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A Global multiplicity result for a very singular critical nonlocal equation

J. Giacomoni∗, T. Mukherjee† and K. Sreenadh‡

Abstract
In this article, we show the global multiplicity result for the following nonlocal singular problem

\[(P_\lambda) : (-\Delta)^s u = u - q + \lambda u^{2^*_s - 1}, \quad u > 0 \text{ in } \Omega, \quad u = 0 \text{ in } \mathbb{R}^n \setminus \Omega,\]

where \(\Omega\) is a bounded domain in \(\mathbb{R}^n\) with smooth boundary \(\partial \Omega\), \(n > 2s\), \(s \in (0, 1)\), \(\lambda > 0\), \(q > 0\) satisfies \(q(2s - 1) < (2s + 1)\) and \(2^*_s = \frac{2n}{n - 2s}\). Employing the variational method, we show the existence of at least two distinct weak positive solutions for \((P_\lambda)\) in \(X_0\) when \(\lambda \in (0, \Lambda)\) and no solution when \(\lambda > \Lambda\), where \(\Lambda > 0\) is appropriately chosen. We also prove a result of independent interest that any weak solution to \((P_\lambda)\) is in \(C^\alpha(\mathbb{R}^n)\) with \(\alpha = \alpha(s, q) \in (0, 1)\). The asymptotic behaviour of weak solutions reveals that this result is sharp.

Key words: Fractional Laplacian, very singular nonlinearity, variational method, Hölder regularity.

2010 Mathematics Subject Classification: 35R11, 35R09, 35A15.

1 Introduction
In this article, we prove the existence, multiplicity and Hölder regularity of weak solutions to the following fractional critical and singular elliptic equation

\[(P_\lambda) : (-\Delta)^s u = u - q + \lambda u^{2^*_s - 1}, \quad u > 0 \text{ in } \Omega, \quad u = 0 \text{ in } \mathbb{R}^n \setminus \Omega,\]

where \(\Omega\) is a bounded domain in \(\mathbb{R}^n\) with smooth boundary \(\partial \Omega\), \(n > 2s\), \(s \in (0, 1)\), \(\lambda > 0\), \(q > 0\) satisfies \(q(2s - 1) < (2s + 1)\) and \(2^*_s = \frac{2n}{n - 2s}\). The fractional Laplace operator denoted by \((-\Delta)^s\) is defined as

\[(-\Delta)^s u(x) = 2C^n_{s}\text{P.V.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} \, dy\]

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A global multiplicity result for singular and critical nonlocal elliptic equation

where P.V. denotes the Cauchy principal value and \( C^m_s = \pi^{-\frac{n}{2}} 2^{2s-1} s^{\frac{n+2s}{4(1-s)}} \), \( \Gamma \) being the Gamma function. The fractional power of Laplacian is the infinitesimal generator of Lévy stable diffusion process and arise in anomalous diffusion in plasma, population dynamics, geophysical fluid dynamics, flames propagation, chemical reactions in liquids and American options in finance, see [3] for instance. The theory of fractional Laplacian and elliptic equations involving it as the principal part has been evolved immensely in recent years. There is a vast literature available on it, however without giving an exhaustive list we cite [7, 10, 14, 16, 19, 21, 22] for motivation to readers.

Nowadays, researchers are inspecting on various forms of singular nonlocal equations. We cite [11, 8, 9] as some contemporary woks related to it. The fractional elliptic equations with singular and critical nonlinearities was first studied by Barrios et al. in [5]. The authors considered the problem

\[
(-\Delta)^su = \lambda \frac{f(x)}{u^\gamma} + Mu^p, \quad u > 0 \text{ in } \Omega, \quad u = 0 \text{ in } \mathbb{R}^n \setminus \Omega,
\]

where \( n > 2s, \ M \geq 0, \ 0 < s < 1, \ \gamma > 0, \ \lambda > 0, \ 1 < p < 2^*_s - 1 \) and \( f \in L^m(\Omega), \ m \geq 1 \) is a nonnegative function. Here, authors studied the existence of distributional solutions using the uniform estimates of \( \{u_n\} \) which are solutions of the regularized problems with singular term \( u^{-\gamma} \) replaced by \((u + \frac{1}{n})^{-\gamma}\). Motivated by their results, Sreenadh and Mukherjee in [15] studied the singular problem

\[
(-\Delta)^su = \lambda a(x)u^{-q} + u^{2^*_s-1}, \quad u > 0 \text{ in } \Omega, \quad u = 0 \text{ in } \mathbb{R}^n \setminus \Omega,
\]

where \( \lambda > 0, \ 0 < q \leq 1 \) and \( \theta \leq a(x) \in L^\infty(\Omega) \), for some \( \theta > 0 \). They showed that although the energy functional corresponding to this problem fails to be Fréchet differentiable, making use of its Gâteaux differentiability the Nehari manifold technique can still be benefitted to obtain existence of at least two solutions over a certain range of \( \lambda \). The significance of \( q \) being less than 1 is the Gâteaux differentiability of the functional corresponding to the problem.

Consider now the case \( q > 1 \). Let

\[
X \overset{\text{def}}{=} \left\{ u \mid u : \mathbb{R}^n \to \mathbb{R} \text{ is measurable}, \ u|_\Omega \in L^2(\Omega) \text{ and } \frac{(u(x) - u(y))}{|x-y|^{\frac{n}{2}+s}} \in L^2(Q) \right\},
\]

where \( Q \overset{\text{def}}{=} \mathbb{R}^n \setminus (C\Omega \times C\Omega) \) and \( C\Omega := \mathbb{R}^n \setminus \Omega \) endowed with the norm

\[
||u||_X \overset{\text{def}}{=} ||u||_{L^2(\Omega)} + [u]_X,
\]

where

\[
[u]_X = \left( \int_Q \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} \, dxdy \right)^{\frac{1}{2}}.
\]

Let \( J : X_0 \to \mathbb{R} \) be the functional defined by

\[
J(u) = \frac{C^m_s}{2} ||u||_{X_0}^2 - \frac{1}{1-q} \int_\Omega |u|^{1-q} \, dx - \frac{\lambda}{2^*_s} \int_\Omega |u|^{2^*_s} \, dx
\]
A global multiplicity result for singular and critical nonlocal elliptic equation

for any $u$ in the Hilbert space $X_0 \overset{\text{def}}{=} \{ u \in X : u = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega \}$ equipped with the inner product

$$\langle u, v \rangle \overset{\text{def}}{=} \int_Q \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}} \, dx dy.$$ 

$J$ may not be defined on the whole space nor it is even continuous on $D(I) \equiv \{ u \in X_0 : I(u) < \infty \}$ and the approach used for $q < 1$ cannot be extended. Besides this, one has that the interior of $D(I) = \emptyset$ because of the singular term. But we notice that if we enforce the condition $q > 1$ satisfies $q(2s - 1) < (2s + 1)$ then we can prove that $D(I)$ is non-empty and Gateaux differentiable on a suitable convex cone of $X_0$.

The existence of weak solutions to $(P_\lambda)$ when $\lambda \in (0, \Lambda)$ and no solution when $\lambda > \Lambda$ has been already obtained by Giacomoni et al. in [12]. But here the multiplicity of solutions has been achieved in $L^1_{loc}(\Omega)$ only, by using non smooth critical point theory, so the questions of existence of solutions in the energy space and of Hölder regularity were still pending. This article is bringing answers to these two issues. For that, we followed the approach of [13] but we notify that the adversity and novelty of this article lies in extending Haitao’s technique in a nonlocal framework. The regularity of weak solution of the purely singular problem

$$( - \Delta )^s u = u^{-q}, \ u > 0, \text{ in } \Omega, \ u = 0 \text{ in } \mathbb{R}^n \setminus \Omega$$

plays a vital role in our study. This has been obtained by Adimurthi, Giacomoni and Santra in [1] in recent times. In the present paper we extend the Hölder regularity results proved in [1, Theorem 1.4] in our framework of weak solutions (see definition 1.1 below) rather than the more restricted classical solutions framework (defined in [1]). It requires additional $L^\infty$-estimates and the use of the weak comparison principle.

Our paper has been organized as follows. Section 2 contains some preliminary results used in the subsequent sections. Section 3 and 4 contain the proof of existence of first and second weak solution to $(P_\lambda)$ respectively (Theorem 1.2). The proof of the Hölder regularity result (Theorem 1.3) is done in Section 4 based on a priori estimates proved in Proposition 4.1. Now we state the main results proved in the paper. First we define the notion of weak solutions.

**Definition 1.1** A function $u \in X_0$ is said to be a weak solution of $(P_\lambda)$ if there exists a $m_K > 0$ such that $u > m_K$ in every compact subset $K$ of $\Omega$, and it satisfies

$$C^+_s \int_Q \frac{(u(x) - u(y))(\phi(x) - \phi(y))}{|x - y|^{n+2s}} \, dx dy = \int_\Omega (u^{-q} + u^{2s-1})\phi \, dx, \text{ for all } \phi \in X_0.$$ 

Given any $\phi \in C^0(\Omega)$ such that $\phi > 0$ in $\Omega$ we define

$$C_\phi(\Omega) \overset{\text{def}}{=} \{ u \in C^0(\Omega) | \exists c \geq 0 \text{ such that } |u(x)| \leq c\phi(x), \forall x \in \Omega \}$$

with the usual norm $\| u \|_{L^\infty(\Omega)}$ and the associated positive cone. We define the following open convex subset of $C_\phi(\Omega)$ as

$$C^+_\phi(\Omega) \overset{\text{def}}{=} \left\{ u \in C_\phi(\Omega) | \inf_{x \in \Omega} \frac{u(x)}{\phi(x)} > 0 \right\}.$$
In particular, $C^+_\phi$ contains all those functions $u \in C_0(\Omega)$ with $k_1 \phi \leq u \leq k_2 \phi$ in $\Omega$ for some $k_1, k_2 > 0$. Let $\phi_{1,s}$ be the first positive normalized eigenfunction ($\|\phi_{1,s}\|_{L^\infty(\Omega)} = 1$) of $(-\Delta)^s$ in $X_0$. We recall that $\phi_{1,s} \in C^s(\mathbb{R}^N)$ and $\phi_{1,s} \in C^+_\phi(\Omega)$ where $\delta(x) = \text{dist}(x, \partial \Omega)$ (see for instance Proposition 1.1 and Theorem 1.2 in [17]). We then define the barrier function $\phi_q$ as follows:

$$
\phi_q \overset{\text{def}}{=} \begin{cases} 
\phi_{1,s} & \text{if } 0 < q < 1, \\
\phi_{1,s} \left( \ln \left( \frac{2}{\phi_{1,s}} \right) \right) \frac{1}{q+1} & \text{if } q = 1, \\
\frac{2}{\phi_{1,s}^2} & \text{if } q > 1.
\end{cases}
$$

We prove the following as the main results:

**Theorem 1.2** There exists $\Lambda > 0$ such that

(i) $(P_\lambda)$ admits at least two solutions in $X_0 \cap C^+_\phi(\Omega)$ for every $\lambda \in (0, \Lambda)$;

(ii) $(P_\lambda)$ admits no solution for $\lambda > \Lambda$;

(iii) $(P_\lambda)$ admits at least one positive solution $u_\Lambda \in X_0 \cap C^+_\phi(\Omega)$.

**Theorem 1.3** Let $\lambda \in (0, \Lambda]$, $q > 0$ satisfies $q(2s - 1) < (2s + 1)$ and $u \in X_0$ is any positive weak solution of $(P_\lambda)$ then

(i) $u \in C^s(\mathbb{R}^n)$ when $0 < q < 1$;

(ii) $u \in C^{s-\epsilon}(\mathbb{R}^n)$ for any small enough $\epsilon > 0$ when $q = 1$;

(iii) $u \in C^{\frac{2s}{q+1}}(\mathbb{R}^n)$ when $q > 1$.

**Remark 1.4** Here, the Hölder regularity for the weak solutions of $(P_\lambda)$ obtained is optimal because of the behavior of the solution near $\partial \Omega$ since we showed that any weak solution of $(P_\lambda)$ lies in $C^+_\phi(\Omega)$.

**Remark 1.5** It follows from Theorem 1.3 that the extremal solution (when $\lambda = \Lambda$), in case of critical growth nonlinearities is a classical solution which extends the results in [1] where in this regard only subcritical nonlinearities are considered.

## 2 Preliminaries

We start by some preliminary results. The energy functional corresponding to $(P_\lambda)$ is given by $I_\lambda : X_0 \to \mathbb{R}$ defined as

$$
I_\lambda(u) \overset{\text{def}}{=} \begin{cases} 
\frac{C_0^s \|u\|_{X_0}^2}{2} - \frac{1}{1-q} \int_\Omega |u|^{1-q} \, dx - \frac{1}{2s} \int_\Omega |u|^{2s} \, dx & \text{if } q \neq 1, \\
\frac{C_0^s \|u\|_{X_0}^2}{2} - \int_\Omega \ln |u| \, dx - \frac{1}{2s} \int_\Omega |u|^{2s} \, dx & \text{if } q = 1.
\end{cases}
$$
Let $q > 0$ satisfies $q(2s - 1) < (2s + 1)$. Then for any $\varphi \in X_0$ and $u \in C^+_{\phi_q}(\Omega)$, by Hardy’s inequality (see [23, Lemma 3.2.6.1, p. 259]) we obtain

$$
\int_{\Omega} u^{-q} \varphi \leq \left( \int_{\Omega} \frac{dx}{\delta(x)^{2(\frac{q}{q+1})}} \right)^{\frac{1}{q}} \left( \frac{\varphi^2}{(\delta(x))^{2s}} \right)^{\frac{1}{2}} < K\|\varphi\| < +\infty \quad (2.1)
$$

where $K > 0$ is a constant. If we define $D(I) = \{ u \in X_0 : \ I_\lambda(u) < \infty \}$ then by virtue of (2.1) we get that $D(I) \neq \emptyset$. This gives an importance of the inequality $q(2s - 1) < (2s + 1)$. From the proof of [1, Theorem 1.2], we know that if $0 < q < 1$ and $u \in X_0$ satisfies $u \geq c \delta^s$ then $I_\lambda$ is Gâteaux differentiable at $u$. In the proposition below, we show the same property of $I_\lambda$ when $q \geq 1$ satisfies $q(2s - 1) < (2s + 1)$.

**Proposition 2.1** If $M = \{ u \in X_0 : \ u_1 \leq u \leq u_2 \}$ where $u_1 \in C^+_{\phi_q}(\Omega)$ and $u_2 \in X_0$ then $I_\lambda$ is Gâteaux differentiable at $u$ in the direction $(v - u)$ where $v, u \in M$.

**Proof.** We need to show that

$$
\lim_{t \to 0} \frac{I_\lambda(u + t(v - u)) - I_\lambda(u)}{t} = C^s \int_{Q} \frac{(v(x) - v(y))(u(x) - (v - u)(y))}{|x - y|^{n+2s}} \, dx dy
$$

$$
- \int_{\Omega} u^{-q}(v - u) \, dx - \lambda \int_{\Omega} u^{2s-1} - (v - u) \, dx.
$$

It is enough to show this for the singular term; for the rest two terms, the proof is standard. For any $t \in (0, 1)$, $u + t(v - u) \in M$ since $M$ is convex. Consider $F(u) = \frac{1}{1-q} \int_{\Omega} u^{1-q} \, dx$ then using Mean Value Theorem we get

$$
\frac{F(u + t(v - u)) - F(u)}{t} = \frac{1}{t(1-q)} \int_{\Omega} ((u + t(v - u))^{1-q} - u^{1-q}) \, dx
$$

$$
= \int_{\Omega} (u + t\theta(v - u))^{-q}(v - u)(v - u) \, dx
$$

for some $\theta \in (0, 1)$. Since $(u + t\theta(v - u)) \in M$ and (2.1), we have

$$
\int_{\Omega} (u + t\theta(v - u))^{-q}(v - u) \, dx \leq \int_{\Omega} u^{-q}(v - u) \, dx < +\infty.
$$

So using Lebesgue Dominated convergence theorem we pass through the limit $t \to 0$ and get

$$
\lim_{t \to 0} \frac{F(u + t(v - u)) - F(u)}{t} = \int_{\Omega} u^{-q}(v - u) \, dx.
$$

This completes the proof. \qed

Let $L(u) := (-\Delta)^s u - u^{-q}$. Then $L$ forms a monotone operator. So we have the following comparison principle:
Lemma 2.2 Let $u_1, u_2 \in X_0 \cap C^+_{\phi_q}(\Omega)$ are weak solutions to

$$L(u_1) = g_1 \text{ in } \Omega, \quad L(u_2) = g_2 \text{ in } \Omega$$

with $g_1, g_2 \in L^2(\Omega)$ such that $g_1 \leq g_2$ a.e. in $\Omega$. Then $u_1 \leq u_2$ a.e. in $\Omega$. Moreover if $g \in L^\infty(\Omega)$ then the problem

$$L(u) = g \text{ in } \Omega, \quad u = 0 \text{ in } \mathbb{R}^n \setminus \Omega$$

has a unique solution in $X_0$.

3 Existence result

Let us define

$$\Lambda := \sup \{ \lambda > 0 : (P_\lambda) \text{ has a weak solution} \}.$$

Also let $w \in C_0(\Omega)$ solves the purely singular problem

$$(-\Delta)^s w = w^{-q}, \quad w > 0 \text{ in } \Omega, \quad w = 0 \text{ in } \mathbb{R}^n \setminus \Omega.$$

Then [1, Theorems 1.2 and 1.4] give us that $w$ is unique, $w \in X_0 \cap C^+_{\phi_q}(\Omega)$ and $w \in C^s(\mathbb{R}^n)$

where $s_q \overset{\text{def}}{=} \begin{cases} s & \text{if } q < 1, \\ s - \epsilon & \text{for any } \epsilon > 0 \text{ if } q = 1, \\ \frac{2s}{q+1} & \text{if } 1 < q \text{ and } q(2s-1) < 2s + 1. \end{cases}$

For sake of clarity we basically focus on the case $1 \leq q$ and $q(2s-1) < (2s + 1)$. Indeed, when $q \in (0, 1)$, the case follows easily along the same line. In this context, the next result is an important lemma for $\Lambda$.

Lemma 3.1 It holds $0 < \Lambda < +\infty$.

Proof. First we prove that $\Lambda < +\infty$. Using $\phi_{1,s}$ as the test function in $(P_\lambda)$ we get

$$\int_{\Omega} (u^{-q}\phi_{1,s} + \lambda u^{2s-1}_2 \phi_{1,s}) \, dx = \int_{\mathbb{R}^n} \phi_{1,s}(-\Delta)^s u \, dx = \int_{\mathbb{R}^n} u(-\Delta)^s \phi_{1,s} \, dx = \lambda_{1,s} \int_{\Omega} u \phi_{1,s} \, dx. \quad (3.1)$$

If we choose a $\lambda > 0$ which satisfies $t^{-q} + \lambda t^{2s-1} \geq 2\lambda_{1,s}t$ for all $t > 0$ then we get a contradiction to (3.1). Therefore it must be $\Lambda < +\infty$. Now to prove $\Lambda > 0$ we need sub and supersolution for $(P_\lambda)$. It is easy to see that $u_\lambda = w$ forms a subsolution of $(P_\lambda)$ and $\overline{u}_\lambda = u_\lambda + Mz$ for $\lambda > 0$ small enough and for a $M = M(\lambda) > 0$ forms a supersolution of $(P_\lambda)$, where $0 < z \in X_0$ solves $(-\Delta)^s z = 1$ in $\Omega$. Now we define the closed convex subset $M_\lambda$ of $X_0$ as

$$M_\lambda := \{ u \in X_0 : \overline{u}_\lambda \leq u \leq u_\lambda \}.$$

Consider the iterative scheme ($k \geq 1$):

$$(P_{\lambda,k}) \begin{cases} (-\Delta)^s u_k - \overline{u}_k^{-q} = \lambda u_{k-1}^{2s-1}, & u_k > 0 \text{ in } \Omega \\ u_k = 0 \text{ in } \mathbb{R}^n \setminus \Omega \end{cases}$$
with \( u_0 = u_\Lambda \). The existence of \( \{u_k\} \) in \( X_0 \cap M_{\Lambda} \cap C^+_\phi(q)(\Omega) \) can be proved by considering the approximated problem corresponding to \((P_{\lambda,k})\), for instance we refer [6, Proposition 2.3]. From Lemma 2.2, it follows that \( \{u_k\} \) is increasing and \( u_k \in M_{\lambda} \) for all \( k \). Let \( \lim_{k \to \infty} u_k = u_\lambda \). Then testing \((P_{\lambda,k})\) by \( u_k \) we get

\[
\|u_k\|^2 \leq 2 \int_{\Omega} u_k^2 \, dx + \lambda \int_{\Omega} |u_k|^{2q} \, dx + \int_{\Omega} \frac{u_k}{\lambda} u_\lambda^{-q} \leq K_\lambda
\]

where \( K_\lambda > 0 \) is a constant depending on \( \lambda \). So, up to a subsequence, \( u_k \to u_\lambda \) in \( X_0 \). Finally using Lebesgue dominated convergence Theorem we pass through the limit in \((P_{\lambda,k})\) to obtain \( u_\lambda \) solves \((P_{\lambda})\) weakly and obviously, \( u_\lambda \in M_{\lambda} \). This proves that \( \Lambda > 0 \). \( \square \)

In the next result, we prove the existence of a weak solution for \((P_{\lambda})\) whenever \( \lambda \in (0, \Lambda) \). In the proof, we use a minimization on a conical shell argument similar as in [2, Lemma 4.1] and in [6, Proposition 3.5]. But here we take advantage of the existence of a strict positive subsolution to control the singular nonlinearity.

**Proposition 3.2** For each \( \lambda \in (0, \Lambda) \), \((P_{\lambda})\) admits a weak solution \( w \in C^+_\phi(q)(\Omega) \).

**Proof.** The proof goes along the line of Perron’s method adapted over a nonlocal framework (see [13, Lemma 2.2]). Let \( \lambda \in (0, \Lambda) \) and \( \lambda' \in (\lambda, \Lambda) \) then it is easy to see that \( u_{\lambda'} \), a weak solution of \((P_{\lambda'})\), forms a supersolution for \((P_{\lambda})\). Such a \( \lambda' \) exists because of the definition of \( \Lambda \) and Lemma 3.1. Let \( u_{\Lambda} \) be the same function as defined in Lemma 3.1 and consider the closed convex subset \( W_{\lambda} \) of \( X_0 \) as

\[
W_{\lambda} = \{ u \in X_0 : u_{\lambda} \leq u \leq u_{\lambda'} \}
\]

Then for each \( u \in W_{\lambda} \), because of fractional Sobolev embedding \( I_{\lambda} \) satisfies

\[
I_{\lambda}(u) \geq \begin{cases} 
\frac{C_2^p \|u\|^2}{2} - \frac{C_2^p \|u\|^{2q}}{q} & \text{if } q > 1, \\
\frac{C_2^p \|u\|^2}{2} - \frac{C_2^p \|u\|^{2q}}{q} - C(\lambda') & \text{if } q \leq 1
\end{cases}
\]

where \( C(\lambda') \) is a positive constant depending solely on \( \lambda' \). Then \( I_{\lambda} \) is bounded from below and coercive over \( W_{\lambda} \). If \( \{u_k\} \subset W_{\lambda} \) be such that \( u_k \to u_0 \) in \( X_0 \) as \( k \to \infty \) then since for each \( k \), \( u_k \geq u_{\lambda} \) for \( q > 1 \) and \( u_k \leq u_{\lambda'} \) for \( q \in (0,1] \), \( \int_{\Omega} u_k^{1-q} \, dx \leq \int_{\Omega} u_0^{1-q} \, dx \), we can use Lebesgue Dominated convergence theorem to get that

\[
\int_{\Omega} u_k^{1-q} \, dx \to \int_{\Omega} u_0^{1-q} \, dx \text{ as } k \to \infty.
\]

Hence from weak lower semicontinuity of norms, it follows that \( I_{\lambda} \) is weakly lower semicontinuous over \( W_{\lambda} \). Moreover, \( W_{\lambda} \) is weakly sequentially closed subset of \( X_0 \). Therefore there exists a \( w \in W_{\lambda} \) such that

\[
\inf_{u \in W_{\lambda}} I_{\lambda}(u) = I_{\lambda}(w).
\]
Claim- \( w \) is a weak solution of \((P_{\lambda})\).

Let \( \varphi \in X_0 \) and \( \epsilon > 0 \) then we define

\[ v_{\epsilon} = \min\{u_{\lambda}, \max\{u_{\lambda}, w + \epsilon \varphi\}\} = w + \epsilon \varphi - \varphi^\epsilon + \varphi_{\epsilon} \]

where \( \varphi^\epsilon = \max\{0, w + \epsilon \varphi - u_{\lambda}\} \) and \( \varphi_{\epsilon} = \max\{0, u_{\lambda} - w - \epsilon \varphi\} \). By construction \( v_{\epsilon} \in W_{\lambda} \) and \( \varphi^\epsilon, \varphi_{\epsilon} \in X_0 \cap L^\infty(\Omega) \). Since \( w + t(v_{\epsilon} - w) \in W_{\lambda} \) for each \( t \in (0,1) \), using (3.2) and Proposition 2.1 we get that

\[
0 \leq \lim_{t \to 0^+} \frac{I_{\lambda}(w + t(v_{\epsilon} - w)) - I_{\lambda}(w)}{t} = \int_Q (v_{\epsilon} - w)(-\Delta)^s w \, dx - \int_Q w^{-q}(v_{\epsilon} - w) \, dx - \int_{\Omega} w^{2^*_s - 1}(v_{\epsilon} - w) \, dx.
\]

This on simplification gives

\[
\int_{\mathbb{R}^n} \varphi(-\Delta)^s w \, dx - \int_{\Omega} (w^{-q} + \lambda w^{2^*_s - 1}) \varphi \, dx \geq \frac{1}{\epsilon}(E_{\epsilon} - E_{\epsilon}) \tag{3.3}
\]

where

\[
E_{\epsilon} = \int_{\mathbb{R}^n} \varphi^\epsilon(-\Delta)^s w \, dx - \int_{\Omega} (w^{-q} + \lambda w^{2^*_s - 1}) \varphi^\epsilon \, dx
\]

\[
= \int_{\mathbb{R}^n} \varphi^\epsilon(-\Delta)^s (w - u_{\lambda}) \, dx + \int_{\mathbb{R}^n} \varphi^\epsilon(-\Delta)^s u_{\lambda} \, dx - \int_{\Omega} (w^{-q} + \lambda w^{2^*_s - 1}) \varphi^\epsilon \, dx
\]

\[
E_{\epsilon} = \int_{\mathbb{R}^n} \varphi_{\epsilon}(-\Delta)^s w \, dx - \int_{\Omega} (w^{-q} + \lambda w^{2^*_s - 1}) \varphi_{\epsilon} \, dx
\]

\[
= \int_{\mathbb{R}^n} \varphi_{\epsilon}(-\Delta)^s (w - u_{\lambda}) \, dx + \int_{\mathbb{R}^n} \varphi_{\epsilon}(-\Delta)^s u_{\lambda} \, dx - \int_{\Omega} (w^{-q} + \lambda w^{2^*_s - 1}) \varphi_{\epsilon} \, dx.
\]

We define \( \Omega^\epsilon = \{x \in \Omega : (w + \epsilon \varphi)(x) \geq u_{\lambda} > w(x)\} \) so that \( \mathcal{L}(\Omega^\epsilon) \to 0 \) as \( \epsilon \to 0^+ \) and also \( \mathcal{C}\Omega^\epsilon := \Omega \setminus \Omega_{\epsilon} \subset \{x \in \Omega : (w + \epsilon \varphi)(x) < u_{\lambda}(x)\} \) which implies that \( \mathcal{L}(\Omega^\epsilon \times \mathcal{C}\Omega^\epsilon) \to 0 \) as
Moreover, using the fact that
\[
\int_{\mathbb{R}^n} \varphi^e(-\Delta)^s (w - u_{\lambda'}) dx
\]
\[
= \int_\Omega \frac{|(w - u_{\lambda'})(x) - (w - u_{\lambda'})(y)|(\varphi^e(x) - \varphi^e(y))}{|x - y|^{n+2s}} dxdy
\]
\[
= \int_{\Omega^e} \int_{\Omega^c} \frac{|(w - u_{\lambda'})(x) - (w - u_{\lambda'})(y)|^2}{|x - y|^{n+2s}} dxdy
\]
\[
+ \epsilon \int_{\Omega^e} \int_{\Omega^c} \frac{(w - u_{\lambda'})(x) - (w - u_{\lambda'})(y))(\varphi(x) - \varphi(y))}{|x - y|^{n+2s}} dxdy
\]
\[
+ 2 \int_{\Omega^e} \int_{C_{\Omega^e}} \frac{(w - u_{\lambda'})^2(x)}{|x - y|^{n+2s}} dxdy + 2\epsilon \int_{\Omega^e} \int_{C_{\Omega^c}} \frac{(w - u_{\lambda'})(x)\varphi(x) - \varphi(x)}{|x - y|^{n+2s}} dxdy
\]
\[
- 2 \int_{\Omega^e} \int_{C_{\Omega^e}} \frac{(w - u_{\lambda'})(x)(w - u_{\lambda'})(y)}{|x - y|^{n+2s}} dxdy + 2\epsilon \int_{\Omega^e} \int_{C_{\Omega^c}} \frac{(w - u_{\lambda'})(x)\varphi(x) - \varphi(x)}{|x - y|^{n+2s}} dxdy
\]
\[
+ 2 \int_{\Omega^e} \int_{C_{\Omega^e}} \frac{(w - u_{\lambda'})(x)\varphi(x)}{|x - y|^{n+2s}} dxdy + 2\epsilon \int_{\Omega^e} \int_{C_{\Omega^c}} \frac{(w - u_{\lambda'})(x)\varphi(x) - \varphi(x)}{|x - y|^{n+2s}} dxdy
\]

where to obtain the last inequality, we use the fact that if \((x, y) \in \Omega^e \times C_{\Omega^e}\) then \((w - u_{\lambda'})(x)(w - u_{\lambda'})(y) \leq \epsilon^2 \varphi(x)\varphi(y)\). Therefore we get
\[
\frac{1}{\epsilon} \int_{\mathbb{R}^n} \varphi^e(-\Delta)^s (w - u_{\lambda'}) dx \geq o(1) \text{ as } \epsilon \to 0^+.
\]

Moreover, using the fact that \(u_{\lambda'}\) is a supersolution of \((P_{\lambda})\), the other terms of \(\frac{1}{\epsilon} E^e\) can be estimated as
\[
\frac{1}{\epsilon} \int_{\mathbb{R}^n} \varphi^e(-\Delta)^s u_{\lambda'} dx - \frac{1}{\epsilon} \int_\Omega (w^q + \lambda w^{2^*s-1})\varphi^e dx
\]
\[
\geq \frac{1}{\epsilon} \int_\Omega (u_{\lambda'}^q - w^q)\varphi^e dx + \frac{1}{\epsilon} \int_\Omega (u_{\lambda'}^{2s-1} - w^{2s-1})\varphi^e dx
\]
\[
\geq -\int_\Omega |u_{\lambda'}^q - w^q|\varphi dx = o(1) \text{ as } \epsilon \to 0^+.
\]

Altogether we get
\[
\frac{1}{\epsilon} E^e \geq o(1) \text{ as } \epsilon \to 0^+
\]
and similarly we obtain
\[
\frac{1}{\epsilon} E^e \leq o(1) \text{ as } \epsilon \to 0^+.
\]
Hence (3.3) gives that for all \( \varphi \in X_0 \)
\[
\int_{\mathbb{R}^n} \varphi (-\Delta)^s w \, dx - \int_\Omega (w^{-q} + \lambda w^{2^*_s-1}) \varphi \, dx \geq o(1) \text{ as } \epsilon \to 0^+
\]
but since \( \varphi \) was arbitrary, this implies that \( w \) is a weak solution of \((P_\lambda)\). This establishes the proof. \( \Box \)

We now prove a special property of \( w \), the weak solution of \((P_\lambda)\) obtained in Proposition 3.2 following the proof in [6, Proposition 3.5]. We also refer [2, Proposition 5.2] where similar ideas were already used.

**Lemma 3.3** Let \( \lambda \in (0, \Lambda) \) and \( w \) denotes the weak solution of \((P_\lambda)\) obtained in Proposition 3.2. Then \( w \) forms a local minimum of the functional \( I_\lambda \).

**Proof.** We argue by contradiction, so suppose \( w \) is not a local minimum of \( I_\lambda \). Then there exists a sequence \( \{u_k\} \subset X_0 \) satisfying
\[
||u_k - w|| \to 0 \text{ as } k \to \infty \text{ and } I_\lambda(u_k) < I_\lambda(w). \tag{3.4}
\]
We define \( \underline{u} = u_\lambda \) and \( \overline{u} = u_\lambda \) as sub and supersolution of \((P_\lambda)\) as defined in the proof of Proposition 3.2. Also we define
\[
v_k = \max\{\underline{u}, \min\{u_k, u\}\} = \begin{cases} 
\underline{u}, & \text{if } u_k < \underline{u}, \\
u_k, & \text{if } \underline{u} \leq u_k \leq \overline{u}, \\
\overline{u}, & \text{if } u_k > \overline{u},
\end{cases}
\]
and \( w_k = (u_k - \underline{u})^-, \overline{w}_k = (u_k - \overline{u})^+ \). Correspondingly, we define the sets \( S_k = \text{Supp}(w_k) \) and \( \overline{S}_k = \text{Supp}(\overline{w}_k) \). Then \( u_k = v_k - w_k + \overline{w}_k \) and \( v_k \in W_\lambda \) where \( W_\lambda \) has been defined in Proposition 3.2. The main idea of the proof is to establish that the measures of \( S_k \) and \( S_k \) tend to 0 as \( k \to \infty \) which together with \( v_k \in W_\lambda \) force \( I_\lambda(u_k) \) to be beyond \( I_\lambda(w) \). First, we have that
\[
\int_{\Omega} (u_k^+)^{1-q} \, dx = \int_{S_k} (u_k^+)^{1-q} \, dx + \int_{\overline{S}_k} (u_k^+)^{1-q} \, dx + \int_{\underline{u} \leq u_k \leq \overline{u}} (v_k)^{1-q} \, dx
\]
\[
= \int_{S_k} ((u_k^+)^{1-q} - \underline{u}^{1-q}) \, dx + \int_{\overline{S}_k} ((u_k^+)^{1-q} - \overline{u}^{1-q}) \, dx + \int_{\underline{u} \leq u_k \leq \overline{u}} (v_k)^{1-q} \, dx
\]
and
\[
\int_{\Omega} (u_k^+)^{2^*_s} \, dx = \int_{S_k} (u_k^+)^{2^*_s} \, dx + \int_{\overline{S}_k} (u_k^+)^{2^*_s} \, dx + \int_{\underline{u} \leq u_k \leq \overline{u}} (v_k)^{2^*_s} \, dx
\]
\[
= \int_{S_k} ((u_k^+)^{2^*_s} - \underline{u}^{2^*_s}) \, dx + \int_{\overline{S}_k} ((u_k^+)^{2^*_s} - \overline{u}^{2^*_s}) \, dx + \int_{\underline{u} \leq u_k \leq \overline{u}} (v_k)^{2^*_s} \, dx.
\]
Then we can express \( I_\lambda(u_k) \) as
\[
I_\lambda(u_k) = I_\lambda(v_k) + \frac{J_0}{2} - \frac{1}{1-q} \left( \int_{S_k} ((u_k^+)^{1-q} - \underline{u}^{1-q}) \, dx + \int_{\overline{S}_k} ((u_k^+)^{1-q} - \overline{u}^{1-q}) \, dx \right)
\]
\[
- \frac{\lambda}{2^*_s} \left( \int_{S_k} ((u_k^+)^{2^*_s} - \underline{u}^{2^*_s}) \, dx + \int_{\overline{S}_k} ((u_k^+)^{2^*_s} - \overline{u}^{2^*_s}) \, dx \right). \tag{3.5}
\]
where $J_0 = C_s^u(||u_k||^2 - ||v_k||^2)$. While denoting $S_k = \{ x \in \Omega : u \leq v_k \leq \overline{u}\}$ and $h_k(x, y) = (u_k(x) - u_k(y))^2 - (v_k(x) - v_k(y))^2$, we get

$$J_0 = \int_{S_k} \int_{S_k} \frac{h_k(x, y)}{|x - y|^{n+2s}} \, dx \, dy + \int_{S_k} \int_{S_k} \frac{h_k(x, y)}{|x - y|^{n+2s}} \, dx \, dy + 2 \int_{S_k} \int_{S_k} \frac{h_k(x, y)}{|x - y|^{n+2s}} \, dx \, dy + 2 \int_{S_k} \int_{S_k} \frac{h_k(x, y)}{|x - y|^{n+2s}} \, dx \, dy.$$

Since $u_k = \overline{u} + \underline{u}$ and $v_k = \underline{u}$ in $S_k$ and $u_k = u - w_k$ and $v_k = \overline{u}$ in $S_k$ we get that

$$\int_{S_k} \int_{S_k} \frac{h_k(x, y)}{|x - y|^{n+2s}} \, dx \, dy = \int_{S_k} \int_{S_k} \frac{(w_k(x) - w_k(y))^2}{|x - y|^{n+2s}} \, dx \, dy$$

$$- 2 \int_{S_k} \int_{S_k} \frac{((w_k(x) - w_k(y))(\overline{u} - \underline{u}))}{|x - y|^{n+2s}} \, dx \, dy$$

$$+ 2 \int_{S_k} \int_{S_k} \frac{(\overline{w}(x) - \overline{w}(y))(\overline{u} - \underline{u}))}{|x - y|^{n+2s}} \, dx \, dy.$$

Also similarly we obtain

$$\int_{S_k} \int_{S_k} \frac{h_k(x, y)}{|x - y|^{n+2s}} \, dx \, dy = \int_{S_k} \int_{S_k} \frac{(w_k(x) + \overline{w}(y))^2}{|x - y|^{n+2s}} \, dx \, dy$$

$$- 2 \int_{S_k} \int_{S_k} \frac{(w_k(x) + \overline{w}(y))(u(x) - \overline{u}(y))}{|x - y|^{n+2s}} \, dx \, dy,$$

$$\int_{S_k} \int_{S_k} \frac{h_k(x, y)}{|x - y|^{n+2s}} \, dx \, dy = \int_{S_k} \int_{S_k} \frac{w_k^2(x)}{|x - y|^{n+2s}} \, dx \, dy - 2 \int_{S_k} \int_{S_k} \frac{w_k(x)(u(x) - u_k(y))}{|x - y|^{n+2s}} \, dx \, dy,$$

$$\int_{S_k} \int_{S_k} \frac{h_k(x, y)}{|x - y|^{n+2s}} \, dx \, dy = \int_{S_k} \int_{S_k} \frac{w_k^2(x)}{|x - y|^{n+2s}} \, dx \, dy + 2 \int_{S_k} \int_{S_k} \frac{w_k(x)(\overline{u}(x) - u_k(y))}{|x - y|^{n+2s}} \, dx \, dy.$$
using all above estimates, we can express $J_0$ as

$$J_0 = C_s^n(\|w_k\|^2 + \|\overline{w}_k\|^2) + 2 \left( \int_{S_k} \int_{S_k} \frac{(w_k(x) + \overline{w}_k(y))^2}{|x - y|^{n+2s}} dxdy - \int_{S_k} \int_{S_k} \frac{w_k^2(x)}{|x - y|^{n+2s}} dxdy \right)$$

$$- \int_{S_k} \int_{S_k} \frac{w_k^2(x)}{|x - y|^{n+2s}} dxdy - 2 \int_{S_k} \int_{S_k} \frac{(w_k(x) - \overline{w}_k(y))(u(x) - \overline{u}(y))}{|x - y|^{n+2s}} dxdy$$

$$+ 2 \int_{S_k} \int_{S_k} \frac{(w_k(x) - \overline{w}_k(y))(\overline{u}(x) - \overline{u}(y))}{|x - y|^{n+2s}} dxdy - 4 \int_{S_k} \int_{S_k} \frac{(w_k(x) + \overline{w}_k(y))(u(x) - \overline{u}(y))}{|x - y|^{n+2s}} dxdy$$

$$- 4 \int_{S_k} \int_{S_k} \frac{w_k(x)(u(x) - \overline{u}(y))}{|x - y|^{n+2s}} dxdy + 4 \int_{S_k} \int_{S_k} \frac{\overline{w}_k(x)(\overline{u}(x) - \overline{u}(y))}{|x - y|^{n+2s}} dxdy. $$

Now we notice that if $(x, y) \in S_k \times S_k$ then $(u(x) - \overline{u}(y)) \leq (u(x) - u(y))$, if $(x, y) \in \overline{S}_k \times S_k$ then $(\overline{u}(x) - \overline{u}(y)) \geq (\overline{u}(x) - \overline{u}(y))$ and

$$\int_{S_k} \int_{S_k} \frac{(w_k(x) + \overline{w}_k(y))^2}{|x - y|^{n+2s}} dxdy - \int_{S_k} \int_{S_k} \frac{\overline{w}_k^2(x)}{|x - y|^{n+2s}} dxdy - \int_{S_k} \int_{S_k} \frac{|w_k(x)|^2}{|x - y|^{n+2s}} dxdy$$

$$= 2 \int_{S_k} \int_{S_k} \frac{w_k(x)\overline{w}_k(y)}{|x - y|^{n+2s}} dxdy. $$

Also using change of variables, we have

$$\int_{S_k} \int_{S_k} \frac{(w_k(x) + \overline{w}_k(y))(u(x) - \overline{u}(y))}{|x - y|^{n+2s}} dxdy$$

$$= \int_{S_k} \int_{S_k} \frac{w_k(x)(u(x) - \overline{u}(y))}{|x - y|^{n+2s}} dxdy - \int_{S_k} \int_{S_k} \frac{\overline{w}_k(x)(\overline{u}(x) - \overline{u}(y))}{|x - y|^{n+2s}} dxdy. $$

Therefore altogether we obtain

$$J_0 \geq C_s^n(\|w_k\|^2 + \|\overline{w}_k\|^2) + 4 \int_{S_k} \int_{S_k} \frac{w_k(x)\overline{w}_k(y)}{|x - y|^{n+2s}} dxdy + 2 \int_{\mathbb{R}^n} \overline{w}_k(\Delta)^s \overline{u} dx - 2 \int_{\mathbb{R}^n} w_k(\Delta)^s u dx$$

$$- 4 \int_{S_k} \int_{S_k} \frac{w_k(x)(u(x) - \overline{u}(y))}{|x - y|^{n+2s}} dxdy + 4 \int_{S_k} \int_{S_k} \frac{\overline{w}_k(x)(u(x) - \overline{u}(y))}{|x - y|^{n+2s}} dxdy$$

$$- 4 \int_{S_k} \int_{S_k} \frac{(w_k(x) + \overline{w}_k(y))(u(x) - \overline{u}(y))}{|x - y|^{n+2s}} dxdy$$

$$\geq C_s^n(\|w_k\|^2 + \|\overline{w}_k\|^2) + 2 \int_{\mathbb{R}^n} \overline{w}_k(\Delta)^s \overline{u} dx - 2 \int_{\mathbb{R}^n} w_k(\Delta)^s u dx$$

where we used the fact that if $(x, y) \in \overline{S}_k \times S_k$ then $\overline{w}_k(x)\overline{w}_k(y) \geq 0$. Now recalling that $u$ and $\overline{u}$ forms sub and supersolution of $(P_{\lambda})$ respectively, inserting the above inequality in (3.5)
we obtain
\[ I_\lambda(u_k) \geq I_\lambda(v_k) + \frac{C_n}{2} \|w_k\|^2 + \frac{C_s}{2} \|\bar{w}_k\|^2 + \int_{S_k} \left( \frac{\bar{u}^{1-q} - (\bar{u} + \bar{w}_k)^{1-q}}{1 - q} + \bar{u}^{-q} w_k \right) dx \]
\[ + \int_{S_k} \left( \frac{u^{1-q} - (u - w_k)^{1-q}}{1 - q} - u^{-q} w_k \right) dx + \lambda \int_{S_k} \left( \frac{\bar{u}^{2s} - (\bar{u} + \bar{w}_k)^{2s}}{2s} + \bar{u}^{2s-1} w_k \right) dx \]
\[ + \lambda \int_{S_k} \left( \frac{u^{2s} - (u - w_k)^{2s}}{2s} - u^{2s-1} w_k \right) dx. \]

Now from Mean Value Theorem it follows that there exists \( \theta \in (0, 1) \) (where \( \theta \) may change its value for different function below) such that
\[
I_\lambda(u_k) \geq I_\lambda(v_k) + \frac{C_n}{2} \|w_k\|^2 + \frac{C_s}{2} \|\bar{w}_k\|^2 - \int_{S_k} ((\bar{u} + \theta \bar{w}_k)^{-q} - \bar{u}^{-q}\bar{w}_k) dx
\]
\[ - \int_{S_k} (u^{-q} - (u + \theta w_k)^{-q}) w_k dx - \lambda \int_{S_k} ((\bar{u} + \theta \bar{w}_k)^{2s-1} - \bar{u}^{2s-1}) \bar{w}_k dx
\]
\[ - \lambda \int_{S_k} (\bar{u}^{2s-1} - (\bar{u} + \theta \bar{w}_k)^{2s-1}) w_k dx \]
\[ \geq I_\lambda(v_k) + \frac{C_n}{2} \|w_k\|^2 + \lambda \int_{S_k} ((\bar{u} + \theta \bar{w}_k)^{2s-1} - \bar{u}^{2s-1}) \bar{w}_k dx
\]
\[ - \lambda \int_{S_k} (\bar{u}^{2s-1} - (\bar{u} + \theta \bar{w}_k)^{2s-1}) w_k dx. \]

Now since \( 2s > 2 \), there exists constant \( C > 0 \) such that (3.6) reduces to
\[
I_\lambda(u_k) \geq I_\lambda(v_k) + \frac{C_n}{2} \|w_k\|^2 + \frac{C_s}{2} \|\bar{w}_k\|^2 - C \int_{S_k} (u^{2s-2} - w_k^{2s-2}) w_k^2 dx
\]
\[ - C \int_{S_k} (\bar{u}^{2s-2} - \bar{w}_k^{2s-2}) \bar{w}_k^2 dx
\]
\[ \geq I_\lambda(v_k) + \frac{C_n}{2} \|w_k\|^2 + \frac{C_s}{2} \|\bar{w}_k\|^2 - C \left( \int_{S_k} |u|^{2s} \right)^{\frac{2s-2}{2s}} \|w_k\|^2
\]
\[ - C \left( \int_{S_k} |\bar{u}|^{2s} \right)^{\frac{2s-2}{2s}} \|\bar{w}_k\|^2. \]

Claim- \( \lim_{k \to \infty} |S_k| = 0 \) and \( \lim_{k \to \infty} |\bar{S}_k| = 0 \).

Let \( \alpha > 0 \) and define
\[ A_k = \{ x \in \Omega : u_k \geq \pi \text{ and } \pi > w + \alpha \}, \ A_k = \{ x \in \Omega : u_k \leq \bar{u} \text{ and } \bar{u} < w - \alpha \}
\]
\[ B_k = \{ x \in \Omega : u_k \geq \bar{u} \text{ and } \bar{u} \leq w + \alpha \}, \ B_k = \{ x \in \Omega : u_k \leq \bar{u} \text{ and } \bar{u} \geq w - \alpha \}.
\]

Since
\[ 0 = \mathcal{L}(\{ x \in \Omega : \bar{u} < w \}) = \mathcal{L}(\cap_{j=1}^{\infty} \{ x \in \Omega : \bar{u} < w + \frac{1}{j} \}) \]
so there exists \( j_0 \geq 1 \) large enough and \( \alpha < 1/j_0 \) such that \( \mathcal{L}(\{x \in \Omega : \pi < w + \alpha\}) \leq \epsilon/2 \). This implies that \( \mathcal{L}(B_k) \leq \epsilon/2 \) and similarly, we obtain \( \mathcal{L}(\bar{B}_k) \leq \epsilon/2 \). From (3.4) we already have \( |u_k - w|_2 \to 0 \) as \( k \to \infty \). So for \( k \geq k_0 \) large enough we get that

\[
\frac{\alpha^2 \epsilon}{2} \geq \int_{\Omega} |u_k - w|^2 \, dx \geq \int_{A_k} |u_k - w|^2 \, dx \geq \alpha^2 \mathcal{L}(A_k)
\]

which implies that \( \mathcal{L}(A_k) \leq \frac{\epsilon}{2} \) for \( k \geq k_0 \). Similarly we obtain \( \mathcal{L}(\bar{A}_k) \leq \frac{\epsilon}{2} \) for \( k \geq k_0 \). Now since \( S_k \subset A_k \cap B_k \) and \( S_k \subset \bar{A}_k \cap \bar{B}_k \) we get that \( \mathcal{L}(S_k) \leq \epsilon \) and \( \mathcal{L}(S_k) \leq \epsilon \) for \( k \geq k_0 \). This proves the claim. Thus

\[
\left( \int_{S_k} |\pi|^{2^*} \right)^{\frac{2^* - 2}{2^*}} \leq o(1) \text{ and } \left( \int_{S_k} |u|^{2^*} \right)^{\frac{2^* - 2}{2^*}} \leq o(1)
\]

which imposing in (3.7) gives that for large enough \( k \)

\[
I_\lambda(u_k) \geq I_\lambda(v_k) \geq I_\lambda(w)
\]

which is a contradiction to (3.4). Therefore \( w \) must be a local minimum of \( I_\lambda \) over \( X_0 \).

**Theorem 3.4** There exists a positive weak solution to \((P_\lambda)\).

**Proof.** Let \( \lambda_m \uparrow \Lambda \) as \( m \to \infty \) and \( \{u_{\lambda_m}\} \) be a sequence of positive weak solutions to \((P_{\lambda_m})\), such that \( u_{\lambda_m} \) forms the local minimum of \( I_{\lambda_m} \) as seen in Lemma 3.3. Since we consider the minimal solutions, we get \( u_m \leq u_{m+1} \) for each \( m \). Then, it is easy to see that \( I_{\lambda_m} < 0 \) in the case \( 0 < q < 1 \) whereas there exists a constant \( K \) independent of \( m \) such that \( I_{\lambda_m} \leq K \) for all \( m \) when \( q > 1 \) but \( q(2s - 1) < (2s + 1) \). This implies that \( \{u_{\lambda_m}\} \) is uniformly bounded in \( X_0 \). Therefore, up to a subsequence there exists \( u_\Lambda \in X_0 \) such that \( u_{\lambda_m} \rightharpoonup u_\Lambda \) weakly and pointwise a.e. in \( X_0 \) as \( m \to \infty \). Also by construction \( u_{\lambda_m} \geq u_\Lambda \) as defined in Lemma 3.1. Therefore, \( u_\Lambda \) is a positive weak solution of \((P_\lambda)\).

### 4 Multiplicity result

We have already obtained the first solution for \((P_\lambda)\) in the previous section when \( \lambda \in (0, \Lambda) \) in \( X_0 \)-topology. We fix \( \lambda \in (0, \Lambda) \) and let \( w \) denotes the first weak solution of \((P_\lambda)\) obtained in Proposition 3.2. In this section, we prove the existence of second solution of \((P_\lambda)\) using the machinery of mountain pass Lemma and with the help of Ekeland variational principle. Precisely, we extend the approach used in [13] to the non local setting and for \( q \geq 1 \). This can be done by using the asymptotic boundary behavior of \( w \in C^+_q(\Omega) \) and the Hardy’s inequality to control the singular nonlinearity in the cone \( T \) defined below:

\[
T \overset{\text{def}}{=} \{x \in X_0 : u \geq w \text{ a.e. in } \Omega\}
\]

and since \( w \) forms a local minimizer of \( I_\lambda \) we get that \( I_\lambda(u) \geq I_\lambda(w) \) whenever \( \|u - w\| \leq \sigma_0 \), for some constant \( \sigma_0 > 0 \). Then one of the following cases holds
(ZA) (Zero Altitude) \( \inf \{ I_\lambda(u) \mid u \in T, \|u - w\| = \sigma \} = I_\lambda(w) \) for all \( \sigma \in (0, \sigma_0) \).

(MP) (Mountain Pass) There exists a \( \sigma_1 \in (0, \sigma_0) \) such that \( \inf \{ I_\lambda(u) \mid u \in T, \|u - w\| = \sigma_1 \} > I_\lambda(w) \).

Before investigating the two distinguished cases (ZA) and (MP), we prove the following regularity result for weak solutions to \( (P_\lambda) \):

**Proposition 4.1** Any weak solution to \( (P_\lambda) \) for \( \lambda \in (0, \Lambda) \) belongs to \( L^\infty(\Omega) \cap C^+_{\phi_0}(\Omega) \).

**Proof.** Let \( u \in X_0 \) denotes a weak solution to \( (P_\lambda) \). We know that \( u_\lambda \in X_0 \cap C^+_{\phi_0}(\Omega) \) (defined in Lemma 3.1) forms a subsolution to \( (P_\lambda) \) satisfying \( (-\Delta)^s u_\lambda = u_\lambda^{-q} \) in \( \Omega \). We first have

Claim : \( u_\lambda \leq u \) a.e. in \( \Omega \).

Let us prove the above claim. Suppose it is not true. First it is easy to see that for any \( v \in X_0 \) it holds

\[
(v(x) - v(y))(v^+(x) - v^-(y)) \geq |v^+(x) - v^+(y)|^2, \text{ for any } x, y \in \mathbb{R}^n.
\]

Therefore using \( (u_\lambda - u)^+ \) as the test function in

\[
(-\Delta)^s (u_\lambda - u) \leq u_\lambda^{-q} - u^{-q} \text{ in } \Omega
\]

we get

\[
0 \leq C_s \int_Q \frac{|(u_\lambda - u)^+(x) - (u_\lambda - u)^+(y)|^2}{|x - y|^{n+2s}} \, dx \, dy \\
\leq C_s \int_Q \frac{((u_\lambda - u)^+(x) - (u_\lambda - u)^+(y))((u_\lambda - u)(x) - (u_\lambda - u)(y))}{|x - y|^{n+2s}} \, dx \, dy \\
\leq \int_\Omega (u_\lambda^{-q} - u^{-q})(u_\lambda - u)^+ \, dx \leq 0.
\]

Hence it must be that \( \operatorname{meas}\{ x \in \Omega : u_\lambda(x) \geq u(x) \} = 0 \) which gives a contradiction. Therefore \( u_\lambda \leq u \) a.e. in \( \Omega \). Let us now prove that \( u \in L^\infty(\Omega) \). We follow the approach in [4, Proposition 2.2]. By virtue of the above claim and Hardy’s inequality, we know that \( \int_\Omega u^{-q} \phi \, dx < \infty \) for any \( \phi \in X_0 \). We aim to show that \( (u - 1)^+ \) belongs to \( L^\infty(\Omega) \) which will imply that \( u \in L^\infty(\Omega) \). If \( f(t) = (t - 1)^+ \) for \( t \in \mathbb{R} \) and \( \psi(t) \in C^\infty(\mathbb{R}) \) be a convex and increasing function such that \( \psi'(t) \leq 1 \) when \( t \in [0, 1] \) and \( \psi'(t) = 1 \) when \( t \geq 1 \) then we can define

\[
\psi_\epsilon(t) = \epsilon \psi(t/\epsilon)
\]

so that \( \psi_\epsilon \to f \) uniformly as \( \epsilon \to 0 \). Also since \( \psi_\epsilon \) is smooth, by regularity results and the uniform convergence of \( \psi_\epsilon \) to \( f \) we get that

\[
(-\Delta)^s \psi_\epsilon(u) \to (-\Delta)^s (u - 1)^+ \text{ as } \epsilon \to 0.
\]

Moreover because \( \psi_\epsilon \) is convex and differentiable, we know that

\[
(-\Delta)^s \psi_\epsilon(u) \leq \psi_\epsilon'(u)(-\Delta)^s u \leq \lambda_{u>1}(-\Delta)^s u
\]
where $\chi_{\{u>1\}}$ denotes the characteristic function over the set $\{x \in \Omega : u(x) > 1\}$. Then passing on the limits $\epsilon \to 0$ in above equation, we obtain

$$(-\Delta)^s(u-1)^+ \leq \chi_{\{u>1\}}(-\Delta)^s u \leq \chi_{\{u>1\}}(u^{-q} + \lambda u^{2s-1}) \leq C(1 + ((u-1)^+)^{2s-1})$$

for some constant $C > 0$. Therefore using [4, Proposition 2.2], we conclude that $(u-1)^+ \in L^\infty(\Omega)$. Finally we show that $u \in C^+_\phi(\Omega)$. Let $z_\lambda$ be the unique solution (refer to [1, Theorem 1.1] with $\delta = q$ and $\beta = 0$) to

$$(\Delta)^sz_\lambda = z_\lambda^{-q} + \lambda c, \quad u > 0 \text{ in } \Omega, \quad u = 0 \text{ in } \mathbb{R}^n \setminus \Omega,$$

with $c = \|u\|^{2s-1}_\infty$ then similarly we can prove that $u \leq z_\lambda$. Therefore using local regularity results in [17, Propositions 2.2 and 2.3] derived from [20, Propositions 2.8 and 2.9], $u \in C^+_\phi(\Omega)$. This completes the proof.

**Lemma 4.2** Let (ZA) holds then for any $\sigma \in (0, \sigma_0)$ there exists a solution $v \in C^+_{\phi}(\Omega)$ to $(P_\lambda)$ such that $0 < w < v \in \Omega$ and $\|v-w\| = \sigma$.

**Proof.** We follow the proof of Lemma 2.6 of [13] in a nonlocal framework. We fix $\sigma \in (0, \sigma_0)$ and $r > 0$ such that $\sigma - r > 0$ and $\sigma + r < \sigma_0$. Let us define the set

$$W = \{u \in T \mid 0 < \sigma - r \leq \|u - w\| \leq \sigma + r\}$$

which is closed in $X_0$ and by (ZA), $\inf_{u \in W} I_\lambda(u) = I_\lambda(w)$. So using Ekeland variational principle, for any minimizing sequence $\{u_k\} \subset X_0$ satisfying $\|u_k\| = \sigma$ and $I_\lambda(u_k) \leq I_\lambda(w) + \frac{1}{k}$, we get another sequence $\{v_k\} \subset W$ such that

$$\begin{align}
I_\lambda(v_k) &\leq I_\lambda(u_k) \leq I_\lambda(w) + \frac{1}{k} \\
\|u_k - v_k\| &\leq \frac{1}{k} \\
I_\lambda(v_k) &\leq I_\lambda(z) + \frac{1}{k}\|z - v_k\|, \text{ for all } z \in W.
\end{align}$$

(4.1)

We can choose $\epsilon > 0$ small enough so that $v_k + \epsilon(z - v_k) \in W$ for $z \in T$. So from (4.1) we obtain

$$I_\lambda(v_k + \epsilon(z - v_k)) - I_\lambda(v_k) \geq -\frac{1}{k}\|z - v_k\|.$$ 

Letting $\epsilon \to 0^+$ and using the fact that $v_k \geq w$ for each $k$, for $z \in T$ we get

$$\int_{\mathbb{R}^n} (-\Delta)^sv_k(z - v_k) - \int_\Omega v_k^{-q}(z - v_k) \, dx - \lambda \int_\Omega v_k^{2s-1}(z - v_k) \, dx \geq -\frac{1}{k}\|z - v_k\|. \quad (4.2)$$

Now since $\{v_k\}$ forms a bounded sequence in $X_0$, we get that there exists a $v \in X_0$ such that, up to a subsequence, $v_k \rightharpoonup v$ weakly in $X_0$ and pointwise a.e. in $\Omega$ as $k \to \infty$. Since $v_k \geq w$ for each $k$, we get $v \geq w$ a.e. in $\Omega$. In what follows, we will prove that $v$ is a weak
solution of \((P_{\lambda})\). For \(\phi \in X_0\) and \(\epsilon > 0\), we set \(\phi_{k,\epsilon} = (v_k + \epsilon \phi - w)^- \in X_0\) which implies that 
\((v_k + \epsilon \phi + \phi_{k,\epsilon}) \in T\). Putting \(z = v_k + \epsilon \phi + \phi_{k,\epsilon}\) in \((4.2)\) we get
\[
\begin{align*}
  C^m_s \int_{Q} & \frac{(v_k(x) - v_k(y))((\epsilon \phi + \phi_{k,\epsilon})(x) - (\epsilon \phi + \phi_{k,\epsilon})(y))}{|x-y|^{n+2s}} \, dx \, dy - \int_{\Omega} v_k^{-q}(\epsilon \phi + \phi_{k,\epsilon}) \, dx \\
  & - \lambda \int_{\Omega} v_k^{-2^*-1}(\epsilon \phi + \phi_{k,\epsilon}) \, dx \geq \frac{1}{k} \| (\epsilon \phi + \phi_{k,\epsilon}) \|.
\end{align*}
\]  
\[(3.3)\]

We define the sets \(\Omega_{k,\epsilon} = \text{Supp} \, \phi_{k,\epsilon}, \Omega_\epsilon = \text{Supp} \, \phi_\epsilon\) and \(\Omega_0 = \{x \in \Omega : v(x) = w(x)\}\). Then we get that \(L(\Omega_\epsilon \setminus \Omega_0) \to 0\) as \(\epsilon \to 0\) and \(L(\Omega_{k,\epsilon} \setminus \Omega_\epsilon) + L(\Omega_\epsilon \setminus \Omega_{k,\epsilon}) \to 0\) as \(k \to \infty\). Also since \(|\phi_{k,\epsilon}| \leq w + \epsilon |\phi|\), using Lebesgue Dominated convergence theorem we get \(\phi_{k,\epsilon} \to \phi_\epsilon = (v + \epsilon \phi - w)^- \) in \(L^m(\Omega)\) for all \(m \in [1, 2^*_s]\). Moreover \(\phi_{k,\epsilon} \to \phi_\epsilon\) weakly in \(X_0\) and pointwise a.e. in \(\Omega\) as \(k \to \infty\). Now we estimate the following integral
\[
\int_{Q} \frac{(v_k(x) - v_k(y))(\phi_{k,\epsilon}(x) - \phi_{k,\epsilon}(y))}{|x-y|^{n+2s}} \, dx \, dy
\]
\[(4.4)\]

We show that \(I_2 \leq o_k(1)\) for which we split the integrals and estimate them separately. Let \(H_k = \Omega_{k,\epsilon} \cap \Omega_\epsilon\) and \(G_k = \Omega_{k,\epsilon} \setminus \Omega_\epsilon \cup \Omega_\epsilon \setminus \Omega_{k,\epsilon}\). Then
\[
\begin{align*}
  \int_{\Omega} \int_{C\Omega} & \frac{(v_k(x) - v_k(y))(\phi_{k,\epsilon} - \phi_\epsilon)(x) - (\phi_{k,\epsilon} - \phi_\epsilon)(y))}{|x-y|^{n+2s}} \, dx \, dy \\
  \leq & \int_{H_k} \int_{H_k} \frac{v(x)(v - v_k)(x)}{|x-y|^{n+2s}} + \int_{G_k} \int_{C\Omega} \frac{v_k(x)(\phi_{k,\epsilon} - \phi_\epsilon)(x)}{|x-y|^{n+2s}} \\
  \leq & \int_{H_k} \int_{H_k} \frac{v(x)(v - v_k)(x)}{|x-y|^{n+2s}} + \int_{G_k} \int_{C\Omega} \frac{v_k(x)(\phi_{k,\epsilon})(x)}{|x-y|^{n+2s}} \\
  = & \int_{H_k} \int_{H_k} \frac{v(x)(v - v_k)(x)}{|x-y|^{n+2s}} + o_k(1)
\end{align*}
\]  
\[(4.5)\]

using the fact that \(L(\Omega_{k,\epsilon} \setminus \Omega_\epsilon) + L(\Omega_\epsilon \setminus \Omega_{k,\epsilon}) \to 0\) as \(k \to \infty\) and Lebesgue Dominated convergence theorem. Similarly
\[
\begin{align*}
  \int_{\Omega} \int_{\Omega} & \frac{(v_k(x) - v_k(y))(\phi_{k,\epsilon} - \phi_\epsilon)(x) - (\phi_{k,\epsilon} - \phi_\epsilon)(y))}{|x-y|^{n+2s}} \\
  \leq & \int_{H_k} \int_{H_k} \frac{(v(x) - v(y))(v - v_k)(x) - (v - v_k)(y)}{|x-y|^{n+2s}} \\
  + & 2 \int_{H_k} \int_{G_k} \frac{(v_k(x) - v_k(y))(\phi_{k,\epsilon} - \phi_\epsilon)(x) - (\phi_{k,\epsilon} - \phi_\epsilon)(y))}{|x-y|^{n+2s}} \\
  + & \int_{G_k} \int_{G_k} \frac{(v_k(x) - v_k(y))(\phi_{k,\epsilon} - \phi_\epsilon)(x) - (\phi_{k,\epsilon} - \phi_\epsilon)(y))}{|x-y|^{n+2s}} + o_k(1)
\end{align*}
\]  
\[(4.6)\]
using again the Lebesgue Dominated convergence theorem with the fact that \( v_k - v \to 0 \) and \( \phi_{k,\epsilon} - \phi_{\epsilon} \to 0 \) pointwise as \( k \to \infty \). Combining (4.5) and (4.6) we obtain that

\[
I_2 \leq \int_{H_k \cup H_{\epsilon} \cup C_{\Omega}} \frac{(v(x) - v(y))((v - v_k)(x) - (v - v_k)(y))}{|x - y|^{n+2s}} + o_k(1) = o_k(1).
\]

Therefore using this in (4.4), we obtain

\[
\int_Q \frac{(v_k(x) - v_k(y))(\phi_{k,\epsilon}(x) - \phi_{k,\epsilon}(y))}{|x - y|^{n+2s}} dxdy \leq \int_Q \frac{(v_k(x) - v_k(y))(\phi_{\epsilon}(x) - \phi_{\epsilon}(y))}{|x - y|^{n+2s}} dxdy + o_k(1).
\]

Moreover, we have that \( |v_k^{-q}(\epsilon \phi + \phi_{k,\epsilon})| \leq w^{-q}(w + 2\epsilon \phi) \in L^1(\Omega) \) using the Hardy’s inequality. Thus using Lebesgue Dominated convergence theorem and passing on the limits \( k \to \infty \) in (4.3) we get

\[
0 \leq C_s^m \int_Q \frac{(v_k(x) - v_k(y))(\epsilon \phi + \phi_{k,\epsilon})(x) - (\epsilon \phi + \phi_{\epsilon})(y))}{|x - y|^{n+2s}} dxdy - \int_{\Omega} (v^{-q} + \lambda v^{2s-1})(\epsilon \phi + \phi_{\epsilon}) dx.
\]

Using the fact that \( w \) is a weak solution of \((P_\lambda)\) and \( v \geq w \), the above inequality implies that

\[
C_s^m \int_Q \frac{(v(x) - v(y))(\phi(x) - \phi(y))}{|x - y|^{n+2s}} dxdy - \int_{\Omega} v^{-q} \phi dx - \lambda \int_{\Omega} v^{2s-1} \phi dx
\]

\[
\geq -\frac{1}{\epsilon} \left( C_s^m \int_Q \frac{(v(x) - v(y))(\phi_{\epsilon}(x) - \phi_{\epsilon}(y))}{|x - y|^{n+2s}} dxdy - \int_{\Omega} v^{-q} \phi_{\epsilon} dx - \lambda \int_{\Omega} v^{2s-1} \phi_{\epsilon} dx \right)
\]

\[
\geq \frac{1}{\epsilon} \left( C_s^m \int_Q \frac{((w - v)(x) - (w - v)(y))(\phi_{\epsilon}(x) - \phi_{\epsilon}(y))}{|x - y|^{n+2s}} dxdy + \int_{\Omega} (v^{-q} - w^{-q}) \phi_{\epsilon} dx \right)
\]

\[
\geq C_s^m \int_{\Omega_{\epsilon}} \int_{\Omega_{\epsilon}} \frac{((w - v)(x) - (w - v)(y))(\phi(x) - \phi(y))}{|x - y|^{n+2s}} dxdy + 2C_s^m \int_{\Omega} \int_{\{w \leq v + \epsilon \phi_{\epsilon}\}} \frac{((w - v)(x) - (w - v)(y)) \phi(x)}{|x - y|^{n+2s}} dxdy
\]

\[
+ 2C_s^m \int_{\Omega} \int_{\lambda \Omega} \frac{(v - w)(x) \phi(x)}{|x - y|^{n+2s}} dxdy + \int_{\Omega_{\epsilon}} (v^{-q} - w^{-q}) \phi dx
\]

\[
= o(1) \text{ as } \epsilon \to 0^+
\]

using the fact that \(|\Omega_{\epsilon} \setminus \Omega_0| \to 0 \) as \( \epsilon \to 0^+ \). From this, we get that

\[
C_s^m \int_Q \frac{(v(x) - v(y))(\phi(x) - \phi(y))}{|x - y|^{n+2s}} dxdy - \int_{\Omega} v^{-q} \phi dx - \lambda \int_{\Omega} v^{2s-1} \phi dx = 0 \text{ for all } \phi \in X_0.
\]

**Claim** - The sequence \( v_k \to v \) strongly in \( X_0 \) as \( k \to \infty \).

From Brezis Lieb lemma we have

\[
\|v_k\|_2^2 - \|v_k - v\|_2^2 = \|v\|_2^2 + o(1)
\]

\[
\int_{\Omega} |v_k|^{2s} dx - \int_{\Omega} |v_k - v|^{2s} dx = \int_{\Omega} |v|^{2s} dx + o(1)
\]
Since $v_k, v \geq w$ a.e. in $\Omega$, we get
\[
\int_{\Omega} |v_k|^{1-q} \, dx - \int_{\Omega} |v|^{1-q} \, dx = \int_{\Omega} (v_k + \theta v)^{-q}(v_k - v) \, dx, \quad \text{for } \theta \in [0, 1].
\]

We know that $(v_k + \theta v)^{-q}(v_k - v) \to 0$ pointwise a.e. in $\Omega$ and $v_k, v \geq w \in C^+_{\phi}(\Omega)$. Therefore for any $E \subset \Omega$, we have
\[
\int_{\Omega} (v_k + \theta v)^{-q}(v_k - v) \, dx \leq C \| \delta^{(1-q)s} (x) \|_{L^2(E)} \| v_k - v \|,
\]
using Hardy’s inequality. \hspace{1cm} (4.7)

Since $q(2s - 1) < (2s + 1)$, for any $\epsilon > 0$, there exists a $\rho > 0$ such that $\| \delta^{(1-q)s} (x) \|_{L^2(E)} < \epsilon$ whenever $L(E) < \rho$. Hence from (4.7) and Vitali’s convergence theorem we obtain
\[
\int_{\Omega} (v_k + \theta v)^{-q}(v_k - v) \, dx \to 0 \text{ as } k \to \infty
\]
that is
\[
\int_{\Omega} |v_k|^{1-q} \, dx \to \int_{\Omega} |v|^{1-q} \, dx \text{ as } k \to \infty.
\]

Taking $v$ as the testing function in (4.2), we deduce
\[
C^n_s \|v_k - v\|^2 \leq \lambda \|v_k - v\|_{L^{2s}(\Omega)}^{2s} + o_k(1). \hspace{1cm} (4.8)
\]

In the other hand, taking $z = 2v_k$ in (4.2) we infer
\[
C^n_s \|v_k\|^2 - \int_{\Omega} v_k^{1-q} \, dx - \lambda \|v_k\|_{L^{2s}(\Omega)}^{2s} \geq o_k(1). \hspace{1cm} (4.9)
\]

Since $v$ is a weak solution, we have that
\[
C^n_s \|v\|^2 - \int_{\Omega} v^{1-q} \, dx - \lambda \|v\|_{L^{2s}(\Omega)}^{2s} = 0. \hspace{1cm} (4.10)
\]

From (4.9) and (4.10),
\[
C^n_s \|v_k - v\|^2 \geq \lambda \|v_k - v\|_{L^{2s}(\Omega)}^{2s} + o_k(1). \hspace{1cm} (4.11)
\]

From (4.8) and (4.11), we have that
\[
C^n_s \|v_k - v\|^2 = \lambda \|v_k - v\|_{L^{2s}(\Omega)}^{2s} + o_k(1). \hspace{1cm} (4.12)
\]

Without loss of generality, we can assume that $I_\lambda(w) \leq I_\lambda(v)$. Then, we easily get
\[
I_\lambda(v_k) - I_\lambda(v) \leq I_\lambda(w) - I_\lambda(v) + o_k(1) \leq o_k(1)
\]
from which it follows that
\[
\frac{C^n_s}{2} \|v_k - v\|^2 - \frac{\lambda}{2s} \|v_k - v\|_{L^{2s}(\Omega)}^{2s} \leq o_k(1). \hspace{1cm} (4.13)
\]
From (4.12) and (4.13), we infer that \( v_k \to v \) strongly in \( X_0 \). This proves the claim. Since \( v_k \in W \) we conclude that \( v \in W \) and \( v \not\equiv w \). Next we prove that \( w < v \) in \( \Omega \). For that, we first observe that from Proposition 4.1 \( w, v \in L^\infty(\Omega) \cap C^+_{\partial \Omega}(\Omega) \). Now suppose that there exists \( x_0 \in \Omega \) such that \( v(x_0) = w(x_0) \). Then, since \( v \geq w \), \( v, w \in C(\mathbb{R}^n) \) and \( v \not\equiv w \), we get
\[
0 > C_s^n \int_{\mathbb{R}^n} \frac{(v-w)(x_0) - (v-w)(y)}{|x_0 - y|^{n+2s}} = v^{-q}(x_0) + \lambda v^{2s-1}(x_0) - (w^{-q}(x_0) + \lambda w^{2s-1}(x_0)) = 0
\]
from which we get a contradiction. Therefore \( v > w \) in \( \Omega \).

\[\Box\]

We define
\[
S_s = \inf_{u \in X_0 \setminus \{0\}} \frac{\int_{Q} |u(x) - u(y)|^2}{\int_{\Omega} |u|^2 x^{\frac{2s}{2s}} \, dx \, dy} = \frac{\int_{Q} |u(x) - u(y)|^2}{\int_{\Omega} |u|^2 x^{\frac{2s}{2s}} \, dx \, dy}
\]
as the best constant for the embedding \( X_0 \hookrightarrow L^{2s}(\Omega) \). Consider the family of minimizers \( \{U_\epsilon\} \) of \( S_s \) (refer [18]) defined as
\[
U_\epsilon(x) = \epsilon^{-(n-2s)/2} u^*(\frac{x}{\epsilon}), \quad x \in \mathbb{R}^n
\]
where \( u^*(x) = \bar{u}\left(\frac{x}{\sqrt{s}}\right) \), \( \bar{u}(x) = \frac{\tilde{u}(x)}{\|\bar{u}\|^2_{L^2}} \) and \( \bar{u}(x) = \alpha(\beta^2 + |x|^2)^{-\frac{n-2s}{2}} \) with \( \alpha \in \mathbb{R} \setminus \{0\} \) and \( \beta > 0 \) are fixed constants. Then for each \( \epsilon > 0 \), \( U_\epsilon \) satisfies
\[
(-\Delta)^s u = |u|^{2s-2} u \text{ in } \mathbb{R}^n.
\]

Let \( \nu > 0 \) be such that \( B_{4\nu} \subseteq \Omega \) and let \( \zeta \in C_0^\infty(\mathbb{R}^n) \) be such that \( 0 \leq \zeta \leq 1 \) in \( \mathbb{R}^n \), \( \zeta \equiv 0 \) in \( \mathbb{R}^n \setminus B_{2\nu} \) and \( \zeta \equiv 1 \) in \( B_{\nu} \). For each \( \epsilon > 0 \) and \( x \in \mathbb{R}^n \), we define \( \Phi_\epsilon(x) := \zeta(x) U_\epsilon(x) \). From [12, Lemma 4.13], we have the following.

**Lemma 4.3** \( \sup \{ I_\lambda(u + t\Phi_\epsilon) : t \geq 0 \} < I_\lambda(u) + \frac{s(C^n_s S_s)^{\frac{n}{n-2s}}}{n\lambda^{\frac{n-2s}{2}}} \), for any sufficiently small \( \epsilon > 0 \).

Now we prove the existence of second solution if \( (MP) \) holds.

**Lemma 4.4** Let \( (MP) \) holds then there exists a \( v \in X_0 \cap C^+_{\partial \Omega}(\Omega) \), verifying \( w < v \) in \( \Omega \), which solves \( (P_\lambda) \) weakly.

**Proof.** From Lemma 4.3, it follows that there exists \( \epsilon > 0 \) and \( R_0 \geq 1 \) such that
\[
(\text{i}) \ I_\lambda(w + RU_\epsilon) < I_\lambda(w) \text{ for } \epsilon \in (0, \epsilon_0) \text{ and } R \geq R_0.
\]
\[
(\text{ii}) \ I_\lambda(w + tR_0 U_\epsilon) < I_\lambda(w) + \frac{s(C^n_s S_s)^{\frac{n}{n-2s}}}{n\lambda^{\frac{n-2s}{2}}} \text{ for } \epsilon \in (0, \epsilon_0) \text{ and } t \in [0, 1].
\]

We define the complete metric space
\[
\Gamma := \{ \eta \in C([0,1], T) : \eta(0) = w, \|\eta(1) - w\| > \sigma_1, I_\lambda(\eta(1)) < I_\lambda(w) \}
\]
with metric defined as $d(\eta', \eta) = \max_{t \in [0,1]} \{\|\eta'(t) - \eta(t)\|\}$ for all $\eta, \eta' \in \Gamma$. From (i) above, we get that $\eta(t) = w + tR_0U_c \in \Gamma$ for large enough $R_0 > 0$. This gives that $\Gamma \neq \emptyset$. Let $
abla = \inf_{\eta \in \Gamma} \max_{t \in [0,1]} I_\lambda(\eta(t))$ then by virtue of (ii) above and condition (MP), we get

$$I_\lambda(w) < \gamma_0 \leq I_\lambda(w) + \frac{s(C_\alpha^n S_n)^{\frac{2}{n}}}{n\lambda}.$$ 

Now let $\Psi(\eta) = \max_{t \in [0,1]} I_\lambda(\eta(t))$ for $\eta \in \Gamma$. Then using Ekeland’s variational principle, we get a sequence $\{\eta_k\} \subset \Gamma$ such that

$$\Psi(\eta_k) < \gamma_0 + \frac{1}{k} \text{ and } \Psi(\eta_k) < \frac{1}{k} \|\Psi(\eta) - \eta(\eta_k)\|_\Gamma, \forall \eta \in \Gamma. \quad (4.14)$$

We define

$$\Lambda_k = \{t \in [0,1] : I_\lambda(\eta_k(t)) = \max_{x \in [0,1]} I_\lambda(\eta_k(x))\}.$$ 

**Claim:** There exists a $t_k \in \Lambda_k$ such that if $v_k = \eta_k(t_k)$ and $z \in T$ then

$$\int_{\mathbb{R}^n} (-\Delta)^s v_k(z - v_k) - \int_{\Omega} (v_k^{-q} + \lambda v_k^{2^* - 1})(z - v_k) \, dx \geq -\frac{1}{k} \max\{1, \|z - v_k\|\}. \quad (4.15)$$

We prove it by contradiction, so assume that for every $t \in \Lambda_k$ there exists a $z_t \in T$ such that

$$\int_{\mathbb{R}^n} (-\Delta)^s \eta_k(t) \left(\frac{z_t - \eta_k(t)}{\max\{1, \|z_t - \eta_k(t)\|\}}\right) \, dx$$

$$- \int_{\Omega} \left(\eta_k(t)\right)^{-q} + \lambda(\eta_k(t))^{2^* - 1} \left(\frac{z_t - \eta_k(t)}{\max\{1, \|z_t - \eta_k(t)\|\}}\right) \, dx < -\frac{1}{k}. \quad (4.15)$$

Since $I_{\lambda}$ is locally Lipschitz in $T$, $z_t$ can be chosen to be locally constant on $\Lambda_t$. Therefore for each $t \in \Lambda_k$ there exists a neighborhood $N_t$ of $t$ in $(0,1)$ such that for each $r \in N_t \cap \Gamma_k$, (4.15) holds that is

$$\int_{\mathbb{R}^n} (-\Delta)^s \eta_k(r) \left(\frac{z_t - \eta_k(r)}{\max\{1, \|z_t - \eta_k(r)\|\}}\right) \, dx$$

$$- \int_{\Omega} \left(\eta_k(r)\right)^{-q} + \lambda(\eta_k(r))^{2^* - 1} \left(\frac{z_t - \eta_k(r)}{\max\{1, \|z_t - \eta_k(r)\|\}}\right) \, dx < -\frac{1}{k}. \quad (4.16)$$

It is possible to choose a finite set $\{r_1, r_2, \ldots, r_m\} \subset \Lambda_k$ such that $\Lambda_k \subset \cup_{i=1}^m J_{r_i}$. For notational convenience, we set $z_i = z_{r_i}$ and denote $\{\kappa_1, \kappa_2, \ldots, \kappa_m\}$ as the partition of unity associated with covering $\{J_{r_1}, J_{r_2}, \ldots, J_{r_m}\}$ of $\Lambda_k$. Now if we define $z(r) = \sum_{i=1}^m \kappa_i(r)z_i$ for $r \in [0,1]$ then $z(r) \in T$ for each $r \in [0,1]$. Therefore from (4.16) we deduce that for all $r \in [0,1]$

$$\int_{\mathbb{R}^n} (-\Delta)^s \eta_k(r) \left(\frac{z(r) - \eta_k(r)}{\max\{1, \|z(r) - \eta_k(r)\|\}}\right) \, dx$$

$$- \int_{\Omega} \left(\eta_k(r)\right)^{-q} + \lambda(\eta_k(r))^{2^* - 1} \left(\frac{z(r) - \eta_k(r)}{\max\{1, \|z(r) - \eta_k(r)\|\}}\right) \, dx < -\frac{1}{k}. \quad (4.16)$$
Let $h : [0, 1] \rightarrow [0, 1]$ be a continuous function such that $h(t) = 1$ in a neighborhood of $\Lambda_k$ and $h(0) = h(1) = 0$. Also we set $\mu_k(t) = \max\{1, \|z(t) - \eta_k(t)\|\}$ and

$$\eta(t) = \eta_k(t) + \frac{h(t)\epsilon}{\mu_k(t)}(z(t) - \eta_k(t)).$$

For $\epsilon \in (0, 1)$, $\eta(t) \in T$ for all $t \in [0, 1]$. Hence (4.14) gives us that

$$\max_{t \in [0, 1]} I_\lambda(\eta_k(t)) \leq \max_{t \in [0, 1]} I_\lambda(\eta(t)) + \frac{\epsilon}{k} \max_{t \in [0, 1]} \left( h(t) \frac{\|z(t) - \eta_k(t)\|}{\mu_k(t)} \right). \tag{4.17}$$

If $t_{k, \epsilon} \in [0, 1]$ denotes the value such that $I_\lambda(\eta(t_{k, \epsilon})) = \max_{t \in [0, 1]} I_\lambda(\eta(t))$ then we can assume that $t_{k, \epsilon_j} \rightarrow t_k$ for some $t_k \in [0, 1]$, where $\epsilon_j$ is a sequence such that $\epsilon_j \rightarrow 0$. Using the continuity of $\eta$, we deduce that

$$\eta(t_{k, \epsilon_j}) \rightarrow \eta_k(t_k) \text{ as } \epsilon_j \rightarrow 0.$$ 

Hence from (4.17) we obtain that $\max_{t \in [0, 1]} I_\lambda(\eta_k(t)) \leq \max_{t \in [0, 1]} I_\lambda(\eta_k(t_k))$ which implies $I_\lambda(\eta_k(t_k)) = \max_{t \in [0, 1]} I_\lambda(\eta(t_k))$. So $t_k \in \Gamma_k$ and $h(t_{k, \epsilon_j}) = 1$ for $j > 0$ large enough, by definition.

If we set $v_k = \eta_k(t_k)$, $v_{k,j} = \eta_k(t_{k, \epsilon_j})$ and $\mu_{k,j} = \max\{1, \|z(t_{k, \epsilon_j}) - v_{k,j}\|\}$ then for large enough $j$ we obtain

$$I_\lambda(v_{k,j}) \leq I_\lambda(v_k) \leq I_\lambda(v_k) + I_\lambda(v_k + \frac{\epsilon_j}{\epsilon_k} (z(t_{k, \epsilon_j}) - v_{k,j}) + \frac{\epsilon_j}{\epsilon_k} \mu_{k,j}.$$ \tag{4.18}

It is easy to see that $\mu_{k,j} \rightarrow \theta_k := \max\{1, \|z(t_k) - v_k\|\}$ and $\|v_k - v_{k,j}\| \rightarrow 0$ as $j \rightarrow \infty$. Let $p_j = v_{k,j} - v_k$ and

$$k_j = p_j + \epsilon_j \left( \frac{z(t_{k,j}) - v_{k,j}}{\mu_{k,j}} - \frac{z(t_k) - v_k}{\theta_k} \right) = p_j + o(1).$$

Then from (4.18), we obtain

$$\frac{1}{\epsilon_j} \left( I_\lambda(v_k + \epsilon_j \left( \frac{z(t_k) - v_k}{\theta_k} \right) + k_j \right) + I_\lambda(v_k + p_j) \geq -\frac{1}{k} \text{ as } j \rightarrow \infty.$$ 

But since $v_k + \epsilon_j \left( \frac{z(t_k) - v_k}{\theta_k} \right) \geq w$ using the fact that $z(t_k) \in T$, from Proposition 2.1 and the above inequality we get

$$\int_{\mathbb{R}^n} (-\Delta)^s v_k \left( \frac{z(t_k) - v_k}{\theta_k} \right) dx - \int_{\Omega} (v_k^{-q} + \lambda v_k^{2^*-1}) \left( \frac{z(t_k) - v_k}{\theta_k} \right) dx \geq -\frac{1}{k}.$$

This is a contradiction to (4.15). Thus, the claim holds. So there exists a sequence $\{v_k\}$ satisfying

$$\left\{ \begin{array}{l} \int_{\mathbb{R}^n} (-\Delta)^s v_k (z - v_k) - \int_{\Omega} (v_k^{-q} + \lambda v_k^{2^*-1})(z - v_k) dx \geq -\frac{c}{k}(1 + \|z\|) \text{ for all } z \in T \\ I_\lambda(v_k) \rightarrow \gamma_0 \text{ as } k \rightarrow \infty \end{array} \right. \tag{4.19}$$
where $c > 0$ is some constant. Setting $z = 2v_k$ in (4.14) and using (4.19) we get
\[ \gamma_0 + o(1) \geq \frac{sC^n}{n} \|v_k\|^2 - \frac{2^* - 1 + q}{2^*} \int_\Omega |v_k|^{1-q} \, dx - \frac{c}{2^*_k}(1 + 2\|v_k\|). \]

Now this implies that $\{v_k\}$ must be bounded in $X_0$, thus up to a subsequence, $v_k \rightharpoonup v$ weakly in $X_0$ as $k \to \infty$. Using similar ideas as in $(ZA)$ case, it can be shown that $v$ is a weak solution of $(P_\lambda)$. Then the remaining part of the proof is similar as in [12, Proposition 4.12] (see also [13, Lemma 2.7] in the local setting) and consists of proving the strong convergence of the sequence $\{v_k\}$ to $v$. To this aim we use that the energy $I_\lambda(v_k)$ is strictly below the first critical level $I_\lambda(w) + \frac{s(C^nS_s)^{\frac{2}{2^*}}}{n\lambda^{\frac{n-2s}{2s}}} - \lambda \lambda^{\frac{n-2s}{2s}} - \frac{2^*}{2^*} \|v_k - v\|_{L^{2^*_s}(\Omega)}^2 \leq \frac{s(C^nS_s)^{\frac{2}{2^*}}}{n\lambda^{\frac{n-2s}{2s}}}$. (4.20)

Now (4.12) and (4.20) and the fact that $S_s\|v_k - v\|_{L^{2^*_s}(\Omega)}^2 \leq \|v_k - v\|^2$ force $\|v_k - v\| \to 0$ as $k \to \infty$. Thus, we infer that $I_\lambda(v) = \gamma_0$ and $v \neq w$ and the proof of $w < v$ in $\Omega$ can be performed as in the proof of lemma 4.2.

**Proof of Theorem 1.2:** The proof follows from Lemma 4.2, Lemma 4.4, Proposition 4.1 and Proposition 3.4 along with Proposition 3.2.

**Proof of Theorem 1.3:** The proof follows directly from Proposition 4.1 and [1, Theorem 1.2] with $\delta = q$ and $\beta = 0$. To see that the regularity result falls into the scope of [1, Theorem 1.2], note that $u$ is a classical solution as defined in [1, Definition 1].

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A global multiplicity result for singular and critical nonlocal elliptic equation

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