Starting with three dimensional Chern–Simons theory with gauge group $SL(N, \mathbb{R})$, we derive an action $S_{\text{cov}}$ invariant under both left and right $W_N$ transformations. We give an interpretation of $S_{\text{cov}}$ in terms of anomalies, and discuss its relation with Toda theory.

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1. Introduction

A few years ago Zamolodchikov made a systematic study of the possible extensions of the Virasoro algebra [1]. Besides the extensions involving Kac-Moody and superconformal currents, he found a new non-linear extension of the Virasoro algebra, based on the occurrence of a spin-three field \( W \). This field, together with the usual stress energy tensor \( T \), forms the so-called \( W_3 \) algebra, which is defined by the following operator product expansions:

\[
\begin{align*}
T(z)T(w) & \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}, \\
T(z)W(w) & \sim \frac{3W(w)}{(z-w)^2} + \frac{\partial W(w)}{z-w}, \\
W(z)W(w) & \sim \frac{c/3}{(z-w)^6} + \frac{2T(w)}{(z-w)^4} + \frac{\partial T(w)}{(z-w)^3} \\
& \quad + \frac{1}{(z-w)^2} \left( \frac{3}{10} \partial^2 T(w) + 2b^2 \Lambda(w) \right) \\
& \quad + \frac{1}{z-w} \left( \frac{1}{15} \partial^3 T(w) + b^2 \partial \Lambda(w) \right), 
\end{align*}
\]

where the non-linear term \( \Lambda(z) \) is defined as

\[
\Lambda(z) = : T(z)T(z) : - \frac{3}{10} \partial^2 T(z),
\]

and the constant \( b^2 \), determined by associativity, reads

\[
b^2 = \frac{16}{22 + 5c}.
\]

The above algebra can be generalized to the case of a \( W_N \) algebra, which contains fields \( W^{(k)} \) of spin \( k \), with \( k \) running from 2 to \( N \). In [2] it was shown that this algebra is intimately related to the affine Kac-Moody algebra associated to \( su(N) \), in the sense that the fields \( W^{(k)} \) can be constructed from the Kac-Moody currents by using the higher-order Casimir invariants of the underlying finite dimensional Lie algebra. Besides this connection with affine Kac-Moody algebras, it is now clear that these non-linear extensions of the Virasoro algebra play a central role in many other areas of two dimensional physics. They have been shown to appear in Toda
theories [3], gauged WZW models [4], reductions of the KP hierarchy [5, 20] and in the matrix model formulation of two dimensional quantum gravity [4].

Despite the relevance of these $W$ algebras for the above mentioned branches of two dimensional physics, which makes it clear that they represent some universal structure, it is fair to say that some aspects of these algebras are still poorly understood. Although by now we have many different realizations of these algebras and most of the algebraic aspects are well sorted out, it is the geometrical interpretation of these algebras which is still missing. Whereas we know that the Virasoro algebra arises after gauge fixing the two dimensional diffeomorphism invariance (conformal gauge), a similar geometrical interpretation for the $W$ case is lacking.

In this paper we will try to uncover some of the mysteries of ‘$W$ geometry’ by constructing an action $S_{\text{cov}}$ which has local $W$ transformations as its symmetries. This action then describes what is commonly denoted as $W$ gravity. $S_{\text{cov}}$ can be viewed as the $W$ generalization of the covariant action for pure gravity, first constructed by Polyakov [8]:

$$S = \frac{c}{96\pi} \int \int R \Box^{-1} R.$$  \hspace{1cm} (1.4)

We will construct $S_{\text{cov}}$ by starting with a topological gauge theory in three dimensions, namely Chern–Simons theory [12], on a three manifold of the form $M = \Sigma \times \mathbb{R}$, where $\Sigma$ is a two dimensional Riemann surface. What we will end up with is then a theory for $W$ gravity on $\Sigma$. The reason for this approach is that we believe that the moduli space of $W_N$ gravity is related (if not equal) to the moduli space of flat $SL(N, \mathbb{R})$ bundles. This latter space appears naturally as the classical phase space of $SL(N, \mathbb{R})$ Chern–Simons theory, which hints towards a possible connection between this theory and $W$ gravity. Indeed, it was shown by H. Verlinde that the above action for pure gravity (1.4) can be obtained from $SL(2, \mathbb{R})$ Chern–Simons theory [13].

There have been previous constructions of actions which admit local $W_3$ symmetries, which all amount to gauging of the $W_3$ transformations. This technique has led to both a chiral action and to a fully covariant action for $W_3$ gravity [19]. As we will show, our construction results in an action closely related to the one of [19]. An important advantage of our method is that we can apply it quite generally to $SL(N, \mathbb{R})$, resulting in the covariant action for $W_N$ gravity. Specializing to the case of $SL(3, \mathbb{R})$ gives our result for $W_3$ gravity, which was already reported in [14]. Another advantage is that formulating the $W_N$ gravity in terms of $SL(N, \mathbb{R})$
Chern–Simons theory should make it more tractable to study its moduli space.

This paper is organized as follows. In section 2 we will review some general- 
esties about Chern–Simons theory. We will introduce the concepts needed for the construction of the covariant action, such as wavefunctions, polarization, the inner- 
product, the Kähler potential etc. Furthermore, we will illustrate our method by first reviewing the construction of the covariant action in the so-called ‘standard po-
larization’ [21, 25]. Next, in sections 3 and 4, we will compute the covariant action $S_{\text{cov}}$ for a different choice of polarization, namely the choice which relates $SL(N, \mathbb{R})$ transformations to $W_N$ transformations. It will turn out that this first result for the covariant action admits a large symmetry group, which can be used to gauge away some of the degrees of freedom. In addition $S_{\text{cov}}$ contains a number of auxiliary fields which can be eliminated by replacing them by their equations of motion. Details will be given for the case of ordinary and $W_3$ gravity. In section 5 we will prove the invariance of $S_{\text{cov}}$ under left and right $W_N$ transformations, discuss its relation with Toda theory, and give an interpretation of $S_{\text{cov}}$ in terms of anomalies. Finally, we will address some open problems and give some concluding remarks in section 6.

2. Chern–Simons theory

Chern–Simons theory on a three manifold $M$ is described by the action

$$S = \frac{k}{4\pi i} \int_M \text{Tr}(\tilde{A} \wedge \tilde{d}\tilde{A} + \frac{2}{3} \tilde{A} \wedge \tilde{A} \wedge \tilde{A}),$$  \hspace{1cm} (2.1)$$

where the connection $\tilde{A}$ is a one form with values in the Lie algebra $\mathfrak{g}$ of some Lie group $G$, and $\tilde{d}$ denotes the exterior derivative on $M$. In this paper $M$ will be of the form $M = \Sigma \times \mathbb{R}$, $\Sigma$ being a Riemann surface, for which $\tilde{A}$ and $\tilde{d}$ can be decomposed into space and time components, i.e. $\tilde{A} = A_0 dt + A$, with $A = A_z dz + A_{\bar{z}} d\bar{z}$, and $\tilde{d} = dt \partial / \partial t + d$. Rewriting the action as

$$S = \frac{k}{4\pi i} \int dt \int_{\Sigma} \text{Tr}(A \wedge \partial_t A + 2A_0(dA + A \wedge A)), \hspace{1cm} (2.2)$$

we recognize that $A_0$ acts as a Lagrange multiplier which implements the constraint $F = dA + A \wedge A = 0$. Furthermore, we deduce from this action the following
non-vanishing Poisson brackets

\[ \{ A^a_z(z), A^b_z(w) \} = \frac{2\pi i}{k} \eta^{ab} \delta(z - w), \]  

(2.3)

where \( A_z = \sum_a A^a_z T^a \), with \( \text{Tr}(T^a T^b) = \eta^{ab} \).

Upon quantizing the theory we have to replace the above Poisson bracket by a commutator, and we have to choose a ‘polarization.’ This simply means that we have to divide the set of variables \( (A^a_z, \bar{A}^a_z) \) into two subsets. One subset will contain fields \( X_i \) and the other subset will consist of derivatives \( \frac{\delta}{\delta X_i} \), in accordance with (2.3). The choice of these subsets is called a choice of polarization. We will denote the subset containing the fields by \( \Pi_i A_z \) and \( \Pi_k \bar{A}_z \), where \( \Pi_i \) and \( \Pi_k \) are certain projections on subspaces of the Lie algebra \( g \). The standard polarization corresponds to the case where \( \Pi_i = 1 \) and \( \Pi_k = 0 \).

Of course we also have to incorporate the Gauss law constraints \( F(A_z, \bar{A}_z) = 0 \). Following [13, 22, 24] we will impose these constraints after quantization. So we will first consider a ‘big’ Hilbert space obtained by quantization of (2.3), and then select the physical wavefunctions \( \Psi \) by requiring \( F(A_z, \bar{A}_z) \Psi = 0 \).

As an example of this construction let us consider the standard polarization, in which the set of fields is given by the \( A^a_z \), so the \( \bar{A}^a_z \) act as derivatives \( -\frac{2\pi}{k} \frac{\delta}{\delta A^a_z} \). This implies that the physical wavefunctions will be functions of \( A_z \). For this choice of polarization it is well known [21] that the solution of the zero-curvature constraint

\[ F \left( A_z, \frac{2\pi}{k} \frac{\delta}{\delta A_z} \right) \Psi(A_z) = - : \frac{2\pi}{k} \partial \frac{\delta}{\delta A_z} + \bar{\partial} A_z + \frac{2\pi}{k} [A_z, \frac{\delta}{\delta A_z}] : \Psi(A_z) = 0, \]  

(2.4)

is given by:

\[ \Psi(A_z) = \exp S(A_z) = \exp -S_{WZW}(g), \]  

(2.5)

where \( S_{WZW}(g) \) is the Wess-Zumino-Witten action:

\[ S_{WZW}(g) = \frac{k}{4\pi} \int d^2 z \ Tr(g^{-1} \partial g g^{-1} \partial g) - \frac{k}{12\pi} \int_B Tr(g^{-1} dg)^3, \]  

(2.6)

and \( A_z \) and \( g \) are related via \( A_z = g^{-1} \partial g \).

If we now want to compute transition amplitudes between some initial physical state \( \Psi_1 \) and some final state \( \Psi_2 \), we have to consider the inner-product between
these states in Chern–Simons theory. (The evolution operator is simply the identity here, as the Hamiltonian vanishes for a topological theory.) The expression for such an inner-product is:

\[
\langle \Psi_1 \mid \Psi_2 \rangle = \int DA e^{K(A_z, A_{\bar{z}})} \bar{\Psi}_1(A_{\bar{z}}) \Psi_2(A_z).
\] (2.7)

This formula should be read as follows. (i) \(DA\) is short for \(DA_z DA_{\bar{z}}\). (ii) \(\bar{\Psi}(A_{\bar{z}})\) is the solution of the zero-curvature constraint, but now with the role of \(A_z\) and \(A_{\bar{z}}\) interchanged. So it is given by:

\[
\bar{\Psi}(A_{\bar{z}}) = \exp \bar{S}(A_{\bar{z}}) = \exp -S_{WZW}(h),
\] (2.8)

with \(A_{\bar{z}} = h \partial h^{-1}\). (iii) The Kähler term \(K(A_z, A_{\bar{z}})\) appears since we want to take the inner-product between wavefunctions depending on different variables, \(A_z\) and \(A_{\bar{z}}\), which are conjugate variables. So we should perform a ‘Fourier’ transformation.

In (2.7) this is automatically taken care off if one does the integral over \(A_{\bar{z}}\) (or \(A_z\)), provided that the Kähler term takes the form:

\[
K(A_z, A_{\bar{z}}) = \frac{k}{2\pi} \int d^2 z \ Tr(A_z A_{\bar{z}}).
\] (2.9)

The inner-product can now be written as:

\[
\langle \Psi_1 \mid \Psi_2 \rangle = \int DA \exp S_{\text{cov}}(A_z, A_{\bar{z}}),
\] (2.10)

where \(S_{\text{cov}}(A_z, A_{\bar{z}}) = S(A_z) + \bar{S}(A_{\bar{z}}) + K(A_z, A_{\bar{z}})\) is a covariant action, i.e. invariant under both left and right transformations given by

\[
\begin{align*}
\delta A_z &= \partial \eta + [A_z, \eta], \\
\delta A_{\bar{z}} &= \bar{\partial} \eta + [A_{\bar{z}}, \eta].
\end{align*}
\] (2.11)

Expressed in terms of the group variables \(g, h\) \(S_{\text{cov}}\) takes the following form:

\[
S_{\text{cov}} = -S_{WZW}(g) - S_{WZW}(h) - \frac{k}{2\pi} \int d^2 z \ Tr(g^{-1} \partial g \bar{\partial} h h^{-1}).
\] (2.12)
This action is invariant under the transformations \( g \rightarrow gf, h \rightarrow f^{-1}h \), which can be easily proven if one makes of the Polyakov–Wiegman formula [11]:

\[
S_{WZW}(gf) = S_{WZW}(g) + S_{WZW}(f) + \frac{k}{2\pi} \int d^2z \, \text{Tr}(g^{-1}\partial g \bar{\partial} ff^{-1}).
\]

(2.13)

(In fact the invariance of \( S_{\text{cov}} \) under \( g \rightarrow gf, h \rightarrow f^{-1}h \) is just the integrated form of (2.11).) From this invariance one suspects that it should be possible to write \( S_{\text{cov}} \) in terms of the invariant product \( G = gh \). Indeed, comparing (2.12) with (2.13), it is evident that the covariant action is given by [9, 25]:

\[
S_{\text{cov}} = -S_{WZW}(G).
\]

(2.14)

In this paper we will repeat the above steps for a different polarization, namely one which relates the \( \text{Sl}(N, \mathbb{R}) \) gauge transformations to \( W_N \) transformations. For this polarization we will solve the Gauss law constraint \( F \Psi = 0 \), compute the inner-product between two wavefunctions \( \Psi_1 = \exp S \) and \( \bar{\Psi}_2 = \exp \bar{S} \), add a Kähler term \( K \), to end up with a covariant action \( S_{\text{cov}} = S + S + K \). \( S_{\text{cov}} \) will depend on two group variables \( g, h \), which are elements of a certain subgroup of \( \text{Sl}(N, \mathbb{R}) \), and on \( 2(N-1) \) parameters \( \mu_i, \bar{\mu}_i \) \( (i = 2, \ldots, N) \) which will play the role of conjugate variables of the \( W_i, \bar{W}_i \) fields of the \( W_N \) algebra. We will show that for this non-standard polarization \( S_{\text{cov}} \) is again invariant under transformations of the form \( g \rightarrow gf, h \rightarrow f^{-1}h \), where \( f \) is now restricted to a \( \text{Sl}(N - 1, \mathbb{R}) \times \mathbb{R} \) subgroup of \( \text{Sl}(N, \mathbb{R}) \). Using this invariance we will be able to rewrite the action in terms of the invariant product \( G = gh \). The resulting action \( S_{\text{cov}}(G) \) is our definition for the covariant action for \( W_N \) gravity. This action is invariant under both left and right \( W_N \) transformations. So, in comparison to the standard polarization, we have somehow split the \( \text{Sl}(N, \mathbb{R}) \) symmetry transformations into \( \text{Sl}(N - 1, \mathbb{R}) \times \mathbb{R} \) and \( W_N, \bar{W}_N \) transformations. Note that this splitting is in agreement with the following dimension formula

\[
\dim sl(N, \mathbb{R}) = \dim sl(N - 1, \mathbb{R}) + 1 + 2(N - 1).
\]

(2.15)
3. The Solution of the Zero-Curvature Constraint

In this section we will solve the zero-curvature constraints of Chern–Simons theory for \( G = Sl(N, \mathbb{R}) \) in a certain, nonstandard polarization, following closely the strategy of [14]. Let us first state the main result of this section: in the polarization where we take as fields \( \Pi_i A_z \) and \( \Pi_k A_{\bar{z}} \), and as derivatives with respect to these fields therefore \( \Pi^\dagger_i A_z \) and \( \Pi^\dagger_k A_{\bar{z}} \), the solutions of the zero-curvature constraints are wave functions of the form:

\[
\Psi[\Pi_i A_z, \Pi_k A_{\bar{z}}] = e^{S(\Pi_i A_z, \Pi_k A_{\bar{z}})}\Psi[\mu_2, \ldots, \mu_N],
\]

(3.1)

where \( \mu_k \) denote the \((1-k, 1)\) differentials that occur naturally in \( W_N \)-gravity (for instance, \( \mu_2 \) is just the well-known Beltrami-differential), and \( \Psi[\mu_2, \ldots, \mu_N] \) solves the Ward identities of the classical \( W_N \)-algebra [20]. The action \( S \) is given in equation (3.24). The nonstandard polarization is given by the projections \( \Pi_i \) and \( \Pi_k \). These are projections on certain subspaces of the Lie algebra \( sl(N, \mathbb{R}) \), that form closed sub-Lie algebras. The subalgebra \( \Pi_k sl(N, \mathbb{R}) \) is the abelian subalgebra that consists of all \( N \times N \) matrices \( M_{ij} \) with \( M_{ij} = 0 \) unless \( i < N \) and \( j = N \). The subalgebra \( \Pi_i sl(N, \mathbb{R}) \) consists of all traceless \( N \times N \) matrices \( M_{ij} \) with \( M_{ij} = 0 \) if \( i = N \) and \( j < N \). More explicitly:

\[
\Pi_i sl(N, \mathbb{R}) = \begin{pmatrix}
* & \cdots & * & * \\
: & \vdots & : & \\
* & \cdots & * & * \\
0 & \cdots & 0 & *
\end{pmatrix}, \\
\Pi_k sl(N, \mathbb{R}) = \begin{pmatrix}
0 & \cdots & 0 & * \\
: & \vdots & : & \\
0 & \cdots & 0 & 0
\end{pmatrix}.
\]

(3.2)

The projections \( \Pi^\dagger_i \) and \( \Pi^\dagger_k \) are defined through \( \Pi^\dagger_i = 1 - \Pi_i \) and \( \Pi^\dagger_k = 1 - \Pi_k \). Note that for arbitrary \( X, Y \in sl(N, \mathbb{R}) \) we have \( \text{Tr}(\Pi_i X \Pi_k Y) = 0 \), so that the Poisson bracket (2.3) of any two fields is indeed zero. Another important property is that

\[
X \in \Pi_i g, \ Y \in \Pi_k g \Rightarrow [X, Y] \in \Pi_k g.
\]

(3.3)

In order to understand why we need solutions of the classical \( W_N \)-Ward identities in (3.1), we will first investigate the relation between the zero curvature equation \( F(A_z, A_{\bar{z}}) = 0 \) and the classical \( W_N \)-Ward identities.
3.1. The $W_N$-Ward identities

The relation between $W_N$-Ward identities and zero-curvature equations is essentially due to Drinfel’d and Sokolov [5], who showed that taking a particular form for the connection $A_z$ gives in a natural way (via Hamiltonian reduction) the second Gelfand-Dickii bracket [6]. In turn, it is known [20] that these brackets reproduce exactly the classical form of the operator expansions of $W_N$-algebras. It is therefore a natural starting point to take the same form for $A_z$ as Drinfel’d and Sokolov did:

\[
A_z^0 = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 0 \\
0 & 0 & 0 & \cdots & 0 & 1 \\
W_N & W_{N-1} & W_{N-2} & \cdots & W_2 & 0
\end{pmatrix}.
\] (3.4)

We will write this also as

\[
A_z^0 = \Lambda + W
\] (3.5)

where $\Lambda$ denotes the matrix with only the one’s next to the diagonal, and $W$ denotes the piece containing only the fields $W_i$, so that $W \in \Pi \hat{\mathbf{g}}_k$. We will sometimes also write $\Lambda = \sum_{i=1}^{N-1} e_{i,i+1}$, where $e_{i,j}$ is the matrix with a one in its $(i,j)$ entry, and zeroes everywhere else. The zero-curvature equation now reads

\[
F = \partial A_z^0 - \bar{\partial} W + [\Lambda + W, A_z^0] = 0.
\] (3.6)

Since $A_z$ and $A_{\bar{z}}$ are conjugate variables with respect to the Poisson bracket (2.3), and $A_z^0$ still contains arbitrary fields $W_i$, we will put the fields conjugate to the $W_i$ in $A_{\bar{z}}$. These fields are precisely the $(1-k, 1)$ differentials $\mu_k$, and we therefore require that

\[
\Pi_k A_{\bar{z}}^0 = \sum_{i=1}^{N-1} \mu_{N+1-i} e_{i,N}.
\] (3.7)

In matrix notation this means that $A_{\bar{z}}^0$ has the form

\[
A_{\bar{z}}^0 = \begin{pmatrix}
* & \cdots & * & \mu_N \\
* & \cdots & * & \mu_{N-1} \\
\vdots & \vdots & \vdots & \vdots \\
* & \cdots & * & \mu_2 \\
* & \cdots & * & * 
\end{pmatrix}.
\] (3.8)
It is now possible to solve the equation $\Pi_i F = 0$ and to determine all other entries in $A^0_\bar{z}$. All other entries in $A^0_\bar{z}$ are polynomials in $\mu_i$, $W_i$, and their derivatives. What remains are the $N-1$ equations $\Pi^\dagger_i F = 0$, and these are precisely the Ward-identities of the $W_N$-algebra, which follows from [5]. The explicit form of $A^0_\bar{z}$ is given in the appendix. For example, in the case of $Sl(2, \mathbb{R})$ one finds

$$A^0_\bar{z} = \begin{pmatrix} \frac{1}{2} \partial \mu & \mu \\ \mu T - \frac{1}{2} \partial^2 \mu & -\frac{1}{2} \partial \mu \end{pmatrix},$$

and the remaining zero-curvature equation reads

$$0 = F_{21} = -\frac{1}{2} \partial^3 \mu + \partial(\mu T) + (\partial \mu)T - \overline{\partial} T.$$  \hspace{1cm} (3.10)$$

Here we used the standard notation $W_2 = T$.

In Chern–Simons theory we have, however, an arbitrary connection $A$, and not a special one like (3.4). It turns out that we need gauge transformations to obtain $W_N$-Ward identities from arbitrary connections $A$.

### 3.2. The Role of Gauge Transformations

If we impose zero-curvature constraints on wave-functions $\Psi$, then we must specify (as in ordinary quantum mechanics) how we quantize the expression $F(A_\bar{z}, A_z) = \partial A_\bar{z} - \overline{\partial} A_z + [A_z, A_\bar{z}]$, if we replace $\Pi^\dagger_i A_\bar{z}$ and $\Pi^\dagger_k A_z$ by functional derivatives with respect to $\Pi_i A_\bar{z}$ and $\Pi_k A_z$. We will simply put all derivatives to the right of the fields. Looking at the expression for the Poisson bracket (2.3), we see that, when acting on wave functions $\Psi[\mu_2, \ldots, \mu_N]$, $W_i$ should be identified with $\frac{2\pi i}{k} \delta_{\mu_i}$. The statement that $\Psi[\mu_2, \ldots, \mu_N]$ solves the $W_N$-Ward identities, is equivalent to

$$F(A^0_\bar{z}, A^0_z) \Psi[\mu_2, \ldots, \mu_N] = 0.$$ \hspace{1cm} (3.11)$$

At this point we make the crucial observation, that if $g$ is an arbitrary $Sl(N, \mathbb{R})$-valued function, independent of the $\mu_i$, equation (3.11) implies

$$F((A^0_\bar{z})^g, (A^0_z)^g) \Psi = g^{-1} F(A^0_\bar{z}, A^0_z) g \Psi = 0,$$

\hspace{1cm} (3.12)$

*A different choice would differ from ours by terms that are of higher order in $1/c$; because our approach is valid only up to the lowest order in $1/c$, in which case the quantum $W_N$-algebra reduces to the classical one, we can completely neglect such differences.*
where \((A_z^0)^g\) and \((A_z^0)^\bar{g}\) denote the gauge transformed connections \(g^{-1}A_z^0 g + g^{-1}\partial g\) and \(g^{-1}A_z^0 g + g^{-1}\bar{\partial} g\). We will assume that this is the most general curvature one can write down, which annihilates those \(\Psi\) that solve the \(W_N\)-Ward identities.

3.3. The Solution

Let us now go back to the original problem, that is, solving the zero-curvature constraint \(F\psi = 0\), where

\[
\Pi^k_i A_z = \frac{2\pi}{k} \frac{\delta}{\delta \Pi^k_i A_z}, \tag{3.13}
\]

and

\[
\Pi^i_j A_z = -\frac{2\pi}{k} \frac{\delta}{\delta \Pi^i_j A_z}, \tag{3.14}
\]

and let us look for solutions of type (3.1), i.e. \(\psi = e^S \Psi\). Multiplying the zero-curvature equation with \(e^{-S}\), it reads \((e^{-S}Fe^S)\Psi = 0\). This equation can also be written as

\[
F \left( A_z + \frac{2\pi}{k} \frac{\delta S}{\delta \Pi^k_i A_z}, A_{\bar{z}} - \frac{2\pi}{k} \frac{\delta S}{\delta \Pi^i_j A_z} \right) \Psi = 0. \tag{3.15}
\]

\(F\) contains, in general, double derivatives, giving rise to terms \((\frac{2\pi}{\kappa})^2 \frac{\delta^2 S}{\delta A_z \delta A_{\bar{z}}}\) when working out \(e^{-S}Fe^S\). However, these terms are of higher order in \(1/c\), and, as discussed previously, we will ignore these. Now we want to show that any \(\Psi\) that solves the \(W_N\)-Ward identities, is a solution of (3.15). Because we assumed that the most general curvature which annihilates such wave functions \(\Psi\) is given by (3.12), we find, upon comparing (3.12) and (3.15), that solutions of the form (3.1) exist if and only if we can find an \(S\) such that

\[
A_z + \frac{2\pi}{k} \frac{\delta S}{\delta \Pi^k_i A_z} = (A_z^0)^g + \text{derivatives not containing } \frac{\delta}{\delta \mu_i}, \tag{3.16}
\]

\[
A_{\bar{z}} - \frac{2\pi}{k} \frac{\delta S}{\delta \Pi^i_j A_z} = (A_{\bar{z}}^0)^g + \text{derivatives not containing } \frac{\delta}{\delta \mu_i}. \tag{3.17}
\]

Restricting (3.16) to the part in \(\Pi_i g\), we find that \(\Pi_i A_z = \Pi_i (g^{-1}\partial g + g^{-1}\Lambda g + g^{-1}Wg)\). The matrix \(W\) contains derivatives with respect to the \(\mu_i\), and we do not want derivatives in the parametrization of our fields. Therefore, \(g\) should satisfy \(\Pi_i (g^{-1}Wg) = 0\). One may easily verify that this restricts \(g\) to be an element of \(G_P = \exp(\Pi_i^1 g)\), the group which has \(\Pi_i^1 g\) as its Lie algebra. This shows that we
must parametrize $\Pi_i A_z$ via

$$\Pi_i A_z = \Pi_i (g^{-1} \partial g + g^{-1} A g) \equiv \Pi_i \Lambda^g,$$  \hfill (3.18)

where we defined $\Lambda^g = g^{-1} \Lambda g + g^{-1} \partial g$. Similarly, taking the $\Pi_k$ of (3.17), we find the parametrization for $\Pi_k A_{\bar{z}}$:

$$\Pi_k A_{\bar{z}} = \Pi_k (g^{-1} A_{\bar{z}}^0 g + g^{-1} \partial g) = \Pi_k (g^{-1} A_{\bar{z}}^0 g).$$  \hfill (3.19)

Due to the fact that $\Pi_k (g^{-1} A_{\bar{z}}^0 g) = \Pi_k (g^{-1} (\Pi_k A_{\bar{z}}^0) g)$, no derivatives will enter in the definition of $\Pi_k A_{\bar{z}}$ either, and we conclude that we have a parametrization of the $N^2 - 1$ fields $\Pi_i A_z$ and $\Pi_k A_{\bar{z}}$ in terms of $N^2 - 1$ independent variables, the $N^2 - N$ components of $g \in G_P$, and the $N - 1$ variables $\mu_i$. The equations, which we still have to solve, are the components of (3.16) in $\Pi_k^* g$, and the components of (3.17) in $\Pi_i^* g$. They read, when separated into pieces which do and do not contain derivatives,

$$\frac{2\pi}{k} \frac{\delta S}{\delta \Pi_k A_{\bar{z}}} = \Pi_k^* \Lambda^g,$$  \hfill (3.20)

$$\frac{2\pi}{k} \frac{\delta S}{\delta \Pi_i A_z} = \Pi_i^* (g^{-1} A_{z,f}^0 g + g^{-1} \partial g),$$  \hfill (3.21)

$$\frac{2\pi}{k} \frac{\delta}{\delta \Pi_k A_{\bar{z}}} = \Pi_k^* (g^{-1} A_{\bar{z}}^0 g) + \text{derivatives not containing } \frac{\delta}{\delta \mu_i},$$  \hfill (3.22)

$$\frac{2\pi}{k} \frac{\delta}{\delta \Pi_i A_z} = \Pi_i^* (g^{-1} A_{z,d}^0 g) + \text{derivatives not containing } \frac{\delta}{\delta \mu_i},$$  \hfill (3.23)

where $A_{\bar{z}}^0 = A_{\bar{z},f}^0 + A_{\bar{z},d}^0$, and $A_{\bar{z},d}^0$ is the part of $A_{\bar{z}}^0$ containing the derivatives. As is shown in the appendix, the last two equations are automatically satisfied in the parametrization (3.18) and (3.19). Thus it remains to solve $S$ from (3.20) and (3.21). This can be done, resulting in

$$S = \frac{k}{2\pi} \int d^2 z \, \text{Tr}(\Pi_k (g^{-1} A_{\bar{z}}^0 g) \Pi_k^* \Lambda^g) - \frac{k}{2\pi} \int d^2 z \, \text{Tr}(\Lambda \partial g g^{-1}) - S_{WZW}(g),$$  \hfill (3.24)

where $S_{WZW}$ is the Wess-Zumino-Witten action defined in (2.6). Equation (3.20) follows straightforwardly from (3.24), because varying $\Pi_k A_{\bar{z}}$ while keeping $\Pi_i A_z$ constant, means we only need to vary $A_{\bar{z}}^0$, or, equivalently, the $\mu_i$. Under such a variation,

$$\delta S = \frac{k}{2\pi} \int d^2 z \, \text{Tr}(\delta (\Pi_k A_{\bar{z}}) \Pi_k^* \Lambda^g),$$  \hfill (3.25)
showing (3.20). To demonstrate (3.21), we have to vary $\Pi_i A_z$ while keeping $\Pi_k A_{\bar{z}}$ constant. If we denote the corresponding variation of $g$ by $\delta g$, we find

$$\delta S = \frac{k}{2\pi} \int d^2 z \; \text{Tr}(\Pi_k (g^{-1} A^0_{\bar{z}} g) \delta (\Pi_k^\dagger \Lambda^g)) - \frac{k}{2\pi} \int d^2 z \; \text{Tr}(\Lambda \delta (\bar{g} g^{-1} \Lambda)) - \delta S_{WZW}(g). \quad (3.26)$$

Using $\Pi_k = 1 - \Pi_i^\dagger$ the first term in (3.26) can be written as

$$\frac{k}{2\pi} \int d^2 z \; \text{Tr}((g^{-1} A^0_{\bar{z}}, f g + g^{-1} \bar{g}) \delta (\Lambda^g) - \Pi_i^\dagger (g^{-1} A^0_{\bar{z}}, f g + g^{-1} \bar{g}) \delta (\Pi_i A_z)). \quad (3.27)$$

The second term of the last expression is already what we want, so we would like the remainder of $\delta S$ to vanish. Using straightforward algebra, this remainder can be written as

$$\delta S = -\frac{k}{2\pi} \int d^2 z \; \text{Tr}(F(\Lambda, g^{-1} A^0_{\bar{z}}, f g + g^{-1} \bar{g})(g^{-1} \delta g)) \quad (3.28)$$

Recall that we constructed $A^0_{\bar{z}}$ in such a way that $\Pi_i F(\Lambda + W, A^0_{\bar{z}}, f + A^0_{\bar{z}}, d) = 0$; restricting this to the piece containing no derivatives, gives $\Pi_i F(\Lambda, A^0_{\bar{z}}, f) = 0$. As $\delta g g^{-1} \in \Pi_i^\dagger g$, it follows immediately that (3.28) vanishes, proving the validity of (3.21).

The wave functions $e^S \Psi$ obtained here are the analogue of (2.5) in a different polarization. In fact, they should be seen as the Fourier transform of (2.5) with respect to $\Pi_i^\dagger A_{\bar{z}}$. The first term of (3.24) can be more or less understood as arising from this Fourier transform. Furthermore, we see that in (3.24) part of the WZW action has survived in the form $S_{WZW}(g)$. The whole action (3.24) bears an interesting similarity with the wave functions introduced in [24]. The wave functions $\Psi$ that solve the classical $W_N$-Ward identities can be obtained from a constrained WZW model as in [21, 28]. Having solved the zero-curvature equations, we can in the next section proceed with the computation of inner-products in this polarization.

4. The Covariant Action

As explained in section 2 we have to compute the inner-product between two wavefunctions $\Psi_1$ and $\bar{\Psi}_2$ to obtain our first result for the covariant action for $W_N$. 

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gravity. The wavefunction $\bar{\Psi}$ can be constructed in a similar way as $\Psi$ was constructed in the previous section. Introducing gauge fields $B_z, B_{\bar{z}}$, where the fields are now in $\Pi^\dagger_k B_z$ and $\Pi^\dagger_i B_{\bar{z}}$, the wavefunction $\bar{\Psi}$ takes the form:

$$\bar{\Psi}[\Pi^\dagger_k B_z, \Pi^\dagger_i B_{\bar{z}}] = e^{\bar{S}(\Pi^\dagger_k B_z, \Pi^\dagger_i B_{\bar{z}})} \bar{\Psi}[\bar{\mu}_2, \ldots, \bar{\mu}_N].$$  (4.1)

Here $\Pi^\dagger_i B_z$ and $\Pi^\dagger_i B_{\bar{z}}$ are parametrized by

$$\Pi^\dagger_i B_z = \Pi^\dagger_k (hB_z^0 h^{-1} - \partial hh^{-1}),$$
$$\Pi^\dagger_i B_{\bar{z}} = \Pi^\dagger_i (h\bar{\Lambda} h^{-1} - \partial hh^{-1}) \equiv \Pi^\dagger_i \bar{\Lambda}^h,$$  (4.2)

with $\bar{\Lambda} = \Lambda^t = \sum_{i=1}^{N-1} e_{i+1,i}$,

$$\Pi^\dagger_i B_z^0 = \sum_{i=1}^{N-1} \bar{\mu}_{N+1-i} e_{N,i},$$  (4.3)

and the other components of $B_z^0$ can be computed from the condition $\Pi^\dagger F(B_z^0, \bar{\Lambda}) = 0$. In (4.2) $h \in \exp(\Pi_i sl(N, \mathbb{R}))$, and $\bar{S}$ appearing in (4.1) is given by:

$$\bar{S} = \frac{k}{2\pi} \int d^2 z \, \text{Tr}(\Pi^\dagger_k (hB_z^0 h^{-1}) \Pi^\dagger_k \bar{\Lambda}^h) + \frac{k}{2\pi} \int d^2 z \, \text{Tr}(\bar{\Lambda} h^{-1} \partial h) - S_{WZW}(h).$$  (4.4)

The last ingredient we need in our construction of the covariant action is the Kähler form. Since this Kähler form should establish the Fourier transformation from $\Pi_i A_z$ to $\Pi^\dagger_i B_z$ and from $\Pi^\dagger_i B_z$ to $\Pi_k A_{\bar{z}}$ (or vice versa), it is given by:

$$K(A, B) = \frac{k}{2\pi} \int d^2 z \, \text{Tr}(\Pi_i A_z \Pi^\dagger_i B_z - \Pi_k A_{\bar{z}} \Pi^\dagger_k B_z),$$  (4.5)

(where the minus sign follows from standard Fourier theory: if one uses $e^{ipx}$ as integration kernel to transform from $x$ to $p$, one should use $e^{-ipx}$ to go from $p$ to $x$).

Using the explicit form of the inner-product

$$\langle \Psi_1 | \Psi_2 \rangle = \int D(\Pi_i A_z) D(\Pi_k A_{\bar{z}}) D(\Pi^\dagger_k B_z) D(\Pi^\dagger_i B_{\bar{z}}) e^{S + \bar{S} + K} \bar{\Psi}_1 [\bar{\mu}] \Psi_2 [\mu]$$
$$\equiv \int D(\Pi_i A_z) D(\Pi_k A_{\bar{z}}) D(\Pi^\dagger_k B_z) D(\Pi^\dagger_i B_{\bar{z}}) e^{S_{\text{cov}}(A, B)},$$  (4.6)

we can now read off the covariant action for $W_N$ gravity. Writing

$$\Psi[\mu_2, \ldots, \mu_N] = \exp - S_{W_N}(\mu),$$  (4.7)
this result reads:

\[
S_{\text{cov}} = \frac{k}{2\pi} \int d^2 z \, \text{Tr}(\Pi_k (g^{-1} A^0_z g) \Pi_k^\dagger A^g) - \frac{k}{2\pi} \int d^2 z \, \text{Tr}(\Lambda \tilde{\partial} gg^{-1}) - S_{\text{WZW}}(g) \\
+ \frac{k}{2\pi} \int d^2 z \, \text{Tr}(\Pi_k (h B_z h^{-1}) \Pi_k \tilde{A}^b) + \frac{k}{2\pi} \int d^2 z \, \text{Tr}(\Lambda h^{-1} \partial h) - S_{\text{WZW}}(h) \\
+ \frac{k}{2\pi} \int d^2 z \, \text{Tr}(\Pi_i A_z \Pi_i^\dagger B_z - \Pi_k A_z \Pi_k^\dagger B_z) - S_{W_N}(\mu) - \bar{S}_{W_N}(\bar{\mu}).
\]

(4.8)

This action depends on \(g, h, \mu, \bar{\mu}\) which amounts to a total number of variables given by: \(2N(N - 1) + 2(N - 1) = 2(N^2 - 1)\). On the other hand, one expects \(W_N\) gravity to be described by the fields \(g_{\mu\nu}, d_{\mu\nu\rho}, \ldots\), which are symmetric tensors of rank 2, 3, \ldots, \(N\), resulting in a total number of degrees of freedom given by: \(3 + 4 + \cdots + (N + 1) = \frac{1}{2}(N - 1)(N + 4)\). The discrepancy between the total number of degrees of freedom in \(S_{\text{cov}}\) as given in (4.8), and the total number of degrees of freedom in \(W_N\) gravity can be partially resolved as follows: (i) \(S_{\text{cov}}\) is invariant under a \(Sl(N - 1, \mathbb{R}) \times \mathbb{R}\) symmetry group, which can be used to gauge away \((N - 1)^2\) degrees of freedom, (ii) in \(S_{\text{cov}}\) there are auxiliary fields (i.e. fields which appear only algebraically in \(S_{\text{cov}}\)), which can be eliminated by replacing them by their equations of motion.

4.1. Symmetries of the Action

To investigate the symmetries of the above action (4.8) we define another projection operator \(\Pi_y = \Pi_i \Pi_i^\dagger\), by decomposing an arbitrary element of \(sl(N, \mathbb{R})\) in the following way:

\[
sl(N, \mathbb{R}) = \Pi_k sl(N, \mathbb{R}) \oplus \Pi_y sl(N, \mathbb{R}) \oplus \Pi_k^\dagger sl(N, \mathbb{R}),
\]

(4.9)

so an element \(f \in \exp(\Pi_y sl(N, \mathbb{R}))\) is of the form

\[
f = \begin{pmatrix} * & \cdots & * & 0 \\ \vdots & \ddots & \vdots & \vdots \\ * & \cdots & * & 0 \\ 0 & \cdots & 0 & * \end{pmatrix},
\]

(4.10)

and they form a \(Sl(N - 1, \mathbb{R}) \times \mathbb{R}\) subgroup. We expect that \(S_{\text{cov}}\) will be invariant under transformations of the form \(g \to gf\) and \(h \to f^{-1}h\). (This is motivated by
the fact that for the standard polarization (where \( \Pi_k = 0 \) so \( \Pi_y = 1 \)) \( f \) would be an arbitrary element of \( SL(N, \mathbb{R}) \), and, as we saw in section 2, for that case these transformations indeed leave the covariant action invariant.) So let us study how \( S \) changes under the transformation \( g \to gf \) and \( h \to f^{-1}h \), with \( f \) as in (1.1).

Note that for any element \( f \in \exp(\Pi_y sl(N, \mathbb{R})) \) we have

\[
\Ad_f(\Pi_{i,k} \in sl(N, \mathbb{R})) \subset \Pi_{i,k} \in sl(N, \mathbb{R}),
\]

(and the same holds for \( \Pi_{i,k}^\dagger \)), where \( \Ad_f(g) = f^{-1}gf \). One easily checks that this implies, that for any \( X \in sl(N, \mathbb{R}) \) we have \( \Pi_{i,k} \Ad_f(X) = \Pi_{i,k} \Ad_f(\Pi_{i,k} X) = \Ad_f(\Pi_{i,k} X) \), i.e.

\[
\Pi_{i,k} \circ \Ad_f = \Ad_f \circ \Pi_{i,k}
\]

(and again the same holds for the \( \Pi_{i,k}^\dagger \)).

Using (4.12) and the fact that \( \Pi_{i,k}^\dagger f^{-1} \partial f = 0 \) we see that the first terms in the first and second line of (4.8) are invariant, whereas the rest of the action changes as:

\[
\delta S_{cov} = \frac{k}{2\pi} \int d^2z \left( g^{-1} \Lambda g \bar{\partial} f f^{-1} - S_{WZW}(f) - \frac{k}{2\pi} \int d^2z \left( g^{-1} \partial g \bar{\partial} f f^{-1} \right) \right) - \frac{k}{2\pi} \int d^2z \left( h \bar{\Lambda} h^{-1} \partial f f^{-1} - S_{WZW}(f^{-1}) + \frac{k}{2\pi} \int d^2z \left( \partial ff^{-1} \bar{\Lambda} h^{-1} \right) \right) + \frac{k}{2\pi} \int d^2z \left( \partial ff^{-1} \bar{\Lambda} + \bar{\partial} f f^{-1} \Lambda + f^{-1} \partial ff^{-1} \bar{\partial} \right),
\]

where we used the Polyakov-Wiegman formula (2.13) and the fact that \( \Tr(X \Pi_{i,k} Y) = \Tr(Y \Pi_{i,k}^\dagger X) \). Since

\[
S_{WZW}(f) + S_{WZW}(f^{-1}) = \frac{k}{2\pi} \int d^2z \left( f^{-1} \partial ff^{-1} \bar{\partial} f \right),
\]

it follows that \( S_{cov} \) is invariant under the above transformations.

In a similar way as was done in section 2 for the standard polarization it is now straightforward to write down the action in terms of the invariant product \( G = gh \). We find

\[
S_{cov} = \Delta S - S_{W_N}(\mu) - \tilde{S}_{W_N}(\tilde{\mu}),
\]

with

\[
\Delta S = \frac{k}{2\pi} \int d^2z \left( \Lambda G \bar{\Lambda} G^{-1} \right) + \frac{k}{2\pi} \int d^2z \left( \bar{\Lambda} G^{-1} \partial G \right) - \frac{k}{2\pi} \int d^2z \left( \Lambda \bar{\partial} GG^{-1} \right) - \frac{k}{2\pi} \int d^2z \left( \Pi_k(A_0^z - \Lambda G) \Pi_y G \Pi_k^\dagger (B_0^z - \Lambda G) \Pi_y G^{-1} \right) - S_{WZW}(G),
\]

(4.15)
where we used the following decomposition for $G$: $G = \Pi_k^\dagger G \Pi_k G$, which in terms of matrices looks like:

$$G = \begin{pmatrix}
1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 1
\end{pmatrix}
\begin{pmatrix}
* & \cdots & * \\
\vdots & \ddots & \vdots \\
* & \cdots & * \\
0 & \cdots & 0
\end{pmatrix}
\begin{pmatrix}
1 & \cdots & * \\
\vdots & \ddots & \vdots \\
* & \cdots & * \\
1 & * & 1
\end{pmatrix}. \quad (4.16)$$

$S_{\text{cov}} = \Delta S - S_{W_N} (\mu) - \bar{S}_{W_N} (\bar{\mu})$ is our final result for the covariant action for $W_N$ gravity. Note that in $S_{\text{cov}}$ only the $\Pi_k$ part of $A_0^z$ and the $\Pi_k^\dagger$ part of $B_0^z$ appear. Thus specifying $A_0^z, B_0^z$ as in (3.7), (4.3) is sufficient to compute the action. As we will see in the next subsection it turns out that $S_{\text{cov}}$ still contains redundant degrees of freedom.

4.2. Auxiliary Fields

At this stage our action depends on $G, \mu_i, \bar{\mu}_i$, so we have reduced the number of degrees of freedom to: $N^2 - 1 + 2(N - 1) = (N + 3)(N - 1)$, which is still more than one would naively expect for $W_N$ gravity. Below we will further reduce this number due to the observation that some fields only appear algebraically in the action, i.e. auxiliary fields, which can in principle be eliminated by replacing them by their equations of motion. Unfortunately, the general expression (4.15) valid for all $\text{Sl}(N, \mathbb{R})$ is not suitable for the determination of auxiliary fields. Instead, we have to use an explicit parametrization for $G$ in (4.15) in order to isolate the auxiliary fields. We will work this out for the case of $\text{Sl}(2, \mathbb{R})$ and $\text{Sl}(3, \mathbb{R})$.

$\text{Sl}(2, \mathbb{R})$:

For this case we parametrize our $G$ as follows:

$$G = \begin{pmatrix}
1 & 0 \\
\omega & 1
\end{pmatrix}
\begin{pmatrix}
e^\phi & 0 \\
0 & e^{-\phi}
\end{pmatrix}
\begin{pmatrix}
1 & -\bar{\omega} \\
0 & 1
\end{pmatrix}. \quad (4.17)$$

Labeling $A_0^z, B_0^z$ as in (3.7), (4.3), respectively, we find $S_{\text{cov}} = \Delta S - S_{W_2} [\mu] - \bar{S}_{W_2} [\bar{\mu}]$ with,
\[
\Delta S = -\frac{k}{2\pi} \int d^2z \left[ \partial \phi \overline{\partial} \phi + \omega (2\partial \phi + \partial \mu) + \bar{\omega} (2\partial \phi + \partial \bar{\mu}) \\
+ \mu \omega^2 + \bar{\mu} \bar{\omega}^2 + 2\omega \bar{\omega} - (1 - \mu \bar{\mu}) e^{-2\phi} \right],
\tag{4.18}
\]

and \( S_{W_2}[\mu] \) is the solution of the Ward-identity

\[
(\bar{\partial} - \mu \partial - 2\partial \mu) \frac{\delta S_{W_2}[\mu]}{\delta \mu} = -\frac{k}{4\pi} \partial^3 \mu.
\tag{4.19}
\]

From (4.18) we recognize that \( \omega, \bar{\omega} \) are auxiliary fields. Replacing these fields by their equations of motion gives:

\[
S_{\text{cov}} = S_L[\phi, \mu, \bar{\mu}] + K[\mu, \bar{\mu}] - S_{W_2}[\mu] - \bar{S}_{W_2}[\bar{\mu}].
\tag{4.20}
\]

Here

\[
S_L = \frac{k}{4\pi} \int d^2z \sqrt{-\hat{g}} \left( \hat{g}^{ab} \partial_a \phi \partial_b \phi + 4 e^{-2\phi} + \hat{R} \right),
\tag{4.21}
\]

is the well-known Liouville action, the metric \( \hat{g} \) is defined by \( ds^2 = |dz + \mu d\bar{z}|^2 \), and \( K[\mu, \bar{\mu}] \) reads

\[
K[\mu, \bar{\mu}] = \frac{k}{4\pi} \int d^2z \left( 1 - \mu \bar{\mu} \right)^{-1} \left( \partial \mu \overline{\partial} \mu - \frac{1}{2} \mu (\partial \bar{\mu})^2 - \frac{1}{2} \bar{\mu} (\partial \mu)^2 \right).
\tag{4.22}
\]

So, finally, we have reduced our set of fields to the three basic ones, namely the three components by which we label the metric \( g = e^{-2\phi} \hat{g} \). \( S_{\text{cov}} \) as given in (4.20) is our final result for the case of \( \text{Sl}(2, \mathbb{R}) \), and is in fact almost equivalent to Polyakov’s action for induced 2D gravity:

\[
S_{\text{cov}} = \frac{c}{96\pi} \int \int R \Box^{-1} R,
\tag{4.23}
\]

written out in components for the metric \( g = e^{-2\phi} \hat{g} \). The only difference is the cosmological term \( \int d^2z \sqrt{-\hat{g}} e^{-2\phi} \), which one usually adds to the induced action. The absolute value of the coefficient in front of the cosmological term is not important, as it can be arbitrarily rescaled by adding a constant to \( \phi \).

Note that the term in the Liouville action (4.21) that is linear in \( \phi \), is proportional to the curvature \( R \). This is due to the fact that the trace or Weyl anomaly
is also proportional to the curvature. Actually, this fact can already be seen from (4.18), where we did not yet eliminate $\omega$ and $\bar{\omega}$. The term linear in $\phi$ in (4.18) is proportional to $\int \phi d\omega$, where $\omega$ is the one form $\omega dz - \bar{\omega} d\bar{z}$. This shows that $\omega$ should be interpreted as being the spin connection [13], since $R$ is the curvature of the spin connection.

$Sl(3, \mathbb{R})$

This case was already extensively discussed in [14]. There it was shown that if we take the following Gauss decomposition for $G$:

$$G = \left( \begin{array}{ccc} 1 & 0 & 0 \\ \omega_1 & 1 & 0 \\ \omega_3 & \omega_2 & 1 \end{array} \right) \left( \begin{array}{ccc} e^{\varphi_1} & 0 & 0 \\ 0 & e^{\varphi_2 - \varphi_1} & 0 \\ 0 & 0 & e^{-\varphi_2} \end{array} \right) \left( \begin{array}{ccc} 1 & -\bar{\omega}_1 & -\bar{\omega}_3 \\ 0 & 1 & -\bar{\omega}_2 \\ 0 & 0 & 1 \end{array} \right),$$

(4.24)

and parametrize $A^i_0$ and $B^0_z$ again as in (3.7) and (4.3) (with $\mu \equiv \mu_2$, $\nu \equiv \mu_3$), the action contains $\omega_3, \bar{\omega}_3$ as auxiliary fields. Substituting their equations of motion results in the following action: $S_{\text{cov}} = \Delta S = S_{W_3}[\mu, \nu] - \bar{S}_{W_3}[\bar{\mu}, \bar{\nu}]$, with

$$\Delta S = \frac{k}{2\pi} \int d^2z \left\{ \frac{1}{2} A^{ij} \partial_i \varphi_j \bar{\partial} \varphi_j + \sum_i e^{-A^{ij}_3 \bar{\varphi}_j} - A^{ij}_3 (\omega_i + \partial \varphi_i) (\bar{\omega}_j + \bar{\partial} \varphi_j) \right\} \right)$$

(4.25)

and $A^{ij}$ is the Cartan matrix of $Sl(3, \mathbb{R})$ $A^{ij} = \left( \begin{array}{cc} 2 & -1 \\ -1 & 2 \end{array} \right)$. $T, W, T, W$ are defined through the following Fateev-Lyukhanov [29] construction:

$$(\partial - \omega_2)(\partial - \omega_1 + \omega_2)(\partial + \omega_1) = \partial^3 + T \partial - W + \frac{1}{2} \partial T,$$

(4.26)

and we shifted $\mu \rightarrow \mu - \frac{1}{2} \partial \nu$, $\bar{\mu} \rightarrow \bar{\mu} + \frac{1}{2} \partial \bar{\nu}$. The first part of $\Delta S$ is precisely a chiral $Sl(3, \mathbb{R})$ Toda action, confirming the suspected relation between $W_3$-gravity and Toda theory, see also section 5.2. Actually, one would expect that in a "conformal gauge", the covariant $W_3$-action will reduce to a Toda action. Indeed, if we put $\nu = \bar{\nu} = 0$ in $\Delta S$, then also $\omega_1, \omega_2, \bar{\omega}_1, \bar{\omega}_2$ become auxiliary fields. Substituting their equations of motion as well, we find that

$$\Delta S = \frac{k}{4\pi} \int d^2z \sqrt{-\hat{g}} \left( \frac{1}{2} \hat{g}^{ab} \partial_a \varphi_i \partial_b \varphi_j A^{ij} + 4 \sum_i e^{-A^{ij}_3 \bar{\varphi}_j} + R \hat{\xi}_j \cdot \varphi \right) + 4K[\mu, \bar{\mu}],$$

(4.27)
where $K[\mu, \bar{\mu}]$ is the same expression as for the $SL(2, \mathbb{R})$ case (4.24), and $\hat{g}$ is again given by $ds^2 = |dz + \mu d\bar{z}|^2$. In the case of $SL(3, \mathbb{R})$, $\overline{\xi} \cdot \overline{\varphi}$, with $\overline{\xi}$ being one half times the sum of the positive roots, is just given by $\varphi_1 + \varphi_2$. The action (4.27) is the same Toda action that was originally present in $\Delta S$ in a chiral form, and the integration over $\omega_1, \omega_1, \omega_2, \omega_2$ has the effect of coupling it to a background metric $\hat{g}$.

Of course, the most interesting part of the action is the part containing $\nu, \bar{\nu}$. Unfortunately, if we do not put $\nu = \bar{\nu} = 0$, we can integrate over either $\rho_1, \bar{\rho}_1$ or over $\rho_2, \bar{\rho}_2$, but not over both at the same time, due to the presence of third order terms in $\Delta S$. Another clue regarding the contents of the action (4.23) can be obtained by treating the second and third line in (4.25) as perturbations of the first line of (4.25). This means that we try to make an expansion in terms of $\mu, \bar{\mu}, \nu, \bar{\nu}$. The saddlepoint of the $\omega$-terms is at $\omega_i = -\partial \varphi_i$ and $\bar{\omega}_i = -\bar{\partial} \varphi_i$. From (4.26) we can now see that $T, W, \bar{T}, \bar{W}$ are, when evaluated in this saddle point, the (anti)holomorphic energy momentum tensor and $W_3$-field that are present in a chiral Toda theory

$$
T = -\frac{1}{2} A^{ij} \partial \varphi_i \partial \varphi_j - \overline{\xi} \cdot \partial^2 \varphi, \\
W = -\partial \varphi_1 (\partial^2 \varphi_2 + \frac{1}{2} \partial^2 \varphi_2 - \partial^2 \varphi_1) + \frac{1}{2} \partial^3 \varphi_1 - (1 \leftrightarrow 2),
$$

(4.28)

and similar expressions for $\bar{T}, \bar{W}$.

This suggests that the full action $\Delta S$ contains the generating functional for the correlators of the energy-momentum tensor and the $W_3$-field of a Toda theory, “covariantly” coupled to $W_3$-gravity. The presence of the third order terms in $W, \bar{W}$ in (4.25) prevents us from computing the action of this covariantly coupled Toda theory. The same structure is also present in the action for $W_N$ gravity, as we will discuss in section 5.2.

In a similar way as for $SL(2, \mathbb{R})$, the terms linear in $\varphi_i$ in (4.25) are expected to be related to the Weyl and $W_3$-Weyl anomaly. The term linear in $\varphi_i$ in (4.27) shows that the Weyl anomaly is related to shifts in $\overline{\xi} \cdot \overline{\varphi} = \varphi_1 + \varphi_2$. This suggests that the $W_3$-Weyl anomaly is related to shifts in directions orthogonal to $\overline{\xi}$, that is, to shifts of $\varphi_1 - \varphi_2$. Writing the terms in (4.25) linear in $\varphi_i$ as

$$
\frac{k}{4\pi} \int (\varphi_1 + \varphi_2) d\omega_+ + 3(\varphi_1 - \varphi_2) d\omega_-,
$$

(4.29)

where the one forms $\omega_+$ and $\omega_-$ are given by

$$
\omega_\pm = (\omega_1 \pm \omega_2) dz - (\bar{\omega}_1 \pm \bar{\omega}_2) d\bar{z},
$$

(4.30)
we see that $\omega_+$ plays the role of the spin connection, whereas $\omega_-$ is some kind of $W_3$-spin connection, whose curvature is presumably related to the $W_3$-Weyl anomaly. These statements may acquire a more precise meaning when comparing (4.25) with a one-loop computation for the induced action of $W_3$-gravity, starting with the action in [13].

5. Properties of the Covariant Action

At this stage the reader may well wonder, what all that we have done so far has to do with covariant $W_N$-gravity. This is probably best explained by looking at the case of ordinary two-dimensional quantum gravity, and before proceeding with general $W_N$-gravity, we will first discuss this much simpler and better understood case.

5.1. 2D Quantum Gravity

Given any action $S(g_{ab}, \varphi)$, where $\varphi$ denotes some set of matter fields, that is both invariant under Weyl transformations $g_{ab} \to e^\rho g_{ab}$ and under general co-ordinate transformations, one can define an induced action for 2D gravity

$$e^{-S_{\text{ind}}(g_{ab})} = \int D\varphi e^{-S(g_{ab}, \varphi)},$$

(5.1)

by integrating out the matter fields $\varphi$. If the theory had no anomalies, $S_{\text{ind}}$ would reduce to an action defined on the moduli space of Riemann surfaces. However, it is well known that there are anomalies, resulting in a non-trivial $g$-dependence of $S_{\text{ind}}$. The precise form of these anomalies depends, of course, on the choice of regularization scheme used in performing the path integral over the matter fields. A scheme often used in conformal field theory is $\zeta$-function regularization [15]. This is a diffeomorphism, but not Weyl invariant regularization method. In [16] it was shown that if $S$ is the action for a $b - c$ system of spin $j$, and one parametrizes $g$ via $ds^2 = e^\rho |dz + \mu d\bar{z}|^2$, one can compute $\frac{\delta S_{\text{ind}}}{\delta \rho}$, which is proportional to the Liouville action, and $\frac{\delta^2 S_{\text{ind}}}{\delta \rho \delta \mu}$, which is also non-vanishing, due to the lack of holomorphic factorization of $S_{\text{ind}}$. We will call this the holomorphic anomaly.
In Chern-Simons theory, the holomorphic wave-functions contain $Ψ[µ]$, and the anti-holomorphic wave-functions contain $Ψ[¯µ]$, which are both solutions of the Virasoro Ward-identities. These wave functions can only carry a holomorphic or anti-holomorphic diffeomorphism anomaly, but not the two anomalies of the type mentioned before. Therefore, these wave-functions do not fit naturally in the $ζ$-function regularization scheme, but in another where the only anomalies are the holomorphic and anti-holomorphic diffeomorphism anomaly. However, it is known that these two schemes are related to each other via a local counterterm $\Delta Γ$. This counterterm consists of two pieces, a Liouville action and another term $K$, to cancel the holomorphic anomaly, which is proportional to (4.22). Thus

$$e^{-S_{ind}} = e^{-S(µ)-\bar{S}(\bar{µ})}. \tag{5.2}$$

All of this strongly suggests that the action $\Delta S = S + \bar{S} + K$ occurring in the inner-product (4.1) is, in the case of $SL(2, ℝ)$, nothing but the local counterterm $\Delta Γ$. That this is indeed the case was shown in section 4.1 (see [13]): upon integrating over the nonpropagating fields $ω, \bar{ω}$, the covariant action becomes precisely equal to the local counterterm $\Delta Γ$. To make the connection more precise, if we write $Ψ(µ) = e^{-S(µ)}$, then (5.2) can be rewritten as

$$e^{-S_{ind}} = e^{\Delta Γ}Ψ(µ)Ψ(\bar{µ}), \tag{5.3}$$

and we see that the partition function of induced gravity is just the inner product as computed in Chern-Simons theory. This also explains the name ‘covariant’, because clearly we have not fixed any gauge in (5.3). For $W_N$, we expect that a similar picture exists, although we have no concrete realization of it. What we do have is the covariant action, and before discussing possible implications this action has for $W_{N}$-gravity, we will first study this action in some more detail.

5.2. Relation to Toda Theory

In the case of ordinary gravity, the covariant action contains the Liouville action. The natural generalization of the Liouville action is the Toda action based on $sl(N, ℝ)$, which is known to be deeply related to $W_{N}$-algebras [3]. Indeed, we will now show that the covariant action is closely related to Toda theory. The cases
$N = 2, 3$ were already dealt with in section 4, and we will now consider the general case. The relation to Toda theory is most easily established by putting $\mu_i = \bar{\mu}_i = 0$. The action is then given by (see (1.13))

$$S = \frac{k}{2\pi} \int d^2 z \ Tr(\Lambda G \bar{G}^{-1}) + \frac{k}{2\pi} \int d^2 z \ Tr(\bar{\Lambda} G^{-1} \partial G) - \frac{k}{2\pi} \int d^2 z \ Tr(\Lambda \partial \bar{G}G^{-1}) - S_{WZW}(G) - \frac{k}{2\pi} \int d^2 z \ Tr(\Pi_k \bar{\Lambda} \Pi_k \Lambda^g),$$

(5.4)

where $G = gh$, as explained in section 4. Under a variation of $h$

$$\delta S = -\frac{k}{2\pi} \int d^2 z \ Tr((h^{-1}\delta h)F(\Lambda^G - h^{-1} \Pi_k^\dagger(\Lambda^g)h, \bar{\Lambda})), \quad (5.5)$$

and under a variation of $g$

$$\delta S = -\frac{k}{2\pi} \int d^2 z \ Tr((\delta g g^{-1})F(\Lambda, \bar{\Lambda} G - g\Pi_k(\bar{\Lambda}^h)g^{-1})). \quad (5.6)$$

Instead of $G = gh$ we will now use a Gauss decomposition $G = n_1 b n_2$ for $G$, where $n_1$ is lower triangular, $b$ is diagonal, and $n_2$ is upper triangular, and restrict ourselves to variations of $n_1$ and $n_2$, for which (5.3) and (5.9) are still valid. In terms of this Gauss decomposition, they can be written as

$$\delta S = -\frac{k}{2\pi} \int d^2 z \ Tr((n_2^{-1}\delta n_2)F(n_2^{-1}(b^{-1}\partial b)n_2 + n_2^{-1}b^{-1}\Pi_i(\Lambda^{n_1})bn_2, \bar{\Lambda})), \quad (5.7)$$

and

$$\delta S = -\frac{k}{2\pi} \int d^2 z \ Tr((\delta n_1 n_1^{-1})F(\Lambda, -n_1(\partial bb^{-1})n_1^{-1} - n_1 b\Pi_i^\dagger(\bar{\Lambda}^{n_2}b^{-1}n_1^{-1}))). \quad (5.8)$$

The equations of motion for $n_2$ and $n_1$ read therefore

$$\Pi_\prec F(n_2^{-1}(b^{-1}\partial b)n_2 + n_2^{-1}b^{-1}\Pi_i(\Lambda^{n_1})bn_2, \bar{\Lambda}) = 0, \quad (5.9)$$

and

$$\Pi_\succ F(\Lambda, -n_1(\partial bb^{-1})n_1^{-1} - n_1 b\Pi_i^\dagger(\bar{\Lambda}^{n_2}b^{-1}n_1^{-1})) = 0, \quad (5.10)$$

where $\Pi_\prec$ and $\Pi_\succ$ denote the projections on the space of lower and upper triangular matrices respectively. The first equation (5.9) can be solved as follows: $\Pi_\prec F(X, \bar{\Lambda}) = 0$ is certainly true if $X$ is upper triangular, and $X$ is upper triangular.
if and only if \( n_2 X n_2^{-1} \) is. Applying this to \( F \) in (5.9), we see that the equations of motion for \( n_2 \) are solved if \( \Pi_i(\Lambda^{n_1 b}) \) is upper triangular. This is the case if there exists an element \( A \in \Pi_k g \) such that

\[
n_1^{-1} \partial n_1 + n_1^{-1} \Lambda n_1 = A - \partial b b^{-1} + \Lambda. \tag{5.11}
\]

In a similar way, (5.10) is solved if there is an element \( B \in \Pi_k g \) such that

\[
- \partial n_2 n_2^{-1} + n_2 \tilde{\Lambda} n_2^{-1} = B + b^{-1} \partial b + \tilde{\Lambda}. \tag{5.12}
\]

If we now replace \( A \) by \(-n_1^{-1} A n_1\) and \( B \) by \(-n_2 B n_2^{-1}\), we see that \( n_1 \) is a gauge transformation relating the connections \( \Lambda - \partial b b^{-1} \) and \( \Lambda + A \), and \( n_2 \) is a gauge transformation relating \( \tilde{\Lambda} + b^{-1} \partial b \) and \( \tilde{\Lambda} + B \). These transformations are well known: they are the Miura transformations that have been used \([29]\) to produce free field expressions for \( W_N \)-algebras. Here, the matrices \( A \) and \( B \) will contain these free field representations. Let us now substitute the equations of motion for \( n_1 \) and \( n_2 \) back into the full covariant action (4.15). After some manipulations, it reads

\[
S = \frac{k}{4\pi} \int d^2 z \, \text{Tr}(b^{-1} \partial b b^{-1} \partial b) + \frac{k}{2\pi} \int d^2 z \, \text{Tr}(b \tilde{\Lambda} b^{-1} \Lambda) - \frac{k}{2\pi} \int d^2 z \, \text{Tr}(A^0_A) - \frac{k}{2\pi} \int d^2 z \, \text{Tr}(B^0_B) - \frac{k}{2\pi} \int d^2 z \, \text{Tr}(\Pi_k(n_1^{-1} A^0_z n_1)b \Pi_k^\dagger(n_2 B^0_z n_2^{-1})b^{-1}). \tag{5.13}
\]

The first two terms are just an expression for a Toda theory in a flat background metric; the more conventional Toda action follows immediately by substituting \( b = \exp(\text{diag}(\phi_1, \phi_2 - \phi_1, \ldots, \phi_{N-1} - \phi_{N-2}, -\phi_{N-1})) \). The first two terms read, when expressed in terms of \( \phi_i \),

\[
\frac{k}{4\pi} \int d^2 z \, A^{ij} \partial \phi_i \bar{\partial} \phi_j + \frac{k}{2\pi} \int d^2 z \, \sum_i e^{-A^{ij} \phi_i}, \tag{5.14}
\]

where \( A^{ij} \) denotes the Cartan matrix of \( sl(N, \mathbb{R}) \). The third and fourth term of (5.13) show that, to lowest order, \( \mu_i \) and \( \bar{\mu}_i \) couple simply the fields \( W_i \) and \( \bar{W}_i \), as one would construct them from the Toda theory. The last, mixing term in (5.13) has no simple interpretation in the Toda theory.

One can in principle go through the same exercise with \( \mu_2 \) and \( \bar{\mu}_2 \) unequal to zero; this was worked out in \([14]\) for the case of \( W_3 \), see also sect. 4.2., and the result in that case is that the Toda theory gets coupled to a non-trivial background...
metric, determined by $\mu_2$ and $\bar{\mu}_2$. We expect that the same thing is true for general $W_N$ algebras, although we have not tried to do the computation. If one also puts other $\mu_i$ or $\bar{\mu}_i$ unequal to zero, it is much more difficult to solve the equations of motion for $n_1$ and $n_2$ in full generality, and we have been unable to do so.

From the previous calculations, one might be tempted to conclude that the covariant action is just the generating function for the correlators of the $W_i$ and $\bar{W}_i$ fields in a Toda theory. This is, however, not true, because most of the fields in $n_1$ and $n_2$ are not just auxiliary fields, and one cannot therefore in general just substitute their equation of motion back into the action. We will make a few more comments about this in the next section.

5.3. $W,\bar{W}$-Invariance of the Covariant Action

There is an interesting analogy between the covariant action for $W_N$-gravity, and the covariant action given in (2.10). If we start with an action $S = \int d^2z \bar{\psi}\gamma^\alpha(\partial_\alpha + A_\alpha)\psi$, then we can repeat the story in section 5.1, and define and induced action by integrating over the fermions [9]. Now $\Delta^\gamma = K(A_z, A_{\bar{z}})$ is a kind of ‘chiral anomaly’, with $K$ given by (2.9). However, from another point of view, $K$ is needed to restore the gauge invariances (2.11). If we adopt this point of view here, the covariant action would arise as an action needed to restore $W$ and $\bar{W}$ invariance. Thus it seems natural to look for the action of the $W$ and $\bar{W}$ algebra on the covariant action, and to check whether the covariant action indeed restores $W$ and $\bar{W}$ invariance.

It turns out that this invariance is indeed present, although the expressions involved are rather cumbersome. We will, therefore, only describe the $W$-transformations, and omit the tedious proof that these leave the full covariant action invariant. We also will not give the $\bar{W}$-transformations, but they can be easily written down, once the $W$-transformations are given.

First consider the wave-functions $\Psi(\mu_2, \ldots, \mu_N)$. As we discussed in section 3.1, the Ward identities that annihilate these wave functions are $F(\Lambda + W, A^0_z)\Psi = 0$. When acting on $\Psi$, $W_i$ is given by $\frac{2\pi}{k} \frac{\delta \Psi}{\delta \mu_i}$. If we substitute these expressions for $W_i$ back into $\Lambda + W$ and $A^0_z$, we will denote the resulting expressions by $\Lambda + W_\Psi$ and

---

*The precise form of the covariant action does, however, not seem to be completely fixed by this requirement alone.*
The Ward identities are then simply
\[ F(\Lambda + W_\psi, A^0_\bar{z},\psi) = 0. \] (5.15)

On the other hand, the counterterm \( S + \bar{S} + K = \triangle S \) induces also certain fields \( W_i \), obtained by differentiating it with respect to \( \mu_i \). The corresponding matrix \( W \) containing these \( W_i \) will be denoted by \( W_{\text{ind}} \) and is easily found from (4.8)
\[ W_{\text{ind}} = -\frac{2\pi}{k} g \Pi^i_k (g^{-1} \partial g + g^{-1} \Lambda g - h B_0^k h^{-1}) g^{-1}. \] (5.16)

In the same way as we constructed \( A^0_\bar{z} \) in section 3.1, we can construct an \( X \in g \) such that
\[ \Pi_i F(\Lambda + W_{\text{ind}}, X) = 0, \] (5.17)

once we specify \( \Pi_k X \). If
\[ \Pi_k X = \begin{pmatrix} 0 & \cdots & 0 & \epsilon_N \\ 0 & \cdots & 0 & \epsilon_{N-1} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \epsilon_2 \\ 0 & \cdots & 0 & 0 \end{pmatrix}, \] (5.18)

we will denote the corresponding solution \( X \) of (5.17) by \( X(\epsilon) \). We now define the following transformation rules for \( G \) and the \( \mu_i \):
\[ \delta_\epsilon G = X(\epsilon) G, \] (5.19)
\[ \delta_\epsilon (\Pi_k A^0_\bar{z},\psi) = -\Pi_k (\bar{\partial} X(\epsilon) + [A^0_\bar{z},\psi, X(\epsilon)]), \] (5.20)

where the \( \epsilon_i \) are the parameters of the \( W_i \)-transformations. One can prove that (i) these transformations form a \( W_N \) algebra, and (ii) that these transformations leave the inner product invariant. The transformation rules (5.19) and (5.20) look like ordinary gauge transformations, although in this case \( X \) is field dependent. All this is closely related to the well-known fact that \( W \)-transformations can be realized as field-dependent \( sl(N, \mathbb{R}) \) gauge transformations \[ [21, 10] \]; more precisely, they are the gauge transformations that preserve the form (3.4) of a connection. Here, the same mechanism is working, although in a different setting. For instance, one of the curious features of the transformation rule (5.20) is that \( \delta_\epsilon \mu_i \) can contain \( \frac{\delta \Psi}{\delta \mu_j} \). Only
for \( W_2 \) this is not the case. In that case the relatively simple transformation rules, leaving invariant the covariant \( W_2 \) action, see (4.18) and (4.19), are given by

\[
\begin{align*}
\delta_\epsilon \mu &= -\bar{\epsilon} \partial \epsilon - \epsilon \partial \mu + \mu \partial \epsilon, \\
\delta_\epsilon \phi &= \frac{1}{2} \partial \epsilon + \epsilon \omega, \\
\delta_\epsilon \bar{\omega} &= -\epsilon \bar{e}^{-2\phi}, \\
\delta_\epsilon \omega &= -\frac{1}{2} \partial^2 \epsilon + \bar{\mu} \epsilon \bar{e}^{-2\phi} - \epsilon \partial \omega - \omega \partial \epsilon.
\end{align*}
\] (5.21)

In \( \delta_\epsilon \mu \) one recognizes the transformation rule for a Beltrami differential under an infinitesimal co-ordinate transformation. In fact, after substituting the equations of motion for \( \omega, \bar{\omega}, \) we have

\[
\delta_\epsilon \mu = -\bar{\epsilon} \partial \epsilon - \epsilon \partial \mu + \mu \partial \epsilon, \\
\delta_\epsilon \phi = \frac{1}{2} \partial \epsilon - \epsilon \omega,
\] (5.22)

which we expect to be the ordinary transformation rules for the components of the metric \( g \), defined by \( ds^2 = e^{-2\phi}|dz + \mu d\bar{z}|^2 \), under general co-ordinate transformations. Under the transformation \( x^\mu \rightarrow x'^\mu = x^\mu - \xi^\mu \) the metric changes as:

\[
\delta g_{ab} = -g_{ac} \partial_d \xi^c - g_{ac} \partial_b \xi^c - \xi^c \partial_c g_{ab}.
\] (5.23)

Writing \( \xi^a = (\xi, \bar{\xi}) \), this implies the following transformation rules for the components of the metric defined by \( ds^2 = e^{-2\phi}|dz + \mu d\bar{z}|^2 \):

\[
\begin{align*}
\delta_{\xi,\bar{\xi}} \phi &= \frac{1}{2} \partial \xi + \frac{1}{2} \partial \bar{\xi} + \frac{1}{2} \mu \partial \xi + \frac{1}{2} \bar{\mu} \partial \bar{\xi} - \xi \partial \phi - \bar{\xi} \partial \bar{\phi}, \\
\delta_{\xi,\bar{\xi}} \mu &= -\bar{\xi} \partial \xi + \mu \partial \xi - \mu \partial \bar{\xi} - \xi \partial \mu + \bar{\xi} \partial \bar{\xi} + \mu^2 \partial \xi, \\
\delta_{\xi,\bar{\xi}} \bar{\mu} &= -\bar{\xi} \partial \bar{\xi} + \bar{\mu} \partial \bar{\xi} - \bar{\mu} \partial \bar{\xi} - \bar{\bar{\xi}} \partial \bar{\mu} - \bar{\bar{\xi}} \bar{\partial} \bar{\mu} + \bar{\bar{\mu}}^2 \partial \xi.
\end{align*}
\] (5.24)

Redefining our parameters as follows: \( \epsilon = \xi + \mu \bar{\xi}, \bar{\epsilon} = \bar{\xi} + \bar{\mu} \xi \), the above rules become

\[
\begin{align*}
\delta_{\epsilon,\bar{\epsilon}} \phi &= \frac{1}{2} \partial \epsilon + \frac{1}{2} \partial \bar{\epsilon} - \frac{\epsilon - \bar{\mu} \epsilon}{1 - \mu \bar{\mu}} (\partial \phi + \frac{1}{2} \partial \bar{\mu}) - \frac{\bar{\epsilon} - \mu \epsilon}{1 - \mu \bar{\mu}} (\bar{\partial} \phi + \frac{1}{2} \bar{\partial} \mu), \\
\delta_{\epsilon,\bar{\epsilon}} \mu &= -\bar{\epsilon} \partial \epsilon + \mu \partial \epsilon - \epsilon \partial \mu, \\
\delta_{\epsilon,\bar{\epsilon}} \bar{\mu} &= -\bar{\epsilon} \partial \bar{\epsilon} + \bar{\mu} \partial \bar{\epsilon} - \bar{\epsilon} \partial \bar{\mu},
\end{align*}
\] (5.25)

which are the same as in (5.22) for the chiral case \( \bar{\epsilon} = 0 \). To find the general case one should of course look at the \( \bar{W} \) analogue of (5.19) and (5.20).
Having established some of the basic properties of the covariant action, we will now discuss the implications the covariant action has for $W$-gravity.

6. Discussion

One of the main gaps in our knowledge of $W$-gravity, is that we do not know what the proper set of fields is, on which the $W$-algebra acts. In other words, what is the counterpart of the metric in the case of $W_N$-gravity? Naively, one might think that one just has to add symmetric tensor fields of rank 3, ..., $N$, as we already mentioned in section 4, to produce fields with the right spin. One has, however, not been able to perform this construction in full detail, and it is still quite a mystery how such a construction should work, in which presumably the conformal factors of the tensor fields should play the role of Toda fields, quantization gives rise to anomalies, the generalized Weyl anomaly gives rise to the Toda action, etc. It is also conceivable that certain auxiliary fields are needed to form a full ‘$W$-multiplet’, and that part of these auxiliary fields become propagating on the quantum level. This shows that it is difficult to count a priori the number of degrees of freedom of $W$-gravity.

It is possible to give an upper limit for the number of degrees of freedom of $W$-gravity, because the covariant action certainly has all degrees of freedom in it. Although we worked only up to lowest order in $1/c$, we expect that the higher order corrections will essentially only give a field and coupling constant renormalization, as is the case for $W_2$ [23] and seems to be the case for $W_3$ as well [26]. Therefore, we can just count the number of degrees of freedom in the covariant action, and it is given by $(N + 3)(N - 1)$. From this we should certainly subtract the number of auxiliary fields in the covariant action. We do not know what this number is for the general case, but for $W_2$ it is two, and in the case of $W_3$ it is at least two (see [14] and section 4.2). However, integrating out more than two fields results in this case in a nonpolynomial action, whose precise meaning is rather obscure. What is certainly not true, is that all the off-diagonal elements of $G$ are auxiliary fields for $N > 2$, so that it is in general impossible to reduce the covariant action to a Toda-like action. This happens only if we put the off-diagonal elements of $G$ on-shell and $\mu_i = \bar{\mu}_i = 0$, as explained in section 5.2.
If we compare the $W_3$-action of $[14]$ with the action for $W_3$ strings given in $[18]$, we see that the two are partially identical, except that the action in $[18]$ does not have terms which couple $\mu_i$ and $\bar{\mu}_i$, nor terms involving exponentials, whereas our action does have both such terms. This is due to the fact that their action is describing a matter system with $W_3$ symmetry, whereas our action describes the induced action for $W_3$-gravity. This relation is similar to the relation between the action for a free boson on the one hand, and the Liouville action on the other hand.

Regarding the question ‘what is the moduli space related to $W$-algebras’, this approach strongly suggests it is just (a component of) the space of flat $Sl(N, \mathbb{R})$-bundles. We know that in the standard polarization the partition function of Chern-Simons theory can be written as the integral of some density over the moduli space of flat $Sl(N, \mathbb{R})$-bundles. The partition function does, of course, not depend on the choice of polarization chosen, and we can therefore in principle also rewrite the partition function of $W_N$-gravity as the integral of some density over the moduli-space of flat $Sl(N, \mathbb{R})$-bundles. This moduli-space should arise by looking at the space of differentials $\{\mu_i\}$ modulo $W$-transformations, but it is difficult to make this relation more precise.

Clearly, there are many problems left in this field, and we hope we will come back to some of those in the near future.

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**A. Appendix**

In this appendix we will give a derivation of (3.22) and (3.23):

$$\frac{2\pi}{k} \frac{\delta}{\delta \Pi_k A_\varepsilon} = \Pi^\dagger_k (g^{-1}W g) + \text{derivatives not containing } \frac{\delta}{\delta \mu_i},$$

(A.1)
\[-\frac{2\pi}{k} \frac{\delta}{\delta \Pi_i A_z} = \Pi_i^\dagger (g^{-1} A^0_{x,d} g) + \text{derivatives not containing } \frac{\delta}{\delta \mu_i}. \hspace{1cm} (A.2)\]

First of all, we will derive an expression for \( A^0_{x,z} \), which was defined in section 3.1. An important object here is the linear operator \( L : g \to g \) defined as follows: one easily verifies that \( X \to \Pi_i(\text{ad}(\Lambda(X)) \) defines an invertible linear operator \( \Pi_i^\dagger g \to \Pi_i g \); \( L \) is the inverse of this map, extended by 0 to an operator \( g \to g \). We also define an element \( F \) of \( g \) by
\[
F = \sum_{i=2}^N \mu_i \Lambda_i^{i-1}. \hspace{1cm} (A.3)
\]
Then
\[
A^0_{x,f} = \frac{1}{1 + L\partial} F, \hspace{1cm} (A.4)
\]
and
\[
A^0_{x,d} = \frac{1}{1 + L\partial} L[F,W], \hspace{1cm} (A.5)
\]
where \((1 + L\partial)^{-1} \) means \( \sum_{i \geq 0} (-L\partial)^i \). To see this, substitute (A.4) and (A.3) into (3.6):
\[
F = -\bar{\partial} W + \partial (1 + L\partial)^{-1} F + \partial (1 + L\partial)^{-1} L[F,W] + \left[ \Lambda + W, (1 + L\partial)^{-1} L[F,W] \right]. \hspace{1cm} (A.6)
\]

We have to show that \( \Pi_i F = 0 \). First consider the term quadratic in \( W \): \([W, (1 + L\partial)^{-1} L[F,W]]\); by definition, \( \text{Im} L \in \Pi_k^i g \), hence \((1 + L\partial)^{-1} L[F,W] \in \Pi_k^i g \). This implies \([W, (1 + L\partial)^{-1} L[F,W]] \in \Pi_k^i g \), (cf. (3.3)), and this term does not contribute to \( \Pi_i F \). By similar reasoning it follows that
\[
\Pi_i [W, (1 + L\partial)^{-1} F] = \Pi_i [W, F]. \hspace{1cm} (A.7)
\]

What remains is
\[
\Pi_i F = \Pi_i \left( (\partial + \text{ad} \Lambda)(1 + L\partial)^{-1} (F + L[F,W]) + [W,F] \right). \hspace{1cm} (A.8)
\]

The definition of \( L \) shows that
\[
\Pi_i (\text{ad} \Lambda(L(X))) = \Pi_i X, \hspace{1cm} (A.9)
\]
and from this we derive that

\[
\Pi_i \left( (\partial + \text{ad}\Lambda)(1 + L\partial)^{-1}X \right) = \Pi_i \left( \sum_{i \geq 0} (-1)^i L^i \partial^{i+1}X + \sum_{i \geq 0} (-1)^{i+1} L^i \partial^{i+1}X + \text{ad}\Lambda(X) \right) = \Pi_i([\Lambda, X]).
\]  

(A.10)

We can use this to evaluate (A.8):

\[
\Pi_i F = \Pi_i([\Lambda, F + L[F, W]] + [W, F]) = \Pi_i([\Lambda, F]) = 0,
\]

(A.11)

where the last line is a trivial consequence of the fact that \( F \) defined in (A.3) commutes with \( \Lambda \). Finally, observe that \( A_0^\alpha \) as defined here is of the required form (3.8). One can also try to compute an expression for \( A_0^\alpha \) by explicitly computing all the entries of the matrix \( A_0^\alpha \) [27], but this is less suitable for general computations as performed in this appendix, and certainly more complicated.

Armed with the expressions (A.4) and (A.5), we can compute the right hand sides of (A.1) and (A.2). Our next task will be to compute the left hand sides of (A.1) and (A.2). Recall that

\[
\Pi_i A_z = \Pi_i(g^{-1} \partial g + g^{-1} \Lambda g),
\]

(A.12)

\[
\Pi_k A_{\bar{z}} = \Pi_k(g^{-1} A_0^\alpha g + g^{-1} \bar{\partial} g) = \Pi_k(g^{-1} F g).
\]

(A.13)

The last line follows easily from the fact that for \( X \in \Pi_i^1 g \), \( \Pi_k(g^{-1} X g) = 0 \), so that, in the expression (A.13), we can always redefine \( A_0^\alpha \) with an arbitrary \( X \in \Pi_i^1 g \). From (A.12) and (A.13), we find that the variations of \( \Pi_i A_z \) and \( \Pi_k A_{\bar{z}} \) in terms of \( \delta g \) and \( \delta F \) are given by

\[
\delta(\Pi_i A_z) = \Pi_i( g^{-1} \partial (\delta gg^{-1}) + g^{-1} [\Lambda, \delta gg^{-1}] g ),
\]

(A.14)

\[
\delta(\Pi_k A_{\bar{z}}) = \Pi_k( g^{-1} \delta F g + g^{-1} [F, \delta gg^{-1}] g ).
\]

(A.15)

Now, in general, given fields \( f_\alpha \) in terms of other fields \( \phi_\beta \), we can express \( \frac{\delta}{\delta f_\alpha} \) in terms of \( \frac{\delta}{\delta \phi_\beta} \), by taking the transpose of the inverse of the matrix \( (\delta f_\alpha / \delta \phi_\beta) \). Here, we must first invert (A.14) and (A.15), and then take the transpose of the resulting expressions. Starting with (A.14), and using once more that for \( X \in \Pi_i^1 g \), \( \Pi_i(g^{-1} X g) = 0 \), (A.14) can be written as

\[
\Pi_i((\partial + \text{ad}\Lambda)(\delta gg^{-1})) = \Pi_i(g \delta(\Pi_i A_z) g^{-1}).
\]

(A.16)
From (A.9) and (A.10) we see that $\Pi_i((\partial + \text{ad}\Lambda)(1 + L\partial)^{-1}LX) = \Pi_iX$. This shows that

$$\delta gg^{-1} = (1 + L\partial)^{-1}L(g\delta(\Pi_iA_z)g^{-1}). \quad (A.17)$$

Proceeding in the same way with (A.15), one finds

$$\delta(\Pi_kF) = \Pi_k(g\delta(\Pi_kA_z)g^{-1} - \text{ad}F((1 + L\partial)^{-1}L(g\delta(\Pi_iA_z)g^{-1}))). \quad (A.18)$$

For equations (A.1) and (A.2), we only need the $\frac{\delta}{\delta g_i}$, or, equivalently, the $\frac{\delta}{\delta F}$ behavior of $\frac{\delta}{\delta \Pi_iA_z}$ and $\frac{\delta}{\delta \Pi_kA_z}$, i.e. we only need to look at (A.18). The transpose $A^T$ of an operator $A$ is in our case defined by requiring that for arbitrary $g$ valued functions $X$ and $Y$, the following identity holds:

$$\int d^2z \ Tr(X(AY)) = \int d^2z \ Tr((A^T X)Y). \quad (A.19)$$

Among other things, this implies that $\partial$ and $L$ are anti-symmetric, $\partial^T = -\partial$ and $L^T = -L$. The computation of the transpose of the operators in the right hand side of (A.18) is now straightforward:

$$\int d^2z \ Tr \left( X\Pi_k(g\delta(\Pi_kA_z)g^{-1} - \text{ad}F((1 + L\partial)^{-1}L(g\delta(\Pi_iA_z)g^{-1}))) \right) = \int d^2z \ Tr \left( \delta(\Pi_kA_z)(g^{-1}\Pi_k^\dagger Xg) \right) - \int d^2z \ Tr \left( \delta(\Pi_iA_z)(g^{-1}(L(1 + L\partial)^{-1}[F, \Pi_k^\dagger X]))g) \right). \quad (A.20)$$

Using $\frac{\delta}{\delta \Pi_k^\dagger X} = \frac{k}{2\pi}W$ we read off from (A.20) that, up to terms containing derivatives with respect to $g$,

$$\frac{\delta}{\delta \Pi_kA_z} = \frac{k}{2\pi}\Pi_k^\dagger(g^{-1}Wg), \quad (A.21)$$

$$\frac{\delta}{\delta \Pi_iA_z} = -\frac{k}{2\pi}\Pi_i^{\dagger} \left( g^{-1}(L(1 + L\partial)^{-1}[F,W])g \right). \quad (A.22)$$

The first equation is just (A.1), and the second one is (A.2), as we see from (A.5). Therefore (A.1) and (A.2) are automatically satisfied in the parametrization (A.12) and (A.13), as claimed.
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