TIGHT CONTACT STRUCTURES ON HYPERBOLIC THREE-MANIFOLDS

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ABSTRACT. We show the existence of tight contact structures on infinitely many hyperbolic three-manifolds obtained via Dehn surgeries along sections of hyperbolic surface bundles over circle.

1. INTRODUCTION

A contact three-manifold is a pair \((M, \xi)\) where \(M\) is a smooth 3–manifold and \(\xi \subset TM\) is a totally non-integrable 2-plane field distribution on \(M\). Here we always assume that \(\xi\) is a co-oriented positive contact structure, that is, \(\xi = \text{Ker}(\alpha)\) for a contact 1–form \(\alpha\) satisfying \(\alpha \land d\alpha > 0\) with respect to a pre-given orientation on \(M\). A disk \(D\) in a contact 3–manifold \((M, \xi)\) is called overtwisted if the boundary circle \(\partial D\) is tangent to \(\xi\) everywhere. A contact structure \(\xi\) is called overtwisted if there is an overtwisted disk in \((M, \xi)\), otherwise it is called tight. It is known that every closed oriented 3–manifold admits an overtwisted contact structure [7, 19]. On the other hand, there are 3–manifolds which do not admit a tight contact structure [10].

There are some classification results on tight contact structures with respect to the geometric speciality of 3–manifolds. Lisca and Stipsicz in [18] proved that a closed oriented Seifert fibered 3–manifold admits a tight contact structure if and only if it is not gotten \((2q - 1)\)–surgery along the \((2, 2q + 1)\) torus knot in \(S^3\) for \(q \geq 1\). In two independent work [2, 16], they showed the existence of tight contact structures on toroidal 3–manifolds. It is known that every irreducible 3–manifold that is neither toroidal nor Seifert fibered is hyperbolic. Kaloti and Tosun in [17] find infinitely many hyperbolic rational homology spheres admitting tight contact structures. Etgül in [9] also explored that infinitely many hyperbolic 3–manifolds that carry tight contact structures. His construction uses Dehn surgeries along sections of hyperbolic torus bundles over \(S^1\). Here we’ll follow similar ideas for surface bundles over \(S^1\) with fiber genus at least two.

Let \(\Sigma_g\) be a closed connected orientable surface with genus \(g\). In this paper assume that \(g\) is always greater than 1. We will denote \(MCG(\Sigma_g)\) by the mapping class group of \(\Sigma_g\), i.e, the group of isotopy classes of orientation preserving homeomorphisms of \(\Sigma_g\). Let \(t_a\) be the positive Dehn twist along a simple closed curve \(a\).

Let \(\phi \in MCG(\Sigma_g)\) be the mapping class representing the homeomorphism

\[
t_{a_1}^m t_{a_2} \cdots t_{a_{2g}} t_{a_{2g+1}}^n
\]

where \(a_i\)'s are simple closed curves on \(\Sigma_g\) as indicated in Figure 1.

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Denote by $M_\phi$ the mapping torus with fibers $\Sigma_g$ and monodromy $\phi$. Let $M_\phi(r)$ be the surgered manifold obtained by performing rational $r$-surgery along a section of $M_\phi$. Clearly, $\phi$ has a fixed point, so such a section exists. The following theorems give examples required:

**Theorem 1.1.** Suppose $g \geq 2$, $m, n \in \mathbb{Z}$, $r \in \mathbb{Q}$ and $\phi$ as indicated in (1). Then $M_\phi(r)$ is hyperbolic for all but finitely many $m$ and $r$.

**Theorem 1.2.** Suppose $g \geq 1$, $r \in \mathbb{Q}$ and $\phi$ as indicated in (1). Then $M_\phi(r)$ admits a tight contact structure $\xi$ for any $m, n \in \mathbb{Z}^+$ and for all $r \neq 2g - 1$.

As a consequence of the theorems we have:

**Corollary 1.3.** Suppose $g \geq 2$, $m, n \in \mathbb{Z}^+$, $r \in \mathbb{Q}$ and $\phi$ as indicated in (1). Then $M_\phi(r)$ is a hyperbolic manifold admitting a tight contact structure for all $r \neq 2g - 1$ and all but finitely many $m \in \mathbb{Z}^+$. □

The proof of Theorem 1.1 and Theorem 1.2 will be given in Section 2 and Section 3.

## 2. Proof of Theorem 1.1

In order to prove the theorem, we’ll focus on pseudo-Anosov homeomorphisms and construct infinitely many hyperbolic 3–manifolds via pseudo-Anosov monodromies. A hyperbolic 3–manifold is a 3–manifold which admits a complete finite-volume hyperbolic structure. Thurston [22] demonstrated that an orientable surface bundle over circle whose fiber is a compact surface of negative Euler characteristic is hyperbolic if and only if the monodromy of the surface bundle is a pseudo-Anosov homeomorphism. Another deep result of Thurston is hyperbolic Dehn surgery theorem which states that a hyperbolic 3–manifold remains hyperbolic after Dehn filling along a link for all slopes except finitely many of them (For details see [23]). In order to apply these results, we need a lemma where we construct infinitely many pseudo-Anosov diffeomorphisms as products of certain Dehn twists:

**Lemma 2.1.** Let $\phi$ be the class in $\text{MCG}(\Sigma_g)$ as described in (1) above. Then $\phi$ is pseudo-Anosov for any integer $n$ and for all but at most 7 consecutive values of $m$.

Denote by $i(\alpha, \beta)$ geometric intersection number of the curves $\alpha$ and $\beta$. We say a set of simple closed curves $\{\gamma_1, \gamma_2, \ldots, \gamma_k\}$ fills $\Sigma_g$ if $\Sigma_g \setminus \{\gamma_1, \gamma_2, \ldots, \gamma_k\}$ is a disjoint union of topological disks. In order to prove Lemma 2.1 we use the following theorem of Fathi:

**Theorem 2.2.** ([12]) Let $f$ be the class in $\text{MCG}(\Sigma_g)$ and let $\gamma$ be a simple closed curve in $\Sigma_g$. If the orbit of $\gamma$ under $f$ fills $\Sigma_g$, then $t_\gamma^m f$ is a pseudo-Anosov class except for at most 7 consecutive values of $m$.

**Proof of Lemma 2.1.** Let $\gamma$ represents the curve $a_1$ and let $f$ be the product of Dehn twists $t_{a_1} t_{a_2} \cdots t_{a_{2g}} t_{a_{2g+1}}^m$. Then conclude that

$$f(\gamma) = t_{a_1} t_{a_2} (a_1) = a_2, \quad f^2(\gamma) = t_{a_1} t_{a_2} t_{a_3} (a_2) = a_3,$$
and inductively,

\[ f^i(\gamma) = a_{i+1} \] for all \( i \in 1, 2, \ldots, 2g - 1 \).

Since the complement \( \Sigma_g \setminus \{a_1, \ldots, a_{2g}\} \) is a topological disk, we can say the orbit of \( \gamma \) under \( f \) fills \( \Sigma_g \). As a result of Theorem 2.2, \( \phi \) is pseudo-Anosov except for at most 7 consecutive \( m \) values.

Now we have a family of pseudo-Anosov monodromies. Using [22] we can say that the surface bundles \( M_\phi \) are all hyperbolic. By hyperbolic Dehn surgery theorem the surgered manifolds \( M_\phi(r) \) are hyperbolic for all \( m, n \in \mathbb{Z} \) and \( r \in \mathbb{Q} \) except 7 values of \( m \) and finitely many “bad” slopes \( r \). This finishes the proof of Theorem 1.1. \( \square \)

![Figure 2](image-url)
3. Proof of Theorem 1.2

We will analyze the proof with respect to the parity of the genus \(g\) of the fiber \(\Sigma_g\). First assume \(g \geq 3\) odd. Note that conjugation of the monodromy by any class of \(\text{MCG}(\Sigma_g)\) does not change the mapping torus up to diffeomorphism. Since

\[
t_{a_2} \cdots t_{a_2} t_{a_2 - 1} \cdots t_{a_2} = t_{a_1}^{-m} \phi t_{a_1}^m
\]

we may replace \(\phi\) in (1) with the mapping class \(t_{a_2} \cdots t_{a_2} t_{a_2 - 1} \cdots t_{a_2} \phi t_{a_1}^m\). Also observe that \(M_\phi(r)\) can be also obtained from a Dehn surgery on the binding of an open book decomposition whose page is \(\Sigma^1_g\) (punctured \(\Sigma_g\)) and monodromy can be still assumed to be \(\phi \in \text{MCG}(\Sigma^1_g)\). We will construct the required contact structure \(\xi\) on \(M_\phi(r)\) via Dehn surgery on the open book decomposition \((\Sigma^1_g, \phi)\) along its binding.

It is known (see [1], [14]) that the contact structure, say \(\xi_0\), (before the surgery along binding) supported by \((\Sigma^1_g, \phi)\) is Stein fillable. More precisely, consider the handlebody diagram of the smooth 4–manifold \(X_\phi\) given in Figure 2-(a) (in the case of genus 3) with “2\(g\)” 1–handles and “\(m + n + 2g - 1\)” 2–handles. Note that Figure 2-(a) describes a Lefschetz fibration structure on \(X_\phi\) with a regular fiber \(\Sigma^1_g\) and the vanishing cycles \(a_1, a_2, \ldots, a_{2g+1}\). There are \(n\) copies for \(a_{2g+1}\) and \(m\) copies for \(a_1\) (not drawn for simplicity). All coefficients (except on \(B\)) are \(-1\) with respect to the framing given by the page \(\Sigma^1_g\). We remark that no handle is attached along the binding of the induced open book \((\Sigma^1_g, \phi)\) on the boundary \(\partial X_\phi\) which is realized as \(B\) in the figure.
Next starting from the topological description in Figure 2(a) of $X_{\phi}$, we’ll get a diagram describing a Stein structure on $X_{\phi}$ inducing $\xi_0$ as follows: First we flip the twisted bands over the 1–handles as pointed out in Figure 2(a) and get Figure 2(b). Figure 3(a) gives another handle description of $X_{\phi}$ obtained by moving the feet of 1-handles as indicated by the dotted arrows in Figure 2(b). Then flip the bands as shown in Figure 2(a) to get rid of one more left half twist for each band (see Figure 3(b)), and obtain Figure 4(a) by flipping the connecting bands over the feet of 1-handles suggested by the dotted arrows in Figure 3(b). Figure 4(b) defines a Stein structure on $X_{\phi}$ obtained by putting the attaching circles in part (a) into Legendrian positions, where a Legendrian realization $L_0$ of $B$ in the tight contact boundary $\partial X_{\phi}$ is also provided. All coefficients (except on $L_0$) are $-1$ with respect to Thurston-Bennequin (contact) framing in $\partial X_{\phi}$ and no handle is attached along $L_0$. Note that $tb(L_0) = 2$ (the case $g = 3$ is shown). In the general case, $tb(L_0) = g - 1$. Finally, we use the trick (“Move 6”) in Figure 20 of [15] to obtain a Legendrian representation $L$ of $B$ with $tb(L) = 2g - 1$ (see Figure 5). Note that Figure 5 describes the same Stein structure on $X_{\phi}$ as in Figure 4(b).

(a)

(b)

Figure 4.
Figure 5. The same Stein structure on $X_\phi$ as in Figure 4-(b), and another Legendrian realization $L$ of the binding $B$ in the tight contact boundary $\partial X_\phi$. $L$ is obtained from $L_0$ by applying “Move 6” (smooth but non-Legendrian isotopy of $L_0$) $g$ times using the left foot of the corresponding 1-handles (when $g = 3$, handles are $B_1, A_4, B_3$). All coefficients (except on $L$) are $-1$ with respect to Thurston-Bennequin (contact) framing in $\partial X_\phi$. No handle attached along $L$. Note that $tb(L) = 5$ (the case $g = 3$ is shown). In the general case, $tb(L) = 2g - 1$.

Now if $g \geq 2$ is even, we replace the monodromy $\phi$ with $t_{a_2 \phi + 1}^n t_{a_2} \cdots t_{a_2} t_{a_1}^m$ since

$$t_{a_2 \phi + 1}^n t_{a_1}^{-m} \phi t_{a_2 \phi + 1}^{-n} t_{a_1}^m = t_{a_2 \phi + 1}^n t_{a_2} \cdots t_{a_2} t_{a_1}^m.$$ 

Then starting from the handlebody diagram given in Figure 6-(a) (where the case $g = 4$ is shown) and following the moves as in the case of odd genus, one can get Figure 6-(b) describing a Stein structure realizing a Legendrian representation $L$ with $tb(L) = 2g - 1$ as in Figure 6. One should note that we need to consider different monodromies (but still giving the same mapping torus) depending on the parity of $g$ to make the contact and the page framing on any attaching circle coincide.
Now (in any case of $g$) we first (Legendrian) slide (Stein) $2$–handle corresponding $a_3$ over the ones represented by the curves $a_1, a_5, a_7, ..., a_{2g+1}$, and then cancel the $2$–handles represented by $a_5, a_7, ..., a_{2g-1}$ with the corresponding $1$–handles. Second, we (Legendrian) slide $2$–handles represented by the curves $a_1$ and $a_{2g+1}$ over a fixed one (chosen from each family in Figure 5/Figure 6(b)), and then cancel $1$–handles $B_1$ and $B_g$ with the chosen $2$–handles corresponding $a_1$ and $a_{2g+1}$ respectively. Also we cancel each $1$–handle $A_i$ with the $2$–handle corresponding
the curve $a_i$ for each $i$ even. As a result, we obtain another (but equivalent) Stein structure on $X_\phi$ which can be also considered as the contact surgery diagram for $\xi_0$ on $\partial X_\phi$. Finally, we set $r' = r - 2g + 1$ and perform $r'$–contact surgery along $L \subset (\partial X_\phi, \xi_0)$ to get a contact structure $\xi$ on $M_\phi(r)$ whose diagram is given in Figure 7 (where we use continued fractions).

First suppose $r' = r - 2g + 1 < 0$. We know any contact surgery with negative contact framing can be converted to a sequence of contact $(-1)$–surgeries and $(-1)$–surgeries preserve Stein fillability ([5], [6], [14]). Thus $(M_\phi(r), \xi)$ is Stein fillable (hence tight).

Now let $r' = r - 2g + 1 > 0$. By Thurston-Winkelnkemper construction ([24]), it is known that the binding $B$ is transverse to the contact structure supported by the open book decomposition. Also since $\partial X_\phi$ is Stein fillable, $\xi_0$ has nonzero contact invariant [21]. As a result of Conway’s work (see [3], Theorem 1.6) if $K$ is a fibered transverse knot in a contact 3–manifold $(M, \eta)$ where $\eta$ has nonvanishing contact class, then $r$-surgery along $K$ preserves the non-vanishing of

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure7.png}
\caption{The contact 3–manifold $(M_\phi(r), \xi)$. (The case $g = 3$ is shown.)}
\end{figure}
the contact class if \( r > 2g - 1 \) where \( g \) is the genus of \( K \). Hence we conclude that \((M_\varphi(r), \xi)\) has nonzero contact invariant (hence tight) through Conway’s result. This finishes the proof of Theorem 1.2.

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