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Abstract

Let $\mathcal{F}_n$ be the set of all cuspidal automorphic representations $\pi$ of $GL_n$ with unitary central character over a number field $F$. We prove the first unconditional zero density estimate for the set $\mathcal{S} = \{ L(s, \pi \times \pi') : \pi \in \mathcal{F}_n \}$ of Rankin–Selberg $L$-functions, where $\pi' \in \mathcal{F}_{n'}$ is fixed. We use this density estimate to establish: (i) a hybrid-aspect subconvexity bound at $s = \frac{1}{2}$ for almost all $L(s, \pi \times \pi') \in \mathcal{S}$; (ii) a strong on-average form of effective multiplicity one for almost all $\pi \in \mathcal{F}_n$; and (iii) a positive level of distribution for $L(s, \pi \times \tilde{\pi})$, in the sense of Bombieri–Vinogradov, for each $\pi \in \mathcal{F}_n$.

1. Introduction and statement of the main result

Let $\mathbb{A}_F$ be the ring of adèles over a number field $F$ with absolute norm $N = N_{F/Q}$ and absolute discriminant $D_F$. Let $\mathcal{F}_n$ be the set of cuspidal automorphic representations $\pi = \bigotimes_v \pi_v$ of $GL_n(\mathbb{A}_F)$, where the (restricted) tensor product runs over all places of $F$ and $\pi$ is normalized so that its central character is trivial on the diagonally embedded copy of the positive reals. Let $\sigma_\pi$ be the arithmetic conductor of $\pi$, $C(\pi) \geq 1$ the analytic conductor of $\pi$ (see (3.4)), and $\mathcal{F}_n(Q) = \{ \pi \in \mathcal{F}_n : C(\pi) \leq Q \}$. The analytic conductor $C(\pi)$ is a useful measure for the arithmetic and spectral complexity of $\pi$. Our normalization for the central characters ensures that $|\mathcal{F}_n(Q)|$ is finite.

Given $\pi \in \mathcal{F}_n$ and $\pi' \in \mathcal{F}_{n'}$, let $L(s, \pi \times \pi')$ be the associated Rankin–Selberg $L$-function, and let $\tilde{\pi} \in \mathcal{F}_{n'}$ and $\tilde{\pi}' \in \mathcal{F}_{n''}$ be the contragredient representations. When $\pi' \in \{ \tilde{\pi}, \tilde{\pi}' \}$, work of Brumley [Hum19, Appendix] and the authors [HT22] shows that there exists an effectively computable constant $c_1 = c_1(n, n') > 0$ such that $L(s, \pi \times \pi')$ has a ‘standard’ zero-free region of the shape

$$\text{Re}(s) \geq 1 - \frac{c_1}{\log(C(\pi)C(\pi')(|\text{Im}(s)| + 3)|F:Q|)}$$

(1.1)

apart from at most one real simple zero. This is comparable to the classical zero-free region for Dirichlet $L$-functions. Brumley (see [Bru06a] and [Lap13, Appendix]) established a much narrower zero-free region for all choices of $\pi$ and $\pi'$. The generalized Riemann hypothesis (GRH) asserts that $L(s, \pi \times \pi') \neq 0$ for $\text{Re}(s) > \frac{1}{2}$. Zeros near the line $\text{Re}(s) = 1$ are typically most damaging in applications, but even a zero-free region of the shape $\text{Re}(s) \geq 1 - \delta$ for some constant $\delta = \delta(n, n', [F : Q]) > 0$ would be sufficient to obtain many spectacular arithmetic consequences.

Since such strong zero-free regions for Rankin–Selberg $L$-functions remain out of reach, it is useful to show that zeros near the line $\text{Re}(s) = 1$ must be ‘sparse’. A suitable quantitative formulation can serve as a proxy for a zero-free region of the shape $\text{Re}(s) \geq 1 - \delta$.

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Famous consequences of this philosophy include Hoheisel’s proof [Hoh30] that \( p_{n+1} - p_n \ll p_n^{1-1/33000} \) (where \( p_n \) is the \( n \)th prime) and Linnik’s proof [Lin44] that if \( \gcd(a, q) = 1 \), then there exists an absolute, effectively computable constant \( B > 0 \) and a prime \( p \leq q^B \) such that \( p \equiv a \pmod{q} \).

To quantify our notion of ‘sparse’, we define for \( \sigma \geq 0 \) and \( T \geq 1 \) the quantity

\[
N_{\pi \times \pi'}(\sigma, T) = \{ \{ \rho = \beta + i\gamma : L(\rho, \pi \times \pi') = 0, \ \beta \geq \sigma, \ |\gamma| \leq T \}. \tag{1.2}
\]

Note that \( N_{\pi \times \pi'}(1/2, T) \) is roughly \( T \log(C(\pi)C(\pi')T) \) via the argument principle and the functional equation, and GRH can be restated as \( N_{\pi \times \pi'}(\sigma, T) = 0 \) for all \( \sigma > 1/2 \). The zero density estimate

\[
N_{\pi \times \pi'}(\sigma, T) \ll_{n, n', [F : \mathbb{Q}]} (C(\pi)C(\pi')T_{(F : \mathbb{Q})})^{107(n'n')^4(1-\sigma)}, \tag{1.3}
\]

follows from work of Soundararajan and Thorner [ST19, Corollary 2.6]. Therefore, while an arbitrary Rankin–Selberg \( L \)-function \( L(s, \pi \times \pi') \) is not yet known to have the standard zero-free region (1.1), the bound (1.2) ensures that the number of zeros in the region (1.1) is \( O_{n,n', [F : \mathbb{Q}]}(1) \).

Let \( \mathcal{S} \subseteq \mathbb{F}_n \), and let \( \mathcal{S}(Q) = \{ \pi \in \mathcal{S} : C(\pi) \leq Q \} \). In this article, we seek a strong averaged form of (1.2), namely

\[
\sum_{\pi \in \mathcal{S}(Q)} N_{\pi \times \pi'}(\sigma, T) \ll_{n, n', [F : \mathbb{Q}], \varepsilon} (Q|\mathcal{S}(Q)|C(\pi')T_{(F : \mathbb{Q})})^{A(1-\sigma)+\varepsilon}, \tag{1.3}
\]

where \( A = A(n, n', [F : \mathbb{Q}]) > 0 \) is a constant and \( \varepsilon > 0 \). The bound (1.3) follows from the works of Brunley, Thorner, and Zaman under at least one of the following hypotheses:

\begin{itemize}
  \item \( \pi' \in \mathbb{F}_1 \) is trivial [TZ21, Theorem 1.2];
  \item \( \max\{n, n'\} \leq 4 \) (see [BTZ22, Theorem 1.3]); or
  \item \( \pi' \) and each \( \pi \in \mathcal{S}(Q) \) satisfy certain unproven partial progress towards the generalized Ramanujan conjecture (GRC) [BTZ22, Hypothesis 1.1 and Theorem 1.3].
\end{itemize}

Here, we prove the first completely unconditional zero density estimate of the form (1.3).

**Theorem 1.1.** Let \( n, n' \geq 1 \) and \( \varepsilon > 0 \). Let \( \mathcal{S} \subseteq \mathbb{F}_n \) and \( \mathcal{S}(Q) = \{ \pi \in \mathcal{S} : C(\pi) \leq Q \} \). If \( 0 \leq \sigma \leq 1, \pi' \in \mathbb{F}_{n'}, \) and \( Q, T \geq 1 \), then

\[
\sum_{\pi \in \mathcal{S}(Q)} N_{\pi \times \pi'}(\sigma, T) \ll_{n, n', [F : \mathbb{Q}], \varepsilon} \left(|\mathcal{S}(Q)|^4(C(\pi')QT_{(F : \mathbb{Q})})^{6.15\max\{n^2, n'n'\}}\right)^{1-\sigma+\varepsilon}. \tag{1.4}
\]

Theorem 1.1 is non-trivial when

\[
\delta_\mathcal{S} = \liminf_{Q \to \infty} \frac{\log|\mathcal{S}(Q)|}{\log Q} > 0. \tag{1.4}
\]

This is important because in applications, it is usually convenient to bound \( Q \) by a power of \( |\mathcal{S}(Q)| \) or vice versa. When \( \mathcal{S} = \mathbb{F}_n \), we have the bounds

\[
Q^{n+1} \ll_{n, F} |\mathbb{F}_n(Q)| \ll_{\varepsilon} D_F^{-n^2} Q^{2n+\varepsilon}. \tag{1.5}
\]

The upper bound in (1.5) was proved by Brunley et al. [BTZ22, Theorem A.1]. The lower bound in (1.5) follows from work of Brunley and Milčević [BM18, Theorem 1.1], who computed a constant \( c_{n, F} > 0 \) such that if \( \mathbb{F}_n(Q) \) is the subset of \( \pi \in \mathbb{F}_n(Q) \) that are spherical at the archimedean places of \( F \), then \( |\mathbb{F}_n(Q)| \sim c_{n, F} Q^{n+1} \). (The lower bound in (1.5) reflects the conjectured order

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1. In [BTZ22, ST19], it is assumed that \( F = \mathbb{Q} \). Uniformity over \( F \neq \mathbb{Q} \) requires minor modifications.
2. If \( \theta_n \) in (3.3) satisfies \( \theta_n \leq 1/2 - \delta_n \) for some \( \delta_n > 0 \), then each \( \pi \in \mathbb{F}_n \) satisfies [BTZ22, Hypothesis 1.1].

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of growth; see [BM18]. Together, Theorem 1.1 and (1.5) imply that
\[ \sum_{\pi \in \mathfrak{F}_n(Q)} N_{\pi \times \pi'}(\sigma, T) \ll_{n, n', \varepsilon} (|\mathfrak{F}_n(Q)|C(\pi')^nT^{|F:Q|})^{1.1}\max\{n, n'\}(1-\sigma)+\varepsilon. \] (1.6)

Furthermore, if \( n = n \geq 3 \) and \( \pi' \in \mathfrak{F}_n' \), then Theorem 1.1 and (1.5) together imply that
\[ \sum_{\pi \in \mathfrak{F}_n(Q)} N_{\pi \times \pi'}(\sigma, T) \ll_{\varepsilon} (C(\pi')QT^{|F:Q|})^{\varepsilon_0 n^2(1-\sigma)+\varepsilon}. \] (1.7)

2. Applications

We now describe some applications of Theorem 1.1. In what follows, we write \( f \ll_{\nu} g, f = O_{\nu}(g) \), and \( g \gg_{\nu} f \) to denote that there exists a constant \( c > 0 \) such that \( |f| \leq c|g| \) in the stated range. The implied constant \( c \), which is effectively computable unless otherwise stated, will depend at most on \( \nu, n, n' \), and \([F : Q]\). The expression \( f \asymp_{\nu} g \) means that \( f \ll_{\nu} g \) and \( g \ll_{\nu} f \). We use \( \varepsilon > 0 \) to denote an arbitrarily small quantity that depends at most on \( n, n' \), and \([F : Q]\).

2.1 Bounds for Rankin–Selberg \( L \)-functions

It is a classical problem for Dirichlet \( L \)-functions to find strong bounds on the critical line \( \text{Re}(s) = \frac{1}{2} \). The Phragmén–Lindelöf convexity principle shows that if \( q_\chi \) is the conductor of a primitive Dirichlet character \( \chi \), then \( L\left(\frac{1}{2}, \chi\right) \ll q^{1/4}; \) improving this bound by replacing \( 1/4 \) with a smaller exponent is known as a subconvex bound. The multiplicative version of the classical large sieve inequality combined with an approximate functional equation shows that for almost all \( \chi \), we have the bound \( L\left(\frac{1}{2}, \chi\right) \ll q_\chi^{\varepsilon} \) for all \( \varepsilon > 0 \), consistent with the generalized Lindelöf hypothesis (GLH).

For \( \pi \in \mathfrak{F}_2 \), GLH predicts that \( L\left(\frac{1}{2}, \pi \right) \ll_{\varepsilon} C(\pi)^{\varepsilon} \). Michel and Venkatesh [MV10] proved that there exists a fixed positive \( \delta > 0 \) such that \( L\left(\frac{1}{2}, \pi \right) \ll_{\nu} C(\pi)^{1/4-\delta} \), the culmination of several decades of research. When \( F = Q \), a sharp mean value estimate for Hecke eigenvalues proved by Deshouillers and Iwaniec [DI82], in conjunction with the approximate functional equation, implies the bound \( L\left(\frac{1}{2}, \pi \right) \ll (qT)^{\varepsilon} \) for almost all \( \pi \in \mathfrak{F}_2 \) of arithmetic conductor \( q \), trivial central character, and archimedean complexity \( \text{Laplace eigenvalue or weight squared} \) lying in the dyadic interval \([T, 2T]\). Note that in this case, \( C(\pi) \asymp qT \).

For \( \pi \in \mathfrak{F}_n \) with \( n \geq 3 \), the best uniform result towards the bound \( L\left(\frac{1}{2}, \pi \right) \ll C(\pi)^{\varepsilon} \) predicted by GLH is that of Soundararajan and Thorner [ST19, Corollary 2.7], namely

\[ L\left(\frac{1}{2}, \pi \right) \ll C(\pi)^{1/4}(\log C(\pi))^{-1/10^{17}n^3}. \] (2.1)

We mention three results that improve upon (2.1) in an average sense, each having complementary strengths. Jana [Jan21, Theorem 6] extended the GLH-on-average bound of Deshouillers and Iwaniec to the family of cuspidal automorphic representations of \( \text{GL}_n(\mathbb{A}_Q) \) of arithmetic conductor 1 and growing analytic conductor. Blomer [Blo23, Corollary 5] proved the corresponding result for the family of cuspidal automorphic representations of \( \text{GL}_n(\mathbb{A}_Q) \) of a large given prime arithmetic conductor \( q \), trivial central character, and whose archimedean components are principal series representations confined to a compact subset of the unitary dual. Thorner and Zaman [TZ21, Theorem 1.3] proved that there exists a constant \( c_2 = c_2(n, [F : Q]) > 0 \) such that if \( \varepsilon > 0 \), then

\[ |\{\pi \in \mathfrak{F}_n(Q) : |L\left(\frac{1}{2}, \pi \right)| \geq c_2C(\pi)^{1/4-\varepsilon/10^{16}n^3}\}| \ll_F |\mathfrak{F}_n(Q)|^{\varepsilon}. \] (2.2)
Unlike the preceding results, (2.2) is uniform in both the arithmetic conductor and spectral aspects and holds over number fields other than \( Q \), but the savings over (2.1) is not comparable to GLH on average.

Given \( \pi \in \mathfrak{F}_n \) and \( \pi' \in \mathfrak{F}_{n'} \), Soundararajan and Thorner [ST19, Corollary 2.7] proved when \( F = Q \) that if \( C(\pi \times \pi') \) is the analytic conductor of \( L(s, \pi \times \pi') \), then

\[
L\left( \frac{1}{2}, \pi \times \pi' \right) \ll |L\left( \frac{3}{2}, \pi \times \pi' \right)|^2 C(\pi \times \pi')^{1/4}(\log C(\pi \times \pi'))^{-1/(10^{17}(n'n')^3)}.
\]

As of now, the best general upper bound for \( |L\left( \frac{3}{2}, \pi \times \pi' \right)|^2 \) is larger than any fixed power of \( \log C(\pi \times \pi') \) (see [Li10, Theorem 2]). The factor of \( |L\left( \frac{3}{2}, \pi \times \pi' \right)|^2 \) can be removed under certain partial progress toward GRC. The bound \( L\left( \frac{1}{2}, \pi \times \pi' \right) \ll C(\pi \times \pi')^\varepsilon \) is predicted by GLH.

In order to improve (2.3) on average with uniformity in \( \pi \) and \( \pi' \), one might first try to mimic the approach that worked well for Dirichlet \( L \)-functions and GL2 \( L \)-functions using trace formulae, approximate functional equations, the spectral large sieve, Voronoï summation, etc. While such methods have seen great success for \( \text{GL}_n \times \text{GL}_{n'} \) with \( n, n' \in \{1, 2\} \), suitably uniform and flexible versions of these tools do not appear to be available yet in the general setting. The special case where \( |n - n'| \leq 1 \) exhibits some nice structural properties, lending itself to approaches via period integrals that completely avoids the aforementioned tools. To describe work in this direction, let \( F = Q \), \( \mathcal{F}_n \subseteq \mathfrak{F}_n \) be the subset of cuspidal automorphic representations of \( \text{GL}_n(\mathbb{A}_Q) \) of arithmetic conductor 1, and \( \mathcal{F}_n(Q) = \{ \pi \in \mathcal{F}_n \colon C(\pi) \leq Q \} \). It follows from work of Jana [Jan22, Corollary 2.2] that if \( \varepsilon > 0 \), \( \pi' \in \mathcal{F}_n \), and the spectral parameters of \( \pi' \) have real part at least \(-1/(n^2 + 1)\) (which is far stronger than the best known unconditional lower bound \(-1/2 + 1/(n^2 + 1)\) due to Luo, Rudnick, and Sarnak [LRS99]), then

\[
\sum_{\pi \in \mathcal{F}_n(Q)} \left| L\left( \frac{1}{2}, \pi \times \pi' \right) \right|^2 \ll_{\pi', \varepsilon} |\mathcal{F}_n(Q)|^{1+\varepsilon}.
\]

Therefore, for fixed \( \pi' \in \mathcal{F}_n \), GLH for \( L\left( \frac{1}{2}, \pi \times \pi' \right) \) holds on average over the \( \pi \in \mathcal{F}_n \). Using Chebyshev’s inequality, we conclude that for all \( \delta > 0 \), there exists a constant \( c_{\pi', \delta} > 0 \) such that

\[
|\{ \pi \in \mathcal{F}_n(Q) \colon |L\left( \frac{1}{2}, \pi \times \pi' \right)| \geq c_{\pi', \delta}C(\pi \times \pi')^{2\delta} \}| \ll_{\pi', \delta} |\mathcal{F}_n(Q)|^{1-\delta}.
\]

See also the work of Blommer [Blo12, Theorem 2], which proves a variant of (2.4) for families of ‘spectrally close’ Hecke–Maass newforms on \( \text{SL}_n(\mathbb{Z}) \).

Along the same lines as (2.2), we use (1.6) to prove the following result.

**Theorem 2.1.** Let \( n, n' \geq 1 \) and \( Q \geq 1 \). If \( \varepsilon > 0 \) and \( \pi' \in \mathfrak{F}_{n'} \), then

\[
|\{ \pi \in \mathfrak{F}_n(Q) \colon |L\left( \frac{1}{2}, \pi \times \pi' \right)| \geq C(\pi \times \pi')^{1/4-\varepsilon/(10^{10}\max\{n,n'\})} \}| \ll_{\varepsilon} (C(\pi')^n|\mathfrak{F}_n(Q)|)^{\varepsilon}.
\]

**Remark 2.2.** Given a subset \( \mathcal{S} \subseteq \mathfrak{F}_n \), a similar result can be proved for \( \pi \in \mathcal{S}(Q) \) using Theorem 1.1. Such a result would depend effectively on \( \delta_{\mathcal{S}} \) in (1.4).

In contrast with Jana’s work in (2.5), Theorem 2.1 provides a smaller power-saving improvement over (2.3), but the improvement is uniform in the arithmetic conductor and spectral aspects as well as in \( \pi' \). The exceptional set in Theorem 2.1 is a much smaller than in (2.5). Theorem 2.1 removes the requirements that \( n = n' \), that \( q_{\pi} = q_{\pi'} = 1 \), and that the spectral parameters of \( \pi' \) have real part at least \(-1/(n^2 + 1)\). Finally, Theorem 2.1 is proved over any number field, while [Jan22] is only proved over \( Q \).

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2.2 Effective multiplicity one

Let \( \pi = \bigotimes_v \pi_v \) and \( \pi' = \bigotimes_v \pi'_v \) be cuspidal automorphic representations in \( \mathfrak{F}_n(Q) \). Under the assumption of GRH for \( L(s, \pi \times \pi') \) and \( L(s, \pi \times \pi') \) and that \( \pi_p \) and \( \pi'_p \) are tempered for all prime ideals \( p \mid q, q' \), it is known that if \( Y = (\log Q)^2 \) and \( \pi_p \cong \pi'_p \) for all \( p \mid q, q' \) with \( Np \ll Y \), then \( \pi = \pi' \) (see [IK04, Proposition 5.22]). Brumley [Bru06a], improving on work of Moreno [Mor85], proved that there exists a constant \( B_n > 0 \) such that this result holds unconditionally with \( Y = Q^{B_n} \). This result makes effective the multiplicity one theorems of Jacquet and Shalika [JS81, Theorem 4.8] and Piatetski-Shapiro [Pia79]. Any fixed \( B_n > 2n \) suffices [LW09].

When \( n = 2 \), we have an average result that nearly achieves what GRH predicts. Specifically, let \( \mathfrak{F}_2 \) be the subset of \( \pi \in \mathfrak{F}_2 \) with squarefree conductor and trivial central character, and let \( \pi' \in \mathfrak{F}_2 \). For all \( \varepsilon > 0 \), there exists an effectively computable constant \( N_\varepsilon > 0 \), depending at most on \( \varepsilon \) and \([F : \mathbb{Q}]\), such that

\[
|\{ \pi \in \mathfrak{F}_2(Q) : \pi_p \cong \pi'_p \text{ for all } p \mid q \text{ with } Np \leq (\log Q)^{N_\varepsilon}\}| \ll_\varepsilon Q^\varepsilon. \tag{2.6}
\]

In particular, the implied constant does not depend on \( \pi' \). This was proved by Duke and Kowalski [DK00, Theorem 3] when \( F = \mathbb{Q} \) in a stronger form under the assumption of GRC. See Brumley’s Ph.D. thesis [Bru04, Corollary 5.2.2] for a proof that does not use GRC.

If \( \pi \in \mathfrak{F}_2 \) and \( \pi \in \mathfrak{F}_2 \) is the contragredient, then \( L(s, \pi \times \pi) = \zeta_F(s)L(s, \pi, \text{Ad}) \), where \( \zeta_F(s) \) is the Dedekind zeta function of \( F \) and \( \text{Ad} \) is the adjoint square lift from a representation of \( GL_2(\mathbb{A}_F) \) to a representation of \( GL_3(\mathbb{A}_F) \). The arguments in [Bru04, DK00] rely on two key results as follows.

1. Gelbart and Jacquet [GJ78] proved that if \( \pi \in \mathfrak{F}_2 \), then \( L(s, \pi, \text{Ad}) \) is the \( L \)-function of an automorphic representation of \( GL_3(\mathbb{A}_F) \), denoted \( Ad \pi \), complete with a criterion by which one can determine whether \( Ad \pi \) is cuspidal (and lies in \( \mathfrak{F}_3 \)).

2. At most \( O_\varepsilon(Q^{1/2+\varepsilon}) \) representations \( \pi \in \mathfrak{F}_2(Q) \) have the same adjoint lift. In addition, Ramakrishnan (see [DK00, Appendix] and [Ram00]) proved that \( \text{Ad} : \mathfrak{F}_2 \to \mathfrak{F}_3 \) is injective.

In attempting to generalize the strategy of Duke and Kowalski to \( GL_n \) for \( n \geq 3 \), one encounters some deep open problems. If \( \pi \in \mathfrak{F}_n \), then for \( \text{Re}(s) \) sufficiently large, \( L(s, \pi \times \pi) \) factors as \( \zeta_F(s)L(s, \pi, \text{Ad}) \), where \( \text{Ad} \) is the adjoint square lift from \( GL_n \) to \( GL_{n^2-1} \). Apart from some special cases, the following obstacles arise.

1. The adjoint lift is not yet known to be automorphic for \( n \geq 3 \), and if it were, there is no known criterion for cuspidality.

2. Let \( \mathcal{H}_n \subseteq \mathfrak{F}_n(Q) \) be the subset of \( \pi \in \mathfrak{F}_n(Q) \) such that \( \text{Ad} \pi \in \mathfrak{F}_{n^2-1} \). It is not known how many \( \pi \in \mathcal{H}_n \) have the same adjoint lift.

Despite these setbacks, we can use (1.7) (more specifically, Corollary 7.1) to prove a \( GL_n \) analogue of (2.6) when \( \pi' \) depends mildly on \( Q \).

**Theorem 2.3.** Let \( n \geq 3 \) and \( Q \geq 1 \). There exists an absolute, effectively computable constant \( c_3 > 0 \) such that if \( 0 < \varepsilon < 1 \) and \( \pi' \in \mathfrak{F}_n((\log Q)^{c_3/(n^2[F:Q]^2)}) \), then

\[
|\{ \pi \in \mathfrak{F}_n(Q) : \pi_p \cong \pi_p' \text{ for all } p \mid q, q' \text{ with } Np \leq (\log Q)^{4n^3/\varepsilon}\}| \ll_\varepsilon Q^\varepsilon.
\]

**Remark 2.4.** Given a subset \( S \subseteq \mathfrak{F}_n \), a similar result can be proved for \( \pi \in S(Q) \) using Theorem 1.1. Such a result would depend effectively on \( \delta_S \) in (1.4).

The proof is very flexible. For example, if \( \varepsilon > 0 \) and \( C(\pi') \ll_\varepsilon Q^{c_32^{2}/(n^2[F:Q]^2)} \), then the same proof with minor changes in choices of parameters produces the bound

\[
|\{ \pi \in \mathfrak{F}_n(Q) : \pi_p \cong \pi'_p \text{ for all } p \mid q, q' \text{ with } Np \leq Q^\varepsilon\}| \ll_\varepsilon Q^\varepsilon.
\]

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While the number of $p$ for which one needs to check that $\pi_p \cong \pi'_p$ is larger, the range of $C(\pi')$ is greatly extended, and the threshold $Np \leq Q^\varepsilon$ (reminiscent of Vinogradov’s conjecture on the size of the least quadratic non-residue) still greatly improves on the unconditional range $Np \ll \varepsilon Q^{2n+\varepsilon}$ from [LW09].

2.3 Automorphic level of distribution
Let $\Lambda(m)$ be the von Mangoldt function, equal to $\log p$ if $m$ is a power of a prime $p$ and zero otherwise. The celebrated Bombieri–Vinogradov theorem states that if $\theta < \frac{1}{2}$ is fixed, then for all $A > 0$, we have

$$\sum_{q \leq x^\theta} \max_{\gcd(a,q)=1} \max_{y \leq x} \left| \sum_{m \leq y \atop \gcd(m,q)=1 \atop m \equiv a \pmod{q}} \Lambda(m) - \frac{y}{\varphi(q)} \right| \ll_A \frac{x}{(\log x)^A}. \quad (2.7)$$

This may be viewed as an average form of GRH for Dirichlet $L$-functions. As part of his proof [Bom65], Bombieri proved a strong form of the zero density estimate in Theorem 1.1 for Dirichlet $L$-functions. We call any $\theta$ for which (2.7) holds a level of distribution for the primes. Elliott and Halberstam conjectured that any fixed $\theta < 1$ is a level of distribution for the primes.

Number theorists have proved several interesting extensions and variations of (2.7). For example, Murty and Murty [MM87] proved that primes in the Chebotarev density theorem have $\theta < \frac{1}{2}$ Halberstam conjectured that any fixed $A > 0$, we have

$$\sum_{q \leq x^\theta} \max_{\gcd(a,q)=1} \max_{y \leq x} \left| \sum_{m \leq y \atop \gcd(m,q)=1 \atop m \equiv a \pmod{q}} \Lambda(m) - \frac{y}{\varphi(q)} \right| \ll_A \frac{x}{(\log x)^A}. \quad (2.7)$$

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Number theorists have proved several interesting extensions and variations of (2.7). For example, Murty and Murty [MM87] proved that primes in the Chebotarev density theorem have a positive level of distribution. To describe a different direction for automorphic representations over $\mathbb{Q}$, we let $n \geq 2$ and consider $\pi \in \mathfrak{g}_n$ with conductor $q_\pi$. Let $\Lambda(m)$ be the von Mangoldt function, and define the numbers $a_\pi(m)$ by

$$-\frac{L'}{L}(s, \pi) = \sum_p \sum_{k=1}^\infty \frac{\alpha_{j,\pi}(p^k) \log p}{p^s} = \sum_{n=1}^\infty \frac{a_\pi(m) \Lambda(m)}{m^s}, \quad \text{Re}(s) > 1. \quad (2.8)$$

Note that $a_\pi(p) = \lambda_\pi(p)$. For fixed $\theta < 1/(n^2 - 2)$, Wong [Won20, Theorem 9] proved that if $\pi$ satisfies GRC and $L(s, \pi \times (\tilde{\pi} \otimes \chi))$ has no Landau–Siegel zero for all Dirichlet characters $\chi$, then for any $A > 0$,

$$\sum_{q \leq x^\theta} \max_{\gcd(a,q)=1} \max_{y \leq x} \left| \sum_{m \leq y \atop \gcd(m,q_\pi)=1 \atop m \equiv a \pmod{q}} |a_\pi(m)|^2 \Lambda(m) - \frac{y}{\varphi(q)} \right| \ll_{A,\pi} \frac{x}{(\log x)^A}. \quad (2.8)$$

This conditionally endows $L(s, \pi \times \tilde{\pi})$ with a positive level of distribution $\theta$. The hypotheses for (2.8) hold for $\pi$ attached to non-CM holomorphic cuspidal newforms on congruence subgroups of $\text{SL}_2(\mathbb{Z})$.

Let $\pi \in \mathfrak{g}_n$. Using (1.7), we unconditionally endow $L(s, \pi \times \tilde{\pi})$ with a notion of positive level of distribution. In particular, we avoid recourse to unproven progress toward GRC or the absence of Landau–Siegel zeros.

**Theorem 2.5.** Let $F = \mathbb{Q}$ and $\pi \in \mathfrak{g}_n$. Fix $\theta < 1/(9n^3)$. If $A > 0$, then

$$\sum_{q \leq x^\theta} \max_{\gcd(a,q)=1} \max_{y \leq x} \left| \sum_{m \leq y \atop \gcd(m,q_\pi)=1 \atop m \equiv a \pmod{q}} |a_\pi(m)|^2 \Lambda(m) - \frac{y}{\varphi(q)} \right| \ll_{A,\pi} \frac{x}{(\log x)^A}. \quad (2.8)$$

The implied constants are ineffective.

**Remark 2.6.** One can prove an analogue of Theorem 2.5 with $F \neq \mathbb{Q}$, replacing residue classes modulo $q$ with ray classes modulo $q$. We restrict to $F = \mathbb{Q}$ for notational simplicity.
Overview of the paper

In §3, we recall basic properties of standard $L$-functions and Rankin–Selberg $L$-functions that we use in our proofs. In §4, we prove a large sieve inequality for the Dirichlet coefficients of $L(s, π \times π')^{-1}$ and a corollary on mean values of Dirichlet polynomials, which we use in our proof of Theorem 1.1 in §5. We then prove Theorem 2.1 in §6, Theorem 2.3 in §7, and Theorem 2.5 in §8.

3. Properties of $L$-functions

We recall some standard facts about $L$-functions arising from automorphic representations and their Rankin–Selberg convolutions. See [Bru06a, GJ72, JPS83, MW89, ST19].

3.1 Standard $L$-functions

Given $π \in \mathcal{F}_n$, let $\bar{π} \in \mathcal{F}_n$ be the contragredient representation and $q_π$ be the conductor of $π$. We express $π$ as a restricted tensor product $\otimes_v \pi_v$ of smooth admissible representations of $\text{GL}_n(F_v)$, where $v$ varies over places of $F$. When $v$ is a non-archimedean place corresponding with a prime ideal $p$, then the local $L$-function $L(s, π_p)$ is defined in terms of the Satake parameters $A_π(p) = \{α_{1,π}(p), \ldots, α_{n,π}(p)\}$ by

$$L(s, π_p) = \prod_{j=1}^{n} (1 - α_{j,π}(p)Np^{-s})^{-1} = \sum_{k=0}^{∞} \frac{λ_π(p^k)}{Np^{ks}}.$$  \hspace{1cm} (3.1)

We have $\alpha_{j,π}(p) \neq 0$ for all $j$ whenever $p \mid q_π$, and when $p \mid q_π$, it might be the case that there exist $j$ such that $α_{j,π}(p) = 0$. The standard $L$-function $L(s, π)$ associated to $π$ is of the form

$$L(s, π) = \prod_p L(s, π_p) = \sum_n \frac{λ_π(n)}{Nn^s}.$$ 

The Euler product and Dirichlet series converge absolutely when $\text{Re}(s) > 1$.

At each archimedean place $v$ of $F$, there are $n$ Langlands parameters $μ_{j,π}(v) \in \mathbb{C}$ such that

$$L(s, π_∞) = \prod_{v \mid ∞} \prod_{j=1}^{n} \Gamma_v(s + μ_{j,π}(v)), \quad Γ_v(s) := \begin{cases} \pi^{-s/2}Γ(s/2) & \text{if } F_v = \mathbb{R}, \\ 2(2π)^{-s}Γ(s) & \text{if } F_v = \mathbb{C}. \end{cases}$$

By combining the work in [BB11, BB13, LRS99, MS04], we know that there exists

$$0 ≤ \theta_n ≤ \begin{cases} 0 & \text{if } n = 1, \\ 7/64 & \text{if } n = 2, \\ 5/14 & \text{if } n = 3, \\ 9/22 & \text{if } n = 4, \\ 1/2 - 1/(n^2 + 1) & \text{if } n ≥ 5, \end{cases} \hspace{1cm} (3.2)$$

such that

$$|α_{j,π}(p)| ≤ Np^{θ_n} \quad \text{and} \quad \text{Re}(μ_{j,π}(v)) ≥ -θ_n.$$ \hspace{1cm} (3.3)

GRC asserts that in (3.2), one may take $θ_n = 0$. We have $q_π = q_{\bar{π}}$, and for each $p$ and each $v$, we have the equalities of sets $\{α_{j,π}(p)\} = \{α_{j,\bar{π}}(p)\}$ and $\{μ_{j,π}(v)\} = \{μ_{j,\bar{π}}(v)\}$.

Let $r_π$ be the order of the pole of $L(s, π)$ at $s = 1$. The completed $L$-function

$$Λ(s, π) = (s(s - 1))^{r_π} (D_F^h Nq_π)^{s/2} L(s, π)L(s, π_∞)$$
is entire of order 1, and there exists a complex number \( W(\pi) \) of modulus 1 such that for all \( s \in \mathbb{C} \), we have the functional equation \( \Lambda(s, \pi) = W(\pi)\Lambda(1 - s, \pi) \). Let \( d(v) = 1 \) if \( F_v = \mathbb{R} \) and \( d(v) = 2 \) if \( F_v = \mathbb{C} \). The analytic conductor of \( \pi \) (see [IS00]) is given by

\[
C(\pi, t) := D_F^n Nq_{\pi} \prod_{v \mid \infty} \prod_{j=1}^n (3 + |it + \mu_{j, \pi}(v)|^{d(v)}), \quad C(\pi) := C(\pi, 0). \tag{3.4}
\]

Since \( \Lambda(s, \pi) \) is entire of order 1, there exist complex numbers \( a_\pi \) and \( b_\pi \) such that

\[
\Lambda(s, \pi) = e^{a_\pi + b_\pi s} \prod_{\Lambda(\rho, \pi) = 0} \left( 1 - \frac{s}{\rho} \right)e^{s/\rho}.
\]

The zeros \( \rho \) in the above Hadamard product are the non-trivial zeros of \( L(s, \pi) \), and the zeros of \( L(s, \pi) \) that arise as poles of \( s^n L(s, \pi_{\infty}) \) are the trivial zeros.

### 3.2 Rankin–Selberg \( L \)-functions

Let \( \pi \in \mathfrak{F}_n \) and \( \pi' \in \mathfrak{F}_{n'} \). At each prime ideal \( p \), Jacquet et al. [JPS83] associate to \( \pi_p \) and \( \pi'_p \) a local Rankin–Selberg \( L \)-function

\[
L(s, \pi_p \times \pi'_p) = \prod_{j=1}^n \prod_{j'=1}^{n'} (1 - \alpha_{j, j', \pi_p \times \pi'_p}(p))Np^{-s} = \sum_{k=0}^{\infty} \frac{\lambda_{\pi_p \times \pi'}(p^k)}{Np^{ks}} \tag{3.5}
\]

and a local conductor \( q_{\pi_p \times \pi'_p} \). If \( p \nmid q_{\pi_p \times \pi'_p} \), then we have the equality of sets

\[
\{\alpha_{j, j', \pi_p \times \pi'_p}(p)\} = \{\alpha_j, \pi_p \alpha_{j', \pi'_p}(p)\}. \tag{3.6}
\]

The Rankin–Selberg \( L \)-function \( L(s, \pi \times \pi') \) associated to \( \pi \) and \( \pi' \) and its arithmetic conductor are

\[
L(s, \pi \times \pi') = \prod_{p} L(s, \pi_p \times \pi'_p) = \sum_n \frac{\lambda_{\pi \times \pi'}(n)}{Nn^s}, \quad q_{\pi \times \pi'} = \prod_p q_{\pi_p \times \pi'_p}.
\]

At an archimedean place \( v \) of \( F \), Jacquet, Piatetski-Shapiro, and Shalika associate \( n \) complex Langlands parameters \( \mu_{j, j', \pi \times \pi'}(v) \) to \( \pi_v \) and \( \pi'_v \), from which one defines

\[
L(s, \pi_{\infty} \times \pi'_{\infty}) = \prod_{v \mid \infty} \prod_{j=1}^n \prod_{j'=1}^{n'} \Gamma_v(s + \mu_{j, j', \pi \times \pi'}(v)).
\]

Using the explicit descriptions of \( \alpha_{j, j', \pi \times \pi'}(p) \) and \( \mu_{j, j', \pi \times \pi'}(v) \) in [Hum19, ST19], one sees that

\[
|\alpha_{j, j', \pi \times \pi'}(p)| \leq Np^{\theta_n + \theta_{n'}}, \quad \text{Re}(\mu_{j, j', \pi \times \pi'}(v)) \geq -\theta_n - \theta_{n'} \tag{3.7}
\]

Let \( r_{\pi \times \pi'} = -\text{ord}_s L(s, \pi \times \pi') \). By our normalization for the central characters of \( \pi \) and \( \pi' \), we have that \( r_{\pi \times \pi'} = 0 \) if and only if \( \pi \neq \pi' \) and \( r_{\pi \times \pi'} = 1 \) otherwise. The function

\[
\Lambda(s, \pi \times \pi') = (s - 1)^{r_{\pi \times \pi'}}(D_F^n Nq_{\pi \times \pi'}^{s/2})L(s, \pi \times \pi')L(s, \pi_{\infty} \times \pi'_{\infty}) \tag{3.8}
\]

is entire of order 1, and there exists a complex number \( W(\pi \times \pi') \) of modulus 1 such that \( \Lambda(s, \pi \times \pi') \) satisfies the functional equation \( \Lambda(s, \pi \times \pi') = W(\pi \times \pi')\Lambda(1 - s, \overline{\pi} \times \overline{\pi'}) \). As with \( L(s, \pi) \), the analytic conductor of \( L(s, \pi \times \pi') \) is given by

\[
C(\pi \times \pi', t) := D_F^n Nq_{\pi \times \pi'} \prod_{v \mid \infty} \prod_{j=1}^n \prod_{j'=1}^{n'} (3 + |it + \mu_{j, j', \pi \times \pi'}(v)|^{d(v)}), \quad C(\pi \times \pi') := C(0, \pi \times \pi'). \tag{3.9}
\]
Zeros of Rankin–Selberg $L$-functions in families

The combined work of Bushnell and Henniart [BH97] and Brumley [Hum19, Appendix] yields

$$C(\pi \times \pi', t) \ll C(\pi \times \pi')(3 + |t|)[F:Q]^{n'/n}, \quad C(\pi \times \pi') \ll C(\pi)^{n'/2} C(\pi)^n. \quad (3.10)$$

Since $\Lambda(s, \pi \times \pi')$ is entire of order 1, there exist complex numbers $a_{\pi \times \pi'}$ and $b_{\pi \times \pi'}$ such that

$$\Lambda(s, \pi \times \pi') = e^{a_{\pi \times \pi'}} + b_{\pi \times \pi'}s \prod_{\Lambda(\rho, \pi \times \pi') = 0} \left(1 - \frac{s}{\rho}\right)e^{s/\rho} \quad (3.11)$$

holds. The zeros $\rho$ in (3.11) are the non-trivial zeros of $L(s, \pi \times \pi')$, and the zeros of $L(s, \pi \times \pi')$ that arise as poles of $s^r \times s' L(s, \pi_\infty \times \pi'_\infty)$ are the trivial zeros.

It follows from work of Li [Li10, Theorem 2] (with minor adjustments when $F \neq \mathbb{Q}$) that there exists an absolute and effectively computable constant $c_4 > 0$, which we assume to be sufficiently large for future convenience, such that

$$\lim_{\sigma_0 \to \sigma} (\sigma_0 - 1)^{r_{\pi \times \pi'}} L(\sigma_0, \pi \times \pi') \ll \exp\left(c_4 n'[F:Q] \frac{\log C(\pi \times \pi')}{\log \log C(\pi \times \pi')}\right), \quad \sigma \in [1, 3]. \quad (3.12)$$

We can change $\pi'$ to $\pi' \otimes |det|^{it}$; at the archimedean places, this has the effect of adding $it$ to each $\mu_{j, j', \pi \times \pi'}(v)$. We then apply functional equation, the Phragmén–Lindelöf convexity principle, and (3.10) to obtain for all $\sigma \geq 0$

$$\lim_{\sigma_0 \to \sigma} \left(\frac{\sigma_0 + it - 1}{\sigma_0 + it + 1}\right)^{r_{\pi \times \pi'}} L(\sigma_0 + it, \pi \times \pi') \ll \varepsilon C(\pi \times \pi', t)^{\max\{1-\sigma, 0\}/2 + \varepsilon/n'[F:Q]}$$

$$\ll \varepsilon (C(\pi)^{n'} C(\pi')^{n}(3 + |t|)[F:Q]^{n'/2})^{\max\{1-\sigma, 0\}/2 + \varepsilon/n'[F:Q]}. \quad (3.13)$$

**Lemma 3.1.** If $\pi \in \mathfrak{f}_n$, $X \geq 3$, and $\varepsilon > 0$, then

$$\sum_{Nn \leq X} \frac{\lambda_{\pi \times \pi'}(n)}{Nn} \ll \varepsilon C(\pi)^{\varepsilon} \log X.$$

**Proof.** Since $\lambda_{\pi \times \pi'}(n) \geq 0$ for all $n$ by [HR95, Lemma a], we observe by (3.12) that

$$\sum_{Nn \leq X} \frac{\lambda_{\pi \times \pi'}(n)}{Nn} \leq \varepsilon \sum_{n} \frac{\lambda_{\pi \times \pi'}(n)}{Nn^{1+1/\log X}} \ll (\log X) \lim_{s \to 1} L(s, \pi \times \pi').$$

The desired bounded now follows from (3.13) with $\sigma = 1, t = 0$, and $\pi' = \pi$. \qed

**Lemma 3.2.** Let $J \geq 1$ be an integer. For all $j \in \{1, \ldots, J\}$, let $t_j \in \mathbb{R}; n_j, n_j' \geq 0$ be positive integers; and $\pi_j \in \mathfrak{f}_{n_j}$ and $\pi'_j \in \mathfrak{f}_{n_j'}$. Let

$$D(s) = \prod_{j=1}^{J} L(s + it_j, \pi_j \times \pi_j'), \quad \Delta(s) = \prod_{j=1}^{J} \Lambda(s + it_j, \pi_j \times \pi_j'), \quad Q = \prod_{j=1}^{J} C(\pi_j)^{n_j'} C(\pi_j')^{n_j}. \quad \text{(3.14)}$$

Let $R = -\operatorname{ord}_{s=1} D(s)$. If $1 < \sigma < 2$ and the $n$th Dirichlet coefficient of $-(D'/D)(s)$ is non-negative when $\gcd(n, \prod_{j=1}^{J} q_{\pi_j, \pi_j'}) = \mathcal{O}$, then

$$\sum_{\Delta(\rho) = 0} \Re \left(\frac{1}{\sigma - \rho}\right) \leq \frac{R}{\sigma - 1} + \sum_{1 \leq j \leq J, \pi_j = \pi_j', t_j \neq 0} \frac{\sigma - 1}{(\sigma - 1)^2 + t_j^2} + \mathcal{O}(\log Q).$$
By (3.3), (3.7), and (3.10), the bound \( \text{Re}((\ell_\mu A) \cdot \text{Stirling's formula}) \) have
ated to
where the sum ranges over all partitions. This yields
\[
\text{Re}((\ell_\mu A) \cdot \text{Stirling's formula}) \leq \sum_{\Delta(\rho) = 0} \left( \frac{1}{\sigma - \rho} + \frac{1}{\rho} \right) + b_D
\]
\[
= \frac{D'}{D}(\sigma) + \frac{\log Nq_D}{2} + \frac{R}{\sigma - 1} + \sum_{1 \leq j \leq J} \left( \frac{1}{\sigma + it_j - 1} + \frac{1}{\sigma + it_j} \right) + \frac{D'}{D}(\sigma).
\]
Since \( \text{Re}(b_D) = -\sum_{\Delta(\rho) = 0} \text{Re}(\rho^{-1}) \) (see [IK04, Proposition 5.7(3)]), we take real parts and obtain
\[
\sum_{\Delta(\rho) = 0} \text{Re} \left( \frac{1}{\sigma - \rho} \right) = \text{Re} \left( \frac{D'}{D}(\sigma) + \frac{\log Nq_D}{2} + \frac{R}{\sigma - 1} + \sum_{1 \leq j \leq J} \frac{\sigma - 1}{(\sigma - 1)^2 + t_j^2} + \frac{D'}{D}(\sigma)\right) + O(1).
\]
By (3.3), (3.7), and (3.10), the bound \( \text{Re}((D'/D)(\sigma)) \ll \log Q \) (respectively, \( (D'_\infty/D_\infty)(\sigma) \ll \log Q \)) follows from our hypothesis on the Dirichlet coefficients of \(- (D'/D)(s)\) (respectively, Stirling’s formula).

### 3.3 Rankin–Selberg combinatorics

A partition \( \mu = (\mu_i)_{i=1}^\infty \) is a sequence of non-increasing nonnegative integers \( \mu_1 \geq \mu_2 \geq \cdots \) with only finitely many non-zero entries. For a partition \( \mu \), let \( \ell(\mu) \) be the number of non-zero \( \mu_i \), and let \( |\mu| = \sum_{i=1}^\infty \mu_i \). For a set \( \{\alpha_1, \ldots, \alpha_n\} \) of real numbers and a partition \( \mu \) with \( \ell(\mu) \leq n \), let \( s_\mu(\{\alpha_1, \ldots, \alpha_n\}) \) be the Schur polynomial \( \det[\alpha_{ij}] \) associated to \( \mu \). If \( |\mu| = 0 \), then \( s_\mu(\{\alpha_1, \ldots, \alpha_n\}) \) is identically one. By convention, if \( \ell(\mu) > n \), then \( s_\mu(\{\alpha_1, \ldots, \alpha_n\}) \) is identically zero.

Let \( \pi \in \mathfrak{S}_n \) and \( \pi' \in \mathfrak{S}_{n'} \). By (3.5), (3.6), and Cauchy’s identity [Bum13, Theorem 38.1], we have
\[
\sum_{k=0}^\infty \frac{\lambda_{\pi \times \pi'}(p^k)}{Np^{k|\mu|}} = L(s, \pi_p \times \pi'_p) = \sum_{\mu} s_\mu(A_\pi(p)) s_\mu(A_{\pi'}(p)) \frac{N^{s|\mu|}}{Np^{s|\mu|}}, \quad p \nmid q \pi q_{\pi'},
\]
where the sum ranges over all partitions. This yields
\[
\lambda_{\pi \times \pi'}(p^k) = \sum_{|\mu| = k} s_\mu(A_\pi(p)) s_\mu(A_{\pi'}(p)), \quad p \nmid q \pi q_{\pi'}.
\]
For an integral ideal \( n \) with factorization \( n = \prod_p p^{\text{ord}_p(n)} \) (with \( \text{ord}_p(n) = 0 \) for all but finitely many \( p \)), the multiplicity of \( \lambda_{\pi \times \pi'}(n) \) tells us that if \( \gcd(n, q \pi q_{\pi'}) = \mathcal{O}_P \), then
\[
\lambda_{\pi \times \pi'}(n) = \prod_p \lambda_{\pi \times \pi'}(p^{\text{ord}_p(n)}) = \sum_{\langle \mu_p \rangle \in \mathbb{Z}[n]} \prod_p s_{\mu_p}(A_\pi(p)) s_{\mu_p}(A_{\pi'}(p)), \quad (3.14)
\]

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where \((\mu_p)_p\) denotes a sequence of partitions indexed by prime ideals and

\[\mu[n] := \{(\mu_p)_p : |\mu_p| = \text{ord}_p(n) \text{ for all } p\}.\]  

(3.15)

Define the numbers \(\mu_{\pi\times\pi'}(n)\) on unramified prime powers by

\[
\sum_{k=0}^{\infty} \mu_{\pi\times\pi'}(p^k) \frac{N^k}{p^k s} = L(s, \pi \times \pi')^{-1} = \prod_{j=1}^{n'} \prod_{j' = 1}^{n} (1 - \alpha_{j,\pi'}(p)\alpha_{j',\pi}(p)Np^{-s}), \quad p \nmid q_{\pi}\pi'.
\]

By multiplicativity, this defines \(\mu_{\pi\times\pi'}(n)\) when \(\gcd(n, q_{\pi}\pi') = \mathcal{O}_F\). For a partition \(\mu = (\mu_i)_{i=1}^{\infty}\), let \(\mu^* = (\mu_i^*)_{i=1}^{\infty}\) be the dual partition defined by \(\mu_i^* = |\{j : \mu_j \geq i\}|.\) It follows from the dual Cauchy identity [Bun13, Chapter 38] and (3.6) that

\[
\sum_{k=0}^{\infty} \mu_{\pi\times\pi'}(p^k) \frac{N^k}{p^k s} = \sum_{\mu} s_{\mu}(A_{\pi}(p))s_{\mu^*}(-A_{\pi'}(p)) \frac{N_p^{s|\mu|}}{p^{|\mu| s}}, \quad p \nmid q_{\pi}\pi',
\]

where \(-A_{\pi'}(p) = \{-\alpha_{1,\pi'}(p), \ldots, -\alpha_{n,\pi'}(p)\}.\) Hence, we have

\[
\mu_{\pi\times\pi'}(n) = \sum_{(\mu_p)_p \in [\mu][n]} \prod_p s_{\mu_p}(A_{\pi}(p))s_{\mu_p^*}(-A_{\pi'}(p)), \quad \gcd(n, q_{\pi}\pi') = \mathcal{O}_F.
\]

(3.16)

**Lemma 3.3.** If \(\gcd(n, q_{\pi}\pi') = \mathcal{O}_F\), then we have \(|\mu_{\pi\times\pi'}(n)| \leq \frac{1}{2}(\lambda_{\pi\times\pi'}(n) + \lambda_{\pi'\times\pi'}(n)).\)

**Proof.** We apply the inequality of arithmetic and geometric means to (3.16):

\[
|\mu_{\pi\times\pi'}(n)| \leq \frac{1}{2} \left( \sum_{(\mu_p)_p \in [\mu][n]} \left| \prod_p s_{\mu_p}(A_{\pi}(p)) \right|^2 + \sum_{(\mu_p)_p \in [\mu][n]} \left| \prod_p s_{\mu_p^*}(-A_{\pi'}(p)) \right|^2 \right).
\]

The first sum equals \(\lambda_{\pi\times\pi'}(n)\) by (3.14). For the second sum, note that since \(|\mu| = |\mu^*|\), we have \((\mu_p)_p \in [\mu][n]\) if and only if \((\mu_p^*)_p \in [\mu][n]\). Hence, by rearranging, we see that

\[
\sum_{(\mu_p)_p \in [\mu][n]} \left| \prod_p s_{\mu_p^*}(-A_{\pi'}(p)) \right|^2 = \sum_{(\mu_p)_p \in [\mu][n]} \left| \prod_p s_{\mu_p}(-A_{\pi'}(p)) \right|^2 = \lambda_{\pi'\times\pi'}(n),
\]

(3.17)

where the last equality holds because \(\alpha_{j,\pi'}(p)\alpha_{j',\pi}(p) = (-\alpha_{j,\pi'}(p))(-\alpha_{j',\pi}(p))\).

\[\square\]

### 4. A new mean value estimate

Our proof of Theorem 1.1 uses the following new mean value estimate for the Dirichlet coefficients of \(L(s, \pi \times \pi')\) and \(L(s, \pi \times \pi')^{-1}\). Let \(S \subseteq \mathfrak{F}_n\), and let \(S(Q) = \{\pi \in S : C(\pi) \leq Q\}\).

**Theorem 4.1.** Let \(b\) be a complex-valued function supported on the integral ideals of \(\mathcal{O}_F\). Let \(n, n' \geq 1\), and let \(\pi' \in \mathfrak{F}_{n'}\). Let \(Q, T \geq 1\), \(\varepsilon > 0\), and \(x \geq 1\). Both

\[
\sum_{\pi \in S(Q)} \sum_{\gcd(nq_{\pi}\pi') = \mathcal{O}_F} \mu_{\pi\times\pi'}(n)b(n)^2 \quad \text{and} \quad \sum_{\pi \in S(Q)} \sum_{\gcd(nq_{\pi}\pi') = \mathcal{O}_F} \lambda_{\pi\times\pi'}(n)b(n)^2
\]

are

\[
\ll_{\varepsilon} Q^\varepsilon \left( \frac{x}{T} + Q^{4n^2\theta_n + nT^{1/2}n[F:Q] + \varepsilon |S(Q)|} \right) \sum_{\gcd(nq_{\pi}\pi') = \mathcal{O}_F} \lambda_{\pi'\times\pi'}(n)b(n)^2,
\]

where \(\theta_n \in [0, \frac{1}{2} - 1/(n^2 + 1)]\) is the best known exponent towards GRC for \(\pi \in S\).

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Remark 4.2. If \( \pi' = 1 \) and \( S = \mathfrak{f}_n \), then Theorem 4.1 recovers [TZ21, Theorem 1.1]:

\[
\sum_{\pi \in \mathfrak{f}_n(Q)} \left| \sum_{\substack{\lambda \in \mathfrak{f}_n \cap \mathfrak{f}_n^{-1} \cap \gcd(n, q_{\pi'}) = 0 \mathcal{O}_F}} \lambda_{\pi'}(n) b(n) \right|^2 \ll_{\epsilon} Q^{\epsilon} \left( \frac{x}{T} + Q^{4n^2\theta_n + nT(1/2)n^2[F:Q] + \epsilon} \right) \sum_{\substack{n \in (x,x+1/T) \cap \gcd(n, q_{\pi'}) = 0 \mathcal{O}_F}} \left| b(n) \right|^2.
\]

One could try to prove Theorem 4.1 starting with this, replacing \( b(n) \) with \( \lambda_{\pi'}(n) \mu_F(n) b(n) \) (where \( \mu_F(n) \) is the \( n \)th Dirichlet coefficient of \( \zeta_F(s)^{-1} \)), and trying to recover a version of Theorem 4.1 with \( \mu_{\pi \times \pi'}(n) \) replaced by \( \lambda_{\pi}(n) \lambda_{\pi'}(n) \mu_F(n) \). Note that \( \mu_{\pi \times \pi'}(n) = \lambda_{\pi}(n) \lambda_{\pi'}(n) \mu_F(n) \) when \( n \) is squarefree and coprime to \( q_{\pi} q_{\pi'} \). Otherwise, equality is not guaranteed. If one wants to approximate \( L(s, \pi \times \pi') \) with \( \sum_{\pi \text{ squarefree}} \lambda_{\pi}(n) \lambda_{\pi'}(n) N \pi^{-s} \) and extend into the critical strip when \( \pi, \pi' \in \mathfrak{f}_n \) and \( n \geq 5 \), then one must have progress towards GRC well beyond what is known unconditionally [Bru06b, DK00]. Such progress would then be a hypothesis for Theorem 1.1.

Theorem 4.1 provides the first non-trivial unconditional mean value estimates of large sieve type for the Dirichlet coefficients \( \lambda_{\pi \times \pi'}(n) \) or \( \mu_{\pi \times \pi'}(n) \) for arbitrary \( n \) and \( n' \). Theorem 4.1 follows from a more general result, Proposition 4.3, for sequences of products of Schur polynomials evaluated on the set \( A_{\pi}(p) \) of Satake parameters of \( \pi \) at \( p \). We begin with a mean value estimate for the Satake parameters of \( \pi \) as \( \pi \in S \) varies. Let

\[ a : \bigcup_{x < n \leq x + 1/T} \mathbb{M}[n] \rightarrow \mathbb{C}, \quad \alpha : \mathcal{S}(Q) \rightarrow \mathbb{C} \]

be functions that are not identically zero. Their \( \ell^2 \) norms \( \| a \|_2 \) and \( \| \alpha \|_2 \) are defined by

\[ \| a \|_2 = \left( \sum_{x < n \leq x + 1/T} \left( \sum_{\mu \mid \mu \in \mathbb{M}[n]} |a((\mu)p)|^2 \right)^{1/2}, \quad \| \alpha \|_2 = \left( \sum_{\pi \in \mathcal{S}(Q)} |\alpha(\pi)|^2 \right)^{1/2}. \]

For convenience, we define \( 1_{(n,q)} \) to equal one when \( \gcd(n, q) = \mathcal{O}_F \) and zero otherwise.

**Proposition 4.3.** Let \( x \geq 1 \) and \( Q, T \geq 1 \). Define

\[ C(Q, T, x) := \sup_{\| a \|_2 = 1} \frac{1}{\| a \|_2^2} \sum_{\pi \in \mathcal{S}(Q)} \left( \sum_{x < n \leq x + 1/T} \left( \sum_{\mu \mid \mu \in \mathbb{M}[n]} \left[ \prod_p \mathfrak{s}_{\mu_p}(A_{\pi}(p)) \right] a((\mu)p) \right)^2. \]

We have the bound \( C(Q, T, x) \ll_{\epsilon} Q^{\epsilon} (T^{-1} x + Q^{4n^2\theta_n + nTn^2[F:Q] + \epsilon} |\mathcal{S}(Q)|) \).

**Proof.** We observe that

\[ C(Q, T, x) = \sup_{\| a \|_2 = 1} \sum_{\pi \in \mathcal{S}(Q)} \left( \sum_{x < n \leq x + 1/T} \left( \sum_{\mu \mid \mu \in \mathbb{M}[n]} \left[ \prod_p \mathfrak{s}_{\mu_p}(A_{\pi}(p)) \right] 1_{(n,q)} a((\mu)p) \right)^2. \]

By the duality principle for bilinear forms, (4.2) is bounded by the supremum over the functions \( \alpha : \mathcal{S}(Q) \rightarrow \mathbb{C} \) such that \( \| \alpha \|_2 = 1 \) of

\[ \sum_{x < n \leq x + 1/T} \left( \sum_{\mu \mid \mu \in \mathbb{M}[n]} \left[ \prod_p \mathfrak{s}_{\mu_p}(A_{\pi}(p)) \right] 1_{(n,q)} \alpha(\pi) \right)^2. \]

\[ \]

\[ \]

\[ ^{3} \text{The factor } Q^{4n^2\theta_n + n} \text{ fixes a minor error in the proof of [TZ21, Theorem 1.1], which led to a factor of } Q^{n^2} \text{ instead.} \]
Let $\phi$ be a fixed smooth test function, supported in a compact subset of $[-2, 2]$, such that $\phi(t) = 1$ for $t \in [0, 1]$ and $\phi(t) \in [0, 1)$ otherwise. Then (4.3) is at most

$$\sum_n \sum_{\langle \mu_p \rangle \in \mathcal{U}[n]} \sum_{\pi \in \mathcal{S}(Q)} \left| \prod_p s_{\mu_p}(A_{\pi}(p)) \right| \sum_{\pi \in \mathcal{S}(Q)} \left| \prod_p s_{\mu_p}(A_{\pi}(p)) \right|^2 \phi \left( T \log \frac{Nn}{x} \right).$$

We expand the square, interchange the order of summation, and apply (3.14) to find that (4.4) equals

$$\sum_{\pi, \pi' \in \mathcal{S}(Q)} \alpha(\pi) \overline{\alpha(\pi')} \sum_n \sum_{\langle \mu_p \rangle \in \mathcal{U}[n]} \left[ \prod_p s_{\mu_p}(A_{\pi}(p)) \right] \left[ \prod_p s_{\mu_p}(A_{\pi'}(p)) \right] \phi \left( T \log \frac{Nn}{x} \right).$$

Let $\kappa_{\pi, \pi'} = \text{Res}_{s=1} L(s, \pi \times \pi') \prod_{q \mid q \neq q_{\pi, \pi'}} L(s, \pi_p \times \pi'_p)^{-1}$. Note that $\kappa_{\pi, \pi'} \geq 0$, with equality if and only if $\pi' \neq \pi$. Since $|\alpha(j, j', \pi \times \pi'(p))| \leq Np$, we have the bound

$$\prod_{p \mid q \neq q_{\pi, \pi'}} |L(1, \pi_p \times \pi'_p)|^{-1} = \prod_{p \mid q \neq q_{\pi, \pi'}} \prod_{j=1}^{n} \prod_{j'=1}^{n'} |1 - \frac{\alpha(j, j', \pi \times \pi'(p))}{Np}| \leq \prod_{p \mid q \neq q_{\pi, \pi'}} 2^{n/n'}.$$

Since $|\{p \mid p \mid n\}| \ll (\log Nn)/\log \log Nn$ (see [Wei83, Lemma 1.13b]), it follows from (3.12) that

$$\kappa_{\pi, \pi'} \ll_c C(\pi)^c.$$

Let $\hat{\phi}(s) = \int_{\mathbb{R}} \phi(t) e^{st} dt$. It follows from a standard contour integral calculation using (3.7) and (3.13) that (4.5) equals

$$\sum_{\pi, \pi' \in \mathcal{S}(Q)} \alpha(\pi) \overline{\alpha(\pi')} \left( \frac{1}{2\pi i T} \int_{3-i\infty}^{3+i\infty} \frac{L(s, \pi \times \pi')}{\prod_{q \mid q \neq q_{\pi, \pi'}} L(s, \pi_p \times \pi'_p)} x^s \hat{\phi}(s/T) ds \right).$$

Recall that $\kappa_{\pi, \pi'} > 0$ when $\pi = \pi'$, and $\kappa_{\pi, \pi'} = 0$ otherwise. Since $\|\alpha\|_2 = 1$ and $\phi$ is fixed, it follows from the inequality of arithmetic and geometric means that (4.7) equals

$$\frac{\hat{\phi}(1/T)}{T} x \sum_{\pi \in \mathcal{S}(Q)} |\alpha(\pi)|^2 \kappa_{\pi, \pi'} + O_{\phi, c} \left( Q^{\log_2 \theta_n + n + \varepsilon T(1/2)n^2[F:Q]} + \varepsilon |\mathcal{S}(Q)| \right).$$

We estimate the maximum using (4.6), and the desired result follows. □

We use Proposition 4.3 to prove Theorem 4.1.
Proof of Theorem 4.1. For the sum involving $\mu_\pi \times \pi'(n)$, we apply Proposition 4.3 with

$$a((\mu_p)_p) = b\left(\prod_p N_p^{\mu_p}\right) 1_{(\prod_p N_p^{\mu_p}, q_{\pi'})} \prod_p s_{\mu_p}(-A_{\pi'}(p)).$$

If $(\mu_p)_p \in \mu[n]$, then by (3.15), we have that $|\mu_p| = \text{ord}_p(n)$ and

$$a((\mu_p)_p) = b(n) 1_{(n, q_{\pi'})} \prod_p s_{\mu_p}(-A_{\pi'}(p)).$$

By (3.16), the left-hand side of Proposition 4.3 becomes

$$\sum_{\pi \in S(Q)} \sum_{x < N n \leq x + 1/T} \sum_{(\mu_p)_p \in \mu[n]} \left[ \prod_p s_{\mu_p}(A_{\pi}(p)) a((\mu_p)_p) \right]^2$$

$$= \sum_{\pi \in S(Q)} \sum_{x < N n \leq x + 1/T} b(n) 1_{(n, q_{\pi'})} \sum_{(\mu_p)_p \in \mu[n]} \prod_p s_{\mu_p}(A_{\pi}(p)) s_{\mu_p}(-A_{\pi'}(p))^2$$

$$= \sum_{\pi \in S(Q)} \sum_{x < N n \leq x + 1/T} \mu_{\pi \times \pi'}(n) b(n)^2.$$

Similarly, the right-hand side of Proposition 4.3 becomes

$$Q^\varepsilon \left(\frac{x}{T} + Q^4 n^2 \theta_n + n T(1/2)n^2[F:Q] + \varepsilon |S(Q)|\right) \sum_{N n \leq x \leq x + 1/T} \sum_{(\mu_p)_p \in \mu[n]} \left[ \prod_p s_{\mu_p}(-A_{\pi'}(p)) \right]^2.$$

The desired result now follows from (3.17). Apart from the new choice of

$$a((\mu_p)_p) = b\left(\prod_p N_p^{\mu_p}\right) 1_{(\prod_p N_p^{\mu_p}, q_{\pi'})} \prod_p s_{\mu_p}(A_{\pi'}(p)),$$

the result for the sum involving $\lambda_\pi \times \pi'(n)$ is handled in the same manner. \qed

Corollary 4.4. Let $\pi' \in \mathfrak{F}_n$, $Q, T \geq 1$, and $\varepsilon > 0$. If $Y \geq e$ and

$$X \geq Q^4 n^2 \theta_n + n + \varepsilon T(1/2)n^2[F:Q] + 1 + \varepsilon |S(Q)|, \quad \log Y \asymp \varepsilon \log X,$$

then

$$\sum_{\pi \in S(Q)} \int_{-T}^T \sum_{N n \geq X} \frac{\mu_{\pi \times \pi'}(n)}{N n^{1+1/\log Y + iv}}^2 dv \ll \varepsilon C(\pi')^\varepsilon Q^\varepsilon \log X,$$

$$\sum_{\pi \in S(Q)} \int_{-T}^T \sum_{N n \leq X} \frac{\mu_{\pi \times \pi'}(n)}{N n^{1/2 + iv}}^2 dv \ll \varepsilon C(\pi')^\varepsilon Q^\varepsilon X \log X.$$

Proof. We prove the first result; the second result is proved completely analogously. A formal generalization of [Gal70, Theorem 1] to number fields tells us that if $c(n)$ is a complex-valued
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function supported on the integral ideals of $\mathcal{O}_F$ with $\sum_n |c(n)| < \infty$, then
\[
\int_{-T}^T \left| \sum_n \frac{C(n)}{N_n(t)} \right|^2 dt \ll T^2 \int_0^\infty \sum_{N_n \in \{x, x+1/T\}} |c(n)|^2 \frac{dx}{x}.
\]
We choose $b(n) = Nn^{-1-1/\log Y}$ if $Nn > X$ and $b(n) = 0$ otherwise, which leads to
\[
\sum_{\pi \in S(Q)} \int_{-T}^T \left| \sum_{\text{gcd}(n,q_{a'})=\mathcal{O}_F} \frac{\mu_{\pi \times \pi'}(n)}{Nn^{1+1/\log Y}} \right|^2 dt \ll T^2 \int_0^\infty \sum_{\pi \in S(Q)} \sum_{N_n \in \{x, x+1/T\}} \frac{\mu_{\pi \times \pi'}(n)b(n)}{x} \frac{dx}{x}.
\]

Theorem 4.1 and the fact that $\lambda_{\pi' \times \pi'}(n) \geq 0$ for all $n$ imply that the above display is
\[
\ll \varepsilon Q^\varepsilon \sum_{N_n > X, \text{gcd}(n,q_{a'})=\mathcal{O}_F} \frac{\lambda_{\pi' \times \pi'}(n)}{Nn^{1+2/\log Y}} \left( 1 + \frac{Q^{4n^2\theta_n+nT(1/2)n^2[F:Q]+1+\varepsilon}|S(Q)|}{Nn} \right)
\ll \varepsilon Q^\varepsilon \sum_{N_n > X} \frac{\lambda_{\pi' \times \pi'}(n)}{Nn^{1+2/\log Y}}.
\]
The desired result now follows from Lemma 3.1 and our choices of $X$ and $Y$. \hfill \qed

5. Proof of Theorem 1.1

Let $n, n' \geq 1$, $\varepsilon > 0$, $T \geq 3$, $Q \geq 3$, $\pi \in S(Q)$, and $\pi' \in \mathfrak{F}_{n'}$. Define
\[
X := C(\pi')^\varepsilon Q^{4n^2\theta_n+n+\varepsilon T(1/2)n^2[F:Q]+1+\varepsilon}|S(Q)|,
\]
\[
Y := \left( (C(\pi'))^{n/2}Q^{n'/2}(C(\pi')Q)^{2n'n \max\{0,\theta_n+\varepsilon,1/2\}}|S(Q)||T^{n'n[F:Q]+1+\varepsilon/4n'\epsilon}|F|X \right)^{1/(3-2\sigma)},
\]
\[
L_{ur}(s, \pi \times \pi') := L(s, \pi \times \pi') \prod_{p \mid q_{a'}} L(s, \pi_p \times \pi_p')^{-1},
\]
\[
M_X(s, \pi \times \pi') := \sum_{\text{gcd}(n,q_{a'})=\mathcal{O}_F} \frac{\mu_{\pi \times \pi'}(n)}{Nn^s},
\]
\[
LM_X(s, \pi \times \pi') := L_{ur}(s, \pi \times \pi') M_X(s, \pi \times \pi').
\]
(5.1)

Note that $\log(C(\pi)C(\pi')T) \ll \log X \sim \log Y$. We assume that $T \geq 2(\log Y)^2$ since the proof does not change appreciably otherwise. In this section, $\varepsilon$ might vary from line to line, and terms of size $(C(\pi')QT[F:Q]|S(Q)|)^\varepsilon$ and polynomials in $\log(C(\pi')QT[F:Q]|S(Q)|)$ might be bounded by $X^\varepsilon$ without mention.

If $t \in [-T,T]$, then
\[
|\{ \rho = \beta + i\gamma: \beta \geq 0, |\gamma - t| \leq 1 \}| \ll \log C(\pi \times \pi', t) \ll \log(C(\pi)C(\pi')T)
\]
by [IK04, Proposition 5.7]. We observe that the rectangle $[\sigma, 1] \times [-T, T]$ is covered by $O(T)$ boxes of the form $[\sigma, 1] \times [y, y + 2(\log Y)^2]$, each containing $O((\log(C(\pi)C(\pi')T))^3)$ zeros. If we
Therefore, we deduce from Stirling's formula that
\[ N_{\pi \times \pi'}(\sigma, T) \ll (\log(C(\pi')QT))^3 n_{\pi \times \pi'}. \]
If \( \pi' = \pi \), then it suffices to assume that \(|\rho - 1| > 1\) since there are \(O(\log(C(\pi')QT))\) zeros \( \rho \) such that \(|\rho - 1| \leq 1\).

Let \( \rho = \beta + i\gamma \) be a zero of \( L(s, \pi \times \pi') \), in which case \( LM_X(\rho, \pi \times \pi') = 0 \). We compute
\[
e^{-1/Y} = \frac{1}{2\pi i} \int_{1-\beta+1/\log Y-i\infty}^{1-\beta+1/\log Y+i\infty} (1 - LM_X(\rho + w, \pi \times \pi')\Gamma(w)Y^w dw \
+ \frac{1}{2\pi i} \int_{1/2-\beta-i\infty}^{1/2-\beta+i\infty} LM_X(\rho + w, \pi \times \pi')\Gamma(w)Y^w dw \
+ k_{\pi \times \pi'}(1 - \rho)Y^{1-\rho} \sum_{\substack{N_n \leq X \\gcd(n, q_n q_{n'})=O_F}} \frac{\mu_{\pi \times \pi'}(n)}{Nn}. \tag{5.2} \]

It follows from (3.13) and Lemmas 3.1 and 3.3 that if Re\( (w) = 1 - \beta + 1/\log Y \), then
\[
|1 - LM_X(\rho + w, \pi \times \pi')| \
= |L_{w}(\rho + w, \pi \times \pi')| \cdot |L_{\pi}(\rho + w, \pi \times \pi')| - M_X(\rho + w, \pi \times \pi')| \
\ll \varepsilon C(\pi \times \pi', |\gamma + \text{Im}(w)| + 1)^\varepsilon \sum_{\substack{N_n \leq X \\gcd(n, q_n q_{n'})=O_F}} \frac{|\mu_{\pi \times \pi'}(n)|}{Nn^{1+1/\log Y}} \
\ll \varepsilon C(\pi \times \pi', |\gamma + \text{Im}(w)| + 1)^\varepsilon \sum_{\substack{N_n \leq X \\gcd(n, q_n q_{n'})=O_F}} \frac{\lambda_{\pi \times \pi'}(n) + \lambda_{\pi' \times \pi}(n)}{Nn^{1+1/\log Y}} \
\ll \varepsilon C(\pi)^\varepsilon C(\pi')^\varepsilon (|\gamma + \text{Im}(w)| + 1)^\varepsilon \log X. \tag{5.3} \]

Therefore, we deduce from Stirling's formula that
\[
\left| \frac{1}{2\pi i} \int_{1-\beta+1/\log Y-i\infty}^{1-\beta+1/\log Y+i\infty} (1 - LM_X(\rho + w, \pi \times \pi')\Gamma(w)Y^w dw \
- \frac{1}{2\pi i} \int_{1-\beta+1/\log Y-i(\log Y)^2}^{1-\beta+1/\log Y+i(\log Y)^2} (1 - LM_X(\rho + w, \pi \times \pi')\Gamma(w)Y^w dw \right| \ll \frac{1}{Y}. \]

The other terms in (5.2) are handled similarly. Since \( e^{-1/Y} = 1 + O(1/Y) \), it follows that
\[
1 \ll \frac{1}{2\pi i} \int_{1-\beta+1/\log Y-i(\log Y)^2}^{1-\beta+1/\log Y+i(\log Y)^2} (1 - LM_X(\rho + w, \pi \times \pi')\Gamma(w)Y^w dw \
+ \frac{1}{2\pi i} \int_{1/2-\beta-i(\log Y)^2}^{1/2-\beta+i(\log Y)^2} LM_X(\rho + w, \pi \times \pi')\Gamma(w)Y^w dw + k_{\pi \times \pi'}(1 - \sigma) \frac{\log X}{\max\{1, |\gamma|^3\}}. \tag{5.4} \]

Write the first integral in (5.4) as \( I_1 \) and the second as \( I_2 \). A simple optimization calculation shows that if \( c \geq 1 \) and \( c^{-1} \leq |I_1| + |I_2| \), then \( c^{-1} \leq 2c(|I_1|^2 + |I_2|) \). We estimate \(|I_1|^2\) using the
Cauchy–Schwarz inequality, thus obtaining the bound
\[
1 \ll (\log Y)^2 Y^{2(1-\sigma)} \int_{\gamma-(\log Y)^2}^{\gamma+(\log Y)^2} \left| 1 - LM_X \left( 1 + \frac{1}{\log Y} + iv, \pi \times \pi' \right) \right|^2 dv \\
+ Y^{1/2-\sigma} \int_{\gamma-(\log Y)^2}^{\gamma+(\log Y)^2} \left| LM_X \left( \frac{1}{2} + iv, \pi \times \pi' \right) \right| dv + \kappa_{\pi \times \pi'} Y^{1-\sigma} \log X \\
\]
Since \( T \geq 2(\log Y)^2 \) by hypothesis, we conclude that
\[
\sum_{S(Q)} N_{\pi \times \pi'}(\sigma, T) \ll (\log C(\pi) C(\pi') T)^3 \sum_{\pi \in S(Q)} n_{\pi \times \pi'} \ll_{\varepsilon} X^{\varepsilon} \left( Y^{2(1-\sigma)} \int_{-T}^{T} \left| 1 - LM_X \left( 1 + \frac{1}{\log Y} + iv, \pi \times \pi' \right) \right|^2 dv \\
+ Y^{1/2-\sigma} \int_{-T}^{T} \sum_{\pi \in S(Q)} \left| LM_X \left( \frac{1}{2} + iv, \pi \times \pi' \right) \right| dv + Y^{1-\sigma} \right). 
\] (5.6)

For the first integral in (5.6), we deduce from (3.13) that
\[
\left| 1 - LM_X \left( 1 + \frac{1}{\log Y} + it, \pi \times \pi' \right) \right|^2 = \left| L^{ur} \left( 1 + \frac{1}{\log Y} + it, \pi \times \pi' \right) \right|^2 \left| L^{ur} \left( 1 + \frac{1}{\log Y} + it, \pi \times \pi' \right) - LM_X \left( 1 + \frac{1}{\log Y} + it, \pi \times \pi' \right) \right|^2 \\
\ll_{\varepsilon} X^{\varepsilon} \left( \sum_{\gcd(n,q,q')=\mathcal{O}_F} \frac{\mu_{\pi \times \pi'}(n)}{N_{\pi \times \pi'}^{1+1/\log Y}} \right)^2.
\]

For the second integral in (5.6), it follows from the Cauchy–Schwarz inequality that
\[
\int_{-T}^{T} \sum_{\pi \in S(Q)} \left| LM_X \left( \frac{1}{2} + iv, \pi \times \pi' \right) \right| dv \leq \left( \int_{-T}^{T} \sum_{\pi \in S(Q)} \left| L^{ur} \left( \frac{1}{2} + iv, \pi \times \pi' \right) \right|^2 dv \right)^{1/2} \\
\times \left( \int_{-T}^{T} \sum_{\pi \in S(Q)} \sum_{\gcd(n,q,q')=\mathcal{O}_F} \frac{\mu_{\pi \times \pi'}(n)}{N_{\pi \times \pi'}^{1/2+iv}} \right)^{1/2}.
\]

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We bound the second moment of \( L^{ur}(\frac{1}{2} + iv, \pi \times \pi') \) trivially as
\[
\int_{-T}^{T} \sum_{\pi \in S(Q)} \left| L^{ur}(\frac{1}{2} + iv, \pi \times \pi') \right|^2 \, dv \\
\ll \varepsilon (QC(\pi'))^{2n} \max \{0, \theta_n + \theta_{n'}, -1/2 + \varepsilon Q^{n'/2}C(\pi')^{n/2}T^{\varepsilon n'}|F:Q|/2 + 1 + \varepsilon |S(Q)| \} \ll \varepsilon \frac{Y^{3-2\sigma}}{X}
\]
using (3.7) (to bound the ramified Euler factors) and (3.13). In summary, we find that
\[
\sum_{\pi \in S(Q)} N_{\pi \times \pi'}(\sigma, T) \ll \varepsilon X^\varepsilon \left( \int_{-T}^{T} \sum_{\pi \in S(Q)} \left| \sum_{n} \frac{\mu_{\pi \times \pi'}(n)}{Nn^{1+1/\log Y+iv}} \right|^2 \, dv \\
+ Y^{1-2\sigma} \left( \frac{Y^{3-2\sigma}}{X} \int_{-T}^{T} \sum_{\pi \in S(Q)} \left| \sum_{n \leq X} \frac{\mu_{\pi \times \pi'}(n)}{Nn^{1/2+iv}} \right|^2 \, dv \right)^{1/2} + Y^{1-\sigma} \right).
\]
We bound the two \( v \)-integrals using Corollary 4.4 and (5.1), thus obtaining
\[
\sum_{\pi \in S(Q)} N_{\pi \times \pi'}(\sigma, T) \ll \varepsilon Y^{2(1-\sigma)} X^\varepsilon.
\]
Theorem 1.1 now follows from (5.1), the bounds for \( \theta_n \) and \( \theta_{n'} \) in (3.2), and (1.5).

### 6. Bounds for Rankin–Selberg \( L \)-functions

In this section, we prove Theorem 2.1. Let \( \pi' \in \mathfrak{F}_n \) and \( Q \geq 1 \). In (1.6), we rescale \( \varepsilon \) to \( \varepsilon/72 \) and define \( \alpha = \varepsilon/(7.2 \max \{n, n'\}) \) so that
\[
|\{ \pi \in \mathfrak{F}_n(Q) : N_{\pi \times \pi'}(1 - \alpha, 6) = 0 \}| \leq \sum_{\pi \in \mathfrak{F}_n(Q)} \sum_{n' \geq \max \{n, n'\}} N_{\pi \times \pi'}(1 - \alpha, 6) \ll \varepsilon \mathfrak{F}_n(Q) |\mathfrak{F}_n(Q)| \varepsilon \quad (6.1)
\]
Let \( \pi \in \mathfrak{F}_n(Q) \). Proceeding as in the proof of [ST19, Theorem 1.1], we obtain\(^4\)
\[
\log \left| L \left( \frac{1}{2}, \pi \times \pi' \right) \right| \leq \left( \frac{1}{4} - \frac{\alpha}{10^9} \right) \log C(\pi \times \pi') + 2 \log \left| L \left( \frac{3}{2}, \pi \times \pi' \right) \right| + \frac{\alpha}{10^7} N_{\pi \times \pi'}(1 - \alpha, 6) + O(1).
\]
By (3.12) and (3.13), we find that there exists an effectively computable constant \( c_5 = c_5(n, n', [F : Q], \varepsilon) > 0 \) such that if \( C(\pi \times \pi') \geq c_5 \), then
\[
\log \left| L \left( \frac{1}{2}, \pi \times \pi' \right) \right| \leq \left( \frac{1}{4} - \frac{9\alpha}{10^{10}} \right) \log C(\pi \times \pi') + \frac{\alpha}{10^7} N_{\pi \times \pi'}(1 - \alpha, 6). \quad (6.2)
\]
Theorem 2.1 now follows from (6.1) and (6.2).

### 7. Effective multiplicity one on average

In this section, we prove Theorem 2.3. We consider the family of cuspidal automorphic representations \( \mathfrak{F}_n(Q) \) over a number field \( F \). Recall our convention that implied constants are allowed to depend on \( n \geq 3 \), \( [F : Q] \), and \( \varepsilon \) unless specifically mentioned otherwise.

\(^4\) Small modifications are required to work over \( F \neq Q \), none of which substantially changes the proof.
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To prove Theorem 2.3, we use Theorem 1.1 to build large zero-free regions for almost all $L$-functions $L(s, \pi \times \pi')$ with $\pi \in \mathfrak{F}_n$ varying. If $\pi' \in \{\pi, \pi', \pi''\}$, then $L(s, \pi \times \pi')$ has the standard zero-free region (1.1) except for a possible Landau–Siegel zero, which must be both real and simple. In all other cases, only Brumley’s much narrower zero-free region is known (see [Bru06a] and [Lap13, Appendix]). We can use Theorem 1.1 to establish a much stronger zero-free region for $L(s, \pi \times \pi')$ for all $\pi \in \mathfrak{F}_n(Q)$ with very few exceptions. Here is a simple example.

**Corollary 7.1.** Let $\varepsilon > 0$, $n \geq 3$, and $\pi' \in \mathfrak{F}_n(Q)$. For all $\pi \in \mathfrak{F}_n(Q)$ with $O_\varepsilon(Q^6)$ exceptions, the $L$-function $L(s, \pi \times \pi')$ is non-zero in the region

$$\Re(s) \geq 1 - \frac{\varepsilon}{20n^2}, \quad |\Im(s)| \leq \log Q.$$

**Proof.** This follows from (1.7) (once we rescale $\varepsilon$ to $\varepsilon/21$) with $C(\pi') \leq Q$, $T = \log Q$, and $\sigma = 1 - \varepsilon/(20n^2)$. □

For $\pi_1, \pi_2 \in \mathfrak{F}_n$ define the numbers $\Lambda_{\pi_1 \times \pi_2}(n)$ by the Dirichlet series identity (for $\Re(s) > 1$)

$$\sum_n \frac{\Lambda_{\pi_1 \times \pi_2}(n)}{n^s} = -\frac{L'}{L}(s, \pi_1 \times \pi_2) = \sum_p \sum_{k \geq 1} \sum_{j=1}^n \sum_{j'=1}^n \alpha_{j,j', \pi_1 \times \pi_2}(p) \log Np \frac{\log Np}{Np^k}.$$  \hspace{1cm} (7.1)

**Lemma 7.2.** Let $\pi' \in \mathfrak{F}_n$. There exist absolute and effectively computable constants $c_6 \in (0, 1)$, $c_7, c_8, c_9, c_{10} \geq 1$, and $c_{11} \in (0, 1)$ such that the following are true.

1. The $L$-function $L(s, \pi' \times \pi')$ has at most one zero, say $\beta_1$, in the region

$$\Re(s) \geq 1 - \frac{c_6}{\log(C(\pi'))n(||\Im(s)|| + c)n^{2[F:Q]}},$$

If $\beta_1$ exists, then it must be real and simple and satisfy $1 - \log(C(\pi'))^{-ctn}$. (2) If $A \geq c_8$, $\log C(\pi') \geq c_{10}n^4[F : Q]^2$, and $x \geq C(\pi')^{c_{10}A^2n^4[F : Q]}$, then

$$\sum_{x/2 < n \leq x} \Lambda_{\pi' \times \pi'}(n) = \begin{cases} \frac{x}{2}(1 - \xi^{\beta_1 - 1})(1 + O(\varepsilon^{-c_{11}A})) & \text{if } \beta_1 \text{ exists}, \\ \frac{x}{2}(1 + O(\varepsilon^{-c_{11}A})) & \text{otherwise}, \end{cases}$$

where $\xi \in [x/2, x]$ satisfies $x^{\beta_1} - (x/2)^{\beta_1} = \beta_1(x/2)\xi^{\beta_1 - 1}$ and the implied constants are absolute and effectively computable.

**Proof.** This is [HT22, Theorem 2.1] with $\delta = 0$ and $x$ replaced with $x/2$. □

Lemma 7.2 informs the following choices of parameters that we will use throughout the rest of this section. Let $0 < \varepsilon < 1$, let $Q$ be sufficiently large with respect to $\varepsilon$, and let

$$c_7 \geq \frac{c_8^2 c_{10}}{3240}, \quad A = \sqrt{\frac{3240c_7}{c_{10}\varepsilon}}, \quad B = \frac{41n^2}{\varepsilon}, \quad x = \frac{1}{2}(\log Q) B,$$

$$\pi \in \mathfrak{F}_n(Q), \quad \pi' \in \mathfrak{F}_n\left(x^{1/2}c_{10}n^4[F : Q]^2\right) = \mathfrak{F}_n\left((2^{-\varepsilon/41n^2}\log Q)^{41/3240c_7n^2[F : Q]^2}\right).$$  \hspace{1cm} (7.2)

**Corollary 7.3.** Under the notation and hypotheses of (7.2), there holds

$$\sum_{x/2 < n \leq x \atop \gcd(n, n', l) = 1} \Lambda_{\pi' \times \pi'}(n) \gg_{\varepsilon} x^{1 - 2\varepsilon/41n^2[F : Q]^2}.$$  \hspace{1cm} (7.3)

**Proof.** Let $\pi, \pi' \in \mathfrak{F}_n(Q)$. In Lemma 7.2, the lower bound on $C(\pi')$ only serves to ensure that the implied constants are absolute. This was pertinent in [HT22], but it is not pertinent here. Thus, we may replace the two conditions $\log \log C(\pi') \geq c_9n^4[F : Q]^2$ and
$x \geq C(\pi')A^2 c_{10} n^3 [F : \mathbb{Q}] \log(en[F : \mathbb{Q}])$ with the single condition $C(\pi') \ll x^{1/(A^2 c_{10}n^3[F : \mathbb{Q}]^2)}$. We want to refine Lemma 7.2 so that one only sums over $n$ such that $\gcd(n, q_{\pi} q_{\pi'}) = \mathcal{O}_F$. Using (3.3) and (3.7), we find that the contribution to Lemma 7.2(2) arising from $n$ such that $\gcd(n, q_{\pi} q_{\pi'}) \neq \mathcal{O}_F$ is

$$\sum_{x/2 < Nn \leq x \atop \gcd(n,q_{\pi} q_{\pi'}) \neq \mathcal{O}_F} \Lambda_{\pi \times \pi'}(n) \ll \sum_{\gcd(n,q_{\pi} q_{\pi'})} \log(x/2) / \log N \leq \log x / \log N \leq \log Q / \log \log Q \ll x^{1-2/(n^2+1)},$$

which is $\ll x^{1-1/(n^2+1)}$. In the worst case, where $\beta_1$ in Lemma 7.2(1) exists, it follows from Lemma 7.2(2) and the above discussion that if $C(\pi') \ll n_{[F : \mathbb{Q}]}^1 x^{1/(A^2 c_{10}n^3[F : \mathbb{Q}] \log(en[F : \mathbb{Q}]^2)}$, then

$$\sum_{x < Nn \leq 2x \atop \gcd(n,q_{\pi} q_{\pi'})} \Lambda_{\pi \times \pi'}(n) \gg x(1-\epsilon\beta_1-1) \gg n_{[F : \mathbb{Q}]} x \min\{1, (1-\beta_1) \log C(\pi')\} \gg x \log C(\pi') / C(\pi') \varepsilon / \tau \frac{n}{20},$$

The desired result now follows.

**Lemma 7.4.** Let $n \geq 3$. Assume the notation and hypotheses in (7.2). Let $\Phi$ be a fixed smooth function supported on a compact subset of $[\frac{1}{4}, 2]$ such that $0 \leq \Phi(t) \leq 1$ for all $t \in [\frac{1}{4}, 2]$ and $\Phi(t) = 1$ for $t \in [\frac{1}{4}, 2]$. If $L(s, \pi \times \pi')$ is entire and does not vanish in the region $\operatorname{Re}(s) \geq 1 - \varepsilon/(20n^2)$ and $|\operatorname{Im}(s)| \leq \log Q$, then

$$\sum_{\gcd(n,q_{\pi} q_{\pi'}) = \mathcal{O}_F} \Lambda_{\pi \times \pi'}(n) \Phi(n/x) \ll_B x^{1+2/B-\varepsilon/20n^2}.$$

**Proof.** Writing $\tilde{\Phi}(s) = \int_0^\infty \Phi(t) t^{s-1} dt$ for the Mellin transform of $\Phi$ (which is an entire function of $s$), we observe via Mellin inversion that

$$\sum_{\gcd(n,q_{\pi} q_{\pi'}) = \mathcal{O}_F} \Lambda_{\pi \times \pi'}(n) \Phi(n/x) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{(L_{\pi \times \pi'}^u)^{-1}}{(L_{\pi \times \pi'}^{-1})} (s, \pi \times \pi')^{-1} \tilde{\Phi}(s) x^s ds,$$

where $L_{\pi \times \pi'}^u(s, \pi \times \pi') = L(s, \pi \times \pi') \prod_{p | q_{\pi} q_{\pi'}} L(s, \pi_p \times \pi'_p)^{-1}$. A standard contour integral calculation using the argument principle shows that the above display equals

$$-\sum_{L_{\pi \times \pi'}^u(\rho, \pi \times \pi') = 0} \tilde{\Phi}(\rho) x^\rho,$$

where $\rho$ ranges over all zeros of $L_{\pi \times \pi'}^u(s, \pi \times \pi')$. Since $\Phi$ is compactly supported and $\tilde{\Phi}$ is entire, it follows that for any $R \geq 2$, we have $|\tilde{\Phi}(s)| \ll R \min\{1, s^{-R} e^{\operatorname{Re}(s)}\}$. Note that the reciprocals of the Euler factors of $L(s, \pi \times \pi')$ at prime ideals $p | q_{\pi} q_{\pi'}$ and all of the trivial zeros of $L(s, \pi \times \pi')$ have real part no larger than $1 - 2/(n^2 + 1)$ per (3.3) and (3.7). Since for any $t \in \mathbb{R}$ there are $\ll \log Q + \log(|t| + 2)$ zeros $\rho = \beta + i\gamma$ of $L_{\pi \times \pi'}^u(s, \pi \times \pi')$ that satisfy $0 < \beta < 1$ and $|\gamma - t| \leq 1$, we find for any $T \geq 1, R \geq 2$, and $\sigma_0 \geq 1 - 2/(n^2 + 1)$ that

$$\sum_{L_{\pi \times \pi'}^u(\rho, \pi \times \pi') = 0} \tilde{\Phi}(\rho) x^\rho = \sum_{\beta \geq \sigma_0 \atop |\gamma| \leq T} O(T^{1-R} \log(QT) x + T \log(QT) x^{\sigma_0}).$$

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We choose \( T = \log Q = (2x)^{1/B} \) and \( \sigma_0 = 1 - \varepsilon/(20n^2) \), in which case our hypotheses imply that the sum over zeros on the right-hand side of (7.3) is empty and

\[
\sum_{L^w(p, \pi \times \tilde{\pi}) = 0} \tilde{\Phi}(p)x^p \ll_B x^{1+(2-R)/B} + x^{2/B+\sigma_0}.
\]

The desired result follows from choosing \( R = \max\{B(1-\sigma_0), 3\} \).

\( \square \)

**Proof of Theorem 2.3.** Suppose to the contrary that \( \pi \neq \pi' \) and \( \pi_p \cong \pi'_p \) for all \( p \nmid q \pi_q \pi' \), with \( Np \leq 2x \). Then \( A_p(\pi) = A_p(\pi') \) for all \( p \nmid q \pi_q \pi' \) with \( Np \leq 2x \). By (3.6) and (7.1), it follows that \( \Lambda_{\pi, \pi'}(n) = \Lambda_{\pi', \pi}(n) \) for all \( n \) such that \( \gcd(n, q \pi_q \pi') = \mathcal{O}_F \) and \( Nn \leq x \). Since \( \Lambda_{\pi, \pi'}(n) \geq 0 \) for all such \( n \), the same holds for \( \Lambda_{\pi, \pi'}(n) \). It follows that

\[
\sum_{x/2 < Nn \leq x} \Lambda_{\pi, \pi'}(n) = \sum_{x/2 < Nn \leq x} \Lambda_{\pi', \pi}(n).
\]

By (7.2), we have that

\[
C(\pi') \ll x^{1/(A_2c_{10}n^3[F:Q])^2}, \tag{7.4}
\]

Therefore, if \( L(s, \pi \times \pi') \neq 0 \) in the region \( \{ s \in \mathbb{C} : \Re(s) \geq 1 - \varepsilon/(20n^2), |\Im(s)| \leq \log Q \} \), then by Corollary 7.3 and Lemma 7.4, we have that

\[
x^{1-2c_7/A_2c_{10}n^3[F:Q]^2} \ll \sum_{x/2 < Nn \leq x} \Lambda_{\pi, \pi'}(n) = \sum_{x/2 < Nn \leq x} \Lambda_{\pi', \pi}(n) \ll_B x^{1+2/B-\varepsilon/20n^2}.
\]

This implies that

\[
1 - \frac{2c_7}{A_2^2c_{10}n^6[F:Q]^2} \leq 1 + \frac{2}{B} - \frac{\varepsilon}{20n^2},
\]

which contradicts our choices of \( A \) and \( B \) in (7.2). The desired result now follows from Corollary 7.1.

\( \square \)

**8. Automorphic level of distribution**

In this section, we prove Theorem 2.5. In what follows, let \( F = \mathbb{Q} \). If \( n = 2 \), then \( L(s, \pi \times \pi) \) is the \( L \)-function of an isobaric automorphic representation of \( GL_4(\mathbb{A}_F) \) whose cuspidal constituents have rank at most 3. Since the \( L \)-function any Dirichlet character twist of any cuspidal constituents of rank 2 or 3 have no Landau–Siegel zero by [Ban97, HR95], a stronger result than Theorem 2.5 follows from a minor variation of the proof of [JLTW23, Theorem 1.1]. Therefore, we may restrict our consideration to \( n \geq 3 \).

Let \( \pi \in \mathfrak{f}_n \) have arithmetic conductor \( q \pi \), and let \( \chi \) be a primitive Dirichlet character modulo \( q \). We allow \( \pi \) to be fixed, so for notational compactness, we introduce \( L_\chi(s) := L(s, \pi \times (\pi \otimes \chi)) \) and \( L_1(s) := L(s, \pi \times \pi) \). If \( q \) and \( q \pi \) are coprime, then \( L_\chi(s) \) is entire if and only if \( \chi \) is non-trivial, since \( \pi \otimes \chi \neq \pi \), which is clear by comparing the arithmetic conductors of \( q \pi \otimes \chi \) and \( q \). If \( \chi \) is trivial, then \( L_\chi(s) = L_1(s) \). We also define \( \Lambda_\chi(s) := \Lambda(s, \pi \times (\pi \otimes \chi)) \) and \( \Lambda_1(s) := \Lambda(s, \pi \times \pi) \).

### 8.1 Preliminaries

We define \( a_\pi(m) \) and \( a_{\pi \times \pi}(m) \) by the Dirichlet series identities

\[
\sum_{n=1}^{\infty} a_\pi(m)A(m)\frac{m^s}{m} = -\frac{L'(s, \pi)}{L(s, \pi)}, \quad \sum_{n=1}^{\infty} a_{\pi \times \pi}(m)A(m)\frac{m^s}{m} = -\frac{L'(s, \pi \times \pi)}{L(s, \pi \times \pi)}, \quad \Re(s) > 1.
\]

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The local calculations in [LRS95, Lemma 2.1] show that if \( \chi \pmod{q} \) is a primitive Dirichlet character and \( \gcd(q, q_\pi) = 1 \), then for all primes \( p \), we have that

\[
L(s, \pi_p \times (\overline{\pi} \otimes \chi)_p) = \prod_{j=1}^{n} \prod_{j'=1}^{n} (1 - \alpha_{j,j',\pi \times \overline{\chi}}(p)\chi(p)p^{-s})^{-1}.
\]

It follows from (3.6) and (8.1) that if \( \gcd(m, q_\pi) = 1 \), then \( a_{\pi \times (\overline{\pi} \otimes \chi)}(m) = |a_{\pi}(m)|^2\chi(m) \). We now provide a convenient expression for \( -L'_\chi(s)/L_\chi(s) \).

**Lemma 8.1.** Let \( \pi \in \mathbb{F}_n \). Let \( \psi \pmod{q} \) be a Dirichlet character with \( q \) such that \( \gcd(q, q_\pi) = 1 \) and \( \chi \) be the primitive Dirichlet character that induces \( \psi \). Let \( \delta(\chi) = 1 \) if \( \chi \) is trivial and \( \delta(\chi) = 0 \) otherwise. There exists a function \( H_\pi(s; \chi, \psi) \) such that in the region \( \Re(s) \geq 1 - 1/n^2 \):

1. \( H_\pi(s; \chi, \psi) \) is holomorphic;
2. \( |H_\pi(s; \chi, \psi)| \ll_\pi \log(q(3 + |\Im(s)|)) \); and
3. we have the identity

\[
\sum_{m=1}^{\infty} \frac{|a_{\pi}(m)|^2\psi(m)\Lambda(m)}{m^s} = \frac{\delta(\chi)}{s-1} - \frac{\Lambda'_{\chi}}{\Lambda_{\chi}}(s) + H_\pi(s; \chi, \psi).
\]

**Proof.** Suppose first that \( \psi \) is primitive, in which case \( \psi = \chi \) and the function \( H_\pi(s; \chi, \psi) \) is

\[
\frac{\log q_{\pi \times (\overline{\pi} \otimes \chi)}}{2} + \frac{L'(s, \pi_\infty \times (\overline{\pi} \otimes \chi)_\infty)}{L(s, \pi_\infty \times (\overline{\pi} \otimes \chi)_\infty)} + \frac{\delta(\chi)}{s} + \sum_{p|q_\pi \chi} \frac{L'(s, \pi_p \times (\overline{\pi} \otimes \chi)_p)}{L(s, \pi_p \times (\overline{\pi} \otimes \chi)_p)}
\]

\[
- \sum_{\substack{p|q_\pi \chi \quad 1 \leq j \leq n \quad 1 \leq j' \leq n \quad \alpha_{j,j',\pi}(p)\alpha_{j',\pi}(p)\chi(p)}} \frac{p^s}{p^s - \alpha_{j,j',\pi}(p)\alpha_{j',\pi}(p)\chi(p)}.
\]

This is holomorphic and bounded as claimed for \( \Re(s) \geq 1 - 1/n^2 \) by (3.7), (3.10), and Stirling’s formula. If \( \psi \) is not primitive and \( \chi \) is the primitive character that induces \( \psi \), then in the same region, we have

\[
\left| \sum_{m=1}^{\infty} \frac{|a_{\pi}(m)|^2\psi(m)\Lambda(m)}{m^s} - \sum_{m=1}^{\infty} \frac{|a_{\pi}(m)|^2\chi(m)\Lambda(m)}{m^s} \right| \leq \sum_{p|q} \sum_{k \geq 1} \frac{|a_{\pi}(p^k)|^2 \log p}{p^k \Re(s)},
\]

which is bounded as desired using (3.7) again. \( \square \)

We use the following zero-free region for \( L_\chi(s) \) and Siegel-type bound on any real exceptional zeros. We prove this theorem later.

**Theorem 8.2.** Let \( Q \geq 3 \) and \( \pi \in \mathbb{F}_n \). There exists an effectively computable constant \( c_{12} = c_{12}(\pi) > 0 \) such that for all primitive Dirichlet characters \( \chi \pmod{q} \) with \( q \leq Q \) and \( \gcd(q, q_\pi) = 1 \), at most one exception, the \( L \)-function \( L(s, \pi \times (\overline{\pi} \otimes \chi)) \) does not vanish in the region

\[
\Re(s) \geq 1 - \frac{c_{12}}{\log(Q(|\Im(s)|) + 3))}.
\]

If the exceptional character \( \chi_1 \pmod{q_1} \) exists, then \( L(s, \pi \times (\overline{\pi} \otimes \chi_1)) \) has at most one zero, say \( \beta_1 \), in this region; \( \beta_1 \) is both real and simple; and \( \chi_1 \) must be quadratic. Moreover, for all \( \varepsilon > 0 \), there exists an ineffective constant \( c_\pi(\varepsilon) > 0 \) such that \( \beta_1 \leq 1 - c_\pi(\varepsilon)q_1^{-\varepsilon} \).
8.2 Proof of Theorem 2.5

We follow Gallagher’s proof of the Bombieri–Vinogradov theorem in [Gal68], with \( n \geq 2 \) and \( \pi \in \mathcal{S}_n \) fixed at the onset. Note that the function

\[
\psi_k(y; q, a) := \frac{1}{k!} \sum_{m \equiv a \pmod{q} \atop m \leq y} |a_\pi(m)|^2 \Lambda(m) \left( \log \frac{y}{m} \right)^k
\]

is monotonically increasing as a function of \( y \) for each \( k \geq 0 \). Thus, if \( 0 < \lambda \leq 1 \), then

\[
\frac{1}{\lambda} \int^y e^{-\lambda y} \psi_{k-1}(t; q, a) \frac{dt}{t} \leq \psi_{k-1}(y; q, a) \leq \frac{1}{\lambda} \int^y e^{\lambda y} \psi_{k-1}(t; q, a) \frac{dt}{t}.
\]

The integrals, which equal \( \psi_k(y; q, a) - \psi_k(e^{-\lambda}y; q, a) \) and \( \psi_k(e^{\lambda}y; q, a) - \psi_k(y; q, a) \), respectively, both have the same asymptotic expansion, namely

\[
(\lambda + O(\lambda^2)) \frac{y}{\varphi(q)} + O\left( \max_{x \leq \epsilon x} |r_k(y; q, a)| \right), \quad r_k(y; q, a) := \psi_k(y; q, a) - \frac{y}{\varphi(q)},
\]

where \( \varphi \) is Euler’s totient function. Thus, we have the bounds

\[
\max_{y \leq x} |r_{k-1}(y; q, a)| \ll \frac{\lambda x}{\varphi(q)} + \frac{1}{\lambda} \max_{y \leq \epsilon x} |r_k(y; q, a)|
\]

and

\[
\sum_{q \leq x^\theta \atop \gcd(q, q_a) = 1} \max_{y \leq x} |r_{k-1}(y; q, a)| \ll k \lambda x \log x + \frac{1}{\lambda} \sum_{q \leq x^\theta \atop \gcd(q, q_a) = 1} \max_{y \leq \epsilon x} |r_k(y; q, a)|.
\]

It follows by induction on \( k \) that

\[
\sum_{q \leq x^\theta \atop \gcd(q, q_a) = 1} \max_{y \leq x} |r_0(y; q, a)| \ll k \lambda x \log x + \frac{1}{\lambda^{2^k - 1}} \sum_{q \leq x^\theta \atop \gcd(q, q_a) = 1} \max_{y \leq \epsilon x} |r_k(y; q, a)|.
\]

**Proposition 8.3.** If \( \theta < 1/(9n^3) \) is fixed and \( k = 9n^2 + 1 \), then for any \( B > 0 \), we have

\[
\sum_{q \leq x^\theta \atop \gcd(q, q_a) = 1} \max_{y \leq x} |r_0(y; q, a)| \ll \pi, B \frac{x}{(\log x)^B}.
\]

**Proposition 8.3 implies Theorem 2.5.** It follows from Proposition 8.3 that

\[
\sum_{q \leq x^\theta \atop \gcd(q, q_a) = 1} \max_{y \leq x} |r_0(y; q, a)| \ll \pi, B \lambda x \log x + \frac{1}{\lambda^{2^k - 1}} \frac{x}{(\log x)^B}.
\]

To finish, we choose \( B = 2^k(A + 1) - 1 \) and \( \lambda = (\log x)^{-B + 1}/2^k \). \( \square \)

8.3 Proof of Proposition 8.3

Let \( \pi \in \mathcal{S}_n \), \( k = 9n^2 + 1 \), \( Q = x^\theta \) for some fixed \( 0 < \theta < 1/(9n^3) \), and \( q \leq Q \). We have the decomposition

\[
\frac{1}{k!} \sum_{m \leq y \atop m \equiv a \pmod{q}} |a_\pi(m)|^2 \Lambda(m) \left( \log \frac{y}{m} \right)^k = \frac{1}{k!} \sum_{\psi \pmod{q}} \frac{\psi(a)}{\varphi(q)} \sum_{y \leq m} |a_\pi(m)|^2 \Lambda(m) \psi(n) \left( \log \frac{y}{m} \right)^k.
\]
By Lemma 8.1 and Mellin inversion, this equals
\[
\frac{1}{\varphi(q)} \sum_{\psi \pmod{q}} \sum_{\chi \text{ primitive}} \frac{\chi(a)}{2\pi i} \int_{3-i\infty}^{3+i\infty} \left( -\frac{\Lambda_{\psi}(s)}{\Lambda_{\chi}(s)} + H_{\pi}(s; \chi, \psi) \right) \frac{y^s}{s^{k+1}} ds.
\]

Since a primitive character \( \chi \pmod{q} \) induces characters to moduli that are a multiple of \( q \), it follows from the bound \( \sum_{f \leq Q} |\varphi(f)|^{-1} \ll (\log Q)/\varphi(q) \) that
\[
\sum_{q \leq Q} \max_{\gcd(a,q)=1} \max_{y \leq x} \frac{\log Q}{\varphi(q)} \sum_{\chi \text{ primitive}} \frac{1}{y^k} \sum_{m \leq y} \left| a_x(m) \right|^2 \Lambda(m) \left( \frac{\log y}{m} \right) \frac{y^s}{s^{k+1}} ds. \tag{8.2}
\]

Observe that by Lemma 8.1 and our range of \( \theta \), (8.2) equals
\[
\ll_{\pi, B} \log Q \sum_{q \leq Q} \max_{\chi \text{ primitive}} \frac{\log Q}{\varphi(q)} \sum_{\lambda \chi \pmod{q}} \frac{1}{\phi(q)} \sum_{\lambda \neq 0} L_{\lambda}(\rho=0) \frac{y^\rho}{|\rho|^{k+1}} + \frac{x}{\log x} ds. \tag{8.3}
\]

where \( \rho = \beta + i\gamma \) ranges over the non-trivial zeros of \( L_{\chi}(s) \). In light of the bounds \( 1/q \leq 1/\varphi(q) \ll (\log(eq))/q \), we dyadically decompose \( [1, Q] \) into \( O(\log Q) \) subintervals and find that (8.3) is
\[
\ll_{\pi, B} x(\log Q)^2 \sup_{3 \leq R \leq Q} \sum_{q \leq R} \frac{1}{\varphi(q)} \sum_{\chi \text{ primitive}} \frac{x^{\beta-1}}{|\rho|^k} + \frac{x}{\log x} B. \tag{8.4}
\]

where \( \beta_1 \) is the exceptional zero in Theorem 8.2. The term \( x^{\beta-1}/\beta_1^k \) is omitted if \( \beta_1 \) does not exist.

If \( \beta_1 \) exists as in Theorem 8.2 and the supremum is achieved when \( R \leq (\log x)^{4B} \), then we apply Theorem 8.2 with \( \varepsilon = 1/8B \) and conclude that the contribution from such a zero is absorbed in our error term. If the supremum is achieved when \( R > (\log x)^{4B} \), then contribution from \( \beta_1 \) is trivially absorbed in our error term. Hence, (8.4) is
\[
\ll_{\pi, B} x(\log x)^3 \sup_{3 \leq R \leq x^\theta} \frac{1}{R} \sum_{q \leq R} \frac{1}{\varphi(q)} \sum_{\chi \text{ primitive}} \frac{x^{\beta-1}}{|\rho|^k} + \frac{x}{\log x} B. \tag{8.5}
\]
Now let us consider the zeros \( \rho \) with \( |\rho| < \frac{1}{4} \). The number of such zeros is \( \ll R^2 \log R \). From the consideration of the corresponding zeros \( 1 - \rho \) of \( L^\chi(s) \), we deduce that \( |\rho| \gg x^{-1(4k)} \). Thus, the contribution from these zeros is \( \ll R^{1\frac{4k}{4k}+1} \log R \ll Qx^{1/2} \log Q \ll B x \log x)^{-B} \). Define \( T_0 = 0 \) and \( T_j = 2^j-1 \) for \( j \geq 1 \). The above discussion shows that (8.5) is

\[
\ll_{\pi,B} x \log x)^3 \sup_{3 \leq R \leq x^\theta} \frac{1}{R} \sum_{q \leq R} \sum_{\chi \mod q, \chi \text{ primitive}} \sum_{|\rho| > \frac{1}{4}} \frac{x^{\beta-1}}{|\rho|^k} + \frac{x}{(\log x)^B}
\]

\[
\ll_{\pi,B} x \log x)^3 \sup_{3 \leq R \leq x^\theta} \frac{1}{R} \sum_{j=1}^{\infty} \sum_{q \leq R} \sum_{\chi \mod q, \chi \text{ primitive}} \sum_{|\rho| \geq \frac{1}{4}} \frac{x^{\beta-1}}{|\rho|^k} + \frac{x}{(\log x)^B}.
\] (8.6)

If \( |\rho| \geq \frac{1}{4} \) and \( T_{j-1} \leq |\gamma| \leq T_j \), then \( |\rho| \geq \max\{T_{j-1}, \frac{1}{4}\} \leq T_j/4 \) and \( |\rho| \gg |\gamma| + 3 \). Therefore, if \( \delta = \min\{1 - 9n^2\theta, \frac{1}{2}\} \), then

\[
x^{\beta-1}|\rho|^{-k} \ll T_j^{-1/2} (|\gamma| + 1)^{-1/2} x^{-1-\beta} R^{1-\delta} (R^{1-\delta} T_j^{-1} )^{-1-\beta} - \delta - 1.
\]

Since \( \rho = \beta + i\gamma \neq \beta_1 \), it follows from Theorem 8.2 that

\[
(|\gamma| + 1)^{-1/2} x^{-\delta(1-\beta)} \leq e^{-\delta \eta(x,R)}, \quad \eta(x,R) := \inf_{t \geq 3} \left[ c \pi \log x \log (Rt) + \log t \right],
\]

so (8.6) is

\[
\ll_{\pi,B} x \log x)^3 \sup_{3 \leq R \leq x^\theta} \frac{e^{-\delta \eta(x,R)}}{R} \sum_{j=1}^{\infty} T_j^{-1/2}
\]

\[
\times \sum_{q \leq R} \sum_{\chi \mod q, \chi \text{ primitive}} \sum_{|\rho| \geq \frac{1}{4}} \frac{x^{\beta-1}}{|\rho|^k} + \frac{x}{(\log x)^B}.
\]

Define

\[
N^*_{\pi}(\sigma,T,R) = \sum_{q \leq R} \sum_{\chi \mod q, \chi \text{ primitive}} |\{\rho = \beta + i\gamma: \beta \geq \sigma, |\gamma| \leq T, L_{\chi}(\rho) = 0\}|.
\]

By (3.10), there exists an effectively computable constant \( c_{13} = c_{13}(n) > 0 \) such that

\[
\{\bar{\pi} \otimes \chi: \chi (\mod q_x) \text{ primitive}, \gcd(q_x, \pi) = 1, q_x \leq Q\} \subseteq \mathcal{F}_n(c_{13}C(\pi)Q^n).
\]

Thus, by (1.7), we have \( N^*_{\pi}(\sigma,T,R) \ll_{\pi,\varepsilon} (R^\alpha T)\alpha^\varepsilon \). Partial summation yields

\[
\ll \sum_{q \leq R} \sum_{\chi \mod q, \chi \text{ primitive}} \sum_{|\rho| \geq \frac{1}{4}} (R^{1-\delta}/\theta T_j^{-1})^{1-\beta}
\]

\[
\ll (R^{1-\delta}/\theta T_j^{-1})^{-1} N^*_{\pi}(0, T_j, R) + \log(RT_j) \int_0^1 (R^{1-\delta}/\theta T_j^{-1})^{-\sigma} N^*_{\pi}(1 - \sigma, T_j, R) d\sigma
\]

\[
\ll_{\pi,\varepsilon} (RT_j)^{\varepsilon} \left( (R^{1-\delta}/\theta T_j^{-1})^{-1} RT_j + \int_0^1 (R^{1-\delta}/\theta T_j^{-1})^{-\sigma} (R^\alpha T_j)^{\alpha^\varepsilon} d\sigma \right).
\] (8.7)
Our choices for $\theta$, $k$, and $\delta$ ensure that (8.7) is $\ll_{\pi,\delta} (RT_j)^\varepsilon$. Thus, (8.6) is

$$
\ll_{\pi, B, \varepsilon} x(\log x)^3 \sup_{3 \leq R \leq x^{1/2}} e^{-\delta \eta_1(x,R)} \frac{R^\varepsilon}{R} \sum_{j=1}^\infty T_j^{-1/2+\varepsilon} + \frac{x}{(\log x)^B}.
$$

(8.8)

A small calculation (cf. [TZ19, §4]) shows that there is a constant $c_{14} = c_{14}(\pi) > 0$ such that $e^{-\delta \eta_1(x,R)} \ll_{\pi, B, \varepsilon} \exp(-c_{14} \log x/\log R) + \exp(-c_{14} \sqrt{\log x})$, and Theorem 2.5 follows.

8.4 Proof of Theorem 8.2

Although the proof of Theorem 8.2 contains only standard techniques, such zero-free regions for $L_\chi(s)$ are new even in the case when $\chi$ is trivial. The case when $\chi$ is trivial was only recently handled unconditionally in [HT22].

Lemma 8.4 [HT22, Theorem 2.1(1)]. Let $\pi \in \mathfrak{F}_n$. There exists an absolute and effectively computable constant $c_{15} > 0$ such that $L_1(s)$ has at most one zero, say $\beta_1$, in the region $\mathrm{Re}(s) \geq 1 - c_{15}/\log(C(\pi)^n(|\mathrm{Im}(s)| + c))$. If $\beta_1$ exists, then it must be real and simple, and there exists an absolute and effectively computable constant $c_{16}$ such that $\beta_1 \leq 1 - C(\pi)^{-c_{16}n}$.

We prove the corresponding result for $L_\chi(s)$. The ideas in [HT22] inform our approach here.

Lemma 8.5. Let $\pi \in \mathfrak{F}_n$. Let $\chi(\bmod q)$ be a non-trivial primitive Dirichlet character such that $\gcd(q, q_\pi) = 1$. There exists an effectively computable constant $c_{17} = c_{17}(n) > 0$ such that $L_\chi(s)$ has at most one zero, say $\beta_1$, in the region

$$
\mathrm{Re}(s) \geq 1 - \frac{c_{17}}{\log(qC(\pi)(|\mathrm{Im}(s)| + 3))}.
$$

(8.9)

If the exceptional zero $\beta_1$ exists, then it is real and simple, and $\chi$ is quadratic.

Proof. Let $\chi(\bmod q)$ be a primitive non-trivial Dirichlet character, let $\psi$ be the primitive character that induces $\chi^2$, and let $\beta + i\gamma$ be a non-trivial zero of $L_\chi(s)$. Define $\Pi_\chi = \pi \boxplus \pi \otimes \chi \cdot | \cdot |^{it} \boxplus \pi \otimes \chi \cdot | \cdot |^{-it}$, and define

$$
D(s) = L(s, \Pi_\chi \times \bar{\Pi}_\chi) = L_1(s)^3 L_\chi(s + i\gamma)^2 L_\chi(s - i\gamma)^2 L_\psi(s + 2i\gamma) L_\psi(s - 2i\gamma).
$$

(8.10)

The factor $L_1(s)^3$ has a pole of order 3 at $s = 1$, and the hypothesis that $\gcd(q, q_\pi) = 1$ ensures that $L_\chi(s + i\gamma)^2 L_\chi(s - i\gamma)^2$ is entire. If $\psi$ is complex, then $L_\psi(s + 2i\gamma) L_\psi(s - 2i\gamma)$ is entire; otherwise, it has poles of order 1 at $s = 1 \pm 2i\gamma$. The additional poles when $\psi$ is real require some additional casework when $\gamma$ is close to zero. For notational compactness, let $Q_\gamma = qC(\pi)(|\gamma| + 3)$. Note that $-(D'/D)(s)$ has non-negative Dirichlet coefficients per [HR95, Lemma a].

The functional equation for $L_\chi(s)$ together with the fact that $L(s, \pi \times \bar{\pi})$ is a self-dual $L$-function (even if $L(s, \pi)$ itself is not self-dual) implies that if $\rho$ is a zero of $L_\chi(s)$, then $\bar{\rho}$ is a zero of $L_\chi(s)$. Thus, we have that

$$
\operatorname{ord}_{s=\beta} D(s) \geq 4.
$$

(8.11)

Let $\omega$ denote a non-trivial zero of $D(s)$, $\delta(\omega) = 1$ if $\psi$ is trivial, and $\delta(\omega) = 0$ otherwise. We apply Lemma 3.2 and (3.10) to (8.10), concluding that if $1 < \sigma < 2$, then

$$
\sum_\omega \Re \left( \frac{1}{\sigma - \omega} \right) < \frac{3}{\sigma - 1} + \delta(\psi) \frac{2(\sigma - 1)}{(\sigma - 1)^2 + 4\gamma^2} + c_{18} \log Q_\gamma,
$$

(8.12)
where $c_{18} = c_{18}(n) > 1$ is a suitable implied constant. Since $\beta < 1$ is, by hypothesis, one of the zeros in the sum in (8.12), we have by (8.11) and non-negativity that

$$
\frac{4}{\sigma - \beta} < \frac{3}{\sigma - 1} + \delta(\psi) \frac{2(\sigma - 1)}{4\gamma^2 + (\sigma - 1)^2} + c_{18} \log Q_\gamma.
$$  \hfill (8.13)

**Case 1:** Either $\chi$ is real and $|\gamma| \geq 1/7c_{18} \log Q_0$ or $\chi$ is complex. If $\sigma = 1 + 1/5c_{18} \log Q_\gamma$, then

$$
\delta(\psi) \frac{2(\sigma - 1)}{4\gamma^2 + (\sigma - 1)^2} \leq \frac{490}{149} c_{18} \log Q_\gamma.
$$
Thus, (8.13) becomes

$$
\frac{4}{1 + 1/5c_{18} \log Q_\gamma - \beta} \leq \frac{2874}{149} c_{18} \log Q_\gamma.
$$
Upon solving for $\beta$, we conclude that $\beta \leq 1 - 1/136c_{18} Q_\gamma$.

**Case 2:** $\chi$ is real and $\gamma = 0$. We start at (8.12) with $\delta(\psi) = 1$, $\gamma = 0$, and $\sigma = 1 + 1/3c_{18} \log Q_0$. If there are $N$ zeros $\beta$ (with multiplicity) of $D(s)$ such that $\beta \geq 1 - 1/96c_{18} \log Q_0$, then by (8.12),

$$
\frac{32}{\Pi} c_{18} N \log Q_0 = \frac{N}{\sigma - (1 - 1/96c_{18} \log Q_0)} \leq \frac{5}{\sigma - 1} + c_{18} \log Q_0 = 16c_{18} \log Q_0.
$$
It follows that (since $N$ is an integer) $N \leq \left[ \frac{16}{32} \right] = 5$. By the bound (8.11), $L_\chi(s)$ has at most one real zero $\beta$, necessarily simple, satisfying $\beta \geq 1 - 1/96c_{18} \log Q_0$.

**Case 3:** $\chi$ is quadratic and $0 < |\gamma| < 1/7c_{18} \log Q_0$. We apply Lemma 3.2 to $L_1(s)L_\chi(s)$. The only singularity is a simple pole at $s = 1$. Both $L_1(s)$ and $L_\chi(s)$ are self-dual, so if $\beta + i\gamma$ is a non-trivial zero of $L_1(s)L_\chi(s) = 0$, then so is $\beta - i\gamma$. By (8.1), the $m$th Dirichlet coefficient of $-L_1'(s)/L_1(s) - L'_\chi(s)/L_\chi(s)$ is $(1 + \chi(m))a_{\pi \otimes \chi}(m)\Lambda(m) \geq 0$, so Lemma 3.2 yields

$$
\frac{2}{\sigma - \beta} \leq \frac{1}{\sigma - 1} + c_{18} \log Q_\gamma.
$$  \hfill (8.14)

If $\sigma = 1 + 1/2c_{18} \log Q_\gamma$ and $0 < |\gamma| < 1/7c_{18} \log Q_0$ in (8.14), then $\beta \leq 1 - 1/8c_{18} \log Q_\gamma$. \hfill \□

We now prove a Siegel-type bound for $\beta_1$ in Lemma 8.5 (if it exists) using the ideas of Hoffstein and Lockhart [HL94]. This is new for all $\pi \in \mathcal{S}_n$ with $n \geq 3$. We begin with an auxiliary calculation. Let $\chi(\text{mod } q)$ and $\chi'(\text{mod } q')$ be distinct non-trivial quadratic Dirichlet characters such that $\gcd(q'q, q_\pi) = 1$, and let $\psi$ be the primitive character that induces $\chi'\chi$ (whose conductor is necessarily coprime to $q_\pi$). These coprimality restrictions ensure that $\pi \neq \pi \otimes \chi$, $\pi \neq \pi \otimes \chi'$, $\pi \neq \pi \otimes \psi$, $q_\pi \otimes \chi = q_\pi q^n$, $q_\pi \otimes \chi' = q_\pi q^n$, and $q_\pi \otimes \psi = q_\pi q^n$. We also have that $\psi|q_\chi q_{\chi'}$. Let

$$
L(s, \Pi^*) = L_1(s)L_\chi(s)L_{\chi'}(s)L_\psi(s).
$$  \hfill (8.15)

By the above discussion, $L(s, \Pi^*)$ is holomorphic apart from a simple pole at $s = 1$. It follows from (8.1) that if $k \geq 1$, then the $p^k$th Dirichlet coefficient of $\log L(s, \Pi^*)$ equals

$$
k^{-1} a_{\pi \otimes \chi}(p^k)(1 + \chi(p^k) + \chi'(p^k) + \psi(p^k)) \geq 0.
$$
The non-negativity of $1 + \chi(p^k) + \chi'(p^k) + \psi(p^k)$ follows from the fact that this sum is a Dirichlet coefficient of the Dedekind zeta function of a biquadratic field. Upon exponentiating, we find that the $m$th Dirichlet coefficient $\lambda_{\Pi^*}(m)$ of $L(s, \Pi^*)$ is non-negative.
Lemma 8.6. If \( \pi \in \mathfrak{S}_n \) and \( \chi \) is a primitive non-trivial real Dirichlet character, then \( L_{\chi}(1) > 0 \) and the Dirichlet coefficients of \( L_1(s)L_{\chi}(s) \) are non-negative.

Proof. Let \( K \) be the quadratic field associated to \( \chi \). If \( \pi_{BC} \) is the base change of \( \pi \) to an automorphic representation of \( GL_n(\mathbb{A}_K) \), then \( L(s, \pi_{BC} \times \pi_{BC}) = L_1(s)L_{\chi}(s) \). Since \( L(s, \pi_{BC} \times \pi_{BC}) \) is holomorphic on \( \mathbb{C} \setminus \{1\} \) apart from a simple pole at \( s = 1 \), the same holds for \( L_1(s)L_{\chi}(s) \). Since \( \gcd(q, q') = 1 \) (hence, \( L_\chi(s) \) is entire), the residue of \( L(s, \pi_{BC} \times \pi_{BC}) \) at \( s = 1 \), which is positive, equals \( L_\chi(1)\text{Res}_{s=1}L_1(s) \). Since \( \text{Res}_{s=1}L_1(s) > 0 \), it follows that \( L_\chi(1) > 0 \). The Dirichlet coefficients of \( L_1(s)L_{\chi}(s) \) are non-negative per case 3 in the proof of Lemma 8.5. □

Let \( 0 < \varepsilon < 1 \), and let \( \beta \in (1 - \varepsilon, 1) \). By (3.13) and the above discussion, we have

\[
L(1/2 + it, \Pi^*) \ll_{\pi, \varepsilon} (q'q)^{n^2/2+\varepsilon}(3 + |t|)^{n^2+\varepsilon}, \tag{8.16}
\]

If \( x \geq 3 \), then since \( \lambda_{\Pi^*}(1) = 1 \), we use (8.16) to deduce that

\[
\frac{1}{e} \leq \sum_{m=1}^{\infty} \frac{\lambda_{\Pi^*}(m)}{m^\beta} e^{-m/x}
= \frac{1}{2\pi i} \int_{3+i\infty}^{3+i\infty} L(s + \beta, \Pi^*)x^s\Gamma(s) \, ds
= \text{Res}_{s=1-\beta} L(s + \beta, \Pi^*)x^s\Gamma(s) + \frac{1}{2\pi i} \int_{1/2-\beta-i\infty}^{1/2-\beta+i\infty} L(s + \beta, \Pi^*)x^s\Gamma(s) \, ds
\ll_{\pi, \varepsilon} L_\chi(1)(q'q)^{\varepsilon}(1 - \beta)^{-1}x^{1-\beta} + L(\beta, \Pi^*) + (q'q)^{n^2+\varepsilon}x^{1/2-\beta}. \tag{8.17}
\]

Proposition 8.7. Recall the notation and hypotheses of Lemma 8.5. If \( \beta_1 \) exists for a primitive character \( \chi \) (mod \( q \)) such that \( \gcd(q, q_\pi) = 1 \), then for all \( \varepsilon > 0 \), there exists an (ineffective) constant \( c'_\varepsilon(\pi) > 0 \) such that \( L_\chi(1) \geq c'_\varepsilon(\pi)q^{-\varepsilon} \).

Proof. It suffices to take \( 0 < \varepsilon < 1 \). Let \( \chi \) (mod \( q \)) and \( \chi' \) (mod \( q' \)) be primitive quadratic Dirichlet characters with, let \( \psi \) be the primitive character that induces \( \chi'\chi \), and recall the definition of \( L(s, \Pi^*) \) from (8.15). Our proof consists of two cases.

First, suppose that there exists no primitive quadratic Dirichlet character \( \omega \) such that \( L_\omega(s) \) does not vanish for \( s \in (1 - \varepsilon/2, 1) \). It then follows that there exists a constant \( c_19 = c_{19}(\pi) > 0 \) such that \( L_1(s)L_{\chi}(s) \neq 0 \) in the interval \( s \in (1 - c_{19}/\log q, 1) \). Since the Dirichlet coefficients of \( L_1(s)L_{\chi}(s) \) are non-negative (using Lemma 8.6) and the residue of \( L_1(s)L_{\chi}(s) \) at \( s = 1 \) is \( L_\chi(1)\text{Res}_{s=1}L_1(s) \), it follows from [HL94, Proposition 1.1] and (3.13) that \( L_\chi(1)\text{Res}_{s=1}L_1(s) \gg_{\pi} (\log q)^{-1} \). Since each term on the left-hand side is positive (using Lemma 8.6), the desired result follows.

Second, suppose that there exists \( \chi' \) (mod \( q' \)) and \( \beta \in (1 - \varepsilon/2, 1) \) such that \( L_{\chi'}(\beta) = 0 \). We may assume that \( q' \) is minimal, subject to this condition. Now, let \( \chi \) be arbitrary. If \( q < q' \), then the minimality of \( q' \) ensures that \( L_{\chi}(s) \neq 0 \) for \( s \in (1 - \varepsilon/2, 1) \), and the preceding case implies the desired result. Suppose now that \( q \geq q' \). If \( L_{\chi}(s) \) has no real zero within a distance of \( c_{17}/\log(3q'qC(\pi)) \) of \( s = 1 \), then since \( q \geq q' \), \( L_{\chi}(s) \) has no real zero within a distance of \( \frac{1}{2}c_{17}/\log(3q'qC(\pi)) \) of \( s = 1 \). Again, the desired result follows by the preceding case. Finally, suppose that \( L_{\chi}(s) \) has a real zero within a distance of \( c_{17}/\log(3q'qC(\pi)) \) of \( s = 1 \). At this stage, we assume that \( \chi \neq \chi' \).
It follows from analysis nearly identical to the second case in Lemma 8.5 (with $L(s, \Pi^*)$ replacing $D(s)$) that $L(s, \Pi^*)$ has at most one real zero within distance $c_{17}/\log(3q'qC(\pi))$ from 1. Since we have supposed that $L_\chi(s)$ has a real zero within distance $c_{17}/\log(3q'qC(\pi))$ of $s = 1$, the above discussion indicates that this is the sole real zero for $L(s, \Pi^*)$ within a distance of $c_{17}/\log(3q'qC(\pi))$ of $s = 1$. It follows that the zero $\beta$ of $L_\chi(s)$ must satisfy

$$\beta \leq 1 - \frac{c_{17}}{\log(3q'qC(\pi))}, \quad \beta \in \left(1 - \frac{\varepsilon}{2}, 1\right).$$

Since $L_\chi(\beta) = 0$, it follows that $L(\beta, \Pi^*) = 0$. Using (8.17), the above bounds on $\beta$, and the fact that $q \geq q'$, we find that

$$1 \ll_{\pi, \varepsilon} L_\chi(1)q^{2\varepsilon}x^{1-\beta} + q^{n^2+2\varepsilon}x^{1/2-\varepsilon} \ll_{\pi, \varepsilon} L_\chi(1)q^{2\varepsilon}x^{\varepsilon/2} + q^{n^2+2\varepsilon}x^{\varepsilon/2-1/2}.$$

Choosing $x = q^{2n}/L_\chi(1)^2$ (which is at least 3 by (3.10) and (3.13)) and solving for $L_\chi(1)$, we find that for all $\varepsilon > 0$, there exists an (ineffective) constant $c_\varepsilon(\varepsilon) > 0$ such that $L_\chi(1) \geq d_\varepsilon(\varepsilon)q^{-\varepsilon(n^2+2)/1-\varepsilon}$. Upon rescaling $\varepsilon$ to $\varepsilon/(n^2 + 2 + \varepsilon)$, we have $L_\chi(1) \geq d_\varepsilon(\varepsilon/(n^2 + 2 + \varepsilon))q^{-\varepsilon}$. As long as $\varepsilon \neq \chi$, the constant $d_\varepsilon(\varepsilon/(n^2 + 2 + \varepsilon))$ is effective. Once we decrease $d_\varepsilon(\varepsilon/(n^2 + 2 + \varepsilon))$ suitably to account for the case where $\chi = \chi'$, the claimed result holds for arbitrary $\chi$. 

**Corollary 8.8.** Recall the notation and hypotheses of Lemma 8.5. If $\beta_1$ exists, then for all $\varepsilon > 0$, there exists an (ineffective) constant $c_\varepsilon(\varepsilon) > 0$ such that $\beta_1 \leq 1 - c_\varepsilon(\varepsilon)q^{-\varepsilon}$.

**Proof.** If $\beta_1$ exists, then there exists $\sigma \in [\beta_1, 1]$ such that $L'_\chi(\sigma)(1-\beta_1) = L_\chi(1) \geq c_\varepsilon(\varepsilon)q^{-\varepsilon}/2$ by Proposition 8.7 and the mean value theorem. The result follows once we establish the bound $L'_\chi(\sigma) \ll_{\pi, \varepsilon} q^{1/2}$ for $\sigma \in [1 - b_n/\log(3qC(\pi)), 1]$, where $b_n > 0$ is a suitable constant depending at most on $n$. To prove this, we observe via Cauchy’s integral formula that

$$L'_\chi(1) = \frac{1}{2\pi i} \int_{|z| = 1} \frac{L_\chi(z)}{(z-1)^2} \, dz \ll (\log q) \max_{|\xi-1| \leq 1/\log q} |L_\chi(\xi)|,$$

in which case the desired bound follows from (3.13). 

We show that among the primitive characters $\chi(\mod q)$ with $q \leq Q$, we encounter very few with the property that $L_\chi(s)$ has an exceptional zero.

**Lemma 8.9.** Let $Q \geq 3$. There exists an effectively computable constant $c_{20} = c_{20}(n) > 0$ such that there is at most one real non-trivial primitive Dirichlet character $\chi_1(\mod q_1)$ with $q_1 \leq Q$ such that $L_{\chi_1}(s)$ has a real zero $\beta_1$ satisfying $\beta_1 > 1 - c_{20}/\log(C(\pi)Q)$.

**Proof.** Suppose to the contrary that $\chi(\mod q)$ and $\chi'(\mod q')$ are two distinct such characters with $q, q' \leq Q$. Let $\Pi = \pi \oplus \pi \otimes \chi \oplus \pi \oplus \chi'$, let $\psi$ be the primitive character that induces $\chi'\chi$, and let

$$F(s) = L(s, \Pi \times \overline{\Pi}) = L_1(s)^3 L_\chi(s)^2 L_{\chi'}(s)^2 L_\psi(s)^2.$$

By [HR95, Lemma a], the Dirichlet coefficients of $-(F'/F)(s)$ are non-negative. By Lemma 3.2, there exists a constant $c_{21} = c_{21}(n) \geq 1$ such that if $\omega$ runs through the nontrivial zeros of $F(s)$ and $1 < \sigma < 2$, then

$$\sum_{\omega} \text{Re} \left( \frac{1}{\sigma - \omega} \right) < \frac{3}{\sigma - 1} + c_{21} \log(C(\pi)Q). \quad (8.18)$$

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If $\sigma = 1 + 1/2c_{21} \log(C(\pi)Q)$ and $M$ is the number (necessarily an integer) of real zeros (counting multiplicity) of $F(s)$ that are at least $1 - 1/14c_{21} \log(C(\pi)Q)$, then it follows from (8.18) that

$$M \leq \frac{1 + 1/2c_{21} \log(C(\pi)Q) - (1 - 1/14c_{21} \log(C(\pi)Q))}{(1 + 1/2c_{21} \log(C(\pi)Q)) - 1} + c_{21} \log(C(\pi)Q).$$

This implies that $M \leq 3$. However, if $L_\chi(s)$ and $L_\chi'(s)$ both have real zeros that are larger than $1 - 1/14c_{21} \log(C(\pi)Q)$, then $F(s)$ has four such zeros, a contradiction. The lemma now follows.

Proof of Theorem 8.2. This follows from Corollary 8.8 and Lemmas 8.4, 8.5, and 8.9.

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Conflicts of Interest
None.

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