1 INTRODUCTION

In this paper, we study a certain class of translation surfaces. The theory of translation surfaces has been a very active research area over the past 40 years with connections to mathematical billiards, dynamical systems, and Teichmüller theory. These applications are, for example, discussed in the survey articles [19], [44], and [46].

The strata $\mathcal{H}_g(k_1 \times a_1, \ldots, k_m \times a_m)$, $a_i, k_i \in \mathbb{Z}^+$, stratify the space of translation surfaces of genus $g$. Here, the stratum $\mathcal{H}_g(k_1 \times a_1, \ldots, k_m \times a_m)$ consists of all translation surfaces of genus $g$ with $k_i$ singularities of multiplicity $a_i + 1$ for $1 \leq i \leq m$. The study of strata has played an important role ever since the fundamental work of Masur, Smillie, and Veech in the 1980s (see [23, 31, 41]). A first essential result was achieved by Kontsevich and Zorich with the classification of the connected components of strata in [24]. A natural $\text{SL}(2, \mathbb{R})$-action on each stratum is crucial in the study of translation surfaces. Eskin and Mirzakhani described in their groundbreaking work [11] the closure of the $\text{SL}(2, \mathbb{R})$-invariant subspaces.

In our paper, we are interested in certain translation surfaces, which we call $p$-origamis. Origamis, also known as square-tiled surfaces, are finite torus covers and form a particularly interesting class of translation surfaces. On the one hand, the set of origamis is dense in each stratum. On the other hand, each origami defines a Teichmüller curve, that is, an algebraic curve induced by the $\text{SL}(2, \mathbb{R})$-orbit of certain translation surfaces. Moreover, each algebraic curve over $\mathbb{C}$ is birational to a Teichmüller curve arising from the $\text{SL}(2, \mathbb{R})$-orbit of an origami (see [10]). In general, the classification of $\text{SL}(2, \mathbb{R})$-orbits of origamis is an unsolved problem. However, in the stratum $\mathcal{H}(2)$, the possible orbits of origamis have been classified in...
Furthermore, counting problems of origamis are related to the study of Masur–Veech volumes of strata (see, e.g., [1, 7, 8, 14]). We refer the interested reader to [37, 38], and [45] as introductions to origamis.

Origamis arising as normal torus covers are called regular or normal origamis. In [32], the homology groups of normal origamis are studied, and results on the Lyapunov exponents of the Kontsevich–Zorich cocycle (which capture certain dynamical properties of a surface) are deduced.

Each normal origami is determined by its deck transformation group $G$ and a particularly chosen pair of generators $(x, y)$ of $G$. This description allows us to examine such origamis using group theory. One question that arises naturally asks in which strata normal origamis occur. It turns out that, if $G$ has order $d$ and the commutator of $x$ and $y$ has order $a$, then the stratum of the corresponding origami is $\mathcal{H}(\frac{d}{a} \times (a-1))$ (see Remark 2.8).

As special cases of normal origamis, we consider those whose deck group is a finite $p$-group, that is, a group of prime-power order. We call such origamis $p$-origamis. A motivating example for studying them is the well-known origami Eierlegende Wollmilchsau (see Example 2.3 and [16]). It is one out of two translation surfaces whose $\text{SL}(2, \mathbb{R})$-orbit induces not only a Teichmüller curve, but also a Shimura curve in the moduli space of abelian varieties. As a consequence, its Teichmüller curve has extraordinary dynamical behavior (see [3] and [34]).

We prove a precise characterization of all strata possible for $p$-origamis. As for many questions in the theory of $p$-groups, the situation is fundamentally different for the even prime 2 and for all other odd primes.

**Theorem A** (Theorem 4.3, Theorem 4.7). Let $n \in \mathbb{Z}_{\geq 0}$. Then, any $p$-origami of degree $p^n$ has either no singularity and genus 1, or lies in one of the following strata:

1. $\mathcal{H}(\frac{2^n-k}{2} \times (2^k-1))$, for $1 \leq k \leq n-2$, if $p = 2$,
2. $\mathcal{H}(\frac{p^n-k}{p} \times (p^k-1))$, for $1 \leq k < \frac{n}{2}$, if $p > 2$.

Moreover, all of these strata occur.

Consider a fixed abelian 2-generated $p$-group. Then, the commutator of any pair of generators is trivial, and hence, so is the stratum of any $p$-origami with the fixed group as its deck group. Two observations can be generalized from this simple example: First, we prove that the possible strata of a $p$-origami only depend on the isoclinism class of its ($p$-)group of deck transformations. Since all abelian groups are isoclinic to the trivial group, this indeed generalized our toy example, while it also implies, for instance, that the dihedral, the semidihedral, and the quaternion group of order $2^k$ for some $k \geq 1$ admit the same possible strata (see Remark 4.2, where we also compute the exact stratum).

Second, we show that far beyond the abelian case, the deck group determines a unique stratum—one which is independent of the choice of generators $x, y$—in many more situations.

**Theorem B** (Theorem 4.11). Many deck groups of prime-power order admit only one possible stratum for their $p$-origamis, including all $p$-groups $G$, which are regular, of maximal class, powerful, or those whose commutator subgroup $G'$ is regular, powerful, or order closed. This includes all $p$-groups of order up to $p^{p+2}$ or of nilpotency class up to $p$.

Our results on strata of $p$-origamis are obtained by an array of group-theoretic methods. Given a 2-generated $p$-group $G$, the possible strata of all $p$-origamis with deck group $G$ depend on the possible orders of the commutators $[x, y] = x^{-1}y^{-1}xy$ as $x, y$ vary over all pairs of generators. We derive various results on the possible exponent of the commutator subgroup, which contains the commutator $[x, y]$ and, in fact, in the case of 2-generated groups, is generated by the set of its conjugates. We then show these exponents can always be realized as commutator orders of pairs of generators. This culminates in a complete characterization of the possible commutator orders $[x, y]$ for each prime-power group order, which forms the group-theoretic analog of Theorem A:

**Theorem C** (Proposition 3.2, Proposition 3.4, Proposition 3.5, Proposition 3.8).

1. For any finite 2-group $G$, $\exp(G') = 1$ if $|G| \leq 2$, or else $\exp(G') \leq \frac{|G|}{4}$.
2. For all integers $n \geq 2$ and $0 \leq k \leq n-2$, there exists a 2-generated 2-group $G$ of order $2^n$ with generators $x, y$ such that $\text{ord}([x, y]) = \exp(G') = 2^k$. 


3. For any nontrivial finite $p$-group $G$ with odd $p$, $\exp(G')^2 < |G|$. 

4. For any odd prime $p$ and any $n, k \in \mathbb{Z}_{\geq 0}$ with $k < \frac{n}{2}$, there exists a 2-generated $p$-group $G$ of order $p^n$ with generators $x, y$ such that 

$$\text{ord}([x, y]) = \exp(G') = p^k.$$ 

To investigate the question, in which cases the deck group already determines the stratum of a $p$-origami (as partially answered in Theorem B), we similarly translate this into a group-theoretic problem. In group-theoretic language, this phenomenon corresponds to the property of a $p$-group, that the commutator order is a fixed number for all pairs of generators. We call this property $(C)$, and we observe that many, but not all $p$-groups have property $(C)$. We solidify this observation by proving property $(C)$ depending on some other known common properties of $p$-groups. The proven implications can be summarized with the following diagram (Theorem 3.29), they form the group-theoretic basis of Theorem B:

![Diagram showing implications between group-theoretic properties](image)

Here, the thick arrows represent the new implications proved in this paper. The remaining implications are established facts in the theory of $p$-groups.

We also construct $p$-groups, which do not have property $(C)$. Such groups have at least order $p^{p+3}$. We construct the examples as subgroups of the Sylow $p$-subgroup of the symmetric group $S_{p^4}$.

It is an open question, which power closed groups, or which groups with a power closed commutator subgroup, have property $(C)$. As it seems hard to find (small) $p$-groups without property $(C)$, their properties or even their classification might be an intriguing problem. Choices of commutators with different orders in such groups will yield normal origamis with isomorphic deck groups lying in different strata.

As an outlook, we consider surfaces that arise as infinite normal torus covers. These surfaces form a special class of infinite translation surfaces. We focus on surfaces with dense subgroups of profinite groups as deck groups, examples include origamis with the infinite dihedral group as deck group. After generalizing the definition of property $(C)$ to profinite and pro-$p$ groups, we transfer results from Section 3.2 on finite $p$-groups to these new situations. As in the finite case, the group-theoretic results have a geometric interpretation concerning the singularities of infinite translation surfaces.

### 1.1 Structure of the paper

The paper is organized as follows. In Section 2, we introduce essential definitions for the geometric point of view and explain the geometric motivation of the questions studied in the subsequent sections.

Developing the group-theoretic results is subject of Section 3. In Section 3.1, we prove Theorem C. For this, we consider the prime 2 separately because the results differ from those for odd primes. Section 3.2 is concerned with the group-theoretic analog of Theorem B. More precisely, we introduce a group-theoretic property called property $(C)$ and prove the above diagram of implications. We further construct examples of $p$-groups that do not have property $(C)$.

The goal of Section 4 is to translate the results for $p$-groups in Section 3 into the language of $p$-origamis. In Section 4.1, we answer the question in which strata $p$-origamis occur by proving Theorem A. The question whether the isomorphism class of the deck group determines the stratum of an origami is addressed in Section 4.2. Theorem B is proven there.

In Section 5, we generalize some results from Section 3 and Section 4 to profinite and pro-$p$ groups, and infinite translation surfaces arising as normal torus covers, respectively.
2 | PREREQUISITES: CONNECTIONS BETWEEN GEOMETRY AND DECK GROUPS

In this section, we recall and explain basic notions and well-known facts from the theory of translation surfaces and origamis. We focus on the interaction between origamis and their group of deck transformations as it plays a key role in this paper. We include proofs whenever we think they help the exposition of the paper.

2.1 | Origamis and deck groups

We begin by recalling the construction of translation surfaces and origamis. A translation surface is constructed from finitely many polygons in the Euclidean plane, where pairs of parallel edges (of the polygons) are identified by translations. For an origami, take finitely many copies of the unit square. Glue them along their edges via translations such that each left edge is glued to exactly one right edge and each upper edge to exactly one lower edge. The resulting surface is a translation surface. We require that the surface is connected. Otherwise one studies the connected components separately. Such a translation surface is called origami or square-tiled surface. An origami \( \mathcal{O} \) naturally defines a covering \( \mathcal{O} \to \mathbb{T} \) of the torus \( \mathbb{T} \) ramified at most over one point denoted by \( \infty \). The number of glued squares is the degree of the covering.

In this section, [46] and [19] are used as main references for well-known facts about translation surfaces. Further, we refer the interested reader to [12] for background knowledge about coverings.

Example 2.1. The following origami is a cover of the torus of degree 4 (see Figure 1). Edges with the same labels are identified.

The concept of monodromy maps is essential for relating the stratum of a normal origami with group-theoretic properties of its deck group. Let \( c: \mathcal{O} \to \mathbb{T} \) be the covering induced by an origami of degree \( d \). Recall that its deck transformation group consists of all homeomorphisms \( f: \mathcal{O} \to \mathcal{O} \) such that \( cf = c \). Consider the corresponding unramified cover of the punctured torus \( \mathcal{O}^* : \mathcal{O}^* \to \mathbb{T}^* \), where \( \mathcal{O}^* = \mathcal{O} \setminus c^{-1}(\infty) \) and \( \mathbb{T}^* = \mathbb{T} \setminus \{\infty\} \). Recall that the fundamental group \( \pi_1(\mathbb{T}^*) \) is the free group \( F_2 \) on two generators. We choose a base point \( q \) on \( \mathbb{T}^* \) and label the preimages of \( q \) (under \( c \)) by \( q_1, \ldots, q_d \). Denote the simple closed horizontal and vertical curve passing through \( q \) by \( a \) and \( b \), respectively. These two curves generate the fundamental group \( F_2 \). Then, the monodromy map \( m: F_2 \to S_d, w \mapsto \sigma_w \) is an antihomomorphism defined as follows: A word \( w \in F_2 \) describes a path \( f \) on \( \mathbb{T}^* \). For \( 1 \leq i \leq d \), one obtains a lifted path \( \tilde{f}_i \) of \( f \) on \( \mathcal{O}^* \) starting at \( q_i \). Set \( \sigma_w(i) := j \), if the path \( \tilde{f}_i \) ends at \( q_j \). This defines a permutation \( \sigma_w \in S_d \).

A normal (or regular) origami is an origami, which is a normal cover of the torus, that is, the deck transformation group acts transitively on the squares. Let \( c: \mathcal{O} \to \mathbb{T} \) be the covering induced by a normal origami. Then, the degree of the cover is the order of the deck transformation group. We obtain a natural bijection between the squares of the tiling and the deck transformations by labeling a fixed square \( S \) with the identity. A square \( S' \) is now labeled with the unique deck transformation sending the square \( S \) to \( S' \). Using this bijection, the monodromy action is described by a map \( m_G: F_2 \to G, w \mapsto g_w \cdot h \) such that the group homomorphism \( G \to G, h \mapsto g_w \cdot h \) induces the permutation \( m(w) \) on the squares of the tiling. Note that the deck group acts from the left on the origami \( \mathcal{O} \).

Lemma 2.2 (see, e.g., [45, Section 4.1]). The following holds:

1. A finite 2-generator group \( G \) together with an (ordered) pair of generators determines a normal origami with deck transformation group \( G \).
A normal origami is uniquely determined by its deck transformation group $G$ and the two deck transformations $m_G(a)$ and $m_G(b)$.

**Proof.**

(i) Given a 2-generator group $G$ of order $d$ together with generators $x, y$, we construct a normal origami of degree $d$ as follows: Take $d$ squares labeled by the group elements. The right and upper neighbor of a square labeled by $g$ in $G$ is the one with label $gx$ and $gy$, respectively.

(ii) Given a normal origami $\mathcal{O} \to \mathbb{T}$ with deck transformation group $G$. Consider the deck transformations $x := m_G(a)$ and $y := m_G(b)$. These deck transformations generate $G$ and the procedure described in (i) reconstructs the origami $\mathcal{O}$ from the data $(G, x, y)$. □

Recall that the monodromy map is an antihomomorphism and induces a left action of the deck group. In the construction described in the proof of Lemma 2.2, we multiply the generators from the right to obtain compatibility with this left action. From now on, we denote a normal origami with deck transformation group $G$ defined by the pair of generators $(x, y)$ by $(G, x, y)$.

**Example 2.3.** We consider the quaternion group $Q_8 := \langle i, j, k \mid i^2 = j^2 = k^2 = ijk \rangle$ with $(i, j)$ as the pair of generators. $Q_8$ can be viewed as a group of units in the quaternion division algebra, where $ijk = -1$. Following the construction described in Lemma 2.2 (i), we obtain a 2-origami $\mathcal{W}$ with deck transformation group $Q_8$. In Figure 2, the corners of the squares with the same color are identified, that is, the cone angles at these points are larger than $2\pi$.

This origami is called **Eierlegende Wollmilchsau**. It is a well-known and extensively studied example (see [16]).

We call normal origamis $c_1 : \mathcal{O}_1 \to \mathbb{T}$ and $c_2 : \mathcal{O}_2 \to \mathbb{T}$ equal if there exists a homeomorphism $\alpha : \mathcal{O}_1 \to \mathcal{O}_2$ such that $c_1 = c_2 \alpha \circ \alpha$. It is natural to ask when different pairs of generators for a given group describe the same origami. The following lemma answers this question.

**Lemma 2.4** (see, e.g., [45, Lemma 4.2]). Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be normal origamis with deck transformation group $G$ defined by pairs of generators $(x_1, y_1)$ and $(x_2, y_2)$, respectively. Let $m_i : F_2 \to G$ with $m_i(a) = x_i$ and $m_i(b) = y_i$ denote the monodromy maps of $\mathcal{O}_i$ for $i = 1, 2$. Then the following are equivalent:

(i) the origamis $\mathcal{O}_1$ and $\mathcal{O}_2$ are equal,

(ii) there exists a group automorphism $\varphi : G \to G$ such that $(\varphi(x_1), \varphi(y_1)) = (x_2, y_2)$.

**Example 2.5.** Recall that the automorphism group of the quaternion group $Q_8$ is isomorphic to the symmetric group $S_4$ and thus has order 24. We further know that there are exactly 24 pairs of generators $(x, y)$. Denote the set of generators by $E$. The restriction on $x, y$ is that $\{x, y\} \subseteq \{\pm i, \pm j, \pm k\}$ is a subset of size two not containing $\pm a$ for $a$ in $Q_8$. The automorphism group $\text{Aut}(Q_8)$ acts on $E$ by applying an automorphism to both components. Note that for each pair $(x, y) \in E$, the stabilizer under this action is trivial. Hence, $\text{Aut}(Q_8)$ acts transitively on $E$, that is, for any two pairs of
generators \((x_1, y_1), (x_2, y_2)\), we find an automorphism \(\varphi\) of \(Q_8\) such that \((\varphi(x_1), \varphi(y_1))\) is equal to \((x_2, y_2)\). By Lemma 2.4, the only 2-origami with deck transformation group \(Q_8\) is the Eierlegende Wollmilchsau (see Example 2.3).

2.2  Singularities and strata

For an origami \(\mathcal{O}\) and a point \(x \in \mathcal{O}\) the cone angle at \(x\) is \(2\pi a\) for some natural number \(a\). The multiplicity of \(x\) is defined as \(a\) and denoted by \(\text{mult}(x)\). A singularity \(s \in \mathcal{O}\) is a point with multiplicity larger than 1. Denote the set of singularities by \(\Sigma\). Let \((a_1, ..., a_m), (k_1, ..., k_m) \in \mathbb{Z}_+^m\). The stratum \(\mathcal{H}(k_1 \times a_1, ..., k_m \times a_m)\) is the set of translation surfaces with \(k_i\) singularities of multiplicity \(a_i + 1\) for \(1 \leq i \leq m\). The set of translation surfaces without a singularity, that is, the set of tori, is denoted by \(\mathcal{H}(0)\).

Each translation surface \(X\) comes with a natural holomorphic 1-form on \(X\). The zeros of this 1-form are the singularities. If the order of a zero is \(a\), the multiplicity of the singularity is \(a + 1\).

Since the squares in the tiling of an origami are glued by translations, only corners of the squares can appear as singularities. In particular, there are only finitely many singularities. We begin with a remark showing that all singularities of a normal origami have the same multiplicity. That gives a first restriction on the strata.

**Remark 2.6.** The deck transformation group of a normal origami acts transitively on the squares of the origami. Hence, all singularities have the same multiplicity, and lie in a stratum of the form \(\mathcal{H}(0)\) or \(\mathcal{H}(k \times a)\) for \(a, k \in \mathbb{Z}_+\).

Further, a normal origami with deck transformation group \(G\) and set of singularities \(\Sigma\) with \(s \in \Sigma\) satisfies the following equation:

\[
\sum_{s' \in \Sigma} \text{mult}(s') = |\Sigma| \cdot \text{mult}(s) = |G|.
\]

Note that either all corner points of a normal origami are singularities or all corner points are regular points.

The next lemma connects the multiplicity of singularities to a statement phrased in the language of group theory. We include a proof for completeness. As in the introduction, we denote the commutator of group elements \(x\) and \(y\) by \([x, y] = x^{-1}y^{-1}xy\).

**Lemma 2.7** (see, e.g., [45, Section 4.1]). Let \(\mathcal{O} = (G, x, y)\) be a normal origami. The cover of the torus induced by the origami is unramified if and only if \([x, y] = 1\). In particular, all normal origamis with abelian deck group lie in the stratum \(\mathcal{H}(0)\). If the cover is ramified, the multiplicity of each singularity of \(\mathcal{O}\) coincides with the order of \([x, y]\) in \(G\).

**Proof.** Let \(S\) denote the square labeled by the group element 1. Then, the deck transformation \([x, y] = x^{-1}y^{-1}xy\) sends the square \(S\) to the one lying \(2\pi\) (with respect to the lower left corner of \(S\)) above \(S\) (Figure 3).

Hence, the deck transformation \([x, y]^m\) sends the square \(S\) to the one lying \(2\pi m\) above \(S\) for \(m \in \mathbb{Z}_+\). We conclude with Remark 2.6 that the cone angle at each corner is \(2\pi \cdot \text{ord}([x, y])\). □
Remark 2.8. By the lemma above and Lemma 2.2, finding a normal origami of degree \( d = (a + 1)k \) in the stratum \( H(k \times a) \) is equivalent to finding a 2-generated group of order \( d \) and a generating set of size two such that the commutator of the generators has order \( k \).

2.3 \hspace{1cm} \textit{p}-origamis

Let \( p \) be a prime number. An origami is called \textit{\( p \)-origami} if it is normal and the deck transformation group of the corresponding cover is a finite \( p \)-group.

We will study in which strata \( p \)-origamis occur.

Remark 2.9. By Remark 2.6, all singularities of a \( p \)-origami have the same multiplicity. Each \( p \)-origami outside the stratum \( H(0) \) satisfies the equation \( d = (a + 1) \cdot k \), where \( d \) is the degree, \( a + 1 \) is the multiplicity of each singularity, and \( k \) is the number of singularities.

This reduces the possible strata significantly. All \( p \)-origamis of degree \( p^n \) (outside the stratum \( H(0) \)) have \( p^{n-k} \) singularities of multiplicity \( p^k \) for \( 1 \leq k \leq n \), that is, they lie in strata of the form \( H \left( p^{n-k} \times (p^k - 1) \right) \).

In Section 4, we will use the connection between group theory and the type of singularities of normal \( p \)-origamis to derive conclusions about the stratum they lie in.

3 \hspace{1cm} RESULTS ON \textit{p}-GROUPS

3.1 \hspace{1cm} Bounds for the exponent of commutator subgroups of \textit{p}-groups

The geometric setting in Section 2 motivates the study of the following problem. Given a finite \( p \)-group \( G \) of order \( p^n \), find a bound for the order of commutators \([x, y]\) with \( x, y \in G \). In this section, we answer a more general question. We give a sharp bound for the exponent of the commutator subgroup. This bound is different for the prime 2 and odd primes.

We recall some basic definitions and facts from the theory of \( p \)-groups. Let \( G \) be a finite \( p \)-group. The \textit{order} \( |G| \) of \( G \) is the number of its elements. For any element \( x \in G \), the \textit{order} \( \text{ord}(x) \) of \( x \) is the smallest positive integer \( n \) such that \( x^n = 1 \). The \textit{exponent} \( \text{exp}(G) \) of \( G \) is the greatest order of any element in \( G \). The \textit{commutator subgroup} \( G' \) of \( G \) is the subgroup generated by all commutators

\[
[x, y] = x^{-1}y^{-1}xy \hspace{0.5cm} \text{for} \hspace{0.5cm} x, y \in G.
\]

The \textit{center} \( Z(G) \) of \( G \) is the subgroup \( \{x \in G \mid xy = yx \ \text{for all} \ y \in G\} \). The \textit{Frattini subgroup} \( \Phi(G) \) of \( G \) is the intersection of all maximal subgroups of \( G \). For each \( i \in \mathbb{Z}_+ \), we recall the definition of the \textit{omega} and \textit{agemo} subgroup (the word “agemo” is the word “omega” spelled backwards),

\[
\Omega_i(G) := \langle g \mid g \in G, g^{p^i} = 1 \rangle,
\]

\[
\Omega^i(G) := \langle g^{p^i} \mid g \in G \rangle.
\]

Lemma 3.1 ([27, Proposition 1.2.4]). Let \( G \) be a finite \( p \)-group.

(1) The Frattini subgroup \( \Phi(G) \) equals \( G'\Omega^1(G) \), the group generated by all commutators and \( p \)-th powers. In particular, \( G/\Phi(G) \) is elementary abelian (i.e., abelian and of exponent \( p \)).

(2) Burnside’s basis theorem: A set of elements of \( G \) is a (minimal) generating set if and only if the images in \( G/\Phi(G) \) form a (minimal) generating set of \( G/\Phi(G) \). In particular, every generating set for \( G \) contains a generating set with exactly \( d(G) \) elements, where \( d(G) \) is the rank of the elementary abelian quotient \( G/\Phi(G) \).

This can be used to establish first bounds for the exponent of a \( p \)-group, which holds for all prime numbers.
Proposition 3.2. For a finite p-group $G$ of order $p^n$, the exponent of $G'$ is 1 if $n \leq 2$, or otherwise at most $p^{n-2}$.

Proof. Any p-group of order $p$ is cyclic, in particular, abelian. Hence, $\text{exp}(G') = 1$, in this case.

Suppose $G$ is a noncyclic p-group of order $p^n$ with a minimal generating set of length $d \geq 2$. By Burnside’s basis theorem, we obtain $|G/\Phi(G)| = p^d$ and thus $|\Phi(G)| = p^{n-d}$. The inclusion $G' \subseteq \Phi(G)$ implies the inequality $|G'| \leq p^{n-d}$. In particular, the inequality $\text{exp}(G') \leq p^{n-d} \leq p^{n-2}$ holds.

3.1.1 2-groups

In this section, we show that the bound in Proposition 3.2 is sharp for the prime 2. What is more, we construct 2-generated 2-groups with certain generators whose commutator has the desired order. These groups will be used to construct 2-origamis in Section 4.1.1. We need the following lemma.

Lemma 3.3 ([21, Hilfssatz III 1.11 a]). For a group $G$ generated by a subset $S$, the commutator subgroup $G'$ is generated by the set $\{g^{-1}[x, y]g | x, y \in S, g \in G\}$.

Proposition 3.4. Let $n, k \in \mathbb{Z}_{\geq 0}$ with $n > 2$ and $k \leq n - 2$. There exists a 2-generated 2-group $G$ of order $2^n$ with generators $x, y \in G$ such that

$$\text{ord}([x, y]) = \text{exp}(G') = 2^k.$$ 

Proof. Let $k$ be a natural number with $0 \leq k \leq n - 2$. We construct a group $G^2_{(n,k)}$ of order $2^n$ and a pair of generators $r, s$ whose commutator is of order $2^k$. The group $G^2_{(n,k)}$ is a semidirect product of two cyclic groups $C_{2^{k+1}} = \langle r \rangle$ and $C_{2^{n-k-1}} = \langle s \rangle$ of order $2^{k+1}$ and $2^{n-k-1}$, respectively. First, define the group automorphism $\alpha : C_{2^{k+1}} \to C_{2^{k+1}}, r^m \mapsto r^{-m}$. Since $\alpha^{2^{n-k-1}}$ is the identity map on $C_{2^{k+1}}$, the map

$$\varphi : C_{2^{n-k-1}} \to \text{Aut}(C_{2^{k+1}}), s^m \mapsto \alpha^m$$

is a group homomorphism. Let $G^2_{(n,k)}$ be the semidirect product

$$C_{2^{k+1}} \rtimes_{\varphi} C_{2^{n-k-1}} = \langle r, s | r^{2^{k+1}} = s^{2^{n-k-1}} = 1, s^{-1}rs = r^{-1} \rangle.$$ 

Then, $G^2_{(n,k)}$ has order $2^n$. Using the defining relations of $G^2_{(n,k)}$, we conclude

$$[r, s] = r^{-1}s^{-1}rs = r^{-2}.$$ 

Hence, the commutator $[r, s]$ has order $2^k$.

Finally we show that the commutator subgroup $(G^2_{(n,k)})'$ has exponent $2^k$. Since $G^2_{(n,k)}$ is generated by $\{r, s\}$, by Lemma 3.3, the commutator subgroup is generated by elements of the form $g^{-1}[r, s]g$ for $g \in G^2_{(n,k)}$. As $[r, s] = r^{-2}$ and $s^{-1}[r, s]s = r^2$, each of the elements $g^{-1}[r, s]g$ is contained in $\langle r^2 \rangle$. Hence, $(G^2_{(n,k)})'$ is cyclic of order $2^k$.

Note that all the 2-groups constructed in the proof of Proposition 3.4 are semidirect products of two cyclic groups.

3.1.2 p-groups for odd primes p

Throughout this section, let $p$ denote an odd prime. In Proposition 3.5, we introduce a much stronger bound on the exponent of the commutator subgroup, which holds for odd primes. This is a generalization of a theorem by van der Waall (see [40, Theorem 1]). There, the order of the commutator subgroup of finite p-groups is bounded under the condition that the commutator subgroup is cyclic.
Subsequently, we show in Proposition 3.8 that this bound is sharp. To this end, we construct 2-generated \( p \)-groups and generators whose commutators have the desired orders. These groups are used to construct certain \( p \)-origamis in Section 4.1.2.

**Proposition 3.5.** For a nontrivial finite \( p \)-group \( G \), \( p \) odd, the following inequality holds:

\[
\exp(G')^2 < |G|.
\]  

*Proof.* As the inequality holds for all cyclic \( p \)-groups, we may use an induction and consider a group \( G \) such that the inequality holds for all groups of smaller order.

By Lemma III 7.5 in [21], a finite \( p \)-group \( G \) for \( p \) odd is either cyclic, or it has a normal subgroup \( N \trianglelefteq G \) isomorphic to \( C_p \times C_p \). In the first case, again (3.1) holds. So without loss of generality, we may assume that there exists a normal subgroup \( N \trianglelefteq G \) isomorphic to \( C_p \times C_p \). Define \( H := G/N \). By the induction hypothesis we have \( \exp(H')^2 < |H| \). Consider the canonical epimorphism \( \varphi : G \rightarrow H \). It maps commutators of \( G \) to commutators of \( H \), so for \( g \in G' \), the image \( \varphi(g) \) lies in \( H' \). Thus, \( \varphi(g) \) has order at most \( \exp(H') \), and \( g \) has order at most \( \exp(N) \cdot \exp(H') = p \cdot \exp(H') \) in \( G \). Hence, we obtain the desired inequality

\[
\exp(G')^2 \leq p^2 \cdot \exp(H')^2 < p^2 \cdot |H| = |N| \cdot |H| = |G|.
\]  

□

**Corollary 3.6.** Let \( G \) be a 2-generated \( p \)-group, \( p \) odd, of order \( p^n \). For generators \( x, y \) of \( G \), the order of their commutator obeys the inequality

\[
\text{ord}([x, y]) < p^{\frac{n}{2}}.
\]

*Proof.* Let \( \exp(G') = p^m \), \( |G| = p^n \) and \( \text{ord}([x, y]) = p^k \). Since \( k \leq \exp(G') \), it is sufficient to show that \( m < \frac{n}{2} \). This is equivalent to \( 2m < n \). By Proposition 3.5, we have \( \exp(G')^2 < |G| \) and thus the inequality \( 2m < n \) holds. □

As in the case of 2-origamis, we construct for natural numbers \( n, k \) with \( k < \frac{n}{2} \) a \( p \)-group, which is a semidirect product of two cyclic groups, in order to show that the proven bound is sharp. The construction given in Proposition 3.8 works similarly as the one for 2-origamis in the proof of Theorem 4.3. However, the group homomorphism defining the semidirect products needs to be chosen more carefully for odd primes.

We begin with a purely number-theoretic observation, which will be useful when constructing the semidirect products.

**Lemma 3.7.** Let \( p \) be an odd prime and let \( k \) be a positive natural number. Then, \( p + 1 \) has order \( p^k \) in \( \left( \mathbb{Z}/p^{k+1}\mathbb{Z} \right)^* \).

*Proof.* This can be proved directly using the binomial expansion (see, for instance, [36, Proof of Theorem 6.7, p. 129]). □

**Proposition 3.8.** Let \( n, k \in \mathbb{Z}_{\geq 0} \) with \( k < \frac{n}{2} \). There exists a 2-generated \( p \)-group \( G \) of order \( p^n \) with generators \( x, y \in G \) such that

\[
\text{ord}([x, y]) = \exp(G') = p^k.
\]

*Proof.* Fix a positive natural number \( n \) and let \( k \) be an integer with \( 0 \leq k < \frac{n}{2} \). The group \( C_{p^{n-k-1}} \) is constructed as a semidirect product of two cyclic groups \( C_{p^{k+1}} = \langle r \rangle \) and \( C_{p^{n-k-1}} = \langle s \rangle \) of order \( p^{k+1} \) and \( p^{n-k-1} \), respectively. First, consider the automorphism group of \( C_{p^{k+1}} \). From elementary group theory, we know that

\[
\text{Aut}(C_{p^{k+1}}) \cong \left( \mathbb{Z}/p^{k+1}\mathbb{Z} \right)^* \cong C_{p^{k+1}p^{k-1}} = C_{p^{k+1}p^{k+1}}.
\]

The map \( \alpha : C_{p^{k+1}} \rightarrow C_{p^{k+1}} \), \( r^m \mapsto r^{m(p+1)} \), defines a group automorphism, since \( p \) and \( p + 1 \) are coprime. Now we consider the map \( \varphi : C_{p^{n-k-1}} \rightarrow \text{Aut}(C_{p^{k+1}}) \), \( s^m \mapsto \alpha^m \). We claim that \( \varphi \) is a well-defined group homomorphism. To see this, we need to show that \( \alpha^{p^{n-k-1}} \) is the identity map on \( C_{p^{k+1}} \). This follows, since \( k \leq \frac{n}{2} \) implies \( n-k-1 \geq k \), so
\[(p + 1)^{p^{n-k-1}} \equiv 1 \mod p^{k+1},\] again using Lemma 3.7. The congruence follows because \(p + 1\) has order \(p^k\) in \((\mathbb{Z}/p^{k+1}\mathbb{Z})^*\) by Lemma 3.7. Write

\[(p + 1)^{p^{n-k-1}} = j \cdot p^{k+1} + 1\]

for some natural number \(j\). Then, we have

\[\alpha^{p^{n-k-1}}(r^i) = r^{i(p+1)^{p^{n-k-1}}} = r^{i \cdot j \cdot p^{k+1}} \cdot r^i = r^i\]

for each \(1 \leq i \leq p^{k+1}\). Since \(r\) has order \(p^{k+1}\) in \(C_{p^{k+1}}\), the last equality follows. We conclude that \(\alpha^{p^{n-k-1}} = \text{id}\).

Let \(G_{(n,k)}^p\) be the semidirect product

\[C_{p^{k+1}} \rtimes C_{p^{n-k-1}} = \langle r, s \mid r^{p^{k+1}} = s^{p^{n-k-1}} = 1, s^{-1}rs = r^{p+1} \rangle.\]

Then, \(G_{(n,k)}^p\) has order \(p^n\). We claim that \(G_{(n,k)}^p\), together with the pair of generators \((r, s)\) has the desired properties. Using the defining relations of \(G_{(n,k)}^p\), we conclude

\[[r, s] = r^{-1}s^{-1}rs = r^p.\]

In particular, the commutator \([r, s]\) has order \(p^k\).

Finally, we show that the commutator subgroup of \(G_{(n,k)}^p\) has exponent \(p^k\). Since \(G_{(n,k)}^p\) is generated by \([r, s]\), the commutator subgroup is generated by elements of the form \(g^{-1}[r, s]g\) for \(g \in G_{(n,k)}^p\) by Lemma 3.3. Each of these elements is contained in \(\langle r^p \rangle\). Hence, \((G_{(n,k)}^p)'\) is cyclic of order \(p^k\).

### 3.2 On the order of certain commutators in 2-generated \(p\)-groups

In this section, we study a second question that arises from the geometric setting in Section 2. Recall that the group of deck transformations of a normal origami is always a finite 2-generated group and that two normal origamis with isomorphic group of deck transformations \((G, x_1, y_1)\) and \((G, x_2, y_2)\) lie in the same stratum if and only if the orders of the commutators \([x_1, y_1]\) and \([x_2, y_2]\) agree.

We first note that the possible strata for normal origamis with a fixed deck group depend only on its isoclinism class. We recall that isoclinism is an equivalence relation for groups generalizing isomorphism. For its definition, we use the observation that the commutator in any group \(G\) induces a well-defined map \(G/Z(G) \times G/Z(G) \to G\).

**Definition 3.9** ([15]). Two groups \(G_1, G_2\) are **isoclinic** if there are isomorphisms \(\phi : G_1/Z(G_1) \to G_2/Z(G_2)\) and \(\psi : G_1' \to G_2'\), which are compatible with the commutator maps in the sense that the following diagram is commutative:

\[
\begin{array}{ccc}
G_1/Z(G_1) \times G_1/Z(G_1) & \longrightarrow & G_1' \\
\phi \times \phi \downarrow & & \psi \downarrow \\
G_2/Z(G_2) \times G_2/Z(G_2) & \longrightarrow & G_2'
\end{array}
\]

In particular, all abelian groups are isoclinic.

**Lemma 3.10.** The set of possible commutator orders \(\text{ord}([x, y])\) for generators \(x, y\) of a 2-generated group \(G\) depends only on the isoclinism class of \(G\).
Proof. Assume $G_1$ and $G_2$ are isoclinic groups and $[x, y]$ has order $n$ for generators $x, y$ of $G_1$. Then, $n$ is also the order of $\psi([x, y]) = [\phi(\bar{x}), \phi(\bar{y})]$, where $\bar{x}, \bar{y}$ are the images in $G_1/Z(G_1)$. Thus, ord($[x', y']$) = ord($[x, y]$) for any $x', y'$ in $G_2$ such that $x' \equiv \phi(x)$ and $y' \equiv \phi(y)$ modulo $Z(G_2)$.

If $Z(G_2)$ does not lie in $\Phi(G_2)$, then a central element can be chosen as one of two generators of $G_2$ (see Lemma 3.1), so $G_2$ is abelian and $G_1$ is abelian, and the only possible commutator order is 1 in each of these groups.

Otherwise, $Z(G_2)$ does lie in $\Phi(G_2)$ and the images of $x', y'$ in $G_2/\Phi(G_2)$ generate the quotient, so $x', y'$ are generators of $G_2$ and the order of $[x', y']$ equals the order of $[x, y]$. Since isoclinism is a reflexive relation, this proves the assertion. \(\square\)

Corollary 3.11. For each $n \geq 1$, the dihedral group, the generalized quaternion group, and the semidihedral group with $2^n$ elements have the same set of possible commutator orders ord($[x, y]$) for generators $x, y$.

Proof. They are well known to be isoclinic, see, for instance, [4, section 29, Exercise 4]. \(\square\)

We will return to these groups, recall their definition, and compute the set of possible commutator orders in Section 3.2.2.

We noted that the possible strata of $p$-origami with a given deck group depends on the possible commutator orders for pairs of generators of this group. We will see that, in fact, for many groups, there is only one stratum possible. To study such groups, we first translate this property into the language of group theory.

Definition 3.12. We say that a finite 2-generated group $G$ has property (C), if there exists a natural number $n$ such that for each 2-generating set $\{x, y\}$ of $G$ the order of $[x, y]$ equals $n$.

Question 3.13. Which finite 2-generated $p$-groups have property (C)?

For a large class of $p$-groups we prove property (C). However, in Section 3.2.5, we construct for each prime $p$ a finite $p$-group with generating sets $\{x, y\}$ and $\{x', y'\}$ such that ord($[x, y]$) $\neq$ ord($[x', y']$).

In a first example, we show that most alternating groups—which are not $p$-groups—do not have property (C). We use this example to construct origamis in Example 4.10.

Example 3.14. For $n \in \mathbb{N}_{\geq 5}$, we consider the alternating group $A_n$ with the pairs of generators: $((1, 2, \ldots, n - 1, n), (1, 2, 3))$ and $((3, 4, \ldots, n - 1, n), (1, 3)(2, 4))$. The orders of the commutators

$$
[(1, 2, \ldots, n - 1, n), (1, 2, 3)] = (1, 2, 4),
$$

$$
[((3, 4, \ldots, n - 1, n), (1, 3)(2, 4))] = (1, 2, 5, 4, 3),
$$

are 3 and 5, respectively. Hence, there are two pairs of generators such that the order of their commutator is different and $A_n$ does not have property (C) for $n \geq 5$. Notice that those $A_n$ are not $p$-groups, since their order is $\frac{n!}{2}$.

Further, note that we multiply permutations from the left because we label the squares of a normal origami by multiplying generators of the deck group from the right.

In the following, we will prove property (C) for certain families of $p$-groups.

### 3.2.1 Regular and order closed $p$-groups

We begin by stating some basic properties of regular $p$-groups. Recall that a $p$-group $G$ is regular if for each $g, h \in G$ and $i \in \mathbb{Z}_+$, there exists some $c \in \langle g, h \rangle$ such that

$$(gh)^{p^i} = g^{p^i}h^{p^i}c.\]$$

Note that the commutator subgroup of a regular $p$-group is regular.

We call a $p$-group $G$ weakly order closed if the product of elements of order at most $p^k$ has order at most $p^k$ for any $k \geq 0$. In the literature, $p$-groups for which all sections (i.e., subquotients) are weakly order closed according to our definition have been studied and are called order closed $p$-groups (see [30]). Clearly, order closed $p$-groups are weakly
order closed, and one can verify that all subgroups of a weakly order closed group are so, as well. Hence, a \( p \)-group is order closed if and only if all its quotients are weakly order closed. The class of \( p \)-groups we call weakly order closed has been called \( \mathcal{O} \) in [43].

**Lemma 3.15** (see [27], Lemma 1.2.13).

1) Any regular \( p \)-group is order closed, and hence also weakly order closed.
2) A 2-group is regular if and only if it is abelian.

**Lemma 3.16.** Let \( G \) be a weakly order closed \( p \)-group. If \( x_1, \ldots, x_r \in G \) generate \( G \), then the exponent of \( G \) is equal to the maximum of the orders \( \text{ord}(x_i) \), \( 1 \leq i \leq r \).

**Proof.** Note that the orders of \( x_i \) and \( x_i^{-1} \) are equal. As every group element can be written as a word in \( \{x_i, x_i^{-1} | 1 \leq i \leq r\} \), the claim follows. □

**Proposition 3.17.** Any finite 2-generated \( p \)-group with weakly order closed commutator subgroup has property (C). In particular, any finite 2-generated \( p \)-group with regular commutator subgroup has property (C).

**Proof.** Let \( G \) be a finite 2-generated \( p \)-group with regular commutator subgroup. Further, let \( x, y \) and \( x', y' \) be two pairs of generators of \( G \). Hence, by Lemma 3.3, the commutator subgroup \( G' \) is generated by each of the sets

\[
\{g^{-1}[x, y]g | g \in G\}, \quad \{g^{-1}[x', y']g | g \in G\}.
\]

We have

\[
\text{ord}(g^{-1}[x, y]g) = \text{ord}([x, y]), \quad \text{ord}(g^{-1}[x', y']g) = \text{ord}([x', y']).
\]

By the previous lemma, \( G' \) being weakly order closed implies that

\[
\text{ord}([x, y]) = \exp(G') = \text{ord}([x', y']).
\]

Hence, the orders of the commutators \([x, y]\) and \([x', y']\) coincide. □

As we will discuss in Section 5, a version of this result holds for pro-\( p \) groups.

### 3.2.2 \( p \)-groups of maximal class

In this section, we prove that property (C) holds for \( p \)-groups of maximal class.

We recall that the **lower central series** of a group \( G \) is the series of subgroups

\[
\gamma_1(G) \supseteq \gamma_2(G) \supseteq \gamma_3(G) \supseteq \ldots,
\]

where \( \gamma_1(G) := G \) and \( \gamma_i(G) := [\gamma_{i-1}(G), G] \) for \( i > 1 \). The **nilpotency class** of \( G \) is \( c \) if

\[
\gamma_c(G) \supseteq \gamma_{c+1}(G) = \langle 1 \rangle.
\]

A \( p \)-group of order \( p^n \) has nilpotency class at most \( n - 1 \), and is called of **maximal class** in that case.

We treat 2-groups separately from the \( p \)-groups for odd primes \( p \). The 2-groups of maximal class are completely classified by the following theorem.

**Proposition 3.18** ([21, Kapitel III, Satz 11.9]). Each 2-group of maximal class and order \( 2^n \) is isomorphic to one of the following groups:
(1) a dihedral group, that is, a group given by the presentation
\[ D_{2n} = \langle r, s \mid r^{2n-1} = s^2 = 1, s^{-1}rs = r^{-1} \rangle. \]

(2) a generalized quaternion group, that is, a group given by the presentation
\[ Q_{2n} = \langle r, s \mid r^{2n-1} = 1, s^2 = r^{2n-2}, s^{-1}rs = r^{-1} \rangle. \]

(3) a semidihedral group, that is, a group given by the presentation
\[ SD_{2n} = \langle r, s \mid r^{2n-1} = s^2 = 1, s^{-1}rs = r^{2n-2} \rangle. \]

We can now strengthen the result in Corollary 3.11 for 2-groups of maximal class. Note that according to the classification in Proposition 3.18, each such group is 2-generated.

**Lemma 3.19.** Any 2-generated 2-group of maximal class and of order \( 2^n \) has property (C) and for each pair of generators \( x, y \), the order of the commutator \([x, y]\) is \( 2^{n-2} \).

**Proof.** Let \( G \) be a 2-group of maximal class and of order \( 2^n \). By Proposition 3.18, it is isomorphic to a dihedral group, quaternion group, or semidihedral group. In each case, we show that the commutator subgroup is regular and that for a pair of generators \( r, s \) the commutator \([r, s]\) has order \( 2^{n-2} \). By Proposition 3.17, this proves the claim.

For the semidihedral group \( SD_{2n} \), the commutator subgroup is generated by the set \( \{ g^{-1}[r, s]g \mid g \in SD_{2n} \} \). Since the commutator \([r, s]\) equals \( r^{2n-2} \), the commutator subgroup is cyclic. In particular, it is abelian and thus regular. The order of the commutator \([r, s]\) is \( 2^{n-2} \).

For the dihedral group \( D_{2n} \) and the generalized quaternion group \( Q_{2n} \), the commutator subgroup is generated by \([r, s] = r^{-2}\). Hence, the commutator subgroup is abelian and thus regular. The commutator \([r, s]\) has order \( 2^{n-2} \). \( \square \)

We turn our attention to the odd primes.

**Lemma 3.20.** Let \( G \) be a 2-generated \( p \)-group of maximal class and order \( p^n \) for \( 5 \leq n \leq p + 1 \) and an odd prime \( p \). Then, \( G \) has property (C) and for each pair of generators \( x, y \), the order of the commutator \([x, y]\) equals \( p \).

**Proof.** Assume \( G \) is a group of order \( p^n \) of maximal class for \( 5 \leq n \leq p + 1 \). Then, the commutator subgroup \( G' \) has exponent \( p \) by [21, Kapitel III, Hilfssatz 14.14]. Hence, all elements of \( G' \) have either order \( p \) or order 1. Let \((x, y)\) be a pair of generators of \( G \). Then, \( G' \) is generated by elements of the form \( g^{-1}[x, y]g \) for \( g \in G \). Since \( G \) is of maximal class, it is nonabelian. We conclude that \([x, y] \neq 1\) has order \( p \). \( \square \)

**Lemma 3.21.** Any finite 2-generated \( p \)-group of maximal class for \( p \) odd has property (C).

**Proof.** Let \( G \) be any group of order \( p^n \). If \( n \leq 4 \), then the order of the commutator subgroup \( G' \) is smaller than \( p^4 \). Hence, we have \(|G'| \leq p^2 \). By [21, Kapitel III, Satz 10.2 b)], the commutator subgroup is regular. Using Lemma 3.15, the claim follows from Proposition 3.17. In Lemma 3.20, we showed the claim for groups of order \( p^n \) of maximal class with \( 5 \leq n \leq p + 1 \).

Now assume \( G \) has order \( p^n \) and is of maximal class for \( n > p + 1 \). Then, there exists a maximal subgroup \( H \) of \( G \), which is regular (see [21, Kapitel III, Satz 14.22]). Recall that the Frattini subgroup \( \Phi(G) \) is the intersection of the set of maximal subgroups. We have the following inclusions:
\[ G' \subseteq \Phi(G) \subseteq H. \]

Since subgroups of regular groups are regular as well, the commutator subgroup \( G' \) is regular. By Proposition 3.17, the claim follows. \( \square \)
We obtain the following proposition from Lemma 3.19 and Lemma 3.21.

**Corollary 3.22.** Any finite 2-generated $p$-group of maximal class has property (C).

### 3.2.3 Powerful $p$-groups

We introduce some further basics from the theory of $p$-groups to show that property (C) holds for the class of powerful $p$-groups.

A $p$-group $G$ is called **powerful** if either $p$ is odd and $G' \subseteq \mathcal{U}^1(G)$, or $p = 2$ and $G' \subseteq \mathcal{U}^2(G) = \langle g^4 \mid g \in G \rangle$. A normal subgroup $N \trianglelefteq G$ is **powerfully embedded** in $G$ if either $p$ is odd and $[N,G] \subseteq \mathcal{U}^1(N)$, or $p = 2$ and $[N,G] \subseteq \mathcal{U}^2(N)$. In particular, any powerfully embedded $p$-group is powerful.

**Lemma 3.23.** Any finite 2-generated powerful $p$-group $G$ has a cyclic commutator subgroup $G'$.

**Proof.** Let $G$ be a powerful $p$-group with 2-generating set $\{x, y\}$. Then the commutator subgroup $G'$ is powerfully embedded in $G$ by [29, Theorem 1.1. and Theorem 4.1.1.]. Further, $G'$ is generated by all elements of the form $g^{-1}[x,y]g$ for $g \in G$ by Lemma 3.3. Now [29, Theorem 1.10. and Theorem 4.1.10.] state that if a powerfully embedded subgroup of a powerful $p$-group is the normal closure of a subset, then it is generated by this subset. Thus, we conclude that $G'$ is generated by $[x,y]$, that is, $G'$ is cyclic. $\square$

**Corollary 3.24.** Any finite 2-generated powerful $p$-group has property (C).

**Proof.** $G'$ is cyclic, so in particular, $G'$ is regular. Using Proposition 3.17, the claim follows. $\square$

**Proposition 3.25.** Let $G$ be a finite 2-generated $p$-group such that $G'$ is powerful. Then $G$ has property (C).

**Proof.** Let $G$ be a finite 2-generated $p$-group such that $G'$ is powerful. Let $\{x, y\}$ be a generating set. Denote the order of $[x, y]$ by $p^m$. Recall that $G'$ is generated by the set $\{g^{-1}[x,y]g \mid g \in G\}$ by Lemma 3.3. Since $G'$ is powerful, we apply [29, Theorem 1.9. and Theorem 4.1.9.] stating that in a powerful $p$-group, any agemo subgroup (as recalled at the beginning of Section 3.1) is generated by the corresponding powers of a given set of generators, and deduce

$$\mathcal{U}^m(G') = \langle g^{p^m} \mid g \in G' \rangle = \langle (g^{-1}[x,y]g)^{p^m} \mid g \in G \rangle = \{1\}.$$ 

In particular, the exponent of $G'$ equals $p^m$, and hence the order of $[x, y]$. As $\{x, y\}$ was an arbitrary pair of generators of $G$, this proves the claim. $\square$

As the commutator subgroup of a powerful $p$-group is powerful by [29, Theorem 1.1], Corollary 3.24 is also a consequence of Proposition 3.25.

We will see in Section 5, that the result of the proposition can be extended to pro-$p$ groups.

### 3.2.4 Power closed $p$-groups

Recall from Section 3.2.1 that we call a $p$-group weakly order closed, if products of elements of order at most $p^k$ have order at most $p^k$ for all $k \geq 0$. We call a $p$-group **weakly power closed**, if products of $p^k$-th powers are $p^k$-th powers for all $k \geq 0$. Such groups generalize **power closed** $p$-groups ([30]), for which all sections (i.e., subquotients) have to be weakly power closed in our sense. As quotients of weakly power closed groups are automatically weakly power closed, a $p$-group is power closed if and only if all its subgroups are weakly power closed. In [43], the class of $p$-groups we call weakly power closed has been called $P_p$.

It is known that order closed $p$-groups generalize regular $p$-groups, and power closed $p$-groups generalize order closed $p$-groups (see [30]).
Recall that in Proposition 3.17, we have shown that 2-generated $p$-groups $G$ for which $G'$ is weakly order closed have property (C).

**Example 3.26.** Using the GAP code in Listing 3 (in the Appendix), we find instances of a 2-group $G$ such that $G'$ is weakly power closed, but which does have generators $x$, $y$ such that $\text{ord}([x, y]) \neq \text{ord}([x, y^3])$. As $G$ is a 2-group, $x, y^3$ is a pair of generators, as well, so $G$ does not have property (C).

An example for such a situation is the case where $x, y \in S_{16}$ are the permutations

$$(1, 13, 2, 14)(3, 16, 4, 15)(5, 9, 7, 11, 6, 10, 8, 12),$$

$$(1, 16, 6, 11, 4, 14, 7, 9, 2, 15, 5, 12, 3, 13, 8, 10),$$

and $G = \langle x, y \rangle$; then

$$[x, y] = (1, 5, 2, 6)(3, 7, 4, 8)(9, 13, 10, 14)(11, 16, 12, 15),$$

$$[x, y^3] = (1, 6)(2, 5)(3, 7)(4, 8)(9, 15)(10, 16)(11, 14)(12, 13)$$

have order 4 and 2, respectively. The group $G$ has order $2^{12}$ and nilpotency class 7.

However, this counterexample is not a power closed $p$-group, since it has subgroups, which are not weakly power closed. Such subgroups can be found using computer algebra codes, as the ones described in the Appendix.

**Corollary 3.27.** There are 2-groups with weakly power closed commutator subgroup which do not have property (C).

However, let us contrast this with the case of minimal non–power closed or minimal non–order closed $p$-groups, where minimal means that all proper sections are power closed or order closed, respectively.

**Lemma 3.28.** Let $G$ be a minimal non–power closed $p$-group or a minimal non–order closed $p$-group. Then, $G$ has property (C) and for each pair of generators $x, y$, the order of the commutator $[x, y]$ equals $p$.

**Proof.** Let $G$ be a minimal non–power closed $p$-groups or minimal non–order closed $p$-group. By [30, Theorem 3 and Theorem 6], the Frattini group $\Phi(G)$ has exponent $p$ and the center $Z(G) \neq G$. As $G'$ is contained in $\Phi(G)$, we conclude that $G'$ has exponent $p$. Hence, $G'$ is regular. Proposition 3.17 implies that $G$ has property (C). For a pair of generators $x, y$ of $G$, the commutator subgroup $G'$ is generated by all conjugates of $[x, y]$. Since $G'$ is not trivial, the commutator $[x, y]$ has exponent $p$.

Let us summarize our results on groups with property (C).

**Theorem 3.29.** The implications and the nonimplication visualized in the following diagram are true for any finite 2-generated $p$-group $G$. In particular, all $p$-groups falling into the classes above the gray line have property (C).

Here, the thick arrows represent the new implications proved in this paper. The remaining implications are established facts.
Proof. The implications regular $\Rightarrow$ order closed $\Rightarrow$ power closed are due to [30]. Each of these properties is inherited by subgroups, so by $G'$ from $G$. By our definition, the properties weakly order closed and power closed generalize order closed and power closed, respectively. $p$-groups of maximal class or powerful $p$-groups have regular commutator subgroups as explained in the proof of Lemma 3.21 and Corollary 3.24. The commutator subgroup of a powerful $p$-group is powerful by [29, Theorem 1.1, Theorem 4.1.1].

The results that $G'$ powerful or weakly order closed imply property $(C)$, are Proposition 3.25 and Proposition 3.17. The fact that $G'$ weakly power closed does not imply property $(C)$ is Corollary 3.27. □

Remark 3.30. We note that, in particular, all 2-generated $p$-groups of order at most $p^{p+2}$ or of nilpotency class at most $p$ have property $(C)$. In the first case, $G'$ has order at most $p^p$, because it lies in the Frattini subgroup, which has index $p^2$ in $G$, so $G'$ is regular by [21, Satz 10.2 b)]. Similarly, in the second case, $G$ is regular by [21, Satz 10.2 a)].

It is an open questions, whether $G$ or $G'$ being power closed implies property $(C)$ for a $p$-group $G$. To study this and similar questions, we offer a reduction argument.

Lemma 3.31. Let $\mathcal{F}$ be a family of finite 2-generated $p$-groups, which is closed under quotients and such that property $(C)$ holds for all members of $\mathcal{F}$ with cyclic center. Then, property $(C)$ holds for all members of $\mathcal{F}$.

Proof. Assume property $(C)$ holds for all abelian groups and all members of $\mathcal{F}$ with cyclic center, and consider a nonabelian group $G \in \mathcal{F}$ whose center is not cyclic such that property $(C)$ holds for any member of $\mathcal{F}$ of smaller cardinality.

Since the center of $G$ has rank of at least two, we can choose two trivially intersecting central cyclic subgroups $N_1, N_2$ of order $p$. We consider pairs of generators $x, y$ and $x', y'$ of $G$, and we define

$$c := [x, y], \quad m := \text{ord}([x, y]),$$
$$c' := [x', y'], \quad m' := \text{ord}([x', y']).$$

Without loss of generality, we may assume $m \geq m' \geq p$. Now $c^m \in N_1$ has order $p$. Since $N_1$ and $N_2$ intersect trivially, $c^m$ lies in at most one of these subgroups. We may assume that $c^m \notin N_1$. Consider the canonical epimorphism $\overline{c} : G \to G/N_1$. The order of $\overline{c}$ is equal to $m$ because $c^m \notin N_1$. Now $\overline{x}, \overline{y}$ and $\overline{x'}, \overline{y'}$ are pairs of generators of the $p$-group $G/N_1$. By our assumptions, $G/N_1$ has property $(C)$, so

$$m = \text{ord}(\overline{c}) = \text{ord}(\overline{c'}).$$

Thus, we conclude that the $m'$ is either $m$ or $m \cdot p$. Since $m \geq m'$ holds, we conclude that $m$ and $m'$ coincide. This shows that $G$ has property $(C)$.

An induction using this argument proves the assertion. □

Corollary 3.32. $G$ being power closed or $G'$ being power closed implies property $(C)$ if it implies property $(C)$ together with the assumption that $G$ has cyclic center.

Proof. The family of power closed $p$ groups is closed under taking arbitrary quotients. The same is true for the family of $p$-groups with power closed commutator subgroup, since for any group, the commutator subgroup of a quotient is a quotient of the commutator subgroup. □

3.2.5 A family of counterexamples

We conclude our discussion of property $(C)$ by constructing 2-generated $p$-groups for all primes $p$, which do not have property $(C)$: For each such group, we exhibit two pairs of generators with different commutator order.

As these will be realized as certain subgroups of a $p$-Sylow subgroup of the symmetric group on $p^r$ letters, for some $r \geq 0$, let us first recall a description of one of these Sylow subgroups, which we will denote $P_{p,r}$. Recall that we multiply permutations from the left.
**Definition 3.33.** For any prime $p$ and any $r \geq 0$, consider the perfect $p$-ary tree with $p^r$ leaves (as in Figure 4 above for $p = r = 3$). Given $1 \leq i \leq r$, $1 \leq j \leq p^{i-1}$, let $N$ be the $j$-th node of this tree at level $i - 1$ (where the root node is at level 0), then we define a tree automorphism by fixing all nodes, which are not descendants of $N$, while rotating the subtrees starting at the $p$ direct descendants of $N$ cyclically to the right. We denote the permutation of the $p^r$ leaves of the tree induced by this graph automorphism by $e_{i,j}$. Let $P_{p,r}$ be the subgroup of $S_{p^r}$ generated by all $e_{i,j}$ for $1 \leq i \leq r, 1 \leq j \leq p^{i-1}$.

In cycle notation, $e_{i,j}$ is a product of disjoint $p$-cycles defined by

$$e_{i,j} := \prod_{k=p^{i-1} - (j-1) + 1}^{p^{i-1} - (j-1) + p^{i-1}} (k, k + p^{i-1}, ..., k + p^{i-1}(p - 1)). \quad (3.2)$$

From Equation 3.2 we get the conjugation relation

$$e_{i,j}^{-1} e_{k,l} e_{i,j} = e_{k,l'}, \quad (3.3)$$

for $i \leq k$, where $l'$ is defined by

$$(j - 1)p^{k-i} < l' \leq j p^{k-i} \quad \text{and} \quad l' - l \equiv p^{k-i} - 1 \mod p^{k-i}$$

if $i < k$ and $(j - 1)p^{k-i} < l \leq j p^{k-i}$, and $l = l'$ otherwise.

This implies that $P_{p,r}$ is, in fact, generated by $(e_{i,1})_{1 \leq i \leq r}$. We can check that $P_{p,r}$ is isomorphic to the $r$-fold wreath product of $C_p$ by identifying the respective generators. So $P_{p,r}$ is indeed a Sylow $p$-subgroup of $S_{p^r}$.

**Lemma 3.34.** For any prime $p$, there exist elements $x, x', y \in P_{p,4}$ such that

$$H_p := \langle x, y \rangle = \langle x', y \rangle \quad \text{and} \quad \text{ord}([y, x']) = p \neq p^2 = \text{ord}([y, x]).$$

In particular, the $p$-group $H_p$ does not have property (C).

**Proof.** We define

$$x := e_{1,1} e_{2,1}, \quad y := e_{1,1} e_{2,1} e_{3,1} e_{4,1} + p^2 \in P_{p,4}.$$

Assume $p > 3$, then by Equation 3.3,

$$x^{-1} y x = e_{2,1}^{-1} (e_{1,1} e_{2,2} e_{3,1} + p e_{4,1} + 2 p^2) e_{2,1} = e_{1,1} e_{2,1} e_{3,1} + p e_{4,1} + 2 p^2,$$

$$x^{-2} y x^2 = e_{2,1}^{-1} (e_{1,1} e_{2,2} e_{3,1} + 2 p e_{4,1} + 3 p^2) e_{2,1} = e_{1,1} e_{2,1} e_{3,1} + 2 p e_{4,1} + 3 p^2,$$

so $[y, x] = y^{-1} x^{-1} y x = e_{4,1}^{-1} e_{3,1} + p e_{4,1} + 2 p^2,$

$$[y, x^2] = y^{-1} x^{-2} y x^2 = e_{4,1}^{-1} e_{3,1} + 2 p e_{4,1} + 3 p^2.$$
We compute the restriction of the permutation \([y, x]\) to the set \(S := \{p^3 + 1, \ldots, p^3 + p^2\}:

\[
[y, x]|_S = (e^{-1}_{4,1+p^2}, e^{-1}_{3,1+p} e_{e_{4,1+2p^2}})|_S = (e^{-1}_{4,1+p^2}, e_{3,1+p})|_S
\]

\[
= (p^3 + 1, p^3 + 2, \ldots, p^3 + p)^{-1}, (p^3 + 1, p^3 + p + 1, \ldots, p^3 + p(p - 1) + 1)
\]

\[
\ldots (p^3 + p, p^3 + 2p, \ldots, p^3 + p^2)
\]

\[
= (p^3 + p, p^3 + 2p - 1, \ldots, p^3 + p^2 - 1,
\]

\[
p^3 + p - 1, p^3 + 2p - 2, \ldots, p^3 + p^2 - 2,
\]

\[
\ldots,
\]

\[
p^3 + 1, p^3 + 2p, \ldots, p^3 + p^2).
\]

a \(p^2\)-cycle. Hence, \(\text{ord}([y, x]) \geq p^2\). As all \(p\)-cycles occurring in \(e_{3,1}\) or \(e_{4,1+2p^2}\) are disjoint from each other and from the set \(S\), indeed, \(\text{ord}([y, x]) = p^2\). At the same time, our formula for \([y, x^2]\) exhibits it as a product of disjoint \(p\)-cycles, hence \(\text{ord}([y, x^2]) = p\). So we have shown

\[
\text{ord}([y, x^2]) = p \neq p^2 = \text{ord}([y, x]),
\]

and since \(p \neq 2\), \(\langle x, y \rangle = \langle x^2, y \rangle\). This proves the assertion with \(x' := x^2\).

It can be verified that the same is true for \(p = 3\), even though the formulas for \(x^{-2}yx^2 = e^{-1}_{2,1}(e_{e_{1,1},e_{2,2}}, e_{e_{3,1+2p},e_{4,1}}) e_{2,1} = e_{1,1} e_{2,1} e_{3,1+2p} e_{4,1+p}\) and \([y, x^2] = e^{-1}_{4,1+p^2}, e_{3,1+p} e_{4,1+p}\).

All conclusions, however, remain valid.

For \(p = 2\), take

\[
x := e_{1,1} e_{2,1} e_{3,1} e_{4,1} = (1, 9, 5, 13, 3, 11, 7, 15, 2, 10, 6, 14, 4, 12, 8, 16),
\]

\[
x' := x^3 = (1, 13, 7, 10, 4, 16, 5, 11, 2, 14, 8, 9, 3, 15, 6, 12),
\]

\[
y := e_{1,1} e_{3,4} e_{4,1} = (1, 9, 2, 10)(3, 11)(4, 12)(5, 15, 7, 13)(6, 16, 8, 14),
\]

then some computations show \(\langle x, y \rangle = \langle x', y \rangle\), but \(\text{ord}([y, x]) = 2 \neq 4 = \text{ord}([y, x'])\).

\[\square\]

4 | RESULTS ON STRATA OF \(p\)-ORIGAMIS

The goal of this section is to derive results about \(p\)-origamis from the results about \(p\)-groups in Section 3. In Section 4.1, we answer the question in which strata \(p\)-origamis occur. Subsequently, we study in Section 4.2 under which conditions \(p\)-origamis with isomorphic deck groups lie in the same stratum.

4.1 | Strata of \(p\)-origamis

The answer to the question in which strata \(p\)-origamis occur depends on whether the considered prime is 2 or not. We begin this section with a fact that holds for all normal origamis. In Section 4.1.1 and Section 4.1.2, we consider 2-origamis and \(p\)-origamis for odd primes \(p\), respectively.

Recall from Lemma 2.7 that all normal origamis with abelian deck group lie in the stratum \(H(0)\). Since all abelian groups are isoclinic, the following can be viewed as a generalization of this observation:
**Corollary 4.1.** The set of possible types of singularities of normal origamis with a given deck group depends only on the isoclinism class of this deck group.

**Proof.** By Lemma 3.10, the set of possible commutator orders for pairs of generators coincides for isoclinic 2-generated groups. The commutator order determines the multiplicity of the singularity. □

**Remark 4.2.** As an example, we have seen that for each \( n \geq 1 \), there is a stratum containing all 2-origamis with the dihedral group, the generalized quaternion group, or the semidihedral group of order \( 2^n \) as the group of deck transformations (see Corollary 3.11). It was computed to be \( H(4 \times (2^{n-2} - 1)) \) in Lemma 3.19.

### 4.1.1 Strata of 2-origamis

In this section, we classify the strata of 2-origamis. We will see in Section 4.1.2 that the occurring strata differ from the ones of \( p \)-origamis for odd primes \( p \).

**Theorem 4.3.** Let \( n \in \mathbb{Z} \geq 0 \). For 2-origamis of degree \( 2^n \), exactly the following strata appear: \( H(0) \) and \( H(2^{n-k} \times (2^k - 1)) \) for \( 1 \leq k \leq n-2 \).

**Proof.** For \( n \leq 2 \), all groups of order \( 2^n \) are abelian, so the corresponding 2-origamis lie in the trivial stratum by Lemma 2.7.

Let \( n, k \in \mathbb{Z} \geq 0 \) with \( n > 2 \) and \( k \leq n-2 \). By Proposition 3.4, there exists a 2-generated group \( G \) of order \( 2^n \) and generators \( x, y \) such that \( \text{ord}([x, y]) = 2^k \). Hence, the origami \((G, x, y)\) lies in the stratum \( H(0) \) for \( k = 0 \) and in \( H(2^{n-k} \times (2^k - 1)) \) for \( k > 0 \).

It remains to prove that other strata cannot occur. Let \( \mathcal{O} = (G, x, y) \) be a 2-origami of degree \( 2^n \). By Remark 2.9, the only possible strata are of the form \( H(k \times (a - 1)) \), where \( a \) is the multiplicity of each singularity and \( k \) is the number of singularities. Using Proposition 3.2, we deduce that the inequality \( \exp(G') \leq 2^{n-2} \) holds. By Lemma 2.7, the multiplicity of each singularity equals \( \text{ord}([x, y]) \). Since \( \text{ord}([x, y]) \leq 2^{n-2} \), the claim follows. □

**Remark 4.4.** We recall the definition of the series of 2-groups in the proof of Proposition 3.4. For \( n, k \in \mathbb{Z} \geq 0 \) with \( k \leq n-2 \), we define the semidirect product

\[
G^2_{(n,k)} := C_{2^{k+1}} \rtimes_{\varphi} C_{2^{n-k-1}} = \langle r, s \mid r^{2^{k+1}} = s^{2^{n-k-1}} = 1, s^{-1}rs = r^{-1} \rangle.
\]

The commutator \([r, s]\) has order \( 2^k \) and thus the 2-origami \((G^2_{(n,k)}, r, s)\) lies in the stratum \( H(2^{n-k} \times (2^k - 1)) \).

In particular, this shows that in each of the occurring strata, there exists a 2-origami with a semidirect product of two cyclic groups as deck transformation group.

**Example 4.5.** For \( n = 3 \) and \( k = 1 \), we obtain the group

\[
G^2_{(3,1)} = C_4 \rtimes_{\varphi} C_2 = \langle r, s \mid r^4 = s^2 = 1, s^{-1}rs = r^{-1} \rangle.
\]

This group is isomorphic to the dihedral group \( D_8 \). We consider the 2-origami \( \mathcal{O} = (G^2_{(3,1)}, r, s) \) (Figure 5). The commutator \([r, s]\) has order 2 and thus \( \mathcal{O} \) lies in \( H(4 \times 1) \).

Choosing \((s, rs)\) as the pair of generators of \( G^2_{(3,1)} \), we obtain the origami \( \mathcal{O}' = (G^2_{(3,1)}, s, rs) \) (Figure 6). Since \([s, rs] = r^{-2}\) has order 2, the origamis \( \mathcal{O} \) and \( \mathcal{O}' \) lie in the same stratum. In the following, we show that the origamis are different. We call a maximal collection of parallel closed geodesics on an origami a cylinder. For a normal origami \((H, x, y)\), the length of each cylinder in horizontal and vertical direction equals the order of \( x \) and \( y \), respectively. We conclude that the horizontal cylinders of \( \mathcal{O}' \) have length 2, whereas the horizontal cylinders of \( \mathcal{O} \) have length 4. Hence, the origamis are different. (Alternatively, one can use Lemma 2.4 and the fact that group isomorphisms preserve orders.)

The example above gives two different 2-origamis with isomorphic deck group. In Section 4.2, we address the question whether the deck transformation group determines the stratum of a normal origami.
Remark 4.6. We recall that the matrix group $\text{SL}(2, \mathbb{Z})$ is generated by $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. There is a natural action of $\text{SL}(2, \mathbb{Z})$ on origamis, where geometrically the action of $S$ corresponds to a rotation by $90^\circ$, while the action of $T$ corresponds to shearing the squares in the tiling of an origami (in the Euclidean plane) by $T$ (Figure 7). One is interested in studying $\text{SL}(2, \mathbb{Z})$-orbits of origamis, because they are connected to the study of Teichmüller curves.

Alternatively, the action can be described with the help of the monodromy map. See [42] for further information. For a normal origami $(H, x, y)$, the actions of $S$ and $T$ are given by

$$S \cdot (H, x, y) = (H, y^{-1}, x), \quad T \cdot (H, x, y) = (H, x, yx^{-1}).$$

In this description, it can be verified easily, that the commutator of the generators $[x, y]$ is sent to a conjugated element in $H$ by the actions of $S$ and $T$, respectively, so in particular, to an element of the same order. Hence, it is clear that the stratum is preserved. It follows that the set of normal origamis in a given stratum decomposes into orbits under the $\text{SL}(2, \mathbb{Z})$-action. This is also known from the geometric description.
We note that Example 4.5 shows an instance of such an \( \text{SL}(2,\mathbb{Z}) \)-action because of the equation

\[
T \cdot \mathcal{O} = (G, s, rs^{-1}) = (G, s, r) = \mathcal{O}.
\]

In particular, \( \mathcal{O} \) and \( \mathcal{O}' \) lie in the same \( \text{SL}(2,\mathbb{Z}) \)-orbit.

### 4.1.2 Strata of \( p \)-origamis for odd primes \( p \)

Throughout this section, let \( p \) denote an odd prime. We study the question in which strata \( p \)-origamis lie. Compared to the case of 2-origamis, fewer strata occur. This is shown in Theorem 4.7.

**Theorem 4.7.** Let \( n \in \mathbb{Z}_{\geq 0} \). For \( p \)-origamis of degree \( p^n \) exactly the following strata appear: \( \mathcal{H}(0) \) and \( \mathcal{H}(p^{n-k} \times (p^k - 1)) \) for \( 1 \leq k < \frac{n}{2} \).

**Proof.** For \( n \leq 2 \), all groups of order \( p^n \) are abelian, so the corresponding \( p \)-origamis lie in the trivial stratum by Lemma 2.7.

Let \( n, k \in \mathbb{Z}_{\geq 0} \) with \( n > 2 \) and \( k < \frac{n}{2} \). By Proposition 3.8, there exists a 2-generated group \( G \) of order \( p^n \) and generators \( x, y \) such that \( \text{ord}([x, y]) = p^k \). Hence, the origami \( (G, x, y) \) lies in the stratum \( \mathcal{H}(0) \) for \( k = 0 \) and in \( \mathcal{H}(p^{n-k} \times (p^k - 1)) \) for \( k > 0 \). It remains to prove that other strata cannot occur. Let \( \mathcal{O} = (G, x, y) \) be a \( p \)-origami of degree \( p^n \). By Remark 2.9, the only possible strata are of the form \( \mathcal{H}(k \times (a - 1)) \) where \( a \) is the multiplicity of each singularity and \( k \) is the number of singularities. By Lemma 2.7, the multiplicity of each singularity equals \( \text{ord}([x, y]) \). Using Corollary 3.6, we deduce that the inequality \( \text{ord}([x, y]) \leq p^k \) holds, where \( k < \frac{n}{2} \).

**Remark 4.8.** We recall the definition of the series of \( p \)-groups in the proof of Proposition 3.8. For \( n, k \in \mathbb{Z}_{\geq 0} \) with \( k < \frac{n}{2} \), we define the semidirect product

\[
G_{(n,k)}^p := C_{p^{k+1}} \rtimes_{\varphi} C_{p^{n-k-1}} = \langle r, s \mid r^{p^{k+1}} = s^{p^{n-k-1}} = 1, s^{-1}rs = r^{p+1} \rangle.
\]

The commutator \([r, s]\) has order \( p^k \), and thus the \( p \)-origami \( (G_{(n,k)}^p, r, s) \) lies in the stratum \( \mathcal{H}(p^{n-k} \times (p^k - 1)) \). As in Remark 4.4, this shows that in each of the occurring strata, there exists a \( p \)-origami with a semidirect product of two cyclic groups as deck transformation group.

**Example 4.9.** For \( p = 3 \), \( n = 3 \), and \( k = 1 \), we obtain the group

\[
G_{(3,1)}^3 = C_9 \rtimes_{\varphi} C_3 = \langle r, s \mid r^3 = s = 1, s^{-1}rs = r^4 \rangle.
\]

We consider the origami \( \mathcal{O} = (G_{(3,1)}^3, r, s) \). The commutator \([r, s]\) has order 3 and thus \( \mathcal{O} \) lies in \( \mathcal{H}(9 \times 2) \). Hence, the origami has nine singularities of angle \( 3 \cdot 2\pi \) (Figure 8).

### 4.2 \( p \)-origamis with isomorphic deck groups

In this section, we study the question whether the deck transformation group determines the stratum of a normal origami. This question was motivated by computer experiments. For \( p \)-origamis, computer experiments suggested that the stratum depends only on the isomorphism class of the deck transformation group. Using Example 3.14, we show that this does not hold for all finite groups.

**Example 4.10.** For \( n \in \mathbb{N}_{\geq 5} \), we consider the alternating group \( A_n \) with the pairs of generators \((1, 2, \ldots, n - 1, n), (1, 2, 3)\) and \((3, 4, \ldots, n - 1, n), (1, 3)(2, 4)\). Recall from Example 3.14 that the order of the commutators \([1, 2, \ldots, n - 1, n], (1, 2, 3)\) and \([(3, 4, \ldots, n - 1, n), (1, 3)(2, 4)]\) are 3 and 5, respectively. Hence, the normal origami \( \mathcal{O}_n := (A_n, (1, 2, \ldots, n - 1, n), (1, 2, 3)) \) has \( \frac{n}{6} \) singularities of multiplicity 3, whereas the normal origami \( \mathcal{O}'_n := (A_n, (3, 4, \ldots, n - 1, n), (1, 3)(2, 4)) \) has \( \frac{n}{3} \) singularities of multiplicity 2.
Figure 8. The 3-origami $(G^3_{(3,1)}, r, s)$. All horizontal cylinders are of length 9 and all vertical cylinders are of length 3.

has $\frac{n!}{10}$ singularities of multiplicity 5. It follows that there are two pairs of generators of $A_n$ defining normal origamis lying in different strata.

Recall that we multiply permutations from the left because we label the squares of a normal origami by multiplying generators of the deck group from the right.

The origami constructed in [2, Example 7.3] is a normal origami with deck group $A_5$. It lies in the same stratum as origami $\mathcal{O}_5'$ and thus could replace $\mathcal{O}_5'$ in the example above for $n = 5$.

Recall that two normal origamis $(G, x_1, y_1)$ and $(G, x_2, y_2)$ with isomorphic group of deck transformation group lie in the same stratum if and only if the orders of the commutators $[x_1, y_1]$ and $[x_2, y_2]$ agree. This is the case for all possible pairs of generators of a group $G$ if and only if the deck transformation group has property (C) (see Definition 3.12). Using Theorem 3.29 and Remark 3.30, we obtain Theorem B from the introduction.

Theorem 4.11. Let $G$ be a finite 2-generated $p$-group. If $G$ satisfies one of the properties (1) to (8), then all $p$-origamis with deck group $G$ lie in the same stratum.

1. $G$ is regular.
2. $G$ has maximal class.
3. $G$ is powerful.
4. $G'$ is regular.
5. $G'$ is powerful.
6. $G'$ is order closed.
7. $G$ has order at most $p^{p+2}$.
8. $G$ has nilpotency class at most $p$.

Proof. By definition, a finite 2-generated group $G$ has property (C), if there exists a natural number $n$ such that for each 2-generating set $\{x, y\}$ of $G$ the order of $[x, y]$ equals $n$. Hence, it is sufficient to show that property (C) holds for all groups satisfying one of the properties (1) to (8). This follows from Theorem 3.29. The connection to groups up to a certain order or nilpotency class is made in Remark 3.30.

For certain $p$-groups with property (C), we studied the constant given by the order of the commutator of a pair of generators in Section 3.2 (see Lemma 3.19, Lemma 3.20, and Lemma 3.28). We deduce the corresponding results for the strata of the respective $p$-origamis.

Corollary 4.12. Any 2-origami of degree $2^n$ whose deck group has maximal class lies in the stratum $H(4 \times (2^{n-2} - 1))$. 
Corollary 4.13. For $5 \leq n \leq p + 1$ and an odd prime $p$, any $p$-origami of degree $p^n$ whose deck group has maximal class lies in the stratum $\mathcal{H} \left( p^{n-1} \times (p - 1) \right)$.

Corollary 4.14. Any $p$-origami of degree $p^n$ whose deck transformation group is a minimal non-power closed $p$-group or a minimal non-order closed $p$-group lies in the stratum $\mathcal{H} \left( p^{n-1} \times (p - 1) \right)$.

In Lemma 3.34, we constructed for each prime $p$ a 2-generated $p$-group that does not have property $(C)$. Hence, we obtain the following proposition.

Proposition 4.15. For each prime $p$, there exist $p$-origamis with isomorphic deck transformation group that lie in different strata.

Proof. In Lemma 3.34, we proved for each prime $p$ the existence of a 2-generated $p$-group $H_p$ which is contained in the Sylow $p$-subgroup of the symmetric group $S_{p^4}$ and does not have property $(C)$. Hence, there exist $p$-origamis with deck transformation group isomorphic to $H_p$ that lie in different strata. □

Remark 4.16. The 2-group $G$ with generating sets $(x, y)$ and $(x, y^3)$ defined in Example 3.26 is weakly power closed, but not power closed. Recall that the orders of the commutators ord$(x, y)$ and ord$(x, y^3)$ are 4 and 2, respectively. We obtain that the 2-origamis $(G, x, y)$ and $(G, x, y^3)$ lie in different strata, namely, $H(2^{10} \times 3)$ and $H(2^{11} \times 1)$. Here, we use that the group $G$ has order $2^{12}$.

5 | OUTLOOK: INFINITE ORIGAMIS AND PRO-$p$ GROUPS

So far, we have considered surfaces that are also called finite translation surfaces, that is, the surface can be described as finitely many polygons with edge identifications via translations. As a generalization, infinite translation surfaces have been studied during the past 10 years (see, e.g., [5]). In contrast to finite translation surfaces, one allows countably many polygons glued by translations. For a detailed introduction to infinite translation surfaces, see [35] and [6].

In this section, we consider a well-known infinite translation surface called staircase origami. Moreover, we generalize the notion of property $(C)$ to pro-$p$ groups, certain infinite analogs of finite $p$-groups. We then transfer some results from Section 3.2 to pro-$p$ groups and deduce conclusions about a class of translation surfaces, which we call infinite normal origamis.

Let $\mathcal{O} \to \mathbb{T}$ be a countably infinite, normal cover of the torus $\mathbb{T}$ ramified over at most one point. Then, the corresponding surface $\mathcal{O}$ is called an infinite normal origami. Among others, these surfaces have been studied by [22], where they are called regular origamis. As in the finite case, they correspond to a special class of infinite translation surfaces where all polygons are squares of the same size. The concepts introduced in Section 2 carry over to infinite origamis. Given a countably infinite group $G$ with 2-generating set $(x, y)$, one constructs an infinite normal origami $(G, x, y)$ as in Lemma 2.2. One has a natural bijection between the squares in the tiling and the elements of the deck group. Singularities of infinite normal origamis can have finite cone angle, that is, $2\pi n$ for $n \in \mathbb{Z}_+$ as in the case of finite origamis, or infinite angle, that is, a neighborhood of the singularity is isometric to a neighborhood of the branching point of the infinite cyclic branched cover of $\mathbb{R}^2$. As in the case of finite normal origamis, the cone angle of all singularities of the origami $(G, x, y)$ coincide and are equal to $2\pi n \cdot a$, where $a$ is the order of the commutator $[x, y]$. If the order of $[x, y]$ is infinite, then the cone angle is infinite as well.

Example 5.1. An example for an infinite normal origami is the staircase origami $\mathcal{O}_\infty$ in Figure 9. It has been studied both from the geometric (see, e.g., [20]) and from the dynamical point of view (see, e.g., [17]).

The deck group of the origami $\mathcal{O}_\infty$ is the infinite dihedral group

$$D_\infty := \langle r, s \mid s^2 = 1, srs = r^{-1} \rangle,$$

for $\mathcal{O}_\infty$, the elements $s, sr$ are chosen as generators. The commutator subgroup of $D_\infty$ is the infinite cyclic group generated by $[r, s] = r^{-2}$. Hence, for any pair of generators $x, y$, the commutator $[x, y]$ has order infinity. We conclude that each infinite normal origami with deck group $D_\infty$ has 4 singularities of infinite cone angle.
The infinite staircase origami $\text{St}_\infty = (D_\infty, s, sr)$ has four singularities of infinite cone angle. All vertical and horizontal cylinders have length 2. Here, the horizontal cylinders are shaded in different colors. Opposite sides are identified unless marked otherwise.

Choosing the generators $r, s$ we obtain a surface different from $\text{St}_\infty$. In contrast to the latter, it has 2 infinite horizontal cylinders, as shown in Figure 10.

The infinite dihedral group is a dense subgroup of the pro-$2$ group $\mathbb{Z}(2) \rtimes C_2$. Pro-$p$ groups have played an essential role in the study of finite $p$-groups (see, e.g., [26]). It is a natural question, whether results of Section 3.2 and Section 4.2 can be transferred to certain infinite groups, in particular profinite and pro-$p$ groups, and infinite normal origamis. To this end, we extend the definition of property (C) to possibly infinite $2$-generated groups. Note that, as we do not consider topological groups yet, the groups under consideration are still countable.

**Definition 5.2.** A (possibly countably infinite) $2$-generated group $G$ has property (C) if there is an element $k \in \mathbb{Z}_+ \cup \{\infty\}$ such that the order of $[x, y]$ equals $k$ for any pair $x, y$ of generators of $G$.

**Example 5.3.** Recall that the infinite dihedral group $D_\infty$ has an infinite cyclic commutator subgroup, that is, $D_\infty$ has property (C) (see Example 5.1).

In the following, we consider $2$-generated profinite groups. For this purpose, recall that an **inverse system** of groups is a collection of groups $(G_i)_{i \in I}$ indexed by a directed poset $I$ and homomorphisms $\psi_{i,j} : G_j \to G_i$ for all $i, j \in I, i \leq j$, such that $\psi_{i,i}$ is the identity and $\psi_{i,j} \circ \psi_{j,k} = \psi_{i,k}$ for all $i \leq j \leq k$. The **inverse limit** of an inverse system $(G_i), (\psi_{i,j})_{i \leq j}$ is the group

$$
\hat{G} := \{(g_i) \in \prod_i G_i : \psi_{i,j}(g_j) = g_i \text{ for all } i \leq j\}
$$

together with projection maps $\pi_i : \hat{G} \to G_i$ onto the components $G_i$. It is the categorical limit for the diagram described by the inverse system in the category of groups, and is distinguished by the corresponding universal property (see Diagram (5.1)).
Recall that a **profinite group** is a Hausdorff, compact, totally disconnected topological group. For any profinite group \( \hat{G} \), the quotients \( \hat{G} / N \) for all normal open subgroups \( N \) of \( \hat{G} \) form an inverse system of finite groups whose inverse limit is isomorphic to \( \hat{G} \). Equivalently, one can define profinite groups as those, which are isomorphic to the inverse limit of an inverse system of finite groups. We refer the interested reader to [9] for background knowledge about profinite groups.

For profinite, or more generally, topological groups, we can consider **topological generating sets**, that is, subsets that generate a dense subgroup. A profinite group is topologically 2-generated if and only if it is isomorphic to the inverse limit of an inverse system of 2-generated finite groups ([9, Proposition 1.5]). This motivates the following definition.

Recall that a **Steinitz number** \( k \) is a formal product \( \prod_p p^{\nu_p(k)} \), where \( p \) ranges over all prime numbers and \( \nu_p(k) \) is a nonnegative integer or \( \infty \) for each prime \( p \).

**Definition 5.4.** We say that a topologically 2-generated profinite group has **property \((C)\pro\)** if there exists a Steinitz number \( k \) such that for all 2-generating sets \( x, y \) the commutator order \( \text{ord}([x, y]) \) equals \( k \).

Using the connection between commutator orders of 2-generating sets and cone angles of singularities for infinite normal origamis, we obtain the following lemma directly from the definition of property \((C)\pro\).

**Lemma 5.5.** Let \( \hat{G} \) be a topologically 2-generated profinite group with property \((C)\pro\). The singularities of all infinite normal origamis whose deck group is a (countable) dense subgroup of \( \hat{G} \) have the same cone angle.

One way of verifying that a given profinite group has property \((C)\pro\) is given by the following lemma.

**Lemma 5.6.** A topologically 2-generated profinite group has property \((C)\pro\) if it is isomorphic to the inverse limit of an inverse system of finite groups, which have property \((C)\).

**Proof.** Consider the profinite group \( \hat{G} \cong \lim_{\to} (G_i, \psi_{i,j}) \) for an inverse system \( ((G_i), (\psi_{i,j}))_{i \leq j} \) of finite groups with property \((C)\). Denote the structural projections of the inverse limit by \( \pi_i : \hat{G} \to G_i \).

From the inverse limit construction, it is clear that the order of any element \( g \in \hat{G} \) is the least common multiple (as a Steinitz number) of the element orders \( \text{ord}([\pi_i(g)]) \) in the respective groups \( G_i \). If \( g \) is the commutator of a pair of generators \( x, y \) of the group \( \hat{G} \), then \( \pi_i(g) \) is the commutator of a pair of generators in \( G_i \) for every \( i \), since \( \pi_i \) is an epimorphism.

By assumption, the group \( G_i \) has property \((C)\) for each \( i \). Hence, the number \( \text{ord}([\pi_i(g)]) \) is independent of the choice of the generators \( x, y \) for each \( i \). We conclude that the least common multiple of the element orders \( \text{ord}([\pi_i(g)]) \) is independent of the choice of the generators \( x, y \) as well. This proves that \( \hat{G} \) has property \((C)\pro\). \( \square \)

**Remark 5.7.** Let \( (G_i), (\psi_{i,j})_{i \leq j} \) be an inverse system of 2-generated finite groups. We call 2-generating sets \( (x_i, y_i) \), **compatible** if \( \psi_{i,j}(x_j) = x_i \) and \( \psi_{i,j}(y_j) = y_i \) for all \( i \). Denote the inverse limit by \((\hat{G}, \pi_i)\). Applying the universal mapping property of the inverse limit to the free group \( F_2 = \langle a, b \rangle \) and the monodromy maps \( m_i : F_2 \to G_i \), one obtains a unique group homomorphism \( \alpha : F_2 \to \hat{G} \) and the following commutative diagram:

\[
\begin{array}{ccc}
F_2 & \xrightarrow{\alpha} & \hat{G} \\
\downarrow m_i & & \downarrow \pi_i \\
G_j & \xrightarrow{\psi_{i,j}} & G_i.
\end{array}
\] (5.1)
The image $\alpha(F_2)$ is a (countable) dense subgroup of $\hat{G}$ with 2-generating set $x := \alpha(a), y := \alpha(b)$ (in the sense of classical group theory). Note that $\pi_i(x) = x_i$ and $\pi_i(y) = y_i$ for all $i \in I$. One obtains an infinite normal origami $(\alpha(F_2), x, y)$ associated with the infinite sequence of finite normal origamis $(G_i, x_i, y_i)$.

Similarly, any given infinite normal origami $(H, x', y')$ for a (countable) dense subgroup $H$ of $\hat{G}$ yields infinitely many finite normal origamis $(G_i, \pi_i(x'_i), \pi_i(y'_i))$. Here, we use that the image $\pi_i(H)$ equals $G_i$ for each $i \in I$ (see [9, Proposition 1.5]).

Recall that for any prime $p$, a pro-$p$ group is a profinite group $\hat{G}$ such that $\hat{G}/N$ is a finite $p$-group for any open normal subgroup $N$ of $\hat{G}$. Equivalently, it is a group, which is isomorphic to the inverse limit of an inverse system of finite $p$-groups. Pro-$p$ groups play a central role in the coclass conjectures by Leedham-Green and Newman ([28]), proved by Leedham-Green ([25]) and Shalev ([39]), concerning a way of classifying all finite $p$-groups.

We can now extend some of our results from finite $p$-groups to pro-$p$ groups. Let us call a pro-$p$ group weakly order closed if products of elements of order $p^k$ have order at most $p^k$, for any $k \geq 0$. Let us also recall that a pro-$p$ group is called powerful if $G/\operatorname{cl}(\langle g^{p^k} : g \in G \rangle)$ is abelian for $k = 1$ if $p > 2$ and for $k = 2$ if $p = 2$, where $\operatorname{cl}(S)$ denotes the minimal closed subgroup generated by a set $S$ (see [9, Definition 3.1]).

We observe that key results for finite $p$-groups, for example, Proposition 3.17 and Corollary 3.24, can be generalized to pro-$p$ groups.

**Lemma 5.8.** A topologically 2-generated pro-$p$ group $\hat{G}$ has property $(C^{pro})$ if either

1. $\hat{G}'$ is weakly order closed, or
2. $\hat{G}'$ is powerful.

**Proof.** Let $x, y$ be (topological) generators of $\hat{G}$. The commutator subgroup $\hat{G}'$ is generated by conjugates of $[x, y]$, all having the same order, say $p^k$ for $k \geq 1$. We will show that assuming (1) or (2) implies this order is the exponent of $\hat{G}'$ and thus is independent of the choice of $x, y$. This proves the assertion that $\hat{G}$ has property $(C^{pro})$.

If we assume (1), then the countable subgroup of $\hat{G}'$ generated (without topological closure) by all conjugates of $[x, y]$ consists of elements of order at most $p^k$. Hence indeed, $p^k$ is the exponent of the countable subgroup, and of $\hat{G}'$ being its closure.

If we assume (2), then $\hat{G}'$ is a powerful pro-$p$ group topologically generated by conjugates of $[x, y]$. Hence, by [9, Proposition 3.6 (iii)], the set of all $p^k$-th powers in $\hat{G}'$ equals the closed subgroup generated by the $p^k$-th powers of the conjugates of $[x, y]$, which is the trivial subgroup. Thus, all $p^k$-th powers in $\hat{G}'$ have to be trivial. This shows that $p^k$ is the exponent of $\hat{G}'$. \qed

As for finite $p$-groups this has implications for families of normal origamis:

**Corollary 5.9.** Let $\hat{G}$ be a topologically 2-generated pro-$p$ group whose commutator subgroup is either powerful or weakly order closed. Then, the singularities of all infinite normal origamis whose deck group is a (countable) dense subgroup of $\hat{G}$ have the same cone angle.

We conclude our exploration into the world of infinite deck groups with some examples.

**Example 5.10.** In this example, we introduce a setup to construct inverse systems from semidirect products of cyclic groups. Such groups appeared several times in Section 3.1 and Section 4 (see, e.g., Proposition 3.4, Proposition 3.8, Remark 4.4, and Remark 4.8). For a prime $p$, let $C_{p,m} = \langle x \rangle$ and $C_{p,\ell'} = \langle y \rangle$ be cyclic groups of order $p^m$ and $p^{\ell'}$, respectively. If $a \in \mathbb{Z}$ is coprime to $p$ and $a^{p^m} \equiv 1 \bmod p^{\ell'}$, then the map $\varphi_a : C_{p,m} \to \operatorname{Aut}(C_{p,\ell'})$, $x \mapsto (y \mapsto y^a)$ is a group homomorphism and defines a semidirect product

$$H(\ell, m, a) := C_{p,\ell'} \rtimes \varphi_a C_{p,m} = \langle x, y \mid x^{p^m} = y^{p^{\ell'}} = 1, x^{-1} y x = y^a \rangle.$$

Now for any $m' \geq m$ and $\ell' \geq \ell$ with $m' - \ell' \geq m - \ell$, we obtain $a^{p^{m'}} \equiv 1 \bmod p^{\ell'}$ and $a^{p^{m'}} \equiv 1 \bmod p^{\ell'}$ as well. Hence, the groups $H(\ell', m', a)$ and $H(\ell, m, a)$ are still well defined and we have epimorphisms $H(\ell', m', a) \to H(\ell, m, a)$ and
Figure 11 The origami $\left(W_2, x, y\right)$ as in Example 5.11 is a cover of the Eierlegende Wollmilchsau of degree 4.

$H_{\ell,m',a} \to H_{\ell,m,a}$ sending $x \mapsto x$ and $y \mapsto y$ in the respective groups. Using these epimorphisms, we get an inverse system $(H_{\ell,m',a})_{m' \geq m}$ with inverse limit $C_{p' \times Z(p)}$, or an inverse system $(H_{\ell',m',a})_{m' \geq m, \ell' \geq \ell, m' - \ell' \geq m - \ell}$ with inverse limit $Z(p) \rtimes Z(p)$.

Choosing compatible 2-generating sets for the groups forming an inverse system, one obtains an infinite sequence of finite normal origams and an infinite normal origami, which has a (countable) dense subgroup of the inverse limit as its deck group (see Remark 5.7). Note, that all groups of the form $H_{\ell,m,a}$ have property $(\mathbb{C})$, since the commutator subgroup is always cyclic (generated by $y^{a-1}$). Hence, both constructions of inverse systems yield pro-$p$ groups with property $(\mathbb{C}^{\text{pro}})$ (see Lemma 5.6). The constructions of inverse systems given above can be applied as well to the inverse systems formed by the groups $G_{(n,k)}^p$ considered in Remark 4.4 and Remark 4.8 for $p = 2$ and $p > 2$, respectively.

The dihedral groups $D_{2n} = \langle r_n, s_n \mid r_n^{2n} = 1, s_n r_n s_n = r_n^{-1} \rangle$, considered in Example 5.1, form an inverse system constructed in a similar, but slightly different way. Here, we fix $m = 1$, $a = -1$, and let $\ell$ vary. The infinite dihedral group $D_{\infty}$ is a 2-generated countable dense subgroup of the inverse limit $D_2 = Z(2) \rtimes C_2$ of the 2-generated 2-groups $(D_{2n})_{n \geq 0}$. By Lemma 3.21, the dihedral groups have property $(\mathbb{C})$ and thus the pro-2 group has property $(\mathbb{C}^{\text{pro}})$ (see Lemma 5.6). For $n \in \mathbb{Z}_+$, the tuples $(r_n, s_n)$ form compatible 2-generating sets. Using the construction in Remark 5.7, we obtain the normal infinite origami $(D_{\infty}, r, s)$ in Figure 10. Choosing $(s_n, s_n r_n)$ as compatible 2-generators, the construction yields the infinite staircase origami in Figure 9.

In the following example, we construct an inverse system taking the quaternion group as a starting point. This yields an infinite series of normal origamis covering the Eierlegende Wollmilchsau (see Example 2.3).

Example 5.11. The groups

$W_n = \langle x, y \mid x^{2n+1} = y^{2n+1} = x^{2n} y^{2n} = 1, x^{-1} y x = y^{-1} \rangle$
are 2-generated 2-groups of order $2^{2n+1}$. For any $n \geq 1$, consider the element $z := y^2 = [x, y]$. It lies in the commutator subgroup $W_n'$, commutes with $y$, and obeys the equality

$$x^{-1}zx = x^{-1}y^2x = y^{-2} = z^{-1}.$$

Hence, by Lemma 3.3, the commutator subgroup $W_n' = \langle z \rangle$ is cyclic of order $2^n$ and the group $W_n$ has property (C). The defining relations of $W_n$ imply that its elements are of the form $x^a y^b$ for $0 \leq a < 2^{n+1}$ and $0 \leq b < 2^n$. The images of $x$ and $y$ in the abelianization $W_n/W_n' = W_n/(y^2)$ have order $2^n$ and 2, respectively, and the abelianization is isomorphic to $C_{2n} \times C_2$.

Note that we have epimorphisms from $W_{n+1}$ to $W_n$ for any $n \geq 1$ sending $x \mapsto x$, $y \mapsto y$ (and $z \mapsto z$) in the respective groups. We obtain an inverse system whose inverse limit has property (C) by Lemma 5.6. The group $W_1$ is isomorphic to the quaternion group with 8 elements and the generators $(x, y)$ (viewed as elements in the groups $W_n$) form a set of compatible generators. Hence, the infinite sequence of normal origamis $(W_n, x, y)$ covers the Eierlegende Wollmilchsau.

We give an example of these origamis for $n = 2$ in Figure 11.

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APPENDIX: GAP CODE

All of the following is GAP4 code (see [13]).
Listing 1. For natural numbers $n, k$ with $1 \leq k \leq n - 2$, the following code defines the group $G^2_{(n,k)}$ as well as the corresponding 2-generating set constructed in Proposition 3.4. A very similar code can be used to define the groups $G^p_{(n,k)}$ and their generating sets as discussed in Proposition 3.8.

```
G2 := function(n,k)
local C1, C2, alpha, phi, G, x, y;
C1 := CyclicGroup(2^(k+1)); C2 := CyclicGroup(2^(n-k-1));
alpha := GroupHomomorphismByImages(C1, C1, [C1.1], [[C1.1]^(-1)]);
phi := GroupHomomorphismByImages(C2, AutomorphismGroup(C1), [C2.1], [alpha]);
G := SemidirectProduct(C2, phi, C1);
x := Image(Embedding(G,2), C1.1); y := Image(Embedding(G,1), C2.1);
return [G, x, y];
end;
```
Listing 2. Variations of the following code were used to find $p$-groups, which do not have property $\mathcal{C}$.

```plaintext
p := 3; n := 4;
g := SylowSubgroup(SymmetricGroup(p^n), p);
repeat x := Random(g); y := Random(g);
   until Order(Comm(x, y)) <> Order(Comm(x, y^2));
```

Listing 3. The following code defines a function to test whether a given $p$-group is weakly power closed (i.e., products of $p^k$-th powers are $p^k$-th powers for any $k \geq 0$), and uses it to find a 2-generated subgroup $G$ of the 2-Sylow subgroup of the symmetric group $S_{24}$ with generators $x, y$ such that $\text{ord}([x, y]) \neq \text{ord}([x, y^3])$ and $G'$ is weakly power closed.

```plaintext
IsWeaklyPowerClosedPGroup := function(g)
   local powers, el;
   if IsTrivial(g) then return true; fi;
   powers := g;
   repeat
      powers := Set(powers, x -> x^PrimePGroup(g));
      if Size(Commutator(powers, powers)) > Size(powers) then return false; fi;
   until IsTrivial(powers);
   return true;
end;

p := 2; n := 4;
gg := SylowSubgroup(SymmetricGroup(p^n), p);
repeat x := Random(gg); y := Random(gg);
   g := Group(x, y); d := DerivedSubgroup(g);
   until Order(Comm(x, y)) <> Order(Comm(x, y^(p+1)))
      and IsWeaklyPowerClosedPGroup(d);
```
