SOME STRUCTURAL AND CLOSURE PROPERTIES OF AN
EXTENSION OF THE $q$-TENSOR PRODUCT OF GROUPS, $q \geq 0$

IVONILDES RIBEIRO MARTINS DIAS, NORAÍ ROMEU ROCCO*,
AND EUNICE CÂNDIDA PEREIRA RODRIGUES

Abstract. In this work we study some structural properties of the group $\eta^q(G, H)$, $q$ a non-negative integer, which is an extension of the $q$-tensor product $G \otimes^q H$, where $G$ and $H$ are normal subgroups of some group $L$. We establish by simple arguments some closure properties of $\eta^q(G, H)$ when $G$ and $H$ belong to certain Schur classes. This extends similar results concerning the case $q = 0$ found in the literature. Restricting our considerations to the case $G = H$, we compute the $q$-tensor square $D_n \otimes^q D_n$ for $q$ odd, where $D_n$ denotes the dihedral group of order $2n$. Upper bounds to the exponent of $\eta^q(G, G)$ are also established for nilpotent groups $G$ of class $\leq 3$, which extend to all $q \geq 0$ similar bounds found by Moravec in [21].

1. Introduction

Let $G$ and $H$ be groups each of which acts upon the other (on the right) and upon themselves by conjugation, in a compatible way, that is,

$$(1) \quad g^{(h_{g_1})} = \left( (g^{h_{g_1}})^h \right)^{g_{g_1}} \quad \text{and} \quad h^{(g_{h_1})} = \left( (h^{g_{h_1}})^g \right)^{h_{h_1}},$$

for all $g, g_1 \in G$ and $h, h_1 \in H$. In this situation, the non-abelian tensor product $G \otimes H$ of $G$ and $H$, as defined by Brown and Loday in [8], is the group generated by the symbols $g \otimes h$, where $g \in G$ and $h \in H$, subject to the defining relations

$$(2) \quad gg_1 \otimes h = (g^{g_1} \otimes h^{g_1})(g \otimes h) \quad \text{and} \quad g \otimes hh_1 = (g \otimes h_1)(g^{h_1} \otimes h^{h_1}),$$

where $g, g_1 \in G$ and $h, h_1 \in H$.

Brown and Loday [8] gave a topological significance for the non-abelian tensor product of groups. They used it to describe the third relative homotopy group of a triad as a non-abelian tensor product of the second relative homotopy groups of appropriate subspaces (see also [2]). When $G = H$ and all actions are by conjugation in $G$, then the group $G \otimes G$ is called the non-abelian tensor square of $G$. The commutator map induces a homomorphism $\kappa : G \otimes G \to G$, such that $g \otimes h \mapsto [g, h]$,
for all $g, h \in G$, whose kernel is usually denoted by $J_2(G)$. Its topological interest is given by the isomorphism (Cf. [8]):

$$\pi_3 SK(G, 1) \cong J_2(G),$$

where $\pi_3 SK(G, 1)$ is the third homotopy group of the suspension of the Eilenberg-MacLane space $K(G, 1)$.

Non-abelian tensor products of groups have been studied by a number of authors. In [27] the second author derived some properties of the non-abelian tensor square of a group $G$ via its embedding in a larger group, $\nu(G)$, defined as follows. Let $G^\varphi$ be an isomorphic copy of $G$ via an isomorphism $\varphi : G \to G^\varphi$, such that $g \mapsto g^\varphi$, for all $g \in G$. Then, $\nu(G)$ is defined to be the group

$$\nu(G) := \langle G \cup G^\varphi \mid [g, h^\varphi]^k = [g^k, (h^k)^\varphi] = [g, h^\varphi]^k \rangle,$$

for all $g, h, k \in G$.

Besides its intrinsic group theoretical interest, the motivation in introducing $\nu(G)$ is that its subgroup $[G, G^\varphi]$ is naturally isomorphic with the non-abelian tensor square, $G \otimes G$. Independently, Ellis e Leonard [15] introduced a similar construction.

Following [27] and [15], Nakaoka [23] extended the operator $\nu$ to an operator $\eta$, for the case of two groups $G$ and $H$ acting compatibly on one another. To this end it is considered an isomorphic copy $H^\varphi$ of $H$, where $\varphi : h \mapsto h^\varphi$, for all $h \in H$. For any $x, x_1 \in G$ and $y, y_1 \in H$, set

$$s_1(x, y, x_1) = [x, y^\varphi]^{x_1} : [x^{x_1}, (y^{x_1})^\varphi]^{-1},$$

$$s_2(x, y, y_1) = [x, y^\varphi]^{y_1} : [x^{y_1}, (y^{y_1})^\varphi]^{-1}.$$

Let $S_1 = \{s_1(x, y, x_1) \mid x, x_1 \in G, \ y \in H\}$, $S_2 = \{s_2(x, y, y_1) \mid x \in G, \ y, y_1 \in H\}$ and $S = S_1 \cup S_2$. Then the group $\eta(G, H)$ is defined by (Cf. [23]):

$$\eta(G, H) = \langle G, H^\varphi \mid S \rangle,$$

the factor group of the free product $G \ast H^\varphi$ by its normal subgroup generated by $S$. It follows from [18, Proposition 1.4] that the map $g \otimes h \mapsto [g, h^\varphi]$ gives rise to an isomorphism from $G \otimes H$ onto the subgroup $[G, H^\varphi] \ast \eta(G, H)$. When $G = H$ and all actions are by conjugation then $\eta(G, H)$ becomes the group $\nu(G)$.

Ellis and Rodríguez-Fernández [16], Brown [5], and Conduché and Rodríguez-Fernández [10] started the investigation of a modular version of the non-abelian tensor product. In [13] Ellis considered the so called $q$-tensor product $G \otimes^q H$, of $G$ and $H$, where $q$ is a non-negative integer, in the case where $G$ and $H$ are normal subgroups of a larger group $L$ and all actions are by conjugation in $L$. In this situation the $q$-tensor product, $G \otimes^q H$, is defined to be the group generated by the symbols $g \otimes h$ and $\tilde{k}$, for $g \in G, \ h \in H$ and $k \in G \cap H$, subject to the following relations (for all $g, g_1 \in G, \ h, h_1 \in H$ and $k, k_1 \in G \cap H$), where for elements $x, y \in L$ the conjugate
of $x$ by $y$ is written $x^y = y^{-1}xy$ and the commutator of $x$ and $y$ is $[x, y] = x^{-1}yx$:

$$g \otimes h_1 = (g \otimes h_1)(g^{h_1} \otimes h^{h_1});$$

(6)

$$gg_1 \otimes h = (g^{g_1} \otimes h^{g_1})(g_1 \otimes h);$$

(7)

$$(g \otimes h)^k = g^{(k^q)} \otimes h^{(k^q)};$$

(8)

$$\kappa k_1 = k \prod_{i=1}^{q-1} (k \otimes (k_1^{-i})^{k^{q-1-i}})k_1;$$

(9)

$$[\kappa, k_1] = k^q \otimes k_1^q;$$

(10)

$$[g, h] = (g \otimes h)^q.$$  

(11)

For $q = 0$ the 0-tensor product $G \otimes^0 H$ is just the non-abelian tensor product $G \otimes H$ (cf. [8]), that is, the group generated by the symbols $g \otimes h$, for $g \in G$, $h \in H$, subject to the relations (6) and (7) only. When $G = H$ then we get the $q$-tensor square, $G \otimes^q G$.

In [5] Brown showed that if $G$ is a $q$-perfect group, that is, $G$ is generated by commutators and $q$-th powers, then the (unique) universal $q$-central extension of $G$ is isomorphic with

$$1 \to H_2(G, \mathbb{Z}_q) \to G \otimes^q G \to G \to 1.$$  

In this paper we address some structural properties of the group $\eta^q(G, H)$, an extension of $G \otimes^q H$ by $G \times H$ which also generalizes in a certain sense the group $\eta(G, H)$ in the particular situation where $G$ and $H$ are normal subgroups of a larger group $L$. The group $\eta^q(G, H)$ first appeared in [14] using a slightly different approach.

Notation is fairly standard (see for instance [25]); as usual we write $x^y$ to mean the conjugate $y^{-1}xy$ of $x$ by $y$; the commutator of $x$ and $y$ is then $[x, y] = x^{-1}y^{-1}xy$. Our commutators are left normed, that is, $[x, y, z] = [[x, y], z]$.

The paper is organized as follows. In Section 2 we briefly describe the group $\eta^q(G, H)$ and treat of some basic structural results. In Section 3 we prove Theorem 3.2 where we address some closure properties of $\eta^q(G, H)$, extending similar results concerning $\nu^q(G)$ and the $q$-tensor square found in [9] and elsewhere. In Section 4 we concentrate on polycyclic groups and present some computations. Finally, in Section 5 we prove Theorem 5.2 were we establish upper bounds to the exponent of $G \otimes^q G$ for nilpotent groups of at most class 3, extending to all $q \geq 0$ similar bounds found by Moravec [21] in the case $q = 0$.

**Acknowledgements:** The authors are very grateful to Raimundo Bastos for the interesting discussions and suggestions on the best approach to these results.

### 2. Basic Structural Results

We begin this section by giving a brief description of the group $\eta^q(G, H)$. To this end we assume that $G$ and $H$ are normally embedded in a larger group $L$ and that all
actions are by conjugation in \( L \). For \( q \geq 1 \) let \( K = G \cap H \) and let \( \hat{K} = \{ \hat{k} | k \in K \} \) be a set of symbols, one for each element of \( K \) (for \( q = 0 \) we set \( \hat{K} = \emptyset \), the empty set). Let \( F(\hat{K}) \) be the free group on \( \hat{K} \) and \( \eta(G, H) \ast F(\hat{K}) \) be the free product of \( \eta(G, H) \) and \( F(\hat{K}) \). Since \( G \) and \( H^x \) are embedded into \( \eta(G, H) \), we shall identify the elements of \( G \) (respectively of \( H^x \)) with their respective images in \( \eta(G, H) \ast F(\hat{K}) \). Let \( J \) be the normal closure in \( \eta(G, H) \ast F(\hat{K}) \) of the following elements, for all \( k, k_1 \in K, g \in G \) and \( h \in H \):

\[
\begin{align*}
(12) & \quad g^{-1} \hat{k} \hat{g}(\hat{k}^g)^{-1}; \\
(13) & \quad (h^x)^{-1} \hat{k} \hat{h}^x(\hat{k}^h)^{-1}; \\
(14) & \quad (\hat{k})^{-1} [g, h^x] \hat{k} \left[ g^{k^x}, (h^{k^x})^x \right]^{-1}; \\
(15) & \quad (\hat{k})^{-1} k \hat{k}_1(\hat{k}_1)^{-1} \left( \prod_{i=1}^{q-1} [k_i, (k_i^{-1})^x]^{k_i^{-1-x}} \right)^{-1}; \\
(16) & \quad [\hat{k}, \hat{k}_1] [k^q, (k_1^{-1})^x]^{-1}; \\
(17) & \quad [\hat{g}, \hat{h}] [g, h^x]^{-q}.
\end{align*}
\]

According to [9] (see also [13]), the group \( \eta^q(G, H) \) is then defined to be the factor group

\[
\eta^q(G, H) := (\eta(G, H) \ast F(\hat{K})) / J.
\]

For \( q = 0 \) the set of relations from (12) to (17) is empty; in this case we have \( (\eta(G, H) \ast F(\hat{K})) / J \cong \eta(G, H) \). Also, for \( G = H = L \) we get that \( \eta^q(G, G) \cong \nu^q(G) \), which becomes the group \( \nu(G) \) if \( q = 0 \).

There is an epimorphism \( \rho : \eta^q(G, H) \rightarrow GH, \ g \mapsto g, h^x \mapsto h, \hat{k} \mapsto k^q \). On the other hand, the immersion of \( G \) into \( \eta(G, H) \) induces a homomorphism \( i : G \rightarrow \eta^q(G, H) \). We have that \( g^{\rho} = g \) and thus \( i \) is injective. Similarly, the immersion of \( H^x \) into \( \eta(G, H) \) induces a homomorphism \( j : H^x \rightarrow \eta^q(G, H) \). Hence, the elements \( g \in G \) and \( h^x \in H^x \) are identified with their respective images \( g^i \) and \( (h^x)^j \) in \( \eta^q(G, H) \).

We write \( K \) to denote the subgroup of \( \eta^q(G, H) \) generated by the images of \( \hat{K} \). By relations (12) and (13), \( K \) is normal in \( \eta^q(G, H) \) and, by relations (13), (14) and (15), the subgroup \( T := [G, H^x] \) is normal in \( \eta^q(G, H) \). Consequently, \( \Upsilon^q(G, H) := [G, H^x]K \) is a normal subgroup of \( \eta^q(G, H) \).

By the above considerations we obtain

\[
\eta^q(G, H) = (\Upsilon^q(G, H) \cdot G) \cdot H^x,
\]

where the dots indicate (internal) semi-direct products.
Besides its intrinsic interest as a group theoretical construction, one of the main motivations to introduce and study the group \( \eta^q(G, H) \) is the canonical “hat” (power) and commutator approach to the \( q \)-tensor product \( G \otimes^q H \).

In effect, an adaptation of the proof of [13] Proposition 2.9] can be easily carried out (see also [13, Theorem 8]) to give us the following:

**Proposition 2.1.** There is an isomorphism \( T^q(G, H) \cong G \otimes^q H \) such that \( [g, h^e] \mapsto g \otimes h \) and \( \hat{k} \mapsto \hat{k} \), for all \( g \in G, h \in H \) and \( k \in K \).

This approach provides us not only with more psychological comfort but it also brings computational advantages by treating tensors as commutators in a larger group (see for instance, [15, 26, 9, 17, 28, 11]).

The \( q \)-exterior product \( G \wedge^q H \) is defined to be the quotient of \( G \otimes H \) by its (central) subgroup \( \nabla^q(G, H) := \langle k \otimes k \mid k \in K \rangle \). According to our approach we write

\[
\Delta^q(G, H) := \langle [k, k^e] \mid k \in K \rangle \quad \text{and} \quad \tau^q(G, H) := \frac{\eta^q(G, H)}{\Delta^q(G, H)},
\]

so that

\[
G \wedge^q H \cong \frac{T^q(G, H)}{\Delta^q(G, H)}.
\]

In the following Lemma we collect some basic consequences of the defining relations of \( \eta^q(G, H) \); their proofs can be easily adapted from [9, Lemma 2.4] and are omitted.

**Lemma 2.2.** [9, Lemma 2.4] Let \( G \) and \( H \) be normal subgroups of a group \( L \) and \( q \geq 0 \). Then the following relations hold in \( \eta^q(G, H) \), for all \( g, g_1 \in G, h, h_1 \in H \) and \( k, k_1 \in K \).

1. \( [g, h^e][g_1, h_1^e] = [g, h^e][g_1, h_1] \);
2. \( [g, h^e, h_1^e] = [g, h, h_1^e] ; [g_1, g, h^e] = [g_1, g, h^e]. \) In particular, \( [g, h^e, k^e] = [g, h^e, k] = [(g, h)^e, k] \);
3. If \( k \in K' \) (or if \( k_1 \in K' \)) then \( [k, k_1^e][k_1, k^e] = 1 \);
4. \( \hat{k}, [g, h] = [\hat{k}, [g, h^e]] = [\hat{k}, [g, h^e]] = [k^q, [g, h^e]] = [(k^q)^e, [g, h^e]] \);
5. \( \hat{k}, [k^e] = [k^q, h^e], [g, \hat{k}] = [g, (k^q)^e] \);
6. If \( [k, k_1] = 1 \) then \( [k, k_1^e] \) and \( [k_1, k^e] \) are central elements in \( \eta^q(G, H) \) and they have the same finite order dividing \( q \). If in addition \( k \) and \( k_1 \) are torsion elements of orders \( o(k) \) and \( o(k_1) \), respectively, then the order of \( [k, k_1^e] \) divides \( \gcd(o(k), o(k_1)) \);
7. \( [k, k^e] \) is central in \( \eta^q(G, H) \), for all \( k \in K \);
8. \( [k, k_1^e][k_1, k^e] \) is central in \( \eta^q(G, H) \);
9. \( [k, k^e] = 1 \), for all \( k \in K' \);
10. If \( [k, g] = 1 = [k, h] \) then \( [g, h, k^e] = 1 = [(g, h)^e, k] \).

The next corollary extends [9, Corollary 2.5] to \( \eta^q(G, H) \).
Corollary 2.3. [27 Corollary 2.5] If $[G, H] = 1$ then $\Upsilon^q(G, H)$ is a central subgroup of $\eta^q(G, H)$. Furthermore, in this case we have

$$\Upsilon^q(G, H) \cong \frac{G}{G^G} \otimes \mathbb{Z} \frac{H}{H^H}.$$  \hspace{1cm} (22)

Proof. It follows directly from parts (ii) and (iv) of Lemma 2.2 that $[G, H^q]$ is central in $\eta^q(G, H)$. Since under our assumptions $K = G \cap H$ is abelian, by relations (12), (13) and (17), we have $[\widehat{k}, \widehat{k_1}] = [k^q, (k_1^q)^q] = [k^q, (k_1)^q]^q = ([k, k_1])^q = 1$ and $(\widehat{k})^q = \widehat{k} = (\widehat{k})^{h^q}$, for all $g \in G, h \in H, k, k_1 \in K$. Hence, the subgroup $K$ is central in $\eta^q(G, H)$, too. Consequently, $\Upsilon^q(G, H)$ is central in $\eta^q(G, H)$. The isomorphism (22) is proved in [10, Theorem 1.24].

We recall the epimorphism $\rho : \eta^q(G, H) \to GH$, where $g \mapsto g, h^q \mapsto h$ and $\widehat{k} \mapsto k^q$. We write $\theta^q(G, H)$ to denote the kernel, ker $\rho$. The following result is essentially an adaptation of [8, Proposition 2.3] to our context.

Proposition 2.4. Let $G$ and $H$ be normal subgroups of $L$ and $q$ a non-negative integer. Then,

(a) The epimorphism $\rho$ induces a homomorphism $\rho' : \Upsilon^q(G, H) \to G \cap H$ such that $\rho'(g, h^q) = [g, h^q]$ for all $g \in G, h \in H$ and $k \in G \cap H$;

(b) $[t, h] = [((t)^q)^q, k^q]$ for all $t \in [G, H^q], g \in G, h \in H$ and $\widehat{k} \in K$;

(c) $\mu^q(G, H) := \ker \rho'$ is a central subgroup of $\eta^q(G, H)$.

Proof. Item (a) is an immediate consequence of the definitions of $\rho$ and $\Upsilon^q(G, H)$. Item (b) follows by using an induction argument based on Lemma 2.2 (ii), (iv) and commutator calculus. To prove Item (c), we first notice that every element $w \in \Upsilon^q(G, H) = TK$ can be written as a product $w = t\widehat{k}$ where $t \in T = [G, H^q]$ and $\widehat{k} \in K$; this follows by an induction argument using defining relations (14) and (15). Now we have

$$[g, w] = [g, t\widehat{k}]$$

$$= [g, \widehat{k}][g, t]$$

$$= [g, (k^q)^q][g, ((t)^q)^q]\widehat{k}$$

by Lemma 2.2 (iv), and Item (b)

$$= [g, (k^q)^q][g, ((t)^q)^q]k^q$$

by Relations (14)

$$= [g, (k^q)^q][g, ((t)^q)^q]k^q$$

by Relations (14)

$$= [g, ((t\widehat{k})^q)^q].$$

With similar arguments we get that $[w, h^q] = [(w)^q, h^q]$ and $[w, \widehat{k}] = [(w)^q, (k^q)^q]$. Consequently, if $w \in \mu^q(G, H) = \ker \rho'$, then $[g, w] = [w, h^q] = [w, \widehat{k}] = 1$, for all $g \in G, h^q \in H^q$, and $k \in K$. Therefore, ker $\rho'$ is central in $\eta^q(G, H)$. \hfill \Box
Remark 2.5. Notice that \( \eta^q(G, H)/\Upsilon^q(G, H) \cong G \times H \), while \( \eta^q(G, H)/\theta^q(G, H) \cong GH \). This implies that there is an isomorphism from \( \eta^q(G, H)/\mu^q(G, H) \) to a subgroup of \( G \times H \times GH \). In addition, \( \eta^q(G, H)/[G, H^p] \cong (K/[G, H]) \times G \times H \), since \( [G, H^p]/K/[G, H^p] \) is generated by the elements \( \hat{k} \), for \( k \in K \), with the relations \( \hat{k} \hat{k}_1 = \hat{k} \hat{k}_1 \) and \( [\hat{k}, \hat{k}_1] = 1 \), according to defining relations (15) and (16). Set \( \mu_0^q(G, H) := [G, H^p] \cap \mu^q(G, H) \). We thus obtain the following exact sequence

\[(24) \quad 1 \to \mu_0^q(G, H) \to \eta^q(G, H) \to (K/[G, H]) \times G \times H \times GH.\]

Now, let \( p \) and \( q \) be non-negative integers with \( p \geq 1 \). Let \( \delta : \eta^p(G, H) \to \eta^q(G, H) \) be defined on the generators of \( \eta^p(G, H) \) by \( (g)\delta := g \), \( (h^p)\delta := h^q \) and \( (\hat{k})\delta := \hat{k}^q \), for all \( g \in G \), \( h^p \in H^p \) and \( k \in K \). It is a routine to check that in this way \( \delta \) preserves the defining relations of \( \eta^p(G, H) \); relations (12), (13), (14), (16) and (17) are easily carried out. However, relation (15) demand tedious calculations and the impatient reader can consult [10, Theorem 1.22]. Thus we obtain a homomorphism from \( \eta^p(G, H) \) to \( \eta^q(G, H) \). Set \( \delta' = \delta|\Upsilon^p(G, H) : \Upsilon^p(G, H) \to \Upsilon^q(G, H) \). The next Proposition generalises [9, Proposition 2.6] and [10, Theorem 1.22].

Proposition 2.6. Let \( p \geq 1 \). There are exact sequences

\[(25) \quad \eta^p(G, H) \xrightarrow{\delta} \eta^q(G, H) \to \frac{K}{[G, H]K^q} \to 1;\]

\[(26) \quad \Upsilon^p(G, H) \xrightarrow{\delta} \Upsilon^q(G, H) \to \frac{K}{[G, H]K^q} \to 1;\]

In particular, if \( q = 0 \) then we have:

\[(27) \quad \eta(G, H) \xrightarrow{\delta} \eta^q(G, H) \to \frac{K}{[G, H]} \to 1;\]

\[(28) \quad \Upsilon(G, H) \xrightarrow{\delta} \Upsilon^q(G, H) \to \frac{K}{[G, H]} \to 1.\]

Proof. According to the definition of \( \delta \) we have

\[\text{Im}(\delta) = \langle g, h^q, \hat{k}^q \mid g \in G, \ h \in H \text{ and } k \in K \rangle.\]

Thus, it follows from Lemma 2.22 that \( \text{Im}(\delta) \) is a normal subgroup of \( \eta^p(G, H) \). Now, as already observed in Remark 2.5, \( \eta^p(G, H)/\text{Im}(\delta) \) is generated by the cosets of the elements \( \hat{k} \in \hat{K} \) with the relations \( \hat{k}\hat{k}_1 = \hat{k}\hat{k}_1 \). Furthermore, as \( \hat{k}^q \in \text{Im}(\delta) \) for all \( k \in K \), it follows that \( (\hat{k})^q \equiv 1 \pmod{\text{Im}(\delta)} \). This proves (25). The sequence (26) is essentially [13, Theorem 6, (ii)] and follows by a similar argument as above, since \( \text{Im}(\delta') \) is also normal in \( \Upsilon^p(G, H) \). The sequences (27) and (28) follow at once, respectively from (25) and (26) with \( q = 0 \). This completes the proof. \( \square \)
The next result shows that the derived group $\eta^q(G, H)'$ has the same formal structure for all $q \geq 0$. In order to avoid any confusion, we write $\mathcal{T}(G, H)$ for the subgroup $[G, H^\varphi] \leq \eta(G, H)$ (case $q = 0$), which is isomorphic with the non-abelian tensor product $G \otimes H$ for all compatible actions of one group upon another. In many places in this paper we write $\mathcal{T}$ for the subgroup $\big[[G, H \varphi] \leq \eta^q(G, H)\big]$ (case $q = 0$), which is isomorphic with the non-abelian tensor product $G \otimes H$ for all compatible actions of one group upon another. In many places in this paper we write $\mathcal{T}$ for the subgroup $\big[[G, H \varphi] \leq \eta^q(G, H)\big]$ (case $q = 0$), which is isomorphic with the non-abelian tensor product $G \otimes H$ for all compatible actions of one group upon another.

Proposition 2.7. Let $G$ and $H$ be normal subgroups of a group $L$. Then, for all $q \geq 0$, $\eta^q(G, H)' = [G, H^\varphi] \cdot G' \cdot (H')^\varphi$. In particular, if $q = 0$ then the non-abelian tensor product $G \otimes H \cong \mathcal{T} K$. If $q = 0$ then $\mathcal{T}(G, H) = \mathcal{T}$.

Proof. With the above discussion, we can write $\eta^q(G, H) = T K G H$, according to (19). Now, $T$ and $TK$ are normal subgroups of $\eta^q(G, H)$, while $[G, H^\varphi] = T$. By defining relations (16) we find that $K' \leq T$. Additionally, from Lemma 2.2 (v) we obtain that $[K, G]$ and $[K, H^\varphi]$ are both contained in $T$. Therefore, $\eta^q(G, H)' = [TKG^\varphi, TKG^\varphi] = TG' (H')^\varphi = [G, H^\varphi]G' (H')^\varphi$. □

3. Some Closure Properties for $\eta^q(G, H)$

A number of authors have studied some closure properties such as finiteness, solubility, polycyclicity and nilpotency, among others, of the non-abelian tensor product of groups and of related constructions (cf. [12, 13, 20, 22, 23, 30]). In the context of $\nu(G)$, $\eta(G, H)$ and $\nu^q(G)$, such closure properties were studied for instance in [3, 4, 26, 27, 9, 11, 1]. In this section we extend these considerations to the scope of $\eta^q(G, H)$, $q \geq 0$. We will consider the following question: Let $\mathfrak{X}$ be a class of groups. If $G, H$ are normal subgroups of a certain group $L$ such that $G$ and $H$ belong to $\mathfrak{X}$, then does $\eta^q(G, H)$ belong to $\mathfrak{X}$?

Recall that a class $\mathfrak{X}$ of groups is called a Schur class if for any group $G$ such that the factor group $G/Z(G)$ belongs to $\mathfrak{X}$, also the derived subgroup $G'$ is a $\mathfrak{X}$-group. Thus, the famous Schur’s theorem just states that finite groups form a Schur class. Other interesting classes of groups (e.g., finite $\pi$-groups, locally (finite $\pi$-groups), polycyclic groups, polycyclic-by-finite groups) are Schur classes. The classical reference to this matter is [24].

In [8] Brown and Loday proved that if $G$ is a finite $\pi$-group, then the non-abelian tensor square $G \otimes G$ is a finite $\pi$-group; in particular, $\nu(G)$ is a finite $\pi$-group. Ellis [13] proved the finiteness of $G \otimes H$ when $G$ and $H$ are finite groups. Moravec [21] showed that if $G$ is a locally (finite $\pi$-groups), then the so is $\nu(G)$. In [20], Lima and Oliveira proved that if $G$ is polycyclic-by-finite, then so is $\nu(G)$. Here we extend these results to the scope of $\eta^q(G, H)$, $q \geq 0$, and give elementary proofs of them by using only the structural properties discussed in Section 2 and the definition of a Schur class, based on Proposition 2.4 and Remark 2.5.
We write $\mathfrak{F}_\pi$ to indicate the class of finite $\pi$-groups and $L\mathfrak{F}_\pi$ for the class of locally (finite $\pi$-groups), where $\pi$ is a set of primes. To ease reference we state Lemma 3.1, which extends Schur’s theorem to the class $L\mathfrak{F}_\pi$.

**Lemma 3.1.** Let $G$ be any group. If $G/Z(G) \in L\mathfrak{F}_\pi$, then $G' \in L\mathfrak{F}_\pi$.

In the next theorem we establish some closure properties on $\eta^q(G, H)$.

**Theorem 3.2.** Let $G$ and $H$ be normal subgroups of a group $L$ and let $q$ be a non-negative integer. Then,

(i) If $G, H \in \mathfrak{F}_\pi$, then $\eta^q(G, H) \in \mathfrak{F}_\pi$;

(ii) If $G, H \in L\mathfrak{F}_\pi$, then $\eta^q(G, H) \in L\mathfrak{F}_\pi$; if furthermore $G$ and $H$ have finite exponents, then the exponent of $\eta^q(G, H)$ can be bound in terms of $q$, $\exp G$, and $\exp H$;

(iii) If $G$ and $H$ are soluble groups of derived lengths $l_1$ and $l_2$, respectively, then $\eta^q(G, H)$ is also soluble, of at most derived length $l_1 + l_2 + 1$;

(iv) If $G$ and $H$ are nilpotent groups of nilpotency classes $c_1$ and $c_2$, respectively, then $\eta^q(G, H)$ is nilpotent of at most class $c_1 + c_2 + 1$;

(v) If $G$ and $H$ is polycyclic-by-finite, then so is $\eta^q(G, H)$.

**Proof.** (i). If $G, H \in \mathfrak{F}_\pi$ then clearly $(\mathcal{K}/[G, H]) \times G \times H \times GH \in \mathfrak{F}_\pi$. Thus, $\eta^q(G, H)/\mu^0_\pi(G, H) \in \mathfrak{F}_\pi$, by Remark 2.3, sequence (2.1). By Schur’s theorem (25, 10.1.4), $\eta^q(G, H)'$ is finite and $\exp(\eta^q(G, H)')$ divides $[G]^2 [H]^2$, that is, $\eta^q(G, H)' \in \mathfrak{F}_\pi$. Since, by Proposition 2.4, $\mu^0_\pi(G, H) \leq [G, H^2] \leq \eta^q(G, H)'$, we get that $\mu^0_\pi(G, H) \in \mathfrak{F}_\pi$ and so, $\eta^q(G, H) \in \mathfrak{F}_\pi$.

(ii). Using a similar argument as in Part (i) and Lemma 3.1 we find that both $\eta^q(G, H)/\mu^0_\pi(G, H)$ and $\mu^0_\pi(G, H)$ are locally (finite $\pi$-groups). Therefore, $\eta^q(G, H) \in L\mathfrak{F}_\pi$, by Reidemeister-Schreier’s theorem (25, 6.1.8)]. If in addition $G$ and $H$ have finite exponents, then by [21, Corollary 5] $\exp \eta(G, H)$ can be bound in terms of $\exp G$ and $\exp H$. Thus, by sequence (27) in Proposition 2.6 we see that the same is true for $\eta^q(G, H)$, with the additional restriction that, due to Lemma 2.2 (vi), such upper bound may also involve $q$ in $\text{Im} \delta$.

Part (iii) follows directly from Proposition 2.4 by the fact that $G \times H \times GH$ is soluble of at most derived length $l_1 + l_2$, while $\mu^q(G, H)$ is abelian.

(iv). Analogously, by Fitting’s theorem (25, 5.2.8) $G \times H \times GH$ is nilpotent of class at most $c_1 + c_2$, while $\mu^q(g, H)$ is central in $\eta^q(G, H)$.

(v). Again, we have that $\eta^q(G, H)/\mu^0_\pi(G, H)$ is polycyclic-by-finite and $\mu^0_\pi(G, H) \leq Z(\eta^q(G, H))$. Therefore, $\eta^q(G, H)'$ is polycyclic-by-finite. It suffices to show that $\mu^0_\pi(G, H)$ is polycyclic. Let $M \leq \eta^q(G, H)'$ be a polycyclic normal subgroup of finite index. Then $M \cap \mu^0_\pi(G, H)$ is a polycyclic normal subgroup of finite index of the abelian group $\mu^0_\pi(G, H)$. Therefore $\mu^0_\pi(G, H)$ is polycyclic. The proof is complete. □
Remark 3.3. Notice that if $G = H = L$ is a soluble group of derived length $l$ (respectively, nilpotent of class $c$), then the bounds in parts (iii) and (iv) of Theorem 3.2 become $l + 1$ (respectively, $c + 1$), according to [9, Theorem 2.8].

4. A polycyclic presentation for the $q$-tensor square of the dihedral group $D_n$, $q$ odd

In this Section we restrict our attention to the group $\nu^q(G)$ (that is, the group $\eta^q(G,G)$ when $G = H = L$), particularly on the computation of the $q$-tensor square of the dihedral groups $D_n$ when $q$ is odd. We begin with a brief description of an algorithm derived in [11] for computing polycyclic presentations for $\nu^q(G)$, when $G$ is a polycyclic group given by a consistent polycyclic presentation.

Let $G$ be a polycyclic group defined by a consistent polycyclic presentation $F/R$, where $F$ is a free group generated by $g_1, ..., g_n$. The relations of a consistent polycyclic presentation $F/R$ are (cf. [29, Section 9.4]):

For some $I \subseteq \{1, ..., n\}$, some exponents $e_i \in \mathbb{N}$ with $i \in I$, $\alpha_{i,j}, \beta_{i,j,k}, \gamma_{i,j,k} \in \mathbb{Z}$ and for all $i, j$ and $k \in \{1, ..., n\}$. Recall that this presentation is refined if all $i \in I$ are prime numbers.

Following [11], we write the relations of $G$ as relators of the form $r_1, ..., r_l$, where every relator $r_j$ is a word in the generators $g_1, ..., g_n$, $r_j = r_j(g_1, ..., g_n)$. Let

$$E_q(G) := \frac{F}{R^q[F,R]}$$

which is a $q$-central extension of $G$.

A presentation for the group $E_q(G)$ can be obtained according to the following construction (see [11]): for each relator $r_i$, we introduce a generator $t_i$; $E_q(G)$ is then the group generated by $g_1, ..., g_n, t_1, ..., t_l$ subject to the relators:

1. $r_i(g_1, ..., g_n)t_i^{-1}$ for $1 \leq i \leq l$,
2. $[t_i, g_j]$ for $1 \leq j \leq n$, $1 \leq i \leq l$,
3. $[t_i, t_j]$ for $1 \leq j < i \leq l$,
4. $t_i^q$ for $1 \leq i \leq l$.

This is a polycyclic presentation of $E_q(G)$, possibly inconsistent (see [11]). The consistency relations can be evaluated in the consistent polycyclic presentation of $E_q(G)$ using a collection system from left to right, given by

1. $r_i(g_1, ..., g_n)t_i^{q_1} ... t_i^{q_l}$ for $1 \leq i \leq l$,
2. $[t_i, g_j]$ for $1 \leq j \leq n$, $1 \leq i \leq l$,
3. $[t_i, t_j]$ for $1 \leq j < i \leq l$,
Proposition 4.1. [17] Proposition 4.2] $G \wedge \epsilon = (E_q(G))' (E_q(G))^\epsilon$. Furthermore, $(E_q(G))' (E_q(G))^\epsilon$ is the subgroup

$$\langle [g_i, g_j]^\epsilon, g_k^\epsilon \mid 1 \leq i < j \leq n, 1 \leq k \leq n \rangle,$$

where $\epsilon = 1$ if $G$ is finite and $\epsilon = \pm 1$ otherwise.

Hence, to obtain a presentation for the $q$-exterior square of a polycyclic group $G$ defined by a consistent polycyclic presentation, we apply the standard methods to determine presentations for subgroups of polycyclic groups (see [17]).

Now, let us consider the dihedral group $G = D_n$ given by the consistent polycyclic presentation

$$D_n = \langle g_1, g_2 \mid g_1^2 = 1, g_1^{-1} g_2 g_1 = g_2^{n-1}, g_2^n = 1 \rangle.$$

We will compute the $q$-exterior square of $D_n$, $q$-odd. To this end we begin with the following proposition. Recall that a group $G$ is called $q$-perfect if $G = G'G^q$.

Proposition 4.2. For $q$ odd, $D_n$ is $q$-perfect.

Proof. Let $G := D_n$. Then we have

$$G'G^q = \langle g_1^q, g_2^q, [g_1, g_2] \rangle = \langle g_1, g_2, g_1^2 \rangle.$$ 

Since $q$ is odd, $qx + 2y = 1$ for some $x, y \in \mathbb{Z}$. Therefore, $g_2 = (g_1^q)^x (g_2^q)^y \in K'K^q$. \square

Notice that if $G$ is $q$-perfect then by Lemma 2.2, $\Delta^q(G) = 1$ and hence $G \otimes \epsilon G \cong G \wedge \epsilon G$. In this case $G \otimes \epsilon G$ can be computed using Proposition 4.1. Now,

$$E_q(D_n) = \langle g_1, g_2, t_1, t_2, t_3 \mid g_1^2 = t_1, g_1^{-1} g_2 g_1 = g_2^{n-1} t_2, g_2^n = t_2, t_3 \rangle.$$ 

Testing the consistency of this presentation we obtain the unique relation:

$$t_2 t_3^{n-2} = 1.$$ 

Computing a consistent polycyclic presentation for $W = E_q(G)' E_q(G)^\epsilon \cong D_n \wedge \epsilon D_n$ we obtain $W = \langle [g_1, g_2], g_1^q, g_2^q \rangle$. Since $[g_1, g_2] = g_2^{-1} t_2^{-1} = (g_2^{n-2} t_2)^{-1} = [g_2, g_1]^{-1}$, we find $W = \langle g_2^{n-2} t_2, g_1^q, g_2^q \rangle$.

Routine computations give us the following result, where we write $(a, b)$ for $\text{lcm}(a, b)$, $[a, b]$ for $\text{lcm}(a, b)$ and $o(g)$ for the order of the element $g$:

Proposition 4.3. Let $q$ be an odd integer. Then, in $E_q(D_n)$ we have:

$$o(g_1) = 2q, \ o(g_2) = [n, q], \ o(t_1) = q, \ o(t_2) = \frac{q}{(n-2, q)} \text{ and } o(t_3) = \frac{q}{(n, q)}.$$ 

Proposition 4.4. For $q$ odd, $D_n \otimes \epsilon D_n \cong D_n$ and $H_2(D_n, \mathbb{Z}_q) = \{1\}$. 

(4) $e_i^d_i$ for $1 \leq i \leq l$ with $d_i \mid q$. 


Proof. As above, we have that \( D_n \otimes^q D_n \cong W = \langle g_2^{n-2}t_2, g_1^{a}, g_2^{b} \rangle \). We need to show that \( W \cong D_n \). Since \( q \) is odd, \( qx + 2y = 1 \) for some \( x, y \in \mathbb{Z} \). Set \( h := (g_2^2)^x(g_2^{n-2}t_2)^{-y} = g_2(t_2t_3)^{-y} \in W \). Thus, \( g_1^q h g_1^q = h^{-1} \) and \( h^n = 1 \). Moreover, since \( q, y \) is odd, it follows that \( g_2^y \neq 1 \) and \( g_2^y \neq t^z \) for all \( l < n \). Therefore, \( o(h) \leq n \) and thus, \( o(h) = n \). On the other hand, setting \( H := \langle h \rangle \) we find that \( g_2^{n-2}t_2^{-y} = (g_2^2)^{-x} h \in H \) and \( g_2^{n-2}t_2 = \{(g_2^{n-2}t_2)^{-y}\}^{-b}(g_2^{n-2}t_2)^{-2a} \in H \), where \( a, b \) are integers such that \( qa + yb = 1 \). Therefore,

\[
W = \langle g_1^{a}, h \mid (g_1^{a})^2 = 1, g_1^{-q}h g_1^{a} = h^{-1}, h^n = 1 \rangle \cong D_n.
\]

\[\square\]

5. Exponents of the \( q \)-tensor square of nilpotent groups of class \( \leq 3 \)

Moravec [22] gives an estimate for \( \text{exp}(G \otimes G) \) in terms of \( \text{exp}(G) \) and \( \text{exp}(G \wedge G) \) and he observed that for finite metabelian groups, \( \text{exp}(G \wedge G) \) divides \( (\text{exp}(G))^2 \text{exp}(G') \); consequently, \( \text{exp}(G \otimes G) \) divides \( (\text{exp}(G))^3 \text{exp}(G') \). For finite nilpotent groups of class \( \leq 3 \) he proved that \( \text{exp}(G \otimes G) \) divides \( \text{exp}(G) \) (cf. [22, Theorem 2]). In this section we show that this upper bound can be extended to the \( q \)-tensor square of finite nilpotent groups of class \( \leq 3 \), \( q \geq 0 \).

Lemma 5.1. Let \( G \) be a nilpotent group of class \( \leq 3 \) and let \( q \geq 1 \). Then

(i) \([K, \gamma_3(\nu^q(G))] = 1\);
(ii) if \( t = \prod_{i=1}^{r}[x_i, y_i^\varphi]^\epsilon_i \) is an arbitrary element in \( T = [G, G^\varphi] \), where \( r \geq 1 \) and \( \epsilon_i = \pm 1 \) for \( i = 1, \ldots, r \), then, for all \( \hat{k} \in \hat{K} \) we have

\[
[t, \hat{k}] = \prod_{i=1}^{r}[x_i, y_i^\varphi, \hat{k}]^{\epsilon_i} = \prod_{i=1}^{r}[x_i, y_i^\varphi, k^\varphi]^{\epsilon_i} = [t, k]^q;
\]

(iii) for all positive integers \( n \) and \( t, \hat{k} \) as in part (ii), we have,

\[
(t \hat{k})^n = t^n[k, k^\varphi]^{-\langle \varphi \rangle}(\hat{k})^n = t^n[k, k^{-\varphi}](\hat{k})^n
\]

Proof. (i). By [3, Proposition 2.7] we know that \( \gamma_j(\nu^q(G)) = [\gamma_{j-1}(G), G^\varphi]\gamma_j(G)\gamma_j(G^\varphi) \), for all \( j \geq 2 \). This implies that \( \nu^q(G) \) has nilpotency class at most 4. Now let \( \hat{k} \in \hat{K} \) be any generator of \( K \). From defining relations [12], [13] and Lemma 2.2 (iv), we see that conjugation of \( \hat{k} \) by any commutator \( [x^\alpha, y^\beta, z^\gamma] \in \gamma_3(\nu^q(G)) \), where \( \alpha, \beta, \gamma \in \{1, \varphi\} \), is the same as conjugating \( \hat{k} \) by the commutator \( [x, y, z] \in \gamma_3(G) \). This shows that \( \gamma_3(\nu^q(G)) \) centralizes \( K \) if \( G \) has nilpotency class \( \leq 3 \).

(ii). The first equality follows from commutator calculus and induction on \( r \), since \([\gamma_3(\nu^q(G)), \gamma_2(\nu^q(G))] \leq \gamma_5(\nu^q(G)) = 1\), as \( \nu^q(G) \) has class \( \leq 4 \). The second equality follows from the identity \( [x, y^\varphi, \hat{k}] = [x, y^\varphi, k^\varphi] \), according to Lemma 2.2 (iv), while the last one is obtained by the way back, making use of part (i) to write \([x_i, y_i^\varphi, k^\varphi] = [x_i, y_i^\varphi, \hat{k}]^q \) for \( i = 1, \ldots, r \).
(iii). We expand \((\hat{t}\hat{k})^n\) by induction on \(n\), collecting commutators in the middle, to get
\[
(\hat{t}\hat{k})^n = t^n \prod_{i=1}^{n-1} ([t, (\hat{k})^{-i}]^{t^{n-1-i}})(\hat{k})^n.
\]
Since \([t, (\hat{k})^{-i}] \in \gamma_3(\nu^q(G))\), we see that \([t, (\hat{k})^{-i}]^{t^{n-1-i}} = [t, (\hat{k})^{-i}], i = 1, \ldots, n - 1\). In addition, by parts (i) and (ii), \([t, (\hat{k})^{-i}] = [t, (\hat{k})]^{-i} = [t, k]^{-i} = ([t, k]^{-q})^i\). Consequently, \(\prod_{i=1}^{n-1} ([t, (\hat{k})^{-i}]^{t^{n-1-i}}) = [t, k]^{-q(\hat{k})^n}\) which, by part (i) and induction, is also equal to \([t, k]^{-q(\hat{k})^n}\) since \([t, k] \in \gamma_3(\nu^q(G))\). This completes the proof. \(\square\)

**Theorem 5.2.** Let \(G\) be a finite nilpotent group of class \(\leq 3\) with \(\exp G = n\) and let \(q \geq 0\). Then,

(i) \(\exp(G \otimes^q G)\) divides \(\exp G\) if either \(n\) is odd or \(4 \mid q\);

(ii) \(\exp(G \otimes^q G)\) divides \(2\exp G\), otherwise.

**Proof.** We use the isomorphism \(G \otimes^q G \cong T^q(G) = TK\), where \(T = [G, G^\nu]\), to work inside \(\nu^q(G)\). As already observed in the proof of Proposition 2.4, by defining relations (11) and (15) we see that \(\hat{K}\) provides a right transversal for \(T\) in \(TK\), that is, \(TK = \{t\hat{k} \mid t \in [G, G^\nu], k \in K\}\). So, all we need is to control the orders of an arbitrary element \(t\hat{k} \in T^q(G)\).

(i). Firstly we consider the case \(q = 0\). Here we have \(\nu^0(G) = \nu(G)\) and \(G \otimes^0 G = G \otimes G\), the non-abelian tensor square of \(G\). So, this case is dead by [21, Theorem 2]: \(\exp(G \otimes G)\) divides \(\exp G\). Thus, suppose \(q \geq 1\) and let \(n := \exp G\). By part (iii) of Lemma 5.1 we have
\[
(\hat{t}\hat{k})^n = t^n \exp([t, k^{-q(\hat{k})}]) = (\hat{k})^n,
\]
if \(n\) is odd or if \(q\) is even. Notice that \(t \in [G, G^\nu]\) and so, \(t^n = 1\) for all \(n\), by the case \(q = 0\) and [21]). Now, an induction on \(n\) using relation (15) gives
\[
1 = (k^n) = (\hat{k})^n[k, k^\nu]^{-q(\hat{k})^n},
\]
which implies that \((\hat{k})^n = [k, k^\nu]^{-q(\hat{k})^n}\). Thus, if \(n\) is odd or if \(4 \mid q\), then \((\hat{k})^n = 1\) and, consequently, \((\hat{t}\hat{k})^n = 1\).

(ii). In the case \(n\) even and \(4 \nmid q\) then certainly we get, from (11) and Lemma 5.1 (i),
\[
(\hat{t}\hat{k})^{2n} = ([t, k^{-q(\hat{k})}])(\hat{k})^{2n} = [t, k^{-q(\hat{k})}]^{2}(\hat{k})^{n-1}[k, k^\nu]^{-q(\hat{k})^{n-1}} = 1.
\]
This completes the proof. \(\square\)

**Example 5.3.** The third of the following simple examples borrowed from [9] Theorem 3.1 and Table 1] shows that the bound in part (ii) of Theorem 5.2 can be attained in the simplest situation, of a cyclic group.

1. \(D_4 \otimes^4 D_4 \cong C_5^2 \times C_4\);
2. \(Q_8 \otimes^4 Q_8 \cong C_2^4 \times C_4\);
\( C_n \otimes^q C_n \cong C_{2n} \times C_s \) if \( q, n \equiv 2 \pmod{4} \) and \( \gcd(q, n) = 2s \).

REFERENCES

[1] R. Bastos, I. N. Nakaoka and N. R. Rocco, Finiteness conditions for the non-abelian tensor product of groups, Monatsh. Math., 187 (2018) 603–615.
[2] R. Bastos, N. R. Rocco and E. R. Vieira, Finiteness of homotopy groups related to the non-abelian tensor product, Annali di Matematica, 198 6 (2019) 2081–2091.
[3] R. D. Blyth, F. Fumagalli and M. Morigi, Some structural results on the non-abelian tensor square of groups, J. Group Theory, 13 (2010) 83–94.
[4] R. D. Blyth and R. F. Morse, Computing the nonabelian tensor squares of polycyclic groups, J. Algebra, 321 (2009) 2139–2148.
[5] R. Brown, \( q \)-perfect Groups and Universal \( q \)-central Extensions, Publ. Mat. 34 (1990) 291–297.
[6] R. Brown, D. L. Johnson and E. F. Robertson, Some computations of non-abelian tensor products of groups, J. Algebra 111 (1987) 177–202.
[7] R. Brown and J.-L. Loday, Excision homotopique en base dimension, C.R. Acad. Sci. Paris S.I Math. 298, No. 15 (1984) 353–356.
[8] R. Brown and J.-L. Loday, Van Kampen Theorems for Diagrams of Spaces, Topology, 26 (1987) 311–335.
[9] T. P. Bueno and N. R. Rocco, On the \( q \)-tensor square of a group, J. Group Theory 14 (2011) 785–805.
[10] D. Conduché and C. Rodríguez-Fernandez, Non-abelian Tensor and Exterior Products modulo \( q \) and Universal \( q \)-central Relative Extensions, J. Pure Appl. Algebra, 78, No.2 (1992) 139–160.
[11] I. R. M. Dias and N. R. Rocco, A polycyclic presentation for the \( q \)-tensor square of a polycyclic group, J. Group Theory, 23, No.1 (2020) 97-120.
[12] G. Donadze, M. Ladra and V. Thomas, On some closure properties of the non-abelian tensor product, J. Algebra, 472 (2017) 399–413.
[13] G. Ellis, The non-abelian tensor product of finite groups is finite, J. Algebra, 111 (1987) 203–205.
[14] G. Ellis, Tensor products and \( q \)-crossed modules, J. London Math. Soc., 2 (51) (2) (1995) 243–258.
[15] G. Ellis and F. Leonard, Computing Schur multipliers and tensor products of finite groups, Proc. Royal Irish Acad., 95A (1995) 137–147.
[16] G. Ellis and C. Rodríguez-Fernández, An exterior product for the homology of groups with integral coefficients modulo \( p \), Cah. Top. Géom. Diff. Cat. 30 (1989) 339–343.
[17] B. Eick and W. Nickel, Computing the Schur multiplicator and the nonabelian tensor square of a polycyclic group, J. Algebra 320, No.2 (2008) 927–944.
[18] N. D. Gilbert and P. J. Higgins, The non-abelian tensor product of groups and related constructions, Glasgow Math. J. 31 (1989) 17–29.
[19] M. Ladra and V. Z. Thomas, Two generalizations of the nonabelian tensor product, J. Algebra, 369 (2012) 96–113.
[20] B. C. R. Lima and R. N. Oliveira, Weak commutativity between two isomorphic polycyclic groups, J. Group Theory, 19 (2016) 239–248.
[21] P. Moravec, The exponents of nonabelian tensor products of groups, J. Pure Appl. Algebra, 212 (2008) 1840–1848.
[22] P. Moravec, *The nonabelian tensor product of polycyclic groups is polycyclic*, J. Group Theory 10 (2007) 795–798.

[23] I. N. Nakaoka, *Non-abelian tensor products of solvable groups*, J. Group Theory, 3 (2000) 157–167.

[24] D. J. S. Robinson, *Finiteness conditions and generalized soluble groups*, Part 1, Springer-Verlag, 1972.

[25] D. J. S. Robinson, *A course in the theory of groups*, 2nd edition, Springer-Verlag, New York, 1996.

[26] N. R. Rocco, *On a construction related to the non-abelian tensor square of a group*, Bol. Soc. Brasil Mat., 22 (1991) 63–79.

[27] N. R. Rocco, *A presentation for a crossed embedding of finite soluble groups*, Comm. Algebra 22 (1994) 1975–1998.

[28] N. R. Rocco and E. C. P. Rodrigues, *The q-tensor square of finitely generated nilpotent groups, q odd*, J. Algebra Appl., 16 no. 11 (2017) 1750211, 16 pp.

[29] C. C. Sims, *Computation With Finitely Presented Groups*, CUP, Cambridge, 1994.

[30] M. P. Visscher, *On the nilpotency class and solvability length of the nonabelian tensor product of groups*, Arch. Math. 73 (1999) 161–171.

Institute of Mathematics and Statistics, Universidade Federal de Goiás, Goiânia-GO, 74001-970 Brazil

E-mail address: ivonildes@ufg.br

Departamento de Matemática-IE, Universidade de Brasília, Brasília-DF, 70910-900 Brazil

E-mail address: norai@unb.br

Institute of Exact and Natural Sciences, Universidade Federal de Rondonópolis, 78730-614 Rondonópolis-MT, Brazil

E-mail address: eunicecpf@hotmail.com