Can slow roll inflation induce relevant helical magnetic fields?

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Abstract. We study the generation of helical magnetic fields during single field inflation induced by an axial coupling of the electromagnetic field to the inflaton. During slow roll inflation, we find that such a coupling always leads to a blue spectrum with $B^2(k) \propto k$, as long as the theory is treated perturbatively. The magnetic energy density at the end of inflation is found to be typically too small to back react on the background dynamics of the inflaton. We also show that a short deviation from slow roll does not result in strong modifications to the shape of the spectrum. We calculate the evolution of the correlation length and the field amplitude during the inverse cascade and viscous damping of the helical magnetic field in the radiation era after inflation. We conclude that except for low scale inflation with very strong coupling, the magnetic fields generated by such an axial coupling in single field slow roll inflation with perturbative coupling to the inflaton are too weak to provide the seeds for the observed fields in galaxies and clusters.

1. Introduction

Cosmic magnetic fields have been observed on all scales ranging from stars to near and far away galaxies and galaxy clusters [1, 2, 3, 4]. The strength of the magnetic fields observed in galaxies and clusters is typically of the order of $\mu$Gauss. Recently, using the absence of extended GeV emission around TeV blazar gamma-rays, a lower limit of $3 \times 10^{-16}$ Gauss on the strength of intergalactic magnetic fields was derived [5, 6, 7, 8].

These observations prompt the question of the origin of cosmic magnetic fields. Have they been generated during structure formation or have primordial magnetic fields been amplified? So far, this question has no clear, satisfactory answer. Several studies [9, 10, 11, 12, 13, 14, 15, 16] of second order perturbation theory have shown that up to recombination only very weak magnetic fields of the order of $10^{-28}$ Gauss can be generated by structure formation. However, it turns out that the magnetic fields from second order perturbation theory are not strong enough to exceed the lower limit derived in Refs. [5, 6, 7, 8]. Whether such fields could have been generated later by the process of galaxy formation and then ejected into intergalactic space remains unclear.
In this work, we pursue the idea that instead magnetic fields are of primordial origin and might have been generated in the early universe. Primordial magnetic fields are interesting as they induce all three kinds of gravitational perturbations i.e. scalar, vector and tensor; all of which contribute to the Cosmic Microwave Background (CMB) temperature and polarisation anisotropies. These primordial fields also lead to non-Gaussian signals in the CMB even at the lowest order contrary to the higher order effect due to inflationary scalar perturbations. Analyses of such effects using recent cosmological data provide an upper limit of a few nGauss on the primordial magnetic fields \[ 17, 18, 19, 20 \]. It has been argued that the electroweak or the QCD phase transitions, if they are first order, lead to the generation of magnetic fields \[ 21, 22, 23, 24 \]. Also non-perturbative processes during preheating can generate significant magnetic fields with, in some cases, a helical component, see for instance \[ 25 \]. However, causality strongly constrains such fields. Their power spectrum is very blue with \( B^2(k) \propto k^2 \), and therefore, their amplitude on large scales is far too small \[ 26, 27 \].

On the other hand, if the magnetic fields are produced during inflation, their power spectrum is a priori not constrained by causality but only by the specific model. Since the standard electromagnetic (EM) action is conformally invariant the fluctuations in the EM field are not amplified in the conformally flat expanding background of inflation. In order to generate magnetic fields, one needs to break conformal invariance of the EM field, e.g. by coupling the EM field to a scalar or a pseudo-scalar field or to a curvature invariant (for an overview see, for instance, Ref. \[ 28 \]). Typically, a term of the form \( fF^2 \) is considered, where \( f \) is a function of time or of the inflaton and \( F \) is the EM field tensor. Depending on the form of the coupling \( f \), this gives rise to different magnetic field power spectra, and even scale-invariant spectra are possible (in this context, see Refs. \[ 29, 30 \]). In this way, the magnetic fields can have sufficient amplitude on large scales to provide seeds for the observed fields in galaxies and clusters. However, the backreaction on the inflation dynamics, the production of gravitational waves and nucleosynthesis bounds on the amplitude of gravitational waves strongly constrain the magnetic energy density \[ 17, 27 \].

Quite a different situation is encountered if a coupling to the parity-violating term \( F\tilde{F} \), i.e. a term \( fF\tilde{F} \), is added to the standard EM action \( F^2 \), where \( \tilde{F} \) is the dual of \( F \). As a consequence, magnetic helicity is generated, which is absent in the case discussed above. This has two interesting consequences: firstly, contrary to non-helical fields, helical fields evolve in the cosmic magnetohydrodynamic (MHD) plasma via inverse cascade \[ 31, 32, 33 \]. This transfers power from small to large scales so that even blue spectra can lead to significant power on large scales. In Ref. \[ 34 \], it was shown that the inverse cascade is not quite sufficient for helical fields generated at the electroweak phase transition \[ 35 \], but it might work for magnetic fields from inflation. Secondly, helical magnetic fields leave a very distinct signature as they violate parity symmetry. This leads to observable effects, e.g. correlations between the anisotropies in the temperature and B-polarisation or in the E- and the B-polarisations in the CMB \[ 36 \]. Furthermore, they induce helical gravitational waves \[ 36 \] which might be observable \[ 37 \].
Some consequences of primordial helical magnetic fields and their generation from primordial helicity have been studied in the past [38, 39]. The interactions of helical magnetic fields and axions have also been investigated [40, 41]. Recently, the generation of helical magnetic fields during inflation in specific models has been studied, e.g. in Ref. [42], helical magnetic fields from N-flation were investigated. In this case, the large number of pseudo-scalar fields driving inflation effectively leads to a large coupling, $f \propto \sqrt{N}$, to the $F \tilde{F}$ term. In Ref. [43], some toy models for the coupling were analysed where $f$ was taken to be a power law function of $kt$ ($k$ being the wave number and $t$ the conformal time).

In this paper, we study magnetic fields generated by an axial coupling of the form $f(\phi)F \tilde{F}$ during inflation, where $\phi$ is the inflaton. We consider two different forms of the coupling function and show that, contrary to a non-helical coupling of the form $f(\phi)F^2$, a helical coupling always leads to a spectral index $n = 1$ for $B^2(k) \propto k^n$, as long as slow roll inflation is considered. We derive the condition for the theory to be perturbative, i.e. the free part of the action dominates over the interaction. Of course this is not necessarily true, but at least naively if not, we can no longer trust our calculation of particle creation out of the quantum vacuum which is based on perturbative quantum field theory. We estimate the magnetic energy density as a function of scale and show that backreaction is typically small. These conclusions are valid for any reasonable coupling function $f(\phi)$. We confirm our analytical results numerically. Furthermore, we study the effects of a short deviation from slow roll on the magnetic field spectrum and show that such deviations, if kept within the bounds permitted by the CMB data, do not strongly modify its shape. Even though the inverse cascade in the radiation dominated era after inflation does move power to larger scales, the final strength of the magnetic field on cosmologically interesting scales is still insufficient to provide seeds for the observed magnetic fields in galaxies and clusters, except if the inflation scale is low, $T_* < 10^4 \text{GeV}$ and the axial coupling is very strong. Even if we assume very efficient dynamo amplification.

The remainder of this paper is organised as follows. In the next section, we introduce the axial coupling of the EM field to the inflaton, derive the field equations, discuss the background evolution and the slow roll approximation and derive the linear perturbation equations of the inflaton and the EM vector potential. In Sec. 3 we discuss the evolution equation of EM quantum fluctuations during inflation and compare the helical to the normal case. In Sec. 4 we derive a condition on the chiral coupling by requiring the theory to be perturbative, investigate different coupling functions in the slow roll approximation, and solve the evolution of the EM fluctuations analytically. In Sec. 5 we discuss the consequences of a brief violation of the slow roll approximation on the magnetic field spectrum. In Sec. 6 we study the evolution of the magnetic field during the radiation era after inflation and determine the final spectrum after the inverse cascade. Finally, in Sec. 7 we conclude with a few comments on our results. Three appendices contain some details on the quantisation of the vector potential, the condition on the coupling function from the perturbativity of the theory and the
asymptotic behaviour of the Coulomb wave functions, respectively.

Notation and units: We work in a metric with signature \((-+++)\). For tensor components, Greek indices take values \(0\ldots3\), while Latin indices run from 1 to 3. The components of spatial 3-vectors with respect to a comoving basis are denoted in bold face. We employ Heaviside-Lorentz units such that \(c = \hbar = k_B = \varepsilon_0 = \mu_0 = 1\). The reduced Planck mass is defined as \(m_P = (8\pi G)^{-1/2}\). We normalise the cosmic scale factor to unity today so that the comoving scales become physical scales today.

2. Axial coupling of electromagnetism to the inflaton

2.1. Action and field equations

We consider a scalar field, \(\phi\), which takes the role of the inflaton and the EM field, \(F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu\), characterised by its four-vector potential \(A_\mu\). The EM field is conformally coupled to the metric and therefore, no fluctuations are generated unless there is either an explicit coupling to the inflaton or conformal symmetry is broken directly, e.g. by coupling \(F\) to a curvature term. Here, we investigate the first possibility and study a helical coupling given by the action

\[
S[\phi, A_\mu] \equiv \int d^4x \sqrt{-g} \left\{ \mathcal{L}_\phi(\phi) + \mathcal{L}_{\text{em}}(A_\mu) + \mathcal{L}_I(\phi, A_\mu) \right\} .
\]

The Lagrangian densities of the free fields are

\[
\mathcal{L}_\phi(\phi) \equiv \frac{1}{2} g^{\alpha\beta}(\partial_\alpha \phi)(\partial_\beta \phi) + V(\phi) \tag{2}
\]

\[
\mathcal{L}_{\text{em}}(A_\mu) \equiv -\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} \tag{3}
\]

and the axial interaction is given by

\[
\mathcal{L}_I(\phi, A_\mu) \equiv \frac{1}{4} f(\phi) F_{\alpha\beta} \tilde{F}^{\alpha\beta} . \tag{4}
\]

It describes a coupling of the scalar field to the parity-violating term, \(F \tilde{F}\), where \(\tilde{F}\) is the dual of the EM field tensor and is defined as

\[
\tilde{F}^{\mu\nu} \equiv \frac{1}{2} \eta^{\mu\nu\alpha\beta} F_{\alpha\beta} . \tag{5}
\]

Here \(\eta^{\mu\nu\alpha\beta}\) is the totally anti-symmetric tensor in four dimensions with \(\eta^{0123} = (-g)^{-1/2}\). For an observer with 4-velocity \(u^\mu\) the electric and magnetic fields are \(E_\mu = F_{\mu\alpha} u^\alpha\) and \(B_\mu = \tilde{F}_{\mu\alpha} u^\alpha\), respectively, and we have \(F_{\alpha\beta} \tilde{F}^{\alpha\beta} = -4E_\alpha B^\alpha\).

If the scale of inflation is above the electroweak scale then, in principle, one should specify whether \(\phi\) couples to \(U(1), SU(2)\) or both of the electroweak \(SU(2)_L \times U(1)_Y\). If the coupling is to both, which seems simplest, the same process that leads to magnetic field helicity also induces a non-zero baryon number which is related to the electroweak Chern-Simons number \([35]\). The electromagnetic Chern-Simons number is equivalent to the helicity. In this work we do not discuss this additional potentially interesting aspect, but concentrate on the magnetic fields remaining after the end of the electroweak phase.
transition. We assume the conversion from $W^0$ and $Y$ into photons to be efficient and not to affect the resulting magnetic field distribution significantly so that we may simply consider the coupling of $\phi$ to the photon field.

The axial coupling is characterised by the scalar function $f(\phi)$. We will see later how this function affects the evolution of the vector potential. Notice that if $f(\phi)$ was a constant, the EM part of the action would still be conformally invariant and therefore, no EM fluctuations could be amplified during inflation. Note also that the term $\mathcal{L}_I$ either breaks parity explicitly if $\phi$ is a normal scalar field or, if $\phi$ is a pseudo-scalar, parity is broken spontaneously by the presence of a background field $\phi \not\equiv 0$. For the discussion in this work, this distinction is not relevant. However, in certain models, it might be relevant for the amount of parity violation generated during reheating.

Varying the action with respect to $\phi$ leads to a sourced equation of motion for the scalar field

$$\nabla^\alpha \partial_\alpha \phi - V'(\phi) = \frac{1}{4} f'(\phi) F_{\alpha\beta} \tilde{F}^{\alpha\beta}. \quad (6)$$

The primes in $V'(\phi)$ and $f'(\phi)$ denote derivatives with respect to $\phi$. The field equations for the EM field follow from varying the action with respect to $A_\mu$:

$$\nabla_\alpha F^{\mu\alpha} = f'(\phi) (\partial_\alpha \phi) \tilde{F}^{\mu\alpha}. \quad (7)$$

Comparison with the usual inhomogeneous Maxwell equation leads us to interpret the source term on the right hand side as an effective axial current. To obtain the above form of the inhomogeneous Maxwell equation, we used the homogeneous Maxwell equation

$$\nabla_\alpha \tilde{F}^{\mu\alpha} = 0 \iff \nabla_\lambda F_{\mu\nu} = 0 \quad (8)$$

which is equivalent to the Bianchi identity, $dF = 0$.

2.2. Background evolution

To describe the universe during inflation, we work in a flat Friedmann-Lemaître (FL) background metric characterised by the line element

$$ds^2 = a^2(t)(-dt^2 + \delta_{ij} dx^i dx^j) \quad (9)$$

where $a$ is the scale factor and $t$ is conformal time which is related to cosmic time $\tau$ by $a \, dt = d\tau$. Derivatives with respect to conformal time are denoted by a dot and the conformal Hubble parameter is $\mathcal{H} \equiv \dot{a}/a = aH$ where $H \equiv (da/d\tau)/a$ is the physical Hubble parameter.

We assume the scalar field to dominate the energy budget of the universe and to drive inflation. We shall check later under what conditions the EM energy density is negligible and this approach is justified. We decompose the scalar field into a background value and a small perturbation: $\phi(x^\mu) \equiv \varphi(t) + \delta\phi(x^\mu)$. At background level, Eq. (6) reduces to the homogeneous evolution equation for the scalar field

$$\ddot{\varphi} + 2\mathcal{H}\dot{\varphi} + a^2 V'(\varphi) = 0. \quad (10)$$

‡ An axial anomaly which also induces a source term of this form has been discussed in Ref. [44].
The evolution of the scale factor is determined by the background Friedmann constraint equation

$$3 m_p^2 H^2 = \frac{1}{2} \dot{\phi}^2 + a^2 V(\phi). \quad (11)$$

During slow roll inflation, the potential of the inflaton field is dominating the energy density of the universe and the first term on the right hand side of Eq. (11) can be neglected. This is quantified by means of the slow roll parameters (see e.g. [45])

$$\epsilon \equiv \frac{m_p^2}{2} \left( \frac{V'}{V} \right)^2 \ll 1, \quad \left| \frac{m_p^2 V''}{3 V} \right| \ll 1. \quad (12)$$

To first order in the slow roll parameters, the evolution equation of $\phi$ and the Friedmann constraint can be reduced to [45]

$$\dot{\phi} \simeq -a^2 \frac{V'}{3H} = \pm \sqrt{2\epsilon} a^2 \frac{V}{3m_p H} \simeq \pm \sqrt{2\epsilon} m_p H. \quad (13)$$

The sign of $\dot{\phi}$ depends on the details of the inflation model, but here we do not assume a specific form of the potential. Using also that $H \simeq |t|^{-1}$ and that $\epsilon$ is roughly constant in the slow roll regime, we can integrate this result to find

$$\phi \simeq \phi_{in} \pm \sqrt{2\epsilon} m_p \ln(t/t_{in}) \quad (14)$$

where $\phi_{in} \equiv \phi(t_{in})$ is the initial value of the inflaton, and $\bar{\epsilon}$ is the average value of $\epsilon$ in the slow roll regime. However, in this approximation, any deviation of $\epsilon$ from a constant value is integrated over time, which can lead to significant deviations in the evolution of $\phi$ towards the end of inflation.

2.3. Linear perturbation equations

The EM field does not contribute to the background expansion but comes into play at the perturbative level. We study the generation of perturbations in the FL background during inflation. We work in longitudinal gauge where the metric perturbation is

$$\delta g_{\mu\nu} dx^\mu dx^\nu = a^2(t) \left( -2\Psi dt^2 - 2\Phi \delta_{ij} dx^i dx^j \right) \quad (15)$$

and the two scalar degrees of freedom, $\Psi$ and $\Phi$, coincide with the gauge-invariant Bardeen potentials [45].

We work in Coulomb gauge throughout, i.e. $A^\mu = (0, A^i)$ with $\partial_i A^i = 0$. To lowest order, the inhomogeneous Maxwell equation (7) with the axial current becomes

$$\ddot{A}_i - \nabla^2 A_i = -f'(\phi) \epsilon_{ijk} \partial_j A_k. \quad (16)$$

where the Euclidean Laplacian is defined as $\nabla^2 \equiv \delta^{ij} \partial_i \partial_j$ and the totally anti-symmetric symbol $\epsilon_{ijk}$ satisfies $\epsilon_{123} = 1$. Note that these equations are like in Minkowski space, there is no coupling to the scale factor. For a constant axial coupling, $f'(\phi) = 0$, the sourced Maxwell equation reduces to the standard free wave equation and no fluctuations are amplified during inflation.

§ Since the EM energy density is quadratic in the fields, one considers the vector potential and the electric and magnetic fields to be at half order in linear perturbation theory.
The evolution of perturbations in the scalar field is also altered by the axial coupling. At linear order, the scalar field equation (6) acquires a source term
\[ \ddot{\delta\phi} + 2H\dot{\delta\phi} - \nabla^2\delta\phi + a^2V''(\phi)\delta\phi - \dot{\phi}(3\dot{\Phi} + \dot{\Psi}) + 2a^2V'(\phi)\Psi = -a^{-2}f'(\phi)\epsilon_{ijk}\dot{A}_i\partial_jA_k. \] (17)

For any scenario where EM perturbations are generated, it is important to investigate the effect of this source term on the generation of scalar perturbations and through these on the primordial curvature perturbations. For instance, this has been studied in the case of natural inflation \[46\]. We leave a general discussion of such effects for future work \[47\] and concentrate here on the generation of magnetic fields.

2.4. Physical properties of electromagnetism in the expanding universe

The four-vector potential is generally covariant and its evolution is independent of the choice of coordinates. However, for an observer, the physical EM field manifests itself in terms of electric and magnetic fields which are intrinsically frame dependent quantities. Measured by an observer with four-velocity \( u^\mu \), with \( u^\alpha u_\alpha = -1 \), the electric and magnetic fields can be covariantly defined as \[48\]
\[ E_\mu = F_{\mu\alpha}u^\alpha \] (18)
\[ B_\mu = \frac{1}{2}\eta_{\alpha\beta\gamma}F^{\alpha\beta}u_\gamma = \tilde{F}_{\mu\alpha}u^\alpha. \] (19)

These are both three-vector fields in the sense that they are orthogonal to the observer velocity, \( E_\alpha u^\alpha = 0 = B_\alpha u^\alpha \). In a perturbed FL metric an observer has the four-velocity \( u^\mu = a^{-1}(1, 0) + O(1) \). As a consequence one finds (in Coulomb gauge)
\[ (E_\mu) = (0, -\frac{1}{a}\dot{A}_i), \quad (B_\mu) = (0, \frac{1}{a}\epsilon_{ijk}\partial_jA_k) \] (20)
up to first order. With respect to an orthonormal basis comoving with the observer (or generally the Hubble flow), we define the Euclidean three-vector fields \( E \) and \( B \) through
\[ (E_\mu) = a(0, E), \quad (B_\mu) = a(0, B). \] (21)

As discussed in Ref. \[28\], in a highly conducting plasma, magnetic fields should scale as \( a^{-2} \) with the expansion, while the electric field is damped away. The three-vector fields \( E \) and \( B \) defined above show exactly this property. We therefore rescale the fields by a factor of \( a^2 \) such that the effect of the expansion is absorbed, i.e. \( \tilde{B} \equiv a^2B. \) In terms of the vector potential, the rescaled fields become
\[ \tilde{E}_i = -\dot{A}_i, \quad \tilde{B}_i = \epsilon_{ijk}\partial_jA_k. \] (22)

Using these expressions and the field equations for \( A_\mu \), one can derive Maxwell’s equations for the rescaled fields \( \tilde{E} \) and \( \tilde{B} \), which take the same form as in a Minkowski space-time.

The physical properties of the EM fields can now be described in terms of the rescaled fields. Here, we are mainly interested in the energy density and the helicity.
density of the fields generated during inflation. The energy density of the magnetic field is
\[ \rho_B \equiv \frac{1}{2} B_\alpha B^\alpha = \frac{1}{2} B \cdot B = \frac{1}{2} a^{-4} \tilde{B} \cdot \tilde{B} \equiv a^{-4} \tilde{\rho}_B \] (23)
and analogously for the electric energy density. The helicity density is given as
\[ \tilde{H} \equiv A_\alpha B^\alpha = a^{-3} \delta^{ij} A_i \tilde{B}_j \equiv a^{-3} \tilde{\tilde{H}}. \] (24)
(Deliberately, we are not denoting the vector-potential in bold face because \( A_i \) are the spatial components of the four-co-vector \( A_\mu \) in the covariant representation with respect to the coordinate basis \( (dx^\mu) \), while e.g. \( B \) are the components of the four-co-vector \( B \) with respect to the orthonormal basis \( (adx^\mu) \) as discussed, for instance, in Ref. [28].)

3. Electromagnetic quantum fluctuations
To investigate the generation of EM fields during inflation, we consider the evolution of quantum fluctuations of the EM vector-potential. The coupling of the vector-potential to the background evolution of the inflaton via the inhomogeneous Maxwell equation (16) can lead to the amplification of EM quantum fluctuations. The amplification depends on the axial coupling \( f(\varphi) \) and therefore on the evolution of \( \varphi \). In Appendix A we review the quantisation of the vector-potential in an expanding background. The result is that Maxwell’s equations lead to an evolution equation for the Fourier modes of the quantised field which reduces to the free wave equation in the absence of the axial coupling. We first discuss this evolution equation and then summarise the physical observables of the EM field expressed in terms of the solutions to the mode equations.

3.1. Evolution of the Fourier modes of the vector potential
We introduce the orthonormal spatial basis as
\[ \left( \varepsilon_1^k, \varepsilon_2^k, \hat{k} \right) \quad \text{with} \quad |\varepsilon_i^k|^2 = 1, \hat{k} = k/k, \] (25)
and
\[ \varepsilon_\pm^k \equiv \frac{1}{\sqrt{2}} (\varepsilon_1^k \pm i \varepsilon_2^k). \] (26)
In radiation gauge, the vector potential then takes the form
\[ A = A_1 \varepsilon_1 + A_2 \varepsilon_2 = A_+ \varepsilon_+ + A_- \varepsilon_. \] (27)
After quantisation of the vector potential, we can study the evolution of the Fourier modes, \( A_h(t,k) \), with respect to the helicity basis for the polarisation states, \( h = \pm \). We find that the helicity modes satisfy the wave equation with a time dependent mass term corresponding to the modified Maxwell equation (16) for the classical vector-potential,
\[ \ddot{A}_h + \left[ k^2 + h k f'(\varphi) \dot{\varphi} \right] A_h = 0. \] (28)
The fact that the sign of the respective helicity mode, \( h = \pm \), appears explicitly in the evolution equation of the Fourier modes, leads to a different evolution of the two helicity
states and therefore, to the generation of magnetic helicity. Also note that the scalar field couples to $k A_h$ and the coupling function itself, $f'(\varphi)\dot{\varphi}$, only depends on time, as $\varphi$ is the background value of the inflaton. The solutions to this mode equation for a given coupling $f(\varphi)$ fully determines the spectrum of the generated EM fields.

Let us compare the mode equation (28) to the non-helical case with the coupling $f(\varphi)F^2$. One obtains a similar evolution equation [28]

$$\ddot{A}_h + 2\frac{f'\dot{\varphi}}{f}A_h + k^2 A_h = 0.$$  (29)

Redefining the mode functions as $\tilde{A}_h \equiv fA_h$, one can rewrite this in the form

$$\ddot{\tilde{A}}_h + \left[k^2 - \frac{\dot{f}}{f}\right] \tilde{A}_h = 0.$$  (30)

We observe two significant differences to the helical case: firstly, the two helicity states couple with the same sign and therefore, no helicity is generated. Secondly, the scalar field couples to $\dot{A}_h$ (or $\tilde{A}_h$) as opposed to $kA_h$ in the helical case. As we shall see, this leads to significant differences in the spectra obtained for the two types of couplings. The reason is that the additional factor $k$ in the helical case leads to a suppression of the coupling term at super-Hubble scales.

To visualise this, let us compare the importance of the different terms in the helical mode equation (28). We define $f_N$ to be the logarithmic derivative of the coupling function, $f$, with respect to the scale factor

$$f_N \equiv \frac{df}{d\ln a} = f'(\varphi)\dot{\varphi} \approx \pm f'(\varphi)\sqrt{2\epsilon}\mpl.$$  (31)

where $N$ stands for the number of e-foldings and is defined as $N = \ln(a/a_\text{in})$. Note that $f_N$ is the dimensionless part of the coupling term appearing in the mode equation (28). First consider super-Hubble scales, $k \ll \cal{H}$, where we can approximate $|\partial_t| \sim \cal{H}$ so that (28) reduces to

$$\left[1 + \frac{k^2}{\cal{H}^2} + hf_N\frac{k}{\cal{H}}\right] A_h \approx 0.$$  (32)

The second term is small by definition and the coupling term can be important on super-Hubble scales only if $f$ is a rapidly varying function of time, i.e. $f_N = \dot{f}/\cal{H} \gg 1$ and then only for as long as $f_N \gg \cal{H}/k$. On sub-Hubble scales, for $k \gg \cal{H}$, we may approximate $\dot{A}_h \sim k^2 A_h$. As long as $k \gg f_N \cal{H}$ the first and second terms in Eq. (28) dominate. Hence the coupling term is typically relevant only at Hubble crossing $k \sim \cal{H}$ during a few Hubble times at best. In Fig. 1, we show the evolution of the two helicity modes for a given wavenumber in case of a power law coupling, $f \propto \varphi^p$. Clearly, the modes only feel the axial coupling around horizon crossing. This fact turns out to be relevant for the resulting spectrum. We will discuss this issue in more detail in Sec. 4.2.
3.2. Vacuum solutions and initial conditions

To study the generation of perturbations during inflation, we need to set the initial conditions when the mass term $\pm kH f_N$ in Eq. (28) can be neglected, i.e. well inside the Hubble horizon at early times. This condition is usually formulated in terms of the variable $x \equiv -kt$ which approaches infinity in this limit. During slow roll inflation $\mathcal{H} \simeq -1/t$, so that $x = -kt \simeq k/\mathcal{H} \gg 1$ if the mode with wavenumber $k$ is well inside the Hubble horizon.

From the mode equation (28), we see that if initially
\[ f_N(x) = f'(x)\dot{x} / H \ll k/H \]  
the axial coupling term can be neglected with respect to the $k^2$ term and the mode equation becomes a free wave equation. Its solutions are plane waves. We match to the incoming vacuum solution described in Appendix A,
\[ A_h(t, k) = A_{\text{free}}(t, k) = (2k)^{-1/2} e^{ikt}, \quad \text{for} \quad -kt \gg 1. \]  
This is used in the following as initial condition for the solutions of the full mode equation. Notice that the free plane wave solution only yields a valid initial condition if Eq. (33) is satisfied.
3.3. Power spectra and physical quantities

The statistical distribution of the EM fields as seen by an observer can now be quantified in terms of a given solution for the helicity modes of the EM vector potential. We define the magnetic power spectrum and relate it to the magnetic energy and helicity density.

If the magnetic field generated by some process is statistically homogeneous and isotropic, its spectrum is determined by two scalar functions $P_S(k)$ and $P_A(k)$. Since the magnetic field is a divergence-free vector field the two-point function of the Fourier components of the magnetic field can be written as

$$\langle \tilde{B}_i(t, k) \tilde{B}_j^*(t, q) \rangle = \frac{(2\pi)^3}{2} \delta(k - q) \left\{ (\delta_{ij} - \hat{k}_i \hat{k}_j) P_S(t, k) - i \epsilon_{ijn} \hat{k}_n P_A(t, k) \right\}$$  \hspace{1cm} (35)

where $P_S$ and $P_A$ are the symmetric and anti-symmetric parts of the power spectrum, respectively. The symmetric part of the spectrum determines the energy density while the anti-symmetric part corresponds to the helicity density:

$$\langle \tilde{A}_i(t, k) \tilde{B}_i^*(t, q) \rangle = (2\pi)^3 \delta(k - q) P_S(t, k)$$  \hspace{1cm} (36)

$$\langle \tilde{A}_i(t, k) \tilde{B}_i^*(t, q) \rangle = ik^{-2}((k \wedge \tilde{B})_i(t, k) \tilde{B}_i^*(t, q))$$
$$= k^{-1}(2\pi)^3 \delta(k - q) P_A(t, k).$$  \hspace{1cm} (37)

With respect to the helicity basis, see Appendix A, the spectra can directly be written as

$$P_{S/A}(t, k) = k^2 \left( |A_+(t, k)|^2 \pm |A_-(t, k)|^2 \right),$$  \hspace{1cm} (38)

where the upper sign corresponds to $P_S$ and the lower sign to $P_A$. Here we use the non-trivial result widely applied in inflationary cosmology that at late times, the vacuum expectation values of the fields generated during inflation can be interpreted as stochastic power spectra.

We define the magnetic energy density per logarithmic wave number via

$$\langle \tilde{\rho}_B(t) \rangle = \int_0^\infty \frac{dk}{k} \frac{d\tilde{\rho}_B}{d\ln k}(t, k),$$  \hspace{1cm} (39)

so that

$$\frac{d\tilde{\rho}_B}{d\ln k}(t, k) = \frac{k^3}{(2\pi)^2} P_S(t, k).$$  \hspace{1cm} (40)

Similarly, we define the magnetic helicity per logarithmic wave number as $\tilde{\mathcal{H}} = A_\alpha \tilde{B}^\alpha$,

$$\frac{d\tilde{\mathcal{H}}}{d\ln k}(t, k) = \frac{k^2}{2\pi^2} P_A(t, k).$$  \hspace{1cm} (41)

Finally, the electric field is given by the time derivative of the vector-potential and thus its contribution to the energy density, $\tilde{\rho}_E = \tilde{E}_\alpha \tilde{E}^\alpha/2$, is computed to be

$$\frac{d\tilde{\rho}_E}{d\ln k}(t, k) = \frac{k^3}{(2\pi)^2} \left( |\dot{A}_+(t, k)|^2 + |\dot{A}_-(t, k)|^2 \right).$$  \hspace{1cm} (42)

Notice that any electric fields produced during inflation will be damped very rapidly after inflation due to the huge conductivity of the primordial plasma.
4. Analytic solutions during slow roll inflation

In this section, we first derive a condition on the coupling function $f(\varphi)$ such that the theory can be treated perturbatively. Even though we cannot prove that the theory does not make sense otherwise, our treatment is perturbative and so we can really trust it only if interactions are small. In our subsequent analysis in Section 6 we shall, however, analyse the results also if this condition is not satisfied. We then investigate two different forms of the coupling function and obtain upper bounds on its parameters without specifying a model of inflation. Finally, we solve the mode equation in slow roll for a constant $f_N$ analytically, derive the resulting magnetic field power spectrum at the end of inflation and discuss the issue of backreaction on the background evolution of the inflaton.

4.1. Condition on the coupling function

In order to derive a condition on the coupling function such that the theory can be treated perturbatively, the interaction between the scalar field and the EM field should be small at all times. For this we demand that the ratio of the actions for the interaction term to the free EM term should be less than unity, i.e.

$$\frac{|S_I[\varphi, A_\mu]|}{|S_{\text{em}}[A_\mu]|} < 1. \quad (43)$$

In Appendix B we show that in a FL background the free EM action can be written as

$$S_{\text{em}}[A_\mu] = V \int dt \langle \tilde{\rho}_E(t) - \tilde{\rho}_B(t) \rangle_V \quad (44)$$

where $\langle \ldots \rangle_V$ denotes the average over volume $V$ in coordinate space and is assumed to be equivalent to the expectation value defined in Eq. (39). Similarly, the interaction part of the action can be expressed in terms of the magnetic helicity as (see Appendix B)

$$S_I[\varphi, A_\mu] = \frac{1}{2} V \int dt \hat{f} \langle \tilde{\mathcal{H}}(t) \rangle_V . \quad (45)$$

To compare the two parts of the action, one has to evaluate the time integrals for a given scenario. For the case of a maximally helical magnetic field, e.g. $|A_+| = |A|$ with $|A_-| = 0$ and assuming a power law form for the spectrum given by

$$P_S = P_A = k^2 |A|^2 = B_0^2 k^n, \quad (46)$$

we find that

$$\langle \tilde{\rho}_B(t) \rangle_V \simeq \frac{B_0^2}{(2\pi)^2} \frac{k_{\text{max}}^{n+3}(t)}{(n+3)} \quad (47)$$

$$\langle \tilde{\mathcal{H}}(t) \rangle_V \simeq \frac{2B_0^2}{(2\pi)^2} \frac{k_{\text{max}}^{n+2}(t)}{(n+2)} \quad (48)$$

where $k_{\text{max}}(t)$ is the cut-off scale which is the smallest scale crossing the Hubble scale at time $t$ when the volume average is computed, hence $k_{\text{max}} \simeq \mathcal{H}(t)$. To estimate the
energy density for the electric field, we can approximate $|\dot{A}| \simeq \mathcal{H}|A|$ at large scales which then leads to

$$
\langle \tilde{\rho}_E(t) \rangle_V \simeq \frac{B_0^2}{(2\pi)^2} \frac{k_{n+1}^{n+1}(t)}{(n+1)^2} \mathcal{H}^2 \tag{49}
$$

After evaluating the volume averages of the two energy densities and the helicity, we can now compare the two actions. Ignoring numerical factors, the comparison condition (43) can be written as

$$
\left| \frac{S_I[\phi, A_\mu]}{S_{\text{em}}[A_\mu]} \right| \simeq \left| \frac{\int dt \, \dot{f} \, \mathcal{H}^{n+2}}{\int dt \, \mathcal{H}^{n+3}} \right| < 1. \tag{50}
$$

Now, for the above inequality to be satisfied, it is sufficient to require $|\dot{f}| < \mathcal{H}$ or equivalently $|f_N| < 1$. We believe that this condition is quite generic and does not depend on the functional form of the coupling function. The above analysis indicates the fact that the condition for the perturbative treatment of the theory does not depend on the coupling function, but on its derivative. This is not surprising as a constant $f$ only yields a surface term which does not affect the dynamics.

In flat spacetime the above condition has to be replaced by $|\dot{f}| < k_{\text{max}}$, where $k_{\text{max}}$ denotes the UV cutoff of the modes under consideration.

### 4.2. Power law and exponential coupling

We first study the special case of a power law coupling,

$$
f(\varphi) = f_0 \left( \frac{\varphi}{m_P} \right)^p. \tag{51}
$$

Here $f_0$ and $p$ are constants. The sign of $f_0$ determines which helicity is amplified by the coupling and, thus, we can take $f_0$ to be positive without loss of generality. Let us first investigate bounds on the parameter space of $f_0$ and $p$ under the condition $|f_N| < 1$.

During slow roll inflation, we use Eq. (13) to express the coupling term in the mode equation (28), $f_N = f'(\varphi) \dot{\varphi}/\mathcal{H}$, in terms of the slow roll parameter $\epsilon$

$$
f_N \simeq \pm f_0 \sqrt{2\epsilon} \left( \frac{\varphi}{m_P} \right)^{p-1} \tag{52}
$$

where, in principle, $\epsilon$ is a slowly varying function of time. If we want to satisfy the condition of perturbativity, $|f_N| < 1$, the above equation leads to an upper bound on the value of $f_0$ as

$$
f_0 < f_0^\text{max} = \frac{1}{|p|} \min_\varphi \left\{ \frac{1}{\sqrt{2\epsilon}} \left( \frac{m_P}{\varphi} \right)^{p-1} \right\}. \tag{53}
$$

For a given inflationary scenario, one can invert the definition of $\epsilon$ to express the above bound only as a function of $\varphi$. Though it turns out to be easy for large field inflationary models, it is not so straightforward in the case of typical models of small field inflation. Once the upper bound on $f_0$ is known, one can calculate the maximal possible value of the coupling term i.e. $f_N^{\text{max}}(p) \equiv f_N(f_0^{\text{max}}(p), p)$.
As a second case, we consider the example of an exponential chiral coupling of the following form

\[ f(\varphi) = f_0 e^{\alpha \varphi/m_P} \]  

(54)

where \( f_0 > 0 \) and \( \alpha \) are constants. Analogously to the power law case above, we calculate the coupling term \( f_N \) in terms of \( \epsilon \) which is given by

\[ f_N \simeq \pm f_0 \alpha \sqrt{2\epsilon} e^{\alpha \varphi/m_P} \]  

(55)

We again apply the constraint \(|f_N| < 1\) to find the bound on \( f_0 \):

\[ f_0 < f_0^{\text{max}} = \frac{1}{|\alpha|} \min_{\varphi} \left\{ \frac{1}{\sqrt{2\epsilon}} e^{-\alpha \varphi/m_P} \right\} \]  

(56)

As before, once \( f_0^{\text{max}} \) is known, the maximum value of the coupling term \( f_N^{\text{max}} \) can be calculated.

In both models, the power law and the exponential chiral coupling, the constraint \(|f_N| < 1\) leads to an upper bound on the overall amplitude of the coupling. The time dependence of \( f_N \), on the other hand, depends on the details of the inflation model. Though, generally we remark that for a given mode \( k \) the coupling is only active for a few e-foldings during which the mode crosses the Hubble scale, and moreover, if \( f_N \) had a significant time variation its overall amplitude would be suppressed very strongly by the bound on \( f_0 < f_0^{\text{max}} \). Therefore, we conclude that \( f_N \) can safely be considered roughly constant (at least during horizon crossing).

4.3. Analytic solution for a constant coupling term

We now solve the mode equation (28) for the case where \( f_N \) is approximately constant. The solution presented here will be valid for any constant value of \( f_N \). In the slow roll regime, the full coupling term is, \( f'(\varphi)\dot{\varphi} = f_N \mathcal{H} \simeq -f_N/t \), i.e. inversely proportional to conformal time and, consequently, the mode equation can be solved analytically. It is convenient to use \( x \equiv -kt \) as the time evolution variable. For each scale, \( k \), the initial condition in the asymptotic past is set well inside the horizon, i.e. for \( x \to \infty \), while inflation is considered to end when \( x \to 0 \). With a prime denoting the derivative with respect to \( x \), the mode equation reads

\[ A''_h(x) + (1 + hf_N x^{-1}) A_h(x) = 0. \]  

(57)

The free solution is a good approximation at early times, \( x \gg |f_N| \). The general solution to the mode equation (57) is \[49\]

\[ A_h(x) = C_1 G_0(y, x) + C_2 F_0(y, x) \]  

(58)

where \( y \equiv -hf_N/2 \) and \( G_0 \) and \( F_0 \) are the irregular and regular Coulomb wave functions of order zero, respectively. As initial condition we require the solution to approach the free solution, Eq. (34), for \( x \to \infty \). It turns out that the combination \( G_0 \pm iF_0 \) has the desired limit, as given in Ref. \[49\]:

\[ G_0(y, x) \pm iF_0(y, x) \xrightarrow{x \to \infty} \exp \pm i [x - y \ln(2x) + \sigma_0(y)] \]  

(59)
with \( \sigma_0(y) \equiv \arg \Gamma(1 + iy) = \gamma_E y + O(y^3) \) and \( \gamma_E \simeq 0.58 \) being the Euler’s constant. For large \( x \), the second term in the exponential, \( y \ln(2x) \), can be neglected with respect to \( x \). Comparison with the free solution shows that the plus sign corresponds to the incoming vacuum solution and we have to choose the initial amplitude

\[
\frac{1}{2}(C_1 - iC_2) = (2k)^{-1/2} \quad \text{and} \quad \frac{1}{2}(C_1 + iC_2) = 0. \quad (60)
\]

The factor with \( \sigma_0 \) only acts as an overall phase and has no physical significance. The normalised full solutions are

\[
A_h(x) = (2k)^{-1/2} \left[ G_0(-hf_N/2, x) + iF_0(-hf_N/2, x) \right] \quad (61)
\]

for \( h = \pm 1 \). This solution of the mode equation has also been found in Ref. [42] for \( N \)-flation. Below we shall argue that it is very general.

To understand the effect of the axial coupling on the growth of EM quantum fluctuations and to compute the magnetic power spectra at the end of inflation, we analyse the late time limit of this solution, i.e. when \( x \to 0 \). Using the approximate expressions derived in Appendix C, we find the asymptotic limit at late times as

\[
A_h(x) \xrightarrow{x \to 0} (2k)^{-1/2} \left[ \frac{\exp(-h\pi f_N/2) \sinh(\pi f_N/2)}{\pi f_N/2} \right]^{1/2}. \quad (62)
\]

The late time behaviour of both helicity modes is independent of \( x \), and therefore of \( t \), and the scale-dependence is not changed with respect to the free solution. The modes are coherently amplified while crossing the horizon, before they saturate outside the horizon. Notice also that we did not use the asymptotic limit given in 14.6.9 – 10 of Ref. [49] because it is not correct. (In this context, see discussion in Appendix C.)

Given the late time behaviour of the helicity mode functions, it is easy to compute the magnetic field power spectra produced at the end of inflation. The symmetric power spectrum is

\[
P_S(k) = k \frac{\sinh(\pi f_N)}{\pi f_N} \quad (63)
\]

while the antisymmetric one is

\[
P_A(k) = k \frac{\cosh(\pi f_N) - 1}{\pi f_N} \quad (64)
\]

Notice that both spectra are proportional to \( k \) and only their amplitude changes with the coupling strength \( f_N \). In Fig. 2 we illustrate the \( k \)-independent amplification factors of \( P_S \) and \( P_A \) as a function of \( f_N \). The larger \( |f_N| \) the smaller the difference between \( P_S \) and \( P_A \), i.e. the more helical the magnetic fields become. As one infers from Eqs. (63) and (64), both amplification factors tend to \((2\pi f_N)^{-1}\exp(\pi f_N)\) for large values of \( f_N \).

The magnetic energy density per logarithmic wave number at the end of inflation can directly be computed to be

\[
\frac{d\rho_B}{d\ln k}(t_{end}, k) = \frac{k^4 \sinh(\pi f_N)}{a_{end}^4 4\pi^3 f_N} \equiv \frac{k^4}{a_{end}^4} S^2(f_N). \quad (65)
\]
Figure 2. The symmetric (solid, red) and antisymmetric (dashed, blue) power spectra of the magnetic fields, $P_{S,A}/k$, are plotted as a function of the effective coupling constant $|f_N|$. For vanishing coupling, the vacuum solution, $P_S/k = 1$ and $P_A/k = 0$ is recovered, while the larger the coupling, the smaller the difference between $P_S$ and $P_A$. As discussed in the main text, for the theory to be perturbative we must require $|f_N| < 1$ and therefore, the amplification of the magnetic fields is small.

Here, we define $S^2(f_N)$ to be the amplitude of magnetic energy density spectrum. Since its spectrum is blue, the magnetic energy density is dominated by the cut-off scale, which is set by the last scale that exits the horizon before the end of inflation, $k_c = \mathcal{H}_{\text{end}} = (aH)_{\text{end}}$. With this we obtain

$$\rho_B(t_{\text{end}}) \simeq \frac{1}{4} \mathcal{H}_{\text{end}}^4 S^2(f_N).$$

By means of the Friedmann equation, we have

$$\Omega_B(t_{\text{end}}) \equiv \frac{\rho_B}{\rho_{\text{tot}}} \simeq \frac{S^2(f_N)}{12} \left( \frac{H_{\text{end}}}{m_P} \right)^2.$$  \hfill (67)

The generic condition for backreaction to be negligible then is

$$S(f_N) \lesssim \frac{m_P}{\mathcal{H}_{\text{end}}}. \hfill (68)$$

If we assume that the reheating phase is short so that $3m_P^2H_{\text{end}}^2 = \rho_{\text{tot}} \simeq T_\ast^4$, where $T_\ast$ is the reheating temperature, we can write the above bound from backreaction as

$$S(f_N) \lesssim \frac{m_P^2}{T_\ast^2}. \hfill (69)$$

For couplings that are perturbative in the naive sense discussed above, $|f_N| < 1$, $S(f_N)$ is of order unity and, therefore, no backreaction on the background evolution is expected if inflation ends well below the Planck scale. However, the results above hold for any
constant value of $f_N$, even if it was larger than unity. Since we cannot prove that perturbativity is strictly required, we shall therefore not restrict ourselves to this case in the following. For instance, if the EM field was coupled to a large number, $\mathcal{N}$, of pseudo-scalar fields then $f_N \rightarrow \sqrt{\mathcal{N}} f_N$, a situation where the effective coupling term can be large without spoiling the perturbativity of the interactions with the individual scalar fields, see Ref. [12].

In the more general case where $f_N$ is time dependent, in principle we have to evaluate $f_N$ at horizon crossing. This would lead to a slight modification of the spectrum but would not spoil the discussion of backreaction above. In the following we neglect this effect, as it is quite irrelevant for the few orders of magnitude in $k$ which we are interested in, see Fig. [14].

At the end of inflation and after reheating, we expect the universe to be filled with relativistic standard model particles, a relativistic highly conducting plasma. In this medium, the MHD approximation is valid and electric fields are rapidly damped away. We therefore do not discuss the electric field spectrum which will not survive reheating.

The helicity density per logarithmic wave number at the end of inflation is simply given by

$$\frac{d\tilde{\mathcal{H}}}{d \ln k} (t_{\text{end}}, k) = \frac{k^3}{a_{\text{end}}^3} \frac{\cosh(\pi f_N) - 1}{2\pi f_N}. \quad (70)$$

Note that $\tilde{\mathcal{H}} \propto \tilde{\rho}_B/k_c$, which therefore has to be constant when helicity is conserved.

We believe that this result is more general than the particular cases studied here: whenever the function $f_N$ is rapidly varying so that $f_N(t) = \text{constant}$ during slow roll is no longer a good approximation, the fact that we require $|f_N| < 1$ during the entire period of inflation means that $f_N$ must be oscillating. Though, because the coupling is active only for the small number of e-foldings during which a mode crosses the Hubble scale, resonant amplification of modes at horizon-crossing is not likely to be efficient. We have checked this statement numerically using different forms of oscillating coupling terms. We, therefore, conclude that as long as the slow roll approximation is valid and the theory can be treated perturbatively, the amplification of helical magnetic fields is always mode independent and consequently leads to a $n = 1$ spectrum for both the magnetic field and the helicity.

Note that this result differs significantly from the non-helical case. There, the source term in the mode equation is of the form $\ddot{f}/f$ which is typically $\propto 1/t^2$. The solutions are then Bessel functions and the Bessel function index, which determines the spectral index at late times, is related to the (nearly arbitrary) pre-factor. The difference comes from the fact that a term of order $1/t^2$ is relevant during all the time when the mode is super-Hubble, $-kt < 1$, while a term of order $(k/t)\mathcal{A}_h$ is relevant only around horizon crossing. On super-Hubble scales, it is dominated by the $\dot{\mathcal{A}}_h \sim \mathcal{A}_h/t^2$ term while on sub-Hubble scales, the $k^2\mathcal{A}_h$ term becomes dominant.
5. Deviations from slow roll

In the previous section, we discuss two different functional forms of the axial coupling of the inflaton to the EM field, namely a power law and an exponential. Within the slow roll approximation we find that $f_N$ can safely be considered constant. This always leads to a magnetic field power spectrum proportional to $k$. In this section, we first confirm our analytical findings numerically. Second, we explore the possibility of obtaining a different magnetic field spectrum by introducing a short deviation from slow roll, motivated by the fact that such deviations can provide a considerably better fit to the angular power spectrum of the CMB anisotropies than the predictions from typical single field inflation models, see for instance Refs. [50, 51, 52, 53, 54, 55].

To compare the analytical result, Eq. (63), to a full numerical solution, we solve the background evolution of $\phi$ with a quadratic potential, $V_0(\phi) \equiv \frac{1}{2}m^2\phi^2$, and integrate the evolution of the modes $A_h(t,k)$ to compute the magnetic power spectrum $P_S(k)$ at the end of inflation.

A short deviation from slow roll can, for example, be achieved by introducing a step in the quadratic inflaton potential as follows [50, 51, 52, 56]

$$V_\beta(\phi) = \frac{1}{2}m^2\phi^2 \left[ 1 + \beta \tanh \left( \frac{\phi - \phi_0}{\Delta \phi} \right) \right].$$

Here $\beta$, $\phi_0$ and $\Delta \phi$ characterise the height, the location and the width of the step, respectively. Such a deviation from slow roll, in general, leads to a burst of oscillations in the primordial power spectrum of curvature perturbations. In what follows, we turn our attention to the possible effects of such a deviation from slow roll on the magnetic field power spectrum.

We perform the comparison with a power law coupling function, $f(\phi) = f_0(\phi/m_P)^p$. In this case we find an exact expression for the slow roll parameter: $\sqrt{2\epsilon} = 2m_P/\phi$. Using this in Eq. (53), the upper bound on the parameter $f_0$ becomes

$$f_0^{\text{max}} = \frac{1}{2|p|} \begin{cases} 
(\phi_{\text{max}}/m_P)^{2-p} & \text{for } p > 2 \\
(\phi_{\text{min}}/m_P)^{2-p} & \text{for } p < 2 
\end{cases}$$

and $\phi_{\text{min}} \leq \phi \leq \phi_{\text{max}}$. With $f_0$ fixed to be $f_0^{\text{max}}(p)$ the coupling term $f_N^{\text{max}} \equiv f_N(f_0^{\text{max}}, p)$ reads

$$f_N^{\text{max}} = \begin{cases} 
(\phi/\phi_{\text{max}})^{p-2} & \text{for } p > 2 \\
(\phi/\phi_{\text{min}})^{p-2} & \text{for } p < 2 
\end{cases}$$

Thus, the most simple choice of parameters is: $p = 2$ such that $f_0^{\text{max}} = 1/4$ and $f_N^{\text{max}}$ is exactly constant in slow roll. Furthermore, we choose $f_0 = (4/5)f_0^{\text{max}} = 1/5$ to make sure that also in the case with the deviation from slow roll the condition $|f_N| < 1$ is respected.

As we discuss earlier, the coupling term $f_N$ is typically relevant only around Hubble crossing for a few Hubble times. The coupling term is, in general, a function of time but this time dependence can be translated into a scale dependence by identifying a time
Figure 3. The coupling term $|f_\mathcal{N}|$ at Hubble crossing of the wave number $k$ is plotted for the power law coupling with $p = 2$, $f_0 = 1/5$. The horizontal dotted blue line represents the constant value of $f_\mathcal{N} = 4/5$ in the slow roll approximation. The dashed black line indicates the behaviour of the coupling term for the potential in Eq. (71) with $\beta = 0$, while the solid red line is the same for $\beta \neq 0$. For the later case, we have used the best fit values of the parameters of the potential as in Ref. [55]. The bump in the coupling term for $\beta \neq 0$ arises due to the short period of deviation from slow roll.

with the corresponding Hubble crossing scale, $k = \mathcal{H}(t)$. In Fig. 3 we plot the coupling term $f_\mathcal{N}$ at Hubble crossing of the mode $k$ for the power law coupling with $p = 2$ and $f_0 = 1/5$. Comparing the slow roll value, $f_\mathcal{N} = 4/5$, with the numerical results for the slowly rolling model and the case departing from slow roll, it is evident from the figure that a deviation from slow roll leads to a bump in the coupling term as compared to the slow roll case and therefore, one can expect an effect in the magnetic field power spectrum on the scales which exit the Hubble radius around the time when the bump in the coupling term occurs.

In Fig. 4 we plot the relative deviation of the magnetic field power spectrum over an exact $k$-spectrum as a function of $k$ for the power law coupling. The exact numerical solution for slow roll deviates slightly from the $P_\mathcal{S} \propto k$ spectrum due to the slight scale dependence of $\mathcal{S}^2(f_\mathcal{N})$. Modulations in the spectrum arise as a result of a deviation from slow roll. We find that for the best fit values of the parameters of the potential (71), the spectrum of the magnetic field is not strongly modified. Indeed, we conclude that even the deviation from slow roll, within the limits required by CMB data, does not significantly modify the magnetic field spectrum. We find that the exponential coupling leads to a similar behaviour for the coupling term and the magnetic field spectrum.
Figure 4. The relative deviation of the magnetic field power spectrum from an exact $k$-spectrum is plotted as a function of the wave number $k$ for the power law coupling with $p = 2$, $f_0 = 1/5$. The dashed black line indicates the numerical solution for slow roll inflation ($\beta = 0$) while the solid red line is the spectrum resulting from a deviation from slow roll. The horizontal dotted blue line is the magnetic field spectrum from the slow roll approximation, $f_N = 4/5$. The deviation from this approximated spectrum is always small. Only the scales which exit the Hubble radius around the time when a deviation from slow roll or equivalently a bump in the coupling term occurs are affected more strongly.

6. The magnetic field at the end of inflation and its further evolution

Our main results are Eqs. (65) and (70) which determine the magnetic energy density and helicity density at the end of inflation. Note, however, that we do not renormalise the energy density. Hence even for $f_N = 0$, we obtain the non-vanishing result

$$\frac{d\tilde{\rho}_B}{d\ln k}(t_{\text{end}}, k) = \frac{k^4}{(2\pi)^2}$$

which comes purely from (not amplified) vacuum fluctuations and may be considered unphysical. However, from Eq. (38) it is clear that $P_S(k) \geq |P_A(k)|$ by definition. Hence the physical result cannot be obtained by a simple subtraction of the vacuum contribution as then $P_S \propto f_N^2$ would become smaller than $P_A \propto f_N$ for small values of $f_N$, see Eqs. (63) and (64). On the other hand, for $f_N \gg 1$, the vacuum contribution becomes subdominant and it is no longer important to subtract it. We shall therefore not perform any renormalisation of the magnetic energy density but just keep in mind that our result becomes dominated by vacuum fluctuations in the limit $f_N \to 0$. 
After inflation, the thermal cosmic plasma contains many relativistic charged particles and can be treated as an MHD plasma. During the process of reheating, the Reynolds number becomes very high and MHD turbulence develops. In the MHD limit the electric field is damped away and the magnetic field evolves by two different processes: it is damped on small scales and it undergoes an inverse cascade due to helicity conservation [31]. Numerical studies have shown that very soon damping on small scales leads to a maximally helical field (for which either \( A_+ \) or \( A_- \) vanishes) which then continues to evolve via an inverse cascade, see [32, 33]. The inverse cascade is active as long as the Reynolds number of the cosmic fluid at the scale under consideration is larger than one and the fluid is therefore turbulent [57]. The damping scale \( k_{\text{diss}}(t) \) is the scale at which the Reynolds number becomes of order unity. On scales smaller than \( k_{\text{diss}}(t) \) the magnetic field and the turbulent motion of the fluid are damped exponentially by viscosity.

In the following we investigate how the spectrum of helical magnetic fields evolves during the turbulent epoch. We first discuss the evolution of the correlation scale of the magnetic field and the duration of the turbulent phase, before computing the magnetic energy spectrum at the end of the inverse cascade. Here we assume that the reheating epoch is relatively short and ends at \( t^* \). This corresponds to the reheating temperature \( T^* \), and because of radiation domination, we can approximately use \( t/t^* \approx T^*/T \) during the turbulent phase.

The helical magnetic field from inflation always has a blue spectrum and is therefore dominated by the largest wavenumber crossing the Hubble scale at the end of inflation or, for simplicity, reheating

\[
\tilde{\rho}_B(t^*) \approx k_*^4 S^2(f_N), \quad \text{with} \quad k_* \approx \mathcal{H}_*.
\]

Accordingly, the correlation scale (which is roughly given by the scale at which the power spectrum peaks) is initially \( k_c(t^*) = k_* \). As an example we consider \( T_* = 10^{14} \text{GeV} \) and find

\[
k_c(t^*) = k_* = \mathcal{H}_* = H_* a_* = \sqrt{\frac{a_{SB}}{3}} g_0^{1/3} g_*^{1/6} \frac{T_0}{m_p} T_*
\]

\[
\approx 10^{-17} \text{GeV} \left( \frac{g_*}{200} \right)^{1/6} \left( \frac{T_*}{10^{14} \text{GeV}} \right)
\]

\[
\approx 2 \times 10^{21} \text{Mpc}^{-1} \left( \frac{g_*}{200} \right)^{1/6} \left( \frac{T_*}{10^{14} \text{GeV}} \right).
\]

Here \( a_{SB} \) is the Stefan-Boltzmann constant, \( a_{SB} = \pi^2 / 15 \) in our units, \( g_* \) and \( g_0 = 2 \) denote the number of relativistic degrees of freedom at \( t_* \) and today, respectively, and \( T_0 = 2.73 \text{K} \) is the present CMB temperature.

Let us assume that the inverse cascade starts at \( t_* \). In Ref. [33] it was found that during the inverse cascade of a maximally helical magnetic field the total rescaled energy density scales like

\[
\tilde{\rho}_B(t) \propto (t/t_*)^{-2/3}
\]
Figure 5. The Reynolds number $\text{Re}(T)$ (black solid) and the comoving correlation scale $k_c(T)/k_{\text{fin}}$ (red dashed) are shown in log scale as a function of $\log(T_*/T)$ with $T_* = 10^{14}$ GeV. Turbulence and with it the inverse cascade terminate at $T_{\text{fin}} \simeq 1$ GeV and $k_{\text{fin}} = k_c(t_{\text{fin}}) \simeq \left(10^{-12}\;\text{Mpc}\right)^{-1}$.

and the comoving correlation scale evolves in the same way

$$k_c(t) = k_c(t_*) (t/t_*)^{-2/3}$$

such that the ratio $\bar{\rho}_B/k_c$ which is proportional to the rescaled helicity density remains constant. This continues until $t_{\text{fin}}$, the time when the damping scale has grown up to the correlation scale, $k_{\text{diss}}(t_{\text{fin}}) = k_c(t_{\text{fin}})$. After $t_{\text{fin}}$ the inverse cascade and turbulence cease and the magnetic field evolves solely by flux conservation on large scales

$$\frac{d\bar{\rho}_B}{d\ln k}(k, t) = \frac{d\bar{\rho}_B}{d\ln k}(k, t_{\text{fin}}) \quad \text{for} \quad t > t_{\text{fin}}, \; k < k_{\text{diss}}(t)$$

and viscosity damping on small scales, $k > k_{\text{diss}}(t)$. At the end of the inverse cascade, the correlation scale of magnetic field spectrum has moved to

$$k_{\text{fin}} \equiv k_c(t_{\text{fin}}) = k_* (T_{\text{fin}}/T_*)^{2/3}$$

and the total energy density is reduced by the same factor.

To compute the ratio $T_{\text{fin}}/T_*$ we need to determine the temperature at which the Reynolds number becomes unity. In Appendix A of Ref. 34 the Reynolds number at very high temperatures is estimated to be

$$\text{Re}(k, T) \propto \frac{aT}{k} \sqrt{\frac{\rho_B(k)}{\rho_f}}$$

for a given scale $k$. Here $\rho_f$ is the energy density of the fluid which contributes to the turbulent motion. In perfect thermal equilibrium $\rho_f = \rho$. More precisely, denoting the
Reynolds number at the beginning of the inverse cascade by \( R_\ast = \text{Re}(k_\ast(t_\ast), T_\ast) \), it is found

\[
\text{Re}(k_\ast(T, T)) = \begin{cases} 
R_\ast \left( \frac{T}{T_\ast} \right)^{1/3} & \text{for } T > T_{\text{ew}} = 100 \text{ GeV} \\
R_\ast \left( \frac{T_{\text{ew}}}{T_\ast} \right)^{1/3} \left( \frac{T}{T_{\text{ew}}} \right)^{11/3} & \text{for } T < T_{\text{ew}}.
\end{cases}
\] (75)

Setting \( g_\ast \simeq 200 \) and \( T_\ast = 10^{14} \text{ GeV} \) yields \( R_\ast \simeq a_\ast T_\ast/k_\ast(t_\ast) \simeq \sqrt{3/(a_{SB}g_\ast)} m_P/T_\ast \sim \mathcal{O}(10^3) \). (For simplicity we have set \( \rho_B/\rho_f \sim 1 \) for this value.) Note that after inflation, until the electroweak transition at \( T_{\text{ew}} \), the Reynolds number of the fluid is actually increasing. This comes from the fact that it is inversely proportional to the comoving mean free path which is constant at early times. After the electroweak phase transition, collisions are much more strongly suppressed and the comoving mean free path grows like \( a^4 \), hence the Reynolds number decreases rapidly. For more details see Ref. [34]. In Fig. 5 we show the evolution of both, the Reynolds number and the correlation scale of the helical magnetic field through the inverse cascade.

Finally, we can now derive the generic scaling of \( T_{\text{fin}} \) with the initial temperature \( T_\ast \). With the scaling \( k_{\text{fin}} = k_\ast(T_{\text{fin}}/T_\ast)^{2/3} \) and the help of Eq. (75) for the evolution of the Reynolds number, we find that the Reynolds number becomes unity and the inverse cascade stops at \( T_{\text{fin}} \) given by

\[
\frac{T_{\text{fin}}}{T_\ast} \simeq 10^{-14} \left( \frac{10^{14} \text{ GeV}}{T_\ast} \right)^{9/11}, \quad \text{for } T_\ast > T_{\text{ew}}
\] (76)

so that

\[
k_{\text{fin}} \simeq 10^{12} \text{ Mpc}^{-1} \left( \frac{T_\ast}{10^{14} \text{ GeV}} \right)^{5/11}.
\] (77)

For \( T_\ast = 10^{14} \text{ GeV} \) we obtain \( T_{\text{fin}} \simeq 1 \text{ GeV} \) and the correlation scale moves by about 9 orders of magnitude from \( k_\ast \) to \( k_{\text{fin}} \), see also Fig. 5.

Now we can trace the magnetic field spectrum through the inverse cascade. The spectral shape on large scales remains unchanged [33]. At late time, \( t > t_{\text{fin}} \) we therefore obtain

\[
\frac{d\tilde{\rho}_B}{d\ln k}(t, k) \simeq \begin{cases} 
\frac{d\tilde{\rho}_B}{d\ln k}(t_\ast, k_\ast) \left( \frac{k}{k_\ast} \right)^4 \left( \frac{T_\ast}{T_{\text{fin}}} \right)^{2} & \text{for } k < k_{\text{fin}} \\
0 & \text{else.}
\end{cases}
\] (78)

With \( T_\ast/T_{\text{fin}} \simeq 10^{14} \) this yields an amplification of the initial amplitude by 28 orders of magnitude on scales larger than \( 1/k_{\text{fin}} \), see the sketch in Fig. 6.

For a generic spectral index \( n \), Eq. (78) is replaced by, see [34],

\[
\frac{d\tilde{\rho}_B}{d\ln k}(t, k) \simeq \begin{cases} 
\frac{d\tilde{\rho}_B}{d\ln k}(t_\ast, k_\ast) \left( \frac{k}{k_\ast} \right)^{n+3} \left( \frac{T_\ast}{T_{\text{fin}}} \right)^{2(n+2)/3} & \text{for } k < k_{\text{fin}}, \ t > t_{\text{fin}} \\
0 & \text{else.}
\end{cases}
\] (79)

Clearly, the smaller \( n \) the less significant is the amplification by the inverse cascade and for \( n = -2 \) there is no amplification at all. For \( n < -2 \) the above result does not apply, see [34].
In our case, where $n = 1$, Eq. (65) can be written as
\[
\frac{d\tilde{\rho}_B}{d \ln k}(t_s, k_s) \simeq 4 \times 10^{-68} \text{GeV}^4 S^2(f_N) \left( \frac{g_s}{200} \right)^{2/3} \left( \frac{T_s}{10^{14} \text{GeV}} \right)^4
\]
with which we arrive at
\[
\frac{d\tilde{\rho}_B}{d \ln k}(t_{\text{fin}}, k_{\text{fin}}) \simeq 2 \times 10^{-77} \text{GeV}^4 S^2(f_N) \left( \frac{T_s}{10^{14} \text{GeV}} \right)^{38/11}
\]
where we assumed $g_s \simeq 200$. This determines the final strength of the magnetic field on large scales.

\[
\tilde{B}(k) \simeq 3 \times 10^{-19} \text{ Gauss } S(f_N) \left( \frac{k}{k_{\text{fin}}} \right)^2 \left( \frac{T_s}{10^{14} \text{GeV}} \right)^{19/11}
\]
for $k \leq k_{\text{fin}}$. With the help of Eq. (77) this can also be written as
\[
\tilde{B}(k) \simeq 3 \times 10^{-19} \text{ Gauss } S(f_N) \left( \frac{k}{10^{12} \text{Mpc}} \right)^2 \left( \frac{T_s}{10^{14} \text{GeV}} \right)^{9/11}
\]
for $k \leq k_{\text{fin}}$.

After the end of the turbulent phase, magnetic fields are damped on small scales by viscosity and evolve by flux conservation, so that $\tilde{B} = \text{const.}$ on large scales. For our typical value of $T_s \simeq 10^{14} \text{GeV}$ hence $k_{\text{fin}} \simeq 10^{12} / \text{Mpc}$, for cosmologically interesting scales of $k \sim 10 / \text{Mpc}$ the magnetic field is of the order of $\tilde{B}(k = 10 / \text{Mpc}) \simeq 10^{-40} \text{ Gauss}$. This is much too small for dynamo amplification. For smaller reheating temperatures, $T_s$, the Reynolds number grows less strongly and turbulence and the associated inverse cascade are of shorter duration. Therefore the value of $\tilde{B}(k)$ at fixed $k < k_{\text{fin}}$ is actually smaller for $T_s < 10^{14} \text{GeV}$ even though $k_s$ is larger for a smaller reheating temperature, see Eq. (81). Considering the lowest value for which our treatment is valid, $T_s \simeq T_{\text{ew}} \simeq 100 \text{GeV}$ we arrive at $T_{\text{fin}} \simeq 8 \text{MeV}$ and $k_{\text{fin}} \simeq 4 \times 10^6 \text{/Mpc}$ but the magnetic field is only
\[
\tilde{B}(k) \simeq 6 \times 10^{-40} \text{ Gauss } S(f_N) \left( \frac{k}{k_{\text{fin}}} \right)^2, \text{ for } k \leq k_{\text{fin}} \simeq 10^6 / \text{Mpc}.
\]
At scales of 0.1 Mpc, this field is by far insufficient for subsequent dynamo amplification which requires seed fields of the order of at least $10^{-20} \text{ Gauss}$ \cite{31}. For an arbitrary reheating temperature $T_s$, we obtain from (81)
\[
\tilde{B}(k = 10 / \text{Mpc}) \simeq 3 \times 10^{-41} \text{ Gauss } S(f_N) \left( \frac{T_s}{10^{14} \text{GeV}} \right)^{9/11}
\]
Let us investigate how large this field can become in the best case in which our treatment may apply. Certainly we want to require backreaction to be unimportant, but we do not insist that $f_N$ be small. Using (69) this requires $S(f_N) < (m_p / T_s)^2$ and hence
\[
\tilde{B}(k = 10 / \text{Mpc}) \leq 10^{-32} \text{ Gauss } \left( \frac{10^{14} \text{ GeV}}{T_s} \right)^{13/11}
\]
|| The magnetic field strength is $B = \sqrt{8\pi \rho_B}$ in Gauss units, where GeV$^2 \simeq 1.4 \times 10^{19}$ Gauss, while in Heaviside-Lorentz units we have $B = \sqrt{2\rho_B}$ and GeV$^2 / \sqrt{4\pi} \simeq 1.4 \times 10^{19}$ Gauss.
To achieve a minimal necessary field for dynamo amplification of $B \sim 10^{-20}$ Gauss we would need $T_* \leq 10^4$ GeV, a somewhat low inflation scale, but not excluded.

Until recombination, fields on scales smaller than about $0.1(B/10^{-9}\text{Gauss})$ Mpc are damped by viscosity [58]. The scales $k \gtrsim 10(10^{-9}\text{Gauss}/B)/\text{Mpc}$ therefore do not survive the linear regime and will not be amplified before being damped. But even on these smallest “surviving scales” the magnetic field generated is in most cases too weak for dynamo amplification.

7. Conclusions

In this work, we have studied the generation of helical magnetic fields during inflation by adding a parity-violating term of the form $f(\phi) F\tilde{F}$ to the action. For two specific choices of the coupling function $f(\phi)$, namely a power law and an exponential form, we found that, during slow roll inflation, the power spectrum of the generated magnetic fields is always blue with spectral index $n = 1$. The aspect responsible for this result...
is that the helical coupling term in the evolution equation for the Fourier modes of the vector potential is active only around horizon crossing of a given mode and it is not varying rapidly during slow roll inflation. As a consequence, in slow roll inflation all Fourier modes are amplified in the same way and the final shape of the power spectrum is unchanged with respect to the initial one.

We have also studied the effects of a short deviation from slow roll. If kept within the bounds allowed by the CMB data, such a departure from slow roll does not strongly modify the overall shape of the magnetic field power spectrum. In principle, a very steep coupling function, e.g. a double-exponential \( f(\phi) = f_0 \exp[\exp(\alpha\phi/m_P)] \), can lead to a different spectrum, \( n \neq 1 \). However, if we want to satisfy perturbativity, \( |f_N| < 1 \), for such a behaviour to be valid throughout inflation, a ridiculously small pre-factor \( f_0 \) is required. In the absence of a convincing physical motivation, we have not investigated such an extreme case.

After inflation and reheating, an inverse cascade sets in and moves power from the correlation scale \( k_* \simeq H_* \) to larger scales, which is schematically represented in Fig. 6. We find that typical values for the reheating temperature, e.g. \( T_* \simeq 10^{14} \) GeV, can only lead to magnetic fields of the order of \( 10^{-40} \) Gauss on scales of 0.1 Mpc and smaller values are obtained for smaller reheating temperatures. Unless \( S(f_N) \gg 1 \) and the reheating temperature is sufficiently low, \( T_* \leq 10^4 \) GeV, these field amplitudes are largely insufficient for dynamo amplification [31].

This result is quite generic. A power law or an exponential coupling to the \( F\tilde{F} \) term during slow roll inflation will generate sufficiently strong seeds for the large scale magnetic fields in galaxies and clusters without also leading to backreaction only if the inflation scale is sufficiently low, of the order of the electroweak scale. Even though for helical fields an inverse cascade does move power to larger scales, this is still insufficient in most cases. For typical inflation scales, to obtain sufficient fields on large scales after the inverse cascade, they must have been so large on small scales after inflation that backreaction cannot be neglected. Another possibility might be a very steep coupling which can lead to a different, \( n \neq 1 \) spectrum. However, only if the spectrum is close to scale invariant, it can have significant amplitudes on large scales without significant backreaction from small scales.

Scenarios of inflationary magnetogenesis are often constrained by requiring that the backreaction of the generated magnetic field on the background evolution is small. Since the perturbations in the scalar field are affected by the presence of a non-minimal coupling as indicated in Eq. (17), it will be interesting to study the backreaction effects of the EM perturbations on the evolution of the primordial curvature perturbations. This may provide another tool to constrain inflationary scenarios of magnetogenesis. This issue will be addressed in a future project [47].
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Appendix A. Quantisation of the vector potential

Always working in Coulomb gauge, we promote the vector potential $A_i$ to a quantum mechanical operator, define conjugate momentum as $\Pi^j \equiv \delta S/\delta \dot{A}_j$, and impose the commutation relation

$$[A_i(t, x), \Pi^j(t, y)] = i \delta_{ij} (x - y). \quad (A.1)$$

Here the transversal Dirac delta, $\delta_{ij}^{\perp}$, is defined as

$$\delta_{ij}^{\perp}(x - y) \equiv \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k} \cdot (x - y)} (\hat{k}_i \hat{k}_m \delta^{mn}) \quad (A.2)$$

with $\hat{k}_i \equiv k_i/k$ and $k \equiv |\mathbf{k}|$. Let us compute the conjugate momentum in terms of the vector potential which we expand in terms of creation and annihilation operators in Fourier space. The canonical commutation relations of the creation and annihilation operators, together with the above commutation relation of $A_i$ and $\Pi^j$ will lead to a Wronskian normalisation condition on the mode functions of the quantised vector potential.

The conjugate momenta in the perturbed FL metric are

$$\Pi^i = a^2 g^{ij} \left( \dot{A}_j + f(\varphi) \epsilon_{jmn} \partial_m A_n \right). \quad (A.3)$$

The non-standard second term arises due to the axial coupling of the EM field to the inflaton. Since it involves the curl of the vector potential it is convenient to expand the operators in Fourier space with respect to a helicity basis. For each comoving wave vector, $\mathbf{k}$, we define the comoving right-handed orthonormal basis

$$e_\mu^0 \equiv a \left( 1, 0 \right), \quad e_{\mu}^{1,2}(\mathbf{k}) \equiv a \left( 0, \epsilon^k_{1,2} \right), \quad e_{\mu}^3(\mathbf{k}) \equiv a \left( 0, \hat{k} \right) \quad (A.4)$$

with

$$\hat{k} \cdot \epsilon^k_{1,2} = 0 = \epsilon^k_1 \cdot \epsilon^k_2, \quad \epsilon^k_1 \wedge \epsilon^k_2 = \hat{k}. \quad |\epsilon^k_1|^2 = 1 \quad (A.5)$$

and

$$\epsilon_{1}^{-k} = -\epsilon^k_1, \quad \epsilon_{2}^{-k} = \epsilon^k_2. \quad (A.6)$$

The two transverse directions are combined to the helicity (or circular) directions

$$e^k_{\pm} \equiv \frac{1}{\sqrt{2}} (e^k_1 \pm ie^k_2). \quad (A.7)$$
These have the following useful properties

\[ \varepsilon^k_{\pm} = \varepsilon^k_{\mp} \quad (A.8) \]

\[ \varepsilon^{-k}_{\pm} = -\varepsilon^k_{\mp} \quad (A.9) \]

\[ i\hat{k} \wedge \varepsilon^k_{\pm} = \pm \varepsilon^k_{\mp} \quad (A.10) \]

\[ \varepsilon^k_h \cdot \varepsilon^{k*}_{h'} = \delta_{hh'} \quad (A.11) \]

\[ \sum_h \varepsilon^k_{h,i} \varepsilon^{k}_{h,j} = \delta_{ij} - \hat{k}_i \hat{k}_j \quad (A.12) \]

where \( h, h' \in \{+, -\} \) and the star denotes complex conjugation. We now expand the vector potential in Fourier space in the helicity basis.

\[ A_j(t, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} \sum_h \left\{ e^h_j(k) b_h(k) A_h(t, k) e^{i\mathbf{k} \cdot \mathbf{x}} + e^{h*}_j(k) b^\dagger_h(k) A^*_h(t, k) e^{-i\mathbf{k} \cdot \mathbf{x}} \right\} \quad (A.13) \]

where the creation and annihilation operators, \( b^\dagger_h(k) \) and \( b_h(k) \), satisfy the canonical commutation relations

\[ \left[ b_h(k), b^\dagger_{h'}(q) \right] = (2\pi)^3 \delta^3(\mathbf{k} - \mathbf{q}) \delta_{hh'} \quad (A.14) \]

\[ \left[ b_h(k), b_{h'}(q) \right] = 0 = \left[ b^\dagger_h(k), b^\dagger_{h'}(q) \right] \quad (A.15) \]

The vacuum, \( \ket{0} \), is defined by \( b_h(k) \ket{0} \equiv 0 \). As a consequence of the commutation relation (A.11) of \( A_i \) and its conjugate momentum, \( \Pi^j \), as given in (A.3), the mode function must satisfy the Wronskian normalisation condition. In terms of the rescaled Fourier modes of the vector potential, \( A_h(t, k) \equiv a A_h(t, k) \), this reads

\[ \dot{A}_h(t, k) A^*_h(t, k) - A^*_h(t, k) \dot{A}_h(t, k) = i \quad (A.16) \]

Finally, the evolution equation for the Fourier modes can now be derived from the forced wave equation of the vector potential, Eq. (16),

\[ \dot{A}_h(t, k) + \left[ k^2 + h k f'(\phi) \right] A_h(t, k) = 0 \quad (A.17) \]

where \( h = \pm \) reflects the two helicity modes.

**Appendix B. Perturbative chiral interaction**

In order to derive a bound on the parameter space of the coupling function \( f(\phi) \), be it a power law, an exponential or any other functional form, we demand the interaction between the scalar field and electromagnetism to be small in order for perturbative quantum field theory to make sense. For example, we want the interaction picture to apply and the first terms in the Feynman-Dyson series for the scattering matrix to be small. Therefore we demand that \( |S_I[\phi, A_\mu]| < |S_{em}[A_\mu]| \).

Let us express the free EM Lagrangian density in terms of the electric and magnetic energy densities (as defined in Sec. 2.4)

\[ \mathcal{L}_{em} = -\frac{1}{4} F^2 = \frac{1}{2} (E^2 - B^2) = \rho_E - \rho_B \quad (B.1) \]
and compute the free EM action in the FL background:

\[ S_{\text{em}}[A_\mu] = \int d^4 x \sqrt{-g} L_{\text{em}} = \int d^4 x \, a^4 (\rho_E - \rho_B) \]

\[ = \int dt \int d^3 x \, (\bar{\rho}_E - \bar{\rho}_B) = V \int dt \, \langle \bar{\rho}_E(t) - \bar{\rho}_B(t) \rangle_V \quad (B.2) \]

where we rescaled the energy densities as \( \bar{\rho}_B \equiv a^4 \rho_B \) and defined the volume average \( \langle \ldots \rangle_V \) in the last step. We assume that for quantum fluctuations the volume average equals the vacuum expectation value, \( \langle \ldots \rangle \), such that

\[ \langle \bar{\rho}_B(t) \rangle_V = \frac{1}{2} \langle \tilde{B}_i \tilde{B}_j^* \rangle \delta^{ij} = \int_0^\infty \frac{k^4 dk}{(2\pi)^2} \left[ |A_+(t, k)|^2 + |A_-(t, k)|^2 \right] \quad (B.3) \]

\[ \langle \bar{\rho}_E(t) \rangle_V = \frac{1}{2} \langle \tilde{E}_i \tilde{E}_j^* \rangle \delta^{ij} = \int_0^\infty \frac{k^2 dk}{(2\pi)^2} \left[ |\dot{A}_+(t, k)|^2 + |\dot{A}_-(t, k)|^2 \right] , \quad (B.4) \]

as we show in Sec. 2.2.

To be able to compute the contribution of the interaction \( f(\phi)F \tilde{F} \) to the action we pass to the language of differential forms by writing the electromagnetic 2-form

\[ F = \frac{1}{2} F_{\mu \nu} dx^\mu dx^\nu = dA = d(A_\mu dx^\mu) , \]

such that

\[ F \tilde{F} \sqrt{-g} d^4 x = 2 F \wedge F \quad (B.5) \]

Then we use the facts that \( F = dA \) and \( dF = d(dA) = 0 \) to integrate by parts

\[ S_I[\phi, A_\mu] = \frac{1}{4} \int d^4 x \sqrt{-g} f(\phi) F \tilde{F} = \frac{1}{2} \int f(\phi) dA \wedge F \]

\[ = \frac{1}{2} \int \left( d[f A \wedge F] - df \wedge A \wedge F \right) = \frac{1}{2} \int df \wedge F \wedge A . \quad (B.6) \]

In the last step we assume the boundary terms to vanish. Clearly, \( S_I[\phi, A_\mu] \) vanishes for a constant \( f(\phi) \). Next, we express \( F \) in terms of \( E \) and \( B \) as measured by an observer with timelike four-velocity \( u \), see Sec. 2.4

\[ F = u \wedge E - 2(u \wedge B) . \quad (B.7) \]

Using this we find

\[ df \wedge F \wedge A = df \wedge u \wedge E \wedge A + 2 f_{,\alpha} u^{[\alpha} B^{\beta]} A_\beta \sqrt{-g} d^4 x . \quad (B.8) \]

In a FL background, the only valid observer is the Hubble flow, \( u = -ad t \), and \( f \) is only a function of time, i.e. \( df = \dot{f} dt \), such that \( df \wedge u = 0 \), \( f_{,\alpha} u^\alpha = a^{-1} \dot{f} \) and \( f_{,\alpha} B^\alpha = 0 \). So we find

\[ df \wedge F \wedge A = \dot{f} B^\beta A_\beta a^3 d^4 x = \dot{f} \tilde{\mathcal{H}} d^4 x \quad (B.9) \]

where \( \tilde{\mathcal{H}} \) is the rescaled helicity density which we introduce in Sec. 2.4. Finally, we can integrate to find the action

\[ S_I[\phi, A_\mu] = \frac{1}{2} \int d^4 x \, \dot{f} \tilde{\mathcal{H}} = \frac{1}{2} V \int dt \, \dot{f} \langle \tilde{\mathcal{H}}(t) \rangle_V . \quad (B.10) \]

And again, the volume average is given by the expectation value

\[ \langle \tilde{\mathcal{H}}(t) \rangle_V = \langle \tilde{B}_i \tilde{A}_j^* \rangle \delta^{ij} = 2 \int_0^\infty \frac{k^3 dk}{(2\pi)^2} \left[ |A_+(t, k)|^2 - |A_-(t, k)|^2 \right] . \quad (B.11) \]
using the results given in Sec. 3.3.

The implications for the coupling function from the condition $|S_f|/A_\mu| < |S_{em}|/A_\mu|$ using the results derived here are discussed in Sec. 4.1.

Appendix C. Asymptotic behaviour of the Coulomb wave functions

To investigate the late time behaviour of the solution to the mode equation for a power law coupling, we need asymptotic expressions for the Coulomb wave function, $G_0(y, x)$ and $F_0(y, x)$, for $x \rightarrow 0$. Based on their expansion in terms of Bessel functions given in Abramowitz & Stegun [49], we derive the asymptotes of $G_0, F_0, G'_0, F'_0$ for $x \rightarrow 0$ for arbitrary but fixed $y$. Actually, we argue that the asymptotic limits for $x \ll 2y$ given in 14.6.9-10 (p. 542) of [49] are incorrect. All statements given here were verified numerically using Mathematica [59] and Maple [60].

We start from the asymptotic expressions in terms of the modified Bessel functions, $K$ and $I$, given in 14.6.7 of [49] for $L = 0$, and $2y \gg x$

\begin{align}
G_0(y, x) &\simeq 2\sqrt{2xy} W(y) K_1(2\sqrt{2xy}) \\
F_0(y, x) &\simeq \frac{x}{\sqrt{2xy}} W(y)^{-1} I_1(2\sqrt{2xy})
\end{align}

where (see $C_0$ given in 14.1.8 of [49])

\[ W(y) \equiv \left[ \frac{e^{\pi y} \sinh(\pi y)}{\pi y} \right]^{1/2}. \]

We notice that the approximations given in 14.6.8 of A&S, where

\[ \sinh \pi y \simeq \frac{1}{2} e^{\pi y} \]

is used, are only accurate to better than 1% if $y \gtrsim 1$. Since in our case $y$ is arbitrary or rather smaller than unity, we cannot employ this approximation. As for the $x$-derivatives of $G_0$ and $F_0$ we have verified numerically that the derivatives of the above expressions are good approximations,

\begin{align}
G'_0(y, x) &\simeq -4yW(y) K_0(2\sqrt{2xy}) \\
F'_0(y, x) &\simeq W(y)^{-1} I_0(2\sqrt{2xy})
\end{align}

Next we use the asymptotic properties of the modified Bessel functions for small arguments. As given on p. 375 in [49]

\begin{align}
K_1(z) &\simeq \frac{1}{z} + \mathcal{O}(z), & K_0(z) &\simeq -\ln(z/2) - \gamma_E + \mathcal{O}(z^2) \\
I_1(z) &\simeq \frac{z}{2} + \mathcal{O}(z^3), & I_0(z) &\simeq 1 + \mathcal{O}(z^2)
\end{align}

for $z = 2\sqrt{2xy} \ll 1$. Using these expressions we find to lowest order in $x$ for $x \ll (8y)^{-1}$

\begin{align}
G_0(y, x) &\simeq W(y), & G'_0(y, x) &\simeq 2yW(y) \ln(2xy) \\
F_0(y, x) &\simeq x/W(y), & F'_0(y, x) &\simeq 1/W(y).
\end{align}

These asymptotic expressions differ significantly from those given in 14.6.9-10 of [49] and it is straightforward to understand why: one arrives at the expressions of [49]
when falsely using the asymptotic expansions of the modified Bessel functions for large arguments, where $I_1(|z| \gg 1) \simeq e^z/\sqrt{2\pi z}$ and $K_1(|z| \gg 1) \simeq e^{-z}\sqrt{\pi/(2z)}$. However, for $x \to 0$ and fixed $y$, $z = 2\sqrt{2}xy$ tends to zero and is not large as assumed for approximations 14.6.9-10 of [49].

References

[1] P. P. Kronberg, Extragalactic magnetic fields, Rept. Prog. Phys. 57 (1994) 325–382.
[2] L. Pentericci, W. Van Reeven, C. L. Carilli, H. J. A. Rottgering, and G. K. Miley, VLA radio continuum observations of a new sample of high redshift radio galaxies, Astron. Astrophys. Suppl. Ser. 145 (July, 2000) 121–159, [astro-ph/0005524].
[3] N. Battaglia, C. Pfenninger, J. L. Sievers, J. R. Bond, and T. A. Emslie, Exploring the magnetized cosmic web through low frequency radio emission, Mon. Not. Roy. Astron. Soc. 393 (Mar., 2009) 1073–1089, [arXiv:0806.3272].
[4] T. E. Clarke, P. P. Kronberg, and H. Boehringer, A New Radio - X-Ray Probe of Galaxy Cluster Magnetic Fields, Astrophys. J. 547 (2001) L111–L114, [astro-ph/0011281].
[5] A. Neronov and I. Vovk, Evidence for strong extragalactic magnetic fields from Fermi observations of TeV blazars, Science 328 (2010) 73–75, [arXiv:1006.3504].
[6] F. Tavecchio et al., The intergalactic magnetic field constrained by Fermi/LAT observations of the TeV blazar 1ES 0229+200, Mon. Not. Roy. Astron. Soc. 406 (2010) L70–L74, [arXiv:1004.1329].
[7] F. Tavecchio, G. Ghisellini, G. Bonnoli, and L. Foschini, Extreme TeV blazars and the intergalactic magnetic field, [arXiv:1009.1048].
[8] K. Dolag, M. Kachelriess, S. Ostapchenko, and R. Tomas, Lower limit on the strength and filling factor of extragalactic magnetic fields, Astrophys. J. 727 (2011) L4, [arXiv:1009.1782].
[9] S. Matarrese, S. Mollerach, A. Notari, and A. Riotto, Lower limit on the strength and filling factor of extragalactic magnetic fields, Phys. Rev. D82 (2010) 083005, [arXiv:0805.0169].
[20] P. Trivedi, K. Subramanian, and T. R. Seshadri, Primordial Magnetic Field Limits from Cosmic Microwave Background Bispectrum of Magnetic Passive Scalar Modes, Phys. Rev. D82 (2010) 123006, arXiv:1009.2724.
[21] K. Enqvist and P. Olesen, On primordial magnetic fields of electroweak origin, Phys. Lett. B319 (1993) 178–185, hep-ph/9308270.
[22] M. Joyce and M. E. Shaposhnikov, Primordial magnetic fields, right electrons, and the Abelian anomaly, Phys. Rev. Lett. 79 (1997) 1193–1196, astro-ph/9703009.
[23] M. M. Forbes and A. R. Zhitnitsky, Primordial galactic magnetic fields from domain walls at the QCD phase transition, Phys. Rev. Lett. 85 (2000) 5268–5271, hep-ph/0004051.
[24] T. Boeckel and J. Schaffner-Bielich, A little inflation in the early universe at the QCD phase transition, Phys. Rev. Lett. 105 (2010) 041301, arXiv:0906.4520.
[25] A. Diaz-Gil, J. Garcia-Bellido, M. G. Perez, and A. Gonzalez-Arroyo, Primordial magnetic fields from preheating at the electroweak scale, JHEP 07 (2008) 043, arXiv:0805.4159.
[26] R. Durrer and C. Caprini, Primordial Magnetic Fields and Causality, JCAP 0311 (2003) 010, astro-ph/0305059.
[27] C. Caprini and R. Durrer, Gravitational wave production: A strong constraint on primordial magnetic fields, Phys. Rev. D65 (2001) 023517, astro-ph/0106244.
[28] K. Subramanian, Magnetic fields in the early universe, Astron. Nachr. 331 (2010) 110–120, arXiv:0911.4771.
[29] J. Martin and J. Yokoyama, Generation of Large-Scale Magnetic Fields in Single-Field Inflation, JCAP 0801 (2008) 025, arXiv:0711.4307.
[30] V. Demozzi, V. Mukhanov, and H. Rubinstein, Magnetic fields from inflation?, JCAP 0908 (2009) 025, arXiv:0907.1030.
[31] A. Brandenburg and K. Subramanian, Astrophysical magnetic fields and nonlinear dynamo theory, Phys. Rept. 417 (2005) 1–209, astro-ph/0405052.
[32] R. Banerjee and K. Jedamzik, The Evolution of Cosmic Magnetic Fields: From the Very Early Universe, to Recombination, to the Present, Phys. Rev. D70 (2004) 123003, astro-ph/0410032.
[33] L. Campanelli, Evolution of Magnetic Fields in Freely Decaying Magnetohydrodynamic Turbulence, Phys. Rev. Lett. 98 (2007) 251302, arXiv:0705.2303.
[34] C. Caprini, R. Durrer, and E. Fenu, Can the observed large scale magnetic fields be seeded by helical primordial fields?, JCAP 0911 (2009) 001, arXiv:0906.4976.
[35] T. Vachaspati, Estimate of the primordial magnetic field helicity, Phys. Rev. Lett. 87 (2001) 251302, astro-ph/0101251.
[36] C. Caprini, R. Durrer, and T. Kahnashvili, The Cosmic Microwave Background and Helical Magnetic Fields: the tensor mode, Phys. Rev. D69 (2004) 063006, astro-ph/0304556.
[37] N. Seto and A. Taruya, Polarization analysis of gravitational-wave backgrounds from the correlation signals of ground-based interferometers: measuring a circular-polarization mode, Phys. Rev. D77 (2008) 103001, arXiv:0801.4185.
[38] J. M. Cornwall, Speculations on primordial magnetic helicity, Phys. Rev. D56 (1997) 6146–6154, hep-th/9704022.
[39] G. B. Field and S. M. Carroll, Cosmological magnetic fields from primordial helicity, Phys. Rev. D62 (2000) 103008, astro-ph/9811206.
[40] L. Campanelli and M. Giannotti, Production of Axions by Cosmic Magnetic Helicity, Phys. Rev. Lett. 96 (2006) 161302, astro-ph/0512458.
[41] L. Campanelli and M. Giannotti, Magnetic Helicity Generation from the Cosmic Axion Field, Phys. Rev. D72 (2005) 123001, astro-ph/0508653.
[42] M. M. Anber and L. Sorbo, N-flationary magnetic fields, JCAP 0610 (2006) 018, astro-ph/0606534.
[43] L. Campanelli, Helical Magnetic Fields from Inflation, Int. J. Mod. Phys. D18 (2009) 1395–1411, arXiv:0805.0575.
[44] M. Giannotti and E. Mottola, *The Trace Anomaly and Massless Scalar Degrees of Freedom in Gravity*, Phys. Rev. D79 (2009) 045014, [arXiv:0812.0351](http://arxiv.org/abs/0812.0351).

[45] R. Durrer, *The Cosmic Microwave Background*. Cambridge University Press, Cambridge, UK, 2008.

[46] M. M. Anber and L. Sorbo, *Naturally inflating on steep potentials through electromagnetic dissipation*, Phys. Rev. D81 (2010) 043534, [arXiv:0908.4089](http://arxiv.org/abs/0908.4089).

[47] R. Durrer, L. Hollenstein, and R. K. Jain, *Effects of helical magnetic field generation on the primordial curvature perturbation*, In preparation.

[48] J. D. Barrow, R. Maartens, and C. G. Tsagas, *Cosmology with inhomogeneous magnetic fields*, Phys. Rept. 449 (2007) 131–171, [astro-ph/0611537](http://arxiv.org/abs/astro-ph/0611537).

[49] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*. Dover, New York, USA, 1972.

[50] L. Covi, J. Hamann, A. Melchiorri, A. Slosar, and I. Sorbera, *Inflation and WMAP three year data: Features have a future!*, Phys. Rev. D74 (2006) 083509, [astro-ph/0606452](http://arxiv.org/abs/astro-ph/0606452).

[51] J. Hamann, L. Covi, A. Melchiorri, and A. Slosar, *New constraints on oscillations in the primordial spectrum of inflationary perturbations*, Phys. Rev. D76 (2007) 023503, [astro-ph/0701380](http://arxiv.org/abs/astro-ph/0701380).

[52] M. J. Mortonson, C. Dvorkin, H. V. Peiris, and W. Hu, *CMB polarization features from inflation versus reionization*, Phys. Rev. D79 (2009) 103519, [arXiv:0903.4920](http://arxiv.org/abs/0903.4920).

[53] R. K. Jain, P. Chingangbam, J.-O. Gong, L. Sriramkumar, and T. Souradeep, *Double inflation and the low CMB multipoles*, JCAP 0901 (2009) 009, [arXiv:0809.3915](http://arxiv.org/abs/0809.3915).

[54] R. K. Jain, P. Chingangbam, L. Sriramkumar, and T. Souradeep, *The tensor-to-scalar ratio in punctuated inflation*, Phys. Rev. D82 (2010) 023509, [arXiv:0904.2518](http://arxiv.org/abs/0904.2518).

[55] D. K. Hazra, M. Aich, R. K. Jain, L. Sriramkumar, and T. Souradeep, *Primordial features due to a step in the inflaton potential*, JCAP 1010 (2010) 008, [arXiv:1005.2175](http://arxiv.org/abs/1005.2175).

[56] J. A. Adams, B. Cresswell, and R. Easther, *Inflationary perturbations from a potential with a step*, Phys. Rev. D64 (2001) 123514, [astro-ph/0102238](http://arxiv.org/abs/astro-ph/0102238).

[57] D. Biskamp, *Magnetohydrodynamic Turbulence*. Cambridge University Press, Cambridge, UK, 2003.

[58] K. Jedamzik, V. Katalinic, and A. V. Olinto, *Damping of Cosmic Magnetic Fields*, Phys. Rev. D57 (1998) 3264–3284, [astro-ph/9606080](http://arxiv.org/abs/astro-ph/9606080).

[59] Wolfram Research, Inc., *Mathematica: version 7.0*, 2008.

[60] Maplesoft, *Maple: version 12.02*, 2009.