Bayesian nonparametric tests for multivariate locations

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Abstract

In this paper, we propose novel, fully Bayesian non-parametric tests for one-sample and two-sample multivariate location problems. We model the underlying distribution using a Dirichlet process prior, and develop a testing procedure based on the posterior credible region for the spatial median functional of the distribution. For the one-sample problem, we fail to reject the null hypothesis if the credible set contains the null value. For the two-sample problem, we form a credible set for the difference of the spatial medians of the two samples and we fail to reject the null hypothesis of equality if the credible set contains zero. We derive the local asymptotic power of the tests under shrinking alternatives, and also present a simulation study to compare the finite-sample performance of our testing procedures with existing parametric and non-parametric tests.

Keywords: Bayesian nonparametrics, Hypothesis testing, credible region, Pitman alternatives.

1. Introduction

Several frequentist testing procedures for multivariate locations are available in the literature, both parametric and non-parametric. The most well-known parametric procedure is the Hotelling’s $T^2$-test, which is based on the multivariate mean vector and the covariance matrix, and it relies on the assumption of multivariate normality. This technique performs well if the assumption of multivariate normality is nearly correct, but suffers heavily otherwise, or in the presence of outliers. Non-parametric and robust alternatives based on signs and ranks have been quite popular over the years [Oja and Randles, 2004].

The notions of signs and ranks are based on the “ordering” of the data points, but in the multivariate setting, there is no objective basis of ordering. The notions are generalized to higher dimensions using $\ell_1$-objective functions (see Section 2). The existing one-sample

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location problem have the following set up. Suppose that, we have \( n \) independent and identically distributed (i.i.d.) observations \( Y = (Y_1, \ldots, Y_n) \) taking values in \( \mathbb{R}^k \) from a distribution \( P(\cdot - \theta) \), a \( k \)-variate continuous distribution centered at \( \theta = (\theta_1, \ldots, \theta_k)^T \). Our objective is to test the hypothesis

\[
H_0 : \theta = \theta_0 \quad \text{vs.} \quad H_1 : \theta \neq \theta_0.
\] (1)

The existing score-based non-parametric test procedures are based on the multivariate spatial sign vector \( U \), multivariate spatial rank \( R \), and multivariate spatial signed rank \( Q \), which are defined respectively as

\[
U(y) = \begin{cases} 
\|y\|_2^{-1}y, & y \neq 0, \\
0, & y = 0,
\end{cases}
\] (2)

\[
R(y; Y) = \frac{1}{n} \sum_{i=1}^{n} U(y - Y_i),
\] (3)

\[
Q(y; Y) = \frac{1}{2} [R(y; Y) + R(y; -Y)].
\] (4)

The estimator of the location associated with spatial sign vector in (2) is the spatial median \( \hat{\theta}_n = \arg \min_{\theta \in \mathbb{R}^k} P_n \|Y - \theta\|_2 \),

where \( P_n = n^{-1} \sum_{i=1}^{n} \delta_{Y_i} \) is the empirical measure. The score functions (3) and (4) give rise to multivariate Hodges-Lehmann estimators \( \hat{\theta}_n \). One drawback of these multivariate sign and rank-based tests is that their \( p \)-values rely on a limiting chi-square distribution of the test statistics, provided the underlying distribution is elliptically symmetric (defined in Section 2). In this paper, we construct Bayesian non-parametric testing procedures for multivariate locations using the spatial median. In other words, here we focus on the score functions of type (2) and propose a non-parametric Bayesian testing procedure. Such a procedure is more attractive because it directly provides a credible set for the spatial median through quick posterior sampling, hence a testing criterion can be formulated without depending on asymptotics. We assume that the observations are drawn from a random distribution \( P \), and we put a Dirichlet process (details given in Section 3) prior on it. From \( P \), we can infer about its spatial median functional

\[
\theta(P) = \arg \min_{\theta \in \mathbb{R}^k} P(\|Y - \theta\|_2 - \|Y\|_2),
\] (6)

where \( Pf = \int f dP \). The exact posterior distribution (modulo the Monte Carlo error) of \( \theta(P) \) can be obtained easily by posterior simulation. Thus, we can form a credible region for \( \theta(P) \) and our decision is based on whether the null value \( \theta_0 \) falls into this credible set. For elliptically symmetric distributions, this testing procedure effectively studies the one-sample location problem described above, but our testing procedure can be used to study a wider
range of distributions $P$, where we study the null hypothesis $H_0 : \theta(P) = 0$. We show that our testing procedure is asymptotically non-parametric, i.e., the limiting type I error does not depend on the true distribution, and we further compute the asymptotic power function under Pitman (contiguous) alternatives along possible directions. The two-sample test has been formulated in a similar way, and its properties have been explored in a similar fashion.

The development of the asymptotic theory for the testing procedures relies on a strengthening of the theory developed in Bhattacharya and Ghosal (2020), which studies the asymptotic properties of a multivariate median, denoted by $\theta(P)$, in a non-parametric Bayesian framework. Precisely, they put a Dirichlet process prior on $P$ and proved a Bernstein-von Mises theorem for $\theta(P)$ (Theorem 3.1 of Bhattacharya and Ghosal (2020)), which we use for the derivation of the theorems in Sections 3 and 4.

The rest of this paper is organized as follows. In Section 2, we give an overview of the existing multivariate testing procedures. In Section 3 we describe our proposed Bayesian non-parametric test procedures. Section 4 gives the local asymptotic power under contiguous alternatives and Section 5 presents a simulation study. All the proofs are given in Section 6, and we close the paper with a brief discussion in Section 7.

2. Overview of existing tests

We begin this section by briefly describing the existing non-parametric testing procedures for one-sample location problems, and later move on to two-sample and several samples problems. Let $Y_1, \ldots, Y_n$ be $n$ i.i.d. observations from a $k$-variate probability distribution $P$. According to Sirkiä et al. (2007), the non-parametric testing methods can be classified as based on a multivariate spatial sign function $U$, a multivariate spatial rank $R$, and a multivariate spatial signed rank $Q$, which are defined as follows.

The test statistic based on the score function $T(Y)$, which is a general notation for the score functions described in Equations (2), (3) and (4), is given by $n^{-1} \sum_{i=1}^{n} T(Y_i)$. Under $H_0$, $n^{-1/2} \sum_{i=1}^{n} T(Y_i) \sim N_k(0, \Sigma)$. The usual estimator for $\Sigma$ is $\hat{\Sigma} = n^{-1} \sum_{i=1}^{n} T(Y_i)T(Y_i)^T$. The appropriate cut-off for constructing the test procedure depends on the assumption of elliptical symmetry of $P$ (Oja and Randles 2004). The underlying distribution is said to be elliptically symmetric if its density is of the form

$$f(y - \theta) = |\Sigma|^{-1/2} g((y - \theta)^T \Sigma^{-1}(y - \theta)),$$

with a symmetry center $\theta$, and a positive definite scatter matrix $\Sigma$. The univariate non-negative function $g(\cdot)$ satisfies the condition $\int_0^\infty u^{k/2-1} g(u)du < \infty$, so that $f$ is a valid density (Gómez Sánchez-Manzano et al 2003). The contours of these densities form concentric ellipses around the center $\theta$. Under $H_0$,

$$V^2 = n \left\| \hat{\Sigma}^{-1/2} \sum_{i=1}^{n} T(Y_i) \right\|^2 \sim \chi^2_k,$$

where $\sim$ denotes convergence in distribution, and $\chi^2_k$ denotes a chi-square distribution with $k$ degrees of freedom (Sirkiä et al. 2007). Note that $V^2$ is $n$ times the squared length of the
average standardized score vectors. For elliptically symmetric distributions, $V^2$ is strictly distribution free (Oja and Randles 2004). An approximate p-value can be obtained from the above limiting chi-square distribution. For small sample sizes, a conditional distribution-free p-value can be obtained under the assumption of directional symmetry (under which $(Y - \theta)/\|Y - \theta\|_2$ has the same distribution as $(\theta - Y)/\|\theta - Y\|_2$). This p-value can be obtained as $E_\delta[\mathbb{I}\{V^2_\delta \geq V^2\}]$, where $E_\delta$ is the expectation for the uniform distribution $\delta$ over the set of $2^k$ k-vectors with each component being $+1$ or $-1$, and $V^2_\delta$ is the value of the test statistic for the data points $\delta_1 Y_1, \ldots, \delta_n Y_n$ (Oja and Randles 2004).

The one sample testing procedure has been naturally extended to two samples. Suppose that, we have two independent random samples $Y^{(j)}_1, \ldots, Y^{(j)}_{n_j}$, from $k$-variate distributions $P(\cdot - \theta^{(j)})$, $j = 1, 2$. We test the hypothesis $H_0 : \theta^{(1)} = \theta^{(2)}$, against $H_1 : \theta^{(1)} \neq \theta^{(2)}$.

Sirkiä et al. (2007) developed a testing procedure using the general score function $T(Y)$ based on the following inner standardization approach. First, a $k \times k$ matrix $H$ and a $k$-vector have to be found such that, for $Z^{(j)}_i = H(Y^{(j)}_i - h)$, $i = 1, \ldots, n_j$, $j = 1, 2$,

$$\frac{1}{n} \sum_{j=1}^2 \sum_{i=1}^{n_j} T(Z^{(j)}_i) = 0,$$

$$\frac{k}{n} \sum_{j=1}^2 \sum_{i=1}^{n_j} T(Z^{(j)}_i)T(Z^{(j)}_i)^T = \left\{ \frac{1}{n} \sum_{j=1}^2 \sum_{i=1}^{n_j} ||T(Z^{(j)}_i)||_2^2 \right\} I_k,$$

where $n = n_1 + n_2$, and $I_k$ denotes the identity matrix of order $k \times k$. The test statistic has the form

$$V^2 = k \sum_{j=1}^2 n_j \sum_{i=1}^{n_j} ||T(Z^{(j)}_i)||_2^2 \bigg/ \frac{1}{n} \sum_{j=1}^2 \sum_{i=1}^{n_j} ||T(Z^{(j)}_i)||_2^2. \quad (7)$$

It has been shown that $V^2$ has a limiting chi-square distribution with $k$ degrees of freedom. Thus, for large samples, a p-value can be constructed using the quantiles of the chi-square distribution. For smaller samples, an approximate p-value can be obtained using a conditionally distribution-free permutation test version (Sirkiä et al. 2007), similar to $V_\delta$ in the one sample problem. This approach has been extended to a general $c$ number of samples as well.

3. Bayesian Non-parametric Tests

3.1. One-sample Problem

Suppose that, we have $n$ observations $Y_1, \ldots, Y_n$ taking values in $\mathbb{R}^k$ from a $k$-dimensional distribution $P$. We choose a non-parametric Bayesian approach, i.e., we impose a prior on the underlying random distribution $P$, and form a credible set based on the posterior distribution of the spatial-median functional

$$\theta(P) = \arg\min_{\theta \in \mathbb{R}^k} P\{||Y - \theta||_2 - ||Y||_2\}. \quad (8)$$
The hypothesis of interest is

\[ H_0 : \theta(P) = \theta_0 \quad \text{vs.} \quad H_1 : \theta(P) \neq \theta_0. \]

The most commonly used prior on \( P \) is a Dirichlet process prior with centering measure \( \beta \) \((\text{DP}(\beta))\) (see Chapter 4, Ghosal and van der Vaart (2017)). A Dirichlet process prior can be alternatively denoted as \( \text{DP}(MG) \), where \( M = |\beta| \), and \( \bar{\beta} = \beta/M \) has cumulative distribution function \( G \). The notations \( \text{DP}(\beta) \) and \( \text{DP}(MG) \) will be used interchangeably in this paper. Precisely, our chosen Bayesian model is given by

\[ Y_1, \ldots, Y_n | P \overset{iid}{\sim} P, \quad P \sim \text{DP}(MG). \] (9)

The process \( \text{DP}(\beta) \) is a conjugate prior for i.i.d. observations from \( P \), and the posterior distribution of \( \theta(P) \) cannot be obtained analytically, but posterior samples can be drawn via the stick-breaking construction of a Dirichlet process (Chapter 4, Ghosal and van der Vaart (2017)). If \( \xi_1, \xi_2, \ldots \overset{iid}{\sim} \bar{\beta} \), and \( V_1, V_2, \ldots \overset{iid}{\sim} \text{Be}(1, M) \) are independent random variables and \( W_j = V_j \prod_{l=1}^{j-1}(1-V_l) \), then \( P = \sum_{j=1}^{\infty} W_j \delta_{\xi_j} \sim \text{DP}(M\bar{\beta}) \). The posterior Dirichlet process can also be written in the form \( \text{DP}(M\bar{\beta}) \) using the updating rule

\[ M \mapsto M + n, \quad \bar{\beta} \mapsto \frac{M}{M+n} \bar{\beta} + \frac{n}{M+n} P_n. \]

The non-informative limit as \( M \to 0 \) of the posterior of \( P \), denoted by \( \text{DP}(nP_n) \), is called the Bayesian bootstrap distribution. Its centering measure is \( P_n \), and a random distribution generated from it is supported on the observation points. It has the representation \( P = \sum_{i=1}^{n} W_i \delta_{Y_i} \), where \( W_i = U_i / \sum_{j=1}^{n} U_j \), with \( U_1, \ldots, U_n \overset{iid}{\sim} \text{Exp}(1) \). If we choose the non-informative limit of the posterior Dirichlet process, we do not need to generate posterior samples from the full Dirichlet process, rather we only need to sample \( n \) independently and identically distributed (i.i.d.) observations from an exponential distribution with parameter \( 1 \), which saves a lot of computational cost. Thus, a posterior 100(1 - \( \alpha \))% credible region can be formed by the following steps.

- For \( b = 1, \ldots, B \), draw \( U_{1b}, \ldots, U_{nb} \overset{iid}{\sim} \text{Exp}(1) \). Thus, we calculate the Bayesian bootstrap weights as \( W_{ib} = U_{ib} / \sum_{j=1}^{n} U_{jb}, i = 1, \ldots, n \).

- Draw posterior samples \( \theta_{ib}, b = 1, \ldots, B \), using the expression

\[ \theta_{ib} = \arg \min_{\theta} \sum_{i=1}^{n} W_{ib} ||Y_{ib} - \theta||_2. \]

- Compute the posterior mean \( \bar{\theta} = B^{-1} \sum_{b=1}^{B} \theta_{ib} \) and the posterior covariance matrix

\[ S = B^{-1} \sum_{b=1}^{B} (\theta_{ib} - \bar{\theta})(\theta_{ib} - \bar{\theta})'. \]
• A 100(1 − α)% credible set for θ(\(P\)) is then constructed as
\[
C(Y_1, \ldots, Y_n; \alpha) = \{\theta : (\theta - \bar{\theta})^T S^{-1} (\theta - \bar{\theta}) \leq r_{1-\alpha}\},
\]
where \(r_{1-\alpha}\) is the 100(1 − α)th percentile of \((\theta_b - \bar{\theta})^T S^{-1} (\theta_b - \bar{\theta})\), \(b = 1, \ldots, B\).

• We reject \(H_0\) if \(\theta_0 \notin C(Y_1, \ldots, Y_n; \alpha)\).

The credible set considered here can be called modulo Monte Carlo error because it is constructed using simulated draws and hence is subject to the Monte Carlo error. However, the Monte Carlo error can be controlled and made arbitrarily small. Next, we investigate the asymptotic properties of this testing procedure. For all the theorems discussed here, we make the following assumptions on the underlying true distributions. Below, \(P^*\) is the general notation for the underlying distributions.

**Assumption 1.** The distribution \(P^*\) has a density that is bounded on bounded subsets of \(\mathbb{R}^k\).

**Assumption 2.** The spatial median of \(P^*\), i.e., \(\bar{\theta} = \theta(P^*)\) is unique.

**Theorem 1.** Suppose that, under the null hypothesis, \(Y \sim P_{\theta_0}\), i.e., \(\theta(P_{\theta_0}) = \theta_0\), and \(P_{\theta_0}\) satisfies Assumptions 1–2. Then, the one-sample Bayesian non-parametric test for \(H_0 : \theta(\bar{\theta}) = \theta_0\) is asymptotically nonparametric, i.e.,
\[
P_{\theta_0}(\theta_0 \in C(Y_1, \ldots, Y_n; \alpha)) \to 1 - \alpha,
\]
as \(n \to \infty\).

As we have already mentioned, the testing procedure has been constructed only using the posterior samples, without relying on any asymptotic properties. The proof of Theorem 1 is based on convergence properties of the posterior mean (\(\bar{\theta}\)) and the covariance matrix (\(S\)) of the spatial median \(\theta(\bar{\theta})\), for the Bayesian model. Let \(P^*\) be the true distribution of \(Y\), and \(\bar{\theta} \equiv \theta(P^*)\). Also define
\[
U_{\theta,P} = P \left\{ \frac{(Y - \bar{\theta})(Y - \theta)^T}{\|Y - \theta\|_2^2} \right\},
\]
\[
V_{\theta,P} = P \left\{ \frac{1}{\|Y - \theta\|_2} \left( I_k - \frac{(Y - \theta)(Y - \theta)^T}{\|Y - \theta\|_2^2} \right) \right\}.
\]

Under Assumptions 1-2 on \(P^*\), the posterior distribution of the spatial median \(\theta(\bar{\theta})\) can be approximated by a Gaussian distribution in the Bernstein-von Mises sense (Theorem 3.1 in Bhattacharya and Ghosal (2020)), i.e., given \(Y_1, \ldots, Y_n\)
\[
\sqrt{n}(\bar{\theta} - \theta) \sim N_k(0, \sqrt{m} \theta_0 - \theta_0, P_{\theta_0,P}^{-1} \theta_0 - \theta_0, \sqrt{m} P_{\theta_0,P}^{-1} \theta_0 - \theta_0, P_{\theta_0,P}^{-1}).
\]
In Lemma 1 (see below), we strengthen the conclusion of Theorem 3.1 in Bhattacharya and Ghosal (2020) to establish the convergence properties of the posterior mean and covariance matrix.

**Lemma 1.** Suppose that, the true distribution of \(Y_1, \ldots, Y_n \in \mathbb{R}^k\) is \(P^*\), and \(P^*\) satisfies Assumptions 1–2. Then, under the Bayesian model, the posterior mean \(\bar{\theta}\) and the covariance matrix \(S\) can be written as \(\bar{\theta} = \bar{\theta}_n + o_P(n^{-1/2})\) and \(nS = V_{\theta_0,P}^{-1} U_{\theta_0,P} V_{\theta_0,P}^{-1} + o_P(1)\), respectively, where \(\bar{\theta}_n\) is the sample spatial median of \(Y_1, \ldots, Y_n\).
3.2. Two Sample Problem

The Bayesian non-parametric testing procedure for two-sample location problem can be constructed generalizing the one-sample procedure. Suppose that, we have \(n_1\) i.i.d. observations \(Y_1^{(1)}, \ldots, Y_{n_1}^{(1)}\) from a \(k\)-variate distribution \(P^{(1)}\), and \(n_2\) i.i.d. observations \(Y_1^{(2)}, \ldots, Y_{n_2}^{(2)}\) from another \(k\)-variate distribution \(P^{(2)}\), independent of \(P^{(1)}\). We want to test the hypothesis

\[
H_0 : \theta(P^{(1)}) - \theta(P^{(2)}) = 0 \quad \text{against} \quad H_1 : \theta(P^{(1)}) - \theta(P^{(2)}) \neq 0.
\]

As we have previously mentioned, if \(P^{(1)} = P(\cdot - \theta^{(1)})\) and \(P^{(2)} = P(\cdot - \theta^{(2)})\) are elliptically symmetric distributions, then this problem boils down to studying the two-sample location problem

\[
H_0 : \theta^{(1)} - \theta^{(2)} = 0 \quad \text{against} \quad H_1 : \theta^{(1)} - \theta^{(2)} \neq 0.
\]

We put a \(\text{DP}(MG)\) prior on both \(P^{(1)}\) and \(P^{(2)}\), for some \(M > 0\) and \(G\), i.e.,

\[
Y_1^{(j)}, \ldots, Y_{n_j}^{(j)} \overset{iid}{\sim} P^{(j)}, \ P^{(j)} \sim \text{DP}(MG), \ j = 1, 2.
\]

Thus \(P^{(1)}\) and \(P^{(2)}\) have stick-breaking representations

\[
P^{(1)} = \sum_{m=1}^{\infty} W_m^{(1)} \xi_m^{(1)}, \ P^{(2)} = \sum_{m=1}^{\infty} W_m^{(2)} \xi_m^{(2)},
\]

respectively, where \(W_m^{(1)} = 1\), \(m = 1, 2, \ldots\), and \(W_m^{(2)} = 1\), \(m = 1, 2, \ldots\), are drawn from \(\text{Be}(1,M)\). Also, \(\xi_m^{(1)}\), \(m = 1, 2, \ldots\), and \(\xi_m^{(2)}\), \(m = 1, 2, \ldots\), are i.i.d. samples from \(G\). The posterior distribution of \(P^{(j)}\) is \(\text{DP}(\beta + \sum_{i=1}^{\infty} \delta_{\theta^{(j)}})\), \(j = 1, 2\). Like before, we consider the Bayesian bootstrap approximations of the posteriors of \(P^{(j)}\), which can be written as

\[
P^{(j)} = \sum_{m=1}^{n_j} W_m^{(j)} \delta_{\theta_m^{(j)}}, \quad \text{where} \quad W_m^{(j)} = U_m^{(j)}/\sum_{j=1}^{n_j} U_j^{(j)} \text{ with } U_1^{(j)}, \ldots, U_{n_j}^{(j)} \overset{iid}{\sim} \text{Exp}(1), j = 1, 2.
\]

We construct a \(100(1 - \alpha)\)% credible set for \(\theta(P^{(1)}) - \theta(P^{(2)})\) by the following steps.

- For \(b \in \{1, \ldots, B\}\) and \(j \in \{1, 2\}\), draw \(U_b^{(j)}, \ldots, U_{n_j}^{(j)} \overset{iid}{\sim} \text{Exp}(1)\). Calculate the Bayesian bootstrap weights as \(W_b^{(j)} = U_b^{(j)}/\sum_{j=1}^{n_j} U_j^{(j)}, j = 1, 2\).

- Draw posterior samples \(\theta_b^{(j)}\), using the expressions

\[
\hat{\theta}^{(j)} = \arg \min_{\theta} \sum_{i=1}^{n_j} W_{ib}^{(j)} \| Y_{ib}^{(j)} - \theta \|_2, \ b \in \{1, \ldots, B\}, \ j \in \{1, 2\}.
\]

- Compute posterior means \(\bar{\theta}^{(j)} = B^{-1} \sum_{b=1}^{B} \theta_b^{(j)}\) and posterior covariance matrices \(S^{(j)} = B^{-1} \sum_{b=1}^{B} (\theta_b^{(j)} - \bar{\theta}^{(j)})(\theta_b^{(j)} - \bar{\theta}^{(j)})'\), for \(j \in \{1, 2\}\).

- A \(100(1 - \alpha)\)% credible set for \(\theta(P^{(1)}) - \theta(P^{(2)})\) is then given by

\[
C(Y_1^{(1)}, \ldots, Y_{n_1}^{(1)}, Y_1^{(2)}, \ldots, Y_{n_2}^{(2)}; \alpha) = \{\theta_1 - \theta_2 : (\theta_1 - \theta_2 - \bar{\theta}^{(1)} + \bar{\theta}^{(2)})' (S^{(1)} + S^{(2)})^{-1} (\theta_1 - \theta_2 - \bar{\theta}^{(1)} + \bar{\theta}^{(2)}) \leq r_{1-\alpha}\},
\]

where \(r_{1-\alpha}\) is the \(100(1 - \alpha)\)th percentile of

\[
(\theta_b^{(1)} - \theta_b^{(2)} - \bar{\theta}^{(1)} + \bar{\theta}^{(2)})' (S^{(1)} + S^{(2)})^{-1} (\theta_b^{(1)} - \theta_b^{(2)} - \bar{\theta}^{(1)} + \bar{\theta}^{(2)}),
\]

for \(b \in \{1, \ldots, B\}\).
We reject $H_0$ if $0 \notin C(Y^{(1)}_1, \ldots, Y^{(1)}_{n_1}, Y^{(2)}_1, \ldots, Y^{(2)}_{n_2}, \alpha)$.

The next theorem shows that the above test is asymptotically non-parametric, provided the underlying true distributions satisfy Assumptions 1-2.

**Theorem 2.** Suppose that, under the null hypothesis, $Y^{(1)} \sim P^{(1)}_{\theta_0}$ and $Y^{(2)} \sim P^{(2)}_{\theta_0}$ independently, such that $\theta(P^{(1)}_{\theta_0}) = \theta(P^{(2)}_{\theta_0}) = \theta_0$ for some fixed $\theta_0 \in \mathbb{R}^k$. Let $P^{(1)}_{\theta_0}$ and $P^{(2)}_{\theta_0}$ satisfy Assumptions 1-2. Then the two-sample Bayesian non-parametric test is asymptotically non-parametric, i.e., for $0 < \alpha < 1$

$$P^{(1)}_{\theta_0} P^{(2)}_{\theta_0} (0 \in C(Y^{(1)}_1, \ldots, Y^{(1)}_{n_1}, Y^{(2)}_1, \ldots, Y^{(2)}_{n_2}, \alpha)) \to 1 - \alpha,$$

for $n_1, n_2$ such that $n_1 \to \infty$, $n_2 \to \infty$, $n_1/(n_1 + n_2) \to \lambda$, and $n_2/(n_1 + n_2) \to 1 - \lambda$, for some fixed $0 < \lambda < 1$.

To prove Theorem 2, we first need to investigate the asymptotic properties of the posterior distribution of $\theta(P^{(1)}_{\theta_0}) - \theta(P^{(2)}_{\theta_0})$. The next lemma gives a Bernstein-von Mises theorem for the difference of spatial medians for two independent samples $Y^{(1)}_1, \ldots, Y^{(1)}_{n_1} \sim P^{(1)}$, and $Y^{(2)}_1, \ldots, Y^{(2)}_{n_2} \sim P^{(2)}$ under the Bayesian model (12). The asymptotic result follows almost immediately from Theorem 3.1 in [Bhattacharya and Ghosal, 2020]. Before stating Lemma 2, we introduce a few more notations. Suppose, $P^{*}(j)$ denotes the true distribution of $Y^{(j)}$, and, $\hat{\theta}^{*}(j) = \theta(P^{*}(j))$ is the spatial median of $P^{*}(j)$, $j = 1, 2$. Next, let $\hat{\theta}^{*}_{n_j}$ denote the sample spatial median constructed from $Y^{(j)}_1, \ldots, Y^{(j)}_{n_j}$, $j = 1, 2$.

**Lemma 2.** Let $P^{*}(j)$, $j = 1, 2$, satisfies Assumptions 1–2. Then under the Bayesian model (12), and $n_1, n_2$ such that $n_1 \to \infty$, $n_2 \to \infty$, $n_1/(n_1 + n_2) \to \lambda$, and $n_2/(n_1 + n_2) \to 1 - \lambda$ for some fixed $0 < \lambda < 1$,

(i) \[ \sqrt{n}(\hat{\theta}^{*}_{n_1} - \theta^{*} - \hat{\theta}^{*}_{n_2} + \theta^{*}) \Rightarrow \mathcal{N}_k(0, \lambda^{-1}V^{-1}_{\theta^{*}(1)} U_{\theta^{*}(1), P^{*}(1)} V^{-1}_{\theta^{*}(1), P^{*}(1)}), \]

(ii) \[ \sqrt{n}(\theta(P^{(1)}_{\theta_0}) - \hat{\theta}^{*}_{n_1} + \hat{\theta}^{*}_{n_2}) \Rightarrow \mathcal{N}_k(0, \lambda^{-1}V^{-1}_{\theta^{*}(1), P^{*}(1)} U_{\theta^{*}(1), P^{*}(1)} V^{-1}_{\theta^{*}(1), P^{*}(1)}), \]

in $P^{*}(1) \times P^{*}(2)$-probability.

4. Asymptotic power Under Contiguous Alternatives

In this section, we analyze the local asymptotic power of the proposed Bayesian non-parametric tests, i.e., the limiting power under a sequence of alternatives converging to the null value. For the one-sample problem, we consider differentiable in quadratic mean (DQM)
densities \( \mathcal{P} = \{ p_\theta = dP_\theta/d\mu: \theta \in \mathbb{R}^k \} \), i.e., there exists a vector valued measurable function \( \dot{\ell}_\theta : \mathbb{R}^k \to \mathbb{R}^k \) such that, for \( h \to 0 \),

\[
\int \left( \sqrt{p_{\theta + h}} - \sqrt{p_\theta} - \frac{1}{2} h^T \dot{\ell}_\theta \sqrt{p_\theta} \right)^2 d\mu = o(h^2).
\]

We consider shrinking alternatives of the form

\[ H_{1n} : \theta = \theta_0 + \frac{h}{\sqrt{n}}, \]  

which are also called Pitman alternatives, for models \( P_\theta \in \mathcal{P} \). Here, we study the limiting power for the sequence of distributions \( P_{\theta_0 + h/\sqrt{n}} \in \mathcal{P} \). As a consequence of the DQM condition, the models \( P_{\theta_0 + h/\sqrt{n}} \) satisfy the local asymptotic normality (LAN) condition, i.e., there exist a matrix \( I_\theta \) and a random vector \( \Delta_n,\theta \xrightarrow{} N_k(0, I_\theta) \) such that, for every converging sequence \( h_n \to h \),

\[
\log \frac{dP_{\theta_0 + h_n/\sqrt{n}}}{dP_{\theta_0}} = h^T \Delta_n,\theta - \frac{1}{2} h^T I_\theta h + o_{P_\theta}(1). 
\]

In this context, specifically \( \Delta_n,\theta = n^{-1/2} \sum_{i=1}^n h^T \dot{\ell}_\theta(Y_i) \), and \( I_\theta = P_\theta \dot{\ell}_\theta \dot{\ell}_\theta^T \). The next theorem gives the limiting power for the one-sample test under a sequence of alternatives of the form \( H_{1n} \) for the DQM models.

**Theorem 3.** Suppose that, \( P_\theta_0 \) satisfies Assumptions 1–2. As \( n \to \infty \), for a sequence of shrinking alternatives of the form (14), i.e., under a sequence of DQM models \( P_{\theta_0 + h/\sqrt{n}} \in \mathcal{P} \), the limiting power of the one-sample Bayesian non-parametric test for \( H_0 : \theta(P) = \theta_0 \) is given by

\[
F_{\chi^2}(x; k, \delta_1(V_{\theta_0,P_{\theta_0}}^{-1} U_{\theta_0,P_{\theta_0}} V_{\theta_0,P_{\theta_0}}^{-1} \delta_1), \delta_1 = P_{\theta_0} \left( - V_{\theta_0,P_{\theta_0}}^{-1} \frac{Y - \theta_0}{\|Y - \theta_0\|_2} \right)^T h),
\]

and \( F_{\chi^2}(x; k, \delta) \) is the distribution function of a non-central chi-square distribution with \( k \) degrees of freedom and non-centrality parameter \( \delta \), with \( \chi^2_{k;\alpha} \) being the 100(1 – \( \alpha \))th percentile of the \( \chi^2_k \) distribution.

For the two sample problem, we again consider DQM models \( P_{\theta_0 + h_1/\sqrt{n_1}}^{(1)}, P_{\theta_0 + h_2/\sqrt{n_2}}^{(2)} \in \mathcal{P} \), i.e., the contiguous alternatives are of the form

\[ H_{1n} : \theta^{(j)}_{n_j} = \theta_0 + \frac{h_j}{\sqrt{n_j}}, \quad j = 1, 2. \]

The following theorem gives the limiting power of the two-sample test under contiguous alternatives of the form (17), and the notations from Theorem 3 directly translate to the next theorem.
Theorem 4. Suppose that, $P^{(1)}_{\theta_0}$ and $P^{(2)}_{\theta_0}$ are mutually independent, and satisfy Assumptions 1–2. For a sequence of shrinking alternatives of the form \((17)\), i.e., for a sequence of DQM models $P^{(1)}_{\theta_0+h_1/\sqrt{n_1}}$, $P^{(2)}_{\theta_0+h_2/\sqrt{n_2}} \in \mathcal{P}$, the asymptotic power of the two-sample Bayesian non-parametric test for testing $H_0 : \theta(P^{(1)}) = \theta(P^{(2)}) = \theta_0$ for any $\theta_0 \in \mathbb{R}^k$, is given by

$$F_{\chi^2}(\chi^2_{\kappa_0}; \delta_1', \delta_2') = F_{\chi^2}(\chi^2_{\kappa_0}; \hat{\delta}_1', \hat{\delta}_2'),$$

where

$$\delta_1 = \frac{1}{\sqrt{\lambda}} P^{(1)}_{\theta_0} \left( - V^{(1)}_{\theta_0} - \lambda \| Y^{(1)} - \hat{\theta}_0 \|^2 \{ \hat{\theta}_0 \}^T h_1 \right) + \frac{1}{\sqrt{1 - \lambda}} P^{(2)}_{\theta_0} \left( - V^{(2)}_{\theta_0} - \lambda \| Y^{(2)} - \hat{\theta}_0 \|^2 \{ \hat{\theta}_0 \}^T h_2 \right),$$

for $n_1, n_2$ such that $n_1 \to \infty$, $n_2 \to \infty$, $n_1/(n_1 + n_2) \to \lambda$, and $n_2/(n_1 + n_2) \to 1 - \lambda$, for some fixed $0 < \lambda < 1$.

5. Simulation Study

We conduct a simulation study to demonstrate the finite sample performance of the proposed one-sample and two-sample Bayesian non-parametric tests, for $k = 2$ and $k = 10$. The choices $k = 2$ and $k = 10$ will help us visualize the power of the tests in relatively lower and higher dimensional scenarios. We compare our tests with the Hotelling’s $T^2$-test, and the spatial sign and rank tests, discussed in Section 2. Recall that, for testing $H_0 : \theta = 0$, we can denote the spatial sign and rank statistics by the general notation

$$V^2 = n \left\| \hat{\Sigma}^{-1/2} \frac{1}{n} \sum_{i=1}^n T(Y_i) \right\|^2,$$

with $T(y) = U(y) = y/\|y\|_2$ for the sign test, and $T(y) = R(y) = n^{-1} \sum_{i=1}^n U(y - Y_i)$, for the rank test. For the two-sample test, the sign and rank statistics are given by \((17)\). The underlying distributions are $k$-variate Gaussian, $k$-variate $t$ with 1 degree of freedom (both elliptically symmetric), and $k$-variate gamma (asymmetric), for $k = 2$ and $k = 10$. For $k = 2$, the covariance and the scale matrices for the normal and the $t$ distributions respectively, have been chosen to be the $k \times k$ identity matrix, denoted by $I_k$. Since correlation structure plays an important role in higher dimensions, for $k = 10$, we choose the covariance and scale matrix to be $\Sigma$ such that $\Sigma_{ij} = 1$ for $i = j$, and 0.7 otherwise, for $i, j \in \{1, \ldots, k\}$.

The $k$-variate gamma distribution is constructed using Gaussian copula [Xue-Kun Song 2000]. To describe the construction briefly, let $Y_1, \ldots, Y_k$ be $k$ many univariate gamma random variables $Ga(s, r)$ with distribution functions and density functions being denoted by $F_j$ and $f_j$, $j = 1, \ldots, k$. Then the joint density of $Y = (Y_1, \ldots, Y_k)$ is given by

$$g(y, s, r, V) = c_\phi \{ F_1, \ldots, F_k | V \} \prod_{j=1}^k f_j(y_j, s, r),$$
where \( c_\phi ( \cdot | V ) \) denotes the density of the \( k \)-dimensional Gaussian copula. For \( k = 2 \), we choose \( s = (3,3)^T \), \( r = (1,1)^T \), and \( V \) such that \( V_{11} = V_{22} = 1 \), and \( V_{12} = 0.5 \). For \( k = 10 \), we choose \( s = r = 1_{10} \) and \( V \) such that \( V_{11} = V_{22} = 1 \), and \( V_{12} = 0.7 \), where \( c_k \) denotes the vector of all \( c \)'s of length \( k \). The usual Hotelling’s \( T^2 \) statistic is based on the assumption of Gaussianity. Hence for comparison, here we choose a general version of Hotelling’s \( T^2 \)-test, where the Gaussian assumption can be relaxed to existence of second moments (Ito 1956). For the general Hotelling’s \( T^2 \)-test, the \( p \)-value is based on a chi-square approximation instead of the usual \( F \)-distribution. For the one-sample test, we consider a sample size of \( n = 100 \). For the two-sample test, we choose \( n_1 = 100 \), \( n_2 = 90 \) for \( k = 2 \), and \( n_1 = 100 \), \( n_2 = 60 \) for \( k = 10 \), to evaluate the performance of the tests where the data sizes are relatively unbalanced. The credible sets are constructed using 5000 posterior draws, and the power is calculated as the proportion of times the null hypotheses are rejected off 2000 replications. The location parameters are chosen suitably to show a good range of powers.

Tables 1 and 2 show the power values for \( k = 2 \), and it can be noted that our test procedures attain the nominal level 0.05, and outperforms all other procedures in most scenarios. When the underlying distributions are not Gaussian, our method performs better than other methods, especially compared with the Hotelling’s \( T^2 \) -test. Note that for the bivariate gamma distribution, the powers for the Hotelling’s \( T^2 \)-test in Table 1 are larger, which may lead us to believe that it performs better compared to the other procedures. However, table 1 shows that the corresponding sizes are also large.

Tables 3 and 4 demonstrate the power for \( k = 10 \), and the Bayesian nonparametric tests perform relatively well here as well. However, the computation of the 10-dimensional spatial median for each posterior distribution is somewhat expensive. Therefore, construction of the credible region takes a significantly longer time for the 10-dimensional scenario. Since the posterior contraction rate for \( \theta (P) \) remains \( n^{-1/2} \) for any finite dimension, the testing procedure does not suffer from the curse of dimensionality.

**Remark 1.** Here we have considered tests for multivariate locations based on spatial medians, but these tests can be constructed using multivariate \( \ell_1 \)-medians (with \( \ell_p \)-norms) as well. For some fixed \( p > 1 \), the \( \ell_1 \)-median for a \( k \)-variate distribution \( P \) can be defined as

\[
\theta_p (P) = \arg \min_{\theta \in \mathbb{R}^k} P \{ \| Y - \theta \|_p - \| \theta \|_p \}.
\]

Bernstein-von Mises theorems of \( \ell_1 \)-medians are available in the literature (Bhattacharya and Ghosal 2020). Hence the expressions for local asymptotic powers under shrinking alternatives can be obtained using those theorems.

**Remark 2.** One may argue that, the Hotelling’s \( T^2 \)-test is designed for testing the mean vector, and hence is unsuitable as a competing method for nonparametric tests based on medians. However, people use Hotelling’s \( T^2 \)-statistic as a testing procedure for the location of a distribution, under the assumption of normality, for which the center of symmetry is same as both mean and median. Naturally, this assumption gets violated for non-normal distributions, for which nonparametric tests are more suitable.
\begin{tabular}{|c|c|c|c|c|}
\hline
\boldmath$\theta$ & NPBayes & Sign Test & Rank Test & Hotelling’s $T^2$ \\
\hline
Bivariate Gaussian Distribution & & & & \\
\hline
(0, 0) & 0.050 & 0.046 & 0.051 & 0.055 \\
(0.05, 0.05) & 0.139 & 0.086 & 0.084 & 0.099 \\
(0.1, 0.05) & 0.169 & 0.125 & 0.141 & 0.156 \\
(0.1, -0.1) & 0.221 & 0.188 & 0.213 & 0.234 \\
\hline
Bivariate t$_1$ Distribution & & & & \\
\hline
(0, 0) & 0.054 & 0.053 & 0.041 & 0.020 \\
(0.05, 0.05) & 0.174 & 0.058 & 0.053 & 0.025 \\
(0.1, 0.05) & 0.179 & 0.094 & 0.082 & 0.018 \\
(0.1, -0.1) & 0.201 & 0.171 & 0.197 & 0.026 \\
\hline
Bivariate Gamma Distribution & & & & \\
\hline
(0, 0) & 0.049 & 0.016 & 0.025 & 0.294 \\
(0.05, 0.05) & 0.027 & 0.021 & 0.039 & 0.528 \\
(0.1, 0.05) & 0.029 & 0.013 & 0.058 & 0.607 \\
(0.1, -0.1) & 0.034 & 0.009 & 0.018 & 0.255 \\
\hline
\end{tabular}

Table 1: Power for testing $H_0: \theta(P) = 0$ for bivariate Gaussian, bivariate t (with 1 degree of freedom), and bivariate gamma distributions with different location parameters ($\theta$), using the nonparametric Bayes (NPBayes) test, spatial sign test, spatial rank test and the Hotelling’s $T^2$-test.

6. Proofs

We start off this section with the proofs of Lemma 1 and 2, and then proceed with the proofs of the main theorems.

Proof of Lemma 1 Define $\theta(\mathbb{B}_n) = \arg\min_{\theta} \mathbb{B}_n \|Y - \theta\|_2$, where $\mathbb{B}_n = DP(n\mathbb{P}_n)$ is the Bayesian bootstrap distribution. It has been shown in Lemma 1 in Bhattacharya and Ghosal (2020) that, asymptotically, $\theta(P)$ is a Bayesian bootstrapped analog of a $Z$-estimator, which implies that, asymptotically, the posterior distribution of $\theta(P)$ is the same as the conditional distribution of $\theta(\mathbb{B}_n)$. Thus, our problem boils down to showing the consistency of the first and second moments of the bootstrap $Z$-estimator $\theta(\mathbb{B}_n)$.

Cheng (2015) proved the consistency of the bootstrap moment estimators for the class of exchangeably weighted bootstrap (see Section 2.2, Cheng (2015)). The Bayesian bootstrap weights fall into the class of the exchangeable bootstrap weights, and we have to show that the $\ell_1$-criterion function $m_\theta(y) = -\|y - \theta\|_2 + \|y\|_2$ satisfies the following two sufficient conditions. Let $G_n = \sqrt{m}(\mathbb{P}_n - \mathbb{P}^*)$ denotes the empirical process and $G^*_n = \sqrt{m}(\mathbb{B}_n - \mathbb{P}_n)$ denotes the bootstrap empirical process. Suppose that the following conditions hold.

1. Let $\Theta$ be the compact parameter space. For any $\theta \in \Theta$,

$$P^*(m_\theta - m_{\theta^*}) \precsim -\|\theta - \theta^*\|_2^2.$$
2. Define $N_\delta = \{ m_\theta - m_\theta_0 : \| \theta - \theta_0 \|_2 \leq \delta \}$. We have to show

$$
\left( E_X \| G_n \|_{N_\delta}^{p'} \right)^{1/p'} \lesssim \delta
$$

(19)

$$
\left( E_{XW} \| G_n^* \|_{N_\delta}^{p'} \right)^{1/p'} \lesssim \delta,
$$

(20)

for some $p' > 2$.

Then the assertion in Lemma 1 holds. First, we need to show that the parameter space can be restricted to a compact subset of $\mathbb{R}^k$ with high probability. In Lemma 2 of Bhattacharya and Ghosal (2020), it has been shown that for some $0 < \epsilon < 1/4$ and $K > 0$ such that $P(\|Y\|_2 \leq K) > 1 - \epsilon$, given $Y_1, \ldots, Y_n, \| \theta(P) \|_2 \leq 3K$ with high $P^*\text{-}n$-probability, which implies that asymptotically, given $Y_1, \ldots, Y_n, \| \theta(P) \|_2 \leq 3K$ with high $P^*\text{-}n$-probability.

After fixing $K > 0$, we choose $\Theta = \{ \theta : \| \theta \|_2 \leq 3K \}$. Since $\Theta$ is compact, Condition 1 can be shown from a Taylor series expansion around $\theta^*$,

$$
P^*m_\theta - P^*m_\theta^* = (\theta - \theta^*)' P^* \dot{m}_{\theta^*} \frac{(\theta - \theta^*)' V_{\theta^*} P^* (\theta - \theta^*)}{2} + o(\| \theta - \theta^* \|^2).
$$

(21)

Since $\theta^*$ is the maximizer of $P^*m_\theta$, $P^* \dot{m}_{\theta^*}$ vanishes. The matrix $V_{\theta^*} P^*$ is negative definite, and hence, the second term in the right hand side of (21) is bounded above by $-c\| \theta - \theta^* \|^2$, for a positive constant $c$. 

Table 2: Power for testing $H_0 : \theta^{(1)} = \theta^{(2)}$ for bivariate Gaussian, bivariate $t$ (with 1 degree of freedom), and bivariate gamma distributions with different location parameters ($\theta^{(1)}$ and $\theta^{(2)}$), using the nonparametric Bayes (NPBayes) test, spatial sign test, spatial rank test and the Hotelling’s $T^2$-test.
| $\theta$ | NPBayes | Sign Test | Rank Test | Hotelling’s $T^2$ |
|---|---|---|---|---|
| | 10-variate Gaussian Distribution | | | |
| $v_0$ | 0.051 | 0.039 | 0.089 | 0.115 |
| $v_1$ | 0.192 | 0.171 | 0.115 | 0.560 |
| $v_2$ | 0.284 | 0.271 | 0.283 | 0.330 |
| | 10-variate $t_1$ Distribution | | | |
| $v_0$ | 0.062 | 0.029 | 0.048 | 0.048 |
| $v_1$ | 0.137 | 0.112 | 0.114 | 0.072 |
| $v_2$ | 0.348 | 0.340 | 0.332 | 0.138 |
| | 10-variate Gamma Distribution | | | |
| $v_0$ | 0.071 | 0.062 | 0.101 | 0.173 |
| $v_1$ | 0.262 | 0.122 | 0.358 | 0.925 |
| $v_2$ | 0.093 | 0.064 | 0.091 | 0.219 |

Table 3: Power for testing $H_0 : \theta(P) = v_0$ for 10-variate Gaussian, 10-variate $t$ (with 1 degree of freedom), and 10-variate gamma distributions for different location parameters $\theta = v_0, v_1, v_2$, with $v_0 = (0,0,0,0,0,0,0,0,0,0)^T$, $v_1 = (0,0,0,0,0,0,1,0,1,0,1)^T$, and $v_2 = (0,0,0,0,0,1,0,1,0,1,-0.1,-0.1)^T$, using the nonparametric Bayes (NPBayes) test, spatial sign test, spatial rank test and the Hotelling’s $T^2$-test.

Before proving Condition 2, we introduce some notations. For any class of functions $A$, and metric $\ell$, its $\epsilon$-bracketing number is denoted as $N_{||}(\epsilon, A, \ell)$. The corresponding bracketing entropy integral is defined as

$$J_{||}(\epsilon, A, \ell) = \int_0^\delta \sqrt{1 + \log N_{||}(\epsilon, A, \ell)} d\epsilon.$$ 

Following Cheng (2015), a simple sufficient condition for (19) is the following global Lipschitz condition

$$|m_\theta(x) - m_{\theta^*}(x)| \leq \|\theta - \theta^*\|_2,$$

for any $\theta \in \Theta$, and

$$J_{||}(1, N_\delta, L_2(P^*)) + \|M\|_{\psi_{p'}} < \infty,$$  

for some $p' > 2$, where $\|\cdot\|_{\psi_{p'}}$ is the Orlicz norm with respect to the convex function $\psi_p(t) = \exp(t^p - 1)$. In our case, (22) holds by the triangle inequality $|m_\theta(y) - m_{\theta^*}(y)| \leq \|\theta - \theta^*\|_2$. Since $M(y) = 1$ for every $y$, we just have to show that $J_{||}(1, N_\delta, L_2(P^*)) < \infty$.

By Example 19.7 of Van der Vaart (1998), since $|m_\theta(y) - m_{\theta^*}(y)| \leq \|\theta - \theta^*\|_2$, for every $\theta, \theta' \in \Theta$, there exists a constant $K$ such that

$$N_{||}(1, N_\delta, L_2(P^*)) \leq \left(\frac{\text{diam } \Theta}{\epsilon}\right)^k,$$

for every $0 < \epsilon < \text{diam } \Theta$.

Then, the entropy is of the order $\log(1/\epsilon)$. By a change of variable, it can be shown that $J_{||}(1, N_\delta, L_2(P^*)) < \infty$. 


| $\theta^{(1)}$ | $\theta^{(2)}$ | NPBayes | Sign Test | Rank Test | Hotelling’s $T^2$ |
|----------------|----------------|--------|-----------|-----------|-----------------|
| $v_0$ | $v_0$ | 0.054 | 0.046 | 0.038 | 0.048 |
| $v_0$ | $v_1$ | 0.172 | 0.087 | 0.109 | 0.138 |
| $v_0$ | $v_2$ | 0.101 | 0.150 | 0.080 | 0.136 |

10-variate Gaussian Distribution
| $v_0$ | $v_0$ | 0.059 | 0.056 | 0.036 | 0.034 |
| $v_0$ | $v_1$ | 0.071 | 0.076 | 0.092 | 0.045 |
| $v_0$ | $v_2$ | 0.074 | 0.075 | 0.072 | 0.039 |

10-variate $t_1$ Distribution
| $v_0$ | $v_0$ | 0.051 | 0.036 | 0.023 | 0.057 |
| $v_0$ | $v_1$ | 0.054 | 0.051 | 0.045 | 0.070 |
| $v_0$ | $v_2$ | 0.132 | 0.064 | 0.125 | 0.037 |

10-variate Gamma Distribution

Table 4: Power for testing $H_0: \theta^{(1)} = \theta^{(2)}$ for 10-variate Gaussian, 10-variate $t$ (with 1 degree of freedom), and 10-variate gamma distributions with different location parameters $v_0 = (0,0,0,0,0,0,0,0,0,0)^T$, $v_1 = (0,0,0,0,0,1,0,1,0,1,0,0,0,0)^T$ and $v_2 = (0,0,0,0,0,0,1,0,1,0,1,1,-0.1,-0.1)^T$, using the nonparametric Bayes (NPBayes) test, spatial sign test, spatial rank test and the Hotelling’s $T^2$-test.

**Proof of Lemma** From Theorem 3.1 of [Bhattacharya and Ghosal (2020)](2020), for $j = 1, 2$,

(i) $\sqrt{n}((\hat{\theta}_{n,j}^{(j)} - \theta^{*}(j))) \sim N_k(0, V_{\theta^{*}(j), p^{*}(j)}^{-1} U_{\theta^{*}(j), p^{*}(j)} V_{\theta^{*}(j), p^{*}(j)}^{-1})$,

(ii) Given $Y_{1}^{(j)}, \ldots, Y_{n_{j}}^{(j)}$,

$\sqrt{n}((\theta(P^{(j)}) - \hat{\theta}_{n,j}^{(j)})) \sim N_k(0, V_{\theta^{*}(j), p^{*}(j)}^{-1} U_{\theta^{*}(j), p^{*}(j)} V_{\theta^{*}(j), p^{*}(j)}^{-1})$,

in $P^{*}(j)$ probability. From the independence of the two samples, the conclusion follows. □

**Proof of Theorem** The probability of accepting the null hypothesis under the null distribution is

$P_{\theta_0}(\theta_0 \in C(Y_1, \ldots, Y_n)) = P_{\theta_0}((\hat{\theta} - \theta_0)^T S^{-1} (\hat{\theta} - \theta_0) \leq r_{1-\alpha})$.

Using Lemma [1] and Theorem 3.1 of [Bhattacharya and Ghosal (2020)](2020),

$n(\hat{\theta} - \theta_0)^T S^{-1} (\hat{\theta} - \theta_0) \sim \chi^2_k,$  \hspace{1cm} (24)

which implies that, under $H_0$, $r_{1-\alpha} = \chi^2_{k,\alpha} + o_{P_{\theta_0}}(1)$. The weak convergence in (24) uses the fact that if $X \sim N_k(0, I_k)$, then $X^T X \sim \chi^2_k$. Next, again using Lemma [1] Theorem 3.1 of [Bhattacharya and Ghosal (2020)](2020) and Slutsky’s theorem,

$P_{\theta_0}(\theta_0 \in C(Y_1, \ldots, Y_n)) = P_{\theta_0}((\hat{\theta} - \theta_0)^T S^{-1} (\hat{\theta} - \theta_0) \leq r_{1-\alpha})$

$= P_{\theta_0}((\hat{\theta}_n - \theta_0)^T (V_{\theta_0, p_0} U_{\theta_0, p_0} V_{\theta_0, p_0}^{-1})^{-1} (\hat{\theta}_n - \theta_0) + o_{P_{\theta_0}}(1) \leq \chi^2_{k,\alpha} + o_{P_{\theta_0}}(1))$

$\to 1 - \alpha$.  

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Proof of Theorem 2 The proof is similar to that of Theorem 1. Using Lemma 1 under $H_0$, $\sqrt{n} (\hat{\theta}_1 - \hat{\theta}_2)$ converges to a Gaussian distribution with mean 0 and covariance matrix $\lambda^{-1} V^{-1}_{\theta_0, P^{(1)}_{\theta_0}} U_{\theta_0, P^{(1)}_{\theta_0}} V^{-1}_{\theta_0, P^{(1)}_{\theta_0}} + (1 - \lambda)^{-1} V^{-1}_{\theta_0, P^{(2)}_{\theta_0}} U_{\theta_0, P^{(2)}_{\theta_0}} V^{-1}_{\theta_0, P^{(2)}_{\theta_0}}$. Using Lemma 1, Lemma 2 and Slutsky’s theorem,

$$n(\hat{\theta}^{(1)} - \hat{\theta}^{(2)})' (S^{(1)} + S^{(2)})^{-1} (\bar{\theta}^{(1)} - \bar{\theta}^{(2)}) \sim \chi^2_k,$$

which implies that, under $H_0$, $r_{1-\alpha} = \chi^2_{k; \alpha} + o_{P_{\theta_0}}(1) + o_{P_{\theta_0}}(1)$. Next, using Lemma 2 under $H_0$,

$$(P_{\theta_0}^{(1)} \times P_{\theta_0}^{(2)}) [ (\bar{\theta}^{(1)} - \bar{\theta}^{(2)})' (S^{(1)} + S^{(2)})^{-1} (\bar{\theta}^{(1)} - \bar{\theta}^{(2)}) ] \leq r_{1-\alpha}$$

$$= (P_{\theta_0}^{(1)} \times P_{\theta_0}^{(2)}) (n(\hat{\theta}_1 - \hat{\theta}_2)' \frac{1}{\lambda} V^{-1}_{\theta_0, P^{(1)}_{\theta_0}} U_{\theta_0, P^{(1)}_{\theta_0}} V^{-1}_{\theta_0, P^{(1)}_{\theta_0}} + \frac{1}{1 - \lambda} V^{-1}_{\theta_0, P^{(2)}_{\theta_0}} U_{\theta_0, P^{(2)}_{\theta_0}} V^{-1}_{\theta_0, P^{(2)}_{\theta_0}}) (\hat{\theta}_1 - \hat{\theta}_2) + o_{P_{\theta_0}^{(1)}}(1) + o_{P_{\theta_0}^{(2)}}(1)$$

$$\leq \chi^2_{k; \alpha} + o_{P_{\theta_0}^{(1)}}(1) + o_{P_{\theta_0}^{(2)}}(1) \rightarrow 1 - \alpha.$$

Proof of Theorem 3 It is well known that the models $P_{\theta_0}^n$ and $P_{\theta_0 + h/\sqrt{n}}^n$ are mutually contiguous for DQM models (Example 6.5, Van der Vaart 1998). Under $H_0$, using Theorem 5.23 of Van der Vaart (1998), $\hat{\theta}_n$ can be written as

$$\sqrt{n} (\hat{\theta}_n - \theta_0) = -\frac{1}{\sqrt{n}} \sum_{i=1}^{n} V^{-1}_{\theta_0, P_0} Y_i - \bar{\theta}_0 + o_{P_{\theta_0}}(1).$$

(25)

Let $L_n$ denote the log-likelihood ratio $L_n = \log(dP_{\theta_0}^n/dP_{\theta_0}^n)$. Using Equation (15), $L_n$ is written as

$$L_n = \frac{1}{\sqrt{n}} h^T \sum_{i=1}^{n} \hat{\theta}_0(Y_i) - \frac{1}{2} h^T I_{\theta_0} h + o_{P_{\theta_0}^n}(1).$$

(26)

Putting together Equations (25) and (26), we have

$$(\sqrt{n} (\hat{\theta}_n - \theta_0), L_n) = \left( -\frac{1}{\sqrt{n}} \sum_{i=1}^{n} V^{-1}_{\theta_0, P_0} Y_i - \bar{\theta}_0, \frac{1}{\sqrt{n}} h^T \sum_{i=1}^{n} \hat{\theta}_0(Y_i) - \frac{1}{2} h^T I_{\theta_0} h \right) + o_{P_{\theta_0}^n}(1).$$

By the central limit theorem, $(\sqrt{n} (\hat{\theta}_n - \theta_0), L_n)$ tends to a $(k + 1)$-dimensional Gaussian distribution with mean zero and covariance

$$\delta_1 = P_{\theta_0} \left( -V^{-1}_{\theta_0, P_0} \frac{1}{2} h^T I_{\theta_0} h \{ \hat{\theta}_0(Y) \}^T h \right).$$
Then by Le Cam’s third lemma (Example 6.7, Van der Vaart (1998)), under \( P_{\theta_0 + h/\sqrt{n}} \), \( \sqrt{n}(\hat{\theta}_n - \theta_0) \) converges weakly to a Gaussian distribution with mean \( \delta_1 \). Following the arguments used in Theorem 1, the local asymptotic power of the test is given by

\[
P_{\theta_0 + h/\sqrt{n}}((\hat{\theta} - \theta_0)')S^{-1}(\hat{\theta} - \theta_0) \leq r_1^{-\alpha} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n_1} V_i^{-1}(\theta_0, P_{\theta_0}) Y_i(1) - \theta_0 + o_{P_{\theta_0}}(1),
\]

Under \( P_{\theta_0 + h/\sqrt{n}} \), \( n(\hat{\theta}_n - \theta_0)'(V_{\theta_0, P_{\theta_0}} U_{\theta_0, P_{\theta_0}} V_{\theta_0, P_{\theta_0}})^{-1}(\hat{\theta}_n - \theta_0) \) tends to a chi-square distribution with the non-centrality parameter \( \delta_1 V_{\theta_0, P_{\theta_0}} U_{\theta_0, P_{\theta_0}} V_{\theta_0, P_{\theta_0}}^{-1} \delta_1 \), which gives us the asymptotic power given in the statement of Theorem 3.

**Proof of Theorem 4.** The proof proceeds along the lines of Theorem 3. The models \( \{P_{\theta_0}^{(1)} \times P_{\theta_0}^{(2)}\} \) and \( \{P_{\theta_0 + h_1/\sqrt{n}}^{(1)} \times P_{\theta_0 + h_2/\sqrt{n}}^{(2)}\} \) are mutually contiguous. From Theorem 5.23 of Van der Vaart (1998), sample spatial medians have the following linearizations,

\[
\sqrt{n_1}(\hat{\theta}_{n_1}^{(1)} - \theta_0) = -\frac{1}{\sqrt{n_1}} \sum_{i=1}^{n_1} V_{i}^{-1}(\theta_0, P_{\theta_0}^{(1)}) Y_{i}(1) - \theta_0 + o_{P_{\theta_0}}(1), \tag{27}
\]
\[
\sqrt{n_2}(\hat{\theta}_{n_2}^{(2)} - \theta_0) = -\frac{1}{\sqrt{n_2}} \sum_{i=1}^{n_2} V_{i}^{-1}(\theta_0, P_{\theta_0}^{(2)}) Y_{i}(2) - \theta_0 + o_{P_{\theta_0}}(1). \tag{28}
\]

Subtracting (28) from (27), under \( H_0 \),

\[
\sqrt{n}(\hat{\theta}_{n_1}^{(1)} - \hat{\theta}_{n_2}^{(2)}) = -\left\{ \frac{1}{\sqrt{n_1 \lambda}} \sum_{i=1}^{n_1} V_{i}^{-1}(\theta_0, P_{\theta_0}^{(1)}) Y_{i}(1) - \theta_0 \right\}
- \frac{1}{\sqrt{n_2(1 - \lambda)}} \sum_{i=1}^{n_2} V_{i}^{-1}(\theta_0, P_{\theta_0}^{(2)}) Y_{i}(2) - \theta_0 + o_{P_{\theta_0}}(1) \tag{29}.\]

Define the log-likelihood ratio \( L'_{n_1, n_2} = \log(dP_{\theta_0 + h_1/\sqrt{n}}^{(1)} dP_{\theta_0 + h_2/\sqrt{n}}^{(2)} / dP_{\theta_0}^{(1)} dP_{\theta_0}^{(2)}) \), which looks like

\[
L'_{n_1, n_2} = \frac{1}{\sqrt{n_1}} h_1^T \sum_{i=1}^{n_1} \dot{Y}_{i}^{(1)}(Y_{i}^{(1)}) - \frac{1}{2} h_1^T \dot{\theta}_{\theta_0}^{(1)} h_1 + \frac{1}{\sqrt{n_2}} h_2^T \sum_{i=1}^{n_2} \dot{Y}_{i}^{(2)}(Y_{i}^{(2)}) - \frac{1}{2} h_2^T \dot{\theta}_{\theta_0}^{(2)} h_2 + o_{P_{\theta_0}}(1).
\]

By central limit theorem, the joint distribution of \( \sqrt{n}(\hat{\theta}_{n_1}^{(1)} - \hat{\theta}_{n_2}^{(2)}) \) and \( L'_{n_1, n_2} \) tends to a \((k + 1)\)-dimensional Gaussian distribution with mean \( \delta_2 \) and covariance

\[
\delta_2 = \frac{1}{\sqrt{\lambda}} P_{\theta_0}^{(1)} \left( -V_{\theta_0, P_{\theta_0}^{(1)}}^{-1} \|Y(1) - \theta_0\|_2 \{\dot{Y}_{\theta_0}^{(1)}(Y(1))^T h_1\} + \frac{1}{1 - \lambda} P_{\theta_0}^{(2)} \left( -V_{\theta_0, P_{\theta_0}^{(2)}}^{-1} \|Y(2) - \theta_0\|_2 \{\dot{Y}_{\theta_0}^{(2)}(Y(2))^T h_2\} \right) \right).
\]
By Le Cam’s third lemma, under $P_{\theta_0+h_1/\sqrt{n_1}}^{(1)} \times P_{\theta_0+h_2/\sqrt{n_2}}^{(2)}$, $\sqrt{n}(\hat{\theta}_n^{(1)} - \hat{\theta}_n^{(2)})$ converges weakly to a Gaussian distribution with mean $\delta_2$. Thus following the arguments used in Theorem 2, the asymptotic power is given by

$$P_{\theta_0+h_1/\sqrt{n_1}}^{(1)} \times P_{\theta_0+h_2/\sqrt{n_2}}^{(2)} \left\{ (\hat{\theta}_n^{(1)} - \hat{\theta}_n^{(2)})'(S^{(1)} + S^{(2)})^{-1}(\hat{\theta}_n^{(1)} - \hat{\theta}_n^{(2)}) \leq r_{1-\alpha} \right\}$$

$$= P_{\theta_0+h_1/\sqrt{n_1}}^{(1)} \times P_{\theta_0+h_2/\sqrt{n_2}}^{(2)} \left\{ n(\hat{\theta}_n^{(1)} - \hat{\theta}_n^{(2)})'(\lambda^{-1}V^{-1}_{\theta_0,P_{\theta_0}^{(1)}}U_{\theta_0,P_{\theta_0}^{(1)}}V^{-1}_{\theta_0,P_{\theta_0}^{(1)}})^{-1}(\hat{\theta}_n^{(1)} - \hat{\theta}_n^{(2)}) + o_{\theta_0,P_{\theta_0}^{(1)}}(1) + o_{\theta_0,P_{\theta_0}^{(2)}}(1) \right\} \leq \chi^2_{\alpha} + o_{\theta_0,P_{\theta_0}^{(1)}}(1) + o_{\theta_0,P_{\theta_0}^{(2)}}(1).$$

Therefore under $P_{\theta_0+h_1/\sqrt{n_1}}^{(1)} \times P_{\theta_0+h_2/\sqrt{n_2}}^{(2)}$,

$$n(\hat{\theta}_n^{(1)} - \hat{\theta}_n^{(2)})'(\lambda^{-1}V^{-1}_{\theta_0,P_{\theta_0}^{(1)}}U_{\theta_0,P_{\theta_0}^{(1)}}V^{-1}_{\theta_0,P_{\theta_0}^{(1)}})^{-1}(\hat{\theta}_n^{(1)} - \hat{\theta}_n^{(2)})$$

tends to a non-central chi-square distribution with the non-centrality parameter $\delta_2^2(\lambda^{-1}V^{-1}_{\theta_0,P_{\theta_0}^{(1)}}U_{\theta_0,P_{\theta_0}^{(1)}}V^{-1}_{\theta_0,P_{\theta_0}^{(1)}})^{-1}(\hat{\theta}_n^{(1)} - \hat{\theta}_n^{(2)})$, which gives the asymptotic power. 

7. Discussion

The nonparametric Bayesian tests constructed here are interesting alternatives to the classical tests for multivariate location, because they are easy to construct and do not require asymptotics to set their critical values. Also, unlike the classical tests, our methods do not need the assumption of elliptical symmetry to determine the rejection criterion.

Theorems 1 and 2 show that the tests are asymptotically nonparametric, which means that the type I errors do not depend on the true distribution for large samples. Also, Theorems 3 and 4 investigate the asymptotic power under shrinking alternatives, which are motivated by arguing that a testing procedure should be powerful at values close to the truth. Besides having useful asymptotic properties, our tests are computationally simple and efficient as well. The Bayesian bootstrap approximation to the posterior Dirichlet process is quite useful since the credible regions can be constructed using simple Monte Carlo methods. We have investigated the performance of the tests in both relatively lower and higher dimensional settings, for balanced and unbalanced data sizes (for the two-sample test), and for correlated data, and the tests exhibit reasonable power in all scenarios.

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