D–INSTANTON CORRECTIONS AS (p,q)–STRING EFFECTS
AND NON–RENORMALIZATION THEOREMS*

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Abstract

We discuss higher derivative interactions in the type IIB superstring in ten dimensions. From the fundamental string point of view, the non-perturbative corrections are due to D-instantons. We argue that they can alternatively be understood as arising from (p, q)-strings. We derive a non-renormalization theorem for eight-derivative bosonic interactions, which states that terms involving either NS-NS or R-R fields occur at tree-level and one-loop only. By using the $SL(2,\mathbb{Z})$ symmetry of M-theory on $T^2$, we show that in order for the possible $R^{3m+1}$ ($m = 1, 2, ...$) interactions in M-theory to have a consistent perturbative expansion in nine dimensions, $m$ must be odd. Thus, only $R^{6N+4}$ ($N = 0, 1, ...$) terms can be present in M-theory and their string theory counterparts arise at $N$ and $2N + 1$ loops. Finally, we treat an example of fermionic term.

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1. Introduction

One of the lessons of the recent developments in string theory is that there is a multitude of theories, whose fundamental objects are of different nature and which however give rise to completely equivalent physics. At the present time, all ten-dimensional superstring theories are related to one another either after being compactified to lower dimensions or being interpreted as different limits of more fundamental theories such as M- or F- theory in eleven and twelve dimensions, respectively. In this paper, we will concentrate on the type IIB string and we will show how the standard interpretation of its effective action in terms of perturbative string and non-perturbative instanton corrections can alternatively be interpreted as arising from the \((p,q)\)-strings. In general, besides these \((p,q)\)-strings \([1]\), the various type IIB branes consist in solitons, which are the self-dual three-brane, an \(SL(2,\mathbb{Z})\) multiplet of five-branes, the seven-brane and finally the D-instanton. These \(p\)-branes play an important role in the theory in dimensions lower than or equal to \(10 - (p + 1)\) since only in these cases can their Euclidean \((p + 1)\)-world volume wrap around cycles of the compact space and contribute to the effective action as standard instantons. However, since we will discuss here the type IIB superstring in dimension nine or ten only, it is natural to think that beside string loop calculations, the non-perturbative corrections can only arise from D-instantons. This is in fact precisely what was shown in \([2]\). Since the D-instanton breaks half of the supersymmetries \([3]\), the sixteen fermionic zero modes, which are generated by the instanton background, can be saturated by eight derivatives. These interactions are of order \(\alpha'^3\) and some of them have been evaluated either in \(\sigma\)-model perturbation theory \([4]\) or by a direct string-amplitude calculation \([5] - [8]\). Besides the perturbative corrections, the form of the non-perturbative ones, which arise from multiply-charged one-instantons (anti-instantons) backgrounds, for the \(R^4\) term, has been discussed in \([2]\) and a non-renormalization theorem has been proved in \([9]\). This particular term has also been conjectured to exist in the eleven-dimensional M-theory \([10, 11]\) and compactifications of
the latter give results consistent with string-theory expectations \([12]–[17]\). The \(SL(2, Z)\) invariant four-point amplitude has been worked out in \([18]\).

The type IIB supergravity has a \(U(1)\) symmetry that rotates the two supercharges \([19]\). Although the \(R^4\) term is neutral under this \(U(1)\), there are other eight-derivative interactions that have a non-zero charge. The form of the non-perturbative corrections to such charged terms has been proposed in \([20]\). More precisely, the tree-level four-point effective action at order \(\alpha'^3\), which contains all the fields in the NS-NS sector, has been completed in an \(SL(2, Z)-\)invariant way. The resulting action involves the R-R fields (except the four-form which is \(SL(2, Z)-\)invariant) and it is consistent with the D-instanton interpretation.

Here, we derive the form of the four-point effective action of \([20]\) and extend it to other eight derivative interactions. This derivation is based on the fact that the tree-level eight-derivative effective action in the NS-NS sector is invariant under a subgroup \(\Gamma_\infty\) of \(SL(2, Z)\). Therefore, \(SL(2, Z)\) invariance must be achieved by taking the orbit of each term under \(\Gamma_\infty \backslash SL(2, Z)\) whose elements are in one-to-one correspondence with the pairs \(p, q\) of integers whose g.c.d. is 1. In this case, the perturbative as well as the non-perturbative corrections can alternatively be viewed as arising from \((p, q)\)-strings rather than fundamental string loops and D-instantons.

In should be noted that although we have started from the tree-level eight derivative interactions of the NS-NS fields only, we expect from the symmetries that there also exist interactions involving \(2r\) \((r = 1, \ldots, 4)\) R-R fields and arising at tree level. Their precise normalization should be evaluated by a string calculation and their orbit under \(\Gamma_\infty \backslash SL(2, Z)\) is then completely determined. By showing that S-duality relates \(n\)-loop to \((1 - n)\)-loop corrections for the eight-derivative interactions, we obtain a non-renormalization theorem in type IIB theory. According to this, an eight-derivative term involving either NS-NS or R-R fields has only tree level and one-loop perturbative corrections.
The method used for the determination of the form of the eight-derivative interactions can also be applied to determine the perturbative and non-perturbative corrections of other terms as well. We discuss as an example the $\lambda^{16}$ fermionic interaction recently considered in the literature, where $\lambda$ is the complex dilatino. Our result is in perfect agreement with the M-theory analysis of [21]. We also consider $6m + 2$-derivative interactions of the form $R^{3m+1}$ which are terms consistent with eleven-dimensional Lorentz invariance in M-theory [12]. By applying the $SL(2,\mathbb{Z})$ symmetry in nine dimensions [22], we find that the allowed interactions are of the form $R^{6N+4}$ ($N = 0, 1, ...$) and that they occur at $N$ and $2N + 1$ loops only.

In the next section, we recall some known results for the type IIB theory. In section 3 we employ the S-duality symmetry in order to determine the perturbative and non-perturbative corrections to eight-derivative interactions. In section 4 we derive a non-renormalization theorem for the eight-derivative bosonic interactions and in section 5 we consider fermionic as well as $R^{3m+1}$ terms. Finally, in Appendix A we explicitly give the $SL(2,\mathbb{Z})$ invariant effective action derived from the tree-level NS-NS sector only and in Appendix B we present some properties of the non-holomorphic modular forms we are using.

2. Tree-level NS-NS sector

The massless bosonic spectrum of the type IIB superstring theory consists in the graviton $g_{MN}$, the dilaton $\phi$ and the antisymmetric tensor $B^1_{MN}$ in the NS-NS sector, while in the R-R sector it contains the axion $\chi$, the two-form $B^2_{MN}$ and the self-dual four-form field $A_{MNPQ}$. The fermionic superpartners are a complex Weyl gravitino $\psi_M$ and a complex Weyl dilatino $\lambda$. The theory has two supersymmetries generated by two supercharges of the same chirality. It has in addition a conserved $U(1)$ charge, which generates rotations of the two supersymmetries and under which some of the fields are charged [19]. In particular, the graviton and the four-form field are neutral, the antisymmetric tensors have charge
$q = 1$, the scalars have $q = 2$, whereas the gravitino and the dilatino have charges $q = 1/2$ and $q = 3/2$, respectively.

The two scalars of the theory can be combined into a complex one, $\tau = \tau_1 + i\tau_2$, defined by

$$\tau = \chi + ie^{-\phi},$$

which parametrizes an $SL(2, \mathbb{R})/U(1)$ coset space. At lowest order in $\alpha'$, the bosonic effective action of the type IIB string theory in the Einstein frame is written as

$$S_{3pt} = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-g} \left[ R - \frac{1}{2\tau_2} \partial_M \partial^M \tau - \frac{1}{12\tau_2} (\tau H^1 + H^2)_{KMN} (\bar{\tau} H^1 + H^2)_{KMN} \right].$$

where $H^a_{KMN} = \partial_K B^a_{MN} + \text{cyclic}$ for $a = 1, 2$, $\kappa_{10}^2 = 2\alpha'\pi^7\alpha'$ and we have set the four-form to zero. It has an $SL(2, \mathbb{R})$ symmetry that acts as

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d}, \quad B^a_{MN} \rightarrow (\Lambda^T)^{-1} B^a_{MN}, \quad g_{MN} \rightarrow g_{MN}, \quad \Lambda = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL(2, \mathbb{R}),$$

while the fermions transform accordingly as

$$\psi_M \rightarrow \left( \frac{c\tau + d}{c\tau + d} \right)^{1/4} \psi_M, \quad \lambda \rightarrow \left( \frac{c\tau + d}{c\tau + d} \right)^{3/4} \lambda.$$ 

The action (2.2) can be written in the string frame,

$$S_{3pt} = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-G} \left\{ e^{-2\phi} \left[ R - \frac{1}{2} (\partial\phi)^2 - \frac{1}{12} (H^1)^2 \right] - \left[ \frac{1}{2} (\partial\chi)^2 + \frac{1}{12} (\chi H^1 + H^2) \right] \right\},$$

where $G_{MN} = e^{\phi/2} g_{MN}$ and $R_{MNPQ}$ are the metric and Riemann tensor in this frame, respectively. From this, we see that the tree level NS-NS sector kinetic terms have a $e^{-2\phi}$ normalization. In order for the R-R fields to have the same tree level normalization, we have to rescale the R-R fields as

$$\chi \rightarrow \chi' = e^\phi \chi, \quad B^2_{MN} \rightarrow B^2_{MN}' = e^\phi B^2_{MN}.$$ 

The next-to-leading order corrections to the tree-level effective action for the NS-NS sector can be calculated either in $\sigma$-model perturbation theory or by string amplitudes. At
the four-point level, they are written as

\[
S_{4\text{pt}}^{\text{tree}} = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-g} \left[ \frac{\alpha'^3}{3 \cdot 2^6 \zeta(3) \tau_2} t_8^{ABCDEFGH} t_8^{MNPQRSTU} + \frac{1}{8 \varepsilon_{10}} t_8^{ABCDEFGH} \varepsilon_{MNPQRSTU} t_{IJ} \right] \times \hat{R}_{ABMN} \hat{R}_{CDPQ} \hat{R}_{EPRS} \hat{R}_{GHTU} ,
\]

(2.7)

where

\[
\hat{R}_{MN}^{\ PQ} = R_{MN}^{\ PQ} + e^{-\phi/2} \nabla_{[M} H_1^{1\ PQ]} - g_{[M}^{[P} \nabla_N] \nabla^Q] \phi ,
\]

(2.8)

and we use the convention \( T_{[MN]} = \frac{1}{2} (T_{MN} - T_{NM}) \). The tensor \( t_8 \) is defined in [5] in terms of \( \eta^{MN} \) and \( \varepsilon_{10} \) is the totally antisymmetric symbol in ten dimensions. However, by general covariance, \( t_8 \) should be written in terms of the metric \( g^{MN} \) and the curved space analogue of \( \varepsilon_{10} \) should be understood.

As has been noted in [7], \( \hat{R}_{MNPQ} \) is the linearized Riemann tensor constructed from the connection

\[
\hat{\omega}_{AB} = \omega_{AB} + \frac{1}{2} e^{-\phi/2} H_1^{1\ AB} + \frac{1}{4} (\delta_{BM} \partial_A \phi - \delta_{AM} \partial_B \phi) ,
\]

(2.9)

where \( \hat{A}, \hat{B} \) are flat space indices. The generalized Riemann tensor defined by

\[
\hat{R}_{BMN} = \partial_M \hat{\omega}_{BN} - \partial_N \hat{\omega}_{BM} + \hat{\omega}_{CM} \hat{\omega}_{BN} - \hat{\omega}_{CN} \hat{\omega}_{BM}
\]

(2.10)

is then

\[
\hat{R}_{MN}^{\ PQ} = R_{MN}^{\ PQ} + e^{-\phi/2} \nabla_{[M} H_1^{1\ PQ]} - g_{[M}^{[P} \nabla_N] \nabla^Q] \phi
\]

\[
- e^{-\phi} H_1^{1\ C[P} H_1^{1\ Q]} + \frac{1}{4} g_{[M}^{[P} \nabla_N] \phi \nabla^Q] \phi - \frac{1}{2} g_{[M}^{[P} g_{N]}^{Q]} \partial_K \phi \partial_K \phi
\]

\[
- \frac{1}{2} e^{-\phi/2} g_{[M}^{[P} \phi H_1^{1\ Q]}_{MN} - \frac{1}{2} e^{-\phi/2} \partial_M \phi H_1^{1\ PQ} - \frac{1}{2} e^{-\phi/2} g_{[M}^{[P} H_1^{1\ Q]}_{N]} \partial_C \phi ,
\]

(2.11)

and the full tree-level interactions at order \( \alpha'^3 \) take the form

\[
S_{\text{tree}}^{\text{tree}} = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-g} \left[ \frac{\alpha'^3}{3 \cdot 2^6 \zeta(3) \tau_2} t_8^{ABCDEFGH} t_8^{MNPQRSTU} + \frac{1}{8 \varepsilon_{10}} t_8^{ABCDEFGH} \varepsilon_{MNPQRSTU} t_{IJ} \right] \times \hat{R}_{ABMN} \hat{R}_{CDPQ} \hat{R}_{EPRS} \hat{R}_{GHTU} .
\]

(2.12)
3. Non-perturbative corrections from S-duality

We will now proceed to determine perturbative and non-perturbative corrections to the tree-level NS-NS sector at $\alpha'^3$ order of the type IIB superstring theory. This will be worked out by employing the S-duality symmetry of the theory, which relates the NS-NS fields to the R-R ones.

It is easily seen from (2.3) that $\nabla_M H_{NPQ}^1$ and $\nabla_M \partial_N \phi$ do not transform covariantly under $SL(2, \mathbb{R})$. It is convenient to introduce $SL(2, \mathbb{R})$-covariant objects and, for this, let us recall that the complex scalar $\tau$ parametrizes a $SL(2, \mathbb{R})/U(1)$ coset space. In general, the group $SL(2, \mathbb{R})$ can be represented by a matrix $V_{\pm}^{\alpha \beta}$ [19, 21]:

$$V = \begin{pmatrix} V_+^1 & V_+^2 \\ V_-^1 & V_-^2 \end{pmatrix} = \frac{1}{\sqrt{-2i \tau_2}} \begin{pmatrix} \bar{\tau} e^{-i \theta} & \tau e^{i \theta} \\ e^{-i \theta} & e^{i \theta} \end{pmatrix}. \quad (3.1)$$

The local $U(1)$ is realized by the shift $\theta \rightarrow \theta + \Delta \theta$ and the global $SL(2, \mathbb{R})$ acts from the left. One may define the quantities

$$P_M = -\epsilon_{\alpha \beta} V_+^\alpha \partial_M V_+^\beta = i e^{2i \theta} \frac{\partial_M \tau}{2 \tau_2}, \quad Q_M = \frac{1}{2i} \epsilon_{\alpha \beta} V_+^\alpha \partial_M V_-^\beta = \partial_M \theta - \frac{\partial_M \tau_1}{2 \tau_2}, \quad (3.2)$$

where $Q_M$ is a composite $U(1)$ gauge connection and $P_M$ has charge $q = 2$. We also define the complex three-form

$$G_{KMN} = -\sqrt{2i} \delta_{\alpha \beta} V_+^\alpha H_{KMN}^{\beta} = -i e^{i \theta} \frac{1}{\sqrt{\tau_2}} \left( \tau H_{KMN}^1 + H_{KMN}^2 \right), \quad (3.3)$$

with charge $q = 1$. We fix the $U(1)$ gauge by choosing $\theta \equiv 0$ from now on. In this case, the global $SL(2, \mathbb{R})$ transformation is non-linearly realized and the various quantities in eqs.(3.2) and (3.3) transform as

$$P_M \rightarrow c \bar{\tau} + d P_M, \quad Q_M \rightarrow Q_M + \frac{1}{2i} \partial_M \ln \left( \frac{c \bar{\tau} + d}{c \tau + d} \right), \quad G_{KMN} \rightarrow \left( \frac{c \tau + d}{c \tau + d} \right)^{1/2} G_{KMN}. \quad (3.4)$$

We also define the covariant derivative $D_M = \nabla_M - iq Q_M$, which transforms under $SL(2, \mathbb{R})$ as

$$D_M \rightarrow \left( \frac{c \bar{\tau} + d}{c \tau + d} \right)^{q/2} D_M. \quad (3.5)$$
The generalized Riemann tensor $\hat{R}_{MNPQ}$ in eq.(2.11) can now be written in terms of the previous covariant quantities as

$$
\hat{R}_{MN}^{\phantom{MN}PQ} = \frac{1}{2} R_{MN}^{\phantom{MN}PQ} + \frac{1}{2} \left( D_{[M} G_{N]}^{\phantom{MN}PQ} + P_{[M} \tilde{G}_{N]}^{\phantom{MN}PQ} \right) - \frac{3}{4} g_{[M}^{\phantom{MN}P} D_{N]}^{\phantom{MN}PQ} - \frac{3}{4} g_{[M}^{\phantom{MN}P} P_{N]}^{\phantom{MN}PQ} \\
- \frac{1}{2} g_{[M}^{\phantom{MN}P} g_{N]}^{\phantom{MN}Q} P_{K}^{\phantom{MN}PQ} + \frac{5}{4} g_{[M}^{\phantom{MN}P} P_{N]}^{\phantom{MN}PQ} - \frac{1}{4} G_{[M}^{\phantom{MN}C[C} G_{N]}^{\phantom{MN}C]}^{\phantom{MN}PQ} - \frac{1}{4} G_{[M}^{\phantom{MN}C[C} G_{N]}^{\phantom{MN}C]}^{\phantom{MN}PQ} \\
- \frac{1}{2} g_{[M}^{\phantom{MN}P} g_{N]}^{\phantom{MN}Q} P_{K}^{\phantom{MN}PQ} - \frac{1}{4} g_{[M}^{\phantom{MN}P} G_{N]}^{\phantom{MN}C]}^{\phantom{MN}C} P_{C} - \frac{1}{4} g_{[M}^{\phantom{MN}P} \tilde{G}_{N]}^{\phantom{MN}C]}^{\phantom{MN}C} P_{C} \\
+ \frac{1}{4} \left( G_{MN}^{\phantom{MN}P} P_{N]}^{\phantom{MN}PQ} + G_{MN}^{\phantom{MN}P} P_{N]}^{\phantom{MN}PQ} \right) + \frac{1}{4} \left( \tilde{G}_{MN}^{\phantom{MN}P} P_{N]}^{\phantom{MN}PQ} + \tilde{G}_{MN}^{\phantom{MN}P} P_{N]}^{\phantom{MN}PQ} \right) + \text{c.c.}
$$

(3.6)

In the first line of the r.h.s. of (3.6), we have kept two terms in parenthesis since this combination involves the $SL(2, \mathbb{R})$-covariant quantity

$$
D_{[M} G_{N]}^{\phantom{MN}PQ} \equiv D_{[M} G_{N]}^{\phantom{MN}PQ} + P_{[M} \tilde{G}_{N]}^{\phantom{MN}PQ} = -\frac{i}{\sqrt{\tau_2}} \left( \tau \nabla_{[M} H_{N]}^{1} + \nabla_{[M} H_{N]}^{2} \right),
$$

(3.7)

which satisfies the Bianchi identity

$$
D_{[M} G_{N]}^{\phantom{MN}PQ} = -D_{[P} G_{Q]}^{\phantom{MN}MN}.
$$

(3.8)

By using the expression (3.6) for $\hat{R}_{MNPQ}$ in the action (2.12), it is clear that the latter is only a part of the full S-duality-invariant effective action. $SL(2, \mathbb{Z})$ transformations generate new terms and the S-duality invariant result must contain the orbit of the $\hat{R}^4$ terms under $SL(2, \mathbb{Z})$. The action (2.12) is invariant under the subgroup

$$
\Gamma_{\infty} = \left\{ \begin{pmatrix} \pm 1 & * \\ 0 & \pm 1 \end{pmatrix} \in SL(2, \mathbb{Z}) \right\},
$$

(3.9)

and thus the $SL(2, \mathbb{Z})$ orbit is obtained by summing over all matrices $\gamma \in \Gamma_{\infty} \setminus SL(2, \mathbb{Z})$. By noticing that

$$
\begin{pmatrix} \pm 1 & * \\ 0 & \pm 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \pm \begin{pmatrix} * & * \\ c & d \end{pmatrix},
$$

(3.10)

and that a matrix belongs to $SL(2, \mathbb{Z})$ if and only if the g.c.d. of its second row is 1, we find that a representative of the coset $\Gamma_{\infty} \setminus SL(2, \mathbb{Z})$ is

$$
\gamma = \pm \begin{pmatrix} * & * \\ p & q \end{pmatrix},
$$

(3.11)
where the g.c.d. \((p, q) = 1\). The effective action \((2.12)\) must then be completed to

\[
S = \frac{1}{2 \kappa^2_{10}} \int d^{10}x \sqrt{-g} \left\{ \frac{\alpha^3}{3 \cdot 2^6} \left( t_8 t_8 + \frac{1}{8} \varepsilon_{10} \varepsilon_{10} \right) \zeta(3) \frac{1}{2} \sum_{(p, q) = 1} \frac{\tau^3}{|p \tau + q|^3} \left[ \frac{1}{2} R_{MPN} g_{PQ} + \frac{1}{2} \left( \frac{p \tau + q}{p \tau + q} \right)^{1/2} \left( D_{[M} G_{N]} P^Q + P_{[M} \tilde{G}_{N]} P^Q \right) - \left( \frac{p \tau + q}{p \tau + q} \right) g_{M[P} D_{N]} P^Q \right] - \frac{3}{4} \left( \frac{p \tau + q}{p \tau + q} \right)^2 g_{M[P} P^Q N] - \frac{1}{2} \left( \frac{p \tau + q}{p \tau + q} \right)^2 g_{M[P} g^Q N] P_K P^K + \frac{5}{4} g_{M[P} P^Q N] \right] \right\} \right\}
\]

where the factor 1/2 in front of the discrete sum is due to the sign ambiguity of eq.\((3.11)\).

Then the sum over \(p, q\) with \((p, q) = 1\) can be extended in an ordinary sum by replacing

\[
\zeta(3) \sum_{(p, q) = 1} \rightarrow \sum'_{p, q}, \quad \text{with } p, q \in \mathbb{Z}^2 - \{0, 0\}.
\]

We may expand the fourth power in the square brackets in eq.\((3.12)\) in terms of \(\left( \frac{p \tau + q}{p \tau + q} \right)^{k/2}\) with \(k = -16, \ldots, 16\) and the result is given in Appendix A. Here, for simplicity, let us consider as an example the contributions that involve three terms in the second line of eq.\((3.12)\). These terms are the only ones that contribute to the four-point string amplitudes.

We find

\[
S = \frac{1}{2 \kappa^2_{10}} \int d^{10}x \sqrt{-g} \left\{ \frac{\alpha^3}{3 \cdot 2^6} \left( t_8 t_8 + \frac{1}{8} \varepsilon_{10} \varepsilon_{10} \right) \sum'_{p, q} \frac{\tau^3}{|p \tau + q|^3} \left[ \frac{1}{2} R_{MPN} g_{PQ} + \frac{1}{2} \left( \frac{p \tau + q}{p \tau + q} \right)^{1/2} \left( D_{[M} G_{N]} P^Q + P_{[M} \tilde{G}_{N]} P^Q \right) - \left( \frac{p \tau + q}{p \tau + q} \right) g_{M[P} D_{N]} P^Q \right] \right\}
\]

\[
= \frac{1}{2 \kappa^2_{10}} \int d^{10}x \sqrt{-g} \left\{ \frac{\alpha^3}{3 \cdot 2^6} \left( t_8 t_8 + \frac{1}{8} \varepsilon_{10} \varepsilon_{10} \right) \times \right\}
\]
\[
\begin{align*}
\left[ \frac{1}{2} f_0(\tau, \bar{\tau}) \left( R^4 + 12R^2 DP D\bar{P} - 6RDP D\bar{G}^2 + 3R^2 DG D\bar{G} 
+ 6DP^2 D\bar{P}^2 + \frac{3}{8} DG^2 D\bar{G}^2 + 6DP D\bar{P} DG D\bar{G} \right) 
+ f_1(\tau, \bar{\tau}) \left( -4R^3 DP + \frac{3}{2} R^2 DG^2 - 12RDP^2 D\bar{P} - 6RDP DG D\bar{G} \right) 
+ 3DP D\bar{P} DG^2 + \frac{3}{2} DP^2 D\bar{G}^2 + \frac{1}{4} DG^3 D\bar{G} \right) 
+ f_2(\tau, \bar{\tau}) \left( 6R^2 DP^2 - 3RDP DG^2 + 4DP^3 DP + 3DP^2 DG D\bar{G} + \frac{1}{16} DG^4 \right) 
+ f_3(\tau, \bar{\tau}) \left( -4RDP^3 + \frac{3}{2} DP^2 DG^2 \right) + f_4(\tau, \bar{\tau}) DP^4 + \text{c.c.} \right] \right), 
\end{align*}
\]

where \( DG \) stands for \((DG)_{MNPQ} \equiv D_M G_{NPQ}\), \( DP \) for \((DP)_{MNPQ} \equiv g_{MP} D_N P_Q\) and similarly for \( D\bar{G} \) and \( D\bar{P} \) of \( U(1) \) charge \( q = -1, -2 \), respectively. This result is in perfect agreement with [20], where the S-duality invariant corrections to the eight-derivative four-point tree-level interactions of the NS-NS sector were conjectured.

The functions \( f_k(\tau, \bar{\tau}) \) are defined as

\[
f_k(\tau, \bar{\tau}) = \sum_{p,q}^\prime \frac{\tau^3/2}{(p\tau + q)^{3/2+k}(p\bar{\tau} + q)^{3/2-k}},
\]

and they transform under \( SL(2, \mathbb{Z}) \) as

\[
f_k(\tau, \bar{\tau}) \rightarrow \left( \frac{c\tau + d}{c\bar{\tau} + d} \right)^k f_k(\tau, \bar{\tau}), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}).
\]

By considering the full expansion in eq.(3.12), the functions \( f_k \) which appear are \( f_0(\tau, \bar{\tau}), f_1(\tau, \bar{\tau}), ..., f_8(\tau, \bar{\tau}) \) and their complex conjugates \( f_0(\tau, \bar{\tau}), f_{-1}(\tau, \bar{\tau}), ..., f_{-8}(\tau, \bar{\tau}) \) only. This is due to the fact that all \( f_k(\tau, \bar{\tau}) \) with \( k \) half-integer vanish, as can be seen from their definition (3.13). There are thus no interactions involving an odd number of two-forms, which is not a surprise since the transformation \(-I_{2 \times 2} \in SL(2, \mathbb{Z})\) acts as

\[
-I_{2 \times 2} : \quad \tau \rightarrow \bar{\tau}, \quad B_{MN}^0 \rightarrow -B_{MN}^0,
\]

and does not allow such terms.
Perturbatively, the type IIB superstring is also invariant under the world-sheet parity operator $\Omega$ whose effect on the massless bosonic fields is

$$\Omega : \tau_1 \rightarrow -\tau_1, \quad B^1_{MN} \rightarrow -B^1_{MN}, \quad A_{MNPQ} \rightarrow -A_{MNPQ}, \quad (3.18)$$

while the graviton, dilaton and R-R two-form are invariant. In the $SL(2,\mathbb{Z})$-invariant action, this symmetry remains unbroken since, once translated in terms of covariant quantities, it takes the form

$$\Omega : f_k(\tau, \bar{\tau}) \rightarrow f_{-k}(\tau, \bar{\tau}), \quad P_M \rightarrow \bar{P}_M, \quad G_{MNP} \rightarrow -\bar{G}_{MNP}, \quad Q_M \rightarrow -Q_M, \quad (3.19)$$

which amounts to a complex conjugation of the effective action.

We have determined above perturbative and non-perturbative corrections to eight-derivative interactions by considering the leading order in their small $\tau_2^{-1/2}$ expansion, which is $2\zeta(3)\tau_2^{3/2}$. In particular, under $SL(2,\mathbb{Z})$, the latter generates perturbative corrections at one loop proportional to $\tau_2^{-1/2}$ only, as can be seen from eq.(B.3). We would like to stress that these two perturbative contributions are actually on an equal footing. By this we mean that if we had started from the $\tau_2^{-1/2}$ contributions, we would have generated the $\tau_2^{3/2}$ corrections in a similar way. As an example, let us explicitly describe how this works on the simplest case, namely the $\tau_2^{-1/2}R^4$ term, which gives

$$\frac{2\pi^2}{3} \tau_2^{-1/2} R^4 \rightarrow \frac{2\pi^2}{3} \frac{1}{2} \sum_{(p,q)=1} \frac{\tau_2^{-1/2}}{|p\tau + q|^{-1}} R^4 = \frac{\pi^2}{3} \sum_{p,q} \tau_2^{-1/2} \frac{1}{|p\tau + q|^{-1}} R^4, \quad (3.20)$$

after considering its orbit under $SL(2,\mathbb{Z})$. Using the alternative expression for $f_k$ given in eq.(B.5), we get

$$\frac{2\pi^2}{3} \tau_2^{-1/2} R^4 \rightarrow f_0(\tau, \bar{\tau}) R^4, \quad (3.21)$$

which is exactly what we obtained before by starting from the $2\zeta(3)\tau_2^{3/2}R^4$ term.

We have seen that $SL(2,\mathbb{Z})$ relates tree and one-loop contributions. We can show now that there are no other perturbative corrections to the eight-derivative interactions.
compatible with S-duality. We will demonstrate this for the \( R^4 \) term and the generalization for interactions of non-zero \( U(1) \) charge is then straightforward. Let us suppose that there exists the \( n \)-loop correction \( \tau_2^{3/2-2n} R^4 \) which under the action of \( SL(2, \mathbb{Z}) \) gives

\[
\tau_2^{3/2-2n} R^4 \rightarrow \frac{1}{2} \sum_{(p,q)=1} \frac{\tau_2^{3/2-2n}}{|p\tau + q|^{3-4n}} R^4 = \frac{1}{2\zeta(3-4n)} \sum_{p,q} \frac{\tau_2^{3/2-2n}}{|p\tau + q|^{3-4n}} R^4
\]

\[= \frac{1}{2\zeta(3-4n)} f_{3/2-2n,0}(\tau, \bar{\tau}) R^4 \tag{3.22} \]

where, in general,

\[f_{s,k}(\tau, \bar{\tau}) = \sum_{m,n} \frac{\tau_2^s}{(m\tau + n)^{s+k}(m\bar{\tau} + n)^{s-k}}, \tag{3.23}\]

are non-holomorphic modular forms of weights \((s + k, s - k)\). Note that \( f_{3/2,k} \) is identical to \( f_k \) introduced before. The \( f_{s,k} \)'s for generic \( k \) are relevant for interactions with non-zero \( U(1) \) charge. For \( n \geq 1 \) we have \( s = 3/2 - 2n < 0 \) and the infinite sum does not converge but can be defined by analytic continuation \([24]\) as

\[f_{s,k}(\tau, \bar{\tau}) = \pi^{2s-1} \frac{\Gamma(1-s+k)}{\Gamma(s+k)} f_{1-s,k}(\tau, \bar{\tau}). \tag{3.24}\]

The small \( \tau_2^{-1} \) expansion of \( f_{3/2-2n,0} \) is then

\[f_{3/2-2n,0}(\tau, \bar{\tau}) = \pi^{2-4n} \frac{\Gamma(2n-1/2)}{\Gamma(3/2 - 2n)} \times \left( 2\zeta(4n-1)\tau_2^{2n-1/2} + \frac{2\sqrt{\pi}\Gamma(2n-1)\zeta(4n-2)\tau_2^{3/2-2n}}{\Gamma(2n-1/2)} + \cdots \right), \tag{3.25}\]

where the dots stand for instanton corrections. We see from the above expansion that S-duality relates the \( n \)-loop contribution to the \((1 - n)\)-loop \( \tau_2^{2n-1/2} R^4 \) term. We conclude then that \( n = 0 \) or \( n = 1 \) are the only perturbative contributions consistent with S-duality.

The formal sums we considered in this section have a physical interpretation. We recall that any contribution in the effective action that has been determined so far by an explicit string or sigma-model calculation is invariant under the abelian group \( \Gamma_\infty \). The invariance under the latter implies that in order to achieve the S-duality symmetry, all contributions
to be added are associated to coprime pairs of integers \( p, q \). This is equivalent to considering them as arising from the \((p, q)\)-strings of the type IIB theory \[1\]. In fact, it is surprising that \((p, q)\)-strings play a role already in ten dimensions. One should expect that these objects are important for non-perturbative physics in dimensions lower than or equal to 8, since their Euclidean world volume can then wrap around two-dimensional cycles of the compact manifold. Here, we see that the perturbative as well as the non-perturbative D-instanton corrections of the fundamental \((0, 1)\)-string have an alternative interpretation in ten dimensions. They can be viewed as a sum of only “perturbative” contributions from all \((p, q)\)-strings. Indeed, due to the fact that these objects are the images of the fundamental string under S-duality, they appear on equal footing in the final result.

4. A Non-renormalization theorem

Here, we discuss in more detail the structure of the terms we collected in Appendix A. At order \( \alpha'^3 \), the contributions that appear involve in the Einstein frame:

\[
NS - NS : \partial g, \partial \phi, e^{-\phi/2}H^1 \\
R - R : e^\phi \partial \tau_1, e^{\phi/2} \left( \tau_1 H^1 + H^2 \right),
\]

and their first derivatives. To determine their string-loop expansion order, we consider them in the string frame. There we find that the interactions involve the independent terms

\[
NS - NS : \partial G, \partial \phi, H^1 \\
R - R : \partial \tau_1, \left( \tau_1 H^1 + H^2 \right),
\]

where \( G_{MN} = e^{\phi/2}g_{MN} \). By using eq.(B.3), the general structure of the perturbative interactions are of the following form

\[
\left( 2\zeta(3)e^{-2\phi}\frac{1}{k_1} + \frac{2\pi^2}{3}k_2 \right) e^{2L\phi} \left( \partial G \right)^{n_1} \left( \partial \phi \right)^{n_2} \left( H^1 \right)^{n_3} \left( \partial \tau_1 \right)^{r_1} \left( \tau_1 H^1 + H^2 \right)^{r_2}
\]
\[ \times (\partial \partial G)^m_1 (\partial \partial \phi)^m_2 (\partial H^1)^m_3 (\partial \partial \tau_1)^n_1 (\tau_1 \partial H^1 + \partial H^2)^s_2, \]

where \( L = \frac{1}{2}(r_1 + r_2 + s_1 + s_2) \) and \( k_1, k_2 \) are rational coefficients depending on the positive integers \( n_i, m_i, r_j, s_j \) \((i = 1, 2, 3; j = 1, 2)\), which satisfy

\[
\begin{align*}
n_1 + n_2 + n_3 + r_1 + r_2 + 2(m_1 + m_2 + m_3 + s_1 + s_2) &= 8 \quad (= \text{number of derivatives}) \\
n_3 + m_3 + r_1 + s_1 &\text{ even} \quad (\Omega \text{ invariance}) \\
n_3 + m_3 + r_2 + s_2 &\text{ even} \quad (-I_{2 \times 2} \text{ invariance}).
\end{align*}
\]

Notice that these conditions imply that \( L = 0, 1, \ldots, 4 \). To determine the loop order this corresponds to, we have to take into account the rescaling (2.6) of the R-R fields. As a result, the perturbative interactions (4.3) arise at tree level and one-loop only. However, it should be stressed that \( k_1 = 0 \) for the terms involving a R-R field. The reason for this is that when we considered the orbit of the tree-level NS-NS sector proportional to \( 2\zeta(3)\tau_2^{3/2} \) in the Einstein frame, we generated only terms proportional to \( \frac{2\zeta(3)}{2} \tau_2^{-1/2} \) or non-perturbative. However, there is no symmetry that excludes terms involving R-R fields at tree level. Therefore, their presence has to be checked by an explicit string calculation and then, their \( SL(2, Z) \) orbit has to be added. However, after performing this procedure, eqs.(4.3, 4.4) remain valid and the only effect of these terms is to change the values of \( k_1 \) and \( k_2 \) while the structure of eq.(4.3) remains unaffected. To show this explicitly, let us consider as an example in the Einstein frame the term

\[
\left( t_8^{ABCDEFGH} t_8^{MNPQSTU} + \frac{1}{8} \epsilon_{10}^{ABCDEFHIJ} MNPQSTU \right) a \tau_2^{3/2} R_{ABMN} R_{CDPQ} g_{ER} \frac{\partial F \tau_1}{\tau_2} \frac{\partial S \tau_1}{\tau_2} g_{GT} \frac{\partial H \tau_1}{\tau_2} \frac{\partial U \tau_1}{\tau_2},
\]

where \( a \) is a number to be determined by a string calculation. By a straightforward calculation following the steps which led us from eq.(2.12) to eq.(A.3), the orbit under \( SL(2, Z) \) of (4.3) is

\[
\left( t_8^{ABCDEFGH} t_8^{MNPQSTU} + \frac{1}{8} \epsilon_{10}^{ABCDEFHIJ} MNPQSTU \right) a \frac{1}{2\zeta(3)} R_{ABMN} R_{CDPQ} \times
\]
\[ f_0(\tau, \bar{\tau}) \left( g_{ER} P_F P g_{GT} \tilde{P}_H \tilde{P}_U + g_{ER} \tilde{P}_F P g_{GT} \tilde{P}_H P_U + g_{ER} \tilde{P}_F P g_{GT} P_H \tilde{P}_U \right) \\
- f_2(\tau, \bar{\tau}) \left( g_{ER} \tilde{P}_F P g_{GT} P_H \tilde{P}_U + g_{ER} P_F \tilde{P} g_{GT} P_H P_U + g_{ER} P_F \tilde{P} g_{GT} \tilde{P}_H P_U \right) \\
+ g_{ER} P_F \tilde{P} g_{GT} P_H \tilde{P}_U \right) + f_4(\tau, \bar{\tau}) \left( g_{ER} P_F P g_{GT} P_H P_U \right) + \text{c.c.} \]

(4.6)

It can now be seen that the interactions induced by the term (4.5) are of the form (4.3) and actually give corrections to \( k_2 \) only.

The previous remarks we demonstrated on the specific example of eq.(4.5) remain valid for any other term involving \( 2r \) R-R fields at tree level and allowed by the symmetries. As a result, we can state a non-renormalization theorem: For the eight-derivative bosonic interactions of the type IIB theory which involve either NS-NS or R-R fields (except the self-dual four-form), there exist only tree level and one-loop perturbative corrections.

We would like to point out that although our procedure infer the structure of the perturbative sector of the type IIB superstring, the non-perturbative one may differ from what we obtained. This is due to the fact that one may add “cusp forms” to the \( f_k \)’s of the same weights without altering their perturbative expansion. By “cusp forms” we mean here modular functions of the form \( \sum'_{m,n \geq 1} a_{m,n}(\tau_2) e^{2\pi m \tau} e^{-2\pi n \bar{\tau}} \). However, this possibility does not affect the above non-renormalization theorem.

5. Fermionic and \( R^{3m+1} \) terms

To illustrate the generality of the procedure that we presented for the eight-derivative bosonic interactions, we now discuss fermionic as well as \( R^{3m+1} \) terms. The fermionic term we will consider is related to \( t_8 t_8 R^4 \) by supersymmetry and is the \( \lambda^{16} \) interaction which should arise at tree level. This term is in fact the product of the 16 components of the complex Weyl spinor whose real and imaginary parts are the two dilatinos of same chirality. The perturbative and non-perturbative corrections associated to it are then determined by
taking into account as before its orbit under $SL(2, \mathbb{Z})$ and the transformation property (2.4),

\[ 2b\zeta(3) \tau^3/2 \lambda^16 \rightarrow 2b\zeta(3) \frac{1}{2} \sum_{(p,q)=1} \frac{\tau^{3/2}}{|p\tau+q|^{3}} \left( \frac{p\bar{\tau} + q}{p\tau + q} \right)^{12} \lambda^16 \]

\[ = b \sum'_{\mu,\nu} \frac{\tau^{3/2}}{(p\tau+q)^{3/2+12}(p\bar{\tau}+q)^{3/2-12}} \lambda^16 \]

\[ = b f_{12}(\tau, \bar{\tau}) \lambda^16, \tag{5.1} \]

where $b$ is a numerical factor. This result is in complete agreement with the M-theory calculation of ref. [21].

Interactions of the form $R^{3m+1}$ of $\alpha'^{3m}$ order with $m = 1, 2, ...$ have been conjectured to exist in M-theory [12]. They are the only terms that can be decompactified in eleven dimensions in a Lorentz-invariant way. In the string frame they can arise at $m$ loops

\[ \int d^{10}x \sqrt{-G} e^{2\phi(m-1)} R^{3m+1}, \tag{5.2} \]

where $R_{MNPQ}$ is the Riemann tensor in this frame. Eventually, there are also lower order perturbative contributions.

M-theory on $T^2$ has an $SL(2, \mathbb{Z})$ symmetry, which is the T-duality group of the torus. It acts on the complex structure $\Omega$, which can be identified with the type IIB complex scalar $\tau$ in nine dimensions. By decompactifying to ten dimensions on the type IIB side and following the procedure described in section 3, we find that eq.(5.2) is promoted to the $SL(2, \mathbb{Z})$-invariant term

\[ \int d^{10}x \sqrt{-g} f_{-m/2,0}(\tau, \bar{\tau}) R^{3m+1}, \tag{5.3} \]

where, $f_{-m/2,0}(\tau, \bar{\tau})$ has been defined in eq.(3.23). The small $\tau_2^{-1}$ expansion of $f_{-m/2,0}$ as follows from eq.(3.23) is then

\[ f_{-m/2,0}(\tau, \bar{\tau}) = \pi^{-(m+1)} \frac{\Gamma(1 + m/2)}{\Gamma(-m/2)} \]

\[ \times \left( 2\zeta(2 + m) \tau_2^{1+m/2} + \frac{2\sqrt{\pi} \Gamma(1/2 + m/2) \zeta(1 + m)}{\Gamma(1 + m/2)} \tau_2^{-m/2} + \cdots \right), \tag{5.4} \]
where the dots stand for instanton corrections. From the expansion (5.4), we see that there exist two perturbative contributions at $m$ and $(m - 1)/2$ loops to the $R^{3m+1}$ terms. Then, a necessary condition for such terms to exist perturbatively is that $m$ be odd. As a result, we conclude that: *Interactions consistent with eleven-dimensional M-theory and SL(2, Z) symmetry in nine dimensions are of the form $R^{6N+4}$ ($N = 0, 1, ...$) and arise at $N$ and $2N + 1$ loops.*

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Appendix A

We present here the S-duality invariant interactions of the type II B effective action, which have been obtained by considering at tree level the NS-NS sector only. However, to obtain the full action at $\alpha'^3$ order, besides these $\hat{R}^4$ terms, it is necessary to extend the analysis to all other eight-derivative contributions that involve $2r$ R-R fields ($r = 1, 2, 3, 4$) and occur at tree level only.

For convenience, we introduce the following quantities $X_i$ ($i = 0, ..., 4$) whose $U(1)$ charges are well defined, namely $q = i,$

\[
\begin{align*}
(X_0)_{MNPQ} &\equiv R_{MNPQ} + \frac{5}{4} g_{MP} \left( P_N \bar{P}_Q + \bar{P}_N P_Q \right) - g_{MP} g_{NQ} P_K \bar{P}^K \\
&\quad - \frac{1}{4} G_{MPC} \bar{G}_{NQ}^C - \frac{1}{4} \bar{G}_{MPC} G_{NQ}^C,
\end{align*}
\]

\[
\begin{align*}
(X_1)_{MNPQ} &\equiv \frac{1}{2} \left( D_M G_{NQP} + P_M \bar{G}_{NPQ} \right) + \frac{1}{4} \left( \bar{G}_{MNP} P_Q + \bar{G}_{PQM} P_N \right) \\
&\quad - \frac{1}{4} g_{MP} \bar{G}_{NQC} P^C,
\end{align*}
\]

\[
\begin{align*}
(X_2)_{MNPQ} &\equiv - g_{MP} D_N P_Q - \frac{1}{4} G_{MPC} G_{NQ}^C,
\end{align*}
\]

\[
\begin{align*}
(X_3)_{MNPQ} &\equiv \frac{1}{4} \left( G_{MNP} P_Q + G_{PQM} P_N \right) - \frac{1}{4} g_{MP} G_{NQC} P^C,
\end{align*}
\]

\[
\begin{align*}
(X_4)_{MNPQ} &\equiv - \frac{3}{4} g_{MP} P_N P_Q - \frac{1}{2} g_{MP} g_{NQ} P_C P^C.
\end{align*}
\]

In addition we will denote

\[
\left( t_8 t_8 + \frac{1}{8} \bar{\varepsilon}_{10} \varepsilon_{10} \right) X_i X_j X_k X_l \equiv \left( t_8^{ABCDEFGH} t_8^{MNPQRSTU} + \frac{1}{8} \bar{\varepsilon}_{10}^{ABCDEFGH} \varepsilon_{10}^{MNPQRSTU} \right)_{ijkl}
\]

\[
\times (X_i)_{ABMN} (X_j)_{CDPQ} (X_k)_{EFRS} (X_l)_{GHTU},
\]

for $i, j, k, l = 0, ..., 4$ and similarly for interactions involving the complex conjugates $\bar{X}_i.$

Then, the action takes the form

\[
\mathcal{S} = \frac{1}{2 \kappa^2_{10}} \int d^{10}\!x \sqrt{-g} \left\{ R - 2 P_M \bar{P}^M - \frac{1}{12} G_{MNP} \bar{G}^{MNP} + \frac{\alpha'^3}{3 \cdot 2^7} \left( t_8 t_8 + \frac{1}{8} \bar{\varepsilon}_{10} \varepsilon_{10} \right) \times \right\}
\]
\[
\left[ \frac{1}{2} f_0 \left( 12 \bar{X}_1 X_0^2 X_1 + 4 \bar{X}_3 X_1^3 + 6 \bar{X}_3^2 X_0^2 + 12 \bar{X}_3 \bar{X}_1 X_2 + X_0^4 \\
+ 24 \bar{X}_3 \bar{X}_2 X_2 X_3 + 12 \bar{X}_2^2 X_1 X_3 + 12 \bar{X}_1^2 X_0 X_2 + 12 \bar{X}_3 X_0^2 X_3 \\
+ 24 \bar{X}_2 \bar{X}_1 X_0 X_3 + 24 \bar{X}_1 \bar{X}_2 X_1 X_2 + 24 \bar{X}_1 \bar{X}_3 X_1 X_3 \\
+ 24 \bar{X}_3 X_0 X_1 X_2 + 6 \bar{X}_2^2 X_2 X_2 + 4 \bar{X}_1^3 X_3 + 6 \bar{X}_1^2 X_1^2 \\
+ 12 \bar{X}_2 X_0^2 X_2 + 12 \bar{X}_2 X_0 X_1^2 + 24 \bar{X}_4 \bar{X}_1 X_1 X_4 \\
+ 24 \bar{X}_4 \bar{X}_1 X_2 X_3 + 24 \bar{X}_3 X_4 \bar{X}_1 X_0 + 6 \bar{X}_4^2 X_4^2 \\
+ 24 \bar{X}_4 \bar{X}_3 X_3 X_4 + 12 \bar{X}_2 X_2^3 X_4 + 12 \bar{X}_3 X_4 X_2 + 12 \bar{X}_1^2 X_4 \bar{X}_2 \\
+ 12 \bar{X}_2^2 X_0 X_4 + 12 X_2 X_4^2 X_4 + 12 X_3 X_4 X_2 + 12 X_1^2 X_4 \bar{X}_4 \\
+ 12 \bar{X}_2^2 X_0 X_4 + 12 X_2^2 X_4 X_0 + 12 X_2 \bar{X}_4 X_1^2 + 12 X_0^2 X_4 \bar{X}_4 \\
+ 24 \bar{X}_3 \bar{X}_2 X_1 X_4 + 24 X_0 \bar{X}_4 X_4 X_3 + 24 X_2 \bar{X}_4 X_4 \bar{X}_2 \right) \\
+ f_1 \left( 12 \bar{X}_2 X_1^2 X_2 + 12 \bar{X}_2 X_0 X_2^2 + 4 \bar{X}_1 X_1^3 + 6 \bar{X}_1^2 X_1^2 \\
+ 24 \bar{X}_3 X_0 X_2 X_3 + 12 \bar{X}_3 \bar{X}_1 X_2^2 + 24 \bar{X}_1 \bar{X}_2 X_2 X_3 + 6 \bar{X}_2^2 X_3^2 \\
+ 12 \bar{X}_3 X_1^2 X_3 + 12 \bar{X}_3 X_1 X_2^2 + 12 \bar{X}_1^2 X_1 X_3 + 12 \bar{X}_1 X_0^2 X_3 \\
+ 24 \bar{X}_2 X_0 X_1 X_3 + 24 \bar{X}_1 X_0 X_1 X_2 + 4 X_2 X_0^3 + 6 X_0^2 X_1^2 \\
+ 12 X_4 X_1^2 X_4 + 6 X_4^2 X_3^2 + 4 X_3 X_4 + 24 X_3 X_4 X_4 X_1 \\
+ 24 X_3 X_4 \bar{X}_3 \bar{X}_2 + 24 \bar{X}_2 \bar{X}_1 X_1 X_4 + 12 \bar{X}_1^2 X_4 \bar{X}_0 \\
+ 12 X_0 X_3^2 \bar{X}_4 + 12 X_4 X_4 \bar{X}_4 X_2 + 12 \bar{X}_2^2 X_2 X_4 + 12 X_0^2 X_4 \bar{X}_2 \\
+ 24 \bar{X}_3 X_0 X_1 X_4 + 24 \bar{X}_4 X_1 X_2 X_3 + 24 \bar{X}_1 X_4 \bar{X}_3 X_2 \\
+ 24 \bar{X}_4 X_4 X_0 X_2 \right) \\
+ f_2 \left( 12 \bar{X}_3 X_2^2 X_3 + 12 X_2 X_0 X_3^2 + 12 X_1 X_1 X_2^2 + 4 X_2 X_3 X_3 + X_1^4 \\
+ 24 \bar{X}_1 X_0 X_2 X_3 + 6 \bar{X}_1 X_3^2 + 12 \bar{X}_3 X_1 X_3^2 + 12 \bar{X}_1 X_1^2 X_3 \right)
\]
\[-20-\]

\[+ 24 \bar{X}_2 \bar{X}_1 X_2 X_3 + 6 X_0^2 X_2^2 + 12 X_0^2 X_1 X_3 + 12 X_0 X_1^2 X_2 \]

\[+ 24 \bar{X}_3 X_1 X_2 X_4 + 12 X_0 X_1^2 \bar{X}_4 + 12 X_2^2 \bar{X}_4 X_4 + 12 \bar{X}_1^2 X_2 X_4 \]

\[+ 12 X_2 \bar{X}_4 X_3^2 + 12 \bar{X}_2 X_1^2 X_4 + 12 X_1^2 \bar{X}_3 X_1 + 6 X_4^2 \bar{X}_2^2 \]

\[+ 4 X_0^3 X_1 + 24 X_0 X_4 \bar{X}_2 X_2 + 24 X_0 X_4 \bar{X}_1 X_1 + 24 X_0 X_4 \bar{X}_3 X_3 \]

\[+ 24 X_1 X_3 \bar{X}_4 X_4 + 24 X_3 X_4 \bar{X}_2 \bar{X}_1 \]

\[+ f_3 \left( 12 X_4^2 \bar{X}_2 X_0 + 12 X_4^2 \bar{X}_3 X_1 + 12 \bar{X}_1 X_2^2 X_3 + 4 X_3 X_3^3 \right) \]

\[+ 12 \bar{X}_2 X_2 X_3^2 + 4 X_0 X_2^3 + 12 \bar{X}_1 X_1 X_3^2 + 4 X_3 X_1^3 + 6 X_0^2 X_3^2 \]

\[+ 24 X_1 X_0 X_2 X_3 + 6 X_1^2 X_2^2 + 24 \bar{X}_1 X_4 X_0 X_3 + 24 \bar{X}_1 X_1 X_2 X_4 \]

\[+ 24 \bar{X}_3 X_3 X_2 X_4 + 12 X_4^2 X_2 \bar{X}_4 + 12 \bar{X}_2 X_2^2 X_4 + 12 \bar{X}_4 X_4 X_3^2 \]

\[+ 12 X_0^2 X_2 X_4 + 12 X_1^2 X_4 X_0 + 6 X_4^2 \bar{X}_1^2 + 24 X_3 X_4 \bar{X}_2 X_1 \]

\[+ f_4 \left( 4 X_1 X_3^3 + X_2^4 + 12 X_0 X_2 X_3^2 + 12 X_1 X_2^2 X_3 + 6 X_1^2 X_3^2 \right) \]

\[+ 12 X_3^2 X_4 \bar{X}_2 + 12 X_4^2 \bar{X}_2 X_2 + 6 X_0^2 X_4^2 + 12 X_3 X_4^2 \bar{X}_3 \]

\[+ 12 X_0 X_4 X_3^2 + 12 X_4^2 \bar{X}_1 X_1 + 12 X_2 X_4 X_1^2 + 24 X_3 X_4 \bar{X}_1 X_2 \]

\[+ 24 X_3 X_4 X_0 X_1 + 4 X_4^2 \bar{X}_4 \]

\[+ f_5 \left( 4 X_4^3 \bar{X}_2 + 12 X_4^2 \bar{X}_1 X_3 + 12 X_4^2 X_0 X_2 + 6 X_4^2 X_1 \right) \]

\[+ 12 X_0 X_4 X_3^2 + 24 X_1 X_3 X_2 X_4 + 4 X_1 X_3^3 + 4 X_3^2 X_1 + 6 X_2^2 X_3^2 \right) \]

\[+ f_6 \left( 4 X_4^3 X_0 + 12 X_4^2 X_1 X_3 + 6 X_4^2 X_2^2 + 12 X_3^2 X_4 X_2 + X_3^4 \right) \]

\[+ f_7 \left( 4 X_4^3 X_2 + 6 X_3^2 X_4^2 \right) + f_8 \left( X_4^4 \right) + \text{c.c..} \right\} . \]  

(A.3)
Appendix B

We present here some properties of the non-holomorphic $f_k$ functions. They have been defined in eq.(3.15) as

$$f_k(\tau, \bar{\tau}) = \sum_{p,q} (p\tau + q)^{3/2+k} (p\bar{\tau} + q)^{-3/2-k}, \quad (B.1)$$

from which we derive the recursion formula for $k \in \mathbb{Z}$

$$\left(k + 2i\tau \frac{\partial}{\partial \tau}\right) f_k = \left(\frac{3}{2} + k\right) f_{k+1}. \quad (B.2)$$

By using the above relation, we find that their small $e^\phi = \tau_2^{-1}$ expansion is

$$f_k(\tau, \bar{\tau}) = 2\zeta(3)\tau_2^{3/2} + \frac{2\pi^2}{3} c_k \tau_2^{-1/2} + 4\pi \sum_{m\geq1, n\geq1} \frac{m^{1/2}}{n^{3/2}} \times \left[ \sum_{r=-k}^{\infty} \frac{C_{k,r}}{(4\pi mn\tau_2)^r} e^{2i\pi mn\tau} + \sum_{r=k}^{\infty} \frac{C_{k,r}}{(4\pi mn\tau_2)^r} e^{-2i\pi mn\bar{\tau}} \right], \quad (B.3)$$

where

$$c_k = (-)^k \frac{\pi}{4} \frac{1}{\Gamma(3/2+k)\Gamma(3/2-k)},$$

$$r \geq l \in \mathbb{Z}, \quad C_{l,r} = \frac{(-)^l}{(r-l)!} \frac{\Gamma(3/2)}{\Gamma(l+3/2)} \frac{\Gamma(r-1/2)}{\Gamma(-r-1/2)}. \quad (B.4)$$

An alternative form of the functions $f_k$ is

$$f_k(\tau, \bar{\tau}) = \frac{4\pi^2}{4k^2 - 1} \sum_{p,q} (p\tau + q)^{-1/2+k} (p\bar{\tau} + q)^{-1/2-k}, \quad (B.5)$$

which is a particular case of eq.(3.24) for $s = 3/2$. In fact, the above sum does not converge but is defined by an analytic continuation similar to the well-known zeta-regularization.

Using eq.(B.1), the functions $f_k$ can be seen to satisfy the differential equation

$$(\tau - \bar{\tau})^2 \partial \bar{\partial} f_k + k(\tau - \bar{\tau}) \partial f_k + k(\tau - \bar{\tau})\bar{\partial} f_k + \frac{3}{4} f_k = 0. \quad (B.6)$$

In fact, it is shown in [24] that any modular function of holomorphic and anti-holomorphic weights $(\frac{3}{2} + k, \frac{3}{2} - k)$, which satisfies the above equation, must be equal to $f_k$. 
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