Abilities and Limitations of Spectral Graph Bisection

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Spectral based heuristics belong to well-known commonly used methods for finding a minimum-size bisection in a graph. The heuristics are usually easy to implement and they work well for several practice-relevant classes of graphs. However, only a few research efforts are focused on providing rigorous analysis of such heuristics and often they lack of proven optimality or approximation quality. This paper focuses on the spectral heuristic proposed by Boppana almost three decades ago, which still belongs to one of the most important bisection methods.

It is well known that Boppana’s algorithm finds and certifies an optimal bisection with high probability in the random planted bisection model – the standard model which captures many real-world instances. In this model the vertex set is partitioned randomly into two equal sized sets, and then each edge inside the same part of the partition is chosen with probability $p$ and each edge crossing the partition is chosen with probability $q$, with $p \geq q$. In our paper we investigate the problem if Boppana’s algorithm works well in the semirandom model introduced by Feige and Kilian. The model generates initially an instance by random selection within the planted bisection model, followed by adversarial decisions. Feige and Kilian posed the question if Boppana’s algorithm works well in the semirandom model and it has remained open so far. In our paper we answer the question affirmatively. We show also that the algorithm achieves similar performance on graph models which generalize the semirandom model. On the other hand we prove some limitations of Boppana’s algorithm: we show that if the density difference $p-q \leq o(\sqrt{p \cdot \ln n} / \sqrt{n})$ then the algorithm fails with high probability in the planted bisection model. This bound is sharp, since under assumption $p-q \geq \Omega(\sqrt{p \cdot \ln n} / \sqrt{n})$ Boppana’s algorithm works well in the model.
1 Introduction

The minimum graph bisection problem is one of the classical NP-hard problems [19]: for an undirected graph $G$ the aim is to partition the set of vertices $V = \{1, \ldots, n\}$ ($n$ even) into two equal sized sets, such that the number of cut edges, i.e. edges with endpoints in different bisection sides, is minimized. The bisection width of a graph $G$, denoted by $bw(G)$, is then the minimum number of cut edges in a bisection of $G$. Due to practical and theoretical importance, the problem has been the subject of a considerable amount of research from different perspectives: approximation complexity [26, 2, 17, 16], parameterized algorithms [24, 31, 13] and average-case complexity [6].

Since the bisection problem is very hard in general – no polynomial-time algorithm is known even for approximating the minimum bisection to within a constant factor – an extensive research has been carried out to explore implementable heuristics that might be applied for wide, practice-relevant classes of graphs. Existing methods range from simple greedy techniques over flow algorithms [6] to spectral graph theory [25] and semidefinite programming [1]. A current survey of various bisection heuristics and common software packages is provided in [7].

In this paper we consider polynomial-time heuristics that for an input graph either output the minimum-size bisection or “fail”. In addition, for the output bisection, they give a proof that the bisection is optimal. The heuristics should work well for all (or almost all, depending on the model) realistic graphs, i.e. provide for them a certified optimum bisection, while for irregular, worst case instances the output can be “fail”, what is justifiable. In this paper we investigate two well-studied graph models which, as commonly believed, capture many real-world instances: The planted random model of Bui, Chaudhuri, Leighton and Sipser [6] and the semirandom model of Blum and Spencer [3] and Feige and Kilian [15]. Moreover, we consider the regular graph model introduced of Bui et al. [6] and a new extension of the semirandom model. For a (semi)random model of graphs we say that some property is satisfied with high probability (w.h.p.) if the probability that the property holds tends to 1 as the number of vertices $n \to \infty$.

In the planted bisection model, denoted as $G_n(p, q)$ with parameters $1 > p = p(n) \geq q(n) = q > 0$, the vertex set $V = \{1, \ldots, n\}$ is partitioned randomly into two equal sized sets $V_1$ and $V_2$, called the planted bisection. Then for every pair of vertices do independently: if both vertices belong to the same part of the bisection (either both belong to $V_1$ or both belong to $V_2$) then include an edge between them with probability $p$; If the two vertices belong to different parts, then connect the vertices by an edge with probability $q$. In the semirandom model for graph bisection [15], initially a graph $G$ is chosen at random according to model $G_n(p, q)$. Then a monotone adversary is allowed to modify $G$ by applying an arbitrary sequence of the following monotone transformations: (1) The adversary may remove from the graph any edge crossing a minimum bisection; (2) The adversary may add to the graph any edge not crossing the bisection. Finally, in the regular random model, denoted as $G_n(r, b)$, with $r = r(n) < n$ and $b = b(n) \leq (n/2)^2$, the probability distribution is uniform on the set of all graphs on $V$ that are $r$-regular and have bisection width $b$.

The main focus of our work is the bisection algorithm proposed by Boppana [5]. Though introduced almost three decades ago, the algorithm belongs still to the most important heuristics in this area. However, several basic questions concerning the algorithm’s performance remain open. Using a spectral approach, Boppana constructs an implementable algorithm
which, assuming the density difference
\[ p - q \geq \Omega(\sqrt{p \ln n / \sqrt{n}}) \] (1)
bisects \( G_n(p, q) \) optimally w.h.p. Remarkably, for a long time this was the largest subclass of graphs \( G_n(p, q) \) for which a minimum bisection could be found. The algorithm works well also on the regular graph model \( G_n(r, b) \), assuming that
\[ r \geq 6 \quad \text{and} \quad b \leq o(n^{1-1/(r+1)/2}) \] (2).

In this paper we investigate the problem if, under assumption (1), Boppana’s algorithm works well for the semirandom model. This question was posed by Feige and Kilian in [15] and remained open so far. Most recently, Makarychev, Makarychev, and Vijayaraghavan [23, Sec. 1.2] have claimed (without a proof) that spectral algorithms do not work for this model. In our paper we answer Feige and Kilian’s question affirmatively which disproves the claim of Makarychev et al. We show also that Boppana’s algorithm achieves similar performance on graph models which generalize the semirandom model of [15].

On the other hand we show some limitations of the algorithm. One of the main results in this direction is that the density difference (1) is tight: we prove that if \( p - q \leq o(\sqrt{p \cdot \ln n / \sqrt{n}}) \) then the algorithm fails on \( G_n(p, q) \) w.h.p. Up to our best knowledge this is the first impossibility result for the random planted bisection model.

Our Results. The motivation of our research was to systematically explore graph properties which guarantee that Boppana’s algorithm outputs a (certified) optimum bisection. Due to [5] we know that random graphs from \( G_n(p, q) \) and \( G_n(r, b) \) satisfy such properties w.h.p. under assumptions (1) and (2) on \( p, q, r, \) and \( b \) as discussed above. But, as we will see later, the algorithm works well also for instances which deviate significantly from such random graphs.

Our first technical contribution is a modification of the algorithm to cope with graphs of more than one optimum bisection, like e.g. hypercubes. The algorithm proposed by Boppana does not manage to handle such cases. Our modification is useful to work on wider classes of graphs.

In this paper we introduce a natural generalization of the semirandom model of Feige and Kilian [15]. Instead of \( G_n(p, q) \), we start with an arbitrary initial graph model \( G_n \), and then apply a sequence of the transformations by a monotone adversary as in [15]. We denote such a model by \( \mathcal{A}(G_n) \). One of our main positive results is that if Boppana’s algorithm outputs the minimum-size bisection for graphs in \( G_n \) w.h.p., then the algorithm finds a minimum bisection w.h.p. for the adversarial graph model \( \mathcal{A}(G_n) \), too. As a corollary, we get that under assumption (1), Boppana’s algorithm works well in the semirandom model, denoted here as \( \mathcal{A}(G_n(p, q)) \), and, assuming (2), in \( \mathcal{A}(G_n(r, b)) \) – the semirandom regular model.

To analyze limitations of the spectral approach we provide structural properties of the space of feasible solutions searched by the algorithm. This allows us to prove that if an optimal bisection contains some forbidden subgraphs, then Boppana’s algorithm fails. Using these tools, we were able to show that if the density difference \( p - q \) is asymptotically smaller than \( \sqrt{p \cdot \ln n / \sqrt{n}} \) then Boppana’s algorithm fails on \( G_n(p, q) \) w.h.p.

Since the behavior of the algorithm on the (common) semirandom model \( \mathcal{A}(G_n(p, q)) \) remained unknown so far, Feige and Kilian proposed in [15] a new semidefinite programming based approach which works for semirandom graphs, assuming (1). For a given graph \( G \) they define a minimization semidefinite program (SDP) with the objective function \( h_p(G) \) and
prove that $h_d(G)$, which is the objective function of the dual maximization SDP, w.h.p. reaches $bw(G)$ on graphs in $G_n(p, q)$. On the other hand they show that $h_p(G) \leq bw(G)$ and that $h_p$ preserves minimal bisection regardless of monotone adversary transformations. The proposed algorithm solves in polynomial time the primal SDP and reconstructs a certified minimum-size bisection of $G$ from a feasible solution attaining optimum $h_p(G)$. A proof that the algorithm works well on $A(G_n(p, q))$ follows from inequalities $h_d(G) \leq h_p(G) \leq bw(G)$ and the property that $h_d(G) = bw(G)$ w.h.p.

The relationship between the performance of the SDP based algorithm and Boppana’s approach was left in [15] as an open problem. Feige and Kilian conjecture that for every $G$, the function $h_p(G)$ and the lower bound computed in Boppana’s algorithm give the same value. In our paper we answer this question affirmatively. To compare the algorithms, we provide a primal SDP formulation for Boppana’s approach and prove that it is equivalent to the dual SDP of Feige and Kilian. Next we give a dual program to the primal formulation of Boppana’s algorithm and prove that the optima of the primal and dual programs are equal to each other. Note that unlike linear programming, for semidefinite programs there may be a duality gap.

An important advantage of the spectral method by Boppana over the SDP based approach by Feige and Kilian is that the spectral method is practically implementable reducing the bisection problem for graphs with $n$ vertices to computing minima of a convex function of $n$ variables while Feige and Kilian’s algorithms needs to solve a semidefinite program over $n^2$ variables.

Related Work. Spectral partitioning goes back to Fiedler [18], who first proposed to use eigenvectors to derive partitions. Spielman and Teng e.g. showed, that spectral partitioning works well on planar graphs [27, 28], although there are also graphs on which purely spectral algorithms perform poorly, as shown by Guattery and Miller [21].

Although spectral partitioning works well, providing a guarantee for the solution to be optimal is hard. Some progress could be made explicitly for the planted bisection random graph model: Coja-Oghlan developed a new spectral algorithm [9] which enables for a wider range of parameters in the planted bisection model than the generic algorithm of Boppana. Latest research by Coja-Oghlan et al. provides better understanding of the planted bisection model [11] and average case behaviour of a minimum bisection. Still the beforementioned algorithms stay the best for certifying partitions.

Also other algorithms have been proven to work on the planted bisection model. Condon and Karp [12] developed a linear time algorithm for the more general $l$-partitioning problem. Their algorithm finds the optimal partition with probability $1 - \exp(-n^{\Theta(\epsilon)})$ in the planted bisection model with parameters satisfying $p - q = \Omega(1/n^{1/2-\epsilon})$. Carson and Impaglizzo [8] show that a hill-climbing algorithm is able to find the planted bisection w.h.p. for parameters $p - q = \Omega((\ln n)/n^{1/4})$.

The paper is organized as follows. The next section contains an overview over Boppana’s algorithm. In Section 3 we propose a modification of the algorithm to deal with non-unique optimum bisections. In Section 4 we define the adversarial graph model and show, that Boppana’s algorithm works well on this class. Next we develop a new analysis of the algorithm and use it to show some limitations of the method. Finally, in Section 5 we compare the algorithm to the SDP approach of Feige and Kilian. We conclude the paper with a discussion. The proofs of most of the propositions presented in Sections 2 through 6 are moved to the appendix (Section 8).
2 Boppana’s Graph Bisection Algorithm

In this section we fix definitions and notations used in our paper and we recall Boppana’s algorithm and known facts on its performance. We need the details of the algorithm to describe its extension in the next section. For a given graph $G = (V, E)$, with $V = \{1, \ldots, n\}$, Boppana defines a function $f$ for all real vectors $x, d \in \mathbb{R}^n$ as

$$f(G, d, x) = \sum_{(i,j) \in E} \frac{1-x_i x_j}{2} + \sum_{i \in V} d_i (x_i^2 - 1).$$ (3)

Call by $S \subset \mathbb{R}^n$ the subspace of all vectors $x \in \mathbb{R}^n$, with $\sum_i x_i = 0$. Based on $f$, the function $g'$ is defined as follows

$$g'(G, d) = \min_{\|x\|^2 = n, x \in S} f(G, d, x),$$ (4)

where $\|x\|$ denotes $L_2$ norm of $x$. Vector $x$ is named a bisection vector if $x \in \{+1, -1\}^n$ and $\sum_i x_i = 0$. Such $x$ determines a bisection of $G$ of the cut width denoted as cutwidth($x$) = $\sum_{(i,j) \in E} \frac{1-x_i x_j}{2}$. For a bisection vector $x$ the function $f$ takes the value (7) regardless of $d$. Minimization over all such $x$ would give the minimum bisection width. Since $g'$ uses a relaxed constraint we get $g'(G, d) \leq bw(G)$ where, recall, $bw(G)$ denotes the bisection width of $G$. To improve the bound, Boppana tries to find some $d$ which leads to a minimal decrease of the function value of $g'$ compared to the bisection width:

$$h(G) = \max_{d \in \mathbb{R}^n} g'(G, d).$$ (5)

It is easy to see that for every graph $G$ we have $h(G) \leq bw(G)$.

In order to compute $g'$ efficiently, Boppana expresses the function in spectral terms. To describe this we need some definitions. Let $I$ denote the $n$-dimensional identity matrix and let $P = I - \frac{1}{n} J$ be the projection matrix which projects a vector $x \in \mathbb{R}^n$ to the projection $Px$ of vector $x$ into the subspace $S$. Here, $J$ denotes an $n \times n$ matrix of ones. For a matrix $B \in \mathbb{R}^{n \times n}$, the matrix $B_S = PBP$ projects a vector $x \in \mathbb{R}^n$ to $S$, then applies $B$ and projects the result again into $S$. Further, for $B \in \mathbb{R}^{n \times n}$ and $d \in \mathbb{R}^n$ we denote the sum of $B$’s elements as $\sum(B) = \sum_{ij} B_{ij}$ and by $\text{diag}(d)$ we denote the $n \times n$ diagonal matrix $D$ with the entries of the vector $d$ on the main diagonal, i.e. $D_{ii} = d_i$.

Now assume $B \in \mathbb{R}^{n \times n}$ is symmetric and let $B_S = PBP$. Denote by $\mathbb{R}^n_{\neq 1}$ the real space $\mathbb{R}^n$ without the subspace spanned by the identity vector $1$, i.e. $\mathbb{R}_x \neq 1 = \mathbb{R}^n \setminus \{c1 : c \in \mathbb{R}\}$. We define $\lambda(B_S) = \max_{x \in \mathbb{R}^n_{\neq 1}} \frac{x^T B_S x}{\|x\|^2}$. It is easy to see that if $\lambda(B_S) \geq 0$ then

$$\lambda(B_S) = \max_{x \in \mathbb{R}^n} \frac{x^T B_S x}{\|x\|^2}$$ (6)

i.e. $\lambda(B_S)$ is the largest eigenvalue of the matrix $B_S$. Vectors $x$ that attain the maximum are exactly the eigenvectors corresponding to the largest eigenvalue $\lambda(B_S)$ of $B_S$.

Let $G$ be an undirected graph with $n$ vertices and adjacency matrix $A$. Let further $d \in \mathbb{R}^n$ be some vector and let $B = A + \text{diag}(d)$, then we define

$$g(G, d) = \frac{\sum(B) - n \lambda(B_S)}{4}.$$ In [7] it is shown that function $g'$ can be expressed as $g'(G, d) = g(G, -4d)$. Since in the definition of $h$ in (5) we maximize over all $d$, we can conclude that

$$h(G) = \max_{d \in \mathbb{R}^n} g(G, d) = \max_{d \in \mathbb{R}^n} \frac{\sum(A + \text{diag}(d)) - n \lambda((A + \text{diag}(d))_S)}{4}.$$ (7)
Boppana’s algorithm that finds and certifies an optimal bisection, works as follows:

1. Compute \( h(G) \): Numerically find a vector \( d^{opt} \) which maximizes \( g(G, d) \). Let \( D = \text{diag}(d^{opt}) \). Use constraint \( \sum d_i^{opt} = 2|E| \) to ensure \( \lambda((A + D)_S) > 0 \).
2. Construct a bisection: Let \( x \) be an eigenvector corresponding to the eigenvalue \( \lambda((A + D)_S) \). Construct a bisection vector \( \hat{x} \) by splitting at the median \( \bar{x} \) of \( x \), i.e. let \( \hat{x}_i = +1 \) if \( x_i \geq \bar{x} \) and \( \hat{x}_i = -1 \) if \( x_i < \bar{x} \). If the partition vector \( \hat{x} \) has no zero sum, move (arbitrarily) some vertices \( i \) with \( x_i = \bar{x} \) to part \(-1\) letting \( \hat{x}_i = -1 \).
3. If \( h(G) = \text{cutwidth}(\hat{x}) \), output certified optimal bisection \( \hat{x} \); otherwise output “fail”.

One can prove that \( g \) is concave and hence, the maximum in Step 1 can be found in polynomial time with arbitrary precision \([20]\). To analyse the algorithm’s performance, Boppana proves the following, for a sufficiently large constant \( c_0 > 0 \):

**Theorem 2.1** (Boppana \([5]\)). Let \( G \) be a random graph from \( G_n(p, q) \), and let \( p - q \geq c_0(\sqrt{\frac{\ln n}{n}}) \). Then with probability \( 1 - O(1/n) \), the bisection width of \( G \) equals \( h(G) \).

From this result one can conclude that the algorithm computes value \( h(G) \) that is, w.h.p., equal to the optimal bisection width of \( G \). However, to guarantee that the algorithm works well one needs additionally to show that it finds and certifies an optimal bisection. Formally, we need:

**Theorem 2.2.** For random graphs \( G \) from \( G_n(p, q) \), with \( p - q \geq c_0(\sqrt{\frac{\ln n}{n}}) \), Boppana’s algorithm certifies the optimality of \( h(G) \) revealing w.h.p. the bisection vector \( \hat{x} \) of cutwidth(\( \hat{x} \)) = \( h(G) \).

To prove this theorem one first has to revise carefully the proof of Theorem 2.1 in \([5]\) and show that w.h.p. the multiplicity of the largest eigenvalue of the matrix \( (A + D)_S \) in Step 1 is 1. This was observed already in \([4]\). Next we need the following property:

**Lemma 2.3.** Let \( G \) be a graph with \( h(G) = \text{bw}(G) \) and let \( d^{opt} \in \mathbb{R}^n \) s.t. \( g(G, d^{opt}) = \text{bw}(G) \) and \( \sum d_i^{opt} \geq 4\text{bw}(G) - 2|E| \). Denote further by \( B^{opt} = A + \text{diag}(d^{opt}) \). Then every optimum bisection vector \( y \) is an eigenvector of \( B^{opt}_S \) corresponding to the largest eigenvalue \( \lambda(B^{opt}_S) \).

(The proof of Lemma 2.3 as the proofs of most of the remaining propositions presented in this paper, are given in Section \([8]\).) This completes the proof that the algorithm works well on random graphs from \( G_n(p, q) \).

### 3 Certifying Non-Unique Optimum Bisections

From the previous section we know that if the bound \( h(G) \) is tight and the bisection of minimum size is unique, or more precisely the multiplicity of the largest eigenvector of \( B_S \) is 1, Boppana’s algorithm is able to certify the optimality of the resulting bisection. We say that a graph \( G \) has a unique optimum bisection if there exists a unique, up to the sign, bisection vector \( x \) such that \( \text{cutwidth}(x) = \text{cutwidth}(-x) = \text{bw}(G) \). In this paper we investigate families of graphs, different than random graphs \( G_n(p, q) \), for which the Boppana’s approach works well. To this aim we first need to show a modification which handles cases such that \( h(G) = \text{bw}(G) \) but for which no unique bisection of minimum size exists. As we will see later hypercubes satisfy these two conditions. We present our algorithm below. Note that the first step is the same as in the original algorithm by Boppana.
1. Compute $h(G)$: Numerically find a vector $d^{\text{opt}}$ which maximizes $g(G, d)$. Let $B^{\text{opt}} = A + \text{diag}(d^{\text{opt}})$. Use constraint $\sum d_{i}^{\text{opt}} = 2|E|$ to ensure $\lambda(B_{S}^{\text{opt}}) > 0$.

2. Let $k$ be the multiplicity of the largest eigenvector of $B_{S}^{\text{opt}}$ and let $M \in \mathbb{R}^{n \times k}$ be the real matrix with $k$ linearly independent eigenvectors corresponding to the largest eigenvalue of $B_{S}^{\text{opt}}$.

3. Transform the matrix to the reduced column echelon form, i.e. there are $k$ rows which form an identity matrix, s.t. $M$ still spans the same subspace.

4. Brute force: for every combination of $k$ coefficients from $\{+1, -1\}$ take the linear combination of the $k$ vectors of $M$ with the coefficients and verify if the resulting vector $x$ is a bisection vector, i.e. $x \in \{+1, -1\}^{n}$ with $\sum i x_{i} = 0$. If yes and if cutwidth($x$) = $h(G)$ then output $x$ and continue. This needs $2^{k}$ iterations.

5. If in Step 4 no bisection vector $x$ is given then output “fail”.

Theorem 3.1. If $h(G) = \text{bw}(G)$ then the algorithm above reconstructs all optimal bisections. Every achieved bisection vector corresponds to an optimal bisection.

The eigenvalues for the family of hypercubes are explicitly known [22]. Hence, it is easy to verify that the bound $h(G)$ is tight and Boppana’s algorithm with the modification above works, i.e. finds an optimal bisection. For a hypercube $H_{n}$ with $n$ vertices we have $h(H_{n}) = g(H_{n}, (2 - \log n)1) = n/2 = \text{bw}(H_{n})$. Since the hypercube with $n$ vertices has $\log n$ optimal bisections and the largest eigenspace of $B_{S}$ has multiplicity $\log n$, the brute force part in our modification of Boppana’s algorithm results in a linear factor of $n$ for the overall runtime. Therefore, the algorithm runs in polynomial time. In the next section we will extend this result to an adversarial model based on hypercubes and show, that Boppana’s algorithm works on that model as well.

4 Bisections in Adversarial Models

We introduce the adversarial model, denoted by $A(G_{n})$, as a generalization of the semirandom model in the following way: Initially a graph $G$ is chosen at random according to the model $G_{n}$. Let $(Y_{1}, Y_{2})$ denote an optimal bisection of $G$. Then, similarly as in [15], a monotone adversary is allowed to modify $G$ by applying an arbitrary sequence of the following monotone transformations:

1. The adversary may remove from the graph any edge $\{u, v\}$ crossing a minimal bisection ($u \in Y_{1}$ and $v \in Y_{2}$);
2. The adversary may add to the graph any edge $\{u, v\}$ not crossing the bisection ($u, v \in Y_{1}$ or $u, v \in Y_{2}$).

We will prove that Boppana’s algorithm works well for graphs from adversarial model $A(G_{n})$ if the algorithm works well for $G_{n}$. First we show that, if the algorithm is able to find an optimal bisection size of a graph, we can add edges within the same part of an optimum bisection and that we can remove cut edges, and the algorithm will still work. This solves the open question of Feige and Kilian [15].

Note that the result follows alternatively from Corollary 4.3 (presented in Section 4) that the SDPs of [15] are equivalent to Boppana’s optimization function and form the property proved in [15] that the objective function of the dual SDP of Feige and Kilian preserves minimal bisection regardless of monotone transformations. The aim of this section is to give a direct proof of this property for Boppana’s algorithm.
Theorem 4.1. Let \( G = (V, E) \) be a graph with \( h(G) = \text{bw}(G) \). Consider some optimum bisection \( Y_1, Y_2 \) of \( G \).

1. Let \( u \) and \( v \) be two vertices within the same part, i.e. \( u, v \in Y_1 \) or \( u, v \in Y_2 \), and let \( G' = (V, E \cup \{ \{u, v\}\}) \). Then \( h(G') = \text{bw}(G') \).

2. Let \( u \) and \( v \) be two vertices in different parts, i.e. \( u \in Y_1 \) and \( v \in Y_2 \), with \( \{\{u, v\}\} \in E \) and let \( G' = (V, E \setminus \{\{u, v\}\}) \). Then \( h(G') = \text{bw}(G) - 1 = \text{bw}(G') \).

Proof. We start by proving the first part, i.e. when we add an edge \( \{u, v\} \). Let \( A \) and \( A' \) denote the adjacency matrices of \( G \) and \( G' \), respectively. It holds \( A' = A + A^\Delta \) with \( A^\Delta_{uv} = A^\Delta_{vu} = 1 \) and zero everywhere else. Since \( h(G) = \text{bw}(G) \), there exists a \( d^{\text{opt}} \) with \( g(G, d^{\text{opt}}) = \text{bw}(G) \). For \( G' \), we set \( d' = d^{\text{opt}} + d^\Delta \) with

\[
d^\Delta_i = \begin{cases} -1 & \text{if } i = u \text{ or } i = v, \\ 0 & \text{else}. \end{cases}
\]

W.l.o.g., we restrict ourselves to solutions, with \( \sum_i d^\Delta_i = 4 \text{bw}(G) - 2|E| \) and hence have \( \lambda(B^\Delta_S) = 0 \) where \( B^\Delta = A + \text{diag}(d^\Delta) \). Since \( \sum_i d'_i = 4 \text{bw}(G) - 2|E| - 2 = 4 \text{bw}(G) - 2|E'| \), we want to show that \( \lambda(B^\Delta_S) = 0 \) holds, where \( B' = A' + \text{diag}(d') \). Since \( B' = A + A^\Delta + \text{diag}(d^{\text{opt}} + d^\Delta) = B^{\text{opt}} + A^\Delta + \text{diag}(d^\Delta) \), we get

\[
\lambda(B^\Delta_S) = \max_{x \in S \setminus \{0\}} \frac{x^T (B^{\text{opt}} + A^\Delta + \text{diag}(d^\Delta)) x}{\|x\|^2} = \max_{x \in S \setminus \{0\}} \frac{x^T B^{\text{opt}} x + x^T (A^\Delta + \text{diag}(d^\Delta)) x}{\|x\|^2}
\]

\[
= \max_{x \in S \setminus \{0\}} \frac{x^T B^{\text{opt}} x + 2x_u x_v - x_u^2 - x_v^2}{\|x\|^2} = \max_{x \in S \setminus \{0\}} \frac{x^T B^{\text{opt}} x - (x_u - x_v)^2}{\|x\|^2}
\]

\[
\leq \max_{x \in S \setminus \{0\}} \frac{x^T B^{\text{opt}} x}{\|x\|^2} = 0.
\]

For the bisection vector of an minimal cut size, we have \( x_u = x_v = 1 \) or \( x_u = x_v = -1 \) and thus the last inequality is equality. Hence, \( \lambda(B^\Delta_S) = 0 \) and \( g(G', d') = \text{bw}(G') \). This completes the proof for the first part.

The proof for the second part is similar to the above one. Assume \( \{u, v\} \), with \( u \in Y_1 \) and \( v \in Y_2 \), is a removed edge from \( G \). We define \( d^\Delta \) as we have done above and we let \( A' = A + A^\Delta \), with \( A^\Delta_{uv} = A^\Delta_{vu} = -1 \) and zero everywhere else. Since \( \sum_i d'_i = 4 \text{bw}(G) - 2|E| - 2 = 4 \text{bw}(G') - 2|E'| \), our aim is to show that \( \lambda(B^\Delta_S) = 0 \) holds with \( B' = A' + \text{diag}(d') \). Indeed we have:

\[
\lambda(B^\Delta_S) = \max_{x \in S \setminus \{0\}} \frac{x^T B^{\text{opt}} x + x^T (A^\Delta + \text{diag}(d^\Delta)) x}{\|x\|^2} = \max_{x \in S \setminus \{0\}} \frac{x^T B^{\text{opt}} x - 2x_u x_v - x_u^2 - x_v^2}{\|x\|^2} = \max_{x \in S \setminus \{0\}} \frac{x^T B^{\text{opt}} x - (x_u + x_v)^2}{\|x\|^2}
\]

\[
= \max_{x \in S \setminus \{0\}} \frac{x^T B^{\text{opt}} x}{\|x\|^2} = 0.
\]
\[ \leq \max_{x \in S \setminus \{0\}} \frac{x^T B_{\text{opt}} x}{||x||^2} = 0. \]

For the bisection vector of an optimal bisection size, we have \( x_u = 1, x_v = -1 \) or \( x_u = -1, x_v = 1 \) and hence the last inequality is equality. We can conclude
\[ g(G', d') = \frac{\text{sum}(B') - n\lambda(B_S')}{4} = \frac{\text{sum}(B') - 0}{4} = \frac{4\text{bw}(G) - 4}{4} = \text{bw}(G) - 1. \]

This completes the proof of the theorem.

\textbf{Theorem 4.2.} If Boppana’s algorithm finds a minimum bisection for a graph model \( G_n \) w.h.p., then the algorithm finds a minimum bisection w.h.p. for the adversarial model \( A(G_n) \), too.

As a direct consequence, we obtain the following corollary regarding the semirandom graph model considered by Feige and Kilian:

\textbf{Corollary 4.3.} Under assumption (1), Boppana’s algorithm computes the minimum bisection in the semirandom model w.h.p.

In [5], Boppana also considers random regular graphs \( G_n(r, b) \), where a graph is chosen uniformly over the set of all \( r \)-regular graphs with bisection width \( b \). He shows that his algorithm works w.h.p. on this graph under the assumption that \( b = o(n^{1-1/(r+1)/2}) \). We can now define the semirandom regular graph model as adversarial model \( A(G_n(r, b)) \). Applying Theorem 4.2, we obtain

\textbf{Corollary 4.4.} Under assumption (1), Boppana’s algorithm computes the minimum bisection in the semirandom regular model w.h.p.

Theorem 4.2 can also be applied on deterministic graph classes, e.g. the class of hypercubes. We then obtain:

\textbf{Corollary 4.5.} Boppana’s algorithm (with our modification for non-unique bisections) finds an optimal bisection on adversarial modified hypercubes.

\textbf{5 The Limitations of the Algorithm}

Boppana shows, that his algorithm works well on some classes of random graphs. However, we do not know which graph properties force the algorithm to fail. For example, for the considered planted bisection model, we require a small bisection width. On the other hand, as we have seen in Section 3 Boppana’s algorithm works for the hypercubes and their semirandom modifications – graphs that have large minimum bisection sizes.

\textbf{5.1 A New Analysis}

In the following, we present newly discovered structural properties from inside the algorithm, which provide a framework for a better analysis of the algorithm itself.

Fixing the sum of \( d \), such that \( \sum_i d_i = 4\text{bw}(G) - 2|E| \), leaves \( h(G) \) unchanged. Thus, every \( d \) with another sum can be shifted and \( g(G, d) \) remains unchanged as well. Therefore, in the following we will only consider vectors \( d \) with sum as above. Furthermore, we can restrict ourselves to cases where \( \lambda(B_S) = 0 \) or \( \lambda(B_S) > 0 \) holds.
Let $y$ be a bisection vector of $G$. We define
\[
d^{(y)} = -\text{diag}(y)Ay.
\]
(8)

An equivalent but more intuitive characterization of $d^{(y)}$ is the following: $d_i^{(y)}$ is the difference between the number of adjacent vertices in other partition as vertex $i$ and the number of adjacent vertices in same partition as $i$. We start with a fact, which has been observed independently in [4].

**Lemma 5.1.** Let $G$ be a graph with $h(G) = \text{bw}(G)$ and let $y$ be the bisection vector of an arbitrary optimum solution. Then for every $d^{\text{opt}}$, with $g(G, d^{\text{opt}}) = \text{bw}(G)$ and $\sum_i d_i^{\text{opt}} = 4 \text{bw}(G) - 2|E|$, there exists some $\alpha(y) \in \mathbb{R}$ such that $d^{\text{opt}} = d^{(y)} + \alpha(y)y$.

**Lemma 5.2.** Let $G$ be a graph with $h(G) = \text{bw}(G)$ and assume there is more than one optimum bisection in $G$. Then (up to constant translation vectors $c1$) there exists a unique vector $d^{\text{opt}}$ with $g(G, d^{\text{opt}}) = \text{bw}(G)$. Additionally, for every bisection vector $y$ of an arbitrary optimum bisection in $G$ there exists a unique $\alpha(y)$ and the corresponding $d^{(y)}$, with $g(G, d^{(y)} + \alpha(y)y) = \text{bw}(G)$.

Thus, if there are two optimum bisections representing by $y$ and $y'$ with $d^{(y)} \neq d^{(y')}$, then the difference of the $d$-vectors in component $i$ is only dependent on $y_i$ and $y_i'$, since we have $d^{(y)} - d^{(y')} = \beta'y' - \beta y$ for some constants $\beta$ and $\beta'$.

However, although the values of $d^{(y)}$ and $d^{(y')}$ have to follow hard constraints, the $d^{(y)}$ is not unique, i.e. they are not necessarily the same. The following graph:

\[
\begin{array}{c}
l_3 \quad l_1 \quad a \quad l_2 \quad b \quad r_3 \quad r_1 \quad r_2 \\
\end{array}
\]

with two optimal bisections $((l_1, l_2, l_3, a), (b, r_1, r_2, r_3))$, resp. $((l_1, l_2, l_3, b), (a, r_1, r_2, r_3))$, is an example where (the modified) Boppana’s algorithm works but the bisections determine two linear independent vectors $d^{(y)} = (-2, -2, -2, +1, -1, -2, -2)$ resp. $d^{(y')} = (-2, -2, -2, -1, +1, -2, -2, -2)$.

### 5.2 Forbidden Substructures

The following fact follows directly from Theorem [4,1].

**Corollary 5.3** (Necessary for single edge). Let $G$ be a graph with $h(G) = \text{bw}(G)$. Let $e$ be some cut edge of an optimal bisection. When we remove all cut edges except $e$, then $h(G) = 1$.

In the other direction: When it does not work for a certain cut edge, it can not get to work with the other cut edges combined. However, combining working cut edges is able to fail the algorithm.

**Lemma 5.4** (Necessary for many edges). Let $G = (V, E)$ be a graph and $y$ an optimal bisection vector of $G$. For $i \in \{+1, -1\}$ let $C_i = \{u \mid y_u = i \land \exists v : y_v = -i \land \{u, v\} \in E\}$ be the set of vertices in part $i$ located at the cut. If there exist non-empty $\hat{C}_i \subseteq C_i$ with $k = \min\{|\hat{C}_+|, |\hat{C}_{-1}|\}$, $k + \delta = \max\{|\hat{C}_+|, |\hat{C}_{-1}|\}$, $l = |V| - (k + \delta)$, s.t. ...
• $3k < l$ and $\delta = 0$

• or $4k < l$ and $\delta < \min\{\frac{4k^2}{14k}, \frac{7}{128}\}$,

• $2|E(\tilde{C}_+ \cup \tilde{C}_-)| \geq |E(\tilde{C}_+ \cup \tilde{C}_- \setminus (\tilde{C}_+ \cup \tilde{C}_-))|$

then $h(G) < bw(G)$.

![Forbidden Graph Structures](image)

**Figure 1:** Forbidden graph structures

**Corollary 5.5.** Let $G$ be a graph, as illustrated in Fig. 1(a) with $n \geq 10$ vertices containing a path segment $\{u', u\}, \{u, w\}, \{w, w'\}$, where $u$ and $w$ have no further edges. If there is an optimal bisection $y$, s.t. $y_u = y_{u'} = +1$ and $y_w = y_{w'} = -1$ (i.e. $\{u, w\}$ is a cut edge), then $h(G) < bw(G)$.

To prove this corollary, we use Lemma 5.4 with parameters $\tilde{C}_+ = \{u\}$ and $\tilde{C}_- = \{w\}$. Then we have $\delta = 0$, $k = 1$, $l = 4$ and it follows directly that $h(G) < bw(G)$.

**Corollary 5.6.** Let $G$ be a graph with $n \geq 10c$ vertices containing a $2 \times c$ lattice with vertices $u_i$ and $w_i$, as illustrated in Fig. 1(b). (The construction is similar to the corollary above, but now we have a lattice instead of a single cut edge.) If there is an optimal bisection $y$, s.t. $y_{u_i} = y_{u'_i} = +1$ and $y_{w_i} = y_{w'_i} = -1$, then $h(G) < bw(G)$.

**Theorem 5.7.** Let $G$ be a graph with $h(G) = bw(G)$. Let $G'$ be the graph $G$ with two additional isolated vertices, then $h(G') \leq h(G) - \frac{4bw(G)}{n^2}$. (Note: $G$ has $n$ vertices and $G'$ has $n + 2$ vertices.)

### 5.3 Limitations for Sparse Planted Partition Model $G_n(p, q)$

In the planted partition model $G_n(p, q)$, if the graphs are dense, e.g. $p = \frac{1}{n^c}$ for a constant $c$ with $0 < c < 1$, the constraints for the density difference $p - q$ assumed in Boppana’s and Coja-Oghlan’s algorithms are essentially the same. However for sparse graphs, e.g. such that $q = O(1)/n$, the situation changes drastically. Now, e.g. $p = \sqrt{\log n}/n$ satisfy Coja-Oghlan’s constraint $p - q \geq \Omega(\sqrt{\rho \ln(pn)/\sqrt{n}})$ but the condition on the difference $p - q$ assumed by Boppana is not true any more.

**Theorem 5.8.** The algorithm of Boppana fails in the subcritical phase w.h.p., i.e. in case $n(p - q) = \sqrt{np \cdot \gamma \ln n}$, for real $\gamma > 0$. 

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The eigenvalues of a symmetric matrix are real. 

The dual program associated with the SDP (for details see e.g. [32]) is the program over the variables \( x \) for all real vectors \( v \) where \( \text{tr}(A) \) denote by \( A \) then \( \text{bw}(A) \) achieves their respective optima. 

for semidefinite programs there may be a duality gap, i.e. the primal and/or dual might not attain their respective optima. 

It is known that the optimal value of the maximization dual SDP is never larger than the optimal value of the minimization primal counterpart. However, unlike linear programming, for semidefinite programs there may be a duality gap, i.e. the primal and/or dual might not attain their respective optima.

6 SDP Characterizations of the Graph Bisection Problem

Feige and Kilian express the minimum-size bisection problem for an instance graph \( G \) as a semidefinite programming problem (SDP) with solution \( h_p(G) \) and prove that the function \( h_d(G) \), which is the solution to the dual SDP, reaches \( \text{bw}(G) \) w.h.p. Since \( \text{bw}(G) \geq h_p(G) \geq h_d(G) \), they conclude that \( h_p(G) \) as well reaches \( \text{bw}(G) \) w.h.p. The proposed algorithm computes \( h_p(G) \) and reconstructs the minimum bisection of \( G \) from the optimum solution of the primal SDP. The authors conjecture in [15, Sec. 4.1.] the following: "Possibly, for every graph \( G \), the function \( h_p(G) \) and the lower bound \( h(G) \) computed in Boppana’s algorithm give the same value, making the lemma that \( h_p(G) = \text{bw}(G) \) w.h.p. a restatement of the main theorem of [15]. In this section we answer this question affirmatively.

The semidefinite programming approach for optimization problems was studied by Alizadeh [1], who as first provided an equivalent SDP formulation of Boppana’s algorithm. Before we give an SDP introduced by Feige and Kilian, we recall briefly some basic definitions and provide an SDP formulation for Boppana’s approach. On the space \( \mathbb{R}^{n \times m} \) of \( n \times m \) matrices, we denote by \( A \bullet B \) an inner product of \( A \) and \( B \) defined as \( A \bullet B = \text{tr}(AB) = \sum_{i=1}^{n} \sum_{j=1}^{m} A_{ij} B_{ij} \), where \( \text{tr}(C) \) is the trace of the (square) matrix \( C \). Let \( A \) be an \( n \times n \) symmetric real matrix, then \( A \) is called symmetric positive semidefinite (SPSD) if \( A \) is symmetric, i.e. \( A^T = A \), and for all real vectors \( v \in \mathbb{R}^n \) we have \( v^T A v \geq 0 \). This property is denoted by \( A \succeq 0 \). Note that the eigenvalues of a symmetric matrix are real.

For given real vector \( c \in \mathbb{R}^n \) and \( m + 1 \) symmetric matrices \( F_0, \ldots, F_m \in \mathbb{R}^{n \times n} \) an SDP over variables \( x \in \mathbb{R}^n \) is defined as

\[
\min_x c^T x \quad \text{subject to} \quad F_0 + \sum_{i=1}^{m} x_i F_i \succeq 0. \tag{9}
\]

The dual program associated with the SDP (for details see e.g. [32]) is the program over the variable matrix \( Y = Y^T \in \mathbb{R}^{n \times n} \):

\[
\max_Y -F_0 \bullet Y \quad \text{subject to} \quad \forall i : F_i \bullet Y = c_i \quad \text{and} \quad Y \succeq 0. \tag{10}
\]

It is known that the optimal value of the maximization dual SDP is never larger than the optimal value of the minimization primal counterpart.
To prove that for any graph $G$ Boppana’s function $h(G)$ gives the same value as $h_p(G)$ we formulate the function $h$ as a (primal) SDP. We provide also its dual program and prove that the optimum solutions of primal and dual are equal in this case. Then we show that the dual formulation of the Boppana’s optimization is equivalent to the primal SDP defined by Feige and Kilian [12].

Below, $G = (V, E)$ denotes a graph, $A$ the adjacency matrix of $G$ and for a given vector $d$, as usually, let $D = \text{diag}(d)$, for short. We provide the SDP for the function $h$ (Eq. (7)) that differ slightly from that one given in [1].

**Proposition 6.1.** For any graph $G = (V, E)$, the objective function

$$h(G) = \max_{d \in \mathbb{R}^n} \frac{\sum(A + D) - n\lambda((A + D)s)}{4},$$

maximized by Boppana’s algorithm can be characterized as an SDP as follows:

$$p(G) = \min_{z \in \mathbb{R}, d \in \mathbb{R}^n} (nz - 1^T d) \quad \text{subject to}\quad zI - A + \frac{J}{n} - \frac{\sum(A)I}{n^2} - D + \frac{1}{n}d1^T - \frac{\sum(D)I}{n^2} \succeq 0,$$

with the relationship $h(G) = \frac{|E|}{2} - \frac{1}{2}p(G)$. The dual program to the program (11) can be expressed as follows:

$$d(G) = \max_{Y \in \mathbb{R}^{n \times n}} \left(A \cdot Y - 1 \cdot \frac{1}{n} \sum_j \deg(j) \sum_i y_{ij} - 1 \cdot \frac{1}{n} \sum_i \deg(i) \sum_j y_{ij} + \frac{1}{n^2} \sum_{i,j} y_{ij}\right)$$

subject to

$$\sum_i y_{ii} = n,$$

$$\forall i \quad y_{ii} - 1 \cdot \frac{1}{n} \sum_j y_{ji} - 1 \cdot \frac{1}{n} \sum_j y_{ij} + \frac{1}{n^2} \sum_{k,j} y_{kj} = 1,$$

$$Y \succeq 0.$$

Using these formulations we prove that the primal and dual SDPs attain the same optima.

**Theorem 6.2.** For the semidefinite programs of Proposition 6.1, the optimal value $p^*$ of the primal SDP (11) is equal to the optimal value $d^*$ of the dual SDP (12). Moreover, there exists a feasible solution $(z, d)$ achieving the optimal value $p^*$.

**Proof.** Consider the primal SDP (11) of Boppana in the form

$$\min_{z \in \mathbb{R}, d \in \mathbb{R}^n} z \quad \text{s.t.} \quad zI - M(d) \succeq 0,$$

with $M(d) = P(A + \text{diag}(d))P - \frac{1}{n}d^T I$ and, recall, $P = I - \frac{d}{n}$. Note that this formulation is equivalent to (11), as we have shown in the proof of Proposition 6.1. We show that this primal SDP problem is strictly feasible, i.e. that there exists an $z'$ and an $d'$ with $z' I - M(d') \succ 0$. To this aim we choose an $z'$ and then some $z' > \lambda(M(d'))$. From [32, Thm. 3.1], it follows that the optima of primal and dual obtain the same value.

To prove the second part of the theorem, i.e. there exists a feasible solution achieving the optimal value $p^*$, consider the following. The function $h(G)$ maximizes $g(G, d)$ over vectors $d \in \mathbb{R}^n$, while $d$ can be restricted to vectors of mean zero. The function $g$ is convex and goes to $-\infty$ for vectors $d$ with some component going to $\infty$. Thus, $g$ reaches its maximum at some finite $d^{\text{opt}}$. Now we choose $d = d^{\text{opt}}$ and $z = \lambda(M(d^{\text{opt}}))$. Clearly, this solution is feasible and obtains the optimal value $p^*$. 

\[ \square\]
For a graph $G = (V, E)$, Feige and Kilian express the minimum bisection problem as an SDP over an $n \times n$ matrix $Y$ as follows:

$$h_p(G) = \min_{Y \in \mathbb{R}^{n \times n}} h_Y(G) \text{ s.t. } \forall i \quad y_{ii} = 1, \sum_{i,j} y_{ij} = 0, \text{ and } Y \succeq 0,$$

where $h_Y(G) = \sum_{\{i,j\} \in E \ i < j} \frac{1-y_{ij}}{2}$. For proving that the SDP takes as optimum the bisection width w.h.p. on $G_n(p, q)$, the authors consider the dual of their SDP:

$$h_d(G) = \max_{x \in \mathbb{R}^n} \left( \frac{|E|}{2} + \frac{1}{4} \sum_i x_i \right) \text{ s.t. } M = -A - x_0 J - \text{diag}(x) \succeq 0,$$

where $A$ is the adjacency matrix of $G$. They show that the dual takes the value of the bisection width w.h.p. and bounds the optimum of the primal SDP. Although we know that their SDP and Boppana’s algorithm both work well on $G_n(p, q)$, it was open so far how they are related to each other. Below we answer this question showing that the formulations are equivalent. We start with the following:

**Theorem 6.3.** The primal SDP (13) is equivalent to the dual SDP (12), with the relationship $h_p(G) = \frac{|E|}{2} - \frac{1}{4} d(G)$.

From Theorems 6.2 and 6.3 we get

**Corollary 6.4.** Let $G$ be an arbitrary graph. Then for the lower bound $h(G)$ of Boppana’s algorithm and for the objective functions $h_p(G)$ of the primal SDP (13), resp. $h_d(G)$ of the dual SDP (13) of Feige and Kilian [15] it is true

$$h(G) = h_p(G) = h_d(G).$$

Thus, the both algorithms provide for any graph $G$ the same objective value. We want to point out another important fact: the bisection algorithm proposed in [15] use an SDP formulation, where the variables are a matrix with dimension $n \times n$. Thus, there are $n^2$ variables for a graph with $n$ vertices. In contrast, Boppana’s algorithm uses $n$ variables in the convex optimization problem. If we consider the dual SDP, we again have only $n+1$ variables. However, due to Corollary 6.4 we can’t be better than Boppana’s algorithm.

### 7 Discussion and Open Problems

Boppana’s spectral method is a practically implementable heuristic. Computing eigenvalues and eigenvectors is well-studied and can be done very efficiently. Falkner, Rendl and Wolkowicz [14] show in a numerical study that using spectral techniques for graph partitioning is very robust and upper and lower bounds for the bisection width can be obtained such that the relative gap is often just a few percentage points apart. In [30] and [29], Tu, Shieh and Cheng present numerical experiments including results for Boppana’s algorithm. They verify that the algorithm indeed has good average case behavior over certain probability distributions on graphs.

We conducted further experiments on the graph model $G_n(r, b)$ which indicated, that Boppana’s algorithm also works for $r = 5$, but not for $r = 3$ and $r = 4$. An interesting question
arising is, which properties of 3- and 4-regular graphs from the planted bisection model let the algorithm fail.

The SDP-based algorithm by Feige and Kilian \cite{FeigeKilian1997} and spectral method by Boppana \cite{Boppana1987} are both proven to work well in theory on the semirandom graph model for the same parameter ranges. However, for graphs with \( n \) vertices Boppana uses a convex function of \( n \) variables to compute his bound, while Feige and Kilian’s algorithms needs to solve a semidefinite program over \( n^2 \) variables.

In the planted partition model \( G_n(p,q) \), Boppana’s assumption \( p - q \geq \Omega(\sqrt{p \ln n}/\sqrt{n}) \) allows a large subclass of \( G_n(p,q) \) for which a certified minimum bisection could be found. This outperforms e.g. the conditions \( p - q \geq \Omega(1/n^{1/2-\varepsilon}) \) by Condon and Karp \cite{CondonKarp1995} and \( p - q \geq \omega((\sqrt{p \ln n})/\sqrt{n}) \) by Carson and Impagliazzo \cite{CarsonImpagliazzo2001}. The algorithm proposed by Coja-Oghlan’s \cite{Coja-Oghlan2011} assumes that \( p - q \geq \Omega(\sqrt{p \ln(np)}/\sqrt{n}) \). If the parameters \( p \) and \( q \) describe non-sparse graphs, this condition is essentially the same as Boppana’s assumption. For sparse graphs, however, Coja-Oghlan’s constraint allows a larger subclass. For example, the algorithm works in \( G_n(p,q) \) for \( q = O(1)/n \) and \( p = \sqrt{\log n}/n \). Due to our results we know that Boppana’s algorithm fails w.h.p. for such graphs. However, a drawback of Coja-Oghlan’s algorithm is that to work well in the planted bisection model with unknown parameters \( p \) and \( q \), the algorithm has to learn in a preprocessing phase the parameters with high precision, since the algorithm is based on the knowledge of values \( p \) and \( q \). Also the performance of the algorithm on other families, like e.g. semirandom graph and the regular random graphs \( G_n(r,b) \), is unknown.

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8 Proofs

Proof of Lemma 2.3

We know
\[ g(G, d^{\text{opt}}) = \text{bw}(G) = \frac{\sum(B^{\text{opt}}) - n\lambda(B^{\text{opt}})}{4} \]
\[ \iff \lambda(B_S^{\text{opt}}) = \frac{\sum(B^{\text{opt}}) - 4 \text{bw}(G)}{n} \geq \frac{\sum(A) + 4 \text{bw}(G) - 2|E| - 4 \text{bw}(G)}{n} = 0 \]

Thus, we conclude that \( \lambda(B_S^{\text{opt}}) \geq 0 \).

We compute the value of the Rayleigh quotient of \( B_S^{\text{opt}} \) and the optimum bisection vector \( y \):
\[ \frac{y^T B_S^{\text{opt}} y}{\|y\|^2} = \frac{y^T P B_S^{\text{opt}} P y}{\|y\|^2} = \frac{y^T B_S^{\text{opt}} y}{n} \]
\[ = \frac{y^T (A + \text{diag}(d^{\text{opt}})) y}{n} = \frac{y^T A y + \sum d_i^{\text{opt}}}{n} \]

We have \( y^T A y = \sum_{i,j} A_{ij} y_i y_j \). According to the definition \( A_{ij} = 1 \) if there is an edge \( \{i, j\} \in E \). Edges with both vertices in the same part contribute (twice) by 1 to the sum. Cut edges on the other hand contribute (twice) by \(-1\). There are \( \text{bw}(G) \) cut edges. Hence, \( y^T A y = \sum(A) - 4 \text{bw}(G) \) and we get:
\[ \frac{y^T A y + \sum d_i^{\text{opt}}}{n} = \frac{\sum(A) - 4 \text{bw}(G) + \sum d_i^{\text{opt}}}{n} = \frac{\sum(B^{\text{opt}}) - 4 \text{bw}(G)}{n} \]

Since the Rayleigh quotient of \( B_S^{\text{opt}} \) and \( y \) takes the value \( \lambda(B_S^{\text{opt}}) \) and \( \lambda(B_S^{\text{opt}}) \geq 0 \), we conclude that \( y \) is an eigenvector of \( B_S^{\text{opt}} \) corresponding to the eigenvalue \( \lambda(B_S^{\text{opt}}) \). \( \square \)

Proof of Theorem 3.1

Due to Lemma 2.3, all optimum bisection vectors \( y \) are found in this subspace. We show even more, namely that non-optimum bisection vectors \( y \) are not in this subspace. For contradiction, assume \( y \) is eigenvector. Consider
\[ \lambda(B_S^{\text{opt}}) = \frac{y^T B_S^{\text{opt}} y}{\|y\|} = \frac{y^T B_S^{\text{opt}} y}{\|y\|} = \frac{y^T (A + \text{diag}(d^{\text{opt}})) y}{\|y\|} = \frac{y^T A y + \sum d_i^{\text{opt}}}{\|y\|} \]

For a bisection vector \( y \), the value \( y^T A y \) counts the number of cut edges. Since this has been minimized, \( y \) has to be an optimal bisection. \( \square \)

Proof of Lemma 5.1

The assumptions imply \( \lambda(B_S^{\text{opt}}) = 0 \) with \( B^{\text{opt}} = A + \text{diag}(d^{\text{opt}}) \). Next, due to a fact stated in Lemma 2.3 \( y \) is an eigenvector of \( B_S^{\text{opt}} \) corresponding to the largest eigenvalue 0. We get the following sequence of equivalent conditions:
\[ B_S^{\text{opt}} y = 0 y = (0, \ldots, 0)^T \]
\[ PB^{\text{opt}} P y = (0, \ldots, 0)^T \]
\[ PB^{\text{opt}} y = (0, \ldots, 0)^T \]  \[ y \text{ has mean zero} \]

Since \( P \) projects into the zero vector only vectors of the subspace spanned by the identity vector, thus we can continue for some \( \alpha \in \mathbb{R} \)
\[ B^{\text{opt}} y = \alpha(1, \ldots, 1)^T \]
\[ (A + D^{\text{opt}}) y = \alpha(1, \ldots, 1)^T \]
\[ Ay + D^{\text{opt}} y = \alpha(1, \ldots, 1)^T \]
\[ D^{\text{opt}} y = -Ay + \alpha(1, \ldots, 1)^T \]

In the next step, we multiply the vectors in the equation with the diagonal matrix \( \text{diag}(y) \).
Since the \( y_i \in \{1, -1\} \), the multiplication is reversible and hence \( \Leftrightarrow \).
\[ \text{diag}(y) D^{\text{opt}} y = -\text{diag}(y) Ay + \alpha \text{diag}(y)(1, \ldots, 1)^T \]
\[ d^{\text{opt}} = -\text{diag}(y) Ay + \alpha y \]  \[ \text{Def. 8} \]

This completes the proof. Note that \( \sum_i y_i = 0 \) and \( \sum_i d_i(y) = 4 \text{bw}(G) - 2|E| \).

Proof of Lemma 5.2
Consider two optimum bisections with bisection vectors \( y \) and \( y' \). (Note that we consider \( y \) and \( -y \) as same bisection.) For contradiction, assume there are two different \( d_1^{\text{opt}} \neq d_2^{\text{opt}} \) (up to a constant transition). Due to Lemma 5.1 we have that for every \( y \) representing an optimum bisection values \( d_1^{\text{opt}} \) and \( d_2^{\text{opt}} \) can be expressed as \( d_1^{\text{opt}} = d(y) + \alpha_1 y \) and \( d_2^{\text{opt}} = d(y) + \alpha_2 y \). The difference is then
\[ d_1^{\text{opt}} - d_2^{\text{opt}} = (\alpha_1 - \alpha_2)y. \]
For \( y' \) representing an optimum bisection, we have analogously \( d_1^{\text{opt}} = d(y') + \beta_1 y' \) and \( d_2^{\text{opt}} = d(y') + \beta_2 y' \) with difference
\[ d_1^{\text{opt}} - d_2^{\text{opt}} = (\beta_1 - \beta_2)y'. \]

We conclude
\[ (\alpha_1 - \alpha_2)y = (\beta_1 - \beta_2)y'. \]
Since \( y \) and \( y' \) are linearly independent, we conclude \( \alpha_1 = \alpha_2 \) and \( \beta_1 = \beta_2 \). This means, if there are two optimum bisections, then there is only one \( d^{\text{opt}} \) and \( \alpha \) is unique!

Proof of Lemma 5.4
For contradiction, we assume \( h(G) = \text{bw}(G) \). For the bisection vector \( y \), we then have \( d^{\text{opt}} = d(y) + \alpha y \) for some \( \alpha \in \mathbb{R} \). Then \( \lambda(B_S) = 0 \) for \( B = A + \text{diag}(d^{\text{opt}}) \). We will contradict this by choosing a vector \( x \) and then show that the Rayleigh quotient for \( x \) and \( B_S \) is larger than 0 (for any \( \alpha \)). W. l. o. g. we assume \( |\hat{C}_{+1}| \geq |\hat{C}_{-1}| \). We choose
We take the larger $\beta$ with $\beta = 1$. We show that by our choice of $\alpha$, will have no effect: First we derive the $x_i$ solution with the +.

\[
x_i = \begin{cases} 
-1 & \text{if } y_i = +1 \land i \notin \tilde{C}_{+1}, \\
z & \text{if } i \in \tilde{C}_{+1} \cup \tilde{C}_{-1}, \\
-\beta z & \text{if } y_i = -1 \land i \notin \tilde{C}_{-1},
\end{cases}
\]

with $\beta = \sqrt{\frac{z \beta}{\delta + l}}$ and $z = \frac{2t + \delta l + 2(2k + \delta)l}{4k^2 + 4k\delta - \delta l}$. Note that for $\delta = 0$, we have $z = l/k > 3$, $\beta = 1/z < 1/3$ and $-\beta z = -1$.

First we derive the $z$ above by enforcing $\sum_i x_i = 0$ and choosing $\beta$ as above:

\[
\begin{align*}
\sum_i x_i &= l(-1) + (k + \delta)z + kz + (\delta + l)(-\beta z) \\
&= -l + (k + \delta)z + kz - (\delta + l)\sqrt{\frac{\delta z^2 + l}{\delta + l}} \\
&= -l + (2k + \delta)z - \sqrt{(\delta + l)(\delta z^2 + l)} \equiv 0
\end{align*}
\]\n
\[
\begin{align*}
\Leftrightarrow & \quad \sqrt{(\delta + l)(\delta z^2 + l)} = (2k + \delta)z - l \\
\Rightarrow & \quad (\delta + l)(\delta z^2 + l) = ((2k + \delta)z - l)^2 \\
\Rightarrow & \quad \delta^2 z^2 + \delta l + \delta lz^2 + l^2 = (2k + \delta)^2 z^2 + l^2 - 2(2k + \delta)lz \\
\Rightarrow & \quad \delta^2 z^2 + \delta l + \delta lz^2 = 4k^2 z^2 + \delta^2 z^2 + 4k\delta z^2 - 4klz - 2\delta lz \\
\Rightarrow & \quad 0 = (4k^2 + 4k\delta - \delta l)z^2 + (-4kl - 2\delta l)z - \delta l \\
\Rightarrow & \quad z = \frac{2kl + \delta l \pm \sqrt{(2kl + \delta l)^2 + \delta l(4k^2 + 4k\delta - \delta l)}}{4k^2 + 4k\delta - \delta l} \\
\Rightarrow & \quad z = \frac{2kl + \delta l \pm \sqrt{4(kl)^2 + 4kl\delta l + \delta l(4k^2 + 4k\delta)}}{4k^2 + 4k\delta - \delta l} \\
\Rightarrow & \quad z = \frac{2kl + \delta l \pm 2\sqrt{kl(kl + \delta l + (k + \delta))}}{4k^2 + 4k\delta - \delta l} \\
\Rightarrow & \quad z = \frac{2kl + \delta l \pm 2\sqrt{kl(kl + \delta)(l + \delta)}}{4k^2 + 4k\delta - \delta l}
\end{align*}
\]

We take the larger $z$-solution with the +.

We show that by our choice of $\beta$, the sum of squares for both parts is the same:

\[
\sum_{i: y_i = +1} x_i^2 - \sum_{i: y_i = -1} x_i^2 = (l(-1)^2 + (k + \delta)z^2) - (kz^2 + \delta l(-\beta z)^2)
\]

\[
= l + (k + \delta)z^2 - kz^2 - (\delta + l)\frac{\delta z^2 + l}{\delta + l} \\
= l + (k + \delta)z^2 - kz^2 - (\delta z^2 + l) = 0
\]

Thus, $\alpha$ will have no effect:

\[
\frac{x^T B x}{\|x\|^2} = \frac{x^T B x}{\|x\|^2}
\]

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\[
\begin{align*}
  &= \frac{x^T(A + \text{diag}(d^{(y)} + \alpha y)x}{\|x\|^2} \\
  &= \frac{x^T(A + \text{diag}(d^{(y)}))x + x^T(\alpha y)x}{\|x\|^2} \\
  &= \frac{x^T(A + \text{diag}(d^{(y)}))x}{\|x\|^2} \\
  &\quad \text{Assumption 4} \\
  &\quad \text{Assumption } 4k < l \\
  &\quad \text{Assumption } \delta < \min\{\frac{4k^2}{(l-4k)}, \frac{7}{128}\} \\
  &\quad \text{Next we show } z > 4. \text{ From } \delta < \frac{4k^2}{4k}, \text{ we get for the denominator of } z \text{ that } 4k^2 + 4k\delta - \delta l > 0. \text{ For the enumerator, we have:}
  \\
  2kl + \delta l + 2\sqrt{kl(k + \delta)(l + \delta)} = 2kl + 5\delta l + 2\sqrt{kl(k + \delta)(l + \delta) - 4\delta l} \\
  \quad \quad > 8k^2 + 20\delta k + 4\sqrt{k^2(k + \delta)(4k + \delta)} - 4\delta l \\
  \quad \quad = 8k(k + \delta) + 12\delta k + 4\sqrt{k^24k^2 - 4\delta l} \\
  \quad \quad > 16k(k + \delta) - 4\delta l \\
  \quad \quad = 4(4k(k + \delta) - \delta l) \quad \text{4 times denominator of } z
\end{align*}
\]

Since the enumerator is more than 4 times larger then the denominator and both are positive, we conclude \( z > 4 \). From \( \delta < \frac{7}{128} \), follows further, that \( \beta < 1/3 \):

\[
\beta^2 = \frac{\delta + l/\sqrt{z^2}}{\delta + l} \leq \frac{1}{9} = \left(\frac{1}{3}\right)^2
\]

\[
\Leftrightarrow 9(\delta + l/\sqrt{z^2}) \leq \delta + l
\]

\[
\Leftrightarrow 8\delta \leq l - \frac{9l}{\sqrt{z^2}}
\]

\[
\Leftrightarrow 8\delta \leq l - \frac{9l}{16} \quad \text{z} > 4
\]

\[
\Leftrightarrow \delta \leq \frac{7}{16} \cdot \frac{l}{8}
\]

Now we want to show that (16) is larger than zero. For this we decompose \( B = A + \text{diag}(d^{(y)}) \) into \( B = \sum_{e \in E} B^e \) and analyze \( x^TB^e x \) for each edge \( e \) separately. Note that \( d^{(y)}_i \) is for vertex \( i \) the number of neighbors in the other part minus the number of neighbors in the same part. For the decomposition, we set \( B^e \) is a cut edge and \( B^e_i = B^e_j = -1 \), if \( e = \{i, j\} \) is a cut edge and \( B^e_i = B^e_j = 1 \), if \( e \) is a inner edge. Further, \( B_{ij} = B_{ji} = 1 \).

If \( e = \{i, j\} \) is a cut edge, we have \( x^TB^e x = 2x_ix_j + x_i^2 + x_j^2 = (x_i + x_j)^2 \). Thus, cut edges always contribute positive. We only consider the edges \( E(\tilde{C}_+ \cup \tilde{C}_-) \). Since \( x_i = x_j = z \), they contribute \( 4z^2 \) each.

If \( e = \{i, j\} \) is an inner edge, we have \( x^TB^e x = 2x_ix_j - x_i^2 - x_j^2 \). For inner edges in \( V \setminus (\tilde{C}_+ \cup \tilde{C}_-) \), \( x_i = x_j \) and the contribution is 0. The same holds for inner edges in \( \tilde{C}_+ \) and \( \tilde{C}_- \). Thus, we only have to consider the edges \( E(\tilde{C}_+ \cup \tilde{C}_-, V \setminus (\tilde{C}_+ \cup \tilde{C}_-)) \). One vertex is \( z \), the other \(-1 \) or \(-\beta z < -1 \). Thus, the contribution is \(-2z - 1 - z^2 \) or \(-2\beta z^2 - \beta^2 z^2 - z^2 = -(3\beta + 1)z^2 \). Since \( 0 < \beta < 1/3 \), both are larger than \(-2z^2 \).
We conclude:

\[ x^T B x > |E(\tilde{C}_{+1}, \tilde{C}_{-1})| \cdot 4z^2 + |E(\tilde{C}_{+1} \cup \tilde{C}_{-1}, V \setminus (\tilde{C}_{+1} \cup \tilde{C}_{-1} ))| \cdot (-2z^2) \]

By the assumption in the Lemma, this is greater or equal to zero. \(\square\)

**Proof of Theorem 5.7**

Let \(A\) be the adjacency matrix of \(G\) and

\[
A' = \begin{pmatrix}
A & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

be the adjacency matrix of \(G'\), where we added two isolated vertices to \(G\). Since \(h(G) = bw(G)\), there exists a \(d'_{opt}\), such that \(g(G, d'_{opt}) = bw(G)\) and \(\lambda(A + d'_{opt})_S) = 0\). It then holds \(\sum_i d'_{opt} = 4 bw(G) - 2|E|\).

\[
h(G) - h(G') = g(G, d'_{opt}) - \max d' g(G', d')
\]

\[
= \frac{\text{sum}(A) + \text{sum}(d'_{opt})}{4} - \max d' \frac{\text{sum}(A') + \text{sum}(d') - (n + 2)\lambda(B'_S)}{4} \quad | B'_S = (A' + \text{diag}(d'))_S \\
= \frac{\text{sum}(d'_{opt})}{4} - \max d' \frac{\text{sum}(d') - (n + 2)\lambda(B'_S)}{4} \quad | \text{sum}(A) = \text{sum}(A')
\]

\[
= \frac{\text{sum}(z) + (d'_{opt})^T - (n + 2)\lambda(B'_S)}{4} \quad | d' = z + (d'_{opt})^T
\]

\[
= -\max z \frac{\text{sum}(z) - (n + 2)\lambda(B'_S)}{4}
\]

\[
= \min z \left( \frac{n + 2}{4} \lambda(B'_S) - \frac{\text{sum}(z)}{4} \right)
\]

\[
= \min z \left( \frac{n + 2}{4} \max_{x \in S \setminus \{0\}} \frac{x^T (A' + \text{diag}(d')) x}{\|x\|^2} - \frac{\text{sum}(z)}{4} \right)
\]

\[
= \min z \left( \frac{n + 2}{4} \max_{x \in S \setminus \{0\}} \left( \frac{x^T (A' + \text{diag}(z + (d'_{opt})^T)) x}{\|x\|^2} \right) - \frac{\text{sum}(z)}{4} \right)
\]

We restrict ourselves two two kinds of vector \(x_a = (x_1, \ldots, x_n, 0, 0)^T\) with \(\sum_{i=1}^n x_i = 0\) and \(x_b = (1, \ldots, 1, -\frac{1}{2}, -\frac{1}{2})\):

\[
\geq \min z \left( \frac{n + 2}{4} \max_{x \in \{x_a, x_b\}} \left( \frac{x^T (A' + \text{diag}(z + (d'_{opt})^T))}{\|x\|^2} \right) - \frac{\text{sum}(z)}{4} \right)
\]

(17)

We want to show that this term is at least \(\frac{4bw(G)}{n^2}\). Therefore, we analyze the max-term separately and then show, for which \(d'\) we have to choose which of the \(x_a\) and \(x_b\).

Firstly, consider vector \(x_a\). Let \(z^{(n)}\) denote the first \(n\) components of vector \(z\). Then

\[
\max_{x_a=(x_1, \ldots, x_n, 0, 0)^T; \sum_{i=1}^n x_i = 0} \left( \frac{x_a^T (A' + \text{diag}(z + (d'_{opt})^T)) x_a}{\|x_a\|^2} \right)
\]

21
\[
\begin{align*} 
&= \max_{\sum_{i=1}^{n} x_i = 0} \left( \frac{x^T (A + \text{diag}(d_{\text{opt}})) x}{\|x\|^2} + \frac{x^T \text{diag}(z_n) x}{\|x\|^2} \right) \\
&= \max_{\sum_{i=1}^{n} x_i = 0} \left( \frac{x^T B x}{\|x\|^2} + \frac{x^T \text{diag}(z_n) x}{\|x\|^2} \right) 
\end{align*}
\]

We choose an optimal bisection vector \( y \) of \( G \):

\[
\sum_{i=1}^{n} x_i = 0 \\
\begin{align*} 
&\geq y^T B y \frac{y^T \text{diag}(z_n) y}{\|y\|^2} = \frac{\sum_{i=1}^{n} z_i}{n} 
\end{align*} 
\]

(18)

Secondly, we consider \( x_b = (1, \ldots, 1, -\frac{n}{2}, -\frac{n}{2})^T \):

\[
\begin{align*} 
&\frac{x_b^T (A' + \text{diag}(z + (d_{\text{opt}}^n, 0, 0)^T)) x_b}{\|x_b\|^2} \\
&= \frac{\text{sum}(A) + \sum_i d_{\text{opt}}^i + x_b^T \text{diag}(z) x_b}{\|x_b\|^2} \\
&= \frac{4 \text{ bw}(G) + x_b^T \text{diag}(z) x_b}{\|x_b\|^2} \quad \left| \sum_{i} d_{i}^2 = 4 \text{ bw}(G) - 2|E| \right| \\
&= \frac{4 \text{ bw}(G)}{(n+2) \left( \frac{n}{4} \right)} + \frac{\sum_{i=1}^{n} z_i + (z_{n+1} + z_{n+2}) \left( \frac{n}{4} \right)^2}{(n+2) \left( \frac{n}{4} \right)} 
\end{align*}
\]

(19)

We insert the result (18) for \( x_a \) and (19) for \( x_b \) into (17):

\[
\begin{align*} 
&\geq \min_{z} \left( \frac{n+2}{4} \max \left( \frac{\sum_{i=1}^{n} z_i}{n}, \frac{4 \text{ bw}(G)}{(n+2) \left( \frac{n}{4} \right)} + \frac{\sum_{i=1}^{n} z_i + (z_{n+1} + z_{n+2}) \left( \frac{n}{4} \right)^2}{(n+2) \left( \frac{n}{4} \right)} - \frac{\text{sum}(z)}{4} \right) \right) 
\end{align*}
\]

We again simplify the terms separately for (18):

\[
\begin{align*} 
\frac{n+2}{4} \sum_{i=1}^{n} z_i & - \frac{\sum_{i=1}^{n+2} z_i}{4} \\
&= \frac{n}{n+2} \sum_{i=1}^{n} z_i - n \sum_{i=1}^{n} z_i - n(z_{n+1} + z_{n+2}) \\
&= \frac{2 \sum_{i=1}^{n} z_i - n(z_{n+1} + z_{n+2})}{4n} \\
&= \frac{\sum_{i=1}^{n} z_i}{2n} - \frac{z_{n+1} + z_{n+2}}{4} = \frac{1}{2} \delta 
\end{align*}
\]

\[
\delta = \frac{\sum_{i=1}^{n} z_i}{n} - \frac{z_{n+1} + z_{n+2}}{2}
\]

and (19):

\[
\begin{align*} 
\frac{n+2}{4} \sum_{i=1}^{n+2} z_i \\
&= \frac{4 \text{ bw}(G)}{2n} + \frac{\sum_{i=1}^{n} z_i + (z_{n+1} + z_{n+2}) \left( \frac{n}{4} \right)^2}{2n} - \frac{\sum_{i=1}^{n} z_i + z_{n+1} + z_{n+2}}{4} \\
&= \frac{4 \text{ bw}(G)}{2n} + \left( \frac{1}{2n} - \frac{1}{4} \right) \sum_{i=1}^{n} z_i + \left( \frac{n}{8} - \frac{1}{4} \right) (z_{n+1} + z_{n+2}) 
\end{align*}
\]

22
\[
\frac{4 \text{bw}(G)}{2n} + \frac{2 - n}{4n} \sum_{i=1}^{n} z_i + \frac{n - 2}{8} (z_{n+1} + z_{n+2}) \\
= \frac{4 \text{bw}(G)}{2n} + \frac{2 - n}{4} \left( \frac{\sum_{i=1}^{n} z_i}{n} - \frac{z_{n+1} + z_{n+2}}{2} \right) \\
= \frac{2 \text{bw}(G)}{n} + \frac{2 - n}{4} \delta.
\]

In both cases, the minimization over \( z \) could be reduced to a minimization over \( \delta \) and we conclude
\[
h(G) - h(G') \geq 17 \geq \min_{\delta} \left( \frac{1}{2} \delta, \frac{2b}{n} + \frac{2 - n}{4} \delta \right).
\]

The first term in the maximum is monotone increasing and the second one monotone decreasing (for \( n \geq 3 \)). Hence, the minimum is at the intersection point of these two lines:
\[
\frac{1}{2} \delta_{\text{min}} = \frac{2 \text{bw}(G)}{n} + \frac{2 - n}{4} \delta_{\text{min}} \\
\frac{2 - 2 + n}{4} \delta_{\text{min}} = \frac{2 \text{bw}(G)}{n} \\
\frac{n}{4} \delta_{\text{min}} = \frac{2 \text{bw}(G)}{n} \\
\delta_{\text{min}} = \frac{8 \text{bw}(G)}{n^2}
\]

It follows
\[
h(G) - h(G') \geq \frac{1}{2} \delta_{\text{min}} = \frac{4 \text{bw}(G)}{n^2}.
\]

**Proof of Theorem 5.8**

Let \( G \) be a graph sampled from the subcritical phase and \((V_1, V_{-1})\) be the planted bisection. Coja-Oghlan [9] defines two sets of vertices:

\[
N_i = \{ v \in V_i : e(v, V_i) = e(v, V_{-i}) \}
\]

\[
N_i^* = \{ v \in N_i : N(v) \setminus \text{core}(G) = \emptyset \}
\]

Let further \((Y_1, Y_{-1})\) be an optimal bisection. Coja-Oghlan claims that, w.h.p., \(\#(Y_i \cap N_i^*) \geq \mu/8\) (eventually swap the parts), where \(\mu = E(\#N_1 + \#N_{-1})\) and \(\mu \geq n^{1-O(\gamma)}\) with \(n(p' - p) = \sqrt{np' \cdot \gamma \ln n}, \gamma = O(1)\) [10, page 122]. Then there are \(\exp(\Omega(\mu))\) many optimal bisections. On the other hand, we will show that, assuming that Boppana works on \(G\), the probability that \(\#(Y_i \cap N_i^*) \geq 2\) will tend to 0, which means that with w.h.p., Boppana will not work on \(G\).

Consider any pair of vertices \(v_1 \in Y_1 \cap N_1^*\) and \(v_{-1} \in Y_{-1} \cap N_{-1}^*\). \(v_1\) and \(v_{-1}\) are not connected by an edge, since they have only neighbors in the core of \(G\). Furthermore, they both have balanced degree. Thus, we can apply Lemma 5.9 and conclude, that \(v_1\) and \(v_{-1}\) have the same neighbors. In direct consequence, all vertices in \(Y_i \cap N_i^*, i \in \{1, -1\}\) have the same neighbors and the same number of edges to each part as well. We denote this number by \(k = e(v_1, V_1)\).
In the following, we will consider sets of 4 vertices, while two are chosen from \( Y_1 \cap N_1^* \) and two from \( Y_{-1} \cap N_{-1}^* \). By our assumption of \( \#(Y_i \cap N_i^*) \geq 2 \), we can choose at least one such set w.h.p.

Let us first rule out two edge cases. In the first case, the vertices have degree \( k = 0 \). Then Boppana does not work due to Theorem 5.7. In the second case, the vertices have maximal many edges, i.e. \( k = n/2 - 2 \) many edges to each part. W.h.p., a graph does not even have two vertices in each part with \( k \) the same neighbors is therefore

\[
\frac{(n/2)^2(n/2 - 1)^2}{4} \cdot (p^{n/2-2}p^{n/2-2})^4(1 - p)^4(1 - p')^2 \to 0
\]

Thus, we have to consider \( 1 \leq k \leq n/2 - 2 \). Let \( C_i^{(k)} = \{ v \in V_i : e(v, V_i) = e(v, V_{-i}) = k \} \) be the set of vertices with a balanced number of exactly \( k \) edges to each part. With the \( k \) from above, we have \( Y_i \cap N_i^* \subseteq C_i^{(k)} \).

We want to estimate the expected number of 4-element sets \( \{ v_1, u_1, v_{-1}, u_{-1} \} \subseteq C_i^{(k)} \cap C_{-i}^{(k)} \) with \( v_1, u_1 \in C_i^{(k)} \) and \( v_{-1}, u_{-1} \in C_{-i}^{(k)} \), where all vertices have the same neighbors. Let us take \( v_1 \) as reference vertex and thus the \( k \) edges from \( v_1 \) to \( V_1 \) as well as \( k \) edges to \( V_{-1} \) are given. Now we estimate the probability, that \( v_{-1}, u_1, u_{-1} \) have exactly the same neighbors. For each vertex and each part, the \( k \) neighbors are chosen independently, since the four vertices are not connected to each other. In both parts, there are \( n/2 - 2 \) possible neighbors. This makes \( \binom{n/2-2}{k} \geq \binom{n/2-2}{1} = n/2 - 2 \) possibilities for the \( k \) edges in one part and only one of them coincides with the edges of \( v_1 \). For 3 vertices to have the same neighbors as \( v_1 \) in two parts each, the probability is at most \( \frac{1}{(n/2-2)^3} \). The expected number of 4 vertices as described with the same neighbors is therefore

\[
E(\#4 - \text{elem} - \text{set}) \leq \binom{n/2}{2} \cdot \frac{1}{(n/2 - 2)^3} \leq \frac{(n/2)^4}{(n/2 - 2)^3} \to 0
\]

This means, w.h.p. we will not find any 4-element set. In consequence, \( \#(Y_i \cap N_i^*) \geq 2 \) may not be true w.h.p. 

\[\square\]

**Proof of Lemma 5.9**

Let \( y \) be the bisection vector corresponding to the optimal bisection in the lemma. Let \( v_1, v_{-1} \in Y_i \), \( i \in \{-1, 1\} \) be vertices as in the lemma, which fulfill \( e(v_i, Y_i) = e(v_{-i}, Y_{-i}) \). We obtain the bisection vector \( y' \) as vector corresponding to \( \{ Y_1 \setminus \{ v_1 \} \cup \{ v_{-1} \}, Y_{-1} \setminus \{ v_{-1} \} \cup \{ v_1 \} \} \). Due to the balanced degree, this bisection is optimal as well.

Hence, we have two optimal bisections and from Lemma 5.2 we know, that the \( d^{\text{opt}} \) is unique and there are unique \( \alpha(y) \) and \( \alpha(y') \) corresponding to \( y \) and \( y' \), resp. It holds

\[
d^{(y)} + \alpha(y)y = d^{(y') + \alpha(y')y'}
\]

\[\Leftrightarrow d^{(y)} - d^{(y')} = \alpha(y')y' - \alpha(y)y
\]

Since \( v_1 \) has balanced degree and is only connected to vertices, which are in the same part in \( y \) and \( y' \), we have \( d^{(y)}_{v_1} - d^{(y')}_{v_1} = 0 \). Furthermore, \( y_{v_1} = 1 \), \( y'_{v_1} = -1 \). Thus we conclude by the equation above, that \( -\alpha^{(y')} - \alpha(y) = 0 \).
Since \( y \) and \( y' \) are optimal bisections and \( e(v_1, Y_i) = e(v_1, Y_{−1}) \), we have

\[
\sum_{i \in Y_1 \setminus \{v_1\}} d_i^{y} - \sum_{i \in Y_1 \setminus \{v_1\}} d_i^{y'} = 0
\]

because

\[
\sum_{i \in Y_1 \setminus \{v_1\}} d_i^{y} = \text{bw}(G) - e(v_1, Y_{−1}) - 2 \cdot |(Y_1 \setminus \{v_1\}) \times (Y_1 \setminus \{v_1\}) \cap E(G)| - e(v_1, Y_1)
\]

\[
= \sum_{i \in Y_1 \setminus \{v_1\}} d_i^{y'}. 
\]

But we have also

\[
\sum_{i \in Y_1 \setminus \{v_1\}} d_i^{y} - d_i^{y'} = (n/2 - 1)(\alpha^{(y')} - \alpha^{(y)}) = -2\alpha^{(y)}(n/2 - 1)
\]

Thus, \( \alpha^{(y)} = \alpha^{(y')} = 0 \). It follows \( d_i^{(y)} - d_i^{(y')} = 0 \), so that each vertex must have no edge to \( v_1 \) and \( v_{−1} \) or must have an edge to both of them. Hence, the \( v_1 \) and \( v_{−1} \) have exactly the same neighbors. \(\square\)

**Proof of Proposition 6.1**

To obtain an SDP formulation we start with Boppana’s function \( h(G) \) and transform it successively as follows:

\[
h(G) = \max_{d \in \mathbb{R}^n} \frac{\text{sum}(A + \text{diag}(d)) - n\lambda((A + \text{diag}(d))_S)}{4} = \max_{d \in \mathbb{R}^n} \frac{J \cdot A + 1^T d - n\lambda(P(A + \text{diag}(d))P)}{4}
\]

\[
= \frac{J \cdot A}{4} + \frac{1}{4} \max_{d \in \mathbb{R}^n} \left(1^T d - n\lambda(P(A + \text{diag}(d))P) - \frac{1^T d}{n} I\right)
\]

\[
= \frac{J \cdot A}{4} - \frac{n}{4} \min_{d \in \mathbb{R}^n} \lambda(P(A + \text{diag}(d))P - \frac{1^T d}{n} I)
\]

\[
= \frac{J \cdot A}{4} - \frac{n}{4} \min_{d \in \mathbb{R}^n} \lambda(M(d)),
\]

where \( M(d) = P(A + \text{diag}(d))P - \frac{1^T d}{n} I \). Hence, we want to solve the following problem: Minimize the largest eigenvalue of the matrix \( M(d) \) for \( d \in \mathbb{R}^n \). For this problem, [32] gives the SDP formulation:

\[
\min z \quad \text{s.t.} \quad zI - M(d) \succeq 0,
\]

with \( z \in \mathbb{R}, d \in \mathbb{R}^n \). Inserting \( M(d) \) and then substituting \( z \) with \( z - \frac{1^T d}{n} \), we get

\[
\min_{z \in \mathbb{R}, d \in \mathbb{R}^n} \left(z - \frac{1^T d}{n}\right) \quad \text{s.t.} \quad zI - P(A + \text{diag}(d))P \succeq 0.
\]
It is easy to see that the constraint matrix above is equal to the constraint matrix of (11), since \( P = I - \frac{J}{n} \). This completes the proof that \( h(G) \) maximized by Boppana’s algorithm gives the same value as the optimum solution of (11) because under the constraints we have

\[
h(G) = \frac{J \cdot A}{4} - \frac{1}{4} \min_{z \in \mathbb{R}, d \in \mathbb{R}^n} (nz - 1^T d).
\]

To obtain the formulation for a dual program, consider the primal SDP in the form:

\[
\min_{z \in \mathbb{R}, d \in \mathbb{R}^n} (nz - 1^T d) \quad \text{s.t.} \quad -PAP + zI - \sum_i d_i PI_i P \succeq 0,
\]

where \( I_i \) denotes the matrix which has a single 1 in the \( i \)th row and the \( i \)th column and zero everywhere else. The dual can be derived by using the rules (10). We obtain:

\[
\max_{Y \in \mathbb{R}^{n \times n}} (PAP) \cdot Y \quad \text{s.t.} \quad I \cdot Y = n, \quad \forall i : \quad -PI_i P \cdot Y = -1, \quad Y \succeq 0.
\]

Thus, since \( P = I - \frac{J}{n} \), we get the following formulation for the dual SDP:

\[
\max_{Y \in \mathbb{R}^{n \times n}} \left( A - \frac{JA + AJ}{n} + \frac{\text{sum}(A)J}{n^2} \right) \cdot Y
\]

under the constraints:

\[
\sum_i y_{ii} = n,
\]

\[
\forall i \quad y_{ii} - \frac{1}{n} \sum_j y_{ji} - \frac{1}{n} \sum_j y_{ij} + \frac{1}{n^2} \sum_{k,j} y_{kj} = 1,
\]

\[
Y \succeq 0.
\]

Here we can note that the second constraint is equal to \( (PYP)_{ii} = 1 \), for all \( i \). Note further that \( (AJ) \cdot Y = \sum_i \deg(i) \sum_j y_{ij} \) and an analogous holds for \( (JA) \cdot Y \). Hence, we can reformulate the objective function as follows:

\[
\max_{Y \in \mathbb{R}^{n \times n}} \left( A \cdot Y - \frac{1}{n} \sum_j \deg(j) \sum_i y_{ij} - \frac{1}{n} \sum_i \deg(i) \sum_j y_{ij} + \frac{1}{n^2} \sum_{i,j} y_{ij} \right).
\]

This completes the proof. \( \square \)

**Proof of Theorem 6.3**

We start with the following fact:

**Claim 8.1.** Let \( X \) be a positive semidefinite matrix. Then the conditions (a) \( \forall i : \sum_j x_{ij} = 0 \) and (b) \( \sum_{i,j} x_{ij} = 0 \) are equivalent.

**Proof.** We show two directions. If (a) holds, it follows directly that (b) is true as well. We proceed with proving of the second direction and assume, that (b) holds.
Each positive semidefinite matrix $X$ can be represented as a Gram matrix, i.e. as matrix of scalar products $x_{ij} = \langle u_i, u_j \rangle$ of vectors $u_i$. Thus, we have

$$\sum_{i,j} x_{ij} = \sum_{i,j} \langle u_i, u_j \rangle = \sum_i \langle u_i, \sum_j u_j \rangle = \langle \sum_i u_i, \sum_j u_j \rangle = 0,$$

where we used condition (b). The scalar product of the vector $\sum_i u_i$ with itself is zero and we conclude that it is the zero vector: $\sum_i u_i = 0$. Now we compute

$$\sum_j x_{ij} = \sum_j \langle u_i, u_j \rangle = \langle u_i, \sum_j u_j \rangle = \langle u_i, 0 \rangle = 0$$

which gives condition (a).

Now we ready to prove Theorem 6.3. For convenience we restate the primal SDP (13) as follows:

$$h_p(G) = \min_Y \left( \frac{|E|}{2} - \frac{1}{4} (A \bullet Y) \right) \text{ s.t. } \forall i \; y_{ii} = 1, \sum_{i,j} y_{ij} = 0, \text{ and } Y \succeq 0,$$

We show that for the following program

$$h'_p(G) = \max_Y A \bullet Y$$

under the constraints:

$$\forall i : y_{ii} = 1,$$

$$\sum_{i,j} y_{ij} = 0,$$

$$Y \succeq 0,$$

we have $h'_p(G) = d(G)$, where recall, $d(G)$ is the objective function of (12). Then we conclude $h_p(G) = \frac{|E|}{2} - \frac{1}{4} h'_p(G) = \frac{|E|}{2} - \frac{1}{4} d(G)$.

Consider an optimal solution matrix $Y$ for the SDP. We show that $Y$ is a solution to the dual program (12) as well, with the value for the objective function equal to (21). Since $y_{ii} = 1$, the first constraint of (12) is fulfilled. Due to Claim 8.1 and since $\sum_{i,j} y_{ij} = 0$, we have $\sum_j y_{ij} = 0$ for all $i$. Hence, the second constraint of (12):

$$\forall i \; y_{ii} - \frac{1}{n} \sum_j y_{ji} - \frac{1}{n} \sum_j y_{ij} + \frac{1}{n^2} \sum_{k,j} y_{kj} = 1$$

is fulfilled as well. In the objective function of (12), the second and third term are zero, since $(AJ) \bullet Y = \sum_i \deg(i) \sum_j y_{ij} = 0$. Obviously, the fourth term is zero due to the constraints as well. Hence, we obtain the same value as $h'_p(G)$.

For the other direction, consider an optimum solution matrix $Y$ of SDP (12). First we show that the first and second constraint of (12) imply $\sum_{i,j} y_{ij} = 0$:

$$\forall i \; y_{ii} - \frac{1}{n} \sum_j y_{ji} - \frac{1}{n} \sum_j y_{ij} + \frac{1}{n^2} \sum_{k,j} y_{kj} = 1 \quad \text{second constraint of (12) for each } i$$

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\[
\sum_i y_{ii} - \frac{1}{n} \sum_{i,j} y_{ji} - \frac{1}{n} \sum_{i,j} y_{ij} + \frac{n}{n^2} \sum_{i,j} y_{ij} = n \quad | \text{sum all } n \text{ constraints}
\]

\[
\Rightarrow \quad n - \frac{1}{n} \sum_{i,j} y_{ji} - \frac{1}{n} \sum_{i,j} y_{ij} + \frac{n}{n^2} \sum_{i,j} y_{ij} = n \quad | \text{use the first constraint of (12)}
\]

\[
\Rightarrow \quad \sum_{i,j} y_{ij} = 0.
\]

Next, due to Claim 8.1 we know that \( \sum_j y_{ij} = 0 \) for all \( i \). Again from the second constraint, of (12) we conclude that \( y_{ii} = 1 \). Hence, the constraints of the SDP (13) are fulfilled. Obviously, the second, third and fourth term in the objective function of (12) are zero again and the objective values of both SDPs are the same as well. \( \square \)