FROM EXACT SYSTEMS TO RIESZ BASES
IN THE BALIAN–LOW THEOREM

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Abstract. We look at the time–frequency localisation of generators of lattice
Gabor systems. For a generator of a Riesz basis, this localisation is described by
the classical Balian–Low theorem. We establish Balian–Low type theorems for
complete and minimal Gabor systems with a frame-type approximation prop-
erty. These results describe how the best possible localisation of a generator is
limited by the degree of control over the coefficients in approximations given by
the system, and provide a continuous transition between the classical Balian–
Low conditions and the corresponding conditions for generators of complete and
minimal systems. Moreover, this holds for the non-symmetric generalisations of
these theorems as well.

1. INTRODUCTION

For \( g \in L^2(\mathbb{R}) \) and \( a, b > 0 \), the Gabor system generated by \( g \) on the lattice
\( a\mathbb{Z} \times b\mathbb{Z} \) is denoted by
\[
G(g, a, b) := \{e^{2\pi ibmt} g(t - an)\}_{(m,n) \in \mathbb{Z}^2}.
\]

Such systems were considered by Gabor \([8]\) and today they play a prominent role in
time–frequency analysis and its applications \([7, 12, 16, 24]\). An interesting general
problem in Gabor analysis is to find “optimal” bounds for the time–frequency
localisation of the window function \( g \), given appropriate constraints on the desired
Gabor system. The Balian–Low theorem gives a precise solution to a version of
this problem for Riesz bases.

In the context of this paper, a system \( G(g, a, b) \) is considered “good” if it is
at least exact (i.e., complete and minimal). It is “better” if, in addition, the
coefficients in the approximations it provides can be, in some sense, controlled
(e.g., it is a Riesz basis). Our main objective is to study the time–frequency
localisation of \( g \) for a scale of systems that lie between Riesz bases and exact

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We extend an uncertainty principle, known as the Balian–Low theorem, to these systems.

Since we are interested in exact systems, we consider systems $G(g,a,b)$ with $a = b = 1$. This is due to the known fact that if $ab > 1$, then $G(g,a,b)$ is not complete in $L^2(\mathbb{R})$, while if $ab < 1$, it is not a minimal system there. (See [21] for the first claim. A modification of the same argument gives the second claim.) Our results can be extended to any lattice $a\mathbb{Z} \times b\mathbb{Z}$ with $ab = 1$ by an appropriate dilation of the generating function $g$.

1.1. Balian–Low type theorems for Riesz bases and exact systems. The Balian–Low theorem [1, 5, 17] is a manifestation of the uncertainty principle in the context of Gabor analysis. It states that if the system $G(g,1,1)$ is a Riesz basis in $L^2(\mathbb{R})$, then the generator $g$ must have much worse time–frequency localisation than allowed by the uncertainty principle. More precisely, if $r \geq 2$, then at least one of the integrals

$$
\int_{\mathbb{R}} |\xi|^r |\hat{g}(\xi)|^2 d\xi, \quad \int_{\mathbb{R}} |t|^s |g(t)|^2 dt,
$$

must diverge, where $\hat{g}$ denotes the Fourier transform of $g$. This result is sharp. That is, for any $r < 2$ there exists a function $g \in L^2(\mathbb{R})$ such that $G(g,1,1)$ is a Riesz basis and both of the integrals in (1) converge [2].

Among the systems that we consider, Riesz bases have the best properties while exact systems have the weakest. For the latter, a Balian–Low type theorem was established in [6]. It states that if the system $G(g,1,1)$ is exact and $r \geq 4$, then at least one of the integrals in (1) must diverge. It follows from our Theorem 1 below that this result is sharp.

More generally, non-symmetric time–frequency conditions for generators of such systems have been considered. Namely, for which $r$ and $s$ can both of the integrals

$$
\int_{\mathbb{R}} |\xi|^r |\hat{g}(\xi)|^2 d\xi, \quad \int_{\mathbb{R}} |t|^s |g(t)|^2 dt,
$$

converge? For simplicity of formulations, and without loss of generality, we assume that $r \leq s$ when discussing this case. For generators of Riesz bases, non-symmetric conditions were found in [4, 9, 10]: If $G(g,1,1)$ is a Riesz basis and

$$
\frac{1}{r} + \frac{1}{s} \leq 1,
$$

then at least one of the integrals in (2) must diverge. As above, this result is sharp [2] (see also [3]).

\footnote{In some sense, the Gaussian $g = \exp(-x^2/2)$ has the best possible time–frequency localisation. In this case, the system $G(g,a,b)$ is a frame in $L^2(\mathbb{R})$ if and only if $ab < 1$ [22]. However, it is always over-complete, i.e., it is never exact. The construction of Gabor orthonormal bases or exact systems is more delicate.}
The non-symmetric conditions for generators of exact systems were studied in [14], where it is shown that if the system \( G(g,1,1) \) is exact, and \( r \leq s \) satisfy
\[
\frac{3}{r} + \frac{1}{s} \leq 1, \tag{4}
\]
then at least one of the integrals in (2) must diverge. Again, the fact that this result is sharp follows from our Theorem 2 below.

The results presented in this paper provide a continuous interpolation between the condition \( r \geq 2 \) for generators of Riesz bases, and the condition \( r \geq 4 \) for generators of exact systems mentioned above. Moreover, these results are extended to the general non-symmetric case, where a similar interpolation is given between the conditions in (3) and in (4). In all cases, the results are sharp.

To obtain these results, we develop some new insights into the connection between the time–frequency localisation of a function and the smoothness of its Zak transform (see Section 3).

1.2. Between Riesz bases and exact systems. We now describe the family of systems that we consider in this work.

Let \( H \) be a separable Hilbert space. A system \( \{f_n\} \) is a Riesz basis in \( H \) if it is an exact frame, i.e., if it is exact and the following inequality holds for every \( f \in H \):
\[
A \|f\|^2 \leq \sum |\langle f, f_n \rangle|^2 \leq B \|f\|^2, \tag{5}
\]
where \( A \) and \( B \) are positive constants not depending on \( f \).

In most cases, the right-hand side inequality in (5), the Bessel property, holds automatically. Therefore, if one wants to relax the frame condition, there is usually no advantage in changing it. The left-hand side inequality in (5) is equivalent to completeness with \( \ell^2 \) control over the coefficients: Every \( f \in H \) can be approximated, with arbitrary small error, by a finite linear combination \( \sum a_n f_n \) with \( \sum |a_n|^2 \leq C \|f\|^2 \), for some positive constant \( C \) not depending on \( f \). We are interested in exact systems with a relaxed version of this property. We use the following definition introduced in [20].

**Definition 1.** Given \( q \geq 2 \), we say that a system \( \{f_n\} \) is a \((C_q)\)-system in \( H \) (complete with \( \ell^q \) control over the coefficients) if every \( f \in H \) can be approximated, with an arbitrary small error, by a finite linear combination \( \sum a_n f_n \) with
\[
\left( \sum |a_n|^q \right)^\frac{1}{q} \leq C \|f\|,
\]
where \( C = C(q) \) is a positive constant not depending on \( f \).

Note that all \((C_q)\)-systems are complete. In addition, if \( q_1 \leq q_2 \), then a \((C_{q_1})\)-system is also a \((C_{q_2})\)-system. Thus, we obtain a range of systems which become “better” the closer \( q \) is to 2. In this extreme case, a system is a Bessel \((C_2)\)-system if and only if it is a frame. The following dual formulation [20] enhances the
analogy between \((C_q)\)-systems and frames: A system is a \((C_q)\)-system if and only if
\[
c\|f\| \leq \left( \sum_p |\langle f, f_n \rangle|^p \right)^{\frac{1}{p}}, \quad \forall f \in H,
\]
where \(1/p + 1/q = 1\) and \(c = c(p)\) is a positive constant not depending on \(f\). This condition should be compared with the left inequality in (5).

In general, frames and \((C_q)\)-systems are not exact. As a system is a Riesz basis if and only if it is an exact frame, exact \((C_q)\)-systems (which are also Bessel systems) can be considered relaxed forms of Riesz bases. See Theorem 3 in this context. In particular, it follows from this theorem that if a Gabor system \(G(g, 1, 1)\) is exact, then it is also a \((C_\infty)\)-system. Therefore, in this case, exact \((C_q)\)-systems provide a continuous scale of systems, ranging from Riesz bases \((q = 2)\) to exact systems \((q = \infty)\).

1.3. The main result. Our main result is Theorem 2. We first discuss a simplified version of it; the symmetric case, where the localisation conditions for the generator \(g\) are the same in time and in frequency.

**Theorem 1** (The symmetric case). Fix \(q > 2\).

(a) Let \(g \in L^2(\mathbb{R})\) and \(r > 4(q - 1)/q\). If \(G(g, 1, 1)\) is an exact \((C_q)\)-system in \(L^2(\mathbb{R})\), then at least one of the integrals in (1) must diverge.

(b) Let \(r < 4(q - 1)/q\). There exists a function \(g \in L^2(\mathbb{R})\) for which \(G(g, 1, 1)\) is an exact (Bessel) \((C_q)\)-system in \(L^2(\mathbb{R})\) while both integrals in (1) converge.

Note that when \(q = 2\), we have \(4(q - 1)/q = 2\). Recall that an exact (Bessel) \((C_2)\)-system is a Riesz basis, and so, in this case, Theorem [1] should be compared with the classical Balian–Low theorem (which studies the condition \(r \geq 2\)). On the other hand, when \(q\) tends to infinity, then \(4(q - 1)/q\) tends to 4, which is the best localisation possible for generators of exact systems according to the corresponding Balian–Low type theorem (which studies the condition \(r \geq 4\)). In particular, part (b) of Theorem [1] implies that this result is sharp.

To state our main result in full generality, we introduce some notation. For \(2 \leq q \leq \infty\), denote by \(\Gamma_q\) the restriction to the area \(0 \leq v \leq u\) of the curve determined by the equations
\[
\frac{3q - 2}{q + 2} \cdot u + v = 1, \quad u + 3v \leq 1,
\]
\[
u + v = \frac{q}{2(q - 1)}, \quad u + 3v > 1.
\]
This curve corresponds to EFG in Figure [4]. As can be seen in the figure, the sector \(0 \leq v \leq u\) can be written as a partition
\[
S_q \cup \Gamma_q \cup W_q; \quad (0, 0) \in S_q, (1, 1) \in W_q,
\]
where \(S_q\) is represented by the shaded area below the curve \(\Gamma_q\) in the Figure, and \(W_q\) by the non-shaded area above the curve. Accordingly, we say that a point \((u, v)\) in the sector \(0 \leq v \leq u\) lies either above, on, or below \(\Gamma_q\).
\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Illustration of Theorem 2 over $Q = [0,1]^2$.}
\end{figure}

We note that in the area discussed, the curve $\Gamma_2$ is given by $u + v = 1$ and the curve $\Gamma_\infty$ by $3u + v = 1$, (the segments BC and AD in Figure 1, respectively). In Theorem 2 the areas below these curves represent the conditions for Riesz bases and exact systems given in (3) and (4).

We are now ready to state and discuss our main result.

**Theorem 2.** Fix $q > 2$ and let $\Gamma_q$ be as above.

(a) Let $g \in L^2(\mathbb{R})$ and $r \leq s$ be such that the point $\left(\frac{1}{r}, \frac{1}{s}\right)$ is below the curve $\Gamma_q$. If $G(g,1,1)$ is an exact $C_q$-system in $L^2(\mathbb{R})$, then at least one of the integrals in (2) must diverge.

(b) Let $r \leq s$ be such that the point $\left(\frac{1}{r}, \frac{1}{s}\right)$ is above the curve $\Gamma_q$. Then there exists a function $g \in L^2(\mathbb{R})$ for which $G(g,1,1)$ is an exact (Bessel) $(C_q)$-system in $L^2(\mathbb{R})$ while both integrals in (2) converge.

For $s = \infty$, the condition that the right-hand integral in (2) is convergent should be replaced by the condition that $g$ has compact support.

We take a closer look at Figure 1. The symmetric case of Theorem 1 is precisely Theorem 2 when restricted to the segment $CD$. In particular, the extremal cases of Riesz bases ($q = 2$) and exact systems ($q = \infty$) correspond to the vertices $C = (1/2,1/2)$ and $D = (1/4,1/4)$ of the segment, respectively.

The non-symmetric Balian–Low type theorems for Riesz bases and exact systems, mentioned in Section 1.1, correspond to the segments BC and AD, respectively. Indeed, if the point $(1/r,1/s)$ is below the segment $BC$, i.e., the curve
\[ \Gamma_2 = \{ u + v = 1 \} \], then it satisfies the condition in (3) and therefore, for a generator of a Riesz basis, at least one of the integrals in (2) must diverge. Similarly, if the point is below \( AD \), i.e., the curve \( \Gamma_\infty = \{ 3u + v = 1 \} \), then the condition in (4) is satisfied, which guarantees that for a generator of an exact system, at least one of these integrals must diverge.

Theorem 2 addresses generators of exact systems which cannot give Riesz bases, and therefore is most interesting for the region \( ABCD \): the area between the curves \( \Gamma_2 \) and \( \Gamma_\infty \). As \( q \) varies from 2 to \( \infty \), the curves \( \Gamma_q \) provide a continuous interpolation between \( \Gamma_2 \) and \( \Gamma_\infty \), covering all of this area. Therefore, the conditions described in Theorem 2 provide a continuous transition between the Balian–Low type conditions for generators of Riesz bases and the corresponding conditions for the generators of exact systems.

Note that Theorem 2 does not address points \((1/r, 1/s)\) which are on the curve \( \Gamma_q \). See also Remark 9 in this context.

1.4. The structure of the paper. We begin by laying the groundwork for our proof of Theorem 2. In Section 2, we relate the \((C_q)\) property of Gabor systems with the regularity of the Zak transform of the generators. It is known that the time–frequency localisation of a function and the regularity of its Zak transform are connected. In Section 3, we study this connection further, obtaining both Lipschitz and integral type estimates.

With this, we give a proof for part (a) of Theorem 2 in Section 4. To prove part (b) of the theorem, we introduce in Section 5 the building blocks for the constructions needed, before completing the proof in Section 6. In Section 7, we give concluding remarks.

2. A REFORMULATION OF THE PROBLEM

We establish some machinery, formulated in Lemma 2, which helps us determine whether a system \( G(g, 1, 1) \) is a \((C_q)\)-system by looking at the Zak transform of its generator.

2.1. Some notation. For \( d \in \mathbb{N} \), the Fourier transform of a function \( g \in L^2(\mathbb{R}^d) \) is denoted by \( \hat{g} \) and defined as the usual extension of the Fourier transform on \( L^1(\mathbb{R}^d) \):

\[ \hat{g}(\xi) = \int_{\mathbb{R}^d} g(t)e^{-2\pi i t \cdot \xi} dt, \quad \xi \in \mathbb{R}^d. \]

We set \( Q = [0, 1]^2 \). The Fourier coefficients of a function \( g \in L^1(Q) \) are given by

\[ \hat{g}(m, n) = \int_Q g(x, y)e^{-2\pi i (mx + ny)} dx dy, \quad (m, n) \in \mathbb{Z}^2. \]

Whenever an \( L^p \) integrable function is almost everywhere equal to a continuous function, we assume that they are equal everywhere. This is possible since the pointwise estimates we make are only used in integral expressions.
For functions defined on some subset of $\Omega \subset \mathbb{R}^d$, we use the notation $f^{(k)}_x$ for the $k$-th partial derivative with respect to a coordinate $x$. By $C^k(\Omega)$, we denote the class of functions whose partial derivatives of order $k$ exist and are continuous on $\Omega$. The functions that satisfy this for every $k \in \mathbb{N}$, is said to be of the class $C^\infty(\Omega)$. Also, by $C$ we denote constants which may change from step to step.

2.2. A characterisation of exact $(C_q)$-systems. A complete system $\{f_n\}$ in a Hilbert space $H$ is called exact if it becomes incomplete when any one of its members is removed. This condition holds if and only if there exists a unique system $\{g_n\} \subset H$ such that $\langle f_m, g_n \rangle = \delta_{m,n}$, where $\delta_{m,n}$ is the Kroenecker delta. In this case, $\{g_n\}$ is called the dual system of $\{f_n\}$.

The following characterization of exact $(C_q)$-systems can be found in [18]. We include a proof for the sake of completeness. Note that if $q = 2$, and the system is in addition a Bessel system, then condition $(c)$ of this theorem coincides with the known characterization of Riesz bases (see, for example, [25]).

**Theorem 3.** Fix $q \geq 2$ and let $\{f_n\}$ be a system in $H$. The following are equivalent.

(a) The system $\{f_n\}$ is an exact $(C_q)$-system.
(b) The system $\{f_n\}$ is exact and

$$\left( \sum |\langle f, g_n \rangle|^q \right)^{\frac{1}{q}} \leq C \|f\|, \quad \forall f \in H,$$

where $\{g_n\}$ is the dual system of $\{f_n\}$.
(c) The system $\{f_n\}$ is complete and

$$\left( \sum |a_n|^q \right)^{\frac{1}{q}} \leq C \left\| \sum a_n f_n \right\|,$$

for every finite sequence of numbers $\{a_n\}$.

**Proof.** $(a) \Rightarrow (b)$: Let $\{g_n\}$ be the dual system of $\{f_n\}$ and choose $f \in H$. Fix an integer $M > 0$. Since $\{f_n\}$ is a $(C_q)$-system, there exists a finite linear combination $\tilde{f} = \sum a_n f_n$ that approximates $f$ in norm, and satisfies

$$\left( \sum_{n=1}^M |\langle \tilde{f}, g_n \rangle|^q \right)^{\frac{1}{q}} = \left( \sum_{n=1}^M |a_n|^q \right)^{\frac{1}{q}} \leq C \|f\|.$$

Since $f$ is approximated by $\tilde{f}$, we have

$$\left( \sum_{n=1}^M |\langle f, g_n \rangle|^q \right)^{\frac{1}{q}} \leq C \|f\|.$$

The conclusion follows.

$(b) \Rightarrow (c)$: This implication is obvious.

$(c) \Rightarrow (a)$: First, if $\{f_n\}$ is not exact then there exists an $n_0$ for which $f_{n_0}$ lies in the closed span of $\{f_n\}_{n \neq n_0}$. So, for $\epsilon > 0$, there exists a finite linear combination
Lemma 1. Let \( f_{n_0} \) such that \( \|f_{n_0} - \tilde{f}\| < \epsilon \). This implies that \( (1 + \sum_{n \neq n_0} |a_n|^q)^{1/q} \leq C\epsilon \). By choosing \( \epsilon \) sufficiently small, we get a contradiction.

Next, let \( f \in H \) and \( \epsilon > 0 \). Since \( \{f_n\} \) is complete, there exists a finite linear combination \( \tilde{f} = \sum a_n f_n \) which approximates \( f \) in norm and satisfies \( \|\tilde{f}\| \leq \|f\| \).

It follows that \( (\sum |a_n|^q)^{1/q} \leq C\|\tilde{f}\| \leq C\|f\| \), and the proof is complete. \( \square \)

Remark 1. In particular, Theorem [3] implies that if a system \( G(g,1,1) \) is exact, then it is also a \((C_\infty)\)-system. This follows from the implication \((b) \implies (a)\) and the fact that for such a system, the dual system also takes the form \( G(h,1,1) \) for some function \( h \in L^2(\mathbb{R}) \).

2.3. Exponential \((C_q)\)-systems and weighted \(L^2\) spaces. Given a weight \( w \in L^1(Q) \) satisfying \( w > 0 \) almost everywhere, the weighted space \( L_w^2(Q) \) is defined by

\[
L_w^2(Q) := \left\{ g : \|g\|_{L_w^2(Q)}^2 = \int_Q |g|^2 w \, dx \, dy < \infty \right\}.
\]

The system of exponentials

\[
E := \left\{ e^{2\pi i(mx+ny)} \right\}_{m,n \in \mathbb{Z}} \tag{7}
\]

is complete in \( L_w^2(Q) \). Moreover, it is easy to check that \( E \) is exact in the space if and only if \( 1/w \in L^1(Q) \). In this case, the dual system of \( E \) consists of the functions

\[
h_{m,n} := \frac{1}{w} e^{2\pi i(mx+ny)} \tag{8}
\]

Lemma 1. Fix \( q > 2 \) and let \( w \in L^1(Q) \) satisfy \( w > 0 \) almost everywhere.

(a) If \( 1/w \in \overline{L^q(Q)} \), then \( E \) is an exact \((C_q)\)-system in \( L_w^2(Q) \).

(b) If there exists a function \( g \in L_w^2(Q) \cap L^1(Q) \) such that

\[
\sum_{m,n \in \mathbb{Z}} |\hat{g}(m,n)|^q = \infty,
\]

then \( E \) is not an exact \((C_q)\)-system in \( L_w^2(Q) \).

Proof. To see (a), we use condition (b) of Theorem [3]. Indeed, we first note that if \( 1/w \in \overline{L^q(Q)} \), then \( 1/w \in L^1(Q) \). Therefore, \( E \) is exact in the space and the dual system is given by (8). For \( g \in L_w^2(Q) \), we evaluate

\[
\sum_{m,n \in \mathbb{Z}} \|g, h_{m,n}\|_{L_w^2(Q)}^q = \sum_{m,n \in \mathbb{Z}} \left\| \int_Q g \frac{1}{w} e^{-2\pi i(mx+ny)} w \, dx \, dy \right\|^q = \sum_{m,n \in \mathbb{Z}} |\hat{g}(m,n)|^q.
\]

By the Hausdorff–Young inequality, the last expression is smaller than \( \|g\|_{L^p(Q)}^q \), where \( 1/p + 1/q = 1 \). We can now use Hölder’s inequality to check that

\[
\|g\|_{L^p(Q)} = \left\| g \sqrt{w} \cdot \frac{1}{\sqrt{w}} \right\|_{L^p(Q)} \leq \|g\|_{L_w^2(Q)} \left\| \frac{1}{w} \right\|_{\overline{L^{1/q}(Q)}},
\]

and (a) follows.
A similar argument can be used to prove (b).

2.4. The Zak transform and Gabor \((C_q)\)-systems. The following definition is commonly used in the study of lattice Gabor-systems (see, for example, [13]).

**Definition 2.** Let \(g \in L^2(\mathbb{R})\). The Zak transform of \(g\) is given by
\[
Zg(x, y) = \sum_{k \in \mathbb{Z}} g(x - k)e^{2\pi iky}, \quad \forall (x, y) \in \mathbb{R}^2.
\]

One can easily verify that, for every \(g \in L^2(\mathbb{R})\), the function \(Zg\) is quasi-periodic on \(\mathbb{R}^2\). That is, for every \((x, y) \in \mathbb{R}^2\), it satisfies
\[
Zg(x, y + 1) = Zg(x, y) \quad \text{and} \quad Zg(x + 1, y) = e^{2\pi iy}Zg(x, y).
\]

This implies that \(Zg\) is determined uniquely by its values on \(\mathbb{Q}\). It is well-known that when restricted to \(\mathbb{Q}\), the Zak transform induces a unitary operator from \(L^2(\mathbb{R})\) onto \(L^2(\mathbb{Q})\). In particular, this means that any quasi-periodic function, for which the restriction to \(\mathbb{Q}\) is square integrable, is the image under the Zak transform of some function \(g \in L^2(\mathbb{R})\). Throughout this paper \(Zg\) denotes either the Zak transform of \(g\) or its restriction to \(\mathbb{Q}\). The use of this notation will be clear from the context.

We now explain how the weighted spaces \(L^2_{|Zg|^2}(\mathbb{Q})\) can be used to study Gabor systems \(G(g, 1, 1)\). First note that
\[
Z \{g(t - n)e^{2\pi imt}\} (x, y) = e^{2\pi i(mx - ny)}Zg(x, y).
\]

Therefore, the system \(G(g, 1, 1)\) is complete in \(L^2(\mathbb{R})\) if and only if \(Zg \neq 0\) almost everywhere. Next, let \(g \in L^2(\mathbb{R})\) be such that \(Zg \neq 0\) almost everywhere and denote by \(U_g : L^2(\mathbb{R}) \to L^2_{|Zg|^2}(\mathbb{Q})\) the operator
\[
U_g : h \mapsto \frac{Zh}{Zg}.
\]

It is clear that \(U_g\) is a unitary bijection, and it follows from [10] that the image of the system \(G(g, 1, 1)\) under this operator is the system \(E\) defined in (7). Hence, \(G(g, 1, 1)\) is an exact system, a \((C_q)\)-system, or a frame in \(L^2(\mathbb{R})\) if and only if the same can be said about the system \(E\) in \(L^2_{|Zg|^2}(\mathbb{Q})\).

The following reformulation of Lemma 1 is now immediate.

**Lemma 2.** Fix \(q > 2\) and let \(g \in L^2(\mathbb{R})\) satisfy \(Zg \neq 0\) almost everywhere.

(a) If \(1/|Zg|^2 \in L^{\frac{2}{q-2}}(\mathbb{Q})\), then \(G(g, 1, 1)\) is an exact \((C_q)\)-system in \(L^2(\mathbb{R})\).

(b) If there exists a function \(f \in L^2_{|Zg|^2}(\mathbb{Q}) \cap L^1(\mathbb{Q})\) with
\[
\sum_{m, n \in \mathbb{Z}} |\hat{f}(m, n)|^q = \infty,
\]
then \(G(g, 1, 1)\) is not an exact \((C_q)\)-system in \(L^2(\mathbb{R})\).
Remark 2. Similarly, one can show that a system $G(g, 1, 1)$ is a Bessel system if and only if the weight $|Zg|^2$ is bounded from above almost everywhere. As follows from Lemma 6 below, this condition holds for all cases we discuss in the context of Theorem 2, i.e., whenever the point $(1/r, 1/s)$ is below the curve $BC = \Gamma_2$ (see (6) and Figure 1).

3. Smoothness properties of the Zak transform

In this section, we study the connection between the time–frequency localisation of a function and the regularity of its Zak transform. This is done both in terms of certain integral estimates as well as pointwise Lipschitz type estimates.

3.1. Smoothness and the Fourier transform. For $h \in \mathbb{R}$ and $k \in \mathbb{N}$, the operator $\tau^k_h : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ is defined by

$$\tau^k_h g(t) = g(t + h) - g(t) \quad \text{and} \quad \tau^k_h g = \tau^k_h \tau^{k-1}_h g.$$ 

We use the convention $\tau^0_h g = g$. Since $\hat{\tau^k_h g}(\xi) = (e^{2\pi i \xi h} - 1)^k \hat{g}(\xi)$, it follows by induction that

$$\hat{\tau^k_h g}(\xi) = (e^{2\pi i \xi h} - 1)^k \hat{g}(\xi). \quad (11)$$

One can now easily deduce the following classical relation (see [23, p. 139–140] for the cases $k = 1, 2$), which connects the smoothness of a function in $L^2(\mathbb{R})$ to the decay of its Fourier transform: For $0 < r < 2k$, there exists a constant $C > 0$ such that

$$\int \int_{\mathbb{R}^2} \frac{|\tau^k_h g(t)|^2}{|h|^{1+r}}
dt dh = C \int \int |\xi|^r |\hat{g}(\xi)|^2 \, d\xi. \quad (12)$$

Indeed, by Parseval’s identity and the equation (11),

$$\int |\tau^k_h g(t)|^2 dt = \int |\tau^k_h g(\xi)|^2 d\xi = \int |e^{2\pi i \xi h} - 1|^{2k} |\hat{g}(\xi)|^2 d\xi.$$ 

Whence, by an appropriate change of variables,

$$\int \int_{\mathbb{R}^2} \frac{|\tau^k_h g(t)|^2}{|h|^{1+r}}
dt dh = \int \int \frac{|e^{2\pi i \xi h} - 1|^{2k}}{|h|^{1+r}}
dh \cdot \int |\xi|^r |\hat{g}(\xi)|^2 d\xi.$$ 

The following two lemmas list some basic properties of the operator $\tau_h$ which are used in later sections.

Lemma 3. For any functions $f$ and $g$ on $\mathbb{R}$, the following relations hold.

(a) $|\tau^k_h g(t)| \leq 2^k \sum_{j=0}^{k} |g(t + jh)|$.

(b) $\tau^k_h (fg)(t) = \sum_{j=0}^{k} \binom{k}{j} \tau^j_h f(t) \tau^{k-j}_h g(t + jh)$. 

Moreover, if \( h \geq 0 \) and \( g \in C^k[t, t + kh] \), then
\[
(c) \quad |\tau_h^k g(t)| \leq |h|^k \sup_{\xi \in [t, t + kh]} |g^{(k)}(\xi)|.
\]
For \( h < 0 \), the same estimate holds over the interval \([t + kh, t]\).

This lemma can be proved easily using an inductive process and the mean value theorem (for estimate \((c)\)). We leave the details to the reader.

**Lemma 4.** Fix \( k \in \mathbb{N} \) and \( 0 < r < 2k \). Suppose \( U \subset \mathbb{R} \) and let \( g \) be a function on \( \mathbb{R} \).

(a) If \( \int_U |g(t + \eta)|^2 dt \) is bounded uniformly for all \( \eta \in \mathbb{R} \), then
\[
\int_{\mathbb{R}} \int_U \frac{|\tau_h^k g(t)|^2}{h^{1+r}} dt dh < \infty \iff \int_{-1}^1 \int_U \frac{|\tau_h^k g(t)|^2}{h^{1+r}} dt dh < \infty.
\]

(b) Suppose that \( U \) is bounded. If \( g \) is locally square integrable and \( \phi \in C^k(\mathbb{R}) \), then
\[
\int_{-1}^1 \int_U \frac{|\tau_h^k g(t)|^2}{h^{1+r}} dt dh < \infty \implies \int_{-1}^1 \int_U \frac{|\tau_h^k (\phi g)(t)|^2}{h^{1+r}} dt dh < \infty.
\]

**Proof.** Throughout this proof we use the notation \( g(t + \eta) = g_\eta(t) \).

(a) : As follows from Lemma 3\((a)\), if \( g \) satisfies the conditions above then
\[
\int_{|h| > 1} \int_U \frac{|\tau_h^k g(t)|^2}{h^{1+r}} dt dh < \infty.
\]
The conclusion follows.

(b) : We prove this by induction on \( k \in \mathbb{N} \). For \( k = 1 \), it is straight-forward since by Lemma 3\((b)\) we have
\[
\tau_h(u v) = \tau_h(u \cdot v + u_h \cdot v).
\]
Indeed, this identity applied to \( u = \phi \) and \( v = g \) yields the inequality
\[
\frac{1}{2} \int_{-1}^1 \int_U \frac{|\tau_h (\phi g)(t)|^2}{h^{1+r}} dt dh \leq \int_{-1}^1 \int_U \frac{|(\tau_h \phi \cdot g)(t)|^2}{h^{1+r}} dt dh + \int_{-1}^1 \int_U \frac{|(\phi_h \cdot \tau_h g)(t)|^2}{h^{1+r}} dt dh.
\]
The second term on the right-hand side is finite since \( \phi \) is bounded on any compact set. To see that the first term is finite, apply Lemma 3\((c)\) to \( \phi \), and conclude using the facts that \( g \) is locally square integrable and \( r < 2 \).

Next, assume that \((b)\) holds for \( k < n \) and that
\[
\int_{-1}^1 \int_U \frac{|\tau_h^n g(t)|^2}{h^{1+r}} dt dh < \infty.
\]
By (13), in the same way as above, we have
\[\tau_h^n(\phi g) = \tau_h^{n-1}\tau_h(\phi \cdot g) = \sum_{A_1} \tau_h^{n-1}(\tau_h \phi \cdot g) + \sum_{B_1} \tau_h^{n-1}(\phi_h \cdot \tau_h g).\]

By the induction hypothesis, \(B_1\) gives rise to a finite term in the corresponding integral. Indeed, note that \(\tau_h^n g = \tau_h^{n-1}\tau_h g\) and apply the induction hypothesis to (14) with \(k = n - 1\). On the other hand, again by (13), now applied with \(u = \tau_h \phi\) and \(v = g\), we have
\[A_1 = \tau_h^{n-2}\tau_h(\tau_h \phi \cdot g) = \sum_{A_2} \tau_h^{n-2}(\tau_h^2 \phi \cdot g) + \sum_{B_2} \tau_h^{n-2}(\tau_h \phi_h \cdot \tau_h g).\]

We apply a version of the relation (13) (replace \(v\) by \(\tau_h v\)):
\[\tau_h u \cdot \tau_h v = \tau_h(u \cdot \tau_h v) - u_h \cdot \tau_h^2 v,\]
and find that
\[B_2 = \tau_h^{n-1}(\phi_h \cdot \tau_h g) - \tau_h^{n-2}(\phi_{2h} \cdot \tau_h^2 g).\]

As above, \(B_2\) gives rise to finite terms in the corresponding integral expressions. Indeed, apply the induction hypothesis to (14) with \(k = n - 1\) and \(k = n - 2\) (use \(\tau_h g\) and \(\tau_h^2 g\) in place of \(g\), respectively).

We iterate this process for \(m \leq n\), applying (15) repeatedly in each step, to get:
\[A_m = \tau_h^{n-m}(\tau_h^m \phi \cdot g)\quad\text{and}\quad B_m = \tau_h^{n-m}(\tau_h^m \phi_h \cdot \tau_h g)\]
\[= \sum_{j=1}^m (-1)^{j-1} \binom{m-1}{j-1} \tau_h^{n-j}(\phi_{jh} \cdot \tau_h^j g).\]

In each step, due to the induction hypothesis, the terms generated by the \(B_m\) yield corresponding finite integrals. The final term \(A_n = \tau_h^n \phi \cdot g\) gives a convergent integral by Lemma 3(c).

3.2. **Smoothness and the Zak transform.** For functions on \(\mathbb{R}^2\), we define the operators \(\Delta_h^k\) and \(\Gamma_h^k\), as analogues to the operator \(\tau_h^k\), by
\[\Delta_h F(x, y) := F(x + h, y) - F(x, y),\]
\[\Gamma_h F(x, y) := F(x, y + h) - F(x, y),\]
and the relations
\[\Delta_h^k F = \Delta_h \Delta_h^{k-1} F \quad\text{and}\quad \Gamma_h^k F = \Gamma_h \Gamma_h^{k-1} F.\]

As above, we use the convention \(\Delta_h^0 F = \Gamma_h^0 F = F\). By setting \(F_g(x) = F(x, y)\), we can write \(\Delta_h^k F(x, y) = \tau_h^k F_g(x)\), and similarly for \(\Gamma_h^k\). This allows us to carry over results on \(\tau_h^k\) to the operators \(\Delta_h^k\) and \(\Gamma_h^k\).

For the operator \(\Delta_h^k\), the identity in (12) takes the following form: For \(0 < r < 2k\), there exists a constant \(C > 0\) such that for every \(F \in L^2(\mathbb{R}^2)\) we have
\[\iint_{\mathbb{R}^2} |\Delta_h^k F(x, y)|^2 \frac{dx dy dh}{|h|^{1+r}} = C \oint_{\mathbb{R}^2} |u|^r |\hat{F}(u, v)|^2 du dv.\]
A similar identity holds for the operator $\Gamma_h^k$.

Remark 3. Lemma 3 remains true in the two variable case when $\tau_h$ is replaced by either $\Delta_h$ or $\Gamma_h$, and the appropriate modifications are made. In what follows, these properties will be referred to as remarks 3(a), 3(b) and 3(c), respectively.

Remark 4. The properties listed in Lemma 4 also hold, under appropriate modifications, for the operators $\Delta_h$ and $\Gamma_h$. In what follows, these properties will be referred to as remarks 4(a) and 4(b), respectively.

A connection between the time–frequency localisation of a function $g$ and the smoothness of its Zak transform is now given in the following lemma. We note that the implications (i) $\Rightarrow$ (iii) and (iv) $\Rightarrow$ (vi), were first proved in [9].

Lemma 5. Let $k \in \mathbb{N}$ and $0 < r, s < 2k$. For every $g \in L^2(\mathbb{R})$ we have:

(a) The following conditions are equivalent.

(i) $\int_{\mathbb{R}} |\xi|^r |\hat{g}(\xi)|^2d\xi < \infty$.

(ii) $\int_{\mathbb{R}} \int_{[0,1]^2} \frac{|\Delta_h^k Zg(x, y)|^2}{|h|^{1+r}} dxdydh < \infty$.

(iii) For every compactly supported function $\psi \in C^k(\mathbb{R})$

$$\int_{\mathbb{R}^2} |u|^s |\hat{\psi}(u, v)|^2dudv < \infty.$$  

(b) Similarly, the following conditions are equivalent.

(iv) $\int_{\mathbb{R}} |t|^s |g(t)|^2dt < \infty$.

(v) $\int_{\mathbb{R}} \int_{[0,1]^2} \frac{|\Gamma_h^k Zg(x, y)|^2}{|h|^{1+s}} dxdydh < \infty$.

(vi) For every compactly supported function $\psi \in C^k(\mathbb{R})$

$$\int_{\mathbb{R}^2} |v|^s |\hat{\psi}(u, v)|^2dudv < \infty.$$  

Proof. (i) $\Leftrightarrow$ (ii) : In fact, an even stronger result holds: there exists a constant $C > 0$ such that for $g \in L^2(\mathbb{R})$ we have

$$\int_{\mathbb{R}} \int_{[0,1]^2} \frac{|\Delta_h^k Zg(x, y)|^2}{|h|^{1+r}} dxdydh = C \int_{\mathbb{R}} |\xi|^r |\hat{g}(\xi)|^2d\xi. \quad (17)$$

To see this, first note that $\Delta_h Zg = Z\tau_h g$. So, by induction $\Delta_h^k Zg = Z\tau_h^k g$. Since the Zak transform is a unitary operator, this implies that

$$\int_{\mathbb{R}} \int_{[0,1]^2} |\Delta_h^k Zg(x, y)|^2dxdy = \int_{\mathbb{R}} |\tau_h^k g(t)|^2dt.$$  

Hence, (17) follows from (12).
(ii) ⇔ (iii) : As follows from the identity (16) and Remark 3(a), it is enough to show that (ii) holds if and only if
\[ \int_{-1}^{1} \int_{\mathbb{R}^2} \frac{|\Delta^k_h(\psi Zg)(x,y)|^2}{|h|^{1+r}} \, dx \, dy \, dh < \infty \] (18)
for every compactly supported function \( \psi \in C^\infty(\mathbb{R}^2) \).

Assume first that (18) is satisfied for some function \( g \in L^2(\mathbb{R}) \). Let \( \psi \in C^\infty(\mathbb{R}^2) \) be a compactly supported function which satisfies \( \psi = 1 \) on \([-k, k+1] \times [0, 1] \), Note that \( \Delta^k_h(\psi Zg) = \Delta^k_h Zg \) for \((x, y) \in Q \) and \( h \in [-1, 1] \). So the integral in (ii) can be written as
\[ \int_{-1}^{1} \int_{[0,1]^2} \frac{|\Delta^k_h(\psi Zg)(x,y)|^2}{|h|^{1+r}} \, dx \, dy \, dh + \int_{|h|>1} \int_{[0,1]^2} \frac{|\Delta^k_h Zg(x,y)|^2}{|h|^{1+r}} \, dx \, dy \, dh. \]
The first integral in this sum converges by (18), while the second integral converges by an application of Remark 3(a) and the quasi-periodicity of \( Zg(x, y) \).

Next, suppose (ii) holds for some \( g \in L^2(\mathbb{R}) \) and let \( \psi \in C^\infty(\mathbb{R}^2) \) be a compactly supported function. It follows by the quasi-periodicity of \( Zg(x, y) \) that for any positive integer \( n \),
\[ \int_{-1}^{1} \int_{[-n,n]^2} \frac{|\Delta^k_h Zg(x,y)|^2}{|h|^{1+r}} \, dx \, dy \, dh < \infty. \] (19)
Choose \( n \in \mathbb{N} \) big enough for the support of \( \psi \) to be included in \([-n + k, n - k] \times [-n, n] \). This allows us to write (18) as
\[ \int_{-1}^{1} \int_{[-n,n]^2} \frac{|\Delta^k_h(\psi Zg)(x,y)|^2}{|h|^{1+r}} \, dx \, dy \, dh. \]
By Remark 3(b), the inequality (19) implies that this is finite.

(iv) ⇔ (v) : Let \( S_h : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \) be the operator defined by
\[ (S_h f)(t) = f(t)(e^{-2\pi i n h t} - 1), \quad \text{for} \quad n \leq t < n + 1. \]
It is easily verified that \( \Gamma_h Zg = ZS_h g \) on \( Q \). So, by induction \( \Gamma_h Zg = ZS_h^k g \). Again, since the Zak transform is a unitary operator, we get
\[ \int_{[0,1]^2} |\Gamma_h^k Zg|^2 \, dx \, dy = \int_{\mathbb{R}} |S_h^k g(t)|^2 \, dt. \]
As above,
\[ \int_{\mathbb{R}^2} \frac{|S_h^k g(t)|^2}{|h|^{1+s}} \, dt \, dh = \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} \frac{|e^{-2\pi i n h t} - 1|^2}{|h|^{1+s}} \, dh \, \int_n^{n+1} |g(t)|^2 \, dt \]
\[ = C \sum_{n \in \mathbb{Z}} |n|^s \int_n^{n+1} |g(t)|^2 \, dt. \]
It is clear that the right-hand side converges if and only if the same is true for the integral \( \int_{\mathbb{R}} |t|^s |g(t)|^2 \, dt. \)
(v) ⇔ (vi): This is proved in a similar way as (ii) ⇔ (iii).

3.3. Lipschitz type conditions and the Zak transform. The following result appears implicitly in [15, Theorem 3.2].

Lemma 6. Let $g \in L^2(\mathbb{R})$ and $r, s > 0$ be such that $1/r + 1/s < 1$. If both integrals in (2) are finite, then $Zg$ is continuous on $\mathbb{R}^2$ and has a zero in $Q$.

Proof. Since this result will be of importance in what follows, we give a short indication of a proof. By combining Proposition 1 and Theorem 1 of [10], one can check that a function for which both integrals in (2) are finite satisfies $\sum_{k \in \mathbb{Z}} \|g\|_{L^\infty(k,k+1)} < \infty$ (i.e., it belongs to the Wiener space). Since, in addition, $\hat{g} \in L^1$ implies that $g$ is continuous, it follows that $Zg$ is also continuous and therefore has a zero in $Q$ (see Lemma 8.2.1 part (c) and Lemma 8.4.2 in [11]).

The next lemma establishes Lipschitz type conditions for the Zak transform of functions satisfying the conditions of Lemma 6. It is of particular interest for us that this lemma describes how “deep” the zero of $Zg$ must be.

Lemma 7. Let $r > 0$ and $s > 0$ satisfy $1/r + 1/s < 1$, and define

$$
\phi_{r,s}(x) = \begin{cases} 
|x|^{2-r}\left(\frac{3}{r} + \frac{1}{s} - 1\right) & \text{if } \frac{3}{r} + \frac{1}{s} > 1, \\
|x|^2 \log\left(1 + \frac{1}{|x|}\right) & \text{if } \frac{3}{r} + \frac{1}{s} = 1, \\
|x|^2 & \text{if } \frac{3}{r} + \frac{1}{s} < 1.
\end{cases}
$$

Suppose that for $g \in L^2(\mathbb{R})$ both integrals in (2) are finite. Then given $(a,b) \in \mathbb{R}^2$ we have

$$
|Zg(x, y) - Zg(a,b)|^2 \leq C\left(\phi_{r,s}(x-a) + \phi_{s,r}(y-b)\right)
$$

on $\mathbb{R}^2$, where $C > 0$ is a constant not depending on $x$ and $y$.

As follows from the proof, and from the fact that $Zg$ is a quasi-periodic function, the constant $C$ can be chosen in such a way that it does not depend on the point $(a,b)$. This, however, is not needed for our purposes.

Proof. By Lemma 6, $Zg$ is continuous, and so it is enough to prove (21) in a neighbourhood of $(a,b)$. Choose a compactly supported function $\psi \in C^\infty(\mathbb{R}^2)$ that satisfies $\psi \equiv 1$ in a neighbourhood $U$ of $(a,b)$. By the Cauchy–Schwarz inequality, the following estimate holds for every $(x, y) \in U$,

$$
|Zg(x, y) - Zg(a,b)| = |(\psi Zg)(x, y) - (\psi Zg)(a,b)| \\
\leq \iint_{\mathbb{R}^2} |1 - e^{2\pi i ((x-a)+(y-b))v}| \left|\hat{(\psi Zg)}(u,v)\right| du dv \\
\leq 2 \left( \iint_{\mathbb{R}^2} (1 + |u|^r + |v|^s) \left|\hat{(\psi Zg)}(u,v)\right|^2 du dv \right)^{1/2}
$$

(1)
to find that, since 

\[ \|H\|_\psi \leq 2 \left( \sum_{\beta=1}^{\infty} C_{\beta}^2 \right)^{1/2} \]

Hence, we get

\[ \left( \sum_{\beta=1}^{\infty} C_{\beta}^2 \right)^{1/2} \]

This induces the splitting of the lemma. As for (II), the symmetry of the integrand and the inequality \( \sin^2(x + y) \leq 2(\sin^2 x + \sin^2 y) \) imply that

\[
(II) \leq 8 \int_{[0, \infty)^2} \frac{\sin^2 \pi(x - a)u + (y - b)v}{1 + |u|^r + |v|^s} \, du \, dv + 8 \int_{[0, \infty)^2} \frac{\sin^2 \pi(y - b)v}{1 + |u|^r + |v|^s} \, du \, dv.
\]

By an appropriate change of variables, we get

\[
(III) = \int_{[0, \infty)^2} \frac{\sin^2 \pi u}{|x - a|^{r-1} + |x - a|^s + |u|^r + |v|^s} \, du \, dv.
\]

To estimate this integral we divide the area of integration into two parts:

\[ Q = [0, 1]^2, \quad Q^c = [0, \infty)^2 \setminus Q. \]

This induces the splitting \((III) = (III_Q) + (III_{Q^c})\). We use the inequalities

\[ c_{\beta}(x + y)^{\beta} \leq x^{\beta} + y^{\beta} \leq C_{\beta}(x + y)^{\beta}, \quad \forall \beta > 0, \ x \geq 0, \ y \geq 0, \quad (22) \]

to find that, since \( s > 1 \),

\[
(III_Q) \leq C \int_Q \frac{u^2}{(|x - a|^{s/2} + u^s + v^s)^s} \, du \, dv
\]

\[
\leq C \left( \int_0^1 \frac{u^2}{(|x - a|^{s/2} + u^s)^{s-1}} \, du \right) + 1
\]

\[
\leq C \left( \int_0^1 \frac{u^2}{(|x - a|^{3/2} + u^3)^{(s-1)/3}} \, du \right) + 1 = C \left( \int_0^1 \frac{w}{w^{(s-1)/3}} \, dw \right) + 1
\]

Hence,

\[
(III_Q) \leq \begin{cases} 
  C & \text{if } \frac{3}{r} + \frac{1}{s} > 1, \\
  C \log \left( 1 + \frac{1}{|x - a|} \right) & \text{if } \frac{3}{r} + \frac{1}{s} = 1, \\
  C|x - a|^{3-r+s/2} & \text{if } \frac{3}{r} + \frac{1}{s} < 1.
\end{cases}
\]

By similar estimates, and the fact that \( 1/r + 1/s < 1 \), it is easy to check that \((III_{Q^c}) \leq C\).

Repeating these arguments for the integral (IV), the lemma is established. \( \square \)

Remark 5. Lemma [7] holds also in the extremal case \( s = \infty \), i.e., when \( g \) is compactly supported. This can be shown using similar arguments. See also [14].
Remark 6. Lemma [7] is sharp. That is, for every $r, s > 0$ as above and $\epsilon > 0$, there exist a function $g$ for which both integrals in (2) converge and a point $(a, b) \in Q$ such that the inequality

$$|Zg(x, y) - Zg(a, b)|^2 \geq C \left( \phi_{r+\epsilon, s+\epsilon}(x - a) + \phi_{r+\epsilon, s+\epsilon}(y - b) \right)$$

(23)

holds in a neighbourhood of $(a, b)$ (note that for $\epsilon > 0$ we have $\phi_{r+\epsilon, s+\epsilon}(x) \leq C\phi_{r, s}(x)$ in $Q$). Indeed, the functions constructed in the proof of part (b) of Theorem 2 provide the required estimates.

4. Theorem 2 – First part

To prove Theorem 2 (a) we will combine Lemma 2 (b) with lemmas 6 and 7. In order to do so, we need to find a family of test functions for which we are able to estimate both $L^2 |Zg| (Q)$ norms and the $\ell^q$ norm of their Fourier coefficients.

4.1. A family of test functions. For $\alpha > 0$ and $\beta > 0$, a suitable family of functions is given by

$$f_{\alpha, \beta}(x, y) := \frac{1}{\left[1 + (1 - |x - \frac{1}{2}|^{\alpha})e^{2\pi i y}\right]^\beta}, \quad (x, y) \in Q. \quad (24)$$

Here $z^\beta = e^{\beta \log z}$, where log $z$ is the principle value of the logarithm on $\mathbb{C} \setminus [-\infty, 0]$. Note that the functions $f_{\alpha, \beta}$ satisfy

$$|f_{\alpha, \beta}(x, y)|^2 \leq \frac{C}{(|x - \frac{1}{2}|^{2\alpha} + |y - \frac{1}{2}|^2)^{\beta}} \quad (25)$$

for some constant $C = C(\alpha, \beta)$. Indeed, for $(x, y) \in Q$ we have

$$\left|1 + \left(1 - \left|x - \frac{1}{2}\right|^\alpha\right)e^{2\pi i y}\right|^2 = \left|x - \frac{1}{2}\right|^{2\alpha} + 2 \left(1 - \left|x - \frac{1}{2}\right|^\alpha\right)(\cos 2\pi y + 1) \geq C \left(\left|x - \frac{1}{2}\right|^{2\alpha} + |y - \frac{1}{2}|^2\right).$$

The following Lemma provides the required estimate for the Fourier coefficients of $f_{\alpha, \beta}$.

Lemma 8. Fix $q > 2$. For every $0 < \alpha < 1$ and $(1 - 1/q)(1 + 1/\alpha) \leq \beta < (1 + 1/\alpha)$, the function $f = f_{\alpha, \beta}$ belongs to $L^1(Q)$ and its Fourier coefficients satisfy

$$\sum_{m,n \in \mathbb{Z}} |\hat{f}(m, n)|^q = \infty.$$

Proof. The fact that $\beta < 1 + 1/\alpha$ implies $f \in L^1(Q)$ follows from (25), as can be easily verified using inequality (22).
For any \( n \geq 2 \) and \( m = 2k \), with \( k \in \mathbb{N} \), we estimate \(|\hat{f}(m,n)|\) from below. We write \( h(x) = 1 - |x - 1/2|^\alpha \), and note that for \( x \neq 1/2 \) we have \( 0 < h(x) < 1 \). With this, we evaluate

\[
\hat{f}(m,n) = \int_0^1 \int_0^1 e^{-2\pi i(mx+ny)}(1 + h(x)e^{2\pi iy})\beta \, dxdy
\]

\[
= \int_0^1 e^{-2\pi imx} \int_0^1 e^{-2\pi ixy} \left( \sum_{j=0}^\infty (-1)^j b_j h^j(x)e^{2\pi ixy} \right) \, dydx
\]

\[
= (-1)^n b_n \int_0^1 h^n(x)e^{-2\pi imx} \, dx,
\]

where \( b_n = \beta(\beta + 1) \cdots (\beta + n - 1)/n! \) are the coefficients of the Taylor expansion of \((1 - z)^{-\beta}\) at the origin. It follows by the product formula for the Gamma function that we have \( c\beta^{-1} \leq b_n \leq C\beta^{-1} \), where \( c \) and \( C \) are positive constants.

By a change of variables and the fact that \((1 - |x|^\alpha)^n\) is even, we get

\[
(I) = 2 \int_0^{1/2} (1 - x^\alpha)^n \cos 2\pi mx \, dx = \frac{1}{k} \int_0^k \left( 1 - \left( \frac{x}{2k} \right)^\alpha \right)^n \cos 2\pi x \, dx.
\]

We integrate by parts and find that the last expression is equal to

\[
C \frac{n}{k^\alpha+1} \int_0^k \left( 1 - \left( \frac{x}{2k} \right)^\alpha \right)^{n-1} x^\alpha \sin 2\pi x \, dx.
\]

The function \((1 - |x/2k|^\alpha)^{n-1}x^\alpha\) is decreasing on \((0,k)\) since \( 0 < \alpha < 1 \). So for any positive integer \( \nu < k \) the integral \( \int_\nu^{\nu+1} (1 - |x/2k|^\alpha)^{n-1}x^\alpha \sin 2\pi x \, dx \) is positive. Using this and \( (27) \), we get

\[
(I) \geq C \frac{n}{k^\alpha+1} \int_0^{1/2} \left( 1 - \left( \frac{x}{2k} \right)^\alpha \right)^{n-1} x^\alpha \sin 2\pi x \, dx.
\]

We use the same type of argument again to find that

\[
(II) \geq \left( \int_0^{1/4} + \int_{1/4}^{1/2} \right) \left( 1 - \left( \frac{x}{2k} \right)^\alpha \right)^{n-1} x^\alpha \sin 2\pi x \, dx
\]

\[
\geq C \left[ \left( 1 - \left( \frac{1}{8k} \right)^\alpha \right)^{n-1} - \left( 1 - \left( \frac{3}{8k} \right)^\alpha \right)^{n-1} \right].
\]

Set \( F(x) = C (1 - (x/k)^\alpha)^n \). There exists a number \( 1/8 < \tau < 3/8 \) such that the right-hand side of \( (29) \) is equal to \( 4^{-1} F'(\tau) \). It now follows that

\[
(II) \geq C \frac{n}{k^\alpha} e^{n \log(1 - (\tau^\alpha))}.
\]
For $0 < x \leq 3/8$, we have $\log(1 - x) \geq -2x$ (actually, even for bigger $x$). Combining this with (26), (28), and the asymptotic behaviour of $b_n$, we find that for constants $C_1 > 0$ and $C_2 > 0$ depending only on $\alpha$ and $\beta$, we have

$$|\hat{f}(m,n)| \geq C_1 \frac{n^{\beta+1}}{k^{2\alpha+1}} e^{-C_2 \frac{n}{k}}$$

whenever $n \geq 2$ is an integer and $m = 2k > 0$ is an even integer.

We are now ready to estimate the $\ell^q$ norm of the Fourier coefficients of $f$. First, we have

$$\sum_{m,n \in \mathbb{Z}} |\hat{f}(m,n)|^q \geq C_1 \sum_{k=1}^{\infty} \frac{1}{k^{q(2\alpha+1)}} \sum_{n=2}^{\infty} n^{q(\beta+1)} e^{-C_2 \frac{n}{k}}.$$  

For a positive number $x$, we denote by $\lceil x \rceil$ the smallest integer $l$ such that $l \geq x$. In this way,

$$\sum_{m,n \in \mathbb{Z}} |\hat{f}(n,m)|^q \geq \sum_{\nu=1}^{\infty} \sum_{n=\nu \lceil k^{\alpha} \rceil}^{\nu+1 \lceil k^{\alpha} \rceil-1} n^{q(\beta+1)} e^{-C_2 \frac{n}{k}}$$

$$\geq k^{\alpha} \sum_{\nu=1}^{\infty} (\nu k^{\alpha})^{q(\beta+1)} e^{-2C_2 q(\nu+1) k^{-\alpha}}$$

$$= k^{\alpha(1+q\beta+q)} \sum_{\nu=1}^{\infty} \nu^{q(\beta+1)} e^{-2C_2 q(\nu+1)}.$$  

Since (**) converges,

$$\sum_{m,n \in \mathbb{Z}} |\hat{f}(n,m)|^q \geq C \sum_{k=1}^{\infty} k^{\alpha+q(\alpha\beta-\alpha-1)}.$$  

The right-hand side is infinite if and only if $\beta \geq (1 + 1/\alpha)(1 - 1/q)$, which gives the desired conclusion.  

4.2. **Proof of Theorem 2, part (a).** Let $r \leq s$ be such that the point $(1/r, 1/s)$ is below the curve $\Gamma_q$, given by (6). This implies that either one of the following conditions holds:

$$\frac{1}{r} + \frac{3}{s} > 1 \quad \text{and} \quad \frac{1}{r} + \frac{1}{s} < \frac{q}{2(q-1)},$$

or

$$\frac{1}{r} + \frac{3}{s} \leq 1 \quad \text{and} \quad \frac{3q-2}{q+2} \cdot \frac{1}{r} + \frac{1}{s} < 1.$$  

Moreover, since $G(g, 1, 1)$ is exact, condition (4) implies that in both of these cases we also have

$$\frac{3}{r} + \frac{1}{s} > 1.$$
To arrive at a contradiction, we assume that the integrals in (2) converge. Since both conditions (30) and (31) imply that the numbers \( r \) and \( s \) satisfy the inequality \( 1/r + 1/s < 1 \), it follows by Lemma 6 that there exists a point \((a, b) \in Q\) such that \( Zg(a, b) = 0 \). Therefore Lemma 7 implies the estimate

\[
|Zg(x, y)|^2 \leq C \left( \phi_{r,s}(x-a) + \phi_{s,r}(y-b) \right), \\
(x, y) \in Q, \tag{33}
\]

where the functions \( \phi_{r,s} \) are defined in (20). Note that the value of \( \phi_{r,s}(x-a) \) is determined by the inequality (32), while the value of \( \phi_{s,r}(y-b) \) is determined by the left-hand inequality in either (30) or (31), depending on the case.

By Lemma 2(b), a contradiction is obtained if we find a function \( h \) that satisfies

\[
h \in L^2(|Zg|)(Q) \tag{34}
\]

and

\[
h \in L^1(Q) \quad \text{with} \quad \sum_{m,n \in \mathbb{Z}} |\hat{h}(m, n)|^q = \infty. \tag{35}
\]

Roughly speaking, we construct a function \( h \) that has a single singularity at the point \((a, b)\), and on the one side, grows fast enough near this singularity for condition (35) to hold, while on the other side, it grows slowly enough for condition (34) to follow from (33). In fact, its size is essentially smaller than some power of \( 1/|Zg|^2 \).

Given \( \alpha, \beta > 0 \) let \( f_{\alpha, \beta} \) be the 1-periodic extension of the function (24) to \( \mathbb{R}^2 \). Set

\[
h_{\alpha, \beta}(x, y) = f_{\alpha, \beta}(x - a + \frac{1}{2}, y - b + \frac{1}{2}). \tag{36}
\]

From (25), we have

\[
|h_{\alpha, \beta}(x, y)|^2 \leq \frac{C}{(|x-a|^2 + |y-b|^2)^{\beta}}, \quad \forall (x, y) \in Q. \tag{37}
\]

In the remainder of the proof, we determine suitable values for the parameters \( \alpha, \beta \) to ensure that (34) and (35) hold for \( h_{\alpha, \beta} \).

First, we assume that (30) holds. The condition \( 1/r + 1/s < q/(2(q - 1)) \) implies that there exists a number \( \lambda \) which satisfies

\[
\frac{2(q-1)}{q} \cdot \left( \frac{1}{r} + \frac{1}{s} \right) < \lambda < 1. \tag{38}
\]

Choose such a \( \lambda \) and define \( h_{\alpha, \beta} \) as in (36) with

\[
\alpha = \frac{r}{s} \quad \text{and} \quad \beta = \frac{s\lambda}{2}. \tag{39}
\]

Since \( r \leq s \), the left inequality in (30) implies that \( 1/r + 1/s > 1/2 \). Combining this with (38) and (39), we have

\[
\left( 1 - \frac{1}{q} \right) \left( 1 + \frac{1}{\alpha} \right) \leq \beta < \left( 1 + \frac{1}{\alpha} \right). \]
So, by Lemma 8, the function \( h_{\alpha,\beta} \) satisfies (35). To show that it satisfies (34), we first note that, in this case, the inequality (33) takes the form

\[
|Zg(x,y)|^2 \leq |x-a|^{2+r(1-\frac{1}{r}-\frac{1}{s})} + |y-b|^{2+(1-\frac{1}{r}-\frac{1}{s})}\]

Combining this with (37) and (39), we use (22) to get

\[
\iint_Q |h_{\alpha,\beta}(x,y)|^2 |Zg(x,y)|^2 \,dxdy \leq C \iint_Q (|x-a|^{\frac{r}{\alpha}} + |y-b|^{[2+(1-\frac{1}{r}-\frac{1}{s})]-s\lambda}) \,dxdy.
\]

The integral on the right-hand side is finite if and only if \( \lambda < 1 \), so (34) follows from (38).

We now assume that (31) holds. Let

\[
\alpha = 1 - \frac{r}{2} \left( \frac{3}{r} + \frac{1}{s} - 1 \right),
\]

and note that \( 0 < \alpha < 1 \). Next, the condition \( \frac{3q-2}{q+2} \cdot \frac{1}{r} + \frac{1}{s} < 1 \) implies that there exists a number \( \beta \) for which

\[
\left( 1 - \frac{1}{q} \right) \left( 1 + \frac{1}{\alpha} \right) \leq \beta < \frac{3}{2} + \frac{1}{2\alpha}.
\]

Choose such a \( \beta \) and let \( h_{\alpha,\beta} \) be the function defined in (36). Since \( 0 < \alpha < 1 \),

\[
\frac{3}{2} + \frac{1}{2\alpha} < 1 + \frac{1}{\alpha},
\]

and so it follows from (41) and Lemma 8 that (35) holds. To check that (34) holds, we first note that, in this case, inequality (33) takes the form

\[
|Zg(x,y)|^2 \leq C \left( |x-a|^{2\alpha} + |y-b|^2 \log \left( 1 + \frac{1}{|y-b|} \right) \right)^{1-\beta} \log \left( 1 + \frac{1}{|y-b|} \right).
\]

Combining this with the estimate in (37), and making an appropriate change of variables, we find that

\[
\iint_Q |h_{\alpha,\beta}(x,y)|^2 |Zg(x,y)|^2 \,dxdy \leq C \iint_Q (x^{2\alpha} + y^{2\alpha})^{1-\beta} \log \left( 1 + \frac{1}{|y|} \right) \,dxdy.
\]

We use (22) to ensure that the last integral is smaller than

\[
C \iint_Q (x + y^2)^{2\alpha(1-\beta)} \log \left( 1 + \frac{1}{|y|} \right) \,dxdy.
\]

This integral is finite if \( \beta < 3/2 + 1/2\alpha \). Hence, (34) follows from the inequality (41).

For \( s = \infty \), use Remark 5 and repeat the previous argument. \( \square \)
5. TWO FAMILIES OF FUNCTIONS

We introduce two families of functions that are used in the next Section to prove part (b) of Theorem 2. The needed estimates are given in lemmas 10 and 12, where we measure the smoothness of these functions near the origin.

5.1. Building blocks for the modulus. Fix $a > 0$. Given $\alpha, \beta, \gamma > 0$, set

$$f_{\alpha, \beta, \gamma}(x, y) := \begin{cases} 
(x^{\alpha/\gamma} + |y|^{\beta/\gamma})^\gamma & \text{for } x \geq 0, \\
((-ax)^{\alpha/\gamma} + |y|^{\beta/\gamma})^\gamma & \text{for } x < 0.
\end{cases}$$

(42)

The following lemma is easily proved by induction.

Lemma 9. Let $\alpha, \beta > 0$ and $k \in \mathbb{N}$. If $0 < \gamma < \min\{\alpha/k, \beta/k, 1\}$, then $f_{\alpha, \beta, \gamma} \in C^k(\mathbb{R}^2 \setminus \{(0, 0)\})$. Moreover, for any $(x, y) \neq (0, 0)$, the partial derivative $(f_{\alpha, \beta, \gamma})^{(k)}_{x}(x, y)$ equals

$$\sum_{m=1}^{k} C_{m,k} (x^{\alpha/\gamma} + |y|^{\beta/\gamma})^{\gamma - m} (-ax)^{m \frac{\alpha}{\gamma} - k} x \geq 0,$$

$$(-a)^k \sum_{m=1}^{k} C_{m,k} ((-ax)^{\alpha/\gamma} + |y|^{\beta/\gamma})^{\gamma - m} (-ax)^{m \frac{\alpha}{\gamma} - k} x < 0,$$

(43)

where $C_{m,k}$ are constants not depending on $(x, y)$.

An explicit estimate for the smoothness of the functions $f_{\alpha, \beta, \gamma}$ near the origin is given in the following lemma. Recall that $\Delta_h$ and $\Gamma_h$ are defined in Section 3.2.

Lemma 10. Let $\alpha, \beta > 0$ and $k \in \mathbb{N}$ be such that $2\alpha + \beta/\gamma + 1 \leq 2k$. If $\gamma < \alpha/k$, then for any $\epsilon > 0$ we have

$$\int_{-b}^{b} \int_{[-c, c]^2} \frac{|\Delta_h^k f_{\alpha, \beta, \gamma}(x, y)|^2}{|h|^{2\alpha + \beta/\gamma + 2 - \epsilon}} \, dx dy dh < \infty,$$

where $b, c$ are any two positive numbers.

Proof. Set $f = f_{\alpha, \beta, \gamma}$. To simplify formulations, we make the assumption $\gamma < \beta/k$ so that Lemma [9] can be applied. Otherwise, a relaxed version of it, where the function $f$ is not necessarily differentiable, can be used. However, in what follows, this extraneous condition holds whenever we refer to Lemma [10].

In the above integral the integrand is even in $y$, so it is enough to show that for $h > 0$,

$$\int_{-c}^{c} \int_{-c}^{c} (|\Delta_h^k f(x, y)|^2 + |\Delta_h^k f(x, y)|^2) \, dx dy \leq Ch^{2\alpha + \beta/\gamma + 1},$$

(44)

where $C = C(f, k)$ does not depend on $h$. Since $f$ is bounded on any compact set, we may assume that $h$ is small enough for the following partition\footnote{To simplify formulations, here and in the following, we allow members of a partition to have intersections of measure zero.} to hold:

$$[-c, c] \times [0, c] = V_1 \cup V_2 \cup V_3 \cup V_4,$$
where (see Figure 2)

\[ V_1 = [- (k+1)h, (k+1)h] \times [0, ((k+1)h)^{\alpha/\beta}] , \]
\[ V_2 = [- (k+1)h, (k+1)h] \times [((k+1)h)^{\alpha/\beta}, c] , \]
\[ V_3 = [(k+1)h, c] \times [0, c] , \]
\[ V_4 = [-c, -(k+1)h] \times [0, c] . \]

To estimate the integral in (44) over \( V_1 \), we use Remark 3 (a) and the inequality (22) to find that it is smaller than some constant times

\[ k \sum_{j=-k}^{k} \int_{V_1} |f(x+jh,y)|^2 \, dx \, dy \leq C \int_0^{(k+1)h} \int_0^{(2k+1)h} |f(x,y)|^2 \, dx \, dy \]
\[ \leq C \int_0^{(k+1)h} \int_0^{(2k+1)h} \left( x^{2\alpha} + y^{2\beta} \right) \, dx \, dy \]
\[ \leq Ch^{2\alpha+\alpha/\beta+1} . \]

For the estimate over the remaining parts, we note that \( \gamma < \min\{\alpha/k, \beta/k\} \) implies \( \gamma < 1 \), and so it follows from Lemma 3 and Remark 3 (c) that

\[ |\Delta_k^h f(x,y)| + |\Delta_x^h f(x,y)| \leq |h|^k \sup_{\xi \in [x-kh, x+kh]} |f_x^{(k)}(\xi, y)| . \]

Hence, to complete the proof we need to show that

\[ I_2 + I_3 + I_4 := \int_{V_2 \cup V_3 \cup V_4} \Omega(x,y)^2 \, dx \, dy \leq Ch^{2\alpha+\alpha/\beta+1-2k} . \]

We do so by estimating the partial derivatives given in (43). For \( (x,y) \in V_2 \) we have

\[ \Omega(x,y) \leq C \sum_{m=1}^{k} y^{\frac{\alpha}{\gamma}(\gamma-m)} h^{m \frac{\alpha}{\gamma} - k} \leq C y^{\beta - k \beta/\alpha} . \]

So,

\[ I_2 \leq Ch(1 + h^{2\alpha+\alpha/\beta-2k}) \leq Ch^{2\alpha+\alpha/\beta+1-2k} , \]

whenever \( h \) is small enough.
To estimate $I_3$, we first note that if $(x, y) \in V_3$, then

$$\Omega(x, y) \leq \sup_{\xi \in [x/(k+1), 2x]} |f^{(k)}(\xi, y)| \leq C \sum_{m=1}^{k} (x^{\alpha/\gamma} + y^{\beta/\gamma})^{\gamma-m} x^{\alpha m - k}.$$  

We apply (22) to the $m$'th term of this sum and find that the corresponding integral is less than some constant times

$$\int_{(k+1)h}^{c} \int_{0}^{c} (x^{\alpha/\beta} + y)^{2(\gamma - m)} x^{2(m \gamma - k)} dy dx \leq C \left(1 + \int_{(k+1)h}^{c} x^{2 \alpha + \beta - 2k} dx \right) \leq Ch^{2 \alpha + \beta + 1 - 2k},$$

whenever $h$ is small enough. This implies the required estimate for $I_3$. In the same way one can show the required estimate for $I_4$, which completes the proof. \[\square\]

### 5.2. Building blocks for the argument.

Let $\phi \in C^\infty(\mathbb{R})$ be a function satisfying $-1 \leq \phi(x) \leq 0$ for all $x \in \mathbb{R}$, and for which

$$\phi(x) = \begin{cases} -1 & x \in (-\infty, 0], \\ 0 & [1, \infty). \end{cases}$$

Given $\lambda > 0$, denote

$$H_\lambda(x, y) = \begin{cases} \phi\left(\frac{y}{x^\lambda}\right) & \text{for } x \geq 0, \text{ and } 0 \leq y \leq x^\lambda, \\ 0 & \text{otherwise}. \end{cases} \quad (45)$$

Such a function was first introduced in [2]. For $\alpha, \beta, \gamma > 0$, set

$$F_{\alpha, \beta, \gamma}(x, y) = f_{\alpha, \beta, \gamma}(x, y)e^{2\pi i H_\lambda(x, y)}, \quad (46)$$

where the functions $f_{\alpha, \beta, \gamma}$ are defined in (42). The following lemma, combined with Lemma[3] provides a preliminary estimate for the smoothness of the functions $F_{\alpha, \beta, \gamma}$. These estimates are easily obtained by an inductive process.

**Lemma 11.** Let $\lambda > 0$. The function $e^{2\pi i H_\lambda(x, y)}$ belongs to $C^\infty(\mathbb{R}^2 \setminus \{0, 0\})$. Moreover, for any $(x, y) \neq (0, 0)$ and $n \in \mathbb{N}$, we have.

\[
(a) \quad \left|\left(e^{2\pi i H_\lambda}\right)^{(n)}(x, y)\right| \leq \begin{cases} C x^{-\lambda - n} & 0 < x, 0 < y < x^\lambda, \\ 0 & \text{otherwise}, \end{cases}
\]

\[
(b) \quad \left|\left(e^{2\pi i H_\lambda}\right)^{(n)}(x, y)\right| \leq \begin{cases} C x^{n \lambda} & 0 < x, 0 < y < x^\lambda, \\ 0 & \text{otherwise}, \end{cases}
\]

where $C = C(n, \lambda)$ does not depend on $x$ and $y$.

An explicit estimate for the smoothness of the functions $F_{\alpha, \beta, \gamma}$ near the origin is given in the following lemma.

**Lemma 12.** Let $\alpha, \beta > 0$ and $k \in \mathbb{N}$ be such that both $2\alpha + \alpha/\beta + 1 \leq 2k$ and $2\beta + \beta/\alpha + 1 \leq 2k$. If $\gamma < \min\{\alpha/k, \beta/k\}$, then for any $\epsilon > 0$ we have

\[
(a) \quad \int_{-b}^{b} \int_{[-c, c]^2} \left|\Delta_h F_{\alpha, \beta, \gamma}(x, y)\right|^2 \frac{dx dy dh}{h^{2 \alpha + \alpha/\beta + 2 - \epsilon}} < \infty,
\]
Figure 3. Partition of $[-c, c]^2$ into $V_1 \cup V_2 \cup V_3$ and $W_1 \cup W_2 \cup W_3$ in the proof of Lemma 12.

\[(b) \quad \int_{-b}^{b} \int_{[-c, c]^2} \frac{|\Gamma_h^k F_{\alpha, \beta, \gamma}(x, y)|^2}{|h|^{2\beta/\alpha + 2 - \epsilon}} \, dx\, dy \, dh < \infty,\]

where $b, c$ are any two positive numbers.

Proof. We show that the integral in (a) converges. For the estimate of the integral in (b), which can be obtained in much the same way, we give a short sketch at the end of this proof.

Set $f = f_{\alpha, \beta, \gamma}$, $F = F_{\alpha, \beta, \gamma}$ and $H = H_{\frac{3}{2}}$. By Remark 3(b), it is enough to show that for every $0 \leq n \leq k$ and $0 < h \leq 1$ we have

$$\iint_{[-c, c]^2} \left( |\Delta_h^n e^{2\pi i H} \cdot \Delta_h^{k-n} f_{nb} |^2 + |\Delta_h^n e^{2\pi i H} \cdot \Delta_h^{k-n} f_{-nh} |^2 \right) \, dx\, dy \leq Ch^{2\alpha+\alpha/\beta+1},$$

(47)

where we use the notation $f_{nb}(x, y) = f(x + nh, y)$ and the constant $C = C(F, k)$ does not depend on $h$. Since the case $n = 0$ follows from Lemma 10, it remains to show that (47) holds for $1 \leq n \leq k$. Fix such an integer $n$.

Since $F$ is bounded on any compact set, we can assume that $h$ is small enough for the following partition to hold

$$[-c, c]^2 = V_1 \cup V_2 \cup V_3,$$

where (see Figure 3)

$$V_1 = [-((k+1)h, (k+1)h) \times [0, ((2k+1)h)^{\alpha/\beta}],$$

$$V_2 = [(k+1)h, c] \times [0, c],$$

$$V_3 = [-c, c]^2 \setminus (V_1 \cup V_2).$$

This induces the splitting $I_1 + I_2 + I_3$ of the integral in (47). The estimate for $I_1$ can be obtained using Remark 3(a), as was done for the estimate over the area $V_1$ in the proof of Lemma 10. To estimate $I_3$, it suffices to observe that $e^{2\pi i H(\xi, \eta)} = 1$
for \((x, y) \in V_3\) and \(\xi \in [x - kh, x + kh]\), whence \(I_3 = 0\) (for example, by Remark \(3c\)).

We estimate \(I_2\). By lemmas \(9\) and \(11\), one can apply Remark \(3c\) in this area. So, it suffices to show that
\[
\int_{V_2} \left( \sup_{|\xi-x| \leq kh} |(e^{2\pi i H(\xi,y)}_x)^n| \right)^2 \left( \sup_{|\xi-x| \leq kh} |f_x^{(k-n)}(\xi,y)|^2 \right) dx dy \leq h^{2\alpha+\alpha/\beta+1-2k}.
\]
Note that for \((x,y) \in V_2\), the condition \(|\xi-x| \leq kh\) implies that \(x/(k+1) \leq \xi \leq 2x\). In addition, if \(y > (2x)^{\alpha/\beta}\), then \((e^{2\pi i H(\xi,y)}_x)^n = 0\). On the other hand, if \(y \leq (2x)^{\alpha/\beta}\), then by Lemmas \(9\) and \(11\) we have
\[
\sup_{|\xi-x| \leq kh} |(e^{2\pi i H(\xi,y)}_x)^n| \sup_{|\xi-x| \leq kh} |f_x^{(k-n)}(\xi,y)| \leq C y x^{\alpha-\alpha/\beta-k}.
\]
Hence, the desired estimate is found by checking that
\[
\int_{(k+1)h}^{c} \int_0^{2\pi} y^2 x^{2\alpha-2\alpha/\beta-2k} dy dx \leq C h^{2\alpha+\alpha/\beta+1-2k}.
\]
This completes the proof of (47).

In a similar way, one can show that the integral in \((b)\) converges. An appropriate partition in this case is \([-c, c]^2 = W_1 \cup W_2 \cup W_3\), where (see Figure \(3\))
\[
W_1 = [0, h^{\beta/\alpha}] \times [-(k+1)h, (k+1)h],
\]
\[
W_2 = [h^{\beta/\alpha}, c] \times [-(k+1)h, c],
\]
\[
W_3 = [-c, c]^2 \setminus (W_1 \cup W_2).
\]
Note that over \(W_2\), the corresponding integral is smaller than the one taken over the area \(h^{\beta/\alpha} < x < c\) and \(-(k+1)x^{\alpha/\beta} < y < (k+1)x^{\alpha/\beta}\), which is easily estimated. This completes the proof.

\[\square\]

6. Theorem \(2\) – Second Part

Here we prove part \((b)\) of Theorem \(2\). Assume that the point \((r, s)\) is above the curve \(\Gamma_q\) (see \(3\) and Figure \(1\)). This implies that either one of the following conditions holds:
\[
\frac{1}{r} + \frac{3}{s} > 1 \quad \text{and} \quad \frac{1}{r} + \frac{1}{s} > \frac{q}{2(q-1)}, \tag{48}
\]
or
\[
\frac{1}{r} + \frac{3}{s} \leq 1 \quad \text{and} \quad \frac{3q - 2}{q + 2} \cdot \frac{1}{r} + \frac{1}{s} > 1. \tag{49}
\]

For each of the conditions (48) and (49), we construct a quasi-periodic function \(G\) on \(\mathbb{R}^2\) that, when restricted to \(Q\), is square integrable. By the surjectivity of the Zak transform, there exists a function \(g \in L^2(\mathbb{R})\) such that \(g = Z^{-1} G\). We prove that this function satisfies all the requirements of Theorem \(2\). Roughly speaking,
we show that on the one hand the functions \( G \) are smooth enough for the time–frequency conditions \([2]\) to follow from Lemma \([3]\), while on the other hand they decrease slowly enough, near their single zero, for the \((C_q)\) property to follow from Lemma \([2]\).

In various stages of our construction we make simple interpolations of functions. To this end, we use of the following auxiliary function. Fix \( 0 < \eta < 1/4 \), and denote by \( \rho(t) \) an even function in \( C^\infty(\mathbb{R}) \) which satisfies \( 0 < \rho(t) < 1 \) on \((-2\eta, 2\eta)\) and

\[
\rho(t) = \begin{cases} 
1 & \text{for } t \in [-\eta, \eta], \\
0 & \text{for } t \in \mathbb{R}\setminus[-2\eta, 2\eta].
\end{cases}
\] (50)

6.1. Proof for the first set of conditions. Fix \( r \leq s \) for which condition \((48)\) holds. We may assume that \( 1/r + 1/s \leq 1 \). Choose \( \epsilon > 0 \) small enough for the numbers

\[
r' = r + \epsilon \quad \text{and} \quad s' = s + \epsilon
\] (51)
to satisfy

\[
\frac{1}{r'} + \frac{1}{s'} > \frac{q}{2(q-1)}.
\] (52)

Set

\[
\alpha = \frac{r'}{2} \left( 1 - \frac{1}{r'} - \frac{1}{s'} \right) \quad \text{and} \quad \beta = \frac{s'}{2} \left( 1 - \frac{1}{r'} - \frac{1}{s'} \right).
\] (53)

In the construction of the function \( G \) described above, we consider the argument and modulus separately. In fact, we construct functions \( \Psi \) and \( \Phi \) such that

\[
G(x, y) = \Phi(x - 1/2, y - 1/2)e^{2\pi i \Psi(x-1/2,y-1/2)}.
\] (54)

To define the argument of \( G \), i.e., the real valued function \( 2\pi \Psi \), we use a minor modification of a construction from \([2]\). That is, instead of a singularity at the origin, we find it more convenient to use an argument with a singularity at \((1/2, 1/2)\). This also accounts for the translation of \( 1/2 \) in the definition \((54)\).

We begin by defining the function \( \Psi(x, y) \) on \([-1/2, 1/2) \times [0, 1)\):

\[
\Psi(x, y) = \begin{cases} 
0 & \text{for } x \in [-1/2, 0], \\
\rho(x)H_{\beta}(x, y) + (1 - \rho(x))(y - 1/2) & \text{for } x \in [0, 1/2).
\end{cases}
\]

where \( H_{\beta} \) is the function defined in \((45)\) (see Figure \([4]\)). We extend \( \Psi \) to the plane according to the rules

\[
\Psi(x + 1, y) = \Psi(x, y) + y - 1/2 \quad \text{for } x \in \mathbb{R}, \ y \in [0, 1),
\]

\[
\Psi(x, y + 1) = \Psi(x, y) \quad \text{for } (x, y) \in \mathbb{R}^2.
\] (55)

Note that the function \( e^{2\pi i \Psi(x,y)} \) is continuous over \( \mathbb{R}^2 \setminus \mathbb{Z}^2 \) since,

\[
\lim_{y \to 1^-} \Psi(x, y) = \begin{cases} 
\Psi(x, 0) & \text{for } x \in [-1/2, 1/2), \\
\Psi(x, 0) + 1 & \text{otherwise}.
\end{cases}
\]
and

\[ \lim_{x \to \frac{1}{2}^-} \Psi(x, y) = y - 1/2 = \Psi(-1/2, y) + y - 1/2, \quad \forall y \in [0, 1). \]

In fact, one can verify that \( e^{2\pi i \Psi(x, y)} \) belongs to \( C^\infty(\mathbb{R}^2 \setminus \mathbb{Z}^2) \) (see also [2]).

We turn to constructing the modulus of \( G \), i.e. the function \( \Phi \). Choose \( k \in \mathbb{N} \) which satisfies

\[ k > \frac{1}{2} \max\{r', s'\}, \quad (56) \]

and a number \( \gamma > 0 \) for which

\[ \gamma < \min \left\{ \frac{\alpha}{k}, \frac{\beta}{k} \right\}. \quad (57) \]

We define the function \( \Phi(x, y) \) on \([-1/2, 1/2)^2 \) by

\[ \Phi(x, y) = \rho(y) \left( \rho(x)(|x|^{\alpha/\gamma} + |y|^{\beta/\gamma})^\gamma + 1 - \rho(x) \right) + 1 - \rho(y), \]

and extend \( \Phi \) to be a 1-periodic function on the plane (See Figure 5). Note that \( \Phi \) is continuous on \( \mathbb{R}^2 \) since

\[ \Phi(-1/2, y) = \lim_{x \to \frac{1}{2}^-} \Phi(x, y) = 1, \quad \forall y \in [-1/2, 1/2), \]
and

$$\Phi(x, -1/2) = \lim_{y \to \frac{1}{2}^-} \Phi(x, y) = 1, \quad \forall x \in [-1/2, 1/2).$$

In fact, using Lemma 9, one can check that $\Phi \in C^k(\mathbb{R}^2 \setminus \mathbb{Z}^2)$. Moreover, $\Phi = 0$ on the lattice $\mathbb{Z}^2$, and only there.

Consider the function $\Phi(x, y)e^{2\pi i \Psi(x, y)}$.

We list its growth and smoothness properties:

(i) It belongs to $C^k(\mathbb{R}^2 \setminus \mathbb{Z}^2)$.

(ii) Its modulus is continuous, bounded, and is equal to zero on $\mathbb{Z}^2$, and only there.

(iii) There exists a neighbourhood of the origin, say $U$, on which it is equal to the function $F_{\alpha, \beta, \gamma}$ defined in (46).

In particular, it follows that the function $G$ defined in (54) is bounded. Moreover, since $\Phi$ is 1-periodic, the condition (55) implies that $G$ is quasi-periodic over $\mathbb{R}^2$. Therefore, there exists a function $g \in L^2(\mathbb{R})$ such that $Zg = G$ on $\mathbb{R}^2$. We prove that $g$ satisfies the requirements of Theorem 2.

To show that $g$ has the required time–frequency localisation, we check that the integrals in (2) are finite. By Lemma 5, we need to show that for $\alpha, \beta, k$ and $\gamma$ chosen above, the following integrals are finite:

$$\int_{-\infty}^{\infty} \int_{[0,1]^2} \frac{|\Delta_h^k G(x, y)|^2}{h^{1+r}} \, dx \, dy \, dh \quad \text{and} \quad \int_{-\infty}^{\infty} \int_{[0,1]^2} \frac{|\Gamma_h^k G(x, y)|^2}{h^{1+s}} \, dx \, dy \, dh. \quad (58)$$
We show that the left-hand integral is finite, the proof for the right-hand integral follows in the same way. As in Remark 3(a), it is enough to show that for some \( \delta > 0 \) the integral

\[
\int_{-\delta}^{\delta} \iint_{[0,1]^2} \frac{|\Delta_h^k G(x,y)|^2}{h^{1+r}} \, dx \, dy \, dh = \int_{-\delta}^{\delta} \iint_{[-\frac{1}{2},\frac{1}{2}]^2} \frac{|\Delta_h^k (\Phi(x,y)e^{2\pi i \Psi(x,y)})|^2}{h^{1+r}} \, dx \, dy \, dh
\]

converges.

So, choose \( \delta > 0 \) such that

\[
[-(2k+1)\delta, (2k+1)\delta]^2 \subset U,
\]

where \( U \) is the neighbourhood of the origin described in property (iii) above. We divide the integral above into the sum of two integrals \( I_1 + I_2 \) according to the following partition of \([-1/2, 1/2]^2\):

\[
\Omega_1 = [-(k+1)\delta, (k+1)\delta]^2 \quad \text{and} \quad \Omega_2 = \left[-\frac{1}{2}, \frac{1}{2}\right]^2 \setminus \Omega_1.
\]

To check that \( I_2 \) is finite, we use Remark 3(c) and property (i) above to get

\[
\int_{-\delta}^{\delta} \iint_{\Omega_2} \frac{|\Delta_h^k (\Phi e^{2\pi i \Psi})|^2}{h^{1+r}} \, dx \, dy \, dh \leq \int_{-\delta}^{\delta} h^{2k-1} \, dh \sup_{(x,y) \in \Omega_2} \sup_{|x-\xi| \leq k\delta} |(\Phi e^{2\pi i \Psi})^k(\xi, y)|,
\]

where the supremum exists and is finite, and so \( I_2 \) is finite since \( 2k > r \) (see (51) and (56)). That \( I_1 \) converges follows from Lemma 12. To see this, first note that (59) and property (iii) above ensure that

\[
\Delta_h^k (\Phi e^{2\pi i \Psi}) = \Delta_h^k F_{\alpha,\beta,\gamma}, \quad \forall (x, y) \in \Omega_1, \quad h \in [-\delta, \delta].
\]

In addition, the conditions (51) and (53) imply that \( 2\alpha + \alpha/\beta + 1 = r' = r + \epsilon \). So the choices of \( k \) and \( \gamma \) ensure that Lemma 12 can be applied (see (56) and (57)).

The system \( G(g,1,1) \) is a Bessel system in \( L^2(\mathbb{R}) \), see Remark 2. It remains to be checked that \( G(g,1,1) \) is an exact \( (C_q) \)-system in \( L^2(\mathbb{R}) \). By Lemma 2, it is enough to show that \( 1/|Zg|^2 \) is in \( L^{\frac{q}{q-2}}([-1/2, 1/2]^2) \). This is equivalent to

\[
\frac{1}{|\Phi(x,y)|^2} \in L^{\frac{q}{q-2}}([-1/2, 1/2]^2).
\]

Note that properties (ii) and (iii) above imply that on \([-1/2, 1/2]^2\) we have

\[
|\Phi(x,y)| \geq C(|x|^\alpha/\gamma + |y|^\beta/\gamma)^\gamma \geq C(|x|^\alpha/\beta + |y|^\beta).
\]

So (61) follows from (52) and (53) in a direct computation. This completes the proof for the first part.

6.2. Proof for the second set of conditions. Next, let \( r < s < \infty \) and assume that the inequalities (19) hold. (The case \( s = \infty \) will be dealt with separately). Choose \( \epsilon > 0 \) small enough for the numbers

\[
r' = r + \epsilon \quad \text{and} \quad s' = s + \epsilon
\]

(62)
to satisfy
\[
\frac{3q - 2}{q + 2} \cdot \frac{1}{r'} + \frac{1}{s'} > 1
\]
and
\[
\frac{1}{r'} + \frac{3}{s'} < 1.
\]
Let the numbers \( \alpha, \beta, k \) and \( \gamma \) be as defined in (53), (56) and (57) respectively.

Our objective is to construct a function \( \Upsilon \), which on \([-1/2, 1/2)^2 \) satisfies
\[
\Upsilon(x,y) = \Theta(x,y) - \Theta(-x,y)e^{2\pi iy}
\]
for some function \( \Theta \), in such a way that the function
\[
G(x,y) := \Upsilon(x - 1/2, y - 1/2)
\]
is a quasi-periodic function with the desired smoothness properties and zero at \((1/2, 1/2)\).

We begin by constructing the function \( \Theta \) on \([-1/2, 1/2)^2 \) in two steps. First, define the function (see Figure 6)
\[
\Theta_0(x,y) := \begin{cases} 
\rho(y)[(2(-x)^{\alpha/\gamma} + |y|^{{\beta/\gamma}})^{\gamma} + 1] + 1 - \rho(x) & \text{for } -1/2 \leq x < 0, \\
\rho(x)[(x^{\alpha/\gamma} + |y|^{{\beta/\gamma}})^{\gamma} + 1] & \text{for } 0 \leq x < 1/2. 
\end{cases}
\]
By Lemma 10 the function \( \Theta_0 \) belongs to \( C^k((-1/2, 1/2)^2 \setminus (0,0)) \). In addition, it has the following properties:

(i) \( \Theta_0(x,y) = \begin{cases} 
1 & \text{for } x \in [-1/2, -2\eta], \\
0 & \text{for } x \in [2\eta, 1/2), 
\end{cases} \)
(ii) It is bounded from below on \( W_1 = [-\eta, \eta] \times [-1/2, 1/2) \).
(iii) The difference \( |\Theta_0(-x,y) - \Theta_0(x,y)| \) is bounded from below on the set \( W_2 = [-1/2, 1/2)^2 \setminus W_1 \).
(iv) On \([-\eta, \eta]^2 \), it is equal to \( f_{\alpha,\beta,\gamma}(x,y) + 1 \), where the function \( f_{\alpha,\beta,\gamma} \) is defined in (42) with a fixed \( a = 2^{\gamma/a} \).

Next, we wish to preserve these properties for the function \( \Theta \) while adding the additional condition
\[(v') \quad \Theta(x, -y) = \Theta(x, y) \quad \text{for} \quad y \in [2\eta, 1/2). \]
To this end, denote by \( \nu(t) \) a function in \( C^\infty([-1/2, 1/2)) \) that satisfies
\[
\nu(t) = \begin{cases} 
1 & \text{for } t \in [-1/2, -2\eta], \\
0 & \text{for } t \in [2\eta, 1/2), 
\end{cases} \]
is bounded from below on \( J = [-\eta, \eta] \), and for which \( |\nu(-t) - \nu(t)| \) is bounded from below on \([-1/2, 1/2) \setminus J \). With this, we make the following definition for \((x, y) \in [-1/2, 1/2)^2 \) (see Figure 6):
\[
\Theta(x, y) := \rho(y)\Theta_0(x, y) + (1 - \rho(y))\nu(x).
\]
One can easily verify that with this interpolation the property \( (v') \), as well as the properties corresponding to \( (i)\)–\( (iv) \), hold for \( \Theta \). For the function \( \Theta \), we refer to these properties as \( (i')\)–\( (v') \).

Define the function \( \Upsilon \) on \( [-1/2, 1/2)^2 \) by (65), and extend it to \( \mathbb{R}^2 \) according to the rules

\[
\Upsilon(x, y + 1) = \Upsilon(x, y), \quad \Upsilon(x + 1, y) = -e^{2\pi iy} \Upsilon(x, y), \quad \forall (x, y) \in \mathbb{R}^2. \tag{67}
\]

The function \( \Upsilon \) is continuous on \( \mathbb{R}^2 \) since, as follows from property \( (i') \),

\[
\lim_{x \to \frac{1}{2}} \Upsilon(x, y) = -e^{2\pi iy} = -e^{2\pi iy} \Upsilon(-1/2, y), \quad \forall y \in [-1/2, 1/2),
\]

while property \( (v') \) implies

\[
\lim_{y \to \frac{1}{2}} \Upsilon(x, y) = \Upsilon(x, -1/2), \quad \forall x \in [-1/2, 1/2).
\]

With a more careful use of properties \( (i') \) and \( (v') \) one can verify that in fact \( \Upsilon \in C^k(\mathbb{R}^2 \setminus \mathbb{Z}^2) \).

Let \( G \) be the function defined in (66). In particular, the conditions above imply that \( G \) is bounded on \( \mathbb{R}^2 \). Moreover, the function \( G \) is quasi-periodic, as follows from the condition \( (67) \) on \( \Upsilon \). As above, this implies that there exists a function \( g \in L^2(\mathbb{R}) \) such that \( Zg = G \) on \( \mathbb{R}^2 \). We prove that \( g \) satisfies the requirements of Theorem 2.

As in the first part, to see that \( g \) has the required time–frequency localisation, we bound the integrals (58) using the partition \( \Omega_1 \cup \Omega_2 \) given in (60). The estimates over \( \Omega_2 \) follow exactly as before, while the estimate for \( \Delta_h \) over \( \Omega_1 \) follows from

\[
\text{Figure 6. Illustration of the functions } \Theta_0 \text{ and } \Theta \text{ on } [-1/2, 1/2)^2.\]
By (53) and (64), we have
\[ |\Delta_h^k \Upsilon(x, y)|^2 \leq 2 \left( |\Delta_h^k \Theta(x, y)|^2 + |\Delta_h^k \Theta(-x, y)|^2 \right), \]
in addition to Lemma 10. The estimate for \( \Gamma_h \) can be obtained in essentially the same way, using Remark 4(b) to compensate for the additional exponential factor.

Again, the system \( G(g, 1, 1) \) is a Bessel system in \( L^2(\mathbb{R}) \), see Remark 2, and so it remains to be checked that \( G(g, 1, 1) \) is an exact \( (C_q) \)-system. As in the first part of the proof, it suffices to check that
\[ \frac{1}{|\Upsilon|^2} \in L^{q/(q-2)}([-1/2, 1/2]^2). \]  
(68)

We claim that on \([-1/2, 1/2]^2\) we have \( |\Upsilon(x, y)|^2 > C(|x|^\alpha + |y|^2) \). This, combined with (53) and (63), implies (68) in a direct computation which completes the proof. So, to verify the estimate above, first calculate
\[
|\Upsilon(x, y)|^2 = |\Theta(x, y) - \Theta(-x, y)e^{2\pi iy}|^2
= \left( \Theta(x, y) - \Theta(-x, y) \right)^2 + 4 \Theta(x, y)\Theta(-x, y) \sin^2 \pi y.
\]

Now, by property (iii'), it follows that \((I)\) is bounded from below on \( W_2 \). In the same way \((II)\) is bounded from below on \( W_1 \setminus [-\eta, \eta]^2 \) due to property (ii'). This leaves the region \([-\eta, \eta]^2\), where, by the above calculation and (iv'), we have
\[
|\Upsilon(x, y)|^2 \geq C \left\{ \left( 2|x|^\alpha + |y|^\beta \right)^\gamma - \left( |x|^\alpha + |y|^\beta \right)^\gamma \right\}^2 + y^2 \}
(III)
\]

By (53) and (64), we have \( \beta > 1 \), and so \( y^2 \geq y^{2\beta} \). In addition, we note that since \( 0 < \gamma < 1 \), we have \( a^\gamma + b^\gamma \geq (a + b)^\gamma \) for positive \( a \) and \( b \). Using these facts, and (22), we obtain
\[
(III) \geq \frac{1}{2} \left[ \left( 2|x|^\alpha + |y|^\beta \right)^\gamma - \left( |x|^\alpha + |y|^\beta \right)^\gamma \right]^2 + y^{2\beta} + y^2 \]
\[
\geq \frac{1}{4} \left[ \left( 2|x|^\alpha + |y|^\beta \right)^\gamma + (y^{2\beta})^\gamma - \left( |x|^\alpha + |y|^\beta \right)^\gamma \right]^2 + y^2 \]
\[
\geq \frac{1}{4} \left[ \left( 2|x|^\alpha + 2|y|^\beta \right)^\gamma - \left( |x|^\alpha + |y|^\beta \right)^\gamma \right]^2 + y^2 \]
\[
\geq C \left[ |x|^{2\alpha} + y^2 \right] \geq C(|x|^\alpha + |y|)^2.
\]

We turn to the case \( s = \infty \). If the point \((1/r, 0)\) is above the curve \( \Gamma_q \), then there exists \( r' \) which satisfies \( r < r' < (3q - 2)/(q + 2) \). Set \( \alpha = (r' - 1)/2 \). The construction of the required example can be done in the same way as above with the following modification of the function \( \Theta \):
\[
\Theta(x, y) = \Theta_0(x, y) = \begin{cases} 
\rho(x)(2(-x)^\alpha + 1) - 1 - \rho(x) & \text{for } -1/2 \leq x < 0, \\
\rho(x)(x^\alpha + 1) & \text{for } 0 \leq x < 1/2,
\end{cases}
\]
and a corresponding change in the formulation and proof of Lemma 10. Note that in this case, the functions \( g \) satisfying \( Zg = G \), can be given explicitly. These functions are essentially the same as the functions \( g_\alpha \) constructed in [14], Section 6.

\[ \square \]

7. Concluding remarks

Remark 7. It is well-known that if a system \( G(g, 1, 1) \) is a frame in \( L^2(\mathbb{R}) \) (i.e., a Bessel \((C_2)\)-system), then it is also exact in the space, and therefore a Riesz basis. For general \((C_q)\)-systems, however, this is not the case: There exists a system \( G(g, 1, 1) \) which is a (Bessel) \((C_q)\)-system, for every \( q > 2 \), but is not exact. This can be shown in much the same way as [19, Theorem 2].

Remark 8. The systems \( G(g, 1, 1) \) constructed in Section 6 prove the following claim: For every \( q_0 > 2 \) there exists a system that is a (Bessel) \((C_q)\)-system whenever \( q > q_0 \), but is not such a system for \( q < q_0 \). Indeed, the first part follows from the construction and the latter part follows by Theorem 2(a).

Remark 9. Fix \( q \geq 2 \). It follows by Theorem 2(a) that if the point \((1/r, 1/s)\) is below the curve \( \Gamma_q \), then a function \( g \), for which both the integrals in (2) converge, cannot generate a \((C_q)\)-system. By Theorem 2(b), on the other hand, if \((1/r, 1/s)\) is above the curve \( \Gamma_q \), then the function can generate a \((C_q)\)-system. However, if \((1/r, 1/s)\) is on the curve \( \Gamma_q \), then Theorem 2 does not determine whether \( g \) can generate a \((C_q)\)-system, or not. We mention this question as a possible problem for future research.

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