On Pentagon And Tetrahedron Equations

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Abstract

We show that solutions of Pentagon equations lead to solutions of the Tetrahedron equation. The result is obtained in the spectral parameter dependent case.
1. Introduction

Yang-Baxter (or triangle, or 2-simplex) equations [1, 2], play a central role in the theory of two-dimensional Integrable Systems of Field Theory and Statistical Mechanics (see for reviews [3, 4]). They also lead to the theory of Quantum Groups [4, 5, 6, 7, 8, 9] and have important applications in low dimensional Topology. In 1980, A. B. Zamolodchikov [10, 11] described a generalization of this equation, the Tetrahedron (or 3-simplex) equation, for three-dimensional Integrable Systems. This equation can be further extended to an arbitrary dimension \( d \) and is called the \( d \)-simplex equation [12]. More recently, the first solution of the Tetrahedron equation, proposed in [11] (see [13] for the proof), has been generalized using results from the two-dimensional Chiral Potts models [14, 15].

The purpose of this letter is to give a construction of unitary solutions of the Tetrahedron equation (depending on spectral parameters) in terms of solutions of Pentagon equations. Our starting point is the geometrical interpretation of these equations given in [16]. It is argued in [16] that the \( d \)-simplex equation can be obtained as a special discretized case of a (generalized) zero holonomy equation for transport operators acting in a space of functionals of \((d-1)\)-dimensional manifolds. In this picture, an \( R \)-matrix \( R_d \) solving the \( d \)-simplex equation is associated to a \( d \)-dimensional parallelepipedic cell, and is interpreted as an operator moving a functional of \( d \) faces to a functional of the \( d \) other faces. The condition for parallel transport (zero holonomy) is then precisely the \( d \)-simplex equation. For \( d = 1 \) it gives Lax type equations, for \( d = 2 \) the Quantum Yang-Baxter equation, for \( d = 3 \) the Tetrahedron equation, etc.

In short, this equation can be described as follows. Let \( \Sigma_d \) and \( \Sigma'_d \) be two oriented \( d \)-dimensional manifolds having the same compact oriented boundary which is divided into two \((d-1)\)-dimensional oriented manifolds \( \Sigma_{d-1} \) (\( \Sigma^* \) meaning the same manifold as \( \Sigma \) but with reversed orientation) and \( \Sigma'_{d-1} \) having also the same boundary. Then we associate to \( \Sigma_{d-1} \) and \( \Sigma'_{d-1} \) respectively two vector spaces \( V_{\Sigma_{d-1}} \) and \( V_{\Sigma'_{d-1}} \). Let \( F(\Sigma_d) \) be a map,

\[
F(\Sigma_d) : V_{\Sigma_{d-1}} \rightarrow V_{\Sigma'_{d-1}}
\]  

(1)
Then $F(\Sigma_d)$ can be interpreted as a transport operator (depending on the manifold $\Sigma_d$) acting on functionals of $(d-1)$-dimensional manifolds. The condition for parallel transport is just,

$$F(\Sigma_d) = F(\Sigma'_d)$$

for any two manifolds $\Sigma_d$ and $\Sigma'_d$ satisfying the above conditions, in particular, $\partial \Sigma_d = \partial \Sigma'_d$.

However as noticed in [16], it is also possible to give another discrete version of such a (generalized) zero holonomy equation in terms of operators $\Phi_d$ attached to $d$-simplices instead of $d$-cells. Equations of this type for $\Phi_d$’s are called the Fundamental $(d+1)$-Simplex Relations (due to the fact that they are written around a $(d+1)$-simplex, each face of it being a $d$-simplex associated to one $\Phi_d$ and that they realize eq. (2) in the minimal (simplicial) way.). Moreover, the operator $R_d$ attached to any $d$-cell can be obtained as an (ordered) product of the $d!$ operators $\Phi_d$ attached to $d!$ $d$-simplices in which this $d$-cell can be decomposed. Given such a formula, the $d$-simplex equation for $R_d$ is a consequence of the fundamental $(d+1)$-simplex relations for the $\Phi_d$’s.

For $d = 2$ this procedure gives the decomposition of a quantum $R$-matrix in terms of $F$ type objects satisfying quadratic equations (a 3-simplex possesses four faces, one $F$ being attached to each of them). In that case, from the algebraic point of view, this procedure gives the geometrical interpretation of the construction, used by V. G. Drinfel’d in [17], of solutions of the Quantum Yang-Baxter equation.

In [17], unitary solutions $R_{12}$, namely $R_{12}(u, v) R_{21}(v, u) = 1$, of the Quantum Yang-Baxter equation,

$$R_{12}(u, v) R_{13}(u, w) R_{23}(v, w) = R_{23}(v, w) R_{13}(u, w) R_{12}(u, v)$$

are obtained in terms of a more fundamental object $F_{12}$ such that,

$$R_{12}(u, v) = F_{21}^{-1}(v, u) F_{12}(u, v)$$

($(u, v)$ being two vectors spectral parameters), $R_{12} \in A \otimes A$, $A$ being a Hopf algebra with co-commutative co-product $\Delta_0$, $F_{12} \in A \otimes A$ and the $F$ objects
satisfy the quadratic (3-simplex) relation,

\[(\Delta_0 \otimes 1) F_{12} = (1 \otimes \Delta_0) F_{23}\]  \hspace{1cm} (5)

The geometrical setting for this construction and its generalizations to non-unitary cases is given in [13]. It is used in [19] to construct from any given classical \(r\)-matrix \(r \in \mathcal{G} \otimes \mathcal{G}\) the corresponding universal quantum \(R\)-matrix as a functional of \(r\), together with the quantized Hopf (quasi-triangular) algebra \(A\), \(R \in A \otimes A\).

Then, for \(d = 3\), the \(R\)-matrix \(R(u, v, w)\) is interpreted as an operator associated to a three-dimensional parallelepipedic cell (depending on three vectors \((u, v, w)\)), and acting in a space of functionals of surfaces. The condition for parallel transport is the Tetrahedron equation. A three-dimensional parallelepipedic cell can be decomposed into six tetrahedrons (and one "hat", see below). We associate one \(\Phi\) to each tetrahedron (and one \(\Gamma\) to the "hat"), such that the \(R\)-matrix decomposes as a product of six \(\Phi\)'s and one \(\Gamma\). Then the 4-simplex relation for \(\Phi\) is in fact a Pentagon equation. The zero holonomy requirement imposes also some consistency relations between \(\Phi\) and \(\Gamma\). These relations for \(\Phi\) and \(\Gamma\) imply that the \(R\)-matrix \(R(u, v, w)\) satisfies the Tetrahedron equation. This will be our main result.

All equations will be given here for vertex models, namely for indices attached to surfaces (plaquettes or triangles). A completely similar description exists for variables on links or on points (or for all these possibilities together) and will be described elsewhere as well as more details on proofs and examples.

This letter is organized as follows. In section 2, we define the objects \(R\), \(\Phi\), and \(\Gamma\) in the three-dimensional case and give their geometrical meaning together with the decomposition of \(R\) in terms of \(\Phi\) and \(\Gamma\). We also give the Tetrahedron equation for \(R\). In section 3, we describe Pentagon equations in this geometrical framework. Our main result is stated in section 4. There we give the skeleton of the proof of the relation between solutions of Pentagon and Tetrahedron equations. Perspectives and conclusions are given in section 5.
It is a great pleasure to dedicate this paper to L. D. Faddeev on the occasion of his 60th birthday.

2. The Tetrahedron equation

For vertex models, the Tetrahedron equation can be written as follows

\[ R_{123}(u, v, w) R_{145}(u, v, t) R_{246}(u, w, t) R_{356}(v, w, t) = R_{356}(v, w, t) R_{246}(u, w, t) R_{145}(u, v, t) R_{123}(u, v, w) \tag{6} \]

where, \( R_{ijk} \in \text{End}(V_i \otimes V_j \otimes V_k) \), \( V_i \) being vector spaces of dimensions \( N_i \) and \( u, v, w, t \) are four arbitrary vectors (say elements of \( \mathbb{C}^n \)) parametrizing the \( R \)-matrices.

As sketched in the Introduction, such \( R \)-matrices can be interpreted as transport operators on a space of functionals of surfaces. So let us first describe this functional space in a discretized case.

We consider an \( n \)-dimensional affine space on \( \mathbb{C} \), with origin \( O \). We denote by \( \Delta(x)(u, v) \) (or equivalently \( \Delta^{(x+u)}(-u-v) \) or \( \Delta^{(x+u+v)}(-u-v,u) \)), \( (u, v, x) \) being vectors in \( \mathbb{C}^n \), the oriented triangle defined by the point \( (O + x) \) and its oriented boundary \( (u, v, -u-v) \). To such a triangle we associate a vector (which is a functional of this triangle) \( h(x)(u, v) \in V^x_u \otimes V^{(x+u)}_v \otimes V^{(x+u+v)}_{-u-v} \otimes A^{(x)}_{(u,v)} \), where \( V^x_u \) is a vector space attached to the oriented link starting at point \( (O + x) \) in direction \( u \), such that its dual vector space \( V^x_u^* \) is equal to \( V^{(x+u)}_v \), and \( A^{(x)}_{(u,v)} \) is a vector space attached to the triangle \( \Delta(x)(u, v) \). Here also, the dual vector space to \( A^{(x)}_{(u,v)} \) is \( A^{(x)}_{(u+v,-v)} \) associated to the same triangle but with reversed orientation. Note also that we have, \( A^{(x)}_{(u,v)} \equiv A^{(x+u)}_{(v,-u-v)} \equiv A^{(x+u+v)}_{(-u-v,u)} \).

We define a composition law for two \( h \)-functionals whenever the two corresponding triangles have (at least) one edge in common with opposite orientation by the evaluation of one \( h \) on the other using the duality bracket on the vector spaces attached to the common edge which are dual to one another. For example, to any two-dimensional parallelepiped \( \square(x)(u, v) \) starting
at point \((O + x)\) with oriented boundary \((u, v, -u, -v)\) we associate a functional

\[
l^{(x)}(u, v) = < h^{(x)}(u, v), h^{(x)}(u + v, -u) >_{V_{u+v}}^{(x)}
\]

(7)

where we have used the natural duality bracket between \(V_{u+v}^{(x)}\) and its dual vector space denoted by \(< , , >_{V_{u+v}}^{(x)}\). There, \(l^{(x)}(u, v)\) is an element of the tensor product, \(V_{u}^{(x)} \otimes V_{v}^{(x+u)} \otimes V_{-u}^{(x+u+v)} \otimes V_{-v}^{(x+v)} \otimes \mathcal{A}_{[u,v]}^{(x)}\), where \(\mathcal{A}_{[u,v]}^{(x)}\) stands for \(\mathcal{A}_{u,v}^{(x)} \otimes \mathcal{A}_{u+v,-u}^{(x)}\) and we will require for simplicity \(\mathcal{A}_{[u,v]}^{(x)}\) not to depend on the vector \(x\). Then it is also possible to define the composition law for \(l\)-functional using their decomposition in terms of the \(h\)'s.

As a useful example we consider the functionals

\[
j^{(x)}(u, v, w) = l^{(x)}(v, u), l^{(x+v)}(w, u), l^{(x)}(w, v) >_{V_{w}^{(x+v)}} \otimes V_{v}^{(x+v)} \otimes V_{u}^{(x)}\]

and

\[
k^{(x)}(u, v, w) = l^{(x+u)}(w, v), l^{(x)}(w, u), l^{(x+w)}(v, u) >_{V_{u}^{(x+w)}} \otimes V_{v}^{(x+w+u)} \otimes V_{w}^{(x+u)}\]

Then, we define the operator \(R^{(x)}(u, v, w) \in \text{End}(\mathcal{A}_{[v,u]}^{(x)} \otimes \mathcal{A}_{[w,u]}^{(x)} \otimes \mathcal{A}_{[w,v]}^{(x)})\) as the map,

\[
R^{(x)}(u, v, w) : j^{(x)}(u, v, w) \mapsto k^{(x)}(u, v, w)
\]

(8)

Here \(R^{(x)}(u, v, w)\) is a functional of the parallelepipedic three-dimensional cell at point \((O + x)\) defined by the three vectors \((u, v, w)\).

We further impose a unitarity condition on this operator, namely, that the map \(R^{(x+u+v+w)}(-u, -v, -w)\) is the inverse map to \(R^{(x)}(u, v, w)\). Then, by considering the two (minimal) ways of mapping the functional,

\[
< l^{(x)}(v, u), l^{(x+v)}(w, u), l^{(x)}(w, v), l^{(x+w+v)}(t, u), l^{(x+w)}(t, v), l^{(x)}(t, w) >
\]

(9)

where the duality bracket evaluation is on,

\[
V_{v}^{(x+v)} \otimes V_{v}^{(x)} \otimes V_{v}^{(x+v)} \otimes V_{v}^{(x+v+w)} \otimes V_{t}^{(x+w)} \otimes V_{w}^{(x+w)} \otimes V_{w}^{(x+v+w)}
\]

\[
\otimes V_{t}^{(x+w)} \otimes V_{v}^{(x)} \otimes V_{v}^{(x+v)} \otimes V_{u}^{(x+v+w)}
\]

(10)

to the functional,

\[
< l^{(x+w+t)}(v, u), l^{(x+t)}(w, u), l^{(x+u+t)}(w, v), l^{(x)}(t, u), l^{(x+u)}(t, v), l^{(x+u+v)}(t, w) >
\]

(11)
where the duality bracket evaluation is on,
\begin{align}
V^{(x+w+t)}_u & \otimes V^{(x+u+w+t)}_v \otimes V^{(x+u+t)}_w \otimes V^{(x+u)}_t \\
& \otimes V^{(x+u+v)}_t \otimes V^{(x+u+v+t)}_w \otimes V^{(x+u+t)}_v \otimes V^{(x+t)}_u
\end{align}
we obtain the following parallel transport condition on the $R$-matrices,
\begin{align}
R^{(x+t)}(u,v,w) R^{(x)}(u,v,t) R^{(x+v)}(u,w,t) R^{(x)}(v,w,t) &= \\
= R^{(x+u)}(v,w,t) R^{(x)}(u,w,t) R^{(x+w)}(u,v,t) R^{(x)}(u,v,w)
\end{align}
If we consider the simplified case where the operator $R^{(x)}(u,v,w)$ do not depend on the shift $(x)$, we obtain the Tetrahedron equation \((1)\), the convention being that the vector spaces $A_{[u,v]}$ are label by numbers $1, 2, 3, 4, 5, 6$ or better here $(1'), (22')$, ... with the correspondence, $[v, u] \equiv (1') ((v, u) \equiv (1)$ and $(u, v) \equiv (1')$) and so on, $[w, u] \equiv 22', [w, v] \equiv 33', [t, u] \equiv 44', [t, v] \equiv 55'$ and $[t, w] \equiv 66'$. Then we have,
\begin{align}
R^{(x)}_{11',22',33'}(u,v,w) &= R^{(x)}(u,v,w)
\end{align}
The (local) unitarity condition is now,
\begin{align}
R^{(x)}_{11',22',33'}(u,v,w) R^{(x+u+v+w)}_{11',22',33'}(-u,-v,-w) &= 1
\end{align}
Note here the exchange of spaces $(i)$ and $(i')$. The (local) Tetrahedron equation is given by,
\begin{align}
R^{(x+t)}_{11',22',33'}(u,v,w) R^{(x)}_{11',44',55'}(u,v,t) R^{(x+v)}_{22',44',66'}(u,w,t) R^{(x)}_{33',55',66'}(v,w,t) &= \\
= R^{(x+u)}_{33',55',66'}(v,w,t) R^{(x)}_{22',44',66'}(u,w,t) R^{(x+w)}_{11',44',55'}(u,v,t) R^{(x)}_{11',22',33'}(u,v,w)
\end{align}
for any set of vectors $(u, v, w, t, x)$.

Let us now define two other transport operators $\Phi$ and $\Gamma$ as the mappings,
\begin{align}
\Phi^{(x)}(u,v,w) : & <h^{(x)}(u,v), h^{(x)}(u+v,w)>_{V^{(x)}} \\
\Gamma^{(x)}(u,v,w) : & <h^{(x)}(u,v+w), h^{(x+u)}(v,w)>_{V^{(x+u)}}
\end{align}
and,

\[ \Gamma^{(x)}(u, v, w) : < h^{(x)}(u, v), h^{(x)}(u + v, -v) >_{V_u^{(x)}} \otimes_{V_v^{(x) + u}} \]

\[ \longmapsto < h^{(x)}(u + v + w, -w), h^{(x)}(u + v, w) >_{V_{u+v+w}^{(x) + u + v} \otimes_{V_w^{(x) + v + w}}} \]  
(17)

where, \( \Phi^{(x)}(u, v, w) \) is a linear map from \( \mathcal{A}_{(u,v)}^{(x)} \otimes \mathcal{A}_{(u+v,w)}^{(x)} \) to \( \mathcal{A}_{(u,v+w)}^{(x)} \otimes \mathcal{A}_{(v,w)}^{(x) + u} \).

Similarly, \( \Gamma^{(x)}(u,v,w) \) is a map from \( \mathcal{A}_{(u,v)}^{(x)} \otimes \mathcal{A}_{(-v,-u)}^{(x) + u + v} \) to \( \mathcal{A}_{(u+v+w,-w)}^{(x)} \otimes \mathcal{A}_{(u,v+w)}^{(x) + u + v} \). Moreover, for simplicity, we will make the identifications (in the above formula for \( \Phi \) and \( \Gamma \)), \( \mathcal{A}_{(u,v)}^{(x)} \equiv \mathcal{A}_{(u,v+w)}^{(x)} \), \( \mathcal{A}_{(u+v,w)}^{(x)} \equiv \mathcal{A}_{(v,w)}^{(x) + u} \) for \( \Phi \) and similarly for \( \Gamma \), \( \mathcal{A}_{(u,v)}^{(x)} \equiv \mathcal{A}_{(u+v,w)}^{(x)} \) and \( \mathcal{A}_{(-v,-u)}^{(x)} \equiv \mathcal{A}_{(u+v+w,-w)}^{(x)} \) such that \( \Gamma \) contains a permutation operator in its definition.

Using these operators it is quite easy to decompose the action of the \( R \)-matrix in terms of \( \Phi \) and \( \Gamma \).

For this purpose, we put indices on \( \Phi \) and \( \Gamma \) to make explicit the vector spaces they are acting upon, namely, using the above conventions and identifications of vector spaces, we obtain for example, \( \Phi^{(x) + u + w}(u, -u - w, -v) \equiv \Phi^{(x)}_{23}(u, -u - w, -v) \), and so on.

We have,

\[
R_{11',22',33'}^{(x)}(u,v,w) = P_{12} P_{13'} P_{2'3'} P_{13} \Phi_{31}^{(x)}(w,u+v,-v) \\
\Phi_{32}^{(x)}(u+w,v,-v-w) \Phi_{32}^{(x)}(w,v,u) P_{12'} P_{2'3'}^{(x)}(u+v+w,-w,-v) \\
\Gamma_{13'}^{(x)}(v,u+w,-v) P_{12'} \Phi_{12}^{(x) + u + v}(-u-v,u+w) \Phi_{23'}^{(x) + v + w}(u,-u-w,-v)
\]
(18)

Note that in this formula each \( \Phi \) is associated to one of the six tetrahedrons decomposing the three-dimensional cell corresponding to the \( R \)-matrix and having always the two points \((O + x)\) and \((O + x + u + v + w)\) among their four vertices. These six tetrahedrons are labelled by the six possible ordered triplets \((a, b, c), a, b, c \in \{u, v, w\}\). Note also that the role of \( \Gamma \) is to create the only vertex of the three-dimensional cell \((O + x, u, v, w)\), namely the point \((O + x + u + w)\), not present in the initial surface.
3. The Pentagon Equation

We are now interested in writing the general equation (1) for the operators Φ and Γ.

Let us first note the useful symmetry relations,

\[ \Phi_{ij}(x)_{uvw} = \Phi_{ji}(x+u+v)_{-u-v-w,u} \] (19)

and for Γ,

\[ \Gamma_{ij}(x)_{uvw} = \Gamma_{ji}^{(x+u+v)}(-v,-u,u+v+w) \] (20)

Then we impose the unitarity relation on Φ,

\[ \Phi_{ij}^{(x+u)}(v,w,-u-v-w) P_{ij} \Phi_{ij}^{(x)}(u,v,w) = 1 \] (21)

and on Γ,

\[ \Gamma_{ij}^{(x)}(u,v,w) \Gamma_{ji}^{(x)}(u+v+w,-w,-v) = 1 \] (22)

We also ask for the following composition law,

\[ \Gamma_{ij}^{(x+u+v)}(w,-u-v-w,t) \Gamma_{ij}^{(x)}(u,v,w) = P_{ij} \Gamma_{ij}^{(x)}(u,v,t-u-v) \] (23)

It means in particular that \( \Gamma_{ij}^{(x)}(u,v,-v) = P_{ij} \).

To obtain the 4-simplex fundamental relation on Φ, we consider the two minimal ways of mapping the functional,

\[ < h^{(x)}(u,v), h^{(x)}(u+v,w), h^{(x)}(u+v+w,t) >_{V_{u+v+w}} \]

to the functional,

\[ < h^{(x)}(u,v+w+t), h^{(x+u)}(v,w+t), h^{(x+u+v)}(w,t) >_{V_{u+v+w+t}} \]

This gives the following Pentagon equation on Φ,

\[ \Phi_{12}^{(x)}(u,v,w+t) \Phi_{23}^{(x)}(u+v,w,t) = \Phi_{23}^{(x+u)}(v,w,t) \Phi_{13}^{(x)}(u,v+w,t) \Phi_{12}^{(x)}(u,v,w) \] (24)
Then using again discretized versions of eq. (1), we obtain the following constraints between $\Phi$ and $\Gamma$,

\[
\begin{align*}
\Phi^{(x+u+v+w)}_{12}(t, -u - v - w - t, u + v) &= \Phi^{(x+u+v)}_{23}(w, -v - w, -u) = \\
&= \Phi^{(x+u+v)}_{23}(w + t, -t, -u - v - w) \Gamma^{(x+u+v)}_{12}(-v, v + w, t) \\
&\quad \Phi^{(x+u)}_{13}(v + w, -u - v - w, u + v) = \\
&= \Phi^{(x+u+v)}_{13}(w, -u - v - w, u + v + w + t) \Gamma^{(x)}_{23}(u, v, w + t) \\
&\quad \Phi^{(x+u+v+w)}_{12}(-u - v - w, u, v)
\end{align*}
\]

and similarly,

\[
\begin{align*}
P_{14} \Gamma^{(x)}_{12}(u, v, w) \Gamma^{(x)}_{34}(u + v + w, -w, -v) &= \Phi^{(x+u)}_{23}(-u, u + v + w, -w) \\
&\quad \Phi^{(x+u)}_{24}(-u, u + v, w) \Phi^{(x)}_{13}(u, v + w, -w) \Phi^{(x)}_{14}(u, v, w)
\end{align*}
\]

This last relation is quite interesting since it allows us to compute the operator $\Gamma$ in terms of $\Phi$'s up to its trace.

4. Tetrahedron equation from Pentagon Equation

We can now state our main result using all the above ingredients,

**Theorem :**

Let $\Phi_{ij}(u, v, w)$ and $\Gamma_{ij}(u, v, w)$ be defined by eqs. (16,17) and satisfying eqs. (19,20, 21, 22,23,24,25,26). Then the $R$-matrix $R^{(x)}_{11',22',33'}(u, v, w)$ defined in terms of $\Phi$ and $\Gamma$ in eq. (18) satisfies the Tetrahedron equation (15) and is unitary.

The proof of this theorem is quite long and will not be given here. In particular it uses 24-times the Pentagon equation for $\Phi$. So, instead of giving
an explicit proof, let us describe the main idea leading to its construction. In fact this idea holds for any of the $d$-simplex equation with regards to their relation to the corresponding fundamental $(d + 1)$-simplex relations when writing the $R$-matrix $R_d$ in terms of the simplicial objects $\Phi_d$. Hence for simplicity let us explain it first in the two-dimensional case, namely for the Yang-Baxter equation.

Geometrically, the $(l.h.s)$ of the quantum Yang-Baxter equation (3) is associated to three faces of a cube, namely to the surface corresponding to the functional $k^{(x)}(u,v,w)$, while the $(r.h.s)$ is associated to the three other faces of the cube, hence to the surface corresponding to the functional $j^{(x)}(u,v,w)$. So the $(l.h.s)$ and the $(r.h.s)$ of the Quantum Yang-Baxter equation can be viewed as related by the (symbolic) action of $R(u,v,w)$. Then in a similar way, the operator $\Phi$ can be consider in the two-dimensional case as the symbolic action relating the $(l.h.s)$ and the $(r.h.s)$ of eq. (5) for $F$, $\Gamma$ being related to unitarity relation for $F$. Now the proof of the Quantum Yang-Baxter equation (3) from eqs. (4,5) is precisely given by the decomposition of $R(u,v,w)$ in terms of $\Phi$ and $\Gamma$. Namely, to each $\Phi$ corresponds the use of eq. (5) for two definite $F$’s, and to $\Gamma$ corresponds the use of the unitarity for $F$. Indeed, using eq. (5) six-times, in the precise order given by the (non-abelian) decomposition (18), and once the unitarity for $F$, we can achieve the proof of eq. (3).

Generalizing this procedure to the $d = 3$ case amounts to decompose an $R$-matrix attached to a four-dimensional cell into its 24 4-simplices. This gives the precise way to use 24-times the Pentagon equation to prove the Tetrahedron equation for $R(u,v,w)$. In fact to achieve the proof of the theorem we also need the compatibility conditions (25,26). A more detailed account of this proof will be given elsewhere.

At this point two remarks are in order. First, It was noticed long ago [21], that any solution of the quantum Yang-Baxter equation (3) leads to solutions of the Tetrahedron equation (we consider here the case with no dependence on shifts $(x)$) (13) as,

$$R_{11',22',33'}(u,v,w) = R_{12}(v,w) R_{1'3}(u,w) R_{2'3}(u,v)$$

(27)

However such solutions are degenerate in the sense that the partition function
of such a model will decompose into the product of three partition functions of two-dimensional models associated to the planes \((u, v), (u, w), (v, w)\). The solutions we propose here are not of this type since the existence of a non-trivial \(\Phi\) ensures precisely that the two-dimensional equations such as (3) and hence (8) are broken.

Second, the restricted Star-Triangle equations proposed in ref. [14] are likely to be very similar to our Pentagon equation. This point deserves further study.

The next step is of course to find solutions \(\Phi\) and \(\Gamma\). In fact, if we consider eqs. (19, 20, 21, 22, 23, 24, 25, 26) for functionals having vector indices on links (and eventually on surfaces), solutions are already at hand. Their are given by Conformal Field Theories or by Topological Field Theories in the sense of Turaev and Viro [22]. In this case \(\Phi\) satisfies a usual Pentagon equation and \(\Gamma\) is the product of Kroenecker delta’s. However in that case the \(R\)-matrix turns out to be non-invertible. Moreover the model is topological [23]. In fact in that situation the proof of the Tetrahedron equation is a trivial consequence of the Turaev-Viro theorem. The problem of finding more general solutions (in particular non-topological one’s, and depending on spectral parameters) is now under study.

4. Conclusion

Using a geometrical interpretation of the \(R\)-matrix solving the Tetrahedron equation as a transport operator acting in a space of functionals of surfaces, we have obtained a decomposition of such an \(R\)-matrix in terms of more fundamental objects \((\Phi, \Gamma)\), \(\Phi\) being the solution of the Pentagon equation (24). This provides an explicit link between Pentagon and Tetrahedron equations. We expect such a relation to be fruitful in the construction of new solutions to the Tetrahedron equation. It also open the possibility of extending the algebraic picture of Quantum Groups as given in [4] to another algebraic structure suitable for Integrable Systems in three dimensions. Finally, as can be expected from the fundamental relation (2), it also relates three-dimensional Topological Field Theories to a special case of the Tetrahedron equation.
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