Thermo-Mechanical Analysis Using a Multiphysics Approach

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Abstract. In the paper the Cell Method, a discrete method for solving partial differential equations, is applied to a time dependent thermo-mechanical problem. The basic equations are developed with a multiphysics approach and results, both in two-dimensional and three-dimensional models, are presented. By means of the comparison with finite element approach, some advantages of the proposed methodology are highlighted, in particular quick model construction, capability to separate thermal strain from the mechanical one and, as a consequence, the capability to model the strain and stress evolution in a time dependent problem, considering possible mutual effects between thermal and mechanical fields.

1. Introduction

Nowadays, more and more applications have to be studied by means of multiphysics approaches, because they are characterised by the interaction of different physical phenomena. An example is the Thermo-Mechanical Fatigue (TMF) where thermal field effects add up to the effects due to mechanical actions given by gravitational or surface forces. Usually, the two problems are separately studied, using two different models, and then combined assuming the superposition principle. This methodology is effective if the non-linearities of the material are not considered, but the residual life assessment of a component undergone TMF is strictly linked to the evaluation of the plastic strain accumulation in the \textit{hot points} of the component. As a consequence, the principle of effects superposition shows its limits, it is then necessary to follow step by step the time evolution of the mechanical and thermal effects and their interactions.

A collaboration between the Department of Electrical Engineering (Dipartimento di Ingegneria Elettrica) and the Department of Mechanics (Dipartimento di Meccanica) of Politecnico di Torino is developing a Matlab based code (named DualLab) that allows the study of several multiphysics problems using the discrete formulation of the physical laws given by Cell Method (CM) approach. A portion of the DualLab code is dedicated to the solution of thermo-mechanical problems, where it is possible to apply in the same time and on the same model the mechanical and thermal boundary conditions.
2. Integrated approach to the thermo-mechanical problem

2.1. Discrete approach to the Physical Laws

The most important contribution provided by the theoretical analysis made by Tonti [1, 2, 3] is the rigorous classification of physical laws. This approach allows the underlining of analogies and common structures of the different physical laws. At the basis of Tonti’s analysis there is the classification of the temporal and geometric structures on which problem variables are defined. Such variables do not present a variation on their own domain, as in the usual differential approach, but they are associated to \( n \)-dimensional elements of the space and time, according to the physical nature of the variable. For example, in the mechanical problem, Tonti’s classification makes reference to the surface forces (expressed in newtons) instead of pressures (expressed in pascals). Variables of this kind are called global in order to distinguish them from the variables linked to density of field quantities that present a punctual variation. The definition of surface forces as a global variable allows its direct use in the equilibrium equation (topological law), without using integral operator acting on the surface. This is the fundamental consideration that is the basis of the Algebraic Formulation of the field problem, because the problem domain is geometrically discretized. As a consequence of the Algebraic Formulation, the applicability of this approach to the solution of a physical problem is easy to implement in numerical solution code.

Referring to [1] for a detailed presentation of the technique, here only its main characteristics are outlined:

- primal and dual meshes, characterised by a one-to-one correspondence between primal and dual geometric entities is used and a simplicial mesh (1-d, 2-d or 3-d with respect to the problem dimension) is taken as primal complex \( K \) while a barycentric one is used as dual complex \( \tilde{K} \);
- standard differential operators are translated in finite form by means of incidence matrices: \( G \) is the discrete equivalent of gradient, \( C \) of curl and \( D \) of divergence operators defined on primal complex \( K \). Similar matrices are defined on dual complex \( \tilde{K} \). Using the duality relationship, it is possible to write: \( \tilde{D} = -G^T \), \( \tilde{C} = C \), \( \tilde{G} = D^T \).
- considering for example the thermal problem, the temperature difference \( \Gamma \), configuration variable, is related to a primal edge \( l \) and the source variable, the thermal flux \( \Phi \), acts on the dual surface \( \tilde{S} \) associated to the primal edge;
- the constitutive equation, for instance on an orthogonal grid, \( \Phi = -\lambda \frac{\tilde{S}}{l} \Gamma \), relates the configuration and source variables and considers the size of the primal edge, the size of the dual surface and the thermal conductivity \( \lambda \) of the material.

In the present paper the above described approach is applied to a thermo-mechanical problem.

2.2. Algebraic formulation of thermal problems

Using Tonti diagram for the thermal problem, depicted in Figure 1, it is possible to identify four global variables: temperature and temperature differences (as configuration variables), thermal flux and heat generation (as source variables).

Primal cell complex is obtained using a triangular mesh. Temperature difference is defined on primal edges. By making reference to Figure 3, temperature gradient along primal edge \( \alpha \) is obtained as difference between nodal temperatures \( \theta_i - \theta_k \). The whole array of temperature differences can thus be obtained from the vector of temperatures, by using edge-to-node incidence matrix \( G \). In this matrix each row corresponds to an edge and is full of 0s but for the positions of the nodes on which edge insists: +1 if edge \( i \) comes out of node \( j \) and -1 if edge \( i \) enters into node \( j \). Construction of matrix \( G \), for the geometrical layout of Figure 4, is highlighted in Figure 5. \( G \) is the discrete counterpart of gradient differential operator and
Figure 1. Tonti diagram for steady state thermal conduction.

Figure 2. Tonti diagram for steady state mechanical problem.

Figure 3. 2-d example of primal (black) and dual (red) complexes: $\theta$ are the temperatures defined on primal nodes, $\Phi$ are thermal fluxes on dual faces and $q$ are the heat produced in dual volumes.

Figure 4. Primal 2-d cell with primal nodes ($n_i$), primal edges ($e_i$) and dual surfaces $\tilde{L}_i$

Figure 5. Definition of incidence matrices

since it only depends on the discretization, so it is the same for all physical problems. Thus,
temperature difference can be computed as:

\[ \Gamma = G\theta, \quad (1) \]

where \( \theta \) are node temperatures, \( \Gamma \) are temperature differences along the *primal* edges.

Again referring to Figure 3, balance equation on dual volume \( i \) is obtained by summing up all thermal fluxes on surfaces which bound it. Orientation of surfaces is translated in a \( \pm 1 \) sign to be applied to each flux. All these \( \pm 1 \) coefficients can be collected in a \( \bar{D} \) which is the incidence matrix that represents the relationship between *dual* volumes and surfaces and that expresses in algebraic form the divergence applied to the *dual* complex.

In this way, balance equation (topological equation written on the *dual* complex) is defined applying a thermal balance operation to the *dual* volumes:

\[ \bar{D}\Phi = q, \quad (2) \]

where \( \Phi \) are thermal fluxes through *dual* surfaces, \( q \) are heat productions defined in *dual* volumes.

The relationship between thermal flux and temperature gradient is the constitutive equation of the problem that represents the discrete Fourier equation:

\[ \Phi = -M_\lambda \Gamma, \quad (3) \]

where \( M_\lambda \) is the constitutive matrix that considers the "size" of the problem and the material features. The constitutive matrix aspect depends on the mesh used to discretize the domain, if a structured mesh with orthogonal lines and surfaces is used, \( M_\lambda \) is a diagonal matrix. In general, if the mesh is not structured, the constitutive matrix is computed by means of local interpolation of the field variables and it becomes a sparse matrix \([6], [7]\).

By operating on Equations (1), (2) and (3), deleting \( \Phi \) e \( \Gamma \) and applying duality principle (\( \bar{D} = -G^T \)), the solutive equation of the thermal problem assumes the form:

\[ G^T M_\lambda G\theta = q, \quad (4) \]

that can be easily assembled by means of sparse linear algebra routines and solved applying the thermal problem boundary conditions.

2.3. Algebraic formulation of mechanical problems

The mechanical problem can be divided in two aspects: elastostatic and elastodynamics problems. The first step is to study the elastostatic problem in order to express the stiffness matrix and to do this the same approach of section 2.2 can be used. The elastostatic problem can be analysed using the Tonti diagram of Figure 2. In analogy with the thermal problem the global variables have to be defined. The configuration variables are displacements \( u \) e their differences \( h \), respectively related to *primal* nodes and to *primal* edges. The source variables are surface forces \( \sigma \), linked to *dual* surfaces, and external forces \( F_e \) acting on the system and related to *dual* volumes.

On the basis of Figure 4, the displacement topological equation can be expressed on the *primal* complex as:

\[ G_3 u = h, \quad (5) \]

The matrix \( G_3 \) has the same form of the incidence matrix \( G \) of the thermal problem, because the discretization is the same, ad it is an expansion of \( G \) for each spatial direction, as shown in Figure 5. Using a different approach from \([1, 5]\), the displacement of a generic point inside a
primal cell is assumed to be affine. As described in Figure 4 for a 2-d problem, the only reported for briefness, the displacement components $u, v$ have the form

$$u(x, y) = H_{xx}x + H_{xy}y + c_u$$  \hspace{0.5cm} (6)
$$v(x, y) = H_{yx}x + H_{yy}y + c_v$$  \hspace{0.5cm} (7)

hence, the relative horizontal displacements are:

$$\Delta u_1 = u_2 - u_1 = H_{xx}(x_2 - x_1) + H_{xy}(y_2 - y_1)$$  \hspace{0.5cm} (8)
$$\Delta u_2 = u_3 - u_1 = H_{xx}(x_3 - x_1) + H_{xy}(y_3 - y_1)$$  \hspace{0.5cm} (9)
$$\Delta u_3 = \Delta u_2 - \Delta u_1$$  \hspace{0.5cm} (10)

Since $\Delta u_3$ is a linear combination of $\Delta u_2$ and $\Delta u_1$, no information is due to this term, that can be neglected. It is useful to rewrite (8) in matrix form:

$$\begin{bmatrix} \Delta u_1 \\ \Delta u_2 \end{bmatrix} = \begin{bmatrix} L_{1x} & L_{1y} \\ L_{2x} & L_{2y} \end{bmatrix} \begin{bmatrix} H_{xx} \\ H_{xy} \end{bmatrix}$$  \hspace{0.5cm} (11)

Then, operating on the matrices written for each considered direction,

$$\begin{bmatrix} H_{xx} & H_{yx} \\ H_{xy} & H_{yy} \end{bmatrix} = H^T = \begin{bmatrix} P_{x1} & P_{x2} \\ P_{y1} & P_{y2} \end{bmatrix} \begin{bmatrix} \Delta u_1 \\ \Delta v_1 \end{bmatrix}$$  \hspace{0.5cm} (12)

Using the classical strain expression coming from continuous mechanics, the (12) becomes:

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma \end{bmatrix} = \begin{bmatrix} P_{x1} & 0 & P_{x2} & 0 & 0 & 0 \\ 0 & P_{y1} & 0 & P_{y2} & 0 & 0 \\ P_{y1} & P_{x1} & P_{y2} & P_{x2} & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta u_1 \\ \Delta v_1 \\ \Delta u_2 \\ \Delta v_2 \\ \Delta u_3 \\ \Delta v_3 \end{bmatrix}$$  \hspace{0.5cm} (13)

and in compact form

$$\varepsilon = \Phi h$$  \hspace{0.5cm} (14)

The same reasoning allows the formulation of the 3-d problem.

The balance topological equation related to the forces acting on the dual volume is:

$$D_3 T = F_e$$  \hspace{0.5cm} (15)

Surface forces are defined on dual faces, see Figure 4. They are linear combination stress:

$$T_x = \sigma_{xx} \tilde{a}_x + \sigma_{xy} \tilde{a}_y$$  \hspace{0.5cm} (16)
$$T_y = \sigma_{yx} \tilde{a}_x + \sigma_{yy} \tilde{a}_y$$  \hspace{0.5cm} (16)

and, writing (16) for each dual face inside a triangle, the relationship between surface force and stress can be written as:

$$\begin{bmatrix} T_{1x} \\ T_{1y} \\ T_{2x} \\ T_{2y} \\ T_{3x} \\ T_{3y} \end{bmatrix} = \delta \begin{bmatrix} \tilde{L}_{1x} & 0 & \tilde{L}_{1y} \\ 0 & \tilde{L}_{1y} & \tilde{L}_{1x} \\ \tilde{L}_{2x} & 0 & \tilde{L}_{2y} \\ 0 & \tilde{L}_{2y} & \tilde{L}_{2x} \\ \tilde{L}_{3x} & 0 & \tilde{L}_{3y} \\ 0 & \tilde{L}_{3y} & \tilde{L}_{3x} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau \end{bmatrix}$$  \hspace{0.5cm} (17)
where $\delta$ is the problem depth. In compact form:

$$T = A\sigma$$  \hspace{1cm} (18)

The constitutive law in the elastostatic problem is usually expressed as a relationship between stress $\sigma$ and strain $\varepsilon$ and, in formula, it can be written as:

$$\sigma = E\varepsilon,$$  \hspace{1cm} (19)

assuming that stress and strain fields are uniform in the cell volume and the material (matrix $E$) is defined as isotropic linear elastic.

As in the thermal problem and in the most physical problems, using the duality principle, the matrix $G$ can be expressed as the transpose of $\tilde{D}$. By assembly together, it is then possible to write the solution equation of the elastostatic problem, in formulae:

$$G_3^T AEPG_3 u = F_e,$$  \hspace{1cm} (20)

and in compact form:

$$G_3^T M_E G_3 u = F_e,$$  \hspace{1cm} (21)

where the product of the three matrices, the two topological and the constitutive, gives the stiffness matrix of the analysed component. As demonstrated in [1, 5], the stiffness matrix obtained using (Algebraic Formulation) is the same deduced using the well-known Finite Element Method, with respect to the same model. By imposing the actual boundary conditions, (21) can be solved and the elastostatic quantities evaluated.

The elastodynamic problem involves the definition of the mass matrix. Using the Algebraic Formulation, the inertial properties of the single dual cell are related to the primal, this consideration allows the expression of the mass matrix as a diagonal matrix, where each element of the diagonal is the inertial property portion that is referred to the related primal node. In the 2-d case, mass matrix can be expressed as:

$$I = \begin{bmatrix}
    m_i & 0 & 0 & 0 & 0 & 0 \\
    0 & m_i & 0 & 0 & 0 & 0 \\
    0 & 0 & m_j & 0 & 0 & 0 \\
    0 & 0 & 0 & m_j & 0 & 0 \\
    0 & 0 & 0 & 0 & m_k & 0 \\
    0 & 0 & 0 & 0 & 0 & m_k
\end{bmatrix}$$  \hspace{1cm} (22)

where $m_{i,j,k}$ are the masses of the dual cells related to $i, j, k$ primal nodes. As a consequence, the elastodynamic problem can be solved using the well-known equation of motion.

### 2.4. Algebraic formulation of the thermo-mechanical problem

By comparing the (4) and (21), it is possible to notice that the unique difference between the two expressions is the constitutive matrix, whereas the model (the topological equations) is the same. In such a way the benefit of the Cell Method in multiphysics approach is clearly evident, it is sufficient to define a unique model. By combining (4) and (21) it is possible to solve a thermo-mechanical problem. In order to due that, it is important to define the mutual effects of thermal and mechanical field. In a first step, the authors only consider the effect of thermal field on the mechanical properties and responses of the studied system. Those effects are related to the variation of the Young modulus with the temperature and the thermal strains of the analysed component due to the nodal temperatures variation. For considering the first effect, it is possible to evaluate the temperature of each node (4), compute a mean temperature
of the dual cell and, by means of a look-up table, evaluate the material feature of this cell; then it is possible to write the elastic constitutive matrix. The second effect can be evaluated considering the actions on the component due to the additional thermal strains. In other words, the thermal expansion effect is transformed in a set of additional nodal forces. In formulae, the thermal strains can be expressed as:

\[
\begin{pmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{pmatrix} = \alpha \begin{pmatrix}
\bar{\theta} - \theta_0 \\
\bar{\theta} - \theta_0 \\
0
\end{pmatrix},
\]

where \(\bar{\theta}\) is the mean temperature of the cell, \(\theta_0\) is the reference temperature and \(\alpha\) is the expansion coefficient of the material.

Then, using the same approach of the elastostatic problem, the additional thermal forces can be written as:

\[
F_{th} = G_3^T A E \varepsilon_{th} = \alpha G_3^T A E \Delta \bar{\theta}.
\]

The thermo-mechanical problem is then formulated as:

\[
I \ddot{u}(t) + G_3^T A E P G_3 u(t) - \alpha G_3^T A E \Delta \bar{\theta} = F_e(t).
\]

2.5. Algebraic Formulation of the time evolution

Algebraic Formulation can be applied also to time evolution. The solution of (25) can be directly written in a discrete form. In fact, if the displacement \(u\) is primal instants, whereas velocity \(v = \dot{u}\) to the primal intervals. It is possible to write that:

\[
\tau_k v_k = u_{k+1} - u_k
\]

\[
\tau_k a_k = v_{k+1} - v_k
\]

where \(k\) is the considered time increment and \(a_k\) is the acceleration.

The motion equation can be then expressed as:

\[
I v_{k+1} = (F_{k+1} + \alpha G_3^T A E \Delta \bar{\theta}) \tau_k + I v_k - G_3^T A E P G_3 u_{k+1} \tau_k.
\]

By knowing that \(u_{k+1} = u_k + \tau_k v_k\), imposing initial condition, the thermo-mechanical problem can be evaluated in its time variation.

3. Results

In the present paper, authors do not consider the non-linearity in the constitutive matrices. So the analysed problems have material features dependent only on the temperature and not on the strain. In order to evaluate the proposed methodology, a 2-d and a 3-d (Figures 6 and 7, respectively) case studies are analysed and compared with numerical solution obtained by a commercial FEM software. As far as the case study mesh is concerned, both CM and FEM use the same discretization, with the same number of elements and nodes. It is worth noting that for FEM models element first order shape functions are chosen and different solution runs are performed for solving the conduction problem and the mechanical problem. In fact the output of the thermal problem solved with FEM are inserted, using a tailored data recovery procedure, as boundary conditions (nodal temperatures) in the mechanical problem in association with mechanical actions. In the CM procedure, the solution run is unique. In addition, CM procedure has the benefit of a quicker creation of the model, in fact, only a model is defined and the boundary conditions, both thermal and mechanical, are at the same time imposed. The case studies prove that results obtained with CM procedure are comparable to those obtained using FEM, confirming that the linear elastic stiffness matrices computed with the two methods are the same. Authors expect relevant differences when the non linearities of the material will be taken into account.
4. Conclusion

In the present paper, thermo-mechanical problem is studied using Cell Method and the solution formulation is proposed. A 2-d and a 3-d case studies are analysed and compared with FEM solution. Some remarks can be done:

- Cell Method proves to be useful in Multiphysics approach, giving coherent results;
- the model construction is quicker than traditional FEM method, only one discretization is needed because it is congruent among the different Physics disciplines;
- Cell Method also provides a low-level coupling of the physical laws, making easy to solve multiphysics problem, also in time dependent problem;
- with the proposed technique, it is possible to evaluate the thermal and mechanical strains for each step of time evolution and a more accurate life assessment can be done.

The present paper is the starting work of a series of technique evaluations with respect to different experimental case studies and different coupling problems, such as magneto-inductive-thermo-mechanical problem and the integration of the stress and strain evaluation in a TMF life assessment postprocessor [8, 9].

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