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Harnack’s Inequality on Homogeneous Spaces

Abstract. We consider a homogeneous space \( X = (X, d, m) \) of dimension \( \nu \geq 1 \) and a local regular Dirichlet form in \( L^2(X, m) \). We prove that if a Poincaré inequality holds on every pseudo-ball \( B(x, R) \) of \( X \), then an Harnack’s inequality can be proved on the same ball with local characteristic constant \( c_0 \) and \( c_1 \).

1. Introduction and Results

We consider a connected, locally compact topological space \( X \). We suppose that a distance \( d \) is defined on \( X \) and we suppose that the balls

\[
B(x, r) = \{ y \in X : d(x, y) < r \}, \quad r > 0,
\]

form a basis of open neighborhoods of \( x \in X \). Moreover, we suppose that a (positive) Radon measure \( m \) is given on \( X \), with \( \text{supp} m = X \). The triple \((X, d, m)\) is assumed to satisfy the following property: There exist some constants \( 0 < R_0 \leq +\infty, \nu > 0 \) and \( c_0 > 0 \), such that

\[
0 < c_0 \left( \frac{r}{R} \right)^\nu m(B(x, R)) \leq m(B(x, r)) \quad \text{(1.1)}
\]

for every \( x \in X \) and every \( 0 < r \leq R < R_0 \). Such a triple \((X, d, m)\) will be called a homogeneous space of dimension \( \nu \). We point out, however, that a given exponent \( \nu \) occurring in (1.1) should be considered, more precisely, as an upper bound of the “homogeneous dimension”, hence we should better call \((X, d, m)\) a homogeneous space of dimension less or equal than \( \nu \). We further suppose that we are also given a strongly local, regular, Dirichlet form \( a \) in the Hilbert space \( L^2(X, m) \) - in the sense of M. Fukushima [2] - whose domain in \( L^2(X, m) \) we shall denote by \( \mathcal{D}[a] \). Furthermore, we shall restrict our study to Dirichlet forms of diffusion type, that is to forms \( a \) that have the following strong local property: \( a(u, v) = 0 \) for every \( u, v \in \mathcal{D}[a] \) with \( v \) constant on \( \text{supp} u \). We recall that the following integral representation of the form \( a \) holds

\[
a(u, v) = \int_X \mu(u, v)(dx), \quad u, v \in \mathcal{D}[a],
\]
where $\mu(u, v)$ is a uniquely defined signed Radon measure on $X$, such that $\mu(d, d) \leq m$, with $d \in D_{loc}[a]$: this last condition is fundamental for the existence of cut off functions associated to the distance. Moreover, the restriction of the measure $\mu(u, v)$ to any open subset $\Omega$ of $X$ depends only on the restrictions of the functions $u, v$ to $\Omega$. Therefore, the definition of the measure $\mu(u, v)$ can be unambiguously extended to all $m$-measurable functions $u, v$ on $X$ that coincide $m$−a.e. on every compact subset of $\Omega$ with some functions of $D[a]$. The homogeneous metric $d$ and the energy form $a$ associated to the energy measure $\mu$, both given on a relatively compact open subset $X_0 \subset X$ with $\overline{\Omega} \subset X_0$, are then assumed to be mutually related by the following basic assumption:

There exists an exponent $s > 2$ and constants $c_1, c_1 > 0$ and $k \geq 1$, such that

1. for every $x \in X_0$ and every $0 < R < R_0$ the following Poincaré inequality holds:

$$\int_{B(x,R)} |u - \bar{u}_{B(x,R)}|^2 \, dm \leq c_1 R^2 \int_{B(x,kR)} \mu(u, u) \, (dx)$$

for all $u \in D[a, B(x, kR)]$, where

$$\bar{u}_{B(x,R)} = \frac{1}{m(B(x,R))} \int_{B(x,R)} u \, dm.$$

2. By 1, for every $x \in X_0$ and every $0 < R < R_0$, the following Sobolev inequality of exponent $s$ can be proved:

$$\left( \frac{1}{m(B(x,R))} \int_{B(x,R)} |u|^s \, dm \right)^{\frac{1}{s}} \leq c_1 R \left( \int_{B(x,R)} \mu(u, u) \, (dx) \right)^{\frac{1}{2}},$$

where $u \in D[a, B(x, kR)]$ and supp $u \subset B(x, R)$. Instead of working with the fixed constants of (1.1) and (1.2), we make this simple generalization

$$c_0 \to c_0(x) \quad \text{and} \quad c_1 \to c_1(r),$$

where $c_0(x), c_0^{-1}(x) \in L^\infty_{loc}(X_0)$ and $c_1(r)$ is a decreasing function of $r$. Then we assume:

3.:

$$\int_{B(x,R)} |u - \bar{u}_{B(x,R)}|^2 \, dm \leq c_1^2(R) R^2 \int_{B(x,kR)} \mu(u, u) \, (dx).$$

By 3) the following Sobolev type inequality may be proved

4.:

$$\left( \frac{1}{m(B(x,R))} \int_{B(x,R)} |u|^s \, dm \right)^{\frac{1}{s}} \leq \tau^3 c_1(R) R \left( \frac{1}{m(B(x,R))} \int_{B(x,R)} \mu(u, u) \, (dx) \right)^{\frac{1}{2}},$$
where \( u \) is as in i) and ii) and where we have defined \( \tau = \left( \frac{\sup_{B(x,2R)} \frac{1}{c_0(x)}}{c_0(x)} \right)^{\frac{1}{2}} \).

Our purpose will be the Harnack’s inequality recovery for Dirichlet forms when the substitution (1.4) is performed.

**Theorem 1 (Harnack).** Let (1.1), (1.3), (1.6), and (1.4) hold, and let \( u \) be a non-negative solution of \( a(u,v) = 0 \). Let \( O \) be an open subset of \( X_0 \) and \( u \in D_{loc}[O] \), \( \forall v \in D_0[\partial O] \) with \( B(x,r) \subset O \), then

\[
\text{ess sup}_{B_{\frac{r}{2}}} u \leq \exp \gamma \mu' \mu^2 \text{ ess inf}_{B_{\frac{r}{2}}} u,
\]

where \( \gamma \equiv \gamma(v,k) \), \( k \) a positive constant, \( \mu' = \tau c_1 \left( \frac{r}{2} \right) \) and \( \mu = \tau^3 c_1 \left( \frac{r}{2} \right) \). A standard consequence of the previous Theorem is the following

**Corollary 1.** Suppose that

\[
(1.7) \quad \int_0^R e^{-\gamma \mu(x,\rho)} \frac{d\rho}{\rho} \to \infty \quad \text{for} \quad r \to 0
\]

then the solution is continuous in the point under consideration. In particular, if \( \mu(x,\rho) \approx o \left( \log \log \frac{1}{\rho} \right) \), then there exists \( c > 0 \) such that

\[
\text{osc}_{B(x,r)} u \leq c \left( \frac{\log \frac{1}{R}}{\log \frac{1}{r}} \right) \text{osc}_{B(x,R)} u.
\]

From the point of view of partial differential equations these results can be applied to two important classes of operators on \( \mathbb{R}^n \):

**a):** *Doubly Weighted uniformly elliptic operators* in divergence form with measurable coefficients, whose coefficient matrix \( A = (a_{ij}) \) satisfies

\[
w(x) |\xi|^2 \leq \langle A\xi, \xi \rangle \leq v(x) |\xi|^2.
\]

Here \( \langle \cdot, \cdot \rangle \) denotes the usual dot product; \( w \) and \( v \) are weight functions, respectively belonging to \( A_2 \) and \( D_\infty \) such that the following *Poincaré inequality*

\[
\left( \frac{1}{|v(B)|} \int_B |f(x) - f_B|^\alpha v dx \right)^{\frac{1}{\alpha}} \leq c \left( \frac{1}{|w(B)|} \int_B |\nabla f|^{2} v dx \right)^{\frac{1}{2}}
\]

holds.

**b):** *Doubly Weighted Hörmander type operators*, whose form is \( L = X^*_\alpha (\alpha^{hk}(x) X_\lambda) \) where \( X_h, h = 1, \ldots, m \) are smooth vector fields in \( \mathbb{R}^n \) that satisfy the Hörmander condition and \( \alpha = (\alpha^{hk}) \) is any symmetric \( m \times m \) matrix of measurable functions on \( \mathbb{R}^n \) such that

\[
w(x) \sum_i (X_i, \xi)^2 \leq \sum_{i,j} \alpha_{ij} (x) \xi_i \xi_j \leq v(x) \sum_i (X_i, \xi)^2,
\]

where \( X_i \xi (x) = \langle X_i, \nabla \xi \rangle \), \( i = 1, \ldots, m \), \( \nabla \xi \) is the usual gradient of \( \xi \) and \( \langle \cdot, \cdot \rangle \) denotes the usual inner product on \( \mathbb{R}^n \). Then the following *Poincaré*
inequality for vector fields

\[
\left( \frac{1}{|v(B)|} \int_B |f(x) - f_B|^q \, v \, dx \right)^{\frac{1}{q}} \leq c \left( \frac{1}{|w(B)|} \int_B \left( \sum_{j} |\langle X_j, \nabla f(x) \rangle|^2 \right)^{\frac{1}{2}} \, w \, dx \right)^{\frac{1}{2}},
\]

holds, with \( w \in A_2 \) and \( v \in D_\infty \).

2. Harnack’s Inequality

We prove the Harnack’s Inequality of Theorem (1) for a non negative solution \( u \geq \delta > 0 \) and with a constant \( C \) independent of \( \delta \). The result of Theorem (1) is obtained passing to the limit \( \delta \rightarrow 0 \).

Lemma 1. Assume that (1.1), (1.5), (1.6) and (1.4) hold.

Let \( u \) be a non-negative subsolution of \( a(u, v) = 0 \), \( u \in D_{loc}[O] \), \( \forall v \in D_0[O] \).

Define \( \tau = \left( \sup_{B(x,2R)} \frac{1}{\eta(x)} \right)^{\frac{1}{2}} \) then

\[
\left( \text{ess sup}_{B_\alpha} u \right)^p \leq \left( cr^3 \frac{C_1 (\frac{1}{\eta})}{(1-\alpha)} \right)^{\frac{2p-\sigma}{\sigma}} \left( \frac{1}{m(B)} \int_B u^p m \, (dx) \right),
\]

where \( \sigma = \frac{q}{2}, p \geq 2 \) and

\( q = \left\{ \begin{array}{ll}
2 & \text{if } \nu > 2 \\
\frac{\nu}{2^\nu-2} & \text{if } \nu \leq 2
\end{array} \right. \)

Proof. We prove the result for a bounded non negative subsolutions. Let \( \beta \geq 1 \) and \( 0 < M < \infty \), define \( H_M(t) = t^\beta \) for \( t \in [0, M] \) and \( H_M(t) = M^\beta + \beta M^{\beta-1} (t - M) \) for \( t > M \). For fixed \( M \) define

\[
\phi_k(x) = \eta(x)^2 \int_0^{u_k(x)} H_M'(t)^2 \, dt
\]

and

\[
a(u, \phi) := \int_X \mu(u, \phi) \, (dx) = \int_X \mu \left( u, \eta(x)^2 \int_0^u H_M'(t)^2 \, dt \right) \, (dx) \leq 0,
\]

then

\[
\int_X \mu \left( u, \eta(x)^2 \int_0^u H_M'(t)^2 \, dt \right) \, (dx) = \int_X \mu \left( u, \eta(x)^2 \int_0^u H_M'(t)^2 \, dt \right) \, (dx)
\]

\[
+ \int_X \mu \left( u, \int_0^u H_M'(t)^2 \, dt \right) \eta(x)^2 \, (dx) = 0,
\]

\[
(2.1)
\]

\[
\int_X 2\eta \mu(u, \eta) \int_0^u H_M'(t)^2 \, dt \, (dx) + \int_X \mu(u, u) H_M'(u)^2 \eta(x)^2 \, (dx) \leq 0,
\]
and therefore

\[(2.2) \quad \int_X \mu(u,u) H'_M(u)^2 \eta(x)^2 \, (dx) \leq 2 \int_X \eta \mu(u,u) \int_0^u H'_M(t)^2 \, dt \, (dx).\]

Taking account of the inequality

\[2 |fg| |\mu(u,v)| \leq f^2 \mu(u,u) + g^2 \mu(v,v)\]

we get

\[2 |\mu(u,\eta)| \int_0^u H'_M(t)^2 \, dt \, (dx) \leq \frac{1}{2} |\mu(u,u)\eta^2 H'_M(u)^2 + 2\mu(\eta,\eta)\left[\frac{1}{H'_M(u)} \int_0^u H'_M(t)^2 \, dt\right]^2 \]

and putting into (2.1) we have

\[\frac{1}{2} \int_X \mu(u,u) H'_M(u)^2 \eta(x)^2 \, (dx) \leq 2 \int_X \mu(\eta,\eta) \left[\frac{1}{H'_M(u)} \int_0^u H'_M(t)^2 \, dt\right]^2 \, (dx) \]

\[(2.3) \quad \leq 2 \int_X \mu(\eta,\eta) \left[uH'_M(u)\right]^2 \, (dx).\]

Let us consider \(\eta\) as a cut-off function s.t. for \(\frac{1}{2} \leq s < t < 1\) \(B(x, sr) \subset B(x, tr) \subset \emptyset \subset X, \eta \equiv 0 \) on \(X - B(x, tr)\), \(\eta \equiv 1 \) on \(B(x, sr)\), \(0 \leq \eta < 1 \) on \(X\). Then

\[\int_{B_r} \mu(H_M(u), H_M(u)) \eta(x)^2 \, (dx) \leq \int_{\partial} \mu(H_M(u), H_M(u)) \eta(x)^2 \, (dx) \]

\[\leq \frac{40}{(t-s)^2 r^2} \int_{B_t} \left[uH'_M(u)\right]^2 m(dx).\]

With the result of (1.2), applied to \(H_M(u)\), one gets

\[\left(\frac{1}{m(B_r)} \int_{B_r} \left[H_M(u) - H_M(u)_{B_r}\right]^2 \, (dx)\right)^{\frac{1}{2}} \leq c'r^2 r \left[\frac{1}{m(B_r)} \int_{B_r} \mu(H_M(u), H_M(u))\right]^{\frac{1}{2}} \]

\[\leq cr^2 c_1 (sr) r \left[\frac{1}{(t-s)^2 r^2} \int_{B_t} \left[uH'_M(u)\right]^2 m(dx)\right]^{\frac{1}{2}} \]

\[\leq cr^2 \sqrt{40} c_1 (sr) \frac{r}{(t-s)} \left(\frac{m(B_t)}{m(B_r)}\right)^{\frac{1}{2}} \left[\frac{1}{m(B_r)} \int_{B_r} \left[uH'_M(u)\right]^2 m(dx)\right]^{\frac{1}{2}} \]

\[\leq cr^3 \sqrt{40} c_1 (sr) \left[\frac{1}{(t-s)} \left(\frac{r}{s}\right)\right]^{\frac{1}{2}} \left[\frac{1}{m(B_t)} \int_{B_t} \left[uH'_M(u)\right]^2 m(dx)\right]^{\frac{1}{2}}.\]
The average of $H_M (u)$ on $B_s$ is defined by:

$$a_B H_M (u) = \frac{1}{m(B_s)} \int_{B_s} H_M (u) \, m (dx) \leq \tau^2 \left( \frac{\lambda}{t} \right)^\nu \frac{2}{m(B_t)} \int_{B_s} H_M (u) \, m (dx)$$

$$\leq \tau \left( \frac{\lambda}{t} \right)^\nu \left( \frac{1}{m(B_t)} \int_{B_s} \left[ u H_M (u) \right]^2 \, m (dx) \right)^\frac{1}{2}$$

Recall that $m (B_t) \leq \tau^2 m (B_s) \left( \frac{\lambda}{t} \right)^\nu$ and $H_M (u) \leq u H_M (u)$. Therefore

$$\left( \frac{1}{m(B_s)} \int_{B_s} |H_M (u)|^q \, dm \right)^\frac{1}{q} = \left( \frac{1}{m(B_s)} \int_{B_s} |H_M (u) - H_M (u)|_{B_s}^q \, dm \right)^\frac{1}{q}$$

$$\leq c \left[ \left( \frac{1}{m(B_s)} \int_{B_s} |H_M (u) - H_M (u)|_{B_s}^q \, dm \right)^\frac{1}{q} + \left( \frac{1}{m(B_t)} \int_{B_t} |H_M (u)|_{B_t}^q \, dm \right)^\frac{1}{q} \right]$$

$$\leq c t^3 \left( \frac{\lambda}{t} \right)^\nu \left( c (1/s^t) + 1 \right) \left( \frac{1}{m(B_t)} \int_{B_t} \left[ u H_M (u) \right]^2 \, m (dx) \right)^\frac{1}{2}$$

where in the last expression we have included all irrelevant constants into $c$. From the fact that $\left( \frac{t^s}{(t-s)} \right) + 1 \leq 2 \frac{t^s}{(t-s)} \geq 1$. Therefore due to $u H_M (u) \leq u \beta u^{\beta-1} = \beta u^\beta, H_M (u) \geq u^{\beta}$, we obtain by letting $M \to \infty$ that

$$\left( \frac{1}{m(B_s)} \int_{B_s} u^{\beta} \, dm \right)^\frac{1}{\beta} \leq c t^3 \left( \frac{t^s}{(t-s)} \right)^\frac{\beta}{s} \left( \frac{\beta}{t} \left( \frac{c (s^t)}{s^t} \right) \left( \frac{1}{m(B_t)} \int_{B_t} \left[ u \right]^{2\beta} \, m (dx) \right) \right)^\frac{1}{\beta}$$

Raise both sides to the power $\frac{1}{\beta}$ and putting $2\beta = \tilde{\beta}$ and $q = 2\sigma$ we have

$$\left( \frac{1}{m(B_s)} \int_{B_s} u^{\sigma \sigma} \, dm \right)^\frac{1}{\sigma} \leq c \tau^3 \left( \frac{t^s}{(t-s)} \right)^\frac{\beta}{s} \left( \frac{\beta}{t} \left( \frac{c (s^t)}{s^t} \right) \left( \frac{1}{m(B_t)} \int_{B_t} \left[ u \right]^{2\beta} \, m (dx) \right) \right)^\frac{1}{\beta}$$

Now, starting from fixed $\alpha$ and $p, \frac{1}{\beta} \leq \alpha < 1, p \geq 2$ iterate this inequality for $t$ and $s$ successive entries in the sequence $s_j = \alpha + \frac{(1-\alpha)}{(j+1)}$, $j = 0, 1 \ldots$, and $\tilde{s}$ and $\tilde{s}^{\tilde{s}}$ successive entries in $\left\{ \sigma^j p \right\}$, recalling that $\sigma > 1$. If we denote $a_j = \frac{s_j}{s_{j+1} - s_j}$, the conclusion of the lemma is a consequence of the estimate of

$$\log \prod_{j=0}^{\infty} \left[ c t^3 (s_j \tilde{s}) \left( \frac{s_{j+1}}{s_j} \right)^\frac{\beta}{s} \left( \sigma^j \right)^{a_j} \right] = \sum_{j=0}^{\infty} \frac{2}{\sigma^j} \log \left[ c t^3 c (s_j \tilde{s}) \left( \sigma^j \right)^{a_j} \right]$$
where we have used the fact that $\frac{s_{j+1}}{s_j} = \frac{\alpha (1 - \alpha)}{\alpha + 1 (\beta - 1)} \approx 1$. Therefore

$$\sum_{j=0}^{\infty} \frac{\alpha}{\alpha + \beta - 1} \log \left( c \frac{r^3 c_1 (s_j r)}{p} \right) + \sum_{j=0}^{\infty} \frac{\beta}{\alpha + \beta - 1} \log \left( \sigma^j a_j \right)$$

$$\leq \frac{2}{p} \sum_{j=1}^{2} \log \left( c \frac{r^3 c_1 (s_j r)}{p} \right) + \sum_{j=0}^{\infty} \frac{\alpha}{\alpha + \beta - 1} \log \sigma^j + \sum_{j=0}^{\infty} \frac{\beta}{\alpha + \beta - 1} \log \left( 2 \log (j + 2) + \log \frac{1}{1 - \alpha} \right)$$

$$\leq \frac{2}{p} \sum_{j=1}^{2} \log \left( c \frac{r^3 c_1 (s_j r)}{p} \right) + \sum_{j=0}^{\infty} \frac{\alpha}{\alpha + \beta - 1} \log \sigma^j + \sum_{j=0}^{\infty} \frac{\beta}{\alpha + \beta - 1} \log \left( 2 \log (j + 2) + \log \frac{1}{1 - \alpha} \right)$$

and at the end

$$\prod_{j=0}^{\infty} \left[ c \frac{r^3 c_1 (s_j r)}{p} \right] \left( \frac{\alpha}{\alpha + \beta - 1} \right)^{s_{j+1}} \left( \sigma^j a_j \right)^{s_j} \leq \left( \frac{c c_1 (r)}{r} \right)^{s_j} \left( \frac{r^3}{\alpha} \right)^{s_{j+1}} \leq \left( \frac{c c_1 (r)}{r} \right)^{s_j} \left( \frac{r^3}{\alpha} \right)^{s_{j+1}} \right.$$

where we used the following inequalities $\log (a_j) \leq 2 \log (j + 2) + \log \frac{1}{1 - \alpha}$, $p^\beta \leq c$ for $p \geq 2$. The general case can be obtained if we apply the above result for $p = 2$ to the truncated subsolutions and we obtain that $u$ is locally bounded. Coming back to the previous result, we obtain the Lemma 1 by an approximation by truncation.

**Lemma 2.** With the same notation and hypothesis as in Lemma 1 and $-\infty < p < +\infty$,

\[
\left( \text{ess sup}_{B_n} u^p \right) \leq \left( c \frac{r^3 c_1 (s_j r)}{p + 1} \right)^{s_j} \left( \frac{1}{m(B_t)} \int_{B_t} u^p m(dx) \right).
\]

**Proof.** It is sufficient to consider $-\infty < p < 2$. Define $\phi(x) = \eta^2(x) u^\beta(x)$, $\eta(x) \geq 0$, $-\infty < \beta < +\infty$. Then

$$0 \leq \int_X \mu(u, \phi) (dx) = \int_X \mu(u, \eta^2 u^\beta) (dx) = \int_X \eta^2 \mu(u, u^\beta) (dx) + \int_X u^\beta \mu(u, \eta^2) (dx)$$

$$= \int_X \beta u^{\beta - 1} \eta^2 \mu(u, u) (dx) + \int_X u^\beta \eta \mu(u, \eta) (dx).$$

Observe that for $\beta \neq -1$, we can write

\[
\frac{4\beta}{(\beta + 1)^2} \int_X \eta^2 \mu \left( \frac{u}{\beta + \eta} \right)^\beta (dx) = \int_X \beta u^{\beta - 1} \eta^2 \mu(u, u) (dx)
\]

and

\[
2 \int_X \frac{u}{\beta + 1} \left( \frac{u}{\beta + \eta} \right) \eta \mu(u, \eta) (dx) = \frac{4}{\beta + 1} \int_X \mu \left( \frac{u}{\beta + 1}, \eta \right) \left( \frac{u}{\beta + 1} \right) (dx).
\]

Then,

\[
- \beta u^{\beta - 1} \eta^2 \mu(u, u) (dx) \leq 2 \int_X u^\beta \eta \mu(u, \eta) (dx),
\]

but

\[
\int_X \beta u^{\beta - 1} \eta^2 \mu(u, u) (dx) = \frac{4\beta}{(\beta + 1)^2} \int_X \eta^2 \mu \left( \frac{u}{\beta + 1}, \frac{u}{\beta + \eta} \right) (dx)
\]

\[
\leq \frac{4}{(\beta + 1)^2} \int_X \mu \left( \frac{u}{\beta + 1}, \eta \right) \frac{u}{\beta + 1} \eta (dx).
\]
Taking absolute values gives

\[
(2.4) \quad \frac{\beta}{(\beta + 1)} \int_X \eta^2 \mu \left( u^{\frac{\beta + 1}{2}}, u^{\frac{\beta + 1}{2}} \right) (dx) \leq \int_X u^{\frac{\beta + 1}{2}} \eta \mu \left( u^{\frac{\beta + 1}{2}}, \eta \right) (dx);
\]

recalling the fundamental inequality $2 |fg| |\mu(u,v)| \leq f^2 \mu(u,v) + g^2 \mu(v,v)$, we have

\[
2 |fg| |\mu(u,v)| = \left| u^{\frac{\beta + 1}{2}} \eta \right| \mu \left( u^{\frac{\beta + 1}{2}}, \eta \right) \leq \frac{|\beta|}{2|\beta + 1|} \mu \left( u^{\frac{\beta + 1}{2}}, u^{\frac{\beta + 1}{2}} \right) \eta^2 + \frac{|\beta + 1|}{2|\beta|} u^{\beta + 1} \mu(\eta, \eta).
\]

Then, from (2.4) it follows that

\[
\frac{|\beta|}{2|\beta + 1|} \int_X \eta^2 \mu \left( u^{\frac{\beta + 1}{2}}, u^{\frac{\beta + 1}{2}} \right) (dx) \leq \frac{|\beta + 1|}{2|\beta|} \int_X u^{\beta + 1} \mu(\eta, \eta) (dx),
\]

that is

\[
\int_X \eta^2 \mu \left( u^{\frac{\beta + 1}{2}}, u^{\frac{\beta + 1}{2}} \right) (dx) \leq \left( \frac{|\beta + 1|}{2|\beta|} \right)^2 \int_X u^{\beta + 1} \mu(\eta, \eta) (dx).
\]

This is the same as (1.2); beginning from the Sobolev inequality applied to $u^{\frac{\beta + 1}{2}}$ with the same meaning and definition of the cut-off functions $\eta$, one gets

\[
\left( \frac{1}{m(B)} \int_{B_s} u^{\frac{\beta + 1}{2}} - \tilde{u}^{\frac{\beta + 1}{2}} \right)^q = \leq c \tau^2 c_1 (sr) s \left[ \int_{B} u^{\beta + 1} \mu(\eta, \eta) (dx) \right]^{\frac{q}{2}} \leq c \tau^2 c_1 (sr) s \left( \frac{|\beta + 1|}{2|\beta|} \right) \left[ \int_{B} u^{\beta + 1} \mu(\eta, \eta) (dx) \right]^{\frac{q}{2}} .
\]

Evaluating the average of $u^{\frac{\beta + 1}{2}}$ on $B_s$, one gets

\[
(2.6) \quad \frac{a u^{\frac{\beta + 1}{2}}}{m(B_s)} = \frac{1}{m(B_s)} \int_{B_s} u^{\frac{\beta + 1}{2}} m(dx) \leq \frac{1}{m(B_s)} \int_{B_s} u^{\frac{\beta + 1}{2}} m(dx).
\]

By Hölder inequality

\[
\frac{\tau^2}{m(B)} \left( \frac{t}{s} \right) \nu \left[ \int_{B_t} u^{\frac{\beta + 1}{2}} m(dx) \right] \leq \left[ \tau^2 \left( \frac{t}{s} \right) \nu \right] \left[ \frac{1}{m(B_t)} \int_{B_t} u^{\beta + 1} m(dx) \right]^{\frac{1}{q}}.
\]

Putting together (2.5) and (2.6), we see that

\[
\left( \frac{1}{m(B_s)} \int_{B_s} u^{\frac{\beta + 1}{2}} \right)^q = \left( \frac{1}{m(B_s)} \int_{B_s} u^{\frac{\beta + 1}{2}} - \tilde{u}^{\frac{\beta + 1}{2}} \right)^q \left( \frac{1}{m(B_s)} \int_{B_s} u^{\beta + 1} m(dx) \right)^{\frac{q}{2}} \leq c \left[ \left( \frac{1}{m(B_s)} \int_{B_s} u^{\frac{\beta + 1}{2}} - \tilde{u}^{\frac{\beta + 1}{2}} \right)^q \left( \frac{a}{m(B_s)} \int_{B_s} \tilde{u}^{\frac{\beta + 1}{2}} m(dx) \right)^{\frac{q}{2}} \right] \leq \left( \frac{c \tau^2 c_1 (sr) s |\beta + 1|}{2|\beta|} + 1 \right) \left( \frac{t}{s} \right) \nu \left[ \frac{1}{m(B_t)} \int_{B_t} u^{\beta + 1} m(dx) \right]^{\frac{1}{q}} .
\]
Setting $\beta + 1 = \tilde{r}$ and $q = 2\sigma$, we see that for any $r$ with $-\infty < \tilde{r} \leq 2, \tilde{r} \neq 0, 1$

\[
\left(\frac{1}{m(B_t)} \int_{B_t} u^{\tilde{r}} |m| \, dx\right)^{\frac{1}{\tilde{r} - \sigma}} \leq \left\{ \left( \frac{c r^3 c_1 (sr) s}{t - s} \frac{\tilde{r}}{2} \frac{\sigma}{\sigma p - 1} + 1 \right) \left( \frac{s_{j+1}}{s_j} \right)^2 \frac{\tilde{r}}{\sigma p} \right\}^{\frac{2}{\sigma p}}
\]

We use the iteration argument with any fixed $p$ as a starting value of $\tilde{r}$, with $-\infty < p \leq 2, p \neq 0, 1$.

a): $p < 0$ \( \tilde{r} = \sigma^j p \rightarrow -\infty \)

\[
\prod_{j=0}^{\infty} \left\{ \left( c r^3 c_1 (s_j r) a_j \frac{\sigma^j}{\sigma p - 1} + 1 \right) \left( \frac{s_{j+1}}{s_j} \right)^2 \frac{\tilde{r}}{\sigma p} \right\}^{\frac{2}{\sigma p}} \leq \prod_{j=0}^{\infty} \left\{ \left( c r^3 c_1 (s_j r) a_j \sigma^j |p| + 1 \right) \right\}^{\frac{2}{\sigma p}}
\]

Then, after the conversion to the log of the previous quantity we have

\[
\log \prod_{j=0}^{\infty} \left\{ \left( c r^3 c_1 (s_j r) a_j \sigma^j |p| + 1 \right) \right\}^{\frac{2}{\sigma p}} = \sum_{j=0}^{\infty} \frac{2}{\sigma^j |p|} \log \left\{ \left( c r^3 c_1 (s_j r) a_j \sigma^j |p| + 1 \right) \right\}
\]

\[
= \sum_{j=0}^{\infty} \frac{2}{\sigma^j |p|} \left[ \log \left( \frac{c r^3 c_1 (s_j r) \tau^j |p| + 1}{c c_1 (s_j r) \tau^j |p| + 1} \right) \right] + \sum_{j=0}^{\infty} \frac{4}{\sigma^j |p|} \left[ \log \left( \frac{c r^3 c_1 (s_j r) \tau^j |p| + 1}{(1 - \alpha) |p|} \right) \right]
\]

\[
\leq \sum_{j=0}^{\infty} \frac{2}{\sigma^j |p|} \left[ \log \left( \frac{c r^3 c_1 (s_j r) \tau^j (j + 2) \sigma^j |p| + 1}{c c_1 (s_j r) \tau^j (j) \sigma^j |p| + 1} \right) \right] \leq \frac{2 \sigma}{(\sigma - 1) |p|} \log \left( \frac{c r^3 c_1 (s_j r) \tau^j |p| + 1}{(1 - \alpha) |p|} \right)
\]

b): $0 < p < 2$

\[
\prod_{j=0}^{\infty} \left\{ \left( c r^3 c_1 \left( \frac{s_j}{2} \right) a_j \sigma^j |p| + 1 \right) \right\}^{\frac{2}{\sigma p}} \leq \left( \frac{c r^3 c_1 (\frac{s_j}{2}) \sigma^j |p| + 1}{(1 - \alpha) |p|} \right)^{\frac{2}{\sigma p}}
\]

**Lemma 3.** Let the hypothesis of Lemma 3 hold, except that now $u$ is a nonnegative supersolution of $a(u, v) = 0$. For $\frac{1}{2} \leq \alpha < 1$, define $k = k(\alpha, u)$ by

\[
\log k = \frac{1}{m(B_\alpha)} \int_{B_\alpha} (\log u) \, dm.
\]

Then for $\lambda > 0$,

\[
m \left( \{ x \in B_\alpha : \log \frac{u(x)}{k} > \lambda \} \right) \leq \frac{c r c_1 (r)}{(1 - \alpha)^{\lambda}} m(B_\alpha).
\]
Proof. Letting \( \phi = \frac{u^2}{u} \), it follows that \( \|\phi\|_0 \) is bounded. We observe that

\[
0 \leq \int_X \mu(u, \phi) \,(dx) = \int_X \mu \left( u, \frac{u^2}{u} \right) \,(dx) = \int_X \eta^2 \mu \left( u, \frac{1}{u} \right) \,(dx) + \int_X \frac{1}{u} \mu(u, \eta^2) \,(dx) = \int_X \eta^2 \mu(u, u) \left( -\frac{1}{u^2} \right) \,(dx) + \int_X \frac{1}{u} \mu(u, \eta) \,2\eta \,(dx)
\]

This implies that

\[
\int_X \eta^2 \mu(u, \eta) \left( \frac{1}{u^2} \right) \,(dx) \leq 2 \int_X \frac{1}{u} \mu(u, \eta) \,\eta \,(dx)
\]

Taking the absolute values and recognizing the gradient of the log to the left side, we have

\[
\int_X \eta^2 \mu (u, \log u) \,(dx) \leq 2 \int_X \frac{1}{u} \mu(u, \eta) \,\eta \,(dx).
\]

Recalling the usual fundamental inequality, we obtain that

\[
2 \left\| \frac{1}{u} \eta \right\| \mu(u, \eta) \leq \frac{1}{2} \left( \eta \right)^2 \mu(u, u) + 2 \mu(u, \eta),
\]

then \( \frac{1}{2} \int_X \eta^2 \mu (u, \log u) \,(dx) \leq 2 \int_X \mu(u, \eta) \,(dx) \). If we choose as a cut-off function \( \eta = 1 \) on \( B_\alpha \) s.t. \( \mu(u, \eta) \leq \frac{b}{(1-\alpha)^2r^2} \), where \( b \) is a constant, we get

\[
\int_{B_\alpha} |\mu(\log u, \log u)| \,(dx) \leq \frac{b}{(1-\alpha)^2r^2} \int_{B_\alpha} m(dx) = \frac{bm(B_\alpha)}{(1-\alpha)^2r^2}.
\]

By the Poincaré inequality,

\[
\int_{B_\alpha} \left| \log u - \log u \right|^2 \, m(dx) \leq cc^2_r(r^2) \int_{B_\alpha} \mu(\log u, \log u) \,(dx)
\]

\[
\leq bcc^2_r(r) \frac{m(B_\alpha)}{(1-\alpha)^2}.
\]

It follows that

\[
a u_{B_\alpha} (\log u) = \frac{1}{m(B_\alpha)} \int_{B_\alpha} (\log u) \, m(dx) \rightarrow \log k.
\]

Therefore, by including in \( c \) every other inessential constant

\[
\int_{B_\alpha} \left| \log u - \log k \right|^2 \, m(dx) \leq cc^2_r(b) \frac{m(B_\alpha)}{(1-\alpha)^2}.
\]

By Chebyshev’s inequality, for \( \lambda > 0 \),

\[
m\left( \left\{ x \in B_\alpha : \left| \log \frac{u(x)}{k} \right| > \lambda \right\} \right) \leq \frac{1}{\lambda} \int_{B_\alpha} \left| \log \frac{u}{k} \right| \, m(dx)
\]

\[
\leq \frac{1}{\lambda} \left( \int_{B_\alpha} \left| \log \frac{u}{k} \right|^2 \, m(dx) \right)^{\frac{1}{2}} \left( m(B_\alpha)^{\frac{1}{2}} \right) \leq \frac{1}{\lambda} \left( cc^2_r \frac{m(B_\alpha)}{(1-\alpha)^2} \right)^{\frac{1}{2}} \left( m(B_\alpha)^{\frac{1}{2}} \right)
\]

\[
\leq \frac{1}{\lambda^7} \left( \frac{k^6 cc^2_r(b)}{(1-\alpha)^2} \right)^{\frac{1}{2}} \left( m(B_\alpha)^{\frac{1}{2}} \right) \leq \frac{1}{\lambda^7} \left( \frac{cc^2_r(b)}{(1-\alpha)^2} \right)^{\frac{1}{2}} \left( m(B_\alpha)^{\frac{1}{2}} \right).
\]
Lemma 4 (Moser). Let \( c_1 (r), c_0 (x) \) and \( f (x) \) be non-negative bounded functions on a ball \( B \) and in particular let \( c_0 (x) \) belong to \( L^\infty_{loc} (X_0) \) together with \( c_0^{-1} (x) \); \( c_1 (r) \) be a decreasing function of \( r \). Assume that there are constants \( D, d \) so that

(a): \( \text{ess sup}_{B_s} (f^p) \leq \left( \frac{\mu}{t-s} \right)^d \frac{1}{m(B_t)} \int_{B_t} f^p m(dx) \) for all \( s, t, p \) with \( 0 < p < \frac{1}{\mu} \) and \( \frac{1}{2} \leq s < t \leq 1, \) where \( \mu = \tau^3 c_1 (\frac{r}{t}) \).

(b): \( m \left( \left\{ x \in B_{\frac{r}{t}} : \log f(x) > \lambda \right\} \right) \leq \exp \left( \frac{\mu}{\lambda} \right) m(B), \forall \lambda > 0, \) where \( \mu = \tau c_1 (\frac{r}{t}) \).

Then, there exists a constant \( \gamma = \gamma (c, d) \) s.t.,

\[
\text{ess sup}_{B_{\frac{r}{t}}} f \leq \exp \gamma \mu^2 \mu'.
\]

Proof. Replacing \( f \) with \( f^\mu \) and \( \lambda \) with \( \lambda \mu \), we simplify the hypothesis to the case \( \mu = 1 \). Similarly, we may assume that \( m(B) = 1 \) and the result will be valid for \( \mu = 1 \) too. Define \( \varphi (s) = \sup_{B_s} \log f \) for \( \frac{1}{2} \leq s < 1 \), which is a nondecreasing function. Decompose \( B_t \) into the sets where \( \log f > \frac{1}{2} \varphi (t) \) and where \( \log f \leq \frac{1}{2} \varphi (t) \) and accordingly estimate the integral

\[
\int_{B_t} f^p m(dx) \leq e^{p \varphi^2} \frac{2c}{\varphi} + e^{p \varphi^2}.
\]

where \( \varphi = \varphi (r) \) and we have used

\[
m \left( \left\{ x \in B_{\frac{r}{t}} : \log f(x) > \lambda \right\} \right) \leq \frac{c}{\lambda}
\]

and the normalization \( m(B) = 1 \). We choose \( p \) so that the two terms on the r.h.s. are equal, i.e.

\[
p = \frac{2}{\varphi} \log \left( \frac{\varphi}{2c} \right),
\]

provided that this quantity is less than 1 so that \( 0 < p < \mu^{-1} = 1 \) holds.

\[
\frac{2}{\varphi} \log \left( \frac{\varphi}{2c} \right) < 1 \text{ means that } c > \frac{\varphi}{2} e^{-\frac{\varphi}{2}}, \text{ but } \frac{\varphi}{2} e^{-\frac{\varphi}{2}} \text{ assumes its maximum at the value } \varphi = 2 \text{ and this means that } \max \left( \frac{\varphi}{2} e^{-\frac{\varphi}{2}} \right) = e^{-1}. \text{ Therefore, if } c > e^{-1} \text{ then } p < 1; \text{ otherwise this requires that } \varphi = \varphi (r) > c_1 \text{ depending only by } c. \text{ In that case, we have }
\]

\[
\int_{B_t} f^p m(dx) \leq 2e^{p \varphi^2}
\]

and hence, by Hp a)

\[
\varphi (s) < \frac{1}{p} \log \left( 2 \left( \frac{\mu}{t-s} \right)^d e^{p \varphi^2} \right) = \frac{1}{p} \log \left( 2 \left( \frac{\mu}{t-s} \right)^d \right) + \frac{\varphi (t)}{2}
\]

and by \([2.11]\)

\[
\varphi (s) < \frac{\varphi (t)}{2} \left\{ \log \left( 2 \left( \frac{\mu}{t-s} \right)^d \right) \log \left( \frac{\varphi}{2c} \right) + 1 \right\}.
\]
If $\varphi(t)$ is so large that the first term in the parentheses is less than $\frac{1}{4}$, i.e. if (2.14)

$$\log \left(2 \left(\frac{\mu}{t-s}\right)^{d}\right) < \frac{1}{2} \log \left(\varphi \frac{2c}{\varphi} \right) \implies \left(2 \left(\frac{\mu}{t-s}\right)^{d}\right)^{2} < \frac{\varphi}{2c} \implies \varphi(t) > 8c \left(\frac{\mu}{t-s}\right)^{2d}$$

then clearly $\varphi(s) < \frac{4}{7} \varphi(t)$. Anyway, let us distinguish the case when $p > 1$ or (2.14) fail. This means that:

1. if $p > 1$, but (2.14) is still valid, then $c < \frac{\varphi(t)}{2}$; therefore

$$c < 4c \left(\frac{\mu}{t-s}\right)^{2d} e^{-\varphi} \implies ce^{\varphi} < 4c \left(\frac{\mu}{t-s}\right)^{2d}$$, but $ce^{\varphi} > \varphi \implies \varphi < 4c \left(\frac{\mu}{t-s}\right)^{2d}$.

2. if (2.14) is violated $\implies \varphi(t) < 8c \left(\frac{\mu}{t-s}\right)^{2d}.$

In any case $\varphi(t) < 8c \left(\frac{\mu}{t-s}\right)^{2d}$. Since $\varphi(s) \leq \varphi(t)$, we have in both cases that

$$\varphi(s) \leq \frac{3}{4} \varphi(t) + \frac{\gamma_{1}}{(t-s)^{2d}},$$

where $\gamma_{1} \equiv \gamma_{1}(c, d)$. For every sequence $\frac{1}{2} \leq s_{0} \leq s_{1} \leq \ldots \leq s_{k} \leq 1$, we iterate the inequality (2.14):

Step 1)

$$\varphi(s_{0}) < \frac{3}{4} \varphi(s_{1}) + \frac{\gamma_{1} \mu^{2}}{(s_{1} - s_{0})^{2d}}$$

1. $\varphi(s_{1}) < \frac{3}{4} \varphi(s_{2}) + \frac{\gamma_{1} \mu^{2}}{(s_{2} - s_{1})^{2d}}$, but $\frac{4}{3} \varphi(s_{0}) - \frac{4}{3} \varphi(s_{0}) - \frac{4}{3} \gamma_{1} \mu^{2} < \varphi(s_{1})$ then

$$\frac{4}{3} \varphi(s_{0}) - \frac{4}{3} \gamma_{1} \mu^{2} < \frac{4}{3} \varphi(s_{2}) + \frac{\gamma_{1} \mu^{2}}{(s_{2} - s_{1})^{2d}}$$

$$\implies \varphi(s_{0}) < \left(\frac{3}{4}\right)^{2} \varphi(s_{2}) + \gamma_{1} \mu^{2} \left(\frac{1}{(s_{1} - s_{0})^{2d}} + \frac{3}{4} \frac{1}{(s_{2} - s_{1})^{2d}}\right)$$

1. $\varphi(s_{2}) < \frac{3}{4} \varphi(s_{3}) + \frac{\gamma_{1} \mu^{2}}{(s_{3} - s_{2})^{2d}}$

$$\implies \left(\frac{3}{4}\right)^{2} \varphi(s_{0}) - \gamma_{1} \mu^{2} \left(\frac{3}{4}\right)^{2} \left(\frac{1}{(s_{1} - s_{0})^{2d}} + \frac{3}{4} \frac{1}{(s_{2} - s_{1})^{2d}}\right) < \varphi(s_{2})$$

$$\implies \varphi(s_{0}) < \left(\frac{3}{4}\right)^{3} \varphi(s_{3}) + \gamma_{1} \mu^{2} \left(\frac{1}{(s_{1} - s_{0})^{2d}} + \frac{3}{4} \frac{1}{(s_{2} - s_{1})^{2d}} + \left(\frac{3}{4}\right)^{2} \frac{1}{(s_{3} - s_{2})^{2d}}\right)$$

By monotonicity, we have $\varphi(s_{k}) \leq \varphi(s_{1}) < \infty$ and letting $k \to \infty$, we obtain

$$\varphi \left(\frac{1}{2}\right) \leq \gamma_{1} \mu^{2} \sum_{j=0}^{\infty} \left(\frac{3}{4}\right)^{j} \frac{1}{(s_{j+1} - s_{j})^{2d}}.$$
The r.h.s. will converge if we choose, for example, \( s_j = 1 - \frac{1}{2 + j} \),

\[
\varphi \left( \frac{1}{2} \right) \leq \gamma \mu^2 \sum_{j=0}^{\infty} \left( \frac{1}{2 + j} \right)^2 (\frac{1}{2 + 2j})^{2d} = \gamma \mu^2
\]

\[
\implies \sup_{B_1} f \leq e^{\gamma \mu^2}.
\]

**Proof of Harnack Inequality.** (for \( \delta > 0 \)\)
Let (1.4), (1.3) and (1.1) hold, \( u \) be a non-negative solution of \( a(u, v) = 0 \), \( u \in D_{\text{loc}}[\Omega] \), \( \forall v \in D_0[\Omega] \). We wish to apply Lemma (4) to both \( u/k \) and \( k/u \), with \( k \) defined by

\[
\log k = \frac{1}{m(B_{r0})} \int (\log u)(dx).
\]

Assumption (b) of the lemma holds by applying Lemma (3) to \( B_{2\alpha} \subset \Omega \); because of the presence of an absolute value in the same Lemma and since \( \log (u/k) = -\log (k/u) \), assumption (b) holds for both \( u/k \) and \( k/u \). Assumption (a) holds by Lemma (1) applied to \( u/k \) with \( d = \sigma \). We obtain from (4) both

\[
\text{ess sup}_{B_{1\frac{r}{2}}} (u/k) \leq e^{\gamma \mu^2 \mu'}, \quad \text{ess sup}_{B_{1\frac{r}{2}}} (k/u) \leq e^{\gamma \mu^2 \mu'}
\]

and the result follows by taking the product of these estimates, that is

\[
\text{ess sup} u \leq e^{\gamma \mu^2 \mu'} \text{ess inf} u.
\]

**Proof of Corollary.** We may assume without loss of generality that \( R \leq R_0/4 \), with \( R_0 < 1 \) and \( r \leq \frac{R}{4} \) in such a way that \( B(x, 4R) \subset B(x, R_0) \subset \Omega \). Let us define

\[
M_R = \sup_{B_R} u, \quad m_R = \inf_{B_R} u, \quad M_r = \sup_{B_r} u, \quad m_r = \inf_{B_r} u.
\]

Then by applying Harnack inequality to the functions \( M_R - u \), \( u - m_R \) in \( B_r \), we obtain

\[
M_R - u \leq \sup_{B_r} (M_R - u) \leq e^{\gamma \mu(x, r)} \inf_{B_r} (M_R - u) = M_R - M_r
\]

and

\[
u - m_R \leq \sup_{B_r} (u - m_R) \leq e^{\gamma \mu(x, r)} \inf_{B_r} (u - m_R) = m_r - m_R
\]

Hence by addition,

\[
M_R - m_R \leq e^{\gamma \mu(x, r)} (M_R - M_r + m_r - m_R)
\]

so that, writing

\[
\omega(r) = \text{osc u} = M_r - m_r
\]

we have

\[
(2.17) \quad \omega(r) \leq \left( 1 - e^{-\gamma \mu(x, r)} \right) \omega(R),
\]
with \( \omega(R) = \omega(4r) \). The application of Lemma 6.5 of [6], gives the following inequality

\[
\omega(r) \leq \exp \left( -c \int_{\frac{r}{4}}^{R} e^{-\gamma \mu(x,\rho)} \frac{d\rho}{\rho} \right) \omega(R),
\]

(2.18)

Let suppose that \( \mu(x, \rho) \approx o \left( \log \log \frac{1}{\rho} \right) \), then

\[
\omega(r) \leq \exp \left( -c \int_{\frac{r}{4}}^{R} e^{-\gamma \mu(x,\rho)} \frac{d\rho}{\rho} - c \int_{r}^{R} e^{-\gamma \mu(x,\rho)} \frac{d\rho}{\rho} \right) \omega(R),
\]

\[
\implies \omega(r) \leq \exp \left( - \int_{\frac{r}{4}}^{R} \frac{1}{\log \frac{1}{\rho}} \frac{d\rho}{\rho} - \int_{r}^{R} \frac{1}{\log \frac{1}{\rho}} \frac{d\rho}{\rho} \right) \omega(R),
\]

that is,

\[
\omega(r) \leq c \frac{\left( \log \frac{1}{r} \right)}{\left( \log \frac{1}{R} \right)} \omega(R),
\]

(2.19)

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