THE IMPLICIT CONTRIBUTION OF SLAB MODES TO THE PERPENDICULAR DIFFUSION COEFFICIENT OF PARTICLES INTERACTING WITH TWO-COMPONENT TURBULENCE

A. Shalchi

Department of Physics and Astronomy, University of Manitoba, Winnipeg, Manitoba R3T 2N2, Canada; andreasm4@yahoo.com

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ABSTRACT

We explore the transport of energetic particles in two-component turbulence in which the stochastic magnetic field is assumed to be a superposition of slab and two-dimensional modes. It is known that in magnetostatic slab turbulence, the motion of particles across the mean magnetic field is subdiffusive. If a two-dimensional component is added, diffusion is recovered. It was also shown before that in two-component turbulence, the slab modes do not explicitly contribute to the perpendicular diffusion coefficient. In the current paper, the implicit contribution of slab modes is explored and it is shown that this contribution leads to a reduction of the perpendicular diffusion coefficient. This effect improves the agreement between simulations and analytical theory. Furthermore, the obtained results are relevant for investigations of diffusive shock acceleration.

Key words: diffusion – magnetic fields – turbulence

1. INTRODUCTION

The interaction between energetic particles and a magnetized plasma is explored analytically. Examples for energetic particles are solar energetic particles and cosmic rays (CRs). If such particles move through space, their motion is usually diffusive and, therefore, a diffusive transport equation has to be used in order to describe their motion. The most important terms in such transport equations are those describing spatial diffusion along and across a mean magnetic field. Parallel and perpendicular diffusion coefficients are mostly controlled by the turbulent magnetic fields of the plasma.

First treatments of perpendicular diffusion were based on quasi-linear theory (see Jokipii 1966), which can be understood as a first-order perturbation theory. However, perturbation theory is usually based on the assumption that there is a small parameter. It is often stated in the literature (see, e.g., Schlickeiser 2002) that this small parameter is the magnetic field ratio $\delta B/B_0$ (here $\delta B$ is the total turbulent field and $B_0$ is the mean magnetic field). Apart from the problem that this magnetic field ratio is usually not small in astrophysical scenarios, it was shown in the literature that a small value of $\delta B/B_0$ alone does not justify the quasi-linear approach (see, e.g., Shalchi 2009 for a detailed discussion of the problems associated with quasi-linear theory).

Because quasi-linear theory is problematic, nonlinear theories have been developed mainly in order to describe perpendicular diffusion. Some work was already presented in the 1970s (see, e.g., Owens 1974). A breakthrough has been achieved by Matthaeus et al. (2003), where the so-called Nonlinear Guiding Center (NLGC) theory was presented. The latter theory agrees well with simulations for a specific turbulent model. However, the NLGC theory often does not provide the correct result. This is particularly the case for slab turbulence, two-component turbulence with a dominant slab contribution, or three-dimensional turbulence with small Kubo numbers$^1$ (see, e.g., Shalchi 2006; Tautz & Shalchi 2011, and Shalchi & Hussein 2014).

In Shalchi (2010), the Unified Nonlinear Transport (UNLT) theory was developed. Although the theory is based on some of the approximations used by Matthaeus et al. (2003), it contains a very different treatment of higher order correlations. UNLT theory uses an approach based on the CR Fokker-Planck equation in order to avoid simple approximations of such correlations. UNLT theory provides a nonlinear integral equation similar compared to the NLGC result, but it contains different terms in the denominator. In particular, for slab and small Kubo number turbulence, NLGC and UNLT theories provide completely different results. Furthermore, UNLT theory contains the Matthaeus et al. (1995) theory for field line random walk as a special limit.

UNLT theory provides the correct subdiffusive result for perpendicular transport in slab turbulence. Furthermore, the theory states that the slab contribution in two-component turbulence damps out subdiffusively even if a dominant two-dimensional component is added (see also Shalchi 2005 and Shalchi 2006). Therefore, slab modes do not explicitly contribute to the perpendicular diffusion coefficient. The purpose of the current paper is to explore the implicit contribution of the slab modes to the perpendicular diffusion coefficient.

The paper is organized as follows. In Section 2, we discuss different analytical theories for perpendicular diffusion. In Section 3, we developed a nonlinear diffusion theory for two-component turbulence which takes into account the implicit contribution of the slab modes. In Section 4, we present some analytical approximations which are useful in order to simplify the new integral equation and in Section 5 we show the perpendicular diffusion coefficients based on different theories and compare them with each other. In Section 6, we summarize and conclude. Furthermore, we point out which theory should

$^1$ The Kubo number occurs in investigations of three-dimensional turbulence and is defined as $K = (\delta B \cdot B_0)/(B_0 \cdot B_0)$. Here we have used characteristic length scales describing the correlation of the turbulent fields in the directions parallel and perpendicular with respect to the mean magnetic field. Furthermore, $\delta B$ is the $x$-component of the turbulent magnetic field vector and $B_0$ is the mean field.
be applied for two-component and three-dimensional turbulence, respectively.

2. DIFFERENT ANALYTICAL THEORIES FOR PERPENDICULAR TRANSPORT

In the current section, we briefly discuss three nonlinear theories for perpendicular diffusion developed in the past. Those are the original NLGC theory of Matthaeus et al. (2003), the Extended Nonlinear Guiding Center (ENLGC) theory of Shalchi (2006), and the UNLT theory of Shalchi (2010).

2.1. The Original NLGC Theory

The original NLGC theory was developed based on different assumptions and approximations. In the following, we briefly rederive this theory. This is necessary to point out the differences between different theories, but some of the assumptions and approximations used here will be employed in Section 3 in order to achieve a further improvement on the analytical description of perpendicular diffusion.

As a starting point, we can use the following equation of motion (see, e.g., Schlickeiser 2002)

\[ v_z(t) = v_z(t) \frac{\delta B_z[x(t)]}{B_0}. \]  

(1)

Strictly speaking, the velocity component \( v_z \) used here is the corresponding component of the guiding center velocity. Matthaeus et al. (2003) introduced a correction parameter \( a \) in Equation (1). In recent numerical investigations, however, it was shown that \( a = 1 \) (see Qin & Shalchi 2016). A more detailed discussion of this matter can be found below.

A diffusion coefficient can be calculated by employing the Taylor–Green–Kubo formula (see Taylor 1922; Green 1951, and Kubo 1957)

\[ \kappa_z = \int_0^\infty dt \langle v_z(t)v_z(0) \rangle \]  

(2)

and with Equation (1), we obtain

\[ \kappa_z = \frac{1}{B_0^2} \int_0^\infty dt \langle v_z(t)v_z(0) \rangle \delta B_z[x(t)] \delta B_z[x(0)]. \]  

(3)

To continue, Matthaeus et al. (2003) have employed the following approximation

\[ \langle v_z(t)v_z(0) \rangle \delta B_z[x(t)] \delta B_z[x(0)] \approx \langle v_z(t)v_z(0) \rangle \langle \delta B_z[x(t)] \delta B_z[x(0)] \rangle. \]  

(4)

It was shown analytically in several papers (see, e.g., Shalchi 2006 and Shalchi 2010) that this type of approximation does not work for slab or slab-like turbulence. Recent numerical tests have shown that this type of approximation works well for two-dimensional dominated turbulence but fails completely for slab dominated turbulence (see Qin & Shalchi 2016).

If approximation (4) is combined with Equation (3), we derive

\[ \kappa_z = \frac{1}{B_0^2} \int_0^\infty dt \langle v_z(t)v_z(0) \rangle \langle \delta B_z[x(t)] \delta B_z[x(0)] \rangle. \]  

(5)

For the parallel velocity correlation function, we employ an isotropic exponential model\(^2\)

\[ \langle v_z(t)v_z(0) \rangle = \frac{v_z^2}{3} e^{-|t|/\lambda_z}. \]  

(6)

It is more difficult to model the magnetic field correlations. First, we replace the turbulent field in Equation (5) by a Fourier representation

\[ \delta B_z(x) = \int d^3k \delta B_z(k)e^{*kx}. \]  

(7)

To proceed, we employ Corrsin’s independence hypothesis (see Corrsin 1959)

\[ \langle \delta B_m(k)\delta B_m^*(k') \rangle = P_{mm}(k)\delta(k - k'), \]  

(8)

as well as the assumption of homogeneous turbulence

\[ \langle \delta B_m(k)\delta B_m^*(k') \rangle = P_{mm}(k)\delta(k - k'), \]  

(9)

where we have used Dirac’s delta. Furthermore, we have used the magnetic correlation tensor

\[ P_{mm}(k) = \langle \delta B_m(k)\delta B_m^*(k) \rangle. \]  

(10)

By combining Equations (5)–(10), we derive

\[ \kappa_z = \frac{v_z^2}{3B_0^2} \int_0^\infty dt \ e^{-|t|/\lambda_z} \times \int d^3k \ P_{xx}(k) \langle e^{ik\Delta x} \rangle \]  

(11)

with \( \Delta x(t) = x(t) - x(0) \). To continue, we combine Equation (11) with an Gaussian distribution with vanishing mean. In this case, the three-dimensional particle distribution function has the form

\[ f(x, t) = \frac{1}{\sqrt{2\pi}((\Delta x)^2)} \frac{1}{\sqrt{2\pi}((\Delta y)^2)} \frac{1}{\sqrt{2\pi}((\Delta z)^2)} \times e^{-\frac{x^2}{2(\Delta x)^2}} e^{-\frac{y^2}{2(\Delta y)^2}} e^{-\frac{z^2}{2(\Delta z)^2}}. \]  

(12)

For the axisymmetric case, this form gives the following characteristic function

\[ \langle e^{ik\Delta x} \rangle = e^{-\frac{1}{2}((\Delta x)^2)k_z^2 - \frac{1}{2}((\Delta y)^2)k_x^2}, \]  

(13)

where we have used cylindrical coordinates for the wave vector. Those are related to Cartesian coordinates via

\[ k_z = k_z, \]  

\[ k_x = \sqrt{k_x^2 + k_y^2}, \]  

\[ \Psi = \arccos(k_z/k_x). \]  

(14)

Furthermore, we assume that the particle motion along and across the mean magnetic field is diffusive and, therefore, \( ((\Delta x)^2) = 2\kappa_z \) and \( ((\Delta y)^2) = 2\kappa_z \). We like to emphasize that for slab turbulence we have \( \kappa_z = 0 \) in Equation (13) and, thus,
no assumption has to be made for the perpendicular motion of the particle as long as slab turbulence is considered. By combining this set of approximations and assumptions, we can derive from Equation (11)

\[ \kappa_{\perp} = \frac{a^2 v^2}{3B_0^2} \int d^3k \frac{P_{\perp}(k)}{v/\kappa_{\perp} k_z^2 + \kappa_{\perp} k_z^2}, \tag{15} \]

which is a nonlinear integral equation for \( \kappa_{\perp} \). Here we have also introduced the correction factor \( a^2 \) as was done in Mattheus et al. (2003). In the latter paper, it was suggested that \( a^2 = 1/3 \). Originally, this parameter was introduced in the equation of motion (1) but it was shown in Qin & Shalchi (2016) that Equation (1) is indeed valid as it is. Below, the reader can find a more detailed discussion of this matter.

### 2.2. The Extended NLGC Theory

One can easily show that, for slab turbulence, Equation (15) provides a finite diffusion coefficient corresponding to normal or Markovian diffusion. However, it is well-known that perpendicular transport in slab turbulence is subdiffusive (see, e.g., Qin et al. 2002a). Therefore, the ENLGC theory was developed in Shalchi (2005) and Shalchi (2006). In the following, we present the latter approach, which was developed for two-component turbulence and cannot be used for any full three-dimensional turbulence model.

Equation (1) with the Fourier representation (7) can be written as

\[ v_x = \frac{1}{B_0} \int d^3k \delta B_x(k) v_x e^{ik \cdot x}. \tag{16} \]

In the slab model, we have by definition \( \delta B_x(x) = \delta B_x(z) \), meaning that the turbulent field depends only on the coordinate along the mean field. For pure slab turbulence, we can use \( k \cdot x = k_z z \) in Equation (16) and, therefore, we can write

\[ \frac{d}{dt} \Delta x = \frac{1}{B_0} \int d^3k \frac{1}{ik_z} \delta B_x(k) \frac{d}{dt} e^{ik_z z}. \tag{17} \]

The latter equation can easily be integrated over time to find

\[ \Delta x = \frac{1}{B_0} \int d^3k \frac{1}{ik_z} \delta B_x(k) [e^{ik_z z(t)} - 1], \tag{18} \]

where we have used \( \Delta x(t) = x(t) - x(0) \) as well as \( z(0) = 0 \). The ensemble averaged square of this formula is

\[ \langle (\Delta x)^2 \rangle = \frac{1}{B_0^2} \int d^3k \ k_z^2 P_\perp(k) \times [2 - \{e^{ik_z z(t)}\} - \{e^{-ik_z z(t)}\}], \tag{19} \]

where we have employed again Equations (8)–(10). We are usually interested in the late time limit of the transport. In this case, and by assuming that the motion in the parallel direction is diffusive in that limit, we can employ the characteristic function of the diffusion equation

\[ \{e^{ik_z z(t)}\} = e^{-\kappa_{\perp} k_z^2 t}. \tag{20} \]

It has to be emphasized that we only assumed that parallel transport is diffusive. No assumption was made concerning the perpendicular motion. Therefore, Equation (19) can be written as

\[ \langle (\Delta x)^2 \rangle = \frac{2}{B_0^2} \int d^3k \ k_z^2 P_\perp(k) (1 - e^{-\kappa_{\perp} k_z^2 t}). \tag{21} \]

The tensor components of the slab modes have the form

\[ P_{mn}(k) = g_{slab}(k) \frac{\delta(k)}{k_z}, \tag{22} \]

with \( m, n = x, y \). Here we have used the Kronecker delta \( \delta_{mn} \) and the Dirac delta \( \delta(k) \). The other tensor components are zero due to the solenoidal constraint. Furthermore, we have used the turbulence spectrum of the slab modes \( g_{slab}(k) \).

If we combine Equations (21) and (22), we derive

\[ \langle (\Delta x)^2 \rangle = \frac{4\pi\kappa_{\perp}}{B_0^2} \int_{-\infty}^{+\infty} dk \ g_{slab}(k) \frac{1 - e^{-\kappa_{\perp} k_z^2 t}}{k_z^2}. \tag{23} \]

The fraction in this integral has the following property: if we consider the limit \( t \to \infty \) the exponential goes to zero and the fraction is finite as long as \( \kappa_{\perp} \neq 0 \). If \( \kappa_{\perp} = 0 \), however, the fraction is directly proportional to \( t \to \infty \). Therefore, the main contribution to the integral comes from very small wavenumbers \( k_z \). Thus we can write in the limit of late times

\[ \langle (\Delta x)^2 \rangle \approx \frac{4\pi\kappa_{\perp}}{B_0^2} g_{slab}(k) = 0 \int_{-\infty}^{+\infty} dk \frac{1 - e^{-\kappa_{\perp} k_z^2 t}}{k_z^2}. \tag{24} \]

The remaining integral yields \( 2\sqrt{\pi t / \kappa_{\perp}} \) and, therefore, we obtain

\[ \langle (\Delta x)^2 \rangle = \frac{8\pi}{B_0^2} g_{slab}(k) = 0 \frac{\sqrt{\pi t / \kappa_{\perp}}}{\sqrt{\pi}}. \tag{25} \]

The field line diffusion coefficient for slab turbulence is given by\(^3\) (see, e.g., Shalchi 2009)

\[ \kappa_{FL} = \frac{2\pi^2}{B_0^2} g_{slab}(k) = 0 \] \tag{26}

and, thus, Equation (25) can be written as

\[ \langle (\Delta x)^2 \rangle = 4\kappa_{FL} \frac{\sqrt{\kappa_{FL}}}{\sqrt{\pi}}. \tag{27} \]

For the spectrum of the slab modes, we employ (see, e.g., Bieber et al. 1994)

\[ g_{slab}(k) = \frac{C(s)}{2\pi} \frac{\delta_{slab}^2}{l_{slab}} \frac{1}{1 + (k/l_{slab})^2} \tag{28} \]

with the normalization function

\[ C(s) = \frac{\Gamma(s/2)}{2\sqrt{\pi} \Gamma((s-1)/2)} \tag{29} \]

where \( \Gamma(z) \) is the Gamma function. Above, we have used the magnetic field strength associated with the slab modes \( \delta B_{slab} \), the slab bendover scale \( l_{slab} \), and the inertial range spectral

\(^3\) For magnetostatic slab turbulence, the theory of field line random walk is exact. A field line diffusion coefficient \( \kappa_{FL} \) is defined via the mean-square displacements of magnetic field lines \( \langle (\Delta x)^2 \rangle = 2\kappa_{FL} \Delta x \) and, therefore, \( \kappa_{FL} \) has length dimensions.
index $s$. For this spectrum, the field line diffusion coefficient (26) becomes

$$\kappa_{FL} = \pi C(s) l_{slab}^2 \frac{\delta B_{slab}^2}{B_0^2}. \quad (30)$$

From Equation (27), we can see that the mean-square displacement increases linearly with $\sqrt{t}$ corresponding to subdiffusion. In the literature, this type of transport is usually called compound diffusion (see, e.g., Kôta & Jokipii 2000; Webb et al. 2006, and Shalchi & Kourakis 2007). Subdiffusion or compound diffusion was also discussed in the work of Getmansev (1963), Fisk et al. (1973), and Chuvilgin & Ptuskin (1993). Furthermore, it was shown via test-particle simulations that this type of transport can indeed be found in slab turbulence (see, e.g., Qin et al. 2002a).

In the slab/2D composite model, we assume that the magnetic field is given by $\delta B(x) = \delta B_{slab}(z) + \delta B_{2D}(x, y)$ and we assume that the two components are uncorrelated, meaning that $\langle \delta B_{slab}(z) \delta B_{2D}(x, y) \rangle = 0$. It was shown before that if a two-dimensional component is added, diffusion is recovered (see, e.g., Qin et al. 2002b). Therefore, we assume the following form for the mean-square displacement

$$\langle (\Delta x)^2 \rangle = 2\alpha \sqrt{t} + 2\kappa_{L} t, \quad (31)$$

where, according to Equation (27),

$$\alpha = 2\kappa_{FL} \frac{\kappa_{L}}{\pi}. \quad (32)$$

Very easily, one can see that for $t \to \infty$, the (subdiffusive) slab contribution can be neglected compared to the second (diffusive) contribution and the diffusion coefficient $\kappa_{L}$ depends only on the properties of the two-dimensional modes. Below, we will show that there can be an implicit contribution due to slab modes.

Within ENLGC theory, the slab contribution is calculated as described above, and the two-dimensional contribution is calculated as within the original NLGC theory. Therefore, within the ENLGC theory, the perpendicular diffusion coefficient in two-component turbulence is given by

$$\kappa_{L} = \frac{v^2}{3B_0^2} \int d^3k \frac{P_{2D}(k)}{v/\lambda_{ij} + \kappa_{L} k_{||}^2}. \quad (33)$$

One can easily see that, for pure two-dimensional turbulence, Equations (15) and (33) are equivalent. As soon as a slab contribution is added, however, both theories provide different results. Such as in the original NLGC theory, one could incorporate a correction factor of $a^2$, but this is not done here.

### 2.3. The UNLT Theory

According to Shalchi (2010), the original NLGC theory fails in general because approximation (4) is not valid. This is, in particular, the case for slab and small Kubo number turbulence. The latter statement was confirmed numerically in Qin & Shalchi (2016). Based on the CR Fokker–Planck equation, Shalchi (2010) developed a nonlinear theory for perpendicular diffusion that no longer requires the usage of approximation (4). The following nonlinear integral equation has been found after lengthy algebra.

$$\kappa_{L} = \frac{v^2}{3B_0^2} \int d^3k \frac{P_{ax}(k)}{v/\lambda_{||} + (4/3)\kappa_{L} k_{||}^2 + F(k||, k_{\perp})} \quad (34)$$

with

$$F(k||, k_{\perp}) = \frac{v^2 k_{||}^2}{3\kappa_{L} k_{\perp}^2}. \quad (35)$$

One can easily see that, for two-dimensional turbulence, Equation (34) agrees with Equations (15) and (33) apart from the factor of $4/3$. For slab turbulence, we find $\kappa_{L} = 0$ and, therefore, UNLT and ENLGC theories agree with each other but not with the original NLGC theory. Whereas ENLGC theory can only be used for two-component turbulence, UNLT theory should also be valid for full three-dimensional turbulence. From Equation (34), one can easily derive the Matthaeus et al. (1995) theory for field line random walk by considering the limit $\lambda_{ij} \rightarrow \infty$. Different asymptotic limits of Equation (34) have been derived and discussed in Shalchi (2015).

### 3. Implicit Contribution of Slab Modes

Within the extended NLGC theory, the mean-square displacement is given by Equation (31). There is obviously a subdiffusive contribution from the slab modes and a diffusive contribution from the two-dimensional modes. In the limit $t \to \infty$, however, only the latter contribution remains.

In previous analytical theories, such as the ones described in Section 2, one usually employs $\langle (\Delta x)^2 \rangle = 2\kappa_{L} t$ in the characteristic function. The idea of the current paper is to use Equation (31) instead of the diffusion approximation. For two-dimensional turbulence, Equation (11) with (13) and (31) becomes

$$\kappa_{L} = \frac{v^2}{3B_0^2} \int d^3k \frac{b_{2D}(k)}{v/\lambda_{ij} + \kappa_{L} k_{||}^2} \times \int_0^\infty dt \ e^{-v/\lambda_{ij} + \kappa_{L} k_{||}^2 t} \sigma. \quad (36)$$

It has to be emphasized that the latter formula is only valid for two-dimensional turbulence and cannot be used for full three-dimensional turbulence. Furthermore, the form (31) is only valid in the late time limit. For earlier times, for instance, one expects a ballistic motion of particles and for intermediate times there could even be a diffusive contribution of the slab modes (see, e.g., Jokipii & Parker 1969; Shalchi 2008; Webb et al. 2009 and Ruffolo et al. 2012 for more details). The model used here is based on the assumption that only late times contribute to the perpendicular diffusion coefficient.
The time integral in Equation (36) is solved by (see, e.g., Gradshteyn & Ryzhik 2000)

\[ \int_0^\infty dt \ e^{-A\tau-Bt} = \frac{1}{B} K(\xi), \quad (37) \]

where we have used the parameters

\[ A = \alpha k^2, \quad (38) \]

\[ B = v/\lambda_l + \kappa_l k^2, \quad (39) \]

and

\[ \xi = \frac{A}{2\sqrt{B}}, \quad (40) \]

as well as the function

\[ K(\xi) = 1 - \sqrt{\pi} \xi e^{\xi^2} \text{erfc}(\xi). \quad (41) \]

Here we have also used the complementary error function. Therewith, Equation (36) becomes

\[ \kappa_l = v^2 \int d^3k \frac{P_{2D}^l(k)}{v/\lambda_l + \kappa_l k^2} K(\xi). \quad (42) \]

The parameter \( \xi \) was defined in Equation (40) and becomes, in our case,

\[ \xi = \frac{\kappa_l k^2}{\sqrt{v/\lambda_l + \kappa_l k^2}} \]

\[ = \frac{1}{\sqrt{3\pi}} \frac{\kappa_l^2 k^4}{\sqrt{1 + \lambda_l k^2}}, \quad (43) \]

where we have used Equation (32) and we have replaced the diffusion coefficients by the corresponding mean-free paths.\(^5\) If we employ Equation (30) in order to replace the field line diffusion coefficient of the slab modes, we can write

\[ \xi = \sqrt{\frac{\pi}{3}} C(s) \frac{l_{\text{slab}} \lambda_l k^2}{\sqrt{1 + \lambda_l k^2}} \frac{\delta B_{\text{slab}}}{B_0^2}. \quad (44) \]

If we replace the diffusion coefficients by the corresponding mean-free paths, Equation (42) can be written as

\[ \lambda_l = \frac{\lambda_l}{B_0^2} \int d^3k \frac{P_{2D}^l(k)}{1 + \lambda_l k^2/3} K(\xi). \quad (45) \]

The two-dimensional turbulence model is defined via

\[ P_{mn}(k) = g^{2D}(k_\perp) \delta(k_\perp) \left[ \delta_{mn} - \frac{k_n k_m}{k^2} \right] \]

where we have used the two-dimensional turbulence spectrum \( g^{2D}(k_\perp) \), which will be discussed below. By combining Equations (45) and (46), we obtain

\[ \lambda_l = \frac{\pi \lambda_l}{B_0^2} \int_0^\infty dk_\perp \frac{g^{2D}(k_\perp)}{1 + \lambda_l k^2/3} K(\xi). \quad (47) \]

Shalchi & Weinhorst (2009) proposed the following form for the spectrum of the two-dimensional modes

\[ g^{2D}(k_\perp) = \frac{2D(s, q)}{\pi} \delta B_{2D}^2 I_{2D} \]

\[ \times \frac{(k_\perp a_{2D})^q}{[1 + (k_\perp a_{2D})^2]^{(s+q)/2}} \]

with the normalization function

\[ D(s, q) = \frac{\Gamma\left(\frac{s+q}{2}\right)}{2\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}. \quad (49) \]

The parameters used in the spectrum are the inertial range spectral index \( s \), the energy range spectral index \( q \), and the bendover scale of the two-dimensional modes \( a_{2D} \). With this spectrum, and by employing the integral transformation \( x = l_{2D} k_\perp \), Equation (47) becomes

\[ \lambda_l = 2D(s, q) \lambda_l \delta B_{2D}^2 \]

\[ \times \int_0^\infty dx \frac{\lambda_l x^q}{[1 + x^2]^{(s+q)/2}} \frac{K(\xi)}{1 + \lambda_l k^2/3}. \quad (50) \]

In Section 5, we shall evaluate Equation (50) for different parameter values and compare our findings with diffusion coefficients obtained from the other theories.

4. FURTHER ANALYTICAL CONSIDERATIONS

The important result of the current paper is Equation (42). Therein the function \( K(\xi) \) is used, which is defined via Equation (41). It should not be problematic to incorporate this function in numerical codes used to evaluate Equation (42). In some cases, however, a further analytical simplification could be convenient. This is done in the following.

The complementary error function has the following asymptotic limits (see, e.g., Abramowitz & Stegun 1974)

\[ \text{erfc}(\xi \ll 1) \approx 1 - \frac{2}{\sqrt{\pi}} \xi \]

\[ \text{erfc}(\xi \gg 1) \approx \frac{1}{\sqrt{\pi}} \xi e^{-\xi^2} \left( 1 - \frac{1}{2\xi^2} \right). \]

Therefore, we find

\[ K(\xi \ll 1) \approx 1 - \sqrt{\pi} \xi \]

\[ K(\xi \gg 1) \approx \frac{1}{2\xi^2}. \]

We can easily see that for increasing \( \xi \), we find a reduction of the perpendicular diffusion coefficient. In order to combine our findings, we use the following approximation

\[ K(\xi) \approx \frac{1}{1 + 2\xi^2}. \]

\(^5\) The mean-free paths are related to the corresponding diffusion coefficients via \( \lambda_l = 3\kappa_l/\nu \) and \( \lambda_l = 3\kappa_l/\nu \).
so that \( K(\xi \ll 1) \approx 1 \) and, for \( \xi \gg 1 \), we recover Equation (54). In Figure 1, we compare the exact form (41) with approximation Equation (55).

If approximation (55) is combined with Equation (42), and if we use Equation (43), we find the following integral equation

\[
\kappa_\| = \frac{v^2}{3B_0^2} \int dk \frac{P_{2D}(k)}{v/\lambda_\| + \kappa_\| k_\|^2 + 2\kappa_\| \kappa_{\text{FL}} k_\|^4/\pi}.
\] (56)

The latter formula has some similarity with the integral equations discussed in Section 2 (see, e.g., Equations (15), (33), and (34)). We like to emphasize that Equation (56) is only valid for slab/2D turbulence and cannot be used for other turbulence models such as full three-dimensional models. The third term in the denominator of Equation (56) contains the factor of \( \kappa_{\text{FL}} \). It has to be pointed out that this is the field line diffusion coefficient \( \kappa_{\text{FL}} \), which is associated with the slab modes as given by Equation (30) and not the total field line diffusion coefficient which would also contain a contribution of the two-dimensional modes.

For numerical investigations, it is useful to rewrite Equation (56) as

\[
\lambda_\| = 2D(s, q) \lambda_\| \frac{\delta B^2_{2D}}{B_0^2} \times \int_0^\infty dx \frac{x^q}{(1 + x^2)^{(s+q)/2}} \left( 1 + \frac{\lambda_\| \kappa_{\text{FL}}}{3l_{2D}} x^2 + \frac{\gamma x^4}{\pi} \right),
\] (57)

where we have used the parameter

\[
\gamma = \frac{2 \lambda_\|^2 \kappa_{\text{FL}}}{3\pi l_{2D}^2} = \frac{2\pi}{3} \left( C(s) \right)^2 \frac{\lambda_\|^2 l_{2D}^2}{l_{2D}^2 l_{2D}^2 B_0^4} \frac{\delta B^4_{2D}}{B_0^4}.
\] (58)

and spectrum (48). Equation (57) with (58) is also used in Section 5 to compute the perpendicular mean-free path and the results are compared with other theoretical results as well.

5. RESULTS

In the current section, we compute the perpendicular mean-free path by employing the original NLGC theory of Matthaeus et al. (2003), the extended NLGC theory of Shalchi (2006), and we use the modified theory developed in the current paper by using different approximations for the function \( K(\xi) \). In all cases, we calculate the perpendicular mean-free path as a function of the parallel mean-free path. In all cases, we have set \( \delta B^2_{\text{slab}} = 0.2B_0^2 \) and \( \delta B^2_{2D} = 0.8B_0^2 \) as originally suggested in Bieber et al. (1994).

5.1. The Case \( q = 0 \) and \( l_{2D} = 0.1l_{\text{slab}} \)

The first set of parameter values is based on those used in Matthaeus et al. (2003). In the latter paper, the original NLGC theory was compared with test-particle simulations. The best agreement was achieved by setting \( a^2 = 1/3 \). In Figure 2, we show the original NLGC theory for \( a^2 = 1 \), the extended NLGC theory, as well as our new results. We can clearly see that the extended NLGC result is smaller than the original NLGC result because there is no contribution from the slab modes. Furthermore, the implicit contribution from the slab modes reduces the perpendicular mean-free path further. This is, in particular, the case for long parallel mean-free paths corresponding to higher particle rigidities/energies. In this case, the perpendicular mean-free path is reduced by about a factor of two compared to the original NLGC result. This finding can explain the value of \( a^2 \approx 1/3 \) suggested by Matthaeus et al. (2003). We also computed the perpendicular mean-free path for \( q = 0 \) and \( l_{2D} = l_{\text{slab}} \) but it did not show a significant effect.

5.2. The Case \( q = 1.5 \) and \( l_{2D} = 0.1l_{\text{slab}} \)

Matthaeus et al. (2007) suggested that the spectrum of the two-dimensional modes is not constant at large scales corresponding to the energy range. Therefore, we set \( q = 1.5 \) and compute the perpendicular mean-free path as was done above. Our findings are shown in Figure 3. Clearly, we can
observe a significant difference between the different theories. For the case considered here, the implicit contribution of the slab modes reduces the perpendicular mean-free path by about a factor of 10 if compared to the original NLGC theory. However, this is only the case for very long parallel mean-free paths corresponding to very high particle energies. We can also see that approximation (55) works well.

5.3. The Case of \( q = 3 \) and \( l_{2D} = 0.1l_{\text{slab}} \)

As shown above, the energy range spectral index seems to be important if the implicit slab contribution is taken into account. Therefore, we further change the parameter \( q \). Our findings for \( q = 3 \) are visualized in Figure 4. Again, we find a significant difference between the different theories. Now the influence of the implicit slab contribution is even larger and, thus, we conclude that for increasing \( q \) we find a stronger effect.

5.4. The Case of \( q = 3 \) and \( l_{2D} = l_{\text{slab}} \)

A further important parameter in the theory of perpendicular diffusion is the scale ratio \( l_{2D}/l_{\text{slab}} \). Above, we have considered the case of \( l_{2D} = 0.1l_{\text{slab}} \) as it was used in Matthaeus et al. (2003). In the current paragraph, we assume that the two bendover scales are equal. Our findings are shown in Figure 5. Clearly, we can see that now the discrepancies between the different theories are much smaller. However, there is still a factor of two or even three between the different theoretical results. Again the observed effect can explain the value \( a^2 = 1/3 \) assumed in Matthaeus et al. (2003). We also made calculations for smaller values of the magnetic field ratio \( \delta B/B_0 \) to find that the effect coming from the implicit slab contribution is weaker.

6. SUMMARY AND CONCLUSION

In the current paper, we have revisited the problem of perpendicular diffusion of energetic particles in two-component turbulence. Whereas it was shown before that the slab modes do not explicitly contribute to the perpendicular diffusion coefficient, we explored the implicit contribution in the current paper. We have derived the modified nonlinear integral Equation (42), which can be approximated by Equation (56). This modification should provide another improvement compared to the original NLGC theory developed in Matthaeus et al. (2003) and the extended NLGC theory of Shalchi (2006). Compared to earlier versions of the NLGC theory, the modified Equations (42) or (56) are not more difficult to evaluate numerically.

In Figures 2–5, we have shown the perpendicular mean-free path versus the parallel mean-free path for different values of the scale ratio \( l_{2D}/l_{\text{slab}} \) and different values of the energy range spectral index \( q \). Both mean-free paths are normalized with respect to the two-dimensional bendover scale \( l_{2D} \). It can easily be seen that, in general, the implicit slab contribution reduces the perpendicular mean-free path.

Matthaeus et al. (2003) used \( q = 0 \) and \( l_{2D} = 0.1l_{\text{slab}} \) in their work and they compared the original NLGC theory with test-particle simulations. They found a difference between analytical theory and simulations but this difference can be balanced out by using the correction factor \( a^2 \) and by setting \( a^2 = 1/3 \). In the current paper, a possible explanation for this value is provided. The implicit contribution from the slab modes reduces the perpendicular mean-free path as required to achieve...
agreement with simulations. The correction factor $a^2$ is no longer needed.

For $q = 1.5$ and $q = 3$, which is in agreement with the values suggested by Matthaeus et al. (2007), we find a stronger effect. In particular, for long, parallel mean-free paths, a strong reduction of the perpendicular mean-free path can be observed.

Another parameter that influences the reduction discussed here is the scale ratio $l_{2D}/l_{slab}$. If this ratio is small, a stronger effect can be observed. For equal bendover scales, however, the perpendicular mean-free path is only about a factor of two shorter than the one computed by using the original NLGC theory.

A further theory for perpendicular diffusion was presented in Shalchi (2010), where the UNLT theory was developed. In the following, we discuss which theory has to be used for which case.

1. Solar wind turbulence is often approximated by a slab/2D composite model, which is also known as two-component turbulence. We suggest using the extended NLGC theory with implicit slab contribution developed in the current paper for this specific turbulence model. This theory is represented by Equation (42), which can be well approximated by Equation (56).

2. For full three-dimensional turbulence, the original NLGC theory, the extended theory, and the approach developed in the current paper cannot be used. For this case, the UNLT theory represented by Equation (34) should provide an accurate description of perpendicular diffusion. In this case, a critical parameter is the Kubo number (see Shalchi 2015 for more details).

The integral equation derived in the current paper should provide an accurate analytical description of perpendicular diffusion in two-component turbulence. It is straightforward to include further effects such as dynamical turbulence. In this case, another term would occur in the denominator of Equation (42) that would be associated with the correlation time of the two-dimensional modes. The situation is more complicated, however, if there is also a dynamical turbulence effect associated with the slab modes because, in this case, the explicit contribution of the slab modes can be diffusive (see, e.g., Shalchi 2014).

The results obtained in the current paper, and in analytical theories for perpendicular diffusion in general, are relevant for several applications.

1. To understand the acceleration of particles due to turbulence (see, e.g., Lynn et al. 2014).
2. For solar modulation studies (see, e.g., Alania et al. 2013; Engelbrecht & Burger 2013; Manuel et al. 2014 and Potgieter et al. 2014).
3. Particle acceleration at interplanetary shocks such as coronal mass ejection driven shocks (see, e.g., Li et al. 2012 and Wang et al. 2012).
4. To describe the motion of CRs in our own and in external galaxies (see, e.g., Berkhuijsen et al. 2013; Buffie et al. 2013; Heesen et al. 2014).
5. To describe diffusive shock acceleration at interstellar shocks (see, e.g., Ferrand et al. 2014).

In particular, for diffusive shock acceleration at supernova shock waves, the fact that the perpendicular mean-free path becomes rigidity independent in the high-energy limit, can help to explain the cosmic-ray spectrum (see Ferrand et al. 2014 for more details). In the current paper, we have shown that the perpendicular diffusion coefficient can even decrease with increasing rigidity in the high-energy regime. To incorporate this effect in simulations of diffusive shock acceleration at interplanetary and interstellar shock waves could be important and should be the subject of future work.

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