Stochastic Second-order Methods for Non-convex Optimization with Inexact Hessian and Gradient

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Abstract

Trust region and cubic regularization methods have demonstrated good performance in small scale non-convex optimization, showing the ability to escape from saddle points. Each iteration of these methods involves computation of gradient, Hessian and function value in order to obtain search direction and adjust the radius or cubic regularization parameter. However, exactly computing those quantities are too expensive in large-scale problems such as training deep networks. In this paper, we study a family of stochastic trust region and cubic regularization methods when gradient, Hessian and function values are computed inexactly, and show the iteration complexity to achieve $\epsilon$-approximate second-order optimality is in the same order with previous work for which gradient and function values are computed exactly. The mild conditions on inexactness can be achieved in finite-sum minimization using random sampling. We show the algorithm performs well on training convolutional neural networks compared with previous second-order methods.

1. introduction

In this paper, we consider the unconstrained optimization problem:

$$\min_{x \in \mathbb{R}^d} f(x) = \frac{1}{n} \sum_{i=1}^{n} f_i(x),$$

where $f_i(x)$ is smooth and not necessarily convex. Such finite-sum structure is increasingly popular in modern machine learning tasks, especially in deep learning, where each $f_i(\cdot)$ corresponds to the loss of a training sample. For large-scale problems, computing the full gradient and Hessian is prohibitive, so Stochastic gradient descent (SGD) has become the most popular method. First order methods such as gradient descent and SGD are guaranteed to converge to stationary points, which can be a saddle point or a local minimum.

It is known that second-order methods, by utilizing the Hessian information, can more easily escape from saddle points. At each iteration, second-order methods typically build a quadratic approximation function around the current solution $x_k$ by

$$m_k(s) = f(x_k) + \langle g(x_k), s \rangle + \frac{1}{2} \langle s, B(x_k)s \rangle,$$

where $g(x_k)$ is the approximated gradient and $B(x_k)$ is the symmetric matrix. To update the current solution, a common strategy is to minimize this quadratic approximation within a small region. Algorithms based on this idea including Trust

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1Some recent analysis indicates SGD can escape from saddle points in certain cases [1, 2], but SGD is not the focus of this paper.
Region method (TR) [3] and Adaptive Regularization using Cubics (ARC) [4, 5] have demonstrated good performance on small-scale non-convex problems.

However, for large-scale optimization such as training deep neural networks, it is impossible to compute gradient and Hessian exactly for every update. As a result, stochastic second-order methods have been studied in the past few years. [6, 7] proposed TR and ARC methods with inexact Hessian, in which the second-order information is approximated by the subsampled Hessian matrix, yet the gradient is still computed exactly. [8] proposed a stochastic version of ARC, but they require a much stronger condition in both gradient and Hessian approximation, thus they need to keep increasing the sample size as iteration goes. More recently, [9] provide stochastic cubic regularization method, but they do not have an adaptive way to adjust the regularization parameters.

In this paper, we consider a simple and practical stochastic version of trust region and cubic regularization methods (denoted as STR and SARC respectively). In the quadratic approximation (2), we replace both $g(x_k)$ and $B(x_k)$ by the approximate gradient and Hessian with a fixed approximation error, which can be achieved using a fixed sample size. Furthermore, we consider the most general case where the trust region radius or cubic regularization parameter is adaptively adjusted by checking the subsampled objective function value.

Note that we do not claim STR and SARC are “new” algorithms, since it is natural to transform TR and ARC to the stochastic setting. The question is whether this simple idea works in theory and in practice, and our contribution is to provide an affirmative answer to this question. Our contribution can be summarized as follows:

- We provide a theoretical analysis of convergence and iteration complexity for STR and SARC. Unlike [8], we do not require the approximation error of Hessian and gradient estimation to be related to $s$ (the update step). Furthermore, even though the proof framework is similar to [7], we provide novel analysis to model the case when both gradient and Hessian are inexact, while [7] does not allow inexact gradient.

- We use the operator-Bernstein inequality to bound the sub-sampled function value, which enables automatically adjusting trust region radius and cubic regularization parameter using subsamples. Even when function value, gradient and Hessian are all inexact, we are able to show that the iteration complexity is in the same order with [7, 8].

- We conduct experiments on CIFAR-10 data with VGG network and show that the proposed algorithms are faster than the existing trust region and cubic regularization methods in terms of running time.

1.1. Our results

We present the iteration complexity for both proposed methods under the Assumptions in Preliminary where the parameters are defined as well.

**STR** The total number of iterations is $O(\max\{(\varepsilon_f - \varepsilon_g)^{-2}, (\varepsilon_f - \varepsilon_B)^{-3}\})^2$.

**SARC** The iteration complexity is the same order of STR. If the condition of the terminal criterion is satisfied, then the total number of iterations is $O(\max\{(\varepsilon_f - \varepsilon_g)^{-3/2}, (\varepsilon_f - \varepsilon_B)^{-3}\})$, which is better than STR.

1.2. Related Work

With the increasing size of data and model, stochastic optimization becomes more and more popular since computing the gradient and Hessian are prohibitively expensive.

For the stochastic first-order optimization, stochastic gradient descent (SGD) [10, 11, 12] is absolutely the main method especially in training deep neural networks and other large-scale machine learning problems, due to its simplicity and effectiveness. However, the estimated gradient will induce the noise such that the variance of the gradient may not approximate zero even when converged to a stationary point. Stochastic variance reduction gradient (SVRG) [13] and SGAG [14] are two typical methods to reduce the variance of the gradient estimator, which lead to faster convergence especially in the convex setting. Several other related variance reduction methods are developed and analyzed for non-convex problems [15, 16]. However, using only first-order information, the saddle point may not be escaped even though [17] prove that SGD with noise can escape but under certain conditions.

For second-order optimization, Newton-typed methods rely on building a quadratic approximation around the current solution, and by exploring the curvature information it can better avoid saddle points in non-convex optimization. The

\[2\text{We use } O(\cdot) \text{ to hide constant factors.}\]
negative eigenvector of the Hessian information provides the decrease direction for the updates. Using exact Hessian is often time consuming, so Broyden-Fletcher-Goldfarb-Shanno (BFGS) and Limited-BFGS [18] are two widely methods that approximate Hessian using first-order information. Another important technique for Hessian approximation is sub-sampling the function \( f_i(\cdot) \) to obtain the estimated Hessian. Both [19] and [20] using the stochastic Hessian matrix to obtain the global convergence, while the former requires \( f_i \) to be smooth and strongly convex. Furthermore, [21, 22] apply the sub-sampled gradient and Hessian to the quadratic model and give the convergence analysis of second-order methods thoroughly and quantitatively.

Trust region Newton method is a classical second-order method that searches the update direction only within a trust region around the current point. The size of trust region is critical to the effectiveness of search direction. The region will be updated based on measuring whether the quadratic approximation is an adequate representation of the function or not. Following the sub-sampling for Hessian matrix as in [21, 22], [6, 7] apply such inexact Hessian to trust region method and also provide the convergence and iteration complexity. Similar to the trust region method, [4, 5] introduced adaptive cubic regularization methods for unconstrained optimization, in which the Hessian metrics can be replaced by the approximate matrix. [8] apply the operator-Bernstein inequality to approximate the Hessian matrix and gradient into a quadratic function with cubic regularization. However, the sample approximate condition is subject to the search direction \( s \), thus they need to increase the sample size at each step. To overcome this issue, [6] provide another approximation condition of Hessian matrix that does not depend on the search step \( s \). However, they assume gradient has to be computed exactly, which is not feasible in large-scale applications. Furthermore, each update of the \( \rho \) that measure the adequacy of the function will need the full computation of the objective function, which will lead to more computation cost.

The rest of paper is organized as follows. Section 2 gives the preliminary about the assumptions and definition. The sub-sampling method for estimating the corresponding function, gradient and Hessian is in Section 3. Section 4 and 5 respectively present the stochastic trust region method and cubic regularization method, and their convergence and iteration complexity. Section 6 gives the experimental results. Section 7 concludes our paper.

2. Preliminary

For a vector \( x \) and a matrix \( X \), we use \( \|x\| \) and \( \|X\| \) to denote the Euclidean norm and the matrix spectral norm, respectively. We use \( S \) to denote the set and \( |S| \) to denote its cardinality. For the matrix \( X \), we use \( \lambda_{\text{min}}(X) \) and \( \lambda_{\text{max}}(X) \) to denote its smallest and largest eigenvalue. In the following, we give assumptions and definition about the characteristic of function, the approximate conditions, related bounds, and optimality definition.

**Assumption 1.** (Lipschitz Continuous) For the function \( f(x) \), we assume that \( \nabla^2 f(x) \) and \( \nabla f(x) \) are Lipschitz continuous satisfying \( \|\nabla^2 f(x) - \nabla^2 f(y)\| \leq L_H \|x - y\| \) and \( \|\nabla f(x) - \nabla f(y)\| \leq L_\nabla f \|x - y\| \), \( \forall x, y \in \mathbb{R}^d \).

**Assumption 2.** (Approximate) For function \( f(x) \), the approximate gradient \( g(x) \) and Hessian matrix \( B(x) \) satisfy

\[ \|\nabla f(x) - g(x)\| \leq \varepsilon_g, \|\nabla^2 f(x) - B(x)\| \leq \varepsilon_B. \]  

with \( \varepsilon_g, \varepsilon_B > 0 \). The approximated function \( h(x) \) at k-iteration satisfies

\[ \|f(x_k) - h(x_k)\| \leq \varepsilon_h \|s_k\|^2, \varepsilon_h > 0. \]  

**Assumption 3.** (Bound) For \( i \in [n] \), the bound assumptions are the function \( f_i(x) \) satisfies \( ||f_i(x)|| \leq \kappa_f, \|\nabla f_i(x)\| \leq \kappa_{\nabla f}, \) and \( \|\nabla^2 f_i(x)\| \leq \kappa_H. \)

**Assumption 4.** (Bound) We assume that \( H_1 \) and \( H_2 \) are the upper bounds on the variance of the \( \nabla f_i(x) \) and \( \nabla^2 f_i(x) \), that is

\[ \frac{1}{n} \sum_{i=1}^{n} \|\nabla f_i(x) - \nabla f(x)\|^2 \leq H_1^2, \frac{1}{n} \sum_{i=1}^{n} \|\nabla^2 f_i(x) - \nabla^2 f(x)\|^2 \leq H_2^2. \]

The following lemmas are important tools for analyzing the convergence of the proposed algorithm, which are used to characterize the variance of random variable decreasing with the factor related to the set size.

**Lemma 1.** If \( v_1, ..., v_n \in \mathbb{R}^d \) satisfy \( \sum_{i=1}^{n} v_i = 0 \), and \( A \) is a non-empty, uniform random subset of \( [n] \), \( A = |A| \), then

\[ \mathbb{E}_A \left\| \frac{1}{A} \sum_{b \in A} v_b \right\|^2 \leq \frac{||A < n||}{A} \frac{1}{n} \sum_{i=1}^{n} v_i^2. \]
Figure 1. Illustration of three situations in analyzing the convergence to first and second critical point.

**Definition 1.** \((\varepsilon_{\nabla f}, \varepsilon_H)\)-Optimality. Given \(\varepsilon_{\nabla f}, \varepsilon_H \in [0, 1]\), \(x\) is an \((\varepsilon_{\nabla f}, \varepsilon_H)\)-Optimality solution to problem (1), if

\[
\| \nabla f(x) \| \leq \varepsilon_{\nabla f}, \quad \lambda_{\min}(\nabla^2 f(x)) \geq -\varepsilon_H.
\]

Furthermore, we introduce three index sets

\[
\begin{align*}
S_{\nabla f} & \overset{\text{def}}{=} \{ x : \| \nabla f(x) \| \geq \varepsilon_{\nabla f} \}, \\
S_H & \overset{\text{def}}{=} \{ x : \| \nabla f(x) \| \geq \varepsilon_{\nabla f} \text{ and } \lambda_{\min}(\nabla^2 f(x)) \leq -\varepsilon_H \}, \\
S_\ast & \overset{\text{def}}{=} \{ x : \| \nabla f(x) \| \leq \varepsilon_{\nabla f} \text{ and } \lambda_{\min}(\nabla^2 f(x)) \geq -\varepsilon_H \}.
\end{align*}
\]

where \(\varepsilon_{\nabla f} > \varepsilon_g\) and \(\varepsilon_H > \varepsilon_B\). In order to clearly classify three situations, we give a simple geometry illustration, as shown in Figure 1.

3. Sub-sampling for finite-sum minimization

For the finite-sum problem (1), we can estimate \(f(x), \nabla f(x), \) and \(\nabla^2 f(x)\) by random sub-sampling, which can drastically reduce the computational complexity. Here, we use \(S_h, S_g\) and \(S_B\) to denote the sample collections for estimating \(f(x), \nabla f(x)\) and \(\nabla^2 f(x)\), respectively, where \(S_h, S_g\) and \(S_B \subseteq [n]\). The approximated functions are formed by

\[
\begin{align*}
h(x) &= \frac{1}{|S_h|} \sum_{i \in S_h} f_i(x), \\
g(x) &= \frac{1}{|S_g|} \sum_{i \in S_g} \nabla f_i(x), \\
B(x) &= \frac{1}{|S_B|} \sum_{i \in S_B} \nabla^2 f_i(x).
\end{align*}
\]

Most papers use operator-Bernstein inequality to probabilistically guarantee such properties, such as [23, 24] use approximate matrix multiplication results as a fundamental primitive in RandNLA [23, 24] to control the approximation error of \(\nabla f(x)\). Furthermore, the vector-Bernstein inequality [25, 26] is applied in [8] to obtain the of sub-sample bound of the gradient, which is different from [23]. However, the number of sub-samples depend on the search direction \(s\) in advance, which will affect the estimation of sub-sampling. Replacing the condition by (3), we can have

**Lemma 2.** Suppose the Assumption 3 holds, if \(|S_g| \geq 16 \log(2d/\delta) L_f^2/\varepsilon_g^2\), then \(g(x)\) formed by (6) satisfies \(\| \nabla f(x) - g(x) \| \leq \varepsilon_g\) with probability \((1 - \delta)\).

Note that, we do not give the proof of above Lemmas as the difference lies on different conditions. We also obtain the approximate gradient \(g(x)\) based on above results.

**Lemma 3.** Suppose the Assumption 3 holds, if \(|S_B| \geq \log(2d/\delta) 16 L_B^2/\varepsilon_B^2\), then \(B(x)\) formed by (7) satisfies \(\| \nabla^2 f(x) - B(x) \| \leq \varepsilon_B\) with probability \((1 - \delta)\).
In order to reduce the computation cost for updating the $\rho$, we also use the sub-sampling combing with operator-Bernstein inequality to obtain the approximate function probabilistically satisfying (4). Note that, before obtain the approximate function $f(x)$, $s$ has been solved. Thus, we can use $|s|$ directly. Furthermore, if the approximate function $h(x)$ satisfies the condition in (4), but could not be guaranteed the bound of $\|\nabla h(x) - \nabla f(x)\|$ and $\|\nabla^2 h(x) - \nabla^2 f(x)\|$, which will be later used to analyze the radius. Thus, we present a important assumption and use Lemma 1 to derive the upper bound.

**Lemma 4.** Suppose the Assumption 3 and 4 hold. If $|S_h| \geq \log(2d/\delta)16n^2_2/(\varepsilon_f^2\|s_k\|^4)$, then $h(x)$ formed by (5) satisfies $\|f(x_k) - h(x_k)\| \leq \varepsilon_f \|s_k\|^2$ with probability $(1 - \delta)$. Furthermore, we can also have the upper bounds with the gradient and the Hessian of $h(x)$, 

$$\|\nabla h(x) - \nabla f(x)\|^2 \leq \frac{\|S_h\| < n}{|S_h|} H_1, \|\nabla^2 h(x) - \nabla^2 f(x)\|^2 \leq \frac{\|S_h\| < n}{|S_h|} H_2.$$ 

**4. Stochastic Trust Region method**

In this section, we consider the stochastic trust region method for solving the constrained optimization problem. At $k$-iteration, the objective function is approximated by a quadratic model within a trust region,

$$\min_{s \in \mathbb{R}^d} m_k(s), \text{ subject to } \|s\| \leq \Delta_k, \quad (9)$$

where $\Delta_k$ is the radius, and $m_k(s)$ is defined in (2). The approximated gradient and Hessian are formed based on the conditions in Assumption 2. The conditions of $\varepsilon_B$ in (3) does not depend on the search direction $s$, which is the same as in [6, 7]. Moreover, we also define a new parameter $\varepsilon_g$, which has the same characteristic as $\varepsilon_B$. In addition, the computations of $f(x)$ and $f(x + s)$ are expensive as the objective function is finite-sum structure. Different from [6] and [8], we replace $f(x)$ and $f(x + s)$ with the approximate function $h(x)$ and $h(x + s)$ under the condition (4). This condition is subjected to the search direction $s$. However, the approximate function $h(x)$ and $h(x + s)$ can be derived after obtaining the solution $s$ through Subproblem-Solver. Algorithm 1 presents the process for updating the $x$ and $\Delta$. This section consists of two parts: Firstly, we analyze the role of radius $\Delta$ to ensure that the radius has the lower bound. Secondly, we derive the corresponding iteration complexity under the assumptions we present in Preliminary 2.

**4.1. Bounds analysis of radius**

First of all, we present three important definition: $\rho_k$, $\tilde{\rho}$ and $\rho$. The first two terms are defined as

$$\rho_k = \frac{f(x_k) - f(x_k + s_k)}{m_k(0) - m_k(s_k)}, \tilde{\rho}_k = \frac{h(x_k) - h(x_k + s_k)}{m_k(0) - m_k(s_k)},$$

**Algorithm 1 STR with Inexact Hessian and Gradient**

**Require:** given $x_0$, $r_2 \geq 1 > r_1$, $1 > \eta > 0$, $\varepsilon > 0$ and $\Delta_0$, $\Delta_{\max} > 0$.

**Ensure:**

1. for $k=1$ to $T$ do
2. Compute the approximate gradient $g(x_k)$ and Hessian matrix $B(x_k)$ based on (6) and (7).
3. $\triangleright$ (if $\|g(x_k)\| \leq \varepsilon_f + \varepsilon_g$, we set $S_g = S_g$)
4. Compute the direction vector $s_k$

$$s_k = \text{Subproblem-Solver}(g_k, B_k, \Delta_k). \quad (8)$$
5. Compute the approximate function $h(x_k)$ and $h(x_k + s_k)$ based on (5).
6. Compute $\tilde{\rho}_k = \frac{h(x_k) - h(x_k + s_k)}{m_k(0) - m_k(s_k)}$
7. Set $\rho = \tilde{\rho}_k - \frac{m_k(0) - m_k(s_k)}{2\rho(\rho)^2}$
8. set $x_{k+1} = \begin{cases} \ x_k + s_k, & \rho \geq \eta, \\ x_k, & \text{otherwise.} \end{cases}$
9. set $\Delta_{k+1} = \begin{cases} \min \{\Delta_{\max}, r_2\Delta_k\}, & \rho > \eta, \\ r_1\Delta_k, & \text{otherwise.} \end{cases}$
10. end for
Based on the inequality (4) in Assumption 2, we have
\[
|\rho_k - \hat{\rho}_k| = \left| \frac{f(x_k) - h(x_k) - (f(x_k + s_k) - h(x_k + s_k))}{m_k(0) - m_k(s_k)} \right| \leq \frac{2\varepsilon_h \|s_k\|^2}{m_k(0) - m_k(s_k)},
\]
that is
\[
\hat{\rho}_k - \frac{2\varepsilon_h \|s_k\|^2}{m_k(0) - m_k(s_k)} \leq \rho_k \leq \frac{2\varepsilon_h \|s_k\|^2}{m_k(0) - m_k(s_k)} + \hat{\rho}_k.
\]
Then, define \( \rho = \hat{\rho}_k - \frac{2\varepsilon_h \|s_k\|^2}{m_k(0) - m_k(s_k)} \). If \( \rho \geq \eta \), we can obtain \( \rho_k \geq \eta \). Thus, in the following analysis, we consider the size of \( \rho \) that derive the desired lower bound of the radius.

Before giving the analyses, we briefly present the processing that why the \( \Delta \) do not approximate to zero. If \( \rho > \eta \), the current iteration will be successful and the radius \( \Delta \) will increase by a factor of \( r_2 \). Thus, we consider that whether there is a constant \( C \) such that \( \Delta < C \) and the current iteration is successful simultaneously. Then we can see that such constant \( C \) is our desired bound of \( \Delta \) due to the fact that \( \Delta \) will increase again under the successful iteration. What’s more, such constant plays a critical role in determining the iteration complexity.

Instead of computing the \( \rho \) directly, we consider another relationship, that is
\[
1 - \rho = \frac{m_k(0) - m_k(s_k) - (h(x_k) - h(x_k + s_k)) + 2\varepsilon_h \|s_k\|^2}{m_k(0) - m_k(s_k)} \tag{10}
\]
As long as \( 1 - \rho < 1 - \eta \), we can see that \( \rho > \eta \). Here, we consider the upper and lower bound of denominator and numerator in (10) in Lemma 5 and Lemma 6, respectively. Moreover, we separately give the corresponding bound under the index sets \( \mathcal{S}_{\mathcal{V}_f} \) and \( \mathcal{S}_H \).

**Lemma 5.** Suppose the Assumption 2 and 3 hold, \( m_k(s) \) is defined in (2). For the case of \( x_k \in \mathcal{S}_{\mathcal{V}_f} \), if \( m_k(s_k) \leq m_k(s'_k) \), where \( s'_k \) is the Cauchy point, then we have
\[
m_k(0) - m_k(s_k) \geq \frac{1}{2}(\varepsilon_{\mathcal{V}_f} - \varepsilon_g) \min \{ \Delta_k, (\varepsilon_{\mathcal{V}_f} - \varepsilon_g)/\kappa_H \}.
\]
For the case of \( x_k \in \mathcal{S}_H \), that is \( \lambda_{\text{min}}(\nabla^2 f(x_k)) \leq -\varepsilon_H \), there exist a vector \( s_k \) such that \( \langle g(x_k), s_k \rangle \leq 0 \), \( s_k^T \nabla^2 f(x_k) s_k < -\nu_0 \Delta_k^2 \), and \( \|s_k\| = \Delta_k \), where \( \nu_0 \geq \varepsilon_H \), then we have
\[
m_k(0) - m_k(s_k) \geq \frac{1}{2}(\varepsilon_H - \varepsilon_B) \Delta_k^2.
\]

The solution \( S_k \) for the Subproblem-Solver is based on subproblem in (2). For the case of \( x_k \in \mathcal{S}_{\mathcal{V}_f} \), we use the Cauchy point [3]; while for the case of \( x_k \in \mathcal{S}_H \), there are many methods to derive the solution, such as Shift-and-invert [27], Lanczos [28] and Negative-Curvature [29]. We do not present the detailed information, which beyond our scope of this paper.

**Lemma 6.** Suppose the Assumption 1, 2 and 4 hold, based on the definition of \( m_k(s) \) in (2), we have
\[
m_k(0) - m_k(s_k) - (h(x_k) - h(x_k + s_k)) \leq 2\left( \frac{\|\mathcal{S}_h\| < n}{|\mathcal{S}_h|} H_1 + \varepsilon_g \right) \Delta_k + \frac{3}{2} \left( \frac{\|\mathcal{S}_h\| < n}{|\mathcal{S}_h|} H_2 + L_H \Delta_k + \varepsilon_B \right) \Delta_k^2.
\]
If \( \mathcal{S}_h = \mathcal{S}_g \), we have
\[
m_k(0) - m_k(s_k) - (h(x_k) - h(x_k + s_k)) \leq \frac{3}{2} \left( \frac{\|\mathcal{S}_h\| < n}{|\mathcal{S}_h|} H_2 + L_H \Delta_k + \varepsilon_B \right) \Delta_k^2.
\]

Note that, in Algorithm 1, for the case of \( \|g(x_k)\| \leq \varepsilon_{\mathcal{V}_f} + \varepsilon_g \), that is
\[
\|\nabla f(x_k)\| \leq \|g(x_k)\| + \|\nabla f(x_k) - g(x_k)\| \leq \varepsilon_{\mathcal{V}_f} + \varepsilon_g = \varepsilon_{\mathcal{V}_f} + 2\varepsilon_g,
\]
we set \( \mathcal{S}_h = \mathcal{S}_g \), \( x \in \mathcal{S}_H \), in which \( \|\nabla f(x_k)\| \leq \varepsilon_{\mathcal{V}_f} \), satisfies such case\(^3\). Thus, we give the \( \Delta_{\text{min}2} \) in (11) based on such implementation, which is key for analyses. The reason we make such implementation is to ensure that there is lower bound of radius. What’s more, the parameters’ setting is more simple. Based on above lemmas, we analyze the minimal radius.

\(^3\)In the case of \( \varepsilon_{\mathcal{V}_f} + 2\varepsilon_g > \nabla f(x_k) > \varepsilon_{\mathcal{V}_f} \), we have \( \mathcal{S}_h = \mathcal{S}_g \) such that the equality (25) in Appendix become \( \frac{4\varepsilon_g + \frac{1}{2}(L_H \Delta_k + 2\varepsilon_B + \frac{3}{2}\varepsilon_g) \Delta_k}{\varepsilon_{\mathcal{V}_f} - \varepsilon_g} \), which is smaller than equality (25), thus \( \Delta_{\text{min}1} \) is also satisfying such case. In this paper, in order to simply the analysis, we consider the case \( \nabla f(x) > \varepsilon_{\mathcal{V}_f} \) without the requirement of \( \mathcal{S}_{\mathcal{V}_f} = \mathcal{S}_H \).
**Lemma 7.** In Algorithm 1, suppose the Assumption 1-4 hold, let $|S_h| = \min\{n, \max\{H_1/\varepsilon_g, H_2/\varepsilon_B\}\}$, $1 > r_1 > 0$, there will be a non-zero radium

$$\Delta_{min} = \min\{\Delta_{min1}, \Delta_{min2}\},$$

where the parameters satisfy

$$\begin{align*}
\Delta_{min1} & = \kappa_1 (\varepsilon_{\nabla f} - \varepsilon_g), \quad \kappa_1 = r_1 \min\left\{ \frac{1}{\kappa_H}, \frac{1}{40} (1 - \eta), \sqrt{\frac{1}{12L_H}} (1 - \eta) \right\}, \\
\Delta_{min2} & = \kappa_2 (\varepsilon_H - \varepsilon_B), \quad \kappa_2 = r_1 \frac{1}{6L_H} (1 - \eta), \\
\varepsilon_g & = \frac{1}{16} (1 - \eta) (\varepsilon_{\nabla f} - \varepsilon_g), \\
\varepsilon_B & = \varepsilon_h = \frac{1}{10} (1 - \eta) (\varepsilon_H - \varepsilon_B).
\end{align*}$$

In particular, $\Delta_{min1}$ belongs to the case of $x \in S_{\nabla f}$ and $\Delta_{min2}$ belongs to the case of $x \in S_H$.

### 4.2. Convergence and iteration complexity

In this section, we present the successful and unsuccessful iteration complexity based on Lemma 7 including $x \in S_{\nabla f}$ and $k \in S_H$, and then provide the total number of iterations.

**Theorem 1.** In Algorithm 1, suppose the Assumption 1-4 hold, let $|S_h| = \min\{n, \max\{H_1/\varepsilon_g, H_2/\varepsilon_B\}\}$, $\{f(x_k)\}$ is bounded below by $f_{low}$, the number of successful iterations $T_{suc}$ is no large than

$$\kappa_3 \max\{(\varepsilon_{\nabla f} - \varepsilon_g)^{-2}, (\varepsilon_H - \varepsilon_B)^{-3}\},$$

where $\kappa_3 = 2 (f(x_0) - f_{low}) \max\{1/(\eta \kappa_1), 1/(\eta \kappa_2^2)\}$, $\kappa_1$ and $\kappa_2$ are defined in (12) and (13). The number of unsuccessful iterations $T_{unsuc}$ is at most

$$\frac{1}{-\log r_1} \left( \log \frac{\Delta_{max}}{\Delta_{min}} - T \log r_2 \right).$$

where $\Delta_{max}$ and $\Delta_{min}$ are defined in (11) and Algorithm 1, $1 > r_1 > 0$ and $r_2 \geq 1$. Thus, the total number of iterations is

$$O\left( \max\{(\varepsilon_{\nabla f} - \varepsilon_g)^{-2}, (\varepsilon_H - \varepsilon_B)^{-3}\} \right).$$

After the $|T_{suc}| + |T_{unsuc}|$ iterations, it will fall into $S_*$ and converge to the stationary point. As can be seen from above Theorems, we make two conclusions: the first is the order of iteration complexities are the same as [5] and [7] if the parameters $\varepsilon_g$ and $\varepsilon_B$ are set properly according to $\varepsilon_{\nabla f}$ and $\varepsilon_H$ respectively; the second is that when $|S_h| < n$, the total number of computation iteration including computing the function is less than that of [5] and [7]; when $|S_h| = n$, our result is equal to [5] and [7]. Thus, our proposed algorithm is more general.

### 5. Stochastic Adaptive Regularization using Cubics

In this section, we consider the stochastic adaptive regularization using Cubics (SARC) method, which solves the following unconstrained minimization problem at each iteration:

$$\min_{s \in \mathbb{R}^d} p_k(s) := m_k(s) + \frac{\sigma_k}{3} \|s\|^3,$$

where $m_k(s)$ is defined in (2), and $\sigma_k$ is an adaptive parameter that can be considered as the reciprocal of the trust-region radius. Algorithm 2 presents the process for updating the $x_k$ and $\sigma_k$. Similar to the analysis as in [4], $\sigma_k$ in the cubic term actually performs one more task, besides accounting for the discrepancy between the objective function and its corresponding second-order Taylor expansion, but also for the difference between the exact and approximate function, gradient and Hessian. The update rules of $\sigma$ is analogous to stochastic region method. $\sigma$ will decrease if sufficient decrease is obtained in some measure of relative objective chance, but increase otherwise. Following the framework of STR, we analyze SARC including two parts: To ensure the existence of the maximal bound of $\sigma$ and present the iterative complexity.
Thus, we can also obtain lower bound of the numerator in (16) and (17). Specifically, we set
\[ s_k = \text{Subproblem-Solver}(g(x_k), B(x_k), \sigma_k). \] (14)

5.1. Bounds analysis of the adaptive parameter

Similar to STR, we present the definition of \( \rho \) directly,
\[ 1 - \rho = \frac{p_k(0) - p_k(s_k) - (h(x_k) - h(x_k + s_k)) + 2\varepsilon_k \|s_k\|^2}{p_k(0) - p_k(s_k)} \] (16)

In order to satisfy inequality (16), we need to obtain the lower bound of the numerator in (16). Firstly, we derived the lower bound from the view of a Cauchy pint, but subject to \( p_k(s_k) \leq p_k(s_k^C) \). Note that the lower bound is almost the same as in [4] and [6], but give the proof from the geometrical explanation.

Lemma 8. Suppose that the step size \( s_k \) satisfying \( p_k(s_k) \leq p_k(s_k^C) \), where \( s_k^C \) is a Cauchy point, defined as
\[ s_k^C = -\alpha_k g(x_k), \alpha_k = \arg \min_{\alpha \in \mathbb{R}_+} \{ p_k(x_k - \alpha g(x_k)) \}, \]
for all \( k \geq 0 \), we have that
\[ p_k(0) - p_k(s_k) \geq \frac{1}{10} \|g_k\| \min \left\{ \|g_k\|/\|B_k\|, \sqrt{\|g_k\|/\|\sigma_k\|} \right\}. \]

Specifically, we set \( \alpha = 2/\|B_k\| + \sqrt{\|B_k\|^2 + 4\sigma_k \|g_k\|} \) and \( s_k = -\alpha g(x_k) \) that satisfy above inequality. Furthermore, we can also obtain the upper bound of the step \( \|s_k\|, k > 0 \), satisfies \( \|s_k\| \leq 11/4 \max\{\|B(x_k)\|/\sigma_k, \sqrt{\|g(x_k)\|/\sigma_k}\} \).

Following the subspace analysis in the cubic model as in [4] and [5], in order to widen the scope of convergence analysis and iteration complexity, we also consider the step size \( s_k \) on the following conditions
\[ \langle g(x_k), s_k \rangle + s_k^2 B_k s_k + \sigma_k \|s_k\|^3 = 0, \] (17)
\[ s_k^2 B_k s_k + \sigma_k \|s_k\|^3 \geq 0, \] (18)
\[ \|\nabla p_k(s_k)\| \leq \theta_k \|\nabla g(x_k)\|, \theta_k \leq \kappa_\theta \min \{1, \|s_k\|\}, \kappa_\theta < 1. \] (19)

Thus, we can also obtain lower bound of the numerator in (16), which will be used for analyzing the convergence and iteration complexity in the case of the saddle point.

Lemma 9. Given the conditions of \( s_k \) in (17) and (18), we have
\[ p(0) - p_k(s_k) \geq \frac{\sigma_k}{6} \|s_k\|^3. \]
Furthermore, suppose Assumption 1 and 2, and the condition (19) hold, for \( \|g(x_{k+1})\| \geq \varepsilon \nabla f - \varepsilon_g \), we have
\[
\|g(x_{k+1})\| \leq \kappa_s \|s_k\|^2,
\]
where
\[
\kappa_s = \min\left\{ \frac{2\varepsilon_B + (L_H + \sigma_k) + 2\eta \varepsilon_g + \kappa_0 \nabla \nabla f}{(1 - \theta_k)}, \frac{L_H + \sigma_k + \kappa_0 \nabla \nabla f}{1 - \theta_k - \zeta_1 - \zeta_2} \right\}, \quad \zeta_1, \zeta_2 < 1
\]

\( \varepsilon_B \leq \zeta_1 (\varepsilon \nabla f - \varepsilon_g), \quad \varepsilon_g \leq \zeta_2 (\varepsilon \nabla f - \varepsilon_g) \)

Different from the STR, we can also derive the relationship between \( g(x_{k+1}) \) and \( \|s_k\| \). The core process of the proof is based on cubic regularization of Newton method [30]. Such a relationship leads to the improved iteration complexity. Besides lower bound of the numerator in (16), we can also obtain the corresponding upper bound of the denominator, which is similar to Lemma 6. Thus, we do not provide the proof.

**Lemma 10.** Suppose the Assumption 1, 2 and 4 hold, at k-iteration, we have
\[
p_k(0) - p_k(s_k) - (h(x_k) - h(x_k + s_k)) \leq 2 \frac{\|\text{I} \{S_h < n\} \| H_1 + \varepsilon_g}{|S_h|} \|s_k\| + \frac{3}{2} \left( \frac{\|\text{I} \{S_h < n\} \| H_2 + \varepsilon_B}{|S_h|} \right) \|s_k\|^2 + \left( \frac{3}{2} L_H - \frac{1}{3} \kappa_0 \right) \|s_k\|^3.
\]

If \( S_h = S_g \), we have
\[
p_k(0) - p_k(s_k) - (h(x_k) - h(x_k + s_k)) \leq \frac{3}{2} \left( \frac{\|\text{I} \{S_h < n\} \| H_2 + \varepsilon_B}{|S_h|} \right) \|s_k\|^2 + \left( \frac{3}{2} L_H - \frac{1}{3} \kappa_0 \right) \|s_k\|^3.
\]

Based on the above lemmas, we can derive the upper bound of adaptive parameter \( \sigma \), which is used to analyze the iteration complexity. Furthermore, the parameters’ setting, such as \( \varepsilon_g, \varepsilon_B \) and \( \varepsilon_h \), are similar to that of Lemma 7.

**Lemma 11.** In Algorithm 2, suppose Assumption 1-4 hold, let \( |S_h| = \min\{n, \max\{H_1/\varepsilon_g, H_2/\varepsilon_B\}\} \), \( r_2 > 1 \), the parameter \( \sigma \) is bounded by
\[
\sigma_{\text{max}} = \max\{\sigma_{\text{max1}}, \sigma_{\text{max2}}\},
\]
where
\[
\sigma_{\text{max1}} = \frac{1}{(\varepsilon \nabla f - \varepsilon_g)}, \quad \sigma_{\text{max2}} = \frac{9}{2} r_2 L_H,
\]
\[
\kappa_4 = r_2 \left\{ \kappa_0 \left( \frac{304 (3 \varepsilon_B + 2 \varepsilon_h)^2}{(1 - \eta)} \right) - \frac{9}{2} (\varepsilon \nabla f - \varepsilon_g) L_H \right\},
\]
\[
\varepsilon_g = \frac{1}{220} (1 - \eta) (\varepsilon \nabla f - \varepsilon_g),
\]
\[
\varepsilon_B = \varepsilon_h = \frac{1}{36} (1 - \eta) (\varepsilon_H - \varepsilon_B).
\]

In particular, \( \sigma_{\text{min1}} \) belongs to the case of \( x \in S_{\nabla f} \) and \( \sigma_{\text{min2}} \) belongs to the case of \( x \in S_H \).

### 5.2. Analysis of convergence and iteration complexity

Based on above lemmas, we present the iteration complexity. Different from STR, we derive two kinds of complexity. The first one has the same order as STR while the second is better or equal to STR. The difference lies that if the criterion conditions in (19) is satisfied, the iteration complexity to the stationary point is improved.

**Theorem 2.** In Algorithm 2, suppose the Assumption 1-4 hold, let \( |S_h| = \min\{n, \max\{H_1/\varepsilon_g, H_2/\varepsilon_B\}\} \), \( \{f(x_k)\} \) is bounded below by \( f_{\text{low}} \), the number of successful iterations \( T_{\text{succ}} \) is no larger than
\[
\kappa_5 \max\{\|\varepsilon \nabla f - \varepsilon_g\|^2, (\varepsilon_H - \varepsilon_B)^3\},
\]
where
\[
\kappa_5 = \frac{1}{\eta^{1/2}} \max\{5/(\eta \kappa_4^{-1/2}), 6 \sigma_{\text{max2}} / \eta\}, \quad \kappa_4 \text{ is defined in (22).}
\]

If the conditions (17)-(19) are satisfied, then the number of successful iterations \( T_{\text{succ}} \) is at most
\[
\kappa_6 \max\{\|\varepsilon \nabla f - \varepsilon_g\|^{-3/2}, (\varepsilon_H - \varepsilon_B)^{-3}\},
\]
where
\[
\kappa_6 = \frac{1}{\eta^{3/2}} \max\{6 \sigma_{\text{min}}^{3/2}/(\eta \sigma_{\text{min}}), 6 \sigma_{\text{max2}} / \eta\}, \quad \kappa_s \text{ is defined (20).}
\]
As can be seen above results, our proposed method has the same order of iteration complexity as in [6] and [7]. However, our algorithm does not require the full computation of function and gradient with the finite-sum structure such that reduce the computation cost properly.

6. Experiment

In this section, we give a comprehensive comparison among trust region and ARC algorithms. Our goal is not to show TR/ARC methods are state-of-the-art compared with other solvers; we are trying to present different variances of TR and ARC, and show using a fixed batch size to estimate both gradient and Hessian is the best choice for large-scale optimization. We compare following variants:

- Stochastic TR with fixed batch size (TR, fixed): Both Hessian and gradient are estimated through a fixed batch size at each iteration. In our experiment, we choose batch size \( |B_g| = |B_H| = 256 \).
- Stochastic TR with growing batch size (TR, inc): As above, both Hessian and gradient are estimated through a batch of samples, except that the sample size is increasing with epochs: At the early stage we feed a crude estimation of \( \nabla f(x) \) and \( \nabla^2 f(x) \) on small batch, then gradually increase the batch size to give better gradient and Hessian information. In practice we multiply the batch size by a factor of 2 for every 10 epochs until memory is used up.
- Stochastic TR with exact gradient and subsampled Hessian (TR, full): We use implementation similar with [6, 7]. At each iteration the gradient is exact, while the Hessian is approximated on a batch of 256 samples.
- Stochastic ARC with fixed batch size (Cubic, fixed): This is similar to stochastic TR, the batch size is fixed to \( |B_g| = |B_H| = 256 \).
- Stochastic ARC with growing batch size (Cubic, inc): Similar to stochastic TR with increasing batch size, we double the batch size once every 10 epochs to estimate both Hessian and gradient.
- ARC with exact gradient and subsampled Hessian (Cubic, full): The gradient is exactly computed while the Hessian is approximated on a batch of 256 samples.

Note that the ARC-inc algorithm is proposed in [8]; TR-inc is a generalization of that to the trust region case; TR-full and ARC-full are proposed in [6]; TR-fixed and ARC-fixed are our method analyzed in this paper.

We train a VGG16 network\(^4\) on CIFAR10 dataset using the above algorithms and evaluate the performance according to training loss and test accuracy with respect to training time. All the experiments are run on a machine with 1 Titan Xp GPU. The result is reported in Figure 2.

Note that TR-full and ARC-full take 3500 seconds while TR-fixed and ARC-fixed only take 40 seconds for each epoch. Thus we omit the full gradient versions on the plot since there will be only one point there. This shows that calculating the full gradient for each update is too expensive for solving large-scale problems like deep learning. Apart from that we notice

\(^4\)Publically available at https://raw.githubusercontent.com/kuangliu/pytorch-cifar/master/models/vgg.py
both Cubic and TR on fixed batch size are faster than their growing batch size version, this validates our guess that sampling a fixed number of data to estimate gradient and Hessian is sufficient to make trust region and ARC work, and this scheme turns out to be more efficient than sample a growing batch over time. Moreover, in our task, the trust region algorithm is faster than ARC algorithm. However, we are not sure whether this phenomenon also applies to other tasks.

7. Conclusion

In this paper, we present a family of stochastic trust region method and stochastic cubic regularization method under inexact gradient and Hessian matrix. Furthermore, in order to reduce the computation cost for the function value of $f(x)$ in evaluating the role of $\rho$, we also present a sub-sample technique to estimate the function. We provide the theoretical analysis of convergence and iteration complexity and obtain that we keep the same order of iteration complexity but reduce the computation cost per iteration. We apply our proposed method to deep learning application which outperforms the previous second-order methods.

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Thus, we have $\sum_{i=1}^{n} v_{i} = 0$, and permutation and combination. For the case that $\mathcal{A}$ is a non-empty, uniformly random subset of $[m]$, we have

$$\mathbb{E}_{\mathcal{A}} \left\| \sum_{b \in \mathcal{A}} v_{b} \right\|^{2} = \mathbb{E}_{\mathcal{A}} \left( \sum_{b \in \mathcal{A}} \left\| v_{b} \right\|^{2} \right) + \frac{1}{C_{n}^{\mathcal{A}}} \sum_{i \in [n]} \left\langle v_{i}, \sum_{i \neq j}^{\mathcal{A} - 1} (A - 1) \sum_{i \neq j}^{\mathcal{A}} \right\rangle$$

$$= \frac{1}{n} \sum_{i=1}^{n} v_{i}^{2} + \frac{A (A - 1)}{n (n - 1)} \sum_{i \in [n]} \left\langle v_{i}, \sum_{i \neq j}^{\mathcal{A} - 1} v_{j} \right\rangle$$

$$= \frac{1}{n} \sum_{i=1}^{n} v_{i}^{2} + \frac{A (A - 1)}{n (n - 1)} \sum_{i \in [n]} \left\langle v_{i}, -v_{i} \right\rangle$$

$$= \frac{A (n - A)}{(n - 1)} \frac{1}{n} \sum_{i=1}^{n} v_{i}^{2} \leq A \| \langle A < n \rangle \frac{1}{n} \sum_{i=1}^{n} v_{i}^{2}.$$

Thus, we have $\mathbb{E}_{\mathcal{A}} \left\| \frac{1}{A} \sum_{b \in \mathcal{A}} v_{b} \right\|^{2} = \frac{1}{A^{2}} \mathbb{E}_{\mathcal{A}} \left\| \sum_{b \in \mathcal{A}} v_{b} \right\|^{2} \leq \frac{(A < n)}{A} \frac{1}{n} \sum_{i=1}^{n} v_{i}^{2}$.

**Proof of Lemma 4**

Proof. Let us define $X_{i} = f_{i}(x) - f(x), i \in S_{b}$. Based on the Assumption 3, we have

$$|X_{i}| = |f_{i}(x) - f(x)| \leq |f_{i}(x)| + |f(x)| \leq 2\kappa_{f} \Rightarrow |X_{i}|^{2} \leq 4\kappa_{f}^{2},$$
which satisfying the conditions $\mathbb{E}[|X_i|] = 0$, $\mathbb{E}[X_i^2] \leq 4\kappa_f^2$. Define the new variable

$$Z_1 = f_s(x) - f(x) = \frac{1}{|S_h|} \sum_{j \in |S_h|} (f_s(x) - f(x)),$$

$$Z_2 = \sum_{j \in |S_h|} (f_s(x) - f(x)).$$

Based on the Operator-Bernstein inequality [31], we give the probability about the condition in Assumption 2, we have $|Z_2| = |S_h| = |Z_1| \geq |S_h| \varepsilon_h \|s_k\|^2$, then $\Pr \left[ |Z_2| > |S_h| \varepsilon_h \Delta_h \right] \leq 2d \exp \left( \frac{|\varepsilon_h| |s_k| \|s_k\|^2}{4|S_h| |\varepsilon_h|^2} \right) \leq \delta$. Thus, the cardinality of $S_h$ should satisfy $|S_h| \geq \frac{16 \kappa_f^2 |s_k|^2}{\delta} \log \left( \frac{2d}{\delta} \right)$.

Furthermore, based on Lemma 1 and Assumption 4, we can also have the upper bounds with respect to the gradient and the Hessian of $h(x)$,

$$\|\nabla h(x) - \nabla f(x)\|^2 \leq \mathbb{I}(\frac{|S_h| < n}{|S_h|}) H_1, \|\nabla^2 h(x) - \nabla^2 f(x)\|^2 \leq \mathbb{I}(\frac{|S_h| < n}{|S_h|}) H_2.$$

\[ \Box \]

B. Proof for Stochastic Trust Region

Proof of Lemma 5

Proof. For the case $x_k \in S_{\nabla f}$, through adding and subtracting the term $\nabla f(x_k)$, we have the lower bound of $\|g(x_k)\|$,

$$\|g(x_k)\| = \|g(x_k) - \nabla f(x_k) + \nabla f(x_k)\| \geq \|\nabla f(x_k)\| - \|g(x_k) - \nabla f(x_k)\| \geq \varepsilon_{\nabla f} - \varepsilon_g,$$

where the last inequality is based on the approximation of $\nabla f(x_k)$ in Assumption 2. Following the lower bound on the decrease of the proximal quadratic function $m_k(s)$ from (2.40) in [32], we have

$$m_k(0) - m_k(s_k) \geq \frac{1}{2} \min \left\{ \Delta_k, \frac{\|g(x_k)\|}{\|B(x_k)\|} \right\} \geq \frac{1}{2} (\varepsilon_{\nabla f} - \varepsilon_g) \min \left\{ \Delta_k, \frac{\varepsilon_{\nabla f} - \varepsilon_g}{\kappa_H} \right\},$$

where the last inequality is from above inequality and the bound of $\|B(x_k)\|$ in Assumption 3.

For the case $x_k \in S_H$, through adding and subtracting the term $\nabla^2 f(x_k)$, we have

$$\frac{s_k^T B(x_k) s_k}{\|s_k\|} \leq s_k^T (B(x_k) - \nabla^2 f(x_k) + \nabla^2 f(x_k)) s_k \leq s_k^T (B(x_k) - \nabla^2 f(x_k)) s_k + s_k^T \nabla^2 f(x_k) s_k \leq \varepsilon_{B_i} - \varepsilon_B \leq \varepsilon_{B_i} - \varepsilon_B = - (\varepsilon_H - \varepsilon_B),$$

where the second inequality is based on the Assumption in 2 and Rayleigh quotient [3]. Using the definition of $m_k(s)$ in (2), we have

$$m_k(0) - m_k(s_k) = - \langle g(x_k), s_k \rangle - \frac{1}{2} s_k^T B(x_k) s_k \geq \frac{1}{2} \varepsilon_{B_i} s_k - \frac{1}{2} \varepsilon_{B_i} - \frac{1}{2} \varepsilon_{B_i} = \frac{1}{2} (\varepsilon_H - \varepsilon_B) \Delta_h^2,$$

where inequality $\odot$ is based on $\langle g(x_k), s_k \rangle \leq 0$, inequality $\odot$ follows from (23), inequality $\odot$ is based on $\|s_k\| = \Delta_k$. \[ \Box \]
Proof of Lemma 6

Proof. Consider the Taylor expansion for \( h(x_k + s_k) \) at \( x_k \),
\[
h(x_k + s_k) = h(x_k) + \langle s_k, \nabla h(x_k) \rangle + \frac{1}{2} s_k^T \nabla^2 h(\xi_k) s_k,
\]
where \( \xi_k \in [x_k, x_k + s_k] \). Based on the definition of \( m_k(s) \) in (2) and the Taylor expansion of the function \( h(x_k + s_k) \) at \( x_k \) above, we have
\[
m_k(0) - m_k(s_k) - (h(x_k) - h(x_k + s_k))
= -\langle g(x_k), s_k \rangle - \frac{1}{2} s_k^T B(x_k) s_k + \left( \langle \nabla h(x_k), s_k \rangle + \frac{1}{2} s_k^T \nabla^2 h(\xi_k) s_k \right)
= \langle \nabla h(x_k) - g(x_k), s_k \rangle + \frac{1}{2} s_k^T \left( \nabla^2 h(\xi_k) - B(x_k) \right) s_k
= \langle \nabla h(x_k) - g(x_k), s_k \rangle + \frac{1}{2} s_k^T \left( \nabla^2 h(\xi_k) - \nabla^2 f(\xi_k) + \nabla^2 f(\xi_k) - \nabla^2 f(x_k) + \nabla^2 f(x_k) - B(x_k) \right) s_k
\leq 2 \left( \|\nabla h(x_k) - g(x_k)\| \cdot \|s_k\| + \|\nabla f(x_k)\| \cdot \|s_k\| \right)
\leq 2 \left( \frac{1}{|S_h|} H_1 + \varepsilon_g \right) \|s_k\| + \frac{3}{2} \left( \frac{1}{|S_h|} H_2 + L_H \|s_k\| + \varepsilon_B \right) \|s_k\|^2
\leq 2 \left( \frac{1}{|S_h|} H_1 + \varepsilon_g \right) \Delta_k + \frac{3}{2} \left( \frac{1}{|S_h|} H_2 + L_H \Delta_k + \varepsilon_B \right) \Delta_k^2,
\]
where inequality (1) follows from the Holder’s inequality, inequality (2) is based on the approximation of \( \nabla f(x) \) and \( \nabla^2 f(x) \) in Assumption 2, and the Lipschitz continuous of Hessian matrix of \( f(x) \) in Assumption 1, inequality (3) follows from the constraint condition as in the objective (9).

Furthermore, for the case \( S_h = S_g \), the first term of (24) is equal to zero, then we have
\[
m_k(0) - m_k(s_k) - (h(x_k) - h(x_k + s_k)) \leq \frac{3}{2} \left( \frac{1}{|S_h|} H_2 + \Delta_k + \varepsilon_B \right) \Delta_k^2.
\]

Proof of Lemma 7

Proof. By setting \( \frac{1}{|S_h|} H_1 \leq \varepsilon_g \), and \( \frac{1}{|S_h|} H_2 \leq \varepsilon_B \), we consider two cases:

For the case \( \nabla f(x) > \varepsilon \nabla f(x) \), we assume that \( \Delta_k \leq \frac{\varepsilon f - \varepsilon_g}{2n} \), which is used for Lemma 5. Combine with (10) and the results in Lemma 5, Lemma 6, and \( \|s_k\|^2 < \Delta_k^2 \), we have
\[
1 - \rho \leq 2 \left( \frac{1}{|S_h|} H_1 + \varepsilon_g \right) \Delta_k + \frac{3}{2} \left( \frac{1}{|S_h|} H_2 + L_H \Delta_k + \varepsilon_B \right) \Delta_k^2 + 2\varepsilon h \Delta_k^2
= \frac{4\varepsilon_g + \frac{3}{2} \left( L_H \Delta_k + 2\varepsilon_B + \frac{\varepsilon h}{2} \right) \Delta_k}{\varepsilon_f - \varepsilon_g}.
\]
In order to have the lower bound radius \( \Delta_k \) such that \( 1 - \rho \leq 1 - \eta \), we consider the parameters’ setting:

- For the first term \( 4\varepsilon_g = \frac{1}{4} (1 - \eta) (\varepsilon_f - \varepsilon_g) \), then we define, \( \varepsilon_g = \frac{1}{16} (1 - \eta) (\varepsilon_f - \varepsilon_g) \).
• For the second term \( \frac{3}{2} (L_H \Delta_k + 2 \varepsilon_B + \varepsilon_h) \Delta_k \leq \frac{1}{4} (1 - \eta) (\varepsilon_{vf} - \varepsilon_g) \), as \( \varepsilon_B < 1, \varepsilon_h < 1, \varepsilon_{vf} - \varepsilon_g < 1 \), Thus, as long as

\[
\Delta_k \leq (\varepsilon_{vf} - \varepsilon_g) \min \left\{ \frac{1}{40} (1 - \eta), \sqrt{\frac{1}{12L_H} (1 - \eta)} \right\}
\]  

(26)

we can obtain that \( 1 - \rho \leq 1 - \eta \).

At the \( k \)-iteration, when the radius \( \Delta_k \) satisfy above condition, the update is successful iteration, as in Algorithm 1, the radius \( \Delta_k \) will increase by a factor \( r_2 \).

For the case \( \lambda_{\text{min}}(\nabla^2 f(x_k)) \leq -\varepsilon_H \), based on the results in Lemma 5 and Lemma 6, we have

\[
1 - \rho \leq \frac{3}{2} \left( \frac{L_H \Delta_k + 2 \varepsilon_B + \frac{\varepsilon_h}{2}}{\varepsilon_H - \varepsilon_B} \right) \Delta_k.
\]

In order to have the lower bound radius \( \Delta_k \) such that \( 1 - \rho \leq 1 - \eta \), we consider the parameters’ setting:

\[
\varepsilon_B = \varepsilon_h = \frac{1}{20} (1 - \eta) (\varepsilon_H - \varepsilon_B), \Delta_k \leq \frac{1}{6L_H} (1 - \eta) (\varepsilon_H - \varepsilon_B).
\]  

(27)

Based on above analysis and combine the assumption bound of \( \Delta \) at beginning, there exist a minimal radius

\[
\Delta_{\text{min}} = \min \{ \varepsilon_{vf} - \varepsilon_g, \varepsilon_H - \varepsilon_B \} \kappa_1, \kappa_1 = r_1 \min \left\{ \frac{1}{\kappa_H} \frac{1}{4} (1 - \eta), \sqrt{\frac{1}{12L_H} (1 - \eta)}, \sqrt{\frac{1}{10L_H} (1 - \eta)} \right\},
\]

where \( 0 < r_1 < 1 \). (Multiply \( r_1 \) due to the fact that (26) and (27) plus a small constant may lead to a successful iteration such that \( \Delta_k \) will be decreased the by a factor \( r_1 \).)

Proof of Theorem 1

**Proof.** Consider two index sets: \( S_{vf} \) and \( S_{\lambda} \), we separately analyze the number of successful iteration based on results in Lemma 5 and 7. And then add both of them to form the most number of successful iterations. Let \( f_{\text{low}} \) is the minimal value of objective, we have two kinds of successful iterations:

• Consider the case of \( \| \nabla f(x) \| \geq \varepsilon_{vf} \), if \( k \)th iteration is successful, then we have

\[
f(x_k) - f(x_k + s_k) \geq \eta (m_k(0) - m_k(s_k)) \geq \frac{1}{2} \eta (\varepsilon_{vf} - \varepsilon_g) \min \left\{ \Delta_k, \frac{\varepsilon_{vf} - \varepsilon_g}{\kappa_H} \right\} \geq \frac{1}{2} \eta (\varepsilon_{vf} - \varepsilon_g) \Delta_{\text{min}}.
\]

where the last two inequalities are based on Lemma 5 and Lemma 7. Let \( T_1 \) denotes the number of successful iterations for \( k \in S_{vf} \). Applying above inequality, we can obtain

\[
f(x_0) - f_{\text{low}} \geq \frac{1}{2} T_1 (\varepsilon_{vf} - \varepsilon_g) \eta \Delta_{\text{min}}.
\]

where \( f_{\text{low}} \) is lower bound of the objective.

• Consider the case of \( \lambda_{\text{min}}(\nabla^2 f(x_k)) \leq -\varepsilon_H \), if \( k \)th iteration is successful, based on Lemma 5 and Lemma 7, we have

\[
f(x_k) - f(x_k + s_k) \geq \eta (m_k(0) - m_k(s_k)) \geq \frac{1}{2} \eta (\varepsilon_H - \varepsilon_B) \Delta_{\text{min}}^2.
\]

Let \( T_2 \) denote the number of successful iteration for \( k \in S_H \), we obtain

\[
f(x_0) - f_{\text{low}} \geq \frac{1}{2} T_2 \eta (\varepsilon_H - \varepsilon_B) \Delta_{\text{min}}^2.
\]
Let $T_{\text{succ}}$ denotes the number of successful iterations, combining above iteration and $\Delta_{\text{min}}$ in Lemma 7, we have

$$T_{\text{succ}} \leq T_1 + T_2 \leq \frac{2 (f(x_0) - f_{\text{low}})}{(\varepsilon f - \varepsilon g) \eta \Delta_{\text{min}}} + \frac{2 (f(x_0) - f_{\text{low}})}{\eta (H - \varepsilon B) \Delta_{\text{min}}} + \frac{2 (f(x_0) - f_{\text{low}})}{\eta (H - \varepsilon B) \Delta_{\text{min}}^2}
$$

$$\leq \frac{2 (f(x_0) - f_{\text{low}})}{(\varepsilon f - \varepsilon g) \eta \kappa_1 (\varepsilon f - \varepsilon g)} + \frac{2 (f(x_0) - f_{\text{low}})}{(H - \varepsilon B) \eta (H - \varepsilon B)^2}
$$

$$\leq \kappa_3 \max \left\{ (\varepsilon f - \varepsilon g)^{-2}, (H - \varepsilon B)^{-3} \right\},$$

where $\kappa_3 = 2 (f(x_0) - f_{\text{low}}) \max \left\{ 1/(\eta \kappa_1), 1/(\eta \kappa_2^2) \right\}$.

Let $T_{\text{unsucc}}$ denotes the number of unsuccessful iteration, we have $r_1 \Delta_k \leq \Delta_{k+1}$; Let $T_{\text{succ}}$ denotes the number of successful iteration, we have $r_2 \Delta_k \leq \Delta_{k+1}$. Thus, we inductively deduce,

$$\Delta_{\text{min}} T_{\text{succ}}^r T_{\text{unsucc}} \leq \Delta_{\text{max}},$$

where $\Delta_{\text{max}}$ is defined in (11) and $\Delta_{\text{min}}$ is defined in Algorithm 1. Thus, the number of unsuccessful index set is at most

$$T_{\text{unsucc}} \geq \frac{1}{\log r_1} \left( \log \left( \frac{\Delta_{\text{max}}}{\Delta_{\text{min}}} \right) - T_{\text{succ}} \log r_2 \right).$$

Combine with the successful iteration, we can obtain the total iteration complexity,

$$T_{\text{succ}} + T_{\text{unsucc}} = T_{\text{succ}} \left( 1 + \log \left( \frac{\Delta_{\text{max}}}{\Delta_{\text{min}}} \right) - \log \left( \frac{\Delta_{\text{max}}}{\Delta_{\text{min}}} \right) \right) \frac{1}{\log r_1}
$$

$$= O \left( \max \left\{ (\varepsilon f - \varepsilon g)^{-2}, (H - \varepsilon B)^{-3} \right\} \right).$$

\[\square\]

C. Proof for Stochastic ARC

Proof of Lemma 8

Proof. Based on the Cauchy-Schwarz inequality and the definition of $m_k(x)$, we have

$$p_k (s_k^C) \leq p_k (0) + \langle g(x_k), -\alpha g(x_k) \rangle + \frac{1}{2} \|B(x_k)(-\alpha g(x_k))\| - \alpha g(x_k)\| + \frac{\sigma_k}{3} \| -\alpha g(x_k)\|^3
$$

$$\leq p_k (0) - \|g(x_k)\|^2 \alpha + \frac{1}{2} \|B(x_k)\| \|g(x_k)\|^2 \alpha^2 + \frac{\sigma_k}{3} \|g(x_k)\|^3 \alpha^3
$$

$$= p_k (0) + q(\alpha),$$

where

$$q(\alpha) = - \|g(x_k)\|^2 \alpha + \frac{1}{2} \|B_k\| \|g(x_k)\|^2 \alpha^2 + \frac{\sigma_k}{3} \|g(x_k)\|^3 \alpha^3
$$

$$= - \|g_k\|^2 \alpha + \frac{1}{2} \|B_k\| \|g_k\|^2 \alpha^2 + \frac{\sigma_k}{3} \|g_k\|^3 \alpha^3
$$

$$= \|g_k\|^2 \left( -\alpha + \frac{1}{2} \|B_k\| \alpha^2 + \frac{\sigma_k}{3} \|g_k\|^3 \alpha^3 \right).$$

For simplicity, we use $g_k = g(x_k)$ and $B_k = B(x_k)$ instead.

In order to show $p_k (s_k^C) \leq f(x_k)$, we should to check that the minimal value of $q(\alpha)$ is negative or not. That is, if the minimal value of $q(\alpha)$ is negative, there exist a $\alpha_0$ such that $q(\alpha_0) \leq 0$, and $p_k (s_k^C) \leq f(x_k) + q(\alpha)$ for all $\alpha > 0$, then, we can obtain $p_k (s_k^C) \leq f(x_k)$). Now, consider the gradient of $p(\alpha)$,

$$\nabla q(\alpha) = \|g_k\|^2 (-1 + \|B_k\| \alpha + \sigma_k \|g_k\|^2 \alpha^2).$$

Let $\nabla q(\alpha) = 0$, we have two solutions,
\[ \alpha_1 = -\|B_k\| - \sqrt{\|B_k\|^2 + 4\sigma_k \|g_k\|} < 0, \quad \alpha_2 = \frac{\|B_k\| + \sqrt{\|B_k\|^2 + 4\sigma_k \|g_k\|}}{2\sigma_k \|g_k\|}. \]

Because we require \( \alpha > 0 \), we do not need to consider \( \alpha_1 < 0 \). Thus, we obtain the geometrical character,

\[
\nabla q(\alpha) = \begin{cases} 
+ & \alpha > \alpha_2 \\
0 & \alpha = \alpha_2 \\
- & \alpha_2 > \alpha \geq 0
\end{cases} \quad \Rightarrow q(\alpha) = \begin{cases} 
\alpha & \alpha \geq \alpha_2 \\
- \alpha_2 & \alpha_2 > \alpha \geq 0
\end{cases},
\]

where we use “+” and “−” to denote the positive and negative of \( \nabla q(\alpha) \), respectively, use “↗” and “↘” to denote the increasing and decreasing functions, respective. From above description, we can obtain that \( \alpha_2 \) is the minimal solution. Putting \( \alpha_2 \) into \( q(\alpha) \), we have

\[
q(\alpha_2) = \|g_k\|^2 \left(-\alpha_2 + \frac{1}{2} \|B_k\| \alpha_2^2 + \frac{1}{3} \sigma_k \|g_k\| \alpha_2^3\right)
\]

\[
= \|g_k\| \left(\alpha_2 \|g_k\|-1 + \|B_k\| \alpha_2 + \sigma_k \|g_k\| \alpha_2^2\right) - \frac{1}{2} \|B_k\| \|g_k\| \alpha_2^2 - \frac{2}{3} \sigma_k \|g_k\| \alpha_2^3
\]

\[
= -\|g_k\|^2 \alpha_2^2 \left(\frac{1}{2} \|B_k\| + \frac{2}{3} \sigma_k \|g_k\| \alpha_2\right)
\]

\[
= \frac{1}{6} \|g_k\|^2 \alpha_2^2 \left(\frac{1}{6} \|B_k\| + \frac{1}{3} \sqrt{\|B_k\|^2 + 4\sigma_k \|g_k\|}\right)
\]

\[
\leq \frac{1}{6} \|g_k\|^2 \alpha_2^2 \left(\|B_k\| + \sqrt{\|B_k\|^2 + 4\sigma_k \|g_k\|}\right),
\]

where the third equality is from

\[
\|g_k\| \left(-1 + \|B_k\| \alpha_2 + \sigma_k \|g_k\| \alpha_2^2\right) = 0.
\]

Because \( \alpha_2 \) can also be expressed as

\[
\alpha_2 = \frac{-\|B_k\| + \sqrt{\|B_k\|^2 + 4\sigma_k \|g_k\|}}{2\sigma_k \|g_k\|} = \frac{2}{\|B_k\| + \sqrt{\|B_k\|^2 + 4\sigma_k \|g_k\|}},
\]

we can obtain

\[
q(\alpha_2) \leq -\frac{1}{3} \|g_k\|^2 \alpha_2 = -\frac{1}{3} \|g_k\|^2 \left(\frac{2}{\|B_k\| + \sqrt{\|B_k\|^2 + 4\sigma_k \|g_k\|}\right)
\]

\[
\leq -\frac{1}{10} \|g_k\| \min \left\{\frac{\|g_k\|}{\|B_k\|}, \sqrt{\frac{\|g_k\|}{\sigma_k}}\right\},
\]

where inequality is based on difference value between \( \|B_k\|^2 \) and \( \sigma_k \|g_k\| \), and \( \frac{1}{10} \) approximates \( \frac{1}{0.103} \approx 10 \) which is close to \( \frac{1}{10} \).

Furthermore, we can also see that Based on Lemma 8 and the definition of \( p_k(x) \) in (15), we have

\[
p_k(s_k) - p(0) = \langle s_k, g(x_k) \rangle + \frac{1}{2} s_k^T B(x_k) s_k + \frac{1}{3} \sigma_k \|s_k\|^3 \leq 0.
\]

Using Cauchy-Schwarz inequality, we have

\[
\frac{1}{3} \sigma_k \|s_k\|^3 \leq -\langle s_k, g(x_k) \rangle - \frac{1}{2} s_k^T B(x_k) s_k \leq \|s_k\| \|g(x_k)\| + \frac{1}{2} \|B(x_k)\| \|s_k\|^2.
\]

Arranging the position of \( \|s_k\| \), we have

\[
\|s_k\| \left(\frac{1}{3} \sigma_k \|s_k\|^2 - \frac{1}{2} \|B(x_k)\| \|s_k\| - \|g(x_k)\|\right) \leq 0.
\]

Because \( \|s_k\| \) is positive, in order to satisfy above equality, \( \|s_k\| \) is upper bounded by,
where the first inequality follows from the solution of a quadratic function in (28), and the second inequality is based on the values between $\|B(x_k)\|$ and $\sigma_k \|g(x_k)\|$.

**Proof of Lemma 9**

**Proof.** Based on the definition of $p_k(s)$ in (15), we have

\[
p(0) - p_k(s_k) = - \langle g(x_k), s_k \rangle - \frac{1}{2} s_k^T B(x_k) s_k - \frac{\sigma_k}{3} \|s_k\|^3
\]

\[
= - \langle g(x_k), s_k \rangle - s_k^T B(x_k) s_k - \frac{\sigma_k}{3} \|s_k\|^3 + \frac{1}{2} s_k^T B(x_k) s_k + \frac{2\sigma_k}{3} \|s_k\|^3
\]

\[
\geq \frac{1}{2} s_k^T B(x_k) s_k + \frac{2\sigma_k}{3} \|s_k\|^3
\]

where $\oplus$ and $\otimes$ follows from (17) and (18), respectively.

Consider the lower bound of $\|g(x_k)\|$: Firstly, based on the definition of $m_k(s)$, for simplicity, we use $g_k = g(x_k)$ and $B_k = B(x_k)$ instead, we have

\[
\|g(x_{k+1})\| = \|g(x_{k+1}) - \nabla p_k(s) + \nabla p_k(s)\|
\]

\[
\leq \|g(x_k)\| + \int_0^1 \left( B(x_k + \tau s_k) s_k d\tau - (g(x_k) + B(x_k) s_k + \sigma_k \|s_k\| s_k) \right) + \|\nabla p_k(s_k)\|
\]

\[
\leq \|g(x_k)\| + \int_0^1 \left( B(x_k + \tau s_k) - H(x_k + \tau s_k) \right) s_k d\tau + \|H(x_k + \tau s_k) s_k - H(x_k) s_k\|
\]

\[
< \|g(x_k)\| + \|H(x_k) s_k - B_k s_k\| + \sigma_k \|s_k\|^2 + \|\nabla p_k(s_k)\|
\]

\[
\leq \|g(x_k)\| + 2\|H(x_k) s_k + (L_H + \sigma_k) \|s_k\|^2 + \theta_k \|g(x_k)\|
\]

\[
= 2\varepsilon_B \|s_k\| + (L_H + \sigma_k) \|s_k\|^2 + \theta_k \|g(x_k)\|
\]

where inequality $\oplus$ is based on the triangle inequality and the Taylor expansion of $g(x)$, equality $\oplus$ is obtained by adding and subtracting the term of $H(x_k) s_k$ and $H(x_k + \tau s_k) s_k$, and triangle inequality; equality $\otimes$ follows from Assumption 1 and 2 and the condition in (19). Secondly, consider $g(x_k)$, we have

\[
\|g(x_k)\|
\]

\[
= \|g(x_k) - \nabla f(x_k) + \nabla f(x_k) + \nabla f(x_k + s_k) - \nabla f(x_k + s_k) + \nabla f(x_k + s_k) - g(x_k + 1) + g(x_k + 1)\|
\]

\[
\leq \|g(x_k) - \nabla f(x_k)\| + \|\nabla f(x_k) - \nabla f(x_k + s_k)\| + \|\nabla f(x_k + s_k) - g(x_k + 1)\| + \|g(x_k + 1)\|
\]

\[
\leq 2\varepsilon_g + L_{\nabla f} \|s_k\| + \|g(x_{k+1})\|
\]

where the first and second inequality are based on the triangle inequality, Lipschitz continuity of gradient $\nabla f(x_k)$ in Assumption 1 and 2.

Finally, replace the term $\|\nabla g(x_k)\|$, we have

\[
(1 - \theta_k) \|g(x_{k+1})\| \leq 2\varepsilon_B \|s_k\| + (L_H + \sigma_k) \|s_k\|^2 + 2\theta_k \varepsilon_g + \theta_k L_{\nabla f} \|s_k\|.
\]

Consider the definition of $\theta_k \leq \zeta_0 \min \{1, \|s_k\|\}$, $\zeta_0 < 1$, we analysis the bound from different rang of $\|s_k\|

- For the case of $\|s_k\| \geq 1$, we have

\[
(1 - \theta_k) \|g(x_{k+1})\| \leq 2\varepsilon_B \|s_k\|^2 + (L_H + \sigma_k) \|s_k\|^2 + 2\kappa_0 \|s_k\|\varepsilon_g + \kappa_0 L_{\nabla f} \|s_k\|^2.
\]
For the case of \( \|s_k\| \leq 1 \), based on the assumption on \( \varepsilon_g \) and \( \varepsilon_g \), that is
\[
\varepsilon_B \leq \zeta_1 (\varepsilon_{vf} - \varepsilon_g) \leq \zeta_1 \|g(x_{k+1})\|, \\
\varepsilon_g \leq \zeta_2 (\varepsilon_{vf} - \varepsilon_g) \leq \zeta_2 \|g(x_{k+1})\|,
\]
where \( \zeta_1, \zeta_2 < 1 \). We have
\[
(1 - \theta_k) \|g(x_{k+1})\| \leq 2\varepsilon_B + (L_H + \sigma_k) \|s_k\|^2 + 2\kappa_0 \varepsilon_B + \kappa_0 L_{vf} \|s_k\|^2 \\
\leq (2\zeta_1 + 2\kappa_0 \zeta_2) \|g(x_{k+1})\| + (L_H + \sigma_k + \kappa_0 L_{vf}) \|s_k\|^2.
\]
Thus, in all, we can obtain
\[
\|g(x_{k+1})\| \leq \kappa_s \|s_k\|^2, \kappa_s = \min \left\{ \frac{2\varepsilon_B + (L_H + \sigma_k) + 2\kappa_0 \varepsilon_B + \kappa_0 L_{vf}}{(1 - \theta_k)}, \frac{L_H + \sigma_k + \kappa_0 L_{vf}}{1 - \theta_k - \zeta_1 - \zeta_2} \right\}.
\]

**Proof of Lemma 11**

*Proof.* We assume that \( \sigma_k \geq \frac{9}{2} L_H \) and \( \sigma_k \geq \frac{\kappa_0^2}{(\varepsilon_{vf} - \varepsilon_g)} \), which are used for Lemma 8 and Lemma 10. By setting \( \frac{\|S_k\|_{\leq n}}{|S_n|} H_1 \leq \varepsilon_g \), and \( \frac{\|S_k\|_{\leq n}}{|S_n|} H_2 \leq \varepsilon_B \), we consider two cases:

1. For the case \( \nabla f(x_k) \geq \varepsilon_{vf} \): Firstly, we consider \( g(x_k) \). Through adding and subtracting the term \( \nabla f(x_k) \), we have the lower bound of \( \|g(x)\| \),
   \[
   \|g(x_k)\| = \|g(x_k) - \nabla f(x_k) + \nabla f(x_k)\| \\
   \geq \|\nabla f(x_k)\| - \|g(x_k) - \nabla f(x_k)\| \\
   \geq \varepsilon_{vf} - \varepsilon_g,
   \]
   where the last inequality is based on the approximation of \( \nabla f(x_k) \) in Assumption 2. Secondly, because of
   \[
   \sigma_k \geq \frac{\kappa_0^2}{(\varepsilon_{vf} - \varepsilon_g)} \geq \frac{\|B(x_k)\|^2}{\|g(x_k)\|} \Rightarrow \|B(x_k)\| \leq \|g(x_k)\| \leq \frac{\|g(x_k)\|}{\sigma_k},
   \]
   Combing with the upper bound of \( s_k \) in Lemma 8, we have \( \|s_k\| \leq \frac{11}{4} \sqrt{\|g(x_k)\|/\sigma_k} \). Finally, based on equality (16), Lemma 8 and Lemma 10, we have
   \[
   1 - \rho \leq 10 \frac{4\varepsilon_g \|s_k\| + 3\varepsilon_B \|s_k\|^2 + 2\varepsilon_k \|s_k\|^2}{\|g_k\| \min \left\{ \|g_k\|/\|B_k\|, \sqrt{\|g_k\|/\sigma_k} \right\}}
   \]
   In order to ensure that there exist a lower bound of \( \sigma \) such that satisfying \( 1 - \rho \leq 1 - \eta \). Combing with the upper bound of \( s_k \) in Lemma 8, if \( \varepsilon_g = \frac{1}{220} (1 - \eta) (\varepsilon_{vf} - \varepsilon_g) \), \( \sigma_k \geq \frac{1}{\varepsilon_{vf} - \varepsilon_g} \left( \frac{304 (3\varepsilon_B + 2\varepsilon_k)^2}{(1 - \eta)} \right) \), we have
   \[
   10 \frac{4\varepsilon_g \|s_k\| + (3\varepsilon_B + 2\varepsilon_k) \|s_k\|^2}{\|g_k\| \min \left\{ \|g_k\|/\|B_k\|, \sqrt{\|g_k\|/\sigma_k} \right\}} \leq 10 \frac{11\varepsilon_g \sqrt{\|g(x_k)\|/\sigma_k} + (3\varepsilon_B + \varepsilon_k) \left( \frac{11}{2} \sqrt{\|g(x_k)\|/\sigma_k} \right)^2 \|g_k\|/\sigma_k \|g_k\|/\sigma_k \}{\|g_k\|/\sigma_k \|g_k\|/\sigma_k \} \\
   \leq \frac{110\varepsilon_g}{\|g_k\|} + 76 (3\varepsilon_B + 2\varepsilon_k) \sqrt{\|g(x_k)\|/\sigma_k} \|g_k\|/\sigma_k \|g_k\|/\sigma_k \} = \frac{110\varepsilon_g}{\|g_k\|} + \frac{76 (3\varepsilon_B + 2\varepsilon_k)}{\sqrt{\|g_k\|/\sigma_k}} \\
   \leq \frac{110\varepsilon_g}{(\varepsilon_{vf} - \varepsilon_g)} \sqrt{\varepsilon_{vf} - \varepsilon_g} \sigma_k \leq \frac{2}{2} (1 - \eta).
   \]
   Thus, we can see that if \( \sigma_{\text{max}} = \frac{1}{(\varepsilon_{vf} - \varepsilon_g)} r_2 \left\{ \frac{\varepsilon_B}{2}, \left( \frac{304 (3\varepsilon_B + 2\varepsilon_k)^2}{(1 - \eta)} + \frac{2}{2} (\varepsilon_{vf} - \varepsilon_g) L_H \right) \right\} \), \( r_2 > 1 \), we can obtain that \( 1 - \rho \leq 1 - \eta \).
For the case $\lambda_{\min}(\nabla^2 f(x)) \leq -\varepsilon_H$: Firstly, based on the Rayleigh quotient [3] that if $H(x)$ is symmetric and the vector $s \neq 0$, then, we have

$$\frac{s_k^T B(x_k) s_k}{\|s_k\|^2} \leq \frac{s_k^T \left( B(x_k) - \nabla^2 f(x_k) + \nabla^2 f(x_k) \right) s_k}{\|s_k\|^2} \leq \frac{s_k^T \nabla^2 f(x_k) s_k}{\|s_k\|^2} + \|\nabla^2 f(x_k) - B(x_k)\|$$

where $\oplus$ is based on the triangle inequality, $\odot$ follows from the Rayleigh quotient [3] and approximation Assumption (3).

Secondly, based on $s_k^T B_k s_k + \sigma_k \|s_k\|^3 \geq 0$, we have $\sigma_k \|s_k\| \geq -\frac{s_k^T B_k s_k}{\|s_k\|^3} \geq (\varepsilon_H - \varepsilon_B)$. Thirdly, based on equality (16), and Lemma 8, Lemma 10, we have

$$1 - \rho \leq \frac{2\varepsilon_h \|s_k\|^2}{\sigma_k \|s_k\|^3} + \frac{3\varepsilon_B \|s_k\|^2}{\sigma_k \|s_k\|^3} = \frac{12\varepsilon_h}{\sigma_k \|s_k\|^3} + 18\varepsilon_B \frac{1}{\sigma_k \|s_k\|^3} \leq \frac{12\varepsilon_h}{(\varepsilon_H - \varepsilon_B)} \leq 18\varepsilon_B \frac{1}{\varepsilon_B - \varepsilon_B},$$

In order to have the lower bound radius $\Delta_k$ such that $1 - \rho \leq 1 - \eta$, we consider the parameters’ setting:

- For the first term, then we define, $\varepsilon_B = \frac{1}{10}(1 - \eta)(\varepsilon_H - \varepsilon_B)$.
- For the second term, we define $\varepsilon_h \leq \frac{1}{20}(\varepsilon_H - \varepsilon_B)(1 - \eta)$.

Thus, we can see that if $\sigma_{\max} = \frac{9}{2}r_2 L_H$, $r_2 > 1$, we can obtain that $1 - \rho \leq 1 - \eta$.

All in all, there is a large $\sigma$ such that lead to the successful iteration, that is

$$\sigma_{\max} = \max\{\sigma_{\max 1}, \sigma_{\max 2}\}.$$

\[\square\]

**Proof of Theorem 2**

**Proof.** We consider two kinds of iteration complexity:

- For the case of $\nabla f(x_k) \geq \varepsilon \nabla f$, based on Lemma 8, if $k$-th iteration is successful, we obtain

$$f(x_k) - f(x_{k+1}) \geq \eta \left( m(0) - m(k(s_k)) \right) \geq \frac{1}{10} \eta (\varepsilon \nabla f - \varepsilon_g) \sqrt{\frac{\varepsilon \nabla f - \varepsilon_g}{\sigma_{\max 1}}} = \frac{1}{10} \eta (\varepsilon \nabla f - \varepsilon_g)^2 \kappa_4^{-1/2},$$

where inequality follow from the inequality $\oplus$ is from $\|f(x_k)\| \geq \varepsilon \nabla f - \varepsilon_g$ and $\sigma_{\max 1}$ in (21). Let $T_3$ is the number of successful iteration, we obtain

$$f(x_0) - f_{\text{low}} \geq \frac{1}{10} T_3 \eta (\varepsilon \nabla f - \varepsilon_g)^2 \kappa_4^{-1/2}.$$

- For the case of $\lambda_{\min}(\nabla^2 f(x_k)) \leq -\varepsilon_H$, based on Lemma 9, if $k$-th iteration is successful, we have

$$f(x_k) - f(x_k + s_k) \geq \frac{\sigma_k}{6} \eta \|s_k\|^3 = \frac{1}{6\sigma_k} \eta \sigma_k^3 \|s_k\|^3 \geq -\frac{1}{6\sigma_{\max 2}} \eta \left( \frac{s_k^T B_k s_k}{\|s_k\|^2} \right)^3 \geq \frac{1}{6\sigma_{\max 2}} \eta (\varepsilon_H - \varepsilon_B)^3,$$
where inequalities are based on the condition of \( s_k \) that satisfies (18), that is \( \sigma_k \| s_k \| \geq \frac{\sigma_k B(x_k) s_k}{\| s_k \|} \), and \( \sigma_{\text{max}2} \) in (21). Let \( T_4 \) be the number of successful iteration, then we obtain
\[
f(x_0) - f_{\text{low}} \geq \frac{1}{6\sigma_{\text{max}2}} T_4 \eta (\varepsilon_H - \varepsilon_B)^3.
\]
Thus the total number of success iteration is
\[
T_3 + T_4 = \kappa_5 \max \left\{ (\varepsilon \nabla f - \varepsilon g)^{-2}, (\varepsilon_H - \varepsilon_B)^{-3} \right\}
\]
where \( \kappa_5 = (f(x_0) - f_{\text{low}}) \max \left\{ 5/\left( \eta \kappa_4^{-1/2} \right), 6\sigma_{\text{max}2}/\eta \right\} \).

- For the case that condition (17)-(19) and \( g(x_k) > (\varepsilon \nabla f - \varepsilon g) \) hold, we have
\[
f(x_k) - f(x_k + s_k) \geq \frac{\sigma_k \| s_k \|}{6} \geq \frac{\sigma_{\text{min}} \kappa_s^{-3/2}}{6} \| g(x_k) \|^{3/2} \geq \frac{\sigma_{\text{min}} \kappa_s^{-3/2}}{6} (\varepsilon \nabla f - \varepsilon g)^{3/2}
\]
where inequality are based on Lemma 9 and \( \sigma_{\text{min}} \) in (21). Let \( T_5 \) is the number of the successful iterations, we obtain
\[
f(x_0) - f_{\text{low}} \geq \frac{\sigma_{\text{min}} \kappa_s^{-3/2}}{6} T_5 \eta (\varepsilon \nabla f - \varepsilon g)^{3/2}.
\]
The total number of success iteration for such case is
\[
T_3 + T_5 = \kappa_6 \max \left\{ (\varepsilon \nabla f - \varepsilon g)^{-3/2}, (\varepsilon_H - \varepsilon_B)^{-3} \right\},
\]
where \( \kappa_6 = (f(x_0) - f_{\text{low}}) \max \left\{ 6\kappa_s^{3/2} / (\eta \sigma_{\text{min}}), 6\sigma_{\text{max}2}/\eta \right\} \).