ON THE INVERSE STURM-LIOUVILLE PROBLEM

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ABSTRACT. We pose and solve an inverse problem of an algebro-geometric type for the classical Sturm-Liouville operator. We use techniques of nonautonomous dynamical systems together with methods of classical algebraic geometry.

1. Introduction. It is well-known that certain ordinary differential operators have the property that their isospectral classes are preserved by the solutions of a corresponding nonlinear evolution equation, and indeed by the solution of a whole commuting hierarchy of such equations. A prototypical and extensively studied example of this phenomenon is the following. Let \( u_t(x) = u(t, x) \) be the solution of the Korteweg-de Vries (K-dV) equation

\[
\frac{\partial u}{\partial t} = 3u \frac{\partial u}{\partial x} - \frac{1}{2} \frac{\partial^3 u}{\partial x^3},
\]

with appropriate initial condition \( u_0(x) = u(0, x) \); then the spectrum \( \Sigma_t \) of the Schrödinger operator

\[
L_t = -\frac{d^2}{dx^2} + u_t(x)
\]

acting in \( L^2(\mathbb{R}) \) does not depend on \( t \) \([14]\). This fact permits one to explicitly solve the K-dV equation for an ample class of initial data \( u_0 \), as a vast literature testifies. The K-dV equation is one of an infinite family of commuting nonlinear evolution equations whose solutions preserve isospectral classes of the Schrödinger operator.

As another example, we mention the (non focusing) nonlinear Schrödinger (NLS) equation:

\[
- \frac{d}{dt} + \frac{1}{2} \frac{\partial^2 u}{\partial x^2} = 2|u|^2 u,
\]

together with the AKNS operator

\[
L = J \left[ \frac{d}{dx} - \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \right], \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\]

acting on \( L^2(\mathbb{R}, \mathbb{C}^2) \). Here, \( a \) and \( b \) are real-valued functions of \( x \). For appropriate solutions \( u_t(x) = u(t, x) \) of the NLS equation, there is a transformation \( u_t \to (u_t, b_t) \)

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such that the spectrum of $L_t = J \left[ \frac{d}{dx} - \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \right]$ does not depend on $t$. See \cite{1} and much additional literature for more information.

Still another example is furnished by the Camassa-Holm equation

\[
4 \frac{\partial u}{\partial t} - \frac{\partial^3 u}{\partial x^2 \partial t} - 2u \frac{\partial^3 u}{\partial x^3} - 4 \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x^2} + 24u \frac{\partial u}{\partial x} = 0.
\]

In this case, the corresponding operator is of Sturm-Liouville type with a non-constant density function. Actually more than one operator can be used. Following \cite{5}, we choose \(-\frac{d^2}{dx^2} + 1\); introducing a positive density function $y$, the relevant spectral problem is

\[
-\frac{d^2 \varphi}{dx^2} + \varphi = \lambda y(x) \varphi. \tag{1}
\]

It turns out that, if $u_t(x) = u(t,x)$ is an appropriate solution of the Camassa-Holm equation, then there is a $t$-dependent density $y_t = y_t(x)$ such that the spectrum of (1) does not depend on $t$.

There are thus good reasons to study the isospectral classes of the above operators. Roughly speaking, this has been done in two cases. The first is that in which methods of inverse scattering theory are applicable. The second is that in which basic techniques of classical algebraic geometry can be effectively exploited.

The inverse scattering theory of the Schrödinger operator was used to solve the K-dV equation in \cite{14} and in many subsequent papers. Algebro-geometric methods were used for the same purpose in the ground-breaking paper \cite{8}.

In this paper, we will use methods of the theory of nonautonomous differential equations and of the classical theory of algebraic curves to formulate and solve an inverse problem for the general Sturm-Liouville problem

\[
- \left( p \varphi' \right)' + q \varphi = \lambda y \varphi, \tag{2}
\]

where $p > 0, y > 0$. This spectral problem includes \cite{11} as a special case. Also, one obtains the Schrödinger operator by setting $p = y = 1$. In this situation our results reduce to those found in \cite{8}.

Here is a somewhat more precise formulation of the problem we study. The differential equation (2) can be written in system form as

\[
\begin{pmatrix} \varphi' \\ q(x) - \lambda y(x) \end{pmatrix} = \begin{pmatrix} 0 & 1/p(x) \\ q(x) - \lambda y(x) & 0 \end{pmatrix} \begin{pmatrix} \varphi \\ \psi \end{pmatrix}.
\]

This system has, in the nonautonomous case, a maximal Lyapunov exponent $\beta = \beta(\lambda)$. We assume that:

- The spectrum $\Sigma$ of (2) is a finite union of intervals:
  
  $\Sigma = [\lambda_0, \lambda_1] \cup [\lambda_2, \lambda_3] \cup \ldots \cup [\lambda_{2g}, \infty)$, where $-\infty < \lambda_0 < \lambda_1 < \ldots < \lambda_{2g}$;
  
- The Lyapunov exponent $\beta$ vanishes on $\Sigma$: $\beta(\lambda) = 0$ for all $\lambda \in \Sigma$.

Here $p, q, y$ are continuous functions and $p, y$ are strictly positive. As we will see, these mild-looking assumptions are sufficient to permit the reconstruction of $p, q,$ and $y$ in terms of algebro-geometric data. In particular, we will see that $\sqrt{py}$ can be described as the restriction of a meromorphic function defined on a generalized
Jacobian of the Riemann surface $\mathcal{R}$ of the algebraic relation

$$w^2 = -\prod_{i=0}^{2g}(\lambda - \lambda_i)$$

to an $x$-motion which is in general nonlinear.

We follow the approach taken in [7] and [15], where analogous inverse problems were posed and solved for the AKNS operator and for the Verblunsky operator arising in the theory of orthogonal polynomials on the unit circle [35]. We note that an inverse spectral problem for an equation of Sturm-Liouville type similar to (1) was studied in [2], [3] and [16]. We offer a systematic study of the more general equation (2). Also, it is our opinion that the use of ideas and methods from the theory of nonautonomous differential equations enables one to obtain more information about the isospectral classes than is present in the cited papers.

The results discussed here form part of the Ph.D. thesis of one of us [38]. In that thesis one can also find a discussion of the solution of the Camassa-Holm equation when the initial data lie in an isospectral class of algebro-geometric type for (1).

2. Preliminaries. In this section, we present some basic material necessary for subsequent developments. We also formulate in a precise way the problem which we will study and solve in Sections 3 and 4.

Let $A$ be a compact metric space, and let \( \{\tau_x \mid x \in \mathbb{R}\} \) be a continuous one-parameter group of homeomorphisms of $A$. As usual, the pair $(A, \{\tau_x\})$ is called a (topological) flow. Let $\mu$ be a fixed $\{\tau_x\}$-ergodic measure on $A$. We suppose (unless stated otherwise) that $A$ is the topological support of $\mu$, or equivalently that $\mu(V) > 0$ for each open set $V \subset A$.

Suppose now that $p, q, y : A \to \mathbb{R}$ are continuous functions, and that $p$ and $y$ are strictly positive. For each $a \in A$, we introduce the matrix function

$$x \mapsto \begin{pmatrix} 0 & 1 \\
q(\tau_x(a)) - \lambda y(\tau_x(a)) & 0 \end{pmatrix}$$

which by abuse of notation we call $a(x)$. We study the family of differential equations

$$\begin{pmatrix} \varphi' \\
\psi' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\
q(\tau_x(a)) - \lambda y(\tau_x(a)) & 0 \end{pmatrix} \begin{pmatrix} \varphi \\
\psi \end{pmatrix} \quad (\text{SL}_a)$$

where $a$ ranges over $A$ and $\lambda$ is a complex parameter. If the function $p' : A \to \mathbb{R} : a \mapsto \frac{d}{dx}p(\tau_x(a))|_{x=0}$ is well defined and continuous, then $(\text{SL}_a)$ is equivalent to

$$-(p\varphi')' + q\varphi = \lambda y \varphi. \quad (2_a)$$

It is convenient to redefine the classical concepts of spectrum and resolvent of the family of spectral problems $(2_a)$ in terms of the idea of exponential dichotomy. We pause for the definition [8]. Let $\Phi_a(x)$ be the fundamental matrix solution of $(\text{SL}_a)$.

**Definition 2.1.** The family of equations $(\text{SL}_a)$ is said to have an exponential dichotomy over $A$ if there are positive constants $K, \gamma$ and a continuous function
\[ a \mapsto P_a : A \to \mathcal{P} = \{ \text{linear projections } P : \mathbb{C}^2 \to \mathbb{C}^2 \} \text{ such that the following estimates hold:} \]

\begin{enumerate}[(i)]
  \item \[ |\Phi_a(x)P_a \Phi_a^{-1}(s)| \leq Ke^{-\gamma(x-s)} \quad x \geq s \]
  \item \[ |\Phi_a(x)(I - P_a) \Phi_a^{-1}(s)| \leq Ke^{\gamma(x-s)} \quad x \leq s \]
\end{enumerate}

It follows from the fact that \( tr(a(x)) = 0 \) for all \( x \in \mathbb{R} \) that both the image \( \text{Im } P_a \) and the kernel \( \text{ker } P_a \) are one-dimensional, i.e., can be viewed as complex lines in \( \mathbb{C}^2 \). If \( p' \) is continuous, then the operator \( L_a \) defined by \( L_a \phi = -(p\phi)' + q\phi \) admits a self-adjoint closure in \( L^2(\mathbb{R}, \sqrt{g}) \). This operator has a spectrum \( \Sigma \subset \mathbb{R} \) which is bounded below. For the proof of the following result see [38]; methods of [15] are used.

**Proposition 2.2.** For \( \mu \)-a.a. \( a \in A \) the spectrum \( \Sigma_a \) equals a closed set \( \Sigma \subset \mathbb{R} \) which does not depend on \( a \). Moreover

\[ \mathbb{C} \setminus \Sigma = \{ \lambda \in \mathbb{C} \mid \text{equations (SL}_a) \text{ admit an exponential dichotomy over } A \}. \]

Thus \( \Sigma_a \) is the “complement of the dichotomy set” for \( \mu \)-a.a. \( a \in A \). We abuse language slightly and call \( \Sigma \) the “spectrum of the family \( (2 \alpha) \)”.

We next give a dichotomy-theoretic interpretation of the Weyl \( m \)-functions \( m_\pm(a, \lambda) \) of equations \( (2 \alpha) \) \( (a \in A) \). We also state a result of Kotani type [24] which will be important in subsequent developments.

To begin with, let \( a \in A \) and \( \lambda \in \mathbb{C} \setminus \Sigma \). Define \( m_+(a, \lambda) \) to be the unique extended complex number such that \( \text{Im } P_a = \text{span} \begin{pmatrix} 1 \\ m_+(a, \lambda) \end{pmatrix} \). Here \( m_+(a, \lambda) = \infty \) if and only if \( \text{Im } P_a = \text{span} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \). It can be shown that \( \exists m_+(a, \lambda) \exists \lambda > 0 \) whenever \( \exists \lambda \neq 0 \). In a similar way, \( \text{ker } P_a = \text{span} \begin{pmatrix} 1 \\ m_-(a, \lambda) \end{pmatrix} \) where \( \exists m_-(a, \lambda) \exists \lambda < 0 \) if \( \exists \lambda \neq 0 \). For each \( a \in A \), the functions \( \lambda \mapsto m_\pm(a, \lambda) \) are holomorphic on \( \mathbb{C} \setminus \mathbb{R} \) and meromorphic on \( \mathbb{C} \setminus \Sigma \); they coincide with the classical Weyl \( m \)-functions of problem \( (2 \alpha) \).

The result referred to above gives information concerning the behavior of the Weyl \( m \)-functions when \( \lambda \) passes through a non-degenerate interval \( I \subset \Sigma \). For each \( \lambda \in \mathbb{C} \), let \( \beta(\lambda) \) be the upper Lyapunov exponent of the family \( (\text{SL}_a) \) with respect to \( \mu \).

**Proposition 2.3.** Let \( I \subset \Sigma \) be an open interval, and suppose \( \beta(\lambda) = 0 \) for a.a. \( \lambda \in I \). Let \( a \in A \). Then both \( m_+ \) and \( m_- \) extend holomorphically through \( I \). Let \( h_+ \) (resp. \( h_- \)) be the extension of \( m_+ \) (resp. \( m_- \)) through \( I \) from the upper half-plane to the lower half-plane. Then

\[ h_+(\lambda) = \begin{cases} m_+(a, \lambda), & \exists \lambda > 0 \\ m_-(a, \lambda), & \exists \lambda < 0 \end{cases} \quad \text{and} \quad h_-(\lambda) = \begin{cases} m_-(a, \lambda), & \exists \lambda > 0 \\ m_+(a, \lambda), & \exists \lambda < 0. \end{cases} \]

A proof of Proposition 2.3 is given in [38]; it uses ideas from [24] and an elaboration of those ideas in [7]. Of course \( \Sigma \) may not contain a non-degenerate interval \( I \) and even if it does it need not to be the case that \( \beta = 0 \) a.e. on \( I \).

Let us now formulate the inverse spectral problem which will be solved in Sections 3 and 4. The point of departure is given by the following
Hypotheses 2.4. (H1) The spectrum $\Sigma$ of the family $(2_\alpha)$ is a finite union of intervals:

$$\Sigma = [\lambda_0, \lambda_1] \cup [\lambda_2, \lambda_3] \cup \ldots \cup [\lambda_{2g}, \infty), \ (0 < \lambda_0 < \lambda_1 < \ldots < \lambda_{2g} < \infty).$$

(H2) The Lyapunov exponent $\beta$ vanishes a.e. on $\Sigma$: $\beta(\lambda) = 0$ for a.a. $\lambda \in \Sigma$.

The problem we pose is to determine all ergodic flows $(A, \{\tau_x\}, \mu)$ such that (H1) and (H2) hold. In other words, we seek all “ergodic triples” $p, q, y$ for which (H1) and (H2) are valid. Sections 3 and 4 are devoted to the solution of this problem. We close the present section with some observations which follow from elementary facts about the Weyl $m$-functions together with Proposition 2.3. These observations will form the starting point of all the subsequent discussion.

We first state a general fact which is independent of Hypotheses (H1) and (H2). Let $I \subset \mathbb{R}$ be an open interval contained in $\mathbb{R} \setminus \Sigma$. Let $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm \infty\}$ be the extended real line with $+\infty$ and $-\infty$ identified to a point. Then for each $x \in A$, the function $\lambda \mapsto m_{+}(a, \lambda) : I \to \overline{\mathbb{R}}$ is strictly monotone increasing. If $\lambda \mapsto m_{+}(a, \lambda)$ has a pole $\lambda_*$ in $I$, then this monotonicity is to be interpreted in the sense that $\lim_{\lambda \to \lambda_*^-} m_{+}(a, \lambda) = \infty$ and $\lim_{\lambda \to \lambda_*^+} m_{+}(a, \lambda) = -\infty$. This follows from the relation

$$\exists m_{+}(a, \lambda) \exists \lambda > 0 \text{ for } \exists \lambda \neq 0 \text{ together with the fact that } m_{+}(a, \cdot) \text{ is meromorphic on } (\mathbb{C} \setminus \mathbb{R}) \cup I. \text{ Similarly, the function } \lambda \mapsto m_{-}(a, \lambda) \text{ is monotone decreasing on } I.$$ 

One can also check that a pole of $m_{+}(a, \cdot)$ (resp. $m_{-}(a, \cdot)$) in $I$ is simple. Clearly $m_{+}(a, \lambda) \neq m_{-}(a, \lambda)$ for all $\lambda \in I$.

Next, suppose that Hypotheses (H1) and (H2) are true. Let us analyze the behavior of the Weyl $m$-functions near a point $\lambda_* \in \{\lambda_0, \lambda_1, \ldots, \lambda_{2g}\}$. Let $a \in A$, and let $D_* \subset \mathbb{C}$ be a punctured disc centered at $\lambda_*$ such that $D_* \cap \mathbb{R}$ is the union of an interval $I_* \subset \mathbb{R} \setminus \Sigma$ and an interval contained in $\Sigma$. If $m_{+}(a, \cdot)$ is continued around a circle in $D_*$ centered in $\lambda_*$, it goes into $m_{-}(a, \cdot)$, and vice-versa. If we introduce the coordinate $z = \sqrt{\lambda - \lambda_*}$, then the inverse image of $D_*$ in the $z$-plane is a punctured disc $D_{\#}$ centered at $z = 0$, and $m_{\pm}(a, \cdot)$ are branches of a function $M_a(z)$ which is meromorphic in $D_{\#}$.

Now, the behavior of $m_{0}(a, \cdot)$ in $I$ has the consequence that $M_a$ omits infinitely many (real) values in $D_{\#}$ (if $I_{\#}$ does not coincide with a resolvent interval...). It follows from the Picard theorem that $M_a$ is meromorphic on the full disc $D_{\#} \cup \{0\}$.

Let us change the meaning of $z$, and write $z = \frac{1}{\sqrt{\lambda - \lambda_*}}$. We can then show that $m_{\pm}(a, \cdot)$ define a meromorphic function $M_a(z)$ on a disc centered at $z = 0$.

These things being said, let $\mathcal{R}$ be the Riemann surface of the algebraic relation

$$w^2 = -\prod_{i=0}^{2g} (\lambda - \lambda_i).$$

We view $\mathcal{R}$ as the union of two copies of the Riemann sphere $\mathbb{C}^2 \cup \{\infty\}$ cut open along $\Sigma$ which are glued together in the standard way. Let $\pi : \mathcal{R} \to \mathbb{C} \cup \{0\}$ be the projection of $\mathcal{R}$ to the Riemann sphere; then $\pi$ is 2-1 except at the branch points $\{\lambda_0, \lambda_1, \ldots, \lambda_{2g}, \infty\}$ where it is 1-1. We abuse notation and write $\lambda_0, \lambda_1, \ldots, \lambda_{2g}, \infty \in \mathcal{R}$.
Let \( \pi^{-1}(0) = \{0^+, 0^-, \} \). Define a meromorphic function \( k \) on \( \mathcal{R} \) by letting \( k(0^+) \) be the positive square root of \( \prod_{i=0}^{2g} \lambda_i \) and letting \( k(P) \) the appropriate square root of \(-\prod_{i=0}^{2g}(\pi(P) - \lambda_i)\) for each \( P \in \mathcal{R} \).

If \( a \in A \), we define a meromorphic function \( M_a \) on \( \mathcal{R} \setminus \{\lambda_0, \lambda_1, \ldots, \lambda_{2g}, \infty\} \) by setting \( M_a(0^+) = m_+(a, 0) \), then using the preceding remarks to extend \( M_a \). It follows from those remarks that \( M_a \) can actually be extended to all of \( \mathcal{R} \) in such a way as to define a meromorphic function. It is easy to show that, for each finite resolvent interval \( (\lambda_{2i-1}, \lambda_{2i}) \), the inverse image \( \pi^{-1}[\lambda_{2i-1}, \lambda_{2i}] \) of its closure contains exactly one pole \( P_i = P_i(a) \) of \( M_a \), and this pole is simple. Also, \( \infty \) is a simple pole of \( M_a \), and \( M_a \) has no poles than those mentioned. Each curve \( \pi^{-1}(\infty, \lambda_0], \pi^{-1}[\lambda_{2i-1}, \lambda_{2i}] \subset \mathcal{R} \) \((1 \leq i \leq g)\) contains exactly one simple zero of \( M_a \), and \( M_a \) has no zeros other than those mentioned. We refer to [38] for more details concerning the construction of \( M_a \) (see also [21] for an analogous discussion in the case of the Schrödinger operator), and finish this section with the following summary:

**Proposition 2.5.** Suppose that (H1) and (H2) hold. Then for all \( a \in A \), the function \( M_a \) defined above is meromorphic on \( \mathcal{R} \). Moreover all poles and zeroes of \( M_a \) are simple. The map \( A \times \mathcal{R} \to \mathbb{C} \cup \{\infty\} : (a, P) \mapsto M_a(P) \) is jointly continuous.

### 3. Reconstruction of \( p, q \) and \( y \)

Let \((A, \{\tau_x\}, \mu)\) be a topological flow together with a \( \{\tau_x\}\)-ergodic measure \( \mu \) on \( A \). We suppose that the topological support of \( \mu \) equals \( A \). Let \( p, q, \) and \( y : A \to \mathbb{R} \) be continuous functions such that \( p \) and \( y \) assume only positive values. We assume that the derivative \( a \mapsto \frac{d}{dx} p(\tau_x(a)) |_{x=0} \) is continuous. A word regarding the notation: sometimes it will be convenient to fix \( a \in A \) and consider \( p, q, y \) as functions of \( x \). In this case \( p(x) \) is identified with \( p(\tau_x(a)) \), and similarly for \( q, y \). Suppose that Hypotheses [24] hold for these data.

Our goal is to determine all such functions \( p, q, y \). Let \( a \in A, \lambda \in \mathbb{C} \setminus \Sigma \), and let \( m_{\pm}(x) = m_{\pm}(\tau_x(a), \lambda) \) be the corresponding \( m \)-functions. They satisfy the Riccati equation

\[
m' + \frac{1}{p}m^2 = q - \lambda y.
\]

It is understood that this equation holds for each \( x \in \mathbb{R} \).

We change the meaning of \( z \) again, and let \( z^2 = -\lambda \) be a parameter near \( \lambda = \infty \) on the Riemann surface \( \mathcal{R} \). We view points \( P \in \mathcal{R} \) near \( \infty \) as being parameterized by \( z \). Write \( M'_a(z) \) for the derivative of the function \( x \mapsto M_{\tau_x(a)}(z) \) in \( x = 0 \): thus \( M'_a(z) = \frac{d}{dx} M_{\tau_x(a)}(z) |_{x=0} \) for each \( a \in A \). Then

\[
M'_a(z) + \frac{1}{p}M_a^2(z) = q + z^2 y. \tag{3}
\]

We expand the function \( z \mapsto M_a(z) \) in a Laurent series near \( z = \infty \):

\[
M_a(z) = \alpha_1 z + \alpha_0 + \sum_{n=1}^{\infty} \alpha_{-n} z^{-n}.
\]
Using the Riccati equation (4) one finds
\[ a_0^2 = yp, \quad a_0 = -\frac{(py)'}{4y}, \quad a'_0 + \frac{a_0^2}{p} + \frac{2\alpha_1 \alpha_{-1}}{p} = q \] (4)

In particular, the above derivatives exist.

Let \( \sigma : \mathcal{R} \to \mathcal{R} \) be the involution \( (\lambda, w) \mapsto (\lambda, -w) \). It is sometimes convenient to think of \( M_a \) as “the meromorphic extension of \( m_+(a, \cdot) \) to \( \mathcal{R} \).” From this point of view, one has \( m_-(a, \cdot) = M_a \circ \sigma(\cdot) \). We then have

\[ M_a(z) = m_+(z) = \sqrt{pyz} - \frac{(py)'}{4y} + \sum_{n=1}^{\infty} \alpha_n z^{-n} \]
\[ M_a(z) \circ \sigma(z) = m_-(z) = -\sqrt{pyz} - \frac{(py)'}{4y} + \sum_{n=1}^{\infty} (-1)^n \alpha_n z^{-n} \] (5)

We adopt the convention that \( \sqrt{py} < 0 \) if \( g \) is even, while \( \sqrt{py} > 0 \) if \( g \) is odd. The derivatives in (4) and (5) are calculated at \( x = 0 \). Thus, for example, \((py)’\) means \( \frac{d}{dx}(py)(\tau_x(a)) |_{x=0} \). Let us now write \( P_i \) (\( 1 \leq i \leq g \)) for the finite poles of \( M_a \). Note that these poles \( P_i \) are functions of \( a \).

We know that \( \pi(P_i) \in [\lambda_{2i-1}, \lambda_{2i}] \) for each \( i \leq i \leq g \). Our aim in the next lines is to obtain an explicit formula for \( M_a \) in terms of the poles and of \( p, q, y \).

Let \( P \) denote a generic point of the Riemann surface \( \mathcal{R} \). Abbreviating \( m_+(a, P) \) to \( m_+(P) \), we first note that \( m_+ - m_- = M_a - M_a \circ \sigma \) is meromorphic on \( \mathcal{R} \) and has zeroes \( \{\lambda_0, \ldots, \lambda_{2g}\} \subset \mathcal{R} \). It has poles at the inverse images \( \pi^{-1}(\pi(P_i)) \) (\( 1 \leq i \leq g \)) and at \( \infty \). Taking account of (4), we have

\[ (m_+ - m_-)(P) = \frac{2\sqrt{pyk(P)}}{H(\lambda)} \] (6)

where we have written
\[ H(\lambda) = \prod_{i=1}^{g} |\lambda - \pi(P_i)|. \]

Recall that \( k(P) \) is defined by the conditions
\[ k(P) = \sqrt{-(\pi(P) - \lambda_0) \ldots (\pi(P) - \lambda_{2g})}, \quad k(0^+) > 0. \] (7)

Next, for each fixed \( a \in A \), the function \( m_+ + m_- \) is meromorphic on the Riemann sphere; i.e., it is a rational function. It has poles at \( \pi(P_1), \ldots, \pi(P_g) \) and moreover tends to \( 2\alpha_0 \) as \( \lambda \to \infty \). This implies that

\[ (m_+ + m_-)(P) = \frac{2Q(\lambda)}{H(\lambda)} \]

where \( \lambda = \pi(P) \) and \( Q(\lambda) = q_g \lambda^g + \cdots + q_1 \lambda + q_0 \) is a polynomial of degree \( g \) in \( \lambda \). It is of degree \( g \) whenever \( \alpha_0 \neq 0 \). Putting together all this information we obtain

\[ M_a(P) = m_+(a, P) = \frac{Q(\lambda) + \sqrt{pyk(P)}}{H(\lambda)} \]
\[ M_a \circ \sigma(P) = m_-(a, P) = \frac{Q(\lambda) - \sqrt{pyk(P)}}{H(\lambda)}. \]
Here \( M_a \) is viewed as a meromorphic function of the generic point \( P \in \mathcal{R} \), and the notation \( m_\pm(a, P) \) is self-explanatory. The polynomials \( Q \) and \( H \) are functions of \( \lambda = \pi(P) \), and so can also be viewed as functions on \( \mathcal{R} \), for each fixed \( a \in A \).

Let us now suppose for a moment that none of the poles are ramification points of \( \mathcal{R} \). Then \( m_-(P_i) \) is finite for each \( 1 \leq i \leq g \), and so we must have

\[
Q(P_i) = \sqrt{\rho y k}(P_i), \quad 1 \leq i \leq g. \tag{8}
\]

It can be shown that these formulas continue to hold even if one or more poles \( P_i \) are ramification points of \( \mathcal{R} \): in fact, if this were not true, such a \( P_i \) would be a non-simple pole of \( M_a \).

Equation (8) give \( g \) linear relations for the \( g + 1 \) coefficients of \( Q \). To get another relation, note that \( \lambda = 0 \) is to the left of \( \Sigma \). Hence \( m_-(a, 0) = m_0^± \) and \( m_+(a, 0) = m_0^± \) have real values. We therefore have

\[
q_0 = Q(0) = \frac{(-1)^g[m_0^± + m_0^±] \prod_{i=1}^g \pi(P_i)}{2}. \tag{9}
\]

Now we can determine the coefficients of \( Q \) from the following van der Monde system:

\[
\begin{pmatrix}
1 & 0 & \ldots & 0 \\
p_1 & \pi(P_1) & \ldots & \pi(P_1)^g \\
\vdots & \vdots & \ddots & \vdots \\
p_g & \pi(P_g) & \ldots & \pi(P_g)^g
\end{pmatrix}
\begin{pmatrix}
q_0 \\
q_1 \\
\vdots \\
q_g
\end{pmatrix} =
\begin{pmatrix}
Q(0) \\
\sqrt{\rho y k}(P_1) \\
\vdots \\
\sqrt{\rho y k}(P_g)
\end{pmatrix}.
\]

In particular, we can determine the highest-order coefficient \( q_g = \alpha_0 = -\frac{(\rho y)'}{4y} \). In fact, letting \( \Delta \) be the determinant of the matrix on the right-hand side of (9), and omitting \( \pi \), we obtain

\[
q_g \Delta = \det
\begin{pmatrix}
P_1 & P_1^2 & \ldots & P_1^{g-1} & \sqrt{\rho y k}(P_1) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
P_g & P_g^2 & \ldots & P_g^{g-1} & \sqrt{\rho y k}(P_g)
\end{pmatrix}

+ \left| (-1)^g q_0 \det
\begin{pmatrix}
P_1 & P_1^2 & \ldots & P_1^{g-1} \\
\vdots & \vdots & \ddots & \vdots \\
P_g & P_g^2 & \ldots & P_g^{g-1}
\end{pmatrix}
\right|.
\]

Working out the determinants, we get

\[
q_g = \frac{(-1)^g q_0}{\prod_{i=1}^g P_i} + \sum_{i=1}^g \frac{\sqrt{\rho y k}(P_i)}{P_i \prod_{r \neq i}(P_i - P_r)}. \tag{10}
\]

Next we obtain a formula for \( \alpha_{-1} \). The Laurent expansions (5) for \( m_± \) together with (6) give

\[
2\sqrt{\rho y z} + \frac{2\alpha_{-1}}{z} + 2 \sum_{k=1}^{\infty} \frac{\alpha - 2k - 1}{z^{2k+1}} = \frac{2\sqrt{\rho y k}(z)}{\prod_{i=1}^g (-z^2 - P_i)}
\]
Multiplying both sides of this relation by $\sqrt{p} \prod_{i=1}^{g} (z^2 + P_i)$ and comparing the coefficients of the $z^{2g}$-terms, we obtain

$$\alpha_{-1} = \sqrt{p} \left( \frac{1}{2} \sum_{i=0}^{2g} \lambda_i - \sum_{i=1}^{g} P_i \right).$$

Let us now fix $a \in A$. In what follows we will “follow the trajectory $\{\tau_x(a)\}$ through $a$”. Thus write $m_\pm(x, z) = m_\pm(\tau_x(a), z), p = p(x), q = q(x), y = y(x), P_1 = P_t(x)$ ($-\infty < x < \infty$), $Q = Q(\lambda, x), H = H(\lambda, x)$. Using the formula for $q$ in (4) we obtain

$$q(x) = y(x) \left( \sum_{i=0}^{2g} \lambda_i - 2 \sum_{i=1}^{g} P_i(x) \right) + q_p(x) + \frac{q_g(x)^2}{p(x)}, \quad (11)$$

Here $q_g$ is given by (10).

Now, consider

$$m'_+ = \left( \frac{\sqrt{p(x)y(x)} k(P) + Q(\lambda, x)}{H(\lambda, x)} \right), \quad \tau' = \frac{d}{dx}$$

From the Riccati type equation for $m_+$ we obtain

$$m'_+ = \left( \frac{\sqrt{p(x)y(x)} k(P) + Q(\lambda, x)}{H(\lambda, x)} \right)' = -\frac{1}{p} m_+^2 + q - \lambda y,$$

hence,

$$m'_+ = \left( \frac{\sqrt{p}y + Q}{H} - H' \frac{\sqrt{p}y + Q}{H^2} \right) = -\frac{1}{p} \left( \frac{\sqrt{p}y + Q}{H} \right)^2 + q - \lambda y,$$

and multiplying by $H^2$ we obtain

$$\left( \frac{\sqrt{p}y + Q}{H} \right)'H - \left( \frac{\sqrt{p}y + Q}{H} \right)'H' + \frac{1}{p} \left( \frac{\sqrt{p}y + Q}{H} \right)^2 = (q - \lambda y)H^2 \quad (12)$$

Let us write $k_r = k(P_r)$. If we compute (12) at a pole $P_r$ we obtain, since $H(P_r) = H_r = 0$ and $\sqrt{p}k_r(P_r) = \sqrt{p}k_r = Q(\pi(P_r))$, $\sqrt{p}k_r$,$$\left( \sqrt{p}k_r \right)'H_r + \frac{1}{p} \left( \sqrt{p}k_r \right)^2 = 0, \quad (13)$$

hence we obtain the following important relation

$$H'_r(x) \frac{1}{k_r(x)} = \frac{2}{p(x)} \sqrt{p(x)y(x)} \quad (14)$$

What we want to do is to express the above fraction (4) as a product of the poles $P_1(x), \ldots, P_g(x)$ of the function $M_{r_\lambda(a)}(P)$, and to do this we compute $m_-$ at $\lambda = 0$, and we have

$$m_-^0(x) = \frac{Q(0, x) - \sqrt{p}y}{(-1)^g \prod_{i=1}^{g} P_i(x)},$$

hence

$$\sqrt{p} = \frac{(-1)^{g+1} m_-^0(x) \prod_{i=1}^{g} P_i(x) + Q(0, x)}{k(0^+)}.$$
and (13) becomes
\[
\frac{H'_r}{k_r} = \frac{2(-1)^{g+1}m_0(x)\prod_{i=1}^{g} P_i(x) + 2Q(0,x)}{p(x)k(0^+)}
\]  (15)

Hence we have
\[
\frac{H'_r(x)}{k_r(x)} = \frac{(-1)^{g+1}[m_0(x) - m_0(t)]\prod_{i=1}^{g} P_i(x)}{p(x)k(0^+)}
\]  (16)

Finally
\[
\frac{H'_r(x)}{P_r(x)k_r(x)} = \frac{(-1)^{g+1}[m_0(x) - m_0(t)]\prod_{r \neq s} P_s(x)}{p(x)k(0^+)}
\]  (17)

We proved the following

**Theorem 3.1.** Suppose that Hypotheses [27] hold. Then the functions \( p, q \) and \( y \) satisfy the following relations involving the \( x \)-motion of the poles of the meromorphic function \( M_u(P) \) on \( \mathcal{R} \),

\[
\frac{2\sqrt{p(x)y(x)}}{p(x)} = \frac{H'_r(x)}{k_r(x)} = \frac{(-1)^{g+1}[m_0(x) - m_0(t)]\prod_{i=1}^{g} P_i(x)}{p(x)k(0^+)}
\]

and

\[
q(x) = y(x)\left(\sum_{i=0}^{2g} \lambda_i - 2\sum_{i=1}^{g} P_i(x)\right) + q'_y(x) + \frac{q^2_y(x)}{p(x)},
\]

where

\[
q'_y(x) = -\frac{(p(x)y(x))'}{4y(x)} = \frac{(-1)^{g}Q(0,x)}{\prod_{i=1}^{g} P_i(x)} + \sum_{j=1}^{g} \frac{\sqrt{p(x)y(x)}k(P_j(x))}{P_j(x)\prod_{r \neq j}(P_j(x) - P_r(x))}
\]

Now we study the nature of the \( x \)-dependence of \( p, q, y \). Clearly this issue is related to that of the pole motion \( x \mapsto \{P_1(x), \ldots, P_g(x)\} \). We will see that a good deal of information about the pole motion can be obtained by introducing certain “nonstandard” Abel-Jacobi coordinates. In what follows, we will use the language of classical algebraic geometry with only limited explanation. For more details concerning the concepts and results we use, the reader is referred to such standard texts as [33, 27]. For the theory of generalized Jacobian varieties, see [13].

Let us introduce the following differential forms on the Riemann surface \( \mathcal{R} \):

\[
\omega_{r-1} = \frac{\lambda^{r-2}d\lambda}{k(P)}, \quad \lambda = \pi(P), \quad 1 \leq r \leq g + 1.
\]

Note that \( \omega_1, \ldots, \omega_g \) are holomorphic differentials, while \( \omega_0 = \frac{d\lambda}{\lambda k(P)} \) is not holomorphic, but is a differential of the third kind with simple poles at \( 0^+ \) and \( 0^- \) with residues

\[
\frac{1}{k(0^\pm)} = \sum_{i=0}^{2g} \frac{\pm 1}{\prod_{i=0}^{g} \lambda_i}
\]

respectively.
Let $a \in A$, and let $P_1(x), \ldots, P_g(x)$ be the finite poles of $M_{r,(a)}$. Abusing notation slightly, write
\[
\omega_{r-1}(x) = \sum_{i=1}^{g} \int_{P_i}^{P_r} \omega_{r-1}, \quad 1 \leq r \leq g + 1.
\]
(18)

Here $P_r$ is a fixed point of $R$ different from $0^+$ and $0^-$. It will turn out that the choice of the paths of integration joining $P_s$ to $P_j(x)$ does not matter.

Differentiating, and simplifying the notation, we obtain
\[
\omega'_{r-1}(x) = \sum_{j=1}^{g} \frac{P_r - P_j}{k(P_j)} = \sum_{j=1}^{g} \frac{P_r - P_j}{P_j k(P_j)}.
\]

We have
\[
H'(P_r) = \left( \prod_{j=1}^{g} (\lambda - P_j) \right)'_{\lambda = P_r} = -P_r' \prod_{s \neq r} (P_r - P_s).
\]
(19)

Note that
\[
\prod_{s \neq r} (P_r - P_s) = \sum_{j=0}^{g-1} \theta_{g-j-1}(r) P_r^j
\]
(20)

where $\theta_{g-j-1}$ is $(-1)^{g-j-1}$ times the $(g-j-1)$-th elementary symmetric function in $(P_1, \ldots, P_r, \ldots, P_g)$.

If we let
\[
A = \begin{pmatrix} 1 & -\theta_1^{(1)} & \ldots & (-1)^{g-1} \theta_1^{(1)} & \ldots & \theta_1^{(g)} & \ldots & (-1)^{g-1} \theta_1^{(g)} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 1 & -\theta_1^{(g)} & \ldots & (-1)^{g-1} \theta_1^{(g)} & \ldots & \theta_1^{(1)} & \ddots & \ddots \end{pmatrix},
\]

then we can rewrite (20) as follows
\[
A \begin{pmatrix} \frac{P_r - P_1}{k(P_1) P_1'} \\ \frac{P_r - P_2}{k(P_2) P_2'} \\ \vdots \\ \frac{P_r - P_g}{k(P_g) P_g'} \end{pmatrix} = \text{diag} \begin{pmatrix} \frac{(-1)^g [m_0^0 - m_+^0] \prod_{r \neq s} P_s}{pk(0^+)} \end{pmatrix}_{r=1 \ldots g}.
\]

Multiplying both sides on the right by $\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$, we obtain
\[
A \begin{pmatrix} \omega_{r-1}' \\ \vdots \\ \omega_0' \end{pmatrix} = \begin{pmatrix} \frac{(-1)^g [m_0^0 - m_+^0] \theta_0^{(1)}}{pk(0^+)} \\ \vdots \\ \frac{(-1)^g [m_0^0 - m_+^0] \theta_0^{(g)}}{pk(0^+)} \end{pmatrix},
\]

where $\theta_0^{(r)} = \prod_{r \neq s} P_s$. Multiplying by $A^{-1}$ we have
curves satisfying the intersection conditions

\[
\begin{pmatrix}
\omega'_{g-1} \\
\vdots \\
\omega'_0
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
\dfrac{[m_0^0(x) - m_{g}^0(x)]}{p(x)k(0^+)}
\end{pmatrix},
\]

hence

\[
\omega'_0(x) = \dfrac{[m_0^0(x) - m_{g}^0(x)]}{p(x)k(0^+)}. \quad (21)
\]

This implies that

\[
\omega_{r-1}(x) = \sum_{j=1}^{g} \int_{P_j(x)}^{P_r(x)} \omega_{r-1} = \begin{cases}
\epsilon_{r-1}, & r = 2, \ldots, g \\
c_0 - \int_{0}^{x} \dfrac{[m_0^0(s) - m_{g}^0(s)]}{p(s)k(0^+)} ds, & r = 1
\end{cases}
\]

From (19) and (16), we get the following important formula describing the pole motion

\[
P'_r(x) = \dfrac{(-1)^gk(P_r(x))[m_0^0(x) - m_{g}^0(x)]\prod_{i=1}^{g} P_i(x)}{p(x)k(0^+)\prod_{s \neq r}(P_r(x) - P_s(x))}, \quad 1 \leq r \leq g \quad (23)
\]

Note that (23) tells that a pole \(P_j(x) \ (1 \leq j \leq g)\) has 0 derivative if and only if it attains one of the values \(\lambda_{2j-1}, \lambda_{2j}\) in the resolvent interval containing it: when it reaches such a value, then it jumps from a sheet to the other in the Riemann surface \(R\), moving towards the opposite direction.

We see that the quantities \(\omega_0(x), \ldots, \omega_{g-1}(x)\) are known once \(p(x) = p(\tau_x(a)), m_{g}^0(x)\), and \(m_0^0(x)\) are given, while \(\omega_g(x)\) is as yet undetermined. At this point we propose to study the pole motion by introducing an appropriate Abel map which sends the pole divisor \(\{P_1(x), \ldots, P_g(x)\}\) into a generalized Jacobian variety. It will turn out that the lack of an explicit formula for \(\omega_g(x)\) is not an accident; rather, this phenomenon is related to the fact the Abel map sends the pole divisor into a nonlinear subvariety of codimension 1 of the generalized Jacobian. Thus, roughly speaking, \(\omega_g(x)\) is determined implicitly by \(\omega_0(x), \ldots, \omega_{g-1}(x)\). For a study of a situation closely related to ours see [21, 33].

To discuss this subvariety, we need some material concerning the solution of the Jacobi inversion problem in the presence of a nonholomorphic differential. The reader is referred to [10] for an excellent exposition of the facts of which we make use.

Let \(\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g\) be a homology basis on \(R\) consisting of simple closed curves satisfying the intersection conditions

\[
\alpha_i \circ \alpha_j = 0, \quad \beta_i \circ \beta_j = 0, \quad 1 \leq i, j \leq g
\]

\[
\alpha_i \circ \beta_j = \delta_{ij} = \begin{cases}
1 & i = j \\
0 & i \neq j
\end{cases}, \quad 1 \leq i, j \leq g.
\]

We require that \(\alpha_i, \beta_i\) do not intersect sufficiently small discs centered at \(0^+\) and \(0^-\) \((1 \leq i \leq g)\). Further let \(\alpha_0\) be a sufficiently small simple closed curve centered
at \(0^+ \in \mathcal{R}\). That is, \(\alpha_0\) is required to bound a disc in \(\mathcal{R}\) centered at \(0^+\), and the support of \(\alpha_0\) is required to be disjoint from the supports of \(\alpha_i\) and \(\beta_i\) \((1 \leq i \leq g)\).

Let \(\Lambda_0 \subset \mathbb{C}^{g+1}\) be the \(Z\)-lattice spanned by all vectors of the form
\[
\int_c (\omega_0, \omega_1, \ldots, \omega_g), \quad c = \alpha_0, \alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g.
\]
Clearly \(\int_c \omega_i = 0\) \((1 \leq i \leq g)\) while \(\int_c \omega_0 = 2\pi i / k(0^+)\). It turns out that the rank of \(\Lambda_0\) is \(2g + 1\).

Define the generalized Jacobian
\[
J_0(\mathcal{R}) = \mathbb{C}^{g+1}/\Lambda_0.
\]

The topological structure of \(J_0(\mathcal{R})\) can be clarified as follows. Let \(\Lambda \subset \mathbb{C}^g\) be the \(Z\)-lattice spanned by all vectors of the form
\[
\int_c (\omega_1, \ldots, \omega_g), \quad c = \alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g.
\]

Then \(\Lambda\) has rank \(2g\). Define the “usual” Jacobian \(J(\mathcal{R})\) to be \(\mathbb{C}^g/\Lambda\); then \(J(\mathcal{R})\) is a torus of complex dimension \(g\). It can now be shown that \(J_0(\mathcal{R})\) is diffeomorphic to \(J(\mathcal{R}) \times \mathbb{C}^*\), where \(\mathbb{C}^* = \mathbb{C} \setminus \{0\}\).

Next let \(S_0^{g+1}\) be the space of unordered \(g + 1\)-tuples of points in \(\mathcal{R} \setminus \{0^+, 0^-\}\). Let \(P_0 \notin \{0^+, 0^-\}\) be a fixed base point in \(\mathcal{R}\). Introduce the Abel map
\[
I_0 : S_0^{g+1} \rightarrow J_0(\mathcal{R}) : I_0(P_0, \ldots, P_g) = \sum_{i=0}^g \int_{P_i} (\omega_0, \omega_1, \ldots, \omega_g),
\]
where the quantity on the right is interpreted \(\text{mod} \ \Lambda_0\). It can be shown that \(I_0\) is a birational isomorphism \([13]\).

We are interested in the restriction of the map \(I_0\) (which by abuse of terminology we again call \(I_0\)) to the set \(S_0^g\) of unordered \(g\)-tuples \(\{P_1, \ldots, P_g\}\) of points in \(\mathcal{R} \setminus \{0^+, 0^-\}\), where \(S_0^g\) is viewed as a subset of \(S_0^{g+1}\) by setting \(P_0 = P_1\). In particular, we wish to describe the image \(I_0(S_0^g) \subset J_0(\mathcal{R})\).

To do this, we introduce normalized differentials \(w_0, w_1, \ldots, w_g\) on \(\mathcal{R}\) with the following properties. First, \(w_1, \ldots, w_g\) are holomorphic differentials and \(\int_{P_i} w_j = \delta_{ij}\) \((1 \leq i, j \leq g)\). Second, \(w_0\) is a differential of the third kind with poles at \(0^+\) and respective residues \(\pm \frac{1}{2\pi i}\), and moreover \(\int_{P_i} w_0 = 0\) \((1 \leq i \leq g)\). We see that
\[
\begin{pmatrix}
\omega_0 \\
\omega_1 \\
\vdots \\
\omega_g
\end{pmatrix}
= C_0
\begin{pmatrix}
w_0 \\
w_1 \\
\vdots \\
w_g
\end{pmatrix}
\]
where \(C_0\) is an invertible \((g + 1) \times (g + 1)\) complex matrix of the form
\[
C_0 =
\begin{pmatrix}
c_{00} & c_{01} & \cdots & c_{0g} \\
0 & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & \cdots & 0
\end{pmatrix}
\]
where \(c_{00} \neq 0\) and \(C\) is invertible.
Next let $Z$ be the period matrix of $R$:

$$Z = \left( \int_{\beta_j} w_i \right)_{1 \leq i \leq q, 1 \leq j \leq g}. $$

It turns out that the complex $q \times q$ matrix $Z$ is symmetric and that the imaginary part $\Im Z$ is positive definite. Introduce the Riemann $\Theta$-function:

$$\Theta(s_1, \ldots, s_g) = \sum_{t \in \mathbb{Z}^g} \exp \{ \pi i (t, \mathbb{Z}l) + 2\pi i (t, s) \}$$

where $(s_1, \ldots, s_g) = s \in \mathbb{C}^g$. The function $\Theta$ is of basic importance in the study of the Jacobi inversion problem, etc.

We need the following generalized version of the Riemann $\Theta$-function:

$$\Theta_0(s_0, s_1, \ldots, s_g) = e^{s_0/2} \Theta(s + q/2) - e^{-s_0/2} \Theta(s + q/2)$$

where $q = \left( \int_{0^-} \omega_1, \ldots, \int_{0^-} \omega_g \right)$ and $s = (s_1, \ldots, s_g)$. A basic fact concerning this function is the following: there is vector $\Delta_0 \subset \mathbb{C}^{q+1}$ of “generalized Riemann constants” such that $\Theta_0(\tilde{s}) = 0$ if and only if

$$\Delta = \Theta_0 + \sum_{i=1}^q \int_{P_i}^1 (w_0, \ldots, w_g), \quad \tilde{s} = (s_0, \ldots, s_g) \in \mathbb{C}^{q+1}$$

for some choice of points $P_1, \ldots, P_g \in R \setminus \{0^+, 0^-\}$. It can be shown that $\Delta_0$ depends only on the choice of the base point $P_1$ and has the following form

$$\Delta_0 = \left( \Delta - \frac{1}{2} \int_{P_1}^0 (w_1, \ldots, w_g) - \frac{1}{2} \int_{P_1}^1 (w_1, \ldots, w_g): \sum_{i=1}^g \int_{0^+}^{P_i} (P_1 w_1) \right)_{\mathbb{C}^{q+1}}$$

where $\Delta$ is the vector of Riemann constants. Moreover it turns out that, if $\Theta_0(\tilde{s}) = 0$, then $\Theta_0(\tilde{s} + C_0^{-1} \lambda) = 0$ for all $\lambda \in \Delta_0$.

Define

$$\Upsilon = \left\{ \sum_{i=1}^g \int_{P_i}^1 (\omega_0, \omega_1, \ldots, \omega_g) \mid P_1, \ldots, P_g \in R \setminus \{0^+, 0^-\} \right\} \subset \mathbb{C}^{q+1}.$$ 

Then $\upsilon \in \Upsilon$ if and only if $\Theta_0(\Delta_0 - C_0^{-1} \upsilon) = 0$. The set $\Upsilon_0 = \{ C_0 \Delta_0 - \upsilon \mid \upsilon \in \Upsilon \}$ is, when viewed as a subset of $J_0(R) = \mathbb{C}^{q+1}/\Lambda_0$, the zero-locus of $\Theta_0$. We will study the pole motion \{ $P_1(x), \ldots, P_g(x) \} \mapsto I_0(P_1(x), \ldots, P_g(x))$ in the affine translate $\mathcal{Y}$ of $\Upsilon_0$, and also in $\Upsilon_0$ itself via the relation \{ $P_1(x), \ldots, P_g(x) \} \mapsto \Delta_0 - C_0^{-1}J_0(P_1(x), \ldots, P_g(x))$.

Let $c_i = \pi^{-1}[\lambda_{2i-1}, \lambda_{2i}] \subset R$ (1 ≤ $i$ ≤ $g$). Then each $c_i$ is a simple closed curve in $R$. The topological product $c_1 \times c_2 \times \cdots \times c_g$ is a real $g$-torus, which embeds into $S_0^g$ and hence into $\Upsilon_0$. If we choose $P_1 \in c_1, \ldots, P_g \in c_g$, then the correspondence

$$\{ P_1, \ldots, P_g \} \mapsto \sum_{i=1}^g \int_{P_i}^1 (\omega_0, \ldots, \omega_g) \text{ “unwinds” } c_1 \times c_2 \times \cdots \times c_g \text{ onto a curvilinear parallelogram in } \mathbb{C}^g.$$ 

The correspondence $\{ P_1, \ldots, P_g \} \mapsto \sum_{i=1}^g \int_{P_i}^1 \omega_g$ can be viewed
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as a function on this parallelogram. This gives a concrete way of viewing \( \omega_g(x) \) as a function of \( \omega_0(x), \ldots, \omega_{g-1}(x) \). See [20] for more information.

Now recall the expression for \( 2\sqrt{p}y \) as in [10], i.e.,

\[
2\sqrt{p(x)y(x)} = \frac{(-1)^{g+1}[m_0^0(x) - m_0^0(x)] \prod_{i=1}^{g} P_i(x)}{k(0^+)}.
\]

If we put

\[
\delta(x) = \frac{(-1)^{g+1}[m_0^0(x) - m_0^0(x)]}{2k(0^+)},
\]

then

\[
\sqrt{p(x)y(x)} = \delta(x) \prod_{i=1}^{g} P_i(x)
\]

(24)

Next, consider the function \( P \mapsto \pi(P) \); it is a meromorphic function having a double pole at \( \infty \) and zeros at \( 0^+ \) and \( 0^- \). Given any \( g + 1 \) points on \( \mathbb{R} \setminus \{0^+, 0^-, \} \), then the symmetric product \( \pi(P_0)\pi(P_1) \cdots \pi(P_g) \) can be expressed by means of the so-called \( \Theta \)-products (see [13, 34]), defining a meromorphic function on \( J_0(\mathbb{R}) \).

It follows that, if we put \( P_0 = P_* \), then (24) is the restriction of a well-defined meromorphic function defined on \( J_0(\mathbb{R}) \), to \( Y_0 \). Note that in general this restriction is no more a meromorphic function on \( Y_0 \).

**Theorem 3.2.** \( \sqrt{p(x)y(x)} \) coincides with the restriction of a meromorphic function, defined on \( J_0(\mathbb{R}) \), to \( Y_0 \).

In particular

\[
\sqrt{p(x)y(x)} = \frac{(-1)^{g+1}[m_0^0(x) - m_0^0(x)] \prod_{i=1}^{g} \pi(P_i(x))}{2k(0^+)} = \delta(x) \prod_{i=1}^{g} P_i(x)
\]

(25)

\[
= \delta(x) \gamma 
\]

\[
\Theta_0 \left( C_0^{-1}[I_0(P_1(x), \ldots, P_g(x))] - \int_{P_*}^{0^+} (w_0, w_1, \ldots, w_g) - \Delta_0 \right) 
\]

\[
\times \Theta^2_0 \left( C_0^{-1}[I_0(P_1(x), \ldots, P_g(x))] - \int_{P_*}^{\infty} (w_0, w_1, \ldots, w_g) - \Delta_0 \right) 
\]

\[
\times \Theta_0 \left( C_0^{-1}[I_0(P_1(x), \ldots, P_g(x))] + \int_{P_*}^{0^-} (w_0, w_1, \ldots, w_g) - \Delta_0 \right) 
\]

\[
\times \Theta^2_0 \left( C_0^{-1}[I_0(P_1(x), \ldots, P_g(x))] - \int_{P_*}^{\infty} (w_0, w_1, \ldots, w_g) - \Delta_0 \right).
\]

Here \( \gamma \) does not depend on the choice of \( P_j(x) \) (1 \( \leq \) \( j \) \( \leq \) \( g \)). The following implicit relation holds between \( (w_0(x), \ldots, w_g(x)) \):

\[
\Theta_0 \left( \Delta_0 - \sum_{i=1}^{g} \int_{P_*}^{P_i(x)} (w_0, \ldots, w_g) \right) = 0.
\]

We now discuss another approach to the study of the pole motion which is based on the Baker-Akhiezer function \( \varphi(x, P) \) of equation (2). Fix \( a \in A \). For each \( \lambda \in \mathbb{C} \) which is not a pole of \( m_+(0, \lambda) \), let \( \varphi_+(x, \lambda) \) be the solution of (2) which satisfies
\[ \varphi_+(0, \lambda) = 1, \quad \varphi'(0, \lambda) = \frac{m_+(0, \lambda)}{p(0)}. \]

Similarly, define \( \varphi_-(x, \lambda) \) for each \( \lambda \in \mathbb{C} \) which is not a pole of \( m_-(0, \lambda) \). It turns out that, for each \( x \in \mathbb{R} \), these two functions glue together to form a single meromorphic function \( \varphi(x, z) \) on the punctured Riemann surface \( \mathcal{R} \setminus \{ \infty \} \). This function has an essential singularity at \( \infty \in \mathcal{R} \). If \( z = \frac{1}{\sqrt{-\lambda}} \) is a local parameter at \( \infty \), then by (5):

\[ \varphi(x, z) = \exp \left( -\frac{1}{z} \int_0^x \sqrt{\frac{y(s)}{p(s)}} \, ds \right) f(x, z) \]

where \( f(x, \cdot) \) is holomorphic on some disc in the \( z \)-plane centered at \( z = 0 \).

We cannot apply the Abel Theorem to \( \varphi(x, \cdot) \) because it is not meromorphic at \( \infty \). However, following [8], we can analyze the differential form \( w = d \ln \varphi \). We make the following observations:

(a) \( w \) has a simple pole at \( P_i(0) \) with residue \(-1\) \( (1 \leq i \leq g) \).

(b) \( w \) has a simple pole at \( P_i(x) \) with residue \(1\) \( (1 \leq i \leq g) \).

(c) \( w \) has a pole of order 2 at \( \infty \), and

\[ w = \left( \int_0^x \sqrt{\frac{y(s)}{p(s)}} \, ds \right) \frac{dz}{z^2} + \ldots \]

near \( z = 0 \).

(d) \( \varphi \) is single-valued on \( \mathcal{R} \setminus \{ \infty \} \), and hence \( \int_{\alpha_j} w = 2\pi i m_j \) and \( \int_{\beta_j} w = 2\pi i n_j \)

where \( m_j, n_j \in \mathbb{Z} \) \( (1 \leq j \leq g) \).

We now follow the arguments of [8], pp.96-98. Let \( \eta \) be the differential of the second kind on \( \mathcal{R} \) which has a pole of order 2 at \( \infty \) and which is normalized so that \( \int_{\alpha_j} \eta = 0 \) \( (1 \leq j \leq g) \). Further, let \( w_1, \ldots, w_g \) be the holomorphic differentials introduced previously; thus \( \{w_1, \ldots, w_g\} \) is a basis for the space of holomorphic differentials on \( \mathcal{R} \), and \( \int_{\alpha_i} w_i = \delta_{ij} \) \( (1 \leq i, j \leq g) \). Let \( z_{jk} = \int_{\beta_j} w_k \) \( (1 \leq j, k \leq g) \) be the components of the period matrix \( Z \) of \( \mathcal{R} \).

Let \( w_{P_i(0), P_i(x)} \) be the differential of the third kind on \( \mathcal{R} \) with simple poles at \( P_i(0) \) and \( P_i(x) \), with residues \(-1\) and \(+1\) respectively \( (1 \leq i \leq g) \); we take these differentials to be normalized so that

\[ \int_{\alpha_j} w_{P_i(0), P_i(x)} = 0, \quad (1 \leq i, j \leq g) \]

By Riemann’s bilinear relation [27]:

\[ \int_{\beta_j} w_{P_i(0), P_i(x)} = \int_{P_i(0)}^{P_i(x)} w_j, \quad (1 \leq i, j \leq g). \]

The equality holds modulo a vector in the period lattice \( \hat{\Lambda} \subset \mathbb{C}^g \), which is the \( \mathbb{Z} \)-span of \( \{\underline{e}_1, \ldots, \underline{e}_g, Z\underline{e}_1, \ldots, Z\underline{e}_g\} \) in \( \mathbb{C}^g \).
Now, \( w = \int_0^x \sqrt{y(s)p(s)} \, ds \eta + \sum_{i=1}^g w_{p_i(0), P_i(x)} + \sum_{j=1}^g \gamma_j w_j \), where \( \gamma_1, \ldots, \gamma_g \) are complex constants. Since \( \int_{\alpha_j} w = 2\pi im_j \), we see that \( \gamma_j = 2\pi im_j \) (\( 1 \leq j \leq g \)). Since \( \int_{\beta_k} w = 2\pi in_k \), we see that \( 2\pi in_k = -\int_0^x \sqrt{y(s)p(s)} \, ds U_k + \sum_{i=1}^g \int_{P_i(0)}^{P_i(x)} w_k + \sum_{i=1}^g 2\pi im_j z_{jk} \)

where \( U_k = -\int_{\beta_k} \eta \) (\( 1 \leq k \leq g \)). We thus find

\[
\sum_{i=1}^g \int_{P_i(0)}^{P_i(x)} w_k = 2\pi i \left( n_k - \sum_{j=1}^g m_j z_{jk} \right) + \left( \int_0^x \sqrt{y(s)p(s)} \, ds \right) U_k, \quad (1 \leq k \leq g).
\]

Let us introduce the standard Abel map \( \tilde{I} : S^g \to \tilde{J}(\mathcal{R}) = \mathbb{C}^g/\tilde{\Lambda} \) given by

\[
\tilde{I}(P_1, \ldots, P_g) = \sum_{i=1}^g \int_{P_i} w_1, \ldots, w_g).
\]

We see that the pole motion \( x \to \{P_1(x), \ldots, P_g(x)\} \) takes a suggestive form under \( \tilde{I} \):

\[
\tilde{I}(P_1(x), \ldots, P_g(x)) = c + \left( \int_0^x \sqrt{y(s)p(s)} \, ds \right) U \quad \text{(26)}
\]

where \( c \) is a constant vector and \( U = (U_1, \ldots, U_g) \).

There is however an unpleasant feature in this formula. In fact, referring to (17) and Theorem 3.1, we see that the function \( \sqrt{y(x)p(x)} \) depends on the product \( \prod_{i=1}^g P_i(x) \). Thus (26) gives the pole motion implicitly. At this point one can follow the suggestion of Alber and Fedorov [2], and introduce a new \( x \)-variable \( \tilde{x} \) via to the relation

\[
\frac{d\tilde{x}}{dx} = \prod_{i=1}^g P_i(x).
\]

With respect to this new variable, the motion \( \tilde{x} \to \{P_1(\tilde{x}), \ldots, P_g(\tilde{x})\} \) defines a motion along a straight line in \( \tilde{J}(\mathcal{R}) \) whose velocity is given by

\[
\frac{(-1)^{g+1}[m_0^+ (\tilde{x}) - m_0^- (\tilde{x})]}{2p(0^+)}.
\]

The implicit relation seen earlier between \( \omega_g \) and \( (\omega_0, \ldots, \omega_{g-1}) \) is reflected here in the relation between \( x \) and \( \tilde{x} \).

We have carried out a more detailed study of the algebro-geometric coefficients \( p(x), q(x), y(x) \) in another place ([20]).

Theorem 3.1 ensures that, if Hypotheses 2.4 hold, then there exist relations of algebro-geometric type for the coefficients \( p(x), q(x), y(x) \) and for the Weyl \( m \)-functions \( m_\pm \) satisfying the Riccati equation (R). Going backwards, we seek to
determine if a fixed algebro-geometric configuration determines the coefficients of a
Sturm-Liouville operator, as well as its Weyl m-functions.

To this end, we fix the Riemann surface $\mathcal{R}$ of genus $g$ of the algebraic relation

$$w^2 = -\prod_{0 \leq i \leq 2g}(\lambda - \lambda_i)$$

where $\lambda_i \in \mathbb{R}^+$ ($0 \leq i \leq 2g$) and are all distinct. Let $\tilde{P}_1, \ldots, \tilde{P}_g$ be points on $\mathcal{R}$ such that $\lambda_{2i-1} \leq \tilde{P}_i \leq \lambda_{2i}$ ($1 \leq i \leq g$). Define

$$H_0(\lambda) = \prod_{i=1}^{g}(\lambda - \pi(\tilde{P}_i))$$

Let $\tilde{A}$ be a compact metric space, let $\{\tilde{x} | x \in \mathbb{R}\}$ be a continuous one-parameter group of homeomorphisms of $\tilde{A}$, and let $\tilde{\mu}$ be a $\{\tilde{x}\}$-ergodic measure on $\tilde{A}$ such that $\tilde{A}$ is the topological support of $\tilde{\mu}$. Let $p : \tilde{A} \to \mathbb{R}$ and $\mathcal{M}_1 : \tilde{A} \to \mathbb{R}$ be positive bounded continuous functions. Suppose further that the function $\tilde{a} \mapsto \mathcal{M}_1(\tilde{x}(\tilde{a}))|_{\tilde{x}=0} : \tilde{A} \to \mathbb{R}$ is defined and continuous. Fix $\tilde{a} \in \tilde{A}$, and write $p(x) = p(\tilde{x}(\tilde{a})), \mathcal{M}_1(x) = \mathcal{M}_1(\tilde{x}(\tilde{a})).$

Let $(\tilde{P}_1(x), \ldots, \tilde{P}_g(x))$ be the solution of the following system of first-order differential equations

$$-P_{\tau}(x) \prod_{s \neq \tau} (\tilde{P}_r(x) - \tilde{P}_s(x)) \prod_{i=1}^{g} P_i(x) \bigg/ k(\tilde{P}_r(x)) = \frac{(-1)^{g+1} \mathcal{M}_1(x) \prod_{i=1}^{g} P_i(x)}{p(x)k(0^+)} , \quad (1 \leq r \leq g) \quad (27)$$

with initial condition $P_r(0) = \tilde{P}_r$ for every $r = 1, \ldots, g$ (i.e. the analogue of (23)), and $k(P)$ is the meromorphic function of $\mathcal{R}$ defined in (17). Note that (27) implies that the $P_r$’s motion behaves as follows:

1. $P_r(x)$ attains the value 0 only at the points $x$ such that $P_r(x)$ is a ramification point;
2. when $P_r(x)$ reaches a ramification point, say at $x = \tilde{x}$, then $P_r(x)$ jumps from a sheet to the other in the Riemann surface $\mathcal{R}$, and begins to move towards the opposite ramification point in the resolvent interval containing it (this can be seen by introducing a parameter $z = \sqrt{\lambda - P_r(x)}$ in a neighborhood of the ramification point $P_r(x)$ of $\mathcal{R}$).

As before, let $c_i = \pi^{-1}[\lambda_{2i-1}, \lambda_{2i}] \subset \mathcal{R}$ ($1 \leq i \leq g$). The product $\tilde{D} = c_1 \times c_2 \times \cdots \times c_g$ is a real analytic $g$-torus which we view as an embedded submanifold of $\mathcal{S}_0^g$. The differential equations (27) define a one-parameter group of homeomorphisms $\{\tau_x\}$ on $\tilde{A} \times \tilde{D}$, as follows: if $\tilde{a} \in \tilde{A}$ and $\tilde{d} = (\tilde{P}_1, \ldots, \tilde{P}_g) \in \tilde{D}$, then

$$\tau_x(\tilde{a}, \tilde{d}) = (\tilde{x}(\tilde{a}), d(x)),$$

where $d(x) = \{P_1(x), \ldots, P_g(x)\}$ is obtained by solving (27) with initial value $\tilde{d}$.

Fix $\tilde{a} \in \tilde{A}, \tilde{d} \in \tilde{D}$. Define a (positive) function $y(x)$ such that

$$2\sqrt{p(x)y(x)} = \frac{(-1)^{g+1} \mathcal{M}_1(x) \prod_{i=1}^{g} P_i(x)}{k(0^+)} . \quad (28)$$

Put

$$H(x, \lambda) = \prod_{i=1}^{g}(\lambda - \pi(\tilde{P}_i(x))).$$
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Consider the non-standard Abel Jacobi coordinates [15]. We repeat the computations after [13] with \( \mathcal{M}_1(x) \) instead of \( m_0^0(x) - m_1^0(x) \). The implicit dependence of \( \omega_g(x) \) with respect to \( \omega_0(x), \ldots, \omega_{g-1}(x) \) is a general fact and does not depend on the particular choice of the surface and the points on it. Thus \( \omega_g(x) \) is a transcendental function of \( \omega_0(x), \omega_1(x), \ldots, \omega_{g-1}(x) \) such that

\[
v(x) = \sum_{i=1}^g \int_{P_i} \left( \omega_0, \omega_1, \ldots, \omega_g \right) \in \Upsilon,
\]

or equivalently, \( \Theta_0(\Delta_0 - C_0^{-1}v(x)) = 0 \).

Let \( D = I_0(\tilde{D}) \subset \Upsilon \). We can identify \( \tilde{A} \times \tilde{D} \) with the product space \( \tilde{A} \times D \subset \hat{A} \times \hat{Y} \). Let \( (\tilde{a}, \tilde{d}) \in \tilde{A} \times \tilde{D} \), then let

\[
\tilde{A} = \text{cls} \left\{ \left( \tilde{\tau}_x(\tilde{a}), I_0(0) + \sum_{i=1}^g \int_{P_i} \left( \omega_0, \omega_1, \ldots, \omega_g \right) \right) \mid x \in \mathbb{R} \right\},
\]

where \( I_0(0) = \sum_{i=1}^g \int_{P_i} \left( \omega_0, \omega_1, \ldots, \omega_g \right) \). The flow \( \tilde{A} \times \tilde{D}, \{ \tau_x \} \) induces a flow on \( \tilde{A} \) which by abuse of notation we denote by \( (A, \{ \tau_x \}) \). Let \( \tilde{a} \in \tilde{A} \) be a point such that the orbit \( \{ \tau_x(\tilde{a}) \mid \tilde{a} \in \tilde{A} \} \) is dense in \( \tilde{A} \) (this holds for \( \mu \)-a.a. \( \tilde{a} \in \tilde{A} \)), then let \( \tilde{\pi} : A \rightarrow \tilde{A} \) be the projection. There exists an ergodic measure \( \mu \) on \( A \) such that \( \tilde{\pi}(\mu) = \mu \); redefine \( A \) to be the topological support of \( \mu \).

Fix \( a = (\tilde{a}, \tilde{d}) \in A \). Put

\[
k_r(x) = k(P_r(x)),
\]

and define a polynomial \( Q(x, \lambda) \) of degree \( g \) in \( \lambda \) and with coefficients depending on \( x \), such that

\[
Q_i(x) = Q(x, \pi(P_i(x))) = \sqrt{p(x)y(x)}k_r(x), \quad (i = 1, \ldots, g). \tag{29}
\]

We also require that the highest term coefficient \( q_g \) is

\[
q_g(x) = -\frac{(p(x)y(x))'}{4y(x)} \tag{30}
\]

Define

\[
M(d(x), P) = \frac{\sqrt{p(x)y(x)}k(P) + Q(x, \lambda)}{H(x, \lambda)}, \quad \lambda = \pi(P). \tag{31}
\]

It’s clear that

\[
M(d(x), \sigma(P)) = \frac{Q(x, \lambda) - \sqrt{p(x)y(x)}k(P)}{H(x, \lambda)},
\]

where \( \sigma \) is the hyperelliptic involution.

Define

\[
h_+(x, P) = M(d(x), P) \quad \text{and} \quad h_-(x, P) = M(d(x), \sigma(P)), \tag{32}\]

and note that

\[
h_+(x, 0) - h_-(x, 0) = -M_1(x). \tag{33}\]

Put

\[
M_2(x) = h_+(x, 0) + h_-(x, 0) = \frac{2Q(x, 0)}{H(x, 0)} \tag{34}\]
Finally, let
\[ q(x) = y(x) \left( \sum_{i=0}^{2g} \lambda_i - 2 \sum_{i=1}^{g} P_i(x) \right) + \frac{q_2(x)}{p(x)} + q'(x). \] (35)

Corresponding to every \( a \in A \subset \mathbb{A} \times \mathbb{Y} \), there is a one parameter family of matrices
\[
\begin{pmatrix}
0 & 1/p(x) \\
q(x) - \lambda y(x) & 0
\end{pmatrix},
\]
where \( y \) and \( q \) are defined as in (28) and (29) respectively.

We are ready to prove the following important result

**Theorem 3.3.** Let \( a \in A, \lambda = \pi^{-1}(P) \in \mathbb{C} \) with \( \Im \lambda \neq 0 \) and let \( p(x), y(x) \) and \( q(x) \) be continuous functions such that the relations (28) and (35) hold. Then \( h_\pm(x, P) \) satisfy the Riccati equation
\[ h_\pm' + \frac{1}{p} h_\pm^2 = q - \lambda y. \] (36)

**Proof.** We will give the proof only for \( h_+ \). We want to prove that, differentiating (28) with respect to \( x \), we obtain
\[
((\sqrt{py})k + Q'H - (\sqrt{py}k + Q)H' + \frac{1}{p}(\sqrt{py}k + Q)^2 - (q - \lambda y)H^2 = 0. \] (37)

We rearrange (37) in the following way
\[
Q'H - QH' + yk^2 + \frac{1}{p}Q^2 - (q - \lambda y)H^2 = k \left( (\sqrt{py})H' - (\sqrt{py})H - 2 \sqrt{py}Q \right). \] (38)

We claim that
\[
F(x, \lambda) = \sqrt{p(x)y(x)}H'(x, \lambda) - (\sqrt{p(x)y(x)})'H(x, \lambda) - \frac{2}{p(x)} \sqrt{p(x)y(x)}Q(x, \lambda) = 0 \] (39)

Such a \( F \) is a polynomial in \( \lambda \) of degree \( g \). It’s easy to observe that \( F(x, \lambda) \) has the points \( \pi(P_i(x)) \) as roots for every \( i = 1, ..., g \), hence it has \( g \) roots. Moreover the \( \lambda^g \) coefficient of \( F(x, \lambda) \) is
\[-(\sqrt{p(x)y(x)})' + \frac{2}{p(x)} \sqrt{p(x)y(x)} \frac{(p(x)y(x))'}{4y(x)} = 0, \]
hence \( F(x, \lambda) = 0 \). To conclude the proof, we examine the remaining term in (38), i.e.
\[
L(x, \lambda) = Q'(x, \lambda)H(x, \lambda) - Q(x, \lambda)H'(x, \lambda) + y(x)k^2(P) \\
+ \frac{1}{p(x)}Q^2(x, \lambda) - (q(x) - \lambda y(x))H^2(x, \lambda), \quad \pi(P) = \lambda. \] (40)

\( L(t, \lambda) \) is polynomial in \( \lambda \) of degree at most \( 2g + 1 \). It is easy to observe that for every \( i = 1, ..., g \), \( \pi(P_i(x)) \) is a root of \( L(x, \lambda) \). As before we compute the coefficient \( \alpha \) of \( \lambda^{2g+1} \) and \( \beta \) of \( \lambda^{2g} \): we have
\[
\alpha = y - y = 0 \\
\beta = y(x) \left( \sum \lambda_i - 2 \sum P_i(x) \right) + \frac{1}{p(x)} q_2^2(x) + q'_g(x) - q(x) = 0,
\]

\[ q(x) = y(x) \left( \sum \lambda_i - 2 \sum P_i(x) \right) + \frac{q_2(x)}{p(x)} + q'(x). \] (35)
hence \( L(x, \lambda) \) has degree at most \( 2g - 1 \). If we show that every \( \pi(P_i(x)) \) ia a double root of \( L(x, \lambda) \), then the proof is complete.

We observe that \( H^2(x, \lambda) \) certainly has all the points \( \pi(P_i(x)) \) as double zeros, so we consider only
\[
V(x, \lambda) = Q'(x, \lambda)H(x, \lambda) - Q(x, \lambda)H'(x, \lambda) + y(x)k^2(P) + \frac{1}{p(x)}Q^2(x, \lambda) = H(x, \lambda)N(x, \lambda), \quad \pi(P) = \lambda
\]  
(41)

where
\[
N(x, \lambda) = Q'(x, \lambda) - Q(x, \lambda)\frac{H'(x, \lambda)}{H(x, \lambda)} + y(x)k^2(P)\frac{H^2(P)}{p(x)H(x, \lambda)} + \frac{Q^2(x, \lambda)}{p(x)H(x, \lambda)}
\]  
(42)

In view of the fact that \( F(x, \lambda) = 0 \), we can write
\[
H'(x, \lambda) = \frac{(p(x)y(x))'}{2p(x)y(x)}H(x, \lambda) + \frac{2}{p(x)}Q(x, \lambda)
\]  
(43)

and substituting this expression in (41) we have
\[
V(x, \lambda) = Q'(x, \lambda)H(x, \lambda) - Q(x, \lambda)\left(\frac{(p(x)y(x))'}{2p(x)y(x)}H(x, \lambda) + \frac{1}{p(x)}Q(x, \lambda)\right)' + y(x)k^2(P).
\]  
(44)

Differentiating \( V \) with respect to \( x \) we have
\[
V'(x, \lambda) = Q''(x, \lambda)H(x, \lambda) + Q'(x, \lambda)H'(x, \lambda) - Q'(x, \lambda)\left(\frac{(p(x)y(x))'}{2p(x)y(x)}H(x, \lambda) + \frac{1}{p(x)}Q(x, \lambda)\right)'
\]  
(45)

and
\[
V'(x, P_r) = Q'_r(x)H'_r(x) - \frac{2}{p(x)}Q_r(x)Q'_r(x) = \frac{(p(x)y(x))'}{2p(x)y(x)}H'_r(x)Q_r(x) \quad \text{and}
\]  
\[
+ \frac{p'}{p^2(x)}Q^2_r(x) + y'(x)k^2_r(x),
\]  
(46)

since \( H_r(x) = H(x, \pi(P_r(x))) = 0 \).

Substituting in the formulas for \( H'_r \) and \( Q_r \) obtained from (27), (28) and (29), we see that \( V'(x, P_r) = 0 \), and we have proved that \( H \) divides \( V' = HN' + H'N \), and so \( H \) divides \( H'N \). Now, \( H'(x, P_r) = -P'_r(x) \prod_{s \neq r}(P_r(x) - P_s(x)) \), and \( H'_r(x) = 0 \) if and only if \( P'_r(x) = 0 \). If \( P'_r \neq 0 \) for every \( r \) then \( H \) divides \( N \) too, and the proof is complete.

Now, suppose that there is an index \( j \) and \( i \) such that \( P'_j(x) = 0 \). In this case we must have necessarily \( \sigma(P_j(x)) = P_j(x) \). In fact, if \( \sigma(P_j(x)) \neq P_j(x) \) for every \( r \), then we see that \( P'_j(x) \neq 0 \), since the (restricted) Abel map \( I_0 \) is a diffeomorphism of \( S_0(R) \) onto \( \Upsilon \) and \( \pi \) is invertible in a neighborhood of each \( P_r \). If then \( \sigma(P_j(x)) = P_j(x) \) for some \( j \in \{1, \ldots, g\} \) and \( \tilde{x} \), we may approximate the divisor \( d(\tilde{x}) \) with an arbitrarily near divisor \( d(x) = P_{1,\varepsilon}(\tilde{x}) + \ldots + P_{g,\varepsilon}(\tilde{x}) \) such that
σ(P_{r_x}(x)) \neq P_{r_x}(x)$ for every $r$. For this divisor, the conclusion of the proof above holds, and, passing to the limit, the same conclusion holds for $d(\hat{x})$. The proof is complete.

We thank the referee for the following alternative proof of the fact that $V(x, \lambda)$

It is enough to prove that $V_\lambda(x, \pi(P_{x}(x))) = 0$, since in our proof we saw that $V(x, \pi(P_{x}(x))) = 0$ (here $V_\lambda$ denotes the derivative of $V$ with respect to $\lambda$). To do this, we differentiate the map $x \mapsto V(x, \pi(P_{x}(x)))$ with respect to $x$, obtaining

\[
V'(x, \pi(P_{x}(x))) + V_\lambda(x, \pi(P_{x}(x)))\dot{\pi}'(P_{x}(x))P_{x}'(x) = 0.
\]

In our proof we showed that $V'(x, \pi(P_{x}(x))) = 0$, hence, since $\dot{\pi}'(P_{x}(x)) \neq 0$, we have either $V_\lambda(x, \pi(P_{x}(x))) = 0$ or $P_{x}'(x) = 0$. If at a point $\hat{x}$ we have $P_{\hat{x}}'(\hat{x}) = 0$, then from Theorem 3.4.

The ergodic dynamical system $(A, \{\tau_x\}, \mu)$ has the property that the spectrum of the family

\[
\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} 0 & 1/p \\ q - \lambda y & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = a(x, \lambda) \begin{pmatrix} u \\ v \end{pmatrix} \quad (a \in A)
\]

of differential equations does not depend on the choice of $a \in A$ and has the form

\[
\Sigma = [\lambda_0, \lambda_1] \cup [\lambda_2, \lambda_3] \cup \cdots \cup [\lambda_{2g}, \infty).
\]

Moreover, $\beta(\lambda) = 0$, for Lebesgue a.e. $\lambda \in \Sigma$ (and surely for every $a \in A$).

Proof. For $\lambda \in \mathbb{C}$, define the flow $\{\tau_x\}$ as the flow on $A \times \mathbb{C}^2$ obtained by

\[
\tau_x(a, u) = (\tau_x(a), \Phi_\lambda(x)u),
\]

where $\Phi_\lambda(x)$ is the fundamental matrix for the equation (SL_{a,\lambda}), i.e. the matrix satisfying

\[
\Phi_\lambda'(x, \lambda) = a(x, \lambda)\Phi_\lambda(x, \lambda)
\]

This flow is clearly induced on $\Omega \times \mathbb{P}(\mathbb{C})$ too.

Given $\lambda \in \mathbb{C}$ with $\mathfrak{R} \lambda \neq 0$, let $P \in \pi^{-1}(\lambda)$ such that $\Im \lambda \mathfrak{R}k(P) > 0$; since $a = (\hat{a}, d) \in \hat{A} \times \mathbb{T}$, define $m_+(\hat{a}, d, \lambda) = M(d, P)$ and $m_-(\hat{a}, d, \lambda) = M(d, \sigma(P))$.

By Theorem 3.3 both $m_+$ and $m_-$ satisfy the Riccati equation (R), hence the sections

\[
S^\pm(\lambda) = \{(\lambda, m_{\pm}(\hat{a}, d, \lambda)) \mid (\hat{a}, d) \in A\} \subset \Omega \times \mathbb{P}(\mathbb{C})
\]

are invariant under the flow $\hat{\tau}$ for every $\lambda$ with $\mathfrak{R} \lambda \neq 0$. By continuity, we can extend $S^\pm(\lambda)$ to the real line, keeping them still invariant, so we can define $m_{\pm}$ for every $\lambda$ with $\mathfrak{R} \lambda = 0$.

Now, take $\lambda \in (\lambda_0, \lambda_1) \cup \cdots \cup (\lambda_{2g}, \infty)$; then it easy to observe that $\Im m_+(\hat{a}, d, \lambda) > 0 > \Im m_-(\hat{a}, d, \lambda)$, so $S^+(\lambda)$ and $S^-(\lambda)$ are distinct. Clearly $A \times \mathbb{P}(\mathbb{R})$ is invariant under the flow $\hat{\tau}$. It can now be proved that every solution of $(\text{SL})_{a,\lambda}$ is bounded for every $a \in A$. Thus $\beta(\lambda) = 0$ and $\lambda$ belongs to the spectrum of the full-line operator $(\text{SL})_{a,\lambda}$. On the other hand, if $\lambda \notin \Sigma$, then the sections $S^\pm(\lambda)$ are real, hence the rotation number $\alpha(\lambda)$ of $\hat{\tau}$ is constant. Following [15, 38], the only
possibility is that \( \lambda \) belongs to the resolvent set (in fact equations \((\text{SL}_a, \lambda)\) admit an exponential dichotomy). For a more detailed discussion about the rotation number and its properties, see [15, 19, 38].

Thus we proved that the spectrum has the form \( \Sigma = [\lambda_0, \lambda_1] \cup \ldots \cup [\lambda_{2g}, \infty) \) for every \( a \in A \), and that \( \beta(\lambda) = 0 \) for Lebesgue a.e. \( \lambda \in \Sigma \).

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