Sparse PCA via Covariance Thresholding

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Abstract

In sparse principal component analysis we are given noisy observations of a low-rank matrix of dimension \( n \times p \) and seek to reconstruct it under additional sparsity assumptions. In particular, we assume here that the principal components \( v_1, \ldots, v_r \) have at most \( k_1, \ldots, k_r \) non-zero entries respectively, and study the high-dimensional regime in which \( p \) is of the same order as \( n \).

In an influential paper, Johnstone and Lu [JL04] introduced a simple algorithm that estimates the support of the principal vectors \( v_1, \ldots, v_r \) by the largest entries in the diagonal of the empirical covariance. This method can be shown to succeed with high probability if \( k_q \leq C_1 \sqrt{n/\log p} \) and to fail with high probability if \( k_q \geq C_2 \sqrt{n/\log p} \) for two constants \( 0 < C_1, C_2 < \infty \). Despite a considerable amount of work over the last ten years, no practical algorithm exists with provably better support recovery guarantees.

Here we analyze a covariance thresholding algorithm that was recently proposed by Krauthgamer, Nadler and Vilenchik [KNV13]. We confirm empirical evidence presented by these authors and rigorously prove that the algorithm succeeds with high probability for \( k_r \) of order \( \sqrt{n} \). Recent conditional lower bounds [BR13] suggest that it might be impossible to do significantly better.

The key technical component of our analysis develops new bounds on the norm of kernel random matrices, in regimes that were not considered before.

1 Introduction

In the spiked covariance model proposed by [JL04], we are given data \( x_1, x_2, \ldots, x_n \) with \( x_i \in \mathbb{R}^p \) of the form:

\[
x_i = \sum_{q=1}^r \sqrt{\beta_q} u_{q,i} v_q + z_i,
\]

(1)

Here \( v_1, \ldots, v_r \in \mathbb{R}^p \) is a set of orthonormal vectors, that we want to estimate, while \( u_{q,i} \sim \mathcal{N}(0,1) \) and \( z_i \sim \mathcal{N}(0, I_p) \) are independent and identically distributed. The quantity \( \beta_q \in \mathbb{R}_{>0} \) quantifies the signal-to-noise ratio. We are interested in the high-dimensional limit \( n, p \to \infty \) with \( \lim_{n \to \infty} p/n = \alpha \in (0, \infty) \). In the rest of this introduction we will refer to the rank one case, in order to simplify the exposition, and drop the subscript \( q = \{1, 2, \ldots, r\} \). Our results and proofs hold for general bounded rank.

The standard method of principal component analysis involves computing the sample covariance matrix \( G = n^{-1} \sum_{i=1}^n x_i x_i^T \) and estimates \( v = v_1 \) by its principal eigenvector \( v_{rc}(G) \). It is a well-known fact that, in the high dimensional asymptotic \( p/n \to \alpha > 0 \), this yields an inconsistent estimate [JL09]. Namely \( \| v_{rc} - v \|_2 \not\to 0 \) in the high-dimensional asymptotic limit, unless \( \alpha \to 0 \) (i.e. \( p/n \to 0 \)). Even worse, Baik, Ben-Arous and Péché [BRAP05] and Paul [Pau07] demonstrate a phase transition phenomenon: if \( \beta < \sqrt{\alpha} \) the estimate is asymptotically orthogonal to the signal \( \langle v_{rc}, v \rangle \to 0 \). On the other hand, for \( \beta > \sqrt{\alpha} \), \( \langle v_{rc}, v \rangle \) remains strictly positive as \( n, p \to \infty \). This phase transition phenomenon has attracted considerable attention recently within random matrix theory [PP07, CDMF09, BCN11, KY13].
These inconsistency results motivated several efforts to exploit additional structural information on the signal \( v \). In two influential papers, Johnstone and Lu [JL04, JL09] considered the case of a signal \( v \) that is sparse in a suitable basis, e.g. in the wavelet domain. Without loss of generality, we will assume here that \( v \) is sparse in the canonical basis \( e_1, \ldots, e_p \). In a nutshell, [JL09] proposes the following:

1. Order the diagonal entries of the Gram matrix \( G_{i(1),i(1)} \geq G_{i(2),i(2)} \geq \cdots \geq G_{i(p),i(p)} \), and let \( J \equiv \{i(1), i(2), \ldots, i(k)\} \) be the set of indices corresponding to the \( k \) largest entries.

2. Set to zero all the entries \( G_{i,j} \) of \( G \) unless \( i, j \in J \), and estimate \( v \) with the principal eigenvector of the resulting matrix.

Johnstone and Lu formalized the sparsity assumption by requiring that \( v \) belongs to a weak \( \ell_q \)-ball with \( q \in (0, 1) \). Instead, here we consider a strict sparsity constraint where \( v \) has exactly \( k \) non-zero entries, with magnitudes bounded below by \( \theta/\sqrt{k} \) for some constant \( \theta > 0 \). Amini and Wainwright [AW09] studied the more restricted case when every entry of \( v \) has equal magnitude of \( 1/\sqrt{k} \).

Within this model, it was proved that diagonal thresholding successfully recovers the support of \( v \) provided \( v \) is sparse enough, namely \( k \leq C \sqrt{n/\log p} \) with \( C = C(\alpha, \beta) \) a constant [AW09]. (Throughout the paper we denote by \( C \) constants that can change from point to point.) This result is a striking improvement over vanilla PCA. While the latter requires a number of samples scaling as the number of parameters, \( n \gtrsim p \), sparse PCA via diagonal thresholding achieves the same objective with a number of samples scaling as the number of non-zero parameters, \( n \gtrsim k^2 \log p \).

At the same time, this result is not as strong as might have been expected. By searching exhaustively over all possible supports of size \( k \) \( ( \text{a method that has complexity of order } p^k \) ) the correct support can be identified with high probability as soon as \( n \gtrsim k \log p \). On the other hand, no method can succeed for much smaller \( n \), because of information theoretic obstructions [AW09].

Over the last ten years, a significant effort has been devoted to developing practical algorithms that outperform diagonal thresholding, see e.g. [MWA05, ZHT06, dEGJL07, dBG08, WTH09]. In particular, d’Aspremont et al. [dEGJL07] developed a promising M-estimator based on a semidefinite programming (SDP) relaxation. Amini and Wainwright [AW09] carried out an analysis of this method and proved that, if \((i)\) \( k \leq C(\beta) n/\log p \), and \((ii)\) if the SDP solution has rank one, then the SDP relaxation provides a consistent estimator of the support of \( v \).

At first sight, this appears as a satisfactory solution of the original problem. No procedure can estimate the support of \( v \) from less than \( k \log p \) samples, and the SDP relaxation succeeds in doing it from –at most– a constant factor more samples. This picture was upset by a recent, remarkable result by Krauthgamer, Nadler and Vilenchik [KNV13] who showed that the rank-one condition assumed by Amini and Wainwright does not hold for \( \sqrt{n} \lesssim k \lesssim (n/\log p) \). This result is consistent with recent work of Berthet and Rigollet [BR13] demonstrating that sparse PCA cannot be performed in polynomial time in the regime \( k \gtrsim \sqrt{n} \), under a certain computational complexity conjecture for the so-called planted clique problem.

In summary, the sparse PCA problem demonstrates a fascinating interplay between computational and statistical barriers.

**From a statistical perspective**, and disregarding computational considerations, the support of \( v \) can be estimated consistently if and only if \( k \lesssim n/\log p \). This can be done, for instance, by exhaustive search over all the \( \binom{p}{k} \) possible supports of \( v \). (See [VL12, CMW+13] for a minimax analysis.)

**From a computational perspective**, the problem appears to be much more difficult. There is rigorous evidence [BR13, MW13] that no polynomial algorithm can reconstruct the support unless \( k \lesssim \sqrt{n} \).

On the positive side, a very simple algorithm (Johnstone and Lu’s diagonal thresholding) succeeds for \( k \lesssim \sqrt{n/\log p} \).

Of course, several elements are still missing in this emerging picture. In the present paper we address one of them, providing an answer to the following question:

\[ f(n) \gtrsim g(n) \text{ as a shorthand of } f(n) \geq C g(n) \text{ for some constant } C = C(\beta, \alpha). \]

Further \( C \) denotes a constant that may change from point to point.
Is there a polynomial time algorithm that is guaranteed to solve the sparse PCA problem with high probability for $\sqrt{n}/\log p \lesssim k \lesssim \sqrt{n}$?

We answer this question positively by analyzing a covariance thresholding algorithm that proceeds, briefly, as follows. (A precise, general definition, with some technical changes is given in the next section.)

1. Form the empirical covariance matrix $G$ and set to zero all its entries that are in modulus smaller than $\tau/\sqrt{n}$, for $\tau$ a suitably chosen constant.
2. Compute the principal eigenvector $\hat{v}_1$ of this thresholded matrix.
3. Denote by $B \subseteq \{1, \ldots, p\}$ be the set of indices corresponding to the $k$ largest entries of $\hat{v}_1$.
4. Estimate the support of $v$ by ‘cleaning’ the set $B$. (Briefly, $v$ is estimated by thresholding $G\hat{v}_B$ with $\hat{v}_B$ obtained by zeroing the entries outside $B$.)

Such a covariance thresholding approach was proposed in [KNV13], and is in turn related to earlier work by Bickel and Levina [BL08]. The formulation discussed in the next section presents some technical differences that have been introduced to simplify the analysis. Notice that, to simplify proofs, we assume $k$ to be known: This issue is discussed in the next two sections.

The rest of the paper is organized as follows. In the next section we provide a detailed description of the algorithm and state our main results. Our theoretical results assume full knowledge of problem parameters for ease of proof. In light of this, in Section 3 we discuss a practical implementation of the same idea that does not require prior knowledge of problem parameters, and is entirely data-driven. We also illustrate the method through simulations. The complete proofs are available in the accompanying supplement, in Sections 4, 5 and 6 respectively.

### 2 Algorithm and main result

#### Algorithm 1 Covariance Thresholding

1: **Input:** Data $(x_i)_{1 \leq i \leq 2n}$, parameters $k_q \in \mathbb{N}$, $\tau, \rho \in \mathbb{R}_{\geq 0}$;
2: Compute the empirical covariance matrices $G = \sum_{i=1}^{n} x_i x_i^T/n$, $G' = \sum_{i=n+1}^{2n} x_i x_i^T/n$;
3: Compute $\hat{\Sigma} = G - I_p$ (resp. $\hat{\Sigma}' = G' - I_p$);
4: Compute the matrix $\eta(\hat{\Sigma})$ by soft-thresholding the entries of $\hat{\Sigma}$:
   $$\eta(\hat{\Sigma})_{ij} = \begin{cases} \hat{\Sigma}_{ij} - \frac{\tau}{\sqrt{n}} & \text{if } \hat{\Sigma}_{ij} \geq \tau/\sqrt{n}, \\ 0 & \text{if } -\tau/\sqrt{n} < \hat{\Sigma}_{ij} < \tau/\sqrt{n}, \\ \hat{\Sigma}_{ij} + \frac{\tau}{\sqrt{n}} & \text{if } \hat{\Sigma}_{ij} \leq -\tau/\sqrt{n}, \end{cases}$$
5: Let $(\hat{v}_q)_{q \leq r}$ be the first $r$ eigenvectors of $\eta(\hat{\Sigma})$;
6: Define $s_q \in \mathbb{R}^p$ by $s_{q,i} = \hat{v}_{q,i} I(|\hat{v}_{q,i}| \geq \theta/2 \sqrt{k_q})$;
7: **Output:** $Q = \{i \in [p] : \exists q \text{ s.t. } |\langle \hat{\Sigma}' s_q, i \rangle| \geq \rho\}.$

For notational convenience, we shall assume hereafter that $2n$ sample vectors are given (instead of $n$): $(x_i)_{1 \leq i \leq 2n}$. These are distributed according to the model (1). The number of spikes $r$ will be treated as a known parameter in the problem.

We will make the following assumptions:

A1 The number of spikes $r$ and the signal strengths $\beta_1, \ldots, \beta_r$ are fixed as $n, p \to \infty$.

Further $\beta_1 > \beta_2 > \ldots \beta_r$ are all distinct.
A2 Let $Q_q$ and $k_q$ denote the support of $v_q$ and its size respectively. We let $Q = \cup Q_q$ and $k = \sum_q k_q$ throughout. Then the non-zero entries of the spikes satisfy $|v_{q,i}| \geq \theta/\sqrt{k_q}$ for all $i \in Q_q$ for some $\theta > 0$. Further, for any $q, q'$ we assume $|v_{q,i}/v_{q',i}| \leq \gamma$ for every $i \in Q_q \cap Q_{q'}$, for some constant $\gamma$.

As before, we are interested in the high-dimensional limit of $n, p \to \infty$ with $p/n \to \alpha$. A more detailed description of the covariance thresholding algorithm for the general model (1) is given in Table I. We describe the basic intuition for the simpler rank-one case (omitting the subscript $q \in \{1, 2, \ldots, r\}$), while stating results in generality.

We start by splitting the data into two halves: $(x_i)_{1 \leq i \leq n}$ and $(x_i)_{n<i \leq 2n}$ and compute the respective sample covariance matrices $G$ and $G'$ respectively. As we will see, the matrix $G$ is used to obtain a good estimate for the spike $v$. This estimate, along with the (independent) second part $G'$, is then used to construct a consistent estimator for the supports of $v$. 

Let us focus on the first phase of the algorithm, which aims to obtain a good estimate of $v$. We first compute $\Sigma = G - I$. For $\beta > \sqrt{n},$ the principal eigenvector of $G$, and hence of $\Sigma$ is positively correlated with $v$, i.e. $\lim_{n \to \infty} \langle \hat{\Sigma} v, v \rangle > 0$. However, for $\beta < \sqrt{n}$, the noise component in $\hat{\Sigma}$ dominates and the two vectors become asymptotically orthogonal, i.e. for instance $\lim_{n \to \infty} \langle \hat{\Sigma} v, v \rangle = 0$. In order to reduce the noise level, we must exploit the sparsity of the spike $v$.

Denoting by $X \in \mathbb{R}^{n \times p}$ the matrix with rows $x_1, \ldots, x_n$, by $Z \in \mathbb{R}^{n \times p}$ the matrix with rows $z_1, \ldots, z_n$, and letting $u = (u_1, u_2, \ldots, u_n)$, the model (1) can be rewritten as

$$X = \sqrt{\beta}uv^T + Z. \tag{2}$$

Hence, letting $\beta' = \beta\|u\|^2/n \approx \beta$, and $w = \sqrt{\beta}v^T u/n$

$$\hat{\Sigma} = \beta'vv^T + vv^T + wv^T + 1/nZ^T Z - I_p, \tag{3}$$

For a moment, let us neglect the cross terms $(vv^T + wv^T)$. The ‘signal’ component $\beta'vv^T$ is sparse with $k^2$ entries of magnitude $\beta/k$, which (in the regime of interest $k = \sqrt{n}/C$) is equivalent to $C\beta/\sqrt{n}$. The ‘noise’ component $Z^T Z/n - I_p$ is dense with entries of order $1/\sqrt{n}$. Assuming $k/\sqrt{n}$ a small enough constant, it should be possible to remove most of the noise by thresholding the entries at level of order $1/\sqrt{n}$. For technical reasons, we use the soft thresholding function $\eta : \mathbb{R} \times \mathbb{R}_{\geq 0} \to \mathbb{R}$, $\eta(z; \tau) = \text{sgn}(z)(|z| - \tau)_+$. We will omit the second argument from $\eta(\cdot; \cdot)$ wherever it is clear from context. Classical denoising theory [DJ94, Joh02] provides upper bounds the estimation error of such a procedure. Note however that these results fall short of our goal. Classical theory measures estimation error by (element-wise) $\ell_p$ norm, while here we are interested in the resulting principal eigenvector. This would require bounding, for instance, the error in operator norm.

Since the soft thresholding function $\eta(z; \tau/\sqrt{n})$ is affine when $z \gg \tau/\sqrt{n}$, we would expect that the following decomposition holds approximately (for instance, in operator norm):

$$\eta(\hat{\Sigma}) \approx \eta(\beta'vv^T) + \eta\left(\frac{1}{n}Z^T Z - I_p\right). \tag{4}$$

The main technical challenge now is to control the operator norm of the perturbation $\eta(\frac{1}{n}Z^T Z - I_p)$. It is easy to see that $\eta(\frac{1}{n}Z^T Z - I_p)$ has entries of variance $\delta(\tau)/n$, for $\delta(\tau) \to 0$ as $\tau \to \infty$. If entries were independent with mild decay, this would imply— with high probability—

$$\left\|\eta\left(\frac{1}{n}Z^T Z - I_p\right)\right\|_2 \lesssim C\delta(\tau), \tag{5}$$

for some constant $C$. Further, the first component in the decomposition (4) is still approximately rank one with norm of the order of $\beta' \approx \beta$. Consequently, with standard linear algebra results on the perturbation of eigenspaces [DK70], we obtain an error bound $\|\tilde{v} - v\| \lesssim \delta(\tau)/C'\beta$. Our first theorem formalizes this intuition and provides a bound on the estimation error in the principal components of $\eta(\Sigma)$.
Theorem 1. Under the spiked covariance model Eq. (1) satisfying Assumption A1, let \( \hat{v}_q \) denote the \( q \)th eigenvector of \( \eta(\Sigma) \) using threshold \( \tau \). For every \( \alpha, (\beta_q)_{q=1}^r \in (0, \infty) \), integer \( r \) and every \( \varepsilon > 0 \) there exist constants, \( \tau = \tau(\varepsilon, \alpha, (\beta_q)_{q=1}^r, \theta) \) and \( 0 < c_* = c_*(\varepsilon, \alpha, (\beta_q)_{q=1}^r, \theta) < \infty \) such that, if \( \sum_q k_q = \sum_q |\text{supp}(v_q)| \leq c_* \sqrt{n} \), then

\[
\mathbb{P} \left\{ \min(\|\hat{v}_q - v_q\|, \|\hat{v}_q + v_q\|) \leq \varepsilon \quad \forall q \in \{1, \ldots, r\} \right\} \geq 1 - \frac{\alpha}{n^k}.
\] (6)

It is clear from the discussion above that the proof of Theorem 1 requires a formalization of Eq. [3]. Indeed, the spectral properties of random matrices of the type \( f(Z^T Z/n - I_p) \), called inner-product kernel random matrices, have attracted recent interest within probability theory [EK10a, EK10b, CS12]. In this literature, the asymptotic eigenvalue distribution of a matrix \( f(Z^T Z/n - I_p) \) is the object of study. Here \( f : \mathbb{R} \to \mathbb{R} \) is a kernel function and is applied entry-wise to the matrix \( Z^T Z/n - I_p \), with \( Z \) a matrix as above. Unfortunately, these results do not suffice to prove Theorem 1 for the following reasons:

- The results [EK10a, EK10b] are perturbative and provide conditions under which the spectrum of \( f(Z^T Z/n - I_p) \) is close to a rescaling of the spectrum of \( (Z^T Z/n - I_p) \) (with rescaling factors depending on the Taylor expansion of \( f \) close to 0). We are interested instead in a non-perturbative regime in which the spectrum of \( f(Z^T Z/n - I_p) \) is very different from the one of \( (Z^T Z/n - I_p) \) (and the Taylor expansion is trivial).

- The authors of [CS12] consider \( n \)-dependent kernels, but their results are asymptotic and concern the weak limit of the empirical spectral distribution of \( f(Z^T Z/n - I_p) \). This does not yield an upper bound on the spectral norm \( \|f(Z^T Z/n - I_p)\| \).

Our approach to prove Theorem 1 follows instead the so-called \( \varepsilon \)-net method: we develop high probability bounds on the maximum Rayleigh quotient:

\[
\max_{y \in S^{p-1}} \langle y, \eta(Z^T Z/n - I_p) y \rangle = \max_{y \in S^{p-1}} \sum_{i,j} y_i y_j \frac{n}{\eta(Z, Z^T)} \frac{\tau}{\sqrt{n}},
\]

where \( S^{p-1} = \{ y \in \mathbb{R}^p : \|y\| = 1 \} \) is the unit sphere. Since \( \eta(Z^T Z/n - I_p) \) is not Lipschitz continuous in the underlying Gaussian variables \( Z \), concentration does not follow immediately from Gaussian isoperimetry. We have to develop more care and concern the weak limit of the empirical spectral distribution of \( f(Z^T Z/n - I_p) \). This does not yield an upper bound on the spectral norm \( \|f(Z^T Z/n - I_p)\| \).

Theorem 2. Consider the spiked covariance model of Eq. [1] satisfying Assumptions A1, A2. For any \( \alpha, (\beta_q)_{q \leq r} \in (0, \infty), \theta, \gamma > 0 \) and integer \( r \), there exist constants \( c_*, \tau, \rho \) dependent on \( \alpha, (\beta_q)_{q \leq r}, \gamma, \theta, r \), such that, if \( \sum_q k_q = |\text{supp}(v_q)| \leq c_* \sqrt{n} \), the Covariance Thresholding algorithm of Table 1 recovers the support of \( v_q \) with high probability.

Explicitly, there exists a constant \( C > 0 \) such that

\[
\mathbb{P} \left\{ \widehat{Q} = \bigcup q \text{supp}(v_q) \right\} \geq 1 - \frac{C}{n^k}.
\] (7)

Before passing to the proofs of Theorem 1 and Theorem 2 (respectively in Sections 6 and 5 of the Supplementary Material), it is useful to pause for a few remarks.

\footnote{Note that [CS12] also provide a non-asymptotic bound for the spectral norm of \( f(Z^T Z/n - I_p) \) via the moment method, but this bound diverges with \( n \) and does not give a result of the type of Eq. [4].}
Remark 2.1. We focus on a consistent estimation of the union of the supports $\bigcup_{q} \operatorname{supp}(v_q)$ of the spikes. In the rank-one case, this obviously corresponds to the standard support recovery. In the general case, once the union is correctly estimated, estimating the individual supports poses no additional difficulty: indeed, since $|\bigcup_{q} \operatorname{supp}(v_q)| = O(\sqrt{n})$ an extra step with $n$ fresh samples $x_i$ restricted to $Q$ yields consistent estimates for $v_q$, hence $\operatorname{supp}(v_q)$.

Remark 2.2. Recovering the signed supports $Q_{q,+} = \{i \in [p] : v_{q,i} > 0\}$ and $Q_{q,-} = \{i \in [p] : v_{q,i} < 0\}$ is possible using the same technique as recovering the supports $\operatorname{supp}(v_q)$ above, and poses no additional difficulty.

Remark 2.3. Assumption A2 requires $|v_{q,i}| \geq \theta / \sqrt{k_q}$ for all $i \in Q_q$. This is a standard requirement in the support recovery literature [Wai09, MB06]. The second part of assumption A2 guarantees that when the supports of two spikes overlap, their entries are roughly of the same order. This is necessary for our proof technique to go through. Avoiding such an assumption altogether remains an open question.

Our covariance thresholding algorithm assumes knowledge of the correct support sizes $k_q$. Notice that the same assumption is made in earlier theoretical, e.g. in the analysis of SDP-based reconstruction by Amini and Wainwright [AW09]. While this assumption is not realistic in applications, it helps to focus our exposition on the most challenging aspects of the problem. Further, a ballpark estimate of $k_q$ (indeed of $\sum_q k_q$) is actually sufficient. Indeed consider the algorithm obtained by replacing steps 7 and 8 as following.

7: Define $s'_q \in \mathbb{R}^p$ by

$$s'_{q,i} = \begin{cases} \tilde{v}_{q,i} & \text{if } |\tilde{v}_{q,i}| > \theta/(2\sqrt{k_0}) \\ 0 & \text{otherwise.} \end{cases}$$

8: Return

$$\hat{Q} = \bigcup_{q} \{i \in [p] : |(\hat{\Sigma}'s'_q)_i| \geq \rho\}.$$  

The next theorem shows that this procedure is effective even if $k_0$ overestimates $\sum_q k_q$ by an order of magnitude. Its proof is deferred to Section 5.

**Theorem 3.** Consider the spiked covariance model of Eq. (1). For any $\alpha, \beta \in (0, \infty)$, let constants $c_*, \tau, \rho$ be given as in Theorem 2. Further assume $k = \sum_q |\operatorname{supp}(v_q)| \leq c_* \sqrt{n}$, and $\sum_q k \leq k_0 \leq 20 \sum_q k_q$.

Then, the Covariance Thresholding algorithm of Table 2, with the definitions in Eqs. (8) and (9), recovers the union of supports of $v_q$ successfully, i.e.

$$\mathbb{P}\left(\hat{Q} = \bigcup_{q} \operatorname{supp}(v_q)\right) \geq 1 - \frac{C}{n^4}.$$  

3 Practical aspects and empirical results

Specializing to the rank one case, Theorems 1 and 2 show that Covariance Thresholding succeeds with high probability for a number of samples $n \gtrsim k^2$, while Diagonal Thresholding requires $n \gtrsim k^2 \log p$. The reader might wonder whether eliminating the log $p$ factor has any practical relevance or is a purely conceptual improvement. Figure 3 presents simulations on synthetic data under the strictly sparse model, and the Covariance Thresholding algorithm of Table 1 used in the proof of Theorem 2. The objective is to check whether the log $p$ factor has an impact at moderate $p$. We compare this with Diagonal Thresholding.

We plot the empirical success probability as a function of $k/\sqrt{n}$ for several values of $p$, with $p = n$. The empirical success probability was computed by using 100 independent instances of the problem. A few observations are of interest: (i) Covariance Thresholding appears to have a significantly larger success probability in the ‘difficult’ regime where Diagonal Thresholding starts to fail; (ii) The curves for Diagonal
Thresholding appear to decrease monotonically with $p$ indicating that $k$ proportional to $\sqrt{n}$ is not the right scaling for this algorithm (as is known from theory); (iii) In contrast, the curves for Covariance Thresholding become steeper for larger $p$, and, in particular, the success probability increases with $p$ for $k \leq 1.1\sqrt{n}$. This indicates a sharp threshold for $k = \text{const} \cdot \sqrt{n}$, as suggested by our theory.

In terms of practical applicability, our algorithm in Table 1 has the shortcomings of requiring knowledge of problem parameters $\beta_q$, $r$, $k_q$. Furthermore, the thresholds $\rho$, $\tau$ suggested by theory need not be optimal. We next describe a principled approach to estimating (where possible) the parameters of interest and running the algorithm in a purely data-dependent manner. Assume the following model, for $i \in [n]$

$$x_i = \mu + \sum_q \sqrt{\beta_q} u_{q,i} v_q + \sigma z_i,$$

where $\mu \in \mathbb{R}^p$ is a fixed mean vector, $u_{q,i}$ have mean 0 and variance 1, and $z_i$ have mean 0 and covariance $I_p$. Note that our focus in this section is not on rigorous analysis, but instead to demonstrate a principled approach to applying covariance thresholding in practice. We proceed as follows:

**Estimating $\mu$, $\sigma$:** We let $\hat{\mu} = \frac{1}{n} \sum_{i=1}^n x_i$ be the empirical mean estimate for $\mu$. Further letting $\bar{X} = X - 1\mu^T$, we see that $pn - (\sum_q k_q)n \approx pn$ entries of $\bar{X}$ are mean 0 and variance $\sigma^2$. We let $\hat{\sigma} = \text{MAD}(\bar{X})/\nu$ where $\text{MAD}(\cdot)$ denotes the median absolute deviation of the entries of the matrix in the argument, and $\nu$ is a constant scale factor. Guided by the Gaussian case, we take $\nu = \Phi^{-1}(3/4) \approx 0.6745$.

**Choosing $\tau$:** Although in the statement of the theorem, our choice of $\tau$ depends on the SNR $\beta/\sigma^2$, we believe this is an artifact of our analysis. Indeed it is reasonable to threshold ‘at the noise level’, as follows. The noise component of entry $i,j$ of the sample covariance (ignoring lower order terms) is given by $\sigma^2(z_i, z_j)/n$. By the central limit theorem, $(z_i, z_j)/\sqrt{n} \stackrel{d}{\Rightarrow} N(0, 1)$. Consequently, $\sigma^2(z_i, z_j)/n \approx N(0, \sigma^2/n)$, and we need to choose the (rescaled) threshold proportional to $\sqrt{\sigma^2} = \sigma^2$. Using previous estimates, we let $\tau = \nu' \cdot \hat{\sigma}^2$ for a constant $\nu'$. In simulations, a choice $3 \leq \nu' \leq 4$ appears to work well.

**Estimating $r$:** We define $\hat{\Sigma} = \bar{X}^T \bar{X}/n - \sigma^2 I_p$ and soft threshold it to get $\eta(\hat{\Sigma})$ using $\tau$ as above. Our proof of Theorem 1 relies on the fact that $\eta(\hat{\Sigma})$ has $r$ eigenvalues that are separated from the bulk of the spectrum. Hence, we estimate $r$ using $\tau$: the number of eigenvalues separated from the bulk in $\eta(\hat{\Sigma})$. The edge of the spectrum can be computed numerically using the Stieltjes transform method as in [CS12].
Estimating $v_q$: Let $\tilde{v}_q$ denote the $q^{th}$ eigenvector of $\eta(\tilde{\Sigma})$. Our theoretical analysis indicates that $\tilde{v}_q$ is expected to be close to $v_q$. In order to denoise $\tilde{v}_q$, we assume $\tilde{v}_q \approx (1 - \delta)v_q + \varepsilon_q$, where $\varepsilon_q$ is additive random noise. We then threshold $\tilde{v}_q$ ‘at the noise level’ to recover a better estimate of $v_q$. To do this, we estimate the standard deviation of the “noise” $\varepsilon$ by $\sigma^2 = \text{MAD}(v_q)/\nu$. Here we set –again guided by the Gaussian heuristic– $\nu \approx 0.6745$. Since $v_q$ is sparse, this procedure returns a good estimate for the size of the noise deviation. We let $\eta_H(\tilde{v}_q)$ denote the vector obtained by hard thresholding $\tilde{v}_q$: set $(\eta_H(\tilde{v}_q))_i = \tilde{v}_{q,i}$ if $|\tilde{v}_{q,i}| \geq \nu \sigma_q$ and 0 otherwise. We then let $\hat{v}_q = \eta(\tilde{v}_q)/\|\eta(\tilde{v}_q)\|$ and return $\hat{v}_q$ as our estimate for $v_q$.

Note that –while different in several respects– this empirical approach shares the same philosophy of the algorithm in Table 1. On the other hand, the data-driven algorithm presented in this section is less straightforward to analyze, a task that we defer to future work.

Figure 1 also shows results of a support recovery experiment using the ‘data-driven’ version of this section. Covariance thresholding in this form also appears to work for supports of size $n$ smaller than $\sqrt{n}$. On the other hand, the data-driven algorithm presented in this section is less straightforward to analyze, a task that we defer to future work.

Figure 1 also shows results of a support recovery experiment using the ‘data-driven’ version of this section. Covariance thresholding in this form also appears to work for supports of size $k \leq \sqrt{n}$. Figure 2 shows the performance of vanilla PCA, Diagonal Thresholding and Covariance Thresholding on the “Three Peaks” example of Johnstone and Lu [JL04]. This signal is sparse in the wavelet domain and the simulations employ the data-driven version of covariance thresholding. A similar experiment with the “box” example of Johnstone and Lu is provided in the supplement. These experiments demonstrate that, while for large values of $n$ both Diagonal Thresholding and Covariance Thresholding perform well, the latter appears superior for smaller values of $n$.

4 Proof preliminaries

In this section we review some notation and preliminary facts that we will use throughout the paper.

4.1 Notation

We let $[m] = \{1, 2, \ldots, m\}$ denote the set of first $m$ integers. We will represent vectors using boldface lower case letters, e.g., $u, v, x$. The entries of a vector $u \in \mathbb{R}^n$ will be represented by $u_i, i \in [n]$. Matrices are represented using boldface upper case letters e.g., $A, X$. The entries of a matrix $A \in \mathbb{R}^{m \times n}$ are represented by $A_{ij}$ for $i \in [m], j \in [n]$. Given a matrix $A \in \mathbb{R}^{m \times n}$, we generically let $a_1, a_2, \ldots, a_m$ denote its rows, and $\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n$ its columns.

For $E \subseteq [m] \times [n]$, we define the projector operator $P_E : \mathbb{R}^{m \times n} \to \mathbb{R}^{m \times n}$ by letting $P_E(A)$ be the matrix with entries

$$
P_E(A)_{ij} = \begin{cases} A_{ij} & \text{if } (i,j) \in E, \\
0 & \text{otherwise.} \end{cases} \quad (11)$$

If $E = E_1 \times E_2$, we write $P_{E_1,E_2}$ for $P_{E_1 \times E_2}$. In the case $E = E_1 \times E_2$ we also define a projection operator $\tilde{P}_{E_1,E_2} : \mathbb{R}^{m \times n} \to \mathbb{R}^{|E_1| \times |E_2|}$ that returns the $E_1 \times E_2$ submatrix. If $m = n$, and $E$ is the diagonal, we write $P_d$ for $P_E$. If instead $E$ is the complement of the diagonal, we write $P_{\bar{d}}$. For a matrix $A \in \mathbb{R}^{m \times n}$, and a set $E \subseteq [n]$, we define its column restriction $A_E \in \mathbb{R}^{m \times n}$ to be the matrix obtained by setting to 0 columns outside $E$:

$$(A_E)_{ij} = \begin{cases} A_{ij} & \text{if } j \in E, \\
0 & \text{otherwise.} \end{cases}$$

Similarly $y_E$ is obtained from $y$ by setting to zero all indices outside $E$. The operator norm of a matrix $A$ is denoted by $\|A\|$ (or $\|A\|_2$) and its Frobenius norm by $\|A\|_F$. We write $\|x\|$ for the standard $\ell_2$ norm of a vector $x$.

We let $Q_q$ denotes the support of the $q^{th}$ spike $v_q$. Also, we denote the union of the supports of $v_q$ by $Q = \cup_q Q_q$. The complement of a set $E \subseteq [n]$ is denoted by $E^c$. 


Figure 2: The results of Simple PCA, Diagonal Thresholding and Covariance Thresholding (respectively) for the “Three Peak” example of Johnstone and Lu [JL09] (see Figure 1 of the paper). The signal is sparse in the ‘Symmlet 8’ basis. We use $\beta = 1.4, p = 4096$, and the rows correspond to sample sizes $n = 1024, 1625, 2580, 4096$ respectively. Parameters for Covariance Thresholding are chosen as in Section 3 with $\nu' = 4.5$. Parameters for Diagonal Thresholding are from [JL09]. On each curve, we superpose the clean signal (dotted).

We write $\eta(\cdot; \cdot)$ for the soft-thresholding function. By $\partial \eta(\cdot; \tau)$ we denote the derivative of $\eta(\cdot; \tau)$ with respect to the first argument, which exists Lebesgue almost everywhere.

In the statements of our results, consider the limit of large $p$ and large $n$ with $p/n \to \alpha$. This limit will be referred to either as “$n$ large enough” or “$p$ large enough” where the phrase “large enough” indicates dependence of $p$ (and thereby $n$) on specific problem parameters.

4.2 Preliminary facts

Let $S^{n-1}$ denote the unit sphere in $n$ dimensions, i.e. $S^{n-1} = \{ x : \| x \| = 1 \}$. We use the following definition (see [Ver12]) of the $\varepsilon$-net of a set $X \subseteq \mathbb{R}^n$:

**Definition 4.1** (Nets, Covering numbers). A subset $T^\varepsilon(X) \subseteq X$ is called an $\varepsilon$-net of $X$ if every point in $X$ may be approximated by one in $T^\varepsilon(X)$ with error at most $\varepsilon$. More precisely:

$$\forall x \in X, \inf_{y \in T^\varepsilon(X)} \| x - y \| \leq \varepsilon.$$
The minimum cardinality of an \( \varepsilon \)-net of \( X \), if finite, is called its covering number.

The following two facts are useful while using \( \varepsilon \)-nets to bound the spectral norm of a matrix. For proofs, we refer the reader to [Ver12].

**Lemma 4.2.** Let \( S^{n-1} \) be the unit sphere in \( n \) dimensions. Then there exists an \( \varepsilon \)-net of \( S^{n-1}, T^\varepsilon(S^{n-1}) \) satisfying:

\[
|T^\varepsilon(S^{n-1})| \leq \left(1 + \frac{2}{\varepsilon}\right)^n.
\]

**Lemma 4.3.** Let \( A \in \mathbb{R}^{n \times n} \) be a symmetric matrix. Then:

\[
\|A\|_2 = \sup_{x \in S^{n-1}} |\langle x, Ax \rangle| \leq (1 - 2\varepsilon)^{-1} \sup_{x \in T^\varepsilon(S^{n-1})} |\langle x, Ax \rangle|.
\]

In particular, if \( A \) is a random matrix, then for \( \Delta > 0 \) we have:

\[
\mathbb{P}\{\|A\| \geq \Delta\} \leq \left(1 + \frac{2}{\varepsilon}\right)^n \sup_{x \in T^\varepsilon(S^{n-1})} \mathbb{P}\{|\langle x, Ax \rangle| \geq \Delta(1 - 2\varepsilon)\}.
\]

Throughout the paper we will denote by \( T_n^\varepsilon \) the minimum cardinality \( \varepsilon \)-net on the unit sphere \( S^{n-1} \), which naturally satisfies Lemma 4.2. Further, for a non-zero vector \( y \in \mathbb{R} \), we define the set \( S_y^{n-1} = \{x : \langle x, y \rangle = 0, |x| = 1\} \) and let its minimum cardinality \( \varepsilon \)-net be denoted by \( T_n^\varepsilon(y) \). Since \( S_y^{n-1} \) is isometric to \( S^{n-2} \), Lemma 4.2 holds for \( T_n^\varepsilon(y) \) as well.

We now state some measure concentration results that we will use at various points in the proofs of Theorems 4 and 6.

**Lemma 4.4.** Consider \( z \sim \mathcal{N}(0, I_N) \) be a vector of \( N \) i.i.d. standard normal random variables on a probability space \( (\Omega, F, \mathbb{P}) \). Suppose \( F : \mathbb{R}^N \to \mathbb{R} \) is a \( \mathbb{R} \)-valued, continuous, a.e. differentiable function and \( G \in \mathcal{B}_{\mathbb{R}^N} \) is a closed convex set satisfying:

\[
\|\nabla F(z)\| 1(z \in G) \leq L \quad \mathbb{P}\text{-a.e.}
\]

\[
\mathbb{P}\{G\} \geq 1 - q.
\]

Then, there exists a function \( F_L : \mathbb{R}^N \to \mathbb{R} \) such that \( F_L \) is \( L \)-Lipschitz throughout and \( F_L \) coincides with \( F \) on the set \( G \). Further for each \( \Delta > 0 \) we have that:

\[
\mathbb{P}\{|F(z) - \mathbb{E}F(z)| \geq \Delta\} \leq q + 2 \exp\left(-\frac{\Delta^2}{2L^2}\right),
\]

where \( \Delta = \Delta - |\mathbb{E}F(z) - \mathbb{E}F_L(z)| \).

**Proof.** For any \( y, y' \in G \) we have that:

\[
F(y') = F(y) + \int_0^1 \langle \nabla F(ty' + (1 - t)y), y' - y \rangle dt.
\]

From this we obtain that \( |F(y') - F(y)| \leq L \|y' - y\| \) using the bound on \( \nabla F \) in \( G \) and the convexity of \( G \). By Kirszbraun’s theorem, there exists an \( L \)-Lipschitz extension \( F_L \) of \( F \) to \( \mathbb{R}^N \). Indeed we may take \( F_L(y) = \inf_{y' \in G} F(y) + L \|y - y'\| \). Then:

\[
\mathbb{P}\{|F(z) - \mathbb{E}F(z)| \geq \Delta\} = \mathbb{P}\{|F(z) - \mathbb{E}F(z)| \geq \Delta ; z \in G\} + \mathbb{P}\{|F(z) - \mathbb{E}F(z)| \geq \Delta ; z \in G^c\}
\]

\[
\leq \mathbb{P}\{|F_L(z) - \mathbb{E}F_L(z)| \geq \Delta\} + \mathbb{P}\{G^c\}
\]

Applying Gaussian concentration of measure [Lec01] to \( F_L \) finishes the proof. \( \square \)
For further reference, we define the following:

**Definition 4.5.** For a function $F : \mathbb{R}^N \to \mathbb{R}$, a constant $L > 0$ and a measurable set $G$, we call $F_L(\cdot)$ the $G, L$-Lipschitz extension of $F(\cdot)$. It is given by:

$$F_L(y) = \inf_{y' \in G} (F(y') + L \|y - y'\|).$$

**Lemma 4.6.** Let $A \in \mathbb{R}^{M \times N}$ be a matrix with i.i.d. standard normal entries, i.e. $A_{ij} \sim \mathcal{N}(0,1)$. Then, for every $t \geq 0$:

$$\mathbb{P}\left\{ \|A\|_2 \geq \sqrt{M} + \sqrt{N} + t \right\} \leq \exp\left( -\frac{t^2}{2} \right).$$

The proof of this result can be found in [Ver12].

## 5 Proof of Theorems 2 and 3

In this section we prove Theorem 2 and Theorem 3 assuming that Theorem 1 holds. The proof of the latter can be found in Section 6.

### 5.1 Proof of Theorem 2

For any fixed $\varepsilon > 0$, and assume $\sum_q k_q \leq \sqrt{n \log \tau / \tau^3}$, where $\tau = \tau(\varepsilon, \beta, \alpha)$ as per Theorem 1. Then we have for every $q$, $\|\tilde{v}_q - v_q\| \leq \varepsilon$ with probability at least $1 - C/n^4$ for some constant $C > 0$.

Throughout the proof, we will work on this favorable event of Theorem 1, namely use

$$\mathbb{P}\left( \tilde{Q} \neq \cup_q \text{supp}(v_q) \right) \leq \mathbb{P}\left( \tilde{Q} \neq \cup_q \text{supp}(v_q); \|\tilde{v}_q - v_q\| \leq \varepsilon^2 \right) + \frac{C}{n^4}, \quad (12)$$

hence focusing on bounding the first term on the right hand side.

It is convenient to isolate the following lemma.

**Lemma 5.1.** Assume $\|\tilde{v}_q - v_q\|^2 \leq \varepsilon^2$ and that $|v_{q,i}| \geq \theta/\sqrt{k_q}$. Let $B_q \equiv \text{supp}(s_q)$ with $s_q$ defined as per Algorithm 1 step 7. Then $|B_q \triangle Q_q| \leq 4\varepsilon^2 k_q / \theta^2$ and hence $|B_q \cap Q_q| \geq (1 - 4\varepsilon^2 / \theta^2) k_q$. (Here $\triangle$ denotes the symmetric set-difference.) Further min$(\|s_q - v_q\|^2, \|s_q + v_q\|^2) \leq 5\varepsilon^2$.

**Proof.** Recall that $s_{q,i} = \tilde{v}_{q,i} I(|\tilde{v}_{q,i}| \geq \theta/\sqrt{k_q})$. Since $|v_{q,i}| \geq \theta/\sqrt{k_q}$:

$$B_q \triangle Q_q \supseteq \left\{ i : |v_{q,i} - \tilde{v}_{q,i}| \geq \frac{\theta}{2\sqrt{k_q}} \right\}.$$

Thus $|B_q \triangle Q_q| \leq 4k_q |\tilde{v}_q - v_q|^2 / \theta^2 \leq 4\varepsilon^2 k_q / \theta^2$.

Now we bound the error $\|s_q - v_q\|$, assuming that $\|\tilde{v}_q - v_q\| \leq \varepsilon$. The other case is handled in an analogous fashion:

$$\|s_q - v_q\|^2 = \sum_{i \in Q_q} (s_{q,i} - v_{q,i})^2 + \sum_{i \in Q_q^c} (s_{q,i})^2 I(|\tilde{v}_{q,i}| \geq \theta/2\sqrt{k_q})$$

$$= \sum_{i \in Q_q} v_{q,i}^2 I(|\tilde{v}_{q,i}| \leq \theta/2\sqrt{k_q}) + \sum_{i \in Q_q} (\tilde{v}_{q,i} - v_{q,i})^2 I(|\tilde{v}_{q,i}| \geq \theta/2\sqrt{k_q}) + \sum_{i \in Q_q^c} (s_{q,i})^2 I(|\tilde{v}_{q,i}| \geq \theta/2\sqrt{k_q})$$

$$\leq \sum_{i \in Q_q} v_{q,i}^2 I(|\tilde{v}_{q,i} - v_{q,i}| \geq |v_{q,i}| - \theta/(2\sqrt{k_q})) + \|\tilde{v}_q - v_q\|^2$$

$$\leq \sum_{i \in Q_q} \frac{v_{q,i}^2}{(|v_{q,i}| - \theta/(2\sqrt{k_q})^2)(\tilde{v}_{q,i} - v_{q,i})^2 + \|\tilde{v}_q - v_q\|^2}$$

$$\leq 5 \|\tilde{v}_q - v_q\|^2 \leq 5\varepsilon^2.$$
The first inequality above follows from triangle inequality as $|\bar{v}_{q,i}| \geq |v_{q,i}| - |\bar{v}_{q,i} - v_{q,i}|$. The second inequality employs $\mathbb{I}(z \geq z') \leq (z/z')^2$. The final inequality uses the fact that $|v_{q,i}| \geq \theta/2\sqrt{K_q}$ implies $|v_{q,i}|/(|v_{q,i}| - \theta/2\sqrt{K_q}) \leq 2$.

Now we are in position to prove the main theorem. Without loss of generality, we will assume that $\langle \bar{v}_q, v_q \rangle > 0$ for every $q$. The other case is treated in the same way.

Recall that $\tilde{\Sigma}'$ was formed from the samples $(x_i)_{n < i < 2n}$, which are independent of $\bar{v}_q$ and hence $B_q$. We let $X' \in \mathbb{R}^{n \times p}$ denote the matrix with rows $(x_i)_{n < i < 2n}$ we have, in the same fashion as Eq. (2), $X' = \sum_q \sqrt{\beta_q}u_q'(v_q)^T + Z'$. We let $\tilde{z}', 1 \leq i \leq p$ denote the columns of $Z'$.

For any $i$:

$$\langle \tilde{\Sigma}'s_i \rangle_i = \frac{\beta_1 \|u_1\|^2}{n} \langle v_1, s_1 \rangle_{v_1,i} + \sum_{q \neq 1} \frac{\beta_q \|u_q\|^2}{n} \langle v_q, s_q \rangle_{v_q,i} + \sum_{q \geq 1} \frac{\sqrt{\beta_q}}{n} \langle (Z'^T u_q', s_1)_{v_q,i} + \langle v_q, s_1 \rangle (Z'^T u_q)_{v_q,i} \rangle$$

$$+ \sum_{q > q'} \frac{\sqrt{\beta_q \beta_{q'}}}{n} \langle u_q', u_{q'}' \rangle \langle v_{q', i} (v_q', s_1) + v_q', i (v_q, s_1) \rangle + \frac{1}{n} \sum_{j \in B_1, j \neq i} (\tilde{z}'_j, \tilde{z}'_s)_{s_1,j} + \left( \frac{\|z'_i\|^2}{n} - 1 \right) s_1,i$$

Let $T_1, T_2, \ldots, T_5$ denote the terms above. Firstly, by a standard calculation $n/2 \leq \|u'_q\|^2 \leq 2n$ and $\max_{q \neq q'} \|u_q', u_{q'}'\| \leq \sqrt{Cn\log n}$ with probability at least $1 - r n^{-10}$ for some constant $C$. Further, using Lemma 5.1 and Cauchy-Schwarz we have that $\langle v_1, s_1 \rangle \geq (1 - 5\varepsilon^2)$ and $\|v_q, s_1\| \leq \|v_1 - s_1\| \leq 3\varepsilon$. This implies that:

$$|T_1| \geq \frac{\beta_1 (1 - 5\varepsilon^2) \|v_{1,i}\|}{2},$$

$$|T_2| \leq 6\varepsilon \sum_{q \geq 1} \beta_q \|v_{q,i}\|,$$

$$|T_4| \leq C((\beta_q)_{q \leq r}) \sqrt{\frac{\log n}{n}}.$$  

Now consider the term $T_5 = \sum_{j \in B_1 \setminus j} (\tilde{z}'_j, \tilde{z}'_s)_{s_1,j}/n = (\tilde{z}'_j, \sum_{j \in B_1 \setminus j} s_1,j \tilde{z}'_s)/n$. Thus, $T_5 \equiv Y_{ij} \equiv (\tilde{z}'_j, \tilde{z}'_s)_{s_1,j}/n$ for $j \neq i$. Conditional on $\tilde{z}'_j$, $Y_{ij} \sim N(0, \|\tilde{z}'_j\|^2 \|s_1\|^2/n^2)$. Using the Chernoff bound, $\|\tilde{z}'_j\|^2 \leq 2n$ with probability at least $1 - \exp(-n/8)$ and, conditional on this event, $|Y_{ij}| \leq \sqrt{C' \log n / n}$ with probability at least $1 - n^{-10}$ for some absolute constant $C'$. It follows from the union bound that $|T_5| \leq \sqrt{C' \log n / n}$ with probability at least $1 - 2n^{-10}$ for $n$ large enough. Using a similar calculation $|T_5| \leq \sqrt{C''((\beta_q)_{q \leq r}) \log n / n}$ with probability exceeding $1 - n^{-10}$. Finally using Proposition 5.4 below, we have that

$$|T_5| \leq \|s_1\| \max_i \left( \frac{\|\tilde{z}_i\|^2}{n} - 1 \right)$$

$$\leq \sqrt{C'' \log n},$$

with probability at least $1 - n^{-10}$. Here we used the fact that $\|s_1\| \leq \|\bar{v}_1\| = 1$. 

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By Assumption A2, and the above estimates, we have with probability at least $1 - n^{-9}$:

For $i \in Q_1$, 

$$|\langle \mathbf{s}_1 \rangle_i|^2 \geq \frac{\beta_1}{2} (1 - 5\varepsilon^2 - 12\varepsilon \gamma \sum_q \beta_q |v_{1,i}|) - \sqrt{C \log n / n}$$

$$\geq \frac{\beta_1}{4} (1 - 5\varepsilon^2 - 12\varepsilon \gamma \sum_q \beta_q) \theta - \sqrt{C \log n / n},$$

For $i \in [p] \setminus \bigcup Q_q$, 

$$|\langle \mathbf{s}_1 \rangle_i|^2 \leq \frac{C \log n}{n}.$$

Choosing $\varepsilon = \varepsilon((\beta_q)_{q \leq r}, r, \theta, \gamma)$ small enough and using threshold $\rho = \min_q (\beta_q \theta / 4 \sqrt{k_q})$ we have that $Q_1 \subseteq \hat{Q}$ and $\hat{Q} \subseteq \cup Q_q$. The analogous guarantees for all $1 \leq q \leq r$ imply Theorem 2.

### 5.2 Proof of Theorem 3

Analogously to the previous proof, we fix $\varepsilon > 0$, and observe that $\sum_q k_q \leq \sqrt{n \log \tau / \tau^3}$, where $\tau = \tau(\varepsilon, \beta, \alpha, \theta)$, and per Theorem 1. Then we have that $\|\mathbf{v}_q - \mathbf{\hat{v}}_q\|^2 \leq \varepsilon / 20$ with probability at least $1 - C/n^4$ for some constant $C > 0$. We then use

$$\mathbb{P}\left(\hat{Q} \neq \cup_q \text{supp}(\mathbf{v}_q)\right) \leq \mathbb{P}\left(\hat{Q} \neq \cup_q \text{supp}(\mathbf{v}_q); \|\mathbf{v}_q - \mathbf{\bar{v}}_q\|^2 \leq \varepsilon / 20^r\right) + \frac{C}{n^4},$$

and bound the first term.

The key change with respect to the proof of theorems 2 is that we need to replace Lemma 5.1 with the following lemma, whose proof follows exactly the same argument as that of Lemma 5.1.

**Lemma 5.2.** Assume $\|\mathbf{v}_q - \mathbf{\bar{v}}_q\|^2 \leq \varepsilon / 20$, and let $\mathcal{B}' = \text{supp}(s')$ with $s'$ defined as per Eq. 8. Further assume $k \leq k_0 \leq 20k$. Then $\|s_q - \mathbf{v}_q\|^2 \leq 5\varepsilon^2$.

The rest of the proof of Theorem 3 is identical to the one of Theorem 2 in the previous section.

### 6 Proof of Theorem 1

Since $\hat{\Sigma} = \mathbf{X}^\top \mathbf{X} / n - I_p$, we have:

$$\hat{\Sigma} = \sum_{q=1}^r \left\{ \frac{\beta_q}{n} \|\mathbf{u}_q\|^2 \mathbf{v}_q (\mathbf{v}_q)^\top + \sqrt{\beta_q / n} \mathbf{v}_q (\mathbf{u}_q)^\top \mathbf{Z} + \mathbf{Z}^\top \mathbf{u}_q \right\}$$

$$+ \sum_{q \neq q'} \left\{ \frac{\sqrt{\beta_q \beta_{q'}}}{n} \mathbf{v}_q (\mathbf{u}_q)^\top \mathbf{v}_q (\mathbf{u}_{q'})^\top \right\} + \frac{\mathbf{Z}^\top \mathbf{Z}}{n} - I_p. \quad (14)$$

We let $\mathcal{D} = \{(i, i) : i \in [p] \setminus \cup_q Q_q\}$ be the diagonal entries not included in any support and $\mathcal{Q} = \cup Q_q$ denote the union of the supports. Further let $\mathcal{E} = \cup_q (Q_q \times Q_q)$, $\mathcal{F} = (Q^c \times Q^c) \setminus \mathcal{D}$, and $\mathcal{G} = [p] \times [p] \setminus (\mathcal{D} \cup \mathcal{E} \cup \mathcal{F})$. Since these are disjoint we have:

$$\eta(\hat{\Sigma}) = \mathcal{P}_E \left\{ \eta(\hat{\Sigma}) \right\} + \mathcal{P}_F \left\{ \eta \left( \frac{1}{n} \mathbf{Z}^\top \mathbf{Z} \right) \right\} + \mathcal{P}_G \left\{ \eta(\hat{\Sigma}) \right\} + \mathcal{P}_D \left\{ \eta(\hat{\Sigma}) \right\}. \quad (15)$$

The first term corresponds to the ‘signal’ component while the last three terms correspond to the ‘noise’ component.

Theorem 1 is a direct consequence of the next four propositions. The first of these proves that the signal component is preserved, while the others demonstrate that the noise components are small.
Proposition 6.1. Let $S$ denote the first term in Eq. (15):

$$S = P_E \{ \eta(\Sigma) \}. \quad (16)$$

Then with probability at least $1 - 3 \exp(-n^{2/3}/4)$:

$$\| S - \sum_{q=1}^r \beta_q v_q (v_q)^T \|_2 \leq \frac{\tau \sum_q k_q}{\sqrt{n}} + \kappa_n.$$

Here $\kappa_n = 16(\sqrt{\tau} + r\sqrt{\beta_1})n^{-1/6}$.

Proposition 6.2. Let $N$ denote the second term of Eq. (15):

$$N = P_F \{ \eta \left( \frac{1}{n} Z^T Z \right) \}.$$

Then there exists $\tau_1 = \tau_1(\alpha)$ such that for any $\tau \geq \tau_1$ and all $n$ large enough, we have

$$\| N \|_2 \leq C_1(\alpha) \sqrt{\frac{\log \tau}{\tau}}, \quad (17)$$

with probability at least $1 - 2 \exp(-c_1(\tau)p)$. The constants can be taken as $\tau_1 = 100 \max(1, \alpha^{2/3} \log \alpha)$, $c_1(\tau) = 1/4\tau$ and $C_1(\alpha) = 5000 \max(1, \alpha^{3/2})$.

Proposition 6.3. Let $R_1$ denote the matrix corresponding to the third term of Eq. (15):

$$R_1 = P_G \{ \eta(\Sigma) \}.$$

Then there exists $\tau_2 = \tau_2(\alpha, \beta_1, r)$ such that for $\tau \geq \tau_2$ and every $n$ large enough we have:

$$\| R_1 \|_2 \leq C_2(\alpha, r, \beta_1) \sqrt{\log \tau \frac{\log \tau}{\tau}}, \quad (18)$$

with probability at least $1 - \exp(-c_2(\tau)p)$. Here we may take $c_2(\tau) = c_1(\tau) = 1/4\tau$.

Proposition 6.4. Let $R_2$ denote the matrix corresponding to the third term of Eq. (15):

$$R_2 = P_D \{ \eta(\Sigma) \}.$$

Then with probability at least $1 - \alpha n^{-C/\alpha + 1}$ for every $n$ large enough:

$$\| R_2 \|_2 \leq \sqrt{\frac{C \log n}{n}}. \quad (19)$$

We defer the proofs of Propositions 6.1, 6.2, 6.3 and 6.4 to Sections 6.1, 6.2, 6.3 and 6.4 respectively.

Proof of Theorem 1. Using these results we now proceed to prove Theorem 1. We will assume that the events in these proposition hold, and control the probability of their complement via the union bound.

Denote by $k$ the sum of the support sizes, i.e. $\sum_q k_q$. From Propositions 6.1, 6.2, 6.3, 6.4 and the triangle inequality we have:

$$\left\| \eta(\Sigma) - \sum_q \beta_q v_q (v_q)^T \right\| \leq \frac{k\tau}{\sqrt{n}} + \max(C_1, C_2) \sqrt{\frac{\log \tau}{\tau}},$$

where $C_1$ and $C_2$ are the constants from Propositions 6.1 and 6.4, respectively.
for every $\tau \geq \max(\tau_1, \tau_2)$ with probability at least $1 - \alpha n^{-4}$. Setting $k \leq \sqrt{n \log \tau / \tau^3}$, the right hand side above is bounded by $\delta(\tau) = 2 \max(C_1, C_2) \sqrt{\log \tau / \tau}$. Further define $\beta \equiv \min_{q \neq q^\prime} \tau(\beta_q, |\beta_q - \beta_{q^\prime}|)$. Employing the Davis-Kahan sin $\theta$ theorem [DK70] we have:

$$\min(||\hat{v}_q - v_q||, ||\hat{v}_q + v_q||) \leq \sqrt{2} \sin \theta(\hat{v}_q, v_q) \leq \frac{\sqrt{2}\delta(\tau)}{\beta - \delta(\tau)}.$$

Choosing $\tau \geq (8 \max(C_1, C_2) / \beta \varepsilon)^4$ yields that $\delta(\tau) / (\beta - \delta(\tau)) \leq \varepsilon$. Letting $\tau$ be the largest of $\tau_1$, $\tau_2$ and $(8 \max(C_1, C_2) / \beta \varepsilon)^4$ gives the desired result.

### 6.1 Proof of Proposition 6.1

The proof proceeds in two steps. In the first lemma we bound $\|E\{S\} - \sum q \beta_q v_q(v_q)^T\|$ and in the second we control $\|S - E\{S\}\|$.

#### Lemma 6.5. Consider $S$ as defined in Proposition 6.1. Then

$$\left\|E\{S\} - \sum q \beta_q v_q(v_q)^T\right\| \leq \frac{\tau \sum_q k_q}{\sqrt{n}}.$$

**Proof.** Notice that $E\{S\}$ is supported on a set of indices $\cup_q Q_q \times \cup_q Q_q$, which has size at most $(\sum_q k_q)^2$. Hence

$$\left\|E\{S\} - \sum q \beta_q v_q(v_q)^T\right\| \leq (\sum_q k_q) \left\|E\{S\} - \sum q \beta_q v_q(v_q)^T\right\|_{\infty},$$

where the last term denotes the entrywise $\ell_\infty$ norm of the matrix. Since $S$ and $\sum q \beta_q v_q(v_q)^T$ have common support and since $|\eta(z; \tau / \sqrt{n}) - z| \leq \tau / \sqrt{n}$ we obtain that:

$$\left\|E\{S\} - \sum q \beta_q v_q(v_q)^T\right\|_{\infty} \leq \left\|E\{\mathcal{P}_E(\eta(\hat{\Sigma}))\} - \sum q \beta_q v_q v_q^T\right\|_{\infty} \leq \frac{\tau}{\sqrt{n}}.$$

The thesis then follows directly.

#### Lemma 6.6. Let $S$ be as defined in Proposition 6.1. Then:

$$\|S - E\{S\}\| \leq \kappa_n,$$

with probability at least $1 - \exp(-n^{2/3}/4)$ where we define $\kappa_n \equiv 16(\sqrt{r\alpha} + r \sqrt{3}) n^{-1/6}$.

Proposition 6.1 follows directly from these two lemmas since we have by triangle inequality:

$$\left\|S - \sum q \beta_q v_q(v_q)^T\right\| \leq \|S - E\{S\}\| + \left\|E\{S\} - \sum q \beta_q v_q(v_q)^T\right\|.$$

This completes the proof of Proposition 6.1 conditional on Lemma 6.6. In the next subsection we prove Lemma 6.6.
6.1.1 Proof of Lemma 6.6

Let \( y \in \mathbb{R}^p \) denote a vector supported on \( \cup_{q} Q_q \). Recall that \( Q = \cup_{q} Q_q \). Fix an \( \ell \in Q \). The gradient of the Rayleigh quotient \( \nabla_{\tilde{z}_\ell}(y, Sy) \) reads:

\[
\nabla_{\tilde{z}_\ell}(y, Sy) = \frac{1}{n} \sum_{i:(i, \ell) \in \cup_{q} Q_q \times Q_q} 2 \partial \eta \left( \tilde{z}_i + \sum_{q} \sqrt{\beta_q} v_q^T u_q \right) y_i y_\ell.
\]

Define the vector \( \sigma^\ell(y) \in \mathbb{R}^p \) as follows:

\[
\sigma^\ell_i(y) = \begin{cases} \partial \eta \left( \tilde{z}_i + \sum_{q} \sqrt{\beta_q} v_q^T u_q \right) y_i, & \text{if } (i, \ell) \in \cup_{q}(Q_q \times Q_q) \\ 0 & \text{otherwise.} \end{cases}
\]

where the left hand side denotes the \( i \)th entry of \( \sigma^\ell(y) \). Recall that \( Z_E \) is the matrix obtained from \( Z \) by setting to zero all columns with indices outside \( E \subseteq [p] \). Using this, we can now rewrite the gradient in the following form:

\[
\nabla_{\tilde{z}_\ell}(y, Sy) = \frac{2y_\ell}{n} (Z_Q + \sum_{q} \sqrt{\beta_q} v_q^T u_q) \sigma^\ell(y).
\]

Since \( \partial \eta(\cdot) \in \{0,1\} \), we see that \( \|\sigma^\ell(y)\| \leq \|y\| = 1 \). Consequently, we have that:

\[
\|\nabla_{\tilde{z}_\ell}(y, Sy)\| \leq \frac{2|y_\ell|}{n} \left( \|Z_Q + \sum_{q} \sqrt{\beta_q} v_q^T u_q\| \|\sigma^\ell(y)\| \right) \\
\quad \leq \frac{2|y_\ell|}{n} \left( \|Z_Q\| + \sum_{q} \sqrt{\beta_q} \|u_q\| \right) \\
\quad = \frac{2|y_\ell|}{n} \left( \|Z_Q\| + \sum_{q} \sqrt{\beta_q} \|u_q\| \right),
\]

Squaring and summing over \( \ell \):

\[
\|\nabla Z_Q(y, Sy)\|^2 \leq \frac{4}{n^2} (\|Z_Q\| + \sum_{q} \beta_q \|u_q\|)^2.
\]

The gradient above is with respect to all the variables \( \tilde{z}_\ell, \ell \in Q_q \) and the norm is the standard vector \( \ell_2 \) norm. Let \( G : \{(Z, (u_q)_{q \leq r}) : \|Z_Q\| \leq (2 + \sqrt{\alpha}) \sqrt{\beta_1}, \|u_q\| \leq 4 \sqrt{n}\} \). Clearly \( G \) is a closed, convex set. Further, using Lemma 4.6 we can bound the probability of \( G^c : \|Z_Q\| \leq (\sqrt{\alpha} + \sqrt{\beta_1}) \leq (2 + \sqrt{\alpha}) \sqrt{\beta_1} \) with probability at least \( 1 - \exp(-n/2) \). Also, with probability at least \( 1 - r \exp(-n/2) \), for every \( q \|u_q\| \leq 4 \sqrt{n} \). Thus, on the set \( G \) we have:

\[
\|\nabla \langle y, Sy \rangle\|^2 \|(Z, u_1 \cdots u_r) \in G\| \leq \frac{64}{n} (2 + \sqrt{\alpha} + \sqrt{\beta})^2 \\
P(G^c) \leq 2 \exp \left( -\frac{n}{4} \right).
\]

Define \( L \) and \( \kappa_n \) as follows:

\[
L \equiv \frac{8(2 + \sqrt{\alpha} + \sqrt{\beta})}{\sqrt{n}} \\
\kappa_n \equiv 16(2 + \sqrt{\alpha} + \sqrt{\beta})n^{-1/6} = 2Ln^{1/3}.
\]

Also let \( F_L(Z_Q) \) denote the \( G, L \)-Lipschitz extension of \( \langle y, Sy \rangle \). We prove the following remark in Appendix A.
Remark 6.7. For every $n$ large enough, $|\mathbb{E}\{\langle y, S y \rangle - F_L(Z_Q)\}| \leq n^{-1}$.

Now employing Lemma 6.3:

$$\mathbb{P} \left\{ |\langle y, S y \rangle - \mathbb{E}\langle y, S y \rangle| \geq \kappa_n/2 \right\} \leq 2 \exp \left( -\frac{n^{2/3}}{2} \right) + 2r \exp \left( -\frac{n}{4} \right) \leq 3 \exp \left( -\frac{n^{2/3}}{2} \right),$$

for every $n$ large enough. Then using $y$ as a vector in the $1/4$-net $T_{|Q|}^{1/4}$ embedded in $\mathbb{R}^p$ via the union of supports $Q$, we use Lemma 6.3 to obtain that:

$$\|S - \mathbb{E}\{S\}\| \leq \kappa_n,$$

with probability at least $1 - 3\cdot|Q| \exp(-n^{2/3}/2) \geq 1 - \exp(-n^{2/3}/4)$ since $|Q| \leq \sum \kappa_i = O(\sqrt{n}) \leq n^{2/3}/2$ for large enough $n$.

6.2 Proof of Proposition 6.2

It suffices to bound the norm of $\tilde{N}$ defined as

$$\tilde{N} = \mathcal{P}_n \left\{ \kappa \left( \frac{1}{n} Z^T Z \right) \right\}.$$

We use a variant of the $\varepsilon$-net argument. For a set of indices $E \subseteq [p]$, recall that $y_E \in \mathbb{R}^p$ denotes the vector coinciding with $y$ on $E$, and zero outside $E$. By decomposing the Rayleigh quotient:

$$\mathbb{P}\left\{ \|\tilde{N}\| \geq \Delta \right\} \leq \mathbb{P}\left\{ \sup_{y \in T_p^p} \langle y, \tilde{N} y \rangle \geq \Delta(1 - 2\varepsilon) \right\} \leq \mathbb{P}\left\{ \sup_{y \in T_p^p} \langle y_E, \tilde{N} y_E \rangle + \langle y_{E^c}, \tilde{N} y_{E^c} \rangle + 2\langle y_E, \tilde{N} y_{E^c} \rangle \geq \Delta(1 - 2\varepsilon) \right\}.$$

We let $E = \{ i \in [p] : |y_i| > \sqrt{A/p}\}$ for the constant $A = A(\tau) = \tau \log \tau$. Since $\|y\| = 1$, it follows that $|E| \leq p/A$. The following lemma allows to bound the term $\langle y_E, \tilde{N} y_E \rangle$ uniformly over all subsets $E$ smaller than $p/A$.

Lemma 6.8. Fix $A \geq 180 \max(\sqrt{\alpha}, 1)$. Then, for every $p$ large enough, the following holds with probability at least $1 - \exp(-p \log A/4A)$:

$$\sup_{E \subseteq [p], |E| \leq p/A} \|\bar{F}_{E,E}(\tilde{N})\|_2 \leq 32 \sqrt{\alpha \log A / A}.$$

The proof of this lemma is provided in subsection 6.2.1. Denoting by $\mathcal{E}$ the favorable event of Lemma 6.8, we obtain:

$$\mathbb{P}\left\{ \|\tilde{N}\| \geq \Delta \right\} \leq \mathbb{P}\{\mathcal{E}^c\} + \mathbb{P}\left\{ \sup_{y \in T_p^p} \left( \langle y_E, \tilde{N} y_E \rangle + \langle y_{E^c}, \tilde{N} y_{E^c} \rangle + 2\langle y_E, \tilde{N} y_{E^c} \rangle \right) \geq \Delta(1 - 2\varepsilon), \mathcal{E} \right\} \leq \mathbb{P}\{\mathcal{E}^c\} + \mathbb{P}\left\{ \sup_{y \in T_p^p} \left( \langle y_{E^c}, \tilde{N} y_{E^c} \rangle + 2\langle y_E, \tilde{N} y_{E^c} \rangle \right) \geq \Delta \right\},$$

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where $$\tilde{\Delta} = \Delta(1 - 2\varepsilon) - 16\sqrt{2\alpha \log A/A}$$.

Further, using the union bound and Lemma 4.2:

$$
P \left\{ \left\| \tilde{N} \right\|_2 \geq \Delta \right\} \leq P \left\{ E^c \right\} + \left| T_p^c \right| \sup_{y \in T_p^c} P \left\{ \langle y_{E^c}, \tilde{N} y_{E^c} \rangle \geq \frac{\tilde{\Delta}}{3} \right\}
$$

$$
+ \left| T_p^c \right| \sup_{y \in T_p^c} P \left\{ \langle y_{E^c}, \tilde{N} y_{E^c} \rangle \geq \frac{\tilde{\Delta}}{3} \right\}.
$$

(20)

$$
P \left\{ E^c \right\} \text{ is bounded in Lemma 6.8. We now proceed to bound the latter two terms. For the second term, the gradient } \nabla \tilde{z}^{\ell} \langle y_{E^c}, \tilde{N} y_{E^c} \rangle \text{ reads, for any fixed } \ell \in E^c:
$$

$$
\nabla \tilde{z}^{\ell} \langle y_{E^c}, \tilde{N} y_{E^c} \rangle = \frac{2y_{E^c}^{\ell}}{n} \sum_{i \in E^c \setminus \ell} \partial \eta \left( \frac{\langle \tilde{z}_i, \tilde{z}^{\ell} \rangle}{n}; \frac{\tau}{\sqrt{n}} \right) y_i \tilde{z}_i.
$$

Let $$\sigma^{\ell}(y) \in \mathbb{R}^p$$ be a vector defined by:

$$
\sigma^{\ell}(y) = \begin{cases} 
\partial \eta \left( \frac{\langle \tilde{z}_i, \tilde{z}^{\ell} \rangle}{n}; \frac{\tau}{\sqrt{n}} \right) y_i & \text{if } i \in E^c \setminus \ell, \\
0 & \text{otherwise.}
\end{cases}
$$

With this definition we can represent the norm of the gradient as:

$$
\left\| \nabla \tilde{z}^{\ell} \langle y_{E^c}, \tilde{N} y_{E^c} \rangle \right\| = 2 \frac{\left| y_{E^c}^{\ell} \right|}{n} \left\| Z_{E^c} \sigma^{\ell}(y) \right\|
$$

$$
\leq 2 \frac{\left| y_{E^c}^{\ell} \right|}{n} \left\| Z_{E^c} \right\| \left\| \sigma^{\ell}(y) \right\|
$$

$$
\leq 2 \frac{\left| y_{E^c}^{\ell} \right|}{n} \left\| Z \right\| \left\| \sigma^{\ell}(y) \right\|.
$$

For $$\sigma^{\ell}(y)$$:

$$
\left\| \sigma^{\ell}(y) \right\|^2 = \sum_{i \in E^c \setminus \ell} \partial \eta \left( \frac{\langle \tilde{z}_i, \tilde{z}^{\ell} \rangle}{n}; \frac{\tau}{\sqrt{n}} \right)^2 y_i^2
$$

$$
\leq \sum_{i \in E^c \setminus \ell} \frac{\langle \tilde{z}_i, \tilde{z}^{\ell} \rangle^2}{n^2 \tau^2} y_i^2
$$

$$
\leq \frac{A}{\tau^2 n^2 p \gamma} \langle \tilde{z}^{\ell}, Z_{E^c \setminus \ell}^T Z_{E^c \setminus \ell} \rangle
$$

$$
\leq \frac{A}{n^2 p \gamma} \left\| \tilde{z}^{\ell} \right\|^2 \left\| Z \right\|^2.
$$

Here the first line follows from $$\partial \eta(x; y) = \mathbb{I}(|x| \geq y) \leq |x|/y$$. The second line follows from the choice of $$E$$ whereby $$|y_i| \leq \sqrt{A/p}$$ and the last line from Cauchy-Schwarz.

For any $$\ell \in E$$, $$\nabla \tilde{z}^{\ell} \langle y_{E^c}, \tilde{N} y_{E^c} \rangle = 0$$. Now, fix $$\Gamma = 5$$, $$\gamma = \Gamma \max(\alpha^{-1}, 1) \geq \Gamma$$ and let $$G = \{ Z : \left\| Z \right\| \leq 2\sqrt{2} \gamma p, \forall \ell, \left\| \tilde{z}^{\ell} \right\| \leq \sqrt{2} \gamma p, \}$$.

Clearly, $$G$$ is a closed, convex set. Furthermore, on the set $$G$$, we obtain from the
gradient and \(\sigma'(y)\) estimates above that:

\[
\|\nabla_z (y_{E^c}, \tilde{N}_{y_{E^c}})\|^2 = \sum_{\ell \in E^c} \|\nabla_{\tilde{z}_\ell} (y_{E^c}, \tilde{N}_{y_{E^c}})\|^2 \\
\leq \sum_{\ell \in E^c} \frac{4g_2^2}{n^2} \|z\|^2 \|\sigma'(y)\|^2 \\
\leq \frac{4}{n^2} \max_{\ell \in E^c} \|\sigma'(y)\|^2 \\
\leq \frac{4A \|z\|^4 \max_{\ell \in E^c} \|\tilde{z}_\ell\|^2}{n^3 p r^2} \\
\leq \frac{128 A \gamma^3 \alpha^3}{p r^2}.
\]  

(21)

Here we treat \(\nabla_z (y_{E^c}, \tilde{N}_{y_{E^c}})\) as a vector in \(\mathbb{R}^{np}\), hence the norm above is the standard \(\ell_2\) norm on vectors. We also write the gradient as \(\nabla_{\tilde{z}_\ell} (y_{E^c}, \tilde{N}_{y_{E^c}})\) to avoid ambiguity in specifying the norm. We now bound \(P\{G^c\}\) as follows. Lemma 4.6 implies that with probability at least 1 \(-\exp(-\Gamma p/2)\):

\[
\|z\| \leq \sqrt{1 + \sqrt{\Gamma} + \alpha^{-1/2}} \sqrt{p} \\
\leq 2 \sqrt{\Gamma p}.
\]  

(23)

since \(\gamma \geq (1+\alpha^{-1/2})^2\). Further, the standard Chernoff bound implies that, for a fixed \(\ell\), \(\|\tilde{z}_\ell\|^2 \leq 2 \gamma n \alpha n = 2 \gamma p\) with probability at least 1 \(-\exp(-\gamma p/2)\). By the union bound, we then obtain that \(P\{G^c\} \leq p \exp(-\gamma p/2) + \exp(-\Gamma p/2) \leq (p + 1) \exp(-\Gamma p/2)\). Define \(K = \sqrt{128 A \gamma^3 \alpha^3 / p r^2}\). Let \(F_K(Z)\) denote the \(G, K\)-Lipschitz extension of \(F(Z) = (y_{E^c}, \tilde{N}_{y_{E^c}})\). We have the following remark for \(F_K(Z)\) which is proved in Appendix A.

Remark 6.9. We have \(E\{y_{E^c}, \tilde{N}_{y_{E^c}}\} = 0\). Further, for every \(p\) large enough, \(|E\{F_K(Z)\}| \leq p^{-1}\).

We can now use Lemma 4.4 for \(F(Z)\). Thus for any \(\Delta_2 \geq 2/p:\)

\[
P \left\{ y_{E^c}, \tilde{N}_{y_{E^c}} \geq \Delta_2 \right\} \leq \exp \left( -\frac{\Delta_2^2}{4 K^2} \right) + 2 p \exp \left( -\frac{\Gamma p}{2} \right).
\]  

(24)

Using \(\Delta_2 = \sqrt{2 \Gamma p K} = 16 \sqrt{A \Gamma \gamma^3 \alpha^3 / \tau}\) we obtain:

\[
P \left\{ y_{E^c}, \tilde{N}_{y_{E^c}} \geq 16 \sqrt{A \Gamma \gamma^3 \alpha^3 / \tau} \right\} \leq (2 p + 2) \exp(-\Gamma p/2).
\]  

(25)

Now we can use the essentially same strategy on the term \(y_{E^c}, \tilde{N}_{y_{E^c}}\). For \(\ell \in E\) we have as before:

\[
\nabla_{\tilde{z}_\ell} (y_{E^c}, \tilde{N}_{y_{E^c}}) = \frac{y_i}{n} \sum_{i \in E^c} \partial_{\eta} \left( \frac{\langle \tilde{z}_\ell, \tilde{z}_i \rangle}{n} \frac{\tau}{\sqrt{n}} \right) y_i \tilde{z}_i, \\
\|\nabla_{\tilde{z}_\ell} (y_{E^c}, \tilde{N}_{y_{E^c}})\|^2 \leq \frac{g_2^2 A \|z\|^4 \max_{i \in E^c} \|\tilde{z}_i\|^2}{\tau^2 p n^3}.
\]

Hence:

\[
\sum_{\ell \in E} \|\nabla_{\tilde{z}_\ell} (y_{E^c}, \tilde{N}_{y_{E^c}})\|^2 \leq \frac{A \|z\|^4 \max_{i \in E^c} \|\tilde{z}_i\|^2}{\tau^2 p n^3}.
\]  

(26)
Analogously, for $\ell \in E^c$:

$$\nabla \tilde{z}_\ell \langle y_E, \tilde{N} y_{E^c} \rangle = \frac{y_n}{n} \sum_{i \in E} \partial \eta \left( \frac{\langle \tilde{z}_i, \tilde{z}_\ell \rangle}{n} \right) y_i \tilde{z}_i$$

$$= \frac{y_n}{n} Z_E \sigma'_E(y),$$

where we define the vector $\sigma'_E(y) \in \mathbb{R}^E$ as:

$$\forall i \in E, \quad \sigma'_E(y)_i = y_i \partial \eta \left( \frac{\langle \tilde{z}_i, \tilde{z}_\ell \rangle}{n} \right).$$

By Cauchy-Schwarz:

$$\|\nabla \tilde{z}_\ell \langle y_E, \tilde{N} y_{E^c} \rangle\|^2 \leq \frac{y_n^2}{n^2} \|Z_E\|^2 \|\sigma'_E(y)\|^2.$$

Summing over $\ell \in E^c$:

$$\sum_{\ell \in E^c} \|\nabla \tilde{z}_\ell \langle y_E, \tilde{N} y_{E^c} \rangle\|^2 \leq \frac{A \|Z\|^2}{n^2} \sum_{\ell \in E^c} y_n^2 \|\sigma'_E(y)\|^2$$

$$= \frac{A \|Z\|^2}{pn^2} \sum_{\ell \in E^c} y_n^2 \partial \eta \left( \frac{\langle \tilde{z}_i, \tilde{z}_\ell \rangle}{n} \right)^2$$

$$\leq \frac{A \|Z\|^2}{pn^2} \sum_{i \in E} \sum_{\ell \in E^c} y_n^2 \left( \frac{\langle \tilde{z}_i, \tilde{z}_\ell \rangle}{n} \right)^2$$

$$= \frac{A \|Z\|^2}{\tau^2 n^3} \sum_{i \in E} y_n^2 (Z_{E^c}^T Z_{E^c} \tilde{z}_i)$$

$$\leq \frac{A \|Z\|^2}{\tau^2 n^3} \max_{i \in [p]} \|\tilde{z}_i\|^2$$

$$\leq \frac{A \|Z\|^4}{\tau^2 n^3} \max_{i \in [p]} \|\tilde{z}_i\|^2.$$

This bound along with Eq. (26) gives:

$$\left\| \nabla \langle \tilde{z}_\ell \rangle_{i \in [p]} \langle y_E, \tilde{N} y_{E^c} \rangle \right\|^2 \leq \frac{2A \|Z\|^4 \max_{i \in [p]} \|\tilde{z}_i\|^2}{\tau^2 np^3}.$$ 

On the set $G$ defined before, we have that:

$$\left\| \nabla \langle \tilde{z}_\ell \rangle_{i \in [p]} \langle y_E, \tilde{N} y_{E^c} \rangle \right\|^2 \leq \frac{64A \gamma^3 \alpha^3}{\tau^2 np^3}.$$

Proceeding as before, applying Lemma 4.4 we have:

$$\mathbb{P} \left\{ \langle y_E, \tilde{N} y_{E^c} \rangle \geq 16 \frac{\sqrt{A \Gamma \gamma^2 \alpha^3}}{\tau} \right\} \leq 2p \exp \left( -\frac{\Gamma p}{2} \right). \quad (27)$$
We can now use Eqs. \[25, 27\] in Eq. \[20\]:

\[
P\left\{ \left\| \tilde{N} \right\|_2 \geq (1 - 2\varepsilon)^{-1} \left( 32 \sqrt{\frac{\alpha \log A}{A}} + 48 \sqrt{\frac{A \Gamma \gamma^2 \alpha^3}{\tau^2}} \right) \right\} \leq \exp \left( -\frac{p \log A}{4A} \right) \\
+ |T_p^\tau| (4p + 4) \exp \left( -\frac{\Gamma p}{2} \right)
\]

We first simplify the probability bound. Since \( A = \tau \log \tau, \log A/A \geq 1/\tau \) when \( \tau \geq \exp(1) \). Further, choosing \( \varepsilon = 1/4 \), with Lemma \[4.2\] we get that \( |T_p^\tau| \leq (1 + 2/\varepsilon)^p = 9^p \). Since \( \log 9 = 2.19 \cdots = \Gamma/2 = 5/2 \), we have \( (4p + 4)|T_p^\tau| \exp(\Gamma p/2) \leq \exp(-p/20) \) for large enough \( p \). Thus the right hand side is bounded above by \( 2 \exp(-p/4 \max(\tau, 5)) \) for every \( p \) large enough.

Now we simplify the operator norm bound. As \( A = \tau \log \tau, \log A/A \leq \log \tau/\tau \) since \( \log z/z \) is decreasing. Further \( \alpha \leq \max(1, \alpha^3) \) and \( \Gamma = 5 \) imply:

\[
(1 - 2\varepsilon)^{-1} \left( 32 \sqrt{\frac{\alpha \log A}{A}} + 64 \sqrt{\frac{A \Gamma \gamma^2 \alpha^3}{\tau^2}} \right) \leq 2(32 + 64\Gamma^2) \sqrt{\frac{\max(1, \alpha^3) \log \tau}{\tau}} \\
\leq 5000 \sqrt{\frac{\max(1, \alpha^3) \log \tau}{\tau}}.
\]

Our conditions on \( \tau \) and \( A \) were: (i) \( \tau \geq \max(4\sqrt{\Gamma \gamma \alpha}, \exp(1)) = 20 \max(1, \sqrt{\alpha}) \) and (ii) \( A \geq 180 \max(\sqrt{\alpha}, 1) \). Using \( \tau \geq 100 \max(1, \alpha^2 \log \alpha) \) satisfies both conditions.

### 6.2.1 Proof of Lemma 6.8

This proof also follows an \( \varepsilon \)-net argument. Let \( a \) denote the size of the set \( E \). For notational simplicity, we will permute the rows and columns of \( \tilde{N} \) to ensure \( E = [a] \) (i.e. \( E \) is the first \( a \) entries of \( |p| \)). For a fixed \( y \in T_p^\tau \), we bound the Rayleigh quotient \( \langle y, \bar{P}_{E,E}(\tilde{N})y \rangle \) with high probability. Note that \( \langle y, \bar{P}_{E,E}(\tilde{N})y \rangle \) is a function of \( \tilde{z}_\ell, \ell \in E \). The gradient of this function with respect to \( \tilde{z}_\ell \) is:

\[
\nabla \tilde{z}_\ell \langle y, \bar{P}_{E,E}(\tilde{N})y \rangle = \frac{2y_\ell}{n} \sum_{i \in E \setminus \ell} \partial \eta \left( \frac{\langle \tilde{z}_i, \tilde{z}_\ell \rangle}{n}; \frac{\tau}{\sqrt{n}} \right) y_i \tilde{z}_i,
\]

where \( \sigma^\ell(y) \in \mathbb{R}^p \) is the vector defined as:

\[
\sigma^\ell_i(y) = \begin{cases} y_i \partial \eta \left( \frac{\langle \tilde{z}_i, \tilde{z}_\ell \rangle}{n}; \frac{\tau}{\sqrt{n}} \right) & \text{when } i \in E \setminus \ell \\ 0 & \text{otherwise.} \end{cases}
\]

The (square of the) total gradient is thus given by:

\[
\left\| \nabla \langle y, \bar{P}_{E,E}(\tilde{N})y \rangle \right\|^2 = \frac{4}{n^2} \sum_{\ell \in E} \left\| Z_E \sigma^\ell(y) \right\|_2^2 y_\ell^2 \\
\leq \frac{4}{n^2} \sum_{\ell \in E} \left\| Z_E \right\|_2^2 \left\| \sigma^\ell(y) \right\|_2^2 y_\ell^2 \\
\leq \left( \frac{2 \left\| Z_E \right\|_2}{n} \right)^2 \sum_{\ell \in E \setminus \ell} \left\| \sigma^\ell(y) \right\|_2^2 y_\ell^2.
\]

Since \( \left| \partial \eta (:; \tau/\sqrt{n}) \right| \leq 1 \) we have that \( \left\| \sigma^\ell(y) \right\|^2 \leq \left\| y \right\|^2 \leq 1 \). Consequently we obtain the bound:

\[
\left\| \nabla \langle y, \bar{P}_{E,E}(\tilde{N})y \rangle \right\|^2 \leq \left( \frac{2 \left\| Z_E \right\|_2}{n} \right)^2.
\]
From Lemma 4.6 we have that:
\[ \|Z_E\|_2 \leq \sqrt{n} + \sqrt{a} + t\sqrt{p}, \]
with probability at least \(1 - \exp(-pt^2/2)\). Let \(G = \{Z_E : \|Z_E\|_2 \leq \sqrt{n} + \sqrt{a} + t\sqrt{p}\}\). Then:
\[ \left\| \nabla \langle y, \tilde{P}_{E,E}(\tilde{N})y \rangle \right\|^2 \leq \frac{4\alpha}{p} \left( 1 + \frac{a\alpha}{p} + t\sqrt{\alpha} \right)^2 = L^2 \]
and \(\mathbb{P}(G^c) \leq e^{-pt^2/2}.\) (28)

We let \(F_L(Z_E)\) denote the \(G, L\)-Lipschitz extension of \(\nabla \langle y, \tilde{P}_{E,E}(\tilde{N})y \rangle\). The following remark is proved in Appendix A.

**Remark 6.10.** Firstly, \(\mathbb{E}\{\langle y, \tilde{P}_{E,E}(\tilde{N})y \rangle\} = 0\). Secondly, for every \(p\) large enough: \(|\mathbb{E}(F_L(Z))| \leq p^{-1}\).

Let \(\tilde{\Delta} = \Delta(1 - 2\varepsilon)\) and \(\nu = 1 + \sqrt{a\alpha}/p\). We choose \(t = \left( \sqrt{\nu^2 + \tilde{\Delta}/2\sqrt{\alpha}} - \nu \right)/2\) and apply Lemma 4.4 and Remark 6.10. This choice of \(t\) ensures that the two unfavorable events of Lemma 4.4 are both bounded above by \(\exp(-pt^2/2)\). Thus,
\[ \mathbb{P}\{\langle y, \tilde{P}_{E,E}(\tilde{N})y \rangle \geq \tilde{\Delta} \} \leq 2e^{-pt^2/2}, \]
for \(p\) large enough. Further, our choice of \(t\) implies:
\[ t^2 = \frac{1}{4} \left( \sqrt{\nu^2 + \frac{\tilde{\Delta}}{2\sqrt{\alpha}}} - \nu \right)^2 \]
\[ = \frac{\nu^2}{2} \left( 1 + \frac{\tilde{\Delta}}{4\nu^2\sqrt{\alpha}} - \sqrt{1 + \frac{\tilde{\Delta}}{2\nu^2\sqrt{\alpha}}} \right) \]
\[ \geq \frac{\tilde{\Delta}^2}{128\nu^2\alpha}, \]
where the last inequality follows from the fact that \(g(x) = 1 + x/2 - \sqrt{1 + x} \geq x^2/16\) when \(x \leq 2\). This requires \(\tilde{\Delta} \leq 4\nu^2\sqrt{\alpha}\). Now, Lemma 4.2 and 4.3 imply:
\[ \mathbb{P}\left\{ \|\tilde{P}_{E,E}(\tilde{N})\|_2 \geq \Delta \right\} \leq 2 \left( 1 + \frac{2}{\varepsilon} \right)^a \exp\left( -\frac{p\tilde{\Delta}^2}{256\nu^2\alpha} \right) \]
\[ \leq \exp\left( -\frac{p\tilde{\Delta}^2}{256\alpha\varepsilon^2} + a \log\left( 2 + \frac{4}{\varepsilon} \right) \right). \]

There are \((p/a)^a\) possible choices for the set \(E\). Using the union bound we have that:
\[ \mathbb{P}\left\{ \sup_{E \subseteq [p], |E| = a} \|\tilde{P}_{E,E}(\tilde{N})\|_2 \geq \Delta \right\} \leq \exp\left( -\frac{p\tilde{\Delta}^2}{256\alpha\varepsilon^2} + a \log\left( 2 + \frac{4}{\varepsilon} \right) + a \log\left( \frac{pe}{a} \right) \right). \]

Since \(a \leq p/A, \nu = 1 + \sqrt{aa}/p \leq 2\) when \(A \geq \max(\sqrt{\alpha}, 1)\). Using \(\varepsilon = 1/4\) we obtain that
\[ \mathbb{P}\left\{ \sup_{E \subseteq [p], |E| = a} \|\tilde{P}_{E,E}(\tilde{N})\|_2 \geq \Delta \right\} \leq \exp\left( -p \left( \frac{\Delta^2}{1024\alpha} - \log\left( \frac{18eA}{A} \right) \right) \right). \]
We required $\tilde{\Delta} \leq 4\nu^2 \sqrt{\alpha}$, and $\Delta = \Delta/2$. Hence we require $\Delta \leq 8\sqrt{\alpha} \leq 8\nu^2 \sqrt{\alpha}$. Choosing $\Delta = 32\sqrt{\alpha \log A/A}$, where $A \geq 180 \max(\sqrt{\alpha}, 1)$ satisfies this condition. Further, with this choice of $A$, $\log(18\varepsilon A) \leq 1.75 \log A$. Consequently:

$$P \left\{ \sup_{E \leq \|p\|, \|E\| = a} \left\| \overline{P}_{E,E}(N) \right\|_2 \geq 32\sqrt{\frac{\alpha \log A}{A}} \right\} \leq \exp \left( -\frac{p \log A}{4A} \right).$$

### 6.3 Proof of Proposition 6.3

We explicitly write the $(i, j)$th entry of $R_1$ (when $(i, j) \in G$) as:

$$(R_1)_{ij} = q \left( \left( \sum_{q} \sqrt{\beta_q} u_q(v_q)_i + \tilde{z}_i, \sum_{q} \sqrt{\beta_q} u_q(v_q)_j + \tilde{z}_j \right) ; \frac{\tau}{\sqrt{n}} \right)$$

Since $G$ is a symmetric set of entries excluding the diagonal, it suffices to consider the case $i < j$ above. Denote by $R$ the upper triangle of $R_1$. Let $g$ denote the number of nonzero rows in $R$. By the definition of $g$, $g \leq \sum_{q} |Q_q| = k$. We wish to bound (with slight abuse of notation) the quantity: $\sup_{N \leq S_{S^2-1} \sup \forall \in S_{S^2-1}} \{ x, Ry \}$

The proof follows an epsilon net argument entirely analogous to the proof of Proposition 6.2. The only difference is the further dependence on the Gaussian random vectors $u_q$. Hence we only give a proof sketch, highlighting the difference with the proof of Proposition 6.2.

Fix a vector $y \in T_p^{1/4}$ and $x \in T_p^{1/4}$, and let $E$ be the subset of indices $E = \{ i \in [p] : |y_i| \geq \sqrt{\Delta/p} \}$ for some constant $A$ to be fixed later in the proof. As before, we split the Rayleigh quotient $\langle x, Ry \rangle = \langle x, R\rangle + \langle x, Ry_e \rangle \leq \| P_{[p] \times E}(R) \| + \langle x, Ry_e \rangle$. By the condition on $E$, we have that $|E| \leq p/A$. Consequently:

$$P \{ \| R \| \geq \Delta \} \leq \sup_{x \in T_p^{1/4}, y \in T_p^{1/4}} P \{ \langle x, Ry \rangle \geq \frac{\Delta}{4} \}$$

We first concentrate on the second term, whose gradient with respect to a fixed $\tilde{z}_i$ is given by:

$$\nabla \tilde{z}_i(x, Ry_e) = \frac{x_i}{n} \sum_{j > i, (i,j) \in G} (y_e)_j \partial \eta \left( \left( \sum_{q} \sqrt{\beta_q} u_q(v_q)_i + \tilde{z}_i, \sum_{q} \sqrt{\beta_q} u_q(v_q)_j + \tilde{z}_j ; \frac{\tau}{\sqrt{n}} \right) \left( \sum_{q} \sqrt{\beta_q} u_q(v_q)_j + \tilde{z}_j \right) \right)$$

Defining $\sigma^i(y)$ and $\sigma^i(x)$ similar to Proposition 6.2, we have by Cauchy Schwarz:

$$\| \nabla \tilde{z}_i(x, Ry_e) \|^2 \leq 2 \frac{\| \sum_q \sqrt{\beta_q} u_q v_q^T + Z \|^2}{n^2} \left( x_i^2 \| \sigma^i(y) \|^2 + (y_e)_i^2 \| \sigma^i(x) \|^2 \right).$$

Summing over $i$:

$$\sum_{i} \| \nabla \tilde{z}_i(x, Ry_e) \|^2 \leq 2 \frac{\| X \|^2}{n^2} \sum_{i} \left( x_i^2 \| \sigma^i(y) \|^2 + (y_e)_i^2 \| \sigma^i(x) \|^2 \right) \leq 2 \frac{\| X \|^2}{n^2} \sup_i \| \sigma^i(y) \|^2 + 2 \frac{\| X \|^2}{n} \sum_{i} (y_e)_i^2 \| \sigma^i(x) \|^2$$

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Let \( G = \{ \mathbf{u} \leq_{r, \tau} \mathbf{z} : \forall q \| \mathbf{u}_q \| \leq C'r / \sqrt{n}, \| \mathbf{z} \| \leq C'(1 / \sqrt{n}, \forall i \| \tilde{z}_i \| \leq C' / \sqrt{n}) \). It is clear that \( G \) is convex, and that \( \mathbb{P}\{G^c\} \leq \rho \exp(-C''r) \) for some \( C'' \) dependent on \( C' \). It is not hard to show that:

\[
\sum_i \| \nabla_{\mathbf{z}_i} \langle \mathbf{x}, \mathbf{Ry}_{E^c} \rangle \|^2 \leq \frac{AC(\alpha, (\beta)_{q \leq r}, r)}{p \tau^2},
\]

for some constant \( C \), when \( C' \) is large enough.

Similarly, taking derivatives with respect to \( \mathbf{u}_q \) for a fixed \( q \), we have:

\[
\nabla_{\mathbf{u}_q} \langle \mathbf{x}, \mathbf{Ry}_{E^c} \rangle = \frac{1}{n} \sum_{(i,j) \in G} \partial q \left( \sum_q \sqrt{\beta_q} \mathbf{u}_q(v_q)_i + \tilde{z}_i, \sum_q \sqrt{\beta_q} \mathbf{u}_q(v_q)_j + \tilde{z}_j ; \tau \sqrt{n} \right)
\cdot \left( x_i(y_{E^c})_j \sqrt{\beta_q} (v_q)_i (\sum_q \sqrt{\beta_q} \mathbf{u}_q(v_q)_j + \tilde{z}_j) + x_j(y_{E^c})_i \sqrt{\beta_q} (v_q)_j (\sum_q \sqrt{\beta_q} \mathbf{u}_q(v_q)_i + \tilde{z}_i) \right)
\]

\[
= \frac{X(\sigma^1_0(\mathbf{x}, \mathbf{y}) + \sigma^2_0(\mathbf{x}, \mathbf{y}))}{n},
\]

where we define the vectors \( \sigma^1_0(\mathbf{x}, \mathbf{y}), \sigma^2_0(\mathbf{x}, \mathbf{y}) \) appropriately. By Cauchy Schwarz:

\[
\| \nabla_{\mathbf{u}_q} \langle \mathbf{x}, \mathbf{Ry}_{E^c} \rangle \|^2 \leq 2 \frac{\| X \|^2}{n^2} (\| \sigma^1_0(\mathbf{x}, \mathbf{y}) \|^2 + \| \sigma^2_0(\mathbf{x}, \mathbf{y}) \|^2).
\]

We now bound the first term above, and the second term follows from a similar argument.

\[
\| \sigma^1(\mathbf{x}, \mathbf{y}) \|^2 = \sum_j \langle y_{E^c} \rangle_j^2 \left( \sum_i \sqrt{\beta_q} x_i(v_q)_i \partial q \left( \sum_q \sqrt{\beta_q} \mathbf{u}_q(v_q)_i + \tilde{z}_i, \sum_q \sqrt{\beta_q} \mathbf{u}_q(v_q)_j + \tilde{z}_j ; \tau \sqrt{n} \right) \right)^2
\]

For simplicity of notation, define \( D_{ij} = \partial q \left( \sum_q \sqrt{\beta_q} \mathbf{u}_q(v_q)_i + \tilde{z}_i, \sum_q \sqrt{\beta_q} \mathbf{u}_q(v_q)_j + \tilde{z}_j ; \tau \sqrt{n} \right) \). The above sum then can be reduced to:

\[
\| \sigma^1(\mathbf{x}, \mathbf{y}) \|^2 = \sum_{i_1, i_2} \beta_q x_{i_1} x_{i_2} (v_q)_{i_1} (v_q)_{i_2} \sum_{j : (i_1, j) \in G \text{ or } (i_2, j) \in G} \langle y_{E^c} \rangle_j^2 D_{i_1,j} D_{i_2,j}.
\]

We first bound the inner summation uniformly in \( i_1, i_2 \) as follows:

\[
\sum_{j : (i_1, j) \in G \text{ or } (i_2, j) \in G} \langle y_{E^c} \rangle_j^2 D_{i_1,j} D_{i_2,j} \leq \frac{A}{p} \sum_j \frac{|(\tilde{x}_{i_1}, \tilde{x}_j) (\tilde{x}_{i_2}, \tilde{x}_j)|}{n^2 \tau^2}
\]

\[
\leq \frac{A}{p} \sum_j (\tilde{x}_{i_1})^2 + (\tilde{x}_{i_2})^2 / 2n^2 \tau^2
\]

\[
\leq \frac{A}{n^2 \tau^2} \| \mathbf{x} \|^2 (\| \tilde{x}_{i_1} \|^2 + \| \tilde{x}_{i_2} \|^2)
\]

Employing a similar strategy for the other term, it is not hard to show that:

\[
\| \sigma^1(\mathbf{x}, \mathbf{y}) \|^2 \leq \frac{A \beta_q \| \mathbf{x} \|^2 \sup_i \| \tilde{x}_i \|^2}{pn^2 \tau^2}.
\]

Thus, on the set \( G \), we obtain that:

\[
\sum_q \| \nabla_{\mathbf{u}_q} \langle \mathbf{x}, \mathbf{Ry}_{E^c} \rangle \|^2 \leq \frac{AC(\alpha, \beta_1, r)}{p \tau^2},
\]

\[\text{(31)}\]
for every \( \tau \) sufficiently large. Indeed the same bound, with a modified value for \( C \) holds for the gradient with respect to all the variables \((u_q, (\bar{z}_i))_{i \leq p}\) using Eqs. (30), (31). Lemma 4.4 then implies that

\[
\sup_{x \in T^{1/4}_n, y \in T^{1/4}_p} P \left\{ (x, R y) \geq \sqrt{AC(\alpha, \beta_1, r) \tau^2} \right\} \leq \exp(-cp),
\]

for an appropriate \( c \). We omit the proof of the following remark that uses similar techniques as above, followed by a union bound.

**Remark 6.11.** For every \( A \geq A_0(\alpha, \beta_1, r) \) we have that:

\[
P \left\{ \sup_{|E| \leq p/A} \mathcal{P}_{|E| \times E}(R) \geq C(\alpha, \beta_1, r) \sqrt{n \log A} \right\} \leq \exp(-c_2(\tau)p).
\]

Here \( c_2(\tau) = 1/4\tau \) suffices.

Using \( A = \tau \log \tau \) for \( \tau \) large enough completes the proof.

### 6.4 Proof of Proposition 6.4

Since \( R_2 \) is a diagonal matrix, its spectral norm is bounded by the maximum of its entries. This is easily done as, for every \( i \in Q^n \):

\[
|(R_2)_{ii}| = \left| \eta \left( \frac{\|\bar{z}_i\|^2}{n} - 1; \frac{\tau}{\sqrt{n}} \right) \right| \leq \left| \frac{\|\bar{z}_i\|^2 - n}{n} \right|.
\]

By the Chernoff bound for \( \chi \)-squared random variables followed by the union bound we have that:

\[
\max_i \left| \frac{\|\bar{z}_i\|^2}{n} - 1 \right| \geq t,
\]

with probability at most \( p(\exp(n(-t + \log(1+t)/2) + \exp(n(t - \log(1-t)/2))) \). Setting \( t = \sqrt{C \log n/n} \) and using \( \log(1+t) \leq t - t^2/3 \), \( \log(1-t) \geq -t - t^2/3 \) for every \( t \) small enough we obtain the probability bound of \( pn^{-C/6} = an^{-C/6+1} \).

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### A Some technical proofs

In this Appendix we prove Remarks 6.7, 6.9 and 6.10. We begin with a preliminary lemma bounding the tail of Gaussian random variables.

**Lemma A.1.** Let \( X = \sum_q \sqrt{\tau_q} u_q v_q^\top + Z \) as according to Eq. (1). Further assume we are given an event \( B \) such that \( P \{ B \} \leq \exp(-bn) \) for some constant \( b \). Then for every \( n \) large enough:

\[
E \left\{ \|X\|_{\ell^p}^p \mathbb{I}(B) \right\} \leq \min(n^{-4}, p^{-4}).
\]

\[
E \left\{ \|X\|_{\ell^p} \mathbb{I}(B) \right\} \leq \min(n^{-2}, p^{-2}).
\]
Proof. Note that $\|X\|_F^2 \leq (r+1)(\sum_q \beta_q \|u_q v_q\|_F^2 + \|Z\|_F^2) = (r+1)\beta \|u_q\|_F^2 + 2\|Z\|_F^2$. Consider the event $\tilde{G} = \{(u, Z) : \|u_q\| \leq 2\sqrt{n}, \|Z\|_F \leq 2\sqrt{n}p\}$. We write:

$$
\mathbb{E} \left\{ \|X\|_F^2 \mathbb{I} (B) \right\} = \mathbb{E} \left\{ \|X\|_F^2 \mathbb{I} (B) \mathbb{I} (\tilde{G}) \right\} + \mathbb{E} \left\{ \|X\|_F^2 \mathbb{I} (B) \mathbb{I} (\tilde{G}^c) \right\}
$$

$$
\leq \mathbb{E} \left\{ (r+1)(\beta \|u\|_F^2 + \|Z\|_F^2) \mathbb{I} (B) \mathbb{I} (\tilde{G}) \right\} + \mathbb{E} \left\{ (r+1)(\beta \|u\|_F^2 + \|Z\|_F^2) \mathbb{I} (\tilde{G}^c) \right\}
$$

$$
\leq 4(r+1)(r\beta n + np)\mathbb{P} \{ B \} + (r+1) \sum_q \beta_q \mathbb{E} \left\{ \|u_q\|_F^2 \mathbb{I} (\tilde{G}) \right\} + 2\mathbb{E} \left\{ \|Z\|_F^2 \mathbb{I} (\tilde{G}^c) \right\}
$$

$$
\leq 8(\beta n + np) \exp(-bn) + (r+1) \sum_q \beta_q \int_{4n}^{\infty} \mathbb{P} \left\{ \|u_q\|_F^2 \geq t \right\} dt + 2 \int_{4np}^{\infty} \mathbb{P} \left\{ \|Z\|_F^2 \geq t \right\} dt.
$$

Here the last line follows from the standard formula $\mathbb{E} \{ X \} = \int_0^\infty \mathbb{P} \{ X \geq t \} dt$ for positive random variables $X$. By the Chernoff bound on $\chi$-squared random variable, $\mathbb{P} \left\{ \|u_q\|_F^2 \geq t \right\} \leq \exp(-t/8)$ and $\mathbb{P} \left\{ \|Z\|_F^2 \geq t \right\} \leq \exp(-t/8)$. Using this we have:

$$
\mathbb{E} \left\{ \|X\|_F^2 \mathbb{I} (B) \right\} \leq (r+1)(r\beta n + np) \exp(-bn) + \sum_q \beta_q \exp \left( -\frac{n}{2} \right) + 16 \exp \left( -\frac{n}{2} \right).
$$

This implies the first claim of the lemma. The second claim follows from Jensen’s inequality. □

Proof of Remark 6.7

Recall the definition of $F_L(Z_Q)$ as the $G, L$-Lipschitz extension of $\langle y, S_y \rangle$ where $G$ is the set:

$$
G = \{ Z : \|\nabla \langle y, S_y \rangle\| \leq L^2 \},
$$

where $L = \frac{8(2 + \sqrt{\alpha + \sqrt{\beta}})}{\sqrt{n}}$.

Further, we have already shown $\mathbb{P} \{ G \} \leq 2 \exp(-n/4)$. It suffices, hence to show that $\langle y, S_y \rangle$ grows at most polynomially in $n$. We have:

$$
|\mathbb{E} \{ \langle y, S_y \rangle \} - \mathbb{E} \{ F_L(Z_Q) \}| \leq \mathbb{E} \{ |\langle y, S_y \rangle - F_L(Z_Q)\| \mathbb{I} (Z_Q \in G^c) \}.
$$

Since $F_L(Z_Q) = \langle y, S_y \rangle$ whenever $Z_Q \in G$. We continue, employing the triangle inequality:

$$
|\mathbb{E} \{ \langle y, S_y \rangle \} - \mathbb{E} \{ F_L(Z_Q) \}| \leq \mathbb{E} \{ (|\langle y, S_y \rangle| + |F_L(Z_Q)|)\| (Z_Q \in G^c) \}.
$$

First consider the term $\mathbb{E} \{ |F_L(Z_Q)| \| (Z_Q \in G^c) \}$. Since $F_L(Z_Q)$ is $L$-Lipschitz,

$$
|F_L(Z_Q)| \leq |F_L(0)| + L \|Z_Q\|_F
$$

$$
\leq |F_L(0)| + L \|Z\|_F
$$

$$
\leq |F_L(0)| + L \|X\|_F.
$$

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since $X = \sum_q \sqrt{\beta_q} u_q v_q^T + Z$. We bound $\langle y, Sy \rangle$ in an analogous manner. As $S = P_{L} \{ \eta(\hat{\Sigma}) \}$ and $|\eta(z; \tau/\sqrt{n})| \leq |z|$: \[
|\langle y, Sy \rangle| \leq \left\| P_{L} \{ \eta(\hat{\Sigma}) \} \right\|_F \\
\leq \left\| \hat{\Sigma} \right\|_F \\
\leq \frac{1}{n} \left\| X^T X - I \right\|_F \\
\leq \frac{1}{n} \left\| X \right\|_F^2 + p.
\]
Consequently: \[
\left| E \{ \langle y, Sy \rangle \} - E \{ F_L(Z_Q) \} \right| \leq E \left\{ \left( \frac{1}{n} \left\| X \right\|_F^2 + p + |F_L(0)| + L \left\| X \right\|_F \right) \mathbb{1}(Z_Q \in G^c) \right\} \\
\leq 3 \min(n^{-2}, p^{-2}),
\]
where the last line follows from an application of Lemma A.1. This completes the proof of the remark.

**Proof of Remarks 6.9 and 6.10**

We only prove Remark 6.9. The proof of Remark 6.10 is similar.

The first claim follows from the fact that $E \{ \tilde{N}_{ij} \} = 0$, by symmetry of the distribution of $\langle \tilde{z}_i, \tilde{z}_j \rangle/n$ and of the soft thresholding function.

As for the second claim, we follow a line of argument similar to that of Remark 6.7. Recall that $F_K(Z_{E^c})$ is the $G, K$-Lipschitz extension of $\langle y_{E^c}, \tilde{N} y_{E^c} \rangle$. Here:

$$G = \left\{ Z_{E^c} : \left\| \nabla \langle y_{E^c}, \tilde{N} y_{E^c} \rangle \right\|_2^2 \leq K^2 \right\},$$

$$K = \sqrt{\frac{256 A \gamma^2 \alpha^3}{p \tau^2}}.$$

Further we have shown that $P \{ G^c \} \leq \exp(-p/2)$. Since $F_K(Z_{E^c})$ is $K$-Lipschitz:

$$|F_K(Z_{E^c})| \leq |F_K(0)| + K \left\| Z_{E^c} \right\|_F \\
\leq K \left\| Z_{E^c} \right\|_F \\
\leq K \left\| Z \right\|_F,$$

since $F_K(0) = 0$. Further, using the fact that $|\eta(z; \tau/\sqrt{n})| \leq |z|$: \[
|\langle y_{E^c}, \tilde{N} y_{E^c} \rangle| \leq \left\| \tilde{N} \right\|_F \\
\leq \left\| \frac{1}{n} Z^T Z \right\|_F \\
\leq \frac{1}{n} \left\| Z \right\|_F^2.
\]

Now we have \[
\left| E \{ \langle y_{E^c}, \tilde{N} y_{E^c} \rangle \} - E \{ F_K(Z_{E^c}) \} \right| \leq E \left\{ \left| \langle y_{E^c}, \tilde{N} y_{E^c} \rangle - F_K(Z_{E^c}) \right| \right\} \\
= E \left\{ \left| \langle y_{E^c}, \tilde{N} y_{E^c} \rangle - F_K(Z_{E^c}) \right| \mathbb{1}(Z_{E^c} \in G^c) \right\} \\
\leq E \left\{ \left| \langle y_{E^c}, \tilde{N} y_{E^c} \rangle \right| + |F_K(Z_{E^c})| \mathbb{1}(Z_{E^c} \in G^c) \right\},
\]

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where the penultimate equality follows from the definition of the Lipschitz extension. Using the above estimates:

\[
|E\{\langle y_{E^c}, \tilde{N}y_{E^c} \rangle \} - E\{F_K(Z_{E^c})\}| \leq E \left\{ \left( \frac{1}{n} \|Z\|^2_F + K \|Z\|_F \right) \mathbb{I}(Z_{E^c} \in G^c) \right\}
\]

\[
\leq E \left\{ \left( \frac{1}{n} \|X\|^2_F + K \|X\|_F \right) \mathbb{I}(Z_{E^c} \in G^c) \right\}
\]

The remark then follows by an application of Lemma A.1.

B Empirical Results

![Graphs showing empirical results](image)

Figure 3: The results of Simple PCA, Diagonal Thresholding and Covariance Thresholding (respectively) for a synthetic block-constant function (which is sparse in the Haar wavelet basis). We use \( \beta = 1.4, p = 4096 \), and the rows correspond to sample sizes \( n = 1024, 1625, 2580, 4096 \) respectively. Parameters for Covariance Thresholding are chosen as in Section 3, with \( \nu' = 4.5 \). Parameters for Diagonal Thresholding are from [JL09]. On each curve, we superpose the clean signal (dotted).
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