EXISTENCE OF PERIODICALLY INVARIANT TORI ON RESONANT SURFACES FOR TWIST MAPPINGS

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(Communicated by Chong-Qing Cheng)

Abstract. In this paper we will prove the existence of periodically invariant tori of twist mappings on resonant surfaces under the low dimensional intersection property.

1. Introduction. In this paper we are concerned with the mapping

\[
\begin{align*}
\theta_1 &= \theta + \omega(r) + X(\theta, r), \\
r_1 &= r + Y(\theta, r),
\end{align*}
\]

(1)

where \( \theta \in \mathbb{T}^m, r \in D \subset \mathbb{R}^n \) with \( D \) being a bounded region, \( \mathbb{T}^m \) is the usual \( m \)-torus obtained from \( \mathbb{R}^m/\mathbb{Z}^m \) and \( X(\theta, r), Y(\theta, r) \) are real analytic with period 1 in \( \theta \). Meanwhile, \( \omega(r) \) is also a real analytic function.

Without the perturbed terms \( X, Y \), the mapping has the form

\[
\begin{align*}
\theta_1 &= \theta + \omega(r), \\
r_1 &= r.
\end{align*}
\]

The frequency vector \( \omega = \omega(r) \) is said to be resonant, or rationally dependent, if there exists a nonzero vector \((k, d) \in \mathbb{Z}^m \times \mathbb{Z}\) such that \( \langle k, \omega \rangle = d \), otherwise \( \omega \) is said to be incommensurate. If there is a nontrivial subgroup \( L \) of \( \mathbb{Z}^m \), whose rank is \( m_2 \), such that for every vector \( k \in L \), \( \langle k, \omega \rangle \in \mathbb{Z} \), then the frequency \( \omega \) is called \( m_2 \)-resonant frequency. For any given subgroup \( L \) with \( \text{rank}(L) = m_2 < n \), the set

\[
O(D, L) = \{ r \in D : \langle k, \omega(r) \rangle \in \mathbb{Z} \text{, } k \in L \}
\]

is an \((n - m_2)\)-dimensional surface, which is called an \( L \)-resonant surface. If let \( \tau_i \in \mathbb{Z}^m, i = 1, \ldots, m_2 \), be the basis for \( L \), then

\[
O(D, L) = \{ r \in D : \langle \tau_i, \omega(r) \rangle \in \mathbb{Z} \text{, } \tau_i \in \{ \tau_1, \ldots, \tau_{m_2} \} \}.
\]

In the ensuing sections we will restrict the action variables \( r \) in the set \( O(D, L) \).

2010 Mathematics Subject Classification. Primary: 37E40; Secondary: 37J40.

Key words and phrases. Periodically invariant tori, twist mappings, resonant surfaces, low dimensional intersection property.

The second author is supported by NSFC (11971059).

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invariant tori of the mapping \( M \) nonlinear perturbations. Meanwhile, notice that the \( q_1Q\omega \)

From [9], when \( r \in \mathcal{O}(D, \mathcal{L}) \), there exists a unimodular matrix \( Q \) such that \( Q\omega(r) = (\omega^1(r), \omega^2)^T \), where \( \omega^2 = \left( \frac{d_1}{q_1}, \ldots, \frac{d_m}{q_m} \right)^T \) with \( d_i \in \mathbb{Z}, q_i \in \mathbb{N} \) and \( q_i \) divides \( q_j \) for \( i < j \), while other \( m_1 = m - m_2 \) components of the vector \( Q\omega \), namely \( \omega^1(r) \) are incommensurate.

After the linear coordinate transformation \( (x, y)^T = Q\theta \), the mapping (1) becomes

\[
\mathfrak{M} : \begin{cases} 
  x_1 = x + \omega^1(z) + X_1(x, y, z), \\
  y_1 = y + \omega^2 + X_2(x, y, z), \\
  z_1 = z + Y(x, y, z),
\end{cases}
\]

where \((X_1(x, y, z), X_2(x, y, z))^T = QX(Q\theta, r), Y(x, y, z) = Y(Q\theta, r) \) and \( Q\theta = (x, y)^T \in \mathbb{T}^{m_1} \times \mathbb{T}^{m_2}, r = z \in \mathcal{O}(D, \mathcal{L}). \)

When \( X_1 = X_2 = Y \equiv 0 \), the mapping (2) takes the form

\[
\mathfrak{M} : \begin{cases} 
  x_1 = x + \omega^1(z), \\
  y_1 = y + \omega^2, \\
  z_1 = z.
\end{cases}
\]

According to [9], the orbit \((x_i, y_i, z_i)^T (i \geq 0)\) of the mapping \( \mathfrak{M} \) with the initial condition \((x_0, y_0, z_0)^T\) will admit \( m_1 - \)dimensional periodically invariant tori

\[
\begin{align*}
  x_i &= x_0 + i \omega^1(z_0), & x_0 & \in \mathbb{T}^{m_1}, \\
  y_i &= y_0 + i \omega^2, & y_0 & \in \mathbb{T}^{m_2}, \\
  z_i &= z_0, & z_0 & \in \mathcal{O}(D, \mathcal{L}),
\end{align*}
\]

which is composed of \( q_0 = \text{lcm}\{q_1, \ldots, q_{m_2}\} \), disjoint \( m_1 - \)dimensional tori that are linked by the iteration. In particular, if \( \mathcal{L} \) is chosen such that \( \omega^2 \in \mathbb{Z}^{m_2} \), the mapping \( \mathfrak{M} \) possesses the invariant torus \( \mathbb{T}^{m_1} \times \{y_0\} \times \{z_0\} \) for any \( y_0 \in \mathbb{T}^{m_2} \) and \( z_0 \in \mathcal{O}(D, \mathcal{L}) \).

We are interested in whether the periodically invariant tori of \( \mathfrak{M} \) will survive the nonlinear perturbations. Meanwhile, notice that the \( m_1 - \)dimensional periodically invariant tori of the mapping \( \mathfrak{M} \) is the \( m_1 - \)dimensional invariant tori of

\[
\mathfrak{M}^{q_0} : \begin{cases} 
  x_{q_0} = x + q_0 \omega^1(z) + f(x, y, z), \\
  y_{q_0} = y + g(x, y, z), \\
  z_{q_0} = z + h(x, y, z),
\end{cases}
\]

where

\[
\begin{align*}
  f(x, y, z) &= X_1(x, y, z) + \sum_{j=1}^{q_0-1} X_1(x_j, y_j, z_j), \\
  g(x, y, z) &= X_2(x, y, z) + \sum_{j=1}^{q_0-1} X_2(x_j, y_j, z_j), \\
  h(x, y, z) &= Y(x, y, z) + \sum_{j=1}^{q_0-1} Y(x_j, y_j, z_j)
\end{align*}
\]
with
\[
\begin{align*}
x_i &= x + i\omega^1(z) + X_1(x, y, z) + \sum_{j=1}^{i-1} X_1(x_j, y_j, z_j), \\
y_i &= y + i\omega^2(z) + X_2(x, y, z) + \sum_{j=1}^{i-1} X_2(x_j, y_j, z_j), \\
z_i &= z + Y(x, y, z) + \sum_{j=1}^{i-1} Y(x_j, y_j, z_j), \quad i = 1, 2, \ldots, q_0 - 1.
\end{align*}
\]

Consequently, instead of investigating the existence of periodically invariant tori of the mapping \(M\), in this paper we are going to prove that the existence of invariant tori of the mapping \(M^0\).

We make the following assumptions

\(H_1\) : \(f, g, h, \omega^1\) are real analytic mappings when \((x, y, z) \in \mathbb{T}^{m_1} \times \mathbb{T}^{m_2} \times \mathcal{O}(D, \mathcal{L})\).

\(H_2\) : \(M^0\) has the low dimensional intersection property, that is, any \(m_1\)-dimensional torus close to the invariant torus of the unperturbed mapping
\[
M^0 \colon \begin{cases} x_{q_0} = x + q_0\omega^1(z), \\
y_{q_0} = y, \\
z_{q_0} = z
\end{cases}
\]
intersects its image under the mapping \(M^0\).

\(H_3\) : In \(\mathcal{O}(D, \mathcal{L})\), \(\omega^1(z)\) satisfies
\[
\text{rank}\left\{ \frac{\partial^{\alpha}\omega^1}{\partial z^\alpha} : \forall \alpha \in \mathbb{Z}_+^n, 0 < |\alpha| \leq L \right\} = m_1, \quad L := \max\{m_1, n\},
\]
where \(\frac{\partial^{\alpha}\omega^1}{\partial z^\alpha} = (\frac{\partial^{\alpha_1}\omega_1}{\partial z^\alpha_1}, \ldots, \frac{\partial^{\alpha_n}\omega_n}{\partial z^\alpha_n})\), \(\alpha = (\alpha_1, \ldots, \alpha_n), |\alpha| = |\alpha_1| + \cdots + |\alpha_n|\). Then we have the result below.

**Theorem 1.1.** Assume that the assumptions \(H_1\) – \(H_3\) hold. Then there exists a positive number \(\varepsilon_*\), for any \(\varepsilon \in (0, \varepsilon_*)\), there is a Cantor set \(G_\varepsilon \subset \mathcal{O}(D, \mathcal{L})\) such that for every \(z_0 \in G_\varepsilon\), if \(|f| + |g| + |h| < \varepsilon\), the mapping \(M\) admits a family of \(m_1\)-dimensional periodically invariant tori
\[
\begin{align*}
x &= \xi + u(\xi, \eta), \\
y &= \eta + v(\xi, \eta), \\
z &= w(\xi, \eta)
\end{align*}
\]
with \(u, v, w\) real analytic functions of period 1 with respect to \(\xi\) and \(\eta\). Moreover, the parametrization is chosen so that the induced mapping on the tori (5) is given by
\[
\xi_{q_0} = \xi + \omega^1_\infty(\eta, z_0), \quad \eta_{q_0} = \eta
\]
with
\[
|\omega^1_\infty(\eta, z_0) - q_0\omega^1(z_0)| < 2\varepsilon_*,
\]
and the functions \(u, v, w\) satisfy
\[
|u| + |v| + |w - z_0| < 6c_0^{1/(\tau + L + 3)} \varepsilon_*^{1/(8(\tau + L + 3))},
\]
the Lebesgue measure
\[
\text{meas}(\mathcal{O}(D, \mathcal{L}) \setminus G_\varepsilon) \leq c \varepsilon_*^{1/(8L(\tau + L + 3))},
\]
where \( c_0 \) and \( c \) are independent of \( \epsilon_* \) and \( \tau > L(L+1) - 1 \).

Throughout this paper, if we do not explicitly state, the norm of a matrix \( p = (p_{ij})_{n_1 \times n_2} \) will be defined by

\[
|p| = \max_{1 \leq i \leq n_1} \sup_{1 \leq j \leq n_2} |p_{ij}(x)|,
\]

where \( n_1, n_2 \geq 1 \) and \( p_{ij} : D \to \mathbb{C} \).

Now we give some remarks. Firstly, since we are seeking for \( m_1 \)-dimensional invariant tori, the low dimensional intersection assumption \((H_2)\) is reasonable. Secondly, when \( m_1 = m \), that is, there is no any resonant relations in frequency vectors, Theorem 1.1 is the same as the corresponding theorem in [7], thus we generalize the result in [7] into the resonant case. Finally, from the proof of Theorem 1.1, if \( m_1 = n \), that is, the number of the nonresonant frequencies \( \omega^1 \) is equal to the number of the action variables, the nonresonant frequencies \( \omega^1 \) can be kept unchanged under the iterative process, thus one can fix the nonresonant frequencies in advance and also do not need the measure estimate, just the same as the classical case in [21].

Before ending the introduction, let us recall some related results. In 1962, Moser [21] proved the existence of invariant closed curves of twist mappings with one angular variable and one action variable. When the number of angular variables is not equal to the number of action variables, the persistence of invariant tori can not be obtained by applying Moser’s result in [21] directly. Cheng and Sun [5] in 1989 successfully proved that there exists a large set of two-dimensional invariant tori under certain nondegeneracy condition for three-dimensional measure preserving mappings. Xia [26] extended the result to the \( n+1 \)-dimensional volume-preserving diffeomorphisms with one action variable. Recently, Cong, Li and Huang [7] further extended the result to the mappings with distinct number of action variables and angular variables. They found that, under the R"ussmann condition and the intersection property, there also exists a large amount of invariant tori.

Since KAM theory was built by Kolmogorov [16], Arnold [1] and Moser [21] in 1960’s, by now there have been many excellent works about the existence of resonant tori or lower dimensional invariant tori in Hamiltonian systems [4, 10, 11, 8, 22, 25, 27, 17, 18, 19, 3, 23, 20] and in symplectic mappings [2, 28], as well as some results on the existence of invariant curves in quasi-periodic or almost periodic twist mappings [12, 13, 14, 15], but the results about resonant tori of twist mappings are relatively few. Only Cheng and Sun [6] proved the existence of periodically invariant curves in 3-dimensional measure-preserving mappings. Simulated by the case of Hamiltonian systems, they made use of normal forms of 3-dimensional measure-preserving mappings and coped with the effect on the resonant frequency by seeking for the zeros of the average part of the perturbation about the resonant angular variable.

Different with the result in [6], on one hand, the mapping we are going to investigate has \( n \) action variables and \( m \) angular variables. On the other hand, we employ the completely different method to prove the existence of periodically invariant tori. Since the resonant angular variables are indeed slow variables on the resonant surface, one can use the low dimensional intersection property not only to deal with the average part of the perturbation about the action variables, which are also slow variables, but also to eliminate the average part of the perturbation about the resonant angular variables. More importantly, periodically invariant curves in
some lemmas used in the preceding sections are stated and proved. In the last section, we will prove Theorem 1.1. The proof of the iterative Lemma is given in Section 3. In the last section, some lemmas used in the preceding sections are stated and proved.

2. The iterative lemma and the proof of theorem 1.1. The proof of Theorem 1.1 follows the traditional line laid out by [21] and [24]. The main step is to make a sequence of successive applications of the Iterative Lemma. In this section, we will state the lemma, and use it to prove Theorem 1.1.

Firstly, the real analytic functions \( f, g, h, \omega^1 \) can be extended to the complex domain \( \{ x \in \mathbb{C}^{m_1} : |3x| < \rho \} \times \{ y \in \mathbb{C}^{m_2} : |3y| < 8\delta \} \times \{ z \in \mathbb{C}^n : \Re z \in \mathcal{O}(D, \mathcal{L}), |z - z_0| < \lambda \} := \Sigma^1 \times \Sigma^2 \times G \), where \( \rho > 0, \lambda = 6\delta^{3/8} \).

\[
\delta = c_0^{1/(\tau + L + 3)} \epsilon^{1/(8(\tau + L + 3))}, \quad c_0 = 3 \left( 2^{m_1-1} \frac{1}{(2\pi)^{m_1+1}} \sum_{j=1}^{\infty} \frac{1}{j^2} + 1 \right),
\]

\[
z_0 \in O := \left\{ z \in \mathcal{O}(D, \mathcal{L}) : |\langle k, \omega^1(z) \rangle + k_0| \geq \delta |k|^{-\tau}, \ 0 \neq k \in \mathbb{Z}^{m_1} \right\}, \quad k_0 \in \mathbb{Z}, |k_0| \leq M_{10} |k|
\]

with \( M_{10} \) being defined in Lemma 4.2 and \( |k| = |k_1| + |k_2| + \cdots + |k_{m_1}| \).

In \( \Sigma^1 \times \Sigma^2 \times G \), we assume \( f, g, h, \omega^1 \) satisfy the following inequalities:

\[
|f| + |g| + |h| < \epsilon,
\]

\[
|\omega^1(\eta, \zeta)| < M_{10} - 1, \quad \left| \frac{\partial \omega^1(\eta, \zeta)}{\partial \zeta} \right| \leq M_{12}, \quad \left| \frac{\partial \omega^1(\eta, \zeta)}{\partial \eta} \right| \leq M_{11},
\]

\[
\left| \frac{\partial^2 \omega^1(\eta, \zeta)}{\partial \eta \partial \zeta} \right| \leq M_{121}, \quad \left| \frac{\partial^2 \omega^1(\eta, \zeta)}{\partial \eta^2} \right| \leq M_{111}, \quad \left| \frac{\partial^2 \omega^1(\eta, \zeta)}{\partial \zeta^2} \right| \leq M_{122},
\]

where we regard \( \omega^1(z) \) as \( \omega^1(y, z) \) for the convenience of iterations, and \( M_{12}, M_{11}, M_{121}, M_{111}, M_{122} \) are constants given in Lemma 4.2.

Hereafter we require \( 0 < \epsilon < \epsilon_* \) with

\[
\epsilon_* = \min \left\{ \frac{\epsilon_0}{2}, \frac{1}{c_0}, \frac{\rho}{28} \left( \frac{1}{24L M_{12}} \right)^{8(\tau + L + 3)/3}, \frac{1}{20L} \left( \frac{1}{M} \right)^{8(\tau + L + 3)/3}, \frac{1}{264(\tau + L + 3)}, \frac{1}{2} \left( \frac{1}{L^2 (M_{122} + M_{111} + 2 M_{121})} \right)^2, \frac{1}{36(M + 4)L^2 M_{12}} \left( \frac{\rho}{8(\tau + L + 3)c_0} \right)^{8/3}, \frac{1}{(M + 4)24L} \left( \frac{\rho}{8(\tau + L + 3)c_0} \right)^{8/3} \right\},
\]

where \( M = (1 + M_{12} + M_{11}) \), and \( \epsilon_0 \) can be found in Lemma 4.3.
The following four complex domains will make it convenient to state the lemma clearly;
\[ \mathcal{A}^{(1)} = \{ (x, y, z) : |\Re x| < \rho + 3\delta, |\Re y| < 8\delta + 3\delta, z \in G - (\lambda - \lambda^+) \}, \]
\[ \mathcal{A}^{(2)} = \{ (x, y, z) : |\Re x| < \rho + 2\delta, |\Re y| < 8\delta + 2\delta, z \in G - (\lambda - \lambda^+ + 2) \}, \]
\[ \mathcal{A}^{(3)} = \{ (x, y, z) : |\Re x| < \rho + \delta, |\Re y| < 8\delta + \delta, z \in G - (\lambda - \lambda^+ + 3) \}, \]
\[ \mathcal{B} = \{ (\xi, \eta, \zeta) : |\Re \xi| < \rho, |\Re \eta| < 8\delta, \zeta \in (G - (\lambda - \lambda^+)) \}, \]
where \( \epsilon_+ = e^{3/8}, \rho^+ = \rho - 7\delta, \delta^+ = c_0^{1/(\tau + L + 3)} \epsilon_+^{3/(8(\tau + L + 3))} \) and \( \lambda^+ = 6\delta^+ \epsilon_+^{3/8} \).

For a set \( X \) and a positive number \( r \), define
\[ X - r := \{ z \in X : \text{dist}(z, \partial X) > r \}, \]
where \( \partial X \) is the boundary set of \( X \).

Now we are in a position to state the iterative lemma.

**Lemma 2.1.** (Iterative Lemma) Under the above assumptions, there exists a coordinate transformation \( U_0 \) of the form
\[
\begin{aligned}
x &= \xi + U_0(\xi, \eta, \zeta), \\
y &= \eta + V_0(\xi, \eta, \zeta), \\
z &= \zeta + W_0(\xi, \eta, \zeta)
\end{aligned}
\]  
with \( U_0, V_0, W_0 \) real analytic functions in \( \mathcal{A}^{(1)} \) of period 1 in \( \xi, \eta, \zeta \), such that the transformed mapping \( U_0^{-1} \mathcal{M}^0 U_0 \) takes the form
\[
\mathcal{M}_0^{(1)} : \begin{cases}
\xi_0 = \xi + g_0(\xi, \eta, \zeta), \\
\eta_0 = \eta + g^+(\xi, \eta, \zeta), \\
\zeta_0 = \zeta + h^+(\xi, \eta, \zeta)
\end{cases}
\]
with \( f^*(\eta, \zeta) = \int_0^1 \cdots \int_0^1 f(\xi, \eta, \zeta) d\xi, \) and \( f^+, g^+, h^+ \) are real analytic functions defined in \( \mathcal{B} \) where they satisfy the estimate
\[ |f^+| + |g^+| + |h^+| < e^{3/8}. \]

To be more exact: \( U_0 \) maps \( \mathcal{B} \) into \( \mathcal{A}^{(3)} \), \( \mathcal{M}^0 \) takes \( \mathcal{A}^{(3)} \) into \( \mathcal{A}^{(2)} \), and \( U_0^{-1} \) takes \( \mathcal{A}^{(2)} \) into \( \mathcal{A}^{(1)} \), so that \( U_0^{-1} \mathcal{M}^0 U_0 \) is well defined in \( \mathcal{B} \). Moreover, in \( \mathcal{A}^{(1)} \) the functions \( U_0, V_0, W_0 \) satisfy the inequality
\[ |U_0| + |V_0| + |W_0| < 3\delta. \]

The proof of Lemma 2.1 will be provided in the next section. Now we will apply Lemma 2.1 to prove Theorem 1.1 on the existence of the periodically invariant
tori. Before proving Theorem 1.1, it is necessary to construct the corresponding parameters of every iteration. For $i = 0, 1, \cdots$, let
$$
\epsilon_{i+1} = \epsilon_{i}^{9/8}, \quad \epsilon_{0} = \epsilon, \quad \delta_{i} = \epsilon_{0}^{\tau/(\tau+L+3)} \epsilon_{i}^{1/(\tau+L+3)},
$$
then we may assume $G_{0} \subset \mathbb{R}^{n} \subset \mathbb{C}^{n}$ with $0 \in \mathbb{C}^{n}$.

1. In order to prove Theorem 1.1, we first need to construct the corresponding $\tau > L$.

2. By Lemma 2.2, we may assume $G_{0}$ is a nonempty set and set
$$
G_{i} := \{z \in \mathbb{C}^{n} : |z - z_{0}| < \lambda_{i}, \exists z \in O(D, L)\}, \quad E_{i} := \Sigma^{1}_{i} \times \Sigma^{2}_{i} \times G_{i},
$$
where $z_{0} \in \bigcup_{j=0}^{i} O_{j}$.

According to Lemma 2.1, the coordinate transformation $U_{0}$, the perturbations of the resulting mapping $\mathcal{M}_{0}^{0}$ can be made smaller than the original mapping $\mathcal{M}^{0}$, and the corresponding domain $E_{1}$ becomes narrower than $E_{0}$. Making a sequence of successive applications of the lemma, we can have a sequence of coordinate transformations $U_{i}$ and the new mapping $\mathcal{M}_{i+1}^{0} = U_{i}^{-1} \mathcal{M}_{i}^{0} U_{i}$ with a narrower domain $E_{i+1}$. With the times of iterations increasing to infinity, the perturbations of the mapping $\mathcal{M}_{i+1}^{0}$ approach to zero. Therefore, when the sequence $\{\epsilon_{i}\}$ converges to $0$, the mapping will be of the form
$$
\mathcal{M}_{\infty}^{0} : \begin{cases}
\xi_{1} = \xi + \omega_{1}^{0}(\eta, \zeta), \\
\eta_{1} = \eta, \\
\zeta_{1} = \zeta
\end{cases}
$$
with $\xi, \eta, \zeta$ restricted to the domain $E_{\infty} = \{\xi : |\Im \xi| < \rho/2\} \times \{\eta : \eta \in \mathbb{T}^{m_{2}}\} \times \{z_{0}\}$. Thus, we get the $m_{1}$-dimensional periodically invariant tori of $\eta$ for every $\eta \in \mathbb{T}^{m_{2}}$.

### 2.1 The existence of the periodically invariant tori

Denote $\mathcal{M}_{i}^{0}$ as the form
$$
\begin{align*}
\begin{cases}
x_{0} &= x + \omega_{1}^{0}(y, z) + f^{i}(x, y, z), \\
y_{0} &= y + g^{i}(x, y, z), \\
z_{0} &= z + h^{i}(x, y, z),
\end{cases}
\end{align*}
$$
where $\omega_{1}^{0}(y, z) = q_{0}^{i} \omega_{1}^{0}(z), \mathcal{M}_{0}^{0} = \mathcal{M}^{0}$ when $i = 0$. Suppose that in $E_{i}$, the perturbations of the mapping $\mathcal{M}_{i}^{0}$ satisfy $|f| + |g| + |h^{i}| < \epsilon_{i}$. By Lemma 4.2, we have
$$
\begin{align*}
|\omega_{1}^{0}(y, z)| &< M_{10} - 1, \\
\left| \frac{\partial \omega_{1}^{0}(y, z)}{\partial z} \right| &\leq M_{12}, \\
\left| \frac{\partial \omega_{1}^{0}(y, z)}{\partial y} \right| &\leq M_{11}, \\
\left| \frac{\partial^{2} \omega_{1}^{0}(y, z)}{\partial y \partial z} \right| &\leq M_{121}, \\
\left| \frac{\partial^{2} \omega_{1}^{0}(y, z)}{\partial y^{2}} \right| &\leq M_{111}, \\
\left| \frac{\partial^{2} \omega_{1}^{0}(y, z)}{\partial z^{2}} \right| &\leq M_{122}.
\end{align*}
$$
Therefore, with respect to $M_{q}^{0}$, it is available to apply Lemma 2.1, thus we can obtain a coordinate transformation

$$
U_{i} : \begin{cases}
x = \xi + U_{i}(\xi, \eta, \zeta), \\
y = \eta + V_{i}(\xi, \eta, \zeta), \\
z = \zeta + W_{i}(\xi, \eta, \zeta),
\end{cases}
$$

transforming $M_{q}^{0}$ into $M_{q}^{0}_{i+1}$ with

$$
|f^{i+1}| + |g^{i+1}| + |h^{i+1}| < \epsilon_{i+1}^{9/8} = \epsilon_{i+1},
$$

$$
\omega_{i+1}^{1}(\eta, \zeta) = \omega_{i}^{1}(\eta, \zeta) + f^{i*},
$$

$$
f^{i*} = \int_{0}^{1} \cdots \int_{0}^{1} f^{i}(\xi, \eta, \zeta) d\xi,
$$

$$
|U_{i}| + |V_{i}| + |W_{i}| < 3\delta_{i}.
$$

Since $U_{i}$ maps the domain $E_{i+1}$ into $E_{i}$ ($i = 0, 1, 2 \cdots$), the transformation $\Gamma_{i} = U_{i}U_{i-1}\cdots U_{1}$ is well defined in $E_{i+1}$ and takes $M_{q}^{0}$ into

$$
M_{q}^{0} = \Gamma_{i}^{-1}M_{q}^{0}\Gamma_{i}.
$$

If we express $\Gamma_{i}$ in the form

$$
\begin{cases}
x = \xi + p_{i}(\xi, \eta, \zeta), \\
y = \eta + q_{i}(\xi, \eta, \zeta), \\
z = \zeta + r_{i}(\xi, \eta, \zeta),
\end{cases}
$$

then from $\Gamma_{i} = \Gamma_{i-1}U_{i}$, it follows that

$$
p_{i} = U_{0} + U_{1} + \cdots + U_{i},
$$

$$
q_{i} = V_{0} + V_{1} + \cdots + V_{i},
$$

$$
r_{i} = W_{0} + W_{1} + \cdots + W_{i}.
$$

By Lemma 2.1 and Lemma 4.1, one obtain that

$$
|p_{i}| + |q_{i}| + |r_{i}| < \sum_{j=0}^{i} 3\delta_{j} < 6\delta_{0},
$$

$$
|\omega_{i}^{1}(y, z) - \omega^{1}(z)| \leq \sum_{j=0}^{i-1} |f^{j*}| < \sum_{j=0}^{i-1} \epsilon_{j} < 2\epsilon,
$$

and

$$
\lim_{i \to \infty} \rho_{i} = \frac{1}{2}\rho, \quad \lim_{i \to \infty} \lambda_{i} = 0,
$$

which imply that

$$
E_{\infty} \subset \{ x : |3x| < \rho/2 \} \times \{ y : y \in \mathbb{T}^{m_{2}} \} \times \{ z_{0} \},
$$

and $p_{i}, q_{i}, r_{i}, \omega_{i}^{1}$ are uniformly convergent in $E_{\infty}$. Denoting the limits of $p_{i}, q_{i}, r_{i},$ and $\omega_{i}^{1}$ by $u(\xi, \eta), v(\xi, \eta), w(\xi, \eta) - z_{0}$, and $\omega_{i}^{1}(\eta, z_{0})$, respectively, we find that a
family of $m_1$-dimensional periodically invariant tori are of the form
\[\begin{align*}
\Gamma : \begin{cases}
x = \xi + u(\xi, \eta), \\
y = \eta + v(\xi, \eta), \\
z = w(\xi, \eta),
\end{cases}
\end{align*}\]
and
\[|u| + |v| + |w - z_0| < 6\delta_0 < 6c_0^{1/(\tau + L+3)}\epsilon, \quad |\omega^1_{\infty}(y, z_0) - \omega^1_0(z_0)| < 2\epsilon.\]
From $\mathfrak{M}_\infty = \Gamma^{-1}\mathfrak{M}$, we have the relations
\[\begin{align*}
\omega^1_{\infty}(\eta, z_0) + u(\xi + \omega^1_{\infty}(\eta, z_0), \eta) = u(\xi, \eta) + q_0\omega^1(w(\xi, \eta)) + f(\xi + u, \eta + v, w), \\
v(\xi + \omega^1_{\infty}(\eta, z_0), \eta) = v(\xi, \eta) + g(\xi + u, \eta + v, w), \\
w(\xi + \omega^1_{\infty}(\eta, z_0), \eta) = w(\xi, \eta) + h(\xi + u, \eta + v, w),
\end{align*}\]
and the induced mapping on the tori $\Gamma$ is expressed by
\[\xi_{00} = \xi + \omega^1_{\infty}(\eta, z_0), \eta_{00} = \eta.\]
This completes the proof of the existence of the periodically invariant torus in the mapping $\mathfrak{M}$. In the next subsection we will work on the measure estimate of the set $G_\epsilon$.

2.2. The measure estimate. Under the assumption $(H_3)$, the measure estimate is similar to that in [7], in which Lemma 4.3 plays a key role.

By implicit function Theorem, $O(D, L)$ is locally diffeomorphic to a surface in $\mathbb{R}^{n-m_2}$ [8]. Without loss of generality, we may assume that $O(D, L)$ is an $n - m_2$-dimensional connected region of $\mathbb{R}^{n-m_2}$. Then the assumption $(H_3)$ becomes: in $O(D, L) \subset \mathbb{R}^{n-m_2}$, $\omega^1(z)$ satisfies
\[\text{rank}\left\{\frac{\partial^{\alpha} \omega^1}{\partial z^\alpha} : \forall \alpha \in \mathbb{N}^{n-m_2}, 0 < |\alpha| \leq l\right\} = m_1, \quad l := \max\{m_1, n - m_2\}.\]
Now we can apply Lemma 4.3 to estimate the measure of $O_i$ by replacing $\omega$ and $\bar{\omega}$ in Lemma 4.3 by $\omega^1$ and $\sum_{i=0}^1 f^{j_i}$, respectively, and regarding $\eta$ as a parameter. In addition, by the choice of $\epsilon_{\ast}$, $\sum_{i=0}^\infty |f^{j_i}| \leq 2\epsilon < \epsilon_{00}$, hence we obtain
\[\text{meas}(O(D, L) \setminus O_i) < c_{00}\delta_1^{1/L},\]
where $c_{00}$ is a constant given in Lemma 4.3.

Set $G_\epsilon = \bigcap_{i=0}^\infty O_i$, then
\[(O(D, L) \setminus G_\epsilon) \subset \bigcup_{i=0}^\infty (O(D, L) \setminus O_i).\]
By trivially imitating the proof in [7], we can find that
\[\text{meas}(O(D, L) \setminus G_\epsilon) \leq \sum_{i=0}^\infty \text{meas}(O(D, L) \setminus O_i) \leq 2c_{00}\delta_1^{1/L} = c\epsilon^{1/(8L(L+\tau+3))},\]
where $c = 2c_{00}\epsilon_0^{1/(\tau+L+3)}$. 
Remark 1. From the above iteration procedure, in order to control uniformly the perturbations $f$, $g$ and $h$, we require that $z_0 \in \mathcal{O}(D,\mathcal{L}) - \lambda_0$. However, this does not influence the conclusion of Theorem 1.1. Indeed, it follows from $6\delta^{3/8} \leq \delta^{1/2}$ that

$$\text{meas}(\mathcal{O}(D,\mathcal{L}) \setminus \mathcal{O}(D,\mathcal{L}) - \lambda_0) = O(\delta^{1/2}).$$

This concludes the proof of Theorem 1.1. In the sequel, we will focus on the proof of Lemma 2.1.

3. The proof of the iterative lemma. In the process of the proof of Theorem 1.1 we made a sequence of successive applications of Lemma 2.1. Noticing that every iteration step is similar, without loss of generality, we only need to check one cycle of this iteration scheme. Here we just check the $i^{th}$ ($i > 0$, $i \in \mathbb{N}$) step to the $(i + 1)^{th}$ step. From the following process we will find that the proof is also available for $i = 0$. Drop the indices "$i$" of corresponding parameters and variables, and replace the indices "$i + 1$", "$i - 1$" with "$+$", "$-$", respectively. Write $M_{q0}^i$ as

$$\begin{cases}
x_{q0} = x + \omega_1(y, z) + f(x, y, z), \\
y_{q0} = y + g(x, y, z), \\
z_{q0} = z + h(x, y, z),
\end{cases}$$

and $\mathcal{U}_i$, $M_{q0}^{i+1}$ will be of the form

$$\begin{cases}
x = \xi + U(\xi, \eta, \zeta), \\
y = \eta + V(\xi, \eta, \zeta), \\
z = \zeta + W(\xi, \eta, \zeta)
\end{cases}$$

and

$$\begin{cases}
\xi_{q0} = \xi + \omega_1^+(\eta, \zeta) + f^+(\xi, \eta, \zeta), \\
\eta_{q0} = \eta + g^+(\xi, \eta, \zeta), \\
\zeta_{q0} = \zeta + h^+(\xi, \eta, \zeta).
\end{cases}$$

Before proving the lemma, it will be convenient to introduce the following notations

$$\omega_1^+(\eta, \zeta) = \omega_1^1(\eta, \zeta) + f^*, $$

$$N = \left[ \frac{L \ln(L)}{\delta} + \frac{L + 1}{\delta} \ln \frac{4}{\delta} + \frac{1}{4\delta} \ln \frac{1}{\epsilon} \right] + 1,$$

$$T_N f = \sum_{0 < |k| < N} f_k e^{2\pi \sqrt{-1}(k, \xi)},$$

$$R_N f = \sum_{|k| \geq N} f_k e^{2\pi \sqrt{-1}(k, \xi)},$$

$$\mathcal{D}^{(1)} = \{(x, y, z) : |3x| < \rho^+ + 3\delta, |3y| < 8\delta^+ + 3\delta, z \in (G - (\lambda - \lambda^+)/4)\},$$

$$\mathcal{D}^{(2)} = \{(x, y, z) : |3x| < \rho^+ + 2\delta, |3y| < 8\delta^+ + 2\delta, z \in (G - (\lambda - \lambda^+)/2)\}.$$
\[ \mathcal{D}^{(3)} = \{ (x, y, z) : |3x| < \rho^+ + \delta, |3y| < 8\delta^+ + \delta, \] 
\[ z \in (G - (\lambda - \lambda^+)3/4) \}, \]
\[ E_+ = \{ (\xi, \eta, \zeta) : |3\xi| < \rho^+, |3\eta| < 8\delta^+, \zeta \in (G - (\lambda - \lambda^+)) \}. \]

Now we are in the position to prove Lemma 2.1. In fact, the first step of the proof of Lemma 2.1 is to find a coordinate transformation \( \mathcal{U} \). In order to obtain the coordinate transformation \( \mathcal{U} \), it is necessary to solve the following difference equations

\[
\begin{cases}
U(\xi + \omega^1(\eta, z_0), \eta, \zeta) - U(\xi, \eta, \zeta) = T_N f + \left( \frac{\partial \omega^1(\eta, \zeta)}{\partial \zeta} , W \right) + \left( \frac{\partial \omega^1(\eta, \zeta)}{\partial \eta} , V \right), \\
V(\xi + \omega^1(\eta, z_0), \eta, \zeta) - V(\xi, \eta, \zeta) = T_N g, \\
W(\xi + \omega^1(\eta, z_0), \eta, \zeta) - W(\xi, \eta, \zeta) = T_N h.
\end{cases}
\]

We solve the equations (9) by means of the Fourier expansion. Substituting \( W = \sum_{0 < |k| < N} W_k e^{2\pi \sqrt{-1}(k, \xi)} \), into the third equation of (9), one obtains

\[ W_k = \frac{h_k}{e^{2\pi \sqrt{-1}(k, \omega^1(\eta, z_0))} - 1}. \]

After obtaining the expression of \( W \), it is necessary to prove that \( W \) is real analytic in \( (\Sigma^1 - \delta) \times (\Sigma^2 - \delta) \times G \). Indeed, for \( 0 < |k| < N \), there exists an integer \( k_0 \) satisfying that \( k_0 \leq M_{10}|k| \), such that

\[ \pi |\langle k, \omega^1(\eta, z_0) \rangle + k_0| \leq \pi/2. \]

Hence

\[ \left| e^{2\pi \sqrt{-1}(k, \omega^1(\eta, z_0))} - 1 \right| = 2 |\sin(\pi (\langle k, \omega^1(\eta, z_0) \rangle + k_0\pi))| \]
\[ \geq \frac{2}{\pi} 2|\pi (\langle k, \omega^1(\eta, z_0) \rangle + k_0\pi)| \]
\[ \geq 2\delta |k|^{-\tau}. \]

On the other hand, since \( h \) is real analytic, the Fourier coefficients

\[ h_k = \int_0^1 \cdots \int_0^1 h(\xi, \eta, \zeta)e^{-2\pi \sqrt{-1}(k, \xi)} d\xi, \quad k \in \mathbb{Z}^{m_1} \]

decay exponentially by Lemma 4.4, that is,

\[ |h_k| \leq |T_N h| e^{-2\pi |k|\rho}, \quad (x, y, z) \in E. \]

By (10) and (11), \( W \) has the following estimate in a narrower domain \( (\Sigma^1 - \delta) \times (\Sigma^2 - \delta) \times G \)

\[ |W| \leq \sum_{0 < |k| < N} \left| \frac{h_k}{e^{2\pi \sqrt{-1}(k, \omega^1(\eta, z_0))} - 1} e^{2\pi \sqrt{-1}(k, \xi)} \right| \]
\[
\frac{|T_N h|}{2\delta} \leq \frac{\sum_{0 < |k| < N} |k|^\tau}{e^{2\pi\delta|k|}} \\
= \frac{|T_N h|}{2\delta} \sum_{j=1}^{N-1} \frac{j^\tau + m_1 - 1}{e^{2\pi\delta j}} \\
\leq \frac{2^{m_1 - 1} |T_N h| (\tau + m_1 + 1)!}{\delta (2\pi\delta)^{\tau + m_1 + 1}} \sum_{j=1}^{N-1} \frac{1}{j^2} \\
\leq \frac{2^{m_1 - 1} |T_N h| (\tau + m_1 + 1)!}{\delta (2\pi\delta)^{\tau + m_1 + 1}} \sum_{j=1}^{N-1} \frac{1}{j^2} \\
\leq \left( \frac{2^{m_1 - 1} (\tau + m_1 + 1)!}{(2\pi\delta)^{\tau + m_1 + 1}} \sum_{j=1}^{N-1} \frac{1}{j^2} \right) \frac{|T_N h|}{\delta^{\tau + m_1 + 2}} \\
\leq \left( \frac{2^{m_1 - 1} (\tau + m_1 + 1)!}{(2\pi\delta)^{\tau + m_1 + 1}} \sum_{j=1}^{N-1} \frac{1}{j^2} \right) \frac{(|h| + |h'| + |R_N h|)}{\delta^{\tau + m_1 + 2}} \\
< \frac{c_0 \delta}{\delta^{\tau + L + 2}} \leq \delta^{7/8} < \delta,
\]

whereby we have made use of \(|h| < \epsilon, |h'| < \epsilon, |R_N h| < \epsilon\) in the domain \((\Sigma^1 - \delta) \times (\Sigma^2 - \delta) \times G\) by the choice of \(N\). According to Lemma 4.6, \(T_N h\) is real analytic, and the conjugate \(h_k = h_{-k}\). It follows that

\[
W(\xi, \eta, \zeta) = \sum_{0 < |k| < N} \frac{h_k}{|k|} \frac{|f_k| + M_{12} |W_k| + M_{11} |V_k|}{e^{2\pi\sqrt{-1}(k, \omega^1(\eta, \zeta))} - 1} e^{2\pi\sqrt{-1}(k, \xi)} = W(\xi, \eta, \zeta),
\]

which implies that \(W(\xi, \eta, \zeta)\) is real in \(T^{m_1} \times T^{m_2} \times O(D, \mathcal{L})\).

The way to solve the other two difference equations of (9) is the same as that of \(W\). It follows that, in the domain \((\Sigma^1 - \delta) \times (\Sigma^2 - \delta) \times G\), \(V\) satisfies

\[
|V| \leq \frac{c_0 |T_N g|}{\delta^{\tau + L + 2}} \leq \delta^{7/8} < \delta,
\]

and in the domain \((\Sigma^1 - 2\delta) \times (\Sigma^2 - 2\delta) \times G\), \(U\) satisfies

\[
|U| \leq \sum_{0 < |k| < N} \left| \frac{|f_k| + M_{12} |W_k| + M_{11} |V_k|}{e^{2\pi\sqrt{-1}(k, \omega^1(\eta, \zeta))} - 1} e^{2\pi\sqrt{-1}(k, \xi)} \right|
\leq \frac{c_0 |T_N f|}{\delta^{\tau + L + 2}} \left| + \frac{(M_{12} |T_N h| + M_{11} |T_N g|) e^2}{\delta^{2(\tau + L + 2)}} \right|
\leq \delta^{7/8} + (M_{12} + M_{11}) \delta^{2} e^{6/8} \\
< \frac{M \delta^{6/8}}{\delta^{5/8}} < \delta^{5/8} < \delta,
\]

where \(e^{1/8} < 1/M\) is guaranteed by (6) and

\[
V_k = \frac{g_k}{e^{2\pi\sqrt{-1}(k, \omega^1(\eta, \zeta))} - 1}.
\]

As Lemma 2.1 asserts, the inequalities (12), (13), (14) verify that \(|U| + |V| + |W| < 3\delta\), by which it also implies that \(U\) maps \(E^+\) into \(\mathcal{D}^{(3)}\). In fact, for \((\xi, \eta, \zeta) \in E^+\),

\[
|\Im x| \leq |\Im \xi| + |U| \leq \rho + \delta \leq \rho - 7\delta + \delta,
\]

\[
|\Im y| \leq |\Im \eta| + |V| \leq 8\delta + \delta,
\]

\[
|\Im z| \leq |\Im \zeta| + |W| \leq 3\delta + \delta.
\]
and it follows from $\delta e^{7/8} < (6\delta e^{3/8} - 12/5\delta e^{3/8})/4 < (\lambda - \lambda^+)/4$ that

$$|z - z_0| \leq |\zeta - z_0| + |W| \leq \lambda^+ + \delta e^{7/8} < \lambda^+ + (\lambda - \lambda^+)/4.$$ 

Analogously, one can check that $M$ maps $D^{(3)}$ into $D^{(2)}$. From the choice of $\epsilon_*$,

$$\epsilon < \min \left\{ \frac{1}{(24LM_{12})^{8/3}}, \left( \frac{c_0^{1/(\tau + L + 3)}}{20L} \right)^{8/7}, \frac{1}{2^{64(\tau + L + 3)}} \right\},$$

and $M_{11} = \frac{1}{mL}$, then $LM_{12}(6\delta e^{3/8}) < \delta/4$, $LM_{11}(8\delta^+ + \delta) < LM_{11}5\delta$, $\epsilon < \delta/4$. Therefore, for $y_0 \in \mathbb{T}^m$,

$$|\Im x_1| \leq |\Im x| + |\Im \omega^1(y_0, z_0)| + L \left| \frac{\partial \omega^1}{\partial z} \right| |z - z_0| + L \left| \frac{\partial \omega}{\partial y} \right| |\Im (y - y_0)| + |f|$$

$$\leq \rho - 6\delta + 0 + 6LM_{12}\delta e^{3/8} + LM_{11}(8\delta^+ + \delta) + \epsilon$$

$$< \rho - 5\delta,$$

and

$$|\Im y_1| \leq |\Im y| + |g|$$

$$\leq 8\delta^+ + (\delta - \delta^+)5/4 + \epsilon$$

$$< 8\delta^+ + 2\delta.$$ 

Implied by (6),

$$\epsilon < \left( \frac{c_0^{1/(\tau + L + 3)}}{20L} \right)^{8/7} < \left( \frac{c_0^{1/(\tau + L + 3)}}{2} \right)^2,$$

then $\epsilon^{1/2} < c_0^{1/(\tau + L + 3)}/2$ and

$$\epsilon < \frac{1}{2} \delta e^{3/8} < (6\delta e^{3/8} - 12\delta e^{3/8})/5 < (\lambda - \lambda^+)/4.$$

Consequently,

$$|z_1 - z_0| \leq |z - z_0| + |h|$$

$$< \lambda^+ + (\lambda - \lambda^+)/4 + \epsilon$$

$$< \lambda^+ + (\lambda - \lambda^+)/4 + (\lambda - \lambda^+)/4.$$

Finally, we prove that $U^{-1}$ takes $D^{(2)}$ into $D^{(1)}$. By the usual iteration scheme, one can construct a solution $(\xi, \eta, \zeta)$ of the equation (8) in $D^{(1)}$ for every $(x, y, z) \in D^{(2)}$. The first step is to construct the series

$$\begin{align*}
\xi_0 &= x, \quad \eta_0 = y, \quad \zeta_0 = z, \\
\xi_{k+1} &= x - U(\xi_k, \eta_k, \zeta_k), \\
\eta_{k+1} &= y - V(\xi_k, \eta_k, \zeta_k), \\
\zeta_{k+1} &= z - W(\xi_k, \eta_k, \zeta_k), \\
k &= 0, 1, \ldots.
\end{align*}$$
By the definition of the series (15) and mean value Theorem, we obtain
\[
\begin{align*}
|\xi_{k+1} - \xi_k| &\leq |U_\xi||\xi_k - \xi_k| + |U_\eta||\eta_k - \eta_k| + |U_\zeta||\zeta_k - \zeta_k|, \\
|\eta_{k+1} - \eta_k| &\leq |V_\xi||\xi_k - \xi_k| + |V_\eta||\eta_k - \eta_k| + |V_\zeta||\zeta_k - \zeta_k|, \\
|\zeta_{k+1} - \zeta_k| &\leq |W_\xi||\xi_k - \xi_k| + |W_\eta||\eta_k - \eta_k| + |W_\zeta||\zeta_k - \zeta_k|.
\end{align*}
\] (16)

In the domain \((\Sigma^1 - 3\delta) \times (\Sigma^2 - 3\delta) \times (G - \delta \epsilon^{3/8}) \supset \mathcal{D}^{(1)}\), it follows from Cauchy’s estimate that
\[
\begin{align*}
\left|\frac{\partial W}{\partial \xi}\right| &\leq \epsilon^{7/8}, & \left|\frac{\partial W}{\partial \eta}\right| &\leq \epsilon^{7/8}, & \left|\frac{\partial W}{\partial \zeta}\right| &\leq \epsilon^{1/2}, \\
\left|\frac{\partial V}{\partial \xi}\right| &\leq \epsilon^{7/8}, & \left|\frac{\partial V}{\partial \eta}\right| &\leq \epsilon^{7/8}, & \left|\frac{\partial V}{\partial \zeta}\right| &\leq \epsilon^{1/2}, \\
\left|\frac{\partial U}{\partial \xi}\right| &\leq M\epsilon^{6/8}, & \left|\frac{\partial U}{\partial \eta}\right| &\leq M\epsilon^{6/8}, & \left|\frac{\partial U}{\partial \zeta}\right| &\leq M\epsilon^{3/8}.
\end{align*}
\] (17)

Assume that \((\xi_k, \eta_k, \zeta_k) \in \mathcal{D}^{(1)}\), then from (16) and (17), we obtain
\[
\begin{align*}
|\xi_{k+1} - x| + |\eta_{k+1} - y| + |\zeta_{k+1} - z| \\
&\leq \sum_{j=0}^{k} (|\xi_{j+1} - \xi_j| + |\eta_{j+1} - \eta_j| + |\zeta_{j+1} - \zeta_j|) \\
&\leq \frac{1}{1 - \epsilon^{7/8}}(|U| + |V| + |W|) \\
&< 2\delta \epsilon^{7/8} + \delta \epsilon^{7/8} + \delta \epsilon^{5/8} \\
&< \delta \epsilon^{3/8} < \min\{\delta, \frac{\lambda - \lambda^+}{4}\}. 
\end{align*}
\] (18)

Based on (18), it is clearly that
\[
\begin{align*}
|\Im\xi_{k+1}| &< |\Im x| + \delta < \rho - 4\delta, \\
|\Im\eta_{k+1}| &< |\Im y| + \delta < 8\delta + 3\delta, \\
|\Im\zeta_{k+1} - \zeta_0| &< |z - z_0| + \frac{\lambda - \lambda^+}{4} < \lambda - \frac{\lambda - \lambda^+}{4},
\end{align*}
\]
which guarantees that \((\xi_{k+1}, \eta_{k+1}, \zeta_{k+1}) \in \mathcal{D}^{(1)}\). Therefore, the convergence of the series and the uniqueness of the solution are obvious. We denote the limits of \((\xi_{k+1}, \eta_{k+1}, \zeta_{k+1})\) by \((\xi, \eta, \zeta)\). Thus we find that there exists a unique solution \((\xi, \eta, \zeta) \in \mathcal{D}^{(1)}\) for every \((x, y, z) \in \mathcal{D}^{(2)}\).

After finding the transformation \(U_t\), we transform \(\mathfrak{M}^{g_0}_{i+1}\) by \(U_t\) to achieve the new mapping \(\mathfrak{M}^{g_0}_{i+1}\). The relation \(U_t\mathfrak{M}^{g_0}_{i+1} = \mathfrak{M}^{g_0}_{i+1}U_t\) means
\[
\begin{align*}
\xi + U + \omega^t(\eta + V, \zeta + W) + f_1 = \xi + \omega^1_+(\eta, \zeta) + f^+ + U(\xi_{g_0}, \eta_{g_0}, \zeta_{g_0}), \\
\eta + V + g_1 = \eta + g^+ + V(\xi_{g_0}, \eta_{g_0}, \zeta_{g_0}), \\
\zeta + W + h_1 = \zeta + h^+ + W(\xi_{g_0}, \eta_{g_0}, \zeta_{g_0}),
\end{align*}
\] (19)
where \( f_1 = f(\xi + U, \eta + V, \zeta + W), \) \( g_1 = g(\xi + U, \eta + V, \zeta + W) \) and \( h_1 = h(\xi + U, \eta + V, \zeta + W). \) Inserting \((9)\) into \((19)\), we have

\[
\begin{align*}
\left\{ 
\begin{array}{l}
f^+ = & \omega^1(\eta + V, \zeta + W) - \omega^1(\eta, \zeta) - \left( \frac{\partial \omega^1(\eta, \zeta)}{\partial \zeta}, W \right) - \left( \frac{\partial \omega^1(\eta, \zeta)}{\partial \eta}, V \right) \\
+ & U(\xi + \omega^1(\eta, z_0), \eta, \zeta) - U(\xi_0, \eta_0, \zeta_0) + f_1 - f + R_N f, \\
g^+ = & V(\xi + \omega^1(\eta, z_0), \eta, \zeta) - V(\xi_0, \eta_0, \zeta_0) + g_1 - g + R_N g + g^*, \\
h^+ = & W(\xi + \omega^1(\eta, z_0), \eta, \zeta) - W(\xi_0, \eta_0, \zeta_0) + h_1 - h + R_N h + h^*.
\end{array}
\right.
\end{align*}
\]

\( (20) \)

In order to obtain the estimate of the new perturbations, every component of the expressions \((20)\) need to be estimated.

The first part about \( U, V, W \) in the right hand side of \((20)\) can be estimated by mean value Theorem. In view of \((17)\), the contribution from the functions \( U, V \) can be bounded by the following inequalities

\[
\begin{align*}
|U(\xi + \omega^1(\eta, z_0), \eta, \zeta)| & \leq m_1 |U_\xi| |\frac{\partial \omega^1}{\partial \zeta}, \zeta - z_0| + m_1 |U_\xi| |f^*| + m_1 |U_\xi| |f^+| \\
& + m_2 |U_\eta||g^+| + n |U_\xi||h^+| \\
& \leq L^2 |U_\xi| |\frac{\partial \omega^1}{\partial \zeta}, \zeta - z_0| + L |U_\xi||f^*| + L |U_\xi||f^+| \\
& + L |U_\eta||g^+| + L |U_\xi||h^+| \\
& \leq 6L^2 M_1 M_2 \delta \epsilon^{9/8} + L M \epsilon^{3/2} + L M \epsilon^{3/8} (|f^+| + |g^+| + |h^+|),
\end{align*}
\]

\( (21) \)

and

\[
\begin{align*}
|V(\xi + \omega^1(\eta, z_0), \eta, \zeta)| & \leq \delta \epsilon^9/8 + L \epsilon^{3/2} + L \epsilon^{3/8} (|f^+| + |g^+| + |h^+|).
\end{align*}
\]

\( (22) \)

Similarly,

\[
|W(\xi + \omega^1(\eta, z_0), \eta, \zeta)| \leq 6L^2 M_1 M_2 \delta \epsilon^{9/8} + L \epsilon^{3/2} + L \epsilon^{3/8} (|f^+| + |g^+| + |h^+|).
\]

\( (23) \)

Again by Cauchy’s estimate, the derivatives of \( f \) in the domain \( E^+ \) satisfy

\[
|f_\xi| \leq \frac{\epsilon}{\delta}, \quad |f_\eta| \leq \frac{\epsilon}{\delta}, \quad |f_\zeta| \leq \frac{\epsilon}{\delta}.
\]

Coupled with mean value Theorem, we get

\[
\begin{align*}
|f(\xi + U, \eta + V, \zeta + W) - f(\xi, \eta, \zeta)| & \leq m_1 |f_\xi||U| + m_2 |f_\eta||V| + n |f_\zeta||W| \\
& \leq L \left( \frac{\epsilon}{\delta} \delta \epsilon^{5/8} + \frac{\epsilon}{\delta} \delta \epsilon^{7/8} + \frac{\epsilon}{\delta} \delta \epsilon^{7/8} \right) \\
& < 3L \epsilon^{3/2}.
\end{align*}
\]

\( (24) \)
By the same way,
\begin{align}
|g(\xi + U, \eta + V, \zeta + W) - g(\xi, \eta, \zeta)| < 3L^{3/2}, \\
|h(\xi + U, \eta + V, \zeta + W) - h(\xi, \eta, \zeta)| < 3L^{3/2}.
\end{align}
\hfill (25)

The contribution associated with frequencies of the new perturbations can be estimated by Lemma 4.5 and Lemma 4.2 in $E^+$, that is,
\begin{align}
\left| \omega^1(\eta + V, \zeta + W) - \omega^1(\eta, \zeta) - \left\langle \frac{\partial \omega^1(\eta, \zeta)}{\partial \zeta}, W \right\rangle - \left\langle \frac{\partial \omega^1(\eta, \zeta)}{\partial \eta}, V \right\rangle \right|
\leq \frac{L^2}{2} \left( \left| \frac{\partial^2 \omega^1(\eta, \zeta)}{\partial \zeta^2} \right| |W|^2 + \left| \frac{\partial^2 \omega^1(\eta, \zeta)}{\partial \eta^2} \right| |V|^2 + 2 \left| \frac{\partial^2 \omega^1(\eta, \zeta)}{\partial \zeta \partial \eta} \right| |W||V| \right)
\leq \frac{1}{2} L^2 (M_{122} \delta^2 e^{7/4} + M_{111} \delta^2 e^{7/4} + 2M_{121} \delta^2 e^{7/4}) < \epsilon^{5/4},
\end{align}
\hfill (26)

where the last inequality is ensured by $\epsilon < \left( \frac{2}{L^2 (M_{122} + M_{111} + 2M_{121})} \right)^2$.

Now we estimate $R_N f, R_N g, R_N h$. By Lemma 4.4, there exists an integer $N = \left[ \frac{L \ln L}{8} + \frac{L + 1}{2} \ln \frac{1}{\epsilon} + \frac{1}{12} \ln \frac{1}{\epsilon} \right] + 1$ such that in $E^+$,
\begin{align}
|R_N f| \leq \epsilon^{5/4}, \quad |R_N h| \leq \epsilon^{5/4}, \quad |R_N g| \leq \epsilon^{5/4}.
\end{align}
\hfill (27)

The novel step in this paper is the way to use the low dimensional intersection property. It can be concluded that the low dimensional intersection property holds at each step of the iteration. In fact, as the transformation $U_0$ is close to the identity, so if $T_0$ is an $m_1$-dimensional torus close to the invariant torus of the mapping $M_0^{\text{tr}},$ then $U_i T_0$ is also an $m_1$-dimensional torus close to the invariant torus of the mapping $M_0^{\text{tr}}.$ From the low dimensional intersection property of the mapping $M_0^{\text{tr}},$
\[M_{i+1}^{\text{tr}} T_0 \cap T_0 = U_i^{-1} M_0^{\text{tr}} U_i T_0 \cap T_0 = U_i^{-1} (M_0^{\text{tr}} U_i T_0 \cap U_i T_0) \neq \emptyset,
\]
therefore, the transformed map $M_{i+1}^{\text{tr}}$ retains the intersection property. By the property, for every $(\eta^0, \zeta^0) \in \mathbb{T}^{m_1}$ \times $\mathcal{O}(D, L),$ the $m_1$-dimensional torus $(\eta, \zeta) = (\eta^0, \zeta^0)$ has to intersect the image of the torus $(\eta, \zeta) = (\eta^0, \zeta^0)$ under $M_i^{\text{tr}}.$ In other words, there exists a $\xi^0$ for the torus $(\eta, \zeta) = (\eta^0, \zeta^0)$ such that both $g^+ (\xi^0, \eta^0, \zeta^0)$ and $h^+ (\xi^0, \eta^0, \zeta^0)$ vanish, which implies that
\begin{align}
\sup_{\xi} |g^+ (\xi, \eta^0, \zeta^0)| \leq \text{osc}_{\xi} \ g^+ (\xi, \eta^0, \zeta^0) \leq 2 \sup_{\xi} |g^+ (\xi, \eta^0, \zeta^0)| - g^+ (\eta^0, \zeta^0),
\sup_{\xi} |h^+ (\xi, \eta^0, \zeta^0)| \leq \text{osc}_{\xi} \ h^+ (\xi, \eta^0, \zeta^0) \leq 2 \sup_{\xi} |h^+ (\xi, \eta^0, \zeta^0)| - h^+ (\eta^0, \zeta^0),
\end{align}
\hfill (28)

where “osc” denotes the oscillation of some function.

From the previous estimates (21)-(27), we have the following estimates of the perturbations in $E^+$
\begin{align}
|f^+| \leq 6L^2 M_{12} \delta e^{9/8} + L M e^{3/2} + L M e^{3/8} (|f^+| + |g^+| + |h^+|)
+ 2\epsilon^{5/4} + 3L e^{3/2},
\end{align}
\[ g^+ - g^* \leq 6L^2M_{12}\delta\varepsilon^{9/8} + Le^{3/2} + Le^{3/8}(|f^+| + |g^+| + |h^+|) + \varepsilon^{5/4} + 3Le^{3/2}, \]
\[ |h^+ - h^*| \leq 6L^2M_{12}\delta\varepsilon^{9/8} + Le^{3/2} + Le^{3/8}(|f^+| + |g^+| + |h^+|) + \varepsilon^{5/4} + 3Le^{3/2}. \]

Then by the intersection property (28), it follows that
\[ |f^+| + |g^+| + |h^+| < |f^+| + 2|g^+ - g^*| + 2|h^+ - h^*| \leq (\overline{M} + 4)Le^{3/8}(|f^+| + |g^+| + |h^+|) + (\overline{M} + 4)4Le^{3/2} + 6\varepsilon^{5/4} + (\overline{M} + 4)6L^2M_{12}\delta\varepsilon^{9/8} \]
will be valid in \( E^+ \). Led by the choice of \( \varepsilon_* \),
\[ \varepsilon < \min \left\{ \frac{1}{(36(\overline{M} + 4)L^2M_{12})^8(\tau + L + 3)^4\varepsilon_0}, \frac{1}{((\overline{M} + 4)24L)^8/3} \right\}, \]
then
\[ (\overline{M} + 4)Le^{3/8} < 1/2, \]
and
\[ (\overline{M} + 4)6L^2M_{12}\delta + 6\varepsilon^{1/8} + (\overline{M} + 4)4Le^{3/8} < 1/2, \]
which yield the desired estimate
\[ |f^+| + |g^+| + |h^+| < \varepsilon^{9/8}. \]
This concludes the proof of the lemma.

4. **Technical lemmas.** In this section, we will give the lemmas used in the preceding sections. The notations in this section are the same as ones of the previous sections, if we do not redefine them.

**Lemma 4.1.** For the sequences \( \{\varepsilon_i\}, \{\delta_i\}, \{\rho_i\} \), the following inequalities hold:
(i) for \( k \in \mathbb{N} \)
\[ \sum_{i=0}^{k} \varepsilon_i < 2\varepsilon_0, \quad \sum_{i=0}^{k} \delta_i < 2\delta_0; \]
(ii) for \( i = 0, 1, \ldots, \rho_i > \rho_\infty > \frac{1}{2} \).

**Proof.** According to the choice of \( \varepsilon_\ast \), \( \varepsilon_0 < \frac{1}{24(\overline{M} + 4)L^2M_{12}} \) and \( \varepsilon_0^{1/8} < 1/2 \), the following inequalities are established
\[ \frac{\varepsilon_{i+1}}{\varepsilon_i} = \varepsilon_i^{1/8} < 1/2, \quad \frac{\delta_{i+1}}{\delta_i} = \varepsilon_i^{1/8(\tau + L + 3)} < 1/2, \]
\[ \sum_{i=0}^{k} \varepsilon_i < \sum_{i=0}^{k} \varepsilon_0^{1/2} < 2\varepsilon_0, \quad \sum_{i=0}^{k} \delta_i < 2\delta_0. \]
Note that \( \varepsilon_0 < (\frac{\rho_0}{28})^{8(\tau + L + 3)} \frac{1}{\varepsilon_0} \), therefore
\[ 14\delta_0 < \frac{\rho_0}{2} \quad \text{and} \quad \rho_i = \rho_0 - \sum_{j=0}^{i-1} 7\delta_j \geq \rho_0 - 14\delta_0 \geq \frac{1}{2}\rho_0, \]
which completes the proof. \( \square \)
Lemma 4.2. There exist positive constants \( M_{10}, M_{12}, M_{11}, M_{111}, M_{121}, \) and \( M_{122} \) such that in \( \Sigma_i^2 \times G_i \),

\[
|\omega_i^1(y, z)| < M_{10} - 1, \quad \left| \frac{\partial \omega_i^1(y, z)}{\partial z} \right| \leq M_{12}, \quad \left| \frac{\partial \omega_i^1(y, z)}{\partial y} \right| \leq M_{11},
\]

\[
\left| \frac{\partial^2 \omega_i^1(y, z)}{\partial y \partial z} \right| \leq M_{121}, \quad \left| \frac{\partial^2 \omega_i^1(y, z)}{\partial y^2} \right| \leq M_{111}, \quad \left| \frac{\partial^2 \omega_i^1(y, z)}{\partial z^2} \right| \leq M_{122}.
\]

Proof. By Cauchy’s estimate and

\[
\epsilon_0 < \epsilon_* \leq \min \left\{ \left( \frac{\epsilon_0^{1/(r+L+3)}}{20L} \right)^{8/7}, \frac{1}{264(r+L+3)} \right\},
\]

from (6), then we have

\[
|\omega_i^1| \leq |\omega^1|_G + \sum_{j=1}^{i-1} |f^j_*| \leq |\omega^1|_G + \sum_{j=0}^{i-1} \epsilon_j < |\omega^1|_G + 1 := M_{10} - 1,
\]

\[
\left| \frac{\partial \omega_i^1(y, z)}{\partial z} \right| \leq \left| \frac{\partial \omega_i^1(z)}{\partial z} \right|_G + \sum_{j=0}^{i-1} \left| \frac{\partial f^j_*}{\partial z} \right|_{\Sigma_i^2 \times G_i},
\]

\[
\leq \left| \frac{\partial \omega_i^1(z)}{\partial z} \right|_G + \sum_{j=0}^{i-1} \epsilon_j^{7/8} 
\]

\[
\leq \left| \frac{\partial \omega_i^1(z)}{\partial z} \right|_G + 2\epsilon_0^{-1/(r+L+3)} := M_{12},
\]

\[
\left| \frac{\partial \omega_i^1(y, z)}{\partial y} \right| \leq \left| \frac{\partial \omega_i^1(z)}{\partial y} \right|_G + \sum_{j=0}^{i-1} \left| \frac{\partial f^j_*}{\partial y} \right|_{\Sigma_i^2 \times G_i},
\]

\[
\leq \sum_{j=0}^{i-1} \epsilon_0^{-1/(r+L+3)} \epsilon_j^{7/8} < 2\epsilon_0^{-1/(r+L+3)} \epsilon_0^{7/8} < \frac{1}{10L} := M_{11},
\]

\[
\left| \frac{\partial^2 \omega_i^1(y, z)}{\partial y^2} \right| \leq \left| \frac{\partial^2 \omega_i^1(z)}{\partial y^2} \right|_G + \sum_{j=0}^{i-1} \left| \frac{\partial^2 f^j_*}{\partial y^2} \right|_{\Sigma_i^2 \times G_i},
\]

\[
\leq \sum_{j=0}^{i-1} \epsilon_j^{7/8} \leq 2\epsilon_0^{-2/(r+L+3)} := M_{111},
\]

\[
\left| \frac{\partial^2 \omega_i^1(y, z)}{\partial z^2} \right| \leq \left| \frac{\partial^2 \omega_i^1(z)}{\partial z^2} \right|_G + \sum_{j=0}^{i-1} \left| \frac{\partial^2 f^j_*}{\partial z^2} \right|_{\Sigma_i^2 \times G_i},
\]

\[
\leq \left| \frac{\partial^2 \omega_i^1(z)}{\partial z^2} \right|_G + \sum_{j=0}^{i-1} \epsilon_j^{7/8} \leq 2\epsilon_0^{-2/(r+L+3)} := M_{12},
\]
Lemma 4.3. [7] Let \( \omega(z) \) and \( \varphi(z) \) be analytic mappings defined on a connected bounded region \( \Omega \subset \mathbb{R}^n \), and satisfy
\[
\text{rank}\left\{ \frac{\partial^n \omega}{\partial z^\alpha} : \forall \alpha \in \mathbb{N}^n, 0 < |\alpha| \leq l \right\} \equiv m, \ l = \max\{m, n\},
\]
then there exists \( \epsilon_{00} > 0 \) such that when \( |\varphi| \leq \epsilon_{00} \) and \( \delta \) is sufficiently small, the set
\[
\Omega_\delta = \left\{ z \in \Omega : |(k, \omega(z) + \varphi(z)) + k_0| \geq \delta|k|^{-\gamma} \right\}
\]
is a nonempty Cantor set, and
\[
\text{meas}\left(\phi \setminus \Omega_\delta \right) \leq c_{00} \delta^1/\ell,
\]
where \( M \) is a positive constant and \( c_{00} \) is independent of \( \alpha \) and \( \delta \).

Lemma 4.4. [1] Assume \( f(x) \) is an analytic function in the domain \( \Sigma_\rho = \{ x \in \mathbb{C}^n : |3\pi| < \rho \} \). Let its Fourier expansion be
\[
f(x) = \sum_{k \in \mathbb{Z}^n} f_k e^{2\pi i (k,x)},
\]
and denote
\[
R_N f = \sum_{|k| \geq N} f_k e^{2\pi i (k,x)}.
\]
Then
(i) in the domain \( \Sigma_\rho \), \( |f_k| \leq |f| \Sigma_\rho e^{-2|k|/\rho} \);
(ii) in the domain \( \Sigma_{\rho-\delta} \), \( 0 < \delta < \rho < 1 \), \( |f(x)| \leq 4^n (\delta^2 \rho)^{-n} |f| \Sigma_\rho \);
(iii) if \( |f_k| \leq |f| \Sigma_\rho e^{-2|k|/\rho} \) and \( 2\delta < \gamma \), \( \delta + \gamma < \rho < 1 \), then in the domain \( \Sigma_{\rho-\delta-\gamma} \),
\[
|R_N f| < \frac{n^2}{2\delta^{n+1}} |f| \Sigma_\rho e^{-2\pi N\gamma}.
\]

Lemma 4.5. [1] If in the segment \( \overline{ab} \) of the space \( x = (x_1, \ldots, x_n) \), the vector-value function \( f = (f_1, \ldots, f_n) \) satisfies the inequality \( |df| \leq C|dx| \), then \( |f(b) - f(a)| \leq C |b - a| \). In particular,
\[
|f(b) - f(a)| \leq cn |b - a|
\]
if \( |\frac{\partial f}{\partial x_i}| \leq c \). If in the domain \( |x_i - a_i| \leq |b_i - a_i| \) the inequality
\[
\left| \frac{\partial^2 f}{\partial x_i \partial x_j} \right| \leq \Theta
\]
is everywhere valid, then
\[
|f(b) - f(a) - \left( \frac{\partial f}{\partial x_i} \bigg|_a , b - a \right) | \leq \frac{\Theta n^2}{2} |b - a|^2.
\]
Lemma 4.6. [7] Let $D \subset \mathbb{R}^n$ be a connected bounded region and $h(x, y, z)$ be an analytic mapping in $\{x \in \mathbb{C}^{m_1} : |x| < \rho \} \times \{y \in \mathbb{C}^{m_2} : |y| < \rho \} \times \{z \in \mathbb{C}^m : |z| < \rho \} = \tilde{D}$, which is continuous in the closure $\Sigma_1 \times \Sigma_2 \times \tilde{D}$. Moreover, $h(x, y, z)$ is real in $\mathbb{T}^{m_1} \times \mathbb{T}^{m_2} \times D$. Then in $\mathbb{T}^{m_2} \times D$, $T_N h(x, y, z)$ and $h^*(y, z)$ are both real analytic.

Acknowledgments. We would like to thank the referee for valuable comments. The work is supported by NSFC (11971059).

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Received January 2019; revised June 2019.

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