Dirac’s monopole, quaternions, and the Zassenhaus formula

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Starting from the quaternionic quantization scheme proposed by Emch and Jadczyk for describing the motion of a quantum particle in the magnetic monopole field, we derive an algorithm for finding the differential representation of the star product generated by the quaternionic Weyl correspondence on phase-space functions. This procedure is illustrated by the explicit calculation of the star product up to the second order in the Planck constant $\hbar$. Our main tools are an operator analog of the twisted convolution and the Zassenhaus formula for the products of exponentials of noncommuting operators.

I. INTRODUCTION

Since the works by Wu and Yang [1–3] and Greub and Petry [4], it has been generally recognized that the theory of fiber bundles provides the most appropriate mathematical framework for the quantum mechanical description of the motion of a charged particle in the field of the Dirac magnetic monopole. The interpretation of the particle wave function as a section of a vector bundle associated with the Hopf bundle reveals the topological nature of Dirac’s charge quantization condition [5, 6] and provides a consistent, singularity-free formulation overcoming the problem of the absence of a globally defined vector potential for the monopole field. There are numerous papers on this subject, but the fiber-bundle description of the motion of a quantum particle in the monopole field continues to attract attention because it is prototypical for many aspects of quantum gauge theory. In an interesting paper, Emch and Jadczzyk [7] have proposed a magnetic monopole model based on using a quaternionic Hilbert space and the concept of a weak projective group representation

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introduced by Adler \cite{8,10}. The feasibility of such an approach stems from the fact that the Hopf fibration $SU(2) \rightarrow SU(2)/U(1) = S^2$ is obtained from the trivial principal fiber bundle $S^2 \times SU(2) \rightarrow S^2$ by reducing the structural group from $SU(2)$ to $U(1)$ (see, e.g., \cite{11} for this reduction).

An interesting question is how to define a generalized Weyl map which associates quaternionic operators to phase-space functions in the Emch and Jadczyk setting. This question was raised in Ref. \cite{12}, where an attempt was made to find an integral representation of the noncommutative star product generated by this map on the phase space. However, for comparison with the general formulation \cite{13} of deformation quantization of Poisson manifolds, a differential form of this product is needed. In the present paper, we develop a regular procedure for finding the differential form of the star product, starting from the multiplier of the quaternionic projective representation \cite{7} of the group of configuration-space translations, and we calculate this product explicitly up to the second order in the Planck constant $\hbar$. Our main tools are an operator analog of the twisted convolution used in the usual Weyl calculus \cite{14,16} and the Zassenhaus formula which is a convenient combinatorial expansion of $e^{X+Y}$ with noncommuting $X$ and $Y$. A detailed description of this formula and its relation to the Baker-Campbell-Hausdorff theorem can be found in Ref. \cite{17}. We also obtain an exact expression for the integral kernel of the star product. This result agrees well with the composition rule \cite{18,19} for gauge-invariant Weyl symbols, which was derived for a quantum particle in a magnetic field having a global vector potential $A(x)$, although the derivation method used in Refs. \cite{18,19} and based on the replacement of the canonical momentum $P$ in the Weyl system by the kinetic momentum $P - (e/c)A(x)$ is inapplicable to the magnetic monopole case.

The paper is organized as follows. In Sec. \ref{II} we recall the Hamiltonian description of the motion of a charged particle in a magnetic field, which uses a noncanonical Poisson structure depending on the magnetic field. In Sec. \ref{III} we consider the quaternionic projective representation of the translation group, introduced by Emch and Jadczyk, and show in particular that this representation is strongly continuous. In Sec. \ref{IV} we define a quaternionic Weyl correspondence for the charged particle-monopole system, using a quaternionic analog of the Fourier transform. In Sec. \ref{V} we prove a lemma on the Weyl symbols of operators of a special type, which provides an optimal way of finding the star product generated by this correspondence. In Sec. \ref{VI} we use the Zassenhaus formula to write the multiplier of the above-mentioned quaternionic projective representation in the form that is most con-
venient for our purposes. In Sec. VII we derive an exact expression for the integral kernel of the emerging star product. In Sec. VIII we present a simple algorithm for finding the differential form of this product and calculate it explicitly up to the second order in \( \hbar \). Section IX contains concluding remarks and a comparison of the obtained star product with the Kontsevich formula [13] for deformation quantization of Poisson structures.

A few words about our notation concerning quaternions: The algebra of quaternions is denoted \( \mathbb{H} \) and its imaginary units are denoted \( \epsilon_1, \epsilon_2, \) and \( \epsilon_3 \). The multiplication in \( \mathbb{H} \) is defined by the relations

\[
\epsilon_i \epsilon_j = \delta_{ij} + \sum_k \epsilon_{ijk} \epsilon_k,
\]

where \( \epsilon_{ijk} \) is the completely antisymmetric Levi-Civita symbol. For each quaternion \( a = a^0 + a^1 \epsilon_1 + a^2 \epsilon_2 + a^3 \epsilon_3 \in \mathbb{H} \), its conjugate is defined as \( a^* = a^0 - a^1 \epsilon_1 - a^2 \epsilon_2 - a^3 \epsilon_3 \) and the conjugation operation is an involution, i.e., \( a^{**} = a \) and \( (ab)^* = b^* a^* \).

II. MAGNETIC POISSON BRACKETS

The motion of a particle with charge \( e \) and mass \( m \) in a magnetic field \( \mathbf{B}(x) \) is described by the equations

\[
m \frac{d\mathbf{v}}{dt} = e \mathbf{v} \times \mathbf{B},
\]

where \( \mathbf{v} = \left( \frac{dx^1}{dt}, \frac{dx^2}{dt}, \frac{dx^3}{dt} \right) \) and the physical units are chosen so that the speed of light \( c = 1 \). It is well known (see, e.g., [20]), that Eqs. (1) can be rewritten in the Hamiltonian form

\[
\dot{x}^i = \{x^i, H\}_B, \quad \dot{p}_i = \{p_i, H\}_B
\]

by taking the kinetic energy as Hamiltonian,

\[
H = \frac{1}{2m} \sum_i p_i^2,
\]

and defining the magnetic Poisson bracket by

\[
\{f, g\}_B = \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial x^i} + \beta_{ij} \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial p_j}, \quad \text{where} \quad \beta_{ij} = e \epsilon_{ijk} B^k.
\]

Then for the coordinate functions \( x^i \) and \( p_i \), we have the relations

\[
\{x^i, x^j\}_B = 0, \quad \{x^i, p_j\}_B = \delta^i_j, \quad \{p_i, p_j\}_B = \beta_{ij}(x),
\]
and the matrix $\mathcal{P}$ of the Poisson structure has the form

$$\mathcal{P} = \begin{pmatrix} \beta(x) & -I_n \\ I_n & 0 \end{pmatrix} \quad (5)$$

if we assume that the momentum coordinates $p_i$ are placed first and the position coordinates $x^i$ are placed second. Accordingly, the phase-space symplectic form is

$$\omega = dp_i \wedge dx^i + \frac{1}{2} \beta_{ij} dx^i \wedge dx^j. \quad (6)$$

(In (3) and (6) and hereafter, we use the summation convention for repeated indices.) If the magnetic field can be written as $\mathbf{B} = \text{curl} \mathbf{A}$, i.e., $\beta_{ij} = e(\partial_i A_j - \partial_j A_i)$, where the magnetic vector potential $\mathbf{A}$ is continuously differentiable, then Eq. (1) is also Hamiltonian relative to the standard symplectic structure $dp_i \wedge dx^i$ with the Hamiltonian function $H_{\mathbf{A}} = \frac{1}{2m}(p - e\mathbf{A})^2$. But such a representation is impossible globally in the case of the magnetic monopole field, which has the form

$$B^k(x) = g \frac{x^k}{|x|^3}, \quad (7)$$

where $g$ is the monopole charge. The flux of the field (7) through any sphere surrounding the monopole is $4\pi g$, while the Stokes theorem gives zero for a field of the form curl $\mathbf{A}$. Therefore, no smooth vector potential for the field (7) exists on its domain of definition $\mathbb{R}^3 \setminus \{0\}$. Such a potential exists only on a smaller domain obtained by removing a curve that begins at the origin and proceeds to infinity in some direction, exhibiting itself as a line of singularity.

III. A WEAK QUATERNIONIC PROJECTIVE REPRESENTATION OF THE TRANSLATION GROUP

Let $L^2(\mathbb{R}^3, \mathbb{H})$ be the Hilbert space of quaternion-valued functions on $\mathbb{R}^3$ with the inner product

$$\langle \Phi, \Psi \rangle = \int_{\mathbb{R}^3} dx \Phi(x)^*\Psi(x).$$

Following Refs. [7, 8], we assume that $L^2(\mathbb{R}^3, \mathbb{H})$ is a right module over $\mathbb{H}$; i.e., the multiplication of vectors by quaternionic scalars is taken to act from the right, while linear operators act from the left. The Emch and Jadczyk approach [7] to the description of the quantum
dynamics of a particle in the monopole field is based on using anti-Hermitian operators \( \nabla_i \) acting in \( L^2(\mathbb{R}^3, \mathbb{H}) \) as follows:

\[
(\nabla_i \Psi)(x) = \left( \partial_i + \frac{1}{2} \epsilon_{ijk} \frac{x^j}{|x|^2} \epsilon_k \right) \Psi(x). \quad (8)
\]

It is easy to verify that they obey the commutation relations

\[
[\nabla_i, \nabla_j] = -\frac{1}{2} J \epsilon_{ijk} \frac{x^k}{|x|^3}, \quad (9)
\]

where \( J \) is the operator of left multiplication by the \( x \)-dependent imaginary unit quaternion

\[
j(x) = \frac{x \cdot \epsilon}{|x|}, \quad \text{with} \quad x \cdot \epsilon \overset{\text{def}}{=} x^i \epsilon_i.
\quad (10)
\]

The operators (8) are defined by a natural \( SU(2) \) connection on the quaternionic line bundle over \( \mathbb{R}^3 \setminus \{0\} \) whose square-integrable sections form the space \( L^2(\mathbb{R}^3 \setminus \{0\}, \mathbb{H}) \) identical to \( L^2(\mathbb{R}^3, \mathbb{H}) \) (for more details we refer the reader to Ref. [7]). As we will soon see, these operators generate “twisted” translations. The operator \( J \) satisfies the relations \( J^\dagger = -J \) and \( JJ^\dagger = JJ^\dagger = I \), i.e., is anti-Hermitian and unitary, and clearly \( J^2 = -I \). It is important that \( J \) commutes with all the operators \( \nabla_i \):

\[
[J, \nabla_i] = 0, \quad i = 1, 2, 3.
\quad (11)
\]

(But it should be noted that if \( g \neq 1 \), then \( J \) does not commute with the operators \( \partial_i + \frac{g}{2} \epsilon_{ijk} \frac{x^j}{|x|^2} \epsilon_k \) considered in Ref. [12] along with \( \nabla_i \).) Clearly, \( J \) commutes with the position operators \( Q^i \) defined as usual, by

\[
(Q^i \Psi)(x) = x^i \Psi(x). \quad (12)
\]

If the quantum Hamiltonian is taken to be \( \mathcal{H} = -\frac{\hbar^2}{2m} \nabla^2 \) and the evolution operator is defined as \( \exp \left( -\frac{J}{\hbar} \mathcal{H} t \right) \), then using the commutation relations (9), we obtain the following evolution equations in the Heisenberg picture:

\[
\dot{Q}_i = \frac{J}{\hbar} [\mathcal{H}, Q_i] = -\frac{J \hbar}{m} \nabla_i, \quad (13)
\]

\[
\ddot{Q}_i = \frac{1}{2m} \left( \frac{\hbar}{2g} \epsilon_{ijk} (\dot{Q}^j B^k + B^k \dot{Q}^j) \right), \quad (14)
\]

where \( B^k \) is the multiplication operator by the function (7). Equations (13) and (14) correspond to the initial classical equations of motion (1) and (2) under the condition that the particle electric charge \( e \) and the monopole magnetic charge \( g \) are connected by the relation

\[
eg e g = \frac{\hbar}{2}.
\quad (15)\]
Now, let \( a \) be a vector in \( \mathbb{R}^3 \) and let \( a \cdot \nabla = a^i \nabla_i \). To construct a quaternionic Weyl correspondence, we consider the family of operators

\[
V(a) = e^{a \cdot \nabla},
\]

treating each one as

\[
V(a) = e^{a \cdot \partial} W(a),
\]

where \( a \cdot \partial = a^i \partial_i \). To find \( W(a) \), we introduce the operator function \( W(a, t) = e^{-t a \cdot \partial} e^{t a \cdot \nabla} \) which becomes \( W(a) \) at \( t = 1 \). Taking its derivative in respect to \( t \) and using that \( e^{-t a \cdot \partial} \) is a shift operator, we obtain

\[
\frac{\partial}{\partial t} W(a, t) = \frac{1}{2} \epsilon_{ijk} a^i (x - ta)^j \frac{a \times x}{|x - ta|^2} e_k W(a, t) = \frac{1}{2} \frac{(a \times x) \cdot \epsilon}{|x - ta|^2} W(a, t).
\]

The unique solution of Eq. (18) subject to the initial condition \( W(a, 0) = I \) is the operator of multiplication by the function

\[
w(a, x, t) = \exp \left( \frac{1}{2} (a \times x) \cdot \epsilon \int_0^t \frac{ds}{|x - sa|^2} \right)
\]

which is well defined for any noncollinear vectors \( a \) and \( x \). For \( t = 1 \), the integral in the exponent equals \( \alpha(x, a)/|a \times x| \), where \( \alpha \) is the angle between \( x \) and \( x - a \), i.e., \( \cos \alpha = (x \cdot (x - a))/|x||x - a| \), and we deduce that \( W(a) \) is the unitary operator of multiplication by

\[
w(a, x) = \exp \left( \frac{1}{2} j (a \times x) \alpha \right) = \cos \frac{\alpha}{2} + j (a \times x) \sin \frac{\alpha}{2}.
\]

As noted in Ref. [7], the unitary operators (17) define a weak projective representation of the translation group in the sense of Adler [8–10]. Namely, it follows from (17) that

\[
V(a)V(b) = V(a + b)M(a, b),
\]

where \( M(a, b) \) is the operator of multiplication by

\[
m(a, b; x) = w(a + b, x)^* w(a, x - b) w(b, x).
\]

The word "weak" means that the quaternion unitary phase (20) has a nontrivial \( x \)-dependence.\(^1\) It is worth noting that \( M(a, b) \) does not commute with \( V(a + b) \) and is

\[^1\text{Adler defined a weak projective group representation by } U_a U_b |f\rangle = U_{ab} |f\rangle \Omega(a, b; f), \text{ where } a \text{ and } b \text{ are group elements, } \{|f\rangle\} \text{ is a (privileged) complete set of states, and } \Omega(a, b; f) \text{ is a phase factor. There are subtleties when the states are unnormalized, but we use the name "weak projective representation" in this case, too, as it was used in Ref. [7].} \]
the right multiplier of this representation, whereas its left multiplier is equal to $M(-b, -a)^\dagger$. The associativity relation $(V(a)V(b))V(c) = V(a)(V(b)V(c))$ implies that $M(a, b)$ satisfies the 2-cocycle condition

$$M(a + b, c)V(c)^{-1}M(a, b)V(c) = M(a, b + c)M(b, c),$$

where $V(c)$ cannot be dropped because of the noncommutativity. If $a \to a_0$, then $w(a, x) \to w(a_0, x)$ almost everywhere and, since $|w(a, x) - w(a_0, x)| \leq 2$, we find that

$$\|W(a)\Psi - W(a_0)\Psi\|_{L^2(\mathbb{R}^3, \mathbb{H})}^2 \to 0$$

by the Lebesgue dominant-convergence theorem. Setting $\Psi_a = W(a)\Psi$ and writing

$$V(a)\Psi - V(a_0)\Psi = e^{a \cdot \partial}(\Psi_a - \Psi_{a_0}) + (e^{a \cdot \partial} - e^{a_0 \cdot \partial})\Psi_{a_0},$$

we conclude that the unitary representation $a \to V(a)$ is strongly continuous; i.e., for each $\Psi \in L^2(\mathbb{R}^3, \mathbb{H})$,

$$V(a)\Psi \to V(a_0)\Psi \quad \text{as} \quad a \to a_0.$$

**IV. QUATERNIONIC WEYL CORRESPONDENCE**

In this section, the phase-space coordinates are denoted $(p, q)$, where $q = (q^1, q^2, q^3)$ is the position vector of a particle and $p = (p_1, p_2, p_3)$ is its momentum vector. To construct a generalized Weyl correspondence for a charged particle in the monopole field, we consider an inclusion $\mathfrak{J}$ of the set of complex phase-space functions into the set of quaternion-valued functions, which is defined by

$$\mathfrak{J} \colon f(p, q) \mapsto \mathfrak{f}(p, q) \overset{\text{def}}{=} \Re f(p, q) + j(x) \Im f(p, q).$$

(22)

We note an important difference between this definition and formula (5.1) of Ref. [12], where $q = x$. Although $\mathfrak{f}$ depends on $x$, hereafter we omit the argument $x$ for brevity, because this dependence is fixed. Clearly, the map (22) is injective, and it follows immediately from the equality $j(x)^2 = -1$ that

$$\mathfrak{J}(f + g) = \mathfrak{f} + \mathfrak{g}, \quad \mathfrak{J}((s + it)f) = (s + j(x)t)\mathfrak{f} \quad (s, t \in \mathbb{R}),$$

(23)

i.e., $\mathfrak{J}$ is a semilinear homomorphism and furthermore it preserves multiplication:

$$\mathfrak{J}(fg) = \mathfrak{f} \mathfrak{g}.$$  

(24)

The set of quaternions of the form $s + j(x)t$ can be regarded as a set of scalars acting on the image of the map $\mathfrak{J}$. 
Assuming that \( f \) has a Fourier transform, we define a “quaternionic” Fourier transform of \( f \) by the relation

\[
\tilde{f}(u, v) = \frac{1}{(2\pi)^3} \int dp dq \, e^{-j(x)(p \cdot u + q \cdot v)} \hat{f}(p, q).
\]  

(25)

If \( f \) depends only on \( p \), or only on \( q \), then its Fourier transform in this variable is also denoted \( \tilde{f} \), when this cannot cause confusion. Assuming that the complex-valued functions of \( u, v \) are embedded in the space of quaternion-valued functions in a manner analogous to (22), we have

\[
\mathcal{J}(\tilde{f}) = \tilde{f},
\]  

(26)

where \( \tilde{f} \) is the ordinary Fourier transform. It is easy to see that familiar formulas of Fourier analysis have analogs for the transform (25). In particular,

\[
\frac{1}{(2\pi)^3} \int dp \, e^{-j(x)p \cdot u} = \delta(u)
\]

and correspondingly

\[
\frac{1}{(2\pi)^3} \int dudv \, e^j(x)(p \cdot u + q \cdot v) \tilde{f}(u, v) = \hat{f}(p, q).
\]

We let \( P_i \) denote the Hermitian momentum operators defined by

\[
P_i = -\hbar J \nabla_i,
\]  

(27)

where \( J \) is the operator of left multiplication by \( j(x) \), as before. The operators \( P_i \) and the position operators \( Q^i \) defined above by (12) satisfy the commutation relations

\[
[Q^i, Q^j] = 0, \quad [Q^i, P_j] = \hbar J \delta^i_j,
\]

\[
[P_i, P_j] = \hbar J \beta_{ij},
\]  

(28)

where \( \beta_{ij} \) is the operator of multiplication by

\[
\beta_{ij}(x) = eg \varepsilon_{ijk} \frac{x^k}{|x|^3}
\]  

and where the charges obey the quantization condition (15). Now we define a quaternionic Weyl correspondence associating operators on the quaternionic Hilbert space \( L^2(\mathbb{R}^3, \mathbb{H}) \) with complex-valued functions on the phase space by the rule

\[
f \mapsto \mathcal{O}_f = \frac{1}{(2\pi)^3} \int dudv \, e^{J(u \cdot P + v \cdot Q)} \tilde{f}(u, v),
\]  

(30)

where the imaginary unit \( j(x) \) occurring in \( \tilde{f} \) acts on wave functions as \( J \), i.e., by left multiplication. The map (30) is semilinear, as is the map (22). Following the usual terminology
of quantization theory, the function \( f \) will be called the Weyl symbol of the operator \( O_f \).

We note that
\[
J(\tilde{\varphi}_{\alpha q}) = (2\pi)^3 j(x)^{\alpha+\gamma} \partial^\alpha \delta(u) \partial^\gamma \delta(v),
\]
where we use the standard multi-index notation
\[
\alpha = (\alpha_1, \alpha_2, \alpha_3), \quad |\alpha| = \alpha_1 + \alpha_2 + \alpha_3, \quad p^\alpha = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3}, \quad \partial^\alpha_u = \partial_u^{\alpha_1} \partial_{u_2}^{\alpha_2} \partial_{u_3}^{\alpha_3}.
\]

Therefore, the monomial \( p^\alpha q^\gamma \) is transformed by the map (30) into the symmetrically ordered product
\[
\{P^\alpha Q^\gamma\}_s \overset{\text{def}}{=} \frac{\partial^{\alpha+\gamma} e^{J(uP+Q)}}{\partial u^\alpha \partial v^\gamma} \bigg|_{u,v=0}.
\]

This is analogous to the usual Weyl correspondence for functions on a phase space with the standard symplectic structure, with the difference that in this case not only the operators \( Q_i \) and \( P_i \) but also the operators \( P_i P_j \), where \( i \neq j \), do not commute with each other. The star product \( \star \) is by definition the operation on the set of symbols which corresponds to the operator product under the map (30), and it easy to see that
\[
q^i \star q^j = q^i q^j, \quad q^i \star p_j = q^i p_j + \frac{i\hbar}{2} \delta^i_j,
\]
\[
p_i \star p_j = p_i p_j + \frac{i\hbar}{2} \beta_{ij}(q)
\]
in accordance with the Poisson structure (4). By way of example, we prove the equality (32).

As has just been said, the function \( p_i p_j \) is transformed by the map (30) into the symmetric product
\[
\frac{1}{2} (P_i P_j + P_j P_i).
\]

Applying the Fourier transform (25) to \( J((i\hbar/2)\beta_{ij}(q)) = (\hbar/2)j(x)\beta_{ij}(q) \), we obtain
\[
\frac{1}{(2\pi)^3} \int dq dp e^{-i(x)(p u + q v)} \frac{\hbar}{2} j(x) \beta_{ij}(q) = \frac{\hbar}{2} (2\pi)^{3/2} j(x) \delta(u) \tilde{\beta}_{ij}(v).
\]

Substituting this expression into (30) instead of \( \tilde{f}(u, v) \) gives
\[
\frac{\hbar}{2} J \frac{1}{(2\pi)^{3/2}} \int dv \tilde{\beta}_{ij}(v) e^{J v Q},
\]
which is the multiplication operator by \((\hbar/2)j(x)\beta_{ij}(x)\) because \((e^{J v Q}\Psi)(x) = e^{j(x) v x}\Psi(x)\) and the integration in (34) yields the inverse Fourier transform. Thus, on account of the commutation relation (28), the function \((i\hbar/2)\beta_{ij}(q)\) is transformed by (30) into the operator \((1/2)[P_i, P_j]\), whose sum with (33) is \(P_i P_j\), which proves the formula (32).
V. AN OPERATOR ANALOG OF THE TWISTED CONVOLUTION

The unitary operators

\[ T(u, v) = e^{J(u \cdot P + v \cdot Q)} = e^{Ju \cdot P} e^{Jv \cdot Q} e^{-Jhu \cdot v/2} \]

(35)

form a weak quaternionic projective representation of the translation group of the phase space. Indeed, using (19) and the commutation relation

\[ e^{Ju \cdot P} e^{Jv \cdot Q} e^{Ju' \cdot P} = e^{Ju' \cdot P} e^{Jv \cdot Q} e^{Ju \cdot P} \]

and letting \( w \) denote for brevity the pair of variables \((u, v)\), we obtain

\[ T(w)T(w') = e^{Ju \cdot P} e^{Jv \cdot Q} e^{Ju' \cdot P} e^{Ju' \cdot Q} e^{-Jh(u \cdot v + u' \cdot v')/2} = T(w + w')M_h(w, w'), \quad (36) \]

where \( M_h \) is the composite multiplier

\[ M_h(w, w') = M(hu, hu') e^{Jh(\dot{u} \cdot v' - \dot{v} \cdot u')/2} \]

(37)

and \( M(hu, hu') \) is the operator of multiplication by the quaternion-valued function \( m(hu, hu'; x) \) defined by (20). We note that \( M_h \) commutes with \( J \) but does not commute with \( T(w + w') \). It is natural to try to find the composition law for the quaternionic Fourier transforms that corresponds to the operator multiplication, as this has been done by von Neumann [21] for the usual Weyl correspondence. It follows from the definition (30) and from the relation (36) that

\[ \mathcal{O}_f \mathcal{O}_g = \frac{1}{(2\pi)^6} \int dw dw' T(w + w') M_h(w, w') \tilde{f}(w) \tilde{g}(w'). \]

(38)

Making the change of variables \( w + w' \rightarrow w \), this can be written as

\[ \mathcal{O}_f \mathcal{O}_g = \frac{1}{(2\pi)^3} \int dw T(w) (\tilde{f} \circ_h \tilde{g})(w), \]

(39)

where

\[ (\tilde{f} \circ_h \tilde{g})(w) = \frac{1}{(2\pi)^3} \int dw' M_h(w - w', w') \tilde{f}(w - w') \tilde{g}(w'). \]

In the case of the usual Weyl correspondence, where the multiplier of a complex projective representation of the translation group is \( \exp(i\hbar(\dot{u} \cdot v' - \dot{v} \cdot u')/2) \), an analogous expression is called the twisted convolution product [14–16] and the inverse Fourier transform converts it
into the Moyal star product \( f \ast_M g \), i.e., into the phase-space function that corresponds to the operator \( \mathcal{O}_f \mathcal{O}_g \). The Moyal product has the form

\[
(f \ast_M g)(p, q) = f(p, q) \exp \left\{ \frac{i\hbar}{2} (\overrightarrow{\partial}_q \cdot \overrightarrow{\partial}_p - \overrightarrow{\partial}_p \cdot \overrightarrow{\partial}_q) \right\} g(p, q)
\]

and the bidifferential operator defining this product is obtained from the multiplier \( \exp (i\hbar(uv' - vu')/2) \) by the simple replacements

\[
u \rightarrow -i\overrightarrow{\partial}_p, \quad v \rightarrow -i\overrightarrow{\partial}_q, \quad u' \rightarrow -i\overrightarrow{\partial}_p, \quad v' \rightarrow -i\overrightarrow{\partial}_q.
\]

This is easily verified by expanding the exponential \( \exp (i\hbar((u - u')v' - (v - v')u')/2) \) in a power series. Then the twisted convolution product becomes a sum of the usual convolution products of functions obtained from \( f \) and \( g \) by multiplication by some monomials in \( u \) and \( v \). Next, it suffices to take into account that the Fourier transform converts the convolution product into pointwise product and converts multiplication by monomials into differentiation and vice versa. The form of (39) outwardly resembles that of the twisted convolution product, but an important difference is that this expression is not a complex-valued but an operator-valued function of \( w = (u, v) \) because of the additional quaternionic factor \( m(hu, hu'; x) \) depending on \( x \). This complicates finding the symbol of the operator \( \mathcal{O}_f \mathcal{O}_g \) from (39). Nevertheless, starting from this formula and using an expansion of the composite multiplier \( M_h \) in powers of \( \hbar \), it is possible to obtain the star product generated by (30), as will be shown in Sec. VIII. As a first step in this direction, we calculate the symbol of the operator \( T(u, v)\mathcal{C} \), where \( \mathcal{C} \) if the multiplication operator by a function depending on \( x \).

**Lemma 1.** Let \( c(x) \) be a complex-valued function and \( \mathcal{C} \) the operator of multiplication by \( \text{Re} c(x) + i(x) \text{Im} c(x) \) in the quaternionic Hilbert space \( L^2(\mathbb{R}^3, \mathbb{H}) \). Then the symbol of \( T(u, v)\mathcal{C} \) under the correspondence (30) is given by \( f_{u,v}(p, q) = e^{i(u - p + v - q)} c(q + hu/2). \)

**Proof.** Let \( c(q) = (\mathcal{J} c)(q) = \text{Re} c(q) + i(x) \text{Im} c(q) \) in accordance with the notation introduced in Sec. IV. By the definition (25), we have

\[
\tilde{f}_{u,v}(u', v') = \frac{1}{(2\pi)^3} \int dpdq e^{-i(x)(u' - p + v' - q) + i(x)(u - p + v - q)} c(q + hu/2)
\]

\[
= (2\pi)^{3/2} \delta(u' - u) \tilde{c}(v' - v) e^{i(x)hu - (v' - v)/2}. \tag{41}
\]

Using (33) and (41), we find that the operator corresponding to \( f_{u,v} \) is given by

\[
\mathcal{O}_{f_{u,v}} = \frac{1}{(2\pi)^3} \int dp' dp e^{Jp' - P} e^{Jv' - Q} e^{-Jhu - (v' - v)/2} \tilde{f}_{u,v}(u', v')
\]

\[
= e^{Ju - (P - hv)/2} \frac{1}{(2\pi)^{3/2}} \int dv' e^{Jv' - Q} \tilde{c}(v' - v).
\]
Applying the operator $(2\pi)^{-3/2} \int dv' e^{Jv' \cdot Q} \tilde{c}(v' - v)$ to $\Psi \in L^2(\mathbb{R}^3, \mathbb{H})$ and integrating over $v'$, we obtain

$$e^{i(x \cdot v)} (\text{Re } c(x) + i(x) \text{ Im } c(x)) \Psi(x) = (e^{Jv \cdot Q} \mathcal{C} \Psi) (x).$$

Hence, $O_{fu,v} = T(u, v) \mathcal{C}$, which completes the proof.

VI. THE ZASSENHAUS FORMULA

To derive the desired differential form of the star product, we will employ, instead of (20), another representation of the multiplier $M(\hbar u, \hbar u')$, which can be obtained by using the Baker-Campbell-Hausdorff formula for products of exponentials of noncommuting variables $X$ and $Y$. More precisely, we prefer to use a modification of this formula which was proposed by Zassenhaus and is written as

$$e^{X+Y} = e^X e^Y \prod_{n=2}^{\infty} e^{C_n(X,Y)}.$$  \hspace{1cm} (42)

Here $C_n$ is a homogeneous Lie polynomial in $X$ and $Y$ of degree $n$, i.e., a linear combination of nested commutators of the form $[Z_1, [Z_2, \ldots, [Z_{m-1}, Z_m]]$, where each of $Z_i$ is $X$ or $Y$. Written out explicitly, the first terms in (42) are

$$C_2(X,Y) = -\frac{1}{2}[X,Y], \quad C_3 = \frac{1}{6}[X,[X,Y]] + \frac{1}{3}[Y,[X,Y]].$$

Several systematic approaches to calculating the Zassenhaus terms $C_n$ have been carried out in the literature; for more details, see [17], where an efficient recursive procedure expressing directly $C_n$ with the minimum number of commutators required at each degree $n$ is proposed.

In our case, $X = J u \cdot P$, $Y = J u' \cdot P$ and it follows from (28) that

$$C_2(J u \cdot P, J u' \cdot P) = \frac{\hbar}{2} J u^i \beta_{ij} u'^j = \frac{\hbar}{2} J u \cdot \beta u'.$$ \hspace{1cm} (43)

Because $J$ commutes with $P$, the calculation of $C_n$ for $n \geq 3$ reduces to differentiation of the functions $\beta_{ij}(x)$. In particular,

$$C_3(J u \cdot P, J u' \cdot P) = -\frac{\hbar^2}{6} J [u \cdot (u \cdot \partial) \beta u' + 2u \cdot (u' \cdot \partial) \beta u'].$$  \hspace{1cm} (44)

Each $C_n(J u \cdot P, J u' \cdot P)$ is the product of $J$ by a function of $x$. Therefore, they commute with each other, and the multiplier $M(\hbar u, \hbar u')$ can be written as

$$M(\hbar u, \hbar u') = \exp \left( -\sum_{n=2}^{\infty} C_n(J u \cdot P, J u' \cdot P) \right).$$  \hspace{1cm} (45)
Using the explicit expressions given by (43) and (44), we get the following representation for the quaternion-valued function $m(hu, hu'; x)$ defining this operator:

$$m(hu, hu'; x) = \exp \left( -\frac{\hbar}{2} j(x) u \cdot \beta(x) u' + \frac{\hbar^2}{6} j(x) [u \cdot (u \cdot \partial) \beta(x) u' + 2u \cdot (u' \cdot \partial) \beta(x) u'] \ldots \right).$$ \hspace{1cm} (46)

VII. THE INTEGRAL FORM OF THE STAR PRODUCT

We let $m(hu, hu'; x)$ denote the complex-valued function obtained from $m(hu, hu'; x)$ by replacing $j(x)$ with $i$. In other words, $\text{Re} \, m(\cdot; x)$ is the scalar part of the quaternion $m(\cdot; x)$, and $\text{Im} \, m(\cdot; x)$ is the scalar part of $-j(x)m(\cdot; x)$. Clearly, $m(\cdot; x)$ is in turn obtained from $m(\cdot; q)$ by applying the mapping (22) and subsequently setting $q = x$. For any fixed $w = (u, v)$, it follows from the definition (39) and relations (23), (24), and (26) that the operator $C_w = (\tilde{f} \circ_h \tilde{g})(w)$ is the multiplication operator by the function $\text{Re} \, c_w(x) + j(x) \text{Im} \, c_w(x)$, where

$$c_w(x) = \frac{1}{(2\pi)^3} \int du' dv' m(h(u - u'), hu'; x) e^{i\hbar(u-v'-v'')/2} \tilde{f}(u - u', v - v') \tilde{g}(u', v').$$

The formula (38) represents the operator $O_f O_g$ as the integral over $w$ of an operator-valued function of the form considered in Lemma 1 of Sec. V. From this lemma and the linearity of the map (30), we immediately deduce that the symbol of $O_f O_g$, i.e., the star product $f \star_h g$ can be written as

$$(f \star_h g)(p, q) = \frac{1}{(2\pi)^6} \int dv dv' e^{i(u-p+q)} m(h(u - u'), hu'; q + hu/2) \times e^{i\hbar(u-v'-v'')/2} \tilde{f}(u - u', v - v') \tilde{g}(u', v') \hspace{1cm} (47)$$

or, equivalently, as

$$(f \star_h g)(p, q) = \frac{1}{(2\pi)^6} \int dv dv' e^{i(u-p+q)} m(hu', h(u - u'); q + hu/2) \times e^{-i\hbar(u-v'-v'')/2} \tilde{f}(u', v') \tilde{g}(u - u', v - v'). \hspace{1cm} (48)$$

Using (35) and the formulas of Sec. IV for the quaternionic Fourier transform, it is easy to verify directly that the map (30) transforms the function (47) into the operator $O_f O_g$ written in the form (38). An integral representation of the star product in terms of the
functions \( f \) and \( g \) themselves can now readily be derived as follows. Making a change of integration variables in (48), we obtain

\[
(f \star_h g)(p, q) = \frac{1}{(2\pi)^6} \int du' du' dv'' dv'' e^{i(u' + u'')q + p + q} m(u', u'') (q + h(u' + u''))/2 \times e^{-i(h(u' - v' - x', u')/2) f(u', v')} \tilde{g}(u'', v') = \frac{1}{(2\pi)^6} \int du' du'' dp' dp'' e^{i(u' + u'')q + p + q} m(u', u'') (q + h(u' + u''))/2 \times f(p', q - hu''/2)g(p'', q + hu'')/2.
\]

Setting \( q' = q - hu''/2 \) and \( q'' = q + hu'/2 \), we arrive at the representation

\[
(f \star_h g)(p, q) = \int dp' dq' dp'' dq'' K(p', q', p'', q''; p, q) f(p', q')g(p'', q''),
\]

where the integral kernel is given by

\[
K(p', q', p'', q''; p, q) = \frac{1}{(\pi\hbar)^6} \exp \left\{ -\frac{2i}{\hbar} \left[ (p - p')(q - q'') - (p - p'')(q - q') \right] \right\} \times m(2(q'' - q), 2(q - q'); q - q' + q'') \quad (49)
\]

The exponential in the first line of (49) is the integral kernel of the Moyal star product, and the additional factor in the second line is caused by the monopole magnetic field. If we keep only the first term \( C_2 \) in the representation (15) of the magnetic multiplier, dropping all other terms, then the corresponding "truncated" expression for \( m(2(q'' - q), 2(q - q'); q - q' + q'') \) is

\[
\exp\left\{ -q \cdot (q' \times q'')/|q - q' + q''|^3 \right\}
\]

and, instead of the exact representation (49) of the integral kernel, we obtain the approximate formula

\[
K_{\text{approx}} = \frac{1}{(\pi\hbar)^6} \exp \left\{ -\frac{2i}{\hbar} \left[ (p - p')(q - q'') - (p - p'')(q - q') \right] - \frac{q \cdot (q' \times q'')}{|q - q' + q''|^3} \right\}.
\]

The star product defined by \( K_{\text{approx}} \) is in agreement with the initial Poisson structure (4) at the first order in \( \hbar \) but is not associative even at the second order.

**VIII. THE DIFFERENTIAL FORM OF THE STAR PRODUCT**

Starting from the Zassenhaus formula (46) and using the Taylor series expansion about the point \( q \), we can express the function \( m(\hbar(u - u'), \hbar u'; q + \hbar u/2) \) in (47) as a series in powers of \( \hbar \) with coefficient functions that are polynomials in the entries of the matrix \( \beta(q) \), their partial derivatives at the point \( q \), and the components of the vectors \( u - u' \) and
u'. On substituting this expansion and the expansion of $\exp(i\hbar((u - u')v' - (v - v')u')/2)$ into (47), the right-hand side is expressed as the inverse Fourier transform in $u$ and $v$ of an infinite sum of terms of the form

$$B(q) \frac{\hbar^n}{(2\pi)^3} \int du' dv' (u - u')^\alpha (v - v')^\gamma u'^\alpha' v'^\gamma' \tilde{f}(u - u', v - v') \tilde{g}(u', v'),$$

(50)

where $B(q)$ is a monomial in the matrix entries $\beta_{ij}(q)$ and their partial derivatives, depending on the multi-indices $\alpha$, $\gamma$, $\alpha'$ and $\gamma'$. Taking the inverse Fourier transform of (50) gives

$$\hbar^n (-i)^{|\alpha|+|\alpha'|+|\gamma|+|\gamma'|} B(q) \partial_p^\alpha \partial_q^\gamma f(p, q) \partial_p'^\alpha' \partial_q'^\gamma' g(p, q)$$

and we obtain a differential representation of the star product which has the form

$$f \star \hbar g = \sum_{n=0}^{\infty} \hbar^n B_n(f, g)$$

(51)

with some bidifferential operators $B_n$. Going from the variables $u - u'$ and $u'$ to $u$ and $u'$, we conclude that the operators $B_n$ in (51) are obtained from the composite multiplier (37) by the following simple algorithm:

1. Expand the multiplier (37) in powers of $\hbar$ using the Zassenhaus formula and replace the imaginary unit quaternion $j(x)$ with the complex imaginary unit $i$.

2. Substitute $q + \hbar(u + u')/2$ for the argument $x$ of the matrix-valued functions $\beta_{ij}(x)$ and their partial derivatives occurring in this expansion, and then expand them in Taylor series about the point $q$.

3. Combine like terms in the resulting expansion, and replace the variables $u$, $u'$, $v$, and $v'$ with differential operators by the rule (40).

We illustrate this procedure by finding explicit expressions for $B_1$ and $B_2$. It follows from (46) that

$$m(\hbar u, \hbar u'; x) = 1 - \frac{i\hbar}{2} u \cdot \beta(x)u' - \frac{\hbar^2}{8} (u \cdot \beta(x)u')^2$$

$$+ \frac{i\hbar^2}{6} [u \cdot (u \cdot \partial)\beta(x)u' + 2u \cdot (u' \cdot \partial)\beta(x)u'] + O(\hbar^3).$$

Using also the expansion

$$\exp\left\{ \frac{i\hbar}{2} (u \cdot v' - v \cdot u') \right\} = 1 + \frac{i\hbar}{2} (u \cdot v' - v \cdot u') - \frac{\hbar^2}{8} (u \cdot v' - v \cdot u')^2 + O(\hbar^3),$$
we find that

\[ m(hu, hu'; x) \exp \left\{ \frac{i}{2} \left( u' \cdot v - v \cdot u' \right) \right\} = 1 + \frac{i}{2} \left[ u \cdot v' - v \cdot u' - u \cdot \beta(x) u' \right] \]

\[ - \frac{\hbar^2}{8} \left[ (u \cdot v' - v \cdot u')^2 - 2(u \cdot v' - v \cdot u')(u \cdot \beta(x) u') + (u \cdot \beta(x) u')^2 \right] \]

\[ + \frac{i \hbar^2}{6} \left[ u \cdot (u \cdot \partial) \beta(x) u' + 2u \cdot (u' \cdot \partial) \beta(x) u' \right] + O(\hbar^3). \] (52)

To calculate the \( \hbar^2 \)-order terms of the star product, it suffices to substitute \( q \) for \( x \) in the second and third lines of (52) and to retain the first-order term of the expansion

\[ \beta(q + \hbar(u + u')/2) = \beta(q) + (\hbar/2)((u + u') \cdot \partial) \beta \big|_{q} + O(\hbar^2) \]

in the first line. This yields two additional terms similar to those in the third line but with different coefficients. Combining the similar terms and making the replacements (40), we obtain the bidifferential operator

\[ 1 + \frac{i \hbar}{2} \left[ \frac{\delta}{\partial p} \cdot \frac{\delta}{\partial q} - \frac{\delta}{\partial q} \cdot \frac{\delta}{\partial p} + \frac{\delta}{\partial p} \cdot \beta \frac{\delta}{\partial p} \right] \]

\[ - \frac{\hbar^2}{8} \left[ (\frac{\delta}{\partial q} \cdot \frac{\delta}{\partial p} - \frac{\delta}{\partial q} \cdot \frac{\delta}{\partial p})^2 + 2(\frac{\delta}{\partial q} \cdot \frac{\delta}{\partial p} - \frac{\delta}{\partial q} \cdot \frac{\delta}{\partial p})(\frac{\delta}{\partial p} \cdot \beta \frac{\delta}{\partial p}) + \left( \frac{\delta}{\partial p} \cdot \beta \frac{\delta}{\partial p} \right)^2 \right] \]

\[ + \frac{\hbar^2}{12} \left[ \frac{\delta}{\partial p} \cdot (\frac{\delta}{\partial q} \cdot \beta \frac{\delta}{\partial p} - \frac{\delta}{\partial p} \cdot (\frac{\delta}{\partial p} \cdot \beta \frac{\delta}{\partial p}) \right]. \]

Hence, up to the second order in \( \hbar \), the star product defined by the quaternionic Weyl correspondence (30) has the form

\[ f \ast_\hbar g = fg + \frac{i \hbar}{2} \left( \partial_{qf} f \partial_{p} g - \partial_{pf} f \partial_{q} g + \beta_{ij} \partial_{pf} f \partial_{pj} g \right) \]

\[ - \frac{\hbar^2}{8} \left( \partial_{qf} \partial_{q} f \partial_{p} g - 2 \partial_{qf} \partial_{pf} g + \partial_{pf} \partial_{q} f \partial_{q} g \right) \]

\[ - \frac{\hbar^2}{4} \beta_{ij} \left( \partial_{qf} \partial_{pf} f \partial_{qj} g - \partial_{pf} \partial_{qj} f \partial_{qf} g \right) - \frac{\hbar^2}{8} \beta_{ij} \beta_{kl} \partial_{p} f \partial_{pf} g \partial_{qj} - \partial_{qf} f \partial_{p} \partial_{pf} g \]

\[ + \frac{\hbar^2}{12} \partial_{qf} \beta_{ij} \left( \partial_{pf} f \partial_{qj} g - \partial_{pf} f \partial_{qj} g \right) + O(\hbar^3). \] (53)

**IX. CONCLUSION**

The formula (53) is in agreement with Kontsevich’s deformation quantization formula [13] which gives an associative star product in the case of a nonconstant Poisson structure, i.e., when its defining matrix \( \mathcal{P}^{ab} \) depends nontrivially on the phase-space coordinates. Up to
the second-order terms, Kontsevich’s formula is written as \(^3\)

\[
f \star g = fg + i\frac{\hbar}{2} \mathcal{P}^{ab} \partial_a f \partial_b g - \frac{\hbar^2}{8} \mathcal{P}^{a_1 b_1} \mathcal{P}^{a_2 b_2} \partial_{a_1} \partial_{a_2} f \partial_{b_1} \partial_{b_2} g - \frac{\hbar^2}{12} \mathcal{P}^{a_1 b_1} \partial_{a_1} \mathcal{P}^{a_2 b_2} (\partial_{a_1} \partial_{a_2} f \partial_{b_2} g - \partial_{a_2} f \partial_{a_1} \partial_{b_2} g) + O(\hbar^3).
\]

We are dealing with the phase space \((\mathbb{R}^3 \setminus \{0\}) \times \mathbb{R}^3\), and it is easy to verify that if \(\mathcal{P}^{ab}\) has the block form \((5)\), then the formula \((54)\) is equivalent to \((53)\). The associativity up to the second order means that

\[
(f \star g) \star h = f \star (g \star h) + O(\hbar^3)
\]
or, which is the same, that the following equalities hold:

\[
B_1(f, g)h + B_1(fg, h) = fB_1(g, h) + B_1(f, gh),
\]

\[
B_2(f, g)h + B_1(B_1(f, g), h) + B_2(fg, h) = fB_2(g, h) + B_1(f, B_1(g, h)) + B_2(f, gh).
\]

We note that these associativity conditions are fulfilled for the star product \((53)\) at any charges \(e\) and \(g\), although the condition \((15)\) is crucial for constructing the quaternionic Weyl correspondence, i.e., for operator quantization. We also remark that in order to construct a quaternionic quantization map at the condition \(eg = -\hbar/2\), a left Hilbert \(\mathbb{H}\)-module could be used; i.e., the convention of left multiplication by scalars and right multiplication by operators could be adopted. Then the anti-Hermitian operators

\[
\mathring{\nabla}_i = \partial_i - \frac{1}{2} \epsilon_{ijk} \frac{x^j}{|x|^2} x_k
\]
satisfy the commutation relations

\[
[\mathring{\nabla}_i, \mathring{\nabla}_j] = \frac{1}{2} J \epsilon_{ijk} \frac{x^k}{|x|^3},
\]

where \(\mathring{J}\) is the operator of right multiplication by the imaginary unit quaternion \((10)\), and this operator commutes with all \(\mathring{\nabla}_i, i = 1, 2, 3\).

A generalized Weyl correspondence for the charged particle-magnetic monopole system can, of course, also be constructed within the complex Hilbert space framework and at any integer \(n\) in the Dirac charge quantization condition \(eg = nh/2, n \in \mathbb{Z}\). This is done in Ref. \([22]\) on the basis a global Lagrangian description \([23]\) of this system as a constrained system with \(U(1)\) gauge symmetry on the total space of the principal bundle

\(^3\) The expansion parameter denoted by \(h\) in Ref. \([13]\) corresponds to \(ih/2\) in our notation.
$\mathbb{C}^2 \setminus \{0\} \rightarrow (\mathbb{C}^2 \setminus \{0\})/U(1) = \mathbb{R}^3 \setminus \{0\}$ whose restriction to the unit sphere of $\mathbb{R}^3$ is the Hopf bundle. The appropriate Hilbert space consists of the square-integrable functions on $\mathbb{C}^2$ satisfying the equivariance condition $\Psi(ze^{i\theta}, \bar{z}e^{-i\theta}) = e^{-i\theta}\Psi(z, \bar{z})$. A comparison of this and the quaternion-based approach to the magnetic monopole quantum mechanics will be given elsewhere.

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