A generalised Kotz type distribution and Riesz distribution

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Abstract
This article derives the distribution of random matrix \( X \) associated with the transformation \( Y = X^*X \), such that \( Y \) has a Riesz distribution for real normed division algebras. Two versions of this distributions are proposed and some of their properties are studied.

1 Introduction
Let \( X_1, \ldots, X_n \) be real independent \( \mathcal{N}_m(\mu, \Sigma) \), that is, \( X_i \) has a \( m \)-dimensional normal distribution with expected value \( \mu \in \mathbb{R}^m \) and \( m \times m \) positive definite covariance matrix \( \Sigma > 0 \). Let \( X \) be the \( n \times m \) \((n \geq m)\), random matrix

\[
X = \begin{bmatrix}
X_1' \\
\vdots \\
X_n'
\end{bmatrix}
\]

and observe that

\[
\mathbb{E}(X) = \begin{bmatrix}
\mu' \\
\vdots \\
\mu'
\end{bmatrix} = 1\mu', \quad \text{where } 1 = (1, \ldots, 1)' \in \mathbb{R}^n
\]

and \( \text{Cov}(\text{vec} X') = I_n \otimes \Sigma \), that is, \( X \) has a \( n \times m \) matrix variate normal distribution, denoted this fact as \( X \sim \mathcal{N}_{n \times m}(1\mu', I_n, \Sigma) \), see [Muirhead (1982, pp. 79-80)], among others.

Now, if \( Y = X'X \), where the \( n \times m \) random matrix \( X \) is \( \mathcal{N}_{n \times m}(0, I_n, \Sigma) \) then \( Y \) is said to have the Wishart distribution with \( n \) degrees of freedom and scale parameter \( \Sigma \).

Based on the theory of Jordan algebras, a family of distributions on symmetric cones, termed the Riesz distributions, were first introduced by [Hassairi and Lajmi (2001)] under the name of Riesz natural exponential family (Riesz NEF); they were based on a special case of the so-called Riesz measure from [Faraut and Korányi (1994, p.137)], going back to [Riesz (1949)]. This Riesz distribution generalises the matrix multivariate gamma and Wishart distributions, containing them as particular cases. Recently, [Díaz-García (2012)] proposes

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two versions of the Riesz distribution and a diverse range of their properties are studied for real normed division algebras.

The main purpose of this article is to introduce the distribution of the random matrix $X$ such that $Y = X^*X$ have one of the two versions of the Riesz distributions and and explores some of their basic properties for real normed division algebras. These distributions shall be termed Kotz-Riesz distributions. Moreover, as can be seen easily, the Kotz-Riesz distribution belong to the matrix multivariate elliptical-spherical distributions, see Fang and Zhang (pp. 102-013, 1990).

This article studies two versions of Kotz-Riesz distributions for real normed division algebras. Section 2 reviews some definitions and notation on real normed division algebras. And also, introduces other mathematical tools as two definitions for the generalised gamma function on symmetric cones. Several integration results for real normed division algebras are found in Section 3. Section 4 introduces Kotz-Riesz distributions for real normed division algebras, and also studies the relationship between the Riesz distributions and the Kotz-Riesz distributions.

2 Preliminary results

A detailed discussion of real normed division algebras may be found in Baez (2002) and Ebbinghaus et al. (1990). For your convenience, we shall introduce some notation, although in general we adhere to standard notation forms.

For our purposes: Let $F$ be a field. An algebra $A$ over $F$ is a pair $(A; m)$, where $A$ is a finite-dimensional vector space over $F$ and multiplication $m : A \times A \to A$ is an $F$-bilinear map; that is, for all $\lambda \in F$, $x, y, z \in A$,

$$m(x, \lambda y + z) = \lambda m(x; y) + m(x; z)$$

$$m(\lambda x + y; z) = \lambda m(x; z) + m(y; z).$$

Two algebras $(A; m)$ and $(E; n)$ over $F$ are said to be isomorphic if there is an invertible map $\phi : A \to E$ such that for all $x, y \in A$,

$$\phi(m(x, y)) = n(\phi(x), \phi(y)).$$

By simplicity, we write $m(x; y) = xy$ for all $x, y \in A$.

Let $A$ be an algebra over $F$. Then $A$ is said to be

1. alternative if $x(xy) = (xx)y$ and $x(yy) = (xy)y$ for all $x, y \in A$,
2. associative if $x(yz) = (xy)z$ for all $x, y, z \in A$,
3. commutative if $xy = yx$ for all $x, y \in A$, and
4. unital if there is a $1 \in A$ such that $x1 = x = 1x$ for all $x \in A$.

If $A$ is unital, then the identity 1 is uniquely determined.

An algebra $A$ over $F$ is said to be a division algebra if $A$ is nonzero and $xy = 0_A \Rightarrow x = 0_A$ or $y = 0_A$ for all $x, y \in A$.

The term “division algebra”, comes from the following proposition, which shows that, in such an algebra, left and right division can be unambiguously performed.

Let $A$ be an algebra over $F$. Then $A$ is a division algebra if, and only if, $A$ is nonzero and for all $a, b \in A$, with $b \neq 0_A$, the equations $bx = a$ and $yb = a$ have unique solutions $x, y \in A$. 

2
In the sequel we assume $F = \mathbb{R}$ and consider classes of division algebras over $\mathbb{R}$ or “real division algebras” for short.

We introduce the algebras of real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$, quaternions $\mathbb{H}$ and octonions $\mathbb{O}$. Then, if $\mathbb{A}$ is an alternative real division algebra, then $\mathbb{A}$ is isomorphic to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$.

Let $\mathbb{A}$ be a real division algebra with identity 1. Then $\mathbb{A}$ is said to be normed if there is an inner product $(\cdot, \cdot)$ on $\mathbb{A}$ such that

$$(x, y) = (x, x)(y, y)$$

for all $x, y \in \mathbb{A}$.

If $\mathbb{A}$ is a real normed division algebra, then $\mathbb{A}$ is isomorphic $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$.

There are exactly four normed division algebras: real numbers ($\mathbb{R}$), complex numbers ($\mathbb{C}$), quaternions ($\mathbb{H}$) and octonions ($\mathbb{O}$), see [Edelman and Rao (2005) and Kabe (1984)]. We take into account that, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ are the only normed division algebras; furthermore, they are the only alternative division algebras.

Let $A$ be a division algebra over the real numbers. Then $A$ has dimension either 1, 2, 4 or 8. Other branches of mathematics used the parameters $\alpha = 2/\beta$ and $t = \beta/4$, see [Edelman and Rao (2005) and Kabe (1984)], respectively.

Finally, observe that

- $\mathbb{R}$ is a real commutative associative normed division algebras,
- $\mathbb{C}$ is a commutative associative normed division algebras,
- $\mathbb{H}$ is an associative normed division algebras,
- $\mathbb{O}$ is an alternative normed division algebras.

Let $L_{m,n}^\beta$ be the set of all $n \times m$ matrices of rank $m \leq n$ over $\mathbb{A}$ with $m$ distinct positive singular values, where $\mathbb{A}$ denotes a real finite-dimensional normed division algebra. Let $\mathbb{A}^{m \times n}$ be the set of all $m \times n$ matrices over $\mathbb{A}$. The dimension of $\mathbb{A}^{m \times n}$ over $\mathbb{R}$ is $\beta mn$. Let $\mathbb{A} \in \mathbb{A}^{m \times n}$, then $\mathbb{A}^* = \mathbb{A}^T$ denotes the usual conjugate transpose.

Table 1 sets out the equivalence between the same concepts in the four normed division algebras.

| Real          | Complex       | Quaternion    | Octonion       |
|---------------|---------------|---------------|----------------|
| Semi-orthogonal | Semi-unitary   | Semi-symplectic| Semi-exceptional type |
| Orthogonal    | Unitary       | Symplectic    | Exceptional type |
| Symmetric     | Hermitian     | Quaternion    | Octonion       |
|               |               | hermitian     | hermitian      |

It is denoted by $\mathcal{S}_m^\beta$, the real vector space of all $S \in \mathbb{A}^{m \times m}$ such that $S = S^\ast$. In addition, let $\mathcal{P}_m^\beta$ be the cone of positive definite matrices $S \in \mathbb{A}^{m \times m}$. Thus, $\mathcal{P}_m^\beta$ consist of all matrices $S = X^\ast X$, with $X \in \mathcal{L}_{m,n}^\beta$, then $\mathcal{P}_m^\beta$ is an open subset of $\mathcal{S}_m^\beta$.

Let $\mathcal{D}_m^\beta$ be the diagonal subgroup of $\mathcal{L}_{m,m}^\beta$ consisting of all $D \in \mathbb{A}^{m \times m}$, $D = \text{diag}(d_1, \ldots, d_m)$. Let $\mathcal{U}_m^\beta(m)$ be the subgroup of all upper triangular matrices $T \in \mathbb{A}^{m \times m}$ such that $t_{ij} = 0$ for $1 < i < j \leq m$.

For any matrix $X \in \mathbb{A}^{m \times n}$, $dX$ denotes the matrix of differentials $(dx_{ij})$. Finally, we define the measure or volume element $(dX)$ when $X \in \mathbb{A}^{m \times n}$, $\mathcal{S}_m^\beta$, $\mathcal{D}_m^\beta$ or $\mathcal{P}_m^\beta$, see [Dumitriu (2002) and Díaz-García and Gutiérrez-Jáimez (2011)].

If $X \in \mathbb{A}^{m \times n}$ then $(dX)$ (the Lebesgue measure in $\mathbb{A}^{m \times n}$) denotes the exterior product of the $\beta mn$ functionally independent variables

$$(dX) = \bigwedge_{i=1}^m \bigwedge_{j=1}^n dx_{ij}$$

where $dx_{ij} = \bigwedge_{k=1}^\beta dx_{ij}^{(k)}$. 

3
If $S \in \mathbb{S}_m^\beta$ (or $S \in \mathbb{T}_m^\beta$) with $t_{ii} > 0$, $i = 1, \ldots, m$ then $(dS)$ (the Lebesgue measure in $\mathbb{S}_m^\beta$ or in $\mathbb{T}_m^\beta$) denotes the exterior product of the $m(m-1)\beta/2 + m$ functionally independent variables,

$$(dS) = \prod_{i=1}^m ds_{ii} \prod_{i<j}^m ds_{ij}^{(k)}.$$  

Observe, that for the Lebesgue measure $(dS)$ defined thus, it is required that $S \in \mathbb{S}_m^\beta$, that is, $S$ must be a non singular Hermitian matrix (Hermitian definite positive matrix).

If $A \in \mathbb{D}_m^\beta$ then $(dA)$ (the Lebesgue measure in $\mathbb{D}_m^\beta$) denotes the exterior product of the $\beta m$ functionally independent variables

$$(dA) = \prod_{i=1}^n \prod_{k=1}^\beta d\lambda_i^{(k)}.$$  

If $H_1 \in \mathbb{V}_{m,n}^\beta$ is such that $H_1 = (h_1, \ldots, h_m)$, where $h_j$, $j = 1, \ldots, m$ are their columns, then

$$(H_1^* dH_1) = \prod_{i=1}^m \prod_{j=i+1}^n h_i^* d h_j,$$  

where the partitioned matrix $H = (H_1 | H_2) = (h_1, \ldots, h_m | h_{m+1}, \ldots, h_n) \in \mathbb{V}^\beta(n)$, with $H_2 = (h_{m+1}, \ldots, h_n)$. It can be proved that this differential form does not depend on the choice of the $H_2$ matrix. When $n = 1$; $\mathbb{V}_{m,1}^\beta$ defines the unit sphere in $\mathbb{R}^m$. This is, of course, an $(m-1)\beta$- dimensional surface in $\mathbb{R}^m$.

The surface area or volume of the Stiefel manifold $\mathbb{V}_{m,n}^\beta$ is

$$\text{Vol}(\mathbb{V}_{m,n}^\beta) = \int_{H_1 \in \mathbb{V}_{m,n}^\beta} (H_1^* dH_1) = \frac{2^m \pi^{mn\beta/2}}{\Gamma_m^\beta [\beta/2]},$$  

where $\Gamma_m^\beta[a]$ denotes the multivariate Gamma function for the space $\mathbb{S}_m^\beta$. This can be obtained as a particular case of the generalised gamma function of weight $\kappa$ for the space $\mathbb{S}_m$ with $\kappa = (k_1, k_2, \ldots, k_m)$, $k_1 \geq k_2 \geq \cdots \geq k_m \geq 0$, taking $\kappa = (0,0,\ldots,0)$ and which for $\text{Re}(\alpha) \geq (m-1)\beta/2 - k_m$ is defined by, see Gross and Richards (1987),

$$\Gamma_m^\beta[a, \kappa] = \int_{A \in \mathbb{S}_m^\beta} \text{etr}(-A)|A|^{a-(m-1)\beta/2-1} q_\kappa(A) (dA)$$  

$$= \pi^{m(m-1)\beta/4} \prod_{i=1}^m \Gamma[a + k_i - (i-1)\beta/2]$$  

$$= [a]^\beta \Gamma_m^\beta[a],$$  

where $\text{etr}(\cdots) = \exp(\text{tr}(\cdots))$, $|\cdots|$ denotes the determinant, and for $A \in \mathbb{S}_m^\beta$

$$q_\kappa(A) = |A_m|^{k_m} \prod_{i=1}^{m-1} |A_i|^{k_i-k_{i+1}}$$  

with $A_p = (a_{rs})$, $r, s = 1, 2, \ldots, p$, $p = 1, 2, \ldots, m$ is termed the highest weight vector, see Gross and Richards (1987). Also,

$$\Gamma_m^\beta[a] = \int_{A \in \mathbb{S}_m^\beta} \text{etr}(-A)|A|^{a-(m-1)\beta/2-1} (dA)$$  

$$= \pi^{m(m-1)\beta/4} \prod_{i=1}^m \Gamma[a - (i-1)\beta/2],$$
and $\text{Re}(a) > (m - 1)\beta/2$.

In other branches of mathematics the highest weight vector $q_\kappa(A)$ is also termed the generalised power of $A$ and is denoted as $\Delta_\kappa(A)$, see Faraut and Korányi (1994) and Hassairi and Lajmi (2001).

Additional properties of $q_\kappa(A)$, which are immediate consequences of the definition of $q_\kappa(A)$ and the following property 1, are:

1. if $\lambda_1, \ldots, \lambda_m$, are the eigenvalues of $A$,
   
   $q_\kappa(A) = \prod_{i=1}^{m} \lambda_i^{k_i}$.

2. $q_\kappa(A^{-1}) = q_{-\kappa}(A) = q_{-\kappa}(A)$,

3. if $\kappa = (p, \ldots, p)$, then
   
   $q_\kappa(A) = |A|^p$,

   in particular if $p = 0$, then $q_\kappa(A) = 1$.

4. if $\tau = (t_1, t_2, \ldots, t_m)$, $t_1 \geq t_2 \geq \cdots \geq t_m \geq 0$, then
   
   $q_{\kappa+\tau}(A) = q_\kappa(A) q_\tau(A)$,

   in particular if $\tau = (p, p, \ldots, p)$, then
   
   $q_{\kappa+\tau}(A) \equiv q_{\kappa+p}(A) = |A|^p q_\kappa(A)$.

5. Finally, for $B \in \mathfrak{S}_m$ in such a manner that $C = B^*B \in \mathfrak{S}_m$,
   
   $q_\kappa(BAB^*) = q_\kappa(C) q_\kappa(A)$

   and
   
   $q_\kappa(B^{-1}AB^{*-1}) = (q_\kappa(C))^{-1} q_\kappa(A)$.

**Remark 2.1.** Let $\mathcal{P}(\mathfrak{S}_m)$ denote the algebra of all polynomial functions on $\mathfrak{S}_m$, and $\mathcal{P}_k(\mathfrak{S}_m)$ the subspace of homogeneous polynomials of degree $k$ and let $\mathcal{P}^\kappa(\mathfrak{S}_m)$ be an irreducible subspace of $\mathcal{P}(\mathfrak{S}_m)$ such that

$$\mathcal{P}_k(\mathfrak{S}_m) = \sum_\kappa \bigoplus \mathcal{P}^\kappa(\mathfrak{S}_m).$$

Note that $q_\kappa$ is a homogeneous polynomial of degree $k$, moreover $q_\kappa \in \mathcal{P}^\kappa(\mathfrak{S}_m)$, see Gross and Richards (1987).

In (4), $[a]_\kappa^{\beta}$ denotes the generalised Pochhammer symbol of weight $\kappa$, defined as

$$[a]_\kappa^{\beta} = \prod_{i=1}^{m} (a - (i - 1)\beta/2)_{k_i}$$

$$= \prod_{i=1}^{m} \frac{\Gamma_{m}[a + k_i - (i - 1)\beta/2]}{\Gamma_{m}[a]}$$

$$= \frac{\Gamma_{m}[a, \kappa]}{\Gamma_{m}[a]}.$$
where \( \text{Re}(a) > (m - 1)\beta/2 - k_m \) and

\[
(a)_i = a(a+1)\cdots(a+i-1),
\]

is the standard Pochhammer symbol.

An alternative definition of the generalised gamma function of weight \( \kappa \) is proposed by Khatri (1966), which is defined as

\[
\Gamma_{\beta \, m}^{a, -\kappa} = \int_{A \in \mathfrak{g}^\beta_m} 	ext{etr}\{-A\} |A|^{a-(m-1)\beta/2-1} q_\kappa(A^{-1})(dA) \tag{13}
\]

\[
= \pi^{m(m-1)\beta/4} \prod_{i=1}^m \Gamma[a - k_i - (m-i)\beta/2] \tag{14}
\]

where \( \text{Re}(a) > (m - 1)\beta/2 + k_1 \).

In addition, observe that by (1) and (2)

\[
(dH_1) = \frac{1}{\text{Vol}(\mathcal{V}^\beta_{m,n})} (H_1^*dH_1) = \frac{\Gamma_{\beta \, m}^{[n\beta/2]}}{2^m \pi^m m^{n\beta/2}} (H_1^*dH_1). \tag{15}
\]

is the normalised invariant measure on \( \mathcal{V}^\beta_{m,n} \) and \((dH)\), i.e., with \((m = n)\), it defines the normalised Haar measure on \( \Omega^\beta(m) \).

Finally, the \( p^F_q \) is the generalised hypergeometric function with matrix argument, defined as

\[
p^F_q(a_1, \ldots, a_p; b_1, \ldots, b_q; X) = \sum_{t=0}^{\infty} \sum_{\tau} \frac{[a_1]_\tau \cdots [a_p]_\tau C^\beta_\kappa(X)}{[b_1]_\tau \cdots [b_q]_\tau t!}
\]

where \( \sum_\tau \) denotes summation over all partition \( \tau = (t_1, t_2, \ldots, t_m) \), \( t_1 \geq t_2 \geq \cdots \geq t_m \geq 0 \), of \( t \), \( \sum_{i=1}^m t_i = t \) and \( t_1, t_2, \ldots, t_m \) are nonnegative integers, and \( C^\beta_\kappa(X) \) is the Jack polynomial of \( X \in \mathfrak{g}^\beta_m \) corresponding to \( \tau \), see Gross and Richards (1987) and Díaz-García (2009).

Jack polynomials for real normed division algebras are also termed spherical functions of symmetric cones in the abstract algebra context, see Sawyn (1997). In addition, in the statistical literature, they are termed real, complex, quaternion and octonion zonal polynomials, or, generically, general zonal polynomials, see James (1964), Muirhead (Chapter 7 in (1982)), Kabe (1984), and Li and Xue (2009). This section is completed, remembering the following result: from Gross and Richards (1987, Equation 4.8(2) and Definition 5.3) is obtained

\[
C^\beta_\kappa(X) = C^\beta_\kappa(I) \int_{H \in \Omega^\beta(m)} q_\kappa(H^*XH)(dH) \tag{15}
\]

for all \( X \in \mathfrak{g}^\beta_m \); where \((dH)\) is the normalised Haar measure on \( \Omega^\beta(m) \), also see Díaz-García (2009).

### 3 Integration

First consider the following result proposed by Zhang and Fang (1990) for real case and extended to real normed division algebras by Díaz-García (2009).
Lemma 3.1. The characteristic function of $H_1 \in \mathcal{V}_{m,n}^\beta$, the normalised invariant measure on $\mathcal{V}_{m,n}^\beta$ is

$$
\phi_{H_1}(T) = \int_{H_1 \in \mathcal{V}_{m,n}^\beta} \det\{iH_1T^T\}(dH_1)
= a P_2^\beta(\beta n/2, -TT^T/4)
= \sum_{\tau=0}^\infty \frac{1}{\beta n/2} C_\tau^\beta(-TT^T/4)/\tau!,
$$

(16)

where, $\sum_{\tau}$, denotes summation over all partition $\tau = (t_1, t_2, \ldots, t_m)$, $t_1 \geq t_2 \geq \cdots t_m \geq 0$, of $t$, $\sum_{i=1}^m t_i = t$ and $t_1, t_2, \ldots, t_m$ are nonnegative integers.

The original version of the following result was obtained for real case in [Xu and Fang (1990)]. Next, we set the version of this result for real normed division algebras.

First consider the following concept: let’s use the complexification $\mathfrak{S}_m^\beta = \mathfrak{S}_m^\beta + i\mathfrak{S}_m^\beta$ of $\mathfrak{S}_m^\beta$. That is, $\mathfrak{S}_m^\beta$ consist of all matrices $Z \in (\mathfrak{S}^\beta)^{m \times m}$ of the form $Z = X + jY$, with $X, Y \in \mathfrak{S}_m^\beta$. It comes to $X = \text{Re}(Z)$ and $Y = \text{Im}(Z)$ as the real and imaginary parts of $Z$, respectively. The generalised right half plane $\Phi_m^\beta = \mathfrak{S}_m^\beta + i\mathfrak{S}_m^\beta$ consists of all $Z \in \mathfrak{S}_m^\beta$ such that $\text{Re}(Z) \in \mathfrak{P}_m^\beta$, see [Gross and Richards (1987, p. 801)].

Theorem 3.1. Let $Z \in \Phi_m^\beta$ and $U \in \mathfrak{S}_m^\beta$. Then

$$
\int_{X \in \mathfrak{P}_m^\beta} f(Z^{1/2}XZ^{1/2})|X|^{a-(m-1)\beta/2-1}C_\tau^\beta(XU)(dX)
= \frac{J(I_m)}{C_\tau^\beta(I_m)} |Z|^{-a} C_\tau^\beta(UZ^{-1}),
$$

(17)

where $Z^{1/2}$ is the positive definite square root of $Z$, i.e. $Z^{1/2}Z^{1/2} = Z$, $\text{Re}(a) > (m-1)\beta/2 - t_m$, $\tau = (t_1, t_2, \ldots, t_m)$, $t_1 \geq t_2 \geq \cdots t_m \geq 0$, $\sum_{i=1}^m t_i = t$ and $t_1, t_2, \ldots, t_m$ are nonnegative integers, and

$$
J(I_m) = \int_{X \in \mathfrak{P}_m^\beta} f(X)|X|^{a-(m-1)\beta/2-1}C_\tau^\beta(X)(dX).
$$

Proof. The proof is a verbatim copy of given by [Xu and Fang (1990)], only take notice that

1. If $Y = H^*XH$ for $H \in \mathfrak{U}_m^\beta$, then $(dY) = (dX)$, and

2. if $B = Z^{1/2}XZ^{1/2}$ with $B, X \in \mathfrak{P}_m^\beta$, then by [Díaz-García and Gutiérrez-Jáimez (2013), Proposition 2], $(dB) = |Z|^{(m-1)\beta/2+1}(dX)$, where $Z^{1/2}$ is the positive definite square root of $Z$, such that $Z^{1/2}Z^{1/2} = Z$.

The following result shows the extension of the Wishart’s integral for real normed division algebras stated in [Díaz-García (2013)].

Theorem 3.2. Let $Y \in \mathfrak{L}_{m,n}^\beta$.

$$
\int_{Y^*Y = R} f(Y^*Y) (dY) = \frac{\pi^{\beta mn/2}}{\Gamma_m^{\beta}[\beta n/2]} |R|^{\beta(m-n+1)/2-1} f(R),
$$

where $\text{Re}(\beta n/2) > (m-1)\beta/2$.
Finally, the Theorem 1 in [1993] is generalised for real normed division algebras. But prior consider the following definition, see Díaz-García and Gutiérrez-Jáimez (2013).

**Definition 3.1.** Let \( X \in \mathbb{S}^\beta_{n,m} \) be a random matrix, then if \( X \overset{d}{=} \Xi X \) for every \( \Xi \in \mathfrak{U}^\beta(n) \), \( X \) is termed left-spherical. Where \( \overset{d}{=} \) signifies that the two sides have the same distribution.

In addition, note that if \( X \) has density, respect to the Lebesgue measure, it has the form \( f(\beta X^*X) \) and \( X \) can be factorised as

\[
X = H_1 A
\]

(18)

where \( A_{m \times m} \) is not unique, and it is independent of \( H_1 \) and \( H_1 \) has a normalised invariant measure on \( \mathbb{V}^\beta_{m,n} \).

**Theorem 3.3.** Suppose that \( X \) is as in Definition 3.1. Then the characteristic function of \( X \) can be expressed as

\[
\phi_X(T) = \sum_{\tau} \sum_{t \geq 0} \frac{C^\beta_\tau(-TT^*/4)}{[\beta n/2]^\tau C^\beta_\tau(I_m)!} E(C^\beta_\tau(R))
\]

where \( R = X^*X \), \( \Re(\beta n/2) > (m-1)\beta/2 - t_m, \tau = (t_1, t_2, \ldots, t_m), t_1 \geq t_2 \geq \cdots t_m \geq 0, \sum_{i=1}^m t_i = t \) and \( t_1, t_2, \ldots, t_m \) are nonnegative integers.

**Proof.** By (18) and Lemma 3.1

\[
\phi_X(T) = E_{\mathbb{X}(\text{etr}(iXT^*))} = E_{H_1A}(\text{etr}(iH_1AT^*))
\]

because \( H_1 \) and \( A \) are independent,

\[
= E_A \left( \sum_{\tau} \sum_{t \geq 0} \frac{C^\beta_\tau(-TT^*/4)}{[\beta n/2]^\tau C^\beta_\tau(I_m)!} E(C^\beta_\tau(R)) \right)
\]

\[
\overset{3.2}{=} \sum_{i=0}^{\infty} \left( \frac{\pi^{\beta mn/2}}{[\beta n/2]^{3\tau} C^\beta_\tau(I_m)!} \int_{R \in \mathfrak{U}^\beta_m} |R|^\beta(n-m+1)/2-1 f(R) C^\beta_\tau(-TT^*/4)(dR) \right)
\]

hence, by Lemma 3.1

\[
= \sum_{i=0}^{\infty} \left( \frac{\pi^{\beta mn/2}}{[\beta n/2]^{3\tau} C^\beta_\tau(I_m)!} \int_{R \in \mathfrak{U}^\beta_m} \frac{\pi^{\beta mn/2}}{[\beta n/2]^{3\tau} C^\beta_\tau(I_m)!} |R|^\beta(n-m+1)/2-1 f(R) C^\beta_\tau(R)(dR) \right)
\]

\[
= \sum_{i=0}^{\infty} \frac{C^\beta_\tau(-TT^*/4)}{[\beta n/2]^\tau C^\beta_\tau(I_m)!} E_R(C^\beta_\tau(R)).
\]

\( \square \)
4 Kotz-Riesz distributions

This section introduce two versions of the Kotz-Riesz distributions. With this purpose in mind, first are defined the spherical Kotz-Riesz distributions.

**Definition 4.1.** Let $\kappa = (k_1, k_2, \ldots, k_m)$, $k_1 \geq k_2 \geq \cdots \geq k_m \geq 0$, $k_1, k_2, \ldots, k_m$ are nonnegative integers.

1. Then it is said that $Z$ has a spherical Kotz-Riesz distribution of type I if its density function is

$$
\frac{\beta \Gamma_m[n\beta/2]}{\pi^{mn\beta/2} \Gamma_m[n\beta/2, \kappa]} \text{etr} \left\{-\beta \text{tr} Z^*Z\right\} q_{\kappa}[\beta Z^*Z] (dZ)
$$

(19)

for $Z \in \Sigma_{n,m}^{\beta}$, $\text{Re}(n\beta/2) > (m - 1)\beta/2 - k_m$; denoting this fact as

$$
Z \sim \mathcal{K}^{\beta,I}_{n \times m}(\kappa, 0, I_n, I_m).
$$

2. Then it is said that $Z$ has a Kotz-Riesz distribution of type II if its density function is

$$
\frac{\beta \Gamma_m[n\beta/2]}{\pi^{mn\beta/2} \Gamma_m[n\beta/2, \kappa]} \text{etr} \left\{-\beta \text{tr} Z^*Z\right\} q_{\kappa}[(\beta Z^*Z)^{-1}] (dZ)
$$

(20)

for $Z \in \Sigma_{n,m}^{\beta}$, $\text{Re}(n\beta/2) > (m - 1)\beta/2 + k_1$; denoting this fact as

$$
Z \sim \mathcal{K}^{\beta,II}_{n \times m}(\kappa, 0, I_n, I_m).
$$

Now the elliptical Kotz-Riesz distribution or simply termed Kotz-Riesz distribution is obtained.

**Theorem 4.1.** Let $\Sigma \in \Phi_n^{\beta}$, $\Theta \in \Phi_n^{\beta}$, $\mu \in \Sigma_{n,m}$ and $\kappa = (k_1, k_2, \ldots, k_m)$, $k_1 \geq k_2 \geq \cdots \geq k_m \geq 0$, $k_1, k_2, \ldots, k_m$ are nonnegative integers. And let $Z \sim \mathcal{K}^{\beta,I}_{n \times m}(\kappa, 0, I_n, I_m)$. Also, defines $X = \mu + \Theta^{1/2}Z\Sigma^{1/2}$, where $\Theta^{1/2}$ is the positive definite square root of $\Theta$, such that $\Theta^{1/2}B^{1/2} = B$.

1. Then it is said that $X$ has a Kotz-Riesz distribution of type I and its density function is

$$
\frac{\beta \Gamma_m[n\beta/2 + \sum_{i=1}^m k_i \Gamma_m[n\beta/2, \kappa][\Sigma]^{n\beta/2}]}{\pi^{mn\beta/2} \Gamma_m[n\beta/2, \kappa][\Sigma]^{n\beta/2}} \times \text{etr} \left\{-\beta \text{tr} [\Sigma^{-1}(X - \mu)^*\Theta^{-1}(X - \mu)]\right\}
$$

$$
\times q_{\kappa}[(\Sigma^{-1/2}(X - \mu)^*\Theta^{-1/2}(X - \mu)^{-1/2})] (dX)
$$

(21)

for $X \in \Sigma_{n,m}$, $\text{Re}(n\beta/2) > (m - 1)\beta/2 - k_m$; denoting this fact as

$$
X \sim \mathcal{K}^{\beta,I}_{n \times m}(\kappa, \mu, \Theta, \Sigma).
$$

2. Then it is said that $X$ has a Kotz-Riesz distribution of type II and its density function is

$$
\frac{\beta \Gamma_m[n\beta/2 - \sum_{i=1}^m k_i \Gamma_m[n\beta/2, -\kappa][\Sigma]^{n\beta/2}]}{\pi^{mn\beta/2} \Gamma_m[n\beta/2, -\kappa][\Sigma]^{n\beta/2}} \times \text{etr} \left\{-\beta \text{tr} [\Sigma^{-1}(X - \mu)^*\Theta^{-1}(X - \mu)]\right\}
$$

(22)
Proof. This is an immediate consequence of Theorem 3 in Díaz-García and Gutiérrez-Jáimez (2013) and the fact is that, for a scalar

\[ q_\kappa(aA) = q_\kappa(aI_m)q_\kappa(A) = a^{\sum_{i=1}^k k_i} q_\kappa(A). \]

\[ \square \]

Observe that, if \( \kappa = (0,0,\ldots,0) \) and \( \Sigma = 2\Sigma \) in two densities in Theorem 4.1 the matrix multivariate normal distribution for real normed division algebras is obtained, see Díaz-García and Gutiérrez-Jáimez (2011). Also, when \( \kappa = (l,l,\ldots,l) \), \( l = a-(m-1)\beta/2-1 \) and \( \Sigma = \Sigma/r \), \( r > 1 \), the original Kotz type distribution are obtained i.e. for \( s = 1 \), in notation of Diaconis (1993).

Next, the main consequence of this article is is achieved by finding the two versions of the Riesz distributions in terms of Kotz-Riesz distributions.

**Theorem 4.2.** 1. Assume that \( X \sim \mathcal{KN}_{n \times m}^{\beta} (\kappa, 0, \Theta, \Sigma) \), and define \( Y = X^* \Theta^{-1} X \). Then its density function is

\[ \frac{\beta^{am + \sum_{i=1}^k k_i}}{\Gamma_m^\beta[a, \kappa]|\Sigma|^a q_\kappa(\Sigma)} \text{etr}\{-\beta \Sigma^{-1} Y\}|Y|^{a-(m-1)\beta/2-1}q_\kappa(Y) (dY) \]  

for \( Y \in \mathcal{R}_{m}^{\beta} \) and \( \text{Re}(a) > (m-1)\beta/2-k_m \); denoting this fact as \( Y \sim \mathcal{KN}_{m}^{\beta,II} (a, \kappa, \Sigma) \).

2. Suppose that \( X \sim \mathcal{KN}_{n \times m}^{\beta,II} (\kappa, 0, \Theta, \Sigma) \), and define \( Y = X^* X \). Then its density function is

\[ \frac{\beta^{am - \sum_{i=1}^k k_i}}{\Gamma_m^\beta[a, -\kappa]|\Sigma|^a} \text{etr}\{-\beta \Sigma^{-1} Y\}|Y|^{a-(m-1)\beta/2-1}q_\kappa(Y^{-1}) (dY) \]  

for \( Y \in \mathcal{R}_{m}^{\beta} \) and \( \text{Re}(a) > (m-1)\beta/2+k_1 \); denoting this fact as \( Y \sim \mathcal{KN}_{m}^{\beta,II} (a, \kappa, \Sigma) \).

Where \( \kappa = (k_1, k_2, \ldots, k_m) \), \( k_1 \geq k_2 \geq \cdots \geq k_m \geq 0 \), \( k_1, k_2, \ldots, k_m \) are nonnegative integers.

**Proof.** 1. By applying (11), the density of \( X \) is

\[ \frac{\beta^{mn\beta/2 + \sum_{i=1}^m k_i \Gamma_m^\beta[n\beta/2]}}{\pi^{mn\beta/2} \Gamma_m^\beta[n\beta/2, \kappa]|\Sigma|^{n\beta/2} q_\kappa(\Sigma) \text{etr}\{ -\beta \text{tr} \Sigma^{-1} X^* \Theta^{-1} X \} q_\kappa(X^* \Theta^{-1} X) (dX). \]

Now, let \( \Theta^{1/2} \) the positive definite square root of \( \Theta \) and define \( V = \Theta^{-1/2} X \). Hence, \( V \sim \mathcal{KN}_{n \times m}^{\beta} (\kappa, 0, I_m, \Sigma) \); moreover, its density is

\[ \frac{\beta^{mn\beta/2 + \sum_{i=1}^m k_i \Gamma_m^\beta[n\beta/2]}}{\pi^{mn\beta/2} \Gamma_m^\beta[n\beta/2, \kappa]|\Sigma|^{n\beta/2} q_\kappa(\Sigma) \text{etr}\{ -\beta \text{tr} \Sigma^{-1} V^* V \} q_\kappa(V^* V) (dV). \]

Finally, defining \( Y = V^* V \) the desired result is obtained by applying the Theorem 4.2.

\[ \text{and denoting } a = n\beta/2. \]
2. It is obtained in analogous way to 1.

Note that (23) and (24) are the density functions of Riesz distributions type I and II, respectively; which were obtained on a special case of the Riesz measure by Faraut and Korányi (1994), Hassairi and Lajmi (2001) and Díaz-García (2012).

Below, are found the characteristic functions of the Kotz-Riesz distribution type I. But before, observe that in order to apply the Theorem 3.3 it is necessary to find the expected value of $C^\beta_r(AY)$ where $Y = X \cdot X$ has a Riesz distribution.

**Lemma 4.1.** Assume that $Z \sim KN^{\beta,I}_{n \times m}(\kappa, 0, I_n, I_m)$, then

$$E(C^\beta_r(AY)) = \frac{\beta^{mn\beta/2 + \sum_{i=1}^{m} k_i} \Gamma_{m}^{\beta} [n \beta/2, \tau + \kappa]}{\Gamma_{m}^{\beta} [n \beta/2, \kappa]} C^\beta_r(A)$$

where $Y \overset{d}{=} Z \cdot Z$; $\tau = (t_1, \ldots, t_m)$, $t_1 \geq t_2 \geq \cdots \geq t_m \geq 0$, $t_1, t_2, \ldots, t_m$ are nonnegative integers; and $\kappa = (k_1, k_2, \ldots, k_m)$, $k_1 \geq k_2 \geq \cdots \geq k_m \geq 0$, $k_1, k_2, \ldots, k_m$ are nonnegative integers.

**Proof.** By (23) it is had that $Y = Z \cdot Z \sim KN^{\beta,I}_{n \times m}(\beta n/2, I_m)$. Then $E(C^\beta_r(AY))$ is

$$E(C^\beta_r(AY)) = \frac{\beta^{mn\beta/2 + \sum_{i=1}^{m} k_i} \Gamma_{m}^{\beta} [n \beta/2, \tau + \kappa]}{\Gamma_{m}^{\beta} [n \beta/2, \kappa]} C^\beta_r(A),$$

where by (15)

$$J(I_m) = \int_{Y \in \mathbb{P}^\beta} \text{etr}(-\beta Y) |Y|^{(n-m-1)\beta/2-1} q_\kappa(Y) C^\beta_r(Y)(dY)$$

$$= \int_{Y \in \mathbb{P}^\beta} \text{etr}(-\beta Y) |Y|^{(n-m-1)\beta/2-1} q_\kappa(Y) \times C^\beta_r(I_m) \int_{H \in U^\beta(m)} q_r(H \cdot Y H)(dH)(dY),$$

thus, by applying (11), $q_r(H \cdot Y H) = q_r(H \cdot H) q_r(Y) = q_r(I_m) q_r(Y) = q_r(Y)$ and $\int_{H \in U^\beta(m)}(dH) = 1$. Then by applying (9)

$$J(I_m) = C^\beta_r(I_m) \int_{Y \in \mathbb{P}^\beta} \text{etr}(-\beta Y) |Y|^{(n-m-1)\beta/2-1} q_\kappa(Y) q_r(Y)(dY)$$

$$= C^\beta_r(I_m) \int_{Y \in \mathbb{P}^\beta} \text{etr}(-\beta Y) |Y|^{(n-m-1)\beta/2-1} q_{\kappa + \tau}(Y)(dY)$$

$$= C^\beta_r(I_m) \Gamma_{m}^{\beta} [\beta n/2, \kappa + \tau].$$

From where the desired result is achieved. 

The following result gives the function the characteristic function of the spherical Kotz-Riesz distribution type I.
Theorem 4.3. Assume that \( Z \sim \mathcal{K}_{n \times m}^{\beta,n}(\kappa, 0, I_n, I_m) \). Then the characteristic function of \( Z \) can be expressed as

\[
\phi_Z(T) = \frac{\beta^{mn/2 + \sum_{i=1}^m k_i}}{\Gamma_m[n\beta/2, \kappa]} \sum_{i=0}^\infty \sum_{\tau} \frac{\Gamma_m[n\beta/2, \tau + \kappa]}{[\beta n/2]_\tau^\kappa} C_\tau(-TT^*/4),
\]

where, \( \text{Re}(\beta n/2) > (m - 1)\beta/2 - t_m, \tau = (t_1, t_2, \ldots, t_m), t_1 \geq t_2 \geq \cdots \geq t_m \geq 0, \sum_{i=1}^m t_i = t \) and \( t_1, t_2, \ldots, t_m \) are nonnegative integers; and \( \kappa = (k_1, k_2, \ldots, k_m) \), \( k_1 \geq k_2 \geq \cdots \geq k_m \geq 0, k_1, k_2, \ldots, k_m \) are nonnegative integers.

Proof. This is immediately from Theorem 3.3 and Lemma 4.1.

Finally, the characteristic function of the elliptical Kotz-Riesz distribution type I is given.

Corollary 4.1. Assume that \( X \sim \mathcal{K}_{n \times m}^{\beta,n}(\kappa, \mu, \Theta, \Sigma) \). Then the characteristic function of \( X \) can be expressed as

\[
\phi_X(T) = \frac{\beta^{mn/2 + \sum_{i=1}^m k_i}}{\Gamma_m[n\beta/2, \kappa]} \text{etr} \left\{i\mu T^*\right\} \sum_{i=0}^\infty \sum_{\tau} \frac{\Gamma_m[n\beta/2, \tau + \kappa]}{[\beta n/2]_\tau^\kappa} C_\tau(-T\Omega^*T^*/4),
\]

where, \( \text{Re}(\beta n/2) > (m - 1)\beta/2 - t_m, \tau = (t_1, t_2, \ldots, t_m), t_1 \geq t_2 \geq \cdots \geq t_m \geq 0, \sum_{i=1}^m t_i = t \) and \( t_1, t_2, \ldots, t_m \) are nonnegative integers; and \( \kappa = (k_1, k_2, \ldots, k_m) \), \( k_1 \geq k_2 \geq \cdots \geq k_m \geq 0, k_1, k_2, \ldots, k_m \) are nonnegative integers.

Proof. Let \( \Sigma \in \Phi^\beta_m, \Theta \in \Phi^\beta_m, \mu \in S^\beta_m \), and let \( Z \sim \mathcal{K}_{n \times m}^{\beta,n}(\kappa, 0, I_n, I_m) \); and defines \( X = \mu + \Theta^{1/2}Z\Sigma^{1/2} \), where \( B^{1/2} \) is the positive definite square root of \( B \), such that \( B^{1/2}B^{1/2} = B \). Then \( X \sim \mathcal{K}_{n \times m}^{\beta,n}(\kappa, \mu, \Theta, \Sigma) \). Therefore

\[
\phi_X(T) = \mathbb{E}(\text{etr}\{iXT^*\}) = \mathbb{E}(\text{etr}\{i(\mu + \Theta^{1/2}Z\Sigma^{1/2})T^*\})
\]

\[
= \text{etr}\{i\mu T^*\} \mathbb{E}(\text{etr}\{i\Theta^{1/2}Z\Sigma^{1/2}T^*\})
\]

\[
= \text{etr}\{i\mu T^*\} \mathbb{E}(\text{etr}\{iZ(\Theta^{1/2}T\Sigma^{1/2})^*\})
\]

\[
= \text{etr}\{i\mu T^*\} \phi_Z(\Theta^{1/2}T\Sigma^{1/2}).
\]

The conclusion is follows as consequence immediately of Theorem 4.3.

Conclusions

There is no doubt of the importance of the Riesz distribution from a theoretical point of view and, by establishing a generalisation of the Wishart distribution in imminent great importance from a practical point of view. For some time, statisticians have been trying to extend the theory of sampling in multivariate analysis for samples of a non-normal population. The introduction of Kotz-Riesz distribution and the distribution of Riesz are an advance in that direction, furthermore, now these results, combined with those obtained in [Diaz-Garcia, 2012; Diaz-Garcia, 2013] allow generalise diverse techniques in multivariate analysis, assuming a Riesz distribution instead an Wishart distribution or equivalently, assuming a Kotz-Riesz distribution instead a matrix multivariate normal distribution. Finally emphasises that Kotz-Riesz distribution belongs to the family of left-elliptical distributions, then all the results obtained for the latter family, as moments, estimation, hypothesis testing, etc., can be quite easily particularised for Kotz-Riesz distribution.
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