GALOIS GROUPS AND CANTOR ACTIONS

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Abstract. In this paper, we study the actions of profinite groups on Cantor sets which arise from representations of Galois groups of certain fields of rational functions. Such representations are associated to polynomials, and they are called profinite iterated monodromy groups. We are interested in a topological invariant of such actions called the asymptotic discriminant. In particular, we give a complete classification by whether the asymptotic discriminant is stable or wild in the case when the polynomial generating the representation is quadratic. We also study different ways in which a wild asymptotic discriminant can arise.

1. Introduction

Let $X$ be a Cantor set, that is, a compact totally disconnected metrizable space with no isolated points. In this paper, Cantor sets arise as spaces of paths in spherically homogeneous rooted trees, see Example 2.1 for details.

Let $G$ be a countably generated discrete group, and suppose $G$ acts on $X$ via the homomorphism $\Phi : G \to \text{Homeo}(X)$. We also denote the action by $(X, G, \Phi)$, and write $g \cdot x$ for $\Phi(g)(x)$.

Let $d_X$ be a metric on $X$, and suppose that the action of $G$ is equicontinuous, that is, for every $\epsilon > 0$ there is $\delta > 0$ so that for every $g \in G$ and $x, y \in X$ with $d_X(x, y) < \delta$ we have $d_X(g(x), g(y)) < \epsilon$. We also assume that the action $(X, G, \Phi)$ is minimal, that is, for every $x \in X$ the orbit of $x$ under $G$ is dense in $X$. For example, if $T$ is a tree with the vertex set $\bigsqcup_{n \geq 1} V_n$, and $G$ acts on $T$ by automorphisms so that the induced action on every vertex set $V_n$ is transitive, then the action of $G$ on the path space of $T$ is minimal and equicontinuous, see Example 2.1 for details.

In the rest of the paper, the term Cantor group action refers to a minimal equicontinuous action of a discrete countably generated group $G$ on a Cantor set $X$. An action $(X, G, \Phi)$ is free if $g \cdot x = x$ for some $x$ implies that $g$ is the identity in $G$. An action $(X, G, \Phi)$ is effective if for any $g \in G$ there exists $x \in X$ such that $g \cdot x \neq x$, that is, every element of $g$ acts non-trivially on $X$.

One of the problems in the study of Cantor group actions is to classify them up to a certain type of equivalence, such as conjugacy, orbit equivalence, return equivalence etc. At the moment most of the results on these topics available in the literature are for the actions of abelian groups [18, 19, 20, 21], or for free actions of non-abelian groups [10]. However, effective but not free actions arise naturally in many areas of mathematics, and, as our work in [11, 12, 25] shows, they exhibit new phenomena not known for the actions of abelian groups. These phenomena are not fully understood and need to be investigated in greater depth. That is the part of what the present work seeks to explore.

More precisely, this paper investigates a recently developed invariant of Cantor group actions, called the asymptotic discriminant, for a special class of examples which arise in number theory and arithmetic dynamics. In most cases such actions are effective but not free. The asymptotic discriminant was developed to study the dynamical properties of Cantor group actions in precisely this setting.

The class of examples we consider is given by representations of Galois groups of fields into the group of automorphisms $\text{Aut}(T)$ of a $d$-ary rooted tree $T$, see Example 2.1 for a definition of a $d$-ary tree.

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Depending on the setting, such representations may be called \textit{profinite iterated monodromy groups} or \textit{arboreal representations}. In this paper we mostly concentrate on the former. We now briefly explain how these representations arise, then we recall some background on Cantor group actions and the asymptotic discriminant, and after that we state our results. In our description of profinite iterated arithmetic and geometric monodromy groups we follow \cite{29}. One can also consult \cite{39}. In this paper, the base field $K$ is always a finite algebraic extension of $\mathbb{Q}$.

Our results in this paper can be considered from two points of view. First, there are classification results for actions of profinite iterated monodromy groups of quadratic polynomials over finite extensions of $\mathbb{Q}$ by whether the asymptotic discriminant of such action is \textit{stable} or \textit{wild}. Theorems \ref{t1} and \ref{t2} provide a complete classification for quadratic polynomials. Also, Theorem \ref{t3} shows that the asymptotic discriminant for actions associated to Chebyshev polynomials of any degree $d \geq 2$ are stable. Second, we study various ways in which an action with wild asymptotic discriminant can arise. It was shown in the author's joint work with Hurder \cite{25} that if the image $\Phi(G)$ contains a non-Hausdorff element in the closure $\overline{\Phi(G)}$. Using this criterion, we determine that the asymptotic discriminant is wild for the representations of Galois groups associated to the polynomial of degree $d = 3$ considered in \cite{27}.

We recall necessary background on profinite iterated monodromy groups.

Let $K$ be a number field, that is, $K$ is a finite algebraic extension of the rational numbers $\mathbb{Q}$. Let $f(x)$ be a polynomial of degree $d \geq 2$ with coefficients in the ring of integers of $K$. Let $t$ be a transcendental element, then $K(t)$ is the field of rational functions with coefficients in $K$.

Denote by $f^n(x)$ the $n$-th iterate of $f(x)$. For $n \geq 1$, consider the solutions of the equation $f^n(x) = t$ over $K(t)$. The polynomial $f^n(x) - t$ is separable and irreducible over $K(t)$ for all $n \geq 1$ \cite[Lemma 2.1]{1}. Therefore, it has $d^n$ distinct roots, and the Galois group $H_n$ of the extension $K_n$ obtained by adjoining to $K(t)$ the roots of $f^n(x) - t$ acts transitively on the roots.

We represent the tower of preimages of $t$ under the iterations of $f$ as a tree $T$ as follows. Let $V_0$ be the set of distinct roots of $f^n(x) = t$. Thus for $n \geq 1$ we have $|V_n| = d^n$. We join $\beta \in V_{n+1}$ and $\alpha \in V_n$ by an edge if and only if $f(\beta) = \alpha$. For each $n \geq 1$, the Galois group $H_n$ acts transitively on the roots of $f^n(x) = t$ by field automorphisms, and so induces a permutation of vertices in $V_n$. Our field extensions satisfy $K_n \subset K_{n+1}$, and so an automorphism of $V_{n+1}$ induces an automorphism of $V_n$, thus defining a group homomorphism $\lambda_{n+1}^n : H_{n+1} \to H_n$. Taking the inverse limit

$$\text{Gal}_{\text{arith}}(f) = \lim_{\leftarrow} \{ \lambda_{n+1}^n : H_{n+1} \to H_n \},$$

we obtain a profinite group called the \textit{arithmetic iterated monodromy group} of the polynomial $f(x)$. In other words, the group $\text{Gal}_{\text{arith}}(f)$ is the Galois group of the extension $K = \bigcup_{n \geq 1} K_n$ obtained by adjoining to $K(t)$ the roots of $f^n(x) - t$ for $n \geq 1$.

For $n \geq 1$, the action of $H_n$ on $T$ preserves the connectedness of paths in $T$, and so $\text{Gal}_{\text{arith}}(f)$ is identified with a subgroup of the automorphism group $\text{Aut}(T)$ of the tree $T$. The tree $T$ is called a \textit{rooted $d$-ary tree}, since $|V_0| = 1$, and for each $n \geq 1$ every vertex in $V_n$ is connected by edges to precisely $d$ vertices in $V_{n+1}$.

**REMARK 1.1.** Given a polynomial $f(x)$ over $K$, one can also consider the extensions $K(f^{-n}(\alpha))/K$ with Galois groups $Y_n$ for some $\alpha \in K$. If all iterates $f^n(x) - \alpha$ are separable and irreducible, by a similar procedure as above one can construct an \textit{arboreal representation} $Y_\infty = \lim_{\leftarrow} \{ Y_{n+1} \to Y_n \}$ of the absolute Galois group of $\overline{K}/K$, where $\overline{K}$ is a separable closure of $K$, into the automorphism
group $Aut(T_\alpha)$ of a tree $T_\alpha$. Since $f^n(x) - \alpha$ is irreducible for $n \geq 1$, the group $Y_\alpha$ acts transitively on every vertex set $V_{\alpha,n}$ in $T_\alpha$. Since $f^n(x) - \alpha$ is separable, for each $n \geq 1$, $|V_{\alpha,n}| = |V_n| = d^n$, and both $T_\alpha$ and $T$ are $d$-ary trees. It follows that $Aut(T)$ and $Aut(T_\alpha)$ are isomorphic, and one may think of the groups $Y_\alpha$ as obtained via the specialisation $t = \alpha$. As explained in [33], Galois groups of polynomials do not increase under such specializations, and certain groups are preserved. So one can think of $Gal_{\text{arith}}(f)$ as $Y_\infty$ for a generic choice of $\alpha$, in a loose sense [33, 29].

Recall that $\overline{K}$ is a separable closure of $K$, and let $L = \overline{K} \cap K$ be the maximal constant field extension of $K$ in $\mathcal{K}$, that is, $L$ contains all elements of $\mathcal{K}$ algebraic over $K$. The Galois group $Gal_{\text{geom}}(f)$ of the extension $\mathcal{K}/L(t)$ is a normal subgroup of $Gal_{\text{arith}}(f)$, and there is an exact sequence [39, 29]

\begin{equation}
1 \longrightarrow Gal_{\text{geom}}(f) \longrightarrow Gal_{\text{arith}}(f) \longrightarrow Gal(L/K) \longrightarrow 1.
\end{equation}

The profinite group $Gal_{\text{geom}}(f)$ is called the geometric monodromy group, associated to the polynomial $f(x)$. The geometric monodromy group $Gal_{\text{geom}}(f)$ does not change under extensions of $L$, so one can calculate $Gal_{\text{geom}}(f)$ over $\mathbb{C}(t)$.

Let $\mathbb{P}^1(\mathbb{C})$ be the projective line over $\mathbb{C}$ (the Riemann sphere), and extend the map $f: \mathbb{C} \to \mathbb{C}$ to $\mathbb{P}^1(\mathbb{C})$ by setting $f(\infty) = \infty$. Then $\infty$ is a critical point of $f(x)$. Let $C$ be the set of all critical points of $f(x)$, and let $P_C = \bigcup_{n \geq 1} f^n(C)$ be the set of the forward orbits of the points in $C$, called the post-critical set. Suppose $P_C$ is finite, then the polynomial $f(x)$ is called post-critically finite. If the polynomial $f(x)$ is post-critically finite, then it defines a partial $d$-to-1 covering $f: M_1 \to M$, where $M = \mathbb{P}^1(\mathbb{C}) \cup P_C$ and $M_1 = f^{-1}(M)$ are punctured spheres. An element $s \in M$ has $d$ preimages under $f$, and $d^n$ preimages under the $n$-th iterate $f^n$. Denote by $\overline{V}_n = f^{-n}(s)$, for $n \geq 1$. Thus in a manner similar to the one above one constructs a rooted $d$-ary tree $\overline{T}$ of preimages of $s$, with vertex sets $\bigcup_{n \geq 1} \overline{V}_n$. The fundamental group $\pi_1(M,s)$ acts on the vertex sets of $\overline{T}$ via path-lifting. Let $\text{Ker}$ be the subgroup of $\pi_1(M,s)$ consisting of elements which act trivially on every vertex set $\overline{V}_n$, $n \geq 1$. The quotient group $\text{IMG}(f) = \pi_1(M,s)/\text{Ker}$, called the iterated monodromy group associated to the partial self-covering $f: M_1 \to M$, acts effectively on the space of paths of the tree $\overline{T}$. The $d$-ary trees $\overline{T}$, equipped with the action of the discrete group $\text{IMG}(f)$, and $T$, equipped with the action of the profinite group $Gal_{\text{geom}}(f)$, are isomorphic. If $f(x)$ is post-critically finite, by [33, Proposition 6.4.2] attributed by Nekrashevych to R. Pink, $Gal_{\text{geom}}(f)$ over $\mathbb{C}(t)$ is isomorphic to the closure of action of $\text{IMG}(f)$ in $Aut(\overline{T}) \cong Aut(T)$.

The actions of discrete iterated monodromy groups $\text{IMG}(f)$ are well-studied, with many results known and many techniques developed, see, for example, Nekrashevych [33]. Thus it is natural to approach the study of the actions of geometric iterated monodromy groups using the machinery of [33]. We are going to use the methods of [33] to prove one of our main theorems, Theorem 1.3.

We now briefly recall the necessary background on the asymptotic discriminant of group actions on Cantor sets. The reader should remember that the notion of the asymptotic discriminant in this article is completely different to the notion of the ‘discriminant of a polynomial’, which is the product of squares of differences of the polynomial roots. These two discriminants should not be confused.

The asymptotic discriminant of a minimal equicontinuous group action $(X, G, \Phi)$ is a local invariant developed by the author in the joint works with Hurder [20, 25]. Briefly, the closure of the equicontinuous group action $\Phi : G \to \text{Homeo}(X)$ in the uniform topology on $\text{Homeo}(X)$ is a profinite group $\Phi(G)$, identified with the Ellis (enveloping) group of the action [15, 4]. The profinite group $\Phi(G)$ acts effectively and transitively on the Cantor set $X$.

For a point $x \in X$, consider the isotropy subgroup $\Phi(G)_x$ of elements in $\Phi(G)$ fixing $x$. This subgroup is closed in $\Phi(G)$, and so it is either a finite group, or a profinite group. Since the action of $\Phi(G)$ on $X$ is transitive, for every $x, y \in X$ the isotropy group $\Phi(G)_x$ and $\Phi(G)_y$ are conjugate. Thus the cardinality of $\Phi(G)_x$ is an invariant of the minimal equicontinuous Cantor action $(X, G, \Phi)$. 
A fundamental property of equicontinuous minimal actions on Cantor sets is that, given \( x \in X \), there exists a descending sequence of clopen neighborhoods \( \{ U_n \}_{n \geq 0} \) of \( x \) in \( X \) with \( U_0 = X \) and \( \bigcap_{n \geq 0} U_n = \{ x \} \), and, for each \( n \geq 1 \), a finite index subgroup \( G_n \subset G \), such that each \( g \in G_n \) preserves \( U_n \), that is, \( \Phi(g)(U_n) = U_n \), and if \( g \notin G_n \), then \( U_n \cap \Phi(g)(U_n) = \emptyset \). This means that for all \( n \geq 1 \) the homomorphism \( \Phi : G \to \text{Homeo}(X) \) restricts to the homomorphism \( \Phi_n : G_n \to \text{Homeo}(U_n) \). In this case we say that the restricted action of \( G \) to \( U_n \) is that of the finite index subgroup \( G_n \).

Let \( U_{n+1} \subset U_n \), and \( g \in G_{n+1} \). The action of \( g \) preserves \( U_{n+1} \), that is, \( \Phi(g)(U_{n+1}) = U_{n+1} \). This implies that the intersection \( \Phi(g)(U_n) \cap U_n \) is non-empty, and so \( G_{n+1} \subset G_n \). Thus associated to a Cantor group action there is a descending chain \( \{ G_n \}_{n \geq 0} \) of finite index subgroups of \( G \), with \( G_0 = G \). The choice of a neighborhood system \( \{ U_n \}_{n \geq 0} \), and so of the group chain \( \{ G_n \}_{n \geq 0} \), is not unique, but all group chains associated to a given action \( (X, G, \Phi) \) satisfy an equivalence relation in Definition 2.6 described in more detail in Section 2 and in 11 13. The existence of group chains, associated to a Cantor group action \( (X, G, \Phi) \) gives a method for computing explicitly the enveloping group \( \Phi(G) \) and the isotropy group \( \Phi(G)_x \). The details of the method are described in detail in Section 2 and in 11. In particular, it turns out that there is an isomorphism

\[
\Phi(G)_x \to D_x = \lim_{\longleftarrow} \{ G_{n+1}/C_{n+1} \to G_n/C_n \},
\]

where for each \( n \geq 1 \) the group \( C_n = \bigcap_{g \in G} gG_ng^{-1} \) is the maximal normal subgroup of \( G_n \) in \( G \). The profinite group \( D_x \) is unique up to an isomorphism, and can be seen as a ‘coordinate representation’ of the abstract group \( \Phi(G)_x \). The group \( D_x \) is called the discriminant group of the action \( (X, G, \Phi) \). The origin of this term is explained in Section 2.

We now return to the system of clopen neighborhoods \( \{ U_m \}_{m \geq 0} \) of \( x \in X \) with associated group chain \( \{ G_m \}_{m \geq 0} \). For each \( m \geq 1 \), the restricted action of \( G \) to \( U_m \) is that of the subgroup \( G_m \), so we may consider a family of minimal equicontinuous group actions \( (U_m, G_m, \Phi_m) \). For each of these actions, there is the enveloping group \( \Phi_m(G_m) \) of the action, and the isotropy group \( \Phi_m(G_m)_x \) of the enveloping group action at \( x \). Using the associated truncated group chain \( \{ G_n \}_{n \geq m} \), one obtains for each \( m \geq 1 \) an isomorphism of the isotropy group

\[
\Phi_m(G_m)_x \to D^m_x = \lim_{\longleftarrow} \{ G_{n+1}/C_{n+1}^m \to G_n/C_n^m, n \geq m \},
\]

where for \( n \geq m \) the group \( C_n^m = \bigcap_{g \in G_m} gG_mg^{-1} \) is the maximal normal subgroup of \( G_n \) in \( G_m \). The profinite group \( D^m_x \) is called the discriminant group of the restricted action \( (U_m, G_m, \Phi_m) \).

The intersection of conjugate groups in the definition of \( C_n^m \) is over a smaller subgroup, and so in general \( C_n \subset C_n^k \subset C_n^m \) for \( k < m \). As a consequence, the groups \( D_x \) and \( D^m_x \) need not be isomorphic. As explained in detail in Section 2 inclusions of subgroups induce a sequence of surjective homomorphisms

\[
D_x \xrightarrow{\psi_{0,1}} D_x^1 \xrightarrow{\psi_{1,2}} D_x^2 \xrightarrow{\psi_{2,3}} \ldots
\]

of the discriminant groups of restricted actions. An equivalence class of such a sequence with respect to an equivalence relation described in Definition 2.10 is called the asymptotic discriminant of the action \( (X, G, \Phi) \).

**DEFINITION 1.2.** A Cantor group action \( (X, G, \Phi) \) is stable, or has stable asymptotic discriminant, if there exists \( m \geq 0 \) such that for all \( n \geq m \), the surjective homomorphisms \( \psi_{n,n+1} \) in the sequence (1) of the discriminant groups, associated to the action, are isomorphisms. If such an \( m \) does not exist, then the action is wild, or is said to have wild asymptotic discriminant.

The asymptotic discriminant was introduced by the author in a joint work with Hurder 20. Examples of stable actions of torsion-free subgroups of \( SL(k, \mathbb{Z}) \), \( k \geq 3 \), were given in 12, and of wild actions in 20. In 25, the notion of stability given by Definition 1.2 was applied to the classification of Cantor group actions up to continuous orbit equivalence.
In the rest of the article, we say that a Cantor group action \((X, G, \Phi)\) is stable with discriminant group \(D\), if in Definition 1.2 for all \(n \geq m\) we have an isomorphism \(D^n_x \cong D\).

If an action has wild asymptotic discriminant, it can arise in a variety of ways. To explain that we need to introduce two more concepts.

Just in this paragraph, let \(G\) be a finite or a profinite group. We say that a minimal equicontinuous action of \(G\) on \(X\) is quasi-analytic if for every \(g \in G\) and every clopen set \(W \subset X\) the restriction \(g|W\) extends uniquely to \(X\). We say that the action of \(G\) is locally quasi-analytic, or LQA, if there exists \(\epsilon > 0\) such that for every open set \(U \subset X\) of diameter less than \(\epsilon\) the restricted action \(G|U\) is LQA.

Now let us again assume that \(G\) is a discrete group, and \(\Phi(G)\) is the profinite enveloping group of the action \((X, G, \Phi)\). The relationship between stability of the action and its LQA properties was studied in [12]. In particular, it was shown there that the asymptotic discriminant of a Cantor group action \((X, G, \Phi)\) is stable if and only if the action of the enveloping group \(\Phi(G)\) on \(X\) is LQA. Since \(G\) is identified with a dense subgroup of \(\Phi(G)\), then the action of \(\Phi(G)\) being LQA implies that the action of \(G\) is LQA. The converse of this statement does not hold if \(G\) acts equicontinuously non-minimally on a path-connected space, see [2] for a counterexample. It is not known at the moment whether the converse holds in the case of minimal actions on Cantor sets. Thus we may need to distinguish between the LQA or non-LQA properties of the action of the discrete group \(G\), and of the closure \(\Phi(G)\).

In the paper [25] the author jointly with Hurder obtained a sufficient condition for an action of a (countable or profinite) group to be non-LQA. This condition is the existence of a so-called non-Hausdorff element in \(G\). Again, just for the purpose of giving a definition, in this paragraph let us allow \(G\) to be discrete or profinite, and let \(\Phi : G \rightarrow \text{Homeo}(X)\) be an action, not necessarily minimal or equicontinuous. Consider the collection \(G(X, G, \Phi) = \{[g]_x \mid g \in G, x \in X\}\), where \([g]_x\) is the germ of \(g\) at \(x\), that is, an equivalence class of \(h \in G\) such that the homeomorphisms \(\Phi(g)\) and \(\Phi(h)\) coincide on an open neighborhood of \(x\). With sheaf topology, \(G(X, G, \Phi)\) forms an étale groupoid, modeled on \(X\). It follows from the result of Winkelkemper [17] that this groupoid is non-Hausdorff if and only if there is an element \(g \in G\) such that the following holds: there is a) \(x \in X\) with \(g(x) = x\), b) a collection of nested open neighborhoods \(\{W_n\}_{n \geq 1}\) of \(x\), such that the restriction \(g|W_n\) is not the identity, and c) for each \(n \geq 1\), there is an open neighborhood \(O_n \subset W_n\) such that \(g|O_n\) is the identity. We call such \(g\) a non-Hausdorff element of \(G\). There exist plenty of examples of actions with non-Hausdorff elements, for example, three out of four generators of the Grigorchuk group, as described in [33] Section 1.6, are non-Hausdorff. Also, the process of the fragmentation of the dihedral groups in [30] is conducted by adding non-Hausdorff elements to the group. It was shown in [23] that if a (discrete or profinite) group \(G\), acting on a Cantor set \(X\), contains a non-Hausdorff element, then the action is non-LQA. As a consequence of our study of the asymptotic discriminant for post-critically finite quadratic polynomials we will obtain examples of non-LQA actions without non-Hausdorff elements. We will also give a sufficient condition under which the closure \(\Phi(G)\) of the action contains a non-Hausdorff element.

Let \(f(x)\) be a polynomial of degree \(d \geq 2\) with coefficients in the ring of integers of the field \(K\), where \(K\) is a finite extension of \(\mathbb{Q}\). Let \(T\) be a \(d\)-ary tree. An infinite path in \(T\) is an infinite sequence \((v_n)_{n \geq 0}\) of vertices in \(\bigcup_{n \geq 0} V_n\), such that for all \(n \geq 0\) the vertices \(v_{n+1}\) and \(v_n\) are joined by an edge. Denote by \(P_d\) the set of all infinite paths in the \(d\)-ary rooted tree \(T\), then \(P_d\) is a Cantor set, see Example 2.1 for details.

Let \(\text{Gal}_{\text{arith}}(f)\) and \(\text{Gal}_{\text{geom}}(f)\) be the arithmetic and the geometric iterated monodromy groups respectively, defined as described earlier in the Introduction. Both \(\text{Gal}_{\text{arith}}(f)\) and \(\text{Gal}_{\text{geom}}(f)\) are inverse limits of finite groups, indexed by natural numbers, so by [46] Proposition 4.1.3 they contain countably generated dense subgroups \(G_{\text{arith}}\) and \(G_{\text{geom}}\) respectively. By a slight abuse of notation, we denote by the same symbols the groups \(G_{\text{arith}}\) and \(G_{\text{geom}}\) with discrete topology. Thus associated to the actions of profinite monodromy groups \(\text{Gal}_{\text{arith}}(f)\) and \(\text{Gal}_{\text{geom}}(f)\), there are actions \((P_d, G_{\text{arith}})\) and \((P_d, G_{\text{geom}})\).
and \((P_d, G_{\text{geom}})\) of discrete groups on the Cantor set \(P_d\), and we may apply the machinery of group chains and asymptotic discriminants to the actions of profinite groups \(\text{Gal}_{\text{arith}}(f)\) and \(\text{Gal}_{\text{geom}}(f)\).

A rigorous proof of the construction described in the previous paragraph, as well as the first examples of actions with stable or wild asymptotic discriminant were given by the author in \cite{32}. Note that for both \(\text{Gal}_{\text{arith}}(f)\) and \(\text{Gal}_{\text{geom}}(f)\) the choice of dense subgroups \(G_{\text{arith}}\) and \(G_{\text{geom}}\) is certainly not unique. However, since all choices of \(G_{\text{arith}}\) (resp. \(G_{\text{geom}}\)) have the same closure \(\text{Gal}_{\text{arith}}(f)\) (resp. \(\text{Gal}_{\text{geom}}(f)\)), and the asymptotic discriminant in \cite{4} is defined as an equivalence relation on the subgroups of the closures of restricted actions, the notion of stable or wild asymptotic discriminant is independent of the choice of a subgroup \(G_{\text{arith}}\) (resp. \(G_{\text{geom}}\)).

In the rest of the article, we say that the action of the profinite iterated monodromy group \(\text{Gal}_{\text{arith}}(f)\) (resp. \(\text{Gal}_{\text{geom}}(f)\)) on a Cantor set \(X\) is stable (resp. wild), if the action of some (and so every) dense subgroup of \(\text{Gal}_{\text{arith}}(f)\) (resp. \(\text{Gal}_{\text{geom}}(f)\)) with discrete topology has stable (resp. wild) asymptotic discriminant.

We now state our results. First, note that if \(f(x)\) is quadratic, then the map \(f: \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})\) of the Riemann sphere has two critical points, the point at infinity \(\infty\), and another point \(c\). Denote by \(r = \#P_C - 1\). Since by definition the point at infinite is fixed, the number \(r\) is the length of the post-critical orbit of \(c\).

**THEOREM 1.3.** Let \(K\) be a finite extension of \(\mathbb{Q}\), and let \(f(x)\) be a post-critically finite quadratic polynomial with coefficients in the ring of integers of \(K\). Then the action of \(\text{Gal}_{\text{geom}}(f)\) has wild asymptotic discriminant, unless \(r = 1\) or the orbit of \(c\) is pre-periodic and \(r = 2\).

More precisely, the following holds:

1. If the orbit of \(c\) is strictly periodic, and \(r = 1\), then the action of \(\text{Gal}_{\text{geom}}(f)\) is LQA with trivial discriminant group.
2. If the orbit of \(c\) is strictly periodic, and \(r \geq 2\), then \(\text{Gal}_{\text{geom}}(f)\) is conjugate in \(\text{Aut}(T)\) to the closure of the action of a discrete group \(\tilde{G}_r\) on \(r\) generators. The action of \(\tilde{G}_r\) is non-LQA, and \(\tilde{G}_r\) does not contain non-Hausdorff elements. Thus the action of \(\text{Gal}_{\text{geom}}(f)\) is non-LQA and has wild asymptotic discriminant.
3. If the orbit of \(c\) is pre-periodic, and \(r = 2\), then the action of \(\text{Gal}_{\text{geom}}(f)\) is LQA with finite discriminant group.
4. If the orbit of \(c\) is pre-periodic, and \(r \geq 3\), then \(\text{Gal}_{\text{geom}}(f)\) conjugate in \(\text{Aut}(T)\) to the closure of the action of a discrete group \(H_r\) on \(r\) generators. The action of \(H_r\) is non-LQA, and \(H_r\) contains non-Hausdorff elements. Thus the action of \(\text{Gal}_{\text{geom}}(f)\) is non-LQA and has wild asymptotic discriminant.

The groups \(\tilde{G}_r\), \(r \geq 2\), and \(H_r\), \(r \geq 3\), in Theorem 1.3 are subgroups of \(\text{Homeo}(P_2)\), where \(P_2\) denotes the path space of the binary tree \(T\). Thus specifying \(\tilde{G}_r\) and \(H_r\) defines an effective action on \(P_2\). The groups \(\tilde{G}_r\) and \(H_r\) are described in detail in Section 5.

If the post-critical set \(P_C\) of \(f(x)\) is infinite, then by \cite{39} Section 1.10 \(\text{Gal}_{\text{geom}}(f) \cong \text{Aut}(T)\), and so by \cite{32} the action of \(\text{Gal}_{\text{geom}}(f)\) is non-LQA and has wild asymptotic discriminant. This together with the results of Theorem 1.3 provides a complete classification of the actions of profinite geometric iterated monodromy groups associated to quadratic polynomials defined over finite extensions of \(K\).

The case (2) of Theorem 1.3 is the case when the critical point \(c\) of a quadratic polynomial \(f(x)\) has a finite strictly periodic orbit of cardinality at least 2. The groups \(\tilde{G}_r\), \(r \geq 2\), in (2) give rise to a family of the actions of discrete groups which are non-LQA and where the groups do not contain non-Hausdorff elements. These are the first examples of this kind, known to the author. Note that a similar statement may or may not be true for the actions of profinite groups. Indeed, although \(\tilde{G}_r\), \(r \geq 2\), in (2) do not contain non-Hausdorff elements, the closure of the action of \(\tilde{G}_r\) might, see Section 5 for more discussion. We summarize this discussion as a corollary of Theorem 1.3.
COROLLARY 1.4. There exist actions \((X, G, \Phi)\) of discrete finitely generated groups \(G\) on a Cantor set \(X\) which are non-LQA and where the groups do not contain non-Hausdorff elements. That is, non-LQA minimal Cantor actions of discrete groups with and without non-Hausdorff elements form two distinct non-empty classes.

Another subtle point of the statement (2) in Theorem 1.3 (and a similar discussion applies to the statement (4)), that although \(\text{Gal}_{\text{geom}}(f)\) is conjugate in \(\text{Aut}(T)\) to the closure of the action of \(\tilde{G}_r\) and, using the conjugacy and Proposition 2.20 we are able to determine that the action of \(\text{Gal}_{\text{geom}}(f)\) is non-LQA, we cannot infer much about the action of a dense subgroup \(G_{\text{geom}}\) of \(\text{Gal}_{\text{geom}}(f)\). Indeed, \(\tilde{G}_r\) need not be mapped onto \(G_{\text{geom}}\) under the conjugacy. Nekrashevych [34] gives examples of actions of discrete non-isomorphic groups on 3 generators whose closures in \(\text{Aut}(T)\) are conjugate. Since the asymptotic discriminant is an invariant of the closures of the actions, we are able to determine if the action is stable or wild without knowing \(G_{\text{geom}}\).

Polynomials described by statement (3) in Theorem 1.3 are examples of quadratic polynomials conjugate to Chebyshev polynomials. Chebyshev polynomials \(T_d\) of degree \(d \geq 2\) are described in Section 4. Chebyshev polynomials are well-studied, with discrete iterated monodromy groups \(\text{IMG}(T_d)\) computed for all degrees \(d \geq 2\). The following theorem computes the asymptotic discriminant for the geometric iterated monodromy groups associated to Chebyshev polynomials.

THEOREM 1.5. Let \(T_d\) be the Chebyshev polynomial of degree \(d \geq 2\) over \(\mathbb{C}\). Then the action of \(\text{IMG}(T_d)\) is stable with discriminant group \(\mathbb{Z}/2\mathbb{Z}\), the finite group of order 2.

Let us now consider the action of the arithmetic iterated monodromy group \(\text{Gal}_{\text{arith}}(f)\) for quadratic polynomials over \(K\), where \(K\) is a finite extension of \(\mathbb{Q}\). Recall that \(\text{Gal}_{\text{geom}}(f)\) is a normal subgroup of \(\text{Gal}_{\text{arith}}(f)\), and so \(\text{Gal}_{\text{arith}}(f)\) is a subgroup of the normalizer \(N\) of \(\text{Gal}_{\text{geom}}(f)\) in \(\text{Aut}(T)\). Properties of the normalizer of \(\text{Gal}_{\text{geom}}(f)\), and how \(\text{Gal}_{\text{arith}}(f)\) sits in the normalizer were studied by Pink [39]. Direct computations based on the results of [39] in (1) and (3), and Lemma 2.19 of this paper in (2) and (4) give the following theorem.

THEOREM 1.6. Let \(K\) be a finite extension of \(\mathbb{Q}\), and let \(f(x)\) be a post-critically finite quadratic polynomial with coefficients in the ring of integers of \(K\). Then the action of \(\text{Gal}_{\text{arith}}(f)\) has wild asymptotic discriminant, unless \(r = 1\) or the orbit of \(c\) is pre-periodic and \(r = 2\).

More precisely, the following holds:

1. If the orbit of \(c\) is strictly periodic, and \(r = 1\), then the action of \(\text{Gal}_{\text{arith}}(f)\) is stable with infinite discriminant group.
2. If the orbit of \(c\) is strictly periodic, and \(r \geq 2\), then the action of \(\text{Gal}_{\text{arith}}(f)\) is non-LQA and so the asymptotic discriminant of the action is wild.
3. If the orbit of \(c\) is pre-periodic, and \(r = 2\), then the action of \(\text{Gal}_{\text{arith}}(f)\) is stable with infinite discriminant group.
4. If the orbit of \(c\) is pre-periodic, and \(r \geq 3\), then the action of \(\text{Gal}_{\text{arith}}(f)\) is non-LQA and so the asymptotic discriminant of the action is wild.

In the case when \(f(x)\) has infinite post-critical set \(P_C\), the action of \(\text{Gal}_{\text{geom}}(f)\) is non-LQA, and so Lemma 2.19 implies that the action of \(\text{Gal}_{\text{arith}}(f)\) is non-LQA and has wild asymptotic discriminant. This together with Theorem 1.6 completes the classification of actions of profinite arithmetic iterated monodromy groups for quadratic polynomials over finite extensions of \(\mathbb{Q}\).

A similar classification as in Theorems 1.3 and 1.6 for polynomials of degree \(d \geq 3\) is currently out of our reach, mostly due to the absence of such comprehensive study of the geometric and arithmetic iterated monodromy groups in this case as was done by Pink for quadratic polynomials in [39]. In some cases the question if a profinite iterated monodromy group or an arboreal representation associated to a certain polynomial has wild asymptotic discriminant can be answered using the following algebraic criterion. The criterion is a sufficient condition for a profinite group acting on the path space \(P\) of a spherically homogeneous rooted tree \(T\) (not necessarily \(d\)-ary, see Example 2.1).
to contain a non-Hausdorff element. The action of such a profinite group on \( \mathcal{P} \) has wild asymptotic discriminant by the results of [25].

**Theorem 1.7.** Let \( T \) be a spherically homogeneous rooted tree, and \( \mathcal{P} \) be the space of paths in \( T \). Let \( H_{\infty} = \varprojlim \{ H_{n+1} \to H_n \} \) be a profinite group, acting on \( \mathcal{P} \), so that for each \( n \geq 1 \) the action of finite groups \( H_n \) on the vertex sets \( V_n \) of \( T \) is transitive. Let \( \{ L_n \}_{n \geq 1} \) be a collection of non-trivial finite groups such that for each \( n \geq 1 \) the group \( H_n \) contains the wreath product \( L_n = L_n \ltimes L_{n-1} \ltimes \cdots \ltimes L_1 \). Then \( H_{\infty} \) contains a non-Hausdorff element.

Using Theorem [1.7] we conclude that the actions of the geometric and arithmetic iterated monodromy groups for the polynomial \( f(z) = -2z^3 + 3z^2 \) studied in [7] have wild asymptotic discriminants, see Example [7.1] in this paper for details.

We finish the introduction with some discussion of motivation for the study of the asymptotic discriminant for actions arising from representations of Galois groups, and of the possible directions of future work.

One of the problems which motivated the study of arboreal representations in arithmetic dynamics is the problem of density of prime divisors in non-linear relations \( a_n = f(a_{n-1}) \). More precisely, let \( f(x) \) be a polynomial of degree \( d \geq 2 \) with coefficients in a ring of integers of a field \( K \), and let \( a_0 \) be a point in \( K \). Let \( Y_\infty \) denote the representation of the absolute Galois group of \( K \) into the group of automorphisms \( Aut(T) \) of the \( d \)-ary tree \( T \), as in Remark [1.1]. Recall that \( \mathcal{P}_d \) denotes the space of paths of the tree \( T \). Consider also the orbit \( \mathcal{O}(a_0) = \{ f^n(a_0) \}_{n \geq 0} \). What is the natural density of prime divisors of the points in \( \mathcal{O}(a_0) \)? Odoni [38] showed that an upper estimate for such density can be obtained by counting the proportion \( \mathcal{F}(Y_\infty) \) of elements in \( Y_\infty \) which fix at least one point in the space of paths \( \mathcal{P}_d \). The proportion \( \mathcal{F}(Y_\infty) \) is computed as the limit of the proportions of the elements with fixed points in the Galois groups \( Y_n \) of finite extensions of \( K \), obtained by adjoining the roots of the \( n \)-th iterate \( f^n(x) = a_0 \) to \( K \). In particular, Odoni [38] showed that if \( Y_\infty \) is isomorphic to the infinite wreath product \( [S_d]^\infty \), where \( S_d \) is a permutation group on \( d \) elements, then \( \mathcal{F}(Y_\infty) = 0 \). Jones [28, 29, 30] developed a method of computing \( \mathcal{F}(Y_\infty) \) using theory of stochastic processes, and, in a series of papers, obtained the values of \( \mathcal{F}(Y_\infty) \) for various classes of arboreal representations.

The discriminant group \( D_x \) of an action \((X, G, \Phi)\) with profinite enveloping group \( \Phi(G) \) counts elements of \( \Phi(G) \) which fix a given point \( x \). For every other point \( y \in X \), the cardinality \( D_y = D_x \). Although the relationship between the cardinality of the asymptotic discriminant and the proportion \( \mathcal{F}(Y_\infty) \) is not direct, since they consider essentially the same objects, it is natural to ask whether the asymptotic discriminant can be related to known phenomena in number theory or arithmetic dynamics. The following question in a slightly different formulation has already been posed by the author in [32].

**Question 1.8.** Is it possible to relate the stability or the wildness of the asymptotic discriminant of group actions on Cantor sets arising from the representations of Galois groups of fields with questions in number theory?

Another motivation to consider the actions associated to representations of Galois groups into the automorphism groups of \( d \)-ary trees comes from the topological point of view. Actions of profinite iterated monodromy groups and of arboreal representations present a large class of examples, and studying them one can gain insights in the properties of general group actions on Cantor sets. For example, one problem we addressed in this paper is different ways in which a wild action can arise.

As a consequence of Theorem [1.3] in Corollary [1.4] we obtained that non-LQA minimal equicontinuous actions of discrete countable groups on Cantor sets may give rise to Hausdorff or non-Hausdorff étale groupoids \( G(X, G, \Phi) \). It is interesting to compare these results to those obtained in different but related settings. For example, Hughes [27] considers germinal groupoids of all local isometries on ultrametric spaces, which include Cantor sets. An interesting property of these groupoids is that if there is a point \( x \in X \) and a local isometry \( \ell \) of \( X \) such that \( \ell(x) = x \), then there exists a local
isometry \( \tilde{\ell} \) with \( \tilde{\ell}(x) = x \) which is, in our terminology, is a non-Hausdorff element. A consequence of this, in particular, is that ultrametric spaces with Hausdorff groupoids only admit local isometries where fixed points have fixed clopen neighborhoods. Hughes’s setting is very different to ours, since in our setting every local map of the space \( X \) must arise as a restriction of an action of an element \( g \in G \). Still, it would be interesting to find out to what extent the properties of groupoids in these two different settings mirror each other.

The rest of the paper is organized as follows. In Section 2 we recall the basics of the method of group chains, and the notion and properties of the asymptotic discriminant. In Section 3 we recall the necessary background on wreath products and actions of self-similar groups as in 13, and study the properties of non-Hausdorff elements in contracting groups. In Section 4 we prove Theorem 1.5. The proof of Theorem 1.3 is in Section 5 and of Theorem 1.6 in Section 6. Finally, the proof of Theorem 1.7 is given in Section 7.

2. The asymptotic discriminant of an equicontinuous Cantor action

In this section, we recall the necessary background on equicontinuous Cantor actions and the asymptotic discriminant. Main references for this section are works 13, 11, 12, 26. Although the standing assumption in 11, 12, 26 was that \( G \) is a finitely generated group, the reason for that was not any restrictions imposed by the proofs or by the properties of the objects considered. The motivation in those papers was to study and classify the dynamics of weak solenoids, and for a group to act on a Cantor fibre of a weak solenoid it must be realizable as a homomorphic image of a fundamental group of a closed manifold. Thus we assumed that the groups under consideration were finitely generated. However, the notion of the Ellis (enveloping) group does not require finite generation, and infinite generation was not used in any of the proofs in 11, 12, 26, so one easily checks that the results apply for the countably generated groups as well.

2.1. Equicontinuous actions on Cantor sets. Let \( X \) be a Cantor set, that is, a compact totally disconnected perfect metrizable space. Recall 15, Section 30) that a space \( X \) is perfect if every point \( x \in X \) is an accumulation point of a non-constant sequence of points in \( X \). In other words, no point of \( X \) is isolated.

Let \( D \) be a metric on \( X \), and suppose \( \Phi : G \to \text{Homeo}(X) \) defines an action of a countably generated discrete group \( G \) on \( X \). The action \((X, G, \Phi)\) is equicontinuous, if for any \( \epsilon > 0 \) there exists \( \delta > 0 \) with the following property: for any \( g \in G \) and any \( x, y \in X \) such that \( D(x, y) < \delta \) we have \( D(\Phi(g)(x), \Phi(g)(y)) < \epsilon \).

**EXAMPLE 2.1.** For readers with non-topological background we explain why the set of infinite paths in a tree \( T \) is a Cantor set, and why the action of \( G \) on the set of paths is minimal and equicontinuous.

A **spherical index** is a sequence \( \ell = (\ell_1, \ell_2, \ldots) \) of positive integers, where \( \ell_n \geq 2 \) for \( n \geq 1 \). Let \( T \) be a tree, defined by \( \ell \). That is, the set of vertices is \( V = \bigcup_{n \geq 0} V_n \), where \( V_0 \) is a singleton, and for \( n \geq 1 \) \( V_n \) contains \( \ell_1 \ell_2 \cdots \ell_n \) vertices. Since \( |V_0| = 1 \), the tree \( T \) is rooted. Every vertex in \( V_n \) is connected by edges to precisely \( \ell_{n+1} \) vertices in \( V_{n+1} \), and every vertex in \( V_{n+1} \) is connected by an edge to precisely one vertex in \( V_n \). A tree with this property is called a **spherically homogeneous** tree. A path in \( T \) is an infinite sequence \((v_n)_{n \geq 0} = (v_0, v_1, v_2, \cdots)\) such that \( v_n \) and \( v_{n+1} \) are connected by an edge, for \( n \geq 0 \). Denote by \( \mathcal{P} \) the set of all such sequences. If \( \ell_n = d \) for all \( n \geq 1 \), where \( d \geq 2 \), then we call \( T \) a **\( d \)-ary rooted tree**, and denote the space of paths in \( T \) by \( \mathcal{P}_d \).

We are now going to put a Cantor set topology on \( \mathcal{P} \). For each \( n \geq 0 \), let \( L_n = \{0, 1, \cdots, \ell_n - 1\} \). If \( T \) is \( d \)-ary, then for all \( n \geq 1 \) \( L_1 = L_n \), and the set \( L_1 \) with \( d \) elements is called the **alphabet**.

For \( n \geq 1 \), let \( \text{pr}_{n-1} : L_1 \times L_2 \times \cdots \times L_n \to L_1 \times L_2 \times \cdots \times L_{n-1} \) be the projection onto the product of the first \( n - 1 \) sets. Define bijections

\[
(5) \quad b_n : L_1 \times L_2 \times \cdots \times L_n \to V_n
\]
in such a way that
\[ \text{pr}_{n-1}(b_n^{-1}(v)) = \text{pr}_{n-1}(b_n^{-1}(w)) \]
if and only if \( v \) and \( w \) are connected by edges to the same vertex in \( V_{n-1} \). Given a vertex \( v \in V_n \), the preimage \( b_n^{-1}(v) = (t_1, t_2, \ldots, t_n) \) is an \( n \)-tuple of integers, which we write as a word, that is, \( b_n^{-1}(v) = t_1t_2 \cdots t_n \). The collection of mappings (5), for \( n \geq 1 \), assigns a label to every vertex in the set \( \bigcup_{n \geq 1} V_n \). Labeling the root will not add any information, so we do not label it.

It follows from the definition of the maps (5) that every word \( t_1t_2 \cdots t_n \) defines a finite path \( (v_k)_{0 \leq k \leq n} \), where \( v_k \) is the unique vertex in \( V_0 \), and \( v_k = b_k(t_1t_2 \cdots t_k) \) for \( 1 \leq k \leq n \). Then every infinite word \( t_1t_2 \cdots t_n \cdots \), where \( 0 \leq t_n \leq t_n - 1 \), defines an infinite path in \( T \), and so there is a bijection
\[
(6) \quad b_\infty : \prod_{n \geq 1} L_n \to \mathcal{P}
\]
such that \( b_\infty|_{L_1 \times \cdots \times L_n} = b_n \). For \( n \geq 0 \), give \( L_n \) discrete topology, then the product \( \prod_{n \geq 1} L_n \) is compact by the Tychonoff theorem [45]. Points are the only connected components in \( \prod_{n \geq 1} L_n \), and so \( \prod_{n \geq 1} L_n \) is totally disconnected.

Open sets in the product topology on \( \prod_{n \geq 1} L_n \) have the form \( \prod_{n \geq 0} U_n \), where \( U_n \subseteq L_n \), and \( U_0 = L_0 \) for all but a finite number of \( n \). For example, given a word \( t_1t_2 \cdots t_k \), let \( U_n = \{t_n\} \) for \( 1 \leq n \leq k \), and \( U_n = L_n \) otherwise. Then \( U = \prod_{n \geq 1} U_n \) is the set of all infinite sequences in \( \prod_{n \geq 1} L_n \) which start with a finite word \( t_1t_2 \cdots t_k \). In \( \mathcal{P} \) this set corresponds to all paths which contain the vertex \( v_k = b_k(t_1t_2 \cdots t_k) \). We denote such a set \( U \) by \( U_k(t_1t_2 \cdots t_k) \).

Note that for any open set \( U = \prod_{n \geq 1} U_n \) its complement in \( \prod_{n \geq 1} L_n \) is also open, and so \( U \) is closed. A set which is open and closed is called a clopen set.

Let \( t = t_1t_2 \cdots \) be an infinite sequence, and consider a descending sequence of open neighborhoods \( U_n(t_1t_2 \cdots t_n) \) for \( n \geq 1 \). Since \( |L_n| \geq 2 \) for all \( n \geq 1 \), every \( U_n(t_1t_2 \cdots t_n) \) is infinite, and it follows that \( \prod_{n \geq 1} L_n \) is a perfect set.

We have shown that \( \prod_{n \geq 1} L_n \) and so \( \mathcal{P} \), is a Cantor set. From now on we identify \( \prod_{n \geq 1} L_n \) and \( \mathcal{P} \), and think of elements in \( \mathcal{P} \) as infinite sequences \( t_1t_2 \cdots \), where \( 0 \leq t_n \leq t_n - 1 \) for \( n \geq 1 \). For \( v_n = b_n(t_1t_2 \cdots t_n) \), we suppress \( b_n \) in the notation, and just write \( v_n = t_1 \cdots t_n \).

Let \( G \) be a countably generated discrete group, and let \( G \) act on the tree \( T \) by permuting vertices in each \( V_n \), \( n \geq 0 \), in such a way that the connectedness of paths in \( T \) is preserved, and the action is transitive on each \( V_n \). Since permutations are bijective, the action of each \( g \in G \) induces a bijective map \( \Phi(g) : \mathcal{P} \to \mathcal{P} \). For each \( n \geq 1 \), the image of an open set \( U_n(t_1t_2 \cdots t_n) \) under \( \Phi(g) \) is an open set \( U_n(g \cdot t_1t_2 \cdots t_n) \), so \( \Phi(g) \) is a homeomorphism. Thus \( G \) acts on \( \mathcal{P} \) by homeomorphisms.

Let \( w = w_1w_2 \cdots \in \mathcal{P} \) be an infinite sequence. Since \( G \) acts transitively on \( V_m \), for every vertex \( t_1t_2 \cdots t_m \in V_m \) there exists \( g \in G \) such that \( g \cdot (w_1w_2w_3 \cdots w_m) = t_1t_2 \cdots t_m \). Thus the image \( g \cdot w \in U_m(t_1 \cdots t_m) \), and the orbit of \( w \) is dense in \( \mathcal{P} \). We obtain that \( G \) acts minimally on \( \mathcal{P} \).

Let \( t = t_1t_2 \cdots \) and \( w = w_1w_2 \cdots \) be two infinite sequences in \( \mathcal{P} \). We define a metric \( D \) on \( \mathcal{P} \) by
\[
(7) \quad D(t, w) = \frac{1}{d^m}, \text{ where } m = \max\{n \mid t_n = w_n\},
\]
that is, \( D \) measures the length of the longest initial finite word contained in both \( t \) and \( w \). Since \( G \) acts bijectively on \( V_n \) for \( n \geq 0 \), \( t \) and \( w \) contain a common word of length \( m \) if and only if the images \( g \cdot t \) and \( g \cdot w \) contain a common word of length \( m \), so the action of \( G \) on \( \mathcal{P} \) is equicontinuous with respect to the metric \( D \), where we can take \( \delta = \epsilon \) for every \( \epsilon > 0 \).

**Definition 2.2.** Let \( G \) be a countably generated discrete group. A nested descending sequence \( \{G_n\}_{n \geq 0} = G_0 \supset G_1 \supset G_2 \supset \cdots \), with \( G_0 = G \), of finite index subgroups of \( G \) is called a group chain.

Any group chain \( \{G_n\}_{n \geq 0} \) gives rise to a Cantor group action as in Example 2.3.
**EXAMPLE 2.3.** Let \( \{G_n\}_{n \geq 0} \) be a group chain as in Definition 2.2. Then for every \( n \geq 0 \) the coset space \( G/G_n \) is a finite set. Define \( V_n = G/G_n \) to be the set of vertices in a tree \( T \). Let \( L_n = G_n/G_{n+1} \), then \( V_n = G/G_n \cong \prod_{1 \leq k \leq n} L_k \) are isomorphic as sets.

Inclusions of cosets induce the mappings
\[
\nu_n^{n+1} : G/G_{n+1} \to G/G_n : gG_{n+1} \to gG_n.
\]

In a tree \( T \), define the set of edges \( E \) by saying that a pair of vertices \( [g_nG_n, g_{n+1}G_{n+1}] \) is an edge if and only if \( g_{n+1}G_{n+1} \subset g_nG_n \). Then it is immediate that the inverse limit space
\[
G_\infty = \lim_{\rightarrow} (G/G_n) = \{ (g_0G_0, g_1G_1, \ldots) | \nu_n^{n+1}(hG_{n+1}) = gG_n \}
\]
can be identified with the space of paths \( \mathcal{P} \cong \prod_{n \geq 1} L_n \) of a rooted tree \( T \) as in Example 2.1. It follows that the inverse limit space \( G_\infty \) is a Cantor set.

The left action of \( G \) on coset spaces \( G/G_n \) induces a natural left action of \( G \) on \( G_\infty \), given by the left multiplication
\[
g \cdot (g_0G_0, g_1G_1, \ldots) = (gg_0G_0, gg_1G_1, \ldots).
\]

Denote by \((G_\infty, G)\) this action. Since \( G_\infty \) is identified with \( \mathcal{P} \), the action (10) induces an action of \( G \) on \( \mathcal{P} \), denoted by \((\mathcal{P}, G)\). The group \( G \) permutes the cosets in \( G/G_n \), and acts transitively on each coset space \( G/G_n \), so by Example 2.1 the action \((G_\infty, G)\) is minimal and equicontinuous. We call \((G_\infty, G)\) the dynamical system associated to a group chain \( \{G_n\}_{n \geq 0} \).

It turns out that every minimal equicontinuous group action on a Cantor set is conjugate to a dynamical system associated to a group chain as in Definition 2.2. This is a consequence of the following statement, which can be found in [8] and [11, Appendix] for the case when \( G \) is finitely generated. The proof carries on to the case of countably generated groups verbatim, so we omit the details.

**PROPOSITION 2.4.** Let \( X \) be a Cantor set, and let \( G \) be a countably generated group acting on \( X \). Suppose that the action of \( G \) on \( X \) is minimal and equicontinuous, and let \( x \in X \) be a point. Then there is a descending chain of clopen sets \( X = U_0 \supset U_1 \supset \cdots \) with \( \bigcap U_n = \{ x \} \) such that:

1. For each \( n \geq 1 \) the collection of translates \( \{ \Phi(g)(U_n) \}_{g \in G} \) is a finite partition of \( X \) into clopen sets.
2. The collection of elements which preserve \( U_n \), that is,
   \[
   G_n = \{ g \in G | \Phi(g)(U_n) = U_n \},
   \]
   is a subgroup of finite index in \( G \).

Thus \( G = G_0 \supset G_1 \supset G_2 \supset \cdots \) is a descending chain of subgroups of finite index, moreover, there is a homeomorphism
\[
\Phi : X \to G_\infty = \lim_{\rightarrow} (G/G_{n+1} \to G/G_n)
\]
equivariant with respect to the action of \( G \) on \( X \), and the natural action of \( G \) on \( G_\infty \) given by \( s \cdot (g_iG_i) = (sg_iG_i) \), and such that \( \Phi(x) = (eG_n) \), the sequence of the cosets of the identity.

Example 2.5 shows how to associated a group chain to a minimal and equicontinuous action of a countably generated discrete group \( G \) on the path space of a tree \( T \).

**EXAMPLE 2.5.** Let \( T \) be a spherically homogeneous rooted tree with minimal and equicontinuous action of a discrete group \( G \) as in Example 2.1. In particular, the action of \( G \) is transitive on each \( V_n, n \geq 1 \). Let \( \mathcal{P} \) be the space of infinite paths in \( T \).

Let \( x = (v_n)_{n \geq 0} \) be a path. Let \( G_n = \{ g \in G | g \cdot v_n = v_n \} \) be the subgroup of elements in \( G \) which fix the vertex \( v_n \), called the stabilizer of \( v_n \), or the isotropy subgroup of the action of \( G \) at \( v_n \). Since \( G \) acts transitively on the finite set \( V_n \), then we have \( |G : G_n| = |G/G_n| = |V_n| \), so \( G_n \) has finite index in \( G \). If \( g \in G \) fixes \( v_n \), then it fixes \( v_i \) for \( 0 \leq i < n \), which implies that \( G_n \subset G_i \) for \( 0 \leq i < n \). So the isotropy subgroups form a nested chain \( \{G_n\}_{n \geq 0} \) of finite index subgroups of \( G \). For each \( n \geq 0 \), the subgroup \( G_n \) fixes the clopen set \( U_n = U_n(v_n) \).
For a given equicontinuous action \((X, G, \Phi)\), the choice of an associated chain \(\{G_n\}_{n \geq 0}\) depends on a choice of a point \(x \in X\), and on a choice of clopen sets \(U_1 \supset U_2 \supset \cdots\). So the choice of a group chain \(\{G_n\}_{n \geq 0}\) is not unique, and distinct group chains can define conjugate actions.

Rogers and Tollefson [43] suggested to use the following notion of equivalence of group chains to study the question when two group chains define conjugate actions.

**Definition 2.6.** [43] In a group \(G\), two group chains \(\{G_n\}_{n \geq 0}\) and \(\{H_n\}_{n \geq 0}\) with \(G_0 = H_0 = G\) are equivalent if there is a group chain \(\{K_n\}_{n \geq 0}\) and infinite subsequences \(\{G_{nk}\}_{k \geq 0}\) and \(\{H_{jk}\}_{k \geq 0}\) such that \(K_{2k} = G_{nk}\) and \(K_{2k+1} = H_{jk}\) for \(k \geq 0\), with \(n_0 = 0\).

Intuitively, two group chains are equivalent if they can be ‘intertwined’ to form a single descending group chain. Fokkink and Oversteegen [13] investigated the equivalence of group chains, determining when two group chains correspond to conjugate actions. A detailed proof of their result, stated below, can also be found in [11].

**Theorem 2.7.** [13] Let \((\Phi, X, G)\) and \((\Psi, X, G)\) be equicontinuous actions on a Cantor set \(X\) with associated group chains \(\{G_n\}_{n \geq 0}\) and \(\{H_n\}_{n \geq 0}\), \(G_0 = H_0 = G\). Then the actions \((\Phi, X, G)\) and \((\Psi, X, G)\) are conjugate if and only if there exists a sequence of elements \(\{g_n\} \subset G\) such that \(g_n G_n g_n^{-1}\) \(\{g_n\}_{n \geq 0}\) and \(\{H_n\}_{n \geq 0}\) are equivalent group chains. Here \(g_n G_m = g_m G_m\) for all \(n \geq m\) and all \(m \geq 0\).

By Theorem 2.7, to study an action \((X, G, \Phi)\) in terms of group chains, it is sufficient to consider the chains of conjugate subgroups \(\{g_n G_n g_n^{-1}\}_{n \geq 0}\). The condition \(g_n G_m = g_m G_m\) for \(n \geq m\) ensures that the chain \(\{g_n G_n g_n^{-1}\}_{n \geq 0}\) is nested.

### 2.2. Ellis group for equicontinuous actions.

The Ellis (enveloping) semigroup associated to a continuous group action \(\Phi : G \to \text{Homeo}(X)\) on a topological space \(X\) was introduced in the papers [17] [14], and is treated in the books [4] [15] [16]. In this section we briefly recall some basic properties of the Ellis group for a special case of equicontinuous minimal systems on Cantor sets.

Let \(X\) be a metric space, and \(G\) be a countably generated group acting on \(X\) via the homomorphism \(\Phi : G \to \text{Homeo}(X)\). Suppose the action \((X, G, \Phi)\) is equicontinuous. Then the closure \(\overline{\Phi(G)} \subset \text{Homeo}(X)\) of the uniform topology on maps is identified with the Ellis group of the action. Each element of \(\overline{\Phi(G)}\) is the limit of a sequence of maps in \(\Phi(G)\), and we use the notation \((g_i)\) to denote a sequence \(\{g_i \mid i \geq 1\} \subset G\) such that the sequence \(\{\Phi(g_i) \mid i \geq 1\} \subset \text{Homeo}(X)\) converges in the uniform topology.

Assume that the action of \(G\) on \(X\) is minimal, that is, for any \(x \in X\) the orbit \(\Phi(G)(x)\) is dense in \(X\). Then the orbit of the Ellis group \(\overline{\Phi(G)}(x) = X\) for any \(x \in X\). That is, the group \(\overline{\Phi(G)}\) acts transitively on \(X\). Denote the isotropy group of the action at \(x\) by

\[
\overline{\Phi(G)}_x = \{(g_i) \in \overline{\Phi(G)} \mid (g_i) \cdot x = x\},
\]

where \((g_i) \cdot x := (g_i(x))\), for a homeomorphism \((g_i)\) in \(\overline{\Phi(G)}\). We then have the natural identification \(X \cong \overline{\Phi(G)}/\overline{\Phi(G)}_x\) of left \(G\)-spaces.

Given an equicontinuous minimal Cantor system \((X, G, \Phi)\), the Ellis group \(\overline{\Phi(G)}\) depends only on the image \(\Phi(G) \subset \text{Homeo}(X)\), while the isotropy group \(\overline{\Phi(G)}_x\) of the action may depend on the point \(x \in X\). Since the action of \(\overline{\Phi(G)}\) is transitive on \(X\), given any \(y \in X\), there is an element \((g_i) \in \overline{\Phi(G)}\) such that \((g_i) \cdot x = y\). It follows that

\[
\overline{\Phi(G)}_y = (g_i) \cdot \overline{\Phi(G)}_x \cdot (g_i)^{-1}.
\]

This tells us that the cardinality of the isotropy group \(\overline{\Phi(G)}_x\) is independent of the point \(x \in X\), and so the Ellis group \(\overline{\Phi(G)}\) and the cardinality of \(\overline{\Phi(G)}_x\) are invariants of \((X, G, \Phi)\).

The definition of the Ellis group, given above, does not provide an easy way to compute it. In [11], we developed a technique for computing the Ellis group and the isotropy group of its action which uses group chains, associated to Cantor group actions by Proposition 2.4.
For every $G_n$ consider the core of $G_n$, that is, the maximal normal subgroup of $G_n$ given by
\begin{equation}
C_n = \text{core}_G G_n = \bigcap_{g \in G} g G_n g^{-1} \subseteq G_n.
\end{equation}

Since $C_n$ is normal in $G$, the quotient $G/C_n$ is a finite group, and the collection $\{C_n\}_{n \geq 0}$ forms a descending chain of normal subgroups of $G$. The inclusions $C_{n+1} \subset C_n$ induce surjective homomorphisms of finite groups, given by
\[ G/C_{n+1} \to G/C_n : g C_{n+1} \mapsto g C_n. \]

The inverse limit space
\begin{equation}
C_\infty = \lim_{\leftarrow \ell} \left\{ \delta^{\ell+1}_f : G/C_{\ell+1} \to G/C_\ell \right\} \subseteq \prod_{\ell \geq 0} G/C_\ell
\end{equation}
is a profinite group. Also, since $G_{n+1} \subset G_n$ and $C_{n+1} \subset C_n$, there are well-defined homomorphisms of finite groups $\delta_n : G_{n+1}/C_{n+1} \to G_n/C_n$, and there is the inverse limit group
\[ D_x = \lim_{\leftarrow \ell} \{ G_{n+1}/C_{n+1} \to G_n/C_n \}, \]
called the discriminant group of this action.

**Theorem 2.8.** \[11\] The profinite group $C_\infty$ is isomorphic to the Ellis group $\Phi(G)$ of the action $(X, G, \Phi)$, and the isotropy group $\Phi(G)_x$ of the Ellis group action is isomorphic to $D_x$.

By Proposition 2.4, the group chain $\{G_n\}_{n \geq 0}$ depends on the choice of a point $x \in X$, and on the choice of a sequence of clopen sets $X = U_0 \supset U_1 \supset \cdots$ such that the isotropy group of the action of $G$ on $U_n$ is $G_n$. Since the groups $C_n$ are normal, they do not depend on the choice of $x \in X$, but they may depend on the choice of the clopen sets $\{U_n\}_{n \geq 0}$. For any choice of $x$ and $\{U_n\}_{n \geq 0}$, the group $C_\infty$ is isomorphic to the Ellis group $\Phi(G)$, so $C_\infty$ is independent of choices up to an isomorphism. One can think of $C_\infty$ as a choice of ‘coordinates’ for the Ellis group $\Phi(G)$.

Similarly, the discriminant group $D_x$ does not depend on choices up to an isomorphism. We note that, since $D_x$ is a closed subgroup of a compact group $C_\infty$, it can either be finite or an infinite profinite group which is topologically a Cantor set.

**Example 2.9.** Suppose the group $G$ is abelian, and let $\{G_n\}_{n \geq 0}$ be a group chain in $G$. Since $G$ is abelian, for $n \geq 1$ the subgroup $G_n$ is normal in $G$, and so $C_n = G_n$. Thus for $n \geq 1$ the quotient space $G_n/C_n$ is a singleton, and it follows that the discriminant group $D_x$ is trivial.

The relationship between the cardinality of $\Phi(G)_x$ and the properties of the action was studied in \[11\] \[12\]. Automorphisms of the Cantor group action $(G_\infty, G)$ (where $G$ acts on the left) are given by the right action of elements of $C_\infty$ on $G_\infty$. It is shown in \[11\] that the automorphism group acts transitively on $G_\infty$ if and only if the isotropy group $\Phi(G)_x \cong D_x$ is trivial. Thus non-triviality of $D_x$ is seen as an obstruction to the transitivity of the action of the automorphism group of $(G_\infty, G)$, and for this reason it was called the discriminant group in \[11\]. The article \[11\] also contains examples of actions with finite non-trivial discriminant group, and examples where the discriminant group is a Cantor group.

### 2.3. The asymptotic discriminant

Let $(X, G, \Phi)$ be a group action on a Cantor set, let $x$ be a point and let $\{G_n\}_{n \geq 0}$ be an associated group chain, that is, the actions $(X, G, \Phi)$ and $(G_\infty, G)$ are conjugate. Recall from Proposition 2.4 that the groups $G_m$, $m \geq 0$, are the isotropy groups of the action of $G$ at the clopen sets $U_m$, that is, the restricted action $\Phi_m = \Phi|_{U_m}$ is the action of $G_m$.

Set $X_m = U_m$, and consider a family of equicontinuous group actions $(X_m, G_m, \Phi_m)$. Then for each $m \geq 0$ we can compute the Ellis group of the action, and the isotropy group at $x$ as follows.

For each $n \geq m \geq 0$, compute the maximal normal subgroup of $G_n$ in $G_m$ by
\begin{equation}
C_n^m = \text{core}_{G_m} G_n = \bigcap_{g \in G_m} g G_n g^{-1} \subseteq G_m.
\end{equation}
Note that $C^m_n$ is the kernel of the action of $G_n$ on the quotient set $G_m/G_n$, and $C^0_n = C_n$. Moreover, for all $n > k \geq m \geq 0$, we have $C^m_n \subset C^k_n \subset G_n \subset G_k \subset G_m$, and $C^m_n$ is a normal subgroup of $G_k$.

Define the profinite group

$$C_{k,\infty}^m \cong \lim_{\leftarrow} \left\{ G_k/C^m_n \to G_k/C^m_{n+1} \mid n \geq k \right\} = \left\{ (g_n C^m_n) \mid n \geq k \; , \; g_k \in G_k \; , \; g_{n+1} C^m_n = g_n C^m_n \right\} .$$

Then $C_{m,\infty}^m$ is the Ellis group of the action $(X_m, G_m, \Phi_m)$, with an associated group chain $\{G_n\}_{n \geq m}$. In particular, $C^0_{0,\infty} = C_{\infty}$, defined by (14).

Since $G_k \subset G_m$, by definition we have that $C^m_{k,\infty} \subset C^m_{m,\infty}$, and so $C^m_{k,\infty}$ is a clopen neighborhood of the identity in $C^m_{m,\infty}$.

The topological discriminant group associated to the truncated group chain $\{G_n\}_{n \geq m}$ is given by

$$D^m_x = \lim_{\leftarrow} \left\{ G_{n+1}/C^m_{n+1} \to G_n/C^m_n \mid n \geq m \right\} \subset C^m_{m,\infty}$$

and

$$D^m_k = \lim_{\leftarrow} \left\{ G_{n+1}/C^m_{n+1} \to G_n/C^m_n \mid n \geq k \right\} \subset C^m_{k,\infty},$$

where we have $D^m_x \subset C^m_{k,\infty}$ since $G_n \subset G_k$ for $n \geq k$. The last statement can be rephrased as saying that the discriminant group $D^m_x$ is contained in any clopen neighborhood of the identity in $C^m_{m,\infty}$.

To relate the discriminant groups $D^m_x$ and $D^m_k$ for $k \geq m$, we define the following maps.

For each $n \geq k \geq m \geq 0$, the inclusion $C^m_n \subset C^k_n$ induces surjective group homomorphisms

$$\phi^m_{n,k} : G_n/C^m_n \to G_n/C^k_n,$$

and the standard methods show that the maps in (20) yield surjective homomorphisms of the clopen neighborhoods of the identity in $C^m_{m,\infty}$ onto the profinite groups $C^k_{k,\infty}$,

$$\hat{\phi}^m_{n,k} : C^m_{k,\infty} \to C^k_{k,\infty},$$

which commute with the left action of $G$. Let $D_{m,k} \subset C^m_{k,\infty}$ denote the image of $D^m_x$ under the map (21). It then follows from (20) that for $k > m \geq 0$, there are surjective homomorphisms,

$$D_x = D^0_x \xleftarrow{\phi^0_{m,m}} D_{0,m} \cong D_x \xrightarrow{\hat{\phi}^m_{m,k}} D^k_x .$$

Thus, given an equicontinuous group action $(X, G, \Phi)$ on a Cantor set $G$, there is an associated sequence of surjective homomorphisms of topological discriminant groups (22), associated to the sequence of truncated group chains $\{G_n\}_{n \geq m}$, $m \geq 0$.

We now define an equivalence relation on such group chains, called the tail equivalence, first introduced by the author in the joint work with Huder (26).

**Definition 2.10.** Let $A = \{\phi_n : A_n \to A_{n+1} \mid n \geq 1\}$ and $B = \{\psi_n : B_n \to B_{n+1} \mid n \geq 1\}$ be two sequences of surjective group homomorphisms. We say that $A$ and $B$ are tail equivalent, and write $A \overset{\text{t}}{\sim} B$, if the sequences of groups $A$ and $B$ are intertwined by a sequence of surjective group homomorphisms. That is, there exists:

1. an increasing sequence of indices $\{p_n \mid n \geq 1 \; , \; p_{n+1} > p_n \geq n \geq 1\}$;
2. an increasing sequence of indices $\{q_n \mid n \geq 1 \; , \; q_{n+1} > q_n \geq n \geq 1\}$;
3. a sequence $C = \{c_n : C_n \to C_{n+1} \mid n \geq 1\}$ of surjective group homomorphisms;
4. a collection of isomorphisms $\Pi_{AC}^n : A_{p_n} \to C_{2n-1} \mid n \geq 1\}$;
5. a collection of isomorphisms $\Pi_{BC}^n : B_{q_n} \to C_{2n} \mid n \geq 1\}$;

such that for all $n \geq 1$, the following diagram commutes:
A sequence $\mathcal{A}$ is constant if each map $\phi_n: A_n \rightarrow A_{n+1}$ is an isomorphism for all $n \geq 1$, and $\mathcal{B}$ is said to be asymptotically constant if it is tail equivalent to a constant sequence $\mathcal{A}$. The following result follows from the usual method of "chasing of diagrams".

**Lemma 2.11.** [26] A sequence $\mathcal{B} = \{\psi_n: B_n \rightarrow B_{n+1} | n \geq 1\}$ of surjective homomorphisms is asymptotically constant if and only if there exists $n_0 \geq 0$ such that $\ker(\psi_n)$ is trivial for all $n \geq n_0$.

We now use the ‘tail equivalence’ of group chains of Definition 2.10 to introduce the notion of the asymptotic discriminant of a Cantor minimal action, and the notions of a stable and a wild action.

**Definition 2.12.** [26] Let $(X, G, \Phi)$ be an action of a countably generated group $G$ on a Cantor set $X$, and let $\{G_n\}_{n \geq 0}$ be an associated group chain. Then the asymptotic discriminant for $\{G_n\}_{n \geq 0}$ is the tail equivalence class $[D_x^m]_\infty$ of the sequence of surjective group homomorphisms

$$[D_x^m]_\infty = \{\psi_{m,m+1}: D_x^m \rightarrow D_x^{m+1} | m \geq 1\}$$

defined by the discriminant groups $D_x^m$ for the restricted actions of $G_m$ on the clopen sets $X_m \subset X$.

The action $(X, G, \Phi)$ is stable if the asymptotic discriminant $[D_x^m]_\infty$ is asymptotically constant, and the action is wild otherwise.

It was shown in [26] that the asymptotic discriminant is invariant under return equivalence of group actions. Intuitively, two actions $(X_1, G, \Phi)$ and $(X_2, G, \Psi)$ are return equivalent, if there are clopen sets $U \subset X_1$ and $V \subset X_2$ such that the collections of local homeomorphisms of $U$ and $V$, induced by the actions, are compatible in a sense made precise in [9]. This notion is analogous to the notion of Morita equivalence for groupoids. In the joint work with Hurder [26], the author constructed an uncountable number of Cantor group actions of the same subgroup of $\text{SL}(n, \mathbb{Z})$ with pairwise distinct asymptotic discriminants. These actions are not return equivalent.

If an action $(X, G, \Phi)$ is stable, then its asymptotic discriminant is asymptotically constant, which means that there exists $m_0 \geq 0$ such that for all $k > m \geq m_0$, group homomorphisms $\tilde{\phi}_{m,k}$, defined in (22), are isomorphisms. It was shown in [12] that every finite group, and every separable profinite group can be realized as the discriminant group of a stable minimal Cantor action.

**Example 2.13.** If $G$ is an abelian group as in Example 2.9 then for all $m \geq 0$ the discriminant group $D_x^m$ is trivial. The maps $\psi_{m,m+1}$ are trivially isomorphisms, and the asymptotic discriminant $[D_x^m]_\infty$ is asymptotically constant. Thus every equicontinuous minimal action of an abelian group $G$ on a Cantor set has stable asymptotic discriminant.

### 2.4. Locally quasi-analytic actions.

In this section we briefly describe the relationship between the stability of the action, as defined in Definition 2.12, and the geometric properties of the action. For details, we refer the reader to [26, 25].

We first recall the notion of a locally quasi-analytic action. Haefliger introduced in [24] the notion of a quasi-analytic action on a topological space, in his study of pseudogroups of local isometries on locally connected spaces. The works [24, 23] by Álvarez-López, Candel and Moreira-Galicia reformulated Haefliger’s definition for the case of topological actions on Cantor spaces as follows:
**DEFINITION 2.14.** [2] Definition 9.4] A topological action \((X, G, \Phi)\) is locally quasi-analytic, or simply LQA, if there exists \(\epsilon > 0\) such that for any non-empty open set \(U \subset X\) with \(\text{diam}(U) < \epsilon\), and for any non-empty open subset \(V \subset U\), and elements \(g_1, g_2 \in G\)

\[(25) \quad \text{if the restrictions } \Phi(g_1)|V = \Phi(g_2)|V, \text{ then } \Phi(g_1)|U = \Phi(g_2)|U.\]

The action is said to be quasi-analytic if \(25\) holds for \(U = X\).

Examples of Cantor group actions which are locally quasi-analytic, but not quasi-analytic, are given in [12, 20]. If an action \((X, G, \Phi)\) is quasi-analytic on an open set \(U\), then for every element \(g \in G\) and every clopen subset \(V \subset U\), the restriction \(g|V\) has a unique extension to \(U\). In other words, the action of \(G\) on \(U\) is locally determined.

Let \((X, G, \Phi)\) be a Cantor action, and let \(\Phi(G)\) be the enveloping group of the action. Recall from Section 2.2 that \(\Phi(G)\) acts transitively on \(X\), and \(G\) embeds as a dense subgroup of \(\Phi(G)\). Thus if the action of \(\Phi(G)\) has the LQA property, then the action of \(G\) has LQA property as well. It is an open problem whether the converse statement holds.

The relationship between the LQA property of group actions and their asymptotic discriminant was studied in [20]. In particular, Proposition 7.4 of [20] can be rephrased as follows.

**PROPOSITION 2.15.** [20] Proposition 7.4] Let \((X, G, \Phi)\) be a minimal equicontinuous action of a countably generated group \(G\) on a Cantor set \(X\), and \(x \in X\) be a point. Let \(\{G_n\}_{n \geq 0}\) be an associated group chain, and let \(\Phi(G)\) be the enveloping group of the action. Let \([D^m_x]\) be the asymptotic discriminant of the action. Thus the asymptotic discriminant \([D^m_x]\) is asymptotically constant if and only if the action of \(\Phi(G)\) on \(X\) is LQA.

The LQA property for a group action \((X, G, \Phi)\) can be interpreted in terms of the properties of the germinal groupoid \(\mathcal{G}(X, G, \Phi)\) associated to the action. This groupoid is fundamental for the study of the \(C^*\)-algebras these actions generate, as discussed for example by Renault in [40, 41]. Recall that for \(g_1, g_2 \in G\), we say that \(\Phi(g_1)\) and \(\Phi(g_2)\) are germinally equivalent at \(x \in X\) if \(\Phi(g_1)(x) = \Phi(g_2)(x)\), and there exists an open neighborhood \(x \in U \subset X\) such that the restrictions agree, \(\Phi(g_2)|U = \Phi(g_2)|U\). We then write \(\Phi(g_1) \sim_x \Phi(g_2)\). For \(g \in G\), denote the equivalence class of \(\Phi(g)\) at \(x\) by \([g]_x\). The collection of germs \(\mathcal{G}(X, G, \Phi) = \{[g]_x \mid g \in G, x \in X\}\) is given the sheaf topology, and forms an étale groupoid modeled on \(X\). We recall the following result.

**PROPOSITION 2.16.** [17] Proposition 2.1] The germinal groupoid \(\mathcal{G}(X, G, \Phi)\) is Hausdorff at \([g]_x\) if and only if, for all \([g']_x \in \mathcal{G}(X, G, \Phi)\) with \(g \cdot x = g' \cdot x = y\), if there exists a sequence \(\{x_n\} \subset X\) which converges to \(x\) such that \([g]_x = [g']_x\) for all \(n\), then \([g]_x = [g']_x\).

For Cantor group actions, the following result was obtained in [25].

**PROPOSITION 2.17.** [25] Proposition 2.5] If an action \((X, G, \Phi)\) is locally quasi-analytic, then \(\mathcal{G}(X, G, \Phi)\) is Hausdorff.

Thus if the groupoid \(\mathcal{G}(X, G, \Phi)\) is non-Hausdorff, then the action \((X, G, \Phi)\) is not LQA.

In Proposition 2.17 consider the composition of maps \(h = g^{-1} \circ g'\). Since \(g \cdot x = g' \cdot x\), then \(h \cdot x = g^{-1} \circ g' \cdot x = x\). Then the statement of the proposition reads as follows: the groupoid \(\mathcal{G}(X, G, \Phi)\) is Hausdorff if and only if for all \([h]_x \in \mathcal{G}(X, G, \Phi)\) with \(h \cdot x = x\), if there exists a sequence \(\{x_n\} \subset X\) which converges to \(x\) such that \([h]_x = [id]_x\) for all \(i\), where \(id\) denotes the identity map, then \([h]_x = [id]_x\).

Taking the contrapositive of this statement, we obtain that \(\mathcal{G}(X, G, \Phi)\) is a non-Hausdorff groupoid if and only if there exists a germ \([h]_x \in \mathcal{G}(X, G, \Phi)\) with \(h \cdot x = x\), and a sequence \(\{x_n\} \subset X\) which converges to \(x\) such that \([h]_{x_n} = [id]_{x_n}\) for all \(i\), and \([h]_x \neq [id]_x\). We call such a germ \([h]_x\), and its representative \(h\), a non-Hausdorff element of \(\mathcal{G}(X, G, \Phi)\).

**DEFINITION 2.18.** Let \((X, G, \Phi)\) be a group action on a Cantor set \(X\). Then \(h\) is a non-Hausdorff element of \(G\) if it has the following property: there exists a point \(x \in X\) and a collection
\{U_n\}_{n \geq 0} of decreasing open neighborhoods of \(x\), with \(\bigcap U_n = \{x\}\), such that for each \(n \geq 0\), an open set \(U_n\) contains an open subset \(V_n\) such that \(h|_{V_n} = \text{id}\), but \(h|_{U_n} \neq \text{id}\).

In the following lemma the groups \(G\) and \(H\) can be either discrete or profinite.

**Lemma 2.19.** Let \((X, G, \Phi)\) be a minimal Cantor action, and let \(H\) be a subgroup of \(G\). If the action of \(H\) on \(X\) is non-LQA then the action of \(G\) on \(X\) is non-LQA.

**Proof.** Since the action of \(H\) is non-LQA, then there exists a descending chain of clopen neighborhoods \(\{U_n\}_{n \geq 0}\) such that \(\bigcap U_n\) a singleton, and, for each \(U_n\), an element \(g_n\) such that the restriction \(g_n|_{U_n} = \text{id}\) and \(g_n|_{U_{n-1}} \neq \text{id}\). Since \(H\) is a subgroup of \(G\), every such \(g_n\) is also in \(G\), and so the action of \(G\) on \(X\) is non-LQA.

We will also need the following proposition, the prove of which is immediate.

**Proposition 2.20.** Let \(\Phi_i(G_i) \subset \text{Homeo}(X)\), \(i = 1, 2\), be actions of two countably generated groups on a path space \(X\). Suppose we have \(\Phi_2(G_2) = w \circ \Phi_1(G_1) \circ w^{-1}\), where \(w \in \text{Homeo}(X)\). Then the following statements are true:

1. The action of \(\Phi_1(G_1)\) on \(X\) is LQA if and only if the action of \(\Phi_2(G_2)\) on \(X\) is LQA.
2. There is a non-Hausdorff element \(g \in \Phi_1(G_1)\) if and only if there is a non-Hausdorff element \(h \in \Phi_2(G_2)\).
3. The action of \(\Phi(G_1)\) on \(X\) is stable with finite discriminant group if and only if the action of \(\Phi(G_2)\) on \(X\) is stable with finite discriminant group.

**Proof.** For (1), we prove the contrapositive. Suppose \((X, G_1, \Phi_1)\) is non-LQA, then there exists a collection of open sets \(\{U_n\}_{n \geq 0}\) with \(\bigcap U_n = \{x\}\), and a collection of elements \(g_n \in \Phi_1(G_1)\) such that \(g_n|_{U_n} \neq \text{id}\), while \(g_n|_{U_{n-1}} = \text{id}\). Let \(y = w(x)\), and \(W_n = w(U_n)\), then \(\{W_n\}_{n \geq 0}\) is a collection of open neighborhoods of \(y\) with \(\bigcap W_n = \{y\}\). Let \(h_n = wg_nw^{-1}\), then \(h_n\) acts non-trivially on \(W_n\), and trivially on \(W_{n+1}\). Thus \((X, G_2, \Phi_2)\) is non-LQA. The converse is obtained by reversing the arrows in this argument, and the proof of (2) is similar.

For (3), let \(y = w(x)\), and note that there is a conjugacy of the isotropy groups \(\Phi_2(G_2)_y = w \circ \Phi_1(G_1)_y \circ w^{-1}\), so \(\Phi_2(G_2)_y\) is finite if and only if \(\Phi_1(G_1)_y\) is finite. Under the maps (24) on discriminant groups, their cardinality can only decrease, so the action of \(\Phi_1(G_1)\) has stable asymptotic discriminant with finite discriminant group if and only if the action of \(\Phi_2(G_2)\) has stable asymptotic discriminant with finite discriminant group.

We note that the conjugacy of the closures \(\Phi_i(G_i)\), \(i = 1, 2\), need not imply the conjugacy of the actions of \(G_i\), \(i = 1, 2\). For instance, Nekrashevych [34] gives examples of the actions of discrete non-isomorphic subgroups of \(\text{Aut}(T)\) whose closures in \(\text{Aut}(T)\) are conjugate. Therefore, in (3) we cannot conclude that the chains of the discriminant groups for the actions of \(\Phi_1(G_1)\) and \(\Phi_2(G_2)\) stabilize to discriminant groups of the same cardinality, but can only make a statement about the finiteness of the discriminant group.

3. Wreath products and self-similar actions

In this section we recall the background on wreath products and automatic groups, which is necessary for the rest of the paper. The main reference here is Nekrashevych [33]. A nice concise exposition of the parts of the theory needed to work with Galois groups of post-critically finite polynomials can be found in [30].
3.1. Wreath products and the automorphism group of a tree. Let $T$ be a $d$-ary rooted tree as in Example 2.1 that is, for $n \geq 1$, $L = L_n$ is a set with $d$ elements. Denote by $L^n = L \times \cdots \times L$ the $n$-fold product. The set of vertices $V_n$ contains $d^n$ elements, every vertex $v \in V_n$ is connected by edges to precisely $d$ vertices in $V_{n+1}$, and every vertex in $V_{n+1}$ is connected by an edge to precisely one vertex in $V_n$. Denote by $\text{Aut}(T)$ the automorphism group of $T$. The elements of $\text{Aut}(T)$ act on the vertex sets $V_n$ by permutations in such a way that the connectedness of tree is preserved, that is, for each $g \in \text{Aut}(T)$ and every pair $v \in V_n$ and $w \in V_{n+1}$ there is an edge $[v, w] \in E$ if and only if there is an edge $[g \cdot v, g \cdot w] \in E$. Thus $\text{Aut}(T)$ acts by homeomorphisms on the path space $\mathcal{P}_d$ of $T$. For completeness, we briefly recall how to compute $\text{Aut}(T)$ from [5].

As in Example 2.1 for $n \geq 1$ let $\text{pr}_{n-1} : L^n \to L^{n-1}$ be the projection on the first $n - 1$ factors in $L^n$, and let $b_n : L^n \to V_n$ be a bijection such that $\text{pr}_{n-1}(b_n^{-1}(v)) = \text{pr}_{n-1}(b_n^{-1}(w))$ if and only if $v$ and $w$ are connected by edges to the same vertex in $V_{n-1}$. Denote by $S_d$ the symmetric group on $d$ elements. Denote by $T_n$ the connected subtree of $T$ with the vertex set $V_0 \sqcup V_1 \sqcup \cdots \sqcup V_n$. The computation is by induction.

Note that $V_1 \cong L$, and so $\text{Aut}(T_1) = S_d$. Suppose $\text{Aut}(T_n)$ is known. Denote by $f : V_n \to S_d$ a function which assigns a permutation of $L$ to each $v \in V_n$, and let $S_d^{[V_n]} = \{ f : V_n \to S_d \}$ be the set of all such functions. Then the wreath product

$$S_d^{[V_n]} \rtimes \text{Aut}(T_n)$$

acts on $V_n \times L$ by

$$(f, s)(v_n, w) = (s(v_n), f(s(v_n)) \cdot w).$$

That is, the action (27) permutes the copies of $L$ in the product $V_n \times L$, while permuting elements within each copy of $L$ independently. Since $V_{n+1} \cong V_n \times L \cong L^n \times L$, it follows that there are isomorphisms

$$\text{Aut}(T_{n+1}) \cong S_d^{[V_n]} \rtimes \text{Aut}(T_n) \cong S_d^{[V_n]} \rtimes \cdots \rtimes S_d^{[V_n]} \rtimes S_d$$

of the group $\text{Aut}(T_{n+1})$ to the $(n + 1)$-fold product $[S_d]^{n+1}$ of the symmetric groups $S_d$.

Next, note that there are natural epimorphisms $\text{Aut}(T_{n+1}) \to \text{Aut}(T_n)$, induced by the projection on the second component in (26). Thus the automorphism group of the $d$-ary rooted tree $T$ is the profinite group

$$\text{Aut}(T) = \lim_{\longleftarrow} \{ \text{Aut}(T_{n+1}) \to \text{Aut}(T_n), n \geq 0 \} \cong \cdots \rtimes S_d^{[V_n]} \rtimes \cdots \rtimes S_d^{[V_1]} \rtimes S_d.$$

We will also denote such infinite wreath product of symmetric groups by $[S_d]^\infty$.

3.2. Self-similarity. The automorphism group $\text{Aut}(T)$ of a $d$-ary tree, and some of its subgroups, have an interesting property called self-similarity. We first describe this property for $\text{Aut}(T)$, and then we define it for the subgroups of $\text{Aut}(T)$.

Recall from Example 2.1 that the bijections $b_n : L^n \to V_n$ assign to each $v_n$ a label $b_n^{-1}(v_n) = t_1 t_2 \cdots t_n$, where $t_i \in L$ for $1 \leq i \leq n$. We suppress the notation for $b_n^{-1}$ and just write $v_n = t_1 \cdots t_n$. The labels are assigned in such a way that $v_{n+1} = t_1 t_2 \cdots t_n t_{n+1}$ is joined by an edge to $v_n \in V_n$ if and only if $v_n = t_1 t_2 \cdots t_n$. Thus an infinite sequence $t = t_1 t_2 \cdots t_n \cdots$ corresponds to an infinite path in $\mathcal{P}_d$.

Let $v = v_1 v_2 \cdots v_m$ be a finite word of length $m$, and denote by $vT$ a subtree of $T$ containing all paths through the vertex $v \in V_m$, that is, all paths which contain the finite subword $v$. The path space of $vT$ is a clopen subset $U_m(v)$ of $\mathcal{P}_d$. Every vertex of $vT \cap V_n$ for $n \geq m$ has a label of the form $w$, where $w$ is a word of length $n - m$ in $L$. Every letter in $v$ or $w$ is a symbol in $L$, so there is a bijection on the sets of vertices

$$\pi_v : vT \cap V \to V : vw \mapsto w,$$

which induces a homeomorphism of path spaces $\pi_v : U_m(v) \to \mathcal{P}_d$. 

Now let $g \in \text{Aut}(T)$, and suppose $g$ maps $v \in V_m$ to a vertex $g(v) \in V_m$. The action of $g$ induces a homeomorphism $\Phi(g) : P_d \to P_d$, and in particular maps the clopen set $U_m(v)$ homeomorphically onto the clopen set $U_m(g(v))$. More precisely, for each vertex $vw \in vT$ there is a unique vertex $g(vw) \in g(v)T$, which is labelled by a word $g(v)w'$ for some finite word $w'$. Composing the bijections $P_d \to P_d$ for $v$ and $g(v)$, we can define the bijection
\[ g|_v = \pi_{g(v)} \circ g \circ \pi_{v^{-1}} : V \to V : w \mapsto w', \]
which induces a homeomorphism $g|_v : P_d \to P_d$, and so defines an automorphism of the tree $T$.

**Remark 3.1.** Note that $g|_v$ is not a restriction in the usual sense. The easiest way to see that is by example. So let $T$ be a binary tree, that is, $L = V_1 = \{0, 1\}$. Then $V_2 = \{00, 01, 10, 11\}$. Define an automorphism of $T$ by setting
\[ g(00v) = 00v, \quad g(01v) = 01v, \quad g(10v) = 11v, \quad g(11v) = 10v. \]
Then $g|_{10}$ defined by (31) is the identity map of the path space $P_2$. On the other hand, if we consider the restriction of $g$ to the clopen set $U_2(10)$ in the usual sense, we see that $g(U_2(10)) = U_2(11)$, in particular, the domain and the range of the restriction $g|U_2(10)$ are disjoint.

We will need to consider both restrictions, so we adopt the following convention for the rest of the article. The notation $g|_v$, with a superscript, denotes maps as in (31). The notation $g|U_m(v)$, or $g|vT$ denotes the usual restriction of a map to a subset. We realize that these two quite similar notations may cause some confusion, but since the notation in (31) became standard in the study of the actions of self-similar groups, and the notation for the restrictions of maps to a set is standard and widely used too, we will continue to use both.

The following definition of self-similar actions is adopted to the action of subgroups of $\text{Aut}(T)$.

**Definition 3.2.** [33 Definition 1.5.3] Let $G$ be a subgroup of $\text{Aut}(T)$. Then $G$ is self-similar if for every $g \in G$ and every vertex $v$ in $T$ the map $g|_v$ defined by (31) is in $G$.

By Definition 3.2 $\text{Aut}(T)$ is self-similar. For self-similar subgroups of $\text{Aut}(T)$, we have the following representation.

Suppose $G \subset \text{Aut}(T)$ is self-similar, and let $g \in G$. Recall that $V_1$ is a set with $d$ vertices. Set $\sigma_g = g|V_1$, that is, $\sigma_g$ is a permutation of vertices in $V_1$ induced by the action of $g$. For every $v \in V_1$ we have $g|_v \in G$, so we can define a function $f_g : V_1 \to G^{[V_1]} : v \mapsto g|_{\sigma^{-1}_g(v)}$. Then $g$ acts on $T$ as the element $(f_g, \sigma_g)$ of the semi-direct product $G^{[V_1]} \rtimes S_d$, where $S_d$ denotes the symmetric group on $d$ elements. More precisely, by formula (27), if $w = (w_1 w_2 \cdots) \in P_d$, then $g$ acts on $w_1$ as $\sigma_g$, and on the infinite sequence $w_2 w_3 \cdots$ as $f_g(\sigma_g(w_1)) = g_{w_1}$. Thus we can represent $g$ as a composition of two maps,
\[ g = (g|_{\sigma^{-1}_g(0)}; g|_{\sigma^{-1}_g(1)}; \cdots; g|_{\sigma^{-1}_g(d-1)}) \circ \sigma_g, \]
\[ \text{where } \sigma_g = (1, \sigma_g) \in G^{[V_1]} \rtimes S_d \text{ and } (g|_{\sigma^{-1}_g(0)}; g|_{\sigma^{-1}_g(1)}; \cdots; g|_{\sigma^{-1}_g(d-1)}) \in G^{[V_1]} \]. Here 1 denotes the trivial function in $G^{[V_1]}$ which assigns to each $v \in V_1$ the identity map of $vT$. When computing (32), we first apply the permutation $\sigma_g$ to the level $V_1$, and then the maps $g|_{\sigma^{-1}_g(v)}$ to the subtrees $vT$, $0 \leq v \leq d - 1$.

Alternatively, we can also write $g$ as the following composition
\[ g = \sigma_g \circ (g|_0; g|_1; \cdots; g|_{d-1}), \]
that is, when computing the action of $g$ we first apply the maps $g|_v$ to the subtrees $vT$, $0 \leq v \leq d - 1$, and then we apply the permutation $\sigma_g$ of $V_1$. Different sources in the literature use one or the other of these two ways to write an automorphism $g \in \text{Aut}(T)$ as a composition of two maps. In particular, [33, 30, 34] use (33), and [29] uses (32). We will follow [39] and mostly use (32), as our results rely on those of [39], and also this notation is consistent with our definition of the wreath product in (27). Formulas (32) and (33) together give the relation
\[ \sigma_g \circ (g|_0; g|_1; \cdots; g|_{d-1}) = (g|_{\sigma^{-1}_g(0)}; g|_{\sigma^{-1}_g(1)}; \cdots; g|_{\sigma^{-1}_g(d-1)}) \circ \sigma_g, \]
Using (32), we can write the elements of \( G \) which one can use to change from one notation to another one.

EXAMPLE 3.3. Let \( \sigma \in S_d \) be a permutation. Then the element

\[
a = (a, 1, \ldots, 1) \sigma
\]

first acts as \( \sigma \) on the set \( V_1 \), and then as \( a \) on the subtree \( 0T \), and as the identity map on every subtree \( sT \), where \( s \in \{1, 2, \ldots, d-1\} \). This means that we have to apply \( \sigma \) to the set \( 0T \cap V_2 \), then \( a \) to \( 00T \) and the identity map to \( 01T \), and then continue inductively, unraveling how \( a \) acts on each next vertex level \( V_n, n \geq 3 \).

If \( \sigma \) is a trivial permutation, then \( a = (a, 1, \ldots, 1) \) is the trivial map.

Next, suppose \( a \) is an element in \( \text{Aut}(T) \) which generates the action of an odometer, or the adding machine, on \( \mathcal{P}_d \). More precisely, start with a word \( w = (w_1 w_2 \cdots w_n \cdots) \), then

\[
(35) \quad a = \begin{cases} 
    a(w) = (w_1 + 1)w_2 \cdots, & \text{if } w_1 \neq d - 1, \\
    a(w) = 0 \cdots 0(w_n + 1)w_{n+1} \cdots, & \text{if } w_i = d - 1 \text{ for } 1 \leq i \leq n - 1, \text{ and } w_n \neq d - 1, \\
    a(w) = 000 \cdots, & \text{if } w_n = d - 1 \text{ for } n \geq 1.
\end{cases}
\]

Clearly, \( a \) acts on \( V_1 \) as a transitive permutation \( \sigma = (0 1 \cdots (d - 1)) \). For \( v \in \{0, 1, \ldots, d - 2\} \) we have \( a|_v = id \), and \( a|_{d-1} = a \). Then, using (32),

\[
a = (a_{\sigma^{-1}(0)}, a_{\sigma^{-1}(1)}, \ldots, a_{\sigma^{-1}(d-1)})\sigma = (a, 1, 1 \cdots, 1)\sigma.
\]

It is convenient to represent the action of an element \( g \in \text{Aut}(T) \) in a diagram, as in Figure 1. In Figure 1, \( d = 2 \), so \( T \) is a binary tree. We think of the labeling as increasing from left to right, that is, if \( w_0, w_1 \) are vertices in \( V_{n+1} \) connected to \( v \in V_n \), with \( w_0 \) on the left and \( w_1 \) on the right in the picture, then \( w_1 = v0 \) and \( w_2 = v1 \). An arc joining two edges emanating from the same vertex \( v \) indicates that the restriction \( a|_v \) is a non-trivial permutation of the set of two elements. Two shorter edges emanating from a vertex \( v \) show that the restriction \( a|_v \) is the identity map. As it was explained in Remark 3.1, although \( a|_v = id \), this does not mean that \( a \) fixes the subtree \( vT \). Indeed, \( a \) may act non-trivially on the vertex set containing \( v \). For example, the diagram in Figure 1 represents a generator of an odometer action, which acts transitively on the vertices of the tree \( T \) at each level. Diagrams as in Figure 1 are called portraits in [33].

Applying (34) to \( a = (a, 1)\sigma \) we can write \( a = \sigma(1, a) \). Then the portrait for \( a \) is the mirror image of the one in Figure 1. Thus the portrait of an element \( g \in \text{Aut}(T) \) depends on the representation of \( g \) as a composition in (32) or (33).

The restrictions (31) satisfy, for any finite words \( v, w \), the relations (33)

\[
(36) \quad g|_{vw} = g|_v|_w, \text{ and } g(vw) = g(v)g|_v(w).
\]
**Definition 3.4.** Let \( G \subset Aut(T) \) be a self-similar subgroup. Then \( G \subset Aut(T) \) is contracting, if there is a finite set \( \mathcal{N} \subset G \) such that for every \( g \in G \) there is \( n_g \geq 0 \) such that for all finite words \( v \) of length at least \( n_g \) we have \( g|_v \in \mathcal{N} \).

The set \( \mathcal{N} \) is called the *nucleus* of the group \( G \), if \( \mathcal{N} \) is the smallest possible set satisfying Definition 3.4. Iterated monodromy groups of post-critically finite polynomials which are the object of the study in this paper are known to be contracting \[33, \text{Theorem 6.4.4}\].

We also consider the following special subsets, introduced in \[30\].

Let \( G \subset Aut(T) \) be a group. Let
\[
\mathcal{N}_0 = \{ g \in G \mid g|_v = g \text{ for some non-empty } v \in \mathcal{P}_d \}.
\]
The set \( \mathcal{N}_0 \) is always non-empty, as it contains the identity of \( G \). It is proved in \[33\], Proposition 3.5] that if \( G \) is contracting, then \( \mathcal{N}_0 \) is finite and the nucleus of \( G \) is given by
\[
\mathcal{N} = \{ h \in G \mid h = g|_v \text{ for some } g \in \mathcal{N}_0 \text{ and } v \in \mathcal{P}_d \}.
\]
Also define
\[
\mathcal{N}_1 = \{ g \in G \mid g|_v = g \text{ and } g(v) = v \text{ for a non-empty word } v \in \mathcal{P}_d \}.
\]
Then \( \mathcal{N}_1 \) contains elements in \( \mathcal{N}_0 \) which fix at least one vertex in \( T \), so \( \mathcal{N}_1 \subset \mathcal{N}_0 \) and \( \mathcal{N}_1 \) is finite for contracting actions. Also, \( \mathcal{N}_1 \) is non-empty as it contains the identity of \( G \). The following statement is proved in the last paragraph of \[30, \text{Section 4}\] on p. 2033.

**Lemma 3.5.** \[30\] Let \( G \subset Aut(T) \) be contracting. Then every \( g \in \mathcal{N}_1 \) is torsion.

It is customary to study contracting actions of self-similar subgroups of \( Aut(T) \) using automata and Moore diagrams. Although we use some results obtained with the help of these techniques in our proofs, we do not use this machinery explicitly, so we are not going to discuss it here. An interested reader may consult \[33\], or, for a concise overview, \[30\].

3.3. Non-Hausdorff elements and contracting actions. In this section, we prove some technical results about non-Hausdorff elements in contracting subgroups of \( Aut(T) \).

Let \( G \subset Aut(T) \) be a finitely generated subgroup acting on a Cantor set \( \mathcal{P}_d \) of infinite paths in a \( d \)-ary tree \( T \). Here we can assume that \( G \) is finitely generated, as the discrete iterated monodromy group \( IM(f) \) associated to a post-critically finite polynomial \( f(x) \) over \( \mathbb{C} \) is always finitely generated \[33\].

**Lemma 3.6.** Let \( G \subset Aut(T) \) be contracting, and suppose \( G \) contains a non-Hausdorff element \( h \). Then there is a non-Hausdorff element \( g \in \mathcal{N}_1 \).

**Proof.** By Definition 2.18 if \( h \in G \) is non-Hausdorff, then there exists a path \( \pi = x_1x_2 \cdots \in \mathcal{P}_d \) and a collection \( \{ U_n \}_{n \geq 0} \) of decreasing open neighborhoods of \( \pi \), with \( \bigcap U_n = \{ \pi \} \), such that \( h(\pi) = \pi \), and for each \( n \geq 0 \) an open set \( U_n \) contains an open subset \( W_n \) such that \( h|W_n = id \), but \( h|U_n \neq id \), where \( h|W_n \) and \( h|U_n \) are restrictions of maps to open sets. Note that this condition implies that \( w \notin W_n \), since no neighborhood of \( \pi \) is fixed by \( h \). In particular, \( W_n \) is properly contained in \( U_n \).

By Example 2.1 without loss of generality we can take \( U_n = U_{i_n}(\pi_n) \), where \( \pi_n = x_1 \cdots x_{i_n} \) is a finite word labelling a vertex in the vertex set \( V_{i_n} \). Then \( U_n \) is the set of all paths containing the vertex \( \pi_n \). Also, we can take \( W_n = U_{k_n}(\pi_n) \), where \( \pi_n = w_1 \cdots w_{k_n} \) is a finite word which labels a vertex in \( V_{k_n} \). Since \( W_n \) is properly contained in \( U_n \), then we have \( k_n > i_n \), and \( w_j = x_j \) for \( 1 \leq j \leq i_n \), that is, \( \pi_n = \pi_n w_{i_n+1} \cdots w_{k_n} \).

Since \( h|W_n = id \), then \( h \) fixes the word \( \pi_n \), and so it fixes the word \( \pi_n \). Note that the map \( h|_{\pi_n} \) defined by \[31\] is a non-Hausdorff element of \( G \). Indeed, set \( \overline{\pi} = x_{i_n+1}x_{i_n+2} \cdots \), that is, \( \overline{\pi} \) is obtained from \( \pi \) by discarding the first \( i_n \) letters. Since \( h(\pi) = \pi \), then \( h(\overline{\pi}) = \overline{\pi} \), and so \( h|_{\pi_n} \) fixes every finite subword of \( \overline{\pi} \). The clopen sets \( U'_m = U_{i_m-\ell}(x_{i_m+1}x_{i_m+2} \cdots x_{i_m}) \) for \( m \geq 1 \) form a descending system of open neighborhoods of \( \overline{\pi} \). Each such set contains a subset \( W'_m =
Now consider the collection of elements \( \{ h|_{\pi_i} \} \), for \( n \geq 1 \). If for some \( n > 0 \) we have \( h|_{\pi_n} = h \), then \( h \in \mathcal{N}_1 \) and we are done. If not, recall that \( G \) is contracting, and so by Definition \([3][4]\) there is a number \( \ell_h \geq 1 \) such that for all finite words \( v \) of length at least \( \ell_h \) the restriction \( h|_v \in \mathcal{N} \). The collection \( \{ h|_{\pi_n} \} \) is infinite, while the nucleus \( \mathcal{N} \) is finite, so there exist indices \( s, t \geq 1 \) such that \( i_s > i_t > \ell_h \) and \( h|_{\pi_s} = h|_{\pi_t} \). Then \( g = h|_{\pi_s} \) is a non-Hausdorff element in \( \mathcal{N}_1 \).

\[ \square \]

Note that if \( h \in \mathcal{N}_1 \) is non-Hausdorff, then necessarily \( h \neq id \), as \( h \) acts non-trivially on neighborhoods of a point \( \pi \in \mathcal{P}_d \).

### 4. The asymptotic discriminant for Chebyshev polynomials

In this section, we prove Theorem \([1.5]\) which computes the asymptotic discriminant for the action of the discrete iterated monodromy group associated to a Chebyshev polynomial \( T_d \) of degree \( d \geq 2 \).

A degree \( d \) Chebyshev polynomial is defined by \( T_d(x) = \cos(d \arccos x) \) for \( x \in [-1, 1] \), or \( T_d(\theta) = \cos(d \theta) \) for \( \theta \in [0, \pi] \). Using the standard trigonometric identity for the sum of cosines, one obtains that the Chebyshev polynomials satisfy the recursive relations

\[ T_0(x) = 1, \quad T_1(x) = x, \quad T_d(x) = 2xT_{d-1}(x) - T_{d-2}(x). \]

Sometimes in the literature Chebyshev polynomials are defined as

\[ \tilde{T}_0(x) = 2, \quad \tilde{T}_1(x) = x, \quad \tilde{T}_d(x) = x\tilde{T}_{d-1}(x) - \tilde{T}_{d-2}(x). \]

The polynomials \([39] \) and \([40] \) are conjugate by a linear map, namely, \( T_d(z) = \frac{1}{2} \tilde{T}_d(2x) \).

By \([33] \) Proposition 6.12.6, the discrete iterated monodromy group \( \text{IMG}(T_d) \) is generated by the following maps:

1. If \( d \) is even, then the generators are

\[ a = \tau, \quad b = \sigma(b, 1, \ldots, 1, a) = (b, 1, \ldots, 1, a)\sigma, \]

where \( \tau = (0, 1)(2, 3) \cdots (d - 2, d - 1) \) and \( \sigma = (1, 2)(3, 4) \cdots (d - 3, d - 2) \). The last equality for \( b \) holds since \( \sigma \) fixes 0 and \( d - 1 \).

2. If \( d \) is odd, then the generators are

\[ a = \tau(1, \ldots, 1, a) = (1, \ldots, 1, a)\tau, \quad b = \sigma(b, 1, \ldots, 1) = (b, 1, \ldots, 1)\sigma, \]

where \( \tau = (0, 1)(2, 3) \cdots (d - 3, d - 2) \) and \( \sigma = (1, 2)(3, 4) \cdots (d - 2, d - 1) \). The last equality for \( a \) holds since \( \tau \) fixes \( d - 1 \), and the last equality for \( b \) holds since \( \sigma \) fixes 0.

Both \( \tau \) and \( \sigma \) have order 2, so \( a \) and \( b \) have order 2. The composition \( \alpha = ba \) has infinite order. Indeed, if \( d \) is even, then \( \tau \sigma = (1, 2, \ldots, d - 1) \) is a \( d \)-cycle, and

\[ \alpha = ba = (b, 1, \ldots, 1, a)\sigma \tau. \]

Denote by \( e \) the identity element in \( \text{IMG}(T_d) \). By the formula \([34] \) we have

\[ \alpha^2 = (b, 1, \ldots, 1, a)\sigma\tau(b, 1, \ldots, 1, a)\sigma\tau = (b, 1, \ldots, 1, a)(a, b, 1, \ldots, 1)(\sigma\tau)^2 = (ba, b, 1, \ldots, 1, a)(\sigma\tau)^2, \]

so \( \alpha^2 \neq e \). Computing inductively \( \alpha^k \), we obtain that if \( 1 \leq k < d \), then \( \alpha^k \) acts on the vertices in the set \( V_1 \) as \( (\sigma\tau)^k \), and so \( \alpha^k \neq e \). Next,

\[ \alpha^d = (ba, ba, \ldots, ba)(\sigma\tau)^d = (a, \alpha, \ldots, a), \]

so \( \alpha^d \) acts as the identity permutation on the set of vertices \( V_1 \), and as \( \alpha \) on every subtree \( vT \) of \( T \), where \( v \in \{0, 1, \ldots, d - 1\} \). So \( \alpha^d \neq e \). Inductively it follows that \( \alpha \) has infinite order. A similar argument shows that \( \alpha = ba \) has infinite order when \( d \) is odd. Summarizing, \( \text{IMG}(T_d) \) is isomorphic to the infinite dihedral group, that is,

\[ \text{IMG}(T_d) \cong \{ a, b \mid a^2 = b^2 = e \} = \{ b, \alpha \mid bab^{-1} = \alpha^{-1}, b^2 = 1 \}. \]
We now compute the asymptotic discriminant for the action of $\text{IMG}(T_d)$, $d \geq 2$, and so prove Theorem 1.5. For the convenience of the reader we re-state this theorem now.

**THEOREM 4.1.** Let $T_d$ be the Chebyshev polynomial of degree $d \geq 2$ over $\mathbb{C}$. Then the action of $\text{IMG}(T_d)$ is stable with discriminant group $\mathbb{Z}/2\mathbb{Z}$, the finite group of order 2.

**Proof.** From the discussion before the theorem it follows that the generator $\alpha$ in (43) acts transitively on $V_n$, for $n \geq 1$.

Denote by $0^n$ the concatenation of $n$ symbols 0, and for $n \geq 1$, consider the vertex $x_n = 0^n$ in $V_n$. By (41) and (42), both when $d$ is even and when $d$ is odd, the generator $b$ fixes $x_n$. Since $\alpha$ acts transitively on $V_n$ and $|V_n| = d^n$, then the smallest power of $\alpha$ which fixes $x_n$ (and any other point in $V_n$) is $\alpha^{d^n}$. Then the isotropy group of the action of $\text{IMG}(T_d)$ at $x_n$ is given by

$$G_n = \{ g \in G_{\text{geom}} \mid g \cdot x_n = x_n \} = \langle b, \alpha^{d^n} \rangle.$$ 

Since $\text{IMG}(T_d)$ acts transitively on $V_n$, then there is a bijection $\text{IMG}(T_d)/G_n \rightarrow V_n$ such that $eG_n \mapsto x_n$, where $e$ is the identity in $\text{IMG}(T_d)$, with the cosets of $\text{IMG}(T_d)/G_n$ represented by the powers $\alpha^s$, $0 \leq s \leq d^n - 1$. In particular, $\alpha G_n \neq \alpha^{d^n - 1} G_n = \alpha^{-1} G_n$ if $n \geq 2$ for any $d \geq 2$.

Denote by $C_n$ the set of elements which act trivially on $V_n$, then $C_n$ is the maximal normal subgroup of $G_n$ in $\text{IMG}(T_d)$.

**LEMMA 4.2.** For $n \geq 1$ the maximal normal subgroup of $G_n = \langle \alpha^{d^n}, b \rangle$ in $\text{IMG}(T_d)$ is $C_n = \langle \alpha^{d^n} \rangle$, and the discriminant group of the action of $\text{IMG}(T_d)$ on the path space $\mathcal{P}_d$ of the tree $T$ is $D_x \cong \mathbb{Z}/2\mathbb{Z}$.

**Proof.** Since $\alpha^{d^n}$ is the smallest power of $\alpha$ which fixes every vertex in $V_n$, then we have $\langle \alpha^{d^n} \rangle \subseteq C_n$.

Consider the action of $b$ on the cosets of $\text{IMG}(T_d)/G_n$. We have

$$b \alpha^s G_n = \alpha^{-s} b^{-1} G_n = \alpha^{-s} G_n = \alpha^{d^n - s} G_n.$$ 

Note that $\alpha^{d^n - s} = \alpha^s$ if and only if $d^n = 2s$. So if $d$ is even, then $b$ fixes the cosets $eG_n$ and $\alpha^{d^n/2} G_n$, and if $d$ is odd, then $b$ fixes only $eG_n$. In both cases for $n \geq 2$ the element $b$ acts non-trivially on $\text{IMG}(T_d)/G_n$, and so $b \notin C_n$.

We show that for $1 \leq t \leq d^n - 1$ the elements $a^t b$ and $b a^t$ are not in $C_n$. Indeed, we have

$$a^t b (eG_n) = a^t G_n \neq eG_n.$$ 

and

$$b a^t (eG_n) = a^{-1} G_n = a^{d^n - t} G_n \neq eG_n,$$

so the action of $a^t b$ and of $b a^t$ on $\text{IMG}(T_d)/G_n$ is non-trivial. It follows that $C_n = \langle \alpha^{d^n} \rangle$.

Then $G_n/C_n = \{ eC_n, bC_n \}$. The coset inclusion maps $G_{n+1}/C_{n+1} \rightarrow G_n/C_n$ are clearly bijective. It follows that

$$D_x = \lim_{\rightarrow} \{ G_{n+1}/C_{n+1} \rightarrow G_n/C_n \} \cong \mathbb{Z}/2\mathbb{Z}.$$

To compute the asymptotic discriminant we consider the restriction of the action of $\text{IMG}(T_d)$ to clopen subsets of $\mathcal{P}_d$. For $k \geq 0$, consider a truncated chain $\{ G_n \}_{n \geq k}$. We have $G_k = \langle \alpha^{d^k}, b \rangle$, and $G_n = \langle \alpha^{d^n}, b \rangle \subseteq G_k$ for $n \geq k$. The group $G_k$ fixes the word $x_k = 0^k$, and so acts on the clopen subset $U_k(x_k)$ of paths through the vertex $x_k$. All such paths are in the subtree $x_k T$ of $T$. Denote by $C_k^n$ the maximal normal subgroup of $G_n$ in $G_k$, that is, $C_k^n$ contains all elements of $G_k$ which act trivially on the coset space $G_k/G_n$.

A computation similar to the one in Lemma 4.2 with $\alpha^{d^n}$ instead of $\alpha$, shows that $C_k^n = \langle (\alpha^{d^n})^{d^n-k} \rangle = \langle \alpha^n \rangle$ is the maximal normal subgroup of $G_n$ in $G_k$, the quotient $G_n/C_k^n = \{ eC_k^n, bC_k^n \} \cong \mathbb{Z}/2\mathbb{Z}$, and the discriminant group of the action of $G_k$ on $U_k(x_k)$ is

$$D_x^k = \lim_{\rightarrow} \{ G_{n+1}/C_{n+1}^k \rightarrow G_n/C_k^n \} \cong \mathbb{Z}/2\mathbb{Z}.$$
Since $C_{n}^{k} = C_{n}$, then the coset maps $\psi_{1,k,n} : G_{n}/C_{n} \to G_{n}/C_{n}^{k}$ are the identity maps, and the induced map $\psi_{1,k} : \mathcal{D}_{x} \to \mathcal{D}_{x}^{k}$ on the inverse limits is an isomorphism. Thus the asymptotic discriminant of the action of $\operatorname{IMG}(T_{d})$ is stable with finite discriminant group, for $d \geq 2$. This finishes the proof of Theorem 4.1. \hfill \square

5. The asymptotic discriminant of the geometric iterated monodromy group of a quadratic polynomial

In this section we study the geometric iterated monodromy group for post-critically finite quadratic polynomials and prove Theorem 1.3. The proof of statement (3) of this theorem follows from Theorem 1.5 in Section 4. Statement (1) follows from Remark 5.1. Before we start the proof of Theorem 1.5 we denote by $\mathcal{D}_{x}$ the action of $\operatorname{IMG}(\psi_{1,k})$. We prove a series of propositions which are used in the proof.

So let $f(x)$ be a quadratic polynomial with coefficients in the ring of integers of a number field $K$. Recall from the Introduction that $f$ induces a map $f : \mathbb{P}^{1}(C) \to \mathbb{P}^{1}(C)$ on the Riemann sphere. Since $f$ is quadratic, it has two critical points in $\mathbb{P}^{1}(C)$. One of them is the point at infinity, denoted by $\infty$, which satisfies $f(\infty) = \infty$. We denote the other point by $c$, so $C = \{c, \infty\}$. As in the introduction, we denote by $P_c$ the post-critical set of $f(x)$, that is, $P_c = \bigcup_{n \geq 1} f^n(C)$. Denote by $P_c = \bigcup_{n \geq 1} f^n(c)$ the orbit of the critical point $c$. Note that $P_c = P_c \cup \{\infty\}$. We assume that $P_c = \{p_1, \ldots, p_r\}$ is finite, such that $f(c) = p_1$, and $f(p_i) = p_{i+1}$ for $1 \leq i < r$. We will consider two cases. In the first case, the post-critical orbit of $c$ is strictly periodic, that is, $f(p_r) = p_1$. Note that since $c$ is critical, $c$ is the only preimage of $p_1$ and so $p_r = c$ and $c \in P_c$. [39 Section 1.9]. In the second case, the orbit of $c$ is pre-periodic, that is, there exists $1 \leq s < r$ such that $f(p_s) = p_{s+1}$. In this case the critical point $c$ is not in $P_c$.

We consider the action of the geometric monodromy group $\operatorname{Gal}_{\text{geom}}(f)$ on the binary tree $T$. See the Introduction for an explanation of how this action arises. Recall that $\mathcal{P}_2$ denotes the space of paths in the binary tree $T$, which is a Cantor set by Example [2.1]. Here the subscript in $\mathcal{P}_2$ refers to the degree of the polynomial $f(x)$.

Denote by $\sigma$ the non-trivial permutation of a set of 2 elements. Given a set $\mathcal{A}_r = \{a_1, \ldots, a_r\}$, with elements $a_i$, $1 \leq i \leq r$, to be specified later, we denote by $\tilde{G}_r = \langle \mathcal{A}_r \rangle$ a countable subgroup of $\operatorname{Aut}(T)$ generated by $\mathcal{A}_r$, and by $\operatorname{CL}(\tilde{G}_r)$ the closure of $\tilde{G}_r$ in $\operatorname{Aut}(T)$. Then $\operatorname{CL}(\tilde{G}_r)$ is the Ellis group of the action of $\tilde{G}_r$.

5.1. Strictly periodic case. We first consider the case when the orbit of $c$ is strictly periodic.

REMARK 5.1. If the critical orbit of $f(x)$ is strictly periodic with $\#P_c = 1$, then by [39 Proposition 1.9.2] $\operatorname{Gal}_{\text{geom}}(f)$ is conjugate in $\operatorname{Aut}(T)$ to the closure of a subgroup generated by an element $a_1 = (a_1, 1)\sigma$. As explained in Example 3.3, the action of $a_1$ on the path space $\mathcal{P}_2$ is an odometer action. Then $\tilde{G}_r \cong \mathbb{Z}$ and is abelian. Then by Examples 2.9 and 2.13 the action of $\operatorname{CL}(\tilde{G}_r)$ is stable with trivial asymptotic discriminant. Then Proposition 2.20(3) implies that the action of $\operatorname{Gal}_{\text{geom}}(f)$ is stable with trivial asymptotic discriminant.

Now suppose the post-critical orbit of $c$ consists of at least two points, that is, $\#P_c \geq 2$. The class of the actions with $\#P_c = 2$ includes those where $\operatorname{IMG}(f)$ is the well-known and well-studied Basilica group, see for example [33] or [22] and references therein. The class of actions with $\#P_c = 3$ includes those associated to the polynomials whose Julia set is the ‘Douady rabbit’ or the ‘airplane’ [33].

PROPOSITION 5.2. Let $\mathcal{A}_r = \{a_1, a_2, \ldots, a_r\}$, $r \geq 2$, be the set of element given by

\[(44)\quad a_1 = (a_r, 1)\sigma, \quad a_i = (a_{i-1}, 1) \quad \text{for } 2 \leq i \leq r,\]

and let $\tilde{G}_r \subset \operatorname{Aut}(T)$ be a discrete group generated by $\mathcal{A}_r$. Then the following is true:

1. The action of $\tilde{G}_r$ on the space of paths $\mathcal{P}_2$ is non-LQA.
2. The group $\tilde{G}_r$ contains no non-Hausdorff elements.
Proof. To show that the action of $\widetilde{G}_r$ is non-LQA, we have to find a descending chain of clopen sets $\{W_n\}_{n \geq 0} \subset P_2$, such that $\text{diam}(W_n) \to n$, 0, and, for each $n \geq 0$, an element $g_n \in \widetilde{G}_r$, such that the restriction $g_n|W_n$ is non-trivial, while the restriction $g_n|W_{n+1}$ is the identity map.

In the arguments below we use the labelling of vertices in $T$ by finite words of 0's and 1's as in Example 2.1. Recall that we denote by $T_n$ the subtree of $T$ with vertex sets $\bigcup_{1 \leq i \leq n} V_i$.

The generators $a_1, a_2, \ldots, a_r$ are defined recursively, so to understand how they act on the tree $T$, one has to ‘uncover’ their action level by level. Let us start with the vertex set $V_1$ of $T$.

By definition in formula (49), the generator $a_1$ acts as a non-trivial permutation (a 2-cycle) on the vertices in $V_1$, and then it acts as $a_r$ on the subtree $0T$ and as the identity on map on the subtree $1T$, see Figure 2 for the portraits of the generators in the case $r = 3$. So to understand how $a_1$ acts on $0T$, we have to understand how $a_r$ acts on the tree $T$.

The generator $a_2$ acts trivially on the vertex set $V_1$, as $a_1$ on the subtree $0T$, and as the identity map on the subtree $1T$. Thus $a_2$ acts as a 2-cycle on the vertices in $0T \cap V_2$, as $a_r$ on the subtree $00T$ and as the identity on the subtree $01T$. Denote by $0^r$ the concatenation of $i$ symbols 0. Continuing by induction, we obtain that for $1 \leq i \leq r$, the generator $a_i$ acts as the identity on the vertex set $V_{i-1}$, as $a_1$ on the subtree $0^{i-1}T$ and as the identity on every subtree $wT$, where $w$ is a word of length $(i - 1)$ in 0’s and 1’s such that at least one letter in $w$ is not 0.

In particular, we write

$$a_r = (a_1, 1, \ldots, 1)1_{r-1}$$

meaning that $a_r$ acts as the identity on vertex set $V_{r-1}$, as $a_1$ on the subtree $0^{r-1}T$ and as the identity on every other subtree starting at a vertex in $V_{r-1}$. Since $a_r$ act trivially on the subtree $T_{r-1}$ and $a_1 = (a_r, 1)\sigma$, we have that $a_1|T_r = (1, 1)\sigma$, that is, $a_1$ acts as $\sigma$ on $V_1$, and then as the identity on the vertices of the finite subtrees $0T \cap T_r$ and $1T \cap T_r$. It follows that for $1 \leq i \leq r$ the element $a_i$ acts on the vertex set $V_i$ as a union of $2^{i-1}$ 2-cycles, interchanging 0 and 1 in the first letter of any word $w$ of length $i$, and fixing the letters from the second to the $i$-th.

In particular, $a_1$ acts on $T_r$ as $2^{r-1}$ 2-cycles. By (34) and using (45) we obtain

$$a_1^2 = (a_r, 1)\sigma(a_r, 1)\sigma = (a_r, 1)(1, a_r)\sigma^2 = (a_r, a_r) = (a_1, 1, \ldots, 1, a_1, 1, \ldots, 1)1_r,$$

so $a_1^2$ acts trivially on the first $r$ levels of the tree $T$, it acts as $a_1$ on the subtrees $0^rT$ and $10^{r-1}T$, and trivially on any other subtree $vT$, where $v$ is any word of length $r$ except $0^r$ or $10^{r-1}$.

Continuing inductively, we obtain that

$$(a_1)^{2^n} = \underbrace{(a_1, 1, \ldots, 1, a_1, 1, \ldots, 1)}_{\text{repeat } n \text{ times } (a_1, 1, \ldots, 1)}1_{nr}.$$
Thus \((a_1)^{2^n}\) acts trivially on the first \(nr\) levels of \(T\). We have \(|V_{nr}| = 2^{nr}\), and the pattern \((a_1,1,\ldots,1)\) of length \(2^r\) is repeated \(n\) times in the formula \([17]\). In particular, \((a_1)^{2^n}\) acts as \(a_1\) on the subtree \(0^rT\), and trivially on the subtree \(0^{nr-1}1T\).

So for \(n \geq 1\), let \(w_n = 0^{nr-1}\), and consider the set \(W_n = U_{nr-1}(w_n)\) which contains all infinite sequences starting from the word \(w_n\) or, alternatively, all infinite paths in \(P_2\) which pass through the vertex \(w_n\). Then \(W_n = U_{nr}(w_n) \cup U_{nr}(w_n1)\), where \(U_{nr}(w_n0)\) contains all paths in the subtree \(0^rT\), and \(U_{nr}(w_n1)\) contains all paths in the subtree \(0^{nr-1}1T\). By the argument above \((a_1)^{2^n}\) acts non-trivially on the clopen set \(U_{nr}(w_n0)\), and trivially on the clopen set \(U_{nr}(w_n1)\).

By \([39]\) Proposition 2.7.1 the composition \(\lambda = a_1a_2 \cdots a_r\) generates an odometer action, and so \(\lambda\) acts transitively on every level of the tree \(T\). Since \(|V_{nr-1}| = 2^{nr-1}\), then the power \(\lambda^{2^{nr-1}}\) fixes every vertex in \(V_{nr-1}\), and acts as \(2^{nr-1}\) 2-cycles on the vertices in \(V_{nr}\). In particular, \(\lambda^{2^{nr-1}}\) fixes \(w_n\), and permutes \(w_n0\) and \(w_n1\). This means that \(\lambda^{2^{nr-1}}\) maps \(U_{nr}(w_n0)\) onto \(U_{nr}(w_n1)\), and \(U_{nr}(w_n1)\) onto \(U_{nr}(w_n0)\). Define

\[
(48) \quad g_n = \lambda^{-2^{nr-1}} \circ (a_1)^{2^n} \circ \lambda^{2^{nr-1}},
\]

then \(g_n\) acts trivially on the clopen set \(U_{nr}(w_n0)\), and non-trivially on the clopen set \(U_{nr}(w_n1)\).

Note that since \(r \geq 2\), then \(W_{n+1} = U_{n+1}(w_{n+1}) \subset U_{nr}(w_n0) \subset W_n\), so \(\{W_n\}_{n \geq 0}\) is a decreasing sequence of clopen sets, such that \(g_n|W_n\) is non-trivial, while \(g_n|W_{n+1}\) is trivial. We conclude that the action of \(\tilde{G}_r\) is non-LQA, which proves (1).

To show (2) note that the group \(\tilde{G}_r\) is contracting. Then by Lemma 3.6 if \(g \in \tilde{G}_r\) is non-Hausdorff then the finite set \(N_1\) defined by \([38]\) contains a non-Hausdorff element. By Lemma 3.5 every element in \(N_1\) is torsion. But the group \(\tilde{G}_r\) is a group of type \(\aleph_0\) in \([6]\) where \(v = 0^{nr-1}\), so by \([6]\) Proposition 3.11 it is torsion free. Therefore, the set \(N_1\) for the action of \(\tilde{G}_r\) contains only the identity element, and so \(\tilde{G}_r\) does not contain any non-Hausdorff elements.

\[\square\]

**Remark 5.3.** Although the group \(\tilde{G}_r\) in Proposition 5.2 does not contain any non-Hausdorff elements, we cannot rule out that the closure \(\text{CL}(\tilde{G}_r)\) of the action does contain them. Indeed, the absence of non-Hausdorff elements in the group \(\tilde{G}_r\) in Proposition 5.2 is a consequence of the fact that \(\tilde{G}_r\) is torsion free. By a celebrated result of Lubotzky [31], profinite completions of torsion free groups may have non-trivial torsion elements. The construction of Lubotzky was used in [12] to construct examples where the action \(\Phi : G \to \text{Homeo}(X)\) is that of a torsion free group, while the closure of the action \(\Phi(G)\) contains torsion elements. We state an open problem.

**Problem 5.4.** Let \(\Phi : G \to \text{Homeo}(X)\) be an action of a countable group \(G\) on a Cantor set \(X\). Suppose that the action is non-LQA. Show that the closure \(\Phi(G)\) contains a non-Hausdorff element, or find a counterexample.

### 5.2. Pre-periodic case.

Now suppose that the post-critical orbit \(P_r = \{p_1, \ldots, p_r\}\) of the critical point \(c\) of \(f(x)\) is pre-periodic, that is, there exists \(s \geq 1\) such that \(f(p_r) = p_{s+1}\). In Proposition 5.5 below we consider the case when \(r \geq 3\).

Recall that we denoted by \(\sigma\) the non-trivial permutation of a set of 2 elements. Given a set \(B_r = \{b_1, \ldots, b_r\}\), with elements \(b_i\), \(1 \leq i \leq r\), to be specified in Proposition 5.5 we denote by \(\tilde{H}_r = (B_r)\) a countable subgroup of \(\text{Aut}(T)\) generated by \(B_r\), and by \(\text{CL}(\tilde{H}_r)\) the closure of \(\tilde{H}_r\) in \(\text{Aut}(T)\). Then \(\text{CL}(\tilde{H}_r)\) is the Ellis group of the action of \(\tilde{H}_r\).

**Proposition 5.5.** Let \(r \geq 3\), let \(1 \leq s < r\), and let \(B_r = \{b_1, b_2, \ldots, b_r\}\) be the set of elements given by

\[
(49) \quad b_1 = \sigma, \quad b_{s+1} = (b_s, b_r), \quad b_i = (b_{i-1}, 1) \quad \text{for} \quad i \neq 1, s + 1,
\]

let \(\tilde{H}_r \subset \text{Aut}(T)\) be a group generated by \(B_r\). Then \(\tilde{H}_r\) contains a non-Hausdorff element, and so the action of \(\tilde{H}_r\) on the space of paths \(P_2\) is non-LQA.
Proof. We will consider two cases, first when \( s + 1 = r \) and so the periodic part of the post-critical orbit of \( c \) is just a fixed point, and second when \( r > s + 1 \), so that the periodic part of the post-critical orbit has length at least 2.

**Lemma 5.6.** Under the conditions of Proposition 5.5, suppose in addition that \( s + 1 = r \). Then \( b_r \) is non-Hausdorff.

Proof. If \( b_r \) is non-Hausdorff, then there exists an infinite path \( \pi \in \mathcal{P}_2 \), a descending collection of clopen neighborhoods \( \{W_n\}_{n \geq 1} \) with \( \bigcap_{n \geq 1} W_n = \{\pi\} \) and, for each \( n \geq 1 \), a clopen subset \( O_n \subset W_n \), such that \( b_r(\pi) = \pi \), \( b_r|O_n \) is the identity, while \( b_r|W_n \) is non-trivial. We will find such \( \pi \), \( \{W_n\}_{n \geq 1} \) and \( \{O_n\}_{n \geq 1} \).

Let us first understand how the generators \( b_i \), \( 1 \leq i \leq s \), act on \( \mathcal{P}_2 \). The generator \( b_1 = \sigma \), so \( b_1 \) acts on \( V_1 \) as a 2-cycle. For \( n \geq 1 \), \( V_n \) contains \( 2^n \) vertices, so \( b_1 \) acts on \( V_n \) as \( n \) 2-cycles. Thus \( b_1 \) has order 2 and no fixed points. Note that since \( r \geq 3 \) and \( s + 1 = r \), then \( s \geq 2 \).

For \( 1 < i \leq s \) we have \( b_i = (b_{i-1}, 1) \). That is, \( b_2 \) acts trivially on the vertex set \( V_1 \), as \( b_1 \) on the subtree \( 0T \) of \( T \), and trivially on the subtree \( 1T \). So \( b_2 \) fixes a clopen set \( U_1(1) \), and acts as \( n - 1 \) 2-cycles on the intersection \( 0T \cap V_n \), where \( V_n \) is the vertex set at level \( n \geq 2 \). Inductively, one obtains that \( b_i \) acts trivially on all vertices in the subtree \( T_{i-1} \), as \( n - i + 1 \) 2-cycles on the vertices in the intersection \( 0^{i-1}T \cap V_n \), for \( n \geq i \), and trivially on the rest of the tree. So \( b_i \) has order 2 for \( 1 < i \leq s \). In particular, \( b_s \) acts non-trivially on a clopen subset of \( U_1(0) \) and trivially on \( U_1(1) \).

The portraits of the generators in \( \mathcal{B} \) for the case \( s + 1 = r = 3 \) are shown in Figure 3.

Now consider \( b_r = (b_s, b_r) \), where \( s + 1 = r \). We will unravel how \( b_r \) acts on the path space \( \mathcal{P}_2 \) by induction on the level \( n \geq 1 \) in the tree \( T \). From the definition, \( b_r \) acts trivially on the vertex set \( V_1 \), as \( b_s \) on \( 0T \) and as \( b_r \) on \( 1T \). Then \( b_r \) acts non-trivially on a clopen subset of \( U_2(00) \), and trivially on \( U_2(01) \). Since \( b_r \) acts as \( b_s \) on \( 1T \), then it acts as \( b_s \) on \( 10T \) and as \( b_r \) on \( 11T \). This means that \( b_r \) acts non-trivially on a clopen subset of \( U_3(100) \) and trivially on \( U_3(101) \).

Inductively, we obtain that \( b_r \) acts as \( b_r \) on the subtree \( 1^nT \), for \( n \geq 1 \). All infinite paths contained in this subtree are in the clopen set \( W_n = U_n(1^n) \), where \( 1^n \) denotes a word obtained by a concatenation of \( n \) copies of 1. Then \( b_r \) acts as \( b_s \) on the subtree \( 1^n0T \), and the clopen set \( U_{n+1}(1^n0) \), containing all paths of \( 1^n0T \). More precisely, the action of \( b_r \) is non-trivial on a clopen subset of \( U_{n+2}(1^n00) \), and it is trivial on the clopen subset \( O_n = U_{n+2}(1^n01) \).
Figure 4. Recursive construction of the generators in the pre-periodic case when $s = 1$ and $r = 3$.

Note that the for $n \geq 1$, we have $\bigcap_{k=1}^{n} W_k = W_n \neq \emptyset$, so $\{W_n\}_{n \geq 1}$ is a family of closed sets in $P_2$ with finite intersection property. Since $P_2$ is compact, then $\bigcap_{n \geq 1} W_n$ is non-empty [15, Section 17]. Any sequence $\mathbf{y}$ which contains at least one letter 0 is not in $W_n$ for $n$ large enough, so it follows that $\bigcap_{n \geq 1} W_n = x = 1^\infty$, where $1^\infty$ denotes an infinite sequence of 1’s. By the discussion above, the action of $b_r$ is non-trivial only on subsets contained in the sets of the form $U_n(1^{n-2}00)$, for $n \geq 1$. Since $x$ does not contain any 0’s, it must be a fixed point of $b_r$. We have shown that $b_r$ is non-Hausdorff. Note that $b_r$ is torsion of order 2. □

We now consider the second case, when $r > s + 1$ and $s \geq 1$.

**Lemma 5.7.** Under the conditions of Proposition 5.5, suppose in addition that $r > s + 1$. Then for $s + 1 \leq i \leq r$, the element $b_i$ is non-Hausdorff.

**Proof.** We have that $b_1 = \sigma$, and for $1 \leq i \leq s$, the generator $b_i = (b_{i-1}, 1)$ acts on the tree $T$ in the same way as in Lemma 5.6. In particular, $b_s$ acts non-trivially on the set $U_1(0)$ if $s \geq 2$, and on the whole space $P_2$ if $s = 1$. The generator $b_s$ has order 2.

Next, we have $b_{s+1} = (b_s, b_r)$, and $b_i = (b_{i-1}, 1)$ for $s + 2 \leq i \leq r$. Inductively, for $s + 1 < i \leq r$ the generator $b_i = (b_{i-1}, 1)$ acts as $b_{i+1}$ on the subtree $0^{r-s-1}T$, and so on the clopen set $U_{i-s-1}(0^{r-s-1})$. Therefore, $b_r$ acts as $b_{s+1}$ on the clopen set $U_{r-s-1}(0^{r-s-1})$. Since $b_{s+1} = (b_s, b_r)$, the generator $b_r$ acts non-trivially (as $b_s$) on the clopen set $U_{r-s}(0^{r-s})$. It acts as $b_r$ on the clopen set $U_{r-s}(0^{r-s-1})$, and so acts trivially on the clopen set $U_{r-s+1}(0^{r-s-1}11)$.

Portraits of generators in the case $s = 1$ and $r = 3$ are presented in Figure 4.

For $n \geq 1$ denote by $(0^{r-s-1}1)^n$ the concatenation of $n$ copies of the word $0^{r-s-1}1$. By induction, we obtain that $b_r$ acts as $b_{s+1}$ on the clopen set $U_{n(r-s)-1}(0^{r-s-1}1^{n-1}0^{r-s-1})$. Then it acts non-trivially (as $b_s$) on the clopen set

$$Z_n = U_n(r-s)((0^{r-s-1}1)^n1^{r-s})$$

and as $b_r$ on the clopen set

$$W_n = U_n(r-s)((0^{r-s-1}1)^n).$$
Then $b_r$ acts trivially on the clopen subset
\[ O_n = U_{n(r-s)+1}((0^{r-s-1})^n1) \subseteq W_n. \]

Note that $Z_n \subseteq W_{n-1}$, so we obtained a nested family of clopen sets $\{W_n\}_{n \geq 1}$ such that for all $n \geq 1$, $b_i|W_n$ is non-trivial, while $b_i|O_n$, for $O_n \subseteq W_n$, is trivial. By an argument similar to the one at the end of Lemma 5.6 we obtain that the intersection $\bigcap_{n \geq 1} W_n$ is a point $\bar{x} = (0^{r-s-1})^\infty$, that is, the sequence $\bar{x}$ is a concatenation of an infinite number of copies of the word $0^{r-s-1}$. By the definition of $b_i$ for $s+1 < i \leq r$ one can see that $b_i$ acts non-trivially only on the subsets of the form $\{b_j\}$; in particular, every sequence in $Z_n$ contains a word $0^{r-s}$. Since $\bar{x}$ does not contain such a word, $\bar{x}$ is fixed by the action of $b_r$. We conclude that $b_r$ is a non-Hausdorff element.

For $s+1 \leq i < r$, by construction the path space $\mathcal{P}_2$ contains a clopen neighborhood $W$ such that $b_i$ acts on $W$ as $b_{i+1} = (b_i, b_r)$. Then $W$ contains a subset $W'$ such that $b_i$ acts on $W'$ as $b_r$. Since $b_r$ is non-Hausdorff, then $b_i$ is non-Hausdorff.

This finishes the proof of Proposition 5.5.

5.3. **Proof of Theorem 1.3**

*Proof.* Statement (1) of the theorem follows immediately from Remark 6.1, that is, if the post-critical orbit of the critical point $c$ of the polynomial $f(x)$ is constant, then the action of $\text{Gal}_{geom}(f)$ is conjugate to the action of the enveloping group of an odometer action. By Proposition 2.20, this implies that the action of $\text{Gal}_{geom}(f)$ is stable with trivial discriminant group.

Let us prove statement (2). That is, suppose the polynomial $f(x)$ has a strictly periodic post-critical orbit of length $r = \#P_c \geq 2$. Then by [39, Theorem 2.4.1] $\text{Gal}_{geom}(f)$ is conjugate in $\text{Aut}(T)$ to the profinite group $\text{CL}(\hat{G}_r)$, where $\hat{G}_r$ is as in Proposition 5.2. Since the action of $\hat{G}_r$ is non-LQA by Proposition 5.2, then the action of the closure $\text{CL}(\hat{G}_r)$ is non-LQA. Then by Proposition 2.20, the action of $\text{Gal}_{geom}(f)$ on the space of paths $\mathcal{P}_2$ in a tree $T$ is non-LQA, and so has wild asymptotic discriminant.

For statement (3), suppose the polynomial $f(x)$ has a pre-periodic orbit of cardinality $\#P_c = 2$. Then by [39, Proposition 3.4.2] $\text{Gal}_{geom}(f)$ is conjugate to the closure of the action of the group $\hat{H}_2$, which is generated by the set $B_2 = \{b_1 = \sigma, b_2 = (b_1, b_2)\}$. The elements in $B_2$ are conjugate to the generating set in [41] for $d = 2$. Indeed, note that $\sigma b_2 \sigma = (b_2, b_1)$ which gives the second generator in [41]. Then by Proposition 2.20(3) it follows from Theorem 4.1 that the action of $\text{Gal}_{geom}(f)$ on the space of paths $\mathcal{P}_2$ in a tree $T$ is stable with finite discriminant group.

For statement (4), suppose the polynomial $f(x)$ has a pre-periodic orbit of cardinality $\#P_c \geq 3$. Then by [39, Theorem 3.4.1] $\text{Gal}_{geom}(f)$ is conjugate in $\text{Aut}(T)$ to the profinite group $\text{CL}(\hat{H}_r)$, where $\hat{H}_r$ is as in Proposition 5.5. Since the action of $\hat{H}_r$ is non-LQA by Proposition 5.5, then the action of the closure $\text{CL}(\hat{H}_r) \subseteq \text{Aut}(T)$ is non-LQA. Then by Proposition 2.20, the action of $\text{Gal}_{geom}(f)$ on the space of paths $\mathcal{P}_2$ in a tree $T$ is non-LQA, and so has wild asymptotic discriminant.

6. **Asymptotic discriminant for the arithmetic iterated monodromy group of post-critically finite polynomials**

Let $f(x)$ be a post-critically finite quadratic polynomial, and denote by $\text{Gal}_{geom}(f)$ and $\text{Gal}_{arith}(f)$ the geometric and the arithmetic iterated monodromy groups respectively. Both groups were defined in the Introduction. In this section, we compute the asymptotic discriminant for the arithmetic iterated monodromy group, thus proving Theorem 1.6.

Recall from the Introduction that $f(x)$ induces the map of the Riemann sphere $\mathbb{P}^1(\mathbb{C})$, which has two critical points, the point at infinity $\infty$ and the point $c$. Recall that we denote by $C = \{c, \infty\}$ the set of critical points, and by $P_C = \bigcup_{n \geq 1} f^n(C)$ the post-critical set. We denote by $P_c$ the orbit of the critical point $c$ of $f(x)$. 


**PROPOSITION 6.1.** Let $K$ be a finite extension of $\mathbb{Q}$, and let $f(x)$ be a quadratic polynomial with coefficients in the ring of integers of $K$. Suppose the post-critical set $P_C$ is finite, and $f(x)$ falls within one of the following categories:

1. either the orbit of the critical point $c$ is strictly periodic with $\#P_c \geq 2$,
2. or the orbit of the critical point $c$ is pre-periodic with $\#P_c \geq 3$.

Then the action of $\text{Gal}_\text{arith}(f)$ is non-LQA, and so has wild asymptotic discriminant.

**Proof.** In both cases the action of $\text{Gal}_\text{geom}(f)$ is non-LQA by Theorem 1.3. By [39] $\text{Gal}_\text{geom}(f)$ is a normal subgroup of $\text{Gal}_\text{arith}(f)$, and both act minimally on the path space $P_2$ of the binary tree $T$. Then it follows by Lemma 2.19 that the action of $\text{Gal}_\text{arith}(f)$ is non-LQA. □

The most work for the proof is required when the critical point $c$ of $f(x)$ has a strictly periodic orbit which is a fixed point, or when it has a pre-periodic orbit of length 2. In both cases the asymptotic discriminant of the action of $\text{Gal}_\text{geom}(f)$ is stable by Theorem 1.3, so we cannot use Lemma 2.19 and have to compute the asymptotic discriminant of the action of $\text{Gal}_\text{arith}(f)$ directly. We do that below in a series of propositions.

**PROPOSITION 6.2.** Let $K$ be a finite extension of $\mathbb{Q}$, and let $f(x)$ be a quadratic polynomial with coefficients in the ring of integers of $K$. Suppose the orbit of the critical point $c$ of $f(x)$ is strictly periodic with $\#P_c = 1$. Then the action of $\text{Gal}_\text{arith}(f)$ is stable with infinite discriminant group.

**Proof.** Under the hypothesis of the proposition, $\text{Gal}_\text{geom}(f)$ is conjugate to the closure of an odometer action, as it is explained in Remark 6.1. More precisely, let $a_1 = (a_1, 1)$, let $G_1 = \langle a_1 \rangle$ be a countable subgroup of $\text{Aut}(T)$ generated by $a_1$, and let $\text{CL}(G_1)$ be the closure of the action of $G_1$ in $\text{Aut}(T)$. Then by [39] Theorem 2.4.1 there is an element $w \in \text{Aut}(T)$, such that

\[(51) \quad \text{CL}(G_1) = w \text{Gal}_\text{geom}(f) w^{-1}.\]

Let $G_\text{geom}$ be a dense subgroup of $\text{Gal}_\text{geom}(f)$ which topologically generates $\text{Gal}_\text{geom}(f)$. We now show that we can choose a conjugating element in (51) in such a way that the subgroup $G_1$ is conjugate to $G_\text{geom}$. We then can assume that $G_\text{geom} = G_1$, and compute the asymptotic discriminant of the action of $\text{Gal}_\text{geom}(f)$.

The closure $\text{CL}(G_1)$ of the odometer action is isomorphic to the profinite group $\mathbb{Z}_2$ of the dyadic integers. Therefore, $\text{Gal}_\text{geom}(f)$ is isomorphic to $\mathbb{Z}_2$, and so it is topologically generated by a single element $z$, which acts transitively on every vertex set $V_n$, $n \geq 1$. It follows that $z \in \text{Gal}_\text{geom}(f)$ also generates an odometer action on the space of paths $P_2$ [39] Proposition 1.6.3]. Then $wzw^{-1}$ also generates an odometer action on $P_2$.

Denote by $N$ the normalizer of $\text{CL}(G_1)$ in $\text{Aut}(T)$. By [39] Theorem 2.7.4] the element $a_1$ is conjugate to $wzw^{-1}$ by an element of $N$, that is, $a_1 = swzw^{-1}s^{-1}$ for $s \in N$. Since $s$ is in the normalizer $N$, then we have

\[
\text{CL}(G_1) = s\text{CL}(G_1)s^{-1} = sw \text{Gal}_\text{geom}(f) w^{-1}s^{-1},
\]

and the conjugation by $sw$ maps the generator $z$ to the generator $a_1$. Relabelling the vertices of the tree $T$ via the conjugating automorphism $sw$, we may assume that $z = a_1$. Then $\text{Gal}_\text{geom}(f) = \text{CL}(G_1)$ and $G_\text{geom} = G_1$.

The normalizer $N$ of $\text{Gal}_\text{geom}(f) \cong \mathbb{Z}_2$ in $\text{Aut}(T)$ is isomorphic to the semi-direct product $\mathbb{Z}_2 \rtimes \mathbb{Z}_2$ [39] Proposition 1.6.3], where $\mathbb{Z}_2 \cong \text{Aut}(\mathbb{Z}_2)$ denotes the multiplicative group of the dyadic integers. By [39] Proposition 1.6.4] this isomorphism can be given explicitly by

\[
\mathbb{Z}_2 \rtimes \mathbb{Z}_2^* \to N : (m, \ell) \mapsto a_1^m z_{\ell},
\]

where $m = (m_i) \in \mathbb{Z}_2$, for $\ell = (\ell_i) \in \mathbb{Z}_2^*$ we have

\[
z_{\ell} = (z_{\ell}, a_1^{-1} z_{\ell}),
\]
and the element $z_\ell$ acts on $\mathbb{Z}_2$ by raising $a_1$ to the $\ell$-th power,
\begin{equation}
(52) \quad z_\ell a_1 z_\ell^{-1} = a_1^\ell.
\end{equation}

Recall that $\text{Gal}_{\text{geom}}(f)$ is a normal subgroup of $\text{Gal}_{\text{arith}}(f)$, so $\text{Gal}_{\text{arith}}(f) \subseteq N$. Recall also that we have an exact sequence (1):
\begin{equation}
0 \longrightarrow \text{Gal}_{\text{geom}}(f) \longrightarrow \text{Gal}_{\text{arith}}(f) \longrightarrow \text{Gal}(K/L) \longrightarrow 0,
\end{equation}
so $\text{Gal}_{\text{arith}}(f)/\text{Gal}_{\text{geom}}(f) \cong \text{Gal}(K/L)$. The field $K$ is the base field of the polynomial $f(x)$, and $L$ is defined in the Introduction. So there is a homomorphism (see [39, Theorem 2.8.4] and [39, Theorem 2.8.4] for $r = 1$)
\begin{equation}
\bar{\rho}: \text{Gal}(K/L) \rightarrow \text{Gal}_{\text{arith}}(f)/\text{Gal}_{\text{geom}}(f) \subseteq N/\text{Gal}_{\text{geom}}(f) \cong \mathbb{Z}_2^k,
\end{equation}
and elements of $\text{Gal}_{\text{arith}}(f)/\text{Gal}_{\text{geom}}(f)$ act on $\text{Gal}_{\text{geom}}(f)$ via (52).

We now use the fact that $K$ is a finitely generated extension of $\mathbb{Q}$. By [39, Theorem 2.8.4] in this case the homomorphism $\bar{\rho}$ is surjective onto $N/\text{Gal}_{\text{geom}}(f)$, that is, $\text{Gal}_{\text{arith}}(f) \cong \mathbb{Z}_2 \times \mathbb{Z}_2^k$. We compute the asymptotic discriminant of the action of $\text{Gal}_{\text{arith}}(f)$ in Lemma 6.3.

In the proposition below, we take $a = a_1$, to simplify the notation. That is, the element $a$ in (53) generates the odometer action of $\text{G}_{\text{geom}}$.

**Lemma 6.3.** Let $k$ be a finite extension of $\mathbb{Q}$. Then $\text{Gal}_{\text{arith}}(f)$ is the closure of the subgroup
\begin{equation}
(53) \quad G_{\text{arith}} \cong \langle a, b, c \mid b^2 = 1, bab^{-1} = a^{-1}, cac^{-1} = a^5, bcb^{-1}c^{-1} = 1 \rangle \subset \text{Aut}(T),
\end{equation}
and the action of $\text{Gal}_{\text{arith}}(f)$ on the path space $\mathcal{P}_2$ of the binary tree $T$ has associated group chain $(G_n)_{n \geq 0}$, where for $n \geq 1$
\begin{equation}
(54) \quad G_n = \langle a^{2^n}, b, c \rangle \subset G_{\text{arith}}.
\end{equation}

**Proof.** As discussed just before the lemma, $\text{Gal}_{\text{arith}}(f) \cong \mathbb{Z}_2 \times \mathbb{Z}_2^k$. We are going to determine the generators and relations for $G_{\text{arith}}$. We start by building a bijection between the path space $\mathcal{P}_2$ and the dyadic integers $\mathbb{Z}_2 = \lim \{\mathbb{Z}/2^{n+1}\mathbb{Z} \rightarrow \mathbb{Z}/2^n\mathbb{Z}\}$.

To this end, recall that vertices in $V_n$ are labelled by words of length $n$ in 0’s and 1’s, and let $x_n = 0^n$ and $x = 0^\infty$. The group $G_{\text{geom}} = \langle a \rangle \cong \mathbb{Z}$ acts transitively on each vertex set $V_n$, $n \geq 1$, so there is a bijection $\kappa_n : V_n \rightarrow \mathbb{Z}/2^n\mathbb{Z}$, such that, given $y_n = t_1t_2 \ldots t_n \in V_n$, we have
\begin{equation}
(55) \quad \kappa_n(y_n) = k + 2^n\mathbb{Z} \text{ if and only if } y_n = x_n \cdot a^k,
\end{equation}
where $\cdot$ denotes the action of $G_{\text{geom}}$ on $V_n$, and $0 \leq k \leq 2^n - 1$. In particular, $x_n$ is mapped onto the coset of 0 in $\mathbb{Z}/2^n\mathbb{Z}$. It is straightforward to check that the maps $\kappa_n$, $n \geq 1$, are compatible with the bonding maps
\begin{equation}
V_{n+1} \rightarrow V_n : t_1t_2 \ldots t_n t_{n+1} \mapsto t_1t_2 \ldots t_n
\end{equation}
and the coset inclusions $\mathbb{Z}/2^n+1\mathbb{Z} \rightarrow \mathbb{Z}/2^n\mathbb{Z}$. That is, for every $y_{n+1} = t_1t_2 \ldots t_n t_{n+1} \in V_{n+1}$ we have
\begin{equation}
(56) \quad \kappa_{n+1}(t_1t_2 \ldots t_n t_{n+1}) \text{ mod } 2^n = \kappa_n(t_1t_2 \ldots t_n).
\end{equation}

Taking the inverse limit of the maps $\kappa_n$, we obtain the bijection $\kappa_{\infty} : \mathcal{P}_2 \rightarrow \mathbb{Z}_2$ of the path space $\mathcal{P}_2$ onto the Cantor set $\mathbb{Z}_2$. Although $\mathbb{Z}_2$ is a group, the map $\kappa_{\infty}$ is only a homeomorphism, since $\mathcal{P}_2$ does not have a group structure.

Recall that $N \cong \mathbb{Z}_2 \times \mathbb{Z}_2^k$, and $\mathbb{Z}_2^k$ acts on $\mathbb{Z}_2$ by (52). Recall [32, Theorem 4.4.7] that $\mathbb{Z}_2^k \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}_2$, where $\mathbb{Z}/2\mathbb{Z}$ is generated by $\langle [-1] \rangle \in \mathbb{Z}_2^k$, where $[-1]$ denotes the equivalence class of $-1$ in $\mathbb{Z}/2\mathbb{Z}$ for $n \geq 1$, and the the second factor is generated by $\langle [5] \rangle \in \mathbb{Z}_2^k$, where $[5]$ is the equivalence class of 5 in $\mathbb{Z}/2\mathbb{Z}$ for $n \geq 1$. Denote these generators by $b$ and $c$ respectively.

By (52), $b$ acts on each $\mathbb{Z}/2^n\mathbb{Z}$ by conjugating an element of the group to its inverse, so $b$ has order 2. Again by (52), $c$ acts on $\mathbb{Z}/2^n\mathbb{Z}$ by conjugating an element of the group to its 5-th power. Since $b$ and $c$ are generators of a direct product of groups, they commute. This gives the relations in (53). Using the identification of $V_n$ with $\mathbb{Z}/2^n\mathbb{Z}$, push forward the action of $b$ and $c$ to $\mathbb{Z}/2^n\mathbb{Z}$, then $b$ acts
on $\mathbb{Z}/2^n\mathbb{Z}$ as multiplication by $-1$, and $c$ as multiplication by $5$. Note that both $b$ and $c$ fix the coset of 0 in $\mathbb{Z}/2^n\mathbb{Z}$, therefore, $b$ and $c$ fix the word $x_n$ in $V_n$.

For $n \geq 1$, denote by $G_n$ the isotropy group of the action of $G_{\text{arith}}$ on $V_n$ at $x_n$. Since $G_{\text{geom}} \subset G_{\text{arith}}$ acts transitively on $V_n$, every coset in $G_{\text{arith}}/G_n$ can be represented by a power of $a$, and so we can extend the maps \([55]\) to the bijections

\[
\pi_n : G_{\text{arith}}/G_n \to \mathbb{Z}/2^n\mathbb{Z},
\]

for $n \geq 1$. The coset of the identity in $G_{\text{arith}}/G_n$ is mapped by \([56]\) onto the coset of 0 in $\mathbb{Z}/2^n\mathbb{Z}$. It follows that both $b$ and $c$ fix the coset of the identity in $G_{\text{arith}}/G_n$, so we obtain that $G_n$ is generated by $a^{n^2}$, $b$ and $c$ as in formula \([54]\). This finishes the proof of Lemma 6.3.

We now compute the discriminant group $D_\pi$ of the action of $G_{\text{arith}}$ on $P_2$. For $n \geq 1$, denote by $C_n$ the maximal normal subgroup $C_n$ of $G_n$ in $G$.

**Lemma 6.4.** Let $G_{\text{arith}}$ be given by \([53]\), and $\{G_n\}_{n \geq 0}$ be a group chain given by \([54]\). Then for $n \geq 1$ the maximal normal subgroup of $G_n$ in $G_{\text{arith}}$ is $C_n = \langle a^{n^2}, c^{2n-2} \rangle \subset G_n$.

**Proof.** The subgroup $C_n$ contains all elements of $G_n$ which act trivially on $G_{\text{arith}}/G_n$. So in particular \(\langle a^{n^2} \rangle \subset C_n\).

Multiplication by $-1$ has order 2 in $(\mathbb{Z}/2^n\mathbb{Z})^\times$, and the generator [5] has order $2^{n-2}$ in $(\mathbb{Z}/2^n\mathbb{Z})^\times$ [44 Theorem 5.44], so the smallest power of $b$ which acts trivially on $G_{\text{arith}}/G_n$ is 2, and the smallest power of $c$ which acts trivially on $G_{\text{arith}}/G_n$ is $2^{n-2}$. Thus \(\langle a^{n^2}, b^2, c^{2n-2} \rangle \subset C_n\).

Applying the relations in \([59]\), every element of $G_{\text{arith}}$ can be written down in the form $a^sb^kc^m$. We have to determine which compositions act trivially on $G_{\text{arith}}/G_n$.

Since $b$ and $c$ fix the coset of the identity, for $s \neq 0 \mod 2^n$ the action of $a^sb^kc^m$ is non-trivial on the coset of the identity $eG_n$, so $a^sb^kc^m$ is not in $C_n$.

Suppose $s = 0 \mod 2^n$, and $1 \leq m \leq 2^{n-2} - 1$. We only need to consider the case when $k = 1$, as $b^2 = 1$. Suppose $bc^m$ acts trivially on $G_{\text{arith}}/G_n$, that is, for all $0 \leq t \leq 2^n - 1$

\[
bc^ma^tG_n = a^tG_n.
\]

Then $c^na^tG_n = ba^tG_n$ for all $0 \leq t \leq 2^n - 1$, and so $-1$ is in the subgroup of $(\mathbb{Z}/2^n\mathbb{Z})^\times$ generated by 5. But that is not the case [44 Theorem 5.44], so $bc^m$ must act non-trivially on $G_{\text{arith}}/G_n$ for all $1 \leq m \leq 2^{n-2} - 1$. Thus we obtain that

\[
C_n = \langle a^{n^2}, b^2, c^{2n-2} \rangle = \langle a^{n^2}, c^{2n-2} \rangle \subset G_n \subset G_{\text{arith}},
\]

which finishes the proof of Lemma 6.4.

It follows from Lemma 6.3 and Lemma 6.4 that for $n \geq 1$ there is an isomorphism

\[
\lambda_n : G_n/C_n \to (\mathbb{Z}/2^n\mathbb{Z})^\times \cong C_2 \times \mathbb{Z}/2^{n-2}\mathbb{Z},
\]

where $C_2 = \{ \pm 1 \}$ is the multiplicative group of order 2. The cosets in $G_n/C_n$ are represented by elements $b^kc^m$, where $k = 0, 1$ and $0 \leq m \leq 2^{n-2} - 1$, and

\[
\lambda_n(b^kc^mC_n) = ((-1)^k, m + 2^{n-2} \mathbb{Z}).
\]

Taking the inverse limit we obtain that the discriminant group is a Cantor group, that is, $D_\pi \leftarrow \lim_{\leftarrow} \{G_n/C_n \to G_n/C_n \approx \mathbb{Z}_2^\times \}$.

We now compute the asymptotic discriminant. To do that, for each $m \geq 1$ we restrict the action to a clopen subset $U_m(x_m)$ of $P_2$ consisting of paths through the vertex $x_m = 0^m$ in $V_n$. The restricted action is that of the group $G_m$ given by \([54]\), and, associated to the action, there is a truncated group chain $\{G_n\}_{n \geq m}$. For each $m \geq 1$, we compute the discriminant group $D^m_\pi$, and show that the natural map $D_\pi \to D^m_\pi$, described in more detail later, is an isomorphism.
By Proposition 2.4, we have a homeomorphism
\[ U_m(x_m) \to G_{m,\infty} = \lim_{\leftarrow} \{ G_m/G_{m+1} \to G_m/G_n, \, n \geq m \}. \]

Recall from the discussion before (56) that the cosets of \( G_{\text{arith}}/G_n \) are represented by the powers of \( a \). Since \( G_m = \langle a^{2^m}, b, c \rangle \), the cosets of \( G_m/G_n \) are represented by powers of \( a^{2^m} \), that is, the cosets are given by
\[ G_n/a^{2^m}G_n, a^{2^{m+1}}G_n, \ldots, a^{(2^m-1)2^m}G_n. \]

Restricting the bijection (56) to the subgroup \( G_m \), we obtain
\[
\pi_n : G_m/G_n \to 2^n\mathbb{Z}/2^n\mathbb{Z} \cong \mathbb{Z}/2^{n-m}\mathbb{Z},
\]
where \( \pi_n(a^{s-2m}G_n) = s \cdot 2^m + 2^n\mathbb{Z} \rightarrow s + 2^{n-m}\mathbb{Z} \). As for (56), the action of \( b \) and \( c \) on \( G_m/G_n \) pushes forward via (59) to multiplication by \(-1\) and by \( 5 \) in \( \mathbb{Z}/2^n\mathbb{Z} \) respectively. Therefore, the order of \( c \) is equal to the order of multiplication by \( 5 \) in \( \mathbb{Z}/2^{n-m}\mathbb{Z} \), which is \( 2^{n-m-2} \). By an argument similar to the one in Lemma 6.4 one obtains that the maximal normal subgroup of \( G_n \) in \( G_m \) is given by
\[ C_n^m = \langle a^{2^m}, b, c^{2^{n-m-2}} \rangle = \langle a^{2^m}, c^{2^{n-m-2}} \rangle \subset G_n \subset G_m, \]
and there is a group isomorphism
\[ \lambda_{m,n} : G_n/C_n^m \to (\mathbb{Z}/2^{n-m}\mathbb{Z})^\times \cong C_2 \times \mathbb{Z}/2^{n-m-2}\mathbb{Z}. \]

The cosets in \( G_n/C_n^m \) are represented by \( bk^jC_n^m \), where \( k = 0, 1 \), and \( 0 \leq s \leq 2^{n-m-2} - 1 \), and
\[ \lambda_{n}(bk^jC_n^m) = ((-1)^k, s + 2^{n-m-2}\mathbb{Z}). \]

Then (60) implies that the discriminant group is a Cantor group, as we have
\[ \mathcal{D}^m_\infty = \lim_{\leftarrow} \{ G_n/C_n^m \to G_n/C_n^m, \, n \geq m \} \cong \mathbb{Z}_2^\omega. \]

The last step is to construct the natural map from \( \mathcal{D}_\infty \) to \( \mathcal{D}^m_\infty \) and to show that it is an isomorphism. For that, consider the group homomorphisms
\[ \psi_n : G_n/C_n \to G_n/C_n^m, \]
given by coset inclusions. Combining these with the maps (58) and (60), for \( n \geq 1 \) we obtain the following commutative diagram
\[
\begin{array}{ccc}
G_n/C_n & \xrightarrow{\psi_n b^k c^j C_n \mapsto b^k c^j C_n^m} & G_n/C_n^m \\
\downarrow \lambda_n & & \downarrow \lambda_{m,n} \\
C_2 \times \mathbb{Z}/2^{n-2}\mathbb{Z} & \xrightarrow{\overline{\psi}_n((\pm 1, s) \mapsto (\pm 1, s \text{ mod } 2^{m}))} & C_2 \times \mathbb{Z}/2^{n-m-2}\mathbb{Z}
\end{array}
\]

The diagram (61) induces the maps of the inverse limits so that the following diagram is commutative
\[
\begin{array}{ccc}
\mathcal{D}_\infty & \xrightarrow{\psi_\infty} & \mathcal{D}^m_\infty \\
\downarrow \lambda_\infty & & \downarrow \lambda_{m,\infty} \\
\mathbb{Z}_2^\omega \cong C_2 \times \mathbb{Z}_2 & \xrightarrow{\overline{\psi}_\infty} & \mathbb{Z}_2^\omega \cong C_2 \times \mathbb{Z}_2
\end{array}
\]
where \( \lambda_\infty \) and \( \lambda_{m,\infty} \) are isomorphisms. So if \( \overline{\psi}_\infty \) is injective, then \( \psi_\infty : \mathcal{D}_\infty \to \mathcal{D}^m_\infty \) is injective.

So let \((s, (r_n)) \neq (t, (y_n)) \in C_2 \times \mathbb{Z}_2\). Since the maps \( \overline{\psi}_n \) in (61) are constant on the first component, if \( s \neq t \), then \( \overline{\psi}_\infty(s, (r_n)) \neq \overline{\psi}_\infty(t, (y_n)) \) and we are done, so assume that \( s = t \). Since the elements are distinct, then there exists \( k \geq 1 \) such that for all \( n \geq k \) we have \( r_n - y_n \neq 0 \) mod \( 2^n \). Choose \( n \) large enough so that \( n - m > k \). Then
\[ r_n - y_n \neq 0 \mod 2^{n-m}, \]
which shows that \( \overline{\psi}_\infty(s, (r_n)) \neq \overline{\psi}_\infty(s, (y_n)) \), and the map \( \overline{\psi}_\infty \) is injective. This finishes the proof of Proposition 6.5. \( \square \)
We now consider the case when \( f(x) \) is post-critically finite quadratic polynomial with pre-periodic orbit of length 2. Recall from Theorem 1.3 that in this case the action of \( \text{Gal}_{\text{geom}}(f) \) is stable with finite discriminant group. Recall that \( \sigma \) denotes the non-trivial permutation of the set with two elements.

**Proposition 6.5.** Let \( K \) be a finite extension of \( \mathbb{Q} \), and let \( f(x) \) be a quadratic polynomial with coefficients in the ring of integers of \( K \). Suppose the orbit of the critical point \( c \) of \( f(x) \) is pre-periodic with \( \#P_c = 2 \). Then the action of \( \text{Gal}_{\text{arith}}(f) \) is stable with infinite discriminant group.

**Proof.** Under the hypothesis of the proposition, \( \text{Gal}_{\text{geom}}(f) \) is conjugate to the closure \( \text{CL}(\tilde{H}_2) \) of the countable subgroup \( \tilde{H}_2 \) generated by the elements \( b_1 = \sigma \) and \( b_2 = (b_1, b_2) \), see the proof of Theorem 1.3. That is, there exists \( w \in \text{Aut}(T) \) such that \( [39] \) Theorem 3.4.1

\[
\text{CL}(\tilde{H}_2) = w \text{Gal}_{\text{geom}}(f) w^{-1}.
\]

Let \( G_{\text{geom}} \) be a dense subgroup which topologically generates \( \text{Gal}_{\text{geom}}(f) \). Under the hypothesis of the proposition the element \( w \in \text{Aut}(T) \) can be chosen in such a way that \( w \text{Gal}_{\text{geom}} w^{-1} = \tilde{H}_2 \) \( [39] \). So relabelling the vertices of the tree \( T \) via the conjugating automorphism \( w \), we may assume that \( \text{Gal}_{\text{geom}}(f) = \text{CL}(\tilde{H}_2) \) and \( G_{\text{geom}} = \tilde{H}_2 \).

The product \( b_0 = b_1 b_2 \) generates an odometer action, and by \( [39] \) Proposition 3.1.9 \( G_{\text{geom}} = \langle b_1 b_2 \rangle \times \langle b_1 \rangle = \langle b_1 \rangle \times \langle b_1 \rangle \) is infinite dihedral, with \( \text{Gal}_{\text{geom}}(f) \cong \mathbb{Z}_2 \times C_2 \), where \( C_2 = \{ \pm 1 \} \) is the multiplicative group of order 2 generated by \( b_1 \), and \( \mathbb{Z}_2 \) is the group of dyadic integers, topologically generated by \( b_0 \).

Denote by \( N \) the normalizer of \( \text{Gal}_{\text{geom}}(f) \) in \( \text{Aut}(T) \). By \( [39] \) Proposition 3.5.2 \( N \) is isomorphic to the semi-direct product \( \mathbb{Z}_2 \rtimes \mathbb{Z}_2^\times \), where \( \mathbb{Z}_2^\times \cong \text{Aut}(\mathbb{Z}_2) \) denotes the multiplicative group of the dyadic integers. This isomorphism is given explicitly by

\[
\mathbb{Z}_2 \rtimes \mathbb{Z}_2^\times \to N : (m, \ell) \mapsto b_0^m w_\ell,
\]

where \( m = (m_i) \in \mathbb{Z}_2 \), for \( \ell = (\ell_i) \in \mathbb{Z}_2^\times \) we have

\[
z_\ell = (b_0^\ell z_\ell, w_\ell) = (b_0^\ell z_\ell, b_0^{-\ell} z_\ell),
\]

and for every \( \ell \in \mathbb{Z}_2^\times \) we have

\[
w_\ell b_0 w_\ell^{-1} = b_0^\ell.
\]

In the rest of the proof we set \( a = b_0 \), so \( a \) is an element generating the odometer action on \( \mathcal{P}_2 \).

Since \( \text{Gal}_{\text{geom}}(f) \) is a normal subgroup of \( \text{Gal}_{\text{arith}}(f) \), we have \( \text{Gal}_{\text{arith}}(f) \subseteq N \cong \mathbb{Z}_2 \rtimes \mathbb{Z}_2^\times \). By a similar argument to the one in Lemma 6.3 we can identify the path space \( \mathcal{P}_2 \) of the binary tree \( T \) with the group of dyadic integers \( \mathbb{Z}_2 \) generated by \( a \). More precisely, since the group of the powers of \( a \) acts transitively on the vertex sets \( V_n \), for each \( n \geq 1 \), there is a bijection \( \kappa_n : V_n \to \mathbb{Z}_2 \rtimes \mathbb{Z}_2^\times \), given by \( [55] \) and, taking the inverse limits as in Lemma 6.3 we obtain a homeomorphism \( \kappa_\infty : \mathcal{P}_2 \to \mathbb{Z}_2 \).

Set \( b = b_1 \), then \( b^2 = 1 \), and we have

\[
G_{\text{geom}} = \langle a \rangle \times \langle b \rangle = \langle a, b \mid b^2 = 1, bab^{-1} = a^{-1} \rangle.
\]

The normalizer of the closure of the odometer action of the subgroup generated by \( a \) in \( \text{Aut}(T) \), which coincides with the normalizer of \( \text{Gal}_{\text{geom}}(f) \) in \( \text{Aut}(T) \), satisfies \( N \cong \mathbb{Z}_2 \rtimes \mathbb{Z}_2^\times \). As in Lemma 6.3 we choose the equivalence classes \( \{ [-1] \} \) and \( \{ [5] \} \) as generators of \( \mathbb{Z}_2^\times \cong C_2 \times \mathbb{Z}_2 \), where \( C_2 = \{ \pm 1 \} \) is the multiplicative group of order 2. Here \( [-1] \) denotes the equivalence class of \(-1 \) in \( \mathbb{Z}/2^n\mathbb{Z} \) for \( n \geq 1 \), and \([5]\) denotes the equivalence class of \( 5 \) in \( \mathbb{Z}/2^n\mathbb{Z} \) for \( n \geq 1 \). Denote by \( b \) and \( c \) respectively the preimages of these generators in \( N \). Then (63) implies that \( N \) is the closure of the action of the group

\[
H \cong \langle a, b, c \mid b^2 = 1, bab^{-1} = a^{-1}, cac^{-1} = a^5, bcb^{-1}c^{-1} = 1 \rangle \subset \text{Aut}(T),
\]

where the last relation follows from the fact that \( b \) and \( c \) correspond to the generators of the different factors of the product \( \mathbb{Z}_2^\times \cong C_2 \times \mathbb{Z}_2 \).
Since $K$ is a finite extension of $Q$, by [39] Corollary 3.10.6 (g) the quotient $\text{Gal}_{\text{arith}}(f)/\text{Gal}_{\text{geom}}(f)$ is a subgroup of finite index in the quotient $Z_2 \times Z_2^\times / Z_2 \times \{\pm 1\}$. This subgroup is generated by a power of $[5]$, and so the corresponding subgroup in $N/\text{Gal}_{\text{geom}}(f)$ is generated by a power $z = c^t$, for some $t \geq 1$. Since the order of $c$ is $2^{n-2}$, the order of $z = c^t$ is $2^{n-2}/t$ which implies that $t = 2^r$ for some $1 \leq r \leq 2^{n-2}$. Using the relation $ca = a^c$, we then obtain that
\[
za^{-1} = c^t a c^{-1} = a^{5^t} = a^{5^r},
\]
and so $\text{Gal}_{\text{arith}}(f)$ has a dense subgroup $G_{\text{arith}}$ with a presentation
\[
G_{\text{arith}} \cong \langle a, b, c \mid b^2 = 1, bab^{-1} = a^{-1}, za^{-1} = a^{5^r}, bzb^{-1}z^{-1} = 1 \rangle \subset Aut(T).
\]

The action of the group $H$ on the path space $P_2$ is the same the one in Lemma 6.3. As in Lemma 6.3 let $x_n = 0^n$ be the vertex labelled by a word of 0’s in $V_n$, and let $\pi = 0^\infty$ be the path containing the vertices $x_n$, $n \geq 1$. We choose $\pi$ as our basepoint. Since the action of $c$ fixes $\pi$, then the action of $z = c^{2^t}$ fixes $\pi$. Arguing further similarly to Lemma 6.3 we obtain that for each $n \geq 1$, the isotropy group of the action of $G_{\text{arith}}$ at $x_n$ is given by
\[
G_n = \langle a^{2^n}, b, z \rangle \subset G_{\text{arith}}.
\]
Thus we obtain a group chain $\{G_n\}_{n \geq 1}$.

Denote by $C_n$ the maximal normal subgroup of $G_n$ in $G_{\text{arith}}$. Using that $z$ is a power of $c$, and arguing similarly to the proof of Lemma 6.4, we obtain that
\[
C_n = \langle a^{2^n}, z^{2^{n-r-2}} \rangle \subset G_{\text{arith}}.
\]
Then by an argument similar to the one after Lemma 6.4 we obtain that for $n \geq 1$ we have an isomorphism
\[
\lambda_n : G_n/C_n \to C_2 \times Z/2^{n-r-2}Z,
\]
and, taking the inverse limits, we obtain an isomorphism $\lambda_{\infty} : D_\pi \to Z_2^\times$. Thus the discriminant group $D_\pi$ is an infinite profinite group.

We now compute the asymptotic discriminant. As in Proposition 6.5, for each $m \geq 1$ we restrict the action to a clopen subset $U_m(x_m)$ of $P_2$ consisting of paths through a vertex $x_m = 0^m$ in $V_m$. The restricted action is that of the group $G_m$ given by (66) for $n = m$, and, associated to the action, there is a truncated group chain $\{G_n\}_{n \geq m}$. For each $m \geq 1$, we compute the discriminant group $D_{\pi}^m$, and show that the natural map $D_\pi \to D_{\pi}^m$ is an isomorphism.

The argument is similar to the one in Proposition 6.5, with small adjustments for the fact that we are now looking at a specific finite index subgroup of $Z_2 \times Z_2^\times$. As before, by Proposition 2.4 we have a homeomorphism
\[
U_m(x_m) \to G_{m, \infty} = \lim_{\longleftarrow} \{G_m/G_{n+1} \to G_m/G_n, n \geq m\}.
\]
Since $\langle a \rangle \subset G_{\text{arith}}$ acts transitively on $V_n$, every coset of $G_{\text{arith}}/G_n$ is represented by a power of $a$, and so there is a bijection $\pi_n : G_{\text{arith}}/G_n \to Z/2^nZ$ as in (50). Since $G_m = \langle a^{2^n}, b, z \rangle$, the cosets of $G_m/G_n$ are represented by powers of $a^{2^n}$, and $G_m/G_n$ bijects under $\pi_n$ onto the subgroup $2^nZ/2^nZ \cong Z/2^nZ$.

By an argument similar to the one in Proposition 6.5, for $c$, we obtain that the order of $z$ is equal to the order of multiplication by $5^2$ in $Z/2^{n-m-2}Z$, which is $2^{n-m-r-2}$. Using an argument similar to the one in Lemma 6.4, one obtains that the maximal normal subgroup of $G_n$ in $G_m$ is
\[
C_n^m = \langle a^{2^n}, b^2, z^{2^{n-m-r-2}} \rangle = \langle a^{2^n}, z^{2^{n-m-r-2}} \rangle \subset G_n \subset G_m,
\]
and there is a group isomorphism
\[
\lambda_{m,n} : G_n/C_n^m \to (Z/2^{n-m-s}Z)^{\times} \cong C_2 \times Z/2^{n-m-r-2}Z.
\]
The it follows by an argument similar to the one in Proposition 6.5 that
\[
D_{\pi}^m \leftarrow \lim_{\longleftarrow} \{G_{n+1}^m/C_{n+1}^m \to G_n/C_n^m, n \geq m\} \cong Z_2^\times.
\]
and the natural maps \( \psi_n : D^m \rightarrow D^m \), given by the inverse limits of coset inclusions \( \psi_n : G_n/C_n \rightarrow G_n/C_n \) are isomorphisms. Thus the asymptotic discriminant of the action is stable with infinite discriminant group.

This finishes the proof of Theorem 1.6

7. Non-Hausdorff elements and the subgroups of profinite groups

In this section we prove Theorem 1.7 that is, given an action of a profinite group \( H_\infty \) on the path space of a spherically homogeneous tree \( T \), we give a condition under which \( H_\infty \) contains non-Hausdorff elements.

Let \( T \) be a spherically homogeneous tree as in Example 2.1 that is, the vertex set of \( T \) is the union of vertices labelled by \( 0 \) and another element labelled by \( 0 \), and each vertex in \( V \) is connected to precisely \( \ell_n \) vertices in \( V_n \). We assume that \( \ell_n \geq 2 \) for \( n \geq 1 \). Recall that the set of paths of \( T \) is a Cantor set by Example 2.1. By assumption of Theorem 1.7 there is a profinite group \( H_\infty = \lim \{ H_n : \} \) which acts on \( T \) in such a way that for each \( n \geq 1 \) the restriction of the action of \( H_\infty \) to \( V_n \) is given by the action of \( H_n \), and that \( H_n \) acts transitively on \( V_n \). By the assumption of Theorem 1.7 there is also a collection of finite groups \( \{ L_n \}_{n \geq 1} \) such that for each \( n \geq 1 \) the wreath product \( L_n \times L_{n-1} \times \cdots \times L_1 \subset H_n \). We are going to prove that \( H_\infty \) contains a non-Hausdorff element.

**Proof of Theorem 1.7** If \( g \) is non-Hausdorff, then there exists an infinite path \( \pi \), a descending collection of clopen neighborhoods \( \{ W_n \}_{n \geq 0} \) with \( \bigcap_{n \geq 1} W_n = \{ \pi \} \) and, for each \( n \geq 1 \), a clopen subset \( O_n \subset W_n \), such that \( g(\pi) = \pi \), \( g|O_n \) is the identity, while \( g|W_n \) is non-trivial. We will construct such \( \pi \) by induction.

Let \( x_0 = 0^n \) denote a word of length \( n \) consisting only of 0's, and let \( \pi = 0^n \) be an infinite sequence of 0's. Then \( \pi \) corresponds to a path in \( P \) containing the vertices labelled by \( x_n \), for \( n \geq 1 \).

For \( k \geq 1 \), let \( W_k = U_{2k}(x_{2k}) = U_{2k}(0^{2k}) \), that is, \( U_{2k}(x_{2k}) \) contains all paths in \( P \) through the vertex in \( V_{2k} \) labelled by \( 0^{2k} \), see Example 2.1 for an explanation of the notation. Since \( \ell_n \geq 1 \) for all \( n \geq 1 \) by assumption, every \( x_{2k} \in V_{2k} \) is connected to at least two vertices in \( V_{2k+1} \), one labelled by \( 0^{2k} \) and another labelled by \( 0^{2k} \). We denote \( O_k = U_{2k+1}(0^{2k+1}) \). We will obtain a non-Hausdorff element \( g \) by induction.

Denote by \( R_n \) a set with \( \ell_n \) elements, and note that there are bijections \( b_n : R_1 \times \cdots \times R_n \rightarrow V_n \), see Example 2.1 for details. Now recall the definition of the wreath product. First, let \( \mathcal{L}_1 = L_1 \), and suppose that \( \mathcal{L}_n \) is defined. Let \( L_{n+1} = L_n \times L_{n+1} \) be the set of all functions from \( V_n \) to \( L_{n+1} \). Then the group

\[
\mathcal{L}_{n+1} = L_{n+1} \times \mathcal{L}_n := L_{n+1} \times L_n \times \cdots \times L_1
\]

acts on the product \( V_n \times R_{n+1} \) by

\[
(f, s)(v_n, w) = (s(v_n), f(s(v_n)) \cdot w).
\]

That is, for each \( v_n \in V_n \) the element \( (f, s) \) maps \( \{ v_n \} \times R_{n+1} \) to \( \{ s(v_n) \} \times R_{n+1} \), and then acts on \( \{ s(v_n) \} \times R_{n+1} \) as \( f(s(v_n)) \in L_{n+1} \).

For each \( n \geq 1 \), denote by \( p_n \) a non-trivial element of \( L_n \). Denote by 1 the identity in \( L_n \).

Let \( k = 1 \), and define \( g \) to act trivially on the vertex sets \( V_1 \) and \( V_2 \). That is, \( g|V_2 = g_2 \), where \( g_2 = (f_1, 1) \in \mathcal{L}_2 \) and \( f_1 : V_1 \rightarrow L_2 \) is the trivial constant function.

Define \( f_2 : V_2 \rightarrow L_3 \) by

\[
f_2(01) = p_3, f_2(w) = 1 \text{ for all } w \neq 01,
\]

and set \( g_3 = g|V_3 = (f_2, g_2) \). Then \( g_3 \in \mathcal{L}_3 \).
For $k > 1$, suppose $g_{2k-1} = g|_{V_{2k-1}} \in \mathcal{L}_{2k-1}$ is defined. Let $f_{2k-1} : V_{2k-1} \to L_{2k}$ be the trivial function, that is, $f_{2k-1}(w) = 1$ for all $w \in V_{2k}$. Define $g_{2k} = (f_{2k-1}, g_{2k-1})$, then $g_{2k} \in \mathcal{L}_{2k}$. Define

$$f_{2k}(0^{2k-1}1) = p_{2k+1}, f_{2k}(w) = 1 \text{ for } w \neq 0^{2k-1}1,$$

and set $g_{2k+1} = g|_{V_{2k+1}} = (f_{2k}, g_{2k})$. Then $g_{2k+1} \in \mathcal{L}_{2k+1}$ by definition of the wreath product. Note that for all $k \geq 1$ the element $g$ is trivial on the sets $O_k = U_{2k+1}(0^{2k}1)$, and non-trivial on the sets $U_{2(k+1)}(0^{2k}1)$. Both $O_k$ and $U_{2(k+1)}(0^{2k}1)$ are subsets of $W_k = U_{2k}(0^{2k})$.

By definition, $g$ acts non-trivially only on clopen sets of paths passing through vertices labelled by a word $0^{2k+1}1$ for some $k \geq 1$. Since the path $\pi = 0^\infty$ clearly does not contain such a vertex, we have $g(\pi) = \pi$. Thus $g$, $\overline{\pi}$, $\{W_k\}_{k \geq 1}$ and $\{O_k\}_{k \geq 1}$ are as desired, and $g$ is a non-Hausdorff element in $H_\infty$.

**EXAMPLE 7.1.** The paper [7] studied profinite iterated monodromy groups and arboreal representations for the polynomial

$$f(z) = -2z^3 + 3z^2$$

over number fields. They showed that for this polynomial, $G = \text{Gal}_{\text{geom}}(f) = \text{Gal}_{\text{arith}}(f)$, and that $G$ contains the infinite wreath product $[C_3]^\infty$, where $C_3$ is a cyclic group of order 3. Our Theorem 1.7 implies then that $G$ contains a non-Hausdorff element, and so the action of $G$ on the path space $P_T$ of the rooted tree $T$ has wild asymptotic discriminant.

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