Dirac observables in the 4-dimensional phase space of Ashtekar’s variables and spherically symmetric loop quantum black holes

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In this paper, we study a proposal put forward recently by Bodendorfer, Mele and Münch and García-Quismondo and Marugán, in which the two polymerization parameters of spherically symmetric black hole spacetimes are the Dirac observables of the four-dimensional Ashtekar’s variables. In this model, black and white hole horizons in general exist and naturally divide the spacetime into the external and internal regions. In the external region, the spacetime can be made asymptotically flat by properly choosing the dependence of the two polymerization parameters on the Ashtekar variables. Then, we find that the asymptotical behavior of the spacetime is universal, and, to the leading order, the curvature invariants are independent of the mass parameter m. For example, the Kretschmann scalar approaches zero as $K \approx A_0 r^{-4}$ asymptotically, where $A_0$ is generally a non-zero constant and independent of $m$, and $r$ the geometric radius of the two-spheres. In the internal region, all the physical quantities are finite, and the Schwarzschild black hole singularity is replaced by a transition surface whose radius is always finite and non-zero. The quantum gravitational effects are negligible near the black hole horizon for very massive black holes. However, the behavior of the spacetime across the transition surface is significantly different from all loop quantum black holes studied so far. In particular, the location of the maximum amplitude of the curvature scalars is displaced from the transition surface and depends on $m$, so does the maximum amplitude. In addition, the radius of the white hole is much smaller than that of the black hole, and its exact value sensitively depends on $m$, too.

I. INTRODUCTION

Loop quantum gravity (LQG) has burgeoned in an effort to quantize gravity. It is a non-perturbative and background independent approach to canonically quantizing Einstein’s general relativity (GR) [1–5]. Loop quantum cosmology (LQC) is an application of the LQG techniques by first performing the symmetry reduction of the homogeneous and isotropic spacetimes at the classical level, and then quantizing it by using the canonical Dirac quantization for systems with constraints, the so-called minisuperspace approach [6]. Singularities are one of the major predictions by GR, which appear (classically) in the very early cosmological epoch and the interior regions of black holes. Classical GR becomes invalid when such singularities appear. One usually expects that in such high curvature regimes quantum gravitational effects will take over and become dominant, whereby the singularities are smoothed out and finally replaced by regions with the Planck scale curvatures. Because of the quantum nature of geometry in LQG, cosmological singularities can be naturally resolved in LQC models, without any additional constraints on matter fields [6]. Although the full theory is still under construction, symmetry reduced models constructed from LQG have received great attention.

Since the Schwarzschild interior is isometric to the homogeneous but anisotropic (vacuum) Kantowski-Sachs cosmological model, techniques of LQC can be used to study black hole(BH) singularities in the spherically symmetric spacetimes. In the treatment of LQC, the full quantum evolution is well approximated by quantum corrected effective equations. Similar treatment is applied to the interior of the Schwarzschild spacetime to get the quantum corrected Schwarzschild spacetime, which cures the black hole singularity. Recently, such works have received lot of attention [7–52].

A particular model proposed recently is the Ashtekar-Olmedo-Singh (AOS) loop quantum black hole (LQBH) [53–55], in which AOS constructed the effective Hamiltonian that governs the dynamics of spherically symmetric loop quantum black holes in the semi-classical limit. This effective Hamiltonian contains two polymerization parameters ($\delta_0$, $\delta_c$), characterizing the quantum gravitational effects. In some of the previous approaches, they were simply taken as constants[7, 11, 31, 33], similar to the $\mu_0$ scheme first introduced in LQC [6]. However, in...
LQC it was found [56] that the $\mu_0$ scheme leads to large quantum geometric effects even in regions much lower than the Planck curvatures. To remand this problem, Ashtekar, Pawlowski and Singh (APS) [56] proposed that the polymerization parameter should depend on phase variables, the so-called $\check{\mu}$ scheme. It turns out that so far this is the only scheme that leads to consistent results in LQC [6].

On the other hand, in the AOS model [53–55], instead of treating $(\delta_b, \delta_c)$ as arbitrary functions of the phase variables, they consider them as Dirac observables, that is, they are particular functions of the phase variables, such that along the trajectories of the effective Hamiltonian equations they become constants. Similar treatments have also been adopted in [9, 14, 15, 23, 24]. But the AOS approach is different as they considered $(\delta_b, \delta_c)$ in the 8-dimensional extended phase space $\Gamma_{ext}$ of the variables $(b, c, p_b, p_c; \delta_b, \delta_c, p_{\delta_b}, p_{\delta_c})$, instead of the 4-dimensional phase space $\Gamma$ of the variables $(b, c, p_b, p_c)$. Another key feature that differentiates the AOS approach is the imposition of the minimum area condition of LQG on the plaquettes that tessellate the transition surface. This treatment helped resolve the long-standing problems in LQBH such as the dependence of the system on the fiducial structure and non-negligible quantum corrections at low curvatures, to name a few.

Despite the success of the AOS model, some questions have been raised [57, 58]. In particular, Bodendorfer, Mele and Münch (BMM) [59] argued that the polymerization parameters in general should depend on both $b$ and $c$, and not just $b$ or $c$ independently. This treatment is also very near to the transition surface, and the ratio of the white and black hole horizon radii is much smaller than one, and sensitively depends on the mass parameter $m$. Finally, in Section V, we summarize our main conclusions.

To distinguish the AOS and BMM/GM approaches, in this paper, we shall refer them as to the extended and canonical phase space approaches, respectively.

Before proceeding further, we would also like to note that parts of the results presented in this paper had been reported in the APS April meeting, April 9 - 12, 2022, New York, as well as in the 23rd International Conference on General Relativity and Gravitation (GR23), Liyang, China, July 3 - 8, 2022.

II. EXTENDED PHASE SPACE APPROACH

The starting point of LQG is the introduction of the Ashtekar variables. In the spherically symmetric spacetimes, they are the metric components $p_b$ and $p_c$ and their moment conjugates $b$ and $c$ with the canonical relations

\[ \{ b, p_b \} = G\gamma, \quad \{ c, p_c \} = 2G\gamma, \tag{2.1} \]

where $\gamma$ is the Barbero-Immirzi parameter and $G$ is the Newtonian gravitational constant.

In terms of $p_b$ and $p_c$, the four-dimensional spacetime line element takes the form,

\[ ds^2 = -N^2dT^2 + \frac{p_b^2}{|p_c|L_o^2}dx^2 + |p_c|d\Omega, \tag{2.2} \]
where \( N \) is the lapse function, and \( L_o \) is a constant, denoting the length of the fiducial cell in the \( x \)-direction with \( x \in (0, L_o) \), and \( d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2 \) with \( \theta \) and \( \phi \) being the two angular coordinates defined on the two spheres \( T, x = \text{Constant} \).

In the internal region of a classical black hole, \((N, p_b, p_c)\) are all functions of \( T \) only (so are \( b \) and \( c \)), and the corresponding spacetimes are of the Kantowski-Sachs cosmological model, which allows one to apply LQC techniques to such homogeneous but anisotropic spacetimes. As a result, the internal region of the Schwarzschild has been extensively studied in the framework of LQC.

On the other hand, in the external region, the coordinates \( T \) and \( x \) exchange their rules, and the spacetime becomes static. However, such changes can be also carried out by the replacement \( N \rightarrow iN \) and \( p_b \rightarrow ip_b \), as shown explicitly below, while keeping the dependence of the Ashtekar variables still only on \( T \).

With the above in mind, we can see that in general the metric (2.2) has the gauge freedom,

\[
T' = T'(T), \quad x' = \alpha x + x_0,
\]

in both external and internal regions, where \( T'(T) \) is an arbitrary function of \( T \) only, and \( \alpha \) and \( t_0 \) are real constants. To see the AOS approach more clearly, let us consider the AOS effective Hamiltonian inside and outside the LQBH, separately.

### A. AOS Internal Solution

With the gauge freedom of (2.3), AOS chose \( T'(T) \) so that

\[
N = \frac{\gamma \delta_b \text{sgn}(p_c) \sqrt{|p_c|}}{\sin \delta_b}.
\]

Then, the effective Hamiltonian in the interior of LQBHs reads [53–55]

\[
H_{\text{eff}} = -\frac{1}{2G\gamma} \left( 2 \frac{\sin \delta_c}{\delta_c} |p_c| \right.
\]

\[
+ \left( \frac{\sin \delta_b}{\delta_b} + \frac{\gamma^2 \delta_b}{\sin \delta_b} \right) p_b \right)
\]  

(2.5)

where \( \delta_b \) and \( \delta_c \) are two Dirac observables, appearing in the polymerizations

\[
b \rightarrow \frac{\sin \delta_b}{\delta_b}, \quad c \rightarrow \frac{\sin \delta_c}{\delta_b}.
\]

That is, replacing \( b \) and \( c \) by Eq.(2.6) in the classical Hamiltonian

\[
H_{\text{cl}} = -\frac{1}{2G\gamma} \left( 2c |p_c| + \left( b + \frac{\gamma^2}{b} \right) p_b \right),
\]

(2.7)

whereby the effective Hamiltonian (2.5) is obtained, provided that the classical lapse function is chosen as

\[
N_{\text{cl}} = \frac{\gamma \text{sgn}(p_c) \sqrt{|p_c|}}{b}.
\]

(2.8)

To fix \( \delta_b \) and \( \delta_c \), AOS first noticed that the above effective Hamiltonian can be written as

\[
H_{\text{eff}} = \frac{L_o}{G} (O_b - O_c),
\]

(2.9)

where

\[
O_b = \frac{-p_b}{2\gamma L_o} \left( \frac{\sin \delta_b}{\delta_b} + \frac{\gamma^2 \delta_b}{\sin \delta_b} \right),
\]

(2.10)

\[
O_c = \frac{|p_c| \sin \delta_c}{\gamma L_o \delta_c},
\]

(2.11)

are two Dirac observables. Then, AOS proceeded as follows:

- First extend the 4-dimensional (4D) phase space \( \Gamma \) spanned by \((b, c, p_b, p_c)\) to 8-dimensional (8D) phase space \( \Gamma_{\text{ext}} \) spanned by \((b, c, \delta_b, \delta_c, p_b, p_c, p_{\delta_b}, p_{\delta_c})\). In \( \Gamma_{\text{ext}} \) the variables \( \delta_b \) and \( \delta_c \) are independent, so they are in particular not functions of \((b, c, p_b, p_c)\) and instead Poisson commute with all of them.
- Lift \( H_{\text{eff}} \) given by Eq.(2.5) to \( \Gamma_{\text{ext}} \), and then consider its Hamiltonian flow. Since \( O_b \) and \( O_c \) are the Dirac observables of this flow, the following choice can be made

\[
\delta_b = \delta_b(O_b), \quad \delta_c = \delta_c(O_c),
\]

(2.12)

so that \((\delta_b, \delta_c)\) are also the Dirac observables.
- Introduce these dependences as two new first-class constraints

\[
\Phi_b \equiv O_b - F_b(\delta_b) \simeq 0, \quad \Phi_c \equiv O_c - F_c(\delta_c) \simeq 0,
\]

(2.13)

so that the four-dimensional reduced \( \hat{\Gamma} \) corresponding to these constraints is symplectomorphic to the original phase space \( \Gamma \). Since \( O_b \) and \( O_c \) are the Dirac observables, Eq.(2.13) implies

\[
\delta_b = F_b^{-1}(O_b), \quad \delta_c = F_c^{-1}(O_c),
\]

(2.14)

are also constants on the trajectories of the effective Hamiltonian \( H_{\text{eff}} \) given by Eq.(2.5).
- To fix \( \delta_b \) and \( \delta_c \), AOS assumed that at the transition surface, \((T = \mathcal{T})\), the physical areas of the \((x, \theta)\)- and \((\theta, \phi)\)-planes are respectively equal to the minimal area \( \Delta \) [53]

\[
2\pi \delta_c b \left| p_b(T) \right| = \Delta,
\]

(2.15)

\[
4\pi \delta_b^2 p_c(T) = \Delta.
\]

(2.16)
With all the above, AOS found that the corresponding Hamilton equations are given by

\[
\dot{b} = -\frac{1}{2} \left( \frac{\sin (\delta_b b)}{\delta_b} + \frac{\gamma^2 \delta_b}{\sin (\delta_b b)} \right), \tag{2.17}
\]

\[
\dot{p}_b = \frac{1}{2} p_b \cos (\delta_b b) \left( 1 - \frac{\gamma^2 \delta_b^2}{\sin^2 (\delta_b b)} \right), \tag{2.18}
\]

and

\[
\dot{c} = -2 \frac{\sin (\delta_c c)}{\delta_c}, \tag{2.19}
\]

\[
\dot{p}_c = 2 p_c \cos (\delta_c c). \tag{2.20}
\]

It is remarkable to note that in the above equations, no cross terms exists between the equations for \((b, p_b)\) and the ones for \((c, p_c)\). As a result, we can solve the two sets of equations independently, and the corresponding solutions are given by [52, 53]

\[
\cos (\delta_b b) = b_o \frac{1 + b_o \tanh \left( \frac{b_o T}{2} \right)}{b_o + \tanh \left( \frac{b_o T}{2} \right)},
\]

\[
= b_o \frac{b_o e^{b_o T} - b_o e^{-b_o T}}{b_o e^{b_o T} + b_o e^{-b_o T}},
\]

\[
p_b = -\frac{m L_o}{2 b_o} (b_o e^{-b_o T} - b_o e^{b_o T}) A, \tag{2.21}
\]

\[
\sin (\delta_c c) = \frac{2 a_o e^{2 T}}{a_o^2 + e^{4 T}},
\]

\[
p_c = 4 m^2 \left( a_o^2 + e^{4 T} \right) e^{-2 T}, \tag{2.22}
\]

where

\[
A = \left[ 2 \left( b_o^2 + 1 \right) e^{b_o T} - b_o^2 - b_o e^{2 b_o T} \right]^{1/2},
\]

\[
a_o \equiv \frac{\gamma \delta_c L_o}{8 m}, \quad b_o \equiv \left( 1 + \gamma^2 \delta_b^2 \right)^{1/2},
\]

\[
b_o \equiv b_o \pm 1, \tag{2.23}
\]

with

\[
\delta_b \in (0, \pi), \quad \delta_c \in (0, \pi),
\]

\[
p_b \leq 0, \quad p_c \geq 0, \quad -\infty < T < 0. \tag{2.24}
\]

The parameter \(m\) is an integration constant, related to the mass parameter of the AOS solution. From the above solution, it can be shown that the two Dirac observables on-shell are given by

\[
O_b = m = O_c. \tag{2.25}
\]

In the large mass limit, \(m \gg m_p\), from Eqs.(2.15) and (2.16) AOS found that

\[
\delta_b = \left( \frac{\sqrt{\Delta}}{\sqrt{2 \pi \gamma^2 m}} \right)^{1/2}, \quad L_0 \delta_c = \frac{1}{2} \left( \frac{\gamma \Delta^2}{4 \pi^2 m} \right)^{1/2}, \tag{2.26}
\]

where \(m_p\) denotes the Planck mass.

It should be noted that in [53] two solutions for \(c\) were given, and here in this paper we only consider the one with "+" sign, as physically they describe the same spacetime.

![Penrose diagram for the AOS LQBH](image)

FIG. 1. The Penrose diagram for the AOS LQBH. The dashed horizontal lines \(ab\) and \(cd\) represent the transition surfaces (throats), and the regions marked with \(B\) is the BH interior, and the regions marked with \(W\) is the WH interior, but there are no spacetime singularities, so the extensions are infinite along the vertical line in both directions. Regions marked with I, I', II, II', III, and III' are asymptotically flat regions but with a falling rate slower than that of the Schwarzschild black hole [55].

From Eq.(2.22), it can be seen that the transition surface is located at \(\partial p_c(T) / \partial T|_{T=T} = 0\), which yields

\[
T = \frac{1}{2} \ln \left( \frac{\gamma \delta_c L_o}{8 m} \right) < 0. \tag{2.27}
\]

There also exist two horizons, located respectively at

\[
T_{BH} = 0, \quad T_{WH} = -\frac{2}{b_o} \ln \left( \frac{b_o + 1}{b_o - 1} \right), \tag{2.28}
\]

at which we have \(A(T) = 0\), where \(T = T_{BH}\) is the location of the black hole horizon, while \(T = T_{WH}\) is the location of the white hole. In the region \(T < T < 0\), the 2-spheres are all trapped, while in the one \(T_{WH} < T < T\), they are all anti-trapped. Therefore, the region \(T < T < 0\) behaves like the BH interior, while the one \(T_{WH} < T < T\) behaves like the WH interior, denoted, respectively, by Region B and Region W in Fig. 1. This explains the reason why we call them the black hole and white hole regions, although the geometric radius \(\sqrt{T}\) of the two-sphere \((T, x = \text{Const})\) is always finite and non-zero, so spacetime singularities never appear.
Finally, we note that in this region the lapse function reads
\[ N = \frac{\gamma \delta_b \text{sgn}(p_c) |p_c|^{1/2}}{\sin(\delta_b b)} \]
\[ = \frac{2m}{\mathcal{A}} \left( b_+ e^{b_+ T} + b_- \right) \left( \alpha^2 e^{-2T} + e^{2T} \right)^{1/2}, \]
where \( \mathcal{A} \) is given in Eq.(2.23).

**B. AOS External Solution**

At the two horizons (2.28), we have \( \mathcal{A}(T) = 0 \), and the metric becomes singular, so extensions beyond these surfaces are needed in order to obtain a geodesically complete spacetime. AOS showed that such extensions can be obtained from (2.5) by the following replacements
\[ b \to ib, \quad p_b \to ip_b, \quad c \to c, \quad p_c \to p_c, \]
(2.30)
for which the canonical relations (2.1) now become
\[ \{b, p_b\} = -G \gamma, \quad \{c, p_c\} = 2G \gamma, \]
(2.31)
while the effective Hamiltonian in the external space of the LQBH is given by
\[ H_{\text{eff}} = -\frac{1}{2G \gamma} \left[ \frac{2}{\delta_c} \sin(\delta_c c) p_c \right. \\
- \left( \sin(\delta_b b) - \frac{\gamma^2 \delta_b}{\sin(\delta_b b)} \right) p_b \]
\[ = \frac{L_o}{G} (O_b - O_c), \]
(2.32)
but now with
\[ O_b \equiv \frac{p_b}{2 \gamma L_o} \left( \frac{\sin(\delta_b b)}{\delta_b} - \frac{\gamma^2 \delta_b}{\sin(\delta_b b)} \right), \]
(2.33)
\[ O_c \equiv \frac{p_c}{\gamma L_o \delta_c}, \]
(2.34)
which can be obtained directly from Eqs.(2.10) and (2.11) with the replacement (2.30). Then, the corresponding Hamilton equations for \( (c, p_c) \) are still given by Eqs.(2.19) and (2.20), while the ones for \( (b, p_b) \) now are replaced by
\[ \dot{b} = -\frac{1}{2} \left( \frac{\sin(\delta_b b)}{\delta_b} - \frac{\gamma^2 \delta_b}{\sin(\delta_b b)} \right), \]
(2.35)
\[ \dot{p}_b = \frac{1}{2} p_b \cosh(\delta_b b) \left( 1 + \frac{\gamma^2 \delta_b^2}{\sinh^2(\delta_b b)} \right). \]
(2.36)
Then, the corresponding solutions of the Hamilton equations are given by
\[ \cosh(\delta_b b) = \frac{1 + b_o \tanh \left( \frac{b_o T}{2} \right)}{b_o + \tanh \left( \frac{b_o T}{2} \right)}, \]
\[ p_b = -2m \gamma L_o \delta_b \frac{\sin(\delta_b b)}{\gamma^2 \delta_b^2 - \sinh^2(\delta_b b)} \]
\[ = -\frac{m L_o}{2b_o} \left( b_+ + b_- e^{-b_+ T} \right) \mathcal{A}, \]
(2.37)
\[ \sin(\delta_c c) = \frac{2a_o e^{2T}}{a_o^2 + e^{4T}}, \]
\[ p_c = 4m^2 \left( e^{2T} + a_o^2 e^{-2T} \right), \]
(2.38)
but now with
\[ \mathcal{A} = \left[ b_+^2 + b_-^2 e^{2b_+ T} - 2 \left( b_+^2 + 1 \right) e^{b_+ T} \right]^{1/2}, \]
(2.39)
which can be obtained from Eq.(2.23) by the replacement \( \mathcal{A} \to i \mathcal{A} \) (or \( \mathcal{A}^2 \to -\mathcal{A}^2 \)), so that \( g_{xx} \to -g_{xx} \), and the coordinate \( x \) now becomes timelike in the external region \( T > 0 \) of the black hole horizon, located at \( T = 0 \). It can be shown that for the above solution, we have \( O_b = m = O_c \), which shows clearly that \( O_b \) and \( O_c \) defined by Eqs.(2.33) and (2.34) are two Dirac observables.

We also note that the replacement of Eq.(2.30) leads to
\[ N^2 = -\frac{\gamma^2 \delta_b^2 |p_c|}{\sinh^2(\delta_b b)} \]
\[ = -\frac{m^2}{4} \left( b_+ e^{b_+ T} + b_- \right)^2 \left( a_o^2 e^{-2T} + e^{2T} \right), \]
(2.40)
so that, in terms of \( N, p_b \) and \( p_c \), the metric now takes the form [53]
\[ ds^2 = -N^2 dT^2 - \frac{p_b^2}{|p_c| L_o} dx^2 + |p_c| d\Omega^2 \]
\[ = -\frac{p_b^2}{|p_c| L_o} dx^2 + \frac{\gamma^2 \delta_b^2 |p_c|}{\sinh^2(\delta_b b)} dT^2 + |p_c| d\Omega^2, \]
(2.41)
which shows clearly that now \( T \) is spacelike, while \( x \) becomes timelike, so the spacetime outside of the LQBH is static.

In addition, AOS showed that the two metrics (2.2) and (2.41) are analytically connected to each other across the two horizons, and as a result, the extensions are unique. The global structure of the spacetime is given by the Penrose diagram of Fig. 1, from which we can see that the extensions along the vertical direction are infinite, quite similar to the charged spherically symmetric Reissner-Nordström solutions [61], but without spacetime singularities, as now the geometric radius \( \sqrt{F} \) never becomes zero.

Before proceeding to the next section, we also note that technically the AOS extended space approach can be realized directly by taking \( \delta_b \) and \( \delta_c \) to be constants.
in the phase space of \((b, c, p_b, p_c)\), and then impose the conditions (2.15) and (2.16), as by definition constants over the whole phase space are also Dirac observables.

III. CANONICAL PHASE SPACE APPROACH

Instead of extending the 4D physical phase space to 8D phase space, and then considering \(\delta_b\) and \(\delta_c\) as the Dirac observables of the extended phase space, Bodendorfer, Mele, and Münch (BMM) pointed out [59] that they can be considered directly as the Dirac observables in the 4D physical phase space of \((b, c, p_b, p_c)\), as those given by Eq.(2.12). Lately, García-Quismondo and Marugán argued [60] that \(\delta_b\) and \(\delta_c\) should in general depend on the four variables, \((b, c, p_b, p_c)\), through Eqs.(2.10) and (2.11) [or Eqs.(2.33) and (2.34) when outside of the LQBH]. Then, the corresponding Hamilton equations are given by [60]

\[
\begin{align*}
\partial_T i &= C_{ij} \left[ s_i \frac{L_o}{G} (i, p_i) \frac{\partial O_i}{\partial p_i} \right], \quad (i = b, c), \\
\partial_T p_i &= C_{ij} \left[ -s_i \frac{L_o}{G} (i, p_i) \frac{\partial O_i}{\partial i} \right],
\end{align*}
\]

where \(i, j = b, c, i \neq j\), \(s_b = 1\), \(s_c = -1\), and

\[
C_{ij} \equiv \frac{1 - \Delta_{ij} - \Delta_{ji}}{(1 - \Delta_{ii}) (1 - \Delta_{jj}) - \Delta_{ij} \Delta_{ji}},
\]

\[
\Delta_{ij} \equiv \frac{\partial O_i}{\partial \delta_i} \frac{\partial f_i}{\partial \delta_j}.
\]

It is interesting to note that, introducing two new variables, \(t_i\), \((i = b, c)\), via the relations

\[
dt_i \equiv C_{ij} dt, \quad (i \neq j),
\]

Eqs.(3.2) and (3.3) take the forms,

\[
\begin{align*}
\partial_t i &= s_i \frac{L_o}{G} (i, p_i) \frac{\partial O_i}{\partial p_i}, \\
\partial_t p_i &= -s_i \frac{L_o}{G} (i, p_i) \frac{\partial O_i}{\partial i},
\end{align*}
\]

which will lead to the same Hamilton equations as those given by AOS, if we replace \(T\) by \(t_b\) in the equations for \(b\) and \(p_b\), and \(T\) by \(t_c\) in the equations for \(c\) and \(p_c\), as first noted in [60]. This observation will significantly simplify our following discussions.

To proceed further, in the rest of this section, let us consider the above equations only in the external region, while the ones in the internal region will be considered in the next section.

A. Dynamics of the external LQBH Spacetimes

In the external region, the Hamilton equations take the form

\[
\begin{align*}
\frac{db}{dt_b} &= -\frac{1}{2} \left( \frac{\sinh (\delta_b b)}{\delta_b} - \frac{\gamma^2 \delta_b}{\sinh (\delta_b b)} \right), \quad (3.9) \\
\frac{dp_b}{dt_b} &= \frac{1}{2} p_b \cosh (\delta_b b) \left( 1 + \frac{\gamma^2 \delta_b^2}{\sinh^2 (\delta_b b)} \right), \quad (3.10)
\end{align*}
\]

for \((b, p_b)\), and

\[
\begin{align*}
\frac{dc}{dt_c} &= -2 \frac{\sin (\delta_c c)}{\delta_c}, \quad (3.11) \\
\frac{dp_c}{dt_c} &= 2 p_c \cos (\delta_c c), \quad (3.12)
\end{align*}
\]

for \((c, p_c)\). Then, the corresponding solutions for \(b\) and \(p_b\) will be given by Eqs.(2.37) and (2.39) by simply replacing \(T\) by \(t_b\), that is,

\[
\begin{align*}
cosh (\delta_b b) &= b_o + b_o \tanh \left( \frac{b_o t_b}{2} \right), \\
p_b &= -\frac{m L_o}{2 b_o^2} \left( b_o + b_c e^{-b_o t_b} \right) A, \\
A &= \left[ b_o^2 + b_o^2 e^{2b_o t_b} - 2 \left( b_o^2 + 1 \right) e^{b_o t_b} \right]^{1/2}
\end{align*}
\]

while the solutions for \(c\) and \(p_c\) will be given by Eqs.(2.38) with the replacement \(T\) by \(t_c\), i.e.

\[
\begin{align*}
\sin (\delta_c c) &= \frac{2 a_c e^{2t_c}}{a_c^2 + e^{4t_c}}, \\
p_c &= 4m^2 \left( e^{2t_c} + a_c^2 e^{-2t_c} \right).
\end{align*}
\]

The relation between \(t_b\) and \(t_c\) is given by Eq.(3.6), from which we find that

\[
\frac{dt_c}{t_{bc}} \equiv \frac{C_{bc}}{dt_b}.
\]

To study the above relation, let us first note that Eqs.(2.33) and (2.34) lead to
\[
\frac{\partial O_b}{\partial \delta_b} = \frac{p_b}{2\gamma L_0 \delta_b^2} \left( 1 + \frac{\gamma^2 \delta_b^2}{\sinh^2 (\delta_b)} \right) \left( \delta_b b \cosh (\delta_b b) - \sinh (\delta_b b) \right),
\]
\[
\frac{\partial O_c}{\partial \delta_c} = \frac{p_c}{\gamma L_0 \delta_c^2} \left( \delta_c c \cos (\delta_c c) - \sin (\delta_c c) \right). \tag{3.16}
\]

Then, we find that
\[
C_{bc} = \frac{1}{D} \left( 1 - \Omega_b \frac{\partial O_b}{\partial \delta_b} \right) = \frac{1}{D} \left( 1 - \frac{\gamma \Omega_c p_c}{\gamma L_0 \delta_c^2} \left( \delta_c c \cos (\delta_c c) - \sin (\delta_c c) \right) \right),
\]
\[
C_{cb} = \frac{1}{D} \left( 1 - \Omega_b \frac{\partial O_b}{\partial \delta_b} \right) = \frac{1}{D} \left( 1 - \frac{\gamma \Omega_b p_b}{2\gamma L_0 \delta_b^2} \left( 1 + \frac{\gamma^2 \delta_b^2}{\sinh^2 (\delta_b)} \right) \left( \delta_b b \cosh (\delta_b b) - \sinh (\delta_b b) \right) \right), \tag{3.17}
\]

where
\[
D = 1 - \omega_{bc} \frac{\partial O_c}{\partial \delta_c} - \omega_{bb} \frac{\partial O_b}{\partial \delta_b} + (\omega_{bc} \omega_{cc} - \omega_{bc} \omega_{cb}) \frac{\partial O_b}{\partial \delta_b} \frac{\partial O_c}{\partial \delta_c},
\]
\[
= 1 - \frac{\omega_{bc} p_c}{\gamma L_0 \delta_c^2} \left( \delta_c c \cos (\delta_c c) - \sin (\delta_c c) \right) - \frac{\omega_{bb} p_b}{2\gamma L_0 \delta_b^2} \left( 1 + \frac{\gamma^2 \delta_b^2}{\sinh^2 (\delta_b)} \right) \left( \delta_b b \cosh (\delta_b b) - \sinh (\delta_b b) \right)
\]
\[
+ \frac{\omega_{bc} \omega_{cc} - \omega_{bc} \omega_{cb}}{2\gamma^2 L_0^2 \delta_b^2 \delta_c^2} p_b p_c \left( 1 + \frac{\gamma^2 \delta_b^2}{\sinh^2 (\delta_b)} \right) \left( \delta_c c \cos (\delta_c c) - \sin (\delta_c c) \right) \left( \delta_b b \cosh (\delta_b b) - \sinh (\delta_b b) \right), \tag{3.18}
\]
\[
\omega_{ij} \equiv \frac{\partial f_i}{\partial O_j}, \quad \Omega_c \equiv \omega_{cc} + \omega_{cb}, \quad \Omega_b \equiv \omega_{bb} + \omega_{bc}, \tag{3.19}
\]

It should be noted that the numerator of $C_{bc}$ is a function of $t_c$ and the one of $C_{cb}$ is a function of $t_b$, where $t_b$ and $t_c$ are related one to the other through Eq.(3.15). In particular, for $t_b, t_c \gg 1$, from Eq.(3.15) we find
\[
t_c - 2\beta m^2 a^3 \rho + O \left(e^{-4t_c} \right) = (1 + \alpha_2) t_b + \frac{\alpha_1}{b_0} e^{b_0 t_b} + \frac{\alpha_3}{2b_0} + O \left(e^{-b_0 t_b} \right), \quad (t_b, t_c \gg 1), \tag{3.20}
\]

where
\[
\alpha_1 = \frac{m(b_0 + 1)^2}{2\gamma b_0^2 \delta_b^2} \left( b_0 \cosh^{-1} b_0 - \gamma \delta_b \right) \Omega_b,
\]
\[
\alpha_2 = -\frac{m \gamma^2 \delta_b}{\gamma^2 \delta_b^2 + 1} \Omega_b, \tag{3.21}
\]

\beta and $\alpha_3$ are other constants, and their explicit expressions will not be given here, as they will not affect our following discussions.

It is interesting to note that for the BMM choice, $f_i = f_i(O_i)$ [cf. Eq.(2.13)], and $\delta_i$ given by Eq.(2.26) together with the fact that on-shell we have $O_b = m = O_c$, we find that
\[
\omega_{bb}^{\text{BMM}} = -\frac{\delta_b}{3m}, \quad \omega_{cc}^{\text{BMM}} = -\frac{\delta_c}{3m}, \quad \omega_{bc}^{\text{BMM}} = \omega_{cb}^{\text{BMM}} = 0,
\]
\[
\Omega_b^{\text{BMM}} = -\frac{\delta_b}{3m}, \quad \Omega_c^{\text{BMM}} = -\frac{\delta_c}{3m}. \tag{3.22}
\]

To study the external spacetimes further, in the following let us consider the two cases, $\alpha_1 = 0$ and $\alpha_1 \neq 0$, separately.

**B. External Spacetimes with $\alpha_1 \neq 0$**

If $\alpha_1 \neq 0$, from Eq.(3.20) we find that
\[
t_c \approx \frac{\alpha_1}{b_0} e^{b_0 t_b}. \tag{3.23}
\]

Then, from Eq.(3.6) we find that $dT = dt_b/C_{bc}$, and in terms of $t_b$ the metric (2.41) becomes
\[
ds^2 = -\frac{p_b^2}{|p_c| L_0^2} dt_b^2 + \frac{\gamma^2 |p_c| \delta_b^2}{\sinh^2 (\delta_b b)} C_{bc}^{-1} dt_b^2 + |p_c| \left( d\theta^2 + \sin^2 \theta d\phi^2 \right), \tag{3.24}
\]
where $C_{bc}$ is given by Eq. (3.17), and

$$g_{xx} \equiv \frac{p^2}{|p_c| L_z} \simeq \left( c_1 e^{2b_0 t_b} + c_2 e^{b_0 t_b} + c_3 + \cdots \right) \exp \left( -\frac{2\alpha_1}{b_0} e^{b_0 t_b} \right),$$

$$g_{t_b t_b} \equiv \frac{\gamma^2 |p_c| \delta^2}{\sinh^2 (b_0 b_c) C_{bc}} \simeq \left( d_1 e^{2b_0 t_b} + d_2 e^{b_0 t_b} + d_3 + \cdots \right) \exp \left( \frac{2\alpha_1}{b_0} e^{b_0 t_b} \right),$$

$$g_{\theta \theta} \equiv |p_c| \simeq 4m^2 \exp \left( \frac{2\alpha_1}{b_0} e^{b_0 t_b} \right),$$

(3.25)

where $(c_1, d_i)$ are constants defined as

$$c_1 \equiv \frac{(b_0 + 1)^4}{16b_0^4}, \quad c_2 \equiv -\frac{(b_0 + 1)^2}{4b_0^2}, \quad c_3 \equiv -\frac{\gamma^2 \delta_b^2 \left( \gamma^2 \delta_b^2 + 4 \right)}{8b_0^4},$$

$$d_1 \equiv \frac{\omega_0 f^2 m^4}{\gamma^2 \delta_b b_0^4} (b_0 + 1)^4, \quad d_2 \equiv \frac{4\omega_0 f m^3}{\gamma^2 \delta_b b_0^4} (b_0 + 1)^2 \left\{ \gamma b_0^2 \delta_b^2 - m \omega_\theta \left( \gamma^3 \delta_\theta^3 - b_0 f \right) \right\},$$

$$d_3 \equiv 2m^2 \left( \frac{m \omega_\theta}{\gamma^2 \delta_b b_0^4} \right) \left( m \omega_\theta \left( 2\gamma \delta_b^2 + f^2 \left( \gamma^4 \delta_b^2 + 4\gamma \delta_b^4 \left( b_0^2 + 1 \right) + 8b_0^2 + 2 \right) + 2\gamma f \delta_b^3 \left( 1 - 4b_0^2 \right) \right) + \right.$$  

$$+8\gamma \delta_b^2 b_0^4 - 4\gamma \delta_b^5 b_0^2 \right) + 2 \right),$$

(3.26)

with

$$f(\gamma \delta_b) \equiv b_0 \cosh^{-1} b_0 - \gamma \delta_b = b_0 \ln (b_0 + \gamma \delta_b) - \gamma \delta_b = \frac{1}{3} \gamma^3 \delta_b^3 + O \left( \gamma^5 \delta_b^5 \right),$$

(3.27)

which is always non-zero for $\gamma \delta_b > 0$, as shown in Fig. 2. The function $f(\gamma \delta_b)$ defined above must not be confused with the Dirac observables $f_i(O_b, O_c)$ ($i = b, c$) introduced in Eq. (3.1).

From the above expressions, it is clear that $\alpha_1$ must be positive, in order to have the spacetime asymptotically flat as $t_b \gg 1$. This is also consistent with Eq. (3.23), as we assumed that $t_b, t_c \gg 1$ asymptotically. Therefore, in the rest of this subsection we assume $\alpha_1 > 0$, which requires

$$\Omega_b > 0.$$  

(3.28)

It is interesting to note that, corresponding to the BMM choices of $f_i = f_i(O_i)$ and $\delta_i$ given by Eq. (2.26), we have

$$\Omega_b^{\text{BMM}} = -\frac{\delta_b}{3m} < 0,$$  

(3.29)

as given in Eq. (3.22). Therefore, the BMM choices cannot be realized in this case.

It is also interesting to note that the spacetimes described by Eqs. (3.24)-(3.27) actually have similar asymptotic behavior as the AOS solution does, although the two metrics, given respectively by Eqs. (3.24) and (2.41), look quite different. To show this claim, let us first introduce a new spacelike coordinate $\xi$ via the relation, $\xi = e^{b_0 t_b}$, and then we find that the metric (3.24) becomes

$$ds^2 \simeq -\left( c_1 \xi^2 + c_2 \xi + c_3 + \cdots \right) e^{-\alpha_0 \xi} dT^2 + \left( d_1 + \frac{d_2}{\xi} + \frac{d_3}{\xi^2} + \cdots \right) e^{\alpha_0 \xi} \frac{d\xi^2}{b_0^2} + 4m^2 e^{\alpha_0 \xi} \left( d\theta^2 + \sin^2 \theta d\phi^2 \right).$$  

(3.30)
where \( T \equiv x \) and \( \alpha_0 \equiv 2\alpha_1/b_o > 0 \). Then, the corresponding curvature invariants of the above metric are given by

\[
g^{\mu\nu} R_{\mu\nu} \simeq \left( \frac{2m}{r} \right)^2 \left[ \frac{1}{2} \left( \frac{1}{m^2} - \frac{b_o^2 \alpha_0^2}{d_1} \right) + \frac{b_o^2 \alpha_0 (2d_1 + d_2 \alpha_0)}{2d_1^2 \xi} + O \left( \frac{1}{\xi^2} \right) \right],
\]

\[
R^{\mu\nu} R_{\mu\nu} \simeq \left( \frac{2m}{r} \right)^4 \left[ \frac{1}{8} \left( \frac{1}{m^4} + \frac{2b_o^4 \alpha_0^4}{d_1^4} \right) - \frac{b_o^2 \alpha_0 (d_1^2 + 3b_o^2 d_1 m^2 \alpha_0^2 + b_o^2 d_2 m^2 \alpha_0^2)}{2(d_1^2 m^2)^2 \xi} + O \left( \frac{1}{\xi^2} \right) \right],
\]

\[
R^{\mu\nu\alpha\beta} R_{\mu\nu\alpha\beta} \simeq \left( \frac{2m}{r} \right)^4 \left[ \frac{d_1^2 - 2b_o^2 d_1 m^2 \alpha_0^2 + 7b_o^4 m^4 \alpha_0^4}{4d_1^2 m^4} \right] + O \left( \frac{1}{\xi^2} \right),
\]

\[
C^{\mu\nu\alpha\beta} C_{\mu\nu\alpha\beta} \simeq \left( \frac{2m}{r} \right)^4 \left[ \frac{(d_1 - 4b_o^2 m^2 \alpha_0^2)^2}{12d_1^2 m^4} + \frac{2b_o^2 \alpha_0 (2d_1 + d_2 \alpha_0) (d_1 - 4b_o^2 m^2 \alpha_0^2)}{3d_1^2 m^2 \xi} + O \left( \frac{1}{\xi^2} \right) \right], \quad (3.31)
\]

where \( r \left( = 2m e^{\alpha_0 \xi/2} \right) \) is the geometric radius of the two spheres \( \xi \), \( T = \text{constant} \). Comparing the above with the ones presented in \[55\], we can see that now the metric approaches asymptotically to the Minkowski spacetime as \( r^{-4} \), which is the same as that of the AOS solution.

It is also remarkable to note that for the AOS choices of \( \delta_b \) and \( \delta_c \) given by Eq.(2.26), we find that \( \alpha_1 \propto m^{2/3} \) and \( d_1 \propto m^{10/3} \). Then, the above expressions show that they are all independent of \( m \) asymptotically. In particular, we have

\[
R^{\mu\nu\alpha\beta} R_{\mu\nu\alpha\beta} \simeq A_0 \xi^4 + O \left( \frac{1}{r^4 \xi^4} \right), \quad (3.32)
\]

where \( \xi = \frac{2}{\alpha_0} \ln \left( \frac{r}{2m} \right) \), and \( A_0 \) is independent of \( m \) given by

\[
A_0 \equiv \frac{28 \Omega^4}{\omega^4} \left( \frac{1}{2} \right) + \frac{8 \Omega^2}{\omega^2} b \text{ } + 4. \quad (3.33)
\]

This is sharply in contrast to the relativistic case, in which the Kretschmann scalar is given by

\[
R^{\mu\nu\alpha\beta} R_{\mu\nu\alpha\beta} \big|_{\text{GR}} = \frac{48 m^2}{r^6}. \quad (3.34)
\]

It is also very interesting to note that the leading order of the Kretschmann scalar of the AOS solution also behaves like \( r^{-4} \) as \( r \to \infty \) \[57\]. In the current case, even the Dirac observables \( f_\iota \) are chosen so that \( A_0 \) given by Eq.(3.33) is zero, the next leading order is \( O \left( \frac{1}{r^4 \xi^4} \right) \), which approaches zero still not as fast as \( r^{-6} \). In fact, it is even slower than \( r^{-5} \).

To understand the solutions further, we first note that to the leading order the metric takes the form

\[
ds^2 \simeq -c_1 \frac{b_o^2}{\alpha_1^2} \left( \ln \left( \frac{r}{2m} \right) \right)^2 dr^2 + \frac{d_1}{4m^2 \alpha_1^2} dr^2 + r^2 d\Omega^2.
\]

(3.35)

for \( r \gg 2m \). On the other hand, the AOS solution takes the asymptotic form \[57\]

\[
ds^2_{\text{AOS}} \simeq -r^{2(\delta_b-1)} dT^2 + dr^2 + r^2 d\Omega^2,
\]

(3.36)

which is identical to the global monopole solution found in a completely different content \[62\]. Then, the corresponding effective energy-momentum tensor is given by

\[
T_{\mu\nu} = u_\mu u_\nu \rho + p r_r r_\nu + p_\perp (\theta_\mu \theta_\nu + \phi_\mu \phi_\nu), \quad (3.37)
\]

where \( u_\mu \) denotes the unit timelike vector along \( T \)-direction, and \( r_\mu, \theta_\mu, \phi_\mu \) are the spacelike unity vectors along, respectively, \( r \)-, \( \theta \)-, and \( \phi \)-directions, and \( \rho, p_r \) and \( p_\perp \) are the energy density and pressures along the radial and tangential directions. To the leading order, they are given by

\[
\rho \simeq \frac{4a_1^2 m^2 - d_1}{d_1 r^2}, \quad p_r \simeq \frac{d_1 + 4a_1^2 m^2}{d_1 r^2}, \quad p_\perp \simeq \frac{4a_1^2 m^2}{d_1 r^2}, \quad (3.38)
\]

which all approach zero as \( r^{-2} \). This is also consistent with the asymptotical behaviors of the quantities given in Eq.(3.31).

It should be also noted that, despite these differences, the spacetimes of the current solutions are also asymptotically flat and the corresponding ADM masses are as well defined as that of the AOS solution \[55\].

C. External Spacetimes with \( \alpha_1 = 0 \)

When \( \alpha_1 = 0 \), from Eq.(3.21) and Fig. 2 we find that this can be the case only when

\[
\Omega_b \equiv \omega_{bb} + \omega_{bc} = 0. \quad (3.39)
\]

It is clear that the BMM choices of \( f_\iota \) and \( \delta_\iota \), given by Eqs.(2.13) and (2.26), are not compatible with this case, too.

Then, to the leading order, Eq.(3.20) yields

\[
t_b \simeq t_c \equiv t, \quad (3.40)
\]
as \( t_c \to \infty \). With Eqs.(3.39) and (3.40) we find that
\[
g_{xx} \simeq e^{-2t} \left( c_1 e^{2b_o t} + c_2 e^{b_o t} + c_3 + \cdots \right)
\]
\[
g_{yy} \simeq 4m^2 e^{2t},
\]
where \( c_n \)'s are still given by Eq.(3.26). Finding the asymptotic limit of \( g_{tt} \) is not so straightforward, and this is mainly because of the term \( C_{bc} \) seen in the expression
\[
g_{tt} = -\frac{\gamma^2 |p_c| b_0^2}{\sinh^2 (b_0 b) C_{bc}^2}.
\]  
(3.42)

The numerator of \( C_{bc} \) in Eq.(3.18) is equal to 1 with the choice of \( \omega_{bb} + \omega_{bc} = 0 \) and the remainder of \( g_{tt} \) is evaluated with the help of Mathematica, and is given by
\[
g_{tt} = e^{2t} \left( d_1 e^{2b_o t} + d_2 e^{b_o t} + d_3 + \cdots \right),
\]
(3.43)

where \( d_n \)'s are also given by Eq.(3.26). Introducing the new coordinates,
\[
r = 2m e^{t}, \quad x = \frac{4b_0^2}{(b_0 + 1)^2} \tau,
\]
we find
\[
ds^2 = -g_{\tau\tau} d\tau^2 + g_{rr} dr^2 + r^2 d\Omega^2,
\]
(3.45)

where
\[
g_{\tau\tau} \simeq \left( \frac{r}{2m} \right)^{2(b_o - 1)} \left( 1 + \frac{c_2}{c_1} \left( \frac{2m}{r} \right)^{b_o} + \frac{c_2}{c_1} \left( \frac{2m}{r} \right)^{2b_o} \right),
\]
\[
g_{rr} \simeq \frac{d_1}{4m^2} \left( \frac{r}{2m} \right)^{2b_o} + \frac{d_2}{4m^2} \left( \frac{r}{2m} \right)^{b_o} + \frac{d_3}{4m^2}.
\]
(3.46)

Then, we find
\[
g^{\mu\nu} R_{\mu\nu} \simeq \left( \frac{2m}{r} \right)^2 \left[ \frac{1}{2m^2} + \frac{2(2b_o - 1)}{d_1 \xi^2} + O \left( \frac{1}{\xi^3} \right) \right],
\]
\[
R^{\mu\nu} R_{\mu\nu} \simeq \left( \frac{2m}{r} \right)^4 \left[ \frac{1}{8m^4} - \frac{b_0 (c_1 d_2 - c_2 d_1)}{2c_1 d_1^2 m^2 \xi^4} + O \left( \frac{1}{\xi^4} \right) \right],
\]
\[
R^{\mu\alpha\beta} R_{\mu\alpha\beta} \simeq \left( \frac{2m}{r} \right)^4 \left[ \frac{1}{4m^4} - \frac{2}{d_1 m^2 \xi^2} + O \left( \frac{1}{\xi^3} \right) \right],
\]
\[
C^{\mu\nu\alpha\beta} C_{\mu\nu\alpha\beta} \simeq \left( \frac{2m}{r} \right)^4 \left[ \frac{1}{12m^2} + \frac{4(b_o - 2)}{3d_1 m^2 \xi^2} + O \left( \frac{1}{\xi^3} \right) \right],
\]
(3.47)

Recall that \( b_0 \equiv \sqrt{1 + \gamma^2 \delta \beta} \geq 1 \). This can be further understood by the analysis of the corresponding effective energy-momentum tensor, which can be also cast in the form of Eq.(3.37), but now with
\[
\rho = \frac{1}{r} \left( \frac{4m^2 (2b_o - 1)}{d_1 \left( \frac{r}{2m} \right)^{2b_o}} \right),
\]
\[
p_r = \frac{1}{r} \left( \frac{4m^2 (2b_o - 1)}{d_1 \left( \frac{r}{2m} \right)^{2b_o}} \right),
\]
\[
p_\perp = -\frac{4m^2 (2b_o - 1)}{d_1^2 r^2 \left( \frac{r}{2m} \right)^{2b_o}},
\]
(3.49)

which are consistent with the behaviors of the quantities given in Eq.(3.47). Following [55], it is not difficult to see that the spacetimes of the current solutions are also asymptotically flat and the corresponding ADM masses are as well defined as that of the AOS solution.
IV. CANONICAL PHASE SPACE APPROACH: INTERNAL SPACETIMES

In the internal region of the LQBH, the dynamical equations (3.2) and (3.3) take the form

\[
\frac{db}{dt_b} = -\frac{1}{2} \left( \sin(\delta_b b) \cdot \frac{\gamma^2 \delta_b}{\delta_b} + \frac{\gamma^2 \delta_b}{\sin(\delta_b b)} \right),
\]

\[
\frac{dp_b}{dt_b} = \frac{1}{2} p_b \cos(\delta_b b) \left( 1 - \frac{\gamma^2 \delta_b^2}{\sin^2(\delta_b b)} \right),
\]

for the variables \((b, p_b)\), and

\[
\frac{dc}{dt_c} = -\frac{2}{\delta_c} \sin(\delta_c c) \quad \text{(4.3)}
\]

\[
\frac{dp_c}{dt_c} = 2p_c \cos(\delta_c c),
\]

for \((c, p_c)\). Eqs.(4.1) and (4.2) are identical with Eqs.(2.17) and (2.18), if we replace \(T\) by \(t_b\), while Eqs.(4.3) and (4.4) are identical with Eqs.(2.19) and (2.20), if we replace \(T\) by \(t_c\). Then, the corresponding solutions can be obtained directly from Eqs.(2.21) - (2.24) by the above replacements, which lead to

\[
\cos(\delta_b b) = b_o \frac{1 + b_o \tanh \left( \frac{b_o t_b}{2} \right)}{b_o + \tanh \left( \frac{b_o t_b}{2} \right)}
\]

\[
= b_o \left[ e^{b_o t_b} - b_o \right] \left[ e^{b_o t_b} + b_o \right]
\]

\[
p_b = -\frac{m L_o}{2b_o^2} \left( b_o + b_o e^{-b_o t_b} \right) \mathfrak{A},
\]

\[
\sin(\delta_c c) = \frac{2a_o e^{2t_c}}{a_o^2 + e^{4t_c}},
\]

\[
p_c = 4m^2 \left( a_o^2 + e^{4t_c} \right) e^{-2t_c},
\]

but now with

\[
\mathfrak{A} \equiv \left[ 2 \left( b_o^2 + 1 \right) e^{b_o t_b} - b_o^2 - b_o^2 e^{2b_o t_b} \right]^{1/2},
\]

where \(a_o\) and \(b_o\) are still given by Eq.(2.23), and the range of the variables is given by Eq.(2.24). Then, it can be seen that the two Dirac observables \(O_b\) and \(O_c\) are also given by Eq.(2.25) along the dynamical trajectories. However, instead of imposing the conditions (2.26), now we shall leave the choice of \(\delta_b\) and \(\delta_c\) open, as we did in the last section. Thus, the corresponding internal spacetimes are described by the metric

\[
ds^2 = -N^2 dT^2 + \frac{p_b}{|p_c| L_o^2} dx^2 + |p_c| d\Omega^2
\]

\[
= -\left( \frac{N}{C_{cb}} \right)^2 dt_c^2 + \frac{p_c}{|p_c| L_o^2} dx^2 + |p_c| d\Omega^2,
\]

where

\[
N \equiv \frac{\gamma \delta_b \text{sgn} (p_c) |p_c|^{1/2}}{\sin(\delta_b b)}
\]

\[
= \frac{2m}{\mathfrak{A}} \left( b_o e^{b_o t_b} + b_o \right) \left( a_o^2 e^{-2t_c} + e^{2t_c} \right)^{1/2}.
\]

In the following, let us study the above spacetimes near the horizons (\(\mathfrak{A} = 0\)) and throat (\(\partial p_c/\partial t_c = 0\)), separately.

A. Spacetimes near the Horizons

The horizons now are located at \(\mathfrak{A} = 0\), which yields two solutions

\[
t_b^{\text{BH}} = 0, \quad t_b^{\text{WH}} = \frac{-2}{b_o} \ln \left( \frac{b_o + 1}{b_o - 1} \right).
\]

Now to find the relation between \(t_b\) and \(t_c\) the following expression has to be integrated

\[
dt_c = \frac{C_{cb}}{C_{bc}} db_a,
\]

where the expressions of \(C_{bc}\) and \(C_{cb}\) in the interior are

\[
C_{bc} = \frac{1}{D} \left( 1 - \Omega_b \frac{\partial O_c}{\partial \delta_c} \right),
\]

\[
C_{cb} = \frac{1}{D} \left( 1 - \Omega_b \frac{\partial O_b}{\partial \delta_b} \right),
\]

but now with

\[
D \equiv 1 - \omega_{cc} \frac{\partial O_c}{\partial \delta_c} - \omega_{bb} \frac{\partial O_b}{\partial \delta_b} + (\omega_{bb} \omega_{cc} - \omega_{bc} \omega_{cb}) \frac{\partial O_b}{\partial \delta_b} \frac{\partial O_c}{\partial \delta_c},
\]

\[
\frac{\partial O_b}{\partial \delta_b} = -\frac{p_b}{2\gamma L_o \delta_b^2} \left( 1 - \frac{\gamma^2 \delta_b^2}{\sin^2(\delta_b b)} \right) \left[ \delta_b b \cos(\delta_b b) - \sin(\delta_b b) \right],
\]

\[
\frac{\partial O_c}{\partial \delta_c} = \frac{p_c}{\gamma L_o \delta_c} \left[ \delta_c c \cos(\delta_c c) - \sin(\delta_c c) \right].
\]

Similar to the previous subsection, in the following section we consider the cases \(\alpha_1 = 0\) and \(\alpha_1 \neq 0\), separately.
1. \( \alpha_1 = 0 \)

In this case, it is remarkable to note that by integrating Eq.\((3.15)\) we find the following explicit solution,

\[
t_b = t_b^0 + t_c + \frac{m \Omega_c}{\delta_c} \left\{ \cosh (2T) \tan^{-1} \left( e^{2T} \right) - \cosh \left[ 2(t_c - T) \right] \tan^{-1} \left[ e^{-2(t_c - T)} \right] \right\},
\]

which holds for any \( t_c \), including the region \( t_c \geq 0 \), outside the black hole horizon, where \( t_c = \mathcal{T} \) is the location of the transition surface, defined by Eq.\((2.27)\). And \( t_b^0 \) is an integration constant which will be set to zero in the following discussions. When \( t_c = 0 \) the second term in the right-hand side of the above expression vanishes identically, and as \( t_c \to \infty \) it goes to zero as \( \mathcal{O} \left( e^{-2t_c} \right) \). This is consistent with Eq.\((3.20)\).

![Graph](image)

**FIG. 3.** Plots of \( t_b \) vs \( t_c \) for \( \alpha_1 = 0 \) defined by Eq.\((4.14)\). Depending on the signs of \( \Omega_c \), the dependence of \( t_b \) on \( t_c \) is different. Curves \( b, c \) and \( d \) are all for \( \Omega_c = 0.5 \) but with different choices of \( (m, \delta) \). In particular, they correspond to \( (m, \delta) = \{ (10^6, 10^{-7}), (10^6, 0.1), (1, 0.1) \} \), respectively. Curves \( b', c' \) and \( d' \) are all for \( \Omega_c = -0.5 \) but with the same choice of \( (m, \delta) \) as that of the unprimed curves in the respective order.

In Fig. 3, we plot the curves of \( t_b \) vs \( t_c \) of Eq. \((4.14)\) for different choices of parameters involved. In particular, we find that the properties of \( t_b \) across the transition surface sensitively depend on the signs of \( \Omega_c \). More specifically, when \( \Omega_c > 0 \), \( t_b \) decreases exponentially right after crossing the transition surface, as \( t_c \) becomes more and more negative, as shown by Curves \( b, c \) and \( d \) with the choice \( \Omega_c = 0.5 \), where the dots on the curves mark the locations of the transition surfaces. On the other hand, when \( \Omega_c < 0 \), \( t_b \) increases exponentially right after crossing the transition surface, as shown by Curves \( b', c' \) and \( d' \) with \( \Omega_c = -0.5 \). However, the locations of the transition surface indeed depend on the choices of the parameters \((m, L_0\delta_c)\), as shown by Eq.\((2.27)\). In particular, Curves \( b, c \) and \( d \) respectively correspond to

\[
\left( \frac{m}{m_p}, \frac{L_0\delta_c}{t_p} \right) = \left\{ (10^6, 10^{-7}), (10^6, 0.1), (1, 0.1) \right\},
\]

while Curves \( b', c' \) and \( d' \) are all for the same choices of \((m, \delta_c)\), as that of the unprimed curves in respective order. Curves \( b \) and \( c \) share the same mass, i.e. \( m/m_p = 10^6 \), but with different \( \delta_c \)'s. Meanwhile, the locations of the throats (the gray dots) move from the left-hand side to the right-hand side in the direction closer to the horizon, which means that the quantum effects increase as \( \delta_c \) increases. Curves \( c \) and \( d \) share the same \( \delta_c = 0.1 \), but different masses. Comparing their throat positions, we find that the smaller mass also means the more significant quantum effects. On the other hand, outside the horizon, no matter what the parameters are, \( t_b \approx t_c \), which is consistent with our previous conclusion for large \( t_b \) and \( t_c \), as shown by Eq. \((3.40)\).

To understand this point further, let us expand the above expression around the horizon, for which we find

\[
t_b = \beta_1 t_c + \beta_2 t_c^2 + \beta_3 t_c^3 + \mathcal{O} \left( t_c \right),
\]

where

\[
\beta_1 = 1 + \frac{m \Omega_c}{a_o \delta_c} \left[ a_o + \left( a_o^2 - 1 \right) \tan^{-1}(a_o) \right],
\]

\[
\beta_2 = -\frac{m \Omega_c}{a_o (a_o^2 + 1) \delta_c} \left[ a_o \left( a_o^2 - 1 \right) + \left( a_o^2 + 1 \right)^2 \tan^{-1}(a_o) \right],
\]

\[
\beta_3 = \frac{2m \Omega_c}{3a_o (a_o^2 + 1)^2 \delta_c} \left[ a_o + 6a_o^3 + a_o^5 \right.
\]

\[
\left. + \left( a_o^2 - 1 \right) \left( a_o^2 + 1 \right)^2 \tan^{-1}(a_o) \right].
\]

For macroscopic black holes, we have \( m/m_p \gtrsim M_\odot \approx 10^{38} \), while the semi-classical limit requires \( L_0\delta_c \ll 1 \). Then, expanding \( \beta_n \) in terms of \( a_o \), we find that

\[
\beta_1 = 1 + \frac{m \Omega_c}{a_o \delta_c} \left[ a_o + \left( a_o^2 - 1 \right) \tan^{-1}(a_o) \right]
\]

\[
\approx 1 + \frac{\gamma^2 L_0^2 \delta_c \Omega_c}{48m} + \mathcal{O} \left( a_o^4 \right) \approx 1,
\]

\[
\beta_2 = -\frac{\gamma^2 L_0^2 \delta_c \Omega_c}{24m} + \mathcal{O} \left( a_o^4 \right) \approx 0,
\]

\[
\beta_3 = \frac{\gamma^2 L_0^2 \delta_c \Omega_c}{18m} + \mathcal{O} \left( a_o^4 \right) \approx 0.
\]

Therefore, for macroscopic black holes, the relation \( t_b \approx t_c \) near the horizon is well justified. Then, we find that the metric components take the form...
FIG. 4. Plots of the relative difference between the Kretschmann scalars $K$ and $K_{GR}$ in the $a_1 = 0$ case, for (a) $m = 10^6$, and (b) $m = 10^{12}$. Here $K_{GR}(\equiv 48m^2/p_c^3)$ is the corresponding Kretschmann scalar given by GR.

To quantify the quantum effects near the horizon, let us compute the Hawking temperature at the horizon. Given a metric of the form

$$ds^2 = -g_{tt}dt^2 + g_{xx}dx^2 + p_c d\Omega^2,$$  \hspace{1cm} (4.20)

the Hawking temperature of the black hole is given by [55],

$$T_H = \frac{h}{k_B P}, \quad P = \lim_{\epsilon \to 0} \frac{4\pi (g_{tt}g_{xx})^{\frac{1}{2}}}{\sqrt{\partial_t g_{xx}}},$$  \hspace{1cm} (4.21)

where $k_B$ is the Boltzmann constant. Then, for the metric coefficients given by Eq.(4.18) we find

$$T_H = \frac{T_H^{GR}}{1 + a_o^2} \left(1 + \epsilon_T\right),$$  \hspace{1cm} (4.22)

where $T_H^{GR} = \hbar/(8k_B \pi m)$ denotes the Hawking temperature of the Schwarzschild black hole calculated in GR, and

$$\epsilon_T \equiv \frac{m \omega_{cc}}{a_o \delta_c} \left[a_o + \left(a_o^2 - 1\right) \tan^{-1} (a_o)\right].$$  \hspace{1cm} (4.23)

For a BH of mass $10^6$, we find that

$$a_o^2 = \left(\frac{\gamma \delta_c L_m}{8m}\right)^2 \approx 10^{-22},$$

and

$$\epsilon_T = \left(\frac{4m \omega_{cc}}{3\delta_c}\right) \left(1 - \frac{2}{3} a_o^2\right) a_o^2 + \mathcal{O}(a_o^6).$$

For the AOS choice of Eq.(2.13), we find that $4m \omega_{cc}/(3\delta_c) \approx \mathcal{O}(1)$, so that $\epsilon_T \lesssim 10^{-44}$, that is, for macroscopic black holes, the quantum effects are negligible. This is consistent with what was concluded by AOS [54, 55].

The above conclusion can be further verified by comparing the Kretschmann scalars $K$ with its relativistic counterpart $K_{GR} \equiv 48m^2/p_c^3$. In particular, in Fig. 4 we plot the relative difference of $K$ and $K_{GR}$ for $m = 10^6 m_{pl}$ and $m = 10^{12} m_{pl}$, which indicate negligible quantum corrections near the horizon for massive LQBHs.

2. $a_1 \neq 0$

When $a_1 \neq 0$, we find that
where $t_b^0$ is an integration constant and will be set to zero as done previously in the $\alpha_1 = 0$ case.

Notice that the $t_c$ part of the above expression is precisely the right-hand side of Eq. (4.14), and we showed explicitly in the last subsection that near the horizon $t_c = 0$ the right-hand side can be well approximated by $t_c$. Now, expanding the $t_b$ part of the above expression around the horizon, we find

$$t_b + \nu_2 t_b^2 + \nu_3 t_b^3 + \mathcal{O} (t_b^4),$$

(4.25)

where

$$\nu_2 = \frac{1}{6} m \gamma^2 \delta_b \Omega_b,$$

$$\nu_3 = \frac{1}{60} m \gamma^2 \delta_b \Omega_b \left( 10 - \gamma^2 \delta_b^2 \right).$$

(4.26)

The above coefficients $\nu_i$ are negligibly small for large black holes. For example, for a BH of mass $10^6$, they are of the order $\sim 10^{-9}$. Hence, for macroscopic black holes Eq. (4.24) can also be well approximated by

$$t_b \approx t_c,$$

(4.27)

near the black hole horizon, similar to the case $\alpha_1 = 0$. This linear relation can be confirmed by the plot of Eq. (4.24) for various values, as seen in Fig. 5. For plotting the curves b, c, and d corresponding to positive $\Omega_c$, the parameters are chosen respectively as,

$$\left( \frac{m}{m_p} \right) = \left( 10^6, 10^8, 10^{10} \right),$$

$$\left( \omega_{cc}, \omega_{cb}, \Omega_c \right) = \left( \frac{\delta_c}{3m}, 0, \frac{\delta_c}{3m} \right),$$

$$\left( \omega_{bc}, \omega_{bc}, \Omega_b \right) = \left( \frac{\delta_b}{3m}, 0, \frac{\delta_b}{3m} \right).$$

(4.28)

where $\delta_i$'s are given by Eq. (2.26).
Since in the current case \((a_1 \neq 0)\) \(t_b \simeq t_c\) also holds near the horizon for macroscopic black holes, the thermodynamics of the black hole horizon is quite similar to the case \(a_1 = 0\). In particular, its temperature is also given by Eqs.\((4.22)\) and \((4.23)\), and the difference to that of the Schwarzschild black hole calculated in GR is negligibly small for macroscopic black holes.

Again, a plot of the relative difference between the Kretschmann scalar \(K\) and \(K_{GR}\) is given in Fig. 6 for the \(a_1 \neq 0\) case, which also shows the negligible quantum effects near the horizons for massive LQBHs.

**B. Spacetimes near Transition Surfaces**

It is evident from Figs. 3 and 5 that the above approximation, \(t_b \simeq t_c\), is no longer valid once we start to probe the spacetime near and to the other side of the transition surface. We break this analysis again into two cases, \(a_1 = 0\) and \(a_1 \neq 0\).

1. \(a_1 = 0\)

In this case, the relation between \(t_b\) and \(t_c\) is given by Eq.\((4.14)\), which is valid everywhere in the interior. Combining this equation with the metric \((4.8)\) we can calculate the curvature invariants to analyze the spacetime near the transition surface. We find that this can be done by xAct[66], a package for tensor computations in Mathematica, although the exact expressions are too complicated to be written down here. For this reason, we only plot out the Kretschmann scalar here for illustration, as other scalars like the Ricci scalar, Ricci tensor squared, have similar features. In particular, in Figs. 7 and 8 we plot the Kretschmann scalar respectively for \(\Omega_c < 0\) and \(\Omega_c > 0\), but all with \(\Omega_b = 0\). In addition, we also provide Table I, in which we show the explicit dependence of the maximal amplitude \(K_m\) of the Kretschmann scalar on the mass \(m\), the location of the maximal amplitude of the Kretschmann scalar, denoted by \(\tau_m\), and the location of the transition surface denoted by \(\tau_{ts}\). To compare with the AOS solution, we also give the maximal amplitudes of the Kretschmann scalar for the AOS solution.

From Figs. 7 and 8 and Table I we can see that the Kretschmann scalar remains finite across the transition surfaces, but the maximal amplitude of the Kretschmann scalar sensitively depends on the mass \(m\), which is in sharp contrast to the AOS solution in which the maximal amplitude \(K_m^{AOS}\) of the Kretschmann scalar remains the same [53–55].

Another unexpected feature is that the maximal point of the Kretschmann scalar usually is not precisely at the transition surface, \(\tau_m \neq \tau_{ts}\). Although this looks strange, a closer examination shows that this is due to two main facts: (1) the appearance of the factor \(1/C_{bc}\) in the lapse function of the metric \((4.8)\), and (2) the dependence of \(t_b\) on \(t_c\), which will lead to the modifications of \(g_{xx}(t_b,t_c)\), in comparison to the corresponding AOS component \(g_{xx}^{AOS}(t_b,t_c)\) in which we have \(t_b = t_c = T\).

In particular, when \(a_1 = 0\), we have \(1/C_{bc} = D\), as can be seen from Eq.\((4.12)\), where \(D\) is defined by Eq.\((4.13)\). In Fig. 9 we plot out the function \(D^2\) for the same choices of the parameters as given in Fig. 7, from which we can see that it changes dramatically near the maximal point \(\tau_m \approx -12.0147\) of the Kretschmann scalar. In Figs. 10 and 11, we plot out the metric components \(g_{tt},g_{bc}\) given in Eq.\((4.8)\) vs \(t_c\), where \(g_{tt} \equiv |g_{tt}|\). From these figures we can see clearly that both of these components change dramatically near the maximal point \(\tau_m\) of the Kretschmann scalar. To compare it with the AOS solution, in each of these two figures, we also plot the corresponding quantities for the AOS solution, from which it can be seen that no such behavior appears in the AOS solution.
TABLE I. The maximal amplitude $K_m$ of the Kretschmann scalar $K$ for the case $\alpha_1 = 0$ with different choices of the mass parameter $m$. Here $\tau_m$ denotes the location of the maximal point of $K$, and $\tau_s$ the location of the corresponding transition surface (throat). To compare it with that given by the AOS solution, we also give the maximal values of $K_m^{AOS}$. Here we choose $\omega_{bb} = -\delta_b/3m$, $\omega_{bc} = \delta_b/3m$, $\omega_{cc} = -\delta_c/3m$, and $\omega_{cb} = 0$, so that $(\Omega_b, \Omega_c) = (0, -\delta_c/3m < 0)$, where $\delta_i$'s are given by Eq.(2.26).

| $m/m_p$ | $\tau_m$ | $K_m$ | $\tau_s$ | $K_m^{AOS}$ |
|---------|----------|-------|----------|-------------|
| $10^6$  | -12.0147 | $2.46 \times 10^{48}$ | -11.6201 | 82188.3628  |
| $10^8$  | -15.0848 | $1.56 \times 10^{52}$ | -14.6902 | 82188.3642  |
| $10^{10}$ | -18.1549 | $3.60 \times 10^{75}$ | -17.7603 | 82188.3642  |
| $10^{12}$ | -21.225  | $2.21 \times 10^{75}$ | -20.8304 | 82188.3642  |
| $10^{14}$ | -24.2951 | $2.87 \times 10^{70}$ | -23.9005 | 82188.3642  |
| $10^{16}$ | -27.3653 | $2.82 \times 10^{69}$ | -26.9706 | 82188.3641  |
| $10^{18}$ | -30.4354 | $9.59 \times 10^{77}$ | -30.0408 | 82188.3618  |

We also study the location of the white horizon and find that it is very near the transition surface. In particular, the ratio between WH and BH horizon radii now is much smaller than 1 and sensitively depends on the mass parameter $m$, as shown explicitly in Table II. Whereas in the AOS model this ratio is very close to 1.

2. $\alpha_1 \neq 0$

In this case, the explicit relation between $t_b$ and $t_c$ is given by Eq.(4.24). This relation allows us to write down the metric and calculate the curvature invariants.

FIG. 9. The function $D^2$ defined by Eq.(4.13) for the case $\alpha_1 = 0$, for which we have $C_{bc} = 1/D$. The vertical (green) line marks the position of the transition surface. When plotting this curve, we have chosen the relevant parameters exactly as those given in Fig. 7.

FIG. 10. The metric component $g_{tc}$ given in Eq.(4.8), where $g_{tt} \equiv |g_{tc}|$. The inserting is the plot of the same quantity for the AOS solution. When plotting this curve, we have chosen the relevant parameters exactly as those given in Fig. 7.

FIG. 11. The metric component $g_{xx}$ given in Eq.(4.8). The inserting is the plot of the same quantity for the AOS solution. When plotting this curve, we have chosen the relevant parameters exactly as those given in Fig. 7.

TABLE II. The ratio of the WH and BH horizon radii for the case $\alpha_1 = 0$ with different choices of the mass parameter $m$. Here we use the same choices as those in 8, except for $m$.

| $m/m_p$ | $\frac{r_{WH}}{r_{BH}}$ |
|---------|------------------------|
| $10^6$  | $5.5872 \times 10^{-5}$ |
| $10^8$  | $2.9462 \times 10^{-6}$ |
| $10^{10}$ | $1.5148 \times 10^{-7}$ |
| $10^{12}$ | $7.6577 \times 10^{-9}$ |
| $10^{14}$ | $3.8242 \times 10^{-10}$ |
| $10^{16}$ | $1.8923 \times 10^{-11}$ |
| $10^{18}$ | $9.2972 \times 10^{-13}$ |
Similar to the $\alpha_1 = 0$ case, the exact expressions of them are too complicated to be written down explicitly here, and instead we find that it sufficient to simply plot them out. Since they all have similar behavior, we plot out only the Kretschmann scalar. In particular, we plot it for $\Omega_c < 0$ and $\Omega_c > 0$, respectively in Figs. 12 and 13. The vertical line in each of these figures represents the location of the transition surface, and is usually different from the maximal point of the Kretschmann scalar, quite similar to the case $\alpha_1 = 0$ and for similar reasons.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig12}
\caption{The Kretschmann scalar for the case $\alpha_1 \neq 0$ with $\Omega_c < 0$. In particular, the parameters are chosen as $m = 10^{6}$, $(\omega_{c}, \omega_{d}, \Omega_{c}) = (-\frac{2b}{3m}, 0, -\frac{2a}{3m})$, $(\omega_{d}, \omega_{c}, \Omega_{d}) = (-\frac{2b}{3m}, 0, -\frac{2a}{3m})$, where $\delta_{i}$'s are given by Eq.(2.26).}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig13}
\caption{The Kretschmann scalar for the case $\alpha_1 \neq 0$ with $\Omega_c > 0$. In particular, the parameters are chosen as $m = 10^{6}$, $(\omega_{c}, \omega_{d}, \Omega_{c}) = (-\frac{2b}{3m}, \frac{a}{3m}, \frac{2a}{3m})$, $(\omega_{d}, \omega_{c}, \Omega_{d}) = (-\frac{2b}{3m}, \frac{a}{3m}, \frac{2a}{3m})$, where $\delta_{i}$'s are given by Eq.(2.26).}
\end{figure}

\section{V. CONCLUSIONS}

In this paper, we studied the 4-dimensional canonical phase space approach, explored respectively by BMM [59] and GM [60] recently, in which the two parameters $\delta_{i}$ (i = $\delta, c$) appearing in the polymerization quantization [35]

$$b \rightarrow \frac{\sin(\delta b)}{\delta b}, \quad c \rightarrow \frac{\sin(\delta c)}{\delta c},$$

are considered as functions of the two Dirac variables $O_{b}$ and $O_{c}$ [60]

$$\delta_{i} = f_{i}(O_{b}, O_{c}), \quad (i = b, c),$$

where $O_{b}$ and $O_{c}$ are given by Eqs.(2.10) and (2.11). Note that BMM only considered the particular case $\delta_{i} = f_{i}(O_{i})$ [59], the same as the AOS choice given in Eq.(2.14), although AOS considered them in the extended 8-dimensional phase space $\Gamma_{ext}$. The corresponding dynamical equations are given by Eqs.(3.7) and (3.8), which allow analytical solutions in terms of $t_{b}$ and $t_{c}$, where $t_{b}$ and $t_{c}$ are all functions of $T$ only, given by Eq.(3.6).

To compare the AOS and BMM/GM approaches, in Section II we first presented the AOS model, and discuss how to uniquely fix the two Dirac observables $\delta_{i}$'s [cf. Eqs.(2.15) and (2.16)] in the extended phase space. In the large mass limit, these conditions lead to $\delta_{i}$'s given explicitly by Eq.(2.26).

In the BMM/GM model, the black and white horizons, in general, all exist, and naturally divide the whole spacetime into the external and internal regions, where $T$ is timelike in the internal region and spacelike in the external region. In Section III, we briefly introduce the BMM/GM approach and focused on studies of the external region of the spacetime. We found that the asymptotical flatness condition of the spacetime requires

$$\Omega_{b} \geq 0,$$

where $\Omega_{b}$ is defined in Eq.(3.19), which excludes the BMM choice $\delta_{i} = f_{i}(O_{i})$ [59], for which we always have $\Omega_{b}^{BMM} < 0$, as shown explicitly by Eq.(3.22). Despite the significant difference of the metrics of the AOS and BMM/GM models, we found that, to the leading order, the asymptotical behavior of the spacetime in the two models is universal and independent of the mass parameter $m$ for the curvature invariants [cf. Eqs.(3.31) and (3.47)]. But, to the next leading order, they are different. In particular, the Kretschmann scalar behaves as

$$K \simeq \frac{A_{0}}{r^{2}} + \mathcal{O}\left(\frac{1}{r^{4+\xi}}\right),$$

as $r \rightarrow \infty$, where $A_{0}$ is a constant and independent of $m$, and $r$ the geometric radius of the two-spheres. For the case $\alpha_{1} \neq 0$, we have $\xi = \frac{2}{\alpha_{1}} \ln \left(\frac{r}{2m}\right)$, and for $\alpha_{1} = 0$, we have $\xi = \left(\frac{r}{2m}\right)^{b_{0}}$. Here $\alpha_{1}$ is defined in Eq.(3.21). The differences from the next leading order can be understood more clearly from the metric and the effective energy-momentum tensor, given, respectively, by Eqs.(3.35), (3.38), (3.48) and (3.49). On the other hand, asymptotically the AOS solution takes the global monopole form (3.36), found previously in a completely different content [62]. Nevertheless, the leading behavior of the Kretschmann scalar in both cases is in sharp contrast to the classical case [55, 57], for which we have $K_{GR} = 48m^{2}/r^{6}$. 


In Section IV, we conducted our studies on the internal region of the spacetime. We first showed that the quantum gravitational effects near the black hole horizon are negligible for massive black holes, and both the Kretschmann scalar and Hawking temperature are indistinguishable from those of GR, as shown explicitly by Figs. 4 and 6, and Eq.(4.22). However, despite the fact that all the physical quantities are finite, and the Schwarzschild black hole singularity is replaced by a transition surface whose radius is always finite and non-zero, the internal region near the transition surface is dramatically different from that of the AOS model in several respects: (1) First, the location of the maxima of the curvature invariants, such as the Kretschmann scalar is considerably modified by polymerization. In particular, we plotted the metric components appearing in Eq.(5.1) before accepting them as viable LQBH models in LQG. In particular, in [49] the consistent gauge-fixing conditions in polymerized gravitational systems were studied, and it would be very interesting to check how these conditions affect the results presented in this paper as well as results obtained in other LQBH models.

(2) The maxima of these curvature invariants depend on the choice of the mass parameter \( m \). In particular, Table I shows such dependence for the Kretschmann scalar, which also shows that such dependence is absent in the AOS model. (3) The location of the white hole horizon is very near to the transition surface, and the ratio of the two horizon radii is much smaller than 1, and depends sensitively on \( m \) as shown in Table II. All these results are significantly different from those obtained in the AOS model.

In review of the results presented in this paper, it is clear that further investigations are highly demanded for LQBH models, in which the two polymerization parameters \( \delta_b \) and \( \delta_c \) appearing in Eq.(5.1) are considered as Dirac observables of the 4-dimensional phase space, spanned by \((b, c, p_b, p_c)\), before accepting them as viable LQBH models in LQG. In particular, in [49] the consistent gauge-fixing conditions in polymerized gravitational systems were studied, and it would be very interesting to check how these conditions affect the results presented in this paper as well as results obtained in other LQBH models.

Notes-in-addition: When we were finalizing our manuscript, we came across three very interesting and relevant articles [63–65]. We will briefly comment on them here. First, in [63] the authors studied the physical meaning of the three integration constants, \( C_1, C_2 \) and \( p_c^0 \), obtained from the integration of the three dynamical equations for the variables \( c, b \) and \( p_c \), respectively, and found that \( C_1 \) is related to the location of the transition surface, \( C_2 \) can be gauged away by the redefinition of time \( t \to t + t_0 \), where \( t_0 \) is a constant, while \( p_c^0 \) is related to the mass parameter. A similar consideration was also carried out in [38] but for the BMM polymer black hole solution [31]. Second, in [64] the authors studied the integrability of \( G_b(t_b) \equiv \int_{t_b}^{t_0} F_{cb}(t'_b) dt'_b = \int_{t_c}^{t_0} F_{bc}(t'_c) dt'_c = G_c(t_c) \), the invertibility of \( t_i = G_i^{-1}[G_j(t_j)] \), and the overlap of the images of \( G_i \). It was shown that \( F_{ij} \)'s are always integrable so that \( G_i \)'s always exist. The images of \( G_i \)'s can be always made overlapping by using the redefinitions of the two-time-variables \( t_i = t_i + t_{i0} \). In addition, \( t_i = G_i^{-1} \) is always invertible except at the zero points \( G_i(t_i^{ext}) = 0 \). Moreover, these zero points never correspond to the same moment \( T \), so at least one of the two \( G_i \)'s is invertible at any given moment \( T \). It must be noted that all the studies carried out in [64] were restricted to the internal region. When restricting our studies to this region, our results are consistent with theirs, whenever the problems of the integrability, invertibility and overlap of the images, all studied in [64], are concerned. Finally, in [65], the authors considered the quantization of the AOS extended phase space model, and found the conditions that guarantee the existence of physical states in the regime of large black hole masses, among other interesting results.

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[1] A. Ashtekar and J. Lewandowski, Background independent quantum gravity: A Status report, Class. Quant. Grav. 21, R53 (2004) [arXiv:gr-qc/0404018 [gr-qc]].
[2] T. Thiemann, Modern canonical quantum general relativity (Cambridge University Press, Cambridge, 2008).
[3] R. Gambini and J. Pullin, A First Course in Loop Quantum Gravity (Oxford University Press, Oxford, 2011).
[4] M. Bojowald, Canonical Gravity and Applications: Cosmology, Black holes, and Quantum Gravity ( Cambridge University Press, Cambridge, 2011).
[5] C. Rovelli and F. Vidotto, Covariant Loop Quantum Gravity: an Elementary Introduction to Quantum Gravity and SpinFoam Theory (Cambridge University Press, Cambridge, 2015).

[6] A. Ashtekar and P. Singh, Loop Quantum Cosmology: A Status Report, Class. Quant. Grav. 28, 213001 (2011) [arXiv:1108.0893 [gr-qc]].

[7] A. Ashtekar and M. Bojowald, Quantum geometry and the Schwarzschild singularity, Class. Quant. Grav. 23 (2006), 391 [arXiv:gr-qc/0509075 [gr-qc]].

[8] L. Modesto, Loop quantum black hole, Class. Quant. Grav. 23, 5587 (2006) [arXiv:gr-qc/0509078 [gr-qc]].

[9] C. G. Böhmer and K. Vandersloot, Loop Quantum Dynamics of the Schwarzschild Interior, Phys. Rev. D76, 104030 (2007) [arXiv:0709.2129 [gr-qc]].

[10] R. Gambini and J. Pullin, Black holes in loop quantum gravity: The Complete space-time, Phys. Rev. Lett. 101, 161301 (2008) [arXiv:0805.1187 [gr-qc]].

[11] M. Campiglia, R. Gambini and J. Pullin, Loop quantization of spherically symmetric midih-superspaces: The Interior problem, AIP Conf. Proc. 977, 52 (2008) [arXiv:0712.0817 [gr-qc]].

[12] J. Brannlund, S. Kloster and A. DeBenedictis, The Evolution of Lambda Black Holes in the Mini-Superspace Approximation of Loop Quantum Gravity, Phys. Rev. D79, 084023 (2009) [arXiv:0801.0010 [gr-qc]].

[13] L. Modesto, Semiclassical loop quantum black hole, Int. J. Theor. Phys. 49, 1649 (2010) [arXiv:0811.2196 [gr-qc]].

[14] D. W. Chion, Phenomenological loop quantum geometry of the Schwarzschild black hole, Phys. Rev. D78, 064040 (2008) [arXiv:0807.0665 [gr-qc]].

[15] D. W. Chion, Phenomenological dynamics of loop quantum cosmology in Kantowski-Sachs spacetime, Phys. Rev. D78, 044019 (2008) [arXiv:0803.3659 [gr-qc]].

[16] A. Perez, The Spin Foam Approach to Quantum Gravity, Living Rev. Rel. 16, 3 (2013) [arXiv:1205.2019 [gr-qc]].

[17] R. Gambini and J. Pullin, Loop Quantization of the Schwarzschild Black Hole, Phys. Rev. Lett. 110, 211301 (2013) [arXiv:1312.5512 [gr-qc]].

[18] R. Gambini, J. Olmedo and J. Pullin, Quantum black holes in Loop Quantum Gravity, Class. Quant. Grav. 31, 095009 (2014) [arXiv:1310.5996 [gr-qc]].

[19] R. Gambini and J. Pullin, Hawking radiation from a spherical loop quantum gravity black hole, Class. Quant. Grav. 31, 115003 (2014) [arXiv:1312.3595 [gr-qc]].

[20] H. M. Haggard and C. Rovelli, Quantum-gravity effects outside the horizon spark black to white hole tunneling, Phys. Rev. D92, 104020 (2015) [arXiv:1407.0989 [gr-qc]].

[21] A. Joe and P. Singh, Kantowski-Sachs spacetime in loop quantum cosmology: bounds on expansion and shear scalars and the viability of quantization prescriptions, Class. Quant. Grav. 32, 015009 (2015) [arXiv:1407.2428 [gr-qc]].

[22] N. Dadhich, A. Joe and P. Singh, Emergence of the product of constant curvature spaces in loop quantum cosmology, Class. Quant. Grav. 32 (2015) 185006 [arXiv:1505.05727 [gr-qc]].

[23] A. Corichi and P. Singh, Loop quantization of the Schwarzschild interior revisited, Class. Quant. Grav. 33, 055006 (2016) [arXiv:1506.08015 [gr-qc]].

[24] J. Olmedo, S. Saini and P. Singh, From black holes to white holes: a quantum gravitational, symmetric bounce, Class. Quant. Grav. 34, 225011 (2017) [arXiv:1707.07333 [gr-qc]].

[25] J. Cortez, W. Cuervo, H. A. Morales-Técotl and J. C. Ruelas, Effective loop quantum geometry of Schwarzschild interior, Phys. Rev. D95, 064041 (2017) [arXiv:1704.03362 [gr-qc]].

[26] C. Rovelli, Planck stars as observational probes of quantum gravity, Nature Astron. 1, 0065 (2017) [arXiv:1708.01789 [gr-qc]].

[27] A. Perez, Black Holes in Loop Quantum Gravity, Rept. Prog. Phys. 80, 126901 (2017) [arXiv:1703.09149 [gr-qc]].

[28] A. Barrau, K. Martineau and F. Moulin, A status report on the phenomenology of black holes in loop quantum gravity: Evaporation, tunneling to white holes, dark matter and gravitational waves, Universe 4, 102 (2018) [arXiv:1808.08857 [gr-qc]].

[29] C. Rovelli and P. Martin-Dussaud, Interior metric and ray-tracing map in the firework black-to-white hole transition, Class. Quant. Grav. 35, 147002 (2018) [arXiv:1803.06330 [gr-qc]].

[30] E. Bianchi, M. Christodoulou, F. D’Ambrosio, H. M. Haggard and C. Rovelli, White Holes as Remnants: A Surprising Scenario for the End of a Black Hole, Class. Quant. Grav. 35, 225003 (2018) [arXiv:1802.04264 [gr-qc]].

[31] N. Bodendorfer, F. M. Mele and J. Münch, Effective Quantum Extended Spacetime of Polymer Schwarzschild Black Hole, Class. Quant. Grav. 36 (2019) 195015 [arXiv:1902.04542 [gr-qc]].

[32] P. Martin-Dussaud and C. Rovelli, Evaporating black-to-white hole, Class. Quant. Grav. 36, 245002 (2019) [arXiv:1905.07251 [gr-qc]].

[33] M. Assanioussi, A. Dapor and K. Liegener, Perspectives on the dynamics in a loop quantum gravity effective description of black hole interiors, Phys. Rev. D101, 026002 (2020) [arXiv:1908.05756 [gr-qc]].

[34] D. Arruga, J. Ben Achour, and K. Noui, Deformed General Relativity and Quantum Black Holes Interior, Universe 6, 39 (2020) [arXiv:1912.02459 [gr-qc]].

[35] A. Ashtekar, Black Hole evaporation: A Perspective from Loop Quantum Gravity, Universe 6, 21 (2020) [arXiv:2001.08833 [gr-qc]].

[36] C. Zhang, Y. Ma, S. Song and X. Zhang, Loop quantum Schwarzschild interior and black hole remnant, Phys. Rev. D102, 041502 (2020) [arXiv:2006.08313 [gr-qc]].

[37] R. Gambini, J. Olmedo and J. Pullin, Spherically symmetric loop quantum gravity: analysis of improved dynamics, Class. Quant. Grav. 37, 205012 (2020) [arXiv:2006.01513 [gr-qc]].

[38] W. C. Gan, N. O. Santos, F. W. Shu and A. Wang, Properties of the spherically symmetric polymer black holes, Phys. Rev. D102, 124030 (2020) [arXiv:2008.09664 [gr-qc]].

[39] J. G. Kelly, R. Santacruz and E. Wilson-Ewing, Effective loop quantum gravity framework for vacuum spherically symmetric spacetimes, Phys. Rev. D102, 106024 (2020) [arXiv:2006.09302 [gr-qc]].

[40] C. Liu, T. Zhu, Q. Wu, K. Jusufi, M. Jamil, M. Azreg-Anou and A. Wang, Shadow and Quasinormal Modes of a Rotating Loop Quantum Black Hole, Phys. Rev. D101, 084001 (2020) [arXiv:2003.00477 [gr-qc]].

[41] N. Bodendorfer, F. M. Mele and J. Münch, (b, v)-type variables for black to white hole transitions in effective loop quantum gravity, Phys. Lett. B819, 136390 (2021) [arXiv:1911.12646 [gr-qc]].
[42] K. Giesel, B. F. Li and P. Singh, Non-singular quantum gravitational dynamics of an LTB dust shell model: the role of quantization prescriptions, Phys. Rev. D104, 106017 (2021) [arXiv:2107.05797 [gr-qc]].

[43] N. Bodendorfer, F. M. Mele and J. Münch, Mass and Horizon Dirac Observables in Effective Models of Quantum Black-to-White Hole Transition, Class. Quant. Grav. 38, 095002 (2021) [arXiv:2112.00774 [gr-qc]].

[44] F. Sartini and M. Geiller, Quantum dynamics of the black hole interior in loop quantum cosmology, Phys. Rev. D103, 066014 (2021) [arXiv:2010.07056 [gr-qc]].

[45] B. F. Li and P. Singh, Does the Loop Quantum Scheme Permit Black Hole Formation? Universe 7 (2021) 406 [arXiv:2110.15373 [gr-qc]].

[46] R. Gambini, J. Olmedo and J. Pullin, Loop Quantum Black Hole Extensions Within the Improved Dynamics, Front. Astron. Space Sci. 8, 74 (2021) [arXiv:2012.14212 [gr-qc]].

[47] Y. C. Liu, J. X. Feng, F. W. Shu and A. Wang, Extended geometry of Gambini-Olmedo-Pullin polymer black hole and its quasinormal spectrum, Phys. Rev. D104, 106001 (2021) [arXiv:2109.02861 [gr-qc]].

[48] M. Han and H. Liu, Improved effective dynamics of loop-quantum-gravity black hole and Nariai limit, Class. Quant. Grav. 39, 035011 (2022) [arXiv:2012.05729 [gr-qc]].

[49] K. Giesel, B. F. Li, P. Singh and S. A. Weigl, Consistent gauge-fixing conditions in polymerized gravitational systems, Phys. Rev. D 105 (2022) 066023 [arXiv:2112.13860 [gr-qc]].

[50] C. Zhang, Y. Ma, S. Song and X. Zhang, Loop quantum deparametrized Schwarzschild interior and discrete black hole mass, Phys. Rev. D105, 024069 (2022) [arXiv:2107.10579 [gr-qc]].

[51] S. Rastgoo and S. Das, Probing the Interior of the Schwarzschild Black Hole Using Congruences: LQG vs. GUP, Universe 8, 349 (2022) [arXiv:2205.03799 [gr-qc]].

[52] W. C. Gan, G. Ongole, E. Alesci, Y. An, F. W. Shu and A. Wang, Understanding quantum black holes from quantum reduced loop gravity, arXiv:2206.07127 [gr-qc].

[53] A. Ashtekar, J. Olmedo and P. Singh, Quantum Transfiguration of Kruskal Black Holes, Phys. Rev. Lett. 121, 241301 (2018) [arXiv:1806.06048 [gr-qc]].

[54] A. Ashtekar, J. Olmedo and P. Singh, Quantum extension of the Kruskal spacetime, Phys. Rev. D98, 126003 (2018) [arXiv:1806.02406 [gr-qc]].

[55] A. Ashtekar and J. Olmedo, Properties of a recent quantum extension of the Kruskal geometry, Int. J. Mod. Phys. D29, 2050076 (2020) [arXiv:2005.02309 [gr-qc]].

[56] A. Ashtekar, T. Pawlowski and P. Singh, Quantum Nature of the Big Bang: Improved dynamics, Phys. Rev. D74, 084003 (2006) [arXiv:gr-qc/0607039 [gr-qc]].

[57] M. Bouhmadi-López, S. Brahma, C. Y. Chen, P. Chen and D. h. Yeom, Asymptotic non-flatness of an effective black hole model based on loop quantum gravity, Phys. Dark Univ. 30, 100701 (2020) [arXiv:1902.07874 [gr-qc]].

[58] M. Bojowald, No-go result for covariance in models of loop quantum gravity, Phys. Rev. D102, 046006 (2020) [arXiv:2007.16066 [gr-qc]].

[59] N. Bodendorfer, F. M. Mele and J. Münch, A note on the Hamiltonian as a polymerisation parameter, Class. Quant. Grav. 36, 187001 (2019) [arXiv:1902.04032 [gr-qc]].

[60] A. García-Quismondo and G. A. M. Marugán, Exploring alternatives to the Hamiltonian calculation of the Ashtekar-Olmedo-Singh black hole solution, Front. Astron. Space Sci. 8, 701723 (2021) [arXiv:2107.00947 [gr-qc]].

[61] S.W. Hawking, G.F.R. Ellis, The Large Scale Structure of Spacetime (Cambridge University Press, Cambridge, 1973).

[62] M. Barriola and A. Vilenkin, Gravitational Field of a Global Monopole, Phys. Rev. Lett. 63, 342 (1989).

[63] B. Elizaga Navascués, A. García-Quismondo and G. A. Mena Marugán, The space of solutions of the Ashtekar-Olmedo-Singh effective black hole model, arXiv:2207.04677 [gr-qc].

[64] A. García-Quismondo and G. A. Mena Marugán, Two-time alternative to the Ashtekar-Olmedo-Singh black hole interior, Phys. Rev. D106, 023532 (2022) [arXiv:2207.04720 [gr-qc]].

[65] B. Elizaga Navascués, A. García-Quismondo and G. A. Mena Marugán, Hamiltonian formulation and loop quantization of a recent extension of the Kruskal spacetime, [arXiv:2208.00425 [gr-qc]].

[66] J. M. Martin-Garcia, xAct: Efficient tensor computer algebra for the Wolfram Language, http://www.xact.es.