Singularities of a surface given by Kenmotsu-type formula in Euclidean three-space

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Abstract
We study singularities of surfaces which are given by Kenmotsu-type formula with prescribed unbounded mean curvature.

Keywords Singularities · Prescribed mean curvature · Wave fronts · Frontals · Cuspidal edges

Mathematics Subject Classification 57R45 · 53A10

1 Introduction

In [14], Kenmotsu gave a formula which describes immersed surfaces in the Euclidean 3-space by prescribed mean curvature and Gauss map. Furthermore, it is generalized to the Lorentz–Minkowski 3-space [1,18]. These formulas are considered important

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in Euclidean or Lorentzian surface theory since they are similar to the Weierstrass representation for minimal surfaces. A coordinate free formula of these formulas is obtained by Kokubu [15], unifying them in a single one, which is called *Kenmotsu-type formula*.

On the other hand, in the recent decades, there are several articles concerning differential geometry of singular curves and surfaces, namely, curves and surfaces with singular points, in the 2 and 3 dimensional Euclidean spaces [3–5, 7, 8, 10, 19, 21–23]. Especially, wave fronts (fronts) is a class of surfaces with singularities, where the unit normal vector is well-defined even on the set of singular points. If a surface $f$ has a singularity, the mean curvature $H$ may diverge. In the case that $f$ is a front, behaviors, in particular boundedness of $H$ near singular points are investigated in [19] and [23].

It is natural to expect that there is a formula which describes singular surfaces by unbounded mean curvature and Gauss map. In this work, we construct a Kenmotsu-type formula for singular surfaces of prescribed unbounded mean curvature and Gauss map by a little modification of Kokubu’s formula (see Theorem 3.1). Furthermore, we study singularities of these surfaces and their geometric invariants.

## 2 Fronts, their geometric invariants and Kossowski metric

### 2.1 Fronts and their mean curvatures

A map-germ $f : (R^2, p) \rightarrow (R^3, 0)$ at $p$ is a called *frontal* if there exists a map (called the *unit normal vector field*) $v : (R^2, p) \rightarrow (R^3, 0)$ satisfying $|v| = 1$ and $\langle df(X), v \rangle = 0$ holds identically, where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product on $R^3$. A frontal is a *front* if the pair $(f, v)$ is an immersion. We call the function

$$\det(f_u, f_v, v)$$

the *signed area density function*, and a function is called *singularity identifier* if it is a non-zero functional multiple of the signed area density function. A *cuspidal edge* is a map-germ $(R^2, p) \rightarrow (R^3, 0)$ at $p$ which is $A$-equivalent to the map-germ $(u, v) \mapsto (u, v^2, v^3)$ at $(0, 0)$. A map-germ $(R^2, p) \rightarrow (R^3, 0)$ which is $A$-equivalent to $(u, v) \mapsto (u, 4v^3 + 2uv, 3v^2 + uv^2)$ (respectively, $(u, v) \mapsto (u, 5v^4 + 2uv, 4v^5 + uv^2 - u^2)$) is called a *swallowtail* (respectively, a *cuspidal butterfly*). Furthermore, a map-germ which is $A$-equivalent to $(u, v) \mapsto (u, 2v^3 + u^2v, 3v^2 + u^2v^2)$ (respectively $(u, v) \mapsto (u, 2v^3 - u^2v, 3v^4 - u^2v^2)$) is called a *cuspidal lips* (respectively, *cuspidal beaks*). These map-germs are fronts and it is known that the generic singularities of fronts are cuspidal edges and swallowtails. Furthermore, cuspidal butterflies, cuspidal lips and beaks in addition above two are the generic singularities of one-parameter families of fronts [2].

Let $f : (R^2, p) \rightarrow (R^3, 0)$ be a front-germ and $v$ the unit normal vector field. The set of singular points of $f$ is denoted by $S(f)$. A singular point $p$ of $f$ is said to be of *corank one* if $\text{rank } df_p = 1$. Let $p$ be a corank one singular point of a front $f$. Then there exists a non-zero vector field $\eta$ near $p$ such that $\eta(q)$ generates the

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kernel of $df_q$ for any $q \in S(f)$. We call $\eta$ the null vector field. A singular point $p$ of $f$ is non-degenerate if $d\lambda_p \neq 0$, where $\lambda$ is a singularity identifier. Otherwise, it is degenerate. Let $p$ be a non-degenerate singular point. Then there exists a regular curve-germ $\gamma: (\mathbb{R}, 0) \to (\mathbb{R}^2, p)$ such that the image of $\gamma$ coincides with $S(f)$. One can easily see that a non-degenerate singular point is corank one. Thus the restriction of the non-zero vector field $\eta|_{S(f)}$ near $p$ can be parameterized by the parameter $t$ of $\gamma$.

We set

$$\varphi(t) = \det(\gamma'(t), \eta(t)). \quad (2.1)$$

A non-degenerate singular point is said to be of the first kind if $\varphi(0) \neq 0$, and a non-degenerate singular point is said to be of the $k$-th kind $k \geq 2$ if $\varphi(0) = 0$, $\varphi^{(j)}(0) = 0$ $(j = 1, \ldots, k-2)$, and $\varphi^{(k-1)}(0) \neq 0$. Then the following fact holds.

**Fact 2.1** ([16, Proposition 1.3], [11, Corollary A.9]) Let $p$ be a non-degenerate singular point of a front $f$. Then

(i) $f$ at $p$ is a cuspidal edge if and only if $p$ is a singular point of the first kind.

(ii) $f$ at $p$ is a swallowtail if and only if $p$ is a singular point of the second kind.

(iii) $f$ at $p$ is a cuspidal butterfly if and only if $p$ is a singular point of the third kind.

On the other hand, cuspidal lips and beaks are examples which are degenerate singularities. The following fact holds.

**Fact 2.2** ([12, Theorem A.1]) Let $p$ be a corank one singular point of a front $f$. Then

(i) $f$ at $p$ is a cuspidal lips if and only if $\det \text{Hess} \lambda(p) > 0$.

(ii) $f$ at $p$ is a cuspidal beaks if and only if $\det \text{Hess} \lambda(p) < 0$ and $\eta \lambda(p) \neq 0$,

where $\eta$ is a null-vector field, and $\lambda$ is a singularity identifier.

For a front $f$, the mean curvature $H$ diverges near non-degenerate singular points. Furthermore, the following fact is known.

**Fact 2.3** ([19, Propositions 3.8, 3.11, 4.2 and (4.6)]) Let $p$ be a non-degenerate singular point of a front $f$. Then $\hat{H} = \lambda H$ is a $C^\infty$-function, and $\hat{H}(p) \neq 0$, where $\lambda$ is a singularity identifier.

For degenerate singularities with some conditions, which include the cuspidal lips and beaks, we have the same statement.

**Proposition 2.4** Let $p$ be a corank one singular point of a front $f$, satisfying $d\lambda_p = 0$, where $\lambda$ is a singularity identifier. Then $\hat{H} = \lambda H$ is a $C^\infty$-function near $p$, and $\hat{H}(p) \neq 0$.

**Proof** Since $p$ is a corank one singular point of $f$, we may assume that $p = (0, 0)$ and $f$ has the form

$$f(u, v) = (u, f_2(u, v), f_3(u, v)), \quad (d(f_2)(0, 0) = d(f_3)(0, 0) = 0).$$

Then we see that we can take $\partial_u$ as a null vector field, and the unit normal vector $v = (v_1, v_2, v_3)$ satisfies $v_1(p) = 0$. Thus, we may further assume that $v_2(p) = 0$, and...
\(v_3(p) = 1\) by a rotation of \(\mathbb{R}^3\). Since \((f_v(u, v), v(u, v)) = (f_2)_v(u, v)v_2(u, v) + (f_3)_v(u, v)v_3(u, v) = 0\), there exists a function \(l(u, v)\) such that

\[
(f_3)_v(u, v) = l(u, v)(f_2)_v(u, v).
\]

We set \(g(u, v) = (0, 1, l(u, v))\) and \(q(u, v) = (f_2)_v(u, v)\). Then the first fundamental matrix \(I\) and the second fundamental matrix \(\mathbb{II}\) are

\[
I = \begin{pmatrix}
1 + (f_2)_u^2 + (f_3)_u^2 & q(f_u, g) \\
q(f_u, g) & q^2(1 + l^2)
\end{pmatrix},
\]

\[
\mathbb{II} = \begin{pmatrix}
-\langle f_u, v_u \rangle & -q\langle g, v_u \rangle \\
-q\langle g, v_u \rangle & -q\langle g, v_v \rangle
\end{pmatrix}.
\]

Hence the mean curvature can be computed as

\[
H = \frac{(1 + (f_2)_u^2 + (f_3)_u^2)(g, v_v) + q*}{2q((1 + (f_2)_u^2 + (f_3)_u^2)(1 + l^2) - \langle f_u, g \rangle^2)},
\tag{2.2}
\]

where \(*\) stands for a function which is not necessary in the later calculations. Since

\[
\det(f_u, f_v, v) = q \det(f_u, g, v),
\]

and \(\det(f_u, g, v)(p) = v_3(p) \neq 0\), we see that \(q\) is a singularity identifier. Thus, the first assertion is shown. By (2.2), to show the second assertion, showing \(\langle g, v_v \rangle(p) \neq 0\) is enough. Since \(\det(f_u, g, v)(p) \neq 0\), the vectors \(f_u(p), g(p), v(p)\) form a basis of \(\mathbb{R}^3\), and by \(\langle f_u, v_v \rangle(p) = \langle f_v, v_u \rangle(p) = 0\) and \(\langle v, v_v \rangle(p) = 0\), we see that \(\langle g, v_v \rangle(p) \neq 0\) is equivalent to \(v_v(p) \neq 0\). Since \(f\) at \(p\) is a front and \(f_v(p) = 0\), we get \(v_v(p) \neq 0\).

We remark that a map-germ \(f : (\mathbb{R}^2, p) \to (\mathbb{R}^3, 0)\) whose set of singular points is nowhere dense is a frontal if and only if the Jacobi ideal of \(f\) is principal [9, Lemma 2.3].

### 2.2 Geometric invariants of fronts

Here we review the known geometric invariants of cuspidal edges and swallowtails. Singular points of cuspidal edges are non-degenerate and several geometric invariants are defined and studied. Let \(f : (\mathbb{R}^2, p) \to (\mathbb{R}^3, 0)\) be a front and \(v\) the unit normal vector.

**Definition 2.5** Let \(p\) be a non-degenerate singular point of \(f\). A positively oriented coordinate system \((u, v)\) centered at \(p\) is said to be \(u\)-singular if \(S(f) = \{(u, v) | v = 0\}\) holds. Let \(p\) be a singular point of the first kind. A \(u\)-singular coordinate system \((u, v)\) is said to be strongly adapted if the null vector on \(S(f)\) is \(\partial_v\).
Existence of $u$-singular and strongly adapted coordinate systems are easily shown. We assume that $(u, v)$ is a strongly adapted coordinate system. We set

$$\kappa_s(u) = \text{sgn} \left( \det(f_{uu}, f_{vv}, v) \right) \frac{\det(f_{uu}, f_{uu}, v)}{|f_u|^3}_{(u,0)},$$

$$\kappa_v(u) = \left( \frac{f_{uu}}{|f_u|^2} \right)_{(u,0)},$$

$$\kappa_t(u) = \frac{\det(f_{uu}, f_{uu}, f_{vvv}) - \det(f_{uu}, f_{uu}, f_{uu}) \langle f_{uu}, f_{vv} \rangle}{|f_u|^5 \langle f_{uu} \rangle^2} |f_u \times f_{vv}|_{(u,0)},$$

$$\kappa_c(u) = \frac{|f_u|^3/2 \det(f_{uu}, f_{vv}, f_{vvv})}{|f_u \times f_{vv}|^{5/2}} |f_{uu}|_{(u,0)}.$$

All these functions are differential geometric invariants, and $\kappa_s$ is called the singular curvature, $\kappa_v$ is called the limiting normal curvature, $\kappa_t$ is called the cuspidal torsion (cusp-directional torsion), and $\kappa_c$ is called the cuspidal curvature. See [19,20,23] for detail. Next we assume that $p$ is a singular point of the $k$-th kind, $k \geq 2$. We take a $u$-singular coordinate system $(u, v)$. Since $p$ is a singular point of the $k$-th kind, $k \geq 2$, $\partial_u$ is a null vector at $p$. Taking $k = 2$, we set

$$\mu_c(p) = -\frac{|f_u|^3 \langle f_{uu}, v_u \rangle}{|f_{uu} \times f_{vv}|^2} |f_{uu}|_p,$$

$$\tau_s(p) = -\frac{|\det(f_{uu}, f_{uu}, v)|}{|f_{uu}|^{5/2}} |f_{uu}|_p.$$ (2.3)

The constant $\mu_c$ is called the normalized cuspidal curvature, and it relates the boundedness of the mean curvature. The constant $\tau_s$ is called the limiting singular curvature, and it measures the wideness of the cusp of the swallowtail. See [19, Section 4] for detail.

### 2.3 Kossowski metric

Let $f : (R^2, p) \to (R^3, 0)$ be a front-germ. Then the induced metric $f^*\langle \langle \cdot , \cdot \rangle \rangle$ on $R^2$ is positive semi-definite. An intrinsic formulation of this metric is called the Kossowski metric and studied [6,17]. Let $g$ be a positive semi-definite metric-germ at $p \in R^2$. A point $q$ is called a singular point of $g$ if the metric $g$ is not positive definite at $q$. The set of singular points of $g$ is denoted by $S(g)$. For a singular point $q \in S(g)$, the subspace

$$N_q = \{ v \in T_q R^2 \mid g(v, w) = 0 \text{ for all } w \in T_q R^2 \}$$

is called the null space at $q$, and a non-zero vector in $N_q$ is called null vector at $q$. A singular point $q$ of $g$ is said to be corank one if the dimension of $N_q$ is one. If $q$ is of corank one, then there exists a non-zero vector field $\eta$ which is a generator of $N_q$. We call $\eta$ a null vector field.
Definition 2.6 ([6, Section 2]) (1) The metric $g$ is admissible if there exists a coordinate system $(u, v)$ on a neighborhood of 0 in $\mathbb{R}^2$ such that

$$F = G = 0, \quad E_v = 2F_u, \quad G_u = G_v = 0$$

hold on $S(g)$, where

$$E = g(\partial_u, \partial_u), \quad F = g(\partial_u, \partial_v), \quad G = g(\partial_v, \partial_v). \quad (2.5)$$

(2) An admissible metric is called a frontal metric if there exists a coordinate system $(u, v)$ on a neighborhood of 0 in $\mathbb{R}^2$ and there exists a function $\lambda$ such that

$$EG - F^2 = \lambda^2 \quad (2.6)$$

holds on $(\mathbb{R}^2, 0)$. (3) A singular point $p$ of an admissible metric $g$ is called non-degenerate if

$$d\lambda_p \neq 0$$

holds, where $\lambda$ is the function as in (2.6). (4) A frontal metric is called a Kossowski metric if all singular points are non-degenerate.

We remark that the definition (1) of Definition 2.6 is not the same as the original. The original definition is coordinate free. See [17, page 103] and [6, Definition 2.3].

Proposition 2.7 ([6, Proposition 2.10]) Let $f : (\mathbb{R}^2, p) \to (\mathbb{R}^3, 0)$ be a front. Then the induced metric $f^* (\langle \cdot, \cdot \rangle)$ is a frontal metric. If $p$ is a non-degenerate singular point, then it is a Kossowski metric.

Let $p$ be a non-degenerate singular point. Like as the case of fronts, we give the following definition. Since $p$ is non-degenerate, there exists a regular curve-germ $\gamma : (\mathbb{R}, 0) \to (\mathbb{R}^2, p)$ such that the image of $\gamma$ coincides with the set of singular points $S(g)$, and there exists a non-zero vector field $\eta$ near $p$ such that $\eta(q)$ is a null vector for $q \in S(g)$. We set

$$\psi(t) = \det(\gamma'(t), \eta(t)). \quad (2.7)$$

A non-degenerate singular point is called an $A_2$-point if $\psi(0) \neq 0$, and a non-degenerate singular point is called an $A_k$-point if $\psi(0) = 0, \psi^{(j)}(0) = 0$ $(j = 1, \ldots, k - 3)$, and $\psi^{(k-2)}(0) \neq 0$. Let $p$ be a corank one but not a non-degenerate singular point (i.e., corank one and $d\lambda_p = 0$) of the frontal metric $g$. Then $p$ is said to be Morse type if $\det \text{Hess} \lambda(p) \neq 0$ and $\eta \eta \lambda(p) \neq 0$, where $\lambda$ is the function defined by (2.6) in Definition 2.6. We remark that the condition of Morse type has an additional condition $\eta \eta \lambda(p) \neq 0$ not only $\lambda$ has a Morse type critical point at $p$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure.png}
\caption{Caption for the figure.}
\end{figure}
3 Kenmotsu-type formula for singular surfaces

3.1 Kenmotsu-type formula

With the terminology in Sect. 2.3, we give a Kenmotsu-type formula following Kokubu [15]. Let \( U \subset \mathbb{R}^2 \) be a simply-connected open set, and \( g \) be a positive semi-definite metric on \( U \). Set \( E, F \) and \( G \) as in (2.5) for a coordinate system \((u, v)\). We assume that \( g \) is a frontal metric, and take a function \( \lambda \) defined by (2.6). Although \( \lambda \) has two choices, in what follows we fix one of them. Let \( H : U \setminus S(g) \to \mathbb{R} \setminus \{0\} \) be a \( C^\infty \)-function which satisfies that

\[
\frac{1}{-2H(u, v)} = \frac{\lambda(u, v)}{\hat{H}(u, v)} \tag{3.1}
\]

for a non-zero function \( \hat{H} \). Let \( \nu : U \to S^2 \) be a unit vector valued function, where \( S^2 = \{X \in \mathbb{R}^3 \mid ||X|| = 1\} \). We assume \( g \) is of corank one at \( p \). Let \( \eta \) be a null vector of \( g \) at \( p \). The pair \((g, \nu)\) is called a front pair if

\[
\eta \nu(p) \neq 0. \tag{3.2}
\]

It is easy to see that if \( g \) is a induced metric of a front \( f \) and \( \nu \) is its unit normal vector, then \((g, \nu)\) is a front pair. The following theorem gives a kind of recipe to obtain a surface with prescribed mean curvature and Kossowski metric. Since Kossowski metric contains information of the class of singularity, one can obtain a desired surface with singular points which has prescribed mean curvature and given types of singular points.

**Theorem 3.1** Let \( g \) be a Kossowski metric (respectively, frontal metric) on a simply-connected open set \( U (\subset \mathbb{R}^2) \), and let \( p \in U \) be an \( A_{k+1} \)-point (respectively, a Morse type singular point) of \( g \). Let \( \lambda \) a function defined by (2.6), and let \( H \) be a function satisfying (3.1) for some non-zero function \( \hat{H} \). Let \( \nu \) be a unit vector valued map so that \((g, \nu)\) is a front pair and \( d\nu \neq 0 \) on \( U \). Assume that \( g, H, \nu \) satisfy the integrability condition

\[
\frac{\partial}{\partial v} \left( \frac{1}{H} (\lambda v_u + F v \times v_u - E v \times v_v) \right) = \frac{\partial}{\partial u} \left( \frac{1}{H} (\lambda v_v + G v \times v_u - F v \times v_v) \right), \tag{3.3}
\]

and set

\[
f(u, v) = \int \frac{1}{H} \left( (\lambda v_u + F v \times v_u - E v \times v_v) \right) du + \left( \lambda v_v + G v \times v_u - F v \times v_v \right) dv. \tag{3.4}
\]

Then \( f : U \to \mathbb{R}^3 \) is a front, and \( p \) is a \( k \)-th kind singular point (respectively, cuspidal lips/beaks). Moreover, the mean curvature of \( f \) on the set of regular points coincides with \( H \), the Gauss map is \( \nu \), it holds that \( S(f) = S(g) \), and the induced metric \( f^*(\langle \cdot, \cdot \rangle) \) is proportional to \( g \).
\textbf{Proof} Since
\begin{equation}
    f_u = \frac{1}{H} \left( \lambda v_u + F v \times v_u - E v \times v_v \right), \quad f_v = \frac{1}{H} \left( \lambda v_v + G v \times v_u - F v \times v_v \right), \quad (3.5)
\end{equation}

\( v \) gives the unit normal vector of \( f \). Setting
\begin{align*}
P &= \langle v_u, v_u \rangle, \quad Q = \langle v_u, v_v \rangle, \quad R = \langle v_v, v_v \rangle, \quad D = \det(v_u, v_v, v), \\
E_f &= \langle f_u, f_u \rangle, \quad F_f = \langle f_u, f_v \rangle, \quad G_f = \langle f_v, f_v \rangle, \quad (3.6)
\end{align*}

and
\begin{equation}
    X = 2\lambda D + GP - 2FQ + ER,
\end{equation}

we have
\begin{align*}
    E_f &= \frac{X}{H^2} E, \quad F_f = \frac{X}{H^2} F, \quad G_f = \frac{X}{H^2} G, \\
    L_f &= -\frac{1}{H} (\lambda P + ED), \quad M_f = -\frac{1}{H} (\lambda Q + FD), \quad N_f = -\frac{1}{H} (\lambda R + GD).
\end{align*}

Thus, we get
\begin{equation}
    H_f = \frac{E_f N_f - 2F_f M_f + G_f L_f}{2(EG - F^2_f)} = -\frac{\hat{H}}{2\lambda} = H
\end{equation}
on the set of non-singular points by using equations \( \lambda^2 = EG - F^2 \) and \( GP - 2FQ + ER = X - 2\lambda D \), and
\begin{equation}
    \det(f_u, f_v, v) = \frac{\lambda}{H^2} X.
\end{equation}

We now consider the function \( X \). Since \( g \) is an frontal metric on \( U \), it follows that \( \eta = \partial_v \), namely, \( E \neq 0 \) and \( \lambda = F = G = 0 \) on \( S(g) \) (cf. Definition 2.6). Since \( (g, v) \) is a front pair, \( \eta v = v_v \neq 0 \) on \( S(g) \), and hence \( R \neq 0 \) on \( S(g) \). Therefore \( X \neq 0 \) on \( S(g) \). Thus we see that \( f \) is a front. Moreover, since \( \eta v \neq 0 \), in particular, \( dv \neq 0 \) on \( U \), the function \( X \) does not vanish on \( U \). Thus \( S(f) = S(g) \) holds on \( U \). Further, it follows that \( \eta f = f_v = 0 \) on \( S(f) \). Thus we can also regard \( \eta \) as a null vector field of \( f \).

We consider types of singularities of \( f \). First we assume that \( g \) is a Kossowski metric on \( U \) and \( p \) is an \( Ak+1 \)-point of \( g \). Then \( d\lambda p \neq 0 \). Thus there exists a regular curve \( \gamma(t) \) such that \( \gamma \) parametrizes \( S(g) = S(f) \) on \( U \). In this situation, the functions \( \phi(t) \) and \( \psi(t) \) as in (2.1) and (2.7), respectively, are the same. By definitions of \( k \)-th kind and \( Ak+1 \)-point, it holds that \( f \) has a \( k \)-th kind singularity at \( p \).
We next assume that $p$ is a Morse type singular point of $g$. Then $d\lambda_p = 0$, $\det\text{Hess}\lambda(p) \neq 0$ and $\eta \eta \lambda(p) \neq 0$ hold. On the other hand, by the above calculation, it holds that $\det(f_u, f_v, v) = \lambda \cdot (\text{non-zero function})$. Thus by Fact 2.2, $f$ has a cuspidal lips or a cuspidal beaks at $p$.

Since the function $\lambda$ defined by (2.6) has the plus-minus ambiguity, by (3.4) the surface $f$ also has two possibilities, say $f_+$ and $f_-$, for a semi-definite metric $g$ and single $H, v$. In this case, considering $(u, -v)$ instead of $(u, v)$, then we see that $-\lambda$ satisfies the integrability condition, and the resulting surface $f_-(u, v)$ coincides with $f_+(u, -v)$.

**Example 3.2** Let us set $J = (-1/2, 1/2)$ and

$$H(v) = -\frac{1}{2 \sinh v}(-3 + \cosh 2v), \quad E(v) = 1/\cosh^2 v, \quad F(v) = 0, \quad G(v) = \sinh^2 v/\cosh^2 v, \quad \lambda(v) = \sinh v/\cosh^2 v$$

for $v \in J$. Then we see $\lambda(v)^2 = E(v)G(v) - F(v)^2$, and $\dot{H}(v) = -2H(v)\lambda(v) = (-3 + \cosh 2v)/\cosh^2 v$ is a non-zero $C^\infty$-function on $J$. Regarding these functions as functions on $\mathbb{R} \times J$, we can see that the null vector field of $g = Edu^2 + 2Fdu dv + Gdv^2$ is $\partial_v$, and each point on $\{(u, 0)\}$ is an $A_2$-singularity. Let us set $v(u, v) = (\cos u \sinh v, \sin u \sinh v, 1)/\cosh v$. Then $\eta v \neq 0$ holds. Moreover, these functions satisfy the condition (3.4). By Theorem 3.1, there exists a surface $f : \mathbb{R} \times J \to \mathbb{R}^3$ whose mean curvature is $H$ and $S(f) = \{(u, 0)\}$. Since $(u, 0)$ is an $A_2$-type singularity of $g$, it holds that $f$ at $(u, 0)$ is a cuspidal edge.

Here, we give an application of Theorem 3.1. Let $I \subset \mathbb{R}$ be an interval, and $S \subset I$ be a discrete subset. Taking two functions $H : I \setminus S \to \mathbb{R} \setminus \{0\}$ and $l : I \to \mathbb{R}$ satisfying $l^{-1}(0) = S$ and $lH$ is a non-zero $C^\infty$ function on $I$. We set $\hat{H} = -2lH$, and $v(u, v) = (\sin \theta(u), -\cos \theta(u), 0)$, where $\theta(u)$ is a primitive function of $\hat{H}$, and $(u, v) \in I \times \mathbb{R}. We also set

$$E(u, v) = l(u)^2, \quad F(u, v) = 0, \quad G(u, v) = 1 \quad ((u, v) \in I \times \mathbb{R}).$$

Then these functions satisfy the condition (3.4). By Theorem 3.1, there exists a surface $f : I \times \mathbb{R} \to \mathbb{R}^3$ whose mean curvature is $H$ and $f = (f_1(u), f_2(u), v)$, where $f_1$ (respectively, $f_2$) is a primitive function of $l(u) \cos \theta(u)$ (respectively, $l(u) \sin \theta(u)$). Hence for any function pair $H : I \setminus S \to \mathbb{R} \setminus \{0\}$, and $l : I \to \mathbb{R}$ satisfying $l^{-1}(0) = S$ and $lH$ is a non-zero $C^\infty$ function on $I$, there exists a surface which is symmetric with respect to a translation, whose mean curvature is $H$ and the singular set is $S \times \mathbb{R}$. Setting $H = -1/\sin u$, and $l = (\sin u)/2$ (respectively, $l = (11/10) \sin u$), then we obtain a surface $f(u, v) = (-(\cos^2 u)/4, (u - \cos u \sin u)/4, v)$ (respectively, $f(u, v) = (11(8 \cos u_1 - 3 \cos u_2))/192, 11(8 \sin u_1 - 3 \sin u_2)/192, v)$, where $u_1 = 6u/5, u_2 = 16u/5$ which is illustrated in Fig. 1, left (respectively, right).
3.2 Invariants of cuspidal edge

In this section, we calculate geometric invariants of $f$ for the case that $f$ is a cuspidal edge. We take $g$, $H$, $v$ satisfying the assumption of Theorem 3.1, and fix a function $\lambda$ defined by (2.6). Let $f$ be the front as in (3.4). We assume further that $p$ is a singular point of the first kind of $f$, namely, $f$ at $p$ is a cuspidal edge. Similar to Definition 2.5, we define coordinate systems.

**Definition 3.3** Let $p$ be a non-degenerate singular point of a metric $g$. A positively oriented coordinate system $(u, v)$ is said to be $u$-singular if the set of singular points satisfies $S(g) = \{(u, v) | v = 0\}$. Let $p$ be an $A_2$-type singular point. A $u$-singular coordinate system $(u, v)$ centered at $p$ is said to be strongly adapted if the null vector on $S(g)$ is $\partial_v$.

See [6, Section 2] for existence of a strongly adapted coordinate system. Then we have the following proposition.

**Proposition 3.4** Let $p$ be an $A_2$-type singular point, and $(u, v)$ a strongly adapted coordinate system. Then the invariants $\kappa_v$, $\kappa_s$, $\kappa_c$ and $\kappa_t$ can be calculated as

$$
\kappa_v(u) = -\frac{D\hat{H}}{ER} \bigg|_{(u,0)},
$$

$$
\kappa_s(u) = \text{sgn} \left( \det(f_u, f_{vv}, v) \hat{H} \right) \frac{\det(v, v_u, v_{uv})\hat{H}}{ER^{3/2}} \bigg|_{(u,0)},
$$

$$
\kappa_c(u) = -\text{sgn} \left( \hat{H} \right) \frac{2|\hat{H}|^{1/2}R^{1/4}}{|\lambda_v|^{1/2}} \bigg|_{(u,0)},
$$

$$
\kappa_t(u) = -\frac{\hat{H}Q}{ER} \bigg|_{(u,0)}.
$$

Here, $E, F, G$ are defined in (2.5), and $P, Q, R, D$ are defined in (3.6).

**Proof** Since $(u, v)$ is a strongly adapted coordinate system, it holds that $f_v = 0$ on the $u$-axis. Thus $F = G = 0$ and $F_u = G_u = G_v = 0$ on the $u$-axis, and $\lambda$ is a non-zero...
functional multiplication of \( v \). So, \( \lambda = \lambda_u = 0 \) on the \( u \)-axis. We set \( 1/\hat{H} = a \). Firstly we consider \( \kappa_v \) and \( \kappa_s \) of \( f \). It holds that \( f_u = -aE v \times v_u \) on the \( u \)-axis. So,

\[
|f_u| = |a|E R^{1/2}
\]

(3.11) on the \( u \)-axis. Since

\[
f_{uu} = (\lambda_u a + \lambda a_u)v_u + \lambda a v_{uu} + (a F_u + a_u F)v \times v_u \\
+ a F(v \times v_u)_u - (a E_u + a_u E)v \times v_v - a E(v \times v_v)_u,
\]

and \( F_u = 0 \) on the \( u \)-axis, we have

\[
f_{uu} = -(a E_u + a_u E)v \times v_v - a E(v_u \times v_v + v \times v_{uv})
\]

(3.12) on the \( u \)-axis. Thus by (3.11) and (3.12), we have \( \kappa_v(u) = -D/(a E R)(u, 0) \), and it proves (3.7). Furthermore, since

\[
f_u \times v = -aE v_v
\]

on the \( u \)-axis, it follows that

\[
\det(f_u, f_{uu}, v) = -\langle f_{uu}, f_u \times v \rangle = a^2 E^2 \det(v, v_v, v_{uv})
\]

on the \( u \)-axis. Therefore,

\[
\kappa_s(u) = \text{sgn}(\det(f_u, f_{uu}, v)) \frac{\det(v, v_v, v_{uv})}{|a|E R^{3/2}} (u, 0),
\]

and it proves (3.8).

Next we consider \( \kappa_c \). By direct calculations, we have:

\[
f_{vv} = (\lambda_v a + \lambda a_v)v_v + \lambda a v_{vv} + (a G_v + a_v G)v \times v_u \\
+ a G(v \times v_u)_v - (a F_v + a_v F)v \times v_v - a F(v \times v_v)_v,
\]

\[
f_{vvv} = (\lambda_{vv} a + 2\lambda_v a_v + \lambda a_{vv})v_v + 2(\lambda a_v + \lambda_v a)v_{vv} + \lambda a v_{vvv} \\
+ (a G_{vv} + 2a_v G_v + a_v G_v) v \times v_u + 2(a G_v + a_v G)(v \times v_u)_v \\
+ a G(v \times v_u)_vv - (a F_{vv} + 2a_v F_v + a_v F_v)v \times v_v \\
- 2(a F_v + a_v F)(v \times v_v)_v - a F(v \times v_v)_vv.
\]

So,

\[
f_{vv} = \lambda_v a v_v - a F_v v \times v_v,
\]

(3.13)

\[
f_{vvv} = (\lambda_{vv} a + 2\lambda_v a_v + \lambda a_{vv})v_v + 2\lambda_v a v_{vv} + a G_{vv} v \times v_u \\
- (a F_{vv} + 2a_v F_v)v \times v_v - 2a F_v(v \times v_v)_v
\]

(3.14)
on the $u$-axis. Therefore, it holds that

$$f_u \times f_{vv} = -a^2 E \lambda_v (v \times v_v) \times v_v = a^2 E \lambda_v R v$$  \hspace{2cm} (3.15)

on the $u$-axis. Noticing that $R = -\langle v, v_{vv} \rangle$, we have

$$\det(f_u, f_{vv}, f_{vvv}) = -2a^3 E \lambda_v^2 R^2.$$

On the other hand, by (3.11) and (3.15), it follows that

$$|f_u|^3/2 = (|a| E R^{1/2})^{3/2}, \quad |f_u \times f_{vv}|^{5/2} = a^4 E^{5/2} \lambda_v^2 R^{5/2} |a||\lambda_v|^{1/2},$$

on the $u$-axis. Hence we obtain

$$\kappa_c(u) = \frac{-2(|a| E R^{1/2})^{3/2} a^3 E \lambda_v^2 R^2}{a^4 E^{5/2} \lambda_v^2 R^5 |a||\lambda_v|^{1/2}} (u, 0)$$

$$= \frac{-2a^3 \lambda_v R^{1/4}}{|a||\lambda_v|^{1/2}} (u, 0) = \frac{-2a^3 \lambda_v R^{1/4}}{|a||\lambda_v|^{1/2}} (u, 0)$$

which proves (3.9). Next we consider $\kappa_t$. Since

$$f_{uv} = a \lambda_v v_u + a F_v v \times v_u - (a E) v \times v_v + \lambda (a v_u + a v_{uv})$$
$$+ F (a v_u \times v_u - a v_v \times v_v + a v \times v_{uv}) - a E v \times v_{vv},$$

using (3.12), (3.15), and noticing that $Q = -\langle v, v_{uv} \rangle$, we obtain

$$\langle f_u, f_{vv} \rangle = a^2 E F_v R$$

$$\det(f_u, f_{vv}, f_{uvv}) = \left( a^2 E \lambda_v R v, 2a \lambda_v v_{uv} - 2a F_v v_u \times v_v - a E v_v \times v_{vv} \right)$$

$$= -a^3 \lambda_v E R (2\lambda_v Q + 2F_v D + E \det(v, v_v, v_{vv})), \hspace{2cm} (3.17)$$

$$\det(f_u, f_{vv}, f_{uu}) = \left( a^2 E \lambda_v R v, -a E v_u \times v_v \right) = -a^3 \lambda_v E^2 RD \hspace{2cm} (3.18)$$

on the $u$-axis. By (3.11), (3.15), (3.16), (3.17) and (3.18), we have

$$\kappa_t(u) = \frac{-a^3 \lambda_v E R (2\lambda_v Q + 2F_v D + E \det(v, v_v, v_{vv}))}{a^4 \lambda_v^2 E^2 R^2} (u, 0)$$

$$+ \frac{(a^3 \lambda_v E^2 RD)(a^2 E F_v R)}{|a^2 E R||a^4 \lambda_v^2 E^2 R^2|} (u, 0)$$

$$= \frac{-2\lambda_v Q - F_v D - E \det(v, v_v, v_{vv})}{a \lambda_v E R} (u, 0). \hspace{2cm} (3.19)$$

On the other hand, by the integrability condition, we have

$$-a_v E v \times v_v + a (\lambda_v v_u + F_v v \times v_u - E_v v \times v_v - E v \times v_{vv}) = 0 \hspace{2cm} (3.20)$$
on the $u$-axis. Taking the inner product with $\nu_v$, we have
\[
\lambda_v Q + F_v D = -E \det(v, v_v, v_{vv}).
\]
This equation together with (3.19) proves the assertion. \qed

We have the following corollary.

**Corollary 3.5** Under the same assumption in Proposition 3.4, the curve $f(u, 0)$ is a part of a straight line if and only if
\[
\det(v_u, v_v, v)(u, 0) = \det(v, v_v, v_{uv})(u, 0) = 0.
\]
The curve $f(u, 0)$ is a plane curve if and only if
\[
\frac{ER^2}{\hat{H} \det(v, v_v, v_{uv})^2 + D^2 R} \chi + \hat{H} - \lambda_v Q + 2E \frac{\det(v, v_v, v_{uv})}{\lambda_v ER} \bigg|_{(u,0)} = 0,
\]
where
\[
V = \frac{\det(v, v_v, v_{uv})}{R^{1/2}} \left( -\frac{D \hat{H}}{ER} \right)_u + \frac{D \hat{H}}{ER} \left( \frac{\det(v, v_v, v_{uv})}{ER^{3/2}} \hat{H} \right)_u.
\]

**Proof** We set $\hat{\gamma} = f(\gamma)$. Since the curvature $\kappa$ of $\hat{\gamma}$ as a space curve satisfies $\kappa^2 = \kappa_s^2 + \kappa_v^2$ ([20, Theorem 4.4]), $\kappa = 0$ is equivalent to $\kappa_s = \kappa_v = 0$. Since $\hat{H} \neq 0$, we have the first assertion by Proposition 3.4.

We show the second assertion. Since the torsion $\tau$ of $\hat{\gamma}$ satisfies
\[
\tau = \frac{\kappa_s \kappa'_v - \kappa'_s \kappa_v}{\kappa_s^2 + \kappa_v^2} + \kappa_l
\]
(see [20, Theorem 4.4], [13, (5.11)]), we have the second assertion by Proposition 3.4. \qed

### 3.3 Bounded of Gaussian curvature case

In this section we study the case $f$ has the bounded Gaussian curvature. According to [19, Corollary 3.12] (see also [23, Theorem 3.1]), it is equivalent to $\kappa_v = 0$ on the set of singular points. We assume that $(U, u, v)$ is an adapted coordinate system. By Proposition 3.4, $\kappa_v = 0$ if and only if $D = 0$. Since $f$ is a front, it is equivalent to that $v_u$ is parallel to $v_v$, say $v_u = \alpha v_v$ on the $u$-axis. If $D_v \neq 0$ and $\alpha \neq 0$, then $v$ is a fold. Here, we shall see the case $\alpha = 0$. This implies that $v_u = 0$, and by the integrability condition, $\det(v, v_v, v_{uv}) = 0$ at the origin. Then by Proposition 3.4, $\kappa_l = 0$ at the origin. Furthermore, $\kappa_s = 0$ if and only if $D_v = \eta D = 0$ at the origin.
3.4 Invariants of swallowtail

In this section, we calculate geometric invariants of $f$ for the case that $f$ is a singular point of the $k$-th kind ($k \geq 2$). We take $g$, $H$, $\nu$ satisfying the assumption of Theorem 3.1, and fix a function $\lambda$ defined by (2.6). Let $f$ be the front as in (3.4). We assume further that $p$ is an $A_{k+1}$-type singular point of $g$ ($k \geq 2$). We take a $u$-singular coordinate system centered at $p$. Then we have the following proposition.

**Proposition 3.6** The invariants $\mu_c$ and $\tau_s$ ($k = 2$) can be calculated as

$$\mu_c(p) = -\text{sgn}(\hat{H}(p)) \frac{G(p) P(p)^{1/2}}{\lambda_v(p) \hat{H}(p)^2},$$

$$\tau_s(p) = \frac{\hat{H}(p)^{1/2} |2F_u(p) \det(\nu, \nu_u, \nu_{uu})(p) - E_{uu}(p) D(p)|}{|F_u(p)|^{3/2} P(p)^{5/4}},$$

where $P$ and $D$ are functions as in (3.6).

**Proof** Since $(u, v)$ is a $u$-singular coordinate system, we see $E = F = E_u = E_v = \lambda_u = 0$ at $p$. Since

$$f_{uv} = -\frac{\hat{H}_v}{\hat{H}} \left( \lambda v_u + F v \times v_u - E v \times v_v \right) + \frac{1}{\hat{H}} \left( \lambda v_v + \lambda v_{uv} + F v \times v_u + F (v_x \times v_u + v \times v_{uv}) 
-E_v v \times v_v - E v \times v_{vv} \right),$$

thus we have

$$f_{uv}(p) = \frac{1}{\hat{H}(p)}(\lambda v(p) v_u(p) + F_v(p)(v \times v_u)(p)).$$

Then by (3.5) and (2.3), we have (3.21). Next, by direct calculations, we have

$$f_{uu}(p) = \frac{F_u(p)}{\hat{H}(p)}(v \times v_u)(p),$$

$$f_{uuv}(p) = -\frac{2\hat{H}_u(p) F_u(p)}{\hat{H}(p)^2}(v \times v_u)(p) + \frac{1}{\hat{H}(p)} \left( F_{uu}(p)(v \times v_u)(p) 
+ 2F_u(p)(v \times v_{uu})(p) - E_{uu}(p)(v \times v_v)(p) \right).$$

Then by (2.4), we have (3.22). \qed
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