Mosco Convergence of Stable-Like Non-Local Dirichlet Forms on Metric Measure Spaces

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Abstract

We prove that stable-like non-local Dirichlet forms converge to local Dirichlet form in the sense of Mosco on metric measure spaces. We prove that subordinated Dirichlet forms converge to the original Dirichlet form in the sense of Mosco on metric measure spaces.

1 Introduction

Let us recall the following classical result.

\[
\lim_{\beta \uparrow 2} (2 - \beta) \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(u(x) - u(y))^2}{|x - y|^{d + \beta}} \, dx \, dy = C(d) \int_{\mathbb{R}^d} |\nabla u(x)|^2 \, dx
\]

for any \( u \in W^{1,2}(\mathbb{R}^d) \), where \( C(d) \) is some positive constant, see [8, Example 1.4.1], that is, stable-like non-local Dirichlet forms can approximate the local Dirichlet form on \( \mathbb{R}^d \). The main purpose of this paper is to consider Mosco convergence of stable-like non-local Dirichlet forms to local Dirichlet form on metric measure spaces.

There are several types of approximations of non-local Dirichlet forms to local Dirichlet form as follows.

The first is pointwise convergence. Similar to (1), Pietruska-Palubla [25] proved that in an \( \alpha \)-regular metric measure space \((K, d, \nu)\) supporting a fractional diffusion (see [1, Definition 3.5] for the definition) that has walk dimension \( \beta^* \) and corresponds to a local regular Dirichlet form \((E_{\text{loc}}, F_{\text{loc}})\) on \( L^2(K; \nu) \), there exists some positive constant \( C \) such that for any \( u \in F_{\text{loc}} \),

\[
\frac{1}{C} E_{\text{loc}}(u, u) \leq \lim_{\beta \uparrow \beta^*} (\beta^* - \beta) \int_K \int_K \frac{(u(x) - u(y))^2}{d(x, y)^{\alpha + \beta}} \nu(dx) \nu(dy) \leq \lim_{\beta \uparrow \beta^*} (\beta^* - \beta) \int_K \int_K \frac{(u(x) - u(y))^2}{d(x, y)^{\alpha + \beta}} \nu(dx) \nu(dy) \leq C E_{\text{loc}}(u, u).
\]

The second is \( \Gamma \)-convergence. Since \( \Gamma \)-convergence is very weak, that is, any sequence of closed forms in the wide sense has a \( \Gamma \)-convergent subsequence, it is usually used to construct local regular Dirichlet form, see [27, 18, 12, 29].

The third is Mosco convergence which is equivalent to strong convergence of corresponding strongly continuous semi-groups or resolvents and implies finite-dimensional distribution convergence. Mosco convergence is much more meaningful in Dirichlet form theory. Barlow, Bass, Chen and Kassmann [5] used Mosco convergence of non-local Dirichlet forms to non-local Dirichlet form to prove heat kernel estimates. On Euclidean spaces, Suzuki and Uemura [28] gave the Mosco convergences of some Dirichlet forms to consider the instability of global properties of Dirichlet forms under Mosco convergence. On the Sierpiński gasket, the author [30] gave the Mosco convergence of stable-like non-local Dirichlet forms to the local Dirichlet form.

A natural question is to what extent Mosco convergence of non-local Dirichlet forms to local Dirichlet form can hold. The following theorem is the main result of this paper.

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Date: December 5, 2019

MSC2010: Primary 60J35; Secondary 28A80, 31C25

Keywords: Mosco convergence, Dirichlet form, subordinated Hunt process, walk dimension

The author was very grateful to Prof. Takashi Kumagai for raising the question of Mosco convergence on the Sierpiński carpet. The author was very grateful to Dr. Eryan Hu for helpful discussions.
Theorem 1.1. Let \((K,d,\nu)\) be an \(\alpha\)-regular metric measure space. Let \((\mathcal{E}_{\text{loc}}, \mathcal{F}_{\text{loc}})\) be a conservative regular Dirichlet form on \(L^2(K;\nu)\) with a heat kernel \(p_t(x,y)\) satisfying
\[
\frac{C_1}{t^{\alpha/\beta^*}} \exp\left(-C_2 \left(\frac{d(x,y)}{t^{1/\beta^*}}\right)^{\frac{\beta^*}{\alpha}}\right) \leq p_t(x,y) \leq \frac{C_3}{t^{\alpha/\beta^*}} \exp\left(-C_4 \left(\frac{d(x,y)}{t^{1/\beta^*}}\right)^{\frac{\beta^*}{\alpha}}\right)
\]
for any \(x, y \in K\), for any \(t \in (0, \infty)\) if \(K\) is unbounded or \(t \in (0, 1)\) if \(K\) is bounded, where \(\beta^* \in [2, \infty)\) is some parameter and \(C_1, C_2, C_3, C_4\) are some positive constants.

For any \(\beta \in (0, \infty)\), let
\[
\mathcal{E}_\beta(u,u) = \int_K \int_K \frac{(u(x) - u(y))^2}{d(x,y)^{\alpha+\beta}} \nu(dx)\nu(dy),
\]
\[
\mathcal{F}_\beta = \left\{ u \in L^2(K;\nu) : \int_K \int_K \frac{(u(x) - u(y))^2}{d(x,y)^{\alpha+\beta}} \nu(dx)\nu(dy) < \infty \right\}.
\]
Then for any \(\beta \in (0, \beta^*)\), we have \((\mathcal{E}_\beta, \mathcal{F}_\beta)\) is a regular Dirichlet form on \(L^2(K;\nu)\) and \(\mathcal{F}_\beta\) satisfies (3). Murugan [22, Corollary 1.8] proved that \((K,d)\) satisfies the chain condition. Grigor’yan and Kumagai [18, Theorem 4.1] proved that \((\mathcal{E}_{\text{loc}}, \mathcal{F}_{\text{loc}})\) on \(L^2(K;\nu)\) is local and corresponds to a diffusion. Grigor’yan, Hu and Lau [9] Corollary 4.9 proved that \(2 \leq \beta^* \leq \alpha + 1\), where \(\alpha\) is indeed the Hausdorff dimension of \((K,d)\) and \(\beta^*\) is called the walk dimension of the diffusion.

(2) The existence of a local regular Dirichlet form with a heat kernel satisfying [3] has been obtained on many fractal spaces, see [6,14] for the Sierpiński gasket (SG), [2, 19, 15] for the Sierpiński carpet (SC), [4] for higher dimensional SCs, [20] for nested fractals and [15, 16] for p.c.f. self-similar sets. Recently, Grigor’yan and the author [12, 22] gave a unified purely analytic construction on the SG and the SC.

(3) The critical exponent
\[
\beta_* := \sup \left\{ \beta \in (0, \infty) : (\mathcal{E}_\beta, \mathcal{F}_\beta) \text{ is a regular Dirichlet form on } L^2(K;\nu) \right\}
\]
is called the walk dimension of the metric measure space \((K,d,\nu)\). Assume that there exists a diffusion, or equivalently, a local regular Dirichlet form \((\mathcal{E}_{\text{loc}}, \mathcal{F}_{\text{loc}})\) on \(L^2(K;\nu)\), with a heat kernel \(p_t(x,y)\) satisfying [3]. Pietruska-Pałuba [23, 27]. Grigor’yan, Hu and Lau [9] proved that \((\mathcal{E}_\beta, \mathcal{F}_\beta)\) is a regular Dirichlet form on \(L^2(K;\nu)\) if \(\beta \in (0, \beta^*)\) and that \(\mathcal{F}_\beta\) consists only of constant functions if \(\beta \in (\beta^*, \infty)\), which implies that \(\beta_* = \beta^*\), that is, the walk dimension of the metric measure space and the walk dimension of the diffusion coincide. Recently, Grigor’yan and the author [11, 12, 22] gave some alternative approaches to determine the walk dimension of the SG and the SC without using diffusion.

(4) We define \(\mathcal{E}_{\text{loc}}(u,u) = +\infty\) for any \(u \in L^2(K;\nu)\setminus \mathcal{F}_{\text{loc}}\). Hence it is meaningful to say that [3] holds for any \(u \in L^2(K;\nu)\) which is an improvement of the result of [22].

Theorem 1.1 is a consequence of the following result.

Theorem 1.3. Let \((K,d,\nu)\) be a metric measure space. Let \((\mathcal{E}, \mathcal{F})\) be a regular Dirichlet form on \(L^2(K;\nu)\) that corresponds to a Hunt process \(\{X_t\}\). For any \(\delta \in (0, 1)\), let \(\left\{ X_t^{(\delta)} \right\}\) be the \(\delta\)-subordinated Hunt process that corresponds to a regular Dirichlet form \((\mathcal{E}^{(\delta)}, \mathcal{F}^{(\delta)})\) on \(L^2(K;\nu)\). Then for any \(\{\delta_n\} \subseteq (0,1)\) with \(\delta_n \uparrow 1\) as \(n \to +\infty\), we have \(\mathcal{E}^{(\delta_n)}\) converges to \(\mathcal{E}\) in the sense of Mosco.

This paper is organized as follows. In Section 2 we give some results about closed form, Mosco convergence and subordinated Hunt process. In Section 3 we prove Theorem 1.1 and Theorem 1.3. In Appendix, we give a counterexample about the domains of local and non-local Dirichlet forms.
In this paper, we always assume that \((K,d,\nu)\) is a metric measure space, that is, \((K,d)\) is a locally compact separable metric space and \(\nu\) is a Radon measure on \(K\) with full support. We use \((\cdot,\cdot)\) to denote the inner product in \(L^2(K;\nu)\). If \((\mathcal{E},\mathcal{F})\) is a closed form on \(L^2(K;\nu)\), then we always define \(\mathcal{E}(u,u) = +\infty\) for any \(u \in L^2(K;\nu)\setminus \mathcal{F}\), hence \(\mathcal{E}\) is defined on the whole \(L^2(K;\nu)\) rather than a dense subspace \(\mathcal{F}\).

## 2 Preparation

First, we list some results about closed form.

Let \((\mathcal{E},\mathcal{F})\) on \(L^2(K;\nu)\) be a closed form that corresponds to a strongly continuous semi-group \(\{T_t: t > 0\}\) on \(L^2(K;\nu)\). By [8, 1.3], there exists a spectral family \(\{E_\lambda: \lambda \in [0, +\infty)\}\) such that

\[
\mathcal{E}(u,u) = \int_{(0, +\infty)} \lambda d(E_\lambda u, u),
\]

and

\[
\mathcal{F} = \left\{ u \in L^2(K;\nu) : \int_{(0, +\infty)} \lambda d(E_\lambda u, u) < +\infty \right\},
\]

and

\[
T_t = \int_{(0, +\infty)} e^{-t\lambda} dE_\lambda \quad \text{for any } t \in (0, +\infty).
\]

For any \(t \in (0, +\infty)\), for any \(u \in L^2(K;\nu)\), let

\[
\mathcal{E}_{\{t\}}(u,u) = \frac{1}{t}(u - T_t u, u).
\]

We have the following result.

**Lemma 2.1.** ([8 Lemma 1.3.4]) For any \(u \in L^2(K;\nu)\), we have \(t \mapsto \mathcal{E}_{\{t\}}(u,u)\) is monotone decreasing in \((0, +\infty)\) and

\[
\mathcal{E}(u,u) = \lim_{t \downarrow 0} \mathcal{E}_{\{t\}}(u,u),
\]

\[
\mathcal{F} = \left\{ u \in L^2(K;\nu) : \lim_{t \downarrow 0} \mathcal{E}_{\{t\}}(u,u) < +\infty \right\}.
\]

**Remark 2.2.** The monotonicity of \(t \mapsto \mathcal{E}_{\{t\}}(u,u)\) for any \(u \in L^2(K;\nu)\) is crucially important in the proof about Mosco convergence.

Second, we give some results about Mosco convergence.

**Definition 2.3.** Let \(\mathcal{E}^n, \mathcal{E}\) be closed forms on \(L^2(K;\nu)\). We say that \(\mathcal{E}^n\) converges to \(\mathcal{E}\) in the sense of Mosco if the following conditions are satisfied.

1. For any \(\{u_n\} \subseteq L^2(K;\nu)\) that converges weakly to \(u \in L^2(K;\nu)\), we have

\[
\lim_{n \to +\infty} \mathcal{E}^n(u_n, u_n) \geq \mathcal{E}(u,u).
\]

2. For any \(u \in L^2(K;\nu)\), there exists \(\{u_n\} \subseteq L^2(K;\nu)\) that converges strongly to \(u\) in \(L^2(K;\nu)\) such that

\[
\lim_{n \to +\infty} \mathcal{E}^n(u_n, u_n) \leq \mathcal{E}(u,u).
\]

We have the following equivalence.

**Proposition 2.4.** ([22 Theorem 2.4.1, Corollary 2.6.1]) Let \(\{T_t: t > 0\}\), \(\{T^n_t: t > 0\}\) be the strongly continuous semi-groups and \(\{G_\alpha: \alpha > 0\}\), \(\{G^n_\alpha: \alpha > 0\}\) the strongly continuous resolvents that correspond to closed forms \(\mathcal{E}, \mathcal{E}^n\), respectively. The followings are equivalent.

1. \(\mathcal{E}^n\) converges to \(\mathcal{E}\) in the sense of Mosco.
2. \(T^n_t u \to T_t u\) strongly in \(L^2(K;\nu)\) for any \(t > 0\), for any \(u \in L^2(K;\nu)\).
3. \(G^n_\alpha u \to G_\alpha u\) strongly in \(L^2(K;\nu)\) for any \(\alpha > 0\), for any \(u \in L^2(K;\nu)\).
We have the following corollary.

**Corollary 2.5.** Let \((E, \mathcal{F})\) be a closed form on \(L^2(K; \nu)\). Then for any \(\{u_n\} \subseteq L^2(K; \nu)\) that converges weakly to \(u \in L^2(K; \nu)\), we have

\[
E(u, u) \leq \lim_{n \to +\infty} E(u_n, u_n).
\]

**Proof.** Let \(E^n = E\) for any \(n \geq 1\). Then by Proposition 2.7, we have \(E^n\) is trivially convergent to \(E\) in the sense of Mosco. By Definition 2.3, the above inequality is obvious.

**Remark 2.6.** It would be tedious to prove this result without using Mosco convergence.

Third, we give some results about subordinated Hunt process.

Let \((E, \mathcal{F})\) be a regular Dirichlet form on \(L^2(K; \nu)\) that corresponds to a Hunt process \(\{X_t\}\). For any \(\delta \in (0, 1)\), let \(\{\xi_t^{(\delta)}\}\) be the \(\delta\)-stable subordinator, that is, the one-dimensional Lévy process whose Laplace transform is given by \(\mathbb{E}e^{-s\xi_t^{(\delta)}} = e^{-ts^\delta}\), let \(\eta_t^{(\delta)}\) be its one-dimensional distribution density. Assume that the processes \(\{X_t\}\) and \(\{\xi_t^{(\delta)}\}\) are independent, then the \(\delta\)-subordinated Hunt process \(\{X_t^{(\delta)}\}\) is given by \(\{X_t^{(\delta)}\}\). Let \(\{T_t^{(\delta)} : t > 0\}\) be the strongly continuous Markovian semi-group and \((E^{(\delta)}, \mathcal{F}^{(\delta)})\) on \(L^2(K; \nu)\) the regular Dirichlet form that correspond to the Hunt process \(\{X_t^{(\delta)}\}\), then

\[
T_t^{(\delta)}u = \int_0^{+\infty} T_s u \bigg( \int_0^s \eta_t^{(\delta)}(\eta)d\eta \bigg)ds \quad \text{for any } t \in (0, +\infty), u \in L^2(K; \nu),
\]

see [7, 26].

The following result is the key ingredient to prove Mosco convergence.

**Proposition 2.7.** For any \(\delta \in (0, 1)\), for any \(u \in L^2(K; \nu)\), we have

\[
E^{(\delta)}(u, u) = \int_{(0, +\infty)} \lambda^\delta d(E\lambda u, u) = \frac{\delta}{\Gamma(1 - \delta)} \int_0^{+\infty} \frac{1}{s^\delta}E(s)(u, u)ds \leq (+\infty).
\]

**Remarks 2.8.**

1. For any \(u \in L^2(K; \nu)\), \(t^{-1}(u - T_t^{(\delta)}u, u)\) is non-negative finite for any \(t \in (0, +\infty)\), monotone decreasing in \(t \in (0, +\infty)\) and \(E^{(\delta)}(u, u)\) is defined as its limit as \(t \downarrow 0\) by Lemma 2.2, which is allowed to be \(+\infty\). \(d(E\lambda u, u)\) is an ordinary measure on \([0, +\infty)\) and \(E(s)(u, u)\) is non-negative finite for any \(s \in (0, +\infty)\), hence the above two integrals are well-defined and allowed also to be \(+\infty\).

2. [25, Equation (3.5)] gave above second equality only for any \(u \in \mathcal{F}\) where the condition \(u \in \mathcal{F}\) is intrinsically used to apply dominated convergence theorem.

**Proof.** For any \(u \in L^2(K; \nu)\), by Lemma 2.1 and Equation 4.1, we have

\[
E^{(\delta)}(u, u) = \lim_{t \downarrow 0} \frac{1}{t} \bigg( u - T_t^{(\delta)}u, u \bigg)
= \lim_{t \downarrow 0} \frac{1}{t} \bigg( (u, u) - \int_0^{+\infty} (T_s u, u) \eta_t^{(\delta)}(s)ds \bigg)
= \lim_{t \downarrow 0} \frac{1}{t} \int_0^{+\infty} (u - T_s u, u) \eta_t^{(\delta)}(s)ds
= \lim_{t \downarrow 0} \frac{1}{t} \int_0^{+\infty} \bigg( \int_0^{+\infty} (1 - e^{-s\lambda})d(E\lambda u, u) \bigg) \eta_t^{(\delta)}(s)ds
= \lim_{t \downarrow 0} \frac{1}{t} \int_0^{+\infty} (1 - e^{-t\lambda})d(E\lambda u, u)
= \lim_{t \downarrow 0} \frac{1}{t} \int_0^{[0, +\infty)} (1 - e^{-t\lambda})d(E\lambda u, u).
\]
Proof. (1) The first equality follows directly from Proposition 2.7. Note that for any \( \lambda \in [0, +\infty) \), by monotone convergence theorem, we have

\[
\mathcal{E}^{(\delta)}(u, u) = \lim_{t \downarrow 0} \frac{1}{t} \int_{[0, +\infty)} \left( 1 - e^{-t\lambda^s} \right) d(E_{\lambda}u, u)
\]

\[
= \frac{1}{t} \int_{[0, +\infty)} \left( 1 - e^{-t\lambda^s} \right) d(E_{\lambda}u, u) = \int_{[0, +\infty)} \lambda^s d(E_{\lambda}u, u).
\]

By (2), we have

\[
\mathcal{E}^{(\delta)}(u, u) = \int_{[0, +\infty)} \lambda^s d(E_{\lambda}u, u) = \int_{[0, +\infty)} \lambda^s d(E_{\lambda}u, u).
\]

Recall that for any \( \delta \in (0, 1) \), we have

\[
\int_0^{+\infty} \frac{1 - e^{-s}}{s^{1+\delta}} ds = \frac{\Gamma(1-\delta)}{\delta}
\]

which implies that for any \( \lambda \in [0, +\infty) \), we have

\[
\int_0^{+\infty} \frac{1 - e^{-s\lambda}}{s^{1+\delta}} ds = \frac{\Gamma(1-\delta)}{\delta} \lambda^s.
\]

Hence

\[
\mathcal{E}^{(\delta)}(u, u) = \int_{[0, +\infty)} \lambda^s d(E_{\lambda}u, u) = \lambda^s \mathcal{E}^{(\delta)}(u, u) = \frac{\delta}{\Gamma(1-\delta)} \int_{[0, +\infty)} \left( \int_0^{+\infty} \frac{1 - e^{-s\lambda}}{s^{1+\delta}} ds \right) d(E_{\lambda}u, u)
\]

\[
= \frac{\delta}{\Gamma(1-\delta)} \int_0^{+\infty} \frac{1}{s^{1+\delta}} \left( \int_{[0, +\infty)} (1 - e^{-s\lambda}) d(E_{\lambda}u, u) \right) ds
\]

\[
= \frac{\delta}{\Gamma(1-\delta)} \int_0^{+\infty} \frac{1}{s^{1+\delta}} (u-T_s u) ds = \frac{\delta}{\Gamma(1-\delta)} \int_0^{+\infty} \frac{1}{s^\delta} \mathcal{E}^{(\delta)}(u, u) ds.
\]

\[
\square
\]

We have some direct corollaries as follows.

**Corollary 2.9.** (1) For any \( \delta \in (0, 1) \), we have

\[
\mathcal{F}^{(\delta)} = \left\{ u \in L^2(K; \nu) : \int_0^{+\infty} \frac{1}{s^\delta} \mathcal{E}^{(\delta)}(u, u) ds < +\infty \right\}
\]

\[
= \left\{ u \in L^2(K; \nu) : \int_0^{+\infty} \frac{1}{s^\delta} \mathcal{E}^{(\delta)}(u, u) ds < +\infty \right\}.
\]

(2) For any \( \delta_1, \delta_2 \in (0, 1) \) with \( \delta_1 < \delta_2 \), we have \( \mathcal{F}^{(\delta_1)} \supseteq \mathcal{F}^{(\delta_2)} \supseteq \mathcal{F} \).

**Remark 2.10.** By (2), we have \( \cap_{\delta \in (0, 1)} \mathcal{F}^{(\delta)} = \mathcal{F} \). One may expect that \( \cap_{\delta \in (0, 1)} \mathcal{F}^{(\delta)} = \mathcal{F} \). However, this is not true and we will give a counterexample in Appendix.

**Proof.** (1) The first equality follows directly from Proposition 2.7. Note that for any \( u \in L^2(K; \nu) \), for any \( s \in (0, +\infty) \), we have \( \mathcal{E}^{(\delta)}(u, u) \leq \frac{\langle u, u \rangle}{s^{1+\delta}} \), hence

\[
\int_1^{+\infty} \frac{1}{s^\delta} \mathcal{E}^{(\delta)}(u, u) ds \leq \int_1^{+\infty} \frac{1}{s^{1+\delta}} \frac{\langle u, u \rangle}{s^{1+\delta}} ds = \frac{1}{\delta} \frac{\langle u, u \rangle}{s^{1+\delta}} < +\infty.
\]

Hence

\[
\int_0^{+\infty} \frac{1}{s^\delta} \mathcal{E}^{(\delta)}(u, u) ds < +\infty
\]

if and only if

\[
\int_0^1 \frac{1}{s^\delta} \mathcal{E}^{(\delta)}(u, u) ds < +\infty.
\]

Hence we have the second equality.

(2) It follows easily from (1).

\[
\square
\]

**Corollary 2.11.** For any \( \delta \in (0, 1) \).
(1) $\mathcal{F}$ is $(\mathcal{E}^{(\delta)}(\cdot, \cdot) + (\cdot, \cdot))$-dense in $\mathcal{F}^{(\delta)}$.

(2) Any core of $(\mathcal{E}, \mathcal{F})$ on $L^2(K; \nu)$ is a core of $(\mathcal{E}^{(\delta)}, \mathcal{F}^{(\delta)})$ on $L^2(K; \nu)$.

**Remark 2.12.** If $(\mathcal{E}^{(\delta)}, \mathcal{F}^{(\delta)})$ on $L^2(K; \nu)$ is defined only as the Dirichlet form corresponding to the strongly continuous Markovian semi-group \( \{ T_t^{(\delta)} : t > 0 \} \) on $L^2(K; \nu)$ which is given by Equation (4), then the regular property of $(\mathcal{E}^{(\delta)}, \mathcal{F}^{(\delta)})$ on $L^2(K; \nu)$ follows also from the regular property of $(\mathcal{E}, \mathcal{F})$ on $L^2(K; \nu)$ and the above result.

**Proof.** (1) For any $t \in (0, +\infty)$, for any $u \in L^2(K; \nu)$, we claim that $T_t^{(\delta)}u \in \mathcal{F}$. We only need to show that

$$\int_{[0, +\infty)} \lambda d(E\lambda T_t^{(\delta)}u, T_t^{(\delta)}u) < +\infty.$$  

Indeed

$$\int_{[0, +\infty)} \lambda d(E\lambda T_t^{(\delta)}u, T_t^{(\delta)}u) = \int_{[0, +\infty)} \int_0^{+\infty} \int_0^{+\infty} \lambda \eta_t^{(\delta)}(r) \eta_t^{(\delta)}(s) d(E\lambda T_t u, T_s u) dr ds = \int_{[0, +\infty)} \lambda e^{-2t\lambda^2} d(E\lambda u, u).$$

Since $\lambda \mapsto \lambda e^{-2t\lambda^2}$ is continuous on $[0, +\infty)$ and $\lim_{\lambda \to +\infty} \lambda e^{-2t\lambda} = 0$, there exists some positive constant $C$ such that

$$0 \leq \lambda e^{-2t\lambda^2} \leq C \text{ for any } \lambda \in [0, +\infty).$$

Hence

$$\int_{[0, +\infty)} \lambda d(E\lambda T_t^{(\delta)}u, T_t^{(\delta)}u) = \int_{[0, +\infty)} \lambda e^{-2t\lambda^2} d(E\lambda u, u) \leq C(u, u) < +\infty.$$

For any $u \in \mathcal{F}$, by [3, Lemma 1.3.3 (iii)], we have $T_t^{(\delta)}u \in \mathcal{F}$ is $(\mathcal{E}^{(\delta)}(\cdot, \cdot) + (\cdot, \cdot))$-convergent to $u$ as $t \downarrow 0$. Hence $\mathcal{F}$ is $(\mathcal{E}^{(\delta)}(\cdot, \cdot) + (\cdot, \cdot))$-dense in $\mathcal{F}^{(\delta)}$.

(2) For any $u \in \mathcal{F}$, by Proposition 2.7, we have

$$\mathcal{E}^{(\delta)}(u, u) = \int_{[0, 1]} \lambda^2 d(E\lambda u, u) + \int_{[1, +\infty]} \lambda d(E\lambda u, u)$$

$$\leq \int_{[0, 1]} d(E\lambda u, u) + \int_{[1, +\infty]} \lambda d(E\lambda u, u)$$

$$\leq \int_{[0, +\infty)} d(E\lambda u, u) + \int_{[0, +\infty)} \lambda d(E\lambda u, u)$$

$$= \mathcal{E}(u, u) + (u, u).$$

Hence

$$\mathcal{E}(u, u) + (u, u) \leq 2(\mathcal{E}(u, u) + (u, u)) \text{ for any } u \in \mathcal{F}.$$  

Let $\mathcal{C}$ be a core of $(\mathcal{E}, \mathcal{F})$ on $L^2(K; \nu)$, that is, $\mathcal{C} = (\mathcal{E}(\cdot, \cdot) + (\cdot, \cdot))$-dense in $\mathcal{F}$ and uniformly dense in $C_c(K)$. We only need to show that $\mathcal{C}$ is $(\mathcal{E}^{(\delta)}(\cdot, \cdot) + (\cdot, \cdot))$-dense in $\mathcal{F}^{(\delta)}$. Indeed, by the above inequality, we have $\mathcal{C}$ is $(\mathcal{E}^{(\delta)}(\cdot, \cdot) + (\cdot, \cdot))$-dense in $\mathcal{F}$. Since $\mathcal{F}$ is $(\mathcal{E}^{(\delta)}(\cdot, \cdot) + (\cdot, \cdot))$-dense in $\mathcal{F}^{(\delta)}$ which is (1), we have $\mathcal{C}$ is $(\mathcal{E}^{(\delta)}(\cdot, \cdot) + (\cdot, \cdot))$-dense in $\mathcal{F}^{(\delta)}$. 

**Corollary 2.13.** Let $p_t(x, dy)$ be the transition density of the regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(K; \nu)$. Then for any $\delta \in (0, 1)$, we have

$$\mathcal{E}_\delta(u, u) = \frac{1}{2} \int_K \int_K (u(x) - u(y))^2 F^{(\delta)}(dx dy) + \int_K u(x)^2 k^{(\delta)}(dx) \text{ for any } u \in L^2(K; \nu),$$
where
\[
J^{(\delta)}(dx,dy) = \frac{\delta}{\Gamma(1-\delta)} \int_0^{+\infty} \frac{1}{s^{1+\delta}} p_s(x,dy)\nu(dx)ds,
\]
\[
k^{(\delta)}(dx) = \frac{\delta}{\Gamma(1-\delta)} \int_0^{+\infty} \frac{1}{s^{1+\delta}} (1 - \int_K p_s(x,dy)) \nu(dx)ds.
\]

**Remark 2.14.** The above result is indeed the Beurling-Deny decomposition of the regular Dirichlet form \((E^{(\delta)}, F^{(\delta)})\) on \(L^2(K;\nu)\) which has only jumping part and killing part, see [8, Theorem 3.2.1, Lemma 4.5.4]. Hence \((E^{(\delta)}, F^{(\delta)})\) on \(L^2(K;\nu)\) is always non-local.

**Proof.** For any \(u \in L^2(K;\nu)\), for any \(s \in (0, +\infty)\), we have
\[
(u - T_s u, u) = \frac{1}{2} \int_K \int_K (u(x) - u(y))^2 p_s(x,dy)\nu(dx) + \int_K u(x)^2 \left(1 - \int_K p_s(x,dy)\right) \nu(dx).
\]
Then the result follows directly from Proposition 2.7. \(\square\)

### 3 Proof

First, we prove Theorem 1.3. We prove the two conditions of Definition 2.3 separately.

Condition 2 of Definition 2.3 is a direct consequence of the following result.

**Proposition 3.1.** For any \(u \in L^2(K;\nu)\), we have
\[
\lim_{\delta \downarrow 1} E^{(\delta)}(u, u) = E(u, u).
\]

**Remark 3.2.** We have the above equality holds for any \(u \in L^2(K;\nu)\) rather than \(u \in F\) which was obtained in [25, Theorem 3.1].

**Proof.** By Proposition 2.7, we have
\[
E^{(\delta)}(u, u) = \int_{[0,1]} \lambda^\delta d(E\lambda u, u) + \int_{[1,+\infty)} \lambda^\delta d(E\lambda u, u).
\]

For any \(\lambda \in [0,1]\), we have \(0 \leq \lambda^\delta \leq 1\). Since
\[
\int_{[0,1]} d(E\lambda u, u) \leq \int_{[0, +\infty)} d(E\lambda u, u) = (u, u) < +\infty,
\]
by dominated convergence theorem, we have
\[
\lim_{\delta \downarrow 1} \int_{[0,1]} \lambda^\delta d(E\lambda u, u) = \int_{[0,1]} \lim_{\delta \downarrow 1} \lambda^\delta d(E\lambda u, u) = \int_{[0,1]} \lambda d(E\lambda u, u).
\]

For any \(\lambda \in [1, +\infty)\), we have \(\delta \mapsto \lambda^\delta\) is non-negative and monotone increasing in \((0,1)\).

By monotone convergence theorem, we have
\[
\lim_{\delta \downarrow 1} \int_{[1, +\infty)} \lambda^\delta d(E\lambda u, u) = \int_{[1, +\infty)} \lim_{\delta \downarrow 1} \lambda^\delta d(E\lambda u, u) = \int_{[1, +\infty)} \lambda d(E\lambda u, u).
\]

Hence
\[
\lim_{\delta \downarrow 1} E^{(\delta)}(u, u) = \lim_{\delta \downarrow 1} \int_{[0,1]} \lambda^\delta d(E\lambda u, u) + \lim_{\delta \downarrow 1} \int_{[1, +\infty)} \lambda^\delta d(E\lambda u, u) = \int_{[0,1]} \lambda d(E\lambda u, u) + \int_{[1, +\infty)} \lambda d(E\lambda u, u) = E(u, u).
\]
\(\square\)
Proof of Condition \[1\] of Definition \[2.3\]. For any \(\{u_n\} \subseteq L^2(K;\nu)\) that converges weakly to \(u \in L^2(K;\nu)\). For fixed \(s \in (0, +\infty)\), by Proposition \[2.7\] we have

\[
\mathcal{E}(\delta_n)(u_n, u_n) = \frac{\delta_n}{\Gamma(1 - \delta_n)} \int_0^{+\infty} \frac{1}{t^n} \mathcal{E}(t)(u_n, u_n) dt \geq \frac{\delta_n}{\Gamma(1 - \delta_n)} \int_0^s \frac{1}{t^n} \mathcal{E}(t)(u_n, u_n) dt \geq \left( \frac{\delta_n}{\Gamma(1 - \delta_n)} \right) s^{1 - \delta_n} \mathcal{E}(s)(u_n, u_n).
\]

Since

\[
\lim_{n \to +\infty} \frac{\delta_n}{\Gamma(1 - \delta_n)} s^{1 - \delta_n} = 1,
\]

letting \(n \to +\infty\), we have

\[
\lim_{n \to +\infty} \mathcal{E}(\delta_n)(u_n, u_n) \geq \lim_{n \to +\infty} \mathcal{E}(s)(u_n, u_n).
\]

It is obvious that \((E(s), L^2(K;\nu))\) is a closed form on \(L^2(K;\nu)\), by Corollary \[2.5\] we have

\[
\lim_{n \to +\infty} \mathcal{E}(s)(u_n, u_n) \geq \mathcal{E}(s)(u, u).
\]

Hence

\[
\lim_{n \to +\infty} \mathcal{E}(\delta_n)(u_n, u_n) \geq \lim_{n \to +\infty} \mathcal{E}(s)(u_n, u_n) \geq \mathcal{E}(s)(u, u) \text{ for any } s \in (0, +\infty).
\]

Letting \(s \downarrow 0\), we have

\[
\lim_{n \to +\infty} \mathcal{E}(\delta_n)(u_n, u_n) \geq \mathcal{E}(u, u). \quad \square
\]

Then, we prove Theorem \[1.1\].

Proof of Theorem \[1.7\] \(\delta = \beta/\beta^*\). Since the heat kernel \(p_t(x,y)\) exists and \((\mathcal{E}_{\text{loc}}, \mathcal{F}_{\text{loc}})\) on \(L^2(K;\nu)\) is conservative, by Corollary \[2.13\] we have \(k(\beta/\beta^*) = 0\) and

\[
J^{(\beta/\beta^*)}(dx, dy) = J^{(\beta/\beta^*)}(x, y)\nu(dx)\nu(dy),
\]

where

\[
J^{(\beta/\beta^*)}(x, y) = \left( \frac{\beta^* - \beta}{\beta^*} \right) \int_0^{+\infty} \frac{1}{t^{1+\beta^*}} p_t(x, y) dt.
\]

By Theorem \[1.3\] we only need to show that there exists some positive constant \(C\) which is uniformly bounded from above and below when \(\beta\) is bounded away from 0 such that

\[
\frac{1}{C} \frac{\beta^* - \beta}{d(x,y)^{\alpha+\beta}} \leq J^{(\beta/\beta^*)}(x, y) \leq C \left( \frac{\beta^* - \beta}{d(x,y)^{\alpha+\beta}} \right).
\]

Then we have \[2\] holds for any \(u \in L^2(K;\nu)\) by Proposition \[3.1\] and we have the Mosco convergence by taking \(\mathcal{E}_\beta = (\beta^* - \beta)^{-1} \mathcal{E}(\beta/\beta^*)\).

Note the following elementary results. For any \(a \in (1, +\infty), b, c \in (0, +\infty)\), we have

\[
\int_0^{+\infty} \frac{1}{t^a} \exp \left( -\frac{c}{b} t \right) dt = \frac{\Gamma \left( \frac{a-1}{b} \right)}{bc^{a-1}}, \quad (5)
\]

\[
\int_0^1 \frac{1}{t^a} \exp \left( -\frac{c}{b} t \right) dt = \frac{1}{bc^{a-1}} \int_1^{+\infty} s^{\frac{a-1}{b}} e^{-s} ds. \quad (6)
\]

For the case that \[3\] holds for any \(t \in (0, +\infty)\) when \(K\) is unbounded. By Corollary \[2.13\] and Equation \[5\], we have

\[
\beta(\beta^* - \beta) \left[ \frac{\beta^* - \beta}{\beta^*} \right]^{\frac{(\beta^* - 1)(\alpha + \beta)}{(\beta^* - 1)}} \frac{C_1}{\Gamma \left( 2\beta^* - \beta \right)} \frac{1}{d(x, y)^{\alpha+\beta}} \leq J^{(\beta/\beta^*)}(x, y).
\]
\[ \leq \beta(\beta^* - \beta) \left[ \frac{\beta^* - 1}{\beta^*} \left( \frac{(\beta^* - 1)(\alpha + \beta)}{\beta^*} \right) \frac{C_3}{C_4} \right] \frac{1}{d(x,y)^{\alpha+\beta}} \text{ for any } x, y \in K, \]

where \([\ldots]\) are uniformly bounded from above and below by some positive constants \(C_5, C_6\) depending only on \(\alpha, \beta^*, C_1, C_2, C_3, C_4\). Hence

\[ C_5 \frac{\beta(\beta^* - \beta)}{d(x,y)^{\alpha+\beta}} \leq J(\beta/\beta^*)(x,y) \leq C_6 \frac{\beta(\beta^* - \beta)}{d(x,y)^{\alpha+\beta}} \text{ for any } x, y \in K. \]

By Theorem 1.3 we have the Mosco convergence.

For the case that (3) holds for any \(t \in (0, 1)\) when \(K\) is bounded. Using semi-group property, we have

\[ p_t(x,y) \leq C_3 \text{ for any } t \in (1, +\infty), x, y \in K, \]

hence by Corollary 2.13 and Equation (6), we have

\[ J(\beta/\beta^*)(x,y) = \frac{\beta}{\Gamma(1 - \frac{\beta}{\beta^*})} \left( \int_0^1 + \int_{+\infty} \right) \frac{1}{t^{1 + \frac{\beta}{\beta^*}}} p_t(x,y) dt \]

\[ \leq \frac{\beta}{\Gamma(1 - \frac{\beta}{\beta^*})} \left( \int_0^1 \frac{1}{t^{1 + \frac{\beta}{\beta^*}}} C_3 \exp \left( -C_4 \left( \frac{d(x,y)}{t^{1/\beta^*}} \right)^{\beta^*/\beta} \right) dt + \int_{+\infty} \frac{1}{t^{1 + \frac{\beta}{\beta^*}}} C_3 dt \right) \]

\[ \leq \frac{\beta}{\Gamma(1 - \frac{\beta}{\beta^*})} \left( \beta^* - \beta \right) \left[ \frac{C_4}{\beta^* \Gamma\left(\frac{\beta^* - 1}{\beta^*}\right)} \left( \frac{\beta(\beta - 1)}{\beta^*} \frac{1}{\Gamma\left(\frac{(\beta^* - 1)(\alpha + \beta)}{\beta^*}\right)} + \operatorname{diam}(K)^{\alpha+\beta} \right) \right] \frac{1}{d(x,y)^{\alpha+\beta}}. \]

On the other hand, by Corollary 2.13 and Equation (6), we have

\[ J(\beta/\beta^*)(x,y) \geq \frac{\beta}{\Gamma(1 - \frac{\beta}{\beta^*})} \int_0^1 \frac{1}{t^{1 + \frac{\beta}{\beta^*}}} p_t(x,y) dt \]

\[ \geq \frac{\beta}{\Gamma(1 - \frac{\beta}{\beta^*})} \left( \int_0^1 \frac{1}{t^{1 + \frac{\beta}{\beta^*}}} C_1 \exp \left( -C_2 \left( \frac{d(x,y)}{t^{1/\beta^*}} \right)^{\beta^*/\beta} \right) dt \right) \]

\[ = \frac{\beta}{\Gamma(1 - \frac{\beta}{\beta^*})} \frac{\beta^* - 1}{\beta^*} \frac{C_1}{\beta^*} \frac{1}{C_2} \int_{+\infty} \frac{1}{C_3d(x,y)^{\alpha+\beta}} \frac{1}{x^{\beta^*/\beta}} \frac{t^{(\beta^* - 1)(\alpha + \beta) - 1}}{x^{\beta^*/\beta}} e^{-t} dt \]

\[ \geq \frac{\beta}{\Gamma(1 - \frac{\beta}{\beta^*})} \frac{\beta^* - 1}{\beta^*} \frac{C_1}{\beta^*} \frac{1}{C_2} \int_{+\infty} \frac{1}{C_3\operatorname{diam}(K)^{\alpha+\beta}} \frac{1}{x^{\beta^*/\beta}} \frac{t^{(\beta^* - 1)(\alpha + \beta) - 1}}{x^{\beta^*/\beta}} e^{-t} dt \]

\[ = (\beta^* - \beta) \left[ \frac{\beta^* - 1}{(\beta^*)^2} \frac{C_1}{\beta^*} \frac{1}{C_2} \int_{+\infty} \frac{1}{C_3\operatorname{diam}(K)^{\alpha+\beta}} \frac{1}{x^{\beta^*/\beta}} \frac{t^{(\beta^* - 1)(\alpha + \beta) - 1}}{x^{\beta^*/\beta}} e^{-t} dt \right] \frac{1}{d(x,y)^{\alpha+\beta}}. \]

Hence

\[ C_7 \frac{\beta^* - \beta}{d(x,y)^{\alpha+\beta}} \leq J(\beta/\beta^*)(x,y) \leq C_8 \frac{\beta^* - \beta}{d(x,y)^{\alpha+\beta}} \text{ for any } x, y \in K, \]

where \(C_7, C_8\) are some positive constants depending on \(\alpha, \beta^*, C_1, C_2, C_3, C_4, \operatorname{diam}(K), \beta\). Note that \(C_7, C_8\) are uniformly bounded from above and below when \(\beta\) is bounded away from 0. By Theorem 1.3 we have the Mosco convergence. \(\square\)

**Remark 3.3.** The calculation for the case that (3) holds for any \(t \in (0, 1)\) when \(K\) is bounded was also given in the proof of [17, Proposition 3.1]. We give the calculation here for completeness.
Appendix

We give a counterexample that \( \cap_{\beta \in (0, \beta^*)} F_\beta \supsetneq F_{\text{loc}} \) as follows.

Let \( K \) be the SG in \( \mathbb{R}^2 \), that is, let \( p_0 = 0, p_1 = (1, 0), p_2 = (\frac{1}{2}, \frac{\sqrt{3}}{2}) \) and \( f_i(x) = \frac{1}{2}(x + p_i), x \in \mathbb{R}^2, i = 0, 1, 2 \), then \( K \) is the unique non-empty compact set in \( \mathbb{R}^2 \) satisfying \( K = \bigcup_{i=0}^{\infty} f_i(K) \), see Figure 1. Let \( d \) be the Euclidean metric in \( \mathbb{R}^2 \) and \( \nu \) the normalized Hausdorff measure of dimension \( \alpha = \frac{\log 3}{\log 2} \) on \( K \). Then \( (K,d,\nu) \) is an \( \alpha \)-regular compact metric measure space.

![Figure 1: The SG in \( \mathbb{R}^2 \)](image)

Denote \( l(S) \) as the set of all real-valued functions on a set \( S \).

We list the construction of local regular Dirichlet form and its heat kernel estimates on the SG as follows, see \[14, 15, 16\] for reference.

Let \( W = \{0, 1, 2\} \) and

\[
W_n = W^n = \{ w = w_1 \ldots w_n : w_i \in W, i = 1, \ldots, n \} \text{ for any } n \geq 1.
\]

For any \( w^{(1)} = w_1^{(1)} \ldots w_m^{(1)} \in W_m, w^{(2)} = w_1^{(2)} \ldots w_n^{(2)} \in W_n \), let

\[
w^{(1)} w^{(2)} = w_1^{(1)} \ldots w_m^{(1)} w_1^{(2)} \ldots w_n^{(2)} \in W_{m+n}.
\]

Let \( V_0 = \{p_0, p_1, p_2\} \) and \( V_{n+1} = \bigcup_{i \in W} f_i(V_n) \) for any \( n \geq 0 \), then \( \{V_n\}_{n \geq 1} \) is an increasing sequence of finite subsets of \( K \) and \( V_n = \bigcup_{n \geq 1} V_n \) is dense in \( K \). For any \( n \geq 1 \), for any \( w = w_1 \ldots w_n \in W_n \), let

\[
V_w = f_{w_1} \circ \ldots \circ f_{w_n}(V_0),
\]

then \( V_n = \bigcup_{w \in W_n} V_w \).

For any \( n \geq 1 \), for any \( u \in l(V_n) \), let

\[
a_n(u) = \left( \frac{5}{3} \right)^n \sum_{w \in W_n} \sum_{p,q \in V_w} (u(p) - u(q))^2,
\]

then for any \( m \leq n \), for any \( u \in l(V_n) \), we have \( a_m(u) \leq a_n(u) \).

There exists a conservative self-similar strongly local regular Dirichlet form \((E_{\text{loc}}, F_{\text{loc}})\) on \( L^2(K;\nu) \) with a heat kernel \( p_t(x,y) \) satisfying \[3\] with \( \beta^* = \frac{\log 5}{\log 2} \). \((E_{\text{loc}}, F_{\text{loc}})\) on \( L^2(K;\nu) \) can be characterized as follows.

\[
E_{\text{loc}}(u,u) = \lim_{n \to +\infty} a_n(u) = \sup_{n \geq 1} a_n(u),
\]

\[
F_{\text{loc}} = \left\{ u \in C(K) : \sup_{n \geq 1} a_n(u) < +\infty \right\}.
\]

We have the characterization of the domain of stable-like non-local Dirichlet form as follows, see \[30\] Theorem 1.1. For any \( \beta \in (\alpha, \beta^*) \), we have

\[
F_\beta = \left\{ u \in C(K) : \sum_{n=1}^{+\infty} 2^{(\beta - \beta^*)n} a_n(u) < +\infty \right\}.
\]
Therefore, to show that \( \cap_{\beta \in (0,\beta^*)} F_\beta \supsetneq F_{\text{loc}}, \) we only need to construct \( u \in C(K) \) such that \( a_n(u) = n \) for any \( n \geq 1. \)

We construct \( u \in l(V_n) \) by induction as follows.

For \( n = 1. \) Let \( u \in l(V_1) \) be given by \( u = \sum_{i=1}^{\infty} 1_{(p_i)}, \) then \( a_1(u) = 1. \)

Assume that we have constructed \( u \in l(V_n) \) satisfying \( a_n(u) = n. \) Then for \( n + 1, \) we only need to extend \( u \in l(V_n) \) to a function on \( V_{n+1} \) still denoted by \( u \in l(V_{n+1}). \)

Recall that

\[
a_{n+1}(u) = \left( \frac{5}{3} \right)^n \sum_{w \in W_{n+1}} \sum_{p,q \in V_w} (u(p) - u(q))^2
\]

\[
= \left( \frac{5}{3} \right)^n \sum_{w \in W_n} \left( \sum_{i \in W, p,q \in V_{wi}} (u(p) - u(q))^2 \right).
\]

For any \( w \in W_n, \) we only need to assign the values of \( u \) on \( \cup_{i \in W} V_{wi} \setminus V_w, \) see Figure 2.

![Figure 2: \( \cup_{i \in W} V_{wi} \)](image)

Denote \( a = u(f_w(p_0)), b = u(f_w(p_1)), c = u(f_w(p_2)), \) let

\[
u(f_{w1}(p_2)) = x = \alpha b + \alpha c + (1 - 2\alpha) a,
\]

\[
u(f_{w2}(p_2)) = y = \alpha c + \alpha a + (1 - 2\alpha) b,
\]

\[
u(f_{w3}(p_1)) = z = \alpha a + \alpha b + (1 - 2\alpha) c,
\]

where \( \alpha \in (0, \frac{1}{2}) \) is some parameter, then

\[
\sum_{i \in W} \sum_{p,q \in V_{wi}} (u(p) - u(q))^2
\]

\[
= (a - z)^2 + (b - z)^2 + (a - y)^2 + (c - y)^2 + (b - x)^2 + (c - x)^2
\]

\[
+ (x - y)^2 + (y - z)^2 + (z - x)^2
\]

\[
= (15\alpha^2 - 12\alpha + 3) \left( (a - b)^2 + (b - c)^2 + (c - a)^2 \right)
\]

\[
= (15\alpha^2 - 12\alpha + 3) \sum_{p,q \in V_w} (u(p) - u(q))^2.
\]

Let \( \varphi(\alpha) = 15\alpha^2 - 12\alpha + 3, \alpha \in (0, \frac{1}{2}) \), then

\[
\min_{\alpha \in (0, \frac{1}{2})} \varphi(\alpha) = \varphi(\frac{2}{5}) = \frac{3}{5} \lim_{\alpha \to 0} \varphi(\alpha) = 3, \lim_{\alpha \uparrow \frac{1}{2}} \varphi(\alpha) = \frac{3}{4},
\]

hence there exists \( \alpha_n \in (0, \frac{2}{5}) \) such that \( \varphi(\alpha_n) = \frac{3}{5}. \) Then we have the definition of \( u \) on \( \cup_{i \in W} V_{wi}. \) Then we have the definition of \( u \) on \( V_{n+1}. \) Moreover, \( a_{n+1}(u) = \frac{n+1}{n} a_n(u) = n+1. \)

By induction principle, we obtain \( u \in l(V_n) \) satisfying \( a_n(u) = n \) for any \( n \geq 1. \) Since \( \alpha \uparrow \frac{2}{5} \) as \( n \to +\infty, \) it is obvious that \( u \) is uniformly continuous on \( V_n, \) hence \( u \) can be extended to a continuous function on \( K \) still denoted by \( u \in C(K). \) Hence \( u \in \cap_{\beta \in (0,\beta^*)} F_\beta \setminus F_{\text{loc}}. \)
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