Collective modes of Dirac and Weyl semimetals in strong magnetic fields

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Dirac and Weyl semimetals provide a new example of three-dimensional electron gases which are sensitive to strong magnetic fields. In this paper we address their collective excitations in the extreme quantum limit in which the Hamiltonian can be projected to the \( N = 0 \) Landau level. We show that the energy of the plasmon collective mode is not shifted by interactions, even though the quasiparticle bands are strongly renormalized, and that both optical and acoustic magnetoplasmons emerge from the particle-hole continuum when the chemical potential is not at the band-crossing point. The rare presence of a sharp acoustic magnetoplasmon excitation can be understood as a consequence of the unusual particle-hole excitation phase space of these pseudo-1D systems, and could prove valuable as a sample characterization tool. We comment on the relationship between the collective mode dispersion and the non-Fermi-liquid character of these interacting electron systems.

I. INTRODUCTION

Three-dimensional Dirac and Weyl semimetals are characterized by linear band crossings at energies close to the Fermi level that lead to a number of interesting and unusual physical properties, including exotic Fermi-arc surface states and positive magneto-conductance. Studies of theoretically proposed candidate materials have identified the expected band crossings and verified both that magneto-transport is anomalous, and that Fermi-arc surface states appear. Despite much recent attention, the influence of electron interactions on the low-energy properties of these systems has just begun to be explored.

The work described in this Article is motivated by the expectation that the discovery and refinement of new Weyl and Dirac semimetal materials will, among other developments, enhance possibilities for the experimental realization of three-dimensional electron gas systems that are in the extreme limit of strong magnetic fields, i.e. the limit in which all electrons at the Fermi energy share a common degenerate Landau level. Because linear band crossings yield widely spaced Landau levels, Dirac and Weyl semimetals are ideal materials for realizing this exotic regime. The role of electron-electron interactions in the three-dimensional electron gas in this extreme limit has not been settled, in spite of long standing interest. In graphite interactions do appear to be responsible for a transition to a charge density wave state, but it is not clear that this is a universal fate. The absence of more spectacular experimental observations makes it clear that the impact of interactions is more subtle than in the two-dimensional case where they give rise to the fractional quantum Hall effect. In this Article we study the combined influences of pseudo-1D dispersion, Landau level degeneracy, and Coulombic electron-electron interactions on the charge-neutral elementary excitations of Dirac and Weyl semimetals in the normal state where no spontaneous symmetry breaking has taken place.

Our paper is organized as follows. In Section II we discuss the model we employ for Dirac and Weyl semimetals at strong magnetic fields. In Section III we describe how the time-dependent Hartree-Fock approximation is applied to this model, highlighting the exchange interaction effects that enhance Dirac and Weyl point quasiparticle velocities and increase the energies of particle-hole excitations. In Section IV we present our results for the collective and incoherent excitation spectra, explaining why the collective mode energy is finite in the long wavelength limit, why an acoustic collective mode emerges when the Fermi level is not at the band-crossing point, and how the collective mode changes at finite transverse wavevectors. Finally in Section V we summarize our findings and present our conclusions.

FIG. 1: (Color online) a) Schematic Landau level spectrum of Cd₃As₂ in a magnetic field oriented along the direction connecting the two Dirac points after Ref. 16. b) The \( N = 0 \) Landau levels overlap in an energy window of width \( 2\Delta \sim 20 \) meV. This illustration emphasizes the discretization of momentum along the field direction that we use to convert the Bethe-Salpeter integral equation for particle-hole excitations to a finite matrix equation. Open circles indicated single-particle states that are empty and closed circles indicate states that are occupied in the non-interacting ground state when the Fermi energy is equal to the Dirac energy. The Green arrow in b) identifies one of the particle-hole transitions contributing to a magnetoplasmon collective modes with wavevector: \( q = (q_x, q_y, 2 \times (2\pi/L_z)) \).
II. THE ZEROTH LANDAU LEVEL MODEL

The model we study is motivated by the prototypical Dirac semimetal, Cd$_3$As$_2$, whose Landau levels have recently been measured using scanning tunneling microscopy. The Landau level structure of Cd$_3$As$_2$ is illustrated schematically in Figure 1a) for the case in which a strong external magnetic field is applied along the direction separating the material’s two Dirac points, which we take to be the $\hat{z}$ direction. We refer to the band crossing points $\pm k_D = \pm (0,0,k_z)$ as the Dirac points and to the band energies at these points, $\epsilon_D = \epsilon_{k_D}$, as the Dirac energy. At zero magnetic field doubly degenerate three-dimensional bands have linear crossings protected by a crystal symmetry at $\pm k_D$. In a strong magnetic field Landau quantization of the transverse band energy leaves a low-energy Hilbert space with only two distinct ($N=0$) Landau levels at energies near $\epsilon_D$ for each wavevector $k_z$ along the field direction. The $N=0$ Landau level states intersect at the Dirac point, as illustrated in Figure 1b), and overlap in an energy window whose value in Cd$_3$As$_2$ is $2\Delta \approx 20$ meV. The strong magnetic field limit applies when $k_D l_B$ is smaller than one and the Landau level separation $\sqrt{2}\hbar v_D l_B$ is large compared to the Coulomb energy scale $\epsilon_D^2/(\kappa l_B)$. The former condition ensures that $N \neq 0$ Landau levels do not appear in the energy window $\pm \Delta$ around the Dirac energy, and the latter condition ensures that virtual transitions to $N \neq 0$ Landau levels can be neglected. Here $l_B = \sqrt{\hbar c/eB}$ is the magnetic length, $v_D$ is the transverse Bloch state group velocity at the Dirac point, and $\kappa$ the effective dielectric constant thought to have a value $\sim 36$ for Cd$_3$As$_2$.

The collective mode calculations presented below start from the following non-interacting electron Hamiltonian, which qualitatively captures the strong field limit of all Dirac semimetals:

$$\mathcal{H}_0 = \sum_{k_z,n} \epsilon_{k_z,\tau} c_{k_z,\tau}^\dagger c_{k_z,\tau}$$

where

$$\epsilon_{k_z,\tau} = \tau \left( \frac{\hbar^2 k_z^2}{2m^*} - \Delta \right),$$

$k_z$ is the wavevector along the direction of magnetic field, $X$ is the guiding-center label of the Landau gauge two-dimensional states within a $N=0$ Landau level, and $\tau = \pm$ labels the two-bands. Note that we are assuming that the low-energy bands separate approximately into $k_z$ and transverse momentum contributions. The two independent material specific parameters in Eq. (1), $m^*$ and $\Delta$, can be fixed by fitting to experimental values for the Dirac velocity, $v_D$, and the $k$-space separation of the two Dirac points, $2k_c$. For Cd$_3$As$_2$ this yields

$$\Delta = \frac{\hbar k_c v_D}{2} \sim 10 \text{ meV},$$

and

$$\frac{m^*}{m} = \frac{\hbar k_c}{mv_D} \sim 0.004,$$

where $m$ is the bare electron mass. Interactions in Cd$_3$As$_2$ are relatively weak and their strength can be characterized by the dimensionless fine structure constant

$$\alpha_{fs} = \frac{e^2}{\hbar k_D} = \frac{e^2}{2k\Delta} \sim 0.065. \quad (5)$$

The minimum magnetic field strength, $B_c$, for using a model which includes only the $N=0$ Landau levels is determined from the condition $k_D l_B \lesssim 1$, which using the aforementioned Cd$_3$As$_2$ material parameters gives $B_c \gtrsim 1$ T.

Eq. (1) applies only when the direction of magnetic field is along the direction separating the two Dirac points. Otherwise the $C_4$ symmetry which protects the Dirac points in Cd$_3$As$_2$ is broken, and a small gap appears at the Dirac point$k-p$ models (with parameters evaluated using density-function theory) show that the $\tau = +1$ and $\tau = -1$ bands consist of $|F_{J=3/2}, J_z = \frac{3}{2}\rangle$ cadmium atomic orbitals and $|S_{1/2}, -\frac{1}{2}\rangle$ arsenic atomic orbitals respectively. Similar band-orbital characteristics have been predicted for the A$_3$Bi (A=Na, K, Rb) family of Dirac semimetals. The calculations we present below are readily generalized to more elaborate $k-p$ models if accurate predictions for a specific material are desired. Our goal in using the simplified model in Eq. (1) is to capture the most interesting qualitative features of the collective excitations of generic Dirac semimetals in strong magnetic fields.

Although the numerical results presented below refer to the material parameters of the Dirac semimetal Cd$_3$As$_2$, we expect the collective excitations of all Weyl and Dirac semimetals to have many of the same qualitative features. Members of the pyrochlore iridates, chalcopyrites, transition-metal monophosphides, and magnetic Heuslers are all predicted to have Weyl points in their bulk electronic band-structure. However, accurate experimentally verified band parameters are still unavailable. Although evidence for both Fermi-arc surface states and anomalous transport has recently been reported in TaAs, for example, it is understood that Weyl Fermions in this material are accompanied by a crescent shaped electron-like Fermi pockets whose band-structure is not yet reliably known. Models similar to the one we analyze can be derived for any Dirac or Weyl semimetal whose electronic structure is accurately known.

Electrons in Weyl and Dirac semimetals occupy relatively small pockets with Fermi volumes that are small compared to the Brillouin zone volume. As emphasized recently in the context of SrTiO$_3$-based two-dimensional electron gases, it is critical in such systems to account for the full long-range Coulomb interaction instead of employing simplified model interactions. Our full model Hamiltonian therefore takes the form

$$\mathcal{H} = \mathcal{H}_0 + \frac{1}{2V} \sum_{q \neq 0} \nu_q (n_{-q} n_q - N) \quad (6)$$
where \( v_g = 4\pi e^2/(\kappa q^2) \), \( \kappa \sim 36 \) is the effective dielectric constant, and \( V = L_x L_y L_z \) is the system volume. The use of this envelope function Coulomb interaction form, which drops structure on an atomic length scale, is justified by small Fermi surface volumes. In Eq. (3) the \( q = 0 \) term in the sum, which is a constant, has been removed to account for the neutralizing positive background and the number operator \( N = n_{q=0} \) term cancels electron self-interactions. The second quantized representation for the ordinary density-operator at wavevector \( q \) is

\[
 n_q = \sum_{k_x} \sum_{k_z' \tau'} \langle k_x \tau | e^{-i q \cdot r} | k_z' \tau' \rangle^* \times e^{i q \cdot r} \delta_{k_x, k_z'} \delta_{\tau, \tau'} ,
\]

where the plane-wave matrix element is

\[
\langle k_x \tau | e^{-i q \cdot r} | k_z' \tau' \rangle = \delta_{\tau, \tau'} \delta_{X, X'} - q \cdot \ell \delta_{k_x, k_z'} - q_z \times e^{-i q \cdot (X+X') / 2 - (q_x'^2 + q_y'^2) \delta_{kB}^2 / 4}. \tag{8}
\]

Notice that we have set plane wave matrix elements between the two-different bands to zero. This is a standard envelope function approximation and follows from the observations that wavefunctions of the two bands are orthogonal when at the same wavevector, and that the range of wavevectors that have important correlations is small compared to the Brillouin-zone size.

### III. TIME-DEPENDENT MEAN-FIELD THEORY IN THE QUANTUM LIMIT

Our theory of collective modes is based on time-dependent mean-field theory, also sometimes known as the generalized random-phase approximation, and as time-dependent Hartree-Fock theory. This approach has the advantage that it is completely systematic and treats direct and exchange interactions on an equal footing. We first explain the details of this calculation for the extreme quantum limit of a three-dimensional electron gas, and then discuss its physical implications. In the following we take the temperature \( T = 0 \).

Our three dimensional system can be viewed as a collection of many Landau levels labelled by different values of \( k_z \) and \( \tau \). One important property of Landau levels is that there is one independent particle-hole excitation at each transverse wavevector for every pair of Landau levels. To take advantage of this we introduce a reduced density-operator,

\[
\Delta_{k_z \tau} (q) = \sum_{\tau'} \left( e^{-i q \cdot X} e^{-\frac{1}{2} q \cdot \ell \delta_{kB}} \right) \epsilon_{k_z \tau}^\dagger \epsilon_{k_z + q \cdot \ell \delta_{kB} x + q \cdot \ell \delta_{kB} \tau} \.
\]

\( \Delta_{k_z \tau} (q) \) is related to the full momentum space density operator by

\[
n_q = e^{-\left(q_x'^2 + q_y'^2\right) \delta_{kB}^2 / 4} \sum_{k_z \tau} \Delta_{k_z \tau} (q) . \tag{10}
\]

Eq. (9) can be inverted to expand individual particle-hole pair amplitudes in terms of the reduced density-operator

\[
\epsilon_{k_z \tau}^\dagger \epsilon_{k_z' \tau'} = \frac{1}{g} \sum_{q} e^{i q \cdot X} \times \delta_{k_z', k_z + q_z} \delta_{X', X + q_y} \delta_{\tau, \tau'} \Delta_{k_z \tau} (q) , \tag{11}
\]

where we have defined the Landau level degeneracy \( g = L_x L_y / (2\pi \delta_{kB}) \) where \( L_x L_y \) is the cross-sectional area of the sample perpendicular to the direction of magnetic field.

Using Eq. (11) the mean-field Hartree-Fock Hamiltonian can be expressed in terms of the reduced density-operator

\[
\mathcal{H}_{HF} = \sum_{k_z \tau} \varepsilon_{k_z \tau} \Delta_{k_z \tau} (q = 0) + \sum_{q} \sum_{k_z \tau} \mathcal{V}_{HF} (q) \Delta_{k_z \tau} (q) , \tag{12}
\]

where the Hartree-Fock potential,

\[
\mathcal{V}_{HF} (q) = \mathcal{V} (q) + \mathcal{V}_{HF} (q) , \tag{13}
\]

has Hartree (electrostatic) contributions,

\[
\mathcal{V} (q) = \frac{\nu}{V} e^{-\left(q_x'^2 + q_y'^2\right) \delta_{kB}^2 / 4} \sum_{k_z' \tau'} \langle \Delta_{k_z' \tau'} (-q) \rangle, \tag{14}
\]

and Fock contributions,

\[
\mathcal{V}_{HF} (q) = \frac{1}{g} \sum_{q'} v_{qq'} e^{-\left(q_x'^2 + q_y'^2\right) \delta_{kB}^2 / 4} e^{i q \cdot \ell \delta_{kB} x + q_y \cdot \ell \delta_{kB} y} \times \sum_{k_z' \tau'} \langle \Delta_{k_z' \tau'} (-q) \rangle \delta_{k_z', k_z + q_z} \, . \tag{15}
\]

In both Eq. (14) and Eq. (15) \( \langle \rangle \) denotes a ground-state expectation value. Assuming that translational symmetry is not broken

\[
\langle \Delta_{k_z' \tau'} (-q) \rangle = g \delta_{q,0} n^{(F)} (\varepsilon_{k_z \tau}) , \tag{16}
\]

where \( n^{(F)} (x) \) is the Fermi-Dirac distribution function. It follows that the Hartree-Fock quasiparticle energies are

\[
\xi_{k_z \tau} = \varepsilon_{k_z \tau} - \frac{1}{\cdots} \sum_{q} v_{q} e^{-\left(q_x'^2 + q_y'^2\right) \delta_{kB}^2 / 4} \times \sum_{k_z} n^{(F)} (\varepsilon_{k_z \tau}) \delta_{k_z', k_z + q_z} . \tag{17}
\]

As illustrated in Fig. exchange interactions lower quasiparticle energies, with the strength of the effect increasing with magnetic field, and do so more strongly for occupied Landau levels. These shifts increase the rate at which the Landau level energies vary with \( k_z \), close to the Fermi level, and therefore increase the energy of charged excitations that add or remove particles at energies close to the Fermi level. A similar effect occurs in two-dimensional electron gases, and we expect that it
be derived in a variety of different ways. Density-response in a Landau level basis has been analyzed in the two-dimensional case using independent-particle time-dependent linear-response theory\textsuperscript{20} and by summing ladder diagrams\textsuperscript{21}, with identical results. Below we follow the linear-response-theory method\textsuperscript{19} in which we begin with the mean-field ground state and add a time-dependent external potential that couples to the reduced density-operator at wavevector $-q$:

$$\mathcal{H} = \mathcal{H}_{HF} + V_{\text{ext}}^{\tau'}(-q, \omega) e^{-i(\omega+in)t} \Delta k_{\tau'}(-q, \omega).$$ \hspace{1cm} (18)$$

where $\lim_{\eta \to 0^+}$ is implied wherever $\eta$ appears. The external potential induces time-dependent variations in the ground-state expectation value of the reduced density-operator, which are captured by linear-response theory when the perturbation is weak, i.e.

$$\delta \langle \Delta k_{\tau'}(q, \omega) \rangle = \chi_{k,\tau,k',\tau'}(q, \omega) V_{k,\tau'}^{\text{ext}}(-q, \omega) .$$ \hspace{1cm} (19)$$

In Eq. (19) the reduced density-operator response-function is

$$\chi_{k,\tau,k',\tau'}(q, \omega) = -\frac{i}{\hbar} \int_0^\infty dt \langle [\Delta k_{\tau'}(q, t), \Delta k_{\tau'}(-q)] \rangle e^{i(\omega+in)t} ,$$ \hspace{1cm} (20)$$

and

$$\Delta k_{\tau'}(q, t) = e^{i\mathcal{H}_{HF}t/\hbar} \Delta k_{\tau'}(q) e^{-i\mathcal{H}_{HF}t/\hbar} .$$ \hspace{1cm} (21)$$

We evaluate the right-hand side of Eq. (20) using the Hartree-Fock quasiparticle energies discussed above to obtain the following proper\textsuperscript{19} reduced-density-operator response function:

$$\Delta k_{\tau'}(q, \omega) = \chi_{k,\tau,k',\tau'}(q, \omega) V_{k,\tau'}^{\text{scr}} \delta_{\tau,\tau'} \delta_{k,\tau+q,\tau'}$$

$$\times \frac{n^{(V)}(\xi_{k,\tau}) - n^{(V)}(\xi_{k',\tau})}{\hbar \omega + \xi_{k,\tau} - \xi_{k',\tau} + i\eta} .$$ \hspace{1cm} (22)$$

which satisfies

$$\delta \langle \Delta k_{\tau'}(q, \omega) \rangle = \chi_{k,\tau,k',\tau'}(q, \omega) V_{k,\tau'}^{\text{scr}}(-q, \omega) ,$$ \hspace{1cm} (23)$$

where the screened interaction is simply the external interaction plus the induced interaction created by fluctuations in the Hartree-Fock potential, $V_{k,\tau'}^{\text{scr}}(-q, \omega) = V_{k,\tau'}^{\text{ext}}(-q, \omega) + V_{k,\tau'}^{\text{ind}}(-q, \omega)$ and

$$V_{k,\tau'}^{\text{ind}}(q, \omega) = \sum_{k',\tau''} \left( \frac{\delta V_{k,\tau'}^{\text{HF}}(q, \omega)}{\delta \langle \Delta k_{\tau''}^{-\tau'}(q) \rangle} \right) \delta \langle \Delta k_{\tau''}^{-\tau'}(q, \omega) \rangle .$$ \hspace{1cm} (24)$$

Eq. (23) and Eq. (24) together yield a matrix equation for the reduced density-matrix response at each wavevector $q$:
\[ \sum_{k'_{z},\tau'} \left( \delta_{k_{z}k'_{z}} \delta_{\tau_{z},\tau'} \hbar \omega - [M]_{k_{z},k'_{z}\tau_{z},\tau'} \right) \delta(\Delta k'_{z},\tau') = \frac{g}{2} \left\{ n^{(F)}(E_{k_{z}+q_{z}q_{y}y}) - n^{(F)}(E_{k_{z}}) \right\} \delta_{k_{z}+q_{z}k'_{z}} \delta_{\tau_{z},\tau'} V_{\tau_{z},\tau'}^{0} \]  

where the excitation-Hamiltonian matrix  

\[ [M]_{k_{z},k'_{z}\tau_{z},\tau'} = \delta_{k_{z},k'_{z}} \delta_{\tau_{z},\tau'} (E_{k_{z}+q_{z}q_{y}y} - E_{k_{z}}) + \frac{1}{g} \sum_{q} \frac{V_{\tau_{z},\tau'}^{0}}{\delta_{k_{z}-q_{z}k'_{z}}} e^{-\left(\Delta k'_{z},\tau' \right)\hbar \omega / 2} e^{i\left(q_{z}q_{y}y - q_{z}q_{y}y\right)\hbar \omega / 2} \]  

has rows labeled by \( \{ k_{z},\tau \} \) and columns labeled by \( \{ k'_{z},\tau' \} \). When the generalized random-phase-approximation term is derived using diagrammatic perturbation theory, the interaction contribution to the matrix \([M]_{k_{z},k'_{z}\tau_{z},\tau'}\) arises from ladder-diagram vertex corrections to response functions.

To solve this equation numerically we apply periodic boundary conditions in the \( \hat{z} \) direction to discretize wavevectors along the field in integer units of \( 2\pi / L_{z} \). (The discretization is emphasized in Figure 1.) For each value of \( Q = \{ q_{x}L_{x}, q_{y}L_{y} / 2, q_{z}L_{z} / 2 \} \), Eq. (20) is a square \( 4Q_{x} \times 4Q_{z} \) matrix whose eigenvalues give the neutral elementary excitations of the system within the time-dependent Hartree-Fock approximation. The excitations include both isolated collective modes and incoherent quasiparticle excitations that approach continuous bands when \( L_{z} \rightarrow \infty \). Because the excitation Hamiltonian is diagonal in \( q_{x} \) and \( q_{y} \), the limits \( L_{x}, L_{y} \rightarrow \infty \), although required to converge integrals, do not increase the excitation Hamiltonian size. The collective mode excitations can be distinguished by the property that their eigenvectors have approximately equal amplitude components from all contributing particle-hole excitations. Because we use values of \( L_{z} \) large enough to reveal the physics of the thermodynamic limit, the filling factor of the highest occupied Landau levels, which could be fractional in principle, plays no role. We therefore assume that that the Fermi factors \( n^{(F)}(E_{k_{z}y}) \) are either zero or one wherever they appear.

IV. RESULTS

In Figure 3 we plot our results for the particle-hole excitation spectrum of Cd_{3}As_{2} at two different magnetic field strengths, \( B = 10 \) Tesla and \( B = 40 \) Tesla. These results were calculated for the Fermi level at the Dirac point and wavevector \( q = q_{z} \hat{z} \) oriented along the direction of magnetic field. The length scales defined by the band structure and by the magnetic field are related by \( k_{c}l_{B} = 0.26 \) at \( B = 10 \) Tesla and by \( k_{c}l_{B} = 0.13 \) at \( B = 40 \) Tesla. It is interesting to compare these results with the analytic expression that can be derived in the

the random phase approximation\(^{22} \)

\[ \hbar \omega_{pl} = \frac{q_{z}}{|q|} e^{-\left(q_{z}^{2} + q_{z}^{2}\right)\hbar \omega / 4} \frac{4e^{2}a_{B}k_{c}}{\pi |k_{c}^{2}l_{B}^{2}|}, \]  

where \( a_{B} \) is the Bohr radius and \( m^{*} = m^{*} / m \).

The most striking aspect of both our numerical generalized random-phase approximation and the the analytic results for the simple random phase approximation is the finite value of the collective excitation energy for \( (q_{x},q_{y}) = (0,0) \) and \( q_{z} \rightarrow 0 \):

\[ \hbar^{2}l_{pl} = 4\pi e^{2} \frac{4e^{2}a_{B}k_{c}}{\pi |k_{c}^{2}l_{B}^{2}|} = 4\alpha_{fs} \times \Delta . \]  

In fact we find that the \( q_{z} \rightarrow 0 \) limit of the generalized random-phase approximation is in perfect agreement with the RPA result in spite of the interaction corrections to the quasiparticle energies and the vertex corrections to the excitation Hamiltonian. This cancellation of beyond-RPA interaction effects is in fact expected and is most simply understood from the fluctuation action derivation of the generalized random-phase approximation explained in Appendix A. For \( (q_{x},q_{y}) = (0,0) \) and \( q_{z} \rightarrow 0 \) particle-hole excitations are made by promoting an electron from an occupied value of \( k_{z} \) to an empty value of \( k_{z} \), while retaining the same \( N = 0 \) Landau level state. These quantum fluctuations are therefore very analogous to those of a one-dimensional electron gas\(^{23} \), which can be described using a bosonization approach that is related to the fluctuation action of the Appendix. For each flavor \( \tau \), the fields constructed from density-fluctuations on the two-sides of the Fermi surface that are in phase and from density-fluctuations that are out of phase are canonically conjugate. The collective-mode energy is therefore the geometric mean of the energy coefficients of the in-phase and out-of-phase fields. For the in-phase fluctuations, the coefficient is dominated at long-wavelengths by the long-range Coulomb interaction energy \( 4\pi e^{2} / \kappa q_{z}^{2} \). In phase fluctuations on the other hand give an energy contribution that is associated with simply shifting the one-dimensional Fermi surface in momentum space and does not involve interactions at all and is proportional to the velocity \( v_{D} \) and to the Landau level degeneracy. Because the square of the plasma
FIG. 3: (Color online) Energy versus wavevector $q_z$ plots for the magnetoplasmon collective mode (green squares) and for the incoherent particle-hole pair (gray crosses) excitations of Cd$_3$As$_2$ when the chemical potential is at the Dirac point. These plots are for $(q_x, q_y) = (0, 0)$ The magnetic field is oriented along the direction that separates the two Dirac points in momentum space and has magnitude 10 Tesla in panel a) and 40 Tesla in panel b). A single collective mode is present when the chemical potential lies at the Dirac point. All calculations of plasmon and electron-hole spectra were converged with respect to $L_z$; these particular calculations were performed for $k_c L_z \sim 150$.

frequency is the product of a factor that is totally dominated by the simple mean-field electrostatic energy, and a factor that does not involve interactions at all, it is given exactly by the random phase approximation result. This property is of course closely related to the analogous property in a three-dimensional electron gas in the absence of a magnetic field. Corrections to the long-wavelength collective mode energy do appear if we allow for the weak flavor dependence of electron-electron matrix elements. In the current model, corrections to the RPA become important at finite $q$ and acting in concert to increase magnetoplasmon energies. At large wavevec-

FIG. 4: (Color online) Energy versus wavevector plots for the coherent optical magnetoplasmon excitation (green squares), the coherent acoustic magnetoplasmon excitation (blue circles), and the incoherent particle-hole excitations (gray crosses) when Cd$_3$As$_2$ is doped to a density of $1.25 \times 10^{19}$ cm$^{-3}$ electrons above the Dirac point. The magnetic field is oriented along the direction separating the Dirac points in momentum space and has magnitudes 10 Tesla and 40 Tesla, in Figure a) and Figure b), respectively. In these plots the excitation wavevector is in the direction of the magnetic field (taken to be the $\hat{z}$ direction). Because particle-hole symmetry is broken in Eq. (1) when the chemical potential is away from the Dirac point two collective magnetoplasmon modes are present. (The chemical potential is above the Dirac point by $\sim 8$ meV in this Figure.) Each inset is the same as the corresponding main figure, but provides a more detailed view of the small wavevector region illustrating the gap in the particle-hole excitation spectrum where the acoustic collective mode resides. All calculations of plasmon and electron-hole spectra were converged with respect to $L_z$; these particular calculations were performed for $k_c L_z \sim 120$. 

tors the magnetoplasmon dispersion is tangential to the top of the particle-hole continuum (see Figure 3 which is increased in energy by the exchange self-energy corrections illustrated in Figure 2). Because the boundaries of the particle-hole continuum are shifted up in energy, the magnetoplasmon merges with the electron-hole excitations at smaller $q$ than at the RPA level.

In Figure 4 we plot the magnetoplasmon mode energies of Cd$_3$As$_2$ doped to a density of $1.25 \times 10^{19}$ cm$^{-3}$ electrons. In this case the chemical potential is raised $\sim$ 8 meV above the Landau level band crossing in Figure 1), and the hole-density in band $\tau = -1$ is much less than the electron-density in band $\tau = +1$. Charge density-fluctuations in the two bands now reveal an out-of-phase collective oscillation which is buried in the particle-hole excitation spectrum when the system has particle-hole symmetry. We refer to the higher energy collective modes as the optical magnetoplasmon, and to the newly apparent lower energy collective mode as the acoustic magnetoplasmon. The latter is gapless but is not a Goldstone mode, as no continuous symmetry has been broken. The acoustic magnetoplasmon is undamped by single electron-hole excitations at long wavelengths, as emphasized in the inset to Figure 4. This feature originates from the unique phase space for electron-hole excitations in models with one-dimensional dispersion as in Eq. 1. In the generalized random phase approximation, both the optical and acoustic magnetoplasmons enter the particle-hole continuum at intermediate wavevectors, where they are strongly damped. When detected experimentally, the presence of an acoustic magneto-plasmon mode demonstrates that the Fermi level is not at the Dirac point.

In Figure 5 we plot excitation spectra calculated at finite transverse wavevector. Comparing Figure 3 and 5 we see that a finite transverse wavevector shifts the magneto-plasmon mode energy down at small $q_z$, but up at intermediate values of $q_z$. At a small but non-zero transverse wavevector, the magnetoplasmon mode energy no longer has a finite limit as $q_z \to 0$. This property is different from the case in the absence of a magnetic field for which the magneto-plasmon mode approaches a finite energy as $q_z \to 0$ at small non-zero transverse wavevectors. The difference can be traced to the property that particle-hole excitation energies depend only on $q_z$ in the strong magnetic field limit, and not on transverse wavevectors. (The zero-field semiclassical magnetoplasmon mode actually evolves into an inter-Landau level transition in the presence of magnetic field which lies outside the Hilbert space considered here.) The increase in the collective mode energy at intermediate wavevectors can be traced to vertex corrections that lower collective mode energies and partially cancel self-energy shifts as discussed above, which are weakened at finite transverse wavevectors.

Also notable in Figure 5 is the absence of a shift in the incoherent particle-hole excitation spectrum with transverse wavevector, which has important implications for interaction effects that lie beyond the reach of the generalized random phase approximation. In the absence of a magnetic field the density-of-states for particle-hole-excitations of the Fermi sea famously vanishes linearly with excitation energy. This vanishing density of states is responsible for a diverging lifetime of low-energy charged excitations of a Fermi sea, for the validity of Landau’s Fermi-liquid ansatz, and for the absence of any phase changes associated with weak repulsive interactions. In the strong field limit here, the particle hole excitation spectrum is independent of transverse wavevector and its density-of-states remains finite as excitation energy.

FIG. 5: (Color online) Energy versus wavevector $q_z$ plots for the magneto-plasmon collective mode (green squares) and for the incoherent particle-hole pair (gray crosses) excitations of Cd$_3$As$_2$ when the chemical potential is at the Dirac point. In panel a) we have fixed $\sqrt{q_x^2 + q_y^2} = 0.1k_c$ and in panel b) we have fixed $\sqrt{q_x^2 + q_y^2} = 0.5k_c$. The magnetic field is oriented along the direction that separates the two Dirac points in momentum space and has magnitude of 40 Tesla. All calculations of plasmon and electron-hole spectra were converged with respect to $L_z$; these particular calculations were performed for $k_cL_z \sim 120$. 
vanishes. This property suggests that in the case of one-dimensional systems, interactions destroy the discontinuity in momentum space occupation probabilities that are present for non-interacting systems and survive interactions in the case of a Fermi liquid. The correlations established in the strong-field limit of Weyl and Dirac semimetals are however three-dimensional and distinctly different from those of a one-dimensional electron system.

V. CONCLUSION

This paper analyses the application of time-dependent mean-field theory, also known as the generalized random-phase approximation and the time-dependent Hartree-Fock theory (TDHF), to the strong field limit of a Dirac semimetal, using Cd$_2$As$_3$ as an example. In this limit, the low-energy Hilbert space can be projected to a single electron-like and a single hole-like Landau level at each value of momentum $q_z$ along the magnetic-field direction. We find that in the long-wavelength limit, self-energy and vertex exchange-interaction corrections to the simple random phase approximation cancel in the collective mode spectrum. Within TDHF this property follows from the long-range of the Coulomb interaction and from the invariance of the interaction energy under a uniform translation in momentum space. In fact, we believe that the random-phase approximation result for the long-wavelength limit of the magneto-plasmon collective mode energy is the exact value for the long-wavelength collective mode energy of the model we study, whose key property in this respect is that the Coulomb interaction in momentum space depends only on momentum transfer and not on individual momenta. Even when all realistic atomic scale corrections to interaction matrix elements are included, it seems unlikely that there will be substantial corrections to the long-wavelength limit, and therefore that this quantity is relatively insensitive to the precise character of the many-electron ground state.

The collective mode dispersion is more interesting when the semimetal is doped so that it does not satisfy particle-hole symmetry and or when it is examined at finite transverse wave-vector. In the former case an acoustic magneto-plasmon collective mode emerges at lower energy than the main optical magneto-plasmon. In the latter case, the magneto-plasmon dispersion has a linear dependence on momentum $q_z$ as $q_z \rightarrow 0$, but stiffens relative to the vanishing transverse wavevector case at intermediate $q_z$. In the GRPA, the typical energy of the particle-hole continuum does not increase with transverse wavevector, as it would in the absence of a magnetic field. This property leads to a particle-hole density-of-states that remains finite as the excitation energy goes to zero, and therefore to a quasiparticle lifetime that vanishes only linearly as the energy of a quasiparticle relative to the Fermi energy $\xi$ vanishes, and to the absence of a discontinuity in momentum-state $k_z$ occupation at the non-interacting Fermi wavevector. We conclude that the ground state is always profoundly altered by interactions, although experimental strong field anomalies in three-dimensional electron gas systems tend to be modest, and there are indications that charge-density-wave states occur in some cases at least. The weak-coupling theory of interaction effects whose application we have described is not able to determine which, if any, accessible observables exhibit evidence of strong correlation physics, except to observe that long-wavelength collective charge-density fluctuations are unlikely to be the smoking gun of unusual electronic properties.

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Appendix A: Variational Approach to Collective mode spectra

The character of the collective magnetoplasmon modes can be further understood by a complementary derivation of the time-dependent Hartree-Fock expressions wherein a variational wave function is considered. In general, small variations on top of a Slater-determinant ground state $|G\rangle$ can be described by the following wave function $|\Psi\rangle$:

$$|\Psi\rangle = \prod_{\alpha} \left[ 1 - \frac{1}{2} \sum_{\beta} |z_{\alpha\beta}|^2 + \sum_{\beta} z_{\alpha\beta} c_{\alpha\beta}^\dagger c_{\alpha\beta} \right] |G\rangle,$$

(A1)

where $\alpha$ and $\beta$ respectively label occupied and unoccupied Hartree-Fock orbitals of a definite $k_z$, $X$ and $\tau$. $z_{\alpha\beta}$ is a variational parameter, which is small for small fluctuations. We construct a Lagrangian using $|\Psi\rangle$ to study fluctuation dynamics:

$$L = \langle \Psi | i\hbar \partial_t - \mathcal{H} | \Psi \rangle.$$

(A2)

The time dynamical part has the following form:

$$\mathcal{E} = \langle \Psi | i\hbar \partial_t | \Psi \rangle$$

$$= \frac{i\hbar}{2} \sum_{(\alpha\beta)} \left[ z_{\alpha\beta}^* \partial_z z_{\alpha\beta} - z_{\alpha\beta} \partial_z z_{\alpha\beta}^* \right]$$

$$= \frac{i\hbar}{2} \sum_{(\alpha\beta)} \left\{ n^{(F)}(\varepsilon_{\alpha}) - n^{(F)}(\varepsilon_{\beta}) \right\} \lambda_{\beta\alpha} \partial_z \lambda_{\alpha\beta},$$

(A3)

where $n^{(F)}(\varepsilon_{\alpha})$ is the occupation number of state $\alpha$ in ground state $|G\rangle$. $(\alpha\beta)$ enforces that $\alpha$ (or $\beta$) is an occupied (or unoccupied) state in $|G\rangle$, while $(\alpha\beta)$ restricts the summation to pairs of $\alpha$ and $\beta$ such that $(n^{(F)}(\varepsilon_{\alpha}) - n^{(F)}(\varepsilon_{\beta}))$
is non-zero. The parameter $\lambda_{\alpha\beta}$ is defined as:

$$\lambda_{\alpha\beta} = \left\{ \begin{array}{ll} z_{\alpha\beta} & \text{for } n^{(F)}(\varepsilon_\alpha) - n^{(F)}(\varepsilon_\beta) = 1 \\
\frac{2}{z_{\alpha\beta}} & \text{for } n^{(F)}(\varepsilon_\alpha) - n^{(F)}(\varepsilon_\beta) = -1 \end{array} \right.. \tag{A4}$$

The energy functional part expanded to second-order in $z_{\alpha\beta}$ is:

$$\delta E^{(2)} = \langle \Psi | H | \Psi \rangle
\approx \frac{1}{2} \sum_{(\alpha\beta)} (\xi_\beta - \xi_\alpha) \left\{ n^{(F)}(\varepsilon_\alpha) - n^{(F)}(\varepsilon_\beta) \right\} \lambda_{\alpha\beta} \lambda_{\beta\alpha}$$
$$+ \frac{1}{2} \sum_{(\alpha\beta)} \sum_{(\alpha'\beta')} (v_{\alpha\alpha'\beta'\beta} - v_{\alpha'\alpha\beta'\beta}) \lambda_{\alpha\beta} \lambda_{\alpha'\beta'}, \tag{A5}$$

where $\xi_\alpha$ is the Hartree-Fock quasiparticle energy of state $\alpha$, and $v_{\alpha\alpha'\beta'\beta}$ is the two-particle interaction matrix element.

To make full use of magnetic translational symmetry, we perform the following transformation:

$$\lambda_{k_x\tau, k_x'\tau'} = \sum_Q \delta_{Q, k_x - k_x} \delta_{Q, k_x' - k_x'} \varepsilon^{Q_x Q_x'} e^{iQ_x (X + X')/2}$$
$$\times \Delta_{k_x \tau}(\vec{Q}). \tag{A6}$$

Because $\lambda_{\alpha\beta} = \lambda_{\beta\alpha}^*$, by the definition in Eq. (A4), we have the relation:

$$\Delta_{(k_x + Q_x)\tau}(\vec{Q}) = \Delta_{k_x \tau}(\vec{Q}). \tag{A7}$$

The Lagrangian expressed in terms of the new parameter $\Delta_{k_x \tau}(\vec{Q})$ is $L = \mathcal{B} - \delta E^{(2)}$ where

$$\mathcal{B} = \frac{i\hbar q}{2} \sum_{k_x \tau} \left\{ n^{(F)}(\varepsilon_{k_x \tau}) - n^{(F)}(\varepsilon_{k_x + Q_x \tau}) \right\}$$
$$\times \Delta_{k_x \tau}^*(\vec{Q}) \partial_{\tau} \Delta_{k_x \tau}(\vec{Q}), \tag{A8}$$

and

$$\delta E^{(2)} = g \sum_{\vec{Q}} \sum_{k_x \tau} \sum_{k_x' \tau'} \left\{ \mathcal{A}_{k_x \tau, k_x' \tau'}(\vec{Q}) \Delta_{k_x \tau}(\vec{Q}) \Delta_{k_x' \tau'}(\vec{Q}) \right\}$$
$$+ \frac{\Gamma_{k_x \tau, k_x' \tau'}(\vec{Q})}{2} |\Delta_{k_x \tau}(\vec{Q})\Delta_{(k_x + Q_x)\tau}^*(\vec{Q}) + c.c.|, \tag{A9}$$

where $[k_x \tau]$ puts the following constraint:

$$n^{(F)}(\varepsilon_{k_x \tau}) - n^{(F)}(\varepsilon_{k_x + Q_x \tau}) = +1. \tag{A10}$$

$\mathcal{A}$ and $\Gamma$ are given by:

$$\mathcal{A}_{k_x \tau, k_x' \tau'}(\vec{Q}) = \mathcal{A}(\mathcal{X} - \mathcal{D})_{k_x \tau, k_x' \tau'}, \tag{A11}$$
$$\Gamma_{k_x \tau, k_x' \tau'}(\vec{Q}) = \mathcal{X}(\mathcal{X} - \mathcal{D})_{k_x \tau, (k_x + Q_x)\tau'}, \tag{A12}$$

The Euler-Lagrange equation of the Lagrangian $L$ is:

$$\frac{\partial L}{\partial (\Delta_{k_x \tau}(\vec{Q}))} - \frac{\partial}{\partial \tau} \frac{\partial L}{\partial (\partial_{\tau} \Delta_{k_x \tau}(\vec{Q}))} = 0, \tag{A13}$$

which leads to the equation of motion for the reduced density matrix:

$$i\hbar \partial_{\tau} \Delta_{k_x \tau}(\vec{Q}) = \sum_{k_x' \tau'} \left\{ \mathcal{A}_{k_x \tau, k_x' \tau'}(\vec{Q}) \Delta_{k_x \tau}(\vec{Q}) \right\}$$
$$\times \left\{ \frac{1}{2L^2} \sum_q \left[ \varepsilon^{Q_x Q_x'} e^{-Q_x^2 \hbar^2/2} - \sum_{\tau'} \delta_{Q_x, k_x' - k_x} v_{Q_x} e^{-Q_x^2 \hbar^2/2} \cos(I_{2Q_x}(\vec{Q} \times \vec{Q} \cdot \hat{z})) \right] \Delta_{k_x' \tau'}(\vec{Q}) \right\}. \tag{A14}$$

We now show that Eq. (A12) can be further simplified. To this end, we reorganize $\mathcal{B}$ and $\delta E^{(2)}$ by utilizing the relation in Eq. (A7):

$$\mathcal{B} = \frac{i\hbar q}{2} \sum_{\vec{Q}} \sum_{k_x \tau} \left\{ \Delta_{k_x \tau}(\vec{Q}) \partial_{\tau} \Delta_{k_x \tau}(\vec{Q}) - \Delta_{k_x \tau}(\vec{Q}) \partial_{\tau} \Delta_{k_x \tau}(\vec{Q}) \right\}, \tag{A15}$$

$$\delta E^{(2)} = \frac{g}{2} \sum_{\vec{Q}} \sum_{k_x \tau} \sum_{k_x' \tau'} \left\{ \mathcal{A}_{k_x \tau, k_x' \tau'}(\vec{Q}) \Delta_{k_x \tau}(\vec{Q}) \Delta_{k_x' \tau'}(\vec{Q}) \right\}$$
$$+ \frac{\Gamma_{k_x \tau, k_x' \tau'}(\vec{Q})}{2} |\Delta_{k_x \tau}(\vec{Q})\Delta_{(k_x + Q_x)\tau}^*(\vec{Q}) + c.c.|, \tag{A16}$$

where $[k_x \tau]$ puts the following constraint:

$$n^{(F)}(\varepsilon_{k_x \tau}) - n^{(F)}(\varepsilon_{k_x + Q_x \tau}) = +1. \tag{A17}$$

$\mathcal{A}$ and $\Gamma$ are given by:

$$\mathcal{A}_{k_x \tau, k_x' \tau'}(\vec{Q}) = \mathcal{A}(\mathcal{X} - \mathcal{D})_{k_x \tau, k_x' \tau'}, \tag{A18}$$
$$\Gamma_{k_x \tau, k_x' \tau'}(\vec{Q}) = \mathcal{X}(\mathcal{X} - \mathcal{D})_{k_x \tau, (k_x + Q_x)\tau'}, \tag{A19}$$
To formally decouple \( \pm \bar{Q} \) contributions in Eq. (A14), we perform the following transformation:

\[
\Delta_{k\tau}^{(+\tau)}(\bar{Q}) = \Lambda_{k\tau}(\bar{Q}) + i\Upsilon_{k\tau}(\bar{Q})
\]

(A17)

\[
\Delta_{-k\tau}^{(-\tau)}(\bar{Q}) = \Lambda_{k\tau}(\bar{Q}) - i\Upsilon_{k\tau}(\bar{Q}).
\]

(A18)

Note that \( \Lambda_{k\tau}(\bar{Q}) \) and \( \Upsilon_{k\tau}(\bar{Q}) \) are still complex variables, and there is a redundancy,

\[
\Lambda_{(-k\tau)}^{(-\tau)}(\bar{Q}) = \Lambda_{k\tau}^{(+\tau)}(\bar{Q}), \quad \Upsilon_{(-k\tau)}^{(-\tau)}(\bar{Q}) = \Upsilon_{k\tau}^{(+\tau)}(\bar{Q}).
\]

(A19)

In terms of \( \Lambda \) and \( \Upsilon \), \( B \) and \( \delta E^{(2)} \) can be expressed as:

\[
B = \frac{\hbar}{2} \sum_{\bar{Q}} \sum_{[k\tau]} (\Lambda^* \partial_t \Lambda + \Upsilon \partial_t \Lambda^*)
\]

\[
- \Lambda^* \partial_t \Upsilon - \Lambda \partial_t \Upsilon^* \bigg)_{k\tau}(\bar{Q}),
\]

\[
\delta E^{(2)} = g \sum_{\bar{Q}} \sum_{[k\tau]} \sum_{[k'\tau']} \left( \Lambda_{k\tau}^{(+\tau)} \nu_{k\tau'}^{(-\tau')} \Lambda_{k'\tau'}^{(\tau')} - \Upsilon_{k\tau}^{(+\tau)} \nu_{k\tau'}^{(-\tau')} \Upsilon_{k'\tau'}^{(\tau')} \right)
\]

\[
+ \Upsilon_{k\tau}^{(+\tau)} \nu_{k\tau'}^{(-\tau')} \Upsilon_{k'\tau'}^{(\tau')} \bigg) \frac{1}{2} (\bar{Q}),
\]

(A20)

Therefore the energy of the collective mode is given by the square root of the eigenvalues of the matrix product \( K^{(\text{+})} K^{(\text{-})} \). Note that the dimension of \( K^{(\text{+})} K^{(\text{-})} \) is half of that of the matrix in Eq. (A12), because of the constraint in Eq. (A13). Thus Eq. (A22) can be viewed as a simplified version of Eq. (A12).