Optimal control of population transfer in Markovian open quantum systems

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Abstract

There has long been interest to control the transfer of population between specified quantum states. Recent work has optimized the control law for closed system population transfer by using a gradient ascent pulse engineering algorithm \cite{1}. Here, a spin-boson model consisting of two-level atoms which interact with the dissipative environment, is investigated. With optimal control, the quantum system can invert the populations of the quantum logic states. The temperature plays an important role in controlling population transfer. At low temperatures the control has active performance, while at high temperatures it has less effect. We also analyze the decoherence behavior of open quantum systems with optimal population transfer control, and we find that these controls can prolong the coherence time. We hope that active optimal control can help quantum solid-state-based engineering.

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1. Introduction

There exists a widespread belief that future technologies will employ typical quantum features of microscopic systems, like entanglement or superpositions of quantum states. The most prominent examples are the rapid development of quantum information theory and computation theory \cite{2,3,4}, including realizing high-speed quantum computation, and high-security quan-
tum communication. Both quantum computation and quantum information are the study of information processing tasks that can be accomplished in quantum mechanical systems. In quantum physics, a system composed of two or more quantum subsystems can be in a state that has no classical counterpart. It is in a superposition of two or more quantum states with a precise phase of the whole system. This property is quantum coherence, which can show entanglement. Control of quantum coherence and entanglement is of great importance [5, 6, 7, 8, 9, 10, 11, 12, 13]. The control of closed quantum systems is well established [16]. Efforts to extend these studies to open quantum systems [17], where the systems of interest interact with their surrounding environments, are now underway. The aim of dynamical control in open quantum systems is to suppress effects of the environment in order to preserve the quantum properties, including quantum coherence and entanglement, etc. The Zeno effect has been used to achieve certain level of control on quantum systems, inhibiting the decay of excited states due to many repeated measurements. This form of quantum control has been studied in several works, including [10, 18, 19, 20].

Superconducting qubits [21] have also been studied as ways to control and interact with naturally formed quantum two-level systems in superconducting circuits. The two-level systems naturally occurring in Josephson junctions constitute a major obstacle for the operation of superconducting phase qubits. Since these two-level systems can possess remarkably long decoherence times, References [22, 23, 24] showed that such two-level systems can themselves be used as qubits, allowing for a well controlled initialization, universal sets of quantum gates, and readout. Thus, a single current-biased Josephson junction can be considered as a multi-qubit register. It can be coupled to other junctions to allow the application of quantum gates to an arbitrary pair of qubits in the system. These results [22, 23, 24] indicate an alternative way to control qubits coupled to naturally formed quantum two-level systems, for improved superconducting quantum information processing. Indeed, these predictions have been found experimentally in [25]. More recently, reference [26] applies quantum control techniques to control a large spin chain by only acting on two qubits at one of its ends, thereby implementing universal quantum computation by a combination of quantum gates on the latter and swap operations across the chain. They [26] show that the control sequences can be computed and implemented efficiently. Moreover, they discuss the application of these ideas to physical systems such as superconducting qubits in which full control of long chains is challenging.
Considerable experimental and theoretical attentions have been paid in this field, especially using superconducting quantum circuits. Reference [28] analyzes the optical selection rules of the microwave-assisted transitions in a flux qubit superconducting quantum circuit. They [28] show that the parities of the states relevant to the superconducting phase in the charge qubit can be controlled via the external magnetic flux. For certain values of the flux, the selection rules are the same as the ones for the electric-dipole transitions in usual atoms. In other cases, the symmetry of the potential of the artificial “atom” is broken, a so-called \( \Delta \)-type “cyclic” three-level atom is formed, where one- and two-photon processes can coexist. They also study how the population of the three states can be selectively transferred by adiabatically controlling the electromagnetic field pulses. Different from \( \Delta \)-type atoms, the adiabatic population transfer in that three-level atom can be controlled not only by the amplitudes but also controlled by the phases of the pulses. Thus, they achieved a pulse-phase-sensitive adiabatic manipulation of quantum states in this three-level artificial atom. Usually, only the amplitude was considered for adiabatic control. This example of control on quantum circuits has been recently studied experimentally [29]. In [30], a novel approach was proposed to coherently transfer populations between selected quantum states in one and two qubit systems by using controllable Stark-chirped rapid adiabatic passages (SCRAPS) (see Fig.1). These time-insensitive evolution transfers, assisted by easily implementable single-qubit phase-shift operations, could offer an attractive approach to implement high-fidelity single-qubit NOT operations for quantum computing. Specifically, this proposal could be conveniently demonstrated by existing Josephson phase qubits.

However, a quantum system can never be isolated from the surrounding environment completely [17, 31]. As a result, a randomization of the phase of
the quantum system takes place, and the initial quantum state ends up in a
classical state. Thus, it is an important subject to analyze the quantum de-
cay induced by the unavoidable interaction with the environment. Population
transfer control is an important problem with application to various quantum
systems, and many results are obtained with closed-loop learning control and
quantum optimal control for large transfer time, respectively. The SCRAP-
based quantum gates proposed in [30] evolve in the closed quantum system,
in which the surrounding environment effect was not considered. Recently, J.
H. Schönfeldt, J. Twamley and S. Rebić considered the closed system popula-
tion transfer control in stark-shift-chirped rapid adiabatic-passage (SCRAP)
technology [1]. Their main result is that by the gradient ascent pulse en-
geniering algorithm the average fidelity of population transfer over a wide
range of detunings for both the ground to excited state detuning and the
ground to target state detuning can be improved. In this paper, we make
use of general optimal control theory to transfer the quantum population for
quantum computing in Markovian open, dissipative quantum systems.

The paper is organized as follows. In Sec. II we introduce the con-
trolled spin-boson model. The Lindblad type master equation for driven
open quantum systems with the noise and dissipation effects are presented
in this section. In Sec. III, we analyze the optimal control of quantum sys-
tem dynamics. Both theoretical analysis and numerical demonstration are
presented in this section. Conclusions and prospective views are given in Sec.
IV.

2. Master equation

A quantum two-level system coupled to an environment is always modeled
by a spin degree of freedom in a magnetic field coupled linearly to an oscillator
bath with Hamiltonian [33]

$$H = H_C + H_B + H_I.$$  

Here the oscillator bath is described by $H_B = \sum_k \hbar \omega_k a_k^\dagger a_k$; the spin’s observ-
able $\sigma_z$ is coupled with the bath “force” operator $X = \sum_k (g_k a_k^\dagger + g_k^* a_k)$. In
the following, the Planck constant $\hbar$ is assumed to be 1. Let the controlled
part

$$H_C = \frac{1}{2} B_z \sigma_z + \frac{1}{2} B_x \sigma_x$$

$$= \frac{1}{2} \Delta E (\cos \eta \sigma_z + \sin \eta \sigma_x),$$  

$$4$$
where the mixing angle $\eta \equiv \arctan(B_x/B_z)$ determines the direction of the effective magnetic field in the $x - z$ plane, and the energy splitting between the eigenstates is $\Delta E = \sqrt{B_x^2 + B_z^2}$. $B_x$, $B_z$ are the external controls. For the free system Hamiltonian $H_0 = \frac{\omega_0}{2} \sigma_z$ with $\omega_0 = 1$ as the norm unit throughout this paper, and $H_0$ can be contained in $H_C$. Usually, identifying the nature of interactions in a quantum system is essential in understanding it. Acquiring information on the Hamiltonian can be difficult for many-body systems because it generally requires access to all parts of the system. Reference [27] showed that if the coupling topology is known, the Hamiltonian identification is indeed possible indirectly even though only a small gateway to the system is used. Surprisingly, even a degenerate Hamiltonian can be estimated by applying an extra field to the gateway. This information can then be used for achieving a better control of the quantum system.

Under the adiabatic approximation, it is natural to describe the evolution of the system in the eigenbasis of $H_C$. From Eq.(2), the eigenvalues are $\pm \frac{1}{2} \Delta E$ and the corresponding instantaneous eigenstates are

$$|\lambda_+(t)\rangle = \cos \frac{\eta}{2} |0\rangle + \sin \frac{\eta}{2} |1\rangle,$$

$$|\lambda_-(t)\rangle = -\sin \frac{\eta}{2} |0\rangle + \cos \frac{\eta}{2} |1\rangle.$$  \hspace{1cm} (3)

The goal of the theoretical treatment is to analyze decoherence and the controlled population transfer, which is given by the elements of the reduced density matrix, defined as $\rho_S(t) = tr_B[\rho_{tot}(t)]$, where $\rho_{tot}$ is the total density matrix for both the system and the environment, $tr_B$ the partial trace taken over the environment. The effect of the environment on the dynamics of the system can be seen as an interplay between the dissipation and fluctuation phenomena.

The simplest quantum system is a two-level system, whose Hilbert space is spanned by two states, an excited state $|e\rangle$ and a ground state $|g\rangle$. The Hilbert space of such a system is equivalent to that of a spin-$\frac{1}{2}$ system. The corresponding Pauli operators are $\sigma_1 = |e\rangle\langle g| + |g\rangle\langle e|$, $\sigma_2 = -i|e\rangle\langle g| + i|g\rangle\langle e|$, $\sigma_3 = |e\rangle\langle e| - |g\rangle\langle g|$, satisfying the commutation relations $[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$, and the anticommutation relations $\{\sigma_i, \sigma_j\} = 2\delta_{ij}$. Then the controlled master equation of two-level system has the following Lindblad form [17] ($\rho$ instead of $\rho_S$ for simplicity),

$$\frac{d\rho(t)}{dt} = -i[H_C, \rho] + \frac{\gamma_0(N + 1)}{2}\left\{2\sigma\rho\sigma^+ - \sigma^+\sigma\rho\right\}.$$  \hspace{1cm} (5)
\[-\rho \sigma^+ \sigma\} + \frac{\gamma_0 N}{2} \left\{ 2 \sigma^+ \rho \sigma - \sigma \sigma^+ \rho - \rho \sigma \sigma^+ \right\}, \quad (5)\]

where \(N = 1/(e^{h \omega_0 / k T} - 1)\) denotes the mean number of quanta in a mode with frequency \(\omega_0\) of the thermal reservoir, \(\gamma_0\) is the spontaneous emission rate. The matrix elements \(\rho_{00} = p_g(t)\) and \(\rho_{11} = p_e(t)\) are the population of the ground and excited state levels, respectively. The off-diagonals \(\rho_{01}(t) = \rho_{10}^*(t)\) are the coherences. Let \(x(t) = Tr[\sigma \rho(t)]\), which is called as Bloch vector and defined by

\[
\begin{align*}
    x_1(t) &\equiv \rho_{01}(t) + \rho_{10}(t), \\
    x_2(t) &\equiv i(\rho_{10}(t) - \rho_{01}(t)), \\
    x_3(t) &\equiv \rho_{00}(t) - \rho_{11}(t).
\end{align*} \quad (6)
\]

Then the Bloch vector form of (5) is

\[
\begin{align*}
    \dot{x}_1(t) &= -\frac{2N+1}{2} \gamma_0 x_1(t) + B_z x_2(t), \\
    \dot{x}_2(t) &= -B_z x_1(t) - \frac{2N+1}{2} \gamma_0 x_2(t) + B_x x_3(t), \\
    \dot{x}_3(t) &= -B_x x_2(t) - (2N+1) \gamma_0 x_3(t) - \gamma_0,
\end{align*} \quad (7)
\]

which can be written compactly as

\[
\dot{x}(t) = A(t)x(t) + B(t) \quad (8)
\]

where

\[
A(t) = \begin{pmatrix}
    -\frac{2N+1}{2} \gamma_0 & \frac{B_z}{2} & 0 \\
    -B_z & -\frac{2N+1}{2} \gamma_0 & B_x \\
    0 & -B_x & -(2N+1) \gamma_0
\end{pmatrix}
\]

and

\[
B(t) = \begin{pmatrix}
    0 \\
    0 \\
    -\gamma_0
\end{pmatrix}
\]

3. Optimal control of population transfer

3.1. Optimal control formalism

Recent work [1] has optimized the control law for closed system population transfer by using a gradient ascent pulse engineering algorithm. They show that the optimized pulses perform at a higher fidelity than the standard Gaussian pulses for a wide range of detunings (i.e., large inhomogeneous broadening). The theory of optimal control was introduced in the theory of
automatic control in the 1960s for electrical engineering applications. Bell-
man and Pontryagin, the two famous scientists, paved the way in this field,
respectively. Prof. Belavkin introduced it to quantum mechanical in 1983
[34]. In quantum chemical, quantum optimal control made great success
[35, 36, 37, 8, 38]. In the optimal control framework, one starts by defining a
cost functional which has the function of optimality criteria. Moreover, the
cost functional will vary from one experimental trial to another, and must
be thought of as a random variable depending on the measurement output.
The strategy is then to minimize this cost functional while satisfying the
constraints of the underlying dynamic equations governing the evolution of
quantum states, e.g., the master equation and the initial state condition.
The calculation of the necessary optimality conditions for this optimization
problem results in a system of coupled equations to be solved. For detail one
can refer to the papers [8, 9, 14, 15]. Obviously, the evolution of the state
variable \( x(t) \) governed by the master equation \( (7) \) depends not only on the
initial state \( x_0 \) but also on the choice of the time-dependent control variable
\( u(t) = (B_x(t), B_z(t))^T \). In this section, we are going to control population
transfer and suppress the unexpected effect of decoherence by optimal con-
trol technique that wants to force the system evolving along some prescribed
cohering trajectories. For some target state \( x^0(t) \), let the cost functional as

\[
J[u(t)] = \Psi[x(t_f), x^0(t_f)] + \int_{t_0}^{t_f} L(x(t), x^0(t_f), u(t))dt, \tag{9}
\]

where the functional \( \Psi[x(t_f), x^0(t_f)] \) represents some distance between the
system and objects at final time and the functional \( \int_{t_0}^{t_f} L(x(t), x^0(t_f), u(t)) dt \) ac-
counts for the transient response with \( L(x(t), x^0(t_f), u(t)) \geq 0 \). The optimal
control problem considered in this paper is to minimize the cost functional
\( J[u(t)] \) with dynamical constraints \( (7) \) and initial state constraint \( x(0) \). Using
the Pontryagin’s maximum principle, the optimal solution to this problem
is characterized by the so-called Hamilton-Jacobi-Bellman (HJB) equation.
We use this method to solve the population transfer and decoherence in the
following.
3.2. Evolution without control

At first we consider the free evolution of the system (5), where the external control field $B_x = 0$, $B_z = \omega_0$. Then the free evolution is

$$\begin{cases}
\dot{x}_1(t) = -\frac{2N+1}{2}\gamma_0 x_1(t) + \omega_0 x_2(t), \\
\dot{x}_2(t) = -\omega_0 x_1(t) - \frac{2N+1}{2}\gamma_0 x_2(t), \\
\dot{x}_3(t) = -(2N+1)\gamma_0 x_3(t) - \gamma_0,
\end{cases}$$

whose solution is

$$x_1(t) = e^{-\frac{2N+1}{2}\gamma_0 t}[x_2(0)\sin(\omega_0 t) + x_1(0)\cos(\omega_0 t)],$$

$$x_2(t) = e^{-\frac{2N+1}{2}\gamma_0 t}[x_2(0)\cos(\omega_0 t) - x_1(0)\sin(\omega_0 t)],$$

$$x_3(t) = e^{-(2N+1)\gamma_0 t}\left(\frac{1}{2N+1} + x_3(0)\right) - \frac{1}{2N+1}.$$  

We observe that the population-component $x_3$ of the Bloch vector decays exponentially with rate $-(2N+1)\gamma_0$, while the coherence $x_{1,2}$ decay with rate $-(2N+1)\gamma_0/2$. The stationary solution is

$$x_1^s = x_2^s = 0, \quad x_3^s = -\frac{1}{2N+1},$$

and the populations of the lower and upper level are found to be

$$p_0(t) \equiv \rho_{00}(t) = \frac{1}{2}(1 + x_3(t)) = \frac{1}{2}e^{-(2N+1)\gamma_0 t}\left[\frac{1}{2N+1} + 2\rho_{00}(0) - 1\right] + \frac{1}{2}\left(1 - \frac{1}{2N+1}\right),$$

Figure 2: (Color online) Time evolution of occupation probabilities for $N \in [0, 10]$ without control.
Figure 3: (Color online) Time evolution of occupation probabilities $\rho_{00}(\text{red})$ and $\rho_{11}(\text{blue})$, with system evolution without control (the dashed line) and control target trajectory (the dotted line).

\[ p_e(t) \equiv \rho_{11}(t) = \frac{1}{2}(1 - x_3(t)) \]

\[ = -\frac{1}{2}e^{-(2N+1)\gamma_0 t} \left[ \frac{1}{2N + 1} + 2\rho_{11}(0) - 1 \right] + \frac{1}{2} \left( 1 + \frac{1}{2N + 1} \right) , \]  

with the stationary populations $\frac{1}{2}(1 \pm \frac{1}{2N+1})$, respectively. As $k_B T \to 0$ the mean number of quanta $N = 1/(e^{\frac{k_B T}{h\omega_0}} - 1) \to 0$, which means that $\rho_{00}(t \to \infty) \to 0$ and $\rho_{11}(t \to \infty) \to 1$. This is a population transfer process. The elementary logic gates in quantum computing networks can be implemented by this transfer. However, when $N > 0$ $\rho_{00}(t \to \infty) = \frac{1}{2}(1 - \frac{1}{2N+1}) > 0$, and $\rho_{11}(t \to \infty) = \frac{1}{2}(1 + \frac{1}{2N+1}) < 1$. Furthermore, as $k_B T \to \infty$ the stationary populations are $\rho_{00}^s(N \to \infty) = \rho_{11}^s(N \to \infty) = \frac{1}{2}$, which fails to transfer the populations. The evolution of populations $\rho_{00} = \frac{1}{2}(1 + x_3(t))$ (left) and $\rho_{11} = \frac{1}{2}(1 - x_3(t))$ (right) is plotted for $N \in [0, 10]$ in Fig. 2, where the dissipation constant $\gamma_0$ is set as 0.1, and the the initial state $x(0) = (\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, 1)$. In the following, choosing the trajectory with $N = 0$ as the target we will use the optimal control technology to make the single qubit rotation that completely inverts the populations of the quantum states.
Figure 4: (Color online) Time evolution of occupation probabilities $\rho_{00}(\text{red})$ and $\rho_{11}(\text{blue})$, with system evolution without control (the dashed line) and control target trajectory (the dotted line).

Figure 5: (Color online) Time evolution of occupation probabilities $\rho_{00}(\text{red})$ and $\rho_{11}(\text{blue})$, with system evolution without control (the dashed line) and control target trajectory (the dotted line).
3.3. Controlled population transfer

Precise design of switching on/off logic gates of quantum systems is a main problem in quantum computer research. In the following we consider the optimal control of population transfer in a dissipative open quantum system. For simplicity let the cost functional be:

$$J[u(t)] = \int_{t_0}^{t_f} [(x(t) - x^0(t))^2 + \theta u^T(t)u(t)]dt$$

(17)

where $\theta > 0$ is a weighting factor used to achieve a balance between the tracking precision and the control constraints. $u(t) = (B_x(t), B_z(t))^T$ is the external control. $x^0(t)$ is the target trajectory

$$x^0(t) = \begin{pmatrix} e^{-\frac{\gamma_0 t}{2}}(x_2(0)\sin(B_z t) + x_1(0)\cos(B_z t)) \\ e^{-\frac{\gamma_0 t}{2}}(x_2(0)\cos(B_z t) - x_1(0)\sin(B_z t)) \\ e^{-\omega t}(x_3(0) + 1) - 1 \end{pmatrix},$$

which is the solution of the master equation (5) in the case of $N = 0$ without control, i.e. $\frac{d\rho}{dt} = -i[\omega_0 \sigma_z, \rho] + \frac{\gamma_0}{2} \{2\sigma \rho \sigma^+ - \sigma^+ \sigma \rho - \rho \sigma \sigma^+ \}. \quad \text{The dissipator of the equation describes spontaneous emission process (rate } \gamma_0 \text{) without thermally induced emission and absorption process (} N = 0, \text{ or } T = 0 \text{). That was the reason we chose it as the target trajectory. We will choose it as the tracking object in both optimal population control in the following.}$

The corresponding Hamiltonian function is

$$\mathcal{H}(x(t), u(t), \psi(t), t) = [(x(t) - x^0(t))^2 + \theta u^T(t)u(t)] + \lambda(t)^T[A(t)x(t) + B(t)]$$

$$= [(x_1(t) - x_1^0(t))^2 + (x_2(t) - x_2^0(t))^2 + (x_3(t) - x_3^0(t))^2 + \theta(B_x^2(t) + B_z^2(t))]$$

$$+ \lambda_1(t)[\frac{2N + 1}{2}\gamma_0 x_1(t) + B_x x_2(t)] + \lambda_2(t)\{-B_z x_1(t) - \frac{2N + 1}{2}\gamma_0 x_2(t)

+ B_z x_3(t)] + \lambda_3(t)[-B_x x_2(t) - (2N + 1)\gamma_0 x_3(t) - \gamma_0],$$

where $\lambda(t) = (\lambda_1(t), \lambda_2(t), \lambda_3(t))^T$ is the so-called Lagrange multiplier. The optimal solution can be solved by the following differential equation with two-sided boundary values,

$$\begin{cases}
\dot{x}(t) = \frac{\partial \mathcal{H}}{\partial \lambda} = A(t)x(t) + B(t), \\
\dot{\lambda}(t) = -\frac{\partial \mathcal{H}}{\partial x} = -2[x(t) - x^0(t)] - A(t)^T\lambda(t), \\
x(0) = x_0, \\
\lambda(t_f) = 0,
\end{cases}$$

(18)
Together with
\[ \frac{\partial H}{\partial u} \bigg|_{*} = \frac{\partial H(x^*(t), u^*(t), \psi(t), t)}{\partial u} = 0, \]
we have
\[ \begin{cases} B_x(t) = \frac{1}{2g} \{ \lambda_3 x_2 - \lambda_2 x_3 \}, \\ B_z(t) = \frac{1}{2g} \{ \lambda_2 x_1 - \lambda_1 x_2 \}. \]

In general, no analytic solution of the complicated nonlinear equations exists. The numerical demonstration to this problem is considered in the following.

### 3.4. Numerical demonstration and discussions

As example we show in Fig.(3-8) the populations as a function of time for the open quantum system with six different values of \( N \) (temperature). In our simulations, the system parameters are chosen as: \( x(0) = (\sqrt{2}, \sqrt{2}, 1), \)
Figure 7: (Color online) Time evolution of occupation probabilities $\rho_{00}$ (red) and $\rho_{11}$ (blue), with system evolution without control (the dashed line) and control target trajectory (the dotted line).

Figure 8: (Color online) Time evolution of occupation probabilities $\rho_{00}$ (red) and $\rho_{11}$ (blue), with system evolution without control (the dashed line) and control target trajectory (the dotted line).
dissipation constant $\gamma_0 = 0.1$, and system frequency $\omega_0 = 1$ as the norm unit. We consider the two-mode control, physically, $B_z(t)$ time-dependent Stark shift and $B_z(t)$ external resonant control field acting on the open system. In our simulations, the time evolutions of the lower level population $p_0(t) \equiv \rho_{00}(t) = \frac{1}{2}(1 + x_3(t))$ are colored red and the time evolutions of the upper level population $p_1(t) \equiv \rho_{11}(t) = \frac{1}{2}(1 - x_3(t))$ are colored blue, and the free evolutions are plotted with dashed line, the control target trajectory with dotted line, optimal populations control with solid line.

More precisely, we study the temperature as a key factor in population transfer control. In fact, $N = 1/(e^{\frac{\hbar \omega_0}{k_B T}} - 1)$, which means that $T = \hbar \omega_0/(k_B \ln(1 + \frac{1}{N}))$, where $k_B = 1.380662 \times 10^{-23} J \cdot K^{-1}$ is the Boltzmann constant and $\hbar = 1.0545887 \times 10^{-34} J \cdot s$ is the reduced Planck constant. When $N$ is from 0 to 10, the temperature changes significantly, $(0 \sim 8.0182) \times 10^{-11} \omega_0 (K)$. The smaller the $N$ the lower the temperature, the larger the $N$ the higher the temperature. Looking at Figs. (3-8) (where we set $N$ by 0.01, 0.2, 0.5, 1, 2, 10, respectively), we can approximately say that when the temperature is low, for example, $N < 0.5$ $(T = 0.6956 \times 10^{-11} \omega_0 (K))$, we can achieve very good population transfer fidelity. And the lower the better. Increasing $N$, smaller population transfer were observed in Fig.(6) $(N = 1)$ and Fig.(7) $(N = 2)$. This is because in higher temperature regime the energy level spacing is decreased, which results in low transfer rate. When $N = 10$, the population is wild oscillations making the optimal control failure. It means that the environment induced fluctuations is large enough, so that the control field is negligible comparing with the high-frequency harmonic oscillators of the reservoir. From Figs. (3-8) we can sum up three rules. (i) The external control field played an important role in population transfer for quantum computing. (ii) The smaller the $N$ (temperature) the better the control performance, especially when $N < 0.2$ the completely population transfer can be achieved. (iii) When temperature is high enough the transfer is uncontrollable.

### 3.5. Decoherence behavior

In the following we make efforts to study the decoherence behavior of the open quantum systems with optimal population transfer control, briefly. The persistence of quantum coherence is relied on in quantum computer, quantum cryptography, quantum teleportation, and it is also fundamental in understanding the quantum world for the interpretation that the emergence
of the classical world from the quantum world can be seen as a decoherence process due to the interaction between system and environment. From (10), one can easily conclude the decoherence factor

\[
\Lambda(t) = \frac{1}{2} \sqrt{x_1^2(t) + x_2^2(t)} \\
= \frac{1}{2} e^{-\frac{2N+1}{2} \gamma_0 t} \sqrt{x_1^2(0) + x_2^2(0)} \\
= e^{-\frac{2N+1}{2} \gamma_0 t} \Lambda(0),
\]

decaying exponentially to 0 with rate \(\frac{2N+1}{2} \gamma_0\). In Figs. (9,10) we plot the time evolution of decoherence for continuous variation of \(N\) (temperature). Fig. (10) shows that at low temperature the decoherence can be well controlled, i.e., the decoherence time can be delayed and its amplitude can be amplified. With increasing \(N\) (the temperature), the environment induced fluctuations will be large enough to neglect the control field. Comparing with Fig. (9) we find that these optimal controls can also prolong the decoherence time and satisfy the controlled quantum logic gate operations, especially in low temperature regime.

4. Conclusions

In the present work, we have studied the controlled population transfer between selected quantum states in the Markovian open quantum system, which can generate universal logic gates for quantum computing. We use the
optimal control method to control the population transfer. Our numerical results indicate that the occupation dynamics behaves differently for the different environmental condition. The result can be summed up in three rules: (i) The external control field plays an important role in population transfer. (ii) The smaller the $N$ (temperature) the better the control performance, especially when $N < 0.2$ the completely population transfer can be achieved. (iii) When temperature is high enough the transfer is uncontrollable. Solid state qubits offer remarkable advantages due to their scalability and controllability. Therefore, optimal control of population transfer perhaps provides an attractive approach to generate universal logic gates. We hope such optimal control techniques to experimentally implement quantum elementary logic gates in quantum computing networks in the near future.

In this paper, we restrict the discussions to Markovian open quantum systems and show the validity of our optimal control strategy to the population transfer, and we analyze corresponding decoherence behavior and we find that these controls can prolong the decoherence time and satisfy the controlled quantum logic gate operation in low temperature regime. To make the model more realistic in the experiments and reduce the low temperature constraints it is worth extending the scope to non-Markovian open quantum systems where the seemingly lost information can return to the system at a later time, which is our further work.

Figure 10: (Color online) Time evolution of decoherence for $N \in [0,10]$ with optimal control.

0.2 0.4 0.6 0.8 1
0 0.2 0.4 0.6 0.8 1
0 10 20 30 40 50
N

Optimal decoherence control
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