QUATERNIONIC SLICE REGULAR FUNCTIONS
AND QUATERNIONIC LAPLACE TRANSFORMS∗

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Abstract The functions studied in the paper are the quaternion-valued functions of a quaternionic variable. It is shown that the left slice regular functions and right slice regular functions are related by a particular involution, and that the intrinsic slice regular functions play a central role in the theory of slice regular functions. The relation between left slice regular functions, right slice regular functions and intrinsic slice regular functions is revealed. As an application, the classical Laplace transform is generalized naturally to quaternions in two different ways, which transform a quaternion-valued function of a real variable to a left or right slice regular function. The usual properties of the classical Laplace transforms are generalized to quaternionic Laplace transforms.

Key words left slice regular function; intrinsic slice regular function; quaternionic Laplace Transform

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1 Introduction

The theory of slice regular functions (originally Cullen-regular functions [1]) of a quaternionic variable was born around 2005, and has rapidly developed since then. Most of the known results and applications regarding slice regular functions can be found in the wonderful book [9]. Many results in the theory of complex analytic functions are generalized to slice regular functions; include the Cauchy Integral Formula, the Maximum Modulus Principles, the Open Mapping Theorem, the Laurent series, etc.

The theories of slice regular functions and slice monogenic functions have now been applied to develop a new functional calculus in a noncommutative setting [4]. The theory of regular functions has also been generalized to real alternative algebras [8]. Some related research can be also be found in [12].

The set of left (resp. right) slice regular functions are not closed under the usual product of functions, but they are closed under the regular product (∗-product). With respect to this regular product and the usual addition and scalar multiplication, the set of left (resp. right) slice regular functions (on some domain of H) becomes a real noncommutative associative algebra.

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The theory of left slice regular functions and that of right slice regular functions are parallel. Sometimes one may encounter left slice regular functions and right slice regular functions at the same time. The Cauchy formulas for the left and for the right slice regular functions involve different Cauchy kernels [2] (as opposed to what happens in the classical case). The quaternionic Cauchy kernel
\[ \phi(q, s) = -\left( |s|^2 - 2q \Re(s) + q^2 \right)^{-1} (q - \overline{s}) \]
is a left slice regular function of \( q \), as well as a right slice regular function of \( s \) [2].

In the paper we will demonstrate that left slice regular functions and right slice regular functions are related by an involution on the space of quaternionic functions.

Let \( \mathbb{H} \) be the real algebra of quaternions. Let \( U \subseteq \mathbb{H} \) be an axially symmetric slice domain and let \( C(U, \mathbb{H}) \) be the set of continuous functions from \( U \) to \( \mathbb{H} \). Let \( \mathcal{R}^0(U) \), \( \mathcal{R}^l(U) \) and \( \mathcal{R}^r(U) \) be the set of intrinsic slice regular functions, left slice regular functions and right slice regular functions on \( U \) respectively. Let
\[ \eta : C(U, \mathbb{H}) \to C(U, \mathbb{H}), \eta(f)(q) = \overline{f(q)}. \]
Then \( \eta \) is an involution on \( C(U, \mathbb{H}) \), which maps left slice regular functions to right slice regular functions and vice versa, thus induces an anti-isomorphism between the associative algebras \( \mathcal{R}^l(U) \) and \( \mathcal{R}^r(U) \).

Another main result of the paper is the isomorphism of associative algebras
\[ \phi : \mathcal{R}^0(U) \times_{\mathbb{R}} \mathbb{H} \to \mathcal{R}^l(U), \]
through which some known results, including the slice derivatives and the regular conjugation of slice regular functions, are better understood; see Theorems 3.2, 3.7 and 3.8.

Next we introduce the quaternionic Laplace transform.

The classical Laplace transform is an integral transform named after its inventor, Pierre-Simon Laplace. It transforms a complex-valued function of a real variable to a complex-valued function of a complex variable. The transform has many applications in mathematics, physics and engineering.

The classical Laplace transform was naturally generalized to quaternions in [10] and [11], and was applied to solve partial differential equations including those of elliptic, parabolic and hyperbolic types [11]. The theory of slice regular functions also allowed for the introduction of a quaternionic functional calculus [4]. The quaternionic Laplace transform can be further defined for a semigroup of quaternionic linear operators, and the classical Hille-Phillips-Yosida theorem can be generalized to the quaternionic setting [3].

As \( \mathbb{H} \) is noncommutative, there are 2 types of quaternionic Laplace transforms (see Section 4); these transform a quaternion-valued function of a real variable to a left or right slice regular quaternion-valued function of a quaternionic variable.

In Section 4, we show that the main properties of the classical Laplace transforms can be generalized to quaternionic Laplace transforms. For example, if \( f(t) \) and \( g(t) \) are quaternion-valued functions defined for all real numbers \( t \geq 0 \), and \( f \circ g \) is the convolution of \( f \) and \( g \). Then the left Laplace transform of \( f \circ g \) is the regular product of the left Laplace transform of \( f \) and the left Laplace transform of \( g \) (Proposition 4.8). The relationship between the 2 types of quaternionic Laplace transforms is also given in Proposition 4.3.
2 Some Preliminary Results on Quaternions and Slice Regular Functions

2.1 The Quaternions

The quaternions $\mathbb{H}$ were introduced by Hamilton in 1843, adding a multiplicative structure to $\mathbb{R}^4$, and is a real division algebra. Let $\{1, i, j, k\}$ be the standard basis of $\mathbb{H}$. Let $\alpha : \mathbb{H} \to \mathbb{H}$ be the usual conjugation, which is an involution of $\mathbb{H}$. In this paper, an involution of an associative algebra means an anti-automorphism of the algebra of order two. Let $\mathbb{R} \cdot 1$ and $\text{Im}(\mathbb{H})$ be the $\mathbb{R}$-span of 1 and the $\mathbb{R}$-span of $\{i, j, k\}$ respectively, which are the 1 and -1 eigenspaces of $\alpha$. One has $\text{Im}(\mathbb{H}) = \{q \in \mathbb{H} | q^2 \in \mathbb{R}$ and $q^2 \leq 0\}$, and the following decomposition of $\mathbb{H}$:

$$\mathbb{H} = \mathbb{R} \cdot 1 \oplus \text{Im}(\mathbb{H}).$$

For $q = a + bi + cj + dk \in \mathbb{H}$, let $\text{Re}(q) = a$ and $\text{Im}(q) = bi + cj + dk$, which are, respectively, the real and imaginary parts of $q$. Let $|q|^2 = q\overline{q}$, which turns $\mathbb{H}$ into a 4-dimensional Euclidean space with the inner product $(p, q) = \text{Re}(p\overline{q})$. The elements in $\mathbb{H}$ with a unit norm constitute a group under multiplication, denoted by $Sp(1)$.

Let $\text{Aut}(\mathbb{H})$ be the automorphism group of $\mathbb{H}$. One knows that $\text{Aut}(\mathbb{H}) \cong SO(3)$.

Let $\tilde{\text{Aut}}(\mathbb{H}) = \text{Aut}(\mathbb{H}) \cup \alpha \cdot \text{Aut}(\mathbb{H})$ be the group generated by $\text{Aut}(\mathbb{H})$ and $\alpha$, which consists of all the automorphisms and anti-automorphisms of $\mathbb{H}$. Since $\alpha$ restricts to minus identity on $\text{Im} \mathbb{H}$, $\tilde{\text{Aut}}(\mathbb{H}) \cong O(3)$. We will identify $\tilde{\text{Aut}}(\mathbb{H})$ with $O(3)$.

Let

$$\mathcal{S} = \{q \in \mathbb{H} | q^2 = -1\},$$

which is the 2-dimensional unit sphere of purely imaginary elements in $\mathbb{H}$ with a unit norm. This is the orbit of $i$ under $\text{Aut}(\mathbb{H})$. For any $I \in \mathcal{S}$, let

$$\mathcal{C}_I = \{x + Iy | x, y \in \mathbb{R}\},$$

which is a subalgebra of $\mathbb{H}$ isomorphic to $\mathbb{C}$. For any $I, J \in \mathcal{S}$, if $J \neq \pm I$, then $\mathcal{C}_I \cap \mathcal{C}_J = \mathbb{R}$. One has that $\mathcal{C}_I = \mathcal{C}_J$ if and only if $J = \pm I$. One has that $\mathbb{H} = \bigcup_{I \in \mathcal{S}} \mathcal{C}_I$. This is called the slice structure of $\mathbb{H}$. For any open set $U$ in $\mathbb{H}$ and $I \in \mathcal{S}$, let

$$U_I = U \cap \mathcal{C}_I.$$  \hspace{1cm} (2.1)

For any $I \in \mathcal{S}$, let $\tau_I : \mathbb{C} \to \mathbb{H}, x + iy \mapsto x + Iy$ be the embedding, whose image is $\mathcal{C}_I$. We also use $\tau_I$ to denote the map $\mathbb{C} \to \mathcal{C}_I, x + iy \mapsto x + Iy$. $\tau_I$ is conjugation equivariant. As $\mathcal{C}_I = \mathbb{C} - I$, there are 2 identifications $\mathbb{C} \to \mathcal{C}_I: x + iy \mapsto x + Iy$ and $x + iy \mapsto x - Iy$.

The Galois group of $\mathbb{C}$ over $\mathbb{R}$ is the order 2 cyclic group generated by the complex conjugation $z \mapsto \bar{z}$, and will be denoted by $C_2$.

One has the following homeomorphism of orbit spaces:

$$\mathbb{H}/SO_3 \cong \mathbb{C}/C_2$$

The map

$$\mathbb{C} \times \mathcal{S} \to \mathbb{H}, (x + iy, J) \mapsto x + Jy,$$

is a surjective continuous map, whose restriction $(\mathbb{C} \setminus \mathbb{R}) \times \mathcal{S} \to \mathbb{H} \setminus \mathbb{R}$ is a double covering.
2.2 Definition of Slice Regular Functions

Recall the $U_I$ defined in (2.1).

**Definition 2.1** Let $U$ be an open set in $\mathbb{H}$ and let $f : U \to \mathbb{H}$ be real differentiable. For every $I \in \mathbb{S}$, let $f_I : U_I \to \mathbb{H}$ be the restriction of $f$ to $U_I$. The function $f$ is said to be left slice regular (left regular) if, for every $I \in \mathbb{S}$,

$$\bar{\partial}_I f_I(x + Iy) := \frac{1}{2} \left( \frac{\partial}{\partial x} f_I(x + Iy) + I \frac{\partial}{\partial y} f_I(x + Iy) \right) = 0$$
on $U_I$. Analogously, a function is said to be right slice regular (right regular) in $U$ if

$$\bar{\partial}_r f_I(x + Iy) := \frac{1}{2} \left( \frac{\partial}{\partial x} f_I(x + Iy) + \frac{\partial}{\partial y} f_I(x + Iy)I \right) = 0$$
on $U_I$.

The class of left (resp. right) regular functions on $U$ will be denoted by $\mathcal{R}^l(U)$ (resp. $\mathcal{R}^r(U)$).

The left (slice) $I$ -derivative of $f \in \mathcal{R}^l(U)$ at a point $q = x + Iy$ is given by

$$\partial_I f_I(x + Iy) = \frac{1}{2} \left( \frac{\partial}{\partial x} f_I(x + Iy) - I \frac{\partial}{\partial y} f_I(x + Iy) \right).$$

Similarly, the right (slice) $I$ -derivative of $f \in \mathcal{R}^r(U)$ at $q = x + Iy$ is given by

$$\partial_r f_I(x + Iy) = \frac{1}{2} \left( \frac{\partial}{\partial x} f_I(x + Iy) - \frac{\partial}{\partial y} f_I(x + Iy)I \right).$$

Let us now introduce the slice derivative for regular functions.

**Definition 2.2** Let $U$ be an open set in $\mathbb{H}$. If $f \in \mathcal{R}^l(U)$, then the left slice derivative $\partial f$ of $f$ is defined by

$$\partial f(q) = \begin{cases} 
\partial_I f_I(q) & \text{if } q = x + Iy, \ y \neq 0, \\
\frac{\partial f}{\partial x}(x) & \text{if } q = x \in \mathbb{R}.
\end{cases}$$

If $f \in \mathcal{R}^r(U)$, then the right slice derivative $\partial^r f$ of $f$ is defined by

$$\partial^r f(q) = \begin{cases} 
\partial_r f_I(q) & \text{if } q = x + Iy, \ y \neq 0, \\
\frac{\partial f}{\partial x}(x) & \text{if } q = x \in \mathbb{R}.
\end{cases}$$

Note that if $f$ is a left regular function, then its derivative is still left regular, because

$$\bar{\partial}_I (\partial f(x + Iy)) = \partial (\bar{\partial}_I f(x + Iy)) = 0.$$

A similar result holds for right regular functions.

Let $r > 0$ and $B(0, r)$ be the open ball centered at 0 with radius $r$. A function $f : B(0, r) \to \mathbb{H}$ is left (resp. right) regular on $B(0, r)$ if and only if $f$ has a series representation of the form

$$f(q) = \sum_{n=0}^{\infty} q^n a_n \ (\text{resp. } f(q) = \sum_{n=0}^{\infty} a_n q^n),$$

where $\{a_n\}, n \in \mathbb{N}$, is a sequence of quaternions satisfying that $(\limsup_{n \to \infty} |a_n|^{1/n})^{-1} \geq r$, see Theorems 2.1.5 and 2.1.6 of [5].

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\section{Intrinsic Slice Regular Functions}

Let $D \subseteq \mathbb{C}$ be a domain symmetric with respect to the real axis. Then $D \cap \mathbb{R} \neq \emptyset$. Let $f : D \rightarrow \mathbb{C}$ be a complex function. Then $f$ is called intrinsic if $f$ is $C_2$-equivariant; i.e.,

$$f(\bar{z}) = \overline{f(z)}.$$  \hfill (2.2)

\textbf{Definition 2.3} Let $U \subseteq \mathbb{H}$. We say that $U$ is axially symmetric if $U$ is $SO(3)$-invariant; i.e., for every $x + iy \in U$, all the elements $x + Sy = \{x + Jy|J \in \mathbb{S}\}$ are contained in $U$. We say that $U$ is a slice domain if it is a connected open set whose intersection with every complex plane $\mathbb{C}_I$ is connected.

If $U \subseteq \mathbb{H}$ is an axially symmetric slice domain, then $U \cap \mathbb{R} \neq \emptyset$.

Let $U \subseteq \mathbb{H}$ be an axially symmetric slice domain and let $C(U, \mathbb{H})$ be the set of continuous functions from $U$ to $\mathbb{H}$. Let

$$\eta : C(U, \mathbb{H}) \rightarrow C(U, \mathbb{H}), \eta(f)(q) = \overline{f(q)}.$$  \hfill (2.3)

Then one has $\eta^2 = 1$ and

$$\eta(f \cdot g) = \eta(g) \cdot \eta(f), \ f, g \in C(U, \mathbb{H}),$$

where $f \cdot g$ is the pointwise multiplication. Thus $\eta$ is an involution on $C(U, \mathbb{H})$; this plays an important role in the paper. Let

$$C(U, \mathbb{H})^\eta = \{f \in C(U, \mathbb{H})|\eta(f) = f\}.$$  

Let $U$ be an axially symmetric slice domain. A function $f : U \rightarrow \mathbb{H}$ is said to be slice-preserving if $f(U \cap \mathbb{C}_I) \subseteq \mathbb{C}_I$ for each $I \in \mathbb{S}$.

If $f$ is slice-preserving, then it is clear that $f$ is left regular if and only if $f$ is right regular.

\textbf{Definition 2.4} A function $f : U \rightarrow \mathbb{H}$ which is slice-preserving and left regular is said to be intrinsic (slice) regular. The set of intrinsic regular functions on $U$ will be denoted by $\mathcal{R}^\eta(U)$.

By Remark 1.31 of [9], one knows that if $f$ is a left regular function on $B(0,r)$ (for some $r > 0$), then $f$ is slice preserving if, and only if, the power series expansion $f(q) = \sum_{n \in \mathbb{N}} q^n a_n$ has real coefficients $a_n \in \mathbb{R}$.

If $f$ is an intrinsic regular function, then $\partial f(q) = \partial f^\eta f(q)$, which will be denoted by $f'(q)$; i.e., for intrinsic regular functions $f$,

$$f'(q) = \partial f(q) = \partial f^\eta f(q).$$

\section{Extension from Complex Intrinsic Holomorphic Functions to Quaternionic Intrinsic Regular Functions}

The following construction is due to Feuter [7]:

Assume that $f(x + iy) = u(x, y) + iv(x, y)$ is an intrinsic complex function on a domain $D \subseteq \mathbb{C}$ symmetric with respect to the real axis, where $u$ and $v$ are real-valued functions. Let $\tilde{D} \subseteq \mathbb{H}$ be the union of the $SO(3)$-orbit of all the points in $D$, i.e., $\tilde{D} = \{x + Sy|x + iy \in D\}$. Then $f$ induces a function, $\widetilde{f}$, from $\tilde{D}$ to $\mathbb{H}$, such that, for any $J \in \mathbb{S}$,

$$\widetilde{f} : \tilde{D} \rightarrow \mathbb{H}, \widetilde{f}(a + Jb) = u(a, b) + Jv(a, b).$$  \hfill (2.4)
This is well-defined because \( f \) is intrinsic. This induced function \( \tilde{f} \) is \( O(3) \)-equivariant and slice-preserving. This construction is later generalized to regular functions.

Assume further that \( f \) is an intrinsic holomorphic function on \( D \). Then \( \tilde{f} \) is left and right regular on \( \tilde{D} \), preserving each \( \tilde{D} \cap C_I \), so it is an intrinsic regular function on \( \tilde{D} \). Let \( H(D, C) \) be the set of intrinsic holomorphic function on \( D \) and define

\[
\text{ext} : H(D, C) \to R_0(\tilde{D}), \ f \mapsto \tilde{f}, \tag{2.5}
\]

which is an isomorphism of commutative and associative algebras. Let \( U = \tilde{D} \). Then the inverse of this map is the restriction of \( g \in R_0(U) \) to some single slice \( U_I = U \cap C_I \) (for some \( I \in S \)), where \( g|_{U_I} \in H(U_I, C_I) \). Write \( H(U_I) = H(U_I, C_I) \). Then (2.5) can also be written as

\[
\text{ext} : H(U_I) \to R_0(U). \tag{2.6}
\]

**Example 2.5** The exponential function

\[
\exp : \mathbb{H} \to \mathbb{H}, \quad q \to e^q = \sum_{n=0}^{\infty} q^n / n!
\]

is an intrinsic regular function induced from the usual complex exponential function.

### 3 Main Results on Regular Functions

In this section, \( U \) will always be an axially symmetric slice domain in \( \mathbb{H} \). Let \( D \subset \mathbb{R}^2 \) be such that \( x + Iy \in U \) whenever \( (x, y) \in D \).

Let us first recall the following result:

**Theorem 3.1** (Corollary 2.2.5. of [5]) Let \( f : U \to \mathbb{H} \). The function \( f \) is slice regular if and only if there exist two differentiable functions, \( \alpha, \beta : D \to \mathbb{H} \) satisfying \( \alpha(x, y) = \alpha(x, -y), \beta(x, y) = -\beta(x, -y) \) and the Cauchy-Riemann system

\[
\begin{align*}
\partial_x \alpha - \partial_y \beta &= 0 \\
\partial_x \beta + \partial_y \alpha &= 0
\end{align*}
\]

such that \( f(x + Iy) = \alpha(x, y) + I\beta(x, y) \).

The following theorem is one of our main results:

**Theorem 3.2** Let \( f : U \to \mathbb{H} \) be a real differentiable function. Then the following statements are equivalent:

1. \( f \in R^l(U) \).
2. For any orthogonal unit vectors \( I_1, I_2, I_3 \in S \) with \( I_1I_2 = I_3 \), there exist 4 uniquely determined intrinsic regular functions \( h_m \) on \( U \), such that \( f(q) = h_0(q) + h_1(q)I_1 + h_2(q)I_2 + h_3(q)I_3 \), \( q \in U \).

**Proof** (2) implies (1) by the definition of left regular functions.

Now assume (1). By the above theorem, there exist two differentiable functions, \( \alpha, \beta : D \to \mathbb{H} \) satisfying \( \alpha(x, y) = \alpha(x, -y), \beta(x, y) = -\beta(x, -y) \) and the Cauchy-Riemann system. The set \( \{I_1, I_2, I_3\} \) is a basis of \( \mathbb{H} \). Let \( I_0 = 1 \). Then \( \alpha \) and \( \beta \) can be written as \( \alpha = \sum_{i=1}^{3} a_i I_i \). 

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\( \beta = \sum_{i=1}^{4} b_i I_i \), where \( a_i \) and \( b_i \) are all differentiable functions from \( D \) to \( \mathbb{R} \). Let

\[
h_i : U \to \mathbb{H}, h_i(x + Ly) = a_i(x, y) + Lb_i(x, y)
\]

for any \( L \in \mathbb{S} \) and \( (x, y) \in D \).

Then, for any \( L \in \mathbb{S} \) and \( (x, y) \in D \),

\[
f(x + Ly) = \alpha(x, y) + L\beta(x, y) = \sum_{i=1}^{4} (a_i(x, y) + Lb_i(x, y))I_i = \sum_{i=1}^{4} h_i(x + Ly)I_i.
\]

As \( f \in \mathcal{R}^l(U) \), one has that

\[
0 = \partial_L f_L = \sum_{i=1}^{4} \partial_L (h_i|_L)I_i,
\]

thus \( \partial_L (h_i|_L) = 0 \) for \( i = 0, 1, 2, 3 \), and for all \( L \in \mathbb{S} \). Therefore, as slice-preserving functions, each \( h_i \) is intrinsic regular, and (2) is proven. \( \Box \)

The following result is a corollary of the Identity Principle (Theorem 1.12 of [9]):

**Proposition 3.3** Assume that \( f, g \in \mathcal{R}^l(U) \) and that \( W \subseteq U \cap \mathbb{R} \) is a nonempty open subset. If \( f(q) = g(q) \) for any \( q \in W \), then \( f = g \).

For any \( f \in \mathcal{R}^l(U) \), one knows that \( f \) can be recovered from its restriction to some single slice \( U_I = U \cap C_I \) with \( I \in \mathbb{S} \); see Corollary 1.16 of [9]. In fact, more than just that is true.

Let \( U \) be an axially symmetric slice domain of \( \mathbb{H} \) and let \( f \in \mathcal{R}^l(U) \). Then \( f \) can be recovered from its restriction to \( U \cap \mathbb{R} \) as follows: by Theorem 3.2, one knows that \( f(q) = f_0(q) + f_1(q)i + f_2(q)j + f_3(q)k \) with \( f_m \in \mathcal{R}^0(U) \). Fix \( I \in \mathbb{S} \). Each \( f_m(q) \) is determined by its restriction, \( f_m \), to \( U_I \). As \( f_m \) is holomorphic on \( U_I \), \( f_m \) is determined by its restriction to \( U_I \cap \mathbb{R} \), which is denoted by \( f_m|_\mathbb{R} \). In fact, \( f_m \) is the analytic continuation of \( f_m|_\mathbb{R} \) to \( U_I \), and \( f_m = \text{ext}(f_m) \) (see (2.4)).

**Proposition 3.4** A left (resp. right) regular function \( f : U \to \mathbb{H} \) is intrinsic regular if and only if \( f(U \cap \mathbb{R}) \subseteq \mathbb{R} \).

**Proof** We will prove the result in the case that \( f \in \mathcal{R}^l(U) \).

If \( f \) is intrinsic regular, then \( f(U \cap C_I) \subseteq C_I \) for each \( I \in \mathbb{S} \). Since \( C_I \cap C_J = \mathbb{R} \) for \( J \neq \pm I \), \( f(U \cap \mathbb{R}) \subseteq \mathbb{R} \).

Conversely, assume that \( f(U \cap \mathbb{R}) \subseteq \mathbb{R} \) and \( f(q) = f_0(q) + f_1(q)i + f_2(q)j + f_3(q)k \) with each \( f_m \in \mathcal{R}^0(U) \). Then \( f_m(q) = 0 \) for \( q \in U \cap \mathbb{R} \) for \( m = 1, 2, 3 \). Since \( f_m(q) \) is complex holomorphic on each slice \( U_I, f_m(q) = 0 \) for \( q \in U_I \) for \( m = 1, 2, 3 \). As \( U = \bigcup_{I \in \mathbb{S}} U_I, f_m(q) = 0 \) for each \( q \in U \) for \( m = 1, 2, 3 \), and \( f = f_0 \) is intrinsic regular. \( \Box \)

The following result was stated in Remark 2.4.1 of [5]:

**Proposition 3.5** A left (resp. right) regular function \( f : U \to \mathbb{H} \) is intrinsic regular if and only if it satisfies that \( f(q) = \overline{f(q)} \) for any \( q \in U \).

**Proof** We will prove the result in the case that \( f \in \mathcal{R}^l(U) \).

Assume that \( f \) is intrinsic regular on \( U \). Then \( f(U \cap \mathbb{R}) \subseteq \mathbb{R} \). Fix \( I \in \mathbb{S} \). Then \( f(U \cap C_I) \subseteq C_I \) and \( f_I : U \cap C_I \to C_I \) is holomorphic. Since \( f(U \cap \mathbb{R}) \subseteq \mathbb{R} \), \( f(q) = \overline{f(q)} \) for any \( q \in U \cap C_I \). Then \( f(q) = \overline{f(q)} \) for any \( q \in U \) as \( U = \bigcup_{I \in \mathbb{S}} U \cap C_I \).
Conversely, assume that $f \in \mathcal{R}^i(U)$ and $f(q) = \overline{f(q)}$ for any $q \in U$. Then $f(U \cap \mathbb{R}) \subseteq \mathbb{R}$. It follows from that last proposition that $f \in \mathcal{R}^b(U)$. \hfill \Box

By Proposition 3.5, one has that

$$\mathcal{R}^0(U) = \mathcal{R}^i(U) \cap C(U, \mathbb{H})^n = \mathcal{R}^r(U) \cap C(U, \mathbb{H})^n.$$ 

For any $f, g \in \mathcal{R}^i(U)$, the function $h(q) = f(q)g(q)$ may not be in $\mathcal{R}^i(U)$. The left regular product $f \ast g$ of $f$ and $g$ (see Definition 1.27 of [9]) is also in $\mathcal{R}^i(U)$. If $f(q) = \sum_{n \in \mathbb{N}} q^n a_n$ and $g(q) = \sum_{n \in \mathbb{N}} q^n b_n$ are left regular functions on $B(0, r)$, then the regular product of $f$ and $g$ is the regular function defined by

$$(f \ast g)(q) = \sum_{n \in \mathbb{N}} q^n \sum_{k=0}^{n} a_k b_{n-k} \tag{3.1}$$

on the same ball $B(0, r)$.

Assume that $f \in \mathcal{R}^i(U)$; for the definition of its regular conjugate $f^c$ and its symmetrization $f^s$, see Definition 1.33 of [9].

Assume that $f \in \mathcal{R}^i(U)$ and $f \neq 0$. The regular reciprocal $f^{-s}$ of $f$ is the function defined on $U \setminus Z_{f^c}$, where $Z_{f^c} = \{ \eta \in \mathbb{H} | f^c(\eta) = 0 \}$, by $f^{-s} = \frac{1}{f} f^c$; see Definition 5.1 of [9].

**Proposition 3.6** Assume that $f, g \in \mathcal{R}^i(U)$. One has that

(1) the left regular product $f \ast g$ is the left regular function in $\mathcal{R}^i(U)$ such that $(f \ast g)(q) = f(q)g(q)$ for $q \in U \cap \mathbb{R}$;

(2) the regular conjugate $f^c$ of $f$ is the function in $\mathcal{R}^i(U)$ such that for $q \in U \cap \mathbb{R}$, $f^c(q) = \overline{f(q)}$;

(3) the symmetrization $f^s$ of $f$ is the intrinsic regular function in $\mathcal{R}^0(U)$ such that for $q \in U \cap \mathbb{R}$, $f^s(q) = f(q) f(\overline{q})$;

(4) assuming that $f \neq 0$. Then regular reciprocal $f^{-s}$ of $f$ is the function in $\mathcal{R}^i(U)$ such that for $q \in (U \cap \mathbb{R}) \setminus Z_{f^c}$, $f^{-s}(q) = f(q)^{-1}$.

**Proof** By Proposition 3.3, we only need to show that the value of the regular functions $f \ast g, f^c, f^s$ and $f^{-s}$ in $q \in U \cap \mathbb{R}$ are as stated in the proposition.

(1) Without loss of generality one can assume that $U \supseteq B(0, r)$ for some $r > 0$. Then, by (3.1), $(f \ast g)(q) = f(q)g(q)$ for $q \in B(0, r) \cap \mathbb{R}$. By Theorem 3.2, the function $U \cap \mathbb{R} \to \mathbb{H}, q \mapsto f(q)g(q)$ is a function of a real variable with each component being real analytic, thus there is a unique left regular function $h$ such that $h(q) = f(q)g(q)$ for $q \in U \cap \mathbb{R}$. As $h(q) = (f \ast g)(q)$ for $q \in B(0, r) \cap \mathbb{R}$, $h = f \ast g$ on $U$, by Proposition 3.3.

(2) It can be proven similarly as in (1).

(3) It follows from (1) and (2) as $f^s = f \ast f^c$.

(4) As $f \neq 0$, $(U \cap \mathbb{R}) \setminus Z_{f^c} \neq \emptyset$. For $q \in (U \cap \mathbb{R}) \setminus Z_{f^c}$, $f(q) \cdot \frac{1}{f(q)} f^c(q) = 1$, thus $f^{-s}(q) = \frac{1}{f(q)} f^c(q) = f(q)^{-1}$. Then (4) follows from Proposition 3.3. \hfill \Box

Now let us review the Splitting Lemma, before we prove the next result. Let $f: U \to \mathbb{H}$ be a real differentiable function with $U$ an axially symmetric slice domain. Then $f$ is left regular if and only if, for any $I \in \mathbb{S}$ and any $J \in \mathbb{S}$ with $I \perp J$, there exist two holomorphic functions, $F, G: U \to \mathbb{C}_I$, such that, for every $z = x + yI$,

$$f_I(z) = F(z) + G(z)J.$$
Furthermore, \( f \) is right regular if and only if, for any \( I \in \mathbb{S} \) and any \( J \in \mathbb{S} \) with \( J \perp I \), there exist two holomorphic functions \( F, G : U_I \to \mathbb{C}_I \) such that, for every \( z = x + yI \),

\[
f_I(z) = F(z) + JG(z).
\]

The space \( R^I(U) \) (resp. \( R^r(U) \)) is a real associative algebra equipped with the regular product.

**Theorem 3.7** Let \( U \) be an axially symmetric slice domain of \( \mathbb{H} \). Then we have that

1. if \( f \in R^I(U) \) then \( \eta(f) \in R^r(U) \) and vice versa.
2. the map

\[
\phi : R^0(U) \otimes \mathbb{H} \to R^I(U), f(q) \otimes \lambda \mapsto f(q)\lambda
\]

is an isomorphism of real associative algebras, where the product of \( f_0 \otimes I \) and \( g_0 \otimes J \) in \( R^0(U) \otimes \mathbb{H} \) is \( f_0 \otimes I \cdot g_0 \otimes J = f_0g_0 \otimes IJ \). One has that \( R^0(U) \) is the center of the associative algebra \( R^I(U) \).

The map \( \phi \) is also an isomorphism of a left (free) \( R^0(U) \)-module and a right \( \mathbb{H} \)-module.

The results for \( R^r(U) \) are symmetric.

**Proof** (1) Let \( f(q) \in R^I(U) \). By the Splitting Lemma, for any \( I \in \mathbb{S} \) and any \( J \in \mathbb{S} \) with \( J \perp I \), there exist two holomorphic functions, \( F, G : \Omega_I \to \mathbb{C}_I \), such that, for every \( z = x + yI \),

\[
f_I(z) = F(z) + G(z)J.
\]

Let \( g(q) = \overline{f(q)} \). Then for every \( z = x + yI \),

\[
g_I(z) = \overline{f_I(\overline{z})} = \overline{\frac{f_I(z)}{J}} = F(\overline{z}) + G(\overline{z})J = \overline{F(z)} + J(-G(\overline{z})).
\]

Since \( F, G : \Omega_I \to \mathbb{C}_I \) are holomorphic functions, \( \overline{F(z)} \) and \( -G(\overline{z}) \) are also holomorphic, thus \( g(q) = \overline{f(q)} \) is right regular.

Similarly, if \( f \in R^r(U) \), then \( \eta(f) \in R^I(U) \).

(2) Theorem 3.2 implies that

\[
R^I(U) = R^0(U) \oplus R^0(U)i \oplus R^0(U)j \oplus R^0(U)k,
\]

which implies that \( \phi \) is an \( \mathbb{R} \)-linear isomorphism of \( R^0(U) \otimes \mathbb{H} \) and \( R^I(U) \). It is directly verified that \( \phi \) preserves multiplications in the two associative algebras. Thus \( \phi \) is an isomorphism of real associative algebras. It is easy to see that \( R^0(U) \) is the center of the associative algebra \( R^I(U) \).

The left \( R^0(U) \)-module structure and right \( \mathbb{H} \)-module structure on \( R^I(U) \) and \( R^0(U) \otimes \mathbb{H} \) are the obvious ones. It is clear that \( \phi \) is an isomorphism of the left \( R^0(U) \)-module and the right \( \mathbb{H} \)-module.

In the rest of this paper, we will identify \( R^I(U) \) with \( R^0(U) \otimes_{\mathbb{R}} \mathbb{H} \) by \( \phi \).

**Theorem 3.8** Let \( U \) be an axially symmetric slice domain of \( \mathbb{H} \). Then we have the following:

1. \( \eta(f \ast g) = \eta(g) \ast \eta(f) \) for \( f, g \in R^I(U) \), where the 1st (resp. the 2nd) product is the left (resp. right) regular product. Thus \( \eta : R^I(U) \to R^r(U) \) is an anti-isomorphism of associative algebras. Similarly, \( \eta : R^r(U) \to R^I(U) \) is also an anti-isomorphism of associative algebras.
2. The map

\[
R^0(U) \otimes \mathbb{H} \to R^0(U) \otimes \mathbb{H}, f \otimes \mu \mapsto f \otimes \overline{\mu}
\]
induces the regular conjugation on $\mathcal{R}^l(U)$, i.e., $(f \otimes \mu)^c = f \otimes T\mu$. A similar result holds for $\mathcal{R}^r(U)$.

(3) Assume $f \in \mathcal{R}^l(U)$. Write $f = \sum_{m=0}^{3} f_m(q) \otimes J_m$, with $\{J_m|m = 0, 1, 2, 3\}$ being a basis of $H$. Then
\[
\partial(\sum_{m=0}^{3} f_m(q) \otimes J_m) = \sum_{m=0}^{3} f_m'(q) \otimes J_m.
\]

Proof (1) Assume $f, g \in \mathcal{R}^l(U)$ with $f(q) = f_0(q) \otimes I$ and $g(q) = g_0(q) \otimes J$. Then
\[
\eta(f \ast g) = \eta(f_0 \otimes I \cdot g_0 \otimes J)
= \eta(f_0g_0 \otimes IJ)
= T\eta g_0 \otimes f_0
= \overline{\eta(f \ast g)} = \eta(g) \ast \eta(f).
\]

Then it follows by linearity that $\eta(f \ast g) = \eta(g) \ast \eta(f)$ for any $f, g \in \mathcal{R}^l(U)$. Similarly, one also has that $\eta(f \ast g) = \eta(g) \ast \eta(f)$ for any $f, g \in \mathcal{R}^r(U)$. Since $\eta^2 = 1$, $\eta: \mathcal{R}^l(U) \to \mathcal{R}^r(U)$ is an anti-isomorphism with inverse $\eta: \mathcal{R}^r(U) \to \mathcal{R}^l(U)$, which is also an anti-isomorphism.

(2) and (3) can be verified directly, by definition.

\section{4 Left and Right Quaternionic Laplace Transforms}

Let $g(t)$ be a complex-valued function, defined for all real numbers $t \geq 0$. Then its classical Laplace transform $L\{g\}$ is defined by
\[
L\{g\}(s) = \int_{0}^{\infty} e^{-ts}g(t)dt
\]
for those complex numbers $s \in \mathbb{C}$ such that the integral converges. This will be referred as the complex Laplace transform. For the main properties of complex Laplace transform, please refer to the classical book [6].

In this section we recall the definition of quaternionic Laplace transforms and show that the main properties of the classical Laplace transforms can be generalized to quaternionic Laplace transforms.

As $\mathbb{H}$ is noncommutative, there are 2 types of quaternionic Laplace transforms; these are defined below.

Let $f(t)$ be a quaternion-valued function defined for all real numbers $t \geq 0$. Note that in the last section, $f(t)$ was a quaternion-valued function of a quaternionic variable, but in this section $f(t)$ is a quaternion-valued function of a real variable!

Define
\[
L^l\{f\}(s) = \int_{0}^{\infty} e^{-ts}f(t)dt
\]
for those quaternions $s \in \mathbb{H}$ such that the integral converges, to be the left quaternionic Laplace transform of $f$, and
\[
L^r\{f\}(s) = \int_{0}^{\infty} f(t)e^{-ts}dt
\]
for those quaternions \( s \in \mathbb{H} \) such that the integral converges, to be the right quaternionic Laplace transform of \( f \).

Note that if \( f \) is real-valued, then \( L^r\{f\} = L^l\{f\} \).

Let \( f(t) \) be a quaternion-valued function defined for all real numbers \( t \geq 0 \). Assume that \( a \geq 0 \). The function \( f(t) \) is said to be of exponential order \( a \) on \( t \geq 0 \) if there exist two positive constants, \( K \) and \( T \), such that for all \( t > T \)

\[
|f(t)| \leq Ke^{at}.
\]

Once we say that \( f(t) \) is of exponential order \( a \), then we always assume that \( a \geq 0 \).

**Theorem 4.1** Let \( f(t) \) be a quaternion-valued function defined for all real numbers \( t \geq 0 \), which is of exponential order \( a \) (with \( a \geq 0 \)), and is continuous or piecewise continuous in every finite interval \((0, T)\). Then,

1. the left Laplace transform \( L^l\{f\}(s) \) of \( f(t) \) convergent absolutely for all \( s \in \mathbb{H} \) provided \( \text{Re}(s) > a \);
2. the left Laplace transform \( L^l\{f\}(s) = \int_0^\infty e^{-st}f(t)dt \) is uniformly convergent with respect to \( s \in \mathbb{H} \), provided \( \text{Re}(s) \geq a_1 \) where \( a_1 > a \);
3. the map \( f \mapsto L^l\{f\} \), where \( f \) is of exponential order \( a \), is right \( \mathbb{H} \)-linear;
4. if \( f(t) \) is real-valued, then \( F(s) \) is intrinsic regular on \( \text{Re}(s) > a \) and \( F'(s) = L^l\{-tf(t)\} \).

The restriction of \( F \) to each \( \mathbb{C}_I \) is just the complex Laplace transform of \( f \);
5. in general, \( F(s) \) is left regular on \( \text{Re}(s) > a \), and \( \partial F(s) = L^l\{-tf(t)\} \).

**Proof** (1) and (2) can be proven as in the complex case.

(3) follows by definition.

(4) Let \( U = \{s \in \mathbb{H} | \text{Re}(s) > a\} \). For any \( I \in \mathbb{S} \), let \( U_I = U \cap \mathbb{C}_I \). Then \( F(U_I) \subseteq U_I \) and \( F|_{U_I} \) is just the usual complex Laplace transform of \( f \), which is holomorphic on \( U_I \). Thus \( F \) is intrinsic regular. For each \( I \in \mathbb{S} \) and \( s \in U_I \) with \( \text{Re}(s) > a \), one has that \( F'(s) = \int_0^\infty e^{-st}(-tf(t))dt \), and for any \( s \in U \), \( F'(s) = L^l\{-tf(t)\} \).

(5) In general, \( f = f_0 + f_1 i + f_2 j + f_3 k \), where \( f_m \) are real valued functions. Then, by (3), \( L^l\{f\} = L^l\{f_0\} + L^l\{f_1\} i + L^l\{f_2\} j + L^l\{f_3\} k \), which is left regular as each \( L^l\{f_m\} \) is intrinsic regular.

**Remark 4.2** The results for right Laplace transform are completely symmetric and are therefore omitted.

For a quaternion-valued function \( f \) defined for all real numbers \( t \geq 0 \), \( \overline{f}(t) \) is the function defined by \( \overline{f}(t) = \overline{f(t)} \). Then \( f \) is real-valued if and only if \( f = \overline{f} \).

The following result gives the relation between the left and right quaternionic Laplace transforms:

**Proposition 4.3** Let \( f(t) \) be a quaternion-valued function defined for all real numbers \( t \geq 0 \). Then \( \eta(L^l\{f\}) = L^r\{\overline{f}\} \) and \( \eta(L^r\{f\}) = L^l\{\overline{f}\} \).

**Proof** Write \( f(t) = \sum_{m=0}^{3} f_m(t) \cdot J_m \), where \( J_0 = 1, J_1 = i, J_2 = j, J_3 = k \) and \( f_m(t) \) are all real-valued functions. Then \( \overline{f}(t) = \sum_{m=0}^{3} \overline{J_m} \cdot f_m(t) \).
One has that
\[ L^1 \{ f \}(s) = \int_0^\infty e^{-ts} f(t) dt = \int_0^\infty e^{-ts} (\sum_{m=0}^{3} f_m(t) \cdot J_m) dt \]
\[ = \sum_{m=0}^{3} (\int_0^\infty e^{-ts} f_m(t) dt) \cdot J_m = \sum_{m=0}^{3} L^1 \{ f_m \}(s) \cdot J_m \]
\[ L^r \{ \overline{f} \}(s) = \int_0^\infty \overline{f}(t) e^{-ts} dt = \int_0^\infty (\sum_{m=0}^{3} \overline{J}_m \cdot f_m(t)) e^{-ts} dt \]
\[ = \sum_{m=0}^{3} \overline{J}_m \cdot (\int_0^\infty e^{-ts} f_m(t) dt) = \sum_{m=0}^{3} \overline{J}_m \cdot L^1 \{ f_m \}(s). \]

So \( \eta(L^1 \{ f \}) = L^r \{ \overline{f} \}. \) The result \( \eta(L^r \{ f \}) = L^1 \{ \overline{f} \} \) can be similarly proven. \( \square \)

**Example 4.4** Let \( b \in \mathbb{H} \) and \( f(t) = e^{bt} \). Let \( F = L^1 \{ f \} \) and \( G = L^r \{ f \} \).

For \( s \in \mathbb{R} \) and \( s > \text{Re}(b) \), \( F(s) = G(s) = \int_0^\infty e^{-ts} e^{bt} dt = (s - b)^{-1}. \)

As \( F \) is left regular, it is the left regular reciprocal of \( s - b \). Thus \( F(s) = (s^2 - 2 \text{Re}(b)s + |b|^2)^{-1} \), where \( \text{Re}(s) > \text{Re}(b) \).

As \( G(s) \) is right regular, it is the right regular reciprocal of \( s - b \). Then \( G(s) = (s - \overline{b})(s^2 - 2 \text{Re}(b)s + \overline{|b|^2})^{-1} \), where \( \text{Re}(s) > \text{Re}(b) \).

For the rest of the paper we will consider only left Laplace transforms, and \( f(t) \) will always be a quaternion-valued function defined for all real numbers \( t \geq 0 \).

**Proposition 4.5** If \( L \{ f(t) \} = F(s) \), then
\[ L^1 \{ e^{-at} f(t) \} = F(s + a), \]
where \( a \) is a real constant.

**Proof** We have, by definition, that
\[ L^1 \{ e^{-at} f(t) \} = \int_0^\infty e^{-(s+a)t} f(t) dt = F(s + a). \]

\( \square \)

Note that one does not have \( L^1 \{ f(t)e^{-at} \} = F(s + a) \), because of the noncommutativity of \( \mathbb{H} \).

Recall the Heaviside function
\[ H(x) = \begin{cases} 0, x < 0; \\ 1, x \geq 0. \end{cases} \]

**Proposition 4.6** If \( L^1 \{ f(t) \} = F(s) \), then, for \( a > 0 \), one has that
\[ L^1 \{ f(t-a)H(t-a) \} = e^{-as} F(s) = e^{-as} L^1 \{ f(t) \}. \]

The proof is as in the complex case, and so is omitted.

**Proposition 4.7** (Laplace transforms of derivatives) Let \( f(t) \) be a differentiable quaternion-valued function defined for all real numbers \( t \geq 0 \). Assume that \( f(t) \) and \( f'(t) \) are both of exponential order \( a \). Let \( U = \{ s \in \mathbb{H} | \text{Re}(s) > a \} \). If \( L^1 \{ f(t) \} = F(s) \), then
\[ L^1 \{ f'(t) \} = sL^1 \{ f(t) \} - f(0+) = sF(s) - f(0+) \]
for those \( s \in U \).

Assume further that \( f(t) \) is \( n \)-times differentiable with \( n \geq 1 \), and that \( f^{(r)}(t) \) is of exponential order \( a \) for \( r = 0, 1, \ldots, n \). Then one has that,

\[
L^l \{ f^{(n)}(t) \} = s^n F(s) - s^{n-1} f(0+) - s^{n-2} f'(0+) - \cdots - s f^{(n-2)}(0+) - f^{(n-1)}(0+)
\]

for those \( s \in U \), where \( f^{(r)}(0+) = \lim_{t \to 0^+} f^{(r)}(t) \), \( r = 0, 1, \ldots, n - 1 \).

**Proof** Integrating by parts, one has that

\[
L^l \{ f'(t) \} = \int_0^\infty e^{-st} f'(t) dt
\]

\[
= \left[ e^{-st} f(t) \right]_0^\infty - \int_0^\infty [d(e^{-st})] f(t)
\]

\[
= \left[ e^{-st} f(t) \right]_0^\infty + s \int_0^\infty e^{-st} f(t) dt
\]

\[
= -f(0+) + sF(s),
\]

which holds for those \( s \in \mathbb{H} \) with \( \text{Re}(s) > a \). The formula for \( L^l \{ f^{(n)}(t) \} \) can be proven similarly.

Let \( f(t) \) and \( g(t) \) be quaternion-valued functions defined for all real numbers \( t \geq 0 \). Let \( f \circ g \) be the convolution of \( f(t) \) and \( g(t) \), which is defined by the integral

\[
(f \circ g)(t) = \int_0^t f(t-\tau)g(\tau)d\tau.
\]

Note that \( (f \circ g)(t) \) may not be equal to \( (g \circ f)(t) \).

**Proposition 4.8** *(Convolution Theorem)*  Assume that \( f(t) \) and \( g(t) \) are of exponential orders \( a \) and \( b \), respectively. Assume that \( L^l \{ f(t) \} = F(s) \) and \( L^l \{ g(t) \} = G(s) \). Let \( c = \max\{a, b\} \) and \( U = \{ s \in \mathbb{H} : \text{Re}(s) > c \} \). Then \( L^l \{ f \circ g \} \in \mathcal{R}^l(U) \) and \( L^l \{ f \circ g \}(s) = (F*G)(s) \) for \( s \in U \). In particular, if \( s \in \mathbb{R} \) and \( s > c \), then

\[
L^l \{ f \circ g \}(s) = L^l \{ f(t) \} L^l \{ g(t) \} = F(s)G(s).
\]

**Proof** It is directly verified that for any \( \varepsilon > 0 \), \( (f \circ g)(t) \) is of exponential order \( c + \varepsilon \). Then \( L^l \{ f \circ g \} \) is defined on \( U = \{ s \in \mathbb{H} : \text{Re}(s) > c \} \), and it is left regular on \( U \).

By definition, for \( s \in \mathbb{R} \cap U \),

\[
F(s)G(s) = \int_0^\infty e^{-s\sigma} f(\sigma) d\sigma \int_0^\infty e^{-s\mu} g(\mu) d\mu
\]

\[
= \int_0^\infty \int_0^\infty e^{-s(\sigma+\mu)} f(\sigma) g(\mu) d\sigma d\mu.
\]

We make the change of variables \( \mu = \tau, \sigma = t - \mu = t - \tau \). Consequently, for \( s \in \mathbb{R} \cap U \), we get that

\[
F(s)G(s) = \int_0^\infty e^{-st} dt \int_0^t f(t-\tau)g(\tau) d\tau
\]

\[
= L^l \left\{ \int_0^t f(t-\tau)g(\tau) d\tau \right\}
\]

\[
= L^l \{ f(t) \circ g(t) \}.
\]
Thus, $L^l\{f \circ g\}$ is the left regular function on $U$ such that, for $s \in \mathbb{R}$ and $s > c$, $L^l\{f \circ g\}(s) = F(s)G(s)$. As $F(s)$ and $G(s)$ are in $\mathcal{R}^l(U)$, $L^l\{f \circ g\}(s) = (F \ast G)(s)$ for $s \in U$, by Proposition 3.6 (1).

\begin{proposition}[Derivatives of the Laplace Transform] Assume that $f(t)$ is of exponential order $a$ and that $U = \{s \in \mathbb{H} | \text{Re}(s) > a\}$. If $L^l\{f(t)\} = F(s)$, then, for $n \in \mathbb{N}$ and $s \in U$,
\[ L^l\{t^n f(t)\} = (-1)^n F^{(n)}(s),\]
where $F^{(n)}(s)$ is the $n$-th left slice derivative of $F(s)$.

This can be proven as in the complex case.
\end{proposition}

\begin{proposition}[The Laplace Transform of an Integral] Assume that $f(t)$ is of exponential order $a$ and that $U = \{s \in \mathbb{H} | \text{Re}(s) > a\}$. If $L^l\{f(t)\} = F(s)$, then, for $s \in U$,
\[ L^l\left\{ \int_0^t f(\tau)d\tau \right\} = s^{-1}F(s).\]
This can also be proved as in the complex case using Proposition 4.7, see Theorem 3.6.4 of [6].

Thus we have seen that most of the properties for the complex Laplace transform can be generalized to the quaternionic Laplace transform.

\section*{References}

[1] Cullen C G. An integral theorem for analytic intrinsic functions on quaternions. Duke Math J, 1965, 32: 139–148
[2] Colombo F, Gentili G, Sabadini I. A Cauchy kernel for regular functions. Annals of Global Analysis and Geometry, 2010, 37: 361–378
[3] Colombo F, Sabadini I. The quaternionic evolution operator. Advances in Mathematics, 2011, 227(5): 1772–1805
[4] Colombo F, Sabadini I, Struppa D C. Noncommutative Functional Calculus. Theory and Applications of Slice Hyperholomorphic Functions. Progress in Mathematics, 289. Basel: Birkhauser/Springer, 2011
[5] Colombo F, Sabadini I, Struppa D C. Entire Regular Functions. Springer Briefs in Mathematics. Cham: Springer, 2016
[6] Debnath L, Bhatta D. Integral Transforms and Their Applications. Third ed. Boca Raton, FL: CRC Press, 2015
[7] Fueter R. Die Funktionentheorie der Differentialgleichungen $\triangle u = 0$ und $\triangle\triangle u = 0$ mit vier reellen Variablen. Comment Math Helv, 1934, 7(1): 307–339
[8] Ghiloni R, Perotti A. Slice regular functions on real alternative algebras. Advances in Mathematics, 2011, 226(2): 1662–1691
[9] Gentili G, Stoppato C, Struppa D C. Regular Functions of a Quaternionic Variable. Springer Monographs in Mathematics. Heidelberg: Springer, 2013
[10] Ludkovsky S V. The two-sided Laplace transformation over the Cayley-Dickson algebras and its applications. Journal of Mathematical Sciences, 2008, 151(5): 3372–3430
[11] Ludkovsky S V. Multidimensional Laplace transforms over quaternions, octonions and Cayley-Dickson algebras, their applications to PDE. Advances in Pure Mathematics, 2012, 2: 63–103
[12] Qiao Y Y. The oblique derivative boundary value problem for generalized regular quaternion-value functions. Acta Math Sci, 1997, 17A(4): 447–451