Abstract. This is the first of a series of papers that investigates the loop space homology of polyhedral products.

To any simplicial complex $K$ on $m$ vertices there corresponds a polyhedral product functor, which associates to $m$ based topological spaces $X = (X_1, \ldots, X_m)$ a certain subspace $X^K$ in the cartesian product $\prod_i X_i$. In this paper we establish a connection between the loop space homology of polyhedral products of any 1-connected spaces and the homology of certain diagonal arrangements associated with $K$. This reduces the problem to the calculation of the Ext-algebra of the exterior Stanley-Reisner algebra of $K$. We illustrate these results by finding the presentation of such loop homology algebras for flag complexes and skeletons of simplices, generalizing results of Panov-Ray, Papadima-Suciu, Lemaire.

Finally, we show that in the case when all the $X_i$'s are suspensions, the homology splitting comes from the stable homotopy splitting of $\Omega(X^K)$.

1. Introduction

Let $K$ be a simplicial complex on the vertex set $[m] = \{1, \ldots, m\}$ and let $X = (X_1, \ldots, X_m)$ be a sequence of based topological spaces. The polyhedral product, or $K$-product, is a natural subspace

$$X^K = (X_1, \ldots, X_m)^K \subset X_1 \times \cdots \times X_m$$

defined by the following condition:

$$(x_1, \ldots, x_m) \in X^K \iff$$

for any $\tau \notin K$ there exists $i \in \tau$ such that $x_i$ is the base point of $X_i$.

This construction was introduced in [1] and generalizes usual wedges (when $K$ is the set of vertices), cartesian products ($K$ is $(m-1)$-dimensional simplex $\Delta^{m-1}$), fat wedges ($K = \partial \Delta^{m-1}$). Our interest in it arises from certain problems in toric topology where $K$-products play an important role: for example, so called Davis-Januszkiewicz spaces $DJ_K$ are constructed in this way as $(\mathbb{C}P^\infty, \ldots, \mathbb{C}P^\infty)^K$. In this paper we consider the problem of calculating the loop homology algebra of this construction.

It is well known that taking homology of the loop space of a given based topological space with field coefficients preserves products and coproducts, which for topological spaces are

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wedges and cartesian products; here homology is regarded as a graded Hopf algebra with Pontryagin multiplication. It turns out that for some strict class of simplicial complexes $K$ (so called flag complexes) the answer is as simple as for wedges and products: the required algebra is the colimit of the corresponding diagram in the category of graded algebras. In general this answer fails due to non-triviality of higher order Whitehead products in $X^K$. In the present paper we investigate the functor for the loop space homology of $K$-products.

Another interesting question raised in [7] concerns possible relations of homology calculations for $\Omega DJ_K$ and for the diagonal arrangement associated with $K$. In this paper we explain this connection. We show that the diagonal arrangement plays an important role in constructing geometric models for the loop spaces for all $K$-products and, moreover, this leads to the corresponding decomposition results in homology.

Our approach is inspired by the theory of labelled configuration spaces. The results obtained are similar to the connection between iterated loop spaces and classical configuration spaces such as Milgram and May results, Snaith stable splitting.

We define the configuration space $C_K = \sqcup_{I \in \mathbb{N}^m} C^I$ of particles in $\mathbb{R}^1$ with labels and partial collisions (see the precise definition in Section 3). This space appears in the configurations space models for our loop spaces:

$$\Omega(X^K) \simeq \sqcup C_K(I) \times \Omega X^I / \sim,$$

which we discuss in Section 6. This model is obtained using methods developed in [10] and originating with Segal.

In the particular case when each space of $X$ is a suspension: $X_i = \Sigma Y_i$, $i \in [m]$, we get a stable splitting (see Theorem 6.6):

$$\Omega(\Sigma Y_1, \ldots, \Sigma Y_m)^K \simeq_s \bigvee_{I=(i_1, \ldots, i_m) \in \mathbb{N}^m} C_K(I) \wedge Y_1^{\wedge i_1} \wedge \cdots \wedge Y^{\wedge i_m},$$

which obviously leads to the homology splitting in this case.

For general $X$, we use the certain algebraic structures on $H_*(C_K)$ and our main homology result states

**Theorem 1.1.** Let $X_1, \ldots, X_m$ be based 1-connected topological spaces. Then for homology with field coefficients the following graded algebra isomorphism holds

$$H_*(\Omega X^K) \cong \bigoplus_{I \in \mathbb{N}^m} H_*(C_K(I)) \otimes H_*(\Omega X)^\otimes I / \sim$$

The equivalence relation is the closure of the relations of the form

$$(\mu_{j,k} y)(\cdots \otimes x_j^k \otimes x_{j_j}^{k+1} \otimes \cdots) \sim y(\cdots \otimes x_j^k \otimes x_{j_j}^{k+1} \otimes \cdots),$$

where $y \in H_*(C_K(I))$, $x_j^k, x_{j_j}^{k+1} \in H_*(\Omega X_j)$ and $\mu_{j,k} : C_K(I) \to C_K(I + e_j)$ are certain doubling operations defined in Section 4.2.
If none of the algebras $H_\ast(\Omega X)$ has torsion, then the isomorphism holds also over the integers.

The algebra structure on the right side is defined by the algebra structure on $H_\ast(C_K)$ constructed in Section 4 and the usual rule for tensor products.

This theorem looks more elegant when formulated in operadic language:

**Theorem 1.1’** If $X$ is as in Theorem 1.1, then

$$H_\ast(\Omega X^K) \cong H_\ast(C_K) \otimes_{\text{Ass}} \tilde{H}_\ast(\Omega X),$$

where Ass is the non-$\Sigma$ associative operad, and $H_\ast(C_K)$ has the natural structure of a left module and a right multi-module over Ass.

This operad description of the theorem will be discussed somewhere else, and trying to make this paper more accessible, we avoid here the operad language.

Due to Theorem 1.1 the original problem is now reduced to the calculation of the homology algebra of $C_K$ and the action of doubling operations $\mu$ on it.

The simple answer for flag complexes has the following explanation: for any flag complex $K$, all connected components of $C_K$ are contractible, and the presentation is this case is as follows (here $\sqcup$ denotes the coproduct of connected graded algebras, see Section 2.1):

**Corollary 1.2.** For any flag $K$ the following isomorphism holds for homology with field coefficients

$$H_\ast(\Omega X^K) \cong \sqcup_{i=1}^n H_\ast(\Omega X_i)/\sim,$$

with $x \cdot y \sim (-1)^{\deg x \deg y} y \cdot x$ for $x \in H_\ast(\Omega X_i)$, $y \in H_\ast(\Omega X_j)$ when $\{i, j\} \in K$.

This corollary was obtained rationally for $X = \mathbb{C}P^\infty$ in [14], and for $X = S^{2n+1}$ in [15].

For all non-flag $K$ this presentation fails to be true due to higher homology of $C_K$. For example, for each minimal non-simplex $\tau$ of $K$ with no less than 3 vertices we have a non-trivial $|\tau| - 2$-dimensional class in $H_\ast(C_K)$. This class gives rise to the non-trivial $s$-product of elements from $H_\ast(\Omega X_i)$, $i \in \tau$ and provides new generators in $H_\ast(\Omega(X^K))$. This higher $s$-product in loop homology corresponds to the higher order Samelson product in $\pi_\ast(\Omega(X^K))$ or, equivalently, to the higher order Whitehead product in $\pi_\ast(X^K)$. We investigate the homology of $C_K$ in terms of higher operations in [11], and apply Theorem 1.1 to the discovered structures.

The connection to the theory of subspace arrangements is provided by the fact that the components $C_K(I)$ can also be expressed as complements of diagonal subspace arrangements associated with $K$. Thus, the problem of calculating the homology algebra $H_\ast(\Omega(X^K))$ is now related to the problem of calculating the homology of the complements of certain diagonal arrangements.

In Section 3.3 we introduce a certain homotopy modification $\tilde{C}_K$ of the space $C_K$ more suitable for homology calculations. We show in Section 4 the isomorphism of its cellular chain
complex with the Adams-Hilton model for the standard cell decomposition of \((S^1, \ldots, S^1)^K\) and with the cobar construction on \(H_*(S^1, \ldots, S^1)^K\). This implies

**Theorem 1.3.** For the configuration space \(C_K\) the following multi-graded algebra isomorphism holds

\[
\oplus_{n,I} H_{n,I}(C_K; \mathbb{Z}) \cong \oplus_{n,I} \text{Ext}^{\wedge(K)}_{|I| - n,I}(\mathbb{Z}; \mathbb{Z}).
\]

Here \(\wedge(K)\) denotes the exterior Stanley-Reisner algebra, which is the factor of the exterior algebra \(\wedge[v_1, \ldots, v_m]\) by the Stanley-Reisner ideal (see Definition 2.10).

In particular, the component \(C_K(1, \ldots, 1)\) corresponds to the complement \(D_K\) of the standard diagonal arrangement associated with \(K\), and this fact leads to the expected relation between \(H_*(D_K)\) and \(H_*(\Omega DJ_K)\) (see Section 5.2).

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## 2. Preliminaries

### 2.1. Polyhedral products.

Here we give the definition of polyhedral product of based topological spaces and outline some known results.

Let \(K\) be an abstract simplicial complex, that is the collection of subsets of \([m] \equiv \{1, \ldots, m\}\) which includes the empty set and is closed under taking subsets. Let \(X = (X_1, \ldots, X_m)\) be a sequence of based topological spaces.

Define \(X^\sigma\) as \(\prod_{i \in \sigma} X_i\) for each \(\sigma \in K\). Then the inclusion \(\sigma \subset [m]\) let us regard \(X^\sigma\) as the subspace in the full product:

\[
X^\sigma \subset \prod_{i \in [m]} X_i.
\]

Now the definition of \(K\)-product in the introduction is equivalent to the following one.

**Definition 2.1.** The polyhedral product, or \(K\)-product, of \(X = (X_1, \ldots, X_m)\) is

\[
X^K = \bigcup_{\sigma \in K} X^\sigma.
\]

In the particular case when \(X_1 \cong \ldots \cong X_m \cong X\), we call \(X^K\) the \(K\)-power of \(X\) and denote it by \(X^K\).
Examples 2.2.  

(1) **Product.** If $K = \Delta[m]$ is the full $(m-1)$-dimensional simplex on $[m]$, then

$$X^K \cong X_1 \times \cdots \times X_m;$$

(2) **Wedge.** If $K$ is the set of disjoint vertices: $K = \{\emptyset, \{1\}, \ldots, \{m\}\}$ then

$$X^K \cong X_1 \vee \cdots \vee X_m;$$

(3) **Fat wedge.** If $K = \partial \Delta[m]$ is the boundary of a $m-1$-dimensional simplex then $X^K$ is the fat wedge of $X$:

$$X^K \cong T_{m-1}(X_1, \ldots, X_m);$$

(4) **Generalized fat wedge.** If $K = \text{skel}_i \Delta[m]$ is the $i$-th skeleton of the full simplex, then $X^K$ is a certain generalization of the fat wedge (see e.g. [16]):

$$X^K \cong T_{i+1}(X_1, \ldots, X_m);$$

(5) **Davis-Janiszkiewicz spaces.** If $K$ is arbitrary, and all $X_i \cong CP^\infty$, then $X^K$ is known as Davis-Januszkiewicz space $DJ_K$ and it plays an important role in toric topology, being homotopy equivalent to the Borel construction of a (quasi)-toric manifold $M^{2n}$:

$$((CP^\infty)^K \cong DJ_K \cong M^{2n} \times_{T^n} ET^n).$$

The definition can be reformulated in categorical language passing to the following diagram. Let $\kappa$ be the small category whose objects are all the simplices of $K$ (including the empty simplex), and the morphisms are the inclusions of simplices.

**Definition 2.3.** The exponential diagram $S_K(X)$ in $\text{Top}_*$ is the functor $\kappa \to \text{Top}_*$ which assigns $X^\sigma$ to each simplex $\sigma$, and the natural inclusion $X^\sigma \hookrightarrow X^\tau$ to each morphism $\sigma \subset \tau$.

Now it is easy to see that

$$X^K = \text{colim}^{\text{Top}_*} S_K(X).$$

The analogous diagrams can be considered in different categories which are augmented and have product and coproduct, for example $\text{DGC}$, $\text{HOPF}$. Throughout the paper we use the following notation for categories.

**Notation 2.4.**  

- $\text{Top}_*$: based topological spaces;  
- coproduct is the wedge, product is the cartesian product;
\begin{itemize}
  \item DGC: coaugmented differential graded coalgebras;
  \hspace{1cm} GC: connected graded coalgebras,
  \hspace{1cm} coproduct is the connected direct sum, product is the tensor product;
  \item HOPF: graded connected cocommutative Hopf algebras;
  \hspace{1cm} coproduct is the free product, product is the tensor product;
  \item GA: graded connected algebras;
  \hspace{1cm} DGA: augmented differential graded algebras;
  \hspace{1cm} coproduct is the free product, and we will not need the product as it is explained below.
\end{itemize}

We will use the same definition of the diagram $S_K$ in these categories with one exception. Writing $S_K(H_*(\Omega X))$ we use the tensor product which is the product in the category of cocommutative Hopf algebras, but when we are interested only in algebra structure on the colimit of this diagram, we write $\text{colim}^{\text{DGA}} S_K(H_*(\Omega X))$. This means that we just forget the coalgebra structure, as coproduct in the category DGA is also the free product.

It is easy to check that
\[ C_*(X^K) \simeq \text{colim}^{\text{DGC}} S_K(C_*(X)), \]
where $C_* : \text{TOP}_* \to \text{DGC}$ is the chain coalgebra functor. The examining differentials in the corresponding Mayer-Vietoris sequence implies that over a field the following isomorphism holds
\[ H_*(X^K) \simeq \text{colim}^{\text{GC}} S_K(H_*(X)), \]
where the colimit is taken in the category GC.

The last isomorphism can be also derived from the stable homotopy splitting proved in [2]:
\[ \Sigma X^K \simeq \bigvee_{\sigma \in K} X^{\wedge \sigma}. \]

It turns out that colimit of the diagrams $S_K$ does not commute with the loop functor. We cannot see that in the simplest cases as the following examples show. Here $R$ denotes a field.

**Examples 2.5.**

1. For the cartesian product $(K = \Delta[m])$ we have the following isomorphism
   \[ H_*(\Omega(X_1 \times \cdots \times X_m); R) \cong H_*(\Omega X_1; R) \otimes \cdots \otimes H_*(\Omega X_m; R) \]
2. [3] For the usual wedge $(K = \{\emptyset, \{1\}, \ldots, \{m\}\})$ we have
   \[ H_*(\Omega(X_1 \vee \cdots \vee X_m); R) \cong H_*(\Omega X_1; R) \sqcup \cdots \sqcup H_*(\Omega X_m; R), \]
   where $\sqcup$ denotes the coproduct for connected graded algebras (free product).

\[ \]
The smallest complex for which the two functors do not commute is the boundary of the 2-simplex.

**Example 2.6.** Let $K = \partial \Delta[3]$ be the boundary of the 2-dimensional simplex, and $X_1 = X_2 = X_3 = S^n$, $n \geq 2$. For any commutative ring $R$ we have

$$H_*(\Omega(X^K); R) \cong R[u_1, u_2, u_3] \sqcup R[w] \cong R[u_1, u_2, u_3] = \colim_{K} S_K(H_*(\Omega X); R),$$

where $\deg u_i = n - 1$, $\deg w = 3n - 2$. Here $R[v_1, \ldots, v_k]$ denotes the polynomial algebra generated by $v_1, \ldots, v_k$.

This paper is therefore devoted to finding the correct functor computing $H_*(\Omega(X^K))$ via $H_*(\Omega X)$ in the category of graded algebras.

2.2. Throughout the paper we will use the following notation.

**Notation 2.7.**

- $[n]$ is the set $\{1, 2, \ldots, n\}$;
- $\mathbb{N}$ is the semigroup of non-negative integers;
- $e_j = \{0\}^{j-1} \times \{1\} \times \{0\}^{m-j} \in \mathbb{N}^m$ for $j \in [m]$;
- for $I = \{i_1, \ldots, i_m\} \in \mathbb{N}^m$, $|I| = i_1 + \cdots + i_m$;
- for $I = \{i_1, \ldots, i_m\} \in \mathbb{N}^m$, $\text{supp} I = \{j \in [m] | i_j \neq 0\}$;
- for topological spaces $X = (X_1, \ldots, X_m)$ and $I = (i_1, \ldots, i_m) \in \mathbb{N}^m$
  $$X^I = X_1^{i_1} \times \cdots \times X_m^{i_m};$$
- for (graded) vector spaces $V = (V_1, \ldots, V_m)$ and $I = (i_1, \ldots, i_m) \in \mathbb{N}^m$
  $$V^I = V_1^{\otimes i_1} \otimes \cdots \otimes V_m^{\otimes i_m};$$
- for two tuples $I, J \in \mathbb{N}^m$ we write $I \prec J$ if $i_k \leq j_k$ for every $k \in [m]$ and $I \neq J$;
- for a commutative ring $R$ the polynomial algebra over $R$ generated by $v_1, \ldots, v_m$ is denoted by $R[v_1, \ldots, v_m]$;
- for a commutative ring $R$ the exterior algebra over $R$ generated by $v_1, \ldots, v_m$ is denoted by $\wedge[v_1, \ldots, v_m]$.

2.3. **Combinatorial definitions and notation.** First define certain special simplicial complexes.

**Notation 2.8.**

- $\Delta[S]$ is the full simplex on the vertex set $S$;
- $\Delta[m]$ denotes $\Delta[S]$ for $S = [m]$. 
• $V[S]$ is the simplicial complex on $S$ consisting of a disjoint union of the vertices; $V[m]$ denotes $V[S]$ for $S = [m]$;
• $\text{skel}_i \Delta[m]$ denotes the simplicial complex which consists of all subsets $\sigma \subset [m]$ of cardinality no more than $i + 1$; geometrically it means that it is the collection of all simplices from $\Delta[m]$ with the dimension $\leq i$.

Let further $K$ be an abstract simplicial complex on $[m]$.

**Notation 2.9.**
• $\bar{K}$ denotes the set of all non-empty simplices of a simplicial complex $K$.
• for a simplex $\sigma$ of a simplicial complex $K$, $|\sigma|$ denotes the number of elements in $\sigma$, and $\dim \sigma = |\sigma| - 1$ its dimension.
• for any subset $V \subset [m]$, $K|_V$ denotes the full subcomplex $\{\sigma \in K | \sigma \subset V\}$;

We associate with each simplicial complex $K$ two algebras in the following way.

**Definition 2.10.** The Stanley-Reisner algebra of a simplicial complex $K$ over a commutative ring $R$ is defined as
\[ R(K) = R[v_1, \ldots, v_m]/I_{SR}(K), \]
where $I_{SR}(K)$ is the Stanley-Reisner ideal defined by
\[ v_I := v_{i_1} \cdots v_{i_s} = 0 \text{ if } I = \{i_1, \ldots, i_s\} \notin K. \]

The exterior Stanley-Reisner algebra $\wedge(K)$ is defined similarly by
\[ \wedge(K) = \wedge[v_1, \ldots, v_m]/I_{SR}(K). \]

Note that in the above definition to construct the ideal $I_{SR}$ it is enough to take only minimal non-simplices of $K$.

**Definition 2.11.** A subset $\tau \subset K$ is called a **missing face** if $\tau \notin K$ but any proper subset $\sigma \subsetneq \tau$ is in $K$. Missing faces are therefore the minimal non-simplices in $K$. Geometrically it means that $\tau \notin K$ but $\partial \tau \in K$.

Now we are ready to define a certain subclass of simplicial complexes, which turned out to be special for the problem considered in the present article.

**Definition 2.12.** $K$ is called **flag** if every its missing face has 2 elements or, equivalently, dimension 1.
Obviously, a flag complex $K$ is completely determined by its 1-skeleton $\text{skel}_1 K$.

**Examples 2.13.** (1) The boundary of any polygon with no less than 4 vertices is flag;
(2) the $n$-dimensional octahedron for any $n$ is flag;
(3) the boundary of the $n$-simplex is non-flag for any $n$;
(4) more generally, the $i$-dimensional skeleton of the $n$-simplex for $n > i \geq 1$ is non-flag.

In certain statements and proofs we use the following construction which assigns to any simplicial complex with $s$ vertices the $s$-functor for simplicial complexes.

**Construction 2.14.** Suppose we have a simplicial complex $K$ on $[s]$, and $s$ simplicial complexes $L = \{L_1, \ldots, L_s\}$ on $V_1, \ldots, V_s$ respectively. This allows to define a functor which we denote by $K(L_1, \ldots, L_s)$, or $K(L)$, which is a simplicial complex on $V(L) = V_1 \sqcup \cdots \sqcup V_s$ constructed by the following rule. Any $I \subset V(L)$ is naturally defined by the data: $\sigma(I) \subset [s]$ and the non-empty sets $S_i(I) \subset V_i$ for each $i \in \sigma(I)$, such that $I = \sqcup_{i \in \sigma(I)} S_i(I)$. Then the defining condition is as follows: $I \in K(L_1, \ldots, L_s)$ if and only if $\sigma(I) \in K$ and each $S_i(I) \in L_i$.

### 3. Configuration space $C_K$ and diagonal arrangements

#### 3.1. Construction of $C_K$ as a configuration space.

In this section we construct the configuration space $C_K$. Informally speaking, this is the unordered configuration space of particles on the real line, each labelled with a nonempty simplex from $K$, and the topology is defined by the following rule: if two particles with labels $\sigma$ and $\tau$ are getting close, they can collide only in case when their labels $\sigma$ and $\tau$ are disjoint and $\sigma \sqcup \tau \in K$, and so the resulting particle gets the label $\sigma \sqcup \tau$.

Also we can regard this space as the configuration space of particles in $\mathbb{R}^1$ with labels from $[m]$, and the condition specifying which subsets can have the same coordinate is determined by the simplicial complex $K$. We give here the strict definition using this second approach.

Let $B(k)$ denotes the classical unordered configuration space of $k$ particles on the real line:

$$B(k) = \{ t = \{t(1), \ldots, t(k)\} \subset \mathbb{R} \mid t(i) < t(j) \text{ for } i < j \}.$$  

Construct the space

$$C_K = \sqcup_{I \in \mathbb{N}^m} C_K(I),$$

where for each $I = \{i_1, \ldots, i_m\}$ the component $C_K(I) \subset B(i_1) \times \cdots \times B(i_m)$ is defined by the following condition:

$$(t_1, \ldots, t_m) \in C_K(I) \Leftrightarrow \cap_{j \in \tau} t_j = \emptyset \text{ for each } \tau \notin K.$$

We call a tuple $I$ the *multi-degree* of a configuration if it is from $C_K(I)$. Note that each $C_K(I)$ is not necessarily connected, $\pi_0(C_K)$ will be calculated in 3.3.
3.2. **$C_K$ as complements of diagonal arrangements.** Given a simplicial complex $K$ on $[m]$ we associate with it the arrangement of diagonal subspaces in $\mathbb{R}^m$ as follows. It consists of the subspaces

$$x_{j_1} = x_{j_2} = \cdots = x_{j_s},$$

one for each missing face $j = \{j_1, \ldots, j_s\}$ of $K$. The complement of this arrangement we denote by $D_K$. The direct comparison of the definitions shows that

$$D_K \cong C_K(1, \ldots, 1).$$

The components of other multi-degrees also admit similar descriptions, we summarize this in the following lemma.

**Proposition 3.1.** The components $C_K(I)$ have the following description in terms of diagonal arrangements:

1. The component $C_K(1, \ldots, 1)$ is homeomorphic to the complement of the diagonal arrangement $D_K$ associated with $K$.
2. The component $C_K(I)$ for $I \prec (1, \ldots, 1)$ is homeomorphic to the complement of the diagonal arrangement $D_L$ associated with the full subcomplex $L = K|_{\text{supp}(I)}$, where $\text{supp}(i_1, \ldots, i_m) = \{j | i_j \neq 0\}$.
3. The component $C_K(I)$ for $I \preceq (1, \ldots, 1)$ is homeomorphic to a certain collection of connected components of $D_{K'}$ associated with the complex $K' = K(V[i_1], \ldots, V[i_m])$. More precisely, this collection is defined by the diagonal inequalities:

$$t_j(1) < t_j(2) < \cdots < t_j(i_j), \; j \in [m]$$

**Examples 3.2.**

1. The diagonal hyperplane arrangement which consists of all hyperplanes $x_i = x_j$ corresponds to the simplicial complex $K = V[m]$ — the disjoint union of vertices from $[m]$.
2. The so called ”$k$-equal arrangement”, consisting of all subspaces of the form:

$$x_{i_1} = \cdots = x_{i_k},$$

corresponds to $(k - 2)$-skeleton of the simplex on $[m]$. Thus its complement, called ”no $k$-equal manifold” is homeomorphic to $C_{\text{skel}_{k-2}\Delta[m]}(1, \ldots, 1)$.
3. The complement of any coordinate subspace arrangement in $\mathbb{R}^m$ is homotopy equivalent to $D_K$ for some $K$: we add one additional coordinate $x_{m+1}$ and replace the equation $x_{i_1} = \cdots = x_{i_k} = 0$ by $x_{i_1} = \cdots = x_{i_k} = x_{m+1}$. Hence, it is homotopy equivalent to the $(1, \ldots, 1)$-component of the corresponding configuration space $C_K$. 
More precisely, if we denote by $Z_K$ the complements of the coordinate subspace arrangement with the subspaces of the form
\[ x_{j_1} = \cdots = x_{j_s} = 0, \]
one for each missing face \( \{j_1, \ldots, j_s\} \), then
\[ Z_K \simeq C_{\Sigma K}(1, \ldots, 1), \]
where $\Sigma K$ denotes the simplicial complex on $[m + 1]$ determined by the following condition: $\tau'$ is a missing face for $\Sigma K$ if and only if $\tau' = \tau \sqcup \{m + 1\}$ for some missing face $\tau$ of $K$.

3.3. Monoid $\tilde{C}_K$. Here we construct a cell complex that is homotopy equivalent to $C_K$ but has the structure of a monoid. Moreover, we will see in Section 4 that it has a nice cellular structure.

**Definition 3.3.** Let $\tilde{C}_K$ be the space of tuples $(c_0, t_1, c_1, \ldots, t_k, c_k)$ for all integer $k \geq -1$ (for $k = -1$ we take an empty tuple), where $c_i \in K$, $t_i \in [0, 1]$, which satisfy the following property:

if $t_j < 1, t_{j+1} < 1, \ldots, t_{j+s} < 1$ for some $j, s$ then $c_j, c_{j+1}, \ldots, c_{j+s}$ are pairwise disjoint and
\[ c_j \sqcup c_{j+1} \sqcup \cdots \sqcup c_{j+s} \in K, \]
and with the following identification:

\[(c_0, \ldots, c_j, 0, c_{j+1}, \ldots, c_k) \sim (c_0, \ldots, c_j \sqcup c_{j+1}, \ldots, c_k).\]

From a more general theory developed by the author in [10] (see Corollary 3.7) the following proposition follows.

**Proposition 3.4.** The space $\tilde{C}_K$ is homotopy equivalent to $C_K$. It has a natural monoid structure:
\[ (c_0, t_1, \ldots, t_k, c_k) \cdot (c'_0, t'_1 \ldots, t'_s, c'_s) = (c_0, t_0, \ldots, c_k, 1, c'_0, t'_0, \ldots, c'_s) \]
and its classifying space is
\[ B\tilde{C}_K \simeq (S^1)^K. \]
It is well-known that monoids $M$ such that $\pi_0(M)$ is a group, satisfy the property $M \simeq \Omega BM$. It obviously fails for $\tilde{C}_K$, and it is not homotopy equivalent to a loop space. However, we will see in Section 4 that its cellular chain complex is isomorphic to the cobar construction on $H_*W_K(S^1)^K$.

In the rest of the section we find the presentation of the monoid $\pi_0(\tilde{C}_K)$. For any $K$ consider the right-angled Artin monoid defined by the following presentation:

$$A^+_K = \langle y_1, \ldots, y_m \mid y_iy_j = y_jy_i \text{ for } \{i,j\} \in K \rangle.$$  

Obviously this monoid depends only on the 1-skeleton of $K$.

**Lemma 3.5.** The following isomorphism of monoids holds:

$$\pi_0(\tilde{C}_K) \cong \pi_0(C_K) \cong A^+_K.$$  

**Proof.** The required homomorphism $\pi_0(\tilde{C}_K) \to A^+_K$ is given by taking product of all labels from the left to the right, and easily can be checked to be an isomorphism. □

Thus, if there are any missing edges in $K$, the spaces $C_K(I)$ can be non-connected. Otherwise, when $\text{skel}_1(K)$ is full, we have that the isomorphism $\pi_0(C_K) \cong \mathbb{Z}_m^+$ is given by taking the multi-degree of a configuration.

### 4. Cellular chain algebra of $C_K$ and doubling operations

**4.1. Cellular chain algebra.** In this paragraph we show that the Adams-Hilton model for $(S^1)^K$, which we denote by $T_K$, is isomorphic to the cellular chain algebra $C_*(\tilde{C}_K)$ for a certain cell decomposition of $\tilde{C}_K$, and thus, $H_*(T_K)$ and $H_*(C_K)$ are isomorphic as algebras. As $T_K$ coincides with the cobar construction for the coalgebra $H_*((S^1)^K)$, we get the description of $H_*(C_K)$ as Ext of the exterior Stanley-Reisner algebra (Theorem 1.3).

**Definition 4.1.** Let $T_K$ be a free tensor algebra generated by all the simplices $\sigma$ in $K$ (the empty simplex corresponds to the unity of the algebra), degree of a generator $\sigma$ is its dimension as a simplex in $K$: $\deg \sigma = \dim \sigma = |\sigma| - 1$. We denote the operation of tensor product by $\otimes$. The differential is defined by the formula:

$$(4.1) \quad d\sigma = \sum_{\sigma = \sigma_1 \sqcup \sigma_2, \sigma \neq \emptyset} (-1)^{\epsilon(\sigma_1, \sigma_2)+|\sigma_1|}(\sigma_1|\sigma_2),$$

where $\epsilon(\sigma_1, \sigma_2)$ is the number of pairs $(i, j)$ with $i \in \sigma_1, j \in \sigma_2$ such that $i > j$.

**Remark 4.2.** The dga-algebra $T_K$ is isomorphic to the Adams-Hilton model constructed from the standard cell decomposition of $(S^1)^K$.  

Now we show that this is the right model for $C_K$.

**Proposition 4.3.** The algebra $C_*(\tilde{C}_K)$ is isomorphic to $T_K$.

*Proof.* We will show that $\tilde{C}_K$ has a cellular decomposition such that the corresponding differential chain algebra is isomorphic to $T_K$.

The definition of $\tilde{C}_K$ gives a natural cubical decomposition in the following way. Consider a sequence $(\tilde{\sigma}_1, \ldots, \tilde{\sigma}_s)$ where each $\tilde{\sigma}_i$ is some simplex $\sigma_i$ together with the order of its vertices. We associate with it the cube in $\tilde{C}_K$ which consists of all tuples $(c_0, t_1, c_1, \ldots, t_k, c_k)$ defined by the following condition: for certain $0 \leq j_1 < \cdots < j_{s-1} < k$ we have

\[
c_j \bigoplus \cdots \bigoplus c_{j_i} = \sigma_i \quad ; \quad t_{j_i+1} = 1
\]

The union of the cubes which are associated with the same underlying tuple of simplices $(\sigma_1, \ldots, \sigma_s)$ for all possible orders of their vertices forms a cell which corresponds to the element $(\sigma_1 | \ldots | \sigma_s)$ in $T_K$. Combinatorially it is the product of $s$ permutohedra of dimensions $|\sigma_1| - 1, \ldots, |\sigma_s| - 1$.

Now it is sufficient to check that the boundary operator on this set of cells is given by the formula which was used to define the differential in $T_K$. This can be shown by the direct calculation the boundary operator on each cube, and then taking their sum. \hfill \Box

Recall that The exterior Stanley-Reisner algebra $\wedge(K)$ is defined as

\[
\wedge(K) = \wedge[v_1, \ldots, v_m]/I_{SR}(K),
\]

where $I_{SR}(K)$ is the Stanley-Reisner ideal (see Definition 2.10).

The exterior Stanley-Reisner coalgebra $\wedge^\vee(K)$ is defined as the dual of $\wedge(K)$. Its basis consists of the elements $v_\sigma$ for each $\sigma \in \tilde{K}$, and the co-multiplication is given by the formula

\[
\Delta(v_\sigma) = \sum_{(\sigma_1, \sigma_2); \sigma=\sigma_1 \sqcup \sigma_2} (-1)^{\epsilon(\sigma_1, \sigma_2)} v_{\sigma_1} \otimes v_{\sigma_2},
\]

where $\epsilon(\sigma_1, \sigma_2)$ was defined in Definition 4.1. This coalgebra has the multi-grading in $\mathbb{N}^m$.

**Remark 4.4.** Obviously, the coalgebra $H_*(\mathbb{S}^1)^K; \mathbb{Z}$ is isomorphic to the exterior Stanley-Reisner colagebra $\wedge^\vee(K)$ over $\mathbb{Z}$. 
The following statement is the direct corollary from the definitions of the algebra $T_K$ and the cobar construction.

**Lemma 4.5.** DGA-algebra $T_K$ is isomorphic to the cobar construction on the coalgebra $\wedge(K)$.
The isomorphism preserves multi-grading.

As the corollary of the results of this section we get the isomorphism

$$H_k(C_K(I), \mathbb{Z}) \cong Cotor_{|I|-k,I}^{\wedge}(\mathbb{Z}, \mathbb{Z}).$$

As this isomorphism respects the algebra structure, this gives the proof of Theorem 1.3.

### 4.2. Doubling operations $\mu$

In this section we define doubling operations which are used in the statement of Theorem 1.1. We give geometric construction of them for the configuration space $C_K$ and the action on the chain algebra $T_K$.

Roughly speaking these operations are replacing one of the particles with two particles on a very small distance $\varepsilon > 0$ and with the same label (colour). It is more convenient to give strict definitions replacing the space $C_K$ by the homotopy equivalent version of labelled small intervals, and to define the action of little interval operad, but here we can proceed as follows. Let the space $C_K^\varepsilon$ be the subspace of $C_K$ with the following condition: the distance of any two particles with the same colour is not less than $\varepsilon$. Fix one of the particles of a configuration of multi-degree $I$. This is equivalent of choosing the pair of integers: $(j, k)$ with $j \in [m]$, $k \in [i_j]$, which means that we fix the $k$-th particle among all the particles with color $j$ taken from left to right.

Now in each configuration of multi-degree $I$ add one more particle of the colour $j$ on the distance $\varepsilon/2$ to the right from the fixed one. This defined the map:

$$C_K^\varepsilon(I) \rightarrow C_K^{\varepsilon/2}(I + \varepsilon_j).$$

As for any $\varepsilon > 0$ the space $C_K^\varepsilon(I)$ is homotopy equivalent to $C_K(I)$, this defines the following map up to homotopy:

$$\phi_{j,k}: C_K(I) \rightarrow C_K(I + \varepsilon_j).$$

Let construct this operations on the level of chains in $T_K$. Fix the multi-degree $I$ and the pair $(j, k)$ as above. Take a basis element $(\sigma(1) | \ldots | \sigma(s)) \in T_K(I)$. Find the minimal $n$ such that exactly $k$ simplices among $\sigma(1), \ldots, \sigma(n)$ contains $\{j\}$.

Set

$$\mu^T_{j,k}(\sigma(1) | \ldots | \sigma(s)) = \sum_{\tau_1 \cup \tau_2 = \sigma(n) - \{j\}, \tau_1 \cup \tau_2 = \sigma(n) - \{j\}} (-1)^{e(\tau_1, \tau_2)}(\sigma(1) | \ldots \{j\} \cup \tau_1 \{j\} \cup \tau_2 \ldots | \sigma(s)).$$

By linearity this map defines the homomorphism of vector spaces

$$\mu^T_{j,k}: T_K(I) \rightarrow T_K(I + \varepsilon_j),$$

and it’s easy to see that it commutes with the differential.
The following statement will be not used in proofs of the results in the present paper, but it gives some geometric feeling of \( \mu \)-operations. We omit its proof here.

**Proposition 4.6.** The map

\[
\phi_{j,k} : C_K(I) \to C_K(I + e_j)
\]

and the homomorphism

\[
\mu_{j,k}^T : T_K(I) \to T_K(I + e_j)
\]

constructed above induce up to sign the same homomorphism in homology:

\[
H_*(C_K(I)) \cong H_*(T_K(I)) \to H_*(C_K(I + e_j)) \cong H_*(T_K(I + e_j))
\]

We denote this homomorphism in homology by \( \mu_{j,k} \).

5. Applications

5.1. Flag complexes. Recall that flag complexes were defined in Definition 2.12.

Nice properties of loops of polyhedral products for flag complexes were noticed by several authors. We give below the examples of that. Here \( T(U) \) denotes the free tensor algebra generated by elements of \( U \).

**Examples 5.1.** If \( K \) is flag, then

1. \( H_*(\Omega(CP_\infty)^K, \mathbb{Q}) \cong T(U)/(u_i^2 = 0, u_iu_j + u_ju_i = 0 \text{ for } \{i, j\} \in K), \) where \( U = \{u_1, \ldots, u_m\} \) with \( \deg u_i = 1 \).
2. \( H_*(\Omega(S^{2k+1})^K, \mathbb{Q}) \cong T(U)/(u_iu_j - u_ju_i = 0 \text{ for } \{i, j\} \in K), \) where \( U = \{u_1, \ldots, u_m\} \) with \( \deg u_i = 2k \).

Here we show that Theorem 1.1 implies that these results hold in more general situation.

**Corollary 5.2.** Over any field

\[
H_*(\Omega X^K) \cong \bigoplus_{i=1}^n H_*(\Omega X_i)/\sim,
\]

with

\[
x \cdot y \sim (-1)^{\deg x \deg y} y \cdot x
\]

for \( x \in H_*(\Omega X_i), y \in H_*(\Omega X_j) \) when \( \{i, j\} \in K \).

In other words, for flag \( K \)

\[
H_*(\Omega X^K) \cong \operatorname{colim}^\text{DGA} S_K(H_*(\Omega X)).
\]

The proof is direct from the following lemma.
Lemma 5.3. For any flag simplicial complex $K$ each connected component of $C_K$ is contractible.

Proof. The statement can be obtained by induction on number of missing edges. We use the description of $C_K$ in terms of diagonal arrangements stated in 3.4.

In the beginning we have the contractible open cone in $\mathbb{R}^{|I|}$, and on each step we will get the union of the interior of disjoint cones which are bounded by hyperplanes passing through 0. Adding one more missing edge, we sect the complement by the corresponding hyperplane. Any cone in the complement either doesn’t intersect this hyperplane, or it is divided into two new cones, and both of them are again contractible. □

5.2. Davis-Januszkiewicz spaces and diagonal arrangements. The important examples of $K$-product are provided by so called Davis-Januszkiewicz spaces which is $K$-power of $\mathbb{C}P^\infty$:

$$DJ_K := (\mathbb{C}P^\infty)^K.$$ 

The spaces $DJ_K$ arises in toric topology as Borel construction for (quasi)-toric manifolds.

In the book [7] Buchstaber and Panov observed that there is some coincidence in cohomology calculations for loops on Davis-Januszkiewicz spaces and for complements of real diagonal arrangements.

Recall that $R(K)$ denotes the Stanley-Reisner algebra of the simplicial complex $K$ over a ring $R$:

$$R(K) = R[v_1, \ldots, v_m]/I_{SR},$$

where $I_{SR}$ is the Stanley-Reisner ideal (see Definition 2.10).

If we set the degree of each generator $v_i$ equal to 2, then the following isomorphism holds for the loops homology of Davis-Januszkiewicz spaces with field coefficients $R$ (see [7]):

$$H^*\Omega DJ_K \cong \text{Tor}^*_R(R, R).$$

From the other side, if the degree of each $v_i$ is set to 1, then we have the following isomorphism for cohomology with field coefficients of the complement of the diagonal arrangement associated with $K$ (see [18]):

$$H^*(D_K) \cong \text{Tor}^{m-\ast}_{R(K)}(R, R)_{(1, \ldots, 1)}.$$ (5.1)

Now using the Theorem 1.1 we can explain this accident.

By Theorem 1.3 we have the multigraded algebra isomorphism $H_{\ast,\ast}(C_K; \mathbb{Z}) \cong \text{Cotor}_{\vert \ast \vert - \ast, \ast}(\mathbb{Z}, \mathbb{Z})$, where the degrees of the generators in $\wedge(K)$ are equal to 1.

As $\Omega\mathbb{C}P^\infty \cong S^1$ has torsion-free homology, applying Theorem 1.1 to $DJ_K = (\mathbb{C}P^\infty)^K$ we get

$$H_*(\Omega DJ_K) \cong \text{Cotor}_{\ast}^{\wedge(K)}(\mathbb{Z}, \mathbb{Z}) \otimes_{\{\mu_{j,k}\}} \wedge[u_1, \ldots, u_m]$$
where $\deg u_i = 1$, $\otimes$ is multidegree-wise tensor product over the action of all the operations $\mu_{j,k}$, and Cotor is regraded as described above.

Now if we use the integral formality of $DJ_K$ proved in [13], and change the multi-grading of $\mathbb{Z}(K)$ setting $\deg v_i = 1$, we get the following statement.

**Corollary 5.4.** There exists the natural epimorphism of multi-graded vector spaces

$$f : A = \text{Ext}_{\wedge(K)}^*(\mathbb{Z}, \mathbb{Z}) \to B = \text{Ext}_{\mathbb{Z}(K)}^*(\mathbb{Z}, \mathbb{Z})$$

such that

1. $f(y \cdot y') = (-1)^\epsilon f(y) \cdot f(y')$, where $y \in A_{n,I}$, $y' \in A_{n',I'}$ and $\epsilon = \sum_{j \leq k} i_j i'_k - n' \sum_j i_j$.

2. $\ker f|_{A_{n,I}} = \bigoplus_{j,k} \mu_{j,k}(A_{n,I-e_j})$.

**Remark 5.5.** Let $\mathbb{Z}_\tau(K) = \mathbb{Z}(K)/\langle v_1^2, \ldots, v_m^2 \rangle$ be the truncated Stanley-Reisner algebra. Then the above epimorphism $f$ is the composition of two homomorphisms

$$\text{Ext}_{\wedge(K)}^*(\mathbb{Z}, \mathbb{Z}) \to \text{Ext}_{\mathbb{Z}(K)}^*(\mathbb{Z}, \mathbb{Z}) \to \text{Ext}_{\mathbb{Z}(K)}^*(\mathbb{Z}, \mathbb{Z})$$

where the first one is the additive isomorphism satisfying (1), and the second one is an algebra epimorphism with the kernel (2). These maps are constructed by applying Theorem 1.1 to the space $(S^{2n})^K$ with $n \geq 1$.

**Examples 5.6.** The simplest examples of finding the kernel of the above epimorphism are as follows.

1. For any generator $x_j$ of multi-degree $e_j$ in $\text{Ext}_{\wedge(K)}^*(\mathbb{Z}, \mathbb{Z})$ we have $\mu_{1,1}(x_j) = x_j^2$. So this element should be in the kernel of $f$, and we get the additional relation in $\text{Ext}_{\mathbb{Z}(K)}^*(\mathbb{Z}, \mathbb{Z})$:

$$x_j^2 = 0.$$ 

2. For any missing face $\tau$ in $K$ we have a generator $\omega_\tau$ of multi-degree $\sum_{j \in \tau} e_j$ in $\text{Ext}_{\wedge(K)}^*(\mathbb{Z}, \mathbb{Z})$. As for $j \in \tau$ we have $\mu_{j,1}(\omega_\tau) = [x_j, \omega_\tau]$, we get that these commutators are also in the kernel of $f$, so the new relations are

$$[x_j, \omega_j] = 0,$$
for \( j \in J \).

**Remark 5.7.** One of the examples of explicit calculations of those two Ext-algebras and the epimorphism of the above Corollary will be given in proofs of Lemma 5.9 and Proposition 5.10 (with topological grading of the generators).

Thus, we see that \( \text{Cotor}^\vee(K)(\mathbb{Z}, \mathbb{Z}) \) is obtained from \( \text{Cotor}^\wedge(K)(\mathbb{Z}, \mathbb{Z}) \) by rescaling and then adding some additional relations. We do not have any relations in multi-degree \((1, \ldots, 1)\) as for the image of any \( \mu_{j,k} \)-operation we have \( i_j \geq 2 \). Thus, we have only the rescaling of homology groups \( H_*(D_J K; \mathbb{Z})_{(2, \ldots, 2)} \) and \( H_*(C_K; \mathbb{Z})_{(1, \ldots, 1)} \cong H_*(D_K; \mathbb{Z}) \), and this explain the connection we were interested to find.

5.3. **Torsion in** \( H_*(\Omega(X^K)) \). For general \( K \) the homology of the space \( C_K \) is not torsion-free. This means that even in case when all \( H_*(\Omega X_i; \mathbb{Z}) \) are torsion free, so that Theorem 1.1 is valid for integer coefficients, the resulting homology \( H_*(\Omega(X^K); \mathbb{Z}) \) can have arbitrary torsion.

To show that we will use Example 3.2(3). Consider the complement \( Z_L \) of the coordinate subspace arrangement associated with a simplicial complex \( L \) on \([m]\). It was shown in [4] that the integral cohomology of its complement can have arbitrary torsion.

Due to Example 3.2 the complement of a coordinate subspace arrangement is homotopy equivalent to \( D_K \) for the certain \( K \). Thus, for that \( K \), \( H_*(C_K(1, \ldots, 1)) \) also can have any torsion.

Nevertheless, for certain classes of simplicial complexes the space \( C_K \) will be torsion free. The simplicial complex is called *shifted* if there is an ordering on its vertices, say \( 1 < \cdots < m \), such that whenever \( \sigma \in K \), \( j \in \sigma \) and \( i < j \), we have \((\sigma - \{j\}) \cup \{i\} \in K \). The skeletons \( K = \text{skel}_i \Delta[m] \) are examples of shifted complexes.

**Proposition 5.8.** For shifted simplicial complexes \( K \) the homology \( H_*(C_K; \mathbb{Z}) \) is torsion-free. Thus, if all \( H_*(\Omega X_i; \mathbb{Z}) \) are torsion-free, then \( H_*(\Omega(X^K)) \) also has no torsion.

**Proof.** The proposition can be proved directly, but we use here another way relying on the homotopy result from [9].

Consider the space \( Y^n_K = (S^{2n+1})^K \) for \( n \geq 1 \). Denote by \( F^n_K \) the homotopy fiber of the embedding \( Y^n_K \hookrightarrow (S^{2n+1})^m \).

Due to Theorem 9.4 from [9] in case when \( K \) is shifted, this space is homotopy equivalent to the following wedge of the form \( F^n_K \cong \vee_{j \in J_i} \Sigma^{k_i}(\wedge_{j \in J_i} \Omega X_j) \) for certain subsets \( J_i \subset [m] \). We have the splitting of the loop spaces \( \Omega Y^n_K \cong (\Omega S^{2n+1})^m \times \Omega F^n_K \), which implies the isomorphism of the vector graded space

\[
H_*(\Omega Y^n_K; \mathbb{Z}) \cong \mathbb{Z}[u_1, \ldots, u_m] \otimes T(V),
\]
where $T$ denotes the free tensor algebra over $\mathbb{Z}$, and $V$ is some set of generators. Thus, $H_*(\Omega Y^n, \mathbb{Z})$ has no torsion. Due to Theorem 1.1 over integers, all algebras $H_*(\Omega Y^n, \mathbb{Z})$ are certain rescalings of the algebra $H_*(C_K, \mathbb{Z})$. So we get that $H_*(C_K, \mathbb{Z})$ has no torsion as well.

5.4. Some calculations. In this subsection we illustrate how to apply Theorem 1.1 for finding the presentations of $H_*(\Omega(X)_K)$. Suppose we know the presentation of $H_*(C_K)$. To derive the presentation for $H_*(\Omega(X)_K)$ we need to investigate the action of $\mu$-operations on $H_*(C_K)$. The easy but helpful observation is that it’s enough to find this action only on generators.

We give the example of such arguments getting results about rational homotopy groups and the homology of $\Omega DJ_K$, when $K$ is the skeleton of a simplex. These calculations are generalized in [11] for more general class of simplicial complexes, but for this particular case we rely on the presentation of $H_*(T_K) \cong H_*(C_K)$ derived from [12]:

Lemma 5.9. Let $K = \text{skel}_{s-2} \Delta[m]$, $s \geq 3$. The algebra $H_*(T_K, \mathbb{Z})$ has the following presentation:

- $m$ generators $x_j$, $j \in [m]$ of degree $0$;
- $C_m^s$ generators $\omega_J$, taken for all $J \subset [m]$ with $|J| = s$, of degree $s-2$.
- relations $x_i x_j = x_j x_i$ for $i, j \in [m]$.
- relations

$$\sum_{j \in S} (-1)^{\epsilon(j, S)} [x_j, \omega_{S \setminus \{j\}}] = 0$$

for $S \subset [m]$ with $|S| = s + 1$.

Proof. Over a field the presentation was obtained in [12]. But $K = \text{skel}_{s-2} \Delta[m]$ is a shifted simplicial complex, so due to Proposition 5.8 $H_*(C_K, \mathbb{Z})$ has no torsion. Moreover, all the generators of the presentation are integral classes: $\omega_J$ corresponds in $T_K$ to the differential $dJ$ of the non-existing element $J \notin T_K$.

Now we need to find action of operations $\mu$ on the generators. As each element $w_J$ is nothing else as $dJ$ in the model $T_K$ constructed in Section 4, the direct verification shows that for $j \in J$ we have

$$\mu_{j,k}(\omega_J) = [x_j, \omega_J].$$

As the algebra $H_*(\Omega CP^\infty) \cong \wedge[u]$ (deg $u = 1$) has no torsion, we obtain the following presentation for integral homology algebra $H_*(\Omega DJ_K)$.
Proposition 5.10. Let $K$ be $(s - 2)$-dimensional skeleton of the full simplex $\Delta[m]$, $s \geq 3$. Then

$$H_*(\Omega DJ_K; \mathbb{Z}) \cong T(u_j, w_J)/R,$$

where $w_J$ are taken for all $J \subset [m]$ with $|J| = s$, and $u_j$ are taken for all $j \in [m]$. The degrees are $\deg w_J = 2s - 2$, $\deg u_i = 1$, and the set of relations $R$ is as follows:

1. $[u_i, u_j] = 0$ for $i, j \in [m]$.
2. $[u_j, w_J] = 0$ for any $j \in J$; 
3. for any $S \subset [m]$ with $|S| = s + 1$

$$\sum_{j \in S} [u_j, w_{S\setminus\{j\}}] = 0.$$

This immediately implies the presentation of the rational homotopy groups for $DJ_K$:

Corollary 5.11. For $K = \text{skel}_{s-2} \Delta[m]$ we have the following presentation for rational homotopy algebra of $DJ_K$:

$$\pi_*(DJ_K) \otimes \mathbb{Q} \cong \text{Lie}(\tilde{u}_j, \tilde{w}_J)/\tilde{R}$$

with

- each classes $\tilde{u}_i$ is the image of the generator $\pi_2(\mathbb{C}P^\infty)$ under the inclusion of $i$-th copy $\mathbb{C}P^\infty \hookrightarrow DJ_K$, and so $\deg \tilde{u}_i = 2$;
- for $J = \{j_1, \ldots, j_s\}$ the homotopy class $\tilde{w}_J = \{u_{j_1}, \ldots, u_{j_s}\}$, which corresponds to the rational higher order Whitehead product of the classes $u_{j_1}, \ldots, u_{j_s}$, and so $\deg \tilde{w}_J = 2s - 1$;
- the relations $\tilde{R}$ are the same as relations $R$ from Proposition 5.10 where $[\cdot, \cdot]$ denotes now the usual Whitehead product.

Proof. All generators obtained in Proposition 5.10 are the images of Hurewicz homomorphism. Moreover, for any space $Y$ the commutator of Pontryagin product on $H_*(\Omega Y)$ corresponds to usual Whitehead product on $\pi_*(Y)$. It is straightforward that the classes $\tilde{w}_J$ are the modified higher order Whitehead products, which are defined similarly to Massey products in cohomology. It is known that they coincide with the classical ones up to rational coefficients and the indeterminacy. $\square$
The interesting corollaries can be obtained by restricting the calculations to the homology of the complements of the corresponding diagonal arrangements. In this particular case the diagonal arrangements are called s-equal arrangements, and their complements — no-s-equal manifolds. We recover the following result about the homology of these manifolds proved in [3] (see also [5]).

**Corollary 5.12.** The integral homology of the no-s-equal manifold in $\mathbb{R}^m$, which is the complement of s-equal arrangement, is torsion free and has the following basis: each element is coded by the sequence of pairwise disjoint subsets of $[m]$: 

$$(I_1, J_1, I_2, J_2, \ldots, J_k, I_{k+1})$$

with the following properties

- $I_i, J_i \subset [m]$;
- $|J_i| = s$ for $i \in [k]$;
- $(\sqcup_i I_i) \sqcup (\sqcup_i J_i) = [m]$;
- $\max\{j \in J_i\} < \max\{j \in I_{i+1}\}$ for each $i \in [k]$.

This element has the degree $(s-2)k$, and is geometrically represented by the product of spheres given by the following system of equations in $\mathbb{R}^m = \{ (x_1, \ldots, x_m) \}$:

\[
\begin{align*}
  x_j &= 3mn + j \text{ for each } n \in [k+1] \text{ and } j \in I_n; \\
  \sum_{j \in J_i} |x_j - (3n + 2)m|^2 &= 1 \text{ and } \sum_{j \in J_i} x_j = (3n + 2)ms \text{ for } n \in [k].
\end{align*}
\]

**Proof.** First, by Proposition 5.8 $H_*(C_K)$ has no torsion. Then its presentation is obtained by combining Lemma 5.9, Proposition 3.1 and Proposition 4.3. The geometrical description follows from the representing the elements $w_J$ as the unit sphere in $\mathbb{R}^J \{0\}$ intersected with the hyperplane $\sum x_i = 0$. \qed

5.5. **Poincare series of $H_*(\Omega(X^K))$.** For example, from Corollary 1.2 it follows that for flag complexes the formulas expressing $P_{\Omega(X^K)}$ via $P_{\Omega X_i}$ have the same form.

**Corollary 5.13.** For flag $K$

\[
P_{\Omega X^K}^{-1}(t) = \sum_{\sigma \in K} \prod_{j \in \sigma} (P_{\Omega X_j}^{-1}(t) - 1)
\]

It turns out that general formula is in some sense deformation of these "flag formulas". The coefficients of those deformations depends only on $K$, and could be calculated by means of commutative algebra. We will discuss the resulting formulas in the forthcoming paper, giving here just one example of such calculations.
Example 5.14. If $K$ is the $(s - 2)$-skeleton of a simplex: $K = \text{skel}_{s-2} \Delta [m]$ the space $X^K$ is a generalized fat wedge $T_{s-1}(X_1, \ldots, X_m)$. For $m \geq 3$ and $s \geq 1$ $K$ fails to be flag, and the formula is

$$P^{-1}_{\Omega T_{s-1}(X_1, \ldots, X_m)}(t) = P^{-1}_{\Omega X_1}(t) \cdots \cdot P^{-1}_{\Omega X_m}(t) + (-t)^{s-1} \sum_{\sigma: |\sigma| \geq s-1} \prod_{i \in \sigma} (P^{-1}_{\Omega X_i} - 1).$$

6. LABELLED CONFIGURATION SPACES AND STABLE SPLITTINGS OF THE LOOP SPACES

The construction of labelled configuration spaces with collisions considered in this section is a particular case of the theory of configuration spaces with labels in partial monoid developed, for example, in [10]. We will use this construction only for partial monoids obtained as $K$-products.

**Definition 6.1.** Let $Y = (Y_1, \ldots, Y_m)$ be a sequence of well-pointed topological spaces. Denote by $C(\mathbb{R}^1, (Y^K))$ the space

$$C(\mathbb{R}^1, Y^K) := \sqcup C_K(I) \times Y^I/\sim,$$

where the equivalence relations are defined by the following condition: if some label $y \in Y_i$ is a basepoint $y = \star$, then the point can be removed from the configuration.

This configuration space $C(\mathbb{R}^1, Y^K)$ can be viewed as the space of unordered configurations in $\mathbb{R}^1$ with labels from $Y^K$ and collisions defined by the following rule: the points with labels $(\sigma, y)$ and $(\tau, z)$ can collide if and only if the simplices $\sigma$ and $\tau$ are disjoint in $K$, and $\sigma \cup \tau \in K$.

**Remark 6.2.** Obviously we have that $C_K \cong C(\mathbb{R}^1, (S^0)^K)$, where $S^0$ is a 0-dimensional sphere.

The similar definition can be given for the sequences of monoids.

**Definition 6.3.** Let $A = (A_1, \ldots, A_m)$ be a sequence of topological monoids. Denote by $\hat{C}(\mathbb{R}^1, A^K)$ the space which as a set of points coincides with the result of applying Definition 6.1 to the sequence of topological spaces $A$ with the base-points — identities of monoids, but with different topology: the points with labels $(\sigma, a)$ and $(\tau, b)$ can collide if and only if the $\sigma \cup \tau \in K$, and the result of their collision is a point with the label $(\sigma \cup \tau, a \cdot b)$, where $a \cdot b$ is coordinate-wise product.

The fundamental result in the theory of labelled configuration spaces is Segal’s theorem ([17]). It was generalized to the case of labelled configuration spaces with collisions, see e.g. Theorem 3.3 in [10]. Here will give the corollaries of the last theorem for our cases.
Corollary 6.4. For a sequence of connected topological spaces $Y = (Y_1, \ldots, Y_m)$ the following homotopy equivalence holds
\[ C(\mathbb{R}^1, Y^K) \simeq \Omega(\Sigma Y)^K, \]
where $\Sigma Y$ denoted the sequence $(\Sigma Y_1, \ldots, \Sigma Y_m)$.

Corollary 6.5. For a sequence of connected topological monoids $A = (A_1, \ldots, A_m)$ the following homotopy equivalence holds
\[ \hat{C}(\mathbb{R}^1, A^K) \simeq \Omega(\Omega X)^K, \]
where $\Omega X$ denoted the sequence of classifying spaces $(\Omega X_1, \ldots, \Omega X_m)$.

The last corollary can be rewritten as the following homotopy equivalence: for 1-connected $X = (X_1, \ldots, X_m)$ we have
\[ \Omega(X^K) \simeq \hat{C}(\mathbb{R}^1, (\Omega X)^K). \]

This fact is related to the isomorphism in Theorem 1.1, as $C(\mathbb{R}^1, (\Omega X)^K)$ is constructed from the space $C_K$ and the monoids $\Omega X$ (see Definition 6.3).

Now let restrict ourself to the second case, when each $X_i$ from $X$ is homotopy equivalent to the suspension $\Sigma Y_i$ for some sequence of $Y = (Y_1, \ldots, Y_m)$ of connected spaces, and so Corollary 6.4 is valid.

Now we can prove the result about stable splittings of such loop spaces.

Theorem 6.6. Let each $X_i \simeq \Sigma Y_i$ for some connected $Y_i$, $i \in [m]$. Then there is a stable homotopy equivalence
\[ \Omega(\Sigma Y)^K \simeq \bigvee_{I \in \mathbb{N}^m} C_K(I)^+ \wedge Y^I \]

Proof. We adapt arguments from the proof in [6] of the classical Snaith splitting.

Consider the configuration space $F = C(\mathbb{R}^1, (Y)^K)$ with collisions. By Corollary 6.4 we have that $D \simeq C(\mathbb{R}^1, (\Omega Y)^K)$.

The space $F$ is constructed as $F = \bigcup_{I \in \mathbb{N}^m} F_I$ for $F_I = C_K(I) \times Y^I/\sim$, where equivalence relations are defined in 6.1.

Construct the new space $\hat{F}$ by imposing on $\sqcup C_K(I) \times Y^I/\sim$ the stronger equivalence relations: if some label $y \in Y_i$ is a basepoint $y = \star$, then the configuration is equivalent to the empty one, or, in other words, the whole configuration disappears.

It has natural multi-grading in $\mathbb{N}^m$ and it splits as a wedge $\hat{F} = \bigvee_{I \in \mathbb{N}^m} \hat{F}_I$. 
There are natural projections $\pi : F_I \rightarrow \hat{F}_I$. The space $\hat{F}_I$ is in fact the filtration quotient for $F_I$: $\hat{F}_I \cong F_I/F_{<I}$, where $F_{<I} = \bigcup_{J : J \prec I} F_J$.

Our goal is to prove the stable homotopy equivalence: $\Sigma^\infty F \simeq \Sigma^\infty \hat{F}$. First, following [6] construct the map $F \rightarrow \Omega^\infty \Sigma^\infty \hat{F}$. As $\hat{F}$ is connected the space $\Omega^\infty \Sigma^\infty \hat{F}$ is homotopy equivalent to the labelled configuration space $C(\mathbb{R}^\infty, \hat{F})$, so in fact we will define a map $P : F \rightarrow C(\mathbb{R}^\infty, \hat{F})$.

We will write the points of $C(\mathbb{R}^\infty, \hat{F})$ in a form $\sum (p, \hat{f})$ for $p \in \mathbb{R}^\infty$ and $\hat{f} \in \hat{F}$.

Let $f \in F_I$, so $f$ can be written as $(c, y)$, where $c \in C_K(I)$ and $y \in Y^I$. Take some subconfiguration $\alpha \subset c$ of multi-degree $J$ (obviously $J \prec I$). As each $C_K(J)$ is a submanifold of $\mathbb{R}^{j_1 + \cdots + j_m}$, fix an embedding of $C_K = \sqcup_{J \in \mathbb{N}^m} C_K(J)$ to $\mathbb{R}^\infty$, and let $\bar{\alpha}$ be an image under this embedding. Then we define

$$P(c, b) = \sum_{\alpha \in c} (\bar{\alpha}, \pi(c, b)).$$

It could be easily checked that this map agrees with the equivalence relations in the construction of $F$.

The rest of the proof which shows that this map is homotopy equivalence repeats the same part in the proof of [6] for the following filtration of $F$ and $\hat{F}$:

$$F_k = \bigcup_{|I| \leq k} F_I \text{ and } \hat{F}_k = \bigvee_{|I| \leq k} \hat{F}_I.$$  

\[\square\]

7. Proof of Theorem 1.1

7.1. Scheme of the proof. Let $A = (A_1, \ldots, A_m)$ be a sequence of DGA-algebra’s. The object of our investigation now is the algebra

$$\widetilde{W}_K(A) := \Omega_* \operatorname{colim}_{\text{DGC}} S_K(B_\ast A),$$

where $B_\ast : \text{DGA} \rightarrow \text{DGC}$ denotes the bar construction, and $\Omega_* : \text{DGC} \rightarrow \text{DGA} —$ the cobar construction, and the diagram $S_K$ is defined in Section 2.1.

For the explicit construction for it we need the following construction of the DGA-algebra $P_K$. Take the $K$-product $\mathbb{N}^K \subset \mathbb{N}^m$ (the role of the base point is played by zero in $\mathbb{N}$),
and consider a tensor algebra on $\mathbb{N}^K$ (again the symbol for tensor product is $\otimes$). Define the differential by its value on a tuple $y \in \mathbb{N}^K$ using the formula:

$$d(y) = \sum_{y=y_1+y_2,y_i \neq 0} (-1)^{\epsilon(y_1,y_2)} (y_1|y_2),$$

where the decomposition into a sum is induced by the additive structure in $\mathbb{N}^m$. The degree is defined by $\deg y = |y| - 1$. The resulting DGA-algebra we will denote by $P_K$.

$P_K$ can be endowed with multi-grading in the following way. One grading is in $\mathbb{N}^m$, and it is calculated by taking the sum of all tuples in the product. The component corresponding to $I \in \mathbb{N}^m$ we denote by $P_K(I)$. The other is an $\mathbb{N}$-grading which corresponds to the number of multipliers in the tensor product. We will denote the components as $P_K = \oplus_{n \in \mathbb{N}} (P_K)_n$.

The total grading is calculated as $|I| - n$.

Notice that this construction is similar to the construction of $T_K$, and so the inclusion $\{0, 1\} \subset \mathbb{N}$ induces the natural monomorphism $T_K \hookrightarrow P_K$, so we will regard $T_K$ as the subalgebra of $P_K$.

**Lemma 7.1.** The DGA-algebra $\widetilde{W}_K(A)$ is isomorphic to the following DGA-algebra:

$$G_K = \oplus_{I \in \mathbb{N}^m} P_K(I) \otimes (B \ast A)^I,$$

with the differential $d$ is the sum of 3 differentials $d = d_P + d_{\text{bar}} + d_A$, where

- $d_P$ corresponds to the differential in $P_K$:
  $$d_P(p \otimes b) = d(p) \otimes b.$$

- $d_{\text{bar}}$ is the differential coming from external bar differentials on $B \ast (A_i)$;

- $d_A$ corresponds to the internal differentials in the DGA-algebras $A_i$’s:
  $$d_A(p \otimes b) = \sum_{j,k} (-1)^{\epsilon_{j,k}p} \otimes d_{j,k}(b),$$

  where $b$ is regarded as a set of $b_{jk} \in A_j$ for $j \in [m]$, $k \in [i_j]$, and $d_{j,k}$ is acting only on $b_{jk} \in A_j$ as the internal differential in $A_j$.

Defining this isomorphism we should be careful with signs as they depend on the gradings in $A$ as well. We skip in the text the careful examining of those signs just giving the resulting formulas.

$G_K$ is graded by usual rule for tensor products. We again split this grading as follows. $G_K$ inherits multi-grading in $\mathbb{N}^m$ from $P_K$. The second grading of the element $p \otimes a$ for $p \in (P_K)_n$ we define as $s = -n + \deg a$ and refer to it as $s$-grading. Respect to the last grading the differentials satisfies the following conditions:

- $d_{\text{bar}}$ preserves $s$-grading;
\begin{itemize}
  \item $d_P$ and $d_A$ decrease the $s$-grading by 1.
\end{itemize}

Sometimes we will write the elements of the canonical basis of $P_K$ in the form $(p(1)|\ldots|p(s))$, where each $p(k)$ is a monomial on $m$ variables $x_1,\ldots,x_m$. We will consider any element from $P_K$ as a linear combination of the basis elements, and refer to them as summands, and to components $p(k)$ of these summands as monomials.

In this terms, the subalgebra $T_K$ defined by the following condition: it is spanned by all such elements of the canonical basis such that all their monomials are square-free. We denote this subalgebra also by $P^0_K = T_K$.

Consider another condition on the powers of monomials: we take $(p(1)|\ldots|p(s))$ if all its monomials are square-free except probably one of them (say, $p(k)$), and this exceptional monomial $p(k)$ has degree 2 respect to one of the variables, and is square-free respect to all the others (so it is of the form $p(k) = x_{j_1}\ldots x_{j_i}^2\ldots x_{j_s}$). The vector subspace spanned by such basis elements and $P^0_K$ we denote by $P^1_K$. Note that it is not a subalgebra of $P_K$.

Define the following operation $\nu_{j,k}$ on $P_K$: up to sign for each pair $(j,k)$ where $j \in [m]$ and $k \in [i_j]$ we replace the $k$-th entrance of $x_j$ by $x_j^2$. More precisely, let $\alpha = (p_1|x_j\tau|p_2) \in (P_K)_n$, where $p_1 \in P_K(I)$ with $i_j = k - 1$, and $p_2 \in P_K$. Then

$$\nu_{j,k}(\alpha) = (-1)^{\deg p_1 + (j,\tau) + n}(p_1|x_j^2\tau|p_2).$$

The definition of $P^1_K$ is equivalent to the following: it is spanned by $P^0_K$ and the summands of the form $\nu_{j,k}(\alpha)$ where $\alpha \in P^0_K$.

Thus, we constructed the inclusions

$$T_K = P^0_K \subset P^1_K \subset P_K.$$

They induce the inclusions

$$G^0_K \subset G^1_K \subset G_K,$$

and $G^0_K$ is a dga-subalgebra of $G_K$.

Now the statement of Theorem 1.1 follows from the following lemma which will be proved in the next subsections.

\textbf{Lemma 7.2.} The homomorphisms $H_*(G^0_K) \rightarrow H_*(G_K)$ and $H_*(G^1_K/G^0_K) \rightarrow H_*(G_K/G^0_K)$ induced by the inclusions are surjective.

\textbf{Proof of the Theorem 1.1.} Denote the sequence of dga-algebras $A = C_*(\Omega X)$.

The first epimorphism of Lemma 7.2 any homology class in $G_K$ has a representative in $G^0_K$. For any element $g_0 \in P^0_K(I) \otimes A^I$ we have $d_{\text{bar}} = 0$, and so the differential splits respect to the splitting $G^0_K \cong P^0_K \otimes A$. Hence, the Künneth isomorphism implies the required splitting of homology classes.

Now we should check that the equivalence relation can be defined by $\mu_{j,k}$-operations.
We have that the differential restricted to $G^1_K/G^0_K$ again splits, and so due to the Künneth isomorphism it is enough to consider images under the differential of elements of the form $g = \nu_{j,k}(p) \otimes a$ for some $p \in [P^0_K(I)]_n$ ($j \in [m]$, $k \in [i_j]$) and $a = (\ldots, a^k_j, a^{k+1}_j, \ldots) \in A^{I+e_j}$ with $d_P(p) = 0$ and $d a = 0$.

Now we have

$$dg = (-1)^n \left( \mu_{i,j}(p) \otimes a - \nu_{j,k}p \otimes (\ldots, a^k_j \cdot a^{k+1}_j, \ldots) \right).$$

This gives the required equivalence relations and finishes the proof of Theorem 1.1 \hfill \Box

The proof of Lemma 7.2 is proved in the rest of this section. The first step is to reduce the statement about surjectivity to the analogous question for the corresponding homomorphisms in $P_K$ (Lemma 7.3). The second step is to show that the surjectivity for the homomorphism $H_*(P^1_K/P^0_K) \to H_*(P_K/P^0_K)$ follows from the surjectivity of the homomorphism $H_*(P^0_K) \to H_*(P_K)$ (Lemma 7.4). And finally we prove that the last homomorphism is an epimorphism geometrically passing to diagonal arrangements (Lemma 7.5).

7.2. Reduction to $P_K$. The aim of this subsection is to prove the implications: the surjectivity of the homomorphism $H_*(T_K) \to H_*(P_K)$ (induced by the inclusion $T_K = P^0_K \hookrightarrow P_K$) implies the surjectivity of the homomorphism $H_*(G^0_K) \to H_*(G_K)$; and the surjectivity of the homomorphism $H_*(P^1_K/P^0_K) \to H_*(P_K/P^0_K)$ implies the surjectivity of the homomorphism $H_*(G^1_K/G^0_K) \to H_*(G_K/G^0_K)$.

Both facts follow from the following technical statement.

Let $(G, d_G)$ be dga-complex with an additional grading $G = \oplus_{k \in \mathbb{N}} G_k$ and the differential split into sum of two: $d_G = d_1 + d_2$ in such a way that they satisfy the following conditions respect to the additional grading:

1. $d_1(G_k) \subseteq G_{k+1}$;
2. $d_2(G_k) \subseteq G_k$.

Further, suppose that $(G, d_G)$ is constructed as $G = \oplus_{I \in \mathbb{N}^m} P(I) \otimes B(I)$ for some differential complexes $(P = \oplus_{I \in \mathbb{N}^m} P(I), d_P)$ and $(B = \oplus_{I \in \mathbb{N}^m} B(I), d_B)$, and the differential $d_1$ also splits:

$$d_1(p \otimes b) = d_P(p) \otimes b + (-1)^{\deg p} p \otimes d_B(b).$$

**Lemma 7.3.** Suppose we are given an embedding dga-subalgebra $P^0 \hookrightarrow P$ such that (a) it induces epimorphism in homology; (b) the restriction of $d_2$ on $P^0$ is zero. Then the induced embedding of dga-algebras $G^0 \hookrightarrow G$ also induces epimorphism in homology.

**Proof.** Given an element $g \in G$ with $dg = 0$, decompose it into the sum $g = \sum_{k \in \mathbb{N}} g_k$ respect to the additional grading, and find the maximal integer $k$ such that the $k$-th homogenous component $g_k$ of $g$ is not in $G^0$. 


As \( g_{k+1} \in \mathcal{G} \) we have \( dg_{k+1} = 0 \). Together with \( dg = 0 \) this implies \( d_{1}g_{k} = 0 \). As the differential \( d_{1} \) splits, we can find such an element \( g' \in \mathcal{G}_{k-1} \) that \( gk + dg' = \sum p'_{i} \otimes b'_{i} \) with \( d_{P}(p'_{i}) = 0 \) and \( d_{B}(b'_{i}) = 0 \) for each \( i \).

Now, by assumption of the lemma, for each \( i \) there exists \( p''_{i} \in \mathcal{P}_{k-1} \) such that
\[
p'_{i} + d_{P}(p''_{i}) \in \mathcal{P}^{0}.
\]
Consider the element \( \tilde{g} = g + d_{G}(g' + \sum p''_{i} \otimes b'_{i}) \in \mathcal{G} \). The element \( \tilde{g} \) has the same components in degrees \( > k \) as \( g \) has, but now we have \( \tilde{g}_{k} \in \mathcal{G}^{0} \). So we decreased the value of \( k \) and proceeding in this way we get the required statement. \( \blacksquare \)

7.3. **Lemmas 7.4 and 7.5.** In this subsection we prove that the embeddings \( T_{K} = \mathcal{P}_{K}^{0} \hookrightarrow \mathcal{P}_{K} \) and \( \mathcal{P}_{K}^{1}/\mathcal{P}_{K}^{0} \hookrightarrow \mathcal{P}_{K}/\mathcal{P}_{K}^{0} \) induce the epimorphisms in homology.

**Lemma 7.4.** The surjectivity of the homomorphism \( H_{\ast}(\mathcal{P}_{K}^{0}) \rightarrow H_{\ast}(\mathcal{P}_{K}) \) implies the surjectivity of the homomorphism \( H_{\ast}(\mathcal{P}_{K}^{1}/\mathcal{P}_{K}^{0}) \rightarrow H_{\ast}(\mathcal{P}_{K}/\mathcal{P}_{K}^{0}) \).

**Proof.** We need to prove that if for \( \alpha \in \mathcal{P}_{K} \) the condition \( d\alpha \in \mathcal{P}_{K}^{0} \) holds, then there exists \( \beta \in \mathcal{P}_{K} \) such that \( \alpha + d\beta \in \mathcal{P}_{K}^{1} \).

Let \( \alpha \in \mathcal{P}_{K}(I) \). We will prove the statement using the induction by \( (\sum_{s=1}^{m} i_{s} - m) \). The base of the induction is trivial as when \( I \cong (1, \ldots, 1) \) the complexes \( \mathcal{P}_{K}(I) \) and \( \mathcal{P}_{K}^{0}(I) \) coincide and so in this case \( \alpha \in \mathcal{P}_{K}^{0} \).

Fix the number \( j \in [m] \) and let \( n = i_{j} \). So in each summand in \( \alpha \) we have exactly \( n \) entries of \( x_{j} \), let enumerate them from left to right by \( x_{j}^{1}, \ldots, x_{j}^{n} \).

First, suppose that there exist an integer \( k \) \( (k \in [n]) \) such that there are no monomials in \( \alpha \) divisible by the product \( x_{j}^{k} \cdot x_{j}^{k+1} \) (let call this assumption a separating condition respect to \( x_{j}^{k} \) and \( x_{j}^{k+1} \)). Then we can rename first \( k \) entries of \( x_{j} \) to \( x'_{j} \), and next \( n - k \) entries of \( x_{j} \) to \( x''_{j} \). The resulting chain \( \alpha' \) will be in \( \mathcal{T}_{K'} \) where \( K' = K(\text{pt}, \ldots, \text{pt}, V[2], \text{pt}, \ldots, \text{pt}) \) (here \( V[2] \) is a disjoint union of vertices \( j' \) and \( j'' \), and it is the \( j \)-th argument in the functor; the functor \( K \) is defined in [2.14]). The differential of \( \alpha' \) satisfies the condition \( d\alpha' \in \mathcal{P}_{K'}^{0} \). Now by induction assumption (for \( \alpha' \) the number \( \sum_{s=1}^{m} i_{s} - m \) is less by 1 than for \( \alpha \))
\[
\alpha' + d\beta' \in \mathcal{P}_{K'}^{0}.
\]
Renaming back \( j' \) and \( j'' \) to \( j \) we get the required statement for \( \alpha \).

Now consider the case when the separation assumption doesn’t hold. So fix some \( k \) and renumber the index set \( \{1, \ldots, k, k+1, k+2, \ldots, n\} \) to \( \{1, \ldots, k, k', k+1, \ldots, n-1\} \).
Let split the complex $P_K$ as $F_1 \oplus F_2$, where $F_1$ is additively generated by the basis elements where $x_j^k$ and $x_j^{k'}$ are not separated, and $F_2$ — by the basis elements satisfying separating condition respect to $j$ and $k$. There is a canonical isomorphism as additive groups $P_K(\hat{I}) := P_K(i_1, \ldots, n-1, \ldots, i_m) \to F_1$, denoted as before by $\nu_{j,k}$, which replace $x_j^k$ by the product $x_j^k x_j^{k'}$ with certain sign. The following holds for the differential:

$$d \nu_{j,k}(\lambda) = \nu_{j,k}(d\lambda) + (-1)\delta \mu_{j,k}(\lambda),$$

where $\nu_{j,k}(d\lambda) \in F_1$, $\mu_{j,k}(\lambda) \in F_2 \subset P_K(I)$ is defined similarly to $\mu_{j,k}^T$ from Section 4.2 and the sign $(-1)^{\delta}$ will be not important here.

Let $\alpha = \alpha_1 + \alpha_2$ be the decomposition corresponding to the splitting constructed above. Then we have $\alpha_1 = \nu_{j,k}(\lambda)$ for some $\lambda \in P_K(i_1, \ldots, n-1, \ldots, i_m)$. The condition $d\alpha = 0$ implies $d\lambda = 0$. By the forthcoming Lemma 7.5 we have $\lambda + d\Lambda = \gamma \in P_K^0$, and, hence,

$$\nu_{j,k}(\lambda) + d\nu_{j,k}(\Lambda) = \nu_{j,k}(\gamma) + (-1)^{\delta} \mu_{j,k}(\Lambda).$$

We know that $d\nu_{j,k}(\gamma) \in P_K^0$ and we have

$$\alpha + d\nu_{j,k}(\Lambda) = \nu_{j,k}(\gamma) + (-1)^{\delta} \mu_{j,k}(\Lambda) + \alpha_2,$$

where (a) $\nu_{j,k}(\gamma) \in P_K^1$; (b) $d\nu_{j,k}(\gamma) \in P_K^0$; (c) $\alpha_2 + (-1)^{\delta} \mu_{j,k}(\Lambda) \in F_2$.

From (b) and the condition that $d\alpha \in P_K^0$ it follows that $d(\alpha_2 + \nu_{j,k}(\Lambda)) \in P_K^0$. So as now $\alpha_2 + (-1)^{\delta} \mu_{j,k}(\Lambda)$ satisfies the condition of the lemma and the separating assumption, by the first part of the proof there exists $\beta$ such that $\alpha_2 + (-1)^{\delta} \mu_{j,k}(\Lambda) + d\beta \in P_K^1$ and so

$$\alpha + d\nu_{j,k}(\Lambda) + d\beta \in P_K^1.$$

This completes the proof. \qed

**Lemma 7.5.** The homomorphism $H_*(P_K^0) \to H_*(P_K)$ is surjective.

**Proof.** Fix the multi-degree $I = \{i_1, \ldots, i_m\}$. Denote $\hat{K} = K(\Delta[i_1], \ldots, \Delta[i_m])$ and $K' = K(V[i_1], \ldots, V[i_m])$. we will use the following notation: $T_0 = T_{K'}(1^{i_1}, \ldots, 1^{i_m})$ and $T = T_{\hat{K}}(1^{i_1}, \ldots, 1^{i_m})$. The inclusion of the simplicial complexes $K' \subset \hat{K}$ induces the inclusion of the differential graded chain complexes: $T_0 \hookrightarrow T$.

Consider the pair $(P_K(I), P_K^0(I))$ and show that it is the retract of the pair $(T, T_0)$. For that we construct the maps of differential chain complexes $Pol : (P_K(I), P_K^0(I)) \hookrightarrow (T, T_0)$, and $Q : (T, T_0) \to (P_K(I), P_K^0(I))$ such that $Q \circ Pol = \mathrm{Id}_{(P_K(I), P_K^0(I))}$. 
Let $\alpha$ be a summand in $P_K(I)$. First define the inclusion $\varphi : P_K(I) \to T$ by the following procedure. For each $j$ we have exactly $i_j$ entries of $x_j$ in $\alpha$, rename them from left to right to $x_{j,1}, \ldots, x_{j,n_j}$, considering new variables as distinct. This map doesn’t commute with the differential.

Now consider all different summands which are obtained from $f(\alpha)$ by permuting the upper indices of $x_j$’s, and define $Pol(\alpha)$ as the sum of them with the appropriate signs. Then we have that $dPol(\alpha) = Pol(d\alpha)$, and $Pol(P^0_K(I)) \subset T_0$.

Introduce the operation $Q'$: it is identical on $f(P_K(I))$, and it maps all the summands not from $P_K(I)$ to zero. Set $Q = f^{-1} \circ Q$. We have $dQ(\beta) = Q(d\beta)$, and $Q(T_0) = P^0_K$.

Obviously, $Q \circ P = f^{-1} \circ Q' \circ P = f^{-1} \circ Q' \circ f = f^{-1} \circ f = Id$. So we constructed the retraction.

Now recall that for any $L$ the differential complex $T_L(1, \ldots, 1)$ is quasi-isomorphic to the chain complex of $D_L$ - the complement of the corresponding diagonal arrangement, and for $L_1 \subset L_2$ on the same set of vertices the inclusion $T_{L_1}(1, \ldots, 1) \hookrightarrow T_{L_2}(1, \ldots, 1)$ corresponds to the inclusion $D_{L_1} \hookrightarrow D_{L_2}$. So by the next Lemma 7.6 we have that $T_0 \hookrightarrow T$ induces an epimorphism in homology. As $(P_K(I), P^0_K(I))$ is the retract of the pair $(T, T_0)$, the same is true for the homomorphism $H_*(P^0_K(I)) \to H_*(P_K(I))$ - it is surjective.

\begin{lemma}
For $\tilde{K} = K(\Delta[i_1], \ldots, \Delta[i_m])$ and $K' = K(V[i_1], \ldots, V[i_m])$ the inclusion $D_{\tilde{K}} \hookrightarrow D_K$ induces an epimorphism in homology.
\end{lemma}

\begin{proof}
Inductive arguments shows that it is enough to prove the following lemma.

**Lemma 7.7.** Let $\{i, j\}$ be an edge in $L$ and let $\hat{L}$ denotes the complex obtained from $L$ by deleting all simplices containing the edge $\{i, j\}$. Let $L$ satisfy the following condition: for any $\sigma \in K$ such that $i \in \sigma$, the subset $\sigma \cup \{j\}$ is also in $K$. Then $D_{\hat{L}} \hookrightarrow D_L$ induces an epimorphism in homology.

Indeed, using this statement we can pass from any simplicial complex of the form $K_1 = K(L_1, \ldots, V[S \cup \{i\}] \star \Delta[T \cup \{j\}], L_m)$ to the simplicial complex $K_1 = K(L_1, \ldots, V[S \cup \{i, j\}] \star \Delta[T], L_m)$, $\star$ denotes the join of simplicial complexes.

So let now prove Lemma 7.7.
Let $\alpha$ is a $k$-dimensional cycle in $D_L$. We can assume that it is transversal to the hyperplane $\pi$ defined by the equation $x_i = x_j$. Let $\alpha_+$ denotes the intersection of $\alpha$ with the halfspace $x_i \geq x_j$, and $\alpha_-$ with the halfspace $x_i \leq x_j$. The intersection $\alpha \cap \pi$ defines a $(k-1)$-dimensional cycle $\lambda$ in $D_L \cap \pi \cong \hat{D}_L$ with $d\alpha_+ = -d\alpha_- = i(\lambda)$, where $i : D_L \cap \pi \to D_L$ is a natural inclusion.

Consider the projection $P : D_L \to \hat{D}_L$ which is forgetting the coordinate $x_j$. Denote $\beta = P_*(\alpha_+)$. As $P_*(i(\lambda)) = \lambda$ we have $d\beta = \lambda$. So $d(\alpha_+ + i(\beta)) = d(\alpha_- - i(\beta)) = 0$. We have the following decomposition of $\alpha$ into the sum of two cycles

$$\alpha = (\alpha_+ + i(\beta)) + (\alpha_- - i(\beta)).$$

But adding a small $\varepsilon$ to the $x_i$-coordinate of all points of the first cycle in the sum, and subtracting also $\varepsilon$ to the $x_i$-coordinates of all points of the second cycle we deform them to homology equivalent cycles which are contained in open halfspaces defined by inequalities $x_i > x_j$ and $x_i < x_j$ respectively.

□

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