BOUND STATES OF DISCRETE SCHRÖDINGER OPERATORS ON ONE AND TWO DIMENSIONAL LATTICES

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Abstract. We study the spectral properties of discrete Schrödinger operator

$$\hat{h}_\mu = \hat{h}_0 + \mu \hat{v}, \quad \mu \geq 0,$$

associated to a one-particle system in $d$-dimensional lattice $\mathbb{Z}^d$, $d = 1, 2$, where the non-perturbed operator $\hat{h}_0$ is a self-adjoint Laurent-Toeplitz-type operator generated by $\hat{r} : \mathbb{Z}^d \to \mathbb{C}$ and the potential $\hat{v}$ is the multiplication operator by $\hat{v} : \mathbb{Z}^d \to \mathbb{R}$. Under certain regularity assumption on $\hat{r}$ and a decay assumption on $\hat{v}$, we establish the existence or non-existence and also the finiteness of eigenvalues of $\hat{h}_\mu$. Moreover, in the case of existence we study the asymptotics of eigenvalues of $\hat{h}_\mu$ as $\mu \to 0$.

1. Introduction

In [6] Klaus studied the eigenvalues of the Schrödinger operator $-d^2/dx^2 + \lambda V$ for $\lambda > 0$ and $V$ obeying

$$\int_\mathbb{R} (1 + |x|)|V(x)| dx < \infty,$$

extending the results of Simon in [11] in case of $d = 1$. Klaus showed that if $\int V(x) dx > 0$, then for small and positive $\lambda$ there is no bound state, and if $\int V(x) dx \leq 0$, then there exists a bound state $E(\lambda)$ and it satisfies

$$(E(\lambda)^{1/2} = -\frac{\lambda}{2} \int V(x) dx - \frac{\lambda^2}{4} \int |V(x)| dx + o(\lambda^2)$$

as $\lambda \to 0$.

In the present paper we replace the Euclidean $d$-dimensional space $\mathbb{R}^d$ by the $d$-dimensional lattice $\mathbb{Z}^d$, $d = 1, 2$, and study the discrete spectrum of a large class of lattice Schrödinger operators $\hat{h}_\mu$ associated to a one-particle system in $\mathbb{Z}^d$ given by

$$\hat{h}_\mu := \hat{h}_0 + \mu \hat{v}, \quad \mu \geq 0,$$

where the non-perturbed operator $\hat{h}_0$ is a Laurent-Toeplitz-type operator with a generating function $\hat{r} \in \ell^1(\mathbb{Z}^d)$ satisfying $\hat{r}(\pm x) = \hat{r}(x) :$

$$\hat{h}_0 f(x) = \sum_{y \in \mathbb{Z}^d} \hat{r}(y) \hat{f}(x + y), \quad \hat{f} \in \ell^2(\mathbb{Z}^d),$$

and the potential $\hat{v}$ is the multiplication operator by a real-valued function $\hat{v} : \mathbb{Z}^d \to \mathbb{R}$ vanishing at infinity.

We also work with the representation of $h_0$ in $L^2(\mathbb{T}^d)$, where $\mathbb{T}^d = (-\pi, \pi]^d$ is the $d$-dimensional torus, equipped with the normalized Haar measure $dp$, i.e. $\int_{\mathbb{T}^d} dp = 1$. The so-called “momentum-space representation” of $h_0$ and $v$ are defined via the standard Fourier transform

$$F: \ell^2(\mathbb{Z}^d) \to L^2(\mathbb{T}^d), \quad F \hat{f}(p) = \sum_{x \in \mathbb{Z}^d} \hat{f}(x) e^{ixp},$$

as

$$h_0 := F \hat{h}_0 F^* \quad \text{and} \quad v := F \hat{v} F^*,$$

where

$$F^*: L^2(\mathbb{T}^d) \to \ell^2(\mathbb{Z}^d), \quad F^* f(x) = \int_{\mathbb{T}^d} f(p) e^{-ixp} dp.$$
where \( \hat{p} \)

Hypothesis 1.1. Assume Hypothesis [12]. Then for any \( \mu > 0 \),

\[
\mathcal{N}^+(\hat{h}_\mu, \epsilon_{\text{max}}) \leq 1 + C_1 \mu \sum_{x \in \mathbb{Z}^d} |x|^{2-d+\gamma} |\hat{\varphi}(x)|
\]

and

\[
\mathcal{N}^-(\hat{h}_\mu, \epsilon_{\text{min}}) \leq 1 + C_2 \mu \sum_{x \in \mathbb{Z}^d} |x|^{2-d+\gamma} |\hat{\varphi}(x)|,
\]

where \( C_1, C_2 > 0 \) are coefficients depending only on \( \epsilon \) and \( \gamma \). In particular, the number of eigenvalues of \( \hat{h}_\mu \) outside the essential spectrum is finite for any \( \mu > 0 \).

We note that Theorem 1.2 improves the upper bound for the number of eigenvalues obtained in [1] Theorem 1.2. To the best of our knowledge, estimates of the form (1.2)-(1.3) are known only for the discrete spectrum. In fact, in \( d = 1 \) sharp bounds for \( \hat{\varphi} \) for the finiteness of bound states of \( -\Delta + \hat{\varphi} \) have been established in [2] using some variational estimates. In \( d = 2 \) an estimate of type (1.3) (with \( \hat{\varphi} \leq 0 \) and with \( \ln(1 + |x|) \) in place of \( |x|^\gamma \)) for \( -\Delta + \hat{\varphi} \) has been obtained in [9] applying Markov processes. Analogous estimate in \( \mathbb{Z}^2 \) (again with \( \hat{\varphi} \leq 0 \) and with \( \ln(1 + |x|) \) in place of \( |x|^\gamma \)) for \( -\Delta + \hat{\varphi} \) has been obtained in [10] using some careful estimates for the two dimensional continuous Schrödinger operators together with interpolation arguments. In this paper we establish (1.2)-(1.3) without using those techniques, rather adapting the methods of Klaus in [9]. Note that in the continuous case (1.2) does not make sense.

Our next results are related to the existence or non-existence and also the uniqueness of eigenvalues of \( \hat{h}_\mu \).
Theorem 1.3. Assume Hypothesis [4]. Then for any \( \mu > 0 \):

(1) if \( \sum_{x \in \mathbb{Z}^d} \hat{v}(x) \geq 0 \), then \( \sigma_{\text{disc}}(\hat{h}_\mu) \cap (\epsilon_{\text{max}}, +\infty) \neq \emptyset \);

(2) if \( \sum_{x \in \mathbb{Z}^d} \hat{v}(x) \leq 0 \), then \( \sigma_{\text{disc}}(\hat{h}_\mu) \cap (-\infty, \epsilon_{\text{min}}) \neq \emptyset \).

Moreover, there exists \( \mu_\circ := \mu_\circ(\epsilon, \hat{v}) > 0 \) such that for any \( \mu \in (0, \mu_\circ) \):

(a) if \( \sum_{x \in \mathbb{Z}^d} \hat{v}(x) > 0 \), then \( \sigma_{\text{disc}}(\hat{h}_\mu) \cap (\epsilon_{\text{max}}, +\infty) \) is a singleton \( \{E(\mu)\} \) and \( \sigma_{\text{disc}}(\hat{h}_\mu) \cap (-\infty, \epsilon_{\text{min}}) = \emptyset \);

(b) if \( \sum_{x \in \mathbb{Z}^d} \hat{v}(x) < 0 \), then \( \sigma_{\text{disc}}(\hat{h}_\mu) \cap (-\infty, \epsilon_{\text{min}}) \) is a singleton \( \{e(\mu)\} \) and \( \sigma_{\text{disc}}(\hat{h}_\mu) \cap (\epsilon_{\text{max}}, +\infty) = \emptyset \);

(c) if \( \sum_{x \in \mathbb{Z}^d} \hat{v}(x) = 0 \), then both \( \sigma_{\text{disc}}(\hat{h}_\mu) \cap (\epsilon_{\text{max}}, +\infty) \) and \( \sigma_{\text{disc}}(\hat{h}_\mu) \cap (-\infty, \epsilon_{\text{min}}) \) are singletons \( \{E(\mu)\} \) and \( \{e(\mu)\} \), respectively.

We remark that the existence of eigenvalues, i.e., assertions (1)-(2) of Theorem 1.3 can also be obtained from [4] Theorem 3.19, however, methods of [4] seem not sufficient to establish the remaining assertions such as non-existence and uniqueness of eigenvalues.

Notice that by the linearity of \( \mu \mapsto \hat{h}_\mu \), being a unique and isolated point of the discrete spectrum, both \( \mu \in (0, \mu_\circ) \mapsto E(\mu) \) and \( \mu \in (0, \mu_\circ) \mapsto e(\mu) \) are analytic. Moreover, \( E(\mu) \searrow \epsilon_{\text{max}} \) and \( e(\mu) \nearrow \epsilon_{\text{min}} \) as \( \mu \searrow 0 \) so that both eigenvalues are absorbed by the essential spectrum as \( \mu \searrow 0 \). Now we study their absorption rate.

Theorem 1.4. Assume Hypothesis [4] and additionally suppose that \( \epsilon \in C^{3,\alpha} \) around \( p_0 \) and \( p_0 \) for some \( \alpha \in (0, 1/8] \). Then there exists \( \mu_1 := \mu_1(\epsilon, \hat{v}) \in (0, \mu_\circ) \) such that for any \( \mu \in (0, \mu_1) \):

(a) if \( \kappa_0 := \sum_{x \in \mathbb{Z}^d} \hat{v}(x) > 0 \), then

\[
E(\mu) - \epsilon_{\text{max}} = \begin{cases}
\mu^2 \left[ \kappa_0 a_1 + \mu^\gamma \Phi_1(\mu) \right]^2 & \text{if } d = 1, \\
\epsilon_{\text{max}} - \frac{1}{\kappa_0} \cosh \left[ c_1 + \Psi_1(\mu) \right] & \text{if } d = 2;
\end{cases}
\]

(b) if \( \kappa_0 := \sum_{x \in \mathbb{Z}^d} \hat{v}(x) < 0 \), then

\[
\epsilon_{\text{min}} - e(\mu) = \begin{cases}
\mu^2 \left[ -\kappa_0 a_2 + \mu^\gamma \Phi_2(\mu) \right]^2 & \text{if } d = 1, \\
\epsilon_{\text{min}} - \frac{1}{\kappa_0} \cosh \left[ c_2 + \Psi_2(\mu) \right] & \text{if } d = 2;
\end{cases}
\]

(c) if \( \sum_{x \in \mathbb{Z}^d} \hat{v}(x) = 0 \), then both integrals

\[
\kappa_1 := \int_{\mathbb{T}^d} \frac{|v(p - p_0)|^2 dp}{\epsilon_{\text{max}} - \epsilon(p)}, \quad \kappa_2 := \int_{\mathbb{T}^d} \frac{|v(p - p_0)|^2 dp}{\epsilon(p) - \epsilon_{\text{min}}}
\]

are finite and

\[
E(\mu) - \epsilon_{\text{max}} = \begin{cases}
\mu^4 \left[ \kappa_1 a_3 + \mu^\gamma \ln^2 \mu \Phi_3(\mu) \right]^2 & \text{if } d = 1, \\
\epsilon_{\text{max}} - \frac{1}{\kappa_1} \cosh \left[ d_3 + \Psi_3(\mu) \right] & \text{if } d = 2,
\end{cases}
\]

and

\[
\epsilon_{\text{min}} - e(\mu) = \begin{cases}
\mu^4 \left[ \kappa_2 a_4 + \mu^\gamma \ln^2 \mu \Phi_4(\mu) \right]^2 & \text{if } d = 1, \\
\epsilon_{\text{min}} - \frac{1}{\kappa_2} \cosh \left[ d_4 + \Psi_4(\mu) \right] & \text{if } d = 2,
\end{cases}
\]

Here \( a_i, b_i, c_i, d_i > 0 \) are constants depending only on \( \epsilon \), and \( \Phi_i, \Psi_i : [0, \mu_1] \to \mathbb{R} \) are continuous.
We remark that the asymptotics for \( e(\mu) \) in \( d = 1 \) corresponds to the continuous counterparts obtained in [6, 11], however, in \( d = 2 \) the asymptotics [13] sharper than the one in [11] Theorem 3.4 obtained in the continuous case.

As in [6, 11] to prove Theorem 1.4 we obtain an asymptotic equation for \( E(\mu) \) and \( e(\mu) \). It turns out that in one dimensional case \( E(\mu) \) and \( e(\mu) \) satisfy
\[
\sqrt{E(\mu) - \epsilon} = [c_1 + g_1(E(\mu) - \epsilon)]\mu^n,
\]
\[
\sqrt{\epsilon - e(\mu)} = [c_2 + g_2(\epsilon - e(\mu))]\mu^n,
\]
where \( c_1, c_2 > 0 \) are explicit constants, \( g_1(z), g_2(z) \to 0 \) as \( z \to 0 \), and \( n = 1 \) or \( n = 2 \) depending on whether \( \sum x \hat{v}(x) \) is nonzero or zero. The equation (1.8) readily gives the first term of the asymptotics of \( E(\mu) \) and \( e(\mu) \). To identify the second term we need to analyse the convergence rates of \( g_1 \) and \( g_2 \). Similarly, in two dimensions the associated equations for \( E(\mu) \) and \( e(\mu) \) read as
\[
\frac{1}{-\mu^n \ln(E(\mu) - \epsilon_{\max})} = c_3 + g_3(E(\mu) - \epsilon_{\max}),
\]
\[
\frac{1}{-\mu^n \ln(\epsilon - e(\mu))} = c_4 + g_4(\epsilon - e(\mu)),
\]
where \( c_3, c_4 > 0 \) are explicit constants, \( g_3(z), g_4(z) \to 0 \) as \( z \to 0 \), and \( n = 1 \) or \( n = 2 \) depending on whether \( \sum x \hat{v}(x) \) is nonzero or zero. Unlike the one dimensional case, (1.9) is not sufficient even to find the first term of the asymptotics of \( E(\mu) \) and \( e(\mu) \), rather it yields only estimates of the form
\[
\exp(-\epsilon) \leq E(\mu) - \epsilon_{\max} \leq \exp(-\epsilon),
\]
and
\[
\exp(-\epsilon) \leq \epsilon - e(\mu) \leq \exp(-\epsilon),
\]
for small \( \epsilon > 0 \) (see e.g., [11] Theorem 3.4).

We prove (1.8), (1.9) obtaining careful estimates for \( g_i \) using the perturbation theory for a (not necessarily self-adjoint) Birman-Schwinger operator \( \hat{v}(z), \ z \in \mathbb{R} \setminus [\epsilon_{\min}, \epsilon_{\max}] \) (see Section 2). The equations for eigenvalues in (1.8) and (1.9) for the case \( \sum x \hat{v}(x) \neq 0 \) is obtained employing the similar arguments to [6]. In this case \( \hat{v}(z) \) is represented as a small perturbation of rank-one operator (Lemma 2.3) that has a unique non-zero eigenvalue. However, the case \( \sum x \hat{v}(x) = 0 \) requires more delicate analysis since in this case the previous perturbation-theory arguments fail. Here we were not able to use the arguments of Klaus and instead we needed to employ the Implicit Function Theorem in Banach spaces and [3, Lemma 3.3] to prove the existence of a unique positive and a unique negative eigenvalues of \( b(z) \) which blows up if \( z \in \mathbb{R} \setminus [\epsilon_{\min}, \epsilon_{\max}] \) approaches to \( [\epsilon_{\min}, \epsilon_{\max}] \). In view of the Birman-Schwinger principle (Lemma 2.4) this allows to establish the uniqueness of the eigenvalue of \( \hat{H}_\mu \) provided \( \mu \) is small enough (Corollary 2.6).

Naturally, to get the further terms of the asymptotics of \( E(\mu) \) and \( e(\mu) \) one needs a further condition on the regularity of \( \epsilon \) and decay of \( \hat{v} \). In the case with analytic \( \epsilon \) and exponentially decaying \( \hat{v} \) one can even obtain convergent expansions as in the continuous setting [7]; such an expansion for \( E(\mu) \) has been obtained, for instance, in [8] in the discrete Laplacian case with zero-range non-positive perturbation.

The present paper is organized as follows. In Section 2 the main technical tool – the Birman-Schwinger operator is introduced and some of its properties are studied. The main results are proven in Section 3. Finally, in Appendix we obtain an asymptotics of a parametrical integral which is frequently used throughout the paper.

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### 2. Birman-Schwinger operator and some of its properties

Given \( z \in \mathbb{C} \setminus [\epsilon_{\min}, \epsilon_{\max}] \), let
\[
b(z) : L^2(\mathbb{T}^d) \to L^2(\mathbb{T}^d), \quad b(z) := s \sqrt{|z - h_0|^{-1}}|v|
\]
be the Birman-Schwinger operator associated to \( h_\mu \), where \( s \) is the sign of \( v \). By the uniqueness of the polar decomposition, \[
|b(z)| = \begin{cases} \sqrt{|v|}(z - h_0)^{-1}\sqrt{|v|} & \text{for } z > \epsilon_{\text{max}}, \\ \sqrt{|v|(h_0 - z)^{-1}\sqrt{|v|}} & \text{for } z < \epsilon_{\text{min}}. 
\end{cases}
\]

Since \( \alpha_x(p) := e^{ixp}, x \in \mathbb{Z}^d \), is the eigenvector of \( v \) associated to \( \hat{v}(x) \),
\[
\text{Tr}(|b(z)|) = \sum_{x \in \mathbb{Z}^d} (|b(z)|\alpha_x, \alpha_x) = |a(z)| \sum_{x \in \mathbb{Z}^d} |\hat{v}(x)|, \quad z \in \mathbb{R} \setminus [\epsilon_{\text{min}}, \epsilon_{\text{max}}],
\]
(2.1)
and
\[
\text{Tr}(b(z)) = \sum_{x \in \mathbb{Z}^d} (b(z)\alpha_x, \alpha_x) = a(z) \sum_{x \in \mathbb{Z}^d} \hat{v}(x), \quad z \in \mathbb{C} \setminus [\epsilon_{\text{min}}, \epsilon_{\text{max}}],
\]
where
\[
a(z) := \int_{\mathbb{T}^d} \frac{dp}{z - \epsilon(p)}.
\]
For shortness write
\[
v^{1/2} := \mathcal{F}(\text{sign}(\hat{v})\sqrt{|v|}) \quad \text{and} \quad |v|^{1/2} := \mathcal{F}(\sqrt{|v|}).
\]
Then \( b(z) \) is the integral operator with the kernel
\[
B(p, q; z) := \int_{\mathbb{T}^d} v^{1/2}(p - t)|v|^{1/2}(t - q)dt, \quad p, q \in \mathbb{T}^d.
\]

Note that for any \( z \in \mathbb{C} \setminus [\epsilon_{\text{min}}, \epsilon_{\text{max}}], \)
\[
\|B(\cdot, ; z)||_{L^\infty(\mathbb{T}^d)^2} \leq \frac{1}{\text{dist}(z, [\epsilon_{\text{min}}, \epsilon_{\text{max}}])} \sum_{x \in \mathbb{Z}^d} |\hat{v}(x)|. 
\]
Therefore, \( b(\cdot) \) is Hilbert-Schmidt.

**Lemma 2.1 (Birman-Schwinger principle).** For any \( \mu \geq 0 \) and \( z \in \mathbb{R} \setminus [\epsilon_{\text{min}}, \epsilon_{\text{max}}], \)
\[
\dim \ker (h_\mu - z) = \dim \ker (1 - \mu b(z)).
\]
(2.3)
Moreover,
\[
\sigma(b(z)) \subset \mathbb{R}, \quad z \in \mathbb{R} \setminus [\epsilon_{\text{min}}, \epsilon_{\text{max}}].
\]
(2.4)

**Proof.** The equality (2.3) is well-known and can be done following the proof of for instance [1, Lemma 2.1]. To prove (2.4), we choose any \( \lambda \in \sigma(b(z)) \). Since \( b(z) \) is compact, either \( \lambda = 0 \) or \( \lambda \in \mathbb{C} \setminus \{0\} \) is an eigenvalue of finite multiplicity. Let \( f_\lambda \in L^2(\mathbb{T}^d) \) be a normalized eigenfunction. Then \( (b(z)f_\lambda, f_\lambda) = \lambda(s_\lambda f_\lambda, f_\lambda) \). Note that if \( (s_\lambda f_\lambda, f_\lambda) = 0 \), then
\[
(b(z)f_\lambda, f_\lambda) = \|\sqrt{|b(z)|}f_\lambda\|^2 = 0,
\]
and therefore, \( \lambda f_\lambda = s|b(z)|f_\lambda = 0 \), i.e., \( f_\lambda = 0 \). Hence, \( (s_\lambda f_\lambda, f_\lambda) \neq 0 \). Since both \( s \) and \( |b(z)| \) are self-adjoint, it follows that \( \lambda = \frac{(b(z)f_\lambda, f_\lambda)}{(s_\lambda f_\lambda, f_\lambda)} \in \mathbb{R} \).

Further we study \( b(z) \) only for \( z > \epsilon_{\text{max}} \); all results straightforwardly apply to the case \( z < \epsilon_{\text{min}} \) considering \(-h_0 + \mu v\).

We start by studying of the largest eigenvalue of \( b(z) \).

**Lemma 2.2.** The function
\[
z \in (\epsilon_{\text{max}}, +\infty) \mapsto \lambda(z) := \sup \sigma(b(z))
\]
is continuous and non-increasing. Moreover, \( \lambda(\cdot) \) is strictly decreasing in \( \{\lambda > 0\} \) and
\[
0 \leq \lambda(z) \leq \|b(z)\|.
\]
(2.6)
Finally \( \lambda(z_0) = 0 \) for some \( z_0 > \epsilon_{\text{max}} \) if and only if \( \hat{v} = 0 \).
Proof. Since $0 \in \sigma(b(z))$ and the spectral radius of $b(z)$ does not exceed $\|b(z)\| = \|b(z)\|$ follows. Let us show that $\lambda(\cdot)$ is continuous. Fix any $z_0 > \epsilon_{\text{max}}$. If there exists $z_k \to z_0$ such that $c := \lim_{k \to \infty} \lambda(z_k) < \lambda(z_0)$, then $\lambda(z_0) > 0$ so that it is an isolated eigenvalue of $b(z_0)$. Then for the spectral projection

$$P(z_k) := \frac{1}{2\pi i} \int_{|\xi - \lambda(z_k)| = \delta_0} (b(z_k) - \xi)^{-1} d\xi,$$

is the spectral projection associated to $\lambda(z_k)$. By the definition of $c$, $\lambda(z_k) \notin \{\xi \in \mathbb{C} : |\xi - \lambda(z_0)| < \delta_0\}$ for all large $k$ so that by the definition of $\lambda(z_k)$, $P(z_k) = 0$. However, by the norm continuity of $b(\cdot)$,

$$0 \neq \frac{1}{2\pi i} \int_{|\xi - c| = \delta_0} (b(z_0) - \xi)^{-1} d\xi = \lim_{k \to \infty} P(z_k) = 0,$$

a contradiction, where $s\lim$ is the strong limit. Analogous contradiction is obtained assuming the existence of $z_k \to z_0$ such that $\lim_{k \to \infty} \lambda(z_k) > \lambda(z_0)$. Thus, $\lambda(\cdot)$ is continuous.

Now we prove that $\lambda(\cdot)$ is non-increasing. It suffices to prove that $\lambda(\cdot)$ strictly decreases in $\{\lambda > 0\}$. Fix any $z_0 \in \{\lambda > 0\}$ and let $\lambda(z_0)$ be an eigenvalue of $b(z_0)$ of multiplicity $n_o$. Since $b(\cdot)$ is analytic and compact, by perturbation theory (see e.g. [5, Chap. II, Par. 4]), there exists $\epsilon > 0$ and $n_o$ differentiable functions $\theta_1, \ldots, \theta_{n_o} : (z_0 - \epsilon, \epsilon) \to (0, +\infty)$ with $\theta_i(z_0) = \lambda(z_0)$ for $i = 1, \ldots, n_o$ and $\theta_i(z)$ is an eigenvalue of $b(z)$ with associated differentiable eigenvectors $\phi_i(z) \in L^2(\mathbb{T}^d)$. Then for any $i = 1, \ldots, n_o$ and $z \in (z_0 - \epsilon, z_0 + \epsilon)$ from the equality

$$b(z)\phi_i(z) = \theta_i(z)\phi_i(z),$$

we get

$$(s\phi_i(z), \phi_i(z)) = \frac{\|b(z_0)\phi_i(z), \phi_i(z)\|}{\theta_i(z)} > 0.$$

Moreover, differentiating (2.7) and scalar multiplying by $s\phi_i(z)$ we obtain

$$-(\sqrt{|v|}(z - h_0)^{-2}\sqrt{|v|}\phi_i(z), \phi_i(z)) + (\phi_i'(z), [b(z)] - \theta_i(z)s)\phi_i(z)) = \theta_i'(z)(s\phi_i(z), \phi_i(z)).$$

By (2.7), $[b(z)] - \theta_i(z)s)\phi_i(z) = 0$, hence,

$$\theta_i'(z) = -\frac{||z - h_0)^{-1}\sqrt{|v|}\phi_i(z)||^2}{(s\phi_i(z), \phi_i(z))}.$$

Since $\theta_i(z)\phi(z) \neq 0$, this implies $\theta_i'(z) < 0$ in $(z_0 - \epsilon, z_0 + \epsilon)$. Hence, each $\theta_i$ is strictly decreasing. Since $\lambda(z) = \max_{1 \leq i \leq n_o} \theta_i(z_0)$, $\lambda(\cdot)$ also strictly decreases in $(z_0 - \epsilon, z_0 + \epsilon)$.

Clearly, if $\epsilon \leq 0$, then $b(z) \leq 0$ for any $z > \epsilon_{\text{max}}$. Thus, $\lambda \equiv 0$. Let us show that $v^+ \neq 0$, then $\lambda(z) > 0$ for any $z > \epsilon_{\text{max}}$. Indeed, if $\lambda(z_0) = 0$ for some $z_0 > \epsilon_{\text{max}}$, then by monotonicity, $\lambda \equiv 0$ in $(z_0, +\infty)$. Thus, for any $z > z_0$ and $\mu > 0$

$$\text{Ker}(h_\mu - z) = \text{Ker}(1 - \mu b(z)) = \{0\}.$$ 

However, since $v^+ \neq 0$, there exists a normalized $f_0 \in L^2(\mathbb{T}^d)$ such that $(vf_0, f_0) > 0$. Thus, if we choose $\mu > \frac{z_0 + 1 - \epsilon_{\text{max}}}{(vf_0, f_0)}$, then

$$(h_\mu f_0, f_0) \geq z_0 + 1,$$

i.e., by the self-adjointness of $h_\mu$ and (1.1), $\text{Ker}(h_\mu - z_0) \neq \{0\}$ for some $z_0 > z_0 + 1$, a contradiction.

Further, without loss of generality we assume that the set $\{z > \epsilon_{\text{max}}\}$ is non-empty so that by Lemma 2.2 $\lambda(z) > 0$ for any $z > \epsilon_{\text{max}}$. The following lemma shows that as $z \to \epsilon_{\text{max}}$, $b(z)$ can be represented as a small perturbation of a rank-one operator.

Lemma 2.3. Assume Hypothesis (1.1) Let

$$Q f(p) = v^{1/2} (p - p^0) \int_{\mathbb{T}^d} |v|^{1/2}(p^0 - q) f(q) dq$$

be the rank-one projection and

$$Q_1(z) := b(z) - a(z) Q.$$ (2.8)
(a) If \( d = 1 \), then there exists \( C_1 := C_1(\varepsilon, \tilde{\nu}, \gamma) > 0 \) such that
\[
\|Q_1(z)\| \leq C_1 a(z)
\]
for any \( z \in (\varepsilon_{\text{max}}, \varepsilon_{\text{max}} + 1) \).

(b) If \( d = 2 \), then there exists \( C_2 := C_2(\varepsilon, \tilde{\nu}, \gamma) > 0 \) such that
\[
\|Q_1(z)\| \leq C_2
\]
for any \( z > \varepsilon_{\text{max}} \) and there exists the operator-norm limit
\[
Q_1(\varepsilon_{\text{max}}) := \lim_{z \to +\infty} Q_1(z).
\]
Moreover, there exists \( C_3 := C_3(\varepsilon, \tilde{\nu}, \gamma) > 0 \) such that
\[
\|Q_1(z) - Q_1(\varepsilon_{\text{max}})\| \leq C_3(z - \varepsilon_{\text{max}})^{\gamma/2}
\]
for any \( z \in (\varepsilon_{\text{max}}, \varepsilon_{\text{max}} + 1) \).

**Proof.** By Hypothesis (b), there exists a ball \( B_{r_0}(p^0) \) such that
\[
c_1|p - p^0|^2 \leq \varepsilon_{\text{max}} - \varepsilon(p) \leq c_2|p - p^0|^2, \quad p \in B_{r_0}(p^0),
\]
and
\[
\varepsilon_{\text{max}} - \max_{p \in \mathbb{T}^d \setminus B_{r_0}(p^0)} \varepsilon(p) \geq c_3,
\]
where \( c_1, c_2, c_3 > 0 \) and \( r_0 \in (0, 1) \) are constants depending only on \( \varepsilon \). We rewrite \( Q_1(z) \) as
\[
Q_1(z) = Q_{11}(z) + Q_{12}(z),
\]
where
\[
Q_{11}(z) f(p) := \int_{\mathbb{T}^d} \int_{\mathbb{T}^d \setminus B_{r_0}(p^0)} \frac{v^{1/2}(p - t)|v|^{1/2}(t - q) dt}{z - \varepsilon(t)} f(q) dq \quad - \int_{\mathbb{T}^d \setminus B_{r_0}(p^0)} \frac{dt}{z - \varepsilon(t)} \int_{\mathbb{T}^d} v^{1/2}(p - p^0)|v|^{1/2}(p^0 - q) f(q) dq
\]
and
\[
Q_{12}(z) f(p) := \int_{\mathbb{T}^d} \int_{B_{r_0}(p^0)} \frac{[v^{1/2}(p - t)|v|^{1/2}(t - q) - v^{1/2}(p - p^0)|v|^{1/2}(p^0 - q)] dt}{z - \varepsilon(t)} f(q) dq.
\]
By the definitions of \( v^{1/2} \) and \( |v|^{1/2} \), and (2.13),
\[
\sup_{z \leq \varepsilon_{\text{min}}} \|Q_{11}(z)\| \leq \frac{2}{c_3} \sum_{x \in \mathbb{Z}^d} |\tilde{\nu}(x)|.
\]
We rewrite \( Q_{12}(z) \) as
\[
Q_{12}(z) := Q_{12}^1(z) + Q_{12}^2(z),
\]
where
\[
Q_{12}^1(z) f(p) := \int_{\mathbb{T}^d} \int_{B_{r_0}(p^0)} \frac{[v^{1/2}(p - t) - v^{1/2}(p - p^0)]|v|^{1/2}(t - q) dt}{z - \varepsilon(t)} f(q) dq
\]
and
\[
Q_{12}^2(z) f(p) := \int_{\mathbb{T}^d} \int_{B_{r_0}(p^0)} \frac{[|v|^{1/2}(t - q) - |v|^{1/2}(p^0 - q)] v^{1/2}(p - p^0) dt}{z - \varepsilon(t)} f(q) dq.
\]
By (2.12) and the Fubini Theorem,
\[
\|Q_{12}(z) f\|_{L^2}^2 = \sum_{x \in \mathbb{Z}^d} |\tilde{\nu}(x)| \left( \int_{\mathbb{T}^d} \left( \int_{B_{r_0}(p^0)} \frac{1 - e^{-ix \cdot (t - p^0)}}{z - \varepsilon(t)} |v|^{1/2}(t - q) dt \right) f(q) dq \right)^2.
\]
Then using the Hölder inequality we estimate

$$
\left| \int_{T_d} \left( \int_{B_{\alpha_0}(p_0)} \frac{1 - e^{-|x|^2(t-p_0)}}{z - \epsilon(t)} |v|^{1/2} (t - q^-) \, dt \right) f(q) \, dq \right|^2 
\leq \left| \int_{B_{\alpha_0}(p_0)} \left| \frac{1 - e^{-|x|^2(t-p_0)}}{z - \epsilon(t)} \right| \left( \int_{T_d} |v|^{1/2} (t - q^-) \, dq \right)^{1/2} \, dt \left( \int_{T_d} |f(q)|^2 \, dq \right)^{1/2} \, dt \right|^2 
= \|f\|^2 \sum_{y \in \mathbb{Z}^d} \left| \hat{v}(y) \right|^2 \left( \int_{B_{\alpha_0}(p_0)} \frac{2|\sin(x \cdot (t - p_0)/2)|}{z - \epsilon(t)} \, dt \right]^2
$$

where we used

$$
\int_{T_d} |v|^{1/2} (t - q^-)^2 \, dq = \int_{T_d} |v|^{1/2} (t - q^-)^2 \, dq = \sum_{y \in \mathbb{Z}^d} |\hat{v}(y)| = \|v\|_{c_1}
$$

for any $t \in T_d$. Note that $\frac{2-d+\gamma}{2} \in (0,1)$, where $\gamma \in (0,1)$ is given in Hypothesis \[1\] thus,

$$
2|\sin(x \cdot (t - p_0)/2)| \leq \frac{2-\delta}{2} |x|^{2-d+\gamma} (t - p_0)^{2-\delta+\gamma}. \quad (2.18)
$$

Hence, by (2.13)

$$
\int_{B_{\alpha_0}(p_0)} \frac{2|\sin(x \cdot (t - p_0)/2)|}{z - \epsilon(t)} \, dt \leq \frac{2-\delta}{2} \int_{B_{\alpha_0}(p_0)} \frac{|t - p_0|^{d-\delta+\gamma}}{c_1 |t - p_0|^{d-\delta+\gamma} + z - \epsilon_{\text{max}}} \, dt 
= \frac{2-\delta}{2} \frac{d\omega_d}{c_1} |x|^{\frac{2-d+\gamma}{2}} \left( \frac{z - \epsilon_{\text{max}}}{c_2} \right)^{1/2}, \quad (2.19)
$$

where in the equality we passed to polar coordinates, $w_d$ is the volume of the unit ball in $\mathbb{R}^d$, i.e., $\omega_1 := 2$ and $\omega_2 := \pi$, and $T_\alpha$ is given in \([A.1]\). Now if $d = 1$, then $\frac{d\omega_1}{c_1} \in (0,1)$ and thus, by Lemma \([A.1]\)

$$
T_d \left( \left( \frac{z - \epsilon_{\text{max}}}{c_2} \right)^{1/2} \right) \leq c_1 \gamma (z - \epsilon_{\text{max}})^{\frac{d-\delta}{\gamma}}, \quad z > \epsilon_{\text{max}}.
$$

for some $c_1, \gamma > 0$. If $d = 2$, then $\frac{d\omega_2}{c_1} \in (1,2)$ and hence

$$
T_d \left( \left( \frac{z - \epsilon_{\text{max}}}{c_2} \right)^{1/2} \right) \leq \frac{2\gamma_0^{1/2}}{\gamma}, \quad z > \epsilon_{\text{max}}.
$$

Hence,

$$
\|Q_{12}^1\| \leq \begin{cases} 
A_1 (z - \epsilon_{\text{max}})^{\frac{d-\delta}{\gamma}} & \text{if } d = 1, \\
A_2 & \text{if } d = 2,
\end{cases} \quad (2.20)
$$

where

$$
A_d := \begin{cases} 
\frac{c_1}{c_1} \left( 2^{3-\gamma} \sum_{y \in \mathbb{Z}^d} |\hat{v}(y)| \sum_{x \in \mathbb{Z}^d} |x|^{2-d+\gamma} |\hat{v}(x)| \right)^{1/2} & \text{if } d = 1, \\
\frac{4\pi \gamma_0^{1/2}}{c_1} \left( 2^{2-\gamma} \sum_{y \in \mathbb{Z}^d} |\hat{v}(y)| \sum_{x \in \mathbb{Z}^d} |x|^{2-d+\gamma} |\hat{v}(x)| \right)^{1/2} & \text{if } d = 2.
\end{cases} \quad (2.21)
$$

Now we estimate $\|Q_{12}^2\|$. By (2.22) and the Fubini Theorem,

\begin{align*}
|Q_{12}^2(z)f(p)| & \leq |v|^{1/2}(p - p_0) \sum_{x \in \mathbb{Z}^d} |\hat{v}(x)| \left| \int_{B_{\alpha_0}(p_0)} \frac{1 - e^{ix \cdot (t-p_0)}}{z - \epsilon(t)} e^{-ix \cdot t} \, dt \int_{T_d} e^{-ix \cdot q} f(q) \, dq \right| \\
& \leq |v|^{1/2}(p - p_0) \sum_{x \in \mathbb{Z}^d} |\hat{v}(x)| \left| \hat{f}(x) \right| \int_{B_{\alpha_0}(p_0)} \frac{1 - e^{ix \cdot (t-p_0)}}{z - \epsilon(t)} \, dt \\
& \leq |v|^{1/2}(p - p_0) \sum_{x \in \mathbb{Z}^d} |\hat{v}(x)| \left| \hat{f}(x) \right| \int_{B_{\alpha_0}(p_0)} \frac{2|\sin(x \cdot (t - p_0)/2)|}{z - \epsilon(t)} \, dt.
\end{align*}
where $\hat{f} = F^* f$. By (2.19) and the H"older inequality
\[
\sum_{x \in \mathbb{Z}^d} \sqrt{\hat{v}(x)} \frac{|\hat{f}(x)|}{|B_{\rho}(p)|} \int_{B_{\rho}(p^0)} 2|\sin [x \cdot (t - t^0)/2]| \, dt \\
\leq \sum_{x \in \mathbb{Z}^d} \left( 2^{d-\gamma} |x|^{2-d+\gamma} |\hat{v}(x)| \right)^{1/2} \frac{|\hat{f}(x)|}{|B_{\rho}(p^0)|} \int_{B_{\rho}(p^0)} \frac{|t - t^0|^{1-\gamma}}{c_1 (t - t^0 + 2) + z - \epsilon_{\max}} \, dt \\
\leq \frac{1}{c_1} \left( 2^{d-\gamma} \sum_{x \in \mathbb{Z}^d} |x|^{2-d+\gamma} |\hat{v}(x)| \right)^{1/2} \|\hat{f}\| \epsilon_{\max}^{1/2} \left( \frac{z - \epsilon_{\max}}{c_1} \right)^{1/2}.
\]

Thus, using $\|\hat{f}\|_{L^2(T^d)} = \|f\|_{L^2(T^d)}$ and (2.17) we get
\[
\|Q_{12}^2(z)\| \leq \begin{cases} 
A_1 (z - \epsilon_{\max})^{1/2} & \text{if } d = 1, \\
A_2 & \text{if } d = 2,
\end{cases} \tag{2.22}
\]
where $A_d$ is given in (2.21). Since,
\[
Q_1(z) = Q_{11}(z) + Q_{12}^1(z) + Q_{12}^2(z),
\]
from (2.15), (2.20) and (2.22) it follows that
\[
\|Q_1(z)\| \leq \frac{2}{c_3} \sum_{x \in \mathbb{Z}^d} |\hat{v}(x)| \left( 2A_1 (z - \epsilon_{\max})^{1/2} \right. \\
\left. \text{if } d = 1, \right) \left. 2A_2 \text{ if } d = 2. \right. \tag{2.24}
\]

(a) Let $d = 1$. Let us estimate $a(z)$ from below. By (2.13)
\[
a(z) \geq \int_{B_{\rho}(p^0)} \frac{dq}{c_2 |p - p^0|^2 + z - \epsilon_{\max}} = \frac{d\omega_d}{c_2} T_{d-1} \left( \frac{z - \epsilon_{\max}}{|q|} \right)^{1/2} \\
= \frac{\pi}{c^2_2 (z - \epsilon_{\max})^{1/2} \left[ 1 - \frac{2}{\pi} \arctan \left( \frac{z - \epsilon_{\max}}{c_2 r_0} \right) \right].
\]
This and (2.24) implies (2.9).

(b) Let $d = 2$. The estimate (2.10) directly follows from (2.24). Now we prove (2.11) - (2.12). By the definition of $Q_1(z)$, the norm limit
\[
Q_{11}(\epsilon_{\max}) := \lim_{z \to \epsilon_{\max}} Q_{11}(z)
\]
exists and
\[
Q_{11}(\epsilon_{\max}) f(p) := \int_{T^d} \int_{T^d \setminus B_\rho(p^0)} \frac{v^{1/2}(p - t)|v|^{1/2}(t - q)}{\epsilon_{\max} - \epsilon(t)} \, dt \, dq \\
- \int_{T^d \setminus B_\rho(p^0)} \frac{dt}{\epsilon_{\max} - \epsilon(t)} \int_{T^d} v^{1/2}(p - p^0)|v|^{1/2}(p^0 - q) \, f(q) \, dq.
\]
It is obvious that
\[
\|Q_{11}(z) - Q_{11}(\epsilon_{\max})\| \leq \tilde{c}_1 (z - \epsilon_{\max}), \quad z > \epsilon_{\max}, \tag{2.25}
\]
for some $\tilde{c}_1 > 0$ independent of $z$. Furthermore, repeating the same proof of the uniform boundedness of $Q_{12}^1(z)$ and $Q_{12}^2(z)$ one can show the boundedness of operators
\[
Q_{12}^1(\epsilon_{\max}) f(p) := \int_{T^d} \int_{B_\rho(p^0)} \frac{\left[ v^{1/2}(p - t) - v^{1/2}(p - p^0) \right] |v|^{1/2}(t - q)}{\epsilon_{\max} - \epsilon(t)} \, dt \, dq
\]
and
\[
Q_{12}^2(\epsilon_{\max}) f(p) := \int_{T^d} \int_{B_\rho(p^0)} \frac{\left[ v^{1/2}(t - q) - v^{1/2}(p^0 - q) \right] v^{1/2}(p - p^0)}{\epsilon_{\max} - \epsilon(t)} \, dt \, dq.
\]
We claim that for any $z > \epsilon_{\max}$
\[
\|Q_{12}^1(z) - Q_{12}^1(\epsilon_{\max})\| \leq \tilde{c}_2 (z - \epsilon_{\max})^{7/2}, \quad \|Q_{12}^2(z) - Q_{12}^2(\epsilon_{\max})\| \leq \tilde{c}_2 (z - \epsilon_{\max})^{5/2}. \tag{2.26}
\]
for some \( \tilde{c}_2 > 0 \) independent of \( z \). We prove only (2.20); the proof of (2.27) being similar. For any \( f \in L^2(T^d) \) let us estimate the \( L^2 \)-norm of

\[
(Q_{12}^1(z) - Q_{12}^1(\epsilon_{\text{max}})) f(p) = -(z - \tilde{c}_{\text{max}}) \int_{T^d} \int_{B_\nu(p')} \frac{|v^{1/2}(p - t) - v^{1/2}(p - p')| |\tilde{v}|^{1/2}(t - q) \, dt}{(z - \epsilon(t))(\epsilon_{\text{max}} - \epsilon(t))} f(q) \, dq.
\]

As in (2.10), (2.15)

\[
\| (Q_{12}^1(z) - Q_{12}^1(\epsilon_{\text{max}})) f \| \leq 2^{2-\gamma} (z - \epsilon_{\text{max}}) \| \tilde{v} \|, \| f \| \times \sum_{x \in \mathbb{Z}^d} |x| |\tilde{v}(x)| \int_{B_{\rho_0}(p')} c_1 |t - p'|^2 |dt| \frac{\tilde{v}(x)}{(c_1 |t - p'|^2 + z - \epsilon_{\text{max}})^{1/2}}.
\]

Then passing to the polar coordinates we obtain

\[
\| (Q_{12}^1(z) - Q_{12}^1(\epsilon_{\text{max}})) f \| \leq \tilde{c}_0 (z - \epsilon_{\text{max}}) \| f \| \int_0^r \frac{r^{2-1} \, dr}{c_1 r^2 + z - \epsilon_{\text{max}}},
\]

where \( \tilde{c}_0 := 2\pi \left( 2^{2-\gamma} \| \tilde{v} \|, \sum_{x \in \mathbb{Z}^d} |x| |\tilde{v}(x)| \right)^{1/2} \). Now using the change of variables \( r = (z - \epsilon_{\text{max}})^{1/2} t \) we get

\[
\| (Q_{12}^1(z) - Q_{12}^1(\epsilon_{\text{max}})) f \| \leq \tilde{c}_0 (z - \epsilon_{\text{max}})^{\gamma/2} \int_0^{+\infty} \frac{t^{2-1} \, dt}{c_1 t^2 + 1}.
\]

Since \( \int_0^{+\infty} \frac{r^{2-1} \, dr}{c_1 r^2 + z - \epsilon_{\text{max}}} < \infty \), (2.20) follows.

Now we set

\[
Q_1(\epsilon_{\text{max}}) := Q_{11}(\epsilon_{\text{max}}) + Q^1_{11}(\epsilon_{\text{max}}) + Q^1_{12}(\epsilon_{\text{max}}).
\]

Then from (2.28), (2.25), (2.20) and (2.24), we conclude (2.11) and (2.12).

Next we study the case of sign-definite potentials.

**Lemma 2.4.** Assume Hypothesis 1.1 and that \( \tilde{v} \geq 0 \). Then there exists \( C_4 := C_4(\epsilon, \gamma) > 0 \) such that

\[
|\text{Tr}(b(z)) - \| b(z) \| | \leq C_4 \sum_{x \in \mathbb{Z}^d} |x|^{2-d+\gamma} |\tilde{v}(x)| < \infty.
\]  

**Proof.** Let \( \nu(z) := \| b(z) \| \); by self-adjointness and (2.25) \( \lambda(z) = \nu(z) \) for any \( z > \epsilon_{\text{max}} \). Since \( \tilde{v} \geq 0 \), one has \( b(z) \geq 0 \) and \( v^{1/2} = |v|^{1/2} \). By Lemma 2.3 the self-adjoint operator \( \frac{b(z)}{\nu(z)} = Q + \frac{Q_1(z)}{\nu(z)} \) is a small perturbation of the self-adjoint rank-one projector

\[
Q := \phi_0(1, \phi_0),
\]

where \( \phi_0(p) = v^{1/2}(p - p^0) \), therefore, by the standard perturbation theory, the eigenvector \( f_z \) associated to \( \nu(z) \) can be written as \( f_z = \phi_0 + \phi_z \), where \( \phi_z \) is orthogonal to \( \phi_0 \). Then using

\[
(\phi_0, \phi_0) = \sum_{x \in \mathbb{Z}^d} \tilde{v}(x)
\]

one has

\[
\nu(z) \phi_0 + \nu(z) \phi_z = a(z) \phi_0 \sum_{x \in \mathbb{Z}^d} \tilde{v}(x) + Q_1(z) \phi_0 + Q_1(z) \phi_z.
\]

Multiplying (2.24) by \( \phi_z \) we get

\[
\nu(z) \| \phi_z \|^2 = (Q_1(z) \phi_0, \phi_z) + (Q_1(z) \phi_z, \phi_z).
\]

Since \( (\phi_z, \phi_0) = 0 \), one has \( (b(z) \phi_z, \phi_z) = (Q_1(z) \phi_z, \phi_z) \geq 0 \). Thus,

\[
\| b(z) \| \| \phi_z \|^2 = (Q_1(z) \phi_0, \phi_z) + (b(z) \phi_z, \phi_z),
\]

and in particular, \( (Q_1(z) \phi_0, \phi_z) \leq 0 \). Multiplying (2.24) by \( \phi_0 \) we get

\[
\nu(z) \sum_{x \in \mathbb{Z}^d} \tilde{v}(x) = a(z) \left( \sum_{x \in \mathbb{Z}^d} \tilde{v}(x) \right)^2 + (Q_1(z) \phi_0, \phi_0) + (Q_1(z) \phi_z, \phi_z).
\]
Since all nonzero eigenvalues of $b(z)$ are positive and $(Q_1(z)\phi_z, \phi_0) \geq 0$, by (2.11) and (2.20),
\[
\left| \text{Tr}(b(z)) - \nu(z) \right| \leq \left( a(z) \sum_{x \in \mathbb{Z}^d} \hat{v}(x) - \nu(z) \right) \sum_{x \in \mathbb{Z}^d} \hat{v}(x) = -(Q_1(z)\phi_0, \phi_0) - (Q_1(z)\phi_z, \phi_0) \leq -(Q_1(z)\phi_0, \phi_0).
\] (2.31)

Note that
\[
-(Q_1(z)\phi_0, \phi_0) = \int_{\mathbb{T}^d} v^{1/2}(p - p^0) v^{1/2}(p^0 - q) - v^{1/2}(p - t) v^{1/2}(t - q) \frac{z - \varepsilon(t)}{z} \, dt \, dp \, dq \, dp
\] 
\[= \int_{\mathbb{T}^d} v(0)^2 - |v(t - p^0)|^2 \frac{z - \varepsilon(t)}{z} \, dt,
\]
where we used
\[
\int_{\mathbb{T}^d} v^{1/2}(p - t) v^{1/2}(s - p) \, dp = v(s - t), \quad s, t \in \mathbb{T}^d.
\]
Since $\hat{v} \geq 0$,
\[
|v(p)|^2 = \sum_{x,y \in \mathbb{Z}^d} \hat{v}(x) \hat{v}(y) \cos([x - y] \cdot p),
\]
and thus, $p = 0$ is the maximum point of $|v(\cdot)|^2$. Therefore, the map $z \in (\varepsilon_{\text{max}}, +\infty) \mapsto -(Q_1(z)\phi_0, \phi_0)$ is decreasing and
\[
-(Q_1(z)\phi_0, \phi_0) = 2 \sum_{x,y \in \mathbb{Z}^d} \hat{v}(x) \hat{v}(y) \int_{\mathbb{T}^d} \sin^2 \left( \frac{|x - y| |t - p^0|}{2} \right) \frac{z - \varepsilon(t)}{z} \, dt. \tag{2.32}
\]
Since $\sin^2 \left( \frac{(x-y)(t-p^0)}{2} \right) \leq \left| \frac{(x-y)(t-p^0)}{2} \right|^{2-d+\gamma}$, from (2.32) it follows that
\[
-(Q_1(z)\phi_0, \phi_0) \leq 2^{d-1-\gamma} \sum_{x,y \in \mathbb{Z}^d} \hat{v}(x) \hat{v}(y) |x - y|^{2-d+\gamma} \int_{\mathbb{T}^d} \frac{|t - p^0|^{2-d+\gamma}}{z - \varepsilon(t)} \, dt.
\]
Using
\[
|x - y|^{2-d+\gamma} \leq 2^{2-d+\gamma} (|x|^{2-d+\gamma} + |y|^{2-d+\gamma})
\]
one can readily check that
\[
\sum_{x,y \in \mathbb{Z}^d} \hat{v}(x) \hat{v}(y) |x - y|^{2-d+\gamma} \leq 2^{3-d+\gamma} \sum_{x \in \mathbb{Z}^d} |\hat{v}(x)| \sum_{x \in \mathbb{Z}^d} |x|^{2-d+\gamma} |\hat{v}(x)|.
\]
Thus, from (2.32) we get
\[
-(Q_1(z)\phi_0, \phi_0) \leq 4 \sum_{x \in \mathbb{Z}^d} |\hat{v}(x)| \sum_{x \in \mathbb{Z}^d} |x|^{2-d+\gamma} |\hat{v}(x)| \int_{\mathbb{T}^d} \frac{|t - p^0|^{2-d+\gamma}}{z - \varepsilon(t)} \, dt.
\]
Let $B_{\rho_0}(p^0)$ be such that (2.13) and (2.14) hold. We write
\[
\int_{\mathbb{T}^d} \frac{|t - p^0|^{2-d+\gamma}}{z - \varepsilon(t)} \, dt = \int_{B_{\rho_0}(p^0)} \frac{|t - p^0|^{2-d+\gamma}}{z - \varepsilon(t)} \, dt + \int_{\mathbb{T}^d \setminus B_{\rho_0}(p^0)} \frac{|t - p^0|^{2-d+\gamma}}{z - \varepsilon(t)} \, dt.
\] (2.33)
By (2.14), $\sup_{z < \varepsilon_{\text{min}}} I_2(z) < \frac{16\pi^2}{c_1}$. Using (2.13) in $I_1(z)$ and passing to polar coordinates we get
\[
I_1(z) = \int_{B_{\rho_0}(p^0)} \frac{|t - p^0|^{2-d+\gamma}}{c_1|t - p^0|^2 + z - \varepsilon_{\text{max}}} \, dt = \frac{d\omega_d}{c_1} T_{1+\gamma}((z - \varepsilon_{\text{max}})/c_1)^{1/2},
\]
where $T_\alpha$ is defined in (A.1). Since $1 + \gamma > 1$, by Lemma (A.1)
\[
I_1(z) \leq \frac{d\omega_d r_0^{\gamma}}{c_1 \gamma} < \infty
\]
for any $z > \varepsilon_{\text{max}}$. Thus, from (2.31) and (2.32),
\[
\left| \text{Tr}(b(z)) - \nu(z) \right| \leq 4 \left( \frac{16\pi^2}{c_1} + \frac{d\omega_d r_0^{\gamma}}{c_1 \gamma} \right) \sum_{x \in \mathbb{Z}^d} |x|^{2-d+\gamma} |\hat{v}(x)|
\]
Moreover:

there exists $C$ methods for the non-selfadjoint operators, the eigenvalue
By Lemma 2.3, 

Let $\lambda_0(z) = \lambda(z) \geq \lambda_1(z) \geq \ldots > 0$ be all positive eigenvalues (if any) of $b(z)$ counted with their multiplicities. Then there exists $C_5 := C_5(\xi, \gamma) > 1$ such that for any $z > \epsilon_{\text{max}}$

$$0 \leq \lambda_k(z) \leq C_5, \quad k \geq 1.$$  

(2.34)

Moreover:

(a) if $\sum_{x \in \mathbb{Z}^d} \hat{v}(x) < 0$, then

$$0 < \sup_{x > \epsilon_{\text{max}}} \lambda(z) \leq C_5;$$

(b) if $\sum_{x \in \mathbb{Z}^d} \hat{v}(x) \geq 0$, then there exists $\delta_0 := \delta_0(\xi, \gamma) > 0$ such that:

(b1) if $\kappa_0 := \sum_{x \in \mathbb{Z}^d} \hat{v}(x) > 0$, then for any $z \in (\epsilon_{\text{max}}, \epsilon_{\text{max}} + \delta_0)$

$$\frac{\lambda(z)}{a(z)} = \kappa_0 + \left\{ \begin{array}{ll}
\frac{g_1(z)(z - \epsilon_{\text{max}})}{\kappa_0 a(z)} & \text{if } d = 1,
\frac{C_6}{\kappa_0 a(z)} + \frac{(z - \epsilon_{\text{max}})^{\gamma/4} g_2(z)}{\kappa_0 a(z)} & \text{if } d = 2,
\end{array} \right.$$  

(2.35)

where $C_6 \in \mathbb{R}$, $g_1, g_2 \in C^0[\epsilon_{\text{max}}, \epsilon_{\text{max}} + \delta_0];$

(b2) if $\kappa_0 := \sum_{x \in \mathbb{Z}^d} \hat{v}(x) = 0$, then

$$\kappa_1 := \int_{\mathbb{R}^d} |v(p - p_0)|^2 dp \quad \epsilon_{\text{max}} = \epsilon(p)$$

is finite and for any $z \in (\epsilon_{\text{max}}, \epsilon_{\text{max}} + \delta_0)$

$$\frac{\lambda(z)^2}{a(z)} = \kappa_1 + \left\{ \begin{array}{ll}
\frac{g_3(z)(z - \epsilon_{\text{max}})^2}{a(z)^{\gamma/2}} & \text{if } d = 1,
\frac{C_7}{a(z)^{\gamma/2}} + \frac{(z - \epsilon_{\text{max}})^{\gamma/4} g_4(z)}{a(z)^{\gamma/4}} & \text{if } d = 2,
\end{array} \right.$$  

(2.36)

where $C_7 \in \mathbb{R}$, $g_3, g_4 \in C^0[\epsilon_{\text{max}}, \epsilon_{\text{max}} + \delta_0].$

Proof. First assume that

$$\kappa_0 := \sum_{x \in \mathbb{Z}^d} \hat{v}(x) \neq 0.$$

Then $\kappa_0$ is the unique nonzero eigenvalue of $Q$ and its associated eigenvector is $\phi_0(p) := v^{1/2}(p - p_0)$. Note that

$$(Q - \xi)^{-1} = -\frac{1}{\xi} - \frac{Q}{\xi(z - \kappa_0)}, \quad \xi \neq 0, \kappa_0.$$  

(2.37)

By Lemma 2.3, $\frac{\partial(v)}{\partial(z)}$ is a small perturbation of $Q$, and hence, by the standard perturbation theory methods for the non-selfadjoint operators, the eigenvalue $\eta(z)$ of $b(z)$ of maximal modulus satisfies

$$\lim_{z \to \epsilon_{\text{max}}} \eta(z) = \kappa_0.$$  

(2.38)

Let $\epsilon := \frac{|\kappa_0|}{8} > 0$. By Lemma 2.3 there exists $\delta_1 \in (0, 1)$ such that

$$\left| \frac{\eta(z)}{a(z)} - \kappa_0 \right| < \epsilon \quad \text{and} \quad \|Q_1(z)(Q - \xi)^{-1}\| < \frac{1}{2}$$

for any $z \in (\epsilon_{\text{max}}, \epsilon_{\text{max}} + \delta_1)$ and any $\xi \in \mathbb{C}$ such that $|\xi - \kappa_0| = \epsilon$. Then for any such $\xi$ one has

$$\left( \frac{b(z)}{a(z)} - \xi \right)^{-1} = (Q - \xi)^{-1} \left( 1 + \frac{Q_1(z)}{a(z)} (Q - \xi)^{-1} \right)^{-1}$$

$$= (Q - \xi)^{-1} + \sum_{n \geq 1} (-1)^n a(z)^n (Q - \xi)^{-1} [Q_1(z)(Q - \xi)^{-1}]^n.$$  

(2.39)
Therefore, if we integrate this equality over the complex circle $|\xi - \kappa_0| = \epsilon$ and insert (2.37), then by the Residue Theorem for analytic functions and the estimates (2.9) and (2.10),

$$P(z) := \frac{1}{2\pi i} \int_{|\xi - \kappa_0| = \epsilon} \frac{b(z)}{a(z) - \xi} \, d\xi = - \frac{Q}{\kappa_0} - \frac{1}{a(z)} \frac{Q_1(z)Q + QQ_1(z)}{\kappa_0^2} + R(z) \frac{1}{a(z)},$$

where $\|R(z)\| = O((z - \epsilon_\text{max})^{\gamma/4})$ as $z \to \epsilon_\text{max}$. By the definition of $\epsilon$, $P(z)$ is the spectral projection of $b(z)$ associated to its eigenvalue $\frac{\eta(z)}{a(z)}$. Hence, using

$$Q\phi_0 = \kappa_0 \phi_0$$

and

$$(Q_1(z)Q + QQ_1(z))\phi_0 = \kappa_0 Q_1(z)\phi_0 + (Q_1(z)\phi_0, \phi_1)\phi_0,$$

where $\phi_1(p) := |v|^{1/2}(p - p^0)$, we deduce that the associated eigenvector $f_z$ is

$$f_z := -P(z)\phi_0 = \phi_0 + \frac{\kappa_0 Q_1(z)\phi_0 + (Q_1(z)\phi_0, \phi_1)\phi_0}{\kappa_0^2 a(z)} + \frac{T(z)\phi_0}{a(z)}.$$

Then scalar multiplying the eigenvalue equation $\eta(z) a(z) f_z = (Q + \frac{Q_1(z)}{a(z)}) f_z$ by $\phi_1$ and using $(\phi_0, \phi_1)_{L^2} = \kappa_0$ we get

$$\left(\frac{\eta(z)}{a(z)} - \kappa_0\right)\kappa_0 + \frac{(Q_1(z)\phi_0, \phi_1)}{\kappa_0^2 a(z)} \left(\frac{2\eta(z)}{a(z)} - 3\kappa_0\right) \frac{\phi_0 - \phi_1}{a(z)} = h_0(z),$$

where $h_0 \in C^0(\epsilon_\text{max}, \epsilon_\text{max} + \delta_1)$ satisfies $|h_0(z)| = O((z - \epsilon_\text{max})^{\gamma/4})$ as $z \to \epsilon_\text{max}$. Hence,

$$\frac{\eta(z)}{a(z)} = \kappa_0 + \frac{(Q_1(z)\phi_0, \phi_1)}{\kappa_0^2 a(z)} + \frac{h_1(z)}{\kappa_0 a(z)}, \quad (2.39)$$

where $h_1 \in C^0(\epsilon_\text{max}, \epsilon_\text{max} + \delta_1)$ satisfies $|h_1(z)| = O((z - \epsilon_\text{max})^{\gamma/4})$ as $z \to \epsilon_\text{max}$. Notice that

$$(Q_1(z)\phi_0, \phi_1) = \int_{T^d} \frac{|v(p - p^0)|^2 - |v(0)|^2}{a(p)} \, dp.$$

Since $v \in C^0(T^d)$, by $v = F\hat{v}$ we have

$$\|v(p - p^0)|^2 - |v(0)|^2\| \leq 2\|v\|_{L^\infty} \|v(p - p^0) - v(0)\|$$

$$\leq 2\|v\|_{L^\infty} \sum_{x \in \mathbb{Z}^d} |\hat{v}(x)| \left|\sin \frac{x \cdot (p - p^0)}{2}\right|$$

$$\leq \begin{cases} 2\|v\|_{L^\infty} \|p - p^0\| \sum_{x \in \mathbb{Z}^d} |x||\hat{v}(x)| & \text{if } d = 1, \\
2\|v\|_{L^\infty} \|p - p^0\|^{\gamma} \sum_{x \in \mathbb{Z}^d} |x|^\gamma |\hat{v}(x)| & \text{if } d = 2. 
\end{cases}$$

Separating integral in (2.40) into integrals over $T \setminus B_{\gamma\eta}^0(p^0)$ and $B_{\gamma\eta}^0(p^0)$ and using (2.13) and (2.14) we obtain

$$|(Q_1(z)\phi_0, \phi_1)| \leq \tilde{c}_1 + \tilde{c}_2 \int_{B_{\gamma\eta}^0(p^0)} \frac{|p - p^0|^{\alpha}}{c_1|p - p^0|^{2\alpha} + z - \epsilon_\text{max}},$$

where $\tilde{c}_1, \tilde{c}_2 > 0$ and $\alpha = 1$ for $d = 1$ and $\alpha = \gamma$ if $d = 2$. Now passing to polar coordinates and using Lemma A.1 for any $z \in (\epsilon_\text{max}, \epsilon_\text{max} + \delta_1)$ we get

$$|(Q_1(z)\phi_0, \phi_1)| \leq \tilde{c}_1 + \tilde{c}_2 d\omega d \frac{T_{d-\alpha+\gamma}}{c_1 \left(\frac{z - \epsilon_\text{max}}{\epsilon_1}\right)^{1/2}}$$

$$\leq \begin{cases} \tilde{c}_1 - \tilde{c}_3 \ln(z - \epsilon_\text{max}) & \text{if } d = 1, \\
\tilde{c}_3 & \text{if } d = 2. 
\end{cases} \quad (2.41)$$

where $T_\alpha$ is defined in (A.1) and $\tilde{c}_3 > 0$. Thus, if $d = 1$, then (2.39) is represented as

$$\frac{\eta(z)}{a(z)} = \kappa_0 + \frac{h_2(z)}{\kappa_0 a(z)}, \quad (2.42)$$
where \( h_2 \in C^0(\epsilon_{\text{max}}, \epsilon_{\text{max}} + \delta_1) \) satisfies \(|h_2(z)| \leq -\tilde{c}_4 \ln(z - \epsilon_{\text{max}})\) for some \( \tilde{c}_4 > 0 \). If \( d = 2 \), then there exists the norm-limit

\[
(Q_1(\epsilon_{\text{max}})\phi_0, \phi_1) := \lim_{z \to \epsilon_{\text{max}}} (Q_1(z)\phi_0, \phi_1).
\]

Repeating the arguments of (2.12) one can show

\[
|((Q_1(z)\phi_0, \phi_1) - (Q_1(\epsilon_{\text{max}})\phi_0, \phi_1))| \leq \tilde{c}_5 (z - \epsilon_{\text{max}})^{\nu/4}
\]

for any \( z \in (\epsilon_{\text{max}}, \epsilon_{\text{max}} + \delta) \). Thus, (2.39) is rewritten as

\[
\eta(z) = \kappa_0 + \frac{(Q_1(\epsilon_{\text{max}})\phi_0, \phi_1)}{\kappa_0 a(z)} + \frac{h_3(z)}{\kappa_0 a(z)}.
\]

where \( h_3 \in C^0(\epsilon_{\text{max}}, \epsilon_{\text{max}} + \delta_1) \) satisfies \(|h_3(z)| = O((z - \epsilon_{\text{max}})^{\nu/4})\) as \( z \to \epsilon_{\text{max}} \). Hence, follows.

Now we consider the case \( \eta_0(z) = \eta_0(z), \eta_1(z), \ldots \) be all nonzero eigenvalues of \( b(z) \) counted with multiplicities as \(|\eta_0(z)| \geq |\eta_1(z)| \geq \ldots > 0 \) and let \( \nu_1(z) = \|b(z)\| \geq \nu_2(z) \geq \ldots > 0 \) be all eigenvalues of \(|b(z)|\), then by [3, Lemma 3.3] we have

\[
0 < |\eta_0(z)| < \nu_1(z) \quad \text{and} \quad 0 < |\eta_0(z)|\eta_1(z) \leq \nu_0(z)\nu_1(z).
\]

By Lemma 2.4

\[
\nu_0(z) = a(z) \left[ \sum_{x \in \mathbb{Z}^d} |\hat{v}(x)| + o(1) \right]
\]

and

\[
\nu_1(z) \leq C_4 \sum_{x \in \mathbb{Z}^d} |x|^{2-d+\gamma} |\hat{v}(x)|,
\]

and by (2.42), (2.44)

\[
\eta_0(z) = a(z) \left[ \sum_{x \in \mathbb{Z}^d} \hat{v}(x) + o(1) \right].
\]

Therefore, there exists \( \delta_2 \in (0, \delta_1) \) such that

\[
|\eta_1(z)| \leq \frac{\nu_0(z)\nu_1(z)}{|\eta_0(z)|} \leq \tilde{C}_5 := C_4 \left| \sum_{x \in \mathbb{Z}^d} \hat{v}(x) \right|^{-1} \sum_{x \in \mathbb{Z}^d} |x|^{2-d+\gamma} |\hat{v}(x)| + 1.
\]

for any \( z \in (\epsilon_{\text{max}}, \epsilon_{\text{max}} + \delta_2) \). Since \(|\eta_1(z)| \leq |\eta_0(z)| \leq \|b(z)\| \) for any \( z > \epsilon_{\text{max}} \) and the map \( z \mapsto \|b(z)\| \) is non-increasing,

\[
\sup_{z \geq \epsilon_{\text{max}} + \delta_2} |\eta_1(z)| \leq \|b(\epsilon_{\text{max}} + \delta_2)\| = \|b(\epsilon_{\text{max}} + \delta_2)\|.
\]

Since \(|\eta_k| \leq |\eta_1| \) for any \( k \geq 1 \), we get

\[
|\eta_k(z)| \leq C_5 := \max\{\tilde{C}_5, \|b(\epsilon_{\text{max}} + \delta_2)\|\}, \quad k \geq 1.
\]

In particular, from (2.35) we get (2.45).

Finally, we observe that

- if \( \kappa_0 = \sum_{x \in \mathbb{Z}^d} \hat{v}(x) < 0 \), then by (2.35), \( \eta_0(z) < 0 \), and hence the assertion (a) follows from (2.43);  
- if \( \kappa_0 = \sum_{x \in \mathbb{Z}^d} \hat{v}(x) > 0 \), then \( \lambda(z) = \eta_0(z) \) and the assertion (b) follows from (2.38), (2.42) and (2.44).

Now we consider the case \( \kappa_0 = \sum_{x \in \mathbb{Z}^d} \hat{v}(x) = 0 \). In this case since \( v(0) = 0 \),

\[
(Q_1(z)f, \phi_1)_{L^2} = \int_{T^d} \int_{T^d} \frac{v(t - s)|v|^{1/2}(t - q) dt}{z - \epsilon(t)} f(q) dq.
\]

Moreover, repeating the same arguments of (2.41) we have

\[
|Q_1(z)\phi_0, \phi_1)_{L^2}| \leq \begin{cases} 
\tilde{c}_3 \|v\|_{L^2} \|f\|_{L^2} \left[ \tilde{c}_3 - \tilde{c}_2 \ln(z - \epsilon_{\text{max}}) \right] & \text{if } d = 1, \\
\tilde{c}_3 \|v\|_{L^2} \|f\|_{L^2} & \text{if } d = 2.
\end{cases}
\]
for some $\tilde{c}_1, \tilde{c}_2, \tilde{c}_3 > 0$. Thus,
\[
\kappa_z := (Q_1(z)\phi_0, \phi_0) = \int_{\mathbb{T}^d} \frac{|v(t-p_0)|^2}{z - \epsilon(t)} dt \in (0, +\infty)
\] (2.48)
and
\[
\kappa_1 := (Q_1(\epsilon_{\text{max}})\phi_0, \phi_0) := \sup_{z > \epsilon_{\text{max}}} (Q_1(z)\phi_0, \phi_0) = \int_{\mathbb{T}^d} \frac{|v(t-p_0)|^2}{\epsilon_{\text{max}} - \epsilon(t)} dt < +\infty.
\] (2.49)

Note that since 0 is not an isolated eigenvalue of $Q$, hence, we cannot directly use the perturbation theory for $b(z)$.

Let us show that if $z - \epsilon_{\text{max}} > 0$ is sufficiently small, then the spectral projection
\[
P_\epsilon(z) := \frac{1}{2\pi i} \int_{|\xi - \kappa_1| = \delta} \left( \frac{\kappa_1(z)}{\xi - \kappa_z} \right)^{-1} d\xi
\] (2.50)
is non-zero for any sufficiently small $|\epsilon|, \delta > 0$. Indeed, since $Q_1(z)$ is rank-one and $(Q_1(z)\phi_0, \phi_0)$ is the unique positive eigenvalue of $Q_1(z)$,
\[
\left( \xi - Q_1(z) \right)^{-1} = \frac{1}{\xi^2 + Q_1(z)} , \quad \xi \in \mathbb{C} \setminus \{0, \kappa_z\}.
\]

Therefore, for $\delta, |\epsilon| < \kappa_1/8$
\[
P_\epsilon(z) := \frac{1}{2\pi i} \int_{|\xi - \kappa_1| = \delta} \left[ 1 - \left( \frac{\kappa_1(z)}{\xi - \kappa_z} \right)^{-1} \right]^{-1} \left( \frac{\kappa_1(z)}{\xi - \kappa_z} \right)^{-1} d\xi
\]
\[
= \frac{1}{2\pi i} \int_{|\xi - \kappa_1| = \delta} \left[ 1 - \left( \frac{1}{\xi^2 + Q_1(z)} \right) \right]^{-1} \left( \frac{1}{\xi^2 + Q_1(z)} \right) d\xi.
\]

Notice that
\[
\|Q_1(z)f\|/\|f\| \leq \|v|^{1/2}\left\{ \begin{array}{ll}
\tilde{c}_4 - \tilde{c}_5 \ln(z - \epsilon_{\text{max}}) & \text{if } d = 1, \\
\tilde{c}_5 & \text{if } d = 2
\end{array} \right.
\]

for some $\tilde{c}_4, \tilde{c}_5 > 0$ independent of $z$. Since $a(z)$ behaves as $(z - \epsilon_{\text{max}})^{-1/2}$ for $d = 1$ and as $-\ln(z - \epsilon_{\text{max}})$ for $d = 2$, by (2.50) and (2.11) we obtain
\[
\left\| \frac{1}{\xi^2 + Q_1(z)} \right\| \leq \left\| \frac{Q_1(z)}{\xi - \kappa_z} \right\| \leq \begin{cases}
-c_3(z - \epsilon_{\text{max}})^{7/4} \ln(z - \epsilon_{\text{max}}) & \text{if } d = 1, \\
-c_4 & \text{if } d = 2
\end{cases}
\]
for some $c_3, c_4 > 0$, where we took also account that $|\xi - \kappa_z| = \delta > 0$. In particular, for all sufficiently small $z - \epsilon_{\text{max}} > 0$
\[
P_\epsilon(z) := \frac{1}{2\pi i} \int_{|\xi - \kappa_1| = \delta} \sum_{n \geq 0} \left( \left( \frac{1}{\xi^2 + Q_1(z)} \right) \frac{Q_1(z)}{\xi - \kappa_z} \right)^{n+1} \left( \frac{1}{\xi^2 + Q_1(z)} \right) d\xi,
\]
and thus,
\[
P_\epsilon(z) = \frac{1}{2\pi i} \int_{|\xi - \kappa_1| = \delta} \left( \frac{1}{\xi^2 + Q_1(z)} \right) d\xi
\]
\[
+ \sum_{n \geq 1} \frac{1}{2\pi i} \int_{|\xi - \kappa_1| = \delta} \left( \left( \frac{1}{\xi^2 + Q_1(z)} \right) \frac{Q_1(z)}{\xi - \kappa_z} \right)^n \left( \frac{1}{\xi^2 + Q_1(z)} \right) d\xi
\]
\[
= \frac{Q_1(z)}{\kappa_z} + \sum_{n \geq 1} \left( \kappa_z + \epsilon \right)^{n/2} \left( \frac{Q_1(z)Q_1(z)}{\kappa_z^{n+1}} + \sum_{0 \leq k \leq n-1} \frac{Q_1(z)^k Q_1(z)^{n-k-1}}{\kappa_z^{n+1}} \right),
\]
where in the second equality we used the Cauchy’s Integral Theorem for analytic functions. Therefore,
\[ P_1(z)\phi_0 = \phi_0 + \psi_z, \] (2.51)
where
\[ \psi_z := \sum_{n \geq 1} \frac{(\kappa_z + 1)^n}{n! a(z)^{n+1}} \left[ \frac{Q_1(z)^n}{\kappa_z^{2n}} \phi_0 + \sum_{0 \leq k \leq n-1} \frac{(Q_1(z)^{n-k+1})}{\kappa_z^{2n+1}} Q_1(z)^{k} \phi_0 \right]. \] (2.52)

Now if \( d = 1 \), then by (2.46) and (2.48) one has
\[ \frac{(\kappa_z + 1)^n}{n! a(z)^{n+1}} \left| \frac{Q_1(z)^n}{\kappa_z^{2n}} \phi_0 + \sum_{0 \leq k \leq n-1} \frac{(Q_1(z)^{n-k+1})}{\kappa_z^{2n+1}} Q_1(z)^{k} \phi_0 \right| \]
\[ \leq -\tilde{C} n a(z)^{-\gamma/2} \ln(z - \epsilon_{\text{max}}) \]
for some \( \tilde{C} > 0 \) depending on \( \kappa_1 \) and \( \delta \). Hence, choosing \( \delta_1 > 0 \) such that \( -\tilde{C} a(z)^{-\gamma/2} \ln(z - \epsilon_{\text{max}}) < \frac{1}{4} \) for \( z \in (\epsilon_{\text{max}}, \epsilon_{\text{max}} + \delta_1) \) we get
\[ \| \psi_z \| \leq -\tilde{C} a(z)^{-\gamma/2} \ln(z - \epsilon_{\text{max}}) < \frac{\| \phi_0 \|}{2}. \] (2.53)

If \( d = 2 \), then by (2.44) \( \| Q_1(z) \| \leq C_2 \) for any \( z \geq \epsilon_{\text{max}} \). Therefore, we can choose \( \delta_1 > 0 \) such that
\[ \| \psi_z \| \leq \frac{\tilde{C} a(z)^{-1/2}}{(1-a(z))^{-1/2}} < \frac{\| \phi_0 \|}{2} \] (2.54)
for any \( z \in (\epsilon_{\text{max}}, \epsilon_{\text{max}} + \delta_1) \). The estimates (2.53) and (2.54) implies that for such \( z \), \( P_1(z) \neq 0 \). Let us show that there exists \( \delta_2 \in (0, \delta_1) \) such that for any \( z \in (\epsilon_{\text{max}}, \epsilon_{\text{max}} + \delta_2) \) there exists a solution \( \epsilon_z \) to the equation
\[ R(z, \epsilon) = 0, \]
where \( R : \langle \epsilon_{\text{max}}, \epsilon_{\text{max}} + \delta_1 \rangle \times (-\kappa_1/8, \kappa_1/8) \to L^2(\mathbb{R}^d) \) is defined as
\[ R(z, \epsilon) := \begin{cases} P_1(z)\phi_0 - \frac{Q_1(z)\phi_0}{\kappa_1 + \epsilon} & \text{if } z \in (\epsilon_{\text{max}}, \epsilon_{\text{max}} + \delta_1), \\ \frac{Q_1(z)\phi_0}{a(z)^{(\kappa_1 + \epsilon)/2}} & \text{if } z = \epsilon_{\text{max}}. \end{cases} \]

Indeed, notice that
\[ \lim_{z \to \epsilon_{\text{max}}} \| R(z, \epsilon) - R(\epsilon_{\text{max}}, \epsilon) \|_{L^2} = 0 \]
so that \( R \in C^0([\epsilon_{\text{max}}, \epsilon_{\text{max}} + \delta_1] \times (-\kappa_1/8, \kappa_1/8); L^2(\mathbb{R}^d)) \) and
\[ \lim_{z \to \epsilon_{\text{max}}} R(z, \epsilon) = R(\epsilon_{\text{max}}, 0) = 0. \]
Moreover, since \( R(z, \cdot) \) is analytic around \( \epsilon = 0 \) and
\[ \frac{\partial R(z, \epsilon)}{\partial \epsilon} \bigg|_{(z, \epsilon) = (\epsilon_{\text{max}}, 0)} = \frac{\phi_0}{\kappa_1} \neq 0, \]
by the Implicit Function Theorem, there exists \( \delta_2 \in (0, \delta_1) \) such that for any \( z \in [\epsilon_{\text{max}}, \delta_2] \) there exists a unique \( \epsilon_z \in (-\kappa_1/8, \kappa_1/8) \) such that
\[ R(z, \epsilon_z) = 0, \quad z \in \epsilon_{\text{max}}, \epsilon_z). \]
Since \( [\epsilon_{\text{max}}, \epsilon_{\text{max}} + \delta_1] \subset \mathbb{R} \), the fact that \( \epsilon_{\text{max}} \) is not an interior point does not affect, since we do not need that any regularity of the implicit function \( \epsilon_z \). However, notice that \( \epsilon_z \to 0 \) as \( z \to \epsilon_{\text{max}} \).

Let us now introduce \( \chi(z) := \sqrt{a(z)[\kappa_z + \epsilon_z]}, \quad z \in \epsilon_{\text{max}}, \epsilon_{\text{max}} + \delta_2); \)
then \( \chi(z) > 0 \) and by the definition of \( R(z, \epsilon) \) and \( \epsilon_z \), we have
\[ \chi(z)^2 \frac{\phi_z}{a(z)} = Q_1(z)\phi_z + \frac{\chi(z)Q_1(z)\phi_z}{a(z)}, \quad z \in \epsilon_{\text{max}}, \epsilon_{\text{max}} + \delta_2), \]
where $\phi_z := P_z(\mathbb{z})\tilde{\phi}_0 \neq 0$. Thus,

$$\phi_z \in \text{Ker} \left[ 1 - \left( a(z) \frac{\chi(z)}{\chi(z)} + a(z)^2 Q(\tilde{z}) \right) \frac{Q(\tilde{z})}{\chi(z)} \right] = \text{Ker} \left[ 1 - \frac{a(z)Q(\tilde{z})}{\chi(z)} \right],$$

(2.55)

where in the equality we used (2.57). By (2.58), this implies that $\phi_z$ is an eigenvector of $b(z)$ associated to its positive eigenvector $\chi(z)$. Recall that $\chi(z) \leq \lambda(z)$.

To establish $\chi(z) = \lambda(z)$ and $\lambda(z)$ is a simple eigenvalue of $b(z)$, we consider the self-adjoint operator

$$\tilde{h}_\mu := h_0 - \mu \nu;$$

notice that $\tilde{h}_\mu$ differs from $h_\mu$ only with the sign of $\nu$. Then associated Birman-Schwinger operator $\tilde{b}(z)$ satisfies $\tilde{b}(z) = -b(z)$. Since $\sum_x \tilde{v}(x) = 0$, by the arguments above, there exists a positive eigenvalue $\xi(z)$ of $\tilde{b}(z)$ which satisfies

$$\tilde{\chi}(z) = \sqrt{a(z)(\delta_z + o(1))} \quad \text{as } z \to \epsilon_{\text{max}}.$$

Recall that $-\tilde{\chi}(z)$ is a negative eigenvalue of $b(z)$. Let us enumerate all nonzero eigenvalues $\eta_0(z), \eta_1(z), \ldots$ (counted with multiplicities) of $b(z)$ as $|\eta_0(z)| \geq |\eta_1(z)| \geq 0$ and also all positive eigenvalues $\nu_0 \geq \nu_1 \geq \ldots$ of $|b(z)|$. By [3] Lemma 3.3,

$$|\eta_0(z)\eta_1(z)\eta_2(z)| \leq \nu_0(z)\nu_1(z)\nu_2(z).$$

(2.56)

Note that by (2.42)–(2.44) applied to $|b(z)|$, we have

$$\frac{\nu_0(z)}{\nu(z)} = \sum_{x \in Z^d} |\tilde{v}(x)| + o(1) \quad \text{as } z \to \epsilon_{\text{max}}.$$

Moreover, by (2.28),

$$\sup_{z > \epsilon_{\text{max}}} |v_i(z)| \leq C_4 \sum_{x \in Z^d} |x|^{2-d+\gamma} |\tilde{v}(x)|, \quad i = 1, 2,$$

(2.57)

and hence, if $b(z)$ has at least two positive eigenvalues with asymptotics $\geq \sqrt{a(z)(\delta_z + o(1))}$, then recalling the definition of $-\tilde{\chi}(z)$ we get

$$|\eta_0(z)\eta_1(z)\eta_2(z)| \geq a(z)^{3/2}(\tilde{c} + o(1))$$

for some $\tilde{c} > 0$ as $z \to +\infty$. Hence, by (2.56)

$$\tilde{c} + o(1) \leq \frac{\hat{c}}{\nu(z)} \to 0 \quad \text{as } z \to \epsilon_{\text{max}},$$

a contradiction. Thus,

$$\lambda(z) = \sqrt{a(z)[\delta_1 + o(1)]} \quad \text{as } z \to \epsilon_{\text{max}}.$$

Analogously,

$$\hat{\lambda}(z) := \sup \sigma(\hat{b}(z)) = \sqrt{a(z)[\delta_1 + o(1)]} \quad \text{as } z \to \epsilon_{\text{max}}.$$

Since $|\eta_0\eta_1| = \lambda(z)\hat{\lambda}(z)$ for small and positive $z - \epsilon_{\text{max}}$, (2.55) implies

$$|\eta_n(z)| \leq |\eta_1(z)| \leq \frac{\nu_0(z)\nu_1(z)\nu_2(z)}{\lambda(z)\hat{\lambda}(z)}$$

for any $n \geq 2$. Now using the asymptotics of $\lambda(z)$, $\nu_0(z)$ and $\hat{\lambda}(z)$ as well as estimate (2.57), we find $\delta_3 \in (0, \delta_2)$ such that

$$\lambda_n(z) \leq \hat{C}_5 := \left( \frac{C_4}{\delta_1} \sum_{x \in Z^d} |\tilde{v}(x)| \right)^2 + 1, \quad n \geq 1,$$

(2.58)

for all $z \in (\epsilon_{\text{max}}, \epsilon_{\text{max}} + \delta_3)$, where $\lambda_0(z) = \lambda(z) \geq \lambda_1(z) \geq \ldots$ are all positive eigenvalues of $b(z)$ (counted with their multiplicities). Since $\lambda_n(z) \leq \lambda_0(z) \leq ||b(z)||$,

$$\sup_{z \geq \epsilon_{\text{max}} + \delta_3} \lambda_n(z) \leq ||b(\epsilon_{\text{max}} + \delta_3)||, \quad n \geq 0.$$

(2.59)

Now (2.58) and (2.59) implies (2.34) with $C_5 := \max \{\tilde{C}_5, ||b(\epsilon_{\text{max}} + \delta_3)||\}$.
It remains to prove (2.36). We set
\[ \epsilon(z) := \frac{\lambda(z)^2}{a(z)} - \kappa_z, \] (2.60)
where \( \kappa_z \) and \( \kappa_1 \) are given in (2.48) and (2.29), respectively. By (2.46), there exists \( \delta_4 \in (0, \delta_3) \) such that
\[ \kappa_z = \kappa_1 + h_4(z), \] (2.61)
where \( h_4 \in C^0(\epsilon_{\text{max}}, \epsilon_{\text{max}} + \delta_4) \) with \( h_4(z) = O((z - \epsilon_{\text{max}})^{\gamma/4}) \) as \( z \to \epsilon_{\text{max}} \). Now we consider the spectral projection \( P_\epsilon(z) \) in (2.50) with \( \epsilon := \epsilon(z) \). By (2.51), \( P_\epsilon(z) \phi_0 = \phi_0 + \psi_\epsilon(z) \). Since
\[ \frac{\lambda(z)^2}{a(z)} P_\epsilon(z) \phi_0 = (QQ_1(z) + \frac{\lambda(z)Q_1(z)}{a(z)}) P_\epsilon(z) \phi_0 \]
(see (2.55)), we have
\[ \epsilon(z) \phi_0 = -[\kappa_z + \epsilon(z)] \psi_\epsilon(z) + \phi_0(Q_1(z) \psi_\epsilon(z), \phi_1) + \frac{Q_1(z) \phi_0 + Q_4(z) \psi_\epsilon(z)}{a(z)} [\kappa_z + \epsilon(z)]^{1/2}. \]
Multiplying this by \( \phi_0 \) and using the definition (2.52) of \( \psi_\epsilon(z) \) we get
\[ \epsilon(z) \| \phi_0 \|_{L^2}^2 = -[\kappa_z + \epsilon(z)](\psi_\epsilon(z), \phi_0)_{L^2} + \| \phi_0 \|_{L^2}^2 (Q_1(z) \psi_\epsilon(z), \phi_1) \]
\[ + \frac{(Q_1(z) \phi_0, \phi_0)_{L^2} + (Q_1(z) \psi_\epsilon(z), \phi_0)_{L^2}}{a(z)} [\kappa_z + \epsilon(z)]^{1/2}. \]
Now if \( d = 1 \), then by (2.27), (2.61) and (2.26),
\[ |\epsilon(z)| \leq \tilde{C} \epsilon(z)^{-\gamma/2} \ln^2 (z - \epsilon_{\text{max}}), \quad z \in (\epsilon_{\text{max}}, \epsilon_{\text{max}} + \delta_4). \]
This, (2.61) and (2.60) implies (2.36).

If \( d = 2 \), then using the definition (2.52) of \( \psi_\epsilon(z) \), (2.61) and (2.26) we get
\[ (\psi_\epsilon(z), \phi_0)_{L^2} = a(z)^{-1/2} \left[ \frac{(Q_1(\epsilon_{\text{max}}) \phi_0, \phi_0)}{\kappa_1^{3/2}} + \frac{\| \phi_0 \|_{L^2}^2 (Q_1(\epsilon_{\text{max}})^2 \phi_0, \phi_0)}{\kappa_1^{5/2}} \right] + a(z)^{-1/2} h_5(z), \]
where \( h_5 \in C^0(\epsilon_{\text{max}}, \epsilon_{\text{max}} + \delta) \) with \( h_5(z) = O((z - \epsilon_{\text{max}})^{\gamma/4}) \). Moreover, by (2.26), (2.24), (2.46),
\[ (Q_1(z) \psi_\epsilon(z), \phi_1) = a(z)^{-1/2} \left[ \frac{(Q_1(\epsilon_{\text{max}})^2 \phi_0, \phi_0)}{\kappa_1^{3/2}} + \frac{(Q_1(\epsilon_{\text{max}})^2 \phi_0, \phi_0)}{\kappa_1^{5/2}} \right] \]
\[ + a(z)^{-1/2} h_6(z), \]
where \( h_6 \in C^0(\epsilon_{\text{max}}, \epsilon_{\text{max}} + \delta) \) with \( h_6(z) = O((z - \epsilon_{\text{max}})^{\gamma/4}) \). Moreover, again by (2.26),
\[ (Q_1(z) \phi_0, \phi_0)_{L^2} + (Q_1(z) \psi_\epsilon(z), \phi_0)_{L^2} [\kappa_z + \epsilon(z)]^{1/2} = \frac{\kappa_z^{1/2}}{a(z)} + \frac{h_7(z)}{a(z)^{1/2}}, \]
where \( h_7 \in C^0(\epsilon_{\text{max}}, \epsilon_{\text{max}} + \delta) \) with \( h_7(z) = O((z - \epsilon_{\text{max}})^{\gamma/4}) \). Hence,
\[ \epsilon(z) = \tilde{C} a(z)^{-1/2} + a(z)^{-1/2} h_8(z), \] (2.62)
where \( h_8 \in C^0(\epsilon_{\text{max}}, \epsilon_{\text{max}} + \delta) \) with \( h_8(z) = O((z - \epsilon_{\text{max}})^{\gamma/4}) \). Finally, since \( a(z)^{1/2}(z - \epsilon_{\text{max}})^{\gamma/4} = o((z - \epsilon_{\text{max}})^{\gamma/4}) \) From (2.60), (2.61) and (2.62) we get (2.36).

Combining Proposition 2.5 and Lemma 2.1 we get

**Corollary 2.6.** Assume Hypothesis 1.7 and let \( C_5 > 1 \) be given by Proposition 2.5. Then for any \( \mu \in (0, \frac{1}{C_5}) \):

(a) if \( \sum_{x \in \mathbb{Z}^d} \hat{v}(x) < 0 \), then \( \sigma_{\text{disc}}(h_\mu) \cap (\epsilon_{\text{max}}, +\infty) = \emptyset \);

(b) if \( \sum_{x \in \mathbb{Z}^d} \hat{v}(x) \geq 0 \), then \( \sigma_{\text{disc}}(h_\mu) \cap (\epsilon_{\text{max}}, +\infty) \) is a singleton \( \{ E(\mu) \} \). Moreover, the map \( \mu \in (0, \frac{1}{C_5}) \mapsto E(\mu) \) is analytic, strictly increasing and \( E(\mu) \to \epsilon_{\text{max}} \) as \( \mu \to 0 \).
Proof. Let $\lambda(z)$ be given by (2.19).
(a) If $\sum_{x \in \mathbb{Z}^d} \tilde{v}(x) < 0$, then by Proposition 2.10 (a) for any $\mu < \frac{1}{\epsilon_0}$ and $z > \epsilon_{\text{max}}$ all positive eigenvalues of $\mu b(z)$ will be less than 1, i.e.,

$$\text{Ker}(1 + \mu b(z)) = 0.$$ 

Thus, by Lemma 2.4 \( \sigma_{\text{disc}}(h_\mu) \cap (\epsilon_{\text{max}}, +\infty) = 0 \).
(b) Assume that $\sum_{x \in \mathbb{Z}^d} \tilde{v}(x) \geq 0$. Given $z > \epsilon_{\text{max}}$, if $\lambda_0(z) = \lambda(z) \geq \lambda_1(z) \geq \ldots > 0$ are all positive eigenvalues of $b(z)$, then by (2.21), $\lambda_k(z) < 1$ for any $\mu < \frac{1}{\epsilon_0}$ and $k \geq 1$. In particular, $\text{Ker}(1 + \mu b(z))$ is at most one-dimensional. By Proposition 2.23 (b), $\lambda(z) \to +\infty$ as $z \to \epsilon_{\text{max}}$. Moreover, by Lemma 2.22 \( \lambda(z) \) is continuous and strictly decreasing in $(\epsilon_{\text{max}}, +\infty)$, and $\lim_{z \to +\infty} \lambda(z) = 0$. Therefore, for any $\mu \in (0, \frac{1}{\epsilon_0})$ there exists a unique $E(\mu) > \epsilon_{\text{max}}$ such that $\mu \lambda(E(\mu)) = 1$. By Lemma 2.4, $E(\mu)$ is the unique eigenvalue of $h_\mu$ in $(\epsilon_{\text{max}}, +\infty)$. By the Implicit Function Theorem in the monotone case, the map $\mu \in (0, \frac{1}{\epsilon_0}) \mapsto E(\mu)$ is strictly increasing and $E(\mu) \to \epsilon_{\text{max}}$ as $\mu \to 0$. \( \square \)

3. PROOFS OF THE MAIN RESULTS

In this section we prove main results of the paper.

Proof of Theorem 1.3. We show only (1.2), and the proof of (1.3) is similar. Recall that $\mathcal{N}^+(\tilde{h}_\mu, \epsilon_{\text{max}}) = \mathcal{N}^+(h_\mu, \epsilon_{\text{max}})$. Since

$$h_\mu \leq h_0 + \mu |v|,$$

by the minmax principle, $\mathcal{N}^+(h_\mu, \epsilon_{\text{max}}) \leq \mathcal{N}^+(h_0 + \mu |v|, \epsilon_{\text{max}})$. Hence, it suffices to establish

$$\mathcal{N}^+(\tilde{h}_0 + \mu |v|, \epsilon_{\text{max}}) \leq 1 + C_4 \mu \sum_{x \in \mathbb{Z}^d} |x|^{2-d+\gamma} |\tilde{v}(x)|$$

for $C_4$ of (2.28). Recall by [1] Lemma 2.1 (iv) that given $z > \epsilon_{\text{max}},$

$$\mathcal{N}^+(h_0 + \mu |v|, z) = \mathcal{N}^+(|b(z)|, 1).$$

Let $\nu_k(z) := \|b(z)|\| |\tilde{v}(z)| \geq \nu_1(z) \geq \ldots > 0$ be all positive eigenvalues of the $|b(z)|$; since $\nu_k(z) \to 0$ as $k \to \infty$, there exists a unique $k_\mu \geq 0$ such that $\mu \nu_{k_\mu} \geq 1$ and $\mu \nu_{k_\mu+1} < 1$. Therefore, by Lemma 2.4

$$\mathcal{N}^+(h_0 + \mu |v|, z) = \mathcal{N}^+(\mu |b(z)|, 1) = 1 + k_\mu \leq 1 + \mu \sum_{i=1}^{k_\mu} \nu_i(z) \leq 1 + \mu \left[ \text{Tr}(|b(z)|) - \nu_0(z) \right] \leq 1 + C_4 \mu \sum_{x \in \mathbb{Z}^d} |x|^{2-d+\gamma} |\tilde{v}(x)|$$

for any $z > \epsilon_{\text{max}}$. Now letting $z \searrow \epsilon_{\text{max}}$ we get (3.1). \( \square \)

Proof of Theorem 1.3. Let $C_4(\tilde{v})$ and $C_4(-\tilde{v})$ be given by Lemma 2.23 applied with $(\tilde{v}, \tilde{v})$ and $(-\tilde{v}, -\tilde{v})$, respectively. Let $\mu_\sigma := \mu_0(\epsilon, \tilde{v}) := \min\{1/C_4(\epsilon, \tilde{v}), 1/C_4(-\epsilon, -\tilde{v})\} > 0$. Now assertions of Theorem 1.3 for small $\mu$, i.e., for $\mu \in (0, \mu_\sigma)$, follows from Corollary 2.8 applied with $(\epsilon, \tilde{v})$ and $(-\epsilon, -\tilde{v})$, respectively.

Now we prove assertion (1) for all $\mu > 0$. Let

$$E_0(\mu) := \|h_\mu\| := \sup \sigma(h_\mu).$$

Note that $E_0(\mu) = E(\mu) > \epsilon_{\text{max}}$ for $\mu \in (0, \mu_\sigma)$, where $E(\mu)$ is the unique eigenvalue of $h_\mu$ given by Corollary 2.8 (b). By (1.1)

$$E(\mu) = \sup_{f \in L^2(\mathbb{T}^d), \|f\| = 1} \max\{\epsilon_{\text{max}}, (h_\mu, f, f)\}.$$ 

Since the map $\mu \in (0, +\infty) \mapsto \min\{\epsilon_{\text{min}}, (h_\mu, f, f)\}$ is nonincreasing for any $f \in L^2(\mathbb{T}^d)$ so is $\mu \in (0, +\infty) \mapsto E_0(\mu)$. In particular, $E_0(\mu) > \epsilon_{\text{max}}$ for all $\mu > 0$. Thus, $E_0(\mu) \in \sigma_{\text{disc}}(h_\mu) \cap (\epsilon_{\text{max}}, +\infty)$.

The proof of assertion (2) follows from applying assertion (1) with $-\epsilon$ and $-\tilde{v}$, respectively. \( \square \)
Proof of Theorem 1.4. We establish only the asymptotics of \( E(\mu) \), i.e., in the case \( \sum x \hat{v}(x) \geq 0 \), then the asymptotics of \( e(\mu) \) follows by applying the established asymptotics with \(-\varepsilon \) and \(-\hat{\varepsilon} \). Let \( \mu_0 > 0 \) be given by Theorem 1.3. Recall that for any \( \mu \in (0, \mu_0) \), \( E(\mu) \) is the unique eigenvalue of \( h_\mu \) in \((\epsilon_{\max}, +\infty)\). By Lemma 2.1,

\[
\mu \lambda(E(\mu)) = 1, \quad \mu \in (0, \mu_0).
\]

We find the asymptotics of \( E(\mu) \) using the asymptotics \((2.35)\) and \((2.36)\) of \( \lambda(\cdot) \). Let us first establish the asymptotics of \( a(z) \) as \( z \to \epsilon_{\max} \). Since \( e(\cdot) \) is has a unique non-degenerate maximum at \( p^0 \) and \( e \) is \( C^{1, \alpha} \) around \( p^0 \), by the Morse Lemma there exists a neighborhood \( U_{p^0} \subset \mathbb{T}^d \) and a \( C^{1, \alpha} \)-diffeomorphism \( \varphi : B_1(0) \subset \mathbb{R}^d \to U_{p^0} \) such that \( \varphi(0) = p^0 \) and \( e(\varphi(u)) = \epsilon_{\max} - u^2 \), \( u \in B_1(0) \).

Without loss of generality, we assume that \( \alpha \in (0, \gamma/8] \). Writing

\[
a(z) = \int_{U_{p^0}} \frac{dq}{e(q) - z} + \int_{\mathbb{T}^d \setminus U_{p^0}} \frac{dq}{e(q) - z} =: I_1(z) + I_2(z),
\]

we observe that \( I_2(\cdot) \) is analytic at \( z = \epsilon_{\max} \). In \( I_1(z) \) we make the change of variables \( q = \varphi(u) : \)

\[
I_1(z) = \int_{B_1(0)} J(\varphi(u)) \frac{du}{u^2 + z - \epsilon_{\max}}
\]

\[
= J(\varphi(0)) \int_{B_1(0)} \frac{du}{u^2 + z - \epsilon_{\max}} + \int_{B_1(0)} \frac{|J(\varphi(u)) - J(\varphi(0))| du}{u^2 + z - \epsilon_{\max}}
\]

\[
= : I_{11}(z) + I_{12}(z),
\]

where \( J \phi > 0 \) is the Jacobian of \( \varphi \). Since \( J \varphi \in C^{0, \alpha}(B_1(0)) \), there exists \( c > 0 \) such that \( |J(\varphi(u)) - J(\varphi(0))| \leq c|u|^\alpha \) for all \( u \in B_1(0) \), and hence, by Lemma 3.1,

\[
|I_{12}(z)| \leq c d \omega_d \int_0^\gamma r^{d + \alpha - 1} \frac{dr}{r^2 + z - \epsilon_{\max}} \leq \begin{cases} \frac{2c(z - \epsilon_{\max})^{\alpha + 1}}{z^2} \int_0^{\infty} r^\alpha dr & \text{if } d = 1, \\ \frac{2 \pi c}{\gamma} + c_1(z - \epsilon_{\max})^\alpha & \text{if } d = 2, \end{cases}
\]

where \( c_2 > 0 \). Moreover, if \( z > \epsilon_{\max} \),

\[
I_{11}(z) = \begin{cases} \frac{\pi J_0}{(z - \epsilon_{\max})^{\alpha/2}} \left( 1 - \frac{2}{\pi} \arctan \left( \frac{z - \epsilon_{\max}}{\gamma} \right) \right) & \text{if } d = 1, \\ -\pi J(\varphi(0)) \ln(E(\mu) - \epsilon_{\max}) \left( 1 + \frac{\ln(z - \epsilon_{\max})}{\ln(\epsilon_{\max})} \right) & \text{if } d = 2. \end{cases}
\]

Thus,

\[
\lim_{z \to \epsilon_{\max}} \frac{I_{12}(z)}{I_{11}(z)} = 0
\]

and

\[
a(z) = \begin{cases} \frac{\pi J_0}{(z - \epsilon_{\max})^{\alpha/2}} \left( 1 + (z - \epsilon_{\max})^\alpha h_1(z) \right) & \text{if } d = 1, \\ -\pi J_0 \ln(z - \epsilon_{\max}) \left( 1 + \frac{C_\alpha}{\ln(z - \epsilon_{\max})} + \frac{(z - \epsilon_{\max})^\gamma h_2(z)}{\ln(\epsilon_{\max})} \right) & \text{if } d = 2, \end{cases}
\]

where \( J_0 := J(\varphi(0)) \), \( h_1, h_2 \in C_0[\epsilon_{\max}, \epsilon_{\max} + \delta_0] \).

Assume that \( \kappa_0 := \sum x \hat{v}(x) > 0 \). Then by \((2.33)\) and \((2.34)\)

\[
\frac{1}{\mu \alpha(E(\mu))} = \kappa_0 + \begin{cases} \frac{g_1(E(\mu)) \ln(E(\mu) - \epsilon_{\max})}{\kappa_0 \alpha(E(\mu))} & \text{if } d = 1, \\ \frac{C_\alpha}{\kappa_0 \alpha(E(\mu))} + \frac{(E(\mu) - \epsilon_{\max})^{\gamma/2} g_2(E(\mu))}{\alpha \kappa_0(E(\mu))} & \text{if } d = 2, \end{cases}
\]

where \( C_\alpha \in \mathbb{R} \), \( g_1, g_2 \in C_0[\epsilon_{\max}, \epsilon_{\max} + \delta_0] \). Therefore, if \( d = 1 \), then by \((3.3)\) and \((3.4)\) we get

\[
1 = \frac{\pi J_0 \kappa_0 \mu}{(E(\mu) - \epsilon_{\max})^{1/2}} \left( 1 + (E(\mu) - \epsilon_{\max})^\alpha h_1(E(\mu)) \right) + \frac{\mu}{\kappa_0} g_1(E(\mu)) \ln(E(\mu) - \epsilon_{\max})
\]

for any \( \mu \in (0, \mu_0) \). Let \( u_1(\mu) \) be such that

\[
(E(\mu) - \epsilon_{\max})^{1/2} = \pi J_0 \kappa_0 \mu (1 + u_1(\mu)).
\]

Then by \((3.5)\)

\[
|u_1(\mu)| \leq \tilde{c}_j \mu^\alpha, \quad \mu \in (0, \mu_0),
\]

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for some $\tilde{c}_1 > 0$. Hence (1.4) in $d = 1$ follows.

If $d = 2$, then by (3.3) and (3.4)
\begin{equation}
1 = -\pi J_0 \kappa_0 \mu \ln(E(\mu) - \epsilon_{\max}) \left(1 + \frac{c_2 h_2(E(\mu))}{\ln(E(\mu) - \epsilon_{\max})}\right)
+ \frac{C_5}{\kappa_0} + \frac{\mu (E(\mu) - \epsilon_{\max})^{1/2} g_2(E(\mu))}{\kappa_0}
\end{equation}
(3.6)
for any $\mu \in (0, \mu_0)$. Let $u_2(\mu)$ be such that
\[ E(\mu) - \epsilon_{\max} = e^{-\frac{C_5}{\kappa_0}} (c + u_2(\mu)), \quad c := e^{-\frac{C_5}{\kappa_0}} - C_5 > 0.\]
Then (3.4) implies
\[ |u_2(\mu)| \leq \tilde{c}_2 \mu, \quad \mu \in (0, \mu_0), \]
for some $\tilde{c}_2 > 0$. Hence (1.4) in $d = 2$ follows.

Now assume that $\kappa_0 := \sum_x \tilde{v}(x) = 0$. By Proposition 2.6, $\kappa_1 \in (0, +\infty)$. Moreover, by (3.2) and (2.36)
\begin{equation}
\frac{1}{\mu^3 u(E(\mu))} = \kappa_1 + \begin{cases}
\frac{g_2(E(\mu))}{u(E(\mu))^{1/2}} & \text{if } d = 1, \\
\frac{(E(\mu) - \epsilon_{\max})^{1/2} g_3(E(\mu))}{u(E(\mu))^{1/2}} & \text{if } d = 2,
\end{cases}
\end{equation}
(3.7)
where $C_7 \in \mathbb{R}$, $g_3, g_4 \in C^0_{\alpha}(\epsilon_{\max}, \epsilon_{\max} + \delta_0)$.

Let $d = 1$. In this case by (3.7) and (3.3) we get
\begin{equation}
\frac{E(\mu) - \epsilon_{\max})^{1/2}}{\pi J_0 \kappa_0 \mu^2 \left(1 + (E(\mu) - \epsilon_{\max})^\alpha h_1(E(\mu))\right)}
= \kappa_1 + \frac{\left(\frac{g_2(E(\mu))}{u(E(\mu))^{1/2}}\right)^\alpha h_1(E(\mu))}{\left[\pi J_0 (1 + (E(\mu) - \epsilon_{\max})^\alpha h_1(E(\mu)))\right]^{1/2}}
\end{equation}
(3.8)
for any $\mu \in (0, \mu_0)$. Let $u_3(\mu)$ be such that
\[ (E(\mu) - \epsilon_{\max})^{1/2} = \pi J_0 \kappa_1 \mu^2 (1 + u_1(\mu)). \]
Then (3.5) implies
\[ |u(\mu)| \leq \tilde{c}_3 \mu^\gamma \ln^2 \mu, \quad \mu \in (0, \mu_0) \]
for some $\tilde{c}_3 > 0$. Hence (1.0) in $d = 1$ follows.

Let $d = 2$. In this case by (3.7) and (3.3) we get
\begin{equation}
-\pi J_0 \mu^2 \ln(E(\mu) - \epsilon_{\max}) \left(1 + \frac{C_5}{\ln(E(\mu) - \epsilon_{\max})} + \frac{(E(\mu) - \epsilon_{\max}) \ln h_2(z)}{\ln(E(\mu) - \epsilon_{\max})}\right)

= \kappa_1 + \frac{C_7 (E(\mu) - \epsilon_{\max})^{1/2} g_4(E(\mu))}{\left[\pi J_0 \ln(E(\mu) - \epsilon_{\max}) \left(1 + \frac{C_5}{\ln(E(\mu) - \epsilon_{\max})} + \frac{(E(\mu) - \epsilon_{\max}) \ln h_2(z)}{\ln(E(\mu) - \epsilon_{\max})}\right)\right]^{1/2}}.
\end{equation}
(3.9)
This equation can be rewritten as
\[ -\pi J_0 \kappa_1 \mu^2 \ln(E(\mu) - \epsilon_{\max}) + C_7 \mu^2 [-\pi J_0 \ln(E(\mu) - \epsilon_{\max})]^{1/2} [1 + o(1)] = 1. \]
Note that the equation
\[ -\pi J_0 \kappa_1 \ln t + C_7 [-\pi J_0 \ln t]^{1/2} = \frac{1}{\mu^2} \]
has a unique solution
\[ t = \exp \left(-\frac{(\sqrt{4\kappa_1 + C_7^2 \mu^2} - C_7 \mu)^2}{4 \pi J_0 \kappa_1^2 \mu^2}\right), \]
hence if we set
\[ E(\mu) - \epsilon_{\max} = \exp \left(-\frac{(\sqrt{4\kappa_1 + C_7^2 \mu^2} - C_7 \mu)^2}{4 \pi J_0 \kappa_1^2 \mu^2}\right) [c + u_4(\mu)], \quad c := e^{-\frac{C_5}{\kappa_0}} - C_5 > 0, \]
for some $u_4(\mu) \in \mathbb{R}$, then from (3.9) we get
\[ |u(\mu)| \leq \tilde{c}_4 \mu, \quad \mu \in (0, \mu_0), \]
for some $\tilde{c}_4 > 0$. Hence \([10]\) in \(d = 2\) follows. \(\square\)

**Appendix A. Asymptotics of some parametric integrals**

In this paper we frequently use the following technical tool.

**Lemma A.1.** Given $\alpha \geq 0$ and $r_0 \in (0, 1)$, consider the integral

$$T_\alpha(\omega) := \int_0^{r_0} \frac{r^n \, dr}{r^2 + \omega^2}, \quad \omega > 0. \quad (A.1)$$

Then for any $\alpha \geq 0$ there exist a polynomial $P_\alpha(\omega)$ such that $P_\alpha \equiv 0$ for $\alpha \in [0, 1]$, $P_\alpha \equiv T_\alpha(0) := \frac{r_0^{\alpha-1}}{\alpha \omega} > 0$ for $\alpha \in (1, 2]$ and $P_\alpha$ is at most of order $[\alpha] - 2$ if $\alpha > 2$, where $[\alpha]$ is the integer part of $\alpha$, and $g_\alpha \in L^\infty(0, +\infty)$ such that

(a) for $\alpha \in [0, 1]$:

$$T_\alpha(\omega) = \begin{cases} \frac{r_0^{\alpha-1}}{\alpha \omega} \left[1 - \frac{2}{\pi} \arctan \frac{r_0}{\omega}\right] & \text{if } \alpha = 0, \\ \frac{g_\alpha(\omega)}{\omega} & \text{if } \alpha \in (0, 1), \\ - \ln \omega \left[1 + \frac{\ln(r_0^2 + \omega^2)}{2 \ln \omega}\right] & \text{if } \alpha = 1; \end{cases}$$

(b) Let $\alpha > 1$. Then

$$T_\alpha(\omega) \leq T_\alpha(0) \quad \text{and} \quad T_\alpha(\omega) = P_\alpha(\omega) + g_\alpha(\omega) \omega^{[\alpha]-1}.$$  

**Proof.** (a) The asymptotics of $T_\alpha$ for $\alpha \in [0, 1]$ is clear. In this case we define $g_\alpha \equiv 1$. If $\alpha \in (0, 1)$, then using the change of variables $r = \omega t$ in the integral we get

$$T_\alpha(\omega) \leq \omega^{\alpha-1} \int_0^{+\infty} \frac{t^n \, dt}{t^2 + 1}.$$  

Hence,

$$g_\alpha(\omega) := \omega^{1-\alpha} T_\alpha(\omega)$$

satisfies $\|g_\alpha\|_{L^\infty(0, +\infty)} \leq \int_0^{+\infty} \frac{t^n \, dt}{t^2 + 1} < +\infty$.

(b) Let $\alpha \in (1, 2]$. Then using the change of variable $r = t \frac{\omega}{\alpha}$ in $T_\alpha$ we get

$$T_\alpha(\omega) = \frac{1}{\alpha - 1} \int_0^{r_0^{\alpha-1} \frac{\omega}{\alpha}} \frac{t^n \, dt}{t^{\frac{\alpha}{\alpha-1}} + \omega^2} = \frac{r_0^{\alpha-1}}{\alpha - 1} + \omega^{2} \int_0^{r_0^{\alpha-1} \frac{\omega}{\alpha}} \frac{dt}{t^{\frac{\alpha}{\alpha-1}} + \omega^2}.$$  

Now using the change of variable $t = \omega^{\alpha-1} s$ we get

$$\int_0^{r_0^{\alpha-1} \frac{\omega}{\alpha}} \frac{dt}{t^{\frac{\alpha}{\alpha-1}} + \omega^2} \leq \omega^{\alpha-1} \int_0^{+\infty} \frac{ds}{s^{\frac{\alpha}{\alpha-1}} + 1}$$

so that

$$g_\alpha(\omega) := \omega^{1-\alpha} \left[T_\alpha(\omega) - T_\alpha(0)\right]$$

satisfies $\|g_\alpha\|_{L^\infty(0, +\infty)} \leq \frac{1}{\alpha - 1} \int_0^{+\infty} \frac{ds}{s^{\frac{\alpha}{\alpha-1}} + 1} < +\infty$.

Let $n := [\alpha] \geq 2$. Note that if $n$ is even, then

$$r^n + (-1)^{[n/4]} \omega^n = (r^2 + \omega^2)(r^{n-2} - r^{n-4}\omega^2 + \ldots + (-1)^{[n/4]} \omega^{n-2}).$$

Thus

$$P_\alpha(\omega) := \int_0^{r_0} \left((r^{n-2} - r^{n-4}\omega^2 + \ldots + (-1)^{[n/4]} \omega^{n-2})\right) r^{\alpha-n} \, dr;$$

is a polynomial of order $n - 2$ and

$$T_\alpha(\omega) = P_\alpha(\omega) - (-1)^{[n/4]} \omega^n \int_0^{r_0} \frac{r^{\alpha-n} \, dr}{r^2 + \omega^2}.$$  

Note that

$$\int_0^{r_0} \frac{r^{\alpha-n} \, dr}{r^2 + \omega^2} \leq \omega^{\alpha-n-1} \int_0^{+\infty} \frac{r^{\alpha-n} \, dr}{r^2 + 1},$$

for some $\tilde{c}_4 > 0$. Hence \([10]\) in \(d = 2\) follows. \(\Box\)
where the last integral is finite since $\alpha - n \in [0,1)$. Hence,

$$g_\alpha(\omega) := \omega^{1-\alpha} [T_\alpha(\omega) - P_{\alpha-3}(\omega)]$$

satisfies $\|g_\alpha\|_{L^\infty(0,\infty)} \leq \int_0^{1/4} r^{n-1} \frac{dr}{r^{n+1}} < +\infty$.

If $n \geq 3$ is odd, then

$$r^{n-1} + (-1)^{\frac{n+1}{2}} \omega^{n-1} = (r^2 + \omega^2)(r^{n-3} - r^{n-5} \omega^2 + \ldots + (-1)^{[n/4]} \omega^{n-3}).$$

Thus,

$$P_{\alpha}(\omega) := \int_0^\alpha \left( r^{n-3} - r^{n-5} \omega^2 + \ldots + (-1)^{[n/4]} \omega^{n-3} \right) r^{1+\alpha-n} dr$$

is a polynomial of order $n-3$ and

$$T_{\alpha}(\omega) = \tilde{P}_{\alpha-3}(\omega) - (-1)^{\frac{n+1}{2}} \omega^{n-1} \int_0^\alpha \left( r^{1+\alpha-n} dr \right).$$

Now as in the case of $\alpha \in (1,2)$,

$$\int_0^\alpha \frac{r^{1+\alpha-n} dr}{r^2 + \omega^2} \leq \omega^{\alpha-n} \int_0^{+\infty} \frac{dr}{r^{\alpha-n+1}},$$

therefore,

$$g_\alpha(\omega) := \omega^{1-\alpha} [T_{\alpha}(\omega) - P_{\alpha-3}(\omega)]$$

satisfies $\|g_\alpha\|_{L^\infty(0,\infty)} \leq \frac{1}{\alpha-n} \int_0^{+\infty} \frac{dr}{r^{\alpha-n+1}} < +\infty$. □

**References**

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