Magnetic Fourier integral operators

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Abstract In some previous papers we have defined and studied a ‘magnetic’ pseudo-differential calculus as a gauge covariant generalization of the Weyl calculus when a magnetic field is present. In this paper we extend the standard Fourier integral operators theory to the case with a magnetic field, proving composition theorems, continuity theorems in ‘magnetic’ Sobolev spaces and Egorov type theorems. The main application is the representation of the evolution group generated by a 1-st order ‘magnetic’ pseudodifferential operator (in particular the relativistic Schrödinger operator with magnetic field) as such a ‘magnetic’ Fourier integral operator. As a consequence of this representation we obtain some estimations for the distribution kernel of this evolution group and a result on the propagation of singularities.

1 Introduction

In a series of papers [13, 14, 16, 17] a gauge covariant formalism has been proposed for associating to any classical observable a (a certain function on the phase space \( \mathbb{R}^{2d} \), usually a standard symbol from \( S^m(\mathbb{R}^d) \)) and any magnetic field \( B = dA \) (with \( A \) a 1-form on \( \mathbb{R}^d \) with \( C^\infty(\mathbb{R}^d) \) coefficients) a quantum observable \( \mathcal{O} \), defined

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as a ‘magnetic’ pseudodifferential operator acting on $S(\mathbb{R}^d)$ in the following way

$$
[Op^A(a)u](x) := \int_{\mathbb{R}^{2d}} e^{i(x-y,\eta)} e^{-i \int_{[x,y]} A} u(y) \, dy \, d\eta,
$$

$$
\forall u \in S(\mathbb{R}^d), \quad \forall x \in \mathbb{R}^d,
$$

(1.1)

with $d\eta := (2\pi)^{-d} d\eta$. The properties of operators of the form (1.1) have been studied in [10], where we also gave some applications. In particular we have proved that for any real elliptic symbol $a \in S^1(\mathbb{R}^d)$, the operator $Op^A(a)$ has a self-adjoint extension $P$ on $L^2(\mathbb{R}^d)$. In particular, the operator $Op^A(\langle \xi \rangle)$ with $\langle \xi \rangle := \sqrt{1 + |\xi|^2}$ for any $\xi \in \mathbb{R}^d$ can be considered as the relativistic Schrödinger operator with magnetic field. Let us remark that $Op^A(\langle \xi \rangle^2) - 1$ is the usual non-relativistic Schrödinger operator with magnetic field. Let us also remark that the usual choice for the relativistic Schrödinger operator with magnetic field $\sqrt{Op^A(\langle \xi \rangle^2)}$, although gauge covariant, has not a reasonable classical counterpart, its symbol being rather complicated.

An important problem is the study of the unitary group $\{e^{-itP}\}_{t \in \mathbb{R}}$ generated by the self-adjoint operator $P$, and for such a study an integral representation may be very useful. Having in mind other versions of pseudodifferential operators we expect that at least for $|t|$ small the operator $e^{-itP}$ should be a ‘Fourier Integral Operator’, where the definition of such an object should depend on the magnetic field $B$ and should be gauge covariant. In fact such a representation has been obtained in a former paper [11], where we have proved that $e^{-itP}$ verifies the hypothesis of a very implicit definition of Fourier integral operators based on commutation properties (in the spirit of Bony [1]). For applications, an explicit integral representation is needed and this is the object of the present paper.

Let us notice first that Proposition 2.13 allows us to replace $a \left( \frac{x+y}{2}, \eta \right)$ in (1.1) with $b(x, \eta)$ with $b$ another symbol of the same order as $a$. Taking this fact into account and the standard definition of Fourier integral operators, it is natural that to a triple $(a, B, \Phi)$ with $a \in S^m(\mathbb{R}^d)$, $B = dA$ a magnetic field and $\Phi : T^*\mathbb{R}^d \to T^*\mathbb{R}^d$ a symplectomorphism with generating function $U$, we shall associate a linear operator

$$
[Op^A_\Phi(a)u](x) := \int_{\mathbb{R}^d} e^{i(U(x,\eta) - \langle y, \eta \rangle) - \int_{[x,y]} A} a(x, \eta) u(y) \, dy \, d\eta,
$$

$$
\forall u \in S(\mathbb{R}^d), \quad \forall x \in \mathbb{R}^d,
$$

(1.2)

that we shall call a magnetic Fourier integral operator.

In the literature there are several procedures that have been elaborated in order to treat oscillatory integrals similar to (1.2) [1–3, 5–7, 15] but without the magnetic factor $e^{-i \int_{[x,y]} A}$. Let us notice that this factor, although being smooth and of modulus 1, its derivatives are unbounded and their growth at infinity is getting worse for higher and higher derivatives; in fact, for a smooth bounded magnetic field with bounded derivatives of all orders, the line integral of its vector potential along the segment $[x, y]$ is proportional with the area of the triangle of vertices 0, $x$ and $y$. In our situation we