On the sequence $\alpha n!$

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Abstract. We prove that there exists $\alpha \in \mathbb{R}$ such that for any $N$ the discrepancy $D_N$ of the sequence $\{\alpha n\}$, $1 \leq n \leq N$ satisfies $D_N = O(\log N)$.

1. Low discrepancy sequences.

Consider a sequence $\xi_k, k = 1, 2, 3\ldots$ of points from the interval $[0, 1)$. The discrepancy $D_N$ of the first $N$ points of the sequence is defined as

$$D_N = \sup_{\gamma \in [0,1)} |N\gamma - \#\{j : 1 \leq j \leq N, \xi_j \leq \gamma\}|.$$ 

According to the famous W.M. Schmidt’s theorem for any infinite sequence one has

$$\liminf_{N \to \infty} \frac{D_N}{\log N} > 0.$$ 

This statement is sharp enough. Consider a real number $\alpha$ with bounded partial quotients in its continued fraction expansion. Then the discrepancy of the sequence

$$\{\alpha k\}, \quad k = 1, 2, 3\ldots$$

(here $\{\cdot\}$ stands for the fractional part) satisfied the inequality

$$D_n \leq M \log N,$$

where $M$ depends on the bound for the partial quotients of $\alpha$. Roughly speaking, this result was obtained long ago by Ostrowski [7] and Khintchine [2]. For further information concerning discrepancy bounds one can see wonderful books [1, 4] and [6].

As for exponentially increasing sequences we would like to refer to Levin’s paper [5]. Given integer $q \geq 2$ Levin proved the existence of real $\alpha$ such that the discrepancy $D_N$ for the sequence

$$\{\alpha q^k\}, \quad k = 1, 2, 3\ldots$$

satisfies the inequality

$$D_N = O((\log N)^2).$$

(1)

(2)

Up to now it is not known if there exists $\alpha$ such that the order of the discrepancy for the sequence (1) is smaller that that from (2). However if the sequence $n_k$ grows faster than exponentially, it is quite easy to construct $\alpha$ such that the discrepancy of the sequence $\{\alpha n_k\}$ is small. This is just the purpose of the present short communication.

Theorem 1. Suppose that a sequence of positive numbers $n_k, k = 1, 2, 3\ldots$ satisfy the condition

$$\inf_k \frac{n_{k+1}}{kn_k} > 0.$$ 

(3)
There exists $\alpha \in \mathbb{R}$ such that for any $N$ the discrepancy $D_N$ of the sequence $\{\alpha n_k\}$, $1 \leq k \leq N$ satisfies $D_N = O(\log N)$.

Corollary 2. Then there exists $\alpha \in \mathbb{R}$ such that for any $N$ the discrepancy $D_N$ of the sequence $\{\alpha n!\}$, $1 \leq n \leq N$ satisfies $D_N = O(\log N)$.

Korobov ([3], Theorem 3 and Example 1) constructed real numbers $\alpha$ for which the sequence $\{\alpha k!\}$ is uniformly distributed. However his construction does not give optimal bounds for the discrepancy.

2. Lemmas.

In the sequel $F_i$ stands for the $i$-th Fibonacci number, so $F_0 = F_1 = 1, F_{i+1} = F_i + F_{i-1}$ and $\phi = \frac{1 + \sqrt{5}}{2}$.

Lemma 3. Any positive integer $N$ can be represented in a form $N = \sum_{i=1}^{r} b_i F_i$, where $b_i \in \{1, 2, 3\}, b_i \in \{1, 2\}, 2 \leq i \leq r$ and $r \leq 1 + \log_\phi N$.

Proof. It is a well-known fact that any positive integer can be represented in a form $N = \sum_{i=1}^{t} a_i F_i$ with $a_i \in \{0, 1\}, t \leq 1 + \log_\phi N$ and in the sequence

\[
a_1, a_1, ..., a_t
\]

there is no two consecutive ones. Now we give an algorithm how to construct from the sequence (4) a sequence $b_1, b_2, ..., b_r$ with all positive $b_i$ and $r \leq t$.

We shall use two procedures.

Procedure 1. If we have two consecutive zeros, that is we have a pattern $a_i, 0, 0, a_{i+3}$, with $a_{i+3} = 1$, we can replace it by the pattern $a_i, 1, 1, 0$. The sum $N = \sum_{i=1}^{t} a_i F_i$ will remain the same as $F_{i+3} = F_{i+2} + F_{i+1}$. But the number of zeros in the sequence will decrease by one.

Procedure 1 enables one to get from the sequence (4) another sequence of ones and zeros without two consecutive zeros.

Procedure 2. If we have a pattern $a_i, 0, a_{i+2}$ we may replace it by the pattern $a_i + 1, 1, a_{i+2} - 1$. If $a_{i+2} = 1$ then there will be no zero in $(i+1)$-th position but a zero will appear in $(i+2)$-th position. The total number of zeros will not change.

The algorithm is as follows. Procedure 1 enables one to get a sequence

\[
a_1', a_1', ..., a_{t'}
\]

of ones and zeros where there in no two consecutive zeros with $t' \leq t$ and $a_{t'} = 1$.

If $a_1' = 0$ then $a_1' = 1$ and we may replace the pattern $a_1', a_2'$ by 2, 0 as $F_2 = 2 = 2F_1$. So we may suppose that in (5) one has $a_1' \in \{1, 2\}$. But it may happen that $a_3' = a_3 = 0$. By applying procedures 1 we obtain a sequence

\[
a_1'', a_1'', ..., a_{t''}
\]

with $t'' \leq t'$ where $a_1' \in \{1, 2\}$ and all other elements are equal to 1 or 0 with no two consecutive zeros. Now we take the zero in the smallest position and apply Procedure 2. The zero will turn into the next position. Then either there are two consecutive zeros (and we can reduce the number of zeros by Procedure 1) or we can move the zero in the next position again. Each "moving to the next position" increases the previous digit by 1. But all the time the previous digit is 1. The only exception is in the very beginning of the process, when $a_1'' = 2$. Then $a_1''$ must turn into 3.

In such a way we get the necessary representation for $N$. □.

From Lemma 3 we immediately deduce

Corollary 4. For any positive integer $N$ the set of the first $N$ positive integers can be partitioned into segments of consecutive integers in the following way:

\[
\{1, 2, ..., N\} = \mathcal{A} \sqcup \bigcup_{i=2}^{r} \bigcup_{j=1}^{b_i} \{R_{i,j}, R_{i,j} + 1, ..., R_{i,j} + F_i - 1\},
\]

(7)
where $\mathcal{A} = \{1\}$ or $\{1, 2\}$ or $\{1, 2, 3\}$, $b_i \in \{1, 2\}$ and

$$R_{i,j} \geq F_i, \quad (8)$$

$$r \leq 1 + \log_2 N. \quad (9)$$

Let $R, i$ be positive integers. We consider the sequence

$$\{\phi k\}, \quad R \leq k < R + F_i \quad (10)$$

The following statement is well-known. It is the main argument of the classical proofs of the logarithmic order of discrepancy of the sequence $\{\alpha k\}$, in the case when $\alpha$ has bounded partial quotients in its continued fraction expansion. It immediately follows from the inequality $||\phi F_i|| \leq 1/F_i$, where $|| \cdot ||$ stands for the distance to the nearest integer. It means that the the set (10) is close to the set

$$\nu \frac{\nu}{F_i}, \quad 1 \leq \nu \leq F_i - 1,$$

and hence discrepancy of the sequence (10) is bounded.

**Lemma 5.** There is a substitution $\sigma_1, \ldots, \sigma_{F_i}$ of the sequence 1, ..., $F_i$ such that

$$||\{\phi(R + k)\} - \frac{\sigma_k}{F_i}|| \leq \frac{1}{F_i}, \quad 0 \leq k \leq F_i - 1.$$

**Lemma 6.** Consider an arbitrary sequence $\xi_k$, $k = 1, 2, 3, \ldots$ from the interval [0, 1). Suppose that a sequence $n_k$ satisfy (2). Then there exist $c > 0$ and $\alpha \in \mathbb{R}$ such that

$$||\alpha n_k - \xi_k|| \leq \frac{c}{k}, \quad k = 1, 2, 3, \ldots$$

(11)

Proof. From (3) we see that $\frac{n_k + 1}{n_k} \geq \kappa k$ for some positive $\kappa$. Fix $k$. Then the set

$$\left\{ \alpha \in \mathbb{R} : \ ||\alpha n_k - \xi_k|| \leq \frac{c}{k} \right\}$$

is a union of segments of the form $\left[\frac{\xi_k + z}{n_k}, \frac{\xi_k + z}{n_k} + \frac{c}{k n_k}\right], \ z \in \mathbb{Z}$. The length of each segment is equal to $\frac{c}{k n_k}$. The distance between the centers of neighboring segments is equal to $\frac{1}{n_k}$. We see that $\frac{c}{k n_k} \geq \frac{\alpha}{n_{k+1}}$. So if $c$ is large enough, we can choose integers $z_k$ to get a sequence of nested segments

$$\left[\frac{\xi_k + z_1}{n_1} - \frac{c}{n_1}, \frac{\xi_k + z_1}{n_1} + \frac{c}{n_1}\right] \supset \ldots \supset \left[\frac{\xi_k + z_k}{n_k} - \frac{c}{k n_k}, \frac{\xi_k + z_k}{n_k} + \frac{c}{k n_k}\right] \supset \left[\frac{\xi_k + z_{k+1}}{n_{k+1}} - \frac{c}{(k+1)n_{k+1}}, \frac{\xi_k + z_{k+1}}{n_{k+1}} + \frac{c}{(k+1)n_{k+1}}\right] \supset \ldots$$

The common point of these segments satisfies (11). $\square$

**3. Proof of Theorem 1.**

We take the sequence $\xi_k = \{\phi k\}, k = 1, 2, 3, \ldots$ and apply Lemma 6. Then we get real $\alpha$. This is just the number what we need. Take positive integer $N$. Then in the decomposition (7) for each segment $\{R_{i,j}, R_{i,j} + 1, \ldots, R_{i,j} + F_i - 1\}$ its right endpoint is $\geq F_i$. So from the inequalities of Lemmas 5, 6 we get

$$\left||\alpha n_k - \frac{\sigma_k}{F_i}\right|| \leq \frac{1}{F_i} + \frac{c}{R_{i,j}} \leq \frac{1 + c}{F_i}, \quad \forall i,j.
This means that each sequence
\[ \{\alpha_n^k\}, \quad R_{i,j} < k \leq R_{i,j} + F_i \] (12)
has discrepancy \( O(1) \). But the sequence \( \{\alpha_n^k\}, 1 \leq k \leq N \) is partitioned into \( O(\log N) \) sequences of the form \( (12) \), as all \( b_i \) are bounded by 3 and we have estimate \( (9) \). So this sequence has discrepancy \( O(\log N) \). □

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