A NOTE ON MULLER’S IRREDUCIBILITY CRITERION FOR GENERALIZED PRINCIPAL SERIES

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Abstract. In this paper, via Casselman–Tadic’s Jacquet module machine, we reprove I. Muller’s irreducibility criterion for principal series, and extend it to generalized principal series. An analogous criterion for covering groups is readily obtained. At last, a conjectural irreducibility criterion for parabolic induction will be discussed in the appendix.

Introduction

One of the aspects of the Langlands program concerns harmonic analysis, especially the analysis of the constituents of parabolic inductions. One of the questions is to see what information one can read out from the Galois side. A notable example is the compatibility of the R-groups on the representation side and the Galois side. To initiate the journey, we may first ask the following question

When is a parabolic induction irreducible?

Recently, we learned the following beautiful conjecture from a seminar talk given by M. Gurevich in the National University of Singapore which says roughly that

$$\pi_1 \times \cdots \times \pi_n \text{ irreducible iff } \pi_i \times \pi_j \text{ irreducible for all } i, j.$$ 

This conjecture rekindles our enthusiasm about the analysis of the structure of parabolic induction, especially Winarsky and Keys’ results about unitary principal series which we have thought at one point to generalize them to covering groups but failed. Back to the conjecture, using the classical Jacquet module machine, we can easily prove the conjecture for non-unitary (generalized) principal series assuming it holds for unitary cases. But we learned that the conjecture for non-unitary principal series of classical groups is a theorem of Tadić via the Jacquet module method. On the other hand, the unitary case fails with the counterexamples in Keys’ paper as follows (cf. [Key82]):

$$SL_3 : \text{Ind}(\omega, \omega^{-1}, 1) \text{ reduces with } R\text{-group equals } \mathbb{Z}/sl \circ sh.l\langle \mathfrak{d} \rangle ft \mathbb{Z}, \text{ where } \omega^3 = 1 \text{ and } \omega \neq 1.$$ 

$$SO_4 : \text{Ind}(\chi_1, \chi_2) \text{ reduces with } R\text{-group equals } \mathbb{Z}/sl \circ sh.l\langle \mathfrak{d} \rangle ft \mathbb{Z}, \text{ where } \chi_i^2 = 1 \text{ and } \chi_i \neq 1, i = 1, 2.$$ 

Note that the second counterexample in some sense can be explained by enlarging $SO_2n$ to the disconnected $O_2n$, while it seems that we have no such surgery to heal the first counterexample. To overcome this phenomenon uniformly, we may try to modify the conjecture using the Knapp–Stein R-group and the set of relative rank-one irreducibility as follows:

$$(\ast) \pi_1 \times \cdots \times \pi_n \text{ irreducible iff } \pi_i \times \pi_j \text{ irreducible for all } i, j, \text{ and } R=\{1\} \text{ for the “unitary” inducing data.}$$

Indeed, an explicit formulation of $(\ast)$ is a theorem of Muller for principal series via the analysis of intertwining operators (cf. [Mul, Proposition 4.2]). To push further, one may ask

How about $(\ast)$ for generalized principal series?

This is exactly what we do in this paper. Unlike the principal series case in which $\mu_\alpha(\lambda) = 0$ iff $\lambda_\alpha = Id$ (cf. [W78, Key82]), (to my best knowledge) there is no such uniform characterization of the relative rank-one irreducibility condition, $\mu_\alpha(\sigma) = 0$, of unitary generalized principal series (cf.

2010 Mathematics Subject Classification. 22E35.

Key words and phrases. Principal series, Generalized principal series, Irreducibility, Jacquet module, R-group, Covering groups.
In such situation, in order to serve an analogous but simple Muller type irreducibility criterion, we introduce a simple subgroup $R_{\sigma}$ of the Knapp-Stein $R$-group and our Main theorem could be stated as follows, please refer to the main content of the paper for the notation.

**Main Theorem.** The following two statements are equivalent

(i) $I(\nu, \sigma)$ is irreducible.

(ii) $R_{\sigma} = \{1\}$, and no relative rank-one reducibility, i.e. $\text{Ind}_{M}^{G}(\sigma \otimes \nu)$ is irreducible for all $\alpha \in \Phi_{M}^{0}$.

Let us end the introduction by saying briefly the structure of the paper. In Section 1, we will first recall some basic notions. In Section 2, we recall Muller’s irreducibility criterion for principal series and reformulate it for generalized principal series, while in the last, we prepare some necessary observations/facts which play an essential role in the proof and emphasize a history on some special cases of the irreducibility criterion. In Section 3, we reprove Muller’s theorem for principal series via Casselman–Tadic’s Jacquet module argument, and extend it to the generalized principal series case. In the appendix, inspired by Gurevich’s talk and Muller type irreducibility criterion, we will propose an ambitious irreducibility criterion conjecture which seems to be a natural generalization of the conjecture in Gurevich’s talk, and then discuss some supportive examples.

1. **Preliminaries**

Let $G$ be a connected reductive group defined over a non-archimedean local field $F$ of characteristic 0. Denote by $|\cdot|_{F}$ the absolute value, by $w$ the uniformizer and by $q$ the cardinality of the residue field of $F$. Fix a minimal parabolic subgroup $B = TU$ of $G$ with $T$ a minimal Levi subgroup and $U$ a maximal unipotent subgroup of $G$, and let $P = MN$ be a standard parabolic subgroup of $G$ with $M$ the Levi subgroup and $N$ the unipotent radical.

Let $X(M)_{F}$ be the group of $F$-rational characters of $M$, and set $a_{M} = \text{Hom}(X(M)_{F}, \mathbb{R})$, $a_{M}^{\ast} = a_{M} \otimes_{\mathbb{Z}} \mathbb{C}$, where $a_{M}^{\ast} = X(M)_{F} \otimes_{\mathbb{Z}} \mathbb{R}$ denotes the dual of $a_{M}$. Recall that the Harish-Chandra homomorphism $H_{P} : M \rightarrow a_{M}$ is defined by

$$q^{-\langle \chi, H_{F}(m) \rangle} = |\chi(m)|_{F}$$

for all $\chi \in X(M)_{F}$.

Next, let $\Phi$ be the root system of $G$ with respect to $T$, and $\Delta$ be the set of simple roots determined by $U$. For $\alpha \in \Phi$, we denote by $\alpha^{\vee}$ the associated coroot, and by $w_{\alpha}$ the associated reflection in the Weyl group $W = W_{G}$ of $T$ in $G$ with

$$W = N_{G}(T)/C_{G}(T) = \{w_{\alpha} : \alpha \in \Phi\}.$$  

Denote by $w_{0}^{G}$ the longest Weyl element in $W$, and similarly by $w_{0}^{M}$ the longest Weyl element in the Weyl group $W^{M}$ of a Levi subgroup $M$.

Likewise, we denote by $\Phi_{M}$ the reduced relative root system of $M$ in $G$, by $\Delta_{M}$ the set of relative simple roots determined by $N$ and by $W_{M} := N_{G}(M)/M$ the relative Weyl group of $M$ in $G$. In general, a relative reflection $\omega_{\alpha} := w_{0}^{M}w_{\alpha}w_{0}^{M}$ with respect to a relative root $\alpha$ does not preserve our Levi subgroup $M$. Denote by $\Phi_{M}^{0}$ the set of those relative roots which contribute reflections in $W_{M}$. It is easy to see that $W_{M}$ preserves $\Phi_{M}$, and further $\Phi_{M}^{0}$ as well, as $\omega_{w_{\alpha}} = w_{\omega_{\alpha}}w_{\alpha}^{-1}$. Note that $W_{M}$ in general is larger than the one generated by those relative reflections.

For our purpose, we define the “small” relative Weyl group $W_{M}^{0} \subset W_{M}$ to be the one generated by those relative reflections, i.e.

$$W_{M}^{0} = \{w_{\alpha} : \alpha \in \Phi_{M}^{0}\}.$$  

Denote by $\Delta_{M}^{0}$ the relative simple roots of $\Phi_{M}^{0}$.

Recall that the canonical pairing

$$\langle \cdot, \cdot \rangle : a_{M}^{\ast} \times a_{M} \rightarrow \mathbb{Z}$$
suggested that each $\alpha \in \Phi_M$ will enjoy a one parameter subgroup $H_{\alpha^\vee}(F^\times)$ of $M$ satisfying: for $x \in F^\times$ and $\beta \in a_M^*$,

$$\beta(H_{\alpha^\vee}(x)) = x^{(\beta, \alpha^\vee)}.$$ 

**Parabolic induction and Jacquet module:** For $P = MN$ a parabolic subgroup of $G$ and an admissible representation $(\sigma, V_\sigma)$ (resp. $(\pi, V_\pi)$) of $M$ (resp. $G$), we have the following normalized parabolic induction of $P$ to $G$ which is a representation of $G$

$$\text{Ind}_P^G(\sigma) := \{\text{smooth } f: G \to V_\sigma \mid f(nmg) = \delta_P(m)^{1/2}\sigma(m)f(g), \forall n \in N, m \in M \text{ and } g \in G\}$$

with $\delta_P$ stands for the modulus character of $P$, i.e., denote by $n$ the Lie algebra of $N$,

$$\delta_P(nm) = |\det Ad_n(m)|_F,$$

and the normalized Jacquet module $J_M(\pi)$ with respect to $P$ which is a representation of $M$.

$$\pi_N := V/\{\pi(n)e - e : n \in N, e \in V_\pi\}.$$ 

Given an irreducible unitary admissible representation $\sigma$ of $M$ and $\nu \in a_M^*$, let $I(\nu, \sigma)$ be the representation of $G$ induced from $\sigma$ and $\nu$ as follows:

$$I(\nu, \sigma) = \text{Ind}_P^G(\sigma \otimes q^{(\nu, H_P(-))}).$$ 

2. GENERALIZED PRINCIPAL SERIES

In this section, we first revisit Muller’s irreducibility criterion for principal series and reformulate it for generalized principal series, then recall some history concerning some special cases and prepare some necessary theory for later use.

In [Mm1], she defines a subgroup $W_\lambda^1$ of the Weyl group $W$ governing the reducibility of the “unitary” part of principal series on the Levi level, which is indeed the Knapp–Stein $R$-group as follows (cf. [Wn78] [Key82]), for the principal series $I(\lambda)$ of $G$,

$$\begin{align*}
\Phi_\lambda^0 := \{\alpha \in \Phi : \lambda_\alpha = I \text{d} \}, \\
W_\lambda^0 := \{w_\alpha : \alpha \in \Phi_\lambda^0\}, \\
W_\lambda^1 := \{w \in W_\lambda : w,(\Phi_\lambda^0)^+ > 0\}, \\
W_\lambda := \{w \in W : w,\lambda = \lambda\}.
\end{align*}$$

In view of [Wal03] Lemma I.1.8, one has

$$W_\lambda = W_\lambda^0 \rtimes W_\lambda^1.$$ 

Following the Knapp–Stein $R$-group theory (cf. [SH79]), we insist to denote by $R_\lambda$ the subgroup $W_\lambda^1$.

In order to generalize the above notions for generalized principal series, we modify some of the notions in what follows. Recall that given a parabolic subgroup $P = MN$ of $G$, a unitary supercuspidal representation $\sigma$ of $M$ and an unramified character $\nu$ of $M$ in $a_M^*$, one forms a parabolic induction

$$I(\nu, \sigma) := \text{Ind}_P^G(\sigma \otimes \nu).$$

Recall that for $\alpha \in \Phi_0^M$, the associated reflection $w_\alpha$ is defined as $w_{M^\alpha}w^M$, where $M^\alpha$ is the relative rank one Levi subgroup determined by $\alpha$, and $w^M$ (resp. $w_{M^\alpha}$) is the longest Weyl element in the Weyl group $W^M$ (resp. $W_{M^\alpha}$) of $M$ (resp. $M^\alpha$). Also recall that the relative Weyl group $W_M$ of $M$ in $G$ is defined to be

$$W_M := N_G(M)/M = \{w \in W : w,M = M\}/W^M,$$

and the “small” relative Weyl group $W_M^0$ is

$$W_M^0 := \{w_\alpha : \alpha \in \Phi_0^M\},$$
where \( \Phi_0^M \) is the set of those reduced relative roots \( \alpha \) which contribute a reflection \( w_\alpha \) preserving \( M \), i.e. \( w_\alpha . M = M \). Given these, we can define the analogous notions as follows:

\[
\Phi_0^{\sigma, \nu}_M := \{ \alpha \in \Phi_0^M : w_\alpha . (\sigma \otimes \nu) = (\sigma \otimes \nu) \},
\]

\[
W_0^{\sigma, \nu}_M := \{ w_\alpha : \alpha \in \Phi_0^{\sigma, \nu}_M \},
\]

\[
W_1^{\sigma, \nu}_M := \{ w \in W_\sigma : w . (\Phi_0^{\sigma, \nu}_M)^+ > 0 \},
\]

\[
W_{\sigma, \nu} := \{ w \in W_M : w . (\sigma \otimes \nu) = (\sigma \otimes \nu) \}.
\]

Likewise, via [Wal03, Lemma I.1.8], we have

\[
W_{\sigma, \nu} = W_0^{\sigma, \nu} \rtimes W_1^{\sigma, \nu},
\]

and we denote \( R_{\sigma, \nu} \) to be \( W_1^{\sigma, \nu} \) following tradition, but it is not the exact \( R \)-group in the sense of Silberger for generalized principal series, even for principal series, for example

\[
\text{Ind}_{M}^{G}(\chi) \text { with } \chi^2 = 1 \text { but } \chi \neq 1,
\]

in such case, we know that

\[
R_{\lambda} \simeq \mathbb{Z}/2\mathbb{Z}, \text { but } R_{\sigma, \nu} = \{1\}.
\]

As all are well-prepared, now we can state the Muller type irreducibility criterion for generalized principal series as follows:

**Main Theorem.** Keep the notions as before. The following two statements are equivalent

(i) \( I(\nu, \sigma) \) is irreducible.

(ii) \( R_{\sigma, \nu} = \{1\} \), and no relative rank-one reducibility, i.e. \( \text{Ind}_{M}^{G} (\sigma \otimes \nu) \) is irreducible for all \( \alpha \in \Phi_0^M \).

Before turning to the proof in the next section, we first observe some facts which play a key role in the next section in what follows.

**Lemma 2.1.** Keep the notation as above. We have

\[
W_M = W_0^M \rtimes W_1^M,
\]

where \( W_1^M \) is defined to be

\[
W_1^M := \{ w \in W_M : w . (\Phi_0^M)^+ > 0 \}.
\]

Moreover \( \Phi_0^M \) is a relative subroot system, may not be irreducible.

**Proof.** The facts that

\[
W_0^M < W_M,
\]

and

\[
\Phi_0^M \text{ is a relative subroot system}
\]

are easy corollaries of the following observation

\[
w_\alpha w^{-1} = w_{w. \alpha} \text{ for } \alpha \in \Phi_0^M \text{ and } w \in W_M.
\]

The remaining part follows from the same argument for the definition of the Knapp–Stein \( R \)-group as above (or cf. [Luo18b, Lemma 4.5]).

As the nature of the Jacquet module argument is to vastly use the induction by stage property of parabolic induction, so we need the following lemma.

Recall that \( Z_M \) is the center of \( M \), where \( M \) is the Levi subgroup of the parabolic subgroup \( P = MN \) in \( G \). For \( \alpha \in \Phi_0^M \), one has the associated coroot \( \alpha^\vee \), then we define

\[
\nu_\alpha(x) = \nu(\alpha^\vee(x)) \text{ for } x \in F^x,
\]

and define

\[
\Delta_1 := \{ \alpha \in \Phi_0^M : \nu_\alpha = 1 \}.
\]

It is easy to see that \( \Delta_1 \) also forms a relative subroot system. Denote by \( W_{\Delta_1} \) the Weyl group generated by \( \Delta_1 \). Then we have, similar to [Mul, Lemma 4.1],

**Lemma 2.2.**

(i) \( \Delta_1 \) admits a base of relative simple roots which is part of the counterpart for \( \Phi_0^M \).
(ii) $W_{\sigma_v}$ is a subgroup of $W_{\Delta_1}$.

Proof. They follows from the fact that

$\Phi^0_M$ is a relative subroot system.

Observe that $Ind^M_{\Delta_1}(\sigma \oplus \nu) \simeq Ind^M_{\Delta_1}(\sigma) \otimes \nu$ is unitary after twisting by the central character $\nu$ of $M_{\Delta_1}$. In view of Lemma 2.2 it is quite natural to define the Knapp–Stein $R$-group of $(G, I(\nu, \sigma))$ in terms of the $R$-group of $(M_{\Delta_1}, Ind^M_{\Delta_1}(\sigma \oplus \nu))$. Then the next question is to see how far $R_{\sigma_v}$ differs from the Knapp–Stein $R$-group. Like the Knapp–Stein theory, we define $W'_{\sigma_v}$ to be the normal subgroup of $W^0_{\sigma_v}$ which governs the unitary rank-one irreducibility of $I(\nu, \sigma)$, i.e.

$$W'_{\sigma_v} := \{ w_\alpha \in \Phi^0_{\sigma_v} : \mu_\alpha(\sigma) = 0 \},$$

where $\mu_\alpha(\cdot)$ is the relative rank-one Plancherel measure associated to $\alpha$ (please refer to [Wal03 Section V.2] for details). Thus we have

$$W_{\sigma_v} = W'_{\sigma_v} \rtimes R'_{\sigma_v},$$

which in turn implies that

Lemma 2.3. Keep the notions as above. We have

$$R'_{\sigma_v} = R^0_{\sigma_v} \rtimes R_{\sigma_v},$$

where

$$R^0_{\sigma_v} \simeq W^0_{\sigma_v}/W'_{\sigma_v} \rightarrow R'_{\sigma_v}.$$

Remark 1. It is easy to see that $R'_{\sigma_v}$ is exactly the Knapp–Stein $R$-group when the parabolic induction datum $\sigma \oplus \nu$ is unitary.

Let us end this section by recalling the history on some special cases of the Main Theorem, especially the regular case and the unitary case.

Theorem. (cf. [Sil79 Theorem 5.4.3.7]) If the inducing datum $\sigma \oplus \nu$ is regular, i.e. $W_{\sigma_v} = \{1\}$. Then

$I(\nu, \sigma)$ is irreducible iff $Ind^M_{\Delta_1}(\sigma \oplus \nu)$ is irreducible for all $\alpha \in \Phi^0_M$.

Theorem. (cf. [Sil79]) If the inducing datum $\sigma \oplus \nu$ is unitary, i.e. $\nu = 0$. Then

$I(\nu, \sigma)$ is irreducible iff $R'_{\sigma_v} = \{1\}$ iff $R_{\sigma_v} = \{1\}$ + relative rank-one irreducibility.

At last, for the unitary regular case, it is always irreducible which is a theorem of Bruhat in [Cas95 Theorem 6.6.1].

3. PROOF OF THE IRREDUCIBILITY CRITERION

In this section, we carry out the proof the Main theorem, i.e. the irreducibility criterion for generalized principal series, following Casselman–Tadic’s Jacquet module argument. Let us first recall the Main theorem as follows:

Main Theorem (Muller type irreducibility criterion). $I(\nu, \sigma)$ is irreducible if and only if the following are satisfied

- $R_{\sigma_v} = \{1\}$
- $Ind^M_{\Delta_1}(\nu, \sigma)$ is irreducible for any $\alpha \in \Phi^0_M$.

Proof. For the necessary part, it follows from the following facts stated in the previous section:

- For each $\alpha \in \Phi^0_M$, under the conjugation of a relative Weyl element $w \in W_M$, we may assume that $\alpha$ is a relative simple root. Therefore, as $Ind^G_\theta(\nu, \sigma)$ and $Ind^G_\theta(\nu, \sigma)^w$ share the same constituents,

$$Ind^M_{\Delta_1}(\nu, \sigma)^w$$

is irreducible,

which in turn implies that $Ind^M_{\Delta_1}(\nu, \sigma)$ is irreducible.
As $W_{\sigma_v}$ is a subgroup of the Weyl group $W_{\Delta_1}$ of the Levi subgroup $M_{\Delta_1}$ determined by $\Delta_1$, same argument as above shows that

$$R_{\sigma_v} = \{1\},$$
given the fact that we can move out the $\nu$ from the inducing data on the $M_{\Delta_1}$-level as follows:

$$\text{Ind}^{M_{\Delta_1}}_M (\sigma \otimes \nu) \simeq \text{Ind}^{M_{\Delta_1}}_M (\sigma) \otimes \nu.$$

As for the sufficient part, one can follow Muller’s intertwining operator argument for principal series based on the following observation which is a corollary of Casselman’s subrepresentation theorem, which has not been pointed out clearly in [Mul],

$$I(\nu, \sigma) \text{ is irreducible iff } \text{Hom}(I(\nu, \sigma), I(\nu, \sigma)^w) \simeq \mathbb{C} \text{ for all } w \in W_M.$$

But we will give a much more classical intuitive argument using Casselman–Tadic’s Jacquet module machine as follows. Without loss of generality, for $\pi \in JH(I(\nu, \sigma))$, assume $\sigma \otimes \nu \in J_M(\pi)$, then

$$I(\nu, \sigma) \text{ is irreducible iff } (\sigma \otimes \nu)^w \in J_M(\pi) \text{ for all } w \in W_M.$$

This follows from two steps:

(i) We first show that the multiplicity appears in $J_M(\pi)$, i.e.

$$|W_{\sigma_v}|(\sigma \otimes \nu) \in J_M(\pi),$$

which follows from the fact used in the proof of the necessary part, i.e. the essentially unitary induction on the $M_{\Delta_v}$-level $\text{Ind}^{M_{\Delta_1}}_M (\sigma)$ as

$$\text{Ind}^{M_{\Delta_1}}_M (\sigma \otimes \nu) \simeq \text{Ind}^{M_{\Delta_1}}_M (\sigma) \otimes \nu.$$

And the fact that $\Phi^0_M$ can be decomposed into irreducible relative subroot systems.

(ii) We then show that each orbit appears in $J_M(\pi)$, i.e.

$$(\sigma \otimes \nu)^w \in J_M(\pi) \text{ for all } w \in W_M/W_{\sigma_v},$$

which follows from the case of regular inducing data after conjugating by a Weyl element in $W_M$ which is hidden in [Luo18b]). To be precise, as

$$W_M = W_M^0 \ast W_M^1,$$

a similar argument as in the proof of the necessary part shows that

$$(\sigma \otimes \nu)^w \in J_M(\pi)$$

for all $w \in W_M^0/W_M^0$. Therefore it remains to show that, for any $w \in W_M^1$, that is not in $W_{\sigma_v}$,

$$(\sigma \otimes \nu)^w \in J_M(\pi).$$

This follows from the same argument as in [Luo18b] Lemma 3.5].

Remark 2. In view of the above argument, it is easy to see that the irreducibility criterion holds for covering groups given the fact that Knapp–Stein $R$-group theory has been established in [Luo17] and the following irreducibility criterion for regular generalized principal series of covering groups.

Recall that $\tilde{G}$ is a finite central covering group of $G$, and $\tilde{P} = MN$ is a parabolic subgroup of $\tilde{G}$. Let $\tilde{\sigma}$ be a genuine regular supercuspidal representation of $\tilde{M}$, and denote by $I(\tilde{\sigma})$ the normalized parabolic induction of $\tilde{\sigma}$ from $\tilde{P}$ to $\tilde{G}$. All other notions are the same as in the non-cover case. Then we have

Lemma 3.1. The following are equivalent

(i) $I(\tilde{\sigma})$ is irreducible.

(ii) $\text{Ind}^{\tilde{M}}_M (\tilde{\sigma})$ is irreducible for all $\alpha \in \Phi^0_M$.

Proof. This follows from the properties of Plancherel measure generalizing to covering groups in [Luo17] and the argument in [Luo18b] Lemma 3.5].
Appendix A. A conjectural criterion of parabolic inductions

In the appendix, we would like to first serve you two simple observations, originating from [Rod81] Luo18b Luo18a, on a conjectural irreducibility criterion of parabolic induction learned from M. Gurevich’s talk in the National University of Singapore. Then we would like to propose an ambitious conjecture for general groups.

In the following, let us first recall the explicit conjectural irreducibility criterion given in M. Gurevich’s talk. Let $P = MN$ be a parabolic subgroup of $GL_n$ with the Levi subgroup $M = \prod_{i\in I} GL_{n_i}$, and denote by $P_{i,j}$ the parabolic subgroup of $GL_{(n_i+n_j)}$ with the Levi subgroup $M_{i,j} = GL_{n_i} \times GL_{n_j}$ for $i \neq j \in I$. For an irreducible admissible representation $\otimes_i \sigma_i$ of $M$, the conjectural irreducibility criterion of parabolic induction for $GL_n$ is as follows:

Conjecture A.1. The parabolic induction $\times_i \sigma_i := Ind_{P}^{GL_n}(\otimes_i \sigma_i)$ is irreducible if and only if the relative rank-one parabolic induction $\sigma_i \times \sigma_j := Ind_{M_{i,j}}^{GL_{(n_i+n_j)}}(\sigma_i \otimes \sigma_j)$ is irreducible for all $i \neq j \in I$.

The first observation comes from the structure theory of regular generalized principal series (cf. Luo18b Rod81) which roughly says that if the supercuspidal support of our induction data is regular, then the above conjecture holds. To be more precise, write the supercuspidal support of $\sigma_i$ as

$$\{\tau_{i,k}\}_{k=1}^{l_i}$$

with $\tau_{i,k}$ supercuspidal.

Then our simple observation can be stated as follows.

Lemma A.2. Assume $\{\tau_{i,k}\}_{i,k}$ is regular, i.e. $\otimes_{i,k} \tau_{i,k}$ is regular. Then the Conjecture A.1 holds.

Proof. The necessary part is obvious, one only has to prove the sufficient part. If $\sigma_i \times \sigma_j$ is irreducible, then

$$\tau_{i,k_1} \times \tau_{j,k_2}$$

is irreducible for all $k_1, k_2$, which follows from [Luo18b Theorem 3.7]). Applying [Luo18b Theorem 3.7] again, one knows that $\times_i \sigma_i$ is irreducible, whence the conjecture A.1 holds. □

Remark 3. If one replaces the condition “$\sigma_i \times \sigma_j$ is irreducible” by “relative rank-one induction is irreducible” in the Conjecture A.1 it is easy to see that the Lemma A.2 still holds for general connected reductive groups as [Luo18b] Theorem 3.7 applies to such generality.

The second observation comes from a “product formula” in Luo18a which roughly says that if the relative rank-one reducibility and the Knapp–Stein R-group conditions lie in a Levi subgroup $L$ of $G$, then

$$\#JH(Ind_{L}^{G}(\sigma)) = \#JH(Ind_{P}^{G}(\sigma))$$

where $\sigma$ is a supercuspidal representation of the Levi subgroup $M$ of $P$.

To be precise, denote by $\Theta_Q$ the associated subset of $\Delta$ which determines the Levi subgroup $Q$ of $G$. We decompose $\Theta_L = \Theta_1 \cup \cdots \cup \Theta_l$ into irreducible pieces, and accordingly $\Theta_M = \Theta_M^0 \cup \cdots \cup \Theta_M^t$. Assume that $R_{\sigma}$ decomposes into $R_{\sigma} = R_1 \times \cdots \times R_l$ with respect to the decomposition of $\Theta_L$, and a similar decomposition pattern holds for the relative rank-one reducibility, i.e. relative rank-one reducibility only occurs within $P_{\Theta_i} = M_{\Theta_i}, N_{\Theta_i}$ for $1 \leq i \leq t$. Then we have

Product formula. (cf. Luo18a Corollary 2.2)

$$\#(JH(Ind_{P}^{G}(\sigma))) = \prod_{i=1}^{t} \#(JH(Ind_{M_{\Theta_i}}^{M_{\Theta_i}}(\sigma)))$$

So an easy corollary of the Product formula is

Lemma A.3. Assume the decomposition pattern of the relative rank-one reducibility and our revised R-group of the supercuspidal support data $\{\tau_{i}\}_i$ of $\sigma$ as above is exactly $L = M$, then the Conjecture A.1 holds.

Remark 4. We learned that Conjecture A.1 is now a theorem of M. Gurevich.
Let $\sigma$ be an irreducible admissible representation of $M$ of the parabolic subgroup $P = MN$ of $G$, we define the revised group $R_\sigma$ of $\sigma$ in $G$ to be our revised $R$-group of its supercuspidal support. Inspired by the Conjecture A.1 and the above two observations, we would like to propose an ambitious conjecture for connected reductive groups in the following.

**Conjecture A.4.** keep the notions as above. Assume $R_\sigma = \{1\}$. Then the following are equivalent

(i) $\text{Ind}_G^P(\sigma)$ is irreducible.

(ii) the relative rank-one induction is irreducible, i.e. $\text{Ind}_{M_\alpha\cap P}^{M_\alpha}(\sigma)$ is irreducible for any $\alpha \in \Phi_M$.

In what follows, let us discuss some supportive examples.

**Example 1.** If $\sigma$ is supercuspidal, then the Conjecture A.4 follows from the Main Theorem, i.e. Muller type irreducibility criterion.

**Example 2.** For classical groups, under the assumption of Lemma A.3 with $L = \text{GL}$-part of $M$, then the Conjecture A.1 implies the Conjecture A.4.

**Remark 5.** By and large the above conjecture may be false. In that case, it would be quite interesting to explore the obstruction.

**Remark 6.** After discussing with A. Minguez, we learned that one direction of our generalized conjecture, i.e. Conjecture A.4 holds which is given by [LT17, Theorem 1.1]. Also [LT17, Conjecture 1.3] holds for groups of types $A_n$, $B_n$ and $C_n$ which follows from our “product formula as the associated revised $R$-group is trivial for groups of types $A_n$, $B_n$ and $C_n$, but may not be trivial for $SO_{2n}$.

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