Hodograph solutions of the wave equation of nonlinear electrodynamics in the quantum vacuum

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The process of photon-photon scattering in vacuum is investigated analytically in the long-wavelength limit within the framework of the Euler-Heisenberg Lagrangian. In order to solve the nonlinear partial differential equations (PDEs) obtained from this Lagrangian use is made of the hodograph transformation. This transformation makes it possible to turn a system of quasilinear PDEs into a system of linear PDEs. Exact solutions of the equations describing the nonlinear interaction of electromagnetic waves in vacuum in a one-dimensional configuration are obtained and analyzed.

I. INTRODUCTION

Perturbation theory has proven to be extremely successful in obtaining a number of prominent results in quantum field theories (QFTs) [1–4]. In spite of these achievements, as is well known, perturbation theory is only valid provided the interaction is weak and thus it cannot provide a full description of a QFT [5, 6]. For this reason the nonperturbative behavior of QFTs has attracted a great deal of attention for decades [7]. As examples of physical objects typical for QFTs and classical mechanics of continuous media whose theoretical description cannot be obtained within the framework of perturbation theory we may list the breaking of nonlinear waves, solitons, instantons, etc. [8–12].

In quantum electrodynamics (QED) perturbation theory breaks in the limit of strong electric fields, when the electric field $E$ approaches the critical field of quantum electrodynamics [13, 14]

$$E_S = \frac{m_e^2 c^3}{\hbar 4\pi}$$

and/or the photon energy becomes substantially large, i.e. for $\alpha \chi^2 \geq 1$ [6] where $\alpha = e^2 / \hbar c$ is the fine structure constant, $\chi = \hbar \gamma (F_{\mu\nu}k^\gamma)^2 / m_c E_S$ is the so called nonlinear quantum parameter (see Refs. [2, 6]). $F_{\mu\nu}$ is the electromagnetic field tensor, and $\hbar k^\gamma$ is the four-momentum of the photon. The electron mass and electric charge are $m_e$ and $e$, respectively, $c$ is the speed of light in vacuum, and $h$ is the Planck constant. The critical field corresponds to the electric field that, acting on the electron charge $e$, would produce a work equal to the electron rest mass energy $m_e c^2$ over a distance equal to the Compton wavelength $\lambda_C = h / m_e c$. Here $h$ is the reduced Planck constant, $e$ and $m_e$ are the electron electric charge and mass, and $c$ is the speed of light in vacuum (see for details Refs. [2, 13–15]). The corresponding wavelength $\lambda_S$ and intensity of electromagnetic radiation are $\lambda_S = 2\pi \lambda_C = 2 \times 10^{-16}$ cm and $I_S = c E_S^2 / 4\pi \approx 10^{29}$ W/cm$^2$, respectively.

One of the most remarkable effects predicted in QED is the vacuum polarization connected with light-light scattering and pair production from vacuum. In classical electrodynamics electromagnetic waves do not interact in vacuum. On the contrary, in QED photon-photon scattering can take place in vacuum via the generation of virtual electron-positron pairs. This interaction gives rise to vacuum polarization and birefringence, to the Lamb shift, to a modification of the Coulomb field, and to many other phenomena [2]. Photon-photon scattering was observed in collisions of heavy ions accelerated in standard particle accelerators (see review article [16] and the results of the experiments obtained with the ATLAS detector at the Large Hadron Collider [17]).

Photon-photon interaction provides a tool for the search for new physics [16, 18]; further studies of this process will make it possible to test extensions of the Standard Model in which new particles contribute to the interaction loop diagrams [19]. Using the Euler-Heisenberg Lagrangian [14, 20], which describes the vacuum polarization and electron-positron pair generation by super-strong electromagnetic field in vacuum [13, 21], also provides one of the most developed approaches for studying non-perturbative processes in QFT, when finding exact solutions of nonlinear problems cannot be underestimated.
The increasing availability of high power lasers has stimulated a growing interest towards the experimental observation of photon-photon scattering processes [22–24] and electron positron pair creation [25]. In addition it has provided strong motivation for their theoretical study in processes such as the scattering of a laser pulse by a laser pulse [26–34], the scattering of XFEL emitted photons [19], and the interaction of relatively long-wavelength, high intensity, laser light pulses with short-wavelength X-ray photons [35].

The process of vacuum polarization can be described within the framework of the approximation using the Euler-Heisenberg Lagrangian [14, 20]. Although this approximation is valid in the limit of colliding photons with relatively low energy and of low amplitude electromagnetic pulses, it allows one to extend consideration over the non-perturbative theory. Its applicability requires the colliding photon energy to be below the electron rest-mass energy, \( E_\gamma < m_e c^2 \), and the electric field of the colliding electromagnetic waves to be below the critical field given by Eq. (1). When writing the condition for the validity of the long-wavelength approximation given above it was assumed that the frequencies of the colliding photons are equal. If the frequencies are different, say \( \omega \) and \( \Omega \) with \( \Omega \neq \omega \), the low-frequency approximation requires that

\[
\omega \Omega < m_e^2 c^4 / \hbar^2.
\]  

(2)

In the limit of electromagnetic fields with extremely large amplitudes approaching the QED critical field \( E_S \), the nonlinear modification of the vacuum refraction index via the polarization of virtual electron-positron pairs leads to the decrease of the propagation velocity of counter-propagating electromagnetic waves [36–38] while, on the contrary, co-propagating waves do not change their propagation velocity because co-propagating photons do not interact, see e.g. Ref. [39].

The nonlinear properties of the QED vacuum in the long-wavelength, low frequency limit can find a counterpart in those of nonlinear dispersionless media, keeping however in mind that in QED there is no preferred frame where the nonlinear medium is at rest. In a material nonlinear medium with a refraction index that depends on the electromagnetic field amplitude an electromagnetic wave can evolve into a configuration with singularities [40, 41]. The evolution of a finite amplitude wave is accompanied by the steepening of its wave front, by the formation of shock-like waves, i.e. it is characterized by a processes leading to gradient catastrophes [4]. In the case of the quantum vacuum, corresponding phenomena have been investigated in Refs. [21, 42, 43] and [38]. The occurrence of singularities in the Euler-Heisenberg electrodynamics has been noticed in Refs. [21, 42], indicated in computer simulations presented in Ref. [43], and thoroughly studied in Ref. [38].

In the present paper, we analyze the interaction of finite amplitude, counter-propagating electromagnetic (e.m.) waves in a one dimensional (1-D) configuration. The interacting waves are assumed to be linearly polarized and to have the same polarization direction. In such a configuration the propagation directions of the two colliding plane waves are collinear, and this collinearity is preserved by Lorentz boosts along the propagation direction. However, the Euler-Heisenberg Lagrangian is invariant under the full Lorentz group. This makes it possible to use the solutions that will be derived in the following sections to construct solutions that describe the interaction of plane waves colliding at an angle, e.g., by considering Lorentz boosts in the direction perpendicular to the direction of the polarization vector of the two colliding waves. This extension of the results presented below may be of interest in an experimental setting.

The hodograph transformation [44] is a useful tool in the study of nonlinear waves as it allows us to obtain a linear system of second order partial differential equations (PDEs) instead of a system of second order quasilinear PDEs. In the case of the e.m. 1-D configuration under study, this transformation makes the electric and the magnetic fields play the role of the independent coordinates. The hodograph transform has been adopted for a non-dispersive formulation of the electromagnetic field equations in a nonlinear material medium, see e.g. Refs. [45, 46].

The analysis described in the following sections allows us to find exact solutions describing the nonlinear interaction of electromagnetic waves in vacuum both in the space-time coordinates and in the hodograph variables, to formulate a perturbative approach that, in the limit of monochromatic waves, does not lead to secularities and to derive the dispersion relation of e.m. waves propagating in vacuum in the presence of steady and uniform, strong e.m. fields.

This article is organized as follows. In Sec. II the Euler-Heisenberg Lagrangian is recalled and in Sec. II A it is specialized to the case of counter-propagating e.m. waves in a 1-D configuration and the corresponding nonlinear wave equation is derived using the so-called light cone coordinates. As an illustration, higher order terms that depend on the sixth power of the e.m. fields are included in the Euler-Heisenberg Lagrangian but, for the sake of algebraic simplicity, the contribution of these terms is neglected in some of the formulae in the present text. In Sec. II B the conservations that arise from the translational and from the Lorentz invariance of the 1-D Euler-Heisenberg Lagrangian are presented. In Sec. II C the linear case of non interacting waves is briefly described and in Sec. II D perturbative solutions are obtained in light cone coordinates. In Sec. II E the derivation of the characteristics of the nonlinear wave equations is outlined, while in Sec. II F exact self-similar solutions are derived. In Sec. II G the dispersion equation of e.m. waves propagating in vacuum perpendicularly to large, steady and uniform, e.m. fields is presented. In Sec. II H the hodograph transform of the equations of nonlinear electrodynamics in vacuum is derived.
II. EQUATIONS OF NONLINEAR VACUUM ELECTRODYNAMICS

The Euler–Heisenberg Lagrangian is given by

\[ \mathcal{L} = \mathcal{L}_0 + \mathcal{L}' \],

where

\[ \mathcal{L}_0 = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} \] (4)

is the Lagrangian in classical electrodynamics, \( F_{\mu\nu} \) is the electromagnetic field tensor

\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \],

with \( A_\mu \) being the 4-vector of the electromagnetic field and \( \mu = 0, 1, 2, 3 \). Here and below a summation over repeating indices is assumed.

In the Euler–Heisenberg theory, the QED radiation corrections are described by \( \mathcal{L}' \) on the right hand side of Eq.(3), which can be written as \[ \mathcal{L}' = -\frac{m^4}{8\pi^2} \int_0^\infty \frac{\exp(-\eta)}{\eta^4} \left[ -\eta a \cot \eta a (\eta b \coth \eta b) + 1 - \frac{\eta^2}{3} (a^2 - b^2) \right] d\eta. \] (6)

Here the invariants \( a \) and \( b \) can be expressed in terms the Poincaré invariants \( F = F_{\mu\nu} F^{\mu\nu} \)

\[ \Phi = F_{\mu\nu} \tilde{F}^{\mu\nu} \] as

\[ a = \sqrt{\sqrt{\Phi^2 + \Phi^2} + \Phi} \] and \[ b = \sqrt{\sqrt{\Phi^2 + \Phi^2} - \Phi}, \]

respectively, where dual tensor \( \tilde{F}^{\mu\nu} = \varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} \) contains \( \varepsilon^{\mu\nu\rho\sigma} \) being the Levi-Civita symbol in four dimensions. Here and in the following text, we use the units \( c = \hbar = 1 \), and the electromagnetic field is normalized on the QED critical field \( E_S \).

As explained in Ref. [2] the Euler–Heisenberg Lagrangian in the form given by Eq.(6) should be used for obtaining an asymptotic series over the invariant electric field \( a \) assuming its smallness.

In the weak field approximation the Lagrangian \( \mathcal{L}' \) is given by (e.g. see [47])

\[ \mathcal{L}' = \kappa \left[ \tilde{\Phi}^2 + \frac{7}{4} \Phi^2 + \frac{90}{315} \frac{\tilde{\Phi}}{\Phi} \left( \tilde{\Phi}^2 + \frac{13}{16} \Phi^2 \right) \right] + \ldots \] (9)

with the constant \( \kappa = (e^4/360\pi^2)m^4 \). In the Lagrangian (9) the first two terms on the right hand side and the last two correspond respectively to four and to six photon interaction.

A. Counter-propagating electromagnetic waves

In the following we consider the interaction of counter-propagating electromagnetic waves with the same linear polarization, in which case the invariant \( \Phi \) vanishes identically. Such a field configuration can be described in a
transverse gauge by a vector potential having a single component, \( A = Ae_z \), with \( e_z \) the unit vector along the \( z \) axis. In terms of the light cone coordinates (see e.g. Ref. \([48]\))

\[
x_+ = (x + t)/\sqrt{2}, \quad x_- = (x - t)/\sqrt{2},
\]

the vector potential \( A \) can be written as

\[
A = a(x_+, x_-).
\]

In these variables the Lagrangian \((3)\) takes the form

\[
\mathcal{L} = -\frac{1}{4\pi} \left[ uw - \epsilon_2(wu)^2 - \epsilon_3(wu)^3 \right]
\]

where the field variables \( u \) and \( w \) are defined by

\[
u = \partial_{x_-} a \quad \text{and} \quad w = \partial_{x_+} a
\]

and are related to the electric field \( E = -\partial_t A \) (along \( z \)) and to the magnetic field \( B = -\partial_x A \) (along \( y \)) by

\[
w = -(E + B)/\sqrt{2}, \quad u = (E - B)/\sqrt{2} \quad \text{and} \quad uw = (B^2 - E^2)/2.
\]

The dimensionless parameters \( \epsilon_2 \) and \( \epsilon_3 \) in Eq. \((12)\) are given by

\[
\epsilon_2 = \frac{2\epsilon^2}{45\pi} = \frac{2}{45\pi} \alpha \quad \text{and} \quad \epsilon_3 = \frac{32\epsilon^2}{315\pi} = \frac{32}{315\pi} \alpha,
\]

where \( \alpha = \epsilon^2/\hbar c \approx 1/137 \) is the fine structure constant, i.e., \( \epsilon_2 \approx 10^{-4} \) and \( \epsilon_3 \approx 2 \times 10^{-4} \), respectively.

The field equations can be found by varying the Lagrangian:

\[
\partial_{x_-} (\partial_u \mathcal{L}) + \partial_{x_+} (\partial_w \mathcal{L}) = 0.
\]

As a result, we obtain the system of equations (see also Appendix \([A]\))

\[
\partial_{x_-} w = \partial_{x_+} u,
\]

\[
\partial_{x_+} [u(1 - 2\epsilon_2 uw - 3\epsilon_3 u^2 w^2)] + \partial_{x_-} [w(1 - 2\epsilon_2 uw - 3\epsilon_3 u^2 w^2)] = 0.
\]

The first of these equations, Eq. \((17)\), is simply a consequence of the symmetry of the second derivatives, \( \partial_{x_-} x_+ a = \partial_{x_+} x_- a \) and it expresses the vanishing of the 4-divergence of the dual tensor \( \tilde{F}^{\mu\nu} \). By rearranging terms and by inserting Eqs. \((13)\) \((17)\), Eq. \((18)\) can be rewritten in the form of a second order, quasi-linear partial differential equation for the potential \( a(x_+, x_-) \):

\[
[1 - uw(4\epsilon_2 + 9\epsilon_3 uw)]\partial_{x_-} x_+ a = u^2(2\epsilon_2 + 3\epsilon_3 uw)\partial_{x_-} x_- a + u^2(2\epsilon_2 + 3\epsilon_3 uw)\partial_{x_+} x_- a,
\]

where \( u(x_+, x_-) \) and \( w(x_+, x_-) \) are defined by Eqs. \((13)\).

### B. Symmetries and conservations

The Lagrangian \((12)\), and thus Eq. \((13)\), are invariant under the discrete transformation \( x_+ \leftrightarrow x_- \) that interchanges \( u \) and \( w \). The Lagrangian \((12)\) is also invariant under translations along \( x \) and \( t \) and under Lorentz boosts along \( x \). In fact the four-vector potential component \( a \) is transverse to the boost and the field product \( uw \) is proportional to the Lorentz invariant \( \beta \). In terms of the light cone coordinates the corresponding infinitesimal transformations can be written with obvious notation as (see also Ref. \([48]\))

\[
x_+ \rightarrow x_+ + \delta_+, \quad x_- \rightarrow x_- + \delta_-,
\]

\[
x_+ \rightarrow (1 - \beta)x_+ \quad \text{and} \quad x_- \rightarrow (1 + \beta)x_-,
\]

(20)
and the product $x^+ x^-$ is invariant under Lorentz boosts along $x$. According to Noether’s theorem these continuous symmetries imply the local conservation of the electromagnetic energy momentum tensor and of the “barycenter” (center of the energy-momentum distribution) which in light cone coordinates takes the form

$$\partial_{x^+} T_{uw} + \partial_{x^-} T_{uw} = 0, \quad \partial_{x^+} T_{uw} + \partial_{x^-} T_{uu} = 0$$
$$\partial_{x^+} (T_{uw} x_+ - T_{wu} x_-) + \partial_{x^-} (T_{uw} x_+ - T_{wu} x_-) = 0,$$  \hspace{1cm} (21)

where

$$T_{ij} = \frac{\partial L}{\partial(\partial_i a)} (\partial_j a) - \delta_{ij} L, \quad i, j = \pm, \quad \text{and} \quad T_{++} = T_{uu}, \quad T_{+-} = T_{wu}, \quad \text{etc.} \quad (22)$$

Neglecting for simplicity the $\epsilon_3$ term, from $L = -(u w - \epsilon_2 u^2 w^2)/4\pi$ we have

$$T_{uu} = T_{uw} = \epsilon_2 u^2 w^2/4\pi,$$
$$T_{wu} = -u^2(1 - 2\epsilon_2 uw)/4\pi, \quad T_{uu} = -w^2(1 - 2\epsilon_2 uw)/4\pi.$$  \hspace{1cm} (23)

where the trace and the determinant are Lorentz invariants.

C. Linear approximation and non-interacting waves

In linear approximation Eqs. (18, 19) take the form

$$\partial_{x^+} u = -\partial_{x^-} w, \quad \partial_{x^+ x^-} a = 0,$$  \hspace{1cm} (24)

where Eq. (19) has reduced to the standard linear wave equation in the light cone coordinates.

The first of Eqs. (24), together with Eq. (17), leads to the general solution $u = f(x_-)$ and $w = g(x_+)$ with $f$ and $g$ arbitrary functions that are determined by the initial conditions. For these solutions the vector potential $a(x_+, x_-)$ takes the factorized form $a(x_+, x_-) = a_+(x_+) + a_-(x_-)$ with $u = \partial_{x_-} a$ and $w = \partial_{x_+} a$. These solutions describe noninteracting electromagnetic waves propagating towards positive and negative directions along the $x$ axis, respectively.

Equations (17, 18) allow for particular solutions for which either $u = 0$ or $w = 0$, in which case $w$ (or $u$) is an arbitrary function depending on the light cone variable $x_+$ (or $x_-$). These solutions describe finite amplitude electromagnetic waves propagating along $x$ from right to left (from left to right) with propagation velocity equal to the speed of light in vacuum. Their shape does not change in time and the electric and magnetic field components are equal $E = B = -w/\sqrt{2}$ and $T_{wu} = -w^2/(4\pi) = -E^2/(2\pi)$, or equal and opposite $E = -B = u/\sqrt{2}$ and $T_{wu} = -u^2/(4\pi) = -E^2/(2\pi)$.

D. Perturbative solutions

In the case of small but finite field amplitudes $u, w$ we can solve Eqs. (17, 18) (or equivalently Eq. (19)) perturbatively by expanding in powers of the field amplitudes, seeking solutions of the form $u(x_-, x_+) = u_0(x_-) + u_1(x_-, x_+)$, $w(x_-, x_+) = w_0(x_+) + w_1(x_+, x_-)$ (or equivalently of the form $a(x_+, x_-) = a_0(x_-) + a_1(x_-, x_+)$).

Keeping only cubic terms in the fields we obtain

$$u_1(x_-, x_+) = \epsilon_2 u_0^2(x_-) w_0(x_+) + \epsilon_2 [\partial_{x_+} u_0(x_-)] \int^{x_+} dx' w_0^2(x'_+),$$
$$w_1(x_+, x_-) = \epsilon_2 u_0^2(x_+) u_0(x_-) + \epsilon_2 [\partial_{x_+} w_0(x_+)] \int^{x_-} dx' u_0^2(x'_-),$$  \hspace{1cm} (25)

where the two integral terms give the net effect of the interaction between two finite length counter-propagating waves after the end of the interaction. Corresponding results can be obtained by integrating directly the wave equation for $a_1(x_+, x_-)$ up to cubic terms

$$\partial_{x_- x_+} a_1(x_+, x_-) = \epsilon_2 \{(\partial_{x_+} a_0)^2 \partial_{x_- x_+} a_0 + (\partial_{x_-} a_0)^2 \partial_{x_+ x_-} a_0\}.$$  \hspace{1cm} (26)
1. Phase shift induced by the interaction with a localized pulse

Taking as an example a monochromatic wave \( u_0(x^-) = U_0 \cos k(x - t) \) interacting with a localized counter-propagating pulse \( u_0 \), such that \( u_0(x^+) = 0 \) both for \( x^+ > L \) and for \( x^- < -L \), we find

\[
\begin{align*}
  u(x^-, x^+ < -L) &= u_0(x^-) = U_0 \cos k(x - t), \quad \text{and} \\
  u(x^-, x^+ > L) &= u_0(x^-) + \varepsilon_2 \left[ \partial x^- u_0(x^-) \right] \int_{-L}^{L} dx^+ u_0^2(x^+) \\
  &= U_0 \left[ \cos(kx - t) - k\varepsilon_2 \sin(k(x - t)) \int_{-L}^{L} dx^+ u_0^2(x^+) \right]
\end{align*}
\]

which, to the considered expansion order, corresponds to a phase shift \( \frac{\varepsilon_2 L}{k} \).

2. Interaction between monochromatic waves and propagation velocity

In the case of two interacting monochromatic waves (independently of their relative frequencies) Eqs. (25) would lead to a secular behavior: in other words, the quadratic terms in the integrands Eqs. (25) do not satisfy in general the integrability conditions. In order to restore integrability, we may uplift an \( \varepsilon_2 \) term in the expansion of the vector potential \( a(x^+, x^-) \) and define the zeroth order solution as

\[
\begin{align*}
  \bar{a}_{+0}(x^+ + \varepsilon_2 s_+(x^+, x^-)), & \quad \bar{a}_{-0}(x^- + \varepsilon_2 s_-(x^+, x^-)).
\end{align*}
\]

To leading order we recover Eq. (24), while two counter-terms are added to Eq. (26) that is changed into

\[
\begin{align*}
  \frac{\partial^2 a_1(x^+, x^-)}{\partial x^+ \partial x^-} &= \varepsilon_2 \left[ (\partial_{x^+} \bar{a}_{0+})^2 \partial_{x^-} \bar{a}_{0-} + (\partial_{x^-} \bar{a}_{0-})^2 \partial_{x^+} \bar{a}_{0+} \right] \\& - \varepsilon_2 \frac{\partial}{\partial x^+} \left[ (\partial_{x^+} \bar{a}_{0+}) \partial_{x^-} s_{+(x^+, x^-)} + (\partial_{x^-} \bar{a}_{0-}) \partial_{x^+} s_{-(x^+, x^-)} \right].
\end{align*}
\]

Neglecting higher order terms in \( \varepsilon_2 \) we have

\[
\begin{align*}
  \frac{\partial^2 a_1(x^+, x^-)}{\partial x^+ \partial x^-} &= \varepsilon_2 \frac{\partial}{\partial x^+} \left[ \left( \partial_{x^+} \bar{a}_{0+} \right) \left( (\partial_{x^-} \bar{a}_{0-})^2 \partial_{x^+} s_+(x^+, x^-) - \partial_{x^-} s_+(x^+, x^-) \right) \right] \\
  &+ \varepsilon_2 \frac{\partial}{\partial x^-} \left[ \left( \partial_{x^+} \bar{a}_{0+} \right) \left( (\partial_{x^-} \bar{a}_{0-})^2 - \partial_{x^-} s_-(x^+, x^-) \right) \right]
\end{align*}
\]

where we take

\[
\begin{align*}
  s_+(x^+, x^-) &= \int_{x^-}^{x^+} dx'^- (\partial_{x^+} \bar{a}_{0-})^2 \approx \int_{x^-}^{x^+} dx'^- (\partial_{x^+} \bar{a}_{0-})^2, \quad \rightarrow \ s_+(x^+, x^-) \sim s_+(x^-) \\
  s_-(x^+, x^-) &= \int_{x^+}^{x^-} dx'^+ (\partial_{x^+} \bar{a}_{0+})^2 \approx \int_{x^+}^{x^-} dx'^+ (\partial_{x^+} \bar{a}_{0+})^2, \quad \rightarrow \ s_-(x^+, x^-) \sim s_-(x^+)
\end{align*}
\]

and set without loss of generality \( a_1 = 0 \). Then to first order in \( \varepsilon_2 \) the renormalized solutions read

\[
\begin{align*}
  a(x^+, x^-) &= a_+ \left( x^+ + \varepsilon_2 \int_{x^-}^{x^+} dx'^- (\partial_{x^+} \bar{a}_{0-})^2 \right) + a_- \left( x^- + \varepsilon_2 \int_{x^+}^{x^-} dx'^+ (\partial_{x^+} \bar{a}_{0+})^2 \right).
\end{align*}
\]

The integrals in the arguments lead to two amplitude dependent, inhomogeneous, propagation velocities with absolute values smaller than the speed of light \( \frac{\varepsilon_2 L}{k} \)

\[
\begin{align*}
  v_-(x^+) &= 1 - \varepsilon_2 (\partial_{x^+} \bar{a}_{0+})^2, \quad v_+(x^-) = 1 - \varepsilon_2 (\partial_{x^-} \bar{a}_{0-})^2.
\end{align*}
\]

and, for localized pulses, to a phase shift at the end of the interaction in agreement with Eq. (27). This amplitude dependent slowing of the wave propagation velocity may lead to self-lensing and wave collapse of two counter-propagating pulses \( \frac{\varepsilon_2 L}{k} \).
3. Perturbed light cone variables

Referring to Eq. (32), we note that the variables
\[ X_+ = x_+ + \epsilon_2 \int x_-^x d\xi_- (\partial_{\xi_-} a_0^-)^2, \quad X_- = x_- + \epsilon_2 \int x_+^x d\xi_+ (\partial_{\xi_+} a_0^+)^2 \]
are “gauge invariant” and transform properly under 1-D Lorentz transformations, see the second line in Eqs. (29). Thus the condition \( X_+ X_- = 0 \) defines a Lorentz invariant perturbed light cone. It is interesting to notice that the causal cone of a wave event is “shrunk” by a counter-propagating wave.

E. Full solutions

The characteristics \( x_{\pm} = \xi_{\pm}(s) \) of Eq. (19), neglecting for the sake of notational simplicity the \( \epsilon_3 \) term, are given by the quadratic equation
\[ \epsilon_2 u^2(s) \left( \frac{d\xi_+}{ds} \right)^2 + \epsilon_2 w^2(s) \left( \frac{d\xi_-}{ds} \right)^2 + [1 - 4\epsilon_2 u(s)w(s)] \left( \frac{d\xi_+}{ds} \right) \left( \frac{d\xi_-}{ds} \right)^2 = 0, \]
and are used in Ref. [12] in order to construct “simple wave” solutions of Eq. (19) and to prove that it admits the formation of discontinuities.

In the following instead we will seek for selfsimilar (scale invariant) solutions of Eq. (19) by reducing it to an ordinary nonlinear differential equation.

F. Lorentz invariant solutions

We look for solutions of the form \( a(x_+, x_-) = a(\rho) \), with \( \rho \equiv x_+ x_- \) i.e. for solutions that are constant along the Lorentz invariant curves \( x_+ x_- = \text{const.} \). Then, from Eq. (19) we obtain
\[ \left[ 1 - 4\epsilon_2 \rho \left( \frac{da}{d\rho} \right)^2 \right] \frac{d}{d\rho} \left( \frac{d}{d\rho} \right) = 2\epsilon_2 \rho^2 \left( \frac{da}{d\rho} \right)^2 \frac{d^2 a}{d\rho^2}, \]
which can be rewritten as
\[ \frac{d}{d\rho} \left( \rho \frac{da}{d\rho} \right) = 2\epsilon_2 \rho \left( \frac{da}{d\rho} \right)^3 \]
and yields the algebraic equation
\[ \frac{da}{d\rho} - 2\epsilon_2 \rho \left( \frac{da}{d\rho} \right)^3 = \frac{C_2}{\rho}. \]
In the limit \( \epsilon_2 \to 0 \) we obtain (with \( C_1, C_2 \) arbitrary constants)
\[ a = C_1 + C_2 \ln |\rho|, \quad w = C_2/x_+, \quad u = C_2/x_- \]
In these solutions the electric and the magnetic fields “cumulate” at \( x = \pm t \) where their amplitude diverges. In this case a power expansion in \( \epsilon_2 \) cannot be used, while the approach of Eq. (28) gives
\[ a = C_1 + C_2 \ln |\tilde{\rho}|, \quad \rho = x_+ x_- = x_+ x_- + \epsilon_2 C_2^2 (x_+ / x_+ + x_- / x_- - 2) \]
which amounts to an amplitude dependent shift in the cumulation coordinates with
\[ w = \frac{C_2 (x_+ + \epsilon_2 C_2^2 / x_+)}{x_+ x_- + \epsilon_2 C_2^2 (x_+ / x_+ + x_- / x_- - 2)} \]
\[ u = \frac{C_2 (x_+ + \epsilon_2 C_2^2 / x_-)}{x_+ x_- + \epsilon_2 C_2^2 (x_+ / x_+ + x_- / x_- - 2)} \]
These Lorentz invariant solutions represent a special case of solutions obtained in the hyperbolic coordinates
\[ \rho = x_+ x_- \quad \psi = (1/2) \ln (x_+ / x_-) \]
that are briefly discussed in Appendix B.
G. Waves in finite amplitude, uniform electric and magnetic fields in vacuum

Let us set

\[ a(x_+, x_-) = W_0 x_+ + U_0 x_- + \bar{a}(x_+, x_-) \]

with \( W_0, U_0 \) uniform background fields and assume the harmonic system of field equations

\[ \dot{W}_0 = \epsilon_2^2 W_0 \sim \dot{U}_0 = \epsilon_2^2 U_0 \sim O(1), \quad W_0, U_0 \gg \partial_{x_+} \bar{a}(x_+, x_-), \partial_{x_-} \bar{a}(x_+, x_-). \]

Then Eq. (19) (with \( \epsilon_3 = 0 \) for the sake of simplicity) becomes

\[ (1 - 4 \dot{U}_0 \dot{W}_0) \partial_{x_+, x_+} \bar{a} = \dot{W}_0^2 \partial_{x_- x_-} \bar{a} + \dot{U}_0^2 \partial_{x_+ x_+} \bar{a}, \]

which is hyperbolic, and thus describes waves, for \( (1 - 4 \dot{U}_0 \dot{W}_0)^2 > 4 U_0^2 W_0^2 \), i.e. for \( \dot{U}_0 \dot{W}_0 < 1/6 \) and for \( \dot{U}_0 \dot{W}_0 > 1/2 \). Taking for the sake of simplicity

\[ \bar{a} = \bar{a}_0 \exp [i(k_+ x_+ + k_- x_-)] = \bar{a}_0 \exp [i(\kappa x - \omega t)], \]

with \( k = (k_+ + k_-)/\sqrt{2} \) and \( \omega = -(k_+ - k_-)/\sqrt{2} \), we obtain the dispersion equation

\[ (1 - 4 \dot{U}_0 \dot{W}_0) k_+ k_- = \dot{W}_0^2 k_+^2 + \dot{U}_0^2 k_-^2, \quad \text{ i.e.} \]

\[ (1 - 4 \dot{U}_0 \dot{W}_0 + \dot{W}_0^2 + \dot{U}_0^2) \omega^2 = (1 - 4 \dot{U}_0 \dot{W}_0 - \dot{W}_0^2 - \dot{U}_0^2) k^2 - 2(\dot{W}_0^2 - \dot{U}_0^2) \omega k, \]

In the two interesting limits of a purely electric (\( W_0 = -U_0, E = \sqrt{2}W_0 \)) and purely magnetic (\( W_0 = U_0, B = -\sqrt{2}W_0 \)) background fields we obtain

\[ \omega^2 = [(1 + \epsilon_2 E^2)/(1 + 3 \epsilon_2 E^2)] k_x^2 \]

\[ \omega^2 = [(1 - 3 \epsilon_2 B^2)/(1 - \epsilon_2 B^2)] k_y^2, \quad \text{for } \epsilon_2 B^2 < 1/3, \]

that correspond to phase velocities smaller than the speed of light in vacuum (see reviews \[37\] and \[54\] and references therein).

III. HODOGRAPH TRANSFORM OF THE EQUATIONS OF NONLINEAR ELECTRODYNAMICS IN VACUUM

A system of quasilinear partial differential equations, i.e. a system linear with respect to the highest order terms in the partial derivatives \( \partial_{x_+} \) and \( \partial_{x_-} \) with coefficients nonlinearly dependent on variables \( u \) and \( w \), admits the hodograph transformation \[44\]. Assuming that both \( u \) and \( w \) are not constant, we perform the hodograph transformation by treating them as coordinates, i.e. we consider \( x_- \) and \( x_+ \) as functions of \( u \) and \( w \):

\[ x_- = x_-(u, w) \quad \text{and} \quad x_+ = x_+(u, w). \]

To transform the system of Eqs. (17) (18) to the new coordinates \( u \) and \( w \) we need to express the partial derivatives with respect to \( x_- \) and \( x_+ \) in terms of derivatives with respect to \( u \) and \( w \). For a function \( \Upsilon(x_-, x_+) \), using the chain rule, we have

\[ \partial_u \Upsilon = \partial_{x_-} \Upsilon \partial_u x_- + \partial_{x_+} \Upsilon \partial_u x_+, \quad \partial_w \Upsilon = \partial_{x_-} \Upsilon \partial_w x_- + \partial_{x_+} \Upsilon \partial_w x_. \]

Solving this system of equations with respect to \( \partial_{x_-} \Upsilon \) and \( \partial_{x_+} \Upsilon \) we obtain

\[ \partial_{x_-} \Upsilon = J^{-1}(\partial_u \Upsilon \partial_u x_- - \partial_w \Upsilon \partial_w x_+), \quad \partial_{x_+} \Upsilon = J^{-1}(\partial_u \Upsilon \partial_u x_+ - \partial_w \Upsilon \partial_w x_-). \]

Here \( J = (\partial_u x_- \partial_w x_+ - \partial_u x_+ \partial_w x_-) \) is the Jacobian of the coordinate transformation, which is assumed not to vanish. Taking \( \Upsilon \) equal either \( u \) or \( w \) we find

\[ \partial_{x_-} u = J^{-1} \partial_u x_- , \quad \partial_{x_+} u = -J^{-1} \partial_w x_-, \quad \partial_{x_-} w = -J^{-1} \partial_u x_+ , \quad \partial_{x_+} w = J^{-1} \partial_w x_. \]

Substituting these relationships to Eqs. (17) (18) yields

\[ \partial_u x_+ = \partial_w x_-, \]

\[ [1 - uw(4 \epsilon_2 + 9 \epsilon_2 w)] \partial_u x_- = -w^2 (\epsilon_2 + 3 \epsilon_2 w) \partial_u x_+ - u^2 (\epsilon_2 + 3 \epsilon_2 w) \partial_u x_. \]

From the system (17) and (18) with coefficients nonlinearly dependent on \( u \) and \( w \) we have obtained a system of linear equations for \( x_- \) and \( x_+ \). Equations (53) (54) are the hodograph transform of Eqs. (17) (18). As is well known the nonlinearity of the original system is shifted from the field equation to the coordinate transformation.
IV. NONLINEAR INTERACTION OF ELECTROMAGNETIC WAVES IN QED VACUUM

Introducing a potential function \( \Phi(u, w) \) such that the functions \( x_- \) and \( x_+ \) are given by

\[
x_- = \partial_u \Phi, \quad \text{and} \quad x_+ = \partial_w \Phi,
\]

we can write Eqs. (53, 54) in the form

\[
[1 - u w (4 \epsilon_2 + 9 \epsilon_3 u w)] \partial_{uw} \Phi = -w^2 (\epsilon_2 + 3 \epsilon_3 u w) \partial_{ww} \Phi - u^2 (\epsilon_2 + 3 \epsilon_3 u w) \partial_{uu} \Phi.
\]

(56)

In Appendix A an equivalent derivation of Eq. (56) involving the momenta of the Lagrangian \( L \) is presented. It is also shown that the function \( \Phi(u, w) \) is related to the Lagrangian function for the hodograph equations.

A. Symmetries and conservations in the hodograph representation

When applying the hodograph transformation \( x_{\pm} = x_{\pm}(u, w) \) a conservation equation of the form

\[
\partial_{x_+} A_+ (x_+, x_-) + \partial_{x_-} A_- (x_+, x_-) = 0,
\]

(57)

becomes (see Appendix A)

\[
\{ A_+ (u, w), x_- \}_{u, w} = \{ A_- (u, w), x_+ \}_{u, w},
\]

(58)

where \( A_\pm (u, w) = A_\pm (x_\pm (u, w), x_\mp (u, w)) \), and

\[
\{ X, Y \}_{u, w} = (\partial X/\partial u)(\partial Y/\partial w) - (\partial Y/\partial u)(\partial X/\partial w),
\]

denotes Poisson brackets with respect to \( u \) and \( w \). Introducing the potential \( \Phi(u, w) \), Eq.(58) can be rewritten as

\[
\{ A_+ (u, w), \partial_u \Phi \}_{u, w} = \{ A_- (u, w), \partial_w \Phi \}_{u, w}.
\]

(59)

Taking either \( A_+ (u, w) = T_{ww} \) and \( A_- (u, w) = T_{uw} \) or \( A(u, w) = T_{wu} \) and \( B(u, w) = T_{wu} \) as given by the expression of the energy-momentum tensor in Eqs.(23) we recover Eq.(56), here for the sake of simplicity we have set \( \epsilon_3 = 0 \). Finally we note that Eq.(59) can be rewritten as a conservation law in \( u-w \) space as

\[
\partial_w \left[ (\partial_u A_+) (\partial_u \Phi) - (\partial_u A_-) (\partial_w \Phi) \right] + \partial_u \left[ (\partial_w A_-) (\partial_u \Phi) - (\partial_w A_+) (\partial_u \Phi) \right] = 0.
\]

(60)

The conservation equation obtained by inserting the components of the energy-momentum tensor in Eqs.(23) into Eq.(60) is related to the invariance of Eq.(56) under the transformation

\[
\Phi(u, w) \to \Phi(u, w) + \delta_+ w + \delta_- u,
\]

(61)

which is the hodograph counterpart of the coordinate translations in Eq.(24). A similar procedure shows that the hodograph counterpart of the conservation of the “barycenter” that is given in Eq.(21) and that arises from the Lorentz invariance, yields a conserved quantity that is quadratic in \( \partial_u \Phi, \partial_w \Phi \), see later Eq.(81).

B. Hodograph transformation in the linear limit

In the linear limit, \( \epsilon_2, \epsilon_3 \to 0 \), Eq.(56) reduces to

\[
\frac{\partial^2 \Phi(u, w)}{\partial u \partial w} = 0, \quad \text{i.e.,} \quad \Phi(u, w) = \mathcal{U}(u) + \mathcal{W}(w).
\]

(62)

Here \( \mathcal{U}(u) \) and \( \mathcal{W}(w) \) correspond to counter-propagating non-interacting electromagnetic waves with

\[
x_- = \frac{\partial \Phi_w(u, w)}{\partial u} = \frac{\partial \mathcal{U}(u)}{\partial u}, \quad x_+ = \frac{\partial \Phi(u, w)}{\partial w} = \frac{\partial \mathcal{W}(w)}{\partial w}.
\]

(63)
The choice that corresponds to counterpropagating monochromatic waves is

\[ \mathcal{U}_{k_u}(u) = \int_0^u du' \left[ -\psi_u + \arcsin \left( u'/\mathcal{A}_u \right) \right]/k_u + \text{const}, \]

\[ \mathcal{W}_{k_w}(w) = \int_0^w dw' \left[ -\psi_w + \arcsin \left( w'/\mathcal{A}_w \right) \right]/k_w + \text{const}, \]

where \( \mathcal{A}_{w,u} \) are amplitudes, \( k_{w,u} \) “frequencies”, \( \psi_{u,w} \) are phases and

\[ \int_0^y dy' \arcsin (y') = y \arcsin y + (1 - y^2)^{1/2} - 1. \]

The definition domain is limited by \( |u/\mathcal{A}_u|, |w/\mathcal{A}_w| \leq 1 \). By properly extending the image domain of the arcsin function, Eqs. (64) can be inverted as

\[ u(x_-) = \mathcal{A}_u \sin (k_u x_- + \psi_u), \quad w(x_+) = \mathcal{A}_w \sin (k_w x_+ + \psi_w), \]

where the expressions inside each oscillation half-periods have been joined smoothly so as to cross over the points where the Jacobian of the hodograph transformation vanishes. By redefining the origin of \( x_{\pm} \) we can set \( \psi_u = \psi_w = 0 \) in agreement with Eq. (61).

In Appendix C the role of the nonlinearity in the inverse hodograph transformation in the case of the superposition of two co-propagating monochromatic solutions is illustrated.

C. Perturbative hodograph solutions

In analogy to the perturbative approach in \((x_+ - x_-)\) space we can search for solutions of Eq. (56) in the form of power series \( \Phi = \Phi_0 + \Phi_1 + \ldots \), where \( \Phi_0 \) satisfies Eq. (62). To the first order to small parameters \( \varepsilon_2 \) and \( \varepsilon_3 \) we obtain

\[ \partial_{ww} \Phi_1 = -u^2 (2 + 3 \xi_3 uw) \partial_{ww} \mathcal{W}(w) - u^2 (2 + 3 \xi_3 uw) \partial_{uu} \mathcal{U}(u), \]

which yields

\[ \Phi_1 = -\varepsilon_2 u \int^w (w')^2 \partial_{ww} \mathcal{W}(w') dw' - \frac{3}{2} \varepsilon_3 u^2 \int^w (w')^3 \partial_{ww} \mathcal{W}(w') dw' \]

\[ -\varepsilon_2 w \int^u (u')^2 \partial_{ww} \mathcal{U}(u') du' - \frac{3}{2} \varepsilon_3 w^2 \int^u (u')^3 \partial_{ww} \mathcal{U}(u') du'. \]

(67)

For the choice of \( \mathcal{W}(w) \) and \( \mathcal{U}(u) \) in Eq. (64) we obtain (for \( \varepsilon_3 = 0 \))

\[ \Phi_1 = -\varepsilon_2 \frac{w \mathcal{A}_u^2}{2k_u} \mathcal{P} \left( \frac{w}{\mathcal{A}_u} \right) - \varepsilon_2 \frac{w \mathcal{A}_w^2}{2k_w} \mathcal{P} \left( \frac{u}{\mathcal{A}_u} \right) + \text{const}, \]

(68)

where \( \mathcal{P}(y) = \arcsin(y) - y(1 - y^2)^{1/2} \). Inserting the zero-order solutions given in Eq. (65) into Eq. (68) and inverting the hodograph transformation we can obtain explicit expressions for \( u(x_+, x_-) \) and \( w(x_+, x_-) \). However, as noted above for the corresponding perturbative solutions in Eqs. (27) \( \pm \), these expressions include a term that exhibits a secular dependence on the \( x_+, x_- \) coordinates. A procedure analogous to the one adopted in Eq. (28) can be used to remove this secular behavior as sketched in Appendix C.

D. Lorentz invariant solutions

Equation (60) admits self-similar solution when the function \( \Phi \) depends on the Lorentz invariant variable \( \xi = uw \) only. These solutions are the hodograph counterpart of the solutions described by Eqs. (37) \( \pm \) \( \xi \) in \( x_+, x_- \) space. For the function \( \Phi(\xi) \) we obtain

\[ (1 - 4 \varepsilon_2 \xi - 9 \xi^2 \Phi' + \xi \Phi'') = -(2 \varepsilon_2 \xi^2 - 6 \xi^3 \Phi') \Phi'', \]

(69)

where \( \Phi' = d\Phi/d\xi \). Introducing the function \( U(\xi) = \Phi' \) Eq. (69) reduces to

\[ U' + \frac{1 - 4 \varepsilon_2 \xi - 9 \xi^2}{\xi(1 - 2 \varepsilon_2 \xi - 3 \xi^2)} U = 0. \]

(70)
Integration of this equation yields

$$U(uw) = \frac{C}{uw(1 - 2\epsilon_2 uw - 3\epsilon_3 u^2 w^2)}.$$  \hfill (71)

For coordinates $x_- = wU$ and $x_+ = uU$ we have

$$x_- = \frac{C}{u(1 - 2\epsilon_2 uw - 3\epsilon_3 u^2 w^2)} \quad \text{and} \quad x_+ = \frac{C}{w(1 - 2\epsilon_2 uw - 3\epsilon_3 u^2 w^2)},$$  \hfill (72)

which in the limit $\epsilon_2 = \epsilon_3 = 0$ coincide with Eq. (37). They can be rewritten as

$$x = \frac{2CB}{(E^2 - B^2)[1 + \epsilon_2(E^2 - B^2) - 3\epsilon_3(E^2 - B^2)^2/4]}$$  \hfill (73)

and

$$t = \frac{2CE}{(E^2 - B^2)[1 + \epsilon_2(E^2 - B^2) - 3\epsilon_3(E^2 - B^2)^2/4]}$$  \hfill (74)

The solution given by Eq. (71) describes two counter-propagating electromagnetic pulses with the electric and magnetic fields “cumulating” at the light cone $x^2 - t^2 = 0$ where the electric and magnetic fields tend to infinity. Note that the position where the cumulation occurs can be shifted by exploiting the translational invariance of the Lagrangian \( \mathcal{L} \), i.e. by looking for solutions (see Eq. (61)) of the form $\Phi(\xi)$.

For all these solutions the Poincaré invariant $\mathfrak{g} = F_{\mu\nu}F^{\mu\nu} = uw$ does not vanish for finite $x$ and $t$. In the case of solutions (72,73) the dependence $\mathfrak{g}$ on $t$ and $x$ is given by

$$x^2 - t^2 = \frac{-4C^2}{\mathfrak{g}(1 + \epsilon_2\mathfrak{g} - 3\epsilon_3\mathfrak{g}^2/4)^2}.$$  

In the vicinity of the lines given by condition $x^2 - t^2 = 0$ in the $(x,t)$ plane the expression (75) cannot be used because here the electromagnetic field amplitude exceeds the critical QED field $E_S$.

The Lorentz invariant solutions derived above represent a special case of solutions obtained by using the hyperbolic coordinates in hodograph space $\xi = uw$ and $\varphi = (1/2) \ln(u/w)$. These solutions are briefly discussed in Appendix \[13\]

### E. Standard form of the hodograph wave equation

The second order linear hyperbolic PDE given by Eq. (60) can be set in the standard form (see e.g., \[53\])

$$\frac{\partial^2 \Phi}{\partial \zeta \partial \theta} + \text{(lower order terms)} = 0,$$  \hfill (75)

by an appropriate redefinition of the independent variables $u$ and $w$. For the sake of simplicity in the following this transformation will be performed here up to linear terms in $\epsilon_2$ and for $\epsilon_3 = 0$. We define the new independent variables

$$\zeta = u(1 - \epsilon_2 uw), \quad \theta = w(1 - \epsilon_2 uw),$$

$$u = \zeta(1 + \epsilon_2 \zeta \theta), \quad w = \theta(1 + \epsilon_2 \zeta \theta),$$  \hfill (76)

and obtain (here and in the following only linear terms in $\epsilon_2$ will be retained)

$$\partial^2 \Phi \partial \zeta \partial \theta = 2\epsilon_2 \left( \zeta \frac{\partial \Phi}{\partial \zeta} + \theta \frac{\partial \Phi}{\partial \theta} \right) \left( 1 - 8\epsilon_2 \zeta \theta \right)^{-1} \sim 2\epsilon_2 \left( \zeta \frac{\partial \Phi}{\partial \zeta} + \theta \frac{\partial \Phi}{\partial \theta} \right).$$  \hfill (77)

Note that the field variables $\sqrt{2} \xi = (E - B)[1 - \epsilon_2(B^2 - E^2)]$ and $\sqrt{2} \theta = -(E + B)[1 - \epsilon_2(B^2 - E^2)]$ are directly related to the perturbed light cone variables $X_+, X_-$ defined in Eq. (31) since

$$\theta = \frac{\partial a}{\partial X_+} \quad \text{and} \quad \zeta = \frac{\partial a}{\partial X_-}.$$  \hfill (78)
Setting now $\Phi(\zeta, \theta) = \Phi_o(\zeta, \theta) (1 + 2\epsilon_2 \zeta \theta)$ we obtain (to first order) the constant coefficient hyperbolic PDE

$$\frac{\partial^2 \Phi_o(\zeta, \theta)}{\partial \zeta \partial \theta} = 2\epsilon_2 \Phi_o(\zeta, \theta),$$

(79)

which is isomorphic to the equation for linear transverse e.m. waves in a uniform plasma.

The solutions of Eq. (79) can be written in the general superposition form

$$\Phi_o(\zeta, \theta) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dk_\zeta dk_\theta \delta(k_\zeta k_\theta + 2\epsilon_2) \tilde{\Phi}_o(k_\zeta, k_\theta) \exp \left[ +i(k_\zeta \zeta + k_\theta \theta) \right] + \mathcal{C}\mathcal{C},$$

(80)

where the condition $\delta(k_\zeta k_\theta + 2\epsilon_2)$ accounts for the “dispersion” in Eq. (79) and $\mathcal{C}\mathcal{C}$ denotes complex conjugate. This dispersion in the hodograph equation can be traced back to the nonlinearity of the wave equation in $(x_+ - x_-)$ space.

1. Conservation equation

If we add the two equations that we derive by multiplying Eq. (79) by $\partial \Phi_o(\zeta, \theta)/\partial \zeta$ and by $\partial \Phi_o(\zeta, \theta)/\partial \theta$ respectively, we obtain the following conservation equation

$$\frac{\partial}{\partial \theta} \left[ \frac{1}{2} \left( \frac{\partial \Phi_o}{\partial \zeta} \right)^2 + \epsilon_2 \Phi_o^2 \right] + \frac{\partial}{\partial \zeta} \left[ \frac{1}{2} \left( \frac{\partial \Phi_o}{\partial \theta} \right)^2 + \epsilon_2 \Phi_o^2 \right] = 0,$$

(81)

which is quadratic in the function $\Phi_o(\zeta, \theta)$, and is related to the Lorentz invariance of the Lagrangian $\mathcal{L}$, see remark below Eq. (61).

V. CONCLUSIONS AND DISCUSSIONS

We have discussed within the framework of the Euler-Heisenberg Lagrangian the main features of the interaction in the quantum vacuum of counterpropagating electromagnetic fields. We have constructed explicit solutions of the nonlinear hyperbolic wave equation obtained from the Euler-Heisenberg Lagrangian within non-perturbative approach. We have used a combination of analytical methods, involving the direct search for solutions in space-time light cone coordinates and the use of the hodograph transformation. With the use of this transformation the role of the dependent and of the independent variables is interchanged and, in the restricted one-dimensional geometry considered here, the wave equation turns out to be a linear hyperbolic equation to which standard solution methods can be applied. When applying the hodograph transformation the nonlinearity of the Euler-Heisenberg Lagrangian shifts to the transformation itself which may be algebraically involved. In addition, in the case of oscillatory fields the implementation of the hodograph transformation requires the smooth joining of piece-wise contributions since the transformation is not globally invertible.

With these analytical methods we have constructed perturbative solutions and have identified exact selfsimilar solutions. The relationship between the properties of the solutions in both approaches has been discussed, with special attention to the different forms that conserved quantities take. These conservations arise from the translational invariance and from the Lorentz invariance of the Euler-Heisenberg Lagrangian.

We have shown that, in accordance with previous results in the literature, the interaction of two counter propagating pulses leads asymptotically only to a cumulative phase shift, a result that can be understood in terms of the energy and momentum conservation of massless particles in a head-on collision. On the contrary, during the interaction of two counterpropagating waves, the propagation velocity of each of them is reduced by a term that depends quadratically on the amplitude on the opposite propagating wave. The phase velocity of linear waves propagating in vacuum in the presence of large, steady and uniform electromagnetic fields (orthogonal to the direction of propagation) has been derived and shown to be smaller than the speed of light in vacuum, again by a term that, to leading order, depends on the square of the amplitudes of the steady electromagnetic fields.

Finally we note that the same analytical methods can be used to find solutions of the so called Born-Infeld Equation, see Ref. [9].
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Appendix A: The hodograph transformation in differential form and Euler-Heisenberg momenta

1. Momenta of the Euler-Heisenberg Lagrangian

We define the field momenta \( \Pi_u, \Pi_w \) in the standard way in terms of the Lagrangian \( \mathcal{L} \)

\[
\Pi_u = \frac{\partial \mathcal{L}}{\partial (\partial u/\partial x_-)} = \frac{\partial \mathcal{L}}{\partial u}, \quad \Pi_w = \frac{\partial \mathcal{L}}{\partial (\partial w/\partial x_+)} = \frac{\partial \mathcal{L}}{\partial w},
\]

and find

\[
\Pi_u = -\frac{w}{4\pi}(1 - 2\varepsilon_2uw - 3\varepsilon_3u^2w^2), \quad \Pi_w = -\frac{u}{4\pi}(1 - 2\varepsilon_2uw - 3\varepsilon_3u^2w^2).
\]

The equations of motion take the form

\[
\partial_{x_+}\Pi_w + \partial_{x_-}\Pi_u = 0,
\]

which leads to Eq. (18) in the main text.

2. Hodograph equations in differential form

Equations (17, 18), or equivalently Eqs. (A7, A8) can be written in the 2-form formalism as

\[
(\partial_{x_+}u) dx_+ \wedge dx_- - (\partial_{x_-}w) dx_+ \wedge dx_- = du \wedge dx_- + dw \wedge dx_+ = 0,
\]

\[
(\partial_{x_+}\Pi_w) dx_+ \wedge dx_- + (\partial_{x_-}\Pi_u) dx_+ \wedge dx_- = d\Pi_w \wedge dx_- - d\Pi_u \wedge dx_+ = 0.
\]

Taking \( u \) and \( w \) as independent variables in Eq. (A4) (assuming that the Jacobian of the transformation is different from zero) we obtain

\[
(\partial_w x_-) du \wedge dw - (\partial_u x_+) du \wedge dw = 0, \quad \Rightarrow \quad \partial_w x_- = \partial_u x_+,
\]

i.e. Eq. (53). Similarly, using \( \Pi_u, \Pi_w \) as the independent variables in Eq. (A5) we obtain

\[
\frac{\partial x_+}{\partial \Pi_w} + \frac{\partial x_-}{\partial \Pi_u} = 0,
\]

which leads to Eq. (54), after \( \Pi_u, \Pi_w \) are expressed in terms of \( u, w \) through Eq. (A2). Conversely, we can express \( u, w \) in terms of \( \Pi_u, \Pi_w \) and write the whole system of the hodograph equations in terms of the momenta \( \Pi_u, \Pi_w \).

Note that the hodograph transformation procedure described above is also applicable to the more general case with vector potential \( A = A(x, y, t) \). In this case however it would lead to nonlinear equations as can be easily seen e.g., by appropriately reformulating Eq. (A4) as a 3-form \((dx \wedge dt \rightarrow dx \wedge dy \wedge dt)\).

3. Conservations and Poisson Brackets

We can rewrite the conservation equation (58) in the differential form

\[
(\partial_{x_+}A_+) dx_+ \wedge dx_- + (\partial_{x_-}A_-) dx_+ \wedge dx_- = 0
\]

\[
\Rightarrow \quad dA_+ \wedge dx_- = dA_- \wedge dx_+
\]

and, imposing the hodograph transformation, we obtain

\[
[(\partial_u A_+) (\partial_u x_-) - (\partial_x x_-) (\partial_w A_+)] du \wedge dw = [(\partial_u A_-) (\partial_w x_+) - (\partial_x x_+) (\partial_w A_-)] du \wedge dw
\]

\[
\Rightarrow \quad \{A_+(u, w), x_-\}_{u, w} = \{A_-(u, w), x_+\}_{u, w}.
\]
4. Hodograph Lagrangian

We introduce an “effective vector potential” \( V(\Pi_+, \Pi_-) \) such that
\[
\frac{\partial a}{\partial x_+} = \frac{\partial V}{\partial \Pi_w}, \quad \frac{\partial a}{\partial x_-} = \frac{\partial V}{\partial \Pi_u}.
\] (A10)

Expressing \( w \) and \( u \) as functions of \( \Pi_\pm \) from Eqs.(A2) with \( \epsilon_3 = 0 \), we obtain
\[
\frac{w}{u} = \frac{\Pi_u}{\Pi_w}, \quad \Pi_u = -\frac{w}{4\pi} [1 - 2\epsilon_2(\Pi_w/\Pi_u)w^2], \quad \Pi_w = -\frac{u}{4\pi} [1 - 2\epsilon_2(\Pi_u/\Pi_w)u^2].
\] (A11)

A perturbative solution of these cubic equations gives
\[
w \sim -\frac{4\pi}{\Pi_u} (1 + 32\epsilon_2^2 \Pi_w/\Pi_u), \quad u \sim -\frac{4\pi}{\Pi_w} (1 + 32\epsilon_2^2 \Pi_u/\Pi_w),
\] (A12)

Using Eq.(A10) we can rewrite Eq.(53) as
\[
x = \frac{\partial a}{\partial w} = \frac{\partial \Phi}{\partial \Pi_w} V, \quad x = -\frac{\partial a}{\partial u} = \frac{\partial \Phi}{\partial \Pi_u} V,
\] (A13)

Finally Eq.(A7) becomes
\[
\frac{\partial}{\partial \Pi_w} \frac{\partial \Phi}{\partial \Pi_w} V + \frac{\partial}{\partial \Pi_u} \frac{\partial \Phi}{\partial \Pi_u} V = 0,
\] (A14)

where the unknown function \( \Phi(\partial \Pi_w, \partial \Pi_u) \) plays the role of the Lagrangian for the equations in the hodograph variables (see also Eq.(56)).

Appendix B: Hyperbolic coordinates

Instead of \( x_+ \) and \( x_- \) we can use the hyperbolic coordinates
\[
\rho = x_+ x_- = x^2 - t^2, \quad \psi = (1/2) \ln \frac{x_+/x_- + 1 + t/x_+}{x_+/x_- - 1+t/x_+} = \arctanh (t/x).
\] (B1)

Under an infinitesimal (finite) Lorentz transformation (see Eq.(20)) we have
\[
\rho \rightarrow \rho, \quad \psi \rightarrow \psi + \beta, \quad \left( \psi \rightarrow \psi + \frac{1}{2} \ln \frac{1+\beta}{1-\beta} = \psi + \arctanh (\beta) \right).
\] (B2)

1. Lagrangian in hyperbolic coordinates

Since the Heisenberg Lagrangian \( L \) is Lorentz invariant, when expressed in hyperbolic coordinates, it cannot depend explicitly on \( \psi \). Starting from the Action in \( x_+, x_- \) variables, bringing it to \( \rho, \psi \) variables ad using the fact that the Jacobian of the transformation is equal to one, the new Lagrangian (with \( \epsilon_3 = 0 \)) reads:
\[
(-4\pi) L_\rho(\rho, \psi) = \rho \left( \frac{\partial a}{\partial \rho} \right)^2 + \frac{1}{4\rho} \left( \frac{\partial a}{\partial \psi} \right)^2 - \epsilon_2 \left[ \rho \left( \frac{\partial a}{\partial \rho} \right)^2 + \frac{1}{4\rho} \left( \frac{\partial a}{\partial \psi} \right)^2 \right]^2.
\] (B3)

The self-similar solution Eq.(36) corresponds to \( \partial a/\partial \psi = 0 \) and can be derived directly from the Lagrangian \( L_\rho(\rho, \psi) \) in the convenient form given by Eq.(37). In the linear limit \( \epsilon_2 = 0 \) the Lagrangian \( L_\rho(\rho, \psi) \) can be expanded into “\( \psi \)-harmonics” and leads to power-law solutions. For \( \epsilon_2 \neq 0 \) these harmonics are coupled.
2. Hodograph equation in hyperbolic coordinates

In terms of the variables $\xi = uw$ and $\varphi = (1/2) \ln (u/w)$ Eq. (56) (with $\epsilon_3 = 0$) becomes

$$(1 - 4\epsilon_2 \xi) \left[ \frac{\partial}{\partial \xi} \left( \xi \frac{\partial \Phi}{\partial \xi} \right) - \frac{1}{4\xi} \frac{\partial^2 \Phi}{\partial \varphi^2} \right] + \epsilon_2 \left( 2\xi^2 \frac{\partial^2 \Phi}{\partial \xi^2} + \frac{1}{2} \frac{\partial^2 \Phi}{\partial \varphi^2} \right) = 0. \quad (B4)$$

Since Eq. (B4) is linear and its coefficients are independent of $\varphi$, its solutions be decomposed into a two sided Poisson expansion i.e., in $\cosh (\alpha \varphi)$ and $\sinh (\alpha \varphi)$ terms with $\alpha$ a real number. We obtain a family of ODEs that, with self evident notation, can be written as

$$(1 - 4\epsilon_2 \xi) \left[ \frac{\partial}{\partial \xi} \left( \xi \frac{\partial \Phi_\alpha}{\partial \xi} \right) - \frac{\alpha^2}{4\xi} \frac{\partial \Phi_\alpha}{\partial \xi} \right] + \epsilon_2 \left( 2\xi^2 \frac{\partial^2 \Phi_\alpha}{\partial \xi^2} + \frac{\alpha^2}{2} \frac{\partial \Phi_\alpha}{\partial \xi} \right) = 0. \quad (B5)$$

In the linear limit ($\epsilon_2 = 0$) the solutions of Eq. (B5) are of the form $\Phi = C_1 u^\alpha + C_2 w^\alpha$ and, for positive integer values of $\alpha$, can be used as a polynomial basis in the non-interaction limit.

Appendix C: Nonlinear inversion of hodograph solutions

As an illustration of the nonlinearity that is intrinsic to the inversion of the hodograph transformation we consider the superposition of two co-propagating waves (marked by the upper index 1 and 2) in the hodograph variables each of which would correspond separately to a monochromatic wave in agreement with Eq. (64):

$$U_{k_u^{(1)}, k_u^{(2)}}(u) = \int_0^u du' \left[ -\psi_u^{(1)}/k_u^{(1)} + \arcsin (u'/A_u) / k_u^{(1)} \right. \\
- \psi_u^{(2)}/k_u^{(2)} + \arcsin (u'/A_u) / k_u^{(2)} \left. \right] + \text{const.} \quad (C1)$$

Using the identity

$$\alpha \arcsin y = -i \ln \left[ i y + (1 - y^2)^{1/2} \right],$$

after some algebraic steps we obtain from Eq. (B5)

$$\exp \left[ i (k_u^{(1)})^{1/2} (x_+ + \psi_u^{(1)}/k_u^{(1)} + \psi_u^{(2)}/k_u^{(2)}) \right] = G(u), \quad (C2)$$

where

$$G(u) = \left[ iu/A_u \right]^{(1)} \left[ (1 - (u/A_u)^2)^{1/2} \right]^{(2)} \left[ iu/A_u + (1 - (u/A_u)^2)^{1/2} \right]^{(2)}$$

to be solved for $u = u(x_+) = G^{-1}(\exp \left[ i (k_u^{(1)})^{1/2} (x_+ + \psi_u^{(1)}/k_u^{(1)} + \psi_u^{(2)}/k_u^{(2)}) \right]).$

1. Renormalized hodograph solutions for interacting waves

In view of Eq. (28) we can rewrite Eq. (B4) as

$$U_{k_u}(u - \epsilon_2 u^2 w, \epsilon_2 w) = \int_0^{u - \epsilon_2 u^2 w} \frac{dv}{k_u} \left[ \frac{\arcsin \left( \frac{v}{A_u} \right)}{1 - 2\epsilon_2 v^2 w} \right] + \left. \text{const.} \right.,$$

$$\epsilon_2 S_u \left( \arcsin \left( \frac{u}{A_u} \right), \arcsin \left( \frac{w}{A_u w} \right) \right) + \text{const.},$$

$$W_{k_w}(w - \epsilon_2 w^2 u, \epsilon_2 u) = \int_0^{w - \epsilon_2 w^2 u} \frac{dv}{k_w} \left[ \frac{\arcsin \left( \frac{v}{A_w} \right)}{1 - 2\epsilon_2 v^2 u} \right] -$$

$$\epsilon_2 S_w \left( \arcsin \left( \frac{u}{A_u} \right), \arcsin \left( \frac{w}{A_w} \right) \right) + \text{const.} \quad (C3)$$
Equation (C3) can be inverted (to first order in $\epsilon_2$) as

$$u(x_-, \epsilon_2 x_+) = A_u \sin (k_w x_- + \epsilon_2 S_w (\arcsin (u/A_u), \arcsin (w/A_w))) =$$

$$= A_u \sin (k_w x_- + \epsilon_2 S_w (k_w x_-, k_w x_+)),$$

$$w(x_+, \epsilon_2 x_-) = A_w \sin (k_w x_+ + \epsilon_2 S_w (\arcsin (u/A_u), \arcsin (w/A_w))) =$$

$$= A_w \sin (k_w x_+ + \epsilon_2 S_w (k_w x_-, k_w x_+)).$$
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