Finite-Sized Plasmas

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When addressing the thermodynamics of finite-sized systems, one must specify whether one wants to fix conserved charges to a sharp value or whether one is content to fix their thermodynamic average. In other words, contrary to the thermodynamic limit, different statistical ensembles are not equivalent. When treating the plasma phases of gauge field theories perturbatively in the canonical ensemble, unexpected new difficulties arise in comparison with the usual grand canonical treatment. The purpose of this paper is to expose these difficulties and show how they can be remedied, thus recovering a well-defined description of plasmas even in a finite-size, canonical ensemble setting. For definiteness, a specific model is considered, namely QCD\textsuperscript{1+1} with SU(2) color; however, the treatment presented should be applicable also to higher-dimensional systems with different types of Coulomb interaction.

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I. INTRODUCTION

The fundamental interactions governing microscopic behavior in nature are described by gauge theories. These theories generically give rise to long-range Coulomb interactions, and for this reason plasma phases represent an almost universal phenomenon in dense systems, whether one is considering quarks and gluons in a heavy ion collision or the electron gas in a metal. Many of the plasmas accessible to experiment are quite small, not only in the case of the heavy-ion collision \cite{1}, but also e.g. in mesoscopic systems \cite{2}, which have become a wealthy source of new phenomena in the past years. In such a setting, finite-size effects become important, and they are indeed one of the causes of the high diversity of mesoscopic physics.

The most obvious effect of a finite configuration space is that the corresponding momentum space becomes discrete. However, there is also a slightly more subtle source of new behavior connected with the ensemble one uses to describe the statistical properties of the system at hand. Commonly, in the presence of a conserved charge \( N \), one uses the grand canonical ensemble to calculate the partition function,

\[
Z_{GC} = \text{Tr} \ e^{-\beta(H - \mu N)}
\]  

(1)

where \( \mu \) is the corresponding chemical potential, \( \beta \) the inverse temperature, and \( H \) the Hamiltonian. One may at the end of the calculation fix \( \mu \) to impose a desired mean value of \( N \). Physically, this corresponds to the system being able to exchange energy and particles with its surroundings. In many cases, however, this is an idealization, and the system is only able to adjust its charge to a limited extent, or not at all. The prime example is the colliding heavy-ion system, which must remain an exact color singlet at all times. In such a situation, the canonical ensemble is more appropriate,

\[
Z_{C} = \int d\alpha \ \text{Tr} \ e^{-\beta(H - i\alpha(N-N_0)/\beta)}
\]  

(2)

This restricts the trace to states in which the charge \( N \) takes exactly the value \( N_0 \). In the following, \( N_0 \) will always be taken to be zero, without loss of generality.

Whereas for very large systems, \( \square \) and \( \square \) give identical results because the fluctuations of \( N \) in the grand canonical ensemble die out as the inverse root of the size, in a finite system \( \square \) and \( \square \) are different. Now, at first glance one may think that the finite size corrections are easy to obtain if one has, as usual, calculated \( \square \); simply put in an imaginary chemical potential \( \mu = i\alpha/\beta \) and do an integration to obtain \( \square \). Such considerations have been made for ideal gases e.g. in \( \square \)-\( \square \). A closer look however reveals that, at imaginary chemical potential, a plasma does

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not conform to the recipes conventionally used in its formal treatment. Specifically, the venerated prescription of \[7\], namely summing the so-called ring diagrams to a logarithm, fails because one runs into the cut of the logarithm. The main formal advance of the present treatment is to show how this difficulty can be remedied. Thus it will be possible to regain a well-defined description of the plasma even in a finite-size, canonical ensemble setting.

The specific model to be considered here is QCD in one space dimension in the high temperature or high baryon density perturbative regime \[8\]. It must be emphasized however that the formal developments presented are quite general and are straightforwardly applied also to higher-dimensional systems with any type of Coulombic interaction. A one-dimensional model is chosen because in this setting, the infrared divergences appear most clearly, uncluttered by the usual ultraviolet problems. The results however are expected to enjoy a wide range of applicability. The goal of the treatment is to evaluate the contribution of the plasmon effect to the thermodynamics of a finite system, exhibiting especially the differences in behavior away from the thermodynamic limit depending on which ensemble is used. Of course, the size of the system is still taken to be large enough to permit a physical interpretation as a plasma, i.e. if \( g \) is the coupling constant in the relevant Lagrangian and \( L \) is the length of the system, then \( gL \) should still be large\[1\]. Otherwise, the concept of screening the Coulomb potential becomes meaningless and the perturbative expansion of the theory is in powers of \( g^2L^2 \). Physically, large \( gL \) means that the plasmons, whose size is controlled by \( 1/g \) in one space dimension, still fit well into the length \( L \). Correspondingly, in three space dimensions, the plasmon size behaves \[9\] like \( 1/gT \) and thus the relevant dimensionless parameter which must remain sizeable would be \( gT L \). Only under such conditions is it meaningful to consider the usual plasmon contribution to the thermodynamic potential and evaluate its behavior as \( 1/gL \) is increased away from zero.

II. THE MODEL

The axial gauge Hamiltonian of QCD\[1+1\] with two colors reads in momentum space

\[
H = \int dp \chi(p) \gamma_5 \chi(p) + m \gamma_0 \chi(p) + \frac{g^2}{4\pi} \int dq j^a(q) j^a(-q) \tag{3}
\]

where the fermion fields \( \chi \) carry Dirac and color indices and the SU(2) currents \( j^a \) are

\[
j^a(q) = \int dp \chi(p) \sigma^a \frac{1}{2} \chi(p+q) \tag{4}
\]

with the Pauli matrices \( \sigma^a \). For later use it is also useful to introduce the U(1) currents

\[
j^0(q) = \frac{1}{2} \int dp \chi(p) \chi(p+q) \tag{5}
\]

The Hamiltonian \( \tag{3} \) conserves the U(1) charge \( j^0(0) \), corresponding to the baryon number, and the SU(2) color charges \( j^a(0) \). For definiteness, it will be assumed that the plasma may exchange baryon number with its surroundings, but not color charge. This will illustrate all the possible complications; all alternative scenarios can be treated along similar lines. In particular, it is straightforward e.g. to specialize to QED in the canonical ensemble. Thus, the goal is to evaluate the partition function

\[
Z = \int_{SU(2)} dG(\alpha) \text{ Tr } \exp \left[ -\beta (H - \mu j^0(0) - i\alpha^a j^a(0)/\beta) \right] \tag{6}
\]

where \( dG(\alpha) \) denotes the Haar measure of SU(2) \( \tag{3} \). Generically, a further simplification is possible at this point. The trace is invariant under a global unitary transformation of the fermion fields,

\[
\chi \rightarrow \chi' = U \chi \tag{7}
\]

and also the Hamiltonian and the baryon number remain unchanged under this transformation (this is a global residual gauge invariance still present after gauge fixing). Therefore, one may choose to diagonalize \( \alpha^a \sigma^a \).

\[\text{Note that } gL \text{ is dimensionless in one space dimension.}\]
\[ U^\dagger \alpha \sigma^a U = \sigma^3 \sqrt{(\alpha^1)^2 + (\alpha^2)^2 + (\alpha^3)^2} \equiv \sigma^3 \alpha \]  

(8)
i.e. the partition function depends only on the length of \( \alpha \), not its direction in color space,

\[ Z = \frac{1}{2\pi} \int_{-2\pi}^{2\pi} d\alpha \sin^2 \frac{\alpha}{2} \text{ Tr } \exp \left[ -\beta (H - \mu j^3(0) - i\alpha j^3(0)/\beta) \right] \equiv \frac{1}{2\pi} \int_{-2\pi}^{2\pi} d\alpha \sin^2 \frac{\alpha}{2} Z(\alpha) \]  

(9)

In general, only the (imaginary) chemical potentials associated with the Cartan subalgebra of the gauge group are relevant. In addition to the symmetry exploited above, there still remains a discrete translational symmetry in \( \alpha \) and also a parity symmetry corresponding to the freedom of permuting the eigenvalues in the diagonalized matrix in (8). In the following, (9) will be referred to as the “canonical” partition function (even though the U(1) charge is treated grand canonically). On the other hand, treating also the SU(2) color charges grand canonically (this will be referred to as the “grand canonical” case) corresponds simply to dropping the average over \( \alpha \) in (9) and setting \( Z = Z(\alpha = 0) \).

It should be mentioned that in the specific setting of QCD\(^1\+\+\), the Hamiltonian (8) is only valid in the color singlet sector. In general, the Hamiltonian of QCD\(^1\+\+) contains additional couplings to residual gauge degrees of freedom (10), the dynamics of which eliminate all non-singlet states from the spectrum. Thus, the partition function of QCD\(^1\+\+) strictly speaking contains only singlet states already by virtue of the dynamics. Of course it is fully equivalent to use the color singlet Hamiltonian (8) and impose the color singlet constraint by hand as in (9). On the other hand, for purposes of comparison below also the grand canonical partition function corresponding to the Hamiltonian (8) will be considered. Strictly speaking, this is not a physically meaningful alternative. Therefore, from the point of view of QCD\(^1\+\+) what is being compared is a physically correct calculation (namely, the canonical one) with the idealized grand canonical one which is often adopted in the literature due to technical convenience. Of course, in other settings such as electrodynamical plasmas in three space dimensions, both alternatives may be physically meaningful, depending on the laboratory conditions.

III. IDEAL GAS

Momentarily neglecting the Coulomb interaction, one can now immediately read off the ideal gas partition function (11),

\[
\ln Z_0(\alpha) = \frac{L}{2\pi} \int dp \sum_i \left[ \ln(1 + e^{-\beta(\epsilon_p + \mu/2 + i\sigma_i^3/2\beta)}) + \ln(1 + e^{-\beta(\epsilon_p - \mu/2 - i\sigma_i^3/2\beta)}) \right] = \frac{L}{2\pi} \int dp \left[ \ln(1 + 2\cos(\alpha/2)e^{-\beta(\epsilon_p + \mu/2)} + e^{-2\beta(\epsilon_p + \mu/2)}) + \ln(1 + 2\cos(\alpha/2)e^{-\beta(\epsilon_p - \mu/2)} + e^{-2\beta(\epsilon_p - \mu/2)}) \right]
\]  

(10)

where \( \epsilon_p = \sqrt{p^2 + m^2} \). In some special cases, this can be evaluated explicitly as long as one still assumes the momenta to be continuous. E.g. in the classical nonrelativistic limit \( m \gg m - \mu/2 \gg T \) one obtains

\[
\ln Z_0(\alpha) = \frac{L}{\pi} \left( \frac{\pi}{3\beta} - \frac{\alpha^2}{4\pi\beta} \right)
\]  

(12)

whereas in the limit \( m = \mu = 0 \) one has (11)

\[
\ln Z_0(\alpha) = L \left( \frac{\pi}{3\beta} - \frac{\alpha^2}{4\pi\beta} \right)
\]  

(13)

Taking now into account the interaction perturbatively, assume that \( \ln Z(\alpha) \) has been calculated to some order in \( g \) (note that one usually calculates directly \( \ln Z_I(\alpha) \) by considering only connected Feynman diagrams),

\[
\ln Z(\alpha) = \ln Z_0(\alpha) + \ln Z_I(\alpha)
\]  

(14)

According to (11), this must now be averaged over \( \alpha \),

\[
\ln Z = \ln \frac{1}{2\pi} \int d\alpha \sin^2 \frac{\alpha}{2} Z_0(\alpha)Z_I(\alpha)
\]  

(15)

\[
= \ln \frac{1}{2\pi} \int d\alpha \sin^2 \frac{\alpha}{2} Z_0(\alpha) + \ln \frac{\int d\alpha \sin^2(\alpha/2)Z_0(\alpha)Z_I(\alpha)}{\int d\alpha \sin^2(\alpha/2)Z_0(\alpha)}
\]  

(16)
Thus $Z_0(\alpha)$, apart from giving the ideal gas contribution in the first term of (16), also acts as a measure for averaging $Z_I(\alpha)$. Now, inspection of (13) and (14), or in general (11), reveals that $Z_0(\alpha)$ falls off as a Gaussian with a width of $1/\sqrt{L}$ in $\alpha$ for large $L$. Therefore, in the thermodynamic limit, only the vicinity of $\alpha = 0$ contributes to $\ln Z$. Thus one recovers the well-known result that the canonical and grand canonical ensembles are equivalent for large systems, or, formulated in another way, color singlet constraints become irrelevant in the thermodynamic limit.

For finite $L$, on the other hand, the two ensembles differ. Whereas the grand canonical ideal gas partition function is obtained simply by setting $\alpha = 0$ in (11), (12), or (13), in the canonical ensemble one must average over $\alpha$. In special cases, this can be accomplished analytically. In the classical nonrelativistic limit, the average over (12) can be carried out to give

$$\ln Z_0 = \ln (2I_1(x)/x)$$

with $x = L/\pi \sqrt{2m\pi/\beta} e^{-\beta(m-\mu/2)}$ (17)

where $I_1(x)$ is a Bessel function of imaginary argument [12]. In the case $m = \mu = 0$ one obtains using (13)

$$\ln Z_0 = \frac{L\pi}{3\beta} + \ln \left[ \frac{1}{4\sqrt{\pi x}} \left( \text{erf}(2\pi \sqrt{x}) - e^{-x} \left( \text{Re erf}(2\pi \sqrt{x} + i\frac{\mu}{2\sqrt{x}}) - \text{Re erf}(i\frac{\mu}{2\sqrt{x}}) \right) \right) \right]$$

(18)

where erf denotes the error function and $x = L/(4\pi \beta)$. The comparison between the ensembles is illustrated for the ideal gas in figures (1) and (2). Also, to give an idea how the effect of discretizing the momenta compares with the effect of varying the ensemble, the free partition function has been calculated using antiperiodic boundary conditions for the fermion fields, i.e.

$$p \rightarrow p_n = \frac{(2n + 1)\pi}{L}, \quad \int dp \rightarrow \frac{2\pi}{L} \sum_{p_n}$$

(19)

as well as periodic boundary conditions,

$$p \rightarrow p_n = \frac{2n\pi}{L}, \quad \int dp \rightarrow \frac{2\pi}{L} \sum_{p_n}$$

(20)

Evidently both figures, though stemming from vastly different regions of the phase diagram, display a dominance of the effect due to varying choice of ensemble except at very low $L$. Formally, what happens is that the difference between an integral and the corresponding Riemann sum behaves as $1/L$, as can be inferred from the Euler-McLaurin summation formula [13]. Thus, merely discretizing the momenta leads to an expansion of the form $\ln Z_0 \sim O(L) + O(1) + \ldots$ for the logarithm of the partition function. On the other hand, the averaging procedure involved in doing a canonical calculation in general introduces power corrections in $L$ into the partition function $Z_0$, as can be explicitly observed in (17) and (18). This leads to an expansion of the form $\ln Z_0 \sim O(L) + O(\ln L) + \ldots$ upon taking the logarithm. At low $L$, the question which effect dominates depends on the details of the dynamics, as evidenced in figures (1) and (2). Whereas in figure (2), the discretization effects are negligible even at $LT = 1$, in the ultrarelativistic case the discretization effects become comparable to the effect of varying the ensemble at sufficiently low $L$. The effect of using a canonical ensemble is reinforced by using antiperiodic boundary conditions and weakened by using periodic ones. A similar comparison will be possible for the plasmon contribution calculated in the next section.

IV. PLASMON CONTRIBUTION

This section is concerned with the main object of the present treatment, the perturbative evaluation of the plasmon contribution to the thermodynamic potential. Whereas this effect arises only at order $O(g^3)$ in gauge theories in three space dimensions [11], it already contributes at order $O(g)$ in one space dimension due to the different volume element in momentum space. Thus it represents the dominant perturbative effect here.

As a side remark, note that in QCD$_{1+1}$, this is in fact the only term of the perturbative expansion which can be obtained in terms of Feynman diagrams. Already in the next order of perturbation theory, the Feynman expansion breaks down due to irremediable ambiguities introduced by the infrared divergences [14]. Formally, one can give convergent sums over selected subclasses of diagrams such that virtually arbitrary fractional powers (greater than one) of the coupling constant $g$ are generated. On the other hand, it is known e.g. in the classical nonrelativistic limit of QCD$_{1+1}$ [15] that expanding the exact equation of state in the coupling constant yields a series made up exclusively of integer powers of $g$. Similar infrared problems arise in QCD$_{3+1}$ at order $O(g^6)$ [14]. In the case of QED,
The infrared divergences are not as severe and the Feynman expansion can be carried out at least to some higher order in $g$.

The perturbative expansion of the expression (3) for the partition function may be obtained by standard methods [11] [17]. The corresponding Feynman rules can be read off from (9) and (3) as follows:

- Fermion propagator with momentum $p$, color $i$ and Matsubara frequency $\omega_s = (2s + 1)\pi/\beta$:
  \[ K_i(p, s) = \frac{i}{\beta} \frac{\omega_s - i\mu/2 + \sigma_i^a\alpha/2\beta + i\gamma_5p - i\gamma_0m}{(\omega_s - i\mu/2 + \sigma_i^a\alpha/2\beta)^2 + p^2 + m^2} \]  
  (21)

- Coulomb interaction with momentum transfer $q$ and energy transfer $2\pi r/\beta$ (cf. figure (3)):
  \[ V_{ij,kl}(q, r) = \frac{g^2\beta}{16\pi} \sum_a \frac{\sigma_{ji}^a\sigma_{kl}^a}{q^2} \]  
  (22)

All internal momenta, frequencies, color and Dirac indices must be integrated or summed over, respectively; fermion loops are associated with an additional factor $(-1)$, and diagrams with $N$ Coulomb interactions have an overall factor $1/N!$. Every connected part of a diagram in addition receives a factor $L/2\pi$; note that these rules correspond to already having used momentum conservation at all vertices. By considering only all connected diagrams, one directly obtains the logarithm of the partition function, an extensive quantity in the infinite volume limit.

The plasmon contribution $\ln Z_R$ is obtained by summing the ring diagrams (cf. figure (3)). Physically, this is interpreted as a screening of the Coulomb interaction by the production of fermion-antifermion pairs; thus the infrared divergences resulting from the long-range character of the interaction are dampened. Since there are $2^{N-1}(N-1)!$ ways of combining $N$ vertices to a ring, the Feynman rules give

\[ \ln Z_R(\alpha) = \frac{L}{4\pi} \sum_r \int dq \Tr_{\text{color}} \sum_{N=1}^{\infty} \frac{1}{N} \left( \frac{\Pi(r, q = 0)}{q^2} \right)^N \]  
  (23)

where $r$ is the integer labeling the frequency flow around the ring, $q$ is the momentum flow around the ring, and the polarization $\Pi$ is a matrix in color space,

\[ \Pi^{ai}(r, 0) = -\frac{g^2}{4\pi\beta} \sum_{i,j} \int dp \sum_s \sigma_{ji}^a \sigma_{kl}^a \frac{(\omega_s - i\mu/2 + \sigma_i^3\alpha/2\beta)(\omega_{s+r} - i\mu/2 + \sigma_j^3\alpha/2\beta)}{[(\omega_s - i\mu/2 + \sigma_i^3\alpha/2\beta)^2 + p^2 + m^2][(\omega_{s+r} - i\mu/2 + \sigma_j^3\alpha/2\beta)^2 + p^2 + m^2]} \]  
  (24)

Note that in (24), only the value of the polarization at $q = 0$ is taken. Higher orders in the Taylor expansion of $\Pi$ around $q = 0$ dampen the infrared divergence and thus ultimately lead only to terms of higher order in the perturbative expansion in $g$ [11]. The sum over $s$ in (24) is readily carried out by standard methods [17],

\[ \Pi^{ai}(r, 0) = g^2 \sum_{i,j} \sigma_{ji}^a \sigma_{ij}^a \frac{\sin \frac{\alpha}{2}(\sigma_j^3 - \sigma_i^3)}{\pi r - \frac{1}{4}(\sigma_j^3 - \sigma_i^3)} G_{ji} \]  
  (25)

\[ G_{ji}(\alpha) = \frac{\beta}{32\pi} \int dp \left[ \frac{1}{\cosh \frac{2\pi}{\beta}(\epsilon_p - \frac{\mu}{2} - i\frac{\sigma_j^3}{2\beta}) \cosh \frac{2\pi}{\beta}(\epsilon_p + \frac{\mu}{2} - i\frac{\sigma_i^3}{2\beta})} + \frac{1}{\cosh \frac{2\pi}{\beta}(\epsilon_p - \frac{\mu}{2} + i\frac{\sigma_j^3}{2\beta}) \cosh \frac{2\pi}{\beta}(\epsilon_p + \frac{\mu}{2} + i\frac{\sigma_i^3}{2\beta})} \right] \]  
  (26)

Note $G_{22} = G_{11}^{\ast}$ and $G_{21} = G_{12} \geq 0$. Furthermore, it is advantageous to diagonalize $\Pi$ in color space, since (24) calls for the trace over arbitrary powers of $\Pi$. One arrives at the eigenvalues

\[ \Pi_1(r) = 2g^2 \Re G_{11}(\alpha) \frac{\sin \frac{\alpha}{2\pi r}}{\pi r - \frac{\mu}{2}} \bigg|_{\epsilon \to 0} \]  
  (27)

\[ \Pi_2(r) = 2g^2 G_{12}(\alpha) \frac{\sin \frac{\alpha}{2\pi r}}{\pi r - \frac{\mu}{2}} \]  
  (28)

\[ \Pi_3(r) = 2g^2 G_{12}(\alpha) \frac{\sin \frac{\alpha}{2\pi r}}{\pi r - \frac{\mu}{2}} \]  
  (29)
Why $\Pi_1$ is left in this form will become clear below. Note that in the case of electrodynamics, the only term which appears is of the form of $\Pi_1$; the SU(2) case thus exhibits additional structures compared with the U(1) case.

Cast as above, it becomes very transparent where the situation at finite imaginary chemical potential differs from the usual one. According to (23), one must now calculate

$$
\ln Z_R(\alpha) = \frac{L}{4\pi} \int dq \sum_i \sum_r \sum_{N=1}^{\infty} \frac{1}{N} \left( \Pi_i \right)_N \tag{30}
$$

Usually, i.e. for $\alpha = 0$, one would observe that (27)-(29) only give contributions for $r = 0$ and are negative there. Thus, the sum over $r$ contains only one term, one may carry out the sum over $N$ to obtain a logarithm, and finally the (now infrared-regular) $q$-integral may be done.

In the more general case considered here, it is not meaningful to sum to a logarithm, since $\Pi_{11}$ contains only one term, one may carry out the sum over $N$ in (24) to obtain a logarithm, and finally the (now infrared-regular) $q$-integral may be done.

There does not seem to be a physical interpretation for this failure. One is simply not doing a good job of defining (30) by proceeding in a different way, namely by exploiting the structure of (27)-(29) in $r$; this is the reason why $\Pi_1$ was not explicitly simplified to $\Pi_1 = -2\delta_{01} \Re G_{11}$. The key observation is that

$$
\sum_r \left( \frac{1}{\pi r - \frac{\alpha}{2}} \right)_N = \frac{1}{2} \sum_{k \neq 0} \frac{1}{|k|} \ii \frac{1}{(N - 1)!} (2i)^N
$$

(this identity is derived in the appendix). In this way, one generates a crucial additional factor $1/(N - 1)!$ which turns the sum over $N$ in (30) into an exponential. Therefore one finally obtains

$$
\ln Z_R(\alpha) = \frac{L}{8\pi} \sum_{k \neq 0} \frac{1}{|k|} \int dq \left[ e^{i\frac{\epsilon}{|k|} (e^{-4\ii i \frac{g}{2 \pi} \Re G_{11} (\alpha)/q^2 - 1})} \right]_{\epsilon \to 0} + 2e^{i\frac{\epsilon}{|k|} (e^{-4\ii i \frac{g}{2 \pi} \Re G_{11} (\alpha)/q^2 - 1})} \tag{32}
$$

This expression is completely well-defined. It represents the main improvement of the present treatment over more conventional ones. Note that the procedure above is reminiscent of the Borel summation method [18], where one introduces a factor $1 = (k!)^{-1} \int_0^{\infty} dt e^{-t/k}$ to generate better convergence of a series. The price one pays is the introduction of an additional integration. Here, on the other hand, the convergence-improving factor is generated simply by rearranging the already present $r$-summation. A certain price is implied by the poor numerical convergence of the $k$-series as it stands. How this is best handled will be commented upon further below.

In general, if the momenta are discrete, one must proceed from equation (32) numerically. As long as the momenta are continuous, one may do the integral over $q$ (due to the reflection symmetry of (32) in $\alpha$ and $\epsilon$ around zero, these parameters will be taken positive in the following):

$$
\ln Z_R(\alpha) = \frac{gL}{8\pi} \sum_{k \neq 0} \frac{1}{|k|} \left[ e^{i\frac{\epsilon}{|k|} \sqrt{|k| \sin \alpha \Re G_{11} (\alpha)/2} (\Re G_{11} (\alpha)/q^2 - 1)} \right]_{\epsilon \to 0} + 2e^{i\frac{\epsilon}{|k|} \sqrt{|k| \sin \alpha \Re G_{11} (\alpha)/2} (\Re G_{11} (\alpha)/q^2 - 1)} \tag{33}
$$

Furthermore, in the case $\alpha \to 0$, one may use

$$
\lim_{\alpha \to 0} \sum_{k \neq 0} \frac{1}{|k|} e^{i\frac{\epsilon}{|k|} \sin \frac{\alpha}{2} (\Re G_{11} (\alpha))} = -2 \int_0^{\infty} dx \frac{\sin(x + \pi/4)}{\sqrt{x}} = -2\sqrt{\pi} \tag{34}
$$
and in this way verify that in a grand canonical treatment, or in a canonical treatment in the thermodynamic limit, one reproduces precisely the result of the usual prescription outlined directly after equation (35). As a side remark, note that taking additionally the classical nonrelativistic limit allows to evaluate $G$ explicitly to give

$$\ln Z_R = -\frac{3}{4}gL \left( \frac{2m\beta}{\pi} \right)^{1/4} e^{-\frac{3}{4}(m-\mu/2)} \quad (35)$$

Combining this with the corresponding ideal gas result, determining $\rho(\mu, T) = \partial P/\partial\mu|_T$, and solving for $\mu$ to obtain the equation of state yields

$$P = \rho T \left( 2 - \frac{3\sqrt{2}}{8} \frac{g}{\sqrt{\rho T}} + \ldots \right) \quad (36)$$

This result was obtained in [15] by a completely different method and thus provides a nice check on the present calculation. Similarly, one obtains in the case $m = 0$ (still in the thermodynamic limit)

$$\ln Z_R = -\frac{3}{4}\sqrt{\frac{2}{\pi}}gL \quad (37)$$

Some further results for finite $\alpha$ are collected in the appendix.

With the help of the general formula (32), one may now again perform a comparison of the different ensembles. Note that in practice, it is advantageous to perform the $\alpha$-averaging over (32) before doing the $k$-sum. Then the $k$-sum converges quite well above $k \sim \sqrt{L}$ for large $L$, as can be inferred from stationary phase considerations. Of course, adopting this procedure forces one to treat rapidly oscillating $\alpha$-integrands for large $k$, but this is numerically the lesser evil. Also, as before, allowance was made for a discretization of the momenta. Note that discretizing the momentum transfer $q$ in (32), which can be interpreted as the plasmon momentum, always means taking

$$q \to q_n = \frac{2\pi n}{L}, \quad \int dq \to \frac{2\pi}{L} \sum_{q_n \neq 0} \quad (38)$$

regardless of whether one is considering antiperiodic or periodic boundary conditions for the fermion fields. Note also that the term $q = 0$ is excluded from the sum. This is due to the fact that this term originates directly from the $q = 0$ term in the original Coulomb interaction (cf. (3)), proportional to $j^a(0)j^a(0)/q^2$. However, $j^a(0)$ is being constrained to be zero and therefore this term may be excluded from the Coulomb interaction. Physically, this simply takes account of the fact that for a neutral system, there is no physical content in a fluctuation translating all charges in the system uniformly.

The comparison between the different cases is illustrated in figures (1) and (2). The pictures are strikingly similar, even though taken from vastly different regions of the phase diagram. Again, at very high $L$, the effect of varying the ensemble dominates. However, in contradistinction to the ideal gas term, the effect of discretizing the momenta takes over already at quite high $L$. This does not come as a complete surprise. The simple argument used in the discussion of the ideal gas term, that the Euler-McLaurin series implies an expansion purely in powers of $1/L$ for the logarithm of the partition function if one discretizes the momenta, is not valid for the integral over the plasmon momentum $q$ due to the singular nature of the integrand in (32) around $q = 0$. Thus, there is no simple argument anymore which would allow one to predict which effect dominates for large $L$. That the strong effect of momentum discretization here is indeed due to the plasmon momentum sum, as opposed to the fermion momenta contained in $\ln Z_0$ and the quantity $G$, is corroborated by the observation that figures (3) and (4) remain identical regardless of whether one uses periodic or antiperiodic boundary conditions for the fermions. Only at very low $L$, not displayed in the figures anymore, could one observe a dependence on the boundary conditions. This happens at such low $L$ that the entire concept of treating the system as a plasma becomes invalid. Note that this is also true for the ideal gas contribution depicted in figures (1) and (2). Below $L/\beta \approx 10$, it is impossible to simultaneously maintain $gL \gg 1$ and $g\beta \ll 1$.

Thus, the only relevant finite-size effects in the plasmon contribution stem from the use of a canonical ensemble and from discretizing the plasmon momentum. The two effects are roughly comparable in magnitude, with the ensemble effect dominating at very high $L$.

V. DISCUSSION

The present investigation has focussed on the different finite-size effects influencing the thermodynamics of a plasma. Special emphasis was placed on the evaluation of the plasmon contribution in a canonical ensemble as opposed to the
usual grand canonical treatment. Here, it turned out that the usual resummation prescription for the ring diagrams is too naive, and it was shown how the resulting ambiguity may be remedied via equation \( (31) \). This latter trick, which in view of its derivation \((A1)-(A5)\) may be suggestively termed “Fourier regularization”, may be useful in quite diverse contexts as a means of properly defining ill-defined series. In the present context, it should be noted that the nontrivial manipulations purely involve the imaginary time direction or the corresponding frequency sums. Thus, while the present treatment focussed on a one-dimensional model, all manipulations are equally applicable for the analogous problem in an arbitrary number of space dimensions. Also, the versatility of the method is evidenced by its application to the more complicated SU(2) Coulomb interaction as opposed to the U(1) one.

Using these formal developments, the partition function of QCD\(_{1+1}\) for SU(2) color was calculated to order \( \mathcal{O}(g) \), exhibiting the effect of varying the ensemble used and of discretizing the momenta. The dominant finite-size effects were identified to be the ones resulting from using a canonical ensemble and from discretizing the plasmon momentum. The effect of discretizing the fermion momenta in comparison was found to be negligible for interesting values of \( L \).

Of the two dominant effects mentioned above, the ensemble effect is strongest at very high \( L \).

Finally, it should be mentioned that there is an alternative interpretation of the imaginary chemical potential \( \alpha \) over which one averages in the canonical ensemble: The configuration space of the model considered here is a cylinder, for finite \( L \) possibly closed to a torus depending on the spatial boundary conditions used. Moving on the cylinder parallel to its axis corresponds to moving in the space direction, moving around the cylinder corresponds to moving in the imaginary time direction. The parameter \( \alpha \) can be interpreted as an (imaginary) background chromoelectric potential as would be produced by a hypothetical solenoid coinciding with the axis of the cylinder. This background potential introduces a phase into Green’s functions which lead around the cylinder, or in more physical terms, the \( \alpha \)-dependence corresponds to the Aharonov-Bohm effect induced by the solenoid on the thermodynamics. A similar situation was investigated recently in connection with the statistical properties of the Gross-Neveu model on a ring \([19]\). One of the interesting results of this latter treatment was the observation of non-analytic behavior of the thermodynamical observables as a function of the Aharonov-Bohm phase. Disregarding for the moment that in the present calculation, the parameter \( \alpha \) is in the end an integration variable, similar behavior is found here in quantities before \( \alpha \)-averaging, e.g. in the first term in the square brackets of the expression \((A7)\) for \( \ln Z_R(\alpha) \). It may be worthwhile to investigate whether there is physical meaning contained in this singularity.

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**APPENDIX: SOME USEFUL DERIVATIONS AND FORMULAS**

Identity \((31)\) can be seen in the following way:

\[
\sum_r \frac{1}{(\pi r - \frac{\alpha}{2})^N} = \frac{2^{N-1}}{(N-1)!} \left( \frac{d}{d\alpha} \right)^{N-1} \sum_r \frac{1}{\pi r - \frac{\alpha}{2}} \tag{A1}
\]

\[
= -\frac{2^{N-1}}{(N-1)!} \left( \frac{d}{d\alpha} \right)^{N-1} \cot(\alpha/2) \tag{A2}
\]

\[
= -\frac{2^N}{(N-1)!} \left( \frac{d}{d\alpha} \right)^{N-1} \sum_{k=1}^{\infty} \sin k\alpha \tag{A3}
\]

\[
= -\frac{2^N}{(N-1)!} \left( \frac{d}{d\alpha} \right)^{N-1} \frac{1}{2i} \sum_{k \neq 0} \text{sgn}(k) e^{ik\alpha} \tag{A4}
\]

\[
= \frac{1}{2} \sum_{k \neq 0} \frac{1}{|k|} e^{ik\alpha} \frac{1}{(N-1)!} (2i)^N \tag{A5}
\]

Note that the manipulations of Fourier series carried out above are well-defined in the context of the theory of distributions \([20]\).
As long as one considers continuous momenta, many quantities connected with the plasmon contribution to the partition function can be evaluated explicitly even for finite parameter $\alpha$, in analogy with the ideal gas expressions given in equations (12) and (13). Thus, in the classical nonrelativistic limit one has

$$G_{12} = \frac{\beta}{8\pi} \sqrt{\frac{2m\pi}{\beta}} e^{\beta(m-\mu/2)}, \quad \text{Re} G_{11} = \cos(\alpha/2) G_{12}$$  \hspace{1cm} (A6)

and therefore, using (33),

$$\ln Z_R(\alpha) = -\frac{gL}{4} \left( \frac{2m\beta}{\pi} \right)^{1/4} e^{-\frac{2}{3}(m-\mu/2)} \left[ \sqrt{\cos(\alpha/2)} \theta(\cos(\alpha/2)) + \sqrt{\frac{8}{\pi}} \sqrt{\sin(\alpha/2)} \sum_{k=1}^{\infty} \frac{\sin(k\alpha + \pi/4)}{\sqrt{k}} \right]$$  \hspace{1cm} (A7)

where $\theta(x)$ denotes the step function. Note that the $k$-sum can be expressed as a generalized zeta function [21]. Similarly, in the limit $m = \mu = 0$ one obtains

$$G_{12} = \frac{1}{4\pi \sin(\alpha/2)}, \quad \text{Re} G_{11} = \frac{1}{4\pi}$$  \hspace{1cm} (A8)

and from this

$$\ln Z_R(\alpha) = -\frac{gL}{\sqrt{8\pi}} \left[ 1 + \sqrt{\frac{4\alpha}{\pi}} \sum_{k=1}^{\infty} \frac{\sin(k\alpha + \pi/4)}{\sqrt{k}} \right]$$  \hspace{1cm} (A9)

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FIG. 1. Ideal gas part of the logarithm of the partition function in the ultrarelativistic limit $m = \mu = 0$. Short-dash-dotted: Grand canonical ensemble with discrete momenta (periodic boundary conditions); long-dash-dotted: Same, with antiperiodic boundary conditions. Solid line: Canonical ensemble with continuous momenta; short dashes and long dashes: Same with discretized momenta, periodic and antiperiodic boundary conditions, respectively. A grand canonical calculation with continuous momenta gives a constant, namely the one all curves converge to at large $L$. 9
FIG. 2. Ideal gas part of the logarithm of the partition function in the classical nonrelativistic limit: $m/T = 10$, $\mu/T = 14$. Short-dash-dotted: Grand canonical ensemble with discrete momenta (periodic boundary conditions); long-dash-dotted: Same, with antiperiodic boundary conditions. Solid line: Canonical ensemble; in this case the curves obtained with continuous momenta and with momenta discretized in the various ways are indistinguishable. A grand canonical calculation with continuous momenta gives a constant, namely the one all curves converge to at large L.

FIG. 3. Coulomb interaction

FIG. 4. Sum of ring diagrams

FIG. 5. Plasmon contribution to the logarithm of the partition function in the ultrarelativistic limit $m = \mu = 0$, with $g/T = 0.1$. Dotted: Grand canonical ensemble with continuous momenta; solid line: Same, with discretized momenta. In the latter case, periodic and antiperiodic boundary conditions for the fermions give the same, shown, curve. Long dashes: Canonical ensemble with continuous momenta; short dashes: Same with discretized momenta. Again, the boundary conditions chosen for the fermions do not make a difference.

FIG. 6. Plasmon contribution to the logarithm of the partition function in the classical nonrelativistic limit: $m/T = 10$, $\mu/T = 14$, with $g/T = 0.1$. Dotted: Grand canonical ensemble with continuous momenta; solid line: Same, with discretized momenta. In the latter case, periodic and antiperiodic boundary conditions for the fermions give the same, shown, curve. Long dashes: Canonical ensemble with continuous momenta; short dashes: Same with discretized momenta. Again, the boundary conditions chosen for the fermions do not make a difference.
M. Engelhardt, Phys. Rev. D, Fig. 5
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M. Engelhardt, Phys. Rev. D, Fig. 4
M. Engelhardt, Phys. Rev. D, Fig. 6
\[ V_{ij,kl}(q,r) = \]

\[ \begin{array}{c}
1, \ s'+r, \ p'+q \\
q, r \\
k, \ s', \ p' \\
p+q, \ s+r, \ i
\end{array} \]

M. Engelhardt, Phys. Rev. D, Fig. 3
M. Engelhardt, Phys. Rev. D, Fig. 1
\frac{1}{L} \ln Z_0

M. Engelhardt, Phys. Rev. D, Fig. 2