The existence and decay rates of strong solutions for Navier-Stokes Equations in Bessel-potential spaces

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Abstract

In this paper, we prove some results on the existence and decay properties of high order derivatives in time and space variables for local and global solutions of the Cauchy problem for the Navier-Stokes equations in Bessel-potential spaces.

§1. Introduction

This paper studies some qualitative properties of mild solutions to the Cauchy problem of the incompressible Navier-Stokes equations (NSE) in the whole space $\mathbb{R}^d$ ($d \geq 2$)

$$
\begin{cases}
\partial_t u = \Delta u - \nabla \cdot (u \otimes u) - \nabla p, \\
\nabla \cdot u = 0, \\
u(0, x) = u_0.
\end{cases}
$$

The unknown quantities are the velocity $u(t, x) = (u_1(t, x), \ldots, u_d(t, x))$ of the fluid element at time $t$ and position $x$ and the pressure $p(t, x)$.

There is an extensive literature on the existence and decay rate of strong solutions of the Cauchy problem for NSE.

The existence of strong solutions: The global well-posedness of strong solutions for small initial data in the critical Sobolev space $\dot{H}^{\frac{d}{2}}(\mathbb{R}^3)$ is due to Fujita and Kato [8], also in [5], Chemin has proved the case of $H^s(\mathbb{R}^3), (s > 1/2)$. In [14], Kato has proved the case of the Lebesgue space $L^3(\mathbb{R}^3)$. In [15], Koch and Tataru have proved the case of the space $BMO^{-1}$ (see also [4, 7]).

Recently, the authors of this article have considered NSE in mixed-norm Sobolev-Lorentz spaces and Sobolev-Fourier-Lorentz spaces, see [16] and [17] respectively. In [19], we prove that NSE are well-posed when the initial datum belongs to the Sobolev spaces $\dot{H}_{\dot{p}}^{\frac{d}{2p}-1}(\mathbb{R}^d)$ with $(1 < p \leq d)$. In [18],

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we considered the initial value problem for the non stationary Navier-Stokes equations on torus $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$ and showed that NSE are well-posed when the initial datum belongs to Sobolev spaces $V_\alpha := D((-\Delta)^{\alpha/2}$ with $\frac{1}{2} < \alpha < \frac{3}{2}$.

*The time-decay of strong solutions:* In [23], Maria E. Schonbek established the decay of the homogeneous $H^m$ norms for solutions to NSE in the two dimension. She showed that if $u$ is a solution to NSE with an arbitrary datum $u_0 \in H^m(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$ with $m \geq 3$ then for $t \geq 1, |\alpha| \leq m$ we have

$$\|D_x^\alpha u\|^2_2 \leq C_\alpha(t + 1)^{-(|\alpha|+1)} \text{ and } \|D_x^\alpha u\|_\infty \leq C_\alpha(t + 1)^{-(|\alpha|+\frac{1}{2})},$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d), |\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_d$ and $D_x^\alpha$ denotes $\partial^{|\alpha|}/\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\ldots\partial x_d^{\alpha_d}$. Further, Zhi-Min Chen [3] showed that if $u_0 \in L^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d), (d \leq p < \infty)$ and $\|u_0\|_1 + \|u_0\|_p$ is small enough then there is a unique solution $u \in BC((0, \infty); L^1 \cap L^p)$, which satisfies the decay property

$$\sup_{t > 0} t^\frac{2}{d}(\|u\|_\infty + t^\frac{1}{2} \sum_{|\alpha| = 1} \|D_x^\alpha u\|_\infty + t^\frac{1}{2} \sum_{|\alpha| = 2} \|D_x^\alpha u\|_d) < \infty.$$

Kato [14] studied strong solutions in the spaces $L^q(\mathbb{R}^d)$ by applying the $L^1 - L^p$ estimates for the semigroup generated by the Stokes operator. He showed that there is $T > 0$ and a unique solution $u$, which satisfies

$$t^{\frac{1}{2}(1-\frac{d}{q})}u \in BC([0, T); L^q), \text{ for } d \leq q \leq \infty,$$

$$t^{\frac{1}{2}(2-\frac{d}{q})}\nabla_x u \in BC([0, T); L^q), \text{ for } d \leq q \leq \infty,$$

as $u_0 \in L^d(\mathbb{R}^d)$. He showed that $T = \infty$ if $\|u_0\|_{L^q(\mathbb{R}^d)}$ is small enough.

Cannone [6] generalized the results of Kato. He showed that if $u_0 \in L^d$ and $\|u_0\|_{B^{-1,\infty}_q, L^q}\cdot (q > d)$ is small enough then there is a unique solution $u$, which satisfies

$$t^{\frac{1}{2}(1-\frac{d}{q})}u \in BC([0, \infty); L^q), \text{ for } q \geq d.$$

Note that the condition on the initial data of Cannone is weaker than that of Kato. In 2002, Cheng He and Ling Hsiao [13] extended the results of Kato. They estimated the decay rates of higher order derivatives in time and space variables for the strong solution to NSE with initial data in $L^d(\mathbb{R}^d)$. They showed that for any integer $l$ there exists a positive constant $C_{l,d}$ which depends only on $l, d$ such that if $\|u_0\|_{L^d(\mathbb{R}^d)} \leq C_{l,d}$ then NSE has a unique mild solution $u$ satisfying

$$t^{\frac{1}{2}(|\alpha| + 2\alpha_0 - \frac{d}{q})}D_x^\alpha D_t^{2\alpha_0} u \in BC([0, \infty); L^q), \text{ for } q \geq d, |\alpha| + 2\alpha_0 \leq l,$$

$$t^{\frac{1}{2}(|\alpha| + 2\alpha_0 - \frac{d}{q})}D_x^\alpha p \in BC([0, \infty); L^q), \text{ for } q \geq d, |\alpha| + 1 \leq l,$$
where \( D_t^{\alpha_0} \) stands for \( \partial_t^{\alpha_0} = \partial_t^{\alpha_0} / \partial t^{\alpha_0} \).

In 2005, Okihiro Sawada \(^{22}\) obtained the decay rate of solutions to NSE with initial data in \( \dot{H}^{\frac{d}{2} - 1}(\mathbb{R}^d) \). He showed that every mild solution in the class
\[
u \in BC([0, T); \dot{H}^{\frac{d}{2} - 1}) \text{ and } t^{\frac{1}{2}(\frac{d}{2} - \frac{1}{p})} \nu \in BC([0, T); \dot{H}^{\frac{d}{2} - 1}_p),
\]
for some \( T > 0 \) and \( p \in (2, \infty] \) satisfies
\[
\|\nu(t)\|_{\dot{H}^{\frac{d}{2} - 1}_q} \leq K_1(K_2\tilde{\alpha}^\frac{d}{2} - 1) \nu \text{ for } q \geq 2, \alpha > \frac{d}{2} - 1, t \in (0, T], \text{ and } \tilde{\alpha} := \alpha + 1 - \frac{d}{q}.
\]
where constants \( K_1 \) and \( K_2 \) depend only on \( d, p, M_1, \) and \( M_2 \) with
\[
M_1 = \sup_{0 < t < T} \|\nu(t)\|_{\dot{H}^{\frac{d}{2} - 1}} \text{ and } M_2 = \sup_{0 < t < T} t^{\frac{1}{2}(\frac{d}{2} - \frac{1}{p})} \|\nu(t)\|_{\dot{H}^{\frac{d}{2} - 1}_p}.
\]

Finally, we recall a known result on globally-defined mild solutions to NSE evolving from possibly large data.

**Theorem 1.** (Decay in \( \mathbb{R}^3 \)). Let \( u \in C([0, +\infty), X) \) be a global solution to NSE for some divergence-free \( u_0 \in X \), where \( X \) is either \( \dot{H}^\frac{d}{2}(\mathbb{R}^3) \) or \( L^3(\mathbb{R}^3) \). Then
\[
\lim_{t \to \infty} \|u(t)\|_X = 0.
\]

This theorem was proved for \( X = \dot{H}^\frac{d}{2}(\mathbb{R}^3) \) in \(^{10}\), and for \( L^3(\mathbb{R}^3) \) in \(^{11}\). Those results will be extended in our paper in the setting of the critical Bessel-potential spaces. More precisely, in this paper, we discuss the existence and decay properties of high order derivatives in time and space variables for local and global solutions of the Cauchy problem for the NSE with initial data in the critical Bessel-potential spaces \( \dot{H}^{\frac{d}{2} - 1}_p(\mathbb{R}^d), (1 < p < \infty) \). By using several tools from harmonic analysis, we obtain decay estimates for derivatives of arbitrary order. The estimate for the decay rate is optimal in the sense that it coincides with the decay rate of a solution to the heat equation. This result improves the previous ones.

The content of this paper is as follows: in Section 2, we state our main theorems after introducing some notations. In Section 3, we first establish some estimates concerning the heat semigroup with differential. We also recall some auxiliary lemmas and several estimates in the Bessel-potential spaces and Besov spaces. Finally, in Section 4, we will give the proof of the main theorems.

### 2. Statement of the results

For \( T > 0 \), we say that \( u \) is a mild solution of NSE on \([0, T]\) corresponding to a divergence-free initial datum \( u_0 \) when \( u \) solves the integral equation
\[
u = e^{t\Delta} u_0 - \int_0^t e^{(t-\tau)\Delta} \mathbb{P} \nabla \cdot (u(\tau, .) \otimes u(\tau, .)) \, d\tau.
\]
Above we have used the following notation: for a tensor \( F = (F_{ij}) \) we define the vector \( \nabla F \) by \( (\nabla F)_i = \sum_{j=1}^{d} \partial_j F_{ij} \) and for two vectors \( u \) and \( v \), we define their tensor product \( (u \otimes v)_{ij} = u_i v_j \). The operator \( \mathbb{P} \) is the Helmholtz-Leray projection onto the divergence-free fields

\[
(\mathbb{P}f)_j = f_j + \sum_{1 \leq k \leq d} R_j R_k f_k,
\]

where \( R_j \) is the Riesz transforms defined as

\[
R_j = \frac{\partial_j}{\sqrt{-\Delta}} \quad \text{i.e.} \quad \hat{R}_j g(\xi) = \frac{i\xi_j}{|\xi|} \hat{g}(\xi)
\]

with \( \hat{\cdot} \) denoting the Fourier transform. The heat kernel \( e^{t\Delta} \) is defined as

\[
e^{t\Delta} u(x) = ((4\pi t)^{-d/2} e^{-|.|^2/4t} * u)(x).
\]

For a space of functions defined on \( \mathbb{R}^d \), say \( E(\mathbb{R}^d) \), we will abbreviate it as \( E \). We denote by \( L^q := L^q(\mathbb{R}^d) \) the usual Lebesgue space for \( q \in [1, \infty] \) with the norm \( \| \|_q \), and we do not distinguish between the vector-valued and scalar-valued spaces of functions. Given a Banach space \( E \) with norm \( \| \|_{\cdot E} \), we denote by \( BC([0,T];E) \), set of bounded continuous functions \( f(t) \) defined on \( (0,T) \) with values in \( E \) such that \( \sup_{0<t<T} \| f(t) \|_E < +\infty \). We define the Bessel-potential space by \( \dot{H}^s_q := \dot{\Lambda}^s L^q \) equipped with the norm \( \| f \|_{\dot{H}^s_q} := \| \dot{\Lambda}^s f \|_q \).

Here \( \dot{\Lambda}^s u := (|\xi|^s \hat{u}(\xi)) \), where \( \hat{\cdot} \) is the inverse Fourier transform. For any collection of Banach spaces \( (X_m)_{m=1}^M \) and \( X = X_1 \cap ... \cap X_m \), we set \( \| g \|_X = (\sum_{m=1}^{M} \| g \|_{X_m}^2)^{1/2} \). Similarly, for a vector-valued function \( f = (f_1, ..., f_M) \), we define \( \| f \|_X = (\sum_{m=1}^{M} \| f_m \|_{X_m}^2)^{1/2} \). Throughout the paper, we sometimes use the notation \( A \lesssim B \) as an equivalent to \( A \leq CB \) with a uniform constant \( C \). The notation \( A \simeq B \) means that \( A \lesssim B \) and \( B \lesssim A \). Now we can state our main results

\textbf{Theorem 2.} Let \( 1 < p < \infty \) be fixed, then

(A) \textit{(Local existence)} For any initial data \( u_0 \in \dot{H}^s_p(\mathbb{R}^d) \) with \( \nabla u_0 = 0 \), there exists a positive \( T = T(u_0) \) such that NSE has a unique mild solution \( u \) satisfying

(i) \( t^{\frac{s+1}{2}-\frac{d}{p}} \dot{\Lambda}^s u \in BC([0,T];L^q), \) for \( q \geq p, s \geq \frac{d}{p} - 1, \)

(ii) \( t^{\frac{1}{2}(s+2n+1-\frac{d}{p})} \dot{\Lambda}^s D_t^n u \in BC([0,T];L^q), \) for \( q \geq p, s \geq \frac{d}{p} - 1, n \in \mathbb{N}, \)

(iii) \( t^{\frac{1}{2}(s+2-\frac{d}{p})} \dot{\Lambda}^s p \in BC([0,T];L^q) \) \( q \geq p, s \geq \max\left\{\frac{d}{p} - 1, 0\right\} \).
(B) (Global existence) For all $\tilde{q} > \max\{p, d\}$ there exists a positive constant $\sigma_{\tilde{q}, d}$ such that if
\[
\|u_0\|_{B^{d-1}_{\tilde{q}, \infty}} \leq \sigma_{\tilde{q}, d},
\]
then the existence time $T$ in (A) for the solution $u$ is equal to $+\infty$. Moreover, we have
\[
\begin{align*}
(i) & \quad \lim_{t \to \infty} t^{\frac{1}{q}(s+1-\frac{d}{q})} \|u(t)\|_{H^{\tilde{q}}_s} = 0, \text{ for } q \geq p, s \geq \frac{d}{p} - 1, \\
(ii) & \quad \lim_{t \to \infty} t^{\frac{1}{q}(s+2n+1-\frac{d}{q})} \|D^nu(t)\|_{H^{\tilde{q}}_s} = 0, \text{ for } q \geq p, s \geq \frac{d}{p} - 1, n \in \mathbb{N}, \\
(iii) & \quad \lim_{t \to \infty} t^{\frac{1}{q}(s+2-\frac{d}{q})} \|p(t)\|_{H^{\tilde{q}}_s} = 0, \text{ for } q \geq p, s \geq \max\left\{\frac{d}{p} - 1, 0\right\}.
\end{align*}
\]

Remark 1. Our result improves the previous ones for $L^d(\mathbb{R}^d)$ and $\dot{H}^{\frac{d}{q} - 1}(\mathbb{R}^d)$. These spaces, studied in [13] and [22], are particular cases of the Bessel spaces $\dot{H}^{\frac{d}{p} - 1}(\mathbb{R}^d)$ with $p = d$ and $p = 2$, respectively. We have the following imbeddings
\[
\dot{H}^{\frac{d}{p} - 1}(\mathbb{R}^d)_{(1 < p < 2)} \hookrightarrow \dot{H}^{\frac{d}{q} - 1}(\mathbb{R}^d) \hookrightarrow L^d(\mathbb{R}^d) \hookrightarrow \dot{H}^{\frac{d}{q} - 1}(\mathbb{R}^d)_{(p > d)}.
\]
Furthermore, here we obtain statements that are stronger than those of Cheng He and Ling Hsiao [13] but under a much weaker condition on the initial data.

The condition (1) on the initial data in Theorem 2 is weaker than the condition in [13]. We have $L^d(\mathbb{R}^d) \hookrightarrow \dot{B}^{\frac{d}{q} - 1, \infty}_{\tilde{q}}(\mathbb{R}^d), (\tilde{q} > d)$, but these two spaces are different. Indeed, we have $|x|^{-1} \notin L^d$ and $|x|^{-1} \in \dot{B}^{\frac{d}{q} - 1, \infty}_{\tilde{q}}$ for all $\tilde{q} > d$.

Remark 2. In the proof of Theorem 2 we also show that every mild solution in the class
\[
u_0 \in \dot{H}^{\frac{d}{p} - 1}_{\tilde{q}}, t^{\frac{1}{q}(s+\frac{d}{q})}u \in BC([0, T]; L^{\tilde{q}}), 1 < p < +\infty,
\]
for some $T > 0$ and $\tilde{q} > \max\{p, d\}$ satisfies
\[
t^{\frac{1}{q}(s+1-\frac{d}{q})}\Lambda^su \in BC([0, T]; L^{\tilde{q}}), \text{ for } q \geq p, s \geq \frac{d}{p} - 1,
\]
In the case $p = 2$, the above condition on the mild solution is weaker than the condition in [22].
Remark 3. From the results in [13, 22], it follows that if the norm of the initial value in the space $\dot{H}^{d-1}_p(\mathbb{R}^d)$ is small enough then there exists a unique strong solution $u$ of NSE satisfying the following decay property

$$\|u(t)\|_{\dot{H}_q^d} = O(t^{-\frac{s}{2}(s+1-\frac{4}{q})})$$

for $q \geq p, s \geq \frac{d}{p} - 1$, where $p = 2$ or $p = d$, these results aren’t optimal. In the point (B) of Theorem 2, we obtain the following decay rates for strong solutions in the space $\dot{H}^{d-1}_p(\mathbb{R}^d)$

$$\|u(t)\|_{\dot{H}_q^d} = o(t^{-\frac{s}{2}(s+1-\frac{4}{q})})$$

for $q \geq p, s \geq \frac{d}{p} - 1$, and $1 < p < +\infty$, and so this our result improves the previous ones.

Theorem 3. Let $d \geq 2$ and $1 < p \leq d$ be fixed, then

(A) For all $\tilde{q} > d$ there exists a positive constant $\delta_{\tilde{q},d}$ such that for all $u_0 \in \dot{H}^{d-1}_p(\mathbb{R}^d)$ with $\text{div}(u_0) = 0$ satisfying

$$\|u_0\|_{\dot{H}^{d-1}_\tilde{q}(\mathbb{R}^d)} \leq \sigma_{\tilde{q},d},$$

NSE has a unique global mild solution $u \in BC([0, +\infty), \dot{H}^{d-1}_p(\mathbb{R}^d))$. Moreover, we have

$$\lim_{t \to \infty} \|u(t)\|_{\dot{H}^{d-1}_p(\mathbb{R}^d)} = 0.$$ 

In the case $d = 3$ we have the following result.

(B)(Decay in $\mathbb{R}^3$) Let $u \in C([0, +\infty), \dot{H}^{d-1}_p(\mathbb{R}^3))$ be a global solution to NSE for some divergence-free $u_0 \in \dot{H}^{d-1}_p(\mathbb{R}^3)$. Then

$$\lim_{t \to \infty} \|u(t)\|_{\dot{H}^{d-1}_p(\mathbb{R}^3)} = 0.$$ 

(2)

Remark 4. Our result extends the previous ones for $L^3(\mathbb{R}^3)$ and $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$. These spaces, studied in [11] and [10], are particular cases of the Bessel spaces $\dot{H}^{d-1}_p(\mathbb{R}^3)$ with $p = 3$ and $p = 2$, respectively.

§3. Tools from harmonic analysis

In this section we prepare some auxiliary lemmas. We first establish the $L^p-L^q$ estimate for the heat semigroup with differential.
Lemma 1. Assume that $d \geq 1$ and $s \geq 0, t > 0$ and $1 \leq p \leq q \leq \infty$. Then for all $f \in L^p$ we have

$$t^{\frac{d}{2}(\frac{1}{p} - \frac{1}{q}) + \frac{s}{2}} \hat{\Lambda}^s e^{t\Delta} f \in BC([0, \infty); L^q(\mathbb{R}^d)) \text{ and } \|\hat{\Lambda}^s e^{t\Delta} f\|_q \leq Ct^{\frac{d}{2}(\frac{1}{p} - \frac{1}{q}) - \frac{s}{2}} \|f\|_p.$$

where $C$ is a positive constant which depends only on $s, p, q,$ and $d$.

Proof. See [22].

In order to obtain our theorems we must establish the estimates for bilinear terms. We thus need a version of the Hölder type inequality in Bessel-potential spaces.

Lemma 2. Let $1 < r, p_1, p_2, q_1, q_2 \leq \infty$ and $s \geq 0$ satisfying $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}$. Then there exists a constant $C = C(d, s, p_1, p_2, q_1, q_2)$ such that for all $f \in \dot{H}^s_{p_1}(\mathbb{R}^d) \cap L^{p_2}(\mathbb{R}^d)$ and for all $g \in \dot{H}^s_{q_1}(\mathbb{R}^d) \cap L^{q_2}(\mathbb{R}^d)$ we have

$$\|fg\|_{\dot{H}^s_r} \leq C(\|f\|_{\dot{H}^s_{p_1}} \|g\|_{q_1} + \|f\|_{p_2} \|g\|_{\dot{H}^s_{q_2}}).$$

Proof. See [12].

Lemma 3. Let $\gamma, \theta \in \mathbb{R}$ and $t > 0$, then

(A) If $\theta < 1$ then

$$\int_0^t (t - \tau)^{-\gamma} \tau^{-\theta} \, d\tau = Ct^{1-\gamma-\theta}, \text{ where } C = \int_0^1 (1 - \tau)^{-\gamma} \tau^{-\theta} \, d\tau < \infty.$$

(B) If $\gamma < 1$ then

$$\int_0^t (t - \tau)^{-\gamma} \tau^{-\theta} \, d\tau = Ct^{1-\gamma-\theta}, \text{ where } C = \int_0^1 (1 - \tau)^{-\gamma} \tau^{-\theta} \, d\tau < \infty.$$

The proof of this lemma is elementary and may be omitted.

Theorem 4. (Calderon-Zygmund theorem).

The Riesz transforms $R_j = \frac{\partial_j}{\sqrt{-\Delta}}$ defined by $\mathcal{F}(R_j f)(\xi) = \frac{i\xi_j}{|\xi|} \hat{f}(\xi)$ are bounded from $\mathcal{H}^1$ to $L^1$, from $L^\infty$ to BMO, and from $L^q$ to $L^q$ for $1 < q < \infty$.

Lemma 4. (Sobolev inequalities).

If $s_1 > s_2, 1 < q_1, q_2 < \infty$, and $s_1 - \frac{d}{q_1} = s_2 - \frac{d}{q_2}$, then we have the following embedding mapping

$$\dot{H}^{s_1}_{q_1} \hookrightarrow \dot{H}^{s_2}_{q_2}.$$
In this paper we use the definition of the homogeneous Besov space $\dot{B}^{s,p}_q$ in [11] [2]. The following lemmas will provide a different characterization of the Besov spaces $\dot{B}^{s,p}_q$ in terms of the heat semigroup and will be one of the staple ingredients of the proof of Theorem 2.

**Lemma 5.** Let $1 \leq p, q \leq \infty$ and $s < 0$ then the two quantities

$$\left( \int_0^\infty (t^{-\frac{s}{2}} \| e^{t\Delta} f \|_q)^p \frac{dt}{t} \right)^{1/p} \text{ and } \| f \|_{\dot{B}^s_q}$$

are equivalent, where $\dot{B}^s_q$ is the homogeneous Besov space.

**Proof.** See ([7], Proposition 3, p. 182), or see ([20], Theorem 5.4, p. 45). □

Let us recall following result on solutions of a quadratic equation in Banach spaces (Theorem 22.4 in [20], p. 227).

**Theorem 5.** Let $E$ be a Banach space, and $B : E \times E \to E$ be a continuous bilinear map such that there exists $\eta > 0$ so that

$$\| B(x, y) \| \leq \eta \| x \| \| y \|,$$

for all $x$ and $y$ in $E$. Then for any fixed $y \in E$ such that $\| y \| \leq \frac{1}{4\eta}$, the equation $x = y - B(x, x)$ has a unique solution $x \in E$ satisfying $\| x \| \leq \frac{1}{2\eta}$.

§4. Proof of Theorems 2 and 3

In this section we shall give the proofs of Theorems 2 and 3. We now need four more lemmas. In order to proceed, we define an auxiliary space $K^{s,q}_T$. Let $s, q, T$ be such that

$$q \in (1, +\infty), s \geq \frac{d}{q} - 1, \text{ and } 0 < T \leq \infty,$$

we set

$$\alpha = \alpha(s, q) = s + 1 - \frac{d}{q}.$$

In the case $T < \infty$, we define the auxiliary space $K^{s,q}_T$ which is made up by the functions $u(t, x)$ such that

$$t^\frac{s}{2} u \in C([0, T]; \dot{H}^s_q)$$

and

$$\lim_{t \to 0} t^\frac{s}{2} \| u(t, \cdot) \|_{\dot{H}^s_q} = 0. \quad (3)$$
In the case \( T = \infty \), we define the auxiliary space \( \mathcal{K}_{q,\infty}^s \) which is made up by the functions \( u(t, x) \) such that

\[
\lim_{t \to 0} t^\frac{s}{2} \|u(t, .)\|_{\dot{H}_q^s} = 0,
\]

and

\[
\lim_{t \to \infty} t^\frac{s}{2} \|u(t, .)\|_{\dot{H}_q^s} = 0.
\]

The auxiliary space \( \mathcal{K}_{q,T}^s \) is equipped with the norm

\[
\|u\|_{\mathcal{K}_{q,T}^s} := \sup_{0 < t < T} t^\frac{s}{2} \|u(t, .)\|_{\dot{H}_q^s}.
\]

In the case \( s = \frac{d}{q} - 1 \) it is also convenient to define the space \( \mathcal{K}_{q,T}^{\frac{d}{q} - 1} \) as the natural space \( C([0, T); \dot{H}_q^{\frac{d}{q} - 1} (\mathbb{R}^d)) \) with the additional condition that its elements \( u(t, x) \) satisfy

\[
\lim_{t \to 0} \|u(t, .)\|_{\dot{H}_q^{\frac{d}{q} - 1}} = 0,
\]

if \( T = \infty \) then its elements \( u(t, x) \) satisfy the additional condition

\[
\lim_{t \to \infty} \|u(t, .)\|_{\dot{H}_q^{\frac{d}{q} - 1}} = 0.
\]

**Remark 5.** The auxiliary space \( \mathcal{K}_q := \mathcal{K}_{q,T}^0 (q \geq d, 0 < T < \infty) \) was introduced by Weissler and systematically used by Kato [14] and Cannone [6]. In the case \( T = \infty \), the space \( \mathcal{K}_q \) of Kato does not require the condition [5].

**Lemma 6.** If \( u_0 \in \dot{H}_p^{\frac{d}{p} - 1} (\mathbb{R}^d), (1 < p < \infty) \) then

(A) For all \( q \) and \( s \) satisfying \( q \geq p \) and \( s \geq \frac{d}{p} - 1 \) we have

\[
t^{\frac{s}{2}(s+1-\frac{d}{2})} e^{t \Delta} u_0 \in BC([0, \infty); \dot{H}_q^s)
\]

and

\[
\lim_{t \to \infty} t^{\frac{s}{2}(s+1-\frac{d}{2})} \|e^{t \Delta} u_0\|_{\dot{H}_q^s} = 0.
\]

(B) For all \( q \) and \( s \) satisfying \( q > p \) and \( s \geq \frac{d}{p} - 1 \) we have

\[
e^{t \Delta} u_0 \in \mathcal{K}_{q,\infty}^s.
\]

(C) For all \( q \) and \( s \) satisfying \( q > p \) and \( s > \frac{d}{q} - 1 \) we have

\[
e^{t \Delta} u_0 \in \mathcal{K}_{q,\infty}^s.
\]
Proof. (A) By applying Lemma \(\mathbb{1}\) for \(\dot{\Lambda}^{-1} u_0 \in L^p\), we have
\[
t^{\frac{1}{2}(s+1-\frac{d}{s})} \dot{\Lambda}^s e^{t\Delta} u_0 = t^{\frac{1}{2}(s+1-\frac{d}{s})} \dot{\Lambda}^{s-\frac{d}{p}+1} e^{t\Delta} \dot{\Lambda}^{-1} u_0 \in BC([0, \infty); L^p).
\]
The relation (8) is equivalent to the relation (9).

We now prove the equality (7), it is easily to prove that
\[
\lim_{n \to \infty} \left\| X_n \dot{\Lambda}^{-1} u_0 \right\|_p = 0,
\]
where \(X_n(x) = 0\) for \(x \in \{ x : \left| x \right| < n \} \cap \{ x : \left| \dot{\Lambda}^{-1} u_0(x) \right| < n \}\) and \(X_n(x) = 1\), otherwise. Let \(p^*\) be such that \(1 < p^* < p\). Applying Lemma \(\mathbb{1}\) we get
\[
t^{\frac{1}{2}(s+1-\frac{d}{s})} \left\| e^{t\Delta} u_0 \right\|_{H^q} = t^{\frac{1}{2}(s+1-\frac{d}{s})} \left\| \dot{\Lambda}^{s-\frac{d}{p}+1} e^{t\Delta} \dot{\Lambda}^{-1} u_0 \right\|_q \leq \left\| X_n \dot{\Lambda}^{-1} u_0 \right\|_p + t^{\frac{1}{2}(s+1-\frac{d}{s})} \left\| \dot{\Lambda}^{-1} \left( (1 - X_n) \dot{\Lambda}^{-1} u_0 \right) \right\|_p,
\]
\[
\leq C \left\| X_n \dot{\Lambda}^{-1} u_0 \right\|_p + t^{\frac{1}{2}(s+1-\frac{d}{s})} \left\| (1 - X_n) \dot{\Lambda}^{-1} u_0 \right\|_p,
\]
\[
\leq C \left\| X_n \dot{\Lambda}^{-1} u_0 \right\|_p + t^{\frac{1}{2}(s+1-\frac{d}{s})} \left\| (1 - X_n) \right\|_p.
\]
For any \(\epsilon > 0\), from (9) we have
\[
C \left\| X_n \dot{\Lambda}^{-1} u_0 \right\|_p < \frac{\epsilon}{2}
\]
for large enough \(n\). Fixed one of such \(n\), there exists \(t_0 = t_0(n) > 0\) satisfying
\[
t^{\frac{1}{2}(s+1-\frac{d}{s})} \left\| (1 - X_n) \right\|_p < \frac{\epsilon}{2}, \text{ for all } t > t_0,
\]
from the inequalities (10), (11), and (12) we deduce that
\[
t^{\frac{1}{2}(s+1-\frac{d}{s})} \left\| e^{t\Delta} u_0 \right\|_{H^q} < \epsilon, \text{ for all } t > t_0.
\]

(B) By (A), we only need to prove that
\[
\lim_{t \to 0} t^{\frac{1}{2}(s+1-\frac{d}{s})} \left\| e^{t\Delta} u_0 \right\|_{H^q} = 0.
\]
Let \(p^*\) be such that \(p < p^* < q\). For any \(\epsilon > 0\), applying Lemma \(\mathbb{1}\), by an argument similar to the previous one, there exist a sufficiently large \(n\) and a sufficiently small \(t_0 = t_0(n)\) such that
\[
t^{\frac{1}{2}(s+1-\frac{d}{s})} \left\| e^{t\Delta} u_0 \right\|_{H^q} \leq \left\| X_n \dot{\Lambda}^{-1} u_0 \right\|_p + t^{\frac{1}{2}(s+1-\frac{d}{s})} \left\| (1 - X_n) \right\|_p < \epsilon \text{ for all } t < t_0.
\]

(C) Let \(\tilde{p}\) be such that \(\max\{p, \frac{d+1}{s+1}\} < \tilde{p} < q\). From Lemma \(\mathbb{4}\) we have \(u_0 \in H^\frac{d}{p}-1 \subset H^\frac{d}{\tilde{p}}-1\). Applying (B) to obtain \(e^{t\Delta} u_0 \in K_{\tilde{p},\infty}^s\). This proves (C). \(\blacksquare\)
In the following lemmas, a particular attention will be devoted to the study of the bilinear operator $B(u, v)(t)$ defined by

$$B(u, v)(t) = \int_0^t e^{(t-\tau)\Delta}P\nabla.(u(\tau) \otimes v(\tau))d\tau. \quad (13)$$

**Lemma 7.** Let $\tilde{s}, \tilde{s} \geq 0$ and $1 < \tilde{q}, \tilde{q} < \infty$ be such that

$$\tilde{s} \geq \frac{d}{\tilde{q}} - 1 \text{ and } \frac{\tilde{s}}{d} < \frac{1}{\tilde{q}} < \min\left\{\frac{1}{2} + \frac{\tilde{s}}{2d}, 1 + \frac{\tilde{s}}{d} - \frac{1}{\tilde{q}}\right\}.$$  

Then the bilinear operator $B(u, v)(t)$ is continuous from

$$(K_{\tilde{q},T}^\tilde{s} \cap K_{\tilde{q},T}^\tilde{q}) \times (K_{\tilde{q},T}^\tilde{s} \cap K_{\tilde{q},T}^\tilde{q})$$

into $K_{\tilde{q},T}^\tilde{s}$, for all $\tilde{q}$ and $\tilde{s}$ satisfying

$$0 < \frac{1}{\tilde{q}} \leq \min\left\{\frac{2}{\tilde{q}} - \frac{\tilde{s}}{d} \frac{1}{\tilde{q}} + \frac{1}{\tilde{q}} - \frac{\tilde{s}}{d}\right\},$$

$$-1 < \tilde{s} - \tilde{s} < 1 - d\left(\frac{1}{\tilde{q}} - \frac{\tilde{s}}{d} \frac{1}{\tilde{q}} + \frac{1}{\tilde{q}} - \frac{1}{\tilde{q}}\right), \tilde{s} \geq \frac{d}{\tilde{q}} - 1, -1 < \tilde{s} - \tilde{s},$$

and the following inequality holds

$$\|B(u, v)\|_{K_{\tilde{q},T}^\tilde{s}} \leq C\|u\|_{K_{\tilde{q},T}^\tilde{s} \cap K_{\tilde{q},T}^\tilde{q}} \|v\|_{K_{\tilde{q},T}^\tilde{s} \cap K_{\tilde{q},T}^\tilde{q}}, \quad (14)$$

where $C$ is a positive constant and independent of $T$.

**Proof.** We split the integral given in (13) into two parts coming from the subintervals $(0, \frac{t}{2})$ and $(\frac{t}{2}, t)$

$$B(u, v)(t) = \int_0^{\frac{t}{2}} e^{(t-\tau)\Delta}P\nabla.(u \otimes v)d\tau + \int_{\frac{t}{2}}^t e^{(t-\tau)\Delta}P\nabla.(u \otimes v)d\tau. \quad (15)$$

Set

$$\frac{1}{h} = 2 - \frac{\tilde{s}}{d} \frac{1}{\tilde{q}}, \frac{1}{\tilde{q}} = \frac{1}{\tilde{q}} - \frac{1}{\tilde{q}}, \text{ and } \frac{1}{r} = \frac{1}{\tilde{q}} + \frac{1}{\tilde{q}}.$$  

To estimate the first term on the right-hand side of the equation (15), applying Lemma 1, Theorem 4, Lemma 2, and Lemma 4, and Lemma 3, we obtain

$$\left\|\int_0^{\frac{t}{2}} e^{(t-\tau)\Delta}P\nabla.(u \otimes v)d\tau\right\|_{\tilde{q}} \leq \int_0^{\frac{t}{2}} \left\|\Lambda^\tilde{s} e^{(t-\tau)\Delta}P\nabla.(u \otimes v)\right\|_{\tilde{q}}d\tau =$$
\[
\int_0^t \left\| \tilde{\Delta}^{\frac{s}{2} + 1} e^{(t-\tau)\Delta} \nabla \cdot (u \otimes v) \right\|_{\dot{H}^\frac{s}{2}} \, d\tau \lesssim \int_0^t \left\| \tilde{\Delta}^{\frac{s}{2} + 1} e^{(t-\tau)\Delta} \nabla \cdot (u \otimes v) \right\|_{\dot{H}^\frac{s}{2}} \, d\tau \\
\int_0^t (t - \tau)^{\frac{s}{2} + 1 + 2(\frac{1}{4} - \frac{s}{2})} \left\| \tilde{\Delta}^{\frac{s}{2}} (u \otimes v) \right\|_{\dot{H}^\frac{s}{2}} \, d\tau \lesssim \int_0^t (t - \tau)^{\frac{s}{2} + 1 + 2(\frac{1}{4} - \frac{s}{2})} \left\| \tilde{\Delta}^{\frac{s}{2}} (u \otimes v) \right\|_{\dot{H}^\frac{s}{2}} \, d\tau \\
\int_0^t (t - \tau)^{\frac{s}{2} + 1 + 2(\frac{1}{4} - \frac{s}{2})} \left( \|u(\tau)\|_{\dot{H}^\frac{s}{2}} \|v(\tau)\|_{\dot{H}^\frac{s}{2}} + \|v(\tau)\|_{\dot{H}^\frac{s}{2}} \|u(\tau)\|_{\dot{H}^\frac{s}{2}} \right) \, d\tau \lesssim \int_0^t (t - \tau)^{\frac{s}{2} + 1 + 2(\frac{1}{4} - \frac{s}{2})} \left( \|u(\tau)\|_{\dot{H}^\frac{s}{2}} \|v(\tau)\|_{\dot{H}^\frac{s}{2}} + \|v(\tau)\|_{\dot{H}^\frac{s}{2}} \|u(\tau)\|_{\dot{H}^\frac{s}{2}} \right) \, d\tau \\
\int_0^t (t - \tau)^{\frac{s}{2} + 1 + 2(\frac{1}{4} - \frac{s}{2})} \left( \|u(\tau)\|_{\dot{H}^\frac{s}{2}} \|v(\tau)\|_{\dot{H}^\frac{s}{2}} + \|v(\tau)\|_{\dot{H}^\frac{s}{2}} \|u(\tau)\|_{\dot{H}^\frac{s}{2}} \right) \, d\tau \lesssim \int_0^t (t - \tau)^{\frac{s}{2} + 1 + 2(\frac{1}{4} - \frac{s}{2})} \left( \sup_{0 < \eta < \tau} \eta^\frac{s+1}{2} \|u\|_{\dot{H}^\frac{s}{2}} \sup_{0 < \eta < \tau} \eta^\frac{s+1}{2} \|v\|_{\dot{H}^\frac{s}{2}} \right) \, d\tau \\
+ \int_0^t (t - \tau)^{\frac{s}{2} + 1 + 2(\frac{1}{4} - \frac{s}{2})} \left( \sup_{0 < \eta < \tau} \eta^\frac{s+1}{2} \|v\|_{\dot{H}^\frac{s}{2}} \sup_{0 < \eta < \tau} \eta^\frac{s+1}{2} \|u\|_{\dot{H}^\frac{s}{2}} \right) \, d\tau \\
\approx t^{-\frac{1}{2}(s+1 - \frac{d}{2})} \left( \sup_{0 < \eta < t} \eta^\frac{1}{2}(s+1 - \frac{d}{2}) \|u(\eta)\|_{\dot{H}^\frac{s}{2}} \sup_{0 < \eta < t} \eta^\frac{1}{2}(s+1 - \frac{d}{2}) \|v(\eta)\|_{\dot{H}^\frac{s}{2}} + \sup_{0 < \eta < t} \eta^\frac{1}{2}(s+1 - \frac{d}{2}) \|v(\eta)\|_{\dot{H}^\frac{s}{2}} \sup_{0 < \eta < t} \eta^\frac{1}{2}(s+1 - \frac{d}{2}) \|u(\eta)\|_{\dot{H}^\frac{s}{2}} \right). \tag{17}
\]
Using the inequalities (16) and (17), we conclude that

\[
\|B(u, v)(t)\|_{\dot{H}^\gamma_q} \lesssim \sup_{0 < \eta < t} \eta^{\frac{1}{2}(s + 1 - \frac{d}{q})} \|u(\eta)\|_{\dot{H}^\gamma_q} \sup_{0 < \eta < t} \eta^{\frac{1}{2}(s + 1 - \frac{d}{q})} \|v(\eta)\|_{\dot{H}^\gamma_q} + \sup_{0 < \eta < t} \eta^{\frac{1}{2}(s + 1 - \frac{d}{q})} \|u(\eta)\|_{\dot{H}^\gamma_q} \sup_{0 < \eta < t} \eta^{\frac{1}{2}(s + 1 - \frac{d}{q})} \|v(\eta)\|_{\dot{H}^\gamma_q} + \sup_{0 < \eta < t} \eta^{\frac{1}{2}(s + 1 - \frac{d}{q})} \|v(\eta)\|_{\dot{H}^\gamma_q} \sup_{0 < \eta < t} \eta^{\frac{1}{2}(s + 1 - \frac{d}{q})} \|u(\eta)\|_{\dot{H}^\gamma_q}.
\]

(18)

The estimate (14) is deduced from the inequality (18). Let us now check the validity of the condition (5) for the bilinear term \(B(u, v)(t)\). Indeed, from the inequality (18) we have

\[
\lim_{t \to 0} t^{\frac{1}{2}(s + 1 - \frac{d}{q})} \|B(u, v)(t)\|_{\dot{H}^\gamma_q} = 0
\]

(19)

whenever

\[
\lim_{t \to 0} t^{\frac{1}{2}(s + 1 - \frac{d}{q})} \|u(t)\|_{\dot{H}^\gamma_q} = \lim_{t \to 0} t^{\frac{1}{2}(s + 1 - \frac{d}{q})} \|v(t)\|_{\dot{H}^\gamma_q} = \lim_{t \to 0} t^{\frac{1}{2}(s + 1 - \frac{d}{q})} \|u(t)\|_{\dot{H}^\gamma_q} = \lim_{t \to 0} t^{\frac{1}{2}(s + 1 - \frac{d}{q})} \|v(t)\|_{\dot{H}^\gamma_q} = 0.
\]

In the case of \(T = \infty\), we check the validity of the condition (5) for the bilinear term \(B(u, v)(t)\). Firstly we estimate the first term on the right-hand side of equation (15). By the estimate (16) and a change variable of the variable \(\tau\) we have

\[
\begin{align*}
t^{\frac{1}{2}(s + 1 - \frac{d}{q})} & \left\| \int_0^t e^{(t - \tau)\Delta} \nabla \cdot (u \otimes v) \right\|_{\dot{H}^\gamma_q} \lesssim \frac{t^{\frac{1}{2}(s + 1 - \frac{d}{q})}}{(1 - \tau)^{\frac{s + 1 - \frac{d}{q}}{2} + \frac{d}{2} - \frac{1}{2}}} \|u(t)\|_{\dot{H}^\gamma_q} \|v(t)\|_{\dot{H}^\gamma_q} = \\
& \int_0^\frac{t}{2} (t - \tau)^{\frac{s + 1 - \frac{d}{q}}{2} + \frac{d}{2} - \frac{1}{2}} \|u(t)\|_{\dot{H}^\gamma_q} \|v(t)\|_{\dot{H}^\gamma_q} \, d\tau = \int_0^\frac{t}{2} (t - \tau)^{\frac{s + 1 - \frac{d}{q}}{2} + \frac{d}{2} - \frac{1}{2}} \|u(t)\|_{\dot{H}^\gamma_q} \|v(t)\|_{\dot{H}^\gamma_q} \, d\tau.
\end{align*}
\]

Applying Lebesgue’s convergence theorem we deduce that

\[
\lim_{t \to \infty} t^{\frac{1}{2}(s + 1 - \frac{d}{q})} \left\| \int_0^t e^{(t - \tau)\Delta} \nabla \cdot (u \otimes v) \right\|_{\dot{H}^\gamma_q} = 0
\]

(20)

whenever

\[
t^{\frac{1}{2}(s + 1 - \frac{d}{q})} u \in BC([0, \infty); \dot{H}^\gamma_q), \quad t^{\frac{1}{2}(s + 1 - \frac{d}{q})} v \in BC([0, \infty); \dot{H}^\gamma_q),
\]

(21)
and
\[ \lim_{t \to \infty} t^{\frac{s}{4} + \frac{1}{2}} \|u(t)\|_{H^s_q} = \lim_{t \to \infty} t^{\frac{s}{4} + \frac{1}{2}} \|v(t)\|_{H^s_q} = 0. \]  
(22)

By an argument similar to the previous one, we have
\[ \lim_{t \to \infty} t^{\frac{s}{4} + \frac{1}{2}} \left\| \int_{\tau}^{t} e^{(t-\tau)\Delta} \mathbf{P} \nabla.(u \otimes v) \, d\tau \right\|_{H^s_q} = 0 \]  
(23)
whenever
\[ t^{\frac{s}{4} + \frac{1}{2}} u \in BC([0, \infty); \dot{H}^s_q), \quad t^{\frac{s}{4} + \frac{1}{2}} v \in BC([0, \infty); \dot{H}^s_q), \]
\[ t^{\frac{s}{4} + \frac{1}{2}} u \in BC([0, \infty); \dot{H}^s_q), \quad \text{and} \quad t^{\frac{s}{4} + \frac{1}{2}} v \in BC([0, \infty); \dot{H}^s_q), \]  
(24)
and
\[ \lim_{t \to \infty} t^{\frac{s}{4} + \frac{1}{2}} \|u(t)\|_{H^s_q} = \lim_{t \to \infty} t^{\frac{s}{4} + \frac{1}{2}} \|v(t)\|_{H^s_q} = 0. \]  
(25)

It follows readily from (20) and (23) that
\[ \lim_{t \to \infty} t^{\frac{s}{4} + \frac{1}{2}} \|B(u, v)(t)\|_{H^s_q} = 0, \]
whenever (24) and (25) are satisfied.

Finally, the continuity at \( t = 0 \) of \( t^{\frac{s}{4} + \frac{1}{2}} B(u, v)(t) \) follows from the equality (19). The continuity elsewhere follows from carefully rewriting the expression \( \int_{0}^{t} - t^{\frac{s}{4} + \frac{1}{2}} \) and applying the same argument. \( \square \)

**Lemma 8.** Let \( p \) be a fixed number in the interval \( (1, +\infty) \). Assume that NSE has a mild solution \( u \in \mathcal{K}_{q,T}^{0} \) for some \( q \in (\max\{p, d\}, +\infty) \) with initial data \( u_0 \in H^{\frac{d}{q} - 1}_p \), then \( u \in \bigcap_{s > \frac{d}{q} - 1} \mathcal{K}_{q,T}^{s} \).

**Proof.** We first prove a claim that if \( u \in \mathcal{K}_{q,T}^{s} \) for some \( s \geq 0 \) then \( u \in \mathcal{K}_{q,T}^{\tilde{s}} \) for all \( \tilde{s} \in (\frac{d}{q} - 1, +\infty) \cap (s - 1, s + 1 - \frac{d}{q}) \). Indeed, from \( u \in \mathcal{K}_{q,T}^{s} \cap \mathcal{K}_{q,T}^{0} \), applying Lemma 7 for \( \tilde{q} = \tilde{q} = q, \tilde{s} = 0, \) and \( \tilde{s} = s, \) we have \( B(u, u) \in \mathcal{K}_{q,T}^{s} \) for all \( \tilde{s} \in [\frac{d}{q} - 1, +\infty) \cap (s - 1, s + 1 - \frac{d}{q}). \) From Lemma 5(C), we have \( e^{t\Delta} u_0 \in \mathcal{K}_{q,T}^{s} \) for all \( \tilde{s} \in (\frac{d}{q} - 1, +\infty). \) Since \( u = e^{t\Delta} u_0 - B(u, u) \), it follows that \( u \in \mathcal{K}_{q,T}^{s} \) for all \( \tilde{s} \in (\frac{d}{q} - 1, +\infty) \cap (s - 1, s + 1 - \frac{d}{q}). \) This proves the result.

We now prove that \( u \in \bigcap_{s > \frac{d}{q} - 1} \mathcal{K}_{q,T}^{s} \). Indeed, from \( u \in \mathcal{K}_{q,T}^{0} \) and the just
proved claim, we have \( u \in \mathcal{K}_{q,T}^s \) for all \( s \in \left( \frac{d}{q} - 1, 1 - \frac{d}{q} \right) \). Applying again the just proved claim, since \( u \in \mathcal{K}_{q,T}^s \) for all \( s \in \left( \frac{d}{q} - 1, 1 - \frac{d}{q} \right) \), it follows that \( u \in \mathcal{K}_{q,T}^s \) for all \( s \in \left( \frac{d}{q} - 1, 2(1 - \frac{d}{q}) \right) \). By induction, we get \( u \in \mathcal{K}_{q,T}^s \) for all \( s \in \left( \frac{d}{q} - 1, n(1 - \frac{d}{q}) \right) \) with \( n \in \mathbb{N} \). Since \( 1 - \frac{d}{q} > 0 \), it follows that \( u \in \mathcal{K}_{q,T}^s \) for all \( s \in \left( \frac{d}{q} - 1, +\infty \right) \). The proof of Lemma \( \mathbf{9} \) is complete. □

**Lemma 9.** Let \( p \) be a fixed number in the interval \( (1, +\infty) \). Assume that NSE has a mild solution \( u \in \bigcap_{s > \frac{d}{q} - 1} \mathcal{K}_{q,T}^s \) for some \( \bar{p} \in \{ \max \{ p, d \}, +\infty \} \) with initial data \( u_0 \in H_{\bar{p}}^{\frac{d}{q} - 1} \), then \( B(u, u) \in \bigcap_{s > \frac{d}{q} - 1} \mathcal{K}_{q,T}^s \).

**Proof.** We consider two cases \( d \leq p < +\infty \) and \( 1 < p < d \) separately.

Firstly, we consider the case \( d \leq p < +\infty \). We begin by proving that \( B(u, u) \in \bigcap_{s > \frac{d}{q} - 1} \mathcal{K}_{q,T}^s \) for all \( \hat{q} \) satisfying \( \frac{1}{q} \in \left( \frac{1}{p}, \frac{2}{\bar{p}} \right] \). Indeed, let \( \hat{q} \) and \( \hat{s} \) be such that \( \frac{1}{q} \in \left( \frac{1}{p}, \frac{2}{\bar{p}} \right] \) and \( q - 1 \leq \hat{s} < +\infty \), applying Lemma \( \mathbf{7} \) for \( \bar{q} = \hat{q} = \bar{p}, \bar{s} = \hat{s} + 1 - \frac{d}{2q}, \hat{s} = 0 \), we have \( B(u, u) \in \mathcal{K}_{q,T}^s \). If \( \frac{2}{\bar{p}} \geq \frac{1}{p} \) then we get the desired statement. Otherwise, in the case \( \frac{2}{\bar{p}} < \frac{1}{p} \), by Lemma \( \mathbf{6}(C) \), we have \( e^{t\Delta}u_0 \in \mathcal{K}_{q,T}^s \) for all \( \hat{q} \) and \( \hat{s} \) satisfying \( \frac{1}{q} \in \left( 0, \frac{1}{\bar{p}} \right) \) and \( q - 1 < \hat{s} < +\infty \). Since \( u = e^{t\Delta}u_0 - B(u, u) \), it follows that \( u \in \bigcap_{s > \frac{d}{q} - 1} \mathcal{K}_{q,T}^s \) for all \( \hat{q} \) be such that \( \frac{1}{q} \in \left( \frac{1}{p}, \frac{2}{\bar{p}} \right] \). By an argument similar to the previous one by replacing \( \bar{p} \) by \( \frac{2}{\bar{q}} \), we obtain \( B(u, u) \in \bigcap_{s > \frac{d}{q} - 1} \mathcal{K}_{q,T}^s \) for all \( \hat{q} \) satisfying \( \frac{1}{q} \in \left( \frac{1}{p}, \frac{2}{\bar{p}} \right] \). By continuing this procedure after a finite number of times we get \( B(u, u) \in \bigcap_{s > \frac{d}{q} - 1} \mathcal{K}_{q,T}^s \) for all \( \hat{q} \) satisfying \( \frac{1}{q} \in \left( \frac{1}{p}, \frac{1}{\bar{p}} \right] \). Therefore, we get the desired conclusion.

Secondly, we consider the case \( 1 < p < d \). By Lemma \( \mathbf{4} \) we have \( u_0 \in L^d \), from the case \( d \leq p < +\infty \) we have already obtained that \( B(u, u) \in \bigcap_{s > \frac{d}{q} - 1} \mathcal{K}_{q,T}^s \) for all \( q \in [d, \bar{p}) \). Moreover, it follows from Lemma \( \mathbf{6}(C) \) that \( e^{t\Delta}u_0 \in \mathcal{K}_{q,T}^s \) for all \( q \) and \( s \) satisfying \( q \in (p, \infty) \) and \( \frac{d}{q} - 1 < s < +\infty \). Therefore, \( u = e^{t\Delta}u_0 - B(u, u) \in \bigcap_{s > \frac{d}{q} - 1} \mathcal{K}_{q,T}^s \) for all \( q \in [d, \bar{p}) \). In exactly the same way as in the case \( d \leq p < +\infty \), since \( u \in \bigcap_{s > \frac{d}{q} - 1} \mathcal{K}_{q,T}^s \) for all \( q \in [d, \bar{p}) \), it follows that \( B(u, u) \in \bigcap_{s > \frac{d}{q} - 1} \mathcal{K}_{q,T}^s \) for all \( q \) satisfying \( \frac{1}{q} \in \left( \frac{1}{p}, \frac{1}{\bar{p}} \right] \). Moreover, it follows from Lemma \( \mathbf{6}(C) \) that \( e^{t\Delta}u_0 \in \mathcal{K}_{q,T}^s \) for all \( q \) and \( s \) satisfying \( s > 0 \). Hence, we conclude that \( B(u, u) \in \bigcap_{s > \frac{d}{q} - 1} \mathcal{K}_{q,T}^s \) for all \( q \in [d, \bar{p}) \) and \( s > 0 \) with \( \frac{1}{q} \in \left( \frac{1}{p}, \frac{1}{\bar{p}} \right] \). Therefore, we get the desired conclusion.
for all \( q \) and \( s \) satisfying \( \frac{1}{q} \in (0, \frac{1}{p}) \) and \( \frac{d}{q} - 1 < s < +\infty \). Therefore \( u = e^{t\Delta}u_0 - B(u, u) \in \bigcap_{s > \frac{d}{q} - 1} K_{s,T}^q \) for all \( q \) satisfying \( \frac{1}{q} \in (\frac{1}{p}, \frac{2}{d}] \cap (0, \frac{1}{p}) \).

We now prove a claim that if \( u \in \bigcap_{s > \frac{d}{q} - 1} K_{s,T}^q \) for some \( q \in (p, d] \) then \( B(u, u) \in \bigcap_{s > \frac{d}{q} - 1} K_{s,T}^q \) for all \( q \) satisfying \( \frac{1}{q} \in (\frac{1}{p}, \frac{2}{d}) \cap (0, \frac{1}{p}] \) and \( u \in \bigcap_{s > \frac{d}{q} - 1} K_{s,T}^q \) for all \( q \) satisfying \( \frac{1}{q} \in (\frac{1}{p}, \frac{3}{d}) \cap (0, \frac{1}{p}] \). By induction, we get \( B(u, u) \in \bigcap K_{s,T}^q \) for all \( q \) satisfying \( \frac{1}{q} \in (\frac{1}{p}, \frac{n}{d}) \cap (0, \frac{1}{p}] \) and \( u \in \bigcap K_{s,T}^q \) for all \( q \) satisfying \( \frac{1}{q} \in (\frac{1}{p}, \frac{n+1}{d}) \cap (0, \frac{1}{p}] \) with \( n \in \mathbb{N} \). However, there exists a sufficiently big \( n \) such that \( (\frac{1}{p}, \frac{n}{d}) \cap (0, \frac{1}{p}] = (\frac{1}{p}, \frac{1}{p}] \). Therefore, we conclude that \( B(u, u) \in \bigcap K_{s,T}^q \) for all \( q \in (p, \bar{p}) \) and \( u \in \bigcap K_{s,T}^q \) for all \( \hat{q} \in (p, \bar{p}) \).

The proof of Lemma 9 is complete. \( \square \)

**Proof of Theorem 2**

(A) To prove (i), we take arbitrary \( \hat{q} \) satisfying \( \hat{q} > \max\{p, d\} \). Applying Lemma 7 for \( \hat{q} = \hat{q} = \hat{q} \) and \( \hat{s} = \hat{s} = \hat{s} = 0 \), we deduce that the bilinear operator \( B \) is bounded from from \( K_{q,q}^0 \times K_{q,q}^0 \) into \( K_{q,q}^0 \). Therefore, applying Theorem 5 to the bilinear operator \( B \), we deduce that there exists a positive constant \( \delta_{q,d} \) such that for all \( T > 0 \) and for all \( u_0 \in H_{p}^{d-1}(\mathbb{R}^d) \) with \( \text{div}(u_0) = 0 \) satisfying

\[
\|e^{t\Delta}u_0\|_{K_{q,T}^q} \leq \delta_{q,d},
\]  

(26)
NSE has a unique mild solution $u$ satisfying

$$u \in K_{q,T}^0.$$  \hfill (27)

By (27), applying Lemmas 8 and 9 we get

$$B(u,u) \in \bigcap_{s \geq \frac{d}{p} - 1} K_{p,T}^s.$$ \hfill (28)

From the relation (28) and Lemma 1 we have

$$B(u,u) \in \bigcap_{s \geq \frac{d}{p} - 1} K_{q,T}^s.$$ \hfill (29)

From $u = e^{\ell \Delta} u_0 - B(u,u)$, the relation (27), the definition of $K_{q,T}^s$, and Lemma 11(A), we deduce that (i) is valid.

(ii) The statement in (i) that we have just proved is (ii) when $n = 0$. We will prove that (ii) is valid for $n = 1$. Applying Lemma 2 and Theorem 4 to obtain

$$t^{\frac{1}{2}(s+2+1-\frac{d}{p})} \| \dot{\Lambda}^s u_t \|_q \leq t^{\frac{1}{2}(s+2+1-\frac{d}{p})} \| \dot{\Lambda}^s u \|_q + t^{\frac{1}{2}(s+2+1-\frac{d}{p})} \| P \nabla \dot{\Lambda}^{s+1} (u \otimes u) \|_q \leq t^{\frac{1}{2}(s+2+1-\frac{d}{p})} \| \dot{\Lambda}^s u \|_q + t^{\frac{1}{2}(s+1+1-\frac{d}{p})} \| \dot{\Lambda}^{s+1} u_{2q} \|_q \lesssim t^{\frac{1}{2}(1-\frac{d}{p})} \| u \|_{2q} < \infty,$$ \hfill (30)

the last inequality in (30) is deduced by applying (ii) for $n = 0$.

We now prove that (ii) is valid for $n = 2$. Applying again Lemma 2 and Theorem 4 to obtain

$$t^{\frac{1}{2}(s+4+1-\frac{d}{p})} \| \dot{\Lambda}^s u_{tt} \|_q \leq t^{\frac{1}{2}(s+2+1-\frac{d}{p})} \| \dot{\Lambda}^s u \|_q + t^{\frac{1}{2}(s+4+1-\frac{d}{p})} \| \dot{\Lambda}^{s+1} D_t (u \otimes u) \|_q \lesssim t^{\frac{1}{2}(s+2+1-\frac{d}{p})} \| \dot{\Lambda}^s u \|_q + t^{\frac{1}{2}(s+1+2-\frac{d}{p})} \| \dot{\Lambda}^{s+1} u_{2q} t^{\frac{1}{2}(1-\frac{d}{p})} \| u \|_{2q} +$$

$$t^{\frac{1}{2}(2+1-\frac{d}{p})} \| u_t \|_{2q} t^{\frac{1}{2}(s+1+1-\frac{d}{p})} \| \dot{\Lambda}^{s+1} u \|_{2q} < \infty,$$ \hfill (31)

the last inequality in (31) is deduced by applying (ii) for $n = 1$.

By continuing this procedure, we can prove that (ii) is valid for all $n \in \mathbb{N}$.

(iii) Applying Lemma 2 and Theorem 4 we have

$$t^{\frac{1}{2}(s+2-\frac{d}{p})} \| \dot{\Lambda}^s p \|_q = t^{\frac{1}{2}(s+2-\frac{d}{p})} \| \nabla \otimes \nabla \dot{\Lambda}^s (u \otimes u) \|_q \lesssim t^{\frac{1}{2}(s+2-\frac{d}{p})} \| \dot{\Lambda}^s (u \otimes u) \|_q \lesssim t^{\frac{1}{2}(s+1-\frac{d}{p})} \| \dot{\Lambda}^s u \|_{2q} t^{\frac{1}{2}(1-\frac{d}{p})} \| u \|_{2q} < \infty.$$ \hfill (32)
The last inequality in (32) is deduced by applying (i).
Finally, we will show that the inequality (26) is valid when \( T \) is small enough. Indeed, applying Lemma 6(C) we have \( u \in K_{s,T}^0 \). From the definition of \( K_{s,T}^0 \), we deduce that the left-hand side of the inequality (26) converges to 0 when \( T \) tends to 0. Therefore the inequality (26) holds when \( T(u_0) \) is small enough.

(B) Applying Lemma 5, we deduce that the two quantities

\[
\|u_0\|_{B_{\tilde{q}}^{\frac{4}{q}-1,\infty}} \quad \text{and} \quad \|e^{t\Delta}u_0\|_{K_{0,\infty}^s}
\]

are equivalent, then there exists a positive constant \( \sigma_{\tilde{q},d} \) such that if \( \|u_0\|_{B_{\tilde{q}}^{\frac{4}{q}-1,\infty}} \leq \sigma_{\tilde{q},d} \) the inequality (26) holds for \( T = \infty \). Therefore, in (A) we can take \( T = +\infty \).

(i) By \( u = e^{t\Delta}u_0 - B(u, u) \) and Lemma 6(A), we only need to prove

\[
\lim_{t \to \infty} t^{\frac{1}{2(s+1)} - \frac{2}{q}} \|w\|_{H_{\tilde{q}}^s} = 0, \quad \text{where} \quad w = B(u, u). \quad (33)
\]

Indeed, this is deduced from the relation (29) and the definition of \( K_{s,\infty}^s \).

(ii) For \( n = 1 \), (ii) is deduced from (i) and the inequality (30). In the case \( n = 2 \), (ii) is deduced by using the inequality (31) and applying (ii) for \( n = 1 \). By continuing this procedure, we can prove that (ii) is valid for all \( n \in \mathbb{N} \).

(iii) Using (i) and the inequality (32), we deduce that (iii) is valid.

\[\square\]

**Proof of Theorem 3**

(A) This is deduced from Theorem 2 and the uniqueness theorem for solutions in the class \( C([0, T); L^d(\mathbb{R}^d)) \) established in [9] (see also [21] for a simplified proof).

(B) From the embedding mapping \( \dot{H}_{\tilde{q}}^{\frac{4}{q}-1,\infty}((\mathbb{R}^3)_{1<p<3}) \hookrightarrow L^3(\mathbb{R}^3) \), we have \( u_0 \in L^3(\mathbb{R}^3) \). Applying Theorem 11 to obtain

\[
\lim_{t \to \infty} \|u(t)\|_{L^3(\mathbb{R}^3)} = 0. \quad (34)
\]

From the equality (34) and the following embedding mapping

\[
L^3(\mathbb{R}^3) \hookrightarrow \dot{B}_{\tilde{q}}^{\frac{4}{q}-1,\infty}(\mathbb{R}^3) \quad \text{for} \quad \tilde{q} > 3,
\]

we have

\[
\lim_{t \to \infty} \|u(t)\|_{\dot{B}_{\tilde{q}}^{\frac{4}{q}-1,\infty}} = 0 \quad \text{for} \quad \tilde{q} > 3. \quad (35)
\]

The equality (2) is deduced from (A) and the equality (35). This proves (B).

\[\square\]

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