THE STRONG BOREL-CANTERLLE PROPERTY
IN CONVENTIONAL AND NONCONVENTIONAL SETUPS

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Abstract. We study the strong Borel-Cantelli property both for events and for shifts on sequence spaces considering both a conventional and a nonconventional setups. Namely, under certain conditions on events \( \Gamma_1, \Gamma_2, \ldots \) we show that with probability one

\[
\left( \sum_{n=1}^{N} \prod_{i=1}^{\ell} P(\Gamma_{q_i(n)}) \right)^{-1} \sum_{n=1}^{N} \prod_{i=1}^{\ell} I_{\Gamma_{q_i(n)}} \to 1 \text{ as } N \to \infty
\]

where \( q_i(n) \), \( i = 1, \ldots, \ell \) are integer valued functions satisfying certain assumptions and \( I_\Gamma \) denotes the indicator of \( \Gamma \). When \( \ell = 1 \) (called the conventional setup) this convergence can be established under \( \phi \)-mixing conditions while when \( \ell > 1 \) (called a nonconventional setup) the stronger \( \psi \)-mixing condition is required. These results are extended to shifts \( T \) of sequence spaces where \( \Gamma_{q_i(n)} \) is replaced by \( T^{-q_i(n)} C_i(n) \) where \( C_i(n), i = 1, \ldots, \ell, n \geq 1 \) is a sequence of cylinder sets. As an application we study the asymptotical behavior of maximums of certain logarithmic distance functions.

1. Introduction

The classical second Borel–Cantelli lemma states that if \( \Gamma_1, \Gamma_2, \ldots \) is a sequence of independent events such that

\[
\sum_{n=1}^{\infty} P(\Gamma_n) = \infty
\]

then with probability one infinitely many of events \( \Gamma_i \) occur, i.e.

\[
\sum_{n=1}^{\infty} I_{\Gamma_n} = \infty \text{ almost surely (a.s.)}
\]

where \( I_\Gamma \) is the indicator of a set (event) \( \Gamma \).

There is a long list of papers, starting probably with [11], providing conditions which replace the independency by a weaker assumption and still yield (1.2) (see, for instance, [4] and references there). On the other hand, it was shown in Theorem 3
of [14] that under \( \phi \)-mixing with a summable coefficient \( \phi \) the condition (1.1) yields the stronger version of the second Borel–Cantelli lemma in the form

\[
\frac{S_N}{\mathcal{E}_N} \to 1 \text{ almost surely (a.s.) as } N \to \infty
\]

where \( S_N = \sum_{n=1}^{N} 1_{\Gamma_n} \) and \( \mathcal{E}_N = \sum_{n=1}^{N} P(\Gamma_n) \).

The same paper [14] started another line of research, known now under the name dynamical Borel–Cantelli lemmas, where (1.3) is proved for \( S_N = \sum_{n=1}^{N} 1_{\Gamma_n} \circ T^n \) where \( T \) is a measure preserving transformation on a probability space \((\Omega, P)\) and \( \Gamma_n, n \geq 1 \) is a sequence of measurable sets. For such \( S_N \)’s the convergence (1.3) was proved, in particular, for the Gauss map \( T x = \frac{1}{2} \pmod{1}, x \in (0, 1) \) preserving the Gauss measure \( P(\Gamma) = \frac{1}{\ln 2} \int_{[0, 1]} \frac{d\gamma}{1+\gamma} \). This line of research became quite popular in the last two decades. In particular, [3] proves (1.3) in the dynamical setup considering \( T \) being the so called subshift of finite type on a sequence space where \( \Gamma_n, n \geq 1 \) is a sequence of cylinders while another series of papers dealt with uniformly and non-uniformly hyperbolic dynamical systems as a transformation \( T \) and with geometric balls as \( \Gamma_n \)'s (see, for instance, [8], [6] and references there).

In this paper we consider, in particular, “nonconventional” extensions of some of the above results aiming to prove that under certain conditions (1.3) holds true with \( S_N = \sum_{n=1}^{N} \prod_{i=1}^{\ell} 1_{\Gamma_{q_i(n)}} \) and \( \mathcal{E}_N = \sum_{n=1}^{N} P(\Gamma_{q_i(n)}) \) where \( q_i(n), i = 1, ..., \ell \) functions taking on positive integer values on positive integers and satisfying certain assumptions valid, in particular, for polynomials with integer coefficients. When \( \ell = 1 \) (conventional setup) the \( \phi \)-mixing with a summable coefficient \( \phi \) suffices for our result, while for \( \ell > 1 \) we have to impose stronger \( \psi \)-mixing conditions.

In the dynamical systems setup we consider \( S_N = \sum_{n=1}^{N} \prod_{i=1}^{\ell} 1_{C_n^{(i)} \circ T^{q_i(n)}} \) and \( \mathcal{E}_N = \sum_{n=1}^{N} \prod_{i=1}^{\ell} P(C_n^{(i)}) \) where \( T \) is the left shift on a sequence space \( \mathcal{A}^\mathbb{Z} \) with a finite or countable alphabet while \( C_n^{(i)} \), \( i = 1, ..., \ell, n \geq 1 \) is a sequence of cylinder sets. As an application we study the asymptotic behaviors of expressions \( M_N = \max_{1 \leq n \leq N} (\min_{1 \leq i \leq \ell} \Phi_{2^{(i)}} \circ T^{q_i(n)}) \) where \( \Phi_{2^{(i)}}(\omega) = -\ln(d(\omega, \tilde{\omega})), \omega, \tilde{\omega} \in \mathcal{A}^\mathbb{N} \) and \( d(\cdot, \cdot) \) is the natural distance on the sequence space.

Our results extend some of the previous work in the following aspects. First, the strong Borel–Cantelli property in the nonconventional setup \( \ell > 1 \) was not studied before at all. Secondly, even in the conventional setup \( \ell = 1 \) considering rather general functions \( q(n) = q_1(n) \) in place of just \( q(n) = n \) seems to be new, as well. Thirdly, we extend for shifts some of the results from [3] considering sequence spaces with countable alphabets and \( \phi \)-mixing invariant measures rather than just subshifts of finite type with Gibbs measures which are exponentially fast \( \psi \)-mixing (see [11]). This allows to apply our results, for instance, to Gibbs-Markov maps and to Markov chains with a countable state space satisfying the Doeblin condition since both examples are exponentially fast \( \phi \)-mixing, see [12] and [2], respectively.

In the next section we will formulate precisely our setups and assumptions and state our main results. In Section 3 we will prove the strong Borel–Cantelli property for events under the \( \phi \)-mixing condition in the conventional setup \( \ell = 1 \) and under \( \psi \)-mixing condition in the nonconventional setup \( \ell > 1 \). In Sections 4 and 5 we extend the strong Borel–Cantelli property to shifts under the \( \phi \)-mixing when \( \ell = 1 \) and under \( \psi \)-mixing when \( \ell > 1 \), respectively. In the last Section 6 we exhibit applications to the asymptotic behaviors of maximums along shifts of logarithmic distance functions.
2. Preliminaries and main results

We start with a probability space \((\Omega, \mathcal{F}, P)\) and a two parameter family of \(\sigma\)-algebras \(\mathcal{F}_{mn}\) indexed by pairs of integers \(-\infty \leq m \leq n \leq \infty\) and such that \(\mathcal{F}_{mn} \subset \mathcal{F}_{m'n'} \subset \mathcal{F}\) if \(m' \leq n \leq n'\). Recall that the \(\phi\) and \(\psi\) dependence coefficient between two \(\sigma\)-algebras \(\mathcal{G}\) and \(\mathcal{H}\) can be written in the form (see [2]),

\[
\phi(\mathcal{G}, \mathcal{H}) = \sup_{\Gamma \in \mathcal{G}, \Delta \in \mathcal{H}} \left\{ \left| \frac{P(\Gamma \cap \Delta)}{P(\Delta)} - P(\Delta) \right|, P(\Gamma) \neq 0 \right\} = \frac{1}{2} \sup \{\|E(g|\mathcal{G}) - Eg\|_{L^\infty} : g \text{ is } \mathcal{H}\text{-measurable and } \|g\|_{L^\infty} \leq 1\}
\]

and

\[
\psi(\mathcal{G}, \mathcal{H}) = \sup_{\Gamma \in \mathcal{G}, \Delta \in \mathcal{H}} \left\{ \left| \frac{P(\Gamma \cap \Delta)}{P(\Delta)} - 1 \right|, P(\Gamma)P(\Delta) \neq 0 \right\} = \frac{1}{2} \sup \{\|E(g|\mathcal{G}) - Eg\|_{L^\infty} : g \text{ is } \mathcal{H}\text{-measurable and } E|g| \leq 1\},
\]

respectively. The \(\phi\)-dependence (mixing) and the \(\psi\)-dependence (mixing) in the family \(\mathcal{F}_{mn}\) is measured by the coefficients

\[
\phi(k) = \sup_m \phi(\mathcal{F}_{-\infty,m}, \mathcal{F}_{m+k,\infty}) \quad \text{and} \quad \psi(k) = \sup_m \psi(\mathcal{F}_{-\infty,m}, \mathcal{F}_{m+k,\infty}),
\]

respectively, where \(k = 0, 1, 2, \ldots\). The probability measure \(P\) is called \(\phi\)-mixing or \(\psi\)-mixing with respect to the family of \(\sigma\)-algebras \(\mathcal{F}_{mn}\) if \(\phi(n) \to 0\) or \(\psi(1) < \infty\) and \(\psi(n) \to 0\) as \(n \to \infty\), respectively.

Our setup includes also functions \(q_1(n), q_2(n), \ldots, q_{\ell}(n)\) with \(\ell \geq 1\) taking on nonnegative integer values on integers \(n \geq 0\) and satisfying

2.1. Assumption. There exists a constant \(K > 0\) such that

(i) for any \(i \neq j, 1 \leq i, j \leq \ell\) and every integer \(k\) the number of integers \(n \geq 0\) satisfying at least one of the equations

\[
q_i(n) - q_j(n) = k \quad \text{and} \quad q_i(n) = k
\]

do not exceed \(K\) (when \(\ell = 1\) only the second equation in (2.4) should be taken into account):

(ii) the cardinality of the set \(\mathcal{N}\) of all pairs \(m > n \geq 0\) satisfying

\[
\max_{1 \leq i \leq \ell} q_i(n) \leq \max_{1 \leq i \leq \ell} q_i(m)
\]

do not exceed \(K\).

Observe that Assumption [2.4] is satisfied if \(q_i, i = 1, \ldots, \ell\) are essentially distinct nonconstant polynomials (i.e. \(|q_i(n) - q_j(n)| \to \infty\) as \(n \to \infty\) for any \(i \neq j\)) with integer coefficients taking on nonnegative values on nonnegative integers. Indeed, \(q_i(n) - q_j(n)\) and \(q_i(n)\) are nonconstant polynomials, and so the number of \(n\)'s solving one of equations in (2.4) is bounded by the degree of the corresponding polynomial. In order to show that (2.4) can hold true in the polynomial case only for finitely many pairs \(m < n\) observe that there exists \(n_0 \geq 1\) such that all polynomials \(q_1(n), q_2(n), \ldots, q_{\ell}(n)\) are strictly increasing on \([n_0, \infty)\). Hence, if \(n > m \geq n_0\) then (2.4) cannot hold true. If \(0 \leq m < n_0\) and \(n \geq n_0\) then there exists \(n_1 \geq n_0\) such that for all \(n \geq n_1\) (2.5) cannot hold true, as well. The remaining case \(0 \leq m < n_0\) and \(0 \leq n < n_1\) concerns less than \(n_0n_1\) pairs \(m < n\).

Next, we will state our result concerning sequences of events. Let \(\Gamma_1, \Gamma_2, \ldots \in \mathcal{F}\) be a sequence of events and each \(\sigma\)-algebra \(\mathcal{F}_{mn}, 1 \leq m \leq n < \infty\) be generated
by the events $\Gamma_m, \Gamma_{m+1}, \ldots, \Gamma_n$. Set also $\mathcal{F}_{mn} = \mathcal{F}_{1n}$ for $-\infty \leq m \leq 0$ and $n \geq 1$, $\mathcal{F}_{mn} = \{\emptyset, \Omega\}$ for $m, n \leq 0$ and $\mathcal{F}_{m,\infty} = \sigma\{\Gamma_m, \Gamma_{m+1}, \ldots\}$. Set

\[
S_N = \sum_{n=1}^{N} \left( \prod_{i=1}^{\ell} \mathbb{1}_{\Gamma_{q_i(n)}} \right) \quad \text{and} \quad \mathcal{E}_N = \sum_{n=1}^{N} \left( \prod_{i=1}^{\ell} P(\Gamma_{q_i(n)}) \right).
\]

2.2. **Theorem.** Let $\phi$ and $\psi$ be dependence coefficients defined by (2.3) for the above $\sigma$-algebras $\mathcal{F}_{mn}$. Assume that $\phi(n), n \geq 0$ is summable in the case $\ell = 1$ and $\psi(n), n \geq 0$ is summable in the case $\ell > 1$. Suppose that the functions $q_1(n), \ldots, q_\ell(n)$ satisfy Assumption (2.7) and

\[
\mathcal{E}_N \to \infty \quad \text{as} \quad N \to \infty.
\]

Then, with probability one,

\[
\lim_{N \to \infty} \frac{S_N}{\mathcal{E}_N} = 1 \quad \text{as} \quad N \to \infty.
\]

Next, we will present our results concerning shifts. Here $\Omega = \mathcal{A}^\mathbb{Z}$ is the space of sequences $\omega = (\ldots, \omega_{-1}, \omega_0, \omega_1, \ldots)$ with terms $\omega_i$ from a finite or countable alphabet $\mathcal{A}$ which is not a singleton with the index $i$ running along integers (or along natural numbers $\mathbb{N}$ which can also be considered requiring very minor modifications). We assume that the basic $\sigma$-algebra $\mathcal{F}$ is generated by all cylinder sets while the $\sigma$-algebras $\mathcal{F}_{mn}, n \geq m$ are generated by the cylinder sets of the form

\[
\{ \omega = (\omega_i)_{-\infty < i < \infty} : \omega_i = a_i \text{ for } m \leq i \leq n \}
\]

for some $a_m, a_{m+1}, \ldots, a_n \in \mathcal{A}$. The setup includes also the left shift $T : \Omega \to \Omega$ acting by $(T\omega)_i = \omega_{i+1}$ and a $T$-invariant probability measure $P$ on $(\Omega, \mathcal{F})$, i.e. $P(T^{-1}\Gamma) = P(\Gamma)$ for any measurable $\Gamma \subset \Omega$. In this setup $\phi$ and $\psi$-dependence coefficients defined by (2.3) will be considered with respect to the family of $\sigma$-algebras $\mathcal{F}_{mn}, m \leq n$ defined above. Without loss of generality we assume that the probability of each 1-cylinder $[a] = \{ \omega = (\omega_i)_{i \in \mathbb{Z}} : \omega_0 = a \text{ is positive} \}$. Given a constant $D > 0$ call an interval of integers $\Lambda = [l, r], l \leq r$, i.e. $C = \{ \omega = (\omega_i)_{-\infty < i < \infty} : \omega_i = a_i \text{ for } i = l, l + 1, \ldots, r \}$ for some $a_l, \ldots, a_r \in \mathcal{A}$. Given a constant $D > 0$ call an interval of integers $\Lambda_1 = [l_1, r_1]$ to be right $D$-nested in the interval of integers $\Lambda_2 = [l_2, r_2]$ if $[l_1, r_1] \subset (-\infty, r_2 + D)$, i.e. $r_1 < r_2 + D$. Such an interval $\Lambda_1$ will be called $D$-nested in $\Lambda_2$ if $[l_1, r_1] \subset (l_2 - D, r_2 + D)$. The latter notion was used also in [3].

Let $C_n^{(j)}, j = 1, \ldots, \ell, n = 1, 2, \ldots$ be a sequence of cylinder sets defined on intervals of integers $\Lambda_n, n = 1, 2, \ldots$ so that $C_n^{(j)}, j = 1, \ldots, \ell$ are defined on $\Lambda_n$ for each $n \geq 1$. Set

\[
S_N = \sum_{n=1}^{N} \left( \prod_{i=1}^{\ell} \mathbb{1}_{C_n^{(i)}} \circ T^{q_i(n)} \right) \quad \text{and} \quad \mathcal{E}_N = \sum_{n=1}^{N} \left( \prod_{i=1}^{\ell} P(C_n^{(i)}) \right).
\]

2.3. **Theorem.** Suppose that the functions $q_1(n), \ldots, q_\ell(n)$ satisfy Assumption (2.7) and

\[
\mathcal{E}_N \to \infty \quad \text{as} \quad N \to \infty.
\]

Let $C_n^{(j)}, j = 1, \ldots, \ell, n \geq 1$ be a sequence of cylinder sets defined on intervals $\Lambda_n \subset \mathbb{Z}$ as described above and $D > 0$ be a constant.
(i) If \( \ell = 1 \) assume that the \( \phi \)-dependence coefficient is summable and that for all \( m < n \) the interval \( \Lambda_m \) is right \( D \)-nested in \( \Lambda_n \). Then, with probability one,

\[
\lim_{N \to \infty} \frac{S_N}{E_N} = 1 \quad \text{as} \quad N \to \infty.
\]

(ii) If \( \ell > 1 \) assume that the \( \psi \)-dependence coefficient is summable and that for all \( m < n \) the interval \( \Lambda_m \) is \( D \)-nested in \( \Lambda_n \). Then with probability one (2.11) holds true, as well.

As in most papers on the strong Borel–Cantelli property both Theorems 2.2 and 2.3 rely on the following basic result.

2.4. Theorem. Let \( \Gamma_1, \Gamma_2, \ldots \) be a sequence of events such that for any \( N \geq M \geq 1 \),

\[
\sum_{m,n=M}^{N} (P(\Gamma_m \cap \Gamma_n) - P(\Gamma_m)P(\Gamma_n)) \leq c \sum_{n=M}^{N} P(\Gamma_n)
\]

where a constant \( c > 0 \) does not depend on \( M \) and \( N \). Then for each \( \varepsilon > 0 \) almost surely

\[
S_N = E_N + O(E_N^{1/2} \log^{1/2} \varepsilon_N)
\]

where

\[
S_N = \sum_{n=1}^{N} I_{\Gamma_n} \quad \text{and} \quad E_N = \sum_{n=1}^{N} P(\Gamma_n).
\]

In particular, if

\[
E_N \to \infty \quad \text{as} \quad N \to \infty
\]

then with probability one

\[
\lim_{N \to \infty} \frac{S_N}{E_N} = 1 \quad \text{as} \quad N \to \infty.
\]

This result (as well as the part of Theorem 2.2 for \( \ell = 1 \) and \( q_1(n) = n \)) appears already in Theorem 3 from [14] and in a slightly more general (analytic) form it is proved as Lemma 10 in §7 of Ch.1 from [16]. Both sources refer to [15] as the origin of this result.

We observe that Theorem 2.3 extends Theorem 2.1 from [3] in several directions. First, for \( \ell = 1 \) we prove the result for arbitrary \( \phi \)-mixing probability measures with a summable coefficient \( \phi \) on a shift space with a countable alphabet and not just for subshifts of finite type with Gibbs measures. Secondly, the case \( \ell > 1 \) and rather general functions \( q_i(n) \) in place of just \( \ell = 1 \) and \( q_1(n) = n \) were not considered before in both the setups of Theorem 2.2 and 2.3.

A direct application of Theorem 2.3 yields corresponding strong Borel–Cantelli property for dynamical systems which have symbolic representations by means of finite or countable partitions, for instance, hyperbolic dynamical systems (see, for instance, [1]) where sequences of cylinders in Theorem 2.3 should be replaced by corresponding sequences of elements of joins of iterates of the partition. By a slight modification (just by considering cylinder sets defined on intervals of nonnegative integers only) Theorem 2.3 remains valid for one-sided shifts and then it can be applied to noninvertible dynamical systems having a symbolic representation via their finite or countable partitions such as expanding transformations, the Gauss map of the interval and more general transformations generated by \( f \)-expansions (see [5]).
In Section 6 we apply Theorem 2.3 to some limiting problems obtaining a symbolic version of results from [7] which dealt with dynamical systems on $\mathbb{R}^d$ or manifolds and not with shifts. Namely, in the setup of Theorem 2.3 we introduce the distance between $\omega = (\omega_i)_{i \in \mathbb{Z}}$ and $\tilde{\omega} = (\tilde{\omega}_i)_{i \in \mathbb{Z}}$ from $\Omega$ by

\[(2.14) \quad d(\omega, \tilde{\omega}) = \exp(-\gamma \min\{i \geq 0 : \omega_i \neq \tilde{\omega}_i \} \text{ or } \omega_{-i} \neq \tilde{\omega}_{-i} \}), \quad \gamma > 0.
\]

Set

\[(2.15) \quad \Phi_\omega(\omega) = -\ln(d(\omega, \tilde{\omega})) \text{ for } \omega, \tilde{\omega} \in \Omega \text{ and }
M_{N,\tilde{\omega}}(\omega) = M_{N,\tilde{\omega}(1),...,\tilde{\omega}(\ell)} = \max_{1 \leq n \leq N} \min_{1 \leq i \leq \ell} (\Phi_{\tilde{\omega}(i)}(\omega) \circ T^{q(n)}(\omega))\]

for some fixed $\ell$-tuple $\tilde{\omega} = (\tilde{\omega}(1),...,\tilde{\omega}(\ell))$, $\tilde{\omega}(i) \in \Omega$, $i = 1,...,\ell$.

2.5. **Theorem.** Assume that the entropy of the partition into 1-cylinders is finite, i.e.

\[(2.16) \quad -\sum_{a \in A} P([a]) \ln P([a]) < \infty.
\]

Then, under the conditions of Theorem 2.3 for almost all $\tilde{\omega}(1),...,\tilde{\omega}(\ell) \in \Omega$ with probability one,

\[(2.17) \quad \frac{M_{N,\tilde{\omega}(1),...,\tilde{\omega}(\ell)}}{\ln N} \to \frac{\gamma}{2\ell h} \quad \text{as } N \to \infty
\]

where $h$ is the Kolmogorov–Sinai entropy of the shift $T$ on the probability space $(\Omega, \mathcal{F}, P)$ and, as in Theorem 2.3 if $\ell = 1$ we assume only $\phi$-mixing with a summable coefficient $\phi$ and if $\ell > 1$ we assume $\psi$-mixing with a summable coefficient $\psi$ (and in both cases $h > 0$ by Lemma 3.1 in [10] and Lemma 3.1 in [9]).

3. **Proof of Theorem 2.2**

3.1. **The case $\ell = 1$.** Let $N \geq M$ and fix an $m$ between $M$ and $N$. By Assumption 2.1 for each $k$ there exists at most $K$ of integers $n$ such that $q(n) - q(m) = k$ where $q(n) = q_1(n)$. If $q(n) - q(m) = k \geq 1$ then by the definition of the $\phi$-dependence coefficient

\[(3.1) \quad |P(\Gamma_{q(m)} \cap \Gamma_{q(n)}) - P(\Gamma_{q(m)})P(\Gamma_{q(n)})| \leq \phi(k)P(\Gamma_{q(m)}).
\]

Hence,

\[(3.2) \quad \sum_{N \geq n \geq M, q(n) > q(m)} |P(\Gamma_{q(m)} \cap \Gamma_{q(n)}) - P(\Gamma_{q(m)})P(\Gamma_{q(n)})| \leq K\sum_{k=1}^{\infty} \phi(k).
\]

Since the coefficient $\phi$ is summable and that similar inequalities hold true when $q(m) > q(n)$ we conclude that the condition (2.14) of Theorem 2.3 is satisfied with $\Gamma_{q(n)}$ in place of $\Gamma_n$, $n = 1,2,...$ there, and so (2.8) follows in the case $\ell = 1$ assuming (2.7).

3.2. **The case $\ell > 1$.** We start with the following counting arguments concerning the functions $q_i, i = 1,...,\ell$ satisfying Assumption 2.1. Introduce

\[q(n) = \min_{1 \leq i \neq j \leq \ell} |q_i(n) - q_j(n)|.
\]
By Assumption 2.1(i) for each pair \( i \neq j \) and any \( k \) there exists at most \( K \) nonnegative integers \( n \) such that \( q_i(n) - q_j(n) = k \), and so

\[
\# \{ n > 0 : q_i(n) = k \} < K \ell^2
\]

where \# stands for "the number of ...". We will need also the following semi-metric between integers \( k, l > 0 \),

\[
\delta(k, l) = \min_{1 \leq i, j \leq \ell} |q_i(k) - q_j(l)|.
\]

It follows from Assumption 2.1(i) that for any integers \( m > 0 \) and \( k \geq 0 \),

\[
\# \{ n > 0 : \delta(m, n) = k \} < 2K^2 \ell^2.
\]

Indeed, the number of \( m \)'s such that \( q_j(m) = q_i(n) - k \) for a fixed \( i, j, n \) and \( k \) does not exceed \( K \) by Assumption 2.1(i) and (3.4) follows since \( 1 \leq i, j \leq \ell \).

In order to prove Theorem 2.2 for \( \ell > 1 \) we will estimate first

\[
|E(X_m X_n) - EX_m EX_n| = |P(\cap_{i=1}^\ell \Gamma_{q_i(n)} \cap \Gamma_{q_j(n)}) - P(\cap_{i=1}^\ell \Gamma_{q_i(m)} \cap \Gamma_{q_j(m)})|
\]

where \( m, n > 0 \) and \( X_k = \prod_{i=1}^\ell \Gamma_{q_i(k)} \). If \( \delta(m, n) = k \geq 1 \) then by Lemma 3.3 in [9] and the definition of the \( \psi \)-dependence coefficient

\[
|E(X_m X_n) - EX_m EX_n| \leq 2^{2\ell+2}\psi(k)(2 - (1 + \psi(k))\ell) - 2EX_m EX_n
\]

where we assume, in fact, that \( k \) is large enough so that \( \psi(k) < 2^{1/\ell} - 1 \). Thus, let \( k_0 = \min\{k : \psi(k) < 2^{1/\ell} - 1\} \). Then by (3.4) and (3.6),

\[
\sum_{M \leq n < M+1} |E(X_m X_n) - EX_m EX_n| \leq cEX_m
\]

where

\[
c = 2K^2 \ell^2 \left( 1 + 2^{2\ell+2}(2 - (1 + \psi(k_0))\ell) \right)^{-2} \sum_{k=k_0}^\infty \psi(k)
\]

where we took into account that

\[
|EX^2_m - (EX_m)^2| \leq EX_m.
\]

Summing in (3.7) in \( m \) between \( M \) an \( N \) we obtain the condition (2.12) of Theorem 2.4 with \( \cap_{i=1}^\ell \Gamma_{q_i(n)} \) in place of \( \Gamma_n \) there. Hence if

\[
\sum_{n=1}^\infty P(\cap_{i=1}^\ell \Gamma_{q_i(n)}) = \infty
\]

then Theorem 2.4 yields that with probability one

\[
S_N / \tilde{E}_N \to 1 \text{ as } N \to \infty
\]

where \( \tilde{E}_N = \sum_{n=1}^N P(\cap_{i=1}^\ell \Gamma_{q_i(n)}) \).

Since we assume (2.10) and not (3.8), it remains to show that under our conditions,

\[
\tilde{E}_N / \tilde{E}_N \to 1 \text{ as } N \to \infty.
\]
By Lemma 3.2 from [9] we obtain when \( q(n) = k \geq 1 \) that

\[
(3.11) \quad |P(\bigcap_{i=1}^{\ell} \Gamma_{q_i}(n)) - \prod_{i=1}^{\ell} P(\Gamma_{q_i}(n))| \leq \left( (1 + \psi(k))^{\ell} - 1 \right) \prod_{i=1}^{\ell} P(\Gamma_{q_i}(n)).
\]

For \( q(n) = 0 \) we estimate the left hand side of (3.11) just by 1. Hence, by (3.3),

\[
(3.12) \quad |\hat{\mathcal{E}}_N - \mathcal{E}_N| \leq K \ell^2 + \sum_{n=1, q(n) \geq 1} \left( (1 + \psi(q(n)))^{\ell} - 1 \right) \prod_{i=1}^{\ell} P(\Gamma_{q_i}(n))
\]

\[
\leq K \ell^2 + \sum_{n=1, q(n) \geq 1} \left( (1 + \psi(q(n)))^{\ell} - 1 \right)
\]

\[
\leq K \ell^2 + K \ell^2 \sum_{n=1}^{\infty} \left( (1 + \psi(q(n)))^{\ell} - 1 \right) \leq C < \infty
\]

for some constant \( C > 0 \), since the coefficient \( \psi \) is summable. Dividing (3.12) by \( \mathcal{E}_N \) and taking into account (2.10) we obtain (3.10) and complete the proof of Theorem 2.2. \( \square \)

4. Proof of Theorem 2.3(i)

Here \( \ell = 1 \), and so we set \( C_n = C_n^{(1)} \) and \( q(n) = q_1(n) \). Consider cylinder sets \( C_m \) and \( C_n \), \( 1 \leq m < n \) defined on intervals of integers \( \Lambda_m = [l_m, r_m] \) and \( \Lambda_n = [l_n, r_n] \) with \( \Lambda_m \) right \( D \)-nested in \( \Lambda_n \) implying that \( r_m < r_n + D \). Let \( k = q(n) - q(m) \).

By Assumption 2.1(i) for each \( m \) and \( k \) this equality can hold true only for at most \( K \) of \( n \)'s and by Assumption 2.1(ii) for no more than \( K \) of \( n \)’s we may have \( q(n) \leq q(m) \). Next, we can write

\[
(4.1) \quad r_n + q(n) > r_m + q(m) + k - D.
\]

Assume first that

\[
(4.2) \quad l_n + q(n) \leq r_m + q(m) \quad \text{and} \quad r_n + q(n) > r_m + q(m).
\]

Let \( C_n = [a_{t_n}, a_{t_n+1}, ..., a_{r_n}] \) and \( \hat{C}_{m,n} = [a_{t_{m,n}}, a_{t_{m,n}+1}, ..., a_{r_n}] \) where we assume that \( r_n > l_n \),

\[
t_{m,n} = s_{m,n} + \left[ \frac{1}{2} (r_n - s_{m,n} + 1) \right] \quad \text{and} \quad a_{m,n} = l_n + (r_m + q(m) - l_n - q(n)) + 1 = r_m + q(m) - q(n) + 1.
\]

It follows that

\[
(4.3) \quad r_n - t_{m,n} + 1 \geq \frac{1}{2}(k - D) \quad \text{and} \quad t_{m,n} + q(n) - r_m + q(m) \geq \frac{1}{2}(k - D) - 1.
\]

Assuming that \( k \geq D + 4 \) we obtain by the definition of the \( \phi \)-dependence coefficient that

\[
(4.4) \quad P(T^{-q(m)}C_m \cap T^{-q(n)}C_n) \leq P(T^{-q(m)}C_m \cap T^{-q(n)}\hat{C}_{m,n})
\]

\[
\leq P(C_m)P(\hat{C}_{m,n}) + \phi(\left[ \frac{1}{2}(k - D) \right] - 1)P(C_m).
\]

To make the estimate (4.4) suitable for our purposes we recall that according to Lemma 3.1 in [10] there exists \( \alpha > 0 \) such that any cylinder set \( C \) defined on an interval of integers \( \Lambda = [l, r] \) satisfies

\[
(4.5) \quad P(C) \leq e^{-\alpha(r-l)},
\]

and so

\[
(4.6) \quad P(\hat{C}_{m,n}) \leq \exp(-\alpha(\left[ \frac{1}{2}(k - D) \right] - 1)).
\]
In addition to (4.4) we can write also

\[ P(C_m)P(C_n) \leq e^{-\alpha(r_n - l_n)}P(C_m) \leq e^{-\alpha(k-D)}P(C_m) \]

where we used that by (4.1),

\[ r_n - l_n \geq r_n - s_{m,n} + 1 = r_n + q(n) - r_m - q(m) > k - D. \]

Observe that by Assumption 2.1 there exists at most \( K(D+1) \) of \( n \)'s for which \( q(n) - q(m) = k \leq D \), and so by (4.1) the second inequality in (4.2) may fail only for at most \( K(D+1) \) of \( n \)'s. For such \( n \)'s we use the trivial estimate

\[ |P(T^{-q(m)}C_m \cap T^{-q(n)}C_n) - P(C_m)P(C_n)| \leq P(C_m). \]

Now if

\[ l_n + q(n) > r_m + q(m) \]

then by the definition of the \( \phi \)-dependence coefficient we can write by (4.1) that

\[ |P(T^{-q(m)}C_m \cap T^{-q(n)}C_n) - P(C_m)P(C_n)| \leq \phi(l_n + q(n) - r_m - q(m))P(C_m) \leq \phi(k - D - (r_n - l_n))P(C_m) \]

but this may not suffice for our purposes when \( r_n - l_n \) is large. In this case we proceed as in (4.4), (4.6) and (4.7) where we take \( \tilde{\mathcal{C}}_n = [a_{\ell_n}, a_{\ell_n+1}, ..., a_{r_n}] \) with \( t_n = l_n + \frac{1}{r}(r_n - l_n) + 1 \). Then

\[ r_n + q(n) - r_m - q(m) > \frac{1}{2}(r_n - l_n) + 1 \quad \text{and} \quad r_n - l_n \geq \frac{1}{2}(r_n - l_n) - 1, \]

and so

\[ P(T^{-q(m)}C_m \cap T^{-q(n)}C_n) \leq P(T^{-q(m)}C_m \cap T^{-q(n)}\tilde{\mathcal{C}}_n) \leq P(C_m)P(\tilde{\mathcal{C}}_n) + \phi(\frac{1}{r}(r_n - l_n) + 1)P(C_m) \leq \left( e^{-\alpha(\frac{1}{r}(r_n - l_n) - 1)} + \phi(\frac{1}{r}(r_n - l_n)) \right)P(C_m). \]

Thus, when (4.9) holds true we use (4.10) if \( r_n - l_n \leq \frac{k-D}{2} \) and (4.11) when \( r_n - l_n > \frac{k-D}{2} \). In both cases we will obtain the estimate

\[ |P(T^{-q(m)}C_m \cap T^{-q(n)}C_n) - P(C_m)P(C_n)| \leq e^{-\alpha(\frac{1}{r}(k-D) - 1)}P(C_m). \]

Finally, taking into account that \( q(n) - q(m) = k \leq D \) can occur only for at most \( K(D+1) \) of \( n \)'s and for each \( k \) the equality \( q(n) - q(m) = k \) may hold true for at most \( K \) of \( n \)'s we conclude from (4.4), (4.6), (4.8), (4.12) and from the summability of the coefficient \( \phi \) that for any \( m = M, M + 1, ..., N \),

\[ \sum_{n=M}^{N} |P(T^{-q(m)}C_m \cap T^{-q(n)}C_n) - P(C_m)P(C_n)| \leq cP(C_m) \]

for some constant \( c > 0 \) independent of \( M \) and \( N \). Summing in \( m \) between \( M \) and \( N \) we conclude that the condition (2.12) of Theorem 2.4 is satisfied with \( \Gamma_n = T^{-q(n)}C_n \), and so assuming (2.10) we obtain (2.11) completing the proof of Theorem 2.3(i).
5. Proof of Theorem 2.3(ii)

Observe that if $\delta(n,m) = k$, $n > m \geq 0$ and the pair $n,m$ does not belong to the exceptional set $\mathcal{N}$ having cardinality at most $K$ then by Assumption 2.1(ii) for some $i_0, j_0 \leq \ell$,

\begin{equation}
q_{j_0} = \max_{1 \leq j \leq \ell} q_j(n) \geq q_{i_0}(m) + k = \max_{1 \leq i \leq \ell} q_i(m) + k.
\end{equation}

Let $C_m$ and $C_n$ be cylinder sets defined on $\Lambda_m = [l_m, r_m]$ and $\Lambda_n = [l_n, r_n]$, respectively. Since $C_m$ is $D$-nested in $C_n$, $r_m \leq r_n + D$, and so by (5.1),

\begin{equation}
r_m + q_{i_0}(m) \leq r_n + q_{j_0}(n) - k + D.
\end{equation}

Assume first that

\begin{equation}
l_n + q_{j_0}(n) \leq r_m + q_{i_0}(m) \quad \text{and} \quad r_n + q_{j_0}(n) > r_m + q_{i_0}(m).
\end{equation}

Let $\hat{C}_{m,n} = [a_{s_m,n}, a_{s_m,n+1}, \ldots, a_r]$ where

\begin{equation}
s_m,n = l_n + (r_m + q_{i_0}(m) - l_n - q_{j_0}) + 1 = r_m + q_{i_0}(m) - q_{j_0}(n) + 1,
\end{equation}

and so $\hat{C}_{m,n}$ is defined on the interval $[s_m,n, r_n]$ of the length

\begin{equation}
r_n - s_m,n - 1 = r_n + q_{j_0}(n) - r_m - q_{i_0}(m) \geq k - D
\end{equation}

where the last inequality follows from (5.2). Hence, by the definition of the $\psi$-dependence coefficient

\begin{equation}
P(\cap_{i=1}^{\ell} (T^{-q_i(m)}C_m \cap T^{-q_i(n)}C_n)) \\
\leq P(\cap_{i=1}^{\ell} (T^{-q_i(m)}C_m \cap T^{-q_i(n)}\hat{C}_{m,n})) \\
\leq (1 + \psi(1))P(\cap_{i=1}^{\ell} T^{-q_i(m)}C_m)P(T^{-q_i(n)}\hat{C}_{m,n}) \\
\leq (1 + \psi(1))e^{-\alpha(k-D)}P(\cap_{i=1}^{\ell} T^{-q_i(m)}C_m)
\end{equation}

where $\hat{C}_{m,n}$ is constructed as above with $C_n = C_{n,j_0}.$

We can write also that

\begin{equation}
P(\cap_{i=1}^{\ell} T^{-q_i(m)}C_m)P(\cap_{i=1}^{\ell} T^{-q_i(n)}C_n) \leq P(C_m^{(1)})P(\cap_{i=1}^{\ell} T^{-q_i(m)}C_m) \\
\leq e^{-\alpha(r_n-l_n)}P(\cap_{i=1}^{\ell} T^{-q_i(m)}C_m).
\end{equation}

Since $r_n - l_n \geq r_n - s_m,n + 1 \geq k - D$, it follows that under the condition (5.3),

\begin{equation}
|P(\cap_{i=1}^{\ell} (T^{-q_i(m)}C_m \cap T^{-q_i(n)}C_n)) \\
- P(\cap_{i=1}^{\ell} T^{-q_i(m)}C_m)P(\cap_{i=1}^{\ell} T^{-q_i(n)}C_n)| \\
\leq (1 + \psi(1))e^{-\alpha(k-D)}P(\cap_{i=1}^{\ell} T^{-q_i(m)}C_m).
\end{equation}

On the other hand, if

\begin{equation}
l_n + q_{j_0}(n) > r_m + q_{i_0}(m),
\end{equation}

then by the definition of the $\psi$-dependence coefficient we obtain similarly to the above that

\begin{equation}
|P(\cap_{i=1}^{\ell} (T^{-q_i(m)}C_m \cap T^{-q_i(n)}C_n)) \\
- P(\cap_{i=1}^{\ell} T^{-q_i(m)}C_m)P(\cap_{i=1}^{\ell} T^{-q_i(n)}C_n)| \\
\leq (1 + \psi(l_n + q_{j_0}(n) - r_m - q_{i_0}(m)))P(\cap_{i=1}^{\ell} T^{-q_i(m)}C_m)P(C_n^{(1)}) \\
\leq (1 + \psi(1))e^{-\alpha(r_n-l_n)}P(\cap_{i=1}^{\ell} T^{-q_i(m)}C_m).
\end{equation}
Let a number \( d_0 \geq 1 \) be such that
\[
(5.11) \quad \psi(d_0) < 2^{1/\ell} - 1 \quad \text{and} \quad k - (r_n - l_n + 2D) > d_0.
\]

Since \( r_n - l_n \geq r_m - l_m - 2D \) by \( D \)-nesting, it follows by \((4.4)\) and Lemma 3.3 from [9] that
\[
(5.12) \quad |P(\cap_{\ell=1}^{\ell=r} (T^{q\alpha}C_m(i) \cap T^{q\alpha}C_n(i)))| \\
- P(\cap_{\ell=1}^{\ell=r} (T^{q\alpha}C_m(i) \cap T^{q\alpha}C_n(i)))| \\
\leq 2^{2d+2}\psi(k - \max(r_n - l_n, r_m - l_m)) \\
\times (2 - (1 + \psi(k - \max(r_n - l_n, r_m - l_m)))^\ell - 2 P(\cap_{\ell=1}^{\ell=r} (T^{q\alpha}C_m(i))) \\
\times P(\cap_{\ell=1}^{\ell=r} (T^{q\alpha}C_n(i))) \leq 2^{2d+2}\psi(k - (r_n - l_n + 2D)(2 - (1 + \psi(d_0))\ell - 2 \\
\times e^{-\alpha(r_n - l_n)} P(\cap_{\ell=1}^{\ell=r} (T^{q\alpha}C_m(i))).
\]

Since the cardinality of \( N \) does not exceed \( K \) we have
\[
(5.13) \quad \sum_{(n,m)\in\mathcal{N}^2} |P(\cap_{\ell=1}^{\ell=r} (T^{q\alpha}C_m(i) \cap T^{q\alpha}C_n(i)))| \\
- P(\cap_{\ell=1}^{\ell=r} (T^{q\alpha}C_m(i) \cap T^{q\alpha}C_n(i)))| \\
\leq KP(\cap_{\ell=1}^{\ell=r} (T^{q\alpha}C_m(i))).
\]

Next, we estimate now the remaining sum
\[
(5.14) \quad \sum_{n>m,(n,m)\notin\mathcal{N}} |P(\cap_{\ell=1}^{\ell=r} (T^{q\alpha}C_m(i) \cap T^{q\alpha}C_n(i)))| \\
- P(\cap_{\ell=1}^{\ell=r} (T^{q\alpha}C_m(i) \cap T^{q\alpha}C_n(i)))|.
\]

For the part of the sum in \( n \)'s satisfying \((5.3)\) we apply the inequality \((5.8)\) which yields the contribution to the total sum estimated using \((5.3)\) by
\[
(5.15) \quad 2K\ell^2(1 + \psi(1))P(\cap_{\ell=1}^{\ell=r} (T^{q\alpha}C_m(i))) \sum_{k=0}^{\infty} e^{-\alpha(k-D)} \\
= 2K\ell^2e^{\alpha(D)(1 + \psi(1))(1 - e^{-\alpha})^{-1}} P(\cap_{\ell=1}^{\ell=r} (T^{q\alpha}C_m(i)).
\]

For the parts of the sum \((5.14)\) which correspond to \( n \)'s satisfying \((5.9)\) but not \((5.11)\) we obtain that
\[
(5.16) \quad e^{-\alpha(r_n - l_n)} \leq e^{-ak} e^{-\alpha(2D-d_0)},
\]
and so taking into account \((3.4)\) the summation in \((5.14)\) over \( n \)'s satisfying \((5.9)\) can be estimated by
\[
(5.17) \quad 2K\ell^2(1 + \psi(1))e^{-\alpha(2D-d_0)} P(\cap_{\ell=1}^{\ell=r} (T^{q\alpha}C_m(i))) \sum_{k=0}^{\infty} e^{-ak} \\
= 2K\ell^2(1 + \psi(1))e^{-\alpha(2D-d_0)(1 - e^{-\alpha})} P(\cap_{\ell=1}^{\ell=r} (T^{q\alpha}C_m(i)).
\]

It remains to estimate the part of the sum \((5.14)\) which corresponds to \( n \)'s satisfying \((5.11)\) where we use \((5.12)\). We observe that
\[
(5.18) \quad \psi(k - (r_n - l_n + 2D))e^{-\alpha(r_n - l_n)} \\
= e^{2\alpha D} \psi(k - (r_n - l_n + 2D))e^{-\alpha(r_n - l_n + 2D)} \\
\leq e^{2\alpha D} \max(\psi([k/2]), \psi(1)e^{-\alpha[k/2])} \leq e^{2\alpha D} (\psi([k/2] + \psi(1)e^{-\alpha[k/2])}
\]
since either \( r_n - l_n + 2D \geq k/2 \) or \( k - (r_n - l_n + 2D) \geq k/2 \). Both summands in the right hand side of \((5.18)\) are summable in \( k \) (the first one by the assumption) which
Hence, it suffices to show that under the condition (2.10) with probability one,

\[ cP(\cap_{i=1}^\ell T^{-q_i(m)}C_m^{(i)}) \]

where \( c > 0 \) does not depend on \( m \). By estimates (5.8), (5.12), (5.13), (5.15) and (5.17), (5.19) above we conclude that the whole sum consisting of the part appearing in (5.13) plus the part displayed by (5.14) can be estimated by the expression (5.19) with another constant \( c > 0 \) independent of \( m \). It follows that there exists \( \tilde{c} > 0 \) such that for all \( N > M \geq 1 \),

\[ \sum_{m=1}^N P(\cap_{i=1}^\ell T^{-q_i(m)}C_m^{(i)}) = \infty \]

then by Theorem 2.4 we obtain that with probability one

\[ \sum_{n=1}^N \frac{\prod_{i=1}^\ell C_n^{(i)} \circ T^{q_i(n)}}{P(\cap_{i=1}^\ell T^{-q_i(n)}C_n^{(i)})} \rightarrow 1 \text{ as } N \to \infty. \]

It remains to show that under the condition (2.10) with probability one,

\[ \sum_{n=1}^N P(\cap_{i=1}^\ell T^{-q_i(n)}C_n^{(i)}) \prod_{i=1}^\ell P(C_n^{(i)}) \rightarrow 1 \text{ as } N \to \infty. \]

Observe again that

\[ P(\cap_{i=1}^\ell T^{-q_i(n)}C_n^{(i)}) \leq P(C_n^{(1)}) \leq e^{-\alpha(r_n-l_n)}. \]

Next, we split the sum in the left hand side of (5.21) into two sums

\[ S_1 = \sum_{n:(r_n-l_n) \leq \frac{1}{\alpha} \ln n} P(\cap_{i=1}^\ell T^{-q_i(n)}C_n^{(i)}) \]

and \( S_2 = \sum_{n:(r_n-l_n) > \frac{1}{\alpha} \ln n} P(\cap_{i=1}^\ell T^{-q_i(n)}C_n^{(i)}). \)

By (5.24),

\[ S_2 \leq \sum_{n=1}^\infty n^{-2} < \infty \text{ and also } \sum_{n:(r_n-l_n) > \frac{1}{\alpha} \ln n} \prod_{i=1}^\ell P(C_n^{(i)}) < \infty. \]

Hence, it suffices to show that under the condition (2.10) with probability one,

\[ \frac{\sum_{n \leq N: (r_n-l_n) \leq \frac{1}{\alpha} \ln n} P(\cap_{i=1}^\ell T^{-q_i(n)}C_n^{(i)})}{\sum_{n \leq N: (r_n-l_n) \leq \frac{1}{\alpha} \ln n} \prod_{i=1}^\ell P(C_n^{(i)})} \rightarrow 1 \text{ as } N \to \infty. \]

Set \( q(n) = \min_{i \neq j} |q_i(n) - q_j(n)|. \) Observe that by Assumption 2.1(i) for each \( k \),

\[ \# \{ n : q(n) = k \} \leq K\ell^2. \]

Consider first \( n \)’s satisfying

\[ q(n) \leq r_n - l_n. \]
In this case by (5.24),
\begin{equation}
P(\bigcap_{i=1}^{\ell} T^{-q_{i}(n)}C_{n}^{(i)}) \leq e^{-a q(n)}
\end{equation}
and relying on (5.26) we conclude that
\begin{equation*}
\sum_{n: q(n) \leq r_{n} - l_{n}} P(\bigcap_{i=1}^{\ell} T^{-q_{i}(n)}C_{n}^{(i)}) \leq K \ell^{2} \sum_{k=0}^{\infty} e^{-\alpha k} = K \ell^{2}(1 - e^{-\alpha})^{-1}
\end{equation*}
and the same estimate holds true for \( \sum_{n: q(n) \leq r_{n} - l_{n}} \prod_{i=1}^{\ell} P(C_{n}^{(i)}) \). Hence, the sum over such \( n \)'s does not influence the asymptotical behavior in (5.23) and (5.25) since the denominators there tend to \( \infty \).

It remains to consider the sums over \( n \)'s satisfying
\begin{equation}
q(n) > r_{n} - l_{n}.
\end{equation}
In this case we can apply Lemma 3.2 from [9] to obtain that
\begin{equation}
|P(\bigcap_{i=1}^{\ell} T^{-q_{i}(n)}C_{n}^{(i)}) - \prod_{i=1}^{\ell} P(C_{n}^{(i)})| \leq ((1 + \psi(q(n) - (r_{n} - l_{n})))^{\ell} - 1) \prod_{i=1}^{\ell} P(C_{n}^{(i)})
\end{equation}
\begin{equation*}
\leq ((1 + \psi(q(n) - (r_{n} - l_{n})))^{\ell} - 1) e^{-\ell \alpha(r_{n} - l_{n})}.
\end{equation*}

Now observe that either \( r_{n} - l_{n} \) or \( q(n) - (r_{n} - l_{n}) \) is greater or equal to \( \frac{1}{2} q(n) \). Denote by \( N_{1} \) the set of \( n \)'s for which \( r_{n} - l_{n} \geq \frac{1}{2} q(n) \) and by \( N_{2} \) the set of \( n \)'s for which \( q(n) - (r_{n} - l_{n}) \geq \frac{1}{2} q(n) \). Taking into account (5.24) and (5.29) we obtain that
\begin{equation}
\sum_{n \in N_{1}} ((1 + \psi(q(n) - (r_{n} - l_{n})))^{\ell} - 1) e^{-\ell \alpha(r_{n} - l_{n})}
\leq ((1 + \psi(1)))^{\ell} - 1) \sum_{n \in N_{1}} e^{-\frac{1}{2} \ell \alpha q(n)}
\leq K \ell^{2}((1 + \psi(1))^{\ell} - 1) \sum_{k=0}^{\infty} e^{-\frac{1}{2} \ell \alpha k}
= K \ell^{2}((1 + \psi(1))^{\ell} - 1)(1 - e^{-\frac{1}{2} \ell \alpha})^{-1} < \infty.
\end{equation}
Next, taking into account that \( \psi(k) \) is summable we see that
\begin{equation}
\sum_{n \in N_{2}} ((1 + \psi(q(n) - (r_{n} - l_{n})))^{\ell} - 1) e^{-\ell \alpha(r_{n} - l_{n})}
\leq \sum_{n \in N_{2}} ((1 + \psi(\max(1, \frac{1}{2} q(n))))^{\ell} - 1)
\leq 2 K \ell^{2} \sum_{k=1}^{\infty} ((1 + \psi(k))^{\ell} - 1) = 2 K \ell^{2} \sum_{k=1}^{\infty} \sum_{m=1}^{\ell} \left( \frac{1}{m!} \right) \psi(k)^{m} < \infty.
\end{equation}
Hence,
\begin{equation}
| \sum_{n=1}^{\infty} (P(\bigcap_{i=1}^{\ell} T^{-q_{i}(n)}C_{n}^{(i)}) - \prod_{i=1}^{\ell} P(C_{n}^{(i)})) | < \infty
\end{equation}
and since \( \sum_{n=1}^{\infty} \prod_{i=1}^{\ell} P(C_{n}^{(i)}) = \infty \), we obtain (5.25), and so (5.23), as well, completing the proof of Theorem 2.2(ii).

6. ASYMPTOTICS OF MAXIMUMS OF LOGARITHMIC DISTANCE FUNCTIONS

In this section we will prove Theorem 2.3. Let \( \check{\omega}^{(j)} = (\check{\omega}_{t}^{(j)})_{i \in \mathbb{Z}} \in \Omega \) and \( C_{n}(\check{\omega}^{(j)}) \), \( j = 1, ..., \ell \), \( n = 1, 2, ... \) be a sequence of cylinder sets such that
\[ C_{n}(\check{\omega}^{(j)}) = \{ \omega = (\omega_{i})_{i \in \mathbb{Z}} \in \Omega : \omega_{i} = \check{\omega}_{i}^{(j)} \text{ provided } |i| \leq r_{n} \} \]
where \( r_n \uparrow \infty \) as \( n \uparrow \infty \) is a sequence of integers. Observe that by the Shannon–McMillan–Breiman theorem (see, for instance, [13]) for almost all \( \tilde{\omega} \in \Omega \),

\[
\lim_{n \to \infty} \frac{1}{2r_n} \ln P(C_n(\tilde{\omega})) = -h
\]

where \( h \) is the Kolmogorov–Sinai entropy of the shift \( T \) with respect to \( P \) since the latter measure is ergodic whether we assume \( \phi \) or \( \psi \)-mixing.

Now suppose that

\[
\sum_{n=1}^\infty \prod_{i=1}^\ell P(C_n(\tilde{\omega}^{(i)})) < \infty.
\]

It follows from (5.32) that (6.2) implies also

\[
\sum_{n=1}^\infty P(\cap_{i=1}^\ell T^{-q_i(n)}C_n(\tilde{\omega}^{(i)})) < \infty
\]

which is, of course, a tautology if \( \ell = 1 \). It follows from the first Borel–Cantelli lemma that for almost all \( \omega \in \Omega \) only finitely many events \( \{T^{q_i(n)}\omega \in C_n(\tilde{\omega}^{(i)}), i = 1, ..., \ell\} \) can occur. But if the latter event does not hold true then

\[
T^{q_j(n)} \not\in C_n(\tilde{\omega}^{(j)}) \text{ for some } 1 \leq j \leq \ell,
\]

and so

\[
d(T^{q_j(n)}\omega, \tilde{\omega}^{(j)}) > e^{-\gamma r_n} \text{ i.e. } \Phi_\omega(T^{q_j(n)}\omega) < \gamma r_n
\]

where the distance \( d(\cdot, \cdot) \) and the function \( \Phi \) were defined in (2.14) and (2.15). It follows that in this case there exists \( N_{\tilde{\omega}}, \tilde{\omega} = (\tilde{\omega}^{(1)}, ..., \tilde{\omega}^{(\ell)}) \) finite with probability one and such that for all \( N > N_{\tilde{\omega}} \),

\[
M_{N,\tilde{\omega}}(\omega) < \gamma r_N,
\]

where \( M_{N,\tilde{\omega}}(\omega) \) was defined in (2.15). Hence,

\[
\limsup_{N \to \infty} \frac{M_{N,\tilde{\omega}}}{\ln N} \leq \gamma \limsup_{N \to \infty} \frac{r_N}{\ln N} \text{ a.s.}
\]

Next, assume that

\[
E_{N,\tilde{\omega}} = \sum_{n=1}^N \prod_{i=1}^\ell P(C_n(\tilde{\omega}^{(i)})) \to \infty \text{ as } N \to \infty
\]

which by (5.32) implies also that

\[
\sum_{n=1}^\infty P(\cap_{i=1}^\ell T^{-q_i(n)}C_n(\tilde{\omega}^{(i)})) = \infty.
\]

Set

\[
L_{n,\tilde{\omega}}(\omega) = \max\{m \leq n : T^{q_i(n)}\omega \in C_m(\tilde{\omega}^{(i)}) \text{ for } i = 1, ..., \ell\}.
\]

It follows from Theorem 2.3 that under (6.6) for almost all \( \omega \in \Omega \),

\[
L_{n,\tilde{\omega}}(\omega) \to \infty \text{ as } n \to \infty.
\]

Observe also that

\[
S_N(\omega) = \sum_{n=1}^N \prod_{i=1}^\ell P(C_n(\tilde{\omega}^{(i)}) \circ T^{q_i(n)}(\omega)) = S_{L_{n,\tilde{\omega}}(\omega)}.
\]
By (4.13), (5.19) and (5.32) we can use (2.13) which yields that for almost all \( \omega \in \Omega \),

\[
0 \leq \mathcal{E}_{N, \omega} - \mathcal{E}_{L_n, \omega} \leq O(\mathcal{E}_{N, \omega}^{1/2} \ln^{2+\varepsilon} \mathcal{E}_{N, \omega}),
\]

and so for almost all \( \omega \),

\[
\lim_{N \to \infty} \frac{\mathcal{E}_{L_n, \omega}(\omega)}{\mathcal{E}_{N, \omega}} = 1.
\]

Next, observe that if \( m = L_n, \omega(\omega) \) then for each \( i = 1, \ldots, \ell \),

\[
d(T^{q_i}(m), \omega(i)) \leq e^{-\gamma r_m} \text{ i.e. } \Phi_{\omega(i)}(T^{q_i}(m), \omega) \geq \gamma r_m,
\]

and so \( M_{m, \omega}(\omega) \geq \gamma r_m \). It follows that

\[
\lim_{N \to \infty} M_{L_n, \omega}(\omega) = 1
\]

and so (6.12) holds.

Next, in order to complete the proof of Theorem 2.5, we will choose sequences \( r_n, n = 1, 2, \ldots \) for appropriate upper and lower bounds. For the upper bound we will take \( r_n = \left[1 + \frac{1+\delta}{2\ell h} \ln n \right] \) for some \( \delta > 0 \). Then by (6.1) for almost all \( \omega(1), \ldots, \omega(\ell) \in \Omega \),

\[
\ln \prod_{i=1}^{\ell} P(C_n(\omega(i))) \sim -(1+\delta) \ln n \text{ as } n \to \infty,
\]

and so the series (6.2) converges as needed. Substituting such \( r_n \)’s to (6.5) and letting \( \delta \to 0 \) we obtain

\[
\limsup_{N \to \infty} \frac{M_{N, \omega}(\omega)}{\ln N} \leq \frac{\gamma}{2\ell h} \text{ a.s.}
\]

Now we deal with the lower bound choosing \( r_n = \left[1 - \frac{4+3}{3\ell h} \ln n \right] \). Then by (6.1) for almost all \( \omega(1), \ldots, \omega(\ell) \in \Omega \) as \( n \to \infty \),

\[
\ln \prod_{i=1}^{\ell} P(C_n(\omega(i))) \sim -(1 - \delta) \ln n,
\]

and so the series (6.6) diverges as needed. For such \( r_n \)’s we have that

\[
\liminf_{N \to \infty} \frac{r_N}{\ln N} = \frac{1 - \delta}{2\ell h}
\]

and letting \( \delta \to 0 \) the proof of Theorem 2.5 will be completed by (6.12), (6.13) and (6.15) once we show that for almost all \( \omega \in \Omega \),

\[
\liminf_{N \to \infty} \ln \frac{L_N, \omega(\omega)}{\ln N} = 1.
\]

By (6.14) there exists a random variable \( n(\omega) < \infty \) a.s. such that if \( n \geq n(\omega) \) then

\[
n^{-(1 - \frac{4+3}{3\ell h})} \leq \prod_{i=1}^{\ell} P(C_n(\omega(i))) \leq n^{-(1 - \frac{1+\delta}{2\ell h})}.
\]
If $L_{N,\omega}(\omega) \geq n(\omega)$ then we obtain from \((6.9)\) and \((6.17)\) that
\[
\left(\frac{4}{36} (N^{\frac{4}{5}} - (L_{N,\omega}(\omega) + 1)^{\frac{4}{5}}) \right) \leq O\left((n(\omega) + \sum_{n=n(\omega)}^{N} n^{-(1-\frac{4}{5}\delta)} \right)
\]
\[
\leq O\left((n(\omega) + \sum_{n=n(\omega)}^{N} n^{-(1-\frac{4}{5}\delta)} \right)^{1/2} \ln^{\frac{1}{2}} \left( n(\omega) + \sum_{n=n(\omega)}^{N} n^{-(1-\frac{4}{5}\delta)} \right) \right).
\]

Dividing these inequalities by $N^{\frac{4}{5}}$, letting $N \to \infty$ and taking into account that $N \geq L_{N,\omega}(\omega)$ by the definition, we see that
\[
\frac{L_{N,\omega}(\omega)}{N} \to 1, \quad \text{and so } \ln N - \ln L_{N,\omega}(\omega) \to 0 \ a.s. \ as \ N \to \infty
\]

implying \((6.16)\) and completing the proof of Theorem 2.5. \qed

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