(Co)isotropic pairs in Poisson and presymplectic vector spaces

Jonathan Lorand*  
Department of Mathematics  
ETH Zurich  
Zurich, Switzerland  
jonathanlorand@gmail.com

Alan Weinstein  
Department of Mathematics  
University of California  
Berkeley, CA 94720 USA  
alanw@math.berkeley.edu

Abstract

We give two equivalent sets of invariants which classify pairs of coisotropic subspaces of a finite-dimensional Poisson vector space. For this it is convenient to dualize; we work with pairs of isotropic subspaces of a presymplectic vector space. We identify ten elementary types which are the building blocks of such pairs, and we write down a matrix, invertible over $\mathbb{Z}$, which takes the one 10-tuple of invariants to the other.

1 Introduction

The problem of classifying coisotropic pairs considered in this note arose from two separate projects.

The first project, by the first author, is to classify, up to conjugation by linear symplectomorphisms, canonical relations (lagrangian correspondences) from a finite-dimensional symplectic vector space to itself. Without symplectic structure, this classification of linear relations was carried out by Towber [13] and is a special case of results of Gelfand and Ponomarev [8]. In the symplectic situation, for the special case of graphs of symplectomorphisms, the classification amounts to identifying the conjugacy classes in the group of symplectic matrices. This classification and the problem of finding associated normal forms has a long history extending from Williamson [16] to Gutt [9]. In the general symplectic case, a result of Benenti and Tulczyjew ([3], Proposizioni 4.4 & 4.5) tells us that any canonical relation $X \leftarrow Y$ is given by coisotropic subspaces of $X$ and $Y$ and a symplectomorphism between the corresponding reduced spaces. When $X = Y$, a first step in the classification up to conjugacy of canonical relations is

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then a classification of the coisotropic pairs giving the range and domain. The further steps of the classification remain as work in progress.

The second project, by the second author, is an extension of the Wehrheim-Woodward theory of linear canonical relations (see [10], [14]) to the case where the set of lagrangian correspondences $X \leftrightarrow Y$ is replaced by the set of coisotropic correspondences, i.e. coisotropic subspaces of $X \times Y$. Each pair of coisotropic subspaces of $X$ gives a WW morphism represented by a diagram of the form $1 \leftrightarrow X \leftrightarrow 1$, and isomorphic pairs correspond to the same WW morphism. There are also inequivalent pairs representing the same WW morphism. The problem is to determine exactly which pairs are “WW equivalent”. This problem is now solved, in the symplectic case, as part of a complete description of the WW categories of (co)isotropic relations (see [15]).

Since coisotropic correspondences are fundamental in Poisson geometry, it is natural to consider the classification and WW problems for linear coisotropic correspondences between any Poisson vector spaces, not just symplectic ones. It turns out to be simpler to replace the Poisson vector spaces by their duals, which are presymplectic (i.e. equipped with a possibly degenerate skew-symmetric bilinear form), and coisotropic subspaces by their annihilators, which are isotropic. Duality provides a complete correspondence between the Poisson/coisotropic and presymplectic/isotropic situations.

We begin, then, with a finite-dimensional vector space $V$, equipped with a skew-symmetric bilinear form $\omega$, the presymplectic structure. We call vectors $v,w \in V$ $\omega$-orthogonal when $\omega(v,w) = 0$ and, for any linear subspace $W \subseteq V$, we call the subspace $W^\omega = \{ v \in V \mid \omega(v,w) = 0 \ \forall w \in W \}$ the $\omega$-orthogonal of $W$. For the radical $V^\omega$ of $V$ we reserve the letter $R$. The term presymplectomorphism will refer to a linear isomorphism $\varphi : (V,\omega) \to (\hat{V},\hat{\omega})$ which is compatible with the presymplectic structures $\omega$ and $\hat{\omega}$ in the sense that $\hat{\omega}(\varphi(u),\varphi(v)) = \omega(u,v)$ for all $u,v \in V$.

An isotropic pair in $V$ is an ordered pair of isotropic subspaces in $V$. Isotropic pairs $(A,B)$ and $(\hat{A},\hat{B})$ in $(V,\omega)$ and $(\hat{V},\hat{\omega})$ respectively are equivalent if there exists a presymplectomorphism $\varphi : V \to \hat{V}$ such that $\varphi(A) = \hat{A}$ and $\varphi(B) = \hat{B}$. In the Poisson setting, where a coisotropic subspace is a subspace annihilated by an isotropic in the dual, this equivalence corresponds to there being an invertible Poisson map which takes one coisotropic pair to the other. In the symplectic situation, when $\omega$ is non-degenerate, any coisotropic subspace is the $\omega$-orthogonal of an isotropic subspace. Clearly, a linear symplectomorphism will take one coisotropic pair to the other if and only if it maps the corresponding isotropic $\omega$-orthogonals to one another.

To obtain invariants for the classification of isotropic pairs, we begin with the spaces $V$, $R$, $A$ and $B$ associated to an isotropic pair $(A,B)$, and construct a decomposition

$$V = \bigoplus_{i=1}^{10} V_i$$  (1)
which determines decompositions

\[ R = \bigoplus_{i=1}^{10} R_i, \quad A = \bigoplus_{i=1}^{10} A_i, \quad B = \bigoplus_{i=1}^{10} B_i \]

such that, for each \( i \), the subspaces \( R_i, A_i \) and \( B_i \) are in \( V_i \). In such a situation we say that the decompositions of \( R, A \) and \( B \) above are subordinated to \( V \). The \textbf{dimension} of a triple or a pair in \( V_i \) will always mean the dimension of \( V_i \).

The decomposition \((1)\) will be such that \( V_i \) is a presymplectic space with radical \( R_i \) and such that \( (A_i, B_i) \) is an isotropic pair in \( V_i \) which has a certain elementary form, different for each \( i \), so that we speak of “types”. Each of the ten types of isotropic pairs is “elementary” in the sense that no isotropic pair of one of these types can be written as the direct sum (in a suitable sense) of isotropic pairs of the other types. Furthermore, each of these elementary types is the direct sum of isotropic pairs which are indecomposable and of the same type, so that the decomposition \((1)\) is analogous to a decomposition into “isotypic components” as is typical in representation theory. The ten elementary types are listed in Definition 3.3 and they are illustrated, in their indecomposable form, in Remark 4.3.

To construct the decomposition \((1)\) we make use of two ways of decomposing \( V \) which arise naturally in the current setting. The first is the decomposition

\[ V = R \oplus U \tag{2} \]

which arises from the choice of any complement \( U \) to \( R \) in \( V \). The restriction \( \omega_U \) of the presymplectic structure \( \omega \) to \( U \) is always non-degenerate (i.e. \( U \) is a symplectic space). Indeed, if this were not the case, and if \( v \) were a non-zero vector in the radical of \((U, \omega_U)\), then \( v \) would be \( \omega \)-orthogonal to both \( U \) and \( R \), and hence to all of \( V \), which would imply that \( v \in R \), a contradiction.

A second way of decomposing \( V \) is with respect to the three subspaces \( R, A \) and \( B \), seen on a purely linear algebraic level, i.e. without taking the presymplectic structure into account. We will see that

\[ V = (R \cap A \cap B) \oplus N_{RA} \oplus N_{RB} \oplus N_{AB} \oplus M_R \oplus M_A \oplus M_B \oplus Q \oplus C \tag{3} \]

where \( N_{RA}, N_{RB} \) and \( N_{AB} \) are complements of \( R \cap A \cap B \) in \( R \cap A, R \cap B \) and \( A \cap B \) respectively, \( M_R, M_A \) and \( M_B \) are respective complements of \( R \cap (A + B) \) in \( R \), \( A \cap (R + B) \) in \( A \) and \( B \cap (R + A) \) in \( B \), \( Q \) is a complement in \( R + A + B \) to the sum of the previous summands, and \( C \) is a complement of \( R + A + B \) in \( V \).

We remark briefly on aspects of this decomposition which are relevant to our classification problem in the presymplectic setting and which, by way of analogy, motivate and illustrate our approach.

First, for each summand \( V' \) in \((3)\), the intersections of \( R, A \) and \( B \) with \( V' \) represent a triple of subspaces \((R', A', B')\) which is of an “elementary type”
in the sense above and which has an accordingly simple form. For example, \( R \cap A \cap B \) represents the situation where \( V' = R' = A' = B' \), the space \( N_{AB} \) represents the situation where \( V' = R' = A' \) and \( B' = 0 \), and so forth. The space \( Q \) represents a type where \( V' = R' = A' \) and \( B' = 0 \), and so forth. The indecomposable types are all 1-dimensional, except for the indecomposable type associated to \( Q \), which is 2-dimensional (it is given by 3 distinct lines in a plane).

Second, any triple \((R', A', B')\) of elementary type is itself decomposable as the direct sum of triples which are of that elementary type and which are indecomposable. The indecomposable types are all 1-dimensional, except for the indecomposable type associated to \( Q \), which is 2-dimensional (it is given by 3 distinct lines in a plane).

Finally, we note that the dimensions of the summands in (3) give a set of invariants which solve the “triple of subspaces problem” of classifying, up to linear isomorphism, three arbitrary (linear) subspaces of a vector space. Indeed, these dimensions can be expressed in terms of the dimensions of \( V, R, A, B \) and subspaces derived by taking certain sums and intersections, so any linear isomorphism \( \psi : V \to \hat{V} \) which maps each member of a triple of subspaces in \( V \) to the corresponding member of a triple in \( \hat{V} \) will also map a decomposition of the form (3) to a corresponding decomposition of \( \hat{V} \) with matching dimensions. Conversely, given triples of subspaces in \( V \) and \( \hat{V} \) respectively, these induce corresponding decompositions of the form (3). If the dimensions of the summands of these decompositions match, it is straightforward to construct a linear isomorphism which maps the triple in \( V \) to the one in \( \hat{V} \).

To classify (co)isotropic pairs we proceed analogously, taking into account the presymplectic structure. Each summand in the ten-part decomposition (1) of an isotropic pair \((A, B)\) is itself the direct sum of copies of indecomposables of a given elementary type; the multiplicities of these indecomposables give (up to an integer factor) the dimension of each summand. We show that these multiplicities are invariants of an isotropic pair – we call them the elementary invariants – and we show that they are equivalent to another set of invariants, which are simple expressions in the dimensions of \( V, R, A \) and \( B \) (and subspaces derived thereof) and which we call the canonical invariants of an isotropic pair \((A, B)\):

\[
\begin{align*}
    k_1 &:= \frac{1}{2}(\dim V - \dim R) \quad & k_6 &:= \dim R \cap A \\
    k_2 &:= \dim R \quad & k_7 &:= \dim R \cap B \\
    k_3 &:= \dim A \quad & k_8 &:= \dim R \cap A \cap B \\
    k_4 &:= \dim B \quad & k_9 &:= \dim R \cap (A + B) \\
    k_5 &:= \dim A \cap B \quad & k_{10} &:= \dim A \cap B.
\end{align*}
\]

The first nine of these invariants correspond roughly to the purely linear algebraic information associated to (3), though \( k_1 \) also reflects the decomposition (2), which is determined by the presymplectic structure. For the tenth invariant, which carries information about the presymplectic structure, one can in fact just as well choose \( \dim B^\omega \cap A \), hence the canonical invariants are “symmetric” with

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\[1\] For the classification of subspace triples, see Brenner [4], p. 109-110, and Etingof, et al. [7], p. 84-86.
respect to $A$ and $B$, up to the order of $k_3$ and $k_4$. This symmetry shows, in passing, that the isotropic pairs $(A,B)$ and $(B,A)$ are equivalent when $\dim A = \dim B$. \footnote{This is the case, for example, when the corresponding coisotropics $A^\omega$ and $B^\omega$ are the range and domain respectively of a linear canonical relation in $V \times \overline{V}$.}

In Section 5 we show that the elementary invariants and the canonical invariants are equivalent, and we write down a matrix, invertible over $\mathbb{Z}$, taking one set of invariants to the other. The main classification result is Theorem 5.5. The key ingredient for this is the decomposition (1), which we construct in Section 3, and in Section 4 we show that each summand of this decomposition is itself the direct sum of isotropic pairs which are indecomposable. Section 2 on preliminaries serves to recall some basic facts and to give a framework which we will apply in our subsequent proofs. The present paper is an extension of our preprint \cite{11}; we have in the meantime found the reference \cite{12}, which, using other means, apparently covers our original results in the symplectic setting.

For convenience, all maps and subspaces are tacitly assumed linear unless otherwise stated, and the letters $A$ and $B$ always denote isotropic subspaces of a finite-dimensional (pre)symplectic vector space $(V,\omega)$. Angled brackets $\langle \cdot \rangle$ indicate the linear span of a vector or a set of vectors. We use the notation $\tilde{\omega}$ for the isomorphism $V \rightarrow V^*$, $v \mapsto \omega(v, \cdot)$ which is induced when the form $\omega$ is non-degenerate, and the symbol $\cong$ denotes a linear isomorphism, not necessarily (pre)symplectic.

2 Preliminaries

We recall briefly some basic facts from (pre)symplectic linear algebra. Let $W, E$ and $F$ be subspaces of the presymplectic space $V$ and denote by $\omega_W$ the restriction of $\omega$ to $W$. One has $\ker \omega_W = W \cap W^\omega$, and the reduced space $W/(W \cap W^\omega)$ admits an induced symplectic form $[\omega]$ given by

$$[\omega](\langle u \rangle, \langle v \rangle) := \omega(u, v) \quad \text{for } u, v \in W.$$

In particular, this form lifts to any choice of complement of $W \cap W^\omega$ in $W$, making it a symplectic space. The decomposition (2) corresponds to the special case when $W = V$.

When $\omega$ is non-degenerate, a different quotient relationship arises in addition to presymplectic reduction $W \rightarrow W/(W \cap W^\omega)$, via the isomorphism $\tilde{\omega} : V \rightarrow V^*$. Namely, $\tilde{\omega}$ composed with the restriction map $V^* \rightarrow W^*$ has kernel $W^\omega$, hence it induces a natural isomorphism $V/W^\omega \rightarrow W^*$. In the special case when $W$ is a lagrangian subspace, $V/W \cong W^*$. If $(L, L')$ is a transversal lagrangian pair in $V$, i.e. lagrangian subspaces such that $V = L \oplus L'$, then $V \cong L \oplus L^*$ symplectically via the natural map

$$\text{id} \oplus \tilde{\omega} : V \rightarrow L \oplus L^*, \quad u + v \mapsto (u, \omega(v, \cdot)) \quad \text{(4)}$$

where the external direct sum $L \oplus L^*$ is endowed with the symplectic form

$$\left(\langle v, \alpha \rangle, \langle w, \beta \rangle\right) \mapsto \beta(v) - \alpha(w). \quad \text{(5)}$$
Under (4), the lagrangians $L$ and $L'$ are mapped to the lagrangians $L \times 0$ and $0 \times L^*$ respectively, and any basis $\{q_1, ..., q_n\}$ of $L$, together with its dual basis $\{q_1^*, ..., q_n^*\}$ in $L^*$, is mapped by the inverse of (4) to a symplectic basis $\{q_1, ..., q_n, p_1, ..., p_n\}$ of $V$, i.e. a basis which satisfies

$$\omega(q_i, p_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \quad \forall \ 1 \leq i, j \leq n.$$ 

In particular, for any two transversal lagrangian pairs $(L, L')$ and $(\hat{L}, \hat{L}')$ in symplectic spaces $V$ and $\hat{V}$ of the same dimension, there always exists a symplectic map $\varphi : V \to \hat{V}$ such that $\varphi(L) = \hat{L}$ and $\varphi(L') = \hat{L}'$.

We return now to the presymplectic setting. In general, if $V = E \oplus F$ and $\hat{V} = \hat{E} \oplus \hat{F}$, we say that a map $\varphi : V \to \hat{V}$ respects the decompositions in $V$ and $\hat{V}$ if

$$\varphi(E) \subseteq \hat{E} \quad \text{and} \quad \varphi(F) \subseteq \hat{F}. \quad \text{(6)}$$

Clearly, if $\varphi$ is a presymplectomorphism which respects the decompositions given above, then $\varphi|_E : E \to \hat{E}$ and $\varphi|_F : F \to \hat{F}$ are also presymplectomorphisms. On the other hand, if $E, F$ are $\omega$-orthogonal, $\hat{E}, \hat{F}$ $\hat{\omega}$-orthogonal, and $\sigma : E \to \hat{E}$, $\rho : F \to \hat{F}$ are presymplectomorphisms, then $\sigma \oplus \rho$ defines a presymplectomorphism $V \to \hat{V}$ which respects the decompositions in $V$ and $\hat{V}$. The $\omega$-orthogonality condition on $E$ and $F$ (and $\hat{E}$ and $\hat{F}$) is tantamount to there being a natural presymplectomorphism between $E \oplus F$ and the external direct sum of two separate presymplectic spaces $(E, \omega_E)$ and $(F, \omega_F)$, endowed with the direct sum presymplectic structure

$$\omega_E \oplus \omega_F : ((e, f), (e', f')) \mapsto \omega_E(e, e') + \omega_F(f, f').$$

These notions naturally extend to any finite number of summands.

The property (6) describes a compatibility between a map and decompositions. A subspace $W \subseteq V$ is compatible with a decomposition $V = E \oplus F$ in the sense of the distributive property

$$W \cap (E \oplus F) = (W \cap E) \oplus (W \cap F) \quad \text{(7)}$$

if and only if $W$ has a decomposition compatible with that of $V$ and the inclusion map. The distributive property does not hold in general, though the weaker modular law

$$W \subseteq E \Rightarrow W \cap (E \oplus F) = E \oplus (W \cap F) \quad \text{(8)}$$

does. When subspaces $W, E$ and $F$ do satisfy (7), we say that the decomposition $E \oplus F$ is distributive with respect to $W$. More generally, if $S \subseteq V$ is the direct sum of subspaces $V_1, ..., V_m$,

$$S = \bigoplus_{i=1}^m V_i, \quad \text{(9)}$$

and \( W \subseteq V \) is such that
\[
W \cap S = \bigoplus_{i=1}^{m} (W \cap V_i),
\]
then we say that the sum (9) is distributive with respect to \( W \) or that it is \textit{\( W \)-distributive}. Similarly, we say that the sum (9) is distributive with respect to a collection of subspaces if it is distributive with respect to each member of that collection.

The decompositions (11) which we will use to obtain our classification result will be constructed so that the decomposition of \( V \) is distributive with respect to \( R, A \) and \( B \). For the remainder of this section, we discuss some useful properties of distributive decompositions, and ways to construct such decompositions.

\textbf{Lemma 2.1} \textit{Suppose that} \( W \subseteq S \) \textit{has a decomposition}
\[
W = \bigoplus_{i=1}^{m} W_i
\]
\textit{which is subordinate to the decomposition (2), i.e.} \( W_i \subseteq V_i \) \textit{for each} \( i \). Then, \textit{the direct sum (9) is} \( W \)-\textit{distributive, and} \( W \cap V_i = W_i \) \textit{for each} \( i \). \textit{In other words, subordinate decompositions are unique.}

\textbf{Proof.} We will show that, for each \( i \), \( W \cap V_i = W_i \); from this it directly follows that (9) is \( W \)-distributive, since then
\[
W \cap S = W = \bigoplus_{i=1}^{m} W_i = \bigoplus_{i=1}^{m} W \cap V_i.
\]
Note first that \( W_i \subseteq W \cap V_i \) because \( W_i \subseteq V_i \) by the assumption that (10) is subordinate to (9). In particular
\[
dim W_i \leq \dim W \cap V_i \quad \forall i.
\]
But \( W \cap V_i \subseteq W \) for each \( i \), so
\[
W = \bigoplus_{i=1}^{m} W_i \subseteq \bigoplus_{i=1}^{m} W \cap V_i \subseteq W
\]
and thus
\[
0 = \sum_{i=1}^{m} \dim W \cap V_i - \sum_{i=1}^{m} \dim W_i = \sum_{i=1}^{m} \left( \dim W \cap V_i - \dim W_i \right).
\]
By (12) the last sum above is a sum of non-negative integers; because they sum to zero, they must each be zero, which in turn implies that \( W_i = W \cap V_i \) for each \( i \), as desired.
Lemma 2.2 Suppose $\varphi : V \to \hat{V}$ is a (linear) map which respects decompositions

$$V = \bigoplus_{i=1}^{m} V_i \quad \text{and} \quad \hat{V} = \bigoplus_{i=1}^{m} \hat{V}_i.$$  

If the decomposition of $V$ is distributive with respect to a subspace $W \subseteq V$, then the decomposition of $\hat{V}$ is distributive with respect to $\varphi(W)$.

**Proof.** By linearity,

$$\varphi(W) = \varphi\left(\bigoplus_{i=1}^{m} W \cap V_i\right) = \bigoplus_{i=1}^{m} \varphi(W \cap V_i), \quad (13)$$

and because $\varphi$ respects the decompositions, the latter sum is subordinate to the decomposition in $\hat{V}$. By Lemma 2.1 the claim follows.

If a direct sum decomposition is distributive with respect to a subspace $W$, it does not follow that it is distributive with respect to every $U \subseteq W$. Distributivity does carry over to sums and intersections of spaces.

Lemma 2.3 Suppose that $S$ has a decomposition (9) which is distributive with respect to both $E$ and $F$. Then,

(i) the decomposition is distributive with respect to $E \cap F$.

(ii) the decomposition is distributive with respect to $E + F$, and

$$(E + F) \cap V_i = (E \cap V_i) + (F \cap V_i) \quad (14)$$

for each $i$.

**Proof.** We set, for each $i$, $E_i := E \cap V_i$ and $F_i := F \cap V_i$. By hypothesis

$$E = \bigoplus_{i=1}^{m} E_i \quad \text{and} \quad F = \bigoplus_{i=1}^{m} F_i \quad (15)$$

and these decompositions are subordinate to (9).

(i) The inclusion

$$\bigoplus_{i=1}^{m} (E \cap F \cap V_i) \subseteq (E \cap F) \cap \bigoplus_{i=1}^{m} V_i$$

is clear; each summand on the left is a subspace of the space given on the right-hand side. For the opposite inclusion, let

$$v \in (E \cap F) \cap \bigoplus_{i=1}^{m} V_i$$
and let
\[ v = v_1 + \ldots + v_m \]  
be the unique decomposition of \( v \) with respect to (9), i.e. with \( v_i \in V_i \) for each \( i \). As an element in \( E \cap F \), \( v \) also has such decompositions
\[ v = e_1 + \ldots + e_m \quad \text{and} \quad v = f_1 + \ldots + f_m \]  
with respect to the direct sums (15), i.e. \( e_i \in E_i \) and \( f_i \in F_i \) for each \( i \). Because the sums (15) are subordinate to (9), \( e_i \) and \( f_i \) are in \( V_i \) for each \( i \). This means that (17) are also decompositions of \( v \) with respect to (9). Thus, by uniqueness of such decompositions,
\[ v_i = e_i = f_i \quad \forall i, \]  
which implies that \( v_i \in E_i \cap F_i \) for each \( i \), and so
\[ v \in \bigoplus_{i=1}^{m} (E \cap F \cap V_i). \]

(ii) Note first that, from (15),
\[ E + F = \sum_{i=1}^{m} (E_i + F_i). \]  
By assumption, \( E_i, F_i \subseteq V_i \) for each \( i \), so also \( E_i + F_i \subseteq V_i \). This implies that the sum (18) is a direct sum and that this decomposition of \( E + F \) is subordinate to (9). Hence, by Lemma 2.1, the decomposition (9) is \( (E + F) \)-distributive and \( (E + F) \cap V_i = E_i + F_i \), which gives (14).

We call a direct sum decomposition \( \text{(9)} \) \( \omega \)-orthogonal if the \( V_i \) are pairwise \( \omega \)-orthogonal. The properties of \( \omega \)-orthogonality and distributivity are compatible in the following sense.

Lemma 2.4 Suppose that \( S \) has a decomposition (9) which is both \( \omega \)-orthogonal and \( W \)-distributive. Then, the decomposition is also \( W^\omega \)-distributive and, for each \( i \), one has
\[ W^\omega \cap V_i = (W \cap V_i)^{\omega_i}, \]  
where \( \omega_i := \omega_{|V_i} \).

Proof. Let \( W_i := W \cap V_i \)

We first show (19). If \( v_i \in W^\omega \cap V_i \) and \( w_i \in W_i \), then
\[ \omega_i(v_i, w_i) = \omega(v_i, w_i) = 0, \]  
because \( W_i \subseteq W \). This shows that \( W^\omega \cap V_i \subseteq W_i^{\omega_i} \). For the opposite inclusion, choose an element \( v_i \in W_i^{\omega_i} \). We have \( v_i \in V_i \) because \( W_i^{\omega_i} \subseteq V_i \). To see that
\( v_i \in W^\omega \), let \( w \in W \cap S \) and let \( w = w_1 + ... + w_m \) be the unique decomposition of \( w \) with respect to the decomposition
\[
W \cap S = \bigoplus_{i=1}^m W \cap V_i.
\]
Then
\[
\omega(v_i, w) = \omega(v_i, w_1) + ... + \omega(v_i, w_m),
\]
which is zero: \( \omega(v_i, w) = 0 \) because \( v_i \in W^\omega \), and \( \omega(v_i, w_j) = 0 \) for all \( j \neq i \) because \( w_j \in V_j \), which, by assumption, is \( \omega \)-orthogonal to \( V_i \) for \( j \neq i \).

We now show that
\[
W^\omega \cap S = \bigoplus_{i=1}^m W^\omega \cap V_i.
\] (20)
The inclusion “\( \supseteq \)” is clear, because each summand in this decomposition is a subspace of \( W^\omega \cap S \). For the opposite inclusion “\( \subseteq \)”, let \( v \in W^\omega \cap S \) and let \( v = v_1 + ... + v_m \) be the unique decomposition of \( v \) with respect to the decomposition
\[
S = \bigoplus_{i=1}^m V_i.
\]
We claim that \( v_i \in W^\omega_i \) for each \( i \). To see this, let \( w_i \in W_i \). On the one hand
\[
\omega(v, w_i) = 0,
\]
because \( w_i \in W_i \). On the other hand
\[
\omega(v, w_i) = \omega(v_1, w_i) + ... + \omega(v_m, w_i) = \omega(v, w_i) = \omega_i(v_i, w_i),
\]
because \( v_j \) is \( \omega \)-orthogonal to \( w_i \) for all \( j \neq i \) (by the \( \omega \)-orthogonality assumption). Hence, \( \omega_i(v_i, w_i) = 0 \), which proves the claim and shows that
\[
W^\omega \cap S \subseteq \bigoplus_{i=1}^m W^\omega_i.
\]
By (19), this is equivalent to the desired inclusion.

\[ \square \]

**Corollary 2.5** *Let \((A, B)\) be an isotropic pair in \( V \) and let
\[
V = \bigoplus_{i=1}^m V_i
\] (21)* be an \( \omega \)-orthogonal decomposition which is distributive with respect to \( A \) and \( B \). Then, for each \( i \), \((A \cap V_i, B \cap V_i)\) is an isotropic pair. Also, any \( \omega \)-orthogonal decomposition (27) is distributive with respect to the radical \( R \) of \( V \) and \( R \cap V_i \) is the radical of \((V_i, \omega|_{V_i})\), for each \( i \). If \( V \) is a symplectic space and \((A, B)\) is a lagrangian pair, then each \( V_i \) is symplectic and each \((A \cap V_i, B \cap V_i)\) is a lagrangian pair in \( V_i \).
Proof. That \( A \) is isotropic means that \( A \subseteq A^\omega \). This implies that, for each \( i \), \( A \cap V_i \subseteq A^\omega \cap V_i \). By Lemma 2.4
\[
A^\omega \cap V_i = (A \cap V_i)^{\omega|V_i}.
\]
Thus, for each \( i \), \( A \cap V_i \subseteq (A \cap V_i)^{\omega|V_i} \), i.e. \( A \cap V_i \) is isotropic in \((V_i, \omega|V_i)\). The same arguments hold for \( B \).

The radical of \( V \) is, by definition, \( R = V^\omega \). Because (21) is distributive with respect to \( V \), by Lemma 2.4 it is also distributive with respect to \( R \), and
\[
R \cap V_i = V^\omega \cap V_i = (V \cap V_i)^{\omega|V_i} = V_i^{\omega|V_i},
\]
which is precisely the radical of \((V_i, \omega|V_i)\).

For the case when \( V \) is symplectic and \((A, B)\) is a lagrangian pair, note first that \( R = R \cap V_i = 0 \) for each \( i \) because \( R = 0 \), so each \( V_i \) is symplectic. Further, note that the argumentation above for isotropics works also with “coisotropic” in place of “isotropic” (and inclusions reversed). Because a lagrangian subspace is one which is both isotropic and coisotropic, the result follows.

\[ \square \]

Definition 2.6 Let \((A, B)\) be an isotropic pair in \( V \) and let
\[
A = \bigoplus_{i=1}^m A_i \quad \text{and} \quad B = \bigoplus_{i=1}^m B_i
\]
be decompositions subordinate to an \( \omega \)-orthogonal decomposition \((21)\) of \( V \). By Lemma 2.4 and Corollary 2.2, each \((A_i, B_i)\) is an isotropic pair; we say that \((A, B)\) is the direct sum of the isotropic pairs \((A_i, B_i)\) and write
\[
(A, B) = \bigoplus_{i=1}^m (A_i, B_i).
\]

Remark 2.7 Given a subspace \( S \subseteq V \), a general strategy for constructing a direct sum decomposition of \( S \) which is distributive with respect to a subspace \( W \subseteq V \) is the following iterative procedure. In each step, find a subspace \( V' \subseteq S \) such that one of the following holds:

(i) \( W \cap V' = 0 \)

(ii) \( W \cap V' = V' \)

If (i) is the case, then there exists a subspace \( C \subseteq S \) such that \( W \cap S \subseteq C \) and \( S = V' \oplus C \). This decomposition of \( S \) is, by construction, distributive with respect to \( W \). If (ii) is the case, i.e. \( V' \subseteq W \), then for any complement \( C \) of \( V' \) in \( S \) one has \( W \cap S = V' \oplus (W \cap C) \) by the modular law, and so the resulting decomposition \( S = V' \oplus C \) is distributive with respect to \( W \).
The following lemma shows that this iterative procedure will indeed achieve the desired result. If in each step $V'$ and $C$ can be chosen to be $\omega$-orthogonal, then the resulting decomposition of $S$ will also be $\omega$-orthogonal.

**Lemma 2.8** Suppose that $S \subseteq V$ has a decomposition of the form \((9)\). For each $l \in \{0, 1, ..., m - 1\}$ set

$$C_l := \bigoplus_{i=l+1}^m V_i.$$

(i) If, for each $l \in \{1, ..., m - 1\}$, the decomposition $V_l \oplus C_l$ is $\omega$-orthogonal, then the decomposition \((9)\) is $\omega$-orthogonal.

(ii) Let $W \subseteq V$ be a subspace. If, for each $l \in \{1, ..., m - 1\}$, the decomposition $V_l \oplus C_l$ is distributive with respect to $W \cap C_{l-1}$, then the decomposition \((9)\) is distributive with respect to $W$.

**Proof.** (i) Choose any two indices $i, j \in \{1, ..., m\}$ such that $i \neq j$. We need to show that $V_i$ and $V_j$ are $\omega$-orthogonal. Because this relation is symmetric with respect to $i$ and $j$, we may assume without loss of generality that $i < j$. Then $i \leq m - 1$ and $V_j \subseteq C_i$. By assumption $V_i$ is $\omega$-orthogonal to $C_i$, so in particular $V_i$ is $\omega$-orthogonal to $V_j$.

(ii) We apply the assumptions iteratively to construct a decomposition of $W \cap S$ composed of the intersections of $W$ with the $V_l$. For $l = 1$, by assumption we have a decomposition

$$W \cap S = W \cap V_1 \oplus W \cap C_1.$$ 

The assumption for $l = 2$, applied to the second summand of this decomposition, gives

$$W \cap S = W \cap V_1 \oplus W \cap V_2 \oplus W \cap C_2.$$ 

Clearly, proceeding in this manner for increasing $l$ will, after $m - 1$ steps, lead to a decomposition

$$W \cap S = W \cap V_1 \oplus W \cap V_2 \oplus ... \oplus W \cap V_{m-1} \oplus W \cap C_{m-1},$$

which, after substitution using the identity $C_{m-1} = V_m$, is the desired result. \(\Box\)

### 3 Decompositions

In this section, we will find, for each isotropic pair, a compatible decomposition of the ambient space into elementary types.
3.1 The symplectic case

In this subsection, \( V \) denotes a symplectic vector space with symplectic form \( \omega \).

**Definition 3.1** Let \((A, B)\) be an isotropic pair in the symplectic space \( V \). We say that \((A, B)\) is of **elementary type** if it is of one of the following types:

- \(\lambda\): \( A \) and \( B \) are lagrangian subspaces, and \( A = B \)
- \(\mu_A\): \( B = 0 \) and \( A \) is a lagrangian subspace
- \(\mu_B\): \( A = 0 \) and \( B \) is a lagrangian subspace
- \(\delta\): \( A \) and \( B \) are lagrangian subspaces, and \( A \cap B = 0 \)
- \(\sigma\): \( A = B = 0 \)

We label these elementary types \( \tau_1, \ldots, \tau_5 \), in the order listed.

**Proposition 3.2** Let \((A, B)\) be an isotropic pair in the symplectic space \( V \). There exists a decomposition

\[
V = \bigoplus_{i=1}^{5} V_i
\]  

which is \( A, B \)-distributive, \( \omega \)-orthogonal, and such that, for each \( i \in \{1, \ldots, 5\} \), the isotropic pair \((A_i, B_i) := (A \cap V_i, B \cap V_i)\) in \( V_i \) is of the elementary type \( \tau_i \).

**Proof.** We proceed step-wise, ‘peeling’ away in each step a symplectic subspace of \( V \) which corresponds to an elementary type.

Step 1. Consider the isotropic subspace \( A \cap B \subseteq V \) and choose a subspace \( C_1 \) such that

\[
(A \cap B)^{\omega} = A^{\omega} + B^{\omega} = (A \cap B) \oplus C_1.
\]  

Then, \( C_1 \) is symplectic (and so is \( C_1^{\perp}\)), and \( A \cap B \subseteq C_1^{\perp} \) is a lagrangian subspace. We set \( V_1 := C_1^{\perp} \). The \( \omega \)-orthogonal decomposition \( V = V_1 \oplus C_1 \) is \( A, B \)-distributive: \( A \subseteq (A \cap B) \oplus C_1 \) implies (by the modular law) that \( V_1 \oplus C_1 \) is \( A \)-distributive when \( (23) \) is, and \( A \cap B \subseteq A \) implies that \( (23) \) is indeed \( A \)-distributive. The same argument holds for \( B \). Because

\[
A_1 = A \cap V_1 = B_1 = B \cap V_1 = A \cap B
\]

is lagrangian in \( V_1 \), we find that \((A_1, B_1)\) is of the elementary type \( \lambda \).

Step 2. In order to decompose \( C_1 \), consider the isotropic subspace

\[
G := A \cap B^{\omega} \cap C_1 \subseteq C_1,
\]

and denote the symplectic form \( \omega|_{C_1} \) by \( \omega_1 \). Using Lemma \( (24) \) with respect to the \( A, B \)-distributive and \( \omega \)-orthogonal decomposition \( V = V_1 \oplus C_1 \), one has

\[
G^{\omega_1} = (A \cap C_1)^{\omega_1} + (B^{\omega} \cap C_1)^{\omega_1} = (A^{\omega} \cap C_1) + (B \cap C_1).
\]
Note that $B \cap C_1 \cap G = 0$, because any element of $B \cap C_1 \cap G$ must be in $A \cap B \cap C_1 = 0$. Thus it is possible to choose a subspace $C_2$, such that

$$G^{\omega_1} = G \oplus C_2$$

and such that $B \cap C_1 \subseteq C_2$. It follows that $C_2$ is symplectic, and we obtain an $\omega$-orthogonal decomposition $C_1 = C_2^{\omega_1} \oplus C_2$, with $G \subset C_2^{\omega_1}$ lagrangian. We set $V_2 := C_2^{\omega_1}$, and note that the decomposition $C_1 = V_2 \oplus C_2$ is $A, B$-distributive. Indeed, $B \cap C_1 \subset C_2$ by construction, which gives distributivity with respect to $B$, and distributivity with respect to $A$ is ensured because $G \subseteq A \cap C_1 \subseteq G_1 \cap C_2$. In $V_2$, $A_2 = A \cap V_2 = G$ is a lagrangian subspace, and $B_2 = B \cap V_2 = 0$, because $B \cap C_1 \subset C_2$. Thus $(A_2, B_2)$ is of the elementary type $\mu_A$.

Step 3. Consider the isotropic subspace

$$H := B \cap A^\omega \cap C_2 \subseteq C_2,$$

and set $\omega_2 = \omega|_{C_2}$. By Lemma 2.4 we have

$$H^{\omega_2} = (B \cap C_2)^{\omega_2} + (A^\omega \cap C_2)^{\omega_2} = (B^\omega \cap C_2) + (A \cap C_2).$$

Because $(A \cap C_2) \cap H = A \cap B \cap C_1 = 0$ and $(A \cap C_2) \subseteq H^{\omega_2}$, we can choose a subspace $C_3$ such that

$$H^{\omega_2} = H \oplus C_3,$$

with $A \cap C_2 \subseteq C_3$. Then $C_3$ is symplectic, giving an $\omega$-orthogonal decomposition $C_2 = C_3^{\omega_2} \oplus C_3$, with $H \cap C_3^{\omega_2}$ as a lagrangian subspace. We set $V_3 := C_3^{\omega_2}$. The decomposition $C_2 = V_3 \oplus C_3$ is $A, B$-distributive, because $A \cap C_2 \subseteq C_3$ and $H \subseteq B \cap C_2 \subseteq H \oplus C_3$. We also see that $A \cap V_3 = 0$ and $B \cap V_3 = H$, so $(A_3, B_3)$ is of the elementary type $\mu_B$.

Step 4. Set $\omega_3 = \omega|_{C_3}$ and consider the two isotropic subspaces

$$A \cap C_2 = A \cap C_3 \quad \text{and} \quad B \cap C_3,$$

which have zero intersection because $A \cap B \cap C_3 \subseteq A \cap B \cap C_1 = 0$ (from step 1). We set

$$V_4 := (A \cap C_3) \oplus (B \cap C_3)$$

and claim that $V_4 \subseteq C_3$ is symplectic. We have

$$V_4^{\omega_3} = (A \cap C_3)^{\omega_3} \cap (B \cap C_3)^{\omega_3} = A^\omega \cap B^\omega \cap C_3;$$

we denote this space by $V_5$. The claim is proved if we show $V_4 \cap V_5 = 0$. To show this, suppose

$$v \in V_4 \cap V_5 = [(A \cap C_3) \oplus (B \cap C_3)] \cap [A^\omega \cap B^\omega \cap C_3].$$

Then $v$ has a decomposition $v = v_A + v_B$, with $v_A \in A \cap C_3$ and $v_B \in B \cap C_3$. Because $v_B \in B \cap C_3 \subseteq B^\omega \cap C_3$ and $v \in B^\omega \cap C_3$, it follows that

$$v_A = v - v_B \in A \cap B^\omega \cap C_3 \subseteq A \cap B^\omega \cap C_1 = G.$$
But, $C_3 \cap G = 0$, so $v_A = 0$. Similarly,

$$v_B \in B \cap A^\omega \cap C_3 \subseteq B \cap A^\omega \cap C_2 = H,$$

and $C_3 \cap H = 0$ implies that $v_B = 0$. This shows that $v = v_A + v_B = 0$ and proves the claim.

In particular, $V_5 = V_4^\omega$ is also symplectic, and we obtain an $\omega$-orthogonal decomposition

$$C_3 = V_4 \oplus V_5. \quad (24)$$

We claim that $A \cap C_3$ and $B \cap C_3$ are each lagrangian subspaces of $V_4$. Because $A \cap C_3$ and $B \cap C_3$ are isotropic subspaces of $V_4$, we have, on the one hand,

$$\dim A \cap C_3 \leq \frac{1}{2} \dim V_4 \quad \text{and} \quad \dim B \cap C_3 \leq \frac{1}{2} \dim V_4. \quad (25)$$

On the other hand,

$$\dim V_4 = \dim A \cap C_3 + \dim B \cap C_3, \quad (26)$$

which implies that the inequalities (25) must be equalities, i.e. $A \cap C_3$ and $B \cap C_3$ are indeed lagrangian. From the definition of $V_4$ it follows that $A_4 = A \cap V_4 = A \cap C_3$ and $B_4 = B \cap V_4 = B \cap C_3$, and hence clearly also that $A_5 = A \cap V_5 = B_5 = B \cap V_5 = 0$. Thus the decomposition (24) is $A,B$-distributive, and we find that $(A_4, B_4)$ is of the elementary type $\delta$ and $(A_5, V_5)$ is of the elementary type $\sigma$.

In total, we have constructed a decomposition

$$V = V_1 \oplus V_2 \oplus V_3 \oplus V_4 \oplus V_5 \quad (27)$$

into symplectic subspaces of the elementary types $\tau_1$ through $\tau_5$. This construction was done with the aid of subspaces $C_1$, $C_2$, $C_3$ and $C_4 := V_5$ such that the assumptions in both point (i) and (ii) of Lemma 2.8 apply. We conclude that the decomposition (27) is both $\omega$-orthogonal and $(A,B)$-distributive, as was to be shown.

$\square$

### 3.2 The general presymplectic case

Let $V$ now be a presymplectic space with presymplectic structure $\omega$. As before, $R$ denotes the radical of $V$.

**Definition 3.3** Let $(A, B)$ be an isotropic pair in the presymplectic space $V$. We say that $(A, B)$ is of **elementary type** if it is of one of the following types:

$\lambda$: $R = 0$, $A$ and $B$ are lagrangian subspaces, and $A = B$

$\mu_A$: $R = 0$, $B = 0$ and $A$ is a lagrangian subspace

$\mu_B$: $R = 0$, $A = 0$ and $B$ is a lagrangian subspace

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\( \delta: R = 0, A \text{ and } B \text{ are lagrangian subspaces, and } A \cap B = 0 \)

\( \sigma: R = 0, A = B = 0 \)

\( \mu_R: 3 \dim R = \dim V, R, A \text{ and } B \text{ have pairwise zero intersection, and } A \oplus R = B \oplus R = A \oplus B \)

\( \zeta: R = V, B = V \text{ and } A = V \)

\( \alpha: R = V, B = 0 \text{ and } A = V \)

\( \beta: R = V, B = V \text{ and } A = 0 \)

\( \rho: R = V, A = 0 \text{ and } B = 0 \)

We label these elementary types \( \tau_1, ..., \tau_{10} \), in the order listed.

In the first five elementary types above, the presymplectic structure is non-degenerate; these five types are precisely the five elementary symplectic types discussed in Section 3.1, and are hence labeled the same. Each type \( \tau \in \{ \lambda, \mu_A, \mu_B, \delta, \sigma \} \) is itself the direct sum of quadruples of type \( \tau \) of dimension 2.

In the last four types, the presymplectic structure is completely degenerate, i.e. zero. These types are the elementary types of pairs of subspaces in a vector space, with no further structure. Each type \( \tau \in \{ \zeta, \alpha, \beta, \rho \} \) is itself the direct sum of pairs of type \( \tau \) of dimension 1.

Only for the type \( \tau_6 = \mu_R \) is the presymplectic structure neither zero nor symplectic. Here \( A \) and \( B \) project onto a single lagrangian subspace of the symplectic space \( V/R \). Every isotropic pair of type \( \mu_R \) is the direct sum of isotropic pairs of this type which are 3-dimensional, in which case the spaces \( A, B \) and \( R \) are three distinct lines which lie in a plane.

**Theorem 3.4** There exists an \( \omega \)-orthogonal and \( R, A, B \)-distributive decomposition

\[
V = \bigoplus_{i=1}^{10} V_i
\]

of the presymplectic space \( V \), such that, for each \( i \in \{1, ..., 10\} \), the isotropic pair \( (A_i, B_i) := (A \cap V_i, B \cap V_i) \) in \( V_i \) is of the elementary type \( \tau_i \).

**Proof.** To construct the decomposition (28), we first construct a decomposition of \( R + A + B \) which is distributive with respect to \( R, A, \) and \( B \), and which we will later extend and modify. We proceed in steps, as in the symplectic case.

Step 1. Set

\[
T := R \cap A \cap B
\]

and note that, for any choice of complement \( C_1 \), the decomposition \( R + A + B = T \oplus C_1 \) is distributive with respect to \( R, A \) and \( B \), because \( T \) is a subspace of

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3The type \( \mu_R \) is closely related to the 2-dimensional minimal elementary type for triples of subspaces in a vector space (all remaining minimal types in that case are 1-dimensional).
each of these three spaces (cf. Remark 2.7). For later reference, we note that $T$, and the implied isotropic pair $(A \cap T, B \cap T)$, is of elementary type $\zeta$.

Step 2. Consider the intersection

$$N_{AB} := A \cap B \cap C_1.$$ 

One has $N_{AB} \cap R = 0$, because any vector in this space must be in $T$, and also in $C_1$, and $T \cap C_1 = 0$. By Remark 2.7, one can thus choose a subspace $C_2 \subseteq C_1$ such that $R \cap C_1 \subseteq C_2$ and

$$C_1 = N_{AB} \oplus C_2.$$ 

This decomposition is then $R, A, B$-distributive.

Step 3. Consider

$$N_{AR} := A \cap R \cap C_2.$$ 

By similar reasoning as above, one has $N_{AR} \cap B = 0$ and one may choose a subspace $C_3$ such that $B \cap C_2 \subseteq C_3$ and such that one obtains a $R, A, B$-distributive decomposition

$$C_2 = N_{AR} \oplus C_3.$$ 

We note here that $N_{AR}$ is a presymplectic space of elementary type $\alpha$.

Step 4. We split off

$$N_{BR} := B \cap R \cap C_3,$$

where $N_{BR} \cap A = 0$ holds and hence a subspace $C_4$ can be chosen such that $A \cap C_3 \subseteq C_4$, and the decomposition

$$C_3 = N_{BR} \oplus C_4$$ 

is $R, A, B$-distributive. Also, we see that $N_{BR}$ is a presymplectic space of elementary type $\beta$.

Up to the present point, we have constructed an $R, A, B$-distributive decomposition

$$R + A + B = T \oplus N_{AB} \oplus N_{AR} \oplus N_{BR} \oplus C_4.$$ 

Let $R', A'$ and $B'$ denote respectively the intersections of $R$, $A$ and $B$ with $C_4$, and note that these spaces have pairwise zero intersection, and $R' + A' + B' = C_4$.

Step 5. We define

$$Q_A := A' \cap (B' + R'),$$

and choose a subspace $M_A$ so that $A' = M_A \oplus Q_A$. We now split off $M_A$, where the fact that $M_A \cap (B' + R') = 0$ ensures that one can choose a complement $C_5$ of $M_A$ in $C_4$ which contains $B' + R'$ (and hence also $Q_A$), so that the resulting decomposition

$$C_4 = M_A \oplus C_5$$ 

is $R, A, B$-distributive. In particular $A \cap C_5 = Q_A$.

Step 6. We decompose $B' \subseteq C_5$ into

$$Q_B := B' \cap (Q_A + R').$$
and a complement $M_B$ of $Q_B$ in $B'$. We split off $M_B$, which has zero-intersection with $Q_A + R'$, choosing a complement $C_6$ in $C_5$ such that $Q_A + R' \subseteq C_6$. The decomposition
\[ C_5 = M_B \oplus C_6 \]
is $R, A, B$-distributive; in particular $B \cap C_6 = Q_B$.

Step 7. We proceed analogously, decomposing $R' \subseteq C_6$ as $R' = M_R \oplus Q_R$, where
\[ Q_R := R' \cap (Q_A + Q_B) \]
and $M_R$ is any choice of complement. Because
\[ M_R \cap (Q_A + Q_B) = 0 \quad (29) \]
and $C_6 = Q_A + Q_B + Q_R$, we can choose $C_7 := Q_A + Q_B$ as a complement of $M_B$ in $C_6$ to obtain an $R, A, B$-distributive decomposition
\[ C_6 = M_B \oplus C_7. \]
In passing, we note here that $A \cap C_6 = Q_A$, $B \cap C_6 = Q_B$ and (29) imply together that $A \cap M_R = 0$ and $B \cap M_R = 0$, and $M_R \subseteq R$ means that $R \cap M_R = M_R$, so $M_R$ is a presymplectic space of elementary type $\rho$.

As will be shown below, the spaces $Q_R, Q_A$ and $Q_B$ are such that
\[ Q_A \subseteq (Q_B + Q_R) \quad Q_B \subseteq (Q_A + Q_R) \quad Q_R \subseteq (Q_A + Q_B), \quad (30) \]
which in turn implies that
\[ Q_A + Q_B = Q_B + Q_R = Q_A + Q_R. \quad (31) \]
Because $Q_R, Q_A$ and $Q_B$ have pairwise zero intersection, these sums are in fact direct, and we conclude that
\[ \dim Q_R = \dim Q_A = \dim Q_B. \quad (32) \]

To see that the inclusions (31) hold, note that $Q_A = A' \cap C_7$, $Q_B = B' \cap C_7$ and $Q_R = R' \cap C_7$, by distributivity of the decompositions above. One then has
\[ Q_A = Q_A \cap C_7 = [A' \cap (B' + R')] \cap C_7 \]
\[ = Q_A \cap [(B' + R') \cap C_7] \]
\[ = Q_A \cap [B' \cap C_7 + R' \cap C_7] \]
\[ = Q_A \cap [Q_B + Q_R], \]
i.e. $Q_A \subseteq Q_B + Q_R$, where in the third line we use Remark 23. An analogous calculation shows that $Q_B \subseteq (Q_A + Q_R)$. Finally, the inclusion $Q_R \subseteq (Q_A + Q_B)$ follows directly from the fact that $C_7 = Q_A + Q_B$.

To summarize: we have constructed a decomposition
\[ R + A + B = T \oplus N_{AB} \oplus N_{AR} \oplus N_{BR} \oplus M_A \oplus M_B \oplus M_R \oplus Q, \quad (33) \]
where $Q := Q_R + Q_A + Q_B$ is a renaming of the space $C_7$ above. That this decomposition is distributive with respect to $A$, $B$, and $R$, follows from Lemma 2.8.

We now aim to extend this decomposition to an $R, A, B$-distributive decomposition of all of $V$.

First, we show

(a) $R + A + B \subset A^\omega + B^\omega$,

(b) $A^\omega + B^\omega \subset Q^\omega$.

For (a), note that the radical $R$ is $\omega$-orthogonal to any subspace in $V$, so $R \subset A^\omega$ and $R \subset B^\omega$, and thus $R \subset A^\omega + B^\omega$. Because $A \subset A^\omega$ and $B \subset B^\omega$, the space $A^\omega + B^\omega$ also contains $A$ and $B$, and hence also $R + A + B$.

For (b), we observe that

$$Q = Q_R + Q_A$$

implies that

$$Q^\omega = Q_R^\omega \cap Q_A^\omega = V \cap Q_A^\omega = Q_A^\omega.$$  

Similarly, $Q = Q_R + Q_B$ implies that

$$Q^\omega = Q_R^\omega \cap Q_B^\omega = V \cap Q_B^\omega = Q_B^\omega,$$

so $Q_A^\omega = Q_B^\omega = Q^\omega$. Noting that $Q_A \subset A$ implies $A^\omega \subset Q_A^\omega = Q^\omega$, and that $Q_B \subset B$ implies $B^\omega \subset Q_B^\omega = Q^\omega$, we see that $(A^\omega + B^\omega) \subset Q^\omega$.

Next, let $L$ be a subspace such that

$$Q^\omega = (R + A + B) \oplus L. \quad (34)$$

Using the decomposition $Q = Q_A \oplus Q_R$ and the fact that

$$R = T \oplus N_{AR} \oplus N_{BR} \oplus Q_R$$

it follows from (34) and (33) that

$$Q^\omega = Q_A^\omega \cap R^\omega = Q_A^\omega = R \oplus Q_A \oplus N_{AB} \oplus M_A \oplus M_B \oplus L. \quad (35)$$

We set

$$V_s := N_{AB} \oplus M_A \oplus M_B \oplus L \quad (36)$$

and note that $\ker \omega|_{Q^\omega} = R \oplus Q_A$, which shows, via presymplectic reduction, that $V_s$ is symplectic. In particular,

$$V = V_s \oplus V_s^\omega. \quad (37)$$

From (35) we know that $R \oplus Q_A \subset V_s^\omega$; we choose a subspace $P$ such that

$$V_s^\omega = R \oplus Q_A \oplus P, \quad (38)$$

which, together with (37) and (36), gives a decomposition

$$V = [T \oplus N_{AR} \oplus N_{BR} \oplus M_R \oplus Q_R] \oplus [Q_A \oplus P \oplus N_{AB} \oplus M_A \oplus M_B \oplus L]. \quad (39)$$
We use the square brackets to view this decomposition as comprised of two main pieces: the radical $R$ on the left side, and the space on the right, which we call $U$. As a complement of $R$ in $V$, $U$ is symplectic.

Note that the decomposition (39) is not $R,A,B$-distributive. Indeed,

$$B \cap (Q_R \oplus Q_A) = Q_R,$$

but $B \cap Q_R = 0$ and $B \cap Q_A = 0$. Recalling that $Q$ is the sum of any two of the spaces $Q_A$, $Q_B$ and $Q_R$, we therefore rewrite (39) in the form

$$V = [T \oplus N_{AR} \oplus N_{BR} \oplus M_R] \oplus [Q \oplus P] \oplus [N_{AB} \oplus M_A \oplus M_B \oplus L], \quad (40)$$

which gives an $R,A,B$-distributive decomposition, because (40) is simply the direct sum of the $R,A,B$-distributive decomposition (33) and the space $L \oplus P$, which is a complement of $R + A + B$ in $V$.

The space bracketed on the right in the decomposition (40) is $V_s$. We call the space bracketed in the middle $V_m$, and the one on the left $V_z$. We have seen that $V_s$ is symplectic, and because $V_z \subseteq R$, we know that $\omega$ restricted to $V_z$ is zero.

To conclude our proof, it remains to show the following:

(i) $V_s$ decomposes as  

$$V_s = V_1 \oplus V_2 \oplus V_3 \oplus V_4 \oplus V_5$$

where, for each $i \in \{1, 2, 3, 4, 5\}$, $V_i$ is a presymplectic space of elementary type $\tau_i$

(ii) $V_6 := V_m = Q \oplus P$ is of type $\tau_6 = \mu_R$

(iii) $V_z$ decomposes as  

$$V_z = V_7 \oplus V_8 \oplus V_9 \oplus V_{10}$$

where, for each $i \in \{7, 8, 9, 10\}$, $V_i$ is a presymplectic space of elementary type $\tau_i$

(iv) The resulting decomposition  

$$V = V_1 \oplus V_2 \oplus V_3 \oplus V_4 \oplus V_5 \oplus V_6 \oplus V_7 \oplus V_8 \oplus V_9 \oplus V_{10}$$

is $\omega$-orthogonal

For (i), Proposition 3.2, applied to the symplectic space $V_s$ and the isotropic subspaces $A \cap V_s$ and $B \cap V_s$, gives the result directly.

For (ii), recall from (39) that  

$$U = Q_A \oplus P \oplus V_s$$

and that $U$ is symplectic. Furthermore, from (38) we have that $Q_A \oplus P$ is $\omega$-orthogonal to $V_s$, so $Q_A \oplus P$ is the $\omega$-orthogonal in $U$ of the symplectic subspace $V_s$ and is hence itself symplectic. Thus  

$$V_m = Q_R \oplus Q_A \oplus P$$
is a presymplectic space with radical $Q_R$. With \( \text{(31)} \) we saw already that

$$Q_A \oplus Q_R = Q_B \oplus Q_R = Q_A \oplus Q_B.$$  

So, the isotropic pair \( (Q_A, Q_B) \) in \( V_m \) is of the type \( \tau_6 = \mu_R \).

For (iii), it suffices to recall that, in the course of the above, we already kept track of the fact that the summands of the decomposition

$$V_z = T \oplus N_{AR} \oplus N_{BR} \oplus M_R$$

are presymplectic spaces of elementary type \( \tau_7 = \zeta, \tau_8 = \alpha, \tau_9 = \beta, \) and \( \tau_{10} = \rho, \) respectively. We set \( V_7 := T, V_8 := N_{AR}, V_9 := N_{BR} \) and \( V_{10} := M_R \) to obtain a decomposition of \( V_z \) of the desired form.

Finally, for (iv), we observe that it is enough to show that the decomposition

$$V = V_z \oplus V_m \oplus V_s$$

is \( \omega \)-orthogonal, because the decompositions of \( V_s \) and \( V_z \) are each \( \omega \)-orthogonal (for \( V_s \), this follows from Proposition 3.2 and for \( V_z \) it is because \( V_z \subseteq R \)). We have that \( V_z \) is \( \omega \)-orthogonal to both \( V_m \) and \( V_s \) because \( V_z \subseteq R \). That \( V_m \) is \( \omega \)-orthogonal to \( V_s \) follows from the decomposition \( V_m = Q_R \oplus Q_A \oplus P \), and the fact that \( Q_R \) and \( Q_A \oplus P \) are both \( \omega \)-orthogonal to \( V_s \): the former is the case because \( Q_R \subseteq R \), and the latter follows from \( \text{58} \).

\[ \square \]

4 Indecomposables

In this section we refine the decompositions of the previous section to ones for which every summand is indecomposable, and we relate these refinements to the decompositions already obtained.

**Definition 4.1** An isotropic pair \( (A, B) \) in a presymplectic space \( V \) is **indecomposable** if, for any direct sum decomposition

\[
(A, B) = (A_1, B_1) \oplus (A_2, B_2) \tag{41}
\]

subordinate to an \( \omega \)-orthogonal decomposition \( V = V_1 \oplus V_2 \), it follows that either \( V_1 = 0 \) or \( V_2 = 0 \).

**Lemma 4.2** The isotropic pairs which are of elementary type and have “minimal dimension” are indecomposable. For the symplectic types \( \lambda, \mu_A, \mu_B, \delta, \sigma \) this minimal dimension is 2, for the mixed type \( \mu_R \) it is 3, and for the zero types \( \zeta, \alpha, \beta, \rho \) it is 1.

**Proof.** Let \( (A, B) \) be an isotropic pair of elementary type \( \tau \) and of minimal dimension for its type. Suppose \( (A, B) \) has a decomposition \( \text{(41)} \) subordinate to an \( \omega \)-orthogonal decomposition \( V = V_1 \oplus V_2 \).
The cases of the zero types and symplectic types are simple and are left to the reader.

Suppose that \( \tau \) is the mixed type \( \mu_R \), so \( \dim V = 3 \). If both \( V_1 \) and \( V_2 \) were non-zero, then one of these spaces would be 1-dimensional and the other 2-dimensional. Without loss of generality assume \( \dim V_1 = 1 \). Then, \( V_1 \) is isotropic and the radical \( R_1 \) of \( V_1 \) is equal to \( V_1 \). By Corollary 2.5, \( R_1 = R \cap V_1 \); by definition of the type \( \mu_R \), \( \dim R = 1/3 \dim V = 1 \). Thus, \( R = V_1 \). The decomposition \( V = V_1 \oplus V_2 \) is, by definition, \( A, B \)-distributive, and by definition of the type \( \mu_R \), \( R, A, \) and \( B \) have pairwise zero intersection. Hence \( A \cap V_1 = A \cap R = 0 \), \( B \cap V_1 = B \cap R = 0 \) and \( A, B \subseteq V_2 \). Thus \( A \oplus B \subseteq V_2 \), and in particular it follows that \( R \cap (A \oplus B) = 0 \), which is in contradiction with the definition of the type \( \mu_R \), according to which \( R \subseteq A \oplus B \).

\[ \square \]

**Remark 4.3** We will see that isotropic pairs of elementary type and minimal dimension are the only indecomposable isotropic pairs. The ten types of indecomposables are illustrated below. According to the dimension of the indecomposable isotropic pair, the ambient presymplectic space is taken to be \( \mathbb{R} \), \( \mathbb{R}^2 \) or \( \mathbb{R}^3 \), equipped with the presymplectic structure defined – with respect to the standard basis vectors \( e_1, e_2 \) and \( e_3 \) – by the matrices

\[
\begin{pmatrix}
0 \\
0 \\
-1
\end{pmatrix}
\quad \text{or} \quad
\begin{pmatrix}
0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

respectively. The isotropics \( A \) and \( B \) are indicated by green and blue, respectively; the radical \( R \) is indicated by red.
Each entry in this listing defines a normal form in the sense that any indecomposable isotropic pair of type $\tau$ is equivalent, as an isotropic pair, to the corresponding pair given here. For the zero types, any isomorphism defines an equivalence of isotropic pairs. For the symplectic types, recall that there always exists a symplectomorphism which maps a given lagrangian pair in one symplectic space to a given lagrangian pair in another. Given an indecomposable isotropic pair of symplectic type, one can choose a decomposition of the ambient symplectic space into a transversal pair of lagrangians such that $A$ and $B$ are each contained in one of these lagrangians, and one can then define a symplectomorphism taking this lagrangian pair to the lagrangian pair $(\langle e_1 \rangle, \langle e_2 \rangle)$ and which is an equivalence to the associated normal form above. Finally, for an isotropic pair $(A, B)$ of the mixed type $\mu_R$, we can construct an equivalence $\phi : \mathbb{R}^3 \to V$ with the normal form above as follows. Choose a complement $U$ of $R$ in $V$ such that $A \subseteq U$, and let $b \in B$ be a vector such that $\langle b \rangle = B$. Because $B \subseteq R \oplus A$, there is a unique decomposition $b = a + r$, with $a \in A \setminus \{0\}$ and $r \in R \setminus \{0\}$. Define $\phi$ on $e_1$ and $e_3$ such that $\phi(e_1) = a$ and $\phi(e_3) = r$; by linearity $\phi(e_1 + e_3) = b$. Because $U$ is symplectic, there exists $a' \in U$ such that $\omega(a, a') = 1$. Set $\phi(e_2) = a'$. Thus defined, $\phi$ maps $\langle e_1 \rangle$ to $A$, $\langle e_3 \rangle$ to $R$, and $\langle e_1 + e_3 \rangle$ to $B$, and it preserves the presymplectic structures.

**Proposition 4.4** Let $(A, B)$ be an isotropic pair of elementary type $\tau$. Then $(A, B)$ decomposes as the direct sum of isotropic pairs which are indecomposable and of type $\tau$.

**Proof.** The cases of the zero types and symplectic types are straightforward. For the zero types, any choice of basis of $V$ induces a decomposition as desired into 1-dimensional indecomposable summands; for the symplectic types, a choice of symplectic basis, adapted to the subspaces $A$ and $B$, similarly induces a decomposition into 2-dimensional symplectic summands. We leave the details to the reader and treat the mixed case in detail.

So let $\tau = \mu_R$; one has $\dim V = 3n$ for some $n \in \mathbb{N}$. Let $Q$ denote the space $R \oplus A = A \oplus B = B \oplus R$ and let $U$ be a complement in $V$ of $R$ such that $A \subseteq U$. For dimension reasons, $A$ is lagrangian in the symplectic subspace $U$; let $P$ be a lagrangian complement of $A$ in $U$ and let $\{a_1, ..., a_n\}$ and $\{p_1, ..., p_n\}$ be bases of $A$ and $P$ respectively which together form a symplectic basis of $U$. Because $A \subseteq R \oplus B$, each $a_i$ has a unique decomposition

$$a_i = r_i + b_i$$

with $r_i \in R \setminus \{0\}$ and $b_i \in B \setminus \{0\}$. Claim: $\{r_1, ..., r_n\}$ and $\{b_1, ..., b_n\}$ are a basis of $R$ and $B$ respectively. Considering dimensions, it is enough to show linear independence. We do this for the $r_i$; the same argument applies to the $b_i$. Assume that

$$\sum_{i=1}^{n} \lambda_i r_i = 0$$

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for scalars $\lambda_i$. Because $r_i = a_i - b_i$ for each $i$,

$$\sum_{i=1}^{n} \lambda_i(a_i - b_i) = \sum_{i=1}^{n} \lambda_i a_i - \left(\sum_{i=1}^{n} \lambda_i b_i\right) = 0.$$  

The sum in parentheses is an element of $B$; it is linearly independent from the $a_i$ because $B \cap A = 0$. The $a_i$ are themselves linearly independent, so it follows that all $\lambda_i$ are zero, which proves the claim.

For each $i$, set $R_i := \langle r_i \rangle$, $A_i := \langle a_i \rangle$, $B_i := \langle b_i \rangle$, $P_i := \langle p_i \rangle$, $Q_i := \langle r_i, a_i \rangle$, and $U_i := \langle a_i, p_i \rangle$. Because $R$, $A$, $B$ and $P$ have pairwise zero intersection, the $R_i$, $A_i$, $B_i$ and $P_i$ do too, and from (42) it follows that $Q_i = R_i \oplus A_i = A_i \oplus B_i = B_i \oplus R_i$. So, for each $i$, $(A_i, B_i)$ is an indecomposable isotropic pair of type $\mu_R$ in the presymplectic space $V_i := R_i \oplus U_i$. The resulting decomposition $V = V_1 \oplus \ldots \oplus V_n$ is $\omega$-orthogonal: the subordinate decomposition $U = U_1 \oplus \ldots \oplus U_n$ is, and each $R_i$, as a subspace of the radical $R$, is $\omega$-orthogonal to every subspace.

□

**Corollary 4.5** Every isotropic pair has a direct sum decomposition into indecomposable isotropic pairs. Any indecomposable isotropic pair is of one of the elementary types and has minimal dimension as given in Lemma 4.2.

**Proof.** By Theorem 3.4 any isotropic pair $(A, B)$ has a decomposition into isotropic pairs of elementary type, and by Proposition 4.4 these in turn decompose as the direct sum of indecomposable pairs of elementary type. If $(A, B)$ is indecomposable, this decomposition into indecomposables of elementary type can have only one non-zero summand, which is equal to $(A, B)$. Thus $(A, B)$ is then of one of the elementary types.

□

The following shows that any direct sum decomposition of an isotropic pair into indecomposables can be simplified to a ten-part sum of the form (28) by summing together summands which are of the same elementary type. The proof is straightforward and is left to the reader.

**Lemma 4.6** If an isotropic pair $(A, B)$ has an $\omega$-orthogonal direct sum decomposition

$$(A, B) = \bigoplus_{i=1}^{m} (A_i, B_i)$$

where every isotropic pair $(A_i, B_i)$ is of the same elementary type $\tau$, then $(A, B)$ is also of the elementary type $\tau$. 

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5 Invariants

In this section we define the elementary invariants and prove the main classification result. The next theorem states that the direct sum decomposition of an isotropic pair into indecomposables is essentially unique.\footnote{This is a Krull-Schmidt type statement, in spirit similar to Theorem 2.19 in [7], for example. It could probably be framed in more abstract terms, such as in [2].}

**Theorem 5.1** Every direct sum decomposition of an isotropic pair \((A, B)\) into indecomposables has, for each elementary type \(\tau\), the same number of summands of type \(\tau\). We call this number the **multiplicity** of \(\tau\) in \((A, B)\).

**Proof.** By Corollary 4.5, every indecomposable isotropic pair is of one of the ten elementary types \(\{\tau_1, \ldots, \tau_{10}\} = \{\lambda, \mu_A, \mu_B, \delta, \sigma, \mu_R, \zeta, \alpha, \beta, \rho\}\), and by Lemma 4.6, any decomposition of an isotropic pair into indecomposables may be consolidated to a ten-part decomposition into isotypic components

\[
(A, B) = \bigoplus_{i=1}^{10} (A_i, B_i),
\]

where we assume the sum to be ordered such that, for each \(i\), the pair isotropic pair \((A_i, B_i)\) is of type \(\tau_i\). Of course, we also obtain associated \(\omega\)-orthogonal decompositions

\[
V = \bigoplus_{i=1}^{10} V_i, \quad R = \bigoplus_{i=1}^{10} R_i, \quad A = \bigoplus_{i=1}^{10} A_i, \quad B = \bigoplus_{i=1}^{10} B_i.
\]

Given an isotropic pair \((A, B)\) and a decomposition into indecomposable isotropic pairs, for each \(i\) let let \(n_i\) denote the multiplicity of \(\tau_i\) in this decomposition, i.e. the number of summands of type \(\tau_i\) in this decomposition. After consolidating the decomposition into the form (43), the dimensions of the isotypic components \(V_i\) are related to the multiplicities \(n_i\) via

\[
n_i = \begin{cases} 
\frac{1}{2} \dim V_i & \text{if } 1 \leq i \leq 5, \quad \text{i.e. } \tau_i \in \{\lambda, \mu_A, \mu_B, \delta, \sigma\} \\
\frac{1}{3} \dim V_i & \text{if } i = 6, \quad \text{i.e. } \tau_i = \mu_R \\
\dim V_i & \text{if } 7 \leq i \leq 10, \quad \text{i.e. } \tau_i \in \{\zeta, \alpha, \beta, \rho\}.
\end{cases}
\]

Consider \(n = (n_1, \ldots, n_{10})\) as a coordinate in the space \(N := \mathbb{Z}_{\geq 0}^{10}\) of all possible ordered 10-tuples arising in this way from decompositions of isotropic pairs into indecomposables. Let \(K\) denote the space of all possible 10-tuples of canonical invariants \(k = (k_1, \ldots, k_{10})\) associated to isotropic pairs. Recall that
we defined the canonical invariants of an isotropic pair \((A, B)\) as

\[ k_1 = \frac{1}{2}(\dim V - \dim R) \quad k_6 = \dim R \cap A \\
 k_2 = \dim R \quad k_7 = \dim R \cap B \\
 k_3 = \dim A \quad k_8 = \dim R \cap A \cap B \\
 k_4 = \dim B \quad k_9 = \dim R \cap (A + B) \\
 k_5 = \dim A \cap B \quad k_{10} = \dim A^\omega \cap B. \]

Clearly these numbers are uniquely determined by \((A, B)\). We will now construct an injective linear map \(M : N \to K\) which maps the numbers \(n = (n_1, \ldots, n_{10})\) associated to a decomposition of a given pair \((A, B)\) to the canonical invariants \(k\) of that pair. In particular, for any two decompositions of \((A, B)\) into indecomposables with associated multiplicities \(n = (n_1, \ldots, n_{10})\) and \(n' = (n'_1, \ldots, n'_{10})\), \(M\) will map both \(n\) and \(n'\) to the canonical invariants \(k\) of \((A, B)\). The injectivity of \(M\) then implies \(n = n'\), which is what is to be shown.

Using the decompositions \((\ref{decomp})\), the definition of the elementary types of isotropic pairs, and the fact that, by Lemmata \(2.3\) and \(2.4\) we can take sums, intersections and \(\omega\)-orthogonals of \(R, A\) and \(B\) term-wise in the decompositions \((\ref{decomp})\), we obtain the following decompositions

\[
\begin{align*}
V &= V_1 \oplus V_2 \oplus \ldots \oplus V_{10} \\
R &= R_0 \oplus V_7 \oplus \ldots \oplus V_{10} \\
A &= A_1 \oplus A_2 \oplus A_4 \oplus A_6 \oplus V_7 \oplus V_8 \\
A^\omega &= A_1 \oplus A_2 \oplus A_3 \oplus A_4 \oplus A_5 \oplus (R_0 \oplus A_6) \oplus V_7 \oplus \ldots \oplus V_{10} \\
B &= B_1 \oplus B_3 \oplus B_4 \oplus B_6 \oplus V_7 \oplus V_9 \\
A \cap B &= A_1 \oplus V_7 \\
R \cap A &= V_7 \oplus V_9 \\
R \cap B &= V_7 \oplus V_9 \\
R \cap A \cap B &= V_7 \\
R \cap (A + B) &= R_0 \oplus V_7 \oplus V_8 \oplus V_9 \\
A^\omega \cap B &= A_1 \oplus B_3 \oplus B_6 \oplus V_7 \oplus V_9.
\end{align*}
\]

Taking dimensions gives the following linear equations for the canonical invariants \(k_i\) in terms of the multiplicities \(n_i\)

\[
\begin{align*}
k_1 &= \frac{1}{2}(\dim V - \dim R) = n_1 + n_2 + n_3 + n_4 + n_5 + n_6 \\
k_2 &= \dim R = n_6 + n_7 + n_8 + n_9 + n_{10} \\
k_3 &= \dim A = n_1 + n_2 + n_4 + n_6 + n_7 + n_8 \\
k_4 &= \dim B = n_1 + n_3 + n_4 + n_6 + n_7 + n_9 \\
k_5 &= \dim A \cap B = n_1 + n_7 \\
k_6 &= \dim R \cap A = n_7 + n_8 \\
k_7 &= \dim R \cap B = n_7 + n_9 \\
k_8 &= \dim R \cap A \cap B = n_7 \\
k_9 &= \dim R \cap (A + B) = n_6 + n_7 + n_8 + n_9 \\
k_{10} &= \dim A^\omega \cap B = n_1 + n_3 + n_6 + n_7 + n_9.
\end{align*}
\]
These equations define a map $M : \mathcal{N} \to \mathcal{K}$, representable with the matrix

$$M = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0
\end{pmatrix}$$

This matrix is non-singular ($\det M = 1$), so $M$ is injective.

\[\square\]

**Definition 5.2** Let $(A, B)$ be an isotropic pair and, for each $i \in \{1, \ldots, 10\}$, let $n_i$ be the multiplicity of the elementary type $\tau_i$ in $(A, B)$. We call these numbers the elementary invariants of $(A, B)$.

**Corollary 5.3** There exists a bijection $M : \mathcal{N} \to \mathcal{K}$ between the space $\mathcal{N}$ of elementary invariants and the space $\mathcal{K}$ of canonical invariants which maps the elementary invariants of an isotropic pair to the canonical invariants of that pair.

**Proof.** We constructed the linear map $M$ in the proof of Theorem 5.1 and showed that $M$ is injective. Any given $k \in \mathcal{K}$ is, by definition of $\mathcal{K}$, realized as the canonical invariants of some isotropic pair. By Corollary 4.5 this pair has a decomposition into indecomposables, and the construction of $M$ showed that the multiplicities of that decomposition are mapped by $M$ to $k$. Hence $M$ is surjective.

\[\square\]

**Remark 5.4** We noted in the introduction that choosing $\dim B^\omega \cap A$ instead of $\dim A^\omega \cap B$ as the tenth canonical invariant would give a new set of invariants $k'$ which is equivalent to the original canonical invariants $k$ associated to an isotropic pair $(A, B)$. To see this, use $B^\omega \cap A$ in place of $A^\omega \cap B$ in the proof of Theorem 5.1 which results in a matrix $M' : \mathcal{n} \mapsto k'$ which is also invertible. Then, the composition $M' \circ M^{-1} : k \mapsto k'$ is a bijection which takes the original canonical invariants $k$ of an isotropic pair $(A, B)$ to the new invariants $k'$ of that pair, which shows that the two choices of invariants are equivalent.
The inverse of the mapping \( M: n \mapsto k \) is

\[
M^{-1} = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 1 & -1 & -1 & 0 & 1 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\
1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
\end{pmatrix}
\]

with which one obtains the linear equations for the \( n_i \) in terms of the \( k_i \):

\[
\begin{align*}
n_1 &= k_5 - k_8 \\
n_2 &= k_3 - k_4 - k_5 + k_7 - k_9 + k_{10} \\
n_3 &= -k_5 + k_6 - k_9 + k_{10} \\
n_4 &= k_4 - k_{10} \\
n_5 &= k_1 - k_3 + k_5 + k_9 - k_{10} \\
n_6 &= -k_6 - k_7 + k_8 + k_9 \\
n_7 &= k_8 \\
n_8 &= k_6 - k_8 \\
n_9 &= k_7 - k_8 \\
n_{10} &= k_2 - k_9.
\end{align*}
\]

In particular, because the elementary invariants take integer values greater or equal to zero, the canonical invariants are subject to the ten inequalities given by the non-negativity of the right-hand sides of the equations above.

We now show that the canonical invariants and the elementary invariants each fully characterize an isotropic pair, up to equivalence.

**Theorem 5.5** Let \((A, B)\) and \((\hat{A}, \hat{B})\) be isotropic pairs in \(V\) and \(\hat{V}\) respectively. The following are equivalent:

(i) \((A, B)\) and \((\hat{A}, \hat{B})\) are equivalent isotropic pairs.

(ii) The canonical invariants of \((A, B)\) and \((\hat{A}, \hat{B})\) are the same.

(iii) The elementary invariants of \((A, B)\) and \((\hat{A}, \hat{B})\) are the same.

**Proof.** We show (i) \(\Rightarrow\) (ii) \(\Rightarrow\) (iii) \(\Rightarrow\) (i).

(i) \(\Rightarrow\) (ii): If \((A, B)\) and \((\hat{A}, \hat{B})\) are equivalent then, by definition, there exists a presymplectomorphism \(\varphi: V \to \hat{V}\) such that \(\varphi(A) = \hat{A}\) and \(\varphi(B) = \hat{B}\). Also, \(\varphi(R) = \hat{R}\) because any presymplectomorphism maps the radical of the source space to the radical of the target. The canonical invariants are built from \(R\),
A and B using operations which are preserved under presymplectomorphisms, hence the canonical invariants of (A, B) must be the same as those of (Å, ˆB).

(ii) ⇒ (iii): This follows directly from the fact that the bijection $M : \mathcal{N} \to \mathcal{K}$ maps the elementary invariants of a given isotropic pair to the canonical invariants of that pair, and vice versa.

(iii) ⇒ (i): The isotropic pairs (A, B) and (Å, ˆB) have, by hypothesis, decompositions into indecomposables which have the same number $n_i$ of summands of each elementary type $\tau_i$. Let $m$ be the total number of summands. Indecomposables of the same type are equivalent, so we can match each indecomposable summand of (A, B) with an indecomposable summand of (Å, ˆB) via an equivalence, choosing a total of $m$ such maps in such a way that no summand is left unmatched. The direct sum $\phi : V \to \hat{V}$ of these maps respects the presymplectic structures because its summands do and because the decompositions of (A, B) and (Å, ˆB) are $\omega$-orthogonal and $\hat{\omega}$-orthogonal, respectively. The summands of $\phi$ map summands of A to summands of Å, so $\phi$ maps A to Å, and similarly so for B. Hence, $\phi$ is an equivalence between (A, B) and (Å, ˆB).

□

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