Spaces of measurable functions

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Abstract: For a metrizable space $X$ and a finite measure space $(\Omega, \mathcal{M}, \mu)$, the space $M_\mu(X)$ of all equivalence classes (under the relation of equality almost everywhere mod $\mu$) of $\mathcal{M}$-measurable functions from $\Omega$ to $X$, whose images are separable, equipped with the topology of convergence in measure, and some of its subspaces are studied. In particular, it is shown that $M_\mu(X)$ is homeomorphic to a Hilbert space provided $\mu$ is (nonzero) nonatomic and $X$ is completely metrizable and has more than one point.

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1. Introduction

In [6] Bessaga and Pełczyński proved that whenever $X$ is a separable completely metrizable topological space having more than one point, then the space $M(X)$ of Borel functions from $[0, 1]$ to $X$ with the topology of convergence in measure is homeomorphic to $\ell_2$. Later it turned out that the topology of $\ell_2$ can be well characterized. This was done by Toruńczyk [17, 18]. After publication of these papers the number of results on spaces homeomorphic to the separable infinite-dimensional Hilbert space has highly risen. Among others, Dobrowolski and Toruńczyk [8] showed that every separable completely metrizable non-locally compact topological group which is an AR is homeomorphic to a Hilbert space. However, the problem whether the assumption of separability in this result may be omitted is still open, see [5].

In the present paper we shall introduce a class of nonseparable completely metrizable topological groups which are homeomorphic to Hilbert spaces. Namely, if $G$ is any (nonzero) completely metrizable topological group and $\mu$ is a (nonzero) finite nonatomic measure (on some set), then the space $M_\mu(G)$ (defined in the abstract) has a natural structure (induced by the one of $G$) of a topological group and is homeomorphic to a Hilbert space. In fact we shall prove the following, quite more general, result: if $X$ is a nonempty metrizable space, $\mu$ is a finite nonatomic measure and $Y = M_\mu(X)$

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is the subspace of $M_{\nu}(X)$ consisting of all (equivalence classes of) functions whose images are contained in $\sigma$-compact subsets of $X$, then $Y$ is an absolute retract such that $Y^w \cong Y$. Since infinite-dimensional Hilbert spaces are the only completely metrizable noncompact AR's homeomorphic to their own countable infinite Cartesian powers [17], the above result may be seen as a generalization of earlier results of Bessaga and Petczynski [6] as well as of Toruńczyk [16]. Our result is a simple consequence of a relatively recent result of Banakh and Bessaga [3] on the Hartman–Mycielski construction [10] which nowadays is a useful tool in topology as well as in topological algebra. Our concept of enlarging a metrizable space $X$ to the space of measurable $X$-valued functions may be seen as a generalization (or extension) of the original concept by Hartman and Mycielski.

The other purpose of the paper is to present the idea of extending maps between metrizable spaces to maps between AR's via functors. Namely, whenever $\mu$ is a finite (nonzero) nonatomic measure, every map $f : X \to Y$ has a natural extension $M_\mu(f) : M_\mu(X) \to M_\mu(Y)$. What is more, the correspondence $f \mapsto M_\mu(f)$ preserves many properties (such as: being an injection, an embedding, a map with dense image).

One more issue that we shall discuss in the paper concerns the question of whether $M_\mu(M_\nu(X))$ is homeomorphic to $M_\nu(X)$. We shall see that the answer is affirmative for a huge class of metrizable spaces (namely, for spaces in which every closed separable subset is absolutely measurable), which contains locally absolutely Borel spaces and (separable) Suslin ones. However, in general we leave this question as an open problem.

We hope this article will help in understanding difficulties which appear when one deals with measurable functions which take values in incomplete spaces (especially in those not absolutely measurable).

The article is organized as follows. In Section 2 we establish notation and terminology, define general spaces of measurable functions and collect fundamental results on them. Most of properties established in this section are stated without proofs which are left to the reader. Section 3 deals with spaces $M_\mu(X)$, defined above. We show there that if $\mu$ and $\nu$ are two homogeneous (nonatomic) measures of the same weight, then the spaces $M_\mu(X)$ and $M_\nu(X)$ are naturally homeomorphic, whatever $X$ is. The fourth part is devoted to spaces of measurable functions over metrizable AM-spaces (i.e. in which every closed separable subset is absolutely measurable). We prove there that if $X$ is an AM-space, then $M_\mu(X) = M_\nu(X)$ for each finite measure $\mu$. We conclude that such spaces over completely metrizable ones are homeomorphic to Hilbert spaces. In the last part we generalize our results from [13] to nonseparable case. Also the idea of extending maps to AR's via the functors $M_\mu$ is presented.

## 2. Preliminaries

In this paper $\mathbb{R}_+$ and $\mathbb{N}$ denote the sets of nonnegative reals and integers, respectively, $I = [0, 1]$ and $m$ stands for the Lebesgue measure on $I$. If $g$ is any function, $\text{im} \ g$ stands for the image of $g$. If, in addition, $g$ takes values in a topological space, $\overline{\text{im}} \ g$ denotes the closure of $\text{im} \ g$ in the whole space. The weight of a topological space $X$ is denoted by $w(X)$ and is understood as an infinite cardinal number (i.e. $w(X) = \aleph_0$ for finite $X$). All topological spaces which appear in the paper are metrizable and all measures are nonatomic, finite and nonzero. For topological spaces $Y$ and $Z$ we shall write $Y \cong Z$ iff $Y$ and $Z$ are homeomorphic. By a map we mean a continuous function. If $X$ is a metrizable space, $X^w$ stands for the countable infinite Cartesian power of $X$, equipped with the Tychonoff topology, and $\text{Metr} \ X$ denotes the family of all bounded metrics on $X$ which induce the given topology of $X$. $\mathfrak{B}(X)$ stands for the $\sigma$-algebra of all Borel subsets of $X$, that is, $\mathfrak{B}(X)$ is the smallest $\sigma$-algebra containing all open subsets of $X$. If $(\Omega_1 \times \Omega_2, \mathfrak{M}, \mu)$ is the product space of measure spaces $(\Omega_1, \mathfrak{M}_1, \mu_1)$ and $(\Omega_2, \mathfrak{M}_2, \mu_2)$, then we shall write $\mathfrak{M}_1 \otimes \mathfrak{M}_2$ and $\mu_1 \otimes \mu_2$ for $\mathfrak{M}$ and $\mu$, respectively.

Whenever $(\Omega, \mathfrak{M})$ is a measurable space and $X$ is a metrizable space, a function $f : \Omega \to X$ is $\mathfrak{M}$-measurable, if $f^{-1}(U) \in \mathfrak{M}$ for each open subset $U$ of $X$. Sets being members of $\mathfrak{M}$ are said to be measurable. We use standard terminology and ideas of measure theory. For details the reader is referred e.g. to [9]. For example, every measurable function $f : \Omega \to X$ with separable image defined on a measure space $(\Omega, \mathfrak{M}, \mu)$ will be identified with its equivalence class (in the set of all measurable functions $\Omega \to X$ with separable images) with respect to the relation of almost everywhere equality mod $\mu$. The subject of our study is introduced below.
**Definition 2.1.**

Let \( (\Omega, \mathcal{M}, \mu) \) be a finite measure space and \( X \) be a metrizable topological space. The space of measurable functions on \( \Omega \) over \( X \) is the set \( M_p(\Omega) \) of all (equivalence classes of all) measurable functions from \( \Omega \) to \( X \) whose images are separable. The set \( M_p(\Omega) \) and all its subsets are equipped with the topology of convergence in measure. In other words, a sequence \( (f_n) \) of elements of \( M_p(\Omega) \) converges to \( f \in M_p(\Omega) \) if, after fixing \( d \in \text{Metr} X \), \( \mu(\{\omega \in \Omega : d(f_n(\omega), f(\omega)) > \varepsilon\}) \to 0 \), \( n \to \infty \), for each \( \varepsilon > 0 \), see e.g. [9, § 22]. Equivalently, the sequence \( (f_n) \) converges in measure to \( f \) iff every its subsequence contains a subsequence \( (f_{n_k}) \) such that \( f_{n_k}(\omega) \to f(\omega) \), \( n_k \to \infty \), for \( \mu \)-almost all \( \omega \in \Omega \). It is well known, see e.g. [6], that if \( q \in \text{Metr} X \), then \( M_p(\Omega) \in \text{Metr}(M_p(\Omega)) \), where \( M_p(\Omega)(f, g) = \int q(f(\omega), g(\omega)) \, d\mu(\omega) \). The subfamilies of \( M_p(\Omega) \) consisting of all measurable functions whose images are, respectively, finite, (at most) countable and contained in \( \sigma \)-compact subsets of \( X \) are denoted by \( M_p^\circ(\Omega) \), \( M_p^c(\Omega) \) and \( M_p^f(\Omega) \).

We call a measure \( \mu \) nonatomic if for every \( B \in \mathcal{M} \) of positive \( \mu \)-measure there is a subset \( A \in \mathcal{M} \) of \( B \) with \( 0 < \mu(A) < \mu(B) \). Otherwise \( \mu \) is said to have atoms. The following elementary fact is of great importance for our further investigations.

**Lemma 2.2.**

Let \( \mu \) be a finite (nonzero) nonatomic measure on \( \mathcal{M} \). There is a collection \( \{A_t\}_{t \in I} \) of measurable sets such that for any \( s, t \in I \),

\[
A_s \subset A_t \quad \text{provided} \quad s < t, \tag{1}
\]

and \( \mu(A_t) = t\mu(\Omega) \).

**Proof.** Since this result is classical, we shall give here only a sketch of the proof. Since \( \mu \) is nonatomic, for every \( A \in \mathcal{M} \) there is \( B \in \mathcal{M} \) such that \( B \subset A \) and \( \mu(B) = \mu(A)/2 \). Taking this into account, construct (using induction) a family \( \{A_t\}_{t \in I_0} \) of measurable sets such that \( I_0 = \{k/2^n : 0 \leq k \leq 2^n, n \geq 1\} \), \( \mu(A_t) = s_j \mu(\Omega) \) for each \( s \in I_0 \) and (1) is satisfied for all \( s, t \in I_0 \). Now for any \( t \in I \setminus I_0 \) just put \( A_t = \bigcup_{s \in I_0, s < t} A_s \).

**Proposition 2.3.**

Let \( \mu \) be a finite (nonzero) nonatomic measure on \( (\Omega, \mathcal{M}) \) having atoms. There are \( \Omega_0 \in \mathcal{M} \) and \( p \in \{1, 2, \ldots, \omega\} \) such that the measure \( \mu_0 = \mu|_{\Omega_0} \) is nonatomic and for every metrizable space \( X \), \( M_p(\Omega) \cong X^p \times M_{\mu_0}(X) \).

**Proof.** Call a set \( A \in \mathcal{M} \) an atom if \( \mu(A) > 0 \) and \( \mu(B) \in \{0, \mu(A)\} \) for every measurable set \( B \subset A \). Two atoms \( A \) and \( B \) are either \( \mu \)-disjoint (that is, \( \mu(A \cap B) = 0 \)) or equal modulo \( \mu \) (i.e. \( \mu(A \setminus B) = \mu(B \setminus A) = 0 \)). Since \( \mu \) is finite, there is a countable (finite or not) maximal family \( \mathcal{A} \) of pairwise disjoint atoms in \( \mathcal{M} \). Then the set \( \Omega_0 = \Omega \setminus \bigcup \mathcal{A} \) is a member of \( \mathcal{M} \) and the measure \( \mu_0 = \mu|_{\Omega_0} \) is nonatomic. Additionally, put \( \Omega_d = \bigcup \mathcal{A} \in \mathcal{M} \), \( \mu_d = \mu|_{\Omega_d} \); and \( p = \text{card} \mathcal{A} \) if \( \mathcal{A} \) is finite and \( p = \omega \) otherwise. Note that the function \( M_{\mu}(\Omega) \) \( f \mapsto (f|_{\Omega_d}, f|_{\Omega_d}) \in M_{\mu_0}(X) \times M_{\mu_0}(X) \) is a homeomorphism for every metrizable space \( X \). Consequently, it suffices to show that \( M_{\mu_0}(X) \cong \mathbb{R}^p \) for any \( X \). But this is immediate: if \( A = \{A_j : j \geq 0, j < p\} \) and \( f : \Omega_d \to X \) is \( \mu \)-measurable, then \( f \) is almost constant on each \( A_j \), which gives the assertion.

Whenever \( \mu \) is a finite measure on a \( \sigma \)-algebra \( \mathcal{M} \), the function \( d_\mu : \mathcal{M} \times \mathcal{M} \ni (A, B) \mapsto \mu(A \setminus B) \cup (B \setminus A) \in \mathbb{R}^+ \) is a pseudometric. The Boolean \( \sigma \)-algebra associated with the measure \( \mu \), denoted by \( \mathcal{B} (\mu) \), cf. [9, § 40], is the quotient space \( \mathcal{M}/(d_\mu = 0) \) equipped with the metric naturally induced by \( d_\mu \). The weight of \( \mathcal{B} (\mu) \) is called by us the weight of \( \mu \) and is denoted by \( w(\mu) \).

A starting point for our investigations is the following

**Theorem 2.4.**

For every metrizable space \( X \) having more than one point and a finite nonatomic measure space \( (\Omega, \mathcal{M}, \mu) \), the space \( N_p(\Omega) \) is an absolute retract for \( N = M, M^c, M^f \) or \( M^r \). Moreover, \( w(N_p(X)) = \max (w(\mu), w(X)) \).
Proof. Let \( Y = N_\mu(X) \). Let \( HM(Y) \subset M(X) \) be the *Harman–Mycielski space over \( Y \) [3, 10]. That is, \( HM(Y) \) consists of all functions \( f : I \to Y \) for which there are numbers \( 0 = t_0 < \ldots < t_p = 1 \) and points \( y_1, \ldots, y_p \in Y \) such that \( f([t_{j-1}, t_j)) = \{y_j\} \). It follows from the results of [3] that \( HM(Y) \) is an absolute retract. According to that, to prove that \( Y \) is an AR as well, it suffices to show that \( f \in HM(Y) \). In what follows we identify elements of \( Y \) with constant functions belonging to \( HM(Y) \). Let \( \{A_i\}_{i \in I} \) be as in Lemma 2.2. Observe that for every \( F \in HM(Y) \) there is a finite system \( 0 = t_0 < \ldots < t_p = 1 \) and functions \( f_1, \ldots, f_p \in Y \) such that

\[
F(x) = f_k \quad \text{whenever} \quad t_{k-1} < x < t_k, \quad k = 1, \ldots, p.
\]

We define \( r : HM(Y) \to Y \) by \( r(F)(\omega) = f_k(\omega) \) for \( \omega \in A_i \backslash A_{i-1} \), and \( k = 1, \ldots, p \), provided (2) is satisfied for \( F \in HM(Y) \). First of all, notice that the formula for \( r(F) \) is independent of the choice of a system \( t_0, \ldots, t_p \) fulfilling (2). Consequently, \( r(F) = F \) for \( F \in Y \). Finally, fix \( d \) in Metr \( X \), put \( \varrho = M_\mu(d) \) and let \( \varrho \) be the metric on \( HM(Y) \) induced by \( \varrho \). For \( F, G \in HM(Y) \) there is a common system \( 0 = t_0 < \ldots < t_p = 1 \) such that \( F(x) = f_k \in Y \) and \( G(x) = g_k \in Y \) for \( x \in [t_{k-1}, t_k) \). But then

\[
\varrho(r(F), r(G)) \leq \sum_{k=1}^p \varrho(f_k, g_k) (t_k - t_{k-1}) \mu(\Omega) = \mu(\Omega) \varrho(F, G)
\]

which implies that \( r \) is continuous.

To convince ourselves that \( w(N_\mu(X)) = \max\{w(\mu), w(X)\} \), first observe that \( N_\mu(X) \) contains subsets homeomorphic to \( X \) as well as to \( \mathcal{A}(\mu) \). (Indeed, constant functions on \( \Omega \) form a set homeomorphic to \( X \); and all functions whose images are contained in a fixed two-point subset of \( X \) form a set homeomorphic to \( \mathcal{A}(\mu) \).) Finally, if \( \mathcal{R} \subset \mathcal{M} \) is an alphabet of subsets of \( \Omega \) which is dense in \( \mathcal{A}(\mu) \) and has size equal to \( w(\mu) \), and \( D \) is a dense subset of \( \mathcal{R} \) such that \( \text{card } D = w(X) \), then the set of all measurable functions \( \nu : \Omega \to D \) such that \( w(\Omega) \) finite and \( \nu^{-1}(\{z\}) \in \mathcal{R} \) for any \( z \in D \) is dense in \( \mathcal{M}_\nu(X) \).

Since the last set is dense in \( M_\nu(X) \), we conclude that \( w(M_\nu(X)) \leq \max\{w(\mu), w(X)\} \).

Our next aim is to prove that if \( (\Omega_j, \mathcal{R}_j, \nu_j), j = 1, 2 \), are two measure spaces, then there is a measure space \( (\Omega, \mathcal{R}, \nu) \) such that \( M_\nu(M_\nu(X)) \) is naturally homeomorphic to \( M_\nu(X) \) for each metrizable space \( X \). To this end, let \( \Omega = \Omega_1 \times \Omega_2 \) and \( \pi : \Omega \to \Omega_2 \) be the natural projection. Let \( \mathcal{R} \) be the \( \sigma \)-algebra of all subsets \( A \) of \( \Omega \) such that \( \pi(A \cap \{(\omega_1, x) \mid \omega_1 \in \Omega_1\}) \in \mathcal{R}_2 \) for each \( \omega_1 \in \Omega_1 \) and the function \( \Omega_1 \ni \omega_1 \mapsto \pi(A \cap \{(\omega_1, x) \mid \omega_1 \in \Omega_1\}) \in \mathcal{R}_2 \) is \( \mathcal{R}_1 \)-measurable and its image is separable. Finally, let \( \nu : \mathcal{R} \to \mathbb{R}_+ \) be given by \( \nu(A) = \int_{\Omega_1} \nu_2(\pi(A \cap \{(\omega_1, x) \mid \omega_1 \in \Omega_1\})) \, d\nu_1(\omega_1) \). It is easy to see that \( \mathcal{R} \) is indeed a \( \sigma \)-algebra and that \( \nu \) is a finite measure on \( \Omega \). Note also that \( \mathcal{R}_1 \otimes \mathcal{R}_2 \subset \mathcal{R} \) and \( \nu \) extends \( \nu_1 \otimes \nu_2 \). We call \( \nu \) the directed product of \( \nu_1 \) and \( \nu_2 \). The \( \sigma \)-algebra \( \mathcal{R} \) and the measure \( \nu \) defined above will be denoted by \( \mathcal{R}_1 \otimes \mathcal{R}_2 \) and \( \nu_1 \otimes \nu_2 \), respectively. We have

**Theorem 2.5.**

Let \( (\Omega_j, \mathcal{R}_j, \nu_j), j = 1, 2 \), be finite measure spaces and \( (\Omega, \mathcal{R}, \nu) = (\Omega_1 \times \Omega_2, \mathcal{R}_1 \otimes \mathcal{R}_2, \nu_1 \otimes \nu_2) \). Let \( (X, d) \) be a bounded metric space and

\[
\Lambda : (M_\nu(X), M_\nu(d)) \to (M_\nu(M_\nu(X)), M_\nu(M_\nu(d)))
\]

be given by the formula \( \Lambda(\nu_1)(\omega_1) = f(\omega_1, \omega_2) \). Then \( \Lambda \) is a well-defined (bijective) isometry.

Proof. To show that \( \text{im } \Lambda \subset M_\nu(M_\nu(X)) \), use the fact that if \( f : \Omega \to X \) is \( \mathcal{A} \)-measurable and \( \text{im } f \) is separable, then there is a sequence of \( \mathcal{A} \)-measurable functions \( f_n : \Omega \to X \) with finite images such that \( \lim_{n \to \infty} f_n(\omega) = f(\omega) \) for each \( \omega \in \Omega \). Further, a direct calculation shows that \( \Lambda \) is isometric. To see the surjectivity, fix an \( \mathcal{A}_1 \)-measurable function \( g : \Omega_1 \to M_\nu(X) \) with separable image. Let \( \overline{X} \) be the completion of \( X \) with respect to \( d \). Since \( M_\nu(X) \) is dense in \( M_\nu(X) \), there is a sequence of \( \mathcal{A}_1 \)-measurable functions \( g_n : \Omega_1 \to M_\nu(X) \) with finite images such that \( \lim_{n \to \infty} g_n(\omega_1) = g(\omega_1) \) for each \( \omega_1 \in \Omega_1 \). It is easy to check that for each \( n \) there is an \( \mathcal{A}_1 \)-measurable function \( f_n : \Omega \to X \) whose image is finite and such that \( f_n(\omega_1, \cdot) \) and \( g_n(\omega_1) \) coincide in \( M_\nu(X) \) for every \( \omega_1 \in \Omega_1 \). Thus (since \( \Lambda \) is isometric), \( (f_n)_n \) is a fundamental sequence in \( M_\nu(\overline{X}) \). This means that there is an \( \mathcal{A} \)-measurable function \( f : \Omega \to \overline{X} \) with separable image which is the limit of \( (f_n)_n \) in \( M_\nu(\overline{X}) \). We conclude
that $\overline{A} = g$, where $\overline{A}$ is the suitable map $\mathcal{A}$ for $X$. So, the set $A' = \{\omega \in \Omega_1 : f(\omega) \neq g(\omega)\}$ belongs to $\mathcal{A}_1$ and $\mathcal{A}(A') = 0$. Now fix $\omega \in \Omega_1 \setminus A'$. Let $h: \Omega_2 \to X$ be an $\mathcal{N}_2$-measurable function with separable image which coincides with $g(\omega)$ in $\mathcal{N}_2(\Omega_2)$. Then the set $A_\omega = \{\omega_2 \in \Omega_2 : f(\omega_2, \omega) \neq h(\omega_2)\}$ belongs to $\mathcal{N}_2$ and $\mathcal{A}(A_\omega) = 0$. Finally, put

$$A = (A' \times \Omega_2) \cup \bigcup_{\omega \in \Omega_1 \setminus \Omega_1' \setminus A'} \{\omega_1 \times (\omega_2 \setminus A_\omega)\} \subset \Omega$$

and let $f_+: \Omega \to X$ be such that $f_+|_A = f|_A$ and $f_+|_{\Omega \setminus A} \equiv b$, where $b$ is a fixed element of $X$. By the construction, $A \in \mathcal{N}$, $f_+ \in \mathcal{M}(X)$, and $\Lambda(f_+) = g$.

Now we shall give a sufficient condition (on a measure $\mu$) under which the space $Y = M_\mu(X)$ is homeomorphic to $Y^\omega$ (for each $X$). To formulate it, we need an additional notion. We say that two measure spaces $(\Omega_1, \mathcal{M}_1, \mu_1)$ and $(\Omega_2, \mathcal{M}_2, \mu_2)$ are pointwise isomorphic if there is a bijection $\psi: \Omega_1 \to \Omega_2$ such that for any $A \subset \Omega_1$, $\psi(A) \in \mathcal{M}_2$ iff $A \in \mathcal{M}_1$ and $\mu_2(\psi(A)) = \mu_1(A)$ for every $A \in \mathcal{M}_1$. In such situation $\psi$ is called an isomorphism. These spaces are said to be almost pointwise isomorphic if there are sets $A_1 \in \mathcal{M}_1$ and $A_2 \in \mathcal{M}_2$ such that $\mu_1(A_1) = 0$, $\mu_2(A_2) = 0$, and the spaces $(\{A_1, \mathcal{M}_{\setminus A_1}, \mu_{|\setminus A_1}(A_1)\})$ and $(\{A_2, \mathcal{M}_{\setminus A_2}, \mu_{|\setminus A_2}(A_2)\})$ are pointwise isomorphic. It is clear that every isomorphism $\varphi : \Omega_1 \to \Omega_2$ induces isometries $(\mathcal{M}_{\varphi}(X), \mathcal{M}_{\varphi}(d)) \ni f \mapsto f \circ \varphi^{-1} \in (\mathcal{M}_\mu(X), \mathcal{M}_\mu(d))$ for any $X$ and $d \in \text{Metr} X$ (the same for $M^{\text{f}}, M^{\text{c}}$, and $M'$-spaces). We now have

**Proposition 2.6.**

If there is a measurable set $A$ such that $0 < \mu(A) < \mu(\Omega)$ and the spaces $(\Omega, \mathcal{N}, \mu/\mu(\Omega))$ and $(\{A, \mathcal{M}_{\setminus A}, \mu_{|\setminus A}(A)\})$ are almost pointwise isomorphic, then $M_\mu(X) \cong [M_\mu(X)]^\omega$ for each metrizable space $X$.

**Proof.** First of all observe that we may assume $\Omega$ and $A$ are pointwise isomorphic. Let $\varphi : \Omega \to A$ be an isomorphism. For a moment we will think of $\varphi$ as of a function from $\Omega$ to $A$. Let $B_0 = \Omega \setminus A$ and $B_n = \varphi^n(B_0)$, $n \geq 1$, where $\varphi^n$ denotes the $n$-th iterate of $\varphi$. Note that $\{B_n\}_{n=0}^\omega$ is a partition of $B = \bigcup_{n=0}^\omega B_n$. What is more, $\mu(\Omega \setminus B) = \mu(B) = 0$. Therefore, again, we may assume that $B = \Omega$. Since $\varphi(B_0) = B_{n+1}$, all the spaces $(\mathcal{M}_{\varphi}(B_n), \mu_{|B_n}(\varphi))$ are pointwise isomorphic. Take a bijection $\kappa : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ and for each $n, l \in \mathbb{N}$ let $\psi_{n,l} : B_n \to B_{\kappa(n,l)}$ be an isomorphism. Finally, for a metrizable space $X$ put

$$h : M_\mu(X) \ni f \mapsto \left(\bigcup_{n=0}^\infty \left(f|_{B_{\kappa(n,l)}} \circ \psi_{l,n}\right)\right)_{l=0}^\infty \in M_\mu(X)^\omega.$$

We leave it as a simple exercise that $h$ is a homeomorphism. }

The above result will be applied in Section 3.

### 3. $M'$-spaces

Let us begin with some notions and a classical result on Boolean $\sigma$-algebras associated with measures. Two Boolean $\sigma$-algebras $\mathfrak{A}(\mu)$ and $\mathfrak{A}(\nu)$ are isomorphic if there is a bijection $\Phi : \mathfrak{A}(\mu) \to \mathfrak{A}(\nu)$ such that $\Phi(\bigcup_{n=0}^\infty A_n) = \bigcup_{n=0}^\infty \Phi(A_n)$ and $\Phi(A_1 \setminus A_2) = \Phi(A_1) \setminus \Phi(A_2)$ for any (equivalence classes of) $A_1, A_2, \ldots \in \mathfrak{A}$. A nonatomic measure $\mu$ is homogeneous if for each $B \in \mathfrak{A}$ of positive $\mu$-measure, $w(\mu) = w(\mu|_B)$, where $|_B : \mu|_B = \mu(\mathfrak{A}_B)$ is a measure on $B$ and $\mathfrak{A}_B = \{A \in \mathfrak{A} : A \subset B\}$. Fix an infinite cardinal number $\alpha$. Each of the sets $\mathfrak{A}'$, where $f$ is countable (finite), will be equipped with the Tychonoff topology. Let $T$ be a set of cardinality $\alpha$. Let $\Omega_\alpha = \{1\} (= I^\alpha)$ and $\Omega_\alpha$ be the $\sigma$-algebra of all subsets $B$ of $\Omega_\alpha$ for which there are a countable infinite set $J \subset T$ and $B_j \in \mathfrak{B}(I)$ such that $B = \{x_j, (j) \in J \in B_j\}$. In other words, $\Omega_\alpha$ is the product of $\alpha$ copies of $\mathfrak{B}(I)$. (Note also that in $\Omega_\alpha$ with the Tychonoff topology not every open subset of $\Omega_\alpha$ is a member of $\mathfrak{A}_\alpha$. Measurable open sets are exactly those which are $\mathcal{F}_T$. Finally, let $m_\alpha : \Omega_\alpha \to I$ be the product measure of $\alpha$ copies of the Lebesgue measure $m$ on $I$. (More on products of probabilistic measures the reader may find in [9, § 38]; especially notes on p. 158 there are helpful.) The following deep result due to Maharam [12] will be applied in the sequel.
Theorem 3.1.
Let \((\Omega, \mathcal{M})\) be a measurable space and \(\mu\) be a nonatomic measure on \(\mathcal{M}\). There are a countable (finite or not) collection \(\{B_j\}_j \in \mathbb{N}\), of pairwise disjoint measurable sets in \(\Omega\) of positive \(\mu\)-measure and a corresponding set of distinct infinite cardinals \(\{\alpha_j\}_j \in \mathbb{N}\), such that \(\Omega = \bigcup_{j \in \mathbb{N}} B_j\) and for each \(j \in \mathbb{N}\) the Boolean -\(\sigma\)-algebras \(\mathcal{A}(\mu|_{B_j})\) and \(\mathcal{A}(\alpha_j)\) are isometrically isomorphic.

Maharam’s work shows that each of the measures \(m_\alpha\) is homogeneous. We need to know a little bit more about the space \((\Omega, \mathcal{M}_\alpha, m_\alpha)\). But first a few necessary definitions.

A Polish space is a separable completely metrizable one. A subset \(B\) of a Polish space \(Y\) is said to be absolutely measurable in \(Y\) if for every probabilistic Borel measure \(\mu\) on \(Y\) there are two Borel subsets \(A\) and \(C\) of \(Y\) such that \(A \subset B \subset C\) and \(\mu(C \setminus A) = 0\). A separable metrizable space \(X\) is absolutely measurable, if for every embedding \(\varphi\) of \(X\) into the Hilbert cube \(Q\), \(\varphi(X)\) is absolutely measurable in \(Q\). Equivalently, \(X\) is absolutely metrizable if there is a \(\sigma\)-compact subset of the whole space. This implies that every finite Borel measure on a (separable) absolutely measurable space is supported on a \(\sigma\)-compact set.

All the above facts yield the following result.

Lemma 3.2.
Let \((\Omega, \mathcal{M}, \mu)\) be a finite measure space and \(X\) be a metrizable space.

(A) If the image of an \(\mathcal{M}\)-measurable function \(f : \Omega \to X\) is contained in a separable absolutely measurable subset of \(X\), then \(f \in M'_\mu(X)\).

(B) If \(\mathcal{R}\) is a \(\sigma\)-subalgebra of \(\mathcal{M}\), \(\nu = \mu|_{\mathcal{R}}\) and a function \(f \in M'_\mu(X)\) belongs to the closure of \(M_\nu(X)\), then \(f \in M'_\nu(X)\), i.e. there is an \(\mathcal{R}\)-measurable function \(g : \Omega \to X\) whose image is separable and which is \(\mu\)-almost everywhere equal to \(f\). In particular, \(M'_\nu(X)\) is closed in \(M'_\mu(X)\).

Proof. (A): Let \(A \subset X\) be a separable absolutely measurable superset of \(\text{im} f\). Let \(\lambda : \mathcal{B}(A) \ni B \mapsto \mu(f^{-1}(B)) \in \mathbb{R}_+\). Since \(\lambda\) is a finite measure, there is a \(\sigma\)-compact subset \(K\) of \(A\) such that \(\lambda(K) = \lambda(A)\). So, \(\mu(U) = \mu(\Omega)\) for \(U = f^{-1}(K)\). Then \(f\) coincides with \(f' \in M'_\mu(X)\) in \(M_\nu(X)\), where \(f'|_U = f'|_U\) and \(f'|_{\Omega \setminus U} \equiv b\) with \(b\) taken from \(K\).

(B): We only need to prove the first claim. We may assume that the image of \(f\) is contained in a \(\sigma\)-compact subset of \(X\), say \(K_0\). By the assumption, there is a sequence of \(\mathcal{R}\)-measurable functions \(f_n : \Omega \to X\) with finite images which is pointwise convergent \(\mu\)-almost everywhere to \(f\). Let \(K = K_0 \cup \bigcup_n \text{im} f_n\). Fix \(d \in \text{Metr} K\) and let \(\bar{K}\) be the completion of \((K, d)\). Note that \(K\) is \(\sigma\)-compact and therefore it is a Borel subset of \(\bar{K}\). Let \(B\) be the set of all those \(\omega \in \Omega\) such that the sequence \(\langle f_n(\omega)\rangle_n\) is convergent in \(\bar{K}\). Since \(f_n\)'s are \(\mathcal{R}\)-measurable, \(B \in \mathcal{R}\). What is more, \(\mu(\Omega \setminus B) = 0\). Thus, after changing each \(f_n\) so that \(f_n|_{\Omega \setminus B} \equiv b\), where \(b \in K\), there is an \(\mathcal{R}\)-measurable function \(\overline{f} : \Omega \to \bar{K}\) such that \(\lim_{n \to \infty} f_n(\omega) = \overline{f}(\omega)\) for each \(\omega\) and \(\overline{f}\) is equal to \(f\) in \(M_\nu(\bar{K})\). This yields that the set \(C = \overline{f}^{-1}(K)\) belongs to \(\mathcal{R}\) and \(\mu(\Omega \setminus C) = 0\). Therefore, to end the proof, it suffices to put \(g = \overline{f}|_C\) and \(g|_{\Omega \setminus C} \equiv b\).

For a metrizable space \(X\) let \(M(X) = M_\mu(X)\), and for an infinite cardinal \(\alpha\) let \(M_\alpha(X) = M_{\alpha\mu}(X)\) (with analogous notation for metrics).

Theorem 3.3.
For every infinite cardinal number \(\alpha\) and each metrizable space \(X\), \(M_\alpha(X) = M'_\alpha(X)\).
Proof. We assume that $\Omega = I^T$. Let $u : \Omega \to X$ be $\mathcal{M}_u$-measurable with separable image. Since $u$ is the pointwise limit of a sequence of $\mathcal{M}_u$-measurable functions with finite images, we conclude that there is a countable infinite set $J \subset \mathbb{T}$ with $u(x) = u(y)$ whenever $x$ and $y$ are elements of $\Omega$ such that $p_j(x) = p_j(y)$, where $p_j : I^T \to I^j$ is the natural projection. This means that there is a Borel function $v : I^j \to X$ such that $u = v \circ p_j$. Let $S = \text{im } v = \text{im } u$. By (Su1) and (Su2), $S$ is absolutely measurable and thus Lemma 3.2 finishes the proof.

And now the main result of the section.

**Theorem 3.4.**
Let $(\Omega_1, \mathcal{M}_1, \mu_1)$ and $(\Omega_2, \mathcal{M}_2, \mu_2)$ be two nonatomic measure spaces such that $\mathfrak{A}(\mu_1)$ and $\mathfrak{A}(\mu_2)$ are isometrically isomorphic. Let $\Phi : \mathfrak{A}(\mu_1) \to \mathfrak{A}(\mu_2)$ be an isometric isomorphism of Boolean $\sigma$-algebras. Then for every metrizable $X$ there is a unique homeomorphism $H : \mathcal{M}_1(X) \to \mathcal{M}_2(X)$ such that for each function $f \in \mathcal{M}_1(X)$ there is a function $g \in \mathcal{M}_2(X)$ such that $g = H(f)$, $\text{im } g = \text{im } f$ and $g^{-1}(\{x\}) = \Phi(f^{-1}(\{x\}))$ in $\mathfrak{A}(\mu_2)$ for every $x \in X$. What is more, for any $d \in \text{Met} X$, $H$ is an isometry with respect to the metrics $\mathcal{M}_1(d)$ and $\mathcal{M}_2(d)$.

Proof. It is clear that the connections between $f \in \mathcal{M}_1(X)$ and $g \in \mathcal{M}_2(X)$ described in the statement of the theorem well (and uniquely) define $H$ on $\mathcal{M}_1(X)$. Moreover, $H$ is a bijection between $\mathcal{M}$-spaces. It is also clear that $H$ is an isometric map with respect to suitable metrics (described in the statement). Fix $d \in \text{Met} X$ and let $(X, d)$ be the completion of $(X, d)$. Since the spaces $(\mathcal{M}_1(X), \mathcal{M}_1(d)), j = 1, 2,$ are complete, there is a unique continuous extension $\overline{H} : \mathcal{M}_1(X) \to \mathcal{M}_2(X)$, which is simultaneously a (bijective) isometry. It is enough to check that $\overline{H}(\mathcal{M}_1(X)) \subset \mathcal{M}_2(X)$ (because then we shall infer an analogous inclusion for $\overline{H}^{-1}$). Take an $\mathcal{M}_1$-measurable function $f : \Omega_1 \to X$ whose image is contained in a $\sigma$-compact subset of $X$. This implies that there is a partition $\{A_n\}_{n=1}^\infty$ of $\Omega_1$ such that $K_n = (\overline{A_n})$ (the closure taken in $X$) is compact for each $n \geq 1$. There is a partition $\{B_n\}_{n=1}^\infty$ of $\Omega_2$ such that $B_n = \Phi(A_n)$ in $\mathfrak{A}(\mu_2)$ for any $n$. For each $l \geq 1$ take a sequence $\{f_l^n : A_l \to K_l\}_{n=1}^\infty$ of $\mathcal{M}_1$-measurable functions with finite images that converges pointwise to $f|_{A_l}$. For every $n$ and $l$ let $g_l^n(f_l^n) = \{x_l^{(n)}\}$ and let $M_l^{(n)} = B_l^{(n)}$ be a partition of $B_l$ such that $\Phi(\{f_l^n\}) = \{B_l^{(n)}\}$ in $\mathfrak{A}(\mu_2)$. Define $g_l^n : B_l \to K_l$ in the following way: $g_l^n(x) = x_l^{(n)}$. Of course $H(\bigcup_{n=1}^\infty f_l^n) = \bigcup_{n=1}^\infty g_l^n$, $n \geq 1$. So, since $f_l = \bigcup_{n=1}^\infty f_l^n$ tends to $f$ in $\mathcal{M}_1(X)$ and $\overline{H}$ is an isometric map, $g_l = \bigcup_{n=1}^\infty g_l^n$ is a fundamental sequence in $\mathcal{M}_2(X)$ and thus also the sequence $\{g_l^n\}$ is fundamental in $\mathcal{M}_2(X)$. But $g_l^n$ is a member of $\mathcal{M}_{2 | \Omega_l}(K_l)$, which is closed in $\mathcal{M}_{2 | \Omega_l}(X)$. This implies that there is $g_l^n \in \mathcal{M}_{2 | \Omega_l}(K_l)$ which is the limit of $\{g_l^n\}$. Then the function $g = \bigcup_{n=1}^\infty g_l^n$ is the limit of $(g_l)$, in $\mathcal{M}_2(X)$. Finally, $g \in \mathcal{M}_2(X)$ and $H(f) = g$.

The above result and Theorem 3.1 give

**Theorem 3.5.**
Let $\mu$ be nonatomic and $Y = \mathcal{M}_2(X)$. Then there is a finite or countable collection $\{a_j\}_{j \in J}$ of different infinite cardinals such that $Y \cong \prod_{j \in J} \mathcal{M}_2(X)$. In particular, $Y^\omega \cong Y$.

Proof. Let $\{B_j\}_{j \in J}$ and $\{a_j\}_{j \in J}, J \subset \mathbb{N},$ be as in Theorem 3.1. Observe that the function

$$M_2(X) \ni f \mapsto \{f|_{B_j}\}_{j \in J} \in \prod_{j \in J} M_{a_j}(X)$$

is a homeomorphism. Further, it follows from Theorem 3.4 that $M_{a_j}(X) \cong M_j(X)$ for each $j \in J$. So, to finish the proof, it suffices to justify that $M_\alpha(X) \cong [M_\alpha(X)]^\alpha$ for every infinite cardinal $\alpha$. When card $S = \alpha$, $S, s_1 \in S$ is fixed and for $s \in S$, $u_s : I \to I$ is the identity map for $s \neq s$, and $u_s(x) = x/2$, then the function $f : I^\alpha \ni (x)_i \in S \mapsto (u_s(x))_{i \in S} \in I^\alpha$ is an isomorphism from $I^\alpha$ onto a set $A \in \mathfrak{M}_\alpha$ such that $m_\alpha(A) = 1/2$. Now an application of Proposition 2.6 completes the proof.
4. AM-class

**Definition 4.1.**
A metrizable space is said to be an AM-space (an A-space) if every its closed separable subset is absolutely measurable (a Suslin space).

Every A-space is an AM-space and all locally absolutely Borel spaces (in particular, completely metrizable spaces) are A-spaces. It is also well known that finite or countable Cartesian products of AM-spaces (A-spaces) are AM-spaces (A-spaces) as well. AM-spaces may be characterized as follows.

**Proposition 4.2.**
For a metrizable space $X$ the following conditions are equivalent:

1. $X$ is an AM-space,
2. $M'_f(X) = M_f(X)$ for every finite measure space $(\Omega, \mathcal{M}, \mu)$,
3. $M'_f(X) = M_f(X)$ for any probabilistic Borel measure $\nu$ on a separable metric space.

**Proof.** Thanks to Lemma 3.2, we only need to prove the implication (iii) $\Rightarrow$ (i). Let $X$ satisfy the claim of (iii) and $A$ be a separable closed subset of $X$. Fix $d \in \text{Metr }A$ and denote by $\hat{A}$ the completion of $(A, d)$. Let $\mu$ be a finite Borel measure on $\hat{A}$. Put $v \colon \mathcal{B}(A) \ni B \mapsto \inf \{\mu(C) : C \in \mathcal{B}(\hat{A}), B \subseteq C\} \in \mathbb{R}_+$. It is well known that $v$ is a measure. By (iii), there is a Borel function $f \colon A \rightarrow X$ whose image is contained in a $\sigma$-compact subset of $X$ and such that $f(a) = a$ for $v$-almost all $a \in A$. Since $A$ is closed in $X$, we may assume that $f$ is contained in a $\sigma$-compact subset of $A$, say $K$. Then $K \in \mathcal{B}(\hat{A})$ and $v(A \setminus K) = 0$. Clearly, there is $B \in \mathcal{B}(\hat{A})$ such that $A \subseteq B$ and $v(A) = \mu(B)$. Then $K \subseteq A \subseteq B$ and $\mu(B \setminus K) = 0$, what shows that $A$ is absolutely measurable.

As an application of the above characterization, we obtain

**Theorem 4.3.**
Let $X$ be an AM-space and $d \in \text{Metr }X$.

(A) For any finite measures $(\Omega_1, \mathcal{M}_1, \nu_1)$ and $(\Omega_2, \mathcal{M}_2, \nu_2)$ the map

$$
\Lambda : \left( M_{\nu_1 \otimes \nu_2}(X), M_{\nu_1 \otimes \nu_2}(d) \right) \rightarrow \left( M_{\nu_1}(M_{\nu_2}(X)), M_{\nu_1}(M_{\nu_2}(d)) \right)
$$

given by $(\Lambda(\nu_1))(\nu_2) = f(\omega_1, \omega_2)$ is a (bijective) isometry.

(B) For every two infinite cardinal numbers $\alpha$ and $\beta$, $M_\alpha(M_\beta(X)) \cong M_\gamma(X)$, where $\gamma = \max(\alpha, \beta)$. In particular, $M(M(X)) \cong M(X)$.

(C) If $(\Omega, \mathcal{M}, \mu)$ is a finite measure space, $\mathcal{M}$ is a $\sigma$-subalgebra of $\mathcal{M}$ and $\nu = \mu|_\mathcal{M}$, then $M_\nu(X)$ is closed in $M_\mu(X)$.

**Proof.** To prove (A), it suffices to show surjectivity of $\Lambda$ (by Proposition 2.6). Let $g \in M_\nu(M_{\nu_2}(X))$. It follows from Proposition 2.6 that there is an $(\mathcal{M}_1 \otimes \mathcal{M}_2)$-measurable function $f : \Omega_1 \times \Omega_2 \rightarrow X$ such that $f(\omega_1, \cdot) = g(\omega_1)$ in $M_{\nu_2}(X)$ for $\nu_1$-almost all $\omega_1 \in \Omega_1$. Since $X$ is an AM-space, we may assume, thanks to Proposition 4.2, that $f$ is contained in a $\sigma$-compact subset of $X$. This implies that $g \in M_\nu(M_{\nu_2}(S))$. Let $(X, \delta)$ be the completion of $(X, d)$ and let $\overline{S}$ denote the closure of $S$ in $X$. Since $\overline{S}$ is complete and $g \in M_\nu(M_{\nu_2}(\overline{S}))$, it follows from the proof of Proposition 2.6 that there is an $(\mathcal{M}_1 \otimes \mathcal{M}_2)$-measurable function $u : \Omega_1 \times \Omega_2 \rightarrow \overline{S}$ such that $u(\omega_1, \cdot) = g(\omega_1)$ in $M_{\nu_2}(\overline{S})$ for $\nu_1$-almost all $\omega_1 \in \Omega_1$. Then the set $Z = u^{-1}(\overline{S} \setminus S)$ is a member of $\mathcal{M}_1 \otimes \mathcal{M}_2$ and for each $(\omega_1, \omega_2) \in Z$, $u(\omega_1, \omega_2) \neq (g(\omega_1))(\omega_2)$, because $g(\omega_1) \in M_{\nu_2}(S)$. This yields that $(\nu_1 \otimes \nu_2)(Z) = 0$. So, to end the proof, it suffices to define an $(\mathcal{M}_1 \otimes \mathcal{M}_2)$-measurable function $v : \Omega_1 \times \Omega_2 \rightarrow X$ by $v(\omega_1, \omega_2) = u(\omega_1, \omega_2)$ for $(\omega_1, \omega_2) \in Z$ and $v(\omega_1, \omega_2) = b$, where $b$ is a fixed element of $S$, otherwise. Then $\Lambda(v) = g$ and we are done.

Notice that item (B) follows from (A) since $m_\alpha \otimes m_\beta$ may be naturally identified with $m_\gamma$, while (C) is a consequence of Proposition 4.2 and Lemma 3.2.
We also obtain a generalization of theorems of Bessaga and Pełczyński [6] and of Toruńczyk [16].

**Theorem 4.4.**
If \( \mu \) is a finite nonatomic (nonzero) measure and \( X \) is a completely metrizable space with more than one point, then \( M_\mu(X) \) is homeomorphic to an infinite-dimensional Hilbert space of dimension \( \alpha = \max(\omega(\mu), \omega(X)) \).

**Proof.** Put \( Y = M_\mu(X) \). By Proposition 4.2 and Theorem 3.5, \( Y^\sim \cong Y \). But \( Y \) is a noncompact AR (by Theorem 2.4) and thus, by [17, Theorem 5.1], \( Y \) is homeomorphic to a Hilbert space of dimension \( \omega(Y) \). So, another application of Theorem 2.4 finishes the proof.

Now repeating the proofs (with \( M_\nu \) replaced by \( M_\nu(G) \)) of [6, Theorem 5.1, Corollary 5.2] we get

**Corollary 4.5.**
Let \( \mathcal{H} \) be a Hilbert space of dimension \( \alpha \geq \aleph_0 \) and \( G \) be a completely metrizable topological group of weight not greater than \( \alpha \). Then the topological group \( G \) is isomorphic to a closed subgroup of a group homeomorphic to \( \mathcal{H} \) and \( G \) admits a free action in \( \mathcal{H} \).

It turns out that the classes of AM-spaces and of A-spaces are invariant under the operators \( M_\mu \), as is shown in the following

**Theorem 4.6.**
If \( X \) is an AM-space (an A-space), then \( M_\mu(X) \) is an AM-space (an A-space) as well for every finite measure space \( (\Omega, \mathcal{M}, \mu) \).

**Proof.** Take a separable and closed subset \( Y \) of \( M_\mu(X) \). Let \( \{f_\alpha\}_{\alpha \in \Omega} \) be a dense subset of \( Y \). Put \( A = \bigcup_{\alpha \in \Omega} \text{im } f_\alpha \) (the closure taken in \( X \)) and let \( \mathcal{A} \) be the smallest \( \sigma \)-subalgebra of \( \mathcal{M} \) such that each of the functions \( f_\alpha \) is \( \mathcal{A} \)-measurable. Then \( A \) is separable and \( \mathcal{A} \) is a countably generated \( \sigma \)-algebra. This means that \( \mathcal{A}(v) \) is separable, where \( v = \mu\vert_\mathcal{A} \). Therefore \( M_\mu(A) \) is separable as well. What is more, by (C) of Theorem 4.3, the space \( M_\mu(A) \) is closed in \( M_\mu(X) \) and thus \( Y \subset M_\mu(X) \). Since closed subspaces of AM-spaces (resp. A-spaces) are AM-spaces (resp. A-spaces) as well, it suffices to show that \( M_\mu(X) \) is an AM-space (an A-space) if so is \( X \). Further, thanks to Proposition 2.3, we may assume that \( v \) is nonatomic. But then, see Proposition 4.2 and Theorem 3.5, \( M_\mu(X) \cong M(X) \). So, we have reduced the proof to showing that \( M(X) \) is an AM-space (an A-space), provided \( X \) is so and \( X \) is separable. First we shall show this for the class of A-spaces.

Suppose \( X \) is a separable nonempty Suslin space. Then there is a continuous surjection \( g: \mathbb{R} \setminus \mathbb{Q} \to X \). Put \( M(g): M(\mathbb{R} \setminus \mathbb{Q}) \ni f \mapsto g \circ f \in M(X) \). By Proposition 5.2 (see the next section), \( M(g) \) is a continuous surjection. So, by complete metrizability and separability of \( M(\mathbb{R} \setminus \mathbb{Q}) \), \( M(X) \) is indeed a Suslin space.

Now assume that \( X \) is a separable absolutely measurable space. Let \( S \) be a separable metrizable space and \( \lambda \) be a probabilistic Borel measure on \( S \). It is enough to prove that \( M_\mu(M(X)) = M_\mu(M(X)) \). Let \( u \in M_\mu(M(X)) \). By (A) of Theorem 4.3, there is a Borel function \( \nu: S \times I \to X \) such that \( u(s) \) and \( \nu(s, \cdot) \) coincide in \( M(X) \) for \( \lambda \)-almost all \( s \) in \( S \). Since \( X \) is absolutely measurable, there is a Borel function \( w: S \times I \to X \) whose image is contained in a \( \sigma \)-compact subset of \( X \) (say \( K \)) and such that \( \nu \) and \( w \) coincide in \( M_\mu(M(X)) \). Put \( \tilde{u}: S \ni s \mapsto w(s, \cdot) \in M(K) \subset M(X) \). Then \( \tilde{u} \) and \( u \) represent the same element of \( M_\mu(M(X)) \). What is more, \( \tilde{u} \in M_\mu(M(K)) \) and \( M(K) \) is a Suslin space, which yields that \( \tilde{u} \in M_\mu(M(K)) \subset M_\mu(M(X)) \). This finishes the proof.

We end the section with the following

**Example 4.7.**
It is well known that there exists a subset \( X \) of the square \( I^2 \) which is not Lebesgue measurable, but for each \( t \in I \) the set
\[
X_t = \{ s \in I : (t, s) \in X \}
\]
is a Borel subset of $I$ and $m(X_i) = 1$. (Such an example under CH was given by Sierpiński; see [14, Counterexamples 7.9(c)].) For reader’s convenience, let us give a short proof of this fact (without CH).

Let $\alpha$ be the least cardinal number such that every subset $A$ of $I$ of cardinality less than $\alpha$ is Lebesgue measurable and $m(A) = 0$. Let $T \subset I$ be a set of cardinality $\alpha$ whose outer measure is positive and let ‘$\infty$’ be an initial well order on $T$. That is, for any $t \in T$ the set $T_t = \{x \in T : x < t\}$ has cardinality less than $\alpha$. In particular,

$$m(T_t) = 0, \quad t \in T.$$  \hspace{1cm} (4)

For any $t \in T$ take a Borel set $B_t \subset I$ such that $m(B_t) = 1$ and $B_t \cap T_t = \emptyset$ (its existence is provided by (4)). Now, put $X = [(I \setminus T) \times I] \cup \bigcup_{t \in T} \{t\} \times B_t$. We see that for each $t \in I$, $X_t$ is Borel and $m(X_t) = 1$ (where $X_t$ is given by (3)). However, $X$ is not Lebesgue measurable. Otherwise it would have two-dimensional Lebesgue measure equal to 1, for almost all $t \in I$ the set $X_t = \{s \in I : \{s, t\} \in X\}$ would be Lebesgue measurable and we would have $m(X') = 1$. But this is impossible, because for $t \in T$ one has $X_t' \subset (I \setminus T) \cup T_t$ and therefore the inner measure of $X_t$ is less than 1 (thanks to (4) and since the outer measure of $T$ is positive).

Now if $X$ is as above, $X \in \mathcal{B}(I) \otimes \mathcal{B}(I)$. So, the map $f : I \to X$ which is the identity on $X$ and constant on its complement is $\mathcal{B}(I) \otimes \mathcal{B}(I)$-measurable. However, since $X$ is nonmeasurable, there is $g \in M_{\text{nonmeas}}(X)$ that coincides with $f$ in $M_{\text{nonmeas}}(X)$; and $f \notin M_{\text{nonmeas}}(X)$. Thus we have obtained that $M_{\text{nonmeas}}(X) \subset M_{\text{nonmeas}}(X)$ and $M_{\text{nonmeas}}(X) \subset M_{\text{nonmeas}}(X)$ as well. The example shows that Theorem 2.5 is in general not true if we put there $\nu = \nu_1 \otimes \nu_2$. It also shows that if $\nu$ is the restriction of $\mu$ to a dense (in $\mathcal{A}(\mu)$) $\sigma$-subalgebra, then $M_\nu(X)$, in spite of its density in $M_\mu(X)$, may differ from $M_\mu(X)$.

We do not know whether $M(M(X)) \cong M(X)$ if $X$ is as in Example 4.7.

5. Extending maps

For each $x \in X$ denote by $\delta_{x,X} \in M_\mu(X)$ the constant function with the only value equal to $x$, and let $\Delta_\mu(X) = \{\delta_{x,X} : x \in X\}$ and $\delta_{x,X} : X \ni x \mapsto \delta_{x,X} \in \Delta_\mu(X) \subset M_\mu(X)$. Observe that $\Delta_\mu(X)$ is closed in $M_\mu(X)$. But $\Delta_\mu(X)$ is not an isometry for each $d \in \text{Metr} X$. In particular, $\Delta_\mu(X) \cong X$. And if $X$ is a group, $\delta_{x,X}$ is a homomorphism.

**Definition 5.1.**

Let $\mu$ be a finite measure and $f : X \to Y$ be a map. Let

$$M_\mu(f) : M_\mu(X) \ni \mu \mapsto f \circ \mu \in M_\mu(Y).$$

$M_\mu(f)$ is said to be the $\mu$-extension of $f$. Additionally, let $M(f) = M_\text{fin}(f)$ and $M_\mu(f) = M_\infty(f)$ for every infinite cardinal $\alpha$.

Note that $M_\mu(f)$ is continuous and that $M_\mu(f)(N_\mu(Y)) \subset N_\mu(Y)$ for $N = M', M'^r, M'^r$. The connection

$$M_\mu(f)(\delta_{x,X}(\mu)) = \delta_{x,Y}(f(\mu)), \quad x \in X,$$  \hspace{1cm} (5)

says that $M_\mu(f)$ extends $f$, when we identify the elements of $Z$ with the ones of $\Delta_\infty$ via $\delta_{x,Z}$ with $Z = X, Y$, which justifies the undertaken terminology. If, in addition, $X$ and $Y$ are topological groups and $f$ is a group homomorphism, so is $M_\mu(f)$.

The reader will easily check that whenever $\mu$ is a fixed finite measure, the operations $X \mapsto M_\mu(X)$ and $f \mapsto M_\mu(f)$ define a functor. This functor has interesting properties, whose proofs are left as exercises (below we assume that $g, g : X \to Y$ are maps):

(F1) $M_\mu(g)$ is an injection [embedding] iff $g$ is so,

(F2) $\overline{\text{im}}(M_\mu(g)) = M_\mu(\overline{\text{im}} g)$.
(F3) the sequence \( \{M_n(g_n)\}_n \) converges pointwise (resp. uniformly on compact sets) to \( M_\mu(g) \) iff the sequence \( \{g_n\}_n \) converges pointwise (uniformly on compact sets) to \( g \).

(F4) for each \( g \in \text{Metr} \) Y the map

\[
(\mathcal{C}(X, Y), d_{\sup}) \ni h \mapsto M_\mu(h) \in \left( \mathcal{C}(M_\mu(X), M_\mu(Y)), M_\mu(g)_{\sup} \right)
\]

is isometric (‘\( \mathcal{C}(A, B) \)’ denotes the collection of all maps from \( A \) to \( B \) and ‘\( d_{\sup} \)’ stands for the supremum metric induced by a bounded metric \( d \)).

It is clear that for each \( f \in \mathcal{C}(X, Y) \), \( \text{im} [M_\mu(f)] \subset \bigcup A M_\mu(f(A)) \) where \( A \) runs over all separable closed subsets of \( X \). We do not know whether the last inclusion can always be replaced by the equality. We are only able to show the following result, the proof of which is similar to the proof of [13, Theorem 4.3].

**Proposition 5.2.**
Whenever \( \mu \) is a finite measure and \( f : X \to Y \) is a map, \( \text{im} [M_\mu(f)] = \bigcup K M_\mu(f(K)) \) where \( K \) runs over all \( \sigma \)-compact subsets of \( X \) and \( M_\mu(f) = M_\mu(f)_\mu(X) \). Moreover, if \( \nu \in M_\mu(f(A)) \), where \( A \) is a (separable) Suslin subset of \( X \), then \( \nu \in \text{im} [M_\mu(f)] \).

**Proof.** We only need to prove the second claim. Put \( C = f(A) \) and let \( L \) be a \( \sigma \)-compact subset of \( C \) such that \( \text{im} \nu \subset L \). Let \( \nu : \mathcal{B}(L) \ni B \to \mu(\nu^{-1}(B)) \in \mathbb{R}_+ \). Put \( K = A \cap f^{-1}(L) \). Then \( K \in \mathcal{B}(A) \) and thus \( K \) is a Suslin space. Now it suffices to apply [11, Theorem XIV.3.1] to obtain a function \( h: L \to K \) such that \( f \circ h \) is the identity map on \( L \) and for every set \( U \subset K \) open in \( K \), \( h^{-1}(U) \) is a member of the \( \sigma \)-algebra generated by the family of all Suslin subsets of \( L \). This implies that for every Borel subset \( B \) of \( K \), \( h^{-1}(B) \) is absolutely measurable and therefore there is a Borel function \( w : L \to K \) and a set \( B_0 \in \mathcal{B}(L) \) such that \( \nu(B_0) = 0 \) and \( w = h \) on \( L \setminus B_0 \). Now put \( u = w \circ v \). By Lemma 3.2, \( u \in M_\mu(X) \). What is more, \( f \circ u \) is \( \mu \)-almost everywhere equal to \( f \circ h \circ v = \nu \), which finishes the proof.

Under the notation of Proposition 5.2 we get

**Corollary 5.3.**

(i) If \( X \) is an \( A \)-space, then \( \text{im} [M_\mu(f)] = \bigcup A M_\mu(f(A)) \) where \( A \) runs over all separable closed subsets of \( X \).

(ii) If for every \( \sigma \)-compact subset \( L \) of \( \text{im} f \) there is a (separable) Suslin subset \( K \) of \( X \) such that \( L \subset f(K) \), then \( \text{im} [M_\mu(f)] = M_\mu(f) \).

Now applying the general scheme and main ideas of [13] (with the same functor \( M \)), thanks to the homeomorphism extension theorem proved in [7], we easily obtain

**Theorem 5.4.**

Let \( \mathcal{H} \) be a nonseparable Hilbert space. Let \( \mathcal{Z} \) be the family of all maps \( \varphi : K \to L \) between \( Z \)-sets \( K \) and \( L \) in \( \mathcal{H} \). There is an assignment \( \mathcal{Z} \ni \varphi \mapsto \hat{\varphi} \in \mathcal{C}(\mathcal{Z}, \mathcal{H}) \) (called the **functor of extension** ) such that for any \( Z \)-sets \( A, B, C \) in \( \mathcal{H} \) and each \( \varphi \in \mathcal{C}(A, B) \),

(a) \( \hat{\varphi} \) extends \( \varphi \),

(b) \( \hat{id} = id_{\mathcal{H}} \) for \( id : A \to A ; \hat{\varphi} \circ \hat{\varphi} = \hat{\varphi} \circ \hat{\varphi} \) for every \( \psi \in \mathcal{C}(B, C) \) (in particular, \( \hat{\varphi} \) is an automorphism of \( \mathcal{H} \) provided \( \varphi \) is a homeomorphism of \( A \) onto \( B \)),

(c) \( \hat{\varphi} \) is an injection (resp. an embedding) iff so is \( \varphi \),

(d) \( \overline{\text{im} \varphi} \subset \mathcal{H} ; \text{im} \varphi \) is dense in \( \mathcal{H} \) iff \( \overline{\text{im} \varphi} \) is dense in \( B \),

(e) for any sequence \( \varphi, \varphi, \ldots \) of members of \( \mathcal{C}(A, B) \), the maps \( \hat{\varphi}_n \) converge to \( \hat{\varphi} \) pointwise (resp. uniformly on compact sets) iff the \( \varphi_n \)’s converge so to \( \varphi \).
(f) every metric \(d \in \text{Metr } \mathcal{B} \) extends to \(\hat{d} \in \text{Metr } \mathcal{H} \) such that the following map is isometric:

\[
(\mathcal{C}(A, B), d_{\text{cup}}) \ni \xi \mapsto \hat{\xi} \in (\mathcal{C}(\mathcal{H}, \mathcal{H}), \hat{d}).
\]

**Proof.** Let \( \mathcal{H}' \) be a topological space homeomorphic to \( \mathcal{H} \) and disjoint from \( \mathcal{H} \). Let \( \mathcal{H} \cup \mathcal{H}' \) denote the topological disjoint union of \( \mathcal{H} \) and \( \mathcal{H}' \). For any \( Z \)-set \( K \) in \( \mathcal{H} \) put \( K^\# = K \cup \mathcal{H}' \subset \mathcal{H} \cup \mathcal{H}' \). Similarly, for a map \( u: K \to L \) between \( Z \)-sets \( K \) and \( L \) in \( \mathcal{H} \) let \( u^\#: K^\# \to L^\# \) be an extension of \( u \) such that \( u(x) = x \) for \( x \in \mathcal{H}' \). Further, it is easy to check that the set \( \delta_{m,K^\#}(K) \) is a \( Z \)-set in \( M(K^\#) \), homeomorphic to \( K \). By Theorem 4.4, \( M(K^\#) \cong \mathcal{H} \). It follows from the \( Z \)-set unknotting theorem [7] that for every \( Z \)-set \( K \) in \( \mathcal{H} \) there is a homeomorphism \( H_K: \mathcal{H} \to M(K^\#) \) which extends \( \delta_{m,K^\#}|_K \). Now for a map \( \varphi: K \to L \) between two \( Z \)-sets in \( \mathcal{H} \), put \( \hat{\varphi} = H_L^{-1} \circ M(\varphi^\#) \circ H_K \). Since \( M \) is a functor, it is easily seen that \( (b) \) is satisfied. Items (c)–(f) follow simply from (F1)–(F4) and Theorem 4.4. Hence it remains to check (a). For each \( x \in K \), we have by (5),

\[
\hat{\varphi}(x) = H_L^{-1}(M(\varphi^\#)(\delta_{m,K^\#}(x))) = H_L^{-1}(\delta_{m,L^\#}(\varphi^\#(x))) = H_L^{-1}(\delta_{m,L^\#}(\varphi(x))) = \varphi(x)
\]

and we are done. \( \square \)

**Remark 5.5.** A similar functor as in Theorem 5.4 can be built using the functor \( \hat{P} \) studied by Banakh [1, 2], and Banakh and Radul [4]. (For a metrizable space \( X \), \( \hat{P}(X) \) is the space of all Borel probabilistic measures supported on \( \sigma \)-compact subsets of \( X \) and for a map \( f: X \to Y \) and \( \mu \in \hat{P}(X) \), \( \hat{P}(f)(\mu) \) is the transport of \( \mu \) under \( f \).) [4, Theorem 2.11] says that \( \hat{P}(X) \) is homeomorphic to an infinite-dimensional Hilbert space, provided \( X \) is completely metrizable and noncompact. Thus it is enough to apply the general scheme of [13] and results of Banakh [1, 2] on extending maps and bounded metrics via the functor \( \hat{P} \).

We end the paper with the following two questions.

(A) Is \( M(M(X)) \) homeomorphic to \( M(X) \) for an arbitrary metrizable space \( X \)?

(B) Is \( M_\mu(X) \) homeomorphic to \( M_\mu'(X) \) for an arbitrary metrizable space \( X \) and any finite measure space \( (\Omega, \mathcal{M}, \mu) \)?

Note that the affirmative answer to (B) implies the affirmative one to (A).

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