Reduced Hamiltonian for intersecting shells

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Abstract

The gauge usually adopted for extracting the reduced Hamiltonian of a thin spherical shell of matter in general relativity, becomes singular when dealing with two or more intersecting shells. We introduce here a more general class of gauges which is apt for dealing with intersecting shells. As an application we give the hamiltonian treatment of two intersecting shells, both massive and massless. Such a formulation is applied to the computation of the semiclassical tunneling probability of two shells. The probability for the emission of two shells is simply the product of the separate probabilities thus showing no correlation in the emission probabilities in this model.

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1 Introduction

A lot of work has been done in the subject of thin spherical shells of matter in general relativity [1, 2] and several applications given [3, 4, 5]. An interesting application of the mechanics of thin shells has been the semiclassical treatment of the black hole radiation [6, 7, 8]. An important role in such a treatment is the choice of the gauge. In the original treatment [6] a gauge was adopted in which the radial component of the metric is equal to the radial coordinate except for an arbitrarily small region around the shell. Such a gauge choice was put in mathematically clear terms in [9, 10] where the reduced canonical momentum is extracted through a well defined limiting process. However such a limit gauge becomes singular when two or more shells are present and intersect. In addition also in the simple instance in which only one shell is present, it would be nice to have a procedure in which no limit process is present as the gauge choice should be completely arbitrary provided it is free of coordinate singularities. Moreover even in the one shell case the procedure for extracting the canonical momentum is usually considered as a very complicated one. Here we shall give a treatment which greatly simplifies the derivation of the reduced action, does not require any limiting procedure and can be applied for the treatment of two or more shells which intersect.

All the problem is to derive the reduced action, i.e. an action in terms of the shell coordinates and some appropriate conjugate momenta. We shall keep the formalism for the two shell case as close as possible to the one shell treatment.

We shall perform all the treatment for a massive dust shell and the simpler case of a null shell can be derived as a particular case. As an application of the developed formalism we rederive the well known Dray-'t Hooft and Redmount relations for the intersection of light-like shells [11, 12].

The main motivation which originated the works [6, 8, 13] is the computation of the semiclassical tunneling amplitude, which is related to the black hole radiation in which one takes into account the effect of energy loss by the black hole. It is remarkable that such a simple model reproduces all the correct features of the Hawking radiation, giving also some corrections due to energy conservation. For approaches more kinematical in nature see e.g. [14]. The adoption of more general gauges allows the dynamical treatment of two intersecting shells without encountering singularities. The formalism developed here is applied to the computation of the semiclassical emission probability of two shells; it was in fact suggested [6] that correlations could show up in the multiple shell emission. We find however that such a model gives no correlation among the probabilities of the two emitted shells.
The paper is structured as follows. In Section 2 we lay down the formalism by exploiting some peculiar properties of a function $F$ strictly related to the momenta canonically conjugate to the metric functions. It is then just a simple matter of partial integrations to extract the reduced canonically conjugate momentum without employing any limit process. This is done in Section 3. A new term containing the time derivative of the mass of the remnant black hole appears; if such a mass is considered as a datum of the problem the result agrees with those of the original massless case of Kraus and Wilczek and with the massive case result obtained through a limiting process in [9]. Then we discuss the derivation of the equations of motion. In [6] and [9] the variational principle was applied by varying the total mass of the system which we shall denote by $H$. In [8] the attitude was adopted of keeping the total mass of the system fixed and varying instead the mass of the remnant black hole. Here we show that both procedures can be applied to obtain the equations of motion; depending however on the choice of the gauge one procedure is far more complicated than the other and we give them both.

In Section 4 we discuss more general gauge choices and derive the equation of motion in the inner gauge. We shall not consider in the present paper complex gauges or complex gauge transformations [15].

In Section 5 we discuss the analytic properties of the conjugate momentum; this is of interest because the imaginary part of the conjugate momentum is responsible for the tunneling amplitude and in determining the Hawking temperature both via a simple mechanical model or more precisely by working out the semiclassical wave functions on which to expand the matter quantum field [16]. We remark that the result for the tunneling probability is independent of the mass of the shell but depends only on the initial and final energy of the black hole.

In Section 6 we extend the treatment to two shells in the outer gauge writing down the explicit expression of the two reduced canonical momenta.

In Section 7 we derive the equations of motion for both shells from the reduced two shell action. For simplicity this is done for the massless case; the general treatment for massive shells is given in Appendix C.

In Section 8 the developed formalism is applied to give a very simple treatment of two shells which intersect. In the massless case we rederive the well known relations of Dray and ’t Hooft [11] and Redmount [12].

In Section 9 we consider the problem of computing the tunneling probability for the emission of two shells which in the process can intersect. To this end one has to compute the imaginary part of the action along the analytically continued solution of the equations of
motion. In this connection a helpful integrability result is proven which allows to compute the action along a specially chosen trajectory on the reduced coordinate space. Such a result allows on one hand to prove the independence of the result from the deformation defining the gauge and on the other hand it allows to compute explicitly such imaginary part. The final outcome is that in all instances (massive or massless shells) the result again depends only on the initial and final values of the masses of the black hole and the expression coincides with the one obtained in the one shell case. The interest in studying the two shell system was pointed out in [6] in order to investigate possible correlations among the emitted shells. Here we simply find that the two shells, even if they interact with an exchange of masses, are emitted with a probability which is simply the product of the probabilities for single shell emissions and thus that the model does not predict any correlation among the emitted shells.

In Section 10 we summarize the obtained results.

2 The action

As usual we write the metric for a spherically symmetric configuration in the ADM form [6, 3, 9]

\[ ds^2 = -N^2 dt^2 + L^2 (dr + N^r dt)^2 + R^2 d\Omega^2. \] (1)

We shall work on a finite region of space time \((t_i, t_f) \times (r_0, r_m)\). On the two initial and final surfaces we give the intrinsic metric by specifying \(R(r, t_i)\) and \(L(r, t_i)\) and similarly \(R(r, t_f)\) and \(L(r, t_f)\).

The complete action in hamiltonian form, boundary terms included is [17, 3, 9]

\[ S = S_{\text{shell}} + \int_{t_i}^{t_f} dt \int_{r_0}^{r_m} dr (\pi_L \dot{L} + \pi_R \dot{R} - N \mathcal{H}_t - N^r \mathcal{H}_r) + \int_{t_i}^{t_f} dt \left( -N^r \pi_L L + \frac{N R R'}{L} \right) \bigg|_{r_0}^{r_m} \] (2)

where

\[ S_{\text{shell}} = \int_{t_i}^{t_f} dt \, \dot{\hat{r}} \, \hat{p}. \] (3)

\(\mathcal{H}_t\) and \(\mathcal{H}_r\) are the constraints which are reported in Appendix A, \(\hat{r}\) is the shell position and \(\hat{p}\) its canonical conjugate momentum. Action (2) is immediately generalized to a finite number of shells. \(S_{\text{shell}}\) as given by eq.(3) refers to a dust shell even though generalizations to more complicated equations of state have been considered [2, 3, 18].

Varying the action w.r.t. \(N\) and \(N^r\) gives the vanishing of the constraints while the variations w.r.t. \(R, L, \pi_R, \pi_L\) give the equations of motion of the gravitational field [3, 9] which for completeness are reported in Appendix A. The functions \(R, L, N, N^r\) are
continuous in \( r \) while \( R', L', N', N'_r, \pi_L, \pi_R \) can have finite discontinuities at the shell position \([9]\). In \([9]\) it was proven that the equation of motion of a massive shell cannot be obtained from a true variational procedure, in the sense that the obtained expression for \( \hat{p} \) is discontinuous at \( r = \hat{r} \). In the same paper it was remarked that for consistency the average value of the r.h.s. of the equation for \( \hat{p} \) has to be taken. For the reader’s convenience we give in Appendix A the explicit proof that the equations of motion of matter i.e. \( \hat{r} \) and \( \hat{p} \) can be deduced from the already obtained equations of motion for the gravitational field combined with the constraints. The equation for \( \hat{r} \) does not pose any problem while for the equation for \( \hat{p} \) one deduces algebraically that the r.h.s. contains the average of the derivatives of \( L, N, N'_r \) across the shell. In Appendix A we also show directly that such discontinuity is absent in the massless case \([10]\).

Already in \([3, 6]\) it was pointed out that in the region free of matter, as a consequence of the two constraints the quantity \( \mathcal{M} \)

\[
\mathcal{M} = \frac{\pi_L^2}{2R} + \frac{R}{2} - \frac{RR'^2}{2L^2} \quad (4)
\]

is constant in \( r \) and this allows to solve for the momenta \([3, 6]\)

\[
\pi_L = R \sqrt{\left( \frac{R'}{L} \right)^2 - 1} + \frac{2\mathcal{M}}{R} \equiv RW \quad (5)
\]

\[
\pi_R = \frac{L[(R/L)(R'/L)'+(R'/L)^2-1+\mathcal{M}/R]}{W} \quad (6)
\]

In the one shell problem we shall call the value of such \( \mathcal{M} \), \( M \) for \( r < \hat{r} \) and \( H \) for \( r > \hat{r} \).

The function

\[
F = RL \sqrt{\left( \frac{R'}{L} \right)^2 - 1} + \frac{2\mathcal{M}}{R} + RR' \log \left( \frac{R'}{L} - \sqrt{\left( \frac{R'}{L} \right)^2 - 1 + \frac{2\mathcal{M}}{R}} \right) - R' f'(R) \quad (7)
\]

has the property of generating the conjugate momenta as follows

\[
\pi_L = \frac{\partial F}{\partial L} \quad (8)
\]

\[
\pi_R = \frac{\delta F}{\delta R} = \frac{\partial F}{\partial R} - \frac{\partial}{\partial r} \frac{\partial F}{\partial R'} \quad (9)
\]

The total derivative \( \frac{\partial f(R)}{\partial r} = R' f'(R) \) of the arbitrary function \( f(R) \) does not contribute to the momenta. The function \( F \) will play a major role in the subsequent developments. A large freedom is left in the choice of the gauge. With regard to \( L \) we shall adopt the usual gauge \( L = 1 \). It will be useful in the following developments to choose the arbitrary function \( f(R) \) such that \( F = 0 \) for \( R' = 1 \). Such a requirement fixes \( f'(R) \) uniquely.

\[
F(R, R', \mathcal{M}) = RW(R, R', \mathcal{M}) + RR'(L(R, R', \mathcal{M}) - B(R, \mathcal{M})) \quad (10)
\]
where
\[ W(R, R', \mathcal{M}) = \sqrt{R'^2 - 1 + \frac{2M}{R}}; \quad \mathcal{L}(R, R', \mathcal{M}) = \log(R' - W(R, R', \mathcal{M})) \] (11)
and
\[ B(R, \mathcal{M}) = \sqrt{\frac{2M}{R} + \log(1 - \sqrt{\frac{2M}{R}})}. \] (12)

The function \( F(R, R', \mathcal{M}) \) has the following useful properties
\[ \frac{\partial F(R, R', \mathcal{M})}{\partial R'} = R(L - B); \quad \frac{\partial L}{\partial R'} = -\frac{1}{W}. \] (13)

Other properties of \( F \) will be written when needed.

In the following section we shall choose \( R = r \) for \( r > \hat{r} \) and also \( R = r \) for \( r < \hat{r} - l \) so that \( F \) vanishes identically outside the interval \( \hat{r} - l < r < \hat{r} \). We shall call this class of gauges “outer gauges”. In Sect. 4 we shall consider more general gauges e.g. the gauge \( R = r \) for \( r < \hat{r} \) and also \( R = r \) for \( r > \hat{r} + l \) which we shall call “inner gauges”. However contrary to what is done in [6, 9] we will not take any limit \( l \to 0 \) and prove that the results are independent of the deformation of \( R \) in the region \( \hat{r} - l < r < \hat{r} \) (or \( \hat{r} < r < \hat{r} + l \) for the inner gauges) provided \( R' \) satisfies the constraint at \( r = \hat{r} \). We shall call these regions \((\hat{r} - l, \hat{r})\) for the outer gauge and \((\hat{r}, \hat{r} + l)\) for the inner gauge, the deformation regions.

The variation of \( S \) which produces the equations of motion has to be taken, as it is well known, by keeping the metric and in particular \( R \) and \( L \) fixed at the boundaries. The variation of the boundary terms gives
\[ -N^r(r_m)\delta\pi_L(r_m) + N^r(r_0)\delta\pi_L(r_0). \] (14)

The \( N, N^r \) can be obtained from the two equations of motion for the gravitational field
\[ 0 = N^r\left[\frac{\pi_L}{R^2} - \frac{\pi_R}{R}\right] + (N^r)'; \quad \hat{R} = -N\frac{\pi_L}{R} + N^r R'. \] (15)

Using these it is easily proved that for \( r \) outside the deformation region \((\hat{r} - l, \hat{r})\) we have \( N^r = N \sqrt{\frac{2H}{r}} \) for \( r > \hat{r} \), \( N = \text{const} \) and \( N^r = N \sqrt{\frac{2M}{r}} \) for \( r < \hat{r} - l \), \( N = \text{const} \) where the two constants as a rule differ. Thus the variation of the boundary term is
\[ -N(r_m)\delta H + N(r_0)\delta M. \] (16)

In the next section we shall connect \( N(r_m) \) with \( N(r_0) \) being \( N(r) \) not constant in the deformation region.
3 The one shell effective action in the outer gauge

As outlined in the previous section we shall choose

\[ R(r, t) = r + \frac{V(t)}{\tilde{r}(t)} \int_0^r \rho(r' - \hat{r}(t))dr' = r + \frac{V(t)}{\tilde{r}} g(r - \hat{r}(t)) \] (17)

having \( \rho \) support \((-l, 0)\), \( \rho(0) = 1 \) and smooth in \(-l\) and

\[ \int_{-l}^0 \rho(r)dr = 0. \] (18)

As a consequence the deformation \( g(r) \) has support in \((-l, 0)\) and \( g'(0 - \varepsilon) = 1 \). Such \( R \) satisfies the discontinuity requirements at \( r = \hat{r} \) which are imposed by the constraints. In fact the constraints (see Appendix A) impose the following discontinuity relations at \( \hat{r} \) (we recall that we chose \( L \equiv 1 \))

\[ \Delta R' = -\frac{V}{R}; \quad V = \sqrt{\hat{p}^2 + m^2} \] (19)

and

\[ \Delta \pi_L = -\hat{p}. \] (20)

In the outer gauge the bulk gravitational action becomes

\[ S_g = \int_{t_i}^{t_f} I_g dt \] (21)

where, keeping in mind that \( L \equiv 1 \)

\[ I_g = \int_{r_0}^{\infty} (\pi_L \dot{L} + \pi_R \dot{R})dr = \int_{r_0}^{\infty} \pi_R \dot{R} dr = \int_{r_0}^{\infty} \left( \frac{\partial F}{\partial R} - \frac{\partial}{\partial r} \frac{\partial F}{\partial R'} \right) \dot{R} dr \]

\[ = \int_{r_0}^{\hat{r}(t)} \frac{dF}{dt} dr - \dot{M}(t) \int_{r_0}^{\hat{r}(t)} \frac{\partial F}{\partial M} dr - \frac{\partial F}{\partial R'} \dot{R} \bigg|_{r_0}^{\hat{r}(t)} \]

\[ = \frac{d}{dt} \int_{r_0}^{\hat{r}(t)} F dr - \dot{M}(t) \int_{r_0}^{\hat{r}(t)} \frac{\partial F}{\partial M} dr - \left[ \dot{r}(t) F + \frac{\partial F}{\partial R'} \dot{R} \right]_{\hat{r}(t) - \varepsilon} \] (22)

where we used the fact that \( F \) vanishes at \( r = r_0 \).

Adding \( I_{shell} = \hat{p} \dot{\hat{r}} \) and neglecting the total time derivative which does not contribute to the equations of motion, we obtain for the reduced action in the outer gauge

\[ \int_{t_i}^{t_f} \left( pc \hat{r} - \dot{M}(t) \int_{r_0}^{\hat{r}(t)} \frac{\partial F}{\partial M} dr + (-N^r \pi_L + N R R') \bigg|_{r_0}^{r_\infty} \right) dt \] (23)
where using \((19,20)\)

\[
p_c = -F(\hat{r}(t) - \varepsilon) - \frac{1}{\hat{r}(t)} \frac{\partial F}{\partial R} \hat{R} \bigg|_{\hat{r}(t) - \varepsilon} + \hat{p} =
\]

\[
= \sqrt{2M \hat{r}} - \sqrt{2H \hat{r}} - \dot{\hat{r}} \log \left( \frac{\hat{r} + \sqrt{\hat{p}^2 + m^2} - \hat{p} - \sqrt{2H \hat{r}}}{\hat{r} - \sqrt{2M \hat{r}}} \right).
\]

\( (24) \)

A few comments are in order: 1) No limit \( l \to 0 \) is necessary for obtaining \( p_c \) of eq.(24) which holds for any deformation \( g \). 2) The \( \dot{M}(t) \) term is important, as we shall see, if we consider the variational problem in which \( M \) is varied. On the other hand if we consider \( M \) as a datum of the problem and vary \( H \) the contribution \( \dot{M}(t) \) is absent. 3) \( \hat{p} \) in eq.(24) is a function of \( \hat{r} \), \( H \) and \( M \) as given by the discontinuity relation \((20)\) equivalent to the implicit equation

\[
H - M = V + \frac{m^2}{2\hat{r}} - \hat{p} \sqrt{\frac{2H}{\hat{r}}}. 
\]

\( (25) \)

We discuss now these issues in more detail. Let us consider at first \( M \) as a datum of the problem and vary \( H \). As shown in Appendix A in order to be consistent with the gravitational equations \( M \) has to be constant in time. This is the situation examined in \([9]\) where the expression \((24)\) for \( p_c \) was derived by a limit process in which \( l \to 0 \). From eq.(23) we see that the equation of motion for \( \dot{\hat{r}} \) is given by

\[
\dot{\hat{r}} \frac{\partial p_c}{\partial H} - N(r_m) = 0
\]

\( (26) \)

where \( N(r_m) \) can be replaced by \( N(\hat{r}) \) being \( N(r) \) in the outer gauge constant for \( r > \hat{r} \). The computation of eq.(26) keeping in mind the implicit definition of \( \hat{p} \) eq.(25) gives the correct equation of motion for the massive shell

\[
\dot{\hat{r}} = \frac{\hat{p}}{V} N(\hat{r}) - N'(\hat{r}) = \left( \frac{\hat{p}}{V} - \sqrt{\frac{2H}{\hat{r}}} \right) N(\hat{r}).
\]

\( (27) \)

The outline of the calculation is done in Appendix B. Alternatively one can consider \( H = \text{const} \) as a datum of the problem and vary \( M(t) \). We remark that as shown in Appendix A the datum \( M \) or \( H \) is consistent with the gravitational equations only if \( H \) and \( M \) are constant in time. Nevertheless in the variational problem \( H \) and \( M \) have to be considered as functions of time, because the constraints tell us only that \( M \) and \( H \) are constant in \( r \). Only after deriving the equation of motion one can insert the consequences of the gravitational equations of motion.

The variation of \( M(t) \) in the outer gauge is a far more complicated procedure, due to the presence of \( \dot{M}(t) \), but gives rise to the same result obtained by varying \( H \) and keeping \( M \)
fixed. As this will be useful to understand the two shell reduced dynamics to be developed in Sect. 6 we go into it with some detail. In this case the \( \dot{M} \) term plays a major role; in fact the equation of motion now takes the form

\[
\dot{r} \frac{\partial p_c}{\partial M} + \frac{d}{dt} \int_{\hat{r}}^{r(t)} \frac{\partial F}{\partial M} \, dr + N^r(r_0) \frac{\partial \pi_L}{\partial M}(r_0) = 0
\]  

(28)

where due to the vanishing of \( g(x) \) for \( x < -l \) we have \( \pi_L(r_0) = \sqrt{2M/r_0} \). \( N^r(r) \) on the other hand is obtained by solving the two coupled equations (15) with the condition that for \( r > \hat{r} \), \( N^r(r) \) equals \( \sqrt{2H/\hat{r}} \), having normalized \( N = 1 \) for \( r > \hat{r} \). One easily finds that for \( r < \hat{r} \)

\[
N^r(r) = W \left[ \int_{\hat{r}}^{r} \dot{\hat{r}} \frac{\partial \pi_R}{\partial M} \, dr + \frac{\sqrt{2H\hat{r}}}{\sqrt{2H\hat{r} + \dot{\hat{r}}}} \right].
\]  

(29)

Taking into account that

\[
- \frac{d}{dr} \left( \frac{\partial^2 F}{\partial R' \partial M} \right) + \frac{\partial^2 F}{\partial M \partial R'} = \frac{\partial \pi_R}{\partial M}
\]  

(30)

we have

\[
\frac{d}{dt} \int_{r_0}^{\hat{r}(t)} \frac{\partial F}{\partial M} \, dr = \int_{r_0}^{\hat{r}} \dot{\hat{r}} \frac{\partial \pi_R}{\partial M} \, dr + \left[ \dot{\hat{r}} \frac{\partial F}{\partial M} + \frac{\partial^2 F}{\partial M \partial R'} \right]_{\hat{r} - \varepsilon}.
\]  

(31)

and eq.(28) becomes

\[
\dot{\hat{r}} \frac{\partial p_c}{\partial M} + \left( \dot{\hat{r}} \frac{\partial F}{\partial M} + \dot{\hat{r}} \frac{\partial^2 F}{\partial R' \partial M} \right)_{\hat{r} - \varepsilon} + \frac{\sqrt{2H\hat{r}}}{\sqrt{2H\hat{r} + \dot{\hat{r}}}} = 0.
\]  

(32)

From the expression for \( p_c \) given by the first line of eq.(24) we obtain

\[
\dot{\hat{r}} \left( \frac{\partial \hat{p}}{\partial M} - \frac{V}{W} \frac{\partial R'}{\partial M} \right)_{\hat{r} - \varepsilon} + \frac{\sqrt{2H\hat{r}}}{\sqrt{2H\hat{r} + \dot{\hat{r}}}} = 0
\]  

(33)

where \( W(\hat{r} - \varepsilon) = \sqrt{R'^2(\hat{r} - \varepsilon) - 1 + 2M/R(\hat{r})} = \sqrt{2H/\hat{r} + \dot{\hat{r}}} \) and \( R'(\hat{r} - \varepsilon) = 1 + V/\hat{r} \).

Using

\[
\frac{\partial \hat{p}}{\partial M} = -\frac{1}{\frac{\dot{\hat{r}}}{\hat{r}} - \sqrt{2H/\hat{r}}}
\]  

(34)

we obtain eq.(27) again. We remark once more that no limit process \( l \to 0 \) is necessary for all these developments.

To summarize, in the present section we have derived the reduced action for the one shell problem in the outer gauge with an arbitrary deformation. One can vary \( H \) (the exterior ADM mass) considering the interior mass as given, or one can vary the interior mass \( M \) considering the exterior mass \( H \) as given, or even one can vary both \( M \) and \( H \) always obtaining the correct equations of motion. Whenever \( M \) is varied the \( \dot{M} \) term in eq.(28) plays a crucial role. All the results do not depend on the deformation \( g \).
4 More general gauges

It is of interest to examine more general gauges given by eq.(17) where \( g \) does not necessarily vanish for positive argument, i.e. we can consider \( g(x) \) with \( g(x) = 0 \) for \( |x| > l \), \( g(0) = 0 \) and \( g'(+0) - g'(-0) = -1 \), thus again satisfying the constraint (19). In this case \( F(R, R', H) \) does not vanish identically for \( r > \hat{r} \) and the bulk action (21) is given by the time integral of

\[
I_g = \frac{d}{dt} \int_{r_0}^{r_m} F dr - \dot{M}(t) \int_{r_0}^{\hat{r}(t)} \frac{\partial F}{\partial M} dr - \dot{H}(t) \int_{\hat{r}(t)}^{r_m} \frac{\partial F}{\partial H} dr + \left[ \dot{\hat{r}}(t) F + \frac{\partial F}{\partial R'} \hat{R} \right]_{\hat{r}(t) - \varepsilon}^{\hat{r}(t) + \varepsilon}
\]

(35)

The \( p_c \) is easily computed using eq.(35) and one gets immediately the general form for the canonical momentum \( p_c \)

\[
p_c = \hat{r}(\Delta L - \Delta B)
\]

(36)

where \( \Delta L = \mathcal{L}(\hat{r} + \varepsilon) - \mathcal{L}(\hat{r} - \varepsilon) \) and similarly for \( \Delta B \) and the reduced action becomes

\[
S = \int_{t_i}^{t_f} \left( p_c \dot{\hat{r}} - \dot{M}(t) \int_{r_0}^{\hat{r}(t)} \frac{\partial F}{\partial M} dr - \dot{H}(t) \int_{\hat{r}(t)}^{r_m} \frac{\partial F}{\partial H} dr + (-N^r \pi_L + N RR') \right|_{r_0}^{r_m} \right) dt.
\]

(37)

We shall call inner gauge the one characterized by \( g(x) = 0 \) for \( x < 0 \).

Due to the similarity with the treatment of Sect.3 we shall go through rather quickly. Now \( F \) vanishes identically for \( r < \hat{r} \) and the action takes the form

\[
S = \int_{t_i}^{t_f} dt \left( p_c^i \dot{\hat{r}} - \dot{H} \int_{\hat{r}(t)}^{r_m} \frac{\partial F}{\partial H} dr + (-N^r \pi_L + N RR') \right|_{r_0}^{r_m} \right)
\]

(38)

where \( p_c^i \) is given by

\[
p_c^i = \left[ F + \frac{\hat{R}}{\hat{r}} \frac{\partial F}{\partial R'} \right]_{\hat{r} + \varepsilon}^{\hat{r} - \varepsilon}
\]

(39)

whose explicit value is

\[
p_c^i = \sqrt{2M} \hat{r} - \sqrt{2H \hat{r}} - \hat{r} \log \left( \frac{\hat{r} - \sqrt{2H \hat{r}}}{\hat{r} + \sqrt{2M \hat{r}}} \right)
\]

(40)

and \( \hat{p} \), again determined by the discontinuity equation (20), is given by the implicit equation

\[
H - M = V - \frac{m^2}{2\hat{r}} - \hat{p} \sqrt{\frac{2M}{\hat{r}}}
\]

(41)

which is different from eq.(25). Now the simple procedure is the one in which one varies \( M \) keeping \( H \) as a fixed datum of the problem. This time the solution of the system eq.(15) gives simply \( N = \text{const} \) and \( N^r = N \sqrt{\frac{2M}{\hat{r}}} \) for \( r_0 < r < \hat{r} \).
5 The analytic properties of \( p_c \)

We saw that in the outer gauge

\[
p_c = \sqrt{2M} \dot{r} - \sqrt{2H} \dot{r} - \dot{r} \log \left( \frac{\dot{r} + V - \hat{p} - \sqrt{2H} \dot{r}}{\dot{r} - \sqrt{2M} \dot{r}} \right).
\] (42)

The solution of eq.(25) for \( \hat{p} \) is

\[
\hat{p} \hat{r} = A \sqrt{\frac{2H}{\dot{r}}} \pm \sqrt{A^2 - (1 - \frac{2H}{\dot{r}}) \frac{m^2}{r^2}}
\] (43)

where

\[
A = \frac{H - M}{\dot{r}} - \frac{m^2}{2 \dot{r}^2}. \quad (44)
\]

If we want \( \hat{p} \) to describe an outgoing shell we must choose the plus sign in front of the square root. Moreover the shell reaches \( r = +\infty \) only if \( H - M > m \) as expected.

The logarithm has branch points at zero and infinity and thus we must investigate for which values or \( \dot{r} \) such values are reached. At \( \dot{r} = 2H \), \( \hat{p} \) has a simple pole with positive residue; then the numerator goes to zero and below \( 2H \) it becomes

\[
1 - \frac{V}{\dot{r}} - \hat{p} \frac{\dot{r}}{\dot{r}} - \sqrt{\frac{2H}{\dot{r}}}.
\] (45)

where here \( V \) is the absolute value of the square root. Expression (45) is negative irrespective of the sign of \( \hat{p} \) and stays so because \( \hat{p} \) is no longer singular. In order to compute the tunneling amplitude, below \( \dot{r} = 2H \) we have to use the prescription \cite{8} \( \dot{r} - 2H \rightarrow \dot{r} - 2H - i\varepsilon \) and as a consequence the \( p_c \) below \( 2H \) acquires the imaginary part \( i\pi \dot{r} \). Below \( \dot{r} = 2M \) the denominator of the argument of the logarithm in eq.(42) becomes negative so that the argument of the logarithm reverts to positive values. Thus the “classically forbidden” region is \( 2M < \dot{r} < 2H \) independent of \( m \) and of the deformation \( g \) and the integral of the imaginary part of \( p_c \) for any deformation \( g \) is

\[
\int \text{Im} \ p_c \, dr = \pi \int_{2M}^{2H} r \, dr = 2\pi (H^2 - M^2) = 4\pi (M + \frac{\omega}{2}) \omega
\] (46)

with \( \omega = H - M \) which is the Parikh-Wilczek result \cite{8}. A more profound way to relate the result for \( p_c \) to the formula for the Hawking radiation is to use (24) to compute semiclassically the modes on which to expand the quantum field, and then proceed as usual by means of the Bogoliubov transformation. This was done in \cite{6,16}. A more direct particle like interpretation of (46) was given e.g. in \cite{19}.
Similarly one can discuss the analytic properties of the conjugate momentum $p^i_c$ in the inner gauge. We have

$$p^i_c = \sqrt{2M\hat{r}} - \sqrt{2H\hat{r}} - \hat{r} \log \left( \frac{1 - \sqrt{2H\hat{r}}}{1 - \frac{V}{\hat{r}} - \sqrt{\frac{2M\hat{r}}{\hat{r}} + \frac{p}{\hat{r}}}} \right).$$  \hspace{1cm} (47)

This time the solution of eq.(41) gives for $\hat{p}$

$$\hat{p} = \frac{\sqrt{2M\hat{r}}}{\hat{r}} \pm \sqrt{A^2 - (1 - \frac{2M\hat{r}}{\hat{r}}) \frac{m^2}{2\hat{r}^2}}$$  \hspace{1cm} (48)

where now

$$A = \frac{H - M}{\hat{r}} + \frac{m^2}{2\hat{r}^2}. \hspace{1cm} (49)$$

Notice that for $m = 0$ we have $p_c = p^i_c$. To describe an outgoing shell the square root in (48) has to be taken with the positive sign.

All the point is the discussion of the sign of the term

$$1 - \frac{V}{\hat{r}} - \sqrt{\frac{2M\hat{r}}{\hat{r}} + \frac{\hat{p}}{\hat{r}}} = R'(\hat{r} + \epsilon) - \sqrt{R'^2(\hat{r} + \epsilon) - 1 + \frac{2H}{\hat{r}}}. \hspace{1cm} (50)$$

where

$$\sqrt{\frac{2M\hat{r}}{\hat{r}} - \frac{\hat{p}}{\hat{r}}} = \sqrt{R'^2(\hat{r} + \epsilon) - 1 + \frac{2H}{\hat{r}}}. \hspace{1cm} (51)$$

For $m = 0$ at $\hat{r} = 2H$ eq.(51) is negative so that at $\hat{r} = 2H$

$$1 - \frac{V}{\hat{r}} - \sqrt{\frac{2M\hat{r}}{\hat{r}} + \frac{\hat{p}}{\hat{r}}} = R'(\hat{r} + \epsilon) + \sqrt{R'^2(\hat{r} + \epsilon) - 1 + \frac{2H}{\hat{r}}}. \hspace{1cm} (52)$$

is positive, being in eq.(52) the square root on the r.h.s. understood as the positive determination of the square root. The same happens for $m \neq 0$ provided $m < H - M$ which is the condition for the shell to be able to reach $\hat{r} = +\infty$. For $\hat{r} < 2H$ as $-1 + \frac{2H}{\hat{r}} > 0$ such a term stays positive irrespective of the sign of $R'$, until $R'$ diverges. This happens at $2M$ where $\hat{p}$ given by eq.(48) has a simple pole with positive residue. Thus below $2M$ eq.(52) reverts to

$$1 - \frac{V}{\hat{r}} - \sqrt{\frac{2M\hat{r}}{\hat{r}} + \frac{\hat{p}}{\hat{r}}} = R'(\hat{r} + \epsilon) - \sqrt{R'^2(\hat{r} + \epsilon) - 1 + \frac{2H}{\hat{r}}}. \hspace{1cm} (53)$$

which is negative irrespective of the sign of $R'(\hat{r} + \epsilon)$. The conclusion is that $p^i_c$ takes the imaginary part $i\pi\hat{r}$ in the interval $2M, 2H$ as it happens for $p_c$.  

11
6 The two shell reduced action

From now on we shall work in the outer gauge. We denote with $\hat{r}_1$ and $\hat{r}_2$ the coordinates of the first and second shell $\hat{r}_1 < \hat{r}_2$. The value of $\mathcal{M}$ for $r < \hat{r}_1$ will be denoted by $M$, for $\hat{r}_1 < r < \hat{r}_2$ by $M_0$ and for $r > \hat{r}_2$ will be denoted by $H$ as before. In extending the treatment to two interacting shells we shall keep the formalism as close as possible to the one developed in Sect. 3. The most relevant difference is that in any gauge the intermediate mass $M_0$ and the total mass $H$ always intervene dynamically. We shall consider the mass $M$ as a datum of the problem.

For the metric component $R$ we shall use

$$R(r) = r + v_2 g (r - \hat{r}_2) + v_1 h (r - \hat{r}_1); \quad v_2 = \frac{V_2}{R(\hat{r}_2)}; \quad v_1 = \frac{V_1}{R(\hat{r}_1)}$$

(54)

where $h(x)$ has the same properties of $g(x)$ described in Sect. 3 (actually we could use the same function). Both $g$ and $h$ vanish for positive argument and thus we are working in an outer gauge according to the definition of Sect. 3.

The action is given by the time integral of

$$I = \hat{p}_2 \dot{\hat{r}}_2 + \hat{p}_1 \dot{\hat{r}}_1 + \int dr \pi_R \dot{R} + b.t.$$  

(55)

Breaking the integration range from $r_0$ to $\hat{r}_1$ and from $\hat{r}_1$ to $\hat{r}_2$ and using eq.(9) for $\pi_R$ and the same technique as used for the one shell case, we reach the expression

$$\hat{p}_2 \dot{\hat{r}}_2 + \hat{p}_1 \dot{\hat{r}}_1 + \frac{d}{dt} \int_{r_0}^{\hat{r}_2} F dr - \hat{M}_0 \int_{\hat{r}_1}^{\hat{r}_2} \frac{\partial F}{\partial M_0} dr - \dot{\hat{r}}_2 F(\hat{r}_2 - \varepsilon) - \left( \dot{\hat{R}} \frac{\partial F}{\partial R'} \right) (\hat{r}_2 - \varepsilon) +$$

$$\dot{\hat{r}}_1 \Delta F(\hat{r}_1) + \Delta \left( \dot{\hat{R}} \frac{\partial F}{\partial R'} \right) (\hat{r}_1)$$

(56)

where $\hat{p}_c^0$ is given by eq.(24) with $M$ replaced by $M_0$, and $\Delta$ stays for the jump across the discontinuity at $\hat{r}_1$. We recall that $F$ depends only on $R,R'$ and $\mathcal{M}$ and the partial derivative w.r.t. these variables will have the usual meaning. For the other quantities appearing in the calculations we recall that the independent variables are $M, M_0, H, \hat{r}_1, \hat{r}_2$ and when taking partial derivatives we shall consider the remaining variables as fixed. In eq.(56) we have, using eq.(13), denoting with $\Delta$ the discontinuity across $\hat{r}_1$ and with
the bar the average at $\hat{r}_1$

$$\dot{r}_1 \dot{p}_1 + \dot{r}_1 \Delta F + \Delta \left( \frac{\partial F}{\partial \dot{r}} \dot{R} \right) =$$

$$= \dot{r}_1 \dot{p}_1 + \dot{r}_1 \Delta F + \frac{\partial F}{\partial \dot{R}} \Delta \dot{R} + \frac{\partial F}{\partial \dot{R}} \Delta \dot{R} =$$

$$\dot{r}_1 [\overline{F} R(\Delta \hat{L} - \Delta \hat{B}) + \Delta R' R(\overline{\hat{L}} - \overline{\hat{B}})] + \Delta \dot{R} R(\overline{\hat{L}} - \overline{\hat{B}}) + \overline{R} R(\Delta \hat{L} - \Delta \hat{B}) =$$

$$= \dot{r}_1 p_{c1} + \dot{r}_2 \tilde{p}_{c2} + \hat{H}(R(\hat{r}_1) - \dot{r}_1) \frac{\partial T}{\partial \hat{H}} \mathcal{D} + \dot{M}_0 (R(\hat{r}_1) - \dot{r}_1) \frac{\partial T}{\partial \dot{M}_0} \mathcal{D}$$

where

$$T = \log v_2; \quad \mathcal{D} = R(\Delta \hat{L} - \Delta \hat{B}); \quad p_{c1} = R'(\dot{r}_1 + \varepsilon) \mathcal{D};$$

$$\tilde{p}_{c2} = -(R'(\dot{r}_1 + \varepsilon) - 1) \mathcal{D} + (R(\hat{r}_1) - \dot{r}_1) \frac{\partial T}{\partial \hat{r}_2} \mathcal{D} = \frac{d}{d \hat{r}_2} (R(\hat{r}_1) - \dot{r}_1) \mathcal{D}$$

having used

$$\Delta \dot{R} = -\dot{r}_1 \Delta R'$$

and

$$\tilde{R} = -\dot{r}_1 \frac{v_1}{2} - \dot{r}_2 (\tilde{R}' - 1) + (R(\hat{r}_1) - \dot{r}_1) \frac{dT}{dt}.$$  

Summing up the reduced action for the two shell system is given, boundary terms included, by the time integral of

$$\dot{r}_1 p_{c1} + \dot{r}_2 \tilde{p}_{c2} + \hat{H}(R(\hat{r}_1) - \dot{r}_1) \frac{\partial T}{\partial \hat{H}} \mathcal{D} + \dot{M}_0 (R(\hat{r}_1) - \dot{r}_1) \frac{\partial T}{\partial \dot{M}_0} \mathcal{D} +$$

$$+ \frac{d}{dt} \int_{r_0}^{\hat{r}_2} F \, dr - \dot{M}_0 \int_{\hat{r}_1}^{\hat{r}_2} \frac{\partial F}{\partial \dot{M}_0} \, dr + (-N \pi L + N R R') |_{r_0}^{\hat{r}_2}$$

where $p_{c2} = \tilde{p}_{c2} + \tilde{p}_{c2}$ given by eq. (21) with $M$ replaced by $\dot{M}_0$ and $\tilde{p}_{c2}$ by eq. (59). With regard to eq. (62) we notice that irrespective of the gauge used both terms in $\dot{M}_0$ and $\hat{H}$ appear in the action. Moreover $p_{c1}$ and $p_{c2}$ depend both on $\dot{r}_1$ and $\dot{r}_2$.

7 The two shell equations of motion

From the reduced action (62) we can derive the equations of motion for the two shells. This is of some importance in order to show the consistency of the scheme. We recall that action (62) has been derived in the outer gauge i.e. $R(r) = r$ for $r > \dot{r}_1$ ($\dot{r}_2 > \dot{r}_1$). While in the one shell problem formulated in the outer gauge $\hat{H}$ does not appear, in the two shell problem it is always present. Again we consider $M = \text{const.}$ as a datum of the problem.
In the variational procedure we can vary \( \hat{r}_1, \hat{r}_2, H \) and \( M_0 \) independently. We shall start varying \( H \) but keeping all other parameters fixed. For the sake of simplicity we shall deal here with the massless case \( m_1 = m_2 = 0 \). In Appendix C we give the general derivation for \( m_1 \) and \( m_2 \) different from zero. The most important fact that occurs when we vary \( H \) keeping \( M_0 \) fixed is that the terms proportional to \( \dot{\hat{r}}_1 \) cancel in the variation. The simplifying feature of the massless case is that \( \Delta \mathcal{L} = 0 \) so that in eq. (58) \( \mathcal{D} = -R\Delta B \), being \( B \) function only of \( R \) and \( M \) and not of \( R' \) and thus only of \( R \) and \( M_0 \) being \( M \) a datum of the problem. The coefficient of \( \dot{\hat{r}}_1 \), taking into account the following relation, easily derived from (13),

\[
\frac{\partial F}{\partial M} + R'R\frac{\partial B}{\partial M} = \frac{1}{W - R'}
\]  

(63)

is found proportional to

\[
\frac{\partial}{\partial H} (R' (\hat{r}_1 + \varepsilon) D) - (R' (\hat{r}_1 + \varepsilon) - 1) \frac{\partial T}{\partial H} D - (R (\hat{r}_1) - \hat{r}_1) \frac{\partial T}{\partial H} \frac{\partial D}{\partial R} R' (\hat{r}_1 + \varepsilon)
\]

(64)

where we used

\[
\frac{dR(\hat{r}_1)}{dt} = \dot{\hat{r}}_1 R' (\hat{r}_1 + \varepsilon) - \dot{\hat{r}}_2 (R' (\hat{r}_1 + \varepsilon) - 1) + \frac{dT}{dt} (R (\hat{r}_1) - \hat{r}_1)
\]

(65)

and we took into account that \( T \) does not depend on \( \hat{r}_1 \) and that on the equations of motion \( \dot{H} = \dot{M}_0 = 0 \). Now employing the relations

\[
\frac{\partial R(\hat{r}_1)}{\partial H} = \frac{\partial T}{\partial H} (R (\hat{r}_1) - \hat{r}_1); \quad \frac{\partial R'(\hat{r}_1 + \varepsilon)}{\partial H} = \frac{\partial T}{\partial H} (R' (\hat{r}_1 + \varepsilon) - 1)
\]

(66)

we see that expression (65) vanishes. Thus we are left only with the \( \dot{\hat{r}}_2 \) terms. We know already that the boundary term and the \( p_{\hat{r}2}^0 \) term give the correct equation of motion and thus we have simply to prove that the \( \dot{\hat{r}}_2 \) terms originating from

\[
- \frac{d}{dt} [(R (\hat{r}_1) - \hat{r}_1) \frac{\partial T}{\partial H} D]
\]

(67)

cancel \( \frac{\partial p_{\hat{r}2}}{\partial H} \) i.e.

\[
\frac{\partial}{\partial H} [- (R' (\hat{r}_1 + \varepsilon) - 1) D + (R (\hat{r}_1) - \hat{r}_1) \frac{\partial T}{\partial H} D] - \frac{\partial}{\partial \hat{r}_2} [(R (\hat{r}_1) - \hat{r}_1) \frac{\partial T}{\partial H} D] = 0.
\]

(68)

Using relations (66) we have that eq. (68) is satisfied. Such a result is expected as the exterior shell parameterized by \( \hat{r}_2 \) moves irrespective of the dynamics which develops at lower values of \( r \) until \( \hat{r}_1 \) crosses \( \hat{r}_2 \).

Now we vary \( M_0 \) keeping \( H \) fixed. We have no boundary term contribution because also \( M \) is constant and we find the equation

\[
\dot{\hat{r}}_1 \frac{\partial p_{\hat{r}1}}{\partial M_0} + \dot{\hat{r}}_2 \frac{\partial p_{\hat{r}2}}{\partial M_0} - \frac{d}{dt} [(R (\hat{r}_1) - \hat{r}_1) \frac{\partial T}{\partial M_0} D] + \dot{\hat{r}}_2 \frac{\partial F(\hat{r}_2 - \varepsilon)}{\partial M_0} - \hat{r}_1 \frac{\partial F(\hat{r}_1 + \varepsilon)}{\partial M_0} +
\]

\[
+ \int_{\hat{r}_1}^{\hat{r}_2} \frac{\partial \pi_R}{\partial M_0} dr + \dot{R} \frac{\partial^2 F}{\partial M_0 \partial R} \bigg|_{\hat{r}_1 + \varepsilon}^{\hat{r}_2 - \varepsilon} = 0.
\]

(69)
Using expression (29) for $N^r(r)$, relation (15) for $N(r)$ and relation (63) we obtain with $\alpha = \sqrt{2H^\hat{r}/(\sqrt{2H^\hat{r}} + \dot{\hat{p}})}$

\[
\left[ R'(\hat{r}_1 + \varepsilon) - W(\hat{r}_1 + \varepsilon) \right]^{-1}(\hat{r}_1 + N^r(\hat{r}_1) - N(\hat{r}_1)) \\
+ \hat{r}_2^\partial \hat{p}_2\partial M_0 + \left[ \hat{r}_2^\partial F\partial M_0 + \frac{\partial^2 F}{\partial M_0 \partial R'} \hat{R} \right]_{\hat{r}_2 - \varepsilon} + \alpha \\
- \left[ R'(\hat{r}_1 + \varepsilon) - W(\hat{r}_1 + \varepsilon) \right]^{-1} \hat{R} \frac{W}{W}(\hat{r}_1 + \varepsilon) \\
+ \hat{r}_2^\partial \hat{p}_2\partial M_0 - \frac{\partial^2 F}{\partial M_0 \partial R'} \hat{R} \Bigg|_{\hat{r}_1 + \varepsilon} - \hat{r}_2 \frac{\partial}{\partial \hat{r}_2} \left[ (R(\hat{r}_1) - \hat{r}_1) \frac{\partial T}{\partial M_0} \right] \quad (70)
\]

The first line is the equation of motion for $\hat{r}_1$, the second line vanishes being the equation of motion for $\hat{r}_2$ (see eq.(32) of the one shell problem) while exploiting the relation

\[
\frac{\partial^2 F}{\partial M \partial R'} + \hat{R} \frac{\partial B}{\partial M} = \frac{1}{W(W - R')} \quad (71)
\]

also derived from (13) and substituting into $\dot{\hat{r}}_2$ the value given by the equations of motion derived previously we find that the third line simplifies with the fourth. Thus we are left with the relation

\[
\dot{\hat{r}}_1 + N^r(\hat{r}_1) - N(\hat{r}_1) = 0 \quad (72)
\]

which is the correct equation of motion for the interior shell.

### 8 Exchange relations

In the present section we consider in the above developed formalism the intersection of two shells during their motion. In the massless case we shall rederive the well known relations of Dray and 't Hooft [11] and Redmount [12] between the masses characteristic of the three regions before and after the collision. We shall denote by $t_0$ the instant of collision, $\hat{r}_0$ the coordinate of the collision. It is well known that some hypothesis has to be done on the dynamics of the collision which should tell us which are the masses of the two shells after the collision. Once these are given the problem is reduced to that of a two particle collision in special relativity. The formulas which we develop below give the intermediate mass $M'_0$ after the collision. The simplest assumption is that of the “transparent crossing” i.e. after the collision shell 1 carries along with unchanged mass $m_1$ and the same for shell 2. A further simplification occurs in the massless case i.e. when the two massless shells go over to two massless shells. We develop first the general formalism which applies also when we have a change in the masses during the collision and then specialize to particular situations.
Just before the collision i.e. at $t = t_0 - \varepsilon$ assuming that $\hat{r}_1(t_0 - \varepsilon) < \hat{r}_2(t_0 - \varepsilon)$ we have for the momentum $\pi_L$

$$\pi_L(\hat{r}_1 - 0, t_0 - \varepsilon) = \pi_L(\hat{r}_1 + 0, t_0 - \varepsilon) + \hat{p}_1 =$$

$$= \pi_L(\hat{r}_2 - 0, t_0 - \varepsilon) + \hat{p}_1 = \pi_L(\hat{r}_2 + 0, t_0 - \varepsilon) + \hat{p}_1 + \hat{p}_2$$

(73)

and after the collision i.e. at $t = t_0 + \varepsilon$ we have with $\hat{r}_2(t_0 + \varepsilon) < \hat{r}_1(t_0 + \varepsilon)$,

$$\pi_L(\hat{r}_2 - 0, t_0 + \varepsilon) = \pi_L(\hat{r}_2 + 0, t_0 + \varepsilon) + \hat{p}'_2 =$$

$$= \pi_L(\hat{r}_1 - 0, t_0 + \varepsilon) + \hat{p}'_2 = \pi_L(\hat{r}_1 + 0, t_0 + \varepsilon) + \hat{p}'_1 + \hat{p}'_2$$

(74)

and at the collision $\hat{r}_2 = \hat{r}_1 = \hat{r}_0$. The main point in treating the crossing is to realize that the sum $V_1 + V_2$ has to be continuous in time as it represents the discontinuity of $R'$ at the time of crossing. In fact just before the crossing we have for $r < \hat{r}_0$, choosing for simplicity in (54) $g(x) = h(x)$,

$$R(r) = r + \frac{(V_1 + V_2)}{\hat{r}_0} g(r - \hat{r}_0)$$

(75)

while immediately after we have

$$R(r) = r + \frac{(V'_1 + V'_2)}{\hat{r}_0} g(r - \hat{r}_0).$$

(76)

If $V_1 + V_2 \neq V'_1 + V'_2$ we would have a discontinuous evolution of the metric for $r < \hat{r}_0$ which is incompatible with the equations of motion of the gravitational field. As $M$ is unchanged during the time evolution $V_1 + V_2 = V'_1 + V'_2$ implies that $\pi_L(\hat{r}_1 - 0, t_0 - \varepsilon) = \pi_L(\hat{r}_2 - 0, t_0 + \varepsilon)$ which combined with the fact that $\pi_L(\hat{r}_2 + 0, t_0 - \varepsilon) = \pi_L(\hat{r}_1 + 0, t_0 + \varepsilon) = \sqrt{2H\hat{r}_0}$ gives the further relation $\hat{p}_1 + \hat{p}_2 = \hat{p}'_1 + \hat{p}'_2$. Thus we have the same kinematics as in the two particle collision in special relativity. In the case of conservation of the masses $m_1$ and $m_2$ we have the same kinematics as that of an elastic collision in $1 + 1$ dimensions. A discussion of special cases with massive shells using a different formalism has been given in [20].

The relation between $\pi_L(\hat{r}_1 - 0, t_0 - \varepsilon)$ and $\pi_L(\hat{r}_2 + 0, t_0 - \varepsilon)$

$$\hat{r}_0 \sqrt{\left(1 + \frac{V_1 + V_2}{\hat{r}_0}\right)^2 - 1 + \frac{2M}{\hat{r}_0}} = \sqrt{2H\hat{r}_0} + \hat{p}_1 + \hat{p}_2$$

(77)

gives the equation

$$m_1^2 + m_2^2 + 2(V_1V_2 - \hat{p}_1\hat{p}_2) + 2(V_1 + V_2)\hat{r}_0 + 2M\hat{r}_0 = 2H\hat{r}_0 + 2(\hat{p}_1 + \hat{p}_2)\sqrt{2H\hat{r}_0}$$

(78)
where $M_0$ is given by

$$H - M_0 = V_2 + \frac{m_0^2}{2\hat{r}_0} - \hat{p}_2 \sqrt{\frac{2H}{r_0}}. \quad (79)$$

From the knowledge of $\hat{p}_1$ one derives $M'_0$ from

$$H - M'_0 = V'_1 + \frac{m'_1}{2\hat{r}_0} - \hat{p}'_1 \sqrt{\frac{2H}{r_0}}. \quad (80)$$

For “transparent” crossing i.e. $m'_j = m_j$ and $\hat{p}'_j = \hat{p}_j$, $M'_0$ is given by (80) with $\hat{p}'_1 = \hat{p}_1$.

The massless case is most easily treated. In this case in order for the shells to intersect they must move in opposite directions, the exterior one with negative and the interior one with positive velocity. The conservation of $V_1 + V_2$ and $\hat{p}_1 + \hat{p}_2$ in the massless case has two solutions which are physically equivalent i.e. $\hat{p}'_1 = \hat{p}_1, \hat{p}'_2 = \hat{p}_2$. The two shells intersect only if $\hat{p}_2 < 0$ and $\hat{p}_1 > 0$ so that $V_2 = -\hat{p}_2$ and $V_1 = \hat{p}_1$. From eq.(79) we have

$$\hat{p}_2 = -\frac{H - M_0}{1 + \sqrt{\frac{2H}{r_0}}} \quad (81)$$

and from $\hat{p}'_1 = \hat{p}_1$

$$\hat{p}_1 = \frac{H - M'_0}{1 - \sqrt{\frac{2H}{r_0}}}. \quad (82)$$

Eq.(78) now becomes

$$H\hat{r}_0 + \sqrt{2H\hat{r}_0(\hat{p}_1 + \hat{p}_2)} = \hat{r}_0(\hat{p}_1 - \hat{p}_2) + M\hat{r}_0 - 2\hat{p}_1\hat{p}_2 \quad (83)$$

and substituting here eqs.(81,82) we obtain

$$H\hat{r}_0 + M\hat{r}_0 - 2HM = M_0\hat{r}_0 - 2M_0M'_0 + M'_0\hat{r}_0 \quad (84)$$

which is the well known Dray-’t Hooft and Redmount relation [11, 12].

9 Integrability of the form $p_{c1}d\hat{r}_1 + p_{c2}d\hat{r}_2$ and the two shell tunneling probability

We are interested in computing the action for the two shell system i.e. the integral

$$\int_{t_i}^{t_f} dt(p_{c1} \dot{\hat{r}}_1 + p_{c2} \dot{\hat{r}}_2) = \int_{r_{1i},r_{2i}}^{r_{1f},r_{2f}} (p_{c1} \, d\hat{r}_1 + p_{c2} \, d\hat{r}_2) \quad (85)$$

on the solution of the equations of motion. This is of interest in the computation of the semiclassical wave function and the tunneling probability in the two shell case. We shall
prove explicitly that the form \( p_{c1} d\hat{r}_1 + p_{c2} d\hat{r}_2 \) is closed i.e. integrable. Such a result will be very useful in the actual computation and in showing the independence of the results of the deformation \( g \). In fact we can reduce the computation to the integral on a simple path on which the two momenta \( p_{c1} \) and \( p_{c2} \) take a simpler form.

The integrability result is similar to a theorem of analytical mechanics \[21, 22\] stating that for a system with two degrees of freedom, in presence of a constant of motion the form \( p_1 dq_1 + p_2 dq_2 \) where the \( p_j \) are expressed in terms of the energy and of the constant of motion, is a closed form. Here the setting is somewhat different as we are dealing with an effective action with two degrees of freedom, arising from the action of a constrained system. We recall that for \( \hat{r}_1 < \hat{r}_2 M_0 \) like \( H \) is a constant of motion in virtue of the equations of motion of the gravitational field. The \( p_{cj} \) are functions of \( H, M_0, \hat{r}_1, \hat{r}_2 \) in addition to the fixed datum of the problem \( M \). The effective action takes the form

\[
\int dt (p_{c1} \dot{\hat{r}}_1 + p_{c2} \dot{\hat{r}}_2 + p_H \dot{H} + p_{M0} \dot{M}_0) + b.t. \tag{86}
\]

where \( b.t. \) is the boundary term (see eq.\[2]\)) which depends only on \( H, N^r \) and not on \( \hat{r}_j \). The value of \( N^r \) is supplied by the solution of the gravitational equations of motion. Thus varying w.r.t. \( \hat{r}_1 \) we have

\[
0 = - \frac{dp_{c1}}{dt} + \dot{\hat{r}}_1 \frac{\partial p_{c1}}{\partial \hat{r}_1} + \dot{\hat{r}}_2 \frac{\partial p_{c2}}{\partial \hat{r}_1} + \frac{\partial p_H}{\partial \hat{r}_1} \dot{H} + \frac{\partial p_{M0}}{\partial \hat{r}_1} \dot{M}_0. \tag{87}
\]

We show in Appendix A that the constraints combined with the equations for the gravitational field impose \( 0 = \dot{H} = \dot{M}_0 \) and thus we have

\[
\frac{\partial p_{c1}}{\partial \hat{r}_2} - \frac{\partial p_{c2}}{\partial \hat{r}_1} = 0. \tag{88}
\]

The meaning of the procedure is that the consistency of the variational principle imposes eq.\[88\]. On the other hand eq.\[88\] can be verified also from the explicit expression of \( p_{c1} \) and \( p_{c2} \) given in Sect. \[6\]. If the crossing of the shells occurs outside the region \( 2M, 2H \) we can choose the path as to keep the two deformations non overlapping in such a region; then \( p_{c1} \) takes the simple form \[21\] with \( H \) substituted by \( M_0 \) and \( p_{c2} \) again the form \[21\] with \( M \) substituted by \( M_0 \) and thus one obtains for the imaginary part of integral \[85\]

\[
\frac{1}{2} \left( (2M_0)^2 - (2M)^2 + (2H)^2 - (2M_0)^2 \right) = \frac{1}{2} \left( (2H)^2 - (2M)^2 \right) \tag{89}
\]

i.e. the two shells are emitted independently. If the crossing occurs at \( \hat{r}_0 \) with \( 2M < \hat{r}_0 < 2H \), with e.g. \( \hat{r}_1 < \hat{r}_2 \) before the crossing, we choose the path \( \hat{r}_1 = \hat{r}_2 - \varepsilon \) before the crossing and \( \hat{r}_2 = \hat{r}_1 - \varepsilon \) after the crossing. For clearness sake we examine at first the problem for the crossing of a null shell with a massive shell, even when the mass of the
massive shell changes. In order to reduce the integration path to the described one, one
must first, given the two initial points with \( R(\hat{r}_{1i}) < 2M \) and \( R(\hat{r}_{2i}) < 2M \), bring them
together ( \( \hat{r}_1 \) will denote the position of the massless shell). For \( \hat{r}_{1i} < \hat{r}_{2i} \), this is done by
integrating along the line \( \hat{r}_2 = \text{const} \) with \( \hat{r}_1 \) varying from \( \hat{r}_{1i} \) to \( \hat{r}_{2i} + \varepsilon \). The contribution is
\[
\int_{\hat{r}_{1i}}^{\hat{r}_{2i} - \varepsilon} p_{c1} d\hat{r}_1 = \int_{\hat{r}_{1i}}^{\hat{r}_{2i} - \varepsilon} R(\hat{r}_1 + \varepsilon) \, D \, d\hat{r}_1. \tag{90}
\]
But \( R'(\hat{r}_1) \) is real and
\[
D = -R(\hat{r}_1) \left[ \sqrt{\frac{2M_0}{R(\hat{r}_1)}} + \log \left( \frac{1 - \sqrt{\frac{2M_0}{R(\hat{r}_1)}}}{1 - \sqrt{\frac{2M_0}{R(\hat{r}_1)}}} \right) \right]. \tag{91}
\]
For \( R(\hat{r}_1) < 2M < 2M_0 \) eq. (91) is real and thus the contribution (90) is real. If on the
other hand \( \hat{r}_{2i} < \hat{r}_{1i} \) we integrate \( p_{c1} \) from \( \hat{r}_{1i} \) to \( \hat{r}_{2i} + \varepsilon \) keeping \( \hat{r}_2 \) fixed. This time we recall that
\[
p_{c1} = p^0_{c1} + \tilde{p}_{c1} \tag{92}
\]
where \( p^0_{c1} \) is real because we are outside the interval \((2M_0, 2H)\), and
\[
\tilde{p}_{c1} = -(R'(\hat{r}_2 + \varepsilon) - 1)D(\hat{r}_2) + (R(\hat{r}_2) - \hat{r}_2) \frac{\partial T}{\partial \hat{r}_1} D(\hat{r}_2) \tag{93}
\]
with \( T \) now given by \( \log v_1 \). Again all the items in eq. (93) are real. Thus we can start
e.g. with \( \hat{r}_{1i} = \hat{r}_{2i} - \varepsilon \). The integration along the path \( \hat{r}_1 = \hat{r}_2 - \varepsilon \) up to \( \hat{r}_0 \) is very simple because
\[
p_{c2} + p_{c1} = \hat{r} \left[ \sqrt{\frac{2M}{\hat{r}}} - \sqrt{\frac{2H}{\hat{r}}} - \log \frac{1 + \frac{V_2 - \hat{p}_2}{\hat{r}} - \sqrt{\frac{2H}{\hat{r}}}}{1 - \sqrt{\frac{2M}{\hat{r}}}} \right] \tag{94}
\]
whose complete discussion has already been done in Sect. 5. The “gap” is \( 2M, 2H \). More
difficult is the analysis for \( \hat{r}_0 < \hat{r} < 2H \). Now we have
\[
p'_{c2} + p'_{c1} = \hat{r} \left[ \sqrt{\frac{2M}{\hat{r}}} - \sqrt{\frac{2H}{\hat{r}}} - \log \frac{1 + \frac{V_2 - \hat{p}_2}{\hat{r}} - \sqrt{\frac{2H}{\hat{r}}}}{1 - \sqrt{\frac{2M}{\hat{r}}}} \right] \tag{95}
\]
with
\[
\hat{p}'_1 = \frac{H - M^{'2}_0}{1 - \sqrt{\frac{2H}{\hat{r}_1}}}. \tag{96}
\]
Moreover
\[
\frac{\hat{p}'_2}{\hat{r}_2} = \frac{AW(\hat{r}_2 + \varepsilon) + R'(\hat{r}_2 + \varepsilon) \sqrt{A^2 - (1 - \frac{2M_0}{\hat{r}_2})\frac{m^2_2}{\hat{r}_2^2}}}{1 - \frac{2M'_0}{\hat{r}_2}} \tag{97}
\]
where

\[ A = \frac{M'_0 - M}{\hat{r}_2} - \frac{m^2}{2\hat{r}_2^2} \]  

(98)

and

\[ W(\hat{r}_2 + \varepsilon) = \sqrt{(1 + \frac{\hat{p}'_2}{\hat{r}_2})^2 - 1 + \frac{2M_0}{\hat{r}_2}}. \]  

(99)

\( \hat{p}'_1 \) diverges with positive residue at \( \hat{r}_1 = 2H \) and thus also \( W(\hat{r}_2 + \varepsilon) \) and \( \hat{p}'_2 \) of eq.(97) diverge like \( \hat{p}'_1 \). As a consequence the analytic continuation of \( V'_2 - \hat{p}'_2 \) below \( \hat{r} = 2H \) is negative which makes the numerator of the argument of the logarithm in eq.(95) negative.

We show now that such numerator stays negative all the way for \( \hat{r} < 2H \). The argument of the square root in (97) never vanishes so that at \( \hat{r}_2 = 2M'_0 \) the numerator in (97) reduces to

\[ A(W(\hat{r}_2 + \varepsilon) + R'(\hat{r}_2 + \varepsilon)). \]  

(100)

We can now explicitly compute \( W(\hat{r}_2 + \varepsilon) \) and \( R'(\hat{r}_2 + \varepsilon) \) at \( \hat{r}_2 = 2M'_0 \). At such a point we have

\[ R'(\hat{r}_2 + \varepsilon) = 1 + \frac{(H - M'_0)/\hat{r}_2}{1 - \sqrt{\frac{2H}{\hat{r}_2}}} = \frac{1}{2} \left( 1 - \sqrt{\frac{H}{M'_0}} \right) \]  

(101)

while

\[ W(\hat{r}_2 + \varepsilon) = \sqrt{\frac{2H}{2M'_0} + \frac{1}{\hat{r}_2}} \frac{H - M'_0}{1 - \sqrt{\frac{2H}{\hat{r}_2}}} = \sqrt{\frac{H}{M'_0}} + \frac{1}{2} \left( \frac{H}{M'_0} - 1 \right) = -\frac{1}{2} \left( 1 - \sqrt{\frac{H}{M'_0}} \right) \]  

(102)

which means that there is no pole in \( \hat{p}'_2 \) at \( \hat{r} = 2M'_0 \). Thus \( \hat{p}'_2 \) is regular below \( 2H \) and \( V'_2 - \hat{p}'_2 \) cannot change sign.

In this way we proved that for the crossing of a null shell and a massive shell, into a null shell and another massive shell even with a change of mass, the imaginary part of the integral (85) is still given by eq.(89).

The reasoning in the general case of both \( m_1 \) and \( m_2 \) different from zero is dealt with similarly. It is sufficient to examine the case \( \hat{r}_1 < \hat{r}_2 \) the other case being now equivalent.

We shall first discuss the integration along the path \( \hat{r}_1 + \varepsilon = \hat{r}_2 \) up to \( \hat{r}_0 \). For \( \hat{r}_1 + \varepsilon = \hat{r}_2 = \hat{r} \) we have

\[ p_{e2} + p_{e1} = \hat{r} \left[ \sqrt{\frac{2M}{\hat{r}}} - \sqrt{\frac{2H}{\hat{r}}} - \log \frac{1 + \frac{V_2 - \hat{p}_2 + V_1 - \hat{p}_1}{\hat{r}} - \frac{2H}{\hat{r}}}{1 - \sqrt{\frac{2M}{\hat{r}}}} \right] \]  

(103)

with

\[ H - M_0 = V_2 + \frac{m^2}{2\hat{r}_2} - \hat{p}_2 \sqrt{\frac{2H}{\hat{r}_2}} \]  

(104)
which as before makes \( \hat{p}_2 \) diverge at \( \hat{r}_2 = 2H \). For \( \hat{p}_1 \) we have

\[
\frac{\hat{p}_1}{\hat{r}_1} = \frac{A_1 W(\hat{r}_1 + 0) + R'(\hat{r}_1 + 0) \sqrt{A_1^2 - (1 - \frac{2M_0}{\hat{r}_1}) \frac{m_1^2}{\hat{r}_1^2}}}{1 - \frac{2M_0}{\hat{r}_1}} \tag{105}
\]

with

\[
A_1 = \frac{M_0 - M}{\hat{r}_1} - \frac{m_1^2}{2\hat{r}_1^2} \tag{106}
\]

\( \hat{p}_2 \) diverges with positive residue at \( \hat{r} = 2H \) and \( \hat{p}_1 \) also diverges like \( \hat{p}_2 \) as

\[
R'(\hat{r}_1 + 0) = 1 + \frac{V_2}{\hat{r}_1} \tag{107}
\]

and \( W(\hat{r}_1 + 0) \) as given by eq. (11) also diverges like \( \hat{p}_2 \). Then for \( \hat{r} \) just below \( 2H \) we have both \( V_2 - \hat{p}_2 < 0 \) and \( V_1 - \hat{p}_1 < 0 \). As before all the point is in proving that \( \hat{p}_1 \) is regular below \( 2H \) i.e. does not diverge at \( \hat{r} = 2M_0 \). To this end we must examine the numerator of eq. (105) at \( \hat{r} = 2M_0 \). We have for \( \hat{r} = 2M_0 \)

\[
W(2M_0 + 0) = \frac{1}{2} \left( \sqrt{\frac{H}{M_0}} - 1 \right) + \frac{m_2^2}{8M_0^2 \left( \sqrt{\frac{H}{M_0}} + 1 \right)} > 0 \tag{108}
\]

as \( H > M_0 \). With regard to \( R'(\hat{r}_1 + 0) \) which is negative for \( \hat{r}_1 = 2H - \epsilon \) it cannot change sign for \( 2M_0 < \hat{r} < 2H \) because at the point where \( R'(\hat{r}_1 + 0) \) vanishes

\[
W(\hat{r}_1 + 0) = \sqrt{R'(\hat{r}_1 + 0)^2 - 1} + \frac{2M_0}{\hat{r}_1} \tag{109}
\]

would become imaginary while from \( W(\hat{r}_1 + 0) = \sqrt{2H/\hat{r}_1 + \hat{p}_2/\hat{r}_1} \) we have that \( W \) is real. Then at \( \hat{r} = 2M_0 \) the numerator in eq. (105) vanishes and \( \hat{p}_1 \) is regular all the way below \( 2H \). Thus we are in the same situation as in the previously discussed case. Below \( \hat{r} = 2M \) the argument of the logarithm in eq. (103) becomes positive again due to the change in sign of the denominator \( 1 - \sqrt{2M/\hat{r}} \). The integration for \( \hat{r} > \hat{r}_0 \) is treated simply by exchanging \( m_1 \) with \( m_2 \).

Finally we deal with the integral

\[
\int_{\hat{r}_1}^{\hat{r}_1 + \epsilon} \frac{p_{c2}}{\hat{r}_2} \, d\hat{r}_2 \tag{110}
\]

which makes the two initial points coalesce. As in the previously discussed case of one massless shell, all the point is in proving that \( D \) is real. In addition to the contribution of \( B \) which we already proved to be real, we have now the contribution of \( L \). We have already proven that at \( r = \hat{r}_2 - \epsilon \)

\[
R'(r) - \sqrt{R'(r)^2 - 1} + \frac{2M_0}{R(r)} = R'(r) - W(r) \tag{111}
\]
equals
\[ 1 + \frac{V_2}{\dot{r}_2} - \frac{\dot{p}_2}{\dot{r}_2} - \sqrt{\frac{2H}{\dot{r}_2}} \] (112)
which is negative for \( \dot{r}_2 < 2H \) and thus in particular for \( \dot{r}_2 < 2M \). In the interval \( \dot{r}_1 < r < \dot{r}_2 < 2M \) the term (111) cannot change sign because \(-1 + 2M_0/R\) is positive. Moreover we have
\[ R'(\dot{r}_1 - \varepsilon) - W(\dot{r}_1 - \varepsilon) = R'(\dot{r}_1 + \varepsilon) - W(\dot{r}_1 + \varepsilon) + \frac{V_1}{R(\dot{r}_1)} - \frac{\dot{p}_1}{R(\dot{r}_1)} . \] (113)
But we proved after eq.(107) that below \( 2H \), the analytic continuation of \( V_1 - \dot{p}_1 \) is negative, implying that (113) is negative, like \( R'(\dot{r}_1 + \varepsilon) - W(\dot{r}_1 + \varepsilon) \). The outcome is that the discontinuity of \( \mathcal{L} \) at \( \dot{r}_1 \), being given by the logarithm of a positive number, is real. This concludes the proof in the case of the emission of two massive shells.

10 Conclusions

The main issue of the present paper is the treatment of two intersecting shells of matter or radiation in general relativity, in a formalism, apt to compute the tunneling amplitude for the emission of two shells. In order to do so it is necessary to adopt a gauge which is more general than the one used in the treatment of a single shell. In the usual treatments of the tunneling amplitude for a single shell, a limit gauge is adopted. Already at the level of a single shell we show that the complete action contains a term in which the mass of the remnant black hole plays a dynamical role. Such a term is unimportant if the variation of the action is taken with respect to the total mass of the system, keeping the mass of the remnant as a datum of the problem, but becomes essential if one varies the mass of the remnant keeping the total mass of the system fixed as done e.g. in [8]. The reduced canonical momentum even in the single shell instance is gauge dependent but the tunneling probability turns out to be independent of such a choice. All the treatment is performed in the general massive case, the massless one being a special case. The tunneling results are independent of the mass of the shell. The adoption of a general non-limit gauge allows the extension of the formalism to two or more shells. In this instance both the intermediate mass and the total mass become dynamical variables in the sense that the reduced action contains terms proportional to the time derivative of them. We show how to derive the equations of motion of both the interior and exterior shell by varying the reduced action and this is done both in the massless and in the more complicated massive case. With regard to the computation of the tunneling amplitude it is possible to prove an integrability theorem which allows to deform the trajectory in
coordinate space to a contour which drastically simplifies the computation. Firstly one proves in such a way that the result is independent of the deformation defining the gauge introduced in Sect. 2, and secondly one finds in the general massive or massless case that the tunneling probability is given simply by the product of the tunneling probabilities for the independent emission of the two shells. Such a circumstance is interpreted \[13\] as the fact that in this model we have no information encoded in the radiation emitted by the black hole.

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**Appendix A**

For completeness we report here some formulas which are useful in the text. The constraints are given by \[3, 6, 9\]

\[
\mathcal{H}_r = \pi_R R' - \pi_L' L - \dot{\hat{p}} \delta(r - \hat{r}),
\]

\[
\mathcal{H}_t = \frac{R R}{L} + \frac{R'^2}{2L} + \frac{L \pi^2_L}{2 R^2} - \frac{R R L'}{L^2} - \frac{\pi_L \pi_R}{R} - \frac{L}{2} + \sqrt{\hat{p}^2 L^{-2} + m^2} \delta(r - \hat{r}).
\]

From the constraints it follows \[3\] that the quantity

\[
\mathcal{M} = \frac{\pi^2_L}{2 R} + \frac{R}{2} - \frac{R (R')^2}{2 L^2}
\]

is constant in the regions of \(r\) where there are no sources as there

\[
\mathcal{M}' = -\frac{R'}{L} \mathcal{H}_t - \frac{\pi_L}{RL} \mathcal{H}_r.
\]

The equations of motion for the gravitational field are \[9\]

\[
\dot{L} = N \left( \frac{L \pi_L}{R^2} - \frac{\pi_R}{R} \right) + (N^r L)',
\]

\[
\dot{R} = -\frac{N \pi_L}{R} + N^r R',
\]

\[
\dot{\pi}_L = \frac{N}{2} \left[ -\frac{\pi^2_L}{R^2} - \left( \frac{R'}{L} \right)^2 + 1 + \frac{2 \hat{p}^2 \delta(r - \hat{r})}{L^3 \sqrt{\hat{p}^2 L^{-2} + m^2}} \right] - \frac{N' R' R}{L^2} + N^r \pi_L'.
\]
\[ \dot{\pi}_R = N \left[ \frac{L\pi_L^2}{R^3} - \frac{\pi_L \pi_R}{R^2} - \left( \frac{R'}{L} \right)' \right] - \left( \frac{N'R}{L} \right)' + (N'\pi_R)' \]  

(121)

The equations of motion for \( \dot{r} \) and \( \dot{p} \) follow from the above equations for \( R, L, \pi_R \) and \( \pi_L \) and the constraints. In fact using the relation

\[ \frac{dR}{dt} (\dot{r}) = \dot{R}(\dot{r} + \epsilon) + R'(\dot{r} + \epsilon)\dot{r} = \dot{R}(\dot{r} - \epsilon) + R'(\dot{r} - \epsilon)\dot{r} \quad (122) \]

we have

\[ \dot{\Delta}R' + \Delta \dot{R} = 0. \quad (123) \]

Using now the equation of motion (119) and the constraints

\[ \Delta R'(\dot{r}) = -\frac{V}{R}, \quad \Delta \pi_L(\dot{r}) = -\frac{\dot{p}}{L} \quad (124) \]

where \( V = \sqrt{\dot{p}^2 + m^2 L^2} \), one obtains

\[ \dot{\hat{r}} = \frac{N(\dot{r})\dot{p}}{L(\dot{r})\sqrt{\dot{p}^2 + m^2 L^2(\dot{r})} - N'(\dot{r}). \quad (125) \]

Using the relation

\[ \frac{dL}{dt} (\dot{r}) = L'(\dot{r} + \epsilon)\dot{r} + \dot{L}(\dot{r} + \epsilon) = L'(\dot{r} - \epsilon)\dot{r} + \dot{L}(\dot{r} - \epsilon) = L'\dot{r} + L \quad (126) \]

and

\[ -\dot{\Delta} \pi_R = -N(\dot{r}) \frac{\Delta R'}{L(\dot{r})} - \frac{\Delta N'}{L(\dot{r})} + N'(\dot{r})\Delta \pi_R \quad (127) \]

derived from (121) and

\[ \dot{\hat{p}} = -\frac{dL}{dt} \Delta \pi_L + L(\dot{r})\frac{d\Delta \pi_L}{dt}. \quad (128) \]

one obtains

\[ \hat{p} = \frac{N(\dot{r})\dot{p}L'}{L^2(\dot{r})\sqrt{\dot{p}^2 + m^2 L^2(\dot{r})} + N'} - \frac{N}{L(\dot{r})}\sqrt{\dot{p}^2 + m^2 L^2(\dot{r})} + \dot{p} (N')'. \quad (129) \]

The occurrence of the average values \( \bar{L} = [L'(\dot{r} + \epsilon) + L'(\dot{r} - \epsilon)]/2 \) etc. in the previous equation, was pointed out and discussed at the level of the variational principle in [9]. In the massless case \( m = 0 \) however, using again relations (122,126,127) one proves that the r.h.s. of eq.(129) has no discontinuity i.e. there is no need to take the average value in the r.h.s. of eq. (129). In fact let us consider the discontinuity of the zero mass version of the r.h.s. of eq. (129)

\[ \dot{\hat{p}}(\frac{\eta N L'}{L^2} - \frac{\eta N'}{L} + N') \quad (130) \]
being $\eta$ the sign of $\hat{p}$, i.e.

$$\hat{p}(\eta \frac{N \Delta L'}{L^2} - \frac{\eta \Delta N'}{L} + \Delta N').$$  

(131)

From eq. (118) and the equation of motion (125) we have

$$\eta \frac{N}{L} \Delta L' = \frac{N}{R} \Delta \pi_R - \frac{NL}{R^2} \Delta \pi_L - L \Delta N'.$$

(132)

and from eq. (121)

$$\eta N \Delta \pi_R = R \Delta N' + N \Delta R'.$$

(133)

Substituting the two into eq. (131) and taking into account that $\Delta \pi_L = -\hat{p}/L$ and $\Delta R' = -\eta \hat{p}/R$ we have that expression (131) vanishes. This fact was discussed at the level of the variational principle in [10].

From the equations of motion (118, 119, 120, 121) it follows that in the region where there are no sources

$$\frac{dM}{dt} = -N \frac{R'}{L^3} \mathcal{H}_t - N' \frac{R'}{L} \mathcal{H}_t - N' \frac{\pi_L}{RL} \mathcal{H}_r$$

(134)

i.e. combining with (117), we have that in the region free of sources $M$ is constant both in $r$ and in $t$.

**Appendix B**

Equation of motion for $\hat{r}$.

1) Outer gauge

In this case

$$p_c = R(\Delta L - \Delta B) = \hat{r} \left(-\mathcal{L}(\hat{r} - \varepsilon) - \sqrt{\frac{2H}{\hat{r}}} + \sqrt{\frac{2M}{\hat{r}}} + \log \left(1 - \sqrt{\frac{2M}{\hat{r}}} \right) \right).$$

(135)

Using

$$\frac{\partial \hat{p}}{\partial H} = \left(1 + \frac{\hat{p}}{\sqrt{2H}} \right) \left(\hat{p} V - \sqrt{\frac{2H}{\hat{r}}} \right)^{-1}$$

(136)

and

$$\frac{\partial \mathcal{L}}{\partial R} (\hat{r} - \varepsilon) = -\frac{1}{W(\hat{r} - \varepsilon)} = -\left(\sqrt{\frac{2H}{\hat{r}}} + \frac{\hat{p}}{\hat{r}} \right)^{-1}$$

(137)

we have

$$\frac{\partial p_c}{\partial H} = -\sqrt{\frac{\hat{r}}{2H}} + \sqrt{\frac{\hat{r}}{2H}} \frac{\hat{p}}{\hat{r}} \frac{1}{\sqrt{\frac{2H}{\hat{r}}}} = \left(\frac{\hat{p}}{V} - \sqrt{\frac{2H}{\hat{r}}} \right)^{-1}$$

(138)
from which
\[ \dot{r} = N(r_m) \left( \frac{\dot{p}}{V} - \sqrt{\frac{2H}{r}} \right). \] (139)

2) Inner gauge
This time we have
\[ H - M = V - \frac{m^2}{2r} - \dot{p} \sqrt{\frac{2M}{r}} \] (140)
and
\[ p_i = \dot{r} \left( \mathcal{L}(\dot{r} + \varepsilon) - \sqrt{\frac{2H}{r}} + \sqrt{\frac{2M}{r}} - \log \left( 1 - \sqrt{\frac{2H}{r}} \right) \right). \] (141)

Using
\[ \frac{\partial \dot{p}}{\partial M} = - \left( 1 - \frac{\dot{p}}{\sqrt{2M}} \right) \left( \frac{\dot{p}}{V} - \sqrt{\frac{2M}{r}} \right)^{-1} \] (142)
and
\[ \frac{\partial \mathcal{L}}{\partial \dot{r}^i} (\dot{r} + \varepsilon) = - \frac{1}{W(\dot{r} + \varepsilon)} = - \left( \sqrt{\frac{2M}{r}} - \frac{\dot{p}}{r} \right)^{-1} \] (143)
we have
\[ \frac{\partial p_i}{\partial M} = \sqrt{\frac{\dot{r}}{2M}} \left[ 1 - \frac{\dot{p}}{V} \left( \frac{\dot{p}}{V} - \sqrt{\frac{2M}{r}} \right)^{-1} \right] = - \left( \frac{\dot{p}}{V} - \sqrt{\frac{2M}{r}} \right)^{-1} \] (144)
i.e.
\[ \dot{r} = \left( \frac{\dot{p}}{V} - \sqrt{\frac{2M}{r}} \right) N(r_0). \] (145)

Appendix C

In this Appendix we derive the equations of motion for the exterior and interior shell from the reduced action (62) for masses \( m_1 \) and \( m_2 \) different from zero.
First we consider the variation \( \delta H \neq 0 \) and \( \delta M_0 = 0 \). Under such a variation the coefficient of the \( \dot{r}_1 \) term is given, using
\[ \frac{\partial R}{\partial H} = (R - \dot{r}_1) \frac{dT}{dH}; \quad \frac{\partial R'}{\partial H} = (R' - 1) \frac{dT}{dH} \] (146)
by \( (W(\dot{r}_1 + \varepsilon)W(\dot{r}_1 - \varepsilon))^{-1} \frac{dT}{dH} \) multiplied by
\[ R' \left[ -W(\dot{r}_1 - \varepsilon)(R' - 1) + W(\dot{r}_1 + \varepsilon)(R' - 1) + (R - \dot{r}_1) \frac{\partial v_1}{\partial R} + (R' - 1) \frac{\partial v_1}{\partial R'} \right] - \]
\[ (R - \dot{r}_1) \left[ -W(\dot{r}_1 - \varepsilon)R'' + W(\dot{r}_1 + \varepsilon)(R'' + \frac{\partial v_1}{\partial R} R' + \frac{\partial v_1}{\partial R'} R'') \right] \] (147)
where $R, R', R''$ stay for $R(\hat{r}_1), R'(\hat{r}_1 + \varepsilon), R''(\hat{r}_1 + \varepsilon)$. From

$$W(\hat{r}_1 - \varepsilon) = W(\hat{r}_1 + \varepsilon) + \frac{\dot{p}_1}{R(\hat{r}_1)}$$

we find

$$W(\hat{r}_1 + \varepsilon) \frac{\partial v_1}{\partial R'(\hat{r}_1 + \varepsilon)} = \frac{\dot{p}_1}{R(\hat{r}_1)}$$

which substituted into eq. (147) makes it vanish. This is an expected result as the motion of the exterior shell must not depend on the dynamics which develops at smaller radiuses, but only on the two masses $H$ and $M_0$.

With regard to the terms proportional to $\dot{\hat{r}}_2$ in addition to $\dot{\hat{r}}_2$ in addition to

$$\dot{\hat{r}}_2 \frac{\partial p_{c2}^0}{\partial H}$$

we have the term given by

$$\frac{1}{W(\hat{r}_1 + \varepsilon) W(\hat{r}_1 - \varepsilon)} \frac{dT}{dH}$$

multiplied by

$$-(R' - 1) + (R - \hat{r}_1) \frac{\partial T}{\partial \hat{r}_2} \left[ -W(\hat{r}_1 - \varepsilon)(R' - 1) +
W(\hat{r}_1 + \varepsilon)(R' - 1 + \frac{\partial v_1}{\partial R}(R - \hat{r}_1) + \frac{\partial v_1}{\partial R'}(R' - 1)) \right] -
(R - \hat{r}_1) \left[ -W(\hat{r}_1 - \varepsilon)(\frac{dT}{d\hat{r}_2}(R' - 1) - R'') + W(\hat{r}_1 + \varepsilon)(\frac{dT}{d\hat{r}_2}(R' - 1) - R'') +
\frac{\partial v_1}{\partial R}(-R' + 1 + \frac{dT}{d\hat{r}_2}(R - \hat{r}_1)) + \frac{\partial v_1}{\partial R'}(-R'' + \frac{dT}{d\hat{r}_2}(R' - 1))) \right]$$

where again $R, R', R''$ stay for $R(\hat{r}_1), R'(\hat{r}_1 + \varepsilon), R''(\hat{r}_1 + \varepsilon)$. As a consequence of eq. (149) the above expression vanishes. Adding the contribution of the boundary term we have

$$\dot{\hat{r}}_2 \frac{\partial p_{c2}^0}{\partial H} - N(r_m) = 0$$

which is the single shell equation of motion (26).

The equation of motion of the interior shell is obtained from the variation $\delta H = 0, \delta M_0 \neq 0$. In this case we have no contribution from the boundary terms. For the coefficient of $\dot{\hat{r}}_1$ we obtain

$$- \frac{1}{W(\hat{r}_1 + \varepsilon)} + R' R \left[ - \frac{1}{W(\hat{r}_1 + \varepsilon)} \frac{dT}{dM_0}(R' - 1) +
\frac{1}{W(\hat{r}_1 - \varepsilon)} \left( \frac{dT}{dM_0}(R' - 1 + \frac{\partial v_1}{\partial R}(R - \hat{r}_1) + \frac{\partial v_1}{\partial R'}(R' - 1)) + \frac{\partial v_1}{\partial M_0} \right) \right] -
(R - \hat{r}_1) \frac{dT}{dM_0} R \left[ - \frac{R''}{W(\hat{r}_1 + \varepsilon)} + \frac{1}{W(\hat{r}_1 - \varepsilon)} (R'' + \frac{\partial v_1}{\partial R} R' + \frac{\partial v_1}{\partial R'} R'') \right].$$
having used the relation (see eq. (13))
\[ R' \frac{\partial^2 F}{\partial M \partial R'} - \frac{\partial F}{\partial M} = -\frac{1}{W}. \] (155)

Substituting in eq. (154) the relation
\[ \frac{\partial v_1}{\partial M_0} = \frac{\hat{p}_1(1 + \frac{\hat{p}_1}{R W(\hat{r}_1 + \varepsilon)})}{R(R'(\hat{r}_1 + \varepsilon)\hat{p}_1 - V_1 W(\hat{r}_1 + \varepsilon))} \] (156)
such a coefficient of \( \dot{\hat{r}}_1 \) becomes simply
\[ \frac{1}{\frac{\hat{p}_1}{V_1} R'(\hat{r}_1 + \varepsilon) - W(\hat{r}_1 + \varepsilon)} \] (157)
which is the non-zero mass generalization of the coefficient of \( \dot{\hat{r}}_1 \) appearing in eq. (70).

Then using eqs. (15, 29) for \( N(\hat{r}_1) \) and \( N'(\hat{r}_1) \) we have
\[
0 = \left[ \frac{\hat{p}_1}{V_1} R'(\hat{r} + \varepsilon) - W(\hat{r} + \varepsilon) \right]^{-1} \left( \dot{\hat{r}}_1 - \frac{\hat{p}_1}{V_1} N(\hat{r}_1) + N'(\hat{r}_1) \right) \\
+ \dot{\hat{r}}_2 \left[ \frac{\partial^2 F}{\partial M_0 \partial R'} \hat{r}_2 - \frac{\partial F}{\partial M_0} \right] + \alpha \\
- \left[ \frac{\hat{p}_1}{V_1} R'(\hat{r} + \varepsilon) - W(\hat{r} + \varepsilon) \right]^{-1} \frac{\hat{p}_1}{V_1} R(\hat{r}_1 + \varepsilon) \\
- \dot{\hat{r}}_2 \frac{\partial \hat{p}_2}{\partial M_0} - \hat{R}(\hat{r}_1 + \varepsilon) \frac{\partial^2 F}{\partial M_0 \partial R'}(\hat{r}_1 + \varepsilon) - \dot{\hat{r}}_2 \frac{d}{d\hat{r}_2} \left[ (R(\hat{r}_1) - \dot{\hat{r}}_1) \frac{dT}{dM_0} \right] \right]
\] (158)
where \( \alpha = \sqrt{2H(\hat{r})/(2H + \hat{p})} \). The first line is simply the equation of motion for \( \dot{\hat{r}}_1 \), the second line vanishes being simply the equation of motion for \( \dot{\hat{r}}_2 \) due to the variation of \( M_0 \) (see eq. (32)) while the sum of the third and fourth line vanishes in virtue of relation (149).
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