HIGHER NAKAYAMA ALGEBRAS I: CONSTRUCTION

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Abstract. We introduce higher dimensional analogues of the Nakayama algebras from the viewpoint of Iyama’s higher Auslander–Reiten theory. More precisely, for each Nakayama algebra \( A \) and each positive integer \( d \), we construct a finite dimensional algebra \( A^{(d)} \) having a distinguished \( d \)-cluster-tilting \( A^{(d)} \)-module whose endomorphism algebra is a higher dimensional analogue of the Auslander algebra of \( A \). We also construct higher dimensional analogues of the mesh category of type \( \mathbb{Z}A_{\infty} \) and the tubes.

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Introduction

Let \( A \) be a finite dimensional algebra over some field, which for simplicity we assume to be algebraically closed. The most basic problem in representation theory is to classify the indecomposable finite dimensional (right) \( A \)-modules. Since the general problem is known to be hopeless, special attention has been paid to finite dimensional algebras of finite representation type. These are the algebras for which there exist only finitely many isomorphism classes of indecomposable finite dimensional \( A \)-modules. Classifying such algebras is still a daunting problem. This lead to the study of algebras satisfying additional homological constraints. An important class of such algebras is that of hereditary algebras of finite representation type, which were classified by Gabriel

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Another significant class is that of selfinjective algebras, which includes the group algebras of finite groups. Selfinjective algebras of finite representation type were classified by Riedtmann in [Rie80].

Yet another important class of algebras of finite representation type is that of Nakayama algebras, see for example Chapter IV.2 in [ARS97] and Chapter V in [ASS06]. In relation to the above, it is worth noting that symmetric Nakayama algebras are stably equivalent to those Brauer tree algebras which arise from blocks of group algebras of finite representation type.

The classification of the Nakayama algebras over an algebraically closed field up to Morita equivalence is straightforward. For each non-negative integer \( n \), the path algebra of the linearly oriented quiver of type \( \mathbb{A}_n \) is the unique (basic, connected) hereditary Nakayama algebra having \( n \) simple modules. Similarly, for each positive integer \( n \) and each \( \ell \geq 2 \) the quotient of the path algebra of the circular quiver with \( n \) vertices and \( n \) arrows modulo the ideal generated by all paths of length \( \ell \) is the unique (basic, connected) selfinjective Nakayama algebra having \( n \) simple modules whose projective covers have Loewy length \( \ell \). The (basic, connected) Nakayama algebras are the admissible quotients of the hereditary and the selfinjective Nakayama algebras.

The purpose of this article is to construct, for each (basic, connected) Nakayama algebra \( A \) and each positive integer \( d \geq 2 \), a finite dimensional algebra \( A^{(d)} \) which we regard as a ‘\( d \)-dimensional analogue’ of \( A \). Our construction is similar in spirit to that of the classical Nakayama algebras.

The first family of algebras was introduced by Iyama in [Iya11] while the second one is constructed using recent results from Darpö and Iyama [DI17]. Every other higher Nakayama algebra is then constructed as an idempotent quotient of a higher Nakayama algebra of the form \( A^{(d)}_n \) or \( \tilde{A}^{(d)}_{n-1,\ell} \) (see Definitions 2.17 and 4.11). The idempotent quotients that arise are determined by the combinatorial data classifying the classical Nakayama algebras, their so-called Kupisch series.

The analogy between the classical Nakayama algebras and the higher Nakayama algebras is better appreciated in the context of Iyama’s higher Auslander–Reiten theory [Iya07a, Iya07b], which we review in Section 1.3. Instead of algebras of finite representation type one considers algebras \( A \) having a so-called \( d \)-cluster-tilting \( A \)-module \( M \). From the viewpoint of this theory, the additive closure of \( M \) in \( \text{mod} A \), denoted by \( \text{add} M \), is considered to be a replacement of the module category. This is justified by the existence of a ‘\( (d+1) \)-dimensional’ Auslander–Reiten theory inside \( \text{add} M \) (see Subsection 1.3 for definitions and precise statements). constructing algebras having a \( d \)-cluster-tilting module is a difficult task, nonetheless higher Auslander–Reiten theory has found applications in subjects such as algebraic geometry, combinatorics, and higher category theory [DJ, IY14, WI13, OT12]. Previous work on this direction has been mostly restricted to algebras of global dimension \( d \) and to selfinjective algebras (see [IO11, HI11] and [DI17] respectively). To our knowledge, the higher Nakayama algebras are the first family of algebras having a \( d \)-cluster-tilting module and which are neither selfinjective nor have global dimension precisely \( d \).

Our higher Nakayama algebras are equipped with distinguished \( d \)-cluster-tilting modules whose additive closures are, by design, higher dimensional analogues of the module categories of their
classical counterparts (see Theorems 2.3 and 4.12). The explicit combinatorial nature of Nakayama algebras makes them particularly suitable for computations. As such, we expect the higher Nakayama algebras to become fertile ground for testing conjectures in higher Auslander–Reiten theory.

Along the way we also construct higher dimensional analogues of the mesh category of type \( Z\mathbb{A}_\infty \) and of the tubes. The latter categories, ubiquitous in representation theory [Rin78, Web82, CB88, Erd95], can be thought of as infinite limits of module categories of Nakayama algebras. Their higher analogues are constructed in a similar vein (see Definitions 3.2 and 4.15). To the best of our knowledge, these categories provide the first non-trivial examples of abelian categories having no non-zero projective objects and no non-zero injective objects and having a cluster-tilting subcategory (see Theorems 3.9 and 4.18).

Further properties of the higher Nakayama algebras and the higher analogues of the mesh category of type \( Z\mathbb{A}_\infty \) and the tubes are investigated in separate works [JKa, JKb].

The article is structured as follows. In Section 1 we collect a few preliminaries which are needed in the construction of the higher Nakayama algebras. This includes an ‘idempotent reduction’ lemma (Lemma 1.18) which is the main technical tool in our constructions. In Section 2 we recall Iyama’s construction of the higher Auslander algebras of type \( \Delta \) and use it to introduce the higher Nakayama algebras of type \( \Delta \). In Section 3 we introduce the higher dimensional analogues of the mesh category of type \( Z\mathbb{A}_\infty \) as well as infinite versions of the higher Nakayama algebras of type \( \Delta \). The latter categories are needed in the construction of the higher Nakayama algebras of type \( \tilde{\Delta} \) in Section 4. The higher dimensional analogues of the tubes are introduced at the end of Section 4. For the convenience of the reader we include an index of symbols at the end of the article.

Conventions

Unless noted otherwise, throughout the article \( d \) denotes an arbitrary positive integer. We fix an arbitrary field \( k \) and write \( D := \text{Hom}_k(-, k) \) for the usual duality on the category of finite dimensional vector spaces. By ‘algebra’ we mean unital associative \( k \)-algebra and by ‘module’ we mean right module. By ‘category’ we mean \( k \)-category, that is category enriched in vector spaces. Thus, for two objects \( x \) and \( y \) in a category \( \mathcal{C} \) there is a vector space of morphisms \( \mathcal{C}(x, y) \) and the composition law in \( \mathcal{C} \) is bilinear. We compose morphisms in an abstract category as functions: if \( f : x \to y \) and \( g : y \to z \) are morphisms in some category, then we denote their composite by \( g \circ f : x \to z \). Given an object \( X \) in an additive category \( \mathcal{C} \), we denote by \( \text{add} X \) the smallest additive subcategory of \( \mathcal{C} \) containing \( X \) and which is closed under direct summands. Recall that the Loewy length of a finite dimensional module \( M \) is the length of its radical filtration; we denote this number by \( \ell\ell(M) \).

1. Preliminaries

In this section we recall some terminology as well as results which are needed in the remainder of the article. Our construction of the higher Nakayama algebras requires us to consider not only finite dimensional algebras but, more generally, locally bounded categories. Of particular interest for us are certain locally bounded categories constructed from partially ordered sets. Therefore we
begin with a brief review of basic aspects of the representation theory of locally bounded categories and partially ordered sets. We also include an overview of higher Auslander–Reiten theory in which we prove a technical result, Lemma \[1.18\] which is the key ingredient in our construction of the higher Nakayama algebras.

1.1. Representation theory of locally bounded categories. Let \( \mathcal{A} \) be a small category. By definition, a (right) \( \mathcal{A} \)-module is a linear functor \( M : \mathcal{A}^{\text{op}} \rightarrow \text{Mod} \mathbb{k} \), that is a functor such that for all \( x, y \in \mathcal{A} \) the induced function

\[
M_{yx} : \mathcal{A}(x, y) \rightarrow \text{Hom}_{\mathbb{k}}(M_y, M_x)
\]

is a linear map. We denote the abelian category of \( \mathcal{A} \)-modules and natural transformations between them by \( \text{Mod} \mathcal{A} \). By Yoneda’s lemma, for every object \( x \in \mathcal{A} \) the representable functor

\[
\mathcal{A}(-, x) : \mathcal{A}^{\text{op}} \rightarrow \text{Mod} \mathbb{k}
\]

is a projective \( \mathcal{A} \)-module. The category of finitely generated projective \( \mathcal{A} \)-modules is the subcategory

\[
\text{proj} \mathcal{A} := \text{add} \{ \mathcal{A}(-, x) \mid x \in \mathcal{A} \} \subset \text{Mod} \mathcal{A}.
\]

By definition, \( \text{proj} \mathcal{A} \) is an idempotent complete additive category.

Remark 1.1. Let \( \mathcal{A} \) be a category with finitely many objects. There is a canonical isomorphism between \( \text{Mod} \mathcal{A} \) and the category of right modules over the unital algebra

\[
\bigoplus_{x,y \in \mathcal{A}} \mathcal{A}(x, y),
\]

whose multiplication is induced by the composition law in \( \mathcal{A} \) (the unit is the tuple \((1_x \mid x \in \mathcal{A})\)). In view of this observation, throughout the article we identify categories with finitely many objects and algebras.

A small category \( \mathcal{A} \) is locally finite dimensional if for every pair of objects \( x, y \in \mathcal{A} \) the vector space of morphisms \( \mathcal{A}(x, y) \) is finite dimensional; note that this property implies that the Krull–Schmidt theorem holds in \( \text{proj} \mathcal{A} \), see for example Corollary 4.4 in [Kra15]. Let \( \mathcal{A} \) be a locally finite dimensional category. In order to simplify the exposition we make the additional assumptions that all objects in \( \mathcal{A} \) have local endomorphism ring and that objects in \( \mathcal{A} \) are pairwise non-isomorphic. These assumptions are equivalent to requiring that the Yoneda embedding \( \mathcal{A} \hookrightarrow \text{Mod} \mathcal{A} \) identifies \( \mathcal{A} \) with a complete set of representatives of the isomorphism classes of indecomposable finitely generated projective \( \mathcal{A} \)-modules. Morita theory guarantees that these assumptions do not result in a loss of generality.

Throughout the article we are mostly concerned with the following class of \( \mathcal{A} \)-modules. An \( \mathcal{A} \)-module \( M \) is finite dimensional if \( \bigoplus_{x \in \mathcal{A}} M_x \) is a finite dimensional vector space. Thus, \( M \) is finite dimensional if for every \( x \in \mathcal{A} \) the vector space \( M_x \) is finite dimensional and \( M_x = 0 \) for all but finitely many objects \( x \in \mathcal{A} \). The finite dimensional \( \mathcal{A} \)-modules form an abelian subcategory \( \text{mod} \mathcal{A} \) of \( \text{Mod} \mathcal{A} \).

\[1\] We remind the reader of our conventions regarding categories.
A locally finite dimensional category $\mathcal{A}$ is called **locally bounded** if for every object $x \in \mathcal{A}$ there are only finitely many objects $y \in \mathcal{A}$ such that $\mathcal{A}(x, y) \neq 0$ and only finitely many objects $w \in \mathcal{A}$ such that $\mathcal{A}(w, x) \neq 0$. Let $\mathcal{A}$ be a locally bounded category. Equivalently, $\mathcal{A}$ is locally bounded if for each object $x \in \mathcal{A}$ the projective $\mathcal{A}$-module $\mathcal{A}(-, x)$ and the projective $\mathcal{A}^{\text{op}}$-module $\mathcal{A}(x, -)$ are finite dimensional. This readily implies that the abelian category $\text{mod} \mathcal{A}$ has enough projectives and a classical argument shows that every finite dimensional $\mathcal{A}$-module has a projective cover in $\text{mod} \mathcal{A}$, see for example Proposition 4.1 in [Kra15]. As a consequence of the duality

$$D : \text{mod} \mathcal{A} \to \text{mod} (\mathcal{A}^{\text{op}}),$$

$$M \mapsto D \circ M$$

the abelian category $\text{mod} \mathcal{A}$ also has enough injectives and every finite dimensional $\mathcal{A}$-module has an injective envelope in $\text{mod} \mathcal{A}$. We denote the category of finitely generated injective $\mathcal{A}$-modules by

$$\text{inj} \mathcal{A} := D(\text{proj}(\mathcal{A}^{\text{op}})).$$

Given a finite dimensional $\mathcal{A}$-module $M$, its Auslander–Bridger transpose $\text{Tr} M$ and its Auslander–Reiten translate

$$\tau(M) := D \text{Tr}(M)$$

can be defined in the usual way, see for example pages 337 and 338 in [AR74]. It is also a classical fact that the abelian category $\text{mod} \mathcal{A}$ has almost-split sequences, see for example page 343 in [AR74].

Let $\mathcal{A}$ be an essentially small category and $\mathcal{X}$ a full subcategory of $\mathcal{A}$. The **idempotent ideal of $\mathcal{A}$ generated by $\mathcal{X}$**, denoted by $[\mathcal{X}]$, is the ideal of morphisms in $\mathcal{A}$ which factor through an object in $\mathcal{X}$. The category $\mathcal{A}_X$ has the same objects as $\mathcal{A}$ but has vector spaces of morphisms $\mathcal{A}_X(x, y) := \mathcal{A}(x, y) [\mathcal{X}](x, y)$.

An object $x \in \mathcal{A}$ becomes a zero object in $\mathcal{A}_X$ if and only if it belongs to $\mathcal{X}$. It is well known that the canonical functor $\pi : \mathcal{A} \to \mathcal{A}_X$ induces a fully faithful exact functor $\pi^* : \text{Mod} \mathcal{A}_X \to \text{Mod} \mathcal{A}$ which restricts to a fully faithful exact functor

$$\pi^* : \text{mod} \mathcal{A}_X \to \text{mod} \mathcal{A},$$

see for example [Ste71] in the case of rings. Note that $\pi^*$ identifies $\text{Mod} \mathcal{A}_X$ with the full subcategory of $\text{Mod} \mathcal{A}$ consisting of those $\mathcal{A}$-modules $M$ such that $M_x = 0$ for all $x \in \mathcal{X}$. To alleviate the notation we leave this identification implicit in the sequel. We also use the standard notations

$$\text{mod} \mathcal{A} := \text{mod} \mathcal{A}_{\text{proj} \mathcal{A}} \quad \text{and} \quad \text{mod} \mathcal{A} := \text{mod} \mathcal{A}_{\text{inj} \mathcal{A}}.$$
\(\tau: \text{mod} A \to \text{mod} A\). In particular, there is a natural isomorphism \(\tau \circ \Omega^- \cong \Omega^- \circ \tau\). Moreover, the classical Auslander–Reiten formula implies the existence of bifunctorial isomorphisms

\[
D\text{Hom}_A(M, N) \cong \text{Ext}^1_A(N, \tau(M)) \cong \text{Hom}_A(N, (\tau \circ \Omega^-)(M))
\]

for all \(M, N \in \text{mod} A\), where the rightmost isomorphism is obtained by the usual dimension-shifting argument.

Let \(T\) be a triangulated category with finite dimensional spaces of morphisms. Recall from [BK89] that an exact autoequivalence \(S: T \to T\) is a Serre functor if there are bifunctorial isomorphisms \(D^T(X, Y) \cong T(Y, S(X))\) for all \(X, Y \in T\). Note that Yoneda’s embedding implies that the Serre functor, if it exists, is unique up to canonical isomorphism. In particular, if \(A\) is a selfinjective locally bounded category, then the Auslander–Reiten formula shows that

\[
S := (\tau \circ \Omega^-) : \text{mod} A \to \text{mod} A
\]

is a Serre functor.

Let \(A\) be a locally finite dimensional category. Although the abelian category \(\text{mod} A\) does not necessarily have enough projectives nor enough injectives, we can always define the extension spaces using equivalence classes of extensions in the sense of Yoneda [Yon60]. With this in mind, we say that the abelian category \(\text{mod} A\) has global dimension \(\delta\) if

\[
\delta = \sup \{ i \geq 0 \mid \exists M, N \in \text{mod} A : \text{Ext}^i_A(M, N) \neq 0 \}
\]

1.2. Representation theory of partially ordered sets. Let \((P, \leq)\) be a poset. The incidence category of \((P, \leq)\) is the category \(\mathcal{P}\) with set of objects \(P\) and vector spaces of morphisms

\[
\mathcal{P}(x, y) := \begin{cases} k f_{yx} & \text{if } x \leq y, \\ 0 & \text{otherwise} \end{cases}
\]

(by convention, \(f_{yx} := 0\) if \(x \not\leq y\)). The composition law in \(\mathcal{P}\) is completely determined by requiring the equation

\[
f_{zx} = f_{zy} \circ f_{yx}
\]

to be satisfied whenever \(x \leq y \leq z\). When convenient, we identify \((P, \leq)\) with its incidence category \(\mathcal{P}\), which is clearly a locally finite dimensional category.

Example 1.2. Let \(n\) be a positive integer. A basic example of a poset, which is of central importance in this article, is the finite linear order

\[
A_n := \{ 0 < 1 < \cdots < n-1 \}.
\]

When viewed as a locally finite dimensional category, the above poset is canonically isomorphic to the path category of the linear quiver

\[
\bar{A}_n : 0 \to 1 \to \cdots n-1.
\]

Given \(x_1, x_2 \in P\), define the closed interval from \(x_1\) to \(x_2\) to be the subset

\[
[x_1, x_2] := \{ x \in P \mid x_1 \leq x \leq x_2 \}.
\]

Quotients by idempotent ideals are easy to describe in this setting.
Lemma 1.3. Let $\mathcal{P} = (P, \leq)$ be a poset, $X$ a subset of $P$ and $\mathcal{X}$ the full subcategory of $\mathcal{P}$ spanned by $X$. Then,

$$\mathcal{P}_\mathcal{X}(x_1, x_2) \cong \begin{cases} 
\mathbb{k} & \text{if } x_1 \leq x_2 \text{ and } [x_1, x_2] \cap X = \emptyset, \\
0 & \text{otherwise}.
\end{cases}$$

Proof. The morphism $f_{x_2, x_1} : x_1 \to x_2$ factors through $x \in X$ if and only if $x_1 \leq x \leq x_2$ if and only if $x \in [x_1, x_2]$. The claim follows. $\square$

Let $\mathcal{P} = (P, \leq)$ be a poset. We are mostly interested in the following class of $\mathcal{P}$-modules.

Definition 1.4. Let $\mathcal{P} = (P, \leq)$ be a poset and $x_1, x_2 \in P$. The interval $\mathcal{P}$-module $M[x_1, x_2]$ is defined by associating to $x \in P$ the vector space

$$M[x_1, x_2]_x := \begin{cases} 
\mathbb{k} & \text{if } x \in [x_1, x_2], \\
0 & \text{otherwise};
\end{cases}$$

and to $x \leq y$ the linear map

$$M[x_1, x_2]_{yx} := \begin{cases} 
1 & \text{if } x, y \in [x_1, x_2], \\
0 & \text{otherwise}.
\end{cases}$$

The combinatorial nature of interval modules makes them particularly suited to explicit calculations. As a first example of this, we describe the space of morphisms between two interval modules in combinatorial terms.

Proposition 1.5. Let $\mathcal{P} = (P, \leq)$ be a poset and $x_1, x_2, y_1, y_2 \in P$. Then,

$$\text{Hom}_\mathcal{P}(M[x_1, x_2], M[y_1, y_2]) \cong \begin{cases} 
\mathbb{k} & \text{if } x_1 \leq y_1 \leq x_2 \leq y_2, \\
0 & \text{otherwise}.
\end{cases}$$

Moreover, if $\eta : M[x_1, x_2] \to M[y_1, y_2]$ is a non-zero morphism of $\mathcal{P}$-modules, then its image is isomorphic to the interval module $M[y_1, x_2]$.

Proof. Unwinding of the definitions, one can verify that a natural transformation $f : M[x_1, x_2] \to M[y_1, y_2]$ is completely determined by its component $f_{x_2}$. This component can be non-zero only if $y_1 \leq x_2 \leq y_2$ in which case it can be canonically identified with a non-zero scalar $\lambda \in \mathbb{k}^\times$. We can also verify that the fact that $f$ is a natural transformation also implies that $x_1 \leq y_2 \leq x_2$. We leave the details to the reader. $\square$

Corollary 1.6. Let $\mathcal{P} = (P, \leq)$ be a poset and $x_1, x_2 \in P$ such that $x_1 \leq x_2$. Then, the $\mathcal{P}$-module $M[x_1, x_2]$ is indecomposable.

Proof. According to Proposition 1.5, the endomorphism algebra of $M[x_1, x_2]$ is isomorphic to the ground field whence it is local. The claim follows. $\square$

Example 1.7. Let $A_n$ be the linear order with $n$ elements, see Example 1.2. It is well known that every indecomposable $A_n$-module is isomorphic to an interval module $M[i, j]$, usually depicted as

$$0 \to \cdots \to 0 \to \mathbb{k} \to \cdots \to \mathbb{k} \to 0 \to \cdots 0$$
where the rightmost \( k \) is at position \( i \) and the leftmost \( k \) is at position \( j \), see for example Section 2.6 in [Bar15]. Proposition 1.5 can be seen as a straightforward generalisation of the corresponding statement for the poset \( A_n \).

One of the main ingredients in our construction of the higher Nakayama algebras are Iyama’s higher Auslander algebras of type \( A \). These algebras were introduced in [Iya11]. A combinatorial approach for describing this algebras was introduced by Oppermann and Thomas in [OT12]. The notation introduced below, a slight modification of theirs, helps us make the analogy between the algebras we construct and the classical Nakayama algebras more transparent.

**Notation 1.8.** Let \( \mathcal{P} = (P, \leq) \) be a poset and \( d \) a positive integer. We endow the cartesian product 
\[
P^d = P \times \cdots \times P
\]
d times with the product order: given tuples \( x = (x_1, \ldots, x_d) \) and \( y = (y_1, \ldots, y_d) \) in \( P^d \), we define \( x \preceq y \) if and only if for each \( i \in \{1, \ldots, d\} \) the relation \( x_i \leq y_i \) is satisfied. We denote the resulting poset by \( P^d := (P^d, \preceq) \).

**Definition 1.9.** Let \( \mathcal{P} = (P, \leq) \) be a poset and \( d \) a positive integer.

(i) Given \( x, y \in P^d \), we say that \( x \) interlaces \( y \) if 
\[
x_1 \leq y_1 \leq x_2 \leq y_2 \leq \cdots \leq x_d \leq y_d.
\]
We use the symbol \( x \rightsquigarrow y \) to signify this relation. By definition, \( x \) is an ordered sequence of length \( d \) if it interlaces with itself. Thus, \( x \) is an ordered sequence of length \( d \) if and only if 
\[
x_1 \leq x_2 \leq \cdots \leq x_d.
\]
We denote the set of ordered sequences of length \( d \) in \( \mathcal{P} \) by \( \text{os}_d(\mathcal{P}) \).

(ii) Let \( \mathcal{P} = (P, \leq) \) be a poset and \( d \) a positive integer. We define the \( d \)-cone of \( \mathcal{P} \) to be the idempotent quotient 
\[
\text{cone}(P^d) := P^d / [P^d \setminus \text{os}_d(\mathcal{P})].
\]

The introduction of the interlacing relation on the set of ordered sequences of length \( d \) in a poset is justified by the following elementary observation.

**Proposition 1.10.** Let \( \mathcal{P} = (P, \leq) \) be a poset and \( d \) a positive integer. Then, for every pair of objects \( x, y \in P^d \) there is an interlacing \( x \rightsquigarrow y \) if and only if \( x \preceq y \) and \( [x, y] \subset \text{os}_d(\mathcal{P}) \). In particular,
\[
\text{cone}(P^d)(x, y) \cong \begin{cases} 
\mathbb{Z} & \text{if } x \rightsquigarrow y, \\
0 & \text{otherwise}.
\end{cases}
\]

**Proof.** When \( d = 1 \) the statement is obvious. Suppose that \( d > 1 \). By Lemma 1.3 it is enough to show that \( x \) interlaces \( y \) if and only if \( x \preceq y \) and \( [x, y] \subset \text{os}_d(\mathcal{P}) \). Suppose that \( x \) interlaces \( y \), which by definition implies \( x \preceq y \). Let \( z \in [x, y] \). Then, for each \( i \in \{1, \ldots, d-1\} \) the inequalities 
\[
x_i \leq z_i \leq y_i \leq x_{i+1} \leq z_{i+1} \leq y_{i+1}
\]
are satisfied (the inequality in the middle uses $x \rightarrow y$ while the others follow from $x \leq z \leq y$). This shows that $z \in \text{os}^d(\mathcal{P})$, as required.

Conversely, suppose that $x \leq y$ and $[x, y] \subset \text{os}^d(\mathcal{P})$. We need to prove that $x$ interlaces $y$. For this, let $j \in \{1, \ldots, d - 1\}$ and consider the auxiliary tuple

$$z := (x_1, \ldots, x_j, y_{j+1}, \ldots, y_d)$$

Given that $x \leq y$, the tuple $z$ belongs to $[x, y] \subset \text{os}^d(\mathcal{P})$ whence it is an ordered sequence of length $d$. In particular, the inequality

$$x_j = z_j \leq z_{j+1} = y_{j+1}$$

is satisfied. This shows that $x$ interlaces $y$.

The $d$-cone can be constructed inductively in representation-theoretic terms. This construction extends the inductive construction of the higher Auslander algebras of type $A$ given in [Iya11] from finite total orders to arbitrary posets.

Notation 1.11. Let $\mathcal{P} = (P, \leq)$ be a poset and $d$ a positive integer. Let $x \in \text{os}^{d+1}(\mathcal{P})$. Note that there is an obvious interlacing

$$(x_1, \ldots, x_d) \rightsquigarrow (x_2, \ldots, x_{d+1})$$

in $\text{os}^d(\mathcal{P})$. Then, in view of Proposition 1.10, the $P^d$-module

$$M(x) := M[(x_1, \ldots, x_d), (x_2, \ldots, x_{d+1})]$$

is in fact a module over $\text{cone}(P^d)$.

We conclude our general discussion on interval modules with an elementary observation. It should be compared with Proposition 3.12 in [OT12] which corresponds to the case of the poset $A_n$ of Examples 1.2 and 1.7.

Proposition 1.12. Let $\mathcal{P} = (P, \leq)$ be a poset and $d$ a positive integer. Then, for every $x, y \in \text{os}^{d+1}(\mathcal{P})$ there is an isomorphism

$$\text{Hom}_{\text{cone}(P^{d+1})}(M(x), M(y)) \cong \begin{cases} \mathbb{K} & \text{if } x \rightsquigarrow y, \\ 0 & \text{otherwise}. \end{cases}$$

Moreover, the image of a non-zero morphism of $P^{d+1}$-modules $M(x) \rightarrow M(y)$ is isomorphic to the interval module

$$M[(y_1, \ldots, y_d), (x_2, \ldots, x_{d+1})].$$

In particular, there is an equivalence of categories

$$P^{d+1} \cong \{ M(x) \in \text{mod cone}(P^d) \mid x \in \text{os}^{d+1}(\mathcal{P}) \} \subset \text{mod cone}(P^d).$$

Proof. In view of Proposition 1.5, it is enough to show that $x$ interlaces $y$ if and only if

$$(x_1, \ldots, x_d) \preceq (y_1, \ldots, y_d) \preceq (x_2, \ldots, x_{d+1}) \preceq (y_2, \ldots, y_{d+1}).$$

But this is tautological.
1.3. Higher Auslander–Reiten theory. We conclude this section with a brief overview of Iyama’s higher Auslander–Reiten theory. We begin by recalling the classical concept of a functorially finite subcategory from [AS80].

**Definition 1.13.** Let $A$ be a category and $X$ a full subcategory of $A$. We say that $X$ is contravariantly finite in $A$ if for every object $a \in A$ there exists an object $x \in X$ and a morphism $f: x \to a$ such that for every object $x' \in X$ and every morphism $g: x' \to a$ there exists a (not necessarily unique) morphism $h: x' \to x$ such that $g = f \circ h$, that is such that the diagram

$$
\begin{array}{ccc}
x' & \xrightarrow{g} & a \\
\downarrow{f} & \swarrow{h} & \\
x & & 
\end{array}
$$

commutes. Such a morphism $f$ is called a right $X$-approximation of $a$. The notions of a covariantly finite subcategory of $A$ and of a left approximation are defined dually. We say $X$ is functorially finite in $A$ if it is both contravariantly finite and covariantly finite in $A$.

One of the central concepts in higher Auslander–Reiten theory is the notion of a $d$-cluster-tilting subcategory which was introduced by Iyama in [Iya07b]. We recall the definition below as well as the stronger notion of a $dZ$-cluster-tilting subcategory which is implicit in [Iya11] and [GKO13] in the triangulated case and was introduced in [IJ16] in the abelian case.

**Definition 1.14.** Let $A$ be an abelian or a triangulated category and $C$ a subcategory of $A$. We call $C$ a $d$-cluster-tilting subcategory if the following conditions are satisfied.

(i) $C$ is a functorially finite subcategory of $A$.

(ii) If $A$ is abelian we require $C$ to be a generating-cogenerating subcategory of $A$, that is for every object $X \in A$ there exist objects $C', C'' \in C$, an epimorphism $C' \to X$, and a monomorphism $N \to C''$.

(iii) There are equalities

$$
C = \{ X \in A \mid \forall i \in \{1, \ldots, d-1\} : \Ext^i_A(X, C) = 0 \} = \{ Y \in A \mid \forall i \in \{1, \ldots, d-1\} : \Ext^i_A(C, Y) = 0 \}.
$$

We call $C$ a $dZ$-cluster-tilting subcategory if the following additional condition is satisfied:

(iv) If $\Ext^i_A(C, C) \neq 0$, then $i \in dZ$.

**Remark 1.15.** Let $A$ be an abelian or triangulated category. We make a few simple observations.

(i) The category $A$ itself is the unique 1-cluster-tilting subcategory of $A$.

(ii) If $A$ is an abelian category with enough projectives and enough injectives, then every subcategory of $A$ satisfying condition (iii) in Definition 1.14 is a generating-cogenerating subcategory.

(iii) If $A$ is a triangulated category with suspension functor $\Sigma$ and $C$ is a full subcategory of $A$ satisfying condition (iii) in Definition 1.14 then $C$ satisfies condition (iv) in Definition 1.14 if and only if $\Sigma^d(C) = C$. 
**Remark 1.16.** We recall the ‘higher homological algebra’ perspective on cluster-tilting subcategories.

(i) Let $\mathcal{A}$ be an abelian category and $\mathcal{C}$ a $d$-cluster-tilting subcategory of $\mathcal{A}$. Then, $\mathcal{A}$ is a $d$-abelian category in the sense of [Jas16], see Theorem 3.16 therein.

(ii) Analogously, if $\mathcal{A}$ is a triangulated category with suspension functor $\Sigma$ and $\mathcal{C}$ is a $d\mathbb{Z}$-cluster-tilting subcategory of $\mathcal{A}$, then $\mathcal{C}$ is a $(d+2)$-angulated category in the sense of [GKO13], see Theorem 1 therein.

**Definition 1.17.** Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and $i: \mathcal{B} \to \mathcal{A}$ a fully faithful exact functor. Let $\mathcal{X}$ be a full subcategory of $\mathcal{B}$.

(i) We say that $i$ is a contravariantly $\mathcal{X}$-relative $(d-1)$-homological embedding if for all $B \in \mathcal{B}$ and for all $i \in \{1, \ldots, d-1\}$ the induced morphism

$$\text{Ext}^i_B(-, B)|_{\mathcal{X}} \to \text{Ext}^i_A(-, B)|_{\mathcal{X}}$$

is an isomorphism.

(ii) We say that $i$ is a covariantly $\mathcal{X}$-relative $(d-1)$-homological embedding if for all $B \in \mathcal{B}$ and for all $i \in \{1, \ldots, d-1\}$ the induced morphism

$$\text{Ext}^i_B(B, -)|_{\mathcal{X}} \to \text{Ext}^i_A(B, -)|_{\mathcal{X}}$$

is an isomorphism.

(iii) We say that $i$ is an $\mathcal{X}$-relative $(d-1)$-homological embedding if it is a covariantly $\mathcal{X}$-relative $(d-1)$-homological embedding and a contravariantly $\mathcal{X}$-relative $(d-1)$-homological embedding.

The following ‘idempotent reduction’ lemma is one of the key technical tools used in our construction of the higher Nakayama algebras.

**Lemma 1.18.** Let $\mathcal{A}$ be a locally bounded category and $\mathcal{M}$ a $d$-cluster-tilting subcategory of $\text{mod}\, \mathcal{A}$. Let $\mathcal{X}$ be a full subcategory of $\mathcal{A}$ such that the following conditions are satisfied.

- All the projective and all the injective $\mathcal{A}$-modules belong to $\mathcal{M}$.
- Every indecomposable $\mathcal{A}$-module $M \in \mathcal{M}$ which does not lie in $\text{mod}\, \mathcal{A}_\mathcal{X}$ is projective-injective.

Define $\mathcal{M}_\mathcal{X} := \mathcal{M} \cap \text{mod}\, \mathcal{A}_\mathcal{X}$. Then, the following statements hold.

(i) The canonical inclusion $\text{mod}\, \mathcal{A}_\mathcal{X} \to \text{mod}\, \mathcal{A}$ is an $\mathcal{M}_\mathcal{X}$-relative $(d-1)$-homological embedding.

(ii) $\mathcal{M}_\mathcal{X}$ is a $d$-cluster-tilting subcategory of $\text{mod}\, \mathcal{A}_\mathcal{X}$.

**Proof.** ([]) We only prove that the canonical inclusion is a covariantly $\mathcal{M}_\mathcal{X}$-relative $(d-1)$-homological embedding, that it is also a contravariantly $\mathcal{M}_\mathcal{X}$-relative $(d-1)$-homological embedding follows by duality. To prove this, we apply Proposition 3.18 in [Jas16] which is a version of the usual ‘dimension shifting’ argument. Let $M \in \mathcal{M}_\mathcal{X} \subseteq \text{mod}\, \mathcal{A}_\mathcal{X}$, $N \in \text{mod}\, \mathcal{A}_\mathcal{X}$ and

$$\cdots \to P^1 \to P^0 \to N \to 0$$
a projective resolution of \( N \) as an \( A_X \)-module. By assumption, for all \( i \in \mathbb{Z} \) the projective \( A_X \)-module \( P^i \) belongs to \( \mathcal{M} \). Given that \( \mathcal{M} \) is a \( d \)-cluster-tilting subcategory of \( \text{mod} \, A \), for all \( i \in \{1, \ldots, d-1\} \) the extension group \( \text{Ext}^i_A(M, M) \) vanishes. Hence, by Proposition 3.18 in [Jas16] for each \( i \in \{1, \ldots, d-1\} \) there is an isomorphism between \( \text{Ext}^i_A(N, M) \) and the cohomology of the induced complex

\[
\text{Hom}_A(P^0, M) \rightarrow \cdots \rightarrow \text{Hom}_A(P^d, M)
\]

at \( \text{Hom}_A(P^i, M) \) which, by definition, is isomorphic to \( \text{Ext}^i_A(N, M) \). This shows that the canonical inclusion is a covariantly \( \mathcal{M}_X \)-relative \((d-1)\)-homological embedding.

By assumption \( \mathcal{M}_X \) contains all projective and all injective \( A_X \)-modules whence it is a generating-cogenerating subcategory of \( \text{mod} A_X \). Let us show that \( \mathcal{M}_X \) is a functorially finite subcategory of \( \text{mod} A_X \). For this, let \( N \) be an \( A_X \)-module and \( f: M \rightarrow N \) a right \( \mathcal{M} \)-approximation of \( N \). Write \( M = Q \oplus M' \) where \( M' \) is the largest summand of \( M \) which lies in \( \mathcal{M}_X \). Note that by assumption \( Q \) is a projective-injective \( A \)-module. Write

\[
f = [f_Q \, f_{M'}]: Q \oplus M' \rightarrow N
\]

and let \( f_P: P \rightarrow N \) a projective cover of \( N \) as an \( A_X \)-module. We claim that

\[
f' = [f_P \, f_{M'}]: P \oplus M' \rightarrow N
\]

is a right \( \mathcal{M}_X \)-approximation of \( N \). Indeed, let \( g: M'' \rightarrow N \) be a morphism with \( M'' \in \mathcal{M}_X \). Since \( f \) is a right \( \mathcal{M} \)-approximation of \( N \) there exists a morphism

\[
h = [h_0 \, h_1]^\top: M'' \rightarrow Q \oplus M
\]

such that \( g = f \circ h \), that is

\[
g = f_Q \circ h_0 + f_{M'} \circ h_1.
\]

Moreover, given that \( p \) is an epimorphism and \( Q \) is a projective(-injective) \( A \)-module, there exists a morphism \( q: Q \rightarrow P \) such that \( f_Q = f_P \circ q \). It readily follows that \( g = f' \circ h' \) where

\[
h' = [q \circ h_0 \, h_1]^\top: M'' \rightarrow P \oplus M.
\]

This shows that \( \mathcal{M}_X \) is contravariantly finite in \( \text{mod} A_X \); that \( \mathcal{M}_X \) is covariantly finite in \( \text{mod} A_X \) follows by duality.

Next, let \( N \) be an \( A_X \)-module such that for each \( i \in \{1, \ldots, d-1\} \) we have

\[
\text{Ext}^i_{A_X}(M_X, N) \cong \text{Ext}^i_A(M_X, N) = 0
\]

where the first isomorphism follows from part \([\text{ii}]\). Since by assumption every indecomposable \( A \)-module \( M \in \mathcal{M} \) which does not lie in \( \text{mod} A_X \) is projective-injective, for each \( i \in \{1, \ldots, d-1\} \) we conclude that \( \text{Ext}^i_A(M, N) = 0 \). Since \( \mathcal{M} \) is a \( d \)-cluster-tilting subcategory of \( \text{mod} A \), we deduce that \( N \in \mathcal{M} \cap \text{mod} A_X = M_X \), which is what we needed to show. By duality, every \( A_X \)-module \( N \) such that for each \( i \in \{1, \ldots, d-1\} \) there is an equality \( \text{Ext}^i_{A_X}(N, M_X) = 0 \) also belongs to \( \mathcal{M}_X \). This shows that \( \mathcal{M}_X \) is a \( d \)-cluster-tilting subcategory of \( \text{mod} A_X \). \( \square \)

Remark 1.19. In the setting of Lemma 1.18 it can be shown that the canonical inclusion \( \text{mod} A_X \rightarrow \text{mod} A \) is moreover a \((d-1)\)-homological embedding, see Appendix A.
Lemma 1.20. Let $A$ be a finite dimensional algebra and $\mathcal{M}$ a $d$-cluster-tilting subcategory of $\text{mod} \ A$. Let $e \in A$ be an idempotent such that the following conditions are satisfied.

- All the projective and all the injective $(A/AeA)$-modules belong to $\mathcal{M}$.
- Every indecomposable $A$-module $M \in \mathcal{M}$ which does not lie in $\text{mod}(A/AeA)$ is projective-injective.

Define $\mathcal{M}_e := \mathcal{M} \cap \text{mod}(A/AeA)$. Then, the following statements hold.

(i) The canonical inclusion $\text{mod}(A/AeA) \to \text{mod} \ A$ is an $\mathcal{M}_e$-relative $(d - 1)$-homological embedding.

(ii) $\mathcal{M}_e$ is a $d$-cluster-tilting subcategory of $\text{mod}(A/AeA)$.

Almost split sequences are the main object of study in classical Auslander–Reiten theory. The analogous concept in higher Auslander–Reiten theory is that of a $d$-almost split sequence, see Definition 3.1 in [Iya07b].

Definition 1.21. Let $\mathcal{M}$ be a Krull–Schmidt additive category. A sequence $\delta$ in $\mathcal{M}$ of the form

$$
\delta : 0 \to L \to M^1 \to \cdots \to M^d \to N \to 0
$$

is $d$-almost split if the following conditions are satisfied.

(i) All morphisms in $\delta$ belong to the Jacobson radical of $\mathcal{M}$.

(ii) For each $X \in \mathcal{M}$ the sequences of abelian groups

$$
0 \to A(X, L) \to A(X, M^1) \to \cdots \to A(X, M^d) \to A(X, N)
$$

and

$$
0 \to A(N, X) \to A(M^d, X) \to \cdots \to A(M^1, X) \to A(L, X)
$$

are exact.

(iii) Every non-split monomorphism $L \to X$ in $\mathcal{M}$ factors through $L \to M^1$.

(iv) Every non-split epimorphism $X \to N$ in $\mathcal{M}$ factors through $M^d \to N$.

Remark 1.22. Note that 1-almost split sequences are nothing but classical almost split sequences.

Definition 1.23. Let $\mathcal{M}$ be a Krull–Schmidt additive category. We say that $\mathcal{M}$ has $d$-almost split sequences if for every non-projective indecomposable object $N \in \mathcal{M}$ (resp. non-injective indecomposable object $L \in \mathcal{M}$) there exists a $d$-almost split sequence $0 \to L \to M^1 \to \cdots \to M^d \to N \to 0$.

It is shown in Theorem 2.3.1 in [Iya07a] that the contravariant functor

$$
\text{Tr}_d := \text{Tr} \circ \Omega^{d-1} : \text{mod} \ A \to \text{mod} \ A^{\text{op}}
$$

induces an adjoint pair of functors

$$
\tau_d := D \circ \text{Tr}_d : \text{mod} \ A \xrightarrow{\sim} \text{mod} \ A : \tau_d^\sim := \text{Tr}_d \circ D
$$
called the $d$-Auslander–Reiten translations. The subsequent result is one of the main motivations for the study of $d$-cluster-tilting subcategories of locally bounded categories. It combines Theorems 2.5.1 and 2.5.3 in [Iya07b] and adapts them to our setting.

**Theorem 1.24.** Let $A$ be a locally bounded category and $\mathcal{M} \subseteq \text{mod} A$ a $d$-cluster-tilting subcategory. Then, the following statements hold.

(i) There are mutually inverse equivalences $$\tau_d:\mathcal{M} \cong \overline{\mathcal{M}}: \tau_d^-.$$ 

(ii) For all $M,N \in \mathcal{M}$ there are bifunctorial isomorphisms $$D\text{Hom}_A(M,N) \cong \text{Ext}^d_A(N,\tau_d M)$$ and $$D\text{Hom}_A(M,N) \cong \text{Ext}^d_A(\tau_d^- N,M).$$ Moreover, if $M$ has projective dimension at most $d$, then there is a bifunctorial isomorphism $$D\text{Hom}_A(M,N) \cong \text{Ext}^d_A(N,\tau_d M),$$ whereas if $N$ has injective dimension at most $d$, then there is a bifunctorial isomorphism $$D\text{Hom}_A(M,N) \cong \text{Ext}^d_A(\tau_d^- N,M).$$

(iii) The additive category $\mathcal{M}$ has $d$-almost split sequences. Moreover, if $$0 \to L \to M^1 \to \cdots \to M^d \to N \to 0$$ is a $d$-almost split sequence in $\mathcal{M}$, then there are isomorphisms $L \cong \tau_d N$ and $N \cong \tau_d^- L$.

In parallel to the introduction of higher Auslander–Reiten theory, Iyama introduced in [Iya07a] the class of weakly $d$-representation-finite algebras which can be thought of as higher analogues of finite dimensional algebras of finite representation type from the viewpoint of Auslander’s correspondence (the precise terminology was introduced in Definition 2.2 of [IO11]). We recall the definition below together with some new terminology which we use in the sequel.

**Definition 1.25.** Let $A$ be a finite dimensional algebra.

(i) We say that $A$ is weakly $d$-representation-finite if there exists a $d$-cluster-tilting $A$-module.

(ii) We say that $A$ is $d$-representation-finite $d$-hereditary if $A$ has global dimension at most $d$ and $A$ is a weakly $d$-representation-finite algebra.

(iii) We say that $A$ is $d\mathbb{Z}$-representation-finite if there exists a $d\mathbb{Z}$-cluster-tilting $A$-module.

**Warning 1.26.** In [IO11] the $d$-representation-finite $d$-hereditary algebras are simply called ‘$d$-representation-finite’. Our choice of terminology is partially motivated by Definition 3.2 in [HIO14] where the notion of ‘$d$-hereditary’ algebra is introduced. Note that, from a homological point of view, $d\mathbb{Z}$-representation-finite algebras are higher analogues of algebras of finite representation type which are not necessarily hereditary.
The subsequent observation gives a restriction on the global dimension of \( d\mathbb{Z}\)-representation-finite algebras.

**Proposition 1.27.** Let \( A \) be a \( d\mathbb{Z}\)-representation-finite algebra of finite global dimension. Then, \( \text{gl.dim} \ A \) is a multiple of \( d \). In particular, if \( A \) is \( d \)-representation-finite \( d \)-hereditary, then either \( A \) has global dimension \( d \) or it is semisimple.

**Proof.** Since \( A \) has finite global dimension, the equality
\[
\text{gl.dim} \ A = \max \{ i \in \mathbb{Z} \mid \text{Ext}^i_A(DA, A) \neq 0 \}
\]
holds. Moreover, given that \( A \) is \( d\mathbb{Z}\)-representation-finite, \( \text{Ext}^i_A(DA, A) \neq 0 \) implies that \( i \in d\mathbb{Z} \). Therefore \( \text{gl.dim} \ A \) is a multiple of \( d \). The second claim is an immediate consequence of the first one. \( \square \)

We recall a result which makes manifest the analogy between \( d \)-representation-finite \( d \)-hereditary algebras and hereditary algebras of finite representation type.

**Theorem 1.28 (Proposition 1.3 in [Iya11]).** Let \( A \) be a \( d \)-representation-finite \( d \)-hereditary algebra. Then,
\[
\text{add} \{ \tau_{-i}^d(A) \mid i \geq 0 \} = \text{add} \{ \tau_{-i}^d(DA) \mid i \geq 0 \} \subseteq \text{mod} \ A
\]
is the unique \( d \)-cluster-tilting subcategory of \( \text{mod} \ A \). In particular, each indecomposable object \( X \in \mathcal{M} \) is of the form \( \tau_{-j}^dP \) for some indecomposable projective \( A \)-module \( P \) and some integer \( 0 \leq j \leq i_P \), where \( i_P < \infty \) is the maximal number \( i \) such that \( \tau_{-i}^dP \neq 0 \).

**Notation 1.29.** Let \( A \) be a \( d \)-representation-finite \( d \)-hereditary algebra. We denote the unique \( d \)-cluster-tilting subcategory of \( \text{mod} \ A \) by \( \mathcal{M}(A) \), see Theorem 1.28. We denote a basic additive generator of \( \mathcal{M}(A) \) by \( M(A) \).

Given a finite dimensional algebra \( A \) of finite global dimension we denote the derived Nakayama functor by
\[
\nu := - \otimes^\mathbb{L}_A DA: D^b(\text{mod} \ A) \to D^b(\text{mod} \ A).
\]
It is well known that \( \nu \) is a Serre functor on \( D^b(\text{mod} \ A) \) and that the autoequivalence \( \nu_1 := \nu[-1] \) is a derived version of the Auslander–Reiten translation, see Section 1.4 in [Hap88] and Theorem 1.2.4 in [RVdB02].

**Notation 1.30.** Let \( A \) be a finite dimensional algebra of finite global dimension and define
\[
\nu_d := \nu[-d]: D^b(\text{mod} \ A) \to D^b(\text{mod} \ A).
\]
This autoequivalence is commonly thought of as a higher analogue of the derived Auslander–Reiten translation \( \nu_1 \). Following [Iya11], we define the subcategory
\[
\mathcal{U}(A) := \text{add} \{ \nu_d^i(DA) \mid i \in \mathbb{Z} \} \subseteq D^b(\text{mod} \ A).
\]

When \( A \) is a \( d \)-representation-finite \( d \)-hereditary algebra, the category \( \mathcal{U}(A) \) plays the role of the derived category of \( \text{add} \mathcal{M}(A) \), as suggested by the following theorem.
Theorem 1.31 (Theorem 1.2.1 in [Iya11]). Let $A$ be a $d$-representation-finite $d$-hereditary algebra. Then,

$$U(A) = \text{add} \{ M \in \mathcal{M}(A)[d] \mid i \in \mathbb{Z} \}$$

is a $d\mathbb{Z}$-cluster-tilting subcategory of $\mathcal{D}^b(\text{mod} \, A)$.

2. The higher Nakayama algebras of type $A$

Let $d$ be a positive integer. In this section we construct the $d$-Nakayama algebras of type $A$. We show that they belong to the class of $d\mathbb{Z}$-representation-finite algebras and establish their basic properties.

2.1. The higher Auslander algebras of type $A$. We begin by recalling Iyama’s original construction of the higher Auslander algebras of type $A$ from [Iya11]. From our viewpoint, these algebras are to be thought of as higher dimensional analogues of the hereditary Nakayama algebras. Oppermann and Thomas gave a combinatorial description of these algebras and their cluster-tilting modules in [OTT2]. Here we give a complementary but closely related description in terms of ordered sequences in a finite linear order. Although the results we need concerning these algebras have all appeared in [OTT2], we give new proofs of most of them using the language of representations of posets.

Setting 2.1. We fix positive integers $d$ and $n$ until further notice.

Consider the linearly ordered set

$$A_n := \{ 0 < 1 < \cdots < n - 1 \}$$

which we view as a locally finite category (or rather as a finite dimensional algebra) as explained in Section 1.2. Therefore we identify $A_n$ with the path algebra of the quiver $0 \to 1 \to \cdots \to n - 1$, see also Examples 1.2 and 1.7.

Definition 2.2. The $d$-Auslander algebra of type $A_n$ is the $d$-cone $A_n^{(d)}$.

A presentation of $A_n^{(d)}$ as a quiver with relations was given Iyama in Definition 6.5 and Theorem 6.7 in [Iya11]. Using the explicit description of the $d$-cone of a poset given in Proposition 1.10 it is elementary to verify that Iyama’s presentation in fact agrees our definition. In order to facilitate this comparison we give an explicit description of the $d$-cone $A_n^{(d)}$ using generators and relations.

By definition, the Gabriel quiver $Q$ of $A_n^{(d)}$ has as vertices the set

$$\text{os}^d_n := \text{os}^d(A_n)$$

of ordered sequences of length $d$ in $A_n$, which are nothing but tuples $\lambda = (\lambda_1, \ldots, \lambda_d)$ of integers satisfying

$$0 \leq \lambda_1 \leq \cdots \leq \lambda_d \leq n - 1.$$ 

For each $\lambda \in \text{os}^d_n$ and each $i \in \{ 1, \ldots, d \}$ such that $\lambda + e_i$ is again an ordered sequence of length $d$ there is an arrow in $Q$ of the form

$$a_i = a_i(\lambda) : \lambda \to \lambda + e_i,$$
where $e_1, \ldots, e_d$ is the standard basis of $\mathbb{Z}^d$. With the above notation, $A^{(d)}_{n}$ is identified with the finite dimensional algebra $\mathbb{k}Q/I$ where $I$ is the two-sided ideal of $\mathbb{k}Q$ generated by the relations

$$a_j(\lambda + e_i)a_i(\lambda) - a_i(\lambda + e_j)a_j(\lambda)$$

for each $\lambda \in \text{os}_d^d$ and each $i, j \in \{1, \ldots, d\}$ such that $i \neq j$. By convention, $a_i(\lambda) = 0$ whenever $\lambda$ or $\lambda + e_i$ are not vertices of $Q$, hence some of the above relations are in fact zero relations. For example, $A^{(1)}_n$ is just the path algebra of the quiver $0 \to 1 \to \cdots \to n-1$. The Gabriel quivers of $A^{(2)}_4$ and $A^{(3)}_4$ are shown in Figure 1.

The choice of terminology in Definition 2.2 is justified by the following theorem, which is a special case of Corollary 1.16 in [Iya11].

**Theorem 2.3.** The algebra $A^{(d)}_{n}$ is $d$-representation-finite $d$-hereditary. In particular, there exists a unique basic $d$-cluster-tilting $A^{(d)}_{n}$-module $M(A^{(d)}_{n})$. Moreover, there is an isomorphism of algebras

$$\text{End}_{A^{(d)}_{n}}(M(A^{(d)}_{n})) \cong A^{(d+1)}_{n}.$$ 

The aim of this subsection is to give a new proof of the following statement, which corresponds to Theorem/Construction 3.4 in [OT12]. It explains our motivation for introducing the inductive construction of the $d$-cone of a poset, see Proposition 1.12.
Theorem 2.4. There is an isomorphism
\[ M(A_n^{(d)}) \cong \bigoplus \{ M(\lambda) \mid \lambda \in \os_n^{d+1} \} \]
where \( M(\lambda) \) denotes the interval module
\[ M[(\lambda_1, \ldots, \lambda_d), (\lambda_2, \ldots, \lambda_{d+1})]. \]
Moreover, for each \( \mu \in \os_n^{d} \) we have
\[ [M(\lambda) : S_\mu] = \begin{cases} 1 & \text{if } \lambda_1 \leq \mu_1 \leq \cdots \leq \lambda_d \leq \mu_d \leq \lambda_{d+1}, \\ 0 & \text{otherwise}. \end{cases} \]

In view of Theorem 2.3 and Theorem 1.28 the proof of Theorem 2.4 amounts to calculating the inverse higher Auslander–Reiten translates of the indecomposable projective modules over the higher Auslander algebras of type \( A \). The following result corresponds to parts (1) and (2) of Theorem 3.6 in [OT12]. It gives an explicit description of the projective and of the injective modules over the higher Auslander algebras of type \( A \).

Proposition 2.5. Let \( \lambda \in \os_n^{d} \). Then, the following statements hold.

(i) The projective \( A_n^{(d)} \)-module at the vertex \( \lambda \) is precisely
\[ P_\lambda = M(0, \lambda_1, \ldots, \lambda_d). \]

(ii) The projective \( A_n^{(d)} \)-module at the vertex \( \lambda \) is precisely
\[ I_\lambda = M(\lambda_1, \ldots, \lambda_d, n-1). \]

Proof. We prove the claim by using the explicit description of the projective \( A_n^{(d)} \)-module \( P_\lambda \) in terms of paths in Gabriel quiver, see for example Lemma III.2.4 in [ASS06]. Proposition 1.10 shows that, up to scaling, there is at most one non-zero path between any two vertices in \( A_n^{(d)} \) and that the existence of such a non-zero path is determined by the interlacing relation. By the definition of the interval module \( M(0, \lambda_1, \ldots, \lambda_d) \), it is enough to show that \( \mu \in \os_n^{d} \) interlaces \( \lambda \) if and only if \( \mu \) belongs to the closed interval
\[ [(0, \lambda_1, \ldots, \lambda_{d-1}), (\lambda_1, \ldots, \lambda_d)] \subset \os_n^{d}. \]
If \( \mu \) interlaces \( \lambda \), then it is clear that \((0, \lambda_1, \ldots, \lambda_{d-1})\) interlaces \( \mu \). But this already implies that \( \mu \) belongs to the above closed interval. On the other hand, since
\[ (0, \lambda_1, \ldots, \lambda_{d-1}) \leadsto (\lambda_1, \ldots, \lambda_d), \]
Proposition 1.10 implies that the closed interval is contained in \( \os_n^{d} \), therefore every \( \mu \) in this interval interlaces \( \lambda \). This proves the statement. Statement (ii) can be shown similarly.

Our next result combines Propositions 3.13 and 3.17 in [OT12]. It gives an explicit description of the higher Auslander translation in \( \text{add} M(A_n^{(d)}) \).

Notation 2.6. Given \( \lambda \in \mathbb{Z}^{d+1} \) we introduce the notation
\[ \tau_d(\lambda) := \lambda - (1, \ldots, 1) \quad \text{and} \quad \tau_d^+(\lambda) := \lambda + (1, \ldots, 1). \]
Proposition 2.7. The following statements hold.

(i) Let $\lambda \in os_{n}^{d+1}$ be such that $\lambda_1 \neq 0$ (that is $M(\lambda)$ is not projective) and

$$0 \to P^{-d} \to P^{-d+1} \to \cdots \to P^0 \to M(\lambda) \to 0$$

a minimal projective resolution of $M(\lambda)$. Then,

$$P^0 \cong P_{\lambda_2,\ldots,\lambda_{d+1}} = M(0, \lambda_2, \ldots, \lambda_{d+1})$$

and for each $i \in \{1, \ldots, d\}$ there is an isomorphism

$$P^{-i} \cong P_{\lambda_{1-1},\lambda_{i+2},\ldots,\lambda_{d+1}}$$

= $M(0, \lambda_1 - 1, \lambda_i - 1, \lambda_{i+2}, \ldots, \lambda_{d+1})$.

(ii) Let $\lambda \in os_{n}^{d+1}$ be such that $\lambda_{d+1} \neq n - 1$ (that is $M(\lambda)$ is not injective) and

$$0 \to M(\lambda) \to I^0 \to \cdots \to I^{d-1} \to I^d \to 0$$

a minimal injective coresolution of $M(\lambda)$. Then,

$$I^0 \cong I_{\lambda_1,\ldots,\lambda_d} = M(\lambda_1, \ldots, \lambda_d, n - 1)$$

and for each $i \in \{1, \ldots, d\}$ there is an isomorphism

$$I^{-i} \cong I_{\lambda_{1-1},\lambda_{d-i-1},\lambda_{d-i+2+1},\ldots,\lambda_{d+1}+1}$$

= $M(\lambda_1, \ldots, \lambda_{d-i-1}, \lambda_{d-i+2+1}, \ldots, \lambda_{d+1} + 1, n - 1)$.

(iii) Let $\lambda \in os_{n}^{d+1}$ be such that $\lambda_1 \neq 0$. Then,

$$\tau_d(M(\lambda)) = M(\tau_d(\lambda)).$$

(iv) Let $\lambda \in os_{n}^{d+1}$ be such that $\lambda_{d+1} \neq n - 1$. Then,

$$\tau_d^{-d}(M(\lambda)) = M(\tau_d^{-d}(\lambda)).$$

Proof. By Proposition 1.12 the morphism

$$M(0, \lambda_2, \ldots, \lambda_{d+1}) \to M(\lambda_1, \ldots, \lambda_d, \lambda_{d+1})$$

is surjective; its kernel is the (interval) submodule of $M(0, \lambda_2, \ldots, \lambda_{d+1})$ whose composition factors are the simple $A_{d}^{(d)}$-modules $S_\mu$ such that

$$(0, \lambda_2, \ldots, \lambda_d) \preceq (\mu_1, \ldots, \mu_d) \preceq (\lambda_2, \ldots, \lambda_{d+1})$$

but

$$(\lambda_1, \ldots, \lambda_d) \npreceq (\mu_1, \ldots, \mu_d).$$

This is precisely the interval module

$$M[(0, \lambda_2, \ldots, \lambda_d), (\lambda_1 - 1, \lambda_3, \ldots, \lambda_{d+1})].$$

Similarly, for each $i \in \{1, \ldots, d\}$ one can verify that the kernel of the morphism

$$M(0, \lambda_1 - 1, \ldots, \lambda_{i-1} - 1, \lambda_{i+1}, \ldots, \lambda_{d+1}) \to M(0, \lambda_1 - 1, \ldots, \lambda_{i-2} - 1, \lambda_i, \ldots, \lambda_{d+1})$$

is the submodule of $M(0, \lambda_1 - 1, \ldots, \lambda_{i-1} - 1, \lambda_{i+1}, \ldots, \lambda_{d+1})$ whose composition factors are the simple $A_{d}^{(d)}$-modules $S_\mu$ such that

$$(0, \lambda_2, \ldots, \lambda_d) \preceq (\mu_1, \ldots, \mu_d) \preceq (\lambda_2, \ldots, \lambda_{d+1})$$

but

$$(\lambda_1, \ldots, \lambda_d) \npreceq (\mu_1, \ldots, \mu_d).$$

This is precisely the interval module

$$M[(0, \lambda_2, \ldots, \lambda_d), (\lambda_1 - 1, \lambda_3, \ldots, \lambda_{d+1})].$$
is precisely the interval module
\[ M[(0, \lambda_1 - 1, \ldots, \lambda_{i-1} - 1, \lambda_{i+1}, \ldots, \lambda_d), (\lambda_1 - 1, \ldots, \lambda_i - 1, \lambda_{i+2}, \ldots, \lambda_{d+1})] \]
which is also the image of the morphism
\[ M(0, \lambda_1 - 1, \ldots, \lambda_i - 1, \lambda_{i+2}, \ldots, \lambda_{d+1}) \to M(0, \lambda_1 - 1, \ldots, \lambda_{i-1} - 1, \lambda_{i+1}, \ldots, \lambda_{d+1}). \]

This proves the claim. The proof of statement (iii) is dual.

Let
\[ 0 \to P^{-d} \to P^{-d+1} \to \cdots \to P^0 \to M(\lambda) \to 0 \]
be the minimal projective resolution of \( M(\lambda) \) constructed in the previous step. By definition, \( \tau_d(M(\lambda)) \) is isomorphic to the kernel of the morphism \( \nu(P^{-d}) \to \nu(P^{-d+1}) \), that is the morphism
\[ M(\lambda_1 - 1, \ldots, \lambda_{d-1} - 1, n - 1) \to M(\lambda_1 - 1, \ldots, \lambda_{d-1} - 1, \lambda_{d+1}, n - 1). \]

By Proposition 2.12 this is precisely the interval module
\[ M(\tau_d(\lambda)) = M(\lambda_1 - 1, \ldots, \lambda_{d-1} - 1, \lambda_{d+1} - 1). \]

The proof of statement (iv) is dual. \( \square \)

We are ready to give our proof of Theorem 2.4

**Proof of Theorem 2.4.** According to Theorem 1.28 it is enough to show that
\[ \bigoplus \{ \tau_d^{-i}(P_{\mu}) \mid \mu \in \mathfrak{os}^{d+n} \text{ and } i \geq 0 \} = \bigoplus \{ M(\lambda) \mid \lambda \in \mathfrak{os}^{d+1} \}. \]

For this, observe that Proposition 2.7 implies that for every \( \lambda \in \mathfrak{os}^{d+1} \) there is an isomorphism
\[ M(\lambda) \cong \tau_d^{-\lambda_1}(P_{\lambda_2 - \lambda_1, \ldots, \lambda_{d+1} - \lambda_1}). \]

Moreover, Proposition 2.7 also shows that the iterated \( \tau_d^{-} \)-translates of the indecomposable projective \( A_n^{(d)} \)-modules are always modules of the form \( M(\lambda) \) for some \( \lambda \in \mathfrak{os}^{d+1} \). This proves the claim. \( \square \)

The following result corresponds to parts (3) and (4) in Theorem 3.6 [OT12]. It shows that \( A_n^{(d)} \) and \( \text{add} M(A_n^{(d)}) \) are, also from a combinatorial perspective, higher dimensional analogues of \( A_n \) and of its module category.

**Proposition 2.8.** Let \( \lambda, \mu \in \mathfrak{os}^{d+1} \). Then, there are isomorphisms
\[ \text{Hom}_{A_n^{(d)}}(M(\lambda), M(\mu)) \cong \begin{cases} \mathbb{k} & \text{if } \lambda \leadsto \mu, \\ 0 & \text{otherwise} \end{cases} \]
and
\[ \text{Ext}_{A_n^{(d)}}^{d}(M(\lambda), M(\mu)) \cong \begin{cases} \mathbb{k} & \text{if } \mu \leadsto \tau_d(\lambda), \\ 0 & \text{otherwise}. \end{cases} \]
Proof. The first isomorphism is proven in Proposition 1.12. The second isomorphism follows from the higher dimensional Auslander–Reiten formulas given in statement ii in Theorem 1.24, taking into account the formulas for the higher Auslander–Reiten translate proven in statements iii and iv in Proposition 2.7 and the fact that $M(\lambda)$ has projective dimension $d$, see Proposition 2.7(i). □

We conclude our study of the higher Auslander algebras of type $A$ with a simple observation.

Lemma 2.9. Let $\lambda \in \mathfrak{so}_n^{d+1}$. Then, the interval module $M(\lambda)$ has Loewy length $\lambda_{d+1} - \lambda_1 + 1$.

Proof. Recall that the interval module $M(\lambda)$ has simple top $S_{\lambda_2,\ldots,\lambda_{d+1}}$ and simple socle $S_{\lambda_1,\ldots,\lambda_d}$. The Loewy length of $M(\lambda)$ is then 1 plus the length of the longest path from $(\lambda_1,\ldots,\lambda_d)$ to $(\lambda_2,\ldots,\lambda_{d+1})$ in $\mathfrak{so}_n^d$ which can easily seen to have length $(\lambda_{d+1} - \lambda_d) + (\lambda_d - \lambda_{d-1}) + \cdots + (\lambda_2 - \lambda_1) = \lambda_{d+1} - \lambda_1$. □

Motivated by Lemma 2.9, we make the following definition.

Definition 2.10. Let $\lambda \in \mathfrak{so}_n^{d+1}$. We define the Loewy length of $\lambda$ to be

$$\ell(\lambda) := \ell(\lambda_{d+1} - \lambda_1 + 1) \in \{1,\ldots,n\}.$$ 

2.2. The higher Nakayama algebras of type $A$. We are ready to introduce the higher Nakayama algebras of type $A$. In the classical case, the Nakayama algebras of type $A$ are admissible quotients of path algebras of the linearly oriented quivers of type $A$. In contrast, the higher Nakayama algebras of type $A$ are idempotent quotients of the higher Auslander algebras of type $A$.

Setting 2.11. We fix positive integers $d$ and $n$ until further notice.

We begin by recalling the notion of a Kupisch series of type $A$, originally introduced in [Kup59].

Definition 2.12. Let $\ell = (\ell_0, \ell_1, \ldots, \ell_{n-1})$ be a tuple of positive integers. We say that $\ell$ is a (connected) Kupisch series of type $A_n$ if $\ell_0 = 1$ and for all $i \neq 0$ there are inequalities

$$2 \leq \ell_i \leq \ell_{i-1} + 1.$$ 

We denote the set of Kupisch series of type $A_n$ by $\mathcal{KS}(A_n)$.

Remark 2.13. Suppose that the ground field is algebraically closed. In this case, it is well known that there is a bijective correspondence between Morita equivalence classes of (connected) Nakayama algebras of type $A$ and Kupisch series of type $A$. In one direction the correspondence is given by associating the tuple

$$(\ell(\ell_0 A), \ell(\ell_1 A), \ldots, \ell(\ell_{n-1} A))$$

to a Nakayama algebra $A$ with Gabriel quiver $0 \rightarrow 1 \rightarrow \cdots \rightarrow n - 1$.

Notation 2.14. Let $\ell$ be a Kupisch series of type $A_n$. We denote the basic Nakayama algebra with Kupisch series $\ell$ by $A^{(1)}_{\ell}$. Note that, by definition, $A^{(1)}_{\ell}$ is an admissible quotient of $A_n$. 

We need some elementary observations concerning Kupisch series which follow by an easy induction from the inequalities \( \ell_i \leq \ell_{i-1} + 1 \).

**Lemma 2.15.** Let \( \ell \) be a Kupisch series of type \( A_n \). Then, the following statements hold.

(i) For all \( i \leq j \) the inequality \( i - j \leq \ell_i - \ell_j \) is satisfied.

(ii) For all \( i \in \{0, 1, \ldots, n - 1\} \) the inequality \( \ell_i \leq i + 1 \) is satisfied.

The product order on \( \mathbb{Z}^n \) endows the set \( \text{KS}(A_n) \) of Kupisch series of type \( A_n \) with the structure of a poset. We need the following simple observations, the proofs of which are also left to the reader.

**Lemma 2.16.** The following statements hold.

(i) The Kupisch series \( (1, 2, \ldots, n) \) is the unique maximal element in \( \text{KS}(A_n) \).

(ii) The Kupisch series \( (1, 2, \ldots, 2) \) is the unique minimal element in \( \text{KS}(A_n) \).

(iii) The Hasse quiver of \( \text{KS}(A_n) \) is connected.

(iv) Two Kupisch series \( \ell \leq \ell' \) are neighbours in the Hasse quiver of \( \text{KS}(A_n) \) then they are also neighbours in \( \mathbb{Z}^n \).

Let \( \ell \) be a Kupisch series of type \( A_n \) and \( \lambda = (\lambda_1, \lambda_2) \in \text{os}^2_n \). It is well known and elementary to verify that the \( A^{(1)}_n \)-module \( M(\lambda) \) is an \( A^{(1)}_\ell \)-module if and only if

\[
\ell(\lambda) = \ell(M(\lambda)) \leq \ell(P_{\lambda_2}) = \ell_{\lambda_2}
\]

where \( P_{\lambda_2} \) is the indecomposable projective \( A^{(1)}_\ell \)-module with top \( S_{\lambda_2} \). This observation motivates our definition of the higher Nakayama algebras of type \( A \).

**Definition 2.17.** Let \( \ell \) be a Kupisch series of type \( A_n \).

(i) The \( \ell \)-restriction of \( \text{os}^{d+1}_n \) is the subset

\[
\text{os}^{d+1}_\ell := \{ \lambda \in \text{os}^{d+1}_n \mid \ell(\lambda) \leq \ell_{\lambda_{d+1}} \}.
\]

(ii) The \((d + 1)\)-Nakayama algebra with Kupisch series \( \ell \) is the finite dimensional algebra

\[
A^{(d+1)}_\ell := A^{(d+1)}_n / \text{os}^{d+1}_n \setminus \text{os}^{d+1}_\ell.
\]

(iii) The \( A^{(d)}_\ell \)-module \( M^{(d)}_\ell \) is by definition

\[
M^{(d)}_\ell := \bigoplus \{ M(\lambda) \mid \lambda \in \text{os}^{d+1}_\ell \}.
\]

Let \( \ell \) be a Kupisch series of type \( A_n \). Note that the condition for \( \lambda \in \text{os}^{d+1}_n \) to belong to \( \text{os}^{d+1}_\ell \) only makes use of the pair \( (\lambda_1, \lambda_{d+1}) \).

The purpose of this section is to prove the following theorem, which establishes the higher Auslander–Reiten theoretical nature of the higher Nakayama algebras of type \( A \).

**Theorem 2.18.** Let \( \ell \) be a Kupisch series of type \( A_n \).

(i) For each \( i \in \{0, 1, \ldots, n - 1\} \) the indecomposable projective \( A^{(d)}_\ell \)-module at the vertex \( (i, \ldots, i) \) has length \( \ell_i \).
Figure 2. The Gabriel quivers of $A^{(2)}_\ell$ (top) and $A^{(3)}_\ell$ (bottom) for $\ell = (1, 2, 2, 3)$.

(ii) $M^{(d)}_\ell$ is a $d\mathbb{Z}$-cluster-tilting $A^{(d)}_\ell$-module. In particular, $A^{(d)}_\ell$ is a $d\mathbb{Z}$-representation-finite algebra.

(iii) For every simple $A^{(d)}_\ell$-module $S$ which is a direct summand of $M^{(d)}_\ell$ the $A^{(d)}_\ell$-module $\tau_i^d(S)$ is either simple or zero for all $i \in \mathbb{Z}$.

Remark 2.19. Statement (i) in Theorem 2.18 can be interpreted as saying that $A^{(d)}_\ell$ has Kupisch series $\ell$. Statement (ii) in Theorem 2.18 shows that $M^{(d)}_\ell$ is a higher dimensional analogue of $M^{(1)}_\ell$, the standard additive generator of $\text{mod} A^{(1)}_\ell$. Finally, statement (iii) in Theorem 2.18 should be compared with Theorem IV.2.10 in [ARS97] which states that an Artin algebra $A$ is a Nakayama algebra if and only if the $\tau$-orbit of each simple $A$-module consists entirely of simple $A$-modules.

We divide the proof of Theorem 2.18 into several parts. Since the case $d = 1$ is classical we restrict our attention to the case $d \geq 2$.

Setting 2.20. We fix integers $d \geq 2$ and $n \geq 1$ and a Kupisch series $\ell$ of type $A_n$ until the end of this section.

Proposition 2.21. The $A^{(d)}_\ell$-module $M^{(d)}_\ell$ is in fact an $A^{(d)}_\ell$-module.

Proof. Let $\lambda \in \text{os}_\ell^{d+1}$ and $\mu \in \text{os}_\ell^d$ be such that $[M(\lambda), S_\mu] \neq 0$; we need to prove that $\mu \in \text{os}_\ell^d$.

Let

$$x := \lambda_{d+1} + 1 - \ell_{\lambda_{d+1}}.$$ 

Note that the inequality $\ell(\lambda) \leq \ell_{\lambda_{d+1}}$, which holds by assumption, is equivalent to

$$\lambda_1 - x = \lambda_1 - (\lambda_{d+1} + 1 - \ell_{\lambda_{d+1}}) \geq 0.$$ 

Therefore $x \leq \lambda_1 \leq \mu_1 \leq \mu_d \leq \lambda_{d+1}$. We claim that there are inequalities

$$\ell(\mu) = \mu_d - \mu_1 + 1 \leq \mu_d - x + 1 \leq \ell_{\mu_d}.$$
The inequality on the left is clear. The inequality on the right is equivalent to
\[ \mu_d - \lambda_{d+1} \leq \ell_{\mu_d} - \ell_{\lambda_{d+1}} \]
and therefore follows from Lemma 2.15. This shows that \( \mu \in \text{os}_d^d \). \( \square \)

The key ingredient in the proof of Theorem 2.18 is Lemma 1.20. To use it we need detailed knowledge of the projective and of the injective modules over the higher Nakayama algebras of type \( \mathbb{A} \).

**Proposition 2.22.** Let \( \lambda \in \text{os}_d^d \). Then, the following statements hold.

(i) The indecomposable projective \( A_\lambda^{(d)} \)-module with top \( S_\lambda \) is precisely
\[ P_\lambda = M(x, \lambda_1, \ldots, \lambda_d) \in \text{add} \ M_\lambda^{(d)} \]
where \( x = \lambda_d + 1 - \ell_{\lambda_d} \). Moreover,
\[ x = \min \left\{ 0 \leq x' \leq \lambda_1 \mid (x', \lambda_1, \ldots, \lambda_d) \in \text{os}_d^{d+1} \right\} . \]

(ii) Let \( \lambda \in \text{os}_d^d \). The indecomposable injective \( A_\lambda^{(d)} \)-module with socle \( S_\lambda \) is precisely
\[ I_\lambda = M(\lambda_1, \ldots, \lambda_d, y) \in \text{add} \ M_\lambda^{(d)}, \]
where
\[ y := \max \left\{ \lambda_d \leq y' \leq n - 1 \mid (\lambda_1, \ldots, \lambda_d, y') \in \text{os}_d^{d+1} \right\} . \]

**Proof.** (i) Note that statement (ii) in Lemma 2.15 implies that \( x \geq 0 \). Moreover, the inequality
\[ \lambda_1 - x = \lambda_1 - \lambda_d - 1 + \ell_{\lambda_d} = -\ell(\lambda) + \ell_{\lambda_d} \geq 0 \]
holds by assumption. It follows that \( (x, \lambda_1, \ldots, \lambda_d) \in \text{os}_d^{d+1} \) and therefore
\[ M(x, \lambda_1, \ldots, \lambda_d) \in \text{add} \ M_\lambda^{(d)}. \]
Suppose now that \( \mu \in \text{os}_d^d \) is such that there exists a non-zero path \( \mu \rightarrow \lambda \). By Proposition 1.10 and the definition of \( A_\lambda^{(d)} \) this is equivalent to the inequalities
\[ \mu_1 \leq \lambda_1 \leq \cdots \leq \mu_d \leq \lambda_d \]
being satisfied together with \( [\mu, \lambda] \subset \text{os}_d^d \). We need to prove that \( x \leq \mu_1 \) or, equivalently
\[ \ell(\mu_1, \lambda_2, \ldots, \lambda_d) = \lambda_d - \mu_1 + 1 \leq \ell_{\lambda_d}. \]
But this follows from the fact that \( (\mu_1, \lambda_2, \ldots, \lambda_d) \in [\mu, \lambda] \subset \text{os}_d^d \). This shows that \( P_\lambda = M(x, \lambda_1, \ldots, \lambda_d) \).

(ii) By construction \( (\lambda_1, \ldots, \lambda_d, y) \in \text{os}_d^{d+1} \) whence
\[ M(\lambda_1, \ldots, \lambda_d, y) \in \text{add} \ M_\lambda^{(d)}. \]
Suppose now that \( \mu \in \text{os}_d^d \) is such that there exists a non-zero path \( \lambda \rightarrow \mu \). By Proposition 1.10 and the definition of \( A_\lambda^{(d)} \) this is equivalent to the inequalities
\[ \lambda_1 \leq \mu_1 \leq \cdots \leq \lambda_d \leq \mu_d \]
being satisfied together with \( [\lambda, \mu] \subset \mathfrak{o}_2^d \). We need to prove that \( \mu_d \leq y \). For this, note that \( (\lambda_1, \ldots, \lambda_{d-1}, \mu_d) \in [\lambda, \mu] \subset \mathfrak{o}_2^d \) which immediately implies that \( (\lambda_1, \mu_d) \in \mathfrak{o}_2^2 \). The claim now follows from the maximality of \( y \).

The following statement settles part \( [\mathfrak{I}] \) in Theorem 2.18.

**Corollary 2.23.** Let \( \lambda \in \mathfrak{o}_2^d \). Then, \( \ell(P_{\lambda}) = \ell_{\lambda_d} \). In particular, for every \( i \in \{0, 1, \ldots, n - 1\} \), the projective cover of the simple \( A_{\mathfrak{L}}^{(d)} \)-module at the vertex \((i, \ldots, i)\) has Loewy length \( \ell_i \).

**Proof.** By Proposition 2.22, the projective cover of the simple \( A_{\mathfrak{L}}^{(d)} \)-module at the vertex \((i, \ldots, i)\) is

\[
P_{(i, \ldots, i)} = M(i + 1 - \ell_i, i, \ldots, i)
\]

whence

\[
\ell(P_{(i, \ldots, i)}) = i - (i + 1 - \ell_i) + 1 = \ell_i
\]

as required.

We continue the proof of Theorem 2.18 by showing that the higher Nakayama algebras are weakly \( d \)-representation-finite.

**Proposition 2.24.** The \( A_{\mathfrak{L}}^{(d)} \)-module \( M_{\mathfrak{L}}^{(d)} \) is \( d \)-cluster-tilting.

**Proof.** We use the partial order on the set of Kupisch series of type \( \mathcal{A}_n \). Suppose that the claim holds for some Kupisch series \( \ell' \geq \ell \) which is a neighbour of \( \ell \) in the Hasse quiver of the poset of Kupisch series of type \( \mathcal{A}_n \). We claim that the statement holds for \( \ell \) in this case. To prove this, it is enough to verify the hypotheses of Lemma 1.20.

**Step 1:** \( \mathfrak{o}_2^d \subset \mathfrak{o}_2^d \). Let \( \lambda \in \mathfrak{o}_2^d \). Then

\[
\lambda_d - \lambda_1 + 1 \leq \ell_{\lambda_d} \leq \ell'_{\lambda_d}
\]

since \( \ell \leq \ell' \). Note that this fact, together with Proposition 2.22, implies that the projective and the injective \( A_{\mathfrak{L}}^{(d)} \)-modules are direct summands of \( M_{\mathfrak{L}}^{(d)} \).

**Step 2:** If \( \lambda \in \mathfrak{o}_2^{d+1} \setminus \mathfrak{o}_2^{d+1} \), then \( M(\lambda) \) is projective-injective as an \( A_{\mathfrak{L}}^{(d)} \)-module. By assumption,

\[
\ell_{\lambda_{d+1}} < \ell(\lambda) \leq \ell'_{\lambda_{d+1}}.
\]

Since \( \ell \) and \( \ell' \) are neighbours, we conclude that \( \ell_{\lambda_{d+1}} + 1 = \ell'_{\lambda_{d+1}} \) and, since \( \ell(\lambda) \) is an integer, that \( \ell(\lambda) = \ell'_{\lambda_{d+1}} \). The last equality is equivalent to

\[
\lambda_1 = \lambda_{d+1} + 1 - \lambda_{d+1} = \lambda_{d+1} + 1 - (\ell_{\lambda_{d+1}} + 1) = \lambda_{d+1} - \lambda_{d+1}.
\]

In view of the rightmost equality, the module \( M(\lambda) \) is a projective \( A_{\mathfrak{L}}^{(d)} \)-module by Proposition 2.22.

Now we prove that \( M(\lambda) \) is an injective \( A_{\mathfrak{L}}^{(d)} \)-module. By Proposition 2.22, it is enough to prove that

\[
\lambda_{d+1} = y := \max \left\{ \lambda_d \leq y' \leq n - 1 \mid (\lambda_1, y') \in \mathfrak{o}_2^2 \right\}.
\]

Suppose that \( \lambda_{d+1} + 1 \leq y \). In view of Theorem 2.4 and the inequalities

\[
\lambda_1 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{d-1} = \lambda_{d-1} \leq \lambda_1 \leq \lambda_{d+1} + 1 \leq y
\]
the simple $A^{(d)}_n$-module at the vertex $(\lambda_1, \ldots, \lambda_{d-1}, \lambda_{d+1} + 1)$ is a composition factor of the $A^{(d)}_n$-module $M(\lambda_1, \ldots, \lambda_{d-1}, \lambda_{d+1})$. In particular, $(\lambda_1, \ldots, \lambda_{d-1}, \lambda_{d+1} + 1) \in \text{os}_\lambda$. We claim that this is impossible. To show this, we observe that the fact that $\ell$ is a Kupisch series implies

$$\ell_{\lambda_{d+1}+1} = \ell_{\lambda_{d+1}} + 1 = \ell_{\lambda_{d+1}} = \ell(\lambda).$$

But this inequality already implies $(\lambda_1, \ldots, \lambda_{d-1}, \lambda_{d+1} + 1) \notin \text{os}_\lambda$. Indeed,

$$\ell_{\lambda_{d+1}+1} \leq \ell\ell(\lambda) < \ell\ell(\lambda) + 1 = \lambda_{d+1} + 2 - \lambda_1.$$

This proves our claim. We conclude that $\lambda_{d+1} = y$, as required.

Since we have verified the hypotheses in Lemma 1.20, we have proven that $M^{(d)}_\lambda$ is a $d$-cluster-tilting $A^{(d)}_\lambda$-module in this case. The general case now follows since the claim holds for the Kupisch series $(1, 2, \ldots, n)$ by Theorems 2.3 and 2.4 and the Hasse quiver of the poset of Kupisch series of type $\mathbb{A}_n$ is connected. \hfill \Box

The following result implies that the higher Nakayama algebras of type $\mathbb{A}$ are moreover $d\mathbb{Z}$-representation-finite. It settles Theorem 2.18.

**Proposition 2.25.** Suppose that $d \geq 2$. Then, add $M^{(d)}_\lambda$ is closed under taking $d$-th syzygies. In particular, $M^{(d)}_\lambda$ is a $d\mathbb{Z}$-cluster-tilting $A^{(d)}_\lambda$-module.

**Proof.** Let $\lambda \in \text{os}_{\lambda}^{d+1}$ be such that $M(\lambda)$ is not projective as an $A^{(d)}_\lambda$-module; by Proposition 2.22 this means that $x < \lambda_1$ where $x = \lambda_{d+1} + 1 - \ell_{\lambda_{d+1}}$. Let

$$0 \to \Omega^{d}(M(\lambda)) \to P^{-d+1} \to \cdots \to P^0 \to M(\lambda) \to 0$$

be part of a minimal projective resolution of $M(\lambda)$. A straightforward calculation analogous to that in the proof of Proposition 2.7 shows that

$$P^0 \cong P_{\lambda_2, \ldots, \lambda_{d+1}} = M(x, \lambda_2, \ldots, \lambda_{d+1}),$$

that for each $i \in \{1, \ldots, d-1\}$ there is an isomorphism

$$P^{-i} \cong P_{\lambda_1-1, \ldots, \lambda_i-1, \lambda_i+2, \ldots, \lambda_{d+1}} = M(x, \lambda_1 - 1, \ldots, \lambda_i - 1, \lambda_{i+2}, \ldots, \lambda_{d+1}),$$

and finally that

$$\Omega^{d}(M(\lambda)) \cong M(x, \lambda_1 - 1, \ldots, \lambda_d - 1).$$

Note that we use Proposition 2.22 to identify the indecomposable projective $A^{(d)}_\lambda$-modules appearing in the exact sequence above. We need to show that

$$(x, \lambda_1 - 1, \ldots, \lambda_d - 1) \in \text{os}_{\lambda}^{d+1}.$$

Indeed, by Lemma 2.15 there is an inequality

$$(\lambda_d - 1) - \lambda_{d+1} \leq \ell_{\lambda_d - 1} - \ell_{\lambda_{d+1}}.$$

This inequality is equivalent to

$$\ell\ell(x, \lambda_1 - 1, \ldots, \lambda_d - 1) = \lambda_d - x = \lambda_d - (\lambda_{d+1} + 1 - \ell_{\lambda_{d+1}}) \leq \ell_{\lambda_d - 1},$$
which is what we needed to prove. The second claim in the proposition follows from Definition-Proposition 2.15 in [J16] which states that a $d$-cluster-tilting subcategory of a module category is $d\mathbb{Z}$-cluster-tilting if and only if it is closed under taking $d$-th syzygies.

Finally, Theorem 2.18(iii) follows from the following more general result.

**Proposition 2.26.** Let $\lambda \in \text{os}_d^{d+1}$ be such that $M(\lambda)$ is not projective as an $A_{\mathbb{L}}^{(d)}$-module. Then, there is an isomorphism $\tau_d(M(\lambda)) \cong M(\tau_d(\lambda))$.

**Proof.** Since by assumption $M(\lambda)$ is not projective as an $A_{\mathbb{L}}^{(d)}$-module, Proposition 2.22 shows that

$$\ell(\lambda) = \lambda_{d+1} - \lambda_1 + 1 \leq \ell_{\lambda_{d+1}} - 1.$$  

Equivalently, in particular $\lambda_1 \neq 0$ (see Lemma 2.15). By Proposition 2.5(i) $M(\lambda)$ is not projective as an $A_n^{(d)}$-module either. Therefore, since the claim holds for $\ell = (1, 2, \ldots, n)$, by Proposition 2.7(ii) there is a $d$-almost split sequence

$$0 \to M(\tau_d(\lambda)) \to M^1 \to \cdots \to M^d \to M(\lambda) \to 0$$

in $\text{add} \ M_n^{(d)}$. In view of Theorem 1.24 it enough to show that this sequence is contained in $\text{add} \ M_{\mathbb{L}}^{(d)}$.

We recall from Theorem 3.8(4) in [OT12] that

$$M(\lambda) \oplus M^1 \oplus \cdots \oplus M^d \oplus M(\tau_d(\lambda)) = \bigoplus \{ M(\mu) \mid \mu \in [\tau_d(\lambda), \lambda] \subset \text{os}_n^{d+1} \}.$$  

Hence we need to show that $[\tau_d(\lambda), \lambda] \subset \text{os}_d^{d+1}$. Note that $\mu \in [\tau_d(\lambda), \lambda]$ must satisfy

$$(\mu_1, \mu_{d+1}) \in \{ (\lambda_1 - 1, \lambda_{d+1} - 1), (\lambda_1, \lambda_{d+1} - 1), (\lambda_1 - 1, \lambda_{d+1}), (\lambda_1, \lambda_{d+1}) \}.$$  

Let $(\mu_1, \mu_{d+1}) = (\lambda_1 - 1, \lambda_{d+1} - 1)$. Then

$$\ell(\tau_d(\lambda)) = \lambda_{d+1} - 1 - (\lambda_1 - 1) + 1 = \ell(\lambda) \leq \ell_{\lambda_{d+1}} - 1 \leq \ell_{\lambda_{d+1}-1}$$

where the rightmost inequality holds since $\ell$ is a Kupisch series. This shows that $\mu \in \text{os}_d^{d+1}$ in this case. It is straightforward to verify that $\mu \in \text{os}_d^{d+1}$ in the remaining cases. \hfill \Box

We conclude this section with an immediate consequence of Proposition 2.26 which should be compared with Corollary 2.9 in [ARS97].

**Corollary 2.27.** Let $M$ be an indecomposable direct summand of $M_n^{(d)}$ of Loewy length $\ell$. Then, every non-zero $A_{\mathbb{L}}^{(d)}$-module in the $\tau_d$-orbit of $M$ also has Loewy length $\ell$.

**Proof.** Let $\lambda \in \text{os}_d^{(d)}$ be such that $M(\lambda)$ is non-projective as an $A_{\mathbb{L}}^{(d)}$-module. By Lemma 2.9 and Proposition 2.26 we have

$$\ell(\tau_d(M(\lambda))) = \ell(\tau_d(\lambda)) = \lambda_{d+1} - 1 - (\lambda_1 - 1) + 1 = \ell(\lambda),$$

as required. \hfill \Box
3. The higher Nakayama algebras of type $\mathbb{A}_\infty$

In this section we introduce a family of categories which are to be thought of as higher dimensional analogues of the mesh category of type $\mathbb{A}_\infty$. We also introduce infinite analogues of the Nakayama algebras of type $\mathbb{A}$. The latter categories are essential for our construction of the higher Nakayama algebras of type $\tilde{\mathbb{A}}$ and the higher dimensional analogues of the tubes in Section 4.

3.1. The mesh category of type $\mathbb{A}^{(d-1)}_\infty$. We begin this section by introducing the higher dimensional analogues of the mesh categories of type $\mathbb{A}^{(d-1)}_\infty$. Setting 3.1. We fix a positive integer $d$ until further notice.

Consider the poset of integer numbers $A_\infty = \{ \cdots, -1, 0, 1, \cdots \}$ and denote the set of ordered sequences of length $d$ in $A_\infty$ by $os^d$. Define $M^{(d)} := \text{add} \{ M(\lambda) \in \text{mod} A^{(d)}_\infty \mid \lambda \in os^{d+1} \} \subseteq \text{mod} A^{(d)}_\infty$.

Definition 3.2. The mesh category of type $\mathbb{A}^{(d-1)}_\infty$ is the $d$-cone $A^{(d)}_\infty$. We also define the subcategory $M^{(d)} := \text{add} \{ M(\lambda) \in \text{mod} A^{(d)}_\infty \mid \lambda \in os^{d+1} \} \subseteq \text{mod} A^{(d)}_\infty$.

A presentation of $A^{(d)}_\infty$ by generators and relations can be given as follows. By definition, the Gabriel quiver $Q$ of $A^{(d)}_\infty$ has as vertices the set $os^d$, that is the set of tuples $\lambda = (\lambda_1, \ldots, \lambda_d)$ of integers satisfying $\lambda_1 \leq \cdots \leq \lambda_d$.

For each $\lambda \in os^d$ and each $i \in \{1, \ldots, d\}$ such that $\lambda + e_i$ is again an ordered sequence there is an arrow in $Q$ of the form $a_i = a_i(\lambda) : \lambda \to \lambda + e_i$.

With the above notation, $A^{(d)}_\infty$ is identified with the (non-unital) algebra $kQ/I$ where $I$ is the two-sided ideal of $kQ$ generated by the relations $a_j(\lambda + e_i)a_i(\lambda) - a_i(\lambda + e_j)a_j(\lambda)$.

Remark 3.3. The choice of terminology in Definition 3.2 can be justified as follows. For $s \in \mathbb{Z}$ define $os^{d+1}(s) := \{ \lambda \in os^{d+1} \mid \lambda_1 = s \}$. 

Figure 3. The Gabriel quivers of $A^{(2)}_{\infty}$ (top) and $A^{(3)}_{\infty}$ (bottom).

It is clear that there is a bijection

$$\bigcup_{s \in \mathbb{Z}} \text{os}^{d+1}(s) = \text{os}^{d+1} \to \text{os}^{d} \times \mathbb{Z}$$

given by $\lambda \mapsto ((\lambda_2 - \lambda_1, \ldots, \lambda_{d+1} - \lambda_1), \lambda_1)$. Note that the subcategory of $A^{(d+1)}_{\infty}$ spanned by $\text{os}^{d+1}(s)$ can be thought of as a higher dimensional analogue of the path category of the quiver

$$(s, s) \to (s, s + 1) \to (s, s + 2) \to \cdots,$$

which corresponds to the case $d = 1$. Also, note that for each $\lambda \in \text{os}^{d+1}$ there is a unique arrow $\lambda \to \lambda + e_1$ precisely when $\lambda_1 + 1 \leq \lambda_2$. Therefore the Gabriel quiver of $A^{(d+1)}_{\infty}$ can equivalently be described as the quiver with vertex set $\text{os}^{d} \times \mathbb{Z}$ and arrows

$$b_i = b_i(\lambda, s): (\lambda, s) \to (\lambda + e_i, s)$$

for each $i \in \{1, \ldots, d\}$, whenever $\lambda + e_i$ belongs to $\text{os}^{d}$ (these arrows correspond to arrows

$$a_{i+1} = a_{i+1}(\lambda): \lambda \to \lambda + e_{i+1}$$

for $i = 1, \ldots, d$ in our original description of $A^{(d+1)}_{\infty}$) as well as ‘connecting arrows’

$$b_0 = b_0(\lambda, s): (\lambda, s) \to (\tau_d(\lambda), s + 1)$$

whenever $\lambda_1 > 0$ (these arrows correspond to arrows

$$a_1 = a_1(\lambda): \lambda \to \lambda + e_1$$

in our original description of $A^{(d+1)}_{\infty}$). Under this bijection, the defining relations of $A^{(d+1)}_{\infty}$ are naturally divided into three types:
(I) For each $(\lambda, s) \in \text{os}_d^d \times \mathbb{Z}$ and each $1 \leq i < j \leq d$ there is a relation
\[ b_j(\lambda + e_i, s)b_i(\lambda, s) - b_i(\lambda + e_j, s)b_j(\lambda, s). \]

(II) For each $(\lambda, s) \in \text{os}_d^d \times \mathbb{Z}$ and each $1 \leq j \leq d$ there is a relation
\[ b_j(\tau_d(\lambda), s + 1)b_0(\lambda, s) - b_0(\lambda + e_j, s)b_j(\lambda, s). \]

(III) For each $(\lambda, s) \in \text{os}_d^d \times \mathbb{Z}$ and each $1 \leq i \leq d$ there is a relation
\[ b_0(\lambda + e_i, s)b_i(\lambda, s) - b_i(\tau_d(\lambda + e_j), s + 1)b_0(\lambda, s). \]

As usual, some of the above commutativity relations are in fact zero relations, depending on whether the arrows involved are present in the quiver or not. Note that the Gabriel quiver of $A_{\infty}^{(d)}$ is just the repetitive quiver $\mathbb{Z}A_{\infty}$ and the above relations reduce to the mesh relations (there are no relations of type (I)).

**Remark 3.4.** The category $A_{\infty}^{(d)}$ is not locally bounded. Indeed, given $\lambda \in \text{os}^d$, for each $i > 0$ there are obvious interlacings $\lambda \rightarrow (\lambda + ie_d)$ and $(\lambda - i e_d) \rightarrow \lambda$. In view of Proposition 1.10 these interlacings correspond to non-zero morphisms $\lambda \rightarrow (\lambda + ie_{d+1})$ and $(\lambda - i e_{d+1}) \rightarrow \lambda$ in $A_{\infty}^{(d)}$.

Although the category $A_{\infty}^{(d)}$ is not locally bounded, it can be ‘approximated’ by locally bounded categories. Let us make this statement precise.

**Notation 3.5.** Let $a \leq b$ be integers and define
\[ \text{os}_{[a,b]}^{d+1} := \{ \lambda \in \text{os}^{d+1} \mid a \leq \lambda_1 \leq \cdots \leq \lambda_d \leq \lambda_{d+1} \leq b \} . \]

By construction, the idempotent quotient
\[ A_{[a,b]}^{(d)} := A_{\infty}^{(d)}/[\text{os}_d^d \setminus \text{os}_{[a,b]}^{d+1}] \]

is isomorphic to $A_n^{(d)}$ where $n = b - a + 1$. The unique $d$-cluster-tilting subcategory of $\mod A_{[a,b]}^{(d)}$ is precisely
\[ \mathcal{M}_{[a,b]}^{(d)} := \text{add} \left\{ M(\lambda) \in \mod A_{[a,b]}^{(d)} \mid \lambda \in \text{os}_{[a,b]}^{d+1} \right\} \]

(compare with Theorem 2.4). As explained in Subsection 1.1, the canonical functor $A_{\infty}^{(d)} \rightarrow A_{[a,b]}^{(d)}$ induces a fully faithful exact functor
\[ \mod A_{[a,b]}^{(d)} \hookrightarrow \mod A_{\infty}^{(d)} \]

which clearly restricts to a fully faithful functor
\[ \mathcal{M}_{[a,b]}^{(d)} \hookrightarrow \mathcal{M}^{(d)} . \]

Finally, since we deal with finite dimensional modules and the difference $b - a$ can be arbitrarily large, we conclude that
\[ \mod A_{\infty}^{(d)} = \bigcup_{a \leq b} \mod A_{[a,b]}^{(d)} \quad \text{and} \quad \mathcal{M}^{(d)} = \bigcup_{a \leq b} \mathcal{M}_{[a,b]}^{(d)} . \]

We begin our study of the category $A_{\infty}^{(d)}$ with a simple observation.

**Proposition 3.6.** The abelian category $\mod A_{\infty}^{(d)}$ contains no non-zero projective objects and no non-zero injective objects.
Proof. Let $M \in \text{mod } A^{(d)}_{\infty}$ be non-zero. Then, there exist integers $a \leq b$ such that $M$ is non-projective as an $A_{[a,b]}^{(d)}$-module. Therefore there exists a projective $A_{[a,b]}^{(d)}$-module $P$ and a non-split epimorphism $f: P \to M$ in $\text{mod } A_{[a,b]}^{(d)}$. Since the canonical inclusion $\text{mod } A_{[a,b]}^{(d)} \hookrightarrow \text{mod } A_{\infty}^{(d)}$ is exact, $f$ is also an epimorphism in $\text{mod } A_{\infty}^{(d)}$ whence $M$ is not projective. A dual argument shows that $\text{mod } A_{\infty}^{(d)}$ contains no non-zero injective objects. □

Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. Following [Psa14], a fully faithful functor $F: \mathcal{A} \hookrightarrow \mathcal{B}$ is called a homological embedding if for all $X, Y \in \mathcal{A}$ and for all $i \geq 1$ the induced homomorphism

$$F: \text{Ext}^i_{\mathcal{A}}(X, Y) \to \text{Ext}^i_{\mathcal{B}}(FX, FY)$$

is an isomorphism, see [Psa14] and [GL91] for further information on this notion.

**Proposition 3.7.** The following statements hold.

(i) Let $c \leq a \leq b \leq d$ be integers. Then, the canonical inclusion $\text{mod } A_{[a,b]}^{(d)} \hookrightarrow \text{mod } A_{[c,d]}^{(d)}$ is a homological embedding. In particular, the canonical exact functor

$$\text{D}^b(\text{mod } A_{[a,b]}^{(d)}) \to \text{D}^b(\text{mod } A_{[c,d]}^{(d)})$$

is fully faithful.

(ii) Let $a \leq b$ be integers. Then, the canonical inclusion $\text{mod } A_{[a,b]}^{(d)} \hookrightarrow \text{mod } A_{\infty}^{(d)}$ is a homological embedding. In particular, the canonical exact functor

$$\text{D}^b(\text{mod } A_{[a,b]}^{(d)}) \to \text{D}^b(\text{mod } A_{\infty}^{(d)})$$

is fully faithful.

Proof. (i) By Proposition 4.9 in [GL91], it is enough to prove that

$$\text{Ext}^i_{A_{[c,d]}^{(d)}}(A_{[a,b]}^{(d)}, A_{[a,b]}^{(d)}) = 0$$

for each $i \geq 0$. But this follows immediately from the fact that $A_{[a,b]}^{(d)} \in \mathcal{M}^{(d)}_{[c,d]}$ and the formulae for the extension spaces given in Proposition 2.8.

(ii) Let $M$ and $N$ be finite dimensional $A_{[a,b]}^{(d)}$-modules and $i \geq 0$. Suppose that an exact sequence $\delta \in \text{Ext}^i_{A_{[a,b]}^{(d)}}(M, N)$ is trivial in $\text{mod } A_{\infty}^{(d)}$ in the sense of Yoneda. Thus, there exist a finite zig-zag of equivalences from $\delta$ to the trivial exact sequence. Let $c \leq a \leq b \leq d$ be integers such that all of the exact sequences appearing in this zig-zag are contained in $\text{mod } A_{[c,d]}^{(d)}$. Thus, $\delta$ is trivial as an element of $\text{Ext}^i_{A_{[c,d]}^{(d)}}(M, N)$. Since, by the previous statement, the canonical embedding $\text{mod } A_{[a,b]}^{(d)} \hookrightarrow \text{mod } A_{[c,d]}^{(d)}$ is homological, $\delta$ is also trivial as an element of $\text{Ext}^i_{A_{[a,b]}^{(d)}}(M, N)$. This shows that the induced homomorphism

$$\text{Ext}^i_{A_{[a,b]}^{(d)}}(M, N) \to \text{Ext}^i_{A_{\infty}^{(d)}}(M, N)$$

is injective. A similar argument can be used to show that it is also surjective. We leave the details to the reader. The fact that the canonical exact functor

$$\text{D}^b(\text{mod } A_{[a,b]}^{(d)}) \to \text{D}^b(\text{mod } A_{\infty}^{(d)})$$

is fully faithful.

□
is fully faithful follows from Theorem 2.1 in [Yao96] (note that in this case we cannot use Proposition 4.9 in [GL91] since mod $A^{(d)}_{\infty}$ does not coincide with the category of finitely presented $A^{(d)}_{\infty}$-modules).

As a consequence of Proposition 3.7 we show that the higher Auslander–Reiten formulae hold in $M^{(d)}_{\infty}$ and thus obtain a combinatorial description of the spaces of degree $d$ extensions in $M^{(d)}_{\infty}$ analogous to that in Proposition 2.8.

**Proposition 3.8.** Let $\lambda, \mu \in \text{os}^{d+1}$. The following statements hold.

(i) There are bifunctorial isomorphisms

\[ D\text{Hom}_{A^{(d)}_{\infty}}(M(\lambda), M(\mu)) \cong \text{Ext}^d_{A^{(d)}_{\infty}}(M(\mu), M(\tau_d(\lambda))) \]

and

\[ D\text{Hom}_{A^{(d)}_{\infty}}(M(\lambda), M(\mu)) \cong \text{Ext}^d_{A^{(d)}_{\infty}}(M(\tau_d(\mu)), M(\lambda)). \]

(ii) There are isomorphisms

\[ \text{Hom}_{A^{(d)}_{\infty}}(M(\lambda), M(\mu)) \cong \begin{cases} \mathbb{k} & \text{if } \lambda \rightsquigarrow \mu, \\ 0 & \text{otherwise}; \end{cases} \]

and

\[ \text{Ext}^d_{A^{(d)}_{\infty}}(M(\lambda), M(\mu)) \cong \begin{cases} \mathbb{k} & \text{if } \mu \rightsquigarrow \tau_d(\lambda), \\ 0 & \text{otherwise}. \end{cases} \]

**Proof.**

(i) Let $a \leq b$ be integers such that $M(\lambda)$ and $M(\mu)$ are mod $A^{(d)}_{[a,b]}$-modules. By Proposition 3.7 there is an isomorphism of vector spaces

\[ \text{Ext}^d_{A^{(d)}_{[a,b]}}(M(\lambda), M(\mu)) \cong \text{Ext}^d_{A^{(d)}_{[a,b]}}(M(\lambda), M(\mu)). \]

Moreover, since the canonical inclusion mod $A^{(d)}_{[a,b]} \hookrightarrow \text{mod} A^{(d)}_{\infty}$ is fully faithful there is an isomorphism

\[ \text{Hom}_{A^{(d)}_{\infty}}(M(\lambda), M(\mu)) \cong \text{Hom}_{A^{(d)}_{[a,b]}}(M(\lambda), M(\mu)) \]

The required isomorphisms follow from Theorem 1.24 applied to the finite dimensional algebra $A^{(d)}_{[a,b]}$, taking into account that every non-zero module in $M^{(d)}_{[a,b]}$ has projective dimension either 0 or $d$, see Proposition 2.7.

(ii) The first isomorphism is proven in Proposition 1.12 while the second one follows immediately from the first one together with statement (i). □

The next theorem describes basic representation-theoretic properties of $A^{(d)}_{\infty}$ from the viewpoint of higher Auslander–Reiten theory. Its proof is postponed to Subsection 3.3 as it relies on the content of the next subsection.

**Theorem 3.9.** The following statements hold.

(i) The abelian category mod $A^{(d)}_{\infty}$ has global dimension $d$. 
(ii) \( M_\infty^{(d)} \) is a \( d \)-cluster-tilting subcategory of \( \text{mod} \ A_\infty^{(d)} \).

(iii) The category \( M_\infty^{(d)} \) has \( d \)-almost split sequences. Moreover, for each \( \lambda \in \text{os}^{d+1} \) there are isomorphisms
\[
\tau_d(M(\lambda)) \cong M(\tau_d(\lambda)) \quad \text{and} \quad \tau_{-d}(M(\lambda)) \cong M(\tau_{-d}(\lambda)).
\]

(iv) For every indecomposable \( A_\infty^{(d)} \)-module \( M \in M_\infty^{(d)} \) and every \( i, j \in \mathbb{Z} \) there is an isomorphism \( \tau_i(M) \cong \tau_j(M) \) if and only if \( i = j \).

(v) For every simple \( A_\infty^{(d)} \)-module \( S \in M_\infty^{(d)} \) and for every \( i \in \mathbb{Z} \) the \( A_\infty^{(d)} \)-module \( \tau_i(S) \) is simple.

3.2. The higher Nakayama categories of type \( \tilde{A}_\infty^\infty \). Let \( d \) be a positive integer. Our proof of Theorem 3 relies on a detailed study of certain idempotent quotients of \( A_\infty^{(d)} \), which we now introduce. These quotients, also essential in the construction of the higher Nakayama algebras of type \( \tilde{A} \) in Section 4, should be thought of as higher dimensional analogues of the admissible quotients of the path category of the infinite quiver
\[
\cdots \rightarrow -1 \rightarrow 0 \rightarrow 1 \rightarrow \cdots.
\]

We begin by extending the definition of a Kupisch series (Definition 2.12) to infinite tuples of positive integers.

**Definition 3.10.** Let \( \ell = (\ldots, \ell_{-1}, \ell_0, \ell_1, \ldots) \) be an infinite tuple of positive integers. We say that \( \ell \) is a (connected) Kupisch series of type \( \tilde{A}_\infty^\infty \) if for all \( i \in \mathbb{Z} \) there are inequalities
\[
2 \leq \ell_i \leq \ell_{i-1} + 1.
\]

We denote the set of Kupisch series of type \( \tilde{A}_\infty^\infty \) by \( KS(\tilde{A}_\infty^\infty) \).

We use the following class of Kupisch series for constructing the higher Nakayama algebras of type \( \tilde{A} \) in Section 4.

**Definition 3.11.** Let \( \ell \geq 2 \) be an integer. A Kupisch series \( \ell \) of type \( \tilde{A}_\infty^\infty \) is \( \ell \)-bounded if for all \( i \in \mathbb{Z} \) there is an inequality \( \ell_i \leq \ell \).

**Setting 3.12.** We fix positive integers \( d \) and \( \ell \geq 2 \) until further notice.

The next definition is an infinite analogue of Definition 2.17.

**Definition 3.13.** Let \( \ell \) be an \( \ell \)-bounded Kupisch series of type \( \tilde{A}_\infty^\infty \).

(i) The \( \ell \)-restriction of \( \text{os}^{d+1} \) is the subset
\[
\text{os}^{d+1}_\ell := \{ \lambda \in \text{os}^{d+1} \mid \ell(\lambda) \leq \ell_{\lambda_{d+1}} \}.
\]

(ii) The \( (d+1) \)-Nakayama algebra with Kupisch series \( \ell \) is the idempotent quotient
\[
A_\ell^{(d+1)} := A_\infty^{(d+1)} / [(\text{os}^{d+1} \setminus \text{os}^{d+1}_\ell)].
\]

(iii) We define the subcategory
\[
M_\ell^{(d)} := \text{add} \left\{ M(\lambda) \in \text{mod} \ A_\infty^{(d)} \mid \lambda \in \text{os}^{d+1}_\ell \right\}.
\]
We make a couple of elementary but important observations.

**Proposition 3.14.** Let $d, \ell \geq 2$ be integers and $\ell$ an $\ell$-bounded Kupisch series of type $\mathbb{A}_\infty\infty$. Then, the category $A^{(d)}_\ell$ is locally bounded.

**Proof.** Let $\lambda, \mu \in \mathcal{O}_d^\ell$ be such that $\lambda \rightarrow \mu$ and $[\lambda, \mu] \subset \mathcal{O}_d^\ell$. By definition, there are inequalities

$$
\lambda_1 \leq \mu_1 \leq \cdots \leq \lambda_d \leq \mu_d \leq \ell + \mu_1 - 1 \leq \ell + \lambda_2 - 1
$$

where the two rightmost inequalities follow from the inequalities

$$
\ell(\mu) = \mu_d - \mu_1 + 1 \leq \ell_\mu \leq \ell.
$$

which hold by assumption. Therefore for fixed $\lambda \in \mathcal{O}_d^\ell$ the set

$$
\{ \mu \in \mathcal{O}_d^\ell \mid \lambda \rightarrow \mu \text{ and } [\lambda, \mu] \subset \mathcal{O}_d^\ell \}
$$

is finite. A similar argument shows that for fixed $\mu \in \mathcal{O}_d^\ell$ the set

$$
\{ \lambda \in \mathcal{O}_d^\ell \mid \lambda \rightarrow \mu \text{ and } [\lambda, \mu] \subset \mathcal{O}_d^\ell \}
$$

is finite. The claim now follows from Proposition 1.10 and the definition of $A^{(d)}_\ell$. □

**Proposition 3.15.** Let $d$ and $\ell \geq 2$ be integers and $\ell$ an $\ell$-bounded Kupisch series of type $\mathbb{A}_\infty\infty$. Then, $M^{(d)}(\ell) \subseteq \text{mod } A^{(d)}_\ell$.

**Proof.** The proof of Proposition 2.21 carries over. We leave the details to the reader. □

The following theorem is an infinite analogue of Theorem 2.3.

**Theorem 3.16.** Let $d$ be a positive integer. The following statements hold.

(i) For each $i \in \mathbb{Z}$ the indecomposable projective $A^{(d)}_\ell$-module at the vertex $(i, \ldots, i)$ has Loewy length $\ell_i$.

(ii) $M^{(d)}_\ell$ is a $d\mathbb{Z}$-cluster-tilting subcategory of $\text{mod } A^{(d)}_\ell$.

(iii) Let $\lambda \in \mathcal{O}_d^{d+1}$ be such that $M(\lambda)$ is non-projective as an $A^{(d)}_\ell$-module. Then, there is an isomorphism of $A^{(d)}_\ell$-modules

$$
\tau_d(M(\lambda)) \cong M(\tau_d(\lambda)).
$$

(iv) Let $\lambda \in \mathcal{O}_d^{d+1}$ be such that $M(\lambda)$ is non-injective as an $A^{(d)}_\ell$-module. Then, there is an isomorphism of $A^{(d)}_\ell$-modules

$$
\tau_d^-(M(\lambda)) \cong M(\tau_d^-(\lambda)).
$$

(v) For every simple $A^{(d)}_\ell$-module $S \in M^{(d)}_\ell$ and for every $i \in \mathbb{Z}$ the $A^{(d)}_\ell$-module $\tau_d^+(S)$ is simple. Moreover, for every pair of integers $i$ and $j$ there is an isomorphism $\tau_d^+(S) \cong \tau_d^+(S)$ if and only if $i = j$. 
The proof of Theorem 3.16 is completely analogous to that of Theorem 2.3. The main difference is that we need to consider a suitable replacement of the finite dimensional algebra $A_n^{(d)}$ for which the claim is essentially known by results of Iyama and Opperman [IO13] (some arguments are needed in order to translate their results to our specific combinatorial framework). The general case then follows by iterated application of Lemma 1.18.

Notation 3.17. To simplify the notation, we write

\[ A^{(d)}_{Z\ell} := A^{(d)}_{\ell, \ell, \ell, \ldots} \]

Remark 3.18. The category $A^{(d+1)}_{Z\ell}$ should be thought of as a higher dimensional analogue of the mesh category of type $Z\kappa$, see Remark 3.3.

Remark 3.19. The following observations should be compared with the discussion in Notation 3.5.

Let $\ell \geq 2$ be an integer. As explained in Subsection 1.1, the canonical functor $A^{(d)}_{\infty} \to A^{(d)}_{Z\ell}$ induces a fully faithful exact functor

\[ \operatorname{mod} A^{(d)}_{Z\ell} \to \operatorname{mod} A^{(d)}_{\infty} \]

which clearly restricts to a fully faithful functor

\[ M^{(d)}_{Z\ell} \to M^{(d)}_{\infty} \].

Moreover,

\[ \operatorname{mod} A^{(d)}_{\infty} = \bigcup_{\ell \geq 2} \operatorname{mod} A^{(d)}_{Z\ell} \quad \text{and} \quad \operatorname{mod} A^{(d)}_{\infty} = \bigcup_{\ell \geq 2} \operatorname{mod} A^{(d)}_{Z\ell} \].

For a finite dimensional algebra $A$, we denote its repetitive category by $\hat{A}$, see Chapter II in [Hap88] for the definition. The next result combines Theorem 6.12 in [Iya11] and Theorem 4.7 in [IO13] applied to our specific setting.

Proposition 3.20. There are equivalences of categories

\[ \operatorname{add} A^{(d+1)}_{Z\ell} \cong \operatorname{add} \hat{A}^{(d+1)}_{\ell-1} \cong \mathcal{U}(A^{(d)}_{\ell}) \].

Proof. We first observe that Proposition 2.8 readily implies that there is an isomorphism of algebras

\[ \operatorname{End}_{A^{(d)}_{\ell}}(M^{(d)}_{\ell}) \cong A^{(d+1)}_{\ell-1} \].

Then, according to Theorem 4.7 in [IO13], there is an equivalence of additive categories

\[ \operatorname{add} A^{(d+1)}_{\ell-1} \cong \mathcal{U}(A^{(d)}_{\ell}) \].

Therefore we only need to prove that there is an equivalence of categories

\[ A^{(d+1)}_{Z\ell} \cong A^{(d+1)}_{\ell-1} \].

Although this is a straightforward verification, we provide some details for the convenience of the reader. We use the main theorem in [Sch99], which describes the quiver with relations of the repetitive category of a finite dimensional algebra defined by a quiver with relations which are zero relations or commutativity relations.
Motivated by Lemma 4.9 in [IO13], for $\lambda \in \mathfrak{os}_{\mathbb{Z}\ell}^{d+1}$ we define

$$S(\lambda) := (\lambda_2, \ldots, \lambda_{d+1}, \lambda_1 + \ell - 1).$$

Note that

$$(\lambda_1 + \ell - 1) - \lambda_{d+1} = \ell - \ell(\lambda) \geq 0$$

and

$$\ell(\lambda) \geq 2,$$

therefore $S(\lambda) \in \mathfrak{os}_{\mathbb{Z}\ell}^{d+1}$. It is easy to verify that this defines a bijection

$$S: \mathfrak{os}_{\mathbb{Z}\ell}^{d+1} \rightarrow \mathfrak{os}_{\mathbb{Z}\ell}^{d+1}$$

which induces an automorphism

$$S: A_{\mathbb{Z}\ell}^{(d+1)} \rightarrow A_{\mathbb{Z}\ell}^{(d+1)}.$$ 

It is also straightforward to verify that $\mathfrak{os}_{\mathbb{Z}\ell-1}^{d+1}$ is a fundamental domain for the action of $S$ on $\mathfrak{os}_{\mathbb{Z}\ell}^{d+1}$, by which we mean that the function

$$\mathfrak{os}_{\mathbb{Z}\ell-1}^{d+1} \times \mathbb{Z} \rightarrow \mathfrak{os}_{\mathbb{Z}\ell}^{d+1},$$

$$(\lambda, i) \mapsto S^i(\lambda)$$

is bijective. We also note that if $\lambda \in \mathfrak{os}_{\mathbb{Z}\ell-1}^{d+1}$ is such that $\lambda_{d+1} = \ell - 2$, then

$$\lambda + e_{d+1} = (\lambda_1, \ldots, \lambda_d, \ell - 1) = S(0, \lambda_1, \ldots, \lambda_d).$$

The above discussion shows that the Gabriel quiver of $A_{\mathbb{Z}\ell}^{(d+1)}$ can be equivalently described as the quiver obtained by gluing together a $\mathbb{Z}$-indexed collection of copies of the Gabriel quiver of $A_{\mathbb{Z}\ell-1}^{(d+1)}$, the $i$-th copy joined to the $(i+1)$-th by ‘connecting arrows’

$$S^i(\lambda) \rightarrow S^i(\lambda) + e_{d+1} = S^i(0, \lambda_1, \ldots, \lambda_d)$$

for each $\lambda \in \mathfrak{os}_{\mathbb{Z}\ell-1}^{d+1}$ such that $\lambda_{d+1} = \ell - 2$. Moreover, the proof of Proposition 2.5 in [IO13] shows that $(0, \lambda_1, \ldots, \lambda_d)$ is the initial vertex of any longest path in $A_{\mathbb{Z}\ell-1}^{(d)}$ ending at $\lambda$. 

**Figure 4.** The isomorphisms $A_{\mathbb{Z}\ell}^{(2)} \cong \overrightarrow{A_3^{(2)}}$ (top) and $A_{\mathbb{Z}\ell}^{(3)} \cong \overrightarrow{A_2^{(3)}}$ (bottom), see Proposition 3.20.
It is now straightforward to verify that the quiver with relations $\hat{A}_{d-1}^{(d+1)}$ obtained by applying the main theorem in [Sch99] to the finite dimensional algebra $A_{d-1}^{(d+1)}$ agrees with the above description of $A_{d}^{(d+1)}$ (see Figure 4 for a couple of examples. We leave the remaining details to the reader. □

**Remark 3.21.** If $d = 1$ and $\ell \geq 2$ is arbitrary, then the equivalence

$$\text{add } A_{Z\ell}^{(2)} \cong U(A_{\ell}^{(1)}) = \text{D}^b(\text{mod } A_{\ell}^{(1)})$$

in Proposition 3.20 is a special case of a result of Happel, see Proposition 5.6 in [Hap88].

The next theorem is essentially a consequence of Proposition 3.20 together with a special case of Corollary 4.10 in [IO13]. It settles Theorem 3.16 in the case $\ell = Z\ell$.

**Theorem 3.22.** The following statements hold.

(i) The locally bounded category $A_{Z\ell}^{(d)}$ is selfinjective. Moreover,

$$\text{proj } A = \text{add } \left\{ M(\lambda) \in \text{mod } A_{Z\ell}^{(d)} \mid \lambda \in os_{Z\ell}^{d+1} \text{ and } \ell \ell(\lambda) = \ell \right\}.$$  

(ii) Let $\lambda \in os_{Z\ell}^{d}$ be such that $\ell \ell(\lambda) < \ell$. Then, there are isomorphisms of $A_{Z\ell}^{(d)}$-modules

$$\tau_{d}(M(\lambda)) \cong M(\tau_{d}(\lambda)) \quad \text{and} \quad \tau_{d}(\lambda) \cong M(\tau_{d}(\lambda)).$$

(iii) There is an equivalence of triangulated categories

$$\text{mod } A_{Z\ell}^{(d)} \cong \text{D}^b(\text{mod } A_{d-1}^{(d+1)})$$

which restricts to an equivalence between the subcategories

$$M_{Z\ell}^{(d)} \cong \text{U}(A_{d-1}^{(d+1)}).$$

In particular, $M_{Z\ell}^{(d)}$ is a $d\mathbb{Z}$-cluster-tilting subcategory of $\text{mod } A_{Z\ell}^{(d)}$.

(iv) For every simple $A_{Z\ell}^{(d)}$-module $S \in M_{Z\ell}^{(d)}$ the $A_{Z\ell}^{(d)}$-module $\tau_{d}(S)$ is simple. Moreover, for every pair of integers $i$ and $j$ there is an isomorphism $\tau_{d}^{i}(S) \cong \tau_{d}^{j}(S)$ if and only if $i = j$.

**Proof.** (i) By Proposition 3.20 there is an equivalence of categories

$$A_{Z\ell}^{(d)} \cong A_{d-1}^{(d+1)}.$$  

It is a general fact the the repetitive category of a finite dimensional algebra is a locally bounded selfinjective category, see Lemma 2.2 in [Hap88]. This proves the first claim.

Now we prove the second claim, concerning the classification of the projective(-injective) $A_{Z\ell}^{(d)}$-modules. Let $\lambda \in os_{Z\ell}^{d}$. We claim that the indecomposable projective $A_{Z\ell}^{(d)}$-module at the vertex $\lambda$ is precisely

$$M(\lambda_{d} - \ell + 1, \lambda_{1}, \ldots, \lambda_{d}),$$

which clearly has Loewy length $\ell$. First, observe that

$$\lambda_{d} - \ell + 1 \leq \lambda_{1}$$

if and only if

$$\lambda_{1} - (\lambda_{d} - \ell + 1) = \ell - \ell(\lambda) \geq 0,$$
which holds by assumption. Thus \( M(\lambda_d - \ell + 1, \lambda_1, \ldots, \lambda_d) \) is a well defined \( A^{(d)}_{\infty} \)-module. In order to show that it is moreover an \( A^{(d)}_{2\ell} \)-module we need to prove that each \( \mu \in os^d \) such that
\[
\lambda_d - \ell + 1 \leq \mu_1 \leq \cdots \mu_d \leq \lambda_d
\]
also satisfies \( \ell \ell(\mu) \leq \ell \). Indeed, given such \( \mu \) we have
\[
\ell \ell(\mu) = \mu_d - \mu_1 + 1 \leq \lambda_d - (\lambda_d - \ell + 1) + 1 = \ell.
\]
To prove that \( M(\lambda) \) is projective as an \( A^{(d)}_{2\ell} \)-module it suffices to show that each \( \mu \in os^d_{2\ell} \) such that \( \mu \sim \lambda \) and \([\mu, \lambda] \subset os^d_{2\ell}\) also satisfies
\[
\lambda_d - \ell + 1 \leq \mu_1 \leq \cdots \mu_d \leq \lambda_d.
\]
Such \( \mu \) by definition satisfies
\[
\mu_1 \leq \lambda_1 \leq \cdots \mu_d \leq \lambda_d.
\]
Suppose that \( \lambda_d - \ell + 1 > \mu_1 \) so that the tuple
\[
\rho := (\lambda_d - \ell, \lambda_2, \ldots, \lambda_d) \in os^d
\]
satisfies \( \rho \in [\mu, \lambda] \). Then, the strict inequality
\[
\ell \ell(\rho) = \lambda_d - (\lambda_d - \ell) + 1 = \ell + 1 > \ell
\]
contradicts the assumption \([\mu, \lambda] \subset os^d_{2\ell}\). Therefore \( \lambda_d - \ell + 1 \leq \mu_1 \), as required.

2. The claim can be proven by direct calculation as in Proposition 2.26. We leave the details to the reader.

3. We follow the proof of Corollary 4.10 in [HOT13]. The general theory of repetitive categories shows that there is an equivalence of triangulated categories
\[
mod A^{(d)}_{2\ell} = \frac{\text{mod } A^{(d)}_{\ell - 1}}{\text{mod } A^{(d+1)}_{\ell - 1}} \rightarrow \text{D}^b(\text{mod } A^{(d+1)}_{\ell - 1})
\]
induced by the tilting object
\[
T := \bigoplus \{ \quad \}
\]
where we use the explicit isomorphism
\[
A^{(d+1)}_{2\ell + 1} \simeq A^{(d+1)}_{\ell - 1}
\]
given in the proof of Proposition 3.20 to identify \( A^{(d+1)}_{2\ell + 1} \) with \( T \), see for example Section II.4 in [Hap88]. Given that the canonical functor \( \text{mod } A^{(d)}_{2\ell} \rightarrow \text{mod } A^{(d)}_{2\ell} \) induces a bijective correspondence between \( d \)-cluster-tilting subcategories of \( \text{mod } A^{(d)}_{2\ell} \) and those of \( \text{mod } A^{(d)}_{2\ell} \), it is enough to show that
\[
\text{mod } A^{(d)}_{2\ell} = \text{add } \{ \quad \}
\]
where \( S_d = S \circ \Omega^d \) and \( S \) is the Serre functor on \( \text{mod } A^{(d)}_{2\ell} \), see Notation 1.30, Theorem 1.31 and recall that \( \Omega \) is the inverse of the suspension functor in \( \text{mod } A \). Therefore, in view of statement 1, it is enough to prove that there are isomorphisms of functors \( S_d \simeq \tau_d \). But this is clear since \( S = \tau \circ \Omega^- \) implies
\[
S_d = S \circ \Omega^d \simeq (\tau \circ \Omega^-) \circ \Omega^d \simeq \tau \circ \Omega^{d-1} \simeq \tau_d.
\]
This proves the claim.

(iv) The claim follows from the explicit formula for the higher Auslander–Reiten translate given in statement (ii).

We conclude this subsection with a sketch of the proof of Theorem 3.16. The argumentation is essentially the same as in the proof of Theorem 2.3.

Sketch of proof of Theorem 3.16. Let $d$ be a positive integer, $\ell \geq 2$, and $\ell$ an $\ell$-bounded Kupisch series of type $\mathcal{A}_k$. The proof of Theorem 3.16 is obtained using Lemma 1.18 by constructing $A_2^{(d)}$ as an iterated idempotent quotient of $A_2^{(d)}$ for which the claim is known by Theorem 3.22. We leave the details to the reader. □

3.3. Proof of Theorem 3.9. We are now ready to prove Theorem 3.9.

Proof of Theorem 3.9 Let $i > d$, $M$ and $N$ two $A_{\infty}^{(d)}$-modules, and

\[ \delta : 0 \to N \to L^1 \to \cdots \to L^i \to M \to 0 \]

and exact sequence in $\text{mod} \ A_{\infty}^{(d)}$. We need to prove that this sequence is trivial in the sense of Yoneda. For this, choose integers $a \leq b$ such that $\delta$ is an exact sequence of $A_{[a,b]}^{(d)}$-modules (this is possible since $\delta$ involves only finitely many finite dimensional $A_{\infty}^{(d)}$-modules). Since $A_{[a,b]}^{(d)} \simeq A_{[a]}^{(d)}$ has global dimension at most $d$ (see Theorem 2.3), the exact sequence $\delta$ is trivial as an exact sequence in $\text{mod} \ A_{[a,b]}^{(d)}$, whence also as an exact sequence in $\text{mod} \ A_{\infty}^{(d)}$. This proves the claim.

We divide the proof into three steps.

Step 1: $\mathcal{M}$ is a generating-cogenerating functorially finite subcategory of $\text{mod} \ A_{\infty}^{(d)}$. Let $N$ be a finite dimensional $A_{\infty}^{(d)}$-module and $\ell \geq 2$ an integer such that $N \in \text{mod} \ A_{\ell}^{(d)}$. Since $\mathcal{M}_{\ell}^{(d)}$ is a generating-(cogenerating) functorially finite subcategory of $\text{mod} \ A_{\ell}^{(d)}$, there exists a surjective right $\mathcal{M}_{\ell}^{(d)}$-approximation $f : M \to N$ of $N$. We claim that $f$ is a right $\mathcal{M}_{\ell}^{(d)}$-approximation of $N$. Indeed, let $\lambda \in \text{os}_{\ell}^{d+1}$ and $g : M(\lambda) \to N$ a morphism of $A_{\infty}^{(d)}$-modules. If $\lambda \in \text{os}_{\ell}^{d+1}$, then we already know that $g$ factors through $f$. Suppose that $\lambda \not\in \text{os}_{\ell}^{d+1}$, that is $\ell := \ell(\lambda) > \ell$. According to Theorem 3.22, $M(\lambda)$ is a projective-injective $A_{\ell}^{(d)}$-module, whence $g$ factors through $f$, as the latter is an epimorphism between $A_{\ell}^{(d)}$-modules. This proves that $f$ is a right $\mathcal{M}_{\ell}^{(d)}$-approximation of $N$. One can prove the existence of injective left $\mathcal{M}_{\ell}^{(d)}$-approximations dually.

Step 2: $\mathcal{M}_{\ell}^{(d)}$ is a $d$-rigid subcategory of $\text{mod} \ A_{\infty}^{(d)}$. Let $\lambda, \mu \in \text{os}_{\ell}^{d+1}$. We need to prove that $\text{Ext}_{A_{\infty}^{(d)}}^i(M(\lambda), M(\mu))$ vanishes for all $i \in \{1, \ldots, d-1\}$. For this, let $i \in \{1, \ldots, d-1\}$ and

\[ \delta : 0 \to M(\mu) \to L^1 \to \cdots \to L^i \to M(\lambda) \to 0 \]

be an exact sequence in $\text{mod} \ A_{\infty}^{(d)}$. Let $a \leq b$ be integers such that all of the modules involved in $\delta$ are $A_{[a,b]}^{(d)}$-modules. Since $\mathcal{M}_{[a,b]}^{(d)}$ is a $d$-rigid subcategory of $\text{mod} \ A_{[a,b]}^{(d)}$, the above sequence is trivial in $\text{Ext}_{A_{[a,b]}^{(d)}}^i(M(\lambda), M(\mu))$ in the sense of Yoneda whence it is also trivial as an element of $\text{Ext}_{A_{[a,b]}^{(d)}}^i(M(\lambda), M(\mu))$, as required.

Step 3: $\mathcal{M}_{\ell}^{(d)}$ is a $d$-cluster-tilting subcategory of $\text{mod} \ A_{\infty}^{(d)}$. Let $N$ be an $A_{\infty}^{(d)}$-module such that for each $i \in \{1, \ldots, d-1\}$ there is an isomorphism

\[ \text{Ext}_{A_{\infty}^{(d)}}^i(M_{\ell}^{(d)}, N) \cong 0. \]
We need to prove that \( N \in \mathcal{M}(d)^{\infty} \). Let \( a \leq b \) be integers such that \( N \) is an \( A_{[a,b]}^{(d)} \)-module. Since \( \mathcal{M}(d)^{a,b} \) is a \( d \)-cluster-tilting subcategory of \( \mod A_{[a,b]}^{(d)} \), it is enough to note that, if \( \lambda \in \text{os}_{[a,b]}^{(d)} \), then for each \( i \in \{1, \ldots, d-1\} \) there is an isomorphism

\[
\text{Ext}^i_{A_{[a,b]}^{(d)}}(M(\lambda), N) \cong \text{Ext}^i_{A_{[a,b]}^{(d)}}(M(\lambda), N) = 0,
\]

(see Proposition 3.7). This shows that \( N \in \mathcal{M}_{[a,b]}^{(d)} \subset \mathcal{M}_\infty^{(d)} \). Dually, one can show that if for each \( i \in \{1, \ldots, d-1\} \) there is an isomorphism

\[
\text{Ext}^i_{A_{\infty}^{(d)}}(N, \mathcal{M}(d)) \cong 0,
\]

then \( N \in \mathcal{M}_\infty^{(d)} \).

(iii), (iv), and (v) These claims are all consequences of the corresponding statements in Theorem 3.22. Indeed, for \( \lambda \in \text{os}^d \) let \( \ell := \ell(\lambda) + 1 \) so that, by Theorem 3.22(ii), the module \( M(\lambda) \) is neither projective nor injective as an \( A_{22}^{(d)} \)-module. We claim that the \( d \)-almost split sequence

\[
\delta: 0 \to \tau_d(M(\lambda)) \to L^1 \to \cdots \to L^d \xrightarrow{f} M(\lambda) \to 0
\]

in \( \mathcal{M}_\infty^{(d)} \) is still \( d \)-almost split in \( \mathcal{M}_\infty^{(d)} \). It is enough to prove that \( f \) is right almost split in \( \mathcal{M}_\infty^{(d)} \). Let \( \mu \in \text{os}^d \) and \( g: M(\mu) \to M(\lambda) \) a non-zero morphism which is not a retraction. If \( \ell(\mu) \leq \ell \), then we already know that \( g \) factors through \( f \). If \( \ell(\mu) > \ell \), then Theorem 3.22(ii) shows that \( M(\mu) \) is projective as \( A_{22}^{(d)} \)-module where \( \ell' := \ell(\mu) \). Therefore \( g \) factors through \( f \), since the latter is an epimorphism between \( A_{22}^{(d)} \)-modules. This shows that \( \delta \) is a \( d \)-almost split sequence in \( \mathcal{M}_\infty^{(d)} \) and also that

\[
\tau_d(M(\lambda)) = M(\tau_d(\lambda))
\]

is the \( d \)-Auslander–Reiten translate of \( M(\lambda) \) in \( \mathcal{M}_\infty^{(d)} \). The remaining statements follow immediately from this identity. This finishes the proof of the theorem. \( \square \)

4. The higher Nakayama algebras of type \( \widetilde{A} \)

Let \( d \) be a positive integer. In this section we construct the \( d \)-Nakayama algebras of type \( \widetilde{A} \). We show that they belong to the class of \( d \mathbb{Z} \)-representation-finite algebras and establish their basic properties.

4.1. The selfinjective higher Nakayama algebras. We begin by constructing certain selfinjective \( d \mathbb{Z} \)-representation-finite algebras which should be thought of as higher dimensional analogues of the selfinjective Nakayama algebras. Our construction uses results of Darpö and Iyama [DI17].

We begin by recalling the definition of the orbit category of a category with a free \( \mathbb{Z} \)-action, see for example page 1136 in [MOS09].

**Definition 4.1.** Let \( \mathcal{A} \) be a small category and \( \varphi: \mathcal{A} \to \mathcal{A} \) an automorphism acting freely on the set of objects of \( \mathcal{A} \) (we say that \( \mathcal{A} \) has a free \( \mathbb{Z} \)-action). Given an object \( x \in \mathcal{A} \) we denote its \( \varphi \)-orbit by

\[
\varphi^\mathbb{Z}(x) := \{ \varphi^i(x) \in \mathcal{A} \mid i \in \mathbb{Z} \}.
\]
The orbit category $\mathcal{A}/\varphi$ has objects the set of $\varphi$-orbits of objects in $\mathcal{A}$. The vector space of morphisms $\varphi^Z(x) \to \varphi^Z(y)$ is defined by the short exact sequence

$$[f - \varphi^k(f) | k \in \mathbb{Z}] \hookrightarrow \bigoplus_{i,j \in \mathbb{Z}} \mathcal{A}(\varphi^i(x), \varphi^j(y)) \twoheadrightarrow (\mathcal{A}/\varphi)(\varphi^Z(x), \varphi^Z(y)).$$

**Remark 4.2.** Let $\mathcal{A}$ be a category and $\varphi: \mathcal{A} \to \mathcal{A}$ an automorphism acting freely on the set of objects of $\mathcal{A}$. Given $x, y \in \mathcal{A}$, there is a canonical isomorphism of vector spaces

$$(\mathcal{A}/\varphi)(\varphi^Z(x), \varphi^Z(y)) \cong \bigoplus_{i \in \mathbb{Z}} \mathcal{A}(x, \varphi^i(y)).$$

**Example 4.3.** Consider the category $\mathcal{A}_{(1)}^\infty$, that is the path category of the infinite quiver

$$\cdots \to -1 \to 0 \to 1 \to \cdots.$$

The quiver automorphism $\tau_0(i \to j) = (i - 1) \to (j - 1)$ extends to an automorphism of $\mathcal{A}_{(1)}^\infty$ acting freely on the set of objects of $\mathcal{A}_{(1)}^\infty$. Therefore for every positive integer $n$ we can form the orbit category

$$\widetilde{\mathcal{A}}_{n-1}^{(1)} := \mathcal{A}_{(1)}^\infty/\tau_0^n$$

which can be identified with the path algebra of the $\widetilde{\mathcal{A}}_{n-1}$ quiver

$$\begin{array}{c}
0 \\
\bullet \\
\vdots \\
n - 1
\end{array}
\begin{array}{c}
1 \\
\circ \\
\downarrow \\
\circ
\end{array}$$

By definition, the basic Nakayama algebras of type $\widetilde{\mathcal{A}}_{n-1}$ are the admissible quotients of $\mathcal{A}_{(1)}^{n-1}$. Note that such admissible quotients correspond to those admissible quotients of $\mathcal{A}_{(1)}^\infty$ which are stable under the action of $\tau_0^n$. These observations are the motivation for our construction of the higher Nakayama algebras of type $\widetilde{\mathcal{A}}$.

**Notation 4.4.** Let $\mathcal{A}$ be a category locally bounded category and $\varphi: \mathcal{A} \to \mathcal{A}$ an automorphism acting freely on the set of objects of $\mathcal{A}$. The canonical functor $F: \mathcal{A} \to \mathcal{A}/\varphi$ induces an exact functor

$$F^* : \text{mod}(\mathcal{A}/\varphi) \to \text{mod}\mathcal{A}$$

which admits an exact left adjoint

$$F_* : \text{mod}\mathcal{A} \to \text{mod}(\mathcal{A}/\varphi),$$

see for example Section 14.3 in [GR97].

We introduce the following modification of Definition 2.12

**Definition 4.5.** Let $n$ be a positive integer and $\ell = (\ell_0, \ell_1, \ldots, \ell_{n-1})$ be a tuple of positive integer numbers. We say that $\ell$ is a (connected) Kupisch series of type $\widetilde{\mathcal{A}}_{n-1}$ if for all $i \in \mathbb{Z}/n\mathbb{Z}$ there are inequalities $2 \leq \ell_i \leq \ell_{i-1} + 1$. Two Kupisch series $\ell$ and $\ell'$ of type $\widetilde{\mathcal{A}}_{n-1}$ are equivalent if there exists a cyclic permutation $\sigma \in \mathfrak{S}_n$ of maximal length such that

$$\ell' = (\ell_{\sigma(0)}, \ell_{\sigma(1)}, \ldots, \ell_{\sigma(n-1)}).$$
We denote the set of equivalence classes of Kupisch series of type $\tilde{\mathcal{K}}_{n-1}$ by $\text{KS}(\tilde{\mathcal{K}}_{n-1})$.

**Remark 4.6.** Suppose that the ground field is algebraically closed. It is well known that there is a bijective correspondence between Morita equivalence classes of (connected) Nakayama algebras of type $\tilde{\mathcal{K}}_{n-1}$ and equivalence classes of Kupisch series of type $\tilde{\mathcal{K}}_{n-1}$. In one direction the correspondence is given by associating the tuple

$$(\ell \ell(e_0A), \ell \ell(e_1A), \ldots, \ell \ell(e_{n-1}A))$$

to a Nakayama algebra $A$ with Gabriel quiver $\tilde{\mathcal{K}}_{n-1}$. Moreover, a non-semisimple (connected) Nakayama algebra of type $\tilde{\mathcal{K}}_{n-1}$ is selfinjective if and only if it has Kupisch series $(\ell, \ldots, \ell)$ for some integer $\ell \geq 2$, see for example Proposition V.3.8 in [ASS06].

**Notation 4.7.** Let $\ell$ be a Kupisch series of type $\tilde{\mathcal{K}}_{n-1}$. We denote the basic Nakayama algebra with Kupisch series $\ell$ by $\tilde{A}_\ell^{(1)}$. By definition, $\tilde{A}_\ell^{(1)}$ is an admissible quotient of $\tilde{A}_{n-1}^{(1)}$.

**Setting 4.8.** We fix positive integers $d, n$ and $\ell \geq 2$ until further notice.

We are ready give the definition of the selfinjective higher Nakayama algebras.

**Definition 4.9.** The selfinjective $(d + 1)$-Nakayama algebra with Kupisch series $(\ell, \ldots, \ell)$ is the orbit category

$$\tilde{A}_{n-1, \ell}^{(d+1)} := A_{d\ell}^{(d+1)}/\tau_d^n.$$

We also define the subcategory

$$\tilde{M}_{n-1, \ell}^{(d)} := F_*(\text{add} \tilde{M}_{n-1, \ell}^{(d)}) \subseteq \text{mod} \tilde{A}_{n-1, \ell}^{(d)}.$$

The following theorem is the starting point of our construction of the higher Nakayama algebras of type $\tilde{\mathcal{K}}_{n-1}$ which we give in the subsequent section. For the most part, it is a direct application of Theorem 2.3 in [DI17].

**Theorem 4.10.** The following statements hold.

(i) The algebra $\tilde{A}_{n-1, \ell}^{(d)}$ is finite dimensional and selfinjective.

(ii) Every indecomposable projective-injective $\tilde{A}_{n-1, \ell}^{(d)}$-module has Loewy length $\ell$.

(iii) There exists an $\tilde{A}_{n-1, \ell}^{(d)}$-module $\tilde{M}_{n-1, \ell}^{(d)}$ such that

$$\tilde{M}_{n-1, \ell}^{(d)} = \text{add} \tilde{M}_{n-1, \ell}^{(d)}.$$

(iv) $\tilde{M}_{n-1, \ell}^{(d)}$ is a $d\mathbb{Z}$-cluster-tilting subcategory of $\text{mod} \tilde{A}_{n-1, \ell}^{(d)}$.

(v) For every indecomposable non-projective $\tilde{A}_{n-1, \ell}^{(d)}$-module $M \in \tilde{M}_{n-1, \ell}^{(d)}$ and every $i, j \in \mathbb{Z}$ there is an isomorphism $\tau_i^d(M) \cong \tau_j^d(M)$ if and only if $i - j \in n\mathbb{Z}$.

(vi) For every simple $\tilde{A}_{n-1, \ell}^{(d)}$-module $S \in \tilde{M}_{n-1, \ell}^{(d)}$ and for each integer $i$ the $\tilde{A}_{n-1, \ell}^{(d)}$-module $\tau_i^d(S)$ is simple.

**Proof.** All claims are well known in the case $d = 1$, hence we may assume that $d \geq 2$. We note that Proposition 3.20 and Theorem 3.22 show that we are in the setting of Theorem 2.3 in [DI17].
and therefore statements (i), (iii), and (iv) hold. Note also that Lemma 3.9(b) in [DI17] implies that the diagram

$$
\begin{array}{c}
\mathcal{M}_n^{(d)} \xrightarrow{F} \tilde{\mathcal{M}}_{n-1,\ell}^{(d)} \\
\downarrow \tau_d & \downarrow \tau_d \\
\mathcal{M}_n^{(d)} \xrightarrow{F} \tilde{\mathcal{M}}_{n-1,\ell}^{(d)}
\end{array}
$$

commutes. The remaining statements follow immediately from the explicit description of the indecomposable projective-injective $A_n^{(d)}$-modules and the formula for the $d$-Auslander–Reiten translation given in Theorem 3.22 taking into account the existence of the above commutative diagram.

4.2. The higher Nakayama of algebras type $\tilde{A}$. We are ready to give the construction of the higher Nakayama algebras of type $\tilde{A}$. As their name suggests, these are to be thought of as higher dimensional analogues of the admissible quotients of basic selfinjective Nakayama algebras.

Definition 4.11. Let $d$ and $n$ be positive integers, $\ell$ a Kupisch series of type $\tilde{A}_{n-1}$, and $\ell = \max \ell$. We make the following definitions.

(i) The $\ell$-restriction of $\os_{n-1,\ell}^{d+1}$ is the subset

$$
\os_{n-1,\ell}^{d+1} := \{ \lambda \in \os_{n-1,\ell}^{d+1} \mid \ell_{\ell+1}(\lambda) \leq \ell_{\lambda_{\ell+1}} \}.
$$

(ii) The $(d + 1)$-Nakayama algebra with Kupisch series $\ell$ is the finite dimensional algebra

$$
\tilde{A}_{n-1,\ell}^{(d+1)} := A_{n-1,\ell}^{(d+1)} / [\os_{n}^{d+1} \setminus \os_{\ell}^{d+1}].
$$

(iii) The $\tilde{A}_{n-1,\ell}^{(d)}$-module $M_{\ell}^{(d)}$ is by definition

$$
M_{\ell}^{(d)} := \bigoplus_{\lambda \in \os_{x}^{d+1}} M(\lambda).
$$

The following theorem and its proof are analogous to Theorem 2.18

Theorem 4.12. The following statements hold.

(i) For each $i \in \{0, 1, \ldots, n-1\}$ the indecomposable projective $\tilde{A}_{\ell}^{(d)}$-module at the vertex $(i, \ldots, i)$ has Loewy length $\ell_i$.

(ii) $M_{\ell}^{(d)}$ is a $d\mathbb{Z}$-cluster-tilting $\tilde{A}_{\ell}^{(d)}$-module. In particular, $\tilde{A}_{\ell}^{(d)}$ is a $d\mathbb{Z}$-representation-finite algebra.

(iii) For every simple $\tilde{A}_{\ell}^{(d)}$-module $S$ which is a direct summand of $M_{\ell}^{(d)}$ the $\tilde{A}_{\ell}^{(d)}$-module $\tau_d(S)$ is simple.

Proof. The proof is completely analogous to that of Theorem 2.18. By Theorem 4.10 the claims are known for the finite dimensional algebra $\tilde{A}_{n-1,\ell}^{(d)}$ were $\ell = \max \ell$. In the general case, the theorem can be proven by iterated application of Lemma 1.20. To see how the lemma applies, note that $\ell$ induces a $\ell$-bounded Kupisch series of type $\tilde{A}_{\ell}^{\infty}$

$$
\ell' := (\cdots, \ell_{n-1}, \ell_0, \ell_1, \ldots, \ell_{n-1}, \ell_0, \ldots).
$$
which is \( n \)-periodic in the obvious sense. Moreover, it is clear that there is a finite sequence of inequalities
\[
\ell' = \ell(0) \leq \cdots \leq \ell(1) \leq \cdots \leq \ell(t) = \mathbb{Z} \ell
\]
in the Hasse quiver of \( KS(\mathbb{A}_{\infty}^n) \) in which every Kupisch series \( \ell(i) \) is also \( n \)-periodic. By the proof of Theorem \ref{4.8}, we know that Lemma \ref{1.18} can be applied inductively to the categories \( \tilde{A}^{(d)}_{\ell(i)} \). It is now straightforward to verify that this implies that Lemma \ref{1.20} can be applied inductively to the sequence of finite dimensional algebras
\[
\tilde{A}^{(d)}_\ell = A^{(d)}_{\ell(0)}/\tau_{d-1}^n, \ A^{(d)}_{\ell(1)}/\tau_{d-1}^n, \ldots, A^{(d)}_{\ell(t)}/\tau_{d-1}^n = \tilde{A}^{(d)}_{n-1,t}.
\]
We leave the details to the reader. \( \Box \)

4.3. The higher dimensional analogues of the tubes. We conclude this article by introducing the higher dimensional analogues of the tubes. We need to introduce further terminology.

Let \( A \) be a small category. A finite dimensional \( A \)-module is nilpotent if there exists a positive integer \( n \) such that \( M \) is an \( (A/\mathfrak{p}^n) \)-module, where \( \mathfrak{p} \) denotes the Jacobson radical of \( A \), see \( \text{[Kel64]} \) for the definition of the Jacobson radical of a category. We denote the category of finite dimensional nilpotent \( A \)-modules by \( \text{nil} A \). It is an abelian subcategory of \( \text{mod} A \).

**Example 4.13.** Let \( n \) be a positive integer. It is well known that the abelian category \( \text{nil} \tilde{A}^{(1)}_{n-1} \) has almost split sequences, see for example Lemma X.1.4(f) in \( \text{[SS07]} \). Moreover, for every indecomposable module \( M \in \text{nil} \tilde{A}^{(1)}_{n-1} \) and for each \( i, j \in \mathbb{Z} \) there is an isomorphism \( \tau^i(M) \cong \tau^j(M) \) if and only if \( i - j \in n\mathbb{Z} \). Thus, the Auslander–Reiten quiver of \( \text{nil} \tilde{A}^{(1)}_{n-1} \) forms a tube of period \( n \). This observation is the motivation for our construction of the higher dimensional analogues of the tubes.

**Setting 4.14.** We fix positive integers \( d \) and \( n \).

**Definition 4.15.** We define the \((d-1)\)-tube of rank \( n \) to be the category \( A^{(d)}_{n-1} := A^{(d)}_{\infty}/\tau_{d-1}^n \). We also define the subcategory
\[
\tau^{(d)}_n := F_*(M^{(d)}_{\infty}) \subset \text{nil} \tilde{A}^{(d)}_{n-1}.
\]

**Remark 4.16.** Note that the 0-tube of rank \( n \) is just the path algebra of the circular quiver with \( n \) vertices and \( n \) arrows. Whence \( \tilde{A}^{(1)}_{n-1} \) is not locally finite dimensional.

**Remark 4.17.** Let \( d \geq 2 \). Then, the category \( \tilde{A}^{(d)}_{n-1} \) is not locally bounded. Indeed, for each \( \lambda \in \mathfrak{o}s^d \) and for each \( i \in \mathbb{Z} \) the are non-zero morphisms \( \lambda \to \lambda + ie_d \) and \( \lambda - ie_1 \to \lambda \) in \( A^{(d)}_{\infty} \) induce non-zero morphisms in \( \tilde{A}^{(d)}_{n-1} \) and for all \( i \neq j \) we have
\[
\lambda + ie_d \neq \lambda + je_d \quad \text{and} \quad \lambda - ie_1 \neq \lambda - je_1
\]
in \( \tilde{A}^{(d)}_{n-1} \).

**Theorem 4.18.** Let \( d \) and \( n \) be positive integers. Then, the following statements hold.

(i) The abelian category \( \text{nil} \tilde{A}^{(d)}_{n-1} \) has global dimension \( d \).

(ii) \( \tau^{(d)}_n \) is a \( d \)-cluster-tilting subcategory of \( \text{nil} \tilde{A}^{(d)}_{n-1} \).

(iii) For every simple \( \tilde{A}^{(d)}_{n-1} \)-module \( S \in \tau^{(d)}_n \) the \( \tilde{A}^{(d)}_{n-1} \)-module \( \tau_d(S) \) is simple.
(iv) For every indecomposable $\tilde{A}_{n-1}^{(d)}$-module $T_n^{(d)}$ and every pair of integers $i$ and $j$ there is an isomorphism $\tau_i^j(M) \cong \tau_j^i(M)$ if and only if $i - j \in n\mathbb{Z}$.

Proof. The proof is completely analogous to that of Theorem 3.9. We leave the details to the reader. \qed

Appendix A. Relative $(d - 1)$-homological embeddings

In this short appendix we prove that the canonical embedding $\mod A \rightarrow \mod A$ in Lemma 1.18 is in fact a $(d - 1)$-homological embedding (and not only a relative one). We recall from Definition 3.6 in [Psa14] that an exact functor $i^*: \mathcal{B} \rightarrow \mathcal{A}$ is an $m$-homological embedding if for all $j \in \{1, \ldots, m\}$ and for all $X, Y \in \mathcal{B}$ the induced morphism

$$\text{Ext}^j_B(X, Y) \rightarrow \text{Ext}^j_A(i^*(X), i^*(Y))$$

is an isomorphism.

Proposition A.1. Consider the recollement situation of abelian categories

$$\begin{array}{ccc}
\mathcal{B} & \xrightarrow{i^*} & \mathcal{A} \\
\downarrow{i^*} & & \downarrow{j_*} \\
\mathcal{C} & \xleftarrow{j^*} & \mathcal{A}
\end{array}$$

where $\mathcal{A}$ has enough projectives. Then, the following statements hold.

(i) $\mathcal{B}$ has enough projectives.

(ii) Denote by $\mathcal{P}_A \subseteq \mathcal{A}$ and $\mathcal{P}_B \subseteq \mathcal{B}$ the corresponding subcategories of projectives. Let $N \subseteq \mathcal{B}$ be a subcategory containing $i^*(P)$ for all $P \in \mathcal{P}_A$. Assume that $i_*: \mathcal{B} \rightarrow \mathcal{A}$ is a contravariantly $N$-relative $m$-homological embedding. Then, $i_*$ is an $m$-homological embedding.

Proof. (i) Let $Y \in \mathcal{B}$. Since $\mathcal{A}$ has enough projectives, there exists an epimorphism $P \rightarrow i_*(Y)$ in $\mathcal{A}$. Since $i^*$ is right exact and sends projective objects in $\mathcal{A}$ to projective objects in $\mathcal{B}$, this yields an epimorphism $i^*(P) \rightarrow i^*i_*(Y) \cong Y$. This shows that $\mathcal{B}$ has enough projectives.

(ii) According to Theorem 3.9 and Proposition 3.3 in [Psa14], for the morphism $i_*: \mathcal{B} \rightarrow \mathcal{A}$ to be an $m$-homological embedding is equivalent to $\text{Ext}^j_A(F(P), i_*(B)) = 0$ for all $B \in \mathcal{B}, P \in \mathcal{P}_A, 0 \leq j \leq m - 1$ where $F(P) := \text{Im}(j_!j^*(P) \rightarrow P)$ is induced by the counit of the adjunction.

Applying $\text{Hom}_A(-, i_*(B))$ to the short exact sequence

$$0 \rightarrow F(P) \rightarrow P \rightarrow i_*i^*(P) \rightarrow 0$$

yields the long exact sequence

$$\begin{array}{ccccccc}
0 & \rightarrow & \text{Hom}_A(i_*i^*(P), i_*(B)) & \rightarrow & \text{Hom}_A(P, i_*(B)) & \rightarrow & \text{Hom}_A(F(P), i_*(B)) \\
& & \rightarrow & \text{Ext}^1_A(i_*i^*(P), i_*(B)) & \rightarrow & 0 & \rightarrow \text{Ext}^1_A(F(P), i_*(B)) \\
& & & & \rightarrow & \cdots & \rightarrow \text{Ext}^m_A(i_*i^*(P), i_*(B)) & \rightarrow
\end{array}$$

$$0$$
As $i_*$ is fully-faithful and $(i_*, i^*)$ is an adjoint pair, it follows that the first map in this long exact sequence is an isomorphism. Hence, the second map is 0. Thus, $\text{Ext}_A^{j+1}(i_*(P), i_*(B)) \cong \text{Ext}_B^{j}(F(P), i_*(B))$ for all $j \geq 0$.

Since $P \in \mathcal{P}_A$, it follows that $i^*(P) \in \mathcal{P}_B$. As, by assumption, $i^*(P) \in \mathcal{N}$, it follows that $\text{Ext}_A^{j+1}(i_*(P), i_*(B)) \cong \text{Ext}_B^{j+1}(i^*(P), B) = 0$ for all $0 \leq j \leq m - 1$. We conclude that $\text{Ext}_A^{j}(F(P), i_*(B)) = 0$ for all $0 \leq j \leq m - 1$. □

**Corollary A.2.** Under the assumptions of Lemma 1.18, the embedding

$$\text{mod} \mathcal{A}_X \hookrightarrow \text{mod} \mathcal{A}$$

is a $(d - 1)$-homological embedding.

**Remark A.3.** In the setting of Corollary A.2 we note that in general the embedding

$$\text{mod} \mathcal{A}_X \hookrightarrow \text{mod} \mathcal{A}$$

is not a $d$-homological embedding.

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**INDEX OF SYMBOLS**

\( k \) the ground field, page 3

\( D \) the usual duality on the category of finite dimensional vector spaces, page 3

\( \ell\ell(M) \) the Loewy length of a module, page 4

\( \text{Mod}_A \) the category of \( A \)-modules, page 4

\( \text{proj}_A \) the category of finitely generated projective \( A \)-modules, page 4

\( \text{mod}_A \) the category of finite dimensional \( A \)-modules, page 5

\( \text{inj}_A \) the category of finitely generated injective \( A \)-modules, page 6

\( \tau(M) \) the Auslander–Reiten translate of \( M \), page 6

\( M[x_1,x_2] \) the interval module with top \( S_{x_2} \) and socle \( S_{x_1} \) over some poset, page 8

\( \mathcal{P}^d \) the \( d \)-fold cartesian product of the poset \( \mathcal{P} \) with itself, page 9

\( x \xrightarrow{x} y \) the interlacing relation between ordered sequences of length \( d \) \( x \) and \( y \) in some poset, page 9

\( \text{os}^d(\mathcal{P}) \) the poset of ordered sequences of length \( d \) in the poset \( \mathcal{P} \), page 9

\( \text{cone}(\mathcal{P}^d) \) the \( d \)-cone of the poset \( \mathcal{P} \), page 9

\( M(x) \) the interval module \( M[(x_1,\ldots,x_d),(x_2,\ldots,x_{d+1})] \) associated to the ordered sequence \( (x_1,\ldots,x_d,x_{d+1}) \) in some poset, page 10

\( \tau^d(M) \) the \( d \)-Auslander–Reiten translate of \( M \), page 16

\( M(A) \) the unique \( d \)-cluster-tilting subcategory in \( \text{mod}_A \), where \( A \) is a \( d \)-representation-finite \( d \)-hereditary algebra, page 17

\( \nu_d \) the \( d \)-th desuspension of the derived Nakayama functor on \( \text{D}^b(\text{mod}_A) \), where \( A \) is a finite dimensional algebra of finite global dimension, page 18

\( \mathcal{U}(A) \) the standard \( d\mathbb{Z} \)-cluster-tilting subcategory in \( \text{D}^b(\text{mod}_A) \), where \( A \) is a \( d \)-representation-finite \( d \)-hereditary algebra, page 18

\( A_n \) the poset \( \{0 < 1 < \cdots < n-1\} \), page 18

\( A_n^{(d)} \) the higher Auslander algebra of type \( A_n \), page 18

\( \text{os}^d_n \) the poset \( \text{os}^d(A_n) \) of ordered sequences of length \( d \) in \( A_n \), page 19

\( M(A_n^{(d)}) \) the unique basic \( d \)-cluster-tilting \( A_n^{(d)} \)-module, page 19
$M(\lambda)$ the interval $A_n^{(d)}$-module $M[(\lambda_1, \ldots, \lambda_d), (\lambda_2, \ldots, \lambda_{d+1})]$ for $\lambda \in \os_n^d$, page 19

$\tau_d(\lambda)$ the tuple $\lambda - (1, \ldots, 1)$ where $\lambda \in \os^d$, page 21

$\ell(\lambda)$ the Loewy length $\lambda_{d+1} - \lambda_1 + 1$ of $\lambda \in \os^{d+1}$, page 23

$\ell$ a Kupisch series, page 24

$\os^d_\ell$ the $\ell$-restriction of $\os_n^d$, page 25

$A_\ell^{(d)}$ the $d$-Nakayama algebra with Kupisch series $\ell$ of type $A_n$, page 25

$M_\ell^{(d)}$ the distinguished $d\Z$-cluster-tilting $A_\ell^{(d)}$-module, page 25

$A_\infty$ the poset of integer numbers $\{\cdots < -1 < 0 < 1 < \cdots\}$, page 31

$\os^d$ the poset of ordered sequences of length $d$ in the poset $A_\infty$, page 31

$A_\infty^{(d)}$ the mesh category of type $\Z A_\infty^{(d-1)}$, page 31

$M_\infty^{(d)}$ the distinguished $d\Z$-cluster-tilting subcategory of $\mod A_\infty^{(d)}$, page 31

$\Z \ell$ the $\ell$-bounded Kupisch series $(\ldots, \ell, \ell, \ell, \ldots)$ of type $A_\infty$, page 38

$\widehat{A}$ the repetitive category of a finite dimensional algebra $A$, page 39

$A_{n-1, \ell}^{(d+1)}$ the selfinjective $(d + 1)$-Nakayama algebra with Kupisch series $(\ell, \ldots, \ell)$, page 46

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