Comments on the Newlander-Nirenberg theorem

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Abstract

The Newlander-Nirenberg theorem says that a necessary and sufficient condition for the complex coordinates associated with a given almost complex structure tensor $I^N_M$ to exist is the vanishing of the Nijenhuis tensor $N^K_{MN}$. In the first part of the paper, we give a simple explicit proof of this fact. In the second part, we discuss a supersymmetric interpretation of this theorem. (i) The condition $N^K_{MN} = 0$ is necessary for certain $N = 1$ supersymmetric mechanical sigma models to enjoy $N = 2$ supersymmetry. (ii) The sufficiency of this condition for the existence of complex coordinates implies that the representation of the supersymmetry algebra realized by the superfields associated with all the real coordinates and their superpartners can be presented as a direct sum of $d$ irreducible representations ($d$ is the complex dimension of the manifold).
1 Introduction

Since 1982, we know that many well-known structures of differential geometry, such as the de Rham complex, allow for a supersymmetric interpretation [1]. For any manifold, one can define a certain supersymmetric quantum mechanical model. The dynamical time-dependent variables of this model include the coordinates and their Grassmann-valued superpartners.

Supersymmetric language is very useful. Besides giving a new unexpected interpretation of known mathematical facts, it allows one to derive many new nontrivial results, which are difficult to derive in a traditional way. Supersymmetry is a standard tool to study geometrical properties of the manifolds used by “physicists” in the papers published in the hep-th section of the arXiv. On the other hand, pure mathematicians are reluctant to use it, preferring traditional methods.

It is an unfortunate fact of our life that a large gap exists between the two communities. The languages in which the papers are written and the ways of thinking derived from these languages are often very different, to the extent that mathematicians and physicists do not often understand each other, even though the subject of their studies could be practically identical.

That is exactly the reason by which I’ve decided to write this methodical paper. Its second half is mainly addressed to mathematicians who might be curious to learn that a certain well-known mathematical fact admits an unexpected interpretation in the supersymmetry framework. And its first half is addressed to physicists who might have heard about the NN theorem, but probably do not know how exactly it is proven. I give here a direct explicit proof of this theorem, which I have not seen in the literature.

2 Geometry

2.1 Preliminaries

Definition 1. A complex manifold is a manifold of even dimension \( D = 2d \) which can be represented as a union of several overlapping charts such that:

1. Such chart is homeomorphic to \( \mathbb{R}^D \).

2. In each chart, one can define complex coordinates \( z^n \) and their conjugates \( \bar{z}^n \) such that the metric acquires the Hermitian form:

\[
|ds|^2 = 2h_{nm} dz^n d\bar{z}^m. \tag{2.1}
\]

3. The transition functions between any two overlapping charts with the coordinates \( \{z^n, \bar{z}^n\} \) and \( \{w^n, \bar{w}^n\} \) are holomorphic, \( z = f(w) \).

\[1\] I give here only one example. The so-called HKT manifolds were first discovered by supersymmetric methods [2] and only then they attracted the attention of pure mathematicians who gave their traditional description [3]. The full classification of HKT metrics was also recently constructed using supersymmetric tools [4, 5].

\[2\] I’ve put here the quotation marks because we are talking in this case about the scholars who may have studied physics at university, but who are now solving pure mathematical problems without much relationship to the physical world.

\[3\] Mathematicians often consider manifolds not endowed with the metric. But we will always assume that our manifold is Riemannian. Then the Hermiticity condition \( h_{nm} = h_{\bar{n}\bar{m}} \) follows from reality of \( ds \). The requirement \( h_{mn} = h_{\bar{m}\bar{n}} = 0 \) is nontrivial, however.
An interesting and important fact is that one can describe complex manifolds without explicitly introducing complex charts, but working exclusively in the real terms. To this end, we introduce first the notion of almost complex manifolds.

**Definition 2.** An almost complex manifold is a manifold of even dimension $D$ endowed with a tensor field $I_{MN}$ satisfying the properties (i) $I_{MN} = -I_{NM}$ and (ii) $I_{M}^{N}I_{N}^{P} = -\delta_{M}^{P}$. The tensor $I_{M}^{N}$ is called the almost complex structure.

To understand why a real tensor is called complex structure, consider first the simplest possible example—flat 2-dimensional Euclidean space. It can be parametrized by the real Cartesian coordinates $x^{1}, x^{2}$ or by the complex coordinate $z = (x^{1} + ix^{2})/\sqrt{2}$. An obvious relation $\partial z/\partial x^{1} = i\partial z/\partial x^{2}$ holds, which can also be presented in the form

$$\frac{\partial z}{\partial x^{A}} - i\varepsilon_{AB} \frac{\partial z}{\partial x^{B}} = 0$$

(2.2)

with

$$\varepsilon = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad (2.3)$$

The tensor $\varepsilon_{AB}$ satisfies both conditions in the definition above and is the complex structure in this case.

If a 2-dimensional manifold is not flat, $I_{M}^{N}$ may have a little bit more complicated form, but its tangent space projection $I_{AB} = I_{MNE}^{N}e_{N}^{A}$ coincides with the matrix $\varepsilon$ or probably with $-\varepsilon$. Indeed, an antisymmetric $2 \times 2$ matrix whose square is $-1$ coincides with (2.3) up to a sign. It describes rotations by $\pi/2$ or by $-\pi/2$.

In the general multidimensional case, one can prove a simple theorem:

**Theorem 1.** Take a tensor $I_{M}^{N}$ satisfying the conditions above. With a proper vielbein choice, its tangent space projection can be brought to the canonical form

$$I_{AB} = \text{diag} (\varepsilon, \ldots, \varepsilon). \quad (2.4)$$

**Proof.** To construct an orthonormal base in the tangent space $E$ where the complex structure acquires the form (2.4), we start with choosing in $E$ an arbitrary unit vector $e_{1}$. It follows from $I = -I^{T}$ and $I^{2} = -1$ that the vector $e_{2} = Ie_{1}$ has also unit length and is orthogonal to $e_{1}$. Obviously, $Ie_{2} = I^{2}e_{1} = -e_{1}$. Consider the subspace $E^{*} \subset E$ that is orthogonal to $e_{1}$ and $e_{2}$. If it is not empty, choose there an arbitrary unit vector $f_{1}$ and consider $f_{2} = If_{1}$. One can easily see that $f_{2}$ also belongs to $E^{*}$. Now consider the subspace $E^{**} \subset E^{*} \subset E$ that is orthogonal to $e_{1,2}, f_{1,2}$ and, if $E^{**}$ is not empty, repeat the procedure. We arrive at the matrix (2.4). \hfill $\square$

Consider the equation system

$$\frac{\partial z^{n}}{\partial x^{M}} - iI_{M}^{N} \frac{\partial z^{n}}{\partial x^{N}} = 0$$

(2.5)

It is convenient (especially, for supersymmetric applications), but not necessary. For example, the popular textbook [6] uses only complex but not real description.

The property (2.2) holds not only for $z$, but for any holomorphic function $f(z)$. In the latter case, the real and imaginary parts of (2.2) are none other than the Cauchy-Riemann conditions.
If not only $I_{AB}$, but also $I_{M}^{N}$ has the form (2.4), a solution to (2.5) can be easily found. It is simply

$$z_{1}^{(0)} = \frac{x^{1} + ix^{2}}{\sqrt{2}}, \quad z_{2}^{(0)} = \frac{x^{3} + ix^{4}}{\sqrt{2}}, \ldots$$  \hspace{1cm} (2.6)$$
or any set of $d$ non-degenerate analytic functions of $z_{n}^{(0)}$.

In a generic case, the solutions to (2.5) are more complicated. Moreover, they do not always exist. The conditions under which they do, is the content of the NN theorem to be proven in the next section. For the time being, we will prove that

**Theorem 2.** If the equation system (2.5) has a solution, the manifold is complex.

**Proof.** We will show first that the metric has a Hermitian form (i.e. the components $g^{nm}$ etc vanish) Let us trade $x^{M}$ for $(z^{n}, \bar{z}^{\bar{n}})$ and write

$$g^{nm} = \frac{\partial z^{n}}{\partial x^{M}} \frac{\partial z^{m}}{\partial x^{N}} g^{MN} = i I_{M}^{P} \frac{\partial z^{n}}{\partial x^{P}} \frac{\partial z^{m}}{\partial x^{N}} g^{MN} = i I_{PN}^{M} \frac{\partial z^{n}}{\partial x^{P}} \frac{\partial z^{m}}{\partial x^{N}} = 0.$$ The vanishing of $g^{\bar{n}\bar{m}}$ follows from the same argument. The properties $g^{\bar{n}\bar{m}} = g^{nm} = 0$ imply also the vanishing of the components $g_{nm}$ and $g_{\bar{n}\bar{m}}$ of the inverse tensor.

Next, we need to show that the transition functions between two overlapping maps with the coordinates $(z^{n}, \bar{z}^{\bar{n}})$ and $(w^{m}, \bar{w}^{\bar{m}})$ are holomorphic. To this end, we express, using (2.5), $I_{M}^{N}$ in the complex frame:

$$I_{m}^{n} = -i \delta_{m}^{n}, \quad I_{\bar{m}}^{\bar{n}} = i \delta_{\bar{m}}^{\bar{n}}, \quad I_{m}^{\bar{n}} = I_{\bar{m}}^{n} = 0$$  \hspace{1cm} (2.7)$$
and consider the transformation of the tensor (2.7) from one chart to another. Knowing that $I$ keeps the form (2.7) after this transformation, one can derive that $\partial w^{m}/\partial \bar{z}^{\bar{n}} = 0$. \hfill \Box

**2.2 NN theorem**

**Theorem 3.** The complex coordinates satisfying the condition (2.5) can be introduced and the manifold is complex iff the condition

$$N_{MN}^{K} = \partial_{[M} I_{N]}^{K} - I_{M}^{P} I_{N}^{Q} \partial_{[P} I_{Q]}^{K} = 0$$  \hspace{1cm} (2.8)$$
holds.

**Proof.**

**Necessity.** Represent the system (2.5) as $D_{M} z^{n} = 0$ with

$$D_{M} = \partial_{M} - i I_{M}^{N} \partial_{N}.$$  \hspace{1cm} (2.9)$$

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\*\*This combination is a tensor, in spite of the presence of the ordinary rather than covariant derivatives. This is so because the terms in the covariant derivatives involving the Christoffel symbols cancel out in this case. Using a sloppy language, we will call the L.H.S. of Eq. (2.8) the **Nijenhuis tensor**. A conventional definition of the Nijenhuis tensor is a little bit different:

$$N_{MN}^{K} (\text{conventional}) = I_{M}^{P} N_{PN}^{K} (\text{this paper})$$

We will do so because the object (2.8) has a more transparent structure, and it is this combination that directly appears in (2.10).
For self-consistency, the conditions \([D_M, D_N]z^n = 0\) should also hold. Bearing in mind that
\[
[D_M, D_N]z^n = [-i\partial_M I_N^Q - I_M^P(\partial_P I_N^Q)] \partial_Q z^n
\]
\[
= [-i\partial_M I_N^K - iI_M^P(\partial_P I_N^Q)I_Q^K] \partial_K z^n - I_M^P(\partial_P I_N^Q)D_Q z^n
\]
(2.10)

(we used \(D_Q z^n = 0\) and \((\partial_P I_N^Q)I_Q^K = -I_N^Q \partial_P I_Q^K\) that follows from \(I^2 = -\mathbb{1}\)), we arrive at the necessary condition (2.8).

**Sufficiency.** This part of the theorem [the proof of existence of the solution to the system (2.5) under the condition (2.8)] is more difficult. We will give here its explicit “physical” proof.

- Let the complex structure \(I_{MN}\) has a canonic form (2.4). Then the solutions to (2.5) exist and one of the solution is given by (2.6).

Suppose now that the complex structure does not coincide with \((I_0)_{MN} = \text{diag}(\varepsilon, \ldots, \varepsilon)\), but is close to it: \(I = I_0 + \Delta, \Delta \ll 1\). We are going to show that, after such an infinitesimal deformation, solutions to (2.5) still exist.

- Let us first do it in the simplest case \(D = 2\). Then the condition (2.8) is fulfilled identically. The condition \(I^2 = -\mathbb{1}\) means that \(\{\Delta, I_0\} = 0\), which is so iff

\[
\Delta_1^1 = -\Delta_2^2, \quad \Delta_2^1 = -\Delta_1^2. \tag{2.11}
\]

(In physical notation, \(\Delta = \alpha \sigma^1 + \beta \sigma^3\), where \(\sigma^a=1,2,3\) are the Pauli matrices.) Look now at the system (2.5). We set \(z = z(0) + \delta z\). The equations acquire the form

\[
\frac{\partial}{\partial x^1}(\delta z) + i\frac{\partial}{\partial x^2}(\delta z) = \frac{1}{\sqrt{2}}(i\Delta_1^1 - \Delta_2^2),
\]

\[
\frac{\partial}{\partial x^2}(\delta z) - i\frac{\partial}{\partial x^1}(\delta z) = \frac{1}{\sqrt{2}}(i\Delta_2^1 - \Delta_2^2). \tag{2.12}
\]

Bearing in mind (2.11), these two equations coincide. They can be expressed as

\[
\frac{\partial(\delta z)}{\partial z(0)} = \frac{i}{2} \Delta_1^{1+i2}, \tag{2.13}
\]

which can be easily integrated on a disk\(^7\)

- The simplest nontrivial case is \(D = 4\). The condition \(\{\Delta, I_0\} = 0\) implies

\[
\Delta_1^1 = -\Delta_2^2, \quad \Delta_2^1 = \Delta_1^2,
\]

\[
\Delta_1^3 = -\Delta_2^4, \quad \Delta_1^4 = \Delta_2^3,
\]

\[
\Delta_3^1 = -\Delta_4^2, \quad \Delta_3^2 = \Delta_4^1,
\]

\[
\Delta_3^3 = -\Delta_4^4, \quad \Delta_3^4 = \Delta_4^3. \tag{2.14}
\]

\(^7\)The whole discussion applies to a particular topologically trivial chart in a set of which a manifold is subdivided.
We pose \( z^1 \to z, z^2 \to w \). A short calculation shows that, bearing the relations (2.14) in mind, the equations (2.5) are reduced to

\[
\frac{\partial (\delta z)}{\partial \bar{z}_{(0)}} = \frac{i}{2} \Delta_1^{1+i^2}, \\
\frac{\partial (\delta z)}{\partial \bar{w}_{(0)}} = \frac{i}{2} \Delta_3^{1+i^2}, \\
\frac{\partial (\delta w)}{\partial \bar{z}_{(0)}} = \frac{i}{2} \Delta_1^{3+i^4}, \\
\frac{\partial (\delta w)}{\partial \bar{w}_{(0)}} = \frac{i}{2} \Delta_3^{3+i^4}.
\] (2.15)

If \( D > 2 \), the conditions (2.8) provide nontrivial constraints. Their linearized version is

\[
\partial_P \Delta_N^M - \partial_N \Delta_P^M = (I_0)_P^Q (I_0)_N^S \left[ \partial_Q \Delta_S^M - \partial_S \Delta_Q^M \right].
\] (2.16)

Again, bearing in mind (2.14), one can show that, for \( D = 4 \), out of 24 real conditions in (2.16), only 4 independent real or 2 independent complex constraints are left. The latter have a simple form

\[
\frac{\partial}{\partial \bar{z}_{(0)}} \Delta_3^{1+i^2} - \frac{\partial}{\partial \bar{w}_{(0)}} \Delta_1^{1+i^2} = 0, \\
\frac{\partial}{\partial \bar{z}_{(0)}} \Delta_3^{3+i^4} - \frac{\partial}{\partial \bar{w}_{(0)}} \Delta_1^{3+i^4} = 0.
\] (2.17)

The first equation in (2.17) is the integrability condition for the system of the first two equations in (2.15). It is necessary and also sufficient for the solution of this system to exist. Indeed, it implies that the (0,1)-form

\[
\omega = \Delta_1^{1+i^2} d\bar{z}_{(0)} + \Delta_3^{1+i^2} d\bar{w}_{(0)}
\]

is closed, \( \partial_0 \omega = 0 \). Bearing in mind the trivial topology of a chart of our complex manifold that we are discussing, \( \omega \) is also exact (see e.g. Theorem 6.1 in [6]), which is tantamount to saying that the solution exists. The second relation in (2.17) is the necessary and sufficient integrability condition for the system of the third and fourth equations in (2.15).

- This reasoning can be translated to the case of higher dimensions. For an arbitrary \( D = 2d \), the equations (2.5) are reduced, bearing in mind \( I^2 = -1 \), to \( d^2 \) conditions similar to (2.15), but with differentiation over each antiholomorphic variable \( \bar{z}^n_{(0)} \) for each complex function \( \delta z^n \). The conditions (2.8) lead to \( d^2(d-1)/2 \) complex constraints which represent integrability conditions of the type (2.17). They imply that the form

\[
\omega_d = \Delta_1^{1+i^2} d\bar{z}^1_{(0)} + \Delta_3^{1+i^2} d\bar{z}^2_{(0)} + \ldots
\]

is closed. Due to the trivial topology of the chart, it also means that this form is exact.
Once the complex coordinates $z = z(0) + \delta z$ satisfying the equations (2.5) are found, the complex structure acquires in these new coordinates the canonical form (2.7) and (2.4). Thus we have actually proven that a small deformation of $I_M^N$ can be brought to the form (2.4) by an infinitesimal diffeomorphism, provided the condition (2.8) is satisfied. But that means that the same statement can be made for a finite deformation representable as a superposition of an infinite number of infinitesimal ones.

3 Supersymmetry

3.1 Preliminaries

To begin with, we present some basic “superfacts”, bearing in mind a reader who is an expert in differential geometry, but may not know much about supersymmetry. We give, however, only the minimal necessary information assuming that our reader knows the basics of Grassmann algebra and, which is not so much necessary but desirable, of classical and quantum mechanics of the systems involving Grassmann dynamical variables. More details can be found in the review [8]. See especially chap. 8.1 there.

The simplest supersymmetry algebra reads

$$Q_1^2 = Q_2^2 = H, \quad \{Q_1, Q_2\}_+ = 0. \quad (3.1)$$

Here $H$ is the Hamiltonian and $Q_{1,2}$ are two different Hermitian operators called supercharges. As follows from (3.1), they commute with $H$. If one introduces a complex supercharge $Q = (Q_1 + iQ_2)/2$, one can also present (3.1) in the form

$$Q^2 = (\bar{Q})^2 = 0, \quad \{Q, \bar{Q}\}_+ = H. \quad (3.2)$$

The algebra (3.1) involves two supercharges and, correspondingly, is usually called the algebra of $\mathcal{N} = 2$ supersymmetric quantum mechanics (SQM). More complicated algebras may involve extra supercharges (the SQM systems enjoying $\mathcal{N} = 4$ or $\mathcal{N} = 8$ supersymmetry are known) or also the momentum operators $P_j$. The latter algebras are relevant for supersymmetric quantum field theories. But in this paper we are going to discuss only the algebra (3.1) and also still more simple $\mathcal{N} = 1$ supersymmetry algebra,

$$Q^2 = H \quad (3.3)$$

with real $Q$. Physically, the latter is too simple to be interesting. After diagonalisation, one can always extract a square root of the Hamiltonian whose spectrum is bounded from below. If some energies in the spectrum are negative, one just redefines $H$ by adding an appropriate positive constant. However, we will use in what follows the algebra (3.3) and its representations as a technical tool.

The algebra (3.1) leads to a double degeneracy of the spectrum. It follows from (3.1) that the eigenvalues of the Hamiltonian are positive or zero. The doublets involving two states $|B\rangle$ and $|F\rangle$ with the properties

$$H|B\rangle = E|B\rangle, \quad H|F\rangle = E|F\rangle, \quad Q|B\rangle = \sqrt{E}|F\rangle, \quad Q|F\rangle = 0, \quad \bar{Q}|B\rangle = 0, \quad \bar{Q}|F\rangle = \sqrt{E}|B\rangle \quad (3.4)$$
represent a simple 2-dimensional irreducible representation of the algebra \((3.2)\). There exist also finite-dimensional representations involving a larger even number of states, but it is easy to show that they are all reducible. In physical language, any set of \(2n\) states providing a representation of \((3.2)\) is split into \(n\) doublets.

The only irreducible finite-dimensional representations of the algebra \((3.3)\) are the trivial singlets—the eigenstates of \(Q\) and \(H\).

We will be interested, however, in more complicated infinite-dimensional representations of the \(\mathcal{N} = 1\) and \(\mathcal{N} = 2\) algebra where the supercharges and the Hamiltonian are realized as linear differential operators acting in superspace.\(^8\)

The \(\mathcal{N} = 1\) superspace includes time \(t\) and a real Grassmann nilpotent variable \(\theta\): \(\theta^2 = 0\). The supercharges and the Hamiltonian are realized as the differential operators:

\[
Q = -i \left( \frac{\partial}{\partial \theta} + i \theta \frac{\partial}{\partial t} \right), \\
H = -i \frac{\partial}{\partial t} 
\]

(3.5)

The Hamiltonian is the generator for the time shifts. The supercharge is the generator for somewhat more complicated transformations:

\[
\theta \rightarrow \theta + \eta, \\
t \rightarrow t + i \eta \theta 
\]

(3.6)

with a real Grassmann parameter \(\eta\).

Consider now \(\mathcal{N} = 1\) superfields (or supervariables) representing functions of \(t\) and \(\theta\). Due to the nilpotency of \(\theta\), they can be presented as

\[
\mathcal{X}(t, \theta) = x(t) + i \theta \psi(t). 
\]

(3.7)

The ordinary real function \(x(t)\) and the Grassmann-odd real function \(\psi(t)\) are called the components of the superfield \((3.7)\). The shifts \((3.6)\) induce the shift

\[
\delta \mathcal{X} = \mathcal{X}(t + i \eta \theta, \theta + \eta) - \mathcal{X}(t, \theta) = i \eta Q \mathcal{X} 
\]

(3.8)

of the superfield \(\mathcal{X}\) implying the following shifts of its components:

\[
\delta x(t) = i \eta \psi(t), \quad \delta \psi(t) = -\eta \dot{x}.
\]

(3.9)

Note that the product of two superfields is also a superfield: \(\delta(\mathcal{X}_1 \mathcal{X}_2) = i \eta Q(\mathcal{X}_1 \mathcal{X}_2)\).

Now we introduce the covariant supersymmetric derivative

\[
\mathcal{D} = \frac{\partial}{\partial \theta} - i \theta \frac{\partial}{\partial t}. 
\]

(3.10)

This operator is Hermitian, nilpotent and anticommutates with \(Q\). The property

\[
\mathcal{D}^2 = -i \frac{\partial}{\partial t} 
\]

(3.11)

holds.

\(^8\)Or, better to say, “supertime” as we do not have any space variables and spatial dependence, but we stick to the terms commonly used in the literature.
Theorem 4. If \( \mathcal{X} \) is a superfield, the same is true for \( \mathcal{D}\mathcal{X} \).

Proof. We have
\[
\delta(\mathcal{D}\mathcal{X}) = \mathcal{D}\delta \mathcal{X} = i\mathcal{D}(\eta \mathcal{Q}\mathcal{X}) = i\eta \mathcal{Q}(\mathcal{D}\mathcal{X})
\]
(do not forget that \( \eta \) anticommutes with \( \mathcal{D} \)). ☐

We understand now why \( \mathcal{D} \) is called the covariant derivative. In the same way as the covariant derivative in Riemannian geometry makes a tensor out of a tensor, the derivative (3.10) make a superfield out of a superfield.

The superfield (3.7) with its transformation law (3.9) defines an infinite-dimensional representation of the algebra (3.3). But it is a reducible representation. Indeed, one can now impose the constraint of reality \( \bar{X} = X \). A real superfield stays real under the variation (3.8).

\( \mathcal{N} = 2 \) superspace and the \( \mathcal{N} = 2 \) superfields are defined in a similar manner. The superspace now includes time \( t \) and a complex Grassmann anticommuting variable \( \theta \): \( \theta^2 = \bar{\theta}^2 = \{\theta, \bar{\theta}\} = 0 \). The supertransformations are
\[
\begin{align*}
\theta & \to \theta + \epsilon, \\
\bar{\theta} & \to \bar{\theta} + \bar{\epsilon}, \\
t & \to t + i(\epsilon \bar{\theta} + \bar{\epsilon}\theta)
\end{align*}
\] (3.12)
with complex Grassmann \( \epsilon \). These transformations are generated by a complex supercharge \( Q \) and its Hermitian conjugate:
\[
\begin{align*}
Q & = -\frac{i}{\sqrt{2}} \left( \frac{\partial}{\partial \theta} + i\bar{\theta} \frac{\partial}{\partial t} \right), \\
\bar{Q} & = -\frac{i}{\sqrt{2}} \left( \frac{\partial}{\partial \bar{\theta}} + i\theta \frac{\partial}{\partial t} \right)
\end{align*}
\] (3.13)
[the factor \( 1/\sqrt{2} \) is added to ensure the validity of (3.2)]. A generic \( \mathcal{N} = 2 \) superfield reads
\[
\Phi(t, \theta, \bar{\theta}) = z(t) + i\theta \chi(t) + i\bar{\theta} \lambda(t) + \theta \bar{\theta} F(t)
\] (3.14)
with Grassmann-even complex \( z(t) \) and \( F(t) \) and Grassmann-odd complex \( \chi(t) \) and \( \lambda(t) \). The supersymmetric variation of \( \Phi \) reads
\[
\delta \Phi = i\sqrt{2}(\epsilon Q + \bar{\epsilon}\bar{Q}) \Phi.
\] (3.15)
The covariant supersymmetric derivatives which are nilpotent and anticommute with \( Q \) and \( \bar{Q} \) are
\[
\begin{align*}
\mathcal{D} & = \frac{\partial}{\partial \theta} - i\bar{\theta} \frac{\partial}{\partial t}, \\
\bar{\mathcal{D}} & = -\frac{\partial}{\partial \bar{\theta}} + i\theta \frac{\partial}{\partial t}.
\end{align*}
\] (3.16)
The operator \( i\bar{D} \) is the Hermitian conjugate of \( iD \). If \( \Phi \) is a superfield, then \( D\Phi \) and \( \bar{D}\Phi \) are also superfields.

The superfield (3.14) defines an infinite-dimensional representation of the algebra (3.2). This representation is reducible. Two different irreducible representations are obtained after imposing the constraints:
• The reality constraint $\Phi = \bar{\Phi}$. If $\Phi$ is real, the variation $\delta \Phi$ is also real.

• The chirality constraints $D\Phi = 0$ or $\bar{D}\Phi = 0$. Again, if $D\Phi$ vanishes, so does $D\delta \Phi$. Note that if $\bar{D}Z = 0$, then $D\bar{Z} = 0$. We will call $Z$ a left chiral superfield and $\bar{Z}$ a right chiral superfield.\(^9\)

In what follows, we will not be interested in the real $\mathcal{N} = 2$ superfields, but exclusively in the chiral ones.

For a chiral superfield, the component expansion (3.14) can be simplified if one introduces “left” and “right” times:

$$t_L = t - i\theta \bar{\theta}, \quad t_R = t + i\theta \bar{\theta}.$$  

The supersymmetric variation of $t_L$ depends only on $\theta$, $\delta t_L = 2i\epsilon \theta$, and the supersymmetric variation of $t_R$ depends only on $\bar{\theta}$.

The set of coordinates $(t_L, \theta)$ describes the holomorphic chiral $\mathcal{N} = 2$ superspace and the set $(t_R, \bar{\theta})$ describes the antiholomorphic chiral $\mathcal{N} = 2$ superspace.

Then, if $\bar{D}Z = 0$, we may write

$$Z = Z(t_L, \theta) = z(t_L) + i\sqrt{2}\theta \chi(t_L),$$  

$$\bar{Z} = \bar{Z}(t_R, \bar{\theta}) = \bar{z}(t_R) + i\sqrt{2}\bar{\theta} \bar{\chi}(t_R).$$  \hspace{1cm} (3.17)

The components of a left chiral superfield are transformed as

$$\delta z = i\sqrt{2}\epsilon \chi, \quad \delta \chi = -\sqrt{2}\epsilon \dot{z}.$$  \hspace{1cm} (3.18)

Let us pose now

$$z = \frac{x_1 + ix_2}{\sqrt{2}}, \quad \chi = \frac{\psi_1 + i\psi_2}{\sqrt{2}}, \quad \epsilon = \frac{\eta + i\tilde{\eta}}{\sqrt{2}}.$$  \hspace{1cm} (3.19)

Suppose first that $\epsilon$ is real, $\tilde{\eta} = 0$. Then we derive

$$\delta x_1 = i\eta \psi_1, \quad \delta \psi_1 = -\eta \dot{x}_1,$$  

$$\delta x_2 = i\eta \psi_2, \quad \delta \psi_2 = -\eta \dot{x}_2.$$  \hspace{1cm} (3.20)

We see that the components $(x_1, \psi_1)$ are not mixed with the components $(x_2, \psi_2)$; each set is transformed in the same way as the components of an $\mathcal{N} = 1$ superfield [see Eq. (3.9)]! In other words, the representation $Z$ is an irreducible representation of the $\mathcal{N} = 2$ superalgebra, but it can also be thought of as a reducible representation of $\mathcal{N} = 1$ superalgebra realized by the transformations (3.18) with real $\epsilon$. When going down from $\mathcal{N} = 2$ to $\mathcal{N} = 1$, the chiral superfield $Z$ is split into two real superfields $X_1$ and $X_2$. To see it quite explicitly, substitute $\theta = (\theta_1 + i\theta_2)/\sqrt{2}$ in (3.17). Then $t_L = t + \theta_2 \theta_1$. We derive

$$Z = \frac{1}{\sqrt{2}} \{X_1(t, \theta_1) + iX_2(t, \theta_1) + i\theta_2[D\chi_1(t, \theta_1) + iD\chi_2(t, \theta_1)]\}.$$  \hspace{1cm} (3.21)

\(^9\)The terms “left” and “right” have a physical origin which is irrelevant for us here.
Look now at the transformations (3.18) when \( \epsilon = i\tilde{\eta}/\sqrt{2} \) is imaginary. We obtain

\[
\begin{align*}
\tilde{\delta}x_1 &= -i\tilde{\eta}\psi_2, \\
\tilde{\delta}x_2 &= i\tilde{\eta}\psi_1,
\end{align*}
\]

\[
\begin{align*}
\tilde{\delta}\psi_1 &= -\tilde{\eta}\dot{x}_2, \\
\tilde{\delta}\psi_2 &= \tilde{\eta}\dot{x}_1
\end{align*}
\]  
(3.22)

or

\[
\tilde{\delta}\mathcal{X}_A = \tilde{\eta}\varepsilon_{AB} \mathcal{D}\mathcal{X}_B
\]  
(3.23)

[with \( \varepsilon \) defined as in (2.3)] in a compact form.

The generators of the transformations (3.20) and (3.22) obey the algebra (3.1). Indeed,

- It is rather evident that the transformations (3.20) and (3.22, 3.23) commute. Indeed, \( \delta\mathcal{X}_A \) is a superfield, and hence \( \delta(\tilde{\delta}\mathcal{X}_A) \) and \( \tilde{\delta}(\delta\mathcal{X}_A) \) coincide, having both the form (3.8) with \( \mathcal{X} \) replaced by \( \tilde{\delta}\mathcal{X}_A \). A corollary of this is the vanishing of the anticommutator \( \{Q, \tilde{Q}\} \) of the corresponding quantum supercharges.

- Bearing in mind (3.11), the Lie bracket of two different tilde-transformations reads

\[
(\tilde{\delta}_1\tilde{\delta}_2 - \tilde{\delta}_2\tilde{\delta}_1)\mathcal{X}_A = -2i\tilde{\eta}_1\tilde{\eta}_2\dot{\mathcal{X}}_A,
\]  
(3.24)

which is tantamount to saying that \( \tilde{Q}^2 \) coincides with the Hamiltonian (the generator of time shifts).

### 3.2 NN theorem: supersymmetric interpretation

The tensor \( \varepsilon_{AB} \) entering (3.23) can be interpreted as the flat complex structure. The components \( x_A \) of the superfields \( \mathcal{X}_A \) can be interpreted as the flat Cartesian coordinates. Suppose now that we have \( 2dN = 1 \) superfields \( \mathcal{X}^M \). One of the supersymmetries follows from the transformations of the superspace coordinates as in (3.20):

\[
\begin{align*}
\delta x^M &= i\eta\psi^M, \\
\delta\psi^M &= -\eta\dot{x}^M.
\end{align*}
\]  
(3.25)

Looking for a generalization of (3.23), we anticipate the presence of the second supersymmetry,

\[
\tilde{\delta}\mathcal{X}^M = \tilde{\eta} I_N^M(\mathcal{X}^P) \mathcal{D}\mathcal{X}^N,
\]  
(3.26)

where

\[
I^2 = -1,
\]  
(3.27)

and ask: under what conditions is it possible? Under what conditions the generators of the transformations (3.25) and (3.26) obey the algebra (3.1)?

**Theorem 5.** The algebra (3.1) holds iff the Nijenhuisen tensor (2.8) vanishes.

**Proof.** The Lie bracket \( [\delta, \tilde{\delta}] \) vanishes by the same reason as in the flat case treated before: the transformation \( \delta \) mixes the components of each multiplet, while the transformation \( \tilde{\delta} \) mixes different superfields and does not bother much about their internal structure. Thus, we only need to explore the Lie bracket \( (\tilde{\delta}_1\tilde{\delta}_2 - \tilde{\delta}_2\tilde{\delta}_1)\mathcal{X}^M \).
Note first that
\[ \tilde{\delta}(\mathcal{D}\chi^N) = \mathcal{D}(\tilde{\delta}\chi^N) = -\tilde{\eta}\mathcal{D}(I_L^N\mathcal{D}\chi^L) = -\tilde{\eta}(\partial_KI_L^N)\mathcal{D}\chi^K\mathcal{D}\chi^L + i\tilde{\eta}I_L^N\dot{\chi}^L. \]

The commutator of two transformations (3.26) is then derived to be
\[
(\tilde{\delta}_1\tilde{\delta}_2 - \tilde{\delta}_2\tilde{\delta}_1)\chi^M = 2i\tilde{\eta}_1\tilde{\eta}_2(I^2)_K^M\chi^K - 2\tilde{\eta}_1\tilde{\eta}_2[I_K^L(\partial_LI_N^M) + (\partial_NI_K^L)I_L^M]D\chi^KD\chi^N.
\]

If we want it to coincide with \(-2i\tilde{\eta}_1\tilde{\eta}_2\partial_t\chi^M\) [as is dictated by Eq.(3.1)] the conditions (3.27) as well as
\[
(\partial_LI_N^M)I_K^L + (\partial_NI_K^L)I_L^M = 0 \quad (3.29)
\]
follow. Using again (3.27), the condition (3.29) can be brought into the form (2.8).

Thus, the condition \(\mathcal{N}_{MN}^K = 0\) is necessary and sufficient for \(\mathcal{N} = 2\) supersymmetry associated with the given complex structure to hold. But the NN theorem is formulated differently: it affirms that the condition (2.8) is necessary and sufficient for the existence of complex coordinates.

Well, as far as necessity is concerned, the equivalence of Theorems 3 and 5 is rather clear. Suppose that complex coordinates \(z^n\) exist. But then each such coordinate can be upgraded to a complex chiral superfield \(Z^n\) whose components are transformed under supersymmetry as in (3.18). Each superfield \(Z^n\) can be expressed via a pair of \(\mathcal{N} = 1\) real superfields as in (3.21). The complex structure tensor \(I_M^N\) has in this case the form (2.4) and does not depend on the coordinates. The tensor \(\mathcal{N}_{MN}^K\) vanishes automatically.

Now, if the Nijenhuis tensor vanishes, we know from Theorem 5 that the algebra of \(\mathcal{N} = 2\) supersymmetry holds. The set of \(2d\) superfields \(\chi^M\) is a representation of this algebra. Then the sufficiency of (2.8) means that, for \(d > 1\), this representation is reducible and can be decomposed in a direct sum of \(d\) irreducible representations realized by the components of the chiral complex superfields \(Z^n\).

This latter statement looks very natural, it is widely used by physicists, but I am not aware of its independent proof. The only known proof of this fact is the proof of the sufficiency part of the NN theorem that we outlined in Sect.2 and that does not resort to supersymmetric description.

### 3.2.1 Invariant actions

Up to now, when talking about the supersymmetric aspects of the NN theorem, we stayed at the purely algebraic level, having discussed only the algebras (3.1), (3.3) and their representations. A reader-mathematician may stop reading this paper at this point.

But, when a physicist thinks of a symmetry, s/he is always interested in dynamical systems that enjoy these symmetries. An industrial method to find supersymmetric dynamical systems is based on the following theorem:
Theorem 6. Let \( \mathcal{X}(t, \theta) \) be an \( \mathcal{N} = 1 \) superfield that vanishes at \( t = \pm \infty \). Then the integral (associated with the physical action)

\[
S = \int d\theta \int_{-\infty}^{\infty} dt \mathcal{X}
\]

is invariant under transformations (3.6).

Proof. We have

\[
\delta S = \int d\theta \int_{-\infty}^{\infty} dt \delta \mathcal{X} = -\epsilon \int d\theta \int_{-\infty}^{\infty} dt \left( \frac{\partial}{\partial \theta} + i\theta \frac{\partial}{\partial t} \right) \mathcal{X}.
\]

The first term vanishes due to the definition (3.30) and the Grassmannian nature of \( \theta \). The second term vanishes due to the condition \( \mathcal{X}(\pm \infty, \theta) = 0 \).

Obviously, the same property holds for the integral

\[
S = \int d\bar{\theta} d\theta \int_{-\infty}^{\infty} dt \Phi
\]

of a \( \mathcal{N} = 2 \) superfield \( \Phi \).

The superfield \( \mathcal{X} \) in Eq. (3.31) and the superfield \( \Phi \) in Eq. (3.32) can be constructed out of certain basic superfields by multiplications, time differentiations and covariant differentiations with the operator \( D \) in the \( \mathcal{N} = 1 \) case and with the operators \( D \) and \( \bar{D} \) in the \( \mathcal{N} = 2 \) case. In particular, one can write

\[
S = \frac{1}{4} \int d\bar{\theta} d\theta dt h_{mn}(Z^k, \bar{Z}^{\bar{k}}) D\bar{Z}^{\bar{n}}(t_R) DZ^m(t_L),
\]

where \( Z^k = 1, \ldots, d \) are left chiral superfields and \( h_{mn} \) is Hermitian. Substituting there the expansions (3.17), not forgetting to expand over \( \theta \) and \( \bar{\theta} \) also \( t_{L,R} = t \mp i\theta \bar{\theta} \) and performing the integral over \( d\bar{\theta} d\theta dt \), one can derive the following expression for the Lagrangian:

\[
L = h_{mn}(z, \bar{z}) \dot{z}^m \dot{\bar{z}}^{\bar{n}} + \text{terms including superpartners } \chi^m(t)
\]

We can now interpret \( z^m \) and \( \bar{z}^{\bar{m}} \) as the coordinates on a complex manifold with the metric \( h_{mn}(z, \bar{z}) \). The displayed term of the Lagrangian can be interpreted as the kinetic energy of a particle with unit mass moving along the manifold. The dynamical system describing such a motion is called sigma model. And the whole Lagrangian [due to Theorem 6] the corresponding action is invariant under (3.18) represents its supersymmetric version.

The same dynamical system can also be described in the \( \mathcal{N} = 1 \) superfield language. Consider the action

\[
S = \frac{i}{2} \int d\theta dt g_{MN}(\mathcal{X}) \dot{\mathcal{X}}^M \mathcal{D}\mathcal{X}^N,
\]

10The symbol \( \int d\theta \) is the Berezin integral,

\[
\int d\theta = \frac{\partial}{\partial \theta}.
\]

11This is not a most general form. The action (3.35) describes in fact only Kähler manifolds; to describe generic complex manifolds, one should add an extra term. But we do not want to plunge into too much details here, addressing an interested reader to Sect. 4 of Ref. [5].
After integration over $d\theta dt$, we obtain the Lagrangian
\[
L = \frac{1}{2}g_{MN} \dot{x}^M \dot{x}^N + \text{terms including superpartners } \psi^M(t),
\]
(3.36)
i.e. $g_{MN}$ has the meaning of the real metric.

By construction, the action (3.35) is invariant under $N = 1$ transformations, but it is also invariant under the extra supersymmetry transformations (3.26) provided the conditions (3.27), (2.8) and the condition $I_{MN} = -I_{NM}$ hold.

Note that, to relate $I_{MN}$ to $I_M^N$, we need the metric. The notion of metric was not used in the proof of Theorem 5, which thus holds also for non-metric manifolds. But we need the metric for the physical applications. And then the condition of the antisymmetry of $I_{MN}$ should be imposed.

Aknowledgements

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References

[1] E. Witten, *Supersymmetry and Morse theory*, J. Diff. Geom. 17 (1982) 661.

[2] G.W. Gibbons, G. Papadopoulos and K.S. Stelle, *HKT and OKT geometries on soliton black hole moduli spaces*, Nucl. Phys. B508 (1997) 623, arXiv:hep-th/9706207.

[3] G. Grantcharov and Y.S. Poon, *Geometry of hyper-Kähler connections with torsion*, Commun. Math. Phys. 213 (2000) 19, arXiv:math/9908015. M. Verbitsky, *Hyperkähler manifolds with torsion, supersymmetry and Hodge theory*, Asian J. Math. 6 (2002) 679, arXiv:math/0112215.

[4] F. Delduc and E. Ivanov, $\mathcal{N} = 4$ mechanics of general $(4, 4, 0)$ multiplets, Nucl. Phys. B855 (2012) 815, arXiv:1104.1429 [hep-th]

[5] S.A. Fedoruk, E.A. Ivanov and A.V. Smilga, *Generic HKT geometries in the harmonic superspace approach*, J. Math. Phys. 59, 083501 (2018), arXiv: 1802.09675 [hep-th].

[6] J. Morrow and K. Kodaira, *Complex manifolds*, AMS Chelsea Publishing, 1971.

[7] A. Newlander and L. Nirenberg, *Complex coordinates in almost complex manifolds*, Ann. of Math. (2) 65 (1957) 391. See also L. Nirenberg, *Lectures on linear partial differential equations*, Amer. Math. Soc., 1973.

[8] F. Cooper, A. Khare and U. Sukhatme, *Supersymmetry and quantum mechanics*, Phys. Rept. 251 (1995) 267.

\footnote{The same applies to the proof of Theorem 3. The equation system (2.5) for the complex coordinates has solutions even when $I_{MN} \neq -I_{NM}$. But the complex metric does not have in this case the Hermitian form (2.4).}
[9] L.I. Nicolaescu, *Notes on Seiberg-Witten theory*, AMS, Providence, 2000, Propositions 1.4.23 and 1.4.25.

[10] E. Ivanov and A. Smilga, *Dirac operator on complex manifolds and supersymmetric quantum mechanics*, Int. J. Mod. Phys. **A27** (2012) 493, arXiv:1012.2069 [hep-th].

[11] R.A. Coles and G. Papadopoulos, *The geometry of the one-dimensional supersymmetric non-linear sigma models*, Class. Quantum Grav. **7** (1990) 427.