The twilight zone in the parametric evolution of eigenstates: beyond perturbation theory and semiclassics

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Considering a quantized chaotic system, we analyze the evolution of its eigenstates as a result of varying a control parameter. As the induced perturbation becomes larger, there is a crossover from a perturbative to a non-perturbative regime, which is reflected in the structural changes of the local density of states. For the first time the full scenario is explored for a physical system: an Aharonov-Bohm cylindrical billiard. As we vary the magnetic flux, we discover an intermediate twilight regime where perturbative and semiclassical features co-exist. This is in contrast with the simple crossover from a Lorentzian to a semicircle line-shape which is found in random-matrix models.

The analysis of the evolution of eigenvalues and of the structural changes that the corresponding eigenstates of a chaotic system exhibit as one varies a parameter $\phi$ of the Hamiltonian $\mathcal{H}(\phi)$ has sparked a great deal of research activity for many years. Physically the change of $\phi$ may represent the effect of some externally controlled field (like electric field, magnetic flux, gate voltage) or a change of an effective-interaction (as in molecular dynamics). Thus, these studies are relevant for diverse areas of physics ranging from nuclear [1, 2] and atomic physics [3, 4] to quantum chaos [5, 6, 7, 8].

Up to now the majority of this research activity was focused on the study of eigenvalues, where a good understanding has been achieved, while much less is known about eigenstates. The pioneering work in this field has been done by Wigner [1], who studied the parametric evolution of eigenstates of a simplified Random Matrix Theory (RMT) model of the type $\mathcal{H} = E + \phi B$. The elements of the diagonal matrix $E$ are the ordered energies $\{E_n\}$, with mean level spacing $\Delta$, while $B$ is a banded random matrix. Wigner found that as the parameter $\phi$ increases the eigenstates undergoes a transition from a perturbative Lorentzian-type line shape to a non-perturbative semicircle line-shape.

For many years the study of parametric evolution for canonically quantized systems was restricted to the exploration of the crossover from integrability to chaos [2, 3]. Only later it has been realized that a theory is lacking for systems that are chaotic to begin with. Inspired by Wigner theory, the natural prediction was that the local density of states (LDOS) should exhibit a crossover from a regime where a perturbative treatment is applicable, to a regime where semiclassical approximation is valid. However, despite a considerable amount of numerical efforts [4], there was no clear-cut demonstration of this crossover. Neither a theory has been developed describing how the transition from the perturbative to the non-perturbative regime takes place.

It is the purpose of this Letter to present, for the first time, a complete scenario of parametric evolution, in case of a physical system that exhibits hard chaos. We explore the validity of perturbation theory and semiclassics, and we discover the appearance of an intermediate regime ("twilight zone") where both perturbative and semiclassical features co-exist. Without loss of generality we consider as an example a billiard system whose classical dynamics is characterized by a correlation time $\tau_c$, which is simply the ballistic time. Associated with $\tau_c$ is the energy scale $\hbar / \tau_c$. Next we look on a similar billiard, but with a rough boundary. This roughness is characterized by a length scale which is $\ell$ times smaller, hence we can associate with it an energy scale $\delta E_{\text{nu}} = (\hbar / \tau_c) \times \ell$. The roughness does not affect the chaoticity: the correlation time $\tau_c$ as well as the whole power spectrum are barely affected. Consequently we explain that $\delta E_{\text{nu}}$ is not reflected in the RMT modeling of the Hamiltonian. Still in the LDOS analysis we find that non-universal (system specific) features appear. The appearance of such features is a generic phenomenon in quantum chaos studies. It introduces a new ingredient into the theory of parametric evolution which goes beyond RMT.

The model that we will use in our analysis is a particle confined to an Aharonov-Bohm (AB) cylindrical billiard (see Fig.1) where one can control the magnetic flux $\Phi$. The cylindrical billiard is constructed by wrapping a 2D billiard with hard wall boundaries. The lower boundary at $y = 0$ is flat, while the upper boundary $y = L_y + W \xi(x)$ is deformed. The deformation is described by $\xi(x) = \sum_{n=1}^{\ell} A_n \cos(nx)$ where $A_n$ are random numbers in the range $[-1,1]$. The illustration in Fig.1 assumes a smooth boundary ($\ell = 1$). The Hamiltonian of a particle in the cylindrical AB billiard is

$$\mathcal{H}(\phi) = \frac{1}{2m} \left[ (p_x - e \frac{\Phi}{L_x})^2 + p_y^2 \right]$$ (1)

![FIG. 1: Left: 2D billiard with $\ell = 1$. Right: Corresponding Aharonov-Bohm cylinder.](image)
supplemented by $L_x$ periodic boundary conditions in the horizontal direction, and hard wall boundary conditions along the lower and upper boundaries. $p_x$ and $p_y$ are the momenta. Later we shall use the notation $\phi = e\Phi/h$. We consider the chaotic $\mathcal{H}(\phi = 0)$ as the unperturbed Hamiltonian.

After conformal transformation \[7\] the billiard is mapped into a rectangular, with a mass tensor which is space dependent. Then it is possible to compute the matrix representation of the Hamiltonian in the plane wave basis $|\nu \mu\rangle$ of the rectangular. The result is:

$$\mathcal{H}_{\nu \mu, \nu' \mu'}(\phi) = \frac{\hbar^2}{2 \pi m} \left\{ \pi \left( \frac{\nu - \phi}{2\pi} \right)^2 \delta_{\nu \nu'} \delta_{\mu \mu'} + \frac{\mu^2}{8\alpha^2} J^{(0,2)}_{\nu' \nu'} + e^2 \frac{J^{(2,2)}_{\nu' \nu'}}{8} \left( \frac{\pi^2 \mu^2}{6} \right) \delta_{\mu \mu'} + (-1)^{\mu+\mu'} \nu \mu' \right\} \times$$

$$\left\{ e^2 \frac{2(\mu^2 + \mu'^2)}{(\mu^2 - \mu'^2)^2} J^{(2,2)}_{\nu' \nu'} - i\epsilon \frac{\nu + \nu' - \phi}{\pi \mu^2 - \mu'^2} J^{(1,1)}_{\nu' \nu'} \right\}$$

where

$$J^{(l,k)}_{\nu' \nu} = \int_0^{L_x} dx e^{i(h\nu' - \nu)x} \left( \frac{d\xi}{dx} \right)^l \frac{1}{(1 + \epsilon \xi(x))^k} \right|_{x = \nu L_x}. \tag{2}$$

The classical dimensionless parameters of the model are the aspect ratio $\alpha = L_y/L_x$, the tilt relative amplitude $\epsilon = W/L_y$, and the roughness parameter $\ell$. Upon quantization we have $\hbar = h \ell$ and $E$ determines the De-Broglie wavelength of the particle, and hence leads to an additional dimensionless parameter $n_E = [L_x L_y/(2\pi h^2)]mE$. For 2D billiards the mean level spacing $\Delta$ is constant, and hence $n_E = E/\Delta \propto 1/h^2$ can be interpreted as either the scaled energy or as the level index. Optionally we define a semiclassical parameter $\hbar_{\text{scale}} = 1/\sqrt{n_E}$.

In the numerical study we have $\epsilon = 0.06$ and $\alpha = 1$, for which the classical dynamics is completely chaotic (for any $\phi$). We consider either $\ell = 1$ for smooth boundary, or $\ell = 100$ for rough boundary. The eigenstates $|n(\phi)\rangle$ of the Hamiltonian $\mathcal{H}(\phi)$ were found numerically for various values of the flux (0.0006 < $\phi$ < 0.6). We were interested in the states within an energy window $\delta E \approx 45$ that contains $\delta n_E \sim 200$ levels around the energy $E \approx 400$. Note that the size of the energy window is classically small ($\delta E \ll E$), but quantum mechanically large ($\delta E \gg \Delta$).

The object of our interest are the overlaps of the eigenstates $|n(\phi)\rangle$ with a given eigenstate $|m(0)\rangle$ of the unperturbed Hamiltonian:

$$P(n|m) = |\langle n(\phi)|m(0)\rangle|^2 = \int \frac{dx dy dp_x dp_y}{(2\pi h)^2} \rho^{(n)}(x) \rho^{(m)}(x) \tag{3}$$

The overlaps $P(n|m)$ can be regarded as a distribution with respect to $n$. Up to some trivial scaling it is essentially the local density of states (LDOS). The associated dispersion is defined as $\delta E = [\sum P(n|m)|E_n - E_m|^2]^{1/2}$ In practice we plot $P(n|m)$ as a function of $r = n - m$ or as a function of $(E_n - E_m)$, and average over the reference state $m$. The second equality in (3) is useful for the semiclassical analysis. It involves the Wigner functions $\rho^{(n)}(x, y, p_x, p_y)$ which are associated with the eigenstates $|n(\phi)\rangle$. The semiclassical approximation is based on the microcanonical approximation $\rho^{(n)} \propto \delta(E_n - H(x, y, p_x, p_y))$. With this approximation the integral can be calculated analytically leading to

$$P(n|m) = \frac{\Delta}{\pi \sqrt{2(2\delta E_n)^2 - [(E_n - E_m) - \delta E_n^2/(2E_m)^2]}} \tag{4}$$

where $\delta E_n = (hv_E/L_x)\phi$ with $v_E = (2E/m)^{1/2}$. It is implicit in (4) that $P_0(n|m) = 0$ outside of the allowed range, which is where the energy under the square root is negative: For large $|E_n - E_m|$ there is no intersection of the corresponding energy surfaces, and hence no classical overlap.

A few words are in order regarding quantum to classical correspondence (QCC). Whenever $P(n|m) \approx P_0(n|m)$ we call it “detailed QCC”, while $\delta E \approx \delta E_n$ is referred to as “restricted QCC” [4]. It is remarkable that (the robust) restricted QCC holds even if (the fragile) detailed QCC fails completely. We have verified [13] that also in the present system $\delta E$ is numerically indistinguishable from $\delta E_n$.

A fixed assumption of this work is that $\phi$ is classically small. But quantum mechanically it can be either ‘small’ or ‘large’. Quantum mechanically small $\phi$ means that perturbation theory do provide a valid approximation for $P(n|m)$. What is the border between the perturbative regime and the non-perturbative regime, we discuss later. First we would like to show that the prediction which is based on perturbation theory, to be denoted as $P_m(n|m)$, is very different from the semiclassical approximation.

In order to write the expression for $P_m(n|m)$ we have first to clarify how to apply perturbation theory in the context of the present model. To this end, we write the perturbed Hamiltonian $\mathcal{H}(\phi)$ in the basis of $\mathcal{H}(\phi = 0)$. Since we assume that the perturbation is classically small, it follows that we can linearize the Hamiltonian with respect to $\phi$. Consequently the perturbed Hamiltonian is written as $\mathcal{H} = \mathcal{H} + \phi \mathcal{F}$, where $\mathcal{F} = \text{diag}(E_n)$ is a diagonal matrix, while $B = (-h/\epsilon)I_{nm}$. The current operator is conventionally defined as

$$\mathcal{I} = -\partial \mathcal{H}/\partial \Phi = (\epsilon/(mL_x))p_x$$

Its matrix elements can be found using a semiclassical recipe [14], namely $I_{nm}[\phi] \approx (\Delta/(2\pi \hbar))^2 C(E_n - E_m)/h$, where $C(\omega)$ is the Fourier transform of the current-current correlation function $C(\tau)$. Conventional condensed matter calculations are done for disordered rings where one assumes $C(\tau)$ to be exponential, with time constant $\tau_0$ which is essentially the ballistic time. Hence $C(\omega) \propto 1/(\omega^2 + (1/\tau_0)^2)$ is a Lorentzian. This Lorentzian approximation works well also for the chaotic ring that we consider. In fact we can do better by exploiting a relation between $\mathcal{I}(t)$ and the force $\mathcal{F}(t) = -\dot{p}_x$, leading to $\mathcal{C}(\omega) = (\epsilon/(mL_x)^2 \mathcal{C}(\omega)/\omega^2$. The force $\mathcal{F}(t)$ is a train of spikes corresponding to collisions with the boundaries. Assuming that the collisions are uncorrelated on short times we have $\mathcal{C}(\omega) \approx (8/3\pi)m^2 v_E^2/L_y$, where $v_E = (2E/m)^{1/2}$.
for $\omega \gg (1/\tau_c)$. This is known as the “white noise” approximation \([12]\). We have checked the validity of this approximation in the present context by a direct numerical evaluation of $\mathcal{C}(\omega)$, and also verified the validity of the above recipe by direct evaluation of the matrix elements of $B$ via Eq.\([2]\), see Fig. 2(a). The classical $\mathcal{C}(\omega)$ was numerically evaluated by Fourier analysis of the fluctuating current $\mathcal{I}(t)$ for a very long ergodic trajectory that covers densely the whole energy surface $\mathcal{H}(0) = E$. 

Perturbation theory to infinite order with the Hamiltonian $\mathcal{H} = E + \phi B$ leads to a Lorentzian-type approximation for the LDOS \([2]\) (see also Section 18 of \([6]\)). It is an approximation because all the higher orders are treated within a Markovian-like approach (by iterating the first order result) and convergence of the expansion is pre-assumed, leading to $P_m(n|m) = |B_{nm}|^2/\Gamma^2 + (E_n - E_m)^2|$. In practice the parameter $\Gamma(\phi)$ can be determined (for a given $\phi$) by imposing the requirement of having $P_m(r)$ normalized to unity. Substituting the expression for the matrix elements we get

$$P_m(n|m) = \frac{8\hbar^2(\hbar v_E)^3/(3\pi mL_y^2E_x)}{(E_n - E_m)^2 + (\hbar/\tau_0)^2} \cdot \frac{\phi^2}{\Gamma^2 + (E_n - E_m)^2}$$ \(5\)

By comparing the exact $P(r)$ to the approximation Eq.\(5\) we can determine the regime $\phi < \phi_m$ for which the approximation $P_m(r) \approx P_m(x)$ makes sense. The practical procedure to determine $\phi_m$ is to plot $\delta E_m(\phi)$ and to see where it departs from $\delta E_x$. The latter is a linear function of $\phi$ while the former becomes sublinear for large enough $\phi$, and even would exhibit saturation if we had a finite bandwidth). In case of Eq.\(5\) this reasoning leads to a crossover when $\delta E_x(\phi) / h/\tau_0$. Hence we get that the border of the perturbative regime (see footnote \([13]\)) is $\phi_m = L_x/(v_E \tau_0) \sim 1$.

What happens to $P(r)$ in practice? If we take the Wigner RMT model as an inspiration, we expect to have at $\phi \sim \phi_m$ a simple crossover from a $P_m$ line-shape to a $P_m$ line-shape. The latter is regarded as the semiclassical analogue of the (artificial) semicircle line shape. Indeed for the smooth billiard ($\ell = 1$) we have verified that this naive expectation is realized for $\delta E_x < \hbar \tau_0$.
random superposition of plane waves with case of billiards it implies that the wavefunction looks like a concentrated in an ergodic-like fashion in the vicinity of the vicinity of the energy shell.

Then the contribution to the overlap comes “collectively” from all the regions of the Wigner (quasi) distribution, not semiclassically and non-semiclassical overlaps. Specifically, if we have non-semiclassical wavefunctions, and then the collective contribution dominates, which give rise to the perturbative-like peak in the LDOS.

Our findings apply to systems, such as the rough billiard, where there is an additional (large) non-universal energy scale $\delta E_{nu}$. This is defined as an energy scale which is not related to the bandprofile, and hence does not emerge in the RMT modeling. Hence in general there is a distinct twilight regime $\hbar/\tau_0 < \delta E_{nu} < \delta E_{sw}$, which is neither “perturbative” nor “semiclassical”. [In our numerics $\ell=100$ is so large that $\delta E_{sw} \sim E$.]

**Summary:** We have analyzed the parametric evolution of the eigenstates of an Aharonov-Bohm cylindrical billiard, as the flux is changed. For the first time the full crossover from the perturbative to the non-perturbative regime is demonstrated. Random matrix theory suggests a simple crossover. Instead, we discover an intermediate twilight regime where perturbative and semiclassical features co-exist. This can be understood by adopting a phase space picture, and taking into account the inapplicability of the Berry conjecture regarding the semiclassical structure of the wavefunctions.

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[15] Optionally $\phi_{nu}$ is determined by $\Gamma(\phi) \sim \hbar/\tau_0$. It should be distinguished from the border of the first order perturbative regime which is determined by $\Gamma(\phi) \sim \Delta$, leading to $\phi_{opt} \sim \phi_{nu}/\sqrt{b}$ where $b = (\hbar/\tau_0)/\Delta \gg 1$. In other words $\phi_{opt}$ is the perturbation which is needed to mix neighboring levels.

![FIG. 4: The probability distribution $|\langle \nu, \mu | n \rangle|^2$ for (a) The $n = 2423$ eigenstate of the smooth ($\ell = 1$) billiard; (b) The $n = 1000$ eigenstate of the rough ($\ell = 100$) billiard. Note that this is essentially the $(p_x, p_y)$ momentum distribution. The state in panel (a), unlike the state in panel (b), is a typical semiclassical state. Namely it is well concentrated on the energy shell.](image-url)