A Boundary Integral Method Based on the Symplectic System

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Abstract. In this paper, we introduce a numerical boundary integral approach to study mechanical problems of curvilinear boundaries, multi-materials and fractures of solids and structures. Based on the equations of the eigensolution expansion and the weight functions of the basic solutions, the integral weighted equations of the undetermined coefficients can be set up by using the boundary conditions and the continuity conditions between different materials. Due to the completeness of the solution space and the independent property between the eigensolutions, the existence and the uniqueness of the solution of the boundary integral equations are seriously guaranteed. Therefore, the problems can be solved numerically by this approach.

1. Introduction

Symplectic system is an effective numerical method in computational mechanics. However, its further application is limited by its analytical properties. On the one hand, this method depends on the regular geometric boundary. Taking rectangular domain as an example, the eigenvalues can be directly obtained from a group of opposite edges, and the boundary conditions of the pair of edges can be directly satisfied [1]. The mechanical problems with curvilinear boundary can not be described by this method. In fact, a large number of mechanical problems in practical engineering do not have regular geometric boundaries, and therefore the symplectic solution scope is limited to a single material. There are many difficulties in the study of multiphase mechanical problems such as composite materials, which are widely used in modern industry. In addition, there are few discussions on the widely existing fracture problems in engineering. Generally, polar coordinate system is established with the crack tip as the center, and special crack problems can only be studied in special areas [2, 3]. For more extensive material and structure fracture, it is difficult to deal with at this stage. It can be seen that although the Hamiltonian system method gives a general solution in the form of eigensolution to satisfy the control equation, there is still a lack of effective means to deal with the boundary conditions. In fact, due to the diversity of mechanical boundary conditions, pure analytical methods are obviously difficult. Therefore, it is necessary to introduce a suitable numerical method in the Hamiltonian system and establish a numerical and analytical solution technology [4–6].

In this paper, the boundary integral method is proposed. We establish the boundary integral equations successfully by using the boundary conditions, and then discuss the solution of the eigenvalues, the boundary conditions and the treatment of the interface continuity conditions. On the one hand, the expression form of the eigenvalue solution is directly integrated on the boundary, and the eigenvalue equation is established to derive the eigenvalue and the eigenvalue solution satisfying the boundary condition in the average sense. Moreover, the eigensolution of each order is taken as the
weight function. Thus the interface between the multi-phase materials, and the algebraic equations about the expansion coefficients are constructed. Because of the completeness of the eigensolution space and the relative independence of the eigensolution function, the existence and uniqueness of the solutions of the equations are ensured. The boundary integral method is based on the combination of analytical and numerical methods, which is not limited by the boundary geometry. In a certain sense, it is not only a supplement to the analytical Hamiltonian mechanical system, but also a new numerical method for the boundary conditions of solid mechanics.

There is no doubt that the existing numerical methods, such as finite element method, have relatively mature solution technology in the field of mechanics, but this does not negate the scientific significance of the symplectic method. The reason is that: first of all, the finite element method is a kind of numerical approximation, and its theoretical basis can not be separated from the analytical method [7, 8]. For example, the starting point of using the finite element to analyze the edge effect of the shell problem, the free boundary of the composite material and its edge singularity analysis with obvious local effect is often the analytical solution. In fact, for some special mechanical problems, the combination of analytical and numerical solution technology has become a subject worthy of discussion. In addition, there are many problems in the field of mechanics, such as the singularity element at the crack tip in fracture mechanics. Because the solution singularity at the crack tip, it is generally necessary to construct special elements to study the stress and strain distribution at the tip by using finite element method. The mesh generation becomes relatively complex, which brings many difficulties to the solution of the problem. The method used in this paper is based on Hamiltonian mechanics system, which can give the analytical solution of the tip region. When constructing the weighted integral equations, the treatment of the crack surface boundary can completely imitate the treatment of the common boundary conditions, so it has certain advantages in the research of this kind of problems.

2. Research contents

Based on the analytical Hamiltonian system, combined with the numerical method of boundary integral, through the study of the mechanical problems of curve boundary, multiphase materials and fracture related mechanics, a solution technology of the combination of analytical and numerical methods is established, which provides a new numerical calculation method and research ideas for the boundary condition problems of solid mechanics. The main research contents include:

(1) Theoretical basis: on the basis of Hamiltonian mechanical system, the boundary integral theory is introduced to study the eigenvalues and boundary conditions of curve boundary problems, the continuity conditions of interface between adjacent media in multiphase materials, and the treatment methods of crack surface boundary conditions in material and structure problems, forming a solution technology combining analytical and numerical methods for further application research Provide necessary theoretical basis.

(2) Solid materials with curvilinear boundary: taking a single solid material as the research object, based on the existing theoretical results of Hamiltonian mechanical system at this stage, combined with the numerical means of boundary integral, the analytical Hamiltonian system is extended to the problem of irregular geometric boundary. In this paper, the boundary integral treatment of plane curvilinear boundary and space curved surface boundary is discussed in detail. Method of boundary conditions.

(3) Mechanical problems of multiphase materials: This paper discusses the eigenvalue problems in multiphase materials and the method of establishing weighted integral equations by using the interface continuity conditions between adjacent media. The boundary conditions of plane and space structures are studied in turn. According to the numerical results, the properties of each component material, the influence of volume content on the overall mechanical properties, and the stress concentration at the interface are discussed.
(4) Mechanical problems related to fracture: taking a single material as a breakthrough point, the mechanical problems related to fracture of composite structures containing multiphase media are further discussed, and the boundary integral treatment method of crack surface conditions is discussed.

3. Boundary condition
In polar coordinates, the mechanical problem shown in Fig. 1 is discussed.

![Figure 1. The model](image)

Taking the continuity condition between adjacent media as the basic equation, and using the basic eigensolution as the weight function, the weighted integral equations can established as

\[-b_k e^{-\mu_j} = \int_\Omega q_k p^{(a)}_k - \int_\Omega \left( \sum_n a_n p^{(a)}_n e^{\mu_j} + \sum_n b_n p^{(\beta)}_n e^{-\mu_j} \right) q^{(a)}_k \]

\[b_n e^{-\mu_j} \int_\Omega q^{(a)}_k p^{(\beta)}_n - b_n e^{-\mu_j} = b_n e^{-\mu_j} \left( \int_\Omega q^{(a)}_k p^{(\beta)} - 1 \right) \]

The stress boundary condition is

\[\int_{-b}^{b} q_0 \cdot p_j dy = \int_{-b}^{b} \left[ \sum_n (a_n \cdot q_n + \hat{a}_n \cdot \hat{q}_n) + q_p \right] \cdot p_j dy \]

\[\int_{-b}^{b} p_0 \cdot q_j dy = \int_{-b}^{b} \left[ \sum_n (a_n \cdot p_n + \hat{a}_n \cdot \hat{p}_n) + p_p \right] \cdot q_j dy \]

The solution of any Saint-Venant problem can always be expressed as

\[\vec{\eta}_c = \sum_{n=1}^{6} \left[ \vec{\alpha}_n \vec{\eta}_n^{(a)} + \vec{b}_n \vec{\eta}_n^{(\beta)} \right] \]

The final solution is

\[\vec{\eta}_i^{(n)} = \vec{\psi}_i^{(n)} + z^2 \vec{\psi}_i^{(n-2)} + \frac{1}{2} z^3 \vec{\psi}_i^{(n-3)} + \frac{1}{6} z^4 \vec{\psi}_i^{(n-4)} + \cdots + \frac{1}{n!} z^n \vec{\psi}_i^{(0)} \]

4. Numerical example
Fig. 2 shows the shear stress distribution in the region. The values of the figure are nearly constants near the end, which demonstrate that zero eigenvectors play a dominate role in the region distant from the displacement constraints, and thus they can provide the approximate solution.
Figure 2. The shear stress

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