Virial expansion for almost diagonal random matrices

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Energy level statistics of Hermitian random matrices $\hat{H}$ with Gaussian independent random entries $H_{i\geq j}$ is studied for a generic ensemble of almost diagonal random matrices with $\langle |H_{ii}|^2 \rangle \sim 1$ and $\langle |H_{i\neq j}|^2 \rangle = b F(|i - j|) \ll 1$. We perform a regular expansion of the spectral form-factor $K(\tau) = 1 + b K_1(\tau) + b^2 K_2(\tau) + \ldots$ in powers of $b \ll 1$ with the coefficients $K_m(\tau)$ that take into account interaction of $(m + 1)$ energy levels. To calculate $K_m(\tau)$, we develop a diagrammatic technique which is based on the Trotter formula and on the combinatorial problem of graph edges coloring with $(m + 1)$ colors. Expressions for $K_1(\tau)$ and $K_2(\tau)$ in terms of infinite series are found for a generic function $F(|i - j|)$ in the Gaussian Orthogonal Ensemble (GOE), the Gaussian Unitary Ensemble (GUE) and in the crossover between them (the almost unitary Gaussian ensemble). The Rosenzweig-Porter and power-law banded matrix ensembles are considered as examples.

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I. INTRODUCTION

Random Matrix Theory (RMT) has proven to be a universal formalism to describe a great variety of complex systems ranging from nuclei to mesoscopic quantum dots\cite{1} to chaotic systems\cite{2}. Of particularly wide application is the Wigner-Dyson RMT\cite{3,4,5}. It is the statistical theory of eigenvalues and eigenfunctions of a random Hermitian matrix $\hat{H}$ whose entries $H_{i\geq j}$ fluctuate as independent Gaussian random variables with zero mean $\langle H_{ij} \rangle = 0$ and a
constant variance $\langle |H_{ij}|^2 \rangle = \text{const}$. There are three Dyson symmetry classes labeled by $\beta$: orthogonal ($\beta = 1$), unitary ($\beta = 2$) and symplectic ($\beta = 4$) that correspond to real, generic complex, and real quaternionic Hermitian matrices. This theory is extremely successful in describing the spectrum of complex nuclei and the statistics of various observable quantities in mesoscopic quantum dots.

The key property of the Wigner-Dyson RMT is that the variance $\langle |H_{ij}|^2 \rangle$ does not change with increasing the distance $|i - j|$ from the diagonal. In order to illustrate the properties of eigenvectors one can invoke one-dimensional chain of sites $1 < i < N$ ($N$ is the matrix size) with on-site energies $H_{ii}$ and hopping matrix elements $H_{ij}$. Then the above property of off-diagonal entries implies the possibility of hopping throughout the entire chain which results in the delocalized character of eigenvectors. If on the contrary the off-diagonal matrix elements $H_{ij}$ are nonzero only inside the band $|i - j| < \lambda$, all the eigenvectors turn out to be localized in the same way as in a quasi one-dimensional disordered wire. This banded random matrix (BRM) theory is also relevant for chaotic systems under a time-periodic perturbation.

One well–known extension of the Wigner-Dyson RMT is the model of Rosenzweig and Porter (RPRM) with the variance $\langle |H_{i\neq j}|^2 \rangle = B^2/N^{2\alpha}$ that does not depend on the distance $|i - j|$ but depends on the matrix size. The spectral statistics of RPRM at different parameter $\alpha$ can range from almost uncorrelated levels (if $\alpha > 1$) to a strong level repulsion which is similar to a weakly perturbed Wigner-Dyson ensemble (if $\alpha = 1/2$), see, for instance, the papers [11,12,13] and references therein. Exact calculation of the two–levels correlation function of RPRM with the complex hopping entries $H_{i\neq j}$ and $\alpha = 1$ shows that the spectral statistics falls neither into the Poissonian nor into the Wigner-Dyson universality classes. It is often called “the regime of crossover”. Much less is known about RPRM with either real or quaternionic hopping elements.

Without lost of generality, RPRM can be presented by an ensemble of the matrices $\hat{H}_{RP} = \hat{A} + (B/N^\alpha)\hat{B}$ with a superposition of a diagonal matrix $\hat{A}$ and a full Wigner-Dyson matrix $\hat{B}$ of any symmetry ($\beta = 1, 2, 4$). If $\hat{A}$ is the random matrix possessing the Poissonian level statistics and $B = 0$ the model describes spectral properties of a classical integrable system. By increasing $\mathcal{B}$ one can explore a transition from integrable to a classically chaotic system with or without the time–reversal symmetry.

Recently another Gaussian RMT that interpolates between the Wigner-Dyson RMT and
BRM theory attracted a considerable attention. This is the power law banded random matrix (PLBRM) theory \cite{14,15,16,17} for which $\langle |H_{ij}|^2 \rangle$ is nearly constant inside the band $|i-j| < \lambda$ and decreases as a power-law function $\langle |H_{ij}|^2 \rangle \sim 1/(|i-j|)^{-2\alpha}$ for $|i-j| > \lambda$. The special case $\alpha = 1$ is relevant for description of critical systems with multifractal eigenstates \cite{14,15,16,17,18,19}, in particular for systems at the Anderson localization-delocalization transition point. The case $\alpha > 1$ corresponds to the power-law localization which can be found in certain periodically driven quantum-mechanical systems \cite{20}.

The progress in BRM and PLBRM theories became possible because of mapping \cite{8,14} onto the nonlinear supersymmetric sigma-model \cite{21} that allowed to obtain rigorous results by using various powerful methods of the field theory. However, such mapping is only justified if the bandwidth $\lambda \gg 1$. In the opposite case where all the off-diagonal matrix elements are parametrically small compared to the diagonal ones, no field-theoretical approach is known so far. Yet such almost diagonal RMT may possess nontrivial properties because of the slow decay of the off-diagonal matrix elements $\langle |H_{ij}|^2 \rangle$ with increasing $|i-j|$. For instance it is of fundamental interest to study the spectral statistics in systems with power-law localization that takes place in the power-law banded random matrix ensembles at $\alpha > 1$. Another problem to study is the critical almost diagonal PLBRM. It is known that the eigenvectors of PLBRM with $\alpha = 1$ remain multifractal for an arbitrary small value of $\lambda$ \cite{19}. This means that the typical eigenfunction is extended though very sparse at small $\lambda$. Thus almost diagonal random matrices may display the localization-delocalization transition with changing the exponent $\alpha$ as well as their large bandwidth counterpart \cite{14}. This transition has been studied numerically \cite{22} for $\lambda \sim 1$ but little is known about it in the limit of almost diagonal random matrices.

The goal of this paper is to develop a formalism that would allow to describe the spectral statistics for a wide class of almost diagonal random matrices with a generic behaviour of $\langle |H_{i\neq j}|^2 \rangle = b^2 F(|i-j|)$ and $\langle |H_{ii}|^2 \rangle \sim 1$. The parameter $b$ may or may not depend on the matrix size $N$. The principle requirement is that $b \ll 1$ is small. It controls the smallness of the off-diagonal matrix elements and will be used as an expansion parameter for the spectral correlation functions.

A natural way to proceed with such an expansion is to develop a perturbation theory in the parameter $b$ starting from the diagonal matrix as a zero-order approximation. However, a naive expansion of the spectral form-factor $K(\tau)$ up to a finite order in $b$ may fail, since
the parameter \( b \) enters in the product \( bN\tau \) with the matrix size \( N \) and the time \( \tau \) measured in units of the Heisenberg time. In this combination, the small parameter \( b \) can be compensated by the (large) product \( N\tau \). In order to be able to obtain the regular expansion in powers of \( b \), \( K(\tau) = 1 + bK_1(\tau) + b^2K_2(\tau) + \ldots \), in the thermodynamic limit \( N \to \infty \) at arbitrary \( N\tau \), we develop a diagrammatic technique that gives \( K_m(\tau) \) in a form of infinite series in powers of \( bN\tau \). It is essentially a kind of locator expansion adjusted to the problem of spectral statistics. It formalizes and generalizes the idea of Ref.\[19\] where the resonance pairs of states \( i,j \) with \( |H_{ii} - H_{jj}| < |H_{ij}| \) have been considered to find the statistics of multifractal eigenstates. In our approach, the coefficients \( K_m(\tau) \) result from an interaction (via the off-diagonal elements \( H_{ij} \)) of \( m+1 \) energy levels. We call this expansion a virial expansion by analogy with the expansion of thermodynamic functions of a dilute system in powers of density with virial coefficients that take into account collisions of \( m+1 \) particles. We stress that what we are doing below is not an approximation of two, three, or more resonant levels. It is rather the classification of exact perturbation series by a number of the interacting levels involved. The question of a contribution from non-resonant levels does not arise, since at a given number of the interacting levels all possible relations between \( |H_{ii} - H_{jj}| \) and \( |H_{ij}| \) are taken into account.

The most serious problems on that route are i) dealing with non-commutative matrices \( \hat{H}_\varepsilon = \text{diag}\{H_{ii}\} \) and \( \hat{V} = \hat{H} - \hat{H}_\varepsilon \); ii) combinatorial coefficients in the infinite series in \( bN\tau \). The first problem is solved by using the Trotter formula (see Section \[III\]). The second problem reduces to a particular case of the general problem of graph coloring that is amenable to an exact solution (see Section \[IV\] and \[V\]).

As a result we present the series for \( K_m(\tau) \) with combinatorial coefficients that do not depend on the particular choice of \( \mathcal{F}(|i - j|) \) in the variance \( \langle |H_{ij}|^2 \rangle = b^2 \mathcal{F}(|i - j|) \). They can thus be applied to any Gaussian ensemble of almost diagonal random matrices. The particularly simple expressions are found for \( m = 1, 2 \) which correspond to two and three interacting levels.

The general formulae obtained in this way are illustrated in Section \[VI A\] for RPRM with \( b = B/N \); \( \mathcal{F}(|i - j|) = 1 \) and in Section \[VII B\] for a critical PLBRM with \( b = B \); \( \mathcal{F}(|i - j|) \sim 1/|i - j|^2 \). In both examples, the constant \( B \) will be taken small: \( B \ll 1 \). As a result, the correction to the Poisson behaviour \( K(\tau) = 1 \) has been obtained in the limit of infinite matrix size \( N \to \infty \). The existence of such a correction can be interpreted
as a rigorous detection of delocalization in the framework of a formalism that starts from the diagonal random matrix where all states are localized. Thus our theory offers a controllable way of obtaining delocalization from localization. This is just complementary to conventional approaches based on the supersymmetric sigma-model that always starts from delocalized (e.g. diffusive) modes.

II. BASIC DEFINITIONS

Let us consider a Hermitian RM of size $N \times N$, $N \gg 1$, from a Gaussian ensemble. We assume that entries of the matrix are random and independent. The RM is the Hamiltonian $\hat{H}$ of the matrix Schrödinger equation

$$\hat{H}\psi_n = \epsilon_n\psi_n$$

where $\epsilon_n$ and $\psi_n$ are the eigenvalues and eigenvectors, respectively. Define statistical properties of the matrix entries:

$$\langle H_{i,j} \rangle = 0; \quad \langle H_{i,i}^2 \rangle = \frac{1}{\beta}; \quad \langle |H_{i,j}|^2 \rangle = b^2 F(|i-j|), \; i \neq j;$$  \hspace{1cm} (1)

where $F(|i-j|) > 0$ is a smooth function of its argument, and the parameter $b$ is small:

$$b \ll 1.$$  

The condition $b \ll 1$ means that RM is almost diagonal. The parameter $\beta$ corresponds to the Dyson symmetry classes: $\beta_{\text{GOE}} = 1$, $\beta_{\text{GUE}} = 2$. The brackets $\langle \ldots \rangle$ denote the ensemble averaging. If the ensemble is Gaussian the averaging of a function $F(A)$ over a random variable $A$ reads:

$$\langle F(A) \rangle_A \equiv \frac{1}{\sqrt{2\pi}A^2} \int_{-\infty}^{+\infty} F(A) \exp\left(-\frac{A^2}{2A^2}\right) dA.$$

We concentrate on the level statistics which is characterized by the density of states $\rho(E) = \sum_n \delta(E - \epsilon_n)$ and its multi-point correlation functions. For example, the two-level correlation function $R(\omega)$ is defined as:

$$R(\omega) = \frac{\langle \rho(\omega/2)\rho(-\omega/2) \rangle}{\langle \rho(0) \rangle^2}; \quad \langle \hat{a} \hat{b} \rangle \equiv \langle \hat{a} \rangle \langle \hat{b} \rangle - \langle \hat{a} \hat{b} \rangle.$$  \hspace{1cm} (2)
The Fourier transform of $R(\omega)$ is known as the spectral form-factor $K(t)$:

$$K(t) = \int_{-\infty}^{+\infty} e^{i\omega t} R(\omega) \, d\omega. \quad (3)$$

We will see below that for almost diagonal RM the representation of spectral statistics in the time domain turns out to be convenient. Therefore from now on we use the form factor instead of the correlation function. We rescale time by the mean level spacing $\Delta \equiv 1 / \langle \rho(0) \rangle$ introducing the dimensionless time $\tau = t \Delta$. Our goal is to develop a regular perturbative expansion for $K(\tau)$ in powers of small parameter $b \ll 1$. In the limit of small time the spectral form-factor $K(\tau \to 0)$ is linked to the other important spectral characteristics called the level compressibility$^{24}$:

$$\chi = \lim_{\tau \to 0} \left( \lim_{N \to \infty} K(\tau) \right). \quad (4)$$

Let us take a window of the width $\delta E$, $\delta E / \Delta \equiv \bar{n} \ll N$, in the energy space centered at $E = 0$ and calculate the number $n$ of levels inside the window at some realization of disorder. The level number variance is $\Sigma_2(\bar{n}) = \langle (n - \bar{n})^2 \rangle$. The level compressibility is by definition the limit

$$\chi = \lim_{\bar{n} \to \infty} \left( \lim_{N \to \infty} \frac{\partial \Sigma_2(\bar{n})}{\partial \bar{n}} \right). \quad (5)$$

The level compressibility contains an information about the localization transition: $\chi$ ranges from $\chi_{WD} = 0$ for the Wigner-Dyson statistics with extended wave functions and a strong levels repulsion to $\chi_P = 1$ in the case of localized wave functions and uncorrelated levels with the Poissonian distribution. The intermediate situation with $0 < \chi_{crit} < 1$ is inherent for the critical regime of multifractal wave functions$^{24}$.

The main object of our further analysis is the following correlation function in the time domain:

$$\tilde{K}_N(\tau) = \frac{1}{N} \langle \langle \text{Tr} e^{-iH\tau/\Delta} \text{Tr} e^{iH\tau/\Delta} \rangle \rangle \equiv \tilde{K}_0(\tau) + b\tilde{K}_1(\tau) + b^2\tilde{K}_2(\tau) + \ldots \quad (6)$$

For the constant mean density of states $\tilde{K}(\tau)$ coincides with $K(\tau)$. However, they are different if $\langle \rho(E) \rangle$ essentially depends on energy $E$. In analogy with Eq.(4) one can define the quantity $\chi_0 \equiv \lim_{\tau \to 0} \left( \lim_{N \to \infty} \tilde{K}_N(\tau) \right)$. It turns out that at small $b$ there is a simple approximate relationship between $\chi$ and $\chi_0$ (see Appendix A):

$$\chi \simeq 1 - \frac{1 - \chi_0}{\Upsilon}, \quad \Upsilon = \frac{\Delta}{N} \int_{-\infty}^{+\infty} \langle \rho(E) \rangle^2 \, dE. \quad (7)$$
The mean density of states for the Gaussian ensemble of almost diagonal RMs with either localized or (sparse) fractal eigenstates is close to the Gaussian distribution of the diagonal entries:

\[
\langle \rho(E) \rangle \simeq N \sqrt{\frac{\beta}{2\pi}} \exp \left( -\frac{\beta E^2}{2} \right) \Rightarrow \Delta \simeq \frac{1}{N} \sqrt{\frac{2\pi}{\beta}}.
\]

Thus with an accuracy of \(O(b)\) the unfolding factor in Eq.(7) can be taken as \(\Upsilon^{-1} \simeq \sqrt{2}\).

III. THE METHOD

A. The Trotter formula

As far as we investigate the properties of almost diagonal RMs the Hamiltonian can be naturally divided into a diagonal part \(\hat{H}_\varepsilon\) and a matrix of hopping elements \(\hat{V}\):

\[
\hat{H} \equiv \hat{H}_\varepsilon + \hat{V}.
\]

It follows from the definition of \(H_{i,j} \equiv V_{i,j} \sim b\) that the hopping elements \(H_{i,i} \equiv \varepsilon_i \sim 1\). However, matrices \(\hat{H}_\varepsilon\) and \(\hat{V}\) do not commute with each other and a direct expansion of the exponentials \(e^{\pm i(\hat{H}_\varepsilon + \hat{V})/\Delta}\) in Eq.(7) in terms of \(\hat{V}\) involves serious difficulties. One possible way to overcome these problems is to represent \(e^{\pm i(\hat{H}_\varepsilon + \hat{V})/\Delta}\) as a product of exponential functions containing matrices \(\hat{H}_\varepsilon\) and \(\hat{V}\) separately.

To do this we use a generalization of the identity \(\exp(a + b) \equiv \exp(a) \exp(b)\) for non-commuting variables known as the the Trotter formula:

\[
e^{\hat{A} + \hat{B}} = \lim_{n \to \infty} \left( e^{\hat{A}/n} e^{\hat{B}/n} \right)^n \Rightarrow e^{\pm i \hat{H}_\varepsilon/\Delta} = \lim_{n \to \infty} \prod_{p=1}^{n} \left( e^{\pm i \hat{H}_\varepsilon/\Delta} e^{\pm i \hat{V}/\Delta} \right).
\]

The Trotter formula is exact for the finite matrices \(\hat{A}\) and \(\hat{B}\). Therefore, correct order of limits reads:

\[
\begin{align*}
1. & \quad n \to \infty ; \\
2. & \quad N \to \infty ; \\
3. & \quad (\text{if necessary}) \quad \tau \to 0 .
\end{align*}
\]

In what follows we will always imply precisely this order of doing limits.

Eq.(10) allows to make a regular expansion in powers \(\hat{V}\). The price for that is the infinite product in Eq.(10). We need a proper selection rule to extract from that product terms of the order of \(O(b^0), O(b^1), O(b^2), \ldots\).
B. Lowest-order terms

For a strictly diagonal matrix the spectral form-factor $\tilde{K}(\tau)$ is calculated straightforwardly:

$$
\tilde{K}(\tau)|_{\hat{V}=0} = \frac{1}{N} \langle \langle \text{Tr} e^{-iH_\varepsilon \tau/\Delta} \text{Tr} e^{iH_\varepsilon \tau/\Delta} \rangle \rangle = \frac{1}{N} \sum_l \left[ 1 - \langle e^{-i\varepsilon_l \tau/\Delta} \rangle^2 \right] = 1 - e^{-\langle \varepsilon^2 \rangle (\tau/\Delta)^2} \quad (12)
$$

For all Gaussian RM of the type Eq.(1) the inverse mean level spacing increases with the matrix size $N$. The inverse mean level spacing $1/\Delta \equiv \tilde{N}$ is proportional to the matrix size: $\tilde{N} \sim N$. For finite $N$ the normalization sum rule requires $\tilde{K}(0) = 0$. If however the limit $N \to \infty$ is done prior to the limit $\tau \to 0$ (see Eq.(11)) the normalization sum rule is violated and we recover the Poisson statistics result $\tilde{K}(\tau) = 1$.

In order to obtain corrections to $\tilde{K}(\tau)$ proportional to $b^k$ one has to expand $e^{\pm i\tilde{N}\tau \hat{V}/n}$ in powers of $\hat{V}$ in the infinite product in the r.h.s. of Eq.(10):

$$
\ldots \times \exp \left( \pm \frac{i\tilde{N}\tau}{n} \hat{H}_\varepsilon \right) \exp \left( \pm \frac{i\tilde{N}\tau}{n} \hat{V} \right) \times \ldots
$$

$n$ pairs of exponentials

and then to perform the Gaussian averaging over $\hat{H}_\varepsilon$ and $\hat{V}$. Note that the Gaussian average is zero for all terms in r.h.s of Eq.(12) which contain an odd number of off-diagonal matrices $\hat{V}$. That is why only terms with even powers of $b$ may result from such an expansion. For the purpose of explaining the details of the new formalism based on the Trotter formula we show how the term $\propto b^2$ arises from it.

1. Gaussian averaging

Each $\text{Tr} e^{\pm iH\tilde{N}\tau}$ in Eq.(12) can be represented using the Trotter formula as $\text{Tr}$ of the infinite product Eq.(13). Therefore we have to expand either two exponentials $e^{\pm i\tilde{V}\tilde{N}\tau/n}$ in Eq.(13) up to $\hat{V}$ or one single exponential up to $\hat{V}^2$ setting $\hat{V} = 0$ in all the other exponentials $e^{\pm i\tilde{V}\tilde{N}\tau/n}$. We will show below that only the first option survives the limit $n \to \infty$.

Expanding two of $\exp \left( \pm \frac{i\tilde{N}\tau p_1}{n} \hat{H}_\varepsilon \right)$ in Eq.(13) up to $\hat{V}$ we obtain:

$$
\exp \left( \pm \frac{i\tilde{N}\tau p_1}{n} \hat{H}_\varepsilon \right) \exp \left( \pm \frac{i\tilde{N}\tau}{n} \hat{V} \right) \times \ldots
$$

We obtain:

$$
\exp \left( \pm \frac{i\tilde{N}\tau p_1}{n} \hat{H}_\varepsilon \right) \exp \left( \pm \frac{i\tilde{N}\tau p_2}{n} \hat{H}_\varepsilon \right) \exp \left( \pm i\frac{\tilde{N}\tau}{n} \hat{V} \right) \sim b^2,
$$

(14)
where \( p_{1,2,3} \) are the numbers of successive exponentials \( \exp \left( \pm \frac{\tilde{N} \tau}{n} \tilde{H} \right) \) ‘fused’ together after we set \( \tilde{V} = 0 \) in all \( \exp \left( \pm \frac{\tilde{N} \tau}{n} \tilde{V} \right) \) but two. We call \( p_s \) the Trotter numbers. They obey the obvious restriction

\[
p_1 + p_2 + p_3 = n, \quad p_{1,2} \geq 1, \quad p_3 \geq 0.
\]

Note that both off-diagonal matrices \( \tilde{V} \) in Eq.(14) must belong to the same trace. If they belong to the different traces the result is zero because \( \text{Tr} \left\{ \exp \left( \pm \frac{\tilde{N} \tau}{n} \tilde{H} \right) \tilde{V} \right\} = 0. \)

In order to find the term \( C \propto O(b^2) \) in Eq.(6) we substitute Eq.(14) into Eq.(6) and perform the Gaussian averaging in accordance with Eq.(1):

\[
C = -\frac{4}{N} \left( \frac{\tilde{N} \tau}{n} \right)^2 \sum_{\{p_s\}} \left\{ D_N \left( \frac{n - p_1 - p_3}{n} \right) D_N \left( \frac{p_2}{n} \right) + D_N \left( \frac{p_1 + p_3}{n} \right) D_N \left( \frac{n - p_2}{n} \right) \right\};
\]

where \( \sum_{\{p_s\}} \) means summation over the Trotter numbers \( \sum_{p_1,p_2,p_3=1}^{n} \delta_{n-(p_1+p_2+p_3)} \) and we denote

\[
D_N(y) \equiv \exp \left[ -\frac{1}{2} \beta \left( \frac{\tilde{N} \tau}{n} \right)^2 y^2 \right]; \quad R_N(k) \equiv \frac{1}{N} \sum_{i>j}^{N} \left( F(|i-j|) \right)^k.
\]

The function \( R_N(k) \) depends on the particular form of decay of the off-diagonal matrix elements in the Gaussian ensemble (see Eq.(11)) and contains the sum over matrix indices which will be referred to below as the summation over the ‘real space’.

It is instructive to give a graphic representation of Eq.(16) before and after averaging. In Fig.1 each circle denotes the trace. The vertices \( V_{i,j} \) and \( V_{j,i} \) in the left circle are uncoupled.
before averaging (Fig.1a). The shadowed box indicates starting point of the Trotter chain Eq.(13). The circle is thus divided into segments (below referred to as the Trotter segments) corresponding to \( e^{-i\frac{\tilde{N}\tau}{n}\varepsilon_i}, e^{-i\frac{\tilde{N}\tau}{n}\varepsilon_j}, \) and \( e^{-i\frac{\tilde{N}\tau}{n}\varepsilon_i}. \) We mark the length of each segment by the Trotter number \( p_s. \) The right circle stands for \( e^{i\tilde{N}\tau\varepsilon_m}. \) The averaging over \( \hat{V} \) connects uncoupled \( V \)-vertices by the correlation function \( |V_{ij}|^2 = b^2 \mathcal{F}(|i-j|) \) denoted by a double line while the averaging over \( \varepsilon_m \) couples \( \varepsilon_m \) to either \( \varepsilon_i \) or \( \varepsilon_j \) (Fig.1b).

2. Integral over Trotter variables

One has to sum over the Trotter numbers in Eq.(16) prior to doing the limit \( n \to \infty. \) To this end we introduce the Trotter variables \( y_s = p_s/n \) which in the limit \( n \to \infty \) can be considered as continuous. Then the summation over the Trotter numbers \( p_s \) can be replaced by integration over the Trotter variables. Switching from the Kronecker-delta \( \delta_p \) to the Dirac \( \delta \)-function \( n\delta_p \to \delta(y) \) we arrive at:

\[
\mathcal{C} = -4 \left( b\tilde{N}\tau \right)^2 \mathcal{R}_N(1) I;
\]

\[
I = \int_{0}^{1} dy_1 dy_2 dy_3 \delta(1 - y_1 - y_2 - y_3) \left\{ D_N(1 - y_1 - y_3) D_N(y_2) + D_N(y_1 + y_3) D_N(1 - y_2) \right\}.
\]  

It is remarkable that as a result of this transformation, which is exact in \( n \to \infty \) limit, all the \( n \)-dependent factors are absorbed by \( y_s. \) This is because the number of independent Trotter numbers \( p_s \) in Eq.(16) coincides with the number of matrices \( \hat{V}, \) each coming with the factor \( 1/n. \) Note that this is not the case if only one exponential \( e^{\pm i\tilde{N}\tau/n} \) in Eq.(13) is expanded up to \( \tilde{V}^2 \) order. Then the number of Trotter summations is less by one which leaves an uncompensated factor \( 1/n \to 0 \) after switching to the Trotter variables. Thus we arrive at an important conclusion that in all orders of \( \tilde{V} \)-expansion one should retain only linear terms of expansion of the exponential \( e^{\pm i\tilde{N}\tau/n} \approx 1 \pm i\tilde{V}\tilde{N}\tau/n \).  

The integral Eq.(18), which will be referred to as ‘the Trotter integral’, can be simplified by observing that the \( \delta \)-function imposes the same arguments in both \( D_N \) functions in the corresponding products. Thus \( D_N \) functions can be fused \( D_N(x) D_N(x) \equiv D_N(\sqrt{2} x) \) and the Trotter integral takes the form:

\[
I = \int_{0}^{1} dy (1 - y) \left\{ D_N(\sqrt{2} y) + D_N(\sqrt{2}[1 - y]) \right\}.
\] (19)
The similar fusion takes place in the Trotter integrals at an arbitrary order of $V$-expansion.

The further simplification is possible only in doing the thermodynamic limit $N \to \infty$. In this limit the function $D_N(y)$ defined in Eq. (17) collapses to the $\delta$-function:

$$D_N(\sqrt{2}y)|_{N \to \infty} \simeq \frac{\sqrt{\pi \beta}}{\tilde{N}|\tau|} \delta(y).$$

(20)

Then the integral (19) is easily calculated and we find finally the expression for $C$:

$$C = -b \sqrt{\beta \pi} (b\tilde{N}|\tau|) \mathcal{R}_N(1),$$

(21)

where $\mathcal{R}_N(1)$ is given by Eq. (17).

Let us look at the calculation of $C$ from a different viewpoint. Namely, we note that in this calculation only two species of levels $i$ and $j$ are involved. We show this situation graphically in Fig. 2. Shadowed boxes mark the energy levels with different shadowing ("colors") for $\varepsilon_i$ and $\varepsilon_j$. They are connected by the interaction line (IL) which is associated with the factor $(\tilde{N}\tau b)^2$. Because of the fusion of two $D_N$-functions, the interaction of two levels brings only one $\delta$-function with the normalization factor $1/(\tilde{N}|\tau|)$. As the result, we get one 'free' parameter $b$ which is decoupled from the combination $\tilde{N}|\tau|\mathcal{R}_N(1)$ that involves the matrix size $N$:

$$C \sim (\tilde{N}\tau b)^2 \frac{\mathcal{R}_N(1)}{\tilde{N}|\tau|} = b^1 \times (\tilde{N}|\tau|b)^1 \mathcal{R}_N(1).$$

It is clear from the above analysis that if we continue expanding in higher powers of $V_{ij} = V_{ji}^*$ but keep the number of colors equal to 2, we will increase the power of the combination $(b\tilde{N}|\tau|)$ but not the power of the free parameter $b$.

C. Selection rule

Consider a generic term of order $b^{2k}$ in the perturbation series:

$$C_{2k}(A,B) = C_N(\{k_i\}) b^A (\tilde{N}|\tau|b)^B; \quad A + B \equiv 2k.$$  

(22)
FIG. 3: Example of diagrams classified in accordance with the selection rule: a) the case of two colors: \( k = k_1 = 2 \); b) and c) two simplest diagrams for the case of three colors: (b) \( k = 2, k_1 = 1, k_2 = 1 \) and (c): \( k = 3, k_1 = 1, k_2 = 1, k_3 = 1 \).

It corresponds to a diagram with \( k = k_1 + k_2 + \ldots \) interaction lines distributed in a certain way \( \{k_i\} = \{k_1, k_2, \ldots\} \) among the links connecting a given number of different energy levels (“colors”).

Let us formulate the selection rule for such a perturbation theory.

- The power \( A \) of the “free” parameter \( b \) is equal to the number of interacting energy levels minus 1: \( A = c - 1 \).

- The sum \( A + B \) is equal to \( 2k \).

We exemplify it in Fig.3 for three different diagrams: the diagram with two colors (a) corresponding to \( k = 2, A = 1, B = 3 \); and two diagrams with three colors: (b) for \( k = 2, A = 2, B = 2 \); and (c) for \( k = 3, A = 2, B = 4 \).

Suppose that we want to derive the perturbative term \( \tilde{K}_A(\tau) \) of the order of \( O(b^A) \), which would be valid at an arbitrary value of \( \tilde{N}|\tau|b \) in the limit \( N \to \infty \). Such physical result can be obtained using the following strategy:

1. Fix the power \( A \) of the “free” small parameter \( b \) by fixing the number of the interacting energy levels, i.e., the number of colors: \( c \);

2. Perform summation of an infinite series in \( \tilde{N}|\tau|b \) at fixed \( c \) over the number(s) \( k_i \) of IL along each link.

3. Analyze the limit \( N \to \infty \) of the corresponding infinite series in \( \tilde{N}|\tau|b \).
If this virial expansion works the series should converge in the limit \( N \to \infty \).

The strategy we suggested is a certain way of summation of perturbative series. Its physical meaning is analogous to the idea of resonant energy levels or locator expansion. However, in contrast to these essentially heuristic approaches our selection rule offers a formalism that allows to implement a regular expansion in the small parameter \( b \). It takes into account resonant as well as non-resonant levels accurately which is impossible in the above heuristic methods.

Concluding this section we would like to note that the coefficient \( C_N(\{k_i\}) \) in Eq.(22) is a product of two different factors:

\[
C_N(\{k_i\}) = \mathcal{R}_N(\{k_i\}) \mathcal{V}(\{k_i\}),
\]

where \( \mathcal{V}(\{k_i\}) \) is a universal coefficient which depends on a combinatorial factor \( \mathcal{K} \) and on the corresponding Trotter integral \( \mathcal{I} \). The factor \( \mathcal{R}(\{k_i\}) \) is not universal. It arises because of the summation of the product of the correlation functions \( \mathcal{F}(|i - j|) \) over the real space and is a generalization of the function \( \mathcal{R}_N(k) \) given by Eq.(17). In the large-\( N \) limit it can be represented as:

\[
\mathcal{R}_N(\{k_i\}) = \mathcal{R}(\{k_i\}) + \frac{1}{N}\mathcal{R}^{(1)}(\{k_i\}) + \ldots
\]

where \( \mathcal{R}(\{k_i\}) \) and \( \mathcal{R}^{(1)}(\{k_i\}) \) are of the same order. In this paper, we study mainly a contribution of the leading term \( \mathcal{R}(\{k_i\}) \). The role of the \( 1/N \) corrections to \( \mathcal{R}(\{k_i\}) \) is discussed below in Section VII B.

**IV. THE CASE OF TWO COLORS**

In this section we calculate the term of order \( b \) in \( \tilde{K}(\tau) \). According to a general scheme outlined in the previous section this correction is governed by the two-color diagrams: which correspond to the following power series:

\[
b\tilde{K}_1 = \sum_{k=1}^{\infty} (-1)^k x^{2k} \mathcal{V}(k) \mathcal{R}_N(k); \quad \mathcal{V}(k) = \mathcal{K}(k) \mathcal{I}(k); \quad x \equiv \tilde{N} |\tau| b.
\]
The combinatorial part of the calculation consists of two tasks. One of them is to distribute colors (denoting $\varepsilon_{i,j}$) over $2k$ segments. Since $\hat{V}$ is the off-diagonal matrix, the adjacent colors in each circle must be different. In general this is a particular case of the famous combinatorial problem of graph coloring\textsuperscript{29}. For the case of only two colors ($\varepsilon_i$ and $\varepsilon_j$) considered in this section the problem is trivial: each circle must contain an even number of the segments while the first segment of the Trotter chain corresponds either to $\varepsilon_i$ or to $\varepsilon_j$ (see Fig.6), so that only the factor of 4 arises from this coloring. The second combinatorial task is to find the number of ways to connect $V$-matrices in pairs making either in-circle or inter-circles connections. Such connections are equivalent to the Gaussian averaging.

In Fig.7 we show three different ways to connect two $V$-matrices. Two links labeled by 1 and 2 are in-circle connections, while the third one (labeled by 3) is the inter-circles connection. The difference between the connections 1 and 2 is that the connection 1 corresponds to the average $\langle V_{ij}V_{ji} \rangle$ while the connection 2 corresponds to the average $\langle V_{ij}V_{ij} \rangle$. For real Hermitian matrices (the orthogonal ensemble, GOE, $\beta = 1$) $V_{ij} = V_{ji}$ both connections

\[
\varepsilon_i \quad O(b^k) \quad \varepsilon_j = \quad b^k (N\tau b)^k + ... \\
k=1 \quad k=2 \quad k=3
\]
are possible and the corresponding averages are equal to each other. However, for complex Hermitian matrices with the same variance of the real and imaginary parts (the unitary ensemble, GUE, $\beta = 2$) only one of the two averages, $\langle V_{ij}V_{ji} \rangle$, is non-zero. In this case the "crossed" connection $2$ is not allowed.

Now one can easily find the combinatorial factors associated with the number of possible connections. In the GOE any of the $2k$ vertices $V$ can be connected with any other $V$-vertex and we get:

$$K_{\text{GOE}}(k) = 4 \times (2k - 1)!!.$$  \hfill (26)

In the GUE the $2k$ $V$-vertices are divided by two groups: one group containing $k$ vertices of $V_{ij}$ and another group containing $k$ vertices $V_{ji}$. The connection is possible only if $V$-vertices belong to different groups. Thus we obtain for the corresponding combinatorial factor:

$$K_{\text{GUE}}(k) = 4 \times k!.$$  \hfill (27)

In order to find the coefficient $I(k)$ we have to fix the number of $\hat{V}$ matrices in each trace to be $2m$ and $2(k - m)$, thereby also fixing the number of the segments in the circles. Each segment corresponds to a Trotter variable to be integrated over. The details of the calculation of the Trotter integral $I(k)$ are presented in the Appendix. The final result is obtained after summation over $m$:

$$I(k) = b \frac{1}{N|\tau|b} \frac{\sqrt{\beta \pi}}{2} \frac{1}{k! (k - 1)!}.$$  \hfill (28)

Substituting the combinatorial factor $K(k)$, Eqs. (26-27), and the Trotter integral (28)

![FIG. 6: Two different ways to put colors in the two-color problem. The first segment is denoted by the shadowed box.](image)
FIG. 7: An example for 3 different connections of $V$-vertices at $k = 4$. 

into the series (25) we find:

$$\tilde{K}_1 = 2 \sqrt{\pi \beta} \sum_{k=1}^{\infty} (-1)^k C_{\beta=1}^{(2)}(k) R_N(k) (\tilde{N} |\tau| b)^{2k-1};$$

$$C_{\beta=1}^{(2)}(k) = \frac{(2k-1)!!}{k!(k-1)!};$$

$$C_{\beta=2}^{(2)}(k) = \frac{1}{(k-1)!}.$$  

(29)  
(30)  
(31)

Eqs. (29-31) correspond to the leading in $b \ll 1$ correction to the Poisson spectral form-factor. They have been obtained in the framework of the two color approximation when we have taken into account only pairs of the interacting energy levels. We note that the transformation (20) of the $D_N$–function into the delta–function is approximate and its sub-leading terms yield the corrections to $b\tilde{K}_1$ of higher orders starting from $\sim b^5 \Delta \tilde{K}_1$.

We emphasize the the formula (29) for $\tilde{K}_1$ is valid for a Gaussian random matrix ensemble with a generic variance of the off-diagonal matrix elements $\langle |V_{ij}|^2 \rangle = b^2 F(|i-j|)$.

V. THE CASE OF THREE COLORS

According to the selection rule explained above, the term of order $b^2$ in $\tilde{K}(\tau)$ results from an interaction of three energy levels, i.e., we have to perform a summation of the three-color diagrams, see Fig. 8. Three shadowed boxes stand for the three independent energy levels $\varepsilon_i$, $\varepsilon_j$, and $\varepsilon_l$ ($l > j > i$), which are connected by ILs. Any two levels can be connected with each other independently of a connection with the third one by an arbitrary number of ILs: $k_1$, $k_2$, and $k_3$. This leads to a three-dimensional power series in $x = \tilde{N} |\tau| b$:

$$b^2 \tilde{K}_2 = \sum_{k_1, k_2, k_3} \infty (i x)^{2(k_1+k_2+k_3)} \mathcal{V}(\{k_i\}) R_N(\{k_i\}); \quad i = 1, 2, 3.$$  

(32)
Here, the universal factor $\mathcal{V}(\{k_i\})$ contains the Trotter integral $\mathcal{I}(\{k_i\})$ and the combinatorial factor $\mathcal{K}(\{k_i\})$. The function $\mathcal{R}_N$ is a generalization of the sum in real space for the case of three colors:

$$
\mathcal{R}_N(\{k_i\}) = \frac{1}{N} \sum_{l=j+1}^{N} \sum_{j=i+1}^{N} \sum_{i=1}^{N} \left[ (\mathcal{F}(|i-j|))^{k_1} (\mathcal{F}(|j-l|))^{k_2} (\mathcal{F}(|l-i|))^{k_3} \right].
$$

(33)

Generically, any energy level interacts with the two others and all numbers of ILs are not zero $k_{1,2,3} > 0$. A diagram describing this case will be referred to as a triangle (see Fig 8a). However, there are configurations where two levels do not interact with each other and one parameter $k_i$ is zero. A corresponding diagram will be named a line (see Fig 8b). We can attribute a certain physical meaning to both of the three-color diagrams. Let us involve a tree-like structure in the real space where a path with loops is prohibited. It will generate only the “lines”. On the contrary, a real space structure where the paths with loops are allowed will yield both the “triangles” and the “lines”.

The combinatorial factor $\mathcal{K}$ includes the number of coloring of segments by three colors (denoting $\varepsilon_{i,j,l}$) and the number of ways to pair $V$-matrices by in-circle and inter-circles connections. The $V$-matrices create three sets of vertices in the circles: $2k_1$ vertices of $\{V_{i,l}, V_{l,i}\}$, $2k_2$ vertices of $\{V_{j,l}, V_{l,j}\}$, and $2k_3$ vertices of $\{V_{i,j}, V_{j,i}\}$. In view of the property

$$
\langle V_{s,p}V_{q,r} \rangle = \langle V_{s,p}^2 \rangle \delta_{s,q} \delta_{p,r} + \langle |V_{s,p}|^2 \rangle \delta_{s,r} \delta_{p,q}
$$

(34)

the vertices belonging to one and the same set must be paired with each other independently of the two other sets. Applying the same arguments that were used in the case of two colors

FIG. 8: The three-color-diagrams: a) a triangle diagram with $k_{1,2,3} > 0$; b) a line diagram with $k_{2,3} > 0$ and $k_1 = 0$. 

we arrive at:

\[ K_{\text{GOE}} = \text{Col}_{\text{GOE}} \prod_{i=1,2,3} (2k_i - 1)!! ; \]  
\[ K_{\text{GUE}} = \text{Col}_{\text{GUE}} \prod_{i=1,2,3} (k_i)! ; \]

where \( \text{Col} \) is the coloring factor. When the number of colors is larger than two the coloring factor is different in GOE and GUE. Details of calculation of \( \text{Col} \) are given in Appendix C.

Here we explain notations and give the results for GOE and GUE.

Denote a total number of the segments for each of the three colors by \( R, G, \) and \( B \). These numbers can be expressed in terms of \( k_{1,2,3} \):

\[ R = k_1 + k_2 ; \quad G = k_2 + k_3 ; \quad B = k_3 + k_1. \]

The colored segments are distributed over the two circles. The numbers of the colored segments in the first circle will be labeled \( r, g, \) and \( b \). Thus the second circle contains \( R - r, G - g, \) and \( B - b \) segments of different colors.

In the case of GOE, the two circles are colored independently under the only condition that any two neighboring segments must be of the different colors (see Appendix C). The factor \( \text{Col}_{\text{GOE}} \) reads

\[ \text{Col}_{\text{GOE}} = \text{Col}^{(3)}(r, g, b) \times \text{Col}^{(3)}(R - r, G - g, B - b) ; \]

\[ \text{Col}^{(3)}(a, b, c) \equiv (a + b + c) \frac{2^{-2k} (k - 1)!}{\sum_{k=\max(a,b,c)}^{a+b+c} (k - a)! (k - b)! (k - c)! (a + b + c - 2k)!}. \]

Before exploring the case of GUE, we remind that the “crossed” connections (see example 2 in Fig.7) are not allowed in GUE. The “crossed” connections can be avoided if each set of the vertices \( \{V_{p,q}, V_{q,p}\} \) contains exactly \( k_i \) of \( V_{p,q} \) vertices and \( k_i \) of \( V_{q,p} \) vertices regardless to their distribution over the circles. Note, that this restriction applies only to the total number of \( V_{p,q} \) and \( V_{q,p} \) while the number of the vertices \( V_{p,q} \) and \( V_{q,p} \) in the same circle is not necessarily balanced. The coloring of the two circles is not independent any longer: the difference \( d = \text{Number}[V_{p,q}] - \text{Number}[V_{q,p}] \) of the conjugated vertices must be the same in magnitude (but opposite in sign) for the two circles. Therefore one has to compute the coloring factors for each ring at a fixed difference \( d \) and then to perform a summation of the product of two coloring factors over \( d \) (or, equivalently, over the related
variable $D = \frac{1}{2}(r + b + g - |d|)$. The result reads (see Appendix for the details):

$$Col_{GUE}(r, g, b, R, G, B) \equiv l_1 l_2 \sum_{D=\text{max}(r,g,b)}^{\frac{1}{2}} \Lambda(D) \times$$

$$\left\{ \sum_{p=\text{max}(r,g,b)}^{D} \frac{(l_1 - D - p, D - p, p - r, p - g, p - b)}{p} \times \sum_{q=\text{max}(R-r,G-g,B-b)}^{l_1-l_2+D} \frac{(l_2, l_2 + D - q, q - (R - r), q - (G - g), q - (B - b))}{q} \right\};$$

$$l_1 \equiv r + g + b, \quad l_2 = R - r + G - g + B - b;$$

$$\Lambda(D) = \begin{cases} 2, & \text{if } D < \frac{l_1}{2}; \\ 1, & \text{if } D = \frac{l_1}{2} \text{ and } \frac{l_1}{2} \text{ is integer}. \end{cases}$$

Brackets $(\alpha_1, \alpha_2, \alpha_3, \ldots) \equiv \frac{(\alpha_1 + \alpha_2 + \alpha_3 + \ldots)!}{\alpha_1! \alpha_2! \alpha_3! \ldots}$ denote the multinomial coefficient.

The coloring factor has much simpler form for the line-diagram. For the example shown in Fig.8.b, where $g = r + b; G = R + B$, $Col$ acquires the following form in GOE and GUE:

$$Col|_{g=r+b, G=R+B} = Col_{\text{line}}(r, b) \ Col_{\text{line}}(R - r, B - b);$$

$$Col_{\text{line}}(a, b) \equiv 2^\frac{(a + b)!}{a! b!}.$$

There is no essential difference in the calculation of the Trotter integral for the case of two and three colors. We follow the algorithm explained in Appendix. For three colors, the integrand contains a product of two “infusible” $D_N$ functions. They are converted to the $\delta$-functions and the result of integration with the accuracy of $1/(\tilde{N}\tau)^2$ is

$$I(r, g, b, R, G, B) = 2\pi \frac{\beta}{\sqrt{3}} \frac{1}{(N\tau)^2} \frac{1}{l_1 l_2} \frac{(R - 2)! (G - 2)! (B - 2)!}{(R + G + B - 4)!} \times$$

$$\frac{1}{(r - 1)! (R - r - 1)!} \frac{1}{(g - 1)! (G - g - 1)!} \frac{1}{(b - 1)! (B - b - 1)!};$$

where the sum over all possible arrangements of the first Trotter segment is done.

To derive the universal factor $V(k_1, k_2, k_3)$ in Eq.32, we have to perform summation over $r, g, \ and b$ accounting for the different distributions of $V$-matrices over the traces. We remind that the matrices $\hat{V}$ from the first trace enter the Trotter chain with the minus sign. Therefore, the summand must be multiplied by the factor $(-1)^{l_1}$. Unlike the case of two colors, both the coloring factor and the Trotter integral depend on the
distribution of the colored segments. Thus we arrive at the sum:

\[ V(k_1, k_2, k_3) = \sum_{r=1}^{R-1} \sum_{g=1}^{G-1} \sum_{b=1}^{B-1} \left\{ (-1)^{r+g+b} \mathcal{K}(r, g, b, R, G, B) \times I(r, g, b, R, G, B) \right\}, \]

which turns out to be rather cumbersome. Amazingly, these summations can be done analytically and we finally find the correction \( \tilde{K}_2 \) to the spectral form-factor:

\[ \tilde{K}_2 = \frac{\sqrt{3} \beta}{3} \sum_{k_1, k_2, k_3=0}^{\infty} (-1)^{k_1+k_2+k_3} C_{\beta=1}^{(3)}(k_1, k_2, k_3) R_N(k_1, k_2, k_3) x^{2(k_1+k_2+k_3)-2}; \] (41)

where \( R_N(k_1, k_2, k_3) \) is given by Eq.(33), \( x \equiv N|\tau|b \) and the coefficients are

\[
C_{\beta=1}^{(3)} = -\frac{\Gamma(k_1 + k_2 + k_3)}{\Gamma(k_1 + k_2 + k_3 - 3/2)} \frac{\Xi_{\beta=1}(k_1) \Xi_{\beta=1}(k_2) \Xi_{\beta=1}(k_3)}{\Gamma(k_1 + k_2 + k_3) \Gamma(k_1 + k_3) \Gamma(k_2 + k_3)}; \quad (42)
\]

\[
\Xi_{\beta=1}(k) = \frac{2^k \Gamma(k - 1/2) \Gamma(k + 1/2)}{\sqrt{\pi} \Gamma(k + 1)};
\]

\[
C_{\beta=2}^{(3)} = (2k_1k_2k_3 - k_1k_2 - k_2k_3 - k_1k_3) \frac{\Xi_{\beta=2}(k_1) \Xi_{\beta=2}(k_2) \Xi_{\beta=2}(k_3)}{\Gamma(k_1 + k_2 + k_3 - 3/2)}; \quad (43)
\]

\[
\Xi_{\beta=2}(k) = \frac{\Gamma(k - 1/2)}{\Gamma(k + 1)}.
\]

The correction \( \tilde{K}_2 \) is governed by the interaction of the three energy levels and results from the summation of “triangles” and “lines”: All terms with \( k_{1,2,3} \geq 1 \) correspond to the “triangle” diagrams while the “line” diagrams are obtained by setting \( k_1 = 0 \) or \( k_2 = 0 \) or \( k_3 = 0 \).

The series in r.h.s. of Eq.(41) is three-dimensional and cannot be reduced to a product of one-dimensional series. Therefore an analysis of its behavior at arbitrary \( x \) is not trivial. In the case of GUE, one can represent the function \( \Gamma^{-1}(k_1 + k_2 + k_3 - 3/2) \) as an integral using the identity:

\[
\frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\exp(a + it)}{(a + it)^{z}} dt, \quad a > 0,
\]

and change the order of summations over \( k_i \) and integration over \( t \). The real-space summation which is implied in the function \( R_N \) has to be done at the last step. This is an effective tool for the asymptotic analysis of the series (41,43). The case of GOE is more complicated. Numerical and semi-numerical methods are useful here. We will demonstrate an application of Eqs.(41,43) to the Rosenzweig-Porter model and to a critical almost diagonal PLBRM in the forthcoming paper.
We end the presentation of the method with the consideration of an almost unitary Gaussian RM ensemble: we explore the case of a crossover between GOE and GUE which is close to GUE. To define the almost unitary ensemble we introduce a parameter \( \eta \) into Eq. (34), which controls an asymmetry for the variance of real and imaginary parts of the hopping elements:

\[
\langle V_{s,p} V_{q,r} \rangle = b^2 F(|s - p|) \left[ \delta_{s,r} \delta_{p,q} + \eta \delta_{s,q} \delta_{p,r} \right].
\] (45)

In GOE, \( V_{s,p} \) is a real number and \( \eta_{GOE} = 1 \). In GUE the entries \( V_{s,p} \) are complex numbers, where \( \Re[V_{s,p}] \) and \( \Im[V_{s,p}] \) are statistically independent random variables with equal variances and therefore \( \eta_{GUE} = 0 \). Hence the parameter \( \beta \) is implicitly linked with \( \eta \). An ensemble of RM is almost unitary if \( 0 < \eta \ll 1 \).

A first (trivial) impact of small \( \eta \) on the form-factor arises from the proportionality: \( \tilde{K}_1 \sim \sqrt{\beta} \); \( \tilde{K}_2 \sim \beta \) (see Eqs. (29, 41)), which comes from the Trotter integrals. We will concentrate on the other outcome of a deviation from GUE where the non-zero parameter \( \eta \) leads to an appearance of new series in powers of \( x \). We will derive a series that yields a leading in \( \eta \) correction to \( \tilde{K}_{1,2} \). When one performs the Gaussian averaging of \( V \)-matrices, \( V \)-vertices in the circles can be paired by either “crossed” or “non-crossed” connections (see Fig. 7). Each “crossed” connection is proportional to \( \eta \) in accordance to the definition (15). Therefore, the “crossed” connections are prohibited in GUE (\( \eta = 0 \)). To explore the leading term in \( \eta \ll 1 \) in the almost unitary case we have to select the diagrams with a minimal number \( Cr \) of “crossed” ILs. Our main task is to calculate their combinatorics.

We start with the two colors. Let us consider a given distribution of \( V \)-matrices over the traces at the fixed parameter \( k \) in GUE. There are sets of conjugated vertices: \( k \) vertices \( V_{ij} \) and \( k \) vertices \( V_{ji} \); and there are \( k \) connections without crossing. Each “non-crossed” connection pairs the vertex \( V_{ij} \) with its conjugated counterpart \( V_{ji} \). A diagram with the minimal number of “crossed” ILs is obtained if we choose two “non-crossed” connections and rearrange them into two “crossed” connections, see an example in Fig. 9. The rest \( k - 2 \) ILs remain “non-crossed”. There are \( \left( k(k - 1)/2 \right)^2 \) ways to choose two \( V_{i,j} \) and two \( V_{j,i} \)-vertices for the two “crossed” connections and \( (k - 2)! \) ways to draw the remaining
FIG. 9: Substitution of the “non-crossed” connections by the “crossed” ones: In GUE, only the dotted “non-crossed” lines are allowed; the diagram with the minimal number of the “crossed” lines is obtained if one substitutes two dotted ILs by the solid ones.

“non-crossed” ILs. Therefore the combinatorial factor for the diagram with the minimal number $Cr = 2$ reads:

$$K_{\eta \ll 1}(k) = 4 \times \frac{k(k - 1)}{4} k!,$$

where the factor 4 accounts for two different ways to color each circle by two colors. Substituting Eq. (46) and the Trotter integral (28) with $\beta = 2$ into the two-color series (25), we find the correction:

$$\Delta \tilde{K}_1 = \sum_{k_1} (-1)^{k_1} \frac{k}{(k-2)!} R_N(k) x^{2k-1}; \quad x \equiv N |\tau| b. \quad (47)$$

Each three-colored diagram in GUE has 3 sets of conjugated vertices, which numbers are balanced: Number[$V_{i,l}$] = Number[$V_{l,i}$] = $k_1$, Number[$V_{j,l}$] = Number[$V_{l,j}$] = $k_2$, and Number[$V_{i,j}$] = Number[$V_{j,i}$] = $k_3$.

To generate the diagram with the minimal $Cr = 2$ we can repeat the arguments explained above for the case of the two colors: a pair of “non-crossed” connections can be rearranged into a pair of “crossed” ones successively in the bonds $V_{i,l} \leftrightarrow V_{i,l}$; $V_{j,l} \leftrightarrow V_{l,j}$; and $V_{i,j} \leftrightarrow V_{j,i}$. This procedure results in the combinatorial factor

$$K_{|\eta \ll 1} = Col_{\text{GUE}}\frac{k_1(k_1-1) + k_2(k_2-1) + k_3(k_3-1)}{4} \prod_{i=1,2,3} (k_i)!; \quad (48)$$

with the coloring factor $Col_{\text{GUE}}$ defined by Eq. (38). Therefore the first correction to GUE takes the form:

$$\Delta \tilde{K}_2 = \frac{2 \sqrt{3}}{3} \sum_{k_1, k_2, k_3=0} (-1)^{k_1+k_2+k_3} C^{(3)}_{\eta \ll 1}(k_1, k_2, k_3) R_N(k_1, k_2, k_3) x^{2(k_1+k_2+k_3)-2}; \quad (49)$$
\[
C_{\eta<1}^{(3)} = \frac{k_1(k_1-1) + k_2(k_2-1) + k_3(k_3-1)}{4} \times C_{\beta=2}^{(3)}.
\] (50)

Another way to insert the “crossed” ILs into the three-colored diagram in GUE is to violate the balance of the vertex numbers \(\text{Number}[V_{i,l}] - \text{Number}[V_{i,l}] = d_1\), \(\text{Number}[V_{j,l}] - \text{Number}[V_{j,l}] = d_2\), \(\text{Number}[V_{i,l}] - \text{Number}[V_{i,l}] = d_3\), \(d_{1,2,3} = \pm 1, \pm 2, \ldots\), and to compensate this violation by the “crossed” connections. One can prove that the integer parameters \(d_i\) are always equal to each other: \(d_1 = d_2 = d_3 \equiv d\) (see Appendix C). Therefore the minimal number of the “crossed” connections obtained in this fashion is \(Cr|_{d=\pm 1} = 3\) which is, however, larger than \(Cr|_{d=0} = 2\) in Eqs.(49, 50).

VII. EXAMPLES

In the last section, we give two examples of the application of the general Eqs.(29-31, 41-43, 47, 49-50) for the corrections \(b\tilde{K}_1\) and \(b^2\tilde{K}_2\) to the Poissonian form-factor \(\tilde{K}_P = 1\). The examples are the Rosenzweig–Porter model with the parameter \(b\) depending on the matrix size \(N\) and the power law banded RM with the size-independent parameter \(b\).

A. The Rosenzweig–Porter model

Let us consider the Rosenzweig–Porter model with

\[
b = \frac{B}{N}, \quad B \ll 1; \quad \mathcal{F}(i - j) = 1.
\] (51)

This definition corresponds to an almost diagonal RPRM where a weak perturbation of the diagonal matrix can nevertheless yield a nontrivial level statistics\(^{10}\).

We note first, that the parameter \(x\) does not depend on the matrix size \(N\):

\[
x = \tilde{N}|\tau|b = \frac{\tilde{N}}{N}|\tau|B \equiv T.
\]

The ratio \(\tilde{N}/N\) is a constant of the order of 1. The product \(b^{c-1}\mathcal{R}_N(\{k_i\})\) (\(c\) is the number of colors) also has the finite limit at \(N \to \infty\). For example, in the case of two colors

\[
\lim_{N \to \infty} \left( b\mathcal{R}_N(k) \right) = \lim_{N \to \infty} \left( \frac{B}{N} \frac{N-1}{2} \right) = \frac{B}{2};
\]

and Eq.(29) takes the form:

\[
b\tilde{K}_1|_{N \to \infty} = \sqrt{\pi B} \sum_{k=1}^{\infty} (-1)^k C_\beta^{(2)}(k) T^{2k-1},
\] (52)
with the coefficients \( C^{(2)}_{\beta}(k) \) from Eqs.\( 30,31 \).

Let us consider for simplicity the complex hopping elements, i.e., \( \beta = 2 \). The sum over \( k \) in Eq.\( 52 \) can be easily calculated and we find:

\[
b\tilde{K}_1(T, \beta = 2)|_{N\to\infty} = -\sqrt{2\pi} B T e^{-T^2}.
\]  

(53)

In a similar manner we treat the case of three colors, where

\[
\lim_{N\to\infty}(b^2 \mathcal{R}_N(k_1, k_2, k_3)) = \frac{B^2}{6};
\]

and consequently

\[
b^2 \tilde{K}_2|_{N\to\infty} = \frac{\sqrt{3}}{18} B^2 \sum_{s=2}^{\infty} (-1)^s \, C^{(3)}(s) \, T^{2s-2}. 
\]  

(54)

The coefficient \( C^{(3)}(s) \) is given by the sum:

\[
C^{(3)}(s) = \sum_{k_1,k_2,k_3=0}^{\infty} C^{(3)}_{\beta}(k_1, k_2, k_3) \delta_{s-(k_1+k_2+k_3)} = \sum_{s',k_3=0}^{s} \sum_{s'=1}^{s'} C^{(3)}_{\beta}(s-s', s'-k_3, k_3).
\]

Here we have introduced the sums of two and three indices: \( s = k_1 + k_2 + k_3 \), \( s' = k_2 + k_3 \); and \( C^{(3)}_{\beta} \) is taken from Eqs.\( 42,43 \). In the case of the complex hopping elements, \( \beta = 2 \), all summations over \( k_3, s' \), and \( s \) in Eq.\( 54 \) can be done analytically. The correction to the form-factor takes a simple form:

\[
b^2 \tilde{K}_2(T, \beta = 2)|_{N\to\infty} = -\frac{2\sqrt{3}\pi}{9} \, B^2 \, T^2 \, (2T^2 - 3) \, e^{-T^2}. 
\]  

(55)

The results \( 53 \) and \( 55 \) are in agreement with the expansion in powers of \( B \) of the exact expression for the form factor \( \tilde{K}(T) \) obtained for \( \beta = 2 \) in the paper \[10\].

The case of the real hopping elements, \( \beta = 1 \), can also be analyzed based on the formulae \( 52,54 \) but the calculations are lengthy and we present them in the forthcoming paper.

**B. Critical PLBRM**

Before giving the example of a critical PLBRM, we would like to discuss generic properties of the virial expansion in the case of the \( N \)-independent parameter \( b \) and a converging real space sum Eq.\( 17 \). In this case, all corrections to the form factor \( \tilde{K}_m \) are given by the series in powers of the parameter \( x = \bar{N}|\tau|b \) that diverges in the thermodynamic limit. In the leading in \( 1/N \) approximation in Eq.\( 24 \), where \( \mathcal{R}_N(\{k_i\}) = \mathcal{R}(\{k_i\}) \), the entire
dependence of the spectral form-factor $\tilde{K}(\tau)$ on $\tau$ comes only through the dependence on the parameter $x$. This means that $\tilde{K}(\tau)$ is either $\tau$-independent or divergent (if the series in $\tilde{N}|\tau|b$ determines a function that does not have a finite limit as $\tilde{N}|\tau|b \to \infty$) in the thermodynamic limit.

In order to obtain a dependence on $\tau$ for a finite limiting spectral form factor $\tilde{K}(\tau)|_{N \to \infty}$ one has to account for $1/N$-corrections to the real space sum $\mathcal{R}(\{k_i\})$ in Eq. (24). Rewriting $1/N \sim b|\tau|/(\tilde{N}|\tau|b)$ and absorbing the factor $(\tilde{N}|\tau|b)$ from the denominator into the infinite series, we obtain the correction of order $b\tau$ to the limiting spectral form factor:

$$\tilde{K}(\tau)|_{N \to \infty} \approx \chi_0 + \chi_1 b|\tau| + \ldots$$

where $\chi_0(b) = 1 + c_{01}b + c_{02}b^2 + \ldots$ is the unfolded level compressibility (see Eq. (7)); $\chi_1(b) = c_{11}b + c_{12}b^2 + \ldots$, and $c_{ij}$ are numerical coefficients. Thus the $1/N$ correction to $\mathcal{R}(\{k_i\})$ determines $\chi_1$ and leads to the $b^2|\tau|$ correction in the limiting form factor. For $b \ll 1$ this correction is small even at the Heisenberg time $\tau \sim 1$. However, it could be important for the tail of the two-level correlation function Eq. (2) at large energy separations $\omega$ because it is non-analytic in $\tau$.

Now, we illustrate an application of the method developed using an example of a critical almost diagonal PLBRM $^{14,16}$:

$$b = B \ll 1; \quad \mathcal{F}(|i-j|) \equiv \frac{1}{2} \left( \frac{1}{(i-j)^2} \right), \quad i \neq j. \quad (56)$$

We restrict ourselves to the case $B\tau \ll 1$ omitting $1/N$-corrections to the real-space sum. The leading term of $\mathcal{R}_N(k)$ takes the following form:

$$\mathcal{R}(k) = \frac{1}{2^k} \lim_{N \to \infty} \sum_{m=1}^{N} \frac{1}{m^{2k}}. \quad (57)$$

Let us substitute expression (57) into Eq. (29) and explore an asymptotic behavior $x = \tilde{N}|\tau|B \to \infty$ of the series (29,30) and (29,31) for GOE and GUE respectively. The simplest route is to change the order of the summations:

$$\tilde{K}_1|_{B\tau \ll 1} \simeq \frac{2\sqrt{3\pi}}{N|\tau|} \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} (-1)^k C_{\beta}^{(2)}(k) \left( \frac{x}{m \sqrt{2}} \right)^{2k}. \quad (58)$$

The summation over $k$ yields:

$$\tilde{K}_1 \simeq -\frac{\sqrt{3\pi}}{N|\tau|} \sum_{m=1}^{\infty} \frac{x^2}{m^2} \exp \left( -\frac{x^2}{2m^2} \right) \left[ I_0 \left( \frac{x^2}{2m^2} \right) - I_1 \left( \frac{x^2}{2m^2} \right) \right], \quad \beta = 1; \quad (59)$$

where $I_0$ and $I_1$ are modified Bessel functions of the first kind.
Here $I_{0,1}(\cdots)$ are the Bessel functions. The sum over $m$ converges at $m \sim x \gg 1$ therefore it can be converted to the integral over $m$. After this integration we find the two-colors correction to the level compressibility $\chi_0 \simeq 1 + B \times c_{01}$:

$$c_{01}|_{\beta=1} = -2; \quad c_{01}|_{\beta=2} = -\pi.$$ (60)

Note, that Eq.(60) can be reproduced with the help of a heuristic renormalization–group method (compare with [16]).

Using our method it is also easy to derive from Eq.(47) the correction in the almost unitary case:

$$\Delta c_{01}|_{\eta \ll 1} = \eta^2 \frac{\pi}{16}.$$

We emphasize that it is the behavior of $F(|i - j|)$ at large $|i - j|$ that affects the factor $R(k)$ and governs the asymptotic behavior of the corresponding series. Depending on it, the two-color correction may either approach a constant limit or diverge as $N \to \infty$. One can show\cite{30} that: 1) for $F(|i - j|)$ decreasing faster than $1/|i - j|^2$ the two-color correction vanishes in the limit $N \to \infty$; 2) for $F(|i - j|)$ decreasing slower than $1/|i - j|^2$ it is divergent. This is the manifestation of the localization-delocalization transition in the spectral statistics. Note that even the case, where the series in r.h.s. of Eq.(29) does not have a finite limit as $N \to \infty$, can be considered by our formalism as long as the correction $\tilde{K}_1(\tau)$ is small at a finite $N$.

A remarkable feature of the critical PLBRM with $F \sim 1/|i - j|^2$ in the three-color approximation is the logarithmic divergence $B^2 \log^2(N|\tau|B)$ of the line- and the triangle-diagrams (see Fig.8) at any $\beta$. However, in the sum of the line- and the triangle-diagrams, the logarithmically divergent terms of order $B^2$ cancel out for both GOE and GUE so that $\lim_{x \to \infty}(\tilde{K}_1(\beta = 1, 2))$ is a finite constant of order 1. The sub-leading diverging term survives strikingly in the three colors correction of the almost unitary ensemble: $B^2 \Delta K_2 \sim -B^2 \eta^2 \log(N|\tau|B)$. This indicates a failure of the virial expansion for the critical PLBRM as $N \to \infty$ in the crossover between the unitary and orthogonal ensembles and raises a question on the cancellation of the logarithmically divergent corrections of higher orders in $b$ in a pure GOE or GUE. This question deserves a separate detailed study elsewhere.
VIII. CONCLUSIONS

In this paper, we have developed a new method to study the spectral statistics of Hermitian Gaussian random matrices with parametrically small hopping elements $H_{i,j}$ as compared to the diagonal ones: $\langle |H_{i\neq j}|^2 \rangle / \langle \epsilon_k^2 \rangle \sim b^2 \ll 1$. We have derived a regular virial expansion of the spectral form-factor in powers of $b$, where the virial coefficient in front of $b^A$ is given by interaction of $A + 1$ levels.

The expansion is represented by diagrams which are generated with the help of the Trotter formula. We have established the rigorous selection rule for the diagrams, which allows to account for exact contributions of a given number of resonant and non-resonant interacting levels. Thus the method offers a controllable way to find an answer to the question when a weak interaction of levels can drive the system from localization toward criticality and delocalization.

The method applies to the spectral properties of the random matrices with uncorrelated entries and with a generic dependence of the variance $\langle |H_{i,j}|^2 \rangle \equiv b^2 F(|i-j|)$ on the distance $|i-j|$ from the main diagonal. We calculated the corrections governed by interaction of two, Eqs.(29-31), and three, Eqs.(41-43), levels in the cases of GOE, GUE and in the almost unitary ensemble, Eqs.(47,49-50), for an arbitrary function $F(|i-j|)$. These equations are the main results of the paper. We have demonstrated their application to an almost diagonal Rosenzweig–Porter model and to an almost diagonal power law banded random matrices.

The calculation of the virial coefficients is based on the solution to the combinatorial problem of the graphs coloring. The solution for the coloring of a single graph is known in a closed form for an arbitrary number of colors. Thus using our method one can, in principle, compute the virial coefficient of an arbitrary order.

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In this Appendix, we derive an approximate relation between the level compressibility and the correlation function $R$. Let us define the two-level correlation function $R_E(\omega)$ centered at an energy $E$:

$$R_E(\omega) = \frac{\langle \sum_{n,m} \delta(E - \epsilon_n) \delta(E - \omega - \epsilon_m) \rangle}{\langle \rho(E) \rangle^2} \tag{A1}$$

In this definition it is assumed that $\omega$ is small compared to the total bandwidth, so that $\langle \rho(E) \rangle \approx \langle \rho(E - \omega) \rangle$.

We set apart the diagonal (singular) part in the double sum in numerator of (A1):

$$R_E(\omega) = R^{(r)}_{E}(\omega) + \frac{\delta(\omega)}{\langle \rho(E) \rangle} \tag{A2}$$

Then the normalization by $\langle \rho(E) \rangle^2$ ensures that the regular part $R^{(r)}_{E}(\omega)$ depends very weakly on the energy $E$, except for regions of the width $\sim \omega$ near the band edges.

The level number variance can be expressed in terms of the correlation function $R(\omega)$ [24]:

$$\Sigma_2(\bar{n}) = \int_{-\infty}^{+\infty} dE dE' \left( \langle \rho(E) \rangle^2 R_E(E - E') \right) \tag{A3}$$

Substituting decomposition (A2) into Eq.(A3), one finds

$$\Sigma_2(\bar{n}) \simeq \bar{n} + \int_{-\bar{n}\Delta}^{\bar{n}\Delta} \frac{\Delta - |\omega|}{\Delta} R^{(r)}(\omega) \frac{d\omega}{\Delta}; \quad R^{(r)}(\omega) \equiv R^{(r)}_{E=0}(\omega) \tag{A4}$$

This expression is valid up to the leading order $O(b)$ with respect to $b \ll 1$.

The definition (5) for the level compressibility yields now the following expression:

$$\chi = 1 + \lim_{\bar{n} \to \infty} \int_{-\bar{n}}^{\bar{n}} ds \lim_{N \to \infty} R^{(r)}(s\Delta) \tag{A5}$$

Using definition (A1), formula (A2) and the identity

$$\int_{-\infty}^{+\infty} dE \int_{-\infty}^{+\infty} d\omega \langle \rho(E) \rho(E - \omega) \rangle e^{i\omega t} \equiv \langle \langle \text{Tr} e^{i\hat{H}t} \text{Tr} e^{-i\hat{H}t} \rangle \rangle$$

we derive a useful relation:

$$\frac{1}{N} \langle \langle \text{Tr} e^{i\hat{H}t} \text{Tr} e^{-i\hat{H}t} \rangle \rangle \simeq 1 + \Upsilon \int_{-\infty}^{+\infty} ds e^{ist} R^{(r)}(s\Delta) \tag{A6}$$

which remains valid also in the limit $N \to \infty$.

At $t = t_0 \equiv 1/(\bar{n}\Delta)$, large energy scale $\omega \equiv s\Delta \gg \bar{n}\Delta$ does not essentially contribute to the integral in Eq.(A6) due to the strong oscillations of the integrand. Hence, we have:

$$\lim_{\bar{n} \to \infty} \int_{-\bar{n}}^{\bar{n}} ds \lim_{N \to \infty} R^{(r)}(s\Delta) = \lim_{\tau_0 \to 0} \int_{-\infty}^{+\infty} ds e^{ist_0} \lim_{N \to \infty} R^{(r)}(s\Delta); \quad \tau_0 \equiv t_0\Delta,$$

and using Eqs.(A5,A6) arrive at the expression (7).
In general, the Trotter integral depends on the number of colors and on the distribution of matrices $\hat{V}$ over the traces. The former defines the number of ‘infusible’ functions $D_N$ in the integrand while the number of the Trotter variables is determined by the latter. In this Appendix, we calculate the integral in the case of 2 colors at an arbitrary number of the colored segments in the circles determined by the parameter $m$. The integrand in Eq.(19) (see the Section III B2) contains two additive parts reflecting two different possibilities to color the first segments. Unlike that example, we will use another procedure with a simplification that we will not pay an attention to a specific color of the first segments. The final result of the calculation is not affected by this simplification.

Let us apply the Trotter formula to both traces in r.h.s. of Eq.(6) involving two infinitely long Trotter chains of the length $n_{1,2} \to \infty$ and let the number of the colored segments in the circles be $2m$ and $2(k - m)$. We remind: correct coloring by 2 colors imposes a restriction that a circle must contain equal number of sectors for each color. Therefore, the total number of the segments for a given color is $k$ with $m$ segments belonging to the first circle and $k - m$ segments in the second one.

Denote the discrete Trotter variables coming from the first trace as $y_1, y_2, \ldots, y_{2m}$ so that the variables $y_1, y_2, \ldots, y_m$ mark the segments of the first color while the rest of variables are attributed to the segments of the second color. The variables from the second trace are denoted as $z_1, z_2, \ldots, z_{k-m}$ for the first color and $z_{k-m+1}, z_{k-m+2}, \ldots, z_{2(k-m)}$ for the second one. Since two neighboring subsegments separated by the origin of the Trotter chain (instead of the $V$-vertex) are of the same color we fuse them and mark by the single Trotter variable.

We rescale the variable introducing the continuous ones: $Y_i = y_i/n_1, Z_i = z_i/n_2$. They are normalized as (see Eq.(15))

$$\sum_{i=1}^{2m} Y_i = \sum_{i=1}^{2(k-m)} Z_i = 1; \quad Y_i \geq 0, \quad Z_i \geq 0.$$  

(B1)

The integrand must be multiplied by the factor:

$$\frac{n_1}{2m} \cdot \frac{n_2}{2(k - m)}.$$  

(B2)

The numerator reflects an invariance with respect to a cyclic permutation of the matrices under the traces while the denominator excludes multiple counting of the equal configurations.
of $V$-vertices in the circle.

We follow further the algorithm explained in the Section III B 2 and, accounting for the property (B1) and fusing two $D_N$ functions as it is done in Eq.(19), the Trotter integral can be presented as:

$$
I(k, m) = \frac{1}{2m} \frac{1}{2(k - m)} \int_0^1 \prod_{i=1}^{2m-1} dY_i \int_0^1 \prod_{j=1}^{2(k-m)-1} dZ_j \times
\theta \left( 1 - \sum_{i=1}^{2m-1} Y_i \right) \theta \left( 1 - \sum_{j=1}^{2(k-m)-1} Z_j \right) D_N \left( \sqrt{2} \sum_{i=1}^{m} Y_i - \sqrt{2} \sum_{j=1}^{k-m} Z_j \right). \tag{B3}
$$

Having integrated over all variables but $\tilde{Y} = \sum_{i=1}^{m} Y_i; \tilde{Z} = \sum_{j=1}^{k-m} Z_j$, we arrive at

$$
I(k, m) = \frac{1}{4} \int_0^1 \tilde{Y}^{m-1} (1 - \tilde{Y})^{m-1} \frac{d\tilde{Y}}{m!(m-1)!} \int_0^1 \tilde{Z}^{k-m-1} (1 - \tilde{Z})^{k-m-1} \frac{d\tilde{Z}}{(k-m)!(k-m-1)!} D_N(\sqrt{2}[\tilde{Y} - \tilde{Z}]) \tag{B4}
$$

$$
\simeq \sqrt{\beta \pi} \frac{1}{N|\tau|} \frac{1}{m!(m-1)!(k-m)!(k-m-1)!} \frac{(k-2)!}{2^k \times (2k-3)!!}. \tag{B5}
$$

At the last step, we sum over the parameter $m$ to account for different distributions of $V$-matrices over the traces:

$$
I(k) = \sum_{m=1}^{k-1} I(k, m) = \sqrt{\beta \pi} \frac{1}{N|\tau|} \frac{1}{k!(k-1)!}. \tag{B6}
$$

APPENDIX C: COMBINATORIAL COEFFICIENT OF COLORING

To calculate the universal factor $V\{k_i\}$ at any number of colors $c$ we need a combinatorial factor of the circles coloring: $V$-vertices divide the circles into the segments. The colors, which denote the statistically independent energy levels $\varepsilon_i$, should be distributed over the segments by all possible ways under the condition that all adjacent segments must bear different colors.

Let us start with the coloring of a single circle. The question of our interest was formulated in Appendix of paper [33] as follows: How many permutations of $L$ objects of $c$ different colors (given $g_1$ objects of the first color, $g_2$ objects of the second color, etc.) are there under the condition that no objects of the same color may stand next to each other? We have to supplement it with two additions: 1) we do not distinguish the segments of the same color; 2) we can do cyclic permutations of the segments in the circle. Thereafter, the generic coloring factor can be obtained easily from Eq.(30) of paper [33]:

$$
P_{g_1, g_2, \ldots, g_c}(c) = L (-1)^{L-c} \sum_{n=0}^{\infty} (-1)^n \frac{\partial^{c-1+L}}{\partial x^{c-1+L}} \left[ \frac{1}{x} \prod_{i=1}^{c} h_{g_i}(x) \right] \bigg|_{x=0}; \tag{C1}
$$
FIG. 10: a) An unrolled circle with \( r = 2 \) and with two clusters of green and blue segments; b) A cluster of odd length; c) A clusters of even length. Pairs of bold letters denote switching of colors at the \( \mathcal{V} \)-vertices.

\[
\mathcal{h}_g(x) = \sum_{s=1}^{g} \binom{g-1}{g-s} \frac{x^s}{s!}.
\]  

(C2)

Eqs. (C1)-(C2) describe any situation with the arbitrary number of colors and the arbitrary number of the segments. Unfortunately, these expressions are cumbersome and are not convenient for a further calculation of the factor \( \mathcal{V} \). In this Appendix, we will derive a simpler formula for the particular case of three colors, \( c = 3 \).

We use the notations from Section V labeling the numbers of the colored segments by \( r = g_1, g = g_2, \) and \( b = g_3 \). The first, the second and the third colors will be denoted as “red”, “green” and “blue”. Let a color of the first segment be, for example, red. There are \( r \) red segments which separate out \( r \) clusters. The clusters contain green and blue segments, which must be arranged in an alternating fashion. A number of the segments in the cluster will be referred as a “length of the cluster”. The are different types of the clusters (see Fig. 10): 1) The clusters with an even length that enclose pairs of sectors; 2) The clusters with an odd length that enclose the pairs as well as one extra segment (either green or blue). We denote a number of these clusters accordingly: \( n_0, n_g, \) and \( n_b \). The parameters \( n_{0,g,b} \) are connected to the numbers of the segments:

\[
n_g + n_0 + n_b = r; \quad g - b = n_g - n_b.
\]  

(C3)

First we have to distribute the clusters of three types over \( r \) positions. The number of the distributions is given by the multinomial coefficient:

\[
(n_0, n_g, n_b) \equiv \frac{r!}{n_0! n_g! n_b!}.
\]  

(C4)
Given that each of \( n_0 \) clusters of even length contains at least one pair we distribute \( b - n_b - n_0 = g - n_g - n_0 \) pairs of the segments over \( r \) clusters. The corresponding combinatorial factor is:

\[
2^{n_0} \left( \frac{r + (b - n_b - n_0)}{r - 1} - 1 \right),
\]

where \( 2^{n_0} \) reflects the possibility to start the clusters of even length with either a green segment or with a blue one.

The coloring factor \( \text{Col}_r(r, g, b) \) for one circle with the red first segment is obtained after summation of the product of Eq.(C4) and Eq.(C5) over the independent number of the clusters, for example, over the parameter \( n_b \) in the range \( 0 \leq n_b \leq \left\lfloor \frac{r+b-g}{2} \right\rfloor \). To find the second \( \text{Col}_g(r, g, b) \) and the third \( \text{Col}_b(r, g, b) \) parts, we have to repeat the same procedure assigning green and blue colors to the first segment of the circle. Eq.(37) has been obtained after the summation of all parts.

\[
\text{Col}^{(3)}(r, g, b) = \text{Col}_r(r, g, b) + \text{Col}_g(r, g, b) + \text{Col}_b(r, g, b).
\]

Eq.(38), that is used to calculate the universal factor \( \mathcal{V}(k_1, k_2, k_3) \) at \( c = 3 \) in GUE, has been obtained under an additional condition. Namely, to exclude the “crossed” connections of the \( V \)-vertices each set \( \{V_{p,q}, V_{q,p}\} \) must contain an equal number of the “conjugated” vertices: \( \text{Number}[V_{p,q}] = \text{Number}[V_{q,p}] \) (see Section V). This restriction applies only to the total number of \( V_{p,q} \) and \( V_{q,p} \) while the number of the conjugated vertices in the same circle is not necessarily balanced. Therefore, coloring of the two circles is not independent.

Let us consider the balance between \( V_{p,q} \) and \( V_{q,p} \) in more detail. The conjugated vertices inside a cluster with an odd length and on its boundary are always balanced (see an example in Fig.10b). Hence, the clusters of odd length cannot violate the total balance. It is not true for a cluster of an even length: the imbalance between the conjugated vertices is \( \pm 1 \) for all three sets: \( \{V_{i,l}, V_{l,i}\}; \{V_{j,l}, V_{l,j}\}; \text{ and } \{V_{i,j}, V_{j,i}\} \) (see an example in Fig.10c). The sign of the imbalance depends on a color of the first segment of the given cluster.

Denote the number of clusters of an even length, which start with green and blue segments, by \( n_0^{(-)} \) and \( n_0^{(+)} = n_0 - n_0^{(-)} \) respectively and introduce a difference \( d = n_0^{(+) - n_0^{(-)} = n_0 - 2n_0^{(-)} \) which will be referred to as “the number of defects”. It gives the number of imbalanced conjugated vertices of each set in a given circle. Below, the upper indices in \( d^{(1)} \) and \( d^{(2)} \) will label the number of defects in the first and in the second circles correspondingly.
Since each cluster of even length produces the same imbalance between $V_{p,q}$ and $V_{q,p}$ in all sets of the vertices, the number of the conjugated vertices is balanced if
\[ d^{(1)} + d^{(2)} = 0. \] (C6)
Equality (C6) must hold true at any combination of the colors of the first segments and expresses the difference of the coloring in GOE and GUE. Eq. (37), which is used in GOE, results from an independent summation over the number of defects $d_1$ and $d_2$ in each circle. In GUE, we calculate the coloring factor $\tilde{Col}_r$ of one circle at the fixed $d$ and perform the summation over the number of defects obeying Eq. (C6):
\[
\sum_{d^{(1)}} \sum_{d^{(2)}} \left\{ \tilde{Col}_r(r, g, b, d^{(1)}) \times \tilde{Col}_r(R - r, G - g, B - b, d^{(2)}) \times \delta_{d^{(1)}, -d^{(2)}} \right\}.
\]
The summand is the product of the coloring factors for the two circles.

To find the expression for $\tilde{Col}_r$ we have to sum the product of Eqs. (C4) and the factor similar to Eq. (C5):
\[
\binom{n_0}{n^{(-)}} \times \binom{r + [b - n_b] - n_0 - 1}{r - 1};
\]
over the parameter $n_b$. The factor $\binom{n_0}{n^{(-)} \quad with \quad n^{(-)} = \frac{n_0 - d}{2} = 0, 1, \ldots \quad accounts for the different distributions of the clusters of the even length starting with green and blue colors. Eq. (38) results from the summation over all combinations of the colors of the first segments.

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1 I.L. Aleiner, P.W. Brouwer, L.I. Glazman, Phys. Reports, 358, 309 (2002).
2 D.R. Grempel, R.E. Prange and S. Fishman, Phys. Rev. A 29, 1639 (1984).
3 E.P. Wigner, Annals of Mathematics 67, 325 (1958). F.J. Dyson, Journal of Mathematical Physics 3, 1191 (1962).
4 M.L. Mehta, Random matrices, San Diego, CA, Academic Press (1991).
5 T. Guhr, A. Muller-Groeling, H.A. Weidenmuller, Physics Reports 299, 189 (1998).
6 M.G. Vavilov, I.L. Aleiner, Phys. Rev. B 64, 085115 (2001); Phys. Rev. B 60, R16311 (1999).
7 M.G. Vavilov, V. Ambegaokar, I.L. Aleiner, Phys. Rev. B 63, 195313 (2001).
8 Y.V. Fyodorov and A.D. Mirlin, Phys. Rev. Lett. 67, 2405 (1991).
9 N. Rosenzweig and C.E. Porter, Phys. Rev. 120, 1698 (1960).
10 H. Kunz and B. Shapiro, Phys. Rev. E 58 400 (1998).
11 A. Pandey, Chaos Solitons and Fractals 5, 1275 (1995). E. Brezin and S. Hikami, Nucl. Phys. B 479, 697 (1996).
12 T. Guhr, Phys. Rev. Lett. 76, 2258 (1996).
13 N. Datta and H. Kunz, cond-mat/0006488.
14 A.D. Mirlin, Y.V. Fyodorov, F.M. Dittes, J. Quezada, and T.H. Seligman, Phys. Rev. E 54, 3221 (1996).
15 V.E. Kravtsov, K.A. Muttalib, Phys. Rev. Lett. 79, 1913 (1997).
16 F. Evers and A.D. Mirlin, Phys. Rev. Lett. 84 3690 (2000); Phys. Rev. B 62, 7920 (2000).
17 V.E. Kravtsov, A.M. Tsvelik, Phys. Rev. B 62, 9888 (2000).
18 B.L. Altshuler and L.S. Levitov, Phys. Rep. 288, 487 (1997).
19 L.S. Levitov, Phys. Rev. Lett. 64, 547 (1990), Annalen der Physik 8, 697 (1999).
20 B.B. Hu, B.W. Li, J. Liu, Y. Gu, Phys. Rev. Lett. 82, 4224 (1999).
21 K. Efetov, Supersymmetry in disorder and chaos, Cambridge, University Press (1997).
22 I. Varga, D. Braun, Phys. Rev. B 61, 11859 (2000). E. Cuevas, V. Gasparian, M. Ortuno, Phys. Rev. Lett. 87, 056601 (2001).
23 J.M. Ziman, Models of disorder: the theoretical physics of homogeneously disordered systems, Cambridge, University Press (1979).
24 J.T. Chalker, V.E. Kravtsov, I.V. Lerner, JETP Lett. 64, 386 (1996).
25 M.L. Ndwana and V.E. Kravtsov, Journal of Physics A 36 (2003) 3639.
26 M. Reed, B. Simon, Methods of modern mathematical physics, New York, Academic Press, (1980).
27 The Expression \( \exp \left( -i \frac{\tilde{N}_x}{n} \hat{H}_\varepsilon \right) \) can be compared with a locator in the locator expansion while \( \exp \left( -i \frac{\tilde{N}_x}{n} \hat{V} \right) \) produces interactors
28 This is similar to a derivation of path integrals technique with the help of the Trotter formula, see, for instance, H. Kleinert, Path integrals in quantum mechanics, statistics, and polymer physics, World Scientific, 1995.
29 B. Bollobas, Modern graph theory, New York, NY, Springer Verlag, 1998.
30 V. Kravtsov and O. Yevtushenko, in progress.

31 I.S. Gradshtejn, I.M. Ryzhik, A. Jeffrey, D. Zwillinger, *Table of integrals, series, and products*, San Diego, CA, Academic Press (2000).

32 V.I. Falko, K.B. Efetov, Phys. Rev. B 50, 11267 (1994).

33 G. Berkolaiko G, J.P. Keating, Journ. of Phys. A 32, 7827 (1999).

34 Formula (37) was obtained with the help of Holger Schanz.