Counting the Resonances in High and Even Dimensional Obstacle Scattering

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Abstract

In this paper, we give a polynomial lower bound for the resonances of $-\Delta$ perturbed by an obstacle in even-dimensional Euclidean spaces, $n \geq 4$. The proof is based on a Poisson Summation Formula which comes from the Hadamard factorization theorem in the open upper complex plane. We take advantage of the singularity of regularized wave trace to give the pole/resonance counting function over the principal branch of logarithmic plane a lower bound.

keywords Weyl’s density theorem/resonances counting/scattering resonances.

1 Introduction

Let $H$ be an embedded hypersurface in $\mathbb{R}^n$ such that

$$\mathbb{R}^n \setminus H = \Omega \cup \mathcal{O}, \text{ with } \mathcal{O} \text{ compact and } \Omega \text{ connected}, n \geq 4,$$

(1.1)

where both $\mathcal{O}$ and $\Omega$ are open. We refer to $\mathcal{O}$ as the obstacle and $\Omega$ as the exterior.

In the exterior domain $\Omega$, we consider the resonance theory of the differential operator

$$P := -\Delta : L^2(\Omega) \rightarrow L^2(\Omega)$$

(1.2)

satisfying the boundary condition either

$$u|_H = 0$$

(1.3)

or

$$\left(\frac{\partial}{\partial n} + \gamma\right)u|_H = 0, \gamma \in C^\infty(H).$$

(1.4)

Let us denote

$$\Lambda_\sharp := \{\lambda \neq 0|0 < \arg \lambda < \pi\}.$$  

(1.5)

It is well-known from spectral analysis that the resolvent operator $(P - \lambda^2)^{-1} : L^2(\Omega) \rightarrow H^2(\Omega)$ is bounded in $\Lambda_\sharp$ except a finite discrete set $\{\mu_1^2, \cdots, \mu_m^2\}$ such that $\{\mu_1, \cdots, \mu_m\}$ are the eigenvalues of $P$. As a special case of the black box formalism in Zworski and Sjöstrand [14], $(P - \lambda^2)^{-1}$ can be meromorphically extended from $\Lambda_\sharp$ to $\Lambda := \{\lambda \neq 0|-\infty < \arg \lambda < \infty\}$, the logarithmic plane, as an operator

$$R(\lambda) := (P - \lambda^2)^{-1} : L^2_{\text{comp}}(\Omega) \rightarrow H^2_{\text{loc}}(\Omega)$$

(1.6)

with poles of finite rank. All such meromorphic poles in $\Lambda$ are called resolvent resonances in mathematical physics literature. Classically, we have Lax-Phillips theory for the meromorphic extension theory of $R(\lambda)$ and of the relative scattering matrix $S(\lambda)$ induced by scatterer $\mathcal{O}$. See Lax and Phillips [11, 12]. For a more complicated boundary setting, we refer to Shenk and Thoe [13].

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A question for the meromorphically extended $R(\lambda)$ would be: how many resonances approximately inside a disc of radius $r$ in the principal sheet of $\Lambda$? Let us array all of the resonances as a sequence $\{\mu, m_2, \cdots \}$ repeated according to their multiplicity. Let us define
\[
\Lambda_1 := \{ \lambda \neq 0 \mid -\pi/2 \leq \arg \lambda < 3\pi/2 \}, \tag{1.7}
\]
where we set the branch cut to be negative imaginary axis. In this paper, we study the resonance counting function
\[
N_1(r) := \sharp \{ \lambda \in \Lambda_1 \mid \text{resonances of } R(\lambda), 0 < |\lambda| \leq r \}, \quad r > 1. \tag{1.8}
\]
From the literature, we define
\[
N(r, a) := \sharp \{ \lambda \in \Lambda \mid \text{resonances of } R(\lambda), 0 < |\lambda| \leq r, |\arg \lambda| \leq a \}, \quad r, a > 1. \tag{1.9}
\]
The optimal upper bound of this counting function in Vodev’s formalism is found in [16, 17] from which we recall for $n \geq 2$,
\[
N(r, a) \leq Ca(r^n + (\log a)^n), \quad \text{where } C \text{ is a constant}, \quad r, a > 1. \tag{1.10}
\]
For $a < \frac{\pi}{2}$, an upper bound of $N(r, a)$ also follows from Sjöstrand and Zworski [14]. That is
\[
N(r, a) = O(r^n). \tag{1.11}
\]
Consequently, (1.10) and (1.11) give a polynomial upper bound on the resonance counting function $N_1(r)$ in $\Lambda_1$ for $n \geq 2$.

To count $N_1(r)$, we begin our analysis with the domain
\[
\Lambda_{\pi/2} := \{ \lambda \neq 0 \mid |\arg \lambda| < \pi/2, |\arg \lambda - \pi| < \pi/2 \}. \tag{1.12}
\]
Defining the scattering determinant
\[
s(\lambda) := \det S(\lambda), \tag{1.13}
\]
where $S(\lambda)$ is the scattering matrix in sense of Zworski [19]. The poles of $s(\lambda)$ coincide with $S(\lambda)$’s. We recall the functional identities for even dimensional scattering matrix in Shenk and Thoe [13, p.468], using the same notation for scattering matrix,
\[
S(\lambda)^{-1} = S(\overline{\lambda})^*, \quad \text{when } \lambda \text{ is not a pole of } S(\lambda) \text{ or } S(\overline{\lambda})^*; \tag{1.14}
\]
\[
S(\overline{\lambda})^* = 2I - S(e^{\pi i} \lambda), \quad \text{when } \lambda \in \Lambda. \tag{1.15}
\]
We define $m_{ij}(R)$ to be the multiplicity of $S(\lambda)$ near the pole $\lambda = \mu$. According to (1.11), we know $\lambda = \overline{\mu}$ is a zero of $S(\lambda)$. There exist finitely many eigenvalues, say, $\mu_1^2, \cdots, \mu_m^2$, repeated according to their multiplicity, such that $\{\mu_j\}_{j=1}^m$ appear as the poles of $S(\lambda)$ in $\Lambda_{\pi/2}$. The result (1.11) is sufficient to give the representation, as a special case in Zworski [19] (2.3)],
\[
s(\lambda) = e^{g_{\pi/2}(\lambda)} \frac{P_{\pi/2}(\lambda)}{P_{\pi/2}(\lambda)}, \quad \text{where } \lambda \in \Lambda_{\pi/2} \cap \{ \Re \lambda > 0 \}. \tag{1.16}
\]
In general, $g_{\pi/2}$ is a symbol on $\mathbb{R}$ such that
\[
|\partial^k g_{\pi/2}(\lambda)| \leq C_{k,\epsilon}(1 + |\lambda|)^{n+\epsilon-k}, \quad \forall \epsilon > 0, \tag{1.17}
\]
and there exists an $m_0$ such that
\[
P_{\pi/2}(\lambda) := \prod_{\{\mu \in \Lambda_{\pi/2} \cap \mathbb{R} \}} E(\lambda, m_0)^{m_{ij}(R)} \tag{1.18}
\]
where
\[
E(z, p) := (1 - z) \exp(1 + \cdots + \frac{z^p}{p}). \tag{1.19}
\]

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In this paper, we set

\[ g_{\pi/2}(-\lambda) = -g_{\pi/2}(\lambda), \lambda > 0. \quad (1.20) \]

We compare this construction to Zworski's in [19] p.3. Combining (1.20) with (1.16), the fact that \( S(\lambda) \) is unitary on \( \lambda > 0 \) implies

\[ |s(\lambda)| = 1, \forall \lambda \in 0i + \mathbb{R}. \quad (1.21) \]

Now, we define

\[ \Lambda^\pm_{\pi/2} := \Lambda_{\pi/2} \cup e^{i\pi/2} \mathbb{R}. \quad (1.22) \]

We describe the analytic behavior of \( s(\lambda) \) on \( e^{i\pi/2} \mathbb{R} \) by the odd reflection defined in (1.20) as follows.

**Lemma 1.1**

\( s(\lambda) \), defined through (1.20), extends from \( \{\lambda|| \arg \lambda \leq \pi/2\} \) to \( \mathbb{C} \) such that it is analytic in the neighborhood along \( e^{i\pi/2} \mathbb{R} \) outside \( \{\mu_1, \cdots, \mu_m\} \).

**Proof**

Let \( V \) be a neighborhood on the imaginary axis \( e^{i\pi/2} \mathbb{R} \) outside \( \{\mu_1, \cdots, \mu_m\} \). Let \( V := V^- \cup e^{i\pi/2} \mathbb{R} \cup V^+ \), where \( V^\pm \) is in the right/left sheet of \( \Lambda_{\pi/2} \).

We claim that \( g_{\pi/2}(\lambda) \) extends to be purely real on the imaginary. From (1.16) and (1.14),

\[ |s(\lambda)| = |e^{g_{\pi/2}i + 3g_{\pi/2}}| = 1, \forall \lambda > 0. \quad (1.23) \]

Hence, \( g_{\pi/2} \) is purely imaginary on real by (1.20). (1.20) says

\[ g_{\pi/2}(-\lambda) = -g_{\pi/2}(\lambda), \forall \lambda \in 0i + \mathbb{R}. \quad (1.24) \]

Hence, using identity theorem, (1.24) implies

\[ g_{\pi/2}(\lambda) = g_{\pi/2}(\lambda), \forall \lambda \in \Lambda_{\pi/2}. \quad (1.25) \]

We extend (1.25) to \( e^{i\pi/2} \mathbb{R} \) by observing

\[ \lim_{\Re \lambda \to 0^-} g_{\pi/2}(\lambda) = \lim_{\Re \lambda \to 0^+} g_{\pi/2}(\lambda) = \lim_{\Re \lambda \to 0^+} g_{\pi/2}(\lambda) = \lim_{\Re \lambda \to 0^-} g_{\pi/2}(\lambda). \quad (1.26) \]

Hence, the claim is proved.

It is obvious that, \( \forall \lambda \in e^{i\pi/2} \mathbb{R} \cup V^+ \),

\[ g_{\pi/2}(\lambda) \to g_{\pi/2}(\lambda), \text{ as } \Re \lambda \to 0^+. \quad (1.27) \]

On the other hand, in \( V^- \), using (1.25) and (1.26),

\[ g_{\pi/2}(\lambda) = g_{\pi/2}(\lambda) \to g_{\pi/2}(\lambda) = g_{\pi/2}(\lambda), \text{ as } \Re \lambda \to 0^-_. \quad (1.28) \]

Hence, for such a \( \lambda \), (1.27) and (1.28) combines to give

\[ \lim_{\Re \lambda \to 0^+} g_{\pi/2}(\lambda) = \lim_{\Re \lambda \to 0^-} g_{\pi/2}(\lambda). \quad (1.29) \]

Therefore, the value of \( s(\lambda) \) meet from both sides up to \( e^{i\pi/2} \mathbb{R} \) outside \( \{\mu_1, \cdots, \mu_m\} \) and \( s(\lambda) \) is analytic in \( V \) by Schwarz reflection. See Lang [8, p.298]. \( \square \)

As far as resonances counting in concerned, \( s(\lambda) \) defined through (1.20) has as much as resonances in \( \Lambda^\pm_{\pi/2} \) as the original scattering determinant in \( \Lambda_1 \). Hence, we do not differentiate these two scattering determinant here.

Let us define the order of a function that is holomorphic in \( \Lambda^1_2 \) to be the greatest lower bound of numbers \( \nu \) for which

\[ \lim_{r \to \infty} \sup \ln |f(re^{i\theta})| = 0, \quad (1.30) \]

uniformly in \( \theta, 0 < \theta < \pi \). See Levin [9]. We observe that

**Lemma 1.2**

\( s(\lambda) \prod_{j=1}^m \frac{\lambda - \mu_j}{\lambda - \bar{\mu}_j} \) is holomorphic in \( \Lambda^\pm_2 \) with zeroes \( \{\bar{\lambda}_j\} \) there such that \( \{\lambda_j\} \) are the poles of \( s(\lambda) \) in \( \Lambda^\pm_2 \). \( s(\lambda) \prod_{j=1}^m \frac{\lambda - \mu_j}{\lambda - \bar{\mu}_j} \) is a holomorphic function of integral order at most \( \nu \) in \( \Lambda^\pm_2 \).
The proof is a direct consequence due to the estimates of Vodev [16] and Sjöstrand and Zworski [13]. Given such a function \( s(\lambda) \prod_{j=1}^{m} \frac{\lambda - \mu_j}{\lambda - \overline{\mu}_j} \) with zeroes \( \overline{\lambda}_j \) of growth order of at most \( n \) in \( \Lambda_\lambda \), we can apply Govorov’s theorem [5], also Levin [9, Appendix VIII], which serves as the Hadamard factorization theorem over \( \Lambda_\lambda \). Govorov’s theorem allows us to write down a Hadamard factorization theorem for an integral function of finite order in terms of its zeroes in an angle.

In particular, we define

\[
P(\lambda) := \prod_{\mu \in \{\lambda_j\}} E(\lambda, \mu)n^{m_\mu(R)} \quad \text{and} \quad \overline{P}(\lambda) := \prod_{\mu \in \{\lambda_j\}} E(\lambda, \mu)n^{m_\mu(R)}
\]

where

\[
E(z, p) := (1 - z)\exp(1 + \cdots + \frac{z^p}{p}).
\]

\( \overline{P}(\lambda) \) is an entire function of finite order at most \( n \) in \( \Lambda_\lambda \). The conjugate is taken as in \( C \). The convergence of this Weierstrass product is guaranteed by (1.10) and (1.11). Then, using Govorov’s theorem,

\[
s(\lambda) \prod_{j=1}^{m} \frac{\lambda - \mu_j}{\lambda - \overline{\mu}_j} = \exp\{i(a_0 + a_1\lambda + \cdots + a_n\lambda^n)\} \exp\{\frac{1}{\pi i} \int_{-\infty}^{\infty} (t\lambda + 1)^n + \sum_{j=1}^{m} \frac{dE(t)}{dt}(\lambda)\} \quad \lambda \in \Lambda_\lambda,
\]

where \( n \) is the order of the \( s(\lambda) \prod_{j=1}^{m} \frac{\lambda - \mu_j}{\lambda - \overline{\mu}_j} \), \( \{a_k\} \) are real constants, \( \{\lambda_j\} \) are the zeroes of \( s(\lambda) \prod_{j=1}^{m} \frac{\lambda - \mu_j}{\lambda - \overline{\mu}_j} \) in \( \Lambda_\lambda \) and, most important of all,

\[
\Sigma(t) := \lim_{y \to 0^+} \int_0^t \ln |s(x + iy)\prod_{j=1}^{m} \frac{x + iy - \mu_j}{x + iy - \overline{\mu}_j}| \, dx.
\]

Since \( n \geq 4 \), \( \lambda = 0 \) is neither an embedded eigenvalue nor a pole. Using (1.21),

\[
\Sigma(t) = \int_0^t \ln \prod_{j=1}^{m} \frac{x - \mu_j}{x - \overline{\mu}_j} |dx| = 0, \forall t \in \mathbb{R}.
\]

Hence, we conclude that

**Lemma 1.3**

\[
s(\lambda) \prod_{j=1}^{m} \frac{\lambda - \mu_j}{\lambda - \overline{\mu}_j} = \exp\{i(a_0 + a_1\lambda + \cdots + a_n\lambda^n)\} \overline{P}(\lambda) \frac{P(\lambda)}{P(\lambda)}, \quad \lambda \in \Lambda_\lambda.
\]

Let us define

\[
\sigma(\lambda) := \frac{1}{2\pi i} \log s(\lambda)
\]

and

\[
g(\lambda) := i(a_0 + a_1\lambda + \cdots + a_n\lambda^n).
\]

Functional analysis and (1.20) give

\[
\sigma'(\lambda) = \frac{1}{2\pi i} \frac{s'(\lambda)}{s(\lambda)}.
\]

In particular, (1.36) and (1.39) give

\[
\sigma'(\lambda) = \frac{1}{2\pi i} g'(\lambda) + \frac{1}{2\pi i} \sum_{\mu_j} \frac{1}{\lambda - \mu_j} - \frac{1}{\lambda - \overline{\mu}_j} + \frac{1}{2\pi i} \sum_{\lambda_j} \frac{1}{\lambda - \lambda_j} - \frac{1}{\lambda - \overline{\lambda}_j} + Q_\lambda(\lambda),
\]

where

\[
Q_\lambda(\lambda) := (\frac{1}{\lambda_j})(1 + (\frac{\lambda}{\lambda_j}) + \cdots + (\frac{\lambda}{\lambda_j})^{n-1}),
\]

which is a polynomial in \( \lambda \) provided \( \{\lambda_j\} \neq 0 \). \( g'(\lambda) \) is also a polynomial of order no greater than \( n - 1 \).

The main theorem of this paper is
Theorem 1.4 Let \( n \geq 4 \) be even. Under the assumption that the set of closed transversally reflected geodesics in \( T^*\Omega \) has measure zero, the resonance counting function for \( P \) in \( \Lambda_1 \),
\[
N_1(r) = \frac{(2\pi)^{-n} \omega_n \text{Vol}(\Omega)}{\Gamma(n+1)} (1 + o(\frac{1}{r})) r^n, \quad \text{as } r \to \infty,
\]
where \( \omega_n \) is the volume of the unit sphere in \( \mathbb{R}^n \).

The Weyl’s asymptotics (1.42) is classical for the eigenvalue counting problem in interior problems. We refer to Ivrii [7] and Melrose [10] for a discussion.

2 A Proof

Firstly, we need a satisfactory Poisson summation formula. We start with a Birman-Krein formula. Such a formula is common in scattering theory for all kinds of perturbation. For a black box formalism setting, we refer to Christiansen [3, 4].

Defining the naturally regularized wave propagator,
\[
u(t) := 2\{\cos t\sqrt{P} - \cos t\sqrt{-\Delta}\} \in \mathcal{D}'(\mathbb{R}; J_1(L^2(\Omega); L^2(\Omega))),
\]
where \( J_1(L^2(\Omega); L^2(\Omega)) \) is the trace class in \( L^2 \). \( \nu(t) \) has a distributional trace. Let \( R^0(\lambda) := (-\Delta - \lambda^2)^{-1} \). By spectral analysis, we have
\[
\text{Tr}\{\nu(t)\} = \int_{\mathbb{R}} e^{it\lambda} 2\lambda \text{Tr}\{R(\lambda) - R^0(\lambda) - R(-\lambda) + R^0(-\lambda)\} d\lambda + 2 \sum_{\Im \mu_j > 0} \cos(t\mu_j).
\]

By Birman-Krein theory, we have
\[
\sigma'(\lambda) = 2\lambda \text{Tr}\{R(\lambda) - R^0(\lambda) - R(-\lambda) + R^0(-\lambda)\} \in S'(\mathbb{R}).
\]

On the other hand, we continue from (1.40)
\[
\sigma'(\lambda) = \frac{1}{2\pi i} g'(\lambda) + \frac{1}{2\pi i} \sum_{\mu_j} \frac{1}{\lambda - \mu_j} - \frac{1}{\lambda - \mu_j} + \frac{1}{2\pi i} \sum_{\lambda_j} \frac{1}{\lambda - \lambda_j} - \frac{1}{\lambda - \lambda_j} + Q_{\lambda}(\lambda_j) - Q_{\lambda}(\lambda_j) \in S'(\mathbb{R}).
\]

In our case, \( g'(\lambda) \) and \( Q_{\lambda}(\lambda_j) \) are at most of order \( n - 1 \). Hence,
\[
\partial^\alpha_n \sigma'(\lambda) = \frac{1}{2\pi i} \sum_{\mu_j} \frac{n!}{(\lambda - \mu_j)^{n+1}} + \frac{n!}{(\lambda - \mu_j)^{n+1}} + \frac{n!}{(\lambda - \mu_j)^{n+1}} - \frac{n!}{(\lambda - \mu_j)^{n+1}}.
\]

Therefore, for \( t \neq 0 \),
\[
et^n \int_{\mathbb{R}} e^{it\lambda} \sigma'(\lambda) d\lambda = \sum_{\mu_j} \frac{1}{2\pi i} \int_{\mathbb{R}} e^{it\lambda} \left\{ \frac{n!}{(\lambda - \mu_j)^{n+1}} - \frac{n!}{(\lambda - \mu_j)^{n+1}} \right\} d\lambda
+ \sum_{\lambda_j} \frac{1}{2\pi i} \int_{\mathbb{R}} e^{it\lambda} \left\{ \frac{n!}{(\lambda - \lambda_j)^{n+1}} - \frac{n!}{(\lambda - \lambda_j)^{n+1}} \right\} d\lambda.
\]

For \( t \neq 0 \), we have
\[
\text{Tr}\{\nu(t)\} = \sum_{\mu_j} \frac{1}{2\pi i} \int_{\mathbb{R}} e^{it\lambda} \left\{ \frac{1}{\lambda - \mu_j} - \frac{1}{\lambda - \mu_j} \right\} d\lambda
+ \sum_{\lambda_j} \frac{1}{2\pi i} \int_{\mathbb{R}} e^{it\lambda} \left\{ \frac{1}{\lambda - \lambda_j} - \frac{1}{\lambda - \lambda_j} \right\} d\lambda
+ 2 \sum_{\Im \mu_j > 0} \cos(t\mu_j).
\]

This cancelling \( t \neq 0 \) technique is due to Zworski [18]. Most importantly, we have the following well-known Poisson integral formula. See Lang [8].
Lemma 2.1 Let $f$ be a bounded holomorphic function over the closed upper half plane. Defining
\[
h_z(\zeta) := \frac{1}{2\pi i} \left( \frac{1}{\zeta - z} - \frac{1}{\zeta - \zeta} \right), \text{ where } \zeta, z \in \Lambda_1,
\]
the following identity holds.
\[
\int_{-\infty}^{\infty} f(\lambda)h_z(\lambda)d\lambda = f(z).
\] (2.9)

Using this lemma, (2.7) becomes, when $t > 0$,
\[
\text{Tr}\{u(t)\} = \sum_{\sigma_j \in \mathcal{Z}} e^{it\sigma_j} + \sum_{\mu_j > 0} e^{-it\mu_j},
\] (2.10)
which we rewrite as
\[
\text{Tr}\{u(t)\} = \sum_{\sigma_j} e^{it\sigma_j}, \text{ in } \mathcal{D}'(\mathbb{R}^+),
\] (2.11)
in which $\sigma_j$ are either $\bar{\sigma}_j$ or $-\mu_j$ such that $\mu_j^2$ are eigenvalues of $P$. Alternatively, we can write
\[
\text{Tr}\{u(t)\} = \int_{\mathcal{Z}} e^{it\sigma} dN_1(r(\sigma)), \text{ in } \mathcal{D}'(\mathbb{R}^+).
\] (2.12)
where
\[
r(\sigma) \text{ is a norm of } \sigma \text{ as in } \mathbb{R}^2; \quad \sigma \in \mathcal{Z} := \{-\mu_1, \ldots, -\mu_m, \bar{\lambda}_1, \bar{\lambda}_2, \ldots\}. \quad (2.13)
\]
We see $N_1(r(\sigma))$ as a function of $\sigma \in \mathcal{Z}$. For example, $N_1(|\sigma_j|) = j$ and the mapping $\sigma_j \mapsto j$ is a well-defined function, so $N_1(|\sigma_j|)$ is seen as a function of $\sigma_j$ not merely as a function of $|\sigma_j|$. Moreover, the summation (2.11) is understood as the complex abstract integration in Ash’s book [1] p.94,p.90(1) from which we quote as Lemma 2.2 below. Hence, the integrand $e^{it\sigma}$ and the integrator $dN_1(r(\sigma))$ in (2.12) are functions to variable $\sigma$.

Lemma 2.2 If $f = (f(\alpha), \alpha \in \mathcal{Z})$ is a real- or complex-valued function on the arbitrary set $\mathcal{Z}$, and $\mu$ is counting measure on subsets of $\mathcal{Z}$, then $\int_{\mathcal{Z}} f d\mu = \sum_{\alpha} f(\alpha)$.

For the left hand side of (2.12), we use the short time asymptotic behavior of $\text{Tr}\{u(t)\}$. It is well-known that
\[
\text{Tr}\{u(t)\} \sim a_0 t^{-n} + a_1 \delta^{(n-2)}(t) + \cdots, \text{ as } t \to 0,
\] (2.15)
where
\[
a_0 = (2\pi)^{-n} \omega_n \text{Vol}(\mathcal{O}), \quad a_1 = \alpha_1 \text{Vol}(\partial\mathcal{O}), \ldots,
\] (2.16)
with $\omega_n$ as the volume of the unit sphere in $\mathbb{R}^n$ and $\omega_n = \frac{n\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}$. The constant $a_0$ here is derived in Ivrii [2]. Additionally, we refer to Branson and Gilkey [2] for a heat propagator version of this formula. Therefore,
\[
\int_{\mathcal{Z}} e^{it\sigma} dN_1(r(\sigma)) \sim (2\pi)^{-n} \omega_n \text{Vol}(\mathcal{O}) t^{-n} + \cdots, \text{ as } t \to 0^+.
\] (2.17)
We want to convert the information in $\mathbb{C}$ to be the one in $\mathbb{R}$. Without loss of generality, we use the sup-norm from now on. That is
\[
N_1 (r(\sigma)) := |\{\sigma_j | 3|\sigma_j| \leq r(\sigma), |\Re \sigma_j| \leq r(\sigma)\}|. \quad (2.18)
\]
Accordingly, we set
\[
N_{3}(r(\sigma)) := |\{\sigma_j | |r(\sigma)| \geq N_{3}(r(\sigma)), |\Re \sigma_j| \leq |\Re \sigma_j| \leq N_{3}(r(\sigma))\}|, \quad (2.19)
\] and
\[
N_{3}(r(\sigma)) := |\{\sigma_j | |\Re \sigma_j| < |\Im \sigma_j| \leq r(\sigma), |\Im \sigma_j| \leq |\Im \sigma_j| \leq N_{3}(r(\sigma))\} |. \quad (2.20)
\]
Let \( \sigma = x + iy \). \( N_\beta(r(\sigma)) \) is a function of \( x \) whenever \( \Re \sigma \geq \Im \sigma \). In particular,

\[
N_\beta(r(\sigma)) = N_\beta(\max\{|x|,|y|\}) = N_\beta(|x|).
\] (2.21)

Similarly, we use \( N_\beta(r(\sigma)) = N_\beta(|y|) \) whenever \( \Re \sigma < \Im \sigma \). Accordingly, we decompose

\[
\int_Z e^{it\sigma} dN_1(r(\sigma)) = \int_{[\Re \sigma] \geq [\Im \sigma]} e^{it\sigma} dN_\beta(r(\sigma)) + \int_{[\Re \sigma] < [\Im \sigma]} e^{it\sigma} dN_3(r(\sigma))
\]

\[
= \int_{[\Re \sigma] \geq [\Im \sigma]} e^{it\sigma} dN_\beta(|x|) + \int_{[\Re \sigma] < [\Im \sigma]} e^{it\sigma} dN_3(|y|).
\] (2.22)

**Lemma 2.3** Let \( \sigma = x + iy \in \mathbb{C} \). We have

\[
\int_{[\Re \sigma] < [\Im \sigma]} (e^{it\sigma} - e^{-ty}) dN_3(|y|) \to 0 \text{ in } D'(\mathbb{R}^+), \text{ as } t \to 0^+; \tag{2.23}
\]

\[
\int_{[\Re \sigma] \geq [\Im \sigma]} (e^{it\sigma} - e^{itz}) dN_\beta(|x|) \to 0 \text{ in } D'(\mathbb{R}^+), \text{ as } t \to 0^+ . \tag{2.24}
\]

**Proof** Without loss of generality, we prove for all \( \varphi(t) \in C_0^\infty(\mathbb{R}^+;[0,1]) \). Since \( e^{-ty} \varphi(t) \) is nonnegative, Tonelli’s theorem gives the product measure

\[
\int e^{-ty} \varphi(t) dN_3(|y|) \times dt = \int \{ \int e^{-ty} \varphi(t) dN_3(|y|) \} dt = \int \{ \int e^{-ty} \varphi(t) dt \} dN_3(|y|). \tag{2.25}
\]

Using Paley-Wiener’s theorem, say, Hörmander [6],

\[
\begin{align*}
&| \int_{[\Re \sigma] < [\Im \sigma]} \int_0^\infty e^{-ty} \varphi(t) dt dN_3(|y|) | \\
\leq & \int_{[\Re \sigma] < [\Im \sigma]} \int_0^\infty e^{-ty} \varphi(t) dt dN_3(|y|) \\
\leq & \int_{[\Re \sigma] < [\Im \sigma]} C_N(1 + |y|)^{-N} e^{H(1-iy)} dN_3(|y|), \forall N \in \mathbb{N}, \\
= & \int_{[\Re \sigma] < [\Im \sigma]} C_N(1 + |y|)^{-N} e^{H(1+y)} dN_3(|y|), \forall N \in \mathbb{N}, \tag{2.26}
\end{align*}
\]

where \( H \) is the related supporting function of \( \varphi(t) \) and \( C_N \) is a constant. Without loss of generality, we assume \( y > 0 \). Hence, the integral (2.26) is convergent for large \( N \) given the estimate of the growth order of \( N_1(r) \) from (1.10) and (1.11). In particular,

\[
e^{it\sigma} \varphi(t) \in L^1(dN_3(|y|) \times dt). \tag{2.27}
\]

Accordingly, Fubini’s theorem gives

\[
\int_0^\infty \int_{[\Re \sigma] < [\Im \sigma]} (e^{it\sigma} - e^{-ty}) \varphi(t) dN_3(|y|) dt = \int_{[\Re \sigma] < [\Im \sigma]} \int_0^\infty (e^{it\sigma} - e^{-ty}) \varphi(t) dt dN_3(|y|). \tag{2.28}
\]

Now, let \( \varphi(t) \in C_0^\infty(\mathbb{R}^+;[0,1]) \) such that \( \int \varphi(t) dt = 1 \). We define \( \varphi_\gamma(t) := \frac{1}{\gamma} \varphi(t/\gamma) \), where \( \gamma > 0 \). To
prove (2.23), we show
\[
\left| \int_{|\Re|<|\Im|} (e^{it\sigma} - e^{-t\gamma})\varphi_\gamma(t)d\mu_N(|\gamma|) \right|, \text{ assuming } y > 0,
\]
\[
\leq \int_{|\Re|<|\Im|} e^{-t\gamma}e^{itx} - 1|\varphi_\gamma(t)d\mu_N(|\gamma|)
\]
\[
= \int_{|\Re|<|\Im|} e^{-t\gamma}\sum_{j\geq 1} \frac{(itx)^j}{j!} |\varphi_\gamma(t)d\mu_N(|\gamma|)
\]
\[
\leq \int_{|\Re|<|\Im|} e^{-t\gamma}\sum_{j\geq 1} \frac{|tx|^j}{j!} |\varphi_\gamma(t)d\mu_N(|\gamma|), \text{ given } |\gamma| > |x|,
\]
\[
< \int_{|\Re|<|\Im|} e^{-t\gamma}(e^{ty} - 1)\varphi_\gamma(t)d\mu_N(|\gamma|)
\]
\[
= \int_{|\Re|<|\Im|} e^{-\gamma y}(e^{\gamma y} - 1)\varphi(s)d\mu_N(|\gamma|), \text{ letting } t := \gamma s,
\]
\[
= \int_{|\Re|<|\Im|} e^{-\gamma y}(e^{\gamma y} - 1)\varphi(s)d\mu_N(|\gamma|), \tag{2.29}
\]
to which we apply the Lebesgue’s dominated convergence theorem, converging to 0 as \(\gamma \to 0\).

The other identity follows similarly. \(\square\)

Using this lemma,
\[
\int \mathbb{Z} e^{it\sigma}dN_1(r(\sigma)) \to \int_{|\Re|\geq|\Im|} e^{itx}dN_\Re(|x|) + \int_{|\Re|<|\Im|} e^{-ty}dN_\Im(|y|), \text{ in } \mathcal{D}'(\mathbb{R}^+), \text{ as } t \to 0^+ \tag{2.30}
\]
For the first integral on \(N_\Re(|x|)\), we set the change of variable
\[
\sigma := x + iy \mapsto y + ix := \sigma'. \tag{2.31}
\]
Let \(\sigma' := x' + iy'\). We see
\[
N_\Re(r(\sigma)) = N_\Re(r(\sigma')) = N_\Re(r(y + ix)) = N_\Re(\max\{|x|, |y|\}) = N_\Re(|y'|). \tag{2.32}
\]
Since \(\sigma'\) is a dummy variable,
\[
\int \mathbb{Z} e^{it\sigma}dN_1(r(\sigma)) \to \int_{\mathbb{R}^+} e^{-ty}d\{N_\Im(|y|) + N_\Re(|y|)\}, \text{ in } \mathcal{D}'(\mathbb{R}^+), \text{ as } t \to 0^+. \tag{2.33}
\]
Please refer to the picture Figure 1 below for the counting measure.

Now we recall the Karamata’s Tauberian theorem. As stated in Taylor’s book [15],

**Proposition 2.4** If \(\mu\) is a positive measure on \([0, \infty)\), \(\alpha \in (0, \infty)\), then
\[
\int_0^\infty e^{-t\lambda}d\mu(\lambda) \sim at^{-\alpha}, t \searrow 0, \tag{2.34}
\]
implies
\[
\int_0^r d\mu(\lambda) \sim br^{-\alpha}, r \nearrow \infty, \text{ with } b = \frac{a}{\Gamma(\alpha + 1)}. \tag{2.35}
\]
In our case, \(a = (2\pi)^{-n}\omega_\nu\Vol(O)\) and \(\alpha = n\). By construction, \(N_\Im(|y|) + N_\Re(|y|)\) is a nondecreasing function of \(|y|\), so \((2.15), (2.33)\) and \((2.34)\) yield
\[
N_1(r) = \frac{(2\pi)^{-n}\omega_\nu\Vol(O)}{\Gamma(n + 1)} \{1 + o\left(\frac{1}{r}\right)\}r^n, \text{ as } r \to \infty. \tag{2.36}
\]
This proves the main theorem.

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The shaded area is the $dN_1(r(y))$

Figure 1: the counting measure in active
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