Appearance of Mobility Edge in Self-Dual Quasiperiodic Lattices

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Within the framework of the Aubry-André model, one kind of self-dual quasiperiodic lattice, it is known that a sharp transition occurs from all eigenstates being extended to all being localized. The common perception for this type of quasiperiodic lattice is that the self-duality excludes the appearance of the mobility edge separating localized from extended states. In this work, we propose a multi-chromatic quasiperiodic lattice model retaining the self-duality identical to the Aubry-André model, and demonstrate numerically the occurrence of the localization-delocalization transition with definite mobility edges. This contrasts with the Aubry-André model. As a result, the diffusion of wave packet exhibits a transition from ballistic to diffusive motion, and back to ballistic motion. We point out that experimental realizations of the predicted transition can be accessed with light waves in photonic lattices and matter waves in optical lattices.

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The concept of Anderson localization-delocalization (LD) transition has been progressively developed from its original scope of solid-state physics to a broad class of physical systems, including light waves in photonic lattices [1,2], matter waves in optical potentials [3,4], sound waves in elastic media [5], and quantum chaotic systems [6]. It is a quantum phase transition caused by disorder where waves experience from delocalized (“metallic” phase) to exponentially localized (“insulator” phase) states [7–10]. This concept predicts a wealth of interesting phenomena. In usual one-dimensional (1D) disordered systems, all eigenfunctions are localized, so there is no LD transition [11]. While in three-dimensional (3D) disordered systems there should exist a transition between delocalized and localized states at a well-defined critical energy called the mobility edge $E_c$ [8].

The absence of the LD transition in usual 1D disordered systems makes itself trivial from the perspective of the physics of disorder-induced localization transition. However, Aubry and André [12] have proposed self-dual quasiperiodic (QP) lattices, the so-called Aubry-André (AA) model (also known as the Harper model [13]), showing localization transition where the QP potential of finite strength mimics the deterministic disorder. One of the key features of the AA model is either all states being extended or localized depending on the modulation strength of the potential [13]. This localization transition in the modulation strength space arises from the self-duality of the AA model. As a result, either ballistic or localized excitations are observed in the AA model [15]. Involved physics in the AA model has been extensively investigated, e.g., metal-insulator transition [16–20], Hofstadter’s butterfly [21], and topological phase transitions [22], just to mention a few. Experimentally, the LD transition of the AA model has been studied in the context of matter waves in bichromatic optical lattices [4] and light waves in fabricated photonic lattices [13]. Both approaches have clearly demonstrated the existence of the localization transition.

The localization transition in the AA model is unique because all states are either localized or extended without energy-dependent mobility edges. However, little attention has been paid to whether or not the mobility edge can exist in QP lattices, especially under the situation of self-duality being retained [23]. Since it has been generally believed that the self-duality keeps all the states being localized or extended, one would expect that the delocalized and localized states do not coexist at the same modulation strength.

In this work, we demonstrate that the above view is not generally true by proposing a new type of self-dual QP model. In this model, the LD transition becomes anomalous, and definite mobility edges can emerge despite the restriction of self-duality. We shall consider the photonic lattices, artificially fabricated arrays of evanescently coupled waveguides, to investigate the transition. The great advantage of photonic lattices lies in the easy fabrication of complex refractive-index landscapes, and in the direct observation of wave function itself during the transport [24].

The AA model is given by the following eigenvalue equation, $E a_n = \lambda \cos(2\pi a_n) a_n + c (a_{n+1} + a_{n-1})$, where $a_n$ is the amplitude of wave function at the lattice site $n$, and $\lambda$ and $c$ denote the strength of the deterministic disorder and the site-to-site hopping energy, respectively. The irrational number $\alpha$ indicates the ratio between the period of the modulation and the underlying periodic lattice. These types of QP lattices possess unique character being intermediate between full periodicity and full disorder. One premier feature of the AA model is its self-duality, which means that the Fourier transformation of the above equation, $E f_k = 2c \cos(2\pi \alpha k) f_k + \frac{\lambda}{2} (f_{k+1} + f_{k-1})$, definitely takes the same form as the original one with the roles of $c$ and $\lambda/2$ being interchanged, transforming localized states into extended states and vice versa. As a result, there exists a critical strength $\lambda/c = 2$ identified as the transition point to separate the localization and delocalization phases. Therefore, the AA model exhibits a transition in parameter space: for $\lambda < 2c$ all the states are extended, while the situation $\lambda > 2c$ makes all of the states exponentially localized. However, this type of self-dual QP system has no emergent energy-dependent mobility edges.

To characterize the localization natures in self-dual QP lat-
FIG. 1: (Color online) (a, b): Amplitude profiles of the eigenstates’ IPR of lattices versus the strength of modulation $\lambda$ (vertical axis). Panel (a) corresponds to the original AA lattice with $\alpha = (\sqrt{5} - 1)/2$ (inverse of golden mean), while panel (b) to the generalized model Eq. (1) with $\alpha = (\sqrt{5} - 1)/2$ and $\sigma = c_2 = 1/3$. We denote more extended states by darker shading while more localized states by lighter shading (see the color bar). (c): Close-up view of the circled area. (d, e, f): Plots of IPR of individual states for different $\lambda$’s in Eq. (1), namely, $\lambda = 0.5, 1.0, 3.0$, to show the mobility edges and anomalous LD transition. All the above calculations are done for a system with $N = 600$ sites, and the mode number is arranged in descending order.

Delocalization

Localization

Pure Delocalization

Coexistence phase I

Coexistence phase II

Delocalization

Localization

Pure Delocalization

Coexistence phase I

Coexistence phase II

As observed from Fig. 1 (e), in the coexistence phase I the extended states in high energy band coexist with the localized states in low energy band. There is a transition from delocalization to localization with decreasing energy. Instead, in the coexistence phase II, the system reverses the transition follow-

$$E_{f_k} = \left[2c_1 \cos(2\pi \alpha k) + 2c_2 \cos(4\pi \alpha k)\right]f_k + \frac{\lambda}{2}(f_{k+1} + f_{k-1}) + \frac{\lambda \sigma}{2}(f_{k+2} + f_{k-2}).$$

(2)

Obviously, the model manifests self-duality under the condition of $\sigma = c_2/c_1$. From the experience of the AA model, one would expect that such a self-dual model undergoes a LD transition at $\lambda = 2c_1$, and the extended and localized states do not coexist. Surprisingly, this is not the case as shown below.

To highlight the new physics involved in the multi-chromatic QP lattices with second-neighbor hopping, we study the light excited in photonic lattices with $\alpha = (\sqrt{5} - 1)/2$. The unit of energy is scaled by the nearest-neighbor hopping constant $c_1$. Figure(1)(b) shows the IPR values of all eigenstates as a function of the potential strength $\lambda$. The phase diagram in Fig. 1(b) provides three parts separated by $\lambda_1$ and $\lambda_2$, i.e., pure delocalization phase, coexistence phase I, and coexistence phase II (29). When the modulation strength $\lambda$ is sufficiently small ($< \lambda_1$), IPR values are approximately vanishing for all states, indicating that all states are extended. As $\lambda$ is increased further, crossovers occur from purely delocalized to coexistence phases (I and further to II). It is remarkable that the states are not simultaneously localized or extended, but depend on their eigenenergies in contrast with the original AA model. The two coexistence phases I and II are separated by the critical value $\lambda_2 = 2$ as seen from Fig. 1(b). Note that in the coexistence phase I two mobility edges come to appear firstly (see the close-up view Fig. 1(c)), followed by a single mobility edge until $\lambda_2$.

Figures (1)(d-f) show the IPR values of states at various $\lambda$’s. As observed from Fig. 1(e), in the coexistence phase I the extended states in high energy band coexist with the localized states in low energy band. There is a transition from delocalization to localization with decreasing energy. Instead, in the coexistence phase II, the system reverses the transition follow-

$$E_{f_n} = \lambda [\cos(2\pi \alpha n) + \sigma \cos(4\pi \alpha n)]a_n + c_1(a_{n+1} + a_{n-1}) + c_2(a_{n+2} + a_{n-2}).$$

(1)

The Fourier transformation of Eq. (1) provides the following equation:

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showing the exponentially decaying tails

\[ \log |\psi_n| \propto \exp(-\gamma x). \]

The parameters are the same as those in Fig. 1(b).

To shed light on the dependence of localization properties on \( c_2 \) and \( \lambda \) for the QP models with mobility edges in Refs. 18, 20, 25 because of the broken self-duality.

Figures 2 summarizes the typical eigenstates at various \( \lambda \)'s in order to confirm that the distinction of IPR does guarantee various phases. In the purely delocalized regime, all states extend over the entire system [Fig. 2(a)]. In the coexistence phase I, the states are extended at higher energies [Fig. 2(b)], while exponentially localized at lower energies [Fig. 2(c)], which is the fingerprint of Anderson localization. However, in the regime \( \lambda > \lambda_c \) the spatial behaviors of the states are just the opposite of coexistence phase I [Fig. 2(e, f)]. At the critical strength \( \lambda = 2 \), the states become spatially fragmented [Fig. 2(d)], which are intermediate between spatially extended and exponentially localized. It should be emphasized that this fragmented feature holds for any states at \( \lambda = 2 \).

To shed light on the dependence of localization properties on the next-nearest-neighbor term, we plot in Fig. 3 the distribution of IPR of several individual states on the \( c_2 - \lambda \) plane. The developed intricate patterns illustrate the nonuniform dependence of localization on \( c_2 \). For example, at a fixed \( \lambda \) smaller than 2, when \( c_2 \) is increased the 500th state is transformed from delocalization to localization, but further growth of \( c_2 \) brings it into the extended regime again.

Diffusion of waves can provide the direct observables on the LD transition and the mobility edges. For light propagation in the photonic lattices with next-nearest-neighbor hopping, equations of motion read

\[ -i \frac{\partial a_n}{\partial z} = \beta_n a_n + c_1(a_{n+1} + a_{n-1}) + c_2(a_{n+2} + a_{n-2}), \]

where \( z \) is the propagation coordinate of light, \( \beta_n \) is the single-site propagation constant of the underlying periodic lattice defined by the multi-chromatic function. The equation is essentially identical to the quantum description of non-interacting ultracold atoms in the optical lattices, especially for shallow lattices, when replacing the time variable \( t \) by the propagation distance \( z \).

The anomalous LD transition and the existence of mobility edge yield a profound effect on the transport properties of the QP systems. We inject a spatially narrow light beam into a single waveguide of the lattice and monitor the time evolution to observe the signatures of transition. Examples of light intensity during the evolution are displayed in Fig. 4(a-d). It is observed that the wave functions grow in qualitatively different manners when \( \lambda \) is varied in different phases. In the regime of pure delocalization the excitation is expanding continuously and the light intensity around the input site is decaying gradually [Fig. 4(a)], similar to the intensity pattern in fully periodic arrays. Therefore, its width increases ballistically, as illustrated by the solid line in the right panel of Fig. 4. In contrast, for the systems belonging to the coexistence phase I and II [Fig. 4(b, d)], a twofold behavior of intensity distribution emerges. One sharp peak localized around the initial position always persists during the spreading, which indicates the existence of localized states. Meanwhile, the ballistic tails contributed by the extended states are superposed on the central peak. Because the fractions of localized states in the spectrum are different, the heights of central peaks are significantly different in phases I and II. For the case of critical strength \( \lambda = 2 \) the intensity structure becomes fragmented and is composed of many separated spikes [Fig. 4(c)].
Another quantity characterizing the transport of light is the mean square width of wave packet \( \Delta x^2(z) = \sum_i (i - \langle i \rangle)^2 |a_i|^2 \). The measure of localization transition can be obtained by plotting \( \langle \Delta x^2(z) \rangle \), where \( \langle ... \rangle \) denotes an average over different excited positions. The right panel in Fig. 4 shows how the mean square widths develop with the propagation distance \( z \) by varying \( \lambda \) in different phases. In agreement with the IPR picture obtained in Fig. 1 in the pure delocalization phase the wave packet exhibits a ballistic spreading (solid line). For the coexistence phase I and II, on the other hand, the widths never flatten off during the evolution even the existence of localized modes, and the asymptotic behaviors also display the ballistic motions. However, the light needs a longer transition period from transient expansion to asymptotical stability, as compared with the pure delocalization phase. Meanwhile, a new transport behavior is observed when \( \lambda = \lambda_{c2} \) (dot line): the mean square width of light is proportional to the propagation distance as the normal diffusion. Globally, increasing the level of modulation leads to the transition from ballistic motion to normal diffusion, and back to ballistic motion. This clearly indicates there exists the LD transition qualitatively different from that of the AA model [15].

Besides the asymptotic behavior of the spreading, we have analyzed the transient process of light waves, especially in the coexistence phases I and II. In both regimes, the quantity \( \langle \Delta x^2(z) \rangle \) approaches asymptotically to be ballistic as \( z \to \infty \). However, the two regimes are contrasted in the transient behaviors about intersection point depending on \( \lambda > \lambda_{c2} \) or \( \lambda < \lambda_{c2} \). In the coexistence phase I, the transient region is very narrow and the slopes of curves decline gradually with the increase of modulation (dashed and dot-dashed lines). On the other hand, we find a wide transient region when \( \lambda \) exceeds \( \lambda_{c2} \). After a comparatively slow evolution, the spreading curve of \( \lambda = 3.0 \) surpasses that of \( \lambda = 2.2 \) (solid lines marked with \( \square \) and \( \bigcirc \)). This leads to an intersection point of curves belonging to the coexistence phase II. This transient behavior is quite abnormal [15], and can be used to distinguish the coexistence phase I and II. Combining the intensity distribution, asymptotic behavior and transient process, all being able to be measured experimentally, constitutes a direct observation of the anomalous transition in the QP photonic lattices.

Herein are some comments. We have also performed the calculations for other incommensurate ratios, such as silver mean and bronze mean, and observed qualitatively the same LD transition. But then we should emphasize that in our QP system the nature of the LD transition with the energy-dependent mobility edges is substantially different from the usual 3D Anderson localization problem. In the 3D disordered systems the phase transition is smooth through the mobility edge \( E_c \), so the states near \( E_c \) are critical. However, in our results the states near the mobility edges present a sudden change from localization to delocalization, not being critical like the 3D Anderson model.

The advances in photonic lattices allow for the realization of QP potentials and the direct observation of light propagation as well [29]. In fact, the QP photonic lattices have been fabricated for demonstrating the transition associated with the AA model [19]. The next-nearest-neighbor interaction can be realized by constructing the zigzag structure of arrays allowing the precise tuning of the hopping [26, 27]. Besides light waves, ultracold atoms loaded into optical lattices with high degree of control have provided another experimental platform to observe the localization-related phenomena [30, 31], where the bichromatic optical lattices were designed to implement the AA model [4]. The QP potential in Eq. (1) can be produced by the three-incommensurate-frequencies generalization of conventional bichromatic lattices: superimposing three optical standing waves with different wavelengths. Illuminating another weaker laser with the wavelengths being doubled into the main lattice, the beams interfere to create a 1D multi-chromatic lattice with a certain amount of incommensuration. The degree of incommensuration in the main lattices can be adjusted by tuning the intensity of the auxiliary lasers. To enter the regime beyond the nearest-neighbor hopping, the optical potential formed by the main lattice should be relatively shallow [32].

In conclusion, we have investigated the self-dual multichromatic quasiperiodic lattices with long-range hopping. Our model definitely realizes the coexistence of extended and localized states, and the LD transition becomes energy-dependent. We have demonstrated that the self-duality of quasiperiodic lattice does not necessarily mean pure spectrum of extended or localized states only. Instead, extended and localized states can coexist at a fixed configuration even if the self-duality persists. AA model is just a special case of the self-dual quasiperiodic models. As a further step, an in-
teresting extension would be to consider the nonlinear inter-
actions, which may lead to anomalous diffusion of quantum
waves particularly in the coexistence phase [33].

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