An enumeration of 1-perfect ternary codes

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Abstract

We study codes with parameters of the ternary Hamming \((n=(3^m-1)/2, 3^{n-m}, 3)\) code, i.e., ternary 1-perfect codes. The rank of the code is defined to be the dimension of its affine span. We characterize ternary 1-perfect codes of rank \(n - m + 1\), count their number, and prove that all such codes can be obtained from each other by a sequence of two-coordinate switchings. We enumerate ternary 1-perfect codes of length 13 obtained by concatenation from codes of lengths 9 and 4; we find that there are 93241327 equivalence classes of such codes.

Keywords: perfect codes, ternary codes, concatenation, switching.

1. Introduction

Perfect 1-error-correcting \(q\)-ary codes are codes with parameters of \(q\)-ary Hamming codes over the Galois field \(GF(q)\) of order \(q\), which exist for every prime power \(q\) and length \(n\) of form \((q^m - 1)/(q - 1), m \in \{2, 3, \ldots\}\). Since the pioneer work of Vasil’ev on 1-perfect binary codes [43] and its \(q\)-ary generalization by Schönheim [34], it is known that the Hamming code is not a unique 1-perfect code and the number of nonequivalent 1-perfect codes grows doubly exponentially in \(n\) (at least \(q^{cn-o(n)}\), where \(c = \frac{1}{q}\) if \(q = 2, 3\) and \(c \simeq \frac{2}{q}\) for large \(q\) [10]).

Perfect codes, including non-binary ones, can be used in different applications, for example, in steganography schemes, see e.g. [44], [30], [11]. As mentioned in [44], the possibility to choose a code from a large variety can increase security of steganographic systems, adding additional difficulties to anyone who wants to hack such a scheme. So, the study of nonlinear 1-perfect codes, especially classes of codes whose structure is well understood (which allows to develop efficient decoding algorithms), is important both from theoretical point of view and for evaluating their potential use in applications.

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The problem of characterization of the class of 1-perfect codes, in any constructive terms, is far from being solved even for \( q = 2 \). However, there are characterizations for 1-perfect codes with some restrictions. One of important restrictions is the restriction on the rank of the code. The rank is the dimension of the affine span of the code. We say that a 1-perfect code is of rank \( +r \) if its rank is \( r \) greater than the dimension of the Hamming code of the same parameters. Binary 1-perfect code of rank at most \( +1 \) and of rank at most \( +2 \) are characterized by Avgustinovich, Heden, and Solov’eva in [2]; in the last case, the characterization is up to the characterization of multiary quasigroups of order 4, which was completed later in [15]. In [10], all 1-perfect codes of rank less than \( n - 1 \) are proven to be decomposed into some independent subsets, so-called \( \mu \)-components, but the characterization of \( \mu \)-components is as hard as of 1-perfect codes in general. However, in special cases (for example, for the case \( q = 2 \) and rank \( +2 \) considered in [2]), the structure of \( \mu \)-components can be well understood, and we use this decomposition to characterize the set of ternary 1-perfect codes of rank \( +1 \), which is the first main result of the current paper.

The second result is computational: we classify ternary concatenated 1-perfect codes of length 13. It can be considered as a ternary analog of the result [27] on the classification of binary concatenated extended 1-perfect codes of length 16; however, in contrast to the results of [27], the number 93241327 of nonequivalent ternary perfect codes found is huge, and processing them in a reasonable time (it took about thirty core-years) required combining and development of classification methods used in [27] for codes and by the second author for equitable partitions and orthogonal arrays [13]. By now, all known computational enumerations of 1-perfect codes were focused on the binary case. In contrast to the \( q \)-ary 1-perfect codes with \( q > 2 \), every binary 1-perfect code has its extended version obtained by appending the all-parity-check bit to every codeword. Although this mapping is bijective, enumerating 1-perfect and extended 1-perfect codes up to equivalence are different tasks. Phelps [27] enumerated partitions of the binary Hamming space of dimension 7 into 1-perfect codes and binary extended 1-perfect codes of length 16 obtained from such partitions by concatenation. V. Zivoniev and D. Zinoviev enumerated binary 1-perfect codes of length 15 and rank 13 in [47], extended binary 1-perfect codes of length 16 and ranks 13 and 14 in [46] and [48], respectively. Finally, Östergård and Pottonen enumerated all binary 1-perfect codes of length 15 and their extended versions in [24]; in the subsequent paper [25], different properties of these codes were studied. Note that, as was mentioned in the later researches [48], [25], the numbers of nonequivalent codes found in [27], [46], [47], and [48] contain mistakes; however, methods developed there are correct and important for further research, including our current study.

In contrast to the most of previous results on non-binary 1-perfect codes, e.g., [29], [19], [20], [10], [31], in the current paper we focus our efforts on the ternary case. It should be noted that this case is of special interest by the following theoretical reasons. For \( q = 3 \), as well as for \( q = 2 \), the group of isometries of the Hamming space is a subgroup of the group of the automorphisms of the corresponding affine space over \( GF(q) \). We hence can ensure that codes that are equivalent combinatorially (the equivalence is defined in Section 2.2) are equivalent algebraically. In particular, algebraic properties such as the rank, the kernel structure (see Section 2.1), maximal affine subspaces are invariant under equivalence. This is not the case for \( q \geq 4 \), where a linear code can be equivalent to a
code that linearly spans the whole space. The last fact is used in [7] to construct special decompositions of a group (the additive group of the space) called tilings, with different parameters (of special interest are so-called full-rank tilings, which can be constructed from 1-perfect codes spanning the space; the kernel size is also important for tilings). However, for \( q \leq 3 \), by the reasons mentioned above, we cannot construct different tilings from equivalent codes. This motivates to study the connection between perfect codes and tilings more closely for \( q = 2 \) and \( q = 3 \). In the binary case, most of related questions have been solved [4], [8], [42], [23], including the characterization of all admissible rank–kernel dimension pairs for 1-perfect codes [3]. For \( q = 3 \) (we mention important results [22], [40], [14] focused on this case), many questions are open, and the table of ranks and kernels for concatenated codes (Table 1) can be considered as a step in this direction.

The structure of the paper is as follows. In Section 2 we define the main concepts and recall some facts we use in our study.

In Section 3, we prove a characterization theorem for 3-ary 1-perfect codes of rank +1 (Theorem 1), count their number (Theorem 2), prove the connectedness of the set of such codes by mean of two-coordinate switching (Theorem 3), and consider the structure of a 3-ary 1-perfect code \( C \) with kernel size \(|C|/3\) (Section 3.4).

In Section 4, we describe the computer-aided enumeration of the concatenated 3-ary 1-perfect codes of length 13 (Theorem 8) and of auxiliary objects including the partitions of the space into 3-ary 1-perfect codes of length 4 (Theorem 7) and partitions of a \((9,3^8,2)_3\) MDS code into \((9,3^6,3)_3\) subcodes (Theorem 6). Enumerating the last partitions was the most resource-intensive step of the computing; it is of independent interest because \((n = q^m, q^{n-m-1}, q)\) subcodes of an \((n,q^{n-1},2)_q\) MDS code form an interesting class of completely regular codes, which share with perfect codes some properties and constructing tools, see e.g. [33]. Moreover, the obtained partitions can further be used for constructing 1-perfect codes of any admissible length larger than 13 by generalized concatenated construction (see [45] for the general approach) as shown in [32]. A database containing representatives of the equivalence classes of the classified objects can be found at https://ieee-dataport.org/open-access/perfect-and-related-codes [12].

2. Preliminaries

In this section, we define main concepts and mention related facts important to our study.

2.1. Graphs and spaces

The Hamming graph \( H(n, q) \) is a graph whose vertices are the words of length \( n \) in the alphabet \( \{0, \ldots, q - 1\} \), two vertices being adjacent if they differ in exactly one symbol. If \( q \) is a prime power, the symbols of the alphabet are associated with the elements of the prime field \( \text{GF}(q) \), and the vertex set of \( H(n, q) \) forms an \( n \)-dimensional vector space \( \mathbb{F}_q^n \) over \( \text{GF}(q) \) with the component-wise addition and multiplication by a constant.

The natural shortest-path distance in \( H(n, q) \) coincides with the Hamming distance, i.e., the distance between two words equals the number of positions they differ. The weight of a vertex \( \bar{x} \) is the distance from \( \bar{x} \) to the all-zero word \( \bar{0} \).
A vertex set $C$ in $H(n, q)$ is called a distance-$d$ code, or an $(n, |C|, d)_q$ code, if there are no two different codewords in $C$ with distance less than $d$. A code forming a linear subspace of $\mathbb{F}_q^n$ is called linear. The rank of a code is the dimension of its affine span (if the code contains the all-zero word, then, equivalently, the dimension of its linear span). The kernel of a code $C \subset \mathbb{F}_q^n$ is the set $\{ \bar{x} \in \mathbb{F}_q^n : \alpha \bar{x} + C = C \ \forall \alpha \in \mathbb{F}_q \}$; if $q$ is prime (in our case, $q = 3$), then the kernel coincides with the set $\{ \bar{x} \in \mathbb{F}_q^n : \bar{x} + C = C \}$ of all periods of $C$.

2.2. Equivalence and automorphisms

The next group of definitions concerns different equivalences and automorphisms of codes. We recall that every automorphism of the graph $H(n, q)$ can be uniquely represented as the composition of a coordinate permutation

$$\pi : (c_0, \ldots, c_{n-1}) \rightarrow (c_{\pi^{-1}(0)}, \ldots, c_{\pi^{-1}(n-1)})$$

and an isotopy $\bar{\theta} = (\theta_0, \ldots, \theta_{n-1})$ that consists of $n$ permutations of the alphabet $\{0, \ldots, q-1\}$, acting independently on the corresponding $n$ symbols of a word of length $n$ over $\{0, \ldots, q-1\}$:

$$\bar{\theta} : (c_0, \ldots, c_{n-1}) \rightarrow (\theta_0(c_0), \ldots, \theta_{n-1}(c_{n-1})).$$

Two sets $C$ and $D$ of vertices of $H(n, q)$ are said to be equivalent if there is an automorphism of $H(n, q)$ that sends $C$ to $D$. The set of automorphism of $H(n, q)$ that send a vertex set $C$ to itself forms the automorphism group $\text{Aut}(C)$ of $C$, with composition in the role of the group operation.

Two sets $C$ and $D$ of vertices of $H(n, q)$ are monomially equivalent if there is an automorphism of $H(n, q)$ that is at the same time an automorphism of the corresponding vector space and sends $C$ to $D$ (in the case $q = 3$, every automorphism of $H(n, 3)$ that fixes the all-zero word is an automorphism of the vector space). Two sets $C$ and $D$ of vertices of $H(n, q)$ are permutably equivalent if there is a permutation of coordinates that sends $C$ to $D$. The monomial automorphism group $\text{MAut}(C)$ and permutation automorphism group $\text{PAut}(C)$ are subgroups of $\text{Aut}(C)$ that correspond to monomial and permutation equivalence, respectively.

2.3. 1-perfect, distance-2 MDS, and Reed–Muller-like codes

A 1-perfect code is an independent set of vertices of $H(n, q)$ (or any other graph) such that every non-code vertex is adjacent to exactly one codeword. If $q$ is a prime power, then a necessary and sufficient condition for the existence of 1-perfect codes is $n = (q^m - 1)/(q-1)$, $m \in \{1, 2, \ldots\}$; so, such codes are $(n, q^{n-m}, 3)_q$ codes. In particular, for every $q$ and $m$, there is a unique (up to equivalence) linear 1-perfect code, called a Hamming code, which has dimension $n - m$, the order of the monomial automorphism group

$$|\text{GL}_m(\mathbb{F}_q)| = (q^m - 1) \cdot (q^m - q) \cdots (q^m - q^{m-1}),$$

and a check matrix consisting of the maximum collection of mutually non-colinear columns of height $m$ (recall that the rows of a check matrix form a basis of the dual space of the
linear code). The Hamming code, obviously, has the minimum rank, $n - m$, among all 1-perfect codes of the same parameters; thus, we say that a 1-perfect code is of rank $+r$ if its rank is $(n - m) + r$.

A code with parameters $(n, q^{n-1}, 2)_q$ is called a distance-2 MDS code (note that we do not require this code to be linear). A function $f : \{0, \ldots, q - 1\}^n \rightarrow \{0, \ldots, q - 1\}$ such that its graph $\{(\bar{x}, f(\bar{x})) : \bar{x} \in \{0, \ldots, q - 1\}^n\}$ is a distance-2 MDS code is called an $n$-ary (multiary) quasigroup of order $q$.

The third special kind of codes that plays a role in our theory is $(n = q^m, q^{n-m-1}, 3)_q$ codes that are subsets of a distance-2 MDS code. We call such codes RM-like codes, because the linear code of this kind is $R_q(qm - m - 2, m)$, a generalized Reed–Muller code (see, e.g., [10, §5.4]) of order $(q - 1)m - 2$. As follows from the following proposition, every RM-like code is a maximum distance-3 subcode of a distance-2 MDS code.

**Proposition 1.** If $C \subset M$, where $C$ is a RM-like code and $M$ is a distance-2 MDS code, then every vertex not in $M$ is adjacent to exactly one codeword of $C$.

**Proof.** Since the minimum distance of $C$ is 3, we see that every vertex is adjacent to at most one codeword of $C$. The number of vertices adjacent to a codeword of $C$ is $|C| \cdot n \cdot (q - 1)$, i.e., $q^n - q^{n-1}$, which is exactly the number of vertices not in $M$.  

By an RM-like partition, we mean a partition of a distance-2 MDS code into RM-like codes.

### 2.4. Concatenation

For any two words or symbols $\bar{x}$ and $\bar{y}$, by $\bar{x}\bar{y}$ we denote their concatenation. For a code $C$ and a symbol or word $\bar{x}$, we denote $C\bar{x} = \{\bar{c}\bar{x} : \bar{c} \in C\}$ and $\bar{x}C = \{\bar{c}\bar{x} : \bar{c} \in C\}$; similarly, $CD = \{\bar{x}\bar{y} : \bar{x} \in C, \bar{y} \in D\}$ for two codes $C$ and $D$.

Next, we define concatenated codes. The following construction of $q$-ary 1-perfect codes suggested by Romanov [31] is a $q$-ary generalization of the Solov’eva–Phelps construction [26, 39] for binary 1-perfect codes.

**Lemma 1** (Romanov [31]). Assume $n = (q^m - 1)/(q - 1)$, $n' = (q^{m-1} - 1)/(q - 1)$, $n'' = q^{m-1}$. Let $(P_0, \ldots, P_{n''-1})$ be a partition of the Hamming space $H(n', q)$ into 1-perfect $(n', q^{n'-m-1}, 3)_q$ codes. Let $(C_0, \ldots, C_{n''-1})$ be a partition of an $(n'', q^{n''-1}, 2)_q$ MDS code into $n''$ codes with parameters $(n'', q^{n''-m}, 3)_q$. And let $\tau$ be a permutation of $\{0, \ldots, n'' - 1\}$. Then the code

$$P = \bigcup_{i=0}^{q^{m-1}-1} C_i P_{\tau(i)}$$  \hspace{1cm} (1)

is a 1-perfect $(n, q^{n-m}, 3)_q$ code.

The role of the permutation $\tau$ in the construction above is technical: since it just changes the order of the codes $P_i$, we will not lose generality by assuming that $\tau$ is identity. However, as in Section 4 we work with concrete representatives of equivalence.
classes of partitions, it is convenient to represent the reordering of the codes in a partition explicitly, as a permutation \( \tau \).

The codes representable in the form (1) are called concatenated. We note that this property is not invariant under equivalence because it depends on the order of coordinates.

**Remark 1.** Another \( q \)-ary generalization of the Solov'eva–Phelps construction was proposed in [21] (see also [3] Theorem 11.4.5); it can be regarded as a special case of the construction in Lemma 1 with a partition \((C_0, \ldots, C_{n''-1})\) explicitly constructed from a partition with the same parameters as \((P_0, \ldots, P_{n''-1})\). In the case \( n'' = 9 \), considered in Section 4, there are 65436 nonequivalent partitions \((C_0, \ldots, C_{n''-1})\) of a \((9, 3^6, 3)_3\) code into \((9, 3^6, 3)_3\) codes (see Theorem 6), while the number of nonequivalent partitions \((P_0, \ldots, P_{n''-1})\) of \( \mathbb{F}_4^3 \) into \((4, 9, 3)_3\) codes is only 2 (see Theorem 7). This shows that the construction in [31] (Lemma 1 above) gives more codes than the one in [21].

**Remark 2.** The construction in Lemma 1, as well as its binary case [26, 39], is very close to the construction of Heden [9]. Namely, if we restrict the choice of the partition \((P_0, \ldots, P_{n''-1})\) by a partition into cosets of the same 1-perfect code and treat the partition \((C_0, \ldots, C_{n''-1})\) as a code in the mixed-alphabet Hamming space over \( \mathbb{F}_q^{n''} \times \{0, \ldots, n''\}\), then the following lemma turns into a special case of [9, Theorem 1]. Finally, we note that the construction can be treated in terms of the generalized concatenation construction [45].

### 3. Codes of rank +1

In this section, we characterize the 3-ary 1-perfect codes of rank +1, count the number of different such codes, and discuss the possibility of switching between such codes.

#### 3.1. Characterization

We will use the result of [10], which states that a code, depending on its rank, is the union of one or more independently defined subsets, called \( \mu \)-components. Below we will show that in the case of ternary 1-perfect codes of rank +1, such \( \mu \)-components are in one-to-one correspondence with multiary quasigroups of order 3.

**Lemma 2** ([10] Th. 2.1, \( r = m - 1, s = 1 \)). Let \( C \) be a \( q \)-ary 1-perfect code of length \( n = (q^m - 1)/(q - 1) \) of rank at most +1 and \( C^* \) be the \( q \)-ary Hamming code of length \( n' = (q^{m-1} - 1)/(q - 1) \). Then for some translation vector \( \bar{v} \) and monomial transformation \( \psi \), it holds

\[
\psi(C + \bar{v}) = \bigcup_{\bar{\mu} \in C^*} K_{\bar{\mu}},
\]

where

\[
K_{\bar{\mu}} = \{(x_0, x_1, \ldots, x_{n-1}) : \bar{\sigma}(x_0, \ldots, x_{n-2}) = \bar{\mu}, \; x_{n-1} = \lambda_{\bar{\mu}}(x_0, \ldots, x_{n-2})\},
\]

\[
\bar{\sigma}(x_0, \ldots, x_{n-2}) = \left( \sum_{i=0}^{q-1} x_i, \sum_{i=q}^{2q-1} x_i, \ldots, \sum_{i=n-1-q}^{n-2} x_i \right).
\]
for some \{0, \ldots, q - 1\}\text{-}valued functions \(\lambda_{\tilde{\mu}}, \tilde{\mu} \in C^*\), defined on
\[
\{(x_0, \ldots, x_{n-2}) : \tilde{\sigma}(x_0, \ldots, x_{n-2}) = \tilde{\mu}\}
\]
and satisfying
\[
d(\bar{x}_s, \bar{y}_s) = 2 \implies \lambda_{\mu}(\bar{x}_s) \neq \lambda_{\mu}(\bar{y}_s). \tag{2}
\]

In the ternary case, the equation \(\bar{\sigma}(x_0, \ldots, x_{n-1}) = \tilde{\mu}\) can be expressed as follows, where \(\tilde{\mu} = (\mu_0, \ldots, \mu_{n'-1})\):
\[
x_2 = -x_0 - x_1 + \mu_0, \quad x_5 = -x_3 - x_4 + \mu_1, \ldots, \quad x_{n-2} = -x_{n-4} - x_{n-3} + \mu_{n'-1}. \tag{3}
\]

**Lemma 3.** If the hypothesis and the conclusion of Lemma 2 hold with \(q = 3\), then
\[
\lambda_{\tilde{\mu}}(x_0, \ldots, x_{n-2}) = \lambda'_{\mu}(x_1 - x_0, x_4 - x_3, \ldots, x_{n-3} - x_{n-4}) \tag{4}
\]
for some \(n'\text{-}ary quasigroup} \(\lambda'_{\mu}\) of order 3, where \(n' = \frac{n-1}{3}\). Moreover, if \(\lambda'_{\mu}\) is an arbitrary \(\frac{n-1}{3}\text{-}ary quasigroup} of order 3 and \(\lambda_{\tilde{\mu}}\) is defined by (4) on any values of arguments satisfying (3), then \(\lambda_{\tilde{\mu}}\) satisfies (2).

**Proof.** The second claim is straightforward from the definition of multiary quasigroups. Let us prove the first one. Any tuple \((x_0, \ldots, x_{n-2})\) satisfying (3) has the form
\[
(x_0, \ldots, x_{n-2}) = (x_0, x_0 + z_0, x_0 - z_0 + \mu_0, \quad \\
x_3, x_3 + z_1, x_3 - z_1 + \mu_1, \quad \\
\ldots, \quad \\
x_{n-4}, x_{n-4} + z_{n'-1}, x_{n-4} - z_{n'-1} + \mu_{n'-1}),
\]
where \(z_i = x_{3i+1} - x_{3i}, i = 0, \ldots, n'-1\). So,
\[
\lambda_{\tilde{\mu}}(x_0, \ldots, x_{n-2}) = \lambda''_{\mu}(x_0, x_3, \ldots, x_{n-4}, z_0, z_1, \ldots, z_{n'-1})
\]
for some function \(\lambda''_{\mu}\). Let us show that \(\lambda''_{\mu}\) does not depend on \(x_0, x_3, \ldots, x_{n-4}\). If \(\bar{x} = (x_0, \ldots, x_{n-2})\) satisfies (3), then \(\bar{x} + \tilde{e}_{012}, \bar{x} + \tilde{e}_{021}, \text{and } \bar{x} + \tilde{e}_{111}\), where \(\tilde{e}_{ijk} = (i, j, k, 0, \ldots, 0)\), also satisfy (3). Since \(\bar{x}, \bar{x} + \tilde{e}_{012}, \text{and } \bar{x} + \tilde{e}_{021}\) are at mutual distance 2 from each other, we see from (2) that \(\{\lambda_{\mu}(\bar{x}), \lambda_{\mu}(\bar{x} + \tilde{e}_{012}), \lambda_{\mu}(\bar{x} + \tilde{e}_{021})\} = \{0, 1, 2\}\). Similarly, \(\{\lambda_{\mu}(\bar{x} + \tilde{e}_{111}), \lambda_{\mu}(\bar{x} + \tilde{e}_{012}), \lambda_{\mu}(\bar{x} + \tilde{e}_{021})\} = \{0, 1, 2\}\). Therefore, \(\lambda_{\mu}(\bar{x}) = \lambda_{\mu}(\bar{x} + \tilde{e}_{111})\) and, in particular, \(\lambda''_{\mu}(x_0, x_3, \ldots, x_{n-4}, z_0, z_1, \ldots, z_{n'-1})\) does not depend on \(x_0\). Similarly, it does not depend on \(x_3, \ldots, x_{n-4}\), and
\[
\lambda''_{\mu}(x_0, x_3, \ldots, x_{n-4}, z_0, z_1, \ldots, z_{n'-1}) = \lambda'_{\mu}(z_0, z_1, \ldots, z_{n'-1})
\]
for some \(\lambda'_{\mu}\), which is an \(n'\text{-}ary quasigroup}, by the definition.

Summarizing Lemmas 2 and 3 we obtain the following.
Theorem 1. Let $C$ be a $3$-ary $1$-perfect code of length $n = (3^m - 1)/2$ of rank at most $+1$ and $C^*$ be the $3$-ary Hamming code of length $n' = (3^{m-1} - 1)/2$. Then for some automorphism $\psi$ of $H(n, 3)$, it holds
$$\psi(C) = \bigcup_{\mu \in C^*} K_{\bar{\mu}},$$
where
$$K_{\bar{\mu}} = \{(x_0, x_1, \ldots, x_{n-1}) : x_2 = \mu_0 - x_0 - x_1, x_5 = \mu_1 - x_3 - x_4, \ldots,$$
$$x_{n-2} = \mu_{(n-4)/3} - x_{n-4} - x_{n-3},$$
$$x_{n-1} = \lambda_{\bar{\mu}}(x_1 - x_0, x_4 - x_3, \ldots, x_{n-3} - x_{n-4})\}$$
for some $(n - 1)/3$-ary quasigroup $\lambda_{\bar{\mu}}$ of order $3$, $\bar{\mu} \in C^*$.

In contrast to mutliary quasigroups of higher orders, all $t$-ary quasigroups of order $3$ (and the corresponding $3$-ary distance-2 MDS codes) are affine:

Proposition 2 ([17, Corollary 13.25, Exercise 13.11]). There are exactly $2 \cdot 3^t$ $t$-ary quasigroups of order $3$. Each of them has the form
$$f(x_0, \ldots, x_{t-1}) = a_0 x_0 + \ldots + a_{t-1} x_{t-1} + a$$
for some $a_0, \ldots, a_{t-1}$ from $\{1, 2\}$ and $a$ from $\{0, 1, 2\}$.

So, the characterization of $3$-ary $1$-perfect codes of rank at most $+1$ in Theorem 1 is constructive.

Corollary 1. The dimension of the kernel of a $3$-ary $1$-perfect code $C$ of length $n$ and rank $+1$ is at least $(n - 1)/3$.

Proof. Without loss of generality, we can assume that the conclusion of Theorem 1 holds with the identity $\psi$. From the proof of Lemma 1 we see that $(1, 1, 1, 0, \ldots, 0)$ is in the kernel of each $K_{\bar{\mu}}$, $\bar{\mu} \in C^*$ and hence belongs to the kernel of $C$. Similarly, the kernel contains $(0, 0, 0, 1, 1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1, 1, 0)$. \square

As we will see in Section 4 (Table 1), the bound is tight for $n = 13$: there are concatenated $(13, 3^{10}, 3)_3$-codes of rank $+1$ with kernel of size $3^4$. The smallest automorphism group of such concatenated codes, however, is twice larger, $162$. We have analyzed 10000 random $(13, 3^{10}, 3)_3$-codes of rank $+1$ and have not found a code with automorphism group of order $81$ (or of any other order that does not occur among concatenated codes of rank $+1$). The most typical values of the order are $162$ (52.2% of the cases), $243$ (29.7%), $486$ (14.8%), and $729$ (2.8%) (the dimension of the kernel is $4$ in 95.6%, $5$ in 4.3%, $6$ in 0.05% of the cases). We conjecture that this is a small-length phenomena, and as $n$ grows, almost all $3$-ary $1$-perfect codes of length $n$ and rank $+1$ have exactly $3^{\frac{n-1}{3}}$ automorphisms.
3.2. The number of rank +1 codes

Lemma 4. In \( H(n = \frac{3^m - 1}{2}, 3) \), the number of 1-perfect codes of rank +1 with the same affine span is

\[
N'(n) = (3 \cdot 2^\frac{n+1}{3} - m + 1 - 6^\frac{n+1}{3} \cdot 3^{-m+2}).
\]

Proof. Assume without loss of generality that one of the 1-perfect codes of rank +1 satisfies the conclusion of Theorem 1 with the identity \( \psi \). Denote by \( S \) its affine span; since \( \psi \) is identity, \( S \) is linear. Then the other 1-perfect codes of rank +1 with affine span \( S \) also satisfy the conclusion of Theorem 1 with the identity \( \psi \) and the same \( C^* \). The number of codes that satisfy the conclusion of Theorem 1 with the identity \( \psi \) is \( Q_{(n-1)/3} \cdot |C^*| \), where \( Q_t = 3 \cdot 2^t \) is the number of \( t \)-ary quasigroups of order 3 (Proposition 2) and \( |C^*| = 3^{(n-1)/3-m+1} \). However, some of these codes are of rank +0, and it remains to find the number of such codes, subsets of \( S \) that are cosets of Hamming codes.

We first count the number of Hamming codes, subsets of \( S \). Let \( H^* \) be a check matrix of the Hamming code \( C^* \). It consists of \( (n-1)/3 \) mutually non-colinear columns of height \( m - 1 \). It is straightforward that a check matrix \( H \) of \( S \) can be constructed by repeating each column of \( H^* \) three times and adding one all-zero column. To complete \( H \) to a check matrix of a Hamming code, we need to add one row that makes all columns mutually non-colinear. There are \( 6^{(n-1)/3} \) ways to do so, assuming without loss of generality that the last symbol is 1. Since adding to the last row a linear combination of the first \( m - 1 \) rows does not change the linear span of the rows, we have \( 6^{(n-1)/3}/3^{m-1} \) different Hamming subcodes of \( S \). Each of them has 3 cosets in \( S \), so the total number of cosets is \( 6^{(n-1)/3}/3^{m-2} \).

Theorem 2. The number of 1-perfect codes of rank +1 in \( H(n = \frac{3^m - 1}{2}, 3) \) is

\[
\frac{n! \cdot 6^n}{|\text{GL}_{m-1}(\mathbb{F}_3)| \cdot 6^{\frac{n+1}{3}} \cdot 3^{n-m+1}} \cdot N'(n),
\]

where \( N'(n) \) is from (6).

Proof. The total number is \( N'(n) \) multiplied by the number of sets equivalent to the subspace \( S \) (we keep the notation from Lemma 4 and its proof). The group of monomial automorphisms of \( S \) has order

\[
|\text{MAut}(S)| = |\text{MAut}(C^*)| \cdot 6^{(n-1)/3} = |\text{GL}_{m-1}(\mathbb{F}_3)| \cdot 6^{(n-1)/3},
\]

where \( 6 = 3! \) is the number of permutations of three coordinates that correspond to three equal columns of \( H \). Hence, \( |\text{Aut}(S)| = |\text{MAut}(S)| \cdot |S| \), and the number of sets (affine spaces) that are equivalent to \( S \) is

\[
\frac{|\text{Aut}(H(n, 3))|}{|\text{Aut}(S)|} = \frac{n! \cdot 6^n}{|\text{GL}_{m-1}(\mathbb{F}_3)| \cdot 6^{(n-1)/3} \cdot 3^{n-m+1}}.
\]

Corollary 2. The number of equivalence classes of 1-perfect codes of rank +1 in \( H(n = \frac{3^m - 1}{2}, 3) \) is not less than

\[
\left[ \frac{N'(n)}{|\text{GL}_{m-1}(\mathbb{F}_3)| \cdot 2^{\frac{n+1}{3}} \cdot 3^{n-m+1}} \right].
\]
**Proof.** The number of equivalence classes is not less than the value of all codes from Theorem 2 divided by the maximum cardinality of an equivalence class. The maximum cardinality of an equivalence class equals \(|\text{Aut}(H(n,3))|\), i.e., \(n! \cdot 6^n\), divided by the minimum order of the automorphism group of a code from the considered family. The minimum order of the automorphism group of a ternary 1-perfect code of rank +1 is not less than \(3^{n-1}\), by Corollary [1].

For example, for \(m = 3\), we have \(N'(n) = 1352605460594256\), the total number of \((13,3^{10},3)\) codes of rank 11 is 9982462029409199967436800, the number of equivalence classes is at least 9942054. Based on experiments with random codes mentioned in the end of Section 3.1, we expect that the real number of equivalence classes is more than 20 millions (and only 1164330 of them can be obtained by concatenation, see Table [1] in Section 4).

### 3.3. Switchings

In this section, we will show that the ternary 1-perfect codes of rank at most +1 can be obtained from each other by a sequence of two-coordinate switchings. A similar result for extended binary 1-perfect codes of rank at most +2 was proved in [16].

Assume that we have two \(q\)-ary 1-perfect codes \(C, C'\) of length \(n\) and an automorphism \(\beta = (\pi, \bar{\theta})\) of \(H(n,q)\) such that the coordinate permutation \(\pi\) fixes all coordinates except maybe the \(i\)th and the \(j\)th coordinates and the isotopy \(\bar{\theta}\) fixes the values of all coordinates except maybe the \(i\)th and the \(j\)th coordinates. We say that \(C'\) is a two-coordinate switching of \(C\), or an \([i,j]\)-switching of \(C\), or, more concrete, a \(\beta\)-switching of \(C\), if

\[C' \subset C \cup \beta(C).\]

(Similarly, three-, four-, etc. coordinate switchings can be defined.) For a given \(\beta\), the process of finding all switchings of \(C\) is rather simple. We construct the inconsistency bipartite graph \(G_{1,2}(C \cup \beta(C))\) on the vertex set \(C \cup \beta(C)\), where two words are adjacent if the distance between them is 1 or 2. We collect in \(C'\) all isolated vertices of \(G_{1,2}(C \cup \beta(C))\) and add a bipartite part of the remaining subgraph. If this subgraph has more than one connected components, then a bipartite part can be chosen in more than two ways and there are switchings different from \(C\) and \(\beta(C)\). The process of finding a new code \(C'\) from \(C\) as described above is also called switching.

**Theorem 3.** The set of codes of rank at most +1 is connected with respect to the two-coordinate switching.

The proof is more or less straightforward from the corollary of the following lemma.

**Lemma 5.** Every two \(t\)-ary quasigroups \(f, f'\) of order 3 can be obtained from each other by a sequence of transformations \(\gamma_{i,a}, i \in \{0, \ldots, t - 1\}, a \in \{0, 1, 2\}\), where \(\gamma_{i,a}\) swaps the values of \(\{0, 1, 2\} \setminus \{a\}\) in the \(i\)th argument of the function.

**Proof.** By Proposition 2 every \(t\)-ary quasigroup \(f\) or order 3 can be written in the worm \([5]\), where \(a_0, \ldots, a_{t-1} \in \{1, 2\}, a \in \{0, 1, 2\}\). To change \(a_i\), we can apply \(\gamma_{i,a}\). To change \(a\), we can apply \(\gamma_{0,0}\gamma_{0,1}\).  

\[\square\]
Corollary 3. Every two different $\mu$-components $K_\mu$ and $K'_\mu$ satisfying, for a given $\mu$, the conclusion of Theorem 4 are obtained from each other by a sequence of two-coordinate switchings.

Proof. By Theorem 1 the $\mu$-components $K_\mu$ and $K'_\mu$ are constructed from some $(n-1)/3$-ary quasigroups $\lambda$ and $\lambda'$ of order 3. By Lemma 2 it is sufficient to prove the claim for two quasigroups that are obtained from each other by the transformation $\gamma_{i,a}$, for some $i \in \{0, \ldots, (n-4)/3\}$ and $a \in \{0, 1, 2\}$. Without loss of generality, assume $i = 0$. Consider two subcases.

Subcase $a = 0$. An arbitrary word from $K_\mu$ has the form

$$\left(x_0, x_0 + z_0, x_0 - z_0 + \mu_0, x_3, x_3 + z_1, x_3 - z_1 + \mu_1, \ldots, x_{n-4}, x_{n-4} + z_{(n-4)/3}, x_{n-4} - z_{(n-4)/3} + \mu_{(n-4)/3}, \lambda(z_0, \ldots, z_{(n-4)/3})\right).$$

After transforming $(z_0, \ldots, z_{(n-4)/3})$ with $\gamma_{0,0}$, the value of the 1st coordinate turns from $x_0 + z_0$ to $x_0 - z_0$, and the value of the 2nd coordinate turns from $x_0 - z_0 + \mu_0$ to $x_0 + z_0 + \mu_0$. This is the same as permuting these two coordinates and adding $(0, -\mu_0, \mu_0, 0, \ldots, 0)$, which is a $\{1, 2\}$-switching by the definition.

Subcase $a = 1$ (similarly, $a = 2$). After transforming $(z_0, \ldots, z_{(n-4)/3})$ with $\gamma_{0,1}$, the value of the 1st coordinate turns from $x_0 + z_0$ to $x_0 - z_0 + 2$, and the value of the 2nd coordinate turns from $x_0 - z_0$ to $x_0 + z_0 + 1$. This is the same as permuting these two coordinates and adding $(0, 2 - \mu_0, 1 + \mu_0, 0, \ldots, 0)$, which is again a $\{1, 2\}$-switching.

Proof of Theorem 3. Utilizing the characterization in Theorem 1 we see that for the identity $\psi$ the claim follows from Corollary 3. It remains to observe that the action of an arbitrary $\psi$ can be represented as a sequence of two-coordinate switchings.

$\square$

3.4. Maximum kernel

In this section, motivated by a question of one of the reviewers, we consider the structure of a nonlinear ternary 1-perfect code with maximum kernel dimension. The following theorem considers only length-13 codes; however, the most part of the proof (except the last paragraph) is applicable to an arbitrary ternary 1-perfect code $C$ with kernel of size $|C|/3$.

Theorem 4. There is only one equivalence class of 1-perfect ternary codes of length 13 with kernel of dimension 9.

Proof. Let $C = K \cup (\bar{a} + K) \cup (\bar{b} + K)$ be a ternary 1-perfect code with kernel $K$. Since $C$ is nonlinear, its rank is $\dim(K) + 2$, i.e., +1.

We claim that $C' = K \cup (\bar{a} + K) \cup (2\bar{a} + K)$ is also a 1-perfect code. It is sufficient to show that there are no two codewords $\bar{x}$ in $K \cup (\bar{a} + K)$ and $\bar{y}$ in $(2\bar{a} + K)$ at distance less than 3 from each other. If $\bar{x} \in K$, then $\bar{x}, \bar{y} \in 2\bar{a} + C$; if $\bar{x} \in (\bar{a} + K)$, then $\bar{x}, \bar{y} \in \bar{a} + C$. In both cases, $\bar{x}$ and $\bar{y}$ belong to the same 1-perfect code (and, moreover, to different cosets of its kernel), and hence the distance between them is at least 3.

Now, we see that $C$ and $C'$ are 1-perfect codes with symmetric difference $(\bar{b} + K) \cup (2\bar{a} + K)$. Moreover, $C'$ is linear. By the definition of a 1-perfect code, every word from
$(\bar{b} + K)$ is at distance 1 from $(2\bar{a} + K)$. It follows that $(\bar{b} + K) = \bar{e} + (2\bar{a} + K)$ for some weight-1 word $\bar{e}$.

We summarize: the code $C$ is obtained from some linear 1-perfect code $C'$ by translating an affine subspace of size $|C'|/3$ with a translation vector of weight 1.

Since all linear 1-perfect codes are equivalent, we can assume without loss of generality that $C'$ has the form from the conclusion of Theorem 1 with identity $\psi$ and $\lambda_\mu(y_0, y_1, \ldots) = y_0 + y_1 + \ldots$ for all $\mu$. Moreover, we can assume that the nonzero value of $\bar{e}$ (say, $e$) is in the last coordinate, i.e., $\bar{e} = (0, \ldots, 0, e)$ (here we utilize the well-known fact that for every two coordinates $i$ and $j$ there is an automorphism of the Hamming code that sends $i$ to $j$).

After translating the subset $(2\bar{a} + K)$ in direction $\bar{e}$, we see that the resulting code $C$ still satisfies the conclusion of Lemma 2 and hence the conclusion of Theorem 1 with identity $\psi$.

The only difference with $C'$ is that for $C$, we have $\lambda_\mu(y_0, y_1, \ldots) = e + y_0 + y_1 + \ldots$ for $\mu \in K^*$, where $K^*$ is some affine subset of $C^*$ of cardinality $|C^*|/3$.

It remains to observe that all affine 3-subsets of the Hamming $(4, 9, 3)_3$ code $C^*$ are equivalent and that the subcases $e = 1$ and $e = 2$ lead to equivalent codes:

$$K \cup (\bar{a} + K) \cup (2\bar{e} + 2\bar{a} + K) = \bar{a} + 2(K \cup (\bar{a} + K) \cup (\bar{e} + 2\bar{a} + K)).$$

\[ \square \]

4. Enumeration of concatenated ternary 1-perfect codes of length 13

In this section, we describe the computer-aided classification of the concatenated ternary 1-perfect codes of length 13. As intermediate steps, of independent interest, we get the classification of RM-like $(9, 3^6, 3)_3$ codes, collections of such codes (including RM-like partitions) that are subsets of the all-parity-check code $M$, and partitions of $\mathbb{F}_3^4$ into 1-perfect codes.

Before we describe the approaches we use on each step of the classification, we briefly discuss recognizing the equivalence, which is a very important and the most time-consuming tool for such classifications.

4.1. Equivalence and graph isomorphism

A usual way to work with the equivalence of codes is to represent them by graphs in such a way that two codes are equivalent if and only if the corresponding graphs are isomorphic, see [12, §3.3.2]. It is easy to adopt such an approach for collections of codes; one of the ways is to represent a collection $(C_i)_{i=0}^{k-1}$ of codes in $\mathbb{F}_q^n$ as a mixed-alphabet code in $\mathbb{F}_q^n \times \{0, \ldots, k-1\}$:

$$C = \{(c_0, \ldots, c_{n-1}, i) : i \in \{0, \ldots, k-1\}, (c_0, \ldots, c_{n-1}) \in C_i\}.$$  

A standard software that helps to recognize the graph isomorphism is nauty&traces [13]; it is realized as a package that can be used in c or c++ programs. With this package,
for a graph one can compute its canonically-labeled version, such that two graphs are isomorphic if and only if the corresponding canonically-labeled graphs are equal to each other. The same procedure computes the automorphism group of the graph, which can be used for the numerical validation of the results. The library suggests two alternatives of such procedure, nauty and traces. According to our experience, traces worked faster on codes with considered parameters.

4.2. Classification of RM-like codes of length 9

Every RM-like code is a subset of a distance-2 MDS code, say $M$. By Proposition 2 in the ternary case such code $M$ is unique up to equivalence, for each length, and we assume without loss of generality that

$$M = \{(x_0, \ldots, x_8) \in \mathbb{F}_3^9 : x_0 + \ldots + x_8 = 0\}.$$  \hspace{1cm} (7)

By $\overline{M}$, we denote the complement of $M$; i.e.,

$$\overline{M} = \{(x_0, \ldots, x_8) \in \mathbb{F}_3^9 : x_0 + \ldots + x_8 \in \{1, 2\}\}.$$  

Our goal, at this stage, is to classify all RM-code subsets of $M$ up to equivalence. Without loss of generality, we take $\bar{0}$ as a codeword. We say that a set $C_i$ of vertices of $H(9, 3)$ is a partial code of level $i$ if

(I) $C_i$ contains $\bar{0}$,

(II) $C_i$ consists of words of weight at most $i$,

(III) every word of weight at most $i − 1$ in $\overline{M}$ is adjacent to exactly one codeword of $C_i$, and

(IV) $C_i$ is a distance-3 code.

The classification algorithm we use is based on the straightforward fact that by removing the weight-$(i + 1)$ codewords from a partial code of level $i + 1$ we obtain a partial code of level $i$.

1. We start with the singleton $\{\bar{0}\}$, which is a unique partial code of level 1 and 2.

2. Assume that at step $i$ we have found representatives of all equivalence classes of partial codes of level $i$. For each representative $C_i$, we can find all partial codes of level $i + 1$ that include $C_i$ in the following way.

- Denote by $W_i$ the set of weight-$i$ words in $\overline{M}$ that are at distance more than 1 from $C_i$.

- Denote by $R_i$ the set of weight-$(i + 1)$ words in $M$ that are at distance at least 3 from $C_i$.

- Let $X = (X_{\bar{x}, \bar{y}})$ be the $\{0, 1\}$-matrix whose rows are indexed by elements of $R_i$ and columns are indexed by elements of $W_i$ such that $\bar{x}$ from $R_i$ and $\bar{y}$ from $W_i$ are adjacent if and only if $X_{\bar{x}, \bar{y}} = 1$.  

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• As follows from (III), for every partial code $C_{i+1}$ of level $i + 1$, the sum of rows of $X$ indexed by the elements of $C_{i+1} \setminus C_i$ equals the all-one row. Finding such collections of rows for a $\{0, 1\}$-matrix is an instance of the well-known exact cover problem, which is usually solved by Donald Knuth’s Algorithm X (already realized as a function in many programming languages).

• After completing $C_i$ by a solution of the exact cover problem with matrix $X$, we need to check property (IV); all solutions that satisfy it correspond to partial codes of level $i + 1$, by the definition.

Finally, all found continuations are checked for equivalence, and we keep only nonequivalent representatives.

We repeat p. 2 for $i = 3, 4, 5, 6, 7, 8, 9$ and obtain the following results:

There are 705600 partial codes of level 3; they form 9 equivalence classes; the orders of the automorphism groups are 864, 108, 32, 24, 18, 6, 6, 4, 4.

**Remark 3.** It is not difficult to observe that every partial code of level 3 consists of the all-zero word, twelve words with three 1s, and twelve words with three 2s. The twelve words from each of the last two groups (to be exact, the sets of indices of nonzero coordinates of these words) form a combinatorial structure known as a Steiner triple system of order 9, SQS(9), see e. g. [6]. There are 840 different SQS(9), and all of them are isomorphic. There are $840^2 = 705600$ pairs of SQS(9), and 9 isomorphism classes of such pairs.

The partial codes of level 3 are continued to, respectively, 4, 4, 0, 4, 0, 0, 0, 0, 0 nonequivalent partial codes of level 4, with automorphism group orders 864, 216, 72, 72, 108, 108, 36, 36, 12, 12, 12, 12. Six of these codes, with automorphism group orders 864, 72, 108, 36, 12, 12, are continued to a partial code of level 5, 6, 7, 8, and 9; at each step the continuation is unique and preserves the automorphism group of the “parent” partial code. The other 6 partial codes of level 4 are continued uniquely to partial codes of level 5, but not to partial codes of level 6.

**Theorem 5** (computational). There are 1428840 $(9, 3^6, 3)_3$ RM-like codes that are subcodes of $M$. 158760 of them contain the all-zero word; they form 4 equivalence classes (the corresponding automorphism group orders are 629856, 78732, 8748, 5832), 6 monomial equivalence classes (the corresponding monomial automorphism group orders are 864, 108, 12, 72, 36, 12), 7 permutation equivalence classes (the corresponding permutation automorphism group orders are 432, 54, 6, 36, 18, 12, 12).

### 4.3. Collections of disjoint RM-like codes

For the concatenation construction, we need a partition of the distance-2 MDS code $M$ [7] into 9 RM-like codes. In this section, we consider the classifications of collections of disjoint RM-like subcodes of $M$; we call such a collection a $k$-collection, where $k$ is the number of codes in it. We classify them recursively. The algorithm is rather straightforward; however, the amount of calculations was huge (it took about 30 core-years to finish it), and the details considered below were essential to make it doable with reasonable computational resources.
We first define the equivalence for \( k \)-collections. Two collections \((C_i)_{i=0}^{k-1}\) and \((D_i)_{i=0}^{k-1}\) of vertex sets of \( H(n, q) \) are \textit{equivalent} if there is an automorphism \( \gamma \) of the graph \( H(n, q) \) and a permutation \( \tau \) of \( \{0, \ldots, k-1\} \) such that \( \gamma(C_i) = D_{\tau(i)}, \ i = 0, \ldots, k-1 \). If, additionally, \( \tau(k-1) = k-1 \), then we will say that \((C_i)_{i=0}^{k-1}\) and \((D_i)_{i=0}^{k-1}\) are \textit{strongly equivalent}. The set or all pairs \((\gamma, \tau)\) such that \( \gamma(C_i) = D_{\tau(i)}, \ i = 0, \ldots, k-1 \), forms the \textit{automorphism group} of \((C_i)_{i=0}^{k-1}\).

From Section 4.2 we know that the number of equivalence classes of RM-like sub-codes of \( M \) is 4. For every \( k \)-collection \((C_i)_{i=0}^{k-1}\), we define its \textit{type} as the sequence \( t_0t_1\ldots t_{k-1} \) where \( C_i \) belongs to the \( t_i \)-th equivalence class, \( t_i \in \{0, 1, 2, 3\} \), \( i = 0, 1, \ldots, k-1 \). The type is \textit{sorted} if \( t_0 \leq t_1 \leq \ldots \leq t_{k-1} \). Obviously, every equivalence class of \( k \)-collections has a representative of sorted type, and this sorted type is uniquely defined for the class. Trivially, removing the last code from a \( k \)-collection \((C_i)_{i=0}^{k-1}, k > 1\), we obtain a \((k-1)\)-collection \((C_i)_{i=0}^{k-2}\), thus, we will say that \((C_i)_{i=0}^{k-1}\) is a \textit{continuation} of \((C_i)_{i=0}^{k-2}\). It is also clear that if two \( k \)-collections are strongly equivalent, then they are continuations of equivalent \((k-1)\)-collections.

For a given sorted type \( t_0t_1\ldots t_{k-1} \), we classify all \( k \)-collections of this type up to equivalence in two steps.

- (I) At first, for each representative \((C_i)_{i=0}^{k-2}\) of \((k-1)\)-collections of type \( t_0\ldots t_{k-2}, \) we construct all possible continuations of type \( t_0\ldots t_{k-2}t_{k-1} \). For this, we consider all RM sub-codes of \( M \) from the \( t_{k-1} \)-th equivalence class (there are 7560, 60480, 544320, and 816480 such codes for \( t_{k-1} = 0, 1, 2, 3 \), respectively). Those codes who are disjoint with all \( C_i, i = 0, \ldots, k-2 \), are used for the role of \( C_{k-1} \) to form a continuation \((C_i)_{i=0}^{k-1}\) of \((C_i)_{i=0}^{k-2}\). The resulting \( k \)-collections are checked for the strong equivalence, and we keep only representatives of strong equivalence classes. This step can be done separately for each initial \((k-1)\)-collection, which allows to process different \((k-1)\)-collections on different machines with relatively small (several gigabytes) amount of memory.

- (II) Next, all representatives of strong equivalence classes kept at step (I) for all different initial \((k-1)\)-collections of the same type are checked for equivalence and representatives of equivalence classes are collected. Because of the huge amount of resulting representatives, this step is processed on a machine with large amount of memory (more than 160 Gb). One of benefits of the two-step approach, apart from the rational use of computational resources, is that comparing for equivalence, especially for big values of \( k (6-9) \), takes much more time than comparing for strong equivalence, and the precalculation made at step (I) minimizes the amount of such operations.

1-collections are essentially RM-like codes, which are classified in Section 4.3. With the two-step algorithm described above, nonequivalent \( k \)-collections are classified subsequently for \( k = 2, \ldots, 9 \).

\textbf{Theorem 6 (computational).} \textit{There are} 4, 131, 10956, 118388, 501915, 945965, 755066, 314833, and 65436 equivalence classes of \( k \)-collections of disjoint RM-like subcodes of the distance-2 MDS code \( M \) \((7)\) for \( k = 1, 2, 3, 4, 5, 6, 7, 8, 9 \), respectively. \textit{The distribution}
of equivalence classes of 9-collections (RM-like partitions of $M$) by type is the following:

- $000000000$: 6
- $000000011$: 11
- $000000111$: 6
- $000000222$: 20
- $000000333$: 41
- $000001111$: 26
- $000011111$: 11
- $000011222$: 107
- $000011333$: 173
- $000111111$: 66
- $000111222$: 41
- $000111333$: 70
- $000222222$: 347
- $000222333$: 990
- $000333333$: 885
- $001111111$: 24
- $001111222$: 199
- $001111333$: 381
- $011111111$: 51
- $011111222$: 112
- $011111333$: 208
- $011222222$: 1205
- $011222333$: 3493
- $011333333$: 3006
- $111111111$: 26
- $111111222$: 99
- $111111333$: 237
- $111222222$: 381
- $111222333$: 1180
- $111333333$: 1126
- $222222222$: 3228
- $222222333$: 14356
- $222333333$: 21405
- $333333333$: 11919

We skip the sorted types that are not represented, e.g., $000000022$: 0.

4.4. 1-perfect partitions of length 4

There are 72 1-perfect $(4, 9, 3)_3$ codes; all of them are equivalent to the 3-ary Hamming code of length 4. Straightforward computations show that from these 72 codes, one can choose 9 pairwise disjoint codes in 104 ways.

Theorem 7 (computational). There are exactly two equivalence classes of partitions of $\mathbb{F}_3^4$ into 1-perfect codes. Each of the 8 partitions from the smallest class consists of the cosets of the same Hamming code, and the order of its automorphism group is 384. Each of the 96 remaining partitions consists of cosets of two different Hamming codes, in the quantity of 6 and 3, and the automorphism group order is 32.

4.5. Concatenated codes

In this section, we describe the final steps of the classification of concatenated 3-ary 1-perfect codes of length 13. As reported in Sections 4.3 and 4.4, we have classified up to equivalence the partitions of the distance-2 MDS code $M$ into RM-like $(9, 3^6, 3)_3$ codes and the partitions of $\mathbb{F}_3^4$ into 1-perfect $(4, 9, 3)_3$ codes. The third ingredient of the concatenation construction is a permutation of 9 codes. There are $9! = 362880$ different permutation, which form the symmetric group Sym(9), and this 9! is the number of different concatenated codes that can be obtained from given partitions of $M$ and $\mathbb{F}_3^4$.

In the following subsection, using the knowledge about the automorphism groups of the used RM-like and 1-perfect partitions, we certify that some of these codes are guaranteedly equivalent; this essentially reduces the number of considered codes.

4.5.1. Double-cosets

The following fact is well known; in particular, similar arguments were used in [27] for the classification of concatenated binary codes.

Lemma 6. Assume that $\tilde{C} = (C_0, \ldots, C_{k-1})$ and $\tilde{P} = (P_0, \ldots, P_{k-1})$ are collections of mutually disjoint codes in $H(n', q)$ and $H(n'' q)$, respectively. Assume that $\alpha$ is a permutation of $\{0, \ldots, k - 1\}$ and we have two automorphisms $(\pi', \theta', \tau')$ and $(\pi'', \theta'', \tau'')$ of $\tilde{C}$ and $\tilde{P}$ respectively. Then the concatenated codes

$$
\bigcup_{i=0}^{k-1} C_i P_{\alpha(i)} \quad \text{and} \quad \bigcup_{i=0}^{k-1} C_i P_{\tau''(\alpha(\tau'(i)))}
$$

are equivalent if and only if $\pi' = \tau''$ and $\theta' = \tau''$. The proof is similar to the one of [27, Lemma 5].
are equivalent.

Proof.

\[
\bigcup_{i=0}^{k-1} C_i \times P_{\tau'\bar{\alpha}(\tau'(i))} \bigcup_{j=0}^{k-1} C_{\tau'\bar{\alpha}(j)} \times P_{\tau'\bar{\alpha}(j)} = \bigcup_{j=0}^{k-1} \pi'\bar{\theta}'\bar{\alpha}(C_j) \times P_{\tau'\bar{\alpha}(j)}
\]

for some \((\pi, \bar{\theta})\), composed from \((\pi', \pi'\bar{\theta}'\bar{\alpha})\) and \((\pi'', \bar{\theta}'\bar{\alpha})\) acting on the corresponding coordinates.

Hence, for given partitions \(\bar{C} = (C_0, \ldots, C_k)\) and \(\bar{P} = (P_0, \ldots, P_k)\), we can restrict our search by considering only permutations that are representatives of the double-cosets from \(T(\bar{P}) \setminus \text{Sym}(9) / T(\bar{C})\), where \(T(\bar{D}) = \{\tau : (\pi, \bar{\theta}, \tau) \in \text{Aut}(\bar{D})\) for some \(\pi, \bar{\theta}\}.

**Corollary 4.** For given partitions \(\bar{C} = (C_0, \ldots, C_k)\) and \(\bar{P} = (P_0, \ldots, P_k)\) all permutations \(\tau\) from the same double-coset in \(T(\bar{P}) \setminus \text{Sym}(9) / T(\bar{C})\) result in equivalent concatenated codes.

The automorphism groups of the 65435 nonequivalent partitions of \(M\) and two nonequivalent partitions of \(\mathbb{F}_3^4\) are found in the way described in Section \ref{sec:automorphism_groups}. Using GAP \cite{GAP}, representatives of all double-cosets were found in several hours (to fasten the process, we group partitions with the same automorphism group and run the double-coset calculation once for each such group). In such a way, we obtain 93278251 concatenated 1-perfect \((13, 3^{10}, 3)_3\) codes. This amount is too huge to check the nonequivalence using the approach described in Section \ref{sec:automorphism_groups} (it takes from less than 1 second to several hours for one code, depending on its symmetric properties). However, as we will see below, more than 99.9% of these codes are guaranteedly nonequivalent, and it remains to process the other 0.1%.

### 4.5.2. Uni-concatenated and multi-concatenated codes

After permuting the coordinates, a concatenated \((13, 3^{10}, 3)_3\) code \(P\) can lose the property to be concatenated. However, if the coordinate permutation \(\pi\) fixes the partition of the coordinates into two groups, \(\{0, \ldots, 8\}\) and \(\{9, 10, 11, 12\}\), then the resulting code \(\pi(P)\) will be surely concatenated. Indeed, the action of such a permutation on the concatenated code can be treated as the actions of two coordinate permutations on the length-9 and length-4 codes \(C_i\) and \(P_i\) in the construction \ref{construction}. If a coordinate permutation \(\pi\) changes the partition \(\{(0, \ldots, 8), \{9, 10, 11, 12\}\}\), and \(\pi(P)\) is still concatenated, then the concatenation representation of \(\pi(P)\) is not derived from the concatenation representation of \(P\); we can say in this case that \(C\) has more than one concatenation structure, or for short,
that it is \textit{multi-concatenated}. Concatenated codes that are not multi-concatenated are called \textit{uni-concatenated}.

The equivalence between uni-concatenated codes can be recognized in an easier way than the equivalence between arbitrary codes. The following lemma is straightforward.

\textbf{Lemma 7.} If two uni-concatenated codes

\[ P = \bigcup_{i=0}^{8} C_i P_{\tau(i)} \quad \text{and} \quad D = \bigcup_{i=0}^{8} A_i B_{\gamma(i)} \]

are equivalent, then \((C_i)_{i=0}^{8}\) is equivalent to \((A_i)_{i=0}^{8}\) and \((P_i)_{i=0}^{8}\) is equivalent to \((B_i)_{i=0}^{8}\).

Since, in our classification, we use only one representative from each equivalence class of RM-like partitions and 1-perfect partitions, we can obtain two equivalent uni-concatenated codes only if the ingredient partitions are the same in the both concatenations.

\textbf{Lemma 8.} Two uni-concatenated codes

\[ P = \bigcup_{i=0}^{8} C_i P_{\tau(i)} \quad \text{and} \quad D = \bigcup_{i=0}^{8} C_i P_{\gamma(i)} \quad (8) \]

are equivalent if and only if \(\tau\) and \(\gamma\) are in the same double-coset from \(T(\bar{P})/\text{Sym}(9)/T(\bar{C})\).

\textit{Proof.} Assume that \(D\) and \(P\) are equivalent, i.e., \(D = \alpha(P)\), where \(\alpha = (\pi, \bar{\theta})\) for some coordinate permutation \(\pi = (\pi(0), \pi(1), \ldots, \pi(12))\) and isotopy \(\bar{\theta} = (\theta_0, \ldots, \theta_{12})\). By the definition of uni-concatenated codes, \(\pi\) fixes the partition \(\{0, \ldots, 8\}, \{9, 10, 11, 12\}\).

Hence, \(\pi' = (\pi(0), \pi(1), \ldots, \pi(8))\) and \(\pi'' = (\pi(9) - 9, \pi(10) - 9, \pi(11) - 9, \pi(12) - 9)\) are valid permutations of \(0, \ldots, 8\) and \((0, 1, 2, 3)\), respectively. Then, denoting \(\bar{\theta}' = (\theta_0, \ldots, \theta_8)\), \(\bar{\theta}'' = (\theta_9, \ldots, \theta_{12})\), \(\alpha' = (\pi', \bar{\theta}')\), \(\alpha'' = (\pi'', \bar{\theta}'')\), we find

\[ D = \alpha(P) = \bigcup_{i=0}^{8} \alpha'(C_i) \alpha''(P_{\tau(i)}). \quad (9) \]

We state that there is a permutation \(\beta\) in \(T(\bar{C})\) such that \(\alpha'(C_i) = C_{\beta(i)}\), \(i = 0, \ldots, 8\). Denote by \(p_i\) the word of weight at most 1 in \(P_i\), \(i = 0, \ldots, 8\). It follows from \(\bar{C}\) that

\[ C_i = \{ \bar{c} \in \mathbb{F}_3^9 : \bar{c}p_{\gamma(i)} \in \bar{D}\}, \]

\[ C_{\gamma^{-1}(i)} = \{ \bar{c} \in \mathbb{F}_3^9 : \bar{c}p_i \in \bar{D}\}. \quad (10) \]

Denote by \(r_i\) the word of weight at most 1 in \(\alpha''(P_i)\), \(i = 0, \ldots, 8\). It follows from \(\bar{D}\) that

\[ \alpha'(C_i) = \{ \bar{c} \in \mathbb{F}_3^9 : \bar{c}r_{\tau(i)} \in \bar{D}\}. \]

Since \(\{p_i\}_{i=0}^{8} = \{r_i\}_{i=0}^{8}\), we have \(r_i = p_{\rho(i)}\) for some permutation \(\rho\), and the last equation turns to

\[ \alpha'(C_i) = \{ \bar{c} \in \mathbb{F}_3^9 : \bar{c}p_{\rho(\tau(i))} \in \bar{D}\}, \]

\[ \alpha'(C_{\rho^{-1}(\tau^{-1}(i))}) = \{ \bar{c} \in \mathbb{F}_3^9 : \bar{c}p_i \in \bar{D}\}. \]

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Comparing with (10), we find

\[ C_{\gamma^{-1}(i)} = \alpha'(C_{\rho^{-1}(\tau^{-1}(i))}) , \]

\[ C_{\gamma^{-1}(\tau(\rho(\rho(i))))} = \alpha'(C_i) , \]

and so (*) holds with \( \gamma^{-1}\tau\rho \), which is in \( T(\bar{C}) \) by the definition of \( T(\bar{C}) \).

(**) We state that there is a permutation \( \lambda \) in \( T(\bar{P}) \) such that \( \alpha''(P_i) = P_{\lambda(i)} , \) \( i = 0, \ldots, 8 \). The proof is similar to (*). Choose a word \( \bar{o} \) in \( \mathbb{F}_3^8 \setminus M \), where \( M = \bigcup_{i=0}^8 C_i \). By Proposition 11 for each \( i \) from \( \{0, \ldots, 8\} \) there is a unique \( \bar{c}_i \) in \( C_i \) at distance 1 from \( \bar{o} \). From (8) we find

\[ P_{\gamma(i)} = \{ \bar{p} \in \mathbb{F}_3^4 : \bar{c}_i\bar{p} \in D \} . \]  

(11)

It is easy to see that \( \bigcup_{i=0}^8 C_i = \bigcup_{i=0}^8 \alpha'(C_i) \), and so for some permutation \( \rho \) we have \( c_{\rho(i)} \in \alpha'(C_i) , \) \( i = 0, \ldots, 8 \). From (9) we find

\[ \alpha''(P_{\tau(i)}) = \{ \bar{p} \in \mathbb{F}_3^4 : \bar{c}_{\rho(i)}\bar{p} \in D \} , \]

\[ \alpha''(P_{\tau(\rho^{-1}(i))}) = \{ \bar{p} \in \mathbb{F}_3^4 : \bar{c}_i\bar{p} \in D \} . \]  

(12)

From (11) and (12) we conclude that (***) holds with \( \lambda = \gamma\rho\tau^{-1} \).

Now, from (5) and (9), we have

\[ \bigcup_{i=0}^8 C_i P_{\gamma(i)} = \bigcup_{i=0}^8 C_{\beta(i)} P_{\lambda(\tau(i))} = \bigcup_{j=0}^8 C_j P_{\lambda(\tau(\beta^{-1}(j)))} \]

with \( \beta \) from \( T(\bar{C}) \) and \( \lambda \) from \( T(\bar{P}) \). We see that \( \gamma = \lambda\tau\beta^{-1} \), which proves the “only if” statement. The “if” statement is straightforward. \[ \square \]

So, among the 93278251 different codes obtained as shown in the end of Section 4.5.1 only multi-concatenated codes can be equivalent. Most of those codes have rank 12 and are uni-concatenated by the following lemma.

**Lemma 9.** A concatenated \((13,3^{10},3)_3\) code has rank at most 12. A multi-concatenated \((13,3^{10},3)_3\) code has rank at most 11.

**Proof.** Let a \((13,3^{10},3)_3\) code \( P \) be represented in the form (11). Assume without loss of generality that \( \bar{0} \in P \). The union \( M = \bigcup_{i=0}^8 C_i \) is a distance-2 MDS codes. Such a code in \( \mathbb{F}_3^8 \) is unique up to equivalence, and it is orthogonal to a word from 1s and 2s. It follows that \( P \) is orthogonal to a word with nonzeros in the first 9 coordinates and zeros in the last 4 coordinates. Hence, the rank of \( P \) is less than 13. If the code is multi-concatenated, then similarly it is orthogonal to another word with another set of nonzero positions. Hence, the rank does not exceed \( 13 - 2 \). \[ \square \]

For the remaining 1164331 codes of rank less than 12, the multi-concatenated property can be checked relatively fast, and we found that the majority of them are uni-concatenated. Recognizing equivalence among the 74464 multi-concatenated codes (it took about 4.5 core-years), 37540 equivalence classes were found. The final results are described in the next section.
4.5.3. Results

**Theorem 8** (computational). *There are exactly $93241327$ equivalence classes of concatenated ternary 1-perfect codes of length 13.*

| dim(kernel) | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 |
|-------------|----|----|----|----|----|----|----|----|----|----|----|
| rank 10     | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  |    |
| rank 11     | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  |
| rank 12     | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  |
| rank 13     | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  |
| rank 10–13  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  |

Table 1: The number of equivalence classes of concatenated ternary 1-perfect codes of length 13 for each admissible rank and kernel dimension.

The distribution of equivalence classes according to the rank and the dimension of the kernel is shown in Table 1. The mark “−” in the table denotes that codes with the corresponding parameters do not exist even without the restriction to be concatenated. In all such cases, there is a theoretical explanation:

- a code of rank 10 is linear and has kernel dimension 10, and vice versa;
- by Corollary 1 a code of rank 11 has kernel dimension at least 4;
- the following argument is a special case of [28, Proposition 5.1]: if a ternary code $C$ has the kernel of size $|C|/3$ (in our case, kernel dimension 9), then the affine span of $C$ has size $3 \cdot |C|$ (i.e., rank 11 in our case);
- codes of rank 13 and kernel dimension 8 do not exist because of the nonexistence of a full-rank tiling of $F_3^5$ [22] (for the connection between tilings and 1-perfect codes, see [14]).

Taking into account recently discovered length 13 perfect ternary codes of full rank and kernel dimension from 3 to 7 [14], only the existence of 1-perfect $(13,3^{10},3)_3$ codes of kernel dimension less than 3 remains open (in Table 1 the corresponding values are grayed). In particular, we see that with concatenation, for $(13,3^{10},3)_3$ codes of rank 11, one can obtain any kernel dimension from 4 to 9. In contrast, by the fixed-coordinate switching from the Hamming code, only kernel dimensions 8 and 9 can be obtained for these parameters, see [28, Table 1]. Examples of 3-ary length-13 perfect codes for each known values of the rank and the kernel dimension (including non-concatenated rank-13 codes) are available in [12].

The distribution of equivalence classes according to the order of the automorphism group is shown in Table 2. Note that the order of the automorphism group was calculated directly (see Section 4.1) for multi-concatenated codes, while for uni-concatenated codes it was found from the automorphism group orders of the partitions $\bar{C}$ and $\bar{P}$ and the size of the corresponding double-coset.

We finalize this section with two particular questions regarding characteristics of unrestricted $(13,3^{10},3)_3$ codes. Note that if the answer to the first question is “no”, then the second one has answer “27”.

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| Aut | #  | Aut | #  | Aut | #  | Aut | #  | Aut | #  | Aut | #  | Aut | #  |
|-----|----|-----|----|-----|----|-----|----|-----|----|-----|----|-----|----|
| 27  | 49195 | 729 | 3034912 | 6561 | 8666 | 59049 | 9 | 472392 | 5 | 11 | 7,8 |
| 54  | 24928 | 972 | 24487 | 8748 | 2601 | 69984 | 1 | 708588 | 10 | 11 | 7,8 |
| 81  | 60630474 | 1296 | 1 | 11664 | 3 | 78732 | 135 | 1062882 | 1 | 12 | 8 |
| 108 | 1887 | 1458 | 222834 | 13122 | 4521 | 104976 | 6 | 1417176 | 3 | 11 | 7,8 |
| 162 | 3120437 | 1944 | 439 | 17496 | 141 | 118098 | 31 | 1889568 | 2 | 11 | 8 |
| 216 | 46 | 2187 | 202868 | 19683 | 167 | 157464 | 13 | 2834352 | 1 | 11 | 8 |
| 243 | 24257914 | 2916 | 10047 | 26244 | 634 | 209952 | 2 | 4251528 | 1 | 11 | 8 |
| 324 | 24277 | 3888 | 3 | 34992 | 4 | 236196 | 21 | 6377292 | 1 | 11 | 9 |
| 486 | 1588122 | 4374 | 30442 | 39366 | 348 | 314928 | 1 | 8503056 | 1 | 11 | 8 |
| 648 | 308 | 5832 | 311 | 52488 | 52 | 354294 | 14 | 663238368 | 1 | 10 | 10 |

Table 2: The number of equivalence classes of concatenated ternary 1-perfect codes of length 13 for each admissible order of the automorphism group (for some codes, the rank $R$ and the kernel dimension $K$ are shown).

**Problem 1.** Do there exist ternary 1-perfect codes of length 13, rank 12 or 13, and with kernel dimension 2, 1, 0? What is the minimum number of automorphisms of a ternary 1-perfect code of length 13?

## 5. Conclusion

In this paper, we studied ternary 1-perfect codes, mainly focusing on the classification results. The two main results of the paper illustrate the two main approaches in constructing nonlinear 1-perfect codes, the switching approach and the concatenation. (It should be noted that there are also algebraic ways to construct 1-perfect codes; for example, one can construct ternary $\mathbb{Z}_4\mathbb{Z}_9$-linear perfect codes as shown in [37].) We theoretically characterized 1-perfect codes of rank $+1$ of any admissible length and obtained a computer-aided enumeration of the equivalence classes of concatenated 1-perfect codes of length 13. The rest of this section contains concluding remarks that concern related questions for further investigation.

Our characterization of ternary 1-perfect codes of rank $+1$ is in the spirit of similar results for binary 1-perfect codes of rank $+2$ in [2]. Based on the connection between 4-ary length-$n$ and binary length-$3n$ perfect codes (for example, by concatenation [45], see, e.g., [36] Remark 2 for the concrete mapping), one can hope that the 4-ary 1-perfect codes of rank $+1$ can also be characterized; this remains actual as an objective for future research. A variant of that problem is to find a characterization of 4-ary 1-perfect codes of small $(+1$ or $+2$) 2-rank, where 2-rank is the dimension of the affine span over the subfield $\mathbb{F}_2$ of $\mathbb{F}_4$. In contrast to the unique linear Hamming code, there are nonequivalent additive (i.e., linear over $\mathbb{F}_2$, or, equivalently, of 2-rank +0) 4-ary 1-perfect codes of the same length [13], which provides additional difficulties to the characterization of 4-ary 1-perfect codes of small 2-rank. A similar question can also be considered for 1-perfect codes in Doob spaces [35], which have much in common with the 4-ary Hamming space.
The evaluation of the number of equivalence classes of 1-perfect codes of length 13 and rank +1 shows that their number (more than 20 millions) is too large to enumerate them computationally using the straightforward approach. However, our experience with concatenated codes shows that combining theoretical and computational approaches can help to enumerate much larger number of equivalence classes of \((13, 3^{10}, 3)_3\) codes. So, we hope that with developing the theory, together with improving the graph isomorphism software and growing the performance of computers, the enumeration of codes of limited rank or even all 1-perfect \((13, 3^{10}, 3)_3\) codes might be possible. For studying ternary 1-perfect codes of larger lengths, only theoretical results can be applied, and among the interesting problems we mention the problem of characterizing all admissible pairs (rank, kernel dimension) of ternary 1-perfect codes, which was done for binary codes in [3]. Another challenge is the problem of existence of an \((n = \frac{q^n - 1}{q - 1} - 1, q^{n-m}, 3)_q\) code that is not a shortened 1-perfect code. Such codes were found for \(q = 4\) [38], but the ternary case, including the parameters \((12, 3^9, 3)_3\), remains unsolved. Finally, agreeing that the binary case is the most interesting among \(q\)-ary 1-perfect codes, we believe that the ternary 1-perfect codes also deserve the separate study.

Data availability

The dataset containing the results of the classifications described in Sections 4.2, 4.3, 4.4, and 4.5 is available in the IEEE DataPort repository [12].

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