Implied volatility of basket options at extreme strikes

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Abstract

In the paper, we characterize the asymptotic behavior of the implied volatility of a basket call option at large and small strikes in a variety of settings with increasing generality. First, we obtain an asymptotic formula with an error bound for the left wing of the implied volatility, under the assumption that the dynamics of asset prices are described by the multidimensional Black-Scholes model. Next, we find the leading term of asymptotics of the implied volatility in the case where the asset prices follow the multidimensional Black-Scholes model with time change by an independent increasing stochastic process. Finally, we deal with a general situation in which the dependence between the assets is described by a given copula function. In this setting, we obtain a model-free tail-wing formula that links the implied volatility to a special characteristic of the copula called the weak lower tail dependence function.

Key words: implied volatility asymptotics, basket options, index options, large/small strikes, time change, copula

1 Introduction

In option markets, prices of vanilla call and put options are commonly quoted in terms of their implied volatility \( I(T, K) \), defined as the value of the volatility parameter which must be substituted into the Black-Scholes option pricing formula to obtain the quoted option price. Similarly, given a risk-neutral model, one can define the function \( (T, K) \mapsto I(T, K) \) from the prices of vanilla options computed for that model. However, since in most stochastic asset price models the implied volatility function is not known explicitly, it becomes important to obtain efficient and accurate asymptotic approximations for it. Such approximations are useful for at least two reasons. First, they may shed light on the qualitative behavior of the implied volatility in the asset price model, and also on the effect of different model parameters on the shape of the model-generated implied volatility surface. Second, they allow to perform an approximate calibration of the model by comparing the market implied volatility with the asymptotic approximation. Such preliminary estimates can be used as intelligent guesses in the construction of a numerical calibration algorithm to accelerate its convergence.

Approximations to the implied volatility have been studied by many authors in a variety of asymptotic regimes, both in specific models and in model-independent settings. One of the early references on the subject is the book by Lewis [25] dealing with stochastic volatility models. Various

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model-free formulas describing the wing behavior of the implied volatility were obtained in the last
decade. To our knowledge, celebrated Lee’s moment formulas were the first model-independent
asymptotic formulas for the implied volatility at extreme strikes (see [24]). Lee’s results were later
refined by Benaim and Friz [16] and Gulisashvili [18, 19]. In Gao and Lee [14], higher order
asymptotic formulas for the implied volatility at extreme strikes were found, and in Tehranchi [32],
uniform estimates for the implied volatility are obtained. Small-time behavior of implied volatility
is analyzed, among other papers, in [9] (in local volatility models), [12] (for the Heston stochastic
volatility model), [26] (for jump-diffusions), and in [11, 27, 29] (for exponential Lévy models).
Formulae for the implied volatility far from maturity are given in [13] (for the Heston model) and
[31] (model-independent). Finally, sharp price and implied volatility approximations for various
models have been obtained as “expansions around the Black-Scholes model” in [8, 16].

Implied volatility is also quoted in the market for options on a basket of stocks (or on a market
index). Note that the Black-Scholes formula can be applied to price a vanilla option by considering
the entire basket (index) as a log-normal random variable. In such a case, finding reliable asymptotic
approximations to the implied volatility can be even more important, since calculating the exact
value numerically can be computationally very expensive due to the large dimension of the basket.
Approximations based on the small-noise asymptotics in multidimensional local volatility models
have been developed in [3] and more recently refined in [5], but in other asymptotic regimes, much
less is known about multi-asset options, than in the single-asset case.

Our main goal in the present paper is to characterize the asymptotic behavior of the implied
volatility of a call option on a basket of stocks (with positive weights) for large and small strikes.

Three different classes of multidimensional risk-neutral models with increasing generality are con-
sidered in the paper. In Section 3 we discuss the case of correlated log-normal assets, in other
words, the assets which follow the multidimensional Black-Scholes model. Using a recent charac-
terization of the tail behavior of sums of correlated log-normal random variables [22], we obtain a
sharp asymptotic formula with error estimates for the implied volatility at small strikes. On the
other hand, the asymptotics of the implied volatility at large strikes can be easily characterized
using the results obtained in [2]. It turns out that for large strikes, the implied volatility of a basket
call option is approximated by the highest volatility among the stocks in the basket.

Section 4 deals with the case, where the assets follow the multidimensional Black-Scholes model
time-changed by an independent increasing stochastic process. It is assumed in this section that
the marginal density of the time-change process decays at infinity like the function \( s \mapsto s^\alpha e^{-\theta s} \) with
\( \alpha \in \mathbb{R} \) and \( \theta > 0 \). The class of such models includes standard multidimensional extensions of various
exponential Lévy models, for instance, of the variance gamma model, the normal inverse Gaussian
model, or the generalized hyperbolic model. To our knowledge, for such a class of multidimensional
models, the tail behavior of marginal distributions has not been studied before. In Section 4 we
provide two-sided estimates for the distribution function of the asset price in the time-changed
multidimensional Black-Scholes model, and use these estimates to find the leading term in the
asymptotic expansion of the implied volatility.

Finally, in section 5 we deal with the case where the assets in the basket are correlated, and
the dependence structure is described by a given copula function. Here we obtain an asymptotic
formula that can be considered as a generalization to the multidimensional setting of one of the
tail-wing formulae established in [7]. The new tail-wing formula uses a special characteristic of the
copula called \textit{weak lower tail dependence function}. This notion was recently introduced in [30].

Remarks on the notation used in the paper
Let \( f \) and \( g \) be functions defined on \( \mathbb{R} \), and let \( a \in [-\infty, \infty] \). Throughout the present paper, we write “\( f \sim g \) as \( x \to a \)” provided that
\[
\lim_{x \to a} \frac{f(x)}{g(x)} = 1.
\]
We also use the notation “\( f \lesssim g \) as \( x \to a \)” if
\[
\limsup_{x \to a} \frac{f(x)}{g(x)} \leq 1,
\]
and write “\( f(x) \approx g(x) \) as \( x \to a \)” if there exist \( c_1 > 0 \) and \( c_2 > 0 \) such that
\[
c_1 g(x) \leq f(x) \leq c_2 g(x)
\]
for all \( x \) in some neighborhood of \( a \).

A positive function \( f \) defined in \( [a, \infty) \) for some \( a > 0 \) is called regularly varying at infinity with index \( \alpha \in \mathbb{R} \) if for any \( \lambda > 0 \),
\[
\lim_{x \to 0} \frac{f(\lambda x)}{f(x)} = \lambda^\alpha.
\]
for all \( \alpha > 0 \). The class of all regularly varying functions with index \( \alpha \) is denoted by \( R_\alpha \). The elements of the class \( R_0 \) are called slowly varying functions. Regularly varying functions at zero can be defined similarly.

The following set will be used in the paper:
\[
\Delta_d := \{ w \in \mathbb{R}^d : w_i \geq 0, i = 1, \ldots, d, \text{and} \sum_{i=1}^d w_i = 1 \}.
\]
Let \( w \in \Delta_d \). We set
\[
\mathcal{E}(w) := -\sum_{i=1}^d w_i \log w_i, \quad (1)
\]
with the convention \( x \log x = 0 \) for \( x = 0 \).

## 2 Model-free formulae for the implied volatility

Let \( X_t \) be a non-negative martingale on a filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\). Consider a stochastic model where the process \( X \) models the price dynamics of an asset. Define the call and put pricing functions in the price model described above by
\[
C(T, K) = \mathbb{E}[(X_T - K)^+] \quad \text{and} \quad P(T, K) = \mathbb{E}[(K - X_T)^+],
\]
respectively. Here \( T > 0 \) is the maturity, while \( K > 0 \) is the strike price.

The implied volatility \( (T, K) \to I(T, K) \) is determined from the following equality:
\[
C(K, T) = C_{BS}(T, K, \sigma = I(T, K)),
\]
where the symbol $C_{BS}$ stands for the Black-Scholes call pricing function. In the sequel, the maturity $T$ will be fixed, and the implied volatility will be considered as a function of only the strike price.

We will next formulate two model-free asymptotic formulas, characterizing the left-wing behavior of the implied volatility in terms of the put pricing function. These formulas will be needed below.

Suppose the initial condition for the price process is $X_0 = 1$. Suppose also that the asset price model does not have atoms at zero. The previous assumption means that $\mathbb{P}(X_T = 0) = 0$. Then the following asymptotic formula (a zero order formula for the implied volatility) holds:

$$I(K) = \sqrt{\frac{2}{T}} \left[ \log \frac{1}{P(K)} - \frac{1}{2} \log \log \frac{K}{P(K)} - \sqrt{\frac{2}{T}} \left( \log \frac{K}{P(K)} - \frac{1}{2} \log \log \frac{K}{P(K)} \right)^{\frac{1}{2}} \right] + O \left( \left( \log \frac{K}{P(K)} \right)^{\frac{1}{2}} \right) \quad (3)$$

as $K \to 0$. Here $\tilde{P}$ is a positive function satisfying the condition $P(K) \approx \tilde{P}(K)$ as $K \to 0$. Formula (3) was established in [17] (see also Theorem 9.29 in [19]). The fact that the absence of atoms is a necessary condition for the validity of formula (3) was noticed in [10] (see also [20]).

The next asymptotic formula (a first-order formula for the implied volatility) can be easily deduced from the results formulated in [19, Sections 9.6 and 9.9]:

$$I(K) = \sqrt{\frac{2}{T}} \left[ \log \frac{1}{P(K)} - \frac{1}{2} \log \log \frac{K}{P(K)} + \log B(K) - \sqrt{\frac{2}{T}} \left( \log \frac{K}{P(K)} - \frac{1}{2} \log \log \frac{K}{P(K)} + \log B(K) \right)^{\frac{1}{2}} \right] + O \left( \log \log \frac{K}{P(K)} \left( \log \frac{K}{P(K)} \right)^{\frac{1}{2}} \right) \quad (4)$$

as $K \to 0$, where

$$B(K) = \frac{\left( \sqrt{\log \frac{1}{P(K)}} - \sqrt{\log \frac{K}{P(K)}} \right)}{2\sqrt{\pi} \sqrt{\log \frac{1}{P(K)}}}. \quad (5)$$

Formula (4) takes into account the results obtained in [14]. It provides more terms in the asymptotic expansion of the implied volatility at small strikes than formula (3) with $\tilde{P} = P$. More information on model free formulas for the implied volatility can be found in [19].

## 3 Basket options in multidimensional Black-Scholes model

Our goal in the present section is to characterize the asymptotic behavior of the implied volatility at small strikes in the case of a basket option of European style in the n-dimensional driftless Black-Scholes model. We assume that the interest rate is equal to zero. Let $S^1, \ldots, S^n$ be a basket of assets such that

$$\log \tilde{S}_t = \log \tilde{S}_0 - \frac{\text{diag}(\mathcal{B})_t}{2} + \mathcal{B}^{\frac{1}{2}} W_t,$$
where $\tilde{S}_t = (S^1_t, \ldots, S^n_t), \tilde{S}_0 = (S^1_0, \ldots, S^n_0)$, $W$ is an n-dimensional standard Brownian motion, $\mathcal{B}$ is the covariance matrix, and $\text{diag}(\mathcal{B})$ stands for the main diagonal of $\mathcal{B}$. We denote by $(\lambda_1, \ldots, \lambda_n) \in \Delta_n$ the weight vector associated with the assets in the basket.

Consider the price process of the following form:

$$S_t = \sum_{i=1}^n \lambda_i S^i_t, \quad t \geq 0. \quad (6)$$

The initial condition for the process $S$ is given by $S_0 = \sum_{i=1}^n \lambda_i S^i_0$, and we will assume in the sequel that $S^i_0 = 1$ for all $1 \leq i \leq n$. The previous condition implies that $S_0 = 1$. Therefore, $S_t = \sum_{i=1}^n \exp\{Y^i_t\}$, where

$$Y^i_t = \log \lambda_i - \frac{b_{ii} t}{2} + \sum_{j=1}^n \beta_{ij} W^j_t, \quad 1 \leq i \leq n. \quad (7)$$

In (7), the symbols $\beta_{ij}$ stand for the elements of the matrix $\mathcal{B}^{-1}$. We also set

$$\mu_{i,t} = \log \lambda_i - \frac{b_{ii} t}{2}, \quad 1 \leq i \leq n. \quad (8)$$

It is clear that the following equality holds: $\exp\{Y^i_t\} = \lambda_i S^i_t, t > 0, 1 \leq i \leq n$.

### 3.1 Asymptotics of put pricing functions in multidimensional Black-Scholes model

The distribution density of the random variable $S_T$ will be denoted by $p_T$. An asymptotic formula for $p_T$ was recently established in [22]. Let us briefly recall the notation used in that paper. Let $\bar{w} \in \Delta_n$ be the unique vector such that $\bar{w}^\top \mathcal{B} \bar{w} = \min_{w \in \Delta_n} w^\top \mathcal{B} w$. \quad (9)

The existence and uniqueness of $\bar{w}$ follows from the non-degeneracy of the matrix $\mathcal{B}$. We let

$$\bar{n} := \text{Card} \{i = 1, \ldots, n : \bar{w}_i \neq 0\}, \quad (10)$$

$$\bar{I} := \{i = 1, \ldots, n : \bar{w}_i \neq 0\} := \{\bar{k}(1), \ldots, \bar{k}(\bar{n})\},$$

$\bar{\mu} \in \mathbb{R}^\bar{n}$ with $\bar{\mu}_i = \mu_{\bar{k}(i)}$, and $\bar{\mathcal{B}} \in M_{\bar{n}}(\mathbb{R})$ with $\bar{\mathcal{B}}_{ij} = \mathcal{B}_{\bar{k}(i), \bar{k}(j)}$. The inverse matrix of $\mathcal{B}$ is denoted by $\mathcal{B}^{-1}$ and its elements and row sums by $\bar{a}_{ij}$ and $\bar{A}_k := \sum_{j=1}^\bar{n} \bar{a}_{kj}$. We refer the interested reader to [22] for more details and explanations on this notation.

It was established in [22] that under a special restriction on the correlation matrix $\mathcal{B}$ (Assumption (A) in [22]), the following asymptotic formula is valid as $x \to 0$:

$$p_T(x) = C_T \left( \log \frac{1}{x} \right)^{-\frac{\bar{n}}{2}} x^{-1 + \frac{1}{2} \sum_{k=1}^\bar{n} \bar{A}_k \left( \log \frac{\bar{A}_1 + \cdots + \bar{A}_{\bar{n}}}{\bar{A}_k} + \bar{\mu}_{k,T} \right)} \exp \left\{ -\frac{1}{2T} (\bar{A}_1 + \cdots + \bar{A}_{\bar{n}}) \log^2 \frac{1}{x} \right\} \left( 1 + O \left( \left( \log \frac{1}{x} \right)^{-1} \right) \right), \quad (11)$$
where the constant $C$ is given by

$$C_T = \frac{1}{\sqrt{2\pi T}} \frac{\sqrt{A_1 + \cdots + A_n}}{|B|^{\frac{1}{2}} \sqrt{A_1 \cdots A_n}}.$$  

Using formula (11), we can characterize the asymptotic behavior of the put pricing function $P$ at small strikes. This can be done as follows. Consider the fractional integral of order two defined by

$$F_2 M(\sigma) = \int_{\sigma}^{\infty} (\tau - \sigma) M(\tau) d\tau,$$  

where $M$ is a positive function on $(0, \infty)$. Since

$$P(K) = \int_{0}^{K} (K - x)p_T(x) dx,$$

it is not hard to see that

$$P(K) = S^{-1}F_2 M(S), \quad \text{where} \quad S = K^{-1} \quad \text{and} \quad M(y) = y^{-3}p_T(y^{-1}).$$

Using (11), we get

$$M(y) = C_T (\log y) \frac{1 + \eta}{2\pi T} y^{-2 - T^{-1}} \sum_{k=1}^{n} \hat{A}_k \left( \log \frac{A_1 + \cdots + A_n}{\hat{A}_k} + \mu_{k,T} \right) \exp \left\{ -\frac{1}{2T} \left( \hat{A}_1 + \cdots + \hat{A}_n \right) \log^2 y \right\} \left( 1 + O \left( (\log y)^{-1} \right) \right)$$

as $y \to \infty$, where $C_T$ is given by (12).

In [21], a general asymptotic formula was obtained for fractional integrals (see also Theorem 5.3 in [19]). We will next formulate this general result. Suppose

$$M(y) = a(y)e^{-b(y)} \quad \text{for all} \quad y \geq c.$$  

where $c > 0$ is some number. Suppose also that the following conditions hold:

1. $y|a'(y)| \leq \gamma a(y)$ for some $\gamma > 0$ and all $y > c$.
2. $b(y) = B(\log y)$, where $B$ is a positive increasing function on $(c, \infty)$ such that $B''(y) \approx 1$ as $y \to \infty$.

Then as $\sigma \to \infty$,

$$F_2 M(\sigma) = \frac{M(\sigma)}{b'(\sigma)^2} (1 + O((\log \sigma)^{-1})).$$

The function $M$ in (13) satisfies the conditions formulated above. Next, using (11), (12), and (15) with

$$B(u) = \frac{1}{2T} (\hat{A}_1 + \cdots + \hat{A}_n) u^2,$$

we establish the following assertion.
Theorem 1. Let $P$ be the price of the put option defined in (2), and suppose Assumption (A) holds for the covariance matrix $B$ (see [22]). Then, as $K \to 0$,

$$P(K) = \delta_0 \left[ \log \sqrt{\frac{1}{K}} \right]^{\delta_1} \left( \frac{1}{K} \right)^{\delta_2} \exp \left\{ -\delta_3 \log^2 \frac{1}{K} \right\} \left( 1 + O \left( \left( \log \frac{1}{K} \right)^{-1} \right) \right),$$

(17)

where

$$\delta_0 = \frac{C_T T^2}{(A_1 + \cdots + A_n)^2}, \quad \delta_1 = -\frac{3 + \bar{n}}{2},$$

$$\delta_2 = -1 - \frac{1}{T} \sum_{k=1}^{\bar{n}} \bar{A}_k \left( \log \frac{\bar{A}_1 + \cdots + \bar{A}_n}{A_k} + \mu_{k,T} \right), \quad \delta_3 = \frac{1}{2T} (\bar{A}_1 + \cdots + \bar{A}_n),$$

and $C_T$ is given by (12).

Formula (17) will be used in the next subsection to characterize the left-wing behavior of the implied volatility associated with a basket option in the multidimensional Black-Scholes model.

3.2 Left-wing asymptotic behavior of the implied volatility associated with basket options

The next statement characterizes the asymptotic behavior of the implied volatility for small strikes.

Theorem 2. Suppose Assumption (A) holds for the covariance matrix $B$. Then, as $K \to 0$,

$$I(K) = \frac{1}{\sqrt{A_1 + \cdots + A_n}} - \frac{2}{2(A_1 + \cdots + A_n)^2} \sum_{k=1}^{\bar{n}} \bar{A}_k \left( \log \frac{\bar{A}_1 + \cdots + \bar{A}_n}{A_k} + \mu_{k,T} \right) + T \left( \log \frac{1}{K} \right)^{-1}$$

$$- \frac{T(\bar{n} - 1)}{2(A_1 + \cdots + A_n)^2} \log \log \frac{1}{K} \left( \log \frac{1}{K} \right)^{-2} + O \left( \left( \log \frac{1}{K} \right)^{-2} \right).$$

(18)

Remark 1. The leading term in the implied volatility expression above can also be written as

$$\lim_{K \to 0} I(K) = \frac{1}{\sqrt{A_1 + \cdots + A_n}} = \sqrt{\min_{w \in \Delta} w^\top B w}.$$

Proof. It follows from (17) that as $K \to 0$,

$$\log \frac{1}{P(K)} = \log \frac{1}{\delta_0} - \delta_1 \log \log \frac{1}{K} - \delta_2 \log \frac{1}{K} + \delta_3 \log^2 \frac{1}{K}$$

$$+ O \left( \left( \log \frac{1}{K} \right)^{-1} \right)$$

(19)

and

$$\log \frac{K}{P(K)} = \log \frac{1}{\delta_0} - \delta_1 \log \log \frac{1}{K} - (\delta_2 + 1) \log \frac{1}{K}$$

$$+ \delta_3 \log^2 \frac{1}{K} + O \left( \left( \log \frac{1}{K} \right)^{-1} \right)$$

(20)
where $\delta_0$, $\delta_1$, $\delta_2$, and $\delta_3$ are such as in Theorem 1. Moreover, the error term in (4) can be represented as follows:

$$O \left( \log \log \frac{1}{K} \left( \log \frac{1}{K} \right)^{-3} \right).$$  \hspace{1cm} (21)

We will next characterize the asymptotic behavior of $\log B(K)$ as $K \to 0$. Denote the functions on the right-hand side of (19) and (20) by $V_1(K)$ and $V_2(K)$, respectively. Then, using (5), (19), and (20), we obtain

$$\log B(K) = \log \frac{1}{2\sqrt{\pi}} + \log \left[ 1 - \sqrt{1 - \frac{V_1(K) - V_2(K)}{V_1(K)}} \right].$$

It is easy to see that $\log(1 - \sqrt{1 - h}) = \log \frac{h}{2} + O(h)$ as $h \to 0$. Put $h = \frac{V_1(K) - V_2(K)}{V_1(K)}$. Then we have

$$\log B(K) = \log \frac{1}{2\sqrt{\pi}} + \log \frac{V_1(K) - V_2(K)}{2V_1(K)} + O \left( \left( \log \frac{1}{K} \right)^{-1} \right),$$

and hence

$$\log B(K) = \log \frac{1}{4\sqrt{\pi} \delta_3} - \log \log \frac{1}{K} + O \left( \left( \log \frac{1}{K} \right)^{-1} \right)$$  \hspace{1cm} (22)

as $K \to 0$.

Our next goal is to simplify formula (4) by taking into account (19), (20), and (22), and replacing the error term by the expression in (21). We can drop the terms $O \left( \left( \log \frac{1}{K} \right)^{-1} \right)$ in (19), (20), and (22), using the mean value theorem. This will introduce an error term $O \left( \left( \log \frac{1}{K} \right)^{-2} \right)$ in the formula that follows from formula (4). Thus

$$I(K) = \frac{\sqrt{T}}{\sqrt{2}} \left\{ \frac{1}{2} \log \tilde{V}_2(K) + \log \frac{1}{4\sqrt{\pi} \delta_3} - \log \log \frac{1}{K} \right\}$$  \hspace{1cm} (23)

as $K \to 0$, where $\tilde{V}_1(K)$ and $\tilde{V}_2(K)$ denote the functions on the right-hand side of (19) and (20), respectively, without the terms $O \left( \left( \log \frac{1}{K} \right)^{-1} \right)$. Next, using the mean value theorem, we see that it is possible to replace $\tilde{V}_2(K)$ in the expression $\log \tilde{V}_2(K)$ in formula (23) by $\delta_3 \log^2 K$. Now, taking
into account the definitions of $\tilde{V}_1(K)$ and $\tilde{V}_2(K)$, we obtain

$$I(K) = \frac{\sqrt{2}}{\sqrt{T}} \sqrt{- \log \left[ 4\sqrt{\pi \delta_3^3 \delta_3^3 \delta_3^3} \right] - (\delta_1 + 2) \log \log \frac{1}{K} - \delta_2 \log \frac{1}{K} + \delta_3 \log^2 \frac{1}{K}$$

$$- \frac{\sqrt{2}}{\sqrt{T}} \sqrt{- \log \left[ 4\sqrt{\pi \delta_0^3 \delta_0^3 \delta_0^3} \right] - (\delta_1 + 2) \log \log \frac{1}{K} - (\delta_2 + 1) \log \frac{1}{K} + \delta_3 \log^2 \frac{1}{K}$$

$$+ O \left( \left( \log \frac{1}{K} \right)^{-2} \right)$$

(24)

as $K \to 0$. Put

$$h_1(K) = \frac{- \log \left[ 4\sqrt{\pi \delta_0^3 \delta_3^3} \right] - (\delta_1 + 2) \log \log \frac{1}{K} - \delta_2 \log \frac{1}{K}}{\delta_3 \log^2 \frac{1}{K}}$$

and

$$h_2(K) = \frac{- \log \left[ 4\sqrt{\pi \delta_0^3 \delta_3^3} \right] - (\delta_1 + 2) \log \log \frac{1}{K} - (\delta_2 + 1) \log \frac{1}{K}}{\delta_3 \log^2 \frac{1}{K}}.$$

It follows from (24) that

$$I(K) = \frac{\sqrt{2} \sqrt{\delta_3}}{\sqrt{T}} \log \frac{1}{K} \left[ \sqrt{1 + h_1(K)} - \sqrt{1 + h_2(K)} \right] + O \left( \left( \log \frac{1}{K} \right)^{-2} \right)$$

(25)

as $K \to 0$. Next, using the formula $\sqrt{1 + h} = 1 + \frac{1}{2} h - \frac{1}{8} h^2 + O(h^3)$ as $h \to 0$ in (25), we get

$$I(K) = \frac{1}{\sqrt{2T \delta_3}} + \frac{1}{4 \delta_3 \sqrt{2T \delta_3}} \left( \log \frac{1}{K} \right)^{-1} + \frac{\delta_1 + 2}{2 \delta_3 \sqrt{2T \delta_3}} \log \log \frac{1}{K} \left( \log \frac{1}{K} \right)^{-2}$$

$$+ O \left( \left( \log \frac{1}{K} \right)^{-2} \right)$$

(26)

as $K \to 0$. Finally, plugging the values of $\delta_1$, $\delta_2$, and $\delta_3$ given in Theorem 1 into formula (26), we obtain formula (18).

This completes the proof of Theorem 2. \qed

**Remark 2 (Implied volatility in the multidimensional Black-Scholes model for large strikes).** From Theorem 1 in [2], it follows that

$$\mathbb{P}[S_t \geq K] \sim \frac{m_{\sigma, \sqrt{t}}}{\sqrt{2\pi \log K}} \exp \left\{ -\frac{(\log K - \mu)^2}{2\sigma^2} \right\}, \quad K \to \infty,$$

where $\sigma^2 = \max_{k=1,...,n} \mathcal{B}_{kk} \mu = \max_{k,t} \mu_{k,t} \mathcal{B}_{kk} = \sigma^2$ and $m_{\sigma} = \# \{ k : \mathcal{B}_{kk} = \sigma^2, \mu_{k,t} = \mu \}$. From this result, we easily deduce that

$$\mathbb{E}[(S_t - K)^+] \approx \frac{K}{\log^2 K} \exp \left\{ -\frac{(\log K - \mu)^2}{2\sigma^2} \right\}, \quad K \to \infty.$$
Applying Corollary 2.4 in [17] (which is nothing but the right-tail version of formula (3)), we conclude that

\[ I(K) = \sigma + O\left(\frac{\psi(K)}{\log K}\right) \]

as \( K \to +\infty \), where \( \psi \) is any function satisfying \( \psi(K) \to +\infty \) as \( K \to +\infty \).

### 3.3 The case where \( n = 2 \)

The detailed discussion of the behavior of the distribution of the sum of two log-normal variables can be found in [15] and [22]. The covariance matrix in this case is as follows:

\[ B = [b_{ij}] \]

where

- \( b_{11} = \sigma_1^2, b_{12} = b_{21} = \rho \sigma_1 \sigma_2, b_{22} = \sigma_2^2 \)

with \( \sigma_1 > 0, \sigma_2 > 0 \), and the correlation coefficient satisfies \(-1 < \rho < 1\). We will also assume \( \sigma_1 \geq \sigma_2 \). Note that the case where \( \rho < \frac{\sigma_2}{\sigma_1} \) is a regular case, and Assumption (A) holds. In the case where \( \rho > \frac{\sigma_2}{\sigma_1} \), we have to rearrange the rows and the columns of \( B \) (see the example in Section 2.1 of [22]). Then \( \bar{B} = (\sigma_2^2) \), and Assumption (A) holds. The case where \( \rho = \frac{\sigma_2}{\sigma_1} \) is exceptional. Here Assumption (A) does not hold.

The following asymptotic formulas for the implied volatility follow from (18):

- Suppose \( \rho > \frac{\sigma_2}{\sigma_1} \). Then

\[ I(K) = \sigma^2 - \sigma^2 \log \lambda_2 \left( \log \frac{1}{K} \right)^{-1} + O \left( \left( \log \frac{1}{K} \right)^{-2} \right) \]

as \( K \to 0 \).

- Suppose \( \rho < \frac{\sigma_2}{\sigma_1} \). Then

\[ I(K) = \sigma_\infty - \sigma_\infty \left( \frac{T}{2} \sigma_\infty^2 + \left[ \log \lambda_1 - \frac{\sigma_1^2 T}{2} - \log \bar{v} \right] \bar{v} \right. \]

\[ + \left[ \log \lambda_2 - \frac{\sigma_2^2 T}{2} - \log(1 - \bar{v}) \right] (1 - \bar{v}) \left( \log \frac{1}{K} \right)^{-1} \]

\[ - \frac{T}{2} \sigma_\infty^3 \frac{\log \log \frac{1}{K}}{\lambda_1} + O \left( \left( \log \frac{1}{K} \right)^{-2} \right) \]

as \( K \to 0 \), where

\[ \sigma_\infty = \frac{\sigma_1 \sigma_2 \sqrt{1 - \rho^2}}{\sqrt{\sigma_1^2 + \sigma_2^2 - 2 \rho \sigma_1 \sigma_2}} \quad \text{and} \quad \bar{v} = \frac{\sigma_2 (\sigma_2 - \rho \sigma_1)}{\sigma_1^2 + \sigma_2^2 - 2 \rho \sigma_1 \sigma_2} \]

Therefore, the behavior of the implied volatility experiences a qualitative change (phase transition) at \( \rho^* = \frac{\sigma_2}{\sigma_1} \). Indeed, for \( \rho < \rho^* \), the expression in formula (28), approximating the left wing of the implied volatility, depends on the correlation coefficient, while for \( \rho > \rho^* \) the left wing is approximated by a correlation-independent expression (see (27)).

We will next discuss the asymptotic behavior of the implied volatility in the exceptional case where \( n = 2 \) and \( \rho = \rho^* \). The following formula holds for the distribution density \( p_T \) in the
exceptional case (see [15]):

\[ p_T(x) \approx x^{-\frac{\mu_2-x}{\sigma_2}} \left( \log \frac{1}{x} \right)^{-\frac{1}{T(\sigma_1^2 - \sigma_2^2)}} \left( \log \log \frac{1}{x} \right)^{-\frac{1}{4}} \]

\[ \exp \left\{ -\frac{1}{2T(\sigma_1^2 - \sigma_2^2)} \left[ \log \left( \frac{1}{\rho^2} - 1 \right) + \log \log \frac{1}{x} - \log \left( \log \left( \frac{1}{\rho^2} - 1 \right) + \log \log \frac{1}{x} \right) + \mu_{1,T} - \mu_{2,T} \right] \right\} \]

\[ \exp \left\{ -\frac{\log^2 x}{2T\sigma_2^2} \right\} \]  

as \( x \to 0 \). Recall that we assume that \( \mu = 0 \). Recall also that \( \mu_{1,T} \) and \( \mu_{2,T} \) are defined in (8).

Remark 3. Formula (29) can be derived from formula (B20) established at the end of the proof of part (ii) of Theorem 2.3 in [15]. Note that in the present paper we assume \( \sigma_1 \geq \sigma_2 \), while in [15], \( \sigma_1 \leq \sigma_2 \).

Set

\[ V_{1,T} = \log \left( \frac{1}{\rho^2} - 1 \right) + \mu_{1,T} - \mu_{2,T} \quad \text{and} \quad V_2 = \log \left( \frac{1}{\rho^2} - 1 \right). \]  

(30)

It is not hard to see using the mean value theorem that

\[ \log^2 \left( V_2 + \log \log \frac{1}{x} \right) - \left( \log \log \log \frac{1}{x} \right)^2 = o(1) \]

as \( x \to 0 \). Hence

\[ \exp \left\{ -\frac{1}{2T(\sigma_1^2 - \sigma_2^2)} \log^2 \left( V_2 + \log \log \frac{1}{x} \right) \right\} \sim \exp \left\{ -\frac{1}{2T(\sigma_1^2 - \sigma_2^2)} \left( \log \log \log \frac{1}{x} \right)^2 \right\} \]

as \( x \to 0 \). In addition,

\[ \exp \left\{ \frac{1}{T(\sigma_1^2 - \sigma_2^2)} \left( \log \log \frac{1}{x} \right) \left( \log \left( V_2 + \log \log \frac{1}{x} \right) \right) \right\} \]

\[ \approx \left( \log \frac{1}{x} \right)^{\frac{1}{T(\sigma_1^2 - \sigma_2^2)}} \exp \left\{ \frac{1}{T(\sigma_1^2 - \sigma_2^2)} \left( \log \log \frac{1}{x} \right) \left( \log \log \log \frac{1}{x} \right) \right\} \]

as \( x \to 0 \). Therefore, (29) implies the following estimate for the density \( p_T \):

\[ p_T(x) \approx \frac{1}{x}^{-\frac{\mu_2-x}{\sigma_2}} \left( \log \frac{1}{x} \right)^{-\frac{V_{1,T}}{T(\sigma_1^2 - \sigma_2^2)}} \left( \log \log \frac{1}{x} \right)^{\frac{V_{1,T}}{T(\sigma_1^2 - \sigma_2^2)} - \frac{1}{4}} \]

\[ \exp \left\{ -\frac{\log^2 \frac{1}{x}}{2T\sigma_2^2} \right\} \exp \left\{ -\frac{1}{2T(\sigma_1^2 - \sigma_2^2)} \left( \log \log \log \frac{1}{x} \right)^2 \right\} \]

\[ \exp \left\{ -\frac{1}{2T(\sigma_1^2 - \sigma_2^2)} \left( \log \log \log \frac{1}{x} \right)^2 \right\} \]

\[ \exp \left\{ \frac{1}{T(\sigma_1^2 - \sigma_2^2)} \left( \log \log \log \frac{1}{x} \right) \left( \log \log \log \frac{1}{x} \right) \right\} \]  

(31)
Our next goal is to obtain a two-sided estimate for the put pricing function $P$, by taking into account formula (31). We will use the ideas employed in the proof of Theorem 1. Let us set

$$B(u) = \frac{u^2}{2T\sigma_2^2} + \frac{\log^2 u}{2T(\sigma_1^2 - \sigma_2^2)} + \frac{(\log \log u)^2}{2T(\sigma_1^2 - \sigma_2^2)} - \frac{1}{T(\sigma_1^2 - \sigma_2^2)}(\log u)(\log \log u)$$

and

$$a(y) = y^{-2 - \frac{\nu_2 T}{2\sigma_2^2}} (\log y)^{-\frac{\nu_1 + T}{\sigma_1^2 - \sigma_2^2}} (\log \log y)^{-\frac{\nu_1 + T}{(\sigma_1^2 - \sigma_2^2)^2}}. $$

It is not hard to see that the restrictions, under which formula (16) is valid, are satisfied. In addition, for the function $b(x) = B(\log x)$, we have $b'(x) \approx \log x$ as $x \to \infty$. Now, reasoning as in the proof of Theorem 1, we obtain the following formula:

$$P(K) \approx \tilde{P}(K) \text{ as } K \to 0, \quad \text{where}$$

$$\tilde{P}(K) = \left(\frac{1}{K}\right)^{-1 - \frac{\nu_2 T}{2\sigma_2^2}} \left(\log \frac{1}{K}\right)^{-\frac{\nu_1 + T}{\sigma_1^2 - \sigma_2^2}}(\log \log \frac{1}{K})^{-\frac{\nu_1 + T}{(\sigma_1^2 - \sigma_2^2)^2}} \exp\left\{\frac{\log^2 \frac{1}{K}}{2T\sigma_2^2} \right\} \exp\left\{-\frac{1}{2T(\sigma_1^2 - \sigma_2^2)} \log(\log \frac{1}{K})^2 \right\} \exp\left\{-\frac{1}{T(\sigma_1^2 - \sigma_2^2)} \log \log \log \frac{1}{K}\right\} \exp\left\{\frac{1}{T(\sigma_1^2 - \sigma_2^2)} \left(\log \log \log \frac{1}{K}\right) \right\}$$

as $K \to 0$. Next, using (3) with $\tilde{P}$ given by (32), and making numerous simplifications, we obtain the following asymptotic formula for the implied volatility in the exceptional case:

$$I(K) = \sigma_2 + O \left(\left(\log \frac{1}{K}\right)^{-1}\right)$$

as $K \to 0$. Comparing formula (33) with formulas (27) and (28), we see that the behavior of the implied volatility at the critical point $\rho = \frac{\sigma_2}{\sigma_1}$, where the qualitative change happens, is similar to that in the case where $\rho > \frac{\sigma_2}{\sigma_1}$.

4 Time-changed multidimensional Black-Scholes model

Recall that in Section 3, we introduced the price process $S$ for a basket of assets (see formula (6)). The present section deals with time changes in such processes. Suppose $\tau, t \geq 0$, is a non-negative non-decreasing stochastic process on $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t \geq 0}, \mathbb{P})$ (a time change). Then, the time-changed process $S$ has the following form: $t \mapsto S_{\tau_t}$. We only consider time changes which are independent of the price process $S$. In the next subsections, two-sided estimates for marginal distribution functions of time-changed price processes such as above will be established. Moreover, the leading term in the asymptotic expansion of the implied volatility associated with a time-changed price process $t \mapsto S_{\tau_t}$ in the $n$-dimensional Black-Scholes model will be found.
4.1 Bounds on distribution functions of sums of log-normal mixtures

The next assertion provides an upper bound for the distribution function of a random variable imitating the random variable $S_{t_i}$ for fixed $t > 0$. The additional drift vector $\tilde{\mu}$ will be needed later to ensure the martingale property.

**Theorem 3** (Upper bound). Let $Y$ be a centered Gaussian vector with covariance matrix $\mathcal{B} = [b_{ij}]_{1 \leq i,j \leq n}$, and let $\mu \in \mathbb{R}^n$ and $\tilde{\mu} \in \mathbb{R}^n$. Suppose $Z$ is a random variable with values in $(0, \infty)$, which has a density $\rho(x)$ satisfying $\rho(s) \leq cs^\alpha e^{-\theta s}$ for $s \geq 1$, where $\theta > 0$, $c > 0$ and $\alpha \in \mathbb{R}$ are constants. Then, there exists $C > 0$ such that as $k \to +\infty$,

$$
\mathbb{P}\left[ \sum_{i=1}^{n} e^{Y_i \sqrt{Z + \mu_i Z + \tilde{\mu}_i}} \leq e^{-k} \right] \leq C k^\alpha e^{-c^* k},
$$

where

$$
c^* = \min_{t \geq 0} \max_{w \in \Delta_n} \left\{ \theta t + \frac{(1 + t \mu^+ w)^2}{2 w^\top \mathcal{B} w t} \right\}. \tag{34}
$$

**Proof.** In this proof, $C$ denotes a constant which may change from line to line. For $k > 0$, set

$$
F_k(t) = \mathbb{P}\left[ \sum_{i=1}^{n} e^{Y_i \sqrt{zt + \mu_i z + \tilde{\mu}_i}} \leq e^{-k} \right].
$$

Fix $w \in \Delta_n$, and let $t$ be such that $1 + t \mu^+ w > 0$. Then, by Jensen’s inequality,

$$
\mathbb{P}\left[ \sum_{i=1}^{n} e^{Y_i \sqrt{zt + \mu_i z + \tilde{\mu}_i}} \leq e^{-k} \right] \leq \mathbb{P}\left[ \sum_{i=1}^{n} w_i Y_i + (1 + t \mu^+ w + \tilde{\mu}^+ w + \mathcal{E}(w)) \leq -k \right]
$$

$$
= N \left( \frac{-k + tk \mu^+ w + \tilde{\mu}^+ w + \mathcal{E}(w)}{\sqrt{w^\top \mathcal{B} w t}} \right) \leq \frac{C \sqrt{t}}{(1 + t \mu^+ w) \sqrt{k}} \exp \left\{ -\frac{(1 + t \mu^+ w)^2}{2 w^\top \mathcal{B} w t} \right\}
$$

$$
\times \exp \left\{ -\frac{\mathcal{E}(w) + \tilde{\mu}^+ w}{w^\top \mathcal{B} w t} \right\} \exp \left\{ -\frac{(\tilde{\mu}^+ w + \mathcal{E}(w))^2}{2 w^\top \mathcal{B} w t} \right\}
$$

$$
\leq \frac{C \sqrt{t}}{(1 + t \mu^+ w) \sqrt{k}} \exp \left\{ -\frac{1}{2 w^\top \mathcal{B} w t} \left( 1 + t \mu^+ w \right)^2 \right\}
$$

$$
\times \exp \left\{ -\frac{\tilde{\mu}^+ w}{w^\top \mathcal{B} w t} \right\} \exp \left\{ -\frac{\tilde{\mu}^+ w}{w^\top \mathcal{B} w t} \right\},
$$

where $\mathcal{E}(w)$ is defined by (1).

Consider the following function:

$$
F(t, w) = \theta t + \frac{(1 + t \mu^+ w)^2}{2 w^\top \mathcal{B} w t}.
$$

The following lemma establishes some properties of this function. The proof is given in the appendix.

**Lemma 1.** There exists a unique couple $(\tilde{t}, \tilde{w})$, with $\tilde{t} \in (0, \infty)$ and $\tilde{w} \in \Delta_n$ such that

$$
F(\tilde{t}, \tilde{w}) = \min_{t \geq 0} \max_{w \in \Delta_n} F(t, w).
$$
In addition, the function

\[ f(t) = F(t, \bar{w}) \]

has a unique minimum at the point \( \bar{t} \).

We clearly have \( 1 + \bar{t} \mu^\perp \bar{w} > 0 \). Indeed, if \( 1 + \bar{t} \mu^\perp \bar{w} < 0 \) then \( f(-\frac{\mu^\perp \bar{w}}{\mu^\perp \bar{w}}) < f(\bar{t}) \) which contradicts the fact that \( \bar{t} \) is the minimizer. If \( 1 + \bar{t} \mu^\perp \bar{w} = 0 \) then \( f'(\bar{t}) = \theta \) which also leads to a contradiction. Let

\[ T' = \begin{cases} \frac{1}{\mu^\perp \bar{w}} & \mu^\perp \bar{w} < 0 \\ +\infty & \text{otherwise,} \end{cases} \]

Remark that if \( T' < \infty \), then \( f(T') = \theta T' > f(\bar{t}) \). Let us also choose \( T \) small enough so that

\[ 1 - |\mu^\perp \bar{w}| T \geq \frac{1}{2} \quad \text{and} \quad \frac{1}{8 \bar{w}^\perp B \bar{w} T} > f(\bar{t}). \]

and assume that \( k \) is large enough so that \( k + 8 \tilde{\mu} \bar{w} > 0 \). We bound the distribution function of the Gaussian mixture from above as follows:

\[
\begin{split}
&\mathbb{P}\left[ \sum_{i=1}^{n} e^{Y_i \sqrt{Z + \mu^i \zeta + \tilde{\mu}^i \zeta} \leq e^{-k}} \right] = \mathbb{E}[F_{Z/k}(k)] = \int_0^\infty \rho(s) F_{s/k}(k) ds = k \int_0^\infty \rho(tk) F_t(k) dt \\
&\leq k \max_{0 \leq t \leq T} F_t(k) + k \int_T^{T'} \frac{C(tk)^\alpha \sqrt{t}}{\sqrt{k(1 + \mu t)^\perp \bar{w}}} e^{-kf(t)} dt + c k \int_{T'}^\infty e^{-tk} (t/k)^\alpha dt. \\
& \quad (36)
\end{split}
\]

Now, by the choice of \( T \), the first term on the right-hand side of the last inequality in (36) satisfies

\[
k \max_{0 \leq t \leq T} F_t(k) \leq C \sqrt{k} e^{-\beta k}
\]

with \( \beta > f(t^*) \). The second term is computed using Laplace’s method. As \( k \to +\infty \), up to a constant,

\[
k \int_T^{T'} \frac{C(tk)^\alpha \sqrt{t}}{\sqrt{k(1 + \mu t)^\perp \bar{w}}} e^{-kf(t)} dt \sim C k^\alpha e^{-k f(t^*)}.
\]

Finally, the last term is negligible by the choice of \( T' \).

The proof of Theorem 3 is thus completed. \qed

Our next goal is to establish a lower estimate complementing the estimate in Theorem 3. Note that the estimates in Theorems 3 and 4 are off by the factor \( k^{-n} \).

**Theorem 4** (Lower bound). Let \( Y \) be a centered Gaussian vector with covariance matrix \( B \) and let \( \mu \in \mathbb{R}^n \) and \( \tilde{\mu} \in \mathbb{R}^n \). Let \( Z \) be a random variable with values in \((0, \infty)\), which has a density \( \rho(x) \) satisfying \( \rho(s) \geq cs^\alpha e^{-\theta s} \) for \( s \geq 1 \), where \( \theta > 0 \), \( c > 0 \) and \( \alpha \in \mathbb{R} \) are constants. Then, there exists \( C > 0 \) such that as \( k \to +\infty \),

\[
\mathbb{P}\left[ \sum_{i=1}^{n} e^{Y_i \sqrt{Z + \mu^i \zeta + \tilde{\mu}^i \zeta} \leq e^{-k}} \right] \geq C k^{\alpha - n} e^{-c^* k},
\]

where \( c^* \) is given by (34).
Proof. It is clear that
\[
\mathbb{P}\left[\sum_{i=1}^{n} e^{Y_i \sqrt{kt} + \mu_i k t + \tilde{\mu}_i} \leq e^{-k}\right] \geq \mathbb{P}[Y_i \sqrt{kt} + \mu_i k t + \tilde{\mu}_i \leq -k - \log n, i = 1, \ldots, n].
\]

By Proposition 3.2 in [23], the above probability can be bounded from below (very roughly) as follows:
\[
\mathbb{P}[Y_i \sqrt{kt} + \mu_i k t + \tilde{\mu}_i \leq -k - \log n, i = 1, \ldots, n] \geq C \frac{(1 + k(1 + t))^{n}}{n^{\alpha_{t}/2}},
\]

where
\[
\alpha_{t} = \min_{x \geq \frac{1}{\sqrt{kt}}((k + \log n)1 + k t \mu + \tilde{\mu})} x^{-1} \mathcal{B}^{-1}(x)
\]
\[
= \max_{w \in \mathbb{R}^n} \left\{ -\frac{1}{2} w^\top \mathcal{B} u + u^\top \frac{1}{\sqrt{kt}}((k + \log n)1 + k t \mu + \tilde{\mu}) \right\}
\]
\[
= \max_{w \in \Delta_n} \frac{1}{2w^\top \mathcal{B} w} \left( k + \log n + k t \mu + \tilde{\mu} \right)^2
\]
\[
\leq \max_{w \in \Delta_n} (1 + t \mu^\top w)^2 + \max_{w \in \Delta_n} \frac{(1 + t \mu^\top w)(\log n + \tilde{\mu}^\top w)}{w^\top \mathcal{B} w} + \max_{w \in \Delta_n} \frac{(\log n + \tilde{\mu}^\top w)^2}{2w^\top \mathcal{B} w}
\]

Finally, we bound the distribution function of the Gaussian mixture from below as follows:
\[
\mathbb{P}\left[\sum_{i=1}^{n} e^{Y_i \sqrt{Z} + \mu_i Z + \tilde{\mu}_i} \leq e^{-k}\right] = k \int_{0}^{\infty} \rho(tk) F(t) dt \geq C k \int_{-1/k}^{1/k} (tk)^{\alpha_{t}} e^{-\theta tk} F(t) dt
\]
\[
\geq \frac{C (tk)^{\alpha_{t}}}{(1 + k(1 + t))^{n}} \int_{-1/k}^{1/k} \exp \left\{ -\theta tk - k \max_{w \in \Delta_n} \frac{(1 + t \mu^\top w)^2}{2w^\top \mathcal{B} w} \right\} dt
\]
\[
\geq \frac{C (tk)^{\alpha_{t}}}{(1 + k(1 + t))^{n}} \exp \left\{ -\theta tk - k \max_{w \in \Delta_n} \frac{(1 + t \mu^\top w)^2}{2w^\top \mathcal{B} w} \right\} = \frac{C k^{\alpha_{t}} e^{-k f(t)}}{(1 + k(1 + t))^{n}}
\]

Remark 4. Theorems 3 and 4 show that under their assumptions, the dominating factor describing the decay of the left tail of the price of a portfolio of assets is exponential with the decay rate equal to the constant \(c^*\). For example, for \(n = 1\), we have
\[
c^* = \min_{t \geq 0} \{ \theta + \frac{(1 + \mu t)^2}{2\sigma^2} \} = \frac{\sqrt{2\theta \sigma^2 + \mu^2 + \mu}}{\sigma^2}.
\]

In symmetric models with \(\mu = 0\), the formula for \(c^*\) simplifies to
\[
c^* = \sqrt{\frac{2\theta}{\min_{w \in \Delta_n} w^\top \mathcal{B} w}}.
\]
4.2 Implied volatility asymptotics

Let $S^1, \ldots, S^n$ be assets such that

$$\log \tilde{S}_t = \log \tilde{S}_0 + \tilde{\mu} t + \mu \tau t + B^\tau \tilde{W}_\tau,$$

where we use the same notation as in the beginning of Section 3. Let $S$ denote the price process of the basket. Fix a maturity $T > 0$, and suppose the random variable $\tau_T$ has a density $\rho_T$. Suppose also that there exist $c_1 > 0$, $c_2 > 0$, $\theta > 0$ and $\alpha \in \mathbb{R}$ such that

$$c_1 s^{\alpha} e^{-\theta s} \leq \rho_T(s) \leq c_2 s^{\alpha} e^{-\theta s}, \quad s \geq 1.$$  \hspace{1cm} (37)

We assume that for every $i = 1, \ldots, n$,

$$\theta > \mu_i + \frac{\mathbb{B}_i}{2}.$$  \hspace{1cm} (38)

This assumption implies that there exists $\varepsilon > 0$ such that

$$\mathbb{E}[(S_i^T)^{1+\varepsilon}] < \infty$$

We then assume further that $\tilde{\mu}_i$ is chosen in such way that

$$\mathbb{E}[S_i^T] = S_i^0.$$  \hspace{1cm} (39)

It follows from Theorems 3 and 4 that there exist $C_1 > 0$, $C_2 > 0$, and $y_0 > 0$ such that

$$C_1 y^{c^*} \left[ \log \frac{1}{y} \right]^{\alpha-n} \leq \mathbb{P}[S_{\tau_T} \leq y] \leq C_2 y^{c^*} \left[ \log \frac{1}{y} \right]^\alpha, \quad y < y_0.$$  \hspace{1cm} (40)

Since we have

$$P(K) = \mathbb{E} \left[ (K - S_{\tau_T})^+ \right] = \int_0^K \mathbb{P}[S_{\tau_T} \leq y] dy,$$

the estimates in (40) imply that there exist $C_3 > 0$, $C_4 > 0$, and $K_0 > 0$ such that

$$C_3 K^{c^*+1} \left[ \log \frac{1}{K} \right]^{\alpha-n} \leq P(K) \leq C_4 K^{c^*+1} \left[ \log \frac{1}{K} \right]^\alpha, \quad K < K_0.$$  \hspace{1cm} (41)

Note that the put pricing pricing in (41) is squeezed between two regularly varying functions with the same index of regular variation at zero. Such estimates allow one to find the leading term in the asymptotic expansion of the implied volatility near zero.

**Theorem 5.** Suppose condition (37) holds for the time-change process $\tau$ and that the assumptions (38) and (39) are satisfied. Then the following asymptotic formula holds for the implied volatility in time-changed $n$-dimensional Black-Scholes model:

$$I(K) \sim \left( \frac{\psi(c^*)}{T} \right)^{\frac{1}{2}} \sqrt{\log \frac{1}{K}},$$

as $K \to 0$, where the function $\psi$ is defined by

$$\psi(u) = 2 - 4(\sqrt{u^2 + u} - u), \quad u > 0$$  \hspace{1cm} (42)

and the constant $c^*$ is given by Formula (34).
Proof. Theorem 5 follows from (41) and Theorem 10.28 in [19].

Remark 5. Condition (37) holds for many processes commonly used as stochastic time changes, e.g., for the gamma process, the inverse Gaussian process, or the generalized inverse Gaussian process. The latter process is used as time change in the generalized hyperbolic Lévy model. Recall that the density of the gamma process is given by

\[ \rho_t(s) = \frac{\lambda^ct}{\Gamma(ct)} s^{ct-1} e^{-\lambda s}, \]

while the density of the inverse Gaussian process is as follows:

\[ \rho_t(s) = \frac{ct}{s^{3/2}} e^{2ct\sqrt{\pi\lambda - \lambda s} - \pi c^2t^2/s}. \]

In the previous formulas, the symbols \( \lambda \) and \( c \) stand for the parameters of the distributions.

We close this section with a counterpart of Theorem 5 for the right tail, which can be deduced from Theorem 10 proved in the next section.

Theorem 6. Suppose condition (37) holds for the time-change process \( \tau \) and that the assumptions (38) and (39) are satisfied. Then the following asymptotic formula holds for the implied volatility in time-changed \( n \)-dimensional Black-Scholes model:

\[ I(K) \sim \left( \frac{\psi(c_{\min})}{T} \right)^{\frac{1}{2}} \sqrt{\log K} \]

as \( K \to +\infty \), where

\[ c_{\min} = \min_{i=1,\ldots,n} \frac{\sqrt{2\theta_{B}B_{ii} + \mu_i^2} - \mu_i}{B_{ii}}. \]

Proof. Let \( G_i(x) = P[\log S_{iT} \geq x] \). By Theorems 3 and 4 there exist constants \( C_1 \) and \( C_2 \) such that

\[ C_1 x^{\alpha} e^{-c_i x} \geq G_i(x) \geq C_2 x^{\alpha-n} e^{-c_i x} \]

as \( x \to +\infty \), where

\[ c_i = \frac{\sqrt{2\theta_{B}B_{ii} + \mu_i^2} - \mu_i}{B_{ii}}. \]

Note that in the single-asset case Theorems 3 and 4 can also be applied to the right tail, by symmetry. It follows that

\[ G_i(x) \sim -c_i x \]

as \( x \to +\infty \), and the proof may be completed by applying Theorem 10.

5 Assets with dependence structure defined by a copula

A popular approach to pricing European style multi-asset options is to calibrate full-fledged models for marginal distributions of asset prices, and then use a copula function from a simple parametric family to model the dependence structure. This is because information about the marginal distributions can be extracted from the prices of single asset options, which are liquidly traded, but the market quotes offer very little information about the dependence.
5.1 A very brief primer on copulas

Recall that the copula of a random vector \((X_1, \ldots, X_n)\) is a function \(C : [0, 1]^n \to [0, 1]\), satisfying the following conditions:

- \(dC\) is a positive measure in the sense of Lebesgue-Stieltjes integration.
- \(C(u_1, \ldots, u_n) = 0\) when \(u_k = 0\) for at least one \(k\).
- \(C(u_1, \ldots, u_n) = u_k\) when \(u_i = 1\) for all \(i \neq k\).

In addition, it is supposed that

\[
\mathbb{P}[X_1 \leq x_1, \ldots, X_n \leq x_n] = C(\mathbb{P}[X_1 \leq x_1], \ldots, \mathbb{P}[X_n \leq x_n]), \quad (x_1, \ldots, x_n) \in \mathbb{R}^n.
\]

A copula exists by Sklar’s theorem and is uniquely defined in the case where the marginal distributions of \(X_1, \ldots, X_n\) are continuous. We refer to [28] for more details on copulas.

The Gaussian copula with correlation matrix \(R\) is the unique copula of any Gaussian vector with correlation matrix \(R\) and nonconstant components (it does not depend on the mean vector and on the variances of the components).

Given a function \(\phi : [0, 1] \to [0, \infty]\) which is continuous, strictly decreasing and such that its inverse \(\phi^{-1}\) is completely monotonic, the Archimedean copula with generator \(\phi\) is defined by

\[
C(u_1, \ldots, u_n) = \phi^{-1}(\phi(u_1) + \cdots + \phi(u_n)).
\]

**Definition 1.** The weak lower tail dependence function \(\chi(\alpha_1, \ldots, \alpha_n)\) of a copula \(C\) is defined by

\[
\chi(\alpha_1, \ldots, \alpha_n) = \lim_{u \to 0} \min_i \log u^{\alpha_i} \log C(u^{\alpha_1}, \ldots, u^{\alpha_n}),
\]

provided that the limit exists and is finite for all \(\alpha_1, \ldots, \alpha_n \geq 0\) such that \(\alpha_k > 0\) for at least one \(k\).

We will next formulate several known assertions (see [30]).

**Theorem 7.** Let \(X_1, \ldots, X_n\) be random variables with state space \((0, \infty)\), marginal distribution functions \(F_1, \ldots, F_n\), and a copula \(C\). Suppose that for every \(k = 1, \ldots, n\), the function \(F_k\) is slowly varying at zero, and there exist constants \(\eta_k, 1 \leq k \leq n\), and a function \(F\) such that

\[
\log F_k(x) \sim \eta_k \log F(x), \quad 1 \leq k \leq n.
\]

Suppose also that the copula \(C\) admits a weak lower tail dependence function \(\chi\). Then,

\[
\lim_{x \downarrow 0} \frac{\log \mathbb{P}[X_1 + \cdots + X_n \leq x]}{\min_i \log \mathbb{P}[X_i \leq x]} = \frac{1}{\chi(\eta_1, \ldots, \eta_n)}.
\]

**Theorem 8.**

- Assume that a copula function \(C\) has strong tail dependence in the left tail, meaning that the limit

\[
\lambda_L = \lim_{u \downarrow 0} \frac{C(u, \ldots, u)}{u},
\]

exists and satisfies \(\lambda_L > 0\). Then, the weak lower tail dependence function of \(C\) satisfies \(\chi(\alpha_1, \ldots, \alpha_n) = 1\).
Let $C$ be a Gaussian copula with correlation matrix $R$ such that $\det R \neq 0$. Then,

$$\chi(\alpha_1, \ldots, \alpha_n) = \max_i \alpha_i \min_{w \in \Delta_n} w^T \Sigma w,$$

for all $\alpha_1, \ldots, \alpha_n > 0$,

where the matrix $\Sigma$ has entries $\Sigma_{ij} = R_{ij} \sqrt{\alpha_i \alpha_j}$, $1 \leq i, j \leq n$.

Let $C$ be an Archimedean copula with a generator function $\phi$ such that $\log \phi^{-1}$ is regularly varying at $\infty$ with index $\lambda > 0$. Then,

$$\chi(\alpha_1, \ldots, \alpha_n) = \max(\alpha_1, \ldots, \alpha_n) \frac{\max(\alpha_1, \ldots, \alpha_n)}{\lambda_1^{1/\lambda} + \cdots + \lambda_n^{1/\lambda}}.$$

### 5.2 Copulas and the implied volatility asymptotics

In this subsection, we study the left-wing behavior of the implied volatility associated with a basket call option. Recall that we denoted by $(Y_1, \ldots, Y_n)$ the vector of logarithmic returns of the risky assets, and by $(\lambda_1, \ldots, \lambda_n)$ the corresponding vector of weights. Let $C$ be the copula of the vector $(Y_1, \ldots, Y_n)$, and $G_i$ be the distribution function of $Y_i$ for $i = 1, \ldots, n$. The implied volatility is considered in this section as a function $k \mapsto I(-k)$ of the variable $-k$, where $k$ is the log-strike defined by $k = \log K$. The tail-wing formulas due to Benaim and Friz (see [7]) play an important role in the sequel.

**Theorem 9.** Let $\alpha > 0$, and assume that the following are true:

- There exists $\varepsilon > 0$ such that $E[e^{-\varepsilon Y_i}] < \infty$, $i = 1, \ldots, n$.
- For every $1 \leq i \leq n$, the function $k \mapsto -\log G_i(-k)$, $k > k_0$, belongs to the class $R_\alpha$ of regularly varying functions, and there exist positive constants $\eta_1, \ldots, \eta_n$ and a function $G$ such that

$$\log G_i(-k) \sim \eta_i \log G(-k) \quad \text{as } k \to \infty. \quad (43)$$

- The copula $C$ admits a weak lower tail dependence function $\chi$.

Then,

$$\frac{I(-k)^2T}{k} \sim \psi \left[ -\frac{\log G(-k)}{k} \frac{\max_i \eta_i}{\chi(\eta_1, \ldots, \eta_n)} \right]$$

as $k \to \infty$, where the function $\psi$ is defined in (42).

**Proof.** The distribution function $F_i$ of the random variable $\lambda_i S_i$ is given by

$$F_i(x) = G_i(\log x - \log \lambda_i).$$

Since the function $\log G_i$ is regularly varying at $-\infty$, it is clear that $\log F_i$ is slowly varying at zero and

$$\log F_i(x) \sim \log G_i(\log x) \sim \eta_i \log G(\log x)$$

as $x \to 0$. It follows from Theorem [7] that

$$\log F(x) \sim \frac{\max_i \eta_i}{\chi(\eta_1, \ldots, \eta_n)} \log G(\log x) \quad \text{as } x \to 0,$$
where \( F \) is the distribution function of \( \sum_{i=1}^{n} \lambda_i S_i \). Equivalently

\[
\log F(e^{-k}) \sim \frac{\max_i \eta_i}{\chi(\eta_1, \ldots, \eta_n)} \log G(-k) \quad \text{as} \quad k \to \infty,
\]

and hence

\[
-\frac{\log F(e^{-k})}{k} \sim -\frac{\log G(-k)}{k} \frac{\max_i \eta_i}{\chi(\eta_1, \ldots, \eta_n)} \quad \text{as} \quad k \to \infty.
\]

It follows from the assumptions in Theorem 9 that \( \log G(-k) \in R_\alpha \) as \( k \to \infty \). Therefore \( \log F(e^{-k}) \in R_\alpha \) as well. Next, using the tail-wing formula of Benaim and Friz (see Theorem 2 in [7]), we obtain

\[
\frac{I(-k)^2 T_k}{k} \sim \psi \left[ -\frac{\log F(e^{-k})}{k} \right] \quad \text{as} \quad k \to \infty.
\]

(46)

We will need the following lemma.

**Lemma 2.** Let \( \psi \) be the function defined by (42), and suppose \( \rho_1 \) and \( \rho_2 \) are positive functions on \((0, \infty)\) such that

\[
\rho_1(x) \to 1 \quad \text{as} \quad x \to \infty.
\]

Then

\[
\frac{\psi(\rho_1(x))}{\psi(\rho_2(x))} \to 1 \quad \text{as} \quad x \to \infty.
\]

(48)

**Proof.** It is not hard to see that for all \( u \geq 0 \),

\[
\psi(u) = \frac{2}{(\sqrt{u + 1} + \sqrt{u})^2}.
\]

(49)

The equality in (49) describes the structure of the function \( \psi \) better than the original definition.

Fix \( \varepsilon > 0 \). Then, using [19] and the inequality \( 1 < \frac{1}{1-\varepsilon} \), we get

\[
\psi((1-\varepsilon)u) \leq \frac{2}{(1-\varepsilon)(\sqrt{u + \frac{1}{1-\varepsilon}} + \sqrt{u})^2} \leq \frac{2}{(1-\varepsilon)(\sqrt{u + 1} + \sqrt{u})^2} = \frac{1}{1-\varepsilon} \psi(u).
\]

Similarly

\[
\psi((1+\varepsilon)u) \geq \frac{1}{1+\varepsilon} \psi(u).
\]

Therefore,

\[
\frac{1}{1+\varepsilon} \psi(u) \leq \psi((1+\varepsilon)u) \leq \psi((1-\varepsilon)u) \leq \frac{1}{1-\varepsilon} \psi(u).
\]

(50)

It follows from (47) that for every \( \varepsilon > 0 \) there exists \( x_\varepsilon > 0 \) such that

\[
(1-\varepsilon)\rho_2(x) \leq \rho_1(x) \leq (1+\varepsilon)\rho_2(x)
\]
for all $x > x_\varepsilon$. Since the function $\psi$ decreases on $(0, \infty)$, we have

$$
\psi((1 + \varepsilon)\rho_2(x)) \leq \psi(\rho_1(x)) \leq \psi((1 - \varepsilon)\rho_2(x))
$$

for all $x > x_\varepsilon$. Now, using (43), we obtain

$$
\frac{1}{1 + \varepsilon}\psi(\rho_2(x)) \leq \psi(\rho_1(x)) \leq \frac{1}{1 - \varepsilon}\psi(\rho_2(x))
$$

for all $x > x_\varepsilon$, and (43) follows.

Finally, it is not hard to see that (45), (46), and Lemma 2 imply (44).

This completes the proof of Theorem 9.

The next example shows that condition (43) does not prevent one from choosing different marginal laws for different components of the process $(Y_1, \ldots, Y_n)$ as long as these laws have a similar tail behavior.

**Example 1.** Let us consider the following multidimensional extension of the example given in Section 5.2 of [7]. We assume that for $i = 1, \ldots, n$, the distribution of the random variable $Y_i$ is normal inverse Gaussian, more precisely, $\text{NIG}(\alpha_i, \beta_i, \mu_i, \delta_i)$. It is also supposed that the parameters satisfy $\alpha_i > |\beta_i| > 0$ and $\delta_i > 0$. This means that the moment generating function of $Y_i$ is given by

$$
M_i(z) = \exp\left(\delta_i \left\{ \sqrt{\alpha_i^2 - \beta_i^2} - \sqrt{\alpha_i^2 - (\beta_i + z)^2} \right\} + \mu_i z \right).
$$

We refer the reader to [4] for more details on the normal inverse Gaussian distribution. In particular, it follows that $Y_i$ has a density $g_i$ which satisfies the following condition:

$$
g_i(k) \sim C_i |k|^{-\frac{3}{2}} e^{-\alpha_i |k| + \beta_i k}, \quad k \to \pm \infty,
$$

where $C_i$ is a constant. Using Theorem 2 in [7], we see that $-\log G_i(-k) \in R_\alpha$ as $k \to +\infty$, and also

$$
-\log G_i(-k) \sim -\log g_i(-k) \sim (\beta_i - \alpha_i)k, \quad k \to +\infty.
$$

Therefore, the condition in (43) holds with $\lambda_i = \alpha_i - \beta_i$ and $G(k) = e^k$.

Assuming that the dependence structure of $(Y_1, \ldots, Y_n)$ is described by the Gaussian copula with correlation matrix $R$, we see that

$$
\frac{I(-k)^2 T}{k} \sim \psi \left[ \frac{1}{\inf_{w \in \Delta_\delta} w^\top \Sigma w} \right], \quad k \to +\infty,
$$

(51)

where the matrix $\Sigma = [\Sigma_{ij}]$ is such that

$$
\Sigma_{ij} = \frac{R_{ij}}{\sqrt{(\alpha_i - \beta_i)(\alpha_j - \beta_j)}}.
$$

In other words, the implied variance is asymptotically linear, with a correlation-dependent limiting slope, which is given by the right-hand side of (51).
For the sake of completeness, we include a proposition that is a counterpart of Theorem 9 in the case of the right tail. This proposition turns out to be somewhat trivial: the leading order of the implied volatility is determined by a single component with the fattest tail, and it does not depend on the copula. Let us denote by $G_i$ the survival function of $Y_i$, i.e., the function $G_i(x) = P[Y_i \geq x]$.

**Theorem 10.** Let $\alpha > 0$, and suppose that the following assumptions hold:

- There exists $\varepsilon > 0$ such that $E[e^{(1+\varepsilon)Y_i}] < \infty$ for $i = 1, \ldots, n$.
- For each $i = 1, \ldots, n$, the function $k \mapsto -\log G_i(k)$ belongs to the class $R_\alpha$ at infinity.

Then,

$$
\frac{I(k)^2 T}{k} \sim \psi \left[ \frac{1}{\kappa} \max_i \log G_i(k) \right] \quad \text{as } k \to +\infty.
$$

(52)

**Proof.** Set $X_i = v_i e^{Y_i}$. Then we get

$$
P[X_1 + \cdots + X_n \geq x] \geq \max_i P[X_i \geq x],
$$

$$
P[X_1 + \cdots + X_n \geq x] \leq P[\exists i : X_i \geq \frac{x}{n}] \leq \sum_{i=1}^n P[X_i \geq \frac{x}{n}] \leq n \max_i P[X_i \geq \frac{x}{n}].
$$

Since for each $i$, the function $\log G_i$ is regularly varying at infinity, it follows that the function $x \mapsto \log P[X_i \geq x]$ is slowly varying, and therefore, for $x$ sufficiently large and any $\varepsilon > 0$,

$$
\max_i \log P[X_i \geq x/n] \leq (1 + \varepsilon) \max_i \log P[X_i \geq x].
$$

Finally,

$$
\lim_{x \to +\infty} \frac{\log P[X_1 + \cdots + X_n \geq x]}{\max_i \log P[X_i \geq x]} = 1,
$$

and formula (52) follows from Theorem 1 in [7] with a similar proof to that of Theorem 9.

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A Proof of Lemma \[1\]

The function $F$ satisfies
\[
F(t, w) = \max_{\lambda > 0} \{ \theta t + \lambda w^\perp (1 + \mu t) - \frac{\lambda^2 w^\perp \mathcal{B} w t}{2} \},
\]
where $1$ stands for the $n$-dimensional vector with all elements equal to 1. Therefore,
\[
\max_{w \in \Delta_n} F(t, w) = \max_{u \in \mathbb{R}_+^n} \tilde{F}(t, u),
\]
with
\[
\tilde{F}(t, u) = \{ \theta t + u^\perp (1 + \mu t) - \frac{u^\perp \mathcal{B} u t}{2} \}.
\]
Since for every $t > 0$, $\tilde{F}(t, u)$ is strictly concave in $u$, there exists a unique $\tilde{u}(t) \in \mathbb{R}_+^n$ with $\tilde{u}(t) \neq 0$ such that $\tilde{F}(t, \tilde{u}) = \max_{u \in \mathbb{R}_+^n} \tilde{F}(t, u)$. This in turn implies that there exists a unique $\tilde{w}(t)$ such that $F(t, \tilde{w}) = \max_{w \in \Delta_n} F(t, w)$. It is also easy to see that $\tilde{u}(t)$ depends continuously on $t$.

Let $\tilde{f}(t) = \tilde{F}(t, \tilde{u}(t))$. We would like to show that $\tilde{f}$ is differentiable in $t$ and compute its derivative. $\tilde{u}(t)$ may be characterized as follows: for $i = 1, \ldots, n$
\[
[1 + \mu t - t \mathcal{B} \tilde{u}(t)]_i = 0 \quad \text{if} \quad \tilde{u}(t)_i > 0 \quad \text{(53)}
\]
\[
[1 + \mu t - t \mathcal{B} \tilde{u}(t)]_i \leq 0 \quad \text{if} \quad \tilde{u}(t)_i = 0. \quad \text{(54)}
\]
Let $I(t)$ denote the set of indices $i \in \{1, \ldots, n\}$ such that $\bar{u}(t)_i > 0$, and, for a vector $x \in \mathbb{R}^n$, let $x_{I(t)}$ denote the subset of components of $x$ with indices in $I(t)$: $x_{I(t)} = \{x_i : i \in I(t)\}$. Furthermore, let $\mathfrak{B}_{I(t),I(t)}$ denote the submatrix of the covariance matrix, containing the elements $b_{ij}$ with $i \in I(t)$ and $j \in I(t)$. Then, the vector $\bar{u}(t)$ satisfies

$$\bar{u}(t)_{I(t)} = \frac{1}{t} \mathfrak{B}_{I(t),I(t)}^{-1} (1 + \mu t)_{I(t)}, \quad v(t)_{I(t)} = 0,$$

where the set $\tilde{I}(t)$ contains the indices $i \in \{1, \ldots, n\}$ which are not in $I(t)$.

Now, fix $t \in (0, \infty)$ and for $t' \in (0, \infty)$, define

$$v(t')_{I(t)} = \frac{1}{t'} \mathfrak{B}_{I(t),I(t)}^{-1} (1 + \mu t')_{I(t)}, \quad v(t)_{\tilde{I}(t)} = 0$$

First, assume that for all $i$ such that $\bar{u}(t)_i = 0$, either $[1 + \mu t - t \mathfrak{B} \bar{u}(t)]_i < 0$ (with strict inequality) or

$$[1 + \mu t' - t' \mathfrak{B} v(t')]_i = 0$$

for all $t' \in (0, \infty)$. We shall call this assumption Assumption 1. Then we can find $\delta > 0$, such that for every $t' \in (0, \infty)$ with $|t' - t| < \delta$, $v(t')$ satisfies the characterization (53)–(54). Therefore, $v(t') = \bar{u}(t')$. This means that

$$\bar{f}(t') = \theta t' + \frac{1}{2t'} (1 + \mu t')_{I(t)}^{1/2} \mathfrak{B}_{I(t),I(t)}^{-1} (1 + \mu t')_{I(t)}.$$

Therefore, $\bar{f}$ is differentiable at $t$ with first derivative given by

$$\bar{f}'(t) = \theta - \frac{1}{2t} \mathfrak{B}_{I(t),I(t)}^{-1} (1 + \mu t)_{I(t)}^{1/2} + \frac{1}{2} \mu t \mathfrak{B}_{I(t),I(t)}^{-1} (1 + \mu t)_{I(t)} = \theta - \frac{1}{2t} \bar{u}(t)_{I(t)}^{1/2} (1 - \mu t)$$

and second derivative

$$\bar{f}''(t) = \frac{1}{t^3} \mathfrak{B}_{I(t),I(t)}^{-1} (1 + \mu t)_{I(t)}^{1/2}.$$

Now assume that there exists at least one $i$ such that $\bar{u}(t)_i = 0$ and $[1 + \mu t - t \mathfrak{B} \bar{u}(t)]_i = 0$, or, equivalently,

$$[1 + \mu t' - t' \mathfrak{B} v(t')]_i = 0$$

with $t' = t$. The case when the above equality holds for all $t'$ is covered by Assumption 1. Since the left-hand side is linear in $t'$, this means that for a given index set $I(t)$ and for a given $i$, there exists only one $t' \in (0, \infty)$ which satisfies the above equality. Since the number of possible index sets is finite, we conclude that there is at most a finite number of elements $t \in (0, \infty)$ which do not satisfy Assumption 1. But then, we can conclude by continuity that $\bar{f}$ is strictly convex (which entails uniqueness of $\bar{t}$) and differentiable for all $t \in (0, \infty)$, with the derivative given by (55) or alternatively by

$$\bar{f}'(t) = \theta - \frac{1}{2t^2 \bar{w}(t)^{1/2} \mathfrak{B} \bar{w}(t)} + \frac{(\bar{w}(t)^{1/2} \mu)^2}{2 \bar{w}(t)^{1/2} \mathfrak{B} \bar{w}(t)}.$$

Comparing this with the derivative of $f$, which is easily computed, we see that at the point $\bar{t}$, these derivatives coincide. Since this point is characterized by the first order condition $\bar{f}'(\bar{t}) = 0$, and the function $f$ is strictly convex, $f$ also attains its unique minimum at $\bar{t}$.