Abundance of invariant and almost invariant pure states of $C^*$-dynamical systems

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Abstract

We show that invariant states of $C^*$-dynamical systems can be approximated in the weak*-topology by invariant pure states, or almost invariant pure states, under various circumstances.

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1 Introduction

The states $E_A$ of a unital $C^*$-algebra $A$ are the weak* closure of the convex envelope of the pure states, i.e., the extremal states, and if $\alpha$ is a *-automorphism of $A$ then the $\alpha$-invariant states $E^\alpha_A$ are the weak* closure of the convex envelope of the extremal $\alpha$-invariant states, i.e., the extremal points of $E^\alpha_A$. These statements are consequences of convexity and compactness arguments. In particular these arguments, and specifically the Krein–Milman theorem, ensure the existence of an abundant set of extremal and extremal $\alpha$-invariant states. This type of reasoning does not, however, establish the existence of $\alpha$-invariant pure states and this is not surprising as there are examples for which no invariant pure states exist. Thus it is of interest to analyze the structural properties of $(A,\alpha)$ which ensure the abundance of invariant pure states.

The state space of a $C^*$-algebra often has much more striking geometric properties. If $A$ is a UHF-algebra, or more generally if $A$ is anti-liminal, then the pure states are weak* dense in $E_A$ (see [Gli61] or [BR87], Example 4.1.31). It is not necessary to take convex envelopes. Fannes, Nachtergaele and Werner [FNW92] have shown that a similar situation can occur for invariant states. If $A = \otimes_{\infty} M_n = M_\infty$ is the UHF-algebra formed as the infinite tensor product of the algebra $M_n$ of all $n \times n$ matrices and $\alpha$ is the shift automorphism then it follows from [FNW92] that the $\alpha$-invariant pure states are dense in $E^\alpha_A$. The proof of this result relies on the construction of a special class of states, the finitely-correlated states, and appears limited to this particular example. In particular it appears difficult to extend the result to the shift action of $\mathbb{Z}^d$ on $\otimes_{\mathbb{Z}^d} M_n$ when $d > 1$ and the question of density of the invariant pure states in this situation appears to remain open. This contrasts to the well known result that the extremal invariant states are weak* dense in the invariant states for this example (see [BR87], Example 4.3.26). In this paper we will demonstrate that this rather surprising density property of the invariant pure states, and related properties, are true for some other classes of $C^*$-dynamical system.

First, in Section 2, we establish a slightly different but related density result. Under general assumptions on the $C^*$-dynamical system $(A,G,\alpha)$ we demonstrate density of the ‘almost’ invariant pure states. The class of systems covered by our assumptions contains the higher dimensional spin systems mentioned in the previous paragraph. It also includes systems where the set of invariant pure states is not weak* dense in the set of invariant states and even systems for which the set of invariant pure states is weak* closed or even empty. In Section 3 we then examine systems for which the group $G$ is compact and demonstrate density of the invariant pure states for a broad class of algebras whenever the action of the group is a quasi-product action [BKR87], [BEEK91], [BEK93]. These actions encompass product type actions of $G$ on UHF-algebras. Finally, in Section 4, we extend the Fannes–Nachtergaele–Werner result to the shift automorphism acting on the infinite tensor product of a prime, unital AF-algebra.

2 Approximation of invariant states by almost invariant pure states for discrete amenable group actions

If $A$ is a $C^*$-algebra let $A_{\mathbb{P}}$ denote the set of positive elements $e \in A$ such that $0 \leq e \leq 1$ and there is a positive non-zero element $p \in A$ with $p e = p$. By spectral modification it is easy to show that the element $p$ can be taken from $A_{\mathbb{P}}$ if it exists and any positive $e \in A$
with $\|e\| = 1$ can be approximated in norm by elements from $A_\mathbb{P}$. These elements will play the role of projections in $A$ if $A$ is not of real rank zero.

Recall from [Ell80], [Kis81] that an automorphism $\alpha$ of a $C^*$-algebra $A$ is said to be properly outer if for any $y \in A_\mathbb{P}$ and any $\varepsilon > 0$ there exists an $x \in A_\mathbb{P}$ with $xy = x$ such that

$$\|x \alpha(x)\| < \varepsilon .$$

A long list of equivalent conditions for proper outerness, when $A$ is separable, is given in Theorem 6.6 in [OP82]. A particularly useful characterization is given in Theorem 2.1 of [Kis82]: $\alpha$ is properly outer if for any $y \in A_\mathbb{P}$, any $a \in A \cup \{1\}$ and any $\varepsilon > 0$ there exists an $x \in A_\mathbb{P}$ such that $xy = x$ and

$$\|xa \alpha(x)\| < \varepsilon .$$

If $A$ is simple and unital then $\alpha$ is properly outer if and only if $\alpha$ is outer.

Theorem 2.1 Let $G$ be an amenable discrete group, $\alpha$ an action of $G$ on a $C^*$-algebra $A$ such that $\alpha_g$ is properly outer for $g \in G \setminus \{e\}$ and assume there exists a faithful $\alpha$-covariant irreducible representation of $A$. Then for any $\alpha$-invariant state $\omega$ on $A$, any finite subset $F \subseteq G$, any finite subset $\mathcal{F} \subseteq A$ and any $\varepsilon > 0$ there exists a pure state $\varphi$ on $A$ such that

$$\|\varphi \circ \alpha_g - \varphi\| < \varepsilon$$

for all $g \in F$ and

$$|\varphi(x) - \omega(x)| < \varepsilon$$

for all $x \in \mathcal{F}$. Moreover, $\varphi$ can be taken to be a vector state in the given $\alpha$-covariant representation.

Remark 2.2 If $G$ is abelian and countable and $A$ is separable and prime then the existence of a faithful $\alpha$-covariant irreducible representation is automatic (see Theorem 3.3 in [Kis87b]).

Remark 2.3 The hypotheses of the theorem do not imply that the set of $\alpha$-invariant pure states is weak*-dense in the set of $\alpha$-invariant states. The set may be weak*-closed (see Example 2.7) and there are even examples satisfying the hypotheses of the theorem for which there are no $\alpha$-invariant pure states (see Examples 2.7 and 2.8).

As a preliminary to the proof of the theorem we establish two lemmas.

Lemma 2.4 Let $A$ be a $C^*$-algebra, $\varphi$ a pure state on $A$ such that the associated representation is faithful, $\varepsilon > 0$ and $\mathcal{F} \subseteq A$ a finite subset of the algebra. Then there exists an element $e \in A_\mathbb{P}$ such that $\varphi(e) = 1$ and

$$\|exe - \varphi(x)e^2\| < \varepsilon$$

for all $x \in \mathcal{F}$.
Proof. By Proposition 3.13.6 in [Ped79], the support projection \( p \) of \( \varphi \) in the bidual \( \mathfrak{A}^{**} \) of \( \mathfrak{A} \) is closed. If, temporarily, we assume that \( \mathfrak{A} \) is separable then there is a decreasing sequence \( z_n \) of positive elements in \( \mathfrak{A} \) such that \( z_n \downarrow p \). Setting \( z = \sum_{n} 2^{-n}z_n \) and taking suitable functions of \( z \) we can construct a sequence \( \{e_n\} \) of positive elements such that \( 0 \leq e_n \leq 1 \),

\[
e_n e_m = e_{\max\{n,m\}}
\]

for all \( n, m \in \mathbb{N} \) and \( e_n \downarrow p \). Thus if \( p_n \in \mathfrak{A}^{**} \) is the eigenprojection of \( e_n \) corresponding to eigenvalue one and \( m > n \) then \( e_n p_n = p_n e_m = e_m \). We also have \( p_n \downarrow p \). We may assume that all elements in \( \mathfrak{F} \) are selfadjoint. For \( x \in \mathfrak{F} \) let \( \sigma(p_n x p_n) \) denote the spectrum of \( p_n x p_n \) as an element in \( p_n \mathfrak{A}^{**} p_n \). Then \( \max\sigma(p_n x p_n) \) is a decreasing sequence, with limit \( a \), and \( \min\sigma(p_n x p_n) \) is an increasing sequence, with limit \( b \leq a \). We argue that \( b = a = \varphi(x) \).

If not there are states \( \omega_a \) and \( \omega_b \) on \( \mathfrak{A} \) with \( \omega_a(x) = a, \omega_a(e_k) = 1, \omega_b(x) = b, \omega_b(e_k) = 1 \) for all \( k \). But then \( \text{supp} \omega_a \leq \lim_n e_n = p \) and \( \text{supp} \omega_b \leq p \). Since \( p \) is a one-dimensional projection in \( \mathfrak{A}^{**} \) by purity of \( \varphi \) one then has \( \omega_a = \omega_b = \varphi \) and \( a = b = \varphi(x) \). Also, as \( \max\sigma(p_n x p_n) \to \varphi(x) \) and \( \min\sigma(p_n x p_n) \to \varphi(x) \) one has \( \|p_n x p_n - \varphi(x) p_n\| \to 0 \). Thus given \( \varepsilon > 0 \) and \( \mathfrak{F} \) we can find an \( n \) with

\[
\|p_n x p_n - \varphi(x) p_n\| < \varepsilon
\]

for all \( x \in \mathfrak{F} \). But if \( m > n \) then

\[
\|e_m x e_m - \varphi(x) e_m^2\| = \|e_m p_n x p_n e_m - \varphi(x) e_m p_n e_m\|
\]

\[
\leq \|p_n x p_n - \varphi(x) p_n\| < \varepsilon
\]

and the lemma is established for separable \( \mathfrak{A} \).

If \( \mathfrak{A} \) is nonseparable the proof of the lemma can be reduced to the separable case by the following argument.

Adopt the hypotheses of the lemma and let \( \mathfrak{B}_1 \) be the separable \( C^* \)-subalgebra of \( \mathfrak{A} \) generated by \( \mathfrak{F} \). We next construct inductively a sequence \( \mathfrak{B}_1 \subseteq \mathfrak{B}_2 \subseteq \ldots \) of separable \( C^* \)-subalgebras of \( \mathfrak{A} \) as follows. Assume \( \mathfrak{B}_n \) has been constructed and let \( \{x_k\} \) be a dense sequence in \( \mathfrak{B}_n \). If \( \Phi \) is the cyclic vector corresponding to \( \varphi \) then \( \{x_k \Phi\} \) is dense in \( [\mathfrak{B}_k \Phi] \). But if \( k, m \) are such that

\[
\|x_k \Phi\| - \|x_m \Phi\| < n^{-1}\|x_m \Phi\|
\]

then by Kadison’s transitivity theorem there is a \( u_{k;m} \in \mathfrak{A} \) such that \( \|u_{k;m}\| \leq 1 \) and

\[
\|x_k \Phi - u_{k;m} x_m \Phi\| < n^{-1}\|x_m \Phi\|
\]

Also, for each \( m \), there is a \( u_m \in \mathfrak{A} \) such that \( \|u_m\| \leq 1 \) and

\[
\varphi(u_m x_m x_m^* u_m^*) \geq (1 - n^{-1})\|x_m x_m^*\|
\]

Now let \( \mathfrak{B}_{n+1} \) be the \( C^* \)-algebra generated by \( \{x_m\}, \{u_{k;m}\}, \{u_m\} \).

Set \( \mathfrak{B} = \bigcup_n \mathfrak{B}_n \). Then \( \mathfrak{B} \) is a separable \( C^* \)-subalgebra of \( \mathfrak{A} \) containing \( \mathfrak{F} \) and having the property that if \( \psi_1, \psi_2 \in [\mathfrak{B} \Phi] \) with \( \|\psi_1\| = \|\psi_2\| \neq 0 \) and \( \varepsilon > 0 \) then there is a \( u \in \mathfrak{B} \) such that

\[
\|\psi_2 - u\psi_1\| \leq \varepsilon\|\psi_1\|
\]
and it follows that the representation of $\mathfrak{B}$ on $[\mathfrak{B} \Phi]$ is irreducible. Furthermore if $x \in \mathfrak{B}$ and $x \neq 0$ there is a $u \in \mathfrak{B}$ with $\varphi(u x x^* u^*) > 0$ and hence this representation is faithful. Thus we may apply the separable case of the lemma, which has already been established, to $\mathfrak{B}$, $\varphi|_{\mathfrak{B}}$ and $\mathfrak{F}$ instead of $\mathfrak{A}$, $\varphi$ and $\mathfrak{F}$ to conclude the general validity of the lemma. □

Lemma 2.5 Let $G$ be a discrete group and $\alpha$ an action of $G$ on a prime $C^*$-algebra such that $\alpha_g$ is properly outer for $g \in G \setminus \{e\}$. Let $\omega$ be an $\alpha$-invariant state. For all finite subsets $\Lambda \subseteq G$, all $\delta > 0$ and all finite subsets $\mathfrak{F} \subseteq \mathfrak{A}$ there exists an element $p \in \mathfrak{A}_{\mathfrak{F}}$ with

(i) if $g \neq h$, $g, h \in \Lambda$ and $x \in \mathfrak{F} \cup \{I\}$ then $\|\alpha_g(p)x \alpha_h(p)\| < \delta$

(ii) if $g \in \Lambda$ and $x \in \mathfrak{F}$ then $\|p \alpha_g^{-1}(x)p - \omega(x) p^2\| < \delta$.

Proof The proof closely follows an argument of [Kis96]. First, by replacing $G$ by the group generated by $\Lambda$ and using an argument similar to that occurring at the end of the proof of Lemma 2.4 one may assume that the algebra $\mathfrak{A}$ is separable. The argument in the present setting is, however, more complicated since one now has to ensure that the restriction of $\alpha$ to the new subalgebra is still properly outer. By Glimm’s theorem (see, for example, [Gli61]), there is for each $\delta > 0$ a pure state $\varphi$ on $\mathfrak{A}$ defining a faithful representation such that

$$|\varphi(\alpha_g(x)) - \omega(\alpha_g(x))| = |\varphi(\alpha_g(x)) - \omega(x)| < \delta$$

for any $x \in \mathfrak{F}$ and $g \in \Lambda^{-1}$. But by Lemma 2.4 there is an element $e \in \mathfrak{A}_{\mathfrak{F}}$ such that $\varphi(e) = 1$ and

$$\|e \alpha_g(x)e - \varphi(\alpha_g(x)e^2\| < \delta$$

for all $x \in \mathfrak{F}$ and $g \in \Lambda^{-1}$. Therefore, slightly modifying $e$ by spectral theory, we may assume that there is a $q' \in \mathfrak{A}_{\mathfrak{F}}$ with $\varphi(q') = 1$ and $\|q'\| = 1$ such that $q'e = q'$. Then the above estimate is still valid with $e$ replaced by any such $q'$.

Since $\alpha_g$ is properly outer for each $g \in G$ there is a $p \in \mathfrak{A}_{\mathfrak{F}}$ with $e p = p$ such that

$$\|\alpha_g(p)x \alpha_h(p)\| = \|\alpha_{g^{-1}}(p)\alpha_{g^{-1}}(x)p\| < \delta$$

for all $g, h \in \Lambda$ with $g \neq h$ and all $x \in \mathfrak{F} \cup \{I\}$. For a single choice of $g, h$ and $x$ the existence of this $p$ follows from proper outerness of $\alpha_{g^{-1}}$ by Lemma 1.1 of [Kis87b] but going through the finite list of $g, h$ and $x$ one successively constructs new $p'$ with $p'p = p'$ satisfying the estimate above with the new $p'$ for the new triple $(g, h, x)$. Then the estimate holds with $p$ replaced by $p'$ for the earlier elements in the list. Continuing with this process one obtains the sought after $p$ and property (i) is immediate. As $pe = p$ one has

$$\|p \alpha_g^{-1}(x)p - \varphi(\alpha_g^{-1}(x)p^2\| < \delta$$

for all $x \in \mathfrak{F}$ and $g \in \Lambda$. But as $\varphi(\alpha_g^{-1}(x))$ is close to $\omega(x)$ one finds

$$\|p \alpha_g^{-1}(x)p - \omega(x)p^2\| < 2\delta$$

and so (ii) is valid. □

Proof of Theorem 2.1 Let $\pi$ be a faithful $\alpha$-covariant irreducible representation of $\mathfrak{A}$ on a Hilbert space $H$ and choose a finite subset $\Lambda \subseteq G$ such that $F \subseteq \Lambda$ and

$$|\Lambda \triangle h\Lambda|/|\Lambda| < \delta$$
for \( h \in F \). This is possible by the amenability of \( G \). Next choose \( p \) as in Lemma 2.5 and \( q \in \mathcal{A}_\Lambda \) with \( pq = q \) where the \( \Lambda \) in the lemma is replaced by \( F \Lambda \cup \Lambda \) and the \( \delta \) by \( \delta / |\Lambda|^2 \). Then choose a unit vector \( \psi \in \pi(q) \mathcal{H} \), i.e., \( \psi = \pi(q) \xi \) for some \( \xi \in \mathcal{H} \). Therefore

\[
\pi(p)\psi = \pi(pq)\xi = \pi(q)\xi = \psi .
\]

Now define

\[
\Phi = |\Lambda|^{-1/2} \sum_{g \in \Lambda} U_g \psi
\]

where \( U \) is the unitary representation of \( G \) on \( \mathcal{H} \) such that

\[
U_g \pi(x) U^*_g = \pi(\alpha_g(x))
\]

for all \( x \in \mathfrak{A} \) and \( g \in G \). Next define

\[
\varphi(x) = (\pi(x) \Phi, \Phi) / \|\Phi\|^2
\]

for all \( x \in \mathfrak{A} \). Then \( \varphi \) is a pure state on \( \mathfrak{A} \) and we will verify that \( \varphi \) has the properties claimed in Theorem 2.1.

First, if \( h \in F \) one has

\[
\|U_h \Phi - \Phi\| = |\Lambda|^{-1/2} \sum_{g \in \Lambda} (U_{hg} \psi - U_g \psi) .
\]

In the latter sum all \( g \in \Lambda \cap h^{-1} \Lambda \) cancel and the remainder is a surface term

\[
|\Lambda|^{-1/2} \sum_{g \in h \Lambda \setminus \Lambda} U_g \psi - \sum_{g \in \Lambda \setminus h \Lambda} U_g \psi .
\]

But if \( g_1, g_2 \in F \Lambda \cup \Lambda \) with \( g_1 \neq g_2 \) then

\[
|(U_{g_1} \psi, U_{g_2} \psi)| = |(U_{g_1} \pi(p) \psi, U_{g_2} \pi(p) \psi)| = |(\pi(\alpha_{g_1}(p)) U_{g_1} \psi, \pi(\alpha_{g_2}(p)) U_{g_2} \psi)| \leq \|\alpha_{g_1}(p) \alpha_{g_2}(p)\| < \delta / |\Lambda|^2
\]

by Lemma 2.5. Thus

\[
\|U_h \Phi - \Phi\|^2 \leq |\Lambda|^{-1} \sum_{g_1, g_2 \in h \Lambda \setminus \Lambda} |(U_{g_1} \psi, U_{g_2} \psi)| = |\Lambda|^{-1} \sum_{g \in h \Lambda \setminus \Lambda} 1 + |\Lambda|^{-1} \sum_{g_1 \neq g_2 \in h \Lambda \setminus \Lambda} |(U_{g_1} \psi, U_{g_2} \psi)| \leq |h \Lambda \setminus \Lambda| / |\Lambda| + \delta |h \Lambda \setminus \Lambda|^2 / |\Lambda|^2 \leq \delta + 4 \delta = 5 \delta .
\]

In addition

\[
|\|\Phi\|^2 - 1| \leq |\Lambda|^{-1} \sum_{g_1, g_2 \in \Lambda} |(U_{g_1} \psi, U_{g_2} \psi)| \leq \delta |\Lambda|^{-3} |\Lambda|^2 \leq \delta |\Lambda|^{-1} .
\]
It follows that
\[ \| \varphi \circ \alpha_h - \varphi \| \leq 2(5\delta)^{1/2}(1 + \delta) \]
for all \( h \in F \) so the first estimate of Theorem 2.1 is valid. Secondly, one checks the second estimate as follows. One has
\[
\| \Phi \|^2 \varphi(x) = (\pi(x)\Phi, \Phi) \\
= |\Lambda|^{-1} \sum_{g, h \in \Lambda} (\pi(x \alpha_g(p))U_g \psi, \pi(\alpha_h(p))U_h \psi) \\
= |\Lambda|^{-1} \sum_{g, h \in \Lambda} (\pi(\alpha_h(p) x \alpha_g(p))U_g \psi, U_h \psi) .
\]
But one also has
\[
\| \Phi \|^2 \omega(x) = (\omega(x)\Phi, \Phi) \\
= |\Lambda|^{-1} \sum_{g, h \in \Lambda} (\pi(\alpha_h(p) \omega(x) \alpha_g(p))U_g \psi, U_h \psi)
\]
and therefore
\[
\| \Phi \|^2 |\varphi(x) - \omega(x)| \leq |\Lambda|^{-1} \sum_{g, h \in \Lambda} |\alpha_h(p) x \alpha_g(p) - \omega(x) \alpha_h(p) \alpha_g(p)| .
\]
If, however, \( x \in \mathfrak{f} \) and \( g \neq h \) it follows from Lemma 2.5 (ii) that
\[
|\alpha_h(p) x \alpha_g(p) - \omega(x) \alpha_h(p) \alpha_g(p)| \leq |\alpha_h(p) x \alpha_g(p)| + |x| \cdot |\alpha_h(p) \alpha_g(p)| \\
\leq \delta(1 + |\mathfrak{f}|)/|\Lambda|^2
\]
where \( |\mathfrak{f}| = \max\{\|x\| : x \in \mathfrak{f}\} \) and if \( g = h \) it follows from Lemma 2.5 (iii) that
\[
|\alpha_g(p) x \alpha_g(p) - \omega(x) \alpha_g(p)|^2 = |p \alpha_g^{-1}(x) p - \omega(x) p^2| \leq \delta/|\Lambda|^2 .
\]
Hence
\[
|\varphi(x) - \omega(x)| \leq |\Lambda|^2 \left( \delta(1 + |\mathfrak{f}|)/|\Lambda|^2 + |\Lambda| \delta/|\Lambda|^2 \right)(1 + \delta)
\]
and the second estimate of Theorem 2.1 follows. \( \square \)

**Example 2.6** It is not true in general that for \( \mathfrak{A}, G \) and \( \alpha \) satisfying the hypotheses of Theorem 2.1, even with \( G = \mathbb{Z} \), that the set of \( G \)-invariant pure states on \( \mathfrak{A} \) is weakly*-dense in the set of \( G \)-invariant states. Let \( \mathfrak{A} \) be the UHF-algebra \( \mathfrak{A} = \bigotimes_{n=1}^{\infty} M_2 \) and let
\[
\alpha = \bigotimes_{n=1}^{\infty} \text{Ad} \left( \begin{array}{cc} 1 & 0 \\ 0 & e^{2\pi i \theta_n} \end{array} \right)
\]
where the irrational numbers \( \theta_i \) are rationally independent. This action is almost periodic with fixed point algebra equal to the maximal abelian subalgebra \( \bigotimes_{n=1}^{\infty} \mathbb{C}^2 \). Thus invariant states correspond to the probability measures on \( \times_{n=1}^{\infty} \{0, 1\} \) and extremal invariant states to point evaluations on the Cantor set. So the extremal invariant states are the pure product measures
\[
\rho(x) = \text{Tr} \left( x \bigotimes_{n=1}^{\infty} \left( \begin{array}{cc} \rho_n & 0 \\ 0 & 1 - \rho_n \end{array} \right) \right)
\]
on \( \mathfrak{A} \) where \( \rho_n \in \{0, 1\} \). Hence all extremal invariant states are pure and the set of extremal invariant states is closed in the weak*-topology.
Example 2.7 We next construct a $C^*$-dynamical system $(\mathfrak{A}, \alpha, \mathbb{Z})$ satisfying the hypotheses of Theorem 2.1 for which there are no invariant pure states! Let $\theta \in [0,1]$ be an irrational number and let $\mathfrak{A}_\theta$ be the universal $C^*$-algebra generated by two unitaries $U, V$ with $VU = e^{2\pi i \theta} UV$, [Kis95]. The algebra $\mathfrak{A}_\theta$ is simple and carries an action $\beta$ of the two-torus given by

$$\beta_{(z_1,z_2)}(U) = z_1 U, \quad \beta_{(z_1,z_2)}(V) = z_2 V$$

for $z_1, z_2 \in \mathbb{T}$. This action is ergodic and the only invariant state for the action $\beta$ is the trace state $\omega$ defined by

$$\omega(U^nV^m) = \begin{cases} 0 & \text{if } (n,m) \neq (0,0) \\ 1 & \text{if } n = m = 0. \end{cases}$$

Note that

$$U^nV^m U(U^nV^m)^* = e^{2\pi i m \theta} U$$

and it can be proved that $\beta_{(z_1,z_2)}$ is inner if and only if

$$z_1, z_2 \in (\theta \mathbb{Z} + \mathbb{Z})/\mathbb{Z} \subseteq \mathbb{R}/\mathbb{Z} = \mathbb{T}$$

(see [ER93], [Kis95], [BKR92]). (One well-known argument for this is the following: If $\beta_{(z_1,z_2)}$ is implemented by a unitary $W \in \mathfrak{A}$ then $WUW^* = z_1 U$ and $WVW^* = z_2 V$. Hence $UWU^* = \pi i \theta$ and $VWV^* = \pi i \theta$. Thus $W$ is an eigenunitary for all the automorphisms $\text{Ad}(U^nV^m)$. But as these are dense in $\beta_{\mathbb{T}^2}$ it follows that $W$ is an eigenunitary for the action $\beta$. Hence $W = \lambda U^n V^m$ for some $\lambda \in \mathbb{T}$ and $n, m \in \mathbb{Z}$. Or just note that as $\beta_{(z_1,z_2)}$ commutes with all $\beta_{(w_1,w_2)}$, one has $\text{Ad} \beta_{w}(W) = \text{Ad} W$ for all $w \in \mathbb{T}^2$, and hence $W$ is an eigenelement for the action $\beta$ by simplicity of $\mathfrak{A}$.) Now choose irrational numbers $\theta_1, \theta_2$ such that $1, \theta, \theta_1$ and $\theta_2$ are rationally independent, put

$$g = (e^{2\pi i \theta_1}, e^{2\pi i \theta_2}) \in \mathbb{T}^2$$

and $\alpha = \beta_{g}$. Since $\{ g | n \in \mathbb{Z} \}$ is dense in $\mathbb{T}^2$ the only $\alpha$-invariant state is the trace state $\omega$. Also, since

$$\{ g^n | n \in \mathbb{Z} \} \cap (e^{2\pi i \theta_1 \mathbb{Z}}, e^{2\pi i \theta_2 \mathbb{Z}}) = 0,$$

all the automorphisms $\alpha^n, n \in \mathbb{Z} \setminus \{0\}$ are outer. (A simple direct argument for this is the following: If $\alpha^n = \text{Ad}(U)$ for some $n \in \mathbb{Z} \setminus \{0\}$ and $U \in \mathfrak{A}$ then any character on $C^*(U)$ extends to a pure $\alpha^n$-invariant state on $\mathfrak{A}$. But the powers $(\alpha^n)^m, m \in \mathbb{Z}$, are dense in $\beta_{\mathbb{T}^2}$ so this state must be the trace. This contradiction shows that all $\alpha^n, n \in \mathbb{Z} \setminus \{0\}$ are outer.) Furthermore, $(\mathfrak{A}, \alpha, \mathbb{Z})$ admits a faithful $\alpha$-covariant irreducible representation by Remark 2.2. Hence all the hypotheses of Theorem 2.1 are satisfied but there are no $\alpha$-invariant pure states on $\mathfrak{A}$.

Example 2.8 There exists a UHF-algebra $\mathfrak{A}$ and an automorphism $\alpha$ of $\mathfrak{A}$ such that the associated action of $\mathbb{Z}$ satisfies all the assumptions of Theorem 2.1 but nevertheless there is no invariant pure state. Take $p,q \in \{2,3,\ldots\}$ and set $\mathfrak{A} = M_p^\infty \otimes M_q^\infty, \alpha = \alpha_1 \otimes \alpha_2$
where $\alpha_1$ is the shift on $M_\infty = \otimes_\infty \mathcal{M}_\rho$ and $\alpha_2$ is the automorphism on $M_\infty$ constructed by Connes in [Con77] with outer invariant $(q, \gamma)$ where $\gamma \neq 1$ is a $q$-th root of unity. This means that $q$ is the smallest natural number such that $\alpha_2$ is inner in the trace representation of $M_\infty$ and if $\alpha_2^q = \text{Ad}(u)$ for a unitary $u$ in the weak closure then $\alpha_2(u) = \gamma u$. (We have also used $\alpha_2$ to denote the extension of the automorphism to the weak closure.) Explicitly, $\alpha_2$ is defined as follows. Set $M_{q^\infty} = \otimes_m M_{q^m}$ where $M_{q^m} \cong M_q$. Then if $\theta$ is the one-sided shift on $M_{q^\infty}$, define the unitaries $U \in M_q^{(1)}$ and $v \in M_q^{(1)} \otimes M_q^{(2)}$ by

$$U = \begin{pmatrix} \gamma & 0 & \cdots & 0 & 0 \\
0 & \gamma^2 & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & \gamma^{q-1} & 0 \\
0 & 0 & \cdots & \gamma^q & 0 \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} 0 & 1 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 1 \\
\theta(U^*) & 0 & \cdots & 0 & 0 \end{pmatrix}$$

and, following [Con77], set

$$\alpha_2 = \lim_{n \to \infty} \text{Ad}(v \theta(v) \theta^2(v) \cdots \theta^n(v)) \ .$$

Then it follows that $\alpha_2^q = \text{Ad} U$ and $\alpha_2(U) = \gamma U$ by Proposition 1.6 in [Con77]. Now if $\varphi$ is an $\alpha$-invariant pure state on $\mathfrak{A}$ it follows from the fact that $\alpha_1$ is asymptotically abelian that $\varphi = \varphi_1 \otimes \varphi_2$ where $\varphi_i$ is an $\alpha_i$-invariant pure state for $i = 1, 2$. But as $\gamma \neq 0$ it follows that $\alpha_2$ is not a quasi-product action of the group $\mathbb{Z}_{rq}$, where $r \in \mathbb{N}$ is the smallest number with $\gamma^r = 1$. This can be seen from the remark prior to Theorem 3.1 below and from this remark it also follows that $(M_{q^\infty}, \alpha_2)$ does not admit an invariant pure state. More specifically, if there exists an $\alpha_2$-covariant irreducible representation, there is a unitary operator $V$ on the representation space with $\alpha_2 = \text{Ad} V$, thus $\alpha_2^q = \text{Ad} V^q = \text{Ad} U$ and $V^q = \eta U$ for an $\eta \in T$ by irreducibility. But then $\alpha_2(V) = VV^* V = V$, so $\alpha_2(V^q) = V^q$ and $\alpha_2(U) = U \neq \gamma U$, where $\alpha_2$ also denotes the extension of $\alpha_2$ to the weak closure. This contradiction establishes that $(M_{q^\infty}, \alpha_2)$ does not admit a covariant irreducible representation and thus it does not admit an invariant pure state. Furthermore, as $\alpha_1$ is asymptotically abelian, $\alpha_1^q$ is outer for all $q \in \mathbb{Z}\setminus\{0\}$. Thus $(\mathfrak{A}, \alpha, \mathbb{Z})$ satisfies all the conditions of Theorem 2.1 but does not admit an $\alpha$-invariant pure state.

## 3 Density of invariant pure states for compact group actions

Let $\alpha$ be an action of a compact group $G$ on a $C^*$-algebra $\mathfrak{A}$ and let

$$x \in \mathfrak{A} \mapsto E(x) = \int_G dg \alpha_g(x) \in \mathfrak{A}^\alpha$$

be the associated projection onto the fixed point algebra $\mathfrak{A}^\alpha$. If $\omega$ is a state on $\mathfrak{A}^\alpha$ then $\omega \circ E$ is an $\alpha$-invariant state on $\mathfrak{A}$ and the map $\omega \mapsto \omega \circ E$ is an affine homeomorphism between the compact convex set $E_{\mathfrak{A}^\alpha}$ of states on $\mathfrak{A}^\alpha$ and the compact convex set $E_{\mathfrak{A}^\alpha}$ of $\alpha$-invariant states on $\mathfrak{A}$. Thus pure states on $\mathfrak{A}^\alpha$ correspond exactly to extremal invariant states on $\mathfrak{A}$. In this section we address the question of when purity of $\omega$ on $\mathfrak{A}^\alpha$ implies purity of $\omega \circ E$ on $\mathfrak{A}$. Simple examples show that it may happen that no extremal invariant
states are pure, e.g., if \( \mathfrak{A} \) is abelian and there are no fixed points in the spectrum of \( \mathfrak{A} \) for the \( \alpha \) action. More interesting examples are the canonical actions of the two-torus \( T^2 \) on the irrational rotation algebras mentioned in Example 2.7. These actions are ergodic and the only invariant state is the trace state. But in this example the automorphisms are inner for a dense set of points in \( T^2 \). The next theorem demonstrates that this situation does not generally arise in the converse case of quasi-product actions.

Recall from [BEK93], [BEEK89] that a faithful action \( \alpha \) of a compact group \( G \neq 0 \) on a \( C^* \)-algebra \( \mathfrak{A} \) is called a quasi-product action if it satisfies any of the following equivalent conditions;

1. If \( x, y \in \mathfrak{A} \) then \( \sup \{ \| x a y \| \mid a \in \mathfrak{A}^\alpha, \| a \| = 1 \} = \| x \| \cdot \| y \| \).
2. There exists a faithful irreducible representation \( \pi \) of \( \mathfrak{A} \) such that \( \pi|_{\mathfrak{A}^\alpha} \) is irreducible.
3. There exists an \( \alpha \)-invariant pure state \( \omega \) on \( \mathfrak{A} \) such that \( \pi|_{\mathfrak{A}^\alpha} \) is a faithful representation for \( \mathfrak{A}^\alpha \).
4. For any sequence \( (\zeta_n) \) of finite-dimensional unitary representations of \( G \) there exists an \( \alpha \)-invariant sub-\( C^* \)-algebra \( \mathfrak{B} \) of \( \mathfrak{A} \) and a closed \( \alpha^*\)-invariant projection \( q \) in the bidual \( \mathfrak{A}^{\ast\ast} \) of \( \mathfrak{A} \) such that \( q \in \mathfrak{B}' \), \( q \mathfrak{A} q = \mathfrak{B} q \), \( q \in \mathfrak{I}^{\ast\ast} \subseteq \mathfrak{A}^{\ast\ast} \) for any non-zero two-sided ideal \( \mathfrak{I} \) of \( \mathfrak{A} \) and the \( C^* \)-dynamical system \( (\mathfrak{B} q, G, \alpha^{\ast\ast}|_{\mathfrak{B} q}) \) is isomorphic to the product system \( (\otimes_{n=1}^\infty M_{\dim(\zeta_n)}, G, \otimes_{n=1}^\infty \Ad \zeta_n) \).

If \( G \) is abelian these conditions are also equivalent to

5. \( \mathfrak{A} \) and \( \mathfrak{A}^\alpha \) are prime and \( \alpha_g \) is properly outer for each \( g \in G \setminus \{0\} \).

It is maybe now not too surprising that these conditions are also equivalent with the density of the invariant pure states in the set of invariant states.

**Theorem 3.1** Let \( \alpha \) be a faithful action of a compact group \( G \neq 0 \) on a separable \( C^* \)-algebra \( \mathfrak{A} \). The following conditions are equivalent:

1. \( \alpha \) is a quasi-product action,
2. The set of \( \alpha \)-invariant pure states on \( \mathfrak{A} \) is weak*-dense in the set of \( \alpha \)-invariant states.

In proving Theorem 3.1 we begin by noting that (2) \( \Rightarrow \) (1) follows straightforwardly from known results as follows.

First, the characterization (3) of quasi-product states is equivalent to

3'. \( \mathfrak{A}^\alpha \) is prime and there exists a family \( \{ \omega_i \} \) of \( \alpha \)-invariant pure states \( \omega_i \) on \( \mathfrak{A} \) such that \( \oplus_i \pi|_{\mathfrak{A}^\alpha} \omega_i \) is a faithful representation of \( \mathfrak{A}^\alpha \).

Clearly (3) \( \Rightarrow \) (3') and the proof of the converse is indicated in [Kis88]. Briefly one has to check the proof of Theorem 3.1 in [BKR87]. But referring to Condition (6) in [BEK93], which is the same as Condition (4) above, one has to prove in addition that one may choose \( q \in \mathfrak{I}^{\ast\ast} \) for any non-zero two-sided ideal \( \mathfrak{I} \) of \( \mathfrak{A} \). This, however, follows once one can show from (3') that \( \mathfrak{A} \) is prime. So suppose \textit{ad absurdum} that \( \mathfrak{A} \) has two ideals \( \mathfrak{I}_1, \mathfrak{I}_2 \) with zero intersection. Then there is an invariant pure state \( \omega \) such that \( \omega|_{\mathfrak{I}_1} \) is a state and \( \omega|_{\mathfrak{I}_2} = 0 \). Since \( \omega \) is also zero on the invariant ideal \( \mathfrak{K}_2 \) generated by \( \mathfrak{I}_2 \) one then has
$\mathfrak{a}_2 \mathfrak{i}_1 = 0$. This remains true if $\mathfrak{i}_1$ is replaced by the invariant ideal generated by $\mathfrak{i}_1$. But intersecting with $\mathfrak{a}^\alpha$ one obtains two non-zero ideals in $\mathfrak{a}^\alpha$ with zero intersection which contradicts the assumption that $\mathfrak{a}^\alpha$ is prime. Thus $\mathfrak{a}$ is prime. Returning to the proof of (2)$\Rightarrow$(1) in Theorem 3.1 it now suffices to show that (2) implies that $\mathfrak{a}^\alpha$ is prime, since the other conditions in (3') obviously follow from (2). But (2) clearly implies that the extremal $\alpha$-invariant states are dense in the set of $\alpha$-invariant states and, by the initial remarks in this section, this means that the pure states on $\mathfrak{a}^\alpha$ are dense in the states on $\mathfrak{a}^\alpha$. This means that $\mathfrak{a}^\alpha$ is prime (and antiliminal).

In the proof of the implication (1)$\Rightarrow$(2) in Theorem 3.1 the characterization (2) of quasi-product actions is the most convenient.

We first focus on the case that $G$ is abelian. For $\gamma \in \hat{G}$ let

$$\mathfrak{a}^\alpha(\gamma) = \{ x \in \mathfrak{a} \mid a_g(x) = \gamma(g)x \}$$

be the corresponding spectral subspace. If $\varphi$ is a state on $\mathfrak{a}^\alpha$ and $x \in \mathfrak{a}^\alpha(\gamma)$ then $x^*x \in \mathfrak{a}^\alpha$ and if $\varphi(x^*y) = 1$ then $y \mapsto \varphi(x^*y) = (x\varphi x^*)(y)$ is a state on $\mathfrak{a}^\alpha$. In this situation let $\mathcal{H} = \mathcal{H}_{\varphi E}$, $\Omega = \Omega_{\varphi E}$, $\pi = \pi_{\varphi E}$ denote the associated Hilbert space, cyclic vector and representation, and $U$ the associated representation of $G$. Thus $U_g = \sum_{\gamma \in \hat{G}} \gamma(g)P_\gamma$ where $P_\gamma = [\pi(\mathfrak{a}^\alpha(\gamma))\Omega]$. If $P_\gamma \neq 0$ let $\rho_\gamma$ be the representation of $\mathfrak{a}^\alpha$ obtained by restricting $\pi|\mathfrak{a}^\alpha$ to $P_\gamma \mathcal{H} = [\pi(\mathfrak{a}^\alpha(\gamma))\Omega]$.

We can now formulate general criteria for purity of an extremal invariant state. Most of this proposition is more or less known.

**Proposition 3.2** Let $\mathfrak{a}$ be a $C^*$-algebra, $\alpha$ an action of a compact abelian group on $\mathfrak{a}$ and $\varphi$ a state on $\mathfrak{a}^\alpha$. The following conditions are equivalent:

1. $\varphi \circ E$ is a pure state on $\mathfrak{a}$,
2. $\varphi$ is a pure state on $\mathfrak{a}^\alpha$ and $P_0 = [\pi(\mathfrak{a}^\alpha)\Omega] \in \pi(\mathfrak{a}^\alpha)''$,
3. $\varphi$ is a pure state on $\mathfrak{a}^\alpha$ and $P_\Omega = [\mathcal{C}\Omega] \in \pi(\mathfrak{a}^\alpha)''$,
4. $\varphi$ is a pure state on $\mathfrak{a}^\alpha$ and $\rho_0 \nsubseteq \rho_\gamma$ for $\gamma \neq 0$,
5. $\varphi$ is a pure state on $\mathfrak{a}^\alpha$ and for all $\gamma \in \hat{G}\setminus\{0\}$ such that $\varphi(\mathfrak{a}^\alpha(\gamma)^*\mathfrak{a}^\alpha(\gamma)) \neq 0$ there is an $x \in \mathfrak{a}^\alpha(\gamma)$ such that $\varphi(x^*x) = 1$ and $x\varphi x^* \nsubseteq \varphi$.

Furthermore, all these conditions imply that the representations $\rho_\gamma$ of $\mathfrak{a}^\alpha$ are irreducible and mutually disjoint whenever they are non-zero.

**Proof** First assume Condition (1) is valid. By covariance of $\pi$ one has

$$\mathfrak{B}(P_\gamma \mathcal{H}) = P_\gamma \mathfrak{B}(\mathcal{H})P_\gamma = P_\gamma \pi(\mathfrak{a})''P_\gamma = \pi(\mathfrak{a}^\alpha)''P_\gamma$$

so the representations $\rho_\gamma$ of $\mathfrak{a}^\alpha$ are irreducible, when they are non-zero. But as $P_\gamma$ is $\text{Ad}(U_g)$-invariant it follows from covariance that $P_\gamma \in \pi(\mathfrak{a}^\alpha)''$. Then it follows that

$$\pi(\mathfrak{a}^\alpha)'' \cap \pi(\mathfrak{a}^\alpha)'' = \{ P_\gamma \mid \gamma \in \hat{G} \}'' .$$

Hence all the representations $\rho_\gamma$ are irreducible and mutually disjoint, when they are non-zero. Therefore Condition (1) implies each of the Conditions (2)–(5) as well as the concluding statement in the proposition.

The proof of the converse implications relies on the following result.
Lemma 3.3 Let $\mathfrak{A}$ be a $C^*$-algebra, $\alpha$ an action of a compact abelian group $G$ on $\mathfrak{A}$ and $\varphi$ a pure state on $\mathfrak{A}^\alpha$. It follows that all the non-zero representations $\rho_\gamma$ are irreducible.

Proof If $\gamma = \dot{e}$ then the corresponding representation $\rho_0$ is irreducible by assumption. Next note that the set of vectors of the form $\pi(x)\omega$ with $x \in \mathfrak{A}^\alpha(\gamma)$ are norm-dense in $P_\gamma \mathcal{H}$, so if one can show that the corresponding vector states on $\mathfrak{A}^\alpha$ are pure then $\rho_\gamma$ is necessarily irreducible. To this end pick an $x \in \mathfrak{A}^\alpha(\gamma)$ and let $\psi$ be a positive functional on $\mathfrak{A}^\alpha$ such that $x\varphi x^* \geq \psi$. Thus there exists a $T \in \rho_\gamma(\mathfrak{A}^\alpha)'$ with support $[\rho_\gamma(\mathfrak{A}^\alpha)\pi(x)\Omega]$ such that $T \geq 0$ and

$$\psi(a) = (\pi(a)\pi(x)\Omega, T\pi(x)\Omega)$$

for all $a \in \mathfrak{A}^\alpha$ (see, for example, [BR87] Theorem 2.3.19). But if $y \in \mathfrak{A}^\alpha(\gamma)$ then

$$y^*x\varphi x^*y \geq y^*y$$

as functionals on $\mathfrak{A}^\alpha$. But as $y^*x \in \mathfrak{A}^\alpha$ it follows that $y^*x\varphi x^*y$ is a vector functional in the $\rho_0$-representation and thus $y^*x\varphi x^*$ is a multiple of a pure state on $\mathfrak{A}^\alpha$, because $\varphi$ is assumed to be a pure state. Thus there exists a $\lambda > 0$ with

$$y^*y = \lambda y^*x\varphi x^*y$$

and then

$$y^*yy^* = \lambda yy^*x\varphi x^*yy^* .$$

Hence

$$(T\pi(yy^*a\varphi y^*)\pi(x)\Omega, \pi(x)\Omega) = \lambda (\pi(yy^*a\varphi y^*)\pi(x)\Omega, \pi(x)\Omega)$$

for any $a \in \mathfrak{A}^\alpha$. In principle the value of $\lambda$ could depend on $y$ but by letting $yy^*$ run through an approximate identity for the ideal $\mathfrak{A}^\alpha(\gamma)\mathfrak{A}^\alpha(\gamma)^*$ in $\mathfrak{A}^\alpha$, or replacing $x$ by $f(xx^*)x$ where $f$ has a plateau at the value one and then choosing $y$ with $yy^*x = x$, one deduces that $\lambda$ is in fact independent of $y$. Therefore one concludes that $T = \lambda I$ on $[\rho_\gamma(\mathfrak{A}^\alpha)\pi(x)\Omega]$. It then follows that $\psi = \lambda x\varphi x^*$ and $x\varphi x^*$ is a multiple of a pure state. Thus $\rho_\gamma$ is irreducible.

Proof of Proposition 3.2 continued We will prove that the conditions of the proposition are also equivalent to

(6) The representations $\rho_\gamma$ of $\mathfrak{A}^\alpha$ are irreducible, whenever they are non-zero, and mutually disjoint.

We have already argued that (1) $\Rightarrow$ (6) and Condition (6) is equivalent to

$$\pi_\varphi E(\mathfrak{A})'' \cap \pi_\varphi E(\mathfrak{A})' = \pi_\varphi E(\mathfrak{A})' = \{P_\gamma | \gamma \in \dot{G}\}'' .$$

Thus if (6) is satisfied

$$\pi_\varphi E(\mathfrak{A})' \subseteq \{P_\gamma | \gamma \in \dot{G}\}'' .$$

But $P_\gamma = [\pi_\varphi E(\mathfrak{A}^\alpha(\gamma))\Omega]$ and hence

$$\pi_\varphi E(\mathfrak{A})'P_0 = P_\gamma \pi_\varphi E(\mathfrak{A}^\alpha(\gamma)) .$$

If $P_\gamma$ is non-zero then both sides of the last equation are non-zero. It now follows that $P_\gamma \not\subseteq \pi_\varphi E(\mathfrak{A})'$ if $P_\gamma \neq 0, 1$ and $\gamma \neq 0$ and one easily extends this argument to show that

$$\pi_\varphi E(\mathfrak{A})' = \pi_\varphi E(\mathfrak{A})' \cap \{P_\gamma | \gamma \in \dot{G}\}'' = CI .$$
Proof of Theorem 3.1. It follows from the introductory remarks to this section that we need to prove that if \( \omega \) is a pure state on \( \mathfrak{A}^0 \), \( x_1, x_2, \ldots, x_n \in \mathfrak{A}^0 \) and \( \varepsilon > 0 \) then there is a pure state \( \varphi \) on \( \mathfrak{A}^0 \) satisfying any of the equivalent conditions of Proposition 3.2 such that

\[
|\varphi(x_i) - \omega(x_i)| < \varepsilon
\]

for \( i = 1, 2, \ldots, n \). So let \( \omega, x_1, \ldots, x_n \) and \( \varepsilon \) be given. By Lemma 2.4 there is a positive element \( e \in \mathfrak{A}_{\mathbb{P}}^0 \) with \( \omega(e) = 1 \) such that

\[
\|e x_i e - \omega(x_i) e^2\| < \varepsilon
\]

for \( i = 1, 2, \ldots, n \).

Let \( \{u_n\} \) be a dense sequence in the unitary group \( \mathcal{U}(\mathfrak{A}^0) \) of \( \mathfrak{A}^0 \), or \( \mathfrak{A}^0 + I \) if \( \mathfrak{A} \) is non-unital. Now \( \mathfrak{A}^0(\gamma) \neq 0 \) for all \( \gamma \in \hat{G} \) since \( G \) acts faithfully. Let \( \{\gamma_n\} \) be a sequence in \( \hat{G} \setminus \{\hat{e}\} \) such that each element in \( \hat{G} \setminus \{\hat{e}\} \) occurs infinitely often and \( \{(\gamma_n, u_n); n = 1, 2, \ldots\} \) is dense in \( \hat{G} \setminus \{\hat{e}\} \times \mathcal{U}(\mathfrak{A}^0) \). Next we construct inductively a sequence \( e_n \) of elements in \( \mathfrak{A}_{\mathbb{P}}^0 \) and elements \( x_\gamma \in \mathfrak{A}^0(\gamma) \) with the three properties

1. \( e_1 e = e_1 \) and \( e_n e_{n-1} = e_n \) \( n = 2, 3, \ldots \)
2. \( e_n x_{\gamma_n} x_{\gamma_n}^* = e_n \)
3. \( \|u_n x_{\gamma_n}^* e_n x_{\gamma_n} u_n^* e_n\| \leq 1/n \) .
If these objects have been constructed for 1, 2, . . . , n − 1 we construct $e_n$ and $x_n$ as follows. If $\gamma_n \in \{\gamma_1, \ldots, \gamma_{n-1}\}$ then $x_{\gamma_n}$ has already been chosen and we keep that choice and define $q_n = e_{n-1}$. If $\gamma_n \notin \{\gamma_1, \ldots, \gamma_{n-1}\}$ then we take some $q_n \in A^n_{\hat{\mathcal{P}}}$ with $q_n e_{n-1} = q_n$ and in addition some $x_{\gamma_n} \in A^a(\gamma_n)$ with $q_n x_{\gamma_n} x^*_{\gamma_n} = q_n$. This is possible since $A^a(\gamma_n)A^a(\gamma_n)^*$ is an ideal in $A^a$ which is prime by assumption. Having now chosen $x_{\gamma_n}, q_n$ choose $e_n$ in $A^n_{\hat{\mathcal{P}}}$ with $e_n q_n = e_n$ such that
\[
\|u_n x^*_n e_n x_{\gamma_n} u^*_n e_n\| \leq 1/n.
\]
This choice is possible by the following reasoning. Since $A$ and $A^a$ are prime and separable and $\alpha_g$ is properly outer for all $g \in G \setminus \{e\}$ it follows from \cite{BEEK89} that there exists an irreducible representation $\pi$ of $A$ on a Hilbert space $\mathcal{H}$ such that $\pi(A^a)^'' = B(\mathcal{H})$. If ad absurdum $e_n$ does not exist then there is an $\varepsilon > 0$ such that
\[
\|u_n x^*_n p x_{\gamma_n} u^*_n p\| \geq \varepsilon
\]
for all $p \in A^n_{\hat{\mathcal{P}}}$ with $pq_n = p$. But if $q$ is the spectral projection of $\pi(q_n)$ corresponding to the eigenvalue one this means that $|\langle \pi(x u_n^*) \psi, \psi \rangle| \geq \varepsilon$ for all $\psi \in q\mathcal{H}$ with $\|\psi\| = 1$ where $x = x_{\gamma_n}$. This conclusion is reached by noting that the orthogonal projection $P_\psi$ onto $\psi$ is the limit of a decreasing net $\pi(p_m)$ with $p_m q_n = p_m$, $p_m \in A^a$, by irreducibility of $\pi$ on $A^a$. But then one concludes that there is a $\theta \in \mathbb{R}$ such that
\[
\text{Re}(e^{i\theta} \pi(x u_n^*) \psi, \psi) \geq \varepsilon
\]
for all $\psi \in q\mathcal{H}$ with $\|\psi\| = 1$. Thus
\[
e^{i\theta} f x u_n^* f + e^{-i\theta} f u_n x^* f \geq 2 \varepsilon f^2
\]
for any positive $f \in A^a$ with $fq_n = f$. But applying $\alpha_g$ to this relation one obtains
\[
\gamma_n(g)e^{i\theta} f x_{\gamma_n} u_n^* f + \gamma_n(g)e^{-i\theta} f u_n x^*_{\gamma_n} f \geq 2 \varepsilon f^2
\]
for all $g \in G$ since $u_n, f \in A^a$ and $x_{\gamma_n} \in A^a(\gamma_n)$. But this is impossible unless $\gamma_n = \hat{\varepsilon}$. Hence the $e_n$ exist.

Now one can complete the proof of Theorem 3.1. Choose $e_n$ and $x_n$ such that (1)–(3) are valid and let $\varphi$ be any pure state on $A^a$ with $\varphi(e_n) = 1$ for all $n$. Then $x^*_n \varphi x_n \notin \varphi$ for all $\gamma \in \hat{G} \setminus \{\hat{e}\}$ by the following reasoning. Let $p \in (A^a)^{**}$ be the closed projection such that $e_n \downarrow_p p$. By Condition (3) it follows that for any unitary $u \in A^a$ and any $\gamma \in \hat{G} \setminus \{\hat{e}\}$ one has
\[
u x^*_n p x_{\gamma} u^* p = 0
\]
but as $\varphi(p) = 1$ this means that $x^*_n \varphi x_n$ is disjoint from $\varphi$ for any $\gamma \in \hat{G} \setminus \{\hat{e}\}$. Now $\varphi \circ E$ is a pure state by Proposition 3.2. But as
\[
\|e x_i e - \omega(x_i) e^2\| < \varepsilon
\]
and $pe = p$ one has
\[
\|px_i p - \omega(x_i) p\| < \varepsilon
\]
and then
\[
|\varphi(x_i) - \omega(x_i)| = |\varphi(px_i p) - \omega(x_i)\varphi(p)| < \varepsilon
\].
This completes the proof of Theorem 3.1 in the case of abelian $G$.

Next we describe the necessary modification of the foregoing argument to handle non-abelian $G$. We first replace the dynamical system $(\mathfrak{A}, G, \alpha)$ by $(\mathfrak{A}, G, \tilde{\alpha})$ where
\[
\tilde{\mathfrak{A}} = \mathfrak{A} \otimes \mathfrak{K}(L_2(G)) \otimes \mathfrak{K},
\]
where $\mathfrak{K}(L_2(G))$ denotes the compact operators on $L_2(G)$ and $\mathfrak{K}$ the compact operators on a separable infinite-dimensional Hilbert space, and
\[
\tilde{\alpha}_g = \alpha_g \otimes \text{Ad} \lambda(g) \otimes \iota
\]
where $\lambda$ is the left regular representation of $G$ on $L_2(G)$. If $\gamma \in \hat{G}$ we identify $\gamma$ with one of its concrete representations and define $\mathfrak{A}^\gamma$ (following [Was81], [BrE86]) as the set of row-matrices $x = (x_1, \ldots, x_d)$, where $x_i \in \mathfrak{A}$ with $d = d(\gamma) = \text{dimension of } \gamma$, such that
\[
\alpha_g(x) = x\gamma(g)
\]
The reason for the replacement of $(\mathfrak{A}, \alpha)$ by $(\tilde{\mathfrak{A}}, \tilde{\alpha})$ is the existence of an $x(\gamma) \in M(\tilde{\mathfrak{A}}^\gamma(\gamma))$ with the properties $x(\gamma)^*x(\gamma) = 1 \otimes 1_{d(\gamma)}$ and $x(\gamma)x(\gamma)^* = 1$ [BEK93]. Thus the problem of proving Theorem 3.1 for $(\mathfrak{A}, G, \alpha)$ reduces to proving the theorem for $(\tilde{\mathfrak{A}}, G, \tilde{\alpha})$ by the following reasoning. Let $\omega$ be a given $\alpha$-invariant state on $\mathfrak{A}$ and assume $x_1, \ldots, x_n \in \mathfrak{A}$ and $\varepsilon > 0$ are specified. Then let $e$ be an $\text{Ad} \lambda(g) \otimes \iota$-invariant one-dimensional projection in $\mathfrak{K}(L_2(G)) \otimes \mathfrak{K}$ and replace $x_1, \ldots, x_n$ by $\tilde{x}_1 = x_1 \otimes e, \ldots, \tilde{x}_n = x_1 \otimes e$ and $1 \otimes e$ in $\tilde{\mathfrak{A}}$ and $\omega$ by $\tilde{\omega} = \omega \otimes \omega'$ where $\omega'$ is the pure state on $\mathfrak{K}(L_2(G)) \otimes \mathfrak{K}$ such that $\omega'(e) = 1$. If Theorem 3.1 is valid for $(\tilde{\mathfrak{A}}, G, \tilde{\alpha})$ let $\tilde{\varphi}$ be a pure $\tilde{\alpha}$-invariant state on $\tilde{\mathfrak{A}}$ such that
\[
|\tilde{\varphi}(\tilde{x}_i) - \tilde{\omega}(\tilde{x}_i)| < \varepsilon/2
\]
\[
|\tilde{\varphi}(I \otimes e) - \tilde{\omega}(I \otimes e)| < \varepsilon/2.
\]
But then $\tilde{\varphi}(I \otimes e) > 1 - \varepsilon/2$ so the state $\varphi$ defined on $\mathfrak{A}$ by
\[
\varphi(x) = \tilde{\varphi}(x \otimes e)/\tilde{\varphi}(I \otimes e)
\]
is an $\alpha$-invariant pure state and
\[
|\varphi(x_i) - \omega(x_i)| < \varepsilon
\]
for $i = 1, \ldots, n$. Thus to prove Theorem 3.1 we may assume there exist $x(\gamma) \in \mathfrak{A}^\gamma(\gamma)$ with $x(\gamma)^*x(\gamma) = I \otimes I_{d(\gamma)}$, $x(\gamma)x(\gamma)^* = I$. One now defines dual endomorphisms $\hat{\alpha}_\gamma : \mathfrak{A}^\gamma \to \mathfrak{A}^\gamma$ by
\[
\hat{\alpha}_\gamma(a) = \sum_{i=1}^{d(\gamma)} x_i(\gamma)ax_i(\gamma)^* = x(\gamma)(a \otimes I_d)x^*(\gamma)
\]
for $a \in \mathfrak{A}^\gamma$. These are indeed endomorphisms of $\mathfrak{A}^\gamma$ and if $\varphi$ is a state on $\mathfrak{A}^\gamma$ one establishes, as in Proposition 3.2, that $\varphi \circ E$ is a pure state on $\mathfrak{A}$ if and only if $\varphi$ is pure on $\mathfrak{A}^\gamma$ and $\varphi \circ \hat{\alpha}_\gamma$ is disjoint to $\varphi$ for all $\gamma \in G \setminus \{e\}$. To construct states with this property one uses the assumption that $\alpha_g$ is properly outer and $\mathfrak{A}$ and $\mathfrak{A}^\gamma$ are prime, which implies $\hat{\alpha}_\gamma$ is properly outer [BEK93], and then one uses the earlier argument to find minimal projections $p \in (\mathfrak{A}^\gamma)^\prime\prime$ such that
\[
\|u\hat{\alpha}_\gamma(p)u^*p\| = 0
\]
for all unitaries $u \in \mathfrak{A}^\gamma + CI$. 

\qed
4 One-dimensional shifts

Let $\mathfrak{A}$ be a unital $C^*$-algebra. Then $\mathfrak{A}$ is nuclear if and only if there exist an increasing sequence $\{k_n\}$ of positive integers and unital completely positive maps $\sigma_n$ of $\mathfrak{A}$ into $M_{k_n}$ and $\tau_n$ of $M_{k_n}$ into $\mathfrak{A}$ such that

$$\lim_{n \to \infty} \|\tau_n \circ \sigma_n(x) - x\| = 0$$

for all $x \in \mathfrak{A}$. We now derive a more specific characterization of prime AF-algebras, which are obviously nuclear.

**Lemma 4.1** Let $\mathfrak{A}$ be a unital $C^*$-algebra. The following conditions are equivalent:

1. $\mathfrak{A}$ is a prime AF-algebra.

2. There exist a sequence $\{k_n\}$ of positive integers, unital completely positive maps $\sigma_n$ of $\mathfrak{A}$ onto $M_{k_n}$, unital completely positive maps $\tau_n$ of $M_{k_n}$ into $\mathfrak{A}$ such that $\tau_n(M_{k_n})$ is a $C^*$-subalgebra of $\mathfrak{A}$, homomorphisms $\iota_n$ (not necessarily unital) of $M_{k_n}$ into $\mathfrak{A}$ such that the restriction of $\tau_n \circ \iota_n$ to $\mathfrak{A} \cap \iota_n(M_{k_n})$ has the form $\tau_n(I_{k_n})\omega_n(\cdot)$ where $\omega_n$ is a pure state on $\mathfrak{A} \cap \iota_n(M_{k_n})$ such that $\sigma_n \circ \iota_n$ is the identity map on $M_{k_n}$ and

$$\lim_{n \to \infty} \|\tau_n \circ \sigma_n(x) - x\| = 0$$

for all $x \in \mathfrak{A}$.

**Proof** (2) ⇒ (1) Since $\tau_n(M_{k_n})$ is a $C^*$-subalgebra of $\mathfrak{A}$, Condition (2) implies that $\mathfrak{A}$ is an AF-algebra by the local characterization of AF-algebras in [Bra71]. That $\mathfrak{A}$ is also prime follows the consequence of Condition (2) given in Proposition 4.2 below (see Remark 4.3).

Let $\{A_n\}$ be an increasing sequence of finite-dimensional $C^*$-algebras such that $\mathfrak{A} \cong \lim\limits_{\rightarrow} A_n$. Let $A_n = \sum_{i=1}^{k_n} A_{ni}$ where the $A_{ni}$ are full matrix algebras, and let $e_{ni}$ be the identity of $A_{ni}$. Since $\mathfrak{A}$ is prime we may suppose that the reduction $A_n \ni x \mapsto xe_{n+1,1}$ is faithful for any $n$, [Bra71].

Suppose that $A_{n+1,1}$ is isomorphic to $M_{k_n}$ and define $\iota_n$ to be an isomorphism of $M_{k_n}$ onto $A_{n+1,1}$. Let $\omega_n$ be a pure state of $\mathfrak{A} \cap A_{n+1,1}$ such that $\omega_n(e_{n+1,1}) = 1$ and define $\sigma_n$ by

$$\sigma_n(x) = (id \otimes \omega_n)(e_{n+1,1}xe_{n+1,1})$$

where $e_{n+1,1}Ae_{n+1,1}$ is identified with $M_{k_n} \otimes \left( e_{n+1,1}Ae_{n+1,1} \cap A_{n+1,1} \right)$ by using $\iota_n \otimes id$. Thus $\sigma_n$ is a unital completely positive map of $\mathfrak{A}$ onto $M_{k_n}$ with $\sigma_n \circ \iota_n = id$ and $\iota_n \circ \sigma_n$ restricted to $\mathfrak{A} \cap \iota_n(M_{k_n})$ is $\iota_n(I_{k_n})$ multiplied by the pure state $\omega_n$. Choose a projection $\varphi_n$ of $A_{n+1,1}$ onto $A_ne_{n+1,1}$ and define $\tau_n$ by $\varphi_n \circ \iota_n$ where $A_ne_{n+1,1}$ is identified with $A_n$, i.e., $\tau_n = \psi_n \circ \varphi_n \circ \iota_n$ where $\psi_n: A_ne_{n+1,1} \hookrightarrow A_n$ is the *-isomorphism given by $\psi_n(xe_{n+1,1}) = x$ for $x \in A_n$. Thus $\tau_n$ is a unital completely positive map and satisfies $\tau_n \circ \sigma_n(x) = x$ for all $x \in A_n$.

**Proposition 4.2** Let $A$ be a unital, prime AF-algebra, $\mathfrak{A} = \otimes_{\infty}^\infty A$ the infinite tensor product of $A$ and $\alpha$ the shift automorphism of $\mathfrak{A}$. Then the $\alpha$-invariant pure states of $\mathfrak{A}$ are dense in the set of $\alpha$-invariant states of $\mathfrak{A}$.
\textbf{Proof} Let $k_n, \sigma_n, \tau_n$ and $\iota_n$ be chosen as in Lemma \ref{lem:4.1} with $A$ in place of $\mathcal{A}$. Then define completely positive maps $\tilde{\sigma}_n$ of $\mathcal{A} = A^{\otimes \infty}$ into $\mathcal{M}_n = M_{k_n}^{\otimes \infty}$ and $\tilde{\tau}_n$ of $\mathcal{M}_n$ into $\mathcal{A}$ by

$$\tilde{\sigma}_n = \sigma_n^{\otimes \infty}, \quad \tilde{\tau}_n = \tau_n^{\otimes \infty}.$$  

Note that $\tilde{\sigma}_n \circ \omega = \omega \circ \tilde{\sigma}_n$ and $\tilde{\tau}_n \circ \omega = \omega \circ \tilde{\tau}_n$ where we use $\omega$ to denote the shift automorphism for both $\mathcal{A}$ and $\mathcal{M}_n$.

Let $e_n = \iota_n(I_{k_n})$ and regard the infinite tensor product $\tilde{e}_n$ of copies of $e_n$ as a closed projection of the second dual $\mathcal{A}^{**}$. Then define a homomorphism $\tilde{\iota}_n$ of $\mathcal{M}_n$ into $\tilde{e}_n \mathcal{A} \tilde{e}_n$ by $\iota_n \circ \alpha = \alpha \circ \iota_n$.

Next let $\omega$ be an $\alpha$-invariant state of $\mathcal{A}$. Since the weak*-limit of $\omega \circ \tilde{\tau}_n \circ \tilde{\sigma}_n$ as $n \to \infty$ is $\omega$ one may approximate $\omega$ by $\alpha$-invariant pure states by the following argument: As $\omega \circ \tilde{\tau}_n$ is an $\alpha$-invariant state of $\mathcal{M}_n$ one can find an $\alpha$-invariant pure state $\varphi$ of $\mathcal{M}_n$ such that $\varphi$ is in a given weak* neighbourhood of $\omega \circ \tilde{\tau}_n$ \cite{FNW92}. Then $\varphi \circ \tilde{\sigma}_n$ is in a given neighbourhood of $\omega \circ \tilde{\tau}_n \circ \tilde{\sigma}_n$. Thus to complete the proof it suffices to show that $\varphi \circ \tilde{\sigma}_n$ is pure.

Since $\varphi \circ \tilde{\sigma}_n = \varphi \circ \tilde{\sigma}_n \circ \tilde{\iota}_n \circ \tilde{\sigma}_n$, the support of $\varphi \circ \tilde{\sigma}_n$ is contained in $\tilde{e}_n$, and

$$\varphi \circ \tilde{\sigma}_n \Big|_{(A \cap \iota_n(M_{k_n}))^{\otimes \infty}} = \omega^{\otimes \infty}_n.$$  

Note also that $\varphi \circ \tilde{\sigma}_n \circ \tilde{\iota}_n = \varphi$. These facts determine $\varphi$ as follows: for $x_i \in M_{k_n}$ and $a_i \in e_n(A \cap \iota_n(M_{k_n}))e_n$ one has

$$\varphi(\iota_n(x_1)a_1 \otimes \iota_n(x_2)a_2 \otimes \ldots \otimes \iota_n(x_k)a_k) = \varphi(x_1 \otimes x_2 \otimes \ldots \otimes x_k)\omega_n(a_1)\omega_n(a_2)\ldots\omega_n(a_k),$$

which is not ambiguous since $\varphi$ and $\varphi \circ \tilde{\sigma}_n$ are $\alpha$-invariant. Hence if $\psi$ is a positive linear functional on $\mathcal{A}$ such that $\psi \leq \varphi \circ \tilde{\sigma}_n$ then it follows first that the support of $\psi$ is contained in $\tilde{e}_n$, and secondly, since $\omega_n$ is pure,

$$\psi \Big|_{(A \cap \iota_n(M_{k_n}))^{\otimes \infty}} = \psi(I)\omega^{\otimes \infty}_n,$$

and finally, since $\varphi$ is pure,

$$\psi \circ \tilde{\iota}_n = \psi(I) \varphi.$$  

Hence it follows that $\psi = \psi(I) \varphi \circ \tilde{\sigma}_n$. Therefore $\varphi \circ \tilde{\sigma}_n$ is pure.  

\hfill $\Box$

\textbf{Remark 4.3} If $A$ is not prime then the conclusion of Proposition \ref{prop:4.2} does not hold because there are then two non-zero ideals $I$ and $J$ of $A$ such that $I \cap J = \{0\}$. If we denote by $\hat{I}$ (respectively $\hat{J}$) the ideal of $\mathcal{A}$ generated by $I$ (respectively $J$) at the position 0 in the factorization $\mathcal{A} = \otimes^\infty A$ there is an $\alpha$-invariant state $\omega$ of $\mathcal{A}$ such that $\omega|_I \neq 0$ and $\omega|_J \neq 0$. But since $\hat{I} \cap \hat{J} = \{0\}$, it follows that $\varphi|_I = 0$ or $\varphi|_J = 0$ for any pure state $\varphi$ on $\mathcal{A}$. Thus $\omega$ cannot be approximated by a pure invariant state, and not even by a pure state.

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References

[BrE86] Bratteli, O., and Evans, D. E., Derivations tangential to compact groups: The non-abelian case, Proc. London Math. Soc. 52 (1986), 369–384.

[BEEK89] Bratteli, O., Elliott, G. A., Evans, D. E., and Kishimoto, A., Quasi-product actions of a compact abelian group on a $C^*$-algebra, Tôhoku Math. J. 41 (1989), 133–161.

[BEK93] Bratteli, O., Elliott, G. A., and Kishimoto, A., Quasi-product actions of a compact group on a $C^*$-algebra, J. Funct. Anal. 115 (1993), 313–343.

[BKR92] Blackadar, B., Kumjian, A., and Rørdam, M., Approximately central matrix units and the structure of non-commutative tori, K-theory 6 (1992), 267–284.

[BKR87] Bratteli, O., Kishimoto, A., and Robinson, D. W., Embedding product type actions into $C^*$-dynamical systems, J. Funct. Anal. 75 (1987), 188–210.

[Bra71] Bratteli, O., Inductive limits of finite-dimensional $C^*$-algebras, Trans. Amer. Math. Soc. 171 (1972), 195–234.

[BR87] Bratteli, O., and Robinson, D. W., Operator algebras and quantum statistical mechanics, vol. 1. Second edition. Springer-Verlag, New York etc., 1987.

[Con77] Connes, A., Periodic automorphisms of the hyperfinite factor of type $II_1$, Acta Sci. Math. (Szeged) 39 (1977), 39–66.

[Ell80] Elliott, G. A., Some simple $C^*$-algebras constructed as crossed products with discrete outer automorphism groups, Publ. RIMS Kyoto Univ. 16 (1980), 299–311.

[ER93] Elliott, G. A. and Rørdam, M., The automorphism group of the irrational rotation $C^*$-algebra, Commun. Math. Phys. 155 (1993), 3–26.

[FNW92] Fannes, M., Nachtergaele, B., and Werner, R. F., Abundance of translation invariant pure states on quantum spin chains, Lett. Math. Phys. 25 (1992), 249–258.

[Gli61] Glimm, J., Type I $C^*$-algebras, Ann. Math. 73 (1961), 572–612.

[Kis81] Kishimoto, A., Outer automorphisms and reduced crossed products of simple $C^*$-algebras, Commun. Math. Phys. 81 (1981), 429–435.

[Kis82] Kishimoto, A., Freely acting automorphisms of $C^*$-algebras, Yokohama Math. J. 30 (1982), 39–47.

[Kis87a] Kishimoto, A., Type I orbits in the pure states of a $C^*$-dynamical system, Publ. RIMS Kyoto Univ. 23 (1987), 321–336.

[Kis87b] Kishimoto, A., Type I orbits in the pure states of a $C^*$-dynamical system II, Publ. RIMS Kyoto Univ. 23 (1987), 517–526.
[Kis88] Kishimoto, A., Compact group actions on C*-algebras, *RIMS Kokyoroko, Kyoto Univ.* **651** (1988), 43-87.

[Kis95] Kishimoto, A., The Rohlin property for automorphisms of UHF-algebras, *J. reine angew. Math.* **465** (1995) 183–196.

[Kis96] Kishimoto, A., A Rohlin property for one-parameter automorphism groups, *Commun. Math. Phys.* (1996) to appear.

[OP82] Olesen, D. and Pedersen, G. K., Applications of the Connes spectrum to C*-dynamical systems III, *J. Funct. Anal.* **45** (1982), 357–390.

[Ped79] Pedersen, G. K., *C*-algebras and their automorphism groups, Academic Press, London, 1979.

[Rie81] Rieffel, M., C*-algebras associated with irrational rotations, *Pac. J. Math.* **93** (1981), 415–429.

[Was81] Wasserman, A., *Automorphic actions of compact groups on operator algebras*, Thesis, University of Pennsylvania, 1981.