ENumerating polytopes

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Abstract. Polytopes are both ordinary and tropical polytopes. Tropical types of polytopes in \( \mathbb{R}^n \) are in bijection with cones of a certain Gröbner fan \( \mathcal{GF}_n \) in \( \mathbb{R}^{n^2-n} \). Unfortunately, even \( \mathcal{GF}_5 \) is too large to be computed by existing software. We show that the entire structure of \( \mathcal{GF}_n \) can be deduced from its restriction to a small cone called the polytrope region. Restricted to this region, we show that \( \mathcal{GF}_n \) equals the refinement of two fans: the fan of linearity of the polytrope map appeared in [18], and the bipartite binomial hyperplane arrangement in \( \mathbb{R}^{n^2-n} \). This gives efficient algorithms for enumerating tropical types of polytopes. We computed types of full-dimensional polytopes for \( n = 4 \), and maximal polytopes for \( n = 5 \) and \( n = 6 \).

1. Introduction

Consider the tropical min-plus algebra \((\mathbb{R}, \oplus, \odot)\), where \( a \oplus b = \min(a, b) \), \( a \odot b = a + b \). A set \( S \subset \mathbb{R}^n \) is tropically convex if \( x, y \in S \) implies \( a \odot x \oplus b \odot y \in S \) for all \( a, b \in \mathbb{R} \). Such sets are closed under tropical scalar multiplication: if \( x \in S \), then \( a \odot x \in S \). Thus, one identifies tropically convex sets in \( \mathbb{R}^n \) with their images in the tropical affine space \( \mathbb{T}P^{n-1} = \mathbb{R}^n \setminus (1, \ldots, 1) \mathbb{R} \). The tropical convex hull of finitely many points in \( \mathbb{T}P^{n-1} \) is a tropical polytope. A tropical polytope is a polytrope if it is also an ordinary convex set in \( \mathbb{T}P^{n-1} \) [11].

Polytopes are important in tropical geometry and combinatorics. They are the alcoved polytopes of type A of Lam and Postnikov [13]. They have appeared in a variety of context, from hyperplane arrangements [13], affine buildings [12], to tropical eigenspaces, tropical modules [3,5], and, semigroup of tropical matrices [8], to name a few. In particular, they are building blocks for tropical polytopes: any tropical polytope can be decomposed into a union of cells, each is a polytrope [7]. Each cell has a type, and together they define the type of tropical polytope. A \( d \)-dimensional polytrope has exactly one \( d \)-dimensional cell, namely, its (relative) interior. This is the basic cell, and its type is the tropical type of the polytrope [11]. We use the word ‘tropical’ to distinguish from the ordinary combinatorial type defined by the face poset. As we shall show, tropical type refines ordinary type.

This work enumerates the tropical types of full-dimensional polytopes in \( \mathbb{T}P^{n-1} \). Since polytopes are special tropical simplices [11] Theorem 7] this number is at most the number of regular polyhedral subdivisions of \( \Delta_{n-1} \times \Delta_{n-1} \) by [7 Theorem 1]. However, this is a very loose bound, the actual number of types of polytopes is much smaller. Joswig and Kulas [11] pioneered the explicit computation of types of polytopes in \( \mathbb{T}P^2 \) and \( \mathbb{T}P^3 \) using the software polymake. They started from the smallest polytrope, proven to be the small tropical simplex [11], and recursively added more vertices in various tropical halfspaces. Their table of results and beautiful figures have been the source of inspiration for this work. Unfortunately, the published table in [11] has errors. For example, there are six, not five, distinct combinatorial types of full-dimensional polytopes in \( \mathbb{T}P^3 \) with maximal number of vertices, as discovered by Jiménez and de la Puente [10]. We recomputed Joswig and Kulas’ result in Table 2.
In contrast to previous works \cite{10,11}, we have a Gröbner approach polytropes. In Section \ref{2} we show that their tropical types are in bijection with a subset of cones in the Gröbner fan $\mathcal{GF}_n$ of a certain toric ideal. While this is folklore to experts, the obstacle has been in characterizing these cones. Without such characterizations, brute force enumeration requires one to compute all of $\mathcal{GF}_n$. Unfortunately, even with symmetry taken into account, $\mathcal{GF}_5$ cannot be handled by leading software such as \texttt{gfan} on a conventional desktop.

We show that the full-dimensional polytrope cones in $\mathcal{GF}_n$ are precisely those contained in a small cone called the polytrope region $\mathcal{P}_n$. Our main result, Theorem \ref{25}, decomposes the fan $\mathcal{GF}_n$ restricted to $\mathcal{P}_n$ as the refinement of two smaller fans. The first is the fan of linearity of the polytrope map $\mathcal{F}_n$, which appeared in \cite{18}. The second is the hyperplane arrangement of bipartite binomial $BB_n$, a novel arrangement which plays a central role for polytropes. In particular, chambers of $BB_n$ are in bijection with polytropes in $\mathcal{TP}_n-1$ with maximal number of vertices. These results elucidate the structure of $\mathcal{GF}_n$ and opens up a line of attack for polytrope enumeration. Specifically, one can either compute the Gröbner fan $\mathcal{GF}_n$ restricted to the given cone $\mathcal{P}_n$, or compute the two smaller fans and take their refinement. With these approaches, we computed representatives for all tropical types of full-dimensional polytropes in $\mathcal{TP}_3$ and all maximal polytropes in $\mathcal{TP}_4$. In $\mathcal{TP}_4$ and $\mathcal{TP}_5$, there are 27248 and 22770 combinatorial types of maximal polytropes, respectively. This is the first result on combinatorial types of polytropes in these dimensions.

Organizations. For self-containment, Section \ref{2} reviews the basics of Gröbner bases and integer programming. Section \ref{3} defines polytropes and their types via the shortest paths program. Section \ref{4} contains our main result, Theorem \ref{25}. Section \ref{5} shows how tropical eigenvalue and eigenspace naturally arise from ideal homogenization. Section \ref{6} presents an algorithm for enumerating full-dimensional polytropes in any dimension, as well as computation results for $\mathcal{TP}_3$ and $\mathcal{TP}_4$. We conclude with discussions and open problems.

Notation. For a positive integer $n$, let $N = n^2 - n$. Let $[N]$ denote the set $\{1, \ldots, N\}$. Index elements of $[N]$ by $ij$, $i, j \in [n], i \neq j$. Identify a monomial $x^u \in \mathbb{R}[x_{ij} : ij \in [N]]$ with its exponent $u \in \mathbb{N}^N$, as well as the multigraph on $n$ nodes where $u_{ij}$ gives the number of edges from $i$ to $j$. The support of $u$ is the set of indices $ij$ where $u_{ij} > 0$. Identify a binomial $x^{u_+} - x^{u_-}$ with $u = u_+ - u_- \in \mathbb{Z}^N$, where $u_+ = \max(u, 0) \in \mathbb{N}^n$, $u_- = -\min(u, 0) \in \mathbb{N}^n$. Where convenient, identify $u \in \mathbb{R}^N$ with the matrix $u \in \mathbb{R}^{n \times n}$ with zero diagonal, viewed as the weight matrix of a graph on $n$ nodes. For a cone $C$, let $C^\circ$ denote its relative interior, $\partial C$ denote its boundary.

2. Background: Gröbner bases and integer programming

This section contains a short exposition on the Gröbner approach to integer programming, adapted from \cite{16} §5 and \cite{17}. For $c \in \mathbb{R}^N$, $A \in \mathbb{R}^{n \times N}$ and $b \in \mathbb{Z}^n$, the primal and dual of an integer program are

$$\begin{align*}
\text{minimize} & \quad c^T u \\
\text{subject to} & \quad Au = b, \quad u \in \mathbb{N}^N \\
\text{maximize} & \quad b^T y \\
\text{subject to} & \quad A^T y \leq c, \quad y \in \mathbb{R}^n.
\end{align*}$$

(P) (D)
Throughout this paper we shall be concerned with a specific $A$, defined in Section 3. Let $Pol(c)$ denote the constraint set of $\{D\}$

$$Pol(c) = \{y \in \mathbb{R}^n : A^\top y \leq c\}.$$  

Associated to this integer program is a toric ideal $I$ with a Gröbner basis whose term orders are induced by $c$. Here are the formal definitions. Consider the polynomial ring $\mathbb{R}[x] = \mathbb{R}[x_{ij} : i \in \{N\}]$. Identify $u \in \mathbb{N}^N$ with the monomial $x^u = \prod_{ij \in \{N\}} x_{ij}^{a_{ij}}$ in $\mathbb{R}[x]$.

Consider the map $\mathbb{R}[x] \to \mathbb{R}[t^{\pm 1}]$, $x_{ij} \mapsto t^{a_{ij}}$, where $a_{ij}$ is the $j$-th column of $A$. The kernel $I$ of this map is a prime ideal in $\mathbb{R}[x]$, called the toric ideal of $A$. Each cost vector $c$ induces a partial ordering $\succ_c$ on $\mathbb{N}^N$, and hence on the monomials in $\mathbb{R}[x]$ via

$$x^u \succ_c x^v, u \succ_c v \text{ if } c \cdot u > c \cdot v.$$  

For generic $c$, $\succ_c$ is a total ordering on the monomials $x^u$, in other words, a term ordering. Given a term order $\succ_c$, every non-zero polynomial $f \in I$ has a unique initial monomial, denoted $in_c(f)$. The initial ideal $in_c$ of $I$ is the ideal generated by $in_c(f)$ for all $f \in I$. Monomials of $I$ which do not lie in $in_c$ are called standard monomials. The Gröbner basis of $I$ with term ordering $\succ_c$ is a finite subset $G_c \subset I$ such that $\{in_c(g) : g \in G_c\}$ generates $in_c$. This set is called reduced if for any two distinct elements $g, g' \in G_c$, no term of $g'$ is divisible by $in_c(g)$. The reduced Gröbner basis is unique for an ideal and a term order.

Each $c \in \mathbb{R}^N$ defines an open cone $G_c$. Say that $c$ and $c'$ are equivalent with respect to $I$, denoted $c \sim c'$, if $G_c = G_{c'}$, or equivalently, $in_c = in_{c'}$. Let $\overline{G}_c$ denote the closure of $G_c$ in the Euclidean topology. Following [9], the Gröbner fan associated with $I$ is defined as

$$G\mathcal{F}_n = \bigcup_{c > 0} \overline{G}_c.$$  

The Gröbner fan is a polyhedral fan consisting of finitely many cones [16]. The following connects the integer program with the Gröbner basis of $I$. It translates to an algorithm for solving integer programs [6,17].

**Theorem 1** (Conti-Traverso). For $c \in \mathbb{R}^N$, let $G_c$ be the reduced Gröbner basis of $I$ with respect to the term ordering $\succ_c$. For $b \in \mathbb{Z}^n$ and any $u$ satisfying $Au = b$, let $x^u$ be the unique remainder obtained by diving $x^u$ by $G_c$. Then $u'$ is an optimal solution to (P).

**Corollary 2.** Suppose $A$ is totally unimodular and that all inequalities in the definition of $Pol(c)$ are facet-defining. For $u'$ in Theorem 1, the face of $Pol(c)$ supported by $b$.

$$\{y \in Pol(c) : (A^\top y)_{ij} = c_{ij} \text{ if } u'_{ij} > 0\}$$  

is the face of $Pol(c)$ supported by $b$.

**Proof.** Since $A$ is totally unimodular, strong duality holds [14, §10]. That is, any solution $y'$ of $\{D\}$ satisfies $b^\top y' = c^\top u'$. But $A^\top y' \leq c$, thus $u^\top A^\top y' = b^\top y' \leq u^\top c$. So one has equality if and only if $(A^\top y')_{ij} = c_{ij}$ for all indices $ij \in \{N\}$ where $u'_{ij} > 0$. \qed

3. Shortest paths, polytropes and tropical simplices

This section defines the shortest paths program, the only integer program we consider in this paper. From now on, objects such as $G\mathcal{F}_n$, $I$, $A$, etc... are implicitly understood to be associated with this program. An important observation in this section is that cones of $G\mathcal{F}_n$, the Gröbner fan associated to the shortest paths program, are in bijection with combinatorial types of standard tropical simplices, of which polytropes are a special case.
3.1. The Shortest Paths Program. Shortest paths is the program \( [P] \) with \( A \in \mathbb{R}^{n \times N} \) defined via

\[
(Au)_i = \sum_{j \neq i, j=1}^{n} u_{ij} - \sum_{j \neq i, j=1}^{n} u_{ji} = b_i \text{ for all } i = 1, \ldots, n. \tag{1}
\]

For \( b \in \mathbb{Z}^n, c \) is the cost matrix, \( u \) defines a transport plan, and the goal of \([P]\) is to minimize the total transport cost subjected to net output \( b_i \) at node \( i \). In this case, each node \( j \) sends out one net unit, and the node \( i \) receives \( n - 1 \) net unit. The optimal plan picks out the shortest path to \( i \).

This program includes the single target shortest path program as a special case. For a fixed target \( i \in [n] \), let \( b = b^i \in \mathbb{Z}^n \) be the vector

\[
b_i^j = -(n - 1), \quad b_j^i = 1 \text{ for } j \in [n], j \neq i. \tag{2}
\]

In this case, each node \( j \neq i \) sends out one net unit, and the node \( i \) receives \( n - 1 \) net unit. The optimal plan picks out the shortest path to \( i \).

The shortest paths program is feasible only if \( \sum_i b_i = 0 \). In this case, the recession cone of the constraint polyhedra in \([P]\) consists of integer flows on the complete graph. Let \( R_n \subseteq \mathbb{R}^{N} \) be the set of \( c \) such that the program is feasible bounded. By classical results \([14, \S 3]\),

\[
R_n = \{ c \in \mathbb{R}^{N} : c \cdot \chi_C \geq 0 \} \tag{3}
\]

where \( \chi_C \) is the incidence vector of the cycle \( C \) and \( C \) ranges over all simple cycles on \( n \) nodes. The lineality space \( V_n \) of \( R_n \) is an \((n - 1)\) dimensional space, consisting of vectors in \( \mathbb{R}^{N} \) with zero-cycle sum. That is,

\[
V_n = \{ c \in \mathbb{R}^{N} : c \cdot \chi_C = 0 \} \tag{4}
\]

One can show that \( R_n \) is also the Gröbner region of the fan \( \mathcal{G} \mathcal{F}_n \) associated with this program. Thus, in this case, \( \mathcal{G} \mathcal{F}_n \) is a fan in \( \mathbb{R}^{N} \) with lineality space of dimension \( n - 1 \).

3.2. Polytopes. Let us consider \( \text{Pol}(c) \), the constraint set of \([D]\). Since \( A \) is totally unimodular, strong duality holds \([14, \S 10]\). Thus, \( \text{Pol}(c) \) is non-empty if and only if \( c \in R_n \). In such cases, we call \( \text{Pol}(c) \) the polytrope of \( c \). Joswig and Kulas \([11]\) first coined the term polytrope to refer to tropical polytopes which are also ordinarily convex. It follows from \([11, \S 3]\) that any polytrope in the sense of Joswig and Kulas arises as \( \text{Pol}(c) \) for some \( c \in R_n \). This justifies our choice of name.

Definition 3. For \( c \in R_n \), call \( \text{Pol}(c) \) the polytrope of \( c \). Explicitly,

\[
\text{Pol}(c) = \{ y \in \mathbb{T}^{n-1} : y_i - y_j \leq c_{ij} \text{ for all } ij \in [N] \}. \tag{5}
\]

Not all of the \( n^2 - n \) inequalities in \([5]\) are facet-defining for \( \text{Pol}(c) \). In particular, for any triple \( i, j, k \), we have \( y_i - y_j = (y_i - y_k) + (y_k - y_j) \). Thus for \( y \in \text{Pol}(c) \), \( y_i - y_j \leq c_{ik} + c_{kj} \).

Definition 4. For \( c \in R_n \), the Kleene star of \( c \) is the vector \( c^* \in \mathbb{R}^{N} \) where \( c^*_{ij} \) is the weight of the shortest path from \( i \) to \( j \).

Lemma 5. The inequality \( y_i - y_j \leq c_{ij} \) is tight if and only if \( c = c^* \). In other words, as sets, \( \text{Pol}(c) = \text{Pol}(c^*) \).
3.3. **Standard tropical simplices.** In the min-plus algebra \((\mathbb{R}, \oplus, \circ)\), one has \(a \oplus b = \min(a, b)\), \(a \circ b = a + b\). A tropical polytope in \(\mathbb{T}\mathbb{P}^{n-1}\) is the tropical convex hull of \(m\) points \(c_1, \ldots, c_m \in \mathbb{R}^n\), that is,
\[
T(c_1, \ldots, c_m) = \{a_1 \circ c_1 \oplus \cdots \oplus a_m \circ c_m : a_1, \ldots, a_m \in \mathbb{R}\}
\]
\[
= \{\min(a_1 + c_1, \ldots, a_m + c_m) : a_1, \ldots, a_m \in \mathbb{R}\}.
\]
If \(m = n\), then \(T(c_1, \ldots, c_n)\) is a tropical simplex. In this paper, we are concerned with a special type of tropical simplices.

**Definition 6.** For \(c \in R_n\), let \(c_1, \ldots, c_n\) be the \(n\) columns of the zero-diagonal matrix \(c \in \mathbb{R}^{n \times n}\). Then \(T(c_1, \ldots, c_n)\), abbreviated \(T(c)\), is called a **standard tropical simplex**.

Develin and Sturmfels \([7]\) pioneered the investigation on tropical polytopes. They showed that a tropical polytope is a union of bounded cells, each has a **cell type**. Together they specify the **type** of the tropical polytope, see \([7]\). The types of tropical simplices are in bijection with cones of a certain Gröbner fan \([7, \text{Theorem 1}]\), which was further shown that a tropical polytope is a union of bounded cells, each has a **type**. By Corollary 2, up to permutation of the vertex labels, \(Pol(c)\) have the same tropical type as \(Pol(c')\) if and only if each of their faces have the same collection of

### Theorem 7 (Special case of Theorem 1 in \([7]\))

For \(c, c' \in R_n\), the tropical simplices \(T(c)\) and \(T(c')\) have the same type, denoted \(T(c) \sim T(c')\), if and only if \(c \sim c'\), that is, \(m_c = m_{c'}\).

The relation between the four objects \(T(c), Pol(c), T(c^*)\) and \(Pol(c^*)\) is a classical result in tropical spectral theory \([5, \S4]\). We collect these facts below, and include a statement on the relation between their types.

**Lemma 8.** Suppose \(c \in R_n\), \(c \neq c^*\).

1. \(Pol(c^*)\) and \(Pol(c)\) are both tropical simplices, and they equal \(T(c^*)\).
2. As sets, \(T(c^*) = Pol(c^*) = Pol(c) \subset T(c)\).
3. \(T(c^*)\) is full-dimensional in \(\mathbb{T}\mathbb{P}^{n-1}\) if and only if \(T(c)\) is, which happens if and only if \(c\) lies in the interior of \(R_n\).
4. \(T(c^*) \not\sim T(c)\).

**Proof.** The first three are classical results on Kleene stars, see \([5, \S4]\). Consider the last statement. Since \(c \neq c^*\), there exists some \(ij \in [N]\) such that \(c_{ij} > c_{ij}^*\), that is, the direct path \(i \rightarrow j\) is not the shortest path from \(i\) to \(j\) in \(c\). Then \(x_{ij}\) is a standard monomial of \(G_c\), while \(x_{ij}\) is not. Thus \(G_c \neq G_{c^*}\), so \(c \not\sim c^*\). By Theorem \([7]\), \(T(c^*) \not\sim T(c)\). \(\square\)

### 3.4. Tropical types of polytopes.

**Definition 9.** Let \(Pol(c), Pol(c')\) be polytopes in \(\mathbb{T}\mathbb{P}^{n-1}\). Say that they have the same **tropical type**, denoted \(Pol(c) \sim Pol(c')\), if \(c^* \sim (c')^*\). Say that they have the same **combinatorial tropical type**, if for some permutation \(\sigma \in S_n\), \((\sigma \cdot c)^* \sim (c')^*\).

**Lemma 10.** If \(Pol(c)\) and \(Pol(c')\) have the same combinatorial tropical type, then they have the same combinatorial type as ordinary polytopes. That is, tropical type refines ordinary type.

**Proof.** By Corollary \([2]\), up to permutation of the vertex labels, \(Pol(c)\) have the same tropical type as \(Pol(c')\) if and only if each of their faces have the same collection of
Lemma 11. If \( c \sim c' \), then \( c^* \sim (c')^* \). In particular, \( T(c) \sim T(c') \Rightarrow Pol(c) \sim Pol(c') \).

Proof. Suppose \( c \sim c' \). By Theorem \( \ref{thm:maximal-cell} \), each bounded cell of \( T(c) \) and \( T(c') \) have the same cell types. But \( T(c^*) \) and \( T((c')^*) \) are the unique full-dimensional polytopes of \( T(c) \) and \( T(c') \), respectively. Thus each cell of \( T(c^*) \) and \( T((c')^*) \) have the type, so \( c^* \sim (c')^* \). \( \square \)

Proposition 12. Define the polytrope region to be the closed cone
\[
\mathcal{P}_n = \{ c \in \mathbb{R}_n : c = c^* \}.
\]

Call the restriction of \( \mathcal{GF}_n \) to \( \mathcal{P}_n \) the polytrope complex \( \mathcal{GF}_n|_\mathcal{P} \)
\[
\mathcal{GF}_n|_\mathcal{P} = \bigcup_{c \in \mathcal{P}_n} \mathcal{G}_c.
\]

Then cones of \( \mathcal{GF}_n|_\mathcal{P} \) are in bijection with the tropical types of polytopes in \( \mathbb{T}^{n-1} \). Cones of \( \mathcal{GF}_n|_\mathcal{P} \) in the strict interior of \( \mathbb{R}_n \) are in bijection with the tropical types of full-dimensional polytopes in \( \mathbb{T}^{n-1} \).

Proof. Consider the map which takes a polytrope \( Pol(c) \) to the cone \( \mathcal{G}_c \). Then \( c \) and \( c' \) are mapped to the same cone if and only if \( c^* \sim (c')^* \). By definition, this is equivalent to \( Pol(c) \sim Pol(c') \). Thus this establishes the first bijection. The second statement follows from Lemma \( \ref{lem:open-cones} \) part (3). \( \square \)

Thus, enumerating tropical types of polytopes equals enumerating cones of \( \mathcal{GF}_n|_\mathcal{P} \) in \( \mathbb{R}_n^\circ \). This is a much smaller polyhedral complex compared to \( \mathcal{GF}_n \). Somewhat surprisingly, if one can enumerate cones of \( \mathcal{GF}_n|_\mathcal{P} \), one can enumerate those of \( \mathcal{GF}_n \) with little extra work (cf. Section \( \ref{sec:open-cones} \)). Thus, \( \mathcal{GF}_n|_\mathcal{P} \) captures the essence of \( \mathcal{GF}_n \). We conclude this section with an interpretation for the open cones of \( \mathcal{GF}_n|_\mathcal{P} \).

Lemma 13. A polytrope \( Pol(c) \) is maximal if and only if \( \mathcal{G}_c \) is an open cone of \( \mathcal{GF}_n|_\mathcal{P} \).

Proof. A polytrope \( Pol(c) \) in \( \mathbb{T}^{n-1} \) is maximal if it has \( \binom{2n-2}{n-1} \) vertices as an ordinary polytope. From \( \ref{thm:maximal-cell} \) Theorem 1, \( T(c) \) has the maximal number of vertices of \( \binom{2n-2}{n-1} \) if and only if \( \mathcal{G}_c \) is an open cone of \( \mathcal{GF} \). But \( T(c) = Pol(c) \) if and only if \( c \in \mathcal{P}_n \), that is, \( \mathcal{G}_c \subset \mathcal{GF}_n|_\mathcal{P} \). Therefore, the open cones of \( \mathcal{GF}_n|_\mathcal{P} \) are precisely the cones of maximal polytopes. \( \square \)

4. The Polytrope Complex

In this section we state and prove our main results on the structure of the polytrope complex \( \mathcal{GF}_n|_\mathcal{P} \). First, we show in Section 4.1 that in \( \mathcal{P}_n^\circ \), \( \mathcal{GF}_n|_\mathcal{P} \) equals the bipartite binomial hyperplane arrangement \( \mathbb{BB}_n \). Thus, open cones of \( \mathcal{GF}_n|_\mathcal{P} \) are indexed by inequalities amongst bipartite binomials. We use this fact to compute the six types of maximal polytopes for \( n = 4 \). In Section 4.2 we recall \( \mathcal{F}_n \), the fan of linearity of the polytrope map in \( \ref{thm:linearity} \), which is a coarsening of \( \mathcal{GF} \). Proposition \( \ref{prop:maximal-fan} \) characterizes the standard monomials in the open cones of \( \mathcal{GF}_n \) in terms of modified bipartite binomials and compatible trees which index cones of \( \mathcal{F}_n \). Altogether these results grant Theorem \( \ref{thm:main} \), which characterizes \( \mathcal{GF}_n|_\mathcal{P} \) as the refinement of \( \mathcal{F}_n \) and \( \mathbb{BB}_n \) restricted to \( \mathcal{P}_n \).
4.1. Interior cones. In this section we consider cones of $\mathcal{GF}_{n}|_P$ which lie in $P_n^\circ$. First we identify the interior polytrope basis, the union of reduced Gröbner bases over all such cones. Since $A$ is totally unimodular, the universal Gröbner basis of $I$ consists of binomials $x^u^+ - x^u^-$, where $u$ is a circuit of $A$, that is, a non-zero primitive vector $u$ in the kernel of $A$ with minimal support with respect to set inclusion [17]. Thus terms in the interior polytrope basis are also of this form. We claim that these terms fall into either one of the following categories: triangle and $m$-bipartite.

Definition 14 ($m$-bipartite monomials and binomials). Let $d = \lfloor n/2 \rfloor$. For $m = 2, \ldots, d$, let $\Sigma(m)$ be the set of cyclic permutations on $m$ letters. Let $K = (k_1, \ldots, k_m)$, $L = (l_1, \ldots, l_m) \subset [n]$ be two lexicographically ordered lists of $m$ distinct indices, $K \cap L = \emptyset$. For $\sigma \in \Sigma_m$, the $m$-bipartite monomial $(K, \sigma, L)$ is the incidence vector of the bipartite graph with sources $K$, sinks $L$, and edges $k_1 \rightarrow \sigma(l_1), \ldots, k_m \rightarrow \sigma(l_m)$.

For $\sigma \neq \sigma'$, the $m$-bipartite binomial $(K, \sigma, \sigma', L)$ is $u^+ - u^-$, where $u^+$ is $(K, \sigma, L)$ and $u^-$ is $(K, \sigma', L)$. Explicitly,

\begin{align*}
u^+ = k_1 \rightarrow \sigma(l_1), \ldots, k_m \rightarrow \sigma(l_m), \quad u^- = k_1 \rightarrow \sigma'(l_1), \ldots, k_m \rightarrow \sigma'(l_m). \tag{6}
\end{align*}

Example 15. For $n = 4$, there are twelve two-bipartite monomials and six two-bipartite binomials. Figure 1 shows the six bipartite binomials, identified with the graphs of $u^+$ and $u^-$. 

\begin{figure}[h]
\centering
\begin{tabular}{cccc}
\hline
1 & 2 & 1 & 2 \\
\hline
3 & 4 & 3 & 4 \\
\hline
\end{tabular}
\begin{tabular}{cccc}
\hline
1 & 2 & 1 & 2 \\
\hline
3 & 4 & 3 & 4 \\
\hline
\end{tabular}
\begin{tabular}{cccc}
\hline
1 & 2 & 1 & 2 \\
\hline
3 & 4 & 3 & 4 \\
\hline
\end{tabular}
\begin{tabular}{cccc}
\hline
1 & 2 & 1 & 2 \\
\hline
3 & 4 & 3 & 4 \\
\hline
\end{tabular}
\end{figure}

Example 17. The arrangement $BB_n^2$ consists of all hyperplanes in $\mathbb{R}^N$ of the form 

\begin{align*}
\{c \in \mathbb{R}^N : c_{ik} + c_{jl} - c_{il} - c_{jk} = 0\}
\end{align*}

for each tuple of distinct indices $i, j, k, l \subset [n]$.

Proposition 18. Let $d = \lfloor n/2 \rfloor$. The interior polytrope basis is the set of binomials of the form $x^u^+ - x^u^-$, where the pair $(u^+, u^-)$, identified with their graphs, ranges over the following sets:

- Triangles: $u^+ = i \rightarrow k \rightarrow j$, $u^- = i \rightarrow j$ for all distinct $i, k, j \in [n]$. 

- m-bipartite: \((K, \sigma, \sigma', L)\) of Definition 14, for all \(m = 2, \ldots, d\), \(\sigma \in \Sigma(m)\), and indexing pairs \((K, L)\).

**Proof.** Let \(\mathcal{G}_c\) be a cone of \(\mathcal{GF}_n|_p\) in \(\mathcal{P}_n^o\). For each distinct triple \(i, j, k \in [n]\), \(c_{ij} < c_{ik} + c_{kj}\). Thus all the triangle binomials are in the interior polytrope basis, with \(x^u = x_{ij}\) the standard monomial. For any m-bipartite binomial \(u_+ - u_-\), one can check that \((u_+, u_-)\) is a circuit of \(A\), and the hyperplane with normal \(u_+ - u_-\) intersects \(\mathcal{P}_n^o\). So the interior polytrope basis contains all triangle and m-bipartite binomials. We need to show the reverse inclusion.

Suppose \(u_+ - u_-\) is a non-triangle binomial in the reduced Gröbner basis \(\mathcal{G}_c\). Either \(u_+\) or \(u_-\) must be a standard monomial. Without loss of generality, suppose it is \(u_-\), that is, \(cu_+ > cu_-\). We shall prove that \(u_+ - u_-\) is m-bipartite.

Suppose, for contradiction, that \(u_-\) contains a path \(i \rightarrow j \rightarrow \ldots \rightarrow k\) of length at least two. One can replace it with the path \(i \rightarrow k\) and form \(u'_-\). Now \((u_-, u'_-)\) lies in the kernel of \(A\), and \(cu_+ > cu'_-\). This contradicts the fact that \(x^u\) is a standard monomial. Therefore all paths in \(u_-\) must be of length 1. Since \((u_+, u_-)\) is not a triangle binomial, the same argument applies to \(u_+\). So \(u_+\) and \(u_-\) contain only paths of length 1, that is, they are bipartite. Since \((u_-, u_+)\) is in the kernel of \(A\), the net outflow at each node must be in the same. Thus \(u_-\) and \(u_+\) have the same number of outgoing and incoming edges for each node, say, \(m\) such edges. So far we have \(u_+ = (K, \sigma, L)\), and \(u_- = (K, \sigma', L)\) for \(\sigma, \sigma' \in S_m\), where \(K\) and \(L\) may have repeated indices. To prove that they are m-bipartite binomials, we need to show that \(K\) and \(L\) have distinct indices, and \(\sigma, \sigma' \in \Sigma_m\), that is, they are cyclic permutations on \(m\) letters.

Since \((u_+, u_-)\) is a circuit, \(u_+\) and \(u_-\) have disjoint supports, thus \(\sigma\) cannot have any fixed elements. If some indices of \(K\) or \(L\) are repeated, they must occur in different cycles of \(\sigma\). Suppose for contradiction that \(\sigma\) has at least two cycles. Then the binomial on each cycle is another bipartite binomial with strictly smaller support. This contradicts the fact that \((u_+, u_-)\) is a circuit. The same argument applies to \(\sigma'\). Therefore, \(\sigma, \sigma' \in \Sigma_m\). So \(K\) and \(L\) do not have disjoint indices. This proves that \(u_+ - u_-\) is m-bipartite. 

**Proposition 19.** Restricted to \(\mathcal{P}_n^o\), \(\mathcal{GF}_n|_p\) equals \(\mathcal{BB}_n\).

**Proof.** On \(\mathcal{P}_n^o\), the term orderings of the triangle binomials are fixed. By Proposition 18, \(\mathcal{P}_n^o\) is subdivided into cones of \(\mathcal{GF}_n|_p\) which are in bijection with all possible partial orderings of the m-bipartite binomials. But these are precisely the cones of \(\mathcal{BB}_n\) over \(\mathbb{R}^N\). Thus we only need to show that every cone of \(\mathcal{BB}_n\) has non-empty intersection with \(\mathcal{P}_n^o\). The lineality space of \(\mathcal{BB}_n\) is

\[
V_n + \text{span}(1, \ldots, 1),
\]

where \(V_n\) is the lineality space of \(\mathcal{P}_n\) defined in 4. Over \(\mathbb{R}^N \setminus V_n\), \(\mathcal{P}_n\) is a pointed cone containing the ray \((1, \ldots, 1)\) in its interior. Modulo the span of this ray, \(\mathcal{BB}_n\) is a central hyperplane arrangement, and \(\mathcal{P}_n^o \setminus \text{span}(1, \ldots, 1)\) is an open neighborhood around the origin. Thus every cone of \(\mathcal{BB}_n\) has non-empty intersection with \(\mathcal{P}_n^o\). 

**Corollary 20.** The number of combinatorial tropical types of maximal polytopes in \(\mathbb{T}\mathbb{P}^{n-1}\) is precisely the number of equivalence classes of open chambers \(\mathcal{BB}_n\) up to action by \(S_n\).

**Example 21** (Maximal polytopes for \(n = 4\)). Number the binomials in Figure 4 from left to right, top to bottom. An open chamber of \(\mathcal{BB}_4\) is a binary vector \(z = \{\pm 1\}^6\), with \(z_i = +1\) if in the \(i\)-th binomial, the left monomial is smaller than the right monomial. For example, \(z_2 = +1\) correspond to the inequality \(c_{12} + c_{34} < c_{14} + c_{32}\). There are at most
2^6 = 64 open chambers in \( BB_4 \). Not all of 64 possible values of \( z \) define a non-empty cone. Indeed, the six normal vectors satisfy exactly one relation:

\[
\begin{align*}
\left( \begin{array}{ccc}
1 & 2 & 1 \\
3 & 4 & 3 \\
\end{array} \right) & \left( \begin{array}{ccc}
1 & 2 & 1 \\
3 & 4 & 3 \\
\end{array} \right) = \left( \begin{array}{ccc}
1 & 2 & 1 \\
3 & 4 & 3 \\
\end{array} \right) + \left( \begin{array}{ccc}
1 & 2 & 1 \\
3 & 4 & 3 \\
\end{array} \right) + \left( \begin{array}{ccc}
1 & 2 & 1 \\
3 & 4 & 3 \\
\end{array} \right) \\
\left( \begin{array}{ccc}
1 & 2 & 1 \\
3 & 4 & 3 \\
\end{array} \right) & \left( \begin{array}{ccc}
1 & 2 & 1 \\
3 & 4 & 3 \\
\end{array} \right) = 0
\end{align*}
\]

**Figure 2.** The relation amongst the two-bipartite binomials for \( n = 4 \).

This means \((1, -1, 1, 1, -1, 1)\) and \((-1, 1, -1, -1, 1, -1)\) define empty cones. Thus there are 62 open chambers of \( BB_4 \), correspond to 62 tropical types of maximal polytropes. The symmetric group \( S_4 \) acts on the vertices of a polytrope \( Pol(c) \) by permuting the labels of the rows and columns \( c \). This translates to an action on the chambers of \( BB_4 \). Up to the action of \( S_4 \), we found six symmetry classes of chambers, corresponds to six combinatorial tropical types of maximal polytropes. Table 1 shows a representative for each symmetry class and their orbit sizes. The first five corresponds to the five types discovered by Joswig and Kulas, presented in the same order in [11, Figure 5]. The class of size 12 was discovered by Jimenez and de la Puente [10].

| Representative                  | Orbit size |
|---------------------------------|------------|
| \((1, 1, 1, -1, 1, 1)\)         | 6          |
| \((-1, 1, 1, -1, -1, 1)\)       | 8          |
| \((1, 1, 1, 1, 1, -1)\)         | 6          |
| \((-1, 1, -1, -1, -1, 1)\)      | 24         |
| \((-1, -1, 1, 1, -1, -1)\)      | 6          |
| \((-1, -1, -1, -1, 1, -1)\)     | 12         |

**Table 1.** Representatives and orbit sizes of the six maximal polytropes in \( \mathbb{TP}^3 \).

4.2. **Boundary cones.** We now define \( \mathcal{F}_n \), the fan of linearity of the polytrope map. We then show how \( \mathcal{G} \mathcal{F}_n \) refines \( \mathcal{F}_n \) in Proposition 24 then state and prove Theorem 25.

Consider the following notion of equivalence on \( R_n \): \( c \) and \( c' \) are equivalent if and only if for each \( i = 1, \ldots, n \), for \( b_i \) given by (2), the affine hyperplane with normal vector \( b_i \) supports the same face of \( Pol(c') \) and \( Pol(c) \). One can show that sets of equivalent \( c \) forms a cone, and collectively they form a fan partition of \( R_n \). (In Section 5, we give a different construction which gives an easy proof that it is a fan.) Denote this fan \( \mathcal{F}_n \).

Since \( b_i \) supports the \( i \)-th column of \( c^* \), \( \mathcal{F}_n \) is precisely the fan of linearity of the Kleene star map \( c \mapsto c^* \) on \( R_n \). That is, in each cone of \( \mathcal{F}_n \), for each entry \( ij \in [N] \), the Kleene star map \( c \mapsto c^*_{ij} \) is given by the same linear functional in the entries of \( c \). The homogenized version \( \mathcal{F}_n^h \) of \( \mathcal{F}_n \) appeared in [18] as the fan of linearity of the polytrope map. In [18], we established the bijection between cones of \( \mathcal{F}_n^h \) and the lattice of complete connected functions. This bijection maps cones of \( \mathcal{F}_n \) to a special subset of complete connected functions, namely, those with a self-loop at each node. Open cones of \( \mathcal{F}_n \) are mapped to sets of compatible trees. We recall these facts below.
Definition 22. For \( i = 1, \ldots, n \), let \( T_i \) be a spanning tree on \( n \) nodes, rooted at \( i \), with edges directed towards the root. A set of trees \( T = (T_1, \ldots, T_n) \) is compatible if for \( i, j \in [n] \), the subtree rooted at \( j \) of \( T_i \) equals the induced subtree of \( T_j \).

Lemma 23. Open cones of \( \mathcal{F}_n \) are in bijection with sets of compatible trees.

Proof. Given an open cone of \( \mathcal{F}_n \), for each pair of nodes \( i, j \in [n], i \neq j \), there exists a unique shortest path from \( i \) to \( j \). Thus cones of \( \mathcal{F}_n \) are indexed by some collection of trees \( (T_1, \ldots, T_n) \), one rooted at each node. A collection of such trees is a solution to some all-pairs shortest path problem if and only if they are compatible [1] §4. Therefore, the sets of compatible trees index the open cones of \( \mathcal{F}_n \). \( \square \)

By construction, \( \mathcal{GF}_n \) refines the fan \( \mathcal{F}_n \). We now make this refinement explicit in Proposition [24]. The resemblance to Proposition [18] should be noted. Indeed, the refinement of \( \mathcal{F}_n \) to \( \mathcal{GF}_n \) is purely governed by the \( m \)-bipartite binomials. Its significance to \( \mathcal{GF}_n|_P \) is as follows: one can check that \( \mathcal{P}_n \) is a cone of \( \mathcal{F}_n \). Its boundary \( \partial \mathcal{P}_n \) in \( \mathcal{F}_n \) is some polyhedral complex. In \( \mathcal{GF}_n \), both this complex \( \partial \mathcal{P}_n \) and the interior \( \mathcal{P}_n^* \) are refined by \( \mathcal{BB}_n \).

Proposition 24. Let \( F \) be an open cone of \( \mathcal{F}_n \), \( \mathcal{G}_c \subset F \) be an open cone of \( \mathcal{GF}_n \). Let \( T = (T_1, \ldots, T_n) \) be the set of compatible trees that index \( F \). Let \( M(c) \) be the set of standard, bipartite monomials of \( \mathcal{G}_c \). Then the standard monomials of the initial monomial ideal \( in_c \) identified with their graphs, are the following:

- \( T_j(i) \), the path from \( i \) to \( j \) in the tree \( T_j \) for each pair \( i, j \in [n], i \neq j \).
- For each \( m \)-bipartite graph \( u \in M(c), 2 \leq m \leq [n/2] \), the graph \( u' \) defined by
  \[
  u'_{ij} = \begin{cases} 
  0 & \text{if } i \rightarrow j \text{ is not in } u \\
  T_j(i) & \text{if } i \rightarrow j \text{ is in } u 
  \end{cases}
  \]

Proof. Note that the statement is independent of choice of \( c \), for if \( c \sim c' \), then \( c^* \sim (c')^* \), so \( \mathcal{G}_c = \mathcal{G}_{c'} \) and \( M(c) = M(c') \). The proof is similar to that of Proposition [18]. The reduced Gröbner basis of \( \mathcal{G}_c \) consists of binomials of the form \( x^{a_+} - x^{a_-} \), where \( (u_+, u_-) \) is a circuit of \( A \). Since \( (u_+, u_-) \) is in the kernel of \( A \), each node in the graph of \( u_+ \) and \( u_- \) must have the same net outflow. This partitions the support of \( u_+ \) and \( u_- \) into three sets: the sources (those with positive net outflow), the sinks (those with negative net outflow), and the transits (those with zero net outflow). Suppose we have precisely one source \( i \) and one sink \( j \). Then \( T_j(i) \) is the standard monomial with source \( i \) and sink \( j \), by definition of shortest path. So suppose we have at least two sources or at least two sinks.

Decompose each of the graph of \( u_+ \) and \( u_- \) into a union of simple paths. Suppose the graph of \( u_+ \) contains paths \( k_1 \rightarrow \ldots \rightarrow l_1, \ldots, k_m \rightarrow \ldots \rightarrow l_m \). Then the paths in \( u_- \) must be \( k_1 \rightarrow \ldots \rightarrow \sigma(l_1), \ldots, k_m \rightarrow \ldots \rightarrow \sigma(l_m) \) for some permutation \( \sigma \). By a similar argument to the proof of Proposition [18] for \( (u_+, u_-) \) to be a circuit, the sources \( (k_1, \ldots, k_m) \) and sinks \( (l_1, \ldots, l_m) \) cannot contain repeated indices, and \( \sigma \) has to be a cycle of length \( m \). Assume without loss of generality that \( u_+ \) is the standard monomial. Then \( u_+ \) is the union of paths \( T_{l_1}(k_1), \ldots, T_{l_m}(k_m) \). Since paths of length one are amongst the shortest paths in \( c^* \), \( v_+ \), \( k_1 \rightarrow l_1, \ldots, k_m \rightarrow l_m \) is a standard \( m \)-bipartite monomial in \( M(c) \). Thus \( u_+ = v'_+ \). Since each fixed set of sources and sinks give a standard monomial of \( \mathcal{G}_c \), we see that all standard monomials on at least two sources and sinks of \( \mathcal{G}_c \) arise this way. \( \square \)

Theorem 25. The fan \( \mathcal{F}_n \wedge \mathcal{BB}_n \) of \( \mathbb{R}^N \), restricted to \( \mathcal{P}_n \), is \( \mathcal{GF}_n|_P \).
Proof. Since $\mathcal{F}_n$ does not subdivide $\mathcal{P}_n^o$, the statement is true on $\mathcal{P}_n^o$ by Proposition 19. Consider a cone $\mathcal{G}_c$ on the boundary $\partial \mathcal{P}_n$. Since $\mathcal{G}_c$ is a cone in $\mathcal{GF}_n$, it can be written as the intersection of a unique set of open cones of $\mathcal{GF}_n$. By Proposition 24, those open cones in $\mathcal{GF}_n|_P$ determine the partial order amongst the bipartite binomials, while those not in $\mathcal{GF}_n|_P$ are contained in a set of open cones in $\mathcal{F}_n$, which determine the shortest paths between any two nodes. But since $\mathcal{P}_n^o$ is the open cone of $\mathcal{F}_n$ where all shortest paths have length one, any shortest path in $\mathcal{G}_c$ can be decomposed into a union of paths of length at most two. Thus, the set of shortest paths define the partial order amongst the triangle terms, and conversely. By Proposition 18, each cone of $\mathcal{GF}_n|_P$ uniquely determines a partial order on the triangle and bipartite binomials. Thus, over $\partial \mathcal{P}_n$, each cone of $\mathcal{GF}_n|_P$ is a cone in the refinement $\mathcal{F}_n \& \mathcal{BB}_n$. □

5. Kleene Star and Homogenization

Identify $c \in \mathbb{R}^N$ with its matrix form in $\mathbb{R}^{n \times n}$, where $c_{ii} = 0$ for all $i \in [n]$. So far, we have only considered $c \in \mathbb{R}^n$. In this section, we shall extend the definition of $\text{Pol}(c)$ in (5) to $c \in \mathbb{R}^{n \times n}$. This leads to the problem of weighted shortest paths. In the tropical linear algebra literature, one often goes the other way around: first consider the weighted shortest path problem, derive polytropes for general matrices $c$, and then restricts to those in $\mathbb{R}^n$ (see [2, 5, 8, 15, 18]). The reverse formulation, from polytrope to weighted shortest path, is not so immediate. However, in the language of Gröbner bases, this is a very simple and natural operation: making a complete Gröbner fan by homogenizing the toric ideal $I$. By doing so, we obtain another construction of $\mathcal{F}_n$ which gives a simple proof that it is a coarsened fan of $\mathcal{GF}_n$.

Let $I$ be the toric ideal of the matrix $A$ appeared in (1). Then

$$I = \langle x_{ij}x_{ji} - 1, x_{ij}x_{jk} - x_{ik} \rangle$$

where the indices range over all distinct $i, j, k \in [n]$. Introduce $n$ variables $x_{11}, \ldots, x_{nn}$, and consider the following homogenized version of $I$ in the ring $\mathbb{R}[x_{ij} : i, j = 1, \ldots, n]$

$$I^h = \langle x_{ij}x_{ji} - x_{ii}x_{jj}, x_{ij}x_{jk} - x_{ik}x_{kk}, x_{ii} - x_{jj} \rangle$$

where the indices range over all distinct $i, j, k \in [n]$. This is the toric ideal of the $(n+1) \times n^2$ matrix

$$A^h = \begin{bmatrix} A & \mathbf{0} \\ 1_N & 1_n \end{bmatrix},$$

where $\mathbf{0}$ is an $n \times n$ zero matrix, $1_N$ is the all-one row vector of length $N$, and $1_n$ is the all-one row vector of length $n$. We obtain a new primal program

$$\text{(P}^h) \quad \begin{array}{l}
\text{minimize } c^\top u \\
\text{subject to } A^h u = b, \quad u \in \mathbb{N}^{n^2-n}.
\end{array}$$

Explicitly, the constraints are

$$\sum_{j \neq i, j=1}^{n} u_{ij} - \sum_{j \neq i, j=1}^{n} u_{ji} = b_i \text{ for all } i = 1, \ldots, n.$$
Compared to \( (D) \), the dual program to \( (P_h) \) has one extra variable. It is helpful to keep track of this variable separately. Let \( \lambda \in \mathbb{R} \). Write \( b^\top = (b_1 \ldots b_n) \). Write \([y, \lambda]\) for the concatenation of the vector \( y \) and \( \lambda \). The dual program to \( (P_h) \) is the following.

\[
\begin{align*}
\text{maximize} & \quad b^\top y + b_{n+1} \lambda \\
\text{subject to} & \quad (A^h)^\top [y, \lambda] \leq c, \quad y \in \mathbb{T}P^{n-1}, \lambda \in \mathbb{R}.
\end{align*}
\]

\((D^h)\)

Explicitly, the constraints are:

\[
\begin{align*}
y_i - y_j - \lambda & \leq c_{ij} \text{ for all } i, j \in [n] \\
\lambda & \geq c_{ii} \text{ for all } i \in [n].
\end{align*}
\]

In fact, \( \lambda \) and \( y \) can be solved separately. For example, by adding the constraints involving \( c_{ij} \) and \( c_{ji} \), we obtain a constraint only in \( \lambda \)

\[
(y_i - y_j) - \lambda + (y_j - y_i) - \lambda \leq c_{ij} + c_{ji}, \iff \lambda \geq \frac{c_{ij} + c_{ji}}{2}.
\]

More systematically, set \( b \) to be the all-zero vector, \( b_{n+1} = 1 \), and view the primal program as a linear program over \( \mathbb{Q} \). (We can always do this, as there are finitely many decision variables). Then the primal program becomes

\[
\begin{align*}
\text{Minimize} & \quad \sum_{i,j=1}^n c_{ij} u_{ij} \\
\text{subject to} & \quad u_{ij} \geq 0 \text{ for all } 1 \leq i, j \leq n, \\
& \quad \sum_{i,j=1}^n u_{ij} = 1 \quad \text{and} \quad \sum_{j=1}^n u_{ij} = \sum_{k=1}^n u_{ki} \text{ for all } 1 \leq i \leq n.
\end{align*}
\]

The constraints require \( (u_{ij}) \) to be a probability distribution on the edges of the graph of \( c \) that represents a flow. The set of feasible solutions is a convex polytope called the normalized cycle polytope. Its vertices are the uniform probability distributions on directed cycles. Let \( \mathcal{N}_n \) denote the normal fan of the normalized cycle polytope. The cone \( R_n \) can be identified with the cone of \( \mathcal{N}_n \) of codimension \( n-1 \), indexed by \( n \) self-loops, one at each node \( i \in [n] \). Since the value of the dual is \( \lambda \), by strong duality, \( \lambda \) is precisely the value of the minimum normalized cycle in the graph weighted by \( c \).

**Definition 26.** For \( c \in \mathbb{R}^{n \times n} \), let \( \lambda(c) \) be the minimum normalized cycle in the graph weighted by \( c \). Define the polytrope of \( c \) to be

\[
\text{Pol}(c) = \{ y \in \mathbb{T}P^{n-1} : y_i - y_j \leq c_{ij} + \lambda(c) \text{ for all } i, j \in [n] \}
\]

This definition reduces to \( ([3]) \) when \( \lambda(c) = 0 \), in particular, when \( c \in R_n \). If \( \lambda(c) \neq 0 \), consider the matrix \( c - \lambda(c) \) obtained from \( c \) by subtracting \( \lambda(c) \) element-wise. Since \( \lambda(c - \lambda(c)) = 0 \), our previous discussions on polytopes and shortest paths apply. Thus, \( \text{Pol}(c) = \text{Pol}(c - \lambda(c)) \) is always convex in both ordinary and tropical sense. Its tropical vertices are the \( n \) column vectors of \( (c - \lambda(c))^* \). The pair \( (\lambda(c), \text{Pol}(c)) \) is the tropical eigenvalue-eigenspace pair of the matrix \( c \), see \([5][15][18]\).

Let \( \mathcal{F}^h_n \) denote the fan in \( \mathbb{R}^{n \times n} \) of linearity of the polytrope map \( c \mapsto (c - \lambda(c))^* \). This is a polytopal fan which refines \( \mathcal{N}_n \) \([18]\). The subfan of \( \mathcal{F}^h_n \) restricted to the cone of \( \mathcal{N}_n \) identified with \( R_n \) is \( \mathcal{F}_n \), the fan of linearity of the polytrope map on \( R_n \) in Section \([4.2]\). This gives a simple proof that \( \mathcal{F}_n \) is a fan in \( R_n \). Analogously, the Gröbner fan \( \mathcal{G}\mathcal{F}_n \) of \( I^h \) refines \( \mathcal{F}^h_n \), and its restriction to \( R_n \) is the fan \( \mathcal{G}\mathcal{F}_n \). The open cones of \( \mathcal{G}\mathcal{F}^h_n \) not in \( \mathcal{G}\mathcal{F}_n \) correspond to matrices \( c \in \mathbb{R}^{n \times n} \) where \( (c - \lambda(c))^* \) are not full rank \([5] \S 4.6, [3] \S 3.2.4\).
6. Polytopes enumeration: algorithms, results and summary

6.1. Algorithms and results. We have two algorithms for enumerating combinatorial tropical types of full-dimensional polytopes in $\mathbb{T}^3$. Recall that we are enumerating cones of $\mathcal{G}F_n|_P$ which are not on $\partial R_n$, up to symmetry induced by $S_n$. The two algorithms differ only in the first step of computing $\mathcal{G}F_n|_P$. The first computes $\mathcal{G}F_n|_P$ using a Gröbner fan computation software such as gfan, while the second computes the refinement of $F_n$ and $BB_n$ over $P_n$. Given $\mathcal{G}F_n|_P$, one can then remove all cones in $\partial R_n$. We find such cones as follows: for each cone, pick a point $c$ in the interior and compute the minimum cycle in the undirected graph with edge weights $c_{ij}$. If the minimum cycle is zero, this point comes from a cone on $\partial R_n$, and thus should be removed. A documented implementation of the first algorithm, with examples for $n=4$ and input files for $n=4, 5$ and 6, is available at https://github.com/princengoc/polytropes.

For $n=4$, we found 1026 symmetry classes of cones in $\mathcal{G}F_n|_P$, of which 13 are in $\partial R_n$. Thus, there are 1013 combinatorial tropical types of polytopes in $\mathbb{T}^3$. Table 2 classifies the types by the number of vertices of the polytrope. This corresponds to the first column of [11, Table 1].

| # vertices | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|------------|---|---|---|---|---|---|----|----|----|----|----|----|----|----|----|----|----|
| # types    | 1 | 1 | 5 | 6 | 34| 38| 81 | 101| 151| 144| 154| 116| 92 | 46 | 28 | 9  | 6  |

Table 2. Combinatorial tropical types of full-dimensional polytopes in $\mathbb{T}^3$, grouped by total number of vertices.

We also implemented the second algorithm for $n=4$. We found 273 equivalence classes of cones of $F_4$. Table 3 groups them by the number of equivalence classes of cones in the refinement $F_4 \land BB_4$ that they contain. Altogether, we obtain 1013 equivalence classes, agreeing with the first output.

| # $F$ | 123 | 10 | 89 | 19 | 2 | 19 | 2 | 3 | 1 | 1 | 1 |
|-------|-----|----|----|---|---|----|---|---|---|---|---|
| # $(F,z)$ | 1 | 2 | 3 | 5 | 6 | 9 | 15 | 18 | 27 | 37 | 42 | 81 |

Table 3. Equivalence classes of cones $F$ of $F_4$, grouped by the number of equivalence classes of cones in $\mathcal{G}F_4|_P$ (which equals $F_4 \land BB_4$ on $P_4$) that they correspond to.

The polytrope complex $\mathcal{G}F_n|_P$ grows large quickly. For $n=5$ and $n=6$ respectively, there are 27248 and 22770 open cones, correspond to combinatorial tropical types of maximal polytopes in $\mathbb{T}^5$ and $\mathbb{T}^6$. This is clearly much bigger than six, the corresponding number for $n=4$. The fan $BB_5$ has $5\binom{5}{2} = 30$ two-bipartite binomial hyperplanes. Up to permutation, there are 11 types of relations analogous to that in Figure 2. We list them on https://github.com/princengoc/polytopes/output/n5relations.txt. We could not compute all cones of $\mathcal{G}F_5|_P$ and $\mathcal{G}F_6|_P$ on a conventional desktop. However, we believe that such computations should be possible on more powerful machines.

6.2. Summary and open problems. Tropical types of polytopes in $\mathbb{T}^{n-1}$ are in bijection with cones of the polyhedral complex $\mathcal{G}F_n|_P$. This complex is the restriction of a certain Gröbner fan $\mathcal{G}F_n \subset \mathbb{R}^{n^2-n}$ to a certain cone $P_n$. We showed that $\mathcal{G}F_n|_P$ equals the refinement of two smaller fans restricted to $P_n$. These are the fan of linearity
of the polytrope map $\mathcal{F}_n$ studied in [18], and the fan of bipartite binomial arrangement $\mathcal{BB}_n$, a rich and interesting arrangement central to polytropes and maximal polytropes. We utilized these results to enumerate all combinatorial tropical types of full-dimensional polytropes in $\mathbb{TP}^3$, and those of maximal polytropes in $\mathbb{TP}^4$ and $\mathbb{TP}^5$.

Both of the fans $\mathcal{F}_n$ and $\mathcal{BB}_n$ are significantly smaller than $\mathcal{GF}_n$, giving a computational advantage over brute force approaches. While $\mathcal{F}_n$ has been studied, we do not understand $\mathcal{BB}_n$, which is the more central object. For example, the number of chambers of $\mathcal{BB}_n$ up to $S_n$ action are precisely the number of combinatorial types of maximal polytropes. Can one obtain an explicit formula for arbitrary $n$?

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