Abstract

The problem about a body in a three dimensional infinite channel is considered in the framework of the theory of linear water-waves. The body has a rough surface characterized by a small parameter \( \varepsilon > 0 \) while the distance of the body to the water surface is also of order \( \varepsilon \). Under a certain symmetry assumption, the accumulation effect for trapped mode frequencies is established, namely, it is proved that, for any given \( d > 0 \) and integer \( N > 0 \), there exists \( \varepsilon(d, N) > 0 \) such that the problem has at least \( N \) eigenvalues in the interval \((0, d)\) of the continuous spectrum in the case \( \varepsilon \in (0, \varepsilon(d, N)) \). The corresponding eigenfunctions decay exponentially at infinity, have finite energy, and imply trapped modes.

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1 Introduction

1.1 Statement of the problem.

Let \( \Gamma \) be a domain on the plane \( \mathbb{R}^2 \ni x' = (x_2, x_3) \) bounded by the line interval \( \gamma_0 = \{x' : |x_2| < l, \ x_3 = 0\} \) and the smooth simple curve \( \gamma \) inside the lower half-plane \( \mathbb{R}^2_- = \{x' : x_3 < 0\} \) which meets \( \gamma_0 \) at the points \( x' = (\pm l, 0) \) with the angles \( \alpha_\pm \in (0, \pi) \) (see Fig. 1 a).

The three-dimensional canal \( \Pi = \mathbb{R} \times \Gamma \ni x = (x_1, x') \) with the horizontal plain surface \( \Lambda = \mathbb{R} \times \gamma_0 \) contains the finite body \( \Theta(\varepsilon) \) (see Fig. 1 b). The shape of the body depends on the small parameter \( \varepsilon > 0 \) so that its upper surface is rough with periodic fine knobs and/or caverns of size \( \varepsilon \) (see Fig. 2 with the three-dimensional image and Fig. 3 with the two-dimensional cross-sections). The body is submerged in the superficial region and the mean distance from \( \Lambda \) to the upper surface of \( \Theta(\varepsilon) \) is of
order \( \varepsilon \) as well. There is no geometrical restriction on the bottom of \( \Theta(\varepsilon) \) but the upper rough horizontal part of the boundary \( \partial \Theta(\varepsilon) \) restricts from below a finite thin rectangular plate-shaped part \( \Omega_\varepsilon \) of the near-surface water layer (see Fig. 1)

\[
\Omega_\varepsilon \subset \Pi(\varepsilon) = \Pi \setminus \Theta(\varepsilon) 
\]

In other words, the upper straight base \( \omega_+ \) of the plate \( \Omega_\varepsilon \) belongs to the horizontal surface \( \Lambda \) of water while the lower base \( \omega_-(\varepsilon) \) of a fine periodic structure repose upon the boundary \( \partial \Theta(\varepsilon) \) of the body.

Although further results are valid for Lipschitz surface \( \omega_-(\varepsilon) \) (see Section 4), we assume in the presentation that this surface is smooth enough though. The assumption crucially simplifies rather cumbersome calculations in Sections 2.4 and 2.5.

To describe the periodic structure of the plate \( \Omega_\varepsilon \) more precisely, we introduce the periodicity cell \( \Sigma \) such that

\[
\sigma \times (-h, 0) \subset \Sigma \subset \sigma \times (-H, 0),
\]

where \( \sigma = \{ y = (y_1, y_2) \in \mathbb{R}^2 : |y_i| < a_i/2, \ i = 1, 2 \} \) is a rectangle \( (a_i > 0) \) and \( 0 < h \leq H \). We introduce another rectangle

\[
\omega = \{ y : |y_i| < A_i/2, \ i = 1, 2 \} 
\]

and assume that the sizes are in the relation

\[
A_i = \varepsilon a_i N_i, \ i = 1, 2, 
\]
where $N_1$ and $N_2$ are large positive integers. We then set

$$\Sigma^\nu = \left\{ x = (y, z) : (\varepsilon^{-1} y_1 - \nu_1 a_1, \varepsilon^{-1} y_2 - \nu_2 a_2, \varepsilon^{-1} z) \in \Sigma \right\},$$

$$(1.3)$$

and

$$\overline{\Pi}_\varepsilon = \bigcup_{\nu : |\nu|_2 \leq N} \Sigma^\nu,$$

$$(1.4)$$

where $\nu = (\nu_1, \nu_2) \in \mathbb{Z}^2$ is a multi-index and $\mathbb{Z} = \{0, \pm 1, \ldots\}$. The domain $\Omega_\varepsilon$, i.e., the interior of the closed set (1.3) is but a thin plate composed from the large number $N_1 \times N_2$ of small periodicity cells (1.3). We do not exclude the case $h = H$ when $\Sigma$ and $\Pi$ imply parallelepipeds of sizes $a_1 \times a_2 \times h$ and $A_1 \times A_2 \times \varepsilon H$, respectively (cf. [1]).

Notice that it is convenient to use different notation for the same Cartesian coordinate system $x = (x_1, x_2, x_3)$, namely, $(x_1, x'_2)$ in the canal $\Pi(\varepsilon)$ and $(y, z)$ in the plate $\Omega_\varepsilon$ while $x' = (x_2, x_3)$ are coordinates on the cross-section $\Gamma$ of $\Pi$ and $y = (y_1, y_2) = (x_1, x_2)$ are coordinates on the upper base $\omega_+ = \{ x : y \in \omega, z = 0 \}$ of $\Omega_\varepsilon$.

In the canal $\Pi$ with the submerged body $\Theta(\varepsilon)$, we consider the spectral problem of the linearized water-wave theory

$$-\Delta_x \Phi_\varepsilon(x) = 0, \quad x \in \Pi(\varepsilon),$$

$$(1.5)$$

$$\partial_n \Phi_\varepsilon(x) = 0, \quad x \in \pi(\varepsilon) := \partial \Pi(\varepsilon) \setminus \Lambda,$$

$$(1.6)$$

$$\partial_z \Phi_\varepsilon(x) = \lambda_\varepsilon \Phi(x), \quad x \in \Lambda$$

$$(1.7)$$

(see, e.g., monographs [2, 3, 4] for physical and mathematical background). Here $\Delta_x = \nabla_x \cdot \nabla_x$ is the Laplace operator, $\nabla_x = \text{grad}$ and $\nabla_{x'} = \text{div}$, while $\partial_n$ is the derivative along the outward normal, in particular, $\partial_n = \partial_z$ on $\Lambda$. Furthermore, $\Phi_\varepsilon$ is the velocity potential and $\lambda_\varepsilon$ the spectral parameter, proportional to square of frequency of harmonic oscillations in the canal.

In addition to the smoothness assumptions introduced above, the whole boundary $\partial \Pi(\varepsilon)$ is Lipschitz. Hence, the normal $n$ and the Neumann (1.6) and the Steklov (1.7) boundary conditions are defined properly for almost all $x \in \partial \Pi(\varepsilon)$. However, the gradient $\nabla_x \Phi_\varepsilon$ can gain singularities, e.g., at edges on the boundary and in Section 1.3 we give a precise definition of an operator $L_\varepsilon$ for problem (1.5)-(1.7) in the Sobolev space $H^1(\Pi(\varepsilon))$. In this framework, being interested to detect trapped modes, i.e., solutions with the exponential decay as $x_1 \to \pm \infty$, we need not to supply the problem with any radiation condition at infinity. We again refer to [2, 3, 4] for formulation of these radiation conditions in similar geometrical situations.

### 1.2 The trapped modes frequencies

In this paper we seek for trapped modes, i.e., solutions $\Phi_\varepsilon \in H^1(\Pi(\varepsilon))$ of problem (1.5)-(1.7) with the finite energy and, therefore, the exponential decay at infinity. Such solutions have been a goal in many investigations (see [5]-[13] and review [14] for much more extensive list of references).
we detect the accumulation effect of trapped mode eigenfrequencies, namely, assuming the geometrical parameter \( \varepsilon > 0 \) sufficiently small, we find out any prescribed number \( N \) of eigenvalues on the given small interval \( (0,d) \), \( d > 0 \), of the continuous spectrum in problem \((1.5)-(1.7)\). We make use of the following issues:

- The artificial Dirichlet boundary conditions on the plane \( \{ x : x_1 = 0 \} \).
- Asymptotic analysis for eigenvalues of a spectral problem in the thin finite domain \( \Omega_\varepsilon \).
- The operator formulation of the problem in Hilbert space.
- The max-min principle.

Let us outline these issues.

First, the artificial Dirichlet boundary conditions on the plane of geometrical symmetry permit to create a positive threshold \( \lambda (\Gamma) > 0 \) in the modified spectral problem so that the continuous spectrum covers the ray \( \left( \lambda (\Gamma), +\infty \right) \) but leaves the gap \( (0, \lambda (\Gamma)) \) for the discrete spectrum. This trick was proposed in \[15\] for detecting trapped modes in a strip with a symmetric obstacle for the Helmholtz equation with the Neumann boundary conditions.

Second, as a subsidiary problem, we investigate sloshing mode eigenfrequencies in the artificially constructed thin finite layer \( \Omega_\varepsilon \) of water (see formula \((1.1)\) and Fig. \ref{fig:4} where it is demonstrated how the plate-shaped layer \( \Omega_\varepsilon \) is cut off and separated by the body \( \Theta (\varepsilon) \)). In other words, we consider the auxiliary Steklov spectral problem

\[
- \Delta_x u_\varepsilon (x) = 0, \quad x \in \Omega_\varepsilon, \tag{1.8}
\]
\[
\partial_n u_\varepsilon (x) = 0, \quad x \in \omega_-(\varepsilon), \quad \partial_n u_\varepsilon (x) = \alpha_x u_\varepsilon (x), \quad x \in \omega_+, \tag{1.9}
\]
\[
u_\varepsilon (x) = 0, \quad x \in \Gamma_\varepsilon = \partial \Omega_\varepsilon \setminus \left( \omega^- (\varepsilon) \cup \omega^+ \right). \tag{1.10}
\]

The asymptotic analysis of \((1.8)-(1.10)\) is rather standard (cf. \[10, 17, 18\] and others). However, a new effect is observed in Theorem \[7\] each entry of the monotone unbounded eigenvalue sequence in problem \((1.8)-(1.10)\)

\[
0 < \alpha^{(1)}_\varepsilon < \alpha^{(2)}_\varepsilon \leq \ldots \leq \alpha^{(N)}_\varepsilon \leq \ldots \rightarrow +\infty \tag{1.11}
\]

becomes infinitesimal when \( \varepsilon \rightarrow 0^+ \). Namely, the eigenvalues \( \alpha^{(1)}_\varepsilon, \alpha^{(2)}_\varepsilon, \ldots, \alpha^{(N)}_\varepsilon \) belong to the interval \((0, d) \subset (0, \lambda (\Gamma))\) if \( \varepsilon \leq \varepsilon (d, N) \), with a certain \( \varepsilon (d, N) > 0 \).

It suffices to prove that the point spectrum of problem \((1.5)-(1.7)\) in the interval \((0, d)\) contain at least \( N \) eigenvalues. This task is fulfilled by applying the max-min principle (see, e.g., \[21, \text{Theorem 10.2.2}\]) to the operator formulation \[7\] of the problem (respectively the fourth and third issues in the above list). We emphasize that the lateral side \( \Gamma_\varepsilon \) of the plate \( \Omega_\varepsilon \) is supplied with the Dirichlet conditions (that is why we call \((1.8)-(1.10)\) the Steklov spectral problem while the complete analogy with sloshing modes is dubious). We again use the geometrical symmetry and reduce the problem \((1.8)-(1.10)\) onto the subdomain \( \Omega^+_\varepsilon = \{ x \in \Omega_\varepsilon : x_2 > 0 \} \). Imposing the Dirichlet condition on the artificial boundary \( \{ x \in \Omega_\varepsilon : x_2 = 0 \} \), we keep the concentration property for eigenvalues \( \alpha^{(p)+}_\varepsilon \) of the Steklov problem in \( \Omega^+_\varepsilon \). The Dirichlet conditions and the inclusion \( \omega^- (\varepsilon) \subset \partial \Theta (\varepsilon) \) permit for the extension of the corresponding eigenfunctions \( u^{(p)+}_\varepsilon \) by zero from \( \Omega^+_\varepsilon \) onto the set

\[
\Pi^+ (\varepsilon) = \{ x \in \Pi (\varepsilon) : x_2 > 0 \}. \tag{1.12}
\]

These extended eigenfunctions are taken as trial functions in the max-min principle which ensure that, for any \( \alpha^{(p)+}_\varepsilon \in (0, \lambda (\Gamma)) \), the point spectrum of the problem in \( \Pi^+ (\varepsilon) \) contains an eigenvalue \( \lambda^{(p)+}_\varepsilon \subset (0, \lambda (\Gamma)) \).

The last step in our consideration is traditional \[15\]: the even extension of eigenfunctions in \( \Pi^+ (\varepsilon) \) through the Dirichlet conditions onto the domain \( \Pi (\varepsilon) \) becomes a trapped mode in the whole problem \((1.5)-(1.7)\).
1.3 Preliminary description of results.

The operator formulation of problem (1.3)-(1.7) given in Section 3.2 permits to deal with its spectrum within the spectral theory of self-adjoint operators in Hilbert space. If \( \lambda \in \mathbb{C} \) is a complex number and \( \lambda \notin \mathbb{R}_+ = [0, +\infty) \), then evidently, the inhomogeneous problem (1.3)-(1.7) with data in the Lebesgue spaces \( L^2(\Pi(\varepsilon)) \) and \( L^2(\partial\Pi(\varepsilon)) \) admits a unique generalized solution in the Sobolev space \( H^1(\Pi(\varepsilon)) \) (see the integral identity (3.3) and cf. \cite{19}). This fact means that \( \mathbb{C} \setminus \mathbb{R}_+ \) implies the resolvent set of the operator \( \mathcal{L}_\varepsilon \) of problem (1.3)-(1.7). In Section 3.3 we show that the closed real positive semi-axis is covered with the continuous spectrum of \( \mathcal{L}_\varepsilon \) (Lemma [8]).

Under the assumption
\[
\Pi(\varepsilon) = \{ x : (x_1, -x_2, x_3) \in \Pi(\varepsilon) \},
\]
which requires the symmetry of domain (1.13) with respect to the middle plane \( \{ x : x_2 = 0 \} \) of the canal (cf. Fig. 3, a, where the dotted line indicates the symmetry axis of the transverse cross-section of the canal), we treat the restriction \( \mathcal{L}_\varepsilon^0 \) of the operator \( \mathcal{L}_\varepsilon \) onto the subspace
\[
\mathcal{H}_0 = \{ \Phi \in H^1(\Pi(\varepsilon)) : \Phi \text{ is odd in } x_2 \}
\]
and associate with the operator \( \mathcal{L}_\varepsilon^0 \) a problem obtained from (1.3)-(1.7) by restricting onto a half of the domain \( \Pi(\varepsilon) \), for definiteness on the right half (1.12), and supplied with the artificial boundary condition
\[
\Phi^+_\varepsilon(x) = 0, \quad x \in \omega^0(\varepsilon),
\]
on the artificially generated surface \( \omega^0(\varepsilon) = \{ x \in \Pi(\varepsilon) : x_2 = 0 \} \). Such the restricted problem is further referred as the problem (1.3)-(1.7), (1.13) on the domain \( \Pi^+(\varepsilon) \).

In Section 3.3 owing to the Dirichlet boundary conditions (1.13), we find out a threshold \( \lambda(\Gamma) > 0 \), depending only on the cross-section \( \Gamma \), such that the continuous spectrum of \( \mathcal{L}_\varepsilon^0 \) implies the ray \( [\lambda(\Gamma), +\infty) \subset \mathbb{R}_+ \), while the segment \( [0, \lambda(\Gamma)) \) contains only the discrete spectrum of \( \mathcal{L}_\varepsilon^0 \).

Note that the odd extension \( \Phi_\varepsilon \) of an eigenfunction \( \Phi^+_\varepsilon \) of the problem in \( \Pi^+(\varepsilon) \) becomes an eigenfunction of the problem in \( \Pi(\varepsilon) \) corresponding to the same eigenvalue \( \lambda_\varepsilon = \lambda^+_\varepsilon \). Based on the above-mentioned observations, we prove in Section 3.4 the main result of the paper.

**Theorem 1** Under the geometrical assumptions (1.4), (1.4) and (1.13), for any \( d > 0 \) and \( N \in \mathbb{N} := \{ 1, 2, \ldots \} \), there exists \( \varepsilon(d,N) > 0 \) such that in the case \( \varepsilon \in (0, \varepsilon(d,N)) \) problem (1.3)-(1.7) has at least \( N \) eigenvalues \( \lambda^{(1)}_\varepsilon, \ldots, \lambda^{(N)}_\varepsilon \) in the interval \( (0, d) \subset \mathbb{R}_+ \). The corresponding eigenfunctions \( \Phi^{(1)}_\varepsilon, \ldots, \Phi^{(N)}_\varepsilon \) decay exponentially at infinity and, therefore, imply so-called trapped modes in the linear theory of water-waves.

We emphasize that the eigenvalues in Theorem 1 lie in the continuous spectrum of the operator \( \mathcal{L}_\varepsilon \).

Our approach does not require any other global geometry assumption on the shape of the body \( \Theta(\varepsilon) \) whilst the symmetric cross-section \( \Gamma \) of the canal is arbitrary. Moreover, any given large number of trapped modes with the frequencies in any preadjusted small interval can be obtained.

2 Asymptotics of eigenvalues of the spectral problem in the thin domain

2.1 Formal asymptotic analysis.

We employ the standard asymptotic expansions of solutions in thin domains (see, e.g., \cite{20, 18, Ch.7})
\[
\alpha_\varepsilon \sim \varepsilon \tau, \quad u_\varepsilon(x) \sim w(y) + \varepsilon w_1(y, \xi) + \varepsilon^2 w_2(y, \xi),
\]
where \( \tau \) and \( w, w_j \) are a number and functions to be determined and \( \xi \) stands for the ”fast” variables
\[
\xi = (\eta, \zeta), \quad \eta = \varepsilon^{-1} y, \quad \zeta = \varepsilon^{-1} z.
\]
We insert formulae (2.1) into equation (1.8) and the boundary conditions (1.9) and gather coefficients on similar powers of the small parameter $\varepsilon$. Since the derivatives in $y_i$ and $z$ of the function $(y, z) \mapsto W(y, \varepsilon^{-1} y, \varepsilon^{-1} z)$ are equal to
\[ \varepsilon^{-1} \frac{\partial W}{\partial \xi_i}(y, \xi) + \frac{\partial W}{\partial y_i}(y, \xi) \] and \[ \varepsilon^{-1} \frac{\partial W}{\partial \zeta}(y, \xi), \]
respectively, we obtain the following problems on the periodicity cell $\Sigma$ with the parameter $y \in \omega$:

\begin{align*}
- \Delta_\xi w(y) &= 0, \quad \xi \in \Sigma, \quad \partial_{n(\xi)} w(y) = 0, \quad \xi \in \sigma^+ \cup \sigma^-; \\
- \Delta_\xi w_1(y, \xi) &= 2 \nabla_\eta \cdot \nabla_y w(y), \quad \xi \in \Sigma, \\
\partial_{n(\xi)} w_1(y, \xi) &= -n^*(\xi) \cdot \nabla_y w(y), \quad \xi \in \sigma^+ \cup \sigma^-; \\
- \Delta_\xi w_2(y, \xi) &= 2 \nabla_\eta \cdot \nabla_y w_1(y) + \Delta_y w(y), \quad \xi \in \Sigma, \\
\partial_{n(\xi)} w_2(y, \xi) &= -n^*(\xi) \cdot \nabla_y w_1(y), \quad \xi \in \sigma^-, \\
\partial_\zeta w_2(y, \xi) &= \tau w(y), \quad \xi \in \sigma^+.
\end{align*}

Here \( n = (n_1, n_2, n_3) \) is the outward unit normal to the upper $\sigma^+$ and lower $\sigma^-$ bases of the cell $\Sigma$ (see Fig. 5 and compare with Fig. 4). Note that $\sigma^-$ is the surface which completes these sides and the rectangular "cover" $\sigma^+$ up to the whole boundary $\partial \Sigma$. We do not write explicitly the periodicity conditions but always deal with solutions which are $a_i$-periodic in the variables $\eta_i, i = 1, 2$.

Equations (2.3) hold true because $w$ does not depend on the fast variables $\xi$ in (2.2). Since, evidently,

\[ \int_{\eta} n_i(\xi) d\eta = \int_{\partial \Sigma} n_i(\xi) d\eta = \int_{\Sigma} \frac{1}{\partial \xi_i} d\xi = 0, \quad i = 1, 2, \]

problem (2.4) admits a solution in the form

\[ w_1(y, \xi) = -\sum_{i=1}^2 W_i(\xi) \frac{\partial w}{\partial y_i}(y), \]

where $W_1$ and $W_2$ raise the standard asymptotic corrector in the theory of homogenization (see, e.g., [10, 17]). Namely, $W_i$ is a (periodic in $\eta$) solution of the model problem

\[ - \Delta_\xi W_i(\xi) = 0, \quad \xi \in \Sigma, \quad \partial_{n(\xi)} W_i(\xi) = n_i(\xi), \quad \xi \in \sigma^+ \cup \sigma^-.
\]
We emphasize that, by definition, \( n_1 = n_2 = 0 \) on \( \sigma^+ \) and, according to the assumed smoothness of the lower base of the cell, the periodic functions \( W_\varepsilon \) are infinitely differentiable.

We now consider problem (2.9). Note that the factor \( \varepsilon \) in the representation (2.11) of the eigenvalue \( \alpha_\varepsilon \) was introduced to fulfil the goal: the main asymptotic term of the right-hand side \( \alpha_\varepsilon u_\varepsilon (x) \) in the spectral boundary condition (1.9) of the Steklov type comes into a problem for the asymptotic term \( \varepsilon^2 u_\varepsilon \) in the expansion for the eigenfunction \( u_\varepsilon \).

The compatibility condition in problem (2.5) reads
\[
0 = 2 \int_{\Sigma} \nabla_y \cdot \nabla_y w_1 (y, \xi) \, d\xi + |\Sigma| \Delta_y w (y) - \int_{\sigma^0} n^* (\xi) \cdot \nabla_y w_1 (y, \xi) \, ds_\xi + \beta |\sigma| w (y),
\] (2.8)
where \( |\Sigma| = \text{meas}_3 \Sigma \) is the volume of the cell \( \Sigma \) and \( |\sigma| = a_1 a_2 \) the area of the cover \( \sigma^+ \). Owing to (2.6), equality (2.8) can be rewritten in the form
\[
B \left( \nabla_y \right) w (y) := - \nabla_y \cdot b \nabla_y w (y) = \tau |\sigma| w (y), \quad y \in \omega.
\] (2.9)
Here \( b \) is a matrix of size \( 2 \times 2 \) with the entries
\[
b_{ik} = - \int_{\Sigma} \left( \frac{\partial W_k}{\partial \eta_i} (\xi) + \frac{\partial W_i}{\partial \eta_k} (\xi) \right) \, d\xi + |\Sigma| \delta_{i,k} + \int_{\sigma} W_k (\xi) \, \partial_{n(\xi)} W_i (\xi) \, ds_\xi =
\] (2.10)
\[
= \int_{\Sigma} \left( \delta_{i,k} - \frac{\partial W_k}{\partial \eta_i} (\xi) - \frac{\partial W_i}{\partial \eta_k} (\xi) + \Delta \xi W_k (\xi) \cdot \nabla \xi W_i (\xi) \right) \, d\xi =
\]
\[
= \left( \nabla \xi (\xi_k - W_k), \nabla \xi (\xi_l - W_l) \right)_\Sigma.
\]
By \((\cdot, \cdot)_\Sigma\) is denoted the natural inner product in the Lebesgue space \( L^2 (\Sigma) \). The vector functions \( \nabla \xi (\xi_1 + W_1) \) and \( \nabla \xi (\xi_2 + W_2) \) are linear independent because \( W_1 \) and \( W_2 \) are periodic in \( \eta \). Thus, the matrix \( b \) with entries (2.10) implies a Gram matrix, i.e. it is positive definite and symmetric and, therefore, \( B \left( \nabla_y \right) \) is a second order elliptic differential operator.

In order to satisfy the Dirichlet conditions (1.10) on the lateral side of the plate \( \Omega_\varepsilon \), we subject the function \( w \) in (2.1) to the boundary condition
\[
w (y) = 0, \quad y \in \partial \omega.
\] (2.11)
We call (2.9), (2.11) the resultant spectral problem.

If \( \beta \) and \( w \) are an eigenvalue and the corresponding eigenfunction of problem (2.9), (2.11), the compatibility condition (2.8) is met and problem (2.5) admits a solution. This completes the asymptotic expansion (2.4).

### 2.2 Spectrum of the resultant problem.

Problem (2.9), (2.11) can be rewritten as the integral identity (19)
\[
(b \nabla_y w, \nabla_y v)_\omega = \tau |\sigma| (w, v)_\omega,
\] (2.12)
the left-hand side of which implies an inner product in the subspace \( H^1 (\omega, \partial \omega) \) of functions \( w \in H^1 (\omega) \) satisfying condition (2.11). Owing to the compact embedding \( H^1 (\omega) \subset L^2 (\omega) \), spectrum of the operator, associated with the bi-linear form \( (b \nabla_y w, \nabla_y v)_\omega \) (see [21 §10.1]), is discrete and form the positive monotone unbounded sequence
\[
0 < \tau^{(1)} < \tau^{(2)} \leq \tau^{(3)} \leq ... \leq \tau^{(\rho)} \leq ... \to +\infty
\] (2.13)
where eigenvalues are repeated according to their multiplicity.

The corresponding eigenfunctions \( w^{(1)}, w^{(2)}, ..., w^{(\rho)}, ... \) can be subject to the normalization and orthogonality condition
\[
(b \nabla_y w^{(p)}, \nabla_y w^{(q)})_\omega + |\sigma| (w^{(p)}, w^{(q)})_\omega = \delta_{p,q}
\] (2.14)
where $p, q \in \mathbb{N}$ and $\delta_{p,q}$ is Kronecker’s symbol. The first eigenvalue $\tau^{(1)}$ is simple due to the strong maximum principle.

An affine transform of the coordinate system $y = (y_1, y_2)$ turns the differential operator $B(\nabla_y)$ on the left of (2.23) into the Laplace operator while the rectangle $\omega$ becomes a parallelogram. A harmonic function, which has the finite Dirichlet integral and vanishes at both sides of an angle with the opening $\psi \in (0, \pi)$, possesses the worst singularity $K r^{\pi/\psi} \sin (\pi \psi^{-1} \varphi)$ \cite[note, see, e.g., 22, and introductory chapters in 23, 24]. Thus, the theory of elliptic boundary value problems in domains with piecewise smooth boundaries, especially, a result in 25, ensures the following assertion.

**Lemma 2** The eigenfunction $w^{(p)} \in \dot{H}^1(\omega)$ of problem (2.24), (2.21) verifies the estimates

$$
\left| \nabla_y^k w^{(p)} (y) \right| \leq c_{p,k} R(x)^{1+p-k}, \quad k \in \mathbb{N}_0 = \{0, 1, 2, \ldots\},
$$

(2.15)

where $\rho \in (0, 1)$ is a number depending on the matrix $b$ with entries (2.11) and $R(x)$ is the distance from a point $x \in \partial \omega$ to the nearest among the tops of the rectangle $\omega$. In particular, $w^{(p)}$ belongs to the Sobolev space $\dot{H}^2(\omega)$ and the Hölder space $C^{1,\rho}(\omega)$.

Recall the definition of the Hölder norm

$$
\|w; C^{1,\rho}(\omega)\| = \sum_{k=0}^1 \sup_{y \in \omega} |\nabla_y^k w(y)| + \sup_{y,y \in \omega} |y - y|^{-\rho} \left| \nabla_y^l w(y) - \nabla_y^l w(y) \right|.
$$

(2.16)

### 2.3 Operator formulation of the problem in $\Omega_\varepsilon$.

Aiming to justify asymptotic expansions constructed in Section 2.1 we endow the Sobolev space

$$
\dot{H}^1(\Omega_\varepsilon, \Upsilon_\varepsilon) = \{ u \in H^1(\Omega_\varepsilon) : u(x) = 0, \quad x \in \Upsilon_\varepsilon \}
$$

(2.17)

with the specific inner product

$$
\langle u, v \rangle_\varepsilon = (\nabla_x u, \nabla_x v)_{\Omega_\varepsilon} + \varepsilon (u, v)_{\omega}.
$$

(2.18)

In the obtained Hilbert space $\mathcal{H}_{\varepsilon}$ we introduce the operator $B_\varepsilon$ by the formula

$$
\langle B_\varepsilon u, v \rangle_\varepsilon = (u, v)_{\omega^+}, \quad u, v \in \mathcal{H}_{\varepsilon}.
$$

(2.19)

This operator is positive continuous and symmetric, therefore, self-adjoint. It is compact due to the compactness of the embedding $H^1(\Omega_\varepsilon) \subset L^2(\partial \Omega_\varepsilon)$. The norm of $B_\varepsilon$ is less than $\varepsilon^{-1}$. Thus, the spectrum of $B_\varepsilon$ is discrete and forms the positive infinitesimal sequence $\{ \beta_\varepsilon^{(j)} \}_{j \in \mathbb{N}}$, $\varepsilon^{-1} > \beta_\varepsilon^{(1)} > \beta_\varepsilon^{(2)} \geq \beta_\varepsilon^{(3)} \geq \ldots \beta_\varepsilon^{(p)} \geq \ldots \rightarrow 0^+$,

(2.20)

where eigenvalues are listed according to their multiplicity and again the first eigenvalue is simple by virtue of the strong maximum principle. The corresponding eigenfunctions $u_\varepsilon^{(p)}$ can be subject to the normalization and orthogonality conditions

$$
\langle u_\varepsilon^{(p)}, u_\varepsilon^{(q)} \rangle_\varepsilon = \delta_{p,q}, \quad p, q \in \mathbb{N}.
$$

(2.21)

**Remark 3** The variational formulation of problem (2.8), (2.10)

$$
\langle \nabla_x u_\varepsilon, \nabla_x v \rangle_{\Omega_\varepsilon} = \alpha_\varepsilon (u_\varepsilon, v)_{\omega^+}, \quad v \in \dot{H}^1(\Omega_\varepsilon; \Upsilon_\varepsilon),
$$

(2.22)
is equivalent to the abstract equation
\[ B_\varepsilon u_\varepsilon = \beta_\varepsilon u_\varepsilon \in \mathcal{H}_\Omega^2 \]  \hfill (2.23)
with the new spectral parameter
\[ \beta_\varepsilon = (\alpha_\varepsilon + \varepsilon)^{-1}. \]  \hfill (2.24)

Formula (2.24) relates only the discrete spectra (2.18) and (1.17). Although the operator \( B_\varepsilon \) has the infinite-dimensional kernel \( H^1(\Omega \varepsilon; Y_\varepsilon \cup \omega^+) \), this kernel does not influence the spectrum of problem (2.12) because \( \beta = 0 \mapsto \alpha = \varepsilon - \beta^{-1} = \infty \).

Justification of asymptotics is based in the next sections on the following classical result known as the lemma on "almost eigenvalues and eigenfunctions", a proof can be found in [26] and [21].

**Lemma 4** Let \( u \in \mathcal{H}_\Omega^2 \) and \( b \in \mathbb{R}_+ \) satisfy
\[ \| u; \mathcal{H}_\Omega^2 \| = 1, \quad \| B_\varepsilon u - bu; \mathcal{H}_\Omega^2 \| = \delta < b. \]

Then at least one eigenvalue \( \beta_\varepsilon^{(q)} \) of the operator \( B_\varepsilon \) verifies the inequality
\[ \left| \beta_\varepsilon^{(q)} - b \right| \leq \delta. \]

Moreover, for any \( \delta_1 \in (\delta, b) \), there exist coefficients \( f_p \) such that
\[ \sum |f_p|^2 = 1, \quad \| u - \sum f_p u_\varepsilon^{(p)}; \mathcal{H}_\Omega^2 \| \leq \frac{2 \delta}{\delta_1} \]
where \( \sum \) means summation over all eigenvalues of the operator \( B_\varepsilon \) in the segment \( [b - \delta_1, b + \delta_1] \) and \( u_\varepsilon^{(p)} \) are corresponding eigenfunctions under condition (2.21).

### 2.4 Approximation solutions.

According to (2.1) and (2.24), we take
\[ b = \varepsilon^{-1} \left( \tau^{(k)} + 1 \right), \quad u_\varepsilon^{(p)}(x) = \left\| U_\varepsilon^{(p)}; \mathcal{H}_\Omega^2 \right\|^{-1} U_\varepsilon^{(p)}(x), \]  \hfill (2.25)
\[ U_\varepsilon^{(p)}(x) = w^{(p)}(y) + \varepsilon X_\varepsilon(y)U_\varepsilon^{(p)}(x), \quad U_\varepsilon^{(p)}(x) = \sum_{i=1}^2 W_i(\varepsilon^{-1}x) \frac{\partial w^{(p)}}{\partial y_i}(y) \]  \hfill (2.26)

as an approximate solution of the spectral abstract equation (2.23). In (2.25) \( \tau^{(k)} \) is an eigenvalue of the resultant problem with multiplicity \( \kappa_k \), i.e.,
\[ \tau^{(k-1)} < \tau^{(k)} = \ldots = \tau^{(k+\kappa_k-1)} < \tau^{(k+\kappa_k)} \]  \hfill (2.27)

in the sequence (2.13), \( X_\varepsilon \) is a smooth cut-off function on \( \omega \) which is equal to 1 outside the \( \varepsilon \)-neighborhood of \( \partial \omega \), \( |\nabla^1_y X_\varepsilon(y)| \leq c \varepsilon^{-1} \), e.g.,
\[ X_\varepsilon(x) = X^1_\varepsilon(x_1)X^2_\varepsilon(x_2), \quad X^i_\varepsilon(x_1) = \begin{cases} 1 & \text{for } |x_1| < \frac{1}{2} a_1 - \frac{\varepsilon}{2}, \\ 0 & \text{for } |x_1| > \frac{1}{2} a_1 - \frac{\varepsilon}{2}. \end{cases} \]  \hfill (2.28)

Furthermore, \( p = k, \ldots, k + \kappa_k - 1 \) and \( w^{(k)}, \ldots, w^{(k+\kappa_k-1)} \) are eigenfunctions of problem (2.14) corresponding to \( \tau^{(k)} \) and verifying conditions (2.14). In other words, formulae (2.25), (2.26) deliver \( \kappa_k \) different approximation solutions of (2.23).
We proceed with calculation of the inner products \( \langle U^{(p)}_\varepsilon, U^{(q)}_\varepsilon \rangle \) \(_\varepsilon\); here and in the sequel \( p, q = k, \ldots, k + \kappa - 1 \). Since \( w^{(p)} \in H^2(\omega) \), \( W^i \in C^1(\Sigma) \) and

\[
\nabla_y U^{(p)}_\varepsilon(x) = \sum_{i=1}^{2} \left( \varepsilon^{-1} \nabla_y W_i(\xi) \frac{\partial w^{(p)}_\varepsilon}{\partial y_i}(y) + W_i(\xi) \nabla_y \frac{\partial w^{(p)}_\varepsilon}{\partial y_i}(y) \right),
\]

we readily obtain

\[
\|w^{(p)}; \mathcal{H}_\Omega\| \leq c\varepsilon^{1/2},
\]

\[
\|U^{(p)}; L^2(\Omega_\varepsilon)\| + \varepsilon^{1/2} \|U^{(p)}; L^2(\omega_+)\| + \varepsilon^{1/2} \|\nabla_x U^{(p)}; L^2(\Omega_\varepsilon)\| \leq c\varepsilon^{1/2}.
\]

Moreover,

\[
\|U^{(p)} \nabla_y X_\varepsilon; L^2(\Omega_\varepsilon)\|^2 \leq c\varepsilon^{-2} \text{meas} \{x \in \Omega_\varepsilon : \text{dist}(y, \partial \omega) \leq c\varepsilon\} \leq c
\]

and analogously

\[
\| (1 - X_\varepsilon) U^{(p)}; L^2(\Omega_\varepsilon)\|^2 \leq c\varepsilon^2.
\]

These inequalities allow to estimate directly certain terms on the right-hand side of the equality

\[
\langle U^{(p)}_\varepsilon, U^{(q)}_\varepsilon \rangle \varepsilon = \left( \nabla_y w^{(p)} + \varepsilon U^{(p)} \nabla_y X_\varepsilon + \varepsilon X_\varepsilon \nabla_y U^{(p)} \right) \nabla_y w^{(q)} + \varepsilon U^{(q)} \nabla_y X_\varepsilon + \varepsilon X_\varepsilon \nabla_y U^{(q)} \right) \Omega_\varepsilon +
\]

\[
+ \left( \varepsilon X_\varepsilon \partial_x U^{(p)} + \varepsilon X_\varepsilon \partial_x U^{(q)} \right) \Omega_\varepsilon + \varepsilon \left( w^{(p)} + \varepsilon X_\varepsilon U^{(p)} + w^{(q)} + \varepsilon X_\varepsilon U^{(q)} \right) \Omega_\varepsilon
\]

and conclude that

\[
\| \langle U^{(p)}_\varepsilon, U^{(q)}_\varepsilon \rangle - \varepsilon \left( w^{(p)} + w^{(q)} \right) \| \leq c\varepsilon^{3/2}.
\]

(2.29)

The formula

\[
J_{pq} - \varepsilon |\sigma|^{-1} \left( B\nabla_y w^{(p)}, \nabla_y w^{(q)} \right) \omega \leq c\varepsilon^{1+\min\{\rho, 1/2\}}
\]

(2.30)

for the integral

\[
J_{pq} = \left( \nabla_y w^{(p)} + \sum_{i=1}^{2} \frac{\partial w^{(p)}}{\partial y_i} \nabla_x W_i, \nabla_y w^{(q)} + \sum_{i=1}^{2} \frac{\partial w^{(q)}}{\partial y_i} \nabla_x W_i \right) \Omega_\varepsilon
\]

follows from the next lemma where it is necessary to put

\[
Z(\xi) = \nabla_\xi (\xi_i + W_i(\xi)) \cdot \nabla_\xi (\xi_k + W_k(\xi)), \quad Y(y) = \frac{\partial w^{(p)}}{\partial y_i}(y) \frac{\partial w^{(q)}}{\partial y_k}(y).
\]

Note that \( \rho \) is the exponent in Lemma 2 and formula (2.14) is used to detect the subtrahend on the left of (2.30). The following result is known (cf. [16] [17]) so that we only adapt a standard proof for the Hölder continuous multiplier \( Y \) in the integrand.

**Lemma 5** Let \( Z \in L^\infty(\Sigma) \) and \( Y \in C^{0, \rho}(\omega) \),

\[
Z = |\Sigma|^{-1} \int_{\Omega} Z(\xi) d\xi.
\]

Then

\[
\left| \int_{\Omega} Z \left( \frac{x}{\varepsilon} \right) Y(y) dx - \varepsilon |\sigma| Z \int_{\omega} Y(y) dy \right| \leq c\varepsilon^{1+\rho}.
\]

(2.32)
Proof. According to (1.4), we have
\[
\int_{\Omega_{\varepsilon}} Z \left( \frac{x}{\varepsilon} \right) Y (y) \, dx = \sum_{\nu : |\nu| \leq N} \int_{\Sigma_{\varepsilon}^\nu} Z \left( \frac{x}{\varepsilon} \right) Y (y) \, dx = \\
= \sum_{\nu : |\nu| \leq N} \int_{\Sigma_{\varepsilon}^\nu} Z \left( \frac{x}{\varepsilon} \right) \left( Y (y') + O (\varepsilon^p) \right) = \varepsilon^3 |\Sigma| Z \sum_{\nu : |\nu| \leq N} \left( Y (y') + O (\varepsilon^p) \right) = \\
= \varepsilon \frac{|\Sigma|}{|\sigma|} Z \sum_{\nu : |\nu| \leq N} \left( \int_{\Sigma_{\varepsilon}^\nu} Y (y) \, dy + O (\varepsilon^2 + |\varepsilon|) \right) = \varepsilon \frac{|\Sigma|}{|\sigma|} \int_{\omega} Y (y) \, dy + O (\varepsilon^2).
\]

Here \( \sigma_{\varepsilon}^\nu = \{ y : |y - \eta_{\varepsilon}^\nu | \leq \varepsilon a_i / 2, \, i = 1, 2 \} \) is the cover of the cell (1.3) and \( y' \) its mass center. The number of the cells \( \Sigma_{\varepsilon}^\nu \) composing the plate \( \Omega_{\varepsilon} \) is less than \( C \varepsilon^{-2} \). Furthermore, we have used twice the relation
\[
|Y (y') - Y (y)| \leq C \varepsilon^p, \quad y \in \sigma_{\varepsilon}^\nu,
\]

inherited from the inclusion \( Y \in C^{0, \rho} (\omega) \) and the definition of the Hölder norm (2.16).■

Now formulae (2.29), (2.32) and (2.14) ensure that
\[
\left( \left( u_{\varepsilon}^{(p)}, u_{\varepsilon}^{(q)} \right) - \varepsilon |\sigma|^{-1} \delta_{p,q} \right) \leq C \varepsilon^{1 + \min \{ p, 1 / 2 \}}.
\]

2.5 Calculating the discrepancy \( \delta \).

According to (2.33), we obtain
\[
\left\| u_{\varepsilon}^{(p)} ; H_{\Omega_{\varepsilon}} \right\| \geq \frac{1}{2} \varepsilon^{1 / 2} |\sigma|^{-1 / 2}
\]
for a small \( \varepsilon > 0 \). Thus, by virtue of (2.23), (2.26), (2.19), (2.18), we have
\[
\delta = \left\| b_{\varepsilon} u_{\varepsilon}^{(p)} - b u_{\varepsilon}^{(p)} ; H_{\Omega_{\varepsilon}} \right\| = \left\| u_{\varepsilon}^{(p)} ; H_{\Omega_{\varepsilon}} \right\|^{-1} \left\| b_{\varepsilon} u_{\varepsilon}^{(p)} - u_{\varepsilon}^{(p)} ; H_{\Omega_{\varepsilon}} \right\| = \\
\leq 2 \varepsilon^{-3 / 2} |\sigma|^{-1 / 2} \left( \tau^{(k)} + 1 \right) \left\| \nabla u_{\varepsilon}^{(p)} ; \Omega_{\varepsilon} \right\| + \varepsilon \tau^{(k)} \left( u_{\varepsilon}^{(p)} ; V \right)_{\omega_{+}} + \varepsilon \tau^{(k)} \left( u_{\varepsilon}^{(p)} ; V \right)_{\omega_{-}},
\]

where the supremum is calculated over all \( V \in H^k_{\Omega_{\varepsilon}} \) such that \( \left\| V ; H^k_{\Omega_{\varepsilon}} \right\| = 1 \). Furthermore,
\[
I^{(p)} = \frac{\nabla u_{\varepsilon}^{(p)} , \nabla y_{\varepsilon}}{\Omega_{\varepsilon}} + \varepsilon \tau^{(k)} \left( u_{\varepsilon}^{(p)} , V \right)_{\omega_{+}} = \\
= - \left( \Delta u_{\varepsilon}^{(p)} , V \right)_{\Omega_{\varepsilon}} + \left( \partial_u u_{\varepsilon}^{(p)} , V \right)_{\omega_{+}} + \varepsilon \tau^{(k)} \left( u_{\varepsilon}^{(p)} , V \right)_{\omega_{+}} + \left( \partial_u u_{\varepsilon}^{(p)} , V \right)_{\omega_{-}(\varepsilon)}.
\]

To examine this expression we need auxiliary inequalities.

Lemma 6 1. Let \( V \in H^k_{\Omega_{\varepsilon}} \) and
\[
\nabla (y) = |\Sigma_{\varepsilon}|^{-1} \int_{\Sigma_{\varepsilon}(y)} V (x) \, dx
\]
where
\[
\Sigma_{\varepsilon}(y) = \{ x = (y, z) \in \Sigma_{\varepsilon}^\infty : |y_i - y_i| \leq \varepsilon a_i / 2, \quad i = 1, 2 \}.
\]
and $V$ is extended by zero from $\Omega_\varepsilon$ onto the thin periodic infinite layer. Then the inequality
\[
\|R^{\varepsilon}_c V^\bullet L^2 (\omega)\| + \|\nabla_y V; L^2 (\omega)\| \leq c \varepsilon^{-1/2} \|V; \mathcal{H}^2_\Omega\|
\]  
holds where $R_c (y) = \varepsilon + \text{dist}(y, \partial \omega)$. Moreover,
\[
\varepsilon^{-1} \|V - \nabla ; L^2 (\Omega_\varepsilon)\| + \varepsilon^{-1/2} \|V - \nabla ; L^2 (\omega_+ \cup\omega_- (\varepsilon))\| \leq c \|V; \mathcal{H}^2_\Omega\|.
\]  
2. A function $V \in \mathcal{H}^2_\Omega$ meets the relation
\[
\|R^{\varepsilon}_c V^\bullet L^2 (\Omega_\varepsilon)\| + \varepsilon^{1/2} \|R^{\varepsilon}_c V; L^2 (\omega_+ \cup\omega_- (\varepsilon))\| \leq c \|V; \mathcal{H}^2_\Omega\|.
\]  
Here all constants depend on neither $V$, nor $\varepsilon \in (0, 1)$.

Proof. First of all, we have
\[
\int_\omega |\nabla (\omega) |^2 dy \leq c \varepsilon^{-6} \int_\omega \left| \int_{\Sigma_\varepsilon (y)} |V (x) |^2 dx \right| dy \leq \varepsilon^{-3} \int_\omega \int_{\Sigma_\varepsilon (y)} |V (y, z) |^2 dxdy \leq \varepsilon^{-3} \int_{\Omega_\varepsilon} \int_{\sigma_\varepsilon (y)} dy |V (y, z) |^2 dxdz \leq c \varepsilon^{-1} \|V; L^2 (\Omega_\varepsilon)\|^2 ,
\]
where
\[
\sigma_\varepsilon (y) = \{y : |y - y_i | < \varepsilon a_i / 2 , \ i = 1, 2\}.
\]
Second,
\[
\frac{\partial \nabla (\omega)}{\partial y_i} (\omega) = |\Sigma_\varepsilon |^{-1} \int_{\sigma^+_\varepsilon (\omega)} V (x) dx - \int_{\sigma^-_\varepsilon (\omega)} V (x) dx \leq c \varepsilon^{-3} \int_{\Omega_\varepsilon} |\nabla V (x) | dx \quad \text{for almost all } y ,
\]
where $\sigma^\pm_\varepsilon (\omega) = \{x \in \partial \Sigma_\varepsilon (\omega) : y_i = y_i \pm \varepsilon a_i / 2\}$ are the opposite lateral faces of the periodicity cell $\Sigma_\varepsilon (\omega)$ (see Fig. 5). Now repeating calculation (2.41) yields the estimate of $\|\nabla \nabla V^\bullet L^2 (\Omega_\varepsilon)\|$ in (2.38).

The support of function (2.37) lies in the rectangle $\{y : |y| \leq a_i (2\varepsilon + 1) / 2 , \ i = 1, 2\}$. Thus, integrating the one-dimensional Hardy inequality
\[
\int_0^\infty t^{-2} |V (t) |^2 dt \leq 4 \int_0^\infty \left| \frac{dV (t)}{dt} \right|^2 dt , \ V \in C^1 [0, \infty) , \ V (0) = 0,
\]
and using the completion argument bring the necessary estimate of the first norm in (2.38).

Dealing with the first norm in (2.39), we compute
\[
\int_{\Omega_\varepsilon} |V (y, z) - \frac{1}{|\Sigma_\varepsilon |} \int_{\Sigma_\varepsilon (y)} V (y, z) dydz |^2 dydz =
\]
\[
= \frac{1}{|\Sigma_\varepsilon |^2} \int_{\Omega_\varepsilon} \left| \int_{\Sigma_\varepsilon (y)} (V (y, z) - V (y, z)) dydz \right|^2 dydz \leq c \varepsilon^{-3} \int_{\Omega_\varepsilon} \int_{\Sigma_\varepsilon (y)} |\nabla_x V (x) |^2 dx dydz \leq c \varepsilon^2 \int_{\Omega_\varepsilon} |\nabla_x V (x) |^2 dx .
\]

For the second norm in (2.39), we need to replace in (2.41) the integration set $\Omega_\varepsilon$ by $\omega_+ \cup\omega_- (\varepsilon)$. As a result, the bound changes for $c \varepsilon \|\nabla_x V; L^2 (\Omega_\varepsilon)\|^2$. 

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Inequality (2.40) is a direct consequence of estimates (2.38), (2.39) together with the evident relations $R_\varepsilon(x)^{-1} \leq \varepsilon^{-1}, \varepsilon^{1/2} R_\varepsilon(x)^{-1} \leq \varepsilon^{-1/2}$ and

\[
\varepsilon^{-1/2} \left\| R_\varepsilon^{-1} \nabla; L^2(\Omega_\varepsilon) \right\| + \left\| R_\varepsilon^{-1} \nabla; L^2(\omega_+ \cup \omega_-(\varepsilon)) \right\| \leq c \left\| R_\varepsilon^{-1} \nabla; L^2(\omega) \right\|. \tag{2.43}
\]

Now we are in position to simplify expression (2.36) and, neglecting inessential terms and changing $V$ for $\overline{V}$, to derive that

\[
\left| I_{(p)} - \left( \Delta_y u_\varepsilon^{(p)} + 2 \sum_{i=1}^2 \nabla_i W_i \cdot \nabla_y \left( \frac{\partial u_\varepsilon^{(p)}}{\partial y_i} x_\varepsilon \nabla \right) + \varepsilon^2 \sum_{i=1}^2 W_i n^* \cdot \nabla_y \left( \frac{\partial u_\varepsilon^{(p)}}{\partial y_i} x_\varepsilon \nabla \right) \omega_-(\varepsilon) \right) \right| \leq c\varepsilon^{p+1/2}.
\]

We start with the simplest term

\[
\left( \partial_z U_\varepsilon^{(k)} + \varepsilon \tau(k) U_\varepsilon^{(k)}, V \right)_{\omega_+} = \left( X \sum_{i=1}^2 \frac{\partial w_\varepsilon^{(p)}}{\partial y_i} \left( \partial_z W_i + \varepsilon^2 \tau(k) W_i \right) + \varepsilon \tau(k) w_\varepsilon^{(p)}, V \right)_{\omega_+} =: I_{(p)}^{(2)}.
\]

Owing to (2.77), we here have $\partial_z W_i = 0$ at $\zeta = 0$, and, hence,

\[
\left| I_{(p)}^{(2)} - \varepsilon \tau(k) \left( w_\varepsilon^{(p)}, x_\varepsilon \nabla \right)_{\omega_+} \right| \leq c\varepsilon \tau(k) \left( \varepsilon \left( X \sum_{i=1}^2 \frac{\partial w_\varepsilon^{(p)}}{\partial y_i} W_i, V \right) + \varepsilon \left( (1 - X) w_\varepsilon^{(p)}, V \right)_{\omega_+} + \left( X w_\varepsilon^{(p)}, V - \nabla \right)_{\omega_+} \right) \leq c\varepsilon \left( \varepsilon \left( \left\| V ; L^2(\omega_+) \right\| \right) + \varepsilon^{1/2} \left\| \left( \varepsilon^{1/2} R_\varepsilon^{-1} V ; L^2(\omega_+) \right) \right\| \right) \leq c\varepsilon^{3/2}.
\]

For the first term (with $W_i$), we readily used the Schwarz inequality. For the second term (with $1 - X_\varepsilon$), we took into account that $R_\varepsilon(x) \leq \varepsilon\varepsilon$ on supp$(1 - X_\varepsilon)$ and applied estimate (2.40). For the third term (with $V - \nabla$), we used estimate (2.39).

Similar argument works for the last term $I_{(p)}^{(3)}$ in (2.36). Recalling boundary conditions in problem (2.77) for the asymptotic correctors $W_i$, we, indeed, obtain

\[
I_{(p)}^{(3)} = \left( n^* \cdot \nabla_y w_\varepsilon^{(p)} + X \sum_{i=1}^2 \left( \frac{\partial w_\varepsilon^{(p)}}{\partial y_i} \partial_n \left( \frac{\partial w_\varepsilon^{(p)}}{\partial y_i} x_\varepsilon \nabla \right) \right) + \varepsilon U_\varepsilon^{(p)} n^* \cdot \nabla_y X_\varepsilon, V \right)_{\omega_-(\varepsilon)} = \left( (1 - X_\varepsilon) n^* \cdot \nabla_y w_\varepsilon^{(p)} + \varepsilon U_\varepsilon^{(p)} n^* \cdot \nabla_y X_\varepsilon + \varepsilon X_\varepsilon \sum_{i=1}^2 W_i n^* \nabla_y w_\varepsilon^{(p)} \right)_{\omega_-(\varepsilon)}
\]

and, therefore,

\[
\left| I_{(p)}^{(3)} - \varepsilon \left( \sum_{i=1}^2 W_i n^* \nabla_y w_\varepsilon^{(p)} x_\varepsilon \nabla \right)_{\omega_-(\varepsilon)} \right| \leq c \left( \left\| \varepsilon \left( \left\{ \text{meas}_2(\text{supp}(1 - X_\varepsilon)) \right\}^{1/2} \varepsilon^{1/2} \left\| \left( \varepsilon^{1/2} R_\varepsilon^{-1} V ; L^2(\omega_-(\varepsilon)) \right) \right\| \right) + \varepsilon \sup_{y \in \text{supp}X_\varepsilon} R(x)^{\rho-1} \left\| V - \nabla \right\| H^0(\omega_-(\varepsilon)) \right) \leq c\varepsilon^{\min\{1, \rho+1/2\}}.
\]

Here we applied inequalities (2.44), (2.35), (2.39) while taking into account the obvious relations

\[
\text{meas}_2(\text{supp}(1 - X_\varepsilon)) = O(\varepsilon), \sup_{y \in \text{supp}X_\varepsilon} R(x)^{\rho-1} = O(\varepsilon^{\rho-1}), \rho \in (0, 1].
\]
It remains to examine the term
\[ I_1^{(p)} = - \left( \Delta_x U_\varepsilon^{(p)}, V \right)_{\Omega_\varepsilon}, \] (2.46)
where, according to (2.13),

\[
\Delta_x U_\varepsilon^{(p)} = \Delta_y w^{(p)}(y) + \varepsilon U_\varepsilon^{(p)}(x) \Delta_y X_\varepsilon + 2\varepsilon \nabla_y U_\varepsilon^{(p)} \cdot \nabla_y X_\varepsilon + \\
+ X_\varepsilon S^2 \sum_{i=1}^2 \left( \varepsilon^{-1} \Delta_\varepsilon W_i(\xi) \frac{\partial w^{(p)}}{\partial y_i}(y) + 2\nabla_y W_i(\xi) \cdot \nabla_y \frac{\partial w^{(p)}}{\partial y_i}(y) + \varepsilon W_i(\xi) \Delta_y \frac{\partial w^{(p)}}{\partial y_i}(y) \right). 
\]

Since \( W_i \) is a harmonics, the first term in the sum vanishes. We now list down estimates permitting to neglect some other terms:

\[
\left| (1 - X_\varepsilon) \Delta_y w^{(p)}, V \right|_{\Omega_\varepsilon} \leq c \int_{\Omega_\varepsilon \setminus \text{supp}(1 - X_\varepsilon)} R(x)^{-1+p} |V(x)| \, dx \leq \\
\leq c \varepsilon \left( \int_{-\varepsilon}^{\varepsilon} \int_{0}^{\varepsilon R - 2+2p} R^d R \, dx \right) \varepsilon \left\| R^{-1}V ; L^2(\Omega_\varepsilon) \right\| \leq c\varepsilon^{p+3/2}, \] (2.47)

\[
\varepsilon \left| \left( U_\varepsilon^{(p)} \Delta_y X_\varepsilon + 2\nabla_y U_\varepsilon^{(p)} \cdot \nabla_y X_\varepsilon, V \right) \right|_{\Omega_\varepsilon} \leq c \varepsilon \int_{\Omega_\varepsilon \setminus \text{supp} \nabla_y X_\varepsilon} \left( \varepsilon^{-1} + R(x)^{-1+p} \right) |V(x)| \, dx \leq \\
\leq c \varepsilon (\text{meas sup} \nabla_y X_\varepsilon)^{1/2} \left\| R^{-1}V ; L^2(\Omega_\varepsilon) \right\| \leq c\varepsilon^2, \]

\[
\left| X_\varepsilon \left( \Delta_y w^{(p)} + 2 \sum_{i=1}^2 \nabla_y W_i \cdot \nabla_y \frac{\partial w^{(p)}}{\partial y_i}, V - \nabla \right) \right|_{L^2(\Omega_\varepsilon)} \leq \\
\leq c \varepsilon^{-1+p} (\text{meas sup} \Omega_\varepsilon)^{1/2} \left\| V - \nabla ; L^2(\Omega_\varepsilon) \right\| \leq c\varepsilon^{p+1/2}. \]

Here we have applied the same arguments as above.

Inequalities (2.47) help to estimate all terms in (2.46) with exception of the first subtrahend on the left of (2.48). Other two subtrahends were exhibited in (2.13) and (2.14); hence, relation (2.15) is verified.

Similarly to the proof of Lemma 5, we now consider the integrals over the cells \( \Sigma_\varepsilon \) and their bases \( \sigma_\varepsilon^+ \) and \( \sigma_\varepsilon^- \), namely,

\[
\int_{\Sigma_\varepsilon^+} \Delta_y w^{(p)}(y) + \sum_{i=1}^2 \nabla_y W_i \left( \frac{x}{\varepsilon}, \frac{y}{\varepsilon} \right) \cdot \nabla_y \frac{\partial w^{(p)}}{\partial y_i}(y) X_\varepsilon(y) V(y) \, dy + \\
+ \varepsilon \int_{\sigma_\varepsilon^-} W_i \left( \frac{x}{\varepsilon}, \frac{y}{\varepsilon} \right) n^* \left( \frac{x}{\varepsilon}, \frac{y}{\varepsilon} \right) \cdot \nabla_y \frac{\partial w^{(p)}}{\partial y_i}(y) X_\varepsilon(y) V(y) \, dy + \varepsilon \tau(k) \int_{\sigma_\varepsilon^+} w^{(p)} y X_\varepsilon(y) \nabla(y) \, dy. \] (2.48)

Freezing here the argument \( y \) at the mass center \( y^* \) of the rectangle \( \sigma_\varepsilon^+ \), we perform integration in \( \xi = \varepsilon^{-1} x \) and, recalling calculation (2.10), we obtain that expression (2.48) turns into the expression

\[
\varepsilon^3 \left( \nabla_y \cdot b \nabla_y w^{(p)}(y^*) + \tau(k) \left| \sigma \right| w^{(p)}(y^*) \right) X_i(y^*) \nabla(y^*)
\]

which vanishes because \( \{ \tau(k), w^{(p)} \} \) is an eigenpair of problem (2.9), (2.11). The error of the freezing procedure does not exceed

\[
c\varepsilon \int_{\sigma_\varepsilon^+} R(y)^{-1+p} |X_\varepsilon(y) \nabla(y) - X_\varepsilon(y^*) \nabla(y^*)| \, dy \] (2.49)
while the weight factor $R^{-1+\rho}$ comes from inequality (2.15) with $k = 2$. Since

$$X_\varepsilon (y) \nabla (y) - X_\varepsilon (y^\rho) \nabla (y^\rho) =$$

$$= \int_{y_1}^{y_2} \frac{\partial (X_\varepsilon \nabla)}{\partial y_1} (y_1, y_2) dy_1 + \int_{y_1}^{y_2} \frac{\partial (X_\varepsilon \nabla)}{\partial y_2} (y_1, y_2) dy_2,$$

quantity (2.39) may be bounded from above by

$$c\varepsilon^2 \int_{\sigma^\varepsilon^+} R^{-1+\rho} \left| \nabla_y (X_\varepsilon \nabla) \right| dy.$$

Summing over $\nu_i \in [-N_i, N_i]$ and applying relation (2.38), we get the following bound for the sum of all integrals (2.49):

$$c\varepsilon^2 \int_{\omega} R^{-1+\rho} \left| \nabla_y (X_\varepsilon \nabla) \right| dy \leq c\varepsilon^{1+\rho} \int_{\omega} \left| \nabla_y \nabla \right|^2 + \left| \nabla_y X_\varepsilon \right|^2 \left| \nabla \right|^2 dy \leq$$

$$\leq c\varepsilon^{1+\rho} \int_{\omega} \left( \left| \nabla_y \nabla (y) \right|^2 + R^{-2} \left| \nabla \right|^2 \right) dx \leq c\varepsilon^{\rho+1/2} \left\| V; \mathcal{H}_\Omega \right\| \leq c\varepsilon^{\rho+1/2}.$$

Collecting the estimates obtained above, we conclude finally that

$$\delta = \left\| B u_\varepsilon^{(p)} - u_\varepsilon^{(p)}; \mathcal{H}_\Omega \right\| \leq c\varepsilon^{\rho-1}. \quad (2.50)$$

### 2.6 The theorem on asymptotics of eigenvalues.

Owing to (2.50), Lemma 1 delivers an eigenvalue $\beta_\varepsilon^{(q)}$ of the operator $\mathcal{B}_\varepsilon$ such that

$$\left| \beta_\varepsilon^{(q)} - \varepsilon^{-1} \left( \tau^{(k)} + 1 \right)^{-1} \right| \leq c\varepsilon^{\rho-1}. \quad (2.51)$$

By (2.24), we have $\beta_\varepsilon^{(q)} = \left( \varepsilon + \alpha_\varepsilon^{(q)} \right)^{-1}$ and, hence,

$$\left| \alpha_\varepsilon^{(q)} - \varepsilon \tau^{(k)} \right| \leq c\varepsilon^{\rho-1} \varepsilon \left( \tau^{(k)} + 1 \right) \left( \varepsilon + \alpha_\varepsilon^{(q)} \right). \quad (2.52)$$

This particularly gives

$$\alpha_\varepsilon^{(q)} \leq \varepsilon \tau^{(k)} + c\varepsilon^{\rho+1} \left( \tau^{(k)} + 1 \right) + c\varepsilon^{\rho} \varepsilon \left( \tau^{(k)} + 1 \right) \alpha_\varepsilon^{(q)}.$$

Thus, with a small $\varepsilon^{(k)} > 0$ and $\varepsilon \in \left( 0, \varepsilon^{(k)} \right]$, we obtain

$$\alpha_\varepsilon^{(q)} \leq 1 - c\varepsilon^{\rho} \left( \tau^{(k)} + 1 \right) \varepsilon \left( \tau^{(k)} + c\varepsilon^{\rho} \left( \tau^{(k)} + 1 \right) \right) \leq c\varepsilon^{\rho} \varepsilon$$

and, according to (2.52),

$$\left| \alpha_\varepsilon^{(q)} - \varepsilon \tau^{(k)} \right| \leq c\varepsilon^{1+\rho}. \quad (2.53)$$

**Theorem 7** Let $\tau^{(k)}$ be an eigenvalue of problem (2.9), (2.11) with multiplicity $\kappa_k$, i.e., (2.27) holds true. There exist $\varepsilon^{(k)} > 0$ and $C_\varepsilon > 0$ such that the eigenvalue sequence (1.11) of problem (1.8)-(1.10) has at least $\kappa_k$ entries $\alpha_\varepsilon^{(Q)}, \ldots, \alpha_\varepsilon^{(Q+k-1)}$ which satisfy inequality (2.53).
**Proof.** It suffices to compute a bound for the number of eigenvalues $\alpha_{\varepsilon}^{(q)}$ in (2.53). Employing again Lemma 4 we set $\delta_1 = T c_k \varepsilon^{-1}$ where $c_k$ is taken from (2.50). Then, for $p = k, \ldots, k + \chi_k - 1$, we get coefficients $f_J^{(p)}$ such that

$$
\left\| u_{\varepsilon}^{(p)} - \sum_{j=J}^{J+X - 1} f_J^{(p)} u_{\varepsilon}^{(j)}; \mathcal{H}_{[1]} \right\| \leq \frac{2 \delta_1}{\delta_1} \leq \frac{2}{T}
$$

(2.54)

where $\beta_{\varepsilon}^{(J)}, \ldots, \beta_{\varepsilon}^{(J+X-1)}$ is the list of all eigenvalues in (2.18) which meet the inequality

$$
\left| \beta_{\varepsilon}^{(q)} - \varepsilon^{-1} \left( \tau^{(k)} + 1 \right)^{-1} \right| \leq T c_k \varepsilon^{1+\rho}.
$$

(2.55)

Recall that the coefficient columns $f_J^{(p)} = \left(f_J^{(p)}(J), \ldots, f_J^{(p)}(J+X-1)\right)^T \in \mathbb{R}^{X(\varepsilon)}$ are of unit length. Moreover, by (2.33) and (2.21), we have

$$
\delta_{p,q} + O(\varepsilon^{\min\{\rho,1/2\}}) = \left\langle u_{\varepsilon}^{(p)}, u_{\varepsilon}^{(q)} \right\rangle \varepsilon = \left\langle \sum_{j=J}^{J+X - 1} f_J^{(p)} u_{\varepsilon}^{(j)}, \sum_{h=J}^{J+X - 1} f_J^{(q)} u_{\varepsilon}^{(h)} \right\rangle + O(T^{-1}) = \left(f_J^{(p)} \right)^T f_J^{(q)} + O(T^{-1}).
$$

Thus, for small $\varepsilon$ and $T^{-1}$, the columns $f_J^{(k)}$, $\ldots$, $f_J^{(k+X-1)}$ are linear independent in $\mathbb{R}^{X(\varepsilon)}$ so that $\chi_k \leq X(\varepsilon)$. Since inequality (2.55) is just of the same kind as inequality (2.54) which has resulted in (2.53), the proof of the theorem is completed. \[\Box\]

### 3 Spectra of the problems

#### 3.1 Variational formulation of problems.

Let $\mathcal{H}$ denote the Sobolev space $H^1(\Pi(\varepsilon))$ equipped with the specific norm

$$
\|\Phi; \mathcal{H}\| = \left(\|\nabla_x \Phi; L^2(\Pi(\varepsilon))\|^2 + \|\Phi; L^2(\Lambda(\varepsilon))\|^2\right)^{1/2}
$$

(3.1)

and the corresponding inner product (cf. (2.22)). We also introduce the weighted Sobolev space $W_\theta$ as the completion of $C_0^\infty(\Pi(\varepsilon))$ (infinitely differentiable functions with compact supports) with respect to the norm

$$
\|\Phi; W_\theta\| = \|R_\theta \Phi; \mathcal{H}\|
$$

(3.2)

where $R_\theta = \exp\left(\theta (1 + x_1^2)^{1/2}\right)$ and $\theta \in \mathbb{R}$. This space consists of all functions $\Phi \in H^1_{\text{loc}}(\Pi(\varepsilon))$ with the finite norm (3.2). Clearly, $W_0 = \mathcal{H}$. If $\theta > 0$, a function $\Phi \in W_\theta$ decays exponentially as $x_1 \to \pm \infty$ but the space $W_\theta$ with $\theta < 0$ includes functions with a certain exponential growth at infinity.

The standard formulation (19) of the spectral problem (1.5)-(1.7) reads: to find $\lambda = \mathbb{C}$ and $\Phi \in \mathcal{H}$ such that

$$
(\nabla_x \Phi, \nabla_x \Psi)_{\Pi(\varepsilon)} = \lambda (\Phi, \Psi)_{\Lambda(\varepsilon)} \, , \quad \Psi \in \mathcal{H}.
$$

(3.3)

For a fixed $\lambda$, we also consider the integral identity

$$
(\nabla_x \Phi, \nabla_x R_\theta^2 \Psi)_{\Pi(\varepsilon)} - \lambda (\Phi, R_\theta^2 \Psi)_{\Lambda(\varepsilon)} = F_\theta (\Psi) \, , \quad \Psi \in W_\theta,
$$

(3.4)

serving for the inhomogeneous problem (1.5)-(1.7) while $F \in \mathcal{H}^*$ is a linear functional in the Hilbert space $\mathcal{H}$. A generalized solution of problem (1.5)-(1.7) in the weighted space $W_\theta$ implies a function $\Phi \in W_\theta$ such that

$$
(\nabla_x \Phi, \nabla_x (R_\theta^2 \Psi))_{\Pi(\varepsilon)} - \lambda (\Phi, R_\theta^2 \Psi)_{\Lambda(\varepsilon)} = F_\theta^* (\Psi) \, , \quad \Psi \in W_\theta,
$$

(3.5)
where $F_{\theta} \in W_{a, \theta}^{*}$. Formally, (3.5) is derived from (3.4) by changing the test function $\Psi$ for the product $R_{\theta}^{2}\Psi$. Notice that the linear space $C^{\infty}_{c}(\Pi(\varepsilon))$ is dense in $W_{\theta}$ with any weight index $\theta$.

By the definition of the weighted norm (3.2), $R_{\theta}^{2}\Psi \in W_{-\theta}$ in case $\Psi \in W_{\theta}$. Hence, $(\cdot, \cdot)_{\Pi(\varepsilon)}$ and $(\cdot, \cdot)_{\Lambda(\varepsilon)}$ stand in (3.3) for extensions of the natural inner products in $L^{2}(\Pi(\varepsilon))$ and $L^{2}(\Lambda(\varepsilon))$ up to the duality between proper weighted Lebesgue spaces.

Let $\theta > 0$ and $F \in W_{a, \theta}^{*} \subset H^{*}$ while $F_{\theta}(\Psi) = F(R_{\theta}^{2}\Psi)$ so that $F_{\theta} \in W_{a, \theta}^{*}$ as well. Then, if $\Phi \in W_{\theta}$ is a solution of problem (3.5), $\Phi$ belongs to $H$ and is a solution of problem (3.4) with an exponential decay at infinity. Viceversa, in the case $\theta < 0$ a solution $\Phi \in H$ of problem (3.4), where $F \in H^{*} \subset W_{a, \theta}^{*}$, becomes a solution of problem (3.5) in $W_{\theta}$.

Under the symmetry assumption (1.13), the same definition works for the problem posed on the set $\Pi^{+}_{c}$ with the artificial Dirichlet conditions (1.14). We use the notation $H^{0}$ and $W_{\theta}^{0}$ for the function space (1.13) and the similar weighted space of odd functions. Moreover, integral identities for this problem on $\Pi^{+}_{c}$ are refereed as the identities (3.4), (3.5) restricted onto the subspaces $H^{0}$ and $W_{\theta}^{0}$, respectively.

### 3.2 The operator formulation of problems.

In the Hilbert space $H$ with the inner product $\langle \cdot, \cdot \rangle$, generated by norm (3.1), we introduce the operator $T_{\varepsilon}$ by the formula

$$
\langle T_{\varepsilon} \Phi, \Psi \rangle = (\Phi, \Psi)_{\Lambda(\varepsilon)}, \quad \Phi, \Psi \in H
$$

(cf. formulae (2.18) and (2.19) in the domain $\Omega_{\varepsilon}$). This operator is continuous with the unit norm, positive and self-adjoint but not compact because the surface $\Lambda(\varepsilon)$ is unbounded. Thus its spectrum lies on the segment $[0, 1]$ of the real axis $\mathbb{R} \subset \mathbb{C}$ and its essential spectrum does not reduce to the single point $\mu = 0$ (see, e.g., [21, Ch. 10]).

The restriction of $T_{\varepsilon}$ on the subspace $H^{0}$ is denoted by $T^{0}_{\varepsilon}$. Clearly, $T^{0}_{\varepsilon}$ acts from $H^{0}$ into $H^{0}$. If $\mu$ is an eigenvalue of the operator $T^{0}_{\varepsilon}$ with the eigenfunction $\Phi^{0}_{\varepsilon} \in H^{0}$, analogously to (2.24),

$$
\lambda = \mu^{-1} - 1
$$

is an eigenvalue of problem (3.3) restricted on $H^{0}$, i.e., of the operator $L^{0}_{\varepsilon}$. Moreover, the odd extension $\Phi_{\varepsilon} \in H$ of the function $\Phi^{0}_{\varepsilon}$ over the plane $\{x : x_{2} = 0\}$ becomes an eigenfunction of problem (3.3) (and problem (1.5)–(1.7) on $\Pi(\varepsilon)$) corresponding to the same eigenvalue (3.7). This observation will be a tool to prove Theorem 1, the main result of the paper.

Formula (3.7) establishes a direct relation between the $\lambda$--spectrum of the operator $L_{\varepsilon}$ of problem (3.3) and the $\mu$--spectrum of $T_{\varepsilon}$. Thus, we only examine the spectra of the operators $T_{\varepsilon}$ and $T^{0}_{\varepsilon}$ in the sequel.

### 3.3 Continuous spectra.

Clearly, the point $\mu = 0$ is an eigenvalue of the operator $T_{\varepsilon}$ with the infinite-dimensional eigenspace

$$
\{ \Phi_{\varepsilon} \in H : \Phi_{\varepsilon} = 0 \text{ on } \Lambda(\varepsilon) \}
$$

A similar conclusion holds true for the operator $T_{\varepsilon}^{+}$.

**Lemma 8** The segment $(0, 1] \subset \mathbb{R} \subset \mathbb{C}$ is filled with the continuous spectrum of the operator $T_{\varepsilon}$.

**Proof.** The assertion follows from general results [27] (see also §5.1 in [29]). For the reader convenience, we show here shortly how to construct a singular Weyl sequence for any $\mu \in (0, 1)$ so that $\mu$ belongs to the essential spectrum of $T_{\varepsilon}$. Since the operator of problem (3.3) regarded as the mapping $W_{\theta} \rightarrow W_{a, \theta}^{*}$ is Fredholm for a sufficiently small negative $\theta$ (see [27], [29] Theorem 5.1.4) and comments on the model problem (3.3) below), the kernel of this operator at $\theta = 0$, regarded as the mapping $H \rightarrow H^{*}$,
is finite-dimensional. Thus, any point \( \mu \in (0, 1] \) of the essential spectrum lies in the continuous spectrum of \( \mathcal{T}_\epsilon \).

Let consider the model problem on the cross-section of the canal \( \Pi \), namely

\[
- \Delta_{x'} \varphi (x') + \eta^2 \varphi (x') = 0, \quad x' \in \Gamma, \\
\partial_{x'} \varphi (x') = \lambda \varphi (x'), \quad x' \in \gamma_0, \quad \partial_{\eta} \varphi (x') = 0, \quad x' \in \gamma.
\]  

(3.9)

Problem (3.9) is obtained by the Fourier transform from the problem of type (3.5)-(3.7) in the cylindrical channel \( \Pi = \mathbb{R} \times \Gamma \) while \( \eta \in \mathbb{R} \) is the dual Fourier variable for \( x_1 \). Let \( \mathcal{A} (\lambda) \) be an unbounded operator in \( L^2 (\Gamma) \) associated (see [21 Ch.10]) with the bi-linear form

\[
Q (\lambda, \varphi, \varphi) = (\nabla_{x'} \varphi, \nabla_{x'} \psi)_{\Gamma} - \lambda (\varphi, \psi)_{\gamma_0}.
\]  

(3.10)

This operator is self-adjoint and bounded from below. Its domain belongs to \( H^1 (\Gamma) \). Since the embedding \( H^1 (\Gamma) \subset L^2 (\Gamma) \) is compact, and

\[
Q (\lambda_1; \varphi, \varphi) \geq Q (\lambda_2; \varphi, \varphi), \quad \lambda_2 \geq \lambda_1, \quad \varphi \in H^1 (\Omega), \\
Q (\lambda, 1, 1) < 0 \quad \text{for} \quad \lambda > 0,
\]

(3.11)

Theorems 10.1.2, 10.1.5, 10.2.4 in [21] ensure that the spectrum of \( \mathcal{A} (\lambda) \) is discrete and form the eigenvalue sequence

\[
\eta_1 (\lambda)^2 < \eta_2 (\lambda)^2 \leq \eta_3 (\lambda)^2 \leq \ldots \leq \eta_k (\lambda)^2 \leq \ldots \to +\infty
\]

(3.12)

while \( \mathbb{R}_+ \ni \lambda \to \eta_1 (\lambda)^2 \) is a continuous, strictly monotone decreasing negative function. The first eigenvalue \( \eta_1 (\lambda)^2 \) is simple due to the maximum principle and \( \eta_1 (\lambda) = \pm i |\eta_1 (\lambda)| \) is imaginary. Let \( \varphi_1 (\lambda, x') \) be the first eigenfunction of problem (3.9). We set

\[
\Phi^m (x) = a_m X_m \left( (2\pi)^{-1} |\eta_1 (\lambda)| x_1 \right) \sin \left( |\eta_1 (\lambda)| x_1 \right) \varphi_1 (\lambda, x'),
\]

(3.13)

where \( a_m \) is a normalization factor, \( X_m \) is the plateau function in Fig. 6

\[
X_m (t) = \chi (t - 2m) \chi (2m+1 - t),
\]

(3.14)

and \( \chi \in C^\infty (\mathbb{R}) \) is a cut-off function, \( \chi (t) = 0 \) for \( t \leq 0 \) and \( \chi (t) = 1 \) for \( t \geq 1 \). The function \( X_m \) is equal to one on the segment

\[
\left[ 2\pi |\eta_1 (\lambda)|^{-1} (2m + 1), 2\pi |\eta_1 (\lambda)|^{-1} (2m+1 - 1) \right] \ni x_1
\]

(3.15)

and both functions (3.14) and (8.13) vanish for

\[
x_1 \notin \left[ 2\pi |\eta_1 (\lambda)|^{-1} 2m, 2\pi |\eta_1 (\lambda)|^{-1} 2m+1 \right].
\]

(3.16)

In the case \( \lambda = 0 \) we simply set \( \Phi^m (x) = a_m X_m (x_1) \). We choose an integer \( m \) such that \( \Theta (\varepsilon) \subset \{ x \in \Pi : x_1 > 2\pi |\eta_1 (\lambda)|^{-1} 2m \} \) and obtain

\[
\langle \Phi^m, \Phi^m \rangle \geq a_m \int_{2\pi |\eta_1 (\lambda)|^{-1} (2m+1)}^{2\pi |\eta_1 (\lambda)|^{-1} (2m+1)} \left( \int_{\Gamma} \left( |\nabla_x \varphi_1 (\lambda; x')|^2 \left[ \sin \left( 2\pi |\eta_1 (\lambda)|^{-1} x_1 \right) \right]^2 + \eta_1 (\lambda)^2 |\varphi_1 (\lambda, x')|^2 \cos \left( 2\pi |\eta_1 (\lambda)|^{-1} x_1 \right) \right)^2 dx' + \int_{\gamma_0} |\varphi (\lambda; x')|^2 dx_2 \left[ \sin \left( 2\pi |\eta_1 (\lambda)|^{-1} x_1 \right) \right]^2 \right) dx_1 = \]

\[
= a_m^2 \pi |\eta_1 (\lambda)|^{-1} (2m+1 - 2m - 2) \left( ||\nabla_x \varphi_1; L^2 (\Gamma)||^2 + |\eta_1 (\lambda)|^2 ||\varphi_1; L^2 (\Gamma)||^2 + ||\varphi_1; L^2 (\gamma_0)||^2 \right).
\]

(3.17)

We fix \( a_m = O (2^{-m/2}) \) such that the last expression equals 1. A similar calculation shows that \( \langle \Phi^m, \Phi^m \rangle \) is bounded from above uniformly in \( m \). The supports of the functions \( \Phi^m \) and \( \Phi^n \)
with $m \neq n$ are disjoint due to (3.16) Hence, $\{\Phi^{(m)}\}$ converges to zero weakly in $H$ as $m \to +\infty$ and $\{\Phi^{(m)}\}$ implies a Weyl sequence provided

$$
\left\| T_2 \Phi^{(m)} - \mu \Phi^{(m)} \right\| \to 0 \text{ as } m \to +\infty.
$$

(3.18)

We have

$$
\left\| T_2 \Phi^{(m)} - \mu \Phi^{(m)} \right\| = \sup \left\| \left( T_2 \Phi^{(m)} - \mu \Phi^{(m)} \right), \Psi \right\| = \mu \sup \left\| \left( \nabla \Phi^{(m)} \right)_{\Pi} - \lambda \left( \Phi^{(m)} \right)_{\Lambda} \right\| = \mu \sup \left\| \left( \Delta \Phi^{(m)} \right)_{\Pi} + \left( \partial_n \Phi^{(m)} \right)_{\Lambda} \right\|.
$$

Here the supremum is calculated over all $\Psi \in H$ such that $\|\Psi; H\| = 1$. By the definition of $\eta_1 (\lambda)$ and $\varphi_1 (\lambda, \cdot)$ as a solution of problem (3.10), function (3.11) satisfies the boundary condition (1.7) on $\Lambda$ and it is harmonic on a part of the cylinder $\Pi$ determined by relation (3.15). Thus, the expression (3.19) reduces to integral over the finite cylinders

$$
\left\{ x \in \Pi : (2\pi)^{-1} |\eta_1 (\lambda)| x_1 \in (2^m, 2^{m+1}) \right\}, \text{ and } \left\{ x \in \Pi : (2\pi)^{-1} |\eta_1 (\lambda)| x_1 \in (2^{m+1} - 1, 2^m) \right\}.
$$

and, therefore, it does not exceed $c \mu_m \|\Psi ; L^2 (\Pi)\| \leq c 2^{-m/2}$. The proof is completed. 

The model problem on the cross section $\Gamma^+ = \{ x' \in \Gamma : x_2 > 0 \}$ of cylinder (1.12)

$$
- \Delta x' \varphi^0 (x') + \eta_x^2 \varphi^0 (x') = 0, \quad x' \in \Gamma^+, \quad \varphi^0 (x') = 0, \quad x' \in \varpi^0,
$$

$$
\partial_{x'} \varphi^0 (x') = \lambda_0^0 \varphi^0 (x'), \quad x' \in \gamma^+, \quad \partial_{n} \varphi^0 (x') = 0, \quad x' \in \gamma_0^+
$$

(3.19)

corresponds to the operator $T^0_2$ of problem (3.5) restricted on $H^0$. Here $\varpi^0 = \{ x' \in \Gamma : x_2 = 0 \}$ and the curves $\gamma^+, \gamma_0^+$ compose the boundary of $\Gamma$.

First of all, we put $\eta = 0$ and denote by $\lambda_1^0$ the first eigenvalue of problem (3.19) with the spectral Steklov boundary condition. In view of the Dirichlet boundary condition, we have $\lambda_1^0 > 0$ and

$$
\mu_0^1 = (1 + \lambda_1^0)^{-1} \in (0, 1),
$$

(3.20)

where $\varphi_1^0$ denotes the corresponding eigenfunction. Notice that the trace inequality in $\Gamma^+$ reads:

$$
\left\| \varphi ; L^2 (\gamma_0^+) \right\| \leq (\lambda_1^0)^{-1} \left\| \nabla x' \varphi ; L^2 (\Gamma^+) \right\|, \quad \varphi \in H^1 (\Gamma^+), \quad \varphi = 0 \text{ on } \varpi^0.
$$

(3.21)

We now examine the spectrum of the operator $T^0_2$ in the space $H^0$ which lies in the segment $[0, 1]$ as well as the spectrum of $T_2$. 

Figure 6: The plateau function.
Lemma 9 The segment \((0, \mu_1^0]\) is covered with the continuous spectrum and the segment \((\mu_1^0, 1]\) contains discrete spectrum of the operator \(T_0^+\). The point \(\mu = 0\) is an eigenvalue with the infinite-dimensional eigenspace \(\{\Phi^+ \in H^0 : \Phi^+_\epsilon = 0 \text{ on } \Lambda (\epsilon)\}\).

Proof. To problem (3.19) with the spectral parameter \(\eta\), we associate the unbounded operator \(A^0 (\lambda)\) in the same way as in the proof of Lemma 8. If \(\lambda > \lambda_1^0\), the operator \(A^0 (\lambda)\) meets the estimate of its solution

\[
\langle A^0 (\lambda) \varphi^1, \varphi^0 \rangle = (\lambda_1^0 - \lambda) \varphi^0, \varphi^0 < 0
\]

and, therefore, the first eigenvalue \(\eta_1^0 (\lambda^2)\) (cf. (3.12)) of problem (3.19) with the fixed parameter \(\lambda > \lambda_1^0\) is negative. The same constructions as in (3.13) form the singular Weyl sequence. If \(\lambda = \lambda_1^0\), we set \(\Phi (\epsilon, x) = a_m X_m (x_1) \varphi^0 (x')\) and again conclude that point (3.20) belongs to the continuous spectrum of \(T_0^+\). It suffices to verify that \((\mu_0^0, 1]\) contains the discrete spectrum only. To this end, we deal with the perturbed problem (3.4) restricted on the subspace \(H^0\), namely

\[
(\nabla x \Phi^+, \nabla x \Psi)_{\Pi^+ (\epsilon)} + \lambda M (\Phi^+, \Psi)_{\Pi^+ (\epsilon, L)} - \lambda (\Phi^+, \Psi)_{\Lambda^+ (\epsilon)} = F^+ (\Psi), \quad \Psi \in H^0,
\]

(3.22)

where \(\Pi^+ (\epsilon, L) = \{x \in \Pi^+ (\epsilon) : |x_1| < L\}\) and \(L\) is chosen such that \(\Theta (\epsilon) \subset \{x \in \Pi (\epsilon) : |x_1| < L\}\). Since the embedding \(H^0 \subset L^2 (\Pi^+ (\epsilon, L))\) is compact, the difference of the operator \(T_0^+\) and the operator \(T_0^+ \Pi_+ L\), given by

\[
\langle T_0^+ \Pi_+ L, \Phi, \Psi \rangle = (\Phi, \Psi)_{\Lambda^+ (\epsilon)} - M (\Phi, \Psi)_{\Pi^+ (\epsilon, L)}, \quad \Phi, \Psi \in H^0,
\]

is a compact operator. If we find \(M\) such that the problem (3.22) is uniquely solvable for \(\lambda \in [0, \lambda_1^0]\) and, thus, the segment \((\mu_1^0, 1]\) is free of the spectrum of \(T_0^+ \Pi_+ L\), then this segment contains only the discrete spectrum of \(T_0^+\). To prove additionally that a solution \(\Phi^+ \in H^0\) of problem (3.22) decays exponentially at infinity, we transform the variational problem (3.22) into the following one which looks similar to (3.25):

\[
(\nabla x \Phi^+, \nabla x (R_0^2 \Psi))_{\Pi^+ (\epsilon)} + \lambda M (\Phi^+, R_0^2 \Psi)_{\Pi^+ (\epsilon, L)} - \lambda (\Phi^+, R_0^2 \Psi)_{\Lambda^+ (\epsilon)} = F^+ (R_0^2 \Psi), \quad \Psi \in W^0, \quad (3.23)
\]

where \(F^+ \in (W_0^0)^*\). We put \(u = R_0 \Phi^+, \quad v = R_0 \Psi\) and compute

\[
(\nabla x \Phi^+, \nabla x (R_0^2 \Psi))_{\Pi^+ (\epsilon)} = (R_0 \nabla x \Phi^+, \nabla x v)_{\Pi^+ (\epsilon)} + (R_0 \nabla x \Phi^+, v R_0^{-1} \nabla x R_0)_{\Pi^+ (\epsilon)} =
\]

\[
= (\nabla x u, \nabla x v)_{\Pi^+ (\epsilon)} - (u R_0^{-1} \nabla x R_0, \nabla x v)_{\Pi^+ (\epsilon)} +
\]

\[
+ (\nabla x u, v R_0^{-1} \nabla x R_0)_{\Pi^+ (\epsilon)} - (u R_0^{-1} \nabla x R_0, v R_0^{-1} \nabla x R_0)_{\Pi^+ (\epsilon)}.
\]

Owing to the Lax-Milgram lemma, the inequality

\[
\|\nabla x u; L^2 (\Pi (\epsilon))\|^2 \leq c I (u, u; \Pi (\epsilon))
\]

(3.25)

for the left-hand side \(I (u, v; \Pi (\epsilon))\) of (3.23) provides the uniqueness and solvability of problem (3.23) together with the estimate of its solution

\[
\|\Phi^+ ; W_0^0\| \leq c \|\nabla x u; L^2 (\Pi (\epsilon))\| \leq c \|F; (W_0^0)^*\|
\]

(3.26)

Let us prove (3.23). First of all, we note that, for \(\Psi = \Phi^+\), the second and third terms on the right of (3.24) cancel each other. Moreover,

\[
|\nabla x R_0 (x)| \leq \theta R_0 (x).
\]

(3.27)

Then we apply the trace inequality (3.21) and the Friedrichs inequality

\[
\|\varphi; L^2 (\Gamma^+)\|^2 \leq C \|\nabla x \varphi; L^2 (\Gamma^+)\|^2, \quad \varphi \in H^1 (\Gamma^+), \quad \varphi = 0 \text{ on } \varphi^0,
\]

20
both integrated over \( x_1 \in \mathbb{R} \setminus [-L, L] \). As a result, we obtain

\[
I (u, u, \Pi^+ (\varepsilon) \setminus \Pi^+ (\varepsilon, L)) \geq \| \nabla_x u; L^2 (\Pi^+ (\varepsilon) \setminus \Pi^+ (\varepsilon, L)) \|^2 - \\
- \theta^2 \| u; L^2 (\Pi^+ (\varepsilon) \setminus \Pi^+ (\varepsilon, L)) \|^2 - \lambda \| u; L^2 (\Lambda^+ \Pi^+ (\varepsilon, L)) \|^2 \geq \\
(1 - \theta^2 - (\lambda^0_1)^{-1} \lambda) \| \nabla_x u; L^2 (\Pi^+ (\varepsilon) \setminus \Pi^+ (\varepsilon, L)) \|^2.
\]

Finally, we use the similar three-dimensional inequalities on a finite part of \( \Pi (\varepsilon) \)

\[
\| \Phi; L^2 (\Pi^+ (\varepsilon, L)) \|^2 \leq c (\varepsilon, L) \| \nabla_x \Phi; L^2 (\Pi^+ (\varepsilon, L)) \|^2,
\]

\[
\| \Phi; L^2 (\Lambda^+ (e) \cap \Pi^+ (\varepsilon, L)) \|^2 \leq t \| \nabla_x \Phi; L^2 (\Pi^+ (\varepsilon, L)) \|^2 + C (t, \varepsilon, L) \| \Phi; L^2 (\Pi^+ (\varepsilon, L)) \|^2,
\]

and we derive that

\[
I (u, u; \Pi^+ (\varepsilon, L)) \geq \| \nabla_x u; L^2 (\Pi^+ (\varepsilon, L)) \|^2 - v^2 \| u; L^2 (\Pi^+ (\varepsilon, L)) \|^2 - \\
- \lambda \left( \| u; L^2 (\Lambda^+ (\varepsilon) \cap \Pi^+ (\varepsilon, L)) \|^2 - M \| u; L^2 (\Pi^+ (\varepsilon, L)) \|^2 \right) \geq \\
(1 - \theta^2 - (\lambda^0_1)^{-1} \lambda) \| \nabla_x u; L^2 (\Pi^+ (\varepsilon, L)) \|^2 + \lambda (M - C (t, \varepsilon, L)) \| u; L^2 (\Pi^+ (\varepsilon, L)) \|^2.
\]

Set \( M = C (t, \varepsilon, L) \) to annul the last term in (3.29) and choose \( \theta \) and \( t > 0 \) sufficiently small. Since \( \lambda \in [0, \lambda^0_1] \), both the factors on norms of \( \nabla_x u \) on the right of (3.28) and (3.29) stay positive. Hence, inequality (3.28) and estimate (3.29) are valid. In terms of the operator \( T_{\varepsilon,L}^0 \) the latter with \( \theta = 0 \) means that

\[
\left\| \left( (T_{\varepsilon,L}^0 - \mu)^{-1} \Phi^+; \mathcal{H}^0 \right) \right\| \leq c (\mu) \| (\Psi; \mathcal{H}^0) \|
\]

for \( \mu \in (\mu^0_1, 1] \). Thus, the operator \( T_{\varepsilon,L}^0 - \mu \) is an isomorphism. 

**Corollary 10** An eigenfunction \( \Phi^+ \in \mathcal{H}^0 \) of the operator \( T_{\varepsilon,L}^0 \), corresponding to an eigenvalue \( \mu \in (\mu^0_1, 1] \), satisfies problems (3.24) and (3.25) with the functional

\[
\Psi \mapsto F^+ (\Psi) = (\mu^{-1} - 1) M (\Phi, \Psi)_{\Pi^+ (\varepsilon, L)}.
\]

This functional is continuous on the weighted space \( \mathcal{W}^0_{\theta} \) with any \( \theta \in \mathbb{R} \) because the integration set \( \Pi^+ (\varepsilon, L) \) in (3.30) is compact. Thus, \( \Phi^+ \in \mathcal{W}^0_{\theta} \) with a small \( \theta > 0 \) so that \( \Phi^+ \) decays exponentially at infinity. 

**3.4 Discrete and point spectra.**

The operator \( -T_{\varepsilon,L}^0 \) is semi-bounded from below and, by Lemma 9, it has the discrete spectrum in the segment \((-1, -\mu^0_1]\). Let order the corresponding eigenvalues:

\[
-\mu^{(1)}_\varepsilon \leq -\mu^{(2)}_\varepsilon \leq \ldots \leq -\mu^{(N)}_\varepsilon.
\]

We cannot exclude the case \( N = 0 \) yet and \( N = + \infty \) is also possible.

The max-min principle (see [21] Theorem 10.2.2) applied for the operator \( -T_{\varepsilon,L}^0 \), reads:

\[
- \mu^{(k)}_\varepsilon = \max_{\varepsilon_k \subset \mathcal{H}^0 \setminus \varepsilon_k \setminus \{0\}} \inf \left\langle -T_{\varepsilon,L}^0 \Psi, \Psi \right\rangle.
\]

Here \( \varepsilon_k \) is any linear subspace of co-dimension \( k - 1 \), i.e., \( \dim (\mathcal{H}^0 \setminus \varepsilon_k) = k - 1 \) and, in particular, \( \varepsilon_1 = \mathcal{H}^0 \).
Thus, the infimum in (3.31) does not exceed these extensions. If \( k > d \) and, owing to (3.7),

\[
\varepsilon \rightarrow \infty \quad \Rightarrow \quad \phi \rightarrow 0 
\]

(3.32)

be the ordered eigenvalue sequence of the formulated problem on \( \Omega_\varepsilon^+ \). The corresponding eigenfunctions \( u^{(k)}_\varepsilon \) satisfy the relation

\[
\delta_{p,q} = \left( \nabla_x u^{(p)}_\varepsilon, \nabla_x u^{(q)}_\varepsilon \right)_{\Omega_\varepsilon^+} = \alpha^{(p)}_\varepsilon \left( u^{(p)}_\varepsilon, u^{(q)}_\varepsilon \right)_{\omega_\varepsilon^+} 
\]

(3.33)

where \( \omega_\varepsilon^+ = \{ x = (y,z) : y \in \omega, y_2 > 0, z = 0 \} \) is the upper base of \( \Omega_\varepsilon^+ \).

Let also

\[
0 < \tau^{(1)}_\varepsilon < \tau^{(2)}_\varepsilon < \cdots < \tau^{(k)}_\varepsilon < \cdots \rightarrow +\infty 
\]

(3.34)

be the eigenvalue sequence of the Dirichlet problem for the equation (2.9) on \( \omega^+ = \{ y \in \omega : y_2 > 0 \} \).

Theorem 4 applied to the problems mentioned above, warrants the inequality

\[
\left| \alpha^{(q)}_\varepsilon - \varepsilon \tau^{(k)}_\varepsilon \right| \leq c_k \varepsilon^{1+\rho} 
\]

for \( \varepsilon \in (0, \varepsilon_k) \) and \( \varepsilon_k > 0, c_k > 0 \) depend on the eigenvalue number \( k \).

We fix \( N \) and put \( \bar{\varepsilon}_N = \min \{ \varepsilon_1, \ldots, \varepsilon_N \} \) and \( \bar{\varepsilon}_N = \max \{ \varepsilon_1, \ldots, \varepsilon_N \} \). Now, for any \( \varepsilon \in (0, \bar{\varepsilon}_N] \) and \( k = 1, 2, \ldots \), there exists \( q = q(k) \) such that \( q(k_1) \neq q(k_2) \) for \( k_1 \neq k_2 \) and

\[
\alpha^{(q(k))}_\varepsilon \leq \varepsilon \tau^{(k)}_\varepsilon + \bar{\varepsilon}_N \bar{c}_N \varepsilon_N^p 
\]

(3.35)

Clearly, \( q(k) \geq k \) and, therefore, \( q(k) \) can be changed for \( k \) in (3.35).

We extend the eigenfunctions \( u^{(1)}_\varepsilon, \ldots, u^{(N)}_\varepsilon \) by zero from \( \Omega_\varepsilon^+ \) to \( \Pi^+ (\varepsilon) \) and keep the notation for these extensions. If \( k \leq N \), any subspace \( \mathcal{E}_k \) in (3.31) contains a non-trivial linear combination

\[
\Psi = a_1 u^{(1)}_\varepsilon + \cdots + a_k u^{(k)}_\varepsilon 
\]

Thus, the infimum in (3.31) does not exceed

\[
\frac{\langle -\mathcal{T}_\varepsilon^0 \psi, \psi \rangle}{\langle \psi, \psi \rangle} = \frac{-\| \psi; L^2 (\Lambda^+ (\varepsilon)) \|^2}{\| \nabla_x \psi; L^2 (\Pi^+ (\varepsilon)) \|^2 \| \psi; L^2 (\Lambda^+ (\varepsilon)) \|^2} = \frac{-1}{1 + \alpha^{(k)}_\varepsilon} \leq \frac{1}{1 + \varepsilon (\tau^{(k)}_\varepsilon + \bar{\varepsilon}_N \bar{c}_N \varepsilon_N^p)} 
\]

(3.36)

Here we have used formulae (3.33) and (3.35). Hence, from the max-min principle (3.31) it follows that

\[
-\mu_k \leq - \left( 1 + \varepsilon \left( \tau^{(k)}_\varepsilon + \bar{\varepsilon}_N \bar{c}_N \varepsilon_N^p \right) \right)^{-1} 
\]

(3.37)

and, owing to (3.7),

\[
\lambda^{(k)}_\varepsilon = \left( 1 + \mu^{(k)}_\varepsilon \right)^{-1} \leq \varepsilon \left( \tau^{(k)}_\varepsilon + \bar{\varepsilon}_N \bar{c}_N \varepsilon_N^p \right)^{-1} 
\]

If \( d > 0 \) and \( N \) are given, we choose \( \varepsilon (d, N) > 0 \) such that \( \varepsilon (d, N) \leq \bar{\varepsilon}_N \) and, with \( k = 1, \ldots, N \) and \( \varepsilon \in (0, \varepsilon (d, N)) \), the bound in (3.37) does not exceed \( d \). Then Theorem 10.2.2 in [21] ensures the existence of, at least, \( N \) eigenvalues \(-\mu^{(k)}_\varepsilon \in [-1, - (1 + d)^{-1}] \) of the operator \(-\mathcal{T}_\varepsilon^0 \) which, as has been explained, belong to the point spectrum of the operator \( \mathcal{T}_\varepsilon \). Corresponding numbers \( \lambda^{(k)}_\varepsilon \in (0, d) \) are nothing but eigenvalues of problem (1.5)-(1.7). Theorem 4 is proved.
4 Concluding remarks

The main feature of the body $\Theta(\varepsilon) \subset \Pi(\varepsilon)$ which provides the accumulation effect of the trapped mode frequencies, is but the thin upper layer $\Omega_\varepsilon$ of water while the shape of the surface $\partial \Theta(\varepsilon) \setminus \partial \Omega_\varepsilon$ has no influence at all (cf. [1] where $\omega_\varepsilon(\varepsilon) = \partial \Theta(\varepsilon) \setminus \partial \Omega_\varepsilon$ is flat). In this way, for the bodies $\Theta(\varepsilon)$ and $\Theta_\| (\varepsilon)$ with cross-section in Fig. 3 and Fig. 4 respectively, Theorem 1 gives the same bound $\varepsilon (d, N)$ for the small parameter $\varepsilon$ in order to provide at least $N$ eigenvalues in the interval $(0, \delta)$ of the continuous spectrum.

If boundary of the periodic layer $\Sigma_\varepsilon^\infty$ is Lipschitz only, the convergence

$$\varepsilon^{-1} \alpha^{(q)} \to \tau^{(q)} \quad \text{for} \ \varepsilon \to 0^+ \quad (4.1)$$

(cf. (2.53)) is valid. However, the homogenization technique to derive (4.1) differs from calculations performed in Section 2 (cf. [16, 17] and others). Formula (4.1) is sufficient to make the same conclusion as in Theorem 1. In the estimate derived we underline the convergence rate $O(\varepsilon^{\rho})$ (cf. (4.1) and (2.53)) caused by singularities of $\nabla^2 y(w(y))$ at the corner point of the rectangle $\omega$ (see problem (2.9), (2.11)).

In [18, Ch. 7], a method of inverse and direct reduction is developed to describe an explicit dependence of constants $c_k$ in estimates of type (2.53) on the eigenvalue number $k$ and other attributes of the limit spectrum (2.13). This method requires rather intricate calculations and we do not apply it here because the estimate (2.53) is sufficient for the main goal of the paper and the explicit dependence mentioned above does not upgrade the result in Theorem 1.

The shape of $\Theta_\| (\varepsilon)$ sketched on Fig. 5 where the rough surface $\partial \Theta_\| (\varepsilon)$ penetrates the water surface, is a possible generalization. Although the plate $\Omega_\varepsilon$ becomes perforated, it is very predictable that convergence (4.1) and, thus, Theorem 1 remain valid though.

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