HOMOTOPY TYPES OF THE COMPONENTS OF SPACES OF EMBEDDINGS OF COMPACT POLYHEDRA INTO 2-MANIFOLDS

TATSUHIKO YAGASAKI

Abstract. Suppose $M$ is a connected PL 2-manifold and $X$ is a compact connected subpolyhedron of $M$ ($X \neq 1$ pt, a closed 2-manifold). Let $\mathcal{E}(X, M)$ denote the space of topological embeddings of $X$ into $M$ with the compact-open topology and let $\mathcal{E}(X, M)_0$ denote the connected component of the inclusion $i_X : X \subset M$ in $\mathcal{E}(X, M)$. In this paper we classify the homotopy type of $\mathcal{E}(X, M)_0$ in terms of the subgroup $G = \text{Im}[i_{X*} : \pi_1(X) \to \pi_1(M)]$. We show that if $G$ is a nontrivial cyclic group and $M \neq \mathbb{T}^2, \mathbb{K}^2$ then $\mathcal{E}(X, M)_0 \simeq \ast$, if $G$ is a nontrivial cyclic group and $M \neq \mathbb{S}^2, \mathbb{T}^2, \mathbb{K}^2$ then $\mathcal{E}(X, M)_0 \simeq \mathbb{S}^1$, and when $G = 1$, if $X$ is an arc or $M$ is orientable then $\mathcal{E}(X, M)_0 \simeq \mathcal{S}(M)$ and if $X$ is not an arc and $M$ is nonorientable then $\mathcal{E}(X, M)_0 \simeq \mathcal{S}(\mathbb{K})$. Here $\mathbb{S}^1$ is the circle, $\mathbb{T}^2$ is the torus, $\mathbb{P}^2$ is the projective plane and $\mathbb{K}^2$ is the Klein bottle. The symbol $\mathcal{S}(M)$ denotes the tangent unit circle bundle of $M$ with respect to any Riemannian metric of $M$ and $\mathbb{K}$ denotes the orientation double cover of $M$.

1. Introduction

The homotopy types of the identity components of homeomorphism groups of 2-manifolds have been classified in [4, 8, 11]. In this article we consider the problem of classifying the homotopy types of embedding spaces into 2-manifolds.

Suppose $M$ is a connected 2-manifold and $X$ is a compact connected subpolyhedron of $M$ with respect to some triangulation of $M$. Let $\mathcal{E}(X, M)$ denote the space of topological embeddings of $X$ into $M$ with the compact-open topology and let $\mathcal{E}(X, M)_0$ denote the connected component of the inclusion $i_X : X \subset M$ in $\mathcal{E}(X, M)$. The purpose of this article is to describe the homotopy type of $\mathcal{E}(X, M)_0$ in term of the subgroup $i_{X*} \pi_1(X) = \text{Im}[i_{X*} : \pi_1(X) \to \pi_1(M)]$.

If $X$ is a point of $M$ then $\mathcal{E}(X, M) \cong M$, and if $X$ is a closed 2-manifold then $X = M$ and $\mathcal{E}(X, M)_0 = \mathcal{H}(M)_0$, whose homotopy type is already known [4, 8]. Below we assume that $X$ is not a point nor a closed 2-manifold.

The followings are the main results of this paper:

**Theorem 1.1.** Suppose $i_{X*} \pi_1(X)$ is not a cyclic subgroup of $\pi_1(M)$.

(1) $\mathcal{E}(X, M)_0 \simeq \ast$ if $M \neq \mathbb{T}^2, \mathbb{K}^2$.

(2) $\mathcal{E}(X, M)_0 \simeq \mathbb{T}^2$ if $M \cong \mathbb{T}^2$.

(3) $\mathcal{E}(X, M)_0 \simeq \mathbb{S}^1$ if $M \cong \mathbb{K}^2$.

**Theorem 1.2.** Suppose $i_{X*} \pi_1(X)$ is a nontrivial cyclic subgroup of $\pi_1(M)$.

(1) $\mathcal{E}(X, M)_0 \simeq \mathbb{S}^1$ if $M \neq \mathbb{P}^2, \mathbb{T}^2, \mathbb{K}^2$.

(2) $\mathcal{E}(X, M)_0 \simeq \mathbb{T}^2$ if $M \cong \mathbb{T}^2$.
(3) Suppose \( M \cong K^2 \).

(i) \( E(X,M)_0 \cong T^2 \) if \( X \) is contained in an annulus which does not separate \( M \).

(ii) \( E(X,M)_0 \cong S^1 \) if \( X \) is not the case (i).

(4) Suppose \( M \cong P^2 \).

(i) \( E(X,M)_0 \cong SO(3)/\mathbb{Z}_2 \) if \( X \) is an o.r. circle in \( M \).

(ii) \( E(X,M)_0 \cong SO(3) \) if \( X \) is not the case (i).

Here \( S^1 \) is the circle, \( T^2 \) is the torus, \( P^2 \) is the projective plane and \( K^2 \) is the Klein bottle. Finally consider the case where \( X \) is null homotopic in \( M \). We choose a Riemannian manifold structure on \( M \) and denote by \( S(TM) \) the unit circle bundle of the tangent bundle \( TM \). Let \( \tilde{M} \) denote the orientation double cover of \( M \).

**Theorem 1.3.** Suppose \( i_X \ast \pi_1(X) = 1 \) (i.e., \( X \simeq * \) in \( M \)).

(1) \( E(X,M)_0 \cong S(TM) \) if \( X \) is an arc or \( M \) is orientable.

(2) \( E(X,M)_0 \cong S(T\tilde{M}) \) if \( X \) is not an arc and \( M \) is nonorientable.

Since \( E(X,M) \) is a topological \( \ell^2 \)-manifold [10, Theorem 1.2], the topological type of \( E(X,M)_0 \) is determined by the homotopy type of \( E(X,M)_0 \) (Theorems 1.1, 1.2, 1.3).

Main theorems are deduced through the following considerations: Section 2 contains some basic facts on 2-manifolds used in this paper. In Section 3 it is shown that, except a few cases, \( E(X,M)_0 \) is homotopy equivalent to the embedding space \( E(N,M)_0 \) of a regular neighborhood \( N \) of \( X \) into \( M \). Since \( X \) is assumed not to be a closed 2-manifold, it follows that \( N \) has a boundary and admits a core \( Y \) which is a wedge (or a one point union) of circles. Theorem 1.1 corresponds to the case where \( Y \) includes at least two independent essential circles. If \( Y \) includes only one independent essential circle, then we have the case of Theorem 1.2. In Sections 4 and 5 we discuss how to eliminate dependent circles from \( Y \) without changing the homotopy type of \( E(Y,M)_0 \). Based on these observations, Theorems 1.1 and 1.2 can be deduced from the homotopy types of homeomorphism groups of 2-manifolds.

On the other hand, Theorem 1.3 follows from the direct comparison with the unit circle bundle \( ST(M) \), and this theorem is regarded as the main result in this article. In the proof we need a lemma on canonical extension of embeddings of \( X \) into a disk, which is deduced from the conformal mapping theorem in the complex function theory. These are discussed in Section 6.

In [9] we stated some partial results on homotopy types of embedding spaces of circles, arcs and disks. This article provides with a complete answer on this problem. In [9] the proof of the arc case depended on some technical arguments using equivariant homotopy equivalences. To avoid them, in this article we make a systematic study on naturality and symmetry property of the canonical extensions of embeddings into a disk.

2. Preliminaries

Throughout the paper we follow the next conventions: Spaces are assumed to be separable and metrizable, and maps are always continuous. \( \text{Fr}_X A, \text{cl}_X A \) and \( \text{Int}_X A \) denote the frontier, closure and
interior of a subset \( A \) in \( X \). On the other hand, \( \partial M \) and \( \text{Int} M \) denote the boundary and interior of a manifold \( M \). The symbol \( \cong \) indicates a homeomorphism and \( \simeq \) denotes a homotopy equivalence. The term orientation preserving (reversing) is abbreviated as o.p. (o.r.).

First we recall some basic facts on the homeomorphism groups of 2-manifolds. Suppose \( M \) is a 2-manifold and \( X \) is a compact subpolyhedron of \( M \) (with respect to some triangulation of \( M \)). Let \( H_X(M) \) denote the group of homeomorphisms \( h \) of \( M \) onto itself with \( h|_X = id \), equipped with the compact-open topology, and let \( H_X(M)_0 \) denote the identity component of \( H_X(M) \).

**Proposition 2.1.** (\[8, 11\])

1. If \( M \) is compact, then \( H_X(M) \) is an \( \ell_2 \)-manifold.
2. If \( M \) is noncompact and connected, then \( H_X(M)_0 \) is an \( \ell_2 \)-manifold.

Here \( \ell_2 \) is the separable Hilbert space consisting of square sumarable real sequences and an \( \ell_2 \)-manifold is a separable merizable space which is locally homeomorphic to \( \ell_2 \). An ANR is a retract of an open set of a normed space and it has the homotopy type of a CW-complex. Every \( \ell_2 \)-manifold is an ANR and its topological type is determined by its homotopy type. (cf. \[9\])

The homotopy type of \( H_X(M)_0 \) is classified as follows: We use the following notations: \( \mathbb{R}^n \) denotes the Euclidean \( n \)-space, \( S^n \) the \( n \)-sphere, \( D^2 \) the 2-disk, \( T^2 \) the torus, \( M^2 \) the Möbius band, \( P^2 \) the projective plane and \( K^2 \) denotes the Klein bottle.

**Proposition 2.2.** Suppose \( M \) is a connected 2-manifold and \( X \) is a compact subpolyhedron of \( M \).

1. Suppose \( M \) is compact \[4, 8, \S 3\]
   i. \( H_X(M)_0 \cong SO(3) \) if \( (M, X) \cong (S^2, \emptyset) \), \( (P^2, \emptyset) \).
   ii. \( H_X(M)_0 \cong T^2 \) if \( (M, X) \cong (T^2, \emptyset) \).
   iii. \( H_X(M)_0 \cong S^1 \) if \( (M, X) \cong (D^2, \emptyset), (D^2, 0), (S^1 \times [0, 1], \emptyset), (M, \emptyset), (S^2, 1pt), (S^2, 2pt\)} or \( (P^2, 1pt) \) or \( (K^2, \emptyset) \).
   iv. \( H_X(M)_0 \cong * \) if \( (M, X) \) is not the cases (i), (ii) and (iii).

2. Suppose \( M \) is noncompact \[11\]
   i. \( H_X(M)_0 \cong S^1 \) if \( (M, X) \cong (R^2, \emptyset), (R^2, 1pt), (S^1 \times R^1, \emptyset), (S^1 \times [0, 1], \emptyset) \) or \( (P^2 \setminus 1pt, \emptyset) \).
   ii. \( H_X(M)_0 \cong * \) if \( (M, X) \) is not the case (i).

We also note that \( H_\partial(D), H_\partial(M) \cong \star [8, \text{Theorem } 3.4] \).

The next proposition is an assertion on relative isotopies on 2-manifolds \[11, \text{Theorem } 3.1\].

**Proposition 2.3.** Suppose \( M \) is a connected 2-manifold and \( N \) is a compact 2-submanifold of \( M \). If \( (M, N) \) satisfies the following conditions, then \( H(M)_0 \cap H_N(M) = H_N(M)_0 \).

1. \( N \) has no connected component which is a disk, an annulus or a Möbius band.
2. \( cl(M \setminus N) \) has no connected component which is a disk or a Möbius band.

**Proposition 2.3’.** (Relative version) Suppose \( M \) is a connected 2-manifold, \( N \) is a compact 2-submanifold of \( M \) and \( X \) is a nonempty subset of \( N \). If \( (M, N, X) \) satisfies the following conditions, then \( H_X(M)_0 \cap H_N(M) = H_N(M)_0 \).
(i) (a) If $H$ is a disk component of $N$, then $\#(H \cap X) \geq 2$.
(b) If $H$ is an annulus or Möbius band component of $N$, then $H \cap X \neq \emptyset$.

(ii) (a) If $L$ is a disk component of $\text{cl}(M \setminus N)$, then $\#(L \cap X) \geq 2$.
(b) If $L$ is a Möbius band component of $\text{cl}(M \setminus N)$, then $L \cap X \neq \emptyset$.

Here $\#X$ denotes the cardinal of the set $X$. In Proposition 2.3, the conditions (i) and (ii) imply that $M \not\equiv S^2, \mathbb{T}^2, \mathbb{P}^2, \mathbb{K}^2$. Hence the condition (i) of [11, Theorem 3.1] is redundant.

Next we recall some fundamental facts on embedding spaces in 2-manifolds. Suppose $M$ is a 2-manifold and $K \subset X$ are compact subpolyhedra of $M$. Let $E_K(X, M)$ denote the space of embeddings $f : X \hookrightarrow M$ with $f|_K = id$, equipped with the compact-open topology, and let $E_K(X, M)_0$ denote the connected component of the inclusion $i_X : X \subset M$ in $E_K(X, M)$.

**Proposition 2.4.** $E_K(X, M)$ is an $\ell_2$-manifold [14].

In the consideration of homotopy types of the embedding space $E(X, M)_0$, we may always assume that $\partial M = \emptyset$ according to the next lemma.

**Lemma 2.1.** Let $\overline{M}$ be a 2-manifold obtained from $M$ by attaching a collar $\partial M \times [0, 1]$ along $\partial M$. Then the inclusions $E(X, \text{Int} M) \subset E(X, M) \subset E(X, \text{Int} \overline{M}) \subset E(X, M)_0 \subset E(X, \text{Int} \overline{M})_0 \subset E(X, \overline{M})_0$ are homotopy equivalences.

**Proof.** Using the collar $\partial M \times [0, 1]$ and a boundary collar of $M$, we can find a non-ambient isotopy $h_t : \overline{M} \rightarrow \overline{M}$ such that $h_0 = \text{id}_{\overline{M}}$, $h_t(M) \subset \text{Int} \overline{M}$, $h_t(M) \subset \text{Int} M$ (0 < $t$ ≤ 1) and $h_1(M) = M$. Then the homotopy $\varphi_t : E(X, \overline{M}) \rightarrow E(X, \overline{M})$, $\varphi_t(f) = h_t f$, satisfies the conditions that $\varphi_0 = \text{id}$, $\varphi_t(E(X, \text{Int} \overline{M})) \subset E(X, \text{Int} \overline{M})$, $\varphi_t(E(X, M)) \subset E(X, \text{Int} M)$ (0 < $t$ ≤ 1) and $\varphi_1(E(X, \overline{M})) \subset E(X, \overline{M})$. Therefore each inclusion mentioned in Lemma 2.1 is a homotopy equivalence with a homotopy inverse $\varphi_1$.

The homeomorphism group $\mathcal{H}_K(M)_0$ and the embedding space $E_K(X, M)_0$ are joined by the restriction map $\pi : \mathcal{H}_K(M)_0 \rightarrow E_K(X, M)_0$, $\pi(f) = f|_X$. In [11] we have investigated some extension property of embeddings of a compact polyhedron into a 2-manifold, based upon the conformal mapping theorem. The result is summarized as follows [11, Theorem 1.1, Corollary 1.1]:

**Proposition 2.5.** Suppose $\partial M = \emptyset$. Then for every $f \in E_K(X, M)$ there exist a neighborhood $\mathcal{U}$ of $f$ in $E_K(X, M)$ and a map $\varphi : \mathcal{U} \rightarrow \mathcal{H}_K(M)_0$ such that $\varphi(g) = g$ for each $g \in \mathcal{U}$ and $\varphi(f) = \text{id}_M$.

**Corollary 2.1.** Suppose $\partial M = \emptyset$.

(i) The restriction map $\pi : \mathcal{H}_K(M)_0 \rightarrow E_K(X, M)_0$ is a principal bundle with the fiber $\mathcal{G} \equiv \mathcal{H}_K(M)_0 \cap \mathcal{H}_X(M)$, where the group $\mathcal{G}$ acts on $\mathcal{H}_K(M)_0$ by the right composition.

(ii) Suppose $K \subset Y \subset X \subset M$ are compact subpolyhedra of $M$. Then the restriction map $p : E_K(X, M)_0 \rightarrow E_K(Y, M)_0$, $p(f) = f|_Y$ is a locally trivial bundle with fiber $\mathcal{F} = E_K(X, M)_0 \cap E_Y(X, M)$. 

[4]
Proposition 2.3 provides with a sufficient condition for the connectivity of the fiber $G$. In Section 6.2 we investigate some naturality and symmetry properties of the extension map $\varphi$ in Proposition 2.5 in the case where $M$ is a disk.

Finally we list some facts on the fundamental groups of 2-manifolds \(^3\). For a group $G$ and $g \in G$, let $\langle g \rangle$ denote the cyclic subgroup of $G$ generated by $g$.

\textbf{Lemma 2.2.} (i)(a) If a simple closed curve $C$ in $M$ is null-homotopic, then it bounds a disk \(^3\).

(b) If $C_1$ and $C_2$ are disjoint simple closed curves in $M$ and they are homotopic in $M$, then they bounds an annulus \(^3\).

(ii) If $M \not\cong \mathbb{P}$, then $\pi_1(M)$ has no torsion elements \(^3\), Lemma 4.3).

(iii) Suppose $N$ is a connected 2-manifold. If there exists $\alpha, \beta \in \pi_1(N)$ such that $\alpha \beta \neq \beta \alpha, \alpha^2 \beta = \beta \alpha^2$ and $\alpha^2 \neq 1$, then $N \cong \mathbb{R}^2$ \(^3\), Lemma 2.3).

(iv) Suppose $M \not\cong \mathbb{T}^2, \mathbb{K}^2$. If $\alpha, \beta \in \pi_1(M)$ and $\alpha \beta = \beta \alpha$, then $\alpha, \beta \in \langle \gamma \rangle$ for some $\gamma \in \pi_1(M)$ \(^3\), Lemma 4.3).

(v) Suppose $M \not\cong \mathbb{K}^2, C$ is a simple closed curve in $M$ which does not bound a disk or a Möbius band in $M$, $x \in C$ and $\alpha \in \pi_1(M, x)$ is represented by $C$. If $\beta \in \pi_1(M, x)$ and $\beta^k = \alpha^\ell$ for some $k, \ell \in \mathbb{Z} \setminus \{0\}$, then $\beta \in \langle \alpha \rangle$ \(^11\), Lemma 3.1).

\textbf{Lemma 2.3.} Suppose $M$ is a connected 2-manifold with $\partial M = \emptyset$ and $X$ is a compact connected subset of $M$. If $X \simeq \ast$ in $M$, then $X$ has a closed disk neighborhood $D$ in $M$.

\textbf{Proof.} Any sufficiently small compact connected 2-manifold neighborhood $N$ of $X$ is null homotopic in $M$. Each boundary circle $C_i$ of $N$ is also null homotopic in $M$ and bounds a disk $E_i$. If $N \subset E_i$ for some $i$, then we set $D = E_i$. Otherwise, $N_1 = N \cup (\cup_i E_i)$ is a closed 2-manifold, so $N_1 = M$. Since $N \simeq \ast$ in $M$, it follows that $\pi_1 M = 1$, so $M = \mathbb{S}^2$ and $X$ has a disk neighborhood. \hfill $\Box$

3. Embedding spaces of regular neighborhoods

Suppose $M$ is a connected 2-manifold with $\partial M = \emptyset$ and $X$ is a compact connected subpolyhedron of $M$ ($X \neq \emptyset, 1$pt). Let $N$ be a regular neighborhood of $X$ in $M$. By Corollary 2.1 (ii) we have the locally trivial fiber bundle

$$\mathcal{F} \equiv \mathcal{E}(N, M)_0 \cap \mathcal{E}_X(N, M) \hookrightarrow \mathcal{E}(N, M)_0 \xrightarrow{p} \mathcal{E}(X, M)_0, \ p(f) = f|_X.$$ 

\textbf{Proposition 3.1.} (1) The fiber $\mathcal{F} = \mathcal{E}_X(N, M)_0 \simeq \ast$ and the map $p$ is a homotopy equivalence exactly in the following cases:

(i) $X$ is not an arc nor a circle, (ii) $X$ is an arc and $M$ is orientable, (iii) $X$ is an o.p. circle.
(2) In the cases (iv) $X$ is an arc and $M$ is nonorientable and (v) $X$ is an o.r. circle, there exists a $\mathbb{Z}_2$-action on $\mathcal{E}(N, M)_0$ for which the map $p$ factors as
\[ p : \mathcal{E}(N, M)_0 \xrightarrow{\pi} \mathcal{E}(N, M)_0/\mathbb{Z}_2 \xrightarrow{q} \mathcal{E}(X, M)_0, \]
where $\pi$ is a double cover and $q$ is a homotopy equivalence.

First we prove the next lemma.

**Lemma 3.1.** (1)(a) $\mathcal{H}_X(M)_0 \cap \mathcal{H}_N(M) = \mathcal{H}_N(M)_0 \simeq *$. (b) $\mathcal{E}_X(N, M)_0 \simeq *$.
(2)(a) $\mathcal{E}(N, M)_0 \cap \mathcal{E}_X(N, M) = \mathcal{E}_X(N, M)_0$ in the cases (i), (ii) and (iii) of Proposition 3.1.
(b) $\mathcal{E}(N, M)_0 \cap \mathcal{E}_X(N, M) = \mathcal{E}_X(N, M) \cong \mathcal{E}_X(N, M)_0 \times \mathbb{Z}_2$ in the cases (iv) and (v) of Proposition 3.1.

*Proof.*

(1) (a) Let $h \in \mathcal{H}_X(M)_0 \cap \mathcal{H}_N(M)$ and $h_t \in \mathcal{H}_X(M)_0$ be an isotopy with $h_0 = id_M$ and $h_1 = h$. Since each $h_t$ does not interchange the two sides of any edges of $X$, by cutting $M$ along Fr $X$, we can reduce the situation to the case where $X = \partial M$ (a finite union of circles) and $N$ is a collar of $\partial M$. Using a boundary collar we can modify $h_t$ so that $h_t|_N = id_N$. This implies that $h \in \mathcal{H}_N(M)_0$.

(b) By Corollary 2.1 (i) and (a) we have the locally trivial bundle
\[ \mathcal{H}_N(M)_0 \hookrightarrow \mathcal{H}_X(M)_0 \xrightarrow{\pi} \mathcal{E}_X(N, M)_0, \quad \pi(h) = h|_N \]
Since $\mathcal{H}_N(M)_0, \mathcal{H}_X(M)_0 \simeq *$ (Proposition 2.2), it follows that $\mathcal{E}_X(N, M)_0 \simeq *$.

(2)(a) Let $f \in \mathcal{E}(N, M)_0 \cap \mathcal{E}_X(N, M)$. Below we show that in each case of (i), (ii) and (iii),

(#) the embedding $f$ does not interchange the two sides of each edge of $X$.

Then we can easily construct an isotopy $f_t \in \mathcal{E}_X(N, M)$ such that $f_0 = id_N$ and $f_1 = f$. This implies that $f \in \mathcal{E}_X(N, M)_0$.

(i) If $X$ is not an arc nor a circle, then $X$ contains a 2-simplex or a triad (a cone over 3 points). Since $f|_X = id_X$, $f$ does not interchange the sides of some edge of $X$. Since $X$ is connected, it follows that $f$ does not interchange the sides of any edge of $X$.

(ii) Suppose $X$ is an arc and $M$ is orientable. Then $N$ is a disk. Since $f \in \mathcal{E}(N, M)_0$, there exists an isotopy $f_t \in \mathcal{E}(N, M)_0$ such that $f_0$ is the inclusion $i_N : N \subset M$ and $f_1 = f$. Choose any point $x \in X$. Then $f_t$ drags the disk $N$ along the loop $f_t(x)$. Since $M$ is orientable, this loop is o.p. and $f_0, f_1 : (N, x) \to (M, x)$ define a same orientation at $x$. Since $f|_X = id_X$, the embedding $f$ does not interchange the sides of $X$.

(iii) Suppose $X$ is an o.p. circle. Then $N$ is an annulus. Choose a point $x \in X$ and an isotopy $f_t \in \mathcal{E}(N, M)_0$ with $f_0 = i_N$ and $f_1 = f$. We note that

(##) $f$ does not interchange the two sides of $X$ iff the loop $f_t(x)$ is o.p.

This claim is verified by taking a disk neighborhood $D$ of $x$ in $N$ and applying the same argument as in (2)(ii) to the isotopy $f_t|_D$. If $M$ is orientable, any loop in $M$ is o.p. and (#) holds. Below we assume that $M$ is nonorientable.

(*) Suppose $M \not\cong \mathbb{R}^2$. 

6
Suppose $X$ does not bound a disk nor a Möbius band. Let $\alpha, \beta \in \pi_1(M, x)$ denote the classes defined by the circle $X$ with some fixed orientation and the loop $f_t(x)$ respectively. Since $f|_X = \text{id}_X$, the isotopy $f_t|_X$ induces a map $F : X \times S^1 \to M$ such that $F|_{X \times 1} = \text{id}_X$ and $F|_{x \times S^1}$ is the loop $f_t(x)$. Since $X \times S^1$ is a torus and $\alpha, \beta \in \text{Im} F_*$, it follows that $\alpha \beta = \beta \alpha$. Hence $\alpha, \beta \in \langle \gamma \rangle$ for some $\gamma \in \pi_1(M, x) \ (\text{Lemma 2.2 (iv)})$ and $\alpha = \gamma^k$ for some $k \in \mathbb{Z} \setminus \{0\}$ since $X$ does not bound a disk. Then $\gamma \in \langle \alpha \rangle \ (\text{Lemma 2.2 (v)})$ and $\beta \in \langle \alpha \rangle$. Since $\alpha$ is o.p., so is $\beta$, and ($\#)$ holds.

($\ast$) Suppose $X$ bounds a disk or Möbius band $E$. Using the bundle $\mathcal{H}(M_0) \to \mathcal{E}(N, M_0)$, we can find an isotopy $\tilde{f}_t \in \mathcal{H}(M_0)$ such that $\tilde{f}_0 = \text{id}_M$ and $\tilde{f}_1|_N = f_t$. Since $f_t|_X = \text{id}_X$ and $M \not\cong S^2$, $\mathbb{K}$, we have $\tilde{f}_1(E) = E$. Hence ($\#)$ holds.

(**) Suppose $M \cong \mathbb{K}^2$. Let $\pi : T \to \mathbb{K}$ denote the natural double covering and let $m, \ell$ denote the meridian and o.p. longitude of $\mathbb{K}$ respectively ($\pi^{-1}(m) = \{ \pm 1 \} \times S^1$, $\pi^{-1}(\ell) = S^1 \times \{ \pm 1 \} \subset T^2$). Since $X$ is an o.p. circle, it follows that $(M, X) \cong (\mathbb{K}, m)$ if $M \setminus X$ is connected, and that $(M, X) \cong (\mathbb{K}, \ell)$ if $M \setminus X$ is not connected.

(***) Suppose $(M, X) = (\mathbb{K}, m)$. Consider the two meridians $m_\pm = \{ \pm 1 \} \times S^1$ in $T^2$ with the orientation induced from $S^1$. Let $\mathcal{T}_\pm \in m_\pm$ denote the point with $\pi(\mathcal{T}_\pm) = x$. Take a unique lift $\tilde{f}_t : m_+ \to T^2$ of the isotopy $f_t : m \to \mathbb{K}$ ($\pi \tilde{f}_t = f_t \pi$) with $\tilde{f}_0 = \text{id}_{m_\pm}$. If the loop $f_t(x)$ is o.r., then its lift $\tilde{f}_t(\mathcal{T}_\pm)$ is a path from $\mathcal{T}_+$ to $\mathcal{T}_-$. Since $f_t|_m = \text{id}_m$, by the definition of $\pi$ it follows that $\tilde{f}_t(\mathcal{T}_\pm) = (m_\pm)^{-1}$ as loops. Hence $m_- \cong m_+ \cong \tilde{f}_t(\mathcal{T}_\pm) = (m_-)^{-1}$ in $T^2$. But this is impossible.

(****) Suppose $(M, X) = (\mathbb{K}, \ell)$. We regard $S^1$ as the unit circle in $\mathbb{C}$. Consider (a) the coverings $\pi : T^2 \to \mathbb{K}$ and $p : S^1 \times \mathbb{R}^1 \to S^1 \times S^1 = T^2$, $p(z, t) = (z, e^{it})$, (b) the covering involutions $r : T^2 \to T^2$, $r(z, w) = (-z, -\overline{w})$ ($r^2 = \text{id}, \pi r = \pi$), and $\overline{\pi} : S^1 \times \mathbb{R}^1 \to S^1 \times \mathbb{R}^1$, $\overline{\pi}(z, \pi/2 + t) = (-z, \pi/2 - t)$ ($\overline{\pi}^2 = \text{id}, p\overline{\pi} = rp$), and (c) the covering transformation $\tau : S^1 \times \mathbb{R}^1 \to S^1 \times \mathbb{R}^1$, $\tau(z, t) = (z, t + 2\pi)$ ($p\tau = p$).

If the loop $f_t(x)$ is o.r., then $f_t|_\ell$ has a unique lift $\tilde{f}_t : \ell \to T^2$ ($\pi \tilde{f}_t = f_t$) such that $\tilde{f}_0(\ell) = S^1 \times \{1\}$ and $\tilde{f}_1(\ell) = S^1 \times \{-1\}$. In turn there is a unique lift $\tilde{f}_t : \ell \to S^1 \times \mathbb{R}^1$ of $f_t$ such that $\tilde{f}_0(\ell) = S^1 \times \{0\}$. Then $\tilde{f}_1(\ell) = S^1 \times \{(2k+1)\pi\}$ for some $k \in \mathbb{Z}$. Consider the embedding $\varphi, \psi : \ell \times [0, 1] \to S^1 \times \mathbb{R}^1 \times [0, 1]$ defined by $\varphi(z, t) = (\tilde{f}_t(z), t)$, $\psi(z, t) = (\tau^k \overline{\pi}\tilde{f}_t(z), t)$. It follows that (i) $\varphi(\ell \times [0, 1])$ separates $S^1 \times \mathbb{R}^1 \times [0, 1]$ into two components, (ii) $\tau^k \overline{\pi}\tilde{f}_t(\ell) = S^1 \times \{(2k+1)\pi\}$, $\tau^k \overline{\pi}\tilde{f}_1(\ell) = S^1 \times \{0\}$, so $\psi(\ell \times [0, 1])$ meets both components, hence (iii) $\varphi(\ell \times [0, 1]) \cap \psi(\ell \times [0, 1]) \neq \emptyset$. This means that $\tilde{f}_t(x) = \tau^k \overline{\pi}\tilde{f}_t(y)$ for some $x, y \in S^1$ and $t \in [0, 1]$. It follows that $\tilde{f}_t(x) = r\tilde{f}_t(y)$, hence $x \neq y$ and $f_t(x) = f_t(y)$. This contradicts that $f_t \in \mathcal{E}(N, \mathbb{K})$. This completes the proof. The above argument is essential since there is a homotopy $h_t : \ell \to \mathbb{K}$ such that $h_0 = h_1 = i_\ell$ and the loop $h_t(x)$ is o.r.

(b) The assertion is verified in the proof of Proposition 3.1 (2). \hfill $\Box$

Proof of Proposition 3.1. (1) The conclusion follows from Lemma 3.1 (1)(b) and (2)(a).

(2) In the cases (iv) and (v) there exists a $h \in \mathcal{H}_X(N)$ such that $h^2 = \text{id}_X$ and $h$ interchanges the two sides of $X$ in $N$. The group $\mathbb{Z}_2 \cong \{ \text{id}_N, h \}$ acts on $\mathcal{E}(N, M_0)$ by the right composition $f \cdot h = fh$. This $\mathbb{Z}_2$-action preserves the fibers of $p$ so that it induces the factorization $p = q\pi$ and also induces an action on $\mathcal{F}$. 

7
We have $\mathcal{F} = \mathcal{E}_X(N, M) = \mathcal{E}_X(N, M)_0 \cup (\mathcal{E}_X(N, M)_0 \cdot h)$ (a disjoint union) $\cong \mathcal{E}_X(N, M)_0 \times \mathbb{Z}_2$ since $\mathcal{E}_X(N, M) \subset \mathcal{E}(N, M)_0$ and $\mathcal{E}_X(N, M)_0 = \{ f \in \mathcal{E}_X(N, M) : f \text{ preserves the two sides of } X \}$. We also note that the local trivializations of $p$ given in the proof of Corollary 2.1 (ii) preserve the $\mathbb{Z}_2$-actions. These observations imply that $\pi$ is a double cover and that $q$ is a locally trivial fiber bundle with fiber $\mathcal{F} / \mathbb{Z}_2 \cong \mathcal{E}_X(N, M)_0 \cong *$ and so $q$ is a homotopy equivalence. \hfill $\square$

4. Simplification of Embedded Polyhedra

Suppose $X = C_1 \cup C_2 \subset M$ is a wedge of two circles with a wedge point $p$ and let $i_X : X \subset M$ denote the inclusion map. The notation $C_1 \simeq_p C_2$ in $M$ means that the curve $C_1$ is homotopic to the curve $C_2$ in $M$ relative to $p$ under some parametrizations (or orientations) of $C_1$ and $C_2$. The notation $(C_1)^2$ denotes a curve which goes around twice along $C_1$ under a parametrization of $C_1$.

**Lemma 4.1.** Suppose $C_1$ and $C_2$ are essential in $M$ and $i_X \ast \pi_1(X, p)$ is a cyclic subgroup of $\pi_1(M, p)$.

1. If both $C_1$ and $C_2$ are o.p., then $C_1 \simeq_p C_2$ in $M$ and $X$ has a compact neighborhood $M_0$ such that $(M_0, C_1, C_2)$ is as in Figure 4.1 (1).

2. If both $C_1$ and $C_2$ are o.r., then $C_1 \simeq_p C_2$ in $M$ and $X$ has a compact neighborhood $M_0$ such that $(M_0, C_1, C_2)$ is as in Figure 4.1 (2).

3. If $C_1$ is o.p. and $C_2$ is o.r., then $C_1 \simeq_p (C_2)^2$ in $M$ and $X$ has a compact neighborhood $M_0$ such that $(M_0, C_1, C_2)$ is as in Figure 4.1 (3).

**Figure 4.1**

\[ \begin{array}{ccc}
\text{(1)} & \text{(2)} & \text{(3)} \\
\text{\includegraphics[width=0.3\textwidth]{annulus}} & \text{\includegraphics[width=0.3\textwidth]{moebius_band}} & \text{\includegraphics[width=0.3\textwidth]{moebius_band}} \\
\text{an annulus} & \text{a Mo\-bius band} & \text{a Mo\-bius band} \\
\end{array} \]

**Proof.** (c.f. Figure 4.2) We choose a small disk neighborhood $N_0$ of $p$ and thin regular neighborhoods $N_1$ and $N_2$ of $C_1$ and $C_2$ which are in general position so that $N = N_1 \cup N_2$ is a regular neighborhood of $X$ and $N_1 \cap N_2 = N_0$. An outer boundary circle $A$ of $N$ means a boundary circle of $N$ which intersects both $N_1$ and $N_2$. The circles $C_1$ and $C_2$ intersect at $p$ either transversely ($*$) or tangentially ($**$) (Figure 4.2 (0)).

Let $\alpha, \beta \in \pi_1(M, p)$ denote the elements represented by $C_1$ and $C_2$ with some fixed orientations. Since $\alpha, \beta$ are contained in the cyclic subgroup $i_X \ast \pi_1(X, p)$, it follows that $\alpha^k = \beta^\ell$ for some $k, \ell \in \mathbb{Z} \setminus \{0\}$.

1. (c.f. Figure 4.2 (1)) In this case $N_1$ and $N_2$ are annuli. In the case ($*$), $N$ is a torus with a hole. Let $\tilde{N}$ be the torus obtained from $N$ by attaching a disk along the boundary circle $\partial N$, and let
\[ r : M \to \hat{N} \text{ be a map with } r|_N = id_N. \] Then \( r_*\alpha \) and \( r_*\beta \) are two generators of \( \pi_1(\hat{N}, p) \cong \mathbb{Z} \oplus \mathbb{Z}, \) while \( r_*\alpha \) and \( r_*\beta \) are contained in the cyclic subgroup \( (ri_X)_*\pi_1(X, p). \) This is a contradiction. Therefore, \( (***) \) holds and \( N \) is a sphere with three holes (Figure 4.2 (1)(a)).

We note the following facts:

(i) \( M \not\cong \mathbb{P}^2. \) In fact, if \( M \not\cong \mathbb{P}^2, \) then \( \alpha \neq 0 \) is the generator of \( \pi_1(M, p) = \mathbb{Z}_2 \) and \( C_1 \) is o.r., a contradiction.

(ii) If \( X \) is contained in an annulus or a Möbius band \( E \) in \( M, \) then \( A \) bounds a disk \( D \) in \( E \) and \( M_0 = N \cup D \) satisfies the required condition.

(iii) Consider the case where \( M \not\cong K. \) If \( C_1 \) does not separate \( M, \) then \( X \) is contained in an annulus. If \( C_1 \) separates \( M, \) then \( C_1 \) is a common boundary circle of two Möbius bands, one of which contains \( X. \) In either case the required conclusion follows from (ii).

Below we assume that

\[ (\#) \quad M \not\cong K \quad \text{and} \quad X \text{ is not contained in any annulus and Möbius band.} \]

and derive a contradiction.

First we show that \( C_1 \simeq_p C_2 \) in \( M \) (i.e., \( \beta = \alpha^\pm). \) Since \( C_1 \) and \( C_2 \) are essential, they do not bound a disk. Suppose \( C_1 \) bounds a Möbius band \( L. \) By the assumption we have \( C_2 \subset cl(M \setminus L). \) Let \( \overline{M} \) denote the 2-manifold obtained from \( M \) by replacing \( L \) by a disk \( E \) and take a map \( r : M \to \overline{M} \) such that \( r = id \) on \( cl(M \setminus L) \) and \( r(L) = E. \) Since \( r(C_1) \simeq \ast \) in \( \overline{M} \) we have \( (r_*\beta)^l = (r_*\alpha)^k = 1 \) in \( \pi_1(\overline{M}), \) and since \( \overline{M} \not\cong \mathbb{K}, \) so \( \overline{M} \not\cong \mathbb{P}^2, \) we have \( r_*\beta = 1 \) by Lemma 2.2 (ii). Hence \( r(C_2) \) bounds a disk \( F \) in \( \overline{M} \) and it follows that \( r^{-1}(F) \) is a disk bounded by \( C_2 \) or a Möbius band bounded by \( C_2 \) and containing \( X. \) Both cases yield contradictions. Hence \( C_1 \) does not band a Möbius band, and similarly \( C_2 \) does not band a Möbius band. Since \( M \not\cong K, \) by Lemma 2.2 (v) we have \( \beta \in \langle \alpha \rangle \) and \( \alpha \in \langle \beta \rangle. \) Since \( M \not\cong \mathbb{P}, \) by Lemma 2.2 (ii) \( \pi_1(M) \) has no torsion, so \( \beta = \alpha^\pm \) as required.

Now we have the cases (b) and (c) in Figure 4.2 (1), depending on the orientations of \( C_1 \) and \( C_2. \) In (b), \( A \simeq C_1 \ast (C_2)^{-1} \simeq \ast \) and \( A \) bounds a disk \( D \) in \( M. \) Then \( N \cup D \) is an annulus containing \( X, \) which contradicts (\#). In (c), \( A \simeq A_2, \) so \( A_1 \) and \( A_2 \) bounds an annulus \( B \) (Lemma 2.2 (i)(b)). Let \( \overline{M} \) denote the 2-manifold obtained from \( M \) by replacing \( F = cl(M \setminus (N \cup B)) \) by a disk \( E \) and take a map \( r : M \to \overline{M} \) such that \( r = id \) on \( N \cup B \) and \( r(F) \subset E. \) Then \( \overline{M} \cong \mathbb{T} \) or \( \mathbb{K} \) and \( C_1 \simeq_p (C_2)^{-1} \) in the annulus \( N \cup E. \) Since \( C_1 \simeq_p C_2 \) in \( \overline{M}, \) it follows that \( (C_2)^2 \simeq \ast. \) Since \( \overline{M} \not\cong \mathbb{P}^2, \) we have \( C_2 \simeq \ast. \) This is impossible since \( C_2 \) is a meridian of \( \overline{M}. \) This completes the proof of (1).

(2) (cf. Figure 4.2 (2)) In this case \( N_1 \) and \( N_2 \) are Möbius bands. In the case \( (***) \) \( N \) is a Klein bottle with a hole. Let \( \hat{N} \) be a Klein bottle obtained from \( N \) by attaching a disk along the boundary circle and let \( r : M \to \hat{N} \) be a map with \( r|_N = id_N. \) It follows that \( r_*\alpha \) and \( r_*\beta \) are the center circles of the two Möbius bands with a common boundary circle. However, \( r_*\alpha \) and \( r_*\beta \) are contained in the cyclic subgroup \( (ri_X)_*\pi_1(X, p). \) This is a contradiction. Hence the case (\#) holds and \( N \) is a Möbius band with a hole (Figure 4.2 (2)(*)).

We note the following facts:

(i) If \( M \cong \mathbb{P}^2, \) then both \( A_1 \) and \( A_2 \) bound disks \( D_1 \) and \( D_2 \) respectively and \( M_0 = N \cup D_1 \) satisfies
the required condition.

(ii) If $M \cong \mathbb{K}$, then one of $A_1$ and $A_2$ bounds a disk $D$ and another one bounds a Möbius band. The Möbius band $M_0 = N \cup D$ satisfies the required condition.

Below we assume that $M \not\cong \mathbb{P}^2$, $\mathbb{K}$. Both $C_1$ and $C_2$ do not bound a disk nor a Möbius band since they are o.r. Since $M \not\cong \mathbb{P}$, by Lemma 2.2 (v) we have $\beta \in \langle \alpha \rangle$ and $\alpha \in \langle \beta \rangle$. Since $M \not\cong \mathbb{P}$, by Lemma 2.2 (ii) $\pi_1(M)$ has no torsion and $\beta = \alpha^\pm$, so we have $C_1 \simeq_p C_2$. Therefore, we have the situation in Figure 4.2 (2). The boundary $\partial N$ consists of two boundary circles $A_1$ and $A_2$, one of which is homotopic to $C_1 \ast (C_2)^{-1} \simeq *$ and the other is homotopic to $C_1 \ast C_2$. The former bounds a disk $D$ and $M_0 = N \cup D$ satisfies the required condition.

(3) (cf. Figure 4.2 (3)) In this case $N_1$ is an annulus and $N_2$ is a Möbius band. In the case $(\ast)$ $N$ is a Klein bottle with a hole. Let $\hat{N}$ be a Klein bottle obtained from $N$ by attaching a disk along the boundary circle, and let $r : M \to \hat{N}$ be a map with $r|_{N} = id_{N}$. Then $r_\ast \alpha$ and $r_\ast \beta$ are two generators of $\pi_1(\hat{N})$, while $r_\ast \alpha$ and $r_\ast \beta$ are contained in the cyclic subgroup $(ri_X)_{\ast} \pi_1(X,p)$. This is impossible. Therefore, the case $(\ast \ast)$ holds and $N$ is a Möbius band with a hole (Figure 4.2 (3) $(\ast \ast)$).

We note the following facts:

(i) $M \not\cong \mathbb{P}^2$ as shown in (1)(i)

(ii) Suppose $M \cong \mathbb{K}$. If $C_1$ does not separate $M$, then $X$ is contained in an annulus and $C_2$ is o.p., a contradiction. Hence $C_1$ separates $M$ and so $C_1$ is a common boundary circle of two Möbius bands, one of which contains $X$. Then $A$ bounds a disk $D$ in this Möbius band and $M_0 = N \cup D$ satisfies the required condition.

Below we assume that $M \not\cong \mathbb{K}$. If $C_1$ does not bound a Möbius band, then by Lemma 2.2 (v) we have $\beta \in \langle \alpha \rangle$. Since $\alpha$ is o.p., so is $\beta$. This is a contradiction. Therefore, $C_1$ bound a Möbius band $L$. Suppose $C_2 \subset cl(M \setminus L)$. Let $\overline{M}$ denote the 2-manifold obtained from $M$ by replacing $L$ by a disk $D$ and let $r : M \to \overline{M}$ denote a map with $r = id$ on $cl(M \setminus L)$ and $r(L) \subset D$. Then $(r_\ast \beta)^k = r_\ast \alpha^k = 1$. Since $\overline{M} \not\cong \mathbb{P}^2$, $\pi_1(\overline{M})$ has no torsion and we have $r_\ast \beta = 1$ and $r(C_2)$ is o.p. By the definition of $r$, this implies that $C_2$ is also o.p., a contradiction. Therefore we have $X \subset L$ and the conclusion follows from an easy argument. $\square$

**Figure. 4.2**

(0) $(\ast)$

(1) $(\ast \ast)$

\[
\begin{array}{cc}
C_1 & C_2 \\
| & |
\end{array}
\quad
\begin{array}{cc}
C_1 & C_2 \\
| & |
\end{array}
\quad
\begin{array}{cc}
C_1 & C_2 \\
| & |
\end{array}
\quad
\begin{array}{cc}
C_1 & C_2 \\
| & |
\end{array}
\]
(1) (a) \[ N \]

(1) (b) \[ A \]

(2) (∗) \[ N \]

(2) (∗) \[ \partial N = 2 \text{ circles } A_1, A_2 \]

(3) (∗) \[ \partial N = \text{ a circle} \]

(3) (∗∗) \[ \text{an annulus} \]

(3) (∗∗) \[ \text{a Möbius band} \]
Lemma 4.2. Suppose $M$ is a 2-manifold and $X$ is a compact connected subpolyhedron of $M$.

1. If $E$ is a disk or a Möbius band in $M$ and $\partial E \subset X$, then the restriction map $p : \mathcal{E}(X \cup E, M)_0 \to \mathcal{E}(X, M)_0$ is a homotopy equivalence.

2. Suppose $X = Y \cup C_1 \cup C_2$ is a one point union of a compact connected subpolyhedron $Y$ (\neq 1pt) and two essential circles $C_1$ and $C_2$. If the pair $(C_1, C_2)$ satisfies one of the conditions listed in Lemma 4.1 (1), (2) and (3), then the restriction map $p : \mathcal{E}(X, M)_0 \to \mathcal{E}(Y \cup C_2, M)_0$ is a homotopy equivalence.

3. Suppose $X = Y \cup C$ is a one point union of a compact connected subpolyhedron $Y$ and a circle $C$. If $C \simeq *$ in $M$ and $Y$ satisfies one of the conditions (i), (ii) and (iii) in Proposition 3.1, then the restriction map $p : \mathcal{E}(X, M)_0 \to \mathcal{E}(Y, M)_0$ is a homotopy equivalence.

Proof. (cf. Figure 4.3)

1. Attaching a collar to $\partial M$, we may assume that $X \cup E \subset \text{Int} M$. Let $N$ denote a regular neighborhood of $X$ in $M$. Then $N \cup E$ is a regular neighborhood of $X \cup E$ in $M$. Consider the diagram

$$
\begin{array}{ccc}
\mathcal{E}(N \cup E, M)_0 & \xrightarrow{p_1} & \mathcal{E}(X \cup E, M)_0 \\
q \downarrow & & \downarrow p \\
\mathcal{E}(N, M)_0 & \xrightarrow{p_2} & \mathcal{E}(X, M)_0.
\end{array}
$$

By the assumption $X$ is not an arc, and if $X$ is a circle, then $X = \partial E$, which is an o.p. circle. Hence by Proposition 3.1 the restriction maps $p_1$ and $p_2$ are homotopy equivalences. It suffices to show that the map $q$ is a homotopy equivalence. By Corollary 2.1 (ii) the map $q$ forms a locally trivial bundle:

$$
\mathcal{E}(N \cup E, M)_0 \cap \mathcal{E}_N(N \cup E, M) \hookrightarrow \mathcal{E}(N \cup E, M)_0 \xrightarrow{q} \mathcal{E}(N, M)_0
$$

Since (i) $N \cap E = \text{cl}(E \setminus \cup_{i=1}^m E_i)$, each $E_i$ is a disk or a Möbius band in $\text{Int} E$ and they are mutually disjoint and (ii) $\mathcal{H}_{\partial}(E_i) \simeq *$, it follows that

$$
\mathcal{E}(N \cup E, M)_0 \cap \mathcal{E}_N(N \cup E, M) = \mathcal{E}_N(N \cup E, M)_0 \cong \mathcal{H}_{N \cap E}(E) \simeq *
$$

and hence the map $q$ is a homotopy equivalence as required.

2. We choose a small disk neighborhood $A$ of the wedge point $x$ of $X$ and thin regular neighborhoods $N(Y)$, $N(C_1)$ and $N(C_2)$ of $Y$, $C_1$ and $C_2$. We may assume that they are in general position and intersect exactly in $A$. Thus, for instance, we have that $N(C_1 \cup C_2) = N(C_1) \cup N(C_2)$ is a regular neighborhood of $C_1 \cup C_2$ and $N(X) = N(Y) \cup N(C_1) \cup N(C_2)$ is a regular neighborhood of $X$, etc. In each case of (1), (2) and (3) in Lemma 4.1, $\text{cl}(M_0 \setminus N(C_1 \cup C_2))$ has a unique disk component, which we denote by $D$. Note that every component of $\text{cl}(D \setminus N(X))$ is a disk. We denote these disk components by $E, D_1, \ldots, D_m (m \geq 0)$, where $E$ is the unique component which meets $N(C_1)$. 


Consider the following commutative diagram:

\[
\begin{array}{ccc}
E(N(X) \cup D, M)_0 & \longrightarrow & E(N(X), M)_0 \\
\downarrow p_1 & & \downarrow p \\
E(N(Y \cup C_2) \cup \bigcup_{i=1}^m D_i, M)_0 & \longrightarrow & E(N(Y \cup C_2), M)_0 \\
\end{array}
\]

The horizontal arrows are homotopy equivalences by (1) and Proposition 3.1. Let \( F = cl(N(C_1) \setminus A) \).

It follows that \( F \) is a disk, \( F \cap E \) is an arc and hence \( F \cup E \) is also a disk, and that

\[
N(X) \cup D = N(X) \cup (\bigcup_{i=1}^m D_i) \cup E = [N(Y \cup C_2) \cup F] \cup [(\bigcup_{i=1}^m D_i) \cup E] = [N(Y \cup C_2) \cup (\bigcup_{i=1}^m D_i)] \cup [F \cup E]
\]

and \( F \cup E \) meets the 2-manifold \( N(Y \cup C_2) \cup (\bigcup_{i=1}^m D_i) \) in an arc. Therefore, the map \( p_1 \) is a homotopy equivalence and the map \( p \) is also a homotopy equivalence.

(3) The proof is essentially same as (2). We choose a small disk neighborhood \( A \) of the wedge point \( x \) of \( X \) and thin regular neighborhoods \( N(Y) \) and \( N(C) \) of \( Y \) and \( C \). The circle \( C \) bounds a disk \( D_0 \) and \( D \equiv cl(D_0 \setminus N(C)) \) is a disk. Every component of \( cl(D \setminus N(X)) \) is a disk. We denote these components by \( E, D_1, \ldots, D_m (m \geq 0) \), where \( E \) is the unique disk component which meets \( N(C) \).

Let \( F = cl(N(C) \setminus A) \). Consider the diagram \((*)\) in (2), where \( Y \cup C_2 \) is replaced by \( Y \). Then the horizontal arrows and the map \( p_1 \) are homotopy equivalences by the same reasons, and therefore, the map \( p \) is also a homotopy equivalence. This completes the proof. \( \square \)

Figure. 4.3
**Proof of Theorem 1.1.** By Lemma 2.1 we may assume that $\partial M = \emptyset$. We show that the restriction map $p : H(M)_0 \to E(X, M)_0$, $p(h) = h|_X$, is a homotopy equivalence.

Let $N$ be a regular neighborhood of $X$ and let $E_i$ ($i = 1, \cdots, m$) and $F_j$ ($j = 1, \cdots, n$) denote the disk or Möbius band components and the other components of $\partial (M \setminus N)$ respectively, and let $N_1 = N \cup (\cup_{i=1}^m E_i)$. By Lemma 4.2 (1) and Proposition 3.1 the following restriction maps are homotopy equivalences:

$$E(N_1, M)_0 \to E(N, M)_0 \to E(X, M)_0.$$ 

Consider the principal bundle

$$H(M)_0 \cap H_{N_1}(M) \to H(M)_0 \xrightarrow{q} E(N_1, M)_0, \quad q(h) = h|_X.$$ 

Since $i_{N_1*} \pi_1(N_1)$ contains the noncyclic subgroup $i_X* \pi_1(X)$, the submanifold $N_1$ is not a disk, an annulus and a Möbius band. Since $\partial (M \setminus N_1) = \cup_{j=1}^n F_j$ and each $F_j$ is not a disk nor a Möbius band, from Proposition 2.3 it follows that $H(M)_0 \cap H_{N_1}(M) = H_{N_1}(M)_0 \simeq *$ and the map $q$ is a homotopy equivalence. Therefore the map $p$ is also a homotopy equivalence.

Since $\pi_1(M)$ contains the noncyclic subgroup $i_X* \pi_1(X)$, it follows that $M \not\simeq S^2$, $\mathbb{P}^2$, $\mathbb{R}^2$, $\mathbb{P}^2 \setminus 1 pt$, $S^1 \times (0, 1)$. Therefore, by Proposition 2.2 $H(M)_0$ is homotopy equivalent to (a) $T^2$ if $M \cong T^2$, (b) $S^1$ if $M \cong \mathbb{K}^2$, and (c) $*$ if $M \not\cong T^2, \mathbb{K}^2$.

---

5. Embedding spaces of a circle

The following is the main result of this section, which implies Theorem 1.2. Suppose $M$ is a connected 2-manifold.

**Theorem 5.1.** Suppose $C$ is an essential circle in $M$.

1. $E(C, M)_0 \simeq S^1$ if $M \not\simeq \mathbb{P}^2, T^2, \mathbb{K}^2$.
2. $E(C, M)_0 \simeq T^2$ if $M \cong T^2$.
3. Suppose $M \cong \mathbb{K}^2$.
   
   (i) $E(C, M)_0 \simeq T^2$ if $C$ is an o.p. nonseparating circle (a meridian).
   
   (ii) $E(C, M)_0 \simeq S^1$ if $C$ is an o.p. separating circle. (an o.p. longitude = a common boundary of two Möbius bands).
   
   (iii) $E(C, M)_0 \simeq S^1$ if $C$ is an o.r. circle (an o.r. longitude).
4. $E(C, M)_0 \cong SO(3)/\mathbb{Z}_2$ if $M \cong \mathbb{P}^2$

Note that $\pi_k(SO(3)) = \pi_k(SO(3)/\mathbb{Z}_2) = \pi_k(S^2)$ ($k \geq 3$), $\pi_2(SO(3)) = \pi_2(SO(3)/\mathbb{Z}_2) = 0$ and $\pi_1(SO(3)) \cong \mathbb{Z}_2$, $\pi_1(SO(3)/\mathbb{Z}_2) \cong \mathbb{Z}_4$.

When a point of $C$ is fixed, we have the following version:

**Proposition 5.1.** Suppose $C$ is an essential circle in $M$ and $x \in C$. Then $E_x(C, M)_0 \simeq *$ if $M \not\cong \mathbb{P}^2$ and $E_x(C, M)_0 \simeq S^1$ if $M \cong \mathbb{P}^2$. 

---

14
Proof. By Lemma 2.1 we may assume that \( \partial M = \emptyset \). Consider the principal bundle

\[
\mathcal{G} \equiv \mathcal{H}_x(M)_0 \cap \mathcal{H}_C(M) \subset \mathcal{H}_x(M)_0 \xrightarrow{p_1} \mathcal{E}_x(C, M)_0, \quad p_1(h) = h|_C.
\]

Below we show that \( \mathcal{G} = \mathcal{H}_C(M)_0 \). Then \( \mathcal{G} \simeq * \) by Proposition 2.2, and the map \( p_1 \) is a homotopy equivalence. Since \( C \) is essential in \( M \), we have \( M \not\equiv \mathbb{P}^2, S^2 \). Therefore, by Proposition 2.2

\[
\mathcal{E}_x(C, M)_0 \simeq \mathcal{H}_x(M)_0 \simeq * \text{ if } M \not\equiv \mathbb{P}^2 \text{ and } \simeq S^1 \text{ if } M \cong \mathbb{P}^2.
\]

Let \( N \) be a regular neighborhood of \( C \) in \( M \). If \( f \in \mathcal{G} \), then (a) \( f \) preserves the local orientation at \( x \) since \( f \simeq_x id_M \), (b) \( f \) does not interchange the sides of \( C \) at \( x \) (and every point of \( C \)) by (a) and \( f|_C = id_C \). Hence \( f \) is isotopic rel \( C \) to \( g \in \mathcal{H}_N(M) \). Since \( g \simeq_C f \simeq_x id_M \), we have \( g \in \mathcal{H}_x(M)_0 \cap \mathcal{H}_N(M) \). If we show that

\[(*) \quad \mathcal{H}_x(M)_0 \cap \mathcal{H}_N(M) = \mathcal{H}_N(M)_0,
\]

then we have \( id_M \simeq_N g \simeq_C f \) and so \( \mathcal{G} = \mathcal{H}_C(M)_0 \).

The claim \((*)\) follows from Proposition 2.3’ as follows:

(I) If \( N \) is an annulus, then we have (a) \( x \in N \), (b) \( cl(M \setminus N) \) has no disk component since \( C \) is essential and (c) \( cl(M \setminus N) \) has either (i) no Möbius band component, (ii) exactly one Möbius band component \( L \), and (iii) two Möbius band components \( L_1 \) and \( L_2 \). In the case (i), \((*)\) follows from Proposition 2.3’. In the case (ii), (a) any \( h \in \mathcal{H}_x(M)_0 \cap \mathcal{H}_N(M) \) is isotopic rel \( N \) to \( k \in \mathcal{H}_N(M) \) (\( N_1 = N \cup L \)) since \( \mathcal{H}_0(L) \simeq * \), and (b) \( \mathcal{H}_x(M)_0 \cap \mathcal{H}_N(M) = \mathcal{H}_N(M) \) by Proposition 2.3’, so (c) \( h \in \mathcal{H}_N(M)_0 \) since \( k \simeq_N h \simeq_x id_M \) and \( k \in \mathcal{H}_N(M)_0 \) \( (k \simeq_N id_M) \). In the case (iii), we have \( \mathcal{H}_N(M) = \mathcal{H}_N(M)_0 \) (any \( h \in \mathcal{H}_N(M) \) is isotopic rel \( N \) to \( id_M \)) since \( M = N \cup L_1 \cup L_2 \). This implies \((*)\).

(II) If \( N \) is a Möbius band, then \( x \in N \) and \( L = cl(M \setminus N) \) is connected. If \( L \) is not a disk nor a Möbius band, then \((*)\) follows from Proposition 2.3’. If \( L \) is a disk or a Möbius band, then \( \mathcal{H}_N(M) = \mathcal{H}_N(M)_0 \) and this implies \((*)\). This completes the proof. \( \square \)

In the proof of Theorem 5.1 we are concerned with the following fiber bundles: Suppose \( C \) is an essential circle in \( M \) and \( x \in C \).

\[
\begin{align*}
(1) \quad & \mathcal{F} \equiv \mathcal{E}(C, M)_0 \cap \mathcal{E}_x(C, M) \subset \mathcal{E}(C, M)_0 \xrightarrow{p} M : p(f) = f(x), \\
(2) \quad & \mathcal{G} \equiv \mathcal{H}_x(M)_0 \cap \mathcal{H}_C(M) \subset \mathcal{H}_x(M)_0 \xrightarrow{p_1} \mathcal{E}_x(C, M)_0 : p_1(h) = h|_C, \\
(3) \quad & \mathcal{H} \equiv \mathcal{H}(M)_0 \cap \mathcal{H}_C(M) \subset \mathcal{H}(M)_0 \xrightarrow{p_2} \mathcal{E}(C, M)_0 : p_2(h) = h|_C, \\
(4) \quad & \mathcal{K} \equiv \mathcal{H}(M)_0 \cap \mathcal{H}_x(M) \subset \mathcal{H}(M)_0 \xrightarrow{p_3} M : p_3(h) = h(x).
\end{align*}
\]

Suppose \( \partial M = \emptyset \) and let \( \alpha \in \pi_1(M, x) \) be the element represented by \( C \) with an orientation.

**Lemma 5.1.** (1) If \( M \not\equiv \mathbb{P}^2 \), then \( \pi_k(\mathcal{E}(C, M)_0) = 0 \) \((k \geq 2)\) and \( p_* : \pi_1(\mathcal{E}(C, M)_0, i_C) \rightarrow \pi_1(M, x) \) is a monomorphism.

(2) \( p_1 \) is a homotopy equivalence.

(3) \( p_2 : \pi_k\mathcal{H}(M)_0 \cong \pi_k\mathcal{E}(C, M)_0 \) \((k \geq 2)\) and \( p_2_* : \pi_1\mathcal{H}(M)_0 \rightarrow \pi_1\mathcal{E}(C, M)_0 \) is a monomorphism.

(4) (a) \( \alpha \in \text{Im} \ p_* \subset \pi_1(M, x) \), (b) \( \alpha \beta = \beta \alpha \) for any \( \beta \in \text{Im} \ p_* \), and (c) If \( M \not\equiv \mathbb{P}^2 \), then \( \langle \alpha \rangle \cong \mathbb{Z} \).
Proof. (1), (3) Since $\mathcal{F}_0 = \mathcal{E}_x(C, M)_0 \simeq \ast$ for $M \not\cong \mathbb{P}^2$ (Proposition 5.1) and $\mathcal{H}_0 = \mathcal{H}_C(M)_0 \simeq \ast$, the assertions follow from the exact sequences of the fibrations $p$ and $p_2$.

(2) The assertion has been verified in the proof of Proposition 5.1.

(4)(b) Every map $\varphi : (S^1, \ast) \to (\mathcal{E}(C, M)_0, i_C)$ induces a map $\Phi : (S \times C, (\ast, x)) \to (M, x)$. Since $p_*[\varphi], \alpha \in \text{Im} \Phi_* \subset \pi_1(M, x)$, we have the conclusion. \hfill \Box

Proof of Theorem 5.1. By Lemma 2.1 we may assume that $\partial M = \emptyset$.

(1) Since $M \not\cong \mathbb{P}^2$, by Lemma 5.1 (1), (4) it suffices to show that $\text{Im} p_* \subset \langle \alpha \rangle$ so that $\text{Im} p_* \cong \mathbb{Z}$. Let $\beta \in \text{Im} p_*$. Since $M \not\cong \mathbb{T}^2, K$ and $\alpha \beta = \beta \alpha$ (Lemma 5.1 (4)), it follows that $\alpha, \beta \in \langle \delta \rangle$ for some $\delta \in \pi_1(M, x)$ and that $\alpha = \delta^k$ and $\beta = \delta^\ell$ for some $k, \ell \in \mathbb{Z}, k \neq 0$, so $\alpha^\ell = \beta^k$.

(i) Suppose $C$ does not bound a Möbius band. Since $M \not\cong K$ and $C$ is essential, we have $\beta \in \langle \alpha \rangle$ (Lemma 2.2 (v)).

(ii) Suppose $C$ bounds a Möbius band $E$. Then $C$ is an o.p. circle, and in the proof of Lemma 3.1 (2)(a) Case (iii), we have already shown that the loop $p f = f_1(x)$ is o.p. for any class $[f] \in \pi_1(\mathcal{E}(C, M)_0, i_C)$ (or for any isotopy $f : [0, 1] \to \mathcal{E}(C, M)_0$ such that $f_0 = f_1 = i_C$). (Note that, for any regular neighborhood $N$ of $C$, using the bundle $\mathcal{E}(N, M)_0 \to \mathcal{E}(C, M)_0$, we can always find an isotopy $f'_t \in \mathcal{E}(N, M)_0$ such that $f'_0 = i_N$ and $f'_1(C) = f_1$.) This observation means that $\beta$ is o.p.

Let $\gamma \in \pi_1(M, x)$ denote the element which is represented by the center circle $A$ of $E$ and satisfies $\gamma^2 = \alpha$. Since $M \not\cong K$ and $A$ does not bound any disk nor Möbius band, and since $\beta^k = \gamma^2$, it follows that $\beta \in \langle \gamma \rangle$ (Lemma 2.2 (v)). Since $\beta$ is o.p. and $\gamma$ is o.r., we have $\beta \in \langle \alpha \rangle$ as required.

(2) If $G$ is a path-connected topological group, then for any point $a \in G$ the map $q : (\mathcal{H}(G)_0, id_G) \to (G, a), q(h) = h(a)$, admits a section $s : (G, a) \to (\mathcal{H}(G)_0, id_G), s(x)(y) = xa^{-1}y$. Hence if $M \cong \mathbb{T}^2$, then $p_* : \pi_1(\mathcal{H}(M)_0, id_M) \to \pi_1(M, x)$ is surjective and so is $p_* : \pi_1(\mathcal{E}(C, M)_0, i_C) \to \pi_1(M, x)$. Hence by Lemma 5.1 (1) $p : \mathcal{E}(C, M)_0 \to M$ is a homotopy equivalence.

(3) Let $a, b \in \pi_1(K)$ denote the classes represented by the meridian $m$ and the o.r. longitude $\ell$ of $K$ respectively. Then $\pi_1(K) = \langle a, b : bab^{-1} = a^{-1} \rangle$ and the center of $\pi_1(K) = \langle b^2 \rangle \cong \mathbb{Z} \ast \mathbb{Z}$ [3,f].

(i) Since $(M, X) \cong (K, m)$, we may assume that $(M, X) = (K, m)$ and $\alpha = a$. By Lemma 5.1 (1) it suffices to show that $\text{Im} p_* \cong \mathbb{Z} \ast \mathbb{Z}$. Note that $a \in \text{Im} p_* , b \not\in \text{Im} p_* $ since $ab \neq ba$ (Lemma 5.1 (4)) and $b^2 = p_*[f] \in \text{Im} p_*$, where the loop $f_t \in \mathcal{E}(m, K)_0$ isotopes $m$ twice along $\ell$. Therefore $\text{Im} p_* = \langle a, b^2 \rangle$ (the subgroup of $\pi_1(K)$ generated by $a$ and $b^2$). Since the natural double cover $\mathbb{T}^2 \to \mathbb{K}^2$ corresponds to $\langle a, b^2 \rangle \subset \pi_1(K)$, we have $\langle a, b^2 \rangle \cong \pi_1(\mathbb{T}^2) \cong \mathbb{Z} \ast \mathbb{Z}$.

(ii) Since $M$ is a union of two Möbius bands with the common boundary circle $C$, from Lemma 4.2 (1) it follows that $\mathcal{E}(C, M)_0 \cong \mathcal{H}(M)_0 \cong S^1$.

(iii) Since $(M, X) \cong (K, \ell)$, we may assume that $(M, X) = (K, \ell)$ and $\alpha = b$. By Lemma 5.1 (1), (4) it suffices to show that $\text{Im} p_* \subset \langle \alpha \rangle$. Given any $\beta = a^r b^s \in \text{Im} p_*$. Since $b \beta = \beta b$, it follows that $a^{2r} = 1$. Since $\pi_1(K)$ has no torsion, we have $r = 0$ and $\beta = b^s \in \langle \alpha \rangle$.

(4) We use the following notations: We regard as $\mathbb{R}^3 = C \times \mathbb{R}$. $S^2$ is the unit sphere of $\mathbb{R}^3$. $C_0 = \{(z, x) \in S^2 \mid x = 0\}$ and $N_0 = \{(z, x) \in S^2 \mid 0 \leq x \leq 1/2\}$. $\pi : S^2 \to \mathbb{P}^2$ denotes the natural double covering, which identifies antipodal points $(z, x)$ and $(-z, -x)$. Since $(M, C) \cong (\mathbb{P}^2, \pi(C_0)),$
we may assume that \((M, C) = (\mathbb{P}^2, \pi(C_0))\). Then \(N = \pi(N_0)\) is a Möbius band with the center circle \(C\).

We will construct the following diagram:

\[
\begin{array}{cccc}
SO(3) & \xrightarrow{\lambda} & \mathcal{H}(\mathbb{P}^2)_0 & \xrightarrow{p_4} & \mathcal{E}(N, \mathbb{P}^2)_0 \\
q_1 \downarrow & & q_2 \downarrow & & q_3 \\
SO(3)/\langle h \rangle & \xrightarrow{\chi} & \mathcal{H}(\mathbb{P}^2)_0/\langle h \rangle & \xrightarrow{\overline{p}_3} & \mathcal{E}(N, \mathbb{P}^2)_0/\langle h \rangle \xrightarrow{\sim} \mathcal{E}(C, \mathbb{P}^2)_0
\end{array}
\]

Each \(f \in SO(3)\) induces a unique \(\overline{f} \in \mathcal{H}(\mathbb{P}^2)_0\) with \(\pi f = \overline{f}\). The map \(\lambda : SO(3) \to \mathcal{H}(\mathbb{P}^2)_0, f \mapsto \overline{f}\), is a homotopy equivalence (Proposition 2.2 (1)(i)). The restriction map \(p_4\) is a homotopy equivalence by Lemma 4.2 (1).

Consider the involution \(h \in SO(3), h(z, x) = (-z, x)\). By right composition, the group \(\langle h \rangle = \{id_{\mathbb{P}^2}, h\}\) acts on \(SO(3)\) and \(\langle \overline{h} \rangle = \{id_{\mathbb{P}^2}, \overline{h}\}\) acts on \(\mathcal{H}(\mathbb{P}^2)_0\). The vertical maps \(q_1\) and \(q_2\) are the associated quotient maps, which are double coverings.

Since \(h(N_0) = N_0\), it follows that \(\overline{h}(N) = N, \overline{h}|_N \in \mathcal{H}(N), (\overline{h}|_N)^2 = id_N\) and \(\langle \overline{h} \rangle = \{id_N, \overline{h}\}\) acts on \(\mathcal{E}(N, \mathbb{P}^2)_0\) by right composition. Since \(\overline{h}\) interchanges the two local sides of \(C\) in \(N\), by Proposition 3.1 (2) the restriction map \(p : \mathcal{E}(N, \mathbb{P}^2)_0 \to \mathcal{E}(C, \mathbb{P}^2)_0\) factors as the composition of the quotient double covering \(q_3\) and the homotopy equivalence \(\overline{p}\).

Since the maps \(\lambda\) and \(p_4\) are equivariant with respect to these \(\mathbb{Z}_2\)-actions, they induce the associated maps \(\overline{\lambda}\) and \(\overline{p}_4\). Since \(\lambda, p_4\) are homotopy equivalences and \(q_i\)'s are covering, the maps \(\overline{\lambda}\) and \(\overline{p}_4\) induce isomorphisms on the \(k\)-th homotopy groups for \(k \geq 2\).

We show that these maps also induce isomorphisms on \(\pi_1\) and so they are homotopy equivalences. Since \(\pi_1(SO(3)) \cong \mathbb{Z}_2\) and \(q_1\) is a double covering, the order \(#\pi_1(SO(3))/\langle h \rangle\) = 4. Consider the loop \(f_t \in SO(3)\), \(f_t(z, x) = (e^{2\pi it}, x)\) \((0 \leq t \leq 1)\). The class \(\alpha = [f_1]\) generates \(\pi_1(SO(3))\). Since \(f_{1/2} = h\), the loop \(q_1 f_t \in SO(3)/\langle h \rangle\) \((0 \leq t \leq 1/2)\) induces a class \(\beta \in \pi_1(SO(3)/\langle h \rangle)\). Since \(f_{1/2} = f_{1/2}(h)\) \((0 \leq t \leq 1/2)\), we have \(\beta^2 = q_1, \alpha \neq 1\). Therefore order \(\beta = 4\) and \(\pi_1(SO(3)/\langle h \rangle) = \langle \beta \rangle \cong \mathbb{Z}_4\).

Same argument applies to show that \(\pi_1(\mathcal{H}(\mathbb{P}^2)_0/\langle h \rangle) = \langle \overline{\beta} \rangle \cong \mathbb{Z}_4\) and \(\pi_1(\mathcal{E}(N, \mathbb{P}^2)_0/\langle h \rangle) = \langle \overline{\beta}|_N \rangle \cong \mathbb{Z}_4\), where \(\overline{\beta} = [q_2 f_t (0 \leq t \leq 1/2)]\) and \(\overline{\beta}|_N = [q_3 f_t (0 \leq t \leq 1/2)]\).

Since \(\overline{\lambda}_* (\beta) = \overline{\beta}\) and \(\overline{p}_4_* (\overline{\beta}) = \overline{\beta}|_N\), it follows that \(\overline{\lambda}\) and \(\overline{p}_4\) induce isomorphisms on \(\pi_1\). This completes the proof.

Proof of Theorem 1.2. When \(X\) is a circle, the assertions follow from Theorem 5.1 directly. Below we assume that \(X\) is not a circle. By Lemma 2.1 we may assume that \(\partial M = \emptyset\).

Let \(N\) be a regular neighborhood of \(X\). Since \(X\) is not a closed 2-manifold, \(N\) is a compact connected 2-manifold with boundary and admits a subpolyhedron \(Y\) such that \(N\) is a regular neighborhood of \(Y\) in \(M\) and \(Y = D \cup (\bigcup_{i=1}^m C_i) \cup (\bigcup_{j=1}^n C'_j)\) is a one point union of a disk \(D\), essential circles \(C_i (i = 1, \cdots m) (m \geq 1)\) and inessential circles \(C'_j (j = 1, \cdots n) (n \geq 0)\). Let \(Y_1 = D \cup (\bigcup_{i=1}^m C_i)\). By Proposition 3.1 and Lemma 4.2 (3) the restriction maps

\[
\mathcal{E}(X, M)_0 \leftarrow \mathcal{E}(N, M)_0 \rightarrow \mathcal{E}(Y, M)_0 \rightarrow \mathcal{E}(Y_1, M)_0
\]
are homotopy equivalences.

Note that \( i_{X*} \pi_1(X) = i_{N*} \pi_1(N) = i_{Y*} \pi_1(Y) = i_{Y_1*} \pi_1(Y_1) \) is a cyclic subgroup of \( \pi_1(M) \). Hence by Lemma 4.1 each pair \((C_k, C_\ell)\) (or \((C_\ell, C_k)\)) \((1 \leq k, \ell \leq m, k \neq \ell)\) satisfies one of the conditions of Lemma 4.1 (1), (2) and (3). By Lemma 4.2 (2) there exists a \( k \) \((1 \leq k \leq m)\) such that the restriction map

\[
\mathcal{E}(Y_1, M)_0 \longrightarrow \mathcal{E}(D \cup C_k, M)_0
\]

is a homotopy equivalence.

Let \( N_1 \) be a regular neighborhood of \( D \cup C_k \). Then \( N_1 \) is an annulus or a Möbius band, which is a regular neighborhood of \( C_k \). We set \( A = C_k \) when \( N_1 \) is an annulus and \( A = \partial N \) when \( N_1 \) is a Möbius band. By Proposition 3.1 and Lemma 4.2 (1) the restriction maps

\[
\mathcal{E}(D \cup C_k, M)_0 \leftarrow \mathcal{E}(N_1, M)_0 \longrightarrow \mathcal{E}(A, M)_0
\]

are homotopy equivalences.

We apply Theorem 5.1 to the circle \( A \). The statements (1), (2) follow from Theorem 5.1 (1), (2) directly.

(3) Suppose \( M \cong \mathbb{R} \).

(i) Suppose \( X \) is contained in an annulus \( N_0 \) which does not separate \( M \). We may assume that \( X \subset Int N_0 \) and \( N \subset N_0 \). Since \( C_k \) is essential and \( C_k \subset N_0 \) it follows that \( C_k \) is an o.p. nonseparating circle. Hence \( A = C_k \) and \( \mathcal{E}(A, M)_0 \cong \mathbb{T}^2 \).

(ii) Suppose \( C_k \) is an o.p. nonseparating circle of \( M \). Each inessential circle \( C'_j \) bounds a disk \( E_j \). Since each \( C_i \) is essential, every disk \( E_j \) does not intersect \( \cup_i C_i \) except the wedge point of \( Y \). Since \( C_k \) is o.p., the choice of \( C_k \) means that each \( C_i \) is also o.p. and each pair \((C_i, C_k)\) satisfies the condition of Lemma 4.1 (1). Hence we can find an annulus neighborhood \( N_0 \) of \( C_k \) with \( Y \subset Int N_0 \). Since \( C_k \) is nonseparating, so is \( N_0 \). Since \( N \) is a regular neighborhood of \( Y \), we can isotope \( N_0 \) so that \( N \subset N_0 \).

This observation means that if \( X \) does not satisfy the condition (3)(i), then \( C_k \) is either (a) o.p. and separating or (b) o.r. In the case (a), \( A = C_k \), and in the case (b), \( N_1 \) is a Möbius band and \( A = \partial N_1 \). In each case, \( A \) is an o.p. separating circle and \( \mathcal{E}(A, M)_0 \simeq \mathbb{S}^1 \).

(4) Suppose \( M \cong \mathbb{P}^2 \). Since \( A \) is an o.p. circle, \( M \) is the union of a disk and a Möbius band with a common boundary \( A \). By Lemma 4.2 (1) and Proposition 2.2 (1)(i) \( \mathcal{E}(A, M)_0 \simeq \mathcal{H}(M)_0 \simeq SO(3) \). This completes the proof of Theorem 1.2.

□

6. Embedding spaces of an arc and a disk

6.1. Main statements.

Suppose \( M \) is a 2-manifold and \( X \) is a compact connected polyhedron \((\neq 1\text{pt})\) in \( M \) with a distinguished point \( x \in X \). In this section we identify the fiber homotopy (f.h.) type of the projection \( p : \mathcal{E}(X, M)_0 \to M, p(f) = f(x) \) in the case where \( X \simeq \ast \) in \( M \).
We choose a smooth structure and a Riemannian metric of $M$ and consider the unit circle bundle $q : S(TM) \to M$ of the tangent bundle $q : TM \to M$. Let $\pi : \tilde{M} \to M$ denote the orientation double cover of $M$, which has a natural orientation and the Riemannian metric induced from $M$. Let $\tilde{q} : S(T\tilde{M}) \to \tilde{M}$ denote the associated unit circle bundle of $\tilde{M}$. The terminology “fiber homotopy equivalence (or equivalent)” is abbreviated as f.h.e.

**Theorem 6.1.** Suppose $X \simeq \ast$ in $M$ and $\partial M = \emptyset$. Then $p : \mathcal{E}(X,M)_0 \to M$ is f.h.e over $M$ to

(i) $q : S(TM) \to M$ if $X$ is an arc or $M$ is orientable,

(ii) $\pi \tilde{q} : S(T\tilde{M}) \to M$ if $X$ is not an arc and $M$ is nonorientable.

Theorem 1.3 follows from Theorem 6.1, Lemma 2.1 and the fact that $S(TM) \simeq S(T\text{Int} M)$. Since $X$ has a disk neighborhood in $M$, Theorem 6.1 is reduced to the following more technical propositions: Suppose $D$ is an oriented disk and $X$ is a compact connected polyhedron ($\neq$ 1pt) in $\text{Int} D$ with a distinguished point $x \in X$. Consider the subspace

$$\mathcal{E}^*(X,M) = \{ f \in \mathcal{E}(X,M) \mid f \text{ admits an extension } \mathcal{T} \in \mathcal{E}(D,M) \},$$

and the projection $p : \mathcal{E}^*(X,M) \to M$, $p(f) = f(x)$. When $M$ is oriented, consider the subspaces

$$\mathcal{E}^\pm(X,M) = \{ f \in \mathcal{E}(X,M) \mid f \text{ admits an o.p./o.r. extension } \mathcal{T} \in \mathcal{E}(D,M) \}.$$

Since $\tilde{M}$ has a natural orientation, this definition applies to spaces of embeddings into $\tilde{M}$. Let $\tilde{p} : \mathcal{E}(X,M) \to \tilde{M}$, $\tilde{p}(f) = f(x)$ denote the projection.

**Proposition 6.1.** (1) Suppose $X$ is not an arc.

(i) When $M$ is oriented, the projection $p : \mathcal{E}^\pm(X,M) \to M$ is f.h.e. to $q : S(TM) \to M$ over $M$.

(ii) When $M$ is nonorientable, $p : \mathcal{E}^*(X,M) \to M$ is f.h.e. to $\pi \tilde{q} : S(T\tilde{M}) \to M$ over $M$.

(2) When $X$ is an arc, the projection $p : \mathcal{E}(X,M) \to M$ is f.h.e. to $q : S(TM) \to M$ over $M$.

When $X$ is an arc and $x$ is an interior point of $X$, we can introduce a $\mathbb{Z}_2$-action: Let $I = [-1,1]$ and let $p : \mathcal{E}(I,M) \to M$, $p(f) = f(0)$ denote the projection. The group $\mathbb{Z}_2 = \{ \pm 1 \}$ admits a f.p. action on $\mathcal{E}(I,M)$ by $(\varepsilon \cdot f)(t) = f(\varepsilon t)$ ($\varepsilon \in \mathbb{Z}_2$, $t \in I$), and a f.p. action on $S(TM)$ by $\varepsilon \cdot v = \varepsilon v$ ($\varepsilon \in \mathbb{Z}_2$, $v \in \mathcal{E}(T_xM)$, $x \in M$).

**Proposition 6.2.** The projection $p : \mathcal{E}(I,M) \to M$ is $\mathbb{Z}_2$-equivariant f.h.e. to $q : S(TM) \to M$ over $M$.

Propositions 6.1 and 6.2 will be verified in the subsequent subsections.

6.2. Extension Lemma.

Let $D(1)$ ($O(1)$) denote the closed (open) unit disk in $\mathbb{R}^2$ and suppose $X$ is a compact connected polyhedron in $O(1)$. In this subsection we apply the conformal mapping theorem in the complex function theory so as to construct a canonical extension map $\Phi : \mathcal{E}^*(X,O(1)) \to \mathcal{H}(D(1))$ and show the naturality and symmetry properties of $\Phi$. The case where $X$ is a tree has been treated in [4].
6.2.1. Canonical parametrizations.

Suppose $Y$ is a compact 1-dim polyhedron. Let $R(Y)$ denote the set of points of $Y$ which have a neighborhood homeomorphic to $\mathbb{R}$, and set $V(Y) = Y \setminus R(Y)$. Each point of $V(Y)$ is called a vertex of $Y$ and the closure of each component of $R(Y)$ in $Y$ is called an edge of $Y$. Therefore, an edge $e$ is an arc or a simple closed curve: in the former case the end points of $e$ are vertices and in the latter case $e$ contains at most one vertex. By $E(Y)$ we denote the set of edges of $Y$. Note that $V(Y)$ and $E(Y)$ are topological invariants of $Y$. An oriented edge $e$ of $Y$ means an edge of $Y$ with a distinguished orientation. By $e^{-1}$ we denote the same edge with the opposite orientation.

Suppose $X$ is a compact connected polyhedron ($\neq 1$ pt) topologically embedded in $O(1)$. It follows that $X$ is a subpolyhedron with respect to some triangulation of $O(1)$ and that $O(1) \setminus X$ is a disjoint union of an open annulus $U^X$ and a finite number of open disks. Let $\Lambda(X)$ and $\Lambda_0(X)$ denote the set of all components and the subset of open disk components of $O(1) \setminus X$.

For each $U \in \Lambda(X)$, $\text{Fr}U = \text{Fr}_{O(1)}U$ is a compact connected 1-dim polyhedron. For the annulus (respectively each disk) component $U \in \Lambda(X)$ let $\mathcal{E}(U)$ denote the set of oriented edges $e$ of $\text{Fr}U$ such that the right (respectively left) hand side of $e$ lies in $U$. Then $\mathcal{E}(U)$ admits a unique cyclic ordering $\mathcal{E}(U) = \{e_U(1), \ldots, e_U(n_U)\}$ ($e_U(n_U + 1) = e_U(1)$) such that

$$(*) \quad e_U(j) \text{ and } e_U(j + 1) \text{ are adjacent when they are seen from } U, \text{ and have compatible orientations for } j = 1, \ldots, n_U.$$

If we move on these edges in this order, we obtain a loop $\ell_U$ which moves on $\text{Fr}U$ in the “counterclockwise” orientation. As a normalization data, for each $U \in \Lambda_0(X)$ we choose an ordered set $a_U = (x_U, y_U, z_U)$ of three distinct points lying on the loop $\ell_U$ in the positive order, while on $C(1)$ we take the ordered set $a_0 = (-i, 1, i)$ (invariant under $\eta$).

The conformal mapping theorem yields a canonical parametrization of each $U \in \Lambda(X)$. Based on the boundary behaviours of these conformal mappings, we obtain the next lemmas:

**Lemma 6.1.** (1) For the annulus component $U = U^X$, there exists a unique $r = r_X \in (0, 1)$ and a unique o.p. map $g = g_X : A(r, 1) \to clU \subset D(1)$ such that $g$ maps $\text{Int} A(r, 1)$ conformally onto $U$ and $g(1) = 1$. Furthermore, $g$ satisfies the following conditions: (a) $g$ maps $C(1)$ homeomorphically onto $C(1)$, (b) $g(C(r)) = \text{Fr}U$ and $g$ satisfies the condition $(\#)_U$ on $C(r)$.

(2) For each disk component $U \in \Lambda_0(X)$, there exists a unique o.p. map $g = g_{(U,a_U)} : D(1) \to clU \subset D(1)$ such that $g$ maps $O(1)$ conformally onto $U$ and $g(a_0) = a_U$. Furthermore, $g$ satisfies the following conditions: $g(C(1)) = \text{Fr}U$ and $g$ satisfies the condition $(\#)_U$ on $C(1)$.

Here, the condition $(\#)_U$ on $C(r)$ is stated as follows:

$(\#)_U$ There exists a unique collection of points $\{u_U(1), \ldots, u_U(n_U)\}$ lying on $C(r)$ in counterclockwise order such that $g$ maps each positively oriented circular arc $u_U(j)u_U(j + 1)$ onto the oriented edge $e_U(j)$ in o.p. way and maps $\text{Int} \left[ u_U(j)u_U(j + 1) \right]$ homeomorphically onto $e_U(j) \setminus V(\text{Fr}U)$. (Here $u_U(n_U + 1) = u_U(1)$, and when $n_U = 1$, we mean that $u_U(1)u_U(1) = C(r)$.)
For $0 < r < 1$ we define a radial map $\lambda_r : A(1/2, 1) \to A(r, 1)$ by $\lambda_r(x) = (2(1-r)(|x|-1)+1)x/|x|$. We set $h_X = g_X \lambda_{r_X} \in C(A(1/2, 1), D(1))$.

6.2.2. Canonical extensions.

Suppose $(X, a)$ and $(Y, b)$ are two compact connected polyhedra ($\neq$ 1 pt) in $O(1)$ with normalization data $a = \{a_U\}_{U \in \Lambda_0(X)}$ and $b = \{b_V\}_{V \in \Lambda_0(Y)}$. In the case where $X$ is not an arc, if $f : X \to Y$ is any homeomorphism which admits an extension $\overline{f} \in \mathcal{H}(D(1))$, then the sign $\delta(f) = \pm$ is defined by $\overline{f} \in \mathcal{H}^\delta(f)(D(1))$, which depends only on $f$. In the case where $X$ and $Y$ are arcs, we consider any pair $(f, \delta)$ of a homeomorphism $f : X \to Y$ and $\delta = \pm$. By abuse of notation, $(f, \delta)$ is simply denoted by $f$, and $\delta$ by $\delta(f)$. In this setting we will construct a canonical extension $\Phi_{a,b}(f) \in \mathcal{H}^\delta(f)(D(1))$ of $f$.

By the choice of $\delta$, we can find an extension $\overline{f} \in \mathcal{H}^\delta(D(1))$ of $f$. The subsequent arguments do not depend on the choice of such an extension $\overline{f}$. The statement $A/B$ mean that $A$ holds for $\delta = +$ and $B$ holds for $\delta = -$.

(1) For each disk component $U \in \Lambda_0(X)$, consider the corresponding disk component $U_f = \overline{f}(U) \in \Lambda_0(Y)$ ($U_f$ is independent of the choice of $\overline{f}$). Lemma 6.1 (2) provides with two maps $g_{(U,a_U)}$ and $g_{(U_f,b_{U_f})}$.

(2) For the annulus component $U_X$, consider the corresponding annulus component $U_Y$. Lemma 6.1 (1) provides with two data $(r_X, g_X, h_X)$ and $(r_Y, g_Y, h_Y)$. For the notational compatibility, for $U = U_X$, let $U_f = U_Y$, $g_U = g_X$ and $g_{U_f} = g_Y$.

For any $U \in \Lambda(X)$, it follows that $\overline{f} : (U, \mathrm{Fr}U) \cong (U_f, \mathrm{Fr}U_f)$ is an o.p./o.r. homeomorphism, and that if $\{e(1), \cdots, e(n)\}$ is the cyclic ordering of $\mathcal{E}(U)$, then $\{f(e(1)), \cdots, f(e(n))\}$ represents the positive/negative cyclic ordering of $\mathcal{E}(U_f)$. In particular, it also follows that $f(\ell_U) = (\ell_{U_f})^\delta$. Thus, by reversing the orientation and order for $\delta = -$, the condition "$(\#)_{U_f}$ on $C(r_f)$" ($r_f = 1$ or $r_Y$) can be restated as follows:

$(\#)_{U_f}$ There exists a unique collection of points $\{v(1), \cdots, v(n)\}$ lying on $C(r_f)$ in counterclockwise/clockwise order such that $g_f$ maps each oriented circular arc $v(j)v(j+1)$ onto the oriented edge $f(e(j))$ in o.p. way and maps $\text{Int} \left[ \overline{v(j)v(j+1)} \right]$ homeomorphically onto $f(e(j) \setminus V(\mathrm{Fr}U))$. (As before, $v(n+1) = v(1)$, and when $n = 1$, we mean that $\overline{v(1)v(1)} = C(r_f)$.)

(1) For each $U \in \Lambda_0(X)$, compare two maps $f g_U$, $g_{U_f} : C(1) \to \mathrm{Fr}U_f$. By the conditions $(\#)_{U_f}$ and $(\#)_{U_f}$, we obtain a unique map $\theta_U(f) \in \mathcal{H}^\delta(C(1))$ such that $g_{U_f} \theta_U(f) = f g_U$. Extend $\theta_U(f)$ conically to

$$\Theta_U(f) \in \mathcal{H}^\delta(D(1)) : \Theta_U(f)(sz) = s \theta_U(f)(z) \quad (z \in C(1), 0 \leq s \leq 1).$$

There exists a unique homeomorphism $\varphi_U(f) : \text{cl}(U) \cong \text{cl}(U_f)$ which satisfies $g_{U_f} \Theta_U(f) = \varphi_U(f) g_U$. Then $\varphi_U(f)$ is an extension of $f : \mathrm{Fr}U \cong \mathrm{Fr}U_f$.

(2) Compare two maps $f h_X$, $h_Y : C(1/2) \to \mathrm{Fr}U_Y$. By the conditions $(\#)_{U_X}$ and $(\#)_{U_Y}$, we obtain a unique map $\theta_X(f) \in \mathcal{H}^\delta(C(1/2))$ such that $h_Y \theta_X(f) = f h_X$. This definition can be also applied
when \( X \) is an arc. Extend \( \theta_X(f) \) radially to
\[
\Theta_X(f) \in \mathcal{H}^\delta(A(1/2,1)) : \quad \Theta_X(f)(sz) = s\theta_X(f)(z/2) \quad (z \in C(1), 1/2 \leq s \leq 1).
\]
There exists a unique homeomorphism \( \varphi_X(f) : cl(U_X) \cong cl(U_Y) \) which satisfies \( h_Y \Theta_X(f) = \varphi_X(f) h_X \).
Then \( \varphi_X(f) \) is an extension of \( f : FrU_X \cong FrU_Y \).

Finally we define \( \Phi(f) \in \mathcal{H}(D(1)) \) by \( \Phi(f) = f \) on \( X \), \( \Phi(f) = \varphi_U(f) \) on \( cl(U) \) \((U \in \Lambda_0(X)) \) and \( \Phi(f) = \varphi_X(f) \) on \( cl(U_X) \).

### 6.2.3. Extension map.

Suppose \((X,a)\) is a compact connected polyhedron \((\neq 1 \text{ pt})\) in \( O(1) \) with a normalization data \( a = \{a_U\}_{U \in \Lambda_0(X)} \). First suppose \( X \) is not an arc. We keep the notations given in §6.1, so that
1. \( \mathcal{E}^\pm(X,O(1)) = \{f \in \mathcal{E}(X,O(1)) \mid f \text{ admits an extension } \overline{f} \in \mathcal{H}^\mp(D(1))\}, \)
2. \( \mathcal{E}^+(X,O(1)) = \mathcal{E}^+(X,O(1)) \cup \mathcal{E}^-(X,O(1)) \) and \( \mathcal{E}^+(X,O(1)) \cap \mathcal{E}^-(X,O(1)) = \emptyset, \)
3. \( \mathcal{E}^+(X,O(1)) = \mathcal{E}(X,O(1))_0 \) and \( \mathcal{E}^+(X,O(1)) = \{\eta f \mid f \in \mathcal{E}^+(X,O(1))\}, \)

where \( \eta : \mathbb{R}^2 \cong \mathbb{R}^2 \) denotes the reflection \( \eta(x,y) = (x,-y) \). In the statement (iii), the former follows from the bundle \( p : \mathcal{H}(D(1))_0 \to \mathcal{E}(X,D(1))_0, p(h) = h|_X \).

For any \( f \in \mathcal{E}^\delta(X,O(1)) \), the image \( f(X) \) is a compact connected polyhedron in \( O(1) \), to which we can assign a normalization data \( a_f = \{(af)_U\}_{U \in \Lambda_0(X)} \) defined by the condition:
\[
(af)_U = f(af)_U = f(a(f(x), f(y), f(z)) \text{ for } \delta = + \text{ and } \equiv (f(z), f(y), f(x)) \text{ for } \delta = -).
\]

Since \( f(\ell_U) = (\ell_U)_f \), \((af)_U\) lies on \( \ell_U \) in the positive order.

By §6.2.2 we obtain (i) the parametrization: \((r_f(X), g_f(X), h_f(X))\) for the annulus component \( U_f(X) \) and \( g_U \), for each disk component \( U_f \in \Lambda_0(f(X)) \), and (ii) the canonical extension \( \Phi_a(f) = \Phi_{a,a(f)}(f) = \mathcal{H}^\delta(D(1)) \) of \( f \).

When \( X \) is an arc, we set \( \mathcal{E}(X,O(1)) = \mathcal{E}(X,O(1)) \times \{\pm\} \) (note that \( \mathcal{E}(X,O(1)) = \mathcal{E}^\pm(X,O(1)) \) in the usual sense). We identify \( \mathcal{E}(X,O(1)) \) with \( \mathcal{E}^+(X,O(1)) \).

By §6.2.2, for any \((f, \delta) \in \mathcal{E}(X,O(1)) \), we obtain (i) the parametrization: \((r_f(X), g_f(X), h_f(X))\) for the annulus component \( U_f(X) \), and (ii) the canonical extension \( \Phi_X(f, \delta) \in \mathcal{H}^\delta(D(1)) \) of \( f \). In the subsequent statements the notation \((X,a)\) simply means \( X \) when \( X \) is an arc. The next assertions follow from the same argument as in [10, Lemma 2.3, Proposition 2.1].

**Lemma 6.2.** The next correspondence is continuous:
\[
\mathcal{E}(X,O(1)) \ni f \mapsto (r_f(X), h_f(X), \{g_U\}_{U \in \Lambda_0(X)}) \in (0,1) \times \mathcal{C}(A(1/2,1), D(1)) \times \prod_{U \in \Lambda_0(X)} \mathcal{C}(D(1), D(1)).
\]

**Proposition 6.3.** The correspondence \( \Phi(X,a) : \mathcal{E}(X,O(1)) \to \mathcal{H}(D(1)) \) is continuous.

The extension map \( \Phi \) has the following naturality and symmetry properties: For notations: As usual we identify \( \mathbb{C} \) with \( \mathbb{R}^2 \). For each \( z \in C(1) \), let \( \theta_z : \mathbb{C} \cong \mathbb{C} \) denote the rotation \( \theta_z(w) = z \cdot w \).
Let \( \eta : \mathbb{R}^2 \cong \mathbb{R}^2 \) denote the reflection \( \eta(x,y) = (x,-y) \). Let \( \eta_+ = id \) and \( \eta_- = \eta \). The restriction of \( \gamma \in O_2 \) to \( E = C(r), D(r), O(r) \) is denoted by the same symbol \( \gamma \).
Proposition 6.4. Suppose $X$ is not an arc.
(1) $\Phi(X,a)(gf) = \Phi(f(X,a)(f)g)\Phi(X,a)(f)$ for any $f \in \mathcal{E}^*(X,O(1))$ and $g \in \mathcal{E}^*(f(X),O(1))$.
(2) $\Phi(X,a)(\gamma|X) = \gamma$ for any $\gamma \in O_2$.
(3) $\Phi(X,a) : \mathcal{E}^*(X,O(1)) \to \mathcal{H}(D(1))$ is left $O_2$-equivariant.

Proposition 6.5. Suppose $X$ is an arc.
(1) $\Phi_X(gf,\varepsilon\delta) = \Phi_f(X)(g,\varepsilon)\Phi_X(f,\delta)$ for any $(f,\delta) \in \mathcal{E}^*(X,O(1))$ and $(g,\varepsilon) \in \mathcal{E}^*(f(X),O(1))$.
(2) $\Phi_X(\gamma|X,\delta(\gamma)) = \gamma$ for any $\gamma \in O_2$.
(3) (i) $\Phi_X : \mathcal{E}^*(X,O(1)) \to \mathcal{H}(D(1))$ is left $O_2$-equivariant.
\hspace{1cm} (b) $\Phi_X(\eta|X,\delta) = \eta \Phi_X(f,\delta)\eta$ for any $(f,\delta) \in \mathcal{E}^*(X,O(1))$.
\hspace{1cm} (ii) If $X = [-1/2,1/2]$, then
\hspace{1.3cm} (a) $\Phi_X : \mathcal{E}^*(X,O(1)) \to \mathcal{H}(D(1))$ is right $\mathbb{Z}_2$-equivariant, and (b) $\Phi_X(i|X,-) = \eta$.

Proof of Propositions 6.4 and 6.5. (1) Since $\delta(gf) = \delta(g)\delta(f)$, $U_{g|f} = (U_f)g$ ($U \in \Lambda(X)$) and $a_{gf} = (a_g)g$, it follows that $\theta_U(gf) = \theta_U(g)\theta_U(f)$, $\Theta_U(gf) = \Theta_U(g)\Theta_U(f)$ and $\varphi_U(gf) = \varphi_U(g)\varphi_U(f)$. This implies that $\Phi(X,a)(gf) = \Phi(f(X,a)(f))\Phi(X,a)(f)$ (or $\Phi_X(gf,\varepsilon\delta) = \Phi_X(f(\varepsilon)\Phi_X(f,\delta)$).
(2) It suffices to show that (a) $\Phi_X(\theta_{z}|X) = \theta_z$, (b) $\Phi_X(\eta|X) = \eta$ and (a) $\Phi_X(\theta_{z}|X,+ = \theta_z$, (b) $\Phi_X(\eta|X,- = \eta$ when $X$ is an arc).
\hspace{1cm} (a) For each $U \in A(X)$, the uniqueness part of Lemma 6.1 (1) implies that $h_{U\theta_z} = \theta_z h_U$. This means that $\Theta_U(\theta_{z}|X) = id$ and $\varphi_U(\theta_{z}|X) = \theta_z$.
\hspace{1cm} For $U = U_X$, let $w \in C(1)$ be the unique point such that $\theta_2 g_X \theta_1^{-1}(w) = 1$. The uniqueness part of Lemma 6.1 (2) implies that $g_{\theta_1}(X) = \theta_2 g_X \theta_1^{-1}w$ and $h_{\theta_2}(X) = \theta_2 h_X \theta_1^{-1}w$. This means that $\Theta_X(\theta_{z}|X) = \theta_w^{-1}z$ and $\varphi_X(\theta_{z}|X) = \theta_z$. Thus we have $\Phi(\theta_{z}|X) = \theta_z$.
\hspace{1cm} (b) For each $U \in A(X)$, the uniqueness part of Lemma 6.1 implies that $\eta h_X = h_{\eta}(X)\eta$. This means that $\theta_U(\eta|X) = \eta$, $\Theta_U(\eta|X) = \eta$ and $\varphi_U(\eta|X) = \eta$. Thus we have $\Phi(\eta|X) = \eta$.
\hspace{1cm} (3) The assertions follow form (1) and (2). Note that $(f |_{\theta_{z}} + 1, \delta) = (f, \delta)(\theta_{z}^{-1}|X,+) + (\eta f, \delta) = (\eta, -)F(\delta)|_{\theta_{z}^{-1}}$ when $X = [-1/2,1/2]$.

6.3. Deformation Lemma.

Suppose $X$ is a compact connected polyhedron ($\neq 1$ pt) in $O(1)$ with a normalization data.

6.3.1. Deformation of $\mathcal{E}^+(X,O(1))$ onto a circle.

In this subsection we use the extension maps $\Phi_X$ to construct an $O_2$-equivariant strong deformation retraction ($O_2$-s.d.r.) of the embedding space $\mathcal{E}^*(X,O(1))$ onto $O_2$. The space $O_2$ is embedded
into $\mathcal{H}(E)$ ($E = C(1), D(1)$) by the restriction $\gamma \mapsto \gamma|_E$ and into $\mathcal{E}^*(X, O(1))$ by $\gamma \mapsto \gamma|_X$ (and $\gamma \mapsto (\gamma|_X, \delta(\gamma))$ when $X$ is an arc).

We need some auxiliary homotopies $G_t, A_t$ and $H_t$ ($0 \leq t \leq 1$):

(i) $\mathcal{H}(C(1))$ has a natural s.d.r. $G_t$ onto $O_2$ defined by

$$G_t : \mathcal{H}(C(1)) \to \mathcal{H}(C(1)), \quad G_t(g(e^{i\theta})) = g(1) \exp[i\delta(g)((1 - t)\tau(\theta) + t\theta)],$$

where $\delta(g) = \pm$ is defined according to $g \in \mathcal{H}^{\pm}(C(1))$ and $\tau : [0, 2\pi] \to [0, 2\pi]$ is a unique map such that $\tau(0) = 0$ and $g(e^{i\theta}) = g(1) \exp[i\delta(g)\tau(\theta)]$.

(ii) The cone extension map $c : \mathcal{H}(C(1)) \to \mathcal{H}(D(1))$, $c(g)(sx) = sg(x)$ ($x \in C(1)$, $0 \leq s \leq 1$), is a section of the restriction map $p : \mathcal{H}(D(1)) \to \mathcal{H}(C(1))$, $p(h) = h|_{C(1)}$.

(iii) The Alexander trick yields a s.d.r. $A_t$ of $\mathcal{H}(D(1))$ onto $\text{Im} \ c$ defined by

$$A_t : \mathcal{H}(D(1)) \to \mathcal{H}(D(1)) : \text{id} \simeq cp, \quad A_t(h)(x) = \begin{cases} |x|h(x/x)| & (s \leq |x| \leq 1, x \neq 0) \\ sh(x/s) & (0 < |x| \leq s) \\ 0 & (x = 0, t = 1) \end{cases} \quad (\text{where } s = 1 - t).$$

(iv) Combining these homotopies we obtain a s.d.r. $H_t$ of $\mathcal{H}(D(1))$ onto $O_2$:

$$H_t : \mathcal{H}(D(1)) \to \mathcal{H}(D(1)) : \ H_t = \begin{cases} A_{2t} & (0 \leq t \leq 1/2) \\ cG_{2t-1}p & (1/2 \leq t \leq 1) \end{cases}.$$

The maps $c, p$ and $A_t$ are easily seen to be left and right $O_2$-equivariant.

Lemma 6.3. $G_t$ and $H_t$ are left $O_2$ and right $\eta$-equivariant.

Proof. $G_t$ : (i) Let $\gamma = \theta_z$ ($z = e^{i\lambda}$). Then

$$(\gamma g)(e^{i\theta}) = (\gamma g)(1)e^{i\delta(g)\tau(\theta)} \quad \text{and} \quad G_t(\gamma g)(e^{i\theta}) = e^{i\lambda}g(1) \exp[i\delta(g)((1 - t)\tau(\theta) + t\theta)] = \gamma G_t(g)(e^{i\theta}).$$

(ii) Since $\eta(e^{i\theta}) = e^{-i\theta} = e^{i(2\pi - \theta)}$ and $\delta(\eta g) = -\delta(g)$, it follows that

$$\begin{align*}
(a) & \quad (\eta g)(e^{i\theta}) = (\eta g)(1)\exp[i\delta(\eta g)\tau(\theta)], \\
(b) & \quad (g\eta)(e^{i\theta}) = g(1)e^{i\delta(g)\tau(2\pi - \theta)} = (g\eta)(1)e^{i\delta(g)(2\pi - (2\pi - \theta))}.
\end{align*}$$

The required $O_2$-s.d.r. $F_t$ of $\mathcal{E}^*(X, O(1))$ onto $O_2$ is defined by

1. $X$ is not an arc : $F_t : \mathcal{E}^*(X, O(1)) \to \mathcal{E}^*(X, O(1))$, $F_t(f) = H_t(\Phi_X(f))|_X$
2. $X$ is an arc : $F_t : \mathcal{E}^*(X, O(1)) \to \mathcal{E}^*(X, O(1))$, $F_t(f, \delta) = (H_t(\Phi_X(f, \delta))|_X, \delta)$

The $O_2$-equivariance follows from Propositions 6.4, 6.5 and Lemma 6.3. When $0 \in X$, we can consider the subspace $\mathcal{E}^{\pm}(X, 0; O(1), 0) = \{ f \in \mathcal{E}^{\pm}(X; O(1)) \mid f(0) = 0 \}$.

Lemma 6.4. If $0 \in X$, then $F_t(\mathcal{E}^{\pm}(X, 0; O(1), 0)) \subset \mathcal{E}^{\pm}(X, 0; O(1), 0)$. 


The circle \( C(1) \) is embedded into \( \mathcal{E}^+(X,O(1)) \) by \( z \mapsto \theta_z \), which corresponds to the embedding \( SO_2 \subset \mathcal{E}^+(X,O(1)) \). The \( O_2 \)-s.d.r. \( F_t \) restricts to the \( SO_2 \)-s.d.r. of \( \mathcal{E}^+(X,O(1)) \) onto \( C(1) \). In the case where \( X \) is an arc, The \( SO_2 \) action on \( \mathcal{E}(X,O(1)) \) extends to the \( O_2 \)-action by left composition \( (\gamma f) = \gamma \circ f \) or \( (\gamma f,+) = (\gamma f,+) \) even if \( \delta(\gamma) = -1 \). (This action should be distinguished from the \( O_2 \)-action on \( \mathcal{E}^*(X,O(1)) \).)

**Lemma 6.5.** When \( X \) is an arc \([a,b]\) \((-1 < a < b < 1)\), the s.d.r. \( F_t : \mathcal{E}(X,O(1)) \to \mathcal{E}(X,O(1)) \) is left \( O_2 \)-equivariant.

**Proof.** Since \( F_t \) is \( SO_2 \)-equivariant, it suffices to show that \( F_t(\eta f) = \eta F_t(f) \) \((f \in \mathcal{E}(X,O(1)))\). Since \( \Phi_X(\eta f,+) = \Phi_X(\eta f|_X,+) = \eta \Phi_X(f,+) \eta \), from Lemma 6.3 it follows that

\[
F_t(\eta f) = H_t(\Phi_X(\eta f,+)|_X) = H_t(\eta \Phi_X(f,+)|_X) = \eta H_t(\Phi_X(f,+)|_X) = \eta F_t(f).
\]

When \( X \) is the arc \( I = [-1/2,1/2] \), we can modify the construction of \( F_t \) in order that \( F_t \) is right \( \mathbb{Z}_2 \)-equivariant: Let \( J = \{ \pm 1 \} \subset C(1) \).

(i) A s.d.r. \( \nu_t \) of \( \mathcal{E}(J,C(1)) \) onto \( C(1) \) is defined by

\[
\nu_t : \mathcal{E}(J,C(1)) \to \mathcal{E}(J,C(1)) : \nu_t(\alpha)(\pm 1) = \alpha(\pm 1) \exp[\mp it(\pi/2 - \theta(\alpha))],
\]

where \( \theta(\alpha) \in (0,2\pi) \) is defined by \( \alpha(-1) = \alpha(1)e^{2\theta(\alpha)} \).

(ii) The cone extension map \( c' : \mathcal{E}(J,C(1)) \to \mathcal{E}([-1,1],D(1)) \) is defined by \( c'(\alpha)(\pm s) = s\alpha(\pm 1) \) \((s \in [0,1])\).

(iii) Let \( q : \mathcal{H}(D(1)) \to \mathcal{E}(J,C(1)) \) denote the restriction map.

The modified s.d.r. \( F_t \) of \( \mathcal{E}(I,O(1)) \) onto \( C(1) \) is defined by

\[
F_t : \mathcal{E}(I,O(1)) \to \mathcal{E}(I,O(1)) : F_t(f) = \begin{cases} A_{2t}(\Phi_I(f,+))|_I & (0 \leq t \leq 1/2) \\
(\nu_t q(\Phi_I(f,+)))|_I & (1/2 \leq t \leq 1) 
\end{cases}
\]

**Lemma 6.6.** (i) \( F_t \) is left \( O_2 \), right \( \mathbb{Z}_2 \)-equivariant.

(ii) \( F_t(\mathcal{E}(I,0,O(1),0)) \subset \mathcal{E}(I,0,O(1),0) \).

**Proof.** (i) The map \( A_t \) is left and right \( O_2 \)-equivariant, and the maps \( c', q, \nu_t \) are easily seen to be left \( O_2 \), right \( \mathbb{Z}_2 \)-equivariant. Since \( \Phi_I(\ ,+ \) is left \( SO_2 \), right \( \mathbb{Z}_2 \)-equivariant (Proposition 6.5 (3)), so is \( F_t \). The left \( \eta \)-equivariance of \( F_t \) follows from Proposition 6.5 (3)(i)(b) and the direct observation:

\[
F_t(\eta f) = \begin{cases} A_{2t}(\eta \Phi_I(f,+))|_I = \eta A_{2t}(\Phi_I(f,+))|_I = \eta A_{2t}(\Phi_I(f,+))|_I \\
(\nu_t q(\eta \Phi_I(f,+)))|_I = (\nu_t q(\Phi_I(f,+)))|_I = \eta c'(\nu_t q(\Phi_I(f,+)))|_I 
\end{cases}
\]

6.3.2. Oriented plane case.

Suppose \( V \) is an oriented 2-dim vector space with an inner product. For \( \varepsilon \in (0,\infty) \), let \( D_V(\varepsilon) = \{ v \in V \mid |v| \leq \varepsilon \} \), \( O_V(\varepsilon) = \{ v \in V \mid |v| < \varepsilon \} \) and \( C_V(\varepsilon) = \{ v \in V \mid |v| = \varepsilon \} \). Consider the subspace

\[
\mathcal{E}^+(X,O_V(\varepsilon)) = \{ f \in \mathcal{E}(X,O_V(\varepsilon)) \mid f \text{ extends to an o.p./o.r. homeomorphism } T : D(1) \cong D_V(\varepsilon) \}.
\]

25
The circle $C_V(1)$ is embedded into $E^+(X, O_V(\varepsilon))$ by $v \mapsto \varepsilon \alpha_v|_X$, where $\alpha_v : \mathbb{R}^2 \cong V$ is the unique o.p. linear isometry with $\alpha_v(1) = v$. The group $SO(V)$ acts on $E^+(X, O_V(\varepsilon))$ by left composition ($\gamma f = \gamma \circ f$). The $SO_2$-s.d.r. $F_t$ of $E^+(X, O(1))$ onto $C(1)$ induces a $SO(V)$-s.d.r. $\varphi^V_t$ of $E^+(X, O_V(1))$ onto $C_V(1)$:

$$\varphi^V_t : E^+(X, O_V(1)) \to E^+(X, O_V(1)), \quad \varphi^V_t(f) = \alpha F_t(\alpha^{-1} f),$$

where $\alpha : \mathbb{R}^2 \cong V$ is any o.p. linear isometry. Since $F_t$ is $SO_2$-equivariant, $\varphi^V_t$ does not depend on the choice of $\alpha$. When $0 \in X$, we have $\varphi^V_t(E^+(X, 0; O_V(1), 0)) \subset E^+(X, 0; O_V(1), 0)$ (Lemma 6.4).

The group $SO(V)$ is canonically isomorphic to $SO_2$ under the isomorphism $\chi_V : SO_2 \cong SO(V)$, $\chi_V(\gamma) = \alpha \gamma \alpha^{-1}$. Here $\alpha = \mathbb{R}^2 \cong V$ is any o.p. linear isometry, and $\chi_V$ does not depend on the choice of $\alpha$. Thus $SO_2$ acts on $E^+(X, O_V(\varepsilon))$ canonically and $\varphi^V_t$ is a $SO_2$-s.d.r.

When $X = [a, b]$, we can include the unoriented case. Suppose $V$ is a 2-dim (unoriented) vector space with an inner product. The circle $C_V(1)$ is embedded into $E(X, O_V(\varepsilon))$ by $v \mapsto \varepsilon \alpha_v|_X$, where $\alpha_v : \mathbb{R}^2 \cong V$ is any one of two linear isometries with $\alpha_v(1) = v$. The s.d.r. $F_t : E(X, O(1)) \to E(X, O(1))$ is $O_2$-equivariant (Lemma 6.5) and it induces a $O(V)$-s.d.r. $\varphi^V_t$ of $E(X, O_V(1))$ onto $C_V(1)$:

$$\varphi^V_t : E(X, O_V(1)) \to E(X, O_V(1)), \quad \varphi^V_t(f) = \alpha F_t(\alpha^{-1} f),$$

where $\alpha : \mathbb{R}^2 \cong V$ is any linear isometry and $\varphi^V_t$ does not depend on the choice of $\alpha$. When $0 \in X$, we have $\varphi^V_t(E(X, 0; O_V(1), 0)) \subset E(X, 0; O_V(1), 0)$. In the unoriented case there is no canonical isomorphism $SO_2 \cong SO(V)$.

When $X$ is the arc $I = [-1/2, 1/2] \subset O(1)$, we can use the modified s.d.r. $F_t$ so that $\varphi^V_t$ is left $O(V)$ and right $\mathbb{Z}_2$-equivariant.

### 6.3.3. Oriented plane bundle case.

The purpose of this subsection is to construct a fiberwise deformation of the fiberwise embedding space $E^+_\pi(X, O_E(\varepsilon))$ onto the unit circle bundle $S(E)$ for any oriented plane bundle $p : E \to B$.

Suppose $\pi : E \to B$ is an oriented 2-dim vector bundle with an inner product. Let $V_b = \pi^{-1}(b)$ for $b \in B$. For any map $\varepsilon : B \to (0, \infty)$, we can associate the subspaces:

$$O_E(\varepsilon) = \cup_{b \in B} O_{V_b}(\varepsilon(b)) \subset E, \quad E^+\pi(X, O_E(\varepsilon)) = \cup_{b \in B} E^+(X, O_{V_b}(\varepsilon(b)) \subset \mathcal{C}(X, E),$$

and the projection: $p : E^+\pi(X, O_E(\varepsilon)) \to B, p(f) = b$ ($f \in E(X, O_{V_b}(\varepsilon(b)))$).

Let $\pi : S(E) \to B$ denote the unit circle bundle of the bundle $\pi : E \to B$. The $SO_2$-actions on $O_{V_b}(1)$ and $C_{V_b}(1)$ ($b \in B$) induce a f.p. $SO_2$-actions on $O_E(\varepsilon), S(E)$ and $E^+\pi(X, O_E(\varepsilon))$. The $SO_2$-embeddings $C_{V_b}(1) \to E^+\pi(X, O_{V_b}(\varepsilon))$ induce a f.p. $SO_2$-embedding $S(E) \to E^+\pi(X, O_E(\varepsilon))$. The $SO_2$-s.d.r.'s $\Phi^b_t = \varphi^b_t$ of $E^+(X, O_{V_b}(1))$ onto $C_{V_b}(1)$ ($b \in B$) induce a f.p. $SO_2$-s.d.r. of $E^+\pi(X, O_E(1))$ onto $S(E)$:

$$\Phi^b_t : E^+\pi(X, O_E(1)) \to E^+\pi(X, O_E(1)), \quad \Phi^b_t(f) = \Phi^b_t(f) \quad (f \in E^+(X, O_{V_b}(1))).$$
When $0 \in X$, we have $\Phi_t(\mathcal{E}_\pi^+(X, O_E(1), 0)) \subset \mathcal{E}_\pi^+(X, O_E(1), 0)$, where $0 \subset O_E(1)$ is the image of the zero-section of $E$.

We define a fiberwise radial homeomorphism $k_\varepsilon : O_E(1) \cong O_E(\varepsilon)$ by $k_\varepsilon(sv) = \varepsilon(b)v$ ($v \in O_{V_b}(1)$). Then $k$ is a f.p. $SO_2$-homeomorphism and induces a f.p. $SO_2$-homeomorphism $K_\varepsilon : \mathcal{E}_\pi^+(X, O_E(1)) \cong \mathcal{E}_\pi^+(X, O_E(\varepsilon))$, $K_\varepsilon(f) = k_\varepsilon \circ f$. When $0 \in X$, we have $K(\mathcal{E}_\pi^+(X, O_E(1), 0)) = \mathcal{E}_\pi^+(X, O_E(\varepsilon), 0)$.

Finally, the required f.p. $SO_2$-s.d.r. $\Phi_t^\varepsilon$ of $\mathcal{E}_\pi^+(X, O_E(\varepsilon))$ onto $S(E)$ is defined by

$$\Phi_t^\varepsilon : \mathcal{E}_\pi^+(X, O_E(\varepsilon)) \to \mathcal{E}_\pi^+(X, O_E(\varepsilon)), \quad \Phi_t^\varepsilon = K_t \Phi_t K_t^{-1}.$$  

When $0 \in X$, we have $\Phi_t^\varepsilon(\mathcal{E}_\pi^+(X, O_E(1), 0)) \subset \mathcal{E}_\pi^+(X, O_E(\varepsilon), 0)$.

When $X = [a, b]$, it follows that $\mathcal{E}_\pi^+(X, O_E(\varepsilon)) = \mathcal{E}_\pi(X, O_E(\varepsilon))$ and for any 2-dim vector bundle $\pi : E \to B$ with an inner product, we obtain the f.p. s.d.r. $\Phi_t^\varepsilon$ of $\mathcal{E}_\pi(X, O_E(\varepsilon))$ onto $S(E)$.

When $X$ is the arc $I = [-1/2, 1/2]$, we can obtain a $\mathbb{Z}_2$-equivariant version. Suppose $\pi : E \to B$ is a 2-dim vector bundle with an inner product. Then $\mathcal{E}_\pi^+(I, O_E(\varepsilon)) = \mathcal{E}_\pi(I, O_E(\varepsilon))$ and the right-$\mathbb{Z}_2$-s.d.r.’s $\Phi_t^\varepsilon = \varphi^B_t$ of $\mathcal{E}(I, O_{V_b}(1))$ onto $C_{V_b}(1)$ ($b \in B$) induces a f.p. right-$\mathbb{Z}_2$-s.d.r. of $\mathcal{E}_\pi(I, O_E(1))$ onto $S(E)$

$$\Phi_t : \mathcal{E}_\pi(I, O_E(1)) \to \mathcal{E}_\pi(I, O_E(1)), \quad \Phi_t(f) = \Phi_t^B(f) \quad (f \in \mathcal{E}(E, O_{V_b}(1))).$$

We have $\Phi_t(\mathcal{E}_\pi(I, O_E(1), 0)) \subset \mathcal{E}_\pi(I, O_E(1), 0)$.

The f.p. homeomorphism $K : \mathcal{E}_\pi(I, O_V(1)) \cong \mathcal{E}_\pi(I, O_V(\varepsilon))$ is right-$\mathbb{Z}_2$-equivariant and $K(\mathcal{E}_\pi(I, O_E(1), 0)) = \mathcal{E}_\pi(I, O_E(\varepsilon), 0)$.

The required f.p. right $\mathbb{Z}_2$-s.d.r. $\Phi_t^\varepsilon$ of $\mathcal{E}_\pi(I, O_E(\varepsilon))$ onto $S(E)$ is defined by

$$\Phi_t^\varepsilon : \mathcal{E}_\pi(I, O_E(\varepsilon)) \to \mathcal{E}_\pi(I, O_E(\varepsilon)), \quad \Phi_t^\varepsilon = K_t \Phi_t K_t^{-1}.$$  

We have $\Phi_t^\varepsilon(\mathcal{E}_\pi(I, O_E(\varepsilon), 0)) \subset \mathcal{E}_\pi(I, O_E(\varepsilon), 0)$.

Replacing $D(1)$ by any oriented disk, we have the following conclusions (the $-$-case is reduced to the $+$-case by reversing the orientation of $D$):

**Proposition 6.6.** (1) Suppose $D$ is an oriented disk and $X$ is a compact connected polyhedron ($\neq 1pt$) in $\text{Int} D$ with a distinguished point $x_0 \in X$. Then for any oriented 2-dim vector bundle $\pi : E \to B$ with an inner product, there exists a f.p. $SO_2$-s.d.r of $\mathcal{E}_\pi^+(X, O_E(\varepsilon))$ (and $\mathcal{E}_\pi^+(X, x_0; O_E(\varepsilon), 0)$) onto $S(E)$.

(2) Suppose $X$ is an arc and $x_0$ is any point of $X$. Then for any 2-dim vector bundle $\pi : E \to B$ with an inner product, there exists a f.p. s.d.r of $\mathcal{E}_\pi(X, O_E(\varepsilon))$ (and $\mathcal{E}_\pi(X, x_0; O_E(\varepsilon), 0)$) onto $S(E)$.

(3) Let $I = [-1, 1]$ and choose 0 as the base point. Then for any 2-dim vector bundle $\pi : E \to B$ with an inner product, there exists a f.p. right $\mathbb{Z}_2$-s.d.r of $\mathcal{E}_\pi(I, O_E(\varepsilon))$ (and $\mathcal{E}_\pi(I, 0; O_E(\varepsilon), 0)$) onto $S(E)$.

6.4. **Proof of Theorem 6.1.**

6.4.1. **Spaces of small embeddings.**

Suppose $M$ is a connected 2-manifold with $\partial M = \emptyset$. We choose a smooth structure and a Riemannian metric on $M$, which induces the path-length metric $d$ on $M$. The tangent bundle $q : TM \to M$
is a 2-dim vector bundle with an inner product and it is oriented when $M$ is oriented. We apply the argument in 6.3.3 to this vector bundle $TM$.

For $x \in M$ and $r > 0$, let $U_x(r) = \{ y \in M \mid d(x, y) < r \}$ and $O_x(r) = OT_xM(r) (= \{ v \in T_xM \mid |v| < r \})$. For any map $\varepsilon : M \to (0, \infty)$, let $U_M(\varepsilon) = \cup_{x \in M} \{ x \times U_x(\varepsilon(x)) \} \subset M \times M$, while $OT_M(\varepsilon) = \cup_{x \in M} O_x(\varepsilon(x))$ by definition. If the map $\varepsilon$ is sufficiently small, then at each $x \in M$ the exponential map, $exp_x$, maps $O_x(\varepsilon(x))$ diffeomorphically onto $U_x(\varepsilon(x))$ (exp$_x$ is o.p. when $M$ is oriented) [4, Theorem 1.6]. Since exp$_x$ is smooth in $x \in M$, we obtain a f.p. diffeomorphism over $M$:

$$\exp : OT_M(\varepsilon) \to U_M(\varepsilon), \quad \exp(v) = (x, \exp_x(v)) \quad (v \in O_x(\varepsilon(x))).$$

In order to connect the space $E^*(X, M)$ with the fiberwise embedding space $E_q^*(X, x_0; OT_M(\varepsilon), 0)$, we introduce spaces of small embeddings.

Suppose $U = \{ U_{\lambda} \}$ is a cover of $M$ by open disks and $\delta : M \to (0, \infty)$ is a map with $\delta \leq \varepsilon$. For any pointed space $(Y, y_0)$ we set

$$E_U(Y, M) = \{ f \in E(Y, M) \mid f(Y) \subset U_{\lambda} \text{ for some } \lambda \}, \quad E_\delta(Y, M) = \{ f \in E(Y, M) \mid f(Y) \subset U_{f(y_0)}(\delta(f(y_0))) \}.$$

Suppose $D$ is an oriented disk and $(X, x_0)$ is a pointed compact connected polyhedron ($\neq 1$pt) in $Int D$ ($x_0$ is also regarded as a base point of $D$). We consider the following subspace of $E^*(X, M)$:

$$E^*_U(X, M) = \{ f \in E(X, M) \mid f \text{ admits an extension } \overline{f} \in E_U(D, M) \}.$$

When $M$ is oriented, we have the subspaces $E^*_U(D, M) = E_U(D, M) \cap E^+(D, M)$ and $E^*_U(D, M) = \{ f \in E(X, M) \mid f \text{ admits an extension } \overline{f} \in E^*_U(D, M) \}$.

The subspaces $E^*_\delta(X, M)$ and $E^\pm(X, M)$ are defined similarly. When $M$ is oriented and $X$ is not an arc, we have $E^*_\delta(X, M) = E^+_U(X, M) \cup E^-_\delta(X, M)$ (a disjoint union). Note that if $(M, X) = (S^2, \text{ a circle})$ and $U$ consists of small open disks, then $E^+_U(X, M) \neq E^+(X, M) \cap E_U(X, M)$.

The projection $p : E(X, M) \to M$, $p(f) = f(x_0)$ restricts to the projections on these subspaces.

**Lemma 6.7.** The f.p. diffeomorphism $\exp$ induces a f.p. homeomorphism over $M$ (which preserves the ±-parts when $M$ is oriented):

$$\exp : E^*_q(X, x_0; OT_M(\varepsilon), 0) \cong E^*_\varepsilon(X, M), \quad \exp(f) = \exp_x \circ f \quad (f \in E^*(X, x_0; O_x(\varepsilon(x)), 0)).$$

Let $I$ denote the interval $[-1, 1]$ with the base point $0$.

**Proposition 6.7.** (i) (a) When $M$ is oriented, the inclusions $E^\pm_\delta(X, M) \subset E^\pm(X, M)$ and $E^+_U(X, M) \subset E^+(X, M)$ are f.h.e.’s over $M$.

(b) When $M$ is nonorientable, the inclusions $E^\pm_\delta(X, M) \subset E^*(X, M)$ and $E^*_U(X, M) \subset E^*(X, M)$ are f.h.e.’s over $M$.

(ii) The inclusions $E_\delta(I, M) \subset E(I, M)$ and $E_U(I, M) \subset E(I, M)$ are $Z_2$ f.h.e.’s over $M$.

First we prove the following assertions.
Lemma 6.8. (1) (i) The inclusion $\mathcal{E}^*_q(X,M) \subset \mathcal{E}^*_e(X,M)$ is a f.h.e. over $M$ (which preserves the $\pm$-parts when $M$ is oriented).

(ii) The inclusion $\mathcal{E}_e(I,M) \subset \mathcal{E}_e(I,M)$ is a $\mathbb{Z}_2$-f.h.e. over $M$.

(2) Suppose $U_\delta \equiv \{U_x(\delta(x))\}_{x \in M}$ refines $U$ (i.e., each $x \in M$ admits a $\lambda$ with $U_x(\delta(x)) \subset U_\lambda$).

(i) When $M$ is oriented, the inclusion $\mathcal{E}_e^+(X,M) \subset \mathcal{E}_e^+(X,M)$ is a f.h.e. over $M$.

(ii) When $M$ is nonorientable, the inclusion $\mathcal{E}_e^{-}(X,M) \subset \mathcal{E}_e^{-}(X,M)$ is a f.h.e. over $M$.

(2) (i) First we consider the case where $X$ is not an arc. We may assume that $D$ is a subdisk of $M$ and that the inclusion $D \subset M$ is o.p. when $M$ is oriented. We treat the cases ($a_\pm$) and (b) simultaneously and use the superscript $\#$ to denote $\pm$ in the case ($a_\pm$) and $*$ in the case (b).

We show that the restriction map $p : \mathcal{E}_e^+(D,M) \to \mathcal{E}_e^+(X,M)$ has a section $s$ in each case of ($a_\pm$) and (b). If $N_0$ is a regular neighborhood of $X$ in $D$, then $\text{cl}(D \setminus N_0)$ is a finite disjoint union of an annulus $A$ and closed disks in $\text{Int}D$. Since $N = N_0 \cup A$ is a regular neighborhood of $X$ in $M$, by Proposition 3.1 (1) and Lemma 4.2 (1) the restriction maps $\mathcal{E}(D,M)_0 \to \mathcal{E}(N,M)_0 \to \mathcal{E}(X,M)_0$ are homotopy equivalences. Thus the restriction map $p_0 : \mathcal{E}(D,M)_0 \to \mathcal{E}(X,M)_0$ is also a homotopy equivalence. Since $p_0$ is a locally trivial bundle (Corollary 2.1 (ii)), it has a section $s_0$. In the cases ($a_\pm$) and (b) it follows that $p = p_0$ (i.e., $\mathcal{E}(D,M)_0 = \mathcal{E}^+(D,M)$, $\mathcal{E}(X,M)_0 = \mathcal{E}^+(X,M)$), so we have done. The ($a_-$) case is deduced by taking $\eta \in \mathcal{H}^-(D)$ and applying ($a_+$) to $(D,\eta(X))$.

Next we show that the restriction map $p : \mathcal{E}_e^+(D,M) \to \mathcal{E}_e^+(X,M)$ has a section $F$ such that $F(\mathcal{E}_e^+(X,M)) \subset \mathcal{E}_e^+(D,M)$. Since $D \setminus X$ consists of a half-open annulus component $A$ with $\partial A = \partial D$ and open disk components $V_i$’s, by shrinking $A$ towards $X$ we can find a s.d.r. $\varphi_\ell$ ($0 \leq \ell \leq 1$) of $D$ onto $\mathcal{X} = X \cup (\cup_i V_i)$ such that $\varphi_\ell \in \mathcal{E}^+(D,D)$ ($0 \leq \ell < 1$) and $\varphi_{\ell_1}(D) \supset \varphi_{\ell_2}(D)$ ($0 \leq \ell_1 \leq \ell_2 \leq 1$).

We construct maps $\mu_1 : \mathcal{E}_e^+(X,M) \to [0,1]$ and $\mu_2 : \mathcal{E}_e^+(X,M) \to [0,1]$ such that $s(f)\varphi_{\mu_1(f)} \in \mathcal{E}_e^+(D,M)$ ($f \in \mathcal{E}_e^+(X,M)$) and $s(f)\varphi_{\mu_2(f)} \in \mathcal{E}_e^+(D,M)$ ($f \in \mathcal{E}_e^+(X,M)$). The maps $\mu_1$ is constructed as follows: Given $f \in \mathcal{E}_e^+(X,M)$, there exists a $\mathcal{T} \in \mathcal{E}_e^+(D,M)$ such that $\mathcal{T}|X = f$. By definition $\mathcal{T}(D) \subset U_\lambda$ for some $\lambda = \lambda_f$. Comparing $s(f) \in \mathcal{E}_e^+(D,M)$ and $\mathcal{T}$, we conclude that $s(f)(V_i) = \mathcal{T}(V_i)$ for each open disk component $V_i$ of $D \setminus X$, so that $s(f)(\mathcal{X}) \subset U_\lambda_f$. Thus, if $t_f \in [0,1]$ is sufficiently close to 1, then $s(f)\varphi_{\lambda_f}(D) \subset U_{\lambda_f}$. If $V_f$ is a sufficiently small neighborhood of $f$ in $\mathcal{E}_e^+(X,M)$, then for each $g \in V_f$ we have $s(g)\varphi_{\lambda_g}(D) \subset U_{\lambda_g}$. Choose a locally finite open covering $\{W_f\}$ of $\mathcal{E}_e^+(X,M)$ such that $W_f \subset V_f$ for each $f$ ($W_f$ may be empty), and then construct a map $\mu_1 : \mathcal{E}_e^+(X,M) \to [0,1]$ such that $\mu_1|W_f \geq t_f$ for each $f$. Then $\mu_1$ satisfies the required condition. (For $\mu_2$, replace $U$ by $\delta$ and $U_{\lambda_f}$ by $U_{\delta(x_0)}(\delta(f(x_0)))$, except that $s(\gamma)\varphi_{\lambda}(D) \subset U_{\lambda}$ is replaced by $s(\gamma)\varphi_{\tau_f}(D) \subset U_{g(x_0)}(\delta(g(x_0)))$.)
We note that $\mathcal{F} = \text{cl} \mathcal{E}^\#_{\delta/2}(X, M) \subset \mathcal{E}^\#_{\delta}(X, M)$, where the closure is taken in $\mathcal{E}^\#(X, M)$. In fact, each $f \in \mathcal{F}$ admits an open neighborhood $\mathcal{U}$ in $\mathcal{E}^\#(X, M)$ and a map $\Phi : \mathcal{U} \to \mathcal{H}(M)_0$ such that $\Phi(f) = \text{id}_M$ and $\Phi(g)f = g (g \in \mathcal{U})$ (Proposition 2.5). Each $g \in \mathcal{U} \cap \mathcal{E}^\#_{\delta/2}(X, M) (\neq \emptyset)$ admits an extension $\overline{g} \in \mathcal{E}^\#_{\delta/2}(D, M)$ and $\overline{T} = \Phi(g)^{-1}\overline{g}$ is an extension of $f$. If $g$ is sufficiently close to $f$, then $\Phi(g)$ is close to $\text{id}_M$ and $\overline{T} \in \mathcal{E}^\#_{\delta}(D, M)$. This means that $f \in \mathcal{E}^\#_{\delta}(X, M)$.

Take a map $\mu : \mathcal{E}^\#_{\delta}(X, M) \to [0, 1]$ with $\mu \geq \mu_1$ and $\mu|_{\mathcal{F}} \geq \mu_2|_{\mathcal{F}}$, and define the section $F$ by $F(f) = s(f)\varphi_{\mu(f)}$.

Finally we construct a f.p. deformation $\Psi_t (t \in [0, 1])$ of $\mathcal{E}^\#_{\delta}(X, M)$ into $\mathcal{E}^\#_{\delta}(X, M)$ such that $\Psi_t(\mathcal{E}^\#_{\delta/2}(X, M)) \subset \mathcal{E}^\#_{\delta}(X, M) (0 \leq t \leq 1)$. Using a cone structure of $D$ with the vertex $x_0$, we can find a s.d.r. $\psi_t (t \in [0, 1])$ of $D$ onto $x_0$ such that $\psi_t \in \mathcal{E}^+(D, D) (0 \leq t < 1)$ and $\psi_{t_1}(D) \supset \psi_{t_2}(D)$ ($0 \leq t_1 \leq t_2 \leq 1$). If $\nu : \mathcal{E}^\#_{\delta}(D, M) \to [0, 1]$ is sufficiently close to 1, then $\dot{h}\psi(h) \in \mathcal{E}^\#_{\delta}(D, M)$ ($h \in \mathcal{E}^\#_{\delta}(D, M)$) and $h\psi_t \in \mathcal{E}^\#_{\delta}(D, M)$ ($h \in \mathcal{E}^\#_{\delta}(D, M), 0 \leq t < 1$). The f.p. deformation $\Psi_t$ is defined by $\Psi_t(f) = F(f)\psi_{\nu(F)}|_{X}$.

It follows from (1) that the inclusion $\mathcal{E}^\#_{\delta/2}(X, M) \subset \mathcal{E}^\#_{\delta}(X, M)$ is a f.h.e. Thus $\Psi_1 : \mathcal{E}^\#_{\delta}(X, M) \to \mathcal{E}^\#_{\delta}(X, M)$ is a f.h.e. inverse of the inclusion $\mathcal{E}^\#_{\delta}(X, M) \subset \mathcal{E}^\#_{\delta}(D, M)$. This completes the proof of the case where $X$ is not an arc.

When $X$ is an arc, there exists a s.d.r. $\psi_t$ of $X$ onto $x_0$ such that $\psi_t \in \mathcal{E}(X, X)$ ($0 \leq t < 1$) and $\varphi_{t_1}(X) \supset \varphi_{t_2}(X)$ ($0 \leq t_1 \leq t_2 \leq 1$). Using $\psi_t$, we can construct a f.p. deformation $\Psi_t (t \in [0, 1])$ of $\mathcal{E}^\#_{\delta}(X, M)$ into $\mathcal{E}^\#_{\delta}(X, M)$ such that $\Psi_t(\mathcal{E}^\#_{\delta/2}(X, M)) \subset \mathcal{E}^\#_{\delta}(X, M) (0 \leq t \leq 1)$.

(2) The proof is similar to the arc case in (1) except that we use the s.d.r. $\psi_t(x) = (1 - t)x$ and the map $\nu_1(h) = \max\{\nu(h), \nu(h\theta_{-1})\}$ instead of $\nu$ itself.

**Proof of Proposition 6.7.** (1) In each case of (a, $\pm$) and (b), if $U_0$ is the covering of $M$ by all open disks, then $\mathcal{E}^\#(X, M) = \mathcal{E}^\#_{U_0}(X, M)$ and $U_0$ refines $U_0$. Therefore, by Lemma 6.7, $\mathcal{E}^\#_{\delta}(X, M) \subset \mathcal{E}^\#(X, M)$ is a f.h.e. and $\mathcal{E}^\#_{\delta}(I, M) \subset \mathcal{E}(I, M)$ is a $\mathbb{Z}_2$-f.h.e. over $M$.

(2) Any $\mathcal{U}$ admits a $\delta$ such that $\mathcal{U}^\delta$ refines $U_0$. By (1) and Lemma 6.8 $\mathcal{E}^\#_{\delta}(X, M) \subset \mathcal{E}^\#(X, M)$ and $\mathcal{E}^\#_{\delta}(X, M) \subset \mathcal{E}^\#_{U_0}(X, M)$ are f.h.e.’s. Thus the inclusion $\mathcal{E}^\#_{\delta}(X, M) \subset \mathcal{E}^\#(X, M)$ is also a f.h.e. Similarly, $\mathcal{E}^\#_{\delta}(I, M) \subset \mathcal{E}(I, M)$ is a $\mathbb{Z}_2$-f.h.e. over $M$.

**6.4.2. Proof of Theorem 6.1.**

Finally, combining Propositions 6.6, 6.7 and Lemma 6.7, we can complete the proof of Propositions 6.1, 6.2 and Theorem 6.1.

**Proof of Proposition 6.1.** (1)(i) By Propositions 6.6 (1)(i), 6.7 (i)(a) and Lemma 6.7 we have the sequence of f.h.e.‘s over $M$:

$$
\mathcal{E}^\pm(X, M) \supset \mathcal{E}^\pm_{\epsilon}(X, M) \cong \mathcal{E}^\pm_q(X, x; O_{TM}(\epsilon), 0) \simeq S(TM).
$$

(ii) The orientation double cover $\tilde{M}$ has a canonical orientation. Let $\mathcal{U}$ be the open covering of $\tilde{M}$ consisting of open disks $U$ on which $\pi : \tilde{M} \to M$ is injective. Each $f \in \mathcal{E}^*(X, M)$ admits a unique lift $\overline{f} \in \mathcal{E}^*_\mathcal{U}(X, \tilde{M})$ and this correspondence induces a f.p. homeomorphism $\mathcal{E}^*(X, M) \cong \mathcal{E}^*_\mathcal{U}(X, \tilde{M})$.
over $M$. By (i) and Proposition 6.7 (i)(a) we have the sequence of f.h.e.’s over $M$:

$$\mathcal{E}^*(X, M) \cong \mathcal{E}^+_{\bar{\partial}}(X, \bar{M}) \subset \mathcal{E}^+(X, \bar{M}) \simeq S(T\bar{M}).$$

(2) By Propositions 6.6 (2), 6.7 (i) and Lemma 6.7 we have the sequence of f.h.e.’s over $M$:

$$\mathcal{E}(X, M) \supset \mathcal{E}_\varepsilon(X, M) \cong \mathcal{E}_q(X, x; O_{TM}(\varepsilon), 0) \cong S(TM).$$

**Proof of Proposition 6.2.** By Propositions 6.6 (3), 6.7 (ii) and Lemma 6.7 we have the sequence of $\mathbb{Z}_2$-f.h.e.’s over $M$:

$$\mathcal{E}(I, M) \supset \mathcal{E}_\varepsilon(I, M) \cong \mathcal{E}_q(I, 0; O_{TM}(\varepsilon), 0) \simeq S(TM).$$

**Proof of Theorem 6.1.** By Lemma 2.3 $X$ has a disk neighborhood $D$ in $M$. When $M$ is orientable, we orient $D$ and $M$ compatibly. From Proposition 6.1 it follows that

(i) if $M$ is orientable or $X$ is an arc, then $\mathcal{E}(X, M)_0 = \mathcal{E}^+(X, M) \simeq S(TM)$.

(ii) if $M$ is nonorientable and $X$ is not an arc, then $\mathcal{E}(X, M)_0 = \mathcal{E}^*(X, M) \simeq S(T\bar{M})$.

**References**

[1] Chavel, I., *Riemannian Geometry: A modern Introduction*, Cambridge Tracts In Math. 108, Cambridge Univ. Press, New York, 1993.

[2] Courant, R., *Dirichlet’s principle, conformal mapping, and minimal surfaces*, Pure and Applied Math., Interscience Publishers Inc., New York, 1950.

[3] Epstein, D. B. A., Curves on 2-manifolds and isotopies, *Acta Math.*, 155 (1966) 83 - 107.

[4] Hamstrom, M. E., Homotopy groups of the space of homeomorphisms on a 2-manifold, *Illinois J. Math.*, 10 (1966) 563 - 573.

[5] Luke, R. and Mason, W. K., The space of homeomorphisms on a compact two-manifold is an absolute neighborhood retract, *Trans. Amer. Math. Soc.*, 164 (1972), 275 - 285.

[6] van Mill, J., *Infinite-Dimensional Topology: Prerequisites and Introduction*, North-Holland, Math. Library 43, Elsevier Sci. Publ. B.V., Amsterdam, 1989.

[7] Pommerenke, Ch., *Boundary Behaviour of Conformal Maps*, GMW 299, Springer-Verlag, New York, 1992.

[8] Scott, G. P., The space of homeomorphisms of 2-manifold, *Topology*, 9 (1970) 97 - 109.

[9] Yagasaki, T., The homeomorphism groups of noncompact 2-manifolds and the spaces of embeddings into 2-manifolds, preliminary report (Topology Atlas, Preprints, Document #paa-08).

[10] ______., Spaces of embeddings of compact polyhedra into 2-manifolds, *Topology Appl.*, 108 (2000) 107 - 122.

[11] ______., Homotopy types of homeomorphism groups of noncompact 2-manifolds, *Topology Appl.*, 108 (2000) 123 - 136.

[12] ______., Embedding spaces and hyperspaces of polyhedra in 2-manifolds, *Topology Appl.*, 121 (2002) 247 - 254.

**Department of Mathematics, Kyoto Institute of Technology, Matsugasaki, Sakyoku, Kyoto 606, Japan**

*E-mail address: yagasaki@ipc.kit.ac.jp*