A free boundary inviscid model of flow-structure interaction

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Abstract. We obtain the local existence and uniqueness for a system describing interaction of an incompressible inviscid fluid, modeled by the Euler equations, and an elastic plate, represented by the fourth-order hyperbolic PDE. We provide a priori estimates for the existence with the optimal regularity $H^r$, for $r > 2.5$, on the fluid initial data and construct a unique solution of the system for initial data $u_0 \in H^r$ for $r \geq 3$. An important feature of the existence theorem is that the Taylor-Rayleigh instability does not occur.

Contents

1. Introduction 1
2. The model and the main results 3
3. A priori bounds 6
   3.1. Basic properties of the coefficient matrix $a$, the cofactor matrix $b$, and the Jacobian $J$ 6
   3.2. The tangential estimate 7
   3.3. Pressure estimates 11
   3.4. The vorticity estimate 14
   3.5. The conclusion of a priori bounds 17
4. Compatibility conditions 18
5. Uniqueness 19
6. The local existence 23
   6.1. Euler equations with given variable coefficients 23
   6.2. The plate equation 29
   6.3. Regularized Euler-plate system 30
   6.4. Applying the a priori estimates to constructed $\nu > 0$ solutions 33
Appendix 38
Acknowledgments 39
References 39

1. Introduction

In this paper, we prove the existence and uniqueness of local-in-time solutions to a system describing the interaction between an inviscid incompressible fluid and an elastic plate. The model couples the 3D incompressible Euler equations with a hyperbolic fourth-order equation that describes the motion of the free-moving interface. We consider a domain that is a channel with a rigid bottom boundary and a top moving boundary which is only allowed to move in the vertical direction according to a displacement function $w$. The function $w$ satisfies a fourth-order hyperbolic equation, with a forcing imposed by the fluid normal stress. The boundary conditions on the Euler equations match the normal component of the fluid velocity with the normal velocity of the plate, while periodic boundary conditions are imposed in the horizontal directions. We also prove that if solutions with the proposed regularity exist, they are...
unique. As far as we know, this is the first treatment of the moving boundary fluid-elastic structure system where the fluid is inviscid.

The viscous model, involving the Navier-Stokes equations, has been treated in the literature by several authors. The earliest known work on the free-moving domain model is by Beirão da Veiga [B], who considered the coupled 2D Navier-Stokes-plate model and established the existence of a strong solution. In [DEGL, CDEG], Desjardins et al. considered the existence of weak solutions to the 3D Navier-Stokes system coupled with a strongly damped plate. Weak solutions to the 2D model without damping were obtained in [G]. In all the treatments mentioned above, the plate equations were considered under clamped boundary conditions at the ends of the interface.

More recent works have considered an infinite plate model with periodic boundary conditions. In [GH], Grandmont and Hillairet obtained global solutions to the 2D model, with lower order damping on the plate. Local-in-time strong solutions for the 2D model were also constructed in [GHL] under different scenarios involving either a plate with rotational inertia (no damping) or a rod instead of a beam (wave equation). Models, where the plate equation on the lower dimensional interface is replaced by the damped wave equation, have also been treated earlier by Lequeurre in both 2D and 3D [L1,L2]. In another recent work, Badra and Takahashi [BT] proved the well-posedness and Gevrey regularity of the viscous 2D model without imposing any damping, rotational inertia, or any other approximation on the plate equations.

Models of viscous Koiter shell interactions which involve coupling the Navier-Stokes equations with fourth-order hyperbolic equations on cylindrical domains were also studied in numerous works [CS2, CCS, GGCC, GGCL, CGH, GM, L, LR, MC1, MC2, MC3]. The considered shell equations are nonlinear and model blood flow inside the arteries. The same 3D model on a cylindrical domain was also studied by Maity, Roy, and Raymond [MRR], who obtained the local-in-time solutions under less regularity on the initial data. For other related works on plate models, see [Bo, BKS, BS1, BS2, C, CK, DEGLT, MS] and for other results on fluids interacting with elastic objects, see [AL,Bo,BST,CS1,IKLT,KOT,KT,RV,TT].

For mathematical treatments of the non-moving boundary viscous models of flow structure interaction, one can find plenty of works on well-posedness and stabilization; cf. for example [AB, AGW, Ch, CR]. On the other hand, inviscid models have been treated mainly through linearized potential flow-structure interaction models on non-moving boundary [CLW, LW, W]. These models are mathematically valuable and physically meaningful if one considers a high order of magnitude in the structure velocity relative to the displacement in which non-moving domains provide a fairly good approximation.

Up to our knowledge, there have been no works in the literature on the well-posedness of the inviscid free boundary model where the Euler equations are considered in place of the Navier-Stokes equations. To address the existence of solutions, we use an ALE (Arbitrary Lagrangian Eulerian) formulation, which fixes the domain and provides the necessary additional regularity for the variables. In particular, we use a change of variable via the harmonic extension of the boundary transversal displacement. The a priori estimates are then obtained using a div-curl type bound on the fluid velocity. Tangential bounds provide control of the structure displacement and velocity, while the pressure term is determined by solving an elliptic problem with Robin boundary conditions on the plate.

The construction of solutions turns out to be a challenging problem. Naturally, we need to first solve the variable coefficients Euler equations with a non-homogeneous type boundary conditions (normal component) but the low regularity of the pressure on the boundary does not allow for the usual fixed point scheme to be carried through. In our construction scheme, we solve the variable coefficients Euler equations with nonhomogeneous type boundary conditions (normal component given) in five stages. In the first step, we solve a linear transport equation under more regular boundary data, where we rely on two new tools: an extension operator allowing to solve the problem on the whole space with no boundary conditions and a specially designed boundary value problem for the pressure that exploits the regularizing effect coming through the boundary data at the interface, and is based on certain cancellations that appear when formulating the Neumann Robin type boundary conditions for the pressure (cf. Remark 3.5). The approximate problem is then solved in the whole space employing a new technique involving Sobolev extensions without imposing any boundary conditions, and without imposing the variable divergence-free condition. In the next stage, the nonlinear problem, still with more regular boundary data, is solved by a fixed point technique using the extension operator and the solution of the linear problem. In the third stage, we prove that the unique fixed point solutions to the Euler equations with given variable coefficients satisfy the boundary conditions and the divergence conditions. In the fourth stage, we employ the vorticity formulation (pressure free) whereby we solve a div-curl type systems and derive estimates for the full regularity of velocity in terms of less regular boundary data. In the final step, we derive solutions to the variable Euler equations under less regular data using a standard density argument and the uniform estimates in the previous step, thus concluding the proof of existence for the variable Euler equations.
For the construction of solutions to the coupled Euler-plate system, the low regularity of the pressure does not allow for a fixed point scheme to be used. Instead, we use the fixed point scheme to obtain solutions to a regularized system that includes a damping term in the plate. Once solutions to the regularized system are obtained, the coupled a priori estimates which involve the cancellation of the pressure boundary terms give rise to estimates uniform in the damping parameter $\nu$ and thus allow us to pass through the limit in the damping parameter to obtain solutions to the original system without damping.

The paper is structured as follows. In Section 2, we introduce the model and restate it in the ALE variables. The first main result, contained in Theorem 2.1, provides a priori estimates for the existence of a local in time solution for the initial velocity in $H^{2.5+\delta}$ (the minimal regularity for the classical Euler equations) and the initial plate velocity in $H^{2+\delta}$, where $\delta > 0$ is arbitrary and not necessarily small. Next, Theorem 2.3 provides the existence of a local solution, i.e., gives a construction of a solution, when $\delta \geq 0.5$. In Section 3, we prove the statement on the a priori estimates. The first part of the proof, stated in Lemma 3.1, contains bounds on the cofactor matrix and the Jacobian. The estimates controlling the tangential components on the boundary are obtained in Lemma 3.2. There, energy estimates performed on the plate equation are derived by exploiting the coupling with the Euler equation to eliminate the pressure term. A characteristic feature of these estimates is that the fluid velocity in the interior appears as a lower order term in these estimates.

The estimates controlling the pressure term are derived in Lemma 3.3 via solving an elliptic problem for the pressure with Robin type boundary conditions on the moving interface. Control of the interior fluid velocity is accomplished using the ALE vorticity formulation and div-curl type estimates (Lemma 3.7) with estimates performed on the whole space using Sobolev extensions. The proof of Theorem 2.1 is then provided in Section 3.5. Next, a short Section 4 provides a discussion on the compatibility conditions imposed on the data at the boundary. Section 5 contains the proof of uniqueness of solutions, in the regularity class $H^{2.5+\delta}$ for the fluid velocity $v$ and $H^{4+\delta} \times H^{2+\delta}$ for plate displacement $w$ and velocity $w_1$, with the additional constraint $\delta \geq 0.5$. This additional constraint on the regularity exponent turns out to be necessary in the uniqueness argument when performing the pressure estimate (cf. the comment below (5.1)) and when bounding the commutator terms on the difference of the two solutions.

Finally, in Section 6, we provide the construction of solutions. We start with the construction of solutions for the variable coefficient Euler equations, where the difficulties are the inflow condition (6.6) on the top and the low regularity of the pressure boundary condition (6.18) due to the first term, $w_{tt}$. In the second step, Sections 6.2–6.4, we construct a local solution for a regularized Euler-plate system. Finally, in the last step of the proof, we pass to the limit in the plate damping parameter $\nu \to 0$, concluding the construction.

## 2. The model and the main results

We consider a flow-structure interaction system, defined on an open bounded domain $\Omega(t) \subseteq \mathbb{R}^3$, which evolves in time $t$ over $[0, T]$, where $T > 0$. The dynamics of the flow are modeled by the incompressible Euler equations

\[
\begin{align*}
    u_t + (u \cdot \nabla) u + \nabla p &= 0 \\
    \nabla \cdot u &= 0
\end{align*}
\]

in $\Omega(t) \times [0, T]$. For simplicity of presentation, we assume that $\Omega(0) = \Omega = T^2 \times [0, 1]$, i.e., the initial domain is $\mathbb{R}^2 \times [0, 1]$, with the 1-periodic boundary conditions on the sides. Denote

\[
\Gamma_1 = T^2 \times \{1\}
\]

and

\[
\Gamma_0 = T^2 \times \{0\}
\]

the initial position of the upper and the lower portions of the boundary. We impose the slip boundary condition on the bottom

\[
u \cdot N = 0 \quad \text{on } \Gamma_0.
\]

A function $w: \Gamma_1 \times [0, T) \rightarrow \mathbb{R}$ satisfies the fourth-order damped plate equation

\[
w_{ttt} + \Delta_2^2 w - \nu \Delta_2 w_t = \rho \quad \text{on } \Gamma_1 \times [0, T],
\]

where $\nu \geq 0$ is fixed; the pressure $p$ is evaluated at $(x_1, x_2, w(x_1, x_2, t))$, with the initial condition

\[
(w, w_t)|_{t=0} = (0, w_1).
\]
solutions with $\nu > 0$ satisfying a uniform in $\nu$ bound in an appropriate solution space, and pass to the limit as $\nu \to 0$. The reason why the parameter $\nu > 0$ is needed is the low regularity of the pressure term forcing the plate equation, while a priori estimates rely on cancellation of the lower regularity term involving the pressure. Since we are mainly interested in the limiting case $\nu = 0$, we always assume $\nu \in [0,1]$. The variable $w$ represents the height of the interface at $t \in [0,T]$. We assume that the plate evolves with the fluid velocity, and $w$ thus satisfies the kinematic condition

$$w_t + u_1 \partial_1 w + u_2 \partial_2 w = u_3. \tag{2.7}$$

Note that (2.7) may be rewritten as

$$[(\partial_1 w, \partial_2 w, -1)u(x_1, x_2, w(x_1, x_2, t)) \cdot n = 0, \tag{2.8}$$

where $n$ is the dynamic normal, asserting matching of the normal velocity components. Denote by $\psi: \Omega \to \mathbb{R}$ the harmonic extension of $1 + w$ to the domain $\Omega = \Omega(0)$, i.e., assume that $\psi$ solves

$$\Delta \psi = 0 \quad \text{on } \Omega$$

$$\psi(x_1, x_2, 1, t) = 1 + w(x_1, x_2, t) \quad \text{on } \Gamma_1 \times [0,T]$$

$$\psi(x_1, x_2, 0, t) = 0 \quad \text{on } \Gamma_0 \times [0,T]. \tag{2.9}$$

Next, we define $\eta: \Omega \times [0,T] \to \Omega(t)$ as

$$\eta(x_1, x_2, x_3, t) = (x_1, x_2, \psi(x_1, x_2, x_3, t)), \quad (x_1, x_2, x_3) \in \Omega, \tag{2.10}$$

which represents the ALE change of variable. Note that

$$\nabla \eta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \partial_1 \psi & \partial_2 \psi & \partial_3 \psi \end{pmatrix}. \tag{2.11}$$

Denote $a = (\nabla \eta)^{-1}$, or in the matrix notation

$$a = \frac{1}{J} b = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\partial_1 \psi / \partial_3 \psi & -\partial_2 \psi / \partial_3 \psi & 1 / \partial_3 \psi \end{pmatrix}, \tag{2.12}$$

where

$$J = \partial_3 \psi \tag{2.13}$$

is the Jacobian and

$$b = \begin{pmatrix} \partial_3 \psi & 0 & 0 \\ 0 & \partial_3 \psi & 0 \\ -\partial_1 \psi & -\partial_2 \psi & 1 \end{pmatrix} \tag{2.14}$$

stands for the cofactor matrix. Since $b$ is the cofactor matrix, it satisfies the Piola identity

$$\partial_i b_{ij} = 0, \quad j = 1, 2, 3, \tag{2.15}$$

which can also be verified directly from (2.14). We use the summation convention on repeated indices; thus, unless indicated otherwise, the repeated indices are summed over 1, 2, 3. Next, denote by

$$v(x, t) = u(\eta(x, t), t)$$

$$q(x, t) = p(\eta(x, t), t) \tag{2.16}$$

the ALE velocity and the pressure. With this change of variable, the system (2.1) becomes

$$\partial_t v_1 + v_1 a_{j1} \partial_j v_1 + v_2 a_{j2} \partial_j v_1 + \frac{1}{\partial_3 \psi} (v_3 - \psi_t) \partial_3 v_1 + a_{k3} \partial_k q = 0, \tag{2.17}$$

in $\Omega \times [0,T]$, where we used $a_{j3} \partial_j v_i = (1/\partial_3 \psi) \partial_3 v_i$. The initial condition reads

$$v|_{t=0} = v_0. \tag{2.18}$$

The boundary condition on the bottom boundary is

$$v_3 = 0 \quad \text{on } \Gamma_0, \tag{2.19}$$

while, using (2.14) and the second equation in (2.9), we may rewrite (2.7) as

$$b_{3i} v_i = w_t \quad \text{on } \Gamma_1. \tag{2.20}$$
On the other hand, the plate equation (2.5) simply reads
\[ w_{tt} + \Delta_2^2 w - \nu \Delta_2 w_t = q, \]  
where the pressure is normalized by the condition
\[ \int_{\Gamma_1} q = 0, \]  
for all \( t \in [0, T] \).

The next theorem, asserting the a priori estimates for the local existence for the flow-structure problem (2.17)–(2.21), is the main result of the paper.

**Theorem 2.1. (A priori estimates for existence)** Let \( 0 \leq \nu \leq 1 \). Assume that \((v, w)\) is a \( C^\infty \) solution on an interval \([0, T]\) with
\[ \|v_0\|_{H^{2+\delta}(\Omega)} \leq M, \]  
where \( M \geq 1 \) and \( \delta > 0 \). Then \( v, w, \psi, \) and \( \nu \) satisfy
\[ \|v\|_{H^{2+\delta}(\Gamma_1)} \leq \|w_1\|_{H^{2+\delta}(\Gamma_1)}, \|w_t\|_{H^{2+\delta}(\Gamma_1)}, \|\psi\|_{H^{2+\delta}(\Gamma_1)}, \|\psi_t\|_{H^{2+\delta}(\Gamma_1)}, \|a\|_{H^{2+\delta}(\Gamma_1)} \leq C_0 M, \quad t \in [0, T], \]  
with
\[ \nu^{1/2}\|w_t\|_{L^2 H^{3+\delta}(\Gamma_1 \times [0, T])} \leq C_0 M \]  
and
\[ \|v_t\|_{H^{1+\delta}(\Gamma_1)}, \|w_t\|_{H^{1+\delta}(\Gamma_1)} \leq K, \quad t \in [0, T], \]  
where \( C_0 > 0 \) is a constant, \( K \) and \( T_0 \) are constants depending on \( M \). In particular, \( C_0, K, \) and \( T_0 \) do not depend on \( \nu \).

The parameter \( \delta > 0 \), which does not have to be small, is fixed throughout; in particular, we allow all the constants to depend on \( \delta \) without mention. All the results in this paper also apply when \( \nu \geq 1 \) with the constants depending on \( \nu \). The proof of Theorem 2.1 is provided in Section 3.

Next, we assert the uniqueness of solutions in Theorem 2.1. For this, we need slightly more regular solutions; namely, we need to assume \( \delta \geq 0.5 \).

**Theorem 2.2. (Uniqueness)** Let \( 0 \leq \nu \leq 1 \) and \( \delta \geq 0.5 \). Assume that two solutions \((u, w)\) and \((\tilde{u}, \tilde{w})\) satisfy the regularity (2.24)–(2.26) for some \( T > 0 \) and
\[ (v(0), w(0)) = (\tilde{v}(0), \tilde{w}(0)). \]  
Then \((u, v)\) and \((\tilde{u}, \tilde{v})\) agree on \([0, T]\).

The theorem is proven in Section 5.

Next, we assert the local existence with initial data \((v_0, w_1)\) in \( H^m(\Omega) \times H^{m-0.5} \) where \( m \geq 3 \) is not necessarily an integer.

**Theorem 2.3. (Local existence)** Let \( 0 \leq \nu \leq 1 \). Assume that initial data
\[ (v_0, w_1) \in H^{2.5+\delta} \times H^{2+\delta}(\Gamma_1), \]  
where \( \delta \geq 0.5 \), satisfy the compatibility conditions
\[ v_0 \cdot N|_{\Gamma_1} = w_1 \]  
and
\[ v_0 \cdot N|_{\Gamma_0} = 0 \]  
with
\[ \text{div } v_0 = 0 \quad \text{in } \Omega, \]  
and
\[ \int_{\Gamma_1} w_1 = 0. \]
Then there exists a unique local-in-time solution \((v, q, w, w_t)\) to the Euler-plate system (2.17)--(2.21) with the initial data \((v_0, w_1)\) such that
\[
\begin{align*}
v &\in L^\infty([0, T]; H^{2.5+\delta}(\Omega)) \cap C([0, T]; H^{0.5+\delta}(\Omega)), \\
v_t &\in L^\infty([0, T]; H^{0.5+\delta}(\Omega)), \\
q &\in L^\infty([0, T]; H^{1.5+\delta}(\Omega)), \\
w &\in L^\infty([0, T]; H^{4+\delta}(\Gamma_1)), \\
w_t &\in L^\infty([0, T]; H^{2+\delta}(\Gamma_1)),
\end{align*}
\] (2.32)
for some time \(T > 0\) depending on the size of the initial data.

The theorem is proven in Section 6 below. Note that (2.32), self-improves to \(v \in C([0, T]; H^{2.5+\delta_0}(\Omega))\) for any \(\delta_0 < \delta\).

3. A priori bounds

This section is devoted to establishing the a priori bounds for the Euler-plate system.

3.1. Basic properties of the coefficient matrix \(a\), the cofactor matrix \(b\), and the Jacobian \(J\). Note that, by multiplying the equation (2.17) \(_2\) with \(J\) and integrating it over \(\Omega\), while using (2.19) and the Piola identity (2.15), we get \(\int_{\Gamma_1} b_{3i} v_t = 0\), which in turn, by (2.20), implies
\[
\int_{\Gamma_1} w_t = 0. \tag{3.1}
\]
Also, since \(a = (\nabla \eta)^{-1}\), we have
\[
a_t = -a \nabla \eta a, \tag{3.2}
\]
where the right-hand side is understood as a product of three matrices. In the proof of the a priori estimates, we work on an interval of time \([0, T]\) such that (2.24) holds, where \(C_0\) is a fixed constant determined in the Gronwall argument below.

**Lemma 3.1.** Let \(\epsilon \in (0, 1/2]\). Assume that
\[
\|v\|_{H^{2.5+\delta}}, \|w\|_{H^{4+\delta}(\Gamma_1)}, \|w_t\|_{H^{2+\delta}(\Gamma_1)} \leq C_0 M, \quad t \in [0, T_0],
\] (3.3)
where \(M \geq 1\) is as in the statement of Theorem 2.1. Then we have
\[
\|a - I\|_{H^{1.5+\delta}}, \|b - I\|_{H^{1.5+\delta}}, \|J - 1\|_{H^{1.5+\delta}} \leq \epsilon, \quad t \in [0, T_0]\]
(3.4)
and
\[
\|J - 1\|_{L^\infty} \leq \epsilon, \quad t \in [0, T_0],
\] (3.5)
where \(T_0\) satisfies
\[
0 < T_0 \leq \frac{\epsilon}{CM^3},
\]
and \(C\) depends on \(C_0\).

Note that, by (3.4), we also have
\[
\|a - I\|_{L^\infty}, \|b - I\|_{L^\infty} \lesssim \epsilon, \quad t \in [0, T_0],
\]
while (3.5) gives
\[
\frac{1}{2} \leq J \leq \frac{3}{2}, \quad t \in [0, T_0];
\] (3.6)
in particular, \(J = \partial_3 \psi\) is positive and stays away from 0. The pressure estimates require \(\epsilon \leq 1/C\), where \(C\) is a constant, while we need \(\epsilon \leq 1/CM\) when concluding the a priori estimates in Section 3.5 below. Therefore, we fix
\[
\epsilon = \frac{1}{CM}, \tag{3.7}
\]
where we assumed for convenience \(M \geq 1\) and work with
\[
T_0 = \frac{1}{CM^3}, \tag{3.8}
\]
where $C$ is a sufficiently large constant. The symbol $C \geq 1$ denotes a sufficiently large constant, which may change from inequality to inequality. Also, we write $A \lesssim B$ when $A \leq C B$ for a constant $C$.

Before the proof, note that by the definitions of $\psi$ and $\eta$ in (2.9) and (2.10) we have
\[
\|\eta\|_{H^{3.5+\delta}} \lesssim \|\psi\|_{H^{4.5+\delta}} \lesssim \|w\|_{H^{4+\delta}(\Gamma_1)}
\]  
(3.9)
and
\[
\|\eta_t\|_{H^{2.5+\delta}} \lesssim \|\psi_t\|_{H^{2.5+\delta}} \lesssim \|w_t\|_{H^{2+\delta}(\Gamma_1)},
\]  
(3.10)
and both far right sides are bounded by constant multiples of $M$.

Also, we have
\[
\|J\|_{H^{3.5+\delta}} = \|\partial_3\psi\|_{H^{3.5+\delta}} \lesssim \|\psi\|_{H^{4.5+\delta}} \lesssim \|w\|_{H^{4+\delta}(\Gamma_1)}
\]  
(3.11)
and
\[
\|J_t\|_{H^{2.5+\delta}} \lesssim \|\psi_t\|_{H^{2.5+\delta}} \lesssim \|\eta\|_{H^{2.5+\delta}} \lesssim \|w_t\|_{H^{2+\delta}(\Gamma_1)},
\]  
(3.12)
with both right sides bounded by a constant multiple of $M$.

**Proof of Lemma 3.1.** By (3.2), we have
\[
\|a - I\|_{H^{1.5+\delta}} \lesssim \left\| \int_0^t a \nabla a ds \right\|_{H^{1.5+\delta}} \lesssim T_0 M^3,
\]
where we used (3.9) and (3.12). Now we only need to choose $T_0 \leq \epsilon/C M^3$, where $C$ is a sufficiently large constant, and the bound on the first term in (3.4) is established. Similarly, we have
\[
\|J - 1\|_{H^{1.5+\delta}} \lesssim \left\| \int_0^t J_t ds \right\|_{H^{1.5+\delta}} \lesssim M T_0.
\]
The bound for $\|b - I\|_{H^{1.5+\delta}}$ follows immediately from those on $\|a - I\|_{H^{1.5+\delta}}$ and $\|J - 1\|_{H^{1.5+\delta}}$ by using $b = J a$. □

As pointed out above, the value of $\epsilon$ in Lemma 3.1 is fixed in the pressure estimates and then further restricted in the conclusion of a priori bounds in Section 3.5.

Note that by the definitions of $a$ and $b$ in the beginning of Section 2, we have
\[
\|a\|_{H^{3.5+\delta}}, \|b\|_{H^{3.5+\delta}} \leq P(\|w\|_{H^{4+\delta}(\Gamma_1)}),
\]  
(3.13)
and
\[
\|a_t\|_{H^{1.5+\delta}}, \|b_t\|_{H^{1.5+\delta}} \leq P(\|w\|_{H^{4+\delta}(\Gamma_1)}, \|w_t\|_{H^{2+\delta}(\Gamma_1)}).
\]  
(3.14)
Above and in the sequel, the symbol $P$ denotes a generic polynomial of its arguments. It is assumed to be nonnegative and is allowed to change from inequality to inequality.

**3.2. The tangential estimate.** Denote
\[
\Lambda = (I - \Delta_2)^{1/2},
\]  
(3.15)
where $\Delta_2$ denotes the Laplacian in $x_1$ and $x_2$ variables. The purpose of this section is to obtain the following a priori estimate.

**Lemma 3.2.** Under the assumptions of Theorem 2.1, we have
\[
\|\Lambda^{4+\delta} w\|_{L^2(\Gamma_1)}^2 + \|\Lambda^{2+\delta} w_t\|_{L^2(\Gamma_1)}^2 + \nu \int_0^t \|\nabla \Lambda^{2+\delta} w_t\|_{L^2(\Gamma_1)}^2 ds \lesssim \|w_t(0)\|_{H^{2+\delta}(\Gamma_1)}^2 + \|v_t(0)\|_{H^{2.5+\delta}}^2 + \|v\|_{L^2} \|v\|_{H^{(4+2\delta)/(2.5+\delta)}} \|w_t\|_{H^{2+\delta}(\Gamma_1)} ds + 
\]
\[
+ \int_0^t P(\|v\|_{H^{2.5+\delta}}, \|w\|_{H^{4+\delta}(\Gamma_1)}, \|w_t\|_{H^{2+\delta}(\Gamma_1)}) ds,
\]
for $t \in [0, T_0]$. 


PROOF OF LEMMA 3.2. Assume that (2.23) holds. We test the plate equation (2.21) with \( \Lambda^{2(\delta)} w_t \), obtaining
\[
\frac{1}{2} \frac{d}{dt} \left( \| \Delta_2 \Lambda^{2+\delta} w \|_{L^2(\Gamma_1)}^2 + \| \Lambda^{2+\delta} w_t \|_{L^2(\Gamma_1)}^2 \right) + \nu \| \nabla_2 \Lambda^{2+\delta} w_t \|_{L^2(\Gamma_1)}^2 = \int_{\Gamma_1} q \Lambda^{2(\delta)} w_t. \tag{3.17}
\]
Integrating in time leads to
\[
\frac{1}{2} \| \Delta_2 \Lambda^{2+\delta} w \|_{L^2(\Gamma_1)}^2 + \frac{1}{2} \| \Lambda^{2+\delta} w_t \|_{L^2(\Gamma_1)}^2 + \nu \int_0^t \| \nabla_2 \Lambda^{2+\delta} w_t \|_{L^2(\Gamma_1)}^2 \, ds = \frac{1}{2} \| \Lambda^{2+\delta} w_t(0) \|_{L^2(\Gamma_1)}^2 + \int_0^t \int_{\Gamma_1} q \Lambda^{2(\delta)} w_t, \tag{3.18}
\]
where we also used \( w(0) = 0 \). Note that, by (2.4), the boundary condition on \( \Gamma_0 \) reads
\[
v \cdot N = 0 \quad \text{on} \ \Gamma_0. \tag{3.19}
\]
To obtain (3.16), we claim that
\[
\frac{1}{2} \int J \Lambda^{1.5+\delta} v_1 \Lambda^{2.5+\delta} v_i \bigg|_{t=0} - \frac{1}{2} \int J \Lambda^{1.5+\delta} v_1 \Lambda^{2.5+\delta} v_i \bigg|_0 + \int J \Lambda^{1.5+\delta} \partial_t v_1 \Lambda^{2.5+\delta} v_i + \bar{I}, \tag{3.20}
\]
where
\[
\bar{I} \leq P(v, w)_{H^{2.5+\delta}}. \tag{3.22}
\]
To show (3.21)–(3.22), first observe that, by the product rule, (3.21) holds with
\[
\bar{I} = \frac{1}{2} \int J \Lambda^{1.5+\delta} v_1 \Lambda^{2.5+\delta} \partial_t v_i - \frac{1}{2} \int J \Lambda^{1.5+\delta} \partial_t v_1 \Lambda^{2.5+\delta} v_i. \tag{3.23}
\]
In order to show that (3.23) has a commutator form, we rewrite
\[
\bar{I} = \frac{1}{2} \left( J \Lambda^{1.5+\delta} v_1 - \Lambda(J \Lambda^{2.5+\delta} v_i) \right) A^{0.5+\delta} \partial_t v_i = \frac{1}{2} \left( J \Lambda^{1.5+\delta} v_1 - \Lambda(J \Lambda^{2.5+\delta} v_i) \right) A^{0.5+\delta} \partial_t v_i + \frac{1}{2} \left( J \Lambda^{1.5+\delta} v_1 - \Lambda(J \Lambda^{2.5+\delta} v_i) \right) A^{0.5+\delta} \partial_t v_i \tag{3.24}
\]
where in the last inequality, we bounded \( v_i \) in terms of \( v \) and \( q \) directly from (2.17), as
\[
\| v_i \|_{H^{0.5+\delta}} \leq P(v, w)_{H^{2.5+\delta}}. \tag{3.25}
\]
Thus (3.21), with the estimate (3.22), is established. Therefore, the equations (2.17) may be rewritten as
\[
J \partial_t v_i + v_1 b_{ij} \partial_j v_i + v_2 b_{ij} \partial_j v_i + (v_3 - \psi_t) \partial_3 v_i + b_{ik} \partial_k q = 0, \tag{3.26}
\]
\[
b_{ki} \partial_k v_i = 0.
\]
Using (3.26) in the second term of (3.21), we obtain
\[
\frac{1}{2} \frac{d}{dt} \int J \Lambda^{1.5+\delta} v_i \Lambda^{2.5+\delta} v_i \nonumber = \frac{1}{2} \int J \Lambda^{1.5+\delta} v_i \Lambda^{2.5+\delta} v_i + \int \left( J \Lambda^{1.5+\delta} (\partial_t v_i) - \Lambda^{1.5+\delta} (J \partial_t v_i) \right) \Lambda^{2.5+\delta} v_i \\
- \sum_{m=1}^2 \int \Lambda^{1.5+\delta} (v_m b_{m,i} \partial_j v_j) \Lambda^{2.5+\delta} v_i - \int \Lambda^{1.5+\delta} \left( (v_3 - \psi_t) \partial_3 v_3 \right) \Lambda^{2.5+\delta} v_i \\
- \int \Lambda^{2.5+\delta} (b_{k,i} \partial_k q) \Lambda^{2.5+\delta} v_i + \bar{I}.
\]
(3.27)

Above and in the sequel, unless indicated otherwise, all integrals and norms are assumed to be over \( \Omega \). For the first two terms, we have
\[
I_1 + I_2 \lesssim \|J\|_{L^\infty} \|v\|_{H^{1.5+\delta}} \|v\|_{H^{2.5+\delta}} + \|\Lambda^{1.5+\delta} J\|_{L^2} \|v\|_{L^2} \|\Lambda^{2.5+\delta} v\|_{L^2} + \|\Lambda J\|_{L^\infty} \|\Lambda^{0.5+\delta} v_i\|_{L^2} \|\Lambda^{2.5+\delta} v_i\|_{L^2} \\
\lesssim \|J\|_{H^{1.5+\delta}} \|v\|_{H^{2.5+\delta}} + \|J\|_{H^{2.5+\delta}} \|v\|_{H^{2.5+\delta}} + \|J\|_{H^{2.5+\delta}} \|v\|_{H^{2.5+\delta}} + \|J\|_{H^{2.5+\delta}} \|v\|_{H^{2.5+\delta}} \\
\leq P(\|v\|_{H^{2.5+\delta}}, \|v\|_{H^{2.5+\delta}}, \|w\|_{H^{2.5+\delta}}, \|w\|_{H^{2.5+\delta}}),
\]
where we used (3.11) and (3.12) in the third inequality and (3.25) in the fourth. For \( I_3 \), we write
\[
I_3 \lesssim \|v_m b_{m,i} \partial_j v_j\|_{H^{1.5+\delta}} \|v\|_{H^{2.5+\delta}} \lesssim \|v\|_{H^{2.5+\delta}} \|b\|_{H^{3+\delta}} \leq P(\|v\|_{H^{2.5+\delta}}, \|w\|_{H^{1+\delta}}),
\]
by (3.13). In the last step, we used the multiplicative Sobolev inequality
\[
\|a b c\|_{H^k} \lesssim \|a\|_{H^l} \|b\|_{H^m} \|c\|_{H^n},
\]
(3.28)
where \( l, m, n \geq k \geq 0 \), which holds when \( l + m + n > 3 + k \) or when \( l + m + n = 3 + k \) and at least two of the parameters \( l, m, n \) are strictly greater than \( k \). Next, we treat \( I_4 \) similarly to \( I_3 \) and write
\[
I_4 \lesssim \|v_3 \partial_3 v\|_{H^{1.5+\delta}} \|v\|_{H^{2.5+\delta}} + \|\partial_3 \eta_3 \|_{H^{1.5+\delta}} \|v\|_{H^{2.5+\delta}} \\
\lesssim \|v_3 \partial_3 v\|_{H^{2.5+\delta}} + \|\partial_3 \eta_3 \|_{H^{2.5+\delta}} \|v\|_{H^{2.5+\delta}} + \|\partial_3 \eta_3 \|_{H^{2.5+\delta}} \|v\|_{H^{2.5+\delta}} \leq P(\|v\|_{H^{2.5+\delta}}, \|w\|_{H^{2.5+\delta}}),
\]
where we used (3.28), and a similar multiplicative Sobolev inequality for two factors
\[
\|ab\|_{H^k} \lesssim \|a\|_{H^l} \|b\|_{H^m},
\]
where either \( l, m \geq k \) and \( l + m > k + 1.5 \) or \( l, m > k \geq 0 \) and \( l + m = k + 1.5 \). Finally, we treat the pressure term \( I_5 \), for which we use integration by parts in \( x_k \) to rewrite it as
\[
I_5 = - \int \Lambda^{2.5+\delta} (b_{k,i} \partial_k q) \Lambda^{2.5+\delta} v_i = \int \Lambda^{2.5+\delta} (b_{k,i} q) \Lambda^{2.5+\delta} \partial_k v_i - \int_{\Gamma_1} \Lambda^{2.5+\delta} (b_{3,i} q) \Lambda^{2.5+\delta} v_i \\
= \int_{\Gamma_1} \Lambda^{1.5+\delta} (b_{k,i} q) \Lambda^{2.5+\delta} \partial_k v_i - \int_{\Gamma_1} \Lambda^{2.5+\delta} (b_{3,i} q) \Lambda^{2.5+\delta} v_i \\
= I_{51} + I_{52},
\]
(3.29)
where we used the Piola identity (2.15) and \( N = (0, 0, 1) \) on \( \Gamma_1 \). Note that the boundary integral over \( \Gamma_0 \) vanishes since
\[
\int_{\Gamma_0} \Lambda^{2.5+\delta} (b_{k,i} q) \Lambda^{2.5+\delta} v_i \|b\|_{H^{2.5+\delta}} = \int_{\Gamma_0} \Lambda^{2.5+\delta} (b_{3,i} q) \Lambda^{2.5+\delta} v_i = \int_{\Gamma_0} \Lambda^{2.5+\delta} (b_{3,i} q) \Lambda^{2.5+\delta} v_i = 0,
\]
where we used (2.14) and (2.9) in the second step, and (3.19) in the third. For the first term in (3.29), we have
\[
I_{51} = \int b_{k,i} \Lambda^{1.5+\delta} q \Lambda^{2.5+\delta} \partial_k v_i + \int \left( \Lambda^{1.5+\delta} (b_{k,i} q) - b_{k,i} \Lambda^{1.5+\delta} q \right) \Lambda^{2.5+\delta} \partial_k v_i \\
= - \int \Lambda^{2.5+\delta} \partial_k (b_{k,i} q) - b_{k,i} \Lambda^{2.5+\delta} \partial_k v_i \Lambda^{1.5+\delta} q + \int \left( \Lambda^{1.5+\delta} (b_{k,i} q) - b_{k,i} \Lambda^{1.5+\delta} q \right) \Lambda^{2.5+\delta} \partial_k v_i \\
= I_{511} + I_{512},
\]
(3.30)
where we used (3.26)2 and the Piola identity (2.15) in the second equality. For the first term, we use the Kato-Ponce commutator inequality to write
\[
I_{511} \lesssim \|b\|_{H^{3.5+\delta}}\|v\|_{L^\infty} + \|b\|_{H^{2.5+\delta}}\|q\|_{H^{2.5+\delta}} \leq P(\|v\|_{H^{2.5+\delta}}, \|w\|_{H^{2+\delta}(\Gamma_1)}),
\]

The term \(I_{512}\) cannot be treated with the Kato-Ponce commutator estimate directly since \(v\) is not bounded in \(H^{3.5+\delta}\). Instead, we use \(\Lambda^2 = I - \sum_{m=1}^2 \partial_m^2\), by (3.15), and write
\[
I_{512} = -\sum_{m=1}^2 \int \partial_m \left( \Lambda^{1.5+\delta} (b_k q) - b_k \Lambda^{1.5+\delta} q \right) \Lambda^{0.5+\delta} \partial_m \partial_k v_i + \int \left( \Lambda^{1.5+\delta} (b_k q) - b_k \Lambda^{1.5+\delta} q \right) \Lambda^{0.5+\delta} \partial_k v_i.
\]

The first term is bounded using the Kato-Ponce commutator estimate, while the second and the third terms are estimated directly. Thus,
\[
I_{512} \lesssim \|b\|_{H^{2.5+\delta}}\|q\|_{H^1} + \|b\|_{H^{2.5+\delta}}\|q\|_{H^{2.5+\delta}} \leq P(\|v\|_{H^{2.5+\delta}}, \|q\|_{H^{2.5+\delta}}).
\]

The boundary term \(I_{52}\) may be rewritten as
\[
I_{52} = -\int_{\Gamma_1} \Lambda^{1+\delta} q (b_k, A^{2+\delta} v_i) - \int_{\Gamma_1} \left( \Lambda^{2+\delta} (b_k q) - b_k \Lambda^{2+\delta} q \right) \Lambda^{2+\delta} v_i,
\]
which may be rewritten as
\[
I_{52} = -\int_{\Gamma_1} \Lambda^{1+\delta} q b_k A^{3+\delta} v_i - \int_{\Gamma_1} \Lambda^{1+\delta} q \left( b_k A^{2+\delta} v_i - b_k A^{3+\delta} v_i \right) - \int_{\Gamma_1} \left( \Lambda^{2+\delta} (b_k q) - b_k \Lambda^{2+\delta} q \right) \Lambda^{2+\delta} v_i
\]
\[
= -\int_{\Gamma_1} \Lambda^{1+\delta} q A^{3+\delta} (b_k v_i) - \int_{\Gamma_1} \Lambda^{1+\delta} q \left( A^{2+\delta} (b_k v_i) - A^{3+\delta} (b_k v_i) \right) - \int_{\Gamma_1} \left( \Lambda^{2+\delta} (b_k q) - b_k \Lambda^{2+\delta} q \right) \Lambda^{2+\delta} v_i
\]
\[
= I_{521} + I_{522} + I_{523} + I_{524}.
\]

The first term is the leading one and, using (2.20), it may be rewritten as
\[
I_{521} = -\int_{\Gamma_1} \Lambda^{1+\delta} q A^{3+\delta} w_i = -\int_{\Gamma_1} q A^{2(2+\delta)} w_i,
\]
which cancels with the second term on the right-hand side of (3.18) upon adding (3.27), integrated in time, to (3.18).

The next three terms are commutators. For the first one, we have
\[
I_{522} \lesssim \|A^{1+\delta} q\|_{L^2(\Gamma_1)} \|A^{3+\delta} b\|_{L^2(\Gamma_1)} \|v\|_{L^\infty(\Gamma_1)} + \|A^{1+\delta} q\|_{L^2(\Gamma_1)} \|A b\|_{L^\infty(\Gamma_1)} \|b\|_{H^{2+\delta}(\Gamma_1)} \|v\|_{H^{2+\delta}(\Gamma_1)}
\]
\[
\lesssim \|q\|_{H^{1+\delta}(\Gamma_1)} \|b\|_{H^{3+\delta}(\Gamma_1)} \|v\|_{H^2(\Gamma_1)} + \|q\|_{H^{1+\delta}(\Gamma_1)} \|b\|_{H^{2+\delta}(\Gamma_1)} \|v\|_{H^{2+\delta}(\Gamma_1)} \|w\|_{H^{2+\delta}(\Gamma_1)}
\]
\[
\leq P(\|v\|_{H^{2.5+\delta}}, \|q\|_{H^{1.5+\delta}}, \|w\|_{H^{1.5+\delta}}),
\]
using the trace inequalities. The second commutator term in (3.31) is estimated similarly as
\[
I_{523} \lesssim \|A^{1+\delta} q\|_{L^2(\Gamma_1)} \|A b\|_{L^\infty(\Gamma_1)} \|A^{2+\delta} v\|_{L^2(\Gamma_1)} \|v\|_{H^{1+\delta}(\Gamma_1)} \|b\|_{H^{2+\delta}(\Gamma_1)} \|v\|_{H^{2+\delta}(\Gamma_1)}
\]
\[
\leq P(\|v\|_{H^{2.5+\delta}}, \|q\|_{H^{1.5+\delta}}, \|w\|_{H^{1.5+\delta}}),
\]
while for the last commutator term \(I_{524}\), we have
\[
I_{524} \lesssim \|A^{2+\delta} b\|_{L^2(\Gamma_1)} \|q\|_{L^\infty(\Gamma_1)} \|A^{2+\delta} v\|_{L^2(\Gamma_1)} + \|A b\|_{L^\infty(\Gamma_1)} \|A^{1+\delta} q\|_{L^2(\Gamma_1)} \|A^{2+\delta} v\|_{L^2(\Gamma_1)}
\]
\[
\lesssim \|b\|_{H^{2+\delta}(\Gamma_1)} \|q\|_{H^{1+\delta}(\Gamma_1)} \|v\|_{H^{2+\delta}(\Gamma_1)} + \|b\|_{H^{2+\delta}(\Gamma_1)} \|q\|_{H^{1+\delta}(\Gamma_1)} \|v\|_{H^{2+\delta}(\Gamma_1)}
\]
\[
\leq P(\|v\|_{H^{2.5+\delta}}, \|q\|_{H^{1.5+\delta}}, \|w\|_{H^{1.5+\delta}}),
\]
Now, we add (3.18) and (3.27), integrated in time, with all the estimates above on the terms $I_1$, $I_2$, $I_3$, $I_4$, $I_5$, and $I$ obtaining
\[
\|\Delta_2 \Lambda^{2+\delta} w \|^2_{L^2(I)} + \| \Lambda^{2+\delta} w_t \|^2_{L^2(I)} + \nu \int_0^t \| \nabla_2 \Lambda^{2+\delta} w_t \|^2_{L^2(I)} \, ds
\leq \| \Lambda^{2+\delta} w_t(0) \|^2_{L^2(I)} - \int J \Lambda^{1.5+\delta} v \Lambda^{2+\delta} v + \int J \Lambda^{1.5+\delta} v_0 \Lambda^{2+\delta} v_0
+ \int_0^t P(\|v\|_{H^{2.5+\delta}}, \|\psi\|_{H^{2.5+\delta}}, \|w\|_{H^{4+\delta}(I)}, \|w_t\|_{H^{2+\delta}(I)}) \, ds.
\]
Next, we estimate the second term on the right-hand side as
\[
- \int J \Lambda^{1.5+\delta} v_t \Lambda^{2+\delta} v_t \lesssim \|J\|_{L^{\infty}} \|\Lambda^{1.5+\delta} v\|_{L^2} \|\Lambda^{2+\delta} v\|_{L^2} \lesssim \|v\|_{L^2} \|\Lambda^{2+\delta} v\|_{L^2}^{1/(2+\delta)} \|v\|_{H^{2.5+\delta}}^{(4+2\delta)/(2.5+\delta)}.
\]
Using the equality
\[
\Delta_2 \Lambda^{2+\delta} w = \Lambda^{2+\delta} w(0) - \Lambda^{4+\delta} w(t) + \int_0^t \Lambda^{2+\delta} w_t \, ds
\]
on the first term on the left-hand side of (3.18), we conclude the proof of (3.20).

3.3. Pressure estimates. In this section, we prove the following pressure estimate.

**Lemma 3.3.** Under the conditions of Theorem 2.1, we have
\[
\|q\|_{H^{4+\delta}} \leq P(\|v\|_{H^{2.5+\delta}}, \|w\|_{H^{4+\delta}(I)}, \|w_t\|_{H^{2+\delta}(I)}).
\]

Applying $b_{ji} \partial_j$ to the Euler equations (2.17) and using the Piola identity (2.15), we get
\[
\partial_j (b_{ji} a_{ki} \partial_k q) = -\partial_j (b_{ji} \partial_i v_i) - \partial_j \left( \sum_{m=1}^2 b_{ji} v_m a_{km} \partial_k v_i \right) - \partial_j (a_{ji} (v_3 - \psi_t) \partial_3 v_i),
\]
where we used $b_{ji}/\partial_3 \psi = a_{ji}$ in the last term. Recall that we use the summation convention over repeated indices unless indicated otherwise (as, for example, in (3.34)). By $\partial_j (b_{ji} \partial_i v_i) = -\partial_j (\partial_i b_{ji} v_i)$, which follows from (3.26), we get
\[
\partial_j (b_{ji} a_{ki} \partial_k q) = \partial_j (\partial_i b_{ji} v_i) - \partial_j \left( \sum_{m=1}^2 b_{ji} v_m a_{km} \partial_k v_i \right) - \partial_j (a_{ji} (v_3 - \psi_t) \partial_3 v_i) = \partial_j f_j \quad \text{in } \Omega.
\]
To obtain the boundary condition for the pressure on $\Gamma_0 \cup \Gamma_1$, we test (2.17) with $b_{3i}$ obtaining
\[
b_{3i} a_{ki} \partial_k q = -b_{3i} \partial_i v_i - b_{3i} v_3 a_{j1} \partial_j v_i - b_{3i} v_2 a_{j2} \partial_j v_i - a_{3i} (v_3 - \psi_t) \partial_3 v_i \quad \text{on } \Gamma_0 \cup \Gamma_1,
\]
where we again employed $b_{ji}/\partial_3 \psi = a_{ji}$ in the last term. On $\Gamma_1$, we use (2.20) and (2.21) and rewrite the first term on the right-hand side of (3.36) by using (2.20) as
\[
-b_{3i} \partial_i v_i = -\partial_i (b_{3i} v_i) + \partial_i b_{3i} v_i = -w_{it} + \partial_i b_{3i} v_i = \Delta_2^2 w - \nu \Delta_2 w_t - q + \partial_i b_{3i} v_i.
\]
Thus, on $\Gamma_1$, the boundary condition (3.36) becomes a Robin boundary condition
\[
b_{3i} a_{ki} \partial_k q + q = \Delta_2^2 w - \nu \Delta_2 w_t + \partial_i b_{3i} v_i - b_{3i} v_3 a_{j1} \partial_j v_i - b_{3i} v_2 a_{j2} \partial_j v_i - a_{3i} (v_3 - \psi_t) \partial_3 v_i = g_1 \quad \text{on } \Gamma_1.
\]
On $\Gamma_0$, we have $a = I$, and then the first term on the right hand side of (3.36) vanishes, and we get
\[
b_{3i} a_{ki} \partial_k q = -b_{3i} v_3 a_{j1} \partial_j v_i - b_{3i} v_2 a_{j2} \partial_j v_i - a_{3i} (v_3 - \psi_t) \partial_3 v_i = g_0 \quad \text{on } \Gamma_0.
\]
The boundary value problem for the pressure can be simplified, as we show in Remark 3.5 below. The form of the equations above suffices for the purpose of obtaining the a priori control, but it is adequate for the construction.

To estimate the pressure, we need the following statement on the elliptic regularity for the Robin/Neumann problem.
Lemma 3.4. Assume that \( d \in W^{1, \infty}(\Omega) \). Let \( 1 \leq l \leq 2 \), and suppose that \( u \) is an \( H^l \) solution of

\[
\partial_t(d_{ij}\partial_j u) = \text{div} f \quad \text{in } \Omega,
\]

\[
d_{mk}\partial_k u N_m + u = g_1 \quad \text{on } \Gamma_1,
\]

\[
d_{mk}\partial_k u N_m = g_0 \quad \text{on } \Gamma_0.
\]

If

\[
\|d - I\|_{L^\infty} \leq \epsilon_0,
\]

where \( \epsilon_0 > 0 \) is sufficiently small, then

\[
\|u\|_{H^l} \lesssim \|f\|_{H^{l-1}} + \|g_1\|_{H^{l-3/2}(\Gamma_1)} + \|g_0\|_{H^{l-3/2}(\Gamma_0)}.
\]

Proof of Lemma 3.4. By interpolation, it is sufficient to establish the inequality (3.41) for \( l = 1 \) and \( l = 2 \). First let \( l = 1 \). Testing (3.39) with \(-u\) and integrating by parts, we obtain

\[
\int d_{ij}\partial_i u \partial_j u + \int_{\Gamma_1} u^2 = -\int u \text{div} f + \int_{\Gamma_0} g_0 u + \int_{\Gamma_1} g_1 u.
\]

Applying the \( H^{-1/2} H^{1/2} \) duality on the boundary, we obtain the result for \( l = 1 \). The inequality (3.41) for \( l = 2 \) is classical for \( d_{mk} = \delta_{mk} \), and then we simply use the perturbation argument and (3.40).

Proof of Lemma 3.3. We apply the elliptic estimate (3.41) in \( H^{1.5+\delta} \) for the equation (3.35) with the boundary conditions (3.37)–(3.38), leading to

\[
\|q\|_{H^{1.5+\delta}} \lesssim \|f\|_{H^{0.5+\delta}} + \|g_0\|_{H^1(\Gamma_0)} + \|g_1\|_{H^1(\Gamma_1)}.
\]

For the interior term, we have

\[
\|f\|_{H^{0.5+\delta}} \leq \sum_{j=1}^{3} \|\partial_j b_{ij} v_i\|_{H^{0.5+\delta}} + \sum_{j=1}^{2} \|b_{ij} v_m a_{km} \partial_k v_i\|_{H^{0.5+\delta}} + \sum_{j=1}^{3} \|a_{ij}(v_3 - \psi_t) \partial_3 v_i\|_{H^{0.5+\delta}}
\]

\[
\leq P(\|v\|_{H^{2, 5+\delta}}, \|a\|_{H^{3, 5+\delta}}, \|b\|_{H^{3, 5+\delta}}, \|b_t\|_{H^{1, 5+\delta}}, \|\psi_t\|_{H^{2, 5+\delta}}, ).
\]

On the other hand, for the boundary terms, we have

\[
\|g_0\|_{H^1(\Gamma_0)} + \|g_1\|_{H^1(\Gamma_1)}
\]

\[
\lesssim \|w\|_{H^{2, 5+\delta}(\Gamma_1)} + \|w_t\|_{H^{2, 5+\delta}(\Gamma_1)} + \|b_t\|_{H^{0.5+\delta}(\partial \Omega)} \|v\|_{H^{0.5+\delta}(\partial \Omega)}
\]

\[
+ \|a\|_{H^{1.5+\delta}(\partial \Omega)} \|v\|_{H^{1.5+\delta}(\partial \Omega)} + \|\psi_t\|_{H^{2, 5+\delta}(\partial \Omega)} \|\nabla v\|_{H^{1, 5+\delta}(\partial \Omega)}
\]

\[
\leq P(\|w\|_{H^{2, 5+\delta}(\Gamma_1)}, \|w_t\|_{H^{2, 5+\delta}(\Gamma_1)}, \|b\|_{H^{3, 5+\delta}}, \|b_t\|_{H^{1, 5+\delta}}, \|\psi_t\|_{H^{2, 5+\delta}}, \|v\|_{H^{2, 5+\delta}}),
\]

where \( \partial \Omega = \Gamma_0 \cup \Gamma_1 \). By combining (3.42)–(3.44) and using (3.13)–(3.14), we obtain (3.33).

Remark 3.5. Note that in the boundary value problem for the pressure, the situation is very different from the pressure value for the classical Euler equations. It is crucial in the construction of solutions below that the equations (3.35), (3.37), and (3.38) may be simplified so that the highest order terms in \( v \) are by one degree more regular.

First, the right hand side of the PDE for the pressure, (3.35), may be rewritten as

\[
\partial_j(\partial_t b_{ij} v_i) - \partial_j \left( \sum_{m=1}^{2} b_{ij} v_m a_{km} \partial_k v_i \right) - \partial_j (J^{-1}(v_3 - \psi_t)b_{ij} \partial_3 v_i)
\]

\[
= \partial_j(\partial_t b_{ij} v_i) - \sum_{m=1}^{2} b_{ij} \partial_j(v_m a_{km}) \partial_k v_i - b_{ij} \partial_j (J^{-1}(v_3 - \psi_t)) \partial_3 v_i
\]

\[
- \sum_{m=1}^{2} v_m a_{km} b_{ij} \partial_j v_i J^{-1}(v_3 - \psi_t) b_{ij} \partial_3 v_i,
\]
which holds in $\Omega$. Therefore, using also the divergence-free condition, we obtain

$$
\partial_j(b_{ji}a_{ki}\partial_k q) = \partial_j(\partial_i b_{ji}v_i) - \sum_{m=1}^{2} b_{ji} \partial_j(v_m a_{km}) \partial_k v_i - b_{ji} \partial_j(J^{-1}(v_3 - \psi)) \partial_3 v_i \\
+ \sum_{m=1}^{2} v_m a_{km} \partial_k b_{ji} \partial_j v_i + J^{-1}(v_3 - \psi) \partial_3 b_{ji} \partial_j v_i \quad \text{in } \Omega.
$$

(3.46)

Next, on $\Gamma_0$, we have $\psi = 0$ from where $b_{3i} = \delta_{3i}$, and by (2.19), the boundary condition (3.38) reduces to

$$
b_{3i}a_{ki}\partial_k q = 0 \quad \text{on } \Gamma_0.
$$

(3.47)

Finally, we simplify the condition (3.36) on $\Gamma_1$. First, we have

$$
a_{3i}v_3 \partial_3 v_i = b_{3i}v_3 \frac{1}{\partial_3 \psi} \partial_3 v_i = b_{3i}v_3 a_{33} \partial_3 \partial_3 v_i = b_{3i}v_3 a_{33} \partial_3 v_i,
$$

(3.48)

and thus we have, on $\Gamma_1$, using (3.48),

$$
-b_{3i}v_1 a_{j1} \partial_j v_i - b_{3i}v_2 a_{j2} \partial_j v_i - a_{3i} (v_3 - \psi) \partial_3 v_i \\
= -b_{3i}v_k a_{jk} \partial_j v_i + a_{3i} \psi \partial_3 v_i = \frac{1}{\partial_3 \psi} \bigl(-b_{3i}v_k b_{jk} \partial_j v_i + b_{3i} \psi \partial_3 v_i \bigr).
$$

(3.49)

The negative of the first term inside the parenthesis equals

$$
b_{3i}v_k b_{jk} \partial_j v_i = b_{3i} \partial_j (v_k b_{jk} v_i) - b_{3i} \partial_j v_k b_{jk} v_i \\
= \sum_{j=1}^{2} \partial_j (b_{3i}v_k b_{jk} v_i) + b_{3i} (v_k b_{jk} \partial_j v_i) - b_{3i} \partial_j v_k b_{jk} v_i \\
= \sum_{j=1}^{2} \partial_j (v_k b_{jk} w_t) + 2v_k b_{3k} \partial_3 (v_i b_{3i}) - b_{3i} \partial_j v_k b_{jk} v_i,
$$

where we used (2.20) in the third equality, and thus

$$
b_{3i}v_k b_{jk} \partial_j v_i = \sum_{j=1}^{2} \partial_j (v_k b_{jk} w_t) + 2w_i \partial_3 (v_i b_{3i}) - \partial_j b_{3i} v_k b_{jk} v_i \\
= 3 \sum_{j=1}^{3} \partial_j (v_k b_{jk} w_t) - \partial_3 (v_k b_{3k} w_t) + 2w_t \partial_3 (v_i b_{3i}) - \partial_j b_{3i} v_k b_{jk} v_i \\
= 3 \sum_{j=1}^{3} v_k b_{jk} \partial_j w_t - \partial_3 (v_k b_{3k} w_t) - v_k \partial_3 (b_{3k} w_t) + 2w_t b_{3i} \partial_3 v_i + 2w_i \partial_3 b_{3i} v_i - \partial_j b_{3i} v_k b_{jk} v_i \\
= w_t b_{3i} \partial_3 v_i + 3 \sum_{j=1}^{3} v_k b_{jk} \partial_j w_t - v_k \partial_3 (b_{3k} w_t) + 2w_t b_{3i} \partial_3 v_i + 2w_i \partial_3 b_{3i} v_i - \partial_j b_{3i} v_k b_{jk} v_i \\
= w_t b_{3i} \partial_3 v_i + 3 \sum_{j=1}^{3} v_k b_{jk} \partial_j w_t - w_t \partial_3 b_{3i} v_i - \partial_j b_{3i} v_k b_{jk} v_i,
$$

where we used $2w_t b_{3i} \partial_3 v_i - \partial_3 v_k b_{3k} w_t = w_t b_{3i} \partial_3 v_i$. We conclude that (3.36) may be written as

$$
b_{3i}a_{ki} \partial_k q = -b_{3i} \partial_3 v_i + b_{3i} v_3 v_i - \frac{1}{\partial_3 \psi} \left( \sum_{j=1}^{2} v_k b_{jk} \partial_j w_t + w_t \partial_3 b_{3i} v_i - \partial_j b_{3i} v_k b_{jk} v_i \right) \quad \text{on } \Gamma_1.
$$

while (3.37) may be rewritten as

$$
b_{3i}a_{ki} \partial_k q + q = \Delta_2^2 w - \nu \Delta_2 w_t + \partial_t b_{3i} v_i - \frac{1}{\partial_3 \psi} \left( \sum_{j=1}^{2} v_k b_{jk} \partial_j w_t + w_t \partial_3 b_{3i} v_i - \partial_j b_{3i} v_k b_{jk} v_i \right) \quad \text{on } \Gamma_1.
$$
3.4. The vorticity estimate. Recall that the Eulerian vorticity $\omega_i = \epsilon_{ijk} \partial_j u_k$, for $i = 1, 2, 3$, solves
$$
\partial_t \omega_i + u_j \partial_j \omega = \omega_j \partial_j u_i, \quad i = 1, 2, 3.
$$

Therefore, the ALE vorticity
$$
\zeta(x, t) = \omega(\eta(x, t), t)
$$
satisfies the equation
$$
\partial_t \zeta_i + v_1 a_{j1} \partial_j \zeta_i + v_2 a_{j2} \partial_j \zeta_i + (v_3 - \psi_i) a_{j3} \partial_j \zeta_i = \zeta_k a_{mk} \partial_m v_i, \quad i = 1, 2, 3. \tag{3.51}
$$

Note that in the ALE variables, the vorticity reads
$$
\zeta_i = \epsilon_{ijk} \partial_m v_k a_{mj}. \tag{3.52}
$$

Since we do not use the Eulerian variables in estimates, we denote the ALE variable, for simplicity of notation, with $x$.

By multiplying (3.51) with $J$, we obtain
$$
J \partial_t \zeta_i + v_1 b_{j1} \partial_j \zeta_i + v_2 b_{j2} \partial_j \zeta_i + (v_3 - \psi_i) b_{j3} \partial_j \zeta_i = \zeta_k b_{mk} \partial_m v_i, \quad i = 1, 2, 3. \tag{3.53}
$$

In order to perform non-tangential estimates, we need to extend functions to $\mathbb{R}^3$ using the classical Sobolev extension operator $f \mapsto \tilde{f}$, which is a continuous operator $H^k(\Omega) \to H^k(\Omega_0)$ for all $k \in [0, 5]$, where
$$
\Omega_0 = T^2 \times [0, 2].
$$

The extension is designed so that $\text{supp} \ f$ vanishes in a neighborhood of $[3/2, \infty)$. For the Jacobian $J$, we need to modify the extension operator to $\tilde{\cdot}: H^k(\Omega) \to H^k(\Omega_0)$ so that we have
$$
\frac{1}{4} \leq \tilde{J}(x) \leq 2, \quad x_3 \leq \frac{4}{3} \tag{3.54}
$$
and $\tilde{J} \equiv 0$ for $x_3 \geq 2$.

First, we verify (3.51) on $\Gamma_0$. By (2.9) and (2.14), we have $b_{31} = b_{32} = 0$ and $\psi_i = 0$, so the left side of (3.56) reduces to $v_3$, which vanishes by the boundary condition (2.19). On $\Gamma_1$, the left side of (3.56) vanishes by (2.20). Thus (3.56) indeed holds. Now, the difference $\sigma = \zeta - \tilde{\theta}$ satisfies
$$
J \partial_t \sigma_i + v_1 b_{j1} \partial_j \sigma_i + v_2 b_{j2} \partial_j \sigma_i + (v_3 - \psi_i) b_{j3} \partial_j \sigma_i = \sigma_k b_{mk} \partial_m v_i \quad \text{on } \Omega, \quad i = 1, 2, 3, \tag{3.57}
$$
using that the extension operators $\tilde{\cdot}$ and $\tilde{\cdot}$ act as an identity in $\Omega$.

We now test (3.57) with $\sigma_i$, on $\Omega$, which leads to
$$
\frac{1}{2} \frac{d}{dt} \int \sigma_i \, ds = - \sum_{m=1}^{2} \int v_m b_{jm} \sigma_i \partial_j \sigma_i - \int (v_3 - \psi_i) b_{j3} \sigma_i \partial_j \sigma_i + \int \sigma_k b_{mk} \partial_m v_i \sigma_i + \frac{1}{2} \int J_i |\sigma|^2
$$
$$
= I_1 + I_2 + I_3 + I_4.
$$
For the first two terms, we write $\sigma_j \partial_j \sigma_i = (1/2) \partial_j (|\sigma|^2)$ and integrate by parts, obtaining

$$I_1 + I_2 = \frac{1}{2} \sum_{m=1}^{2} \int \partial_j v_m b_{jm} \sigma_i \sigma_i + \frac{1}{2} \int \partial_j (v_3 - \psi_t) b_{j3} \sigma_i \sigma_i$$

$$- \frac{1}{2} \sum_{m=1}^{2} \int \partial_j v_m b_{jm} \sigma_i \sigma_i N_j - \frac{1}{2} \int \partial_j (v_3 - \psi_t) b_{j3} \sigma_i \sigma_i N_j,$$

where we used the Piola identity. The boundary terms vanish by (3.56) and $N = (0, 0, \pm 1)$. Therefore,

$$I_1 + I_2 \leq P(||v||_{H^{2.5+\delta}} , ||b||_{H^{3.5+\delta}} , ||\psi_t||_{H^{2.5+\delta}} ) ||\sigma||^2_{L^2}.$$

Note that also $I_3$ and $I_4$ are bounded by the right side of (3.58). Using (3.6), we get

$$\frac{1}{2} \frac{d}{dt} \int J|\sigma|^2 \leq P(||v||_{H^{2.5+\delta}} , ||b||_{H^{3.5+\delta}} , ||\psi_t||_{H^{2.5+\delta}} ) \int J|\sigma|^2$$

$$\leq P(||v||_{H^{2.5+\delta}} , ||w||_{H^{4+\delta}(\Gamma_1)} , ||w_t||_{H^{2+\delta}(\Gamma_1)} ) \int J|\sigma|^2,$$

where we used (3.12) and (3.13) in the last step. The lemma then follows a standard Gronwall argument. □

By the properties of the extension operator and since the equation for $\theta$ is of transport type (note that (3.54) holds), we have

$$\theta(x, t) = 0, \quad (x, t) \in (0, 3/2) \times [0, T].$$

The main result of this section is the following estimate on the Sobolev norm of the vorticity.

**Lemma 3.7.** Under the assumption (3.3), the quantity

$$Y = \int_{\Omega_0} J|\Lambda_3^{1.5+\delta} \theta|^2$$

satisfies

$$||\zeta||_{H^{1.5+\delta}} \lesssim Y$$

with

$$||\zeta(0)||_{H^{1.5+\delta}} \lesssim Y(0) \lesssim ||\zeta(0)||_{H^{1.5+\delta}}$$

and

$$\frac{d}{dt} Y \lesssim M P(||v||_{H^{2.5+\delta}} , ||w||_{H^{4+\delta}(\Gamma_1)} , ||w_t||_{H^{2+\delta}(\Gamma_1)}) Y,$$

for all $t \in [0, T]$, where $M$ is as in (2.23).

**Proof of Lemma 3.7.** Denote $\Lambda_3 = (I - \Delta)^{1/2}$ on the domain $\mathbb{T}^2 \times \mathbb{R}$. We apply $\Lambda_3^{1.5+\delta}$ to the equation (3.55) and test it with $\Lambda_3^{1.5+\delta} \theta$, obtaining

$$\frac{dY}{dt} = -\sum_{m=1}^{2} \int_{\Omega_0} \tilde{v}_m \tilde{b}_{jm} \partial_j \Lambda_3^{1.5+\delta} \theta_i \Lambda_3^{1.5+\delta} \theta_i - \int_{\Omega_0} \left( \tilde{v}_3 - \tilde{\psi}_t \right) \tilde{b}_{j3} \partial_j \Lambda_3^{1.5+\delta} \theta_i \Lambda_3^{1.5+\delta} \theta_i$$

$$+ \int_{\Omega_0} \theta_k \tilde{b}_{km} \partial_m \Lambda_3^{1.5+\delta} \tilde{v}_i \Lambda_3^{1.5+\delta} \theta_i$$

$$- \sum_{m=1}^{2} \int_{\Omega_0} \left( \Lambda_3^{1.5+\delta} \left( \tilde{v}_m \tilde{b}_{jm} \partial_j \theta_i \right) - \tilde{v}_m \tilde{b}_{jm} \partial_j \Lambda_3^{1.5+\delta} \theta_i \right) \Lambda_3^{1.5+\delta} \theta_i$$

$$- \int_{\Omega_0} \left( \Lambda_3^{1.5+\delta} \left( \tilde{v}_3 - \tilde{\psi}_t \right) \tilde{b}_{j3} \partial_j \theta_i \right) - \left( \tilde{v}_3 - \tilde{\psi}_t \right) \tilde{b}_{j3} \partial_j \Lambda_3^{1.5+\delta} \theta_i \right) \Lambda_3^{1.5+\delta} \theta_i$$

$$+ \int_{\Omega_0} \left( \Lambda_3^{1.5+\delta} \left( \theta_k \bar{b}_{km} \partial_m \tilde{v}_i \right) - \theta_k \bar{b}_{km} \partial_m \Lambda_3^{1.5+\delta} \tilde{v}_i \right) \Lambda_3^{1.5+\delta} \theta_i$$

$$+ \frac{1}{2} \int_{\Omega_0} J|\Lambda_3^{1.5+\delta} \theta|^2 + \int_{\Omega_0} \left( \Lambda_3^{1.5+\delta} (J \partial_j \theta_i) - J \Lambda_3^{1.5+\delta} (\partial_j \theta_i) \right) \Lambda_3^{1.5+\delta} \theta_i$$

$$= I_1 + \cdots + I_8.$$
For the first two terms on the right-hand side of (3.62), we integrate by parts in $x_j$ obtaining

$$I_1 + I_2 = \sum_{m=1}^{2} \int_{\Omega} \partial_j(\tilde{v}_m \tilde{b}_{jm}) \Lambda_3^{1.5+\delta} \theta_i \Lambda_3^{1.5+\delta} \theta_i + \int_{\Omega} \partial_j\left((\tilde{v}_3 - \tilde{\psi}_3) \tilde{b}_{3j}\right) \Lambda_3^{1.5+\delta} \theta_i \Lambda_3^{1.5+\delta} \theta_i + \int_{\partial\Omega} \tilde{v}_m \tilde{b}_{jm} N_3 \Lambda_3^{1.5+\delta} \theta_i \Lambda_3^{1.5+\delta} \theta_i - \int_{\partial\Omega} (\tilde{v}_3 - \tilde{\psi}_3) \tilde{b}_{3j} N_3 \Lambda_3^{1.5+\delta} \theta_i \Lambda_3^{1.5+\delta} \theta_i,$$

(3.63)

Since the extension operators are the identity on $\Omega$, the last two terms in (3.63) equal

$$- \sum_{m=1}^{2} \int_{\partial\Omega} \tilde{v}_m \tilde{b}_{jm} N_3 \Lambda_3^{1.5+\delta} \theta_i \Lambda_3^{1.5+\delta} \theta_i - \int_{\partial\Omega} (\tilde{v}_3 - \tilde{\psi}_3) \tilde{b}_{3j} N_3 \Lambda_3^{1.5+\delta} \theta_i \Lambda_3^{1.5+\delta} \theta_i = 0,$$

where the last equality follows by (3.56). Using that the sum of the last two terms in (3.63) vanishes, we get

$$I_1 + I_2 \lesssim \|\tilde{v}\|_{H^{2.5+\delta}(\Omega)} \|\tilde{b}\|_{H^{2.5+\delta}(\Omega)} \|\theta\|_{H^{1.5+\delta}(\Omega)}^2 + (\|\tilde{v}\|_{H^{2.5+\delta}(\Omega)} + \|\tilde{\psi}\|_{H^{2.5+\delta}(\Omega)}) \|\tilde{b}\|_{H^{2.5+\delta}(\Omega)} \|\theta\|_{H^{1.5+\delta}}^2 \lesssim \|\tilde{v}\|_{H^{2.5+\delta}(\Gamma_1)} \|\tilde{w}\|_{H^{2.5+\delta}(\Gamma_1)} \|\theta\|_{H^{1.5+\delta}}^2,

(3.64)

where we used multiplicative Sobolev inequalities in the first step, the continuity properties of the Sobolev extension operator in the second, and (3.52) in the last. Therefore,

$$I_1 + I_2 \leq P(\|v\|_{H^{2.5+\delta}(\Gamma_1)}, \|w\|_{H^{2.5+\delta}(\Gamma_1)}, \|v_t\|_{H^{2.5+\delta}(\Gamma_1)}) \|\theta\|_{H^{1.5+\delta}}^2.

(3.65)

For the third term in (3.62), we have

$$I_3 \lesssim \|\tilde{b}\|_{H^{2.5+\delta}(\Omega)} \|\tilde{v}\|_{H^{2.5+\delta}(\Omega)} \|\theta\|_{H^{1.5+\delta}}^2 \lesssim \|\tilde{b}\|_{H^{2.5+\delta}(\Gamma_1)} \|\tilde{w}\|_{H^{2.5+\delta}(\Gamma_1)} \|\theta\|_{H^{1.5+\delta}}^2 \leq P(\|v\|_{H^{2.5+\delta}(\Gamma_1)}, \|w\|_{H^{2.5+\delta}(\Gamma_1)}, \|v_t\|_{H^{2.5+\delta}(\Gamma_1)}) \|\theta\|_{H^{1.5+\delta}}^2,

(3.66)

which is bounded by the right-hand side of (3.65). For the next term, we use Kato-Ponce type estimate to write

$$I_4 \lesssim \left(\|\Lambda_3^{1.5+\delta}(\tilde{v}_m \tilde{b}_{jm})\|_{L^6} \|\Lambda_3 \theta_i\|_{L^3} + \|\Lambda_3(\tilde{v}_m \tilde{b}_{jm})\|_{L^\infty} \|\Lambda_3^{1.5+\delta} \theta_i\|_{L^2}\right) \|\Lambda_3^{1.5+\delta} \theta_i\|_{L^2}

\lesssim \left(\|\tilde{v}_m \tilde{b}_{jm}\|_{H^{2.5+\delta}(\Omega)} \|\theta_i\|_{H^{1.5+\delta}} + \|\tilde{v}_m \tilde{b}_{jm}\|_{H^{2.5+\delta}(\Omega)} \|\Lambda_3^{1.5+\delta} \theta_i\|_{L^2}\right) \|\Lambda_3^{1.5+\delta} \theta_i\|_{L^2}

\leq P(\|v\|_{H^{2.5+\delta}(\Gamma_1)}, \|\tilde{b}\|_{H^{2.5+\delta}(\Gamma_1)} \|\Lambda_3^{1.5+\delta} \theta_i\|_{L^2},

(3.67)

which is also bounded by the right-hand side of (3.65). The terms $I_5$ and $I_6$ are treated similarly, following the Kato-Ponce and Sobolev inequalities, we get

$$I_5 + I_6 \leq P(\|v\|_{H^{2.5+\delta}(\Gamma_1)}, \|w\|_{H^{2.5+\delta}(\Gamma_1)}, \|w_t\|_{H^{2.5+\delta}(\Gamma_1)}) \|\theta\|_{H^{1.5+\delta}}^2.

(3.68)

For the seventh term in (3.62), we also have

$$I_7 \leq P(\|v\|_{H^{2.5+\delta}(\Gamma_1)}, \|w\|_{H^{2.5+\delta}(\Gamma_1)}, \|w_t\|_{H^{2.5+\delta}(\Gamma_1)}) \|\theta\|_{H^{1.5+\delta}}^2.

(3.69)

Finally, for the eight term, we write

$$I_8 \lesssim \left(\|\Lambda_3^{1.5+\delta} \tilde{J} \right) \|\theta_i\|_{L^3} + \|\Lambda_3 \tilde{J}\|_{L^\infty} \|\Lambda_3^{0.5+\delta} \theta_t\|_{L^2}\right) \|\Lambda_3^{1.5+\delta} \theta_i\|_{L^2}

\lesssim \|\tilde{J}\|_{H^{3.5+\delta}(\Omega)} \|\Lambda_3^{1.5+\delta} \theta_t\|_{L^2} \|\Lambda_3^{0.5+\delta} \theta_t\|_{L^2}.

(3.70)

In order to treat the last factor $\|\Lambda_3^{0.5+\delta} \theta_t\|_{L^2}$, we divide (3.55) by $\tilde{J}$ and use the fractional Leibniz rule to estimate

$$\|\Lambda_3^{0.5+\delta} \theta_t\|_{L^2} \leq P(\|v\|_{H^{2.5+\delta}(\Gamma_1)}, \|J\|_{H^{3.5+\delta}(\Omega)} \|\tilde{b}\|_{H^{3.5+\delta}(\Gamma_1)} \|\psi\|_{H^{2.5+\delta}(\Gamma_1)} \|\Lambda_3^{1.5+\delta} \theta\|_{L^2},

(3.71)

where we also used (3.54) and (3.59). Employing (3.71) in (3.70), we then obtain

$$I_8 \leq P(\|v\|_{H^{2.5+\delta}(\Gamma_1)}, \|J\|_{H^{3.5+\delta}(\Omega)} \|\tilde{b}\|_{H^{3.5+\delta}(\Gamma_1)} \|\psi\|_{H^{2.5+\delta}(\Gamma_1)} \|\Lambda_3^{1.5+\delta} \theta\|_{L^2}.

(3.72)

Combining (3.62) and the upper bounds (3.64), (3.65), (3.66), (3.67), (3.68), (3.69), and (3.72), we get

$$\frac{dY}{dt} \leq P(\|v\|_{H^{2.5+\delta}(\Gamma_1)}, \|w\|_{H^{2.5+\delta}(\Gamma_1)}, \|w_t\|_{H^{2.5+\delta}(\Gamma_1)} \|\Lambda_3^{1.5+\delta} \theta\|_{L^2},

and then, using (3.54) and (3.59), we obtain (3.61).
3.5. The conclusion of a priori bounds. Now, we are ready to conclude the proof of the main statement on a priori estimates for the system.

Proof of Theorem 2.1. Using the pressure estimate (3.33) in the tangential bound (3.16), we get

$$
\|\mathbf{\nabla}^4 + \delta u\|^2_{L^2(\Gamma_1)} + \|\mathbf{\nabla}^2 + \delta u_t\|^2_{L^2(\Gamma_1)} + \nu \int_0^t \|\nabla^2 + \delta u_t\|^2_{L^2(\Gamma_1)} ds
\lessgtr \|w_t(0)\|^2_{H^{2.5+\delta}(\Gamma_1)} + \|\nu(0)\|^2_{H^{2.5+\delta}(\Gamma_1)} + \|v\|_{L^2}^{(1+\delta)/(2.5+\delta)} \|v\|_{H^{2.5+\delta}}^{(4+\delta)/(2.5+\delta)}
+ \int_0^t P(\|\mathbf{v}\|_{H^{2.5+\delta}}, \|w\|_{H^{2.5+\delta}}, \|w_t\|_{H^{2.5+\delta}}) ds.
$$

(3.73)

By the div-curl elliptic estimate (see [BB]), we have

$$
\|\mathbf{v}\|_{H^{2.5+\delta}} \lessgtr \|\text{curl } v\|_{H^{1.5+\delta}} + \|\text{div } v\|_{H^{1.5+\delta}} + \|\nu \cdot \mathbf{N}\|_{H^{2.5+\delta}(\Gamma_1)} + \|\mathbf{v}\|_{L^2}.
$$

(3.74)

We bound the terms on the right-hand side in order. Using the formula (3.52) for the ALE vorticity \( \zeta \), we may estimate

$$
\|(\text{curl } v)_i\|_{H^{1.5+\delta}} \lessgtr \|\varepsilon_{ijk} \partial_{ij} v_k\|_{H^{1.5+\delta}} + \|\zeta_i\|_{H^{1.5+\delta}} \lessgtr \epsilon \|v\|_{H^{2.5+\delta}} + \|\zeta_i\|_{H^{1.5+\delta}},
$$

for \( i = 1, 2, 3 \), where we used (3.4) in the last step. Therefore, applying the vorticity bound (3.61), integrated in time, with (3.60), in (3.75), we get

$$
\|\text{curl } v\|_{H^{1.5+\delta}} \lessgtr \|\zeta(0)\|_{H^{1.5+\delta}} + \epsilon \|v\|_{H^{1.5+\delta}} + \int_0^t P(\|v\|_{H^{2.5+\delta}}, \|w\|_{H^{2.5+\delta}}, \|w_t\|_{H^{2.5+\delta}}) Y ds.
$$

(3.76)

For the divergence term in (3.74), we use (2.17) to estimate

$$
\|\text{div } v\|_{H^{1.5+\delta}} = \|(a_{ki} - \delta_{ki}) \partial_{kj} v_k\|_{H^{1.5+\delta}} \lessgtr \epsilon \|v\|_{H^{2.5+\delta}}.
$$

(3.77)

The part on \( \Gamma_0 \) of the final term in (3.74) vanishes, while on \( \Gamma_1 \), we use (2.20). We get

$$
\|\nu \cdot \mathbf{N}\|_{H^{2.5+\delta}((\Gamma_1)} \lessgtr \|b_{3i} - \delta_{3i} v_i\|_{H^{2.5+\delta}(\Gamma_1)} + \|w_t\|_{H^{2.5+\delta}(\Gamma_1)}
\lessgtr \epsilon \|w_t\|_{H^{2.5+\delta}(\Gamma_1)} + \|w_t\|_{H^{2.5+\delta}(\Gamma_1)}.
$$

(3.78)

Note that

$$
b - I_{H^{2.5+\delta}(\Gamma_1)} \lessgtr \|b - I\|_{H^{2.5+\delta}} \lessgtr \|b - I\|_{H^{2.5+\delta}}^{1-\alpha} \|b - I\|_{H^{2.5+\delta}}^{\alpha}
\lessgtr \epsilon (1 + \|b\|^{1-\alpha}_{H^{2.5+\delta}(\Gamma_1)}) \lessgtr \epsilon (1 + \|w_t\|^\alpha_{H^{2.5+\delta}(\Gamma_1)}),
$$

(3.79)

where \( \alpha \in (0, 1) \) is the exponent determined by \( 2.5 + \delta = (1 - \alpha)(1.5 + \delta) + \alpha(3.5 + \delta) \). Using (3.79) in (3.78), we get

$$
\|\nu \cdot \mathbf{N}\|_{H^{2.5+\delta}(\Gamma_1)} \lessgtr \epsilon (1 + \|w_t\|^\alpha_{H^{2.5+\delta}(\Gamma_1)}) |v|_{H^{2.5+\delta}} + \|w_t\|_{H^{2.5+\delta}(\Gamma_1)},
$$

and thus

$$
\|\nu \cdot \mathbf{N}\|_{H^{2.5+\delta}(\Gamma_1)} \lessgtr \epsilon \|w\|_{H^{2.5+\delta}(\Gamma_1)} |v|_{H^{2.5+\delta}} + \epsilon \|v\|_{H^{2.5+\delta}} + \|w_t\|_{H^{2.5+\delta}(\Gamma_1)}.
$$

(3.80)

Now, we use (3.76), (3.77), and (3.80) in (3.74), while also absorbing the term \( \epsilon \|v\|_{H^{2.5+\delta}} \), obtaining

$$
\|v\|^2_{H^{2.5+\delta}} \lessgtr \|\zeta(0)\|^2_{H^{2.5+\delta}} + \|w_t\|^2_{H^{2.5+\delta}} + \epsilon^2 \|w\|_{H^{2.5+\delta}}^2 + \|w_t\|^2_{H^{2.5+\delta}}
+ \int_0^t P(\|v\|_{H^{2.5+\delta}}, \|w\|_{H^{2.5+\delta}}, \|w_t\|_{H^{2.5+\delta}}) Y ds.
$$

(3.81)

Next, we combine (3.81) with the tangential estimate (3.73). Multiplying (3.81) with a small constant \( \epsilon_0 \in (0, 1] \) and adding the resulting inequality to (3.73), we obtain

$$
\epsilon_0 \|v\|^2_{H^{2.5+\delta}} + \|w_t\|^2_{H^{2.5+\delta}} + \|w_t\|^2_{H^{2.5+\delta}}
\lessgtr \|w_t(0)\|^2_{H^{2.5+\delta}(\Gamma_1)} + \|v(0)\|^2_{H^{2.5+\delta}} + \|v\|^2_{L^2}^{(4+\delta)/(2.5+\delta)} \|v\|_{H^{2.5}}^{(4+\delta)/(2.5+\delta)}
+ \epsilon_0 \|w_t\|^2_{H^{2.5+\delta}(\Gamma_1)} + \epsilon_0 \epsilon^2 \|w\|^2_{H^{2.5+\delta}(\Gamma_1)} \|v\|^2_{H^{2.5+\delta}}
+ \int_0^t P(\|v\|_{H^{2.5+\delta}}, \|w\|_{H^{2.5+\delta}}, \|w_t\|_{H^{2.5+\delta}}) Y ds,
$$

(3.82)

with the implicit constant independent of \( \nu \). Now, first choose and fix \( \epsilon_0 \) so small that the fourth term on the right-hand side is absorbed in the third term on the left. Then choose \( \epsilon \) as in (3.7) with a sufficiently large constant \( C \) so that the
fifth term in (3.82) is absorbed in the second term on the left. This choice as requires (3.8) for \( T_0 \). For the third term on the right-hand side of (3.73), we use
\[
\left\| v \right\|_{L^2}^{1/(2.5+\delta)} \left\| v \right\|_{H^{2.5+2\delta}}^{(4+\delta)/(2.5+\delta)} \leq \epsilon_1 \left\| v \right\|_{H^{2.5+\delta}}^2 + C_{\epsilon_1} \left\| v \right\|_{L^2}^2 \\
\leq \epsilon_1 \left\| v \right\|_{H^{2.5+\delta}}^2 + C_{\epsilon_1} \left\| v \right\|_{L^2}^2 + C_{\epsilon_1} \int_0^t \left\| v \right\|_{L^2}^2 \, ds \\
\leq \epsilon_1 \left\| v \right\|_{H^{2.5+\delta}}^2 + C_{\epsilon_1} \left\| v \right\|_{L^2}^2 + C_{\epsilon_1} \int_0^t P \left( \| v \|_{H^{2.5+\delta}}, \| w \|_{H^{4+\delta}(\Gamma_1)}, \| w_t \|_{H^{2+\delta}(\Gamma_1)} \right) Y \, ds,
\]
by (3.25) and (3.33), where \( \epsilon_1 \in (0, 1) \) is a small constant to be determined. Using (3.83) in (3.82), and choosing \( \epsilon_1 \) sufficiently small, we obtain
\[
\left\| v \right\|_{H^{2.5+\delta}}^2 + \left\| w \right\|_{H^{4+\delta}(\Gamma_1)}^2 + \left\| w_t \right\|_{H^{2+\delta}(\Gamma_1)}^2 + \nu \int_0^t \left\| \nabla A^{2+\delta} w_t \right\|_{L^2(\Gamma_1)}^2 \, ds \\
\leq \left\| w_t(0) \right\|_{H^{2+\delta}(\Gamma_1)}^2 + \left\| w(0) \right\|_{H^{2.5+\delta}}^2 + \int_0^t P \left( \| v \|_{H^{2.5+\delta}}, \| w \|_{H^{4+\delta}(\Gamma_1)}, \| w_t \|_{H^{2+\delta}(\Gamma_1)} \right) Y \, ds.
\]

A standard Gronwall argument on (3.84) and (3.61) then implies a uniform in \( \nu \) estimate
\[
\left\| v \right\|_{H^{2.5+\delta}} + \left\| w \right\|_{H^{4+\delta}(\Gamma_1)} + \left\| w_t \right\|_{H^{2+\delta}(\Gamma_1)} + Y \lesssim M
\]
on \([0, T_0]\), where \( T_0 \) is independent of \( 0 \leq \nu \leq 1 \), and the proof of Theorem 2.1 is concluded.

### 4. Compatibility conditions

From (2.20), we obtain \( b_{3i}(0) v_i(0) = w_1(0) \), and since \( b_{3i}(0) = (0, 0, 1) \), we get the compatibility condition
\[
v_3(0) = w_1(0) \quad \text{on } \Gamma_1.
\]
Next, the divergence-free boundary condition gives the compatibility condition
\[
\int_{\Gamma_1} v_3(0) = 0,
\]
which results from integrating the divergence-free condition \( b_{ij} \partial_j v_j = 0 \) and evaluating it at \( t = 0 \). By the condition (2.20), we also obtain
\[
\int_{\Gamma_1} w_1(0) = 0,
\]
but this also follows from (4.1) and (4.2).

Assume that \( u = q \) solves
\[
\partial_i (d_{ij} \partial_j u) = \text{div } f \quad \text{in } \Omega, \\
d_{mk} \partial_k u N_m + u = g_1 \quad \text{on } \Gamma_1, \\
d_{mk} \partial_k u N_m = g_0 \quad \text{on } \Gamma_0.
\]
Integrating (4.3) over \( \Omega \), we get
\[
\int_{\Gamma_0} d_{ij} \partial_j u N_i + \int_{\Gamma_1} d_{ij} \partial_j u N_i = \int_{\Gamma_0 \cup \Gamma_1} f_i N_i,
\]
from where, using (4.3)\(_2\) and (4.3)\(_3\),
\[
\int_{\Gamma_0} g_0 + \int_{\Gamma_1} (g_1 - u) = \int_{\Gamma_0 \cup \Gamma_1} f_i N_i,
\]
and thus
\[
\int_{\Omega} u = \int_{\Gamma_0} g_0 + \int_{\Gamma_1} g_1 - \int_{\Gamma_0 \cup \Gamma_1} f_i N_i.
\]
Therefore, every solution of (4.3) satisfies (4.4). We apply this to the equation (3.35) with the boundary conditions (3.37) and (3.38). We have
\[
d_{ij} = h_{ik} a_{jk}.
and
\[
f_j = \partial_j (\partial_i b_{ji} v_i) - \partial_j (b_{ji} (v_3 - \psi_i) \partial_3 v_i) - \partial_j \left( \sum_{m=1}^{2} b_{jim} v_m \right) \quad \text{in } \Omega
\]
\[
g_0 = -b_{3i} v_1 a_{ji} \partial_j v_i - b_{3i} v_2 a_{j2} \partial_j v_i - b_{3i} (v_3 - \psi_i) \partial_3 v_i \quad \text{on } \Gamma_0
\]
\[
g_1 = \Delta_2^3 w - \nu \Delta_2 w_t + \partial_i b_{ji} v_i - b_{3i} v_1 a_{ji} \partial_j v_i - b_{3i} v_2 a_{j2} \partial_j v_i - b_{3i} (v_3 - \psi_i) \partial_3 v_i \quad \text{on } \Gamma_1
\]

Thus the equation (4.4) reads
\[
\int_{\Gamma_1} q = -\int_{\Gamma_0} b_{3i} v_1 a_{ji} \partial_j v_i - \int_{\Gamma_1} b_{3i} v_2 a_{j2} \partial_j v_i - \int_{\Gamma_1} b_{3i} (v_3 - \psi_i) \partial_3 v_i
\]
\[
+ \int_{\Gamma_1} \Delta_2^3 w - \nu \int_{\Gamma_1} \Delta_2 w_t + \int_{\Gamma_1} \partial_i b_{ji} v_i - \int_{\Gamma_1} b_{3i} v_1 a_{ji} \partial_j v_i - \int_{\Gamma_1} b_{3i} v_2 a_{j2} \partial_j v_i - \int_{\Gamma_1} b_{3i} (v_3 - \psi_i) \partial_3 v_i
\]
\[
- \int_{\Gamma_0 \cup \Gamma_1} \partial_i b_{ji} v_i N_j + \int_{\Gamma_0 \cup \Gamma_1} (b_{ji} v_3 - \psi_i) \partial_3 v_i N_j + \int_{\Gamma_0 \cup \Gamma_1} \sum_{m=1}^{2} b_{jim} v_m a_{km} \partial_k v_i N_j,
\]
from where
\[
\int_{\Gamma_1} q = \int_{\Gamma_1} \Delta_2^3 w - \nu \int_{\Gamma_1} \Delta_2 w_t = 0,
\]
where the last equality follows from the periodic boundary conditions imposed on \(w\) and \(w_t\) in \(x_1\) and \(x_2\).

5. Uniqueness

For simplicity, we only consider the case \(\nu = 0\); the uniqueness result is the same for other values of \(\nu\). To obtain uniqueness, we need to assume
\[
d \geq \frac{1}{2}.
\]
The main reason for this restriction is that when we apply the elliptic estimate (3.41) to (5.13)–(5.15) below: Since we use it with \(k = 0.5 + \delta\) and Lemma 3.4 requires \(k \geq 1\), this imposes the condition (5.1).

PROOF OF THEOREM 2.2. Assume that \((v, q, w, a, \eta)\) and \((\tilde{v}, \tilde{q}, \tilde{w}, \tilde{\eta}, \tilde{a})\) are solutions of the system on an interval \([0, T_0]\), both satisfying the bounds in Theorem 2.1. Denote by
\[
(W, V, Q, E, A, \Psi) = (w, v, q, a, \psi) - (\tilde{w}, \tilde{v}, \tilde{q}, \tilde{a}, \tilde{\psi})
\]
the difference, and assume that
\[
(W, V, Q, E, A, \Psi)(0) = 0.
\]
We start with tangential estimates by claiming that
\[
\|\Lambda^{3+\delta} W\|^2_{L^2(\Gamma_1)} + \|\Lambda^{1+\delta} W_t\|^2_{L^2(\Gamma_1)}
\]
\[
\lesssim \|V\|^{1/(1.5+\delta)}_{L^2} \|V\|^{(2+2\delta)/(1.5+\delta)}_{H^{1.5+\delta}} + \int_0^t \|V\|_{H^{1.5+\delta}} + \|Q\|_{H^{0.5+\delta}} + \|W\|_{H^{3+\delta}(\Gamma_1)} + \|W_t\|_{H^{1+\delta}(\Gamma_1)}^2 \, ds,
\]
where, in this section, we allow all the implicit constants to depend on the norms of \((v, q, w)\) and \((\tilde{v}, \tilde{q}, \tilde{w})\). To prove this, we start by subtracting the equation (2.21) and the analogous equation for \(\tilde{w}\) and get
\[
W_{tt} + \Delta_2^3 W = Q.
\]
We test this equation with \(\Lambda^{2(1+\delta)} W_t\) obtaining
\[
\frac{1}{2} \frac{d}{dt} \left( \|\Delta_2 \Lambda^{1+\delta} W\|^2_{L^2(\Gamma_1)} + \|\Lambda^{1+\delta} W_t\|^2_{L^2(\Gamma_1)} \right) = \int_{\Gamma_1} Q \Lambda^{2(1+\delta)} W_t,
\]
where we used (5.2). Subtracting the velocity equation (3.26)\(_1\) and its analog for \(\tilde{v}\), we get
\[
J \partial_i V_i + (J - \tilde{J}) \partial_i \tilde{v}_i + V_1 b_{ji} \partial_j v_i + \tilde{V}_1 b_{ji} \partial_j \tilde{v}_i + \tilde{v}_1 B_{ji} \partial_j V_i + V_2 b_{ji} \partial_j v_i + \tilde{V}_2 B_{ji} \partial_j \tilde{v}_i + \tilde{v}_2 B_{ji} \partial_j \tilde{v}_i
\]
\[
+ (V_3 - \Psi_e) b_{ji} \partial_j v_i + (\tilde{v}_3 - \tilde{\Psi}_e) b_{ji} \partial_j \tilde{v}_i + (\tilde{v}_3 - \tilde{\Psi}_e) \tilde{b}_{ji} \partial_j \tilde{v}_i + b_{ki} \partial_k \tilde{v}_i + b_{ki} \partial_k \tilde{q} = 0,
\]
while the difference of divergence-free conditions gives
\[
b_{ki} \partial_k \tilde{v}_i = -B_{ki} \partial_k \tilde{v}_i.
\]
As in (3.21)–(3.24), we have
\[
\frac{1}{2} \frac{d}{dt} \int J A^{0.5+\delta} V_i A^{1.5+\delta} V_i = \frac{1}{2} \int J A^{0.5+\delta} V_i A^{1.5+\delta} V_i + \int J A^{0.5+\delta} \partial_t V_i A^{1.5+\delta} V_i + \tilde{I},
\]
where
\[
\tilde{I} \lesssim \|V\|_{H^{1.5+\delta}} \|V_i\|_{H^{-0.5+\delta}}; \tag{5.7}
\]
recall that $\delta \geq 0.5$ and that the constants depend on the norms of $(v, q, w)$ and $(\tilde{v}, \tilde{q}, \tilde{w})$. Note that (5.7) is obtained analogously to (3.24) by writing
\[
\tilde{I} = \frac{1}{2} \int \left( \Lambda^2 (J A^{0.5+\delta} V_i) - \Lambda (J A^{1.5+\delta} V_i) \right) \Lambda^{-0.5+\delta} \partial_t V_i
\]
\[
= \frac{1}{2} \int \left( \Lambda^2 (J A^{0.5+\delta} V_i) - J A^{2.5+\delta} V_i \right) \Lambda^{-0.5+\delta} \partial_t V_i + \int \left( J A^{2.5+\delta} V_i - \Lambda (J A^{1.5+\delta} V_i) \right) \Lambda^{-0.5+\delta} \partial_t V_i,
\]
and estimating the commutators by employing Kato-Ponce inequalities.

Next, we apply $\Lambda^{0.5+\delta}$ to (5.5) and test with $\Lambda^{1.5+\delta} V$ obtaining
\[
\frac{1}{2} \frac{d}{dt} \int J A^{0.5+\delta} V_i A^{1.5+\delta} V_i
\]
\[
= \frac{1}{2} \int J A^{0.5+\delta} (J \partial_t V_i) A^{1.5+\delta} V_i - \int \left( \Lambda^{0.5+\delta} (J \partial_t V_i) - J \Lambda^{0.5+\delta} (\partial_t V_i) \right) \Lambda^{1.5+\delta} V_i
\]
\[
- \int \Lambda^{0.5+\delta} ((J - \tilde{J}) \partial_t v_i) \Lambda^{1.5+\delta} V_i - \sum_{m=1}^2 \int \Lambda^{0.5+\delta} (\tilde{v}_m b_m \partial_j v_i) \Lambda^{1.5+\delta} V_i
\]
\[
- \sum_{m=1}^2 \int \Lambda^{0.5+\delta} (\tilde{v}_m b_m \partial_j v_i) \Lambda^{1.5+\delta} V_i - \sum_{m=1}^2 \int \Lambda^{0.5+\delta} (\tilde{v}_m b_m \partial_j v_i) \Lambda^{1.5+\delta} V_i
\]
\[
- \int \Lambda^{0.5+\delta} ((V_3 - \Psi_I) \partial_3 v_i) \Lambda^{1.5+\delta} V_i - \int \Lambda^{0.5+\delta} ((\tilde{v}_3 - \tilde{v}_t) \partial_3 v_i) \Lambda^{1.5+\delta} V_i
\]
\[
- \int \Lambda^{0.5+\delta} (b_{k_1} \partial_k q_i) \Lambda^{1.5+\delta} V_i - \int \Lambda^{0.5+\delta} (b_{k_1} \partial_k Q) \Lambda^{1.5+\delta} V_i + \tilde{I}
\]
\[
= I_1 + \cdots + I_{10} + \tilde{I}.
\]
All the terms are treated similarly as those in (3.27). We show a detailed treatment of the tenth (and the most essential) term $I_{10}$. We first rewrite it as
\[
I_{10} = \int \Lambda^{0.5+\delta} (b_{k_1} Q) \Lambda^{1.5+\delta} \partial_k V_i - \int \Lambda^{1+\delta} (b_{k_1} Q) \Lambda^{1+\delta} V_i = J_1 + J_2. \tag{5.9}
\]
For the first term in (5.9), we proceed as in (3.30) and write
\[
J_1 = \int \Lambda^{0.5+\delta} Q \Lambda^{1.5+\delta} (b_{k_1} \partial_k v_i) - \int \Lambda^{1+\delta} (\partial_k (b_{k_1} v_i) - b_{k_1} \Lambda^{1+\delta} \partial_k V_i) \Lambda^{0.5+\delta} Q
\]
\[
+ \int \Lambda^{0.5+\delta} (b_{k_1} Q) - b_{k_1} \Lambda^{0.5+\delta} Q \Lambda^{1.5+\delta} \partial_k V_i
\]
\[
= J_{11} + J_{12} + J_{13}. \tag{5.10}
\]
Note that $J_{11} = - \int \Lambda^{0.5+\delta} Q \Lambda^{1.5+\delta} (B_{k_1} \partial_k \tilde{v}_1)$, due to (5.6). Since $0.5 + \delta \geq 1$ by (5.1), we have
\[
J_{11} + J_{12} \lesssim \|Q\|_{H^{0.5+\delta}} \|B\|_{H^{1.5+\delta}} \|v\|_{H^{2.5+\delta}} + \|\tilde{b}\|_{H^{3.5+\delta}} \|V\|_{H^{1.5+\delta}} \|Q\|_{H^{0.5+\delta}} \lesssim \|Q\|_{H^{0.5+\delta}} \|W\|_{H^{3.5+\delta}} \|V\|_{H^{1.5+\delta}} \|Q\|_{H^{0.5+\delta}},
\]
recalling the agreement on constants. For the third term in (5.10), we write
\[
\Lambda^{1.5+\delta} = \Lambda^{0.5+\delta} - \partial_t T_1 - \partial_2 T_2, \tag{5.11}
\]
where
\[
T_j = \partial_j (I - \Delta_2)^{\delta/2 - 0.25}, \quad j = 1, 2.
\]
are tangential operators of order $0.5 + \delta$. Using (5.11) and integrating by parts, we have

$$J_{13} = \sum_{j=1}^{2} \int \left( \partial_{j} \Lambda^{0.5+\delta}(b_{kj} Q) - b_{kj} \partial_{j} \Lambda^{0.5+\delta} Q \right) T_{j} \partial_{k} V_{i} + \sum_{j=1}^{2} \int \partial_{j} b_{kj} \Lambda^{0.5+\delta} Q T_{j} \partial_{k} V_{i}$$

$$+ \sum_{j=1}^{2} \int \left( \Lambda^{0.5+\delta}(b_{kj} Q) - b_{kj} \Lambda^{0.5+\delta} Q \right) \Lambda^{\delta-0.5} \partial_{k} V_{i}$$

$$\lesssim \|b\|_{H^{3.5+\delta}} \|Q\|_{H^{0.5+\delta}} \|V\|_{H^{1.5+\delta}} \lesssim \|Q\|_{H^{3.5+\delta}} \|V\|_{H^{1.5+\delta}}.$$

The boundary term $J_{2} = -\int_{\Gamma_{1}} \Lambda^{1+\delta}(b_{3i} Q) \Lambda^{1+\delta} V_{i}$ in (5.9) is rewritten as

$$J_{2} = -\int_{\Gamma_{1}} \Lambda^{\delta} Q b_{3i} \Lambda^{2+\delta} V_{i} - \int_{\Gamma_{1}} \Lambda^{\delta} Q \left( \Lambda(b_{3i} \Lambda^{1+\delta} V_{i}) - b_{3i} \Lambda^{2+\delta} V_{i} \right)$$

$$- \int_{\Gamma_{1}} \Lambda^{1+\delta}(b_{3i} Q) - b_{3i} \Lambda^{1+\delta} Q \Lambda^{1+\delta} V_{i}$$

$$= -\int_{\Gamma_{1}} \Lambda^{\delta} Q \Lambda^{2+\delta}(b_{3i} V_{i}) + \int_{\Gamma_{1}} \Lambda^{\delta} Q \left( \Lambda^{2+\delta}(b_{3i} V_{i}) - b_{3i} \Lambda^{2+\delta} V_{i} \right)$$

$$- \int_{\Gamma_{1}} \Lambda^{\delta} Q \left( \Lambda(b_{3i} \Lambda^{1+\delta} V_{i}) - b_{3i} \Lambda^{2+\delta} V_{i} \right) - \int_{\Gamma_{1}} \Lambda^{1+\delta}(b_{3i} Q) - b_{3i} \Lambda^{1+\delta} Q \Lambda^{1+\delta} V_{i}$$

$$= J_{21} + J_{22} + J_{23} + J_{24}.$$ 

For the first term, we use (2.20), which for the differences of solutions reads as

$$b_{3i} V_{i} = W_{i} - B_{3i} \tilde{v}_{i}.$$ 

We obtain

$$J_{21} = -\int_{\Gamma_{1}} \Lambda^{\delta} Q \Lambda^{2+\delta} W_{i} + \int_{\Gamma_{1}} \Lambda^{\delta} Q \Lambda^{2+\delta}(B_{3i} \tilde{v}_{i}) = -\int_{\Gamma_{1}} \Lambda^{2(1+\delta)} W_{i} + \int_{\Gamma_{1}} \Lambda^{\delta} Q \Lambda^{2+\delta}(B_{3i} \tilde{v}_{i})$$

$$= J_{211} + J_{212}.$$ 

The first term $J_{211}$ cancels with the right side of (5.4) after adding (5.4) and (5.8), while the second term $J_{212}$ may be bounded as

$$J_{212} \lesssim \|Q\|_{H^{3.5+\delta}(\Gamma_{1})} \|B\|_{H^{2+\delta}(\Gamma_{1})} \|\tilde{v}\|_{H^{2+\delta}(\Gamma_{1})} \lesssim \|Q\|_{H^{3.5+\delta}} \|B\|_{H^{2.5+\delta}} \|\tilde{v}\|_{H^{2.5+\delta}}$$

$$\lesssim \|Q\|_{H^{0.5+\delta}} \|W\|_{H^{3.5+\delta}(\Gamma_{1})},$$

using the agreement on constants. The last three terms in (5.12) are commutators and the sum is estimated easily as

$$J_{22} + J_{23} + J_{24} \lesssim \|Q\|_{H^{0.5+\delta}} \|V\|_{H^{1.5+\delta}},$$

employing the Kato-Ponce and trace inequalities. Finally, we add (5.4) and (5.8), observing that $J_{211}$ and the right-hand side of (5.4) cancel, we obtain (5.3).
With the tangential estimates completed, we now estimate the difference of the pressures, $Q = q - \tilde{q}$. Subtracting the pressure equation (3.35) and its analog for $\tilde{q}$, we have

$$
\begin{align*}
\partial_j (b_{ji} a_{ki} \partial_k Q) &= -\partial_j (B_{ji} a_{ki} \partial_k \tilde{q}) - \partial_j (\tilde{b}_{ji} A_{ki} \partial_k q) + \partial_j (\partial_t b_{ji} v_i) + \partial_j (\partial_t B_{ji} \tilde{v}_i) \\
&\quad - \partial_j \sum_{m=1}^{2} B_{ji} v_m a_{km} \partial_k v_i - \partial_j \sum_{m=1}^{2} \tilde{b}_{ji} v_m a_{km} \partial_k v_i \\
&\quad - \partial_j \sum_{m=1}^{2} \tilde{b}_{ji} \tilde{v}_m A_{km} \partial_k v_i - \partial_j \sum_{m=1}^{2} \tilde{b}_{ji} \tilde{v}_m \tilde{a}_{km} \partial_k V_i \\
&\quad - \partial_j (A_{ji} (v_3 - \partial_t \psi) \partial_3 v_i) - \partial_j (\tilde{a}_{ji} (V_3 - \Psi_t) \partial_3 v_i) - \partial_j (\tilde{a}_{ji} (\tilde{v}_3 - \tilde{\psi}_t) \partial_3 V_i) \\
&= \partial_j f_j \quad \text{in } \Omega.
\end{align*}
$$

(5.13)

Subtracting (3.37) and the same equation for $\tilde{q}$ gives

$$
\begin{align*}
b_{3i} a_{ki} \partial_k Q + Q &= -B_{3i} a_{ki} \partial_k \tilde{q} - \tilde{b}_{3i} A_{ki} \partial_k q + \Delta^{2}_{3} W + \partial_t B_{3i} v_i + \partial_t \tilde{b}_{3i} V_i \\
&\quad - B_{3i} v_1 a_{j1} \partial_j v_i - \tilde{b}_{3i} V_1 a_{j1} \partial_j v_i - \tilde{b}_{3i} \tilde{v}_1 A_{j1} \partial_j v_i - \tilde{b}_{3i} \tilde{v}_1 \tilde{a}_{j1} \partial_j V_i \\
&\quad - B_{3i} v_2 a_{j2} \partial_j v_i - \tilde{b}_{3i} V_2 a_{j2} \partial_j v_i - \tilde{b}_{3i} \tilde{v}_2 A_{j2} \partial_j v_i - \tilde{b}_{3i} \tilde{v}_2 \tilde{a}_{j2} \partial_j V_i \\
&\quad - A_{3i} (v_3 - \psi_t) \partial_3 v_i - \tilde{a}_{3i} (V_3 - \Psi_t) \partial_3 v_i - \tilde{a}_{3i} (\tilde{v}_3 - \tilde{\psi}_t) \partial_3 V_i = g_1 \quad \text{on } \Gamma_1,
\end{align*}
$$

(5.14)

while from (3.38), we get

$$
\begin{align*}
b_{3i} a_{ki} \partial_k Q &= -B_{3i} a_{ki} \partial_k \tilde{q} - \tilde{b}_{3i} A_{ki} \partial_k q \\
&\quad - B_{3i} v_1 a_{j1} \partial_j v_i - \tilde{b}_{3i} V_1 a_{j1} \partial_j v_i - \tilde{b}_{3i} \tilde{v}_1 A_{j1} \partial_j v_i - \tilde{b}_{3i} \tilde{v}_1 \tilde{a}_{j1} \partial_j V_i \\
&\quad - B_{3i} v_2 a_{j2} \partial_j v_i - \tilde{b}_{3i} V_2 a_{j2} \partial_j v_i - \tilde{b}_{3i} \tilde{v}_2 A_{j2} \partial_j v_i - \tilde{b}_{3i} \tilde{v}_2 \tilde{a}_{j2} \partial_j V_i \\
&\quad - A_{3i} (v_3 - \psi_t) \partial_3 v_i - \tilde{a}_{3i} (V_3 - \Psi_t) \partial_3 v_i - \tilde{a}_{3i} (\tilde{v}_3 - \tilde{\psi}_t) \partial_3 V_i = g_0 \quad \text{on } \Gamma_0.
\end{align*}
$$

(5.15)

Applying the elliptic estimate (3.41) with $l = 0.5 + \delta$, we get

$$
\|Q\|_{H^{0.5+\delta}} \lesssim \|V\|_{H^{1.5+\delta}} + \|W\|_{H^{2+\delta}(\Gamma_1)} + \|W_t\|_{H^{1+\delta}(\Gamma_1)}.
$$

(5.16)

This concludes the pressure estimates.

Next, we obtain the vorticity bound for the difference $Z = \zeta - \tilde{\zeta}$. We use the approach from Section 3.4 by extending $\zeta$ and $\tilde{\zeta}$ to $\theta$ and $\tilde{\theta}$, respectively, with the extensions defined on $T^2 \times \mathbb{R}$. For simplicity of notation, we do not distinguish between functions defined in $\Omega$ and their extension, i.e., we assume that the quantities $b, \tilde{b}, J, \tilde{J}, v,$ and $\tilde{v}$ are already extended to $T^2 \times \mathbb{R}$ and that the Jacobian $J$ is bounded as in (3.54).

The equation for $\Theta = \theta - \tilde{\theta}$ then reads

$$
\begin{align*}
J \partial_i \Theta_i + v_1 b_{j1} \partial_j \Theta_i + v_2 b_{j2} \partial_j \Theta_i + (v_3 - \psi_t) b_{j3} \partial_j \Theta_i &= F_i,
\end{align*}
$$

(5.17)

where

$$
\begin{align*}
F_i &= -(J - \tilde{J}) \partial_i \tilde{\theta}_i - V_1 b_{j1} \partial_j \tilde{\theta}_i - \tilde{v}_1 B_{j1} \partial_j \tilde{\theta}_i - V_2 b_{j2} \partial_j \tilde{\theta}_i - \tilde{v}_2 B_{j2} \partial_j \tilde{\theta}_i \\
&\quad - (V_3 - \Psi_t) b_{j3} \partial_j \tilde{\theta}_i - (\tilde{v}_3 - \tilde{\psi}_t) B_{j3} \partial_j \tilde{\theta}_i - \theta_k b_{mk} \partial_m \tilde{v}_i + \Theta_k b_{mk} \partial_m \tilde{v}_i + \tilde{b}_k B_{mk} \partial_m \tilde{v}_i, \quad i = 1, 2, 3.
\end{align*}
$$
We proceed as in (3.62), except that we use $\Lambda_3^{0.5+\delta}$ instead of $\Lambda_3^{1.5+\delta}$. We get
\begin{align*}
\frac{1}{2} \frac{d}{dt} \int_{\Omega_0} \tilde{J} |\Lambda_3^{0.5+\delta} \Theta|^2 &= -\sum_{m=1}^{2} \int_{\Omega_0} \tilde{\eta}_m \tilde{b}_{j_m} \partial_j \Lambda_3^{0.5+\delta} \Theta_i \Lambda_3^{0.5+\delta} \Theta_i - \int_{\Omega_0} (\tilde{v}_3 - \psi_t) \tilde{b}_{j_3} \partial_j \Lambda_3^{0.5+\delta} \Theta_i \Lambda_3^{0.5+\delta} \Theta_i \\
&\quad - \sum_{m=1}^{2} \int_{\Omega_0} \Lambda_3^{0.5+\delta} (\tilde{v}_m \tilde{b}_{j_m} \partial_j \Theta_i) - \tilde{v}_m \tilde{b}_{j_m} \partial_j \Lambda_3^{0.5+\delta} \Theta_i \Lambda_3^{0.5+\delta} \Theta_i \\
&\quad - \int_{\Omega_0} \Lambda_3^{0.5+\delta} (\tilde{v}_3 - \psi_t) \tilde{b}_{j_3} \partial_j \Theta_i - \tilde{v}_3 - \psi_t) \tilde{b}_{j_3} \partial_j \Lambda_3^{0.5+\delta} \Theta_i \Lambda_3^{0.5+\delta} \Theta_i \\
&\quad + \frac{1}{2} \int_{\Omega_0} \tilde{J} |\Lambda_3^{0.5+\delta} \Theta|^2 + \int_{\Omega_0} \Lambda_3^{0.5+\delta} (\tilde{J} \partial_t \Theta_i - \tilde{J} \Lambda_3^{0.5+\delta} (\partial_t \Theta_i)) \Lambda_3^{0.5+\delta} \Theta_i \\
&\quad + \int_{\Omega_0} \Lambda_3^{0.5+\delta} F_i \Lambda_3^{0.5+\delta} \Theta_i \\
&= I_1 + \cdots + I_7.
\end{align*}

For the first five terms in (5.18), we have
\begin{equation}
I_1 + \cdots + I_5 \lesssim \|\Theta\|_{H^{0.5+\delta}},
\end{equation}
where, as above, the constant depends on $\|v\|_{H^{2.5+\delta}}, \|\tilde{v}\|_{H^{2.5+\delta}}, \|w\|_{H^{4+\delta}(\Gamma_1)}, \|\tilde{w}\|_{H^{4+\delta}(\Gamma_1)}, \|w_t\|_{H^{2+\delta}(\Gamma_1)}, and \|\tilde{w}_t\|_{H^{2+\delta}(\Gamma_1)}$. For the sixth term, which involves the time derivative of the vorticity, we have
\begin{equation}
I_5 \lesssim \|\tilde{J}\|_{H^{3.5+\delta}} \|\Theta_t\|_{H^{-0.5+\delta}} \|\Theta\|_{H^{0.5+\delta}},
\end{equation}

since we assumed (5.1). To estimate the right-hand side, we use (5.17), obtaining
\begin{align}
I_6 &\lesssim \|\tilde{J}\|_{H^{3.5+\delta}} \|\Theta\|_{H^{0.5+\delta}} \|\tilde{J}\|_{H^{3.5+\delta}} \|F\|_{H^{-0.5+\delta}} \|\Theta\|_{H^{0.5+\delta}} \\
&\lesssim (\|v\|_{H^{1.5+\delta}} + \|\Theta\|_{H^{0.5+\delta}} + \|W\|_{H^{3+\delta}(\Gamma_1)} + \|W_t\|_{H^{1+\delta}(\Gamma_1)}) \|\Theta\|_{H^{0.5+\delta}},
\end{align}

where we also used (3.6). The last term in (5.18) is estimated similarly to the first five, using the fractional product rule, leading to
\begin{align}
I_7 &\lesssim (\|v\|_{H^{1.5+\delta}} + \|\Theta\|_{H^{0.5+\delta}} + \|W\|_{H^{3+\delta}(\Gamma_1)} + \|W_t\|_{H^{1+\delta}(\Gamma_1)}) \|\Theta\|_{H^{0.5+\delta}}.
\end{align}

Using the estimates (5.19), (5.20), and (5.21) in (5.18), we get
\begin{align}
\frac{d}{dt} \int_{\Omega_0} \tilde{J} |\Lambda_3^{0.5+\delta} \Theta|^2 &\lesssim (\|v\|_{H^{1.5+\delta}} + \|\Theta\|_{H^{0.5+\delta}} + \|W\|_{H^{3+\delta}(\Gamma_1)} + \|W_t\|_{H^{1+\delta}(\Gamma_1)}) \|\Theta\|_{H^{0.5+\delta}},
\end{align}

and then, using $1/4 \leq \tilde{J} \leq 2$, we obtain
\begin{align}
\frac{d}{dt} \int_{\Omega_0} \tilde{J} |\Lambda_3^{0.5+\delta} \Theta|^2 &\lesssim (\|v\|_{H^{1.5+\delta}} + \|\Theta\|_{H^{0.5+\delta}} + \|W\|_{H^{3+\delta}(\Gamma_1)} + \|W_t\|_{H^{1+\delta}(\Gamma_1)}) \left(\int_{\Omega_0} \tilde{J} |\Lambda_3^{0.5+\delta} \Theta|^2 \right)^{1/2},
\end{align}

concluding the vorticity estimates.

Finally, we apply a standard barrier argument to (5.3), (5.16), and (5.22). \hfill \Box

6. The local existence

In this section, we construct a solution to the Euler-plate model, thus proving Theorem 2.3.

6.1. Euler equations with given variable coefficients. We start by assuming that the function $w$ on the top boundary $\Gamma_1$ is given, and consider the Euler equations with given variable coefficients
\begin{align}
\partial_t v_i + v_j \tilde{a}_{ji} \partial_j v_i + v_2 \tilde{a}_{j2} \partial_j v_i + (v_3 - \psi_t) \tilde{a}_{33} \partial_3 v_i + \tilde{a}_{k1} \partial_k g = 0, \\
\tilde{a}_{k1} \partial_k g = 0.
\end{align}
Here \( \tilde{a} \) is defined as the inverse of the matrix \( \nabla \tilde{\eta} \) where \( \tilde{\eta} = (x_1, x_2, \psi) \), and \( \psi \) is a harmonic function satisfying the boundary value problem

\[
\Delta \psi = 0 \quad \text{on } \Omega \\
\psi(x_1, x_2, 1, t) = 1 + w(x_1, x_2, t) \quad \text{on } \Gamma_1 \times [0, T] \\
\psi(x_1, x_2, 0, t) = 0 \quad \text{on } \Gamma_0 \times [0, T].
\] (6.2)

More explicitly, we have

\[
\tilde{a} = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
-\partial_1 \psi / \partial_3 \psi & -\partial_2 \psi / \partial_3 \psi & 1 / \partial_3 \psi
\end{pmatrix},
\] (6.3)

and \( \tilde{b} \) is the cofactor matrix

\[
\tilde{b} = (\partial_3 \psi) \tilde{a} = \begin{pmatrix}
\partial_3 \psi & 0 & 0 \\
0 & \partial_3 \psi & 0 \\
-\partial_1 \psi & -\partial_2 \psi & 1
\end{pmatrix}.
\] (6.4)

Note that, since \( \tilde{b} \) is a cofactor matrix (or by a direct verification), it satisfies the Piola identity

\[
\partial_i \tilde{b}_{ij} = 0, \quad j = 1, 2, 3.
\]

We impose the boundary condition

\[
v_3 = 0 \quad \text{on } \Gamma_0
\] (6.5)

on the bottom boundary \( \Gamma_0 \) and

\[
\tilde{b}_{3i} v_i = w_i \quad \text{on } \Gamma_1
\] (6.6)

on \( \Gamma_1 \). Assume that we have

\[
(w, w_t, w_{tt}) \in L^\infty([0, T]; H^{4+\delta}(\Gamma_1) \times H^{2+\delta}(\Gamma_1) \times H^\delta(\Gamma_1))
\] (6.7)

with

\[
\int_{\Gamma_1} w_t = 0
\] (6.8)

and \( w_0 = 0 \), so that \( \psi(0, x) = x_3 \) and \( \tilde{a}(0) = I \). We further assume that the matrix \( \nabla \tilde{\eta} \) is non-singular on \([0, T]\) with a well-defined inverse \( \tilde{a} \) (i.e., \( \partial_3 \psi \neq 0 \)) and is such that

\[
\| \tilde{a} - I \|_{L^\infty([0, T]; H^{1.5+\delta}(\Omega))} \leq \epsilon
\] (6.9)

and

\[
\| \tilde{b} - I \|_{L^\infty([0, T]; H^{1.5+\delta}(\Omega))} \leq \epsilon,
\] (6.10)

for some \( \epsilon > 0 \) sufficiently small. Note that we have the estimate

\[
\| \tilde{a} \|_{L^\infty([0, T]; H^{3-1/2}(\Gamma_1))} \leq \| w \|_{L^\infty([0, T]; H^r(\Gamma_1))},
\] (6.11)

for \( t \in [0, T] \) and \( s > 1/2 \).

We prove the following theorem pertaining to the above Euler system with given coefficients.

**Theorem 6.1.** Assume that \( v_0 \in H^{2.5+\delta} \), where \( \delta \geq 0.5 \), satisfies (6.5)–(6.6), and suppose

\[
(w, w_t, w_{tt}) \in L^\infty([0, T]; H^{4+\delta}(\Gamma_1) \times H^{2+\delta}(\Gamma_1) \times H^\delta(\Gamma_1))
\]

with \( w_0 = 0 \) and the compatibility condition (6.8), as well as (2.28)–(2.30). Then, there exists a local-in-time solution \((v, q)\) to the system (6.1) with the boundary conditions (6.5) and (6.6) such that

\[
v \in L^\infty([0, T]; H^{2.5+\delta}(\Omega))
\]

\[
v_t \in L^\infty([0, T]; H^{0.5+\delta}(\Omega))
\]

\[
q \in L^\infty([0, T]; H^{1.5+\delta}(\Omega)),
\]

for some time \( T > 0 \) depending on the initial data and \((w, w_t)\). The solution is unique up to an additive function of time for the pressure \( q \). Moreover, the solution \((v, q)\) satisfies the estimate

\[
\| v(t) \|_{H^{2.5+\delta}} + \| q(t) \|_{H^{0.5+\delta}} \leq \| v_0 \|_{H^{2.5+\delta}} + \| w\|_{L^\infty([0, T]; H^{4+\delta}(\Gamma_1))} \int_0^t \| w(t) \|_{H^{4+\delta}(\Gamma_1)} \, ds,
\] (6.12)
for $t \in [0, T)$.

In the proof of the theorem, we shall employ the generalized vorticity corresponding to a given velocity (see (6.34) below). In order to estimate the velocity from the vorticity, we use the following div-curl theorem.

**Lemma 6.1.** For a fixed time $t \in [0, T]$, consider the system

\[
\begin{align*}
&\epsilon_{ijk} \hat{b}_{mj} \partial_m v_k = \zeta_i \quad \text{in } \Omega, \quad i = 1, 2, 3 \\
&\hat{b}_{mj} \partial_m v_j = 0 \quad \text{in } \Omega \\
&\hat{b}_{3j} v_j = \psi_i \quad \text{on } \Gamma_0 \cup \Gamma_1,
\end{align*}
\]

where $\zeta \in H^{1.5+\delta}$ with $\hat{b} \in H^{2.5+\delta}$ and $\|\hat{b} - I\|_{L^\infty} \leq \epsilon_0$. If $\epsilon_0 > 0$ is sufficiently small, then $v$ satisfies the estimate

\[
\|v\|_{H^{2.5+\delta}} \lesssim \|\zeta\|_{H^{1.5+\delta}} + \|w_t\|_{H^{2+\delta}(\Gamma_1)} + \|v\|_{L^2},
\]

(6.13)

where the implicit constant depends on the bound on $\hat{b}$.

**Proof of Lemma 6.1.** (sketch) The proof is standard and is obtained by rewriting the system as

\[
\epsilon_{ijk} \delta_{mj} \partial_m v_k = \zeta_i + \epsilon_{ijk} (\delta_{mj} - \hat{b}_{mj}) \partial_m v_k \quad \text{in } \Omega \times [0, T]
\]

\[
\delta_{mj} \partial_m v_j = (\delta_{mj} - \hat{b}_{mj}) \partial_m v_j \quad \text{in } \Omega \times [0, T]
\]

\[
v_3 = \psi_i + (\delta_{3j} - \hat{b}_{3j}) v_j \quad \text{on } (\Gamma_0 \cup \Gamma_1) \times [0, T].
\]

The rest depends on the classical div-curl estimates as in [BB] and the smallness assumption $\|\hat{b} - I\|_{L^\infty} \leq \epsilon_0$. \hfill $\square$

**Proof of Theorem 6.1.** We prove the theorem in three steps.

**Step 1: Linear Problem.** Assume that $w, w_t, w_{tt}$ satisfy the assumptions in the theorem, but with the additional regularity

\[
(w, w_t, w_{tt}) \in L^\infty([0, T]; H^{6+\delta}(\Gamma_1) \times H^{4+\delta}(\Gamma_1) \times H^{2+\delta}(\Gamma_1)),
\]

(6.14)

and $\tilde{a}$ as defined above. Denote by $E$ the Sobolev extension $H^k(\Omega) \to H^k(\Omega_0)$ for all $k \in [0, 5]$, where $\Omega_0 = \mathbb{T}^2 \times [-1, 2]$ (which is different than in Section 3.4). We consider the linear transport equation

\[
\partial_t v_i + E(\tilde{v}_1) E(\tilde{a}_{j1}) \partial_{j} v_i + E(\tilde{v}_2) E(\tilde{a}_{j2}) \partial_{j} v_i + E(\tilde{v}_3 - \psi) E(\tilde{a}_{33}) \partial_{3} v_i + E(\tilde{a}_{k1}) E(\partial_k \tilde{q}) = 0 \quad \text{in } \mathbb{T}^2 \times \mathbb{R}
\]

(6.15)

with $\tilde{v} \in L^\infty([0, T]; H^{2.5+\delta}(\Omega))$ a given periodic function in the $x_1, x_2$ directions. In (6.15), the pressure function $\tilde{q}$ is given as the solution to the elliptic problem

\[
\begin{align*}
\partial_j (\tilde{b}_{ji} \tilde{a}_{kji} \partial_k \tilde{q}) = \partial_j (\partial_t \tilde{b}_{ji} \tilde{v}_i) - & \sum_{m=1}^{2} \tilde{b}_{ji} \partial_j (\tilde{v}_m \tilde{a}_{km}) \partial_k \tilde{v}_i - \tilde{b}_{ji} \partial_j (J^{-1}(\tilde{v}_3 - \psi)) \partial_3 \tilde{v}_i \\
& + \sum_{m=1}^{2} \tilde{v}_m \tilde{a}_{km} \partial_k \tilde{b}_{ji} \partial_j \tilde{v}_i + J^{-1}(\tilde{v}_3 - \psi) \partial_3 \tilde{b}_{ji} \partial_j \tilde{v}_i + \mathcal{E} = \tilde{f} \quad \text{in } \Omega,
\end{align*}
\]

(6.16)

with the Neumann boundary conditions

\[
\tilde{b}_{3i} \tilde{a}_{k3i} \partial_k \tilde{q} = 0 = \tilde{g}_0 \quad \text{on } \Gamma_0,
\]

(6.17)

and

\[
\tilde{b}_{3i} \tilde{a}_{k3i} \partial_k \tilde{q} = -w_{tt} + \partial_t \tilde{b}_{3i} \tilde{v}_i - \frac{1}{\partial_3 \psi} \left( \sum_{j=1}^{2} \tilde{v}_k \tilde{b}_{jk} \partial_j (w_t) + w_t \partial_3 (\tilde{b}_{3k}) \tilde{v}_i + \tilde{v}_k \tilde{b}_{jk} \partial_j (\tilde{b}_{3k}) \tilde{v}_i \right) = \tilde{g}_1 \quad \text{on } \Gamma_1.
\]

(6.18)
Note that (6.16) and (6.18) are suggested by Remark 3.5. The function of time
\[
E = \frac{1}{|\Omega|} \int \partial_t(\tilde{a}_km \tilde{v}_m) \tilde{b}_{j_1} \partial_{j_1} \tilde{v}_i - \frac{1}{|\Gamma|} \int \tilde{v}_m \tilde{a}_{3m} \tilde{b}_{j_1} \partial_{j_1} \tilde{v}_i - \frac{1}{|\Omega|} \int \partial_3(\tilde{a}_{33} \psi) \tilde{b}_{j_1} \partial_{j_1} \tilde{v}_i \\
+ \frac{1}{|\Omega|} \int \psi \tilde{a}_{33} \tilde{b}_{j_1} \partial_{j_1} \tilde{v}_i + \frac{1}{|\Omega|} \int \frac{1}{\partial_3} \tilde{b}(\tilde{b}_k \tilde{v}_k - \psi) \tilde{b}_{3j} \partial_{3j} \tilde{v}_i + \frac{1}{|\Omega|} \sum_{k=1}^2 \tilde{a}_{km} \tilde{v}_m \partial_k(\tilde{b}_{3j} \tilde{v}_i - \psi) \\
+ \frac{1}{|\Gamma_o|} \int \frac{1}{\partial_3} \tilde{b}(\tilde{b}_k - \psi) \partial_3 \tilde{b}_{3j} \tilde{v}_i - \frac{1}{|\Gamma_o|} \int \tilde{v}_k \tilde{a}_{j_k} \partial_{j_k} \tilde{b}_{3j} \tilde{v}_i,
\]
(6.19)
where
\[\Gamma = \Gamma_0 \cup \Gamma_1,\]
is introduced to insure the validity of the compatibility condition
\[
\int_{\Gamma} \tilde{f} = \int_{\Gamma_1} \tilde{g}_i;
\]
(6.20)
see Appendix for the verification of (6.20). The condition (6.20) is necessary and sufficient for the existence of the solution \(\tilde{q}\) to the Neumann boundary value problem (6.16)–(6.18), which satisfies the estimate
\[
\|\nabla \tilde{q}\|_{H^{2.5+\delta}} \lesssim \|\tilde{f}\|_{H^{1.5+\delta}} + \|\tilde{g}_0\|_{H^{2+\delta}} + \|\tilde{g}_1\|_{H^{2+\delta}}.
\]
and is determined up to a constant.

Estimating \(\tilde{f}\) defined in (6.16) in \(H^{1.5+\delta}\), we have
\[
\|\tilde{f}\|_{H^{1.5+\delta}} \lesssim \|\partial_t \tilde{b}\|_{H^{2.5+\delta}} \|\tilde{v}\|_{H^{2.5+\delta}} + \|\tilde{b}\|_{H^{2.5+\delta}} \|\tilde{v}\|_{H^{2.5+\delta}} + \|\psi\|_{H^{2.5+\delta}} \|\tilde{a}\|_{H^{2.5+\delta}} + \|\tilde{v}\|_{H^{2.5+\delta}} + \|\tilde{a}\|_{H^{2.5+\delta}} + \|\tilde{v}\|_{H^{2.5+\delta}} + \|\tilde{a}\|_{H^{2.5+\delta}} + \|\tilde{v}\|_{H^{2.5+\delta}}.
\]
while \(\tilde{g}_1\) from (6.18) may be bounded as
\[
\|\tilde{g}_1\|_{H^{2+\delta}} \lesssim \|w_i\|_{H^{2+\delta}} + \|\partial_t \tilde{b}\|_{H^{2+\delta}} + \|\tilde{b}\|_{H^{2+\delta}} + \|\tilde{a}\|_{H^{2+\delta}} + \|\tilde{v}\|_{H^{2+\delta}} + \|\tilde{a}\|_{H^{2+\delta}} + \|\tilde{v}\|_{H^{2+\delta}}.
\]
Therefore,
\[
\|\nabla \tilde{q}\|_{H^{2.5+\delta}} \lesssim \|w_i\|_{H^{2+\delta}} + \|\tilde{a}\|_{H^{2+\delta}} + \|\tilde{b}\|_{H^{2+\delta}} + \|\tilde{v}\|_{H^{2+\delta}}.
\]
(6.21)
Since \(\tilde{q}\) is given, the linear equation (6.15) has the structure of a transport system
\[
\partial_t v_i + E(K) \cdot \nabla v_i = F_i \quad \text{in } T^2 \times \mathbb{R}, \quad i = 1, 2, 3,
\]
where \(K \in L^\infty([0, T]; H^{2.5+\delta})\) and \(F \in L^\infty([0, T]; H^{2.5+\delta})\). The existence of a solution \(v \in L^\infty([0, T]; H^{2.5+\delta})\) is standard, and in addition we have the estimate
\[
\|v\|_{L^\infty([0, T]; H^{2.5+\delta})} \lesssim \|v_0\|_{H^{2.5+\delta}} + \int_0^T P(\|w_i\|_{H^{2+\delta}}, \|\tilde{a}\|_{H^{2.5+\delta}}, \|\tilde{b}\|_{H^{2.5+\delta}}, \|\tilde{v}\|_{H^{2.5+\delta}}, \|\tilde{a}\|_{H^{2.5+\delta}}, \|\tilde{v}\|_{H^{2.5+\delta}}) \, ds.
\]

Step 2: Local-in-time solution of the nonlinear problem with more regular boundary data. In the second step we still assume (6.14) and aim to solve the nonlinear problem
\[
\partial_t v_i + v_1 \partial_{j_1} v_i + v_2 \partial_{j_2} v_i + v_3 \partial_{j_3} v_i + \tilde{a}_{3k} \partial_k q = 0 \quad \text{in } \Omega, \quad i = 1, 2, 3,
\]
\[
v_3 = 0 \quad \text{on } \Gamma_0,
\]
\[
\tilde{b}_{3l} v_l = w_l \quad \text{on } \Gamma_1,
\]
(6.22)
using the iteration
\[
\partial_t v_i = (v_{i_1}^{(n+1)} + E(v_1^{(n)}) E(\tilde{a}_{j_1}) \partial_{j_1} v_i^{(n+1)} + E(v_2^{(n)}) E(\tilde{a}_{j_2}) \partial_{j_2} v_i^{(n+1)} + E(v_3^{(n)}) E(\tilde{a}_{j_3}) \partial_{j_3} v_i^{(n+1)} + E(\tilde{a}_{k_1}) E(\partial_{k_1} q^{(n+1)}) = 0 \quad \text{in } \Omega_0, \quad i = 1, 2, 3,
\]
(6.23)
where \(q^{(n)}\) is obtained by solving the system (6.16)–(6.18) with \(\tilde{v}\) replaced by \(v^{(n)}\).

Note that, given \(\tilde{v} = v^{(n)}\), we solve for \(\tilde{q} = q^{(n+1)}\) and then obtain \(v^{(n+1)}\) as in Step 1. We now proceed by using a fixed point argument. We first choose \(M > 0\) sufficiently large and a time \(T\) sufficiently small so that \(\|v^{(n)}\|_{L^\infty([0, T]; H^{2.5+\delta})} \leq M\) for all \(n\) and we establish that the mapping \(v^{(n)} \mapsto v^{(n+1)}\) is a contraction in the norm of
$L^\infty([0, T]; L^2)$. For $n \in \mathbb{N}_0$, denote $V^{(n)} = u^{(n)} - u^{(n-1)}$ and $Q^{(n)} = q^{(n)} - q^{(n-1)}$. Note that the function $V^{(n+1)}$ satisfies

$$
\partial_t V_i^{(n+1)} + E(v_i^{(n)})E(\tilde{a}_{jk})\partial_j V_i^{(n+1)} + E(V_k^{(n)})E(\tilde{a}_{jk})\partial_j v_i^{(n)} - E(q_{it})E(\tilde{a}_{33})\partial_3 V_i^{(n+1)} + E(\tilde{a}_{ki})E(\partial_k Q^{(n+1)}) = 0 \quad \text{in } \Omega_0.
$$

Applying the differential operator $(I - \Delta)^{1/2}$, multiplying with $(I - \Delta)^{1/2}V^{(n+1)}$, and integrating in time and space, we may then estimate the norm of $V$ in $H^1$ as

$$
\|V^{(n+1)}(t)\|_{H^1}^2 \lesssim \int_0^t \|V^{(n+1)}\|_{H^2}^2 \|\tilde{a}\|_{H^{2.5+\delta}}(\|v(t)\|_{H^{2.5+\delta}} + \|\psi(t)\|_{H^{2.5+\delta}}) \, ds
+ \int_0^t \|V^{(n+1)}\|_{H^1} \|\tilde{a}\|_{H^{2.5+\delta}} \|v(t)\|_{H^{2.5+\delta}} \, ds
+ \int_0^t \|\nabla Q^{(n+1)}\|_{H^1} \|V^{(n+1)}\|_{H^1} \|\tilde{a}\|_{H^{2.5+\delta}} \, ds.
$$

We now use a similar elliptic estimate to (6.21) to bound the difference of two solutions $Q$ to the pressure equation. Namely,

$$
\|\nabla Q^{(n+1)}\|_{H^1} \leq P(\|\tilde{a}\|_{H^{3.5+\delta}}, \|\tilde{b}\|_{H^{3.5+\delta}}, \|\tilde{\psi}\|_{H^{2.5+\delta}}, \|\psi(t)\|_{H^{2.5+\delta}}, M) \|V^{(n)}\|_{H^1}.
$$

Note that we used a bound on the error term $\mathcal{E}^{(n)} - \mathcal{E}^{(n-1)}$ by $\|V^n\|_{H^1}$ with constants depending on $w$, $w_t$, and $M$. Hence, we have

$$
\|V^{(n+1)}(t)\|_{H^1}^2 \lesssim \int_0^t \|V^{(n+1)}\|_{H^1}^2 \, ds + \int_0^t \|V^n\|_{H^1}^2 \, ds,
$$

where all constants are allowed to depend on the norms of $w$. Using Gronwall’s inequality and taking $T$ sufficiently small, we obtain the desired contraction estimate

$$
\|V^{(n+1)}\|_{L^\infty([0, T]; H^1)} \leq \frac{1}{2} \|V^n\|_{L^\infty([0, T]; H^1)}.
$$

Thus there exist $v \in L^\infty([0, T]; H^{2.5+\delta})$ and a pressure function $q \in L^\infty([0, T]; H^{2.5+\delta})$, which is defined up to a constant, which are the fixed point for the iteration scheme. The couple $(v, q)$ then satisfies the first equation of the nonlinear system (6.22). We next show that the divergence condition and the boundary conditions in (6.22) are satisfied by the pair $(v, q)$.

**Step 3: Reconstruction of Divergence and Boundary Conditions of the nonlinear problem (6.22).**

It follows that the fixed point $v$ of the iteration scheme defined in (6.23) solves the problem

$$
\partial_t v_i + v_m \tilde{a}_{km} \partial_k v_i - \psi \tilde{a}_{33} \partial_3 v_i + \tilde{a}_{ki} \partial_k q = 0 \quad \text{in } \Omega.
$$

Using the equations (6.16)–(6.18), the corresponding pressure $q$ satisfies the elliptic boundary value problem

$$
\partial_j (\tilde{b}_{j,k} \tilde{a}_{ki} \partial_k q) = \partial_j (\tilde{b}_{j,k} \psi) = \partial_j (v_m \tilde{a}_{km} \partial_k v_i) + \partial_j (\tilde{a}_{33} \tilde{\psi}) \partial_j \psi + \tilde{a}_{ki} \psi \partial_j \tilde{b}_{j,k} \tilde{b}_{j,k} \partial_j v_i + \tilde{a}_{33} \partial_3 \psi \partial_j \tilde{b}_{j,k} \tilde{b}_{j,k} \partial_j v_i + \mathcal{E} \quad \text{in } \Omega,
$$

with the Neumann type boundary conditions

$$
\tilde{b}_{3i} \tilde{a}_{ki} \partial_k q = 0 \quad \text{on } \Gamma_0
$$

and

$$
\tilde{b}_{3i} \tilde{a}_{ki} \partial_k q = -w_t + \tilde{b}_{3i} \psi - \frac{1}{\partial_3 \psi} \left( \sum_{j=1}^3 v_k \tilde{b}_{jk} \partial_j w_t + w_t \partial_3 \tilde{b}_{3i} v_i - v_k \tilde{b}_{jk} \partial_j \tilde{b}_{j,k} \tilde{b}_{3i} v_i \right) \quad \text{on } \Gamma_1.
$$

Applying the variable divergence $\tilde{b}_{j,k} \partial_j$ to (6.24) and using the expression for $q$ from (6.25), we obtain

$$
\partial_l (\tilde{b}_{j,k} \partial_j v_i) + v_m \tilde{a}_{km} \partial_k (\tilde{b}_{j,k} \partial_j v_i) - \psi \tilde{a}_{33} \partial_l (\tilde{b}_{j,k} \partial_j v_i) = -\mathcal{E}.
$$
Also, multiplying (6.24) by \( \tilde{b}_{3i} \), restricting to \( \Gamma_1 \), then substituting the expression for \( \tilde{b}_{3i} \tilde{a}_{ki} \partial_k q \) from (6.27) into the equation we obtain
\[
\partial_t (\tilde{b}_{3i} v_i - w_t) + \sum_{j=1}^{2} v_k \tilde{a}_{jk} \partial_j (\tilde{b}_{3k} v_i - w_t) = -\frac{1}{\partial_3 \psi} \partial_3 (\tilde{b}_{3k} v_k) (\tilde{b}_{3i} v_i - w_t) \quad \text{on } \Gamma_1.
\]

Note this equation on \( \Gamma_1 \) is a transport equation on \( \mathbb{R}^2 \) with periodic boundary conditions, satisfied by \( \tilde{b}_{3i} v_i - w_t \). Since \( \tilde{b}_{3i} v_i - w_t = 0 \) at time 0, this implies \( \tilde{b}_{3i} v_i - w_t = 0 \) for all \( t \). Indeed, testing the equation \( \tilde{b}_{3i} v_i - w_t \) on \( \Gamma_1 \) leads to this conclusion.

Similarly, multiplying (6.24) by \( \tilde{b}_{3i} \), restricting to \( \Gamma_0 \), then using the fact that \( \tilde{b} = I \) on \( \Gamma_0 \) while \( \psi = 0 \) and the boundary condition (6.26) for \( q \) on \( \Gamma_0 \), we obtain the transport equation
\[
\partial_t v_3 + v_k \partial_k v_3 = 0 \quad \text{on } \Gamma_0,
\]
which may be rewritten as
\[
\partial_t v_3 + \sum_{k=1}^{2} v_k \partial_k v_3 = -(\partial_3 v_3) v_3 \quad \text{on } \Gamma_0.
\]
Since \( v_3 = 0 \) at time 0, we conclude that
\[
\begin{align*}
v_3 &= 0 \quad \text{on } \Gamma_1, \\
v_3 &= 0 \quad \text{on } \Gamma_0.
\end{align*}
\]
and the boundary conditions satisfied by \( v \) are recovered.

We next recover the divergence condition. Now, use that in (6.19) all the integrals over \( \Gamma \) vanish and integrating by parts in the remaining two, we get
\[
\mathcal{E} = \frac{1}{|\Omega|} \int \sum_{m=1}^{2} v_m a_{km} \partial_k (\tilde{b}_{ji} \partial_j v_i) + \frac{1}{|\Omega|} \int \frac{1}{\partial_3 \psi} (v_3 - \psi) \partial_3 (\tilde{b}_{ji} \partial_j v_i).
\]
By (6.28) and (6.30), the ALE divergence
\[
\mathcal{D} = \tilde{b}_{ji} \partial_j v_i
\]
satisfies the PDE
\[
\partial_t \mathcal{D} + A \cdot \nabla \mathcal{D} = \int B \cdot \nabla \mathcal{D},
\]
where \( A \cdot N = 0 \) on \( \Gamma \) and \( A, B \in L^\infty([0, T]; H^{2.5+\delta}) \). Using an \( H^1 \) estimate on \( \mathcal{D} \) and employing \( \mathcal{D}(0) = 0 \), we get \( \mathcal{D} = 0 \) recovering the divergence condition
\[
\tilde{b}_{ji} \partial_j v_i = 0,
\]
for all \( t \in [0, T] \).

**Step 4: Regularity of the vorticity with more regular boundary data.** Still under the assumption (6.14), we apply the variable curl, \( \epsilon_{ijk} \tilde{b}_{mj} \partial_m (\cdot)_k \), to (6.22), with \( k \) replaced by \( m \) and \( i \) replaced by \( k \), and obtain the system
\[
\begin{align*}
\partial_t \zeta_i + v_1 \tilde{a}_{j1} \partial_j \zeta_i + v_2 \tilde{a}_{j2} \partial_j \zeta_i + (v_3 - \psi) \tilde{a}_{33} \partial_3 \zeta_i - \zeta_1 \tilde{a}_{j1} \partial_j v_i - \zeta_2 \tilde{a}_{j2} \partial_j v_i - \zeta_3 \tilde{a}_{33} \partial_3 v_i &= 0, \quad i = 1, 2, 3,
\end{align*}
\]
for \( i = 1, 2, 3 \), in \( \Omega \), for \( i = 1, 2, 3 \), where the ALE vorticity \( \zeta \) is given by
\[
\zeta_i = \epsilon_{ijk} \tilde{a}_{mj} \partial_m v_k, \quad i = 1, 2, 3
\]
and where \( \tilde{a} \) is defined as before in (6.2)–(6.3), depending on given functions \( (w, w_t) \), and such that
\[
\tilde{b}_{3i} v_i - w_t = 0 \quad \text{on } \Gamma_0 \cup \Gamma_1.
\]

Based on (6.33), we claim that
\[
\begin{align*}
\|\zeta(t)\|_{H^{1.5+\delta}} &\lesssim \|\zeta_0\|_{H^{1.5+\delta}} + \int_0^t P(\|v\|_{H^{2.5+\delta}}, \|w\|_{H^{4+\delta}(\Gamma_1)}, \|w_t\|_{H^{2+\delta}(\Gamma_1)}) \, ds, \quad \text{(6.36)}
\end{align*}
\]
where \( P \) always denotes a generic polynomial. Note that (6.33) is a transport equation of the form \( \zeta + A \nabla \zeta + B \zeta = 0 \) such that \( A \in L^\infty([0, T]; H^{2.5+\delta}(\Omega)) \) and \( B \in L^\infty([0, T]; H^{1.5+\delta}(\Omega)) \) with \( A \cdot N|_{\Gamma_0 \cup \Gamma_1} = 0 \). The regularity assumptions hold since \( \tilde{a} \in L^\infty([0, T]; H^{3.5+\delta}(\Omega)) \), \( \psi_t \in L^\infty([0, T]; H^{2.5+\delta}(\Omega)) \), and \( v \in L^\infty([0, T]; H^{2.5+\delta}(\Omega)) \). On the other hand, the boundary condition \( A \cdot N|_{\Gamma_0 \cup \Gamma_1} = 0 \) is satisfied by (6.35). The norms of \( \tilde{a} \) and \( \tilde{b} \) are estimated using (6.11) in terms of \( \psi \), which in turn depends on the boundary data \( w \). This concludes the proof of (6.36).
Now, we use (6.13) and (6.36), as well as $\|v\|_{L^2} \lesssim \|v_0\|_{L^2} + \int_0^t \|v_t\|_{L^2} \, ds$, to estimate
\[
\|v\|_{H^{2.5+\delta}} \lesssim \|v_0\|_{H^{2.5+\delta}} + \|w_t\|_{H^{2.5+\delta}(\Gamma_1)} + \int_0^t P(\|v\|_{H^{2.5+\delta}}, \|w\|_{H^{2.5+\delta}(\Gamma_1)}, \|w_t\|_{H^{2.5+\delta}(\Gamma_1)}) \, ds.
\]
(6.37)

Applying the Gronwall inequality on (6.37), we consequently obtain
\[
\|v\|_{H^{2.5+\delta}} \lesssim \|v_0\|_{H^{2.5+\delta}} + \|w_t\|_{H^{2.5+\delta}(\Gamma_1)} + \int_0^t P(\|w\|_{H^{2.5+\delta}(\Gamma_1)}, \|w_t\|_{H^{2.5+\delta}(\Gamma_1)}) \, ds,
\]
(6.38)
for small times $t \in (0, T]$. This inequality provides a bound on $v$ in terms of lower norms (see (6.7)) of the boundary data, under the assumption of higher regularity (6.14). In the statement above and in the rest of the paper, we continue to use the convention that the domain in norms is $\Omega$ unless otherwise indicated.

**Step 5: Solution to the nonlinear problem with less regular boundary data.** Now, assume only (6.8). We approximate $(w, w_t, w_{tt}) \in L^\infty([0, T]; H^{4+\delta} \times H^{2+\delta} \times H^\delta)$ by a sequence of more regular data $(w^{(m)}, w^{(m)}_t, w^{(m)}_{tt}) \in L^\infty([0, T]; H^{4+\delta} \times H^{2+\delta} \times H^\delta)$. From Step 2, we can find a sequence of solutions $v^{(m)} \in L^\infty([0, T]; H^{2.5+\delta}(\Omega))$ and $q^{(m)} \in L^\infty([0, T]; H^{5.5+\delta}(\Omega))$ determined up to a constant employing the given boundary data $(u^{(m)}, u^{(m)}_t, u^{(m)}_{tt})$. Using the estimate (6.38), we have a uniform bound on the sequence $(v^{(m)})$. Therefore, we may extract a sequence $(w^{(m)})$ which converges weak-* to some $v \in L^\infty([0, T]; H^{2.5+\delta}(\Omega))$. Moreover, we may also obtain a uniform bound on $\nabla q^{(m)}$ in $L^\infty([0, T]; H^{0.5+\delta}(\Omega))$ in terms of the data $(w, w_t, w_{tt})$ by considering the elliptic problem (6.16) with the Neumann boundary conditions (6.17)–(6.18), from which one can derive the estimate
\[
\|\nabla q^{(m)}\|_{H^{0.5+\delta}} \lesssim P(\|w_t\|_{H^{4}(\Gamma_1)}, \|\tilde{a}\|_{H^{2.5+\delta}}, \|\tilde{b}\|_{H^{2.5+\delta}}, \|\tilde{b}_t\|_{H^{2.5+\delta}}, \|\psi_t\|_{H^{2.5+\delta}}, \|v\|_{H^{2.5+\delta}}).
\]

In addition, we may adjust the pressure $q^{(m)}$ by an appropriate constant so that $\int_{\Gamma_1} q^{(m)} = 0$, which then in turn implies
\[
\|q^{(m)}\|_{H^{1.5+\delta}} \lesssim \|\nabla q^{(m)}\|_{H^{0.5+\delta}}.
\]
It then follows that we have a uniform bound on $q^{(m)}$ in $L^\infty([0, T]; H^{1.5+\delta}(\Omega))$, and we can thus extract a further weak-* convergent subsequence with a limit $q \in L^\infty([0, T]; H^{1.5+\delta}(\Omega))$. Consequently, the corresponding sequence of time derivatives $v^{(m)}$ is uniformly bounded in $L^\infty([0, T]; H^{0.5+\delta}(\Omega))$, which can be directly deduced from the equation. Using a standard Aubin-Lions compactness argument, we may pass to the limit in (6.1) and boundary conditions (6.5) and (6.6) satisfied by $v^{(m)}$ and $q^{(m)}$ as $m \to \infty$ to obtain a solution $v \in L^\infty([0, T]; H^{2.5+\delta}(\Omega))$ and $q \in L^\infty([0, T]; H^{0.5+\delta}(\Omega))$ satisfying the equations (6.1) and the boundary conditions (6.5)–(6.6), given $(w, w_t, w_{tt}) \in L^\infty([0, T]; H^{4+\delta} \times H^{2+\delta} \times H^\delta)$. Note that the resulting pressure $q$ is periodic in $x_1, x_2$ and has zero average on $\Gamma_1$.

**6.2. The plate equation.** Next, we provide the existence theorem for the plate equation.

**Lemma 6.2.** Consider the damped plate equation
\[
w_{tt} + \Delta_2^2 w - \nu \Delta_2 w_t = d \quad \text{on} \quad \Gamma_1 \times [0, T],
\]
(6.39)
where $\nu > 0$, defined on the domain $\Gamma_1 = [0, 1] \times [0, 1] \subseteq \mathbb{R}^2$ with periodic boundary conditions. Given the initial data $w(0, \cdot) = w_0 \in H^{4+\delta}(\Gamma_1)$ and $w_t(0, \cdot) = w_t \in H^{2+\delta}(\Gamma_1)$ such that (2.31) holds and the forcing term $d \in L^2([0, T]; H^{2+\delta}(\Gamma_1))$ with $\delta > 0$, there exists a unique solution $w \in L^\infty([0, T]; H^{4+\delta}(\Gamma_1))$ such that $w_t \in L^\infty([0, T]; H^{2+\delta}(\Gamma_1)) \cap L^2([0, T]; H^{5/2}(\Gamma_1))$ and $w_{tt} \in L^2([0, T]; H^{3/2}(\Gamma_1))$ with (3.1) for all $t \in [0, T]$. Moreover, we have the estimate
\[
\|w\|_{L^\infty([0, T]; H^{4+\delta}(\Gamma_1))} + \|w_t\|_{L^\infty([0, T]; H^{2+\delta}(\Gamma_1))} + \|w_{tt}\|_{L^2([0, T]; H^{5/2}(\Gamma_1))} + \nu \|w\|_{L^2([0, T]; H^{5/2}(\Gamma_1))} \lesssim \|w_0\|_{H^{4+\delta}(\Gamma_1)} + \|w_t\|_{H^{2+\delta}(\Gamma_1)} + \|d\|_{L^2([0, T]; H^{2+\delta}(\Gamma_1))},
\]
(6.40)
where the constant depends on $\nu$.

**Proof of Lemma 6.2.** We provide a necessary a priori estimate for (6.40). Since the equation is linear, it is straight-forward to justify it using a truncation in the Fourier variables. With $\Lambda$ as in (3.15), we test (6.39) with...
6.2. Given in $$\mathcal{W}$$ on the bottom, and on the top. The coefficient matrices $$a$$ side.

where $$\delta$$ prove next, establishes the existence of the solution to the above system.

$$\Gamma$$ damped plate equation $$T >$$ for some time $$t$$ and the estimate $$(\delta + 2\epsilon)$$ where $$\nu$$ obtaining

$$\int_{\Gamma_1} \delta^{4+2\delta} w_t \leq \|\delta^{1+\delta} d\|_{L^2(\Gamma_1)} \|\delta^{3+\delta} w_t\|_{L^2(\Gamma_1)} \leq \frac{\nu}{2} \|\delta^{3+\delta} w_t\|_{L^2(\Gamma_1)} + \frac{1}{2\nu} \|d\|_{L^{1+\delta}},$$

and the estimate (6.40) follows upon absorbing the first term on the far-right side into the third term on the far-left side.

$$\Box$$

6.3. Regularized Euler-plate system. We now consider the regularized Euler-plate system consisting of the Euler equations

$$\partial_t v_1 + v_1 a_{11} \partial_j v_1 + v_2 a_{12} \partial_j v_1 + (v_3 - \psi_t) \partial_k v_1 + a_{ki} \partial_k q = 0,$$

$$a_{ki} \partial_k v_1 = 0$$

in $$\Omega \times [0, T]$$, with the boundary condition

$$v_3 = 0 \quad \text{on } \Gamma_0$$

on the bottom, and

$$b_3 v = w_t \quad \text{on } \Gamma_1$$

on the top. The coefficient matrices $$a$$ and $$b$$ are defined as in (6.2)–(6.4) in terms of $$(w, w_t)$$ solving the regularized damped plate equation

$$w_{tt} - \nu \Delta_2 w_t + \Delta^2 w = q \quad \text{on } \Gamma_1 \times [0, T]$$

(6.42) defined on a domain $$\Gamma_1 = [0, 1] \times [0, 1] \subseteq \mathbb{R}^2$$ with periodic boundary conditions. The following theorem, which we prove next, establishes the existence of the solution to the above system.

**Theorem 6.2.** Let $$\nu > 0$$. Assume that initial data

$$(v_0, w_0, w_1) \in H^{2,5+\delta} \times H^{4,5+\delta}(\Gamma_1) \times H^{2+\delta}(\Gamma_1),$$

where $$\delta \geq 0.5$$, satisfy the compatibility conditions (2.28)–(2.31). Then there exists a unique local-in-time solution $$(v, q, w, w_t)$$ to the system (6.41)–(6.42) such that

$$v \in L^\infty([0, T); H^{2,5+\delta}(\Omega)) \cap C([0, T]; H^{0,5+\delta}(\Omega)),$$

$$v_t \in L^\infty([0, T]; H^{0,5+\delta}(\Omega)),$$

$$q \in L^\infty([0, T]; H^{1,5+\delta}(\Omega)),$$

$$w \in L^\infty([0, T]; H^{4,5+\delta}(\Gamma_1)),$$

$$w_t \in L^\infty([0, T]; H^{2+\delta}(\Gamma_1)),$$

for some time $$T > 0$$ depending on initial data as well as on $$\nu$$ and $$\epsilon$$.

**Proof of Theorem 6.2.** Given $$\nu > 0$$ and a regularization parameter $$\epsilon > 0$$, we shall construct a solution to the above system using the iteration scheme

$$\partial_t v_1^{(n+1)} + v_1^{(n+1)} a_{11} \partial_j v_1^{(n+1)} + v_2^{(n+1)} a_{12} \partial_j v_1^{(n+1)} + (v_3^{(n+1)} - \psi_t^{(n)}) \partial_k v_1^{(n+1)} + a_{ki}^{(n)} \partial_k q^{(n+1)} = 0,$$

$$a_{ki}^{(n)} \partial_k v_1^{(n+1)} = 0,$$

where $$a^{(n)}$$ is determined from

$$\Delta \psi^{(n)} = 0 \quad \text{on } \Omega$$

$$\psi^{(n)}(x_1, x_2, 1, t) = 1 + \psi^{(n)}(x_1, x_2, t) \quad \text{on } \Gamma_1$$

$$\psi^{(n)}(x_1, x_2, 0, t) = 0 \quad \text{on } \Gamma_0$$

by

$$a^{(n)} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
-\partial_1 \psi^{(n)}/\partial_3 \psi^{(n)} & -\partial_2 \psi^{(n)}/\partial_3 \psi^{(n)} & 1/\partial_3 \psi^{(n)}
\end{pmatrix}$$

(6.43)
and where $w$ satisfies
\[ w_{tt}^{(n+1)} + \Delta^2 w^{(n+1)} - \nu \Delta_2 w_t^{(n+1)} = q^{(n+1)} \quad \text{on } \Gamma_1 \times [0, T]. \] (6.44)

With the initial data as in Theorem 6.2, let
\[ (w_t^{(n)}, w_t^{(n)}, w_t^{(n)}) \in L^\infty([0, T]; H^{4+\delta}(\Gamma_1) \times H^{2+\delta}(\Gamma_1)) \times H^\delta(\Gamma_1) \]

with $w_t^{(n)} \in L^2([0, T]; H^{3+\delta}(\Gamma_1))$ such that (6.8) and the bound
\[ \|w_t^{(n)}\|_{L^\infty([0, T]; H^{4+\delta}(\Gamma_1))} + \|w_t^{(n)}\|_{L^\infty([0, T]; H^{2+\delta}(\Gamma_1))} + \|w_t^{(n)}\|_{L^\infty([0, T]; H^\delta(\Gamma_1))} \leq M \]

holds, where $M = C(\|v_0\|_{H^{2+\delta}} + \|w_0\|_{H^{4+\delta}} + \|w_1\|_{H^{2+\delta}(\Gamma_1)}$ with a sufficiently large constant $C \geq 1$. We now invoke Theorem 6.1 to obtain $(v^{(n+1)}, q^{(n+1)})$. The functions $\psi^{(n)}$ and $a^{(n)}$ are obtained as in (6.43), while $b^{(n)} = \partial_t^3 \psi^{(n)} a^{(n)}$. The coefficients $a^{(n)}$ and $b^{(n)}$ satisfy (6.9) and (6.10) for some time $T > 0$ only depending on $M$ and the initial data. Hence, Theorem 6.1 guarantees the existence of a solution $(v^{(n+1)}, q^{(n+1)}) \in L^\infty([0, T]; H^{2.5+\delta}(\Omega) \times H^{1.5+\delta}(\Omega))$ for a time $T > 0$ depending on $M$ and $v_0$. Moreover, from (6.12) we have the estimate
\[ \|q^{(n+1)}(t)\|_{H^{2.5+\delta}} + \|\nabla q^{(n+1)}(t)\|_{H^{0.5+\delta}} \lesssim \|v_0\|_{H^{2.5+\delta}} + \int_0^T P(\|w_t^{(n)}\|_{H^{4+\delta}(\Gamma_1)}, \frac{\|w^{(n)}\|_{H^{2+\delta}(\Gamma_1)}}{d.s.}. \] (6.45)

Invoking Lemma 6.2, we then solve the plate equation and obtain $(w^{(n+1)}, w_t^{(n+1)})$ given $q^{(n+1)}$. We need to adjust the pressure by an appropriate function of time to insure $w_t^{(n+1)}$ satisfies the compatibility condition (6.8). To achieve this, we adjust $q^{(n+1)}$ with an additive function of time such that $\int_{\Gamma_1} q^{(n+1)} = 0$. Since $w_t^{(n+1)}$ and $w_t^{(n+1)}$ are periodic, we have $\int_{\Gamma_1} \Delta_2 w_t^{(n+1)} = - \int_{\Gamma_1} \Delta_2 w_t^{(n+1)} = 0$ and hence, by (6.44), we obtain $\int_{\Gamma_1} w_t^{(n+1)} = 0$. From $\int_{\Gamma_1} w_t^{(n+1)} = 0$ and $\int_{\Gamma_1} w_t^{(n+1)} = 0$, we obtain
\[ \int_{\Gamma_1} w_t^{(n+1)} = 0. \]

Also, from (6.40) we have the estimate
\[ \|w_t^{(n+1)}\|_{L^\infty([0, T]; H^{4+\delta}(\Gamma_1))} + \|w_t^{(n+1)}\|_{L^\infty([0, T]; H^{2+\delta}(\Gamma_1))} \lesssim \|v_0\|_{H^{4+\delta}(\Gamma_1)} + \|w_1\|_{H^{2+\delta}(\Gamma_1)} + \|q^{(n+1)}\|_{L^2([0, T]; H^{1+\delta}(\Gamma_1))}. \] (6.46)

Since $\int_{\Gamma_1} q^{(n+1)} = 0$, we have
\[ \|q^{(n+1)}\|_{L^2([0, T]; H^{1+\delta}(\Gamma_1))} \lesssim \|\nabla q^{(n+1)}\|_{L^2([0, T]; H^{0.5+\delta}(\Omega))}, \]

where we used the standard trace inequality. We now estimate the pressure term using (6.45), so that (6.46) becomes
\[ \|w_t^{(n+1)}\|_{L^\infty([0, T]; H^{4+\delta}(\Gamma_1))} + \|w_t^{(n+1)}\|_{L^\infty([0, T]; H^{2+\delta}(\Gamma_1))} \lesssim \|v_0\|_{H^{4+\delta}(\Gamma_1)} + \|w_1\|_{H^{2+\delta}(\Gamma_1)} + T^{1/2} \|v_0\|_{H^{2.5+\delta}} + T^{1/2} \int_0^T P(\|w_t^{(n)}\|_{H^{4+\delta}(\Gamma_1)}, \frac{\|w^{(n)}\|_{H^{2+\delta}(\Gamma_1)}}{d.s.}. \] (6.47)

From (6.47), it is standard to obtain
\[ \|w_t^{(n+1)}\|_{L^\infty([0, T]; H^{4+\delta}(\Gamma_1))} + \|w_t^{(n+1)}\|_{L^\infty([0, T]; H^{2+\delta}(\Gamma_1))} \leq M = C M_0, \]

where
\[ M_0 = \|v_0\|_{H^{4+\delta}(\Gamma_1)} + \|w_1\|_{H^{2+\delta}(\Gamma_1)}, \]

provided $T$ is chosen so that $T \leq 1/P(M_0)$, where $P(M)$ is a certain polynomial depending on $P$ in (6.47). This estimate establishes the iteration map, which takes a ball of appropriate size $M$ into itself.

Now we proceed by obtaining a contraction estimate for the sequence. We denote the differences between two iterates by $W^{(n+1)} = u^{(n+1)} - u^{(n)}, A^{(n+1)} = a^{(n+1)} - a^{(n)}, B^{(n+1)} = b^{(n+1)} - b^{(n)}, V^{(n+1)} = q^{(n+1)} - q^{(n)}$, and $Q^{(n+1)} = q^{(n+1)} - q^{(n)}$. Consider the vorticity formulation of the $(n+1)$-th iterate, which reads
\[ \partial_t \zeta^{(n)} + u_k^{(n+1)} a_{jk}^{(n)} \frac{\partial_j \zeta^{(n+1)}}{\partial_t} - \psi^{(n)} a_{ijk} \frac{\partial_j \zeta^{(n+1)}}{\partial_t} - \zeta_k^{(n)} a_{ijk} \frac{\partial_j \zeta^{(n+1)}}{\partial_t} = 0 \quad \text{in } \Omega \]
\[ \epsilon_{ijk} a_{m}^{(n)} \frac{\partial_m \zeta^{(n+1)}}{\partial_t} = \zeta_i^{(n+1)} \quad \text{in } \Omega \]
\[ a_{m}^{(n)} \frac{\partial_m v_i^{(n+1)}}{\partial_t} = 0 \quad \text{in } \Omega \]
We now consider the equation satisfied by the difference \( Z^{(n+1)} = \zeta^{(n+1)} - \zeta^{(n)} \). Denote the corresponding extension defined on \( \mathbb{T}^2 \times \mathbb{R} \) by \( \Theta^{(n+1)} = \theta^{(n+1)} - \theta^{(n)} \). Then we have
\[
\begin{align*}
\partial_t \Theta^{(n+1)} + V_k \partial_i \phi^{(n+1)} &+ \psi_k \partial_j \phi^{(n+1)} + \psi_k a_{jk} \partial_i \phi^{(n+1)} - \partial_t \Theta^{(n+1)} + \psi_k a_{jk} \partial_i \phi^{(n+1)} \\
&- \psi_i \partial_j \phi^{(n+1)} - \theta_k a_{jk} \partial_i \phi^{(n+1)} = 0 \quad \text{in } \Omega.
\end{align*}
\]
Combining \((6.48)\) with \((6.49)\), we obtain the inequality \((5.22)\), with the constants depending on \( M \), i.e.,
\[
\| (\Theta^{(n+1)}(t))_{H^{0.5+\delta}} \leq P(M) \int_0^t \left( \| V^{(n+1)} \|_{H^{1.5+\delta}} + \| \Theta^{(n+1)} \|_{H^{0.5+\delta}} \\
+ \| W^{(n)} \|_{H^{3+\delta}(\Gamma_1)} + \| W_t^{(n)} \|_{H^{3+\delta}(\Gamma_1)} \right) ds.
\]
Choosing \( T \) sufficiently small compared to \( P(M) \), we may absorb the term containing \( \Theta^{(n+1)} \) and obtain
\[
\| (\Theta^{(n+1)} \|_{L^\infty H^{0.5+\delta}} \leq TP(M) \left( \| V^{(n+1)} \|_{L^\infty H^{1.5+\delta}} + \| W^{(n)} \|_{L^\infty H^{3+\delta}} + \| W_t^{(n)} \|_{L^\infty H^{3+\delta}} \right),
\]
where the norms of the terms involving \( W^{(n)} \) are over \( \Gamma_1 \times (0, T) \), while others are over \( \Omega \times (0, T) \). We next invoke the div-curl estimates on the system
\[
\begin{align*}
\epsilon_{ijk} a_{mi} \partial_m v^{(n+1)} &+ Z^{(n+1)} - \epsilon_{ijk} A_{mi} \partial_m v^{(n+1)} \quad \text{in } \Omega \times [0, T] \\
\partial_a \partial_m v^{(n+1)} &+ -A_{mn} \partial_m v^{(n+1)} \quad \text{in } \Omega \times [0, T] \\
\partial_{3j} v^{(n+1)} &- \Psi_t^{(n)} = -B_{3j} v^{(n+1)} \quad \text{on } (\Gamma_0 \cup \Gamma_1) \times [0, T]
\end{align*}
\]
to obtain, for any \( t \in [0, T] \),
\[
\| V^{(n+1)} \|_{H^{1.5+\delta}} \leq \| (\Theta^{(n+1)} \|_{H^{0.5+\delta}} + \| A^{(n)} \|_{H^{0.5+\delta}} + \| \nabla v^{(n)} \|_{H^{1.5+\delta}} \\
+ \| \Psi_t^{(n)} \|_{H^{1.5+\delta}} + \| B^{(n)} \|_{H^{1.5+\delta}} + \| (\Theta^{(n+1)} \|_{H^{0.5+\delta}}.
\]

We next use the fact that the pressure term \( Q^{(n+1)} \) satisfies an elliptic boundary value problem with the Neumann type boundary conditions. In particular, the term \( Q^{(n+1)} \) satisfies the equation with the structure of \((5.13)\) with the Neumann boundary condition similar to \((5.15)\) on both \( \Gamma_0 \) and \( \Gamma_1 \) (but not the Robin boundary condition as in \((5.14)\)), with the usual extra terms on \( \Gamma_1 \). Omitting the details, as they are similar to those in \((5.13)-(5.16)\), we obtain the elliptic estimate
\[
\| \nabla Q^{(n+1)} \|_{H^{0.5-\delta}} \leq \| V^{(n+1)} \|_{H^{1.5+\delta}} + \| W^{(n)} \|_{H^{3+\delta}(\Gamma_1)} + \| W_t^{(n)} \|_{H^{3+\delta}(\Gamma_1)},
\]
where the constant depends on \( M \). Therefore, using the fundamental theorem of calculus, we have
\[
\| Q^{(n+1)} \|_2 = \left\| Q^{(n+1)} \right\|_{L^2} = \left\| \frac{1}{\| \Gamma_1 \|} \int_{\Gamma_1} Q^{(n+1)} \right\|_{L^2} \leq \| \partial_3 Q^{(n+1)} \|_{L^2}.
\]
Combining \((6.52)\) and \((6.53)\), we get
\[
\| Q^{(n+1)} \|_{H^{0.5-\delta}} \leq \| V^{(n+1)} \|_{H^{1.5+\delta}} + \| W^{(n)} \|_{H^{3+\delta}(\Gamma_1)} + \| W_t^{(n)} \|_{H^{3+\delta}(\Gamma_1)}.
\]
The energy estimate for the plate equation yields
\[
\|W^{(n+1)}(t)\|_{H^{3+\delta}(\Gamma_1)}^2 + \|W_t^{(n+1)}(t)\|_{H^{1+\delta}(\Gamma_1)}^2 + \|W_{tt}^{(n+1)}(t)\|_{H^{2+\delta}(\Gamma_1)}^2 + \nu \int_0^t \|W_t^{(n+1)}(s)\|_{H^{2.5+\delta}(\Gamma_1)}^2 \, ds
\]
\[
\leq \int_0^t \|Q^{(n+1)}(s)\|_{H^{3+\delta}}^2 \, ds \leq \int_0^t \|Q^{(n+1)}(s)\|_{H^{3+0.5}}^2 \, ds \lesssim T\|Q^{(n+1)}\|_{L^\infty([0,T];H^{3+0.5}([\Omega])},
\]
for \( t \in [0,T] \). Using (6.54) in (6.55), we get
\[
\|W^{(n+1)}(t)\|_{L^\infty(H^{3+\delta}(\Gamma_1) \times (0,T))}^2 + \|W_t^{(n+1)}(t)\|_{L^\infty(H^{1+\delta}(\Gamma_1) \times (0,T))}^2 + \nu \|W_{tt}^{(n+1)}(t)\|_{L^2(H^{2.5+\delta}(\Gamma_1) \times (0,T))}^2
\]
\[
\lesssim T\|V^{(n+1)}(t)\|_{L^\infty(H^{3+\delta}(\Gamma_1) \times (0,T))}^2 + T\|W^{(n)}(t)\|_{L^\infty(H^{3+\delta}(\Gamma_1) \times (0,T))}^2 + T\|W_t^{(n)}(t)\|_{L^\infty(H^{1+\delta}(\Gamma_1) \times (0,T))}^2 + T\|W_{tt}^{(n)}(t)\|_{L^\infty(H^{2.5+\delta}(\Gamma_1) \times (0,T))}^2,
\]
which then, choosing \( T \) sufficiently small showing the contractivity property for the plate component. On the other hand, using (6.56), with \( n_1 \) replaced by \( n \) in (6.51) and choosing \( T \) sufficiently small compared to \( P(M) \), we get that the velocity component is contractive too. Hence, there exists a unique solution \( (w, u_t) \in L^\infty([0,T];H^{4+\delta}(\Gamma_1) \times H^{2+\delta}(\Gamma_1)) \) satisfying (6.41)--(6.42).

To obtain the contractive property, denote
\[
\alpha_n = \|W^{(n)}(t)\|_{L^\infty(H^{3+\delta}(\Gamma_1) \times (0,T))}^2 + \|W_t^{(n)}(t)\|_{L^\infty(H^{1+\delta}(\Gamma_1) \times (0,T))}^2 + \|W_{tt}^{(n)}(t)\|_{L^\infty(H^{2.5+\delta}(\Gamma_1) \times (0,T))}^2
\]
and
\[
\beta_n = \|V^{(n)}(t)\|_{L^\infty(H^{3+\delta}(\Omega) \times (0,T))}^2.
\]
The inequality (6.56) may then be written as
\[
\alpha_{n+1} \leq C_0 \alpha_n + \beta_{n+1},
\]
while
\[
\beta_{n+1} \leq C_0 \alpha_n,
\]
where \( C_0 \geq 1 \) is a fixed constant. With \( \epsilon_0 > 0 \) to be determined, we have
\[
\alpha_{n+1} + \epsilon_0 \beta_{n+1} \leq C_0 \alpha_{n+1} + \alpha_n + C_0 \epsilon_0 \alpha_n.
\]
To obtain a contractive property, it is sufficient to require \( C_0 \) to be less than or equal to \( 1/2 \). Note that it is possible to achieve these two inequalities if \( T = \epsilon_0/C_0 \) and \( 1 + C_0 \epsilon_0 \leq 1/2 \). With the two choices, we obtain \( \alpha_{n+1} + \epsilon_0 \beta_{n+1} \leq (1/2)(\alpha_n + \epsilon_0 \beta_n) \). Hence, there exists a unique solution \( (w, u_t) \in L^\infty([0,T];H^{4+\delta}(\Gamma_1) \times H^{2+\delta}(\Gamma_1)) \) satisfying (6.41)--(6.42). The regularity and uniqueness of the corresponding pair \((v, q)\) can be deduced from Theorem 6.1. This establishes Theorem 6.2.

6.4. Applying the a priori estimates to constructed \( \nu > 0 \) solutions. Now that we have constructed solutions given \( \nu > 0 \), the uniform bounds from a priori estimates in Section 3.5 are used to pass through the limit as \( \nu \to 0 \). However, the constructed solutions are not sufficiently regular to justify a direct application of the a priori estimates in Section 2. Instead, we perform the a priori estimates on partial difference quotients of solutions.

Proof of Theorem 2.3. Denote by
\[
D_{h,l} f(x) = \frac{1}{h} (f(x + he_l) - f(x)), \quad x \in \Omega, \quad l = 1, 2, \quad h \in \mathbb{R} \setminus \{0\}
\]
the difference quotient of a function \( f \) by \( h \in \mathbb{R} \setminus \{0\} \) in the direction \( e_l \). We start with the analog of the plate estimate (3.17), which reads
\[
\frac{1}{2} \frac{d}{dt} \left( \|\Delta_2 A^{1+\delta} D_{h,l} w \|_{L^2(\Gamma_1)}^2 + \|\Lambda^{1+\delta} D_{h,l} w_{l} \|_{L^2(\Gamma_1)}^2 \right) + \nu \|\nabla_2 A^{1+\delta} D_{h,l} w_{l} \|_{L^2(\Gamma_1)}^2
\]
\[
= \int_{\Gamma_1} A^{1+\delta} D_{h,l} q \Lambda^{1+\delta} D_{h,l} w_{l},
\]
(6.57)
where we assume \( h \in \mathbb{R}\setminus\{0\} \) and \( l \in \{1, 2\} \) throughout. Since \( h \) and \( l \) are fixed for most of the proof, we denote
\[
D = D_{h,l}.
\]
The identity (6.57) is obtained by applying \( \Lambda^{1+\delta} D_{h,l} \) to the plate equation (2.21) and testing the resulting equation with \( \Lambda^{1+\delta} D_{h,l} w_l \), for \( l = 1, 2 \). Integrating (6.57) in time, we obtain
\[
\frac{1}{2} \| \Delta_2 \Lambda^{1+\delta} D w_l \|^2_{L^2(\Gamma_1)} + \frac{1}{2} \| \Lambda^{1+\delta} D w_l \|^2_{L^2(\Gamma_1)} + \nu \int_0^t \| \nabla_\nu \Lambda^{1+\delta} w_l \|^2_{L^2(\Gamma_1)} \, ds
\]
\[
= \frac{1}{2} \| \Lambda^{1+\delta} D w_l(0) \|^2_{L^2(\Gamma_1)} + \int_0^t \int_{\Gamma_1} \Lambda^{1+\delta} D q \Lambda^{1+\delta} D w_l \, d\sigma \, ds.
\]
For the tangential estimate for the Euler equations, we start, in analogy with (3.21), by
\[
\frac{1}{2} \frac{d}{dt} \int J \Lambda^{0.5+\delta} D v_l \Lambda^{1.5+\delta} D v_i = \frac{1}{2} \int J_t \Lambda^{0.5+\delta} D v_i \Lambda^{1.5+\delta} D v_i + \int J \Lambda^{0.5+\delta} \partial_t D v_i \Lambda^{1.5+\delta} D v_i + \bar{I}, \tag{6.58}
\]
where
\[
\bar{I} = \frac{1}{2} \int J \Lambda^{0.5+\delta} D v_l \Lambda^{1.5+\delta} \partial_t D v_i - \frac{1}{2} \int J \Lambda^{0.5+\delta} \partial_t D v_l \Lambda^{1.5+\delta} D v_i
\]
\[
= \frac{1}{2} \int \left( (J \Lambda^{0.5+\delta} D v_l) - J \Lambda^{2.5+\delta} D v_l \right) \Lambda^{-0.5+\delta} \partial_t D v_i
\]
\[
+ \frac{1}{2} \int \left( J \Lambda^{2.5+\delta} D v_l - J (J \Lambda^{1.5+\delta} D v_l) \right) \Lambda^{-0.5+\delta} \partial_t D v_i
\]
\[
\lesssim \| J \|_{H^2} \| v \|_{H^{2.5+\delta}} \| v_i \|_{H^{0.5+\delta}} \leq P(\| v \|_{H^2}, \| w \|_{H^{4+\delta}(\Gamma_1)}),
\]
recalling that \( \delta \geq 0.5 \).

For the second term in (6.58), we use the Euler equations (3.26)_1, which leads to
\[
\frac{1}{2} \frac{d}{dt} \int J \Lambda^{0.5+\delta} D v_l \Lambda^{1.5+\delta} D v_i
\]
\[
= \frac{1}{2} \int J_t \Lambda^{0.5+\delta} D v_l \Lambda^{1.5+\delta} D v_i - \int \left( \Lambda^{0.5+\delta} D (J \partial_t v_l) - J \Lambda^{0.5+\delta} D (\partial_t v_l) \right) \Lambda^{1.5+\delta} D v_i
\]
\[
+ \int \Lambda^{0.5+\delta} D (v_m b_{jm} \partial_j v_i) \Lambda^{1.5+\delta} D v_i - \int \Lambda^{0.5+\delta} D \left( (v_3 - \partial_t \eta_3) \partial_3 v_i \right) \Lambda^{1.5+\delta} D v_i
\]
\[
= I_1 + I_2 + I_3 + I_4 + I_5 + \bar{I}.
\]
The first term satisfies
\[
I_1 \lesssim \| J_t \|_{H^{1.5+\delta}} \| v \|_{H^{1.5+\delta}} \| v \|_{H^{2.5+\delta}}.
\]
For the next commutator term \( I_2 \), we use the product rule
\[
D_{h,l}(fg) = D_{h,l} f g + \tau_{h,l} f D_{h,l} g,
\]
where we denote by
\[
\tau_{h,l} g = g(x+h e_l), \quad x \in \Omega, \quad l = 1, 2, \quad h \in \mathbb{R}\setminus\{0\}
\]
the translation operator and abbreviate \( \tau = \tau_{h,l} \). We get
\[
I_2 = \int \left( \Lambda^{0.5+\delta} (J D (\partial_t v_l)) - J \Lambda^{0.5+\delta} D (\partial_t v_l) \right) \Lambda^{1.5+\delta} D v_i + \int \Lambda^{0.5+\delta} (D J \tau (\partial_t v_l)) \Lambda^{1.5+\delta} D v_i.
\]
Using commutator estimates, we get
\[
I_2 \lesssim \| J \|_{H^{2.5+\delta}} \| v_l \|_{H^{0.5+\delta}} \| v \|_{H^{2.5+\delta}} \leq P(\| v \|_{H^{2.5+\delta}}, \| v_i \|_{H^{0.5+\delta}}, \| w \|_{H^{4+\delta}(\Gamma_1)}),
\]
where we used $\delta \geq 0.5$. The third term $I_3$ may be estimated using the product rule as

$$I_3 \lesssim \|D(v_m b_m \partial_j v_i)\|_{H^{\frac{5}{2} + \delta}} \|Dv\|_{H^{1.5 + \delta}} \lesssim \|v\|^3_{H^{2.5 + \delta}} \|b\|_{H^{3.5 + \delta}} \leq P(\|v\|_{H^{2.5 + \delta}}, \|w\|_{H^{4.5 + \delta}}).$$

Similarly,

$$I_4 \lesssim \|D(v_m - \psi_t)\partial_j v\|_{H^{\frac{5}{2} + \delta}} \|Dv\|_{H^{1.5 + \delta}} \lesssim \|v\|^3_{H^{2.5 + \delta}} + \|\eta_i\|_{H^{2.5 + \delta}} \|v\|^2_{H^{2.5 + \delta}} \leq P(\|v\|_{H^{2.5 + \delta}}, \|w\|_{H^{4.5 + \delta}}).$$

Next, $I_5$ can be expressed as

$$I_5 = \int I_1 A^{0.5+\delta} D(b_k) \Lambda^{1.5+\delta} \partial_k D v_i - \int I_1 A^{1+\delta} D(b_{3i}) \Lambda^{1+\delta} D v_i,$$

$$= I_{51} + I_{52}.$$

Using the product rule (6.59), we rewrite $I_{51}$ as

$$I_{51} = \int b_k A^{0.5+\delta} Dq A^{1.5+\delta} \partial_k D v_i + \int (A^{0.5+\delta} (b_k Dq) - b_k A^{0.5+\delta} Dq) A^{1.5+\delta} \partial_k D v_i + \int (A^{0.5+\delta} (b_k Dq) - b_k A^{0.5+\delta} Dq) A^{1.5+\delta} \partial_k D v_i + \int A^{0.5+\delta} D(b_{ki}) \tau q A^{1.5+\delta} \partial_k D v_i,$$

and thus

$$I_{51} = \int (b_k A^{1.5+\delta} \partial_k D v_i - A^{1.5+\delta} \partial_k (b_k D v_i)) A^{0.5+\delta} D q + \int (A^{0.5+\delta} (b_k Dq) - b_k A^{0.5+\delta} Dq) A^{1.5+\delta} \partial_k D v_i + \int A^{0.5+\delta} D(b_{ki}) \tau q A^{1.5+\delta} \partial_k D v_i.$$

We treat these four terms using Kato-Ponce commutator estimates, and proceeding as for $I_2$, we obtain $I_{51} \leq P(\|v\|_{H^{2.5+\delta}}, \|q\|_{H^{1.5+\delta}}, \|b\|_{H^{3.5+\delta}})$. As for $I_{52} = -\int_{\Gamma_1} A^{1+\delta} D(b_{3i}) A^{1+\delta} D v_i$, we write

$$I_{52} = -\int_{\Gamma_1} (A^{\delta} D q) b_{3i} A^{2+\delta} D v_i - \int_{\Gamma_1} A^{\delta} D q \left( A(b_{3i} A^{1+\delta} D v_i) - b_{3i} A^{2+\delta} D v_i \right)$$

$$- \int_{\Gamma_1} (A^{1+\delta} (b_{3i} D q) - b_{3i} A^{1+\delta} D q) A^{1+\delta} D v_i - \int_{\Gamma_1} A^{1+\delta} (D b_{3i} \tau q) A^{1+\delta} D v_i$$

$$= -\int_{\Gamma_1} (A^{\delta} D q) A^{2+\delta} D(b_{3i} D v_i) + \int_{\Gamma_1} A^{\delta} D q \left( A^{2+\delta} (b_{3i} D v_i) - b_{3i} A^{2+\delta} D v_i \right)$$

$$- \int_{\Gamma_1} (A^{1+\delta} (b_{3i} D q) - b_{3i} A^{1+\delta} D q) A^{1+\delta} D v_i - \int_{\Gamma_1} A^{1+\delta} (D b_{3i} \tau q) A^{1+\delta} D v_i$$

$$- \int_{\Gamma_1} A^{1+\delta} (D b_{3i} \tau q) A^{1+\delta} D v_i + \int_{\Gamma_1} A^{\delta} D q A^{2+\delta} (D b_{3i} \tau v_i).$$

After integration in time, the first boundary term cancels with the boundary integral on the right hand side of (6.57). The remaining terms are estimated as above by $P(\|v\|_{H^{2.5+\delta}}, \|q\|_{H^{1.5+\delta}}, \|b\|_{H^{3.5+\delta}}).$
We also need to justify applying the vorticity estimate (3.61) for the constructed solutions. We apply $\Lambda_3^{0.5+\delta} D$ to the equation (3.55) and test with $\Lambda_3^{0.5+\delta} D\theta$, obtaining

$$
\frac{1}{2} \frac{d}{dt} \int_{\Omega_0} \bar{J}|\Lambda_3^{0.5+\delta} D\theta|^2
$$

\begin{align*}
&= - \sum_{m=1}^2 \int_{\Omega_0} \tilde{v}_m \tilde{b}_{jm} \partial_j \Lambda_3^{0.5+\delta} D\theta_i \Lambda_3^{0.5+\delta} D\theta_i - \int_{\Omega_0} (\tilde{v}_3 - \tilde{\psi}_t) \tilde{b}_{jm} \partial_j \Lambda_3^{0.5+\delta} D\theta_i \Lambda_3^{0.5+\delta} D\theta_i \\
&\quad + \int_{\Omega_0} \theta_k \tilde{b}_{mk} \partial_m \Lambda_3^{0.5+\delta} D\tilde{v}_i \Lambda_3^{0.5+\delta} D\theta_i \\
&\quad - \sum_{m=1}^2 \int_{\Omega_0} \Lambda_3^{0.5+\delta} \left( (\tilde{v}_m - \tilde{\psi}_t) \tilde{b}_{jm} \partial_j \Lambda_3^{0.5+\delta} D\theta_i \right) \Lambda_3^{0.5+\delta} D\theta_i \\
&\quad - \int_{\Omega_0} \Lambda_3^{0.5+\delta} \left( D((\tilde{v}_3 - \tilde{\psi}_t) \tilde{b}_{jm} \partial_j \Lambda_3^{0.5+\delta} D\theta_i \right) \Lambda_3^{0.5+\delta} D\theta_i \\
&\quad + \int_{\Omega_0} \Lambda_3^{0.5+\delta} \left( \partial_k \tilde{b}_{mk} \partial_m \Lambda_3^{0.5+\delta} D\tilde{v}_i \right) \Lambda_3^{0.5+\delta} D\theta_i \\
&\quad + \frac{1}{2} \int_{\Omega_0} \bar{J}|\Lambda_3^{1.5+\delta} D\theta|^2 - \int_{\Omega_0} \Lambda_3^{0.5+\delta} \left( \bar{J} D\partial_i \theta_i \right) \Lambda_3^{0.5+\delta} D\theta_i \\
&\quad - \int_{\Omega_0} \Lambda_3^{0.5+\delta} \left( D\bar{J} \partial_i \theta_i \right) \Lambda_3^{0.5+\delta} D\theta_i.
\end{align*}

The first two terms are treated as above by integrating by parts in $x_j$ and noting that the boundary term vanishes. The rest of the terms are estimated also as above using commutator estimates and Sobolev inequalities to conclude

$$
\frac{1}{2} \frac{d}{dt} \int_{\Omega_0} \bar{J}|\Lambda_3^{0.5+\delta} D_{h,l}\theta|^2 \leq P(\|v\|_{H^{2.5+\delta}}, \|b\|_{H^{3.5+\delta}}, \|\psi\|_{H^{2.5+\delta}}, \|J\|_{H^{3.5+\delta}}, \|J_i\|_{H^{1.5+\delta}}) \left( \int_{\Omega_0} |\Lambda_3^{0.5+\delta} D_{h,l}\theta|^2 \right)
$$

for all $h \in \mathbb{R} \setminus \{0\}$ and $l \in \{1, 2\}$. Then the a priori estimates (2.24) and (2.25) can then be applied directly to the constructed $\nu > 0$ solutions.

We now pass to the limit as $\nu \to 0$ with solutions for which we have uniform bounds. For a sequence $\nu_1, \nu_2, \ldots \to 0$, denote the corresponding solution to the damped system by $(v^{(n)}, q^{(n)}, w^{(n)}, w_1^{(n)})$ and the corresponding matrix coefficient by $a^{(n)}$. We then have the uniform bound

$$
\|v^{(n)}\|_{L^\infty([0,T];H^{2.5+\delta})} + \|q^{(n)}\|_{L^\infty([0,T];H^{1.5+\delta})} + \|w^{(n)}\|_{L^\infty([0,T];H^{1.5+\delta}(\Gamma_1))} + \|w_1^{(n)}\|_{L^\infty([0,T];H^{2.5+\delta}(\Gamma_1))} \lesssim \|v_0\|_{H^{2.5+\delta}} + \|w_0\|_{H^{1.5+\delta}(\Gamma_1)} + \|w_1\|_{H^{2.5+\delta}(\Gamma_1)},
$$

(6.60)
for all $n \in \mathbb{N}$, for a uniform time $T > 0$ depending on the initial data, independent of $\nu$. Consequently, $a^{(n)}$ is also uniformly bounded in $L^{\infty}([0, T]; H^{3+\delta})$. We may now pass to a subsequence for which

$$ v^{(n)} \rightharpoonup v \quad \text{ weakly-* in } L^{\infty}([0, T]; H^{2.5+\delta}) $$

$$ q^{(n)} \rightharpoonup q \quad \text{ weakly-* in } L^{\infty}([0, T]; H^{1.5+\delta}) $$

$$ w^{(n)} \rightharpoonup w \quad \text{ weakly-* in } L^{\infty}([0, T]; H^{4+\delta}(\Gamma_1)) $$

$$ w_t^{(n)} \rightharpoonup w_t \quad \text{ weakly-* in } L^{\infty}([0, T]; H^{2.5+\delta}(\Gamma_1)) $$

$$ \nu \cdot w_t^{(n)} \rightharpoonup \chi \quad \text{ weakly in } L^{2}([0, T]; H^{3+\delta}(\Gamma_1)) $$

$$ a_t^{(n)} \rightharpoonup a_t \quad \text{ weakly-* in } L^{\infty}([0, T]; H^{3.5+\delta}) $$

To pass through the limit in the nonlinear terms, we need a strong convergence. Given that we also have

$$ v_t^{(n)} \rightarrow v_t \quad \text{weakly-* in } L^{\infty}([0, T]; H^{0.5+\delta}) $$

by the a priori estimates, the Aubin-Lions lemma yields

$$ v^{(n)} \rightarrow v \quad \text{in } C([0, T]; H^s), $$

for any $s < 2.5 + \delta$. Similarly, we can conclude that

$$ a^{(n)} \rightarrow a \quad \text{in } C([0, T]; H^r), $$

for any $r < 3.5 + \delta$ since

$$ a_t^{(n)} \rightarrow a_t \quad \text{weakly-* in } L^{\infty}([0, T]; H^{1.5+\delta}). $$

We are now ready to pass to the limit in both equations. Starting with the Euler equations, and denoting the duality pairing by $\langle \cdot, \cdot \rangle$, we pass to the limit as $n \rightarrow \infty$ by

$$ \langle v_t^{(n)} - v_t, \phi \rangle \rightarrow 0, \quad \phi \in C_0^{\infty}(\Omega \times (0, T)). $$

We next pass through the limit in the nonlinear terms using

$$ |\langle (v_j)^{(n)}(a_{kj})^{(n)} \partial_k (v_i)^{(n)} - v_j a_{kj} \partial_k v_i, \phi \rangle| $$

$$ \lesssim \|v^{(n)} - v\|_{L^{\infty}([0, T]; H^2)}\|v^{(n)}\|_{L^{\infty}([0, T]; H^{2.5+\delta})}\|\nabla v^{(n)}\|_{L^{\infty}([0, T]; H^{1.5+\delta})}\|\phi\|_{L^1([0, T]; H^{-0.5-\delta})} $$

$$ + \|v\|_{L^{\infty}([0, T]; H^{0.5+\delta})}\|a^{(n)} - a\|_{L^{\infty}([0, T]; H^{1.5+\delta})}\|\nabla v^{(n)}\|_{L^{\infty}([0, T]; H^{1.5+\delta})}\|\phi\|_{L^1([0, T]; H^{-0.5-\delta})} $$

$$ + \|v\|_{L^{\infty}([0, T]; H^{1.5+\delta})}\|a\|_{L^{\infty}([0, T]; H^{1.5+\delta})}\|\nabla v^{(n)} - \nabla v\|_{L^{\infty}([0, T]; H^{0.5+\delta})}\|\phi\|_{L^1([0, T]; H^{-0.5-\delta})}, $$

for $\phi \in C_0^{\infty}(\Omega \times (0, T))$. By the strong convergence result above, the right-hand side goes to zero as $n \rightarrow \infty$. For the other nonlinear term, the argument is similar. It remains to pass through the limit in the pressure term, for which we have

$$ |\langle a_{ki}^{(n)} \partial_k q^{(n)} - a_{ki} \partial_k q, \phi \rangle| \lesssim \|a^{(n)} - a\|_{L^{\infty}([0, T]; H^{1.5+\delta})}\|\nabla q^{(n)}\|_{L^{\infty}([0, T]; H^{0.5+\delta})}\|\phi\|_{L^1([0, T]; H^{-0.5-\delta})} $$

$$ + |\langle \partial_k q^{(n)} - \partial_k q, a_{ki}\phi \rangle|. $$

The first term on the right hand side again converges to zero by strong convergence of $a^{(n)}$ to $a$ for all test functions $\phi \in L^1([0, T]; H^{-0.5+\delta})$. The second term goes to zero as well for all $\phi \in L^1([0, T]; H^{-0.5+\delta})$ by the weak-* convergence of $q^{(n)} \rightarrow q$ since the element $a\phi \in L^1([0, T]; H^{1.5+\delta})$ which easily follows from $a \in L^{\infty}([0, T]; L^{\infty})$. Passing to the limit in the plate equation, we have

$$ \langle w_t^{(n)} - w_{tt}, \psi \rangle_{\Gamma_1} \rightarrow 0, \quad \psi \in L^1([0, T]; H^{-\delta}(\Gamma_1)) $$

$$ \langle \Delta_2 w^{(n)} - \Delta_2 w, \psi \rangle \rightarrow 0, \quad \psi \in L^1([0, T]; H^{-\delta}(\Gamma_1)). $$
Moreover, since $(1/n)w_{it}^{(n)}$ converges weakly in $L^2([0,T];H^6(\Gamma_1))$, by the Aubin-Lions Lemma, we have additionally

$$\nu_n w_{it}^{(n)} \to \chi \quad \text{in} \quad L^2([0,T];H^s(\Gamma_1))$$

(strongly) for all $s < 3 + \delta$. Since we also have

$$w_{t}^{(n)} \to w_t \quad \text{in} \quad L^\infty([0,T];H^r(\Gamma_1))$$

for all $r < 2 + \delta$, we get

$$\nu_n w_{i}^{(n)} \to 0 \quad \text{in} \quad L^2([0,T];H^r(\Gamma_1))$$

for all $r < 2 + \delta$. By uniqueness of the limit, this implies $\chi = 0$ and

$$|\nu_n (\Delta_2 w_{t}^{(n)}, \psi)_{\Gamma_1} | \lesssim \|\nu_n w_{t}^{(n)}\|_{L^\infty([0,T];H^{3+\delta}(\Gamma_1))} \|\psi\|_{L^1([0,T];H^{-s}(\Gamma_1))} \to 0, \quad \psi \in L^1([0,T];H^{-\delta}(\Gamma_1)).$$

Finally, by weak-* convergence of the pressure terms on the boundary $\Gamma_1$ in $L^\infty([0,T];H^1(\Gamma_1))$, we obtain

$$(q^{(n)} - q, \psi)_{\Gamma_1} \to 0, \quad \psi \in L^1([0,T];H^{-\delta}(\Gamma_1)).$$

For the divergence term we have

$$|\langle (a_{ki})^{(n)} \partial_k(v_{i})^{(n)} - a_{ki} \partial_k v_i, \rho \rangle| \lesssim \|a^{(n)} - a\|_{L^\infty([0,T];H^{2.5+\delta})} \|v^{(n)}\|_{L^\infty([0,T];H^{2.5+\delta})} \|\rho\|_{L^1([0,T];H^{-1.5-\delta})}$$

for all $\rho \in L^1([0,T];H^{-1.5-\delta})$. The first term converges to zero as $n \to \infty$ by the strong convergence of $a^{(n)}$ to $a$, while the second term goes to zero as $n \to \infty$ by the weak-* convergence $\nabla v^{(n)} \to \nabla v$ in $L^\infty([0,T];H^{1.5+\delta})$, since $a \in L^\infty([0,T];L^\infty)$ and thus $\rho a \in L^1([0,T];H^{-1.5-\delta})$.

We finally pass to the limit in the boundary condition on $\Gamma_1$, to obtain

$$|\langle (a_{3k})^{(n)} (v_{i})^{(n)} - a_{3k}v_i, \xi \rangle_{\Gamma_1} | \lesssim \|a^{(n)} - a\|_{L^\infty([0,T];H^{2.5+\delta})} \|v^{(n)}\|_{L^\infty([0,T];H^{2.5+\delta})} \|\xi\|_{L^1([0,T];H^{-2-\delta}(\Gamma_1))}$$

for all $\xi \in L^1([0,T];H^{-2-\delta}(\Gamma_1))$, by the strong convergence of $a^{(n)}$ to $a$. The second term also converges to 0 by weak star convergence of $v^{(n)}|_{\Gamma_1}$ in $L^\infty([0,T];H^{2.5+\delta}(\Gamma_1))$ since $a\xi \in L^1([0,T];H^{-2-\delta}(\Gamma_1))$ which is a consequence of $a \in L^\infty([0,T];L^\infty(\Gamma_1))$.

\[\square\]

**Appendix**

Here we provide the proof of the compatibility condition (6.20). Computing the integral of $\tilde{f}$ over $\Omega$, we have

$$\int_{\Omega} \tilde{f} = \int_{\Omega} \partial_j (\partial_i b_{ji} \tilde{v}_i) - \int_{\Omega} \tilde{b}_{ji} \partial_j (\tilde{v}_m \tilde{a}_{km}) \partial_k \tilde{v}_i + \int_{\Omega} \tilde{b}_{ji} \partial_j (J^{-1} \psi) \partial_k \tilde{v}_i + \int_{\Omega} \tilde{v}_m \tilde{a}_{km} \partial_k \tilde{b}_{ji} \partial_j \tilde{v}_i - \int_{\Omega} J^{-1} \psi \partial_j \partial_k \tilde{b}_{ji} \partial_j \tilde{v}_i + \int_{\Omega} \mathcal{E}. $$

This can be rewritten using the divergence theorem and the product rule as

$$\int_{\Omega} \tilde{f} = \int_{\Gamma_1} \partial_i \tilde{b}_{3i} \tilde{v}_i - \int_{\Omega} \partial_j (\tilde{b}_{ji} \tilde{v}_m \tilde{a}_{km} \partial_k \tilde{v}_i) + \int_{\Omega} \partial_j (\tilde{b}_{ji} J^{-1} \psi) \partial_k \tilde{v}_i$$

$$+ \int_{\Omega} \tilde{v}_m \tilde{a}_{km} \partial_k (\tilde{b}_{ji} \partial_j \tilde{v}_i) - \int_{\Omega} J^{-1} \psi \partial_j \partial_k (\tilde{b}_{ji} \partial_j \tilde{v}_i) + \int_{\Omega} \mathcal{E}. $$

Noting that $b = 1$ on $\Gamma_0$ and using the divergence theorem again, this can be expressed as

$$\int_{\Omega} \tilde{f} = \int_{\Gamma_1} \partial_i \tilde{b}_{3i} \tilde{v}_i - \int_{\Omega} \tilde{b}_{ji} \tilde{v}_m \tilde{a}_{km} \partial_k \tilde{v}_i + \int_{\Omega} \tilde{b}_{ji} J^{-1} \psi \partial_k \tilde{v}_i$$

$$- \int_{\Omega} \partial_k (\tilde{v}_m \tilde{a}_{km}) \tilde{b}_{ji} \partial_j \tilde{v}_i + \int_{\Omega} \tilde{v}_m \tilde{a}_{3m} \tilde{b}_{ji} \partial_j \tilde{v}_i + \int_{\Omega} \partial_j (\tilde{J}^{-1} \psi) \tilde{b}_{ji} \partial_j \tilde{v}_i - \int_{\Omega} J^{-1} \psi \tilde{b}_{ji} \partial_j \tilde{v}_i + \int_{\Omega} \mathcal{E}. $$
The last five integrals cancel with terms in $\mathcal{E}$. The boundary integral $\int_\Gamma \tilde{b}_{3i}\tilde{v}_m \tilde{a}_{km} \partial_k \tilde{v}_i$ can be expressed using a similar calculation as in (3.50) as
\[
\int_\Gamma \tilde{b}_{3i}\tilde{v}_m \tilde{a}_{km} \partial_k \tilde{v}_i = \int_\Gamma \left( \tilde{a}_{3k}\tilde{v}_k \partial_k \tilde{v}_i + \sum_{j=1}^{2} \tilde{v}_k \tilde{a}_{jk} \partial_j (\tilde{b}_{3i} \tilde{v}_i) + \frac{1}{\partial_{\psi}^2} \tilde{b}_{3k}\tilde{v}_k \partial_3 \tilde{b}_{3i} \tilde{v}_i - \frac{1}{\partial_{\psi}^2} \partial_j \tilde{b}_{3i} \tilde{v}_k \partial_j \tilde{b}_{3i} \tilde{v}_i \right).
\]
In particular,
\[
\tilde{b}_{3i}\tilde{v}_k \partial_k \partial_j \tilde{v}_i = \tilde{v}_k \tilde{b}_{jk} \partial_j (\tilde{b}_{3i} \tilde{v}_i) - \tilde{v}_k \tilde{b}_{jk} \partial_j \tilde{b}_{3i} \tilde{v}_i,
\]
\[
= \sum_{j=1}^{2} \tilde{v}_k \partial_j \partial_j (\tilde{b}_{3i} \tilde{v}_i) + \tilde{v}_k \tilde{b}_{jk} \partial_j (\tilde{b}_{3i} \tilde{v}_i) - \tilde{v}_k \tilde{b}_{jk} \partial_j \tilde{b}_{3i} \tilde{v}_i.
\]
From the definition of $\mathcal{E}$, (6.19), we get cancellation of the first three terms, while only the integral over $\Gamma_1$ of the last term remains. Hence, we get
\[
\int_{\Omega} \tilde{f} = \int_{\Gamma_1} \partial_t \tilde{b}_{3i} \tilde{v}_i - \int_{\Gamma} \sum_{j=1}^{2} \tilde{v}_k \tilde{a}_{jk} \partial_j (\psi_t) - \int_{\Gamma} \frac{1}{\partial_{\psi}^2} \psi_t \partial_3 \tilde{b}_{3i} \tilde{v}_i + \int_{\Gamma_1} \frac{1}{\partial_{\psi}^2} \partial_j \tilde{b}_{3i} \tilde{v}_k \partial_j \tilde{v}_i.
\]
Noting that $\psi_t = 0$ on $\Gamma_0$ and $\psi_t = w_t$ on $\Gamma_1$ while $\int_{\Gamma_1} w_{tt} = 0$, this is precisely the integral of $\tilde{g}_1$ on $\Gamma_1$.

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References

[AB] G. Avalos and F. Bucci, Rational rates of uniform decay for strong solutions to a fluid-structure PDE system, J. Differential Equations 258 (2015), no. 12, 4398–4423.

[AL] H. Abels and Y. Liu, On a fluid-structure interaction problem for plaque growth, arXiv:2110.00042.

[AGW] G. Avalos, P.G. Geredeli, and J.T. Webster, A linearized viscous, compressible flow-plate interaction with non-dissipative coupling, J. Math. Anal. Appl. 477 (2019), no. 1, 334–356.

[B] H. Beirão da Veiga, On the existence of strong solutions to a coupled fluid-structure evolution problem, J. Math. Fluid Mech. 6 (2004), no. 1, 21–52.

[BB] J.P. Bourguignon and H. Brezis, Remarks on the Euler equation, J. Functional Analysis 15 (1974), 341–363.

[BKS] B. Benešová, M. Kampschulte, and Sebastian Schwarzacher, A variational approach to hyperbolic evolutions and fluid-structure interactions, arXiv:2008.04796.

[BST] M. Boulakia, E.L. Schwandt, and T. Takahashi, Existence of strong solutions for the motion of an elastic structure in an incompressible viscous fluid, Interfaces Free Bound. 14 (2012), no. 3, 273–306.

[BS1] D. Breit and S. Schwarzacher, Compressible fluids interacting with a linear-elastic shell, Arch. Ration. Mech. Anal. 228 (2018), no. 2, 495–562.

[BS2] D. Breit and S. Schwarzacher, Navier-Stokes-Fourier fluids interacting with elastic shells, arXiv:2101.00824.

[BT] M. Badra and T. Takahashi, Gevrey regularity for a system coupling the Navier-Stokes system with a beam equation, SIAM J. Math. Anal. 51 (2019), no. 6, 4776–4814.

[Bo] M. Boulakia, Existence of weak solutions for an interaction problem between an elastic structure and a compressible viscous fluid, J. Math. Pures Appl. (9) 84 (2005), no. 11, 1515–1554.

[C] J.-J. Casanova, Existence of time-periodic strong solutions to a fluid-structure system, Discrete Contin. Dyn. Syst. 39 (2019), no. 6, 3291–3313.

[Ch] I. Chueshov, Interaction of an elastic plate with a linearized inviscid incompressible fluid, Commun. Pure Appl. Anal. 13 (2014), no. 5, 1759–1778.

[CGH] J.-J. Casanova, C. Grandmont, and M. Hillairet, On an existence theory for a fluid-beam problem encompassing possible contacts, J. Éc. polytech. Math. 8 (2021), 933–971.

[CDEG] A. Chamboille, B. Desjardins, M.J. Esteban, and C. Grandmont, Existence of weak solutions for the unsteady steady-state of a viscous fluid with an elastic plate, J. Math. Fluid Mech. 7 (2005), no. 3, 368–404.

[CCS] C.H. Arthur Cheng, D. Coutand, and S. Shkoller, Navier-Stokes equations interacting with a nonlinear elastic biofluid shell, SIAM J. Math. Anal. 39 (2007), no. 3, 742–800.

[CK] A. Celik and M. Kyed, Fluid-plate interaction under periodic forcing, arXiv:2103.00795.

[CS1] D. Coutand and S. Shkoller, Motion of an elastic solid inside an incompressible viscous fluid, Arch. Ration. Mech. Anal. 176 (2005), no. 1, 25–102.

[CS2] C.H. Arthur Cheng and S. Shkoller, The interaction of the 3D Navier-Stokes equations with a moving nonlinear Koiter elastic shell, SIAM J. Math. Anal. 42 (2010), no. 3, 1094–1155.

[CLW] I. Chueshov, I. Lasiecka, and J. Webster, Flow-plate interactions: well-posedness and long-time behavior, Discrete Contin. Dyn. Syst. Ser. S 7 (2014), no. 5, 925–965.
I. Chueshov and I. Ryzhikova, A global attractor for a fluid-plate interaction model, Commun. Pure Appl. Anal. 12 (2013), no. 4, 1635–1656.

B. Desjardins, M.J. Esteban, C. Grandmont, and P. Le Tallec, Weak solutions for a fluid-elastic structure interaction model, Rev. Mat. Complut. 14 (2001), no. 2, 523–538.

G. Guidoboni, R. Glowinski, N. Cavallini, and S. Canic, Stable loosely-coupled-type algorithm for fluid-structure interaction in blood flow, J. Comput. Phys. 228 (2009), no. 18, 6916–6937.

B. Muha and S. Schwarzacher, Existence and regularity of weak solutions for a fluid-interacting with a non-linear shell in 3D.

B. Muha and S. Čanić, Existence of weak solutions for the unsteady interaction of a viscous fluid with an elastic plate, SIAM J. Math. Anal. 40 (2008), no. 2, 716–737.

B. Muha and M. Hillairet, Existence of global strong solutions to a beam-fluid interaction system, Arch. Ration. Mech. Anal. 220 (2016), no. 3, 1283–1333.

C. Grandmont, M. Hillairet, and J. Lequeurre, Existence of local strong solutions to fluid-beam and fluid-rod interaction systems, Ann. Inst. H. Poincaré C Anal. Non Linéaire 36 (2019), no. 4, 1105–1149.

M. Ignatova, I. Kukavica, I. Lasiecka, and A. Tuffaha, Small data global existence for a fluid-structure model, Nonlinearity 30 (2017), 848–898.

I. Kukavica, W.S. Ozafiński, and Amjad Tuffaha, On the global existence for a fluid-structure model with small data, arXiv:2110.15284.

I. Kukavica and A. Tuffaha, Regularity of solutions to a free boundary problem of fluid-structure interaction, Indiana Univ. Math. J. 61 (2012), no. 5, 1817–1859.

I. Lasiecka and J. Webster, Generation of bounded semigroups in nonlinear subsonic flow—structure interactions with boundary dissipation, Math. Methods Appl. Sci. 36 (2013), no. 15, 1995–2010.

D. Lengeler, Weak solutions for an incompressible, generalized Newtonian fluid interacting with a linearly elastic Koiter type shell, SIAM J. Math. Anal. 46 (2014), no. 4, 2614–2649.

D. Lengeler and M. Růžička, Weak solutions for an incompressible Newtonian fluid interacting with a Koiter type shell, Arch. Ration. Mech. Anal. 211 (2014), no. 1, 205–255.

J. Lequeurre, Existence of strong solutions to a fluid-structure system, SIAM J. Math. Anal. 43 (2011), no. 1, 389–410.

J. Lequeurre, Existence of strong solutions for a system coupling the Navier-Stokes equations and a damped wave equation, J. Math. Fluid Mech. 15 (2013), no. 2, 249–271.

D. Maity, J.-P. Raymond, and A. Roy, Maximal-in-time existence and uniqueness of strong solution of a 3D fluid-structure interaction model, SIAM J. Math. Anal. 52 (2020), no. 6, 6338–6378.

B. Muha and S. Čanić, Existence of a weak solution to a nonlinear fluid-structure interaction problem modeling the flow of an incompressible, viscous fluid in a cylinder with deformable walls, Arch. Ration. Mech. Anal. 207 (2013), no. 3, 919–968.

B. Muha and S. Čanić, Existence of a weak solution to a fluid-elastic structure interaction problem with the Navier slip boundary condition, J. Differential Equations 260 (2016), no. 12, 8550–8589.

B. Muha and S. Čanić, Fluid-structure interaction between an incompressible, viscous 3D fluid and an elastic shell with nonlinear Koiter membrane energy, Interfaces Free Bound. 17 (2015), no. 4, 465–495.

B. Muha and S. Schwarzacher, Existence and regularity of weak solutions for a fluid interacting with a non-linear shell in 3D, arXiv:1906.01962.

J.-P. Raymond and M. Vanninathan, A fluid-structure model coupling the Navier-Stokes equations and the Lamé system, J. Math. Pures Appl. (9) 102 (2014), no. 3, 546–596.

T. Takahashi and M. Tucsnak, Global strong solutions for the two-dimensional motion of an infinite cylinder in a viscous fluid, J. Math. Fluid Mech. 6 (2004), no. 1, 53–77.

J.T. Webster, Weak and strong solutions of a nonlinear subsonic flow-structure interaction: semigroup approach, Nonlinear Anal. 74 (2011), no. 10, 3123–3136.

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