Abstract. For the sake of hyperkähler SYZ conjecture, finding holomorphic Lagrangian fibrations becomes an important issue. Toric hyperkähler manifolds are real dimension 4\(n\) non-compact hyperkähler manifolds which are quaternion analog of toric varieties. The \(n\) dimensional residue circle action on it admitting a hyperkähler moment map. We use the complex part of this moment map to construct a holomorphic Lagrangian fibration with generic fiber diffeomorphic to \((\mathbb{C}^*)^n\), and study the singular fibers.

1. Introduction

In\cite{SY96}, Strominger, Yau and Zaslow conjectured that Mirror Symmetry of Calabi-Yau manifolds comes from real Lagrangian fibrations. Let \(Y\) be a compact, Kähler, holomorphic symplectic manifold. By Calabi-Yau theorem, such a manifold admits a hyperkähler metric(see \cite{Bes87}, or \cite{Huy99}). A complex Lagrangian subvariety of \((Y, I_1)\) is special Lagrangian with respect to \(I_2\), which is clear from the linear algebra. The hyperkähler SYZ conjecture asserts the existence of holomorphic Lagrangian fibrations on the compact hyperkähler manifolds. Although the original version is concerning the compact hyperkähler manifold, finding holomorphic Lagrangian fibration in non-compact hyperkähler manifold is also an interesting problem. For example, Hitchin has constructed holomorphic Lagrangian fibration in the moduli space of rank-2 stable Higgs bundles of odd degree with fixed determinant over a Riemann surface(cf. \cite{Hit87}).

Another important fact must be mentioned is that, in the quest of examples of special Lagrangian fibration, the physicists set up Mirror Symmetry in toric Calabi-Yau manifold(cf. \cite{AKMV05}, \cite{Mar10}), and similar topics were also studied by Batyrev(cf. \cite{Bat94}, \cite{Bat98}). Thus
it is also natural to consider the fibration in some hyperkähler manifold with large symmetry group.

Toric hyperkähler manifolds (some authors call them hypertoric varieties ([Pro08])) are another important class of non-compact hyperkähler manifolds, which are quaternion analogues of toric varieties. They can be obtained as symplectic quotients of level sets of the holomorphic moment maps, and themselves admit residue hyperkähler moment maps. Using symplectic quotient technique, in [BD00], Bielawski and Dancer studied their moment maps, cores, cohomologies, etc. While Hausel and Sturmfels study the toric hyperkähler varieties from a more algebraic viewpoint ([HS02]). Then Konno study them as GIT quotients in [Kon03] and [Kon08].

We first use the complex residue moment map to find holomorphic Lagrangian fibration in toric hyperkähler manifold, then study the type of generic and singular fibers. Namely, let $Y(\alpha, \beta)$ be a toric hyperkähler manifold, then

**Theorem.** The map $\bar{\mu}_C : Y(\alpha, \beta) \to n_C^* \cong \mathbb{C}^n$ defines a holomorphic Lagrangian fibration, i.e. for any $b \in n_C^*$, $F_b = \bar{\mu}_C^{-1}(b)$ is a complex Lagrangian subvariety.

To study the detail of the generic fiber and singular fiber, for simplicity, we let $\beta = 0$. Define a wall structure $\{W_i\}_{i=1}^d$ on the dual Lie algebra $n_C^*$ (see the precise definition in section 4), there follows

**Theorem.** The generic fiber $F_b$ of $Y(\alpha, \beta)$ over $n_C^* \setminus \bigcup_{i=1}^d W_i$ is diffeomorphic to complex torus $(\mathbb{C}^*)^n \cong T^n \times \mathbb{R}^n$.

In the case of $\beta = 0$, it is easy to check that the most singular central fiber $F_0 = \bar{\mu}_C^{-1}(0)$ is the extended core of the toric hyperkähler manifold $Y(\alpha, 0)$, constituted by toric varieties intersecting together (cf. [Pro08]). In general, the singular fiber on the discriminant locus is a little complicated, which can be described by the shrinking torus procedure.

**Theorem.** Consider the singular fiber $F_b$ of $Y(\alpha, \beta)$. When $b$ lies in the generic position of $W_i$, then $F_b$ diffeomorphic to shrinking the real torus $T^1$ generated by $u_i$ in the complex torus $(\mathbb{C}^*)^n \cong \mathbb{R}^n \times T^n$ over the real hyperplane $H_i \subset n^* \cong \mathbb{R}^n$. When $b$ lies in the intersection of $s$ walls $\{W_i\}_{i=1}^s$, then the singular fiber is given by shrinking $T^1$ due to $u_i$ over $H_i$ respectively, and at the intersection of $H_i$, $i = 1, \ldots, q$, shrinking a torus of real dimension $\dim(\{u_i\}_{i=1}^q)$ generated by $\{u_i\}_{i=1}^q$.

The structure of the article is as follows. In section 2, we introduce some facts of Calabi-Yau and hyperkähler geometry, the special
Lagrangian and holomorphic Lagrangian fibration, and background of Mirror Symmetry.

We present the basic properties of toric hyperkähler manifold in section 3. Mainly focus on the symplectic quotient and the GIT quotient construction. It has close relation with toric variety, namely the extended core of toric hyperkähler manifold are all constituted by toric varieties and compact toric varieties respectively, and the cotangent bundle of toric variety in the extended core is a dense open set of toric hyperkähler manifold.

The essential part of this paper is section 4, where we show that the complex moment map \( \bar{\mu}_C \) defines a holomorphic Lagrangian fibration. Then we study the type of generic fiber and singular fiber in great detail.

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2. CALABI-YAU AND HYPERKÄHLER GEOMETRY

A Calabi-Yau manifold is a Kähler manifold of complex dimension \( n \) with a covariant constant holomorphic \((n, 0)\)-form \( \Omega \) called the holomorphic volume form, which satisfies

\[
\frac{\omega^n}{n!} = (-1)^{n(n-1)/2}(\sqrt{-1}/2)^n\Omega \wedge \overline{\Omega},
\]

where \( \omega \) is the Kähler form. It is a Riemannian manifold with holonomy contained in \( \text{SU}(n) \). A (real) Lagrangian subvariety \( S \) of an \( n \)-dimensional Calabi-Yau manifold is called special Lagrangian if it is calibrated by \( \text{Re}(e^{i\theta}\Omega) \), where \( \theta \) is a constant. This condition is equivalent to \( \omega|_S = 0 \) and \( \text{Im}(e^{i\theta}\Omega)|_S = 0 \)(cf. [Joy01]).

In 1996, Strominger, Yau and Zaslow [SYZ96] suggested a geometrical interpretation of Mirror Symmetry between Calabi-Yau 3-folds in terms of dual fibrations by special Lagrangian 3-tori, now known as the SYZ Conjecture.

A 4n-dimensional manifold is hyperkähler if it possesses a Riemannian metric \( g \) which is Kähler with respect to three complex structures \( I_1; I_2; I_3 \) satisfying the quaternionic relations \( I_1I_2 = -I_2I_1 = I_3 \) etc, thus has three forms \( \omega_1, \omega_2, \omega_3 \) corresponding to the three complex structures. It has holonomy group contained in \( \text{Sp}(n)/\text{SU}(2n) \), a prior is Calabi-Yau. With respect to the complex structure \( I_1 \) the
form \( \omega_C = (\omega_2 + \sqrt{-1}\omega_3) \) is a holomorphic symplectic form. If \( X \) is a complex Lagrangian submanifold (\( X \) is a complex submanifold and \( \omega_C \) vanishes on \( X \)), then the real and imaginary parts of \( \omega_C \), namely \( \omega_2 \) and \( \omega_3 \), vanish on \( X \). Thus \( \omega_2 \) vanishes and if \( n \) is odd (resp. even), the real (resp. imaginary) part of \( \Omega = (\omega_3 + \sqrt{-1}\omega_1)^n \) vanishes. Using the complex structure \( I_2 \) instead of \( I_1 \), we see that \( X \) is special Lagrangian (see also [Hit97]). A fibration of hyperkähler manifold with complex lagrangian fibers is called holomorphic Lagrangian fibration, which is also very important in Mirror Symmetry. This is because of examples of special Lagrangian fibrations are very rare, all known examples are derived from holomorphic Lagrangian fibrations on \( K3 \), torus, or other hyperkähler manifolds. In the weakest form, the hyperkähler SYZ conjecture is stated as follows.

**Conjecture 2.1.** Let \( Y \) be a hyperkähler manifold. Then \( Y \) can be deformed to a hyperkähler manifold admitting a holomorphic Lagrangian fibration.

For a more precise form of hyperkähler SYZ conjecture, see [Ver10].

### 3. Toric Hyperkähler Manifold

One of the most powerful techniques for constructing hyperkähler manifolds is the hyperkähler quotient method of Hitchin, Karlhede, Lindström and Roček ([HKLR87]). We specialized on the class of hyperkähler quotients of flat quaternionic space \( \mathbb{H}^d \) by subtori of \( T^d \). The geometry of these spaces turns out to be closely connected with the theory of toric varieties.

Since \( \mathbb{H}^d \) can be identified with \( T^d \), it has three complex structures \( \{I_1, I_2, I_3\} \). The real torus \( T^d = \{(\zeta_1, \zeta_2, \ldots, \zeta_d) \in \mathbb{C}^d, |\zeta_i| = 1\} \) acts on \( \mathbb{C}^d \) induce a action on \( T^d \) keeping the hyperkähler structure,

\[
(z, w)\zeta = (z\zeta, w\zeta^{-1}).
\]

Denote \( M \) the \( m \)-dimensional connected subtorus of \( T^d \) whose Lie algebra \( m \subset t^d \) is generated by integer vectors (which is always taken to be primitive), then we have the following exact sequences

\[
0 \rightarrow m \xrightarrow{\iota} t^d \xrightarrow{\pi} n \rightarrow 0,
\]

\[
0 \leftarrow m^* \xleftarrow{\imath^*} (t^d)^* \xleftarrow{\pi^*} n^* \leftarrow 0,
\]

where \( n = t^d/m \) is the Lie algebra of the \( n \)-dimensional quotient torus \( N = T^d/M \) and \( m + n = d \).

Let \( \{e_i\}_{i=1}^d \) be the standard basis of \( t^d \) and \( \{\theta_i\}_{i=1}^m \) some basis span \( m \). Denote \( \{e_i^*\}_{i=1}^d \) and \( \{\theta_i^*\}_{i=1}^m \) the dual basis. The subtorus \( M \) action
admits a hyperkähler moment map: \( \mu = (\mu_R, \mu_C) : \mathbb{H}^d \to m^* \times m_C^* \),
given by,

\[
\mu_R(z, w) = \frac{1}{2} \sum_{i=1}^{d} (|z_i|^2 - |w_i|^2) \iota^*_i e_i^*,
\]

\[
\mu_C(z, w) = \sum_{i=1}^{d} z_i w_i \iota^*_i e_i^*.
\]

The complex moment map \( \mu_C : \mathbb{H}^d \to m_C^* \) is holomorphic with respect to \( I_1 \). Bielawski and Dancer introduced the definition of toric hyperkähler varieties, and generally speaking, they are not toric varieties.

**Definition 3.1** (BD00). A toric hyperkähler variety \( Y(\alpha, \beta) \) is a hyperkähler quotient \( \mu^{-1}(\alpha, \beta)/M \) for \( (\alpha, \beta) \in m^* \times m_C^* \).

A smooth part of \( Y(\alpha, \beta) \) is a 4n-dimensional hyperkähler manifold, whose hyperkähler structure is denoted by \( (g, I_1, I_2, I_3) \). The quotient torus \( N = T/M \) acts on \( Y(\alpha, \beta) \), preserving its hyperkähler structure. This residue circle action admits a hyperkähler moment map \( \bar{\mu} = (\bar{\mu}_R, \bar{\mu}_C) \),

\[
\bar{\mu}_R([z, w]) = \frac{1}{2} \sum_{i=1}^{d} (|z_i|^2 - |w_i|^2) \iota^*_i e_i^*,
\]

\[
\bar{\mu}_C([z, w]) = \sum_{i=1}^{d} z_i w_i \iota^*_i e_i^*.
\]

Differs from the toric case, the map \( \bar{\mu} \) to \( n^* \times n_C^* \) is surjective, never with a bounded image.

In this article, we always assume that \( Y(\alpha, \beta) \) is a smooth manifold(readers could consult the regularity argument for [BD00] and [Kon08]).

4. \((\mathbb{C}^*)^n \cong T^n \times \mathbb{R}^n\) FIBRATION OVER \( \mathbb{C}^n \)

We focus on the complex moment map \( \bar{\mu}_C \) of the residue circle action. The first task of this section is to prove the general theorem.

**Theorem 4.1.** The map \( \bar{\mu}_C : Y(\alpha, \beta) \to n_C^* \cong \mathbb{C}^n \) defines a holomorphic Lagrangian fibration, i.e. for any \( b \in n_C^* \), \( F_b = \bar{\mu}_C^{-1}(b) \) is a complex Lagrangian subvariety.
Proof. First of all, $\bar{\mu}_C([z, w]) = \sum_{i=1}^d z_i w_i e^*_i$ is obviously a holomorphic map from $Y(\alpha, 0)$ to $n^*_C$. Let $X$ be the tangent vector space of $F_b$, for $\bar{\mu}_C(F_b) = b$, we have $d\bar{\mu}_C(X) = 0$. Denote $V$ the $n$ dimensional space of tangent vectors to the orbit of residue circle action $N$, this is equivalent saying $\omega_C(V, X) = \omega_2(V, X) + \sqrt{-1}\omega_3(V, X) = 0$, i.e. $g\langle I_2 V, X \rangle = 0$ and $g\langle I_3 V, X \rangle = 0$. At a smooth point of $F_b$, $X$ has real dimension $2n$, thus is the orthogonal component of $I_2 V \oplus I_3 V$, thus $X$ must be $V \oplus I_1 V$. By the quaternionic relation, we have $g\langle I_2 X, X \rangle = 0$ and $g\langle I_3 X, X \rangle = 0$, hence $\omega_C|_X = 0$, which means that $F_b$ is a complex Lagrangian subvarieties. □

For the study of the singularity of the fibers, we have to investigate the regularity of the complex moment map $\bar{\mu}_C$ of toric hyperkähler manifold $Y(\alpha, \beta)$. We need to reinterpret the dual of Lie algebra $n^*$ and $n^*_C$ first. For $\alpha$ lies in $m^*$, there is some $x \in \mathbb{R}^d$, such that

$$
(4.1) \quad \alpha = \sum_{i=1}^d x^i \theta_i^*.
$$

Assume $\iota^* e_i^* = a_k^i \theta_k^*$, and $\alpha = \alpha^k \theta_k^*$, above equation turns to a linear equation system

$$
(4.2) \quad Ax = \alpha,
$$

where $A$ is $m \times d$ matrix with entry $a_k^i$, and $\alpha$ represents the column vector $\{\alpha^k\}_{k=1}^m$. Its $n$-dimensional solution space is denoted as $\mathfrak{N}_\alpha$, a $n$-plane in $\mathbb{R}^d$. We can identify $\mathfrak{N}_\alpha$ with $n^*$. A hyperplanes arrangement $\{H_i\}_{i=1}^d$ is defined by $H_i = \mathfrak{N}_\alpha \cap \{x_i = 0\}$, where $\{x_i = 0\}$ are the coordinate hyperplanes in $\mathbb{R}^d$. Similarly, we could define the $\mathfrak{N}_\beta$ the complex solution space which identifies to $n^*_C$, and a complex hyperplanes arrangement by $W_i = \mathfrak{N}_\beta \cap \{y_i = 0\}$, where $\{y_i = 0\}$ is the coordinate hyperplanes in $\mathbb{C}^d$. We call the union $\bigcup_{i=1}^d W_i$ a wall structure on $\mathfrak{N}_\beta$.

Lemma 4.2. The set of regular value of the moment map $\bar{\mu}_C$ of toric hyperkähler manifold $Y(\alpha, 0)$ is

$$
\quad n^*_C\text{reg} = n^*_C \setminus \bigcup_{i=1}^d W_i.
$$

Proof. Let $f : \mathbb{C}^2 \to \mathbb{C}$ be a map defined by $f(a, b) = ab$. We can easily observe that $(a, b) \in \mathbb{C}^2$ is a regular point of $f$ if and only if $(a, b) \neq 0$. If one of $(z_i, w_i) = 0$, constrained by equation $\sum_{i=1}^d z_i w_i \iota^* e_i^* = \beta$, the image of $(d\bar{\mu}_C)([z, w]) = \sum_{i=1}^d (df)(z_i, w_i) \otimes e_i^*$
at the point \(([z, w]) \in Y(\alpha, \beta)\) can not span the whole \(\mathfrak{H}_C\). Thus the point for some \((z_i, w_i) = 0\) is a critical point of \(\bar{\mu}_C\).

Immediately, we have

**Theorem 4.3.** The generic fiber \(F_b\) of \(Y(\alpha, \beta)\) over \(\mathfrak{n}_C^* \setminus \bigcup_{i=1}^{d} W_i\) is diffeomorphic to complex torus \((\mathbb{C}^*)^n \cong T^n \times \mathbb{R}^n\).

**Proof.** By the regularity of the moment map \(\bar{\mu}_C\), \(F_b\) is a smooth manifold. We claim that the residue circle action \(N\) acting on \(F\) freely.

To see this, lift the \(Y(\alpha, \beta)\) to \(\mathbb{H}^d\). Then \([z, w] \in Y(\alpha, \beta)\) has non-trivial isotropy group in \(N\) if and only if the orbit of a subgroup of \(N\) through \((z, w)\) lies in the \(M\)-orbit. For \(N\) is the quotient group, the only possibility is that some \((z_i, w_i)\) equals to zero, which can not hold if \(b \in \mathfrak{n}_C^* \setminus \bigcup_{i=1}^{d} W_i\). At another hand, the real moment map \(\bar{\mu}_R\) restricted to \(F_b\) is still surjective on \(\mathbb{R}^n\), moreover since \((z_i, w_i) \neq 0\), \(\bar{\mu}_R\) is also regular, thus \(F_b\) must diffeomorphic to \(T^n \times \mathbb{R}^n \cong (\mathbb{C}^*)^n\).

It is natural to ask what the singular fiber looks like, the picture will be a little bit complicated. Suggested by the above proof, we need to investigate the isotropy group in detail.

We first check the simplest case, where \(b\) lies in generic position of \(W_i\). Fixing \(i\), based on above discussion, the point \([z, w] \in F_b\) where \(z_i w_i = 0, z_j w_j \neq 0\) for \(j \neq i\) but \((z_i, w_i) \neq 0\), is with trivial isotropy, thus a smooth point on \(F_b\). And the real torus \(N\) acts on the complex subvariety \(P_{b,i}\) of \(F_b\) defined by \((z_i, w_i) = 0\) must has a 1 dimensional isotropy subgroup. For the real moment map restricted to \(F_b\) is always surjective, this is equivalent to shrinking the torus \(T^1\) whose Lie algebra is \(u_i\) over the \(n-1\) dimensional real subvariety \(\mu_{\mathbb{R}}(P_{b,i}) = H_i \subset \mathfrak{n}^* \cong \mathbb{R}^n\), where \(H_i\) is the hyperplane in the real arrangement.

Consider \(b\) lies in the intersection of \(s\) walls \(\{W_i\}_{i=1}^s\), the situation becomes more complicated. On the subvariety \(P_{b,l} = \{[z, w] \in F_b| (z_l, w_l) = 0\}\), we shrink the torus \(T^1\) corresponding to \(u_l\) over the hyperplane \(H_l\). Some of these subvarieties may intersect, or equivalent saying, the hyperplanes will intersect via the real residue moment map. For simplicity, let first \(q \leq s\) subvariety intersects. Their normals \(\{u_l\}_{l=1}^{q}\) may be linear dependent, thus we denote \(\dim(\{u_l\}_{l=1}^{q})\) the dimension of the subspace they spanning. By the Equation (3.4), we know that the image of the real residue moment map is the intersection of \(H^l, l = 1, \ldots, q\). Over this intersection, we shrink the real \(\dim(\{u_l\}_{l=1}^{q})\) dimensional torus generated by \(\{u_l\}_{l=1}^{q}\).

Finally, we get the conclusion

**Theorem 4.4.** Consider the singular fiber \(F_b\) of \(Y(\alpha, \beta)\). When \(b\) lies in the generic position of \(W_i\), then \(F_b\) diffeomorphic to shrinking the...
real torus $T^1$ generated by $u_i$ in the complex torus $(\mathbb{C}^*)^n \cong \mathbb{R}^n \times T^n$ over the real hyperplane $H_t \subset n^* \cong \mathbb{R}^n$. When $b$ lies in the intersection of $s$ walls $\{W_i\}_{i=1}^s$, then the singular fiber is given by shrinking $T^1$ due to $u_i$ over $H_l$ respectively, and at the intersection of $H_l$, $i = 1, \ldots, q$, shrinking a torus of real dimension $\dim(\{u_i\}_{l=1}^q)$ generated by $\{u_i\}_{l=1}^q$.

Theoretically, using the data defining the $Y(\alpha, \beta)$, checking all the intersections of the singular subvarieties, we can identify the type of the singular fiber. As we know, when $s$ grows bigger, the computation becomes overwhelming.

**Example 4.5.** Let the subgroup $M$ generated by $\iota \theta_1 = e_1 + e_2$ and $\iota \theta_2 = e_1 + e_3$, then $\iota^* e_1 = \theta_1^* + \theta_2^*$, $\iota^* e_2 = \theta_1^*$, $\iota^* e_3 = \theta_2^*$. If we take $\alpha = \iota^* e_1 - \frac{1}{2} \iota^* e_2 = \frac{1}{2} \theta_1^* + \theta_2^*$ and $\beta = 0$, consider the toric hyperkähler manifold $Y(\alpha, \beta)$ with generic fiber $\mathbb{C}^*$. The complex equation becomes

$$
\begin{pmatrix}
1 & 1 & 0 \\
1 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
y_1 \\
y_2 \\
y_3
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix},
$$

has solution $y = (1, -1, -1)^T$.

The solution space $\mathcal{N}_C$ only intersects the coordinates hyperplanes in the origin, thus the central fiber is the only singular fiber. We investigate it closer. For the residue circle action is 1-dimensional, the point will be fixed point if it has nontrivial isotropy. By the defining Equation (3.2), (3.3) and (3.5), the central fiber $F_0$ satisfies

$$
|z_1|^2 - |w_1|^2 + |z_2|^2 - |w_2|^2 = \frac{1}{2}
$$

$$
|z_1|^2 - |w_1|^2 + |z_3|^3 - |w_3|^2 = 1.
$$

For 0 is the intersection of 3 walls, consider them respectively: if $z_1 = w_1 = 0$, then $z_2$ and $w_3$ must be nonzero, similarly, if $z_2 = w_2 = 0$, then $z_1$ and $w_3$ must be nonzero, if $z_3 = w_3 = 0$, then $z_2$ and $w_1$ must be nonzero. These are the only 3 fixed points of $N$. Shrinking a real torus $T^1$ in these 3 points, we get $\mathbb{C}^1$, $\mathbb{C}P^1$, $\mathbb{C}P^1$ and $\mathbb{C}^1$ intersecting sequentially, which is nothing but the toric varieties in the extended core.

Recall that in the category of $T^n$ fibration of toric varieties, the $T^n$ fibers degenerate at the boundary of the Delzant polytopes of the toric varieties(cf. [Bon07]). Our theorem can be viewed as the hyperkähler analog. Using moment map to construct special Lagrangian variety had already been studied by Joyce intensively in [Joy02].
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