Interaction of massless Dirac field with a Poincaré gauge field

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In this paper we consider a model of Poincaré gauge theory (PGT) in which a translational gauge field and a Lorentz gauge field are actually identified with the Einstein’s gravitational field and a pair of “Yang-Mills” field and its partner, respectively. In this model we re-derive some special solutions and take up one of them. The solution represents a “Yang-Mills” field without its partner field and the Reissner-Nordström type spacetime, which are generated by a PGT-gauge charge and its mass. It is main purpose of this paper to investigate the interaction of massless Dirac fields with those fields. As a result, we find an interesting fact that the left-handed massless Dirac fields behave in the different manner from the right-handed ones. This can be explained as to be caused by the direct interaction of Dirac fields with the “Yang-Mills” field. Accordingly, the phenomenon can not happen in the behavior of the neutrino waves in ordinary Reissner-Nordström geometry. The difference between left- and right-handed effects is calculated quantitatively, considering the scattering problems of the massless Dirac fields by our Reissner-Nordström type black-hole.

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I. INTRODUCTION

Poincaré gauge theory was first founded by Utiyama [1] and Kibble [2], and later developed by Hayashi [3], Hehl and his collaborators [4]. Since then, PGT has been studied by many people. Most of them have adopted a model with nine independent parameters. The physical meaning of those parameters has been examined until now only in the weak-field approximation [5]. The good choice for parameters and also PGT itself will, of course, have to be tested eventually through some experiments. However, before doing so we need to know the behavior of strong Poincaré gauge fields and therefore to have some exact solutions of PGT with nine independent parameters. But it is very hard to solve exactly such the full-theory. Thus various models have been proposed and solved by many authors [6]. In those solutions there seems to exist such ones that the property depends strongly on the structure of the gauge group and therefore have little dependence on the choice of parameters, i.e., on a peculiarity of the models. For example, in a few years ago [7] one of the authors has obtained exact solutions (monopole solutions) in a model where PGT can be actually identified with complex Einstein-Yang=Mills theory. The idea is based on an universal property which is an extension of ’t Hooft’s original idea [8]: any non-Abelian theory can have a monopole solution if it has a compact covering group in its static limit. Accordingly, the investigations based on the solutions seem to give us some universal results.

In this paper we shall re-derive some special solutions of above ones, but in the different method from Ref [7]. For, owing to this method we can more naturally interpret two kinds of integral (vector) constants, $\mathbf{Q}_2$, $\mathbf{Q}_1$ as the “gauge charges” which creat the “Yang-Mills” field and its partner field, respectively. And the magnitude $|\mathbf{Q}_2|$ can be identified with an electric charge, according to Kalb’s anzatz [9]. On the other hand, our gravitational field is generated...
by not only the mass $M$ but also two gauge charges $\vec{Q}_1$, $\vec{Q}_2$ through a combination $q^2 = -2(a_1 + a_3)(\vec{Q}_1^2 - \vec{Q}_2^2)$. And the spacetime structures can be classified by the signature of $q^2$ as follows: (I) if $q^2 = 0$, then the spacetime is just Schwarzschildian, (II) if $q^2 > 0$, Reissner-Nordstr"omian, and (III) if $q^2 < 0$, then Schwarzschildian-like. The difference between (I) and (III) may, for example, be clarified by considering the correction of the classical Kepler orbits. The details will be discussed in the forthcoming paper.

By the way, it is well-known in the weak-field approximation that the Poincaré gauge fields interact with the Dirac field through only the axial part of the Lorentz gauge fields. In our solutions the Dirac fields are also able to interact with the gravitational field through the gauge charge $q^2$. In fact, in a region $\Delta = r^2 - 2Mr + q^2 > 0$, our Dirac equation can be written as $[\gamma^k \partial_k + \frac{1}{2} \Delta_{mnk} \gamma^k \sigma^{mn} - \frac{1}{2\sqrt{\Delta}} \gamma^a \gamma_5 + im] \Psi = 0$, where $\gamma^k$ and $\sigma^{mn}$ are Dirac matrices and their combinations, respectively. And $\Delta_{kmn}$ are Ricci's rotation coefficients and $Q_{2[a]}$ is an a-component of $\vec{Q}_2$. From this we can easily see that the Dirac fields can directly interact with "Yang-Mills" field through the gauge charge $\vec{Q}_2$, and not directly interact with the partner field but only through the gravitational field. This means that the direction of $\vec{Q}_1$ has no effect on the Dirac fields. And therefore, if we put $\vec{Q}_1 = 0$ we shall be able to consider the behavior of the Dirac field in the gravitating "pure Yang-Mills" field which is generated by the gauge charge $\vec{Q}_2$ and the mass $M$. This is the motivation of this paper. Incidentally, in this case the gravity is a type of the Reissner-Nordström spacetime with the gauge charge $\vec{Q}_2$ in place of ordinary charges.

The investigation is done by using the Newman-Penrose formalism, in terms of which the problems on the behavior of the neutrino waves in Kerr geometry have been discussed mainly by Chandrasekhar [10]. In this paper we are discussing our problems following to him as similarly as possible. As a result, we find an interesting fact that the left-handed massless Dirac fields behave in a different manner from the right-handed ones. This can be considered as the direct interaction of Dirac fields with the "pure Yang-Mills" field. Accordingly, the phenomenon of the neutrino waves in Kerr geometry have been discussed mainly by Chandrasekhar [10]. In this paper we are considering the equations of massless Dirac fields interacting with our Poincaré gauge fields in terms of spinor forms. In Sec. IV we shall investigate the scattering problems of massless Dirac fields by a Reissner-Nordström type black-hole with a gauge charge in place of normal charges. And the final section is devoted to conclusions.

II. SOME SPECIAL SOLUTIONS IN PGT

We start with the following Lagrangian for gauge fields [3]

$$L_g = \alpha T_{kmn} T_{C_{k\mu\nu}} + \beta V_{C_{k\mu\nu}} C_{k\mu\nu} + \gamma A_{C_{k\mu\nu}} A_{k\mu\nu} + a_1 A_{kmnp} A_{k\mu\nu} + a_2 B_{kmnp} B_{k\mu\nu} + a_3 C_{kmnp} C_{k\mu\nu} + a_4 E_{km} E_{k\mu} + a_5 G_{km} G_{k\mu} + a_6 F^2 + a F.$$  

Here $T_{kmn}$, $A_{kmnp}$, $B_{kmnp}$, $C_{kmnp}$, $E_{km}$, $E_{k\mu}$, $G_{km}$, $G_{k\mu}$, $F$ are the irreducible components of translational and Lorentz gauge field strengths which are defined in terms of tetrad $b_{\mu}^\nu$ and Lorentz gauge fields $A_{km\mu}$ as

$$C_{km\mu} = b_{\mu}^n C_{km\mu} = 2b_{k\mu\nu} b_{\nu}^n b_{\mu}^m + 2A_{k\mu\nu},$$

$$F_{km\mu} = b_{\mu}^n b_{\nu}^p F_{km\mu\nu} = 2(A_{km\mu\nu} + A_{k\mu\nu} A_{\nu\mu} b_{\mu}^n b_{\nu}^p).$$

And $a, \alpha, \beta, \gamma$ and $a_i (i = 1, 2, \cdots, 6)$ are ten constant parameters and it is well-known that only five of six $a_i$ are independent [3].

In this paper we make slightly more loose choice than in Ref. [3] as follows: $\alpha + \frac{3}{2} a = \beta - \frac{3}{2} a = \gamma + \frac{3}{2} a = 0$ and $4a_1 = 3a_2 = 2a_3 = 24a_4 = 24a_5 = 2a_6$. In this choice action integral can be rewritten as

$$I_g = \int d^4 x b L_g = \int d^4 x b (aR + L_F)$$

with

$$L_F = a_1 (A_{kmnp} A_{k\mu\nu} + a_2 B_{kmnp} B_{k\mu\nu} + a_3 E_{km} E_{k\mu} + \frac{1}{6} F^2) + a_5 (C_{kmnp} C_{k\mu\nu} + 2G_{km} G_{k\mu}).$$

1We here use the same notations as of Ref. [3].
where $R$ is a Riemann scalar curvature defined by the metric $g_{\mu\nu} = b_{k\mu} b^{k\nu}$, and then $\frac{1}{2a}$ can be plausibly interpreted as Einstein’s gravitational constant and $a_1$ and $a_3$ only are parameters to be determined by experiments.

From this action we can derive the following complex Einstein-Yang-Mills equations (CEYM):

\[
G^{\mu\nu} = \frac{1}{2a} T_{(L)}^{\mu\nu},
\]

\[
\bar{F}^{\mu\nu} - i \bar{A}_\nu \times \bar{F}^{\mu\nu} = 0,
\]

\[
\bar{F}^{\dagger\mu\nu} - i \bar{A}_\nu \times \bar{F}^{\dagger\mu\nu} = 0.
\]

Here $G^{\mu\nu}$ is the Einstein’s tensor and $T_{(L)}^{\mu\nu}$ is an energy-momentum tensor for the Lorentz gauge field:

\[
T_{(L)}^{\mu\nu} = b_{k\mu} b^m \eta^{km} (-2 F_{pqn}^{\quad k} \frac{\partial L_F}{\partial F_{pqmn}} + \eta^{km} \cdot \bar{F}).
\]

And $\bar{A}_\mu$ and $\bar{F}_{\mu\nu}$ are a complex field and its strength, respectively, which are made from $A_{km\mu}$ and $F_{km\mu\nu}$ as follows:

\[
\bar{A}_\mu = \bar{v}_\mu + i \bar{\alpha}_\mu, \quad \bar{F}_{\mu\nu} = \bar{F}_P^{\mu\nu} + i \bar{F}_A^{\mu\nu},
\]

where

\[
A_{0\mu} \equiv v_{(a)\mu} \rightarrow \bar{v}_\mu, \quad \frac{1}{2} \epsilon_{abc} A_{bc\mu} \equiv a_{(a)\mu} \rightarrow \bar{\alpha}_\mu
\]

\[
F_{0\mu\nu} \equiv F_P^{\mu\nu} \rightarrow \bar{F}_P^{\mu\nu} \quad \frac{1}{2} \epsilon_{abc} F_{bc\mu\nu} \equiv F_A^{\mu\nu} \rightarrow \bar{F}_A^{\mu\nu}.
\]

The symbol (·) means the ordinary covariant derivatives with Christoffel connection, and the quantities with † symbol mean the duals to the corresponding ones.

Now we put

\[
\bar{A}_\mu = \bar{\beta} A_\mu
\]

where $\bar{\beta}$ is a constant vector in 3D complex space. Then we get

\[
\bar{F}_{\mu\nu} = \bar{\beta} F_{\mu\nu} \quad \text{with} \quad F_{\mu\nu} = A_{\nu\mu} - A_{\mu\nu},
\]

and therefore we are led to the complex Einstein-Maxwell equations for the complex field $A_\mu$:

\[
F_{\mu\nu} = 0,
\]

\[
F^{\dagger\mu\nu} = 0.
\]

From these equations it is very easy to find a solution similar to the ordinary Coulomb field. In fact, we can get from those equations the following exact solution with a complex vector gauge charge $Q_*(= \bar{Q}_1 + i\bar{Q}_2)$:

\[
\bar{A}_\mu = \bar{\beta} A_0 \delta^0_\mu = \frac{Q_*}{r} \delta^0_\mu,
\]

or

\[
\bar{\alpha}_\mu = \frac{\bar{Q}_2}{r} \delta^0_\mu, \quad \bar{v}_\mu = \frac{\bar{Q}_1}{r} \delta^0_\mu.
\]

The energy-momentum tensor of this field is calculated by (8) to be

\[
T_{(L)}^{\mu\nu} = \text{diag}\left[ \frac{\bar{Q}_2^2}{r^4}, \frac{\bar{Q}_2^2}{r^4}, -\frac{\bar{Q}_2^2}{r^4}, \frac{\bar{Q}_2^2}{r^4} \right]
\]

with

\[
\bar{Q}_2^2 = -2(a_1 + a_3)(\bar{Q}_1^2 - \bar{Q}_2^2).
\]
Substituting above results to the Einstein equation (15) and according to the similar procedure to the real one (10), we can easily get the following spacetime:

\[
    ds^2 = \frac{\Delta}{r^2} (dx^0)^2 - \frac{\Delta}{r^2} (dr)^2 - r^2 \left\{ (d\theta)^2 + (d\phi)^2 \sin^2 \theta \right\} \quad \text{in a region of } \Delta > 0, \quad (17)
\]
or

\[
    ds^2 = \frac{4|\Delta|}{r^2} dudv - r^2 \left\{ (d\theta)^2 + (d\phi)^2 \sin^2 \theta \right\} \quad \text{in a region of } \Delta < 0, \quad (18)
\]

where \( \Delta = r^2 - 2Mr + q_*^2 \). As seen from the relation between \( \Delta \) and \( r \), above spacetime is classified by the value of \( q_*^2 \): (I) If \( q_*^2 = 0 \), then we have just Schwarzschild spacetime. (II) If \( 0 < q_*^2 < M^2 \), then the spacetime is like Reissner-Nordström (RN spacetime). (III) If \( q_*^2 < 0 \), then we get a spacetime having the similar structure to Schwarzschild. The difference between the cases (I) and (III) may, for example, be clarified by considering the correction of the classical Kepler orbits. In fact, under post-Newtonian approximation \( \mu \ll 1 \), \( \lambda \ll 1 \), \( \kappa \ll 1 \), we can find that the case (III) gives the advance in the perihelion per revolution \[12\],

\[
    \delta \phi \sim 2\pi \left\{ 3\mu + \frac{1}{2} \kappa + 3(1 + \frac{1}{4}e^2)\lambda \right\}, \quad (19)
\]

where \( \mu = \frac{M}{r} \), \( \lambda = \frac{|q_*^2|}{L} \), \( \kappa = \frac{|q_*^2|}{L^2} \), and \( \ell \), \( L \) and \( e \) are in turn the latus rectum, the angular momentum and the eccentricity of the orbit.

By the way, in the previous paper \[7\] we saw the fields \( \vec{a}_\mu, \vec{v}_\mu \) could be interpreted as “Yang-Mills” field and its partner field, respectively. Furthermore, according to Kalb \[9\] the following quantity \( F_{\mu\nu} \) was interpreted as the electromagnetic field tensor:

\[
    \vec{a}_{\nu,\mu} - \vec{a}_{\mu,\nu} + \vec{a}_\mu \times \vec{a}_\nu = F_{\mu\nu} \frac{\vec{a}_0}{|\vec{a}_0|} \quad (20)
\]

Following this interpretation the magnitude of the gauge charge \( |\vec{Q}_2| \) can be identified with an electric charge by which an electric field \( E_k = F_{0k} = \frac{|\vec{Q}_2|}{r^2}\delta^1_k \) is created.

In this paper we take up the case (II) and consider the scattering problems of the massless Dirac fields by the “pure Yang-Mills” fields, when the Dirac field approaches to the outer event horizon \( r_+ \) of the R-N type black-hole far from the outside. To this purpose we choose as \( \vec{Q}_1 = 0, \vec{Q}_2 = (0, 0, Q_2[3]) \) and \( a_1 + a_3 > 0 \), and also the tetrad \( b_k^{\mu} \) as

\[
    b_0^{\mu} = \left( \frac{3r^2 - 2Mr + q_*^2}{2\sqrt{2}\Delta}, \frac{r^2 + 2Mr - q_*^2}{2\sqrt{2}r^2}, 0, 0 \right), \quad b_1^{\mu} = \left( 0, 0, 1, 0 \right),
\]

\[
    b_2^{\mu} = \left( 0, 0, 0, \frac{1}{r\sin \theta} \right), \quad b_3^{\mu} = \left( \frac{r^2 + 2Mr - q_*^2}{2\sqrt{2}\Delta}, \frac{3r^2 - 2Mr + q_*^2}{2\sqrt{2}r^2}, 0, 0 \right). \quad (21)
\]

### III. DIRAC EQUATION

The gauge-invariant Lagrangian for the Dirac fields can be obtained from the original one by means of the replacement of the ordinary derivative \( \Psi_{,k} \) by the covariant one \( D_k \Psi = b_k^{\mu} (\Psi_{,\mu} + \frac{i}{2} A_{mn\mu} S^{mn} \Psi) \). Thus action is given as

\[
    I_{Dirac} = \int d^4x \left[ \frac{1}{2} \bar{\Psi} \gamma^k D_k \Psi - \frac{1}{2} D_k \bar{\Psi} \gamma^k \Psi + im \bar{\Psi} \Psi \right]. \quad (22)
\]

Here \( \gamma^k (k = 0, 1, 2, 3) \) are Dirac matrices and we shall adopt the representations:

\[
    \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_i = \begin{pmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad \gamma_5 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

where \( \sigma_i (i = 1, 2, 3) \) are Pauli matrices. Six generators \( S^{km} \) are now given in terms of \( \gamma^k \) as \( S^{km} = -\frac{1}{2} \sigma^{km} = -\frac{i}{2} [\gamma^k, \gamma^m] \).
Applying the variational principle to (22) we can derive the following equation:

$$\left[ \gamma^k (\nabla_k - \frac{3}{4} i^A C_k \gamma_5) + i m \right] \Psi = 0. \tag{24}$$

Here $^A C_k$ is an axial vector part of the translational gauge field strength: $^A C_k = \frac{1}{2} \epsilon_{kmpq} C^{mpq}$. It should be also remarked that a new covariant derivative $\nabla_k \Psi$ is introduced here. This is defined in terms of Ricci’s rotation coefficients $\Delta_{kmp}$ instead of $A_{kmp}$ in $D_k \Psi$ as $\nabla_k \Psi = b_k^\mu (\partial_\mu \Psi + \frac{i}{2} \Delta_{mpq} \sigma^{mpq} \Psi)$.

It is convenient for our purpose to resolve above the 4-components Dirac equation (24) to two 2-components equations. Using the well-known technique [13] these are given as

$$\partial_{AB} (\psi_L)^B + \Gamma^B_{CAB} (\psi_L)^C - \frac{3}{4} i^A C_{AB} (\psi_L)^B + \frac{i m}{\sqrt{2}} (\bar{\psi}_R)^A = 0, \tag{25a}$$

$$\partial_{AB} (\psi_R)^B + \Gamma^B_{CAB} (\psi_R)^C - \frac{3}{4} i^A C_{AB} (\psi_R)^B + \frac{i m}{\sqrt{2}} (\bar{\psi}_L)^A = 0. \tag{25b}$$

Here we put

$$\Psi = \left( \begin{array}{c} \psi_L \\ \psi_R \end{array} \right), \quad \partial_{AB} = b_k^\mu \sigma^k_{AB} \partial_\mu, \quad ^A C_{AB} = b_k^\mu \sigma^k_{AB} C_\mu,$$

$$\Gamma_{ABC} = -\frac{1}{2} \Delta_{kmp} b_n^\mu \sigma^k_{AB} \sigma^{m} E_A^B \sigma^n C_D, \tag{27}$$

where $\sigma^k_{AB}(k = 0, 1, 2, 3)$ are multiples of identity and Pauli matrices by $\frac{i}{\sqrt{2}}$.

Substituting the results of previous section for above equations (23) and putting the dependency on $t$ and $\phi$ as $e^{i(\sigma t + \vec{m} \phi)}$ we can get a set of equations

$$D_0 f_1 + \frac{1}{\sqrt{2}} L_\frac{1}{2} f_2 = 0, \tag{28a}$$

$$\Delta (D_{\frac{1}{2}} - \frac{r}{\Delta} Q_{2[3]}) f_2 - \sqrt{2} L_{\frac{1}{2}} f_1 = 0, \tag{28b}$$

$$D_0 g_2 - \frac{1}{\sqrt{2}} L_{-\frac{1}{2}} g_1 = 0, \tag{28c}$$

$$\Delta (D_{\frac{1}{2}} + \frac{r}{\Delta} Q_{2[3]}) g_1 + \sqrt{2} L_{-\frac{1}{2}} g_2 = 0. \tag{28d}$$

Here we define

$$f_1 = r(\psi_L)^0, \quad f_2 = (\psi_L)^1, \quad g_1 = (\bar{\psi}_R)^1, \quad g_2 = -r(\bar{\psi}_R)^0,$$

$$D_n = \partial_n + i \frac{\delta r^2}{\Delta} + 2n \frac{r - M}{\Delta}, \quad D_{\frac{1}{2}} = \partial_{\frac{1}{2}} - i \frac{\delta r^2}{\Delta} + 2n \frac{r - M}{\Delta},$$

$$L_n = \partial_n + n \cot \theta + \vec{m} \csc \theta, \quad L_{\frac{1}{2}} = \partial_{\frac{1}{2}} + n \cot \theta - \vec{m} \csc \theta. \tag{29c}$$

In order to solve the equations (28) we put

$$f_1 = R_{-\frac{1}{2}} (r) S_{-\frac{1}{2}} (\theta), \quad f_2 = R_{+\frac{1}{2}} (r) S_{+\frac{1}{2}} (\theta),$$

$$g_1 = \tilde{R}_{-\frac{1}{2}} (r) \tilde{S}_{-\frac{1}{2}} (\theta), \quad g_2 = \tilde{R}_{+\frac{1}{2}} (r) \tilde{S}_{+\frac{1}{2}} (\theta). \tag{30}$$

Then the equations (28a) and (28b) are separated to produce

$$D_0 R_{-\frac{1}{2}} = \lambda_1 R_{+\frac{1}{2}}, \tag{31a}$$

$$\Delta (D_{\frac{1}{2}} - i \frac{r}{\Delta} Q_{2[3]}) R_{+\frac{1}{2}} = \lambda_2 R_{-\frac{1}{2}}, \tag{31b}$$

$$L_{\frac{1}{2}} S_{+\frac{1}{2}} = -\sqrt{2} \lambda_1 S_{-\frac{1}{2}}, \tag{31c}$$

$$L_{\frac{1}{2}} S_{-\frac{1}{2}} = \frac{1}{\sqrt{2}} \lambda_2 S_{+\frac{1}{2}}. \tag{31d}$$
where \( \lambda_1 \) and \( \lambda_2 \) are separation constants. Similarly the following equations can be derived from \((28c)\) and \((28d)\):

\[
\begin{align*}
\mathcal{D}_0 \hat{R}_{-\frac{1}{2}} &= \lambda_3 \hat{R}_{-\frac{1}{2}}, \\
\Delta[\mathcal{D}^\dagger_{\frac{1}{2}} + i \frac{r}{\Delta} Q_{2[3]}] \hat{R}_{+\frac{1}{2}} &= \lambda_4 \hat{R}_{-\frac{1}{2}},
\end{align*}
\]

\((32a)\) and \((32b)\), respectively, from the left-hand side we get in turn

\[
\begin{align*}
\mathcal{L}^\dagger \hat{S}_{+\frac{1}{2}} &= -\frac{1}{\sqrt{2}} \lambda_3 \hat{S}_{-\frac{1}{2}}, \\
\mathcal{L}^\dagger \hat{S}_{-\frac{1}{2}} &= \sqrt{2} \lambda_3 \hat{S}_{+\frac{1}{2}}.
\end{align*}
\]

\((32c)\) and \((32d)\)

We first consider the angular parts of the equations \((31)\) and \((32)\). Operating \(\mathcal{L}^\dagger \) and \(\mathcal{L} \) on \((31a)\) and \((31b)\), respectively, from the left-hand side we get in turn

\[
\begin{align*}
\mathcal{L}^\dagger \mathcal{L} \hat{S}_{+\frac{1}{2}}(\theta) &= -\lambda_1 \lambda_2 S_{+\frac{1}{2}}(\theta), \\
\mathcal{L}^\dagger \mathcal{L} \hat{S}_{-\frac{1}{2}}(\theta) &= -\lambda_1 \lambda_2 S_{-\frac{1}{2}}(\theta).
\end{align*}
\]

\((33a)\) and \((33b)\)

In the same way we shall get the following equations from \((32c)\) and \((32d)\):

\[
\begin{align*}
\mathcal{L}^\dagger \mathcal{L} \hat{S}_{+\frac{1}{2}}(\theta) &= -\lambda_3 \lambda_4 \hat{S}_{+\frac{1}{2}}(\theta), \\
\mathcal{L}^\dagger \mathcal{L} \hat{S}_{-\frac{1}{2}}(\theta) &= -\lambda_3 \lambda_4 \hat{S}_{-\frac{1}{2}}(\theta).
\end{align*}
\]

\((34a)\) and \((34b)\)

Replacing \(\theta\) by \(\pi - \theta\) in \((33a)\) and \((34a)\) and using a relation \(\mathcal{L}^\dagger n(\theta) = -\mathcal{L} n(\pi - \theta)\), and comparing the results with \((33b)\) and \((34b)\) we can conclude

\[
S_{+\frac{1}{2}}(\pi - \theta) = S_{-\frac{1}{2}}(\theta), \quad \hat{S}_{+\frac{1}{2}}(\pi - \theta) = \hat{S}_{-\frac{1}{2}}(\theta).
\]

\((35)\)

Furthermore, using these relations in \((31c)\) and \((32d)\) and comparing the results with \((31d)\) and \((32c)\) we get the relations

\[
\lambda_2 = 2\lambda_1, \quad \lambda_4 = 2\lambda_3.
\]

\((36)\)

Then the comparison of \((33b)\) and \((34b)\) with an equation satisfied by spin(-1/2)-weighted spherical harmonics:

\[
\mathcal{L}^\dagger \mathcal{L} \hat{Y}_{-\frac{1}{2}} Y_{\ell m} = -\left(\ell + \frac{1}{2}\right)^2 \frac{1}{2} \hat{Y}_{-\frac{1}{2}} Y_{\ell m}
\]

with \(\lambda_1 = \frac{1}{2} \lambda_2 = \frac{\sqrt{2}}{\sqrt{2}} (\ell + \frac{1}{2})\) and \(\hat{S}_{-\frac{1}{2}} = \frac{1}{2} \hat{Y}_{-\frac{1}{2}} Y_{\ell m}\) with \(\lambda_3 = \frac{1}{2} \lambda_4 = \frac{1}{\sqrt{2}} (\ell + \frac{1}{2})\).

We are now in a position to consider the radial parts of equations \((31)\) and \((32)\). We first notice the relations \(\Delta^\dagger \mathcal{D}^\dagger_{\frac{1}{2}} = \mathcal{D}^\dagger_{0} \Delta^\dagger_{\frac{1}{2}}\) and \(\Delta \mathcal{D} = \mathcal{D} \Delta\). Applying the former to \((31b)\) and putting as

\[
R_{\pm \frac{1}{2}} = \frac{2^{\pm \frac{1}{2}}}{2} (Z_+ \pm Z_-) \Delta^{-(\pm \frac{1}{2})} \exp \left[\frac{1}{2} i \int \frac{r}{\Delta} Q_{2[3]} dr\right]
\]

we can derive the following equations from \((31a)\) and \((31b)\):

\[
\left(\frac{d}{dr^\ast} \mp W\right) Z_\pm = i \sigma Z_\mp.
\]

\((38)\)

Here we have introduced a new variable \(r^\ast\) and a new quantity \(W\), which are defined by

\[
\begin{align*}
&\quad r^\ast = r + \frac{r_+(r_+ + \frac{Q_{2[3]}}{2r_+})}{r_+ - r_-} \ln |r - r_+| - \frac{r_-(r_- + \frac{Q_{2[3]}}{2r_-})}{r_+ - r_-} \ln |r - r_-|
\end{align*}
\]

and

\[
W = \frac{\left(\ell + \frac{1}{2}\right) \Delta^\dagger_{\frac{1}{2}}}{1 + \frac{Q_{2[3]}}{2r^\ast}}
\]

\((39)\) and \((40)\)

In the same way we can also get the following equations from \((32a)\) and \((32b)\):
\[
\left( \frac{d}{dr_*} + \tilde{W} \right) \tilde{Z}_\pm = -i \tilde{\alpha} \tilde{Z}_\mp. \tag{41}
\]

Here we have also introduced new quantities \( \tilde{Z}_\pm \) which are defined through

\[
\tilde{R}_\pm = \frac{2^{\pm 1/2}}{2} (\tilde{Z}_+ \pm \tilde{Z}_-) \Delta^{-1/2} \exp \left[ \frac{1}{2i} \int \frac{r}{\Delta} Q_{2[3]} dr \right],
\]

\[
\tilde{r}_* = r + \frac{r_+ (r_+ - Q_{2[3]})}{r_+ - r_-} \ln |r - r_+| - \frac{r_- (r_+ - Q_{2[3]})}{r_+ - r_-} \ln |r - r_-| \tag{42}
\]

and

\[
\tilde{W} = \frac{(\ell + \frac{1}{2}) \Delta^{1/2}}{1 - \frac{Q_{2[3]}}{2r_+}} \tag{43}
\]

For later discussion we would like to note here that the gauge charge \( Q_{2[3]} \) gives a different contribution to the left-handed massless Dirac fields from the right-handed ones.

### IV. SCATTERING OF DIRACフィールド BY A BLACK-HOLE WITH A PGT-GAUGE CHARGE

As stated in final part of section II, we consider here a scattering problem of the massless Dirac fields by a Reissner-Nordström type black-hole having a gauge charge \( Q_{2[3]} \) and a mass \( M \). However, we shall constrain ourselves to discuss only such a case that the massless Dirac fields approach to the event horizon \( r_+ \) of the black-hole far from the outside.

To this purpose we first note a fact that the following two Schrödinger equations can be made from (38) and (41):

\[
\left( -\frac{d^2}{dr_*^2} + V_\pm \right) Z_\pm = \bar{\delta}^2 Z_\pm \quad \text{with} \quad V_\pm = W^2 \pm \frac{dW}{dr_*}, \tag{45}
\]

\[
\left( -\frac{d^2}{dr_*^2} + \tilde{V}_\pm \right) \tilde{Z}_\pm = \bar{\delta}^2 \tilde{Z}_\pm \quad \text{with} \quad \tilde{V}_\pm = \tilde{W}^2 \pm \frac{\tilde{dW}}{dr_*}. \tag{46}
\]

From these equations we can get at once the following conserved Wronskians:

\[
\frac{d[Z_\pm, Z^*_\pm]}{dr_*} = 0, \quad \frac{d[\tilde{Z}_\pm, \tilde{Z}^*_\pm]}{d\tilde{r}_*} = 0, \tag{47}
\]

where the Wronskians are defined as

\[
[Z_\pm, Z^*_\pm] = Z_\pm \frac{dZ^*_\pm}{dr_*} - Z^*_\pm \frac{dZ_\pm}{dr_*}, \quad [\tilde{Z}_\pm, \tilde{Z}^*_\pm] = \tilde{Z}_\pm \frac{d\tilde{Z}^*_\pm}{d\tilde{r}_*} - \tilde{Z}^*_\pm \frac{d\tilde{Z}_\pm}{d\tilde{r}_*}. \tag{48}
\]

And also the following relations are obtained using (45) and (46):

\[
Z^*_+ Z^- + Z^-* Z^+_+ = \frac{1}{-i\tilde{\alpha}} [Z_+, Z^*_+], \quad \tilde{Z}_+ \tilde{Z}^- + \tilde{Z}^-* \tilde{Z}_+ = \frac{1}{-i\tilde{\alpha}} [	ilde{Z}_+, \tilde{Z}^*_+]. \tag{49}
\]

Before going forward, we must discuss about the conserved current of massless Dirac field. First, it should be remarked that the Dirac action (22) is invariant under the phase transformation \( \Psi' = e^{i\alpha} \Psi \quad (\alpha = \text{real constant}) \), so that we can get a continuity equation

\[
\partial_{\mu} (b J^\mu) = 0, \tag{50}
\]

where \( J^\mu = b_k^\mu \bar{\Psi}_{\gamma^k} \Psi \). \( J^\mu \) can also be written, using the representation (23) for Dirac gamma matrices, as

\[
J^\mu = J^\mu_L + J^\mu_R, \tag{51}
\]

where \( J^\mu_L = b_k^\mu \psi_L^\dagger \sigma^k \psi_L \), \( J^\mu_R = b_k^\mu \psi_R^\dagger \bar{\sigma}^k \psi_R \) with \( \sigma^k = \{ I, \sigma_i (i = 1, 2, 3) \} \) and \( \bar{\sigma}^k = \{ I, -\sigma_i (i = 1, 2, 3) \} \). Thus the conserved net current of massless Dirac particles is calculated on account of the spherical symmetry to be
\[
\frac{\partial N}{\partial t} = - \int_0^{2\pi} \int_0^\pi (J_L^r + J_R^r) \sin \theta d\theta d\phi = \frac{i\pi}{\sigma} \left\{ [Z_+, Z^*_+] + [\tilde{Z}_+, \tilde{Z}^*_+] \right\},
\]

where the first term of the right-hand side represents the contribution of the left-handed Dirac fields and the second of the right-handed ones.

Hereafter, we assume \(Q_{2[3]} < 0\). This assumption can be done without the lose of generality, because the change of sign means only the exchange of a role of the left- and right-handed fields. To go ahead, we further need to distinguish the following two cases: (1) \(\hat{\sigma} > \sigma_s\) and (2) \(0 < \hat{\sigma} < \sigma_s\), where \(\sigma_s\) is defined as \(\sigma_s = \frac{Q_{2[3]}}{2r_+}\). In (1) both \(r_s\) and \(\hat{r}_s\) are single-valued functions over all range of \(r_+ < r < \infty\) and both potentials \(V_+\) and \(\hat{V}_+\) are continuous and short-range. On the other hand, in (2) \(\hat{r}_s\) is also a single-valued function and the potential \(\hat{V}_+\) is continuous over all range of \(r_+ < r < \infty\). However, \(r_s\) has a minimum \(r_{s\text{min}}\) at \(r = \frac{Q_{2[3]}}{2r_+}\). Therefore \(r_s\) is a double-valued function over the range \(r_+ < r < \infty\) and the potentials \(V_+\) become singular at \(r_{s\text{min}}\). From this fact we may expect the existence of the so-called super-radiance which is well-known in a scattering problem of an electromagnetic field by a Kerr black-hole. However, this being not so can be easily shown in exactly the same way as a case of scattering of massless Dirac fields by a Kerr black-hole [10].

In this paper we pick up only the case (1) and concretely calculate the reflection and transmission coefficients for both left- and right-handed incident massless Dirac fields. In doing so we first remark that the wave functions \(\vec{Q}_\pm\) give the same reflection and transmission coefficients because of the relations (38) (or (41)). Therefore it is enough for us to consider only the scatterings by the potentials \(V_\pm\) and \(\hat{V}_\pm\). These potentials tend to decrease whenever the frequency of incident fields \(\hat{\sigma}\) is increased. (A prototype of these potentials is drawn in Fig.1 and 2.) Accordingly, we can see, before the calculation, that the reflection coefficients decrease (and the transmission coefficients increase), in accordance with the increase of the frequencies of incident fields.

The calculations were performed by using the 6th order Runge-Kutta method and putting as \(\ell = \frac{1}{2}\), \(M = 1\), \(Q_{2[3]} = -0.675470\), \(h = 1\), \(c = 1\), \(q_s = 0.5\) and Einstein’s gravitational constant being unity. The results are summarised in a table [1].

V. SUMMARY AND CONCLUSIONS

In this paper we have adopted a model of Poincaré gauge theory which can be actually identified with complex Einstein-Yang-Mills theory. And we have derived again a solution which can be interpreted as the “Yang-Mills” fields generated by the gauge charge \(Q_2\) and the spacetime curved by Reissner-Nordström type black-hole. However, the solution is special one of more general set of solutions [3] which are derived from the universal idea following to ‘tHooft [8]. Accordingly, we can expect that the investigations will give us the qualitative but universal results.

Then we have investigated the behavior of the massless Dirac fields in the external Poincaré gauge field mentioned above, namely the scattering problems of massless Dirac fields by the gravitating “Yang-Mills” fields generated by the gauge charge \(Q_2\) and the mass \(M\).

The investigation has been done by using the standard spinor techniques and following to Chandrasekhar’s method [10]. As a result, we have found an interesting fact that the left-handed massless Dirac fields behave in a different manner from the right-handed ones, owing to the gauge charge \(Q_{2[3]}\). The fact has been also examined concretely by setting in turn for \(Q_{2[3]}\), \(q_s\), and \(M\) to \(-0.675470\), \(0.5\) and \(1\). The results are summarised in the table [10], showing that more right-handed massless Dirac fields can come near the outer event-horizon of the Reissner-Nordström type black-hole than the left-handed ones if \(Q_{2[3]} < 0\).

Although in this paper we have restricted ourselves to a massless mode of Dirac fields, it is an open question whether the same procedures can be applied to the massive modes or not. This problem should be considered before too long.

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TABLE I. Reflection coefficients for $\ell = 0.5$ massless left- and right-handed Dirac particles incident on a Reissner-Nordström black-hole with a gauge charge $q_s = 0.5$, $\sigma_s = -\frac{Q_{CG}}{M^2} = 0.18099$ and a mass $M = 1$.

| $\sigma$   | $R_{\text{left}}$ | $R_{\text{right}}$ | $\sigma$   | $R_{\text{left}}$ | $R_{\text{right}}$ |
|------------|-------------------|---------------------|------------|-------------------|-------------------|
| 0.181987   | 0.97524           | 0.033468            | 0.220987   | 0.94419           | 0.0094065         |
| 0.182987   | 0.97477           | 0.032379            | 0.230987   | 0.92996           | 0.0068388         |
| 0.183987   | 0.97427           | 0.031338            | 0.240987   | 0.91178           | 0.0049839         |
| 0.184987   | 0.97377           | 0.030319            | 0.250000   | 0.89127           | 0.0037564         |
| 0.185987   | 0.97326           | 0.029334            | 0.250987   | 0.88874           | 0.0036428         |
| 0.186987   | 0.97274           | 0.028392            | 0.260987   | 0.85994           | 0.0026686         |
| 0.187987   | 0.97220           | 0.027471            | 0.300000   | 0.67973           | 0.0008104         |
| 0.188987   | 0.97165           | 0.026580            | 0.350000   | 0.34543           | 0.00001834        |
| 0.189987   | 0.97109           | 0.025719            | 0.400000   | 0.11622           | 0.00000430        |
| 0.200987   | 0.96384           | 0.017949            | 0.450000   | 0.03212           | 0.00000104        |
| 0.210987   | 0.95525           | 0.012978            |            |                   |                   |