RECENT DEVELOPMENTS 
in NONPERTURBATIVE QUANTUM GRAVITY

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Abstract

New results from the new variables/loop representation program of nonperturbative quantum gravity are presented, with a focus on results of Ashtekar, Rovelli and the author which greatly clarify the physical interpretation of the quantum states in the loop representation. These include:

1) The construction of a class of states which approximate smooth metrics for length measurements on scales, $L$, to order $l_{\text{Planck}}/L$. 2) The discovery that any such state must have discrete structure at the Planck length. 3) The construction of operators for the area of arbitrary surfaces and volumes of arbitrary regions and the discovery that these operators are finite. 4) The diagonalization of these operators and the demonstration that the spectra are discrete, so that in quantum gravity areas and volumes are quantized in Planck units. 5) The construction of finite diffeomorphism invariant operators that measure geometrical quantities such as the volume of the universe and the areas of minimal surfaces.

These results are made possible by the use of new techniques for the regularization of operator products that respect diffeomorphism invariance.

Several new results in the classical theory are also reviewed including the solution of the hamiltonian and diffeomorphism constraints in closed form of Capovilla, Dell and Jacobson and a new form of the action that induces Chern-Simon theory on the boundaries of spacetime. A new classical discretization of the Einstein equations is also presented.
Contents

1 Introduction 3
  1.1 Why is quantum gravity such a hard problem? ............... 3

2 A picture of quantum geometry 8
  2.1 Preliminary remarks about nonperturbative quantum geometry . 8
  2.2 Quantum states as functions of loops .......................... 10
  2.3 Quantum geometry at the kinematical level .................... 11
  2.4 Quantum geometry at the diffeomorphism invariant level ...... 18
  2.5 Physical states .............................................. 21

3 Basics of the loop representation and the Ashtekar variables 22
  3.1 The loop representation ....................................... 22
  3.2 The Ashtekar variables ......................................... 28

4 Results about quantum geometry 32
  4.1 Why the metric at a point is not a good operator ............ 32
  4.2 More about representations of quantum theories with discrete norms ................................................. 38
  4.3 The construction of the area operator ......................... 43
  4.4 The action of operators on intersecting loop states and the extension of the area operator to the same ................................................. 48
  4.5 The construction of the volume operator ....................... 50
  4.6 The construction of the operator $Q[\omega]$ .................. 55
  4.7 More about weaves and the semiclassical equivalence of quantum states to classical geometries .................... 56

5 Some recent developments in the classical theory 59
  5.1 How to solve the classical constraint equations exactly .... 60
  5.2 New lagrangians and a connection with Chern-Simon theory .. 61
  5.3 Distributional frame fields as the classical analogue of quantum geometry ................................................. 62
  5.4 A new classical discretization of the Einstein equations .... 64

6 Three open questions 66
  6.1 Solutions to the Hamiltonian constraint ....................... 66
  6.2 The problem of the choice of the inner product ............... 69
  6.3 The question of the existence of a ground state ............... 72

7 Conclusion 76
1 Introduction

This review is devoted to the new variables/loop representation approach to quantum gravity which has been under development for the last five years. It particularly focuses on several developments of 1991 which have expanded greatly our understanding of several related issues: What is the structure of space and time on the Planck scale? How is the classical limit to be understood from a nonperturbative point of view? How are operators representing observables to be regulated and constructed nonperturbatively without breaking the diffeomorphism invariance of the theory?

The central result I review here is the construction of a class of nonperturbative quantum states which approximate classical metrics on scales, \( L \), large compared to the Planck length, but have discrete structure when probed on the Planck length[1, 2]. This is closely related to the discovery that several operators that characterize the spatial geometry, such as the area of an arbitrary surface and the volume of an arbitrary region, are finite when constructed nonperturbatively. Furthermore, their spectra are quantized in multiples of the natural Planck units.

This review is organized into seven chapters. In the remainder of this introduction I attempt to set the scene with a general discussion of the problem of quantum gravity. The second chapter is devoted to a description of quantum geometry that follows from the new results; this is done with a minimum of formalities and no calculations. The loop representation and Ashtekar variables, which underlie this picture are introduced in chapter three and chapter four contains the details of the calculations that support the picture given in chapter two. Chapter five reviews some recent developments in classical relativity which complement and illuminate the new picture of quantum geometry. Three key open issues are discussed in chapter six, after which the review closes with a brief conclusion.

1.1 Why is quantum gravity such a hard problem?

As physicists, why should we be interested in the problem of quantum gravity? It is a problem that is, if we are honest with ourselves, far removed from the frontiers of experimental and observational science, and it is also a problem that, quite conceivably, will not be solved during any of our lifetimes. Yet, in spite of a rate of progress slower than that of even the search for a cure for cancer, more than a few of us continue to devote most of our energies to it. Why do we do it and what are we hoping to achieve?

The main reason, of course, is that the present situation in theoretical physics is untenable. We have two theories that work very well in two different observational and experimental regimes[3]. Somehow, in the world around us, general relativity and quantum mechanics coexist; indeed, more than that, they must be different aspects of a single theory. The problem of quantum gravity is to
discover that theory.

The problem is that, as there are no obvious experimental clues as to the shape of that theory\(^2\), it is anyone’s guess which postulates and formal structures from the two theories will survive the unification and which should be discarded. As a result there has been, in the last twenty years, a great proliferation of approaches to the problem. Here is a (partial) list: supergravity, higher derivative theories, nonpolynomial lagrangians, large \(N\) expansions, quantum field theory in curved spacetimes, lattice quantum gravity, induced gravity, twistor theory, spacetime codes, asymptotic safety, Kaluza-Klein theories, Euclidean quantum cosmology, nonperturbative hamiltonian methods, nonperturbative Monte Carlo calculations, wormholes, decoherence, perturbative string theory, nonperturbative closed string theory, matrix models,... Certainly, we have learned something interesting from most of these about how to go about trying to construct a quantum theory of gravity. I would not want to argue with anyone who enthusiastically advocates any particular direction. I have my own directions and I believe that the healthiest thing for the field as a whole is to let a thousand theories bloom so that we can all learn from the success and failures of the different directions. But it is rather sobering, in my opinion, to realize that only one of these has, so far, led to a firm prediction about nature. This is the black hole evaporation of Hawking\(^4\), discovered in 1974. It is further, at least to me, worrying, that since 1974 we have come no closer to uncovering the connection between quantum mechanics, relativity and thermodynamics, at whose existence black hole thermodynamics is believed to be hinting\(^5\).

In this situation, perhaps it is not inappropriate to reflect a little on what we are trying to do when we work on an approach to quantum gravity. What follows are some remarks in this direction.

Thomas Kuhn, in his irritatingly influential\(^3\) "The Structure of Scientific Revolutions"\(^6\) draws a distinction between what he calls "normal science" and "revolutionary science". Normal science is supposed to be what we are doing when we know and accept the basic principles and methods of a subject (which make up what he calls the "paradigm"); revolutionary science is what happens when those basic foundations change. It is then commonplace to say that the first third of this century—the period of the development of relativity and quantum theory—was a "revolutionary" period for physics, and that since then we have been mostly engaged in "normal" science. As exciting as the rest of the twentieth century has been for physics, even in potentially revolutionary fields such as particle physics and cosmology, over the last fifty years almost all work has accepted as given the basic principles of relativity and quantum mechanics.

Against this background quantum gravity is an anomaly; for this field is nothing if it is not "revolutionary" science. From this point of view one can,

\(^2\)I say obvious, because it surely will be the case that when we have the right theory we will realize that there were clues all around us.

\(^3\)At least in English speaking academic circles.
perhaps, see a basis for the slow rate of progress: to work on quantum gravity now may be a mistimed attempt to do "revolutionary science" during a period of "normal science."

I believe, however, that this point of view results from too narrow of an historical focus. To get some perspective, let's look at the last great revolutionary period in science: the transition from Aristotelian to Newtonian science that is usually called, for short, the Copernican revolution. Two things are most impressive about this period, given the modern view of scientific revolutions: first, how long it took and second, how clueless all of the major participants were as to the shape of the theory and world view that were to be the outcomes of the revolution[7]. The Copernican revolution was only begun by the publication of *De Revolutionibus* in 1543, in the early 1600’s Kepler and Galileo made great contributions, but the synthesis of Newton did not appear until 1687. And, as great as they were, neither Copernicus, Kepler nor Galileo believed any of the following basic tenants of the Newtonian world view: that the universe is infinite, that the sun is a star, that there are laws of physics which govern both motion on Earth and in the heavens and, moreover, that these laws are deterministic. Indeed, Kepler and Galileo did not, apparently, feel themselves very much threatened when Bruno, who advocated the first two of these ideas, was burned at the stake in the Campo di Fiori in Rome in 1600[8].

This, perhaps, is the reason for the long period of transition. The change from the Aristotelian world view to the Newtonian world view was perhaps too much, even for geniuses such as Copernicus, Galileo or Kepler. At the same time, some of the basic features of the new theory could only be glimpsed by a certifiable mystic such as Bruno. Thus, the change had to occur over several generations during which the groundwork for the overthrow of Aristotle was laid by people who were, in their outlooks and expectations, mainly Aristotelian themselves. Only after that could Descartes propose a radically new world view and only after that could a social outcast in a scientifically out of the way place-self taught from his readings of Descartes- carry out the final steps of the revolution.

With this example in mind, let me suggest that, rather than think of ourselves as living in a period following a great scientific revolution, we should think of ourselves as living in the middle period of a revolution that is taking a long time for the same reason the Copernican revolution took so long. We are, perhaps, in the position of Galileo and Kepler: we are sure that quantum mechanics and relativity are more right than Newtonian physics, but all of our expectations about what physics is and how it works are at heart Newtonian. This is as it must be, because quantum mechanics and relativity have, neither separately or together, given us a world view or a view of science that is coherent and complete enough to replace the Newtonian views. For exactly this reason we are convinced of the need to replace the current situation of having two, apparently incompatible, but allegedly fundamental, theories with one synthesis. This indeed, is the problem of quantum gravity. But we are having a hard time
doing this and I, at least, suspect that the reason is that the final synthesis, when it comes, will be as far from our expectations as to what a physical theory should be as Newtonian mechanics was to the expectations of Copernicus, Kepler and Galileo.

In this situation, what can we usefully do? I think, first of all, that we should lower a little our expectations. We should stop trying every five years to invent a candidate for the final theory of everything. Let me put forward the proposition that almost anything that we can now invent, educated as we are mostly in a classical framework, is unlikely to be radical enough. For what changes during a scientific revolution is not only the answers to questions, but the questions themselves. Those brought up on Aristotle, including Copernicus, Kepler and Galileo, were stuck on trying to answer the problem: what is the shape of the orbits of the planets. It never occurred to them that this question was to become much less important and that the new physics would center around completely different questions: what are the laws of motion and what are the forces. Similarly, by trying to invent "The" lagrangian and "The" symmetry we are, perhaps, acting out of our Newtonian instincts; we are trying to answer the important questions of the old science. Mathematics will do us little good if we have not yet stumbled upon the right new questions; had Copernicus known Fourier analysis he could have made a much better epicycle theory (indeed, he could have used it, there were more epicycles in his theory than in Ptolemy’s) but he never would have hit on the idea of a law of motion.

The main problem, then, is what to do while we are waiting to stumble upon the right questions. One approach is to try to go out and reinvent physics. This is, indeed, the most likely approach to succeed in the long run and I believe that we should try to spend as much time and energy as we can doing this. But it’s hard, and, except for those rare individuals with strong imaginations and no fear of failure, it costs at least as much in terms of our own sense of self-confidence as it does in terms of our professional life. But is there then anything else to do?

One thing that we can do is to take the laws and principles of physics as we have them and try as hard as we can to make them work in this new domain. That is, let us take quantum mechanics, as given by Heisenberg, Schroedinger and Dirac, and general relativity, as given by Einstein, and try to put them together, making as few ad-hoc hypothesis and approximations as we can get away with. The aim is not to invent a new fundamental theory, but to try to learn as much as we can about what the problem with putting them together is. By doing this we may gather clues that could help the eventual invention of a new theory. We may also learn how to speak about nature in a language which is completely consistent with both quantum mechanics and relativity. By doing so we may find ourselves asking new questions that we would not have asked had we not first rid ourselves of the pervasive influence of notions which are clearly wrong.

This is, then, the philosophy of the work that I will be presenting here. The
approach I will be describing is a nonperturbative approach that some of us have been following for about the last five years. By nonperturbative I mean that no use is made of any classical background metrics or connections, so that all the degrees of freedom of the gravitational field are treated fully quantum mechanically. Thus, this work differs from some other approaches to quantum gravity in which one expands the fields around a classical background and then quantizes, not the whole field, but only the deviations from the background.

Indeed, one of the themes of this review will be that the idea that there is a classical spatial or spacetime geometry is one of the vestiges of the old Newtonian physics that we have to leave behind if we are to understand how quantum theory can incorporate gravitation. Classical metrics and connections may appear as approximate, coarse grained, descriptions of some quantum states, when one is studying phenomena on scales much larger than the Planck scale. But in a fundamental theory these can have no place.

The question that any nonperturbative approach must then answer is what description of geometry is to replace the differential geometry of the classical theory. The results that are described in the following chapters address this question. We find that consistent application of quantum mechanics to general relativity reveals a rather simple picture of quantum geometry. In this picture, quantum geometry is discrete at Planck scales and all physical quantities are topological and combinatorial. Local quantities, such as the metric at a point, no longer have any meaning, but non-local quantities, such as the areas and volumes of arbitrary surfaces and regions, are well defined. Indeed, these quantities are represented by finite operators whose spectra are quantized in Planck units. Classical metrics do play a role, but only as approximate descriptions of the physics of certain states, at scales much larger than the Planck scale.

These results have been found using a new formulation of general relativity which is due to Ashtekar and is called the new, or self-dual, variables. It also involves a new representation for quantum field theories which is called the loop representation. The new variables and the loop representation have led to a number of interesting results, for example, large classes of exact solutions to all the constraints of quantum gravity have been found.

The basics of the new variables and loop representation will be presented in chapter three. However, because the new picture which has emerged of quantum geometry is so simple, I will begin in the next section to describe it without a lot of formalities and equations. The rest of the review will then be devoted to explaining the details of the calculations that give rise to this picture and to presenting other results which illuminate it.

There is good reason to believe that any description of quantum geometry that is valid at the Planck scale must be nonperturbative, because, by the dimensional character of the gravitational constant, the conventional perturbation theory blows up at the Planck scale. More generally, a general conclusion we can draw from all of the work on quantum gravity from the 1950’s to the present is that perturbative approaches simply don’t work, so that any successful approach to quantum gravity must be nonperturbative.
2 A picture of quantum geometry

In this chapter a new, and completely nonperturbative, description of quantum geometry is given. The calculations that support it are described in chapter 4; as we go along a series of footnotes will tell the reader in which sections to find particular calculations.

2.1 Preliminary remarks about nonperturbative quantum geometry

To confront quantum gravity nonperturbatively, we must first forget a great deal that we have learned from conventional quantum field theory. This is because all conventional quantum field theory depends on a fixed background metric. Thus, the first thing we must do is to identify the familiar structures from conventional field theory that will be absent in a nonperturbative formulation of quantum gravity or any diffeomorphism invariant quantum field theory:

A) **There is no background metric, connection, or any other structures given on space or spacetime besides (at most) the topological and differentiable structure of a three manifold.** Further, if diffeomorphism invariance is to be maintained no such classical structures may be introduced by the quantization procedure. This poses a great challenge to us as quantum field theorists, because literally all of the regularization and renormalization procedures we have in our toolbox rely on the presence of a fixed background metric. We thus need to introduce new kinds of regularization procedures in nonperturbative quantum gravity. In practice, we so far make use of a kind of a compromise: we introduce background metrics for the purposes of constructing regulated operators. Than we insist that any dependence on them disappears when we take the limits in which the regulators are removed. We must do this to insure that the final, nonperturbative operators have no dependence on arbitrary background structures.

In chapter 4 will see several examples of regularization procedures in which this can be carried out.

B) **There are no N-point functions.** In general there are no observables associated with the values of fields at points of the manifold. This is because the diffeomorphism invariance means that points have no meaning. For those used to perturbation theory it is important to stress that in a nonperturbative context, diffeomorphism invariance has a rather different effect than it does in the linearized or perturbative theory. It has the full force of its original meaning, which is that the manifold of points has no meaning. The only meaningful objects are the equivalence classes of manifolds under all diffeomorphisms. In the context of the quantum theory is important to stress that diffeomorphisms must be thought of as active transformations. Although their action is related to the action of general coordinate transformations, they are not the same thing.
Under a general coordinate transformation the points remain the same, but have
different labels. Under a diffeomorphism points are taken to different points. If a point, \( p \), of a manifold has no meaning, then neither can a field, \( \phi(p) \), evaluated at that point. As a result, no local observables are diffeomorphism invariant. It is then a non-trivial problem to construct diffeomorphism invariant observables. Indeed it is fair to say that most of the difficulties presently faced by nonperturbative quantum gravity are connected with this issue.

Nonperturbative diffeomorphism invariant quantum field theories are then
going to be very different from conventional quantum field theories as, in the latter, essentially all observables we use have to do with metrical relations among local observables. Since there is no background metric and there are no local observables these will not exist in a nonperturbative diffeomorphism invariant theory. The question is then: what new kinds of observables are to replace these? One answer, which has emerged from the recent work on nonperturbative quantum gravity, and which will be developed at length below, is that all diffeomorphism invariant observables measure topological relations of non-local observables.

In these remarks I have been referring, implicitly, to the effects of spatial
diffeomorphisms. But there are also additional problems associated with diffeo-
morphisms that change the time coordinate. This is because the effect on the
fields of a diffeomorphism that changes the time coordinate is indistinguishable
from the effect of the Hamiltonian, that gives the change of the fields under
evolution in time. Indeed, locally, they are generated by the action of the
same object: the Hamiltonian constraint. Thus, in contrast to ordinary gauge
theories, the gauge symmetry is deeply interwoven with the dynamics and the
problem of finding spacetime diffeomorphism invariant observables is a dynam-
ical problem. This is the essence of the problem of time in quantum gravity,
a deep and central issue about which a great deal has been written. I will
touch on this problem only in section 6.2.

It may be useful if I describe in a more formal way some of the distinc-
tions I have been raising informally in the last few paragraphs. In the work that I
will be reviewing here, which is based on what is called the Dirac quantiza-
tion procedure, it will be important to distinguish three stages in the construc-
tion of a nonperturbative quantum theory of gravity. The first stage is the
kinematical level, where we construct the states as a representation of a certain
algebra of observables which are defined in terms of easily accessible local quan-
tities such as metrics and connections. This state space, which will be denoted,
\( S^{\text{kin}} \), is not physically meaningful, but in what is called the Dirac approach to
the quantization of a gauge theory it is the starting point for the construction
of the physical states.

The second stage is achieved by finding those states which can be constructed
from the kinematical states which are spatially diffeomorphism invariant. This
is done by constructing the operator that generates spatial diffeomorphisms
within the kinematical state space and finding its kernel. The resulting space
of diffeomorphism invariant states will be denoted $S^{\text{diff}}$.

Finally, we must impose also the remaining part of the spacetime diffeomorphism invariance—that associated with changing the definition of the time coordinate. We do this by defining, in either $S^{\text{kin}}$ or $S^{\text{diff}}$, an operator to represent the Hamiltonian constraint that classically generates reparametrization of the time coordinate. The resulting simultaneous kernel of the Hamiltonian and diffeomorphism constraints is called the physical state space and is denoted $S^{\text{phys}}$.

An important issue is how we choose the inner product of the state spaces at each of the three levels. At present this is one of the issues that is not settled, indeed, this question is intimately related to the problem of time. However, in the last two years a definite point of view and a program for resolving it has developed. Because of the importance of this issue, I will deal with it separately, in section 6.2.

Finally, I want to emphasize that I will be discussing primarily the Hamiltonian approach to quantization. Because of this there is not only not a classical metric in the picture, there is not even in any sense a four dimensional classical manifold. The four dimensional spacetime manifold has gone to the same place in quantum gravity as the trajectory of the electron in ordinary quantum mechanics: classical physicists’ heaven. What remains is a spatial manifold, $\Sigma$, on which there is given a fixed topological and differentiable structure. Associated with each $\Sigma$ we then have the three state spaces $S^{\text{kin}}_\Sigma$, $S^{\text{diff}}_\Sigma$ and $S^{\text{phys}}_\Sigma$.

2.2 Quantum states as functions of loops

Having set the scene with the discussion of these preliminaries, I can now return to my goal, which is to describe in simple terms the state spaces of quantum gravity. The description I will be employing is based on a particular representation of the quantum theory, which is called the loop representation. I will be saying much more about it later; for this first look it is enough to know that in this representation the kinematical states, $S^{\text{kin}}_\Sigma$, are described as functions over a certain space of loops. To describe this adequately I need to say exactly what loops are involved in these spaces.

Conventionally, the loop space of a manifold, $\Sigma$, is taken to be the space of maps, $\gamma$, from $S^1$ into $\Sigma$. We will require that the loops be, in addition,
piecewise smooth. As has been discussed in many places, the space of such maps is an infinite dimensional differentiable manifold.

However, when we are dealing with the loop representation we are interested, not directly in these maps, but in the equivalence classes of the maps under the following three operations:

i) Reparametrization invariance.

ii) Equivalence under retracings: if $\eta$ is a curve originating on a loop $\alpha$, then $\alpha \circ \eta \circ \eta^{-1} \approx \alpha$, where $\circ$ means composition of loops.

iii) Inversion: $\alpha \equiv \alpha^{-1}$.

The space of loops defined modulo these relations will be called the space of nonparametric loops. It will be denoted $\mathcal{HL}_\Sigma$. It is the quotient of a differentiable manifold by the operations i) and ii) but it is not, in itself, a differentiable manifold\(^1\).

It is natural when considering loops with the identifications i) - iii) to extend the definition to include multi-loops. Multi-loops are taken to be countable unordered sets of loops. We will generally use the word loop to refer either to a single loop or to a multi-loop, but we will always mean loops defined modulo the relations i) - iii).

Generally, a loop will be denoted by a lower case Greek letter, such as $\alpha$, $\beta$ or $\gamma$. When we need to we will put on labels to distinguish the single loops (or components) of a multi-loop, as in $\gamma_i$, $i = 1, \ldots, N$. I should stress that the space of loops includes loops with arbitrary intersections, self-intersections, multiple retracings and nondifferentiable points. These will be important for the physics, because certain important operators act specially at such singular points.

However, in this introductory sketch, we will mostly restrict attention to loops without intersections or multiple retracings. These will be called simple loops.

In the loop representation, the quantum states are taken to be functions on loops and denoted $\Psi[\gamma]$. The meaning of this loop representation will become clearer as we use it.

### 2.3 Quantum geometry at the kinematical level

Having taken care of the preliminaries, I can now describe the state spaces of nonperturbative quantum gravity. I will begin at the kinematical level.

A good intuitive way to describe a quantum state space is to give a basis that consists of eigenstates of a familiar set of observables. Thus, one thing that one might like to do in quantum gravity at the kinematical level is describe the basis of eigenstates of the three metric. This is exactly what I cannot do here, and for a good reason, which is that in the representation I am describing there

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\(^6\) The $H$ stands for holonomy. Another way to define the space of loops is to say that two loops, $\alpha$ and $\beta$ are equivalent whenever their $U(1)$ holonomies $U_\alpha = \exp \oint_\alpha A$ are equal for all connections $A_\alpha$ on $\Sigma$.\(^2\)
simply is no well defined operator which corresponds to the three metric at a point. The absence of any operator that measures anything about the metric at a point is a key lesson and I will spend considerable time below\[7\] to convince the reader that this is the case\[8\].

While the metric makes no sense at a point, there are other operators which do measure metric information that can be defined in the loop representation. Moreover these operators, not only exist, they are, as I will show in detail in chapter 4, finite when they are defined through a proper regularization procedure. In this review I will be discussing three such operators.

The first measures the area of any given two dimensional surface, \( S \), in \( \Sigma \). It is denoted \( \hat{A}[S] \).

The second measures the volume of any three dimensional region \( R \) in \( \Sigma \). It is denoted \( \hat{V}[R] \).

The third measures the integrated norm of any one form \( \omega \) on \( \Sigma \). It is a little funny looking, written in terms of the classical three metric \( q_{ab} \) it is

\[
Q[\omega] = \int_{\Sigma} \sqrt{\det(q)} q^{ab} \omega_a \omega_b.
\]

Note that the square root is a density and is thus integrable.

I will now describe a basis in \( S_{\Sigma}^{kin} \) that is constructed of simultaneous eigenstates of these three operators. It is composed of characteristic states of loops. That is, given any loop \( \alpha \), we associate a state \( \Psi_{\alpha}[\gamma] \). In the loop representation this state has a simple representation when the loops \( \alpha \) and \( \gamma \) are both simple. In that case,

\[
\Psi_{\alpha}[\gamma] = 1 \text{ if } \gamma = \alpha \text{ and zero otherwise.}
\]

The value of such a state on loops \( \gamma \) that are not simple is more complicated; this is discussed in sections 4.3 and 4.4.

The reader may object that such states seem very formal, we are defining a state to be a kronocker delta on the space of loops, which is a continuous space. Certainly there are norms we could put on the state space with respect to which this state, if it could be defined at all, would have either vanishing or infinite norm. Among these is the Fock inner product, which can be defined in the loop representation\[21, 22, 23\]. Now, the Fock inner product is the appropriate norm to use if one is defining the state space of Maxwell theory or of free gravitons. But its use makes absolutely no sense in nonperturbative quantum gravity, because it depends on a fixed background metric and thus breaks diffeomorphism invariance.

To construct the non-perturbative theory we would like to use a norm on the state space that depends on no fixed background structure and is therefore invariant under the action of the diffeomorphisms on the space of loops. To my

\[7\] This is the subject of section 4.1

\[8\] There are, of course, other representations, such as the metric representation, in which such an operator can be constructed, at least at the formal level.
knowledge, there is only one class of inner products we can put on the space of functions over loops that satisfies this requirement. This is the class which make use of the discrete topology on the loop space. (The reader may recall that the discrete topology of a point set is the one in which every point constitutes an open set. This topology, and the corresponding discrete measure, exists on every continuous topological space.)

We thus define a norm on $S^{\text{kin}}_{\Sigma}$ which is given by
\[ |\Phi|_{\text{discrete}} = \sum_{\alpha} |\Phi[\alpha]|^2 \] (3)
where the sum is over all loops $\alpha$ on which the state $\Phi$ has support.

States which are normalizable under (3) can have support only on a countable set of loops. They can therefore be expressed in the form
\[ \Phi[\alpha] = \sum_I c_I \Psi_{\gamma_I}[\alpha] \] (4)
where $\gamma_I$ are any countable set of loops, indexed by $I$ and
\[ \sum_I |c_I|^2 < \infty. \] (5)

This may seem a weird sort of state space on which to base a quantum field theory. However, one of the themes of this review is that such a state space is adequate to serve as the Hilbert space for nonperturbative quantum gravity and for diffeomorphism invariant quantum field theories in general, at the kinematical level. Moreover, it is not only adequate, but it has some definite advantages. I will make, as we go along, three kinds of arguments for its use.

First of all, to demonstrate its adequacy one can show that it carries a faithful representation of an algebra of observables that completely coordinatize the phase space of the classical theory. Second, it carries an unbroken representation of the spatial diffeomorphism group and is, apparently, the only kind of representation of the kinematical algebra of classical observables that does so.

Third, the space $S^{\text{kin}}_{\Sigma}$ with the norm (3) contains within it states that are semiclassical, in the sense that all measurements performed on them give, to a certain degree of approximation, the same result as we would obtain with the flat space metric of the classical theory.

9There are some complications in this definition in the case that the loop is not simple. These will be discussed below in section 4.3 and 6.2. Nothing we are saying here will be changed by this.

10This kind of representation has been discussed earlier in [24, 25] and has recently been studied rigorously by Ashtekar and Isham [20].
We will then take the state space \( S^{\text{kin}}_{\Sigma} \) with the norm (3) as a basis, at least provisionally of the kinematics for non-perturbative quantum gravity\(^{11}\). We may then go on and consider the problem of diagonalizing the three observables \( \hat{A}(S) \), \( \hat{V}(R) \) and \( \hat{Q}(\omega) \) within this space.

We start with the areas. A large set of eigenstates of \( \hat{A}(S) \), for all surfaces, \( S \), in \( \Sigma \) is given by the characteristic states \( \Psi_{\gamma} \) of simple loops defined by (2)\(^{12}\). The associated eigenvalues are very simple: the spectrum is, in fact, discrete! The eigenvalue of \( \hat{A}(S) \), associated with the eigenstate \( \Psi_{\gamma} \) is equal to \( \sqrt{6} \) times the Planck area times the number of times the loop \( \gamma \) pierces the surface \( S \).

Thus, in quantum gravity, the area of any surface can only be a discrete multiple of the Planck area. This is, to my knowledge, the first time that a quantum theory of gravity gives a simple answer to the simple question: what is it that is quantized in quantum gravity?

Furthermore, the area observable allows us to give an interpretation to the characteristic states \( \Psi_{\gamma} \). They are quantum flux tubes, analogous to electric flux tubes in Yang-Mills theory. However, they are not carrying flux of electric field-they are carrying quantum flux of area. That is, each line is carrying a unit of Planck area which it will contribute to the area of any surface it crosses.

The characteristic states (2) are also all eigenstates of the operator \( \hat{Q}(\omega) \) corresponding to the classical observable defined in (1)\(^{13}\). The eigenvalue associated with the characteristic state of a loop \( \gamma \) is simply \( \sqrt{6} \) times the integral of the form \( \oint_{\gamma} \omega \).

Already with these two operators we can completely characterize the classical limit of the theory. Let us take a moment to discuss this, as the semiclassical limit in quantum gravity is somewhat different from semiclassical limits taken in quantum field theories with only dimensionless coupling constants. The reason is that to define the semiclassical limit we must take into account that \( \hbar \) is in the Planck length. This means that a semiclassical limit in quantum gravity must involve a limit of large distances, so that if we make a measurement that involves a length scale \( L \), the quantum result must agree with the classical one to order \( \hbar G / c^3 \) is the Planck length.

It is easy to describe states in \( S^{\text{kin}}_{\Sigma} \) which approximate a flat metric \( h^{0}_{ab} \) in this sense. Essentially all we have to do is to distribute a set of loops, \( \gamma \), so that any flat surface \( S \) in \( \Sigma \) is crossed by one of the loops on the average of \( \sqrt{6} \) times per Planck area-where the notions of flatness-and the areas of the surfaces are measured according to \( h^{0}_{ab} \). Such a set of loops will be called a weave. The weave approximates \( h^{0}_{ab} \) in the sense that for any surface whose

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\(^{11}\)I should point out that while I use the loop representation in this review, everything that is done in this paper at the kinematical level could be done in the connection representation, where the states are functions of the Ashtekar connection \( A^{a}_{a} \) (see section 3.2). In this case the characteristic functions of loops are defined as \( \Psi_{\gamma}(A) = \text{Tr} [\exp (\oint_{\gamma} A \cdot d\gamma)] \) as in ref. [2].

\(^{12}\)The details of the construction and diagonalization of the area operator is given in section 4.3. There are other eigenstates associated with intersecting loops, which are given in section 4.4.

\(^{13}\)With all intersections counted positively.

\(^{14}\)The details of this calculation are in section 4.6.
radius of curvatures are large compared to \( l_P \) the state is an eigenstate of the operator that measures the area of that surface, and the eigenvalue will agree with the area according to the classical metric \( h_0^{ab} \) up to terms of the order of the ratio of the Planck area to the area.

The observables \( Q[\omega] \) can also be used to demonstrate the correspondence between the weave and the smooth metric \( h_0^{ab} \). This will be discussed below in section 4.7.

The semiclassical limit includes not only the classical metric, but the quantum states of linearized gravitons that propagate on that classical metric. Although I will not discuss it here, it is possible to show that the exact quantum theory can be linearized in a neighborhood of the state space of a weave state and that the result is exactly the Fock space of linearized gravitons.

The weave states then provide a prototype for understanding quantum geometry as well as a number of issues related to both the semiclassical and short distance limit of quantum gravity. In order to discuss this, we must first emphasize that, at the non-perturbative level, with no background metric, there is no intrinsic notion of distance in the theory. The Planck length is a parameter of the theory, but without a metric there is nothing to tell us what lengths, areas or volumes are Planck scale. A notion of distance can only be associated with a particular state of the system.

On the other hand, most quantum states describe quantum geometries that have no classical equivalent. If we consider, for example, the quantum geometry of a simple unknotted loop, we see that most surfaces have zero area. There also do exist surfaces with area of \( n\sqrt{l_P^2} \), for any \( n \). However, there is no smooth and everywhere nondegenerate metric that can give this assignment of areas to the surfaces.

Thus, classical notions of geometry can only be recovered from special quantum states that have the property that they approximate a classical metric in the sense we have just described. Using the picture that each line carries one Planck unit of area we can construct such states by spacing the lines by Planck units according to the classical metric we want to recover.

This means that all states that have semiclassical limits—that approximate classical metrics at large scales—necessarily have discrete structure at the Planck scale. The two things are tied together because if the state does not have a classical limit then there is no metric which enables us to talk about the Planck length structure. At the same time from the way the states are constructed it is impossible to pack the loops more tightly than we have describe here because the quantum operator \( A[S] \) measures areas by counting intersections in Planck units. If we put the loops ten times closer together, according to some background coordinate, then nothing really changes, because the metric that the state will approximate will not be the original one \( h_0^{ab} \), but one hundred times \( h_0^{ab} \).

Thus, there is no reason for, and no possibility of, taking a limit in which
the lines of the weave are taken closer and closer together. Because the discrete Planck scale structure is essential to the existence of a classical limit, the states that have classical limits are those that involve only countable numbers of loops. This means that the state space we have been working in, which are states which are normalizable with respect to the discrete norm (3) is completely adequate for discussing the classical limit of the theory.

We have still said nothing about the third operator mentioned above: the one that measures the volume of arbitrary regions. This operator turns out to act nontrivially only on states that have support on intersecting loops. Thus, to describe its action I need to say a little more about the role of the intersecting loops in the state space $S_{\text{kin}}$.

We must first introduce some nomenclature to describe the intersection loops. Given any loop $\gamma_0$ with intersections, there is a finite set of $M$ other loops $\gamma_i$, $i = 1, \ldots, M$ which differ from $\gamma_0$ by rearranging the routings through each intersection points. The equivalence class of these loops under reroutings is called the graph $\Gamma$ of the loops $\gamma_i$.

Let us begin by thinking of a particular graph $\Gamma$ with one self-intersection point, $p$. Let a finite number, $N$, of lines intersect at $p$. Associated with this graph are a finite number of loops, which I will denote $\Gamma_I$, $I = 1, \ldots, P$, which differ by changes of the routing at $p$.

In the loop representation the value of the states on these different reroutings are not all independent. This is because there are imposed on the states on intersecting loops a set of relations which code the fact that loops represent holonomies of an $SU(2)$ connection. These relations are:

$$\Psi[\alpha \cup \beta] = \Psi[\alpha \circ \beta] + \Psi[\alpha \circ \beta^{-1}]$$  \hspace{1cm} (6)

where $\alpha$ and $\beta$ are any two loops that share a common point, so that their composition may be defined. These are called the spin network identities because they make the loop functionals valued on spin networks, which are discrete models of quantum geometry introduced by Penrose\[29\].

Taking into account these identities, there are then $R < P$ independent characteristic states associated with the graph $\Gamma$. We will label these also by $\Gamma_I$, where from now on the index $I = 1, \ldots, R$ labels independent loops. These span an $R$ dimensional subspace of $S_{\text{kin}}$ which I will denote $S_{\Gamma}^{\text{kin}}$.

As I will discuss in chapter 4, these subspaces play an important role in the representation of a large class of operators. The first example of these is the volume operator $V[\mathcal{R}^P]$. This operator, first of all, has a simple action when evaluated on states with support only on non-intersecting loops: it annihilates them. To see what its action is on states with support on intersecting loops, let us consider acting on an arbitrary state $\Phi[\alpha]$ and then evaluating the result on one of the intersecting loops $\Gamma_I$. The action is zero unless the region $\mathcal{R}$ contains

\footnote{The details concerning the regularization of the volume operator are in section 4.5.}
the intersection point $p$. When it does, the action is given by

$$V[R]\Phi[\Gamma_J] = i\hbar^3 M_J^{\dagger} \Phi[\Gamma_J]. \quad (7)$$

where $M_J$ is an $R \times R$ dimensional real matrix whose entries are dimensionless numbers. That is, the effect of the volume operator at the intersection is simply to act with a matrix which rearranges the routings and then multiply the result by the Planck volume.

It is easy to generalize this to the evaluation of states on graphs with an arbitrary number of intersection points. The result is the sum of the action (7) at each intersection point in $R$. From this we can see that the eigenstates of the volume operator are linear combinations of the characteristic states associated with each graph and the eigenvalues are all equal to the Planck volume times arbitrary sums of the eigenvalues of the rearrangement matrices. Thus, just like the areas, in nonperturbative quantum gravity the volume of a given region has a discrete spectrum of eigenvalues proportional to the Planck volume.

The reader may notice an apparent contradiction between this result and the results I described earlier about the eigenstates of $A[S]$ and $Q[\omega]$. This is that one of the weave states, which correspond to the flat metric when measurements are made on scales larger than the Planck scale, can have zero volume. This will be the case when the weave has no intersections.

This is true, but the contradiction is only apparent. A quantum state can indeed approximate a smooth nondegenerate classical metric when probed at large scales and still have zero volume when it is measured with the operator $V[R]$. This is because the latter is a completely microscopic operator, it measures something about the connectivity of the weave, while the correspondence to a classical metric only measures a long distance property of it. A zero volume weave is then something like a fractal geometry: it gives us a picture of geometry which is indistinguishable from a classical metric at large scales, but if one probes at Planck scales one discovers no metric structure at all, just some one dimensional structures associated with the loops.

If the reader still finds this disturbing, let me remind him or her that something like this is exactly what must happen if quantum general relativity is to exist. If the exact quantum states looked like semiclassical states built around a classical metric at arbitrarily short distances, then perturbation theory would be reliable. But we know that perturbation theory in fact gives only nonsense. Conversely, if, as we have shown, any state that has a classical description at large scales has discrete, completely non-classical, structure at the Planck scale then perturbation theory, which does not recover this feature, must be unreliable at Planck scales. Therefore the short distance structure of the theory cannot be studied perturbatively and the fact that the perturbation theory is nonsense cannot be taken as an argument that the exact quantum theory does not exist.

This discussion brings us to the end of our introduction to nonperturbative quantum geometry at the kinematical level. We now go on to the next stage,
which is the nonperturbative description of spatially diffeomorphism invariant states.

2.4 Quantum geometry at the diffeomorphism invariant level

One of the early successes of the loop representation is that the space of diffeomorphism invariant quantum states of the gravitational field can be exactly represented\[13\]. To construct this representation, let us note that an unbroken representation of the group of diffeomorphisms of $\Sigma$ exists on the state space $S_{\Sigma}^{\text{kin}}$, which is given by

$$\hat{U}(\phi)\Phi[\alpha] = \Phi[\phi^{-1} \circ \alpha].$$

(8)

where, here, $\phi \circ \alpha$ denotes the action of the diffeomorphism $\phi$ on the loop $\alpha$. One can check that under the norm (3) $\hat{U}(\phi)$ is a unitary operator\[16\]. The diffeomorphism constraint operators are then defined as\[17\]

$$\hat{D}(v)\Phi[\alpha] = \frac{d}{dt} U(\phi_t)\Phi[\alpha]|_{t=0}$$

(9)

where $\phi_t$ is a one parameter group of diffeomorphisms generated by the vector field $v$.

It is then straightforward to write down the exact solutions to the diffeomorphism constraints,

$$D(v)\Psi[\alpha] = 0$$

(10)

The solutions are that $\Psi$ must be a function of the diffeomorphism equivalence classes of $\alpha$. These will be denoted as follows: $\{\gamma\}$ is the diffeomorphism equivalence class of the loop $\gamma$. The set of these classes are called the generalized link classes of $\Sigma$.

Similarly, $\{\Gamma\}$ is the diffeomorphism equivalence class of the graph $\Gamma$. These will be called, following earlier literature, knotted graphs. Just as the case of graphs and loops, given any knotted graph $\{\Gamma\}$ there are a finite set of generalized link classes $\{\Gamma_i\}$ which differ by rearrangements of lines through intersection points.

The space of diffeomorphism invariant quantum states of the gravitational field, denoted $S_{\Sigma}^{\text{diff}}$ then consists of all functionals $\Psi[\{\alpha\}]$. The generalized link classes are countable, so this space has a countable basis, given by the characteristic functions of the classes. These will be denoted $\Psi[\gamma]$; this state is

\[16\] This representation is reducible. The irreducible representations are classified by the knot and link classes of $\Sigma$\[30\].

\[17\] It is not hard to show that this form of the operator is exactly equivalent to the one which is found by beginning with the classical expression for the constraint and translating it into quantum operators through a suitable regularization prescription\[31\].
equal to one when evaluated on the class \( \{ \gamma \} \) and zero when it is evaluated on all other classes\(^{18}\).

Unfortunately, although we know the exact state space of the theory at the diffeomorphism invariant level, we know rather less about the theory at this level than we would like to. For example, the exact form of the inner product on this space is not presently known. In addition, at present we only know how to represent a few observables on this space.

It may seem strange that there is a problem constructing spatially diffeomorphism invariant observables, as an infinite number of them may immediately be written down at the classical level, by integrating local densities constructed from the classical fields over \( \Sigma \). Furthermore, at the quantum level, as we have a countable basis for the state space \( S_{\Sigma}^{\text{diff}} \), we may immediately write down an infinite number of diffeomorphism invariant quantum operators. The problem is that to construct a physically meaningful quantum observable we must construct a relationship between a classical observable and a quantum operator.

Unfortunately, to translate an expression for a classical observable into a quantum operator, while preserving diffeomorphism invariance, is a far from trivial operation. As the classical expressions for the observables involve products of fields this translation necessarily involves a regularization procedure. During the regularization procedure extra structure is introduced which breaks diffeomorphism invariance. The challenge is to remove the dependence on this extra structure as one takes the limit in which the regulator is removed.

I will mention here two diffeomorphism invariant operators that we know how to construct, the details of their construction are given below in section 4. These allow us to give a partial physical interpretation for the states in \( S_{\Sigma}^{\text{diff}} \).

The first operator is based on extending the definition of surface areas in the kinematical theory to an operator that measures the areas of minimal surfaces. To describe this I need to consider the case that \( \Sigma \) has nontrivial \( \pi_2 \). Then let us consider a noncontractible surface \( S \) in \( \Sigma \) and its homotopy class \( \{ S \} \). We can define, both classically and quantum mechanically, a diffeomorphism invariant observable, which is the minimal value of the areas of surfaces in \( \{ S \} \). This will be denoted \( A[\{ S \}] \). It is clearly a diffeomorphism invariant function of the three metric.

The operator for the area of a minimal surface is constructed by taking the operator for the area of a surface I described above and minimizing it over all surfaces within the homotopy class \( \{ S \} \). Given this, it is not hard to see that the eigenstates of the minimal surface area operator are exactly the characteristic states, \( \Psi_{\{ \gamma \}} \) of generalized link classes. The eigenvalues are given by \( \sqrt{6l^2} \) times the minimal number of intersections (without regard to sign so that all intersections are counted positively) of the loops in \( \{ \gamma \} \) with the surfaces in \( \{ S \} \).

\(^{18}\)This is for diffeomorphism equivalence classes of simple loops. As at the kinematical level, for nonsimple loops the situation is more complicated due to the spin network relations (6).
There are two interesting things about this result. First, the result is essentially topological: the area of the eigenstate is gotten by counting the intersections of the loops associated with the eigenstate with the surfaces. This is an example of a general situation which I noted above: a property which is metrical classically is translated into an operator which measures a topological property of the label of the quantum mechanical state. Second, as in the kinematical theory, the spectrum is quantized. The minimal area of any class must be an integral multiple of $\sqrt{6}$ times the Planck area.

Thus, we see that as in the kinematical case, the lines of the link classes carry quantized area, in the sense that they contribute a Planck area to the minimal area of any class of surfaces that they minimally intersect. To build up a diffeomorphism invariant quantum geometry that is large compared to the Planck scale, we must have a complicated link class involving lots of lines. For example, the simplest quantum geometry we can build in the three torus consists of unknotted loops wrapped around the three $S^1$'s. The areas of the minimal surfaces wrapping each of the three two toruses are given by the numbers of the loops in each direction in Planck units.

We have already introduced the second diffeomorphism invariant operator, it is simply the volume of the whole manifold $\Sigma$. The action of this operator, given by (7), is diffeomorphism invariant, so it can be extended directly to the states in $S_{diff}^{\Sigma}$. For example, its action evaluated on graph classes with one intersection is given by the extension of (7):

$$\hat{V}[\Sigma]\Phi[\{\Gamma_I\}] = i_P^3 \mathcal{M}_{I'}^{I} \Phi[\{\Gamma_{I'}\}], \quad (11)$$

where $\mathcal{M}_{I'}^{I}$ are the same rearrangement matrices as in the kinematical case. As in that case, this result extends trivially to graph classes with more than one intersection, one gets the sum of rearrangement matrices acting at each intersection. The eigenstates then consist of linear combinations of the characteristic states of the generalized link classes $\{\Gamma_I\}$ and are found by diagonalizing the rearrangement matrices. Further, we see that the volume of the Universe is quantized: its eigenvalues are sums of all the possible eigenvalues of the rearrangement matrices taken in Planck units. Thus, the volume of the universe in an eigenstate is roughly proportional to the number of intersections in the graph that labels that state.

Four comments before we go on to physical states: First, again as in the kinematical case, at the quantum level the volume of the universe and the area of minimal surfaces are decoupled, in a way that indicates that the area is a more macroscopic measure of the quantum geometry, while the volume is more microscopic. Second, we see that acting on the space $S_{diff}^{\Sigma}$ of diffeomorphism invariant states the volume operator is block diagonal: it is represented by simple matrices of numbers acting in each of the finite dimensional blocks subspaces associated with different rearrangements of routings within a graph class. This is an example of a general property of diffeomorphism invariant operators that
are local in the sense that they are constructed by a single integral over a density: all these operators are either block diagonal or near block diagonal, in the sense that they mix one graph class with graph classes that differ by the addition or subtraction of a finite number of loops.

Third, note that the diffeomorphism invariant states in $S^{\text{diff}}_\Sigma$ are not normalizable with respect to the kinematical inner product (3). This is a general feature of Dirac quantization, the states in the kernel of a constraint are not normalizable in the kinematical inner product. This necessitates the choice of new inner products at the diffeomorphism invariant and physical levels. This will be discussed in more detail in section 6.2.

Finally, all of the operators I have described here are both finite and background independent (by which I mean independent of the background structure that must be used during the regularization procedure.) The reader will see in the details of the constructions of the operators described in chapter 4 how this comes about. However, it is interesting to mention that in general, while a finite operator need not be background independent, it is the case that a background independent operator must always be finite. This is because the regulator scale and a background metric are always introduced together in the regularization procedure. This is necessary, because the scale that the regularization parameter refers to must be described in terms of some metric and, since none other is available, it must be described in terms of a background metric or coordinate chart introduced in the construction of the regulated operator. Because of this the dependence of the regulated operator on the cutoff, or regulator, parameter, is related to its dependence on the background metric (this can be formalized into a kind of renormalization group equation \[ \square \). When one takes the limit of the regulator parameter going to zero one isolates the nonvanishing terms. If these have any dependence on the regulator parameter (which would be the case if the term is blowing up) then it must also have a dependence on the background metric. Conversely, if the terms that are nonvanishing in the limit the regulator is removed have no dependence on the background metric, they must be finite.

This point has profound implications for the whole discussion of finiteness and renormalizability of quantum gravity theories. It means that any nonperturbative and diffeomorphism invariant construction of the observables of the theory must be finite. A particular approach could fail in that there could be no way to construct the diffeomorphism invariant observables as quantum operators. But if it can be done, without breaking diffeomorphism invariance, those operators will be finite.

2.5 Physical states

So far everything I have described is based on only two facts about general relativity, first that the gauge symmetry includes spatial diffeomorphism invariance and second, that there is a representation in which the states of the theory are
functions of loops. As I will discuss in the next section, such a representation exists whenever the theory has a canonical coordinate which is a connection. But I have not yet used anything about the dynamics of general relativity. Thus, all of the results I have described so far, including the existence of the classical limit and the quantization of areas and volumes should be true in a wider class of theories.

The real surprise of the whole development, so far, is that the Hamiltonian constraint has a simple action in the same representation in which the diffeomorphism constraints can be solved. This, as far as I understand it, did not have to happen and exactly why it happens is not really satisfactorily understood.

Be that as it may, the basic fact is that acting on states in $S_{\text{kin}}^\Sigma$ or $S_{\text{diff}}^\Sigma$ the action of the Hamiltonian constraint is concentrated at intersections of loops. The result is that in $S_{\text{kin}}^\Sigma$ one can find an infinite dimensional space of exact solutions to the Hamiltonian constraint. These include an infinite space of solutions which consists of all the states have support on loops that have no intersections. There are, in addition, a large number of states with support on intersecting loops. Further, all of these solutions spaces are diffeomorphism invariant, so that we can simultaneously solve the Hamiltonian and diffeomorphism constraints. The result is that we have, explicitly, an infinite dimensional space of exact physical states of the theory.

These results will be further described in section 6.1 below. There I will also describe some very recent work concerning the form of the Hamiltonian constraint in the loop representation that leads to new (as of 1991) kinds of solutions to all of the constraints.

3 Basics of the loop representation and the Ashtekar variables

The purpose of this chapter is to give the basics which are necessary to understand the derivations of the results just described. The reader desiring a more detailed introduction should consult one of the reviews or the original references cited.

3.1 The loop representation

The loop representation is a representation, in the sense of Dirac, of a gauge theory in which the Yang-Mills gauge invariance is automatically implemented. In the loop representation quantum states are represented as functions of sets of loops, of the form $\Psi[\gamma, \alpha, \beta]$, where the loops $\alpha, \gamma, \beta,...,$ live in the space $\mathcal{H}_\Sigma$, defined in section 2.2. It can be constructed for any quantum field theory which
can be expressed in a form in which the canonical coordinate is a connection. The loop representation was invented independently by Gambini and Trias\textsuperscript{12} for Maxwell and Yang-Mills theory and by Rovelli and the author for quantum general relativity\textsuperscript{13}. It differs from many of the representations of quantum theories we are familiar with in that, as there is no classical variable corresponding to the position of the loops, it is not expressed as a space of functions over a configuration space. In this sense it is analogous to using the energy eigenfunctions for the hydrogen atom as a basis for one particle quantum theory, in that there are no classical variables associated with \(n, l\) and \(m\).

Let us consider a theory based on a Yang-Mills connection \(A^a\) and a conjugate electric field \(\tilde{E}^b\). We assume that these satisfy the standard Poisson bracket relation

\[
\{A^i_a(x), E^b_j(y)\} = \delta^b_a \delta^i_j \delta^3(x, y).
\]  

(12)

There are two ways to construct the loop representation of such a theory. The first is to construct a transform from a representation in which the states are functions of the connection \(A^a\). The basic form for the transform is,

\[
\Psi[\gamma] = \int d\mu[A] T[\gamma, A] \psi[A]
\]  

(13)

Here, \(\psi[A]\) is the state in the connection representation and \(T[\gamma, A]\) is the trace of the holonomy of the connection \(A\) around the loop \(\gamma\). This is commonly denoted the Wilson loop variable by particle physicists, it can be written

\[
T[\gamma, A] = Tr U_\gamma(0, 1)
\]  

(14)

where

\[
U_\gamma(s, t) = Pe^{\int_s^t A \cdot d\gamma}
\]  

(15)

is the parallel transport of the connection along the loop \(\gamma\) from the point \(s\) to \(t\). Here \(c\) is the coupling constant of the theory. For example, in the case of general relativity, \(c\) is Newton’s gravitational constant, \(G\).

Note that in (13) we have written the transform for the case that we evaluate \(\Psi\) on a single loop \(\gamma\). In the general case in which the argument is a set of loops \(\{\alpha, \beta, ...\}\) the kernel of the transform is the product of the Wilson loops variables for the loops.

In the definition of the transform \(d\mu[A]\) is an integration measure on the space of connections. In order to turn this equation from a formal statement into an actual definition of the loop representation it is necessary to give a precise definition of this measure. It turns out that in a number of cases this can be done and the loop representation constructed completely from (13).

\textsuperscript{20}i and \(j\) denote internal Yang-Mills indices while \(a, b, c, ...\) denote spatial indices

\textsuperscript{21}Since \(G\) has dimensions, this means that the connection, \(A^a_\gamma\), in the case of gravity does not have the standard dimensions of inverse length. We will see that this is an important fact for all of the developments of chapter 4.
In the last several years the loop representation has been studied in several cases in which the quantum field theory is already well understood. These include Maxwell theory\[12, 21, 22\], linearized quantum gravity\[23\], Yang-Mills theory on a lattice\[36\], abelian Chern-Simon theory\[37\] and 2 + 1 gravity\[38\]. In each of these cases the loop representation is found to be equivalent to the standard representation in which the states are functions of the connection. There can thus be no doubt that the loop representation is a well defined and reliable approach to the quantization of field theories involving connections.

The case of Maxwell theory is particularly instructive. There it is possible to explicitly compute the transform and express the full Fock space of photons as functions \(\Psi[\gamma]\) on the loop space\[21\]. I will not describe the process, but I will write down the answer because it is so simple. The vacuum is given by the state \(\Psi[\gamma] = 1\). A state with one photon of momentum \(p^a\) and polarization \(m^a\) is given by

\[
\Psi_{p,m}[\gamma] = \oint dse^{ip_a\gamma^a(s)}\dot{\gamma}^a(s) \quad (16)
\]

Multi photon states are given by products of these factors. Furthermore, the inner product can be expressed simply in terms of these factors.

The second way to construct the loop representation is to come to it by seeking representations of a certain algebra of classical observables. This algebra is called the loop algebra. It consists of the \(T\) variables that I defined above in (14) and a complementary set of variables that involve both the connection and the conjugate electric field \(E^a\). These variables are parametrized by both a loop and a point on the loop. They are defined by

\[
T^a[\gamma](s) \equiv \text{Tr}[\tilde{E}^a(\gamma(s))U_\gamma(s, 2 + 2\pi)] \quad (17)
\]

It is then not hard to show that, given (12), the variables \(T[\gamma]\) defined by (2) and \(T^a[\gamma](s)\) defined by (17) have a closed algebra. The basic Poisson bracket

\[\{\tilde{E}^a, E^b\} = 0\]

Perhaps of interest also to particle physicists is some recent work which uses the loop representation as the starting point for an alternative approach to solving lattice QCD numerically\[36\]. The main idea is that for a fixed lattice one can truncate the representation by cutting the loops off at some large, but finite size. The Hamiltonian of QCD is then represented as a sparse matrix, which is sparser and sparser the larger the set of loop considered. For small lattices, and in the 2 + 1 dimensional theory, the Hamiltonian can then be diagonalized by sparse matrix methods\[36, 39\]. The results are equivalent to those found by other methods. Not surprisingly, it is clear that for weaker and weaker coupling it is necessary to truncate the theory at larger and larger loops to have a good approximation. For the real case of 3 + 1 QCD, with coupling in the scaling region, it is necessary to use a more sophisticated algorithm to construct a good truncation of the Hamiltonian. Work on this direction is proceeding and it is not out of the question that this could lead to a method that is competitive with Monte Carlo methods for the solution of real problems. This is especially true as the inclusion of fermions is not such a great problem for these methods as it is for the Monte Carlo procedure. For example, calculations based on the loop representation have recently been done for QED with fermions in 3 + 1 dimensions\[39\].

\[\triangleq\]
is given by

\[
\{T[\gamma], T^a[\alpha](s)\} = c \int dt \delta^3(\alpha(s), \gamma(t)) \dot{\gamma}^a(t) \left[ T[\gamma \circ \alpha] - T[\gamma \circ \alpha^{-1}] \right]
\]  

(18)

where \(\circ\) means to combine the loops at the intersection point. (The expression vanishes if there is no intersection point.) This algebra is, furthermore, a complete algebra, in the sense that any gauge invariant function on the phase space of \(A_a\) and \(E^b\) can be expressed in terms of them (alternatively, it can be shown that except for points of measure zero, they coordinatize the physical phase space). Thus, it is possible to take this algebra as the starting point for the quantization of the theory rather than the standard canonical algebra (12). This can be done for the cases already mentioned and the result is, in each case, equivalent to the representation constructed by means of the transform (13).

It is easy to write down the operators that represent \(T[\alpha]\) and \(T^a[\alpha](s)\) in the loop representation. They are defined by

\[
(\hat{T}[\alpha]\Psi)[\gamma] \equiv \Psi[\alpha \cup \gamma]
\]

(19)

\[
(\hat{T}^a[\alpha](s)\Psi)[\gamma] \equiv \hbar c \int dt \delta^3(\alpha(s), \gamma(t)) \dot{\gamma}^a(t) \left[ \Psi[\gamma \circ \alpha] - \Psi[\gamma \circ \alpha^{-1}] \right]
\]

(20)

In equation (19) \(\cup\) stands for union in the set of loops. It is because of this definition that the loop states take values on sets of loops. (19) tells us that the Wilson loop is represented by an operator that is a kind of a shift operator in the space of sets of loops.

For the calculations we will do in the next chapter, we will find it useful to employ two elaborations of notation concerning the loop representation.

The first elaboration concerns a pretty geometrical expression of the loop algebra (18). The expressions (18) and (20) contain distributional factors. This is just like the local expression (12), the difference is that the distributional factors have support on curves rather than points. The remedy is the same as in ordinary local field theories, at both the classical and the quantum mechanical level the expressions are to be integrated against smooth test functions. There is a very pretty way to do this, which is the following. As the distributional factors in (18) and (20) are already one dimensional, the integral over test functions must be two dimensional. It is natural to make this an integral over a surface. This can be done very naturally, because the \(\tilde{E}^{ai}\) is dual to a two form. We may define,

\[
E^a_{bc}(x) \equiv \epsilon_{abc} \tilde{E}^{ai}(x).
\]

(21)

and, from (17),

\[
T^a_{bc}[\gamma](s) \equiv \text{Tr} [\tilde{E}^a_{bc}(\gamma(s)) U_{\gamma}(s, s + 2\pi)].
\]

(22)

Let us have a one parameter continuous family of loops \(\gamma^a(s, u)\) where \(u \in [0, 1]\) and for each \(u\), \(\gamma^a(s, u) \equiv \gamma^a(s, u)\) is a closed loop. We call such a family a
strip, which we will denote with a hat, as in $\hat{\gamma}$, $s$ and $t$ then coordinatize the two dimensional surface of the strip. We may then define an observable associated the strip $\hat{\gamma}$,

$$T^1[\hat{\gamma}] \equiv \int du \int ds \frac{\partial \gamma^a}{\partial s} \frac{\partial \gamma^b}{\partial t} T_{ab}[\gamma_u](s)$$

(23)

The 1 refers to the fact that one factor of $\tilde{E}^{ai}$ is inserted in the trace.

The Poisson bracket of this observable with the holonomy, $T[\gamma, A]$ is expressed in terms of the intersection number of the loop $\alpha$ and the surface $S$. Denoted,

$$I[S, \gamma] \equiv \int d^2 S_{ab} \oint d\gamma^c \delta^3(S, \gamma) \epsilon_{abc}$$

(24)

this is equal to $\pm 1$, depending on orientation, if the loop intersects the surface, and zero otherwise. It is then true that,

$$\{T[\gamma], T^1[\hat{\alpha}]\} = cI[\hat{\alpha}, \gamma] [T[\gamma \circ \alpha^*] - T[\gamma \circ (\alpha^*)^{-1}]]$$

(25)

where $\alpha^*$ is the curve in the strip that intersects $\gamma$. This gives us a geometrical representation of the quantum operator.

$$(\hat{T}^1[\hat{\alpha}]\Psi)[\gamma] = \hbar cI[\hat{\alpha}, \gamma] [\Psi[\gamma \circ \alpha^*] - \Psi[\gamma \circ (\alpha^*)^{-1}]]$$

(26)

This strip regularization of the loop algebra is a prototype for the background independent regularizations that will be introduced in chapter 4.

The second elaboration of notation involves employing a dual basis to the representation space to write

$$\Psi[\alpha] = <\alpha|\Psi>$$

(27)

Here $<\alpha|$ is a basis in the dual space of loop functionals, which is parametrized by the loops. By definition, these basis states must satisfy the identities i) to iii) introduced in section 2.2, so that $<\alpha|$ is parametrization independent and

$$<\alpha \circ \eta \circ \eta^{-1}| = <\alpha|$$

(28)

and

$$<\alpha| = <\alpha^{-1}|.$$  

(29)

In addition, for the application to gravity or $SU(2)$ Yang-Mills theory we impose the $SU(2)$ spin network relations,

$$<\alpha \cup \beta| = <\alpha \circ \beta| + <\alpha \circ (\beta^{-1})|.$$  

(30)

We may note that, by (28) $\alpha$ and $\beta$ are always equivalent to loops with a common base point (by expressing $\beta$ as $\beta \circ \eta \circ \eta^{-1}$, with $\eta$ an arbitrary segment that connects to $\alpha$.) Thus, the relation (30) is always meaningful.
The relations satisfied by the states $\langle \alpha |$ may be summarized by saying that a linear combination $\sum_i c_i \langle \alpha_i | = 0$ whenever it is the case that $\sum_i c_i T[\alpha_i, A] = 0$ for all values of the SU(2) connection $A_i^a$.

We may then express the defining relations of the loop operators in terms of the loop basis,

$$< \gamma | \hat{T}[\alpha] \equiv < \alpha \cup \gamma |$$  \hspace{1cm} (31)

and

$$< \gamma | \hat{T}^1[\hat{\alpha}] = \hbar c I[\hat{\alpha}, \gamma] \left[ < \gamma \circ \alpha^* | - < \gamma \circ (\alpha^*)^{-1} | \right]$$  \hspace{1cm} (32)

We may note that this representation on the dual space is then a cyclic representation, as all states may be built up by repeated application of $\hat{T}[\gamma]$ acting on the state $< \cdot |$, where $\cdot$ stands for the trivial loop in which all of $S^1$ is mapped to an arbitrary (by (28)) point in $\Sigma$. Then the representation may be defined by the algebra (25), together with the relation

$$< \cdot | \hat{T}^1[\hat{\alpha}] = 0$$  \hspace{1cm} (33)

It is important to stress that the definition of the dual basis uses only the linear structure of the state space and does not involve the inner product. Without the inner product, we do not know what ket state $| \alpha >$ corresponds to the dual bra state $< \alpha |$. Thus, without the inner product, we cannot write $< \alpha | \beta > \equiv \Psi_\beta[\alpha]$.

The existence of a quantization based on the loop algebra (18) (or (25)) raises questions about the relationship between this quantization and the conventional quantization based on the canonical algebra (12). A rigorous study of this question has recently been carried out by Ashtekar and Isham for the case of Maxwell theory and SU(2) Yang-Mills theory. Their results are not simple to state completely without using the language of the rigorous representation theory. However, simply put, they find that there exist several kinds of representations of the algebra (25) as function spaces on the loop space $\mathcal{H}_L$. Some of these are equivalent to the standard Fock representation of Maxwell theory, but others are not equivalent to previously known representations. In

23On the other hand, in the connection representation, we know how to write the bra state corresponding to a single loop $\alpha$, it is $\Psi_\alpha[A] \equiv < A | \alpha > = T[\alpha, A]$.  \hspace{1cm} (34)

Written in terms of Dirac notation, the transform (13) is

$$< \gamma | \Psi > = \int d\mu[A] < \gamma | A > < A | \Psi > .$$  \hspace{1cm} (35)

Thus, we see that the dual space to the loop representation, consisting of states of the form $\Psi = \sum c_i < \alpha_i |$ corresponds to the connection representation states of the form $\Psi[A] = \sum c_i T[\alpha_i, A]$. Indeed, at the kinematical level, all of the results we report in the next chapter for the dual basis of the loop representation are true also for the corresponding states in the connection representation.
the case of the representations which are equivalent to Fock representations, one can show \cite{20} that the equivalence is expressed by an expression of the form of the transform (13).

Among the representations that Ashtekar and Isham show are not equivalent to the Fock representation are those based on the discrete norm (3). As the reader may check, the operators $\hat{T}[\alpha]$ and $\hat{T}^a[\beta](s)$ are well defined through (19) and (20) on the space $S_{\Sigma}^{\text{kin}}$ with the norm (3). This is the basis for my claim there that the full kinematics of the theory is well represented on that space.

To summarize, in Maxwell theory, linearized general relativity, Yang-Mills theory and $2 + 1$ gravity the loop representation has been constructed and in each case it has been found to give results that are equivalent to the results found by other methods. Further, in each of these cases the connection representation is also known and the loop representation can be constructed by means of the transform.

The reader encountering all of this for the first time will no doubt be wondering what Yang-Mills fields have to do with gravity. This is explained in the next section.

3.2 The Ashtekar variables

When a person first encounter the Einstein equations, they often have two strong impressions. The first is how beautiful they are. The second, which comes about as one tries to find a solution, or compute something, is how complicated they are. This seems, somehow, unfair. One feels that the geometrical beauty behind the equations should serve some useful purpose when we try to do physics with the theory.

This impression is even more strongly born out when one realizes that, contrary to the case with many other nonlinear partial differential equations, a great many exact solutions to Einstein’s equations have been found. Indeed, although the theory is, presumably, not an integrable system, there exist several different restrictions of the theory that lead to integrable systems. Two of these are the restrictions to spacetimes with two symmetries (two killing fields) and the restriction to self-dual solutions.

Einstein’s equations and their solutions thus have a lot of mathematical structure. We may then ask, if the space of solutions has non-trivial structure, should this not play a role somewhere in the quantum theory? In order to investigate this question, it is necessary first to express the classical theory in a set of variables in which the structure of the space of solutions is more apparent. This was the idea that led Ashtekar to look for, and find, what are called the new variables.

Now, we know that the self-dual Einstein equations can be solved exactly and the space of solutions characterized in terms of certain free data by using the twistor theory of Penrose\cite{41} and the related heavenly methods of Newman\cite{19}. 
Can we use this information somewhere in the construction of the quantum theory? To do this we should express the theory in a language in which the solvability of the self-dual sector is more transparent. This may be done in the following way.

Consider the Palatini variational principle for the Einstein equations,

$$ S(e, \Gamma) = \int \epsilon_{ijkl} e^i \wedge e^j \wedge R^{kl}(\Gamma). $$

Here $e^i$ is the frame field, $\Gamma^i_j$ is the $O(3,1)$ connection, whose (Yang-Mills) curvature is $R^{kl}$. I am using form notation, the explicit indices, $i, j, k, l = 0, 1, 2, 3$, are internal $O(3,1)$ indices. The metric, $g_{\alpha\beta}$ (where now $\alpha, \beta$ are the four dimensional spacetime indices), is obtained from the $e^i$ by

$$ g_{\alpha\beta} = e^i_{\alpha} e^j_{\beta} \eta_{ij} $$

where $\eta_{ij}$ is the Minkowski metric.

It has been known for a long time that Einstein’s equations are recovered if one varies both $e^i$ and $\Gamma^i_j$ in (36). What has not been known is that, actually we need only half of this action in order to get the Einstein equations. Given any antisymmetric pair of internal Lorentz indices, such as those on $\Gamma^{ij}$, we may split them into self-dual and antself-dual parts, by

$$ A_{ij}^\pm \equiv \frac{1}{2} (\Gamma_{ij} \pm i \Gamma^*_{ij}) $$

$$ F_{ij}^\pm \equiv (R_{ij} \pm i F^*_{ij}). $$

It then follows from the fact that the Lie algebra of $SO(3,1)$ splits into two commuting $SU(2)$ subalgebras that $F^+_i$ is a function only of $A_i^+$ (and similarly for the minus ones). We may then consider taking only the self-dual half of the action (36)

$$ S^{sd}(e, A^+) = \int \epsilon_{ijkl} e^i \wedge e^j \wedge F^{+\, kl}(A^+). $$

Now, this is a complex action. But, as is not hard to show, it leads to the same field equations as the Palatini action (36). Actually this is not quite true. It leads to the same field equations under the assumption that the frame fields $e^i$ have non-vanishing determinant, so that a non-degenerate metric can be formed from them by (37).

\footnote{The fact that new variables are only equivalent to the standard formulation of general relativity when the $e^i$ are nondegenerate is important for some of the developments I will be describing in chapter 5.}
The field equations that follow from (40) are

\[
\frac{\delta S^{sd}}{\delta A^j_i} = (\epsilon_{ijkl}D(e^k \wedge e^l))^+ = 0 \tag{41}
\]

\[
\frac{\delta S^{sd}}{\delta e^i} = 2\epsilon_{ijkl}e^j \wedge F^{+ \; kl} = 0 \tag{42}
\]

In the first equation \(D\) is the gauge covariant exterior derivative and the overall + means to take the self-dual part. It is straightforward to show that the equation then says that \(A^+\) is the self-dual part of the Christoffel connection of the \(e^i\). The second equation then is equivalent to the Einstein equations. This can be seen because when the first equation is used, its imaginary part vanishes by virtue of the three index antisymmetric identity of the curvature tensor\[42\].

Now, we wanted a formulation of the theory in which the simplification of the theory when it is restricted to the self-dual sector is manifest. It is easy to restrict these equations to the self-dual sector, that can be done by simply setting \(F^+ = A^+ = 0\) (at least locally). But then the field equations become

\[
(\epsilon_{ijkl}d(e^k \wedge e^l))^+ = 0 \tag{43}
\]

It is not hard to show that these are equivalent to the self-dual Einstein equations\[43, 44\].

Given the action, there is a standard procedure to construct the Hamiltonian formulation of a diffeomorphism invariant theory. This may be done by choosing a spacelike surface, \(\Sigma\), in the spacetime, and then constructing a canonical formalism to generate the changes of the fields that come from evolving that surface in the spacetime. I will give here only the results of applying this procedure to the self-dual action (40)\[42\]. The canonical coordinate of the theory turns out to be the self-dual connection \(A^+\), or rather its pullback into \(\Sigma\). I will denote that simply by \(A^i_a\), where \(a, b, c, \ldots\) will from now on refer to spatial indices in \(\Sigma\) and \(i, j, k\) will be \(SU(2)\) indices. I should note that this \(SU(2)\) is the left handed (or self-dual) part of \(SL(2, C)\), not the spatial rotation group. So \(A^i_a\) is still the spacetime connection (or its spatial components) and its curvature, denoted \(F^i_{ab}\), is the self-dual piece of the \textit{spacetime} curvature tensor, pulled back from spacetime into the three manifold \(\Sigma\).

The fact that there is a canonical formalism in which all of the components of the left handed part of the \textit{spacetime} connection commute with each other is the basic discovery of Ashtekar that makes everything that follows possible\[9\]. The canonically conjugate field to \(A^i_a\) turns out to be the pullback into \(\Sigma\) of \(e^j \vee e^k \epsilon_{ijk}\). This is easy to see from (40), it is the coefficient of the time derivative of \(A^i_a\). It is convenient to raise its spatial index, so that we have a variable,

\[
\tilde{E}^a_i \equiv e^{abc}\epsilon_{ijk}e^j_b e^k_c \tag{44}
\]
where $e^a_i$ denotes the pullback of the one form $e^i$ into $\Sigma$ and $\epsilon^{abc}$ denotes the Levi-Civita density. Thus, $\tilde{E}^a_i$ is a triplet of vector densities. (In relativity it is conventional to denote densities by a tilde, with one tilde for each weight.) All the components of $\tilde{E}^a_i$ commute with each other, moreover looking at the action we now see that the only occurrence of time derivatives is in

$$S^{ad} = \iota \int d^3x dt \tilde{E}^a_i \frac{dA^i_a}{dt} + \ldots$$

(45)

so that the canonical commutation relations are,

$$\{A^i_a(x), \tilde{E}^b_j(y)\} = \iota \delta^i_b \delta^j_a \delta^3(x,y)$$

(46)

The $\iota$ is important, it says that it is really the imaginary part of $A^i_a$ that fails to commute with $\tilde{E}^a_i$.

Recall, from the definition (38) that for real spacetimes $A^i_a$ is complex. However, it is not an arbitrary complex connection, by its definition it is a certain function of the frame fields $e^i$ and their derivatives. From the Hamiltonian point of view this is embodied in a set of conditions which are called the reality conditions. They are conditions on the $\tilde{E}^a_i$ and $A^i_a$ that guarantee that the three metric $\tilde{q}^{ab}$ and its time derivatives are real. These are a set of fourth order polynomial equations [11].

When we are working in the classical theory we must impose the reality conditions on the initial data. Once imposed, they are maintained for the whole evolution. The question then arises as to how to deal with the reality conditions in the quantum theory. The key point is that to state the reality conditions quantum mechanically requires the use of the inner product, because complex conjugation is translated into Hermitian conjugation, which uses the inner product. The problem of the quantum reality conditions is then intimately connected with the inner product. For this reason, we put off further discussion of the reality conditions till section 6.2, which discusses the inner product.

Except for that $\iota$ we can interpret $A^i_a$ and $\tilde{E}^a_i$ as being a Yang-Mills connection and its conjugate electric fields. Indeed, because $\tilde{E}^a_i$ is a density, it is well defined on our three manifold $\Sigma$, without referring to any background metric. Because of this, general relativity can be interpreted as a special kind of Yang-Mills field, which is constructed on a three manifold without any background metric or connection structure.

The dynamics of general relativity is then defined by a set of constraints. There are three sets, corresponding to the three gauge invariances of the theory. They generate, respectively, $SU(2)$ gauge transformations, diffeomorphisms of the three surface $\Sigma$, and evolution of the surface $\Sigma$ in the spacetime. They are called, correspondingly, the Gauss’s law, diffeomorphism and hamiltonian constraints. They are expressed as,

$$G^i \equiv D_a \tilde{E}^{ai} = 0$$

(47)
\[ C_a \equiv F_{ab}^i \tilde{E}_i^b = 0 \quad (48) \]
\[ C \equiv \epsilon_{ijk} F_{ab}^k \tilde{E}_i^a \tilde{E}_j^b = 0 \quad (49) \]

We may note that the first is familiar from Yang-Mills theory and the second and third are the simplest gauge invariant combinations of \( F_{ab}^i \) and \( \tilde{E}_i^a \) that could be written down. That is, if someone had started out trying to write down a canonical formulation of a Yang-Mills theory on a three manifold that has no background structure, that someone would have reinvented general relativity.

Indeed, the other simple terms that one could add to (47-49) all correspond to well known modifications of general relativity. For example, the cosmological constant can be added by changing (49) to

\[ C^\Lambda \equiv \epsilon_{ijk} F_{ab}^k \tilde{E}_i^a \tilde{E}_j^b + \Lambda \epsilon_{abc} \epsilon^{ijk} \tilde{E}_i^a \tilde{E}_j^b \tilde{E}_k^c = 0 \quad (50) \]

4 Results about quantum geometry

We now have the background to describe the calculations that went into the description of quantum geometry I gave in chapter 2. This chapter is organized into a series of seven sections, each of which contains the details of the calculations of a result used in that sketch.

The results described in this chapter were found in collaboration with Abhay Ashtekar and Carlo Rovelli. They are new, and have not been reviewed previously. They are described in references [1, 2] by Ashtekar, Rovelli and the author.

4.1 Why the metric at a point is not a good operator

In the Ashtekar formulation, the metric is most simply expressed in terms of the doubly densitized inverse metric \( \tilde{q}_{ab} \), which is given by,

\[ \tilde{q}_{ab}(x) = \tilde{E}_i^a(x)\tilde{E}_i^b(x) \quad (51) \]

The problem is that, as a product of elementary fields at a point, there is, in the loop representation, no elementary operator that corresponds to this quantity or to any other function of the metric at a point. In order to construct an operator corresponding to \( \tilde{q}_{ab} \) we must build it by means of a regularization procedure. This means that we must find a sequence of observables \( \tilde{q}^{ab}_\epsilon \) such that classically \( \tilde{q}_{ab} = \lim_{\epsilon \to 0} \tilde{q}^{ab}_\epsilon \), and such that each one of them, for \( \epsilon > 0 \), has a representation as a well defined operator. The purpose of this section is to show that in the loop representation any such procedure is bound to fail when applied to a measurement of the metric at a point. It will fail because

\[ ^{25}\text{Except for some comments on the weaves in the review of Rovelli. [14]} \]
the resulting operator will necessarily depend on auxiliary background structure introduced in the regularization procedure. In a nonperturbative quantization, in which there are no such background structures, such a result is unacceptable.

We will proceed by studying a class of regularization and renormalization procedures that can be used to define an operator corresponding to $\tilde{q}^{ab}(x)$. From the results we will be able to draw some general conclusions as to why any such procedure must fail.

The regularization procedures that we will study are based on the old idea of point splitting. That is, we want to pull apart the two $\tilde{E}^i_a$’s in (51) and use

$$
\tilde{q}^{ab}(x) = \lim_{y \to x} \tilde{E}^a_i(x) \tilde{E}^b_i(y).
$$

Because in the loop representation we keep manifest the internal $SU(2)$ gauge invariance we will consider a gauge invariant version of this. To do this we make use of a useful set of observables which generalize the $T^a$ defined in (17). Given a loop $\gamma$ and $n$ points on it $s_1, \ldots, s_n$ we may define,

$$
T^{a_1 \cdots a_n}[\gamma](s_1, \ldots, s_n) \equiv Tr \left[ \tilde{E}^{a_1}(\gamma(s_1)) U(\gamma(s_1, s_2)) \cdots \tilde{E}^{a_n}(\gamma(s_n)) U(\gamma(s_n, s_1)) \right]
$$

These are called the $T^n$ observables, where $n$ labels the number of insertions of the $\tilde{E}^a_i$’s. The set of all the $T^n$ form a graded Poisson algebra that generalizes (18) with $\{ T^n, T^m \} \approx T^{n+m-1}$ (see ref. [13]). The whole algebra can be defined as operators on the loop representation, which we have already defined as a representation of the subalgebra of the $T^0$ and $T^1$’s.

In order to write their actions in a simple form we must take a moment and expand our notation as to how we denote loops formed from combinations of loops that intersect. First of all, in the case that there is more than one intersection point, we denote $\alpha \circ_s \beta$ to be the loop which combines $\alpha$ and $\beta$ at the intersection point labeled by the parameter $s$ of the first loop. If there is no intersection point at $\alpha(s)$ then $\alpha \circ_s \beta$ will denote the trivial loop. Second, if the two loops intersect at more than one point, we have the possibility of breaking the two loops simultaneously at more than one intersection point and putting them together in the various reroutings. We will denote the loops formed by breaking and joining $\alpha$ and $\beta$ at two intersection points and then rerouting by $(\alpha \circ_{s_1} \circ_{s_2} \beta)_r$ where $r$ labels the possible reroutings. If there is no ambiguity, we can drop the labels on the intersection points and write the reroutings as $(\alpha \circ \beta)_r$. In general, the act of breaking and joining at $n$ points will be denoted by $n \circ$’s.

We may now write the definition of the general $\tilde{T}^n$

$$
\tilde{T}^{a_1 \cdots a_n}[\gamma](s_1, \ldots, s_n) \Psi[\alpha] = \int_{\mathbb{P}^n} \oint dt_1 \delta^3(\gamma(t_1), \alpha(s_1)) \dot{\alpha}(s_1)^{a_1} \cdots \times \oint dt_n \delta^3(\gamma(t_n), \alpha(s_n)) \dot{\alpha}(s_n)^{a_n}
$$
The sum over $r$ is over the $2^n$ loops obtained by rearranging the routings at the $n$ intersections in all the possible ways. $q_r$ is the number of segments between intersection and intersection that must change orientation in order to have a consistent orientation in the loop obtained by combining the loop $\gamma$ in the operator and the loop $\alpha$ in the state.

An informal nomenclature has developed in conjunction with these operators: the points where the $\tilde{E}_{ai}$’s are represented on the loop of the operator are called the ”hands” of the operator. We say that a hand acts by grasping a loop in the state. The result of such a grasp is to multiply the state by a distributional factor which is,

$$l_p^2 \int dt \delta^3 (\gamma(s), \alpha(t)) \dot{\alpha}^a(t)$$

(55)

where $\gamma$ is the loop associated with the operator, $s$ is the loop parameter at the hand and $\alpha$ is the loop in the state. Thus, we see that the action is zero unless every hand grasps the loop in the argument of the state.

The state is then evaluated on a loop which is gotten by combining the two loops $\gamma$ and $\alpha$ in the following way. One breaks them apart at each of the hands where they meet. Then there are two ways to rejoin them at each hand: so that their orientations agree or disagree. The result is gotten by summing over the evaluation of the state at these $2^n$ new loops, with a phase factor given by $(-1)^{q_r}$.

The appearance of $l_p^2 = \hbar G_{Newton}$ in the action (54) has profound consequences, as we will see below and so deserves some comment here. First of all, from dimensional analysis, the action of a hand must be dimensionless because it represents an insertion of a frame field, which, being a square root of a metric, is dimensionless. Thus, there must be a factor with units of area in front of (55). Why is this the Planck area? The $G_{Newton}$ comes from the fact that there is a $G_{Newton}$ in the definition of the parallel propagator (15). It is there because Ashtekar’s connection $A^a_i$ actually has dimensions of $(\text{length})^{-3}$. Thus, the action of an $\tilde{E}_{ai}$ in a classical poisson bracket on a holonomy brings down, by (12), a factor of $G_{Newton}$. The $\hbar$ is then there in the definition of the actions (20) and (54) on states so that the quantum commutators of these operators are equal to $\hbar$ times the corresponding Poisson brackets.

Let us now return to the problem at hand, which is to construct an operator to represent the spatial metric $\tilde{q}_{ab}$ at a point $x$. What we want to do is to represent this as the limit of a $T^2$ as the points where the two $\tilde{E}_{ai}$’s live are brought together. The regulated observable will then depend on a loop $\gamma$ that contains these two points. The choice of this loop is completely arbitrary, for simplicity we may introduce a uniform way of choosing it.
We may introduce an arbitrary flat background metric $h^0_{ab}$ in a neighborhood, $\mathcal{U}$, of $x$. Given any two points $y$ and $z$ in $\mathcal{U}$ let us define a loop, $\gamma_{y,z}$ which is a metric circle according to $h^0_{ab}$ and which satisfies $\gamma_{y,z}(0) = y$ and $\gamma_{y,z}(\pi) = z$. It then follows that the limit $z \to y$ $\gamma_{y,z}$ is the degenerate loop in which the whole circle is mapped to the point $y$.

For short, we will define,

$$T^{ab}[y, z] = T^{ab}[\gamma_{y, z}(0, \pi)] \quad (56)$$

We need one final element for our regularization procedure: a smearing function. We introduce a regulation of the delta function by means of a smooth function of compact support $f_\epsilon(x, y)$ such that

$$\lim_{\epsilon \to 0} f_\epsilon(x, y) = \delta^3(x, y) \quad (57)$$

and

$$\int d^3x f_\epsilon(x, y) = 1. \quad (58)$$

For detailed calculations we will take for $f_\epsilon(x, y)$ a gaussian,

$$f_\epsilon(x, y) = \frac{\sqrt{h^0(x)}}{\pi^{3/2} \epsilon^3} e^{-\frac{1}{2\epsilon^2} |x - y|^2}, \quad (59)$$

where the norm $|...|$ is defined with respect to the background metric $h^0_{ab}$.

We now may define our regularization of the densitized inverse metric. It is

$$G^{ab}_\epsilon(x) = \int d^3y \int d^3zf_\epsilon(x, y)f_\epsilon(x, z)T^{ab}[y, z]. \quad (60)$$

For finite $\epsilon$ each of the corresponding operators is well defined on the kinematical representation space $S^\text{kin}_{\Sigma}$. Furthermore, when evaluated on smooth geometries

$$\lim_{\epsilon \to 0} G^{ab}_\epsilon(x) = \tilde{q}^{ab}(x). \quad (61)$$

What we need to do now is to study the same limit in the quantum theory. Let us first act with the operator version of (60), on a quantum state $\Psi[\alpha]$. The result is

$$G^{ab}_\epsilon(x)\Psi[\alpha] = \mathcal{L}^a_{\tilde{q}} \int ds \int dt \dot{\alpha}(s) \ddot{\alpha}(t) f_\epsilon(x, \alpha(s)) f_\epsilon(x, \alpha(t)) \quad (62)$$

$$\times \left( \sum_r (-1)^r \Psi[\gamma_{\alpha(s)\alpha(t)} \circ \alpha_r] \right).$$

We now want to consider what happens to this as $\epsilon \to 0$. There are two factors to consider: what happens in the sum over states inside the parenthesis.
in (62) and what happens to the c-number factor multiplying it. We consider first the factor in the loops, as this will be practice for similar calculations we will have to do later.

In the sum over the reroutings in (62) there are two kinds of terms. The first are terms of the form \( \Psi[\alpha \cup \eta_\epsilon] \) where \( \eta_\epsilon \) is a loop that is shrinking to a point as \( \epsilon \to 0 \). The second kind are terms of the form \( \Psi[\beta_\epsilon] \) where \( \lim_{\epsilon \to 0} \beta_\epsilon = \alpha \).

The reader is invited to write out this sum to verify that there are two terms of each kind and that all have, by the rules given after equation (54) for signs, positive signs.

We then have to evaluate the following limits

\[
\lim_{\epsilon \to 0} \Psi[\alpha \cup \eta_\epsilon] \quad \text{and} \quad \lim_{\epsilon \to 0} \Psi[\beta_\epsilon]
\]

We will take the meaning of the limit to be such that

\[
\lim_{\epsilon \to 0} \Psi[\beta_\epsilon] = \Psi[\alpha]
\]

whenever \( \lim_{\epsilon \to 0} \beta_\epsilon = \alpha \). This interpretation of the limit involves a certain subtlety, which requires discussion. This will be the subject of the next section. For now, the reader may verify that given this last relation, (28) and (30) together imply that

\[
\lim_{\epsilon \to 0} \Psi[\alpha \cup \eta_\epsilon] = 2\Psi[\alpha]
\]

It then follows that the sum over the reroutings in (62) yields after the limit a factor of 6 times \( \Psi[\alpha] \).

We now come to the remaining c-number factor. It is easy to see that this is divergent as \( \epsilon \to 0 \). With the particular case that the smearing function is chosen as (59) it is not hard to show that the leading divergent factor is

\[
\lim_{\epsilon \to 0} \left[ \int ds \int dt \dot{\alpha}^a(s) \dot{\alpha}^b(t)f_\epsilon(x, \alpha(s))f_\epsilon(x, \alpha(t)) \right]
\]

\[
= \frac{l_\text{P}^4 \sqrt{\hbar_0(x)}}{\epsilon^2} \int \frac{ds}{|\dot{\alpha}(s)|} \delta^3(x, \alpha(s)) \dot{\alpha}^a(s) \dot{\alpha}^b(s)
\]

This divergence, of course, reflects the fact that we are trying to multiply two distributions. In order to define the product, we must modify the definition of the limit (61). This can be done by \textit{renormalizing} the observable. We thus define,

\[
(\hat{G}^{\text{ren}})_{ab}(x) = \lim_{\epsilon \to 0} Z \frac{\epsilon^2}{l_\text{P}^4} \hat{G}^{\epsilon}_{ab}(x)
\]

where \( Z \) is an arbitrary renormalization constant. This limit is finite. We find,

\[
(\hat{G}^{\text{ren}})_{ab}(x)\Psi[\alpha] = 6l_\text{P}^2 \sqrt{\hbar_0(x)} Z \int \frac{ds}{|\dot{\alpha}(s)|} \delta^3(x, \alpha(s)) \dot{\alpha}^a(s) \dot{\alpha}^b(s) \Psi[\alpha].
\]

36
We have thus succeeded in defining an operator to represent the inverse three metric on the loop states. But at a cost: the final form is not really satisfactory, because it depends on the background metric \( h^{0}_{ab} \). That dependence occurs in two places, in the factor of \( |\dot{\alpha}| = \sqrt{h^{0}_{ab}\dot{\alpha}^{a}\dot{\alpha}^{b}} \) in the denominator and in the overall factor of \( \sqrt{h^{0}} \) that multiplies the result. Since the background metric \( h^{0}_{ab} \) is completely arbitrary, this means that the observable \( \tilde{q}^{ab} \) has only been determined up to an overall conformal factor.

We must then ask whether this background dependence is a consequence of the particular regularization method we have used, or whether it is a general consequence of trying to define an operator to represent \( \tilde{q}^{ab}(x) \) through a regularization and renormalization procedure. A simple argument makes a strong case for the latter conclusion. Because the state depends only on the loop \( \alpha \), it is natural to expect that a renormalized operator which represents \( \tilde{q}^{ab}(x) \) must have an action which depends only on the loop. Thus, the action should be proportional to a distribution that has support on \( \dot{\alpha}^{a} \). Furthermore, geometrically, the only vector field in the problem is \( \dot{\alpha}^{a}(s) \). Thus, the two indices in the action of \( G_{ren}^{ab}(x) \) must be proportional to \( \dot{\alpha}^{a}(s) \dot{\alpha}^{b}(s) \). Thus, making use of the way the loops collapsed in the argument of the state, if \( G_{ren}^{ab}(x) \) can be defined at all it must be of the form,

\[
G_{ren}^{ab}(x)\Psi[\alpha] = \tilde{F}(x) \int \frac{ds}{J(s)} \delta^{3}(x, \alpha(s))\dot{\alpha}^{a}(s)\dot{\alpha}^{b}(s)\Psi[\alpha].
\]

Let us note, first, that the factor \( J(s) \) must transform under reparametrizations of the curve like a one dimensional density, so that the overall expression preserves the reparametrization invariance of the formalism. In the regularization we have just discussed this factor is equal to \( \sqrt{\vert \dot{\alpha}(s) \vert} \). However, in general, we know that some background structure will be needed to construct such a one dimensional density, because there is none that can be constructed from the geometrical information at hand.

The external factor \( \tilde{F}(x) \) turned out in our regularization to be equal to \( 6l_{p}^{2}\sqrt{h^{0}}Z \). This is, as we pointed out above, a free function. This means that the observable has no more information in it other than that it lives in the \( \dot{\alpha}^{a}\dot{\alpha}^{b} \) subspace. We may ask, however, whether it is possible that another regularization method will leave us with a constant \( F \). The answer is that this is impossible. The reason is because the delta function \( \delta^{3}(x, y) \), in addition to being a distribution, is a density in its first argument\(^{26}\). The \( E^{ai}(x) \) is a density and its product \( \tilde{q}^{ab} \) is thus a double density. Now, any method that defines the product must give a rule to show how the product of two distributions is proportional to a single distribution. However, we see that the factor of proportionality must be a density, in order to preserve the fact that the product is only well defined in the presence of a flat background metric.

\(^{26}\)This is why it is written \( \delta^{3}(x, y) \), rather than \( \delta^{3}(x - y) \). The latter expression is only well defined in the presence of a flat background metric.
a double density. This is why the $\tilde{F}(x)$ came out to be a density. Further, as there is no natural density in the problem, if the extension is to be well defined, the overall density factor must come from the background structure that is used in the regularization procedure.

From this general situation there follows an important lesson about diffeomorphism invariant quantum field theories. The lesson is that renormalization will not normally be an acceptable way to define diffeomorphism invariant observables. A renormalization procedure is a procedure to multiply two well defined local operators, defined at a single point, and get another local operator. Since operators are distribution valued we must also, in the absence of a background metric, take into account the fact that any such procedure must either change the density character of the observable or introduce an arbitrary density as an overall factor in the definition of the renormalized operator. Either way, the diffeomorphism invariance of the theory is compromised.

Thus, to conclude: we have shown in this section how to define a regularization and renormalization procedure for the inverse metric at a point. We succeeded, in that the result was a well defined operator on the kinematical representation space $\mathcal{S}_{\text{kin}}^\Sigma$. However, the resulting form of the operator was infected with background dependence. This background dependence renders the result meaningless in a nonperturbative context. Furthermore, we argued that this background dependence is bound to infect any attempt to construct an operator to represent the metric at a point in the loop representation.

All, however, is not lost. In the next several sections we will show that the problem is not with the loop representation, it is with the idea that the metric can be observed at a point of space. We will do this by showing that several different operators that measure metric information on surfaces and regions may be defined through regularization procedures that result in operators that are finite and independent of any unphysical background fields. However, first, some clarification about our state space.

### 4.2 More about representations of quantum theories with discrete norms

This section is devoted to the explication of a certain subtlety associated with the use of characteristic states and discrete measures in quantum field theories. It may be skipped on a first reading.

In the previous section we made an assumption about the how the limits in the regularization of operator products are to be taken. For the evaluation of (67), and for later calculations, we will need it to be true that the limits that will appear in the regularization of operators in the loop representation are taken so that the following conditions are satisfied. If $\eta_\epsilon$ is a loop of radius $\epsilon$, in some background coordinates, then it should be true that

$$\lim_{\epsilon \to 0} \langle \alpha \cup \eta_\epsilon \rangle = 2 \langle \alpha \rangle$$

(70)
and
\[ \lim_{\epsilon \to 0} < \alpha \circ \eta \epsilon | = < \alpha |. \quad (71) \]

As I mentioned above, the former expression actually follows from the latter, together with the identities (28) and (30) we are imposing on the loop space. Now, (70) and (71) are guaranteed if the loop functionals satisfy some condition of continuity or differentiability on the loop space. For example, suppose that
\[ \lim_{\epsilon \to 0} (\Psi[\alpha \circ \eta \epsilon] - \Psi[\alpha]) \epsilon^2 = \nabla \eta \Psi[\alpha] \quad (72) \]
exists, as is often the case in applications of the loop representation to conventional quantum field theories. (This is the definition of the loop derivative. For more information about it see \cite{31} and references contained therein.) Then we have
\[ \Psi[\alpha \circ \eta \epsilon] = \Psi[\alpha] + \epsilon^2 \nabla \eta \Psi[\epsilon] \quad (73) \]

There are, however, two problems with this when it comes to the application to quantum gravity. The first is that the diffeomorphism invariant states are not loop differentiable. They are not even continuous on the loop space. The reason is that \( \Psi[\alpha] \) and \( \Psi[\alpha \circ \eta \epsilon] \) are in different topological classes for all \( \epsilon \neq 0 \) and that, moreover, \( \Psi[\alpha \circ \eta \epsilon] \) does not depend on \( \epsilon \) as long as \( \epsilon \neq 0 \). Thus, taken on diffeomorphism invariant states, the limit in (72) diverges as \( 1/\epsilon^2 \).

The second problem is that, even at the kinematical level, none of the states in the space \( \mathcal{S}^\text{kin}_\Sigma \) with the norm (3) are loop differentiable.

As the calculations discussed in this chapter are done at the kinematical level, we will discuss the second of these issues. The first, having to do with the application of the loop derivative to diffeomorphism invariant states has not yet been resolved.

It will be helpful if we begin by talking about an analogous problem in one dimensional quantum mechanics. Although it is not usually done, we can introduce characteristic states and discrete measures in there if we relax a little bit what we require of the state space\(^\text{27}\).

Suppose we wanted a representation of one dimensional quantum mechanics in which \( \hat{x} \) and \( \hat{T}(\epsilon) \) were well defined operators, where \( \hat{T}(\epsilon) \) is the translation operator defined by \( \hat{T}(\epsilon) \psi(x) = \psi(x + \epsilon) \), but one did not require that there exists an operator \( \hat{p} \) such that \( \hat{T}(\epsilon) = e^{i\hat{p}\epsilon} \). Instead, one wants to construct the quantum theory to be a representation of the algebra
\[ \hat{T}(-\epsilon) \hat{x} \hat{T}(\epsilon) = \hat{x} + \epsilon \quad (74) \]

In this case, because one does not insist that \( \hat{p} \) exist, the usual uniqueness theorem does not apply and one can construct a representation which is inequivalent to the usual \( L^2 \) hilbert space. This representation consists of states which are
\[ ^\text{27} \text{Here I am following I line of thought that I learned from Abhay Ashtekar} \]
normalizable with respect to the discrete measure, which is based on the discrete topology on \( R \), with respect to which each point is its own open set. Given this discrete topology, we may construct the discrete measure

\[
\int_{\mathbb{R}} d\mu(x)_{\text{discrete}} F(x) = \sum_{x \in \mathbb{R}} F(x)
\]

(75)

The normalizable functions under this measure are those that have support on only a countable set of points \( x \in \mathbb{R} \), so that the integral in (75) converges. A normalizable basis for this Hilbert space is then given by the characteristic states,

\[
\psi_y(x) = \delta_{xy} \equiv 1 \text{ if } x = y, \text{ and } 0 \text{ otherwise.}
\]

(76)

It is straightforward to show that this space of functions is a representation\(^{28}\) for the algebra defined by (74) and that, with respect to the discrete measure, \( \hat{x} \) and \( \hat{T}(\epsilon) \) are hermitian and unitary operators, respectively. Further (although it is not important for the one dimensional quantum mechanics), it is interesting to note that this representation does not depend on a background metric on \( R \), so that a unitary representation of the one dimensional diffeomorphism group exists on this space.

If we use the loop variables (14) and (17), with the algebra (18) as the basis for quantization of gravity or a gauge theory, then we are in an analogous situation to the case we have just considered because we want a representation in which there exists an operator for \( \tilde{E}^{ai} \), (or for a gauge invariant operator polynomial in it) but we do not require there to exist any operator for its canonical conjugate \( A_i^a \). Instead, the operators which do not commute with polynomials in \( \tilde{E}^{ai} \) are non-local operators, of the form of \( \hat{T}[\gamma] \), which are exponential in \( A_i^a \).

Now, the algebra (74) of \( \hat{x} \) and \( \hat{T}(\epsilon) \) can be represented both in terms of the standard \( L^2(R) \), in which case the operator \( \hat{p} \) exists, or it can be represented on the space of states normalized with respect to (75), in which case, as can easily be shown, no operator for \( \hat{p} \) exists. Similarly, as shown recently by Ashtekar and Isham\(^{20}\), the loop algebra (25) has several kinds of representations. There are Fock representations, in which operators linear in \( A_i^a \) also exist. But there are also representations which are defined on the space \( \mathcal{S}_{\mathbb{R}^n}^{\mathbb{C}} \) of functions which are normalizable with respect to the discrete measure (3). (The reader may verify that the defining relations (19) and (26) are well defined on this space of loop functionals.) These representations do not admit any operator for \( A_i^a \) and they are inequivalent to the Fock representation.

There are cases, such as Maxwell theory and linearized gravity, where the existence of gravitons and photons requires that there be operators linear in \( A_i^a \). In this case, even if one begins with a loop representation defined by (19) and (26) one is forced back to the Fock representation. However, in the nonperturbative context there should be no graviton operator, as these are only

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\(^{28}\)This is given by \( \hat{T}[\epsilon] \psi_y(x) = \psi_{y - \epsilon}(x) \) and \( \hat{x} \psi_y(x) = y \psi_y(x) \).
defined with respect to a fixed classical background, and we have no reason to insist that $A^n_i$ itself be represented by an operator. In this case we could choose one of these new alternative representations.

Given a choice between different inequivalent representations, the choice must be made on physical grounds. There are, in my opinion, two reasons for choosing these non-Fock representations at the kinematical level for the construction of quantum gravity. First, as I have already mentioned, the Fock representations necessarily break diffeomorphism invariance, while the discrete measure on loop space is diffeomorphism invariant. The second reason, which is based on results that we will derive in the rest of this chapter, is that the classical limit of the theory is well defined in the case that we restrict the state space to a non-Fock representation. Thus, it is not necessary to consider a larger class of states to recover known physics.

We can now return to the technical problem at hand, which is how to take the limits of shrinking small loops involved in the regularization procedure inside of this state space. We will find it helpful to use the one dimensional example to understand the source of the problem.

The problem is to specify the topology in which the limit $\epsilon \to 0$ is to be taken in expressions such as (67). Let me first show that the first, guess, which is to use the pointwise topology on the loop space, cannot work. For, using the pointwise topology on the loop space to define the limit, we have

$$\left( \lim_{\eta \to \beta} \Psi_\eta[\alpha] \right)_{\text{pointwise}} = 0 \neq \Psi_\beta[\alpha] \quad (77)$$

As a result, the topology based on the discrete norm (3) is also useless. For if we use an inner product based on the discrete measure on loop space, such as

$$\langle \beta | \alpha \rangle = \delta_{\alpha\beta}, \quad (78)$$

where $\delta_{\alpha\beta}$ is the kronecker delta, we have

$$\lim_{\eta \to \beta} \langle \eta | \beta \rangle = \lim_{\eta \to \beta} \delta_{\eta\beta} = 0 \quad (79)$$

This is a general problem which arises when one uses a state space based on a discrete norm. It occurs also in the one dimensional example. Using the pointwise topology we have

$$\left( \lim_{z \to y} \psi_z[x] \right)_{\text{pointwise}} = 0 \neq \psi_y[x] \quad (80)$$

A characteristic problem which occurs as a result has to do with the operator $\hat{T}(\epsilon)$. According to the definition $\hat{T}(\epsilon)\psi(x) = \psi(x + \epsilon)$ this operator is well

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29 We will see in the next section that this is the form of the inner product when $\alpha$ and $\beta$ are simple loops.
defined on the space of functions normalizable with respect to (75) for all \( \epsilon \in R \). In particular, \( \hat{T}(0) \), is well defined and is equal to the identity operator. However, if we use the pointwise topology to define the limit it is not true that \( \lim_{\epsilon \to 0} \hat{T}[\epsilon] = I \). Instead, if we define the limit in the pointwise topology we have

\[
\lim_{\epsilon \to 0} \left( \hat{T}(\epsilon) \psi_y(x) \right) = \lim_{\epsilon \to 0} \psi_{y-\epsilon}(x) = 0
\]

(81)

The problem reflects the fact that on the space of states which are normalizable with respect to the discrete norm, (75), the operator \( \hat{T}(\epsilon) \) is not differentiable at \( \epsilon = 0 \). This is, of course, necessary, because if the derivative did exist it would be equal to \( \hat{p} \), which cannot exist in this representation.

It is then clear that to preserve the property that \( \lim_{\epsilon \to 0} \hat{T}(\epsilon) = 1 \), even when its action is extended to states that are normalizable under the discrete norm (75), we have to use a different topology to define the limit. What we need is to induce on the states the standard continuous topology on \( R \) such that

\[
\left( \lim_{z \to y} \psi_z(x) \right)_{\text{continuous}} = \psi_y(x).
\]

(82)

We can do this just as a matter of definition. We may also observe that the use of such a topology is equivalent to using the topological dual of the state space to define the limits of operators. Let us note, first of all, that under the discrete norm (75) normalizable functions are dual to the smooth bounded functions, in the sense that for any smooth and bounded \( f(x) \)

\[
\int d\mu_{\text{discrete}}(x) f(x) \left( \sum_i a_i \psi_{y_i}(x) \right) = \sum_i a_i f(y_i) < \infty
\]

(83)

because \( \sum_i |a_i|^2 < \infty \). We may then define the limit of the translation operators such that

\[
\sum_{x \in R} f(x) \hat{T}(0) \psi_y(x) = \lim_{\epsilon \to 0} \left( \sum_{x \in R} f(x) \hat{T}(\epsilon) \psi_y(x) \right)
\]

(84)

for all smooth \( f(x) \). By this definition it follows that \( \hat{T}(0) = I \).

We can then use these ideas to resolve the problem with the limits of our regulated operators in the loop representation. In the remainder of this chapter will use the hilbert space of states \( S^\text{fin}_\Sigma \) which are normalizable with respect to the discrete measure on the loop space \( \mathcal{H[L[\Sigma]]} \). However, when we have to define an operator on this space through a regularization procedure, we will do this not by using the pointwise topology on the space of states but the continuous topology which is defined so that

\[
\left( \lim_{\eta \to \beta} \Psi_\eta[\alpha] \right)_{\text{continuous}} = \Psi_\beta[\alpha].
\]

(85)
We can define this limit in terms of the topological dual, as we did for the one dimensional case. This requires that we define smooth functions on \( \mathcal{H}[\Sigma] \). These will be defined to be those functions on \( \mathcal{H}[\Sigma] \) which are infinitely differentiable under the loop derivatives (72) defined in [1]. We then require that a sequence of operators \( \hat{O}(\epsilon) \) will be said to converge to an operator \( \hat{O}_0 \) if, for all smooth functions \( \Phi[\alpha] \) on \( \mathcal{H}[M] \) and all normalizable states \( \Psi[\alpha] \) in \( S_{kin}^{\Sigma} \) we have

\[
\lim_{\epsilon \to 0} \sum_{\alpha \in \mathcal{H}[M]} \Phi[\alpha] \hat{O}(\epsilon) \Psi[\alpha] = \sum_{\alpha \in \mathcal{H}[M]} \Phi[\alpha] \hat{O}_0 \Psi[\alpha]
\] (86)

### 4.3 The construction of the area operator

In this section, I show how to define a quantum operator that measures \( A[S] \), the area of an arbitrary surface \( S \), in \( \Sigma \).

The key idea involved in this construction is really an idea about regularization of the action of the \( T^n \) operators defined by (54). Recall that the action of each hand at a point \( \gamma(s) \) on a state \( \Psi[\alpha] \) is given by (55). Recall also from (21) that the dual of an \( \tilde{E}_{ab} \) is a two form. As in the construction of the ribbon operator (23) we can integrate this two form against a surface. Thus, if we let the coordinates of the surface be \( S^a \) we have

\[
\int_S d^2S^{bc}...\tilde{E}^*_{bc}(S)...\Psi[\alpha] = \frac{l_P^2}{\hbar} \int_S d^2S^{bc} \int dt \delta^3(S, \alpha(t)) \dot{\alpha}^a(s) \epsilon_{abc}...
\] (87)

Here \( I[S, \alpha] \) is the intersection number defined by (24) and the ...’s indicate other factors that make the result gauge invariant.

This makes sense of the factor of the Planck area sitting in from of the action (55). It tells us that the hands are naturally associated with quanta of area; integrated over a surface, the hand wants to give a unit of the Planck area to each line it meets in its action on a state.

The idea is to make use of this pretty fact by finding a way to write the area of a surface \( S \) in terms of the two forms \( E^*_{ab} \)’s integrated over the surface.

To show how to do this we begin by writing the usual expression for \( A[S] \) in the classical theory,

\[
A[S] = \int_S \sqrt{\hbar},
\] (88)

to be little choice if we are to use a state space based on the discrete measure. The example of taking the limit of \( \hat{T}(\epsilon) \) as \( \epsilon \to 0 \) shows us that when dealing with state spaces based on discrete norms, we must use such a topology if we are not to get nonsensical results for the action of operators. Further, we should recall that the purpose of a regularization procedure is only to define a certain operator on the state space in question. We are free to do this in any way we like, as long as the operator defined by the limit exists and has reasonable properties.
where \( h \) is the determinant of the induced two metric, \( h_{ab} = q_{ab} - n_a n_b \), where \( n^a \) is the unit normal to the surface. A simple calculation shows that \( h = \tilde{q}_{ab} n_a n_b \).

Now, to avoid the problem that \( \tilde{q}_{ab} \) cannot itself be extended to distributional loop geometries, we must construct the area (88) through a limit that does not need this extension. To do this, let us divide the surface up into \( N \) disjoint regions \( S_i \), such that \( S = \cup_i S_i \). We then have

\[
A[S] = \sum_i A[S_i] \tag{89}
\]

We will proceed by introducing an approximation for the square of \( A[S_i] \) which becomes exact in the limit of infinitesimal surfaces. This is,

\[
A^2_{\text{approx}}[S_i] \equiv \int d^2S_i^{ab} \int d^2S_i'{}^{cd} T^{**}(S, S')_{ab}{}^{cd} \tag{90}
\]

where \( T^{**}(x, y)_{ab}{}^{cd} \equiv \epsilon_{abc} \epsilon_{cdf} \tilde{T}^{ef}(x, y) \) and the latter quantity is that we defined in eq. (56). To show that this approximates the area of the surface element for small surfaces, we use the facts that in the limit \( T^{ab}(S, S') \approx \tilde{q}^* (S) \). We may invert the relation \( h = \tilde{q}_{ab} n_a n_b \) to find that \( \tilde{q}^* = h n^a n^b - r_{ab} \) where \( r_{ab} n_b = 0 \). An infinitesimal element of area is given by \( dA = d^2S^{ab} \sqrt{hn^a \epsilon_{abc}} \), from which it follows that,

\[
dA^2 = d^2S^{ab} d^2S^{cd} \epsilon_{abc} \epsilon_{cdf} \tilde{q}^{*f} \tag{91}
\]

For smooth fields this is then equal to (90) in the limit of small areas. We may then consider the limit in which we divide the surface up into smaller and smaller elements, so that \( N \to \infty \). It then follows that,

\[
A[S] = \lim_{N \to \infty} \sum_{i=1}^N \sqrt{A^2_{\text{approx}}[S_i]} \tag{92}
\]

For smooth, nondegenerate, metrics this is a long way round to go to define the area. But, because it incorporates the approach to regularization I described above, this particular definition can be carried over to the quantum theory.

I will then introduce an operator for the area of an arbitrary surface, \( S \), which I will call \( A[S] \) by following the same procedure that we found just gave the operator classically. We define the operator by equation (92), with \( A^2_{\text{approx}}[S_i] \) defined by equation (90), but this time with the \( T^{**} \) taken as the operator. Let us assume that in dividing up the surface we have taken \( N \) big enough that a given curve, \( \alpha \), intersects each \( S_i \) in the partition at most once. (For each \( \alpha \) there is an \( N \) such that this is the case.)

Let us then compute

\[
\hat{A}^2_{\text{approx}}[S_i] \Psi[\alpha] = t^4 P \int d^2S_i^{bc} \oint dt \delta^3(S, \alpha(t)) \dot{\alpha}^a(t) \epsilon_{abc} \tag{93}
\]
Using (24), we see that the c-number factors give the square of the intersection number. As \( \alpha \) intersects the surface element \( S_i \) at most once the sum over loop factors is easy. Let us for the moment restrict ourselves to the case that the loop \( \alpha \) is simple, which means it has no intersections or retracings. Then both hands actually grasp on to the loop \( \alpha \) at the same point. In this case the routings may be defined by taking the curve joining the hands, \( \gamma_{x,x} \) to be first a circle (with respect to some background Euclidean coordinates, of radius \( \delta \) and then shrinking \( \delta \) (and hence the circle) to zero. A factor of 6 comes out because the counting is then the same as in (62), in our attempt to construct a renormalized metric operator.

We then have,

\[
\hat{A}_2^{\text{approx}}[S]|\Psi[\alpha]\rangle = (l_p^2 I[S_i, \alpha])^2 6 \Psi[\alpha]\]  \tag{94}

It is then trivial to take the square root, to sum up the contributions from the different subsurfaces and then take the limit \( N \to \infty \). The result is,

\[
\hat{A}[S]|\Psi[\alpha]\rangle = \sqrt{6} l_p^2 I^+[S, \alpha]|\Psi[\alpha]\]  \tag{95}

where \( I^+[S, \alpha] \) is the oriented intersection number, which is simply the number of intersections of \( \alpha \) with \( S \), with all intersections counted positively.

From this result we can deduce immediately from (27) that for simple \( \alpha \) the bra states \( <\alpha| \) are eigenstates of \( \hat{A}[S] \):

\[
<\alpha|\hat{A}[S]|\Psi[\alpha]\rangle = \sqrt{6} l_p^2 I^+[S, \alpha]|\Psi[\alpha]\]  \tag{96}

Note that the result is finite, with no need of renormalization. Further, we see that the spectrum of the area operator in the loop representation contains a discrete sector, in which the eigenvalues are quantized in integer multiples of the Planck area. In the next section we will argue that, at least on the space of states normalizable under (3), the full spectrum is discrete.

We may also solve the eigenspace equation on the right and find a state \( \Psi_\alpha \), which has the property that for any loop \( \gamma \)

\[
\hat{A}[S]\Psi_\alpha[\gamma] = \sqrt{6} l_p^2 I^+[S, \alpha]\Psi_\alpha[\gamma]\]  \tag{97}

If we assume the existence of an inner product with respect to which \( \hat{A}[S] \) is a self-adjoint operator, then this state must be the hermitian conjugate of \( <\alpha| \). That is, we must have \( |\alpha \rangle \equiv <\alpha| = |\Psi_\alpha \rangle \). From this we can find some, but not complete, information about the inner product, as I will now show.
In particular, we may first determine $\langle \alpha | \beta \rangle$ when both loops are simple. This follows directly from the condition that $\hat{A}[S]$ be self-adjoint. For we have, for both $\alpha$ and $\beta$ simple,

\[
\begin{align*}
\left( \langle \alpha | \hat{A}[S] \right) | \beta \rangle & = \sqrt{6} l_p^2 I^+ [S, \alpha] \langle \alpha | \beta \rangle \\
= \left( \langle \beta | \hat{A}[S] \right) | \alpha \rangle & = \sqrt{6} l_p^2 I^+ [S, \beta] \langle \beta | \alpha \rangle
\end{align*}
\] (98)

Thus we see that unless $I^+ [S, \alpha] = I^+ [S, \beta]$ for all surfaces $S$ we must have $\langle \alpha | \beta \rangle = 0$. For this to be the case the support of $\alpha$ and $\beta$ must be identical, however, since they are both assumed simple we have,

\[
\langle \beta | \alpha \rangle = D(\alpha) \delta_{\alpha \beta}
\] (99)

where $D(\alpha)$ is real. If we normalize the eigenstates so that $D(\alpha) = 1$ we then have

\[
\Psi_\alpha [\beta] = \langle \beta | \alpha \rangle = \delta_{\alpha \beta}
\] (100)

when $\beta$ is also simple.

With a little more work we can extend this result to find the representation of the eigenstate $\Psi_\alpha [\beta] = \langle \beta | \alpha \rangle$ for a general $\beta$ (but holding $\alpha$ still simple.) Let us begin with an example. We will find $\Psi_\alpha [\alpha \circ \beta]^2$, where $\beta$ is an arbitrary simple loop not equal to $\alpha$. We begin with the computation of the action of $\hat{A}^2 [S]$. For simplicity, we will assume that $S_i$ is small enough that it intersects each segment of the loop at most once. After a short calculation we find,

\[
\hat{A}^2 [S] \Psi [\alpha \circ \beta]^2 = 6 l_p^4 I [S_i, \alpha]^2 \Psi [\alpha \circ \beta]^2 + 12 l_p^4 I [S_i, \beta]^2 \Psi [\alpha \circ \beta]^2 + 4 l_p^4 I [S_i, \beta]^2 (\Psi [\alpha] + \Psi [\alpha \circ \beta \cup \beta])
\] (101)

Note that the second line comes from terms where both hands of the $T^2$ act on the same $\beta$, while the last line comes from terms where each hand acts on a different $\beta$. The effect of this last term is to induce two terms, one where the flow along the two $\beta$’s is switched and one in which they are joined crosswise, so that they retract via the retracing identity, leaving only $\alpha$.

If we use the spin network identity (30) we can write this result as (restricting to the eigenstate):

\[
\hat{A}^2 [S] \Psi_\alpha [\alpha \circ \beta]^2 = 6 l_p^4 I [S_i, \alpha]^2 \Psi_\alpha [\alpha \circ \beta]^2 + 12 l_p^4 I [S_i, \beta]^2 \Psi_\alpha [\alpha \circ \beta]^2 + 4 l_p^4 I [S_i, \beta]^2 (2 \Psi_\alpha [\alpha \circ \beta]^2 + \Psi_\alpha [\alpha])
\] (102)

We already know that the eigenvalue is $6 l_p^2 I [S_i, \alpha]^2$, so that the term proportional to $I [S_i, \beta]^2$ must vanish. This gives us

\[
\Psi_\alpha [\alpha \circ \beta]^2 = - \frac{1}{2} \Psi_\alpha [\alpha] = - \frac{1}{2}
\] (103)

46
Having understood this example, it is not hard to do the general case. Let us consider a general loop, \( \gamma \), which is a particular routing of a graph \( \Gamma \) and which may be considered to be made out of segments \( \eta_I \) such that \( \gamma = \prod_I \eta_I \). Along segments which are traced more than once, \( \hat{A}[S_i] \) will act to produce a term in which two of the tracings of that segment are eliminated. Let us call that loop \( C_{IJ}(\gamma) \), which indicates that the segments \( \eta_I \) and \( \eta_J \), which have the same support, have been eliminated by the action of the \( \hat{A}[S_i] \). Now, let us cut the segment in this manner in all possible places. If there are \( M \) cuttings possible, we may denote this by \( C_M(\gamma) \). This loop will have no multiple segments. (If it is simple then it will also be unique, which is the case we are interested in.) It may then be demonstrated that if and only if \( C_M(\gamma) \) is equal to \( \alpha \) then \( \Psi_\alpha[\gamma] \neq 0 \).

To show this, and to compute the explicit components, we need to show that the value of \( \Psi_\alpha[\gamma] \), for any \( \gamma \), is determined by its value on all the loops that result from cutting \( \gamma \). To show this, let us denote by \( X_{IJ}(\gamma) \) the other loop that arises from the action of \( \hat{A}[S_i] \) on the doubled segment \( \eta_I = \eta_J \). This is the one in which the two lines are switched, but their orientations are not reversed, so they are not retraced. (In the example above \( X(\alpha \circ \beta^2) = \alpha \circ \beta \cup \beta \).) For notation, let us divide the segments \( \eta_I \) into a set, labeled \( \eta_\rho \), which are not multiple, and cannot be cut, and the rest, labeled by \( I' \) and \( J' \), which are multiple. We then have

\[
\hat{A}[S_i] \Psi_\alpha[\gamma] = 6I_P^2 \sum_{\rho} I[S_i, \eta_\rho]^2 \Psi_\alpha[\gamma] \\
+ 8I_P^4 \sum_{I', J', \text{s.t. } \eta_{I'} = \eta_J} \left( [6I[S_i, \eta_{I'}]^2 + 6I[S_i, \eta_{J'}]^2] \Psi_\alpha[\gamma] \\
+ I[S_i, \eta_{I'}]I[S_i, \eta_{J'}] \left[ 4\Psi_\alpha[C_{I', I'}(\gamma)] + \Psi_\alpha[X_{I', I'}(\gamma)] \right] \right)
\]

It is clear that this process continues until one can cut no more. The resulting loop is either equal to \( \alpha \), or equal to a loop that we know \( \Psi_\alpha \) on gives zero. In the latter case \( \Psi_\alpha[\gamma] = 0 \). In the former case, we must solve the equations at each level to find \( \Psi_\alpha[\gamma] \). To do this we should note that since we know the eigenvalue is proportional to \( I[S_i, \alpha] \), all the terms which are proportional to the intersection numbers of cut segments must vanish. This gives from (104) a linear system to solve for each cut segment. As the reader may show, each of these can be expressed as a matrix equation and inverted. The result is that all the nonvanishing coefficients of \( \Psi_\alpha[\gamma] = \langle \gamma | \alpha \rangle \) can be found, for the case of simple \( \alpha \), but general \( \gamma \).

In the next section we will see that the complete set of eigenstates, in the dual representation can be found for general \( \alpha \). However, the complete inner product, including all the values of \( \langle \alpha | \beta \rangle \) when neither \( \alpha \) nor \( \beta \) are simple is a more complicated problem, which has not yet been solved.
4.4 The action of operators on intersecting loop states and the extension of the area operator to the same

We have two purposes in this section. First, we describe some technology that is useful when we describe operators acting on intersecting loops. This will be useful particularly for the construction of the volume operator in the following section. Then I will show, as an example, how to extend the area operator described in the previous section to the intersecting case and find, in particular, all the dual eigenstates of the areas.

Let us then consider a graph $\Gamma$ with $N$ intersection points $p_{\alpha}$, $\alpha = 1, \ldots, N$. Each intersection point $p_{\alpha}$ has $n_{\alpha}$ lines going through it and $M_{\alpha} = \{2^{n_{\alpha}}\}$ different ways to route through it. Let us have a label, $I_{\alpha} = 1, \ldots, M_{\alpha}$ which labels the different ways to route through each of the $p_{\alpha}$. We can then label the different loops which result from a graph $\Gamma$ by choosing the $N$ different routings as $\Gamma_{I_1, \ldots, I_N}$.

Now, because of the spin network relations (30), not all of the values of a loop state on different members of a graph are independent. Instead, a general loop state, evaluated at an intersecting loop must satisfy a set of linear relations, one for each intersection point, which may be written as

$$
\sum_{I_{\alpha} = 1}^{M_{\alpha}} P_{I_{\alpha} n_{\alpha}} \Psi[\Gamma_{I_1, \ldots, I_\alpha, \ldots, I_N}] = 0 \quad (105)
$$

where the coefficients $P_{I_{\alpha} n_{\alpha}}$ come from repeated application of (30). For each $n$-fold intersection point, one can then choose $w(n)$ routings, which will be a maximal set on which the loop states are independent. We may then work instead with the smaller set of loops, which are labeled by such an independent, complete set. We will denote these loops by $\Gamma_{I'_1, \ldots, I'_N}$, where $I'_{\alpha} = 1, \ldots, w(n_{\alpha})$.

We now can show that the action of $A[S]$ on intersecting loops can cause rearrangement of the routings. We need only check the case that the surface $S$ crosses exactly one of the intersection points of a graph $\Gamma$. We will consider first the simplest case, in which a surface $S$ crosses a simple intersection point of two curves. The two curves will be labeled by $\alpha$ and $\beta$ and the two independent routings through the intersection will be denoted by $\Gamma_{I_1, I_2}$, where $I_1$ takes the two values $\times$ and $\triangleright\triangleright$ (the symbols indicate what happens at the intersection point)\(^{23}\). Using (93) we find that,

$$
\tilde{A}_{approx}^{2} [S] \Psi[\Gamma \times] = \int d^2 S_1 \int d^2 S_2 \tilde{A}^{2} \tilde{T}^{**} \tilde{A} \Psi[\Gamma \times] \quad (106)
$$

$$
= \int d^2 S_1 \int d^2 S_2 \epsilon_{a_1 a_2} \epsilon_{b_1 b_2} \Psi[\Gamma \times] \quad (106)
$$

$$
\times \left[ \int ds \delta^3(S_1, \alpha(s)) \dot{\alpha}^a(s) \int dt \delta^3(S_2, \alpha(t)) \dot{\alpha}^b(t) \right]
$$

\(^{23}\)Recall that in section 2.3 we introduced notation for graphs.
\[
\sum_r (-1)^q \Psi[(\alpha \circ_s \gamma_{\alpha(s)}(t))_r \cup \beta] \\
+ (\alpha < - > \beta) \\
+ 2 \int ds \delta^3(S_1, \alpha(s)) \dot{\alpha}^3(s) \int dt \delta^3(S_2, \beta(t)) \dot{\beta}^3(t) \\
\times \left( \sum_r (-1)^q \Psi[(\alpha \circ_s \gamma_{\alpha(s)}(t) \circ_t \beta)\_r] \right)
\]

At the intersection point the curves \(\gamma_{\alpha(s)}\alpha(t)\) and \(\gamma_{\alpha(s)\beta(t)}\) are trivial but their orientations give us an ordering with which to define the action of the splittings and reroutings. It then follows that,

\[
\sum_r (-1)^q \Psi[(\alpha \circ_s \gamma_{\alpha(s)}(t) \circ_t \beta)\_r] = 6\Psi[\alpha \cup \beta] = 6\Psi[\Gamma_x] \tag{107}
\]

and

\[
\sum_r (-1)^q \Psi[(\alpha \circ_s \gamma_{\alpha(s)}(t) \circ_t \beta)\_r] = 4\Psi[\Gamma_>\_] - 2\Psi[\Gamma_x] \tag{108}
\]

Using the formula (24) for the intersection numbers we then have,

\[
\hat{A}^2_{\text{approx}}[S_i] \Psi[\Gamma_x] = 6I_{p}^{2} (I[S_i, \alpha]^{2} + I[S_i, \beta]^{2}) \Psi[\Gamma_x] \\
+ I_{p}^{2} I[S_i, \alpha] \Psi[\Gamma_x] (4\Psi[\Gamma_>\_] - 2\Psi[\Gamma_x]) \\
= I_{p}^{2} (10\Psi[\Gamma_x] + 4\Psi[\Gamma_>\_]) \tag{109}
\]

We have dropped the intersection numbers in the last line, because we assume that the surface element goes through the intersection point. Similarly, one can show that

\[
\hat{A}^2_{\text{approx}}[S_i] \Psi[\Gamma_>\_] = I_{p}^{2} (10\Psi[\Gamma_>\_] + 4\Psi[\Gamma_x]) \tag{110}
\]

These results may be summarized by introducing a rerouting matrix, \(M_{I,J}\), whose entries are just numbers. We have thus shown that

\[
< \Gamma| \hat{A}^2_{\text{approx}}[S_i] = I_{p}^{2} M_{I,J} < \Gamma| \tag{111}
\]

It is straightforward to generalize this to the general case that a surface goes through an arbitrary intersection point where \(n\) lines cross. If \(S_i\) is the section of the surface that goes through the first intersection point of a general graph \(\Gamma\), we have

\[
\hat{A}^2_{\text{approx}}[S_i] \Psi[\Gamma_{I_1,\ldots,I_N}] = I_{p}^{2} M(A)^n_{I_1,J} \Psi[\Gamma_{J,\ldots,I_N}] \tag{112}
\]

Here we have shown that the matrix \(M\) depends on the number \(n\) of lines that go through the intersection point and we have also shown that it arises from the operator \(A\). It is now easy to take the square root (the matrixes being finite dimensional) and the limit (92) to find that,

\[
< \Gamma_{I_1,\ldots,I_N} | \hat{A}^2[S] = I_{p}^{2} (M(A)^n_{I_1,J} \hat{M}_{J,I} \Psi[\Gamma_{J,\ldots,I_N}] \tag{113}
\]
We thus see that the effect of the operator on the intersection is to act with a finite dimensional matrix to rearrange the routings through the point of intersection. There are thus additional eigenstates of $\hat{A}[S]$ that come from diagonalizing the rearrangement matrices. They will be linear combinations of the characteristic states associated with the routings through intersections. As a result the area operator has additional eigenvalues, but they are still a discrete set.

It is not hard to see that these are all the eigenstates that can be constructed in terms of the dual basis $<\gamma|$. There is one more class we have to study, which is when $\gamma$ has retraced segments. However, from the results of the previous section, it is clear that the result of the action of $\hat{A}[S]$ on a dual basis state $<\gamma|$ of a loop with retraced segments gives, when the surface intersects a retraced segment, a mixing with other routings through the segment plus terms involving the dual bases of graphs in which the segment has been cut as in (102) and (104).

However, when it acts on a state with support on simple loops it produces a state which only has support on simple loops. Thus, it is not a symmetric operator. One can then show that because of this no eigenstate of $\hat{A}[S]$ can have a term involving $<\gamma|$, where $<\gamma|$ has retraced segments. Thus, all dual eigenstates must be linear combinations of $<\gamma|$, where $\gamma$ have no retraced segments, but are only simple loops, or loops with intersections. It then follows that in the space of states which are normalizable under the discrete measure the complete set of eigenstates are the characteristic states of simple loops together with the eigenstates of the rearrangement matrices of intersecting graphs.\footnote{It then follows that the complete set of eigenstates of $\hat{A}[S]$ do not span the representation space $S^{kin}_\Sigma$ of the loop algebra. As a result, $\hat{A}[S]$ cannot be a Hermitian operator on this whole space. However, it does have an invariant subspace, which consists of states with support only on simple loops and loops with ordinary intersections with no retracings. What we have done in the previous section is then to choose the inner product such that on its invariant subspace $\hat{A}[S]$ is an hermitian operator. Similar remarks apply to the operator $\hat{Q}[\omega]$.}

\subsection{4.5 The construction of the volume operator}

The construction of the operator which measures the volume of a region quantum mechanically uses the same two ideas that were basic to the construction of the area operator: first, integrate the hands against two dimensional surfaces and second, rather than integrate against a density divide the region up into pieces and approximate the volume of the pieces.

Classically, in terms of the new variables the volume of a region $\mathcal{R}$ is equal to,

$$V[\mathcal{R}] = \int_{\mathcal{R}} d^3x \sqrt{\det(\mathring{E}^{ai}(x))} \tag{114}$$

where

$$\det(\mathring{E}^{ai}(x)) = \frac{1}{3!} \epsilon_{abc} \epsilon_{ijk} \mathring{E}^{ai}(x) \mathring{E}^{bj}(x) \mathring{E}^{ck}(x) \tag{115}$$
Before describing the right way to define an operator for $V[R]$, let us take a moment to explain why the obvious approach does not work. The obvious approach is to define first an operator for $\text{det}(\tilde{E}^a_i(x))$ at a point, $x$, and then integrate the resulting operator (which must be a density) over the region $R$. The problem with this procedure is that it is impossible to define an operator which measures the determinant of the metric at a point. This is true for the same reason we found in section 4.1 that it was impossible to define an operator to measure the densitized inverse metric, $\tilde{\tilde{q}}^{ab}(x)$ at a point.

To see this let us consider a multi-loop $\rho$, with three components, $\alpha, \beta$ and $\gamma$ that intersect only at a single intersection point $p = \alpha(s_0) = \beta(t_0) = \gamma(u_0)$. A naive unregulated "calculation" gives

$$\text{det}(\tilde{E}^a_i(x))\Psi[\rho] = \int_0^\beta ds \int_0^\gamma dt \int_0^\alpha du$$

$$\times \delta^3(x,\alpha(s))\delta^3(x,\beta(t))\delta^3(x,\gamma(u))$$

$$\times \epsilon_{abc}\dot{\alpha}^a(s)\dot{\beta}^b(t)\dot{\gamma}^c(u) \Psi[\rho]$$

This is clearly undefined, as it involves, where it does not vanish, the product of the three delta functions. As in the case of the inverse metric, the problem is then to define the product of the three delta functions through a regularization procedure. It is clear that, however this is done, any right answer should be expected to vanish except at the point $p$ where there are three independent tangent vectors. A regularization procedure for $\text{det}(\tilde{E}^a_i)$ can be constructed along the lines of the one we introduced in section 4.1 for the inverse metric.

To do this we again make use of a flat background metric $h^a_{0b}$ in the neighborhood, $U$, of $x$ and we use the smearing function (59). For each three points $x, y, z$ of $U$ we may use the euclidean coordinates based on $h^a_{0b}$ to define a coordinate circle $\gamma^a_{xyz}(s)$ which passes through the points $x, y$ and $z$ at the parameter values, $0, 2\pi/3$ and $4\pi/3$, respectively. We then define a generalization of (56)

$$T^{abc}(x, y, z) = T^{abc}[\gamma_{xyz}](0, 2\pi/3, 4\pi/3)$$

We now may define a regularized version of $\text{det}(\tilde{E}^a_i)$ by

$$H_\epsilon(x) \equiv \int d^3y \int d^3z \int d^3w f_\epsilon(x, y)f_\epsilon(x, z)f_\epsilon(x, w)\epsilon_{abc}T^{abc}(y, z, w)$$

Interestingly enough, when we act with this operator on the state $\Psi$ and evaluate the result on the intersecting loop $\rho$, this operator is finite in the limit $\epsilon \to 0$. But it is hopelessly regularization dependent.

The result is of the form,

$$\lim_{\epsilon \to 0} \hat{H}_\epsilon(x)\Psi[\rho] = \delta^3(x, p)\frac{\text{det}(h^0)^a(\epsilon)\epsilon_{abc}\dot{\alpha}^a(s_0)\dot{\beta}^b(t_0)\dot{\gamma}^c(u_0)}{[\dot{\alpha}(s_0)][\dot{\beta}(t_0)][\dot{\gamma}(u_0)]} \Psi[\rho]$$
where, as in (66), the norm $|...|$ is taken with respect to the background metric $h^0_{ab}$.

Just as in the case of our attempt to define $\tilde{q}^{ab}(x)$, the regularization dependence comes in two places. First, in order to preserve reparameterization invariance the factor $\epsilon_{abc}\dot{\alpha}(s_0)\dot{\beta}(t_0)\dot{\gamma}(u_0)$ in the numerator must be balanced by the factor linear in each of the tangent vectors in the denominator. In the calculation based on the smearing function (59) this factor is $|\dot{\alpha}(s_0)||\dot{\beta}(t_0)||\dot{\gamma}(u_0)|$. However, the key point is that any such factor must depend on additional background structure. Second, because $\det(\tilde{E}^{ai})$ has density weight two, there must be in the answer a factor of density weight two because the density weight of the $\delta^3(x,p)$ is balanced by the $\epsilon_{abc}$. Again, in the particular calculation we have done this weight two function turns out to be $\det(h^0)$, but in general some such density factor must emerge from the calculation and, because there is no density that can be defined only from the geometry of the problem, this can only come from some arbitrary, auxiliary, structure used in the regularization.

As the determinant of the metric is only a single function, it is clear that we get absolutely zero information from this operator. The moral of the story is that we cannot measure the determinant of the metric at a point in the loop representation.

How are we then to measure the volume of a region? The only way to do it is to find a procedure for defining the volume of a region that does not involve ever defining the volume element at a point. This can be done by mimicking the procedure by which we defined the area of a surface. We divide the region $\mathcal{R}$ up into $N$ subregions $\mathcal{R}_i$, $i = 1, ..., N$ and for each of these we define an approximate expression for the square of the volume, $V^2_{\text{approx}}[\mathcal{R}_i]$, that is exact in the limit of infinitesimal volume. We then take the limit in which we take $N$ to infinity, making the volume of each subregion $\mathcal{R}_i$ infinitesimal. We then have,

$$V[\mathcal{R}] = \lim_{N \to \infty} \sum_{i=1}^{N} \sqrt{|V^2_{\text{approx}}[\mathcal{R}_i]|}. \quad (120)$$

We may do this in the following way. Let us introduce a euclidean coordinate system, $\hat{y}^a$, on $\mathcal{R}$ and consider a cubic lattice with $N$ cells, defined with a coordinate lattice spacing $d \approx N^{-1/3}$. We will consider the cells to be the subregions $\mathcal{R}_i$. We can define an approximation to the volume of the $i$'th cell in the following way. Take three of the faces of the cell that share a common vertex and call them $S^a_i$, where $a = 1, 2, 3$. We then define,

$$V^2_{\text{approx}}[\mathcal{R}_i] \equiv \int_{S^a_i} d^2S_1^{a_1,a_2} \int_{S_2^b} d^2S_2^{b_1,b_2} \int_{S_3^c} d^2S_3^{c_1,c_2} \epsilon_{a_1a_2b_1b_2c_1c_2} T^{abc}(S_1, S_2, S_3) \quad (121)$$

Now, classically, it is not hard to show that for smooth fields,

$$T^{abc}(x, y, z) = \det(\tilde{E}^{ai})(x_i)\epsilon^{abc} + O(d\tilde{E}) \quad (122)$$

52
where $x_i$ is a point in the (coordinate) center of the cube. If we now consider
infinitesimal boxes and define $s_{\alpha}^a = \epsilon_{\alpha a_1 a_2} d^2 S_{\alpha}^{a_1 a_2}$, we see that

$$|\mathcal{V}_2^{\text{approx}}[\mathcal{R}_i]| = |det(q)(x_i)| e^{abc} s_{\alpha}^a s_{\beta}^b s_{\gamma}^c + O(d) = \mathcal{V}^2[\mathcal{R}_i] + O(d) \quad (123)$$

The relation (120) then follows, as the limit $N \to \infty$ is to be taken in such a
way that $d \to 0$.

Let us now translate $\mathcal{V}_2^{\text{approx}}[\mathcal{R}_i]$ into a quantum operator by writing it as
(121) with the $T^3$ now an operator. We may now evaluate it on a quantum
state. The calculation is similar to the evaluation of (93). We will evaluate
the action on a multi-loop $\rho$, which will have components that we will denote $\rho_I$.
For any multi-loop $\rho$ there will be a lattice spacing $d$ which is small enough that
each wall face, $S_{\alpha}^i$ of each cell is pierced by each component of $\rho$ at most once.
Let us assume that $d$ is taken at least that small. Using the definition of the
intersection number (24) of a curve and a surface, we then find,

$$\hat{\mathcal{V}}_2^{\text{approx}}[\mathcal{R}_i]\Psi[\rho] = i_P \sum_I \sum_J \sum_K I(S_I^1, \rho_I) I(S_J^2, \rho_J) I(S_K^3, \rho_K) \quad (124)$$

$$\times \left( \sum_r (-1)^r \Psi[(\rho \circ \circ \circ \gamma_{\rho_I(s_I^1)} \rho_J(s_J^2) \rho_K(s_K^3))_r] \right).$$

Here $s_{\alpha}^a$ is the intersection point for the $I$'th component $\rho_I$ with the $\alpha$'th surface.
(We can take these coordinates to vanish when there is no intersection.)

In order to take the limit (120) we need to be able to define the square root of
this action. However, we only need do it for an arbitrarily small lattice spacing
$d$. Let us then consider what happens to the factors in (124) as we take the limit
$d \to 0$, making the boxes smaller and smaller. There are again two factors to
to consider: the c-number factors which are composed of the intersection numbers
and the rearranging of the loops.

Let us first consider the c-number factors. The main thing to see is that in
the limit that we shrink the boxes down there will only survive one term for
each intersection point at which there are three independent tangent vectors.
Clearly, a box that none of the curves pass through gives zero. Consider next a
box that only one of the three curves, say $\alpha = \Gamma_1$, passes through. In this case
the result is zero, because it can pass through at most two walls of the box.

Consider then a box that contains a double intersection point. There are
then two curves in the box, which we will call $\alpha$ and $\beta$. The nonvanishing terms
in the intersection numbers in (124) will come from terms in which one of them,
say $\alpha$, passes through two walls and the second, $\beta$, passes through the third
wall. However, for this case, as the reader may show, the factor coming from
the rearrangement of loops will vanish.

This leaves only the possibility that the box contains a triple intersection
point $p$ at which at least three curves meet. To get a nonvanishing contribution,
the tangent vectors of three of the curves at the intersection point must be
independent because only if this is the case will three different curves pass each through a different of the three walls of the box, no matter how small we take the box. Each of these meetings of a wall and a curve then contributes a separate intersection number. It remains only to figure out the effect of rearranging the loops.

If we use the continuity conditions (70) and (71) to define the limits, then the result is just a rearrangement matrix, which reshuffles the routings at the intersection point $p$. To write this, we use the notation we developed in the previous section and write the graph containing the loops with the intersection $p$ as $\Gamma$. Let us assume, to begin with, that $p$ is the only intersection point of $\Gamma$. A particular choice of rearrangements at $p$ gives a loop which will be labeled, as before, $\Gamma_I$, where $I = 1, \ldots, N_p$. Here $N_p$ is the dimension of the finite dimensional state space $V_\Gamma$ associated with the intersection.

We then find that in the limit in which the box becomes arbitrarily small we will have

$$\hat{V}^{approx}_{\text{area}}[R_i]\Psi[\Gamma_I] = l_p^n \mathcal{M}(\mathcal{V})_{IJ}\Psi[\Gamma_J],$$

where $\mathcal{M}(\mathcal{V})_{IJ}$ is an $N_p \times N_p$ matrix gotten by summing over the rearrangements in (124) and then shrinking the loops down as the boxes shrink.

Now, in the limit that $N \to \infty$ each triple intersection point has a unique box surrounding it such that each of the three lines pass through a separate wall. We can then take the square root and compute the limit in equation (120) to find,

$$\hat{V}[R_i]\Psi[\Gamma_I] = l_p^n \mathcal{M}(\mathcal{V})^{\frac{1}{2}}_{IJ}\Psi[\Gamma_J]$$

It is then straightforward to extend this to the general case that the graph $\Gamma$ has $m$ such intersection points, $p_\rho$, with $\rho = 1, \ldots, m$. We have then a separate rearrangement matrix $\mathcal{M}(\mathcal{V})_{IJ}$ at each intersection point. We can label the different routings with a separate index at each intersection point. Thus a choice of routings will be given by $\Gamma_{I_1, \ldots, I_m}$ with $I_\rho$ labeling the independent choices of routings through $p_\rho$. The result is then,

$$\hat{V}[R_i]\Psi[\Gamma_I] = l_p^n \sum_{\rho=1}^{m} \mathcal{M}_{IJ}^{\rho}(\mathcal{V})^{\frac{1}{2}}_{IJ} \Psi[\Gamma_{I_1, \ldots, I_m}]$$

We can make some comments on this result.

1) The eigenvalues of the volume operator are also quantized in Planck units, being given by the Planck volume times the square roots of the eigenvalues of the matrices $\mathcal{M}_{IJ}^{\rho}(\mathcal{V})$. At present, nothing is known about these matrixes except the simplest examples. The eigenstates of the volume operator are then constructed from linear combinations of the characteristic functions of intersecting loops, using the eigenvectors of the matrices $\mathcal{M}_{IJ}^{\rho}(\mathcal{V})$. We thus see that, roughly, the volume operator seems to count the number of triple (or greater) intersections, in units of the Planck volume.
2) We see that the operator is, as in the case of the area operator, finite and background independent. Its action in the basis of characteristic states is block diagonal, in that it induces a matrix in the finite dimensional subspaces $V_\Gamma$ associated with each graph. This is actually a general property: a large class of operators that are expressed classically as single integrals over densities express themselves as rearrangement matrices in the finite dimensional blocks associated with rearranging the routings through graphs.

3) By taking the region $R$ to be the whole three manifold $\Sigma$ we find a diffeomorphism invariant operator: the volume of the universe. As discussed above in section 2.4, this operator acts directly on diffeomorphism invariant states, by (127). This is the first example we have of an explicit construction of a spatially diffeomorphism operator nonperturbatively.

4.6 The construction of the operator $Q[\omega]$

In this section we study the construction of an operator to represent the observable $Q[\omega]$ introduced in section 2.3. Let $\omega_\alpha(x)$ be an arbitrary smooth one form on $M$. Classically, we defined

$$Q[\omega, \tilde{E}] \equiv \int d^3 x \sqrt{\tilde{E}^{ai}(x)\tilde{E}_a^i(x)\omega_a(x)\omega_b(x)}$$

(128)

For every $\omega$, the observable $Q[\omega, \tilde{E}]$ is a well defined observable on the configuration space of frame fields $\tilde{E}^{ai}$. Moreover, the collection of $Q[\omega]$ for all smooth $\omega$ provide a good coordinate system on the space of these frame fields: if we know $Q[\omega]$ for every smooth one form $\omega$, then we can reconstruct the frame fields $\tilde{E}^{ai}$, up to local $SU(2)$ gauge transformations.

To construct a quantum operator to represent $Q[\omega, \tilde{E}]$ we must first regulate this classical expression. We thus write

$$Q[\omega] = \lim_{\epsilon \to 0} Q[\omega]$$

(129)

where

$$Q[\omega] \equiv \int d^3 x \sqrt{W(\omega, x)}$$

(130)

and the factor inside the square root is

$$W(\omega, x) \int d^3 y \int d^3 z f_\epsilon(x, y)f_\epsilon(x, z)T^{ab}(y, z)\omega_a(y)\omega_b(z),$$

(131)

(131)

where $f_\epsilon(x, y)$ is a smearing function which satisfies (57) and (58). We will proceed to define the action of this operator on a state $\Psi[\alpha]$ through the following steps. 1) Define the action of the regulated factor $W(\omega, x)$. 2) Isolate the leading term of the loop factors of the regulated operator, as $\epsilon \to 0$. 3) Take the square root. 4) Integrate. 5) Take the limit in which $\epsilon$ goes to zero.
Beginning with the first step, we use a definition which follows the classical one, so that,

$$
\hat{W}_\epsilon(\omega, x)\Psi[\alpha] = \int d^3 y \int d^3 z f_\epsilon(x, y) f_\epsilon(x, z) \hat{T}^{ab}(y, z) \omega_a(y) \omega_b(z) \Psi[\alpha] \\
= l_P^4 \int ds f_\epsilon(x, \alpha(s)) \dot{\alpha}^a(s) \omega_a(\alpha(s)) \int dt f_\epsilon(x, \alpha(t)) \dot{\alpha}^b(t) \omega_b(\alpha(t)) \\
\times \left( \sum_r (-1)^r \Psi((\alpha \circ \gamma_{\alpha(s)}\alpha(t))_r) \right)
$$

We want to take the square root of this action for small $\epsilon$. To do this we use the conditions (70) and (71), so that the operator is expressed in terms of a leading, $\epsilon$ independent piece, whose square root we can take, and a correction proportional to $\epsilon$. We then have, at a point where the loop is nonintersecting,

$$
\hat{W}_{\epsilon/2}(\omega, x)\Psi[\alpha] = 6 \left( l_P^2 \int ds f_\epsilon(x, \alpha(s)) \dot{\alpha}^a(s) \omega(\alpha(s))_a \right)^2 \Psi[\alpha] + O(\epsilon)
$$

The 6 comes from the sum over routings in (132), as in previous cases. The square root is then the square root of the leading piece plus a correction proportional to $\epsilon$ that we don’t need to compute. It is thus,

$$
\hat{W}_{\epsilon/2}(\omega, x)\Psi[\alpha] = \sqrt{6l_P^2} \int ds f_\epsilon(x, \alpha(s)) \dot{\alpha}^a(s) \omega(\alpha(s))_a \Psi[\alpha] + O(\epsilon)
$$

We may now integrate and take the limit to find,

$$
(\hat{Q}\Psi)[\alpha] \equiv \lim_{\epsilon \to 0} \hat{W}_{\epsilon}(\omega, x)\Psi[\alpha] = 6l_P^2 \int d\alpha^a \omega_a \Psi[\alpha]
$$

Note that the resulting operator is, again, finite and independent of the background structure that went into the definition of the regularization.

### 4.7 More about weaves and the semiclassical equivalence of quantum states to classical geometries

In section 2.3 I gave an introduction to the idea of a weave. The main idea uses the fact that the characteristic states are eigenstates of the operators that measure metric information and that, in particular, the lines of the loops carry a quantized flux of area, that they contribute to the area of any surface they cross. We can then construct a quantum state that approximates a given classical metric $h_{ab}$ by building a characteristic state on a set of loops that are distributed so that one (or more properly $\sqrt{6}$) loops per Planck area cross each surface, as measured by that background metric.
Here I would like to be more precise about what is meant by approximating a classical solution, after which I will describe the construction of one example of such a weave state.

It will be convenient to study the equivalence of a quantum state with a classical metric by using the operator $Q[\omega]$. Let me first note that, classically, if we know this observable for all smooth one forms $\omega_a$ the classical metric $q_{ab}$ is completely determined. This can be shown by taking $\omega_a$ of arbitrarily small compact support; the different components of $\tilde{q}_{ab}$ are then gotten by studying $Q[\omega]$ for $\omega$ that are linear combinations of some set of basis one forms.

Now, what if we only know the value of this observable for $\omega$ that are slowly varying on some scale $L$? The idea is then that we only determine the metric to within that scale $L$. It is a little tricky to say this precisely, because we need a metric to describe which one forms are slowly varying. A good formulation turns out to be the following:

We must first give a definition of a slowly varying one form $L$. We say that a one form $\omega_a$ is slowly varying on a scale $L$ with respect to a metric $q_{ab}$ if at all points of $\Sigma$

$$\frac{\left|\nabla_a \omega_b\right|^2}{|\omega_a|^2} < \frac{1}{L^2},$$

(136)

where $\nabla_a$ is the metric covariant derivative of $q_{ab}$ and the norms are taken using $q_{ab}$. We will also use normalized one forms so that

$$\int_{\Sigma} \sqrt{q}|\omega_a|^2 = 1$$

(137)

We then can use such slowly varying $\omega_a$ to state a criteria for when another metric $q'_{ab}$ approximates $q_{ab}$ on scales larger than $L$.

We say that $q'_{ab}$ $Q$-approximates $q_{ab}$ at scales larger than $L$, to an accuracy $\epsilon$, if for all one forms $\omega$ which satisfy (136) and (137)

$$|Q[\omega, q'] - Q[\omega, q]| < \epsilon$$

(138)

More details of this notion of one metric approximating another one at large scales are given in reference [2].

We then can apply this notion to the quantum theory in the following way. Let $\Psi$ be an eigenstate of the operator $\hat{Q}[\omega]$ so that

$$\hat{Q}[\omega]|\Psi\rangle = \lambda[\omega, \Psi]|\Psi\rangle.$$ 

(139)

We will then say that the state $\Psi$ $Q$-approximates a classical metric $q_{ab}$ on a scale $L >> l_P$ if for all slowly varying $\omega_a$ on the scale $L$

$$|\lambda[\omega, \Psi] - Q[\omega, q]| < \frac{l_P}{L}$$

(140)

Note that there is no need to introduce a separate parameter $\epsilon$ to measure the error, that will naturally come out to be of order $l_P/L$. 

57
I will now introduce a specific state that satisfies this criteria. Let us fix a flat metric $h_{ab}^0$ on $\Sigma$. Let $x$ be a coordinate system in which the metric $h_{ab}^0$ is locally euclidean. We will construct a collection, $\Delta$, of loops $\alpha_{\vec{n}}$. Each loop $\alpha_{\vec{n}}$, is a circle of radius $r$ (defined with respect to $h_{ab}^0$). The center of the loop $\alpha_{\vec{n}}$ is placed at the vertex $\vec{n}$ of a square lattice with lattice spacing $d$. The orientation of the circles is random (each circle being oriented in a different direction, chosen from a uniform probability distribution on the unit sphere, again defined with respect to $h_{ab}^0$). We will call such a collection of loops, defined with respect to a smooth metric, $h_{ab}^0$ as an $h$-weave, or weave, for short.

It is convenient to describe this weave in terms of the "average density of loops", $l$, which is defined by

$$l^2 = \frac{\text{Volume}}{\text{Length of the loops}} = \frac{d^3}{2\pi r} \quad (141)$$

and the dimensionless quantity

$$z = \frac{r}{d}. \quad (142)$$

We may then consider the characteristic state of this collection of loops, which will be denoted $\Psi_{\text{weave}}$. Using (135) this is an eigenstate of $\hat{Q}[\omega]$ with eigenvalue,

$$\lambda[\omega, \Psi_{\text{weave}}] = \sqrt{6} l^2 \sum_{\vec{n}} \oint_{\alpha_{\vec{n}}} |ds^a \omega_a| \quad (143)$$

Now, we must compute what this is for slowly varying $\omega$, assuming that the parameters $d$ and $r$ that govern the weave are much smaller than the scale $L$ on which $\omega_a$ is slowly varying. We find that

$$\lambda[\omega, \Psi_{\text{weave}}] = \sqrt{6} \pi l^2 d^{-3} r \int d^3 x \sqrt{h^0} |\omega| + R = \frac{\sqrt{6} l^2}{2l^2} Q[\omega, h^0] + R \quad (144)$$

Here $R$ is the error, which can be shown to satisfy

$$|R| < \frac{r + 2d \sqrt{6} l^2}{L} Q[\omega, \omega^a] \quad (145)$$

We can then satisfy the condition (140) if we take the density parameter $l$ for the loops in the weave to satisfy

$$l^2 = \frac{\sqrt{6} l^2}{2} \quad (146)$$

and the error will be of the order of the ratio $l_P/L$ if the two parameters $r$ and $d$ are separately taken to be of the order of $l_P$.

We close this section with some comments on these results:

1) The definition of $Q$-approximation, given by (138) at the classical level can be shown to be equivalent to other notions that can be defined using the
integrated norms of fields of different tensorial character, or by directly integrating the metric against a tensorial test density $\tilde{f}_{ab}$, as in $q(\tilde{f}) = \int q_{ab}q_{ab}$. We then use the notion of $Q$-approximate quantum mechanically, because the $Q$ operator is well defined in the loop representation, whereas not all observables which are functions of the metric extend to the quantum theory.

2) One can check that a similar notion of approximation can be defined using the areas of an appropriate set of surfaces and that the resulting notion agrees with the one we have defined here both classically and quantum mechanically.

3) There are many weave configurations for a given classical metric $h_{0ab}$. By construction, the definition of approximation on large scales constrains nothing about the behavior of the weave at short distances.

4) Thus, as noted before, a state $\Psi_{\text{weave}}$ can be an eigenstate of the volume operator with zero volume and still approximate a flat metric $h_{0ab}$ on a scale $L >> l_P$ when measured by the operators for $Q[\omega]$ or areas. There is no problem with this, as the volume operator measures something about the connectivity of the weave it has to do with the short distance structure and has, thus, nothing to do with the semiclassical limit.

5) Finally, we should emphasize that a weave state $\Psi_{\text{weave}}$ is not going to be the vacuum state of the theory, either perturbatively or nonperturbatively. It is an eigenstate of the three metric, whereas the vacuum presumably must minimize the product of the uncertainty between the three metric and its evolution in time. The problem of the construction of the ground state is a key problem, which is discussed in some detail below, in section 6.3.

5 Some recent developments in the classical theory

In this chapter I would like to review some very interesting recent developments in the classical formulation of general relativity. These build on the Ashtekar formulation and concern four subjects. In section 5.1 I describe an important result of Capovilla, Dell and Jacobson. They show that it is possible, to solve the classical Hamiltonian and diffeomorphism constraints exactly, except for a set of configurations of measure zero. This makes it possible to write new forms for the action for general relativity, one of which is described in section 5.2. This leads to an interesting connection between general relativity and Chern-Simon theory.

Since the beginning of work on the Ashtekar formulation, we have known that the new forms of the constraints and equations of motion admit solutions that are not also solutions to the Einstein’s equations because the metric is either degenerate or distributional. Recently, a lot has been learned about such solutions and it is becoming increasingly clear that they play a role in the quantum theory. Essentially, the quantum geometry we uncovered in chapter
4 can be described in terms of classical distributional geometries. This is the subject of section 5.3. Finally, these classical distributional geometries make possible a new kind of discretization of general relativity, analogous to the Regge calculus. This is the subject of section 5.4.

5.1 How to solve the classical constraint equations exactly

One of the more striking developments of the last two years is that the Hamiltonian and diffeomorphism constraints can be solved, in closed form for all but a measure zero of cases. Indeed, the solutions are so simple that it is surprising in retrospect that they were not discovered earlier. To construct these solutions we do something that may seem at first strange. We treat the gravitational magnetic fields, 

\[ \tilde{B}^{ai} = \frac{1}{2} \epsilon^{abc} B_{bc} \]

as frame fields for the three manifold \( \Sigma \). That is, we assume that,

\[ \det(\tilde{B}^{ai}) \neq 0 \]

so that any vector field can be expanded in terms of them. In particular, we can so expand the \( \tilde{E}^{ai} \)'s,

\[ \tilde{E}^{ai} = M_{ij} \tilde{B}^{aj} \]

It is straightforward to show that the diffeomorphism and hamiltonian constraints, (48) and (50) correspond, respectively, to the conditions that the inverse of \( M_{ij} \), which we will denote \( \phi_{ij} \), is symmetric and that the trace is fixed by the condition

\[ \phi_{ii} = -3\Lambda \]

Thus, the theory can be expressed completely in terms of \( A_{a}^{i} \) and a symmetric matrix of scalar fields \( \phi_{ij} \) restricted by the condition (150). In terms of these fields the only remaining constraint is the Gauss law constraint, which now, however, takes a new form

\[ G^{i} = \tilde{B}^{aj} \mathcal{D}_{a}[\phi^{-1}]^{i}_{j} = 0 \]

Given a fixed \( A_{a}^{i} \) this is a first order differential equation for \( \phi_{ij} \). Its solutions give pairs \((A_{a}^{i}, \phi_{ij})\) that are the free data for general relativity.

One should note that this form of the theory requires that (148) be true everywhere. Thus, there is a set of measure zero of solutions in which \( \tilde{B}^{ai} \) are degenerate that cannot be expressed this way. Among these solutions is, of course, flat spacetime. Whether this is a good or a bad thing depends on what one wants to use the formalism for. Whether it is a good or a bad thing for the quantum theory is not known.

\[ \text{33} \text{Partial results in the right direction were found by Renteln and Ashtekar.} \]

60
5.2 New lagrangians and a connection with Chern-Simon theory

Using the solutions of Capovilla, Dell and Jacobson one can construct a Lagrangian form of the theory that does not involve the metric or the frame field. There are actually several of these, some of which just involve $A^i_a$ and a single scalar density. These were found by Capovilla, Dell and Jacobson, in the same paper in which the solutions to the constraints were found. One very nice form, which involves $A^i_a$ and $\phi^i_j$ is,

$$S(A, \phi) = \int F^i \wedge F^j [\phi^{-1}]_{ij}$$

In this form of the variational principle, $\phi^i_j$ is to be varied respecting its symmetry and the trace condition (150). It is then straightforward to show that if we define $\tilde{E}^{ai}$ by

$$\tilde{E}^{ai} = [\phi^{-1}]_1^i \tilde{B}^{ai}$$

the constraints (47), (48) and (50) are satisfied.

Using the results of Capovilla, Dell and Jacobson a variety of results can be derived about the solution space to general relativity. I will mention two here. The first, due to Samuel, is that in the presence of a cosmological constant, every self-dual solution to the Einstein equations comes from a self-dual $SU(2)$ Yang-Mills field that satisfies,

$$\tilde{E}^{ai} = -\Lambda \tilde{B}^{ai}$$

This means that the self-dual sector of the theory corresponds to $\phi^i_j$ that satisfy

$$\phi^i_j = -\Lambda \delta^i_j$$

The second result, due to Torre, is that for $\Lambda > 0$ and $\Sigma$ compact, the moduli space of gravitational instantons is discrete. There are no continuous parameters as in the case of Yang-Mills theory.

Related to the topics I have been discussing is an interesting connection between general relativity and Chern-Simon theory. Suppose we wanted to base a quantization of the theory on the action (152). Then we would likely need to define the variational principle on a four manifold $M$ with boundary $\Sigma$. A boundary is a good thing to have if one wants to construct observables for the theory or give a meaning to the path integral. We must then ask what kind of boundary conditions are allowed and what boundary terms need to be added to the action so that the variational principle is still well defined. It turns out (the details will be presented elsewhere) that there is a very pretty variational principle in which we demand that the connection be self-dual on the boundary $\Sigma$. Making use of Samuel’s result (154) it is then easy to see that the variational
principle in this case is

\[ S(A, \phi) = \int_{\mathcal{M}} F^i \wedge F^j [\phi^{-1}]_{ij} + \int_{\Sigma} Y_{CS}(A) \tag{156} \]

where \( Y_{CS} \) is the Chern-Simon action. Thus, if we construct a variational principle for general relativity, with a non-vanishing cosmological constant, in which the connection is restricted to be self-dual on the boundary, we induce the Chern-Simon action for the dynamics of the connection on the boundary.\(^\text{[34]}\)

5.3 Distributional frame fields as the classical analogue of quantum geometry

One of the main points that we developed in the last chapter is that there are quantum states which approximate classical metrics, as long as we probe on large scales. These states emerged from diagonalizing certain operators that are functions only of the three metric. Now, one might ask the following question: usually in quantum mechanics when we diagonalize a complete set of commuting operators corresponding to the coordinates of a configuration space, the resulting set of eigenvalues are a subset of that classical configuration space. However, what we found was not quite this, because the resulting eigenvalues have a distributional character. This is natural given the fact that the eigenstates are characteristic functions of particular loops. But we may ask what happened to the expected connection between the eigenvalues and the classical configurations? Is there only a connection to the classical theory in the sense of approximation at large scales?

The answer is that the eigenvalues of the operators we studied in chapter 4 can be related directly to classical configurations of the field, but these are distributional configurations.\(^\text{[3]}\) Normally, the Einstein equations do not allow distributional configurations, because those equations are non-polynomial. But, in the Ashtekar form, all equations are polynomial and, as I will now describe, distributional solutions to the equations exist.

Let us then consider configurations of the frame fields which have exactly the form of the eigenvalues of the operators we studied in chapter 4. That is, given a loop \( \alpha \), let us consider

\[ \tilde{E}_a^i(x) \equiv a^2 \int ds \, \delta^3(x, \alpha(s)) \dot{\alpha}^a(s) e_a^i(s) \tag{157} \]

We see that this is naturally a vector density. The \( e_a^i(s) \) is valued in the lie algebra of \( SU(2) \) and will be taken to be dimensionless. Then the constant \( a \)

\(^{34}\)This result is perhaps also related to a result of Kodama who showed that \( \exp[\Lambda^{-1} \int Y_{CS}(A)] \) is an exact solution to all the constraints of general relativity, in the connection representation in which states are functions of \( A^a_i \), in the presence of the cosmological constant \( \Lambda \).
has dimensions of length. If we compare this with the expression (55), which is the result of applying an operator containing an insertion of \( \hat{E}_{\alpha}(x) \) on a loop state, evaluated at \( \alpha \), we see that in any such expression arising from the quantum theory we will have \( a \approx 1/\lambda \).

Given a collection of loops \( \Delta = \bigcup \alpha_i \), we can construct a distributional frame field by adding the contributions from the separate loops:

\[
\hat{E}_{\Delta}(x) \equiv \sum_i \hat{E}_{\alpha_i}(x).
\] (158)

I would now like to show that the configuration \( \hat{E}_{\alpha}(x) \) is the point of the classical configuration space associated with the eigenstate \( |\alpha\rangle \) of the observables we studied in chapter 4. To accomplish this one can show that the observables \( Q[\omega], A[S] \) and \( V[R] \) can be evaluated on \( \hat{E}_{\alpha}(x) \), and that the values are equal to the corresponding eigenvalues of \( |\alpha\rangle \), when \( a = 6\frac{1}{4}/\lambda \) and \( |e_\alpha(s)| = 1 \).

I will not give the details of the calculations here, they are very similar to the quantum calculations that were described in chapter 4. To begin with, one may try to define the inverse metric \( \tilde{q}^{\alpha b} \) by taking the square of (157); one will find that just as in the quantum case the product cannot be defined without introducing an arbitrary density into the problem. However, the other operators can be defined through the same regularization procedures that we found worked in the quantum theory. We find that,

\[
Q[\omega, \hat{E}_\alpha] = a^2 \oint ds |\dot{\alpha}(s)\omega(\alpha(s))| |e_\alpha(s)|
\] (159)

and

\[
A[S, \hat{E}_\alpha] = a^2 \sum_i |e_\alpha(s_i^*)|
\] (160)

where the sum is over the intersection points of the curve \( \alpha \) with the surface \( S \) and \( s_i^* \) are the parameters at the intersection points.

For the volumes there is a result analogous to (127). The volume of a configuration \( \hat{E}_{\alpha}(x) \) is zero unless it contains intersection points at which there are three independent tangent vectors. In the simplest case, where three segments \( \alpha_i, i = 1, 2, 3 \) meet at a point \( p \), the volume in any region containing \( p \) is,

\[
V[R, \hat{E}_\alpha] = a^3 Tr[e_{\alpha_1} e_{\alpha_2} e_{\alpha_3}]
\] (161)

If there is more than one intersection point the volume is the sum of the contributions from the intersection points.

Thus we see that in spite of having support on a set of measure zero, distributional frame fields of the form of (157) carry finite values of geometrical quantities such as areas and volumes. Furthermore, one may repeat the arguments of section 4.7 to show that any smooth metric \( h_{ab}^0 \) can be approximated by a distributional frame field, where the notion of approximation is defined.
also by equation (138). Thus, the classical configuration (157), where the set of loops is given by the weave $\Delta$ defined in section 4.7 (with $|e_\alpha(s)| = 1$), is a classical distributional geometry that approximates the flat metric $h_{\alpha\beta}^0$, up to terms of order $a/L$. The distributional frame fields are thus the closest we can get in the classical theory to a description of quantum geometry. Indeed, with $a$ set equal to the Planck length (times $6^{\frac{1}{4}}$), they are something like the Bohr orbits of quantum gravity: they give a classical picture that we can use to see to a first approximation what quantum gravitational effects are like.

In the next section we will give a further argument for taking these distributional field configurations seriously. We will see that the constraints and equations of motion of the theory can be extended to them.

5.4 A new classical discretization of the Einstein equations

I would like to bring together the topics of the last three sections to show that the dynamical equations of general relativity, in the Ashtekar form, can be completely solved for distributional configurations of the form (157). To begin, let us return to the ansatz of Samuel, equation (154). The reader may verify that if we plug this into the constraint equations (47), (48) and (50), it solves them with a fixed value of the cosmological constant, as long as $\det(\tilde{E}_{\alpha\beta}^i) \neq 0$.

However, notice also that if the determinant vanishes we still have a solution, but now for any cosmological constant, including zero.

Since the determinant of $\tilde{E}_{\alpha}^i$, defined by (157) vanishes, at least for simple loops $\alpha$, we might try to find a solution consisting of such a frame field and $\tilde{B}_{\alpha}^i$ given by a distribution with the same support,

$$\tilde{B}_{\alpha}^i(x) = \int ds \delta^3(x, \alpha(s))\dot{a}^\alpha(s)\tilde{b}_{\alpha}^i(s).$$

Such solutions exist, if the constraints are defined through a regularization process analogous to that we introduced in section 4.5 for the volume operator[51, 52]. I will now describe a large class of these.

To get nontrivial solutions we should take field configurations with support on loops with intersections. Let us consider one familiar example of such a set of loops, which is a cubic lattice. Given a coordinate chart on $\Sigma$ let us define a standard cubic lattice with coordinate lattice spacing $a$. The vertices will be labeled by three integers $\vec{n}$ and the links by the pair $(\vec{n}, \hat{a})$. Thus, $\gamma_{\vec{n}\hat{a}}(s)$, with $s \in (0, 1)$ will be taken to refer to the link leaving the vertex $\vec{n}$ in the positive $\hat{a}$ direction.

We will then take the frame field and curvature to be of the suggested forms

$$\tilde{E}_{\alpha}^i(x) = a^2 \sum_{\vec{n}, \hat{a}} \int_0^1 ds \delta^3(x, \gamma_{\vec{n}\hat{a}}(s))\dot{a}^{\alpha}(s)\tilde{b}_{\alpha}^i(s) e_i^{\vec{n}\hat{a}}$$

(163)
\[ F_{ab}^i(x) = \frac{1}{G} \sum_{\vec{n}, \hat{a}} \int_0^1 ds \delta^3(x, \gamma_{\vec{n}\hat{a}}(s)) \hat{e}_{\vec{n}\hat{a}}(s) \epsilon_{abc} b_{\vec{n}\hat{a}}^i. \]  

(164)

Here, the \( G \) is put in for dimensions. We would like the free factors \( b_{\vec{n}\hat{a}}^i \) to be dimensionless. In Ashtekar’s formalism it is \( GF_{ab}^i \) that has the dimensions of curvature, which is inverse length squared.

For this to be a valid ansatz, we must find a connection \( A_{\vec{n}\hat{a}}^i \) whose curvature is (164). This is a standard problem from Chern-Simon theory, where we sometimes have to consider connections whose curvatures are valued on curves [51]. There the usual thing is to introduce an auxiliary background metric to define the connection as in the usual solenoid problems of first year electromagnetism. However, there is an alternative which is very natural in this context, which is to pick \( A_{\vec{n}\hat{a}}^i \) that are also distributional. A natural choice turns out to be to pick \( A_{\vec{n}\hat{a}}^i \) that have support on the faces of the lattice. If we call \( S_{\vec{n}\hat{a}\hat{b}} \) the two dimensional face which leaves the vertex \( \hat{n} \) in the positive \( \hat{a} > \hat{b} \) directions, we can write a distributional connection as

\[ A_{\vec{n}\hat{a}}^i(x) = \frac{1}{G} \sum_{\vec{n}\hat{a}\hat{b}} d^2 S_{\vec{n}\hat{a}\hat{b}} \epsilon_{abc} \delta^3(x, S_{\vec{n}\hat{a}\hat{b}}) a_{\vec{n}\hat{b}}^i \]  

(165)

The curvatures may be expressed in terms of these connections in a simple way. Any link \( \gamma_{\vec{n}\hat{a}} \) is the meeting point of four faces. Let us label these for simplicity face one through face four (choosing one as the arbitrary starting point.) Then, using the nonabelian stokes theorem, one can show that,

\[ Tr[e^b] = Tr[e^{a_1}e^{a_2}e^{a_3}e^{a_4}] \]  

where \( b = b^i \tau_i \) and similarly for the \( a^i \)’s.

A field configuration is then given by associating a Lie algebra element \( a^i \) to every face and another one, \( e^i \) to every link of the lattice.

We may then extend the constraints to these configurations[52]. The Gauss’s law constraint, (47), is a straightforward calculation. The result is that

\[ G^i(\hat{n}) = \sum \hat{n} \left( e_{\hat{a}\hat{n}}^i - e_{\hat{i}(\hat{n}-\hat{a})\hat{a}}^i \right) 
+ \frac{1}{8} \epsilon_{ijk} \hat{a}^b \left( e_{\hat{a}\hat{n}}^i + e_{\hat{i}(\hat{n}-\hat{a})\hat{a}}^j \right) \left( a_{\hat{n}bc}^k + a_{\hat{i}(\hat{n}-\hat{a})\hat{b}c}^k + a_{\hat{i}(\hat{n}-\hat{b})\hat{b}c}^k + a_{\hat{i}(\hat{n}-\hat{a}-\hat{b})\hat{b}c}^k \right) = 0. \]  

(167)

The Hamiltonian and diffeomorphism constraints involve products of distributions and must be regularized. This can be done by following the same procedure developed in section 4.5 for the volume operator. The volume is divided up into boxes and an approximation for the constraint integrated over the box is developed as an expression in which the frame fields and curvatures are...
integrated over the walls of the box. The resulting integrals are all intersection numbers and one finds in the end that,

\[ C(\vec{n}) = \epsilon_{ijk} \epsilon^i_{\alpha \beta} e^j_{\alpha \gamma} e^k_{\beta \delta} \tag{168} \]

\[ C(\vec{n}) = \epsilon_{\alpha \beta \gamma} e^i_{\alpha \delta} e^j_{\beta \epsilon} e^k_{\gamma \zeta} \tag{169} \]

These last two constraints may be solved directly, using the Capovilla-Dell-Jacobson trick (149), which is just algebraic. The result is that we are left with three sets of difference equations given by the discrete analogue of (151).

Thus, we see that there are distributional solutions to the initial data constraints in the Ashtekar form. As we have seen that such distributional configurations can approximate any smooth configuration, what we have is a discrete approximation to the initial data equations of general relativity.

Finally, the evolution equations may be developed for these distributional solutions. This may be done by taking the equations of motion and writing them as evolution equations for the \( e^i_{\alpha \delta} \)'s and \( a^i_{\alpha \beta} \)'s. Equivalently, one can make a Hamiltonian system as follows. Take as the phase space the collection of all \( (e^i_{\alpha \delta}, a^i_{\alpha \beta}) \) pairs and define the Poisson brackets,

\[ \{ a^i_{\alpha \beta}, e^j_{\beta \epsilon} \} = \frac{iG}{a^2} \delta^i_j \delta_{\alpha \beta} \epsilon_{\epsilon \delta \epsilon} \tag{170} \]

One may then show that for test fields \( f_{ai}(x) \) and \( \tilde{g}^{bij}(y) \), slowly varying on a scale \( L \),

\[ \{ \int \tilde{E}^{ai} f_{ai}, \int \tilde{g}^{bij} A_{bij} \} = i \int \tilde{g}^{ai} f_{ai} + O(a/L). \tag{171} \]

The equations of motion may then be found by taking Poisson brackets with the Hamiltonian constraint, I will not write them down here. Their exact form, and more details about these distributional solutions, may be found in [52].

6 Three open questions

6.1 Solutions to the Hamiltonian constraint

As I mentioned in section 2.5, we know how to construct an infinite dimensional space of solutions to the Hamiltonian constraint. This was, indeed, one of the first results achieved in the program of nonperturbative quantization based on the Ashtekar variables. The solutions were first found in the connection representation (in which the states are functions of the connection \( A^i_\alpha \)) [26] before the invention of the loop representation.

Since the discovery of these solutions in 1986 there has been disagreement as to whether the set of solutions found then was a complete set, or whether they were, in some sense, accidental of unphysical. Recently there has been
progress in several directions which sheds light on this issue. I would like to briefly describe these developments.

First, a few words of background, as the problem of finding solutions to the Hamiltonian constraint was not reviewed in detail above. The key point, which makes it possible to find exact solutions to the Hamiltonian constraint, is that when the Hamiltonian constraint acts on a loop function, all of the action happens at singularities of loops. By a singularity of a loop I mean here a nondifferentiable point of a loop, which may be a kink or an intersection point.

To be more specific, the Hamiltonian constraint (49) involves a product of operators at a point and thus needs to be regulated. It is then represented by a sequence of operators, $C^\delta(x)$, which are, for $\delta$ strictly positive, well defined operators. The condition that the Hamiltonian constraint annihilate a state $\Psi$ then becomes

$$\lim_{\delta \to 0} \left( C^\delta(N) \Psi[\gamma] \right) = 0.$$  \hfill (172)

Here $C^\delta(N) \equiv \int d^3 x N(x) C^\delta(x)$ is the constraint smeared with a smooth lapse $N$. The topology in which the limit (172) is taken is usually taken to be the pointwise topology, which is to say the expression must vanish when it is evaluated at every loop $\gamma$, as well as for every smooth $N$. This is, of course, the same thing as asking that it vanish in terms of the norm (3).

What is meant by saying that the action is concentrated on loops with singular points is then that $C^\delta(N) \Psi[\gamma]$ is of order $\delta$ unless the loop $\gamma$ has a singular point. If it does then there are terms of order $\delta^{-2}$, $\delta^{-1}$ and order 1, which do not go away when we take the limit $\delta \to 0$.

Now, the first implication of this is the following \cite{26, 13}:

Any loop functional $\Psi[\gamma]$ with support only on nonsingular loops is a solution to the Hamiltonian constraint.

If $\Psi[\gamma]$ is also diffeomorphism invariant then it is a function only on the link classes of the loop (which don’t intersect because they have no singular points.) We can then state the result another way \cite{13}:

Associated to every link invariant $I[\{\gamma\}]$ is an exact quantum state of the gravitational field which is defined by $\Psi[\gamma] = I[\{\gamma\}]$ if $\gamma$ is simple and $\Psi[\gamma] = 0$ if $\gamma$ is nonsimple.

As there are an infinite number of link classes, we have an infinite dimensional space of exact quantum states of general relativity.

The question we would like to ask is then whether this set of solutions is complete. A great deal of work has been done on this question recently, mostly by Berndt Bruegman, Rodolfo Gambini and Jorge Pullin \cite{31, 34, 35}. I would like to discuss what is known, presently, about this question.

First of all, what do we mean by completeness of the space of solutions of the constraints? As far as physics is concerned, what we need is that the space of physical states carry a nontrivial, and perhaps faithful, representation of the algebra of physical observables.
Unfortunately, this criteria is very hard to evaluate at the present time, given our ignorance about the physical observables. Given this situation, we can look to other criteria for guidance. However, we must keep in mind that any other criteria we consider must be in the end secondary to the one just enunciated.

We can consider criteria for completeness of the solutions to the Hamiltonian which are mathematical: do we have, in fact, the complete set of solutions to the equation as an infinite dimensional linear differential equation? In order to be able to investigate this question we must be specific about both the operator in question and the space of states within which we are asking for the complete set of solutions. Let us first consider the space of states in question to be the space $S_{\text{kin}}^\infty$ with the discrete norm (3) that has been the subject of most of this paper. We must, further, be specific about the operator ordering and regularization procedure that goes into the construction of the operator $\mathcal{O}^\delta(N)$. Let me take this to be the original ordering and regularization as was given in the papers [26, 13]. This is an ordering in which, in essence, the two $\tilde{E}_a$’s are put to the right and hence act on the state before the $F_{ab}$.

The answer in this case is that the solutions based on the nonintersecting link classes are not complete. There are, in addition, a large number of solutions based on intersecting loops. In the language developed in section 2.3 the leading term, as $\delta \to 0$ of the action of the Hamiltonian constraint, evaluated on a loop $\Gamma$ which is part of a graph $\Gamma$ is proportional to $1/\delta$ times a matrix which acts in the finite dimensional subspace $V_\Gamma$ of rearrangements of the routings through the graph. The problem of finding solutions has been studied for the case that two [26], three [33], four or five [34] lines meet at a point. In each case solutions are found which are linear combinations of routings through the points of intersection.

Unfortunately, little is known generally about these intersection solutions. We have neither a proof of existence for an arbitrarily complicated intersection nor a general characterization of the solutions. More importantly, we are still missing a proof of completeness. However, there is an important consideration, which bears on this question.

This is that, as was suggested by Ted Jacobson [53] and demonstrated recently by Bruegman and Pullin [34], the method of finding solutions employed in constructing all of the intersecting solutions does not use all of the information in the Hamiltonian constraint operator. All that is used is the antisymmetry of the action of the two $\tilde{E}_a$’s, which are on the right in the ordering used. That is, all of the solutions described in [26, 13, 33, 34] are also annihilated by any operator of the form

$$\hat{O} = \mathcal{W}^{ck} \epsilon_{abc} \epsilon_{ijk} \tilde{E}_a \tilde{E}_b$$

where $\mathcal{W}^{ck}$ is an arbitrary operator. The fact that $\mathcal{W}^{ck}$ is actually the curvature is not used. This means, in particular, that all of these states are annihilated by the operator (115) for the determinant of the metric, which is of the form of (173) with $\mathcal{W}^{ck} = \tilde{E}^{ck}$.

68
We should mention that this does not mean that the states in the loop representation cannot have a good physical interpretation. Indeed, one of the main themes of this review is that states which are in the kernel of the operator that represents the determinant of a metric can, and do, have a good classical limit in terms of observables measured over scales larger than the Planck length. If all states are annihilated by the determinant of the metric at a point, this points to a breakdown of the classical picture at short scales, rather than of the theory. Furthermore, another objection that has been offered—that the theory is thereby insensitive to the value of the cosmological constant—is also not fatal. It is certain that whether or not the physical state space is insensitive to \( \Lambda \), the physical observables, and through them the inner product, will certainly depend on the its value.

However, it is still very interesting to investigate whether there could be another class of solutions to the Hamiltonian constraint which are not annihilated by the determinant of the metric or other operators. This is a problem which is currently under investigation, from several different points of view.

To study this problem it is necessary to understand better the action of the Hamiltonian constraint operator. Recently a lot of progress has been made on this problem. As this is a rather technical area, I will refer the reader to the original papers[13, 54, 34, 35]. As a byproduct of this work a new kind of solution to all the constraints was constructed by Bruegmann, Gambini and Pullin[35]. These solutions cannot be found by constructing the Hamiltonian constraint operator in the kinematical representation space \( S^\text{kin}_N \), they are based on a representation of the loop observables developed by Gambini and Leaf[56] that is, apparently, inequivalent at the kinematical level to the one developed here. Furthermore, this new method suggests connections between quantum gravity, on the one hand, and knot theory and Chern-Simon theory[56], on the other hand, which are rather suggestive. This is an extremely interesting discovery, and we can expect further developments in this direction in the near future.

6.2 The problem of the choice of the inner product

As mentioned many times in the previous pages, the problem of the inner product is perhaps the key unsolved problem in nonperturbative quantum gravity. Even if we have the complete set of physical states we cannot compute anything in quantum mechanics without an inner product. Here I would like to briefly discuss three aspects of it. First, I will explain why it is such a hard problem. Second, I will summarize what we learned about it in chapter 4. Third, I will propose a strategy for attacking it.

In free and perturbative quantum field theories the inner product is determined by the symmetries of the theory, and in particular by Poincare invariance. The problem is that in nonperturbative quantum gravity there is no Poincare symmetry, so we have to find another criteria. There is a criteria which could
work in this context: this is to fix the inner product by the reality conditions of
the theory. Indeed, this criteria is available generally to fix the inner product of
any quantum mechanical system. The classical algebra of observables is a star
algebra, which means that the complex conjugate of any observable is defined
within the algebra. One can then proceed to quantize a classical system in two
steps. First, one finds a representation of the classical algebra of observables
as an algebra of linear operators on some representation space $\mathcal{S}$. Second, one
chooses an inner product on $\mathcal{S}$ such that the reality conditions on the classical
observables are represented in terms of hermiticity conditions on their quantum
representatives.

This procedure has been developed formally by Ashtekar\cite{11} and many ex-
amples have been worked out for finite dimensional systems by Ashtekar and
Tate\cite{15,57}. It also works in the case of linear quantum field theories, such
as Maxwell\cite{21,22} theory and linearized gravity\cite{23}. In these cases it yields
the Poincare inner product. It is especially useful in nonstandard quantizations
of Maxwell and linearized gravity in which one chooses to quantize complex
combinations of the field variables, such as the self-dual variables.$^{35}$

In section 4.3 we found that the reality conditions did give us nontrivial
information about the inner product at the kinematical level. By requiring that
the area observable, $\mathcal{A}[\mathcal{S}]$, be hermitian$^{36}$ we found that the inner product is
diagonal on simple loops. However, it is non-trivial when the loops involve inter-
sections and retracings. To completely determine it we must lift the degen-
eracies of the area operator by studying the reality conditions for some operator that
does not commute with it. This is a straightforward problem, that is currently
under attack.

However, when we come to the problem of the physical inner product we
must face another difficulty. For each of the three representation spaces, the
kinematical, diffeomorphism invariant and the physical, we must determine the
inner product using the reality conditions. Further, normally the states which
solve constraints are not normalizable in the inner product of the kinematical
theory. This means that the inner product of the physical state space should be
determined in terms of the reality conditions on the physical observables.

The difficulty is that in general relativity we know almost nothing at the
classical level about the physical observables. Indeed, except for a handful in
the asymptotically flat case we know not a single physical observable explicitly
in the full theory. This is because the Hamiltonian is a constraint, so that the
problem of finding time reparametrization invariant observables is a dynamical
problem. This is, indeed, the key problem that separates the quantization of

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$^{35}$The representations in which the quantum state is a function of the self-dual part of the
linearized field are unusual, in that the self-dual part of the field is the positive frequency part
of the left helicity plus the negative frequency part of the right handed helicity. Thus the right
handed part of the field must be quantized in an anti-Bargmann representation. At first sight
this may seem a problem, but it can be done$^{22}$

$^{36}$At least on a subspace of the state space.
a gravitational theory from a nongravitational theory. The result, and the problem, is that since the physical inner product should be chosen to realize the reality conditions of the physical observables its choice is a dynamical problem.

The problem of finding and interpreting the physical observables in quantum gravity is exactly the problem of time. The problem of time is currently very controversial and has been the subject of a number of recent articles\[17, 58\]. I will say only a few words about it here.

Because physical observables must commute with the Hamiltonian constraint, which generates time reparametrization, they cannot depend on a background time coordinate. However this does not mean they do not contain information about evolution in time. As first stressed by DeWitt\[59\] and recently emphasized by Rovelli\[58\], the physical time evolution must be specified in terms of physical clocks, that are dynamical degrees of freedom of the theory. One must pick some degree of freedom of the theory, perhaps arbitrarily to call a clock. There are then physical, that is time coordinate invariant, observables that tell us the value of other observables as a function of the physical clock degree of freedom. For example, if one couples the gravitational field dynamically to a point particle carrying a physical clock, one can construct a physical observable that will tell us what the Ricci scalar is at the position of the particle for every reading of the clock.

In practice, the construction of such an observable as an explicit function of the canonical coordinates of the theory involves using solutions of the equations of motion of the theory. In a theory such as general relativity, which is not integrable, and for which we do not expect to be able to write the general solution to the equations of motion in closed form, this will always be a difficult problem. It clearly must be approached through some systematic approximation procedure. But there is no objection, in principle, to the construction of such observables at the classical level.

The problems of time and of the interpretation of physical observables are then essentially difficulties of the classical theory. Once the problem is solved there, one can proceed to the quantum theory. As argued in several recent papers by Rovelli\[58\], if the problem can be solved at the classical level, there is no objection in principle to using the results at the quantum level. That is, given the classical physical observables at the classical level, perhaps expressed in some systematic approximation, one can seek operators which represent them acting on the space of physical quantum states. This may be a difficult technical problem, involving ordering and regularization difficulties, but there is no reason this should be a more difficult problem than finding representatives for the spatially diffeomorphism invariant observables. If this can be done then the reality conditions can be expressed in terms of these physical operators and an inner product found on the physical state space which represents them.

This, then, is a program for solving the problem of the physical inner product. I should mention that this is a controversial proposal, particularly cogent objections have been put forward by Kuchar\[17\]. I suggest that the reader con-
sult the review of Kuchar for the statement of his objections. One point Kuchar makes is the, entirely reasonable, assertion that at some level difficulties in practice, such as operator ordering difficulties may become difficulties in principle. Further, as Kuchar describes, there are other proposals for solving the problem of time in quantum gravity. In the absence of a solution to this key and fundamental problem, it seems to me one should not argue too strongly for any one point of view. Real results on any of these proposals would be most welcome.

As I indicated in passing above, if we are to make progress on constructing the physical observables, the only hope seems to be to invent a systematic approximation procedure. If, as is strongly suggested by the results I described in chapter 4, the problems of the divergences of the theory are solved at the kinematical level it is perhaps appropriate now to return to some form of perturbation theory. But what is required is a purely quantum mechanical perturbation theory, based on expansion around a nonperturbative quantum state such as the weave, rather than around a classical background.

6.3 The question of the existence of a ground state

It used to be that the key issue to be resolved by a successful quantum theory of gravity was renormalizability. Certainly, if there was going to be a successful perturbative theory this was the main problem. Or rather, the problem was to find a theory that is both perturbatively renormalizable and has a stable ground state. General relativity is perturbatively nonrenormalizable, and by now all hopes of fixing the problem through a modification of perturbation theory, such as a $1/N$ expansion, are dead. It is not hard to invent theories that are perturbatively renormalizable and much work in the 1970’s and 1980’s was put into studying such theories. The problem is that none of them have a ground state. This is true for a simple reason: perturbative renormalizability requires that the Lagrangian include all terms of dimension four consistent with the gauge symmetry, but as the metric is dimensionless these will necessarily include terms with four derivatives, leading to the well known instabilities of higher derivative theories.

For this reason, many people began working on string theories in the 1980’s. The main idea of string theory is that if the theory is well defined it will be finite perturbatively, because of modular invariance. The main problem faced by string theory is then stability. The bosonic string theories are unacceptable because of tachyon instabilities. The superstring theories seem to solve this problem, at least perturbatively. However, as emphasized by Eliezer and Woodard, stability and the existence of a ground state are potential problems for nonperturbative formulations of string theory.

What I would like to argue here is that the situation is much the same for quantum general relativity, nonperturbatively. If the theory is well defined at all it will be finite. But there are serious questions about whether the theory has a ground state.
One of the key themes of this review has been that if a spatially diffeomorphism invariant observable can be translated into a quantum operator, without breaking the diffeomorphism invariance, that operator must be finite. I gave an argument for this at the end of section 2.4. The key idea of this argument is that i) any diffeomorphism invariant operator must be constructed through a regularization procedure, ii) that regularization procedure introduces both a background metric, $h_{ab}^0$, and a regulator or cutoff scale $\epsilon$, and that iii) the later is measured in units determined by the former, so that if the operator, in the limit the cutoff is taken away, has no dependence on the background metric (which must be the case if diffeomorphism invariance is to be preserved) it can have no dependence on the cutoff.

It is then a nontrivial problem to invent regularization procedures which result in finite and background independent operators. We saw in chapter four examples of both unsuccessful and successful ways to do this. In accordance with the argument, we did find that the resulting operators were finite whenever they were background independent.

Let me then turn to the question of the existence of a ground state. This problem must be approached differently than conventional quantum field theories, because gravitational theories, in general, do not have Hamiltonians. Because of the inseparability of evolution and time reparametrization, the classical evolution is generated by the Hamiltonian constraint. However, there are special cases in which Hamiltonians exist. These cases are when the spacetime diffeomorphism invariance is broken by the imposition of boundary conditions.

One way that this can be done is to require that the gravitational field be asymptotically flat. Classically, this means that the spatial topology is taken noncompact and that on a region of $\Sigma$, meant to be a neighborhood of spatial infinity\footnote{This is normally taken to be the exterior of a compact region of $\Sigma$. For a full statement of the conditions for asymptotic flatness, see \cite{10, 11}.} one can impose a flat metric $h_{ab}$ such that, given a radial coordinate, $r$ defined with respect to it, the spatial metric $q_{ab}$ approaches $h_{ab}$ as $r \to \infty$. One can show that this means that ”at infinity” there is a lorentz invariance so that we can imagine there exists a family of lorentz observers sitting at infinity watching what is going on inside of the spacetime. We can then define a Hamiltonian which evolves the whole spacetime according to the time of one of these observers. As we have broken the diffeomorphism invariance by the imposition of $h_{ab}$ that Hamiltonian is no longer just a constraint, it has an additional term. This additional term is, however, a boundary term. Thus, the Hamiltonian looks like

$$H(N) = \int_\Sigma NC + \int_{\partial \Sigma} NB$$

(174)

where $N$ is the lapse function that measures the increase in coordinate time. In the new variables formalism, the boundary term, $B$, is very simple, when $C$
vanishes the Hamiltonian is just

\[ H(N) = \int_{\partial \Sigma} d^2 S_N A_{abi} \sigma_{abi}, \]

(175)

where we have assumed an asymptotic form for the frame field, \( \tilde{E}^{ai} = \delta^{ai} \).

We can now begin to address the problem of whether the theory has a ground state. Before attacking this problem quantum mechanically, however, we must discuss the situation classically.

Equation (174) is not a sum of squares and it is a nontrivial problem to show that it is positive. Indeed, the positive energy theorem is a justly celebrated result that was open for a long time before being proved by Schoen and Yau\[62\] and Witten\[63\]. The proof of Witten is, in fact, closely related to the Ashtekar formalism. It uses the self-dual connection \( A^s \) and was prefigured in a paper by Sen\[64\] that was also a precursor of Ashtekar’s work. However, there is an important point, which was noticed only recently. This is that the proof of Witten assumes the nondegeneracy of the frame field \( \tilde{E}^{ai} \).

As I discussed in sections 5.3 and 5.4, the space of solutions of the Ashtekar form of the constraints includes solutions in which \( \tilde{E}^{ai} \) is degenerate\[65, 51\]. Is there an extension of the positive energy theorem to these configurations? The answer, for better or worse, is no: an explicit class of counterexamples was discovered by Varadarajan\[66\]. These are spherically symmetric solutions in which \( \tilde{E}^{ai} \) is nondegenerate and asymptotically flat in a neighborhood of infinity, nondegenerate in a neighborhood of the origin, but degenerate in an intermediate region. They can have any value for the Hamiltonian, positive or negative. Furthermore, these solutions are everywhere nonsingular, so that even for the positive energy ones there is no singularity as there is in the Schwarzschild solution. Because of the crucial role of the degenerate regions in these solutions, they are called degenerons.

The existence of the degeneron solutions raises many problems for the theory. For example, the transition between the lagrangian theory and the hamiltonian theory, sketched in section 3.2, depends on the nondegeneracy of \( \tilde{E}^{ai} \). When this breaks down their solution spaces are not the same. Thus, the degenerons are not solutions to the four dimensional field equations (41-42). Indeed, one can prove a positive energy theorem for the solutions to (41-42) which includes the case that the four dimensional \( e^i \)'s are degenerate\[67\]. The crucial difference between the field equations (41-42) and the constraint equations (47-49) are the density weights. The existence of the degeneron solutions turn out to depend on the fact that the hamiltonian constraint (49) is a density of weight two.

The existence of the degenerons certainly suggests that the problem of the existence of the ground state in the quantum theory is going to be non-trivial. This is a problem that has been the subject of some discussion during the last year. There are three possibilities which have emerged from these discussion.

The first possibility is that the degeneron solutions destabilize any asymptotically flat quantum state, so that the theory has no ground state. If this is
the case then we will have to abandon the attempt to quantize general relativity nonperturbatively using the Ashtekar variables.

The second possibility is that their effect may not be very serious because the degeneron solutions may be in a region of phase space or configuration space which is disconnected from the region of phase space corresponding to non-singular solutions, for which positive energy applies. Although this has not been shown, it is plausible that this is the case. It may then be possible to restrict the quantum states to the non-degenerate region. For example, in $2+1$ gravity for compact spaces the phase space has several disconnected regions and the quantum theory can be constructed by restricting attention to just one of them[38].

There is, however, a third possibility. This is that the quantum Hamiltonian may be bounded from below in spite of the fact that the classical Hamiltonian is not. Certainly, there are many cases in quantum mechanics in which the classical Hamiltonian is not bounded from below, but the quantum Hamiltonian is. Of course, the inner product plays a crucial role in this problem because even in cases, such as the harmonic oscillator, where the classical Hamiltonian is bounded from below, the quantum Hamiltonian is only bounded from below for states that are normalized with respect to the inner product. Thus, this problem cannot be attacked until we know more about the inner product.

Given our present state of ignorance about this problem, perhaps I can conclude with an intuitive argument. If it is the case that the full quantum Hamiltonian is bounded from below then the degeneron solutions may play an important role: at a semiclassical level they may destabalize the perturbative vacuum of the theory. This is because in examples in which the quantum Hamiltonian is bounded, but the classical one is not, the ground state gives a lot of amplitude to those regions of the phase (or configuration) space where the energy is unbounded. But this is exactly the region of the degenerate solutions.

This means that if the theory exists any attempt to construct a semiclassical construction of the ground state may fail because the vacuum will fill up with a condensate of degenerate configurations. Thus, the short distance structure of the theory will be dominated by such configurations-which means that there will be no smooth metric if we look to short enough distances.

However, the ground state of the asymptotically flat theory must also approximate a smooth metric at large distances: the flat metric. Otherwise it would not be a good state of the asymptotically flat theory. The question to ask is then: is it possible that the ground state could have support on degenerate configurations and still be asymptotically flat when probed on scales much larger than the Planck scale?

The key conclusion of all the work I have described here is that such states do exist at the nonperturbative level: they are the weave states we discussed in section 4.7. Moreover, as I showed in section 5.3, their classical analogue are distributional configurations that are also, except for sets of measure zero, completely degenerate.
We may then make the following conjecture: the degenerons are the connection between the semiclassical and nonperturbative approaches to the theory. If one attempts to construct the theory semiclassically one will find that one cannot do it because the perturbative vacuum is unstable and space will fill up with a condensate of degenerate solutions. If a stable ground state exists this must happen at the nonperturbative level. However, at the nonperturbative level all states look like the result of a condensate of degenerons: they are degenerate almost everywhere. Furthermore, if such a state is to be asymptotically flat we found that its structure is not completely free: it must have discrete structure fixed at the Planck scale. This is, if we are optimistic, the signal that the theory can be stable nonperturbatively.

7 Conclusion

In my introductory remarks, I indicated that a great obstacle to our being able to invent a quantum theory of gravity is the difficulty of seeing past our Newtonian preconceptions of the nature of physical reality and physical theory. One of the themes of this review has been that the classical spacetime background is one of the features of Newtonian theories that we must transcend. In retrospect, it might be argued that the difficulties faced by many perturbative and semiclassical approaches to quantum gravity are due to the fact that they do not sufficiently achieve this. A major virtue, and the key difficulty, of a purely nonperturbative approach to quantum gravity is that one eschews the use of classical backgrounds at the beginning. As the development of this work has shown, this leads to two kinds of difficulties. First, how do we construct a description of the geometry of spacetime that is purely quantum mechanical, and makes no use of a classical background? Second, how do we construct well behaved quantum operators without the classical background sneaking in the back door through the regularization and renormalization procedures?

The chief interest of the results I have reviewed here is that they begin to show us the way to answering these two questions. Attention here has been restricted to the kinematical and diffeomorphism invariant level and to only one set of observables: those that measure the spatial geometry. However, the lessons of the work are more general and there is reason to believe that the same set of ideas and technical tricks may help us to extend this work to observables that measure the evolution of the three geometry and, after that, to physical observables.

It should also be emphasized that all of the new results reported in chapter 4 make use only of the spatial diffeomorphism invariance, as well as the basic kinematics of the new variables and loop representation. Thus, the results such as the quantization of areas and volumes will hold in a large class of theories, including general relativity with arbitrary matter couplings.

Finally, although I have not touched on it here, it is worth mentioning that
the problem of constructing a physical theory without the use of fixed background structures is a very old problem\cite{7}. It was at the heart of Leibniz’s and Mach’s criticisms of Newtonian mechanics, which were the starting point for Einstein’s construction of general relativity. If general relativity, to a large extent, achieves this, by making the spacetime geometry dynamical, a large part of the problem of quantum gravity is to achieve this for the quantum theory\cite{68}. The results presented here indicate that at least a small part of this problem, that connected with the kinematics of the theory, can be successfully attacked. Whether the direction described here will develop into a complete solution to this problem, at the dynamical level, and by doing so give rise to a consistent and useful quantum theory of gravity, remains an open question.

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References

\[1\] A. Ashtekar, C. Rovelli and L. Smolin \textit{How to weave a classical geometry from Planck scale quantum threads} Pittsburgh and Syracuse preprints, February (1992).

\[2\] A. Ashtekar, C. Rovelli and L. Smolin \textit{Low energy physics from loop-space quantum gravity: Part I: Finite operators, the "weave", and the emergence of a Planck-scale structure}. Pittsburgh and Syracuse preprints, February (1992).
[3] The reader seeking an in to the current discussion about quantum gravity could peruse the contents of several recent conference proceedings, including *Conceptual Problems in Quantum Gravity* ed. A. Ashtekar and J. Stachel (Birkhauser, Boston, 1991).

[4] S. W. Hawking, Commun. Math. Phys. 43 (1975) 199.

[5] J. Beckenstein, Phys. Rev. D 12 (1975) 3077-85; S. W. Hawking, Phys. Rev. D 14 (1976) 2460; R. M. Wald, in *Quantum Theory of Gravity: Essays in honor of the 60th birthday of Bryce S. DeWitt* ed. by. S. Christensen (Adam Hilger, Bristol, 19854); W. G. Unruh, Phys. Rev. D 14 (1976) 970. Beckenstein. For a recent discussion of the thermodynamics of the gravitational field, see C. Rovelli, Pittsburgh and Trento preprint, (1991).

[6] T. Kuhn *The Structure of Scientific Revolutions* University of Chicago Press, Chicago, 1962). For an antidote see P. Feyerabend *Against Method: Outline of an anarchistic* theory of knowledge (Verso, London, 1975).

[7] Provocative accounts of the Copernican revolution may be found in J. B. Barbour *Absolute of Relative Space, Volume I: The Discovery of Dynamics* (Cambridge University Press, Cambridge, 1991 Press, 1989) and A. Koestler *The Sleepwalkers* (Arkana, London, 1959).

[8] See A. Koestler (previous reference) footnote 22a on pages 599-600.

[9] A. Ashtekar. *New variables for classical and quantum gravity*. Physical Review Letters 57, 2244–2247 (1986). *New Hamiltonian formulation of general relativity*, Physical Review D36, 1587–1602 (1987).

[10] A. Ashtekar (with invited contributions), *New Perspectives in Canonical Gravity*. Lecture Notes. Bibliopolis, Napoli, Italy, February 1988.

[11] A. Ashtekar, *Non-perturbative canonical gravity*. Lecture notes prepared in collaboration with Ranjeet S. Tate. (World Scientific Books, Singapore, 1991).

[12] R. Gambini and A. Trias, Phys. Rev. D23 (1981) 553, Lett. al Nuovo Cimento 38 (1983) 497; Phys. Rev. Lett. 53 (1984) 2359; Nucl. Phys. B278 (1986) 436; R. Gambini, L. Leal and A. Trias, Phys. Rev. D39 (1989) 3127.

[13] C. Rovelli and L. Smolin, *Knots and quantum gravity* Phys. Rev. Lett. 61, 1155 (1988); *Loop representation for quantum General Relativity*, Nucl. Phys. B133 (1990) 80.;

[14] C. Rovelli, Classical and Quantum Gravity, 8 (1991) 1613-1676.

[15] G. Horowitz, to appear in the proceedings of the conference on Strings and Symmetries, held at Stony Brook, May 1991.
A good discussion of this, in the context of Einstein’s struggles over the invention of general relativity is in J. Stachel, *Einstein’s search for general covariance 1912-1915* in *Einstein and the History of General Relativity* ed. by D. Howard and J. Stachel, Einstein Studies, Volume 1 (Birkhauser,Boston,1989).

For a recent review see K. V. Kuchar *Time and interpretations of quantum gravity* in the *Proceedings of the 4th Canadian Conference on General Relativity and Relativistic Astrophysics* eds. G. Kunstatter, D. Vincent and J. Williams (World Scientific, Singapore,1992).

P. A. M. Dirac, *Lectures on Quantum Mechanics* Belfer Graduate School of Science Monographs, no. 2 (Yeshiva University Press, New York,1964). Note that Dirac did not address certain issues that arise in field theories such as the choice of an inner product and the effect of regularization. For this reason the procedure as specified by Dirac is ambiguous. We follow a particular version of the Dirac program, described in refs. [11, 13, 14].

E. T. Newman, in *General Relativity and Gravitation*, (ed. G. Sharir and J. Rosen) (Wiley, New York,1975); M. Ko, M. Ludvigsen, E. T. Newman and K. P. Tod, Phys. Rep. 71 (1981) 51-139; M. Ludvigsen, E. T. Newman and K. P. Tod, J. Math. Phys. 22 (1981) 818-823.

A. Ashtekar and C. J. Isham, *Inequivalent observer algebras: A new ambiguity in field quantization and Representations of the holonomy algebra of gravity and non-abelian gauge theories*, Syracuse University and Imperial College preprints (1991).

A. Ashtekar and C. Rovelli, *Quantum Faraday lines: Loop representation of the Maxwell theory*, Syracuse preprint 1991.

A. Ashtekar, C. Rovelli and L. Smolin, *Self duality and quantization*, Syracuse preprint (1991) to appear in J. Geometry and Physics, Penrose Festschrift issue.

A. Ashtekar, C. Rovelli and L. Smolin *Gravitons and Loops*, Phys. Rev. D 44 (1991) 1740-1755.

D. Rayner, Class. and Quant. Grav. 7 (1990) 651.

L. Smolin, *The $G_{\text{Newton}} \to 0$ limit of Euclidean quantum gravity*, Class. and quant. grav. (1992), in press.

T. Jacobson and L. Smolin, Nucl. Phys. B 299 (1988) 295.

A. Ashtekar, C. Rovelli and L. Smolin, *Low energy physics from loop-space quantum gravity: Part II: Gravitons from knots* Syracuse, Pittsburg and Trento preprint (1992) in preparation.
[28] J. Zegwaard, *Gravitons in Loop Quantum Gravity* Utrecht preprint, THU-91/13.

[29] R. Penrose, in *Quantum Theory and Beyond* ed. T. Bastin (Cambridge University Press, Cambridge, 1971); and in *Advances in Twistor Theory* ed. L. P. Hughston and R. S. Ward (Pitman, San Francisco, 1979).

[30] L. Smolin, Chapter V6 of *New Perspectives in Canonical Gravity*, [10] above.

[31] B. Bruegmann and J. Pullin, *On the Constraints of Quantum Gravity in the Loop Representation*, Syracuse University preprint, (1991).

[32] C. Rovelli, in preparation.

[33] V. Husain, Nucl. Phys. B313 (1989) 711-724.

[34] B. Bruegmann and J. Pullin, Nucl. Phys. B 363 (1991) 221.

[35] B. Bruegmann, R. Gambini and J. Pullin, Phys. Rev. Lett. 68 (1992) 431-434; *Knot invariants as nondegenerate states of four dimensional quantum gravity* Syracuse University Preprint (1991), to appear in the proceedings of the XXth International Conference on Differential Geometric Methods in Physics, ed. by S. Catto and A. Rocha (World Scientific, Singapore, in press).

[36] R. Gianvittorio, R. Gambini and A. Trias, Phys. Rev. D38 (1988) 702; C. Rovelli and L. Smolin. *Loop representation for lattice gauge theory*, 1990 Pittsburgh and Syracuse preprint; B. Bruegmann, Physical Review D 43 (1991) 566; J.M.A. Farrerons, *Loop calculus for SU(3) on the lattice* Phd. thesis, Universitat Autonoma de Barcelona (1990); R. Loll *A new quantum representation for canonical gravity and SU(2) Yang-Mills theory*, University of Bonn preprint, BONN-HE-90-02 (1990).

[37] M. Li, Nuovo Cimento 105B (1990) 1113.

[38] A. Ashtekar, V. Husain, C. Rovelli, J. Samuel and L. Smolin. *2+1 quantum gravity as a toy model for the 3+1 theory* Class. and Quantum Grav. L185-L193 (1989); A. Ashtekar, in ref. [1], above; L. Smolin, *Loop representation for quantum gravity in 2+1 dimensions*, in the proceedings of the John’s Hopkins Conference on *Knots, Tolopoly and Quantum Field Theory* ed. L. Lusanna (World Scientific, Singapore, 1989).

[39] H. Fort and R. Gambini *Lattice QED with light fermions in the P representation* IFFI preprint, 90-08.

[40] J. Goldberg, J. Lewendowski and C. Stornaiolo *Degeneracy in the loop variables* Syracuse Preprint (1991).
[41] R. Penrose, Gen. Rel. and Grav. 7 (1976) 31; R. Penrose and W. Rindler, 
_ Spinors and spacetime_, Vol. 2, Cambridge University Press, and references 
contained within.

[42] J. Samuel, Pramana-J Phys. 28 (1987) L429; T. Jacobson and L. Smolin, 
Phys. Lett. B 196 (1987) 39; Class. and Quant. Grav. 5 (1988) 583.

[43] A. Ashtekar, T. Jacobson and L. Smolin, Commun. Math. Phys. 115 (1988) 
631-648.

[44] D. Robinson, in A. Ashtekar, reference [10], above.

[45] R. Capovilla, J. Dell and T. Jacobson, Phys. Rev. Lett. 63 (1989) 2325; 
Class. and Quant. Grav. 8 (1991) 59; R. Capovilla, J. Dell, T. Jacobson 
and L. Mason, Class. and Quant. Grav. 8 (1991) 41. For a related work see 
P. Peldan, Class. Quantum Grav. 8 (1991) 1765.

[46] P. Renteln, Phd. thesis, Harvard Univerity (1988).

[47] L. Smolin, _Quantum gravity in a box_, Syracuse preprint in preparation 
(1992).

[48] J. Samuel, Class. and Quant. Grav. 5 (1988) L123; _Self-duality in classical 
gravity_, to appear in the Newman Festschrift, ed. by A. Janis and J. Porter, 
(Birkhauser,Boston,in press).

[49] C. G. Torre, Phys. Lett. B 252 (1990) 242, Phys. Rev. D41 (1990) 3620; J. 
Math. Phys. 31 (1990) 2983.

[50] H. Kodama, Phys. Rev. D 42 (1990) 2548-2565.

[51] L. Smolin, Modern Phys. Lett. A (1989)1091.

[52] L. Smolin, _A new discretization of the classical Einstein equations_, Syracuse 
preprint in preparation (1992).

[53] T. Jacobson, personal communication.

[54] M. Blencowe, Nuclear Physics B 341 (1990) 213.

[55] R. Gambini, _Loop space representation of quantum general relativity and 
the group of loops_, preprint University of Montevideo 1990, Physics Letters 
B 255 (1991) 180. R. Gambini and L. Leal, _Loop space coordinates, linear 
representations of the diffeomorphism group and knot invariants_ preprint, 
University of Montevideo, 1991.

[56] E. Witten _Quantum field theory and the Jones polynomial_ in the Proceed-
ings of the 1988 IAMP Congress, Swansea; Commun. Math. Phys. 121 
(1989)351.

81
[57] R. S. Tate, *Constrained Systems and Quantization, Lectures at the Advanced Institute of Gravitational Theory, December 1991, Cochin, India*, Syracuse University preprint (1991).

[58] C. Rovelli Phys. Rev. D 42 (1990) 2638-46; D 43 (1991) 442-456; in *Conceptual Problems of Quantum Gravity* ed. A. Ashtekar and J. Stachel, (Birkhauser,Boston,1991); Class. Quantum Grav. 8 (1991) 317-331.

[59] B. S. DeWitt, Physical Review 160 (1967) 1113.

[60] E. Tomboulis, Phys. Lett. 70B (1977) 361; 97B (1980) 77; A. Strominger Phys. Rev. D 24 (1981) 3082; L. Smolin, Nucl. Phys. B 208 (1982) 439.

[61] D. A. Eliezer and R. P. Woodard, Nucl. Phys. B 325 (1989) 389-469.

[62] R. Schoen and S.-T. Yau, Phys. Rev. Lett. 48 (1982) 369.

[63] E. Witten, Commun. Math. Phy. 80 (1981) 381.

[64] A. Sen, J. Math. Phys. 22 (1981) 1781-1786; Phys. Lett. B 119 (1982) 89-91.

[65] I. M. Bengstrom, Class. and Quant. Grav. 5 (1988) L139; 7 (1990) 27; preprint 1991; S. Koshti and N. Dadhich, Class. Quant. Grav. 6 (1989) L223 V. Husain and L. Smolin, Nucl. Phys. B 327 (1989) 205.

[66] M. Varadarajan, Class. and Quant. Grav. 8 (1991) L235-L240; L. Smolin and M. Varadarajan, Syracuse preprint in preparation (1992).

[67] J. Samuel, Raman Institute preprint, in preparation (1992).

[68] L. Smolin, *Space and Time in the Quantum Universe in Conceptual Problems of Quantum Gravity* ed. by A. Ashtekar and J. Stachel, (Birkhauser,Boston,1991).