FRactal Dimensions for Continuous Time Random Walk Limits

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Abstract. In a continuous time random walk (CTRW), each random jump follows a random waiting time. CTRW scaling limits are time-changed processes that model anomalous diffusion. The outer process describes particle jumps, and the non-Markovian inner process (or time change) accounts for waiting times between jumps. This paper studies fractal properties of the sample functions of a time-changed process, and establishes some general results on the Hausdorff and packing dimensions of its range and graph. Then those results are applied to CTRW scaling limits.

1. Introduction

Continuous time random walks have attracted a lot of attention in recent years. They provide flexible models for anomalous diffusion phenomena in a wide range of scientific areas including physics, finance and hydrology. Consider a random walk \( S(n) = J_1 + \cdots + J_n \) on \( \mathbb{R}^d \), where \( \{ J_n, n \geq 1 \} \) model the particle jumps. The continuous time random walk (CTRW) imposes a random waiting time between jumps. Let \( T_n = W_1 + \cdots + W_n \), where \( \{ W_n, n \geq 1 \} \) are nonnegative random variables. The CTRW jumps to location \( S(n) \) at time \( T_n \). The number of jumps by time \( t \geq 0 \) is given by the counting process \( N_t = \max\{ n \geq 0 : T_n \leq t \} \), where \( T_0 = 0 \). The time-changed process \( S(N_t) \) represents the location of a random walker at time \( t \geq 0 \). A standard assumption in the literature is that \( \{(J_n, Y_n), n \geq 1\} \) are iid. In recent years CTRW with dependent jumps or/and waiting times have also been considered, see for example [12, 26, 41].

The scaling limit of a CTRW \( \{S(N_t), t \geq 0\} \) is a time-changed (or iterated) process \( X = \{X(t), t \geq 0\} \) of the form \( X(t) = Y(E_t) \), where the outer process \( \{Y(t), t \geq 0\} \) is the scaling limit of the random walk \( \{S_n, n \geq 0\} \) and the inner process \( \{E_t, t \geq 0\} \) accounts for the random waiting times \( \{W_n, n \geq 1\} \). This has been proved by Meerschaert and Scheffler [29], and Becker-Kern, Meerschaert and Scheffler [3, 4] under the assumption that \( \{(J_n, W_n), n \geq 1\} \) are iid, the jumps \( \{J_n, n \geq 1\} \) belong to

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the domain of attraction of an operator stable law and the waiting times \( \{W_n, n \geq 1\} \) belong to the strict domain of attraction of a positive stable random variable \( D \) with index \( \beta \in (0, 1) \). In this case, the outer process \( \{Y(t), t \geq 0\} \) is an operator stable Lévy process with values in \( \mathbb{R}^d \) and the inner process \( \{(E_t, t \geq 0)\} \) is the inverse of a \( \beta \)-stable subordinator \( \{D(x), x \geq 0\} \) with \( D(1) = D \). Namely,

\[
E_t = \inf \{x \geq 0 : D(x) > t\}, \quad \forall t \geq 0.
\]

The aforementioned authors further proved that the density function \( p(t, x) \) of \( X(t) = Y(E_t) \) solves fractional partial differential equations; see [1, 2, 25] and the references therein for further information on PDE connections of CTRW limits.

When the independence assumption on the jumps \( \{J_n, n \geq 1\} \) is removed, Meerschaert, Nane and Xiao [26] showed that the outer process \( Y \) can be taken as a fractional Brownian motion, a stable Lévy process or a linear fractional stable motion. More general inner processes may also be possible if the waiting times are dependent.

In general, a CTRW limiting process \( X \) is non-Markovian, non-Gaussian and satisfies a form of self-similarity. Figure 1 illustrates a typical trajectory of the time-changed process \( X(t) = Y(E_t) \), in the case where the outer process \( Y \) is a Brownian motion. The graph resembles that of a Brownian motion, interrupted by long resting periods. This process \( X \) is the long-time scaling limit of a CTRW with mean zero, finite variance jumps and heavy tailed waiting times in the domain of attraction of a \( \beta \)-stable subordinator, see [29].

This paper is concerned with fractal properties of the CTRW limiting process \( X = \{X(t), t \geq 0\} \) defined by \( X(t) = Y(E_t) \) for \( t \geq 0 \). In particular we determine the Hausdorff and packing dimensions of the range \( X([0, 1]) = \{X(t) : t \in [0, 1]\} \) and the graph \( \text{Gr}X([0, 1]) = \{(t, X(t)) : t \in [0, 1]\} \). There has been a large literature on sample path and fractal properties of Lévy processes [40, 43], and Gaussian or stable
random fields \[19, 44\]. Several methods have been developed for computing the Hausdorff dimensions of the range and graph of stochastic processes under “minimal” conditions. To give a brief description of the general method, let \( U = \{ U(t), t \geq 0 \} \) be a stochastic process with values in \( \mathbb{R}^d \) (for simplicity we assume that the components of \( U(t) \) are independent). If there exist positive constants \( C \) and \( H \in (0, 1) \) such that

\[
\mathbb{E} \left( \sup_{0 \leq h \leq T} |U(t + h) - U(t)| \right) \leq C T^H, \quad \forall T \in (0, 1),
\]

then one can prove

\[
\dim_H U([0, 1]) \leq \min \left\{ d, \frac{1}{H} \right\} \quad \text{a.s.}
\]

and

\[
\dim_H \text{Gr} U([0, 1]) \leq \min \left\{ \frac{1}{H}, 1 + (1 - H)d \right\} \quad \text{a.s.}
\]

In the above and sequel, \( \dim_H \) denotes Hausdorff dimension. Moreover, if there exist positive constants \( C \) and \( H \in (0, 1) \) such that

\[
P \left( |U(t) - U(s)| \leq |t - s|^H x \right) \leq C \min \{ 1, x^d \}, \quad \forall s, t \in [0, 1],
\]

then equalities hold in both (1.3) and (1.4). The above method can be applied to a wide class of stochastic processes, including self-similar processes with stationary increments such as stable Lévy processes, fractional Brownian motion and the iterated Brownian motion \((9, 10)\). See \[14, 45, 38\] for further information.

However, the time-changed process \( X = \{ Y(E_t), t \geq 0 \} \) considered in this paper does not satisfy (1.5). In fact, if we consider the process \( X(t) = Y(E_t) \) in Figure 1, where \( Y \) is a Brownian motion in \( \mathbb{R}^d \) and \( E_t \) is the inverse of a \( \beta \)-stable subordinator \( D \) defined by (1.1), then the inner process \( E_t \) remains constant over infinitely many intervals, corresponding to the jumps of the stable subordinator \( D \). Hence the graph of the process \( X \) remains flat over these resting intervals, as evidenced by Figure 1. More precisely, it can be proved by using Proposition 2 in Chapter III of Bertoin \[5\] that, for any \( s < t \), \( P \{ E_s = E_t \} > 0 \). This implies that \( P \{ Y(E_s) - Y(E_t) = 0 \} > 0 \). Hence \( X \) does not satisfy (1.5). As we will see from Propositions 3.1 and 3.6, the actual value of \( \dim_H \text{Gr} X([0, 1]) \) may be strictly smaller than what is suggested by (1.4).

Packing dimension was introduced in 1980’s by Tricot \[42\] as a dual concept to Hausdorff dimension, and has become a useful tool for analyzing fractal sets and sample paths of stochastic processes. It is known that Hausdorff and packing dimensions of a set \( E \) characterize different geometric aspects of \( E \) and many random fractals arising in studies of stochastic processes have different Hausdorff and packing dimensions. See \[40, 43\] and the references therein for more information and \[22, 20, 38\] for recent development. A fractal set \( E \subseteq \mathbb{R}^d \) with the property \( \dim_H E = \dim_p E \) is
usually called a regular fractal. We will see that the range and graph of CTRW limits considered in Section 3 are often regular fractals.

The rest of this paper is organized as follows. In Section 2 we prove under quite general conditions that

\begin{align}
\dim_H X([0,1]) &= \dim_H Y([0,1]) \quad \text{and} \quad \dim_p X([0,1]) = \dim_p Y([0,1]), \quad \text{a.s.,} \\
\dim_H \text{Gr} X([0,1]) &= \max \{1, \dim_H Z([0,1])\} \quad \text{a.s.,} \\
\dim_p \text{Gr} X([0,1]) &= \max \{1, \dim_p Z([0,1])\} \quad \text{a.s.,}
\end{align}

where \( Z = \{Z(x), x \geq 0\} \) is the \( \mathbb{R}^{d+1} \)-valued process defined by \( Z(x) = (D(x), Y(x)) \) (see (2.12) and (2.13) below). These results are applied in Section 3 to the scaling limits of continuous time random walks. First we consider the uncoupled case, in which the iid waiting times \( \{W_n, n \geq 1\} \) are independent of the iid particle jumps \( \{J_n, n \geq 1\} \). Then we treat certain coupled examples, where the jump depends on the previous waiting time. We also consider triangular array CTRW limits, which lead to general inverse subordinators. Finally we examine the case of correlated jumps. In all these cases, the outer process \( Y \) is either a Lévy process or a fractional Brownian motion.

2. General Results

In this section we prove some general results on the Hausdorff and packing dimensions of the range and graph of the time-changed process \( X = \{X(t), t \geq 0\} \) defined by \( X(t) = Y(E_t) \) for \( t \geq 0 \). We assume that \( Y = \{Y(x), x \geq 0\} \) is a stochastic process with values in \( \mathbb{R}^d \) and \( E = \{E_t, t \geq 0\} \) is a process with \( E_0 = 0 \) and nondecreasing continuous sample functions. Both processes \( Y \) and \( E \) are defined on a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \), and they are not necessarily independent. In the following section, the results in this section will be applied to CTRW scaling limits. There, \( E_t \) will be taken as the inverse of a strictly increasing subordinator \( D \), defined by (1.1). For a coupled CTRW, where the jump variable depends on the waiting time, the inner process \( E_t \) and the outer process \( Y(x) \) in the scaling limit are dependent.

First we recall briefly the definitions of Hausdorff and packing dimension. More detailed information together with their applications to stochastic processes and other areas can be found in Falconer [13], Kahane [19], Taylor [40] and Xiao [43]. For any \( \alpha > 0 \), the \( \alpha \)-dimensional Hausdorff measure of \( F \subseteq \mathbb{R}^d \) is defined by

\begin{equation}
\inf_{\varepsilon \to 0} \lim s^\alpha - m(F) = \inf_{\varepsilon \to 0} \left\{ \sum_{i=1}^{\infty} (2r_i)^\alpha : F \subseteq \bigcup_{i=1}^{\infty} B(x_i, r_i), r_i < \varepsilon \right\},
\end{equation}

where \( B(x, r) \) denotes the open ball of radius \( r \) centered at \( x \). The sequence of balls satisfying the two conditions on the right-hand side of (2.1) is called an \( \varepsilon \)-covering of
F. It is well-known that $s^\alpha$-m is a metric outer measure and every Borel set in $\mathbb{R}^d$ is $s^\alpha$-m measurable. The Hausdorff dimension of $F$ is defined by

$$\dim_H F = \inf \{ \alpha > 0 : s^\alpha \text{-} m(F) = 0 \} = \sup \{ \alpha > 0 : s^\alpha \text{-} m(F) = \infty \}.$$ 

It is easily verified that $\dim_H$ satisfies the $\sigma$-stability property: For any $F_n \subseteq \mathbb{R}^d$, one has

$$\dim_H \left( \bigcup_{n=1}^\infty F_n \right) = \sup_{n \geq 1} \dim_H F_n. \tag{2.2}$$

Similarly to (2.1), the $\alpha$-dimensional packing measure of $F \subseteq \mathbb{R}^d$ is defined as

$$s^\alpha \text{-} p(F) = \inf \left\{ \sum_n s^\alpha \text{-} P(F_n) : F \subseteq \bigcup_n F_n \right\},$$

where $s^\alpha \text{-} P$ is the set function on subsets of $\mathbb{R}^d$ defined by

$$s^\alpha \text{-} P(F) = \lim_{\varepsilon \to 0} \sup \left\{ \sum_i (2r_i)^\alpha : B(x_i, r_i) \text{ are disjoint, } x_i \in F, \ r_i < \varepsilon \right\}.$$ 

The packing dimension of $F$ is defined by $\dim_p F = \inf \{ \alpha > 0 : s^\alpha \text{-} p(F) = 0 \}$. It can be verified that $\dim_p$ also satisfies the $\sigma$-stability property analogous to (2.2).

The packing dimension can also be defined through the upper box-counting dimension. For any $\varepsilon > 0$ and any bounded set $F \subseteq \mathbb{R}^d$, let $N(F, \varepsilon)$ be the smallest number of balls of radius $\varepsilon$ needed to cover $F$. The upper box-counting dimension of $F$ is defined as

$$\overline{\dim}_\varepsilon F = \limsup_{\varepsilon \to 0} \frac{\log N(F, \varepsilon)}{-\log \varepsilon}. \tag{2.3}$$

Tricot [42] proved that the packing dimension of $F$ can be obtained from $\overline{\dim}_\varepsilon$ by

$$\dim_p F = \inf \left\{ \sup_n \overline{\dim}_\varepsilon F_n : F \subseteq \bigcup_{n=1}^\infty F_n \right\}, \tag{2.4}$$

see also Falconer [13, p.45]. It is well known that for every (bounded) set $F \subseteq \mathbb{R}^d$,

$$0 \leq \dim_H F \leq \dim_p F \leq \overline{\dim}_\varepsilon F \leq d. \tag{2.5}$$

The following theorem determines the Hausdorff and packing dimension of the range $X([0,1]) = \{X(t) : t \in [0,1]\}$ in terms of the range of $Y$.

**Theorem 2.1.** Let $X = \{X(t), t \geq 0\}$ be the iterated process with values in $\mathbb{R}^d$ defined by $X(t) = Y(E_t)$, where the processes $Y$ and $E$ satisfy the aforementioned conditions. If $E_1 > 0 \ a.s.$ and there exist constants $c_1$ and $c_2$ such that for all constants $0 < a < \infty$

$$\dim_H Y([0, a[) = c_1, \quad \dim_p Y([0, a[) = c_2 \quad a.s., \tag{2.6}$$
then almost surely

\[
\dim_h X([0,1]) = c_1 \quad \text{and} \quad \dim_p X([0,1]) = c_2.
\]

**Proof.** Since the process \( t \mapsto E_t \) is non-decreasing and continuous, the range \( E([0,1]) \) is the random interval \([0, E_1]\). Hence \( X([0,1]) = Y([0, E_1]) \).

It follows from the \( \sigma \)-stability of \( \dim_h \) and \( \dim_p \) that \( \dim_h Y([0, \infty)) = c_1 \) a.s. Hence \( \dim_h X([0, 1]) \leq c_1 \) almost surely. On the other hand, \( \dim_p \) implies

\[
P\left\{ \dim_h Y([0, q]) = c_1, \ \forall q \in \mathbb{Q}_+ \right\} = 1,
\]

where \( \mathbb{Q}_+ \) denotes the set of positive rational numbers. Since \( E_1 > 0 \) almost surely, we see that there is an event \( \Omega' \subset \Omega \) with \( P(\Omega') = 1 \) such that for every \( \omega \in \Omega' \) we have \( E_1(\omega) > 0 \) and \( \dim_h Y([0, q], \omega) = c_1 \) for all \( q \in \mathbb{Q}_+ \). Since for every \( \omega \in \Omega' \) there is a \( q \in \mathbb{Q}_+ \) such that \( 0 < q < E_1(\omega) \), we derive that

\[
\dim_h X([0, 1], \omega) = \dim_h Y([0, E_1(\omega)], \omega) \geq \dim_h Y([0, q], \omega) = c_1.
\]

Combining the upper and lower bounds for \( \dim_h X([0, 1]) \) yields the first equation in \( (2.7) \). The proof of the second equation in \( (2.7) \) is very similar and is omitted. \( \square \)

Applying Theorem 2.1 to the space-time process \( x \mapsto (x, Y(x)) \) with values in \( \mathbb{R}^{d+1} \), one obtains immediately the following corollary.

**Corollary 2.2.** Let \( X = \{X(t), t \geq 0\} \) be the iterated process with values in \( \mathbb{R}^d \) as in Theorem 2.1. If \( E_1 > 0 \) a.s. and there exist constants \( c_3 \) and \( c_4 \) such that for all constants \( 0 < a < \infty \)

\[
\dim_h \text{Gr} Y([0, a]) = c_3 \quad \text{and} \quad \dim_p \text{Gr} Y([0, a]) = c_4, \quad \text{a.s.},
\]

then

\[
\dim_h \{ (E_t, Y(E_t)) : t \in [0, 1] \} = \dim_h \text{Gr} Y([0, 1]), \quad \text{a.s.}
\]

and

\[
\dim_p \{ (E_t, Y(E_t)) : t \in [0, 1] \} = \dim_p \text{Gr} Y([0, 1]), \quad \text{a.s.}
\]

The random set in the left hand side of \( (2.10) \) may be interesting, but it is quite different than the graph of \( X \). In order to determine the Hausdorff and packing dimension of the graph set of \( X \), we will make use of the \( \mathbb{R}^{d+1} \)-valued process \( Z = \{Z(x), x \geq 0\} \) defined on the probability space \((\Omega, \mathcal{F}, P)\) by

\[
Z(x) = (D(x), Y(x)), \quad \forall x \geq 0,
\]

where \( D = \{D(x), x \geq 0\} \) is defined by

\[
D(x) = \inf \{ t > 0 : E_t > x \}.
\]

Since \( t \mapsto E_t \) is nondecreasing and continuous, it can be verified that the function \( x \mapsto D(x) \) is strictly increasing and right continuous, thus can have at most countably many jumps. Moreover, one can verify that \( D(E_t) \geq t \) for all \( t \geq 0 \) and \( E_{D(x)} = x \) for all \( x \geq 0 \).
Theorem 2.3. Let \( X = \{X(t), t \geq 0\} \) be the iterated process with values in \( \mathbb{R}^d \) as in Theorem 2.1 and let \( Z = \{Z(x), x \geq 0\} \) be the \( \mathbb{R}^{d+1} \)-valued process defined by (2.12) and (2.13). If \( E_1 > 0 \) a.s. and there exist constants \( c_5 \) and \( c_6 \) such that for all constants \( 0 < a < \infty \)

\[
(2.14) \quad \dim_{\text{H}} Z([0, a]) = c_5 \quad \text{and} \quad \dim_{\text{p}} Z([0, a]) = c_6 \quad \text{a.s.,}
\]

then

\[
(2.15) \quad \dim_{\text{H}} \text{Gr}_X([0, 1]) = \max \{1, \dim_{\text{H}} Z([0, 1])\}, \quad \text{a.s.}
\]

and

\[
(2.16) \quad \dim_{\text{p}} \text{Gr}_X([0, 1]) = \max \{1, \dim_{\text{p}} Z([0, 1])\}, \quad \text{a.s.}
\]

Proof. We only prove (2.15), and the proof of (2.16) is similar. The sample function \( x \mapsto D(x) \) is a.s. strictly increasing and we can write the unit interval \([0, 1]\) in the state space of \( D \) as

\[
(2.17) \quad [0, 1] = D([0, E_1)) \cup \bigcup_{i=1}^{\infty} I_i,
\]

where for each \( i \geq 1 \), \( I_i \) is a subintervals on which \( E_t \) is a constant. Using \( D \) we can express \( I_i = [D(x_i^-), D(x_i)] \), which is the gap corresponding to the jumping site \( x_i \) of \( D \), except in the case when \( x_i = E_1 \). In the latter case, \( I_i = [D(x_i^-), 1] \).

Notice that \( I_i (i \geq 1) \) are disjoint intervals and

\[
E_t = E_s \quad \text{if and only if} \quad s, t \in I_i \quad \text{for some} \quad i \geq 1.
\]

Thus, over each interval \( I_i \), the graph of \( X \) is a horizontal line segment. More precisely, we can decompose the graph set of \( X \) as

\[
\text{Gr}_X([0, 1]) = \{(t, Y(E_t)) : t \in [0, 1]\}
\]

(2.18)

\[
= \{(t, Y(E_t)) : t \in D([0, E_1))\} \cup \bigcup_{i=1}^{\infty} \{(t, Y(E_t)) : t \in I_i\}.
\]

Hence, by the \( \sigma \)-stability of \( \dim_{\text{H}} \), we have

\[
(2.19) \quad \dim_{\text{H}} \text{Gr}_X([0, 1]) = \max \{1, \dim_{\text{H}} \{(t, Y(E_t)) : t \in D([0, E_1))\}\}.
\]

On the other hand, every \( t \in D([0, E_1)) \) can be written as \( t = D(x) \) for some \( 0 \leq x < E_1 \) and \( E_t = E_{D(x)} = x \), we see that

\[
(2.20) \quad \{(t, Y(E_t)) : t \in D([0, E_1))\} = \{(D(x), Y(x)) : x \in [0, E_1]\}, \quad \text{a.s.}
\]

It follows from (2.14) that

\[
(2.21) \quad \mathbb{P}\{\omega : \dim_{\text{H}} \{(D(x, \omega), Y(x, \omega)) : x \in [0, q]\} = c_5, \quad \forall q \in \mathbb{Q}_+\} = 1.
\]
Combining this with the assumption that $E_1(\omega) > 0$ almost surely, we can find an event $\Omega''_2$ such that $\mathbb{P}(\Omega''_2) = 1$ and for every $\omega \in \Omega''_2$ we derive from (2.21) that
\begin{equation}
\dim_h \{ (D(x, \omega), Y(x, \omega)) : x \in [0, E_1(\omega)) \} = c_5,
\end{equation}

since $q_1 < E_1(\omega_2) < q_2$ for some $q_1, q_2 \in \mathbb{Q}_+$, and $U \subseteq V$ implies $\dim_h(U) \leq \dim_h(V)$. Combining (2.20) and (2.22) yields
\begin{equation}
\dim_h \{ (t, Y(E_t)) : t \in D([0, E_1)) \} = c_5, \quad \text{a.s.}
\end{equation}

Therefore, (2.15) follows from (2.19) and (2.23).

Many self-similar processes $Y$ with stationary increments satisfy (2.6) and (2.14), hence Theorems 2.1 and 2.3 have wide applicability. To apply the above theorems to CTRW scaling limits, we now take $E_t$ to be the inverse of a subordinator $D = \{D(x), x \geq 0\}$. We assume that $D$ has no drift, $D(0) = 0$ and its Laplace transform is given by
\[ \mathbb{E}[e^{-sD(x)}] = e^{-x\psi_D(s)}, \]

where the Laplace exponent
\begin{equation}
\psi_D(s) = \int_0^\infty (1 - e^{-sy})\nu_D(dy).
\end{equation}

We also assume that the Lévy measure $\nu_D$ of $D$ satisfies $\nu_D(0, \infty) = \infty$, so that the sample function $x \mapsto D(x)$ is a.s. strictly increasing.

Let $E_t$ denote the inverse of $D$ defined by (1.1). Since the sample function of $D$ is strictly increasing, we see that the function $t \mapsto E_t$ is almost surely continuous and nondecreasing. Moreover, $\mathbb{P}\{E_1 > 0\} = 1$.

**Corollary 2.4.** Let $X = \{X(t), t \geq 0\}$ be the iterated process with values in $\mathbb{R}^d$ as in Theorem 2.1, where $E_t$ is the inverse (1.1) of a strictly increasing subordinator $D$ with $D(0) = 0$. Then the conclusions of Theorem 2.1, Corollary 2.2, and Theorem 2.3 hold.

### 3. Continuous Time Random Walk Limits

In the following, we compute the Hausdorff and packing dimensions of the range and graph of the sample path of scaling limits of continuous time random walks.

#### 3.1. CTRW with iid jumps: The uncoupled case.

Consider a CTRW whose iid waiting times $\{W_n, n \geq 1\}$ belong to the domain of attraction of the positive $\beta$-stable random variable $D(1)$, and whose iid jumps $\{J_n, n \geq 1\}$ belong to the strict domain of attraction of the $d$-dimensional stable random vector $Y(1)$. We assume that $\{W_n\}$ and $\{J_n\}$ are independent; that is, the CTRW is uncoupled.

It follows from Theorem 4.2 in Meerschaert and Scheffler [29] that the scaling limit of this CTRW is a time-changed process $X(t) = Y(E_t)$, where $E_t$ is the inverse (1.1) of a $\beta$-stable subordinator $D$. Since $D$ is self-similar with index $1/\beta$, its inverse $E$ is
self-similar with index $\beta$. Since $Y$ is independent of $E$, the CTRW scaling limit $X$ is self-similar with index $\frac{\beta}{\alpha}$.

**Proposition 3.1.** Let $X = \{Y(E_t) : t \geq 0\}$, where $Y = \{Y(x) : x \geq 0\}$ is a stable Lévy motion of index $\alpha \in (0, 2]$ with values in $\mathbb{R}^d$ and $E_t$ is the inverse of a stable subordinator of index $0 < \beta < 1$, independent of $Y$. Then

\begin{equation}
\dim H X([0, 1]) = \dim P X([0, 1]) = \min\{d, \alpha\}, \quad \text{a.s.}
\end{equation}

and

\begin{equation}
\dim H \text{Gr} X([0, 1]) = \dim P \text{Gr} X([0, 1])
\end{equation}

\begin{equation}
= \begin{cases}
\max\{1, \alpha\} & \text{if } \alpha \leq d, \\
1 + \beta(1 - \frac{1}{\alpha}) & \text{if } \alpha > d = 1,
\end{cases} \quad \text{a.s.}
\end{equation}

**Proof.** The result (3.1) follows from Theorem 2.1 (or Corollary 2.4) and the known results on the Hausdorff and packing dimension of the range of the stable Lévy process $Y$. The former is due to Blumenthal and Getoor [6, 7], and the latter is due to Pruitt and Taylor [36].

In order to prove (3.2), we first recall from Pruitt and Taylor [35] their result on the Hausdorff dimension of the range of the Lévy process $Z(x) = (D(x), Y(x))$ with independent stable components: for any constant $a > 0$,

\begin{equation}
\dim H Z([0, a]) = \begin{cases}
\beta & \text{if } \alpha \leq \beta, \\
\alpha & \text{if } \beta < \alpha \leq d, \\
1 + \beta(1 - \frac{1}{\alpha}) & \text{if } \alpha > d = 1,
\end{cases} \quad \text{a.s.}
\end{equation}

Theorem 3.2 in Meerschaert and Xiao [32] (see also Khoshnevisan and Xiao [22] for more general results) shows that $\dim P Z([0, a])$ also equals the right hand side of (3.3). Therefore, (3.2) follows from the above and Theorem 2.3. □

If the CTRW jumps $(J_n)$ have finite second moments, then the limiting process is $X(t) = B(E_t)$, where $B$ is a Brownian motion, and Proposition 3.1 with $\alpha = 2$ gives

\begin{equation}
\dim H \text{Gr} X([0, 1]) = \dim P \text{Gr} X([0, 1]) = \begin{cases}
1 + \frac{\beta}{2} & \text{if } d = 1, \\
\frac{2}{d} & \text{if } d \geq 2, \quad \text{a.s.}
\end{cases}
\end{equation}

The Hausdorff dimension of the graph of a stable Lévy process $Y$ in $\mathbb{R}^d$ was determined by Blumenthal and Getoor [8] when $d = 1$ and $Y$ is symmetric, by Jain and Pruitt [17] when $Y$ is transient (i.e. $d > \alpha$) and by Pruitt and Taylor [35] in general. The packing dimension of the graph of $Y$ was determined by Rezakhanlou and Taylor [37]. Combining their results with Corollary 2.2, we obtain

\begin{equation}
\dim H \{(E_t, Y(E_t)) : t \in [0, 1]\} = \dim P \{(E_t, Y(E_t)) : t \in [0, 1]\}
\end{equation}

\begin{equation}
= \begin{cases}
\max\{1, \alpha\} & \text{if } \alpha \leq d, \\
2 - \frac{1}{\alpha} & \text{if } \alpha > d = 1.
\end{cases} \quad \text{a.s.}
\end{equation}
Clearly, this is different from (3.2). Moreover, we notice that the results (3.1) and (3.5) do not depend on $\beta$, because the set $\{E_t(\omega) : t \in [0, 1]\}$ is a.s. a closed interval, so the range dimension is the same after the time change.

3.2. CTRW with iid jumps: The coupled case. In some applications, it is natural to consider a coupled CTRW where $\{(J_n, W_n), n \geq 1\}$ are iid, but the jump $J_n$ depends on the preceding waiting time $W_n$. We can also extend the results of the last section to certain coupled CTRW limits. In the coupled case, the CTRW $S(N_t)$ has scaling limit $Y(E_t)$ and the so-called oracle CTRW $S(N_t + 1)$ has scaling limit $Y(E_t)$, see [16, 18]. If $Y, D$ are independent, they have a.s. no simultaneous jumps, and the two limit processes are the same. The proof of Theorem 2.3 extends immediately to the process $Y(E_t)$ in this case, with the same dimension results. This is because the graphs of $Z(x) = (D(x), Y(x))$ and $Z'(x) = (D(x), Y(x-))$ have the same Hausdorff and packing dimension, as they differ by at most a countable number of discrete points. In the following, we discuss examples for $Y(E_t)$, with the understanding that the same dimension results hold for $Y(E_t-)$.

The simplest case is $W_n = J_n$, in which case $X(t) = D(E_t)$. This process is self-similar with index 1, see for example Becker-Kern et al. [3]. It follows from Theorems 2.1, 2.3 and the fact that for any constant $a > 0$,

$$\dim_h D([0, a]) = \dim_h \{(D(x), D(x)) : x \in [0, a]\} = \beta, \text{ a.s.}$$

that

$$\dim_h X([0, 1]) = \dim_p X([0, 1]) = \beta, \text{ a.s.} \quad (3.6)$$

and

$$\dim_h \text{Gr} X([0, 1]) = \dim_p \text{Gr} X([0, 1]) = 1, \text{ a.s.} \quad (3.7)$$

We should also mention that by applying the “uniform” Hausdorff and packing dimension results for the $\beta$-stable subordinator $D$ (see Perkins and Taylor [33]), which states that almost surely

$$\dim_h D(F) = \beta \dim_h F \quad \text{and} \quad \dim_p D(F) = \beta \dim_p F \quad \text{for all Borel sets } F \subseteq \mathbb{R},$$

we obtain (3.6) directly by choosing $F = [0, E_1]$.

Shlesinger et al. [39] consider a CTRW where the waiting times $W_n \geq 0$ are iid with the $\beta$-stable random variable $D$ and $\mathbb{E}(e^{-sD}) = e^{-s^\beta}$ and, conditional on $W_n = t$, the jump $J_n$ is normal with mean zero and variance $2t$. Then $J_n$ is symmetric stable with index $\alpha = 2\beta$. This model was applied to stock market prices by Meerschaert and Scalas [27]. Becker-Kern et al. [3] show that the CTRW limit is $X(t) = Y(E_t)$ ($t \geq 0$), where $Y$ is a real-valued stable Lévy process with index $\alpha = 2\beta$ and $E_t$ is the inverse of a $\beta$-stable subordinator. Then $X(t)$ is self-similar with index 1/2, the same as Brownian motion. However, the Hausdorff dimensions of the range and graph of $X$ are completely different than those for Brownian motion.
Note that here $E_t$ is not independent of $Y(t)$. Theorem 2.1 gives that $\dim_{\mu} X([0, 1]) = \min\{1, 2\beta\}$ a.s. To determine the Hausdorff dimension of the graph of $X(t)$, we first verify that the Fourier-Laplace transform of $(D(1), Y(1))$ is
\[
\mathbb{E}\left(e^{i\xi Y(1) - \eta D(1)}\right) = \mathbb{E}\left[e^{-\eta D(1)} \mathbb{E}(e^{i\xi Y(1)} | D(1))\right] = \mathbb{E}\left(e^{-(\eta + \xi^2)D(1)}\right) = e^{-(\eta + \xi^2)^\beta}.
\]
It follows that the Lévy process $Z(x) = (D(x), Y(x))$ is operator stable \[28\] with the unique exponent
\[
C = \begin{pmatrix}
\beta^{-1} & 0 \\
0 & (2\beta)^{-1}
\end{pmatrix},
\]
which has eigenvalues $\beta^{-1}$ and $(2\beta)^{-1}$.

By applying Theorem 3.2 from Meerschaert and Xiao \[32\], we derive that for any $a > 0$,
\[
\dim_{\mu} Z([0, a]) = \dim_{\nu} Z([0, a]) = \begin{cases}
2\beta & \text{if } 2\beta \leq 1, \\
\frac{1}{2} + \beta & \text{if } 2\beta > 1,
\end{cases}
\]
a.s.
Consequently, we use Theorem 2.3 to derive
\[
\dim_{\mu} \text{Gr} X([0, 1]) = \dim_{\nu} \text{Gr} X([0, 1]) = \max\{1, \beta + \frac{1}{2}\}, \quad \text{a.s.}
\]
which is quite different from the corresponding result (3.4) in the uncoupled case.

3.3. **CTRW with iid jumps: Triangular array limits.** Proposition 3.1 and (3.10) rely on the Hausdorff and packing dimension results for sample functions of stable or, more generally, operator stable Lévy processes. The Hausdorff dimensions and potential theoretic properties of general Lévy processes have been studied by several authors \[34, 21, 23\] and the packing dimension results have been proved by Khoshnevisan and Xiao \[22\] and Khoshnevisan, Schilling and Xiao \[20\]. These results are useful for studying fractal properties of the CTRW limits under more general settings such as triangular array schemes.

In particular, for any Lévy process $Z = \{Z(x), x \geq 0\}$ with values in $\mathbb{R}^p$ and characteristic exponent $\Phi$ (i.e., $\mathbb{E}(e^{i\xi Z(x)}) = e^{-x\Phi(\xi)}$), Corollary 1.8 in \[23\] shows that for any $a > 0$,
\[
\dim_{\mu} Z([0, a]) = \sup \left\{ \gamma < p : \int_{\{\xi \in \mathbb{R}^p : \|\xi\| \geq 1\}} \text{Re} \left( \frac{1}{1 + \Phi(\xi)} \right) \frac{d\xi}{\|\xi\|^\gamma} < \infty \right\}, \quad \text{a.s.}
\]
where $\| \cdot \|$ is the Euclidean norm on $\mathbb{R}^p$. On the other hand, Theorem 1.1 in Khoshnevisan and Xiao \[22\] shows that for any $a > 0$,
\[
\dim_{\nu} Z([0, a]) = \sup \left\{ \gamma \geq 0 : \liminf_{r \to 0^+} \frac{W(r)}{r^\gamma} = 0 \right\}, \quad \text{a.s.}
\]
where, for all $r > 0$, $W(r)$ is defined by

$$W(r) = \int_{\mathbb{R}^d} \text{Re} \left( \frac{1}{1 + \Phi(\xi/r)} \right) \prod_{j=1}^{p} \frac{1}{1 + \xi_j^2} d\xi.$$  

(More precisely, Theorem 1.1 in [22] is proved for $a = 1$, but its proof works for arbitrary $a > 0$. Another way to get (3.12) from Theorem 1.1 in [22] is to use the stationarity of increments of $Z$.)

By combining (3.11) and (3.12) with Theorems 2.1 and 2.3 we extend the results in the previous sections to more general time-changed processes.

**Proposition 3.2.** Let $X = \{Y(E_t), t \geq 0\}$, where $Y = \{Y(x) : x \geq 0\}$ is a Lévy process with values in $\mathbb{R}^d$ and characteristic exponent $\psi$ and let $E_t$ be the inverse of a subordinator $D = \{D(x), x \geq 0\}$ with characteristic exponent $\sigma$. If $Z = \{(D(x), Y(x)), x \geq 0\}$ is a Lévy process in $\mathbb{R}^{1+d}$ and its characteristic exponent $\Phi$ satisfies

$$K^{-1} \text{Re} \left( \frac{1}{1 + \sigma(\eta) + \psi(\xi)} \right) \leq \text{Re} \left( \frac{1}{1 + \Phi(\eta, \xi)} \right) \leq K \text{Re} \left( \frac{1}{1 + \sigma(\eta) + \psi(\xi)} \right)$$

for all $(\eta, \xi) \in \mathbb{R}^{1+d}$ with $|\eta| + \|\xi\|$ large, where $K \geq 1$ is a constant. Then almost surely,

$$\dim_H X([0,1]) = \sup \left\{ \gamma < d : \int_{\mathbb{R}^d : \|\xi\| \geq 1} \text{Re} \left( \frac{1}{1 + \psi(\xi)} \right) \frac{d\xi}{\|\xi\|^\gamma} < \infty \right\}$$

and $\dim_H \text{Gr} X([0,1]) = \max\{1, \chi\}$ almost surely, where

$$\chi = \sup \left\{ \gamma < 1 + d : \int_{\{\|\eta\| + \|\xi\| \geq 1\}} \text{Re} \left( \frac{1}{1 + \sigma(\eta) + \psi(\xi)} \right) \frac{d\eta d\xi}{(\|\eta\| + \|\xi\|)^\gamma} < \infty \right\}.$$

The packing dimensions of $X([0,1])$ and $\text{Gr} X([0,1])$ are given as follows, which may be different from the Hausdorff dimensions given in Proposition 3.2.

**Proposition 3.3.** Let $X = \{Y(E_t), t \geq 0\}$ be the same as in Proposition 3.2. If $Z = \{(D(x), Y(x)), x \geq 0\}$ is a Lévy process in $\mathbb{R}^{1+d}$ and its characteristic exponent $\Phi$ satisfies (3.13), then

$$\dim_p X([0,1]) = \sup \left\{ \gamma \geq 0 : \liminf_{r \to 0^+} \frac{W(r)}{r^\gamma} = 0 \right\}, \quad \text{a.s.,}$$

where $W(r)$ is defined by

$$W(r) = \int_{\mathbb{R}^d} \text{Re} \left( \frac{1}{1 + \psi(\xi/r)} \right) \prod_{j=1}^{d} \frac{1}{1 + \xi_j^2} d\xi,$$
and \( \text{dim}_p \text{Gr}X([0,1]) = \max\{1, \chi'\} \) almost surely, where

\[
\chi' = \sup \left\{ \gamma \geq 0 : \lim_{r \to 0^+} \frac{\widetilde{W}(r)}{r^{\gamma}} = 0 \right\}
\]

and where

\[
\widetilde{W}(r) = \int_{\mathbb{R}^{1+d}} \text{Re} \left( \frac{1}{1 + \sigma(\eta/r) + \psi(\xi/r)} \right) \frac{1}{1 + \eta^2} \prod_{j=1}^d \frac{1}{1 + \xi_j^2} d\eta d\xi.
\]

Next we consider a generalized CTRW limit, as in [31], obtained by using a triangular array scheme. This limit can be applied as a stochastic model for ultraslow diffusion (cf. [30, 11]).

At each scale \( c > 0 \) we are given iid waiting times (\( W_n^c \)) and iid jumps (\( J_n^c \)). Assume the waiting times and jumps form triangular arrays whose row sums converge in distribution. More specifically, let \( S^c(n) = J_1^c + \cdots + J_n^c \) and \( T^c(n) = W_1^c + \cdots + W_n^c \), we require that \( S^c(cu) \Rightarrow Y(t) \) and \( T^c(cu) \Rightarrow D(t) \) as \( c \to \infty \), where the limits \( Y(t) \) and \( D(t) \) are independent Lévy processes. Letting \( N_t^c = \max\{n \geq 0 : T^c(n) \leq t\} \), the CTRW scaling limit \( S^c(N_t^c) \Rightarrow Y(E_t) \) [31, Theorem 2.1].

A power law mixture model for waiting times was proposed in [30]: Take a n iid sequence of random variables \( \{B_i\} \) with \( 0 < B_i < 1 \) and assume \( P\{W_i^c > u \mid B_i = \beta\} = c^{-1}u^{-\beta} \) for \( u \geq c^{-1/\beta} \), so that the waiting times are power laws conditional on the mixing variables. The waiting time process \( T^c(cu) \Rightarrow D(t) \), which is a subordinator with Laplace transform \( E[e^{-sD(t)}] = e^{-t\psi_D(s)} \), where (2.24) holds. The Lévy measure of \( D \) is given by

\[
(3.14) \quad \nu_D(t, \infty) = \int_0^1 t^{-\beta} \mu(d\beta),
\]

where \( \mu \) is the distribution of the mixing variable [30, Theorem 3.4 and Remark 5.1].

A computation [30, Eq. (3.18)] using \( \int_0^\infty (1 - e^{-st})\beta t^{-\beta - 1} dt = \Gamma(1 - \beta)s^\beta \) shows that

\[
(3.15) \quad \psi_D(s) = \int_0^1 s^\beta \Gamma(1 - \beta)\mu(d\beta).
\]

Then \( c^{-1}N_t^c \Rightarrow E_t \) the inverse subordinator [30, Theorem 3.10].

Now we take \( \mu(d\beta) = \sum_{k=1}^n \delta_{\beta_k}^d \Gamma(1 - \beta_k)\delta_{\beta_k}(d\beta) \), where \( 0 < \beta_1 < \beta_2 < \cdots < \beta_n < 1 \) are constants and \( \delta_a \) is the unit mass at \( a \). In this case, the subordinator is \( D(t) = \sum_{k=1}^n \delta_k D_k(t) \), which is a mixture of independent \( \beta_k \)-stable subordinators \( D_k(t) \) \( (k = 1, \ldots, n) \). See Chechkin et al. [11] for some applications of such CTRW and its scaling limit \( X(t) = Y(E_t) \).

In order to apply Propositions 3.2 and 3.3 to establish Hausdorff and packing dimension results for the above time-changed process \( X \), we will make use of the following technical result.
Lemma 3.4. Let \( D(x) = \sum_{k=1}^{n} d_k D_k(x) \), where \( d_k > 0 \) are constants and \( D_k(x) \) are independent stable subordinators of index \( \beta_k \) and \( 0 < \beta_1 < \beta_2 < \cdots < \beta_n < 1 \). Let \( Y = \{ Y(x), x \geq 0 \} \) be a strictly stable Lévy motion of index \( \alpha \in (0, 2] \) with values in \( \mathbb{R}^d \). We assume \( D \) and \( Y \) are independent and let \( \Phi \) be the characteristic exponent of the Lévy process \( Z(x) = (D(x), Y(x)) \). Then for all \((\eta, \xi) \in \mathbb{R}^{1+d}\) that satisfies \(|\eta| + \|\xi\| > 1\)

\[
\begin{align*}
K^{-1} \frac{K^{-1}}{|\eta|^{\beta_n} + \|\xi\|^\alpha} \leq \text{Re} \left( \frac{1}{1 + \Phi(\eta, \xi)} \right) \leq \frac{K}{|\eta|^{\beta_n} + \|\xi\|^\alpha},
\end{align*}
\]

where \( K \geq 1 \) is a constant which may depend on \( n, \alpha, \beta_k, d_k \) for \( k = 1, 2, \ldots, n \).

**Proof.** For simplicity, we assume that the characteristic exponent of \( Y \) is \( \psi(\xi) = \|\xi\|^\alpha \). Then the Lévy process \( Z(x) = (D(x), Y(x)) \) has characteristic exponent

\[
\Phi(\eta, \xi) = \sum_{k=1}^{n} (-id_k \eta)^{\beta_k} + \|\xi\|^\alpha
\]

\[
= \sum_{k=1}^{n} |d_k \eta|^{\beta_k} [\cos(\pi \beta_k/2) - i \sin(\pi \beta_k/2)] + \|\xi\|^\alpha
\]

\[=: f(\eta, \xi) - ig(\eta).\]

Since \( \beta_k \in (0, 1) \), we have \( f(\eta, \xi) \geq 0 \) for all \( \eta, \xi \in \mathbb{R}^{1+d} \). Moreover, \( 0 \leq g(\eta) \leq K f(\eta, \xi) \) for some constant \( K > 0 \). Hence

\[
\frac{1}{(1 + K^2)(1 + f(\eta, \xi))} \leq \text{Re} \left( \frac{1}{1 + \Phi(\eta, \xi)} \right) \leq \frac{1}{1 + f(\eta, \xi)}.
\]

From here it is elementary to verify (3.16). \( \square \)

By using Lemma 3.4, Propositions 3.2 and 3.3, we derive the following proposition. Since the proof is similar to that of Proposition 4.1 in [32] (see also Proposition 7.7 in [21]), we omit the details.

**Proposition 3.5.** Let \( X = \{ Y(E_t), t \geq 0 \} \), where \( Y = \{ Y(x), x \geq 0 \} \) is a strictly stable Lévy motion of index \( \alpha \in (0, 2] \) with values in \( \mathbb{R}^d \) and \( E_t \) is the inverse of a subordinator \( D(t) = \sum_{j=1}^{n} d_j D_j(t) \), where \( d_k > 0 \) and \( D_k(t) \) are independent stable subordinators of index \( \beta_k \) and \( 0 < \beta_1 < \beta_2 < \cdots < \beta_n < 1 \). Suppose also that \( E \) is independent of \( Y \). Then

\[
\dim_{\text{H}} X([0, 1]) = \dim_{\text{p}} X([0, 1]) = \min\{d, \alpha\}, \quad \text{a.s.}
\]

and

\[
\dim_{\text{H}} \text{Gr} X([0, 1]) = \dim_{\text{p}} \text{Gr} X([0, 1])
\]

\[
= \begin{cases} 
\max\{1, \alpha\} & \text{if } \alpha \leq d, \\
1 + \beta_\alpha (1 - \frac{1}{\alpha}) & \text{if } \alpha > d = 1, \end{cases}
\quad \text{a.s.}
\]
3.4. **CTRW with correlated jumps.** Now consider an uncoupled CTRW whose jumps \( \{J_n\} \) form a correlated sequence of random variables, and whose waiting times \( \{W_n\} \) are iid and belong to the domain of attraction of a positive \( \beta \)-stable random variable \( D(1) \).

We further assume that \( \{J_n\} \) and \( \{W_n\} \) are independent. In this case, Meerschaert, et al. [26] show that, under certain conditions on the correlation structure of \( \{J_n\} \), the CTRW scaling limit is the \((H\beta)\)-self-similar process \( X = \{Y(E_t) : t \geq 0\} \), where \( Y \) is a fractional Brownian motion with index \( H \in (0, 1) \), and \( E_t \) is the inverse of a \( \beta \)-stable subordinator \( D \) which is independent of \( Y \).

The following proposition determines the Hausdorff and packing dimension of the sample path of \( X \).

**Proposition 3.6.** Let \( X = \{Y(E_t) : t \geq 0\} \), where \( Y \) is a fractional Brownian motion with values in \( \mathbb{R}^d \) of index \( H \in (0, 1) \) and \( E_t \) is the inverse of a \( \beta \)-stable subordinator \( D \) which is independent of \( Y \). Then

\[
\dim_H X([0,1]) = \dim_P X([0,1]) = \min \left\{ \frac{1}{H}, \frac{1}{\beta} + (1 - H\beta)d \right\}, \quad \text{a.s.}
\]

and

\[
\dim_H Gr X([0,1]) = \dim_P Gr X([0,1])
\]

\[
= \left\{ \begin{array}{ll}
\frac{1}{H} & \text{if } 1 \leq Hd, \\
\frac{1}{\beta} + (1 - H\beta)d & \text{if } 1 > Hd,
\end{array} \right. \quad \text{a.s.}
\]

**Proof.** Eq. (3.20) follows from Theorem 2.1 and the well known result on Hausdorff and packing dimension of the range of fractional Brownian motion (see, e.g., Chapter 18 of [19]). In order to prove (3.21), by Theorem 2.3 it is sufficient to prove that for the \( \mathbb{R}^{d+1} \)-valued process \( Z = \{Z(x), x \geq 0\} \) defined by \( Z(x) = (D(x), Y(x)) \), \( x \geq 0 \) and for any constant \( a > 0 \), we have

\[
\dim_H Z([0,a]) = \dim_P Z([0,a])
\]

\[
= \left\{ \begin{array}{ll}
\frac{1}{H} & \text{if } 1 \leq Hd, \\
\frac{1}{\beta} + (1 - H\beta)d & \text{if } 1 > Hd,
\end{array} \right. \quad \text{a.s.}
\]

Thanks to (2.5), we can divide the proof of (3.22) into proving the upper bound for \( \dim_P Z([0,a]) \) and the lower bound for \( \dim_H Z([0,a]) \) separately. These are given as Lemmas 3.7 and 3.9 below.

**Lemma 3.7.** Let the assumptions of Proposition 3.6 hold and let \( a > 0 \) be a constant. Then

\[
\dim_P Z([0,a]) \leq \left\{ \begin{array}{ll}
\frac{1}{H} & \text{if } 1 \leq Hd, \\
\frac{1}{\beta} + (1 - H\beta)d & \text{if } 1 > Hd,
\end{array} \right. \quad \text{a.s.}
\]

In order to prove Lemma 3.4, we will make use of the fact that for every \( \varepsilon > 0 \) the function \( Y(x) \) (\( 0 \leq x \leq a \)) satisfies the uniform Hölder condition of order \( H - \varepsilon \) and
the following Lemma 3.8 which is an immediate consequence of Lemma 3.2 in Liu and Xiao [24]. It can also be derived from Lemma 6.1 in Pruitt and Taylor [35].

Let $c_7 > 0$ be a fixed constant. A collection $\Lambda(b)$ of intervals of length $b$ in $\mathbb{R}$ is called $c_7$-nested if no interval of length $b$ in $\mathbb{R}$ can intersect more than $c_7$ intervals of $\Lambda(b)$. Note that for each integer $n \geq 1$, the collection of dyadic intervals $I_{n,j} = [j/2^n, (j + 1)/2^n]$ is $c_7$-nested with $c_7 = 3$.

**Lemma 3.8.** Let $\{D(x), x \geq 0\}$ be a $\beta$-stable subordinator and let $\Lambda(b)$ be a $c_7$-nested family. Denote by $M_u(b, s)$ the number of intervals in $\Lambda(b)$ which intersect $D([u, u + s])$. Then there exists a positive constant $c_8$ such that for all $u \geq 0$ and all $0 < b^\beta \leq s$,

$$
(3.24) \quad \mathbb{E}(M_u(b, s)) \leq c_8 sb^{-\beta}.
$$

If one takes $b = s \leq 1$, then we have

$$
(3.25) \quad \mathbb{E}(M_u(a, s)) \leq c_8.
$$

Now we are ready to prove Lemma 3.7.

**Proof of Lemma 3.7.** The proof is based on a moment argument. We divide the interval $[0, a]$ into $([a] + 1)2^n$ dyadic intervals $I_{n,j}$ of length $2^{-n}$.

First we construct a covering of the range $Z([0, a])$ by using balls in $\mathbb{R}^{d+1}$ of radius $2^{-Hn}$ as follows. Define $t_{n,j} = j/2^n$ so that for each $I_{n,j} = [t_{n,j}, t_{n,j} + 2^{-n}]$, the image $Y(I_{n,j})$ is contained in a ball in $\mathbb{R}^{d}$ of radius $\sup_{s \in I_{n,j}}\|Y(s) - Y(t_{n,j})\|$ and can be covered by at most

$$
(3.26) \quad N_{n,j} = c_9 \left( \frac{\sup_{s \in I_{n,j}}\|Y(s) - Y(t_{n,j})\|}{2^{-Hn}} \right)^d
$$

balls of radius $2^{-Hn}$. By the self-similarity and stationarity of increments of $Y$, we have

$$
(3.27) \quad \mathbb{E}(N_{n,j}) = c_9 2^{Hdn} \mathbb{E}\left( \left( \sup_{s \in I_{n,j}}\|Y(s) - Y(t_{n,j})\| \right)^d \right) = c_{10} < \infty,
$$

where the last inequality follows from the well known tail probability for the supremum of Gaussian processes (e.g., Fernique’s inequality).

In order to get a covering for $D(I_{n,i})$, let $\Gamma(2^{-n})$ be the collection of dyadic intervals of order $n$ in $\mathbb{R}_+$. Let $M_{n,j}$ be the number of dyadic intervals in $\Gamma(2^{-n})$ which intersect $D(I_{n,j})$. Applying (3.26) in Lemma 3.8 with $b_n = s_n = 2^{-n}$, we obtain that

$$
(3.28) \quad \mathbb{E}(M_{n,j}) \leq c_8, \quad \forall \quad 1 \leq j \leq ([a] + 1)2^n.
$$

Since $2^{-n} < 2^{-Hn}$, we see that $Z(I_{n,j}) = \{(D(x), Y(x)) : x \in I_{n,j}\}$ can be covered by at most $M_{n,j}N_{n,j}$ balls in $\mathbb{R}^{d+1}$ of radius $2^{-Hn}$. Denote by $N(Z([0, a]), 2^{-Hn})$ the
smallest number of balls in \( \mathbb{R}^{d+1} \) of radius \( 2^{-Hn} \) that cover \( Z([0, a]) \), then

\[
N(Z([0, a]), 2^{-Hn}) \leq \sum_{j=1}^{(\lfloor a \rfloor + 1)2^n} M_{n,j} N_{n,j}.
\]

It follows from (3.27), (3.28) and the independence of \( Y \) and \( D \) that

\[
\mathbb{E}\left[ N(Z([0, a]), 2^{-Hn}) \right] \leq (\lfloor a \rfloor + 1)c_8c_{10} 2^n.
\]

Hence, for any \( \varepsilon > 0 \),

\[
P\left\{ N(Z([0, a]), 2^{-Hn}) \geq (\lfloor a \rfloor + 1)c_8c_{10} 2^{n(1+\varepsilon)} \right\} \leq 2^{-n\varepsilon}.
\]

It follows from the Borel-Cantelli lemma that almost surely

\[
N(Z([0, a]), 2^{-Hn}) < (\lfloor a \rfloor + 1)c_8c_{10} 2^{n(1+\varepsilon)}
\]

for all \( n \) large enough. This and (2.3) imply that \( \dim_p Z([0, a]) \leq 1/H \) a.s. Since \( \varepsilon > 0 \) is arbitrary, we obtain from the above and (2.5) that \( \dim_p Z([0, a]) \leq 1/H \) almost surely.

Next we construct a covering for the range \( Z([0, a]) \) by using balls in \( \mathbb{R}^{d+1} \) of radius \( 2^{-n/\beta} \). Let \( \Gamma(2^{-n/\beta}) \) be the collection of intervals in \( \mathbb{R}^+ \) of the form \( I'_{n,k} = [k2^{-n/\beta}, (k+1)2^{-n/\beta}] \), where \( k \) is an integer. Then the class \( \Gamma(2^{-n/\beta}) \) is 3-nested. Let \( M'_{n,j} \) be the number of intervals in \( \Gamma(2^{-n/\beta}) \) that intersect \( D(I'_{n,j}) \). By Lemma 3.8 with \( b_n = 2^{-n/\beta} \) and \( s_n = 2^{-n} \), we derive \( \mathbb{E}(M'_{n,j}) \leq c_8 \). Thus \( D(I'_{n,j}) \) can almost surely be covered by \( M'_{n,j} \) intervals of length \( 2^{-n/\beta} \) from \( \Gamma(2^{-n/\beta}) \).

On the other hand, the image \( Y(I'_{n,j}) \) can be covered by at most

\[
N'_{n,j} = c_9 \left( \sup_{s \in I'_{n,j}} \|Y(s) - Y(t_{n,j})\| \right)^d
\]

balls of radius \( 2^{-n/\beta} \), where \( t_{n,j} = j/2^n \), and then similar to (3.27) we derive (3.29)

\[
\mathbb{E}(N'_{n,j}) = c_{10} 2^n(\frac{1}{\beta} - H)d.
\]

Denote by \( N(Z([0, a]), 2^{-n/\beta}) \) the smallest number of balls in \( \mathbb{R}^{d+1} \) of radius \( 2^{-n/\beta} \) that cover \( Z([0, a]) \), then

\[
N(Z([0, a]), 2^{-n/\beta}) \leq \sum_{j=1}^{(\lfloor a \rfloor + 1)2^n} M'_{n,j} N'_{n,j}.
\]

By (3.29) and the independence of \( Y \) and \( D \) we have

\[
\mathbb{E}\left[ N(Z([0, a]), 2^{-n/\beta}) \right] \leq (\lfloor a \rfloor + 1)c_8c_{10} 2^{n(1+(\frac{1}{\beta} - H)d/2)}.
\]

Hence, for any \( \varepsilon > 0 \), the Borel-Cantelli Lemma implies that a.s.

\[
N(Z([0, a]), 2^{-n/\beta}) < (\lfloor a \rfloor + 1)c_8c_{10} 2^{n(1+(\frac{1}{\beta} - H)d + \varepsilon)}
\]
for all \( n \) large enough. This and (2.3) imply that \( \dim_{H}Z([0,a]) \leq \beta + (1 - \beta H)d + \beta \varepsilon \) almost surely which, in turn, implies \( \dim_{p}Z([0,a]) \leq \beta + (1 - \beta H)d \) a.s.

Combining the above we have

\[
\dim_{p}Z([0,a]) \leq \min \left\{ \frac{1}{H}, \beta + (1 - \beta H)d \right\} \quad \text{a.s.}
\]

This proves (3.23).

\[ \square \]

**Lemma 3.9.** Under the assumptions of Proposition 3.6, we have

(3.30) \( \dim_{n}Z([0,1]) \geq \begin{cases} \frac{1}{H} & \text{if } 1 \leq Hd, \\ \beta + (1 - H\beta)d & \text{if } 1 > Hd, \end{cases} \) a.s.

**Proof.** Since the projection of \( Z([0,1]) \) into \( \mathbb{R}^{d} \) is \( Y([0,1]) \) and \( \dim_{n}Y([0,1]) = \frac{1}{H} \) a.s. when \( 1 \leq Hd \). This implies the first inequality in (3.30).

To prove the inequality in (3.30) for the case \( 1 > Hd \), by Frostman’s theorem (cf. [19, p.133]) along with the inequality

\[
\|Z(x) - Z(y)\| \geq \frac{1}{2} \left[ |D(x) - D(y)| + \|Y(x) - Y(y)\| \right],
\]

it is sufficient to prove that for every constant \( \gamma \in (0, \beta + (1 - H\beta)d) \), we have

(3.31) \[ \mathbb{E} \int_{0}^{a} \int_{0}^{a} \frac{dx \ dy}{\|D(x) - D(y)| + \|Y(x) - Y(y)\|^{\gamma}} < \infty. \]

Since \( 1 > Hd \), we have \( \beta + (1 - H\beta)d > d \). We only need to verify (3.31) for every \( \gamma \in (d, \beta + (1 - H\beta)d) \).

For this purpose, we will make use of the following easily verifiable fact (see, e.g., Kahane [19, p.279]): If \( \Xi \) is a standard normal vector in \( \mathbb{R}^{d} \), then there is a finite constant \( c_{11} > 0 \) such at for any constants \( \gamma > d \) and \( \rho \geq 0 \),

\[
\mathbb{E} \left[ \frac{1}{(\rho + \|\Xi\|)^{\gamma}} \right] \leq c_{11} \rho^{-(\gamma - d)}. \]

Fix \( x, y \in [0,a] \) such that \( x \neq y \). We use \( \mathbb{E}_{1} \) to denote the conditional expectation given the subordinator \( D \), apply the above fact with \( \rho = |D(x) - D(y)||x - y|^{-H} \) and
use the self-similarity of $D$ to derive

$$
\mathbb{E}\left(\frac{1}{[|D(x) - D(y)| + \|Y(x) - Y(y)\|]^{\gamma}}\right)
= |x - y|^{-H\gamma}\mathbb{E}_1\left(\frac{1}{(\rho + \|\Xi\|)^{\gamma}}\right)
\leq c_{11}|x - y|^{-H\gamma}\mathbb{E}\left(\frac{|x - y|^{H(\gamma - d)}}{|D(x) - D(y)|^{\gamma - d}}\right)
= c_{12}|x - y|^{Hd + (\gamma - d)/\beta},
$$

(3.32)

where the last equality follows from the $1/\beta$-self-similarity of $D$ and the constant $c_{12} = c_{11}\mathbb{E}(D(1)^{-(\gamma - d)})$. Recall from Hawkes [15, Lemma 1] that, as $r \to 0+$,

$$
\mathbb{P}(D(1) \leq r) \sim c_{13}^{-\beta/(2(1-\beta))}\exp\left(- (1 - \beta)^{\beta/(1-\beta)} r^{-\beta/(1-\beta)}\right),
$$

where $c_{13} = [2\pi(1-\beta)^{\beta/(2(1-\beta))}]^{-1/2}$. We verify easily $c_{12} < \infty$.

It follows from Fubini’s theorem and (3.32) that

$$
\mathbb{E}\int_0^a \int_0^a \frac{dx\,dy}{[|D(x) - D(y)| + \|Y(x) - Y(y)\|]^{\gamma}}
\leq c_{12}\int_0^a \int_0^a \frac{dx\,dy}{|x - y|^{Hd + (\gamma - d)/\beta}} < \infty,
$$

(3.33)

the last integral is convergent because $Hd + (\gamma - d)/\beta < 1$. This proves (3.31) and thus the lemma. \[\square\]

Remark 3.10. Other iterated processes can arise as scaling limits of CTRW with dependent and/or heavy-tailed jumps or waiting times. For example, the process $Y$ can be taken as a linear fractional stable motion, see [26]. Our main theorems in Section 2 are applicable to these self-similar processes too. However, the problems for determining the Hausdorff dimensions of the range and graph sets of the processes $Y$ and $Z(x) = (D(x), Y(x))$ have not been satisfactorily solved, see [38, 45] for partial solutions.

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