Excited-state phase transition leading to symmetry-breaking steady states in the Dicke model

Ricardo Puebla,1 Armando Relaño,2 and Joaquín Retamosa1

1Grupo de Física Nuclear, Departamento de Física Atómica, Molecular y Nuclear, Universidad Complutense de Madrid, Av. Complutense s/n, 28040 Madrid
2Departamento de Física Aplicada I and GISC, Universidad Complutense de Madrid, Av. Complutense s/n, 28040 Madrid

We study the phase diagram of the Dicke model in terms of the excitation energy and the radiation-matter coupling constant $\lambda$. Below a certain critical value $\lambda_c$ of the coupling constant $\lambda$ all the energy levels have a well-defined parity. For $\lambda > \lambda_c$, the energy spectrum exhibits two different phases separated by a critical energy $E_c$, that proves to be independent of $\lambda$. In the upper phase, the energy levels have also a well-defined parity, but below $E_c$ the energy levels are doubly degenerated. We show that the long-time behavior of appropriate parity-breaking observables distinguishes between the different phases of the energy spectrum of the Dicke model. Steady states reached from symmetry-breaking initial conditions restore the symmetry only if their expected energies are above the critical. This fact makes it possible to experimentally explore the complete phase diagram of the excitation spectrum of the Dicke model.

PACS numbers:

Introduction.- The rapid development of experimental techniques controlling ultra-cold atoms has given rise to a great breakthrough in the physics of quantum many-body systems. A logical outcome has been the increase of the interest in certain phenomena, such as non-equilibrium dynamics and quantum phase transitions (QPTs). Also, it has entailed the revival of well-known physical models as that formulated by R. Dicke, which describes the interaction between an ensemble of two-level atoms with a single electromagnetic field mode, as a function of the radiation-matter coupling [1]. Its most representative features are a second order QPT, which leads the system from a normal to a superradiant phase, characterized by a macroscopic population of the upper atomic level [2], the emergence of quantum chaos and the spontaneous symmetry breaking [3,4]. Although this model has been extensively studied from many points of view, there still exists a heated controversy about its significance in real physical systems. A no-go theorem was formulated in the seventies, stating that the super-radiant phase transition cannot occur in a general system of atoms or molecules interacting with a finite number of radiation modes in the dipole approximation [5]. In addition, it is not clear if this theorem also forbids the superradiant transition in other realizations of the Dicke model, like in circuit QED [6]. On the contrary, this transition has been experimentally observed with a superfluid gas in an optical cavity, giving rise to a self-organized phase [7]. All these facts have turned the Dicke model into a multidisciplinary hot topic, involving different branches of physics. As a consequence, there exists an intense theoretical research; a few representative examples concern non-equilibrium QPTs [8], thermal phase transitions in the ultrastrong-coupling limit [9], or equilibration and macroscopic quantum fluctuations [10].

In this Letter we explore the phase diagram of the Dicke model as a function of two control parameters: the radiation-matter coupling constant $\lambda$ and the energy $E$ of its eigenstates. We show that the energy spectrum can be divided into three different sectors or phases separated by certain critical values $\lambda_c$ and $E_c$. For $\lambda < \lambda_c$ we find that parity is a well-defined quantum number at any excitation energy. The situation is rather different if $\lambda > \lambda_c$. Below a certain critical energy $E_c$, all the energy levels of the system are doubly degenerated, and, as a consequence, the parity symmetry of each level can be broken. Above the critical energy $E_c$ there are no such degeneracies and parity is again a good quantum number. To some extent we can say that beyond $\lambda_c$ the excited energy levels up to energy $E_c$ inherit the properties of the superradiant phase, characteristic of the ground state. We also show that this phase diagram entails measurable effects in the long-time dynamics of certain observables. Indeed, if one prepares the system in a symmetry-breaking ground state of the superradiant phase and performs a quench to a non-equilibrium state, then the symmetry of the final steady state remains broken if its energy is below $E_c$, whilst it is restored in the opposite case. As a consequence, parity non-conserving observables relax to steady values different from zero only if the energy of the non-equilibrium state is below the critical. This fact constitutes an unheralded characteristic of the Dicke model, that can be accessible to experiments and shed some light over the current controversy about the relevance of the critical behavior of this model in real physical systems.

The Dicke model.- The Dicke Hamiltonian can be written as follows

$$H = \omega_0 J_z + \omega a^\dagger a + \frac{2\lambda}{\sqrt{N}} (a^\dagger + a) J_x,$$  \hspace{1cm} (1)

where $a^\dagger$ and $a$ are the usual creation and annihilation operators of photons, $\mathbf{J} = (J_x, J_y, J_z)$ is the angular momentum, with a pseudo-spin length $J = N/2$, and $N$
is the number of atoms. The frequency of the cavity mode is represented by \( \omega \) and the transition frequency by \( \omega_0 \). Finally, the parameter \( \lambda \) is the radiation-matter coupling. Throughout all this Letter, we take \( \hbar = 1 \), and \( \omega = \omega_0 = 1 \). The parity \( \Pi = e^{i\pi(J_x+J_y)} \) is a conserved quantity, due to the invariance of \( H \) under \( J_x \rightarrow -J_x \) and \( a \rightarrow -a \) [2], and thus all the eigenstates are labeled with positive or negative parities. The system undergoes a second-order QPT at \( \lambda_c = \sqrt{\omega_0/2} \), which separates the so-called normal phase (\( \lambda < \lambda_c \)) from the superradiant phase (\( \lambda > \lambda_c \)) [2]. In the latter the ground state becomes doubly degenerated and parity can be spontaneously broken —because of the fluctuations, the system can evolve into one particular ground state without a well-defined parity [4].

It is important to note that the system has a finite number of atoms but infinite photons, reason why it is mandatory to set in numerical calculations a cutoff in the photon Hilbert space. The convergence of our results is tested, checking their stability against small increases of this cutoff.

**Phase diagram.** As previously commented, two different phases, separated by \( \lambda_c \), are found in Dicke model at zero temperature. The normal phase, where parity is a well-defined quantum number, and the superradiant phase characterized by a degenerated ground state and a spontaneous parity-symmetry breaking. A convenient method to see if this phenomenon is also present in excited states is to analyze the difference

\[
\Delta E_i(\lambda, N) = \frac{E_i^{\Pi=+1}(\lambda, N) - E_i^{\Pi=-1}(\lambda, N)}{E_i^{\Pi=+1}(\lambda, N)}
\]  

(2)

between the \( i \)-th excited states of both parity sectors \( \Pi = \pm 1 \). If \( \Delta E_i \) is different from zero, the corresponding eigenstates have well-defined parity; if it is zero, they are degenerated and one can perform a rotation that mixes both parity values. Results for \( N = 40 \) atoms are shown in Fig. 1. For \( \lambda > \lambda_c \) there exists an abrupt change from \( \Delta E_i \approx 0 \) to \( \Delta E_i > 0 \) at a certain critical energy \( E_c(\lambda, N) \). A quantitative estimate of this energy can be obtained as the first eigenvalue \( E_i \) for which \( \Delta E_i > k_{\text{err}} \), where \( k_{\text{err}} \) is a given error bound. For all the results shown below we have set \( k_{\text{err}} = 10^{-6} \); similar ones are obtained with different bounds. Since the actual phase transition occurs in the thermodynamical limit, it is mandatory to infer how this critical line evolves as \( N \rightarrow \infty \). To do so, we assume that for each value of \( N \) the critical line obeys the linear law

\[
\frac{E_c(\lambda, N)}{J} = A_N + B_N \lambda, \quad \lambda > \lambda_c,
\]  

(3)

where the coefficients \( A_N \) and \( B_N \) are numerically determined by means of a least-squares fit. The inset of Fig. 1 displays the dependence of these coefficients on \( N \). Solid circles represent the numerical points corresponding to \( B_N \); solid squares, the corresponding to \( A_N \), and solid lines the fits to power laws \( N^{-\alpha} \). It is clearly seen that \( A_N \rightarrow -1 \) and \( B_N \rightarrow 0 \) as \( N \rightarrow \infty \), and hence we conclude that \( E_c(\lambda) = E_c(\lambda, \infty) = -J \). It is worth mentioning that this value coincides with that recently obtained in the study of the connection of an excited-state quantum phase transition (ESQPT) with the development of quantum chaos and the critical decay of the survival probability [11].

**FIG. 1:** (Color online). Intensity plot showing the decimal logarithm of the relative difference \( \Delta E(E, \lambda, N) \) in terms of \( E/J \) and \( \lambda \) for a system with \( N = 40 \) atoms. Lighter regions correspond to the conserving-parity region, while darker regions represents the spontaneously broken-parity phase. The inset shows the finite-size scaling of parameters \( A_N \) and \( B_N \) in Eq. (3).

We can also use the mean-field approximation, that gives the exact ground-state energy in the thermodynamic limit, to estimate \( E_c(\lambda) \). Let us introduce for \( \mu, \nu \in \mathbb{R} \) the coherent ansatz \( |\mu, \nu\rangle = |\mu\rangle \otimes |\nu\rangle \), where

\[
|\mu\rangle = (1 + \mu^2)^{-J/2} e^{\mu J_+} |J, -J\rangle,
\]

\[
|\nu\rangle = e^{\bar{\nu}^2/2} e^{\nu a^\dagger} |0\rangle,
\]

(4)

correspond to the atomic and the photonic parts of the state, respectively. The resulting energy surface is

\[
E_{\text{var}}(\mu, \nu, \lambda) = \langle \nu, \mu | H | \nu, \mu \rangle = \omega_0 J \left( \frac{\mu^2 - 1}{\mu^2 + 1} \right) + \omega \nu^2 + \lambda \sqrt{2J} \left( \frac{4\mu \nu}{\mu^2 + 1} \right).
\]

(5)
It is plotted in Fig. 2, the upper panel shows the case with \( \lambda = 0.25 \) (below \( \lambda_c \)), and the lower panel, the case with \( \lambda = 2.0 \) (above \( \lambda_c \)); in both panels a number of level curves are drawn with solid lines. The geometry of this surface reveals that the level curve \( E = -J \) plays an especial role. For \( \lambda < \lambda_c \), it reduces to a single point at \( (\mu, \nu) = (0,0) \), which is the absolute minimum of the energy surface; for \( \lambda > \lambda_c \), it changes abruptly to a non-analytic level curve containing a saddle point. Moreover, the shape of the energy surface is quite different depending on whether \( E \) is below or above \( E = -J \). In the former case, the energy surface exhibits two symmetric wells below \( E = -J \), so that level curves are disjoint. On the contrary, for \( E > -J \) there is a single well with connected level curves for any value of \( \lambda \). This behavior supports that \( E_c(\lambda) = -J \), as our previous numerical estimations for finite \( N \).

![Figure 2](image)

**FIG. 2**: (Color online). Contour plot of the energy surface \( E_{\text{eq}}(\mu, \nu, \lambda)/J \) for two different values of \( \lambda \), one above and the other below the critical coupling \( \lambda_c \). Upper panel, \( \lambda = 0.25 \), and lower panel \( \lambda = 2.0 \). Solid lines represent level curves.

**Dynamical symmetry breaking.** Baumann and co-workers [4] explored in real time the spontaneous parity breaking of the ground state at the superradiant phase transition, by measuring the behavior of \( \langle J_x(t) \rangle \) as the coupling constant \( \lambda \) increases in time and crosses the critical point. Here, we follow an analogous procedure to study the different phases of the excited spectrum when \( \lambda > \lambda_c \). We study the non-equilibrium dynamics and the relaxation to a steady state of certain physical observables, like \( J_x \) and \( \hat{q} \equiv (a^\dagger + a)/\sqrt{2} \). They are physically measurable operators [11,12], which change the parity of the state on which they operate. Thus, they give rise to qualitatively different steady expectation values, depending on whether the energy of the non-equilibrium state is above or below \( E_c \). Although we only report results for \( J_x \), the behavior of \( \hat{q} \) is completely similar.

![Figure 3](image)

**FIG. 3**: (Color online). Contour plot of the energy surface \( E(\lambda_i, \lambda_f)/J \) as function of \( \lambda_i \) and \( \lambda_f \). The critical energy line is placed at \( E/J = -1 \). The darker region corresponds to energies greater that the critical one, while the lighter zone corresponds to lower energies. Solid line represent level curves.

Let us take as our initial condition a symmetry-breaking ground state \( |\Psi(0)\rangle = |\mu_i, \nu_i\rangle \), where \( (\mu_i, \nu_i) \) is the minimum of the energy surface corresponding to a coupling constant \( \lambda_i \) inside the superradiant phase. Then, we perform a *adiabatic* change of \( \lambda \), i.e., a quench \( \lambda_i \rightarrow \lambda_f \), so that the ground state \( |\mu_i, \nu_i\rangle \) becomes a non-equilibrium state of \( H(\lambda_f) \). Its energy \( E(\lambda_i, \lambda_f) = \langle \mu_i, \nu_i|H(\lambda_f)|\mu_i, \nu_i\rangle \) can be written as

\[
E(\lambda_i, \lambda_f) = -\omega_0 J \left( \frac{\lambda_i^2}{\lambda_f^2} \right) + 2 J \frac{\lambda_i^4 - \lambda_f^4}{\omega^2 \lambda_f^4} - 4 J \frac{\lambda_i^4 - \lambda_f^4}{\omega^2 \lambda_f^4} \frac{\lambda_f^4 - \lambda_f^4}{\omega^2 \lambda_f^4},
\]

The contours of \( E(\lambda_i, \lambda_f) \) are shown in Fig. 3. It is clearly seen that choosing \( \lambda_i \) and \( \lambda_f \) properly, one can explore the different phases of the excited spectrum. In particular, from any initial initial condition satisfying that \( \lambda_i \gtrsim 1.5 \lambda_c \), both phases can be reached by just quenching the system to different final coupling parameters \( \lambda_f \).

After performing the quench, we study the time evolution of \( \langle J_x(t) \rangle = \langle \Psi(t)|J_x|\Psi(t)\rangle \), where \( |\Psi(t)\rangle = e^{-iH(\lambda_f) t}|\Psi(0)\rangle \). If one expands the initial state in the eigenstate basis of \( H(\lambda_f) \), denoted here as \( \{E_i\} \), the expectation value of \( J_x \) reads

\[
\langle J_x(t) \rangle = \sum_{i,j} C_i^* C_j e^{-i(E_i - E_j)t} \langle E_j|J_x|E_i\rangle, \tag{7}
\]

being \( C_i = \langle E_i|\Psi(0)\rangle \).
by the diagonal approximation which it fluctuates \cite{14}. Moreover, when the energy initial condition relaxes to a certain steady state, arounding. For long-enough time evolutions, almost any ini-

\[ \langle E \rangle > E \]

quickly to a non-zero value. The same result is obtained

\[ \sum_i |C_i|^2 \langle E_i | J_x | E_i \rangle = 0. \]

On the contrary, in the broken-parity phase, the energy eigen-

values are doubly degenerated with opposite parities, so that the diagonal approximation is not valid. Thus, one can find expectation values \( \langle J_x(t) \rangle \propto 0 \) in this phase.

Consequently, the steady expected value of \( J_x \) provides a neat signature of the two phases of the excited spectrum whenever \( \lambda > \lambda_c \). In fact, it acts like an order parameter of the ESPQT, as it is equal to zero if \( E > E_c \), and different from zero if \( E < E_c \). Therefore, it suffices to follow the long-time dynamics of a parity-changing operator to infer whether the energy of the initial state is above or below the critical energy. Furthermore, as this is already true for small values of \( N \), finite precursors of this phase transition could be clearly observed in experiments. In particular, the setup used in\cite{14} is a good candidate for covering this aim.

\textbf{Conclusions.}- We have studied the phase diagram of the Dicke model in terms of the coupling constant \( \lambda \) and the energy \( E \). Using numerical calculations and the mean-field approximation, we have found different phases in the excitation spectrum, separated by certain critical values \( E_c \) and \( \lambda_c \), where the latter also defines the critical point of the superradiant transition of the ground state. For \( \lambda < \lambda_c \) we find a single phase where parity is a well defined quantum number. On the contrary, for \( \lambda > \lambda_c \) there exists a critical energy \( E_c = -J \), such that below \( E_c \) all the energy levels of the system are doubly degenerated and composed of states with opposite parities. As a consequence, fluctuations can entail a spontaneous parity breaking —the system can evolve into a state without a definite value of the parity. In some sense the excited energy levels up to energy \( E_c \) inherit the properties of the ground state in the superradiant phase. This fact leads to measurable dynamical consequences. Starting from a symmetry-breaking ground state in the superradiant phase and performing a quench to a non-equilibrium state, the relaxed expected value of certain observables, like \( J_x \) or \( q \), is different from zero only if the energy of the non-equilibrium state is below \( E_c \). This constitutes a new feature of the Dicke model, which could be observed in experiments similar to that of Ref.\cite{14}. We think that the results contained in this Letter might shed some light about the significance of the critical behavior of the Dicke model in real physical systems.

\textbf{Acknowledgments.}- The authors thank Borja Peropadre for his valuable comments. This work is supported in part by Spanish Government grants for the research projects FIS2009-11621-C02-01, FIS2009-07277, CSDP-2007-00042-Ingenio2010, and by the Universidad Complutense de Madrid grant UCM-910059.
[1] R. H. Dicke, Phys. Rev. Lett. 93, 99 (1954).
[2] K. Hepp and E. H. Lieb, Ann. Phys. (N.Y.) 76, 360 (1973); Y. K. Wang and F. T. Hioe, Phys. Rev. A 7, 831 (1973); H. J. Carmichael, C. W. Gardiner, and D. F. Walls, Phys. Lett. 46A, 47 (1973).
[3] C. Emary, and T. Brandes, Phys. Rev. Lett. 90, 044101 (2003); Phys. Rev. E 67, 066203 (2007).
[4] K. Baumann, R. Motl, F. Brennecke, and T. Esslinger, Phys. Rev. Lett. 107, 140402 (2011).
[5] K. Rzazewsky, K. Wodkiewicz, and W. Zakowicz, Phys. Rev. Lett. 35, 432 (1975); K. Rzazewsky and W. Wodkiewicz, Phys. Rev. A 13, 1967 (1976); J. M. Knight, Y. Aharonov, and G. T. C. Hsieh, Phys. Rev. A 17, 1454 (1978); I. Bialynicki-Birula and K. Rzazewsky, Phys. Rev. A 19, 301 (1979).
[6] P. Nataf and C. Ciuti, Nature Commun. 1, 72 (2010); O. Viehmann, J. von Deft, and F. Marquardt, Phys. Rev. Lett. 107, 113602 (2011).
[7] K. Baumann, C. Guerlin, F. Brennecke, and T. Esslinger, Nature 464, 1301 (2010).
[8] V. M. Bastidas, C. Emary, B. Regler, and T. Brandes, Phys. Rev. Lett. 108, 043003 (2012).
[9] M. Aparicio Alcalde, M. Bucher, C. Emary, and T. Brandes, Phys. Rev. E 86, 012101 (2012).
[10] A. Altland and F. Haake, N. J. Phys 14 (2012) 073011.
[11] P. Pérez-Fernández, P. Cejnar, J. M. Arias, J. Dukelsky, J. E. García-Ramos, and A. Relaño, Phys. Rev. A 83, 033802 (2011); P. Pérez-Fernández, A. Relaño, P. Cejnar, J. M. Arias, J. Dukelsky, and J. E. García-Ramos, Phys. Rev. E 83, 046208 (2011).
[12] A. Relaño, J. M. Arias, J. Dukelsky, J. E. García-Ramos, and P. Pérez-Fernández, Phys. Rev. A 78, 060102(R) (2008); P. Pérez-Fernández, A. Relaño, J. M. Arias, J. Dukelsky, and J. E. García-Ramos, Phys. Rev. A 80, 032111 (2009).
[13] K. Banaszek, C. Radzewicz, and K. Wódkiewicz, Phys. Rev. A 60, 674-677 (1999); S. Deléglise, I. Dotsenko, C. Sayrin, J. Bernu, M. Brune, J. Raimond, and S. Haroche, Nature 455, 510-514 (2008).
[14] P. Reimann and M. Kastner, New. J. Phys 14 (2012) 043020; A. J. Short, New. J. Phys 13 (2011) 053009.