On the structure of the solution set to Killing-type equations and LRL conservation

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Abstract

If we impose infinitesimal gauge-invariance of the action functional for Lagrangian ordinary differential equations, we are led to Killing-type equations, which are related to first integrals through Noether theorem. We discuss the structure of the solution set of two interpretations of Killing-type equations. In particular, we give an explicit solution formula for the “on-flow” interpretation, in terms of the associated first integral. As applied to the Laplace-Runge-Lenz vector conservation for the Kepler system, we propose new solutions, some of which have zero gauge terms.

Keywords: Noether variational theorem; Killing-type equations; Laplace-Runge-Lenz vector.
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1 Introduction

Suppose we are given a smooth Lagrangian function \( L(t, q, \dot{q}) \), with \( t \in \mathbb{R} \), \( q, \dot{q} \in \mathbb{R}^n \). The variational principle for Lagrangian dynamics requires that

\[
\delta \int_{t_1}^{t_2} L(t, q(t), \dot{q}(t)) \, dt = 0, \tag{1}
\]

which is equivalent to the Euler-Lagrange equation

\[
\frac{d}{dt} \partial_{\dot{q}} L(t, q(t), \dot{q}(t)) - \partial_q L(t, q(t), \dot{q}(t)) = 0. \tag{2}
\]

We will assume that the Euler-Lagrange equation can be put into normal form

\[
\ddot{q} = g(t, q, \dot{q}), \tag{3}
\]

Following closely the notation of Sarlet and Cantrijn [7] (except that their \( \tau \) is replaced by our \( T \), and that \( \dot{q} \) is an independent variable from the outset), we consider an infinitesimal transformation in the \((t, q)\) space given by

\[
\bar{t} = t + \varepsilon T(t, q, \dot{q}), \quad \bar{q} = q + \varepsilon \xi(t, q, \dot{q}). \tag{4}
\]

This transformation is said to leave the action integral invariant up to gauge terms, if a function \( f(t, q, \dot{q}) \) exists, such that for each smooth curve \( t \mapsto q(t) \) we have

\[
\int_{t_1}^{t_2} L(\bar{t}, \bar{q}(\bar{t}), \frac{d\bar{q}}{d\bar{t}}(\bar{t})) \, d\bar{t} = \int_{t_1}^{t_2} L(t, q(t), \dot{q}(t)) \, dt + \varepsilon \int_{t_1}^{t_2} \frac{df}{dt}(t, q(t), \dot{q}(t)) \, dt + O(\varepsilon^2), \tag{5}
\]

which is equivalent to the following Killing-type equation for ODEs:

\[
\partial_t LT + \partial_q L \cdot \xi + \partial_{\dot{q}} L \cdot (\dot{\xi} - \dot{q} \dot{T}) + L \dot{T} = \dot{f}. \tag{6}
\]

This is basically the same formula as Sarlet and Cantrijn [7], formula (9) p. 471. We write \( \partial_t, \partial_q, \partial_{\dot{q}} \) for the partial derivative and gradients, \( \dot{x} \) for the total time derivative of the function \( x \), and \( x \cdot y \) will denote the ordinary scalar product of \( x, y \in \mathbb{R}^n \).

Noether’s theorem states that equation (6) is a sufficient condition so that the function

\[
N = LT + \partial_q L \cdot (\xi - \dot{q} T) - f \tag{7}
\]

([7] p. 471, formula (11)) multiplied by \(-1\) be a constant of motion for the solutions to the Lagrange equation (2).

Equation (6) is referred to as “Killing-type” because of the important particular case when the Lagrangian function is a quadratic form in the \( \dot{q} \) variable:

\[
L = \frac{1}{2} \dot{q} \cdot A(q) \dot{q}, \text{ with } A(q) \text{ symmetric } n \times n \text{ non-singular matrix. The Lagrange}
\]
equation (3) reduces to the equation of geodesics. Equation (6) with $T \equiv 0$, $f \equiv 0$, and $\xi(q)$ as a function of $q$ only, becomes $\partial_\xi L \cdot \xi(q) + \partial_q L \cdot \xi'(q)\dot{q} = 0$, which is quadratic homogeneous in $\dot{q}$: if we equate to zero the coefficients, we get the well-known Killing equations of Riemannian Geometry. The first integral (7) simplifies to $\partial_q L \cdot \xi(q) = A(q)\dot{q} \cdot \xi(q) = \dot{q} \cdot A(q)\xi(q)$.

Back to general Killing-type equation (6), the function $L(t, q, \dot{q})$ will be given, and we solve for the triple $(T, \xi, f)$. The equation in the terse form (6) is open to at least three interpretations that we know of, differing on what the independent variables are and on how to treat the $\ddot{q}$ terms.

The most restrictive approach is when we seek $T, \xi, f$ as functions of $(t, q)$ only: $(T(t, q), \xi(t, q), f(t, q))$. In this case $\ddot{q}$ does not appear, the independent variables are $t, q, \dot{q}$, and we want the equation to hold identically. We will not be concerned with this interpretation in the sequel.

If we instead allow full dependence of $T, \xi, f$ on $\dot{q}$, the expanded-out form of the total time derivatives $\dot{\xi}, \dot{T}, \dot{f}$ will contain $\ddot{q}$. We distinguish two alternatives:

- **Strong form**: we treat $\ddot{q}$ as just another independent variable and require the equation to hold for all $t, q, \dot{q}, \ddot{q}$. This way the Hamiltonian action functional will be infinitesimally invariant along all smooth trajectories $q(t)$. The strong form was introduced by Djukic [3], and studied also by Kobussen [5].

- **On-flow form**: we replace every occurrence of $\ddot{q}$ with $g(t, q, \dot{q})$ from the normalized Lagrange equation (3), and require the resulting equation to hold for all $t, q, \dot{q}$. This way the Hamiltonian action functional will be infinitesimally invariant as in (5) only along the Lagrangian motions $q(t)$, a condition which is however enough for the expression (7) to be a first integral.

In Section 2 of this work we investigate the structure of the solution sets of the Killing-like equation. For both the strong and the on-flow form we give a multiplicity result (Theorem 2), in the sense that each solution gives rise to a whole family of other solutions, parameterized by an arbitrary scalar function, yielding the same first integral. For the on-flow form we give a complete explicit description of the set of solutions associated to a given first integral (Theorem 3), a solution set which is nonempty (Theorem 1). For both forms, each solution leads to another for which $T = 0$ and to another for which $f = 0$ (what we call “trivialization” of time change or gauge; Corollary 2). The on-flow existence result (Theorem 1, which can be subsumed into Theorem 3) is a possible approach to the inverse Noether theorem: each first integral can be deduced from an infinitesimal transformation that leaves the action invariant.

In Sections 3 and 4 we test our theory on the well-known example of the Laplace-Runge-Lenz (or RLR) vector conservation for the Kepler problem. For the on-flow version, all the literature that we found on the subject gives the very same explicit formulas for the triple $(T, \xi, f)$, up to a constant factor. In Section 3 we will show other solutions, at least one of which looks to us more
elegant in the formula and easier in the calculations:
\[ T \equiv 0, \quad \vec{\xi} = (\vec{r} \times \vec{v}) \times \vec{u}, \quad f = \frac{\mu}{\|\vec{r}\|} \vec{r} \cdot \vec{u}, \] (8)
and we will also show an on-flow solution with zero gauge term. In Section 4 we exhibit a strong solution with trivial gauge term, based on the solution we found in Sarlet and Cantrijn [7]. We also provide computer algebra code to help give and independent check of the validity of the solutions.

2 Structure of the solution sets

We will tacitly assume that \( L \in C^3 \) and that the Legendre condition is satisfied:
\[ \det \partial^2_{\dot{q}, q} L(t, q, \dot{q}) \neq 0, \] (9)
where \( \partial^2_{\dot{q}, q} L \) is the matrix of the second derivatives of \( L \) with respect to \( \dot{q} \). This will ensure that the Lagrange equation can indeed be put into normal form (3), and there is existence and uniqueness of the solutions to the Cauchy problems.

If we know a first integral \( N \) of the Lagrangian system, we easily deduce a corresponding solution to the on-flow Killing-type equation:

**Lemma 1** (Basic on-flow existence). Let \( L(t, q, \dot{q}) \) be a Lagrangian function. Suppose that the Lagrange equation has the \( C^1 \) first integral \( N(t, q, \dot{q}) \). Then the on-flow version of the Killing-type equation (6) is satisfied by the triple
\[ T(t, q, \dot{q}) = \frac{N(t, q, \dot{q})}{L(t, q, \dot{q})}, \quad \xi(t, q, \dot{q}) = \dot{q} \frac{N(t, q, \dot{q})}{L(t, q, \dot{q})}, \quad f \equiv 0. \] (10)

The corresponding first integral (7) is precisely \( N \).

**Proof.** The on-flow total derivative \( \dot{N} \) is identically zero because \( N \) is constant along the Lagrange motions. Hence
\[ \dot{T} = -\frac{N \dot{L}}{L^2}, \quad \dot{\xi} = \dot{q} \frac{N}{L} - \dot{q} \frac{N \dot{L}}{L^2}. \]
So the Killing-type equation (6) with \( f \equiv 0 \) is satisfied:
\[ \partial_t LT + \partial_q L \cdot \dot{\xi} + \partial_{\dot{q}} L \cdot (\dot{\xi} - \dot{q} \dot{T}) + L \dot{T} = \]
\[ = \partial_t L \frac{N}{L} + \partial_q L \cdot \dot{q} \frac{N}{L} + \partial_{\dot{q}} L \cdot \left( \dot{q} \frac{N}{L} - \dot{q} \frac{N}{L^2} \dot{L} + \dot{q} \frac{N}{L} \dot{L} \right) - L \frac{N}{L^2} \dot{L} = \]
\[ = \frac{N}{L} (\partial_t L + \partial_q L \cdot \dot{q} + \partial_{\dot{q}} L \dot{q} - \dot{L}) = 0 = \dot{f}. \]
The first integral given by (7) is
\[ LT + \partial_q L \cdot (\dot{\xi} - \dot{q} T) = L \frac{N}{L} + \partial_q L \cdot \left( \dot{q} \frac{N}{L} - \dot{q} \frac{N}{L} \right) = N. \]
Around points where \( L = 0 \) we can take \( T = N/(L + c) \), \( \xi = N/(L + c) \), for a constant \( c \neq 0 \).

**Lemma 2** (On-flow solutions with trivial \( N \)). Let \( L(t, q, \dot{q}) \) be a Lagrangian function. Let \( R(t, q, \dot{q}) \) be an arbitrary smooth function with values in \( \mathbb{R}^n \). Then the on-flow version of the Killing-type equation (10) is satisfied by the triple

\[
T(t, q, \dot{q}) = -\frac{\partial_q L \cdot R}{L}, \quad \xi(t, q, \dot{q}) = R - \dot{q} \frac{\partial_q L \cdot R}{L}, \quad f \equiv 0.
\]  

(11)

The corresponding first integral (7) is the trivial constant 0.

**Proof.** We can compute, using the Lagrange equation (2),

\[
\frac{d(LT)}{dt} = -\frac{d}{dt}(\partial_q L \cdot R) = -R \cdot \frac{d}{dt} \partial_q L - \partial_q L \cdot \dot{R} = -R \cdot \partial_q L - \partial_q L \cdot \dot{R},
\]

(12)

\[
\dot{\xi} = \dot{R} + \dddot{q} T + \dot{q} \dot{T}.
\]

(13)

Hence the on-flow Killing-like equation becomes

\[
\partial_t LT + \partial_q L \cdot \xi + \partial_q L \cdot (\dot{\xi} - \dot{q} \dot{T}) + L \dot{T} =
\]

\[
= \partial_t LT + \partial_q L \cdot (R + T \dot{q}) + \partial_q L \cdot (\dot{R} + \dddot{q} T + \dot{q} \dot{T} - \dot{q} \dot{T}) + L \dot{T} =
\]

\[
= \partial_t LT + \partial_q L \cdot (T \dot{q}) + \partial_q L \cdot \dot{q} T + \partial_q L \cdot (R) + \partial_q L \cdot (\dot{R}) + L \dot{T} =
\]

\[
= LT + L \dot{T} + \partial_q L \cdot R + \partial_q L \cdot \dot{R} = \frac{d(LT)}{dt} + \partial_q L \cdot R + \partial_q L \cdot \dot{R} =
\]

\[
= 0 = f.
\]

It is clear that formula (7) gives 0 in this case.

**Theorem 1** (On-flow existence). Let \( L(t, q, \dot{q}) \) be a Lagrangian function. Suppose that the Lagrange equation has the \( C^1 \) first integral \( N(t, q, \dot{q}) \). Let \( R(t, q, \dot{q}) \) be an arbitrary smooth function with values in \( \mathbb{R}^n \). Then the on-flow version of the Killing-type equation (6) is satisfied by the triple

\[
T(t, q, \dot{q}) = \frac{N - \partial_q L \cdot R}{L}, \quad \xi(t, q, \dot{q}) = R + \dot{q} \frac{N - \partial_q L \cdot R}{L}, \quad f \equiv 0.
\]

(14)

The corresponding first integral (7) is precisely \( N \).

**Proof.** Since the Killing-type equation is linear in \( T, \xi, f \), we can simply take the sum of solution triples (10) and (11). The associated first integral is the sum \( N + 0 = N \).

A solution triple was given by Candotti, Palmieri, and Vitale [2] in a different setting. When translated into our language it becomes the following:

**Corollary 1.** Suppose that the Lagrange equation has the \( C^2 \) first integral \( N(t, q, \dot{q}) \). Let

\[
R := (\partial^2_{q \dot{q}} L)^{-1} \partial_q N.
\]

(15)
Then the on-flow version of the Killing-type equation \(6\) is satisfied by the triple
\[
T(t, q, \dot{q}) = \frac{N - \partial_q L \cdot R}{L}, \quad \xi(t, q, \dot{q}) = R + \dot{q} \frac{N - \partial_q L \cdot R}{L}, \quad f \equiv 0.
\] (16)

The corresponding first integral \(7\) is precisely \(N\).

If we know a solution to the Killing-type equation, either on-flow or strong, we can easily generate infinitely many others, parameterized by an arbitrary function:

Theorem 2 (Multiplicity for both on-flow and strong equation). Let \(L(t, q, \dot{q})\) be a Lagrangian function. Suppose that the triple \((T(t, q, \dot{q}), \xi(t, q, \dot{q}), f(t, q, \dot{q}))\) satisfies the Killing-type equation \(6\) in either the strong or the on-flow version. Take an arbitrary smooth function \(h(t, q, \dot{q})\). Then also the following triple
\[
\tilde{T} = T + \frac{h - f}{L}, \quad \tilde{\xi} = \xi + \dot{q} \frac{h - f}{L}, \quad \tilde{f} = h
\] (17)
satisfies the Killing-type equation of the same form. The corresponding first integral \(7\) is the same.

Proof. It is a simple computation, under the assumption that the Killing-type equation holds for all values of the independent variables \(t, q, \dot{q}, \ddot{q}\):

\[
\partial_t L \hat{T} + \partial_q L \cdot \hat{\xi} + \partial_q L \cdot (\dot{\xi} - \dot{q} \hat{T}) + L \hat{T} =
\]
\[
= \partial_t L \left( T + \frac{h - f}{L} \right) + \partial_q L \cdot \left( \xi + \dot{q} \frac{h - f}{L} \right) +
\]
\[
+ \partial_q L \cdot \left( \dot{\xi} + \dot{q} \frac{h - f}{L} + \dot{q} \frac{d}{dt} \left( \frac{h - f}{L} \right) - \dot{q} \frac{d}{dt} \left( \frac{h - f}{L} \right) \right) +
\]
\[
+ L \hat{T} + L \frac{d}{dt} \left( \frac{h - f}{L} \right) =
\]
\[
= \partial_t L T + \partial_q L \cdot \xi + \partial_q L \cdot \left( \dot{\xi} - \dot{q} \hat{T} \right) + L \hat{T} +
\]
\[
+ \partial_t L \frac{h - f}{L} + \partial_q L \cdot \dot{q} \frac{h - f}{L} +
\]
\[
+ \partial_q L \cdot \left( \dot{q} \frac{h - f}{L} + \dot{q} \frac{d}{dt} \left( \frac{h - f}{L} \right) - \dot{q} \frac{d}{dt} \left( \frac{h - f}{L} \right) \right) +
\]
\[
+ L \frac{d}{dt} \left( \frac{h - f}{L} \right) =
\]
\[
= \dot{f} + L \frac{h - f}{L} + L \frac{d}{dt} \left( \frac{h - f}{L} \right) = \dot{f} + \frac{d}{dt} \left( L \frac{h - f}{L} \right) = \dot{f} + \frac{d}{dt} (h - f) =
\]
\[
= \dot{h}.
\]

The reader can easily check that the first integral is the same. \(\square\)
Within the family of solutions given by Theorem 2 there is always one with trivial (i.e., zero) time change and another one with trivial gauge. This simple fact was already established in a more general setting (including, for example, nonlocal constants of motion) and different notations by the authors [4, Theorem 10].

**Corollary 2** (Trivializing either time-change or gauge). Let $L(t, q, \dot{q})$ be a Lagrangian function. Suppose that the triple $(T(t, q, \dot{q}), \xi(t, q, \dot{q}), f(t, q, \dot{q}))$ satisfies the Killing-type equation (11) in either the strong or the on-flow form. Then also the following two triples are solutions:

$$
(0, \xi - \dot{q} T, f - LT), \quad \left( T - \frac{f}{L}, \xi - \dot{q} \frac{f}{L}, 0 \right).
$$

The corresponding first integrals are the same.

**Proof.** Simply take either $h = f - LT$ or $h = f$ in Theorem 2.

Theorem 1 combined with Theorem 2 yield a solution formula where the gauge term $f$ is easily calculated from arbitrary $T$ and $\xi$:

**Theorem 3** (General solution of the on-flow equation). Let $L(t, q, \dot{q})$ be a Lagrangian function. Suppose that the Lagrange equation has the $C^1$ first integral $N(t, q, \dot{q})$. Let $\xi(t, q, \dot{q})$ be a smooth function with values in $\mathbb{R}^n$ and $T(t, q, \dot{q})$ an arbitrary smooth functions with values in $\mathbb{R}$. Then the following triple is a solution of the on-flow version of the Killing-type equation:

$$
(T, \xi, TL - N + \partial_q L \cdot (\xi - T\dot{q})),
$$

always with $N$ as the associated first integral. If $f_1$ is another gauge term such that $(T, \xi, f_1)$ is an on-flow solution too, then $f - f_1$ is a first integral.

**Proof.** Take the triple of formula (14)

$$
\left( \frac{N - \partial_q L \cdot R}{L}, R + \dot{q} \frac{N - \partial_q L \cdot R}{L}, 0 \right),
$$

use it as the starting triple of Theorem 2 to obtain the solution triple

$$
\left( \frac{N - \partial_q L \cdot R}{L} + \frac{h - 0}{L}, R + \dot{q} \frac{N - \partial_q L \cdot R}{L} + \dot{q} \frac{h - 0}{L}, h \right).
$$

Specializing the choices of $R, h$ as $R := \xi - T\dot{q}$ and $h = TL - N + \partial_q L \cdot (\xi - T\dot{q})$ we obtain precisely formula (19).

If another $f_1$ makes $(T, \xi, f_1)$ into a solution triple of equation (6), then $\dot{f} = f_1$, so that the difference $f - f_1$ is a first integral.

From formula (19) we can express $T$ as a function of $f, \xi$, at least when $L - \partial_q L \cdot \dot{q} \neq 0$. 

7
3 On-flow solutions for LRL conservation

Consider the Lagrangian function and Lagrange equation of Kepler’s problem in dimension 3

\[ L(t, \vec{r}, \vec{v}) = \frac{1}{2} \| \vec{v} \|^2 + \frac{\mu}{\| \vec{r} \|}, \quad \vec{r} \in \mathbb{R}^3 \setminus \{0\}, \]  \hfill (22)

\[ \ddot{\vec{r}} = -\frac{\mu}{\| \vec{r} \|^3} \vec{r}. \]  \hfill (23)

The vector product “×” for 3-dimensional vectors will allow more compact formulas than what we get in the otherwise equivalent 2-dimensional treatment we gave in an earlier paper [4].

Besides energy and angular momentum, the Kepler system has the LRL vector first integral

\[ \vec{A} := \vec{v} \times (\vec{r} \times \vec{v}) - \frac{\mu}{\| \vec{r} \|} \vec{r}. \]  \hfill (24)

Fix an arbitrary vector \( \vec{u} \in \mathbb{R}^3 \). Our purpose is to obtain \( N := \vec{u} \cdot \vec{A} \) as the first integral from equation (7).

Formula (19) of Corollary 3 gives infinitely many solutions, one of which is for example

\[ T_0 = \frac{\vec{u} \cdot \vec{v} \times (\vec{r} \times \vec{v})}{L}, \quad \vec{\xi}_0 = \frac{\vec{u} \cdot \vec{v} \times (\vec{r} \times \vec{v})}{L} \vec{v}, \quad f_0 = \frac{\mu}{\| \vec{r} \|} \vec{r} \cdot \vec{u}. \]  \hfill (25)

Formula (19) with \( R = (\partial^2_{\eta_4} L)^{-1} \partial_4 N \), as in Corollary 1 but with \( T = 0 \) gives the solution triple

\[ T_L = 0, \quad \vec{\xi}_L = \partial_{\vec{v}} N = 2(\vec{r} \cdot \vec{u})\vec{v} - (\vec{v} \cdot \vec{u})\vec{r} - (\vec{v} \cdot \vec{r})\vec{u}, \]  \hfill (26)

\[ f_L = -N + \partial_{\vec{v}} L \cdot \partial_{\vec{v}} N = \frac{\mu}{\| \vec{r} \|} \vec{r} \cdot \vec{u} + \vec{r} \cdot \vec{v} \times (\vec{u} \times \vec{v}) \]  \hfill (27)

which is exactly the one given by Levy-Leblond [6], except for a constant factor and the dimension being 3 here, instead of 2. Boccaletti and Pucacco in their textbook [1] arrive at this same solution starting from a suitable ansatz.

To find different solutions with trivial first order time variation \( T \equiv 0 \), we write equation (8) within the current setting:

\[ \partial_{\vec{v}} L \cdot \vec{\xi} + \partial_{\vec{v}} L \cdot \dot{\vec{\xi}} = \dot{f}, \]  \hfill (28)

and we impose that the first order space variation \( \vec{\xi} \) be such that

\[ \partial_{\vec{v}} L \cdot \vec{\xi} = \vec{u} \cdot \vec{v} \times (\vec{r} \times \vec{v}). \]  \hfill (29)

Our favourite way to satisfy this condition is

\[ \vec{\xi}_Z = (\vec{r} \times \vec{v}) \times \vec{u} = (\vec{r} \cdot \vec{u})\vec{v} - (\vec{v} \cdot \vec{u})\vec{r}. \]  \hfill (30)
This choice is not the only possible: \( \xi = 2\xi_L \) satisfies the same condition:
\[
\partial_t \xi_L \cdot 2\xi_L = \ddot{v} \cdot (\ddot{v} \times (\ddot{r} \times \ddot{v})) = \|v\|^2 \dddot{u} \cdot \dddot{r} - (\dddot{v} \cdot \dddot{r})(\ddot{v} \cdot \dddot{u}).
\] (31)

The two alternatives \( \xi_L, \xi_Z \) cannot be obtained from each other through the multiplicity Theorem because their difference \( \xi_Z - \xi_L \) is not a multiple of \( v \).

Using Lagrange equation (23) we have
\[
\dot{\xi}_Z = (\partial_r \xi_Z) \dddot{v} + \partial_v \xi_Z \left( -\frac{\mu}{\|r\|^3} \dddot{r} \right) = \ddot{u}.
\] (32)

Using equation (24), the left-hand side of equation (28) becomes
\[
\partial_t \xi_L \cdot \xi_Z + \partial_v \xi_L \cdot \dot{\xi}_Z = \partial_t \xi_L \cdot \xi_Z + \partial_v \xi_L \cdot \dddot{u} =
\]
\[
= -\frac{\mu}{\|r\|^3} \dddot{r} \times (\dddot{r} \times \dddot{v}) \cdot \dddot{u} =
\]
\[
= -\frac{\mu}{\|r\|^3} \dddot{u} \cdot (\dddot{r} \cdot \dddot{v}) - (\dddot{v} \cdot \dddot{r})^2 = \dddot{v} \cdot \partial_r \left( \frac{\mu}{\|r\|^3} \dddot{r} \cdot \dddot{u} \right) =
\]
\[
= \frac{d}{dt} \left( \frac{\mu}{\|r\|^3} \dddot{r} \cdot \dddot{u} \right) = \dot{f}_0.
\]

We can complete the solution triple as follows:
\[
T_Z = 0, \quad \xi_Z = (\dddot{r} \times \dddot{v}) \times \dddot{u}, \quad f_Z = f_0 = \frac{\mu}{\|r\|^3} \dddot{r} \cdot \dddot{u}.
\] (33)

The resulting first integral of formula (7) is what we expected:
\[
\partial_t \xi_Z - f_Z = \ddot{v} \cdot (\dddot{r} \times \dddot{v}) \times \dddot{u} - \frac{\mu}{\|r\|^3} \dddot{r} \cdot \dddot{u} = \left( \dddot{v} \times (\dddot{r} \times \dddot{v}) - \frac{\mu}{\|r\|^3} \dddot{r} \right) \cdot \dddot{u} = \dddot{A} \cdot \dddot{u}.
\]

The triple (33) belongs to the family of Theorem as can be checked by direct computation.

Theorem formula (17), applied to the triple (33), gives a whole family of solution triples, depending on an arbitrary function \( h(t, r, \dddot{v}) \):
\[
T = \frac{1}{T} \left( h - \frac{\mu}{\|r\|^3} \dddot{r} \right), \quad \dddot{\xi} = (\dddot{r} \times \dddot{v}) \times \dddot{u} + \frac{1}{T} \left( h - \frac{\mu}{\|r\|^3} \dddot{r} \right) \dddot{v}, \quad f = h.
\] (34)

As in Corollary the choice \( h \equiv 0 \) will trivialize the gauge function.

Using a computer algebra system the reader can directly check all these solutions to the Killing-type equation, independently of the theorems in Section 2. For example here is some simple code written for Wolfram Mathematica that implements the solution triple (34) and then checks that the on-flow Killing-type equation is satisfied and that the first integral is the LRL vector:

(*Defining the variables*)
\[
r = \{r1, r2, r3\};
\]
v = {v1, v2, v3};
L = v.v/2 + mu/Sqrt[r.r];
A = Cross[v, Cross[r, v]] - mu*r/Sqrt[r.r];
u = {u1, u2, u3};
arbitrary = h[t, r1, r2, r3, v1, v2, v3];
f0 = mu*(u.r)/Sqrt[r.r];
T = (arbitrary - f0)/L;
Xi = Cross[Cross[r, v], u] + T*v;
rDotDot = -mu*(u.r)*r/(r.r)^(3/2);
Tdot = D[T, t] + D[T, {r}].v + D[T, {v}].rDotDot;
XiDot = D[Xi, t] + D[Xi, {r}].v + D[Xi, {v}].rDotDot;
f = arbitrary;
fDot = D[f, t] + D[f, {r}].v + D[f, {v}].rDotDot;

(*checking Killing-type equation*)
Simplify[
  D[L, t]*T + D[L, {r}].Xi +
  D[L, {v}].(XiDot - v*Tdot) + L*Tdot == fDot]

(*checking LRL vector as first integral*)
Simplify[
  L*T + D[L, {v}].(Xi - T*v) - f == A.u]

Upon evaluation, the code gives True and True in an instant.

4 Strong solutions for LRL conservation

The strong form of Killing-type equations

\[
\partial_t LT + \partial_q L \cdot \xi + \partial_q L \cdot (\dot{\xi} - \dot{q} \dot{T}) + L \ddot{T} = \ddot{f},
\]

as we mentioned, corresponds to expanding the total derivatives \(\dot{T}, \dot{\xi}, \dot{f}\), and treating the resulting \(\dot{q}\) as just another independent variable, together with \(t, q, \dot{q}\). It corresponds to requiring the Hamiltonian action functional to be infinitesimally invariant along all smooth paths \(q(t)\), and not just along the Lagrangian flow. Equation (35) is linear with respect to \(\ddot{q}\). If we wish, we could eliminate the dependence on \(\ddot{q}\) by equating to 0 all the coefficients of the components \(\ddot{q}_j\) and the remaining known term, obtaining a system of \(n + 1\) scalar equations with \(t, q, \dot{q}\) as independent variables (equations (63) and (64), p. 480, in Sarlet and Cantrijn [7]). However, we will work here with the original single equation (35).

Sarlet and Cantrijn [7, Sec. 6] exhibited an explicit elementary solution triple to the strong Killing-like equation, leading again to the Laplace-Runge-Lenz vector as first integral. We will not reproduce their formulas here. It will suffice to mention that their setting is 2-dimensional, the first order variation of time is a function of \(q\) only, the first order variation of space \(\xi(q, \dot{q})\) is linear in \(\dot{q}\), and
the gauge-function \( f(q, \dot{q}) \) is quadratic in \( \dot{q} \). In our recent paper [4, Sec. 12] we have already given a variant of that solution with trivializes the gauge function, still in dimension 2. Here we apply the full multiplicity Theorem 2 and also increase the dimension to 3, where the vector cross product again leads to elegant formulas for the solution triple \( (T, \Xi, f) \):

\[
\vec{b} := -\vec{u}(\vec{r} \cdot \vec{v}) - \vec{r}(\vec{v} \cdot \vec{u}) + \vec{v}(\vec{u} \cdot \vec{r}),
\]

\[
T = \frac{1}{L}\left(h - \vec{u} \cdot \left(\vec{v} \times (\vec{r} \times \vec{v}) + \frac{\mu}{\|r\|} \vec{r}\right)\right),
\]

\[
\vec{\Xi} = \frac{1}{L}\left(h \vec{v} + \frac{1}{2} \vec{v} \times \left(\vec{b} \times \vec{v}\right) + \frac{\mu}{\|r\|} \vec{b}\right), \quad f = h,
\]

where \( \vec{u} \in \mathbb{R}^3 \) is an arbitrary parameter vector, as in the previous section. The first integral associated to the triple through Noether’s theorem (7) is the same as before:

\[
LT + \partial_\vec{L} \cdot (\vec{\Xi} - T\vec{v}) = \vec{u} \cdot \left(\vec{v} \times (\vec{r} \times \vec{v}) - \frac{\mu}{\|r\|} \vec{r}\right) = \vec{u} \cdot \vec{A}.
\]

Again we provide below some Mathematica code that implements the solution triple and checks that it solves the Killing-type equation and that it gives the Laplace-Runge-Lenz first integral.

(*Defining the variables*)
\[
r = \{r_1, r_2, r_3\};
\]
\[
v = \{v_1, v_2, v_3\};
\]
\[
L = v.v/2 + \mu/\text{Sqrt}[r.r];
\]
\[
A = \text{Cross}[v, \text{Cross}[r, v]] - \mu r/\text{Sqrt}[r.r];
\]
\[
u = \{u_1, u_2, u_3\};
\]
\[
arbitrary = h[t, r_1, r_2, r_3, v_1, v_2, v_3];
\]
\[
T = (\text{arbitrary} - u.(\text{Cross}[v, \text{Cross}[r, v]] + \mu r/\text{Sqrt}[r.r]))/L;
\]
\[
b = -u.(r.v) - r.(u.v) + v.(r.u);
\]
\[
\Xi = (\text{arbitrary} v + \text{Cross}[v, \text{Cross}[b, v]]/2 + \mu b/\text{Sqrt}[r.r])/L;
\]
\[
r\text{DotDot} = \{a_1, a_2, a_3\};
\]
\[
T\text{dot} = D[T, t] + D[T, \{r\}].v + D[T, \{v\}].r\text{DotDot};
\]
\[
\Xi\text{dot} = D[\Xi, t] + D[\Xi, \{r\}].v + D[\Xi, \{v\}].r\text{DotDot};
\]
\[
f = \text{arbitrary};
\]
\[
f\text{dot} = D[f, t] + D[f, \{r\}].v + D[f, \{v\}].r\text{DotDot};
\]

(*checking Killing-type equation*)
\[
\text{Simplify}\[D[L, t]*T + D[L, \{r\}].Xi + D[L, \{v\}].(Xi\text{dot} - v\text{dot}) + L\text{dot} == f\text{dot}]
\]

(*checking LRL vector as first integral*)
\[
\text{Simplify}\[L*T + D[L, \{v\}].(Xi - T*v) - f == A.u\]
\]

The evaluation gives True, as expected.
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