Classical Nonlinear Response of a Chaotic System: Langevin Dynamics and Spectral Decomposition

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(Dated: October 1, 2018)

We consider the classical response of a strongly chaotic Hamiltonian system. The spectrum of such a system consists of discrete complex Ruelle-Pollicott (RP) resonances which manifest themselves in the behavior of the correlation and response functions. We interpret the RP resonances as the eigenstates and eigenvalues of the Fokker-Planck operator obtained by adding an infinitesimal noise term to the first-order Liouville operator. We demonstrate how the deterministic expression for the linear response is reproduced in the limit of vanishing noise. For the second-order response we establish an equivalence of the spectral decomposition with infinitesimal noise and the long-time asymptotic expansion for the deterministic case.

PACS numbers: 78.20.Bh, 05.45.-a, 05.10.Gg

I. INTRODUCTION

Vibrational molecular motion is an interesting and important example of complex dynamics. Due to nonlinearity, nonadiabatic effects and interactions with the environment, the behavior of such systems is remarkably rich, and their theoretical description away from the equilibrium becomes a challenge. In many cases, e.g. when the frequencies of the vibrational modes do not exceed the temperature, a completely classical description is adequate. The dynamics far from equilibrium is typically irregular and can exhibit chaotic features. In the extreme case of strong chaos all trajectories in phase space are unstable, and the phase-space dynamics shows mixing features.

Spectroscopic methods represent powerful tools for obtaining detailed information on vibrational dynamics. The outcome of spectroscopic experiments can be naturally interpreted in terms of optical response functions, either in the time or in frequency domain. The multi-time (nonlinear) response functions that constitute the main objects in multi-dimensional time-domain spectroscopy are usually analyzed using their spectral decompositions, i.e. the Fourier transforms with respect to the time intervals. These time intervals can be viewed as the time delays between the exciting pulses, provided the latter are short compared to the typical time scales of the system dynamics. Periodic features of response functions are routinely interpreted as signatures of periodic motions. The central frequencies of diagonal and off-diagonal (cross) peaks are associated with some coherent vibrations, whereas the peak shapes contain information on the system-bath coupling. Smooth behavior of response functions in the case of almost harmonic vibration of the primary system is achieved due to its coupling to a macroscopically large number of the bath motions. The bath is often considered in the harmonic approximation: such model is known in the spectroscopic literature as a multi-mode Brownian oscillator.

The situation is totally different in the case of chaotic motion of the primary system. The spectrum of a strongly chaotic system is known to consist of so-called Ruelle-Pollicott (RP) resonances that represent the eigenmodes of the Perron-Frobenius operator \cite{1,2}. The imaginary part of any particular resonance describes an oscillating feature in the system correlations, whereas its real part is responsible for the correlations decay. An individual RP resonance is not directly related to any particular periodic motion, although their positions can be expressed in terms of all periodic orbits in a very collective way via the dynamical $\zeta$-function. These collective resonances in chaotic systems should be distinguished from the signatures of stable periodic motions.

Nonlinear response functions of quantum systems can be conveniently represented in terms of spectral decompositions via the system’s stationary states. These spectral decompositions that are often referred to as Bloembergen’s expressions allow to relate resonances in the spectroscopic measurement data to the transitions between the stationary states. A natural question that arises in the context of approaching the classical limit is what would be the classical counterpart of the quantum Liouville space spectral decomposition for the linear and nonlinear response functions? In quantum mechanics the system state can be described by a density matrix, whereas the evolution is governed by the quantum Liouville operator. One can try to reach the classical limit by replacing the quantum density matrix by the classical phase-space distribution using the Wigner transformation. The quantum Liouville operator should be naturally replaced by its classical counterpart.

This conceptually straightforward approach, however, faces certain major difficulties. Quantum Liouville operators are second-order elliptic operators, whose eigenstates

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belong to certain natural Hilbert spaces. In particular in the case of compact (restricted) coordinate spaces the corresponding spectra are discrete, and finding spectral decompositions does not face conceptual difficulties, at least on the physical level of rigor. The situation with the classical limit is much more involved since the classical Liouville operator is a first-order non-elliptic operator that describes the propagation of phase-space distributions along classical trajectories. In particular any non-periodic classical trajectory generates a family of eigenstates of the classical Liouville operator, concentrated on the trajectory with all possible eigenvalues. Such a spectrum contains no useful information on the system relaxation.

A meaningful spectrum that provides detailed information on the relaxation of a strongly chaotic system is known to be represented by the RP resonances. One of the ways to reproduce the RP resonances as eigenvalues of the classical Liouville operator is based on the appropriate simultaneous definition of the functional space where the operator acts. These functional Hilbert spaces, often referred to as rigged spaces, are very different from “standard” Hilbert spaces involved in spectral decompositions of quantum operators. They are usually chosen on the case-to-case basis.

To avoid these difficulties we follow a more physical approach introduced in Ref. 3. The approach is based on introducing weak Langevin noise, followed by considering the limit of vanishing noise. It has several major advantages. First of all, this describes a realistic situation since any system is at least weakly coupled to some environment, and in many cases the system-bath coupling can be described on the level of Langevin noise. Second, it allows to avoid dealing with a problem of choice of the appropriate Hilbert space. The stochastic Langevin processes can be described by adding a diffusion operator to the Liouville operator \( \hat{L} \), which results in the Fokker-Planck operator \( \hat{L} = -(\kappa/2)\nabla^2 + \hat{L} \). The Fokker-Planck operator \( \hat{L} \) is a second-order elliptic operator, and its spectral decomposition is free of the aforementioned difficulties that arise in the case of classical Liouville operator. The RP resonances are obtained from the spectrum of the Fokker-Planck operator by taking the limit \( \kappa \to 0 \). This allows to interpret introducing infinitesimal noise as a regularization procedure similar to coarse graining.

The dependence of the eigenfunctions of the Fokker-Planck operator \( \hat{L} \) on the noise strength \( \kappa \) is nonanalytical. In the limit \( \kappa \to 0 \) of the vanishing noise strength the smooth eigenfunctions turn into generalized functions (distributions). The rigged Hilbert spaces are reproduced from the “standard” Hilbert spaces in the limit \( \kappa \to 0 \). They are spanned on the generalized eigenfunction obtained from the eigenfunctions of \( \hat{L}(\kappa) \) by applying the noiseless limit.

The calculation of the linear response function in terms of the generalized functions does not pose a problem, since the expression involves an expansion of the smooth function over generalized functions. The calculation of nonlinear response functions is essentially more complicated because one needs to define expansions of the generalized functions over similar generalized functions. We avoid dealing with the latter problem by performing all calculations for weak, yet finite noise, followed by applying the limit \( \kappa \to 0 \) to the final expressions.

In our previous work 4 we demonstrated the convergence of the second-order response function for strongly chaotic systems. As an example of the strong chaos, we considered free motion on a compact surface of constant negative curvature. The model has been studied since more than a century and served as a prototype for quantum chaos 5, 6, 7. The motion on a surface of genus 2 is in particular known as a Hadamard billiard. Using the dynamical symmetry (DS) of the system we found analytical expressions for the response functions. The long-time asymptotic expansion of the second-order response function turned out to have the form of the double decomposition in the resonances that appear in the linear response.

In the present work we find the decomposition of the linear and nonlinear response in Ruelle-Pollicott resonances for this classical strongly chaotic model by introducing weak Langevin noise. The spectral decomposition in this case is conceptually straightforward. We further demonstrate that the decomposition coefficients converge in the noiseless limit \( \kappa \to 0 \), and the resulting spectral decompositions reproduce the asymptotic expansions for the purely classical response functions computed in our earlier work 4. To the best of our knowledge, this is the first calculation of the classical nonlinear response for a chaotic flow that uses spectral decomposition based on the noise regularization.

Our paper is organized as follows. In Sec. II we summarize the geometry and dynamical symmetry of free motion on a compact surface of constant negative curvature. Statistical description of the response is presented in Section III. We regularize the dynamics by adding noise to the Liouville operator and derive general forms of the spectral decompositions in Section IV. The eigenstates of the resulting Fokker-Planck operator are found in Section V. Explicit forms of spectral decompositions are obtained in Section VI for the linear response and in Section VII for the second-order response.

II. CHAOTIC MODEL SYSTEM: LIOUVille-SPACE PICTURE AND DYNAMICAL SYMMETRY

Following 4 we consider free motion on a 2D compact surface \( M^2 \) of constant negative curvature (Gaussian curvature). This strongly chaotic system is described by the classical free-particle Hamiltonian

\[
H(x, \zeta) = \frac{1}{2m} g^{ik} p_i p_k = \frac{\kappa^2}{2m},
\]

(1)
that depends on the absolute value $\zeta$ of the momentum $p$ only. The curvature of the 2D configuration space is expressed in terms of the metric tensor $g^{ik}$. The Hamiltonian classical dynamics preserves the smooth compact 3D manifold $M^3$ that corresponds to a fixed value of energy. Points $x \in M^3$ of the reduced phase space are described by two coordinates $r \in M^2$ and the momentum direction angle $\theta$. Hereafter, we will use dimensionless units so that the mass $m = 1$ and the curvature $K = -1$. Despite the complexity of the flow due to its chaotic nature, its strong dynamical symmetry (DS) enables an analytical treatment [4].

The geodesic flow $\dot{\eta} = \hat{\mathcal{L}} \eta$, where $\eta = (r, p) = (x, \zeta)$ denotes a point in phase space, is generated by the Liouville operator

$$\hat{\mathcal{L}} = \{H, \zeta\} = \frac{\partial H}{\partial p} \frac{\partial}{\partial r} - \frac{\partial H}{\partial r} \frac{\partial}{\partial p}.$$  

(2)

Hereafter, we identify any vector field with the corresponding first-order operator of the derivative in the vector field direction. We denote by $\sigma_l$ the vector field that determines the phase space velocity: $\hat{L} = \zeta \sigma_l$. We further introduce the second vector field $\sigma_2 = \partial / \partial \theta$ in the tangent space, and finally set $\sigma_2 = [\sigma_1, \sigma_2]$ (see e.g. Refs. [4, 8] for the details). A simple local calculation yields $[\sigma_2, \sigma_2] = -\sigma_1$ and $[\sigma_1, \sigma_2] = -K \sigma_2$, which implies that in the constant negative curvature case, the vector fields $\sigma_1, \sigma_2$ and $\sigma_2$ form the Lie algebra $so(2,1)$. The group $SO(2,1)$ action in the reduced phase space $M^3$ is obtained by integrating the $so(2,1)$ algebra action. DS with respect to the action of the group $G \cong SO(2,1)$ does not mean symmetry in a usual sense, i.e. that the system dynamics commutes with the group action, but rather reflects the fact that the vector field $\hat{L}$ that determines the classical dynamics is represented by an element of the corresponding Lie algebra $so(2,1)$, whereas the stable and unstable directions of our hyperbolic flow are determined by $\sigma_2 \mp \sigma_2$. The algebra generators $\sigma_l$, $l = 1, 2, 3$ are anti-Hermitian, i.e. satisfy the equalities $\sigma_l^\dagger = -\sigma_l$ with respect to the natural scalar product $(\varphi, \psi) = \int dx \varphi^*(x) \psi(x)$ in the Poisson bracket of two functions $f(x, \zeta)$ and $g(x, \zeta)$ reads

$$\{f, g\} = \frac{\partial f}{\partial \zeta}(\sigma_1 g) - (\sigma_1 f) \frac{\partial g}{\partial \zeta} + \frac{1}{\zeta} (\sigma_2 f)(\sigma_2 g) - (\sigma_2 f)(\sigma_2 g).$$  

(3)

The smooth action of $G = SO(2,1)$ in $M^3$ can be interpreted as that the space $\mathcal{H}$ of smooth functions in $M^3$ constitutes a representation of $G$, which turns out to be a unitary representation (see Refs. [8, 9, 10]), and therefore can be decomposed into a direct sum of irreducible representations of $G$. Stated differently, any distribution in the reduced phase space $M^3$ can be decomposed in irreducible representations. DS implies that the distributions in different representations evolve independently. We focus on the principal series representations of $SO(2,1)$ since only these provide experimentally interesting contributions to the linear and second-order response [4]. A principal series representation $\mathcal{H}_s$ is labeled by an imaginary number $s$, $\text{Im} s > 0$. Eigenfunctions $\psi_k(x; s)$ of the momentum rotation operator $\sigma_2$, hereafter also referred to as angular harmonics, form a convenient basis set in the irreducible representation $\mathcal{H}_s$. The functions $\psi_0(x; s)$ do not depend on the momentum direction and can be viewed as eigenfunctions of the Laplacian operator on our compact Riemann surface $M^2$ [4, 10, 11]. The Laplacian eigenvalues provide a set of numbers $s \in \text{Spec}(M^2)$ according to the equation $\nabla^2 \psi_0(x; s) = (s^2 - 1/4) \psi_0(x; s)$. The angular harmonics have the following properties:

$$\sigma_2 \psi_k(x; s) = i k \psi_k(x; s),$$  

(4)

$$\sigma_\pm \psi_k(x; s) = \left( \pm k + \frac{1}{2} - s \right) \psi_{k \pm 1}(x; s),$$

where we introduced the raising and lowering operators $\sigma_\pm = \sigma_1 \pm i \sigma_2$. The operators $\sigma_\pm$ are anti-Hermitian conjugated: $\sigma_\dagger_\pm = -\sigma_\pm$.

Using the dynamical symmetry (DS) of the problem, one can calculate the response functions. Two-point correlations that are related to the linear response via the fluctuation-dissipation theorem have been also considered in Ref. [11]. Detailed calculations of the response functions are presented in our previous paper [4]. Since the zero harmonics $\psi_0(x; s)$ are momentum-independent, the expansion of the dipole $f(r)$ in irreducible representations has the form

$$f = \sum_{s \in \text{Spec}(M^2)} B_s \psi_0(x; s),$$  

(6)

where only the principle series contributions are retained. Typically the dipole is represented by a smooth distribution, and, therefore, only a small number $N_f$ of representations are relevant in the expansion. For the sake of the presentation clarity we focus on the case $N_f = 1$. Generalizations to $N_f > 1$ are straightforward and can be easily performed.
III. LANGEVIN STOCHASTIC DYNAMICS: FROM LIOUVILLE TO FOKKER-PLANCK EQUATION

A. Liouville equation and response functions

Strongly chaotic systems are characterized by complicated irregular dynamics\[12\]. Exponential divergence of initially close trajectories and mixing make the description in terms of individual trajectories complicated and inadequate. An alternative picture, based on Liouville representation of classical mechanics, involves propagating phase space densities. Suppose that the system initially occupies a certain region in the phase space which correspond to a range of initial phase space variables. In the course of the evolution of a chaotic system, the shape of the region becomes stretched along the unstable directions and contracted along the stable ones.

The response functions characterize the system response to a time-dependent external driving field $\mathcal{E}(t)$ and constitute the basic outcome of spectroscopic measurements. The external field is coupled to the system via the polarization $f(r)$, also often referred to as the dipole. The function $f(r)$ of the system coordinates is the classical counterpart of the polarization operator. The total classical Hamiltonian of the system coupled to the driving field has the form

$$H_T = H - f\mathcal{E}(t).$$

(7)

The initial equilibrium phase space distribution $\rho_0$ starts to evolve once the external field is turned on. The phase space density $\rho(\eta, t)$ at time $t$ depends on the values of the external field at preceding times. In most cases the observed signal at time $t$ is directly related to the same polarization $f(r)$, which describes the system-field coupling. Within the classical mechanics formalism the observed polarization is given by $\int d\eta f(r)\rho(\eta, t)$. The evolution of the phase space distribution $\rho$ is governed by the classical Liouville equation

$$\left(\delta_t + \hat{L}\right)\rho = \mathcal{E}(t)\{f, \rho\},$$

(8)

where the Liouville operator $\hat{L}$ acts as the Poisson bracket with the Hamiltonian according to Eq. (2).

The sign of the Poisson bracket is defined by a convention that $\{p_i, r_j\} = \delta_{ij}$ for canonically conjugate coordinates $r$ and momenta $p$. Note that the Liouville operator is a first-order linear differential operator. As stated earlier, it can be identified with the vector field $\hat{L} = \zeta\sigma \mathcal{E}$ in the phase space. The Liouville operator is anti-Hermitian with respect to the conventional scalar product, so that $\hat{L}^\dagger = -\hat{L}$. This makes the evolution operator $e^{-\hat{L}t}$ unitary and leads to the conservation of phase space volume (Liouville theorem).

One can solve Eq. (8) iteratively considering the right-hand side as a small perturbation and taking $\rho_0$ for the zero approximation. Then the phase space density, and hence the response (observed signal) can be naturally represented in the form of the functional Taylor expansion over the external field. The $n$-th order response function $S_n$ is defined as the coefficient (more precisely, an integral kernel) in front of the $n$-th power of the driving field. The response function $S_n$ depends on $n + 1$ time moments, and if the unperturbed system is initially at equilibrium, it actually depends on $n$ time intervals.

For the linear and second-order response functions we have the following expressions:

$$S^{(1)}(t_1) = \int d\eta f(\eta)e^{-\hat{L}t_1}\{f(\eta), \rho_0\},$$

(9)

$$S^{(2)}(t_1, t_2) = \int d\eta f(\eta)e^{-\hat{L}t_2}\{f(\eta), e^{-\hat{L}t_1}\{f(\eta), \rho_0\}\}.$$  

(10)

Here $e^{-\hat{L}t}$ is the the phase space density evolution operator, often referred to as the Perron-Frobenius operator. We understand $e^{-\hat{L}t}g(\eta)$ as the solution $\rho(\eta, t)$ of the equation $(\delta_t + \hat{L})\rho = 0$ with the initial condition $\rho(x, 0) = g(x)$. Since $\hat{L}$ is a first-order differential operator, the solution can be readily found using the method of characteristics. In other words, the first-order operator $\hat{L}$ allows to write $e^{-\hat{L}t}g(\eta) = g(e^{-\hat{L}t}\eta)$.

B. Langevin dynamics and Fokker-Plank equation

Phase space trajectories found from the Hamilton equations of motion are invariant with respect to the time reversal. This property is easily observed in the behavior of integrable systems. Namely, any integrable dynamics can be represented by a set of quasiperiodic motions, which implies reversibility and recurrence in the values of physical observables. The behavior of chaotic systems is quite different. Although described by the same formalism based on Eqs. (2) and (3) that possess time-reversal symmetry, one observes obviously irreversible features such as relaxation phenomena.

The apparent paradox and its solution are well known in statistical mechanics. In the course of evolution of a chaotic system, more and more fine features develop in the phase space distribution. The distribution width decreases exponentially along the stable directions. At the same time, physical quantities are represented by smooth functions in phase space. Therefore, fine features of the distribution are actually irrelevant for the smooth observables. In particular, this results in the exponential damping of the linear and nonlinear response functions in a strongly chaotic system[4].

Only coarse grained properties of the phase space density remain relevant. For instance, in the long-time asymptotic, the strongly chaotic system may be found in the neighborhood of any point of the phase space with equal probability, which is reflected in the homogeneity of the stationary phase space density. A physically useful and meaningful definition of the evolution operator which relates initial and final distributions must rely on...
some kind of regularization that eliminates unnecessary distracting details. Coarse graining can be introduced either directly in phase space or, alternatively, in the functional space of phase space distributions. Sometimes coarse graining is inevitable, e.g. in computer simulations, where numerical errors determine the precision.

The finest scale of classical dynamics is limited by the onset of quantum effects. One of well-known examples is the quasiclassical calculation of the entropy. Quantum effects were also shown to remove unphysical power-law divergences in the nonlinear response functions of integral systems.13,14 It has been demonstrated that the limit $\hbar \to 0$ should be taken after calculating the long-time asymptotics. This property resembles the characteristic feature of our approach to the second-order response in a chaotic system, as presented below.

Besides, real physical systems are never isolated. The influence of an environment shows up as noise or random forces at the level of equations of motion, and as unobservable degrees of freedom and diffusive behavior of relevant variables in the statistical description. Although in many cases interaction with the environment is sufficiently weak and can be neglected, it is often utilized as a convenient and physically meaningful way to perform calculations.

The effect of the diffusion in the chaotic system is to regularize the long-time dynamics and to introduce irreversible leveling of gradients in the stable direction. We regularize the operator generating the density evolution by adding a small diffusive term in the form of the second-order differential operator:

$$\hat{\mathcal{L}} = \hat{\sigma}_1 - \frac{\kappa}{2} \hat{\sigma}_z^2,$$

(11)

Since $\hat{\sigma}_z$ generates rotations of the momentum, $\kappa/2 > 0$ has the meaning of the angular diffusion coefficient. Factor $1/2$ is chosen for reasons of convenience. We will be interested in the case $\kappa \ll 1$. As we will see, the limit $\kappa \to 0$ does exist in the spectral decomposition, therefore in the limit the factor does not play any role.

The resulting Fokker-Planck operator $\hat{\mathcal{L}}$ defines the evolution of the phase space density. This statistical description for the density is equivalent to the description in terms of the Langevin equations for the dynamical variables. Such equations contain a random force which tends to change the momentum direction keeping the energy constant. This can be formally described by a stochastic Liouville operator

$$\hat{\mathcal{L}}_{st}(t) = \zeta (\hat{\sigma}_1 + \gamma(t) \hat{\sigma}_z)$$

(12)

where $\gamma(t)$ is a random Gaussian Markovian process with the zero mean. The noise intensity is determined by $\kappa$, and the two-point correlation function reads

$$\langle \gamma(t)\gamma(t') \rangle = \kappa \delta(t - t').$$

(13)

The addition of the diffusion term with the second derivative defines the spectrum of the resulting operator in the space of smooth functions. The eigenfunctions become regular, differentiable functions. As we will see, they are still singular at $\kappa = 0$, and in the limit $\kappa \to 0$ they turn into generalized functions, which is natural for the case of a small parameter in front of the highest derivative.

The mixing property of chaos makes the type of noise irrelevant. Random force exerted on any of the phase space variables affects other variables if they are not fixed by conservation laws. Thus, in the reduced phase space represented by the shell of constant energy the mixing leads to the fast and irregular randomization of the position variables $r$.

As a result, the Liouville equation that describes the phase space density evolution in a system perturbed by an external field $\mathcal{E}$, is replaced by the Fokker-Planck equation:

$$\langle \partial_t + \zeta \hat{\mathcal{L}} \rangle \rho = \mathcal{E}(t) \{ f, \rho \},$$

(14)

The evolution operator in the unperturbed system can be written symbolically as $e^{-\zeta \mathcal{L} t}$, in a full analogy with the noiseless case. Similarly, the iterative solution of Eq. (14) yields the phase space density in a form of an expansion in powers of $\mathcal{E}$. The response functions are obtained from Eqs. (9) by replacing $\hat{\mathcal{L}}$ with $\zeta \mathcal{L}$:

$$S^{(1)}(t) = \int d\zeta \zeta \langle f e^{-\zeta \mathcal{L} t} f \rho_0 \rangle,$$

$$S^{(2)}(t_1, t_2) = \int d\zeta \zeta \langle f e^{-\zeta \mathcal{L} t_2} f e^{-\zeta \mathcal{L} t_1} f \rho_0 \rangle,$$

(15, 16)

where angular brackets stand for the integral over the reduced phase space, and we defined the action of the operator $f_-$ on a function $g(\eta)$ as the Poisson bracket of $f$ and $g(\eta)$ so that $f_- g = \{ f, g \}$.

IV. SPECTRAL DECOMPOSITION OF RESPONSE FUNCTIONS: GENERAL FORMALISM

In the well-known situation with quantum response, the response functions can be readily represented in the form of spectral decompositions, since the infinitesimal evolution is determined by a Hermitian operator. The conjugation property with respect to a simple scalar product facilitates expansions in the basis of the eigenstates.

We have a natural scalar product for functions in $M^3$ which allowed to implement the unitary presentations in $3D$ space by functions on the circle. With respect to this scalar product $\hat{L}$ is an anti-Hermitian first-order differential operator. However, the Fokker-Planck operator $\hat{\mathcal{L}}$ is obtained by adding a second-order Hermitian contribution to $\hat{L}$. This makes the resulting operator $\hat{\mathcal{L}}$ neither Hermitian nor anti-Hermitian. Namely, its adjoint is

$$\hat{\mathcal{L}}^\dagger = -\sigma_1 - \frac{\kappa}{2} \hat{\sigma}_z^2.$$
It seems that there is no natural choice of the scalar product that would make our Fokker-Planck operator simply related to its adjoint.

The expressions for the response functions include integrations over the reduced phase space. The convolution \( \langle \varphi(x) \psi(x) \rangle = \int dx \varphi(x) \psi(x) \) of two functions \( \varphi(x) \) and \( \psi(x) \) can be recast in the form of the natural scalar product as \( \langle \varphi \psi \rangle = \langle \varphi^{*} \psi \rangle \).

As shown by the construction of eigenmodes in Section V, the operator \( \hat{L} \) is diagonalizable in \( \mathcal{H}_{s} \), and the Jordan-block structures that are possible in a general case do not appear in our spectral decompositions. Therefore, any function in \( \mathcal{H}_{s} \) can be expanded in the eigenfunctions \( \varphi_{\lambda}(x; s) \) that obey the equation

\[
\hat{L} \varphi_{\lambda} = \lambda \varphi_{\lambda}.
\]

The expansion coefficients are linear functionals of \( \varphi(x; s) \) and can be represented by scalar products of \( \varphi(x; s) \) with certain functions denoted by \( \varphi_{\lambda}(x; s) \):

\[
\varphi(x; s) = \sum_{\mu} \langle \varphi_{\mu}^{*}(x; s) \varphi_{\lambda}(x; s) \rangle \varphi_{\mu}(x; s).
\]

It follows from the expansion of \( \varphi_{\lambda}(x; s) \) and \( \hat{L} \varphi_{\lambda}(x; s) \) that the functions \( \varphi_{\lambda}(x; s) \) and \( \hat{L} \varphi_{\lambda}(x; s) \) are the eigenfunctions of the operator \( \hat{L} \) which satisfy the following properties:

\[
\hat{L}^{\dagger} \varphi_{\lambda}(x; s) = \lambda^{*} \varphi_{\lambda}(x; s),
\]

\[
\langle \varphi_{\mu}^{*}(x; s) \varphi_{\lambda}(x; s) \rangle = \delta_{\mu \lambda}.
\]

We focus our detailed treatment on the simplest case \( N_{f} = 1 \) of a single irreducible representation contributing to the dipole moment \( f(r) \). This corresponds to a single contribution in the expansion (6), and below we imply \( f = \psi_{0}(x, s) \). A generalization to an arbitrary linear combination of \( N_{f} > 1 \) such terms with different \( s \) is straightforward for both response functions under consideration 4.

We can successively apply the expansion procedure (19) and obtain any response or correlation function as a spectral decomposition. Based on Eq. (19) we get the following form for the linear response function:

\[
S^{(1)}(t) = \int d\zeta \sum_{\lambda} e^{-\lambda \zeta} \langle f \varphi_{\lambda} \rangle \langle \varphi_{\lambda} f - \rho_{0} \rangle.
\]

For the calculation of the second-order response starting from Eq. (16), we introduce a double expansion in the basis of the angular harmonics that are defined by Eqs. (4):

\[
S^{(2)}(t_{1}, t_{2}) = \int d\zeta \sum_{\mu} e^{-\mu \zeta} \sum_{\lambda} \langle f \varphi_{\mu} \psi_{n} \rangle \langle \varphi_{n} f - \psi_{m} \rangle \langle \psi_{m} e^{-\lambda \zeta_{1}} \varphi_{\lambda} \rangle \langle \varphi_{\lambda} f - \rho_{0} \rangle.
\]

The double expansion in the angular harmonics yields a geometric matrix element \( \langle \psi_{n} \psi_{m} \rangle \) studied in our previous work 4. Note that \( f_{-} \) in the middle angular brackets contains a derivative \( \partial / \partial \zeta \) acting on all functions of the momentum to the right of it. This complication occurs due to the absence of the integration over \( \zeta \) in the angular brackets which correspond to projections onto the basis set in \( \mathcal{H}_{s} \).

\[V. \text{ EIGENSTATES OF FOKKER-PLANK OPERATOR}\]

As stated earlier the Fokker-Planck operator (11) is neither Hermitian nor anti-Hermitian with respect to any natural scalar product and, therefore, not necessarily diagonalizable. In what follows, we show that it actually is: We apply the representation on the circle to find its eigenstates and demonstrate that they constitute a basis set.

The eigenfunctions and eigenvalues of \( \hat{L} \) can be found by solving the 1D eigenvalue problem

\[
\hat{L} \Phi_{\lambda} = \lambda \Phi_{\lambda}.
\]

on a circle with the Fokker-Planck operator [see Eqs. (19) and (11)]

\[
\hat{L} = \frac{k}{2} \frac{d^{2}}{du^{2}} + \sin u \frac{d}{du} + \frac{1 - 2s}{2} \cos u.
\]

The first derivative in the operator can be eliminated by redefining the functions

\[
\Phi_{\lambda}(u) = e^{-\frac{\sin u}{k}} \xi_{\lambda}(u).
\]

In terms of the functions \( \xi_{\lambda}(u) \), the eigenvalue problem (23) assumes the form of a stationary Schrödinger equation

\[
\hat{H} \xi_{\lambda}(u) = \lambda \xi_{\lambda}(u)
\]

with the effective Hamiltonian

\[
\hat{H} = \frac{k}{2} \frac{d^{2}}{du^{2}} + \frac{\sin^{2} u}{2k} - s \cos u.
\]

The Hamiltonian describes a quantum particle in a complex-valued potential on the circle. The operator in not Hermitian because of the imaginary part of the potential. However, since the Hamiltonian does not contain first derivatives, we can define a symmetric (non-Hermitian) scalar product \( V(\xi, \phi) = \int du \xi(u) \phi(u) \) so that the Hamiltonian is self-adjoint with respect to it: \( V(\hat{H} \xi, \phi) = V(\xi, \hat{H} \phi) \). Eigenfunctions of \( \hat{H} \) corresponding to different eigenvalues are orthogonal and can be normalized:

\[
\int_{0}^{2\pi} du \xi_{\lambda}(u) \xi_{\mu}(u) = \delta_{\lambda \mu}.
\]

The Hamiltonian \( \hat{H} \) also possesses certain symmetries that simplify the analysis. First, it is invariant with respect to the sign change of \( u \), and therefore all its eigenfunctions are either even or odd functions of \( u \). As we
will see, only even eigenstates contribute to the spectral decompositions of response functions. Therefore, we present the detailed expressions only for even eigenstates on the half-circle $0 < u < \pi$.

Since the potential contains a nonzero imaginary part, its eigenvalues can be complex. The second symmetry involves the complex conjugation of the Hamiltonian: $H^*(u) = H(u + \pi)$. Consequently, the eigenfunctions corresponding to the complex conjugate eigenvalues are related by

$$\xi_{\lambda^*}(u) = (\xi_{\lambda}(u + \pi))^*.$$  \hspace{1cm} (30)

Detailed analysis of the Schrödinger equation in the weak noise case $\kappa \ll 1$ is presented in Appendix A. The small parameter $\kappa$ allows to solve the Schrödinger equation using the WKB method. The imaginary part of the potential energy is small compared to its real part. This supports the use of such terms as the “minimum” of the potential and the “under the barrier” wave function. Since the potential minima become deeper for smaller $\kappa$, the eigenfunctions are concentrated near $u = 0$ or $u = \pi$. These states do not mix together since $s$ is imaginary.

Due to the compact nature of the circle, the spectrum of the Fokker-Planck operator is discrete. Its real part is positive and unbounded from above. For such large energies that $\text{Re} \lambda \gg \kappa^{-1}$, the spectrum can be adequately approximated by $\lambda_{\nu} \sim \kappa \nu^2/2$ (with the eigenfunctions $\xi_{\lambda_{\nu}} \sim e^{i\nu u}$) of a free particle on the circle.

In the limit $\kappa \to 0$ even low-energy eigenstates of the first set are concentrated near $u = 0$ and have energies

$$\lambda_{\nu} = \nu - s + \frac{1}{2},$$  \hspace{1cm} (31)

with nonnegative even numbers $\nu$ ($\nu = 0, 2, 4, \ldots$). This discrete equidistant spectrum of the Fokker-Planck operator in the noiseless limit $\kappa \to 0$ is a quite fascinating property. Infinitesimal noise regularizes the Liouvillean dynamics to yield a physical spectrum in the space of smooth functions.

The normalized eigenfunctions that correspond to the specified above eigenvalues are concentrated near $u = 0$:

$$\xi_{\lambda_{\nu}}(u) = A_{0,\nu} e^{-u^2/2\kappa} H_{\nu} \left( \frac{u}{\sqrt{\kappa}} \right),$$  \hspace{1cm} (32)

$$A_{0,\nu} = (\pi \kappa)^{-1/4} (2\nu!)^{-1/2},$$  \hspace{1cm} (33)

where $H_{\nu}$ are Hermite polynomials. The eigenfunctions $\Phi_{\lambda_{\nu}}(u) = e^{\cos u/\kappa} \xi_{\lambda_{\nu}}(u)$ are singular at $u = 0$ in the limit $\kappa \to 0$, as they should form a basis set of distributions with large gradients in the stable direction.

In the limit $\kappa \to 0$ the eigenvalues in Eq. (31) are determined by the single potential minimum at $u = 0$. The form of the eigenfunctions is appropriate in the region where the potential may be approximated by a harmonic well. Outside the region, at $|u| \gtrsim 1$, the values of the eigenfunction are exponentially small and do not influence its normalization.

A more rigorous treatment in Appendix provides the WKB approximation for the even eigenfunctions $\xi_{\nu}(u)$ on the whole circle. Under the potential barrier, if $u \gtrsim \kappa^{1/2}$ and $(\pi - u) \gtrsim \kappa^{1/2}$ the approximate solution at the energy $\lambda_{\nu} = \nu - s + 1/2$ is found to be

$$\xi(u) = A_{2,\nu} e^{-(\nu - s)^2/2\kappa} H_{\nu} \left( \frac{u - \pi}{\sqrt{\kappa}} \right)e^{-\frac{1}{2} e^{\cos u}},$$  \hspace{1cm} (34)

$$A_{2,\nu} = A_{0,\nu} e^{-\frac{1}{2} 2^{s-1} \kappa^{-\frac{1}{2}}}.\hspace{1cm} (35)$$

In the vicinity of $u = \pi$, at $\pi - u \ll 1$, we obtain:

$$\xi(u) = A_1 e^{-(u-s)^2/2\kappa} \Gamma_{\nu_1} \left( \frac{u - \pi}{\sqrt{\kappa}} \right),$$  \hspace{1cm} (36)

$$+ B_1 e^{(u-s)^2/2\kappa} \Gamma_{\nu_1-1} \left( \frac{u - \pi}{\sqrt{\kappa}} \right),$$

where

$$\nu_1 = \nu - 2s = \frac{1}{2} - \frac{1}{2} s,$$  \hspace{1cm} (37)

$$A_1 = -B_1 i 2^{-\nu_1 - 1/2} \Gamma(-\nu_1) \cos \frac{\pi \nu_1}{2},$$  \hspace{1cm} (38)

$$B_1 = A_{2,\nu} e^{-\frac{1}{2} 2^{s+1} \kappa^{-1}} e^{-\frac{1}{2} \Gamma(s+1)}.$$  \hspace{1cm} (39)

Even low-lying eigenstates of the second set concentrated near $u = \pi$ have eigenenergies $\lambda_{\nu}^* = \nu + s + 1/2$, and their eigenfunctions can be obtained by using Eq. (35) which follows from the symmetry of the Hamiltonian.

Eqs. (32)-(37) provide a zero-order approximation for the eigenfunction of the Hamiltonian (25) on the circle. In the limit $\kappa \ll 1$ one can distinguish two types of corrections to the specified above eigenvalues and eigenfunctions of low-lying states. The first one is due to the omitted terms in the potential in the vicinity of its minima. Such corrections are important in the treatment of states with $\nu > 0$. Corrections of the second type originate from the tunneling through the potential barriers and therefore are exponentially small $\sim e^{-2/\kappa}$.

Details of computing the eigenfunctions are presented in Appendix A. It turns out that a straightforward calculation of the higher-mode contributions to the spectral decomposition requires knowledge of eigenfunctions with increasing accuracy in $\kappa$. In the next two sections we present a more elegant approach for which the approximation given by Eqs. (32)-(37) is sufficient.

VI. SPECTRAL DECOMPOSITION FOR LINEAR RESPONSE

Spectral decompositions of the response functions in the eigenmodes of the evolution operator are obtained in Eqs. (22, 23) in the general form. The expressions are not really useful for the calculation, since it would involve several three-dimensional integrations that involve the functions not explicitly known. Nevertheless, the calculation can be carried out by implementing the representation in terms of functions on the circle.
We start with the spectral decomposition of the linear response, and employ the correspondence between the functions $\varphi$ in $M^3$ and $\Phi(u)$ on the circle, to obtain the following expansion:

$$\Phi(u) = \sum_{\lambda} \langle \tilde{\varphi}_{\lambda} \varphi \rangle \Phi_{\lambda}(u).$$  \hspace{1cm} (40)

Using the representation $\Phi_{\lambda}(u) = e^{-\cos u / \kappa} \xi_{\lambda}(u)$ and the orthonormality of the functions $\xi_{\lambda}(u)$ with respect to the symmetric scalar product [29], we can specify a rule to find the projection of $\varphi$ on $\tilde{\varphi}_{\lambda}$:

$$\langle \tilde{\varphi}_{\lambda} \varphi \rangle = \int_{-\pi}^{\pi} du \Phi(u) e^{i \pi \kappa} \xi_{\lambda}(u).$$  \hspace{1cm} (41)

This implies that $\tilde{\varphi}_{\lambda}$ is implemented on the circle by the function

$$\tilde{\Phi}_{\lambda}(u) = 2\pi e^{i \pi \kappa} \xi_{\lambda}(u).$$  \hspace{1cm} (42)

To proceed with the calculation of the response functions, we introduce a notation

$$R_{\lambda,n} \equiv \langle \tilde{\varphi}_{\lambda} \psi_n \rangle = \int_{-\pi}^{\pi} du \, e^{i n u} e^{i \pi \kappa} \xi_{\lambda}(u)$$  \hspace{1cm} (43)

for the coefficient of the expansion of $\psi_n(x; s)$ in $\varphi_{\lambda}(x; s)$. The expansion of $\varphi_{\lambda}(x; s)$ in angular harmonics $\psi_n(x; s)$ contains coefficients

$$\langle \psi_n^* \varphi \rangle = \int_{-\pi}^{\pi} du \, e^{-i n u} e^{-i \pi \kappa} \xi_{\lambda}(u).$$  \hspace{1cm} (44)

The symmetry property [30] allows to relate them to $R_{\lambda,n}$:

$$\langle \psi_n^* \varphi \rangle = \frac{(-1)^n}{2\pi} (R_{\lambda,n}^*)^*. \hspace{1cm} (45)$$

Therefore, the convolution in the last angular brackets in Eq. (22) is equal to $(R_{\lambda,n})^*/2\pi$. The action of the $f_-$ in the last angular brackets, represented by a Poisson bracket [4], creates the following convolution

$$\langle \tilde{\varphi}_{\lambda} \sigma_1 \psi_0 \rangle = \frac{1 - 2s}{2} \int_{-\pi}^{\pi} \frac{du}{2\pi} \cos u \, e^{i \pi \kappa} \xi_{\lambda}(u),$$  \hspace{1cm} (46)

where we have used the representation of $\sigma_1$ on the circle given by Eq. (5). Bearing in mind that $\xi_{\lambda}$ is the eigenfunction of Hamiltonian [28] and integrating by parts, we find the following relation:

$$\langle \tilde{\varphi}_{\lambda} \sigma_1 \psi_0 \rangle = \lambda R_{\lambda,0}. \hspace{1cm} (47)$$

We conclude that the linear response function can be recast in the form of the time derivative of the two-point correlation function, in agreement with the fluctuation-dissipation theorem (FDT):

$$S^{(1)}(t) = \frac{\partial}{\partial t} \sum_{\lambda} \int_0^\infty d\zeta \, Q_{\lambda,0} e^{-\zeta \lambda} \frac{\partial \rho_0}{\partial \zeta},$$  \hspace{1cm} (48)

where we introduced

$$Q_{\lambda,n} = \frac{(-1)^n}{2\pi} R_{\lambda,0} (R_{\lambda,n}^*)^*. \hspace{1cm} (49)$$

Following the approach developed in Ref. [4] for the purely deterministic situation, we introduce the matrix elements of the evolution operator between $n$-th and zero harmonics:

$$A_n(\zeta; t) = \int_{s, s=0} dx \, \psi_n^* (x; s) e^{-\zeta t} \psi_0 (x; s). \hspace{1cm} (50)$$

Spectral decomposition

$$A_n(\zeta; t) = \sum_{\lambda} Q_{\lambda,n} e^{-\zeta \lambda} \hspace{1cm} (51)$$

appears in the response functions of the system with noise.

As we show below, the noiseless limit $\kappa \to 0$ of the series $A_n(t; s)$ reproduces the matrix element $A_0(t; s)$ of the deterministic evolution operator, calculated in Ref. [4]. These series and their coefficients $Q_{\lambda,n}$ play an important role in the calculation of the second-order response function.

We are now in a position to proceed with an explicit calculation of the coefficients in the spectral decomposition of the linear response function. We focus on the first set of modes with the energies $\lambda = \nu - s + 1/2$, whose eigenfunctions are concentrated in the neighborhood of $u = 0$ for $\kappa \ll 1$. The coefficients $Q_{\lambda,0}$ for the other set of modes, with the energies $\lambda^* = \nu + s + 1/2$, can be then easily found by employing the symmetry, described by Eq. (30):

$$Q_{\lambda^*,0} = (Q_{\lambda,0})^*,$$

which also reflects the fact that the response function is real.

The first two terms in the spectral expansion originate from the lowest RP resonances with the energies $\lambda_0 = 1/2 - s$ and $\lambda_0^* = 1/2 + s$. The eigenfunction with $\lambda_0 = 1/2 - s$ concentrated in the vicinity of $u = 0$, is given by $\xi_{\lambda_0}(u) = (\nu \kappa)^{-1/4} \exp(-u^2/2\kappa)$. The first integral, $R_{\lambda_0,0}$, is calculated without much effort. This is done by noticing that the main contribution to the integral comes from $|u| \lesssim \sqrt{\pi}$. Expanding $\cos u = 1 - u^2/2$ in the exponential results in a Gaussian integral:

$$R_{\lambda_0,0} = \int_{-\pi}^{\pi} du \, e^{i \pi \kappa} \xi_{\lambda_0}(u) = e^{\pi \kappa} A_{0,0}^{-1}, \hspace{1cm} (52)$$
where the normalization factor $A_{0,0}$ is related to $\kappa$ via Eq. (53).

The situation, however, is more complicated for higher modes with $\nu > 0$. Due to the orthogonality of the eigenfunctions $\xi_\lambda$ and $\xi_{\lambda'}$ with $\nu > 0$, the integral vanishes if we employ the lowest-order approximation for the wave functions. We find an explicit expression for the latter are computed by applying the standard quantum mechanical perturbation theory to the Schrödinger equation. The procedure is feasible for few low modes only, since the calculation for the higher modes requires higher orders of the perturbation theory.

However, it is possible to overcome this difficulty and find an explicit expression for $R_{\lambda,0}$ for all $\nu$ by using the following trick. We notice that the dominant contribution to the integral

$$\int du \, e^{iu/\kappa} \xi_\lambda(u) \approx A_{0,\nu} e^{i\nu u/\kappa} \int du \, e^{-u^2/\nu} \frac{1}{\sqrt{\kappa}} = 0. \tag{53}$$

This actually means that more accurate expressions are needed to find the first nonvanishing term. Note that one should use better approximations for both the exponential $e^{iu/\kappa}$ and the wave function. The corrections to the latter are computed by applying the standard quantum mechanical perturbation theory to the Schrödinger equation. The procedure is feasible for few low modes only, since the calculation for the higher modes requires higher orders of the perturbation theory.

Our goal is to calculate the integral $R_{\lambda,0} = P_{0,\lambda}$. This will be achieved by expressing it via the known integral $P_{\nu/2,\lambda}$ for $\kappa \ll 1$. Consider $\lambda P_{j,\lambda}$. Making use of $\lambda \xi_\lambda = \mathcal{H} \xi_\lambda$, we integrate in Eq. (55) by parts, and neglect higher-order terms in $\kappa$. This results in the recurrence relation

$$P_{j,\lambda} = \int du \, e^{iu/\kappa} \xi_\lambda(u) = 2^{-\nu/2} (-1)^{\nu/2} A_{0,\nu} e^{i\nu u/\kappa}. \tag{54}$$

does not involve higher-order perturbative calculations described above. The integral is dominated by the region $|u| \lesssim \sqrt{\kappa}$, where we can approximate $(\cos u - 1)^{\nu/2} = (-1/2)^{\nu/2} u^\nu$ and expand it in the Hermite polynomials $H_j(u/\sqrt{\kappa})$ with $j = 0, 2, \ldots, \nu$. All terms except for the last one vanish due to orthogonality of the Hermite polynomials. Further approximations lead to negligible corrections in the limit $\kappa \to 0$. We proceed by introducing

$$P_{j,\lambda} = \int du \, e^{iu/\kappa} \cos(u - 1)^{\nu/2} \xi_\lambda(u). \tag{55}$$

Our goal is to calculate the integral $R_{\lambda,0} = P_{0,\lambda}$. This will be achieved by expressing it via the known integral $P_{\nu/2,\lambda}$ for $\kappa \ll 1$. Consider $\lambda P_{j,\lambda}$. Making use of $\lambda \xi_\lambda = \mathcal{H} \xi_\lambda$, we integrate in Eq. (55) by parts, and neglect higher-order terms in $\kappa$. This results in the recurrence relation

$$P_{j,\lambda} = \frac{j - s + \frac{1}{2}}{\lambda - 2j + s - \frac{1}{2}} P_{j+1,\lambda}. \tag{56}$$

which being applied $\nu/2$ times and followed by setting $\lambda = \lambda_\nu$ yields:

$$P_{0,\lambda_\nu} = 2^{-\nu/2} \frac{\Gamma \left( \frac{\nu+1}{2} \right)}{\nu/2! \Gamma \left( \frac{\nu}{2} - \frac{1}{2} \right)} P_{\nu/2,\lambda_\nu}. \tag{57}$$

and we arrive at

$$R_{\lambda_\nu,0} = e^{i\pi \nu u/2} 2^{-\nu} A_{0,\nu} (\nu/2) \Gamma \left( \frac{\nu}{2} - \frac{1}{2} \right) \frac{\Gamma \left( \frac{\nu+1}{2} \right)}{\nu! \Gamma(1/2)}. \tag{58}$$

The second integral $\langle \psi_0 | \varphi_{\lambda,0} \rangle$ in the linear response is expressed via $(R_{\lambda_\nu,0})^*$ [see Eqs. (52) and (53)]. Making use of the complex conjugation symmetry specified in Eq. (39) this can be recast as

$$\langle \psi_0 | \varphi_{\lambda_\nu,0} \rangle^* = \int du \, e^{-iu/\kappa} \xi_{\lambda_\nu}(u). \tag{59}$$

The obtained integral looks more complicated compared to $R_{\lambda_\nu,0}$, even for the lowest resonance with $\nu = 0$. We first point out the problem and present its straightforward solution for $\nu = 0$. Afterwards we apply a more elegant method, similar to the one developed above, which provides the solution for all modes.

Before starting we emphasize that, due to the eigenfunction symmetry, the discussion can be limited to the interval $0 < u < \pi$. Three different analytical approximate forms for the eigenfunction near $u = 0$, $u = \pi$, and under the barrier are specified in Eqs. (32) - (37). As opposed to the case of $R_{\lambda_\nu,0}$, the integrand in $(R_{\lambda_\nu,0})^*$ does not experience exponential dependence on $u$, and the exponential part of its dependence on $\kappa$ in the contributions from all three regions amounts to a common factor $\exp(-1/\kappa)$.

In the region $\sqrt{\kappa} \ll (\pi - u) \ll 1$, where the approximate forms, given by Eqs. (41) and (42) match, the solution behaves $\sim (\pi - u)^{-\nu-1/2}$. This region, where the two solutions match, provides an essential contribution to the integral. We can break up the integral into two parts by introducing an intermediate point $\nu = u_1$ that belongs to the matching region. Since for small $\kappa \ll 1$ one can always choose $\kappa^{1/2} \ll u_1 \ll 1$, the approximate forms provide good approximations to the left and to the right of the intermediate point, respectively. We intend to show that the sum of two integrals is independent of $u_1$.

The contribution from the region nearby $u = \pi$ has the form:

$$\int_{\pi - u_1}^\pi du \, e^{-iu/\kappa} \xi_\lambda(u) = \frac{e^{i\sqrt{\kappa}}}{u_1/\sqrt{\kappa}} \int_0^u dz \, (A_1 e^{-z^2/\nu} H_{\nu}(z) + B_1 H_{-\nu-1}(iz)) =$$

$$= -B_1 e^{i\pi \nu/2} \frac{\sqrt{\kappa} e^{-i\pi \nu/2} H_{-\nu-1}(i u_1/\sqrt{\kappa}) - H_{-\nu-1}(-i u_1/\sqrt{\kappa})}{4 u_1 \sin(\pi u_1/2)}. \tag{60}$$

We chose $u_1$ so that $\kappa^{1/2} \ll u_1 \ll \kappa^{1/4} \ll 1$ and retained first two terms in the expansion of $\cos u$. The integral is calculated by expressing $e^{-z^2} H_{\nu}(z)$ in terms
of $H_{-\nu_1-1}(iz)$ and $H_{-\nu_1-1}(-iz)$, since all three functions are solutions of the same second-order differential equation (see Ref. [16]). We also employ the solution symmetry and the recurrence relation $\frac{d}{dz}H_{-\nu_1}(iz) = -2i\nu_1 H_{-\nu_1-1}(iz)$ for Hermite functions.

The integration with the under-barrier function can be safely extended to $u = 0$, since the deviation of the approximate integrand from the actual function in a short region of length $\sim \sqrt{\kappa}$ is finite. Then the integration is performed exactly in terms of the hypergeometric function:

$$\int_0^{\pi} du \left( \sin \frac{u}{2} \right)^\nu \left( \cos \frac{u}{2} \right)^{s-\nu-\frac{1}{2}} = \frac{\Gamma \left( \frac{\nu+1}{\nu} \right) \Gamma \left( \frac{-\nu}{s+1} \right)}{\Gamma \left( s+\frac{1}{2} \right)} +$$

$$\frac{2}{\nu_1} \frac{F_1(1, s+\frac{1}{2}, 2-\nu_1, \sin^2 \frac{u}{2})(\cos \frac{u}{2})^{\nu+1}}{\nu_1 \left( \sin \frac{u}{2} \right)^{\nu_1}}.$$  \hfill (61)

We further expand Eq. (60) in $\sqrt{\kappa}/u_1 \ll 1$ and Eq. (61) in $u_1 \ll 1$. In the sum of two pieces of the integral [59], the strongest dependence on $u_1$ cancels out. For the lowest resonance with $\nu = 0$ there are no other contributions comparable with the constant in Eq. (61), which is not the case if $\nu > 0$.

As $\nu$ increases the divergence of the under-barrier solution becomes stronger, and one should take into account all $u_1$-dependent terms with non-positive real parts of the exponents that appear in Eqs. (60) and (61). For large $\nu$ this requires higher order of the perturbative calculations. One might have argued that, since the corrections to the wave functions have the same structure as the initial approximations for them, all relevant $u_1$-dependent terms should cancel out, and the corrections to the constant are small in $\kappa$.

However, we can avoid exact unpleasant calculations by applying the method developed above. This is achieved by relating $(R_{\nu'_0,0}^*)$ to the integral that can be calculated using the principal approximation for the eigenfunction specified in Eqs. (32) - (37). According to the definition (35) of $P_{\lambda,\nu}$ and due to the wavefunction symmetry (30), we have $(R_{\nu'_0,0}^*) = (P_{\lambda,\nu}^*)^*$. The recurrence relation (30) yields

$$(P_{\lambda,\nu}^*)^* = \frac{(s + \frac{\nu-1}{2})}{(-2)^{1+\nu/2} \Gamma(1+\nu)} P_{\lambda,\nu}^*.$$  \hfill (62)

Next, we consider the integral

$$(P_{\lambda,\nu}^*)^* =$$

$$\left( -1 \right)^{1+\nu/2} \int du e^{-\frac{\nu-1}{2}} (1 + \cos u)^{1+\nu/2} \xi_{\nu}(u).$$  \hfill (63)

and notice that for its calculation it is sufficient to take the under-barrier approximation [54] for $\xi_{\nu}$ on the whole half-circle. The resulting integral can be easily calculated:

$$(P_{1+\nu/2,\lambda}^*)^* =$$

$$A_{2,\nu}(-1)\frac{(1+\nu/2)(\nu+2s+3)/2}{\Gamma(1+\nu/2)} \frac{(\nu+1)}{\Gamma((2s + \nu + 3)/2)}.$$  \hfill (64)

The integral converges, and the differences between the approximate and exact integrands in the narrow regions of width $\sim \sqrt{\kappa}$ near $u = 0$ and $u = \pi$ are smaller than $\sim A_{2,\nu}k^{\nu'/2}$ and $\sim A_{2,\nu}k^{\nu'+1/2}$, respectively. The calculation of the second integral is completed by combining Eqs. (62) and (64) followed by expressing $A_{2,\nu}$ via $A_{0,\nu}$, according to Eq. (33):

$$(R_{\nu'_0,0}^*) = A_{0,\nu}e^{-\frac{\nu}{2}\kappa^{\nu'/2}}1+2\nu \frac{(\nu+1)}{\Gamma((2s + \nu + 3)/2)}.$$  \hfill (65)

The method is naturally also applicable to the case $\nu = 0$, which has been treated earlier, using a more straightforward approach, with much more effort involved.

We complete the calculation by inserting Eqs. (58) and (55) into Eq. (49) to find that $Q_{\nu_0,0}$ and $Q_{\nu'_0,0}$ have finite limits at $\kappa \to 0$, namely

$$\lim_{\kappa \to 0} Q_{\nu_0,0} = \frac{\Gamma(1+\nu/2)}{\pi^{\nu/2} \Gamma(1+\nu/2)} \cot(\pi s),$$

$$\lim_{\kappa \to 0} Q_{\nu'_0,0} = (\lim_{\kappa \to 0} Q_{\nu_0,0})^*.$$  \hfill (66)

We conclude the section by demonstrating that the spectral decomposition of the linear response in the noiseless limit $\kappa \to 0$ reproduces the asymptotic expansion of its purely deterministic counterpart presented in Ref. [4]. To that end we show that in the noiseless limit $Q_{\nu_0,0}$ and $Q_{\nu'_0,0}$ coincide with the coefficients in the expansion

$$A_n(t; s) = \sum_{\nu=0,2,...} \left( Q_{\nu_0,0} e^{st} + \bar{Q}_{\nu_0,0} e^{-st} \right) e^{-\nu(1/2)t}$$  \hfill (67)

at $n = 0$, where $A_n(t; s)$ is defined as a matrix element of the Liouvillian deterministic evolution:

$$A_n(\zeta; s) = \int dx \psi_n^* (x; s) e^{-\frac{s}{2}t} \psi_0 (x; s).$$  \hfill (68)

The integral can be calculated using the representation on the circle by solving the 1D Liouville equation $\partial_t e^{-\sigma t} \Psi_0 (u) = -\sigma e^{-\sigma t} \Psi_0 (u)$ [see Refs. [1], [2]].

$$A_n(t; s) = \frac{2}{\sqrt{\pi t^{1/2}} \Gamma(1 - s - t)} e^{-2t s} \frac{\Gamma(s) e^{st}}{\Gamma(n + \frac{1}{2} + s)} F_1 \left( n + \frac{1}{2} - s, n + \frac{1}{2}, 1 - s, e^{-2t} \right).$$  \hfill (69)

The expression is substantially simplified in the relevant for the linear response case $n = 0$. Expanding the Gauss hypergeometric function $F_1$ in $A_0(t; s)$ into the hypergeometric series of $e^{-2t}$ to obtain $Q_{\nu,0}$ and $Q_{\nu'_0,0}$, we see
directly that they are indeed reproduced by the noiseless limit of the coefficients $Q_{\lambda_0,0}$ and $Q_{\lambda^*_0,0}$ given by Eq. (69), and in the limit $\kappa \to 0$, i.e.

$$Q_{\lambda_0,0} = Q_{\kappa,0} \quad \text{and} \quad Q_{\lambda^*_0,0} = \tilde{Q}_{\kappa,0}. \quad (70)$$

VII. SPECTRAL DECOMPOSITION FOR THE SECOND-ORDER RESPONSE: NOISE REGULARIZATION

Spectral decomposition of nonlinear response functions in the case of finite noise is conceptually straightforward. The general expression for $S^{(2)}(t_1, t_2)$, decomposed in the eigenmodes of the Fokker-Planck operator $\hat{L}(\kappa)$ is given by Eq. (23). In this section we demonstrate that in the noiseless limit $\kappa \to 0$ the expansion coefficients converge to the coefficients of the long-time asymptotic expansion of the purely deterministic $\kappa = 0$ response that have been derived in our previous work [4]. For the sake of simplicity we focus on the $N_f = 1$ case when only one irreducible representation contributes to the dipole, i.e. $f = \psi_0(\mathbf{x}; s)$. The expression (23) for the second-order response contains four matrix elements calculated in the previous section. The fifth matrix element in the middle angular brackets requires a careful treatment since $f_-$ includes the operator $\sigma_3 \psi_0 \partial / \partial \xi$ that acts on all momentum-dependent functions to the right. We first perform integration over the reduced phase space, represented by the middle angular brackets, which results in a $\xi$-dependent expression that also includes derivatives. Integrating over $\xi$ by parts, and employing the symmetries, we obtain the second-order response function in the form of the following spectral decomposition:

$$S_2(t_1, t_2) = \sum_{\lambda} \int d\xi \frac{\partial \rho_0}{\partial \xi} e^{-\xi \lambda_1 - \xi \mu_2} \lambda \sum_{n=0}^{\infty} (-1)^n \times$$

$$(a_n - a_{n+1}) \left( \left( n + s + \frac{1}{2} \right) [\mu_2 + n] Q_{\mu, n} Q_{\lambda, n+1} \right. - \left. \left( n - s + \frac{1}{2} \right) [\mu_2 + n - 1] Q_{\mu, n+1}^* Q_{\lambda, n} \right), \quad (71)$$

where $Q_{\lambda, n}$ is given by Eq. (39). The symmetric $a_n = a_{-n}$ coefficients

$$a_n = \int_{M^3} dx \psi_0^*(x; s) \psi_0(x; s) \psi_n(x; s) \quad (72)$$

are purely geometrical factors that do not depend on the dynamics and are [4]. They are all related to $a_0$ with the help of the recurrence relations

$$a_{n+1} = \frac{8s^2 + 1 - 4s^2}{(2n + 1)^2 - 4s^2} a_n - \frac{(2n - 1)^2 - 4s^2}{(2n + 1)^2 - 4s^2} a_{n-1} \quad (73)$$

and, therefore, can be expressed in terms of the Laplacian operator eigenfunctions $\psi_0(x; s)$.

The spectral decomposition [7] is an exact expression for any given $\kappa$. The diffusion coefficient $\kappa$ enters the expression via the eigenvalues as well as via $Q_{\lambda, n}$, expressed in terms of the eigenfunctions as specified in Eq. (54). In what follows we determine the leading contributions in the noiseless limit $\kappa \to 0$.

Compared to the linear case, noise plays a more delicate role for the spectral decomposition of the second-order response function. It provides convergence of the series over angular harmonics in the expression for the response function that does not appear in the linear case [compare Eqs. (48) and (71)].

To retrieve the asymptotic behavior of $Q_{\lambda, n}$ for $\kappa \ll 1$ and $n \gg 1$ we apply the method developed in Section VI to derive the following recurrence relation:

$$Q_{\lambda, n+1} = \frac{2n + 1 - 2s}{2n + 1 + 2s} Q_{\lambda, n-1} - \frac{4\lambda - 2\kappa n^2}{2n + 1 + 2s} Q_{\lambda, n}. \quad (75)$$

Specifically we replace $\lambda \xi$ with $\hat{H} \xi$ in the integral representation [Eq. (43)] for $\lambda R_{\lambda, n}$ followed by integrating by parts. Eq. (74) results in another exact recurrence relation

$$Q_{\lambda, n+1} = \frac{2n + 1 - 2s}{2n + 1 + 2s} Q_{\lambda, n-1} - \frac{4\lambda - 2\kappa n^2}{2n + 1 + 2s} Q_{\lambda, n}. \quad (75)$$

Since only symmetric eigenfunctions contribute to the spectral decomposition, we have $Q_{\lambda, n} = Q_{\lambda, n}$, which allows to express all quantities $Q_{\lambda, n}$ via $Q_{\lambda, 0}$ specified by Eqs. (49).

We start with the limit $\kappa \to 0$ for fixed $n$. Although the dependence of $R_{\lambda, n}$ on $\kappa$ is singular, the coefficients $Q_{\lambda, n}$ have well-defined limits at $\kappa \to 0$ that are equal to the corresponding coefficients in the expansion (67) of the purely deterministic matrix elements $A_n(t; s)$.

This can be established in the following way. Viewing $A_n(t; s)$ as a matrix element of the evolution operator between zero and $n$th harmonic, we employ the identity $\sigma_1 = (\sigma_+ + \sigma_-)$ and the relations between the neighboring harmonics that follow from Eqs. (5) to express $\partial_t A_n(t; s)$ in terms of $A_{n-1}(t; s)$ and $A_{n+1}(t; s)$. Finally, the result is expanded in $e^{-2t}$ according to Eq. (67). For the components oscillating as $e^{-\kappa t/2}$ we arrive at a relation:

$$-\lambda_\nu Q_{\nu, n} = \frac{2n + 1 + 2s}{4} Q_{\nu, n+1} - \frac{2n - 1 + 2s}{4} Q_{\nu, n-1}, \quad (76)$$

which that can be viewed as the noiseless limit of the recurrence relation (73). Combined with the already established equivalence $\lim_{\kappa\to 0} Q_{\nu, 0} = Q_{\nu, 0}$ for the zero term, this implies the equivalence $\lim_{\kappa\to 0} Q_{\lambda, n} = Q_{\lambda, n}$ for any given $n$.

So far the equivalence has been established for the RP resonances with $\lambda_\nu = \nu - s + 1/2$. The coefficients for the other set of RP resonances can be easily found by
employing the symmetry of the recurrence relation (75) combined with \( Q_{\lambda^*,0} = (Q_{\lambda,0})^* \), which results in:

\[
Q_{\lambda^*,n} = \frac{\Gamma(n + \frac{1}{2} - s)\Gamma(\frac{1}{2} + s)}{\Gamma(n + \frac{1}{2} + s)\Gamma(\frac{1}{2} - s)}(Q_{\lambda,n})^*. \tag{77}
\]

in the limit \( \kappa \to 0 \). This establishes the equivalence \( \lim_{\kappa \to 0} Q_{\lambda^*,n} = \tilde{Q}_{\nu,n} \) for the other set of resonances.

We are now in a position to demonstrate that the series in angular harmonics \( n \) for any coefficient in the spectral decomposition converges for small \( 0 < \kappa \ll 1 \) and in the noiseless \( \kappa \to 0 \) limit reproduces the corresponding coefficient in the long-time asymptotic series for the purely deterministic response function \( S^{(2)}(t_1, t_2) \).

Explicit expressions for the coefficients \( Q_{\nu,n} \) and \( \tilde{Q}_{\nu,n} \) that enter the expansion (67) for \( A_n(t;s) \) become increasingly lengthy as \( \nu \) grows. The coefficients can be obtained by expanding the deterministic evolution matrix element in Eq. (69) in powers of \( e^{-2t} \). The simplest expressions can be found for the lowest modes with \( \nu = 0 \) whose energies are \( \lambda = \pm s + 1/2 \). For example, for the mode with \( \lambda = \lambda_0^s = s + 1/2 \), the recurrence relation (70) implies \( Q_{0,0} = (1)^n Q_{0,n} \), where \( Q_{0,n} = Q_{0,n}^* \) and \( Q_{0,n} \) is specified in the r.h.s. of Eq. (66) taken at \( \nu = 0 \).

To analyze the limit \( n \to \infty \) for fixed finite \( \kappa \ll 1 \) we view \( R_{\lambda,n} \) as the Fourier coefficients of the smooth function \( \Phi_\lambda(u) \). Consequently, \( Q_{\nu,n} \) decay faster than any power of \( n \) for \( n \to \infty \), and indeed we find from the recurrence relations the intermediate asymptotic of \( Q_{\lambda,n} \) at \( 1 \ll n \ll \kappa^{-1} \):

\[
Q_{\lambda,n} \propto (-1)^n n^{\lambda-s-\frac{1}{2}} e^{-\kappa n^2/4}. \tag{78}
\]

For larger \( n \) the recurrence relation implies \( Q_{\lambda,n}/Q_{\lambda,n-1} = 1/(\kappa n) \), which also leads to the decay faster than any power law as \( n \) increases. Finally we compare Eq. (78) with the asymptotic \( n \gg 1 \) of the deterministic coefficients \( Q_{\nu,n} \) to derive

\[
Q_{\lambda,n} = Q_{\nu,n} e^{-\kappa n^2/4} \tag{79}
\]

which is valid in the range \( 1 \ll n \ll \kappa^{-1} \) for small \( \kappa \). Eq. (79) means that noise provides a homogeneous cut-off scale for all RP resonances and the series over angular harmonics for the spectral decomposition coefficients converges at \( n \sim \kappa^{-1/2} \ll \kappa^{-1} \) where Eq. (79) holds.

It remains to be demonstrated that the resulting spectral decomposition in the \( \kappa \to 0 \) limit of vanishing noise coincides with the asymptotic long-time expansion of \( S^{(2)}(t_1, t_2) \) in the deterministic case. The latter expansion has been derived in Ref. [4].

First of all, we observe that the eigenvalues of the Perron-Frobenius operator converge in the \( \kappa \to 0 \) to the factors that appear in the expansion of \( A_n(t;s) \). The same factors that appear in the spectral decomposition of linear response have been interpreted in Ref. [11] as RP resonances. While the linear response function may be represented in the form of a converging series, the expansion of \( S^{(2)} \) appears to be more involved. If \( \kappa = 0 \), for a pair of resonances, the series (71) over \( n \) diverges, as clearly seen from the power-law growth in Eq. (78).

In the deterministic case, the time dependence of the second-order response is determined by the converging series which contains the matrix elements \( A_n(t;s) \). The expansion should be formally performed after the series summation. The long-time expansion of \( S^{(2)}(t_1, t_2) \) is an asymptotic, rather than a converging expansion in \( e^{-2t_1} \) and \( e^{-2t_2} \). We have developed a method, equivalent to regrouping, which allowed to approximate the infinite sum by a sum of a finite number of terms. Due to the alternating character of the series, any smooth cut-off effective for larger term numbers did not influence the result.

The noise \( \kappa > 0 \) actually introduces a smooth cutoff in the sum over \( n \) in Eq. (77) for a given pair of resonances \( \lambda, \mu \). The suppressive exponential factor is present for arbitrarily small positive \( \kappa \). The series over angular harmonics is almost alternating and converging. Therefore, the sum of the series does not depend on the value of \( \kappa \) as long as it is small. It is intuitively clear that another form of regularizing noise would lead to the same results.

For the second-order response function we have demonstrated the equivalence of the spectral decomposition in the limit of vanishing noise \( \kappa \to 0 \) with the asymptotic expansion in the case of deterministic dynamics. In short, this follows from the equality \( \lim_{\kappa \to 0} A_n(t;s) = A_n(t;s) \) for the evolution operator matrix elements and the property that the asymptotic expansion of \( S^{(2)}(t_1, t_2) \) originates from the expansion of \( A_n(t;s) \) and the subsequent proper summation of the apparently diverging series.

VIII. DISCUSSION

In the present manuscript we have studied linear and second-order nonlinear response of a stochastic system obtained by adding Langevin noise to a deterministic system whose dynamics is strongly chaotic. The deterministic system is represented by a free particle moving on a 2D compact Riemann surface \( M^2 \) with constant negative curvature (geodesic flow). We chose the random Langevin force to be orthogonal to the particle momentum, so that the noise does not change the energy, and random walk occurs on the energy shell represented by the 3D reduced phase space \( M^3 \). The stochastic dynamics has been analyzed in terms of the Fokker-Planck operator \( \hat{\mathcal{L}}(\kappa) = -\left(\kappa/2\right)\hat{\sigma}_x^2 + L \), where the second term stands for the deterministic component (advection), whereas the first term describes the Langevin noise in the form of diffusion in the momentum space with \( \kappa \) being the diffusion coefficient.

The \( \kappa = 0 \) case that corresponds to purely deterministic dynamics has been studied in our earlier work [4], where we have employed strong dynamical symmetry (DS) in the system to find the analytical solution to the problem. The Langevin noise added to the system has been chosen in a way that it does not break down the
DS. Similar to the deterministic case, where the space of reduced phase space distributions has been decomposed into simpler components [irreducible representations of the DS group SO(2, 1)] invariant with respect to the Liouville operator $\hat{L}$, the same components form invariant subspaces for our Fokker-Planck operator $\hat{L}(\kappa)$.

Here we have considered noise as a regularization to construct spectral decompositions of the Perron-Frobenius operator and focused on the noiseless $\kappa \to 0$ limit of the stochastic dynamics. We have employed the DS to analyze the eigenvalue problem for the Fokker-Planck operator $\hat{L}$ whose eigenfunctions are nice smooth functions for finite values of $\kappa$. In each irreducible component the distributions can be represented by functions on the circle, whereas the Fokker-Planck operator is represented by a sum of the first-order Liouville operator and a 1D diffusion operator $-(\kappa/2)\partial^2_t$. The deterministic dynamics has the stable $u = \pi$ and unstable $u = 0$ fixed points that represent the dynamical processes along the stable and unstable directions, respectively. Mapping the original problem onto much simpler 1D stochastic dynamics on the circle allowed for an explicit analysis of the eigenvalue problem. Being interested in the noiseless limit we focused on small finite values $\kappa \ll 1$ and found the relevant eigenfunctions of $\hat{L}(\kappa)$ analytically using the WKB method where the diffusion coefficient $\kappa$ plays the role of the square $\hbar^2$ of the Planck constant.

The fluctuation-dissipation theorem relates the linear response function to the two-point correlation function calculated earlier [4], the latter being interpreted as an expansion in RP resonances using the language of rigged Hilbert spaces. The long-time asymptotic expression for the second-order response functions, has the form of a spectral decomposition over the same set of resonances [4]. Interpretation of the asymptotic expansion in the nonlinear case is more involved due to the following reasons. The eigenmodes that correspond to the RP resonances are represented by generalized, rather than smooth functions. In the linear case the initial smooth distribution should be decomposed in the RP modes. The signal is computed by convoluting the RP modes with the smooth dipole function. Both operations are well-defined for generalized functions. In the nonlinear case the second interaction with the driving field involves applying a differential operator to a generalized function followed by projecting it onto another generalized function. The legitimacy of the latter operation is less obvious, and has been related to the cancellation of apparently dangerous terms [3].

Interpretation of nonlinear response in terms of RP resonances has been addressed in this manuscript by considering the noiseless limit of the Langevin dynamics. In the nonzero noise case $\kappa > 0$, the spectral decomposition of the response functions of any order is conceptually more or less straightforward. We have demonstrated explicitly for geodesic flows that in the $\kappa \to 0$ limit the relevant eigenvalues of $\hat{L}(\kappa)$ converge to the RP resonances, whereas the coefficients in the spectral decompositions of the response functions converge to the coefficients of the asymptotic series in the purely deterministic $\kappa = 0$ expressions derived in Ref. [4].

Summarizing, RP resonances can be interpreted as the noiseless limit for the eigenvalues of the Fokker-Planck operator $\hat{L}(\kappa)$, and the spectral decompositions in this limit reproduce the long-time asymptotic series for the response functions. Note that the dynamical $\zeta$-function can be also reproduced as the limit $\zeta(z) = \lim_{\kappa \to 0} Z^{-1} \det(z - \hat{L}(\kappa))$. In our model the RP decomposition for the linear response is represented by a converging series, whereas the nonlinear response is given by a non-converging asymptotic series. The converging character of the spectral decomposition is lost when the limit $\kappa \to 0$ is applied. Computation of the expansion coefficients in the nonlinear case requires a delicate summation of almost sign-alternating series whose convergence is ensured thanks to the noise.

Applications of our results are not limited to interpretations of purely deterministic quantities. Irreversibility that shows itself in the decaying correlations appears only when the deterministic chaotic dynamics is regularized by some kind of coarse graining. Full “physical” mixing always requires some diffusion mechanism. The difference between stable and chaotic deterministic dynamics is that the diffusion-induced “physical” mixing is much more efficient in a chaotic system. We consider a small fraction of the phase space that represent the initial conditions. As the ball of initial conditions is stretched and folded back, the shape becomes elongated along unstable directions and contracted along stable ones, while the phase space volume remains constant. The diffusion time scales as a square of the blurring size. Therefore, a purely diffusive relaxation in the stable system occurs on the time scale of $\tau_{\text{reg}} \sim l^2/\kappa$. In a chaotic system the scale of the density inhomogeneity decreases with time exponentially. Considering for simplicity a 3D reduced phase space of a chaotic Hamiltonian system, a ball of size $a$ of initial conditions becomes a fettucine-like shape. The fettucine length grows as $ae^{\lambda t}$. The size of the system $l$ induces the folding of the unstable manifold. We estimate the number of folds as $\sim ae^{\lambda t}/l$. Then the characteristic “physical” mixing time $\tau_c \sim \lambda^{-1} \ln(\ell^2 a^{-2}(\lambda/\kappa)^{1/2})$ in a chaotic system with weak diffusion is very short compared to the stable case. Note that the “physical” mixing time $\tau_c$ can be measured in photon echo experiments, where it represents the characteristic time scale of the photon echo decay as a function of the delay between the exciting and the dynamics-reversing pulses.

**APPENDIX A: EIGENMODES OF THE FOKKER-PLANCK OPERATOR**

In this section we calculate the relevant eigenmodes of the Fokker-Planck operator in a given representation of $SO(2, 1)$ labeled by $s$. They are given by symmetric solutions of the Schrödinger equation on a circle with the
effective Hamiltonian (28):
\[ -\frac{\kappa}{2} \frac{d^2}{du^2} \xi_\lambda(u) + \left( \frac{\sin^2 u}{2\kappa} - s \cos u \right) \xi_\lambda(u) = \lambda \xi_\lambda(u). \tag{A1} \]

The Hamiltonian is self-adjoint with respect to a natural symmetric scalar product.

For our purposes we focus on the weak limit \( \kappa \ll 1 \). The low lying states with \(|\lambda| \ll \kappa^{-1}\) are concentrated near the potential minima, in the vicinity of \( u = 0 \) and \( u = \pi \). The principal approximation for the solutions of Eq. (A1) in the regions \(|u| \ll 1 \) or \(|u - \pi| \ll 1 \) can be found by retaining up to second order terms in the expansion of the potential.

For \(|u| \ll 1 \) we arrive at the approximate equation:
\[ -\frac{\kappa}{2} \frac{d^2}{du^2} \xi_\lambda(u) + \frac{u^2}{2\kappa} \xi_\lambda(u) = (\lambda + s) \xi_\lambda(u). \tag{A2} \]

that reproduces the well-known Schrödinger equation for a linear harmonic oscillator. One can easily identify the characteristic scales \( \sqrt{\kappa/\kappa} = 1 \) and \((\kappa\kappa)^{1/4} = \sqrt{\kappa}\) of the energy and length, respectively. A general solution of Eq. (A2) can be represented in terms of the Hermite functions as
\[ \xi(u) = A_0 e^{-u^2/2\kappa} H_\nu \left( \frac{u}{\sqrt{\kappa}} \right) + B_0 e^{u^2/2\kappa} H_{-\nu-1} \left( i \frac{u}{\sqrt{\kappa}} \right), \tag{A3} \]

with \( A_0 \) and \( B_0 \) being complex constants. The parameter \( \nu \) is related to the energy \( \lambda \) by
\[ \nu = \lambda - \frac{1}{2} + s. \tag{A4} \]

As opposed to the case of a harmonic oscillator, the general solution (A3) contains both decaying and growing waves. Their relative amplitude can be determined only by solving the equation on the whole circle. The functions \( H_\nu(z) \) and \( e^{z^2} H_{-\nu-1}(iz) \) that are linearly independent for any parameter \( \nu \) can be simply related to confluent hyperbolic functions and parabolic cylinder functions, the latter being the solutions of the original Schrödinger equation with the harmonic potential. We prefer to deal with the Hermite functions rather than with other special functions since the former reproduce Hermite polynomials in the case of integer order.

The general solution of Eq. (A2) near \( u = \pi \) has the form similar to Eq. (A3):
\[ \xi(u) = A_1 e^{-(u-\pi)^2/2\kappa} H_\nu \left( \frac{u - \pi}{\sqrt{\kappa}} \right) \tag{A5} \]
\[ + B_1 e^{(u-\pi)^2/2\kappa} H_{-\nu-1} \left( i \frac{u - \pi}{\sqrt{\kappa}} \right), \]

where
\[ \nu_1 = \nu - 2s = \lambda - \frac{1}{2} - s, \tag{A6} \]

and \( A_1 \) and \( B_1 \) are two complex constants.

We begin with the construction of a symmetric solution near \( u = \pi \). The requirement of the solution to be invariant with respect to \((u - \pi) \leftrightarrow (\pi - u)\) combined with the relations between Hermite functions leads to an identity
\[ \frac{B_1}{A_1} = -i \frac{2\nu_1 \sqrt{\pi}}{\Gamma(-\nu_1) \cos \frac{2\nu_1}{\pi}}, \tag{A7} \]

and after some transformations of the Hermite functions we arrive at an explicit form of a symmetric solution for (A5):
\[ \xi(u) = A_1 2^{\nu_1} \Gamma(\nu_1 + 1) e^{(u-\pi)^2/2\kappa} e^{-i\nu_1/2} \times \]
\[ \left( H_{-\nu_1} \left( i \frac{u - \pi}{\sqrt{\kappa}} \right) \right. + H_{-\nu_1-1} \left( -i \frac{u - \pi}{\sqrt{\kappa}} \right). \tag{A8} \]

Therefore, in the region \(|u - \pi| \ll 1 \) the symmetric solution is determined by two unknown parameters \( \nu \equiv \nu_1 + 2s \) and \( A_1 \); both can assume complex values.

The harmonic approximation for the potential can be used if \(|u - \pi| \ll 1 \), hence it is the region where the solution given by Eqs. (A5) \( \Box \) is valid.

The general WKB solution of Eq. (A1) under the barrier and around \( u = \pi/2 \) is represented by a superposition of two waves:
\[ \xi(u) = \frac{A_2}{\sqrt{p(u)}} \exp(-S(u)) + \frac{B_2}{\sqrt{p(u)}} \exp(S(u)), \tag{A9} \]

where \( p(u) \) and \( S(u) \) are defined by
\[ p(u) = \sqrt{\sin^2 u - 2\kappa s \cos u - 2\kappa \lambda}, \tag{A10} \]
\[ S(u) = \frac{1}{\kappa} \int \limits_{\pi/2}^{u} dw p(w). \tag{A11} \]

The quasiclassical expansion over \( \kappa \) for the wavefunction phase can be employed when the wavelength \( \kappa/p(u) \) does not change too fast:
\[ \left| \frac{d}{du} \left( \frac{\kappa}{p(u)} \right) \right| \ll 1. \tag{A12} \]

This holds when we are not too close to the classical turning points where \( p(u) \) turns to zero. Small value of \( \kappa \) ensures that the classical turning points lie close to \( u = 0 \) or \( u = \pi \). The WKB solutions under the barriers are valid at least for \(|u| \gg \sqrt{2\kappa(\lambda + s)} \) and \(|u - \pi| \gg \sqrt{2\kappa(\lambda - s)} \). The right-hand sides of these inequalities contain expressions for the classical turning points. If \(|\lambda|, |s| \sim 1 \), the validity of the quasiclassical approximation \( \Box \) conditions is limited to the regions
\[ |u|, |u - \pi| \gg \sqrt{\kappa}. \tag{A13} \]

We can invoke a further approximation in the expressions for the solutions under the barrier. For \( \kappa \ll 1 \) and
\begin{align*}
|\sin u| \gg \sqrt{\kappa} \quad (\text{i.e. in the regions determined by the same inequalities (A13)), we arrive at a simplified expression for } S(u), \text{ where only those terms that are small compared to one have been neglected:}
\end{align*}

\begin{align*}
S(u) &= \frac{1}{\kappa} \int_{\pi/2}^{u} du \left( \sin w - \kappa \cos u \frac{\cos u}{\sin u} - \frac{\kappa \lambda}{\sin u} \right) = \quad (A14) \\
&= - \frac{\cos u}{\kappa} - s \ln \sin u - \lambda \ln \tan \frac{u}{2}. \quad (A15)
\end{align*}

In approximating the pre-exponential factor of Eq. (A9) it is sufficient to set \( p(u) = \sin u \).

We can see that in all intervals the wave function consists of two components that exponentially grow in the opposite directions under the barrier. A crude estimate for the magnitudes of the two components is presented in Fig. 1. Only the dominant part \( \sin^2 u/(2\kappa) \) of the potential contributes to the exponential dependence of the barrier transparency on \( \kappa \).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1.png}
\caption{Schematic picture of the amplitudes that result from the tunneling through the potential barrier.}
\end{figure}

We further notice that for \( u \) close to \( \pi \) (actually in the interval \( \kappa^{1/2} \ll (\pi - u) \ll \kappa^{1/4} \ll 1 \)) we can use the expansion
\begin{align*}
S(u) &= \frac{1}{\kappa} - \frac{(u - \pi)^2}{2\kappa} + (\lambda - s) \ln(\pi - u) - \lambda \ln 2,
\end{align*}
where the neglected terms are small compared to one.

This is the interval where the WKB solution under the barrier and the solution in the harmonic well both represent a good approximation and can be matched. We use the asymptotic expressions for the Hermite functions valid for \( |u - \pi| \gg \sqrt{\kappa} \). In fact, taking into account Eqs. (A14)—(A17), we can see that the term with \( A_2 \) in Eq. (A9) and the solution given by Eq. (A16) contain the same quadratically increasing argument in the exponential, and the same power-like pre-exponential dependence:
\begin{align*}
A_2(\pi - u)^{-\lambda+s-\frac{1}{2}2^\lambda} \exp \left[ - \frac{1}{\kappa} + \frac{(u - \pi)^2}{2\kappa} \right] = \quad (A16) \\
B_1 \left( \frac{2(\pi - u)}{\sqrt{\kappa}} \right)^{-\nu_1-1} \exp \left[ - \frac{i\pi(\nu_1 + 1)}{2} + \frac{(u - \pi)^2}{2\kappa} \right].
\end{align*}

In Eq. (A16) we have also used the symmetry \( \xi(u) = \xi(2\pi - u) \), which implies that for positive \( (u - \pi) \gg \sqrt{\kappa} \) the asymptotic form of \( \xi(u) \) includes the only term that is proportional to \( B_1 \).

Therefore, the matching near \( u = \pi \) yields
\begin{align*}
B_1 = A_2 e^{-\frac{1}{2}2^{\nu_1+\lambda+1} \kappa} e^{i\pi(\nu_1+1)}.
\end{align*}

The value of \( A_1 \) is then found from Eq. (A7). The component of \( \xi(u) \) in Eq. (A5) with \( B_2 \) in front of it decreases rapidly \( \exp(-\cos u/\kappa) \) under the barrier as \( u \) deviates from \( \pi \). Its magnitude can be estimated as
\begin{align*}
B_2 \propto A_1 e^{-\frac{u}{\kappa}}. \quad (A18)
\end{align*}

The procedure of solving the Schrödinger equation near \( u = 0 \) is completely similar: In the region \( \sqrt{\kappa} \ll |u| \ll 1 \) both approximate solutions (A9) and (A9) are adequate and can be matched. To that end we expand \( S(u) \) in the interval \( |u| \ll \kappa^{1/4} \), neglecting the terms that are small compared to one:
\begin{align*}
S_2(u) &= \frac{1}{\kappa} + \frac{u^2}{2\kappa} - (\lambda + s) \ln u + \lambda \ln 2,
\end{align*}
and also employ the asymptotic forms of the Hermite functions in the overlap region. With \( u \) approaching \( 0 \) the second term in Eq. (A9) decreases and hence matches the second term in Eq. (A3),
\begin{align*}
B_0 \propto B_2 e^{-\frac{u}{\kappa}} \propto A_1 e^{-\frac{u}{\kappa}}. \quad (A19)
\end{align*}

The component with \( A_2 \) in front of it matches under the barrier with the component that has \( A_0 \) in front of it and decreases as \( u \) deviates from \( 0 \):
\begin{align*}
A_2 = A_0 e^{-\frac{1}{2}2^{\nu_1+\lambda+1} \kappa} e^{i\pi(\nu_1+1)}. \quad (A20)
\end{align*}

First we assume that the ratio \( B_1/A_1 \) given by Eq. (A7) does not contain exponentially small term \( \sim e^{-1/\kappa} \). Then Eqs. (A17)—(A20) imply that the second component in Eq. (A3) scales \( B_0 \propto e^{-4/\kappa}A_0 \) near \( u = 0 \). Its presence is related to the nonresonant tunneling along the circle from the potential minimum into itself. The argument in the suppressive exponential factor is evaluated as
\begin{align*}
\frac{1}{\kappa} \int_{0}^{2\pi} du p(u) \approx \frac{1}{\kappa} \int_{0}^{2\pi} du |\sin u| \approx \frac{4}{\kappa}. \quad (A21)
\end{align*}

The tunneling is nonresonant, since the two wells near \( u = 0 \) and \( u = \pi \) are offset by \( s \) which is not an integer number.

Performing the same procedure in the region with \( \sin u < 0 \) as it has been done for \( \sin u > 0 \), we find the symmetric solution
\begin{align*}
\xi(u) = A_0 e^{-u^2/2\kappa} H_\nu \left( \frac{|u|}{\sqrt{\kappa}} \right), \quad (A22)
\end{align*}
valid for \( |u| \ll 1 \). For arbitrary \( \nu \) the function may have a cusp at \( u = 0 \). The solution is smooth if \( \nu \) is a non-negative even integer. Then the Hermite functions \( H_\nu \)
reduce to Hermite polynomials of even order. The dependence $\lambda_\nu = \nu + (1 - s)/2$ on $\nu$ with $\nu = 0, 2, \ldots$ of the eigenstate energy for the states concentrated near $u = 0$ is given by Eq. (A1). Exponentially small corrections $\propto e^{-4/\kappa}$, induced by the tunneling can be neglected. On the other hand, the power-like in $\kappa$ corrections to the energies, play a crucial role for $\nu > 0$. The latter corrections appear because of the deviation of trigonometric functions from their harmonic approximations. However the method developed in Section VI allows to avoid these calculations completely.

Complex conjugation of the Hamiltonian that corresponds to the change of sign of $s$, or to the shift $u \to u + \pi$ implies the quantization condition $\nu_1 = 2k$ with $k = 0, 1, 2, \ldots$ for the other set of eigenstates. They have energies $\lambda = 2k + (1 + s)/2$ and are concentrated near $u = \pi$. The eigenfunctions satisfy the relations similar to the specified above.

The eigenfunctions of the Hamiltonian (28) found above significantly differ from those of the linear harmonic oscillator only far from the interval $|u| \lesssim \kappa^{1/4}$. Outside the interval the exponentially decaying functions take negligibly small values so that the normalization factor has the form typical to the harmonic oscillator:

$$A_{0, \nu} = (\pi \kappa)^{-1/4} (2^\nu \nu!)^{-1/2}. \quad (A23)$$

The phase of the normalized eigenfunctions of Hermitian operators can be chosen arbitrarily. For our Hamiltonian and scalar product (29) this reduces to the freedom in the sign choice.

Finally, for illustrative purposes, we calculate the correction to the energy $\lambda_\nu = \nu - s + 1/2$ of the eigenstate that originates from the vicinity of $u = 0$ (note that the result is not used anywhere):

$$\lambda^{(1)}_\nu = \int_\pi^{-\pi} du \left( \frac{su^2}{2} - \frac{u^4}{6\kappa} \right) \xi^2_{\lambda_\nu}(u) = \kappa (2s(1 + 2\nu) - (1 + 2\nu + 2\nu^2)) \xi^2_{\lambda_\nu}(u). \quad (A24)$$

One of the ways derive Eq. (A24) is to consider the next term in the WKB expansion under the barrier. The WKB wavefunction contains the state energy. One can simply require that the wavefunction does not contain logarithmic terms due to their absence near $u = 0$.

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