dimensional stochastic differential equation (SDE) such that

\[
X_t^{N,i} - X_0^{N,i} = B_t^i + \frac{\beta}{2} \int_0^t \sum_{j \neq i} \frac{1}{X_u^{N,i} - X_u^{N,j}} \, du - \frac{\beta}{2N} \int_0^t \frac{1}{X_u^{N,i}} \, du.
\]  
(1.2)
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If $\beta = 2$, then SDE (1.2) describes the dynamics of the eigenvalues of Gaussian unitary ensembles of order $N \in \mathbb{N}$ [3, 14]. Spohn [27] constructed the limit dynamics as the $L^2$-Markovian semi-group given by the Dirichlet form

$$E(f, g) = \int_S \mathbb{D}[f,g] d\mu$$

on $L^2(\mathcal{S}, \mu)$, where $\mathcal{S}$ is the configuration space over $\mathbb{R}$, $\mathbb{D}$ is the standard carré du champ on $\mathcal{S}$, and $\mu$ is the sine random point field. Furthermore, the domain of the Dirichlet form is taken to be the closure of the polynomials on $\mathcal{S}$.

Let $\mu$ be the sine-$\beta$ random point field. If $\beta = 2$, then $\mu$ becomes a determinantal random point field whose $m$-point correlation function $\rho^m$ with respect to the Lebesgue measure is given by

$$\rho^m(x) = \det[K_{\sin,2}(x^i, x^j)]_{i,j=1}^m.$$

Here, $K$ is the sine kernel given by

$$K(x, y) = \frac{\sin(\theta \sqrt{2}(x - y))}{\pi(x - y)}.$$

Spohn [27] proved the closability of $E$ on $L^2(\mathcal{S}, \mu)$ with a predomain consisting of polynomials on $\mathcal{S}$ for $\beta = 2$.

In [17], the first author proved that $(E, D^\mu_\circ)$ is closable on $L^2(\mathcal{S}, \mu)$, and that its closure is a quasi-regular Dirichlet form. Here, $D_\circ$ is the set consisting of local and smooth functions on $\mathcal{S}$ and $D^\mu_\circ$ is given by

$$D^\mu_\circ = \{ f \in D_\circ : E^1(f,f) < \infty \}.$$

Thus, Osada constructed the $L^2$-Markovian semi-group as well as the diffusion

$$\mathcal{X}(t) = \sum_{i \in \mathbb{Z}} \delta_{X_i(t)}$$

associated with the Dirichlet form $(E, D)$ on $L^2(\mathcal{S}, \mu)$. We call $\mathcal{X}$ the unlabeled dynamics or unlabeled diffusion because the state space of the process is $\mathcal{S}$. The unlabeled diffusion can be constructed for $\beta = 1, 4$ [21], and the associated labeled process $\mathcal{X} = (X^i)_{i \in \mathbb{N}}$ satisfies ISDE (1.1) for $\beta = 1, 2, 4$ [20]. These cases have been proved as examples of the general theory developed in various papers [19, 20, 21, 22]. In [20], the meaning of a solution to an ISDE is a weak solution; the uniqueness of a weak solution of an ISDE and the Dirichlet form is left open in [20, 21]. (See [8] for the concept of strong and weak solutions of stochastic differential equations).

Tsai [29] solved ISDE (1.1) for all $\beta \in [1, \infty)$. He proved the existence of a strong solution and the path-wise uniqueness of this solution. The method used by Tsai depends on an artistic coupling specific to Dyson’s model. A non-equilibrium solution is obtained in the sense that the ISDE is solved by starting at each point in an explicitly given subset $\mathcal{S}_0 \subset \mathcal{S}$ such that $\mu(\mathcal{S}_0) = 1$. 

The $\mu$-reversibility of the associated unlabeled diffusion is left open in [29]. Combining [20] and [29], we find that the unlabeled process given by the solution of (1.2) obtained in [29] is reversible with respect to $\mu$ for $\beta = 1, 4$. For a general $\beta > 0$, note that the reversible probability measure of the unlabeled diffusion given by the solution to ISDE (1.1) is expected to be a $\text{sine}_\beta$-random point field. This remains an open problem, except for $\beta = 1, 2, 4$ [21].

One of the authors and Tanemura [24] also proved the existence of a strong solution and the path-wise uniqueness of this solution for $\beta = 1, 2, 4$. Their method can be applied to quite a wide range of examples. Using the result in [24], Kawamoto et al. proved the uniqueness of Dirichlet forms [11]. They checked the infinite system of finite-dimensional SDEs with consistency (IFC) condition in [12], which plays an important role in the theory developed in [24]. Kawamoto and the second author of [12] derive a solution to the ISDE from $N$-particle systems [9, 10].

The goal of this paper is to prove that the solution of (1.3) for $\beta = 2$ is irreducible (Theorem 1.1). In the remainder of this paper, we consider the case $\beta = 2$. Hence, we take $\mu$ to be the $\text{sine}_2$ random point field.

By definition, the configuration $\mathcal{S}$ over $\mathbb{R}$ is given by

$$\mathcal{S} = \left\{ s = \sum_i \delta_{s_i}; s(K) < \infty \text{ for any compact } K \right\}.$$  

We endow $\mathcal{S}$ with the vague topology. Under the vague topology, $\mathcal{S}$ is a Polish space. A probability measure on $(\mathcal{S}, \mathcal{B}(\mathcal{S}))$ is called a random point field (also called a point process). Let

$$\mathcal{S}_{s,i} = \left\{ s \in \mathcal{S}; s(\{s\}) \leq 1 \text{ for all } s \in \mathbb{R}, s(\mathbb{R}) = \infty \right\}.$$  

In [21] [18], we proved that the $\text{sine}_2$ random point field $\mu$ satisfies

$$\text{Cap}((\mathcal{S}_{s,i})^c) = 0.$$  

Furthermore, $\mu$ is translation invariant and tail trivial [23] [15]. Hence, by the individual ergodic theorem, we have that, for $\mu$-a.s. $s$,

$$\lim_{R \to \infty} \frac{s([-R, R])}{R} = \int_{\mathcal{S}} s([-1, 1])d\mu.$$  

Then, we set

$$\mathcal{S}_n = \left\{ s \in \mathcal{S}; \frac{1}{n} \leq \frac{s([-R, R])}{R} \leq n \text{ for all } R \in \mathbb{N} \right\}.$$  

Using the argument in the proof of Theorem 1 in [17, p.127], we see that

$$\text{Cap}\left( \bigcup_{n=1}^{\infty} \mathcal{S}_n \right)^c = 0.$$  

(1.5)
We write \( s = (s^i)_{i \in \mathbb{Z}} \in \mathbb{R}^\mathbb{Z} \), and we set
\[
\mathbb{R}^\mathbb{Z}_s = \{ s = (s^i)_{i \in \mathbb{Z}} \in \mathbb{R}^\mathbb{Z}; s^i < s^{i+1} \text{ for all } i \}.
\]
Let \( u \) be a map on \( \mathbb{R}^\mathbb{Z}_s \) such that \( u(s) = \sum_{i \in \mathbb{Z}} \delta_{s^i} \). Let \( I : \mathcal{G}_{s,i} \to \mathbb{R}^\mathbb{Z}_s \) be a function such that \( u \circ I = \text{id.} \). We call \( u \) an unlabeling map and \( I \) a labeling map. There exist many labeling maps. We can take \( I \) in such a way that \( l_0(s) = \min\{ s^i; s^i \geq 0, s = \sum_{i \in \mathbb{Z}} s^i \} \) and \( l_i(s) < l_i+1(s) \) for all \( i \in \mathbb{Z} \), where \( I(s) = (l(s))(i)_{i \in \mathbb{Z}} \). This choice of \( I \) is just for convenience and has no specific meaning. Let
\[
W = C([0, \infty); \mathbb{R}^\mathbb{Z}_s),
\]
(1.7)
Let \( I_{\text{path}} = \{ I_{\text{path}}(w)_t \}_{t \in [0, \infty]} \) be the label path map generated by \( I \) (see [24, pp. 1148-1149] and (2.6) in [12]). By definition, \( I_{\text{path}} \) is the map from \( C([0, \infty); \mathcal{S}_{s,i}) \) to \( W \) such that \( I_{\text{path}}(w)_0 = I(w)_0 \), where \( w = \{ w_t \}_{t \in [0, \infty]} \in C([0, \infty); \mathcal{S}_{s,i}) \).

Let \( X = (X^i)_{i \in \mathbb{Z}} \) be a solution to ISDE (1.1) with \( \beta = 2 \) defined on a filtered space \((\Omega, \mathcal{F}, P, \{ \mathcal{F}_t \})\). We set
\[
\mu_\infty = \mu \circ I^{-1}
\]
(1.8)
and assume that
\[
\mu_\infty = P(X_0 \in \cdot).
\]
(1.9)
The associated unlabeled process
\[
\mathcal{X} = \sum_{i \in \mathbb{Z}} \delta_{X^i}
\]
is a \( \mu \)-reversible diffusion given by the Dirichlet form \((\mathcal{E}, \mathcal{D})\) in (1.3) [24]. Note also that the labeled process \( X = I_{\text{path}}(\mathcal{X}) \) obtained by the Dirichlet form in [20-24] coincides with the solution obtained by Tsai [29].

From (1.4) and \( X = I_{\text{path}}(\mathcal{X}) \) we find that
\[
P(X \in W) = 1.
\]
We set \( w = (w^i)_{i \in \mathbb{Z}} \in \mathbb{W} \) and
\[
P^\infty = P \circ \mathcal{X}^{-1}, \quad P^\infty_x = P^\infty(\cdot | w_0 = x).
\]
(1.10)

**Theorem 1.1.** \( \{ P^\infty_x \} \) is irreducible. That is, if \( A \) and \( B \in \mathcal{B}(\mathbb{R}^\mathbb{Z}_s) \) satisfy
\[
P^\infty(\cdot | w_0 \in A, w_t \in B) = 0,
\]
(1.11)
then \( P^\infty(\cdot | w_0 \in A) = 0 \) or \( P^\infty(\cdot | w_t \in B) = 0 \).

We do not know whether \( \mathcal{X} \) has an invariant probability measure that is absolutely continuous with respect to \( \mu_\infty \). If this is the case, then Theorem 1.1 implies that \( \mathcal{X} \) is irreducible in the usual sense.

From Theorem 1.1, (1.8), and (1.10), we immediately have the following.
Corollary 1.2. The solution of (1.1) with $\beta = 2$ is irreducible in the sense that, if $A$ and $B \in \mathcal{B}(\mathbb{R}_+^\mathbb{Z})$ satisfy
\[ P(X_0 \in A, X_t \in B) = 0, \]
then $P(X_0 \in A) = 0$ or $P(X_t \in B) = 0$.

To prove Theorem 1.1, we prepare two results, Theorem 1.3 and Theorem 1.4. For $x \in \mathbb{R}^2$, we set $x^m = (x^i)_{|i| < m}$ and $x^{m*} = (x^i)_{|i| \geq m}$. We set
\[ \mathbb{R}^m = \{x^m = (x^i)_{|i| < m}; x^i < x^{i+1} \text{ for all } -m < i < m - 1\}, \]
\[ \mathbb{R}^{m*} = \{x^{m*} = (x^i)_{|i| \geq m}; x^i < x^{i+1} \text{ for all } i < -m, m \leq i\}. \]
Let $X^m = (X^i)_{|i| < m}$ and $X^{m*} = (X^i)_{|i| \geq m}$. Let $w^{m*} = (w^i)_{|i| \geq m}$ for $w = (w^i) \in \mathbb{R}$. We introduce the regular conditional probabilities such that
\[ P_{w}^m = P(X^m \in \cdot | X^{m*} = w^{m*}), \]
\[ P_{x,w}^m = P(X^m \in \cdot | X_0^m = x^m, X^{m*} = w^{m*}). \]

By construction, $X^m$ under $\{P_{x,w}^m\}$ is a time-inhomogeneous diffusion. The heat equations describing the transition probability density are given by (3.20) and (3.21).

Theorem 1.3. Let $P^{m*} = P \circ (X^{m*})^{-1}$. For each $m \in \mathbb{N}$, $\{P_{x,w}^m\}$ is irreducible for $P^{m*}$-a.s. $w$. That is, if $A$ and $B \in \mathcal{B}(\mathbb{R}_+^\mathbb{Z})$ satisfy
\[ P_{w}^m(\omega_0^m \in A, \omega_t^m \in B) = 0 \quad \text{for } P^{m*}\text{-a.s. } w, \]
then $P_{w}^m(\omega_0^m \in A) = 0$ for $P^{m*}$-a.s. $w$ or $P_{w}^m(\omega_t^m \in B) = 0$ for $P^{m*}$-a.s. $w$.

Theorem 1.4. Let $P^m = P \circ (X^m)^{-1}$. For each $m \in \mathbb{N}$, the process $P^m$ is irreducible. That is, if $A$ and $B \in \mathcal{B}(\mathbb{R}_+^\mathbb{Z})$ satisfy
\[ P^m(\omega_0^m \in A, \omega_t^m \in B) = 0, \]
then $P^m(\omega_0^m \in A) = 0$ or $P^m(\omega_t^m \in B) = 0$.

Let $\Phi : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ and $\Psi : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ be measurable functions. A stochastic process given by a solution $X = (X^i)_i$ of the ISDE
\[ X^i_t - X^i_0 = B^i_t + \frac{1}{2} \int_0^t \nabla \Phi(X^i_u)du + \frac{1}{2} \int_0^t \sum_{j \neq i} \nabla \Psi(X^i_u, X^j_u)du \]
is called an interacting Brownian motion (in infinite dimensions) with potential $(\Phi, \Psi)$. Here, $(\nabla \Psi)(x,y) = \nabla_x \Psi(x,y)$. The study of interacting Brownian motions was initiated by Lang [13, 14], who solved the ISDE for $(0, \Psi)$ with $\Psi \in C_0^\infty(\mathbb{R}^d)$ such that $\Psi$ is of Ruelle’s class in the sense that it is super stable and regular. Fritz [5] constructed non-equilibrium solutions for the same potentials as in [13, 14] with a further restriction that the dimension $d \leq 4$. Tanemura
solved the ISDE for the hard-core potential [28], while Fradon–Roelly–Tanemura solved the ISDE for the hard-core potential with long range interactions, but still of Ruelle’s class [4]. Various ISDEs with logarithmic interaction potentials have also been solved [7, 11, 20, 22, 24, 25, 29].

There are fewer results for the irreducibility and ergodicity of solutions of interacting Brownian motions. Albeverio–Ma–Röckner [1] proved the equivalence of the ergodicity of Dirichlet forms and the extremal property of the associated (grand canonical or canonical) Gibbs measures with potentials of Ruelle’s class [26]. Corwin-Sun [2] proved the ergodicity of the Airy line ensembles, for which the dynamics are related to the Airy$_2$ random point field. A general result concerning the ergodicity of Dirichlet forms can be found in [6].

The remainder of this paper is organized as follows. In Section 2 we recall the concept of the $m$-labeled process and the Lyons–Zheng decomposition for interacting Brownian motions. In Section 3 we prove Theorem 1.3 and Theorem 1.4. Finally, in Section 4 we prove Theorem 1.1.

2 The $m$-labeled process and the Lyons–Zheng decomposition

We introduce the $m$-labeled process $X^{[m]} = (X^m, X^{m*})$, where

$$X_t^{m*} = \sum_{|i| \geq m} \delta_{X_i^t}.$$  

The process $X^{[m]}$ is given by the Dirichlet form $(\mathcal{E}^{[m]}, \mathcal{D}^{[m]})$ on $L^2(\mathbb{R}^m_\times \mathcal{G}, \mu^{[m]})$ such that

$$\mathcal{E}^{[m]}(f, g) = \int_{\mathbb{R}^m_\times \mathcal{G}} \mathcal{D}^{[m]}(f, g) \, d\mu^{[m]}.$$  

Here, $\mathcal{D}^{[m]}$ is the standard carré du champ on $\mathbb{R}^m_\times \mathcal{G}$ [24], $\mathcal{D}^{[m]}$ is the closure of

$$\{f_1 \otimes f_2 \in C_0^\infty \otimes \mathcal{D}_\mathcal{G}; \mathcal{E}^{[m]}_1(f_1 \otimes f_2, f_1 \otimes f_2) < \infty\},$$

and $\mu^{[m]}$ is the $m$-reduced Campbell measure such that

$$\mu^{[m]}(A \times B) = \int_A \rho^m(x^m) \mu_{x^m}(B) \, dx^m.$$  

Moreover, $\rho^m$ is the $m$-point correlation function of $\mu$, and $\mu_{x^m}$ is a reduced Palm measure conditioned at $x^m = (x^i)_{|i| < m}$ given by

$$\mu_{x^m} = \mu(\cdot - \sum_{|i| < m} \delta_{x^i} |f(\xi) = x^i, \text{ for all } |i| < m).$$ (2.1)
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The standard definition of the reduced Palm measure $\mu_{x^m}$ is

$$
\mu_{x^m} = \mu(\cdot - \sum_{|i| < m} \delta_{x^i} \mathbb{1}(\{|i| \geq 1\text{ for all }|i| < m\}))
$$

Because the state space of process $X$ is $\mathbb{R}_{<}^Z$, we take $\mu_{x^m}$ given by (2.1). We set

$$
P_{x,s}^{[m]} = P(\mathbb{X}_0^{[m]} = (x,s)).
$$

From [19, 24], we have that $\{P_{x,s}^{[m]}\}$ is a diffusion associated with the Dirichlet form $(\mathcal{E}^{[m]}, \mathcal{D}^{[m]})$ on $L^2(\mathbb{R}^m_{<} \times \mathcal{S}, \mu^{[m]}).$ By construction, $\{P_{x,s}^{[m]}\}$ is $\mu^{[m]}$-symmetric and $\mu^{[m]}$ is an invariant measure of $\{P_{x,s}^{[m]}\}.$ One of the most critical properties of $\{P_{x,s}^{[m]}\}$ is its consistency. To explain the consistency, we prepare some notations.

Let $l_{\text{path}}$ be the path label introduced in Section 1. We write $l_{\text{path}} = (l_{i_{\text{path}}}^{(w)}).$ We set $l_{\text{path}}^{[m]}$ from $l_{\text{path}}$ as follows:

$$
l_{\text{path}}^{[m]}(w) = ((l_{i_{\text{path}}}^{(w)}), |i| < m, \sum_{|j| \geq m} \delta_{l_{i_{\text{path}}}^{(w)}}).
$$

For an $\mathbb{R}^{[m]}$-valued path $w^{[m]}$ such that $w_t^{[m]} = ((w_i^t), |i| < m, \sum_{|j| \geq m} \delta_{w_i^t}),$ we set

$$
u_{\text{path}}^{[m]}(w^{[m]}) = \sum_{i \in \mathbb{Z}} \delta_{w_i^t}. 
$$

Clearly, $u_{\text{path}}^{[m]}(w^{[m]})_t = u(w_t),$ where $u$ is the unlabeling map defined in Section 1. Additionally, $u(x,s) = \sum_i \delta_{x^i} + s$ for $x = (x^i).$

We have the following consistency.

**Lemma 2.1.** For each $m \in \{0\} \cup \mathbb{N},$

$$
P_{x,s}^{[m]} \circ \nu_{\text{path}}^{[m]} = P_{u(x,s)}^{[0]}, \quad P_s^{[0]} \circ \nu_{\text{path}}^{[m]} = P_{1_{\text{path}}^{[m]}}^{[m]}.
$$

**Proof.** Applying Theorem 2.4 in [19] to Dyson’s model, we obtain Lemma 2.1.

Let $x^{[m]} = (x^m, x^{m*})$ for $x = (x^i)_{i \in \mathbb{Z}} \in \mathbb{R}^Z_{<},$ where $x^{m*} = \sum_{|i| \geq m} \delta_{x^i}.$

**Lemma 2.2.** For each $m \in \{0\} \cup \mathbb{N}$

$$
P_{x}^{[\infty]} \circ \nu_{\text{path}}^{[m]} = P_{x^{[m]}}^{[m]}.
$$

**Proof.** Lemma 2.2 follows from (1.10) and Lemma 2.1.
Note that \( w^j \) under \( P[m] \) is a solution to SDE (1.1) for \( |j| < m \). Thus, the martingale term of the Fukushima decomposition of \( w^j \) describes Brownian motion. Hence, applying the Lyons–Zheng decomposition to \( w^j \) under \( P[m] \), we obtain

\[
\mathbb{P}_m \{ w^j_t - w^j_u = \frac{1}{2} \{ B^j_t - B^j_u + \hat{B}^j_t - \hat{B}^j_u \} \mid \text{ for } 0 \leq t, u \leq T, \tag{2.2}
\]

where \( \hat{B}^j_t = B^j_{T-t} \). Because \( P[m] \) is a symmetric diffusion, \( \hat{B}^j_t \) describes Brownian motion. Furthermore, \( \{ B^j_t \}_{|j|<m} \) is a sequence of independent Brownian motions under \( P[m] \). Because \( \hat{B}^j_t \) is a time reversal of \( B^j_t \), \( \{ \hat{B}^j_t \}_{|j|<m} \) is a sequence of independent Brownian motions under \( P[m] \). Because of the consistency in Lemma 2.2, we have that (2.2) holds for all \( j \in \mathbb{Z} \) under \( P_\infty \). Furthermore, \( \{ B^j_t \}_{j \in \mathbb{Z}} \) and \( \{ \hat{B}^j_t \}_{j \in \mathbb{Z}} \) are sequences of independent Brownian motions under \( P_\infty \). Thus, \( \{ B^j_t - B^j_u \}_{j \in \mathbb{Z}} \) and \( \{ \hat{B}^j_t - \hat{B}^j_u \}_{j \in \mathbb{Z}} \) are sequences of increments of independent Brownian motions. Collecting these statements together, we obtain the following.

**Lemma 2.3.** (1) For each \( j \in \mathbb{Z} \), we have (2.2) for \( P_\infty \)-a.s.

(2) \( \{ B^j_t - B^j_u \}_{j \in \mathbb{Z}} \) and \( \{ \hat{B}^j_t - \hat{B}^j_u \}_{j \in \mathbb{Z}} \) are sequences of increments of independent Brownian motions under \( P_\infty \).

### 3 Proof of Theorem 1.3 and Theorem 1.4

Let \( \log \mu \) be the logarithmic derivative of \( \mu \). By definition, \( \log \mu \) is a function defined on \( \mathbb{R} \times \mathcal{S} \) such that \( \log \mu \in L^1_{\text{loc}}(\mu[1]) \) and

\[
\int_{\mathbb{R} \times \mathcal{S}} \log \mu(s, s) \varphi(s, s) d\mu[1] = -\int_{\mathbb{R} \times \mathcal{S}} \nabla \varphi(s, s) d\mu[1]
\]

for all \( \varphi \in C_0^\infty(\mathbb{R}) \otimes \mathcal{D}_0^b \), where \( \mathcal{D}_0^b \) is the set consisting of bounded, local, and smooth functions on \( \mathcal{S} \). We write \( s = \sum_i \delta_{s_i} \). In [21], it is proved that \( \mu \) has a logarithmic derivative such that

\[
\log \mu(s, s) = 2 \lim_{R \to \infty} \sum_{s' \in S_R} \frac{1}{s - s'} \text{ in } L^2_{\text{loc}}(\mu[1])
\]

\[
= 2 \lim_{R \to \infty} \sum_{|s-s'|<R} \frac{1}{s - s'} \text{ in } L^2_{\text{loc}}(\mu[1]).
\]

The sums in (3.1) converge because \( \mu \) is translation invariant, \( d = 1 \), and the variance of \( s([-R, R]) \) under \( \mu \) increases logarithmically as \( R \to \infty \). The second equality in (3.1) comes from \( d = 1 \).
Note that the Ginibre random point field $\mu_{\text{gin}}$ satisfies the following [20]:

$$d\mu_{\text{gin}}(s, s) = -2s + 2 \lim_{R \to \infty} \sum_{s' \in S_R} \frac{s - s'}{|s - s'|^2} \quad \text{in } L^2_{\text{loc}}(\mu_{\text{gin}})$$  \hspace{1cm} (3.2)

$$= 2 \lim_{R \to \infty} \sum_{|s - s'| < R} \frac{s - s'}{|s - s'|^2} \quad \text{in } L^2_{\text{loc}}(\mu_{\text{gin}}).$$

The Ginibre random point field $\mu_{\text{gin}}$ is the counterpart of $\mu$ in $\mathbb{R}^2$, because $\mu_{\text{gin}}$ is rotation- and translation-invariant, and the interaction potential of $\mu_{\text{gin}}$ is the logarithmic potential with an inverse temperature of $\beta = 2$. Compare (3.1) and (3.2). The first equalities in (3.1) and (3.2) have different expressions according to the dimension $d$.

Recall that $\beta = 2$. Then, the ISDE in question is given by

$$X_i^t - X_i^0 = B_i^t + \int_0^t \lim_{R \to \infty} \sum_{|s - s'| < R} \frac{1}{X_u - X_a} \, du \quad (i \in \mathbb{Z}).$$  \hspace{1cm} (3.3)

Using (3.1) and (3.3), we have

$$X_i^t - X_i^0 = B_i^t + \frac{1}{2} \int_0^t \delta^\beta(X_u, \sum_{j \neq i} \delta X_u) \, du \quad (i \in \mathbb{Z}).$$  \hspace{1cm} (3.4)

From (3.1), (3.3), and (3.4), it is easy to see that, $i \in \mathbb{Z},$

$$X_i^t - X_i^0 = B_i^t + \int_0^t \sum_{|j| < m, j \neq i} \frac{1}{X_u - X_a} \, du + \int_0^t \lim_{R \to \infty} \sum_{m \leq |j| \neq i} \frac{1}{X_u - X_a} \, du$$

$$= B_i^t + \int_0^t \sum_{|j| < m, j \neq i} \frac{1}{X_u - X_a} \, du + \int_0^t \lim_{n \to \infty} \sum_{m \leq |j| \neq n} \frac{1}{X_u - X_a} \, du.$$

Taking this equation into account, we set $b_{m}^w = (b_{m,i}^w)_{|i| < m}$ such that

$$b_{m,i}^w(x^m, t) = \sum_{|j| < m, j \neq i} \frac{1}{x^i - x^j} + \lim_{n \to \infty} \sum_{m \leq |j| \neq n} \frac{1}{x^i - w^j}. \hspace{1cm} (3.5)$$

Let $x^m = (x^i)_{|i| < m}, x = (x^i)_{i \in \mathbb{Z}},$ and $y = (y^i)_{i \in \mathbb{Z}}.$ For $y \in \mathbb{R}^Z_{<}$, we set

$$\mathbb{R}^m_{<}(y) = \{x^m \in \mathbb{R}^m_{<} : y^{-m} < x^{-m+1}, x^{-m-1} < y^m\}.$$ 

Let $O_{T, w}^m$ be a time-dependent open set in $\mathbb{R}^m_{<}$ such that

$$O_{T, w}^m = \{(x^m, t) \in \mathbb{R}^m_{<} \times [0, T) : x^m \in \mathbb{R}^m_{<}(w_t)\}.$$
Proof. Let $(\mathbf{x}^m, t)$, we set $\mathbf{x}^m_i = (x^m_i)_{|i|<m}$ such that $(\mathbf{x}^m, t) = (\mathbf{x}^m, t)$. For $\epsilon \geq 0$, $m, T \in \mathbb{N}$, and $\mathbf{w} \in \mathbb{W}$, we set

$$\mathcal{O}_{T, \mathbf{w}}^{m, \epsilon} = \{(\mathbf{x}^m, t) \in \mathcal{O}_{T, \mathbf{w}}^m; |x^m_i - x^m_{i+1}| > \epsilon, -m < i < m - 1$$

$$|x^m_{i+1} - w^m_i| > \epsilon, |x^m_{i-1} - w^m_i| > \epsilon\}.$$  

(3.6)

Suppose that $\epsilon > 0$ and that $\mathcal{O}_{T, \mathbf{w}}^{m, \epsilon}$ is nonempty and connected. For $P^\infty$-a.s. $\mathbf{w}$, we find a connected open set $\mathcal{D}_{T, \mathbf{w}}^{m, \epsilon}$ in $\mathbb{R}^m_+ \times [0, T]$ with smooth boundary such that

$$\mathcal{O}_{T, \mathbf{w}}^{m, \epsilon} \subset \mathcal{D}_{T, \mathbf{w}}^{m, \epsilon} \subset \mathcal{D}_{T, \mathbf{w}}^{m, \epsilon/2}. \quad (3.7)$$

Lemma 3.1. For each $T, m \in \mathbb{N}$, and $P^\infty$-a.s. $\mathbf{w}$, the following hold.

1. $b^m_\mathbf{w}(\mathbf{x}^m, t)$ is Hölder continuous in $t$ in $\mathcal{D}_{T, \mathbf{w}}^{m, \epsilon}$ for each $\mathbf{x}^m$.
2. $b^m_\mathbf{w}(\mathbf{x}^m, t)$ is Lipschitz continuous in $\mathbf{x}^m$ in $\mathcal{D}_{T, \mathbf{w}}^{m, \epsilon}$.

Proof. Let $(\mathbf{x}^m, t), (\mathbf{x}^m, u) \in \mathcal{D}_{T, \mathbf{w}}^{m, \epsilon}$ and fix $i$ such that $|i| < m$. Then from (3.5)

$$b^m_\mathbf{w}(\mathbf{x}^m, t) - b^m_\mathbf{w}(\mathbf{x}^m, u) = \sum_{|i| \geq m} \frac{1}{x^m_i - w^m_i} - \sum_{|i| \geq m} \frac{1}{x^m_i - w^m_i}$$

$$= \sum_{|i| \geq m} \frac{w^m_i - w^m_i}{(x^m_i - w^m_i)(x^m_i - w^m_i)}.$$  

(3.8)

From Lemma 2.3, we can deduce for $P^\infty$-a.s. that

$$w^m_i - w^m_i = \frac{1}{2} \{B^m_i - \hat{B}^m_i + \hat{B}^m_i - \hat{B}^m_i\}, \quad (3.9)$$

where $\{B^m_i - \hat{B}^m_i\}_{j \in \mathbb{Z}}$ and $\{\hat{B}^m_i - \hat{B}^m_i\}_{j \in \mathbb{Z}}$ are sequences of increments of independent Brownian motions under $P^\infty$. From (3.8) and (3.9), we have that

$$b^m_\mathbf{w}(\mathbf{x}^m, t) - b^m_\mathbf{w}(\mathbf{x}^m, u) = \frac{1}{2} \sum_{|i| \geq m} \frac{B^m_i - \hat{B}^m_i + \hat{B}^m_i - \hat{B}^m_i}{(x^m_i - w^m_i)(x^m_i - w^m_i)}$$

$$= \frac{1}{2} \sum_{|i| \geq m} \frac{B^m_i - \hat{B}^m_i}{(x^m_i - w^m_i)(x^m_i - w^m_i)} + \frac{1}{2} \sum_{|i| \geq m} \frac{B^m_i - \hat{B}^m_i}{(x^m_i - w^m_i)(x^m_i - w^m_i)}.$$  

(3.10)

Let $\mathbf{W}$ be as in (17). To control the denominator in (3.11), we set

$$A_n = \{\mathbf{w} \in \mathbf{W}; \{\min_{t \in [a, b]} |x^m_i - w^m_i| \geq \frac{|j|}{n} \text{ for all } |j| \geq m\}. \quad (3.11)$$
Using (3.11), we deduce that, for $P^\infty$-a.s. $w \in A_n$,

$$\sup_{t \in [a, b]} \left\{ \sum_{|j| \geq m} \frac{|j|}{|x^i - w^j_t|^3} \right\} \leq \left\{ \sum_{|j| \geq m} \frac{|j|}{\min_{t \in [a, b]} |x^i - w^j_t|^3} \right\} \leq \left\{ \sum_{|j| \geq m} \frac{|j|}{(|j|^2)^3} \right\} \text{ by (3.11)}$$

$$= n^{3} \left\{ \sum_{|j| \geq m} \frac{1}{|j|^2} \right\} < \infty.$$

We set $Q(x^m) = \{ w \in W : e^{m,c/2}_{T,w} \cap (\{x^m\} \times [0, T]) \neq \emptyset \}$. Then using (1.6), (3.6), and (3.7), we deduce

$$P^\infty((\bigcup_{n \in \mathbb{N}} A_n)^c : Q(x^m)) = 0. \tag{3.13}$$

Hence, from (3.12) and (3.13), we obtain, for $P^\infty$-a.s. $w \in Q(x^m)$,

$$\mathcal{T}(w) := \sup_{t \in [a, b]} \left\{ \sum_{|j| \geq m} \frac{|j|}{|x^i - w^j_t|^3} \right\} < \infty. \tag{3.14}$$

Using Young’s inequality and (3.14), we have

$$\sum_{|j| \geq m} \left| \frac{B^j_t - B^j_u}{(x^i - w^j_t)(x^i - w^j_u)} \right| \leq \left( \sup_{t \in [a, b]} \sum_{|j| \geq m} \frac{|j|}{|x^i - w^j_t|^3} \right)^{1/3} \left( \sup_{u \in [a, b]} \sum_{|j| \geq m} \frac{|j|}{|x^i - w^j_u|^3} \right)^{1/3} \left( \sum_{|j| \geq m} \frac{|B^j_t - B^j_u|^3}{|j|^2} \right)^{1/3}$$

$$= \mathcal{T}(w)^{2/3} \left( \sum_{|j| \geq m} \frac{|B^j_t - B^j_u|^3}{|j|^2} \right)^{1/3}. \tag{3.15}$$

Similarly, for $P^\infty$-a.s. $w \in Q(x^m)$, we have that

$$\sum_{|j| \geq m} \left| \frac{\hat{B}^j_t - \hat{B}^j_u}{(x^i - w^j_t)(x^i - w^j_u)} \right| \leq \mathcal{T}(w)^{2/3} \left( \sum_{|j| \geq m} \frac{|\hat{B}^j_t - \hat{B}^j_u|^3}{|j|^2} \right)^{1/3}. \tag{3.16}$$

Recall that $c_1(w) < \infty$ for $P^\infty$-a.s. $w \in Q(x^m)$ from (3.13). Note that $\{B^j_t\}_{j \in \mathbb{Z}}$ and $\{\hat{B}^j_t\}_{j \in \mathbb{Z}}$ are sequences of independent Brownian motions. Then, we obtain Lemma 3.1 (1) from (3.10), (3.15), and (3.16).
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Let \((x^m, t)\) and \((y^m, t)\) be elements of \(\mathcal{Q}^m, \epsilon, w\). From (3.5), we have that

\[
\begin{align*}
  b_{w}^{m, i}(x^m, t) - b_{w}^{m, i}(y^m, t) &= \\
  &= \sum_{j \neq i, |j| < m} \frac{1}{x^j - x^i} - \sum_{j \neq i, |j| < m} \frac{1}{y^j - y^i} + \sum_{|j| \geq m} \frac{1}{x^j - w^j} - \sum_{|j| \geq m} \frac{1}{y^j - w^j} \\
  &= \sum_{j \neq i, |j| < m} \frac{1}{x^j - x^i} - \sum_{j \neq i, |j| < m} \frac{1}{y^j - y^i} + \sum_{|j| \geq m} \frac{1}{x^j - x^i} - \sum_{|j| \geq m} \frac{1}{y^j - y^i} + \sum_{|j| \geq m} \frac{1}{x^j - w^j} - \sum_{|j| \geq m} \frac{1}{y^j - w^j}.
\end{align*}
\]

Then, using (1.6) and (3.17), we obtain (2).

We define the probability measure on \(\mathbb{R}^m\) by

\[
P^m_w(t) = P^\infty(w^m \in \cdot | w^m_0).
\]

Let \(\mathcal{O}^{m, \epsilon}_{T, w}\) be as in (3.6). Let \(\mathcal{O}^{m, \epsilon}_{T, w}(t)\) be the cross section of \(\mathcal{O}^{m, \epsilon}_{T, w}\) such that

\[
\mathcal{O}^{m, \epsilon}_{T, w}(t) = \{x^m \in \mathbb{R}^m_\leq : (x^m, t) \in \mathcal{O}^{m, \epsilon}_{T, w}\}.
\]

Proof of Theorem 1.3. We consider the time-inhomogeneous heat equation on \(\mathcal{Q}^m, \epsilon, w\) such that the associated backward equation is given by

\[
\begin{align*}
\frac{\partial}{\partial t} + \frac{1}{2} \sum_{|i| < m} \left( \frac{\partial}{\partial x^i} \right)^2 + \sum_{|i|, |j| < m, i \neq j} \frac{1}{x^i - x^j} \frac{\partial}{\partial x^i} - \sum_{|i| \geq m} \frac{1}{x^i - w^i} \frac{\partial}{\partial x^i} \right) p(t, x, y, u) &= 0
\end{align*}
\]

and the forward equation is given by

\[
\begin{align*}
\frac{\partial}{\partial u} \frac{1}{2} \sum_{|i| < m} \left( \frac{\partial}{\partial x^i} \right)^2 - \sum_{|i|, |j| < m, i \neq j} \frac{1}{x^i - x^j} \frac{\partial}{\partial x^i} - \sum_{|i| \geq m} \frac{1}{x^i - u^i} \frac{\partial}{\partial x^i} \right) p(t, x, y, u) &= 0.
\end{align*}
\]

From Lemma 3.1, we have constants \(c_2\) and \(\alpha\) such that \(0 < \alpha < 1\) and

\[
|b_{w}^{m, i}(x^m, t) - b_{w}^{m, i}(y^m, u)| \leq c_2 \{ |x^m - y^m| + |t - u|^{\alpha} \}.
\]

From (3.22), we can apply a general theorem of heat equations to determine that the fundamental solution (the transition probability density) of (3.20) and (3.21) on \(\mathcal{O}^{m, \epsilon}_{T, w}\) under a Dirichlet boundary condition on the boundary is positive and continuous. Taking \(\epsilon \to 0\) and using the obvious inequality such that the heat kernel dominates that with the Dirichlet boundary condition, we find
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that the heat kernel \( p(t, x, u, y) = p^{m,0}_{T,\mathbf{w}}(t, x, u, y) \) on \( \mathcal{E}^{m,0}_{T,\mathbf{w}}(t) \times \mathcal{E}^{m,0}_{T,\mathbf{w}}(u) \) is a positive density of the transition probability with respect to the Lebesgue measure. Using (1.13), we find

\[
\int_{A \times B} p(0, x, t, y) dxdy = 0. \tag{3.23}
\]

From (3.23) and positivity of \( p \), we deduce that \( A \) or \( B \) have Lebesgue measure zero. Hence, either of the following hold:

\[
P^m_w(w_0^m \in A) = \int_{A \times \mathcal{E}^{m,0}_{T,\mathbf{w}}(t)} p(0, x, t, y) dxdy = 0 \quad \tag{3.24}
\]

or

\[
P^m_w(w_t^m \in B) = \int_{\mathcal{E}^{m,0}_{T,\mathbf{w}}(0) \times B} p(0, x, t, y) dxdy = 0. \tag{3.25}
\]

We thus obtain Theorem 1.3.

**Proof of Theorem 1.4** Using (1.14) and Fubini’s theorem, we deduce (1.13). Then, applying Theorem 1.3 we have that

\[
P^m_w(w_0^m \in A) = 0 \text{ for } P^m^*\text{-a.s. } w
\]

or

\[
P^m_w(w_t^m \in B) = 0 \text{ for } P^m^*\text{-a.s. } w.
\]

Integrating these with respect to \( P^m^* \), we conclude that Theorem 1.4 holds from (1.12).

**4 Proof of Theorem 1.1**

Let \( W^{m^*} = C([0, \infty); \mathbb{R}^{m^*}_<) \). Let \( \varpi^{m^*} : W \to W^{m^*} \) be the projection such that \( w = (w_i)_{i \in \mathbb{Z}} \mapsto w^{m^*} = (w^i)_{|i| \geq m^*} \). Let

\[
T = \{ t = (t_1, \ldots, t_l) : 0 < t_k < t_{k+1} (1 \leq k < l), \ l \in \mathbb{N} \}.
\]

We set \( \varpi^m_t(w) = w^m_t = (w^i_t)_{|i| \geq m}, \) where \( w^i_t = (w^i_{t_1}, \ldots, w^i_{t_l}), \) and

\[
\mathcal{C}_\text{path}^{\infty} = \bigvee_{t \in T} \bigcap_{m=1}^{\infty} \sigma[\varpi^m_t].
\]

We know that \( \mu \) is tail trivial [23, 115]. That is, \( \mu(A) \in \{0, 1\} \) for each \( A \in \mathcal{T}(\mathcal{G}) \), where

\[
\mathcal{T}(\mathcal{G}) = \bigcap_{R=1}^{\infty} \sigma[\pi^R].
\]
The tail triviality of $\mu$ can be refined to the triviality of $\psi^\infty_{\text{path}}$ with respect to $P^\infty$ using Lemma 4.1. The triviality of $\psi^\infty_{\text{path}}$ with respect to $P^\infty$ is one of the critical properties in the proof of the uniqueness of solutions to ISDEs in [24].

We require a rather difficult argument for the proof of this fact.

**Lemma 4.1.** $\psi^\infty_{\text{path}}$ is trivial with respect to $P^\infty$. That is,

$$P^\infty(\mathcal{A}) \in \{0, 1\} \text{ for each } \mathcal{A} \in \psi^\infty_{\text{path}}.$$

**Proof.** Lemma 4.1 follows directly from Theorem 5.3 in [24].

**Proof of Theorem 1.1.** Recall that $\Psi$ is the tail triviality of $\hat{\psi}_{\text{path}}^\infty$. We require a rather difficult argument for the proof of this fact.

Let $A_0 = \varpi(\mathcal{A})$, $A^m_0 = \varpi^0(\mathcal{A})$, $A^m_0 = \varpi^0_{\text{path}}(\mathcal{A})$, $B^m_0 = \varpi^0_{\text{path}}(\mathcal{A})$, $B^m_0 = \varpi^m_{\text{path}}(\mathcal{A})$.

Let $A$ and $B$ be as in the statement of Theorem 1.1. We take $\mathcal{A} = \varpi^{-1}(A)$ and $\mathcal{B} = \varpi^{-1}(B)$. Then we find $A = A_0$ and $B = B_t$. Noting $A_0^{m_0}, B_0^{m_0} \in \mathcal{F}_{t_0}^m$ and using (1.12), we deduce that, for $P^{m_0}$-a.s. $w$,

$$P^m_w(w_0^m \in A_0^m, w_t^m \in B_t^m | \mathcal{F}_{t_0}^m) = P^m_w(w_0^m \in A_0^m, w_t^m \in B_t^m, w_0^{m_0} \in A_0^{m_0}, w_0^{m_0} \in A_0^{m_0}, w_0^{m_0} \in A_0^{m_0}, w_0^{m_0} \in A_0^{m_0}, w_0^{m_0} \in A_0^{m_0}) \quad (4.1)$$

From (4.1) and (1.11), we have

$$\int_W P^m_w(w_0^m \in A_0^m, w_t^m \in B_t^m | \mathcal{F}_{t_0}^m) 1_{A_0^m} | \mathcal{F}_{t_0}^m) 1_{B_t^m} | \mathcal{F}_{t_0}^m) P^m(dw) = P^m_w(w_0^m \in A_0^m, w_t^m \in B_t^m, w_0^{m_0} = 0) \quad (4.2)$$

Using this, we obtain, for $P^{m_0}$-a.s. $w$,

$$P^m_w(w_0^m \in A_0^m, w_t^m \in B_t^m | \mathcal{F}_{t_0}^m) 1_{A_0^m} | \mathcal{F}_{t_0}^m) 1_{B_t^m} | \mathcal{F}_{t_0}^m) P^m(dw) = 0.$$
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Using \(4.2\) and Theorem 1.3 we deduce

\[
P^m(w_0^m \in A_0^m)1_{A_0^m}(w_0^m)1_{B_0^m}(w_0^m) = 0 \quad \text{for } P^m\text{-a.s. } w.
\] (4.3)

or

\[
P^m(w_t^m \in B_t^m)1_{A_0^m}(w_0^m)1_{B_t^m}(w_t^m) = 0 \quad \text{for } P^m\text{-a.s. } w.
\] (4.4)

Suppose (4.3). Then, using \(P_m^\ast = P^{\infty} \circ (w_0^m)^{-1}\), we obtain

\[
P^m(w_0^m \in A_0^m)1_{A_0^m}(w_0^m)1_{B_0^m}(w_0^m) = 0 \quad \text{for } P^{\infty}\text{-a.s. } w.
\] (4.5)

Taking \(B = W\) in (4.1) and using \(P_m^\ast = P^{\infty} \circ (w_0^m)^{-1}\), we obtain

\[
P^m(w_0^m \in A_0^m)1_{A_0^m}(w_0^m) = P^{\infty}(w_0 \in A_0; F_{0,t}^m) \quad \text{for } P^{\infty}\text{-a.s. } w.
\]

Hence, we deduce

\[
P^m(w_0^m \in A_0^m)1_{A_0^m}(w_0^m)1_{B_0^m}(w_t^m) = 0 \quad \text{for } P^{\infty}\text{-a.s. } w.
\] (4.6)

From (4.5) and (4.6), we obtain

\[
P^{\infty}(w_0 \in A_0; F_{0,t}^m)1_{B_0^m}(w_t^m) = 0 \quad \text{for } P^{\infty}\text{-a.s. } w.
\] (4.7)

Integrating (4.7) with respect to \(P^{\infty}\), we obtain

\[
\int_W P^{\infty}(w_0 \in A_0; F_{0,t}^m)1_{B_0^m}(w_t^m)P^{\infty}(dw) = 0.
\] (4.8)

Next, suppose (4.4). Then, similarly as (4.8), we obtain

\[
\int_W P^{\infty}(w_t \in B_t; F_{0,t}^m)1_{A_0^m}(w_0^m)P^{\infty}(dw) = 0.
\] (4.9)

Thus, we see either (4.8) or (4.9) holds for each \(m \in \mathbb{N}\). Hence, (4.8) holds for infinitely many \(m \in \mathbb{N}\) or (4.9) holds for infinitely many \(m \in \mathbb{N}\).

Note that the sequence of \(\sigma\)-fields \(\{F_{0,t}^m\}_{m \in \mathbb{N}}\) is decreasing. Furthermore, the sequences of sets

\[
\{(w_0^m)^{-1}(A_0^m)\}_{m \in \mathbb{N}} \quad \text{and} \quad \{(w_t^m)^{-1}(B_t^m)\}_{m \in \mathbb{N}}
\]

are increasing and the limits

\[
\mathcal{A} := \bigcup_{m=1}^{\infty} (w_0^m)^{-1}(A_0^m) \quad \text{and} \quad \mathcal{B} := \bigcup_{m=1}^{\infty} (w_t^m)^{-1}(B_t^m)
\]
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are $\mathcal{C}_\infty^\ast$-measurable. The sets $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{B}}$ contain $\mathcal{A}$ and $\mathcal{B}$, respectively. Hence, using the martingale convergence theorem and the Lebesgue convergence theorem, we find that, $P^\infty$-a.s. and in $L^1(W, P^\infty)$,

$$\lim_{m \to \infty} P^\infty(w_0 \in A_0 | \mathcal{F}_{0,t}^m) 1_{B_t^\ast}(w_t^m) = P^\infty(w_0 \in A_0 | \bigcap_{m=1}^\infty \mathcal{F}_{0,t}^m) 1_B(w), \quad \text{(4.10)}$$

$$\lim_{m \to \infty} P^\infty(w_t \in B_t | \mathcal{F}_{0,t}^m) 1_{A_t^\ast}(w_0^m) = P^\infty(w_t \in B_t | \bigcap_{m=1}^\infty \mathcal{F}_{0,t}^m) 1_{\tilde{\mathcal{A}}}(w). \quad \text{(4.11)}$$

From Lemma 4.1 and $\bigcap_{m=1}^\infty \mathcal{F}_{0,t}^m \subset \mathcal{C}_\infty^\ast$, we deduce $P^\infty(\tilde{\mathcal{A}}) \in \{0, 1\}$. Furthermore, $\{w; w_0 \in A_0\} \subset \tilde{\mathcal{A}}$ by construction. Hence, \begin{equation}
\int_{\tilde{\mathcal{A}}} P^\infty(w_0 \in A_0 | \bigcap_{m=1}^\infty \mathcal{F}_{0,t}^m) dP^\infty = P^\infty(\tilde{\mathcal{A}}) P^\infty(w_0 \in A_0). \quad \text{(4.12)}
\end{equation}

Similarly, we have

$$\int_{\tilde{\mathcal{B}}} P^\infty(w_t \in B_t | \bigcap_{m=1}^\infty \mathcal{F}_{0,t}^m) dP^\infty = P^\infty(\tilde{\mathcal{B}}) P^\infty(w_t \in B_t). \quad \text{(4.13)}$$

Suppose $P^\infty(\tilde{\mathcal{A}}) = 0$. Then $P^\infty(w_0 \in A_0) = 0$ because $\{w; w_0 \in A_0\} \subset \tilde{\mathcal{A}}$. Suppose $P^\infty(\tilde{\mathcal{A}}) = 1$. If, in addition, \begin{equation} \text{(4.8)} \end{equation} holds for infinitely many $m \in \mathbb{N}$, then from \begin{equation} \text{(4.8)}, \text{(4.10)}, \text{and} \text{(4.12)}, \end{equation} we deduce $P^\infty(w_0 \in A_0) = 0$.

Similarly, $P^\infty(\tilde{\mathcal{B}}) = 0$ implies $P^\infty(w_t \in B_t) = 0$. If $P^\infty(\tilde{\mathcal{B}}) = 1$ and \begin{equation} \text{(4.9)} \end{equation} holds for infinitely many $m \in \mathbb{N}$, then $P^\infty(w_t \in B_t) = 0$ from \begin{equation} \text{(4.9)}, \text{(4.11)}, \text{and} \text{(4.13)}. \end{equation}

Combining these and recalling $A = A_0$ and $B = B_t$ complete the proof. \qed

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