A relationship between the ideals of \( \mathbb{F}_q[x, y, x^{-1}, y^{-1}] \) and the Fibonacci numbers

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Abstract. Let \( C_n(q) \) be the number of ideals of codimension \( n \) of \( \mathbb{F}_q[x, y, x^{-1}, y^{-1}] \), where \( \mathbb{F}_q \) is the finite field with \( q \) elements. Kassel and Reutenauer [1] proved that \( C_n(q) \) is a polynomial in \( q \) for any fixed value of \( n \geq 1 \). For \( q = \frac{3 + \sqrt{5}}{2} \), this combinatorial interpretation of \( C_n(q) \) is lost. Nevertheless, an unexpected connexion with Fibonacci numbers appears.

Let \( f_n \) be the \( n \)-th Fibonacci number (following the convention \( f_0 = 0 \), \( f_1 = 1 \)). Define the series\n\[
F(t) = \sum_{n \geq 1} f_{2n} t^n.
\]

We will prove that for each \( n \geq 1 \),\n\[
C_n \left( \frac{3 + \sqrt{5}}{2} \right) = \lambda_n \left( f_{2n} \frac{3 + \sqrt{5}}{2} - f_{2n-2} \right),
\]
where the integers \( \lambda_n \geq 0 \) are given by the following generating function\n\[
\prod_{m \geq 1} (1 + F(t^m)) = 1 + \sum_{n \geq 1} \lambda_n t^n.
\]

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1 Introduction

Let \( \mathbb{F}_q \) be the finite field with \( q \) elements. We recall that the codimension of the ideal \( I \) of the algebra \( \mathbb{F}_q[x, y, x^{-1}, y^{-1}] \) is the dimension of the quotient \( \mathbb{F}_q[x, y, x^{-1}, y^{-1}]/I \) viewed as a vector space over \( \mathbb{F}_q \). Kassel and Reutenauer [1] computed the number of ideals of codimension \( n \) of \( \mathbb{F}_q[x, y, x^{-1}, y^{-1}] \), denoted \( C_n(q) \). The ideals of codimension \( n \) of \( \mathbb{F}_q[x, y, x^{-1}, y^{-1}] \) are the \( \mathbb{F}_q \)-points of the Hilbert scheme\n\[
H^n = \text{Hilb}^n \left( \mathbb{A}_F^1 \times \mathbb{A}_F^1 \setminus \{0\} \right)
\]
of \( n \) points of the two-dimensional torus (i.e., of the affine plane minus two distinct straight lines). This scheme is smooth and quasi-projective (see [2]).
The local zeta function of $H^n$, denoted $Z_{H^n/F_q}(t)$, is related to $C_n(q)$ by means of the formula
\[
C_n(q) = \frac{d}{dt} \left. \frac{Z_{H^n/F_q}(t)}{Z_{H^n/F_q}(q)} \right|_{t=0}.
\]

Kassel and Reutenauer [2] computed $Z_{H^n/F_q}(t)$, obtaining that $C_n(q)$ is a self-reciprocal polynomial of degree $2n$, satisfying several number-theoretical properties, e.g.,

\[
\lim_{q \to 1} \frac{C_n(q)}{(q - 1)^2} = \sum_{d|n} d,
\]
\[
C_n(-1) = \# \{(x, y) \in \mathbb{Z}^2 : x^2 + y^2 = n\},
\]
\[
|C_n(\sqrt{-1})| = \# \{(x, y) \in \mathbb{Z}^2 : x^2 + 2y^2 = n\}.
\]

The aim of this paper is to show a rather unexpected relationship between $C_n(q)$ and the Fibonacci numbers. Let $f_n$ be the $n$th Fibonacci number (following the convention $f_0 = 0$, $f_1 = 1$). Consider the series
\[
F(t) = \sum_{n \geq 1} f_{2n} t^n
\]
and the sequence of integers $\lambda_n \geq 0$ given by
\[
\prod_{m \geq 1} (1 + F(t^m)) = 1 + \sum_{n \geq 1} \lambda_n t^n.
\]

We will prove the following result.

**Theorem 1.** For each integer $n \geq 1$,
\[
C_n \left( \frac{3 + \sqrt{5}}{2} \right) = \lambda_n \left( f_{2n} \frac{3 + \sqrt{5}}{2} - f_{2n-2} \right).
\]

Furthermore, we will derive a formula for $\lambda_n$ in terms of the distribution of the divisors of $n$ on short intervals.

## 2 Proof of the main result

The main ingredient in the proof of Theorem 1 will be the identity
\[
\prod_{m \geq 1} \frac{(1 - t^m)^2}{1 - (q + q^{-1}) t^m + t^{2m}} = 1 + \sum_{n \geq 1} \frac{C_n(q)}{q^n},
\]
due to Kassel and Reutenauer (see Corollary 1.4 in [1]).
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Proof (Proof of Theorem 1). It is well-known that the generating function of \( f_{2n} \) is

\[
F(t) = \frac{t}{1 - 3t + t^2}.
\]

(5)

Using the identity

\[
\frac{(1 - t^m)^2}{1 - (q + q^{-1}) t^m + t^{2m}} = 1 + \frac{(q + q^{-1} - 2) t^m}{1 - (q + q^{-1}) t^m + t^{2m}},
\]

(6)

computing

\[
\left( \frac{3 + \sqrt{5}}{2} \right) + \left( \frac{3 + \sqrt{5}}{2} \right)^{-1} = 3
\]

(7)

and applying (5), we obtain that

\[
\prod_{m \geq 1} \frac{(1 - t^m)^2}{1 - (q + q^{-1}) t^m + t^{2m}} \Bigg|_{t = \frac{3 + \sqrt{5}}{2}} = \prod_{m \geq 1} (1 + F(t^m)).
\]

(8)

Combining (8) and (1), we conclude that

\[
C_{\ell_n}\left( \frac{3 + \sqrt{5}}{2} \right)^n = \lambda_n.
\]

(9)

Applying the identity \( \left( \frac{3 + \sqrt{5}}{2} \right)^n = f_{2n}\frac{3 + \sqrt{5}}{2} - f_{2n-2} \) to (9) we obtain (3).

3 Applications

Let \( \ell_n \) be the \( n \)-th Lucas number, i.e. \( \ell_0 = 2, \ell_1 = 1 \) and \( \ell_n = \ell_{n-1} + \ell_{n-2} \). The following result can be used to compute \( \lambda_n \) in terms of the divisors of \( n \).

Corollary 2. For each \( n \geq 1 \),

\[
\lambda_n = a_{n,0} + \sum_{k=1}^{n-1} a_{n,k} \ell_{2k},
\]

(10)

where \( a_{n,k} \) the number of \( d | n \) satisfying

\[
k + \sqrt{k^2 + 2n} < d \leq k + \sqrt{k^2 + 2n}.
\]

Proof. It follows by Theorem 1.2. in [2] that

\[
\frac{C_n(q)}{q^n} = (q + q^{-1} - 2) \left( a_{n,0} + \sum_{k=1}^{n-1} a_{n,k} (q^k + q^{-k}) \right).
\]

(11)
The identity
\[
\left(\frac{3 + \sqrt{5}}{2}\right)^k + \left(\frac{3 + \sqrt{5}}{2}\right)^{-k} = \ell_{2k}.
\] (12)
is well-known. Combining (11), (12), (7) and (9) we obtain (10).

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**References**

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