Darboux Transformation for the Hirota equation

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Abstract

The Hirota equation is an integrable higher order nonlinear Schrödinger type equation which describes the propagation of ultrashort light pulses in optical fibers. We present a standard Darboux transformation for the Hirota equation and then construct its quasideterminant solutions. The multisoliton and breather solutions of the Hirota equation are given explicitly.

Keywords: Hirota equation; Darboux transformation; Quasideterminants.

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1 Introduction

There exists a large class of nonlinear evolution equations which can be solved analytically. Such equations are called integrable. Integrable equations constitute an important part of the nonlinear wave theory. The simplest integrable equation which describes the dynamics of deep-water gravity waves is the nonlinear Schrödinger (NLS) equation

$$i q_t + q_{xx} + 2 |q|^2 q = 0.$$  \quad(1.1)

In 1967, it was first discussed in the general context of nonlinear dispersive waves by Benney and Newell [3]. In 1968, this equation was also derived by Zakharov in his study of modulational stability of deep water waves [35]. In 1972, Zakharov and Shabat found that the NLS equation had a Lax pair and could be solved by the inverse scattering transform (IST) method [37]. This equation plays an important role in different physical systems as wide as plasma physics [36], water waves [3, 4, 35], and nonlinear optics [12, 13]. One of the most interesting applications of the NLS equation is that it can be employed to model for short soliton pulses in optical fibres [16]. However, as the pulses get shorter, various additional effects become important and the NLS model is no longer appropriate. In order to understand these additional effects, Kodama and Hasegawa [17,18] suggested a higher-order NLS equation

$$i q_t + \alpha_1 q_{xx} + \alpha_2 |q|^2 q + i \beta \left[ \gamma_1 q_{xxx} + \gamma_2 |q|^2 q_x + \gamma_3 q \left(|q|^2ight)_x \right] = 0,$$ \quad(1.2)

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where the $\alpha_i, \gamma_i$ are real constants, $\beta$ is a real spectral parameter and $q$ is a complex-valued function of $x$ and $t$. By choosing $\beta = 0$ and $\alpha_2 = 2\alpha_1 = 2$ in this equation, we can easily see that the first three terms form the standard NLS equation (1.1). Generally, the Kodama-Hasegawa higher-order NLS equation (1.2) may not be completely integrable if some restrictions are not imposed on the real constants $\gamma_i$ ($i = 1, 2, 3$). Until now it is known that, besides the NLS equation (1.1) itself, there are four cases in which integrability can be proved via the IST. These are the Chen-Lee-Liu [5] derivative NLS equation ($\gamma_1 : \gamma_2 : \gamma_3 = 0 : 1 : 0$), the Kaup-Newell [15] derivative NLS equation ($\gamma_1 : \gamma_2 : \gamma_3 = 0 : 1 : 1$), the Hirota [14] NLS equation ($\gamma_1 : \gamma_2 : \gamma_3 = 1 : 6 : 0$) and the Sasa-Satsuma [30] NLS equation ($\gamma_1 : \gamma_2 : \gamma_3 = 1 : 6 : 3$).

In this paper, we consider the Hirota [14] NLS equation

$$iq_t + \alpha (q_{xx} + 2|q|^2q) + i\beta (q_{xxx} + 6|q|^2q_x) = 0, \quad \alpha, \beta \in \mathbb{R},$$

in which $\alpha_2 = 2\alpha_1 = 2\alpha$. This equation is commonly known as the Hirota equation (HE), and we will denote it as such from now on. The HE (1.3) can be used to describe the wave propagation of ultrashort light pulses in optical fibers [1, 17, 18, 21, 24, 33]. It is very interesting to see that the Hirota equation (1.3) is the sum of the NLS (1.1) equation ($\alpha = 1, \beta = 0$) and the complex version of the modified Korteweg-de Vries (mKdV) equation ($\alpha = 0, \beta = 1$)

$$q_t + q_{xxx} + 6|q|^2q_x = 0$$

which is completely integrable [14,31]. In the resent years, there has been some interest in solutions of the HE (1.3) obtained by Darboux - type transformations [2,20,29]. These solutions are often written in terms of determinants. These methods have been used to prove various solutions of the HE (1.3) such as multisolitons, breathers and rogue waves.

In 1882, the French mathematician Jean Gaston Darboux [6] introduced a method to solve the Sturm-Louville equation, which is called Darboux transformation (DT) afterwards. Almost a century later, in 1979, Matveev [22] realised that the method given by Darboux for the spectral problem of second order ordinary differential equations can be extended to some important soliton equations. Darboux transformations are one of important tools in studying integrable systems. They provide a universal algorithmic procedure to derive exact solutions of integrable systems.

In the present article, we construct for the first time a standard Darboux transformation for the Hirota equation (1.3). We underline that the method we use here is based on Darboux’s [6] and Matveev’s original ideas [22,23]. Therefore, our approach should be considered on its own merits. Furthermore, our solutions for the HE are written in terms of quasideterminants [7,8] rather than determinants. It has been proved that quasideterminants are very useful for constructing exact solutions of integrable equations [9,10,19,27,28,32,38], enabling these solutions to be expressed in a simple and compact form.

This paper is structured as follows. In Section 1.1 below, we give a brief review of quasideterminants. In Section 2, we establish a $2 \times 2$ eigenfunction and corresponding constant $2 \times 2$ square matrix for the eigenvalue problems of the Hirota equation (1.3) using two symmetries of the Lax pair of the HE. In Section 3, we state a standard Darboux theorem for the Hirota system. We review the reduced DTs for the HE, which can be considered as a dimensional reduction from $(2 + 1)$ to $(1 + 1)$ dimensions. In Section 4, we present the quasideterminant solutions for the HE constructed by the DT. In Section 5, the multisoliton and breather solutions of the Hirota equation are given for both zero and non-zero seed solutions as particular solutions of the HE. The conclusion is given in the final Section 6.
1.1 Quasideterminants

In this short section we will list some of the key elementary properties of quasideterminants used in the paper. The reader is referred to the original papers [7, 8] for a more detailed and general treatment.

Let $M = (m_{ij})$ be an $n \times n$ matrix with entries over a ring (noncommutative, in general) has $n^2$ quasideterminants written as $|M|_{ij}$ for $i, j = 1, \ldots, n$. They are defined recursively by

$$ |M|_{ij} = m_{ij} - r^j_i (M^{ij})^{-1} c^i_j, \quad (1.5) $$

where $r^j_i$ represents the row vector obtained from $i^{th}$ row of $M$ with the $j^{th}$ element removed, $c^i_j$ is the column vector obtained from $j^{th}$ column of $M$ with the $i^{th}$ element removed and $M^{ij}$ is the $(n-1) \times (n-1)$ submatrix obtained by deleting the $i^{th}$ row and the $j^{th}$ column from $M$. Quasideterminants can be also denoted as shown below by boxing the entry about which the expansion is made

$$ |M|_{ij} = \left| \begin{array}{cc} M^{ij} & c^i_j \\ r^j_i & m_{ij} \end{array} \right|. \quad (1.6) $$

If the entries in $M$ commute, then the quasideterminant $|M|_{ij}$ can be expressed as a ratio of determinants

$$ |M|_{ij} = (-1)^{i+j} \frac{\det M}{\det M^{ij}}. \quad (1.7) $$

2 Hirota equation

Let us consider the couple Hirota equations

$$ q_t - i\alpha (q_{xx} + 2q^2 r) + \beta (q_{xxx} + 6qrq_x) = 0, \quad (2.1) $$
$$ r_t + i\alpha (r_{xx} + 2qr^2) + \beta (r_{xxx} + 6qrr_x) = 0, \quad (2.2) $$

where $q = q(x,t)$ and $r = r(x,t)$ are complex valued functions. Equations (2.1) and (2.2) reduce to the Hirota equation (1.3) when $r = q^*$. Here the asterisk superscript on $q$ denotes the complex conjugate.

The Lax pair [29] for the couple Hirota equations (2.1)-(2.2) is given by

$$ L = \partial_x + J\lambda - R, \quad (2.3) $$
$$ M = \partial_t + 4\beta J\lambda^3 + 2U\lambda^2 - 2V\lambda - W, \quad (2.4) $$

where $J, R, U, V$ and $W$ are $2 \times 2$ matrices

$$ J = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad R = \begin{pmatrix} 0 & q \\ -r & 0 \end{pmatrix}, \quad U = \begin{pmatrix} i\alpha & -2\beta q \\ 2\beta r & -i\alpha \end{pmatrix}, \quad (2.5) $$
$$ V = \begin{pmatrix} i\beta qr & aq + i\beta qx \\ -ar + i\beta rx & -i\beta qr \end{pmatrix}, \quad W = \begin{pmatrix} iaqr + \beta (qr_x - rq_x) & iaq_x - \beta (qx + 2q^2r) \\ iar + \beta (r_{xx} + 2qr^2) & -iaqr - \beta (qr - rq_x) \end{pmatrix}. \quad (2.6) $$
Here $\lambda$ is a spectral parameter. It can be seen that the potential matrix $R$ in (2.5) has two symmetry properties. One is that it is skew-Hermitian: $R^\dagger = -R$. The other one is that $SRS^{-1} = R^*$, where

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (2.7)$$

Let $\phi = (\varphi, \psi)^T$ be a vector eigenfunction for (2.3)-(2.4) for eigenvalue $\lambda$ so that $L_\lambda(\phi) = M_\lambda(\phi) = 0$. Using the second symmetry, it may be seen that $\tilde{\phi} = S\phi = (\psi^*, -\varphi^*)^T$ is another eigenfunction for eigenvalue $\lambda^*$ such that $L_{\lambda^*}(\tilde{\phi}) = M_{\lambda^*}(\tilde{\phi}) = 0$. Using these vector eigenfunctions we can define a square $2 \times 2$ matrix eigenfunction $\theta$ with $2 \times 2$ eigenvalue $\Lambda$

$$\theta = \begin{pmatrix} \varphi & \psi^* \\ \psi & -\varphi^* \end{pmatrix}, \quad \Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^* \end{pmatrix}, \quad (2.8)$$

satisfying

$$\theta_x + J\theta\Lambda - R\theta = 0, \quad (2.9)$$

$$\theta_t + 4\beta J\theta\Lambda^3 + 2U\theta\Lambda^2 - 2V\theta\Lambda - W\theta = 0. \quad (2.10)$$

3 Darboux transformations and Dimensional reductions

3.1 Darboux transformation

Let us consider the linear operators

$$L = \partial_x + \sum_{i=0}^n u_i \partial_i^y, \quad M = \partial_t + \sum_{i=0}^n v_i \partial_i^y, \quad (3.1)$$

where $u_i, v_i$ are $m \times m$ matrices. The standard approach to Darboux transformations [6, 22, 23] involves a gauge operator $G_\theta = \theta \partial_y \theta^{-1}$, where $\theta = \theta(x, y, t)$ is an invertible $m \times m$ matrix solution to a linear system

$$L(\phi) = M(\phi) = 0. \quad (3.2)$$

If $\phi$ is any eigenfunction of $L$ and $M$ then $\tilde{\phi} = G_\theta(\phi)$ satisfies the transformed system

$$\tilde{L}(\tilde{\phi}) = \tilde{M}(\tilde{\phi}) = 0, \quad (3.3)$$

where the linear operators $\tilde{L} = G_\theta LG_\theta^{-1}$ and $\tilde{M} = G_\theta MG_\theta^{-1}$ have the same forms as $L$ and $M$:

$$\tilde{L} = \partial_x + \sum_{i=0}^n \tilde{u}_i \partial_i^y, \quad \tilde{M} = \partial_t + \sum_{i=0}^n \tilde{v}_i \partial_i^y. \quad (3.4)$$

3.2 Dimensional reduction of Darboux transformation

Here, we describe a reduction of the Darboux transformation from $(2 + 1)$ to $(1 + 1)$ dimensions. We choose to eliminate the $y$-dependence by employing a ‘separation of variables’ technique. The reader is referred to the paper [25] for a more detailed treatment. We make the ansatz

$$\phi = \phi^r(x, t)e^{\lambda y}, \quad (3.5)$$
\[ \theta = \theta^r(x, t)e^{\Lambda y}, \quad (3.6) \]

where \( \lambda \) is a constant scalar and \( \Lambda \) an \( N \times N \) constant matrix and the superscript \( r \) denotes reduced functions, independent of \( y \). Hence in the dimensional reduction we obtain \( \partial_y^i (\phi) = \lambda^i \phi \) and \( \partial_y^i (\theta) = \theta \Lambda^i \) and so the operator \( L \) and Darboux transformation \( G \) become

\[ L^r = \partial_x + \sum_{i=0}^n u_i \lambda^i, \quad (3.7) \]
\[ G^r = \lambda - \theta^r \Lambda (\theta^r)^{-1}, \quad (3.8) \]

where \( \theta^r \) is a matrix eigenfunction of \( L^r \) such that \( L^r (\theta^r) = 0 \), with \( \lambda \) replaced by the matrix \( \Lambda \), that is,

\[ \theta^r_x + \sum_{i=0}^n u_i \theta^r \Lambda^i = 0. \quad (3.9) \]

Below we omit the superscript \( r \) for ease of notation.

### 3.3 Iteration of reduced Darboux Transformations

In this section we shall consider iteration of the Darboux transformation and find closed form expressions for these in terms of quasideterminants.

Let \( L \) be an operator, form invariant under the reduced Darboux transformation \( G_{\theta} = \lambda - \theta \Lambda \theta^{-1} \) discussed above.

Let \( \phi = \phi(x, t) \) be a general eigenfunction of \( L(\phi) = 0 \). Then

\[
\tilde{\phi} = G_{\theta} (\phi) = \lambda \phi - \theta \Lambda \theta^{-1} \phi
= \begin{vmatrix} \theta & \phi \\ \theta \Lambda & \lambda \phi \end{vmatrix}
\]

is an eigenfunction of \( \tilde{L} = G_{\theta} L G_{\theta}^{-1} \) so that \( \tilde{L} (\tilde{\phi}) = \lambda \tilde{\phi} \). Let \( \theta_i \) for \( i = 1, \ldots, n \), be a particular set of invertible eigenfunctions of \( L \) so that \( L(\theta_i) = 0 \) for \( \lambda = \Lambda_i \), and introduce the notation \( \Theta = (\theta_1, \ldots, \theta_n) \). To apply the Darboux transformation a second time, let \( \theta_{[1]} = \theta_1 \) and \( \phi_{[1]} = \phi \) be a general eigenfunction of \( L_{[1]} = L \). Then \( \phi_{[2]} = G_{\theta_{[1]}}(\phi_{[1]}) \) and \( \theta_{[2]} = \phi_{[2]} |_{\phi \to \theta_2} \) are eigenfunctions for \( L_{[2]} = G_{\theta_{[1]}} L_{[1]} G_{\theta_{[1]}}^{-1} \).

In general, for \( n \geq 1 \), we define the \( n \)th Darboux transform of \( \phi \) by

\[
\phi_{[n+1]} = \lambda \phi_{[n]} - \theta_{[n]} \Lambda_n \theta_{[n]}^{-1} \phi_{[n]}, \quad (3.10)
\]

in which

\[ \theta_{[k]} = \phi_{[k]} |_{\phi \to \theta_k} \cdot \]

For example,

\[
\phi_{[2]} = \lambda \phi - \theta_1 \Lambda_1 \theta_1^{-1} \phi = \begin{vmatrix} \theta_1 & \phi \\ \theta_1 \Lambda_1 & \lambda \phi \end{vmatrix},
\]

Below we omit the superscript \( r \) for ease of notation.
\[ \phi_{[3]} = \lambda \phi_{[2]} - \theta_{[2]} \Lambda_2 \theta_{[2]}^{-1} \phi_{[2]} \]
\[ = \begin{vmatrix} \theta_1 & \theta_2 & \phi \\ \theta_1 \Lambda_1 & \theta_2 \Lambda_2 & \lambda \phi \\ \theta_1 \Lambda_1^2 & \theta_2 \Lambda_2^2 & \lambda^2 \phi \end{vmatrix} . \]

After \( n \) iterations, we get

\[ \phi_{[n+1]} = \begin{vmatrix} \theta_1 & \theta_2 & \ldots & \theta_n & \phi \\ \theta_1 \Lambda_1 & \theta_2 \Lambda_2 & \ldots & \theta_n \Lambda_n & \lambda \phi \\ \theta_1 \Lambda_1^2 & \theta_2 \Lambda_2^2 & \ldots & \theta_n \Lambda_n^2 & \lambda^2 \phi \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ \theta_1 \Lambda_1^n & \theta_2 \Lambda_2^n & \ldots & \theta_n \Lambda_n^n & \lambda^n \phi \end{vmatrix} . \] (3.11)

### 4 Constructing Solutions for Hirota Equation

In this section we determine the specific effect of the Darboux transformation \( G_\theta = \lambda - \theta \Lambda \theta^{-1} \) on the operator \( L \) given by (2.3). Corresponding results hold for the operator \( M \) given by (2.4). Here the eigenfunction \( \theta \) is the solution of the linear system (2.9)-(2.10) is given explicitly with the eigenvalue \( \Lambda \) in (2.8). From \( \tilde{L}G_\theta = G_\theta L \), the operator \( L = \partial_x + J\lambda - R \) is transformed to a new operator \( \tilde{L} \) in which \( J \) is unchanged and

\[ \tilde{R} = R - [J, \theta \Lambda \theta^{-1}] . \] (4.1)

For notational convenience, we introduce a \( 2 \times 2 \) matrix \( Q \) such that \( R = [J, Q] \), and hence

\[ Q = \frac{1}{2i} \begin{pmatrix} r & q \\ \theta & \theta A \end{pmatrix} . \] (4.2)

where the entries left blank are arbitrary and do not contribute to \( R \). From (4.1) it follows that

\[ \tilde{Q} = Q - \theta \Lambda \theta^{-1} \] (4.3)

which can be written in a quasideterminant structure as

\[ \tilde{Q} = Q + \begin{vmatrix} \theta & I_2 \\ \theta A & 0_2 \end{vmatrix} . \] (4.4)

We rewrite (4.3) as

\[ Q_{[2]} = Q_{[1]} - \theta_{[1]} \Lambda_1 \theta_{[1]}^{-1} \] (4.5)

where \( Q_{[1]} = Q, Q_{[2]} = \tilde{Q}, \theta_{[1]} = \theta_1 = \theta \) and \( \Lambda_1 = \Lambda \). Then after \( n \) repeated Darboux transformations, we have

\[ Q_{[n+1]} = Q_{[n]} - \theta_{[n]} \Lambda_n \theta_{[n]}^{-1} \] (4.6)
in which \( \theta[k] = \phi[k] \mid \phi \to \theta \). We express \( P_{[n+1]} \) in quasideterminant form as

\[
Q_{[n+1]} = Q + \begin{vmatrix}
\theta_1 & \theta_2 & \ldots & \theta_n & 0_2 \\
\theta_1 \Lambda_1 & \theta_2 \Lambda_2 & \ldots & \theta_n \Lambda_n & 0_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\theta_1 \Lambda_1^{n-2} & \theta_2 \Lambda_2^{n-2} & \ldots & \theta_n \Lambda_n^{n-2} & 0_2 \\
\theta_1 \Lambda_1^{n-1} & \theta_2 \Lambda_2^{n-1} & \ldots & \theta_n \Lambda_n^{n-1} & 0_2 \\
\theta_1 \Lambda_1^n & \theta_2 \Lambda_2^n & \ldots & \theta_n \Lambda_n^n & 0_2
\end{vmatrix},
\]

(4.7)

where each \( \theta_i, \Lambda_i \) as a \( 2 \times 2 \) matrix

\[
\theta_i = \begin{pmatrix}
\varphi_i & \psi_i^* \\
\psi_i & -\varphi_i^*
\end{pmatrix}, \quad \Lambda_i = \begin{pmatrix}
\lambda_i & 0 \\
0 & \lambda_i^*
\end{pmatrix}
\]

(4.8)
in which \( i = 1, \ldots, n \). Now let \( \Theta^{(n)} \) be a \( 2 \times 2n \) matrix such that

\[
\Theta^{(n)} = (\theta_1 \Lambda_1^n, \ldots, \theta_n \Lambda_n^n) = \begin{pmatrix}
\varphi^{(n)} \\
\psi^{(n)}
\end{pmatrix}
\]

(4.9)

where

\[
\varphi^{(n)} = (\lambda_1^n \varphi_1, \lambda_1^n \psi_1^*, \ldots, \lambda_n^n \varphi_n, \lambda_n^n \psi_n^*), \\
\psi^{(n)} = (\lambda_1^n \psi_1, -\lambda_1^n \varphi_1, \ldots, \lambda_n^n \psi_n, -\lambda_n^n \varphi_n^*)
\]
denote \( 1 \times 2n \) row vectors. Thus, (4.7) can be written as

\[
Q_{[n+1]} = Q + \begin{vmatrix}
\hat{\Theta} & \begin{pmatrix}
E_{2n-1} & 0 \\
E_{2n} & 0
\end{pmatrix}
\end{vmatrix},
\]

(4.10)

where \( \hat{\Theta} = (\theta_i \Lambda_j^{i-1}) \) and \( E = (e_{2n-1}, e_{2n}) \) denote \( 2n \times 2n \) and \( 2n \times 2 \) matrices respectively, in which \( e_i \) represents a column vector with 1 in the \( i^{th} \) row and zeros elsewhere. Hence, we obtain

\[
Q_{[n+1]} = Q + \begin{vmatrix}
\hat{\Theta} & \begin{pmatrix}
\varphi^{(n)} \\
\psi^{(n)}
\end{pmatrix} \\
\varphi^{(n)} & 0 \\
\psi^{(n)} & 0
\end{vmatrix}.
\]

(4.11)

Here we immediately see that a quasideterminant solution \( q_{[n+1]} \) of the Hirota equation (1.3) along with its complex conjugate \( r_{[n+1]} \) can be expressed as

\[
q_{[n+1]} = q + 2i \begin{vmatrix}
\hat{\Theta} & \begin{pmatrix}
e_{2n-1} \\
0
\end{pmatrix}
\end{vmatrix}, \quad r_{[n+1]} = r + 2i \begin{vmatrix}
\hat{\Theta} & \begin{pmatrix}
e_{2n-1} \\
0
\end{pmatrix}
\end{vmatrix},
\]

(4.12)

where it can be easily shown that the reduction \( r_{[n+1]} = q_{[n+1]}^* \) holds.
4.1 Explicit solutions

In order to construct explicit solutions for the Hirota equation (1.3), we consider the quasideterminant solution given by (4.12) in which we obtain

\[
q_{n+1} = q + 2i \begin{vmatrix}
\varphi_1 & \psi_1^* & \cdots & \varphi_n & \psi_n^* & 0 \\
\psi_1 & -\varphi_1^* & \cdots & \psi_n & -\varphi_n^* & 0 \\
\varphi_1 \lambda_1 & \psi_1 \lambda_1^* & \cdots & \varphi_n \lambda_n & \psi_n \lambda_n^* & 0 \\
\psi_1 \lambda_1 & -\varphi_1 \lambda_1^* & \cdots & \psi_n \lambda_n & -\varphi_n \lambda_n^* & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\varphi_1 \lambda_1^{n-1} & \psi_1 \lambda_1^{n-1} & \cdots & \varphi_n \lambda_n^{n-1} & \psi_n \lambda_n^{n-1} & 0 \\
\psi_1 \lambda_1^{n-1} & -\varphi_1 \lambda_1^{n-1} & \cdots & \psi_n \lambda_n^{n-1} & -\varphi_n \lambda_n^{n-1} & 1 \\
\end{vmatrix}. \tag{4.13}
\]

Here \( \varphi_j \) and \( \psi_j \) are scalar functions such that the eigenfunction \( \phi_j = (\varphi_j, \psi_j)^T \) denotes \( n \) distinct solutions of the spectral problem \( L(\phi_j) = M(\phi_j) = 0 \) with the associated eigenvalue \( \lambda_j \), where the operators \( L, M \) are given by (2.3)-(2.4) so that

\[
\phi_{j,x} + J \phi_j \lambda_j - R \phi_j = 0, \\
\phi_{j,t} + 4\beta J \phi_j \lambda_j^3 + 2U \phi_j \lambda_j^2 - 2V \phi_j \lambda_j - W \phi_j = 0, \tag{4.14}
\]

in which \( j = 1, \ldots, n \) and \( J, R, U, V, W \) are \( 2 \times 2 \) matrices given by (2.5)-(2.6). In the next section we will present some explicit solutions of the equation (1.3) for the cases \( n = 1, \ldots, 3 \). For the one-fold \( (n = 1) \), two-fold \( (n = 2) \) and three-fold \( (n = 3) \) Darboux transformations, the solution (4.13) yields

\[
q_{[2]} = q + 2i \begin{vmatrix}
\varphi_1 & \psi_1^* & 0 \\
\psi_1 & -\varphi_1^* & 1 \\
\varphi_1 \lambda_1 & \psi_1 \lambda_1^* & 0 \\
\end{vmatrix}, \tag{4.15}
\]

\[
q_{[3]} = q + 2i \begin{vmatrix}
\varphi_1 & \psi_1^* & \varphi_2 & \psi_2^* & 0 \\
\psi_1 & -\varphi_1^* & \psi_2 & -\varphi_2^* & 0 \\
\varphi_1 \lambda_1 & \psi_1 \lambda_1^* & \varphi_2 \lambda_2 & \psi_2 \lambda_2^* & 0 \\
\psi_1 \lambda_1 & -\varphi_1 \lambda_1^* & \psi_2 \lambda_2 & -\varphi_2 \lambda_2^* & 1 \\
\varphi_1 \lambda_1^2 & \psi_1 \lambda_1^{2^*} & \varphi_2 \lambda_2^2 & \psi_2 \lambda_2^{2^*} & 0 \\
\end{vmatrix}, \tag{4.16}
\]

and

\[
q_{[4]} = q + 2i \begin{vmatrix}
\varphi_1 & \psi_1^* & \varphi_2 & \psi_2^* & \varphi_3 & \psi_3^* & 0 \\
\psi_1 & -\varphi_1^* & \psi_2 & -\varphi_2^* & \psi_3 & -\varphi_3^* & 0 \\
\varphi_1 \lambda_1 & \psi_1 \lambda_1^* & \varphi_2 \lambda_2 & \psi_2 \lambda_2^* & \varphi_3 \lambda_3 & \psi_3 \lambda_3^* & 0 \\
\psi_1 \lambda_1 & -\varphi_1 \lambda_1^* & \psi_2 \lambda_2 & -\varphi_2 \lambda_2^* & \psi_3 \lambda_3 & -\varphi_3 \lambda_3^* & 0 \\
\varphi_1 \lambda_1^2 & \psi_1 \lambda_1^{2^*} & \varphi_2 \lambda_2^2 & \psi_2 \lambda_2^{2^*} & \varphi_3 \lambda_3^2 & \psi_3 \lambda_3^{2^*} & 0 \\
\varphi_1 \lambda_1^3 & \psi_1 \lambda_1^{3^*} & \varphi_2 \lambda_2^3 & \psi_2 \lambda_2^{3^*} & \varphi_3 \lambda_3^3 & \psi_3 \lambda_3^{3^*} & 0 \\
\end{vmatrix}, \tag{4.17}
\]

respectively. The quasideterminant solutions (4.15)-(4.16) can be expanded as

\[
q_{[2]} = q - 2i (\lambda_1 - \lambda_1^*) \frac{\varphi_1 \psi_1^*}{|\varphi_1|^2 + |\psi_1|^2} \tag{4.18}
\]
and

\[ q_{[3]} = q - 2i \frac{\Lambda_{11} \left( \Pi_{12} |\varphi_2|^2 + \Pi_{12}^* |\psi_2|^2 \right) \varphi_1 \psi_1^* + \Lambda_{22} \left( \Lambda_{12} |\varphi_1|^2 + \Lambda_{12}^* |\psi_1|^2 \right) \varphi_2 \psi_2^*}{|\lambda_1 - \lambda_2|^2 (|\varphi_1 \varphi_2^* + \psi_1 \psi_2^*|^2 + |\lambda_1 - \lambda_2|^2 (|\varphi_1 \psi_2 - \varphi_2 \psi_1|^2)}, \quad (4.19) \]

where

\[ \Lambda_{11} = \lambda_1 - \lambda_1^*, \quad \Lambda_{22} = \lambda_2 - \lambda_2^*, \quad \Lambda_{12} = (\lambda_1 - \lambda_2) (\lambda_1 - \lambda_2^*), \quad \Pi_{12} = (\lambda_1 - \lambda_2) (\lambda_1^* - \lambda_2). \]

Moreover, the solution (4.17) can be expressed in terms of determinants such that

\[ q_{[4]} = q - 2i \frac{D}{\Delta}, \quad (4.20) \]

in which

\[ D = \begin{vmatrix} \varphi_1 & \psi_1^* & \varphi_2 & \psi_2^* & \varphi_3 & \psi_3^* \\ \psi_1 & -\varphi_1^* & \psi_2 & -\varphi_2^* & \psi_3 & -\varphi_3^* \\ \varphi_1 \lambda_1 & \psi_1 \lambda_1^* & \varphi_2 \lambda_2 & \psi_2 \lambda_2^* & \varphi_3 \lambda_3 & \psi_3 \lambda_3^* \\ \psi_1 \lambda_1 & -\varphi_1 \lambda_1^* & \psi_2 \lambda_2 & -\varphi_2 \lambda_2^* & \psi_3 \lambda_3 & -\varphi_3 \lambda_3^* \\ \varphi_1 \lambda_1^2 & \psi_1 \lambda_1^2 & \varphi_2 \lambda_2^2 & \psi_2 \lambda_2^2 & \varphi_3 \lambda_3^2 & \psi_3 \lambda_3^2 \\ \varphi_1 \lambda_1^3 & \psi_1 \lambda_1^3 & \varphi_2 \lambda_2^3 & \psi_2 \lambda_2^3 & \varphi_3 \lambda_3^3 & \psi_3 \lambda_3^3 \end{vmatrix}, \quad (4.21) \]

\[ \Delta = \begin{vmatrix} \varphi_1 & \psi_1^* & \varphi_2 & \psi_2^* & \varphi_3 & \psi_3^* \\ \psi_1 & -\varphi_1^* & \psi_2 & -\varphi_2^* & \psi_3 & -\varphi_3^* \\ \varphi_1 \lambda_1 & \psi_1 \lambda_1^* & \varphi_2 \lambda_2 & \psi_2 \lambda_2^* & \varphi_3 \lambda_3 & \psi_3 \lambda_3^* \\ \psi_1 \lambda_1 & -\varphi_1 \lambda_1^* & \psi_2 \lambda_2 & -\varphi_2 \lambda_2^* & \psi_3 \lambda_3 & -\varphi_3 \lambda_3^* \\ \varphi_1 \lambda_1^2 & \psi_1 \lambda_1^2 & \varphi_2 \lambda_2^2 & \psi_2 \lambda_2^2 & \varphi_3 \lambda_3^2 & \psi_3 \lambda_3^2 \\ \varphi_1 \lambda_1^3 & \psi_1 \lambda_1^3 & \varphi_2 \lambda_2^3 & \psi_2 \lambda_2^3 & \varphi_3 \lambda_3^3 & \psi_3 \lambda_3^3 \end{vmatrix}. \quad (4.22) \]

5 **Particular solutions**

5.1 **Solutions for zero seed**

For \( q = r = 0 \), the spectral problem (4.14) becomes

\[ \phi_{j,t} + J \phi_j \lambda_j = 0, \quad \phi_{j,t} + \left( 4\beta \lambda_j^3 + 2\alpha \lambda_j^2 \right) J \phi_j = 0, \quad (5.1) \]

which has solution \( \phi_j = (\varphi_j, \psi_j)^T \) such that

\[ \varphi_j (x, t, \lambda_j) = e^{-i \left[ \lambda_j x + (2\alpha \lambda_j^2 + 4\beta \lambda_j^3) t \right]}, \]

\[ \psi_j (x, t, \lambda_j) = e^{i \left[ \lambda_j x + (2\alpha \lambda_j^2 + 4\beta \lambda_j^3) t \right]}, \quad (5.2) \]

where \( j = 1, \ldots, n \).
Case I \((n = 1)\)

By letting \(\lambda_1 = \xi + i \eta\) and substituting the functions \(\varphi_1\) and \(\psi_1\) given by (5.2) into (4.18), we obtain the one-soliton solution of the Hirota equation (1.3) as

\[
q_2 = 2\eta e^{-2i(\xi x + 2[\alpha\xi - \eta^2] + 2\beta[\xi^3 - 3\xi \eta^2]) t]} \sech \left( 2\eta x + 8 [\alpha\xi \eta + \beta (3\xi^2 \eta - \eta^3) t] \right)
\]

which yields

\[
|q_2|^2 = 4\eta^2 \sech^2 \left( 2\eta x + 8 [\alpha\xi \eta + \beta (3\xi^2 \eta - \eta^3) t] \right).
\]

This solution is plotted in Fig. 1.

![Fig. 1](image-url) (Color online) One-soliton solution \(|q_2|\) of the HE (1.3) when \(\alpha = \beta = 1, \xi = 0.8, \eta = 1.6\). Figure (a) describes its surface and (b) gives its profiles at different times \(t = -1.8\) (red), \(t = 0\) (blue), \(t = 1.8\) (green).

Case II \((n = 2)\)

Let \(\lambda_1 = \xi + \eta_1\) and \(\lambda_2 = \xi + \eta_2\) such that \(\eta_1 \eta_2 \neq 0\). By substituting the corresponding eigenfunctions \(\varphi_1, \psi_1\) and \(\varphi_2, \psi_2\), given by (5.2), into (4.19), we obtain the two-soliton solution of the Hirota equation (1.3) as

\[
q_3 = 4 (\eta_1^2 - \eta_2^2) \frac{\eta_1 e^{-ig_1} \cosh f_2 - \eta_2 e^{-ig_2} \cosh f_1}{(\eta_1 - \eta_2)^2 \cosh F_1 + (\eta_1 + \eta_2)^2 \cosh F_2 - 4\eta_1 \eta_2 \cos F_3}
\]

which yields

\[
|q_3|^2 = 16 (\eta_1^2 - \eta_2^2)^2 \frac{\eta_2^2 \cosh^2 f_1 + \eta_1^2 \cosh^2 f_2 - 2\eta_1 \eta_2 \cosh f_1 \cosh f_2 \cos F_3}{[(\eta_1 - \eta_2)^2 \cosh F_1 + (\eta_1 + \eta_2)^2 \cosh F_2 - 4\eta_1 \eta_2 \cos F_3]^2},
\]

where

\[
f_1 = 2\eta_1 [x + 4 (\alpha \xi + \beta [3\xi^2 - \eta_1^2]) t],
\]
\[ f_2 = 2\eta_2 \left[ x + 4 (\alpha \xi + \beta \left[ 3\xi^2 - \eta_2^2 \right] \right] t, \]
\[ g_1 = 2\xi x + 4 \left[ \alpha (\xi^2 - \eta_1^2) + 2\beta \xi (\xi^2 - 3\eta_1^2) \right] t, \]
\[ g_2 = 2\xi x + 4 \left[ \alpha (\xi^2 - \eta_2^2) + 2\beta \xi (\xi^2 - 3\eta_2^2) \right] t \]

and \( F_1 = f_1 + f_2, F_2 = f_1 - f_2, F_3 = g_1 - g_2 \) such that

\[ F_1 = 2 (\eta_1 + \eta_2) \left[ x + 4 (\alpha \xi + \beta \left[ 3\xi^2 + \eta_1 \eta_2 - \eta_1^2 - \eta_2^2 \right] \right] t, \]
\[ F_2 = 2 (\eta_1 - \eta_2) \left[ x + 4 (\alpha \xi + \beta \left[ 3\xi^2 - \eta_1 \eta_2 - \eta_1^2 - \eta_2^2 \right] \right] t, \]
\[ F_3 = 4 (\eta_2^2 - \eta_1^2) \left[ \alpha + 6\beta \xi \right] t. \]

By choosing appropriate parameters, the two-soliton solution of the Hirota equation (1.3) is plotted in Fig. 2.

![Fig. 2.](image)

**Fig. 2.** (Color online) Two-soliton solution \(|q_{[3]}|\) of the HE (1.3) when \(\alpha = \beta = 1, \xi = 0.5, \eta_1 = 0.7\) and \(\eta_2 = 1.1\). (a) Surface diagram. (b) Contour diagram.

**Case III \((n = 3)\)**

In this case, we have three eigenvalues \(\lambda_1, \lambda_2\) and \(\lambda_3\). Let us choose \(\lambda_1 = i, \lambda_2 = 2i\) and \(\lambda_3 = 3i\). By substituting the corresponding eigenfunctions \((\varphi_1, \psi_1)^T, (\varphi_2, \psi_2)^T\) and \((\varphi_3, \psi_3)^T\), given by (5.2), into (4.20), we obtain the three-soliton solution of the Hirota equation (1.3). By choosing appropriate parameters, this solution is plotted in Fig. 3.

**5.2 Solutions for non-zero seed**

In this subsection, for \(q, r \neq 0\) and \(r = q^*\), we take \(q = ce^{i\mu}\) as a plane wave solution of the Hirota equation (1.3), where \(\mu = ax + bt\) in which \(a, b, c \in \mathbb{R}\) under the condition \(b = \alpha (2c^2 - a^2) + \beta (a^3 - 6ac^2)\). We use this as a seed solution. Substituting \(q = ce^{i\mu}\) into the linear system (4.14)
and then solving for the eigenfunction $\phi_j = (\phi_j, \psi_j)^T$, we obtain

$$
\begin{align*}
\phi_j(x, t, \lambda_j) &= e^{\frac{i}{2} \mu} \left( c_j e^{\frac{i}{2} \gamma_j} + e_j e^{-\frac{i}{2} \gamma_j} \right), \\
\psi_j(x, t, \lambda_j) &= e^{-\frac{i}{2} \mu} \left( \tilde{c}_j e^{\frac{i}{2} \gamma_j} + \tilde{e}_j e^{-\frac{i}{2} \gamma_j} \right),
\end{align*}
$$

(5.7)

where

$$
\gamma_j = s_j (x + k_j t),
$$

$$
\tilde{c}_j = \frac{i c_j}{2c} \left( a + 2\lambda_j + s_j \right),
$$

$$
\tilde{e}_j = \frac{i e_j}{2c} \left( a + 2\lambda_j - s_j \right)
$$

in which $s_j = \sqrt{(a + 2\lambda_j)^2 + 4c^2}$, $k_j = \alpha (2\lambda_j - a) + \beta \left( a^2 - 2a\lambda_j + 4\lambda_j^2 - 2c^2 \right)$ and $c_j, e_j$ are arbitrary constants such that $j = 1, \ldots, n$.

**Case IV** ($n = 1$)

Let the eigenvalue $\lambda_1 = \xi + i\eta$. For simplicity, choose $a = -2\xi$ and $c_1 = e_1 = c$. Substituting the seed solution $q = ce^{i\mu}$ and the functions $\phi_1, \psi_1$ given by (5.7), into (4.18), we obtain the following breather solution

$$
q[2] = ce^{i\mu} \left( 1 - 2\eta \cosh(\Omega t) - i\omega \sinh(\Omega t) + \eta \cos[2\omega(x + \Gamma t)] + \omega \sin[2\omega(x + \Gamma t)] \right) \left( e^2 \cosh(\Omega t) + \eta^2 \cos[2\omega(x + \Gamma t)] + \eta \omega \sin[2\omega(x + \Gamma t)] \right),
$$

(5.8)

where

$$
\mu = -2\xi x + \left[ 2a \left( e^2 - 2\xi^2 \right) + 4\beta \left( 3c^2 \xi - 2\xi^3 \right) \right] t,
$$

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Fig. 4. (Color online) Breather solution $|q_{[2]}|$ of the HE (1.3) when $\alpha = \beta = 1$, $\xi = 0.04$ and $\eta = 0.76$. (a) Surface diagram. (b) Density diagram.

\[
\begin{align*}
\omega &= \sqrt{c^2 - \eta^2}, \\
\Omega &= 4\eta\omega(\alpha + 6\beta\xi), \\
\Gamma &= 4\alpha\xi + 2\beta(6\xi^2 - 2\eta^2 - c^2).
\end{align*}
\]

Thus, we have

\[|q_{[2]}|^2 = c^2 \frac{F^2 + G^2}{H^2},\]

where

\[\begin{align*}
F &= (2\eta^2 - c^2) \cosh(\Omega t) + \eta^2 \cos[2\omega(x + \Gamma t)] + \eta\omega \sin[2\omega(x + \Gamma t)], \\
G &= 2\eta\omega \sinh(\Omega t), \\
H &= c^2 \cosh(\Omega t) + \eta^2 \cos[2\omega(x + \Gamma t)] + \eta\omega \sin[2\omega(x + \Gamma t)].
\end{align*}\]

Fig. 4 shows the dynamical evolution of the breather solution of the Hirota equation (1.3).

6 Conclusion

In conclusion, we have studied a standard Darboux transformation to construct quasideterminant solutions for the Hirota equation (1.3). These quasideterminants are expressed in terms of solutions of the linear partial differential equations given by (4.14). It should be highlighted that these quasideterminant solutions arise naturally from the Darboux transformation we present here. Furthermore, the multisoliton and breather solutions for zero and non-zero seeds have been given as particular examples for the HE. Examples of these particular solutions are plotted in the figures 1 – 4 with the chosen parameters. Finally, we point out that the method we have presented in this paper allows us to construct exact solutions for other integrable nonlinear evolution equations such as [11, 26, 34].
References

[1] G. P. Agrawal, Nonlinear Fiber Optics (Academic Press, 2007).

[2] A. Ankiewicz, J.M. Soto-Crespo and N. Akhmediev, Rogue waves and rational solutions of the Hirota equation, Phys. Rev. E 81 (2010), 046602.

[3] D. J. Benney and A. C. Newell, The propagation of nonlinear wave envelopes, J. Math. Phys. 46 (1967), 133–139.

[4] D. J. Benney and G. J. Roskes, Wave instabilities, Stud. Appl. Math. 48 (1969), 377–385.

[5] H. H. Chen, Y. C. Lee and C. S. Liu, Integrability of nonlinear Hamiltonian systems by inverse scattering method, Physica Scripta 20 (1979), 490–492.

[6] G. Darboux, Comptes Rendus de l’Académie des Sciences 94 (1882), 1456–1459.

[7] I. Gelfand and V. Retakh, Determinants of the matrices over noncommutative rings, Funct. Anal. App. 25 (1991), 91–102.

[8] I. Gelfand, S. Gelfand, V. Retakh, and R. L. Wilson, Quasideterminants, Adv. Math. 193 (2005), 56–141.

[9] C.R. Gilson and J.J.C. Nimmo, On a direct approach to quasideterminant solutions of a noncommutative KP equation, J. Phys. A: Math. Theor. 40 (2007), 3839–3850.

[10] C. R. Gilson, M. Hamanaka and S.C. Huang and J. J. C. Nimmo, Soliton Solutions of Noncommutative Anti-Self-Dual Yang-Mills Equations, arXiv:2004.01718.

[11] C.R. Gilson and S.R. Macfarlane, Dromion solutions of noncommutative Davey-Stewartson equations, J. Phys. A: Math. Theor. 42 (2009), 235232.

[12] A. Hasegawa and F. Tappert, Transmission of stationary nonlinear optical pulses in dispersive dielectric fibres I. Anomalous dispersion, Appl. Phys. Lett. 23 142 (1973).

[13] A. Hasegawa and F. Tappert, Transmission of stationary nonlinear optical pulses in dispersive dielectric fibres II. Normal dispersion, Appl. Phys. Lett. 23 171 (1973).

[14] R. Hirota, Exact envelope-soliton solutions of a nonlinear wave equation, J. Math. Phys. 14 (1973), 805–809.

[15] D. J. Kaup and A. C. Newell, An exact solution for a derivative nonlinear Schrödinger equation, J. Math. Phys. 19 (1978), 798–801.

[16] Y. Kivshar and G. Agrawal, Optical Solitons: From fibers to photonic crystals, Academic Press (2003).

[17] Y. Kodama, Optical solitons in a monomode fiber, J. Stat. Phys. 39 (1985), 597–614.

[18] Y. Kodama and A. Hasegawa, Nonlinear pulse propagation in a monomode dielectric guide, IEEE J. Quantum Electron. 23 (1987), 510–524.
C.X. Li and J.J.C. Nimmo, Darboux transformations for a twisted derivation and quasideterminant solutions to the super KdV equation, *Proc. R. Soc. A* **466** (2010), 2471–2493.

L. Li, Z. Wu, L. Wang and J. He, Higher-order rogue waves for the Hirota equation, *Ann. Phys.* **334** (2013), 198–211.

B.A. Malomed and D. Mihalache, Nonlinear waves in optical and matter-wavemedia: A topical survey of recent theoretical and experimental results, *Rom. J. Phys.* **64** (2019), 106.

V.B. Matveev, Darboux transformation and explicit solutions of the Kadomtsev-Petviashvily equation, depending on functional parameters, *Lett. Math. Phys.* **3** (1979), 213–216.

V.B. Matveev and M.A. Salle, Darboux transformations and solitons (Springer Series in Nonlinear Dynamics, Springer-Verlag, Berlin, 1991).

D. Mihalache, N. Truta, and L.-C. Crasovan, Painlevé analysis and bright solitary waves of the higher-order nonlinear Schrödinger equation containing third-order dispersion and self-steepening term, *Phys. Rev. E* **56** (1997), 1064–1070.

J. J. C. Nimmo, C. R. Gilson and Y. Ohta, Applications of Darboux transformations to the self-dual Yang-Mills equations, *Theor. Math. Phys.* **122** (2000), 239–246.

J.J.C. Nimmo and H. Yilmaz, On Darboux Transformations for the derivative nonlinear Schrödinger equation, *J. Nonlinear Math. Phys.* **21** (2014), 278–293.

J.J.C. Nimmo and H. Yilmaz, Binary Darboux transformation for the Sasa-Satsuma equation, *J. Phys. A: Math. Theor.* **48** (2015), 425202.

HWA. Riaz, Noncommutative coupled complex modified Korteweg-de Vries equation: Darboux and binary Darboux transformations, *Mod. Phys. Lett. A* **34** (2019), 1950054.

Y. Tao and J. He, Multisolitons, breathers and rogue waves for the Hirota equation generated by the Darboux transformation, *Phys. Rev. E* **85** (2012), 026601.

N. Sasa and J. Satsuma, New type of soliton solutions for a higher-order nonlinear Schrödinger equation, *J. Phys. Soc. Japan* **60** (1991), 409–417.

M. Wadati, The exact solution of the modified Korteweg-de Vries equation, *J. Phys. Soc. Japan* **32** (1972), 1681.

H. Wu, J. Liu and C. Li, Quasideterminant solutions of the extended noncommutative Kadomtsev-Petviashvili hierarchy, *Theor Math Phys* **192** (2017), 982–999.

Z. Yan and C. Dai, Optical rogue waves in the generalized inhomogeneous higher-order nonlinear Schrödinger equation with modulating coefficients, *J. Opt.* **15** (2013), 064012.

H. Yilmaz, Exact solutions of the Gerdjikov-Ivanov equation using Darboux transformations, *J. Nonlinear Math. Phys.* **22** (2015), 32–46.

V. E. Zakharov, Stability of periodic waves of finite amplitude on the surface of a deep fluid, *J. Appl. Mech. Tech. Phys.* **9** (1968), 190–194.
[36] V. E. Zakharov, Collapse of Langmuir waves, *Sov. Phys. JETP* **35** (1972), 908–914.

[37] V. E. Zakharov and A. B. Shabat, Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media, *Sov. Phys. JETP* **34** (1972), 62–69.

[38] H.Q. Zhang, Y. Wang and W.X. Ma, Binary Darboux transformation for the coupled Sasa-Satsuma equations, *Chaos* **27** (2017), 073102.