INVERSE LITTLEWOOD-OFFORD PROBLEMS FOR QUASI-NORMS

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Abstract. Given a star-shaped domain $K \subseteq \mathbb{R}^d$, $n$ vectors $v_1,\ldots,v_n \in \mathbb{R}^d$, a number $R > 0$, and i.i.d. random variables $\eta_1,\ldots,\eta_n$, we study the geometric and arithmetic structure of the set of vectors $V = \{v_1,\ldots,v_n\}$ under the assumption that the small ball probability

$$
\sup_{x \in \mathbb{R}^d} \mathbb{P}\left( \sum_{j=1}^n \eta_j v_j \in x + RK \right)
$$

does not decay too fast as $n \to \infty$. This generalises the case where $K$ is the Euclidean ball, which was previously studied in $[NV11,TV12]$.

1. Introduction

1.1. Background. A body $K \subseteq \mathbb{R}^d$ is said to be a star-shaped domain if for every $x \in K$, $tx \in K$ for every $t \in [0,1]$. In this note, $K$ will always assumed to be compact. Given a random vector $X$ in $\mathbb{R}^d$ and $R > 0$, define the small-ball probability

$$
\rho^K_R(X) = \sup_{x \in \mathbb{R}^d} \mathbb{P}\left( X \in x + RK \right).
$$

(1.1)

In particular, if $V = \{v_1,\ldots,v_n\} \subseteq \mathbb{R}^d$ is a set of $n$ fixed vectors, $\eta_1,\ldots,\eta_n$ are i.i.d. random variables, then one can consider the following random vector,

$$
X_V = \sum_{j=1}^n \eta_j v_j.
$$

(1.2)

It is known that the asymptotic behaviour of $\rho^R_{B^d_2}(X_V)$ as $n \to \infty$ is closely related to the various structural aspects of the set $V$. Here and in what follows, $B^d_2$ denotes the Euclidean ball in $\mathbb{R}^d$. We refer the reader to $[Erd45,FF88,NV11,NV13,RV08,TV09,TV10,TV12]$ to name just a few, where this type of questions is discussed, as well as some interesting applications. In particular, we refer the reader to $[NV11]$, which includes some enlightening remarks and examples of the relation between the behaviour of $\rho^R_{B^d_2}(X_V)$ and and the structure of $V$, as well as to $[NV13]$, which gives a broad introduction to the topic.

In the results of $[NV11,TV12]$, one always assumes that the norm on $\mathbb{R}^d$ is the Euclidean norm. One of the key technical tools in the proofs is Esseen type estimates, which relate the small ball probability to the behaviour of the characteristic function of $X_V$. See for example $[Ess66]$ and $[TV06$, Section 7.3$]$. Esseen’s inequality for a general random vector $X$
saris that for every \( \varepsilon > 0 \),
\[
\rho_R^B(X) \leq C^d \left( \frac{R}{\sqrt{d}} + \frac{\sqrt{d}}{\varepsilon} \right)^d \int_{\xi \in B_2} |\mathbb{E} \exp(i \langle X, \xi \rangle)| d\xi.
\] (1.3)

In (1.3) and in what follows, \( C \) denotes an absolute constant. In [FGG14], based on previous work from [FG11], an Esseen type estimate was obtained for a general quasi-norm. If \( K \subseteq \mathbb{R}^d \)
is a centrally symmetric star-shaped domain in \( \mathbb{R}^d \), then the functional
\[
\|x\|_K = \inf \{ t > 0 \mid x \in tK \},
\] (1.4)
is a quasi-norm, that is, \( \| \cdot \|_K \) behaves like a norm, with the only exception that instead of the triangle inequality, there exists a number \( C_K \geq 1 \) such that for every \( x, y \in \mathbb{R}^d \),
\[
\|x + y\|_K \leq C_K (\|x\|_K + \|y\|_K).
\]
The case \( C_K = 1 \) corresponds to the case when \( \| \cdot \|_K \) is a norm and \( K \) is convex. If we omit the assumption that \( K \) is centrally symmetric then we do not have \( \|x\|_K = ||-x||_K \). In this note we do not need to assume that \( K \) is centrally symmetric. The following Esseen type estimate was shown in [FGG14].
\[
\rho_R^K(X) \leq (\kappa(K)R)^d \int_{\mathbb{R}^d} |\mathbb{E} \exp(i \langle X, \xi \rangle)| e^{-\frac{2d|\xi|^2}{\gamma_d d}} d\xi = \kappa(K)^d \cdot \mathcal{I} \left( \frac{1}{R} \right) X, \quad (1.5)
\]
where we deonte
\[
\kappa(K) = C_K \sqrt{\frac{2}{\pi}} \left( \frac{\mu_d(K)}{\gamma_d(K)} \right)^{1/d}, \quad \mathcal{I}(X) = \int_{\mathbb{R}^d} |\mathbb{E} \exp(i \langle X, \xi \rangle)| e^{-\frac{|\xi|^2}{\gamma_d d}} d\xi,
\]
\( \gamma_d(K) \) being the \( d \)-dimensional gaussian measure of \( K \), and \( \mu_d(K) \) its Lebesgue measure. In particular, if \( X = X_V \) as defined in (1.2), we have
\[
\mathcal{I}(X_V) \equiv \int_{\mathbb{R}^d} \left| |\mathbb{E} \exp \left( i \left( \sum_{j=1}^n \eta_j v_j, \xi \right) \right) \right| e^{-\frac{|\xi|^2}{2}} d\xi, \quad (1.6)
\]
where in (*) we used the fact the \( \eta_j \)'s are independent. Inequality (1.5), as well as inequality (1.3), imply that there is a relation between the behaviour of \( \rho_R^K(X_V) \) and the arithmetic behaviour of the vector \( X_V \). Note also that (1.6) implies that it is natural to consider random variables \( \eta_j \)'s that satisfy some anti-concentration property. See Section 2.1 and Section 2.2, and in particular the anti-concentration conditions (2.2) and (2.7). Therefore, given (1.5) and (1.6), it is natural to consider the following type of problems, also known as Inverse Littlewood-Offord Problems:

\textbf{Assume that} \( \rho_R^K(X_V) \) is large. \textbf{Show that the set} \( \{v_1, \ldots, v_n\} \subseteq \mathbb{R}^d \) \textbf{is well-structured.}

Clearly, the term ‘large’ should be formulated quantitatively, and the term ‘well-structured’ can have different meanings. In this note, we discuss two ways to obtain ‘well-structured’ sets. One way is to consider sets whose elements are all found near a given subspace of \( \mathbb{R}^d \) (again, the term ‘near’ can be made precise). In Section 2.1, we show that if \( \rho_R^K(X_V) \) does
not decay too fast as \( n \to \infty \), then many of the vectors in the set \( \{v_1, \ldots, v_n\} \subseteq \mathbb{R}^d \) are ‘well-concentrated’ around a given hyperplane. See Section 2.1 for the exact formulation. Then, in Section 2.2, we show that if \( \rho_K^R(X_V) \) does not decay too fast, then the set \( \{v_1, \ldots, v_n\} \subseteq \mathbb{R}^d \) can be approximated with a set which has some arithmetic structure. See Section 2.2 for the exact definitions and formulation. Finally, Section 3 and Section 4 are dedicated to the proofs of the main theorems.

One point which is worth emphasising is the following. In the study of many asymptotic problems, including the ones discussed in [NV11, TV12], one is primarily interested in the asymptotic behaviour as \( n \to \infty \). In particular, since all norms in \( \mathbb{R}^d \) are equivalent, any Euclidean result trivially yields a result for a general norm. For a quasi-norm, trivial bounds can also be easily obtained. For a quasi-norm, trivial bounds can also be deduced from the Euclidean results. The main purpose of this note is to obtain an estimate which is better than these trivial conclusions and to extend the results of [NV11, TV12] to a non-Euclidean setting. See Section 2.3, for a comparison of the previously obtained results with the results of this note.

**Notations.** For a star-shaped body \( K \subseteq \mathbb{R}^d \), we let \( \| \cdot \|_K \) be defined as in (1.4). In the special case of the \( \ell^d_p \) norm, for \( p \in (0, \infty] \) we denote

\[
|x|_p = \|x\|_{B^d_p} = \left( \sum_{j=1}^d |x_j|^p \right)^{1/p}.
\]  

(1.7)

Note that if \( p \geq 1 \), (1.7) gives a norm and for \( p \leq 1 \), (1.7) gives a quasi-norm with \( C_{B^d_p} = 2^{1/p-1} \).

For a set \( S \subseteq \mathbb{R}^d \) and a vector \( v \in \mathbb{R}^d \), denote

\[
\text{dist}_K(v, S) = \inf \{ \|x - s\|_K \mid s \in S \}.
\]

In particular, \( \text{dist}_2(v, S) = \text{dist}_{B^d_2}(v, S) \), and \( \text{dist}_\infty(v, S) = \text{dist}_{B^d_\infty}(v, S) \).

Given a star-shaped domain \( K \subseteq \mathbb{R}^d \) and \( p \in (0, \infty] \), denote

\[
\omega_p(K) = \inf \{ t > 0 \mid B^d_p \subseteq tK \},
\]  

(1.8)

and also

\[
\frac{1}{W_p(K)} = \sup \{ t > 0 \mid tK \subseteq B^d_p \}.
\]  

(1.9)

Note that since we have

\[
\frac{1}{\omega_p(K)} B^d_p \subseteq K \subseteq W_p(K) B^d_p,
\]

it follows that for every \( x \in \mathbb{R}^d \),

\[
\frac{1}{W_p(K)} |x|_p \leq \|x\|_K \leq \omega_p(K) |x|_p.
\]  

(1.10)

In this note, \( C \) always denotes an absolute constant. If an implied constant depends on a parameter, say \( \gamma \), we write \( C(\gamma) \). Also, if \( F \) is a finite set and \( k \) is a positive integer, denote

\[
kF = \left\{ \sum_{j=1}^k v_j \mid v_j \in F \right\}.
\]
If \( \alpha \) is a real number which is not an integer, then \( \alpha F \) denotes the dilation of \( F \), that is \( \alpha F = \{ \alpha x \mid x \in F \} \). \( |F| \) denotes the cardinality of any finite set \( F \). If \( F \) is any set, for example, if \( F \) is a star-shaped body, then \( \mu_d(F) \) denotes its Lebesgue measure, while \( \gamma_d(F) \) denotes its \( d \)-dimensional gaussian measure.

2. Statement of the main results

2.1. Concentration near a hyperplane. The first result in this note shows that if the concentration function \( \rho^K_R(X_V) \) is asymptotically large, then many vectors are necessarily close to a given hyperplane in \( \mathbb{R}^d \). We begin by fixing some notation. For a real number \( a \), let

\[
\|a\|_T = \inf_{z \in \mathbb{Z}} |a - \pi z|,
\]

(2.1)

where \( T = \mathbb{R}/\pi \mathbb{Z} \). As mentioned above, from (1.6) it is natural to assume some bound on \( |\mathbb{E} \exp(i\eta_j v_j, \xi)| \). For the first theorem, we will use the following condition. There exists a number \( c_\eta > 0 \) such that for every \( a \in \mathbb{R} \), we have

\[
|\mathbb{E} \exp(i\eta a)| \leq \exp \left( -c_\eta \|a\|^2_T \right).
\]

(2.2)

Condition (2.2) can be thought of as an anti-concentration assumption. Note that, for example, symmetric Bernoulli random variables satisfy (2.2), since in this case we have

\[
|\mathbb{E} \exp(i\eta a)| = |\cos(a)| \leq 1 - \frac{2}{\pi^2} \|a\|^2_T \leq \exp \left( -\frac{2}{\pi^2} \|a\|^2_T \right).
\]

The main tool in the proof of Theorem 2.1 is the following proposition.

**Proposition 2.1.** Let \( k \leq n \) be integers. Let \( V = \{v_1, \ldots, v_n\} \subseteq \mathbb{R}^d \) be a set of fixed vectors. Assume that \( \eta_1, \ldots, \eta_n \) are i.i.d. random variables that satisfy (2.2). Assume also that for every hyperplane \( H \subseteq \mathbb{R}^d \), there exists at least \( n - k \) vectors satisfying \( \text{dist}_2(v_j, H) \geq R \). Then

\[
\mathcal{I}(X_V) \leq \left( 80 \frac{R + 1}{R} \sqrt{\frac{d}{d + c_\eta k}} \right)^d.
\]

(2.3)

The main result of this section is the following.

**Theorem 2.1.** Let \( V = \{v_1, \ldots, v_n\} \subseteq \mathbb{R}^d \) and \( \eta_1, \ldots, \eta_n \) be i.i.d. random variables satisfying (2.2). Assume that there exists \( k \leq n \) such that

\[
\rho^K_R(X_V) \geq (40 \kappa(K))^d \left( \frac{d}{d + c_\eta k} \right)^{d/2}.
\]

Then there exists a hyperplane \( H \) in \( \mathbb{R}^d \) for which at least \( n - k \) vectors from \( V \) satisfy

\[
\text{dist}_2(v_j, H) \leq R.
\]

In particular, using (1.10), we have

\[
\text{dist}_K(v_j, H) \leq \omega_2(K)R.
\]

(2.4)

Theorem 2.1 implies the following.
Corollary 2.1. Let $A > 0$ be a positive constant. Assume that for all $n$ sufficiently large, we have
\[ \rho^K_n(X_V) \geq n^{-A}. \]
Then there exist at least $n - k$ vectors in $V$ satisfying
\[ \text{dist}_K(v_j, H) \leq \omega_2(K) R, \]
and $k$ satisfies
\[ k \leq Cd \frac{\kappa(K)^2 n^{2A/d} - 1}{c_\eta}. \]

Remark 2.1. The Euclidean case of Theorem 2.1 is a key ingredient in the proof of the main theorem of [TV12]. More specifically, assuming that the $\eta_j$’s are Bernoulli random variables and defining
\[ P_{R^d}^\text{ld}(n) = \sup \left\{ \rho_{[R]}^d(V) \left| V \subseteq \mathbb{R}^d, |V| = n, |v|_2 \geq 1 \forall v \in V \right\}, \quad (2.5) \]
the authors prove that
\[ P_{R^d}^\text{ld}(n) = (1 + o(1)) 2^{-n} S(n, [R] + 1), \quad (2.6) \]
where $S(n, m)$ is the sum of the $m$ largest binomial coefficients $\binom{n}{m}$. Here the error term tends to 0 as $n \to \infty$. The authors also show that if $R$ is sufficiently close to an integer, then the error term in (2.6) can be removed. This problem had previously been studies in the one-dimensional case in [Erd45] and in the multi-dimensional case in [FF88] (again with the Euclidean norm). Similarly to (2.5), one could define
\[ P_{R}^K(n) = \sup \left\{ \rho_{[R]}^K(V) \left| V \subseteq \mathbb{R}^d, |V| = n, \|v\|_K \geq 1 \forall v \in V \right\}, \]
and ask whether an estimate similar to (2.6) could be obtained in some non-Euclidean setting. However, the proof of (2.6) in [TV12] makes heavy use of the rotation invariance of the Euclidean norm, and therefore it is not clear how (2.6) could be generalised.

2.2. Approximate arithmetic progression. We begin with the following definition.

Definition 2.1 (General arithmetic progression, GAP). A set $Q \subseteq \mathbb{R}^d$ is said to be general arithmetic progression (GAP), if there exist integers $r, L_1, \ldots, L_r$ and vectors $g_1, \ldots, g_r \in \mathbb{R}^d$ such that $Q$ can be written in the following way.
\[ Q = \left\{ \sum_{j=1}^r x_j g_j \left| x_j \in \mathbb{Z}, |x_j| \leq L_j, j \leq r \right\}. \]
The number $r$ is said to be the rank of $Q$, and is denoted by $\text{rank}(Q)$. $Q$ is said to be proper if we have
\[ |Q| = \prod_{j=1}^r L_j. \]
Finally, the vectors $g_1, \ldots, g_r \in \mathbb{R}^d$ are said to be generators of $Q$.

Remark 2.2. For every GAP, we have $|Q| \leq \prod_{j=1}^r L_j$. A GAP which is proper is a GAP in which no cancellation between the generators occurs.
A set which is GAP clearly has an additive structure. Hence, in the context of Littlewood-Offord problem, one could expect that if $\rho^K_R(X_V)$ does not decay too fast as $n \to \infty$, then $V$ should have additive structure, which is given by Definition 2.1. This problem has been studied in [NV11]. Here we consider the non-Euclidean setting.

As in Theorem 2.1, we need some anti-concentration condition to assure that we get efficient bounds in (1.5). Here we use the following: that if $\eta_1, \eta_2$ are independent copies of a random variable $\eta$, then there exists a number $C_\eta > 0$ such that

$$P\left(1 \leq |\eta_1 - \eta_2| \leq C_\eta\right) \geq \frac{1}{2}. \quad (2.7)$$

Note that Bernoulli variables satisfy (2.7), for example with $C_\eta = 2$. We can now state the second main result of this note.

**Theorem 2.2.** Fix absolute positive real numbers $A$ and $\varepsilon$. Let $K$ be a star-shaped domain in $\mathbb{R}^d$, and let $\eta_1, \ldots, \eta_n$ be i.i.d. random variables that satisfy (2.7). Assume that

$$\rho^K_R(X_V) \geq n^{-A}.$$

Let $n' \in [n^\varepsilon, n]$ be a positive integer, and assume that $n$ is sufficiently large compared to $d$, $A$, $\varepsilon$ and $\kappa(K)$. Then there exists a GAP $Q \subseteq \mathbb{R}^d$, a positive integer $k$ satisfying

$$\sqrt{\frac{n'}{640\pi^2 \sqrt{d \log (n^\varepsilon \kappa(K))}}} \leq k \leq \sqrt{n'},$$

and a number $\alpha$ which depends only on the constant $C_\eta$ from (2.7), such that

1. **$Q$ has small rank and cardinality:**

   $$\text{rank}(Q) \leq C\left(d + \frac{A}{\varepsilon}\right),$$

   $$|Q| \leq C(A,d,\varepsilon) \frac{(n')^{d-\text{rank}(Q)}}{\rho^K_R(X_V)}.$$

2. **$Q$ approximates $V$ in the $K$ quasi-norm:** At least $n - n'$ elements of $v \in V$ satisfy

   $$\text{dist}_K(v, Q) \leq C(\eta)\frac{\omega_\infty(K)R}{dk}.$$

3. **$Q$ has full dimension:** There exists $C' \leq C d \alpha$ such that

   $$\{-1, 1\}^d \subseteq C'kR Q.$$

4. **The generators of $Q$ have bounded $K$ quasi-norm:**

   $$\max_{1 \leq j \leq r} \|g_j\|_K \leq C(A, d, \varepsilon) C^K_{\kappa + 1} \left(\frac{dk}{R} \max_{v \in V} \|v\|_K + \omega_\infty(K)\right). \quad (2.8)$$

**Remark 2.3.** Note that when $K$ is not convex, that is, when $\| \cdot \|_K$ is a quasi-norm but not a norm, we have $C_K > 1$, in which case (2.8) does not give a sublinear bound (in $n$) on the norm.
2.3. **Comparing previous and new results.** As discussed in Section 1.1, the main purpose of this note is to show that in some cases, one can obtain estimates which are better than estimates which are trivially obtained from using the results in the Euclidean setting. This is true for both Theorem 2.1 and for Theorem 2.2. Recall that by (1.8) and (1.9), we have that

$$\frac{1}{\omega_2(K)} B^d_2 \subseteq K \subseteq W_2(K) B^d_2.$$  \hfill (2.9)

Now, (2.9) implies that

$$\rho^K_R(X_V) \leq \rho^{B^d_2}_{W_2(K)R}(X_V).$$

If, in addition, we use the fact that \(\| \cdot \|_K \leq \omega_2(K) \cdot 2\), we can use the Euclidean version of Theorem 2.1 to conclude that

$$d_K(v_j, H) \leq \omega_2(K) W_2(K) R.$$ \hfill (2.10)

If, for example, we assume that \(K\) is convex, that is, \(\| \cdot \|_K\) is a norm, then by taking a linear transformation of \(K\), we may assume that the Euclidean unit ball is the ellipsoid of maximal volume contained in \(K\), in which case \(B^d_2 \subseteq K \subseteq \sqrt{d} B^d_2\), see for example [Bal97]. This implies that we have \(\omega_2(K) = 1\) and \(W_2(K) \leq \sqrt{d}\). Thus, in general, (2.10) can be a worse bound than (2.4).

Similarly, by using the Euclidean version of Theorem 2.2, if we assume that \(\rho^K_R(X_V) \geq n^{-A}\) then we have \(\rho^{B^d_2}_{W_2(K)}(X_V) \geq n^{-A}\). Then using the Euclidean version of the theorem gives an approximating GAP, but in this case, by Part 2 and the fact that \(\omega_\infty(B^d_2) = \sqrt{d}\), the Euclidean approximation is

$$|v - q|_2 \leq \frac{CW_2(K) R}{\sqrt{d} k},$$

where \(v \in V\) and \(q \in Q\). Again, we have that \(\| \cdot \|_K \leq \omega_2(K) \cdot 2\), which means that the approximation in the \(K\) norm is of order \(\omega_2(K) W_2(K) R / \sqrt{dk}\). On the other hand, the approximation obtained from directly using Theorem 2.2, is of order \(\omega_\infty(K) R / \sqrt{dk}\). If again we assume that \(K\) is in a position such that \(B^d_2 \subseteq K \subseteq \sqrt{d} B^d_2\), then we have that \(\omega_2(K) W_2(K) \in [1, \sqrt{d}]\), while \(\omega_\infty(K) / \sqrt{d} \leq 1\). This means that the bound obtained in Part 2 of Theorem 2.2 is generally better. Note however that if \(K = B^d_2\) the two bounds coincide.

**Remark 2.4.** For every \(t > 0\), we have \(\rho^K_R(V) = \rho^K_{R/t}(V)\). Thus, (1.5) gives

$$\rho^K_R(V) \leq \inf_{t > 0} \left[ \kappa(tK)^d \cdot \mathcal{I} \left( \frac{t}{R} X_V \right) \right].$$

Therefore, in order to find good bounds on \(\rho^K_R(V)\), one possible approach would be to study the behaviour of \(\kappa(tK)\), where \(t > 0\). Note that in the case \(K\) is convex, that is, when \(\| \cdot \|_K\) is a norm, the results of [CEFM04] imply that \(\kappa(tK) = \sqrt{\frac{d}{t}} e^{\varphi_K(t)}\), where \(\varphi_K\) is convex. However, in general we do not seem to have enough information about \(\varphi_K\) to obtain meaningful results. Also, it could be of interest to study bodies for which \(\kappa(K)\) is a constant, that is, does not depend on \(d\). By (1.5), this would again yield good bounds on \(\rho^K_R(V)\).
3. Proof of Proposition 2.1 and Theorem 2.1

We begin with the following lemma, which is a simple variant of a result that appeared in [TV12].

**Lemma 3.1.** Let \( \lambda > 0 \) and \( w \neq 0 \). Then for every \( \alpha \in \mathbb{R} \), we have

\[
\int_{\mathbb{R}} \exp \left( -\lambda \| \xi w + \alpha \|_T^2 - \frac{|\xi|^2}{2} \right) d\xi \leq \frac{40(|w| + 1)}{|w| \sqrt{1 + \lambda}}.
\]

**Proof.** Since by definition (2.1), for every real number \( w \) we have \( \| w \|_T = \| -w \|_T \), we may assume without loss of generality that \( w > 0 \). Using the change of variables \( t = \xi w + \alpha \) and the fact that \( w > 0 \), we get

\[
\int_{\pi/0} \exp \left( -\lambda \| t \|_T^2 \right) dt = 1 \quad \text{and} \quad \int_{\pi/0} \exp \left( -\lambda \| t \|_T^2 \right) dt \leq \exp \left( -\lambda \| t \|_T^2 \right) dt.
\]

Let \( N = \lfloor w \rfloor + 1 \). Then \( w \leq N \leq w + 1 \), and so we have

\[
\int_{0 \leq t - \alpha \leq \pi} \exp \left( -\lambda \| t \|_T^2 \right) dt \leq \int_{0 \leq t - \alpha \leq \pi} \exp \left( -\lambda \| t \|_T^2 \right) dt
\]

\[
= \sum_{s=0}^{N-1} \int_{\pi s + \alpha}^{\pi(s+1)+\alpha} \exp \left( -\lambda \| t \|_T^2 \right) dt \leq N \max_{0 \leq s \leq N-1} \left[ \int_{\pi s + \alpha}^{\pi(s+1)+\alpha} \exp \left( -\lambda \| t \|_T^2 \right) dt \right]
\]

\[
\leq (w + 1) \max_{0 \leq s \leq N-1} \left[ \int_{\pi s + \alpha}^{\pi(s+1)+\alpha} \exp \left( -\lambda \| t \|_T^2 \right) dt \right].
\]

Plugging (3.2) into (3.1),

\[
\int_{0}^{\pi} \exp \left( -\lambda \| \xi w + \alpha \|_T^2 \right) d\xi \leq \frac{w + 1}{w} \max_{0 \leq s \leq N-1} \left[ \int_{\pi s + \alpha}^{\pi(s+1)+\alpha} \exp \left( -\lambda \| t \|_T^2 \right) dt \right].
\]

Consider first the integral

\[
\int_{0}^{\pi} \exp \left( -\lambda \| t \|_T^2 \right) dt.
\]

This integral is trivially bounded by \( \pi \). Also,

\[
\int_{0}^{\pi} \exp \left( -\lambda \| t \|_T^2 \right) dt = \int_{0}^{\pi/2} \exp \left( -\lambda \| t \|_T^2 \right) dt + \int_{\pi/2}^{\pi} \exp \left( -\lambda \| t \|_T^2 \right) dt
\]

\[
= \int_{0}^{\pi/2} \exp \left( -\lambda t^2 \right) dt + \int_{\pi/2}^{\pi} \exp \left( -\lambda |t - \pi|^2 \right) dt
\]

\[
= 2 \int_{0}^{\pi/2} \exp \left( -\lambda t^2 \right) dt = \frac{2}{\sqrt{\lambda}} \int_{0}^{\frac{\pi^2}{\sqrt{\lambda}}} e^{-x} dx \leq \frac{2}{\sqrt{\lambda}} \sqrt{\frac{\pi^2}{\lambda}} \leq \frac{6}{\sqrt{\lambda}}.
\]

Altogether, we get

\[
\int_{0}^{\pi} \exp \left( -\lambda \| t \|_T^2 \right) dt \leq \min \left\{ \pi, \frac{6}{\sqrt{\lambda}} \right\} \leq \frac{10}{\sqrt{1 + \lambda}}.
\]
Since the function $\| \cdot \|_T^2$ is $\pi$-periodic, it follows that for every $\alpha \in \mathbb{R}$,
\[
\int_{\pi s + \alpha}^{\pi(s+1) + \alpha} \exp \left( -\lambda \| t \|_T^2 \right) dt = \int_{\alpha}^{\pi + \alpha} \exp \left( -\lambda \| t \|_T^2 \right) dt \leq \int_0^{2\pi} \exp \left( -\lambda \| t \|_T^2 \right) dt \leq \frac{20}{\sqrt{1 + \lambda}}. \tag{3.5}
\]

Plugging (3.5) into (3.3), we get
\[
\int_0^\pi \exp \left( -\lambda \| \xi w + \alpha \|_T^2 \right) d\xi \leq \frac{20(w + 1)}{w \sqrt{1 + \lambda}}. \tag{3.4}
\]

Since $\| \cdot \|_T^2$ is $\pi$-periodic, we also have for every $s \in \mathbb{Z}$,
\[
\int_{\pi s}^{\pi(s+1)} \exp \left( -\lambda \| \xi w + \alpha \|_T^2 \right) d\xi \leq \frac{20(w + 1)}{w \sqrt{1 + \lambda}}.
\]

Hence,
\[
\int_{\mathbb{R}} \exp \left( -\lambda \| \xi w + \alpha \|_T^2 - \frac{|\xi|^2}{2} \right) d\xi = \sum_{s=-\infty}^{\infty} \int_{\pi s}^{\pi(s+1)} \exp \left( -\lambda \| \xi w + \alpha \|_T^2 - \frac{|\xi|^2}{2} \right) d\xi
\]
\[
\leq \sum_{s=-\infty}^{\infty} e^{-s^2/2} \int_{\pi s}^{\pi(s+1)} \exp \left( -\lambda \| \xi w + \alpha \|_T^2 \right) d\xi \leq \left( \sum_{s=-\infty}^{\infty} e^{-s^2/2} \right) \frac{20(w + 1)}{w \sqrt{1 + \lambda}}
\]
\[
\leq \frac{40(w + 1)}{w \sqrt{1 + \lambda}},
\]
which completes the proof. \qed

We are now in a position to prove Proposition 2.1.

**Proof of Proposition 2.1.** By (1.6) and (2.2), we have
\[
\mathcal{I}(X_V) \leq \int_{\mathbb{R}^d} \exp \left( -c_\eta \sum_{j=1}^{n} \| \langle v_j, \xi \rangle \|_T^2 - \frac{|\xi|^2}{2} \right) d\xi. \tag{3.6}
\]

Assume first that $k = d\ell$, $\ell \in \mathbb{N}$. The general case will be considered at the end of the proof. Let $v_{0,1}, \ldots, v_{0,\ell}$ be $\ell$ elements of $V$, and set $V^{(1)} = V \setminus \{v_{0,1}, \ldots, v_{0,\ell}\}$. Then, we can write
\[
\int_{\mathbb{R}^d} \exp \left( -c_\eta \sum_{j=1}^{n} \| \langle v_j, \xi \rangle \|_T^2 - \frac{|\xi|^2}{2} \right) d\xi
\]
\[
= \int_{\mathbb{R}^d} \exp \left( -c_\eta \sum_{u \in V^{(1)}} \| \langle v, \xi \rangle \|_T^2 \right) \prod_{j=1}^{\ell} \exp \left( -c_\eta \| \langle v_{0,j}, \xi \rangle \|_T^2 \right) \exp \left( -\frac{|\xi|^2}{2} \right) d\xi
\]
\[
= \int_{\mathbb{R}^d} \prod_{j=1}^{\ell} \exp \left( -c_\eta \| \langle v_{0,j}, \xi \rangle \|_T^2 - \frac{1}{\ell} \left( c_\eta \sum_{u \in V^{(1)}} \| \langle v, \xi \rangle \|_T^2 + \frac{|\xi|^2}{2} \right) \right) d\xi. \tag{3.7}
\]


It follows from Hölder’s inequality that if $f_1, \ldots, f_\ell$ are positive functions, then
\[
\int_{\mathbb{R}^d} \prod_{j=1}^\ell f_j \leq \prod_{j=1}^d \left( \int_{\mathbb{R}^d} f_j^\ell \right)^{\frac{1}{\ell}}.
\]
In particular, there exists $j_0 \in \{1, \ldots, \ell\}$ such that
\[
\int_{\mathbb{R}^d} \prod_{j=1}^\ell f_j \leq \int_{\mathbb{R}^d} f_{j_0}^\ell.
\]
Therefore, applying Hölder’s inequality to the right side of (3.7), we conclude that there exists an index $j_0 \in \{1, \ldots, \ell\}$ such that
\[
\int_{\mathbb{R}^d} \prod_{j=1}^\ell \exp \left( -c_\eta \|\langle v, \xi \rangle\|^2_T - \frac{1}{\ell} \left( c_\eta \sum_{u \in V^{(1)}} \|\langle v, \xi \rangle\|^2_T + \frac{|\xi|^2_2}{2} \right) \right) \, d\xi
\]
\[
\leq \int_{\mathbb{R}^d} \exp \left( -c_\eta \sum_{v \in V^{(1)}} \|\langle u, \xi \rangle\|^2_T \right) \exp \left( -c_\eta \ell \|\langle v, \xi \rangle\|^2_T \right) \exp \left( - \frac{|\xi|^2_2}{2} \right) \, d\xi. \quad (3.8)
\]
Plugging (3.8) into (3.7) gives
\[
\int_{\mathbb{R}^d} \exp \left( -c_\eta \sum_{j=1}^n \|\langle v_j, \xi \rangle\|^2_T - \frac{|\xi|^2_2}{2} \right) \, d\xi
\]
\[
\leq \int_{\mathbb{R}^d} \exp \left( -c_\eta \sum_{v \in V^{(1)}} \|\langle u, \xi \rangle\|^2_T \right) \exp \left( -c_\eta \ell \|\langle v, \xi \rangle\|^2_T \right) \exp \left( - \frac{|\xi|^2_2}{2} \right) \, d\xi.
\]
Set $w_1 = v_{0,j_0}$. If $d = 1$, then we stop at this point. Otherwise, by the assumption on the vectors in $V$, we can choose $\ell$ elements $v_{1,1}, \ldots, v_{1,\ell}$ of $V^{(1)}$ which lie at a distance at least $R$ from the span $\{w_1\}$. Set $V^{(2)} = V^{(1)} \setminus \{v_{1,1}, \ldots, v_{1,\ell}\}$, and as before, find $j_1$ such that
\[
\int_{\mathbb{R}^d} \exp \left( -c_\eta \sum_{j=1}^n \|\langle v_j, \xi \rangle\|^2_T - \frac{|\xi|^2_2}{2} \right) \, d\xi
\]
\[
\leq \int_{\mathbb{R}^d} \exp \left( -c_\eta \sum_{v \in V^{(1)}} \|\langle u, \xi \rangle\|^2_T \right) \exp \left( -c_\eta \ell \|\langle w_1, \xi \rangle\|^2_T \right)
\]
\[
\cdot \exp \left( -c_\eta \ell \|\langle v_{1,j_1}, \xi \rangle\|^2_T \right) \exp \left( - \frac{|\xi|^2_2}{2} \right) \, d\xi.
\]
Now, set $w_2 = v_{1,j_1}$, and repeat this procedure $d - 1$ times, eventually obtaining
\[
\int_{\mathbb{R}^d} \exp \left( -c_\eta \sum_{j=1}^n \|\langle v_j, \xi \rangle\|^2_T \right) \, d\xi
\]
\[
\leq \int_{\mathbb{R}^d} \exp \left( -c_\eta \sum_{v \in V^{(d)}} \|\langle u, \xi \rangle\|^2_T \right) \exp \left( -c_\eta \sum_{j=1}^d \|\langle w_j, \xi \rangle\|^2_T \right) \exp \left( - \frac{|\xi|^2_2}{2} \right) \, d\xi.
\]
for some $w_1, \ldots, w_d$, now with the property that
\[
\inf_{1 \leq j \leq n} \left[ \text{dist}_2 \left( w_j, \text{span}\{w_1, \ldots, w_{j-1}\} \right) \right] \geq R, \tag{3.9}
\]
for all $1 \leq j \leq d$, and $V^{(d)}$ is a subset of $V$ with at least $n - k$ vectors. Note also that we can choose $w_1$ such that $|w_1|^2 \geq R$. In addition, the following trivial bound holds,
\[
\exp \left( -c_\eta \sum_{v \in V^{(d)}} \| \langle u, \xi \rangle \|^2_T \right) \leq 1.
\]
Thus, we have
\[
\int_{\mathbb{R}^d} \exp \left( -c_\eta \sum_{j=1}^n \| \langle v_j, \xi \rangle \|^2_T - \frac{\| \xi \|^2_T}{2} \right) d\xi \leq \int_{\mathbb{R}^d} \exp \left( -c_\eta \ell \sum_{j=1}^d \| \langle w_j, \xi \rangle \|^2_T - \frac{\| \xi \|^2_T}{2} \right) d\xi. \tag{3.10}
\]
To bound the right side of (3.10), use induction on $d$. Consider first the case $d = 1$. By Lemma 3.1,
\[
\int_{\mathbb{R}} \exp \left( -c_\eta \| \xi w + \alpha \|^2_T - \frac{\| \xi \|^2_T}{2} \right) d\xi \leq \frac{40(|w| + 1)}{|w| \sqrt{1 + c_\eta \ell}}. \tag{3.11}
\]
In the case $d = 1$, we the assumption on the vectors $\{v_1, \ldots, v_n\}$ implies that $|w| \geq R$, and so (3.11) gives
\[
\int_{\mathbb{R}} \exp \left( -c_\eta \ell \| \xi w + \alpha \|^2_T - \frac{\| \xi \|^2_T}{2} \right) d\xi \leq \frac{40(R + 1)}{R \sqrt{1 + c_\eta \ell}}. \tag{3.12}
\]
To handle the general case, use Fubini’s Theorem and induction on $d$. By following a Gram-Schmidt process, find an orthonormal basis $\{e_1, \ldots, e_d\}$ of $\mathbb{R}^d$, such that span$\{w_1, \ldots, w_j\} = \text{span}\{e_1, \ldots, e_j\}$, for all $1 \leq j \leq d$. Suppose that the desired claim holds for dimension $d - 1$, and for a vector $\xi \in \mathbb{R}^d$, write $\xi = \xi' + \xi_d e_d$, where $\xi' \in \text{span}\{e_1, \ldots, e_{d-1}\}$ and $\xi_d \in \mathbb{R}$. This gives
\[
\sum_{j=1}^d \| \langle w_j, \xi \rangle \|^2_T = \sum_{j=1}^{d-1} \| \langle w_j, \xi' \rangle \|^2_T + \| \langle w_d, \xi_d \rangle \|^2_T.
\]
Note that (3.9) implies that
\[
|\langle w_d, e_d \rangle| \geq R. \tag{3.13}
\]
Thus, we have
\[
\int_{\mathbb{R}^d} \exp \left( -c_{\eta} \ell \sum_{j=1}^{d} \| \langle w_j, \xi \rangle \|_2^2 - \frac{\| \xi \|_2^2}{2} \right) d\xi \\
= \int_{\mathbb{R}^{d-1}} \exp \left( -c_{\eta} \ell \sum_{j=1}^{d-1} \| \langle w_j, \xi' \rangle \|_2^2 - \frac{\| \xi' \|_2^2}{2} \right) \int_{\mathbb{R}} \exp \left( -c_{\eta} \ell \| \xi_d \|_2^2 \right) d\xi' \\
\leq \frac{40(R + 1)}{R \sqrt{1 + c_{\eta} \ell}} \int_{\mathbb{R}^{d-1}} \exp \left( -c_{\eta} \ell \sum_{j=1}^{d-1} \| \langle w_j, \xi' \rangle \|_2^2 - \frac{\| \xi' \|_2^2}{2} \right) d\xi' \\
\leq \left( \frac{40(R + 1)}{R \sqrt{1 + c_{\eta} \ell}} \right)^d ,
\]
where in (*) we used the induction hypothesis. Combining (3.6), (3.10) and (3.14) gives that when \( k = d\ell \),
\[
\mathcal{I}(X_V) \leq \left( \frac{40(R + 1)}{R \sqrt{1 + c_{\eta} \ell}} \right)^d .
\]
In general, assume that \( d(\ell - 1) \leq k \leq d\ell \). Then we know that there are at least \( n - d\ell \) vectors which satisfy \( \text{dist}_2(v_j, H) \geq R \) for every hyperplane \( H \subseteq \mathbb{R}^d \). In such case, since \( \ell \geq k/d \) we have
\[
\mathcal{I}(X_V) \leq \left( \frac{40(R + 1)}{R \sqrt{1 + c_{\eta} \ell}} \right)^d \leq \left( \frac{40(R + 1)}{R \sqrt{1 + c_{\eta}(k/d)}} \right)^d = \left( 40 \frac{R + 1}{R} \sqrt{\frac{d}{d + c_{\eta}k}} \right)^d ,
\]
which completes the proof. \( \square \)

Now with Proposition 2.1 in hand, we can prove Theorem 2.1.

**Proof of Theorem 2.1.** By (1.5), we have
\[
\rho_R^K(X_V) \leq \kappa(K)^d \mathcal{I} \left( \frac{1}{R} X_V \right) = \kappa(K)^d \mathcal{I}(X_{V_R}) ,
\]
where \( V_R = \{ R^{-1} v_1, \ldots, R^{-1} v_n \} \). In particular, by the assumptions on the set \( V \), we know that for at least \( n - k \) vectors \( v_j' \in V_R \), we have \( \text{dist}_2(v_j', H) \geq 1 \). Hence, using Proposition 2.1 with \( R = 1 \), we have
\[
\rho_R^K(X_V) \leq \kappa(K)^d \left( 80 \sqrt{\frac{d}{d + c_{\eta}k}} \right)^d = \left( 80\kappa(K) \right)^d \left( \frac{d}{d + c_{\eta}k} \right)^{d/2} ,
\]
which completes the proof. \( \square \)
4. Proof of Theorem 2.2

Recall again that by (1.5), we have

$$\rho^K_R(X_V) \leq (\kappa(K) R)^d \int_{\mathbb{R}^d} \prod_{j=1}^n |\mathbb{E} \exp(i \langle \eta_j v_j, \xi \rangle)| \exp(-\frac{R^2|\xi|_2^2}{2}) d\xi.$$ 

For a random variable $\eta$ and a real number $a$, define

$$\|a\|_\eta = \frac{2}{\pi} \left( \mathbb{E} \|a(\eta_1 - \eta_2)\|_T^2 \right)^{1/2},$$

where $\eta_1$ and $\eta_2$ are independent copies of $\eta$, and $\| \cdot \|_T$ is as defined in (2.1). The constant $\frac{2}{\pi}$ is simply a normalisation constant which makes some of the computations simpler. Now, it is easy to show that we have

$$\left| \mathbb{E} \exp(i a\eta) \right|^2 = \left| \mathbb{E} \cos(a(\eta_1 - \eta_2)) \right| \leq 1 - \frac{2}{\pi^2} \mathbb{E} \|a(\eta_1 - \eta_2)\|_T^2 \leq \exp\left(-\frac{1}{2} \|a\|_\eta^2\right).$$

Hence, (1.5) implies in fact that we have

$$\rho^K_R(X_V) = \rho^K_1(X_{V_R}) \leq (\kappa(K) d \int_{\mathbb{R}^d} \exp\left(-\frac{1}{2} \sum_{v \in V_R} \|\langle v, \xi \rangle\|_\eta^2 - \frac{1}{2} |\xi|_2^2\right) d\xi. \quad (4.2)$$

The first step in the proof of Theorem 2.2 is to find a large subset of $\mathbb{R}^d$ on which the sum $\sum_{v \in V_R} \|\langle v, s \rangle\|_\eta^2$ is relatively small. Such a set should then have some arithmetic structure. The proof is similar to [NV11] and we present it for the sake of completeness, while making the required modifications.

**Proposition 4.1.** Assume that $A > 0$ is an absolute constant, and assume that for every $n$ sufficiently large, we have

$$\rho^K_R(X_V) \geq n^{-A}.$$ 

Then there exists a positive integer $m$ satisfying

$$m \leq 10\sqrt{d \log(\kappa(K)n^A)},$$

such that for any sufficiently large integer $N$, there exists a finite set $S \subseteq \mathbb{R}^d$

$$|S| \geq \left( \frac{N}{2\sqrt{dK}} \right)^d \rho^K_R(X_V),$$

and if we let $V_R = \{R^{-1}v_1, \ldots, R^{-1}v_n\}$, then there exists a real number $\alpha$ which depends only on $C_\eta$ such that the set $S$ satisfies

$$\frac{1}{|S|} \sum_{s \in S} \left[ \sum_{v \in V_R} \|\alpha \langle v, s \rangle\|_\eta^2 \right] \leq 8\pi^2 m.$$

**Proof.** First, notice that we have $\rho^K_R(X_V) = \rho^K_1(X_{V_R})$. Let $\rho = \rho^K_1(X_{V_R})$. We have

$$\int_{|\xi| > M} \exp\left(-\frac{1}{2} \sum_{v \in V_R} \|\langle v, \xi \rangle\|_\eta^2 - \frac{1}{2} |\xi|_2^2\right) d\xi \leq \int_{|\xi| > M} \exp\left(-\frac{1}{2} |\xi|_2^2\right) d\xi \leq (2\pi)^{d/2} e^{-M^2/4},$$
where we used the fact the a normal distribution has a subgaussian tail. In particular, assuming that $n \geq \kappa(K)^{-1/A}$, and choosing $M$ to be
\[
M = \sqrt{-4 \log \left( \frac{n^{-A}(2\pi)^{d/2}}{2\kappa(K)^d} \right)} \leq 10\sqrt{d \log (\kappa(K)n^A)},
\]
it follows that
\[
\int_{|\xi| > M} \exp \left( -\frac{1}{2} \sum_{v \in V_R} \|\langle v, \xi \rangle\|_\eta^2 - \frac{1}{2} ||\xi||_2^2 \right) d\xi \leq (2\pi)^{d/2} e^{-M^2/2} = \frac{n^{-A}}{2\kappa(K)^d} \leq \frac{\rho}{2\kappa(K)^d}. \tag{4.3}
\]
Using (4.2), it follows that
\[
\frac{\rho}{\kappa(K)^d} \leq \int_{\mathbb{R}^d} \exp \left( -\frac{1}{2} \sum_{v \in V_R} \|\langle v, \xi \rangle\|_\eta^2 - \frac{1}{2} ||\xi||_2^2 \right) d\xi. \tag{4.4}
\]
Thus, combining (4.3) and (4.4), we have
\[
\int_{|\xi| \leq M} \exp \left( -\frac{1}{2} \sum_{v \in V_R} \|\langle v, \xi \rangle\|_\eta^2 - \frac{1}{2} ||\xi||_2^2 \right) d\xi \geq \frac{\rho}{2\kappa(K)^d}. \tag{4.5}
\]
For $m \in \{0, 1, \ldots, M\}$, define
\[
T_m = \left\{ \xi \in \mathbb{R}^d \left| \sum_{v \in V_R} \|\langle v, \xi \rangle\|_\eta^2 + ||\xi||_2 \leq m \right. \right\}.
\]
Then (4.5) implies that
\[
\sum_{m=0}^M \mu_d(T_m) e^{-m/2} \geq \frac{\rho}{2\kappa(K)^d},
\]
where $\mu_d(\cdot)$ denotes the Lebesgue measure on $\mathbb{R}^d$. It follows that there exists $m \leq M$ such that $\mu_d(T_m) \geq \frac{\rho e^{m/4}}{2\kappa(K)^d}$. Now, since clearly we have $T_m \subseteq B_2^d(0, \sqrt{m})$, by the pigeon-hole principle, there exists $x \in \mathbb{R}^d$ such that $B_2^d(x, 1/2) \subseteq B_2^d(0, \sqrt{m})$ and
\[
\mu_d\left( B_2^d(x, 1/2) \cap T_m \right) \geq \mu_d(T_m) m^{-d/2} \geq \frac{\rho e^{m/4} m^{-d/2}}{2\kappa(K)^d}, \tag{4.6}
\]
where we recall that for $\alpha > 0$, $B_2^d(x, \alpha) = x + \alpha B_2^d$. Next, let $\xi_1, \xi_2 \in B(x, 1/2) \cap T_m$. Since $\|\cdot\|_\eta$ satisfies the triangle inequality, we have
\[
\sum_{v \in V_R} \|\langle v, \xi_1 - \xi_2 \rangle\|_\eta^2 \leq 2 \sum_{v \in V_R} \|\langle v, \xi_1 \rangle\|_\eta^2 + 2 \sum_{v \in V_R} \|\langle v, \xi_2 \rangle\|_\eta^2 \overset{(*)}{=} 2m + 2m = 4m,
\]
where in $(*)$ we used the fact that $\xi_1, \xi_2 \in T_m$. Now, we have $\xi_1 - \xi_2 \in B_2^d$. Also, we have
\[
\mu_d\left( B_2^d(x, 1/2) \cap T_m - B_2^d(x, 1/2) \cap T_m \right) \geq \mu_d\left( B_2^d(x, 1/2) \cap T_m \right), \tag{4.7}
\]
Hence, defining

\[ T = \left\{ \xi \in B_2^d \left| \sum_{v \in V_R} \| (v, \xi) \|_\eta^2 \leq 4m \right. \right\}, \]

it follows that we have

\[ \mu_d(T) \stackrel{(4.6) \wedge (4.7)}{\geq} \frac{\rho e^{m/4} m^{-d/2}}{2 \kappa(K)^d}. \]

Now, use the fact that for every positive integer \( m \), \( e^{m/4} m^{-d/2} \geq e^{d/2}(2d)^{-d/2} \) to conclude that

\[ \mu_d(T) \geq \frac{1}{2} \left( \frac{\sqrt{e}}{\sqrt{2dk(K)}} \right)^d \rho \geq \frac{\rho}{\left(2\sqrt{d\kappa(K)}\right)^d}. \]  

(4.8)

Next, for a given positive integer \( N \), let \( B_0 \) be the discrete box

\[ B_0 = \{(k_1/N, \ldots, k_d/N) \in \mathbb{R}^d \mid -N \leq k_j \leq N\}. \]

Consider all the boxes \( x + B_0 \) with \( x \in [0, 1/N] \). Since

\[ \lim_{N \to \infty} \frac{1}{N^d} \mathbb{E} \left| (x + B_0) \cap T \right| = \mu_d(T), \]

it follows that there exists \( x_0 \in \mathbb{R}^d \) and a positive integer \( N \), such that

\[ \left| (x_0 + B_0) \cap T \right| \geq N^d \mu_d(T). \]

Fix \( \xi_0 \in x_0 + B_0 \). For any \( \xi \in (x_0 + B_0) \cap T \), we have

\[ \sum_{v \in V_R} \| (v, \xi_0 - \xi) \|_\eta^2 \leq 2 \sum_{v \in V_R} \| (v, \xi_0) \|_\eta^2 + 2 \sum_{v \in V_R} \| (v, \xi) \|_\eta^2 \leq 16m. \]  

(4.9)

We have,

\[ \xi_0 - \xi \in B_0 - B_0 = \{(k_1/N, \ldots, k_d/N) \in \mathbb{R}^d \mid -2N \leq k_j \leq 2N\}. \]

This means that there exists a subset \( S \subseteq (B_0 - B_0) \cap T \) of size at least \( N^d \mu_d(T) \) such that for any \( s \in S \),

\[ \sum_{v \in V_R} \| (v, s) \|_\eta^2 \leq 16m. \]  

(4.9)

Hence, we have

\[ \frac{1}{|S|} \sum_{s \in S} \left[ \sum_{v \in V_R} \| (v, s) \|_\eta^2 \right] \leq 16m. \]  

(4.10)

By combining (4.10) with (4.1), it follows that

\[ \frac{1}{|S|} \sum_{s \in S} \left[ \sum_{v \in V_R} \mathbb{E} \| (\eta_1 - \eta_2)(v, s) \|_\eta^2 \right] \leq 4\pi^2 m. \]  

(4.11)
Now, there exists $\alpha \in [1, C^\eta]$, such that

$$\mathbb{E}\| (\eta_1 - \eta_2)\langle v, s \rangle \|^2 \geq \mathbb{P} (1 \leq |\eta_1 - \eta_2| \leq C^\eta) \| \alpha \langle v, s \rangle \|^2 \geq \frac{1}{2} \| \alpha \langle v, s \rangle \|^2 .$$ \hspace{1cm} (4.12)

Therefore, combining (4.11) and (4.12), we have

$$\frac{1}{|S|} \sum_{s \in S} \left[ \sum_{v \in V_R} \| \alpha \langle v, s \rangle \|^2 \right] \leq 8\pi^2 m ,$$

which completes the proof.

Fix $\varepsilon \in (0, 1)$. Let $n'$ be an integer between $n^\varepsilon$ and $n$. Let $S \subseteq \mathbb{R}^d$ be the set from Proposition 4.1. Say that a vector $v \in V_R$ is said to be bad if

$$\frac{1}{|S|} \sum_{s \in S} \| \alpha \langle v, s \rangle \|^2 \geq \frac{8\pi^2 m}{n'} .$$

Let $V'_R$ denote all the vectors in $V_R$ which are not bad. It follows that $|V'_R| \geq n - n'$. Recall also that if $A \subseteq \mathbb{R}^d$ and $k$ is a positive integer, then we denote

$$kA = \left\{ \sum_{j=1}^k a_j \mid a_j \in A \right\} .$$

**Lemma 4.1.** Let $n' \in [n^\varepsilon, n]$ and choose

$$k = \sqrt{\frac{n'}{64\pi^2 m}} . \hspace{1cm} (4.13)$$

If $N$ is sufficiently large, then

$$\mu_d \left( k(V'_R \cup \{ 0 \}) + B_{\infty} \left( 0, \frac{1}{256d\alpha} \right) \right) \leq 36\pi^2 \left( \frac{2\sqrt{d\kappa(K)}}{\alpha} \right)^d \left( \frac{1}{\rho_K^R(X_V)} \right) .$$

**Proof.** Let $v \in V'_R$. By definition of vectors which are not bad, we have

$$\sum_{s \in S} \| \alpha \langle v, s \rangle \|^2 \leq \frac{8\pi^2 m |S|}{n'} .$$

Since $\| \cdot \|_{\infty}$ satisfies the triangle inequality, for any $w \in k(V'_R \cup \{ 0 \})$ we have

$$\sum_{s \in S} \| \langle s, \alpha w \rangle \|^2 \leq k^2 \sup_{v \in V'_R} \left[ \sum_{s \in S} \| \langle s, \alpha v \rangle \|^2 \right] \leq \frac{8\pi^2 m |S| k^2}{n'} \leq \frac{|S|}{4} . \hspace{1cm} (4.13)$$

Since for any real number $a$, $\cos(2a) \geq 1 - 2\|a\|^2_{\infty}$,

$$\sum_{s \in S} \cos(2\langle s, \alpha w \rangle) \geq |S| - \sum_{s \in S} 2\| \langle s, \alpha v \rangle \|^2 \geq \frac{|S|}{2} .$$
Note that if \( \|x\|_\infty \leq \pi / 256d \) and \( \|s\|_\infty \leq 2 \), then \( \cos (2 \langle x, s \rangle) \geq 1/2 \) and \( \sin (2 \langle x, s \rangle) \leq 1/12 \). Thus, for any \( x \) with \( \|x\|_\infty \leq \pi / 256d \),

\[
\sum_{s \in S} \cos (2 \langle s, \alpha w + x \rangle) = \sum_{s \in S} \cos (2 \langle s, \alpha w \rangle) \cos (2 \langle s, x \rangle) - \sum_{s \in S} \sin (2 \langle s, \alpha w \rangle) \sin (2 \langle s, x \rangle) \geq \frac{|S|}{4} - \frac{|S|}{12} = \frac{|S|}{6}.
\]

On the other hand, we have

\[
\int_{[0, \pi N]^d} \left( \sum_{s \in S} \cos (2 \langle s, x \rangle) \right)^2 \, dx \leq \sum_{s_1, s_2 \in S} \int_{x \in [0, \pi N]^d} \exp (2i \langle s_1 - s_2, x \rangle) \, dx = \pi^2 N^2 |S|,
\]

and so

\[
\mu_d \left( \left\{ x \in [0, \pi N]^d \left| \left( \sum_{s \in S} \cos (2 \langle s, x \rangle) \right)^2 \geq \left( \frac{|S|}{6} \right)^2 \right. \right\} \right) \leq \frac{\pi^2 N^2 |S|}{(|S|/6)^2} = 36 \pi^2 \frac{N^d}{|S|}.
\]

Let \( V''_R = k (V'_R \cup \{0\}) \). If \( N \) is chosen such that \( \alpha V''_R + B_\infty (0, \pi / 256d) \subseteq [0, \pi N]^d \), then it follows that

\[
\mu_d \left( \alpha V''_R + B_\infty \left( 0, \frac{\pi}{256d} \right) \right) \leq \mu_d \left( \left\{ x \in [0, \pi N]^d \left| \left( \sum_{s \in S} \cos (2 \pi \langle s, x \rangle) \right)^2 \geq \left( \frac{|S|}{6} \right)^2 \right. \right\} \right) \leq 36 \pi^2 \frac{N^d}{|S|}.
\]

Now, let \( N \) be large enough so that Proposition 4.1 holds. The result now follows from homogeneity of \( \mu_d \) and Proposition 4.1. □

**Remark 4.1.** Since \( k = \sqrt{\frac{n'}{64 \pi^2 m}} \), using the estimate on \( m \) from Proposition 4.1 it follows that

\[
\sqrt{\frac{n'}{64 \pi^2 \sqrt{d} \log (\kappa \langle K \rangle / n^A)}} \leq k \leq \sqrt{n'}.
\]

The remaining main tools in the proof of Theorem 2.2 are the following. 

**Theorem 4.1 ([NV11]).** Assume that \( X \) is a discrete subset of a torsion free group. Assume that there exists an integer \( k \) such that \( |kX| \leq k^\gamma |X| \) for some positive number \( \gamma \). Then there exists a proper GAP \( Q \) with rank \( \text{rank}(Q) \leq C\gamma \) and cardinality \( |Q| \leq C(\gamma) k^{-\text{rank}(Q)} |kX| \), such that \( X \subseteq Q \).

We are now in a position to prove Theorem 2.2.

**Proof of Theorem 2.2. Proof of Part 1.** Let \( D = 512 d \alpha \). Also, as before, let \( \rho = \rho_{R}^K(X_v) = \rho_{R}^K(X_{v_{0}}) \), where \( V_{R} = \{ R^{-1}v_{1}, \ldots, R^{-1}v_{n} \} \). For \( v' \in V'_{R} \), find \( z \in \mathbb{Z}^d \) such that \( z/Dk \) is the closest vector to \( v' \) in the \( \ell_{\infty} \) norm. That is, we can choose \( z \in \mathbb{Z}^d \) such that

\[
\|v' - \frac{z}{Dk}\|_{\infty} \leq \frac{1}{Dk}.
\] (4.14)
Denote the set of all z’s satisfying (4.14) by F. Let \( \sum_{j=1}^{k} v_j' \in k(V'_k \cup \{0\}) \). For each \( j \leq k \), let \( z_j \) be the approximation to \( v_j' \) as in (4.14). Then,

\[
\left\| x - \sum_{j=1}^{k} v_j' \right\|_{\infty} \leq \left\| x - \sum_{j=1}^{k} \frac{z_j}{Dk} \right\|_{\infty} + \sum_{j=1}^{k} \left\| \frac{z_j}{Dk} \right\|_{\infty} \leq \left\| x - \sum_{j=1}^{k} \frac{z_j}{Dk} \right\|_{\infty} + \frac{1}{D},
\]

which implies that

\[
k \left( \frac{1}{Dk} F \right) + B_{\infty} \left( 0, \frac{1}{D} \right) \subseteq k \left( V'_k \cup \{0\} \right) + B_{\infty} \left( 0, \frac{2}{D} \right).
\]

Recall that here \( \frac{1}{Dk} F = \{ x/(Dk) \mid x \in F \} \). Thus, by the choice of \( D \) and Lemma 4.1, we have

\[
\mu_d \left( k \left( \frac{1}{Dk} F \right) + B_{\infty} \left( 0, \frac{1}{D} \right) \right) \leq 36\pi^2 \left( \frac{2\sqrt{d}\kappa(K)}{\alpha} \right)^d \rho^{-1}.
\]

Therefore, by homogeneity,

\[
\mu_d (k F + B_{\infty}(0, k)) \leq 36\pi^2 \left( \frac{2\sqrt{dDk}\kappa(K)}{\alpha} \right)^d \rho^{-1} \leq (C\kappa(K)d^{3/2})^d k^d \rho^{-1},
\]

which implies that there exists an absolute constant \( C \) such that

\[
|k \left( F + \{ -1, 1 \}^d \right)| \leq \left( C\kappa(K)d^{3/2} \right)^d k^d \rho^{-1}.
\]

Using the choice \( k = \sqrt{\frac{n'}{64\pi^2m}} \), the fact that \( n' \in [n^\varepsilon, n] \) and the fact that \( m \leq 10\sqrt{d\log(n^A\kappa(K))} \), it follows that for sufficiently large \( n \),

\[
\frac{1}{8\pi} \left( \frac{n^\varepsilon}{Ad\log(\kappa(K))} \right)^{1/4} \leq \sqrt{\frac{n^\varepsilon}{64\pi^2m}} \leq k \leq \sqrt{n'} \leq \sqrt{n}.
\]

Hence, we have

\[
\rho^{-1} \leq n^A \leq \left( Ad\log(\kappa(K)) \right)^{\frac{A}{\varepsilon}} (8\pi k)^{\frac{4A}{\varepsilon}}.
\]

Plugging this into (4.15),

\[
|k \left( F + \{ -1, 1 \}^d \right)| \leq \left( C\kappa(K)d^{3/2} \right)^d (8\pi)^{\frac{4A}{\varepsilon}} \left( Ad\log(\kappa(K)) \right)^{\frac{A}{\varepsilon}} k^{d+\frac{4A}{\varepsilon}} = k^{d+\frac{4A}{\varepsilon}} \left( \frac{\log(Ad\log(\kappa(K))) + \frac{4A}{\varepsilon} \log(8\pi) + \frac{A}{\varepsilon} \log(Ad\log(\kappa(K)))}{\log k} \right).
\]

Assuming that \( d, A, \varepsilon \) and \( \kappa(K) \) are given constant and \( k \to \infty \), it follows that

\[
d + \frac{4A}{\varepsilon} + \frac{\log(Ad\log(\kappa(K))) + \frac{4A}{\varepsilon} \log(8\pi) + \frac{A}{\varepsilon} \log(Ad\log(\kappa(K)))}{\log k} \leq d + \frac{8A}{\varepsilon}.
\]

Using the trivial bound \( |F + \{ -1, 1 \}^d| \geq 1 \), and choosing \( \gamma = d + \frac{8A}{\varepsilon} \), gives

\[
|k \left( F + \{ -1, 1 \}^d \right)| \leq k^\gamma \left| F + \{ -1, 1 \}^d \right|.
\]
Now use Theorem 4.1 with the set $F + \{-1,1\}^d$ to deduce that there exists a proper GAP

$$Q' = \left\{ \sum_{j=1}^{r} x_j g_j \mid x_j \in \mathbb{Z}, \ |x_j| \leq L_j \right\},$$

which contains $F + \{-1,1\}^d$, and has rank

$$\text{rank}(Q') \leq C \left( d + \frac{A}{\varepsilon} \right),$$

and cardinality

$$|Q'| \leq C(\gamma)k^{-\text{rank}(Q')} |k \ (F + \{-1,1\}^d)| \leq C(A,d,\varepsilon) \left( C_K(K)^{d^3/2} \right)^d k^{d-\text{rank}(Q')} \rho^{-1}.$$

Define

$$Q = \frac{R}{Dk}Q' = \left\{ \frac{R}{Dk} q' \mid q' \in Q' \right\}.$$

Then $Q$ has the same rank and cardinality of $Q'$. This completes the proof of Part 1.

**Proof of Part 2.** By (4.14) and the fact that $\frac{1}{\omega_\infty(K)}D_k^d \subseteq K$, we have for every $v' \in V'_R$,

$$\left\| v' - \frac{z}{Dk} \right\|_K \leq \frac{\omega_\infty(K)}{Dk}.$$

Since $V'_R \subseteq V_R = \{ R^{-1}v_1, \ldots, R^{-1}v_n \}$ and since $|V'_R| \geq n - n'$ it follows that for at least $n - n'$ elements of $V$, there exists $q \in Q$ with

$$\| v - q \|_K \leq \frac{\omega_\infty(K)R}{Dk},$$

which proves Part 2.

**Proof of Part 3.** Follows from the fact that $Q = \frac{R}{Dk}Q'$.

**Proof of Part 4.** By (4.14) it follows that

$$\left\| v' - \frac{z}{Dk} \right\|_K \leq \frac{\omega_\infty(K)}{Dk}. \quad (4.16)$$

Also, if $\| v' \|_\infty \leq \frac{1}{Dk}$, can choose $z = 0$ as the approximation in $\mathbb{Z}^d$. This means that whenever $\| v' \|_K \leq \frac{\omega_\infty(K)}{Dk}$, we can choose $z = 0$ as the approximation. Otherwise, if $\| v' \|_K \geq \frac{\omega_\infty(K)}{Dk}$, we have

$$\| z \|_K \leq C_K \left( \omega_\infty(K) + Dk \| v' \|_K \right) \leq 2C_K Dk \| v' \|_K.$$

Since $v' \in V'_R \subseteq V_R$, it follows that $\max_{z \in F} \| z \|_K \leq \frac{2Dk}{R} \max_{v \in V} \| v \|_K$. Now by the results of [NV11, SV06, Tao10], it follows that there exist $C = C(A,d,\varepsilon)$ such that

$$kQ \subseteq C(A,d,\varepsilon) \left[ k(F + \{-1,1\}^d) \right].$$
In particular, for every $1 \leq j \leq r$, we have

$$\|g_j\|_K \leq C(A, d, \varepsilon) C_K^{k+1} \left( \max_{z \in F} \|z\|_K + \max_{z \in \{-1,1\}^d} \|z\|_K \right),$$

(4.17)

where we use the fact that in a quasi-normed space, we have

$$\left\| \sum_{j=1}^k v_j \right\|_K \leq C_K^k \sum_{j=1}^k \|v_j\|_K.$$

Now, (4.17) implies,

$$\|g_j\|_K \leq C(A, d, \varepsilon) C_K^{k+1} \left( \frac{Dk}{R} \max_{v \in V} \|v\|_K + \omega_\infty(K) \right),$$

which completes the proof of Part 4 and of Theorem 2.2. □

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