A New Structure of 2-State Number-Conserving Cellular Automata

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SUMMARY Two-state number-conserving cellular automaton (NCCA) is a cellular automaton of which cell states are 0 or 1, and the total sum of all the states of cells is kept for any time step. It is a kind of particle-based modeling of physical systems. We introduce a new structure of its value-1 patterns, which we call a “bundle pair” and a “bundle quad”. By employing this structure, we show a relation between the neighborhood size \( n \) and \( n-2 \) NCAs.

key words: cellular automata, number-conservation

1. Introduction

A cellular automaton (CA), which was introduced by von Neumann as a biological self-reproducing model, is a discrete dynamical system that evolves in discrete space and time [16]. A CA consists of the state of cells, the tiling of cells, its neighborhood, and its rules. The state of each cell is a finite integer, which changes according to a rule that depends on the cell states in the neighborhood. There are many variations of CAs, depending on the dimension, state, and neighborhood, and they can also be applied to a wide variety of fields [4], [5], [14].

A number-conserving CA (NCCA) is a CA whose states are integers with the characteristic that the total sum of all cells is conserved at any time step. The NCCA is widely used to the research of lattice gas, traffic flow, etc. [14], [18] and it has been continuously studied until recently [17].

Number-conserving condition of CA was first discussed by Hattori et al. [6]. And Boccara et al. studied NCCA on circular conditions in [2]. Durand et al. studied the two-dimensional case. They also studied the relation between several boundary conditions and showed that the condition of number-conservation is equivalent for both finite and infinite configurations [3].

Two-state NCCA with only 0 or 1 status is always maintained by the number of 1s on the configuration. In other words, all the 1s on the configuration move without disappearing or appearing on any time step. Moreira et al. used motion representation to effectively express the 1 movement of each two-state NCCA rule [1], [2], and we introduced hierarchical motion representation, which gives complexity to each motion to express the rules more simply and systematically in [9].

The movement of 1 in each two-state NCCA naturally results from patterns in which the next state is 1 called value-1 patterns. For us who analyze value-1 patterns to find the characteristics of NCCA rules, it is important to keep content while simply expressing value-1 patterns. Thus we proposed the relation bundle between \( n \)-cell and \((n-1)\)-cell patterns, and introduced the relation between the value-1 pattern sets of \( n \)-cell and \((n-1)\)-cell NCCA rules [10], [11].

In this paper, we proposed the new structure of each \( n \)-cell rule using the \((n-2)\)-cell pattern “bundle quad” and \((n-1)\)-cell pattern “bundle pair”. Any NCCA rule can be represented by a combination of bundle pairs and bundle quads. Also, the number of \( n \)-cell NCCA rules with \( 2^{n-3} \) bundle quads is the same as the number of \((n-2)\)-cell NCCA rules. And we introduced some propositions about \( n \)-cell NCCA rules with \( 2^{n-2} \) bundle pairs.

2. Number-Conserving Cellular Automata

Definition 1 (one-dimensional two-state Cellular Automata): A one-dimensional two-state cellular automaton \( A \) is defined as \( A = (n, f) \), where its neighborhood size \( n \) is a non-negative finite integer and \( f: \{0,1\}^n \to \{0,1\} \) is a mapping called the local function of \( A \). A configuration over \( \{0,1\} \) is a mapping \( c: Z \to \{0,1\} \), where \( Z \) is the set of all integers. Then, \( \text{Conf}(\{0,1\}) = \{c: Z \to \{0,1\}\} \) is the set of all configurations over \( \{0,1\} \). The global function \( F \) of \( A \) is defined as \( F: \text{Conf}(\{0,1\}) \to \text{Conf}(\{0,1\}) \), i.e.,

\[
\forall c \in \text{Conf}(\{0,1\}), \forall i \in Z : F(c)(i) = f(c(i) \cdots c(i+n-1)).
\]

Note that we use the Wolfram numbering [19] \( W(f) \) to represent a local function \( f: W(f) = \Sigma f(a_1 \cdots a_n)2^{n-1}a_1 + \cdots + 2^{n-1}a_n \) where the sum is applied on \( a_1 \cdots a_n \in \{0,1\}^n \). To represent a CA, we also use a pair of its neighborhood size and its Wolfram number instead of its local function. The local function \( f \) is also referred to as the rule of \( A \).

Definition 2 (Number Conserving CA): A cellular automaton \( A = (n, f) \) is said to be number-conserving if \( F(\alpha_0) = \alpha_0 \) and

\[
\lim_{m \to \infty} \frac{\mu_m(F(\alpha))}{\mu_m(\alpha)} = 1 \text{ for all } \alpha \in \text{Conf}(\{0,1\}) - \alpha_0
\]

where \( \mu_m(F(\alpha)) \) is the number of \( 1 \)-states of the \( m \)-th iteration of \( \alpha \).
where $F$ is the global function of $A$, $a_0$ is zero configuration, i.e., the value of every cell is 0, and $\mu_m(\alpha) = \sum_{i=-m}^{m} \alpha(i)$.

Since Durand et al. [3] showed that finite-number-conserving is equivalent to the general infinite case, it is enough to show the number-conservation of a CA even for the case of infinite number of nonzero cells.

Figure 1 is the rule table and its space-time diagram of Rule 184 3-cell NCCA. State-1 (state-0) cells are shown in black (white), respectively in the diagram. During the time evolution, the sum of all 1s (black cells) is not changed. This NCCA is famous as a simple model of traffic flow which describes the property that each vehicle moves forward only if there is a space in front of it [14].

**Definition 3 (Pattern):** A pattern $p = a_1 a_2 \cdots a_n$ is a sequence of $a_i \in \{0, 1\}$ of a finite length $n$.

We also use the notation of concatenation of two or more patterns to represent another pattern. For example, if $p = 010$, then $0p = 0010$ and $p1 = 0101$. In addition, a pattern containing the wildcard character “_” which represents both 0 and 1 is called an extended pattern. For example, _010 means two patterns: 1010 and 0010.

**Definition 4 (Bundle):** For a length $n(\geq 1)$ pattern $r$, if length $n + 1$ patterns $p$ and $q$ satisfy the condition

$$p = 0r, \ q = 1r $$ (resp. $p = r0, \ q = r1$),

then we call $p(q)$ l-bundle (resp. r-bundle) of $r$. When $r, p$ and $r, q$ satisfy both cases, we call $r$ the bundle pattern of $p$ and $q$.

**Definition 5 (Bundle pair):** Two patterns $p$ and $q$ are a bundle pair if $p$ and $q$ are either l-bundle or r-bundle of a certain pattern $r$.

For example, the two patterns 1010 and 1011 are a bundle pair because the two patterns are r-bundle of 101. It is also denoted by the extended pattern, 101_. The place of _ shows its extending direction and we also express it by the symbol ‘l’ or ‘r’. Thus, the r-bundle of 101 is denoted as 101, and its extending direction is ‘r’ and the l-bundle case is _101 and its direction is ‘l’.

**Definition 6 (Value-1 pattern set):** For a CA $A = (n, f)$, we call the pattern set $P_A = \{p \mid f(p) = 1\}$ the value-1 pattern set of $A$.

Since we only consider two-state CAs, the local function of each CA is identified with its value-1 pattern set. In addition, we call the elements of $P_A$ the value-1 patterns. We have shown the following results in our previous studies [10], [11].

**Lemma 1:** [11] For a pattern $p$ in the value-1 pattern set of an NCCA, there exists a pattern $q$ in the set and a pattern $r$ s.t. $p$ and $q$ are either l-bundle or r-bundle of $r$.

**Lemma 2:** [11] Let $P_A$ be the value-1 pattern set of an NCCA $A(n, f)$. If three patterns $a_1 \cdots a_n, a_1 \cdots a_{n-1} a_n$, and $\bar{a}_1 a_2 \cdots a_n$ are in $P_A$, then pattern $\bar{a}_1 a_2 \cdots a_{n-1} \bar{a}_n$ is also in $P_A$.

**Theorem 1:** [11] For an $n$-cell NCCA $A (n \geq 2)$ with $|P_A| = |P_A|/2$, an $(n-1)$-cell CA $B$ satisfying $P_B = P_A$ is an $(n-1)$-cell NCCA when $P_A$ is a multiset such that each element is a bundle pattern of two patterns of $P_A$.

This theorem shows that the value-1 pattern set of each NCCA is characterized by the combination of bundle pairs.

In the following section, we will introduce another structure of the value-1 patterns of NCCAs, a bundle quad. We will use it to describe a relation between $(n-2)$-cell and $n$-cell NCCA rules.

3. Bundle Pair and Bundle Quad

In this section, we use the following corollary of theorem 1, i.e., the value-1 pattern set of an NCCA is a set of bundles.

**Corollary 1:** Let $Q$ be the value-1 pattern set of an $(n+1)$-cell NCCA with $2^n$ elements. There exists a pattern set $P = \{p_i | 1 \leq i \leq 2^{n-1}\}$, which is the value-1 pattern set of an $n$-cell NCCA, where for each element $p_i$, two patterns $\alpha, \beta \in Q$ are bundle of $p_i$, and $\alpha, \beta$ are not for any other $p_j (j \neq i)$.

In brief, corollary 1 is directly derived from theorem 1 and means that $(n-1)$-cell NCCA with the number of elements of the value-1 pattern set is $2^{n-2}$ can be obtained from $n$-cell NCCA with the number of elements of the value-1 pattern set of elements is $2^{n-1}$.

Because theorem 1 is a characterization of the value-1 pattern set by its bundle patterns, the following corollary is also derived:

**Corollary 2:** Let $P = \{p_i | 1 \leq i \leq 2^{n-1}\}$ be the value-1 pattern set of an $n$-cell NCCA. There exists a pattern set $Q = \{q_j | 2 \leq j \leq 2^n + 1\}$, which is the value-1 pattern set of an $(n+1)$-cell NCCA, where $q_2, q_{2i+1}$ are a bundle of $p_i$ for $1 \leq i \leq 2^{n-1}$.

Corollary 2 means that $n$-cell NCCAs can be obtained from each $(n-1)$-cell NCCA, in contrast to corollary 1.

The bundle patterns from the value-1 pattern set of an NCCA forms a multiset, but its multiplicity is at most two by the following lemma:

**Lemma 3:** Let $Q = \{q_j | 1 \leq j \leq 2^n\}$ be the value-1 pattern set of an $(n+1)$-cell NCCA. There always exists a multiset $M = \{m_i | 1 \leq i \leq 2^{n-1}\}$ s.t. $m_i$ is a bundle pattern of two
patterns $q_p, q_r$ in $Q$, and $q_s, q_t$ are not for any $m_i(j \neq i)$. Each element of $M$ has a multiplicity at most two.

Proof: Let $Q$ be the value-1 pattern set of an $(n+1)$-cell NCCA. Then, each element in $Q$ is one of a bundle pair by lemma 1. If there are three patterns $p, q, r$ such that $p$ and $q$ are $l$-bundle of a certain pattern and $q$ and $r$ are $r$-bundle of a certain pattern. Then there must be a pattern $s$ that can be $r$-bundle with $p$ or $l$-bundle with $r$ by lemma 2. Then, a multiset $M = \{m_i | 1 \leq i \leq 2^{n-1}\}$ always exists when $m_i$ is a bundle pattern of two patterns $p, q$ in $Q$.

For example, there are sixteen 4-cell CAs of which bundle pattern set is the value-1 pattern set of Rule 184. Eight cases among them are NCCA. Figure 2 shows one of them, 4-cell NCCA Rule 60200.

Until now, we show the relation between $n$-cell and $(n+1)$-cell NCCAs using bundle. From now, we introduce the relation between $n$-cell and $(n+2)$-cell NCCAs using either $l$-bundle or $r$-bundle. Lemma 4 is the description of the relation between $n$-cell and $(n+1)$-cell NCCAs using only $l$-bundle.

Lemma 4: Let $P = \{p_i | 1 \leq i \leq 2^{n-1}\}$ be the value-1 pattern set of an $n$-cell NCCA. If $Q = \{q_i | 2 \leq j \leq 2^n + 1\}$ is a pattern set, where $q_{2i}, q_{2i+1}$ are $l$-bundle of $p_i$ for $1 \leq i \leq 2^{n-1}$, then $Q$ is the value-1 pattern set of an $(n+1)$-cell NCCA.

Proof: Each element of $Q$ is a pattern in $P$ prefixed with a number 0 or 1. Then, the number of elements of $Q$ is $2^n$ without any duplicated patterns. Hence, $Q$ is the value-1 pattern set of an $(n+1)$-cell NCCA by corollary 2.

When $q_{2i}, q_{2i+1}$ are $r$-bundle of $p_i$, in lemma 4, $Q$ is also the value-1 pattern set of an $(n+1)$-cell NCCA.

Next, we think about the relation between $n$-cell and $(n+2)$-cell NCCAs using the combination of $l$- and $r$-bundle.

Lemma 5: Let $P = \{p_i | 1 \leq i \leq 2^{n-1}\}$ be the value-1 pattern set of an $n$-cell NCCA and $R = \{r_i | 2 \leq k \leq 2^{n+1} + 3\}$ be a pattern set satisfied that $r_{4i}, r_{4i+1}, r_{4i+2}, r_{4i+3}$ are $l$-bundle of $r$-bundle of $p_i$ for $1 \leq i \leq 2^{n-1}$. Then, $R$ is the value-1 pattern set of an $(n+2)$-cell NCCA.

Proof: Each element in $R$ is a pattern in $P$ prefixed with a number, 0 or 1, and suffixed with a number, 0 or 1. Then, the number of elements of $R$ is $2^{n+1}$ without any duplicated patterns. Hence, $R$ is the value-1 pattern set of an $(n+2)$-cell NCCA by corollary 2.

When $r_{4i}, r_{4i+1}, r_{4i+2}, r_{4i+3}$ are $r$-bundle of $l$-bundle of $p_i$ in lemma 5, $R$ is also the value-1 pattern set of an $(n+2)$-cell NCCA.

By lemma 5, the value-1 pattern set can be extended from $n$-cell NCCA to $(n+2)$-cell NCCA without overlapping elements using the operations of generating $l$-bundle and $r$-bundle as shown in Fig. 3.

Although NCCA rules should be recursively generated by extending bundle pairs, enumeration or indexing of NCCA rules is still difficult, because a similar value-1 pattern set could be generated by some different extending operations. To figure out the difficulty, we introduce another structure on the value-1 patterns, a bundle quad.

Definition 7: [Bundle quad] Four patterns $p, q, r, s$ are a bundle quad if $p$ (resp. $s$) is a bundle pair with $q$ and $r$ simultaneously.

For example, in Fig. 4, the four patterns 0010, 1010, 0011, and 1011 are a bundle quad because 0010 (resp. 1011) and 1010 are $l$-bundle (resp. $r$-bundle) of 010 (resp. 101). Also, 0010 (resp. 1011) and 0011 are $r$-bundle (resp. $l$-bundle) of 001 (resp. 011). Further, those four patterns are $l$-bundle of $r$-bundle (or $r$-bundle of $l$-bundle) of 01.

Let four patterns $p, q, r, s$ of length $n+2$ be a bundle quad and two patterns $a, \beta$ of length $n+1$ be the bundle patterns of $p, q$ and $r, s$, respectively. Moreover, let a length $n$ pattern $\gamma$ be a bundle pattern of $a, \beta$. Then, $p, q, r, s$ are $l$-bundle of $r$-bundle of $\gamma$. We call the pattern $\gamma$, the seed of the bundle quad, and we denote the bundle quad as $[\gamma]$. For example, in Fig. 4, the seed of 0010, 1010, 0011, 1011 is 01, and we denote them as $[01]$. In the previous section, we showed that the elements of the value-1 pattern set of any NCCA $A$ form bundle pairs. Moreover, some pairs of the bundle pairs may form bundle quads. Therefore, we can represent the value-1 pattern set of $A$ by the set of bundle quads and bundle pairs. We call it
the quad and pair set of A. The following theorem relates to the case where value-1 patterns only form bundle quads.

**Theorem 2:** The number of n-cell NCCAs with \(2^{n-3}\) bundle quads is not shown in general. The following properties are of their value-1 patterns as illustrated in Fig. 6. From the number of NCCA rules by combination of bundle pairs and found so far:

**Proposition 1:** For any n-cell NCCA A = (n, fA) (n ≥ 2) of full pair with its quad and pair set U, the following properties hold:

1. Let U′ be the set of all symmetric elements in U, i.e., for each extended pattern \(a_1a_2 \cdots a_n \in U, a_{n+1} \cdots a_2a_1 \in U'\). U′ is also a quad and pair set of full pair.
2. Either 1 \(\cdots\) 1 \(\in U \text{ or } 1 \cdot \lfloor 1 \cdot \lfloor 1 \text{ holds. If } 1 \cdot \lfloor 1 \cdot \lfloor 1 \text{ is also a quad and pair set of an n-cell NCCA.}}

### Table 1

| Neighborhood size | Structure of value-1 patterns | The number of rules |
|-------------------|-------------------------------|---------------------|
| 1-cell            |                               | 1                   |
| 2-cell            | 1 bundle pair                 | 2                   |
| 3-cell            | 1 bundle quad                 | 4                   |
|                   | 0 bundle quad & 2 bundle pairs | 3                   |
|                   | Total                         | 5                   |
| 4-cell            | 2 bundle quads                | 2                   |
|                   | 1 bundle quad & 2 bundle pairs | 12                  |
|                   | 0 bundle quad & 4 bundle pairs | 8                   |
|                   | Total                         | 22                  |
| 5-cell            | 4 bundle quads                | 5                   |
|                   | 3 bundle quads & 2 bundle pairs | 64                  |
|                   | 2 bundle quads & 4 bundle pairs | 185                 |
|                   | 1 bundle quad & 6 bundle pairs | 150                 |
|                   | 0 bundle quad & 8 bundle pairs | 23                  |
|                   | Total                         | 428                 |
| 6-cell            | 8 bundle quads                | 21                  |
|                   | 7 bundle quads & 2 bundle pairs | 20                  |
|                   | 6 bundle quads & 4 bundle pairs | 56                  |
|                   | 5 bundle quads & 6 bundle pairs | 23416               |
|                   | 4 bundle quads & 8 bundle pairs | 46256               |
|                   | 3 bundle quads & 10 bundle pairs | 40956               |
|                   | 2 bundle quads & 12 bundle pairs | 14632               |
|                   | 1 bundle quads & 14 bundle pairs | 1780                |
|                   | 0 bundle quads & 16 bundle pairs | 64                  |
|                   | Total                         | 133184              |

**Fig. 5** A 5-cell NCCA rule 3485519808 having the same space-time diagram as Rule 184

**Fig. 6** The number of NCCA rules by combination of bundle pairs and bundle quads.
The first property in Proposition 1 is trivial because the symmetric value-1 pattern set of an NCCA is also that of another NCCA and \( U' \) is also a full pair set because the operation just changes \( r \)- or \( l \)-bundle pattern to its opposite direction.

The outline of a proof of the second property is as follows: \( f_A(1 \cdots 11) = f_A(1 \cdots 10) = 1 \) and \( f_A(01 \cdots 1) = 0 \) holds by the assumption. For a sub-configuration \( s_i = 1 \cdots 10 \) (with \( f_A(s_i) = 1 \)) of a configuration \( c_i \), the evolved sub-configuration is \( s_{i+1} = 1 \cdots 10 \). In the case of \( A' = (n, f_{A'}) \), where \( f_{A'}(1 \cdots 11) = f_{A'}(01 \cdots 1) = 1 \) and \( f_{A'}(1 \cdots 10) = 0 \), thus the evolved sub-configuration is changed to \( s'_{i+1} = 01 \cdots 1 \). But there are only two cases of its left-extension either \( 0s_1 \) or \( 1s_1 \). If \( 0s_1 \), the evolved configuration is \( 1s'_1 \) and the application of \( f_{A'} \) only switches these positions 0 and 1. If \( 1s_1 \), we repeatedly check the above extension whether \( 01 \cdots 1s_1 \) or \( 11 \cdots 1s_1 \). Because we start from \( s_1 \), the case \( 01 \cdots 1s_1 \) should appear for any finite cyclic configuration including \( s_1 \). Because \( A \) is a full-pairs NCCA, \( A' \) is also a full-pairs NCCA.

Figure 7 shows quad and pair sets of two 4-cell full pair rules 60200 and 65280. By Proposition 1, we get the other six rules from the above two rules in 4-cell full pair cases as in Fig. 7.

**Proposition 2:** For the quad and pair set \( U \) of any \( n \)-cell NCCA (\( 2 \leq n \leq 4 \)), the derived set by deleting ‘.’ from all elements of \( U \) is the value-1 pattern set of a certain \((n-1)\)-cell NCCA rule. For \( n \geq 5 \), there exist \( n \)-cell NCCA rules which do not satisfy the property.

Proposition 2 is easily verified for the case of \( 2 \leq n \leq 4 \) by the enumeration result of NCCA rules.

As Fig. 7, deleting ‘.’ form each element of their quad and pair sets, we get \{111, 101, 011, 100\}, \{111, 101, 110, 001\}, \{111, 110, 101, 100\}, and \{111, 011, 101, 001\}. These sets are the value-1 pattern sets of 3-cell full pair rules 184, 226, 240, and 170. In the case of \( n \)-cell full pair (\( n \geq 5 \)), the most of the rules, such as the rule 4294901760 in Fig. 8, satisfy Proposition 2 but there are some other rules which do not satisfy it, e.g., the rule 4021231776 in Fig. 8. There are several extended patterns which share a same pattern from second to \((n-1)\)-th numbers but different extending direc-

\[
\begin{align*}
60200: \{111, 101, 011, 100\} & \quad \Rightarrow \{111, 101, 011, 100\} \quad \text{(3)} \\
65280: \{111, 110, 101, 100\} & \quad \Rightarrow \{111, 110, 101, 010\} \quad \text{(3)}
\end{align*}
\]

**Fig. 7** Six 4-cell rules derived from two NCCA rules 60200, 65280

Nevertheless, the following Proposition holds for any full pair case in general.

**Proposition 3:** For any \( n \)-cell NCCA (\( n \geq 2 \)) of full pair, its quad and pair set \( U \) has the following form:

\[
U = \{u_i | u_i = \text{either } a_ip_i \text{ or } -p_i a_i\}
\]

where \( p_i \) is the zero-padded binary representation of \( i \) of the length \( n-2 \) and \( a_i \in \{0,1\} \) for \( 0 \leq i \leq 2^{n-2}-1 \).

**Proof:** Let \( U \) be the quad and pair set of any \( n \)-cell NCCA (\( n \geq 2 \)) of full pair and let \( p_i \) and \( p_j \) be the length \( n-2 \) sub-patterns from second to \( n-1 \) positions of two different element \( u_i \) and \( u_j \) of \( U \) (\( i \neq j \)). Suppose \( p_i = p_j \). There are only two cases according to the extending directions of \( u_i \) and \( u_j \). First, if they are the same then \( u_i = \_p_i a_i, u_j = \_p_j a_j \) (or \( u_i = a_i p_i, u_j = a_p a_j \) with \( a_i \neq a_j \)). These two bundles pair forms a bundle quad, but \( U \) is a quad and pair set of full pair. Secondly, if they are different, then \( u_i = \_p_i a_i, u_j = a_i p_j, u_j = \_p_d a_j \) is shown in the same way and thus \( p_i \neq p_j \).

For example, Fig. 9 shows a set derived from the quad and pair sets of the 6-cell 16987224761694323240.
directions accumulate the recursive information of \( n - 1 \) or less NCCA rules.

4. Conclusion

In this paper, we introduced a new structure of the value-1 patterns of each NCCA, a bundle pair and a bundle quad. Any NCCA rule can be represented by some combinations of bundle pairs and bundle quads. The structure reflects the relation between NCCAs of different neighborhood sizes and we show the relation that the number of \((n - 2)\)-cell NCCA rules is equal to the number of \(n\)-cell NCCA rules only composed of bundle quads. In addition, we show some propositions of full pair case and the full pair case can be derived from all kinds of \((n - 2)\)-cell patterns, unlike the full quad. If the rules of large-neighborhood NCCAs are represented by the quads and pairs of smaller-neighborhood NCCA, we can analyze large-neighborhood NCCAs more easily.

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