TRIPLICATE DUAL SERIES OF DOUGALL–DIXON THEOREM

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Abstract. Applying the triplicate form of the extended Gould–Hsu inverse series relations to Dougall’s summation theorem for the well–poised $7\mathcal{F}_6$-series, we establish, from the dual series, several interesting Ramanujan–like infinite series expressions for $\pi^2$ and $\pi^{\pm 1}$ with convergence rate $\frac{1}{729}$.

1. Introduction and Motivation

For an indeterminate $x$, the shifted factorial is defined by $(x)_0 \equiv 1$ and

$$(x)_n = x(x+1) \cdots (x+n-1) \quad \text{for} \quad n \in \mathbb{N}.$$ 

This can also be expressed as a quotient $(x)_n = \Gamma(x+n)/\Gamma(x)$, where the $\Gamma$-function (see [28], §8 for example) is given by the beta integral

$$\Gamma(x) = \int_0^\infty u^{x-1} e^{-u} \, du \quad \text{for} \quad \Re(x) > 0,$$

which admits Euler’s reflection property

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x} \quad \text{with} \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

and the following asymptotic formula

$$\Gamma(x+n) \approx n^x (n-1)! \quad \text{as} \quad n \to \infty.$$ (2)

This last formula is simpler than Stirling formula and utilized frequently to evaluate limits of $\Gamma$-function quotients.

About one century ago, Dougall [20] discovered a very general summation theorem for the terminating well–poised $7\mathcal{F}_6$-series. By making use of the triplicate form of the extended Gould–Hsu inverse series relations, we shall investigate the dual series of Dougall’s well–poised sum, that will lead to a large class of summation formulae for $\pi$-related infinite series of convergence rate $\frac{1}{729}$. According to the bisection series method, a number of these series can be reduced to simpler ones.
with convergence rate $-\frac{1}{27}$. Five elegant formulae are anticipated as follows:

\[
\frac{9\sqrt{3}}{\pi \sqrt{16}} = \sum_{k=0}^{\infty} \left(\frac{-1}{27}\right)^k \left[\frac{1}{3}, \frac{2}{7}, \frac{1}{6}\right]_k \{2 + 21k\}, \tag{Example 21}
\]

\[
\frac{27\sqrt{3}}{\pi \sqrt{32}} = \sum_{k=0}^{\infty} \left(\frac{-1}{27}\right)^k \left[\frac{1}{3}, \frac{2}{7}, \frac{5}{6}\right]_k \{5 + 42k\}, \tag{Example 22}
\]

\[
\frac{3\sqrt{3}}{\pi \sqrt{4}} = \sum_{k=0}^{\infty} \left(\frac{-1}{27}\right)^k \left[\frac{1}{3}, \frac{1}{7}, \frac{2}{6}\right]_k \{1 - 63k^2\}, \tag{Example 23}
\]

\[
\frac{\pi \Gamma^2(\frac{1}{3})}{6\Gamma^2(\frac{1}{6})} = \sum_{k=0}^{\infty} \left(\frac{-1}{27}\right)^k \left[\frac{1}{3}, \frac{4}{7}, \frac{2}{3}\right]_k \{3 + 7k\}, \tag{Example 34}
\]

\[
\frac{48\pi \Gamma^2(\frac{1}{3})}{\Gamma^2(\frac{1}{4})} = \sum_{k=0}^{\infty} \left(\frac{-1}{27}\right)^k \left[\frac{1}{3}, \frac{1}{7}, \frac{1}{3}\right]_k \{9 + 28k + 21k^2\}. \tag{Example 35}
\]

They resemble the so–called Ramanujan–like series, mainly discovered one century ago by Ramanujan [29] and recently by Guillera [25–27].

The rest of the paper will be organized as follows. The next section will serve as the theoretical part, where the main theorems and proofs will be included. Then in Section 3, we shall present 35 infinite series expressions for $\pi^2$ and $\pi^{\pm 1}$ as applications.

Throughout the paper, the following abbreviated notations will be adopted for product and quotient of shifted factorials:

\[
[a, b, \ldots, \gamma]_n = (a)_n(b)_n \cdots (\gamma)_n,
\]

\[
[A, B, \ldots, C]_n = (A)_n(B)_n \cdots (C)_n.
\]

### 2. TriPLICATE INVERSION OF DOUGALL’S \(\mathbb{F}_6\)-SERIES

In 1973, Gould and Hsu [24] discovered a useful pair of inverse series relations, which can equivalently be reproduced below. Let \(\{a_i, b_i\}\) be any two complex sequences such that the $\phi$-polynomials defined by

\[
\phi(x; 0) \equiv 1 \quad \text{and} \quad \phi(x; n) = \prod_{k=0}^{n-1} (a_k + xb_k) \quad \text{for} \quad n \in \mathbb{N} \tag{3}
\]

differ from zero for $x, n \in \mathbb{N}_0$. Then there hold the inverse series relations

\[
f(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \phi(k; n) \ g(k), \tag{4}
\]

\[
g(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{a_k + kb_k}{\phi(n; k + 1)} \ f(k). \tag{5}
\]
The Gould–Hsu inversions have the following extended form (cf. [2,5,11]):

\[
\begin{align*}
\Omega_n(a; b, c, d) & := \left[ 1 + a, 1 + a - b - c, 1 + a - b - d, 1 + a - c - d \right] \\
& \quad \times \left[ 1 + a - b, 1 + a - c, 1 + a - d, 1 + a - b - c - d \right]_n, \\
& = \sum_{k=0}^{n} \frac{a + 2k}{a} \left[ a, b, c, d, e, -n \right]_k, \\
& \quad \left[ 1, 1 + a - b, 1 + a - c, 1 + a - d, 1 + a - e, 1 + a + n \right]_k,
\end{align*}
\]

where the series is 2-balanced because \(1 + 2a + n = b + c + d + e\).

For all \(n \in \mathbb{N}_0\), it is well known that \(n = \left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{1+n}{3} \right\rfloor + \left\lfloor \frac{2+n}{3} \right\rfloor\), where \(\lfloor x \rfloor\) denotes the greatest integer not exceeding \(x\). Then it is not difficult to check that Dougall’s formula (8) is equivalent to the following one

\[
\Omega_n(a; b + \left\lfloor \frac{n}{3} \right\rfloor, c + \left\lfloor \frac{1+n}{3} \right\rfloor, d + \left\lfloor \frac{2+n}{3} \right\rfloor) = \frac{(1 + a)_n}{(b + c + d - a)_n}
\]

\[
\times \frac{(1 + a - c - d)_{\left\lfloor \frac{n}{3} \right\rfloor}}{(b - a)_{\left\lfloor \frac{n}{3} \right\rfloor}} \frac{(1 + a - b - d)_{\left\lfloor \frac{1+n}{3} \right\rfloor}}{(c - a)_{\left\lfloor \frac{1+n}{3} \right\rfloor}} \frac{(1 + a - b - c)_{\left\lfloor \frac{2+n}{3} \right\rfloor}}{(d - a)_{\left\lfloor \frac{2+n}{3} \right\rfloor}}
\]

\[
\times \frac{(b + c - a)_{\left\lfloor \frac{n}{3} \right\rfloor}}{(b + d - a)_{\left\lfloor \frac{n}{3} \right\rfloor}} \frac{(b + d - a)_{\left\lfloor \frac{1+n}{3} \right\rfloor}}{(c + d - a)_{\left\lfloor \frac{1+n}{3} \right\rfloor}} \frac{(c + d - a)_{\left\lfloor \frac{2+n}{3} \right\rfloor}}{(2 + 2a)_{\left\lfloor \frac{2+n}{3} \right\rfloor}} \frac{(2 + 2a)_{\left\lfloor \frac{2+n}{3} \right\rfloor}}{(1 + a - c)_{\left\lfloor \frac{2+n}{3} \right\rfloor}} \frac{(1 + a - b)_{\left\lfloor \frac{2+n}{3} \right\rfloor}}{(2 + 2a)_{\left\lfloor \frac{2+n}{3} \right\rfloor}}
\]

with its parameters subject to \(1 + 2a = b + c + d + e\).
Reformulate the above equality as a binomial sum
\[
\sum_{k=0}^{n} (-1)^{k} \binom{n}{k} \frac{a + 2k}{(a + k + 1)} \left[ a, b, c, d, 1 + 2a - b - c - d \right]_{k} \\
\times [b + k, b - a - k]_{\frac{1}{2}a + 1} [c + k, c - a - k]_{\frac{1}{2}a + 1} \left[ d + k, d - a - k \right]_{\frac{1}{2}a + 1} \\
= [b, 1 + a - c - d]_{\frac{1}{2}a + 1} [c, 1 + a - b - d]_{\frac{1}{2}a + 1} [d, 1 + a - b - c]_{\frac{1}{2}a + 1} \\
\times \frac{(a)_{n}}{(b + c + d - a)_{n}} \left[ b + c - a \right]_{\frac{1}{2}a + 1} \left[ b + d - a \right]_{\frac{1}{2}a + 1} \left[ c + d - a \right]_{\frac{1}{2}a + 1} \\
\times \left[ 1 + a - b, 1 + a - c, 1 + a - d, b + c + d - a \right]_{k}.
\]}

This equality matches exactly to (9) under the assignments \( \lambda \to a \) and
\[
\phi(x; n) \to (b - a + x)_{\frac{1}{2}a + 1} (c - a + x)_{\frac{1}{2}a + 1} (d - a + x)_{\frac{1}{2}a + 1}
\]
as well as
\[
f(n) \to n! (a)_{n} \times \mathcal{F}(n),
\]
\[
g(k) \to \left[ a, b, c, d, 1 + 2a - b - c - d \right]_{k}.
\]
where
\[
\mathcal{F}(n) = \frac{b + c - a}{1 + a - d} \left[ b + d - a \right]_{\frac{1}{2}a + 1} \left[ c + d - a \right]_{\frac{1}{2}a + 1} \\
\times \frac{[b, 1 + a - c - d]_{\frac{1}{2}a + 1} [c, 1 + a - b - d]_{\frac{1}{2}a + 1} [d, 1 + a - b - c]_{\frac{1}{2}a + 1}}{n! (b + c + d - a)_{n}}.
\]

For the sake of brevity, we introduce the \( \psi \)-polynomials by
\[
\psi(x; n) = \phi(a + x; n)\phi(-x; n) = (b + x)_{\frac{1}{2}a + 1} (c + x)_{\frac{1}{2}a + 1} (d + x)_{\frac{1}{2}a + 1} \\
\times (b - a - x)_{\frac{1}{2}a + 1} (c - a - x)_{\frac{1}{2}a + 1} (d - a - x)_{\frac{1}{2}a + 1}.
\]

Then the dual relation corresponding to (7) can explicitly be stated, after some simplifications, in the following lemma.

**Lemma 1.** For the \( \mathcal{F} \)-quotient of shifted factorials and the \( \psi \)-polynomials defined respectively in (10) and (11), we have the summation formula
\[
\sum_{k=0}^{n} \mathcal{F}(k) \frac{\psi(k; k + 1)}{\psi(k; k)} \left( -n \right)_{k} \left( a + n \right)_{k}.
\]

Observe that \( \psi; n; k + 1 \) is a polynomial of degree \( 2k + 2 \) in \( n \) with the leading coefficient equal to \( (-1)^{k+1} \). Now multiply by \( -n^{2k} \) across the binomial relation in Lemma 1 and then let \( n \to \infty \). We may evaluate the limits of the left member by (2) and of the corresponding right member through Weierstrass’s \( M \)-test on uniformly convergent series (cf. Stromberg [30] §3.106). After some routine simplifications, the resultant limiting relation can be expressed explicitly as follows.

**Proposition 2.** Let \( \Gamma(a, b, c, d) \) stand for the quotient of the \( \Gamma \)-function given by
\[
\Gamma(a, b, c, d) = \frac{\Gamma(1 + a - b)\Gamma(1 + a - c)\Gamma(1 + a - d)\Gamma(b + c + d - a)}{\Gamma(b) \Gamma(c) \Gamma(d) \Gamma(1 + 2a - b - c - d)}.
\]


Then for the $\mathcal{F}$-quotient of shifted factorials and the $\psi$-polynomials defined respectively in (9) and (10), the following infinite series identity holds:

$$\Gamma(a,b,c,d) = -\sum_{k=0}^{\infty} \frac{\psi(k; k+1)}{\psi(k; k)} \mathcal{F}(k).$$

Let $\varepsilon = 0, 1, 2$ and $T(k)$ be the summand with index $k$ in the last series. Putting the initial $\varepsilon$ terms aside and then classifying the remaining terms with respect to their indices modulo 3, we get the expressions

$$\sum_{k=0}^{\infty} T(k) = \sum_{k=0}^{\varepsilon-1} T(k) + \sum_{k=0}^{\infty} \sum_{i=0}^{2} T(\varepsilon + i + 3k)$$

$$= \sum_{k=0}^{\varepsilon-1} T(k) + \sum_{k=1}^{\infty} \sum_{j=1}^{3} T(\varepsilon - j + 3k).$$

Denote further by $\sigma(\varepsilon)$, $\Delta_k(\varepsilon)$ and $\nabla_k(\varepsilon)$ the sum of initial $\varepsilon$-terms and the “weight functions” (where the latter are clearly a rational function of $k$):

$$\sigma(\varepsilon) = \sum_{k=0}^{\varepsilon-1} \left\{ \frac{\psi(k; k+1)}{-\psi(k; k)} \right\} \mathcal{F}(k), \quad (12)$$

$$\Delta_k(\varepsilon) = \sum_{i=0}^{2} \left\{ \frac{\psi(\varepsilon + i + 3k; 1 + \varepsilon + i + 3k)}{-\psi(\varepsilon + i + 3k; \varepsilon + i + 3k)} \right\} \frac{\mathcal{F}(\varepsilon + i + 3k)}{\mathcal{F}(3k)} \quad (13)$$

$$\nabla_k(\varepsilon) = \sum_{j=1}^{3} \left\{ \frac{\psi(\varepsilon - j + 3k; 1 + \varepsilon - j + 3k)}{-\psi(\varepsilon - j + 3k; \varepsilon - j + 3k)} \right\} \frac{\mathcal{F}(\varepsilon - j + 3k)}{\mathcal{F}(3k)}. \quad (14)$$

Then the identity in Proposition 2 can be restated in the theorem below.

**Theorem 3.** Assume that $\Gamma(a,b,c,d)$, $\sigma(\varepsilon)$, $\Delta_k(\varepsilon)$ and $\nabla_k(\varepsilon)$ are as in (11), (12), (13) and (14) respectively. Then for $\varepsilon = 0, 1, 2$ and the $\mathcal{F}$-quotient of shifted factorials defined in (9), the following infinite series identities hold:

$$\Gamma(a,b,c,d) = \sigma(\varepsilon) + \sum_{k=0}^{\infty} \Delta_k(\varepsilon) \mathcal{F}(3k)$$

$$= \sigma(\varepsilon) + \sum_{k=1}^{\infty} \nabla_k(\varepsilon) \mathcal{F}(3k).$$

In the above theorem, the series is expressed in two different manners because it happens frequently that a series with its summation index initiating at $k = 0$ has better looking than that at $k = 1$, or vice versa. This will be seen from our examples in the next section. In the above series, $\mathcal{F}(3k)$ results in the dominant part

$$\mathcal{F}(3k) = \left[ \frac{b + c - a, b + d - a, c + d - a}{1 + a - b, 1 + a - c, 1 + a - d} \right]_{2k} \times \left[ \frac{b, c, d, 1 + a - b - c, 1 + a - b - d, 1 + a - c - d}{(3k)! (b + c + d - a)_{3k}} \right]_{k},$$

which determines the convergence rate of the series to be “$\frac{1}{729}$”. Instead, both $\Delta_k(\varepsilon)$ and $\nabla_k(\varepsilon)$ are perturbing parts consisting of only a few terms. Therefore for specific values of $\varepsilon$ and $\{a,b,c,d\}$, in order to find the infinite series identity, it is enough to compute the corresponding $\sigma(\varepsilon)$ and $\Delta_k(\varepsilon)$ (or $\nabla_k(\varepsilon)$), and then to simplify the resultant expression.
3. Infinite Series of Ramanujan Type Involving $\pi$

By specifying the parameters $\{a, b, c, d\}$, we can derive numerous infinite series identities of convergence rate $\sim \frac{1}{729}$ from Theorem 3 with $\varepsilon = 0, 1, 2$. However in general, there are complicated weight polynomials appearing in the summands of these series. Two examples are illustrated as follows. Letting

$$\varepsilon = 0 \quad \text{and} \quad \{a, b, c, d\} = \left\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}\right\},$$

we can compute with Mathematica commands

$$F(3k) = \left[\frac{1}{2}, \frac{1}{4}, \frac{1}{2}\right]_{2k} \left[\frac{1}{1}, \frac{1}{3}, \frac{1}{2}\right]_{(3k)!} \left(\frac{1}{2}\right)^{3k} k = \left[\frac{1}{2}, \frac{1}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{5}{11}\right]_{k} \left(\frac{1}{729}\right)^{k},$$

$$\Delta_k(0) = \frac{(1 + 8k)(93184k^4 + 154432k^3 + 85840k^2 + 17484k + 855)}{768(1 + 3k)(2 + 3k)(7 + 12k)},$$

$$\sigma(0) = 0 \quad \text{and} \quad \Gamma\left(\frac{1}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) = \frac{1}{4\pi}.$$  

Substituting them into Theorem 3 and then multiplying by $\sim 4 \times 2688$ across the resultant equation, we derive the following infinite series identity.

**Example 1** ($\varepsilon = 0 : \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}$).

$$\frac{2688}{\pi} = \sum_{k=0}^{\infty} \left[\frac{1}{2}, \frac{1}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{5}{11}\right]_{k} \frac{93184k^4 + 154432k^3 + 85840k^2 + 17484k + 855}{729^k}.$$

Analogously, we have another infinite series identity of similar type.

**Example 2** ($\varepsilon = 1 : \frac{3}{2}, 1, 1, \frac{5}{6}$).

$$\frac{20\pi}{3\sqrt{3}} = 10 + \sum_{k=1}^{\infty} \left[\frac{1}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{3}, \frac{3}{4}, \frac{4}{5}, \frac{7}{9}, \frac{10}{9}\right]_{k} \frac{19656k^4 - 7749k^3 - 3150k^2 + 613k + 118}{729^k}.$$

When $b + c + d - a$ equals to a half integer, the corresponding series in Theorem 3 can be reformulated, by means of the bisection series method, as a simpler series with convergence rate $\sim \frac{1}{27}$. In order to show how this approach works, we present demonstrations in details for two infinite series identities.

We start with the following strange evaluation of a hypergeometric $\,_{3}F_{2}$-series.

**Example 3** ($\varepsilon = 0 : \frac{5}{3}, \frac{5}{3}, \frac{2}{3}, \frac{5}{6}$).

$$\,_{3}F_{2}\left[\frac{2}{3}, \frac{7}{3}, \frac{1}{1}, \frac{-1}{2}, \frac{1}{27}\right] = \sum_{k=0}^{\infty} \left[\frac{2}{3}, \frac{7}{3}, \frac{1}{1}, \frac{-1}{2}, \frac{1}{27}\right]_{k} (-1)^{k} \left(\frac{1}{27}\right)^{k} = \frac{81\sqrt{3}}{28 \cdot 2^{2/3} \pi}.$$

**Proof.** By specifying the parameters in Theorem 3

$$\varepsilon = 0 \quad \text{and} \quad \{a, b, c, d\} = \left\{\frac{5}{3}, \frac{5}{3}, \frac{2}{3}, \frac{5}{6}\right\}.$$
We claim that the above series is the bisection one of the series below. Therefore, we can evaluate the following simpler series by specializing the parameters in Theorem 3.

\[
\sigma(0) = 0 \quad \text{and} \quad \Gamma\left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right) = -\frac{15\sqrt{3}}{8}\cdot\frac{\pi}{2}\cdot\frac{1}{27}
\]

which lead us to the following identity:

\[
\frac{59049\sqrt{3}}{14 \cdot 2^{2/3}\pi} = \sum_{k=0}^{\infty} \left[ \frac{\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}}{1, 1, 2, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}} \cdot \frac{1}{27} \right] _k \frac{\Gamma(5 + 12k)}{729^k}.
\]

We claim that the above series is the bisection one of the series below for \( \Lambda_k \quad \text{for} \quad \Lambda_k := \left[ \frac{\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}}{1, 1, 2, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}} \cdot \frac{1}{27} \right] _k \).

This can be justified by computing

\[
\Lambda_{2k} + \Lambda_{2k+1} = \left[ \frac{\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}}{1, 1, 2, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}} \cdot \frac{1}{27} \right] _k \frac{5616k^3 + 11358k^2 + 7233k + 1465}{1458 \cdot 729^k}.
\]

Therefore, we can evaluate the following simpler series

\[
\sum_{k=0}^{\infty} \sum_{k=0}^{\infty} \left\{ \Lambda_{2k} + \Lambda_{2k+1} \right\} = \frac{1}{1458} \times \frac{59049\sqrt{3}}{14 \cdot 2^{2/3}\pi} = \frac{81\sqrt{3}}{28 \cdot 2^{2/3}\pi}.
\]

Next, we prove the following elegant formula for a Ramanujan-like series.

**Example 4 \( \varepsilon = 0 : \frac{4}{3}, 1, \frac{5}{6} \).**

\[
\frac{\pi\Gamma^2\left(\frac{4}{3}\right)}{6\Gamma^2\left(\frac{4}{3}\right)} = \sum_{k=0}^{\infty} \left( \frac{-1}{27} \right) ^k \left[ \frac{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}}{1, 2, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}} \right] _k \left\{ 3 + 7k \right\}.
\]

**Proof.** By specializing the parameters in Theorem 3

\[
\varepsilon = 0 \quad \text{and} \quad \{a, b, c, d\} = \{\frac{4}{3}, 1, 1, \frac{5}{6}\}
\]

we can compute

\[
\int(3k) = \left[ \frac{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}}{1, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}} = \left[ \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right] _k \left( \frac{1}{729} \right) ^k.
\]

\[
\Delta_k(0) = \frac{(1 + 4k)(4368k^4 + 9742k^3 + 7799k^2 + 2588k + 283)}{72(2 + 3k)^3},
\]

\[
\sigma(0) = 0 \quad \text{and} \quad \Gamma\left(\frac{4}{3}, \frac{4}{3}, \frac{4}{3}\right) = \frac{\pi\Gamma^2\left(\frac{4}{3}\right)}{36\Gamma^2\left(\frac{4}{3}\right)};
\]

which give rise to the following identity

\[
\frac{16\pi\Gamma^2\left(\frac{4}{3}\right)}{\Gamma^2\left(\frac{4}{3}\right)^2} = \sum_{k=0}^{\infty} \left[ \frac{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}}{1, 2, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}} \cdot \frac{1}{27} \right] _k \frac{4368k^4 + 9742k^3 + 7799k^2 + 2588k + 283}{729^k}.
\]
For the sequence defined by
\[ \Lambda_k := (3 + 7k) \left[ 1, \frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{5}{3} \right] \left( \frac{1}{27} \right)^k, \]
it is routine to compute the sum of its two consecutive terms
\[ \Lambda_{2k} + \Lambda_{2k+1} = \left[ 1, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}, \frac{5}{3}, \frac{5}{3} \right]_{k}^{4368k^4 + 9742k^3 + 7799k^2 + 2588k + 283} \cdot \frac{1}{96 \cdot 729^k}. \]
Hence the afore–displayed series is equivalent to the following simpler series
\[ \sum_{k=0}^{\infty} \Lambda_k = \sum_{k=0}^{\infty} \{ \Lambda_{2k} + \Lambda_{2k+1} \} = \frac{1}{96} \times \frac{16\pi \Gamma \left( \frac{1}{2} \right)^2}{\Gamma \left( \frac{5}{6} \right)^2} = \frac{\pi \Gamma \left( \frac{1}{2} \right)^2}{6\Gamma \left( \frac{5}{6} \right)^2}. \]
This completes the proof of the formula given in Example 4. \( \square \)

By carrying out the same procedure, we shall evaluate further 31 Ramanujan–like series in closed forms. Compared with the other existing \( \pi \)-related series with convergence rate \( \frac{1}{27} \) in the literature (see [16,17,19] for example), all the formulae recorded below are believed to be new, except for Examples 21 and 22. They are divided into four classes according to their values and exhibited as examples. In each example, the parameter setting \( \varepsilon : a, b, c, d \) and eventual references will be highlighted in the header. Furthermore, all the formulae are experimentally checked by an appropriately devised Mathematica package in order to ensure the accuracy.

§3.1. Series for \( \pi^2 \).

Example 5 (Chu and Zhang [19]: \( \varepsilon = 0 : \frac{4}{3}, 1, 1, 1 \)).
\[ \frac{\pi^2}{2} = \sum_{k=0}^{\infty} \left( \frac{-1}{27} \right)^k \left[ 1, \frac{1}{3}, \frac{5 + 7k}{1 + 2k} \right]_{k} \]

Example 6 (\( \varepsilon = 1 : \frac{3}{2}, 1, 1, 1 \)).
\[ 9\pi^2 = 89 \sum_{k=1}^{\infty} \left( \frac{-1}{27} \right)^k \left[ \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{5}{2}, \frac{7}{2}, \frac{7}{2} \right]_{k} \{ 17 + 14k \}. \]

Example 7 (\( \varepsilon = 2 : \frac{5}{2}, 1, 2, 1 \)).
\[ \frac{1575\pi^2}{8} = 1960 - \sum_{k=0}^{\infty} \left( \frac{-1}{27} \right)^k \left[ \frac{5}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{5}{2}, \frac{5}{2}, \frac{5}{2}, \frac{5}{2} \right]_{k} \{ 17 + 7k \}. \]

Example 8 (\( \varepsilon = 2 : \frac{3}{2}, 1, 1, 1 \)).
\[ 675\pi^2 = 6600 + \sum_{k=0}^{\infty} \left( \frac{-1}{27} \right)^k \left[ \frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2} \right]_{k} \{ 63 + 59k + 14k^2 \}. \]

§3.2. Series for \( \pi^2/\Gamma^3 \).

Example 9 (\( \varepsilon = 0 : \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \)).
\[ \frac{2\pi^2}{\Gamma^3 \left( \frac{1}{3} \right)} = \sum_{k=0}^{\infty} \left( \frac{-1}{27} \right)^k \left[ \frac{2}{3}, \frac{2}{3}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6} \right]_{k} \{ 8 + 21k \}. \]
Example 10 \( \{ \varepsilon = 1 : 1/3, 2/3, -1/3, 2/3 \} \).

\[
\frac{2\pi^2}{\Gamma^3\left(\frac{1}{3}\right)} = \sum_{k=0}^{\infty} \left(\frac{-1}{27}\right)^k \left[ \frac{1}{3}, \frac{4}{3}, -\frac{1}{3} \right]_k \{ 1 + 21k \}.
\]

Example 11 \( \{ \varepsilon = 0 : 1/3, 2/3, 2/3, 5/3 \} \).

\[
\frac{45\pi^2}{\Gamma^3\left(\frac{1}{3}\right)} = \sum_{k=0}^{\infty} \left(\frac{-1}{27}\right)^k \left[ \frac{1}{3}, \frac{7}{3}, -\frac{1}{3} \right]_k \{ 23 + 42k \}.
\]

Example 12 \( \{ \varepsilon = 1 : 1/3, 1/3, 1/3, 1/3 \} \).

\[
\frac{5\pi^2}{3\Gamma^3\left(\frac{1}{3}\right)} = \sum_{k=0}^{\infty} \left(\frac{-1}{27}\right)^k \left[ \frac{1}{3}, \frac{1}{3}, -\frac{5}{6} \right]_k \{ 5 - 42k \}.
\]

Example 13 \( \{ \varepsilon = 0 : 1, -1, 2/3, 5/3 \} \).

\[
\frac{55\pi^2}{21\Gamma^3\left(\frac{1}{3}\right)} = \sum_{k=0}^{\infty} \left(\frac{-1}{27}\right)^k \left[ \frac{10}{3}, \frac{2}{3}, -\frac{7}{6} \right]_k \{ 16 + 21k \}.
\]

Example 14 \( \{ \varepsilon = 2 : 1, 1, 1, 1 \} \).

\[
\frac{91\pi^2}{16\Gamma^3\left(\frac{2}{3}\right)} = \sum_{k=0}^{\infty} \left(\frac{-1}{27}\right)^k \left[ \frac{11}{2}, -\frac{1}{2}, -\frac{3}{6} \right]_k \{ 23 + 21k \}.
\]

Example 15 \( \{ \varepsilon = 1 : 1, 1, 1, 1 \} \).

\[
\frac{175\pi^2}{36\Gamma^3\left(\frac{2}{3}\right)} = \sum_{k=0}^{\infty} \left(\frac{-1}{27}\right)^k \left[ \frac{5}{3}, \frac{5}{3}, -\frac{5}{6}, -\frac{7}{6} \right]_k \{ 25 + 42k \}.
\]

Example 16 \( \{ \varepsilon = 0 : 1, 1, 1, 1 \} \).

\[
\frac{825\pi^2}{8\Gamma^3\left(\frac{2}{3}\right)} = \sum_{k=0}^{\infty} \left(\frac{-1}{27}\right)^k \left[ \frac{7}{3}, \frac{7}{3}, -\frac{1}{6}, -\frac{1}{6} \right]_k \{ 53 + 42k \}.
\]

Example 17 \( \{ \varepsilon = 1 : 1/3, 1/3, 1/3, 1/3 \} \).

\[
\frac{60\pi^2}{\Gamma^3\left(\frac{1}{3}\right)} = \sum_{k=0}^{\infty} \left(\frac{-1}{27}\right)^k \left[ \frac{4}{3}, -\frac{2}{3}, -\frac{1}{6} \right]_k \{ 32 + 111k + 126k^2 \}.
\]

Example 18 \( \{ \varepsilon = 0 : 1/3, 1/3, 1/3, 1/3 \} \).

\[
\frac{21\pi^2}{\Gamma^3\left(\frac{1}{3}\right)} = \sum_{k=0}^{\infty} \left(\frac{-1}{27}\right)^k \left[ \frac{5}{3}, -\frac{1}{3}, \frac{1}{6} \right]_k \{ 83 + 195k + 126k^2 \}.
\]

Example 19 \( \{ \varepsilon = 1 : 1/3, -1/3, 5/3, 2/3 \} \).

\[
\frac{715\pi^2}{12\Gamma^3\left(\frac{1}{3}\right)} = \sum_{k=0}^{\infty} \left(\frac{-1}{27}\right)^k \left[ \frac{13}{3}, -\frac{5}{3}, -\frac{13}{6} \right]_k \{ 13 + 51k - 126k^2 \}.
\]
Example 20 \( (\varepsilon = 1 : \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}) \).

\[
\frac{35\pi^2}{12 \Gamma^3(\frac{1}{4})} = \sum_{k=0}^{\infty} \left( \frac{-1}{27} \right)^k \begin{bmatrix} 1, & \frac{7}{3}, & -\frac{7}{6} \\ \frac{1}{3}, & \frac{2}{3}, & \frac{1}{6} \end{bmatrix}_k \frac{7 - 75k - 126k^2}{(1 - 6k)(5 + 6k)}
\]

§3.3. Series for \( \pi^{-1} \).

Example 21 (Chu [17]): \( (\varepsilon = 0 : \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}) \).

\[
\frac{9\sqrt{3}}{2^2 \pi} = \sum_{k=0}^{\infty} \left( \frac{-1}{27} \right)^k \begin{bmatrix} \frac{1}{3}, & \frac{4}{3}, & \frac{1}{6} \\ 1, & 1, & 1 \end{bmatrix}_k \{ 2 + 21k \}.
\]

Example 22 (Chu [17]): \( (\varepsilon = 0 : \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}) \).

\[
\frac{27\sqrt{3}}{2^3 \pi} = \sum_{k=0}^{\infty} \left( \frac{-1}{27} \right)^k \begin{bmatrix} \frac{1}{3}, & \frac{4}{3}, & \frac{2}{3} \\ 1, & 1, & 1 \end{bmatrix}_k \{ 5 + 42k \}.
\]

Example 23 \( (\varepsilon = 1 : \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}) \).

\[
\frac{72\sqrt{3}}{20^2 \pi} = \sum_{k=0}^{\infty} \left( \frac{-1}{27} \right)^k \begin{bmatrix} 1, & \frac{8}{3}, & \frac{1}{6} \\ \frac{1}{3}, & 1, & 1 \end{bmatrix}_k \{ 16 + 21k \}.
\]

Example 24 \( (\varepsilon = 2 : \frac{5}{3}, -\frac{1}{3}, \frac{5}{3}, \frac{5}{3}) \).

\[
\frac{2673\sqrt{3}}{14^3 \pi} = \sum_{k=0}^{\infty} \left( \frac{-1}{27} \right)^k \begin{bmatrix} 2, & \frac{13}{3}, & \frac{5}{3}, & -\frac{7}{6} \\ 1, & 1, & 3, & \frac{17}{6} \end{bmatrix}_k \{ 65 + 42k \}.
\]

Example 25 \( (\varepsilon = 1 : \frac{2}{3}, \frac{2}{3}, -\frac{1}{6}, -\frac{2}{3}) \).

\[
\frac{3\sqrt{3}}{\pi \sqrt{4}} = \sum_{k=0}^{\infty} \left( \frac{-1}{27} \right)^k \begin{bmatrix} 1, & \frac{1}{3}, -\frac{1}{3}, -\frac{1}{6} \\ 1, & 1, & 1 \end{bmatrix}_k \{ 1 - 63k^2 \}.
\]

Example 26 \( (\varepsilon = 0 : \frac{4}{3}, \frac{1}{3}, \frac{4}{3}, \frac{1}{3}) \).

\[
\frac{81\sqrt{3}}{2^4 \pi} = \sum_{k=0}^{\infty} \left( \frac{-1}{27} \right)^k \begin{bmatrix} \frac{1}{3}, & -\frac{1}{3}, & \frac{7}{6} \\ 1, & 1, & 2 \end{bmatrix}_k \{ 35 + 90k + 63k^2 \}.
\]

Example 27 \( (\varepsilon = 1 : \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{5}{3}) \).

\[
\frac{2187\sqrt{3}}{10^3 \pi} = \sum_{k=0}^{\infty} \left( \frac{-1}{27} \right)^k \begin{bmatrix} \frac{7}{3}, & -\frac{3}{4}, & \frac{11}{6} \\ 1, & 1, & 2 \end{bmatrix}_k \{ 77 + 144k + 63k^2 \}.
\]

Example 28 \( (\varepsilon = 1 : \frac{1}{3}, \frac{4}{3}, \frac{1}{3}, \frac{1}{3}) \).

\[
\frac{2187\sqrt{3}}{14^2 \pi} = \sum_{k=0}^{\infty} \left( \frac{-1}{27} \right)^k \begin{bmatrix} \frac{4}{3}, & -\frac{1}{3}, & \frac{13}{6} \\ 1, & 1, & 2 \end{bmatrix}_k \{ 65 + 195k + 126k^2 \}.
\]

Example 29 \( (\varepsilon = 1 : \frac{4}{3}, \frac{1}{3}, \frac{1}{3}, \frac{6}{3}) \).

\[
\frac{6561\sqrt{3}}{40^2 \pi} = \sum_{k=0}^{\infty} \left( \frac{-1}{27} \right)^k \begin{bmatrix} \frac{4}{3}, & 11, & \frac{1}{6} \\ 1, & 1, & 3 \end{bmatrix}_k \frac{143 + 285k + 126k^2}{(1 - 3k)(2 + 3k)}.
\]
Example 30 \( \varepsilon = 1 : \frac{1}{3}, \frac{1}{4}, \frac{1}{7}, \frac{1}{7} \).
\[
\frac{2187\sqrt{3}}{440\sqrt{2}\pi} = \sum_{k=0}^{\infty} \left( \frac{1}{27} \right)^k \left[ \frac{1, \frac{14}{3}, -\frac{5}{6}}{1, 2, 3} \right]_k 196 + 333k + 126k^2
\]
\{(7 + 6k)(13 + 6k)\}.

§3.4. Series for \( \pi \).

Example 31 \( \varepsilon = 1 : \frac{1}{3}, 1, 1, \frac{1}{6} \).
\[
\frac{9\pi\Gamma^2(\frac{2}{3})}{\Gamma^2(\frac{1}{6})} = 1 + \sum_{k=1}^{\infty} \left( \frac{1}{27} \right)^k \left[ \frac{3, \frac{1}{2}, -\frac{2}{3}}{\frac{2}{3}, \frac{5}{3}, \frac{5}{3}} \right]_k \{13 + 21k\}.
\]

Example 32 \( \varepsilon = 1 : \frac{1}{3}, 1, 1, \frac{5}{6} \).
\[
\frac{4\pi\Gamma^2(\frac{1}{3})}{\Gamma^2(\frac{5}{6})} = 71 + \sum_{k=1}^{\infty} \left( \frac{1}{27} \right)^k \left[ \frac{3, \frac{1}{2}, \frac{2}{3}}{\frac{4}{3}, \frac{7}{3}, \frac{7}{3}} \right]_k \{23 + 21k\}.
\]

Example 33 \( \varepsilon = 0 : \frac{2}{3}, 1, 1, \frac{1}{6} \).
\[
\frac{3\pi\Gamma^2(\frac{2}{3})}{\Gamma^2(\frac{1}{6})} = \sum_{k=1}^{\infty} \left( \frac{1}{27} \right)^k \left[ \frac{1, \frac{1}{4}, -\frac{2}{3}}{\frac{4}{3}, \frac{5}{3}, \frac{7}{3}} \right]_k \{13 + 21k\}.
\]

Example 34 \( \varepsilon = 0 : \frac{4}{5}, 1, 1, -\frac{1}{5} \).
\[
\frac{\pi\Gamma^2(\frac{4}{5})}{12\Gamma^2(\frac{3}{5})} = \sum_{k=0}^{\infty} \left( \frac{1}{27} \right)^k \left[ \frac{1, \frac{1}{5}, -\frac{4}{3}}{\frac{4}{3}, \frac{5}{3}, \frac{7}{3}} \right]_k \{21k^2 + 8k - 3\}.
\]

Example 35 \( \varepsilon = 0 : \frac{5}{6}, 1, 1, \frac{5}{6} \).
\[
\frac{48\pi\Gamma^2(\frac{2}{3})}{\Gamma^2(\frac{1}{6})} = \sum_{k=0}^{\infty} \left( \frac{1}{27} \right)^k \left[ \frac{1, \frac{1}{6}, \frac{1}{6}}{\frac{5}{3}, \frac{5}{3}, \frac{5}{3}} \right]_k \{9 + 28k + 21k^2\}.
\]

Concluding Comments. There exist different ways to invert Dougall’s \( \tau F_6 \)-sum through \( \mathbf{10} \) and \( \mathbf{17} \). However, all the dual series that we detected by \textit{Mathematica} are ugly because of the presence of very complicated weight polynomials. Here is a couple of discouraging examples.

By examining another triplicate form of Dougall’s \( \tau F_6 \)-sum
\[
\Omega_n(a; b + \left[ \frac{1+n}{3} \right], c, d + \left[ \frac{1+2n}{3} \right]) = \left[ \frac{1+a-c-d, b+c-a}{1+a-d, b-a} \right]^{\frac{1+n}{3}}
\]
\[
\times \left[ \frac{1+a, b+d-a}{1+a-c, b+c+d-a} \right]_n \left[ \frac{1+a-b-c+d-a}{1+a-b, d-a} \right]^{\frac{1+n}{3}},
\]
we can arrive, under the parameter setting \( \varepsilon = 1 \) and \( \{a, b, c, d\} = \left\{ \frac{5}{6}, 2, 1, \frac{2}{3} \right\} \) and after a long and tedious computations, at the following series for \( \pi \):
\[
\frac{75\pi}{8} = 30 + \sum_{k=1}^{\infty} \left( \frac{16}{729} \right)^k \left[ \frac{1, -\frac{2}{3}, \frac{1}{3}, -\frac{1}{3}, \frac{1}{3}, -\frac{5}{6}, -\frac{5}{6}, -\frac{5}{6}}{\frac{1}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{7}{12}, \frac{12}{12}, \frac{13}{12}, \frac{11}{12}} \right]_k \times \{60 - 101k + 1075k^2 - 4840k^3 - 49360k^4 + 136896k^5\}.
\]
Analogously, by specifying parameters $\varepsilon = 0$ and $\{a, b, c, d\} = \{1/2, 1/2, 1/3\}$, we get another series for $\pi^{-1}$:

\[
\frac{1485\sqrt{3}}{\pi} = \sum_{k=0}^{\infty} \left( \frac{16}{729} \right)^k \left[ \frac{1/2, 1/3, 1/4, 1/5}{1, 1, 1, 1} \right]_{2k} \times \left\{ 812 + 20373k + 169774k^2 + 634857k^3 + 1091016k^4 + 693036k^5 \right\}.
\]

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