Construction and Assignment of Orthogonal Sequences and Zero Correlation Zone Sequences Over GF(p)

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This work was supported in part by the National Natural Science Foundation of China under Grant 62072054 and Grant 11901049; in part by the Key Research and Development Program of Shaanxi under Grant 2021GY-047; in part by the Fundamental Research Funds for the Central Universities, CHD, under Grant 300102240103; in part by the Project of High-Level Talent Scientific Research of Shihezi University under Grant RCZK202027; and in part by the Guangdong provincial basic and applied basic research fund project–regional joint fund–Youth Fund under Project 2021A1515110605.

ABSTRACT Orthogonal sequences can be assigned to a regular tessellation of hexagonal cells, typical for synchronised code-division multiple-access (S-CDMA) systems. The sequences within any cell should both be orthogonal in the adjacent cells and require a sufficient number of users in each cell. However, for binary sequences, the capacity of communication system is limited. So, a new family of non-binary orthogonal sequences is constructed, which has more sequences per cell than binary sequences in the network. Next, an efficient assignment of these orthogonal sequence sets to a regular tessellation of hexagonal cells is given, and also the sequence sets have large re-use distance. Finally, based on above the construction, we construct a family of orthogonal sequences with zero correlation zone (ZCZ) property which can reduce the interference among users in a multi-path environment. The constructed ZCZ sequences can be applied to the quasi-synchronous CDMA (QS-CMDA) spread spectrum systems. And the assignment of sequences has the same re-use distance.

INDEX TERMS Orthogonal sequences, zero correlation zone, Boolean functions, semi-bent functions.

I. INTRODUCTION

Orthogonal sequences are used in many applications, such as synchronous code division multiple access (S-CDMA) spread spectrum systems, detection signals in satellite communications and mobile devices in cloud computing [1], [2]. To prevent interference from the neighboring cells, the regular tessellation of hexagonal cells be used as a model [3], [4], [5]. The sequences within any cell should both be orthogonal in the adjacent cells and require a sufficient number of users in each cell [6]. Employing correlation-constrained sets of Hadamard matrices to construct spreading codes in these systems are a usual way such that the maximal cross-correlation of sequences lies in the range \([2^{m/2}, 2^{(m+2)/2}]\) [7], [8]. Akansu et al. presented a class of Walsh-like nonlinear binary orthogonal sequence sets which can be used in asynchronous direct sequence CDMA communications system [9]. However, the number of binary sequences in each cell is \(2^{m-3}\) or \(2^{m-2}\) and it is difficult to increase in S-CDMA systems. This raises a challenge to how to increase the capacity of each cell in S-CDMA systems. In order to increase the number of users in each cell, we propose a class of sequences over GF(p) such that each cell has more sequences. To more sufficient using the sequence sets, the re-use distance was proposed. The so-called re-use distance \(D\) reflects the ability to use the same codewords in non-adjacent cells that are far away from the cell where these codewords have originally been placed. At the same time, to prevent interference from the users in...
neighbouring cells, a standard requirement for the assignment of orthogonal sequences in the network is that the sequences within any cell should be orthogonal to the sequences in the neighbouring cells.

In S-CDMA systems, orthogonal sequences, such as well-known Walsh sequences, can guarantee that the correlation of any pairwise sequence is zero without shifting. However, whenever there is a relative shift between the sequences which will lose their orthogonality. To reduce the interference among users in a multipath environment or within any cell should be orthogonal to the sequences in the interfering cells.

Wang et al. obtained a class of M-sequences, which have good autocorrelation properties [21], [22]. Zhou et al. proposed a class of optimal sequences based on interleaving technique [15]. Li and Xu presented a construction of ZCZ sequence sets over Gaussian integers [16], [17]. Zhou et al. proposed a class of optimal sequences based on interleaving technique and perfect nonlinear functions [18], [19]. Gu et al. constructed a new family of polyphase sequences with low correlation [20]. Wang et al. obtained a class of M-sequences, which have good auto-correlation properties [21], [22].

The main contribution this paper is an efficient method to construct a large set of p-phase orthogonal sequences, using (vectorial) semi-bent functions such that there are zero correlation properties [21]. [22]. Let \( \lambda \in \mathbb{F}_p^m \). If \( |W_f(\lambda)| = \{0, p^{(m+1)/2}\} \) for all \( \lambda \in \mathbb{F}_p^m \), then \( f \) is called semi-bent function.

Let \( f_1, f_2, \cdots, f_p \) be \( p \) functions with length \( p^m \). The concatenation of \( f_1, f_2, \cdots, f_p \), denoted by \( f = f_1 \| f_2 \| \cdots \| f_p \), is a function with length \( p^{m+1} \). The true table of the function \( f \) is divided into \( p \) equipartitions such that the first part of function \( f_1 \) corresponds to \( f_1 \), the second part to \( f_2 \) and the last part to \( f_p \), the rest can be done in the same manner.

The sequence of a function \( f \in \mathbb{B}_m \) is defined as

\[
 f = \left( \omega f(0,\cdots,0), \omega f(0,\cdots,1), \cdots, \omega f(p-1,p-1,p-1) \right),
\]

where the period of sequence is \( p^m \).

**Definition 1:** Let \( f_1, f_2 \) be the sequences of functions \( f_1 \) and \( f_2 \) over the vector space \( \mathbb{F}_p^m \), sequences \( f_1 \) and \( f_2 \) are called orthogonal, denoted by \( f_1 \perp f_2 \), if

\[
 f_1 \cdot f_2^* = \sum_{x \in \mathbb{F}_p^m} \omega f_1(x) f_2^*(x) = 0,
\]

where \( f^* \) denotes the conjugate of \( f \).

Noticing that

\[
 W_{f_1-f_2}(0_m) = 0 \iff \sum_{x \in \mathbb{F}_p^m} \omega f_1(x) f_2^*(x) = 0,
\]

so we have

\[
 W_{f_1-f_2}(0_m) = 0 \iff f_1 \cdot f_2^* = 0,
\]

then the following important characterization of orthogonal sequences is obtained.

**Lemma 1:** Let \( f_1, f_2 \) be the functions over the vector space \( \mathbb{F}_p^m \), then \( f_1 \perp f_2 \) if and only if \( W_{f_1-f_2}(0_m) = 0 \). According to the Lemma 1, it is easily deducing that the set of linear functions \( \mathcal{L}_m \) over the vector space \( \mathbb{F}_p^m \) is a set of orthogonal sequences.

**Definition 2:** Let \( a = (a_0, a_1, \cdots, a_{N-1}) \) and \( b = (b_0, b_1, \cdots, b_{N-1}) \) be two complex sequences, the aperiodic correlation function of \( a \) and \( b \) at shift \( \tau \) is given as follows:

\[
 R_{a,b}(\tau) = \sum_{\tau=0}^{N-1} a_\tau b_{\tau+\tau}, \quad 0 \leq \tau \leq N - 1,
\]

\[
 R_{a,b}(\tau) = 0, \quad \tau \geq N.
\]

\( R_{a,b}(\tau) \) is called aperiodic cross-correlation function (ACCF) if \( a \neq b \); otherwise, it is called the aperiodic auto-correlation function (AACF). For simplicity, the ACCF of \( a \) will be written as \( R_a(\tau) \).

**Definition 3:** Let \( \mathcal{A} = \{A_1, A_2, \cdots, A_K\} \) be a sequences set, let the length of each sequence of \( \mathcal{A} \) is \( L \),

\[
 A_i = (a_0^i, a_1^i, \cdots, a_{L-1}^i), \quad 1 \leq i \leq K.
\]
A is said to be a zero correlation zone (ZCZ) sequence set with ZCZ width Z if and only if the set A satisfies the following two conditions: i) 
1) \( R_A(\tau) = 0 \) holds for any \( 1 \leq i \leq K \) and \( 1 \leq |\tau| \leq Z \);  
2) \( R_{A,j}(\tau) = 0 \) holds for any \( i \neq j \) and \( 0 \leq |\tau| \leq Z \).

III. CONSTRUCTION OF A LARGE FAMILY OF \( p \)-ARY ORTHOGONAL SEQUENCES

In this section we obtain the orthogonal sequences such that the number of users equals \( p^{m-1} \) per cell based on linear functions.

Construction 1: Let \( m, s \) and \( t \) be three positive integers with \( m = s + t \), where \( s = \lfloor (m-1)/2 \rfloor \) and \( t = \lfloor (m+2)/2 \rfloor \). Let \( p \) be a prime with \( p \geq 3 \). Let \( y \) be a primitive element of \( \mathbb{F}_{p^m-1} \) and \( \{ 1, y, \cdots, y^{t-2} \} \) be a polynomial basis of \( \mathbb{F}_{p^t-1} \) over \( \mathbb{F}_p \). Define the isomorphism \( \pi: \mathbb{F}_{p^t-1} \mapsto \mathbb{F}_{p^m-1} \) by

\[
\pi(b_1 + b_2 y + \cdots + b_{t-1} y^{t-2}) = (b_1, b_2, \cdots, b_{t-1}).
\]

Let \( \phi_c: \mathbb{F}_p \to \mathbb{F}_{p^t-1} \) be a mapping defined by

\[
\phi_c(y) = \begin{cases} 
0_{t-1}, & y = 0^t, \\
(y^{\beta+c}, & y \in \mathbb{F}_p \setminus 0^t 
\end{cases}
\]

where \( [y] \) denotes the integer representation of \( y \), and \( c \in \{ 1, 2, \cdots, p^{t-1} \} \) is an integer. For \( y \in \mathbb{F}_p \), \( x \in \mathbb{F}_p \), the semi-bent functions can be obtained as follows,

\[
f_c(y, x) = \phi_c(y) \cdot x = 0_t \cdot x \cdot (\pi(y^c), 0) \cdot x \\
\times (\pi(y^{c+1}), 0) \cdot x \cdots (\pi(y^{c+t-2}), 0) \cdot x.
\]

Let \( \beta, \alpha, \gamma \in \mathbb{F}_p \), for any fixed \( \alpha \in \mathbb{F}_p \), define

\[L_\alpha = \{1 = (\beta, \alpha, \gamma, (y, x) \mid \beta \in \mathbb{F}_p, \alpha \in \mathbb{F}_{p^t-1} \} \]

We denote \( H_0 = L_0, H_1 = L_1, \cdots, H_{p^m-1} = L_{p^m-1} \). For any \( c \in \{ 1, 2, \cdots, p^{t-1} \} \), \( p^t \) disjoint sequence sets are constructed as follows:

\[S_{c,i} = \{ f_c + l \mid l \in H_i \}, \quad \text{for } i \in \{0, 1, \cdots, p-1\}.
\]

According to (7), note that the \( f_c(y, x) \) is the concatenation of \( p^t \) \( t \)-variable functions, the length of the function \( f_c(y, x) \) is \( p^m \cdot p^t = p^{m+t} \). These sequences can be easily assigned to hexagonal cells in such a way that the sequences assigned to adjacent cells is orthogonal, while the correlation between sequences assigned to non-adjacent cells is small.

Theorem 1: Let the sequence set \( S_{c,i} \) be defined by (9) as in Construction 1. Then, we always have

1) For \( c \in \{ 1, 2, \cdots, p^{t-1} \} \), \#\( S_{c,i} \) = \( p^{m-1} \), where \# denotes the number of a sequence set \( S_c \);
2) All the sequences in \( S_{c,i} \) are semi-bent sequences;
3) Let \( c, c' \in \mathbb{F}_{p^t-1}, i, i' \in \{0, 1, \cdots, p-1\} \). If \( i \neq i' \), \( S_{c,i} \cap S_{c',i'} \) holds.

Proof: 1) Note that \#\( L_\alpha \) = \( p^{m-1} \), which implies that 1) hold.
2) For the sequence set \( S_{c,i} \), without loss of generality, we only consider the case of \( i = 0 \). Then the sequence set \( S_{c,0} = \{ f_c + l \mid l \in H_0 \} \), where \( f_c \) is the sequence of a function \( f_c \), and \( l \) is the sequence of a linear function \( l \).

According to (7), note that the \( f_c(y, x) \) is the concatenation of \( p^t \) \( t \)-variable linear functions, the length of the function \( f_c(y, x) \) is \( p^t \cdot p^t = p^{m+t} \).

For any \( (\beta, \alpha, \gamma) \in \mathbb{F}_p \times \mathbb{F}_{p^t-1} \times \mathbb{F}_p \), we have

\[
|W_{f_c}(\beta, \alpha, \gamma)| = |\sum_{(y, x) \in \mathbb{F}_p \times \mathbb{F}_{p^t-1} \times \mathbb{F}_p} \alpha \cdot y \cdot (\beta+y-(\alpha \cdot x))| \tag{10}
\]

From (10), \( \beta, \alpha, \gamma \) are an \( m \)-variable linear function, where \( \beta, \alpha, \gamma \in \mathbb{F}_p \). An \( m \)-variable linear function \( \beta, \gamma \cdot (\alpha, \gamma) \cdot x \) can be regarded as the concatenation of a \( t \)-variable linear function \( (\alpha, \gamma) \cdot x \) and the corresponding affine functions \( (\alpha, \gamma) \cdot x + c \), where \( c \in \mathbb{F}_p \).

In order to get the magnitude of the Fourier transform of the function \( f_c(y, x) \), we just need to know if the linear function \( (\alpha, \gamma) \cdot x \) appears in the function \( f_c(y, x) \). When \( \alpha = \pi(y^{c+k}) \), and \( \alpha = 0 \), where \( k \in \{0, 1, \cdots, p^t-2\} \), \( |W_{f_c}(\beta, \alpha, \gamma)| = p^t \) holds; otherwise, \( |W_{f_c}(\beta, \alpha, \gamma)| = 0 \). Then all the sequences in \( S_{c,i} \) are semi-bent sequences.

3) Let \( f_c + l \in S_{c,i} \) and \( f_c + l' \in S_{c,i'} \), where \( l = \beta \cdot y + (\alpha, \gamma) \cdot x \in (T_i \) and \( l' = \beta' \cdot y + (\alpha', \gamma') \cdot x \in T_{i'} \). To analyze the orthogonality between \( f_c + l \) and \( f_c + l' \), we consider

\[h = (f_c + l) - (f_c + l') = f_{c-e} + (l - l').
\]

From the analysis of the 2), when \( i \neq i' \), we can obtain \( \alpha \neq \alpha' \), then \( W_h(0_{n}) = 0 \). From Lemma 1, we have \( S_{c,i} \cap S_{c,i'} \) holds.

Remark 1: The assignment of the sequences is consistent with the above assignment such that the correlation between sequences assigned to adjacent cells is zero.

Example 1: Let \( m = 4, p = 3 \) and \( y \in \mathbb{F}_{3^2} \) be a root of the primitive polynomial \( x^2 + z + 2 \). Define the isomorphism \( \pi: \mathbb{F}_{3^2} \mapsto \mathbb{F}_3^2 \) by

\[
\pi(b_1 + b_2 y) = (b_1, b_2).
\]

For \( y \in \mathbb{F}_{3^2}, x \in \mathbb{F}_3^2 \), then the function \( f_c(y, x) \) can be denoted as follows,

\[
f_c(y, x) = 0_3 \cdot x \cdot (\pi(y^c), 0) \cdot x \cdot (\pi(y^{c+1}), 0) \cdot x,
\]

where \( c \in \{1, 2, \cdots, 9\} \). We can get 9 semi-bent functions. For simplicity, we only list two functions \( f_1 \) and \( f_2 \) as follows,

\[
f_1(y, x) = 0_3 \cdot x \cdot (x^2), \quad f_2(y, x) = 0_3 \cdot x \cdot (x^2) \cdot (2x^2 + x^2),
\]

where \( y \in \mathbb{F}_3 \), \( x = (x_1, x_2, x_3) \in \mathbb{F}_3^3 \).

For \( c \in \{1, 2, \cdots, 9\} \) and \( i \in \mathbb{F}_3 \), a set of orthogonal sequences can be defined as follows,

\[S_{c,i} = \{ f_c + l \mid l \in H_i \}.
\]

where \( H_0 = \{\alpha_1 x_1 + \alpha_2 x_2 | \alpha_1, \alpha_2 \in \mathbb{F}_3^3\}, H_1 = \{\alpha_1 x_3 + \alpha_2 x_2 + x_1 | \alpha_1, \alpha_2 \in \mathbb{F}_3^3\}, \) and \( H_2 = \{\alpha_1 x_3 + \alpha_2 x_2 + 2x_2 | \alpha_1, \alpha_2 \in \mathbb{F}_3^3\} \).
The orthogonality between \( f_c \) and \( H_i \) is shown in the following Table 1. Then we construct 27 sets of orthogonal sequences \( \{S_{c,i} \mid c \in \{1, 2, \cdots, 9\}, i \in F_3\} \). All the 27 sets of orthogonal sequences can be used to get an assignment with the re-use distance \( D = \sqrt{27} \) as depicted in Fig. 1.

**Table 1. Orthogonality between \( f_c \) and \( H_i \).**

| \( H_0 \) | \( H_1 \) | \( H_2 \) |
|---------|---------|---------|
| \( f_{00} \) | \( \perp \) | \( \perp \) |
| \( f_{01} \) | \( \perp \) | \( \perp \) |
| \( f_{02} \) | \( \perp \) | \( \perp \) |
| \( f_{10} \) | \( \perp \) | \( \perp \) |
| \( f_{11} \) | \( \perp \) | \( \perp \) |
| \( f_{12} \) | \( \perp \) | \( \perp \) |
| \( f_{20} \) | \( \perp \) | \( \perp \) |
| \( f_{21} \) | \( \perp \) | \( \perp \) |
| \( f_{22} \) | \( \perp \) | \( \perp \) |

**Construction 2:** Let \( m, k \geq 2 \) be two positive integers with \( m = 2k + 2 \). Let \( y \) be a primitive element of \( \mathbb{F}_p^k \), and \( \{1, \gamma, \cdots, \gamma^{k-1}\} \) be a polynomial basis of \( \mathbb{F}_p \) over \( \mathbb{F}_p \). Define the isomorphism \( \pi: \mathbb{F}_p^k \rightarrow \mathbb{F}_p^k \) by

\[
\pi(b_1 + b_2 \gamma + \cdots + b_k \gamma^{k-1}) = (b_1, b_2, \cdots, b_k).
\]

For \( i = \{1, \cdots, p^k\} \), let \( \phi_i: \mathbb{F}_p^k \rightarrow \mathbb{F}_p^k \) be a bijective mapping defined by

\[
\phi_i(y) = \begin{cases} 0_k, & y = 0_k \\ \pi(y[y]+), & y \in \mathbb{F}_p^{k+2} \end{cases},
\]

where \([y]\) denotes the integer representation of \( y \). Let \( y \in \mathbb{F}_p^k \), \( x \in \mathbb{F}_p^{k+2} \). For \( i = 1, \cdots, k \), let

\[
f_i(y, x) = (\phi_i(y), 00) \cdot x.
\]

For any fixed \( \alpha \in \mathbb{F}_p^k \), linear function can be defined as follows,

\[
L_\alpha = \{ l_\beta = (\beta, \alpha) \cdot (y, x) \mid \beta \in \mathbb{F}_p^{m-2} \}.
\]

Let \( H_0 = L_{00} \cup L_{01} \cup \cdots \cup L_{0(p-1)} \), \( H_1 = L_{10} \cup L_{11} \cup \cdots \cup L_{1(p-1)} \), \( H_{p-1} = L_{(p-1)0} \cup L_{(p-1)1} \cup \cdots \cup L_{(p-1)(p-1)} \).

We construct disjoint sequence sets as follows:

\[
S_{i,j} = \{ f_i + l \mid l \in H_j \}, \quad i \in \mathbb{F}_p^k, \quad j \in \{0, 1, \cdots, p-1\},
\]

**Theorem 2:** Let the sequence set \( S_{c,i} \) be defined by (17) as in Construction 2. Then, we always have

1) For \( i \in \mathbb{F}_p^k \), \( \#S_{i,j} = p^{m-1} \), where \( j \in \{0, 1, \cdots, p-1\} \);
2) All the sequences in \( S_{i,j} \) are semi-bent sequences, where \( c \) is nonzero;
3) Let \( i, i' \in \mathbb{F}_p^{m-1}, j, j' \in \{0, 1, \cdots, p-1\} \). \( S_{i,j} \perp S_{i',j'} \) if and only if \( j \neq j' \).

**Proof:** 1) Note that \( \#L_\alpha = p^{m-2} \cdot p = p^{m-1} \), and \( \#S_{i,j} = p^{m-2} \cdot p = p^{m-1} \), which implies that 1) holds.

2) For the sequence set \( S_{i,j} \), without loss of generality, considering the case of \( i = j = 0 \), we have \( S_{0,0} = \{f_0 + l \mid l \in H_0\} \).

\[
f_0(y, x) = (\phi_0(y), 00) \cdot x.
\]

For any \( (\beta, \alpha_1, \alpha_2) \in S_{i,j} \times S_{i,j} \times S_{i,j} \), we have

\[
W_{f_0}(\beta, \alpha_1, \alpha_2) = \sum_{(y, x_1, x_2) \in \mathbb{F}_p^k} \omega^{\beta(y) - \alpha_1(x_1) - \alpha_2(x_2)} \cdot \mathbf{F}_{x_1} \cdot \mathbf{F}_{x_2}.
\]

Noticing that

\[
| \sum_{x_2 \in \mathbb{F}_p^k} \mathbf{F}_{x_2} | = \left\{ \begin{array}{ll} p^2, & \alpha_2 = 0, \\ 0, & \text{otherwise}. \end{array} \right.
\]

Note that \( \pi \) is bijective and there exists a unique \( y \in \mathbb{F}_p^k \) such that \( \phi_0(y) = \alpha_1 \), we have \( \sum_{x_1 \in \mathbb{F}_p^k} \omega^{\phi_0(y) - \alpha_1} = p^k \). For \( \alpha_2 = 0 \), then \( |W_{f_0}(\beta, \alpha_1, \alpha_2)| = p^{k+2} \), holds, which implies that 2) holds.

3) Let \( f_i + l \in S_{i,j} \) and \( f_j + l' \in S_{i,j'} \), where the Boolean function corresponding to the sequence \( l = (\beta, \gamma, \cdots, \gamma^{k-1}) \) is \( \mathbf{F}_l \), and the Boolean function corresponding to the sequence \( l' = (\beta', \gamma, \cdots, \gamma^{k-1}) \cdot x \in H_j \). To analyze the orthogonality between \( f_i + l \) and \( f_j + l' \), we consider

\[
h = (f_i + l) - (f_j + l') = f_{i-j} + (l - l').
\]

When \( j \neq j' \), then \( W_{h}(0, m_0) = 0 \) holds. From Lemma 1, we have \( S_{i,j} \perp S_{i,j'} \).

The orthogonal sequences can be applied to the S-CDMA systems successfully. But orthogonal sequences can not be applied to the Q-S-CDMA systems directly. This problem is solved in the next section by constructing large sets of orthogonal sequences with ZCZ property.

**IV. A FAMILY OF ORTHOGONAL SEQUENCES WITH ZCZ**

Based on the orthogonal sequences obtained in Section III, we next construct a family of sequences satisfying ZCZ properties, which implies that the sequences can be applied to Q-S-CDMA systems. Furthermore, the re-use distance \( D \) is the same with the orthogonal sequences obtained in Section III.

**Definition 4:** Let \( b = (b_0, b_1, \cdots, b_{N-1}) \) be a sequence with length \( N \), and \( A = (a_0, a_1, \cdots, a_{N-1}) \) be a row vector consisting of \( N \) sequences \( A_i \), \( 0 \leq i \leq N \). We define

\[
b \otimes A = (b_0 \cdot A_0, b_1 \cdot A_1, \cdots, b_{N-1} \cdot A_{N-1})
\]

is the Kronecker product operation.

**Definition 5 ([24], [25]):** A length \( N \) sequence \( A = (a_0, a_1, \cdots, a_{N-1}) \) is a 3-phase sequence if for \( 0 \leq i \leq N, a_i \in \{1, \omega, \omega^2\} \), where \( \omega = e^{2\pi i/3} \). The set of three 3-phase sequence triad \( \{A, B, C\} \) is a Golay complementary triad (GCT) if \( R_A(\tau) + R_B(\tau) + R_C(\tau) = 0 \), for \( 0 \leq \tau \leq N - 1 \).
Table 2 gives the total number of 3-phase GCTs and total number of sequences available till date for various lengths. For example, for $N = 5$, $A = \{1 1 1 \omega \omega \} \cap B = \{1 \omega \omega^2 \omega^2 \omega \}$ and $C = \{1 1 \omega^2 \omega^2 \omega^2 \}$, we have $R_A(\tau) + R_B(\tau) + R_C(\tau) = 0$, for $1 \leq \tau \leq 4$; For $N = 6$, $A = \{1 \omega^2 1 1 \omega^2 \}$, $B = \{1 \omega^2 \omega^2 \omega^2 \omega^2 \omega \}$ and $C = \{1 \omega \omega \omega \}$, we have $R_A(\tau) + R_B(\tau) + R_C(\tau) = 0$, for $1 \leq \tau \leq 5$.

![Figure 1](image_url)

**FIGURE 1.** Assignment of orthogonal sets to a lattice of regular hexagonal cells.

Then we can obtain $R_{A_1A_2}(\tau) + R_{B_1B_2}(\tau) + R_{C_1C_2}(\tau) = 0$, for $0 \leq \tau \leq 6$.

**Construction 3:** Let $S_{c, \alpha}$ be the sequence set generated by the section III, where $p = 3$. Let $\{A_1, B_1, C_1\}$ be a GCT with length $N$ defined as in Definition 5. The sequence set $A = \{A_1, A_2, \cdots , A_i\}$ is $p^m$-tuples, $B = \{B_1, B_2, \cdots , B_i\}$, $C = \{C_1, C_2, \cdots , C_i\}$. We construct $3k^2 + 1$ disjoint sequence sets of $(3mN + 2N - 2)$-length with ZCZ width $N$ as follows:

$$T_{c, \alpha} = \{T_{c, \alpha}^i | 0 \leq i \leq 3^{m-1} - 1\}$$

$$= \{(S_{c, \alpha}^i \otimes A, 0_{N-1}, S_{c, \alpha}^i \otimes B, 0_{N-1}, S_{c, \alpha}^i \otimes C) | \sum_{i=0}^{k} c_i = k\}$$

(21)

**Theorem 3:** Let $p = 3$, for $c \in \{1, 2, 3, \cdots , 3^k\}$, $\alpha \in \mathbb{F}_3$, let the sequence set $T_{c, \alpha}$ be defined by (21) as in Construction 3. Then $T_{c, \alpha}$ forms a $(3mN + 2N - 2)$-length ZCZ sequence set with ZCZ $N$. Let $T_{c, \alpha}^i, T_{c', \alpha'}^i$ be two sequences taken from the sequence sets $T_{c, \alpha}$ and $T_{c', \alpha'}$, respectively. Then we can obtain

$$R_{T_{c, \alpha}^i, T_{c', \alpha'}^j}(\tau) = 0, \text{ if } 0 < \tau < N,$$

$$R_{T_{c, \alpha}^i, T_{c', \alpha'}^j}(0) = 0, \text{ if } c = c', \alpha = \alpha' \text{ and } i \neq j.$$
If \( T_{c,a} \) and \( T_{c,a'} \) are in two non-adjacent cells, then
\[
|R_{T_{c,a}T_{c,a'}}(0)| \leq 3^{(m+2)/2} + 1 N.
\]

When \( \tau \neq 0 \), we consider the following case, \( T_{c,a}^i = (S_{c,a}^i \otimes A, \alpha, 0 N \alpha, S_{c,a}^i \otimes B, 0 N \alpha, S_{c,a}^i \otimes C) \) and \( T_{c,a'}^j = (S_{c,a'}^j \otimes A, 0 N \alpha, S_{c,a'}^j \otimes B, 0 N \alpha, S_{c,a'}^j \otimes C) \).

Note that
\[
R_{T_{c,a}^iT_{c,a'}^j}(\tau) = 3^{m-1} - 1 \sum_{i=0}^{3^{m-1}} s_i s_i^* R_A(\tau) + \sum_{i=0}^{3^{m-2}} s_{i+1} s_{i+1}^* R_B(\tau) + R_C(\tau) = 0.
\]

We can determine that \( T_{c,a} \) is a ZCZ sequence set with \( Z_{c,a} = N \). □

**Remark 2:** The \( p^{k+1} \) sets \( T_{c,a}^i, c \in \{1, 2, \ldots, p^k\} \), \( \alpha \in \mathbb{F}_p \), can be arranged as Fig. 1 in a similar way, where we just replace \( S_{c,a} \) with \( T_{c,a} \). At the same time, the orthogonality is updated to ZCZ property accordingly. Note that the number of sequences is \( p^{m-1} \) in each \( T_{c,a} \).

**Example 2:** Let \( N = 6 \), the set of three 3-phase Golay triad, for \( 0 < \tau < N \), and \( 0 < N - \tau < N, R_A(\tau) + R_B(\tau) + R_C(\tau) = 0 \) holds. Then for \( 0 < \tau < N \), we can deduce
\[
R_{T_{c,a}^iT_{c,a'}^j}(\tau) = 0.
\]

Table 3: Parameters of ZCZ sequences.

| literature | length | ZCZ | \( p \)-phase sequences |
|------------|--------|-----|-------------------------|
| [12]       | \( 2^{m+1} \cdot N + N - 1 \) | \( N \) | 2-phase |
| [23]       | \( 2^{m+1} \cdot N + N - 1 \) | \( N \) | 2-phase |
| Th. 3      | \( 3^{m+1} \cdot N + 2N - 2 \) | \( N \) | 3-phase |
| Col. 1     | \( 3^{m+1} \cdot N + 4N - 2 \) | \( 2N \) | 3-phase |
| Col. 2     | \( p^{m+2} + (p - 1)^2 \) | \( p \) | \( p \)-phase, \( p \geq 3 \) |

We construct \( 3^{k+1} \) disjoint sequence sets of \((3m+1)N + 4N - 2\)-length with ZCZ width \( 2N \) as follows:
\[
T_{c,a} = \{ T_{c,a}^i | 0 \leq i \leq 3^{m-1} - 1 \}
= \{ (S_{c,a}^i \otimes A, 0 N \alpha, S_{c,a}^i \otimes B, 0 N \alpha, S_{c,a}^i \otimes C) | 0 \leq i \leq 3^{m-1} - 1 \}.
\]

Corollary 1 has a similar proof to Theorem 3 and is therefore omitted here.

**Remark 3:** Corollary 1 obtain a class of ZCZ sequence sets with \( Z_{c,a} = 2N \), which has larger ZCZ width compared with Theorem 3.

**Definition 6:** ( [26], [27]) Let the sequence set \( A = \{ A_1, A_2, \ldots, A_{M-1} \} \), and \( A_j = \{ a_{j0}, a_{j1}, \ldots, a_{j(M-1)} \} \), where \( a_{ij} \) is a sequence with length \( N \). We call \( A \) a \( (M, M, N) \)-complete complementary code (CCC), if
\[
R_{A} = \sum_{l=0}^{M-1} R_{A_{l}A_{j}}(\tau) = \begin{cases} 0, & 0 < \tau \leq N - 1, i = j, \\ 0, & 0 \leq \tau \leq N - 1, i \neq j. \end{cases}
\]

**Corollary 2:** Let \( S_{c,a} \) be the sequence set generated by the section III. Let \( A = \{ A_0, A_1, \ldots, A_{p-1} \} \) be a \((p, p, p)\)-CCC, and \( A_j = \{ a_{j0}, a_{j1}, \ldots, a_{j(M-1)} \} \), where \( a_{ij} \) is a sequence with length \( p \) and \( 0 \leq j, l \leq p - 1 \). Defining \( A_{j} = \{ a_{j0}, a_{j1}, \ldots, a_{ji} \} \) with \( p^{m} \)-tuples, we construct \( 3^{k+1} \) disjoint sequence sets of \((p^{m+2} + (p - 1)^2)\)-length with ZCZ as follows:
\[
T_{c,a} = \{ T_{c,a}^i | 0 \leq i \leq p^{m-1} - 1, 0 \leq j, l \leq p - 1 \}
= \{ (S_{c,a}^i \otimes A_{j0}, 0 p - 1, S_{c,a}^i \otimes A_{j1}, 0 p - 1, \ldots, \times S_{c,a}^i \otimes A_{j, p - 1}) | 0 \leq i \leq p^{m-1} - 1, 0 \leq j, l \leq p - 1 \}.
\]

In each ZCZ sequence set, we have \( p^m \) constituent sequence. Corollary 2 has a similar proof to Theorem 3 and is therefore omitted here.

**Remark 4:** Corollary 2 obtains a class of ZCZ sequence sets with \( Z_{c,a} = N \), and the number of constituent sequences is \( p^m \) in each sequence set, which has a larger number of sequences compared with Theorem 3.

**Remark 5:** In Table 3, we compare the main parameters of ZCZ sequence. Comparison with the literature [12] and [23]. A class of \( p \)-phase ZCZ sequence sets is constructed, which the length of sequence is \( 3^{m+1} \cdot N + 2N - 2 \) in Theorem 3. This is the first time we obtain a ZCZ sequence with new length. In Corollary 1, a class of ZCZ sequence sets is presented with ZCZ width \( 2N \) of \( 3^{m+1} \cdot N + 4N - 2 \) length. The ZCZ sequence sets in Corollary 1 have larger ZCZ width than sequence sets.
in Theorem 3. In Corollary 2, a family of ZCZ sequence sets is proposed based on \( p \)-phase CCC, which has \( p^{m+2} + (p - 1)^2 \) length.

In regular tessellation of hexagonal cells, the constructed ZCZ sequence sets have the same reuse distance as orthogonal sequence sets constructed in Theorem 1.

V. CONCLUSION

In this article, we construct a class of orthogonal sequences over \( GF(p) \), and assign them to a tessellation of hexagonal cells. The method increases the number of sequences to be \( p^{m-1} \) per cell in the network. Compared with binary sequences, non-binary sequences have more sequences in each sequence set. Next we extended the orthogonal sequences to ZCZ sequences which can be applied to QS-CDMA system, and analyzed the properties of the new sequence sets. For the sequence set in each cell, the ZCZ sequence sets not only have the same \( m \)-use distance as the orthogonal sequences, but also have large zero correlation zone which can be applied to QS-CDMA system.

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