GRAPH FILTERING OVER EXPANDING GRAPHS

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1. INTRODUCTION

Graph filters are parametric information processing architectures for network data with wide applicability in signal de-noising, recommender systems, semi-supervised learning, and graph-based dictionary representations. By relying on information exchange between neighbouring nodes, graph filters extend the convolution operation to the graph domain. In turn, by learning the convolution parameters, they can account for the network data-topology coupling to solve the task at hand. However, graph filters are used only for graphs with a fixed number of nodes despite the evidence that practical graphs often grow in size. Filtering network data in this setting is challenging not only because of the increase in graph dimension but also because we do not know how the new nodes attach to the graph.

The importance of processing data over expanding graphs and the challenges arising when learning a filter on them have been recently recognized in a few works. Authors in [4] focus on semi-supervised learning with incoming nodes. First, a filter is learned to solve the task on the existing nodes, and then the filter output is used as a feature vector to predict the label of a new incoming data-point node. The work in [15] learns a graph filter-based neural network over a sequence of growing graphs, which are generated from a common graphon model. However, the generated graphs are not related to each other. The works in [17, 18] perform regression over a sequence of growing graphs, which are generated from a common graphon model. However, the generated graphs are only because of the increased dimensions but also because the connectivity is known only up to an attachment model. We propose a filter learning scheme for data over expanding graphs by relying only on such a model. By characterizing the filter stochastically, we develop an empirical risk minimization framework inspired by multi-kernel learning to balance the information inflow and outflow at the incoming nodes. We particularize the approach for denoising and semi-supervised learning (SSL) over expanding graphs and show near-optimal performance compared with baselines relying on the exact topology. For SSL, the proposed scheme uses the incoming node information to improve the task on the existing ones. These findings lay the foundation for learning representations over expanding graphs by relying only on the stochastic connectivity model.

2. PROBLEM FORMULATION

Consider a graph $G = (\mathcal{V}, \mathcal{E})$ with node set $\mathcal{V} = \{v_1, \ldots, v_N\}$, edge set $\mathcal{E}$, and adjacency matrix $A$. An incoming node $v_n$ attaches to $G$ and forms two sets of directed edges: a set $\{v_n, v_+\}$ starting from $v_+$ and landing at the existing nodes $v_o$, whose weights are collected in vector $b^+_o \in \mathbb{R}^N$; and another set of $\{v_+, v_n\}$ starting from the existing nodes $v_n$ in $\mathcal{V}$ and landing at $v_+$, whose weights are collected in vector $a^+_n \in \mathbb{R}^N$. We represent these connections with two directed graphs $G^+_{v_+}$ and $G^+_n = (\{v \cup v_+\}, \mathcal{E} \cup \{(v_n, v_+)\})$ whose adjacency matrices are

$$A^+ = \begin{bmatrix} A & b^+_o \\ 0^\top & 0 \end{bmatrix} \quad \text{and} \quad A^+_n = \begin{bmatrix} A & 0 \\ a^+_n^\top & 0 \end{bmatrix}$$

respectively and where $^\top$ denotes the transpose and 0 the all-zero vector. A conventional way to model the attachment of incoming nodes is via stochastic models in which node $v_n$ connects to $v_+$ with probability $p_i$ and weight $w^+_i$ in graph $G^+_{v_+}$, and probability $p_j$ and weight $w^+_j$ in graph $G^+_n$. Hence, $b^+_o$ and $a^+_n$ are random vectors with expected values $\mu = w^+_o \Sigma^o$ and $\mu = w^+_n \Sigma^n$, and covariance matrices $\Sigma^o$ and $\Sigma^o$, respectively. Here we define $w = [w_1^+, \ldots, w_N^+]^\top$, $w^o = [w_1^o, \ldots, w_N^o]^\top$ and denote by $\circ$ the Hadamard product.
While analyzing expanding graphs is an important topic, in this paper we are interested in processing data defined over the nodes of these graphs. Let then $x_+ = [x_1^+, \ldots, x_N^+]^\top \in \mathbb{R}^{N+1}$ be a set of signal values over nodes $\mathcal{V}$ in which vector $x = [x_1, \ldots, x_N]^\top \in \mathbb{R}^N$ collects the signals for the existing nodes $\mathcal{V}$ and $x_+$ is the signal at the incoming node $v_+$. Processing signal $x_+$ amounts to designing graph filters that can capture its coupling w.r.t. the underlying directed graphs $\mathcal{G}_+^a$ and $\mathcal{G}_+^o$. To do so, we consider a filter bank of two convolutional filters (2), one operating on graph $\mathcal{G}_+^a$ and one on graph $\mathcal{G}_+^o$. Mathematically, with $h^a = [h_0^a, \ldots, h_d^a]^\top$ and $h^o = [h_0^o, \ldots, h_d^o]^\top$ representing the vector of coefficients for filters $H^a(A_+^a)$ and $H^o(A_+^o)$, respectively, the filter bank output is

$$y_+ = y_+^a + y_+^o := \sum_{l=0}^L h_l^a [A_+^a]^l x_+ + \sum_{m=0}^M h_m^o [A_+^o]^m x_+$$

(2)

where without loss of generality we consider $y_+ = [y_+^a, y_+^o]^\top$ and $y_+^a$ and $y_+^o$ are the outputs over graphs $\mathcal{G}_+^a$ and $\mathcal{G}_+^o$, respectively.

The stochastic nature of the attachment yields a random $y_+$. It is also notoriously challenging to compute statistical moments of or higher-order moments of the attachment pattern of the incoming node $v_+$. However, because of the decoupled nature between the incoming and outgoing edges at node $v_+$, the $k$th powers of the adjacency matrices have the block structure

$$[A_+^a]^k = \begin{bmatrix} A^k & A^{k-1} b^a \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad [A_+^o]^k = \begin{bmatrix} A^k & 0 \\ a_0^o A^{k-1} & 0 \end{bmatrix}$$

(3)

which facilitate the stochastic analysis of the filter output [cf. Sec. 3].

Given the filter bank in (2), our goal translates into estimating the filter coefficients $h^a$ and $h^o$ to solve specific learning tasks in a statistical fashion (21). Specifically, we consider a training set $\mathcal{T} = \{(v_+, x_+, t_+)\}$ comprising a set of incoming nodes $v_+$ w.r.t. a fixed existing graph $\mathcal{G}$, an expanded graph signal $x_+$ (e.g., noisy or partial observations), and a target output signal $t_+ = [t_1^+, \ldots, t_L^+]^\top$ (e.g., true signal, or class labels). Then, we learn the filter by solving

$$\min_{h^a, h^o} \frac{1}{2\gamma} \mathbb{E}[f_r(h^a, h^o, t_+)] + \frac{1}{2\alpha} g(h^a) + \frac{1}{2(1-\alpha)} j(h^o)$$

(4)

where $\mathbb{E}[f_r(h^a, h^o, t_+)]$ is the expected task-specific loss with the expectation taken w.r.t. both the graph attachment vectors $b^a_+, a^o_+$, and the data distribution; and $g(\cdot)$, $j(\cdot)$ are filter-specific regularizers (e.g., norm two of coefficient vectors) to avoid overfitting. The regularization weight $\gamma > 0$ controls the trade-off between fitting and regularization and scalar $0 < \alpha < 1$ balances the impact between the two filters inspired by multi-kernel learning (22).

In the next section, we shall particularize problem (4) to graph signal de-noising (Sec. 2) and graph-based semi-supervised learning (Sec. 3.3). For both settings, we shall characterize the filter output stochastically, use its first- and second-order moments in (4), and show the role played by the attachment models on $\mathcal{G}$ and $\mathcal{G}_+^a$.

3. FILTERING WITH INCOMING NODES

Before defining the filter bank in (2) for the two tasks, we first rearrange it in a compact form, instrumental for our analysis. This form will also show the influence of the incoming node connectivity on the filter output.

3.1. Compact Form

The goal of this section is to isolate the filter coefficients $h = [h^a_1, h^o_1]^\top$ and write (2) as $y_+ = W h$, where $W^a_+$ contains the coupling between the stochastic expanded graphs and the signal.

Analyzing filter $H^a(A_+^a)$, it is possible to write its output as

$$y_+^a = [x_+, A_+^a x_+, \ldots, [A_+^a]^d x_+] [h^a_1]$$

(5)

Then, leveraging the structure of the input $x_+ = [x^T, x_+]^\top$ and the block-structure of $[A_+^a]^k$ in (3), we can write (5) as

$$y_+ = \frac{L_x}{x_L} h^a = \left[ \sum_{x_L} x_L x^L \right] h^a$$

(6)

where $\frac{L_x}{x_L} = x_L + x_L \mathcal{T}$, $x_L = [x, A x, \ldots, A^L x]^\top$ and $\mathcal{T} = [0, b^a_0, A^{d-1} b^a_1] \in \mathbb{R}^{L+1}$. Eq. (6) shows that the output $y^a_+$ on the existing nodes $v_+$ is influenced by propagating their own signal $x$ [cf. $L_x$] and by propagating signal $x_+$ of $v_+$ w.r.t. the incoming attachments $b^a_0$ [cf. $\mathcal{T}$]. Instead, the filter output at the incoming node $y^o_+$ is just a scaled version of the input $x_+$ by coefficient $b^o_0$. The latter is because edges on graph $G$ leave node $v_+$ and land on $V$; hence, governing the direction of the signal propagation.

Likewise, analyzing filter $H^o(A_+^o)$, we can write its output as

$$y_+^o = [x_+, A_+^o x_+, \ldots, [A_+^o]^d x_+] [h^o_1]$$

(7)

Leveraging again the structure of the input and that of the matrix powers $[A_+^o]^k$ in (3), allows writing (7) as

$$y_+^o = \frac{M_x}{x_M} h^o = \left[ \sum_{x_M} x_M x^M \right] h^o$$

(8)

where $M_x = [x, A x, \ldots, A^M x]$ is an $N \times (M + 1)$ matrix, $\frac{M_x}{x_M} = \left[ \sum_{x_M} x_M x^M \right] \in \mathbb{R}^{N+1}$ and $\mathcal{T} = [x_0, x_1, \ldots, A^{M-1} x_0]$ are of dimensions $(M + 1) \times 1$ and $N \times (M + 1)$, respectively, and $x_M = [x_0, x_1, \ldots, x_M] \in \mathbb{R}^{M+1}$. That is, the output $y_+^o$ on the existing nodes $v_+$ is influenced only by propagating their own signal $x$. Instead, the output $y_+^o$ at node $v_+$ comprises: (i) the match between the attachment pattern $a^o_+^T$ and the signal shifted over the existing graph $\mathcal{G}$; i.e., $a^o_+^T M_x$; and (ii) a scaled version of its own signal $x_+$ by coefficient $b^o_0$. The output on the existing nodes $v_+$ is not influenced by signal $x_+$ because the edges in $G$ leave those nodes and land on $v_+$. The latter is also justified by the structure of matrix $M_x$; i.e., the existing signal $x$ is first percolated over $G$ and then mapped onto $v_+$ through its attachment pattern $a_+$ in a matched filtering principle (23).

Bringing together (6) and (8), leads to the desired compact form

$$y_+ = y_+^a + y_+^o = W h$$

(9)

where $W = \left[ \begin{array}{c} \frac{L_x}{x_L} M_x \\ \frac{M_x}{x_M} \end{array} \right]$. The block trace operator takes as arguments matrix $Z$ and an $N \times N$ matrix $Y$ to yield a matrix $U = \text{blktr}(Z, Y)$ with $(i, j)$ entry $U_{ij} = \text{tr}(Y Z_{ij})$ and $\text{tr}(-)$ being the trace operator.

3.2. Statistical Identity

Throughout the statistical analysis of the filter output $y_+$, we will deal with expectations of the form $\mathbb{E}[\tilde{L}^a_+ C \tilde{M}^o_+]$ for some $N \times N$ square matrix $C$. In the remainder of this section, we derive a handy formulation for it by using the compact form (9). For this, we will need the block-trace operator as defined next.

Definition 1 (Block trace). Let $Z$ be a block matrix comprising $N \times N$ sub-matrices $Z_{ij}$. The block trace operator takes as arguments matrix $Z$ and an $N \times N$ matrix $Y$ to yield a matrix $U = \text{blktr}(Z, Y)$ with $(i, j)$ entry $U_{ij} = \text{tr}(Y Z_{ij})$ and $\text{tr}(-)$ being the trace operator.
Lemma 1. Given an existing graph $G$ with a noisy signal graph $x = t + n$, where $t$ is the true signal and $n$ a Gaussian noise $N(0, \sigma^2 I)$. Consider also matrices $L_x$ and $M_x$ [cf. (6) and (8)], which can be expanded further as

$$L_x = L(L_{t+1} \otimes x) \quad \text{and} \quad M_x = M(I_{M+1} \otimes x)$$

where $L = [I, A, \ldots, A^L]$, $M = [I, A, \ldots, A^M]$, $I_N$ is the $N \times N$ identity matrix, and $\otimes$ is the Kronecker product. Then, for any $N \times N$ square matrix $C$ the following identity holds:

$$E[|L_x^T CM_x|^2] = L_x^T CM_x + \sigma^2 \text{blktr}(L^T CM_x, I_N)$$

where $L_x = L_x|_{x=0}$, $M_x = M_x|_{x=0}$, and blktr() is the block operator in Def. 7.

Proof. See the appendix in the Supplementary Material. \qed

3.2. Signal Denoising

The first task we are interested in is recovering a true signal $t_x$ from its noisy observations $x_n$ by knowing only the stochastic attachment pattern of the incoming node. For this, we consider as cost the mean squared error $E[\|f_x^h(t_x, t_n)\|_2^2] = E[\|W_h t_n - t_x\|_2^2]$, where $D = \text{diag}(d_1, \ldots, d_{N+1}) \in \{0, 1\}^{N+1 \times N+1}$ is a diagonal matrix with $d_n = 1$ only if account for the MSE at node $n$ and zero otherwise. The following proposition quantities the latter.

Proposition 1. Given a graph $G = (V, E)$ with adjacency matrix $A$ and an incoming node $v_{i+1}$ connecting to $G$ with random attachment vectors $b_l$ and $b_o$ with respective means $\mu_l$ and $\mu_o$ and covariance matrices $\Sigma_l, \Sigma_o$ [cf. (1)]. Consider a noisy signal $x_n = t_n + n_n$ over the nodes $V \cup v_{i+1}$, with $n = [x_n^T, x_{i+1}]$ and $t_n \sim N(0, \sigma^2 I_{N+1})$. Let also $y_n = W_h t_n$ [cf. (2)] be the filtered output. Then, the MSE of the filter output $MSE_D(h) = E[\|W_h t_n - t_x\|_2^2]$ computed on a set of nodes sampled by the diagonal matrix $D = \text{diag}(d_1, \ldots, d_{N+1}) \in \{0, 1\}^{N+1 \times N+1}$ is

$$MSE_D(h) = h^T \Delta h + 2h^T \Theta + \|t_x\|^2_D$$

where $\Delta = [\Delta_{11}, \Delta_{12}; \Delta_{21}, \Delta_{22}]$ is a $2 \times 2$ block matrix with:

$$\Delta_{11} = L^T L_D L_t + \sigma^2 \text{blktr}(L^T L_D L_I, I_N) + t_x L_D \text{Diag}(\mu_l) + t_o L_D \text{Diag}(\mu_o) + t_x^T \mu_o L_D \text{Diag}(\mu_o) + t_o^T \mu_l L_D \text{Diag}(\mu_l) + \text{blktr}(L^T L_D \Sigma_l) + \text{diag}(t_x^T + \sigma^2, 0_L)$$

where $D = \text{diag}(d_1, \ldots, d_{N+1})$ contains the indices of the sampled nodes in $G$, $L = [I, \ldots, A^L]$, $L_t = [t, A, \ldots, A^L t]$, $L_I = [0, I, \ldots, A^L I]$, $L_o = [t, A, \ldots, A^L o]$, and $L_o = [t, A, \ldots, A^L o]$ with $(t_x^T + \sigma^2)$ in location $(1, 1)$ and zero elsewhere.

$$\Delta_{12} = \Delta_{21} = L^T L_M t_o + t_o^T \mu_l L_M t_o + \text{blktr}(L^T L_M \Sigma_o, I_N) + \text{diag}(t_o^T + \sigma^2)$

where $M = [I, \ldots, A^M]$, $M_t = [t, A, \ldots, A^M t]$, $M_o = [t, A, \ldots, A^M o]$, $M_{L,M} = [t_x, o, \ldots, t_x, o]$ with $(t_x^T + \sigma^2)$ in location $(1, 1)$ and zero elsewhere.

$$\Delta_{22} = M_x^T D_M x + \sigma^2 \text{blktr}(M^T D_M x, M_I, N_I) + \text{diag}(t_x^T + \sigma^2, 0_M)$$

where $M_x = [0, I, \ldots, A^M - 1]$, $M_o = [t, A, \ldots, A^M - 1]$, $M_L = [t_x, o, \ldots, t_x, o]$ with $(t_x^T + \sigma^2)$ in location $(1, 1)$ and zero elsewhere.

Proof. See the appendix in the Supplementary Material. \qed

The MSE in (12) is governed by the interactions between the statistics of the attachment vectors, and those of the percolated signals $t$ and $\mu$. Using it as the cost, problem (4) becomes

$$\min_{h \in [h^0, h^{N+1}], \Theta} \frac{1}{2}\text{MSE}_D(h) + \frac{1}{2\alpha} \|h\|^2_2 + \frac{1}{2(1 - \alpha)} \|h^o\|^2_2$$

Scalar $\gamma > 0$ controls how much we want to reduce the MSE over the nodes in $D$; for $\gamma \to 0$ the importance of minimizing the MSE increases, while for $\gamma \to \infty$ it decreases. Instead, scalar $\alpha \in [0, 1]$ controls the role of the filters $H(A^I)$ and $H_o(A^o)$ in (2). For $\alpha \to 0$ we prioritise more the filter over graph $G^I$; i.e., leverage the information on the existing nodes $V$ towards the incoming node $v_1$. And for $\alpha \to 1$ we prioritise the filter over graph $G^o$; i.e., leverage the information on the incoming node $v_1$ towards the existing nodes $V$. Note that such a formulation would work for any additive noise or attachment model as long as respective parameters are known.

Problem (17) is quadratic and convex only if matrix $A$ is positive semi-definite (PSD). However, proving the latter is challenging because of the structure of this matrix; hence, we can find local minimum via descent algorithms [24]. But since we estimate $\Delta$ from the training set $\mathcal{T}$, we can check if it is PSD and for a positive outcome we can find the closed-form solution for (17)

$$h^* = (\Delta + 2\gamma A)^{-1} \Theta$$

where matrix $A = [1/2\alpha I_{L+1}, 0, 0, 1/2(1 - \alpha) I_{M+1}] \in \mathbb{R}^{(M+L+2) \times M+L+2}$ and subscript $f$ indicates that these quantities are estimated from data.
filter output smoothness w.r.t. graphs $G^+_a$ and $G^+_o$. For $\beta \to 0$, we bias filter $H'(A'_+)$ to give an output $y_{t+}^+$ that is smooth over graph $G^+_a$ and to ignore the behavior of filter output $y_{t+}^+$ over graph $G^+_o$. This may be useful when the connectivity model of $v_t$ respects the clustering structure of $G$. The opposite trend is observed for $\beta \to 1$.

The MSE in (19) is of the form (17) and encompasses both SSL cases with clean labels ($\sigma^2 = 0$) and noisy labels ($\sigma^2 > 0$). In (19) also have the expected signal 2–Dirichlet form that influences the filter behavior. The following proposition quantifies it.

**Proposition 2.** Given the setting of Proposition 2 and considering that $x_{t+} = L x_t$, the 2–Dirichlet forms of the filter outputs $y_{t+}^+$ and $y_{t+}^o$ over graphs $G^+_a$ and $G^+_o$ respectively

$$\mathbb{E}[S_2(y_{t+}^+)] = h^T \Psi h^+$$

and

$$\mathbb{E}[S_2(y_{t+}^o)] = h^T \Psi h^o$$

where

$$\Psi^+ = (L + t \bar{L}_{\mu})^T \Gamma (L + t \bar{L}_{\mu}) + t \bar{L}_{\mu} \text{blktr}((L + t \bar{L}_{\mu})^T \Gamma L, \Sigma) - 2t \bar{L}_{\mu} \text{blktr}(L \Sigma) - t \bar{L}_{\mu} \text{blktr}(L \Sigma) + w^T \mathbf{1}^T$$

and

$$\Psi^o = M^+ + t \bar{M}_{\mu} \text{blktr}((L + t \bar{L}_{\mu})^T \Gamma L, \Sigma) - t \bar{M}_{\mu} \text{blktr}(L \Sigma) + w^T \mathbf{1}^T$$

are matrices that capture the attachment patterns and label propagation on the edges of the incoming node and $\Gamma = (I - A)^T (I - A)$.

**Proof.** See the appendix in the Supplementary Material. □

The expected 2–Dirichlet forms depend on the attachment statistics in two ways: first, the expected attachments $\mu$ and $\mu^o$ control the label percolation from and towards the incoming node; second the in-attachment covariance $\Sigma$ and the out-attachment covariance $\Sigma^i$ influence the percolated labels through $\bar{L}$ and $\mathbf{A}$ and $M$; $\mathbf{M}_{t}$ and $\bar{M}_{t}$. Using then (20) in (19), we get

$$\min_{h} \frac{1}{2} \mathbb{E} \Sigma y^+ (h) + h^T \mathbf{A} h + h^T \mathbf{A} \Omega h$$

where $h = (h^+, h^o)^T$, and $\Sigma = [1/2 \Psi^+, 0, 0, 1/2 (1 - \beta) \Psi^o]$ is an $(M + L + 2) \times (M + L + 2)$ matrix. As for (19) proving convexity for (23) is challenging but solvable with descent algorithms. And if empirically we observe that the matrix in the quadratic form of $h$, $\Delta_T + \Lambda + \Omega$ is PSD, the solution of (23) is given by

$$h^+ = \left(\Delta_T + 2\gamma (\Lambda + \Omega_T + \Omega_T^+)\right)^{-1} \theta_T$$

where again the subscript $T$ indicates that the respective quantities are estimated from data.

**4. NUMERICAL RESULTS**

This section compares the proposed method with competing alternatives to illustrate the trade-offs inherent to graph filtering over expanding graphs with synthetic and real data. Our numerical tests have been focused to answer the following research questions:

**RQ.1.** How does the proposed approach compare with baselines that utilize the known attachment?

That is, we want to understand to what extent the proposed empirical learning framework compensates for the ignorance of the true connection.

To answer this question, we compare with two baselines:

1. **Single filter with known connectivity ($KC_1$)**: This is the intuitive solution where the incoming node $v_t$ connects to the nodes in $V$ forming a single graph $G^+_o = (V \cup v_t, E^+_o)$, in which set $E^+_o$ collects both the known incoming and outgoing edges w.r.t. $v_t$. Then, a single filter is trained on $G^+_o$ as conventionally done by the state-of-the-art. This comparison validates the proposed scheme over the conventional strategy.

2. **Filter bank with known connectivity ($KC_2$)**: This is the proposed filter bank scheme in (2) with the known connectivity of node $v_t$. The rationale behind this choice is to factorize the filter degrees of freedom since $KC_1$ employs a single filter and to highlight better the role of the topology.

**RQ.2.** How much does the information of the attached node contribute to the task performance over the existing graph?

We want to understand if the proposed approach exploits the incoming node signal without knowing the topology to improve the task over the existing graph instead of ignoring such information.

**RQ.3.** How does the proposed model compare on the incoming node w.r.t. inductive graph filtering?

Since graph filters have inductive bias capabilities [10], they can be learned on the existing graph $G$ and then transferred to expanded graphs without retraining. We want to understand if learning with a stochastic model is more beneficial than transference. To answer RQ.2 and RQ.3, we compare with:

3. **Inductive transference (IT)**: I.e., we employ a single filter to solve the task over the existing graph and transfer it on the expanded graph under the same attachment model.

For all experiments, the incoming node attaches to the existing nodes ($G^+_a$) uniformly at random with $p^i = 1/N$, and have edges landing at itself ($G^+_o$) with a preferential attachment $p^o = d/\mathbf{1}^T d$ where $d$ is the degree vector; i.e., they are likely to form links with nodes having a higher degree. These standard attachment rules have been observed in the study of evolving real world networks. The covariance matrices are estimated from 10, 000 generated samples of their respective attachment vectors. For simplicity, we set the expanded graph weights $w^i = w^o = w\mathbf{1}$ with $w$ being the median of the non-zero existing edge weights in $G$. We fixed the filter orders to $L = M = 4$ and considered also a filter order of four for $KC_1$ and IT. We performed a 70 – 30 train-test data split and selected parameters $\gamma \in [10^{-3}, 10], \alpha, \beta \in [0, 1]$ via five-fold cross-validation. We averaged the testing performance over 100 realizations per test node.

**4.1. Denoising**

Following the paper outline, we first answer the RQs for the denoising task over a Barabasi-Albert (BA) graph model and the NOAA temperature data-set [28].

**Experimental setup.** For the BA model, we considered an existing graph of 100 nodes and 1000 incoming node realizations. For each realization, we generated a bandlimited graph signal by randomly mixing the first ten eigenvectors with the smallest variation of $A_{\alpha} \in \mathbb{R}^{101 \times 101}$ [7]. For the NOAA data set, we considered hourly temperature recordings over 109 stations across the continental U.S. in 2010. We built a five nearest neighbors (5NN) graph $G$ of $N = 100$ random stations as in [29][30]. We treated the remaining nodes as incoming, each forming 5NN on $G$ and 5NN on $G^o$. We considered 200 hours, yielding 1800 incoming data samples.

We corrupted the true signals with Gaussian noise of SNRs $\in \{5\text{dB}, 10\text{dB}, 20\text{dB}\}$. We measured the recovery performance over all existing and incoming nodes through the normalized mean square error (NMSE) and the normalized mean absolute error (NMAE).
squared error $\text{NMSE} = \frac{1}{|\mathcal{V}_{x}|} \sum_{v \in \mathcal{V}_{x}} \left| \frac{y_{v} - t_{v}}{\sqrt{t_{v}} - \sqrt{t_{v}}} \right|$ and we also measured the NMSE only at the incoming node and denote it as $\text{NMSE}_{+}$.

**Observations.** Table 1 reports the demeaning performance on both datasets. Overall, we observe that the proposed approach compares well with the two baselines relying on the exact topology ($\text{KC}_1$ and $\text{KC}_2$). As regards the performance at the incoming node $\text{NMSE}_{+}$, we see that not knowing the topology leads to a worse performance. However, we see that in the NOAA dataset the gap is much smaller; a potential explanation for this is may be in the NN nature of the graph. Regarding then the last two research questions, we see that the proposed approach performs comparably well w.r.t. IT on the existing graphs but outperforms it by a margin when it comes to the performance of the incoming node (NMSE$_{+}$). In turn, such findings show the advantages of the proposed scheme to keep a comparable performance with baselines relying on the exact topology and to improve substantially w.r.t. methods relying only on transfereuce.

### 4.2. Semi-supervised Learning

For SSL, we consider a synthetic sensor network graph from the GSP toolbox [31] and the political blog network [32].

**Experimental setup.** For the sensor network, the existing graph $\mathcal{G}$ has $N = 200$ nodes that are clustered into two classes ($\pm 1$) via spectral clustering to create the ground-truth. The training set $T$ comprises 500 realizations of incoming nodes each making the same number of incoming and outgoing as the median degree of $\mathcal{G}$. The ground-truth label at the incoming node is assigned based on the class that has more edges with $v_x$. For the blog network, we considered 1222 blogs as nodes of a graph with directed edges being the hyperlinks between blogs and labels being their political orientation ($+1$ conservative vs. $-1$ liberal). We built a connected existing graph $\mathcal{G}$ of $N = 622$ blogs with a balanced number of nodes per class. The remaining 600 blogs are treated as incoming nodes.

In both settings, we use only 10% of the labels in $\mathcal{G}$ and aim at inferring the missing labels in this graph by using also the information from the incoming node. These labels act also as the graph signal $|x_{+}|_{n} = \pm 1$ for a labelled node and $|x_{+}|_{n} = 0$ if unlabelled. We also considered two settings: first, all incoming nodes in the training set have labels (fully labelled), which allows identifying if the additional label contributes to the SSL task on $\mathcal{G}$; second, only half of the incoming nodes have labels (50% labelled), which adheres more to a real scenario where some of the incoming nodes are unlabelled. For the IT baseline, we solve the corresponding filters using [7], while for $\text{KC}_1$ and $\text{KC}_2$ we use the true connections. During training, standard SSL requires evaluating the loss at the nodes with available labels. Hence, when an incoming node has no label, we cannot account for its importance during training. Consequently, SSL models cannot predict labels when we do not know the connectivity. Thus, we measure only the performance of the existing nodes.

**Observations.** Tables 2 and 3 report the classification errors for the sensor and blog networks, respectively. The proposed approach achieves a comparable statistical performance with the two baselines ($\text{KC}_1$ and $\text{KC}_2$) that rely on the exact topology. This suggests that controlling the information in-flow and out-flow with a filter bank compensates effectively for the exact topology ignorance. The proposed approach reduces the error substantially compared to IT.

From these results, we also observe the models tend to perform better when 50% of the labels are present. We have identified two factors for this. First, some of the incoming nodes form misleading connections with both clusters. Hence, when their label diffuses it hampers the classification performance on the opposite cluster. Instead, when these nodes have no label they do influence the opposite class. This trend is observed also for $\text{KC}_1$ and $\text{KC}_2$, which shows that these wrong connections are present in the dataset. In the blog network, these are blogs with an unclear political position and have linked both with liberals and conservative groups [32]. Instead, in the sensor network, we do not see such a trend because nodes are better clustered. Second, this two-class classification problem has labels $\pm 1$ and we use the MSE as a criterion. Hence, the term $t_{v}^{\pm} = 1$ affects the costs when the incoming node is present [c.f. Prop[12] and does not help discriminating irrespective of the class. Thus, we conclude that when dealing with SSL classification in expanding graphs, the connectivity model plays also a central role in the performance.

### 5. CONCLUSION

We proposed a method to filter signals over expanding graphs by relying only on their attachment model connectivity. We used a stochastic model where incoming nodes connect to the existing graph, forming two directed graphs. A pair of graph filters, one for each graph, then process the expanded graph signal. To learn the filter parameters, we performed empirical risk minimisation for graph signal de-noising and graph semi-supervised learning. Numerical results over synthetic and real data show the proposed approach compares well with baselines relying on exact topology and outperforms the current solution relying on filter transference. However, the performance is strongly dependent on a fixed attachment model, prone to model mismatch. Hence, potential future works may consider a joint filter and graph learning framework for expanding graphs.
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Supplementary Material

This document contains the proofs of the main claims of the paper "Graph filtering over expanding graphs."

Proof of Lemma 1

Substituting \(L_x = L(I_{L+1} \otimes x)\) and \(M_x = M(I_{M+1} \otimes x)\) into \(E[L_x^T CM_x]\), we get
\[
E[L_x^T CM_x] = E[(I_{L+1} \otimes x)^T L_x^T CM(I_{M+1} \otimes x)]. \tag{S1}
\]
Substituting further \(L = [I, A, \ldots, A^{L-1}], M = [I, A, \ldots, A^M]\), the \((i, j)\)th block of \(L^T CM\), is
\[
(L^T CM)_{ij} = A^{-1}CA^{-1} \quad \text{for } i = 1, \ldots, L, j = 1, \ldots, M. \tag{S2}
\]
Incorporating the Kronecker products involving \(x\), we further write the \((i, j)\)th entry of \(S2\) as
\[
E[L_x^T CM_x]_{ij} = E[x^T A^{-1}CA^{-1}x] \tag{S3}
\]
since the expectation acts element-wise. Since the expectation argument is a scalar, we bring in the trace operator and leverage its cyclic property \(tr(XYZ) = tr(ZXY)\) to write
\[
E[x^T A^{-1}CA^{-1}x] = E[tr(xx^T A^{-1}CA^{-1})] \tag{S4}
\]
where remark the only random variable in \(S4\) is \(x\). Substituting then \(E[xx^T] = tt^\top + \sigma^2I_N\) in \(S3\), we get
\[
E[L_x^T CM_x]_{ij} = tr(tt^\top A^{-1}CA^{-1}) + \sigma^2 tr(A^{-1}CA^{-1}) \tag{S5}
\]
The first term in the R.H.S. of \(S3\) is \(L_x^T CM_x)_{ij} = t^\top A^{-1}CA^{-1}t\) with \(L_t = L_x \mid_{x=t}\) and \(M_t = M_x \mid_{x=t}\). Instead for the second term \(\sigma^2 tr(A^{-1}CA^{-1})\), we leverage \(S2\) and Def. 1 and note that it is the \((i, j)\)th element of \(blkr(L^T CM, \sigma^2I_N)\). Thus, the \((i, j)\)th element of \(E[L_x^T CM_x]\) is the sum of the \((i, j)\)th element of these two matrices, proving the Lemma.

Proof of Proposition 1

Expanding the MSE definition we get
\[
E[||W^+ h - t_+||_2^2] = h^\top E[W^+ D W^+] h + 2h^\top E[W^+ D t_+] + t_+^\top D t_+ \tag{S6}
\]
The first term contains the matrix \(\Delta := E[W^+ D W^+]\). Substituting \(W^+\) [cf. S2], we expect the argument becomes
\[
W^T D W_x = \begin{bmatrix} L_x^T & M_x^T \\ \tilde{M}_x & \tilde{M}_x \end{bmatrix} \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} L_x & M_x \\ \tilde{M}_x & \tilde{M}_x \end{bmatrix} = \begin{bmatrix} L_x D L_x + d_{N+1} x_L \tilde{M}_x & L_x D M_x + d_{N+1} x_L \tilde{M}_x \\ M_x D L_x + d_{N+1} \tilde{M}_x x_L & M_x D M_x + d_{N+1} \tilde{M}_x \tilde{M}_x \end{bmatrix} \tag{S7}
\]
which are related to the four blocks appearing in \(\Delta\) and where \(D = \text{diag}(d_1, \ldots, d_N)\).

\(\Delta_{11}\). The first block matrix in \(S7\) is \(\Delta_{11} := E[L_x^T D L_x + d_{N+1} x_L \tilde{M}_x]\). Substituting \(L_x = L_+ + x_+ \tilde{L}_x\) we get
\[
E[L_x^T D L_x] = E[(L_+ + x_+ \tilde{L}_x)^T D (L_+ + x_+ \tilde{L}_x)] \tag{S8}
\]
which is further composed of the following four terms:
- \(E[(L_+^T D L_+) + \sigma^2 \text{blkr}(L_+^T DL, I_N)]\), which follows directly from Lemma 1;
- \(E[x_+ L_+^T DL_+] = t_+ L_+^T DL_+\) given the noise and the attachments are independent of each other;
- \(E[x_+ L_+^T DL_+] = t_+ L_+^T DL_+\) under the same independence considerations;
- \(E[x_+ L_+^T DL_+] = t_+ L_+^T DL_+\). Under the independence between \(x_+\) and \(b_+\), and by using Lemma 1 on \(L_+^T DL_+\), we get
\[
E[x_+ L_+^T DL_+] = (t_+^2 + \sigma^2) (I + \tilde{M}_x^T \tilde{M}_x + \text{blkr}(\tilde{T}_x^T D \tilde{T}_x, \Sigma')) \tag{S9}
\]
where we also used the identity \(E[b_+ b_+^T] = \mu \mu^T + \Sigma'\).

In the expression of \(\Delta_{11}\) we also have the term \(E[d_{N+1} x_L x_L^T] = d_{N+1} \text{diag}(t_+^2 + \sigma^2, 0_L)\), which holds because \(x_L = [x_+, 0_L]\). Combining these, we get expression (13) for \(\Delta_{11}\).

\(\Delta_{12}\). The second block matrix in \(S7\) is \(\Delta_{12} := E[L_x^T D M_x + d_{N+1} x_L \tilde{M}_x]\). Substituting \(L_x \tilde{M}_x\) and \(\tilde{M}_x \tilde{M}_x\), we get
\[
\Delta_{12} = E[(L_+ + x_+ \tilde{L}_x)^T D M_x + d_{N+1} x_L \tilde{M}_x] \tag{S10}
\]
which is in turn composed of the following terms:
- \(E[L_+^T D M_x] := L_+^T D M_x + \sigma^2 \text{blkr}(L_+^T DL, I_N)\) which yields from Lemma 1;
- \(E[x_+ L_+^T DL_+] = t_+ L_+^T DL_+\) under the independence consideration;
- \(E[d_{N+1} x_L x_L^T] = d_{N+1} \text{diag}(t_+^2 + \sigma^2, 0_L)\) again under independence and where \(t_+ = [t_+^2, 0_L]\);
- \(E[x_L x_L^T] = E[t_+^2 + \sigma^2]\). Here, note that \(x_L = [x_+, 0_L]\) and \(x_M = [x_+, 0_M]\). Hence, \(E[x_L x_L^T]\) equals \(E[t_+^2 + \sigma^2]\) in position (1, 1) and zero elsewhere. Defining then matrix \(T_{12} \in R^{M+1 \times M+1}\) with \(t_+^2 + \sigma^2\) in location (1, 1) and zero elsewhere, we can write \(E[d_{N+1} x_L x_L^T] = d_{N+1} T_{12}\).

Combining then these derivations, we get expression (13) for \(\Delta_{12}\).

\(\Delta_{22}\). The third block matrix in \(S7\) is \(\Delta_{22} := E[M_x^T D L_x + d_{N+1} x_L \tilde{M}_x]\). It is easy to see that \(\Delta_{21} = \Delta_{12}\); hence, (13).

\(\Delta_{22}\). The fourth block matrix in \(S7\) is \(\Delta_{22} := E[M_x^T D M_x + d_{N+1} x_L \tilde{M}_x]\) for the first term on the R.H.S. of the latter we have
\[
E[M_x^T D M_x] = M_x^T D M_x + \sigma^2 \text{blkr}(M_x^T D M_x, I_N) \tag{S11}
\]
which yields from Lemma 1. Regarding the second term on the R.H.S., we substitute \(\tilde{M}_x\) and write it out as
\[
E[d_{N+1} x_L \tilde{M}_x]^T = d_{N+1} E[(M_x^T a_{x}^T + x_M)(M_x^T a_{x}^T + x_M)^T] = d_{N+1} [M_x^T a_{x}^T M_x + M_x^T a_{x}^T x_M + x_M a_{x}^T M_x + x_M x_M^T]. \tag{S12}
\]

We proceed in the same way and elaborate on each terms within the expectation on the R.H.S. of \(S12\); respectively:
- \(E[M_x^T a_{x}^T M_x] = \text{blkr}(M_x^T R_M^T M_x, (t_+^2 + \sigma^2 I_N))\) which holds from Lemma 1 and where \(R = \Sigma' + \mu \mu^T\);
- \(E[M_x^T a_{x}^T x_M] = M_x^T \mu a_{x}\);
- \(E[x_M a_{x}^T M_x] = t_+ E[\tilde{M}_x^T \tilde{M}_x]\);
- \(E[x_M a_{x}^T x_M] = (t_+^2 + \sigma^2, 0_M)\).
Combining all these terms and \((\text{S11})\) yields expression \((\text{S15})\) for \(\Delta_{22}\).

Next, we focus on the second expectation on the R.H.S. of \((\text{S6})\):
\[
\theta := \mathbb{E}[W_+^\top D_+ t_+].
\]
Substituting once again \(W_+\) [cf. \((\text{S9})\)], we can write the expectation argument as
\[
W_+^\top D_+ t_+ = \begin{bmatrix}
L_+ & x_t \\
M_+ & \tilde{m}_t \\
\end{bmatrix}
\begin{bmatrix}
D & 0 \\
0 & d_{N+1}
\end{bmatrix}
\begin{bmatrix}
t \\
t_+
\end{bmatrix}
= \begin{bmatrix}
L_+^\top D_t + t_d x_L \\
\end{bmatrix}
\begin{bmatrix}
M_+ & \tilde{m}_t \\
\end{bmatrix}
\begin{bmatrix}
t \\
t_+
\end{bmatrix}
.
\]
(S13)

Upon substituting \(\tilde{L}_t\) and \(\tilde{m}_t\) and applying the expectation, expression \((\text{S16})\) for \(\theta\) follows, completing the proof. \(\square\)

**Proof of Proposition 2**

**Graph \(G^\alpha_\beta\).** The expected discrete-2 Dirichlet form is
\[
\mathbb{E}[S_2(y^\alpha_\beta)] = \mathbb{E}[y^\top (I - A^\alpha_\beta) y^\beta_+]
\]
whilst substituting the adjacency matrix \(A^\alpha_\beta\) [cf. \((\text{I})\)] becomes
\[
\mathbb{E}[S_2(y^\alpha_\beta)] = \mathbb{E}[-\Gamma^\top b^\top (I - A) b^\beta_+ + (b^\alpha x + 1) t_+ t_+^\top h^\top].
\]
(S14)

with \(\Gamma = (I - A)^\top (I - A)\). Substituting further \(y^\beta_+\) [cf. \((\text{S5})\)] with \(x^\beta = t_+\) we can write \((\text{S15})\) as
\[
\mathbb{E}[S_2(y^\alpha_\beta)] = h^\top \mathbb{E}[-\Gamma^\top b^\top (I - A) b^\beta_+ + (b^\alpha x + 1) t_+ t_+^\top h^\top].
\]
(S15)

The third term is the transpose of the second one, i.e.,
\[
\mathbb{E}[(I - A) (I - A)^\top t_+ t_+] = \mathbb{E}[t_+ t_+^\top (I - A) (I - A)^\top].
\]
(S16)

The final term is
\[
\mathbb{E}[(b^\alpha x + 1) t_+ t_+^\top h^\top] = \sum_{n=1}^N \mathbb{E}_{[b^\alpha_+] N+1}^{\text{diag}}(t_+^2, 0).
\]
(S23)

Combining \((\text{S16})\) and \((\text{S23})\), we get
\[
\mathbb{E}[(b^\alpha x + 1) t_+ t_+^\top h^\top] = \sum_{n=1}^N \mathbb{E}_{[b^\alpha_+] N+1}^{\text{diag}}(t_+^2, 0).
\]

Combining these together we get expression \((\text{S17})\) for \(\Psi^1\).

**Graph \(G^\alpha_\beta\).** By substituting \(A^\alpha_\beta\) [cf. \((\text{I})\)] and \(y^\beta_+ = [M, h^\beta_], \tilde{m}_t^\top h^\top\) in \(y^\beta_+ (I - A^\alpha_\beta)^\top (I - A^\beta_\alpha) y^\beta_+\) the expected 2-Dirichlet form is
\[
\mathbb{E}[S_2(y^\alpha_\beta)] = \mathbb{E}[-\Gamma^\top b^\top (I - A) b^\beta_+ + (b^\alpha x + 1) t_+ t_+^\top h^\top].
\]
(S17)

Using \(\bar{m}_t = M^\top a^\alpha + t_+ M\), we get
\[
\mathbb{E}[S_2(y^\alpha_\beta)] = h^\top \mathbb{E}[-\Gamma^\top b^\top (I - A) b^\beta_+ + (b^\alpha x + 1) t_+ t_+^\top h^\top].
\]
(S25)

Equation \((\text{S26})\) comprises the following terms:
\[
\mathbb{E}[M^\top (I + a^\beta_+ a^\alpha_+) M] = M^\top (I + R^\top) M = \mathbb{E}[a^\beta_+ a^\alpha_+].
\]

The third term is the transpose of the above, hence it has the expectation \(M^\top R M + M^\top L M\).

The fourth term is random here and \(R = \mathbb{E}[a^\beta_+ a^\alpha_+]\).

The fifth term is the transpose of the above, hence it has the expectation \(M^\top R M + M^\top L M\).

Combining these together, we get expression \((\text{S2})\) for \(\Psi^o\); hence, completing the proof. \(\square\)