ZS_n-MODULES AND POLYNOMIAL IDENTITIES WITH INTEGER COEFFICIENTS

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Abstract. We show that, like in the case of algebras over fields, the study of multilinear polynomial identities of unitary rings can be reduced to the study of proper polynomial identities. In particular, the factors of series of ZS_n-submodules in the ZS_n-modules of multilinear polynomial functions can be derived by the analog of Young’s (or Pieri’s) rule from the factors of series in the corresponding ZS_n-modules of proper polynomial functions.

As an application, we calculate the codimensions and a basis of multilinear polynomial identities of unitary rings of upper triangular 2 × 2 matrices and infinitely generated Grassmann algebras over unitary rings. In addition, we calculate the factors of series of ZS_n-submodules for these algebras.

Also we establish relations between codimensions of rings and codimensions of algebras and show that the analog of Amitsur’s conjecture holds in all torsion-free rings, and all torsion-free rings with 1 satisfy the analog of Regev’s conjecture.

Polynomial identities and their numeric and representational characteristics are well studied in the case of algebras over fields of characteristic zero (see e.g. [3, 5]). However, polynomial identities in rings also play an important role [12]. The systematic study of multilinear polynomial identities started in 1950 by A.I. Mal’cev [9] and W. Specht [14]. In his paper, W. Specht considered polynomial identities with integer coefficients. This article is devoted to numeric and representational characteristics of polynomial identities with integer coefficients.

In Propositions 1–3, we prove basic facts about codimensions of polynomial identities with integer coefficients in the case when the ring is an algebra over a field. In Theorem 1, we consider algebras obtained from rings by the tensor product of the ring and a field. As a consequence of Theorem 1, we derive the analog of Amitsur’s conjecture for all torsion-free rings and the analog of Regev’s conjecture for all torsion-free rings with 1. In Theorem 2, we show that the ordinary codimensions of rings with 1 can be calculated using their proper codimensions by the same formula as in the case of algebras over fields [3, Theorem 4.3.12 (ii)]. In Theorem 3, we prove an analog of Drensky’s theorem [3, Theorem 12.5.4] that establishes a relation between ZS_n-modules corresponding to proper and ordinary polynomial identities of unitary rings. In order to apply Theorem 3, we show that the analog of Young’s (or Pieri’s) rule holds for ZS_n-modules too (Theorem 4).

In the case of algebras over fields, few examples are known where the numeric and representational characteristics can be precisely evaluated. Among those are the algebras of upper triangular matrices [7, 10, 11, 13] and the Grassmann algebra [8]. In Sections 5 and 6, we apply the results obtained in the preceding sections, to evaluate the basis of multilinear polynomial identities with integer coefficients and calculate their numeric and representational characteristics for unitary rings of upper triangular 2 × 2 matrices and infinitely generated Grassmann algebras over unitary rings.

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1. Introduction

Let $R$ be a ring. If $R$ has a unit element $1_R$, then the number

$$\text{char } R := \min\{ n \in \mathbb{N} \mid n1_R = 0 \} = \min\{ n \in \mathbb{N} \mid na = 0 \text{ for all } a \in R \}$$

is called the characteristic of $R$. (As usual, if $n1_R \neq 0$ for all $n \in \mathbb{N}$, then $\text{char } R := 0$.)

Let $\mathbb{Z}(X)$ be the free associative ring without 1 on the countable set $X = \{ x_1, x_2, \ldots \}$, i.e., the ring of polynomials in non-commuting variables from $X$ without a constant term.

Let $I$ be an ideal of a ring $R$. We say that $I$ is a $T$-ideal of $R$ if $\varphi(I) \subseteq I$ for all $\varphi \in \text{End}(R)$. We say that $f \in \mathbb{Z}(X)$ is a polynomial identity of $R$ with integer coefficients if $f(a_1, \ldots, a_n) = 0$ for all $a_i \in R$. In other words, $f$ is a polynomial identity if $\psi(f) = 0$ for all $\psi \in \text{Hom}(\mathbb{Z}(X), R)$. Note that the set $\text{Id}(R, \mathbb{Z})$ of polynomial identities of $R$ with integer coefficients is a $T$-ideal of $\mathbb{Z}(X)$.

Let $P_n(\mathbb{Z})$ be the additive subgroup of $\mathbb{Z}(X)$ generated by $x_\sigma(1)x_\sigma(2) \ldots x_\sigma(n), \sigma \in S_n$. (Here $S_n$ is the $n$th symmetric group, $n \in \mathbb{N}$.) Then $P_n(\mathbb{Z})/\text{Id}(R, \mathbb{Z})$ is a finitely generated Abelian group which is the direct sum of free and primary cyclic groups:

$$\frac{P_n(\mathbb{Z})}{P_n(\mathbb{Z}) \cap \text{Id}(R, \mathbb{Z})} \cong \bigoplus_{c_n(0,R)} \mathbb{Z} \oplus \bigoplus_{p \text{ is a prime number}} \bigoplus_{k \in \mathbb{N}} \left( \mathbb{Z}_{p^k} \oplus \cdots \oplus \mathbb{Z}_{p^k} \right).$$

We call the numbers $c_n(0,R,q)$ the codimensions of polynomial identities of $R$ with integer coefficients.

Note that the symmetric group $S_n$ is acting on $P_n(\mathbb{Z})/\text{Id}(R, \mathbb{Z})$ by permutations of variables, i.e., $P_n(\mathbb{Z})/\text{Id}(R, \mathbb{Z})$ is a $\mathbb{Z}S_n$-module. We refer to $P_n(\mathbb{Z})/\text{Id}(R, \mathbb{Z})$ as the $\mathbb{Z}S_n$-module of ordinary multilinear polynomial functions on $R$.

Denote by $\Gamma_n(\mathbb{Z})$ the subgroup of $P_n(\mathbb{Z})$ that consists of proper polynomials, i.e linear combinations of products of long commutators. (All long commutators in the article are left normed, e.g. $[x, y, z, t] := [[[x, y], z], t]$.) Then $\Gamma_n(\mathbb{Z})$ is a $\mathbb{Z}S_n$-submodule of $P_n(\mathbb{Z})$. Obviously, $\Gamma_1(\mathbb{Z}) = 0$.

Analogously, we define the codimensions $\gamma_n(R,q)$ of proper polynomial identities of $R$:

$$\frac{\Gamma_n(\mathbb{Z})}{\Gamma_n(\mathbb{Z}) \cap \text{Id}(R, \mathbb{Z})} \cong \bigoplus_{\gamma_n(0,R)} \mathbb{Z} \oplus \bigoplus_{p \text{ is a prime number}} \bigoplus_{k \in \mathbb{N}} \left( \mathbb{Z}_{p^k} \oplus \cdots \oplus \mathbb{Z}_{p^k} \right).$$

If $R$ has a unit element $1_R$, then, by the definition, $\gamma_0(R,q)$ is the number of $\mathbb{Z}_p$ in the decomposition of the cyclic additive subgroup of $R$ generated by $1_R$. We refer to $\Gamma_n(\mathbb{Z})/\text{Id}(R, \mathbb{Z})$ as the $\mathbb{Z}S_n$-module of proper multilinear polynomial functions on $R$.

If $A$ is an algebra over a field $F$, then we can consider codimensions $c_n(A,F) := \dim \frac{P_n(F)}{\text{Id}(A,F)}$ of polynomial identities of $A$ with coefficients from $F$. (See [5, Definition 4.1.1].) Here $\text{Id}(A,F) \subset F(X)$ is the set of polynomial identities of $A$ with coefficients from $F$, and $P_n(F)$ is the subspace of $F(X)$ generated by $x_\sigma(1)x_\sigma(2) \ldots x_\sigma(n), \sigma \in S_n$. The subspace of $P_n(F)$ consisting of proper polynomials, is denoted by $\Gamma_n(F)$.

We say that $\lambda = (\lambda_1, \ldots, \lambda_s)$ is a (proper or ordered) partition of $n$ and write $\lambda \vdash n$ if $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s > 0$, $\lambda_i \in \mathbb{N}$, and $\sum_{i=1}^{s} \lambda_i = n$. In this case we write $\lambda \vdash n$. For our convenience, we assume $\lambda_i = 0$ for all $i > s$.

We say that $\mu = (\mu_1, \ldots, \mu_s)$ is an unordered partition of $n$ if $\mu_i \in \mathbb{N}$ and $\sum_{i=1}^{s} \mu_i = n$. In this case we write $\mu \vDash n$. Again, for our convenience, we assume $\mu_i = 0$ for all $i > s$.

For every ordered or unordered partition $\lambda$ one can assign the Young diagram $D_\lambda$ which contains $\lambda_k$ boxes in the $k$th row. If $\lambda$ is unordered, then $D_\lambda$ is called generalized. A Young
diagram filled with numbers is called a Young tableau. A tableau corresponding to \( \lambda \) is denoted by \( T_\lambda \).

In the representation theory of symmetric groups, partitions and their Young diagrams are widely used. (See [3, 5] for applications to PI-algebras.) Let \( a_{T_\lambda} = \sum_{\pi \in R_{T_\lambda}} \pi \) and \( b_{T_\lambda} = \sum_{\sigma \in C_{T_\lambda}} (\text{sign} \, \sigma) \sigma \) be symmetrizers corresponding to a Young tableau \( T_\lambda \), \( \lambda \vdash n \). Then \( S(\lambda) := (\mathbb{Z}S_n)_n b_{T_\lambda} a_{T_\lambda} \) is the corresponding Specht module. Moreover, modules \( S(\lambda) \) that correspond to different \( T_\lambda \) but the same \( \lambda \), are isomorphic too. (The proof is analogous to the case of fields.) Though \( S(\lambda) \) are not irreducible over \( \mathbb{Z} \) and even contain no irreducible \( \mathbb{Z}S_n \)-submodules (it is sufficient to consider the submodule \( 0 \neq 2M \subseteq M \) for any submodule \( M \subseteq S(\lambda) \)), we will use them in order to describe the structure of \( \frac{P_n(\mathbb{Z})}{P_n(\mathbb{Z}) \cap \text{Id}(R, \mathbb{Z})} \).

### 2. Codimensions of algebras over fields

Every algebra over a field can be treated as a ring. Therefore, we have to deal with two different types of codimensions. Here we establish a relation between them.

**Proposition 1.** Let \( A \) be an algebra over a field \( F \). Then \( \text{c}_n(A, q) = 0 \) for all \( n \in \mathbb{N} \) and \( q \neq \text{char} \, F \).

**Proof.** Note that \((\text{char} \, F) f \in \text{Id}(R, \mathbb{Z}) \) for all \( f \in \mathbb{Z}(X) \). Hence \( \text{char} \, F > 0 \) implies \( \text{c}_n(A, q) = 0 \) for all \( n \in \mathbb{N} \) and \( q \neq \text{char} \, F \). If \( \text{char} \, F = 0 \), then every \( q = p^k \neq 0 \) is invertible and \( \text{c}_n(A, q) = 0 \) for all \( n \in \mathbb{N} \) and \( q \neq 0 \) too. \( \square \)

**Proposition 2.** Let \( A \) be an algebra over a field \( F \), \( \text{char} \, F = 0 \). Then \( \text{c}_n(A, F) \leq \text{c}_n(A, 0) \) for all \( n \in \mathbb{N} \). Moreover, \( \text{c}_n(A, Q) = \text{c}_n(A, 0) \) for all \( n \in \mathbb{N} \).

**Proof.** By Proposition \([\underline{4}]\) \( \frac{P_n(\mathbb{Z})}{P_n(\mathbb{Z}) \cap \text{Id}(A, \mathbb{Z})} \) is a free Abelian group. Let \( f_1, \ldots, f_s \) be the preimages of its free generators in \( P_n(\mathbb{Z}) \). Note that \( P_n(\mathbb{Z}) \subseteq P_n(\mathbb{Q}) \subseteq P_n(F) \) and for every \( \sigma \in S_n \) the monomial \( x_{\sigma(1)}x_{\sigma(2)} \cdots x_{\sigma(n)} \) can be expressed as a linear combination with integer coefficients of \( f_1, \ldots, f_s \) and an element of \( P_n(\mathbb{Z}) \cap \text{Id}(A, \mathbb{Z}) \). Hence the images of \( f_1, \ldots, f_s \) generate \( \frac{P_n(F)}{P_n(F) \cap \text{Id}(A, F)} \) and \( \text{c}_n(A, F) \leq \text{c}_n(A, 0) = s \).

Suppose \( f_1, \ldots, f_s \) are linearly dependent modulo \( \text{Id}(A, \mathbb{Q}) \). In this case \( \frac{\pi}{q_1} f_1 + \cdots + \frac{\pi}{q_s} f_s \in \text{Id}(A, \mathbb{Q}) \) for some \( q_i \in \mathbb{N} \), \( r_i \in \mathbb{Z} \). Thus

\[
\begin{align*}
f &:= r_1 \left( \prod_{i=2}^{s} q_i \right) f_1 + r_1 q_1 \left( \prod_{i=3}^{s} q_i \right) f_2 + \cdots + r_s \left( \prod_{i=1}^{s-1} q_i \right) f_s \\
& \in \text{Id}(A, \mathbb{Q}).
\end{align*}
\]

However, \( f \in \mathbb{Z}(X) \). Hence \( f \in \text{Id}(A, \mathbb{Z}) \) and all \( r_i = 0 \) since \( f_i \) are linearly independent modulo \( \text{Id}(A, \mathbb{Z}) \). Therefore, the images of \( f_1, \ldots, f_s \) form a basis of \( \frac{P_n(\mathbb{Q})}{P_n(\mathbb{Q}) \cap \text{Id}(A, \mathbb{Q})} \) and \( \text{c}_n(A, \mathbb{Q}) = \text{c}_n(A, 0) = s \). \( \square \)

The next example shows that in the case \( F \supsetneq \mathbb{Q} \) we could have \( \text{c}_n(A, F) < \text{c}_n(A, \mathbb{Q}) = \text{c}_n(A, 0) \).

**Example 1.** Note that \( P_3(\mathbb{Q}) \cong \mathbb{Q}S_3 \cong S^3(3) \oplus S^3(2, 1) \oplus S^3(2, 1) \oplus S^3(1^3) \). Let \( a \in \mathbb{Q}S_3 \) such that \( S^3(2, 1) = \mathbb{Q}S_3 a \). Denote by \( f_1 \) and \( f_2 \) the polynomials that correspond to \( a \) in the copies of \( S^3(2, 1) \) in \( P_3(\mathbb{Q}) \). Let \( F = \mathbb{Q}(\sqrt{2}) \). Consider the \( T \)-ideal \( I \) of \( F(X) \) generated by \((f_1 + \sqrt{2}f_2)\). We claim that \( \text{c}_3(F(X)/I, F) = 4 < \text{c}_3(F(X)/I, \mathbb{Q}) = 6 \).

**Proof.** First we notice that \( P_3(F) \cap \text{Id}(F(X)/I, F) = F S_3 \cdot (f_1 + \sqrt{2}f_2) \cong S^3(2, 1) \). Hence by the hook formula, \( \text{c}_3(F(X)/I, F) = 6 - 2 = 4 \). However, \( P_3(\mathbb{Q}) \cap \text{Id}(F(X)/I, \mathbb{Q}) = P_3(\mathbb{Q}) \cap F S_3 (f_1 + \sqrt{2}f_2) = 0 \). Indeed, suppose \( f = b(f_1 + \sqrt{2}f_2) \in P_3(\mathbb{Q}) \) for some \( b \in F S_3 \). Note that \( b = b_1 + \sqrt{2}b_2 \) where \( b_1, b_2 \in \mathbb{Q}S_3 \). Therefore, \( f = (b_1 + \sqrt{2}b_2)(f_1 + \sqrt{2}f_2) = (b_1f_1 + \sqrt{2}b_2f_1 + \sqrt{2}b_1f_2 + 2b_2f_2) \).

form a basis of \( f \) of \( \mathbb{Q}S_3 \) and \( \mathbb{Q}S_2 \) is the direct sum of \( \mathbb{Q}S_3 \)-submodules. Hence \( b_1 f_2 = b_2 f_1 \) too. However, \( \mathbb{Q}S_3 f_1 \cong \mathbb{Q}S_2 f_2 \). Thus \( b_1 f_1 = b_2 f_2 = 0 \) too, \( f = 0 \), \( P_3(\mathbb{Q}) \cap \text{Id}(F(\mathcal{X})/I, \mathbb{Q}) = 0 \) and \( c_3(F(\mathcal{X})/I, \mathbb{Q}) = 6 \). \( \square \)

The result, analogous to Proposition [2] holds in a positive characteristic.

**Proposition 3.** Let \( A \) be an algebra over a field \( F \), char \( F = p \). Then \( c_n(A, F) \leq c_n(A, p) \) for all \( n \in \mathbb{N} \). Moreover, \( c_n(A, \mathbb{Z}_p) = c_n(A, p) \) for all \( n \in \mathbb{N} \).

**Proof.** By Proposition [1] \( \frac{P_n(\mathbb{Z})}{P_n(\mathbb{Z}) \cap \text{Id}(A, \mathbb{Z})} \) is the direct sum of copies of \( \mathbb{Z}_p \). Let \( f_1, \ldots, f_s \) be the preimages of their standard generators in \( P_n(\mathbb{Z}) \). Note that \( P_n(\mathbb{Z}_p) \) is an image of \( P_n(\mathbb{Z}) \) under the natural homomorphism, \( P_n(\mathbb{Z}_p) \subseteq P_n(A) \) and for every \( \sigma \in S_n \) the monomial \( x_{\sigma(1)}^{m_1}x_{\sigma(2)}^{m_2} \ldots x_{\sigma(n)}^{m_n} \) can be expressed as a linear combination with integer coefficients of \( f_1, \ldots, f_s \) and an element of \( P_n(\mathbb{Z}) \cap \text{Id}(A, \mathbb{Z}) \). Hence the images of \( f_1, \ldots, f_s \) generate \( \frac{P_n(F)}{P_n(F) \cap \text{Id}(A, F)} \) and \( c_n(A, F) \leq c_n(A, p) \).

Suppose \( f_1, \ldots, f_s \) are linearly dependent modulo \( \text{Id}(A, \mathbb{Z}_p) \). In this case \( \bar{m}_1 f_1 + \cdots + \bar{m}_s f_s \in \text{Id}(A, \mathbb{Z}_p) \) for some \( m_1 \in \mathbb{Z} \). Thus \( m_1 f_1 + \cdots + m_s f_s \in \text{Id}(A, \mathbb{Z}) \) and all \( m_i \in \mathbb{P} \mathbb{Z} \) since \( f_i \) generate modulo \( \text{Id}(A, \mathbb{Z}) \) the direct sum of copies of \( \mathbb{Z}_p \). Therefore, the images of \( f_1, \ldots, f_s \) form a basis of \( \frac{P_n(\mathbb{Z}_p)}{P_n(\mathbb{Z}_p) \cap \text{Id}(A, \mathbb{Z}_p)} \) and \( c_n(A, \mathbb{Z}_p) = c_n(A, p) = s \). \( \square \)

The next result is concerned with the extension of a ring to an algebra over a field.

**Theorem 1.** Let \( R \) be a ring and let \( F \) be a field. Then

\[
c_n\left(R \otimes \mathbb{Z}[F, F]\right) = \begin{cases} c_n\left(R/\text{Tor} R, 0\right) & \text{if char } F = 0, \\ c_n\left(R/pR, p\right) & \text{if char } F = p 
\end{cases}
\]

where \( \text{Tor} R := \{r \in R \mid mr = 0 \text{ for some } m \in \mathbb{N}\} \) is the torsion of \( R \).

First, we prove the following lemma

**Lemma 1.** Let \( R \) be a ring and let \( F \) be a field. Then

\[
R \otimes \mathbb{F} \cong \begin{cases} R/\text{Tor} R & \text{if } F = \mathbb{Q}, \\ R/pR & \text{if } F = \mathbb{Z}_p
\end{cases}
\]

where \( R \otimes \mathbb{F} \subseteq R \otimes \mathbb{Z}[F, F] \) is a subring.

**Proof.** Consider the natural homomorphism \( \varphi : R \to R \otimes \mathbb{F} \) where \( \varphi(a) = a \otimes 1_F, a \in R \).

Suppose \( F = \mathbb{Q} \). If \( ma = 0 \) for some \( m \in \mathbb{N} \) and \( a \in R \), then \( \varphi(a) = a \otimes 1_\mathbb{Q} = ma \otimes 1_m = 0 \). Hence \( \text{Tor} R \subseteq \ker \varphi \). We claim that \( \ker \varphi = \text{Tor} R \).

Let \( a \in \ker \varphi \), i.e., \( a \otimes 1_\mathbb{Q} = 0 \). By one of the definitions of the tensor product,

\[
(a, 1_\mathbb{Q}) = \sum_i \ell_i((a_i + b_i, q_i) - (a_i, q_i) - (b_i, q_i)) + \sum_i m_i((c_i, s_i + t_i) - (c_i, s_i) - (c_i, t_i)) + \sum_i n_i((k_i d_i, u_i) - (d_i, k_i u_i))
\]

holds for some \( a_i, b_i, c_i, d_i \in R, k_i, \ell_i, m_i, n_i \in \mathbb{Z} \), and \( q_i, s_i, t_i, u_i \in \mathbb{Q} \) in the free \( \mathbb{Z} \)-module \( H_{R \times \mathbb{Q}} \) with the basis \( R \times \mathbb{Q} \). We can find such \( m \in \mathbb{N} \) that all \( mq_i, ms_i, mt_i, mu_i \in \mathbb{Z} \). Then

\[
(a, m) = \sum_i \ell_i((a_i + b_i, mq_i) - (a_i, mq_i) - (b_i, mq_i)) + \sum_i m_i((c_i, ms_i + mt_i) - (c_i, ms_i) - (c_i, mt_i)) + \sum_i n_i((k_i d_i, mu_i) - (d_i, k_i mu_i))
\]
holds in the free $\mathbb{Z}$-module $H_{R \times Z}$ with the basis $R \times Z$. Note that in the right hand side of the latter equality we have a relation in $R \otimes \mathbb{Z}$. Hence $a \otimes m = 0$ in $R \otimes \mathbb{Z} \cong R$ and $ma = 0$. Thus $a \in \text{Tor } R$. Therefore, $\varphi = \text{Tor } R$ and $R \otimes 1_Q \cong R/\text{Tor } R$.

Suppose $F = \mathbb{Z}_p$. Then $\varphi(pR) = R \otimes p1_{\mathbb{Z}_p} = 0$ and $pR \subseteq \ker \varphi$. Let $a \in \ker \varphi$, i.e., $a \otimes 1_{\mathbb{Z}_p} = 0$. Then

$$(a, 1_{\mathbb{Z}_p}) = \sum_i q_i((a_i + b_i, \ell_i) - (a_i, \ell_i) - (b_i, \ell_i)) + \sum_i s_i((c_i, \bar{m}_i - n_i)) - (c_i, \bar{n}_i) + \sum_i t_i((k_i d_i, \bar{u}_i) - (d_i, k_i \bar{u}_i))$$

holds for some $a_i, b_i, c_i, d_i \in R$ and $k_i, \ell_i, m_i, n_i, q_i, s_i, t_i, u_i \in \mathbb{Z}$ in the free $\mathbb{Z}$-module $H_{R \times \mathbb{Z}_p}$ with the basis $R \times \mathbb{Z}_p$. Note that $H_{R \times \mathbb{Z}_p}$ is the factor module of $H_{R \times Z}$ by the subgroup $\langle (a, m) - (a, m + p) \mid a \in R, m \in \mathbb{Z} \rangle$. Hence

$$(a, 1_{\mathbb{Z}}) = \sum_i q_i((a_i + b_i, \ell_i) - (a_i, \ell_i) - (b_i, \ell_i)) + \sum_i s_i((c_i, m_i + n_i) - (c_i, m_i) - (c_i, n_i)) + \sum_i t_i((k_i d_i, u_i) - (d_i, k_i u_i)) + \sum_i \alpha_i((r_i, \beta_i) - (r_i, \beta_i + p))$$

holds in $H_{R \times Z}$ for some $r_i \in R$ and $\alpha_i, \beta_i \in \mathbb{Z}$. Thus $a \otimes 1_{\mathbb{Z}} = \sum_i \alpha_i r_i \otimes p$. Now we use the isomorphism $R \otimes \mathbb{Z} \cong R$ and get $a = \sum_i \alpha_i r_i p \in pR$. Therefore, $\ker \varphi = pR$ and $R \otimes 1_{\mathbb{Z}_p} \cong R/pR$.

**Proof of Theorem 4.2.** Recall that $R \otimes 1_F$ is a subring of $R \otimes \mathbb{Z}$. Hence $P_n(\mathbb{Z}) \cap \text{Id}(R \otimes \mathbb{Z} : \mathbb{Z}) \subseteq P_n(\mathbb{Z}) \cap \text{Id}(R \otimes 1_F : \mathbb{Z})$. Conversely, $P_n(\mathbb{Z}) \cap \text{Id}(R \otimes \mathbb{Z} : \mathbb{Z}) \supseteq P_n(\mathbb{Z}) \cap \text{Id}(R \otimes 1_F : \mathbb{Z})$ since $R \otimes 1_F$ generates $R \otimes \mathbb{Z}$ as an $F$-vector space. Therefore, $c_n(R \otimes 1_F, \text{char } F) = c_n(R \otimes \mathbb{Z}, \text{char } F)$ and we get Theorem 4.2 for $F = \mathbb{Q}$ and $F = \mathbb{Z}_p$ from Lemma 4.2 and Propositions 4.2, 4.3. The general case follows from the fact that $(R \otimes \mathbb{Z}) \otimes_F K \cong R \otimes \mathbb{Z} K$ (as a $K$-algebra) for any field extension $K \supseteq F$ and, by [5, Theorem 4.1.9],

$$c_n(R \otimes \mathbb{Z}, K) = c_n((R \otimes \mathbb{Z}) \otimes_F K, K) = c_n(R \otimes \mathbb{Z}, F).$$

**Corollary.** Let $R$ be a torsion-free ring satisfying a non-trivial polynomial identity. Then

1. either $c_n(R, 0) = 0$ for all $n \geq n_0$, $n_0 \in \mathbb{N}$, or there exist $d \in \mathbb{N}$, $C_1, C_2 > 0$, $q_1, q_2 \in \mathbb{R}$ such that $C_1 n^{q_1} d^n \leq c_n(R, 0) \leq C_2 n^{q_2} d^n$ for all $n \in \mathbb{N}$; in particular, polynomial identities of $R$ satisfy the analog of Amitsur’s conjecture, i.e., there exists $\lim_{n \to \infty} \sqrt[n]{c_n(R, 0)} \in \mathbb{Z}_+$;

2. if $R$ contains $1$, then there exist $C > 0$ and $q \in \mathbb{Z}$ such that $c_n(R, 0) \sim C n^{q} d^n$ as $n \to \infty$, i.e., the analog of Regev’s conjecture holds in $R$. (We write $f \sim g$ if $\lim \frac{f}{g} = 1$.)

**Proof.** By Theorem 4.2, $c_n(R, 0) = c_n(R \otimes \mathbb{Q}, \mathbb{Q})$. Now we apply [5, Theorem 6.5.2] and [2, Theorem 4.2.2].

**Remark.** If $R$ is a torsion-free ring, then $c_n(R, q) = 0$ for all $q \neq 0$ since $f \in \text{Id}(R, \mathbb{Z})$ for all $f \in \mathbb{Z}(X)$ such that $mf \in \text{Id}(R, \mathbb{Z})$ for some $m \in \mathbb{N}$.

We conclude the section with an example.

**Example 2.** Let $R = \bigoplus_{k=1}^{\infty} \mathbb{Z}_{2^k}$. Then $c_n(R, 0) = 1$ and $c_n(R, q) = 0$ for all $q \neq 0$ and $n \in \mathbb{N}$. Although $mR \neq 0$ for all $m \in \mathbb{N}$, $R \otimes \mathbb{Q} = 0$ and $c_n(R \otimes \mathbb{Q}, \mathbb{Q}) = 0$ for all $n \in \mathbb{N}$. 
Proof. The ring $R$ is commutative. Hence all monomials from $P_n(Z)$ are proportional to $x_1x_2 \ldots x_n$ modulo $\text{Id}(R, Z)$. However, $\text{max}(x_1, x_2, \ldots, x_n) \notin \text{Id}(R, Z)$ for all $m \in \mathbb{N}$. (It is sufficient to substitute $x_1 = x_2 = \cdots = x_n = 1_{Z^k}$ for $2^k > m$.) Thus $\frac{P_n(Z)}{P_n(Z) \cap \text{Id}(R, Z)} \cong \mathbb{Z}$ and $c_n(R, 0) = 1$ and $c_n(R, q) = 0$ for all $q \neq 0$ and $n \in \mathbb{N}$. However $a \otimes q = 2^k a \otimes \frac{q}{2^k}$ for all $a \in R, q \in \mathbb{Q}$, and $k \in \mathbb{N}$. Choosing $k$ sufficiently large, we get $a \otimes q = 2^k a \otimes \frac{q}{2^k} = 0$. Thus $R \otimes \mathbb{Q} = 0$ and $c_n(R \otimes \mathbb{Q}, \mathbb{Q}) = 0$ for all $n \in \mathbb{N}$. □

3. Relation between $\mathbb{Z}S_n$-modules of proper and ordinary polynomial functions

First, we describe the relation between proper and ordinary codimensions.

Theorem 2. Let $R$ be a unitary ring. Then $c_n(R, q) = \sum_{j=0}^{n} (\binom{n}{j}) \gamma_j(R, q)$ for every $n \in \mathbb{N}$ and $q \in \{p^k \mid p, k \in \mathbb{N}, \text{ p is prime} \} \cup \{0\}$.

Proof. First, we notice that

$$P_n(Z) = \bigoplus_{k=1}^{n-1} \bigoplus_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} x_{i_1}x_{i_2} \cdots x_{i_k} \sigma_{i_1, \ldots, i_k} \Gamma_{n-k}(Z) \quad \text{(direct sum of Z-modules) (1)}$$

where $\Gamma_0(Z) := Z$ and $\sigma_{i_1, \ldots, i_k} \in S_n$ is any permutation such that $\sigma((n-k) + j) = i_j$ for all $1 \leq j \leq k$.

One way to prove (1) is to use the Poincaré — Birkhoff — Witt theorem for Lie algebras over rings [1, Theorem 2.5.3].

Another way is to show this explicitly in the spirit of Specht [14]. Using the equalities $yx = [y, x] + xy$ and $[\ldots, [x, x], \ldots] + [[\ldots, x, \ldots], x]$, we can present every polynomial from $P_n(Z)$ as a direct combination of polynomials $x_{i_1}x_{i_2} \cdots x_{i_k} f$ where $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ and $f$ is a proper multilinear polynomial of degree $(n - k)$ in the variables from the set $\{x_1, x_2, \ldots, x_n\} \setminus \{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\}$. In other words, $f \in \sigma_{i_1, \ldots, i_k} \Gamma_{n-k}(Z)$. In order to check that the sum in (1) is direct, we consider a linear combination of $x_{i_1}x_{i_2} \cdots x_{i_k} \sigma_{i_1, \ldots, i_k} f$ where $f \in \Gamma_{n-k}(Z)$, for different $k$ and $i_j$ and choose the term $g := x_{i_1}x_{i_2} \cdots x_{i_k} \sigma_{i_1, \ldots, i_k} f$ with the greatest $k$ among the terms with a nonzero coefficient. Then we substitute $x_{i_1} = x_{i_2} = \cdots = x_{i_k} = 1$ and $x_j = x_j^2$ for the rest of the variables. (We assume that we are working in the free ring with 1 on the set $X = \{x_1, x_2, \ldots, \}$. All the other terms vanish and we get $f = 0$. Therefore, the sum is direct and (1) holds.

Substituting $x_{i_1} = x_{i_2} = \cdots = x_{i_k} = 1_R$ and arbitrary elements of $R$ for the other $x_j$, we obtain

$$P_n(Z) \cap \text{Id}(R, Z) = (\text{char } R)Zx_1x_2 \cdots x_n \oplus \bigoplus_{k=1}^{n-2} \bigoplus_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} x_{i_1}x_{i_2} \cdots x_{i_k} \sigma_{i_1, \ldots, i_k} (\text{Id}(R, Z) \cap \Gamma_{n-k}(Z)). \quad \text{(2)}$$

Combining (1) and (2), we get

$$\frac{P_n(Z)}{P_n(Z) \cap \text{Id}(R, Z)} \cong \bigoplus_{k=0}^{n} \bigoplus_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \frac{\Gamma_{n-k}(Z)}{\Gamma_{n-k}(Z) \cap \text{Id}(R, Z)}$$

(direct sum of $\mathbb{Z}$-modules) for an arbitrary ring $R$ with the unit $1_R$. (We define $\frac{\Gamma_{n}(Z)}{\Gamma_{n}(Z) \cap \text{Id}(R, Z)} := (1_R)_Z \subseteq R$.) Calculating the number of the components, we obtain Theorem 2. □

Corollary. Let $R$ be a unitary ring. Then all multilinear polynomial identities of $R$ are consequences of proper multilinear polynomial identities of $R$ and the identity $(\text{char } R)x \equiv 0$.

Proof. This follows from (2). □
Corollary. Let $R$ be a unitary ring and let the sequence $(c_n(R,q))_{n=1}^{\infty}$ be polynomially bounded for some $q$. Then $c_n(R,q)$ is a polynomial in $n \in \mathbb{N}$.

Proof. If the sequence $(c_n(R,q))_{n=1}^{\infty}$ is polynomially bounded, then by Theorem 2 there exists $j_0 \in \mathbb{N}$ such that $\gamma_j(R,q) = 0$ for all $j \geq j_0$. Now we apply Theorem 2 once again. \qed

If $H$ is a subgroup of a group $G$ and $M$ is a left $\mathbb{Z}H$-module, then $M \uparrow G := \mathbb{Z}G \otimes_{\mathbb{Z}H} M$. The $G$-action on $\mathbb{Z}G \otimes_{\mathbb{Z}H} M$ is induced as follows: $g_0(g \otimes x) := g_0 g \otimes x$ for $a \in M$, $g, g_0 \in G$.

Now we prove an analog of Drensky’s theorem [3, Theorem 12.5.4]:

Theorem 3. Let $R$ be a unitary ring, $\text{char} R = \ell \in \mathbb{Z}_+$. Consider for every $n \in \mathbb{N}$ the series of $\mathbb{Z}S_n$-submodules

$$M_0 := \frac{P_n(\mathbb{Z})}{P_n(\mathbb{Z}) \cap \text{Id}(R, \mathbb{Z})} \supseteq M_2 \supseteq M_3 \supseteq \cdots \supseteq M_n \cong \frac{\Gamma_n(\mathbb{Z})}{\Gamma_n(\mathbb{Z}) \cap \text{Id}(R, \mathbb{Z})}$$

where each $M_k$ is the image of $\bigoplus_{i=1}^{n-k} \mathbb{Z}S_n(x_1, \ldots, x_{n-i}, \Gamma_i(\mathbb{Z}))$ and $M_{n+1} := 0$. Then $M_0/M_2 \cong \mathbb{Z}_\ell$ (trivial $S_n$-action),

$$M_t/M_{t+1} \cong \left( \frac{\Gamma_t(\mathbb{Z})}{\Gamma_t(\mathbb{Z}) \cap \text{Id}(R, \mathbb{Z})} \otimes_{\mathbb{Z}} \mathbb{Z} \right) \uparrow S_n := \mathbb{Z}S_n \otimes_{\mathbb{Z}(S_t \times S_{n-t})} \left( \frac{\Gamma_t(\mathbb{Z})}{\Gamma_t(\mathbb{Z}) \cap \text{Id}(R, \mathbb{Z})} \otimes_{\mathbb{Z}} \mathbb{Z} \right)$$

for all $2 \leq t \leq n$ where $S_{n-t}$ is permuting $x_{t+1}, \ldots, x_n$ and $\mathbb{Z}$ is a trivial $\mathbb{Z}S_{n-t}$-module.

Proof. First we notice that $M_0/M_2$ is generated by the image of $x_1 x_2 \ldots x_n$. Suppose the image of $kx_1 x_2 \ldots x_n$ belongs to $M_2$ for some $k \in \mathbb{N}$. All the polynomials in $M_2$ vanish under the substitution $x_1 = \cdots = x_n = 1_R$ since each of them contain at least one commutator. Hence we get $k 1_R = 0$, $\ell | k$, and $M_0/M_2 \cong \mathbb{Z}_\ell$.

Note that $\frac{\Gamma_t(\mathbb{Z})}{\Gamma_t(\mathbb{Z}) \cap \text{Id}(R, \mathbb{Z})} \otimes_{\mathbb{Z}} \mathbb{Z} \cong \frac{\Gamma_t(\mathbb{Z})}{\Gamma_t(\mathbb{Z}) \cap \text{Id}(R, \mathbb{Z})}$ where $S_{n-t}$ acts trivially. Consider the bilinear map

$$\varphi: \mathbb{Z}S_n \times \frac{\Gamma_t(\mathbb{Z})}{\Gamma_t(\mathbb{Z}) \cap \text{Id}(R, \mathbb{Z})} \rightarrow M_t/M_{t+1}$$

defined by $\varphi(\sigma, f) = x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(n)}(\sigma f)$ for $\sigma \in S_n$, $f \in \frac{\Gamma_t(\mathbb{Z})}{\Gamma_t(\mathbb{Z}) \cap \text{Id}(R, \mathbb{Z})}$. Note that $\varphi(\sigma \pi, f) = \varphi(\sigma, \pi f)$ for all $\pi \in S_1 \times S_{n-t}$ and $M_t/M_{t+1}$ is generated by all $\varphi(\sigma, f)$ for $\sigma \in S_n$ and $f \in \Gamma_t(\mathbb{Z})$.

Suppose $L$ is an Abelian group and $\psi: \mathbb{Z}S_n \times \frac{\Gamma_t(\mathbb{Z})}{\Gamma_t(\mathbb{Z}) \cap \text{Id}(R, \mathbb{Z})} \rightarrow L$ is a $\mathbb{Z}$-bilinear map and $\psi(\sigma \pi, f) = \psi(\sigma, \pi f)$ for all $\pi \in S_1 \times S_{n-t}$. First we define $\tilde{\psi}: M_t \rightarrow L$ on the elements that generate $M_t$ modulo $M_{t+1}$:

$$\tilde{\psi}(x_1, x_2 \ldots x_{n-t}) = \psi(\sigma, \sigma^{-1} f)$$

where $\sigma \in S_n$ and $\sigma^{-1} f \in \frac{\Gamma_t(\mathbb{Z})}{\Gamma_t(\mathbb{Z}) \cap \text{Id}(R, \mathbb{Z})}$ (e.g. we can take $\sigma(k) = i_k$ for $1 \leq k \leq n-t$). Clearly, $\tilde{\psi}(x_1, x_2 \ldots x_{n-t}, f)$ does not depend on the choice of $\sigma$. Suppose the image $f_0$ of a polynomial

$$f_0 = \sum_{i_1 < \cdots < i_{n-t}} x_{i_1} x_{i_2} \cdots x_{i_{n-t}} f_{i_1, \ldots, i_{n-t}}$$

belongs to $M_{t+1}$ for some $f_{i_1, \ldots, i_{n-t}} \in \Gamma_t(\mathbb{Z})$. Substituting

$x_{i_1} = x_{i_2} = \cdots = x_{i_{n-t}} = 1_R$

and arbitrary values for the other $x_j$, we get zero for every $i_1 < \cdots < i_{n-t}$. Hence $f_{i_1, \ldots, i_{n-t}} \in \text{Id}(R, \mathbb{Z})$ and $\tilde{\psi}(f_0) = 0$. Thus we can define $\tilde{\psi}$ to be zero on $M_{t+1}$ and we may assume that $\tilde{\psi}: M_t/M_{t+1} \rightarrow L$. 

Note that $\bar{\psi} \varphi = \psi$. Hence $M_t/M_{t+1} \cong \mathbb{Z}S_n \otimes_{\mathbb{Z}(S_t \times S_{n-t})} \left( \frac{\mathbb{F}_n(\mathbb{Z})}{\Gamma_n(\mathbb{Z}) \otimes \mathbb{Z} \mathbb{Z}} \right)$ (isomorphism of Abelian groups) where $\varphi(\sigma, f) \mapsto \sigma \otimes f$. Therefore, this is an isomorphism of $\mathbb{Z}S_n$-modules too. \hfill $\square$

4. A PARTICULAR CASE OF THE LITTLEWOOD — RICHARDSON RULE

Let $\mu \vdash n$, $\lambda \vdash n'$, $n' \leq n$. Suppose $\lambda_i \leq \mu_i$ for all $i \in \mathbb{N}$. Denote by $M(\mu)$ the free Abelian group generated by all $\mu$-tabloids. Now we treat $D_\lambda$ as a Young subdiagram in $D_\mu$. Later on we always assume that in a pair $(\lambda; \mu)$ we have $\lambda_1 = \mu_1$.

Following [6, Definition 17.4], we define a $\mathbb{Z}S_n$-submodule $S(\lambda, \mu) \subseteq M(\mu)$ where

$$S(\lambda; \mu) := \langle e^{\lambda, \mu}_{T_\mu} \mid T_\mu \text{ is a tableau of the shape } \mu \rangle_{\mathbb{Z}}$$

and $e^{\lambda, \mu}_{T_\mu} := \sum_{\sigma \in C_{T_\mu}} (\text{sign} \sigma) \sigma[T_\mu]$. Here $T_\mu$ is the subtableau of $T_\mu$ defined by the partition $\lambda$ and $C_{T_\mu} \subseteq S_n$ is the subgroup that leaves the numbers out of $T_\mu$ invariant and puts every number from each column of $T_\mu$ to the same column. By $[T_\mu]$ we denote the tabloid corresponding to $T_\mu$. We assume $S(0; 0) = 0$ for the zero partitions $0 \vdash 0$. Note that $S(\lambda; \lambda) \cong S(\lambda)$. (The proof is completely analogous to the case when the coefficients are taken from a field.)

Let $F$ be a field and let $M^F(\mu)$ be the vector space over $F$ with the formal basis consisting of all $\mu$-tabloids. In other words, $M^F(\mu) = M(\mu) \otimes_F F$. We define $S^F(\lambda; \mu)$ as the subspace in $M^F(\mu)$ generated by $S(\lambda; \mu) \otimes 1$.

**Lemma 2.** Let $\mu \vdash n$, $\lambda \vdash n'$, $n' \leq n$. Suppose $\lambda_i \leq \mu_i$ for all $i \in \mathbb{N}$. Then $M(\mu)/S(\lambda; \mu)$ has no torsion.

**Proof.** Recall that $M(\mu)$ is a finitely generated free Abelian group and $S(\lambda; \mu)$ is its subgroup. Hence we can choose a basis $a_1, a_2, \ldots, a_i$ in $M(\mu)$ such that $m_1a_1, m_2a_2, \ldots, m_ka_k$ is a basis of $S(\lambda; \mu)$ for some $m_i \in \mathbb{N}$. We claim that all $m_i = 1$. First, we notice that $a_1 \otimes 1, a_2 \otimes 1, \ldots, a_i \otimes 1$ form a basis of $M^F(\mu)$ and $m_1a_1 \otimes 1, m_2a_2 \otimes 1, \ldots, m_ka_k \otimes 1$ generate $S^F(\lambda; \mu)$ for any field $F$. Thus $\dim_F S^F(\lambda; \mu) = k$ for char $F = 0$ and $\dim_F S^F(\lambda; \mu) < k$ if char $F \mid m_i$ for at least one $m_i$. However, by [6, Theorem 17.13 (III)], $\dim_F S^F(\lambda; \mu)$ does not depend on the field $F$. Therefore all $m_i = 1$ and $M(\mu)/S(\lambda; \mu)$ is a free Abelian group. \hfill $\square$

Let $c \geq 2$ be a natural number satisfying the following conditions: $\mu_{c-1} = \lambda_{c-1}$ and $\mu_c > \lambda_c$. Then we define the operators $A_c$ (“adding”) and $R_c$ (“raising”) in the following way:

1. if $\lambda_c = \lambda_{c-1}$, then $A_c(\lambda; \mu) = (0; 0)$ where $0 \vdash 0$ is a zero partition, otherwise $A_c(\lambda; \mu) = (\lambda; \mu)$ where $\lambda_i = \mu_i$ for $i \neq c$ and $\lambda_c = \lambda_c + 1$;

2. $R_c(\lambda; \mu) = (\lambda; \mu)$ where $\mu_i = \mu_i$ for $i \neq c - 1, c$; $\mu_c = \lambda_c$, $\mu_{c-1} = \mu_{c-1} + (\mu_c - \lambda_c)$, $\lambda_1 = \mu_1$ and $\lambda_i = \mu_i$ for $i > 1$.

Fix $i \in \mathbb{N}$ and $0 \leq v \leq m_{i+1}$. Let $\nu_k \equiv \nu_{k+1} \equiv \nu_{k+2} \equiv \cdots \equiv \nu_{k+j} = m_k$ for $j \neq i$, $\nu_{i+1} = \mu_i + \mu_{i+1} - v$, $\nu_{i+1} = v$. Then we define $\psi_{c,v} \in \text{Hom}_{\mathbb{Z}S_n}(M(\mu), M(\nu))$ in the following way: $\psi_{c,v}[T_\mu] = \sum [T_\nu]$ where the summation runs over the set of all tabloids $[T_\nu]$ such that $[T_\nu]$ agrees with $[T_\mu]$ in all the rows except the $i$th and the $(i+1)$th, and the $(i+1)$th row is a subset of size $v$ of the $(i+1)$th row in $[T_\mu]$. Analogously, we define $\psi_{c,v}^F \in \text{Hom}_{\mathbb{F}S_n}(M^F(\mu), M^F(\nu))$ for any field $F$.

**Lemma 3.**

1. $\psi_{c-1,\lambda_c} S(\lambda; \mu) = S(R_c(\lambda; \mu));$

2. $\ker \psi_{c-1,\lambda_c} \cap S(\lambda; \mu) = S(A_c(\lambda; \mu)).$

**Proof.** The proof of the first part of the lemma and of the embedding $\ker \psi_{c-1,\lambda_c} \supseteq S(\lambda; \lambda) \otimes 1 \subset M^Q(\lambda)$. By [6, Theorem 17.13], $\ker \psi_{c-1,\lambda_c} \cap S(\lambda; \mu) = \psi_{c-1,\lambda_c} \cap S(\lambda; \mu) = \psi_{c-1,\lambda_c} \cap S(\lambda; \lambda)$.
where $S(\lambda; \mu)$. Thus if $\psi_{c-1,\lambda} a = 0$ for some $a \in S(\lambda; \mu)$, then $ma \in S(A_c(\lambda; \mu))$ for some $m \in \mathbb{N}$ and $a \in S(A_c(\lambda; \mu))$ since $M(\mu)/S(A_c(\lambda; \mu))$ is torsion-free by Lemma 2.

\section*{Lemma 4} Let $n \in \mathbb{N}$, $\lambda \vdash n'$, $\mu \preceq n$, $n' \leq n$, $\lambda_i \leq \mu_i$ for all $i \in \mathbb{N}$. Then $S(\lambda; \mu)$ has a chain of submodules

$$S(\lambda; \mu) = M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots \supseteq M_t = 0$$

with factors $M_i/M_{i+1}$ isomorphic to Specht modules. Moreover, $S(\lambda; \mu)/M_i$ is torsion-free for any $i$.

\begin{proof}
If $\mu = \lambda$, then $S(\lambda; \mu) = S(\lambda)$ and there is nothing to prove. If $\mu \neq \lambda$, then we find $c \in \mathbb{N}$ such that $\lambda_i = \mu_i$ for all $1 \leq i \leq c - 1$ and $\lambda_c < \mu_c$. Since we always assume $\lambda_1 = \mu_1$, we have $c \geq 2$. Now we apply Lemma 3. Note that $\lambda_c > \lambda_c$ where $A_c(\lambda; \mu) = (\lambda; \hat{\mu})$ and $R_c$ moves the boxes of $D_\mu$ upper. Applying Lemma 3 many times, we get the first part of Lemma 4 by induction.

Suppose $S(\lambda; \mu)/M_i$ is not torsion-free and $ma \in M_i$ for some $a \in S(\lambda; \mu)$, $a \notin M_i$, and $m \in \mathbb{N}$. Then we can find an index $0 \leq k < i$ such that $a \in M_k$, $a \notin M_{k+1}$. However $ma \in M_i \subseteq M_{k+1}$. i.e., the Specht module $M_k/M_{k+1}$ is not torsion-free either. We get a contradiction since all Specht modules are subgroups in finitely generated free Abelian groups.
\end{proof}

Now we can prove the $\mathbb{Z}$-analog of the particular case of the Littlewood — Richardson rule that sometimes is referred to as Young’s rule [5, Theorem 2.3.3], [3, Theorem 12.5.2] and sometimes as Pieri’s formula [3, (A.7)].

\section*{Theorem 4} Let $t, n \in \mathbb{N}$, $m \in \mathbb{Z}_+$, $t < n$, and $\lambda \vdash t$ and let $\mathbb{Z}$ be the trivial $\mathbb{Z}S_{n-1}$-module. Then

$$(S(\lambda)/mS(\lambda)) \uparrow S_n := \mathbb{Z}S_n \otimes_{\mathbb{Z}(S_t \times S_{n-t})} ((S(\lambda)/mS(\lambda)) \otimes \mathbb{Z})$$

has a series of submodules with factors $S(\nu)/mS(\nu)$ where $\nu$ runs over the set of all partitions $\nu \vdash n$ such that

$$\lambda_n \leq \nu_n \leq \lambda_{n-1} \leq \nu_{n-1} \leq \cdots \leq \lambda_2 \leq \nu_2 \leq \lambda_1 \leq \nu_1.$$ 

(Each factor occurs exactly once.)

\begin{proof}
Suppose $\lambda = (\lambda_1, \ldots, \lambda_s)$, $\lambda_s > 0$. Then $S(\lambda) \uparrow S_n \cong S(\lambda; \mu)$ where $\mu = (\lambda_1, \ldots, \lambda_s, n-t)$. Now Lemma 3 implies the theorem for $m = 0$.

Suppose $m > 0$. Then $(S(\lambda)/mS(\lambda)) \uparrow S_n \cong (S(\lambda) \uparrow S_n)/(mS(\lambda) \uparrow S_n))$. Let

$$S(\lambda) \uparrow S_n = M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots \supseteq M_t = 0$$

where $M_{i-1}/M_i \cong S(\lambda^{(i)})$, $\lambda^{(i)} \vdash n$, $1 \leq i \leq t$.

Hence

$$(S(\lambda \uparrow S_n)/(m(S(\lambda) \uparrow S_n)) = \overline{M_0} \supseteq \overline{M_1} \supseteq \overline{M_2} \supseteq \cdots \supseteq \overline{M_t} = 0$$

where $\overline{M_i} \cong (M_i + m(S(\lambda) \uparrow S_n))/m(S(\lambda) \uparrow S_n)$ and

$$\overline{M_{i-1}/M_i} \cong (M_{i-1} + m(S(\lambda) \uparrow S_n))/(M_i + m(S(\lambda) \uparrow S_n)) \cong \overline{M_{i-1}/M_i} \cap (M_i + m(S(\lambda) \uparrow S_n)) = M_{i-1}/(M_i + M_{i-1} \cap m(S(\lambda) \uparrow S_n))) \cong (M_{i-1}/M_i)/((M_i + M_{i-1} \cap m(S(\lambda) \uparrow S_n))/M_i).

By Lemma 4 $(S(\lambda) \uparrow S_n)/M_{i-1}$ is torsion-free. Hence $M_{i-1} \cap m(S(\lambda) \uparrow S_n) = mM_{i-1}$ and

$$\overline{M_{i-1}/M_i} \cong (M_{i-1}/M_i)/((M_i + mM_{i-1})/M_i) = (M_{i-1}/M_i)/(m(M_i/M_i)) \cong S(\lambda^{(i)})/mS(\lambda^{(i)}).$$

The description of $\lambda^{(i)}$ is obtained from the proof of Lemma 4.
\end{proof}
5. Algebras of upper triangular matrices

5.1. Codimensions and multilinear identities. Let $M$ be an $(R_1, R_2)$-bimodule for commutative rings $R_1$, $R_2$ with 1 and let $R = \left( \begin{array}{cc} R_1 & M \\ 0 & R_2 \end{array} \right)$.

In this section, we calculate $c_n(R, q)$ for all $q = p^k$ and $q = 0$, describe the structure of the $\mathbb{Z}S_n$-module $P_n(\mathbb{Z})/\text{Id}(R, \mathbb{Z})$ and find such multilinear polynomials that elements of $\text{Id}(R, \mathbb{Z}) \cap P_n(\mathbb{Z})$ are consequences of them.

Remark. If $F$ is a field of characteristic 0 and $A = \text{UT}_2(F) := \left( \begin{array}{cc} F & F \\ 0 & F \end{array} \right)$, then $c_n(A, F)$ and generators of $\text{Id}(A, F)$ as a $T$-ideal can be found, e.g., in [3, Theorem 4.1.5]. The structure of the $FS_n$-module $P_n(\mathbb{Z})/\text{Id}(A, F)$ can be determined using proper cocharacters [3, Theorem 12.5.4].

**Theorem 5.** All polynomials from $P_n(\mathbb{Z}) \cap \text{Id}(R, \mathbb{Z})$, $n \in \mathbb{N}$, are consequences of the left hand sides of the following polynomial identities in $R$:

\[ [x, y][z, t] \equiv 0, \quad \ell x \equiv 0, \quad m[x, y] = 0 \]

where $[x, y] := xy - yx$, \[ \ell := \min \{ n \in \mathbb{N} \mid na = 0 \text{ for all } a \in R_1 \cup R_2 \}, \]

\[ m := \min \{ n \in \mathbb{N} \mid na = 0 \text{ for all } a \in M \}. \]

(If one of the corresponding sets is empty, we define $\ell = 0$ or $m = 0$, respectively. Note that $m \mid \ell$.)

Moreover, $P_n(\mathbb{Z})/\text{Id}(R, \mathbb{Z}) \cong \mathbb{Z}_\ell \oplus (\mathbb{Z}_m)^{(n-2)2^{n-1}+1}$ where $\mathbb{Z}_0 := \mathbb{Z}$.

Remark. Now $c_n(R, q)$ can be easily computed. If $R_1 = R_2 = M$ and $R_1 = R_2$ is a field, we obtain the same numbers as in [3, Theorem 4.1.5].

**Proof of Theorem 5.** Denote by $e_{ij}$ the matrix units. Then $R = R_1e_{11} \oplus R_2e_{22} \oplus Me_{12}$ (direct sum of subspaces), $[R, R] \subseteq Me_{12}$, and (3)–(5) are indeed polynomial identities of $R$.

Now we consider an arbitrary monomial from $P_n(\mathbb{Z})$ and find the first inversion among the indexes of its variables. We replace the corresponding pair of variables with the sum of their commutator and their product in the right order. Note that $[x, y][u][z, t] = [x, y][z, t][u] + [x, y][u, [z, t]] \equiv 0$ is a consequence of (3). Therefore, we may assume that all the variables to the right of the commutator have increasing indexes. For example:

\[ x_3x_1x_4x_2 = x_1x_3x_4x_2 + [x_3, x_1]x_4x_2 \]

\[ = x_1x_3x_2x_4 + x_1x_3[x_4, x_2] + [x_3, x_1]x_2x_4 \]

\[ = x_1x_2x_3x_4 + x_1[x_3, x_2]x_4 + x_1x_3[x_4, x_2] + [x_3, x_1]x_2x_4. \]

Continuing this procedure, we present any element of $P_n(\mathbb{Z})$ modulo the consequences of (3) as a linear combination of polynomials $f_0 := x_1x_2 \ldots x_n$ and

\[ x_{i_1} \ldots x_{i_k} [x_{s}, x_{r}]x_{j_1} \ldots x_{j_{n-k-2}} \text{ for } i_1 < \cdots < i_k < s, \quad r < s, \quad j_1 < \cdots < j_{n-k-2}. \]

(6)

Denote the set of polynomials (6) by $\Xi$.

Consider the free Abelian group $\mathbb{Z}(\Xi \cup \{ f_0 \})$ with the basis $\Xi \cup \{ f_0 \}$. Now we have the surjective homomorphism $\varphi: \mathbb{Z}(\Xi \cup \{ f_0 \}) \to P_n(\mathbb{Z})/\text{Id}(R, \mathbb{Z})$ where $\varphi(f)$ is the image of
Let $f \in \Xi \cup \{f_0\}$ in $\frac{P_n(Z)}{\text{Id}(R, Z)}$. We claim that $\ker \varphi$ is generated by $\ell f_0$ and all $mf$ where $f \in \Xi$.

Suppose that a linear combination $f_1$ of $f_0$ and elements from $\Xi$ is a polynomial identity, however $f_1$ is not a linear combination of $\ell f_0$ and $mf$, $f \in \Xi$. If we substitute

$$x_1 = x_2 = \cdots = x_n = 1_R e_{ii} \text{ where } i \in \{1, 2\},$$

all $f \in \Xi$ vanish. Therefore, the coefficient of $f_0$ is a multiple of $\ell$. Now we find $f_2 := x_{i_1} \cdots x_{i_k}[x_{s_1}, x_{r_1}]x_{j_1} \cdots x_{j_{n-k-2}} \in \Xi$ with the greatest $k$ such that the coefficient $\beta$ of $f_2$ in $f_1$ is not a multiple of $m$. Then we substitute $x_{i} = \cdots = x_{i_k} = x_s = 1_R e_{11}$, $x_r = ae_{12}$, $x_{j_1} = \cdots = x_{j_{n-k-2}} = 1_R e_{11} + 1_R e_{22} = 1_R$ where $a \in M$ and $\beta a \neq 0$. Our choice of $f_2$ implies that $f_2$ is the only summand in $f_1$ that could be nonzero under this substitution. Hence $f_1$ does not vanish and we get a contradiction. Therefore, $\ker \varphi$ is generated by $\ell f_0$ and $mf$, $f \in \Xi$. In particular, $\frac{P_n(Z)}{\text{Id}(R, Z)} \cong \mathbb{Z}_\ell \oplus (\mathbb{Z}_m)^{|\Xi|}$ and every multilinear polynomial identity of $R$ is a consequence of (3)–(5).

Note that

$$|\Xi| = \sum_{k=2}^{n} (\begin{pmatrix} n \\ k \end{pmatrix}) = \sum_{k=2}^{n} \frac{n!}{(k-1)! (n-k)!} - \sum_{k=2}^{n} \frac{n}{k} \frac{(n-1)!}{k!(n-k-1)!} - (2^n - n - 1) = n(2^{n-1} - 1) - (2^n - n - 1) = (n-2)2^{n-1} + 1$$

and the theorem follows.

**Corollary.** Multilinear polynomial identities of $\text{UT}_2(\mathbb{Q})$ as a ring are generated by (3).

5.2. $\mathbb{Z}S_n$-modules. Note that the Jacobi identity and (3) imply that $\frac{\Gamma_n(Z)}{\text{Id}(\text{UT}_2(\mathbb{Q}), Z)}$ is generated as a $\mathbb{Z}$-module by $[x_1, x_n, x_1, x_2, \ldots, \hat{x}_i, \ldots, x_{n-1}]$ where $1 \leq i \leq n-1$.

**Lemma 5.** Let $R$ be the ring from Subsection 5.1 and $T_\lambda = \begin{array}{c} 1 \newline n \newline 2 \newline \vdots \newline n-1 \end{array}$. Then

$$b_{T_\lambda} a_{T_\lambda} [x_1, x_n, x_2, x_3, \ldots, x_{n-1}] \equiv n(n-2)! [x_1, x_n, x_2, x_3, \ldots, x_{n-1}] \text{ (mod } P_n(Z) \cap \text{Id}(R)). \text{ (7)}$$

**Proof.** Indeed,

$$b_{T_\lambda} a_{T_\lambda} [x_1, x_n, x_2, x_3, \ldots, x_{n-1}] \equiv b_{T_\lambda} (n-2)! \sum_{i=1}^{n-1} [x_1, x_n, x_1, x_2, \ldots, \hat{x}_i, \ldots, x_{n-1}] =$$

$$(n-2)! \sum_{i=2}^{n-1} ([x_1, x_n, x_1, x_2, \ldots, \hat{x}_i, \ldots, x_{n-1}] - [x_1, x_n, x_2, \ldots, \hat{x}_i, \ldots, x_{n-1}]) + 2(n-2)! [x_1, x_n, x_2, x_3, \ldots, x_{n-1}] \equiv n(n-2)! [x_1, x_n, x_2, x_3, \ldots, x_{n-1}]$$

since, by the Jacobi identity, $[x_1, x_1, x_n] = [x_1, x_n, x_1] + [x_n, x_1, x_1]$.

First, we determine the structure of $\frac{\Gamma_n(Z)}{\text{Id}(\text{UT}_2(\mathbb{Q}), Z)}$ for $R = \text{UT}_2(\mathbb{Q})$.

**Lemma 6.** Let $T_\lambda = \begin{array}{c} 1 \newline n \newline 2 \newline \vdots \newline n-1 \end{array}$. Then $\frac{\Gamma_n(Z)}{\text{Id}(\text{UT}_2(\mathbb{Q}), Z)} \cong (\mathbb{Z}S_n) b_{T_\lambda} a_{T_\lambda}$.

**Proof.** We claim that if $ub_{T_\lambda} a_{T_\lambda} = 0$ for some $u \in \mathbb{Z}S_n$, then

$$u[x_1, x_n, x_2, x_3, \ldots, x_{n-1}] \in \Gamma_n(Z) \cap \text{Id}(\text{UT}_2(\mathbb{Q}), Z).$$

Indeed, by (4),

$$n(n-2)! u[x_1, x_n, x_2, x_3, \ldots, x_{n-1}] \equiv ub_{T_\lambda} a_{T_\lambda} [x_1, x_n, x_2, x_3, \ldots, x_{n-1}] = 0.$$
Since $U_{T_2}(Q)$ has no torsion, \( u[x_1, x_n, x_2, x_3, \ldots, x_{n-1}] \equiv 0 \) is a polynomial identity of $U_{T_2}(Q)$.

Thus we can define the surjective homomorphism $\varphi: (ZS_n)_{bT_2}a_{T_2} \rightarrow \Gamma_n(Z)$ by $\varphi(\sigma bT_2a_{T_2}) = \sigma[x_1, x_n, x_2, x_3, \ldots, x_{n-1}]$ for $\sigma \in S_n$.

Analogously, we can define the surjective homomorphism

$$\varphi_0: (QS_n)_{bT_2}a_{T_2} \rightarrow \Gamma_n(Q)\cap Id(U_{T_2}(Q), Q)$$

by $\varphi_0(\sigma bT_2a_{T_2}) = \sigma[x_1, x_n, x_2, x_3, \ldots, x_{n-1}]$ for $\sigma \in S_n$. Since $(QS_n)_{bT_2}a_{T_2}$ is an irreducible $QS_n$-module, $\varphi_0$ is an isomorphism of $QS_n$-modules. We claim that $\varphi$ is an isomorphism of $ZS_n$-modules.

Indeed, suppose

$$u[x_1, x_n, x_2, x_3, \ldots, x_{n-1}] \in \Gamma_n(Z)\cap Id(U_{T_2}(Q), Z)$$

for some $u \in ZS_n$. Then $\varphi_0(u^bT_2a_{T_2}) = u[x_1, x_n, x_2, x_3, \ldots, x_{n-1}] \in \Gamma_n(Z)\cap Id(U_{T_2}(Q), Z)$ and $u^bT_2a_{T_2} = 0$. Hence $\varphi$ is an isomorphism and the lemma is proven. \(\square\)

**Theorem 6.** Let $R$ and $m$ be, respectively, the ring and the number from Subsection 6.1. Then \(\frac{\Gamma_n(Z)}{\Gamma_n(Z)\cap Id(U_{T_2}(Q), Z)}\) $\cong S(\lambda)/mS(\lambda)$ where $\lambda = (n - 1, 1)$, for all $n \geq 2$.

**Proof.** Recall that \(\frac{\Gamma_n(Z)}{\Gamma_n(Z)\cap Id(U_{T_2}(Q), Z)}\) is generated as a $Z$-module by $[x_1, x_n, x_2, x_3, \ldots, x_{n-1}]$ where $1 \leq i \leq n - 1$. We exploit the same trick as in the proof of Theorem 5. Using the substitution $x_1 = \cdots = x_{i-1} = x_{i+1} = \cdots = x_n = 1_{R_1}e_{i1}, x_i = ae_{i2}$ where $a \in M$, we obtain that $\frac{\Gamma_n(Z)}{\Gamma_n(Z)\cap Id(U_{T_2}(Q), Z)}$ is the direct sum of $n - 1$ cyclic groups isomorphic to $Z_m$ and generated by $[x_1, x_n, x_2, x_3, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n-1}]$ where $1 \leq i \leq n - 1$.

By Theorem 5 and its corollary, we have the natural surjective homomorphism $\frac{\Gamma_n(Z)}{\Gamma_n(Z)\cap Id(U_{T_2}(Q), Z)} \rightarrow \frac{\Gamma_n(Z)}{\Gamma_n(Z)\cap Id(U_{T_2}(Q), Z)}$. The remarks above imply that the kernel equals $m^{\frac{\Gamma_n(Z)}{\Gamma_n(Z)\cap Id(U_{T_2}(Q), Z)}}$. Now the theorem follows from Lemma 6. \(\square\)

Applying Theorems 3, 4, and 6 we immediately get

**Theorem 7.** Let $R$, $\ell$, and $m$ be, respectively, the ring and the numbers from Subsection 6.1. Then there exists a chain of $ZS_n$-submodules in $\frac{\Gamma_n(Z)}{\Gamma_n(Z)\cap Id(U_{T_2}(Q), Z)}$ with the set of factors that consists of one copy of $Z_\ell$ and $(\lambda_1 - \lambda_2 + 1)$ copies of $S(\lambda_1, \lambda_2, \lambda_3)/mS(\lambda_1, \lambda_2, \lambda_3)$ where $(\lambda_1, \lambda_2, \lambda_3) \mid n, \lambda_2 \geq 1, \lambda_3 \in \{0, 1\}$.

6. Grassmann algebras

Let $R$ be a commutative ring with a unit element $1_R$, char $R = \ell$ where either $\ell$ is an odd natural number or $\ell = 0$. We define the Grassman algebra $G_R$ over a ring $R$ as the $R$-algebra with a unit, generated by the countable set of generators $e_i, i \in \mathbb{N}$, and the anti-commutative relations $e_i e_j = -e_j e_i, i, j \in \mathbb{N}$. Here we consider the same questions as for the upper triangular matrices.

6.1. Codimensions and polynomial identities. This lemma is known but we provide its proof for the reader’s convenience.

**Lemma 7.** The polynomial identity $[y, x][z, t] + [y, z][x, t] \equiv 0$ is a consequence of $[x_1, x_2, x_3] \equiv 0$. In particular, $[x, y]u[z, t] + [x, t]u[z, y] \equiv 0$ for all $u \in \mathbb{Z}(X)$.

**Proof.** Note that

$$[x, y, t, z] = [[x, y][t, z] + [y][x, t], z] = [x, y][t, z] + [y][x, t] + [y, z][x, t] + [y, z][x, t] \equiv 0$$

(8)
modulo \([x_1, x_2, x_3] \equiv 0\). (Here we have used Jacobi’s identity too.) Hence
\[
[x, y]u[z, t] + [x, t]u[z, y] = [x, y][u, [z, t]] + [x, y][z, t]u +
[x, t][u, [z, y]] + [x, t][z, y]u \equiv [x, y][z, t]u + [x, t][z, y]u \equiv 0.
\] (9)

\[\square\]

**Theorem 8.** All polynomials from \(P_n(\mathbb{Z}) \cap \text{Id}(G_R, \mathbb{Z})\), \(n \in \mathbb{N}\), are consequences of the left hand sides of the following polynomial identities in \(R\):
\[
[x, y, z] \equiv 0, \tag{10}
\]
\[
\ell x \equiv 0. \tag{11}
\]

Moreover, \(\frac{P_n(\mathbb{Z})}{P_n(\mathbb{Z}) \cap \text{Id}(G_R, \mathbb{Z})} \cong (\mathbb{Z}_\ell)^{2^n-1}\).

**Proof.** Define \(G_R^{(0)} = \langle e_i e_{i_2} \ldots e_{i_{2k}} \mid k \in \mathbb{Z}_+ \rangle R\) and \(G_R^{(1)} = \langle e_1 e_2 \ldots e_{i_{2k+1}} \mid k \in \mathbb{Z}_+ \rangle R\).

Clearly, \(G_R = G_R^{(0)} \oplus G_R^{(1)}\) (direct sum of \(R\)-submodules), \([G_R, G_R] \subseteq G_R^{(0)}, G_R^{(0)} = Z(G_R)\).

Hence \([x_1, x_2, x_3] \equiv 0\) is a polynomial identity. Obviously, (11) is a polynomial identity too.

Let \(\Xi = \{x_{i_1} \ldots x_{i_k} [x_{j_1}, x_{j_2}] \ldots [x_{j_{2m-1}}, x_{j_{2m}}] \mid i_1 < \cdots < i_k, j_1 < \cdots < j_{2m}, k + 2m = n, k, m \in \mathbb{Z}_+ \} \subset P_n(\mathbb{Z})\).

By Lemma 7, every polynomial from \(P_n(\mathbb{Z})\) can be presented modulo (10) as a linear combination of polynomials from \(\Xi\). For example,
\[
x_3 x_2 x_4 x_1 = -[x_2, x_3] x_4 x_1 + x_2 x_3 x_4 x_1 =
\]
\[
([x_2, x_3][x_1, x_4] - [x_2, x_3] x_1 x_4] + (x_2 x_3 x_1 x_4 - x_2 x_3 [x_1, x_4]) \equiv
\]
\[
-x_2 x_1 [x_3, x_4] - x_1 x_4 [x_2, x_3] + x_2 x_1 x_3 x_4 - x_2 [x_1, x_3] x_4 - x_2 x_3 [x_1, x_4] \equiv
\]
\[
[x_1, x_2] [x_3, x_4] - x_1 x_4 [x_2, x_3] + x_1 x_2 x_3 x_4 - [x_1, x_2] x_3 x_4 - x_2 x_4 [x_1, x_3] - x_2 x_3 [x_1, x_4] \equiv
\]
\[
[x_1, x_2] [x_3, x_4] - x_1 x_4 [x_2, x_3] + x_1 x_2 x_3 x_4 - x_3 x_4 [x_1, x_2] - x_2 x_4 [x_1, x_3] - x_2 x_3 [x_1, x_4].
\]

Consider the free Abelian group \(\Xi\Xi\) with the basis \(\Xi\). Now we have the surjective homomorphism \(\varphi: \Xi\Xi \rightarrow \frac{P_n(\mathbb{Z})}{P_n(\mathbb{Z}) \cap \text{Id}(G_R, \mathbb{Z})}\) where \(\varphi(f)\) is the image of \(f \in \Xi\) in \(\frac{P_n(\mathbb{Z})}{P_n(\mathbb{Z}) \cap \text{Id}(G_R, \mathbb{Z})}\). We claim that \(\ker \varphi\) is generated by \(\ell f\) where \(f \in \Xi\).

Suppose that a linear combination \(f_1\) of elements from \(\Xi\) is a polynomial identity, however \(f_1\) is not a linear combination of \(\ell f\), \(f \in \Xi\). Now we find
\[
f_2 := x_{i_1} \ldots x_{i_k} [x_{j_1}, x_{j_2}] \ldots [x_{j_{2m-1}}, x_{j_{2m}}] \in \Xi
\]
with the greatest \(k\) such that the coefficient \(\beta\) of \(f_2\) in \(f_1\) is not a multiple of \(\ell\). Then we substitute \(x_{i_1} = \cdots = x_{i_k} = 1\) in \(G_R\), \(x_{j_i} = e_i\), \(1 \leq i \leq 2m\). Our choice of \(f_2\) implies that \(f_2\) is the only summand in \(f_1\) that could be nonzero under this substitution. Hence the value of \(f_1\) equals \((2^m \beta) 1_R e_1 e_2 \ldots e_m = 0\). However, \(G_R\) is a free \(R\)-module and \(e_1 e_2 \ldots e_m\) is one of its basis elements. Therefore \(2^m \beta 1_R = 0\), \(\ell \mid (2^m \beta)\) and \(\ell \mid \beta\) since \(2 \nmid \ell\). We get a contradiction.

Thus \(\ker \varphi\) is generated by \(\ell f\), \(f \in \Xi\). In particular, \(\frac{P_n(\mathbb{Z})}{P_n(\mathbb{Z}) \cap \text{Id}(G_R, \mathbb{Z})} \cong (\mathbb{Z}_\ell)^{\Xi}\) and every multilinear polynomial identity of \(G_R\) is a consequence of (10) and (11).

We now calculate \(|\Xi|\). The number of these polynomials equals the number of choices of \(x_{i_1}, \ldots, x_{i_k}\). If \(n\) is odd, this number equals \(\binom{n}{1} + \binom{n}{3} + \cdots + \binom{n}{n}\). If \(n\) is even, the number equals \(\binom{n}{0} + \binom{n}{2} + \cdots + \binom{n}{n}\). But the both are equal to \(2^{n-1}\). Indeed, denote \(s_0 = \sum_{i \text{ even}} \binom{n}{i}\) and \(s_1 = \sum_{i \text{ odd}} \binom{n}{i}\). Then \(2^n = (1 + 1)^n = s_0 + s_1\) and \(0 = (1 - 1)^n = s_0 - s_1\). So \(|\Xi| = s_0 = s_1 = 2^{n-1}\). \(\square\)
6.2. $\mathbb{Z}S_n$-modules. First we determine the structure of $\mathbb{Z}S_n$-modules of proper polynomial functions.

**Theorem 9.** Let $G_R$ be the Grassmann algebra over $R$. Let $\lambda = (1^{2m})$ and $T_\lambda = \begin{bmatrix} 1 \\ 2 \\ \vdots \\ 2m \end{bmatrix}$. Then

$$\frac{\Gamma_{2m}(Z)}{\Gamma_{2m}(Z)\cap \text{Id}(G_R;Z)} \cong S(\lambda)/\ell S(\lambda) \text{ for all } m \in \mathbb{N}, \text{ where } \ell = \text{char } R,$$

and

$$\frac{\Gamma_{2m+1}(Z)}{\Gamma_{2m+1}(Z)\cap \text{Id}(G_R;Z)} = 0 \text{ for all } m \in \mathbb{Z}_+.$$

**Proof.** Note that $\Gamma(\lambda)$ is a free cyclic group generated by $bT_\lambda[T_\lambda]$ and $\sigma bT_\lambda[T_\lambda] = (\text{sign } \sigma)bT_\lambda[T_\lambda]$ for all $\sigma \in S_n$.

The proof of Theorem 8 implies that $\frac{\Gamma_{2m}(Z)}{\Gamma_{2m}(Z)\cap \text{Id}(G_R;Z)} \cong \mathbb{Z}_+$ is a cyclic group generated by $[x_1,x_2]\ldots[x_{2m-1},x_{2m}]$. By Lemma 8,

$$\sigma[x_1,x_2]\ldots[x_{2m-1},x_{2m}] = (\text{sign } \sigma)[x_1,x_2]\ldots[x_{2m-1},x_{2m}] \text{ for all } \sigma \in S_n.$$

Hence $\frac{\Gamma_{2m}(Z)}{\Gamma_{2m}(Z)\cap \text{Id}(G_R;Z)} \cong S(\lambda)/\ell S(\lambda)$. The first assertion is proved. The second assertion is evident since every long commutator of length greater than 2 is a polynomial identity of $G_R$. \hfill $\square$

**Theorem 10.** Let $G_R$ be the Grassmann algebra over the $R$. Then there exists a chain of $\mathbb{Z}S_n$-submodules in $\frac{\Gamma_{2m}(Z)}{\Gamma_{2m}(Z)\cap \text{Id}(G_R;Z)}$ with factors $S(n-k,1^k)/\ell S(n-k,1^k)$ for each $0 \leq k \leq n-1$ (each factor occurs exactly once) where $\ell = \text{char } R$.

**Proof.** Now we apply Theorems 8 and 9. By Theorem 8, a diagram consisting of a single column can generate only diagrams $D_{(n-k,1^k)}$. Since we have diagrams of an even length only, each factor occurs only once. \hfill $\square$

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