Heat Kernel estimates for some elliptic operators with unbounded diffusion coefficients

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Abstract

We prove heat kernel bounds for the operator $(1 + |x|^\alpha)\Delta$ in $\mathbb{R}^N$, through Nash inequalities and weighted Hardy inequalities.

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1 Introduction and preliminary results

In this paper we prove heat kernel estimates for the operator

$$L = m(x)(1 + |x|^\alpha)\Delta$$

in the whole space $\mathbb{R}^N$, under the assumption that $m$ is a bounded, locally Hölder continuous function with $\inf m > 0$. The solvability of the elliptic and parabolic problem associated with $L$, either in spaces of continuous functions or in $L^p$ spaces, has been widely investigated in literature. If $\alpha \leq 2$, $L$ generates an analytic semigroup both in $L^p(\mathbb{R}^N)$ and in $C_0(\mathbb{R}^N)$, see [10]. If $\alpha > 2$, the generation results depend upon the space dimension $N$. If $N = 1, 2$, $L$ generates a semigroup in $C_b(\mathbb{R}^N)$, the space of all continuous and bounded functions on $\mathbb{R}^N$, but $C_0(\mathbb{R}^N)$ and $L^p(\mathbb{R}^N)$ are not preserved. If $N \geq 3$, the resolvent and the semigroup map $C_b(\mathbb{R}^N)$ into $C_0(\mathbb{R}^N)$ (see [10]) but $L^p(\mathbb{R}^N)$ is preserved if and only if $p > N/(N-2)$. For $N/(N-2) < p < \infty$ and under the additional assumption that $m$ admits a finite limit at infinity, the semigroup is also analytic in $L^p(\mathbb{R}^N)$ and, for $m \equiv 1$, it is contractive if and only if $\alpha \leq (N-2)(p-1)$. We refer the reader to [3] for all these results, as well as for domain characterization and spectral properties of $L$. Here we only recall that the domain of $L$ in $L^p(\mathbb{R}^N)$ coincides with the maximal one

$$D_{p,\text{max}}(L) = \{u \in W^{2,p}(\mathbb{R}^N) : (1 + |x|^\alpha)\Delta u \in L^p(\mathbb{R}^N)\}$$

and that the resolvents in $L^p(\mathbb{R}^N)$ and $L^q(\mathbb{R}^N)$ are consistent, provided that $p, q > N/(N-2)$. Finally, the resolvent is compact if and only if $\alpha > 2$ and in this case the spectrum consists of a sequence of negative eigenvalues $\lambda_n$ diverging to $-\infty$.

Due to the local regularity of the coefficients, the semigroup $(T(t))_{t \geq 0}$ generated by $L$ admits an integral kernel $p(x, y, t)$, with respect to the Lebesgue measure (see e.g. [10]), for which the following representation holds

$$T(t)f(x) = \int_{\mathbb{R}^N} p(x, y, t)f(y)dy.$$
However, the operator $L$ is symmetric with respect to the measure $d\mu(x) = (1 + |x|^\alpha)^{-1} \, dx$ and it is more convenient to express $T(t)$ through a kernel with respect to $d\mu$, namely

$$T(t)f(x) = \int_{\mathbb{R}^N} p_\mu(x, y, t) f(y) d\mu(y)$$

where

$$p_\mu(x, y, t) = (1 + |y|^\alpha)p(x, y, t).$$

Our goal consists in obtaining upper bounds for the integral kernel $p_\mu$ by working in $L^2_\mu$ spaces and then deducing upper bounds for $p$. This will be done by using the well-known equivalence between Nash inequalities and ultracontractivity for symmetric Markov semigroups, see [13, Section 6.1].

Throughout the paper the dimension $N$ will be always assumed to be greater than or equal to 3 and $\alpha$ will be a positive real number.

We shall prove that for small $t$

$$p_\mu(x, y, t) \leq \frac{C}{t^{\frac{N}{2}}(1 + |x|^\alpha)^{\frac{2-2N}{4}}(1 + |y|^\alpha)^{\frac{2-2N}{4}}}$$

for $0 < \alpha \leq 4$, or

$$p_\mu(x, y, t) \leq \frac{C}{t^{\frac{N}{2}}\phi(x)\phi(y)}$$

for $2 < \alpha \leq 4$ and

$$p_\mu(x, y, t) \leq \frac{C}{t^{\frac{N}{2}}\phi(x)\phi(y)}$$

for $\alpha \geq 4$. Here $\phi$ is the first eigenfunction of $L$ and satisfies the bounds $C_1(1 + |x|)^{2-N} \leq \phi(x) \leq C_2(1 + |x|)^{2-N}$ for suitable $C_1, C_2 > 0$.

We also show that the powers of $t$ appearing in the bounds above are optimal. Estimates for large $t$ easily follow from the semigroup law, since the semigroup decays exponentially at infinity. Observe that for $2 < \alpha < 4$ both the first and the second estimate hold (see also Remark 2.16).

1.1 Definition of $L$ via the quadratic form methods

Consider the Hilbert spaces $L^2_\mu$, where $d\mu(x) = (m(x)(1 + |x|^\alpha))^{-1} \, dx$, endowed with its canonical inner product. Note that the measure $\mu$ is finite if and only if $\alpha > N$. Consider also the Sobolev space

$$H = \{ u \in L^2_\mu : \nabla u \in L^2 \}$$

endowed with the inner product

$$(u, v)_H = \int_{\mathbb{R}^N} (u \bar{v} \, d\mu + \nabla u \cdot \nabla \bar{v} \, dx)$$

and let $V$ be the closure of $C_0^1$ in $H$, with respect to the norm of $H$. Observe that Sobolev inequality

$$\|u\|_2^2 \leq C_2^2 \|\nabla u\|_2^2$$

(1)

holds in $V$ but not in $H$ (consider for example the case where $\alpha > N$ and $u = 1$). Here $2^* = 2N/(N - 2)$ and $C_2$ is the best constant for which the equality above holds.

Next we introduce the continuous and weakly coercive symmetric form

$$a(u, v) = \int_{\mathbb{R}^N} \nabla u \cdot \nabla \bar{v} \, dx$$

(2)
for \( u, v \in V \) and the self-adjoint operator \( \mathcal{L} \) defined by

\[
D(\mathcal{L}) = \{ u \in L^2_\mu : \text{there exists } f \in L^2_\mu : a(u, v) = -\int_{\mathbb{R}^N} f \bar{v} \, d\mu \text{ for every } v \in V \} \quad \mathcal{L}u = f.
\]

Since \( a(u, u) \geq 0 \), the operator \( \mathcal{L} \) generates an analytic semigroup of contractions \( e^{t\mathcal{L}} \) in \( L^2_\mu \). An application of the Beurling-Deny criteria shows that the generated semigroup is positive and \( L^\infty \)-contractive. For our purposes we need that the resolvents and the semigroups generated by \( \mathcal{L} \) and of \( (L, D_{p,\text{max}}(L)) \) are coherent. This is stated in the following proposition. We refer to [8, Proposition 7.4] for its proof.

**Proposition 1.1**

\[
D(\mathcal{L}) \subset \{ u \in V \cap W^{2,2}_{\text{loc}} : (1 + |x|^\alpha) \Delta u \in L^2_\mu \}
\]

and \( \mathcal{L}u = (1 + |x|^\alpha) \Delta u \) for \( u \in D(\mathcal{L}) \). If \( \lambda, t > 0 \) and \( f \in L^p \cap L^2_\mu \), then

\[
(\lambda - \mathcal{L})^{-1} f = (\lambda - L)^{-1} f
\]

and

\[
e^{t\mathcal{L}} f = T(t) f.
\]

According with the above Proposition we write \( T(t) \) for the semigroup in \( L^p \) with respect to the Lebesgue measure or in \( L^2_\mu \). It admits a positive integral kernel \( p(x, y, t) \) with respect to the Lebesgue measure, see [10, Theorem 4.4], for which the following representation holds

\[
T(t)f(x) = \int_{\mathbb{R}^N} p(x, y, t)f(y)dy.
\]

Clearly we have also

\[
T(t)f(x) = \int_{\mathbb{R}^N} p_\mu(x, y, t)f(y)d\mu
\]

with

\[
p_\mu(x, y, t) = m(y)(1 + |y|^\alpha)p(x, y, t).
\]

**1.2 Eigenfunctions and eigenvalues of \( L (\alpha > 2) \)**

Spectral properties of \( L \) have also been investigated in [8, Section 7] where the following result has been proved. To unify the notation, when \( p = \infty \), \( L^p \) stands for \( C_0 \). We recall that the resolvent of \( (L, D_{p,\text{max}}(L)) \) is compact in \( L^p \) if and only if \( \alpha > 2 \), see [8], a condition that we assume throughout this section.

**Proposition 1.2** If \( N/(N-2) < p \leq \infty, 2 < \alpha < \infty \), then the spectra of \( (L, D_{p,\text{max}}(L)) \) and \( \mathcal{L} \) coincide and lie in \( ]-\infty, 0[ \). They consist of a sequence \( \lambda_n \) of eigenvalues, which are simple poles of the resolvent and tend to \( -\infty \). Each eigenspace is finite dimensional and independent of \( p \).

The next propositions give a lower and upper bound of the first eigenfunction of \( L \). Better results will follow later from kernel estimates.

**Proposition 1.3** Let \( \lambda < 0 \) be the first eigenvalue of \( L \) and \( \phi \) be the corresponding eigenfunction. Then there exists a positive constant \( C \) such that

\[
\phi(x) \geq C(1 + |x|)^{2-N}
\]

for every \( x \in \mathbb{R}^N \).
Proof. Since the kernel $p$ is positive, $T(t)$ is irreducible and from \cite{[3] Proposition 1.4.3} it follows that the eigenspace relative to the first eigenvalue is one-dimensional and admits a strictly positive eigenfunction $\phi$. Therefore $c = \min_{B(1)} \phi > 0$. Since $\phi \in D_{p, \max}(L)$ for every $N/(N - 2) < p \leq \infty$ and

$$\Delta \phi = \frac{\lambda \phi}{m(x)(1 + |x|^\alpha)},$$

from \cite{[8] Section 4} we may write

$$\phi(x) = \frac{1}{N(2 - N)\omega_N} \int_{\mathbb{R}^N} \frac{\lambda \phi(y)}{m(y)|x - y|^{N-2}(1 + |y|^\alpha)} dy$$

$$\geq \frac{\lambda c}{N(2 - N)\omega_N} \int_{B(1)} \frac{1}{m(y)|x - y|^{N-2}(1 + |y|^\alpha)} dy$$

$$\geq C \int_{B(1)} \frac{1}{|x - y|^{N-2}} dy \geq C(1 + |x|)^{2-N}.$$

\[\square\]

Next we find upper bounds for the other eigenfunctions. Since the spectrum is independent of $p$, the eigenfunctions $\phi_i$ belong to $C_0$ and therefore are bounded.

**Proposition 1.4** Let $\phi_i$ be an eigenfunction of $L$ corresponding to the eigenvalue $\lambda_i$. Then there exists a positive constant $C$ such that

$$|\phi_i(x)| \leq C|\lambda_i|^k \|\phi_i\|_\infty (1 + |x|)^{2-N}$$

where $k = \left[\frac{N}{\alpha - 2}\right] + 1$.

The proof is based upon the following lemma whose proof can be found in \cite{[8] Lemma 6.1].

**Lemma 1.5** Let $\gamma, \beta > 0$ such that $\gamma < N$ and $\gamma + \beta > N$. Set

$$J(x) = \int_{\mathbb{R}^N} \frac{dy}{|x - y|^{\gamma}(1 + |y|^{\beta})}.$$

Then $J$ is bounded in $\mathbb{R}^N$ and has the following behaviour as $|x|$ goes to infinity

$$J(x) \approx \begin{cases} 
c_1|x|^{-N-(\gamma + \beta)} & \text{if } \beta < N \\
c_2|x|^{-\gamma} \log |x| & \text{if } \beta = N \\
c_3|x|^{-\gamma} & \text{if } \beta > N
\end{cases}$$

for suitable positive constants $c_1, c_2, c_3$.

**Proof (Proposition 1.4).** As in Proposition 1.3 we write

$$\phi_i(x) = \int_{\mathbb{R}^N} \frac{\lambda_i \phi_i(y)}{m(y)|x - y|^{N-2}(1 + |y|^\alpha)} dy.$$ (3)

It follows that

$$|\phi_i(x)| \leq C|\lambda_i||\phi_i||_\infty \int_{\mathbb{R}^N} \frac{1}{|x - y|^{N-2}(1 + |y|^\alpha)} dy.$$
By Lemma [1.5]
\[ |\phi_i(x)| \leq \begin{cases} c_1 |\lambda_i| \phi_i(x) \| \phi_i \|_\infty (1 + |x|)^{2-\alpha} & \text{if } \alpha < N \\ c_2 |\lambda_i| \phi_i(x) \| \phi_i \|_\infty (1 + |x|)^{2-N} \log |x| & \text{if } \alpha = N \\ c_3 |\lambda_i| \phi_i(x) \| \phi_i \|_\infty (1 + |x|)^{2-N} & \text{if } \alpha > N \end{cases} \]

If \( \alpha > N \), we immediately deduce the claim. Otherwise, we iterate the procedure as follows. Assuming \( |\phi_i(x)| \leq c_1 |\lambda_i| \phi_i(x) \| \phi_i \|_\infty (1 + |x|)^{2-\alpha} \) and inserting this inequality in (1), we deduce
\[ |\phi_i(x)| \leq C |\lambda_i|^2 \| \phi_i \|_\infty \int_{\mathbb{R}^N} \frac{(1 + |y|)^{2-\alpha}}{|x - y|^{N-2}(1 + |y|^\alpha)} dy \leq C |\lambda_i|^2 \| \phi_i \|_\infty \int_{\mathbb{R}^N} \frac{1}{|x - y|^{N-2}(1 + |y|^{2-\alpha})} dy \]
and, by Lemma [1.5] again, \( |\phi_i(x)| \leq C |\lambda_i|^2 \| \phi_i \|_\infty (1 + |x|)^{2(2-\alpha)} \). We iterate this procedure \( k \)-times obtaining \( |\phi_i(x)| \leq C |\lambda_i|^k \| \phi_i \|_\infty (1 + |x|)^{k(2-\alpha)} \) until \( k(\alpha - 2) \leq N \). The claim then follows since at the step \( k + 1 \).

2 Kernel estimates

In this section we will prove kernel estimates through weighted Nash inequalities involving suitable Lyapunov functions. The main tool will be [2, Theorem 2.5] whose original formulation is due to Wang, see [14, Theorem 3.3].

**Definition 2.1** A Lyapunov function is a positive function \( V \) such that
\[ T(t)V(x) = \int_{\mathbb{R}^N} p_\mu(x,y,t)V(y)d\mu(y) \leq c^t V(x) \]
for every \( x \in \mathbb{R}^N \), \( t > 0 \) and some positive constant \( c \) (called a Lyapunov constant).

By following [2, Definition 2.2] we introduce weighted Nash inequalities.

**Definition 2.2** Let \( V \) be a positive function on \( \mathbb{R}^N \) and \( \psi \) be a positive function defined on \( (0, \infty) \) such that \( \frac{\psi(x)}{x} \) is increasing. A Dirichlet form \( a \) on \( L^2_\mu \) satisfies a weighted Nash inequality with weight \( V \) and rate function \( \psi \) if
\[ \psi \left( \frac{\| u \|_{L^2_\mu}^2}{\| uV \|_{L^2_\mu}^2} \right) \leq \frac{a(u,u)}{\| uV \|_{L^2_\mu}^2} \]
for all functions \( u \) in the domain of the Dirichlet form such that \( \| u \|_{L^2_\mu} > 0 \) and \( \| uV \|_{L^2_\mu} < \infty \).

Next we state [2, Theorem 2.5], adapted to our situation.

**Theorem 2.3** Let \( (T(t))_{t \geq 0} \) be a symmetric Markov semigroup on \( L^2_\mu \) with generator \( L \). Assume that there exists a Lyapunov function \( V \) with Lyapunov constant \( c \geq 0 \) and that the Dirichlet form associated to \( L \) satisfies a weighted Nash inequality with weight \( V \) and rate function \( \psi \), integrable near infinity and not integrable near zero. Then
\[ \| T(t)f \|_{L^2_\mu} \leq K(2t)c^t \| fV \|_{L^2_\mu} \]
for all functions \( f \in L^2_\mu \) such that \( fV \in L^1_\mu \). Here the function \( K \) is defined by
\[
K(t) = \sqrt{U^{-1}(t)}
\]
where
\[
U(t) = \int_{\infty}^t \frac{1}{\psi(u)} du.
\]
From [2, Proposition 2.1], the following corollary follows.

**Corollary 2.4** Under the assumptions ons of Theorem 2.3, \( T(t) \) has a kernel \( p_\mu \) which satisfies
\[
p_\mu(x,y,t) \leq K(t)^2 e^{ct} V(x)V(y)
\]
for all \( t > 0, x, y \in \mathbb{R}^N \times \mathbb{R}^N \).

**Remark 2.5** In [2, Theorem 2.5], \( V \) is required to be in the domain of the operator and to satisfy \( LV \leq cV \). By the proof, however, it is evident that such an assumption is only used to ensure the validity of the inequality
\[
\int_{\mathbb{R}^N} VT(t)f d\mu \leq e^{ct} \int_{\mathbb{R}^N} V f d\mu \tag{4}
\]
for all positive functions \( f \in L^2_\mu \) such that \( fV \in L^1_\mu \).

2.1 Intrinsic ultracontractivity

We show kernel estimates through weighted Nash inequalities with respect to the first eigenfunction of \( L \). More precisely we will prove that Definition (2.2) holds with Lyapunov function \( V \) given by the first eigenfunction \( \phi \) of \( L \) and rate functions \( \psi(t) = t^{1+\frac{\alpha}{2}} \) or \( \psi(t) = t^{1+\frac{\alpha-2}{N+\alpha-4}} \).

**Theorem 2.6** Assume that \( \alpha > 2 \). Then the kernel \( p_\mu \) of the semigroup generated by \( L \) satisfies
\[
p_\mu(x,y,t) \leq \frac{C}{t^{\frac{\alpha-2}{N+\alpha-4}}} (1 + |x|)^{2-N} (1 + |y|)^{2-N} \quad 2 < \alpha \leq 4
\]
\[
p_\mu(x,y,t) \leq \frac{C}{t^{\frac{\alpha}{2}}} (1 + |x|)^{2-N} (1 + |y|)^{2-N} \quad \alpha \geq 4
\]
or, equivalently,
\[
p_\mu(x,y,t) \leq \frac{C}{t^{\frac{\alpha-2}{N+\alpha-4}}} \phi(x)\phi(y) \quad 2 < \alpha \leq 4
\]
\[
p_\mu(x,y,t) \leq \frac{C}{t^{\frac{\alpha}{2}}} \phi(x)\phi(y) \quad \alpha \geq 4
\]
for every \( 0 < t \leq 1, x, y \in \mathbb{R}^N \).
Proof. Let \( u \in \mathcal{V} \) such that \( \|u\phi\|_{L^1_\mu} < \infty \). We treat separately the cases \( 2 < \alpha \leq 4 \) and \( \alpha \geq 4 \).

**Case** \( 2 < \alpha \leq 4 \). Set \( \theta = \frac{2(N+\alpha-4)}{\alpha-2} > 2 \). By Hölder’s inequality (with \( p = (\theta + 2)/ (\theta - 2) \)), we obtain

\[
\int_{\mathbb{R}^N} |u|^2 d\mu = \int_{\mathbb{R}^N} |u|^{\frac{2\theta}{\theta-2}} \phi^{\frac{\theta}{\theta-2}} d\mu \\
\leq \left( \int_{\mathbb{R}^N} |u|^{\frac{2\theta}{\theta-2}} \phi^{\frac{1}{\theta-2}} d\mu \right)^{\frac{\theta-2}{2\theta}} \left( \int_{\mathbb{R}^N} |u|^{\phi d\mu} \right)^{\frac{2}{\theta}}
\]

and, by recalling that

\[
C_1(1 + |x|)^{-N} \leq \phi(x) \leq C_2(1 + |x|)^{-N},
\]

(5)

\[
\left( \int_{\mathbb{R}^N} |u|^2 d\mu \right)^{\frac{1+\theta}{\theta}} \leq C \left( \int_{\mathbb{R}^N} |u|^{2\theta} (1 + |x|)^{(4(N-2))^{-\alpha}} dx \right)^{\frac{\theta-2}{\theta}} \left( \int_{\mathbb{R}^N} |u|^{\phi d\mu} \right)^{\frac{2}{\theta}}.
\]

Set now

\[
q = \frac{2\theta}{\theta-2} = \frac{2(N+\alpha-4)}{N-2}, \quad \beta = \frac{(\alpha-4)(N-2)}{2(N+\alpha-4)}.
\]

\[
\gamma = 0, \quad p = 2, \quad \nu = -\alpha
\]

and observe that, for \( 2 < \alpha \leq 4 \), the assumptions in Proposition 3.5 are satisfied. Therefore

\[
\left( \int_{\mathbb{R}^N} |u|^2 d\mu \right)^{1+\frac{\theta}{\theta}} \leq C \left( \int_{\mathbb{R}^N} |\nabla u|^2 dx + \int_{\mathbb{R}^N} |u|^2 d\mu \right)^{\frac{\theta}{\theta}} \times \left( \int_{\mathbb{R}^N} |u|^{\phi d\mu} \right)^{\frac{\theta}{\theta}}
\]

or, equivalently,

\[
\|u\|_{L^2_\mu}^{2+\frac{\theta}{\theta}} \leq C \tilde{a}(u, u) \|u\phi\|^\frac{\theta}{\theta}_{L^1_\mu}
\]

where

\[
\tilde{a}(u, u) = \int_{\mathbb{R}^N} |\nabla u|^2 dx + \int_{\mathbb{R}^N} |u|^2 d\mu
\]

is the quadratic form associated with the operator \( L + I \). By Theorem 2.3, we obtain

\[
\hat{p}_\mu(x, y, t) \leq \frac{C}{t^{\frac{N+\alpha-4}{\alpha-2}}} \phi(x)\phi(y)
\]

for every \( x, y \in \mathbb{R}^N, 0 < t \leq 1 \) and where \( \hat{p}_\mu = e^{-t} p_\mu \) is the kernel associated with \( L + I \). It follows

\[
p_\mu(x, y, t) \leq \frac{C}{t^{\frac{N+\alpha-4}{\alpha-2}}} \phi(x)\phi(y)
\]
for every \( x, y \in \mathbb{R}^N \), \( 0 < t \leq 1 \).

**Case \( \alpha \geq 4 \).** As before, by Hölder’s inequality with \( p = (N + 2)/(N - 2) \),

\[
\left( \int_{\mathbb{R}^N} |u|^2 \, d\mu \right)^{\frac{1}{N}} \leq \left( \int_{\mathbb{R}^N} |u|^{\frac{2N}{N-2}} \phi^{\frac{4}{N-2}} \, d\mu \right)^{\frac{N-2}{N}} \left( \int_{\mathbb{R}^N} |\phi| \, d\mu \right)^{\frac{1}{N}}
\]

\[
\leq C \left( \int_{\mathbb{R}^N} |u|^{\frac{2N}{N-2}} (1 + |x|)^4 \frac{1}{1 + |x|^\alpha} \, dx \right)^{\frac{N-2}{N}} \left( \int_{\mathbb{R}^N} |u|(1 + |x|)^{2-N} \, d\mu \right)^{\frac{1}{N}}
\]

\[
\leq C \left( \int_{\mathbb{R}^N} |u|^{\frac{2N}{N-2}} \, dx \right)^{\frac{N-2}{N}} \left( \int_{\mathbb{R}^N} |u|(1 + |x|)^{2-N} \, d\mu \right)^{\frac{1}{N}}.
\]

By Sobolev embedding

\[
\left( \int_{\mathbb{R}^N} |u|^2 \, d\mu \right)^{\frac{1}{N}} \leq C \left( \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right)^{\frac{1}{N}} \left( \int_{\mathbb{R}^N} |u| \, d\mu \right)^{\frac{1}{N}}
\]

and the kernel estimates follow as in the first part of the proof.

Large time estimates follow from the semigroup law and small times estimates.

**Proposition 2.7** There exists a positive constant \( C \) such that

\[
p_\mu(x, y, t) \leq C e^{\lambda t} \phi(x) \phi(y)
\]

for every \( x, y \in \mathbb{R}^N \), \( t \geq 1 \), where \( \phi \) is the first eigenfunction of \( L \) and \( \lambda < 0 \) is the first eigenvalue.

**Proof.** Let \( t \geq 1 \). By the semigroup law,

\[
p_\mu(x, y, t) = \int_{\mathbb{R}^N} p_\mu(x, z, t-1) p_\mu(z, y, 1) \, d\mu(z).
\]

By **Theorem 2.6**, we obtain

\[
p_\mu(x, y, t) \leq C \int_{\mathbb{R}^N} p_\mu(x, z, t-1) \phi(z) \phi(y) \, d\mu(z) = C \phi(y) \int_{\mathbb{R}^N} p_\mu(x, z, t-1) \phi(z) \, d\mu(z)
\]

\[
= C \phi(y) e^{(t-1)\lambda} \phi(x) = C_1 e^{\lambda t} \phi(x) \phi(y).
\]

**Remark 2.8** By recalling that the kernel \( p \) with respect to the Lebesgue measure \( dx \) satisfies \( p_\mu(x, y, t) = (1 + |y|^{\alpha}) p(x, y, t) \), we can reformulate **Theorem 2.6** as follows

\[
p(x, y, t) \leq \frac{C}{t^\frac{\alpha}{N+2}} (1 + |x|)^{2-N} (1 + |y|)^{2-N-\alpha} \quad \alpha \geq 4
\]

\[
p(x, y, t) \leq \frac{C}{t^\frac{N+\alpha-4}{N+2}} (1 + |x|)^{2-N} (1 + |y|)^{2-N-\alpha} \quad 2 < \alpha \leq 4
\]

for \( 0 < t \leq 1 \). A similar remark holds also for **Proposition 2.7**.
Remark 2.9 The estimate

\[ p_{\mu}(x, y, t) \leq \frac{C}{t^{\frac{N}{2}}} (1 + |x|)^{2-N} (1 + |y|)^{2-N}, \]

which holds for \( \alpha \geq 4 \), is optimal among the estimates of the form \( c(t)\psi(x)\psi(y) \), since the space profile is that of the first eigenfunction and the factor \( t^{-N/2} \) cannot be improved (by the local arguments of the proof of Theorem 3.4). Concerning the estimate

\[ p_{\mu}(x, y, t) \leq \frac{C}{t^{\frac{N+\alpha-4}{\alpha-2}}} (1 + |x|)^{2-N} (1 + |y|)^{2-N} \]

which holds for \( 2 < \alpha \leq 4 \), the same argument as before shows its optimality with respect to the space variable. The optimality with respect to time (among all estimates of the form \( t^{-\beta}(1 + |x|)^{2-N}(1 + |y|)^{2-N} \)) is proved by the following argument. Assume that

\[ p_{\mu}(x, y, t) \leq \frac{C}{t^{\beta}} (1 + |x|)^{2-N} (1 + |y|)^{2-N} \]

holds for \( \beta > 0 \) and \( 0 < t \leq 1 \). By the argument of Proposition 2.7 it holds for every \( t > 0 \) and then by [2, Theorem 2.10] the weighted Nash inequality

\[ \left( \int_{\mathbb{R}^N} |u|^2d\mu \right)^{\frac{1}{1+\frac{2}{\alpha}}} \leq Ca(u, u) \left( \int_{\mathbb{R}^N} |u|(1 + |x|)^{2-N}d\mu \right)^{\frac{2}{\alpha}} \]  

is valid for every \( u \in \mathcal{V} \). By replacing \( u(x) \) with \( u(\lambda x) \), \( \lambda > 0 \), in (6) we obtain

\[ \lambda^{\alpha-2+N/4} \left( \int_{\mathbb{R}^N} \frac{|u|^2}{\lambda^\alpha + |x|^{\alpha}}dx \right)^{\frac{1}{1+\frac{2}{\alpha}}} \leq Ca(u, u) \left( \int_{\mathbb{R}^N} |u| \frac{dx}{\lambda^\alpha + |x|^{\alpha+N-2}} \right)^{\frac{2}{\alpha}} \]

for every \( u \in \mathcal{V} \) and \( \lambda > 0 \). Letting \( \lambda \) to zero, such an inequality can be true only if the exponent of \( \lambda \) is nonnegative, that is if \( \beta \geq (N + \alpha - 4)/(\alpha - 2) \). Observe also that \( (N + \alpha - 4)/(\alpha - 2) > N/2 \) if and only if \( 2 < \alpha < 4 \).

2.2 Kernel estimates for \( \alpha \leq 4 \)

In this subsection we find different kernel estimates which hold for \( 0 < \alpha \leq 4 \). In the case where \( 2 < \alpha < 4 \), these estimates are better than those of the preceeding subsection with respect to the time variable, but worse with respect to the space variables. We emphasize that the case \( 0 < \alpha \leq 2 \) was not covered by the previous computations.

In the following proposition we show weighted Nash inequalities with respect to the weight \( V = (1 + |x|^\alpha)^{\frac{2-N}{\alpha}} \).

Proposition 2.10 Let \( u \in \mathcal{V} \). Then

\[ \left( \int_{\mathbb{R}^N} |u|^2d\mu \right)^{\frac{1}{1+\frac{2}{\alpha}}} \leq a(u, u) \left( \int_{\mathbb{R}^N} |u|(1 + |x|^{\alpha})^{\frac{2-N}{\alpha}}d\mu \right)^{\frac{2}{\alpha}}. \]
Proof. Let $u \in V$. Then, by Hölder’s inequality,
\[
\int_{\mathbb{R}^N} |u|^2 \, d\mu \leq C \int_{\mathbb{R}^N} |u|^2 (1 + |x|^\alpha)^{-1} \, dx = C \int_{\mathbb{R}^N} |u|^{\frac{2N}{N+2}} |u|^{\frac{N}{N+2}} (1 + |x|^\alpha)^{-1} \, dx
\]
\[
\leq C \left( \int_{\mathbb{R}^N} |u|^{\frac{2N}{N+2}} \, dx \right)^{\frac{N+2}{N}} \left( \int_{\mathbb{R}^N} |u|^2 (1 + |x|^\alpha)^{-\frac{N+2}{2}} \, dx \right)^{\frac{1}{N}}
\]
\[
= C \left( \int_{\mathbb{R}^N} |u|^2 \, dx \right)^{\frac{1}{N+2}} \left( \int_{\mathbb{R}^N} |u|(1 + |x|^\alpha)^{\frac{2-N}{2}} \, d\mu \right)^{\frac{1}{N+2}},
\]
where $2^* = 2N/(N-2)$. By Sobolev embedding,
\[
\int_{\mathbb{R}^N} |u|^2 \, d\mu \leq C \left( \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right)^{\frac{N}{N+2}} \left( \int_{\mathbb{R}^N} |u|(1 + |x|^\alpha)^{\frac{2-N}{2}} \, d\mu \right)^{\frac{1}{N+2}}
\]

hence
\[
\left( \int_{\mathbb{R}^N} |u|^2 \, d\mu \right)^{1+\frac{\alpha}{N}} \leq C a(u, u) \left( \int_{\mathbb{R}^N} |u|(1 + |x|^\alpha)^{\frac{2-N}{2}} \, d\mu \right)^{\frac{1}{N}}.
\]

Next we show that $V$ is a Lyapunov function.

Lemma 2.11 The function $V(x) = (1 + |x|^\alpha)^\beta$ satisfies $LV(x) = m(x)(1 + |x|^\alpha)\Delta V(x) \leq 0$ for $\frac{2-N}{\alpha} \leq \beta < 0$.

Proof. In fact
\[
(1 + |x|^\alpha)\Delta V = \alpha \beta |x|^\alpha - 2V(x) \left[ \alpha(\beta - 1) \frac{|x|^\alpha}{1 + |x|^\alpha} + \alpha - 2 + N \right]
\]
\[
\leq \alpha \beta(\alpha - 2 + N)|x|^\alpha - 2V \leq 0
\]
if $\frac{2-N}{\alpha} \leq \beta < 0$.

Corollary 2.12 If $\alpha \leq 4$, the function $V(x) = (1 + |x|^\alpha)^{\frac{2-N}{4}}$ satisfies $LV \leq 0$.

Lemma 2.13 If $0 < \alpha \leq 4$, then $V(x) = (1 + |x|^\alpha)^{\frac{2-N}{4}}$ is a Lyapunov function for $L$ with a Lyapunov constant $c = 0$.

Proof. Observe that $V \in C_0(\mathbb{R}^N)$. Let $\lambda > 0$ and set $u = R(\lambda, L)V$, where the resolvent is understood in $C_0(\mathbb{R}^N)$, see [10] or [8, Section 6]. Since $LV \leq 0$ then
\[
\lambda \left( \frac{V}{\lambda} - u \right) - L \left( \frac{V}{\lambda} - u \right) \geq 0.
\]

By the maximum principle it follows that $V \geq \lambda R(\lambda, L)V$. By iteration the last inequality implies
\[
V \geq \lambda^n R(\lambda, L)^n V
\]
for every $n \in \mathbb{N}$. Then
\[
T(t)V = \lim_{n \to \infty} \left[ \frac{n}{t} R \left( \frac{n}{t}, L \right) V \right]^n \leq V
\]
in $C_0(\mathbb{R}^N)$.
From Proposition 2.10, Lemma 2.13 and [2, Corollary 2.8] the following result follows.

**Theorem 2.14** If $0 < \alpha \leq 4$ the semigroup generated by $L$ has a kernel $p_\mu$ with respect to the measure $\mu$ that satisfies the following bounds

$$p_\mu(x, y, t) \leq \frac{C}{t^{\frac{\alpha}{2}}} (1 + |x|^\alpha)^{\frac{2-\alpha}{4}} (1 + |y|^\alpha)^{\frac{2-\alpha}{4}}$$

for every $t > 0$, $x, y \in \mathbb{R}^N$.

**Remark 2.15** By recalling that the kernel $p$ with respect to the Lebesgue measure $dx$ satisfies

$$p_\mu(x, y, t) = (1 + |y|^\alpha)p(x, y, t),$$

we deduce for $0 < \alpha \leq 4$

$$p(x, y, t) \leq \frac{C}{t^{\frac{\alpha}{2}}} (1 + |x|^\alpha)^{\frac{2-\alpha}{4}} (1 + |y|^\alpha)^{\frac{2-\alpha}{4} - 1}$$

for every $t > 0$, $x, y \in \mathbb{R}^N$.

**Remark 2.16** When $2 < \alpha < 4$, writing $p_\mu = p^\theta p_\mu^{1-\theta}$ for every $0 \leq \theta \leq 1$, we can combine the estimates of Theorems 2.6, 2.14 thus obtaining a family of bounds depending on a parameter $\theta$.

### 3 Some consequences

In this section we assume $\alpha > 2$ and deduce further properties of the eigenvalues and eigenfunctions of $L$ from kernel estimates. Let us denote by

$$\lambda_1 \geq \lambda_2 \geq ...$$

the eigenvalues of $L$, repeated according to their multiplicity, and by $\phi_n$ the corresponding eigenfunctions, which we assume to be normalized in $L^2_\mu$. We write $\phi$ for $\phi_1$.

**Proposition 3.1** The following estimates hold.

$$|\phi_n(x)| \leq C|\lambda_n|^\frac{N}{2} |\phi(x)|$$

for every $x \in \mathbb{R}^N$ and for $\alpha \geq 4$.

$$|\phi_n(x)| \leq C|\lambda_n|^{\frac{N+\alpha-4}{4}} |\phi(x)|$$

and

$$|\phi_n(x)| \leq C|\lambda_n|^\frac{N}{2} (1 + |x|^\alpha)^{\frac{2-N}{4}}$$

for every $x \in \mathbb{R}^N$ and $2 < \alpha \leq 4$.

**Proof.** Let $\hat{T}(t)f = \frac{1}{t}T(t)(Vf)$ with $V = \phi$ or $V = (1 + |x|^\alpha)^{\frac{2-N}{4}}$. Assume e.g. $\alpha \geq 4$. Theorem 2.6 and Proposition 2.7 show that $\hat{T}(t)$ is a bounded by $Ct^{-N/2}$ from $(L^1, V^2 d\mu)$ into $L^\infty$ and [4] shows that it is a contraction from $(L^1, V^2 d\mu)$ into itself. Then $\hat{T}(t)$ is bounded by $Ct^{-N/4}$ from $(L^2, V^2 d\mu)$ into $L^\infty$ and hence

$$\left\| \frac{1}{t^{\frac{\alpha}{2}}} \left( \frac{\phi_n}{\phi} \right) \right\|_\infty \leq \frac{C}{t^{\frac{\alpha}{2}}} \left\| \frac{\phi_n}{\phi} \right\|_{(L^2, \phi^2 d\mu)} = \frac{C}{t^{\frac{\alpha}{2}}}. $$

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Since
\[
\tilde{T}(t) \left( \frac{\phi_n}{\hat{\phi}} \right) = \frac{1}{\hat{\phi}} T(t) \phi_n = e^{\lambda_n t} \phi_n,
\]
it follows that
\[
\left| \phi_n(x) \overline{\phi(x)} \right| \leq Ce^{-\lambda_n t} \frac{t^n}{n!}
\]
for every \( x \in \mathbb{R}^N \) and \( t > 0 \). Minimizing over \( t \) we obtain
\[
|\phi_n(x)| \leq C|\lambda_n|^{\frac{n}{N}} |\phi(x)|
\]
for every \( x \in \mathbb{R}^N \). The estimates for \( 2 < \alpha \leq 4 \) follow in a similar way.

For \( \lambda > 0 \) let \( N(\lambda) \) be the number of \( \lambda_j \) such that \( |\lambda_j| \leq \lambda \). The kernel estimates allow to deduce some information on the distribution of the eigenvalues. The following result is usually obtained as a corollary of the classical Mercer’s Theorem. We refer to [9, Proposition 4.1] for a simple proof based on the semigroup property. The convergence of the integral below is easily verified using Theorem 2.6 for \( \alpha \geq 4 \) and Theorem 2.14 for \( 2 < \alpha < 4 \).

**Proposition 3.2** Let \( t > 0 \). Then
\[
\sum_{n=1}^{\infty} e^{\lambda_n t} = \int_{\mathbb{R}^N} p_{\mu}(x, x, t) \, d\mu(x) < \infty.
\]
The following proposition, which is a weaker (and elementary) version of Karamata’s Theorem, allows to deduce informations on \( N(\lambda) \). For its proof we refer to [9, Proposition 7.2].

**Proposition 3.3** Let \( r > 0 \), \( C_1 > 0 \) such that
\[
\limsup_{t \to 0} t \sum_{n \in \mathbb{N}} e^{\lambda_n t} \leq C_1.
\]
Then
\[
\limsup_{\lambda \to \infty} \lambda^{-r} N(\lambda) \leq C_1 e^r r^r.
\]
Moreover if (7) holds and
\[
\liminf_{t \to 0} t \sum_{n \in \mathbb{N}} e^{\lambda_n t} \geq C_2
\]
for some \( C_2 > 0 \) then
\[
\liminf_{\lambda \to \infty} \lambda^{-r} N(\lambda) \geq C_3
\]
for some positive \( C_3 \).

**Theorem 3.4** Let \( N(\lambda) \) be defined as before. Then
\[
\limsup_{\lambda \to \infty} \lambda^{-\frac{\alpha}{2}} N(\lambda) \leq C_1
\]
and
\[
\liminf_{\lambda \to \infty} \lambda^{-\frac{\alpha}{2}} N(\lambda) \geq C_2
\]
for some positive \( C_1, C_2 \).
Proof. By Theorems 2.6, 2.14 we deduce

\[ \limsup_{\lambda \to \infty} \lambda^{-\frac{N}{2}} N(\lambda) \leq C_1 \]

for some positive constant \( C_1 \). On the other hand, for \(|x| \leq 1\) let \( \Omega = B(x, 1) \) and denote by \( p_\Omega \) the heat kernel of the restriction of \( L \) to \( \Omega \) with Dirichlet boundary conditions. Since we have (see \cite{7}, Remark 6)

\[ p_\mu(x, x, t) \geq p_\Omega(x, x, t) \geq \frac{C}{t^{N/2}} \]

for \( 0 < t \leq 1 \) and constant \( C \) independent of \(|x| \leq 1\), it follows that

\[ \int_{\mathbb{R}^N} p_\mu(x, x, t) d\mu(x) \geq \int_{B(0,1)} p_\mu(x, x, t) d\mu(x) \geq \frac{C}{t^{N/2}} \]

for \( 0 < t \leq 1 \). Proposition 3.3 implies that

\[ \liminf_{\lambda \to \infty} \lambda^{-\frac{N}{2}} N(\lambda) \geq C_2 \]

for some positive \( C_2 \).

Appendix: some weighted Sobolev inequalities

The weighted Sobolev inequalities below have been a major tool to prove kernel estimates. For reader’s convenience we provide a direct proof following the methods of \cite{1}). We point out that such type of inequalities follow also by more general results about Sobolev inequalities with respect to weights satisfying Muckenhoupt-type conditions, see \cite{11}, \cite{12}.

**Proposition 3.5** Let \( \beta, \gamma, \nu \in \mathbb{R}, \gamma - 1 \leq \beta \leq \gamma, 1 < p \leq q < \infty \) satisfy

\[ 0 \leq \frac{1 - \frac{1}{p}}{q} = \frac{1 - \gamma + \beta}{N}, \quad N + p(\gamma - 1) \neq 0, \quad p \leq q \leq p^*, \quad p < N. \]

Then there exists a positive constant \( C \) such that for every \( u \in C_c^\infty (\mathbb{R}^N) \)

\[ \left( \int_{\mathbb{R}^N} (1 + |x|)^{q\beta} |u(x)|^q dx \right)^{\frac{1}{q}} \leq C \left( \int_{\mathbb{R}^N} (1 + |x|)^{\gamma p} |\nabla u(x)|^p dx \right)^{\frac{1}{p}} + C \left( \int_{\mathbb{R}^N} (1 + |x|)^{\nu} |u(x)|^p dx \right)^{\frac{1}{p}}. \]  

(9)

First we prove an homogeneous version of the previous inequality.

**Lemma 3.6** Let \( \beta, \gamma \in \mathbb{R}, \gamma - 1 \leq \beta \leq \gamma, 1 < p \leq q < \infty \) satisfy

\[ 0 \leq \frac{1 - \frac{1}{p}}{q} = \frac{1 - \gamma + \beta}{N}, \quad N + p(\gamma - 1) \neq 0, \quad p \leq q \leq p^*, \quad p < N. \]

Then there exists a positive constant \( C \) such that for every \( u \in C_c^\infty (\mathbb{R}^N \setminus \{0\}) \)

\[ \left( \int_{\mathbb{R}^N} |x|^{q\beta} |u(x)|^q dx \right)^{\frac{1}{q}} \leq C \left( \int_{\mathbb{R}^N} |x|^{\gamma p} |\nabla u(x)|^p dx \right)^{\frac{1}{p}}. \]  

(10)
Proof. We follow [1]. By applying the divergence theorem to suitable vector fields, we prove (10) in correspondence of $\beta = \gamma - 1$ and $p = q$, then by classical embeddings theorems we prove (10) for $\gamma = \beta$ and $q = p^*$ and finally, by interpolation, we deduce (10) in the general case. Let $u \in C^\infty_c(\mathbb{R}^N \setminus \{0\})$. Since

$$\text{div} \left( \frac{x|x|^{(\gamma-1)p}}{N + p(\gamma - 1)} \right) = |x|^{(\gamma-1)p}$$

by the divergence theorem and Hölder’s inequality,

$$\int_{\mathbb{R}^N} |x|^{(\gamma-1)p} |u(x)|^p dx = \int_{\mathbb{R}^N} \text{div} \left( \frac{x|x|^{(\gamma-1)p}}{N + p(\gamma - 1)} u(x) \right)^p dx =$$

$$- \int_{\mathbb{R}^N} \frac{x|x|^{(\gamma-1)p}}{N + p(\gamma - 1)} \nabla(|u(x)|^p) \leq \frac{p}{N + p(\gamma - 1)} \int_{\mathbb{R}^N} |x|^{(\gamma-1)(p-1) + \gamma} |u(x)|^{p-1} |\nabla u(x)| dx$$

$$\leq C \left( \int_{\mathbb{R}^N} |x|^{(\gamma-1)p} |u(x)|^p dx \right)^\frac{1}{p} \left( \int_{\mathbb{R}^N} |x|^{\gamma p} |\nabla u(x)|^p dx \right)^\frac{1}{p}.$$

It follows

$$\left( \int_{\mathbb{R}^N} |x|^{(\gamma-1)p} |u(x)|^p dx \right)^\frac{1}{p} \leq C \left( \int_{\mathbb{R}^N} |x|^{\gamma p} |\nabla u(x)|^p dx \right)^\frac{1}{p}. \quad (11)$$

Let us now consider the case $q = p^*$. Setting $f(x) = |x|^\gamma u(x)$, by Sobolev inequality we have

$$\int_{\mathbb{R}^N} |x|^{p^*} |u(x)|^{p^*} dx = \int_{\mathbb{R}^N} |f(x)|^{p^*} dx \leq C \left( \int_{\mathbb{R}^N} |\nabla f(x)|^p dx \right)^{\frac{p^*}{p}}$$

$$\leq C \left( \int_{\mathbb{R}^N} |x|^{(\gamma-1)p} |u(x)|^p dx \right)^\frac{p^*}{p} + C \left( \int_{\mathbb{R}^N} |x|^{\gamma p} |\nabla u(x)|^p dx \right)^\frac{p^*}{p}.$$

Using (11) to estimate the first addendum in the right hand side of the previous expression, we arrive at

$$\left( \int_{\mathbb{R}^N} |x|^{p^*} |u(x)|^{p^*} dx \right)^\frac{1}{p^*} \leq C \left( \int_{\mathbb{R}^N} |x|^{\gamma p} |\nabla u(x)|^p dx \right)^\frac{1}{p}. \quad (12)$$

Let now $p \leq q \leq p^*$ and $\gamma - 1 \leq \beta \leq \gamma$. There exists $\theta \in [0, 1]$ such that $\beta = (1 - \theta)(\gamma - 1) + \theta \gamma$. By the assumption $\frac{1}{p} - \frac{1}{q} = \frac{1 - \gamma + \beta}{\gamma}$ it follows that $q = \frac{Np}{N - \theta p} = (1 - c)p + cp^*$ with $c = \frac{\theta(N - p)}{N - \theta p} \in [0, 1]$. Therefore, Hölder’s inequality yields

$$\int_{\mathbb{R}^N} |x|^\beta |u(x)|^q dx = \int_{\mathbb{R}^N} |x|^{(1-c)(\gamma-1)p + \gamma cp^*} |u(x)|^{(1-c)p + cp^*} dx$$

$$= \int_{\mathbb{R}^N} \left( |x|^{(\gamma-1)p} |u(x)|^p \right)^{1-c} \left( |x|^{\gamma p} |u(x)|^{p^*} \right)^c dx$$

$$\leq \left( \int_{\mathbb{R}^N} |x|^{(\gamma-1)p} |u(x)|^p dx \right)^{1-c} \left( \int_{\mathbb{R}^N} |x|^{\gamma p} |u(x)|^{p^*} dx \right)^c.$$

By (11) and (12), we obtain

$$\left( \int_{\mathbb{R}^N} |x|^{\beta q} |u(x)|^q dx \right)^\frac{1}{q} \leq C \left( \int_{\mathbb{R}^N} |x|^{\gamma p} |\nabla u(x)|^p dx \right)^\frac{1}{p}. \quad \square$$
Proof. (Proposition 3.5) Let $u \in C_c^\infty(\mathbb{R}^N)$ and fix $\eta \in C_c^\infty(\mathbb{R}^N)$, $0 \leq \eta \leq 1$ such that $\eta = 1$ in $B(1)$ and $\eta = 0$ in $\mathbb{R}^N \setminus B(2)$. By Sobolev embedding,

$$
\left( \int_{B(2)} (1 + |x|)^{\gamma\beta} |\eta(x)u(x)|^q \, dx \right)^{\frac{1}{q}} \leq C \left( \int_{B(2)} |\eta(x)u(x)|^p \, dx \right)^{\frac{1}{p}} \leq C \left( \int_{B(2)} |\nabla(\eta(x)u(x))|^p \, dx \right)^{\frac{1}{p}}
$$

By combining with (10), we deduce

$$
\left( \int_{\mathbb{R}^N} (1 + |x|)^{\gamma\beta} |u(x)|^q \, dx \right)^{\frac{1}{q}} = \left( \int_{\mathbb{R}^N} (1 + |x|)^{\gamma\beta} |\eta(x)u(x) + (1 - \eta(x))u(x)|^q \, dx \right)^{\frac{1}{q}}
\leq C \left( \int_{\mathbb{R}^N} (1 + |x|)^{\gamma\beta} |\nabla u(x)|^p \, dx \right)^{\frac{1}{p}} + C \left( \int_{B(2)} (1 + |x|)^{\gamma\beta} |\nabla \eta(x)|^p |u(x)|^p \, dx \right)^{\frac{1}{p}}.
$$

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