Random Words, Toeplitz Determinants and Integrable Systems. I.

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Abstract

It is proved that the limiting distribution of the length of the longest weakly increasing subsequence in an inhomogeneous random word is related to the distribution function for the eigenvalues of a certain direct sum of Gaussian unitary ensembles subject to an overall constraint that the eigenvalues lie in a hyperplane.

1 Introduction

A class of problems—important for their applications to computer science and computational biology as well as for their inherent mathematical interest—is the statistical analysis of a string of random symbols. The symbols, called letters, are assumed to belong to an alphabet $\mathcal{A}$ of fixed size $k$. The set of all such strings (or words) of length $N$, $W(\mathcal{A}, N)$, forms the sample space in the statistical analysis of these strings. A natural measure on $W$ is to assign each letter equal probability, i.e. $1/k$, and to define the probability measure on words by the product measure. Thus each letter in a word occurs independently and with equal probability. We call such random word models homogeneous.

Of course for some applications, each letter in the alphabet does not occur with the same frequency and it is therefore natural to assign to each letter $i$ a probability $p_i$. If we again use
the product measure for the words (letters in a word occur independently), then the resulting random word models are called \textit{inhomogeneous}.

Fixing an ordering of the alphabet $\mathcal{A}$, a \textit{weakly increasing subsequence} of a word

$$w = \alpha_1 \alpha_2 \cdots \alpha_N \in \mathcal{W}$$

is a subsequence $\alpha_{i_1} \alpha_{i_2} \cdots \alpha_{i_m}$ such that $i_1 < i_2 < \cdots < i_m$ and $\alpha_{i_1} \leq \alpha_{i_2} \leq \cdots \leq \alpha_{i_m}$. The positive integer $m$ is called the \textit{length} of this weakly increasing subsequence. For each word $w \in \mathcal{W}$ we define $\ell_N(w)$ to equal the \textit{length of the longest weakly increasing subsequence} in $w$. We now define the fundamental object of this paper:

$$F_N(n) := \text{Prob} \left( \ell_N(w) \leq n \right)$$

where Prob is the inhomogeneous measure on random words. Of course, Prob depends upon $N$ and the probabilities $p_i$.

Our results are of two types. To state our first results, we order the $p_i$ so that

$$p_1 \geq p_2 \geq \cdots \geq p_k$$

and decompose out alphabet $\mathcal{A}$ into subsets $\mathcal{A}_1, \mathcal{A}_2, \ldots$ such that $p_i = p_j$ if and only if $i$ and $j$ belong to the same $\mathcal{A}_a$. Setting $k_\alpha = |\mathcal{A}_\alpha|$, we show that the limiting distribution function as $N \to \infty$ for the appropriately centered and normalized random variable $\ell_N$ is related to the distribution function for the eigenvalues $\xi_i$ in the \textit{direct sum} of mutually independent $k_\alpha \times k_\alpha$ Gaussian unitary ensembles (GUE)\footnote{A basic reference for random matrices is Mehta’s book \cite{Mehta}.} conditional on the eigenvalues $\xi_i$ satisfying $\sum \sqrt{p_i} \xi_i = 0$. In the case when one letter occurs with greater probability than the others, this result implies that the limiting distribution of $(\ell_N - Np_1)/\sqrt{N}$ is Gaussian with variance equal to $p_1(1 - p_1)$. In the case when all the probabilities $p_i$ are distinct, we compute the next correction in the asymptotic expansion of the mean of $\ell_N$ and find that

$$\text{E}(\ell_N) = Np_1 + \sum_{j > 1} \frac{p_j}{p_1 - p_j} + O \left( \frac{1}{\sqrt{N}} \right), \quad N \to \infty.$$
Poincaré index 1 irregular singular point at infinity. In Part II we will show that the finite $N$ inhomogeneous model is represented by the isomonodromy deformations of the $2 \times 2$ matrix linear ODE which has $m+1$ simple poles in the finite complex plane and, again, one Poincaré index 1 irregular singular point at infinity. The number $m$ is the total number of the subsets $\mathcal{A}_\alpha$, and the poles are located at zero point and at the points $-p_{i\alpha}$ ($i_\alpha = \max \mathcal{A}_\alpha$). The integers $k_\alpha$ appear as the formal monodromy exponents at the respective points $-p_{i\alpha}$. We will also analyse the monodromy meaning of the asymptotic results obtained in this part.

The results presented here are part of the recent flurry of activity centering around connections between combinatorial probability of the Robinson-Schensted-Knuth (RSK) type on the one hand and random matrices and integrable systems on the other. From the point of view of probability theory, the quite surprising feature of these developments is that the methods came from Toeplitz determinants, integrable differential equations of the Painlevé type and the closely related Riemann-Hilbert techniques. The first to discover this connection at the level of distribution functions was Baik, Deift and Johansson \[1\] who showed that the limiting distribution of the length of the longest increasing subsequence in a random permutation is equal to the limiting distribution function of the appropriately centered and normalized largest eigenvalue in the GUE \[13\]. This result has been followed by a number of developments relating random permutations, random words and more generally random Young tableaux to the distribution functions of random matrix theory \[2, 3, 4, 5, 7, 9, 11, 13, 16\].

2 Random Words

2.1 Probability Measure on Words and Partitions

The Robinson-Schensted-Knuth (RSK) algorithm is a bijection between two-line arrays $w_\mathcal{A}$ (or generalized permutation matrices) and ordered pairs $(P, Q)$ of semistandard Young tableaux (SSYT)\footnote{For a detailed account of the RSK algorithm see Stanley, Chp. 7 \[14\]. We use without further reference various results from symmetric function theory all of which can be found in Stanley.} When the two-line arrays have the special form

$$w_\mathcal{A} = \begin{pmatrix} \alpha_1 & 2 & \cdots & N \\ \alpha_1 & \alpha_2 & \cdots & \alpha_N \end{pmatrix},$$

$\alpha_i \in \mathcal{A} = \{1, 2, \ldots, k\}$, we identify each $w_\mathcal{A}$ with a word $w = \alpha_1 \alpha_2 \cdots \alpha_N$ of length $N$ composed of letters from the alphabet $\mathcal{A}$; furthermore, in this case the insertion tableaux $P$ have shape $\lambda \vdash N$, $\ell(\lambda) \leq k$, with entries coming from $\mathcal{A}$ and the recording tableaux $Q$ are standard Young tableau (SYT) of the same shape $\lambda$. As usual, $f^\lambda$ denotes the number of SYT of shape $\lambda$ and $d_\lambda(k)$ the number of SSYT of shape $\lambda$ whose entries come from $\mathcal{A}$.

We define a probability measure, Prob, on $W(\mathcal{A}, N)$, the set of all words $w$ of length $N$ formed from the alphabet $\mathcal{A}$, by the two requirements:
1. For each word $w$ consisting of a single letter $i \in \mathcal{A}$, $\Pr(w = i) = p_i$, $0 < p_i < 1$, with $\sum p_i = 1$.

2. For each $w = \alpha_1 \alpha_2 \cdots \alpha_N \in \mathcal{W}$ and any $i_j \in \mathcal{A}$, $j = 1, 2, \ldots, N$,

$$\Pr(\alpha_1 \alpha_2 \cdots \alpha_N = i_1 i_2 \cdots i_N) = \prod_{j=1}^{N} \Pr(\alpha_j = i_j) \quad \text{(independence)}.$$ 

Of course, $\Pr$ depends both on $N$ and the probabilities $\{p_i\}$.

Under the RSK correspondence, the probability measure $\Pr$ induces a probability measure on partitions $\lambda \vdash N$, which we will again denote by $\Pr$. This induced measure is expressed in terms of $f^\lambda$ and the Schur function. To see this we first recall that a tableau $T$ has type $\alpha = (\alpha_1, \alpha_2, \ldots)$, denoted $\alpha = \text{type}(T)$, if $T$ has $\alpha_i$ parts equal to $i$. We write

$$x^T = x_1^{\alpha_1(T)} x_2^{\alpha_2(T)} \cdots.$$

The combinatorial definition of the Schur function of shape $\lambda$ in the variables $x = (x_1, x_2, \ldots)$ is the formal power series

$$s_\lambda(x) = \sum_T x^T$$

summed over all SSYT of shape $\lambda$. The $p = \{p_1, \ldots, p_k\}$ specialization of $s_\lambda(x)$ is $s_\lambda(p) = s_\lambda(p_1, p_2, \ldots, p_k, 0, 0, \ldots)$.

For each word $w \leftrightarrow (P, Q)$, the $N$ entries of $P$ consist of the $N$ letters of $w$ since $P$ is formed by successive row bumping the letters from $w$. Because of the independence assumption,

$$p^P = p_1^{\alpha_1(P)} p_2^{\alpha_2(P)} \cdots p_k^{\alpha_k(P)}$$

gives the weight assigned to word $w$. From the combinatorial definition of the Schur function, we observe that its $p$ specialization is summing the weights of words $w$ that under RSK have shape $\lambda \vdash N$. The recording tableau $Q$ keeps track of the order of the letters in the word. The weights of any words with the same number of letters of each type are equal (independence), so we need merely count the number of such $Q$, i.e. $f^\lambda$, and multiply this by the weight of any given such word to arrive at the induced measure on partitions,

$$\Pr(\lambda) = s_\lambda(p) f^\lambda, \quad (2.1)$$

which satisfies the normalization $\sum_{\lambda \vdash N} \Pr(\lambda) = 1$. For the homogeneous case $p_i = 1/k$, the measure reduces to

$$\Pr(\lambda) = s_\lambda(1/k, 1/k, \ldots, 1/k) f^\lambda = \frac{d_\lambda(k) f^\lambda}{k^N}, \quad \lambda \vdash N.$$

The Poissonization of this homogeneous measure is called the Charlier ensemble in \cite{10}.
If $\ell_N(w)$ equals the length of the longest weakly increasing subsequence in the word $w \in W(A, N)$, then by the RSK correspondence $w \leftrightarrow (P, Q)$, the number of boxes in the first row of $P$, $\lambda_1$, equals $\ell_N(w)$. Hence,

$$\text{Prob}(\ell_N(w) \leq n) = \sum_{\lambda \vdash N, \lambda_1 \leq n} s_\lambda(p) f^\lambda.$$  \hfill (2.2)

### 2.2 Toeplitz Determinant Representation

Gessel’s theorem [6] is the formal power series identity\footnote{Precisely, we use the dual version of Gessel’s Theorem, see §II in [17] whose notation we follow.}

$$\sum_{\lambda \vdash N, \lambda_1 \leq n} s_\lambda(x)s_\lambda(y) = \det(T_n(\varphi))$$

where $T_n(\varphi)$ is the $n \times n$ Toeplitz matrix whose $i, j$ entry is $\varphi_{i-j}$, where $\varphi_i$ is the $i^{th}$ Fourier coefficient of

$$\varphi(z) = \prod_{n=1}^{\infty} (1 + y_n z^{-1}) \prod_{n=1}^{\infty} (1 + x_n z), \quad z = e^{i \theta}.$$  

If we define the (exponential) generating function

$$G_I(n; \{p_i\}, t) = \sum_{N=0}^{\infty} \text{Prob}(\ell_N(w) \leq n) \frac{t^N}{N!},$$

then an immediate consequence of Gessel’s identity with $p$ specialization of the $x$ variables and exponential specialization of the $y$ variables and the RSK correspondence is

$$G_I(n; \{p_i\}, t) = \det(T_n(f_I)) \hfill (2.3)$$

where

$$f_I(z) = e^{t/z} \prod_{j=1}^{k} (1 + p_j z).$$  \hfill (2.4)

### 3 Limiting Distribution

We start with the probability distribution \hfill (2.1) on the set of partitions $\lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_k\} \vdash N$. For $f^\lambda$ we use the formula

$$f^\lambda = \frac{N! \Delta(h)}{h_1! h_2! \cdots h_k!}.$$
where
\[ h_i = \lambda_j + k - i \]
and
\[ \Delta(h) = \Delta(h_1, h_2, \ldots, h_k) = \prod_{1 \leq i < j \leq k} (h_i - h_j). \tag{3.1} \]

Equivalently,
\[ f^\lambda = \frac{\Delta(h)}{\prod_{i=1}^{k-1} \prod_{j=i}^{k-1} (\lambda_i + k - j)} \left( \begin{array}{c} N \\ \lambda_1 & \lambda_2 & \cdots & \lambda_k \end{array} \right). \]
The (classical) definition of the Schur function is
\[ s^\lambda(p) = \frac{\det (p_i^{h_j})}{\Delta(p)} = \frac{1}{\Delta(p)} \sum_{\sigma \in S_k} (-1)^\sigma p_1^{h_{\sigma(1)}} p_2^{h_{\sigma(2)}} \cdots p_k^{h_{\sigma(k)}}. \tag{3.2} \]

This holds when all the \(p_i\) are distinct but in general the two determinants require modification, which we now describe. We order the \(p_i\) so that
\[ p_1 \geq p_2 \geq \cdots \geq p_k. \tag{3.3} \]
and decompose our alphabet \(A = \{1, 2, \ldots, k\}\) into subsets \(A_1, A_2, \ldots\) such that \(p_i = p_j\) if and only if \(i\) and \(j\) belong to the same \(A_\alpha\). Set \(i_\alpha = \max A_\alpha\). Think of the \(p_i\) as indeterminates and for all indices \(i\) differentiate the determinant \(i_\alpha - i\) times with respect to \(p_i\) if \(i \in A_\alpha\). Then replace the \(p_i\) by their given values. (That this is correct follows from l'Hôpital’s rule.) If we set \(k_\alpha = |A_\alpha|\) and write \(p_\alpha\) for \(p_i^\alpha\) then we see that \(\Delta(p)\) becomes
\[ \Delta'(p) = \prod_\alpha (1! \cdots (k_\alpha - 1)! \prod_{\alpha < \beta} (p_\alpha - p_\beta)^{k_\alpha k_\beta} \tag{3.4} \]
and (after performing row operations) that the \(i\)th row of \(\det (p_i^{h_j})\) becomes \(h_{\sigma(i)}^{i_\alpha - i} p_i^{h_{\sigma(i)} - i_\alpha + i}\). Equivalently, the partial product \(\prod_{i \in A_\alpha} p_i^{h_{\sigma(i)}(i)}\) from the summand in (3.3) gets multiplied by
\[ \prod_{i \in A_\alpha} (h_{\sigma(i)}^{i_\alpha - i} p_i^{i_\alpha - i_\alpha + i}) = \left( \prod_{i \in A_\alpha} h_{\sigma(i)}^{i_\alpha - i} \right) p_\alpha^{k_\alpha (k_\alpha - 1)/2}. \tag{3.5} \]
In the case of distinct \(p_i\) we write our formula as
\[ \text{Prob}(\lambda) = s^\lambda(p_1, \ldots, p_k) f^\lambda \]
\[ = \frac{\Delta(h)}{\Delta(p)} \frac{1}{\prod_{i=1}^{k-1} \prod_{j=i}^{k-1} (\lambda_i + k - j)} \sum_{\sigma \in S_k} (-1)^\sigma p_1^{k-\sigma(1)} \cdots p_k^{k-\sigma(k)} \prod_{\alpha=1}^k \lambda_\sigma^{\lambda_\sigma(k)} \left( \begin{array}{c} N \\ \lambda_1 & \lambda_2 & \cdots & \lambda_k \end{array} \right). \]
Let $M_q(\lambda)$ denote the multinomial distribution associated with a sequence $q = \{q_1, \ldots, q_k\}$,

$$M_q(\lambda) = q_1^{\lambda_1} \cdots q_k^{\lambda_k} \left( \frac{N}{\lambda_1 \lambda_2 \cdots \lambda_k} \right).$$

If $p_\sigma$ denotes the sequence $\{p_{\sigma^{-1}(1)}, \ldots, p_{\sigma^{-1}(k)}\}$, then the above may be written

$$\operatorname{Prob}(\lambda) = \frac{\Delta(h)}{\Delta(p)} \frac{1}{\prod_{i=1}^{k-1} \prod_{j=i}^{k-1} (\lambda_i + k - j)} \sum_{\sigma \in S_k} (-1)^\sigma p_1^{k-\sigma(1)} \cdots p_k^{k-\sigma(k)} M_{p_\sigma}(\lambda). \quad (3.6)$$

This is the formula for distinct $p_i$. In the general case we must replace $\Delta(p)$ by $\Delta'(p)$ and each partial product $\prod_{i \in A_\alpha} p_i^{k-\sigma(i)}$ appearing in the sum on the right must be multiplied by the factor $\{i\}$.

The multinomial distribution $M_q(\lambda)$ has the property that the total measure of any region where $|\lambda_i - Nq_i| > \epsilon N$ for some $i$ and some $\epsilon > 0$ tends exponentially to zero as $N \to \infty$. All the other terms appearing in (3.6) or its modification are uniformly bounded by a power of $N$. Since $\lambda_{i+1} \leq \lambda_i$ for all $i$ it follows that the contribution of the terms involving $M_q(\lambda)$ in (3.6) will tend exponentially to zero unless $q_{i+1} \leq q_i$ for all $i$. Since $q_i = p_{\sigma^{-1}(i)}$ this shows that the contribution to (3.6) of the summand corresponding to $\sigma$ is exponentially small unless $\sigma$ leaves each of the sets $A_\alpha$ invariant. It follows that if we denote the set of such permutations by $S'_k$ then we may restrict the sum in (3.6) to the $\sigma \in S'_k$ without affecting the limit. Observe that when $\sigma \in S'_k$ all the $M_{p_\sigma}(\lambda)$ appearing in (3.6) equal $M_p(\lambda)$.

Write

$$\lambda_i = Np_i + \sqrt{Np_i} \xi_i.$$

In terms of the $\xi_i$ the multinomial distribution $M_p(\lambda)$ converges to

$$(2\pi)^{-k(k-1)/2} e^{-\sum \xi_i^2/2} \delta(\sum \sqrt{q_i} \xi_i). \quad (3.7)$$

(See section 3.1) Here $\delta(\sum \sqrt{q_i} \xi_i)$ denotes Lebesgue measure on the hyperplane $\sum \sqrt{q_i} \xi_i = 0$.

We now consider the contribution of the other terms in (3.6) as modified. Again, they are uniformly bounded by a power of $N$ and the total measure of any region where $|\lambda_i - Np_i| > \epsilon N$ for some $i$ and some $\epsilon > 0$ tends exponentially to zero as $N \to \infty$. Thus in determining the asymptotics of the other terms we may assume that $\lambda_i \sim Np_i$ for all $i$.

The constant $\Delta'(p)$ is given by (3.4). As for $\Delta(h)$, observe that the factor

$$h_i - h_j = \lambda_i - \lambda_j - i + j$$

in the product in (3.1) is asymptotically equal to $N (p_i - p_j)$ when $i$ and $j$ do not belong to the same $A_\alpha$ and to $\sqrt{Np_\alpha} (\xi_i - \xi_j)$ if $i, j \in A_\alpha$. It follows that

$$\Delta(h) \sim N^k (k-1)/2 - \sum_\alpha k_\alpha (k_\alpha - 1)/4 \prod_{\alpha} p_\alpha^{k_\alpha(k_\alpha - 1)/4} \prod_{\alpha < \beta} (p_\alpha - p_\beta)^{k_\alpha k_\beta} \prod_\alpha \Delta_\alpha(\xi),$$

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where $\Delta_\alpha(\xi)$ is the Vandermonde determinant of those $\xi_i$ with $i \in A_\alpha$.

The next factor in (3.6), the reciprocal of the double product, is asymptotically

$$N^{-(k-1)/2} \prod_{i=1}^{k-1} p_i^{i-k}.$$  

As for the sum in (3.6) as modified, observe that since each $\sigma$ now belongs to $S'_k$, each product appearing there is equal to $\prod p_i^{k-i}$. Each such product is to be multiplied by

$$\prod_{\alpha} \left[ \left( \prod_{i \in A_\alpha} h^{i_\alpha-1}_{\sigma(i)} \right) p_\alpha^{-k_\alpha(k_\alpha-1)/2} \right].$$

(See (3.5).) Hence the sum itself is equal to

$$\prod_{i} p_i^{k-i} \prod_{\alpha} p_\alpha^{-k_\alpha(k_\alpha-1)/2} \sum_{\sigma \in S'_k} (-1)^\sigma \prod_{\alpha} \prod_{i \in A_\alpha} h^{i_\alpha-1}_{\sigma(i)}.$$

Since each $\sigma \in S'_k$ is uniquely expressible as a product of $\sigma_\alpha \in S(A_\alpha)$ (where $S(A_\alpha)$ is the group of permutations of $A_\alpha$) we have

$$\sum_{\sigma \in S'_k} (-1)^\sigma \prod_{\alpha} \prod_{i \in A_\alpha} h^{i_\alpha-1}_{\sigma(i)} = \prod_{\alpha} \Delta_\alpha(h) \sim N^{\sum k_\alpha(k_\alpha-1)/4} \prod_{\alpha} \left( p_\alpha^{k_\alpha(k_\alpha-1)/4} \Delta_\alpha(\xi) \right).$$

Putting all this together shows that the limiting distribution is

$$(2\pi)^{-(k-1)/2} \prod_{\alpha} (1!2! \cdots (k_\alpha-1)!)^{-1} \prod_{\alpha} \Delta_\alpha(\xi)^2 e^{-\sum \xi_i^2/2} \delta(\sum \sqrt{p_i} \xi_i). \quad (3.8)$$

This has a random matrix interpretation. It is the distribution function for the eigenvalues in the direct sum of mutually independent $k_\alpha \times k_\alpha$ Gaussian unitary ensembles, conditional on the eigenvalues $\xi_i$ satisfying $\sum \sqrt{p_i} \xi_i = 0$.

It remains to determine the support of the limiting distribution. In terms of the $\xi_i$ the inequalities $\lambda_{i+1} \leq \lambda_i$ are equivalent to

$$\xi_{i+1} \leq N(p_i - p_{i+1}) \sqrt{Np_i} + \sqrt{p_i p_{i+1} \xi_i}.$$  

In the limit $N \to \infty$ this becomes no restriction if $p_{i+1} < p_i$ but becomes $\xi_{i+1} \leq \xi_i$ if $p_{i+1} = p_i$. Otherwise said, the support of the limiting distribution is restricted to those $\{\xi_i\}$ for which
\[ \xi_{i+1} \leq \xi_i \text{ whenever } i \text{ and } i+1 \text{ belong to the same } A_\alpha. \text{ (In the random matrix interpretation it means that the eigenvalues within each GUE are ordered.)} \] We denote this set of \( \xi_i \) by \( \Xi \).

It now follows from (2.2) and (3.8) (also recall the ordering (3.3)) that

\[
\lim_{N \to \infty} \text{Prob}\left( \frac{\xi_N - Np_1}{\sqrt{Np_1}} \leq s \right) = (2\pi)^{-(k-1)/2} \prod_{\alpha}(1! \cdot \cdots \cdot (k_\alpha - 1)!)^{-1} \times \]

\[
\int \cdots \int \prod_{\alpha} \Delta_\alpha(\xi)^2 e^{-\frac{1}{2} \sum \xi_i^2 / 2} \delta(\sum \sqrt{p_i} \xi_i) d\xi_1 \cdots d\xi_k
\]

When the probabilities are not all equal this may be reduced to a \( k_1 \)-dimensional integral as follows. Let \( i \) denote the indices in \( A_1 \) and \( j \) the other indices. We have to integrate

\[
\prod_{\alpha} \Delta_\alpha(\xi)^2 e^{-\frac{1}{2} \sum \xi_i^2 / 2} \delta(\sum \sqrt{p_i} \xi_i + \sum \sqrt{p_j} \xi_j)
\]

over the subset of \( \Xi \) where \( \xi_1 \leq s \). Since \( \xi_1 = \max \xi_i \) and since the integrand is symmetric in the \( \xi_i \) and the \( \xi_j \) within their groups we may (by changing the normalization constant) integrate over all \( \xi_i \leq s \) and all \( \xi_j \). We first fix the \( \xi_i \) and integrate over the \( \xi_j \). These have to satisfy

\[
\sum \sqrt{p_j} \xi_j = -\sum \sqrt{p_i} \xi_i = -\sqrt{p_1} \sum \xi_i.
\]

If we write

\[
\xi_j = \eta_j + x \sqrt{p_j}
\]

where \( \{ \eta_j \} \) is orthogonal to \( \{ \sqrt{p_j} \} \) then

\[
x = \frac{\sum \sqrt{p_j} \eta_j}{\sum p_j} = -\frac{\sqrt{p_1}}{1 - k_1 p_1} \sum \xi_i.
\]

(Recall that \( A_1 \) has \( k_1 \) indices.) For each \( \alpha > 1 \) we have \( \Delta_\alpha(\xi) = \Delta_\alpha(\eta) \) since the \( p_j \) within groups are equal and

\[
\sum \xi_i^2 = \sum \eta_j^2 + x^2 \sum p_j = \sum \eta_j^2 + \frac{p_1}{1 - k_1 p_1} (\sum \xi_i)^2.
\]

So the distribution function is equal to a constant times

\[
\int_{-\infty}^{s} \cdots \int_{-\infty}^{s} \Delta(\xi)^2 e^{-\frac{1}{2} \sum \xi_i^2 + \frac{p_1}{1 - k_1 p_1} (\sum \xi_i)^2} d\xi_1 \cdots d\xi_{k_1} \int \prod_{\alpha>1} \Delta_\alpha(\eta)^2 e^{-\frac{1}{2} \sum \eta_j^2} d\eta,
\]

where the \( \eta \) integration is over the orthogonal complement of \( \{ \sqrt{p_j} \} \). The \( \eta \) integral is just another constant. Therefore the distribution function equals

\[
\frac{1}{\epsilon_{k_1, p_1}} \int_{-\infty}^{s} \cdots \int_{-\infty}^{s} \Delta(\xi)^2 e^{-\frac{1}{2} \sum \xi_i^2 + \frac{p_1}{1 - k_1 p_1} (\sum \xi_i)^2} d\xi_1 \cdots d\xi_{k_1},
\]
where $c_{k_1, p_1}$ is the integral over all of $\mathbb{R}^{k_1}$.
To evaluate this we make the substitution (3.10), but with $j$ replaced by $i$ and each $p_j$ replaced by $1/\sqrt{k}$. The integral becomes
\[
\int \prod_j \Delta(\eta) e^{-\frac{1}{2} \sum \eta_i^2} d\eta \int e^{-\frac{x^2}{2} \left( \frac{1}{k_1} + \frac{p_1}{1 - k_1 p_1} \right)} \, dx,
\]
taken over $x \in \mathbb{R}$ and $\eta$ in hyperplane $\sum \eta_i = 0$ with Lebesgue measure. The $x$ integral equals $\sqrt{2\pi k_1 (1 - k_1 p_1)}$ while the first integral equals $(2\pi)^{(k_1 - 1)/2} 1! \cdots k_1!$. (For the last, observe that the right side of (3.9) must equal 1 when $s = \infty$.) Hence
\[
c_{k_1, p_1} = (2\pi)^{k_1/2} 1! \cdots k_1! \sqrt{k_1 (1 - k_1 p_1)}.
\]

3.1 Distinct probabilities—the next approximation

If all the $p_i$ are different then $P(\lambda) := \text{Prob}(\lambda)$ equals

\[
\frac{\Delta(h)}{\Delta(p)} \left( \prod_{i=1}^{k-1} \frac{1}{\prod_{j=1}^{k-1} (\lambda_i + k - j)} \right) \prod_{i=1}^{k} p_i^{k-i} M_p(\lambda)
\]

plus an exponentially small correction. We recall that

\[
\lambda_j = Np_j + \sqrt{Np_j} \xi_j
\]

and compute the Fourier transform of the measure $P$ with respect to the $\xi$ variables. Beginning with $M_p$, we have

\[
\hat{M}_p(x) = \int e^{i \sum x_j \xi_j} dM_p(\lambda) = e^{-i \sum \sqrt{Np_j} x_j} \int e^{i \sum x_j \lambda_j / \sqrt{Np_j}} dM_p(\lambda)
\]

\[
= e^{-i \sum \sqrt{Np_j} x_j} \left( \sum p_j e^{ix_j / \sqrt{Np_j}} \right)^N
\]

since $M_p$ is the multinomial distribution. An easy computation gives

\[
\hat{M}_p(x) = \left( 1 + \frac{i}{\sqrt{N}} Q(x) + O\left( \frac{1}{N} \right) \right) e^{-\frac{1}{2} \sum x_j^2 + \frac{1}{2} \left( \sum \sqrt{p_j} x_j \right)^2},
\]

where $Q(x)$ is a homogeneous polynomial of degree three. (In particular the limit of $M_p$ is the inverse Fourier transform of the exponential in the above formula, which equals (3.7).)

As for the other nonconstant factors in (3.11), we have

\[
\prod_{i=1}^{k-1} \prod_{j=1}^{k-1} (\lambda_i + k - j) = \prod_{i=1}^{k-1} \left( Np_i + \sqrt{Np_i} \xi_i + O(1) \right)^{k-1}
\]

10
\[ N^{k(k-1)/2} \prod_{i=1}^{k-1} p_i^{k-i} \left( 1 + \frac{1}{\sqrt{N}} \sum_{i=1}^{k-1} (k-i) \frac{\xi_i}{\sqrt{p_i}} + O\left( \frac{1}{N} \right) \right). \]

and

\[
\Delta(h) = \prod_{i<j} \left[ N(p_i - p_j) + \sqrt{N}(\sqrt{p_i} \xi_i - \sqrt{p_j} \xi_j) + O(1) \right]
\]

\[ = N^{k(k-1)/2} \triangle(p) \left( 1 + \frac{1}{\sqrt{N}} \sum_{i<j} \frac{\sqrt{p_i} \xi_i - \sqrt{p_j} \xi_j}{p_i - p_j} + O\left( \frac{1}{N} \right) \right). \]

Thus the factors in (3.11) aside from \( M_p \) contribute

\[ 1 + \frac{1}{\sqrt{N}} \left( \sum_{i<j} \frac{\sqrt{p_i} \xi_i - \sqrt{p_j} \xi_j}{p_i - p_j} \right) + O\left( \frac{1}{N} \right) = 1 + \frac{1}{\sqrt{N}} \left( \sum_{i<j} \sqrt{p_j} \xi_i - \sqrt{p_j} \xi_j \right) + O\left( \frac{1}{N} \right). \]

Using the fact that multiplication by \( \xi_j \) corresponds, after taking Fourier transforms, to \(-i\partial_{x_j}\) and combining this with the preceding we deduce that \( \hat{P}(x) \), the Fourier transform of \( P(\lambda) \) with respect to the \( \xi \) variables, equals

\[
\left( 1 + \frac{i}{\sqrt{N}} \sum_{i<j} \frac{\sqrt{p_j} \xi_i - \sqrt{p_j} \xi_j}{p_i - p_j} + \frac{i}{\sqrt{N}} Q(x) + O\left( \frac{1}{N} \right) \right) e^{-\frac{i}{2} \sum x_j^2 + \frac{i}{2} \sum \sqrt{p_j} x_j^2}
\]

plus a correction which is exponentially small in \( N \).

### 3.1.1 The mean

We have

\[
E(\xi_1) = \int \xi_1 dP(\lambda) = -i \partial_{x_1} \hat{P}(x) \bigg|_{x=0}.
\]

From the above we see that this equals

\[ \frac{1}{\sqrt{N}p_1} \sum_{j>1} \frac{p_j}{p_1 - p_j} + O\left( \frac{1}{N} \right). \]

Hence

\[ E(\ell_N) = E(\ell_1) = Np_1 + \sum_{j>1} \frac{p_j}{p_1 - p_j} + O\left( \frac{1}{\sqrt{N}} \right), \quad N \to \infty. \quad (3.12) \]

This last formula is, in fact, an accurate approximation for \( E(\ell_N) \) (for distinct \( p_i \)) for moderate values of \( N \). Table II summarizes various simulations of \( \ell_N \) and compares the means of these simulated values with the asymptotic formula. We remark that even though the proof assumed distinct \( p_i \), we expect the asymptotic formula to remain valid for \( p_1 > p_2 \geq \cdots \geq p_k \).

(See the last set of simulations in Table II.)
3.1.2 The variance

Let us write our approximation as $P = P_0 + N^{-1/2}P_1 + O(N^{-1})$ with corresponding expected values $E = E_0 + N^{-1/2}E_1 + O(N^{-1})$. (In fact $P_1$ is a distribution, not a measure, but the meaning is clear.) Then the variance of $\lambda_1$ is equal to

$$Np_1 [E(\xi_1^2) - E(\xi_1)^2]$$

$$= Np_1 \left[ E_0(\xi_1^2) - E_0(\xi_1)^2 + \frac{1}{\sqrt{N}}E_1(\xi_1^2) - \frac{2}{\sqrt{N}}E_0(\xi_1)E_1(\xi_1) + O(\frac{1}{N}) \right].$$

Of course $E_0(\xi_1) = 0$, but also

$$E_1(\xi_1^2) = -\frac{2}{\sqrt{N}}E_0(\xi_1)E_1(\xi_1)\bigg|_{x_1 = 0} = 0.$$

Since

$$E_0(\xi_1^2) - E_0(\xi_1)^2 = 1 - p_1$$

we find that the variance of $\lambda_1$ equals $Np_1(1 - p_1) + O(1)$ and so its standard deviation equals $\sqrt{Np_1(1 - p_1)} + O(N^{-1/2})$.

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Table 1: Simulations of the length of the longest weakly increasing subsequence in inhomogeneous random words of length $N$ for two- and three-letter alphabets. $N_S$ is the sample size. The last column gives the asymptotic expected value (1.12).
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