CONNES’ EMBEDDING PROBLEM AND LANCE’S WEP

NATHANIAL P. BROWN

Abstract. A II$_1$-factor embeds into the ultraproduct of the hyperfinite II$_1$-factor if and only if it satisfies the von Neumann algebraic analogue of Lance’s weak expectation property (WEP). This note gives a self contained proof of this fact.

1. Introduction

On page 105 in [2] Connes suggested that every separable II$_1$-factor ought to be embeddable into the ultraproduct, $R^\omega$, of the hyperfinite II$_1$-factor $R$. Largely due to work of Kirchberg, Voiculescu and, most recently, Haagerup this seemingly technical question has received more and more attention in recent years. Indeed, Kirchberg proved in [5] that this problem can be reformulated in an unexpected variety of ways (see [7] for a wonderful exposition of Kirchberg’s work), this problem turns out to be a necessary condition for Voiculescu’s ‘Unification Problem’ (i.e. if the microstates and non-microstates approaches to free entropy yield the same quantity then every II$_1$-factor is embeddable) and, finally, Haagerup has shown that this problem is nearly sufficient for resolving the relative invariant subspace problem for II$_1$-factors (he showed that every operator in an embeddable II$_1$-factor which satisfies a mild non-degeneracy condition has invariant subspaces – see [4]).

In [6] Lance introduced the weak expectation property (WEP) for $C^*$-algebras. Blackadar shifted the point of view to von Neumann algebras with the following definition.

Definition 1.1. Let $M \subset B(H)$ be a von Neumann algebra acting on some Hilbert space $H$ and let $A \subset M$ be a weakly dense $C^*$-subalgebra. Then $M$ has a weak expectation relative to $A$ if there exists a unital, completely positive map $\Phi : B(H) \to M$ such that $\Phi(a) = a$ for all $a \in A$.

This notion was inspired by injectivity; $M \subset B(H)$ is injective if there exists a unital, completely positive map $\Phi : B(H) \to M$ such that $\Phi(x) = x$ for all $x \in M$.

It follows from Arveson’s Extension Theorem that a $C^*$-algebra $A$ has the WEP if and only if the enveloping von Neumann algebra $A^{**}$ has a weak expectation relative to $A \subset A^{**}$.

In [1] we observed that the $W^*$-version of the WEP is closely related to Connes’ embedding problem.

Theorem 1.2. Let $M$ be a separable II$_1$-factor. Then $M$ is embeddable into $R^\omega$ if and only if $M$ has a weak expectation relative to some weakly dense subalgebra.

A simple corollary of this result states that many well known II$_1$-factors which are “far from being hyperfinite” (in the sense that they exhibit vastly different properties than $R$ – no Cartan subalgebras, prime, property T, etc.) are in fact built out of $R$ in a way which naturally mixes von Neumann algebraic and operator space notions. More precisely, we have the following approximation property.

Supported by an NSF Postdoctoral Fellowship. 2000 MSC number: 46L05.
Corollary 1.3. Let $M \subset R^\omega$ be a II$_1$-factor, $\mathfrak{F} \subset M$ be a finite set and $\epsilon > 0$ be given. Then there exists a subspace $X \subset M$ such that $X$ nearly contains $\mathfrak{F}$ (within $\epsilon$ in 2-norm) and $X \cong R$ (as operator systems).

In other words, free group factors (and $L(\Gamma)$ for any other residually finite group) are the weak closure of (operator space isomorphic) copies of $R$. It turns out that $R$ is a II$_1$-factor in standard form (i.e., GNS, on the $L^2$-space coming from its unique trace) and $\tau : A \to M \subset B(L^2(M))$ is a *-homomorphism with weakly dense range then we may, thanks to uniqueness of GNS representations, identify $\pi$ with the GNS representation of $A$ coming from $\tau \circ \pi$, where $\tau$ is the unique trace on $M$.

Throughout this paper $A$ will denote a separable unital C*-algebra. Separability is really not necessary, but it is convenient. We will use the abbreviation u.c.p. for unital completely positive maps. If $\tau$ is a state on a C*-algebra $A$ then $\pi_\tau : A \to B(L^2(A, \tau))$ will denote the GNS representation. Note that if $M$ is a II$_1$-factor in standard form (i.e. acting, via GNS, on the $L^2$-space coming from its unique trace) and $\pi : A \to M \subset B(L^2(M))$ is a *-homomorphism with weakly dense range then we may, thanks to uniqueness of GNS representations, identify $\pi$ with the GNS representation of $A$ coming from $\tau \circ \pi$, where $\tau$ is the unique trace on $M$.

Finally, recall that if $R$ denotes the hyperfinite II$_1$-factor and $\omega \in \beta(\mathbb{N})\setminus\mathbb{N}$ is a free ultrafilter then the ultraproduct $R^\omega$ is defined to be $l^\infty(R) = \{(x_n) : x_n \in R, \sup_n \|x_n\| < \infty\}$ modulo the ideal $I_\omega = \{(x_n) : \lim_{n \to \omega} \|x_n\|_2 = 0\}$, where $\|x\|_2^2 = \tau(x^*x)$ and $\tau$ is the unique trace on $R$. It turns out that $R^\omega$ is a II$_1$-factor with tracial state $\tau_\omega((x_n)) = \lim_{n \to \omega} \tau(x_n)$.

2. Invariant Means on C*-algebras

In [2, Remark 5.35] Connes points out that a hypertrace can be regarded as the analogue of an invariant mean on a group. We essentially take this as the definition of an invariant mean on a C*-algebra.

Definition 2.1. Let $A \subset B(H)$ be a C*-algebra. A (tracial) state $\tau$ on $A$ is called an invariant mean if there exists a state $\psi$ on $B(H)$ which is (1) invariant under the action of the unitary group of $A$ on $B(H)$ (i.e. $\psi(uTu^*) = \psi(T)$ for all $T \in B(H)$ and unitaries $u \in A$) and (2) extends $\tau$ (i.e. $\psi|_A = \tau$). We will denote by $T(A)_M$ the set of all invariant means on $A$.

The main result of this section gives an important characterization of invariant means. There are several other ways to characterize invariant means (cf. [1, Theorem 3.1], [7, Theorem 6.1]) but we only present the ones we need. The main step in the proof ($(1) \implies (2)$) is essentially due to Connes in the unique trace case and Kirchberg in general. We will isolate the main technical aspects in a lemma.

Below, $\text{Tr}(\cdot)$ will denote the canonical (unbounded) trace on $B(H)$ and, if $H$ is finite dimensional, $\text{tr}(\cdot)$ will denote the (unique) tracial state on $B(H)$. Also, $\mathcal{T} \subset B(H)$ will be the trace class operators (i.e. the predual of $B(H)$) and $\|\cdot\|_{1,\text{Tr}}$ (resp. $\|\cdot\|_{2,\text{Tr}}$) will denote the $L^1$-norm (resp. $L^2$-norm) on $\mathcal{T}$. Recall that the Powers-Størmer inequality states that if $h, k \in \mathcal{T}$ are positive then $\|h-k\|_{2,\text{Tr}}^2 \leq \|h^2-k^2\|_{1,\text{Tr}}$. In particular, if $u \in B(H)$ is a unitary and $h \geq 0$ has finite rank then $\|uh^{1/2} - h^{1/2}u\|_{2,\text{Tr}} = \|uh^{1/2}u^* - h^{1/2}\|_{2,\text{Tr}} \leq \|uhu^* - h\|_{1,\text{Tr}}^{1/2}$.

Lemma 2.2. Let $h \in B(H)$ be a positive, finite rank operator with rational eigenvalues and $\text{Tr}(h) = 1$. Then there exists a u.c.p. map $\phi : B(H) \to M_q(\mathbb{C})$ such that $\text{tr}(\phi(T)) = \text{Tr}(hT)$.
for all $T \in B(H)$ and $|\text{tr}(\phi(uu^*) - \phi(u)\phi(u^*))| < 2\|uhu^* - h\|_1^{1/2}$ for every unitary operator $u \in B(H)$.

**Proof.** This proof is taken directly from the proof of [7, Theorem 6.1] which, in turn, is based on work of Haagerup.

Let $v_1, \ldots, v_k \in H$ be the eigenvectors of $h$ and $\frac{p_1}{q}, \ldots, \frac{p_k}{q}$ the corresponding eigenvalues. Note that $\sum p_j = q$ since $\text{Tr}(h) = 1$. Let $\{w_m\}$ be any orthonormal basis of $H$ and consider the following orthonormal subset of $H \otimes H$:

$$\{v_1 \otimes w_1, \ldots, v_1 \otimes w_{p_1}\} \cup \{v_2 \otimes w_1, \ldots, v_2 \otimes w_{p_2}\} \cup \ldots \cup \{v_k \otimes w_1, \ldots, v_k \otimes w_{p_k}\}.$$ 

Let $P \in B(H \otimes H)$ be the orthogonal projection onto the span of these vectors. We encourage the reader to write down the matrices of $h^{1/2}T$, $h^{1/2}T^*$ and $h^{1/2}Th^{1/2}T^*$ (in any orthonormal basis which begins with $\{v_1, \ldots, v_k\}$). Having done so one immediately sees that, letting $T_{i,j} = \langle Tv_j, v_i \rangle$,

$$\text{Tr}(h^{1/2}Th^{1/2}T^*) = \sum_{i,j=1}^k \frac{1}{q}(p_ip_j)^{1/2}|T_{i,j}|^2.$$ 

Hence, if we define a u.c.p. map $\phi : B(H) \to M_q(\mathbb{C})$ by $\phi(T) = P(T \otimes 1)P$ then $\text{tr}(\phi(T)) = \text{Tr}(hT)$ for all $T \in B(H)$ and, moreover, we have the following estimates:

$$|\text{Tr}(h^{1/2}Th^{1/2}T^*) - \text{tr}(\phi(T)\phi(T^*))| = \sum_{i,j=1}^k \frac{1}{q}|T_{i,j}|^2 \left( (p_ip_j)^{1/2} - \min\{p_i, p_j\} \right)$$

$$\leq \sum_{i,j=1}^k \frac{1}{q}|T_{i,j}|^2 \left| p_i^{1/2} - p_j^{1/2} \right|$$

$$\leq \left( \sum_{i,j=1}^k \frac{1}{q}|T_{i,j}|^2 p_i \right)^{1/2} \left( \sum_{i,j=1}^k \frac{1}{q}|T_{i,j}|^2 \left( p_i^{1/2} - p_j^{1/2} \right)^2 \right)^{1/2}$$

$$= \|Th^{1/2}\|_{2,\text{Tr}}\|h^{1/2}T - Th^{1/2}\|_{2,\text{Tr}}.$$ 

Now if $T$ happens to be a unitary operator then $\|Th^{1/2}\|_{2,\text{Tr}} = \|h^{1/2}\|_{2,\text{Tr}} = 1$ and $\|h^{1/2}T - Th^{1/2}\|_{2,\text{Tr}} = \|Th^{1/2}T^* - h^{1/2}\|_{2,\text{Tr}}$ and hence we can apply the Powers-Størmer inequality after the inequalities above to get:

$$|\text{Tr}(h^{1/2}Th^{1/2}T^*) - \text{tr}(\phi(T)\phi(T^*))| \leq \|Th^* - h\|_{1,\text{Tr}}^{1/2}.$$
Finally, the Cauchy-Schwartz inequality applied to the Hilbert-Schmidt operators implies that for every unitary operator $T \in B(H)$,
\[
\text{tr}(\phi(TT^*) - \phi(T)\phi(T^*)) \leq |1 - \text{tr}(h^{1/2}Th^{1/2}T^*)| + \|ThT^* - h\|_{1,Tr}^{1/2}
\]
\[
= |\text{tr}(ThT^*) - \text{tr}(h^{1/2}Th^{1/2}T^*)| + \|ThT^* - h\|_{1,Tr}^{1/2}
\]
\[
= |\text{tr}((Th^{1/2} - h^{1/2}T)h^{1/2}T^*)| + \|ThT^* - h\|_{1,Tr}^{1/2}
\]
\[
\leq \|h^{1/2}T\|_{2,Tr}\|Th^{1/2} - h^{1/2}T\|_{2,Tr} + \|ThT^* - h\|_{1,Tr}^{1/2}
\]
\[
\leq 2\|ThT^* - h\|_{1,Tr}^{1/2}.
\]

\[\square\]

**Theorem 2.3.** Let $\tau$ be a tracial state on $A$. Then the following are equivalent:

1. $\tau \in T(A)_{\text{IM}}$.
2. There exists a sequence of u.c.p. maps $\phi_n : A \to M_{k(n)}(\mathbb{C})$ such that $\|\phi_n(ab) - \phi_n(a)\phi_n(b)\|_{2,T} \to 0$ and $\tau(a) = \lim_{n \to \infty} \text{tr} \circ \phi_n(a)$, for all $a, b \in A$, where $\|x\|_{2,T}^2 = \text{tr}(x^*x)$ for every $x \in M_{k(n)}(\mathbb{C})$.
3. For any faithful representation $A \subset B(H)$ there exists a u.c.p. map $\Phi : B(H) \to \pi_\tau(A)^\omega$ such that $\Phi(a) = \pi_\tau(a)$.

**Proof.** (1) $\implies$ (2). Let $A \subset B(H)$ be a faithful representation. Since $\tau \in T(A)_{\text{IM}}$ we can find a state $\psi$ on $B(H)$ which extends $\tau$ and such that $\psi(uT\nu^*) = \psi(T)$ for all unitaries $u \in A$ and operators $T \in B(H)$. Since the normal states on $B(H)$ are dense in the set of all states on $B(H)$ we can find a net of positive operators $h_\lambda \in T$ such that $\text{tr}(h_\lambda T) \to \psi(T)$ for all $T \in B(H)$. Since $\psi(u^*Tu) = \psi(T)$ it follows that $\text{tr}(h_\lambda T) - \text{tr}((uh_\lambda u^*)T) \to 0$ for every $T \in B(H)$ and unitary $u \in A$. In other words, $h_\lambda - uh_\lambda u^* \to 0$ in the weak topology on $T$. Hence, by the Hahn-Banach theorem, there are convex combinations which tend to zero in $L^1$-norm. In fact, taking finite direct sums (i.e. considering $n$-tuples $(u_1h_\lambda u_1^*, \ldots, u_nh_\lambda u_n^*)$) one applies a similar argument to show that if $\mathfrak{F} \subset A$ is a finite set of unitaries then for every $\epsilon > 0$ we can find a positive trace class operator $h \in T$ such that $\text{tr}(h) = 1$, $|\text{tr}(uh) - \tau(u)| < \epsilon$ and $\|h - u\|_{1} < \epsilon$ for all $u \in \mathfrak{F}$. Since finite rank operators are norm dense in $T$ we may further assume that $h$ is finite rank with rational eigenvalues.

Applying Lemma 2.2 to bigger and bigger finite sets of unitaries and smaller and smaller epsilon’s we can construct a sequence of u.c.p. maps $\phi_n : B(H) \to M_{k(n)}(\mathbb{C})$ such that $\text{tr}(\phi_n(u)) \to \tau(u)$ and $|\text{tr}(\phi_n(\nu^*) - \phi_n(u)\phi_n(\nu^*))| \to 0$ for every unitary $u$ in a countable set with dense linear span in $A$. Since $\phi_n(\nu^*) - \phi_n(u)\phi_n(\nu^*) \geq 0$ we have
\[
\|1 - \phi_n(u)\phi_n(\nu^*)\|_{2,T}^2 \leq \|1 - \phi_n(u)\phi_n(\nu^*)\|\text{tr}(\phi_n(\nu^*) - \phi_n(u)\phi_n(\nu^*)) \to 0.
\]
It follows that $\|\phi_n(ab) - \phi_n(a)\phi_n(b)\|_{2,T} \to 0$ for every $a, b \in A$. Indeed, defining $\Phi = \oplus \phi_n : A \to \prod M_{k(n)}(\mathbb{C}) \subset l^\infty(R)$ we can compose with the natural quotient map $l^\infty(R) \to R^\omega$ and it follows that every unitary such that $\|\phi_n(\nu^*) - \phi_n(u)\phi_n(\nu^*)\|_{2,T} \to 0$ and $\|\phi_n(\nu^*) - \phi_n(u)\phi_n(\nu^*)\|_{2,T} \to 0$ will fall in the multiplicative domain of the composition. However we have arranged that such unitaries have dense linear span and hence all of $A$ falls in the multiplicative domain.

(2) $\implies$ (3). Let $\phi_n : A \to M_{k(n)}(\mathbb{C})$ be a sequence of u.c.p. maps with the properties stated in the theorem. Identify each $M_{k(n)}(\mathbb{C})$ with a unital subfactor of $R$ and we can define a u.c.p. map $A \to l^\infty(R)$ by $x \mapsto (\phi_n(x)) \in \prod M_{k(n)}(\mathbb{C}) \subset l^\infty(R)$. Since the $\phi_n$’s are asymptotically
multiplicative in 2-norm one gets a $\tau$-preserving $*$-homomorphism $A \to R^\omega$ by composing
with the quotient map $l^\infty(R) \to R^\omega$. The weak closure of $A$ under this mapping will be
isomorphic to $\pi_\tau(A)^\prime\prime$ and we can extend the mapping on $A$ to all of $B(H)$ because (a) $l^\infty(R)$
is injective (hence we first extend into $l^\infty(R)$) and (b) there exists a conditional expectation
$R^\omega \to \pi_\tau(A)^\prime\prime$.

(3) $\implies$ (1). Note that $A$ falls in the multiplicative domain of $\Phi$ and hence $\Phi$ is a bimodule
map; i.e. $\Phi(aTb) = \pi_\tau(a)\Phi(T)p_\tau(b)$ for all $a, b \in A$ and $T \in B(H)$. From this observation
one easily checks that if we let $\tau''$ denote the vector trace on $\pi_\tau(A)^\prime\prime$ then $\tau'' \circ \Phi$ is a state
on $B(H)$ which extends $\tau$ and which is invariant under the action of the unitary group of $A$
on $B(H)$. Hence $\tau$ is an invariant mean. 

3. \( \Pi_1 \)-factor representations of \( C^*(\mathbb{F}_\infty) \)

In this section we observe that every separable \( \Pi_1 \)-factor contains a weakly dense copy of
the universal $C^*$-algebra generated by a countably infinite set of unitaries (i.e. $C^*(\mathbb{F}_\infty)$). Since
every separable \( \Pi_1 \)-factor $M$ is generated by a countable number of unitaries it follows from
universality that there is always a $*$-homomorphism $C^*(\mathbb{F}_\infty) \to M$ with weakly dense range.
However, the next proposition completes the \( \Pi_1 \)-factor representation theory of $C^*(\mathbb{F}_\infty)$; it
is not particularly deep but rather amounts to some universal trickery.

**Proposition 3.1.** Let $M$ be a $\Pi_1$-factor. There exists a $*$-monomorphism $\rho : C^*(\mathbb{F}_\infty) \hookrightarrow M$
such that $\rho(C^*(\mathbb{F}_\infty))$ is weakly dense in $M$.

**Proof.** We first need to write $C^*(\mathbb{F}_\infty)$ as an inductive limit of free products of itself. That
is, we define

$$A_1 = C^*(\mathbb{F}_\infty), A_2 = A_1 \ast C^*(\mathbb{F}_\infty), \ldots, A_n = A_{n-1} \ast C^*(\mathbb{F}_\infty), \ldots,$$

where $\ast$ denotes the full (i.e. universal) free product (with amalgamation over the scalars).

Letting $A$ denote the inductive limit of the sequence $A_1 \to A_2 \to \cdots$ it is easy to see (by
universal considerations) that $A \cong C^*(\mathbb{F}_\infty)$. Since $A$ is residually finite dimensional (c.f. [3])
we can find a sequence of integers $\{k(n)\}$ and a unital $*$-monomorphism $\sigma : A \hookrightarrow \Pi M_{k(n)}(\mathbb{C})$.

Note that we may naturally identify each $A_i$ with a subalgebra of $A$ and hence, restricting
$\sigma$ to this copy of $A_i$, get an injection of $A_i$ into $\Pi M_{k(n)}(\mathbb{C})$.

To construct the desired embedding of $A$ into $M$, it suffices to prove the existence of a
sequence of unital $*$-homomorphisms $\rho_i : A_i \to M$ with the following properties:

1. Each $\rho_i$ is injective.
2. $\rho_{i+1}|_{A_i} = \rho_i$, where we identify $A_i$ with the ‘left side’ of $A_i \ast C^*(\mathbb{F}_\infty) = A_{i+1}$.
3. The (increasing) union of $\{\rho_i(A_i)\}$ is weakly dense in $M$.

To this end, we first choose an increasing sequence of projections $p_1 \leq p_2 \leq \cdots$ from $M$
such that $\tau_M(p_i) \to 1$. Then define orthogonal projections $q_2 = p_2 - p_1$, $q_3 = p_3 - p_2$, \ldots
and consider the $\Pi_1$-factors $Q_j = q_j M q_j$ for $j = 2, 3, \ldots$. As is well known and not hard to
construct, we can, for each $j \geq 2$, find a unital embedding $\Pi M_{k(n)}(\mathbb{C}) \hookrightarrow Q_j \subset M$ and thus
we get a sequence of (orthogonal) embeddings $A \hookrightarrow \Pi M_{k(n)}(\mathbb{C}) \hookrightarrow Q_j \subset M$ which will be
denoted by $\sigma_j$.

We are almost ready to construct the $\rho_i$'s. Indeed, for each $i \in \mathbb{N}$ let $\pi_i : C^*(\mathbb{F}_\infty) \to p_i M p_i$
be a (not necessarily injective!) $*$-homomorphism with weakly dense range. We then define
\[ \rho_1 = \pi_1 \oplus \left( \bigoplus_{j \geq 2} \sigma_j|_{A_1} \right) : A_1 \hookrightarrow p_1 M p_1 \oplus \left( \Pi_{j \geq 2} Q_j \right) \subset M. \]

Note that this is a unital \(\ast\)-monomorphism from \(A_1\) into \(M\) (since each \(\sigma_j\) is already faithful on all of \(A\)). Now define a \(\ast\)-homomorphism \(\theta_2 : A_2 = A_1 \ast C^*(\mathbb{F}_\infty) \to p_2 M p_2\) as the free product of the \(\ast\)-homomorphisms \(A_1 \to p_2 M p_2, x \mapsto p_2 \rho_1(x) p_2\), and \(\pi_2 : C^*(\mathbb{F}_\infty) \to p_2 M p_2\). We then put
\[ \rho_2 = \theta_2 \oplus \left( \bigoplus_{j \geq 3} \sigma_j|_{A_2} \right) : A_2 \hookrightarrow p_2 M p_2 \oplus \left( \Pi_{j \geq 3} Q_j \right) \subset M. \]

Note that \(\rho_2|_{A_1} = \rho_1\). Hopefully it is now clear how to proceed. In general, we construct a map (whose range is dense in \(p_{n+1} M p_{n+1}\)) \(\theta_{n+1} : A_{n+1} = A_n \ast C^*(\mathbb{F}_\infty) \to p_{n+1} M p_{n+1}\) as the free product of the cutdown (by \(p_{n+1}\)) of \(\rho_n\) and \(\pi_{n+1}\). This map need not be faithful and hence we take a direct sum with \(\bigoplus_{j \geq n+2} \sigma_j|_{A_{n+1}}\) to remedy this deficiency. It is then easy to see that these maps have all the required properties and hence the proof is complete. \(\square\)

4. Proof of main result

With Theorem 2.3 and Proposition 3.1 in hand we can now prove the main result.

**Theorem 4.1.** Let \(M\) be a separable \(\text{II}_1\)-factor. Then \(M\) is embeddable into \(R^\omega\) if and only if \(M\) has a weak expectation relative to some weakly dense subalgebra.

**Proof.** (\(\implies\)) First assume that \(M \subset R^\omega\). By Proposition 3.1 we may identify \(C^*(\mathbb{F}_\infty)\) with a weakly dense subalgebra of \(M\). Letting \(\tau\) denote the unique trace on \(M\) we first claim that \(\tau|_{C^*(\mathbb{F}_\infty)}\) is an invariant mean. To see this we note that since matrix algebras are weakly dense in \(R\) we can find a sequence \(M_k(n, \mathbb{C}) \subset R\) such that each unitary in \(C^*(\mathbb{F}_\infty) \subset M \subset R^\omega\) lifts to a unitary in \(\Pi M_k(n, \mathbb{C}) \subset l^\infty(R)\). In other words, there is a \(\ast\)-homomorphism \(\sigma : C^*(\mathbb{F}_\infty) \to \Pi M_k(n, \mathbb{C})\) such that \(\pi(\sigma(x)) = x\) for all \(x \in C^*(\mathbb{F}_\infty)\), where \(\pi : l^\infty(R) \to l^\infty(R)/I_0 = R^\omega\) is the canonical quotient mapping. By definition of the trace on \(R^\omega\) it follows that \(\tau|_{C^*(\mathbb{F}_\infty)}\) is the weak-\(\ast\) limit of traces on matrix algebras composed with homomorphisms \(C^*(\mathbb{F}_\infty) \to M_k(n, \mathbb{C})\) and hence \(\tau|_{C^*(\mathbb{F}_\infty)}\) is an invariant mean. Now, if we move \(M\) to its left regular representation coming from \(\tau\) then we can apply Theorem 2.3 and conclude that \(M\) has a weak expectation relative to \(C^*(\mathbb{F}_\infty)\).

(\(\Longleftarrow\)) Now suppose that there exists a weakly dense \(C^*\)-algebra \(A \subset M \subset B(H)\) and a u.c.p. map \(\Phi : B(H) \to M\) such that \(\Phi(a) = a\) for all \(a \in A\). If \(\tau\) is the unique trace on \(M\) then it follows that \(\tau|_A\) is an invariant mean just as in the proof of (3) \(\implies\) (1) from Theorem 2.3. From Theorem 2.3 it follows that we can find a sequence of u.c.p. maps \(\phi_n : A \to M_k(n, \mathbb{C})\) which are asymptotically multiplicative (in 2-norm) and which asymptotically recover \(\tau|_A\) after composing with the traces on \(M_k(n, \mathbb{C})\). Hence the u.c.p. mapping \(A \to l^\infty(R)\) given by \(x \mapsto (\phi_n(x)) \in \Pi M_k(n, \mathbb{C}) \subset l^\infty(R)\) induces a \(\tau|_A\)-preserving \(\ast\)-monomorphism \(A \to R^\omega\) by composing with the quotient map \(l^\infty(R) \to R^\omega\). It follows (essentially due to uniqueness of GNS representations) that the weak closure of \(A\) in \(R^\omega\) is isomorphic to \(M\) and the proof is complete. \(\square\)

Finally we give the proof of the approximation property stated in the introduction. Note that a consequence of this result is that if Connes’ embedding problem is true (i.e. every separable \(\text{II}_1\)-factor is embeddable) then \(R\) is the basic building block for all \(\text{II}_1\)-factors. It is
hard for us to imagine that every II$_1$-factor is built up from the inside by the nicest possible II$_1$-factor, however a counterexample remains elusive.

**Corollary 4.2.** Let $M \subset R^\omega$ be a II$_1$-factor, $\mathfrak{F} \subset M$ be a finite set and $\varepsilon > 0$ be given. Then there exists a complete order embedding $\Phi : R \hookrightarrow M$ (i.e. $\Phi$ is an operator system isomorphism between $R$ and $\Phi(R)$ – that is, $\Phi$ is completely positive and $\Phi^{-1} : \Phi(R) \to R$ is also completely positive) such that for each $x \in \mathfrak{F}$ there exists $r \in R$ such that $\|x - \Phi(r)\|_2 < \varepsilon$.

**Proof.** Let a finite set $\mathfrak{F} \subset M$ and $\varepsilon > 0$ be given. Choose a projection $p \in M$ such that $\tau(p) > 1 - \varepsilon$. Note that the corner $pMp$ is also embeddable into $R^\omega$ (the fundamental group of $R^\omega$ is $\mathbb{R}_+$).

Now let $C^*(\mathbb{F}_\infty) \subset R$ be an identification with a dense subalgebra of $R$ and $\pi : C^*(\mathbb{F}_\infty) \hookrightarrow pMp$ be a $*$-monomorphism with weakly dense range. By Theorem 2.3 we can find a u.c.p. map $\Psi : R \to pMp$ which extends $\pi$. Since we can also find a unital $*$-homomorphism $\nu : R \hookrightarrow (1-p)M(1-p)$ we get the desired complete order embedding by defining $\Phi : R \to M$ by $\phi(r) = \Psi(r) \oplus \nu(r)$.

**REFERENCES**

1. N.P. Brown, *Invariant means and finite representation theory of C*-algebras*, preprint 2003.
2. A. Connes, *Classification of injective factors: cases II$_1$, II$_\infty$, III$_\lambda$, $\lambda \neq 1$*, Ann. Math. **104** (1976), 73–115.
3. K.R. Davidson, *C*-algebras by example*, Fields Institute Monographs 6, American Mathematical Society, Providence, RI, 1996.
4. U. Haagerup, *Spectral decomposition of all operators in a II$_1$-factor which is embeddable in $R^\omega$*, MSRI notes, 2001.
5. E. Kirchberg, *On nonsemisplit extensions, tensor products and exactness of group C*-algebras*, Invent. Math. **112** (1993), 449–489.
6. E.C. Lance, *On nuclear C*-algebras*, J. Funct. Anal. **12** (1973), 157–176.
7. N. Ozawa, *About the QWEP conjecture*, preprint 2003.