HYPERBOLIC POLYHEDRAL SURFACES WITH REGULAR FACES

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Abstract. We study hyperbolic polyhedral surfaces with faces isometric to regular hyperbolic polygons satisfying that the total angles at vertices are at least $2\pi$. The combinatorial information of these surfaces is shown to be identified with that of Euclidean polyhedral surfaces with negative combinatorial curvature everywhere. We prove that there is a gap between areas of non-smooth hyperbolic polyhedral surfaces and the area of smooth hyperbolic surfaces. The numerical result for the gap is obtained for hyperbolic polyhedral surfaces, homeomorphic to the double torus, whose 1-skeletons are cubic graphs.

1. Introduction

The combinatorial curvature for planar graphs, as the generalization of the Gaussian curvature for surfaces was introduced by [1, 2, 3, 4]. Many interesting geometric and combinatorial results have been obtained since then, see e.g. [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23].

Let $(V,E)$ be an undirected, locally finite, simple graph with the set of vertices $V$ and the set of edges $E$. The graph $(V,E)$ is called semiplanar if it is topologically embedded into the surface $S$, see [24]. We write $G = (V,E,F)$ for the combinatorial structure, or the cell complex, induced by the embedding where $F$ is the set of faces, i.e. connected components of the complement of the embedding image of the graph $(V,E)$ in the target. Two elements in $V,E,F$ are called incident if the closures of their images have non-empty intersection. We say that a graph $G$ is a tessellation of $S$ if the following hold, see e.g. [21]:

(1) Every face is homeomorphic to a disk whose boundary consists of finitely many edges of the graph.

(2) Every edge is contained in exactly two different faces.

(3) For all two faces whose closures have non-empty intersection, the intersection is either a vertex or an edge.

In this paper, we only consider tessellations and call them semiplanar graphs for the sake of simplicity. For each semiplanar graph, we always assume that for each vertex $x$ and face $\sigma$,

$$\deg(x) \geq 3, \quad \deg(\sigma) \geq 3$$

Key words and phrases. critical area; gap; hyperbolic tiling with regular hyperbolic polygons.

The first author is partially supported by JSPS KAKENHI [grant Number JP16K05247].
where $\deg(\cdot)$ denotes the degree of a vertex or a face. For each semiplanar graph $G$, the combinatorial curvature at the vertex is defined as

$$\Phi(x) = 1 - \frac{\deg(x)}{2} + \sum_{\sigma \in F: x \in \sigma} \frac{1}{\deg(\sigma)}, \quad (x \in V),$$

where the summation is taken over all faces $\sigma$ incident to $x$. To digest the definition, we endow the ambient space of $G$ with a canonical piecewise flat metric and call it the (regular) Euclidean polyhedral surface, denoted by $S(G)$: Replace each face by a regular Euclidean polygon of same facial degree and of side length one, glue them together along their common edges, and define the metric on the ambient space via gluing metrics, see [25, Chapter 3]. It is well-known that the generalized Gaussian curvature on a Euclidean polyhedral surface, as a measure, concentrates on the vertices. And one is ready to see that the combinatorial curvature at a vertex is in fact the mass of the generalized Gaussian curvature at that vertex up to the normalization $2\pi$, see e.g. [26, 24]. For each finite semiplanar graph embedded into a surface $S$, the Gauss-Bonnet theorem, see e.g. [15, Theorem 1.2], reads as

$$\sum_{x \in V} \Phi(x) = \chi(S),$$

where $\chi(\cdot)$ is the Euler characteristic of the surface.

We denote by

$$\mathcal{NC}_{<0} := \{G = (V,E,F): \Phi(x) < 0, \forall x \in V\}$$

the class of semiplanar graphs with negative combinatorial curvature everywhere. There are many examples in the class $\mathcal{NC}_{<0}$.

We review some known results on the class $\mathcal{NC}_{<0}$. For each semiplanar graph $G$ in $S = \mathbb{R}^2$, there hold various isoperimetric inequalities, see e.g. [5, 6, 7, 9, 10, 11, 20, 21]. The following proposition is proved by Higuchi.

**Proposition 1** ([7, Proposition 2.1]).

$$x \in V, \Phi(x) < 0 \Rightarrow \Phi(x) \leq -\frac{1}{1806}.$$  

Equality holds if and only if $\deg x = 3$ and $x$ is incident to $3$, $7$, and $43$-gons.

2. **Hyperbolic polyhedral surfaces and main results**

In this paper, we study hyperbolic polyhedral surfaces with faces isometric to regular hyperbolic polygons. Let $\mathbb{H}^2$ be the simply connected hyperbolic surface of constant curvature $-1$. We only consider regular hyperbolic polygons in the hyperbolic space $\mathbb{H}^2$ with at least three sides. For each $n \geq 3$ and $a > 0$, there is a regular hyperbolic $n$-gon in $\mathbb{H}^2$ of side length $a$, denoted by $\Delta_a(n)$, which is unique up to the hyperbolic isometry. Analogous to Euclidean polyhedral surfaces, we define hyperbolic polyhedral surfaces associated to a semiplanar graph. For each semiplanar graph $G = (V,E,F)$ and $a > 0$, we replace each face by a regular hyperbolic polygon in $\mathbb{H}^2$ of side length $a$, and glue them together along their common edges. This induces a metric structure on the ambient space of $G$, called hyperbolic polyhedral surface of $G$ with side length $a$ and denoted by $S_a^g(\mathbb{H}^2)(G)$. In the literature, hyperbolic polyhedral surfaces are well studied for circle packings on triangulations of surfaces whose side lengths are induced by the radii of circles, which indicates that the triangles are not necessarily regular in that setting, see
In our setting, we consider general polyhedral surfaces with arbitrary faces, but restrict to those with constant side length everywhere, focusing on the combinatorics of such surfaces.

We denote by $\text{Area}_a(G)$ the area of $S^H_a(G)$, which is possibly infinite. For each $x \in V$, the total angle at $x$ measured in $S^H_a(G)$ is denoted by $\theta_a(x)$, and the angle defect at $x$ is defined as

$$K_a(x) = 2\pi - \theta_a(x), \quad a > 0.$$  

By the hyperbolic geometry, we yield the Gauss-Bonnet theorem on hyperbolic polyhedral surfaces. Since the argument is standard, we omit the proof, see e.g. [32].

**Theorem 2.** For each finite semiplanar graph $G = (V, E, F)$ embedded into a surface $S$ and $a > 0$,

$$-\text{Area}_a(G) + \sum_{x \in V} K_a(x) = 2\pi \chi(S). \quad (4)$$

We say that a geodesic metric space $(X, d)$ locally has the (sectional) curvature bounded above by $-1$ in the sense of Alexandrov if it satisfies the Toponogov triangle comparison property with respect to the hyperbolic space $H^2$, and denote by $\text{CAT}_{\text{loc}}(-1)$ the set of such spaces, see e.g. [25]. It is well-known that $S^H_a(G) \in \text{CAT}_{\text{loc}}(-1)$ if and only if $K_a(x) \leq 0$, $\forall x \in V$.

We denote by

$$\mathcal{NC}_{\leq -1} := \{G = (V, E, F): S^H_a(G) \in \text{CAT}_{\text{loc}}(-1) \quad \text{for some} \quad a > 0\}$$

the class of semiplanar graphs whose hyperbolic polyhedral surface of certain side length has the curvature locally bounded above by $-1$ in the Alexandrov sense. We will prove that for each finite semiplanar graph $G$,

$$G \in \mathcal{NC}_{\leq -1} \iff G \in \mathcal{NC}_{< 0},$$

see Proposition [10]. This suggests a possible way to study the class $\mathcal{NC}_{\leq -1}$ by known results on the class $\mathcal{NC}_{< 0}$, where the latter refers to the Euclidean setting.

From now on, we only consider finite semiplanar graphs.

**Definition 3.** Let $G = (V, E, F) \in \mathcal{NC}_{\leq -1}$.

1. We define the pattern of $x \in V$ by

$$\text{Pttn}(x) := (\deg(\sigma_1), \deg(\sigma_2), \cdots, \deg(\sigma_N)),$$

where $\{\sigma_i\}_{i=1}^N$ are the faces incident to $x$ ordered by $\deg(\sigma_1) \leq \deg(\sigma_2) \leq \cdots \leq \deg(\sigma_N)$, and $N = \deg(x)$.

2. We define the critical side length of a vertex $x \in V$, the critical side length of $G$, and the critical area of $G$, respectively by

$$a_c(x) := \max\{a > 0: K_a(x) \leq 0\},$$

$$a_c(G) := \min_{x \in V} a_c(x), \quad \text{and}$$

$$\text{Area}^{\text{cri}}(G) := \text{Area}_{a_c}(G).$$

One is ready to see that $\text{Area}_a(G)$ is monotonically increasing in $a$, which implies that

$$\text{Area}^{\text{cri}}(G) = \max\{\text{Area}_a(G): S^H_a(G) \in \text{CAT}_{\text{loc}}(-1)\}.$$
Let $G$ be a semiplanar graph embedded into $S_g$, the closed orientable surface of genus $g \geq 2$. In particular, for $g = 2$, $S_2$ is called the double torus. We say that a semiplanar graph $G$ admits a hyperbolic tiling with regular hyperbolic polygons if there is a hyperbolic tiling with regular hyperbolic polygons of $S_g$ equipped with a smooth hyperbolic metric, whose semiplanar graph structure is isomorphic to $G$, and denote by $T_{S_g}$ the set of such semiplanar graphs. One is ready to prove the following proposition.

**Proposition 4.** For each semiplanar graph $G$ embedded in $S_g$, the following are equivalent:

1. $G \in T_{S_g}$.
2. There is some $a > 0$ such that $S^{H^2}_a(G)$ is isometric to $S_g$ equipped with a smooth hyperbolic metric.
3. $a_c(x) = a_c(y)$ for all $x, y \in V$.
4. $\text{Area}_{cri}(G) = 2\pi(2g - 2)$.

For $g \geq 2$, we denote by $\mathcal{NC}^g_{\leq -1} := \{G = (V, E, F) \in \mathcal{NC}_{\leq -1} : S(G) = S_g, \#V < \infty\}$ the set of finite semiplanar graphs embedded into $S_g$ whose hyperbolic polyhedral surface $S^{H^2}_a(G) \in \text{CAT}_{loc}(-1)$, for some $a > 0$.

**Example 5.** The Bolza surface \cite{33} is a compact Riemann surface of genus 2.

1. The Bolza surface is a hyperbolic surface that can be defined by a subgroup of the $(2, 3, 8)$ triangle group \cite{34} Section 3 (See also \cite{35} Lemma 2.2). That is, the hyperbolic triangle of inner angles $\frac{\pi}{2}, \frac{\pi}{3},$ and $\frac{\pi}{8}$ tiles the Bolza surface by reflection of the triangle on the edges. The set of the centers of the incircles of the triangles induces a tessellation $G_B$ of $S_2$. Actually, $G_B \in T_{S_2}$, because the side length of the tessellation is the diameter $a$ of the incircles and $K_v(a) = 0$ for each vertex $v$ of $G_B$.

2. We consider the fundamental domain of the Bolza surface in the Poincaré disk where the opposite sides of the octagon are identified in the octagon in Figure 1. Figure 1 is a Delaunay triangulation of the Bolza surface where the points of the same name are identified. Lemma 18 proves that the Voronoi tessellation corresponding to the Delaunay triangulation is in $\mathcal{NC}^2_{\leq -1} \setminus T_{S_2}$. Figure 1 designates the degrees of the 15 vertices of the triangulation, and enumerates the 34 faces of the triangulation.

Figure 1 is drawn by a program the first author wrote with “2D Periodic Hyperbolic Triangulations” package \cite{36} of CGAL 5.4.

We are interested in critical areas of semiplanar graphs in $\mathcal{NC}^g_{\leq -1}$. In particular, we propose the following problem.

**Problem 6.** What is the constant

$$\widetilde{\text{Area}}_{\text{max}} := \sup_{G \in \mathcal{NC}^g_{\leq -1} \setminus T_{S_g}} (\text{Area}_{cri}(G)) ?$$

By Proposition 4

$$\text{Area}_{cri}(G) = 2\pi(2g - 2), \text{ for each } G \in T_{S_g}.$$
One is ready to see that for each $G \in \mathcal{NC}_{\leq -1}$ by Gauss-Bonnet formula (4),

$$\text{Area}^{\text{cri}}(G) \leq 2\pi(2g - 2).$$

Hence graphs in the class $\mathcal{T}_{S_g}$ attain the maximal critical area in the class $\mathcal{NC}_{\leq -1}$. The problem is to determine the maximum of critical areas in the class $\mathcal{NC}_{\leq -1}$ except hyperbolic tilings $\mathcal{T}_{S_g}$. We prove the following gap for the areas.

**Theorem 7.** For each $g \geq 2$, there is a constant $\epsilon > 0$, depending on $g$, such that

$$\tilde{\text{Area}}_{\text{max}} \leq 2\pi(2g - 2) - \epsilon.$$

**Proof.** We will show that there are only finitely many graphs in $\mathcal{NC}_{\leq -1}$, see Corollary 14 in Section 4. Hence the gap for the areas exists by the assertion (4) in Proposition 4. \hfill $\square$

Concerning with the result in Theorem 7 we aim to obtain a quantitative estimate of $\tilde{\text{Area}}_{\text{max}}$ in the following.
**Definition 8.** Let

\[ A_g(N) := \{ G = (V,E,F) \in NC^{g}_{\leq -1} \setminus T_{S_g} : \deg(x) \leq N, \forall x \in V \}, \]

\[ \text{Area}^N_{\text{max}} := \sup_{G \in A_g(N)} \text{Area}^{\text{cri}}(G) \quad (3 \leq N \leq \infty). \]

Lemma 18 proves that Figure 3 is indeed a member of \( A_2(3) \). In Corollary 14, we establish \( \# \left( NC^{g}_{\leq -1} \right) < \infty \) for each \( g \geq 2 \). Note that

1. \( \text{Area}^N_{\text{max}} \) is nondecreasing in \( N \), and
2. \( \text{Area}_{\text{max}}^{\infty} = \text{Area}_{\text{max}} \).

**Theorem 9.** The following is the computation result when we represent numbers with 40 decimal digit in the computation: For \( g = 2 \),

\[ \text{Area}^{\infty}_{\text{max}} \leq 4\pi - 4.96239 \cdots \times 10^{-10}. \]

Let \( g = 2 \). Our strategy is to reformulate the problem to a new one which can be estimated by local arguments. We rewrite

\[ \text{Area}^N_{\text{max}} = 4\pi - \epsilon^N_{\text{max}}, \]

where

\[ \epsilon^N_{\text{max}} := \inf_{G \in A_2(N)} \left( 4\pi - \text{Area}^{\text{cri}}(G) \right). \]

We call this the gap between the maximal critical area \( 4\pi \) and other critical areas. By the Gauss-Bonnet theorem, \( [4] \), we have

\[ \epsilon^N_{\text{max}} = \inf_{G \in A_2(N)} \sum_{x \in V} (-K_{a_c}(G)(x)). \]

Hence for the upper bound estimate in Theorem 9, it suffices to obtain the lower bound estimate of the absolute value of total angle defect for these semiplanar graphs. This new problem fits to local arguments, and we prove the results by enumerating all cases, see Section 5 and Appendix.

The paper is organized as follows: In next section, we introduce basic properties of hyperbolic polyhedral surfaces and prove Proposition 4. We bound combinatorial quantities of semiplanar graphs in the class \( NC^{g}_{\leq -1} \), which are embedded in general surfaces, in Section 4. In Section 5, we refine the above estimates for cubic graphs embedded in the double torus and prove Theorem 9. In Section 6, we propose some further works. We present the algorithm for Theorem 9 in Appendix.

3. Preliminaries

Let \( G = (V,E,F) \) be a semiplanar graph. Two vertices are called neighbors if there is an edge connecting them. We denote by \( \deg(x) \) the degree of a vertex \( x \), i.e. the number of neighbors of a vertex \( x \), and by \( \deg(\sigma) \) the degree of a face \( \sigma \), i.e. the number of edges incident to a face \( \sigma \) (equivalently, the number of vertices incident to \( \sigma \)).

For each regular hyperbolic \( n \)-gon of side length \( a > 0, \Delta_n(a) \), in \( \mathbb{H}^2 \), we denote by \( \beta = \beta_{n,a} \) the inner angle at corners of the \( n \)-gon. By the hyperbolic geometry,

\[ \cosh \frac{a}{2} \sin \frac{\beta}{2} = \cos \frac{\pi}{n}. \]
One is ready to see that $\beta_{n,a}$ is monotonely decreasing in $a$ and

$$\lim_{a \to 0} \beta_{n,a} = \frac{n-2}{n}\pi, \quad \lim_{a \to \infty} \beta_{n,a} = 0$$

where the right side of the first equation is the inner angle of a regular $n$-gon in the plane. The area of $\Delta_n(a)$ is given by

$$\text{Area}(\Delta_n(a)) = (n-2)\pi - n\beta.$$ 

**Proposition 10.** For each finite semiplanar graph $G$,

$$G \in NC_{\leq -1} \iff G \in NC_{< 0},$$

**Proof.** For each $G \in NC_{\leq -1}$, there is $a > 0$ such that $S^H_{\alpha}(G) \in \text{CAT}_{loc}(-1)$. For each face $\sigma \in F$ with $\text{deg}(\sigma) = n$, we know that the inner angle of $\Delta_n(a)$ is less than that of a Euclidean $n$-gon. Consider the Euclidean polyhedral surface $S(G)$. Note that the total angle at each vertex in $S(G)$ is greater than that in $S^H_{\alpha}(G)$. This yields that $G \in NC_{< 0}$. Thus $NC_{\leq -1} \subset NC_{< 0}$.

For the other direction, let $G \in NC_{< 0}$. Note that by (7), for each vertex $x \in V$, there is a small constant $a(x)$ such that the total angle $\theta_{a(x)}(x) > 2\pi$. Since the graph is finite, we can choose a small constant $a$ such that $S^H_{\alpha}(G) \in \text{CAT}_{loc}(-1)$, which proves that $G \in NC_{\leq -1}$. This proves the proposition. \hfill \Box

For each $a > 0$, the total angle at the vertex $x$ in $S^H_{\alpha}(G)$ for some graph $G$ is given by

$$\theta_{\alpha}(x) = \sum_{i=1}^{N} 2\arcsin \left( \frac{\cos \frac{\pi}{f_i}}{\cosh \frac{a}{2}} \right), \quad ((f_1, \ldots, f_N) = \text{Pttn}(x)).$$

As $a > 0$, $\cosh(a/2) > 1$. Thus $\arcsin(\cos(\pi/f_i)/\cosh(a/2))$ is defined. To determine the critical side length of the vertex, $a_{c}(x)$ is the unique solution to the following equation

$$\theta_{\alpha}(x) = 2\pi.$$

Now we prove Proposition [4]

**Proof of Proposition [4]** (1) $\iff$ (2) : This is trivial.

(2) $\Rightarrow$ (3) : This follows from the monotonicity of $\theta_{\alpha}(x)$ in $a$ for each $x \in V$.

(3) $\Rightarrow$ (4) : For $S_{a_{c}(G)}(G)$, it is smooth at each vertex, hence it is locally isometric to a domain in $H^2$. This implies that $S_{a_{c}(G)}(G)$ is isometric to a hyperbolic surface. By the Gauss-Bonnet formula [4], $\text{Area}^{\text{H}(2)}(G) = 2\pi(2g-2)$.

(4) $\Rightarrow$ (2) : We know that the area of $S_{a_{c}(G)}(G)$ is $2\pi(2g-2)$. By the Gauss-Bonnet formula [4], $K_{a_{c}(G)}(x) = 0$ for all $x \in V$. Hence $S_{a_{c}(G)}(G)$ is a smooth hyperbolic surface. \hfill \Box
Definition 11. Let \( p = (f_1, \ldots, f_N) \) and \( q = (g_1, \ldots, g_M) \) be two nondecreasing integer sequences with \( f_i, g_j \geq 3 \).

\[
\Phi(p) := 1 - \sum_{i=1}^{N} \left( \frac{1}{2} - \frac{1}{f_i} \right).
\]

\[
K_a(p) := 2\pi - \sum_{i=1}^{N} 2 \arcsin \frac{\pi}{\cosh \frac{a}{2}} \quad (a > 0).
\]

\[
a_c(p) := \max\{a > 0 : K_a(p) \leq 0\}.
\]

Let \( \prec \) be the strict part \( \leq \setminus = \).

Lemma 12. Let \( p, q \) be as in Definition 11

1. \( K_a(p) \) is a strictly increasing, continuous function of \( a \).
2. Suppose \( \Phi(p) < 0 \). Then \( a_c(p) \) is well-defined, and is the unique solution \( a \) such that \( K_a(p) = 0 \) and \( a > 0 \).
3. Suppose that \( 3 \leq f_1 \leq f_2 \leq f_3 \) are nondecreasing integers, \( l_i = \cos(\pi/f_i) \) \((i = 1, 2, 3)\), and \( H = (l_1 + l_2 + l_3)(l_1 + l_2 - l_3)(l_1 - l_2 + l_3)(-l_1 + l_2 + l_3) \).
   Then
   \[
a_c(f_1, f_2, f_3) = \arccosh \left( \frac{8\Pi^3}{H} - 1 \right).
   \]
4. \( p \prec q \Rightarrow a_c(p) < a_c(q) \).
5. \( f_i \leq f_i' \ (1 \leq i \leq N) \Rightarrow \Phi(f_1, \ldots, f_N) \geq \Phi(f_1', \ldots, f_N') \).

Proof. The assertion (1) is clear.

The assertion (2) follows from the assertion (1) and \( \lim_{a \to \infty} K_a(p) = 2\pi \) and \( \lim_{a \to 0} K_a(p) = 2\pi \Phi(p) < 0 \).

The assertion (3). The denominator in the argument of \( \text{arccosh} \) is well-defined, because

\[
\frac{1}{2} = \cos \frac{\pi}{3} \leq \cos \frac{\pi}{f_i} < 1.
\]

The equation (8) is proved as follows: By Lemma 12 (2), \( a = a_c(p) \) satisfies

\[
2\pi = \sum_{i=1}^{3} 2 \arcsin \left( \frac{\cos(\pi/f_i)}{\cosh(a/2)} \right).
\]

Hence,

\[
\cos \left( \pi - \arcsin \frac{l_1}{\cosh \frac{a}{2}} \right) = \cos \left( \arcsin \frac{l_2}{\cosh \frac{a}{2}} + \arcsin \frac{l_3}{\cosh \frac{a}{2}} \right).
\]

By the addition formula for cosines,

\[
-\sqrt{1 - \frac{l_1^2}{\cosh^2 \frac{a}{2}}} = \sqrt{\left( 1 - \frac{l_2^2}{\cosh^2 \frac{a}{2}} \right) \left( 1 - \frac{l_3^2}{\cosh^2 \frac{a}{2}} \right) - \frac{l_2l_3}{\cosh^2 \frac{a}{2}}}.
\]

By multiplying both sides by \( \cosh^2(a/2) \),

\[
-\cosh \frac{a}{2} \sqrt{\cosh^2 \frac{a}{2} - l_1^2} = -l_2l_3 + \sqrt{\left( \cosh^2 \frac{a}{2} - l_2^2 \right) \left( \cosh^2 \frac{a}{2} - l_3^2 \right)}.
\]
By squaring both sides and then moving terms,

\[-2l_2l_3^2 + (l_2^3 + l_2^2 - l_2^1) \cosh^2 \frac{a}{2} = -2l_2l_3 \sqrt{l_2^2 - \cosh^2 \frac{a}{2}} (l_3^2 - \cosh^2 \frac{a}{2}).\]

By subtracting the square of the right side from the square of the left side,

\[-H \cosh^4 \frac{a}{2} + 4l_2^3 l_2^2 l_2^1 \cosh^2 \frac{a}{2} = 0.\]

Because \(a\) is a real number, \(\cosh \frac{a}{2} > 0\). Thus, \(\cosh^2 \frac{a}{2} = 4 \prod_{i=1}^{3} l_i^2 / H\). From \(\cosh^2 \left(\frac{a}{2}\right) = (1 + \cosh a) / 2\), we obtain the desired equation. By Heron’s area formula for Euclidean triangles, we have the last sentence.

The assertion \(4\) is obvious.

\[\square\]

4. Number of vertices, vertex degrees and face degrees

Let \(S = S_g\) be a closed orientable surface of genus \(g \geq 0\). Let \(G\) be a tessellation of \(S\). For each \(G \in \mathcal{NC}_{\leq -1}\), by the Gauss-Bonnet formula (Theorem 2), \(g \geq 2\). Then Gauss-Bonnet formula \(2\), the result of Higuchi, and Proposition \(1\) yield the following result.

**Proposition 13.** For \(G \in \mathcal{NC}_{\leq -1}^g\),

\[\#V \leq 3612(g - 1).\]

**Proof.** By \(2\) and \(3\),

\[\#V \frac{1}{1806} \leq \sum_{x \in V} |\Phi(x)| \leq 2g - 2.\]

This yields the result. \[\square\]

By this proposition, we have the following corollary.

**Corollary 14.** For each \(g \geq 2\), \(\# (\mathcal{NC}_{\leq -1}^g) < \infty\).

Moreover, we can estimate the vertex degree.

**Proposition 15.** Let \(G \in \mathcal{NC}_{\leq -1}^g\) with \(g \geq 2\). For each \(x \in V\),

\[\deg(x) \leq 12g - 7.\]

**Proof.** Suppose it is not true. Let \(x_0\) be a vertex such that \(\deg(x_0) \geq 12g - 6\). Then

\[\Phi(x_0) \leq 1 - \frac{\deg(x_0)}{2} + \frac{1}{\deg(\sigma)} \leq 1 - \frac{\deg(x_0)}{2} + \frac{\deg(x_0)}{3},\]

since \(\deg(\sigma) \geq 3\) (\(\forall \sigma \in F\)). Thus, \(\Phi(x_0) \leq 1 - \frac{\deg(x_0)}{6} \leq 2 - 2g\). Since there are at least two vertices in the graph, by Proposition \(10\) the Gauss-Bonnet formula \(2\) yields a contradiction to \(G \in \mathcal{NC}_{\leq -1}^g\). \[\square\]

We recall the result of [7, Table 1].
Lemma 16 ([7] Table 1). For \( f_1, \ldots, f_N \) \((3 \leq f_1 \leq f_2 \leq \cdots \leq f_N)\), \( \Phi(f_1, \ldots, f_N) < 0 \) if and only if

- \( N \geq 7 \);
- \( N = 6 \) and \( (f_1, \ldots, f_N) \geq (3, 3, 3, 3, 3, 4) \);
- \( N = 5 \) and \( (f_1, \ldots, f_N) \geq p \) for some \( p = (3, 3, 3, 3, 7), (3, 3, 3, 4, 5), (3, 3, 4, 4, 4) \);
- \( N = 4 \) and \( (f_1, \ldots, f_N) \geq p \) for some \( p = (3, 3, 4, 13), (3, 3, 5, 8), (3, 3, 6, 7), (3, 4, 4, 7), (3, 4, 5, 5), (4, 4, 4, 5) \);
- \( N = 3 \) and \( (f_1, f_2, f_3) \geq p \) for some \( p = (3, 7, 43), (3, 8, 25), (3, 9, 19), (3, 10, 16), (3, 11, 14), (3, 12, 13), (4, 5, 21), (4, 6, 13), (4, 7, 10), (4, 8, 9), (5, 5, 11), (5, 6, 8), (5, 7, 7), \) or \((6, 6, 7)\).

Proposition 17. For \( G \in \mathcal{NC}_{\leq -1}^g \) with \( g \geq 2 \),
\[
\deg(\sigma) \leq 84g - 43, \quad \forall \sigma \in F.
\]

Proof. Let \( \sigma \) be the face with maximal facial degree \( f = \deg(\sigma) \).

On the one hand, by the tessellation properties (1) and (2) in the introduction, \( \# F \geq 2 \) and there is at least one vertex \( y \) which is not on the boundary of \( \sigma \). By Proposition 10, \( \Phi(y) < 0 \). Thus, by the Gauss-Bonnet formula (2),
\[
\sum_{x \in \sigma} \Phi(x) > 2 - 2g.
\]

On the other hand, by Table 1 in [7], i.e., Lemma 16 we have:
\[
\text{Pttn}(x) \geq (3, \ldots, 3, f), (3, 3, 4, f), (3, 7, f), \text{ or } (4, 5, f), \quad (x \in \sigma).
\]

Indeed, for \( N := \deg(x) = 3 \), \( \text{Pttn}(x) \geq (3, 7, f) \) for \( \text{Pttn}(x) = (3, \ldots) \), and \( \text{Pttn}(x) \geq (4, 5, f) \) otherwise. For \( N = 4 \), \( \text{Pttn}(x) \geq (3, 3, 4, f) \), and for \( N \geq 5 \), \( \text{Pttn}(x) \geq (3, \ldots, 3, f) \) as in (11).

The maximum \( \Phi(p) \) where \( p \) ranges over the four tuples in the right side of (11), is \( \Phi(3, 7, f) \). By (11) and Lemma 12 (3),
\[
\Phi(x) \leq \Phi(3, 7, f) = -\frac{1}{42} + \frac{1}{f}, \quad (x \in \sigma).
\]

By summing both sides over the \( f \) vertices \( x \in \sigma \),
\[
\sum_{x \in \sigma} \Phi(x) \leq -\frac{f}{42} + 1.
\]

Thus, the conclusion follows from (10). \( \square \)

5. Cubic graphs embedded in the double torus

In this section, we study cubic graphs, i.e. regular graphs of vertex degree 3, embedded into the double torus \( S_2 \).

Lemma 18. The Delaunay triangulation of Bolza surface (Figure 7) induces a Voronoi tessellation which is a member of \( A_2(3) \).

Proof. We can check that the vertex patterns of the Voronoi tessellation are as in Table 1. Here, the numbering of the vertices in Table 1 corresponds to the numbering of the faces of the Delaunay triangulation in Figure 7. All the vertices have negative combinatorial curvature, so the graph is in \( \mathcal{NC}_{\leq 0} \) by Lemma 16.
and thus in $\mathcal{NC}^2_{\leq -1}$. The vertex pattern $(6, 7, 8)$ of vertex no. 1 is strictly greater than the vertex pattern $(6, 7, 7)$ of the vertex no. 2, so the two vertices have different critical side lengths by Lemma 12 (4). Therefore, the graph is not in $T_{S_2}$ by Proposition 4. □

Definition 19. For $g \geq 2$,

$$B_g(3) := \max\left\{ \deg \sigma : (V, E, F) \in \mathcal{NC}^g_{\leq -1}, \ \deg x = 3 \ (\forall x \in V), \ \sigma \in F \right\}.$$ 

Proposition 20. $B_g(3) \geq 7$.

Proof. Owing to Proposition 10, $\mathcal{NC}^2_{\leq -1} \subseteq \mathcal{NC}_{< 0}$. By Lemma 16, every vertex pattern $(f_1, f_2, f_3)$ of $G \in \mathcal{NC}^g_{\leq -1} \subseteq \mathcal{NC}_{< 0}$ is greater than or equal to one of the 14 triples $(3, 7, 43), \ldots, (6, 6, 7)$. Hence, $f_3 \geq 43, \ldots, \text{or} \ 7$. Therefore, the maximum facial degree $B_g(3)$ of $G$ is the maximum of such $f_3$, and is greater than or equal to 7. □

By Example 5 (1), the tessellation $G_B \in T_{S_2}$ of the Bolza surface provides a lower bound 16 on $B_2(3)$. Moreover, for cubic graphs, we can improve the upper bound $84g - 43$ on the facial degree that Proposition 17 provides, to $40g - 21$.

Proposition 21. For $g \geq 2$, $B_g(3) \leq 40g - 21$.

Proof. Suppose that $G = (V, E, F) \in \mathcal{NC}^g_{\leq -1}$ and $\deg x = 3$ for all $x \in V$. Suppose that $\sigma$ is a face of maximal facial degree $f$. By Lemma 16 we know that for each $x \in \sigma$, which is not of pattern $(3, 7, f)$ or $(3, 8, f)$,

$$\Phi(x) \leq -\frac{1}{20} + \frac{1}{f}. \tag{12}$$

Set

$$W := \{ y \in \sigma : \text{Ptn}(y) = (3, 7, f) \ \text{or} \ (3, 8, f) \}.$$ 

Let $y \in W$. We denote by $w$ the vertex satisfying $w \in \sigma$, $w \sim y$ and $w, y$ are incident to a common triangle, see Figure 2. We define a map,

$$T : W \to V \cap \sigma,$$

$$y \mapsto w.$$ 

We denote by $z$ the vertex adjacent to $w$ and $y$, by $\tau$ the face incident to $w$, which is neither $\sigma$ nor the triangle $\Delta_{w, z, w}$. Since $\Phi(z) < 0$, by Lemma 16

$$\deg(\tau) \geq 25. \tag{13}$$
Therefore, by $\text{Pttn}(w) = (3, \deg \tau, f)$, $w \notin W$. Hence

$$TW \cap W = \emptyset.$$  

Let $u$ be the vertex on $\sigma$ adjacent to $w$, which is not $y$. By (13), $\text{Pttn}(u) = (\ldots, \deg \tau, f)$ is neither $(3, 7, f)$ nor $(3, 8, f)$. Thus $u \notin W$. Hence $T^{-1}(w) = y$ and the map $T$ is injective. By (13),

$$\Phi(Ty) = \Phi(w) = -\frac{1}{2} + \frac{1}{3} + \frac{1}{\deg(\tau)} + \frac{1}{f} \leq -\frac{19}{150} + \frac{1}{f} \quad (y \in W).$$

Obviously

$$\Phi(y) \leq -\frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \frac{1}{f} \quad (y \in W).$$

By (14),

$$\sum_{x \in \Sigma} \Phi(x) = \sum_{y \in W} (\Phi(y) + \Phi(Ty)) + \sum_{x \in \Sigma \setminus (W \cup TW)} \Phi(x).$$

Hence by (16), (15), and (12), $\sum_{x \in \Sigma} \Phi(x)$ is at most

$$\left(\#W\right) \left(-\frac{1}{42} + \frac{1}{f}\right) + \left(\#(TW)\right) \left(-\frac{19}{150} + \frac{1}{f}\right)$$

$$+ (f - 2(\#W)) \left(-\frac{1}{20} + \frac{1}{f}\right) \leq -\frac{53(\#W)}{1050} - \frac{f}{20} + 1,$$

by (14) and by $\#W = \#(TW)$ that follows from the injectivity of $T$. In addition, by the tessellation properties (1) and (2) in the introduction, there is at least one vertex which is not on the boundary of $\sigma$, as in the proof of Proposition 17. This yields that

$$\sum_{x \in V} \Phi(x) < \sum_{x \in \Sigma} \Phi(x).$$
Therefore by the Gauss-Bonnet formula (2),

\[ 2 - 2g = \sum_{x \in V} \Phi(x) < \sum_{x \in \mathcal{P}} \Phi(x) \leq -\frac{53(#W)}{1050} - \frac{f}{20} + 1. \]

By \( #W \geq 0, (17) \), and \( f \in \mathbb{Z} \), we have \( f \leq 40g - 21 \). \( \Box \)

**Definition 22.** A nondecreasing integer sequence \( p = (f_1, f_2, f_3) \) with \( f_i \geq 3 \) is called admissible, if \( \Phi(p) < 0 \) and \( f_3 \leq B_2(3) \). The set of admissible \( p \) is denoted by \( \text{AdP} \).

By the finiteness of the set \( \text{AdP} \), we can define the following:

**Definition 23.**

\( \kappa := \max \{ K_{\alpha_c}(p) : K_{\alpha_c}(p) < 0, p \in \text{AdP}, q \in \text{AdP}, \alpha_c(p) < \alpha_c(q) \} \).

Recall from Section 2

\[ \epsilon_3^N = \inf_{G \in A_d(N)} (4\pi - \text{Area}^{\text{cri}}(G)) = \inf_{G \in A_d(N)} \sum_{x \in V} (-K_{\alpha_c(G)}(x)). \]

**Lemma 24.** For \( g \geq 2 \), \( \epsilon_3^N \geq -\kappa \).

**Proof.** By Corollary 14 there are only finitely many \( G \) in \( A_d(3) \). Suppose \( G = (V, E, F) \in A_d(3) \) and \( \alpha_c(G) = \alpha_c(x) \) with \( x \in V \). Then there is \( y \in V \) such that \( \alpha_c(y) > \alpha_c(x) \), by Proposition 4 and Definition 3. By Lemma 12 \( K_{\alpha_c(G)}(y) < K_{\alpha_c(G)}(x) = 0 \). Thus,

\[ \epsilon_3^N \geq \inf \{ -K_{\alpha_c(G)}(y) : G \in A_d(3), y \in V, \alpha_c(y) > \alpha_c(x) \} = -\max \{ K_{\alpha_c(G)}(y) : G \in A_d(3), y \in V, \alpha_c(y) > \alpha_c(x) \}. \]

Suppose that \( y \in V \) attains the maximum. By \( G \in A_d(3) \) and \( \alpha_c(x) < \alpha_c(y) \), \( G \in N\mathcal{C}_{\leq 1} \). By Proposition 14 \( G \in N\mathcal{C}_{<0} \) and thus \( \Phi(x), \Phi(y) < 0 \). Therefore both of \( \text{Pttn}(x) \) and \( \text{Pttn}(y) \) are in \( \text{AdP} \). Hence, the conclusion follows from Definition 24.

**Proof of Theorem 2.** By (5), (6) and Lemma 24, it suffices to calculate \( \kappa \). By Example 3 (1) and Proposition 21 the constant \( B_2(3) \) is in an interval \([16, 59]\). If \( B_2(3) \) is estimated large, so are \( \text{AdP} \) and \( \kappa \) by Definition 22 and Definition 23. For each \( B_2(3) \in [16, 59] \), we computed \( \kappa \) by using Maple with the algorithm given in Appendix. The result is Table 2. The largest \( \kappa \) is

\[ \kappa = K_{\alpha_c((29,55,55))}((32, 43, 55)) = -4.96239 \cdots \times 10^{-10}. \]

This proves the theorem. \( \Box \)

6. Future work

We recall the results in \([32]\) for regular spherical polyhedral surfaces. For each finite planar graph, it associates with some metric spaces, called regular spherical polyhedral surfaces, by replacing faces with regular spherical polygons in the unit sphere and gluing them edge-to-edge. We consider the class of planar graphs which admit spherical polyhedral surfaces with the curvature bounded below by 1 in the sense of Alexandrov, i.e. the total angle at each vertex is at most \( 2\pi \). We classify all spherical tilings with regular spherical polygons, i.e. total angles at vertices are exactly \( 2\pi \). We prove that for each graph in this class which does not admit a
spherical tiling, the area of the associated spherical polyhedral surface with the curvature bounded below by 1 is at most $4\pi - 1.6471 \cdots \times 10^{-5}$. In other words, $\widetilde{\text{Area}}_{\text{max}} \leq 4\pi - 1.6471 \cdots \times 10^{-5}$ for regular spherical polyhedral surfaces.

In the spherical case, there are vertex types $p$ such that the combinatorial curvature $\Phi(p)$ of $p$ is positive and $K(a_c(p), p) > 0$. In the hyperbolic case, $K(a_c(p), p) = 0$ for every vertex type $p$ with $\Phi(p) < 0$. We conjecture that $\widetilde{\text{Area}}_{\text{max}}$ for regular hyperbolic polyhedral surfaces with $g = 2$ is greater than $\widetilde{\text{Area}}_{\text{max}}$ for regular spherical polyhedral surfaces, and is close to $4\pi$.

For $\mathcal{PC}_{>0} := \{ G = (V, E, F) : \Phi(x) > 0, \forall x \in V \}$, DeVos and Mohar [15] proved that any graph $G \in \mathcal{PC}_{>0}$ is finite, which solves a conjecture of Higuchi [7], see [2, 12] for early results. It would be interesting to classify $\mathcal{PC}_{>0}$. For the set

$$P := \{ G : G \in \mathcal{PC}_{>0} \text{ is neither a prism nor an antiprism } \},$$

DeVos and Mohar proved that $\#P < \infty$ and asked the number

$$C_{S^2} := \max_{(V, E, F) \in P} \#V.$$

For the lower bound estimate of $C_{S^2}$, large examples in this class are constructed [13 37 28 38], and finally some examples possessing 208 vertices were found. DeVos and Mohar [15] showed that $C_{S^2} \leq 3444$, which was improved to $C_{S^2} \leq 380$ by Oh [22]. By a refined argument, in [23], Ghidelli completely solved the problem.

**Theorem 25.**

(1) $C_{S^2} = 208$. [23 37]

(2) $\max\{\deg(\sigma) : \sigma \in F, (V, E, F) \in P\} \leq 41$. [23]

Comparing with these results, we propose to determine

$$\max_{(V, E, F) \in \mathcal{NC}_{g}^\circ} \#V \text{ and } B_g(3) \text{ for } g \geq 2.$$

Hyperbolic polyhedral surfaces enjoy more combinatorial structures than the spherical ones by Proposition [10]. It is a challenge to compute effective numerical results for “almost” hyperbolic tilings, i.e., non-smooth hyperbolic polyhedral surfaces.

| Range of $B_2(3)$ | $\kappa = K(a_c(p), q)$ | $p$ | $q$ |
|-------------------|----------------------|-----|-----|
| 16–16             | $-7.85456 \cdots \times 10^{-6}$ | (12, 12, 16) | (11, 14, 15) |
| 17–18             | $-3.49781 \cdots \times 10^{-6}$ | (11, 13, 15) | (11, 12, 17) |
| 19–20             | $-1.46541 \cdots \times 10^{-6}$ | (11, 16, 16) | (10, 17, 19) |
| 21–34             | $-1.38186 \cdots \times 10^{-8}$ | (16, 16, 21) | (14, 20, 20) |
| 35–40             | $-1.12992 \cdots \times 10^{-8}$ | (15, 18, 19) | (11, 26, 35) |
| 41–44             | $-4.70209 \cdots \times 10^{-9}$ | (9, 36, 41) | (10, 23, 26) |
| 45–54             | $-1.30776 \cdots \times 10^{-9}$ | (35, 38, 43) | (31, 44, 45) |
| 55–59             | $-4.96239 \cdots \times 10^{-10}$ | (29, 55, 55) | (32, 43, 55) |

Table 2.
Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

The second author thanks Feng Luo for many helpful suggestions on hyperbolic polyhedral surfaces. We thank the anonymous referee for valuable comments and suggestions to improve the writing of the paper.

Appendix A. The algorithm

We will try to numerically compute the following constant of Definition 23:

\[ \kappa = \max \{ K_{a,c}(p) : (p,q) \in \mathcal{A} \mathcal{D} \mathcal{P}^2 \text{, } K_{a,c}(p) < 0 \text{, } a_c(p) < a_c(q) \} . \]

We note two difficulties:

1. It is expensive to check the conditions for all the \((p,q) \in \mathcal{A} \mathcal{D} \mathcal{P}^2\).
2. Since \(|\kappa|\) would be minute, the numerical computation of \(\kappa\) would suffer from loss of significance (so-called catastrophic cancellation) [39].

We do not know whether the algorithm with 40 decimal digit representation of numbers is enough to confirm the value of \(\kappa\). If the true value of \(\kappa\) has modulus less than \(10^{-40}\), the algorithm misses the true value of \(\kappa\), because it computes \(\kappa\) as \(2\pi\) minus a total angle \(\vartheta\) where the decimal representations of \(2\pi\) and \(\vartheta\) coincide up to the first 40 digits. By the same reasoning, if the true value of \(\kappa\) is indeed \(K_{a,c}(29,55,55)(32,43,55)\), then the number of significant digits of \(\kappa = -4.96239\cdots \times 10^{-10}\) is at most 40 - 10.

To go around the first difficulty, i.e., to save the computation time, we first note that \(\kappa\) is the maximum of the following two:

\[ \max \{ K_{a,c}(p) : (p,q) \in \mathcal{A} \mathcal{D} \mathcal{P}^2 \text{, } K_{a,c}(p) < 0 \text{, } a_c(p) < a_c(q) \text{, } p \prec q \} \]
\[ \max \{ K_{a,c}(p) : (p,q) \in \mathcal{A} \mathcal{D} \mathcal{P}^2 \text{, } K_{a,c}(p) < 0 \text{, } a_c(p) < a_c(q) \text{, } p \not\prec q \} \]

(18) (19)

Then, we will prove that the following attains [18]:

\[ \mathcal{A} \max \{ K_{a,c}(p) : (p,q) \in \mathcal{A} \mathcal{D} \mathcal{P}^2 \text{, } K_{a,c}(p) < 0 \text{, } a_c(p) < a_c(q) \} . \]

(18)

\[ \mathcal{A} \max \{ K_{a,c}(p) : (p,q) \in \mathcal{A} \mathcal{D} \mathcal{P}^2 \text{, } K_{a,c}(p) < 0 \text{, } a_c(p) < a_c(q) \} . \]

(19)

To ease the computation for [19], the following inexpensive constraint is a necessary condition of \(K_{a,c}(p)(q) < 0\) where \(p = (f_1,f_2,f_3)\) and \(q = (g_1,g_2,g_3)\):

Definition 26. Let \(cncv(f_1,f_2,f_3,g_1,g_2,g_3)\) be:

\[ \neg (f_1 \geq g_1 \text{ and } f_2 - g_2 \geq g_3 - f_3 > 0) \]
\[ \text{and } \neg (f_2 \geq g_2 \text{ and } f_1 - g_1 \geq g_3 - f_3 > 0) \]
\[ \text{and } \neg (f_3 \geq g_3 \text{ and } f_1 - g_1 \geq g_2 - f_2 > 0) . \]

The proofs of Lemma 28 and Lemma 29 both depend on mean value theorem for the following function:
Definition 27.

\[ \beta(x, a) := 2 \arcsin \frac{\cos \frac{x}{a}}{\cosh \frac{a}{2}} \quad (3 \leq x \leq B_2(3), a > 0). \]

For each integer \( x \geq 3 \), \( \beta(x, a) \) is the inner angle \( \beta_{x,a} \) of a regular hyperbolic \( x \)-gon of side length \( a \).

Lemma 28. Let \( 3 \leq x \leq B_2(3) \) and \( a > 0 \). Then

1. \( \beta(x, a) \) is increasing and concave in \( x \).
2. \( \frac{\partial \beta}{\partial x} (x, a) \) is positive and decreasing in \( a \).

Proof. \( \frac{\partial \beta}{\partial x} \) is \( 2 \pi \sin \left( \frac{x}{a} \right) x^{-2} u(x, a)^{-1/2} > 0 \) where

\[ u(x, a) := - \left( \cos \frac{\pi}{x} \right)^2 + \left( \cosh \frac{a}{2} \right)^2 > 0. \]

As \( u(x, a) \) is increasing in \( a \), \( \frac{\partial \beta}{\partial x} \) is decreasing in \( a \). This establishes the second assertion. \( \frac{\partial^2 \beta}{\partial x^2} (x, a) \) is the product of a positive number \( u(x, a)^{-3/4} x^{-4} > 0 \) and

\[ -4 \sin \left( \frac{\pi}{x} \right) u(x, a) \pi x - 2 \cos \left( \frac{\pi}{x} \right) \pi x \left( \cosh \frac{a}{2} \right) - 1 < 0. \]

Thus, \( \frac{\partial^2 \beta}{\partial x^2} (x, a) < 0 \). This establishes the first assertion and completes the proof of Lemma 28. \( \square \)

By Definition 11, \( K_{a_c(p)}(q) \) is represented in the following form, because Lemma 12 implies \( K_{a_c(p)}(p) = 2\pi - \sum_{i=1}^{N} \beta(f_i, a_c(p)) = 0 \).

Lemma 29. For \( p = (f_1, \ldots, f_N), \; q = (g_1, \ldots, g_M) \in \mathcal{A}d\mathcal{P}, \)

\[ K_{a_c(p)}(q) = \sum_{i=1}^{N} \beta(f_i, a_c(p)) - \sum_{i=1}^{M} \beta(g_i, a_c(p)). \]

These two lemmas and mean value theorem for \( \beta(x, a) \) totally dispense with the computation for (18), and ease the computation for (19), as follows:

Lemma 30.

(18) \( \max \{ K_{a_c(p)}(q) : K_{a_c(p)}(q) < 0, (p, q) \in \mathcal{A}d\mathcal{P}^2, p \prec q \} = K_{a_c(\tilde{p})}(\tilde{q}). \)

Proof. In the argument of max, we have only to consider \( \prec \)-maximal \( p \) such that \( p \prec q \), by Lemma 12. Since \( q \in \mathcal{A}d\mathcal{P} \), there is a strictly increasing sequence \( r_1 < \cdots < r_k \) (\( 1 \leq k \leq 3 \)) of integers in an interval \([3, B_2(3)]\) and positive integers \( n_1, \ldots, n_k \) such that \( q \) consists of \( n_i \) number of \( r_i \) \( (1 \leq i \leq k) \). Then \( \tilde{p} \in \mathcal{A}d\mathcal{P} \) is obtained from \( \tilde{q} \in \mathcal{A}d\mathcal{P} \) by replacing exactly one component \( j \) of \( \tilde{q} \) with \( j - 1 \), because \( \tilde{p} \) is maximal. Thus, by Lemma 29, \( K_{a_c(\tilde{p})}(\tilde{q}) = \beta(j - 1, a_c(\tilde{p})) - \beta(j, a_c(\tilde{p})) \), which is \( -\frac{\partial \beta}{\partial x}(\xi, a_c(\tilde{p})) \) for some \( \xi \) \((j - 1 < \xi < j)\) by mean value theorem for \( \beta \). By Lemma 28 and Lemma 12, \( j = B_2(3) \) and \( a_c(\tilde{p}) \) is the maximum among such \( \tilde{p} \). This completes the proof. \( \square \)

Lemma 31. For \( p = (f_1, f_2, f_3), \; q = (g_1, g_2, g_3) \in \mathcal{A}d\mathcal{P}, \)

\[ K_{a_c(f_1, f_2, f_3)}(g_1, g_2, g_3) < 0 \Rightarrow \text{ncv}(f_1, f_2, f_3, g_1, g_2, g_3). \]
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Proof. By Lemma [29] and \(a_c(p) > 0\), we only need to verify the following:

(21) \[
\{ l, m, n \} = \{ 1, 2, 3 \}, \ m < n, \ f_l \geq g_l \text{ and } f_m - g_m \geq g_n - f_n > 0
\]

\[
\Rightarrow \sum_{i=1}^{3} (\beta(f_i, a) - \beta(g_i, a)) > 0 \ (a > 0).
\]

We prove (21) as follows: Suppose \(l = 1\). Then, by Lemma [28] (1), the premise \(f_i \geq g_i\) of (21) implies

(22)
\[
\sum_{i=1}^{3} (\beta(f_i, a) - \beta(g_i, a)) \geq \sum_{i=2}^{3} (\beta(f_i, a) - \beta(g_i, a)).
\]

By mean value theorem,

(23) \[
\sum_{i=2}^{3} (\beta(f_i, a) - \beta(g_i, a)) = \frac{\partial \beta}{\partial x} (\xi_2, a)(f_2 - g_2) - \frac{\partial \beta}{\partial x} (\xi_3, a)(g_3 - f_3)
\]

for some \(\xi_2\) (\(g_2 < \xi_2 < f_2\)) and \(\xi_3\) (\(f_3 < \xi_3 < g_3\)), where the premise of (21) implies \(f_2 > g_2\) and \(g_3 > f_3\). By \(p, q \in \text{AdP}\), \(f_2 \leq f_3\). Thus, \(\xi_2 < \xi_3\). By Lemma [28] (1), \(\frac{\partial \beta}{\partial x}(\xi_2, a) > \frac{\partial \beta}{\partial x}(\xi_3, a) > 0\). By the assumption \(f_2 - g_2 \geq g_3 - f_3 > 0\), we have (23) > 0. Hence, by (22), we have (21). The cases \(l \neq 1\) are proved similarly. \(\square\)

To sum up, the algorithm for \(\kappa\) saves the computer time as follows: to find \((p, q)\) that attains \(\kappa\), the algorithm first sets \((\hat{p}, \hat{q})\) and then checks all the \((p, q) \in \text{AdP}^2\) satisfying the four conditions in (19) with updating \((p, q)\) whenever \(0 > K_{a_c(p)}(q)\) is greater than the tentative maximum \(K_{a_c(p)}(q)\). When the computed value of \(K_{a_c(p)}(q)\) is less than the tentative maximum, the algorithm dispenses computing \(K_{a_c(p)}(q)\) for \(p' < p\), because of Lemma [12] (1) and (4).

Next, in order to eradicate the second difficulty, i.e., the round-off difficulty,

- we will consider \(\vartheta\) instead of \(K_{a_c(p)}(q) = 2\pi - \vartheta\), and
- we will compute the fraction in the argument of \(\text{arccosh}\) in (8), based on Kahan’s method [10, 11] for Heron’s formula for Euclidean triangles. Kahan’s method is robust against loss of significant digits according to the numerical experiment [10, Table 1].

Now, our algorithm for \(\kappa\) consists of ‘Initialization and subroutines’ and ‘Main routine’.

**Initialization and subroutines:**

1. For \(3 \leq i \leq B_2(3), 3 \leq j < k \leq B_2(3)\), allocate memory for \(C_i, S_i, D_{j,k}\), compute the values and let \(C_i := \cos(\pi/i), \ S_i := C_i^2, \ D_{j,k} := C_k - C_j\); \(8S_iS_jS_k\)

2. \(\alpha(i, j, k): \quad \text{return} \quad \frac{(C_k + (C_j + C_i))(C_i - D_{j,k})(C_i + D_{j,k})(C_k + D_{i,k})}{8S_iS_jS_k}\);

3. \(\vartheta(A, i, j, k):\)

4. \(\text{return} \quad \left(\text{arccos} \left(1 - \frac{4S_i}{A}\right) + \text{arccos} \left(1 - \frac{4S_j}{A}\right)\right) + \text{arccos} \left(1 - \frac{4S_k}{A}\right)\); By [40, Section 4] and [9], at most one bit of \(D_{f_1,f_2}\) and \(D_{f_2,f_3}\) is rounded off. Because of the first line of the algorithm, the algorithm computes \(\cos(\pi/f_i)\), \(\cos^2(\pi/f_i)\), \(\cos(\pi/f_i) - \cos(\pi/f_j)\) at most once.

The subroutines \(\alpha\) and \(\vartheta\) of the algorithms have the following properties:
Lemma 32. For \( p, q \in \mathcal{A} \mathcal{D} \mathcal{P} \) and \( A \geq 2 \),

1. The subroutine \( \theta(A,p) \) returns a real number.
2. \( \alpha(p) = \cosh a_c(p) + 1 > 2 \).
3. \( K_{\text{arccosh}(\alpha(p)-1)}(q) = 2\pi - \vartheta(\alpha(p), q) \).

Proof. Let \( p = (f_1, f_2, f_3) \in \mathcal{A} \mathcal{D} \mathcal{P} \). By \( A \geq 2 \), \( -1 < 4 \cos^2(\pi/f_i)/A - 1 < 2 \cos^2(\pi/f_i) - 1 = \cos(2\pi/f_i) \leq 1 \). Thus, \( \text{arccos}(4\cos^2(\pi/f_i)/A - 1) \) is a real number. Therefore, \( 1 \) holds. \( 2 \) is due to Lemma 12 and \( a_c(p) > 0 \). \( 3 \). Let \( q = (g_1, g_2, g_3) \in \mathcal{A} \mathcal{D} \mathcal{P} \) and let \( \varphi \) be \( \text{arcsin}(\cos(\pi/g_i)/\cosh(a_c(p)/2)) > 0 \). Then \( \cos 2\varphi = 1 - 2 \sin^2 \varphi = 1 - 2 \cos(\pi/g_i)/\cosh(a_c(p)/2))^2 = 1 - 2 \cos^2(\pi/g_i)/(\cosh a_c(p) + 1/2) \). By \( 2 \) of this Lemma, \( \cos 2\varphi = 1 - 4 \cos^2(\pi/g_i)/\alpha(p) = 1 - 4 \cos^2(\pi/f_i) \). Therefore,

\[
2 \text{arcsin} \frac{\cos(\pi/g_i)}{\cosh(a_c(p)/2)} = \text{arccos} \left( 1 - 4 \frac{\cos^2(\pi/g_i)}{\alpha(p)} \right).
\]

The conclusion follows from Definition 11. This completes the proof. \( \square \)

In computing the maximum \( \kappa \), the main routine runs over many of \( (p, q) \in \mathcal{A} \mathcal{D} \mathcal{P}^2 \), so it uses the following function to run over all \( \{q : g_i \in \mathcal{A} \mathcal{D} \mathcal{P} \}_{i,j} \), \( \emptyset \).

Recall Proposition 16

Definition 33. For \( 3 \leq i \leq B_2(3) \), let

\[
m_2(i) := 7 \quad (i = 3); \quad 5 \quad (i = 4, 5); \quad 6 \quad (i = 6); \quad \text{otherwise}.
\]

For \( i, j \) \( 3 \leq i \leq B_2(3) \), \( m_2(i) \leq j \leq B_2(3) \), let

\[
m_3(i,j) := \begin{cases} 43 \quad (i = 3, j = 7); & 25 \quad (i = 3, j = 8); & 19 \quad (i = 3, j = 9); \\ 16 \quad (i = 3, j = 10); & 14 \quad (i = 3, j = 11); & 13 \quad (i = 3, j = 12); \\ 21 \quad (i = 4, j = 5); & 13 \quad (i = 4, j = 6); & 10 \quad (i = 4, j = 7); \\ 9 \quad (i = 4, j = 8); & 11 \quad (i = 5, j = 5); & 8 \quad (i = 5, j = 6); \\ 7 \quad (i = 5, j = 7); & 7 \quad (i = 6, j = 6); & j \left( 1 - \frac{1}{2} + \frac{1}{7} + \frac{2}{7} < 0 \right); \\ \infty \quad \text{otherwise}. 
\end{cases}
\]

We set \( \min \emptyset = \infty \).

Lemma 34. Let \( i, j, k \) be integers. Then, \( (i,j,k) \in \mathcal{A} \mathcal{D} \mathcal{P} \) if and only if

\[
3 \leq i \leq B_2(3), \quad m_2(i) \leq j \leq B_2(3), \quad \text{and} \quad m_3(i,j) \leq k \leq B_2(3).
\]

Proof. Proposition 16 implies that \( m_2(i) \) is \( \min \{j : \exists k. (i,j,k) \in \mathcal{A} \mathcal{D} \mathcal{P} \} \), and that \( m_3(i,j) \) is \( \min \{k : (i,j,k) \in \mathcal{A} \mathcal{D} \mathcal{P} \} \).

The main routine to compute the constant \( \kappa \) is the following:

Main routine:

5 \( \hat{P} := (B_2(3) - 1, B_2(3), B_2(3)) \); \( \quad \text{cf. Lemma 30} \)
6 \( \hat{Q} := (B_2(3), B_2(3), B_2(3)) \); \( \quad \text{cf. Lemma 30} \)
7 \( T := \theta(\alpha(\hat{P}), \hat{Q}) \); \( \quad \text{Tentative minimum of} \ \theta \)
8 \( \text{for} \ g_1 = 3, \ldots, B_2(3) \ \text{do} \)
9 \( \text{for} \ g_2 = m_2(g_1), \ldots, B_2(3) \ \text{do} \); \( \quad \text{cf. Lemma 34} \)
10 \( \text{for} \ g_3 = m_3(g_1, g_2), \ldots, B_2(3) \ \text{do} \); \( \quad \text{cf. Lemma 34} \)
11 \( \text{B_is_computed} := \text{false} \); \( \quad \text{B =} \ \alpha(g_1, g_2, g_3) \) \( \text{is not computed yet} \)
for $f_1 = g_3 - 1, g_3 - 2, \ldots, 3$ do // cf. remark just after the algorithm
for $f_2 = B_2(3), B_2(3) - 1, \ldots, m_2(f_1)$ do
  $m := \max(g_1 + 1, m_3(f_1, f_2))$; // (18) requires $p \neq q$
for $f_3 = B_2(3), B_2(3) - 1, \ldots, m$ do
  if $\exists i,j. (f_i < g_i \& f_j > g_j)$ & $cnv(f_1, f_2, f_3, g_1, g_2, g_3)$ // cf. Lemma 31
    then
      if $B_{is\_computed} = false$ then
        $B := \alpha(g_1, g_2, g_3)$; // $\cosh(a_c(q)) + 1$
        $B_{is\_computed} := true$
      end if
      $A := \alpha(f_1, f_2, f_3)$; // $\cosh(a_c(p)) + 1$
      if $A < B$ then // (18) requires $a_c(p) < a_c(q)$
        $k := \partial(A, g_1, g_2, g_3)$;
        if $k < T$ then
          $T := k; \hat{P} := (f_1, f_2, f_3); \hat{Q} := (g_1, g_2, g_3)$;
          end if // if $k < T$
      end if // if $A < B$
    end if // if $\exists \cdots$
end do // for $f_3$
end do // for $f_2$
end do // for $f_1$
end do // for $g_3$
end do // for $g_2$
end do // for $g_1$
return $2\pi – T, \hat{P}, \hat{Q}$ // $\kappa, p, q$ such that $K_{a_c(p)}(q) = \kappa$

The for-loop of $f_1$ of the main routine decreases $f_1$ by $-1$, and thus does not consider case $g_3 \leq f_1 \leq B_2(3)$. However, this algorithm correctly computes the maximum $\kappa$ of $K_{a_c(p)}(q) < 0$ such that both of $p$ and $q$ are in $A\partial P$ and $a_c(p) < a_c(q)$, because of the following: If $f_1 \geq g_3$, then $(f_1, f_2, f_3) \succeq (g_1, g_2, g_3)$. Thus $a_c(f_1, f_2, f_3) \geq a_c(g_1, g_2, g_3)$ by Lemma 12 [4]. Because $\kappa$ is greater than or equal to (18), the algorithm does not have to check the case $g_3 \leq f_1 \leq B_2(3)$.

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