From de Sitter to de Sitter

A non-singular inflationary universe driven by vacuum

Saulo Carneiro*

Instituto de Física, Universidade Federal da Bahia, 40210-340, Salvador, BA, Brazil
International Centre for Theoretical Physics, Trieste, Italy†

Abstract

A semi-classical analysis of vacuum energy in the expanding spacetime suggests that the cosmological term decays with time, with a concomitant matter production. For early times we find, in Planck units, $\Lambda \approx H^4$, where $H$ is the Hubble parameter. The corresponding cosmological solution has no initial singularity, existing since an infinite past. During an infinitely long period we have a quasi-de Sitter, inflationary universe, with $H \approx 1$. However, at a given time, the expansion undertakes a phase transition, with $H$ and $\Lambda$ decreasing to nearly zero in a few Planck times, producing a huge amount of radiation. On the other hand, the late-time scenario is similar to the standard model, with the radiation phase followed by a dust era, which tends asymptotically to a de Sitter universe, with vacuum dominating again.

* saulo@fis.ufba.br, saulocarneiro@yahoo.com
† Associate member
The cosmological constant problem has been a theme of theoretical discussion for decades, and has turned into a central point of modern cosmology since recent observations suggested the existence of a negative-pressure component in the cosmic energy content \cite{1}. 

The problem arises when we try to associate such a component with the vacuum energy density predicted by quantum field theories. In the case, for example, of a massless scalar field, the energy density associated to its quantum fluctuations is given by

\[ \Lambda_0 \approx \int_0^\infty \omega^3 d\omega. \] (1)

This divergent integral can be regularized by imposing a superior cutoff \( m \), leading to \( \Lambda_0 \approx m^4 \). This may also be performed by introducing a bosonic distribution function in (1),

\[ \Lambda_0 \approx \int_0^\infty \frac{\omega^3 d\omega}{e^{\omega/m} - 1} \approx m^4. \] (2)

The regularization procedure is thus equivalent to assume a thermal distribution of vacuum fluctuation modes, at a characteristic temperature \( m \).

A natural choice for \( m \) is the energy scale of QCD condensation, the latest cosmological vacuum transition we know, since vacuum fluctuations above this cutoff - which has the order of the pion mass - would generate quark de-confinement. Unfortunately, even with this value (many orders of magnitude below the Planck mass, also usually taken as a natural cutoff), the obtained vacuum density is around 40 orders of magnitude higher than the presently observed cosmological constant. That is the problem.

We should observe, however, that the above reasoning is based on QFT in flat spacetime. In this case, the energy-momentum tensor appearing in Einstein’s equations must be zero, and, therefore, (2) should be exactly canceled by a bare cosmological constant. Such a cancelation should occur for any vacuum contribution derived in flat spacetime.

Now, what would happen if we could calculate the vacuum density in the expanding background? The regularized result would depend on the curvature, and, after subtracting
Λ₀, we should obtain a renormalized, time-dependent cosmological term Λ, decaying from high initial values to smaller ones, as the universe expands [2. 3. 4]. This renormalization is similar to what happens in the Casimir effect, where the important thing is not the vacuum density itself, but the difference between its values inside a bounded region and in unbounded space.

The variation in the vacuum density leads, on the other hand, to matter production, in order to preserve the conservation of total energy implied by Einstein’s equations. Indeed, in the realm of a spatially homogeneous and isotropic spacetime, the Bianchi identities lead to the conservation equation

\[ \dot{\rho}_T + 3H(\rho_T + p_T) = 0. \] (3)

Here, \( H = \dot{a}/a \) is the Hubble parameter, while \( \rho_T \) and \( p_T \) are, respectively, the total energy and pressure of cosmic fluid. By introducing the matter density and pressure, and writing \( \rho_T = \rho_m + \Lambda \) and \( p_T = p_m - \Lambda \), we have\(^1\)

\[ \dot{\rho}_m + 3H(\rho_m + p_m) = -\dot{\Lambda}. \] (4)

This shows that matter is not conserved - the decaying vacuum acting as a source of entropy.

But how to evaluate the vacuum contribution in the expanding spacetime? A possible answer is suggested by a semi-classical analysis of the equation of motion of a minimally coupled massless scalar field, \( D^\mu D_\mu \phi = 0 \), where \( D \) denotes the covariant derivative. In a FLRW spacetime, it assumes the form

\[ 3H \frac{\partial \phi}{\partial t} + \frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi = 0. \] (5)

Since the space is isotropic, let us consider a plane wave solution propagating in the radial direction. As for any plane wave, the wavelength is supposed very small compared to the cosmological scale. Therefore, \( H \) changes very slowly compared to the wave function, and the solution has the form

\[ \phi \approx \phi_0 e^{-br} e^{-i(\omega t - kr)}, \] (6)

\(^1\) Since the vacuum has the same symmetry as spacetime, its energy-momentum tensor has the form \( T^{\mu \nu}_\Lambda = \Lambda g^{\mu \nu} \), where \( g^{\mu \nu} \) is the metric tensor, and \( \Lambda \) is a scalar function of coordinates (in the FLRW spacetime, just a function of time). Therefore, it has the same structure as for a perfect fluid in co-moving observers, with \( p_\Lambda = -\Lambda \).
with
\[ k = \frac{\sqrt{2}\omega}{2} \left[ 1 + \sqrt{1 + \left(\frac{3H}{\omega}\right)^2} \right]^{1/2}, \]  
(7)

\[ b = \frac{3\sqrt{2}H}{2} \left[ 1 + \sqrt{1 + \left(\frac{3H}{\omega}\right)^2} \right]^{-1/2}. \]  
(8)

As one can see, the wave amplitude decreases with \( r \), with a depth length equal to \( b^{-1} \).

The energy-momentum tensor of this scalar field is
\[ T^{\mu\nu} = \partial_\mu \phi \partial_\nu \phi^\dagger - \frac{1}{2} \partial_\sigma \phi \partial_\sigma \phi^\dagger \delta^{\nu\mu}. \]  
(9)

Taking its time component, and using (6), we derive
\[ \rho \approx \left(\omega^2 + b^2\right) \phi \phi^\dagger. \]  
(10)

One can interpret this result by saying that the scalar particle performs, superposed to the mode of frequency \( \omega \), a thermal motion of temperature \( b \).

On this basis, we may evaluate the vacuum fluctuations by doing the shift \( m \to m + b \) in (2). The dominant contributions to the integral will be given by modes with \( \omega \approx m + b \).

For \( b << m \), the dominance occurs for \( \omega \approx m \), which implies, through (8), that \( b \approx H \). For \( b \sim m \) or \( b >> m \), the dominance occurs for \( \omega \approx b \), leading again, through (8), to \( b \approx H \).

In this case, however, the plane-wave approximation (3)-(8) cannot be used anymore, and the identity between \( b \) and \( H \) will be taken as an \textit{ad hoc} assumption.

Then, after subtracting \( \Lambda_0 \), we obtain
\[ \Lambda \approx (m + H)^4 - m^4. \]  
(11)

This has a similar structure as in the Casimir effect. Actually, for \( H >> m \) we have the same cutoff-independent result, \( \Lambda \approx H^4 \), with \( H^{-1} \) playing the role of a distance between Casimir plates.

Therefore, in the limit of very early times, the cosmological term scales as \( \Lambda \approx H^4 \), while for late times (\( H << m \)) it scales as \( \Lambda \approx m^3H \) (we should, however, be careful with this last conclusion, as discussed below). Let us investigate the corresponding cosmological scenarios. For simplicity, we will only consider the spatially flat case.

Leading the Friedmann equation \( \rho_T = 3H^2 \) into the conservation equation (3), using for matter the equation of state of radiation, \( p_m = \rho_m/3 \), and taking for the vacuum our
early-time result $\Lambda = 3H^4$ (the constant factor is not important, being taken three for convenience), we obtain the evolution equation

$$\dot{H} + 2H^2 - 2H^4 = 0.$$  \hspace{1cm} (12)

Apart an integration constant which determines the origin of time, its solution is

$$2t = \frac{1}{H} - \tanh^{-1} H.$$  \hspace{1cm} (13)

The evolution of $H$ is plotted in Figure 1. As one sees, this universe has no initial singularity, existing since an infinite past, when $H$ approaches asymptotically the Planck value $H = 1$. During an infinitely long period we have a quasi-de Sitter, inflationary expansion, with $H \approx 1$. But at a given time (chosen around $t = 0$) we have a huge phase transition, with a characteristic time scale of a few Planck times, during which $H$ (and so $\Lambda$) falls to nearly zero.

The transition can also be understood in terms of the energy content. The energy density of radiation is $\rho_m = \rho_T - \Lambda$, and its relative energy density is $\Omega_m = 1 - H^2$. Therefore, the transition leads from an empty, vacuum-dominated universe to a radiation-dominated phase, with $\Omega_m$ approaching 1 asymptotically (see Figure 2, where we also plot the relative energy density of vacuum, $\Omega_\Lambda = H^2$). This behavior can also be described with help of the deceleration factor we obtain from (13). It is $q = 1 - 2H^2$, and suddenly changes from $-1$, in the quasi-de Sitter phase, to 1, in the radiation one (Figure 3).

Let us now consider the limit of late times, for which $\Lambda = \sigma H$, with $\sigma \approx m^3$. We have shown elsewhere that, in the radiation phase, $a \propto t^{1/2}$, with $\rho_m = 3/(4t^2)$, as in the standard model. On the other hand, in the dust phase we have

$$a = C \left( e^{\sigma t/2} - 1 \right)^{2/3},$$  \hspace{1cm} (14)

where $C$ is an integration constant.

For early times ($\sigma t << 1$), we have $a \propto t^{2/3}$, as in the Einstein-de Sitter model. This decelerated phase follows until very recently, when vacuum begins dominating again and the expansion reenters in an accelerated phase, which, as one can see from (14), tends asymptotically to a de Sitter universe. We have analyzed the redshift-distance relation for supernovas Ia in this model, obtaining a fit of observational data as good as in the $\Lambda$CDM model. The obtained present values of $H$ and $\Omega_m$ and the universe age are also in
good accordance with other observations \[\text{[5]}\]. Finally, the analysis of evolution of density perturbations until the present time shows no important difference compared to the ΛCDM model.

In the de Sitter limit, the Hubble parameter is given, as we know, by \(H = \sqrt{\Lambda/3}\). Therefore, using \(\Lambda \approx m^3 H\), one derives the results \(H \approx m^3\) and \(\Lambda \approx m^6\). The former is an expression of the famous Eddington-Dirac large number coincidence, provided we take \(m\) of the order of the pion mass, i.e., the order of the energy scale of the QCD chiral transition, as initially supposed. The last relation, on the other hand, was suggested by Zel’dovich four decades ago, on the basis of different arguments.

Nevertheless, we should be careful before concluding that the present universe evolves as described above. Our approach is based on a macroscopic, semi-classical reasoning, and we still do not have a microscopic description of vacuum decay. At late times the decay probably depends on the mass of the produced particles, and so we have no guarantee that vacuum is still decaying. If it stopped decaying at some earlier time, we just have, after the primordial transition, a ΛCDM universe.

To conclude, some words about the entropy of this universe. If the vacuum fluctuations are thermally distributed, as suggested by \(\text{[2]}\), the number of states inside a volume \(V\) may be estimated as

\[
N \approx V \int_0^\infty \frac{\omega^2 d\omega}{e^{\omega/(m+H)} - 1}.
\]

For late times we have \(H \ll m\), and \(N \approx V m^3\). In the de Sitter limit, taking \(V\) as the Hubble volume, and \(m^3 \approx H\), we obtain \(N \approx H^{-2} \approx 10^{120}\). That is, in the final de Sitter phase, the entropy inside the Hubble sphere is equal to the area of its surface, which is an expression of the holographic conjecture \(\text{[6, 7]}\).

On the other hand, during the primeval quasi-de Sitter phase, we have \(H \gg m\), and \(\text{[15]}\) leads to \(N \approx VH^3\). Now, by taking the Hubble volume we obtain \(N \approx 1\), which is, again, equal to the area of the Hubble surface. In this way, one may conclude that the primordial phase transition leads a universe of very low entropy into a state of very high entropy. The thermodynamic and time arrows coincide.

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FIG. 1: The Hubble parameter as a function of time (in Planck units)
FIG. 2: The relative energy densities of radiation and vacuum as functions of time

FIG. 3: The deceleration parameter as a function of time