Quantum Creation of a Universe with varying speed of light: $\Lambda$-problem and Instantons

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One of the most interesting development trends of a modern cosmology is the analysis of models of a modified gravitation. Without exaggeration it is possible to say that Sergei Odintsov is one of the leaders of this direction of researches (see [1]). This article is dedicated to cosmologies with variable speed of light (VSL) - models, which one can consider as a particular case of models of a modified gravitation.

In quantum cosmology the closed universe can spontaneously nucleate out of the state with no classical space and time. The semiclassical tunneling nucleation probability can be estimated as

$$P \sim \exp(-\alpha^2/\Lambda)$$

where $\alpha=$const and $\Lambda$ is the cosmological constant.

In classical cosmology with varying speed of light $c(t)$ it is possible to solve the horizon problem, the flatness problem and the $\Lambda$-problem if $c = sa^n$ with $s=$const and $n < -2$. We show that in VSL quantum cosmology with $n < -2$ the semiclassical tunneling nucleation probability is $P \sim \exp(-\beta^2\Lambda^k)$ with $\beta=$const and $k > 0$. Thus, the semiclassical tunneling nucleation probability in VSL quantum cosmology is very different from that in quantum cosmology with $c=$const. In particular, it can be strongly suppressed for large values of $\Lambda$. In addition, we propose two instantons that describe the nucleation of closed universes in VSL models. These solutions are akin to the Hawking-Turok instanton in sense of $O(4)$ invariance but, unlike to it, are both non-singular. Moreover, using those solutions we can obtain the probability of nucleation which is suppressed for large value of $\Lambda$ too. We also discuss some unusual properties of models with $n > 0$.

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I. INTRODUCTION

One of the major requests concerning the quantum cosmology is a reasonable specification of initial conditions in early universe, that is in close vicinity of the Big Bang. There are known the three common ways to describe quantum cosmology: the Hartle-Hawking wave function [2], the Linde wave function [3], and the tunneling wave function [4]. In the last case the universe can tunnel through the potential barrier to the regime of unbounded expansion. Following Vilenkin [5] lets consider the closed ($k = +1$) universe filled with radiation ($w = 1/3$) and $\Lambda$-term ($w = -1$). One of the Einstein’s equations can be written as a law of a conservation of the (mechanical) energy: $P^2 + U(a) = E$, where $P = -a\dot{a}$, $a(t)$ is the scale factor, the “energy” $E = $ const and the potential

$$U(a) = c^2a^2 \left(1 - \frac{\Lambda a^2}{3}\right),$$

where $c$ is the speed of light; see Fig.1. The maximum of the potential $U(a)$ is located at $a_e = \sqrt{3/2\Lambda}$ where $U(a_e) = 3c^2/(4\Lambda)$. The tunneling probability in WKB...
approximation can be estimated as

\[ P \sim \exp \left( -\frac{2c^2}{8\pi G \hbar} \int_{a_i'}^{a_i} da \sqrt{U(a) - E} \right), \]  

(1)

where \( a_i' < a_i \) are two turning points. The universe can start from \( a = 0 \) singularity, expand to a maximum radius \( a_i' \), and then tunnel through the potential barrier to the regime of unbounded expansion with the semiclassical tunneling probability \( \hat{P} \). Choosing \( E = 0 \) one gets \( a_i' = 0 \) and \( a_i = \sqrt{3/\Lambda} \). The integral in (1) can be calculated. The result can be written as

\[ P \sim \exp \left( -\frac{2c^3}{8\pi G \hbar \Lambda} \right). \]  

(2)

For the probability to be of reasonable value, for example \( P = 1/e \sim 0.368 \), one has to put \( \Lambda \sim 0.3 \times 10^{53} \text{ cm}^{-2} \) (see \[2\]). In other words, the \( \Lambda \)-term must be large. However, despite this problem, we do acquire one prise: Once nucleated, the universe immediately begins a de Sitter inflationary expansion. Therefore the tunneling wave function results in inflation. And the \( \Lambda \)-term problem, which arises in this approach is usually being gotten rid of via the anthropic principle.

Now let’s return to the Fig. 1. We have two Lorentzian regions \((0 < a < a_i', a > a_i)\) and one Euclidean region \((a_i' < a < a_i)\). The second turning point \( a = a_i \) corresponds to the beginning of our universe. If \( \Lambda = 0 \) then \( U(a) \) has the form of parabola and we get only one Lorentzian region. In this case, the universe can start at \( a = 0 \), expand to a maximum radius and then recollapse. If \( E \to 0 \) then the single Lorentzian region is contract to the point. This, of course, comes to an agreement with the tunnelling nucleation probability: \( P \to 0 \) as \( \Lambda \to 0 \). In this article, however, we’ll show that in quantum cosmological VSL models the situation can be opposite, viz: the probability to find the finite universe short after it’s tunneling through the potential barrier is

\[ P \sim \exp(-\beta(n)\Lambda^{\alpha(n)}) \]  

with \( \alpha(n) > 0 \) and \( \beta(n) > 0 \) when \( n < -2 \) or for \(-1 < n < -2/3 \). After the tunneling one get the finite universe with "initial" value of scale factor \( a_1 \sim \Lambda^{-1/2} \), so the probability to find the universe with large value of \( \Lambda \) and small value of \( a_1 \) is strongly suppressed. The reason of this lies in the behavior of potential \( U(a) \), which, for the case \( \Lambda \to 0 \), is being transformed into the hyperbola, located under the abscissa axis. As a result, such a universe can start at \( a \sim 0 \) the regime of unbounded expansion. Therefore, we get the single Lorentzian region which is not contract to the point at \( E \to 0 \).

This new property of VSL quantum cosmology will be discussed in the Sec. II for the case \( w = 1/3 \). But there arouse the two new questions which have to be answered. First at all, the effective potential in VSL models can be unbounded from below at \( a \to 0 \). What possible meaning of such potentials can be? The second question is the geometric interpretation of the quantum creation of a Universe with varying speed of light. We know that universe can be spontaneously created from nothing (in model where \( c = \text{const} \)) and this process can be described with the aid of the instantons solutions possessing \( O(5) \) (if \( V(\phi) \) has a stationary point at some nonzero value \( \phi_0 = \text{const} \)) or \( O(4) \) (as Hawking-Turok instanton \[3\]) invariance. So, what can be said about instantons in the VSL models?

The whole plan of the paper looks as follows: in the next Section we’ll consider the simplest VSL model: model of Albrecht-Magueijo-Barrow. Then we show that in framework of tunneling approach to quantum cosmology with VSL the semiclassical tunneling nucleation probability can be estimated as

\[ P \sim \exp(-\beta^2 \Lambda^k) \]  

with \( \beta = \text{const} \) and \( k > 0 \). All corresponding calculations will be done for the case of the universe filled with radiation \( (w = 1/3) \) and vacuum energy. The case of any \( w \) will be considered in the Sec. III, where we will discuss the problem of potentials unbounded from below at \( a \to 0 \). We’ll use the following naive procedure: By analogy with Schrödinger equation we apply the Heisenberg uncertainty relation to the Wheeler-DeWitt equation in order to find the potentials which admit the ground state, even being unbounded from below at \( a \to 0 \).

In the Section IV we’ll propose the non-singular instanton solutions possessing only \( O(4) \) invariance (so the Euclidean region is a deformed four sphere). These solutions can in fact lead to inflation after the analytic continuation into the Lorentzian region. We will discuss these results in Sec. V. Some unusual properties of VSL models with \( n > 0 \) (including "big rip" and "big trip") are discussed in Appendix.

\section{Albrecht-Magueijo-Barrow VSL Model}

Lets start with the Friedmann and Raychaudhuri system of equations with \( k = +1 \) (we assume the \( G = \text{const} \)): 

\[ \frac{\dot{a}}{a} = -\frac{4\pi G}{3} \left( \rho + \frac{3p}{c^2} \right) + \frac{\Lambda c^2}{3}, \]

\[ \left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G \rho}{3} - k \left( \frac{c}{a} \right)^2 + \frac{\Lambda c^2}{3}, \]  

(3)

\[ c = c_0 \left( \frac{a}{a_0} \right)^n = sa^n, \quad p = wc^2 \rho, \]

where \( a = a(t) \) is the expansion scale factor of the Friedmann metric, \( p \) is the fluid pressure, \( \rho \) is the fluid density, \( k \) is the curvature parameter (we put \( k = +1 \)), \( \Lambda \) is the cosmological constant, \( c_0 \) is some fixed value of speed of light which corresponds some fixed value of scale factor \( a_0 \).

\textbf{Remark 1.} In \[2\] George F.R. Ellis and Jean-Philippe Uzan has pointed out at the fact, that the system of Friedmann-Raychaudhuri equations (system \[3\]), as denoted in our article) is not consistent from the point.
of view of a field theory. This is because the system (3) couldn’t be derived via the standard variation procedure, despite the contrary claims, given in [7]-[10] (see the references in [7]). Variation of action (36) (see [7]) leads instead to the system of equations (39)-(42) (see [7]), and not to the Friedmann-Raychaudhuri. Therefore, since our article is initially based on [7], it seems that the present article is just a formal exercise based on unreliable grounds (Eqs. (3)) so that its conclusions cannot tell us much about the effect of varying constants in Quantum cosmology.

We don’t think so. The fact, that the equations (39)-(42) (of [7]) are distinct from set (3) doesn’t immediately mean that the system (39)-(42) (of [7]) is correct while (3) is not. Moreover, even assuming that George F.R. Ellis and Jean-Philippe’s claim is indeed true and it is impossible to get (3) from (36) (see [7]), this would not have any consequences for our further results, barring the instanton chapter. In our article (except the mentioned chapter) we use neither the action (36) (of [7]) nor the fact that (3) can be derived from (36). Contrary to that, we just use (3) as a basic phenomenological model. This, of course, can be regarded as a flaw of a model, but same can be said about the system (39)-(42) (of [7]), which is also but a phenomenological model, based on assumption, that the VSL models are in essence the particular example of the scalar-tensor theories (see for example [7]).

Hence, in order to be able to make the preference of (39)-(42) (of [7]) over equation (3) (or vice versa) we have to gain the better comprehension of the physical principles, lying behind the variability of the speed of light (in assumption, that it really changes at all, which is still far from obvious!). That, in turn, would be possible only after we will understand the origins of “c”. It has been shown in reviewing part of (3), that it is possible to define not just one, but a couple of “speeds of light” (c_{ST}, c_{EM}, c_{E}). Beyond any doubts, there should exist a fundamental physical reason for those (generally different) values being perfectly equal. Such reason can only be discovered in the future fundamental physical theory (string theory?). Only such theory would verify which equations are true: (39)-(42) (in [7]), (3) or maybe some other ones.

Returning to system (39)-(42) (in [7]), which may be considered as a more reliable one than Friedmann and Raychaudhuri system (3), we shall in turn remind about some of the assumptions, that has been used by George F.R. Ellis and Jean-Philippe’s in order to get to (39)-(42).

For example, authors consider the value of c_{E} (which is used in Einstein’s equation in a form 8\pi G/c^4) as a variable, while assuming that the value of c_{ST}, that appears in the integral
\[ ds^2 = -c_{ST}^2 dt^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2, \]
satisfies c_{ST} = 1 (see page. 8, in the last line above the formula (39) in [7]). But if we’ll assume that all “speeds of light” are in fact the representations of just one value (an assumption, that we strongly believe to be true), then the value of c = \psi^{1/4} will require to be multiplied by the lagrangian (d^4x = cdtd^3x), including the various fields of matter and shall also be taken into account in the multiplier \sqrt{\psi}. With all this in mind, it becomes really doubtful that the VSL model (36) (of [7]) will remain being equal to some scalar-field theory. On the other hand, if we’ll insist on difference between the values of c_{ST} and c_{E}, while requiring the variability of both of them, it doesn’t seem impossible to choose the c_{ST} in a way, that leads to results, similar to the ones, received in [7]-[10] (see the reference list of [7]), which is sufficiently what we need for the IV chapter of our work, devoted to instantons (and is necessary for this chapter only!). Of course, this might look quite artificial, but it just serves to show that both (39)-(42) of [7] and (3) shall be regarded as phenomenological models.

Therefore, it appears, that studying of equations (3) are no more (or less) well-grounded, than studying of equations (39)-(42) of [7]. As follows from our work, system (3) can be quite interesting in the quantum cosmology, since it allows for an unexpected solution of a cosmological constant problem. This, with regard to the overall difficulty of the problem, can be considered as an additional indication that the model (3) does indeed deserve the further examinations. We aren’t making any further claims; it is of course quite possible that the system (39)-(42) of [7], if being applied to the quantum cosmology is also capable of producing of the interesting results, but that is a topic of a different research.

In conclusion, let us note that the equations (39)-(42) of [7] are in fact completely incompatible with our model. Our article is based on assumption that there exist a one-to-one correlation between c and the scalar factor a: c = sa^n. In other words, the speed light (or \psi = c^4 from [7]) is a function of a Ricci scalar \bar{R}, and, hence, is NOT an independent variable. But then we come to contradiction with the basic assumption of [7], which one basically allowed to get the system (39)-(42). Therefore, it would even formally be a mistake to use the (39)-(42) of [7] in a way, similar to the one, that we adopted for (3) - at least without special pre-given restrictions.

**Remark 2.** One can consider more general model. Let
\[ c = sa^n + \frac{m}{a^m}, \]
with n > 0, m > 0, r > 0, s > 0. In this case the flatness and A-term problems can be solved if
\[ n > m, \quad n > 1, \quad m > \frac{3(w + 1)}{2}, \]
for the weak energy condition is valid: w > -1. If a → ∞ then the Hubble root
\[ H^2 \rightarrow \frac{\Lambda s^2(1 + w)a^{2n}}{3(1 + w) + 2n}. \]
Choosing a substitution gives us the potential (8) in the form
\[ \rho = \frac{M}{a^{3(n+1)}} + \frac{3ks^2na^{2(n-1)}}{4\pi G(1 + 3w + 2n)} + \frac{3ksr(n - m)a^{n-m-2}}{4\pi G(1 + 3w + n - m)} \frac{\Lambda r^2m}{\Lambda r^2m} - \frac{3kr^2m}{4\pi G(1 + w - 2m)a^{2(1+m)}} - \frac{\Lambda s^2na^{2n}}{4\pi G(1 + w + n - m)}. \]

After that extensive digression let’s return to the equations (3). Using these ones
\[ \dot{\rho} = -\frac{3a}{a} (\rho + \frac{p}{c^2}) + \frac{\dot{c}c(3 - a^2\Lambda)}{4\pi Ga^2}. \] (5)
Choosing \( w = 1/3 \) one can solve (3) to receive
\[ \rho = \frac{M}{a^4} + \frac{3s^2na^{2(n-1)}}{8\pi G(n+1)} - \frac{s^2n\Lambda a^{2n}}{8\pi G(n+2)}. \] (6)
where \( M > 0 \) is a constant characterizing the amount of radiation. It is clear from the (8) that the flatness problem can be solved in a radiation-dominated early universe by an interval of VSL evolution if \( n < -1 \), whereas the problem of \( \Lambda \)-term can be solved only if \( n < -2 \). The evolution equation for the scale factor \( a \) (the second equation in system (3)) can be written as
\[ \rho^2 + U(a) = E, \] (7)
where \( p = -a\dot{a} \) is the momentum conjugate to \( a \), \( E = 8\pi GM/3 \) and
\[ U(a) = \frac{s^2a^{2n+2}}{n+1} - \frac{2s^2\Lambda a^{2n+4}}{3(n+2)}. \] (8)
The potential (8) has one maximum at \( a = a_e = \sqrt{3/(2\Lambda)} \) such that
\[ U_e \equiv U(a_e) = \frac{s^23^{n+1}}{2n+1\Lambda^{n+1}}(n+1)(n+2), \] (9)
so \( U_e > 0 \) if (i) \( n < -2 \) or (ii) \( n > -1 \). The first case allows us to solve the flatness and “Lambda” problems. The surplus dividend of the model is the presence of finite time region during which universe has accelerated expansion.

A. The semiclassical tunneling probability in VSL models with \( n < -2 \); the case \( E \ll U_e \)

One can choose \( n = -2 - m \) with \( m > 0 \). Such a substitution gives us the potential (8) in the form
\[ U_m(a) = \frac{s^2}{a^{2(n+1)}} \left( \frac{2\Lambda a^2}{3m} - \frac{1}{m+1} \right). \] (10)

The expression \( \rho \) which has the form:

The equation (7) is quite similar to equation for the particle of energy \( E \) that is moving in potential (10), hence the universe in quantum cosmology can start at \( a \sim 0 \), expand to the maximum radius \( a'_i \) and then tunnel through the potential barrier to the regime of unbounded expansion with "initial" value \( a = a_i \). The semiclassical tunneling probability can be estimated as
\[ P \sim \exp \left( -2 \int_{a_i}^{a'_i} | \tilde{p}(a) | da \right), \] (11)
with
\[ | \tilde{p}(a) | = \frac{c^2(t)}{8\pi G} | p(a) |, \quad | p(a) | = \sqrt{U_m(a) - E}, \]
where \( E < U_e \). It is convenient to write \( E = U_e \sin^2 \theta \), with \( 0 < \theta < \pi/2 \).

For the case \( E \ll U_e \) one can choose
\[ a'_i \sim a_1 = \sqrt{\frac{3m}{2(1+2\Lambda)}}, \quad a_i \sim \sqrt{\frac{3}{2\Lambda} \left( \frac{\sqrt{m+1}}{\sin \theta} \right)^{1/m}}, \] (12)
and evaluate the integral (11) as
\[ P \sim \exp \left( -\frac{s^3\Lambda^{2+3m/2}I_m(\theta)}{4\pi Gh} \right), \] (13)
where
\[ I_m(\theta) = \int_{z_i(\theta)}^{z_i'(\theta)} dz z^{-5-3m} \sqrt{\frac{2z^2}{3m} - \frac{1}{m+1}}, \] (14)
with
\[ z_i'(\theta) = \sqrt{\frac{3m}{2(1+2\Lambda)}}, \quad z_i(\theta) = \sqrt{1.5 \left( \frac{\sqrt{m+1}}{\sin \theta} \right)^{1/m}}. \]
The integral (14) can be calculated for the \( m \in Z \). For example
\[ I_1(\theta) = \frac{\sqrt{3}}{17010} (3 + \cos^2 \theta) \left( 191 - 78 \cos^2 \theta + 15 \cos^4 \theta \right) \sqrt{6 + 2 \cos^2 \theta} \sim 0.148 + O(\theta^6), \]

\[ I_2 \sim 0.025, \quad I_3 \sim 0.007, \quad I_4 \sim 0.002, \]

and so on. One can further show that \( I_m(\theta) > 0 \) at \( 0 < \theta < 1 \). Thus, it is easy to see from (13) that the semiclassical tunneling probability \( P \to 0 \) for large values of \( \Lambda > 0 \) and \( P \to 1 \) at \( \Lambda \to 0 \).

Note, that the case \( c = \text{const} \) can be obtained by substitution \( m = -2 \) into the (13). Not surprisingly, this case will get us the well known result \( P \sim \exp(-1/\Lambda) \) (see [3]).

B. The semiclassical tunneling probability with \( n < -2 \) and \( n > -1 \)

In the case of general position the semiclassical tunneling probability with \( n = -2 - m \) has the form

\[ P_m \sim \exp \left( - \frac{s^3 \Lambda(3m+4)^2}{4\pi G h^2} \sqrt{m(m+1)} \int_{z_1'}^{z_2'} \frac{dz}{z^{3m+5}} \right) \]

where

\[ F_m(z, \theta) = -2^{m+1} \sin^2 \theta z^{2(m+1)} + 2 \times 3^m(m+1)z^2 - m3^{m+1}, \]

(15)

This expression can be calculated exactly:

\[ P_1 \sim \exp \left( - \frac{s^3 \Lambda^2 \sin \theta J(\theta)}{6\sqrt{2\pi G h}} \right), \]

with

\[ z_1' = \frac{\sqrt{3}}{2 \cos(\theta/2)}, \quad z_2' = \frac{\sqrt{3}}{2 \sin(\theta/2)}. \]

where \( \mu^2 = \cos \theta / \cos^2(\theta/2), \lambda = \cot(\theta/2), \Pi \) and \( K \) are complete elliptic integral of the first and the third kinds correspondingly [9].

Similarly, \( P \sim \exp(-S) \), with

\[ S = \frac{s^3 \Lambda^5 \sin \theta}{18\pi G h} \int_{z_1'}^{z_2'} \frac{dz}{z^{11}} \sqrt{(z^2 + z_1'^2)(z^2 - z_1'^2)(z^2 - z_2^2)}, \]

where

\[ z_1 = \sqrt{\frac{3}{\sin \theta}} \cos \left( \frac{\theta}{2} - \frac{\pi}{6} \right), \quad z_2 = \sqrt{\frac{3}{\sin \theta}} \sin \left( \frac{\theta}{2} \right), \]

\[ z_1' = \sqrt{\frac{3}{\sin \theta}} \cos \left( \frac{\theta}{3} + \frac{\pi}{6} \right), \]

and so on.

Therefore the probability to obtain (via quantum tunneling through the potential barrier) the universe in the regime of unbounded expansion is strongly suppressed for large values of \( \Lambda \) and small values of the initial scale factor \( a_i = \sqrt{3}/(2 \sin(\theta/2) \sqrt{\Lambda}) \). In other words, overwhelming majority of universes which are nascent via quantum tunneling through the potential barrier ([5]) have large initial scale factor and small value of \( \Lambda \).

Now, let us consider the case (ii), when \( n > -1 \). The "quantum potential" has the form

\[ U(a) = s^2 a^{2m} \left( \frac{1}{m} - \frac{2\Lambda a^2}{3(m+1)} \right), \]

(17)

where \( m = n + 1 > 0 \). The points of intersection with the abscissa axis \( a \) are \( a_0 = 0 \) and \( a_1 = \sqrt{3(m+1)/2\Lambda m} \).

Choosing \( E = 0 \) in equation (17) and substituting (17)
into the (11) we get

\[ P \sim \exp \left( -\frac{s^3 \Lambda (1-3m)/2}{4\pi G \hbar} \int_0^{z_1} z^{2m-2} \sqrt{\frac{1}{m} - \frac{2z^2}{3(m+1)}} \, dz \right) , \]

with \( z_1 = \sqrt{3(m+1)/2m} \) (The starting value \( z = 0 \) means that the Universe tunneled from "nothing" to a closed universe of a finite radius \( a_1 = z_1/\sqrt{\Lambda} \)). Thus, we have the same effect as if \( 0 < m < 1/3 \).

### C. Peculiar cases with \( n = -1 \) and \( n = -2 \)

At last, let's consider the cases of \( n = -1 \) and \( n = -2 \). The formula (15) is not valid in these cases (\( m = -1 \) and \( m = 0 \)) so we shall consider these models separately.

If \( n = -1 \) (\( m = -1 \)) then

\[ \rho = \frac{M}{a^4} + \frac{s^2 \Lambda^2}{8\pi Ga^2} - \frac{3s^2}{4\pi Ga^4} \log \frac{a}{a_*} , \]

therefore

\[ U(a) = s^2 \left( 2 \log \left( \frac{a}{a_*} \right) - \frac{2a^2 \Lambda}{3} + 1 \right) , \quad (18) \]

where \( a_* \) is constant and \( |a_*| = c_m \). The potential (18) has one maximum at \( a = a_c = \sqrt{3/(2\Lambda)} \) such that \( U_c = U(a_c) = 2s^2 \log(a_c/a_*) \), so if \( a_c > a_* \) then \( U_c > 0 \). We choose \( a_* = \Lambda^{-1/2} \). This gives us \( U_c = 0.41s^2 > 0 \). For the case \( E \ll U_c \) the semiclassical tunneling nucleation probability is

\[ P_{-1} \sim \exp \left( -\frac{s^3 \sqrt{\Lambda}}{4\pi G \hbar} \int_{z_c'}^{z_1} \frac{dz}{z^2} \sqrt{\frac{2(z^2) - 2/3 + 1}{z^2}} \right) \sim \exp \left( -\frac{s^3 \sqrt{\Lambda}}{10\pi G \hbar} \right) , \quad (19) \]

where the turning points are \( z_c' = 0.721 \), \( z_1 = 1.812 \). As we can see from the (19), when \( n = -1 \) we receive the aforementioned effect again.

If \( n = -2 \) (\( m = 0 \)) then

\[ \rho = \frac{M}{a^4} + \frac{s^2 \Lambda}{2\pi Ga^4} \log \left( \frac{a}{a_*} \right) + \frac{3s^2}{4\pi Ga^8} . \]

We choose \( a_* = 1/(\alpha \sqrt{\Lambda}) \), where \( \alpha \) is a dimensionless quantity. Thus

\[ U(a) = -s^2 \left( \frac{1}{a^2} + \frac{4\Lambda}{3} \log \left( \alpha a \sqrt{\Lambda} \right) + \frac{\Lambda}{3} \right) . \quad (20) \]

The maximum of potential (20) is located at the same point \( a_c \) and

\[ U_c = -s^2 \frac{\Lambda}{3} \left( 3 + \log \left( \frac{9a_c^4}{4} \right) \right) . \]

Therefore, \( U_c > 0 \) if \( \alpha < 2e^{-3/4}/\sqrt{6} \sim 0.386 \). Choosing \( \alpha = 0.286 \) and \( E \ll U_c \) gets us the turning points \( z_c' \sim 0.77 \) and \( z_1 \sim 2.391 \).

At last, the semiclassical tunneling nucleation probability is

\[ P_{-2} \sim \exp \left( -\frac{s^3 \sqrt{\Lambda}}{4\pi G \hbar} \int_{z_c'}^{z_1} \frac{dz}{z^2} \sqrt{\frac{1}{z^2} - \frac{4}{3} \log(\alpha z) - \frac{1}{3}} \right) \sim \exp \left( -\frac{0.084s^3 \sqrt{\Lambda}}{\pi G \hbar} \right) . \]

### III. AN EXISTENCE OF GROUND STATES FOR SINGULAR POTENTIALS

It is interesting to ask: what can be said about the value of \( U(a) \) when \( a = 0^+ \)? What can be the possible meaning of the potential which at \( a \to 0 \) is unbounded from below? It seems that such universe is able to just roll down towards small values of \( a \) (where the potential is tending to minus infinity) instead of any tunneling to large values.

This situation can in fact be alleviated if the considered potential \( U(a) \) has the ground state. Indeed, one can imagine the fictitious particle with some energy and coordinate \( a(t) \) in the potentials (3) (or (7)) rolling down towards small values of \( a \). The main problem is: whether the quantum mechanical energy spectrum of \( U(a) \) is unbounded below? If not, then it does admit the ground state and hence can have the physical meaning.

To find such a potential lets suppose that our fictitious particle is located in a small region \( a \) near the singularity \( a = 0 \), with the momentum \( P \). We will consider the case of arbitrary \( w \). In this case the flatness problem can be solved in early universe by an interval of VSL evolution if \( n < n_{fl} \) (\( w = -(1 + 3w)/2 \)), whereas the problem of \( \Lambda \)-term can be solved only if \( n < n_{\Lambda}(w) = n_{fl}(w) - 1 = -3(w+1)/2 \).

One can use the Heisenberg uncertainty relation as

\[ P a \sim (8\pi G \hbar)^{1-n_{fl}(w)/2} c_{(3n_{fl}(w)-1)/2} . \quad (21) \]

Using (21), and (3) (or (17)) one get for the \( a \to 0 \)

\[ E = P^2 + U(a) \to \frac{Z^2}{a^{2-n(3n_{fl}(w)-1)} + (n_{fl}(w) - n)a^{2(n_{fl}(w)-n)}}, \]

where \( Z^2 = (8\pi G \hbar)^{1-n_{fl}} s^{3n_{fl}-1} \), and

\[ n < n_{fl}(w) < 0 . \quad (22) \]

Therefore the energy spectrum will be bounded below if

\[ (3n + 2) (n_{fl}(w) - 1) < 0 . \quad (23) \]
FIG. 2: The ground state exists for $w$ and $n$ from the interior of the triangle ABC.

and (22) are valid. This situation is represented graphically on the Fig. 2. It is easy to see that the conditions of ground state existence can be satisfied if $n_s(w) < n < n_\mu(w)$. (The additional restriction $w > -1/3$ is just a condition of existence of maximum of the potential $U(a)$.) For more detailed examination of the general case see [10]). In the case of universe filled with radiation (as above) one get $-2 < n < -1$.

### IV. INSTANTONS

If we are going to describe the quantum nucleation of universe we should find the instanton solutions, simply putted as a stationary points of the Euclidean action. The instantons give a dominant contribution to the Euclidean path integral, and that is the reason of our interest in them.

First at all, lets consider the $O(4)$-invariant Euclidean spacetime with the metric:

$$ds^2 = c^2(\tau)d\tau^2 + a^2(\tau) (d\psi^2 + \sin^2 \psi d\Omega_2^2).$$ (24)

In the case $c = const$ one can construct the simple instantons, which are the $O(5)$ invariant four-spheres. Then one can introduce the scalar field $\phi$, whose (constant) value $\phi = \phi_0$ is chosen as the one providing the extremum of potential $V(\phi)$. The scale factor will be $a(\tau) = H^{-1} \sin H\tau$ and after the analytic continuation into the Lorentzian region one will get the de Sitter space or inflation. Many other examples of non-singular and singular instantons were presented in [11]

Now, lets consider the VSL model with scalar field.

Here we get the following Euclidean equations:

$$\phi'' + 3\frac{\alpha'}{\alpha} \phi' = \frac{c^2 V'}{\phi'} + \frac{c^2 c'(\Lambda a^2 - 3)}{4\pi Ga^2 \phi'} + \frac{2\phi' c'}{c} - \frac{2c V c'}{\phi'}. \quad (25)$$

where primes denote derivatives with respect to $\tau$.

At the next step we represent the potential $V$ in factorized form:

$$V = F(a)U(\phi). \quad (26)$$

Indeed, lets for example consider the power-low potential $\sim \phi^k$. If the coupling $\lambda$ is dimensionloss one then we get

$$V \sim \frac{\lambda}{k} G^{k/2 - 2} c^{1 - 2k} \phi^k. \quad$$

Since $c = sa^n$ then in the simplest case we come to (26).

Let $\phi = \phi_0 = const$ be solution of the (25). (Note, that we don’t require the $\phi_0$ to be the extremum of potential.) Using the first equation of system (25) and (26) we get the equation for the $F(a)$,

$$\frac{dF(a)}{da} - \frac{2n}{a} F(a) = \frac{3ns^4}{4\pi GU_0^2} a^{4n-3} - \frac{ns^4 \Lambda}{4\pi GU_0^2} a^{4n-1}, \quad (27)$$

where $U_0 = U(\phi_0) = const$. The integration of the (27) results in

$$F(a) = a^{2n} \left( C - \frac{3ns^4}{8\pi G(1-n)U_0} a^{2(n-1)} - \frac{s^4 \Lambda}{8\pi GU_0^2} a^{2n} \right), \quad (28)$$

where $C$ is the constant of integration and by assumption $n \neq -1$ and $n \neq 0$. Substitution of (28) into the second equation of the system (25) transforms it into the the model of nonlinear oscillator the integration of which result in

$$\frac{a'^2}{2} + u(a) = 0, \quad (29)$$

where

$$u(a) = \omega^2 a^2 - \frac{s^2 a^{2n}}{2(1-n)}, \quad (30)$$

with $\omega^2 = 8\pi GU_0 C/(3s^2)$ and with the choice $C > 0$ made. We can see that for $c = const$ (i.e. $n = 0$) the equation (30) turns out to be just the usual harmonic oscillator and we come to the well-known $O(5)$ solution (but in this case $\phi_0$ must be the stationary point of $V$).

The equation (29) naturally describes the “movement of a classical particle” with zero-point energy in mechanical potential (29). Depending on value of $n$ this potential can take one of four distinct forms (excluding the well-known classical case $n = 0$, which lies beyond the scoop of this article).
Case 1: \( n < 0 \). Potential \( u(a) \) has the form, depicted on the Fig. 3. Here we have one Euclidean (0 \( \leq a \leq a_1 \)) and one Lorentzian (\( a > a_1 \)) regions where

\[
a_1 = \left( \frac{s}{\omega \sqrt{1-n}} \right)^{1/(1-n)}.
\]

On the bound between Euclidean and Lorentzian regions (\( a = a_1 \)) we have \( a' = 0 \).

This mechanical potential is unbounded from below at \( a \to 0 \). With this in mind, we’ll have to ascertain that the Euclidean action for our solution will stay finite. The gravitation action has the form

\[
S_{\text{grav}} = -\int d^4x \frac{c^3}{8\pi G} \sqrt{g} R.
\]

We are using the dimensionless variables \( x^0 = c_0 \tau/a_0 \), \( x^1 = \psi \) and so on. Calculating \( R \) we get

\[
R = \frac{6}{c_0 a^2} \left[ c_0^2 - \frac{a_0}{a} \right]^{2n} \left[(1-n)a^2 + aa''\right],
\]

so we do have the potential divergence at \( a = 0 \). Multiplying (22) on the \( \sqrt{g} \) and \( c^3 \) and using the equation of motion we get the expression:

\[
R \sqrt{g} c^3 \sim 6c_0 \left( (2-n)\omega^2 a_0^{2n+3} a_0^{2n-1} - \frac{n c_0^2 a_0^{4n+1}}{1-n a_0^{4n-1}} \right), \tag{33}
\]

where the most dangerous multiplier factor is \( a^{1+4n} \). But if \( -1/4 \leq n < 0 \) then the Euclidean action becomes finite and therefore, we end up with the legitimate gravitation instanton. In a similar manner, using (20) and (28) we get for the scalar field (in dimensionless \( x^i \)):

\[
\sqrt{g} \phi_{0} \sim \frac{c_0 a_0^{-1-3n}}{8\pi G} (3\omega^2 a_0^{1+4n})\]

\[
+ \frac{3nc_0^2}{(n-1)a_0^{2n}} \frac{\Lambda c_0^2 a_0^{3+5n}}{a_0^{2n}}.
\]

Therefore the instanton exists for \( n > -1/5 \). This demand is more powerful than what we got for the gravitation instanton where \( n > -1/4 \) (see (33)).

Case 2. \( 0 < n < 1 \). Here the potential \( u(a) \) suffers no singularity at \( a = 0 \), but \( u(0) = 0 \). Also this potential has a minimum at

\[
a_0 = \left( \frac{s}{\omega \sqrt{1-n}} \right)^{1/(1-n)},
\]

and is equal to zero at \( a_1 \), hence, once again creating one Euclidean and one Lorentzian regions, separated by \( a_1 \).

Case 3. \( n = 1 \). This case is somehow special, since for such \( n \) the solution of (27) shall be

\[
F(a) = a^{2n} \left( C - \frac{3s^4}{4\pi G U_0} \ln a - \frac{s^4 A}{8\pi G U_0} a^2 \right),
\]

instead of (28), and hence, the equation of (30) shall be substituted by

\[
u(a) = a^{2n} \left( \frac{\omega^2}{2} - s^2 \ln a \right). \tag{34}
\]

It is easy to see that this function has two zeros (at \( a_1 = 0 \) and \( a_2 = \exp(\frac{\omega^2}{s}) \)), is strictly positive at the interval (\( a_1, a_2 \)) and strictly negative outside of it. Therefore, this case doesn’t admit the instanton.

Case 4. \( n > 1 \). The potential \( u(a) \) is strictly positive. The instanton doesn’t exist either.

Both of a newly founded solutions possess only \( O(4) \) invariance just like Hawking-Turok instanton (so the Euclidean region is a deformed four sphere) but, unlike to it, they are all non-singular. Note that if the value \( a \) is sufficiently large then one can neglect the second term in (30) (after the analytic continuation into the Lorentzian region) therefore, as in the case of the usual \( O(5) \) instanton, one can get the de Sitter universe, i.e. the inflation.

The equation (29) has no terms with \( \Lambda \). In other words, the scale factor \( a(\tau) \) doesn’t depend on the value \( \Lambda \) (although being dependant on the \( U_0 \)). Therefore, the full Euclidean action \( S_E = S_{\text{grav}} + S_{\text{field}} \) has the form,

\[
S_E = S_0 - \Lambda S_1,
\]

where \( S_0 \) and \( S_1 \) are both independent of the \( \Lambda \). Returning to what has been said in Introduction, there exist three common ways to describe the quantum cosmology: the Hartle-Hawking wave function \( \exp(-S_E/\hbar) \), the Linde wave function \( \exp(+S_E/\hbar) \) and the tunneling wave function. In the second Section we have been working with the tunneling wave function. In case of instantons situation becomes slightly different. If \( S_1 > 0 \) then (as a first, tree semiclassical approximation) we should choose the Linde wave function, whereas for the case \( S_1 < 0 \) the Hartle-Hawking wave function seems more naturally.

In conclusion, we note that another choice of \( C \) (\( C < 0 \) and \( C = 0 \)) eliminates any possible instantons.
V. DISCUSSION

VSL models contain both some of the promising positive features [12] and some shortcomings and unusual (unphysical?) features as well [13]. But, as we have shown, application of the VSL principle to the quantum cosmology indeed results in amazing previously unexpected observations. The first observation is that the semiclassical tunneling nucleation probability in VSL quantum cosmology is quite different from the one in quantum cosmology with \( c = \text{const} \). In the first case this probability can be strongly suppressed for large values of \( \Lambda \) whereas in the second case it is strongly suppressed for small values of \( \Lambda \). This is interesting, although we still can’t say that VSL quantum cosmology definitely results in solution of the \( \Lambda \)-mystery. The problem here is the validity the WKB wave function. And what is more, throughout the calculations we have been omitting all preexponential factors (or one loop quantum correction) which can be essential ones near the turning points. Another troublesome question is the effective potentials in VSL models, being unbounded from below at \( a \to 0 \). The naive way to solve this problem is to use the Heisenberg uncertainty relation to find those potentials with the ground state. However, this is just a crude estimation. To describe the quantum nucleation of universe we have to find the instanton solution which, being a stationary point of the Euclidean action, gives the dominant contribution to the Euclidean path integral. As we have seen, such solutions indeed exist in VSL models. Those instantons are \( O(4) \) invariant, are non-singular, and provide an inflation as well. They describe the quantum nucleation of universe from ”nothing” and, what is more, upon usage of these solutions we can obtain the probability of a universe from “nothing” and, what is more, upon usage of these solutions we can obtain the probability of a universe which is suppressed for large value of \( \Lambda \) (as in see Sec. II) using either Linde or Hartle-Hawking wave function.

Note, that we can weaken the condition \( n > -1/5 \) to obtain a singular instanton suffering the integrable singularity (i.e. such that the instanton action will be finite) in the way of the Hawking-Turok instanton. However, there exist some arguments [14], that such singularities, even being integrable, still lead to serious problems with solutions.

In conclusion, we note that obtained instantons both have a free parameter (\( \omega^2 \)) so we are free to use the anthropic approach to find the most probable values of \( \Lambda \) too.

Acknowledgments

After finishing the first version of this work, we learned that T.Harko, H.Q.Lu, M.K.Mak and K.S.Cheng [13], have independently considered the VSL tunneling probability in both Vilenkin and Hartle-Hawking approaches. The interesting conclusion of their work is that at zero scale factor the classical singularity is no longer isolated from the Universe by the quantum potential but instead classical evolution can start from arbitrarily small size. In contrast to [13], we attract attention to the problem of \( \Lambda \)-term and instantons in VSL quantum cosmology.

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APPENDIX A: THE CASE \( n > 0 \)

Usually those of authors, who apply an idea of variability of the light speed to cosmology, are restricting themselves to studying the models with negative values of \( n \) (upon doing so they became able to solve some of the cosmological problems without involvement of inflation). On the other hand, as we have shown in the last section, the models with positive \( n \) involve a nonsingular instanton, and hence are sensible from the point of view of quantum cosmology. Moreover, as we shall see, such models can also describe our observed universe; in particular, they provides a solution to the flatness problem and describe the present acceleration of universe. Finally, the models with positive \( n \) can naturally result in Big Rip, as will be shown below.

1. The dust case: flatness and acceleration

The integration of (9) (with assumption \( p = wc^2 \rho \)) results in

\[
\rho = \frac{M}{a^{3(w+1)}} + \frac{3nk^2a^{2(n-1)}}{4\pi G(3w + 2n + 1)} - \frac{ns^2\Lambda a^{2n}}{4\pi G(3w + 2n + 3)},
\]

where \( M = \text{const} \). The pressure is

\[
p = \frac{w\rho^2}{a^{3w+3-2n}} + \frac{3kns^4w^4a^{4n-2}}{4\pi G(3w + 2n + 1)} - \frac{s^3n\Lambda w^4a^{4n}}{4\pi G(3w + 2n + 3)}.
\]

Therefore

\[
\frac{\ddot{a}}{a} = \frac{4\pi GM(3w + 1)}{3a^{3(w+1)}} - \frac{s^2nk(3w + 1)a^{2(n-1)}}{3w + 2n + 1} + \frac{s^2\Lambda (w + 1)(n + 1)a^{2n}}{3w + 2n + 3}.
\]

An equation on the Hubble constant \( H = \dot{a}/a \) is

\[
H^2 = \frac{8\pi GM}{3a^{3(w+1)}} - \frac{(3w + 1)k^2s^2a^{2(n-1)}}{3w + 2n + 1} + \frac{\Lambda a^{2(w + 1)a^{2n}}}{3w + 2n + 3}.
\]
If \( w = 0 \) then equations (\( A2 \)) and (\( A3 \)) are reduced to
\[
\frac{\ddot{a}}{a} = -\frac{4\pi GM}{3a^3} - \frac{s^2 n k a^{2n-1}}{2n+1} + \frac{s^2 \Lambda (n+1) a^{2n}}{2n+3},
\]
(\( A4 \))
\[
H^2 = \frac{8\pi GM}{3a^3} - \frac{k s^2 a^{2n-1}}{2n+1} + \frac{\Lambda s^2 a^{2n}}{2n+3}.
\]
(\( A5 \))
The modern observations result in
\[
\frac{\ddot{a}_0}{a_0} \sim \frac{7H_0^2}{10},
\]
(A5)
where \( H_0 = h \times 0.324 \times 10^{-19} \) s\(^{-1} \), with \( h = 45 \div 75 \),
\( a_0 \sim 10^{28} \) cm. Thus we have
\[
\frac{\ddot{a}_0}{a_0} = -\frac{4\pi GM}{3a_0^3} - \frac{n k c_0^2 a_0^2}{a_0^2 (1+2n)} + \frac{c_0^2 \Lambda (n+1)}{3+2n},
\]
(A6)
and
\[
H^2_0 = -\frac{k c_0^2}{(1+2n)a_0^2} + \frac{\Lambda c_0^2}{3+2n} + \frac{8\pi GM}{3a_0^3}.
\]
(A7)
Using (\( A5 \)), (\( A6 \)) and (\( A7 \)) we get to
\[
M = 3a_0 \left[ a_0^2 H_0^2 (1+2n)(3+10n) + 10k c_0^2 \right],
\]
(A8)
\[
\Lambda = \frac{5k c_0^2 + 12H_0^2 a_0^2}{5a_0^2 c_0^2},
\]
(A9)

hence, concluding that such model are indeed able to describe our observable universe.

Let \( n \) be positive albeit very small quantity: \( n \ll 1 \) (and \( w = 0 \)). In this case the equations (\( A1 \), (\( A8 \)), (\( A4 \))
will be reduced to
\[
\rho = \frac{M}{a^3} - \frac{ns^2 \Lambda a^{2n}}{12\pi G} + \frac{3n k s^2}{4\pi G a^2},
\]
\[
M = \frac{a_0 (3H_0^2 a_0^4 + 10k c_0^2)}{40\pi G},
\]
\[
H^2 = \frac{8\pi GM}{3a^3} - \frac{k s^2 a^2}{a^2} + \frac{\Lambda s^2 a^{2n}}{3},
\]
(A10)
\[
\frac{\ddot{a}}{a} = -\frac{4\pi GM}{3a^3} - \frac{s^2 n k a^2}{a^2} + \frac{\Lambda s^2 a^{2n}}{3}.
\]
(A11)

It easily follows from the (\( A11 \)) that the evolution of the universe can be divided into three stages:

Stage I. For values of \( a \) that are close enough to the initial singularity (i.e. \( a \rightarrow 0 \)) the first term in (\( A11 \)) dominates above the second and the third ones. The very young universe has been a flat one.

Stage II. The second term in (\( A11 \)) dominates above the first and third ones. Here we can’t consider the universe as flat. But we can do it on the last stage.

Stage III. The last term in (\( A11 \)) dominates above the first and second ones. This stage is the one best fit to describe the universe we resides in. In order to estimate the starting time of this phase we shall use (\( A11 \)). In fact, it is easy to show that last stage will start only if
\[
z < \frac{1}{\sqrt{n}} - 1,
\]
(A12)
where \( z = a_0/a - 1 \) is a red shift. Therefore, the modern astronomical observations can give us the upper bound on \( n \).

2. Big Rip and Big Trip

The integrals, appearing in (\( A10 \)) (and (\( A11 \))) in case of general position can be extremely hard to solve. However, we can make everything a lot more easier, noting that at final stage both of the first terms in equation (\( A10 \)) can actually be omitted. Keeping this in mind, we come to
\[
a(t) = \left( \frac{1}{n s \sqrt{\frac{3}{\Lambda}}} \right)^{1/n} \frac{1}{(t_* - t)^{1/n}}.
\]
(A13)
here \( t_* = \text{const} \). The value of \( t_* \) (i.e. a time of a big rip occurrence, see [4]) can be obtained via integration of unabridged equation (\( A10 \)). Thus, quite surprisingly, we get the big rip (See [14] about big rip in "usual cosmology" with \( c = \text{const} \)).

Remark 3. In the case of "usual cosmology" the big rip singularity can exist because of phantom energy. There are few ways to escape of future big rip singularity: (i) to consider phantom energy just as some effective models (see [17], [18], [19]); (ii) to use quantum effects to delay the singularity [20]; (iii) to use new time variable such that the big rip singularity will be point at infinity \((t \rightarrow \infty)\) [21]; (iv) to avoid big rip via another cosmological "Big": big trip (see below).

In [22] Pedro F. González-Díaz had shown that phantom energy can lead to an achronal cosmic future where the wormholes become infinite before the arisement of the big rip singularity. Soon after that, with the continuing accretion of a phantom energy, any wormhole becomes the Einstein-Rosen bridge (see also [23], [24]). Pedro F. González-Díaz has suggested that such objects can be used by an advanced civilization as the means of escape from the big rip, but, via usage of Bekenstein Bound we have shown it to be impossible due to the very strong restrictions laid on the total amount of information which can be sent through this bridge [25]. It is very interesting that in VSL models with big rip the escape is still possible.
To show this, we shall use Babichev, Dokuchaev and Eroshenko (BDE) equation for the throat radius of a Morris-Thorne wormhole [26] in the dust universe with VSL during the last stage of evolution with

$$\rho \sim - \frac{s^2 n \Lambda a^{2n}}{12 \pi G}.$$ 

In this case the BDE equation takes the form

$$\frac{1}{b(t)} = \frac{1}{b_0} + \frac{\pi D \sqrt{3 \Lambda}}{6} \log \left| \frac{t_s - t}{t_s - t_0} \right|,$$

where $b$ is a throat radius and $D$ is a positive constant. Hence, in assumption that the equation of BDE remains correct in VSL models, we conclude that there exist a big trip at $t = T < t_*$, where

$$T = t_* - (t_* - t_0) \exp \left( \frac{2}{\pi Db_0 \sqrt{\frac{3}{\Lambda}}} \right),$$

and $b_0$ is initial value of $b$. Now we can calculate the horizon distance in such universe:

$$R_c(t) = a(t) \int_t^{t_*} \frac{c(t') dt'}{a(t')}.$$ 

Substituting (A13) into this expression we get

$$R_c(t) = \sqrt[3]{\frac{3}{\Lambda}}$$

at the last stage of evolution. Thus, we have no information bound like the one taking place in models with phantom fields (where $R_c \to 0$ when $t \to t_*$) and so, the advanced civilization can in principle use such objects as a means of escape from the big rip singularity.

Another interesting situation is connected with the quantum gravity effects. Such effects take place at a scales $a \sim L_{Pl}$. Since

$$L_{Pl}^2 = \frac{G h}{c^3} = \frac{G h}{c^3} \left( \frac{a_0}{a} \right)^{3n},$$

we can see that $L_{Pl} \to \infty$ at $a \to 0$ (or, in other words, the value of red shift $z \to \infty$). It now follows, that the universe in vicinity of a Big Bang has been quantum. In other words, in order to successfully describe the universe for large $z$ one needs the quantum gravity theory. On the other hand, close to the big rip we get $L_{Pl} \to 0$! Therefore, in VSL models, the big rip is a purely classical effect and we don’t need a quantum gravity (either string theory or loop quantum gravity) in order to describe big rip in such universe.

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[1] Of course, in close vicineway initial singularity it will be more correctly to consider $w = 1/3$ rather then $w = 0$. But the result will be the same: the universe was flat.