Fourier transforms of Lorentz invariant functions

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Abstract

Fourier transforms of Lorentz invariant functions in Minkowski space, with support on both the timelike and the spacelike domains are performed by means of direct integration. The cases of $1 + 1$ and $1 + 2$ dimensions are worked out in detail, and the results for $1 + n$ dimensions are given.

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I Introduction

The main goal of this paper is to perform spherical averages over the hyperbolic coordinates in Minkowski spacetime by means of direct integration. We work with Lorentz invariant (or “radial”) functions, defined on $\mathbb{R}^{1,n}$, which depend only on the distance to the light-cone $s^2 = (x^0)^2 - (\vec{x})^2$. ($x^0$ is the time coordinate and $\vec{x}$ stands for the spatial coordinates). In particular, we will compute their Fourier transforms:

$$F(k) := \int_{\mathbb{R}^{1,n}} dx^{n+1} f(\eta_{\mu\nu} x^\mu x^\nu) \exp(-2\pi ik_{\mu}x^\mu),$$

where the metric $\eta_{\mu\nu}$ is of the form

$$\eta_{\mu\nu} = \text{diag}(1, -1, \ldots, -1)$$

(1)

(We set the $2\pi$ in the exponential to avoid factors of $2\pi$ appearing in front of the integral. Here it is just for convenience but this strategy becomes crucial when integrating over infinite-dimensional spaces.)

This type of integral is ubiquitous in Quantum Field Theory (QFT), since the fields can be decomposed in terms of their Fourier components, and the correlation functions depend only on the radial distance $s^2$. Although QFT is defined in Minkowski space, a common procedure is to perform a Wick rotation at an early stage (this procedure can be found in standard textbooks, e.g., Ref. 2) in which all integrals over spacetime are reduced to Euclidean integrals.

There are several obvious disadvantages to relying on the Wick rotation procedure, in which the time coordinate $t$ is rotated to imaginary values $t \rightarrow -it$, and Minkowski spacetime is transformed into Euclidean space. First, the special structure of Minkowski spacetime in which there is a preferred direction (i.e., time direction) is lost. Second, Wick rotation might not be a valid procedure in all circumstances, e.g., in cases where the arc at infinity does not vanish.

The motivation for this work is to provide tools for performing calculations in QFT while working directly in Minkowski space.

In analogy with Euclidean space, we introduce a (pseudo-) spherical coordinate atlas, i.e., one of the angles is hyperbolic. Integrating over all the angular variables leaves us with a one-dimensional radial integral. The advantage of this procedure is that any possible light-cone singularities of the integrand are mapped to point singularities in the radial variable and therefore easier to deal with.

In this paper we will not specify the function space for which the integrals are convergent. We will assume that only such functions have been chosen.

Below, we will discuss the cases $n = 1$ and $n = 2$ in detail. For the case of general $n$ we only quote the results.

After preparing an earlier version of this paper we found an article by Codelupi\textsuperscript{3} which seems to be largely unknown. Codelupi studied Fourier transforms of Lorentz invariant functions $f(s)$, using related methods, and found the same results. To find the case of arbitrary $n$ he proved a recursion relation, that relates Fourier transform of $n + 2$ spatial dimensions to the Fourier transform of $n$ spatial dimensions. In Section IV we adapt Codelupi’s elegant method to our case.

II Case $\mathbb{R}^{1,1}$

II.1 Pseudospherical coordinate atlas for $\mathbb{R}^{1,1}$

A characteristic feature of manifolds with an indefinite metric is that a global spherical coordinate system does not exist.\textsuperscript{4} To cover $\mathbb{R}^{1,1}$ we use four patches and four different parametrizations. The parametrizations coincide on the boundary between the domains, i.e., on the light-cone. (See Fig. 1).

Denote by $t, x$ the global Cartesian coordinate system with distance $s^2 = t^2 - x^2$. We parametrize the different patches as follows:
Figure 1: Standard division of $\mathbb{R}^{1,1}$. I is the forward light-cone, III the backward light-cone, and II and IV the spacelike domains.

- **Patch I:**
  \[ t = s \cosh \psi, \quad x = s \sinh \psi, \]
  where $s \in [0, +\infty]$ and $\psi \in ]-\infty, +\infty[.$
  Volume element: $dt \wedge dx = s \, ds \wedge d\psi$.
  Line element: $s^2 = t^2 - x^2$.

- **Patch II:**
  \[ t = is \sinh \psi, \quad x = is \cosh \psi, \]
  where $s \in ]i\infty, i0]$ and $\psi \in ]-\infty, +\infty[.$
  Volume element: $dt \wedge dx = s \, ds \wedge d\psi$.
  Line element: $s^2 = t^2 - x^2$.

- **Patch III:**
  \[ t = s \cosh \psi, \quad x = s \sinh \psi, \]
  where $s \in [0, -\infty]$ and $\psi \in ]-\infty, +\infty[.$
  Volume element: $dt \wedge dx = s \, ds \wedge d\psi$.
  Line element: $s^2 = t^2 - x^2$.

- **Patch IV:**
  \[ t = is \sinh \psi, \quad x = is \cosh \psi, \]
  where $s \in ]-i\infty, i0]$ and $\psi \in ]-\infty, +\infty[.$
  Volume element: $dt \wedge dx = s \, ds \wedge d\psi$.
  Line element: $s^2 = t^2 - x^2$.

The limits of integration in each patch are chosen to yield a positive result when the volume element is integrated over a small, finite volume.

### II.2 Fourier transform of radial functions

We integrate the function $f(s^2)$ separately in the timelike and spacelike domain for the cases of timelike and spacelike momenta. To simplify the calculation, we set $k_x = 0$ when the momentum is timelike, and $k_t = 0$ when the momentum is spacelike. This can always be achieved with a Lorentz transformation and is not a restriction on the results.

- **Patch I+III:**
(i) Timelike momentum: $k_x = 0$,

\[ I_{III}(k_t) = \int_{I+II} dt \, dx \, f(t^2 - x^2) \exp(-2\pi i k_t t) \]

\[ = \int_0^\infty ds \, s \, f(s^2) \int_{-\infty}^{+\infty} d\psi \, \exp(-2\pi i k_t s \cosh \psi) \]

\[ + \int_0^{-\infty} ds \, s \, f(s^2) \int_{-\infty}^{+\infty} d\psi \, \exp(-2\pi i k_t s \cosh \psi) \]

\[ = 2 \int_0^\infty ds_0 \, s_0 \, f(s_0^2) \int_{-\infty}^{+\infty} d\psi \, \cos(2\pi k_t s_0 \cosh \psi) \]

\[ = -2\pi \int_0^\infty ds_0 \, s_0 \, f(s_0^2) \, N_0 (2\pi k_t s_0) , \quad (3) \]

where $N_0$ is a Bessel function of zeroth order and

\[ s_0 = \sqrt{t^2 - x^2} . \quad (4) \]

In the last line we have used formula 3.868(2) from Ref. 5, after a change of variable to $x = e^\psi$.

(ii) Spacelike momentum: $k_t = 0$,

\[ I_{III}(k_x) = \int_{I+II} dt \, dx \, f(t^2 - x^2) \exp(-2\pi i k_x x) \]

\[ = \int_0^\infty ds \, s \, f(s^2) \int_{-\infty}^{+\infty} d\psi \, \exp(-2\pi i k_x s \sinh \psi) \]

\[ + \int_0^{-\infty} ds \, s \, f(s^2) \int_{-\infty}^{+\infty} d\psi \, \exp(-2\pi i k_x s \sinh \psi) \]

\[ = \int_0^\infty ds_0 \, s_0 \, f(s_0^2) \int_{-\infty}^{+\infty} d\psi \, \cos(2\pi k_x s_0 \sinh \psi) \]

\[ = 4 \int_0^\infty ds_0 \, s_0 \, f(s_0^2) \, K_0 (2\pi k_x s_0) , \quad (5) \]

where $K_0$ is a Bessel function of zeroth order and $s_0$ as before. In the last line we have used formula 3.868(4) from Ref. 5, after a change of variable to $x = e^\psi$.

• Patch II+IV:

(i) Timelike momentum: $k_x = 0$,

\[ I_{IV}(k_t) = \int_{I+IV} dt \, dx \, f(t^2 - x^2) \exp(-2\pi i k_t t) \]

\[ = \int_{-\infty}^{+\infty} ds \, s \, f(s^2) \int_{-\infty}^{+\infty} d\psi \, \exp(2\pi k_t s \sinh \psi) \]

\[ + \int_{-\infty}^{+\infty} ds \, s \, f(s^2) \int_{-\infty}^{+\infty} d\psi \, \exp(2\pi k_t s \sinh \psi) \]

\[ = \int_0^\infty ds_1 \, s_1 \, f(s_1^2) \int_{-\infty}^{+\infty} d\psi \, \cos(2\pi k_t s_1 \sinh \psi) \]

\[ = 4 \int_0^\infty ds_1 \, s_1 \, f(s_1^2) \, K_0 (2\pi k_t s_1) , \quad (6) \]
where $K_0$ is a Bessel function of zeroth order and

$$s_1 = \sqrt{x^2 - t^2}.$$  \hspace{9cm} (7)

In the last line we have used formula 3.868(2) from Ref. 5, after a change of variable $x = e^\psi$.

(ii) Spacelike momentum: $k_t = 0$,

$$I_{II+IV}(k_x) = \int_{II+IV} dt dx f(t^2 - x^2) \exp(-2\pi i k_x x)$$

$$= \int_{-\infty}^{\infty} ds s f(s^2) \int_{-\infty}^{\infty} d\psi \exp(2\pi k_x s \cosh \psi)$$

$$+ \int_{-\infty}^{\infty} ds s f(s^2) \int_{-\infty}^{\infty} d\psi \exp(2\pi k_x s \cosh \psi)$$

$$= 2 \int_{0}^{\infty} ds_1 s_1 f(s_1^2) \int_{-\infty}^{\infty} d\psi \cos(2\pi k_x s_1 \cosh \psi)$$

$$= -2\pi \int_{0}^{\infty} ds_1 s_1 f(s_1^2) N_0(2\pi k_x s_1), \hspace{9cm} (8)$$

where $N_0$ is a Bessel function of zeroth order and $s_1$ as before. In the last line we have used formula 3.868(2) from Ref. 5, after a change of variable to $x = e^\psi$.

In summary

$$I(k_t) = -2\pi \int_{0}^{\infty} ds_0 s_0 f(s_0^2) N_0(2\pi k_t s_0) + 4\int_{0}^{\infty} ds_1 s_1 f(s_1^2) K_0(2\pi k_t s_1)$$

$$I(k_x) = 4\int_{0}^{\infty} ds_0 s_0 f(s_0^2) K_0(2\pi k_x s_0) - 2\pi \int_{0}^{\infty} ds_1 s_1 f(s_1^2) N_0(2\pi k_x s_1)$$

**II.3 Example: Fourier transform of a Gaussian**

In this section we apply the results from above to a specific test function and show that the correct answer is obtained. Whenever necessary, we define the integral $\int_0^\infty dx f(x)$ as $\lim_{\epsilon \to 0^+} \int_0^\infty dx e^{-\epsilon x^2} f(x)$. With this proviso, we can directly compute the Fourier transform of a “Gaussian” (where the quotation marks remind us that the exponential is imaginary):

$$\int dt dx e^{i(t^2-x^2)} e^{-2\pi ik_t t} = \pi e^{-i\pi k_t^2}.$$

We can now check that our Fourier transform integrals give the same result,

$$I_1(k_t) = 4\int_0^\infty dr e^{-(r+i)^2} K_\nu(2\pi r k_t)$$

$$= \frac{1}{\pi k_t} \frac{1}{e^\epsilon + 1} \Gamma(1 + \frac{\nu}{2}) \Gamma(1 - \frac{\nu}{2}) e^{\frac{\pi^2 k_t^2}{4}} W_{1,\nu} \frac{\pi^2 k_t^2}{2(\epsilon + i)},$$

$$I_2(k_t) = -2\pi \int_0^\infty dr e^{-(r-i)^2} N_\nu(2\pi r k_t)$$

$$= -\frac{1}{k_t} \frac{1}{\sqrt{\epsilon - i}} \frac{1}{\sin(\frac{\pi}{2} r)} e^{\frac{-\pi^2 k_t^2}{2(\epsilon - i)}} \left[ W_{\frac{i}{2},\nu} \frac{\pi^2 k_t^2}{2(\epsilon - i)} - \cos(\frac{\pi}{2} \nu) \frac{\Gamma(1 + \frac{\nu}{2})}{\Gamma(1 + \nu)} M_{\frac{i}{2},\nu} \frac{\pi^2 k_t^2}{2(\epsilon - i)} \right].$$

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where we have used 6.631(2,3) from Ref. 5. Each integral has a pole as \( \nu \to 0 \); the \( W \) function with negative first argument has a simple pole in \( \nu \), and there is an inverse sine in \( \nu \) in the second integral. However, as we will see, this singularity exactly cancels between the two integrals. Thus we may, in fact, take \( \nu \to 0 \) in the sum. First we convert the \( M \) function to \( W \) functions,

\[
M_{1, \nu}^{1, \nu}(z) = i \frac{\Gamma(1 + \nu)}{\Gamma(1 + \zeta)} W_{1, \nu}^{1, \zeta}(z) + i \frac{\Gamma(1 + \nu)}{\Gamma(1 + \zeta)} e^{-i\pi\nu/2} W_{1, \nu}^{1, \zeta}(z)
\]

using formula 9.233(1) in Ref. 5. Then we make use of the following identities (9.234(1,2) and 9.235 in Ref. 5):

\[
W_{1, \nu}^{1, \nu}(z) = \sqrt{\frac{\pi}{2}} W_{0, 1+\nu}(z) + \sqrt{\frac{\pi}{2}} W_{0, 1-\nu}(z),
\]

\[
W_{-1, \nu}^{1, \nu}(z) = \frac{2\sqrt{\pi}}{\nu} W_{0, 1+\nu}(z),
\]

\[
W_{0, 1}^{1, \nu}(z) = e^{-z/2}.
\]

Expanding the \( \Gamma \) functions and \( 1/\sin z = 1/z + z/6 + O(z^2) \), we can explicitly verify that the singularity cancels:

\[
I_1 + I_2 = -\frac{2i}{\nu} + \frac{2i}{\nu} + O(1) = 0 + O(1)
\]

We have verified numerically that the constant term is, in fact,

\[
I_1 + I_2 = \pi e^{-i\pi^2 k^2}.
\]

III Case \( \mathbb{R}^{1,2} \)

III.1 Pseudospherical coordinate atlas for \( \mathbb{R}^{1,2} \)

Global coordinate system: \( t, x, y \), distance \( s^2 = t^2 - x^2 - y^2 \).

- **Patch I:**
  \[
t = s \cosh \psi, \quad x = s \sinh \psi \cos \theta, \quad y = s \sinh \psi \sin \theta,
\]
  where \( s \in [0, +\infty[ \), \( \psi \in ]-\infty, +\infty[ \) and \( \theta \in [-\pi/2, \pi/2] \).

To avoid problems when \( \sinh \psi \) switches sign at 0, the integral over \( \psi \) needs to be split up into two integrals: \( \psi \in [0, +\infty] \) and \( \psi \in [0, -\infty] \).

Volume element: \( dt \wedge dx \wedge dy = s^2 \sinh \psi ds \wedge d\psi \wedge d\theta \).

Line element: \( s^2 = t^2 - x^2 - y^2 \).

- **Patch II:**
  \[
t = i s \sinh \psi, \quad x = i s \cosh \psi \cos \theta, \quad y = i s \cosh \psi \sin \theta,
\]
  where \( s \in [0, i\infty[ \), \( \psi \in ]-\infty, +\infty[ \) and \( \theta \in [-\pi/2, \pi/2] \).

Volume element: \( dt \wedge dx \wedge dy = is^2 \cosh \psi ds \wedge d\psi \wedge d\theta \).

Line element: \( s^2 = t^2 - x^2 - y^2 \).

- **Patch III:**
  \[
t = s \cosh \psi, \quad x = s \sinh \psi \cos \theta, \quad y = s \sinh \psi \sin \theta,
\]
  where \( s \in [0, -\infty[ \), \( \psi \in ]-\infty, +\infty[ \) and \( \theta \in [-\pi/2, \pi/2] \).

To avoid problems when \( \sinh \psi \) switches sign at 0, the integral over \( \psi \) needs to be split up into two integrals: \( \psi \in ]-\infty, 0[ \) and \( \psi \in ]-\infty, 0[ \).

Volume element: \( dt \wedge dx \wedge dy = s^2 \sinh \psi ds \wedge d\psi \wedge d\theta \).

Line element: \( s^2 = t^2 - x^2 - y^2 \).
• Patch IV:

\[ t = is \sinh \psi, \quad x = is \cosh \psi \cos \theta, \quad y = is \cosh \psi \sin \theta, \]

where \( s \in [-i \infty, i0], \psi \in (-\infty, +\infty] \) and \( \theta \in [-\pi/2, \pi/2] \).

Volume element: \( dt \wedge dx \wedge dy = is^2 \cosh \psi \, ds \wedge d\psi \wedge d\theta \).

Line element: \( s^2 = t^2 - x^2 - y^2 \).

The limits of integration in each patch are chosen to yield a positive result when the volume element is integrated over a small, finite volume.

### III.2 Fourier transform of radial functions

As before, we integrate the function \( f(s^2) \) separately in the timelike and spacelike domain for the cases of timelike and spacelike momenta. To simplify the calculation, we set \( k_x = k_y = 0 \) when the momentum is timelike, and \( k_t = k_y = 0 \) when the momentum is spacelike. This can always be achieved with a Lorentz transformation and is not a restriction on the results.

• Patch I+III:

(i) Timelike momentum: \( k_x = k_y = 0 \),

\[
I_{I+III}(k_t) = \int_{I+III} dt \, dx \, dy \, f(t^2 - x^2 - y^2) \exp(-2\pi ik_t t)
\]

\[
= \pi \int_0^\infty ds \, s^2 \, f(s^2) \left[ \int_0^\infty d\psi \, \sinh \psi \exp(-2\pi ik_t s \cosh \psi) \right.
\]

\[
+ \int_0^{-\infty} d\psi \, \sinh \psi \exp(-2\pi ik_t s \cosh \psi) \left. \right]
\]

\[
+ \pi \int_0^\infty ds \, s^2 \, f(s^2) \left[ \int_0^\infty d\psi \, \sinh \psi \exp(-2\pi ik_t s \cosh \psi) \right.
\]

\[
+ \int_0^{-\infty} d\psi \, \sinh \psi \exp(-2\pi ik_t s \cosh \psi) \left. \right]
\]

\[
= \int_0^\infty ds \, s^2 \, f(s^2) \frac{\exp(-2\pi ik_t s)}{ik_t s} - \int_{-\infty}^0 ds \, s^2 \, f(s^2) \frac{\exp(-2\pi ik_t s)}{ik_t s}
\]

\[
= -\frac{2}{k_t} \int_0^\infty ds_0 \, s_0 \, f(s_0^2) \sin(2\pi k_t s_0),
\]

where

\[
s_0 = \sqrt{t^2 - x^2 - y^2}.
\]

The angular integrals have been computed as follows:

\[
\int_0^\infty d\psi \, \sinh \psi \exp(-ia \cosh \psi) + \int_0^{-\infty} d\psi \, \sinh \psi \exp(-ia \cosh \psi)
\]

\[
= \lim_{\epsilon \to 0} 2 \int_{-\infty - i\epsilon}^{+\infty - i\epsilon} d\psi \, \sinh \psi \exp(-ia \cosh \psi)
\]

\[
= \lim_{\epsilon \to 0} \frac{2}{ia} \left[ -\exp(-ia \cosh(\infty - i\epsilon)) + \exp(-ia \cosh(0 - i\epsilon)) \right]
\]

\[
= \frac{2}{ia} \exp(-ia),
\]

where \( a = 2\pi k_t s \), and in the last line \( \cosh(\psi \pm i\epsilon) = \cosh \psi \pm i\epsilon \sinh \psi \) has been used.
(ii) Spacelike momentum: \( k_t = k_y = 0, \)

\[
I_{I+III}(k_x) = \int_{I+III} dt \, dx \, dy \, f(t^2 - x^2 - y^2) \exp(-2\pi ik_x x)
\]

\[
= \int_0^\infty ds \, s^2 \, f(s^2) \int_{-\pi/2}^{\pi/2} d\theta \left[ \int_0^\infty d\psi \, \sinh \psi \exp(-2\pi ik_x s \sinh \psi \cos \theta) \right.
\]

\[
+ \int_0^\infty d\psi \, \sinh \psi \exp(-2\pi ik_x s \sinh \psi \cos \theta) \right]
\]

\[
+ \int_0^\infty ds \, s^2 \, f(s^2) \int_{-\pi/2}^{\pi/2} d\theta \left[ \int_0^\infty d\psi \, \sinh \psi \exp(-2\pi ik_x s \sinh \psi \cos \theta) \right.
\]

\[
+ \int_0^\infty d\psi \, \sinh \psi \exp(-2\pi ik_x s \sinh \psi \cos \theta) \right]
\]

\[
= \int_0^\infty ds \, s^2 \, f(s^2) \frac{\exp(-2\pi k_x s)}{k_x s} + \int_{-\infty}^0 ds \, s^2 \, f(s^2) \frac{\exp(-2\pi k_x s)}{ik_x s}
\]

\[
= \frac{2}{k_x} \int_0^\infty ds \, s \, f(s^2) \exp(-2\pi k_x s).
\] (11)

The angular integrals have been computed as follows (\( a = 2\pi k_x s \)):

\[
\int_{-\pi/2}^{\pi/2} d\theta \left[ \int_0^\infty d\psi \, \sinh \psi \exp(-ia \sinh \psi \cos \theta) \right. \]

\[
+ \left. \int_0^\infty d\psi \, \sinh \psi \exp(-ia \sinh \psi \cos \theta) \right]
\]

\[
= 4 \int_0^\infty d\psi \, \sinh \psi \int_{-\pi/2}^{\pi/2} d\theta \cos(a \sinh \psi \cos \theta)
\]

\[
= 2\pi \int_0^\infty d\psi \, \sinh \psi \, J_0 (a \sinh \psi)
\]

\[
= \frac{2\pi}{a} \exp(-a).
\]

To get to the third line we have used formula 3.753(2) from Ref. 5, after a change of variable to \( x = \cos \theta \). The last line is obtained using 6.554(1) from Ref. 5, after a change of variable to \( y = \sinh \psi \).

- **Patch II+IV:**
  
  (i) Timelike momentum: \( k_x = k_y = 0, \)

\[
I_{II+IV}(k_t) = \int_{II+IV} dt \, dx \, dy \, f(t^2 - x^2 - y^2) \exp(-2\pi ik_t t)
\]

\[
= i \int_0^{i\infty} ds \, s^2 \, f(s^2) \int_{-\pi/2}^{\pi/2} d\theta \int_{-\infty}^{+\infty} d\psi \, \cosh \psi \exp(2\pi k_t s \sinh \psi)
\]

\[
+ i \int_{-i\infty}^0 ds \, s^2 \, f(s^2) \int_{-\pi/2}^{\pi/2} d\theta \int_{-\infty}^{+\infty} d\psi \, \cosh \psi \exp(2\pi k_t s \sinh \psi)
\]

\[
= -\pi \int_0^{-i\infty} ds' \, s'^2 \, f\left(-s'^2\right) \int_{-\infty}^{+\infty} d\psi \, \cosh \psi \exp(-2\pi ik_t s' \sinh \psi)
\]

\[
-\pi \int_0^{i\infty} ds' \, s'^2 \, f\left(-s'^2\right) \int_{-\infty}^{+\infty} d\psi \, \cosh \psi \exp(-2\pi ik_t s' \sinh \psi)
\]

\[
= 0.
\] (12)
The angular integrals have been computed as follows (with $a = 2\pi k_s$):

$$\int_{-\infty}^{+\infty} d\psi \ c\cosh \psi \exp(-ia \sinh \psi)$$

$$= \lim_{\epsilon \to 0} \int_{-\infty - i\epsilon}^{+\infty - i\epsilon} d\psi \ c\cosh \psi \exp(-ia \sinh \psi)$$

$$= \lim_{\epsilon \to 0} \frac{i}{a} \left[ \exp(-ia\infty) \exp(-\epsilon a\infty) - \exp(i\epsilon a\infty) \exp(-i\epsilon a\infty) \right]$$

$$= 0.$$  

(ii) Spacelike momentum: $k_t = 0,$

$$I_{II+IV} = \int_{II+IV} dt \ dx \ dy \ f(t^2 - x^2 - y^2) \ \exp(-2\pi ik_x x)$$

$$= i \int_0^{+\infty} ds \ s^2 \ f(s^2) \ \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \ \int_{-\infty}^{+\infty} d\psi \ \cosh \psi \ \exp(2\pi k_x s \cosh \psi \cos \theta)$$

$$+ i \int_{-\infty}^{0} ds \ s^2 \ f(s^2) \ \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \ \int_{-\infty}^{+\infty} d\psi \ \cosh \psi \ \exp(2\pi k_x s \cosh \psi \cos \theta)$$

$$= 4 \int_0^{+\infty} ds_1 \ s_1^2 \ f(-s_1^2) \ \int_{0}^{\frac{\pi}{2}} d\theta \ \int_{0}^{+\infty} d\psi \ \cosh \psi \ \exp(2\pi i k_x s_1 \cosh \psi \cos \theta)$$

$$+ 4 \int_0^{+\infty} ds_1 \ s_1^2 \ f(s_1^2) \ \int_{0}^{\frac{\pi}{2}} d\theta \ \int_{0}^{+\infty} d\psi \ \cosh \psi \ \exp(-2\pi i k_x s_1 \cosh \psi \cos \theta)$$

$$= 8 \int_0^{+\infty} ds_1 \ s_1^2 \ f(s_1^2) \ \int_{0}^{\frac{\pi}{2}} d\theta \ \int_{0}^{+\infty} d\psi \ \cosh \psi \ \cos(2\pi k_x s_1)$$

$$= \frac{2}{k_x} \int_{0}^{+\infty} ds_1 \ s_1 \ f(s_1^2) \ \cos(2\pi k_x s_1),$$  \hspace{1cm} (13)

where

$$s_1 = \sqrt{x^2 + y^2 - t^2}.$$  

The angular integral has been computed as follows (with $a = 2\pi k_x s_1$):

$$\int_{0}^{+\infty} d\psi \ \cosh \psi \ \int_{0}^{\frac{\pi}{2}} d\theta \ \cos(a \cosh \psi \cos \theta)$$

$$= \frac{\pi}{2} \int_{0}^{+\infty} d\psi \ \cosh \psi \ J_0(a \cosh \psi)$$

$$= \frac{\pi}{a} \cos(a),$$

where in the second line formula 3.715(19) from Ref. 5, and in the last line formula 6.554(3) from Ref. 5 have been used, after a change of variable to $x = \cosh \psi.$

In summary
\[ I(k_t) = -\frac{2}{k_t} \int_0^\infty ds_0 \ f(s_0^2) \ \text{sin}(2\pi k_t s_0) \]
\[ I(k_x) = \frac{2}{k_x} \int_0^\infty ds_0 \ f(s_0^2) \ \exp(-2\pi k_x s_0) + \frac{2}{k_x} \int_0^\infty ds_1 \ f(s_1^2) \ \text{cos}(2\pi k_x s_1). \]

**IV Case \( \mathbb{R}^{1,n} \)**

In this section we follow Codelupi’s derivation\(^3\) of the \( 1 + n \) dimensional case. The idea is to derive a recursion relation between the Fourier transform in \( n \) and \( n + 2 \) spatial dimensions, and then use the explicit expressions found before to construct the general case.

**IV.1 Recursion relation**

Define the radius in \( n \) spatial dimensions as follows:

\[ r^2 = \sum_{i=1}^n x_i^2. \]

Formally, the function \( f(s) = f(\sqrt{t^2 - r^2}) \) always looks the same, independent of the number of spatial dimensions. Suppose we have spaces with spatial dimensions \( n = 1 \) to \( n = m \). For each of these spaces exists a transform

\[ F^{(n)}(k, k_0) = \int_0^\infty dr \ \chi_n(r, k) \ G(r, k_0), \quad (14) \]

where (proof given in the appendix)

\[ \chi_n(r, k) = 2\pi \frac{r^{n/2}}{k^{n/2-1}} J_{n/2-1}(2\pi rk) \]

and

\[ G(r, k_0) = \int_{-\infty}^{+\infty} dt \ f\left(\sqrt{t^2 - r^2}\right) \ \exp(-2\pi ik_0 t). \quad (16) \]

But formally, \( G(r, k_0) \) looks the same for all cases, for example,

\[ G(r, k_0) = \int_0^\infty dk \ \chi_m(k, r) \ F^{(m)}(k, k_0), \quad (17) \]

assuming that in the inverse Fourier transform, the angular contribution can also be integrated out.

Substitute in the line before,

\[ F^{(n)}(k, k_0) = \int_0^\infty dr \ \chi_n(r, k) \int_0^\infty du \ \chi_m(u, r) \ F^{(m)}(u, k_0) \]
\[ = \int_0^\infty du \ F^{(m)}(u, k_0) \int_0^\infty dr \ \chi_n(r, k) \ \chi_m(u, r). \quad (18) \]

We can explicitly evaluate the second integral using formula 6.575(1) from Red. 5. The result is

\[ \int_0^\infty dr \ \chi_n(r, k) \ \chi_m(u, r) = \frac{2\pi h}{\Gamma(h)} u (u^2 - k^2)^{h-1} \Theta(u - k) \quad (19) \]
where \( h = (m - n)/2 \), \( \Gamma(x) \) Euler’s Gamma function and \( \Theta \) the step function. Equation (18) now takes on the form

\[
F^{(n)}(k, k_0) = \frac{2\pi^h}{\Gamma(h)} \int_k^\infty du \ F^{(n+2h)}(u, k_0) \ u \ (u^2 - k^2)^{h-1}. \tag{20}
\]

Considering the special case \( h = 1 \), i.e. \( m = n + 2 \), and taking the derivative with respect to \( k \) of both sides of this equation leads to the recursion formula

\[
F^{(n+2)}(k, k_0) = -\frac{1}{2\pi k} \frac{\partial}{\partial k} F^{(n)}(k, k_0). \tag{21}
\]

So the problem is solved, at least in principle, once we find the explicit expressions for \( n = 1 \) and \( n = 2 \). But as we will show in the next section (again following Codelupi\(^3\)), the recursion relation above will also allow us to find explicit formulas for the case of general \( n \).

### IV.2 Explicit expressions for \( R^{1,n} \)

Let us define

\[
l_0 = \sqrt{k_0^2 - k^2}, \\
l_1 = \sqrt{k^2 - k_0^2}.
\]

The recursion relation Eq.(21) can be rewritten in terms of \( l_0 \) and \( l_1 \),

\[
F^{(n+2)}(l_0) = -\frac{1}{2\pi k} \frac{\partial}{\partial k} F^{(n)}(l_0) = \frac{1}{2\pi l_0} \frac{d}{dl_0} F^{(n)}(l_0), \tag{22}
\]

\[
F^{(n+2)}(l_1) = -\frac{1}{2\pi k} \frac{\partial}{\partial k} F^{(n)}(l_1) = -\frac{1}{2\pi l_1} \frac{d}{dl_1} F^{(n)}(l_1). \tag{23}
\]

To find expressions for general \( n \) consider the following two cases

(a) \( n \)-even,

\[
F^{(n)}(l_0) = \left( \frac{1}{2\pi l_0} \frac{d}{dl_0} \right)^{n-1} F^{(2)}(l_0) = (-1)^{n} 2\pi \int_0^\infty ds_0 \ f(s_0) \frac{s_0^{1/2}}{l_0^{1/2}} \ J_{n-1} \left( 2\pi s_0 l_0 \right). \tag{24}
\]

The last equation is proved be iteratively applying the derivative to \( F^{(2)}(l_0) \) (result of Section III.2) and using formula 8.472(2) from Ref. 5. Similarly, we find

\[
F^{(n)}(l_1) = 4 \int_0^\infty ds_0 \ f(s_0) \frac{s_0^{1/2}}{l_1^{1/2}} \ K_{n-1} \left( 2\pi s_0 l_1 \right) - 2\pi \int_0^\infty ds_1 \ f(s_1) \frac{s_1^{1/2}}{l_1^{1/2}} \ N_{n-1} \left( 2\pi s_1 l_1 \right), \tag{25}
\]

where the formulae 8.472(2) and 8.486(13) from Ref. 5 have been used.
(b) $n$-odd,

$$F^{(n)}(l_0) = \left(\frac{1}{2\pi l_0}\right)^{\frac{n+1}{2}} F^{(1)}(l_0)$$

$$= (-1)^{\frac{n+1}{2}} 2\pi \int_0^\infty ds_0 f(s_0) \frac{s_0^{\frac{n+1}{2}}}{l_0^{\frac{n+1}{2}}} N_{\frac{n-1}{2}} (2\pi s_0 l_0)$$

$$+ (-1)^{\frac{n+1}{2}} 4 \int_0^\infty ds_1 f(s_1) \frac{s_1^{\frac{n+1}{2}}}{l_0^{\frac{n+1}{2}}} K_{\frac{n-1}{2}} (2\pi s_1 l_0) \quad (26)$$

and

$$F^{(n)}(l_1) = 4 \int_0^\infty ds_0 f(s_0) \frac{s_0^{\frac{n+1}{2}}}{l_1^{\frac{n+1}{2}}} K_{\frac{n-1}{2}} (2\pi s_0 l_1)$$

$$-2\pi \int_0^\infty ds_1 f(s_1) \frac{s_1^{\frac{n+1}{2}}}{l_1^{\frac{n+1}{2}}} N_{\frac{n-1}{2}} (2\pi s_1 l_1) \quad , \quad (27)$$

using the results of Section II.2, and, again, formulas 8.472(2) and 8.486(13) from Ref. 5.

For even $n$, the Fourier transform with a timelike momentum has no contribution from the spacelike region of spacetime.

We can summarize both cases in the following formulas, now valid for arbitrary $n$:

| $F^{(n)}(l_0)$ | $= -2\pi \int_0^\infty ds_0 f(s_0) \frac{s_0^{\frac{n+1}{2}}}{l_0^{\frac{n+1}{2}}} \left[ N_{\frac{n-1}{2}} (2\pi s_0 l_0) \cos \left(\frac{\pi n - 1}{2}\right) \right]$ |
|----------------|----------------------------------------------------------------------------------------------------------------------------------|
|                 | $+ J_{\frac{n-1}{2}} (2\pi s_0 l_0) \sin \left(\frac{\pi n - 1}{2}\right) \right]$ |
|                 | $+ 4 \int_0^\infty ds_1 f(s_1) \frac{s_1^{\frac{n+1}{2}}}{l_0^{\frac{n+1}{2}}} K_{\frac{n-1}{2}} (2\pi s_1 l_0) \cos \left(\frac{\pi n - 1}{2}\right)$ |

| $F^{(n)}(l_1)$ | $= 4 \int_0^\infty ds_0 f(s_0) \frac{s_0^{\frac{n+1}{2}}}{l_1^{\frac{n+1}{2}}} K_{\frac{n-1}{2}} (2\pi s_0 l_1)$ |
|----------------|----------------------------------------------------------------------------------------------------------------------------------|
|                 | $-2\pi \int_0^\infty ds_1 f(s_1) \frac{s_1^{\frac{n+1}{2}}}{l_1^{\frac{n+1}{2}}} N_{\frac{n-1}{2}} (2\pi s_1 l_1)$ |

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Appendix : Derivation of Eq.(15)

Equation (15) can be derived in several different ways. For direct integration see e.g., Ref. 6, Chap. 4. We follow here a derivation presented in Ref. 7.

We define the Fourier transform on $\mathbb{R}^n$ by
\[
F(k_1, \ldots, k_n) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} dx_1 \ldots dx_n \ f(x_1, \ldots, x_n) \ \exp(-2\pi i (k_1 x_1 + \ldots + k_n x_n)),
\] (A1)
or short
\[
F(\vec{k}) = \int_{\mathbb{R}^n} d^n x \ f(\vec{r}) \ \exp(-2\pi i \vec{k} \cdot \vec{r})
\] (A2)
and the inverse Fourier transform by
\[
f(\vec{r}) = \int_{\mathbb{R}^n} d^n k \ F(\vec{k}) \ \exp(2\pi i \vec{k} \cdot \vec{r}).
\] (A3)

One property we will need is
\[
\Delta^2 f(\vec{r}) = \sum_m \frac{\partial^2 f}{\partial x_m^2} = -4\pi^2 \int_{\mathbb{R}^n} d^n k \ k^2 F(\vec{k}) \ \exp(2\pi i \vec{k} \cdot \vec{r}).
\] (A4)

Now assume we have a radial function, and we work in spherical coordinates. We would like to write the Fourier transformation as
\[
F(k) = \int_0^\infty dr \ f(r) \ \chi_n(r, k)
\] (A5)
and the inverse Fourier transform as
\[
f(r) = \int_0^\infty dk \ F(k) \ \chi_n(k, r) dk,
\] (A6)
where the $\chi$’s contain the integration over the compact $n - 1$ angular coordinates. From Eq.(A4) we also know
\[
\Delta^2 f(r) = -4\pi^2 \int_0^\infty dk \ k^2 \ \chi_n(k, r) \ F(k).
\] (A7)
Calculating $\Delta^2 f(r)$ (now starting with Eq.(A6)),
\[
\frac{\partial^2 f(r)}{\partial x_m^2} = \int_0^\infty dk \ F(k) \ \frac{\partial^2}{\partial x_m^2} \ \chi_n(k, r),
\]
where
\[
\frac{\partial^2}{\partial x_m^2} \chi_n(k, r) = \left( \frac{1}{r} - \frac{x_m^2}{r^3} \right) \frac{\partial}{\partial r} \chi_n(k, r) + \frac{x_m^2}{r^2} \frac{\partial^2}{\partial r^2} \chi_n(k, r),
\]
yields
\[
\Delta^2 f(r) = \sum_m \frac{\partial^2 f(r)}{\partial x_m^2} = \int_0^\infty dk \ F(k) \ \left( \frac{n-1}{r} \frac{\partial}{\partial r} \chi_n(k, r) + \frac{\partial^2}{\partial r^2} \chi_n \right).
\] (A8)
But with Eq.(A7),
\[
\int_0^\infty dk \ F(k) \left[ \frac{\partial^2}{\partial r^2} \chi_n(k, r) + \frac{n-1}{r} \frac{\partial}{\partial r} \chi_n(k, r) + 4\pi^2 k^2 \chi_n(k, r) \right] = 0.
\] (A9)
This equation is valid for arbitrary $F(k)$, so the expression in brackets has to be zero. This ODE (in $r$) has the general solution (see Ref. 8, p.146)
\[
\chi_n(k, r) = A_n(k) r^{1-n/2} Z_p(2\pi kr),
\] (A10)
where $A_n(k)$ is determined by the initial conditions, $Z$ is a Bessel function of order $p$, and $p = \pm (1 - n/2)$. Computing the inverse Fourier transform explicitly in the cases of $n = 1$ and $n = 2$, determines $p$ to be $n/2 - 1$.

To find $A_n$, consider $f(r)$ at $r = 0$,

$$f(0) = \int_0^\infty dk \, F(k) \, \chi_n(k,0).$$

From Eq. (A10) we have

$$\chi_n(k,0) = \lim_{r \to 0} A_n(k) r^{1-n/2} J_{n/2-1}(2\pi rk)$$

$$= A_n(k) \frac{r^{1-n/2}(\pi rk)^{n/2-1}}{\Gamma(n/2)}$$

$$= A_n(k) \frac{(\pi k)^{n/2-1}}{\Gamma(n/2)}.$$

where 9.1.7 from Ref. 9 has been used.

But according to the definition of the inverse Fourier transform

$$f(0) = \int_{\mathbb{R}^n} dk \, F(k)$$

$$= \frac{\pi^{n/2} n!}{\Gamma(1 + n/2)} \int_0^\infty dk \, k^{n-1} F(k).$$

The factor in front of the integral is the volume of the unit $n-1$-sphere. Equating both expressions for $f(0)$, which are valid for arbitrary $F(k)$, yields

$$A_n(k) = 2\pi k^{n/2},$$

and therefore as the final result

$$\chi_n(k,r) = 2\pi k^{n/2} r^{1-n/2} J_{n/2-1}(2\pi rk). \quad (A11)$$

Because of the symmetry of the transformation,

$$\chi_n(r,k) = 2\pi r^{n/2} k^{1-n/2} J_{n/2-1}(2\pi rk). \quad (A12)$$

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