TRAIN TRACK MAPS FOR GRAPHS OF GROUPS

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ABSTRACT. We define train track maps for graphs-of-groups and show that a fundamental finiteness property (which allows one to control the growth of illegal turns) for classical train track maps extends to this generalization.

1. INTRODUCTION

Train track theory has first been introduced by Thurston for surface homeomorphisms, and later carried over by Bestvina-Handel in [1] to free group automorphisms. It has turned out to be a central tool in the study of outer automorphisms of free groups $F_N$ of finite rank $N \geq 2$.

A self-map $f : \Gamma \to \Gamma$ of a finite graph $\Gamma$ is a train track map if every edge $e$ of $\Gamma$ is “legal”: for any $t \geq 1$ the edge path $f^t(e)$ is reduced.

One of the fundamental properties which makes expanding train track maps such a valuable tool is the fact that for every path or loop $\gamma$ in $\Gamma$ there is an exponent $t \geq 0$ such that $f^t(\gamma)$ is pseudo-legal: it is a legal concatenation of legal paths and so called INP paths. There are only finitely many INP paths in $\Gamma$, and they can be determined easily from the defining data for $f$.

The purpose of this note is to show that the analogous statement is true for any train track map $f : \mathcal{G} \to \mathcal{G}$ of a graph-of-groups $\mathcal{G}$ with trivial edge groups (see Theorem 2.5). The proof presented below in §3 is elementary but a bit intricate; it goes back to the proof of Lemma 3.2 of [4]. The result presented here plays a crucial role in deriving from a given relative train track map an absolute train track map, modulo polynomially growing edges (see [5, 6]).

2. TERMINOLOGY AND PRECISE STATEMENT OF THE RESULT

Throughout this section let $\mathcal{G}_X$ be a graph-of-groups with trivial edge groups, built on a finite connected graph. Let $\mathcal{G}$ be the topological realization of the graph-of-groups $\mathcal{G}_X$ as a graph-of-spaces, by which we mean that every vertex group $G_v$ of $\mathcal{G}_X$ is realized by a vertex space $X_v$ with $\pi_1 X_v = G_v$. We can thus think of $\mathcal{G}_X$ as obtained from contracting every connected component $X_v$ of the relative part $X = \cup X_v$ of $\mathcal{G}$ to a single vertex $v$, and in turn providing $v$ with a non-trivial vertex group $G_v$. This gives

$$\pi_1 \mathcal{G} = \pi_1 \mathcal{G}_X,$$

where the term on the left hand side is an ordinary fundamental group, while on the right hand side we have the classical fundamental group of a graph-of-groups. The language for graph-of-groups used here is standard; it is summarized for instance in §2.1 of [8].

Even though in praxis the vertex spaces $X_v$ may well also contain edges, we will consider them only as “local edges”; by an edge of $\mathcal{G}$ we always mean an edge in $\mathcal{G} \setminus X$. Indeed, the concrete shape of the vertex spaces never plays a role, and we will consider paths in any vertex space $X_v$ only up to homotopy relative endpoints.

An edge path in $\mathcal{G}$ is a path $\gamma = e_1 \circ \chi_1 \circ e_2 \circ \chi_2 \circ \ldots \circ \chi_{r-1} \circ e_r$, where each $e_i$ is an edge in $\mathcal{G} \setminus X$, while every connecting path $\chi_i$ is contained in the vertex space $X_{v(i)}$ which contains the terminal
endpoint of the edge $e_i$ and the initial endpoint of the edge $e_{i+1}$. The “identification” of $G$ with $G_X$ transforms $\gamma$ into a “connected” word $W_\gamma = e_1 x_1 e_2 x_2 \ldots x_{r-1} e_r$ in the Bass group $\Pi(G_X)$, i.e. each $x_i$ is an element of the vertex group $G_{v(i)}$, for $v_i = \tau(e_i) = \tau(\tau_{i+1})$. Here $\tau$ denotes the edge $e$ with reversed orientation, and $\tau(e)$ is the terminal vertex of $e$. The length $|\gamma|$ of an edge path $\gamma$ is equal to the number of edges (from $G \setminus X$) traversed by $\gamma$.

**Convention 2.1.** In this paper we use the convention that every edge path starts and finishes with an edge from $G$.

A path $\gamma'$ in $G$ is not necessarily an edge path, but $\gamma'$ is always part of an edge path $\gamma$ as above, where some (possibly trivial) initial segment from the first and terminal segment from the last edge of $\gamma$ is missing in $\gamma'$. The edge path $\gamma$ is called the **canonical vertex-prolongation** of the path $\gamma'$. We define the length of $\gamma'$ through $|\gamma'| = |\gamma|$.

We define a turn in $G_X$ to be a word $e x e' \in \Pi(G_X)$ with $x \in G_{\tau(e)} = G_{r(\tau)}$. It corresponds to an edge path $\gamma = e \circ \chi \circ e'$ of length 2 in $G$. This definition is a slight variation of classical train track terminology, where one has always $x = 1$ and hence prefers to present the path $e x e'$ as pair $(\tau, e')$.

A turn $e x e'$ is degenerate if $x = 1 \in G_{\tau(e)}$ and $\tau = e$. Equivalently, for the corresponding edge path $e \circ \chi \circ e'$ one has $\tau = e'$ and $\chi$ is a contractible loop in $X_{\tau(e)}$. An edge path $\gamma$ in $G$ is reduced if any subpath of length 2 defines a non-degenerate turn. A path $\gamma'$ in $G$ is reduced if the canonical vertex-prolongation $\gamma$ of $\gamma'$ is reduced. Every non-reduced path $\gamma$ can be transformed by iterative reductions (i.e. cancellation of degenerate turns) into a reduced path $[\gamma]$ with same endpoints as $\gamma$. The order of the reductions is irrelevant: the reduced path $[\gamma]$ is uniquely determined by $\gamma$, and the two paths are homotopic in $G$ relative endpoints. If $\gamma$ is a path but not an edge path, then we have to include the special case where $\gamma$ has both endpoints $P$ and $Q$ on the same edge $e$ of $G$ and reduces to the segment $[P, Q]$ of $e$.

**Convention 2.2.** Since the transition from $G$ to $G_X$ (and conversely) is completely canonical, we will from now on simplify notation and denote both, the graph-of-spaces and the corresponding graph-of-groups by $G$. We will freely use both languages, according to whichever is better suited to the circumstances.

A map $f : G \to G$ is always assumed to map vertex spaces to vertex spaces, and edges to edge paths. The map $f$ is expanding if some power $f^t$ maps every edge $e$ to an edge path of length $|f^t(e)| \geq 2$.

We always assume that $f$ induces an (outer) automorphism of $\pi_1 G$. As a consequence we observe that any vertex $v$ of $G$ which is not $f$-periodic must have a trivial vertex group $G_v$. We call such non-periodic vertices (and also the corresponding vertex groups or vertex spaces) inessential, while the periodic ones are called essential.

**Remark 2.3.** (1) If $\pi_1 G$ is a free group of finite rank (which is the case we are most interested in), then there is an upper bound $C \geq 0$ to the length of any backtracking path in the $f$-image of any reduced path. This is classical for the special case where all vertex groups are trivial, and can be reduced to this case through realizing every vertex group $G_v$ by a “local graph” $X_v$ with $\pi_1 X_v = G_v$.

(2) In the general case the author is not aware of the analogous result. However, in this paper we only consider the special case where $f$ is a train track map (defined just below). In this case the existence of a cancellation bound $C$ as above can be derived by elementary methods similar to those presented below in the proof of Lemma 3.3.

A path $\gamma$ in $G$ is legal if the image path $f^t(\gamma)$ is reduced, for any integer $t \geq 1$. The map $f$ is a train track map if every edge (understood as edge path of length 1) is legal. A turn $e \circ \chi \circ e'$ is illegal if the path $e \circ \chi \circ e'$ is not legal.
Remark 2.4. From the hypothesis that $f$ induces an automorphism on $\pi_1 \mathcal{G}$, and hence is injective on $\pi_1 X_v$ for every vertex $v$, we deduce directly that for any two edges $e, e'$ of $\mathcal{G}$ there is at most one connecting path $\chi$ so that $e \circ \chi \circ e'$ is an illegal turn. This shows that there are only finitely many illegal turns in $\mathcal{G}$. [Recall our convention that paths in $X$ count as “equal” if they are homotopic in $X$ rel. endpoints.]

A concatenation $\gamma \circ \chi \circ \gamma'$ of two edge paths $\gamma$ and $\gamma'$ is legal if the turn at the concatenation vertex is legal.

A non-trivial path (not necessarily an edge path!) $\eta = \gamma \circ \chi \circ \gamma'$ is an INP path if both, $\gamma$ and $\gamma'$ are legal, and if for some exponent $t \geq 1$ the reduced path $[f^t(\eta)]$ is equal to $\eta$. In this case the turn at the concatenation vertex of $\eta$, called the tip of the INP path $\eta$, is necessarily illegal, if we assume that $f$ is expanding.

We can now state the main result of this note:

Theorem 2.5. Let $\mathcal{G}$ be a graph-of-groups with trivial edge groups, and let $f : \mathcal{G} \to \mathcal{G}$ be an expanding train track map. Then the following holds:

1. There are only finitely many INP paths in $\mathcal{G}$.
2. For every edge path of loop $\gamma$ in $\mathcal{G}$ there is an exponent $t \geq 0$ such that $f^t(\gamma)$ is a legal concatenation of legal and INP subpaths. The exponent $t$ depends only on the number of illegal turns in $\gamma$ and not on the particular choice of $\gamma$ itself.
3. More precisely, there is an exponent $\hat{t} \geq 1$ such that for every edge path or loop $\gamma$ the number of illegal turns in $f^{\hat{t}}(\gamma)$ is at most half the number of illegal turns in $\gamma$, where one doesn’t count the illegal turns at the tip of any INP subpath of $\gamma$ or of $f^{\hat{t}}(\gamma)$.

3. INP PATHS IN $\mathcal{G}$

Definition 3.1. (1) For any integer $t \geq 1$ a turn in $\mathcal{G}$ is called $t$-special if $\gamma$ is a subpath of any $f^t(e_0)$ or of any $f^t(e_1e_2)$, where $e_0, e_1$ and $e_2$ are edges of $\mathcal{G}$ and the vertex $\tau(e_1) = \tau(\overline{e}_2)$ is inessential.

2. Any path $\gamma$ (not necessarily an edge path) in $\mathcal{G}$ is called pre-$t$-special if every turn in the canonical vertex-prolongation of $f^t(\gamma)$ is $t$-special.

The following is a direct consequence of Definition 3.1 and our assumption that $f$ induces an automorphism of $\pi_1 \mathcal{G}$.

Lemma 3.2. (1) For any $t \geq 1$ there are only finitely many $t$-special turns.

2. For any integers $t \geq 1$ and $k \geq 0$ there are only finitely many edge paths $\gamma$ in $\mathcal{G}$ of length $|\gamma| \leq k$ which are pre-$t$-special.

Lemma 3.3. Let $\gamma$ and $\gamma'$ be two legal paths with common initial vertex space but distinct first edges (or first edge segments), and endpoints that may lie in the interior of an edge. Assume that for some integer $t \geq 1$ one has $f^t(\gamma) = f^t(\gamma')$, and assume that this is a proper edge path. Then every turn in $f^t(\gamma) = f^t(\gamma')$ is $t$-special.

Proof. We consider lifts of $\gamma$ and $\gamma'$ to the Bass-Serre tree $\tilde{\mathcal{G}}_X$, such that these lifts have a common initial vertex. Since any lift $\tilde{f} : \tilde{\mathcal{G}}_X \to \tilde{\mathcal{G}}_X$ of the map $f^t$ acts as bijection on the essential vertices of $\tilde{\mathcal{G}}_X$, we observe directly that every turn in $f^t(\gamma) = f^t(\gamma')$ must be $t$-special.

Lemma 3.4. Let $\mathcal{V}(f)$ be the set of edge paths $\eta = \overline{\gamma}_1 \circ \chi \circ \gamma_2$ with the following properties:

Assume that $\gamma_1$ and $\gamma_2$ be two legal edge paths with common initial vertex space but distinct first edges, and define $\gamma'_{1}$ and $\gamma'_{2}$ be the initial subpaths of $\gamma_1$ and $\gamma_2$ respectively which satisfy $f(\gamma'_1) = f(\gamma'_2)$, and are maximal with respect to this property. We require furthermore that each $\gamma'_{i}$
is non-trivial and contains all vertices of $\gamma_i$ except for the terminal one, and that none of the $\gamma'_i$ terminates in a vertex of $\gamma_i$. Finally, the turn at $\chi$ must be illegal.

Then the set $\mathcal{V}(f)$ is finite.

**Proof.** By hypothesis every vertex on each $\gamma_i$ except for the final one belongs to $\gamma'_i$. From Lemma 2.3 we know that the $\gamma'_i$ are both pre-1-special. Since each $\gamma_i$ is the canonical vertex-prolongation of $\gamma'_i$, it follows that that both $\gamma_i$ are also pre-$1$-special.

Furthermore, the length of the $\gamma_i$ is bounded, by Remark 2.3. Hence Lemma 3.2 (2) shows that there are only finitely many choices for $\gamma_1$ and $\gamma_2$. The hypothesis that the turn at $\chi$ is illegal thus proves (see Remark 2.4) the finiteness of the set $\mathcal{V}(f)$. \hfill \Box

We will now consider more systematically reduced paths $\eta = \gamma \circ \chi \circ \gamma'$ in $\mathcal{G}$ where $\gamma$ and $\gamma'$ are legal paths which start at points (not necessarily vertices) of $\mathcal{G}$ and end at a common vertex space $\tau(\gamma) = \tau(\gamma')$, which contains the connecting path $\chi$. If the turn of $\eta$ at this vertex (called the tip of $\eta$) is illegal, we say that $\eta$ is a pseudo-INP path. If furthermore $f(\eta)$ reduces to

$$[f(\eta)] = \eta,$$

then $\eta$ is an INP path. The backtracking subpath of $f(\eta)$, which starts at the tip of $[f(\eta)]$, runs up to the tip of $f(\eta)$, and then doubles back to the tip of $[f(\eta)]$, is in fact the maximal backtracking subpath of $f(\eta)$. The subpath $\eta_1 = \gamma_1 \circ \chi \circ \gamma'_1$ of $\eta$ which is mapped by $f$ to this maximal backtracking subpath of $f(\eta)$ will be called the $f$-backtracking subpath of $\eta$. We denote the $f'$-backtracking subpath of $\eta$ by $\eta_t = \gamma_t \circ \chi \circ \gamma'_t$, and observe:

**Definition-Remark 3.5.** Let $\eta$ be a pseudo-INP path in $\mathcal{G}$.

1. For any $t' \geq t \geq 1$ the path $\eta_{t'}$ contains $\eta_t$ as subpath.
2. For any $t \geq 1$ the path $[f^t(\eta)]$ is legal if and only if $\eta_t = \eta_{t'}$ for any $t' \geq t$.
3. We denote by $\eta_\infty$ the subpath of $\eta$ which is the union of all $\eta_t$ for any integer $t \geq 1$.

From the existence of a cancellation bound as in Remark 2.3 it follows exactly as for classical expanding train track maps (see for instance Remark 6.2 and Lemma 6.3 of [2]) that for any expanding train track map $f : \mathcal{G} \to \mathcal{G}$ the length of $\eta_\infty$ is bounded, independently of the choice of $\eta$. Indeed, one has:

**Proposition 3.6.** Let $f : \mathcal{G} \to \mathcal{G}$ be an expanding train track map. Then there exists integers $\hat{t}(f) \geq t(f) \geq 1$ such that for any pseudo-INP path $\eta$ in $\mathcal{G}$ one of the following statements (1) or (2) is true:

1. The reduced path $[f^{\hat{t}(f)}(\eta)]$ is legal.
2. (a) The two legal branches $\gamma_\infty$ and $\gamma'_\infty$ of the subpath $\eta_\infty = \gamma_\infty \circ \chi \circ \gamma'_\infty$ of $\eta$ are pre-$t(f)$-special.
   
   (b) The path $f^{\hat{t}(f)}(\eta_\infty)$ is an INP path.

**Proof.** Assume that claim (1) is false for any $\hat{t}(f) \geq 1$. From the definition of $\eta_\infty$ it follows directly that every vertex $v$ on $\eta_\infty$ which is not a boundary vertex must be contained in $\eta_{t(v)}$ for some $t(v) \geq 1$. From the boundedness of the length of $\eta_\infty$ we deduce that there are only finitely many such vertices, so that there is some upper bound $t(v) \in \mathbb{N}$ to all $t(v)$. It follows that every turn on $\gamma_\infty$ or $\gamma'_\infty$ is pre-$t(v)$-special, which proves assertion (2)(a) for $t_\eta$ instead of $t(\eta)$.

For any $k \geq 0$ let $E_k$ be the set of subpaths $\eta_t$ of any INP path $\eta$ for $f$, for any $t \geq 1$, with the condition that $\eta_t$ contains at most $k$ vertices. We now fix any $k \geq 0$ and assume that there is some integer $t_k$ with $t(v) \leq t_k$ for any vertex $v$ on any of the paths from $E_k$.

We now consider any pseudo-INP path $\eta$ and some subpath $\eta_t$ (for any $t \geq 1$) with $k + 1$ vertices. Then either we have $t \leq t_k$, or else the canonical vertex-prolongation of $\eta_{t_k}$ contains $\eta_t$ as subpath
and belongs to $V(f^{t_k})$. It follows from the finiteness result in Lemma 3.4 that in the latter case there is an exponent $t_{k+1}$, which only depends on $V(f^{t_k})$ and not on our choice of $\eta_t$, such that for every vertex $v$ of $\eta_t$ we have $t(v) \leq t_{k+1}$. Since for $k = 0$ the existence of $t_0$ is trivially true, the existence of an integer $t(f)$ which satisfies claim (2)(a) follows from the boundedness of the length of any $\eta_t$.

The claim (2)(b) is now a direct consequence of (2)(a): Since the length of the canonical vertex-prolongation $\hat{\eta}$ of any $\eta_{\infty}$ is bounded, and each legal turn is pre-$t(f)$-special, it follows from Lemma 3.2 (2) that there are only finitely many such $\hat{\eta}$ and thus only finitely many $\eta_{\infty}$ in $G$. Hence some power $f^t(\eta)$ of $f$ will eventually map any such $\eta_{\infty}$ to an $f$-periodic path, which is thus an INP path.

To finish the proof we still need to show that if claim (1) is true for some $t \geq 1$, then it must be true for $t = \hat{t}(f)$. But we have just concluded that, if $f^t(\eta)$ is not legal, then it contains an INP subpath: In this case, however, no $f$-iterate of $f^t(\eta)$ could possibly be legal. \(\Box\)

From Proposition 3.6 one deduces directly all statements from Theorem 2.5: The arguments used here are exactly the same as the ones used previously in more than one occasion for classical train track maps, see for instance [7], Propositions 4.12 and 4.18 or [3], Lemma 6.1.

References

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