Low-Input Accurate Periodic Motion of an Underactuated Mechanism: Mass Distribution and Nonlinear Spring Shaping

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Abstract—This work presents a control-oriented structural design approach for a 2-DOF underactuated mechanical system, with the purpose of generating an optimal oscillatory behavior of the end-effector. To achieve the desired periodic motion, we propose to adjust the dynamic response of the mechanism by selecting its mass distribution and the characteristic of a nonlinear spring. In particular, we introduce a two-step optimization strategy to shape the system’s zero dynamics, obtained via input-output linearization. The first part of the procedure aims to minimize the root-mean-square value of the input torque by optimizing the mechanism’s mass distribution. In this context, we show that a perfect matching with the desired trajectory can be reached by assuming the ability to design an arbitrary shape of the system’s elastic properties. Then, in order to favor a simpler physical implementation of the structure, we dedicate the second optimization step to the piecewise linear approximation of the previously defined stiffness characteristic. The proposed procedure is finally tested in detailed numerical simulations, confirming its effectiveness in generating a complex and efficient periodic motion.

Index Terms—Underactuated Mechanical Systems, Periodic Motion Planning, Optimal Design.

I. INTRODUCTION

In recent years, the design of high-performance and sustainable mechanical systems has become an ever-growing challenge. This topic is especially relevant in industrial applications, where it is crucial to obtain an accurate motion of the end-effector while minimizing as much as possible the realization costs and the energy consumption. However, in such a context, a prevalent practice consists of making the engineering design a sequential procedure, so that the control problem is approached only when the structure is fully established. Inevitably, this approach leads to oversized components and a high impact on costs and efficiency.

Due to the severe disadvantages of sequential strategies, this work focuses on a more integrated approach, where the control requirements are evaluated at the early design stage. In particular, our objective is to achieve high-performance motion planning for multibody systems through structural optimization. This challenging topic has received, in recent years, widespread interest by the scientific community. We refer to [1], [2] for detailed overviews regarding structural and topological optimization of multibody systems.

Within the domain of multibody systems, we specialize the analysis of this work to underactuated Euler-Lagrange systems, i.e., mechanisms having fewer control inputs than the Degrees of Freedom (DOF). The specific properties of these systems and the related control techniques are examined, e.g., in [3] and [4]. These mechanisms are often desirable because they involve a reduced overall actuation. However, not all DOF can be controlled simultaneously; as a consequence, such systems present internal dynamics whose properties cannot be entirely influenced by the control inputs. Therefore, structural design becomes an essential tool to impose a desirable behavior of the mechanism. In this respect, [5] poses the attention to shaping the system’s zero dynamics, i.e., the internal dynamics in conditions of perfect reference tracking. There, the author defines an integrated structure-control optimization procedure aimed at obtaining a minimum phase system, i.e., stable zero dynamics. In [6], an integrated method is proposed to minimize both the feedforward control input and the internal dynamics motion of underactuated manipulators.

In this paper, we present an optimal design strategy for a 2-DOF underactuated mechanism, with the purpose of following a periodic reference trajectory. The proposed approach consists of finding a minimum control effort by optimizing some relevant structural parameters so that the resulting system oscillates according to a desired periodic reference trajectory. A closely related problem has been considered in [7]. However, here we do not focus on constructing a comprehensive control law stabilizing the system on the desired orbit. Instead, we perform input-output linearization and we approach the structural design in conditions of perfect output tracking. One of the main contributions of this paper, derived from the formalism in [8], is showing the connection between the system’s stiffness and its resulting zero dynamics. In particular, we point out that the presence of at least one customizable compliant joint is sufficient to shape the zero dynamics arbitrarily. Then, inspired by the possibility of realizing nonlinear springs [9] and by the solutions found in [10], [11], we employ a piecewise linear characteristic as a means to approximate arbitrary stiffness models.

The proposed optimization approach is based on the following two-step solution. Firstly, we optimize the system’s mass distribution to minimize the root-mean-square (RMS)
value of the input torque required to maintain the system on the desired trajectory. Contextually, we also derive an ideal nonlinear stiffness characteristic associated with a compliant joint. Then, the second step consists of designing a piecewise linear spring that closely matches the ideal one.

The paper is organized as follows. In Section II, we formalize the problem statement and derive a minimal form model useful for the proposed solution. In Section III we also derive the explicit form of the zero dynamics and highlight how the system’s elasticity properties influence it. In Section IV the two-step optimization strategy is presented. Then, in Section V numerical simulation results are shown for validation, while some final considerations are reported in Section VI.

II. CONTROL-ORIENTED STRUCTURAL DESIGN FOR PERIODIC MOTION OPTIMIZATION

A. Problem Statement

We begin by introducing a precise mathematical framework for the proposed optimization strategy. Although our approach is compatible, in principle, with any underactuated 2-DOF Euler-Lagrange system, we specialize the discussion to the mechanism in Fig. 1. This structure, employed in [7] as a case of study for a closely related structure-control design, is a modified 4-bar-linkage where the rocker is replaced with two links. In particular, the purpose of the mechanism is to achieve a periodic horizontal motion of the end-effector, given by the position of joint \( J_3 \) in the \((e_x, e_y)\) frame, applying a torque \( u \) to joint \( J_1 \) (while all other joints are passive). Let \( \theta \in \{q_2, q_3, q_4\}^\top \in \mathbb{R}^4 \) denote the angles associated with joints \( J_i, i \in \{1, \ldots, 4\} \). In order to appropriately optimize the behavior of the mechanism, not only the input \( u \) but also some structural elements are available as degrees of freedom. In particular, we suppose that the system can be shaped through the following features:

- two masses attached to links \( L_3 \) and \( L_4 \) (in red in Fig. 1), with tunable values \( m_{Ai} \) and offsets \( \delta_i, i \in \{3, 4\} \). The masses can be placed on the longitudinal axis of the link and \( \delta_i \) denotes the offset between the mass position and the geometric center of the link. Such offset is positive when the mass is closer to joint \( J_i \). The design parameters are thus collected as:
  \[
p_m = (m_{A3} \ m_{A4} \ \delta_3 \ \delta_4)^\top; \tag{1}\n\]

- a torsional spring placed in joint \( J_5 \) (in green in Fig. 1), with a characteristic that can be customized according to a desired (finite-dimensional) class of functions \( \mathcal{S} \). In particular, we employ a model of the form \( \tau_s = S(p_s, \theta(q)) \), where \( \tau_s \) indicates the spring output torque and \( \theta(q) = q_4 + \pi \) is the angular displacement. Finally, it holds \( S(p_s, \cdot) \in \mathcal{S} \), where \( p_s \in \mathbb{R}^d \) is used to parameterize the set \( \mathcal{S} \). The choice of \( \mathcal{S} \) will be specified in Section III-B.

In view of the above definitions, the model of the mechanism is given by the following differential algebraic equation (DAE):

\[
M(q)p\ddot{q} + C(q)p\dot{q}\dot{q} + G(q)p, q, q) = Bq\dot{u} + u_{\phi}(q, \lambda) \phi(q) = 0, \quad u_{\phi}(q, \lambda) = \left[ \frac{\partial \phi}{\partial q}(q) \right]^\top \lambda, \tag{2}\n\]

where \( M(p_m, q) \in \mathbb{R}^{4\times4} \) is the inertia matrix, \( C(p_m, q, \dot{q}) \in \mathbb{R}^{4\times4} \) denotes the centrifugal and Coriolis effects, \( G(p_m, p_s, q) \in \mathbb{R}^4 \) is the term accounting for the potential energy contributions (gravity and elasticity), while \( B_q = (1 \ 0 \ 0 \ 0)^\top \) is the input matrix. In addition, \( \lambda \in \mathbb{R}^2 \) are the Lagrange multipliers associated with the kinematic loop constraints \( \phi(q) = 0 \), where \( \phi : \mathbb{R}^4 \to \mathbb{R}^2 \) is a smooth map. For convenience in the design, we let \( p_m \in \mathcal{P}_m \) and \( p_s \in \mathcal{P}_s \), where \( \mathcal{P}_m \subset \mathbb{R}^4 \) and \( \mathcal{P}_s \subset \mathbb{R}^d \) are compact and convex sets. Notice that no friction nor damping effects are considered.

In the following, we consider an output for control \( \chi \) given by the workspace coordinates of the end-effector (see Fig. 1):

\[
\chi = h(q) = \begin{pmatrix} h_1(q) \\ h_2(q) \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}, \tag{3}\n\]

where \( h : \mathbb{R}^4 \to \mathbb{R}^2 \) is a smooth map representing the direct kinematics of the system. Notice that the kinematic properties of the mechanism are not influenced by \( p_m \) and \( p_s \). The design objectives are formally collected in the following statement.

**Problem 1:** Consider a periodic reference in the workspace, with period \( T_r > 0 \):

\[
r(t) = \begin{pmatrix} r_x(t) \\ \bar{r}_y \end{pmatrix} \tag{4}\n\]

with \( \bar{r}_y \) a constant scalar and \( r_x(\cdot) \in C^2 \). Design:

- a set of parameters \( p_m \in \mathcal{P}_m \);
- a set of parameters \( \bar{p}_s \in \mathcal{P}_s \);
• an input torque of the form

\[ \tilde{u} = \mu(\tilde{p}_m, \tilde{p}_s, q, \dot{q}), \]  

where \( \mu \) is an appropriate map;

such that the following properties are satisfied:

• there exists a solution \((\tilde{q}(t), \dot{\tilde{q}}(t))\) of the closed-loop system \([2], [5]\) that is periodic, with period \(T_q\) and satisfies

\[ (\tilde{x}(0), \dot{\tilde{x}}(0), \tilde{y}(0), \dot{\tilde{y}}(0)) = (r_x(0), \dot{r}_x(0), r_y(0), 0), \]

where \((\tilde{x}(t) \dot{\tilde{y}}(t)) = h(\tilde{q}(t));

• \((\tilde{q}(t), \dot{\tilde{q}}(t))\) is such that the set \(\Gamma_\chi = \{\xi \in \mathbb{R}^4 : \xi = (\tilde{x}(t), \tilde{y}(t), \tilde{y}(t), t \in [0, T_q]\}\) approaches, as much as possible, the set \(\Gamma_r = \{\xi \in \mathbb{R}^4 : \xi = (r_x(t), \dot{r}_x(t), \tilde{r}_y(t), 0), t \in [0, T_r]\};

• the RMS value of \(\tilde{u}\):

\[ \tilde{u}_{\text{RMS}}(\tilde{p}_m, \tilde{p}_s) = \sqrt{\frac{1}{T_q} \int_0^{T_q} \mu^2(\tilde{p}_m, \tilde{p}_s, \tilde{q}(s), \dot{\tilde{q}}(s)) \, ds} \]  

is minimized.

Remark 1: We highlight that the feedback law \([5]\) should not be intended as a full control solution ensuring convergence of the trajectories \((\tilde{q}(t), \dot{\tilde{q}}(t))\) to \((q(t), \dot{q}(t))\) or to its orbit, given by \(\Gamma_q = \{\xi_1, \xi_2 \in \mathbb{R}^4 : \xi_1 = \tilde{q}(t), \xi_2 = \dot{\tilde{q}}(t), t \in [0, T_q]\}\). In order to ensure this feature, it is necessary to design an additional input control \(u_{\text{ad}}\) such that the control law \(u = \tilde{u} + u_{\text{ad}}\) renders either \((\tilde{q}(t), \dot{\tilde{q}}(t))\) or \(\Gamma_q\) attractive from a given set of initial conditions, with appropriate stability properties. We leave this problem out of the purpose of this work, while we refer to \([7]\) for a solution based on the orbital stabilization techniques found in \([12]\).

In a fully actuated scenario (i.e., if another joint were actuated), the computation of the input torque \(\tilde{u}\) could be achieved, e.g., via full-state feedback linearization. However, in the considered underactuated scenario, we lack a complete control authority on both output coordinates \(x\) and \(y\). Therefore, the input \(\tilde{u}\) cannot be designed to exactly impose the reference trajectory \(r\), unless such a trajectory is compatible with the behavior of the resulting internal dynamics. The main intuition of this paper, formally described in the following, is to shape the input torque to keep the end-effector precisely on the reference height \(\tilde{r}_y\), while shaping the spring characteristic \(S\) so that the orbit of \((x(t), \dot{x}(t))\) closely matches the one associated with \((r_x(t), \dot{r}_x(t))\). In this perspective, the selection of \(p_m\) is aimed at minimizing \(\tilde{u}_{\text{RMS}}\) without affecting \((x(t), \dot{x}(t))\).

B. Representation of the Mechanism in Minimal Form

The procedure to convert model \([3]\) to the minimal form model is quite standard and is presented in detail in \([5]\) Section 2.1.5. This procedure consists of mapping the coordinate vector \(q\) in some independent coordinates \(q_1 \in \mathbb{R}^2\) and some dependent coordinates \(q_d \in \mathbb{R}^2\). In this work, we follow the same arguments in \([7]\) to specialize the general approach to the selection \(q_1 = \chi\), so that we can express the model in the workspace coordinates \((x, y)\).

Assumption 1: The maps \(\phi\) and \(h\) are such that:

• there exists a differentiable map \(q_r : \mathbb{R}^2 \rightarrow \mathbb{R}^4\) such that \(h(q_r(r(t))) = r(t)\) and \(\phi(q_r(r(t))) = 0, \text{ for all } t \in [0, T_r]\);

• there exist open sets \(U, V \subset \mathbb{R}^4\) and a diffeomorphism \(\rho : U \rightarrow V\), satisfying \(q = \rho(x, \psi)\) (for some \(\psi \in \mathbb{R}^2\)) and \(q_r(r(t)) \in V\), for all \(t \in [0, T_r]\);

• the matrix

\[ \frac{\partial \phi}{\partial q} \frac{\partial \rho}{\partial \psi}(x, \psi) \]

is non-singular in \((x, \psi) = \rho^{-1}(q_r(r(t)), \forall t \in [0, T_r].\)

This way, it is straightforward to show that there exists an open set, \(\mathcal{N} \subset \mathbb{R}^2\), such that \(r(t) \in \mathcal{N}, \text{ for all } t \in [0, T_r]\), and such that system \([2]\) can be rewritten as

\[ M(p_m, \chi) \ddot{x} + C(p_m, \chi, \dot{x}) \ddot{x} + G(p_m, p_s, \chi) = B(\chi)u, \chi \in \mathcal{N} \]  

where we express the data as follows:

• inertia matrix:

\[ M(p_m, \chi) = \begin{bmatrix} M_{11}(p_m, \chi) & M_{12}(p_m, \chi) \\ M_{12}(p_m, \chi) & M_{22}(p_m, \chi) \end{bmatrix}; \]

• Coriolis and centrifugal torques: \(C(p_m, \chi, \dot{x}) \ddot{x} = [C_1(p_m, \chi)]^T C_2(p_m, \chi)\), where:

\[ C_1 = \frac{1}{2} \frac{\partial M_{11}}{\partial x} \dot{x}^2 + \frac{\partial M_{11}}{\partial y} \dot{y} \dot{x}, \quad C_2 = \frac{\partial M_{22}}{\partial x} \dot{x}^2 + \frac{\partial M_{22}}{\partial y} \dot{y} \dot{x}; \]

• potential energy derivative, accounting for the gravity and the elasticity contributions:

\[ G(p_m, p_s, \chi) = [G_1(p_m, p_s, \chi)]^T; \]

\[ G_1 = S(p_s, \theta) \frac{\partial q_4}{\partial x} + U_{\tilde{z}} \frac{\partial q_4}{\partial x}; \quad G_2 = S(p_s, \theta) \frac{\partial q_4}{\partial y} + U_{\tilde{z}} \frac{\partial q_4}{\partial y}; \]

where \(U_{\tilde{z}}(p_m, \chi)\) denotes the gravity potential energy. Notice that \(\theta\), previously defined as \(\theta = \pi + q_4\), can be equivalently expressed, exploiting the inverse kinematics \((q = h^{-1}(\chi))\), as a function of the workspace coordinates: \(\theta = \eta(x, y)\).

• the input distribution vector \(B(\chi) = [B_1(\chi) \quad B_2(\chi)]^T\), where \(B_1 = \partial q_1/\partial x\) and \(B_2 = \partial q_1/\partial y\).

C. Input-Output Feedback Linearization

As anticipated, our strategy consists of employing the input torque to maintain the \(y\) coordinate on the reference \(\tilde{r}_y\). The resulting zero dynamics, i.e., the residual system obtained under conditions of perfect tracking for the height \(y\), can be written in terms of only \(x\) and \(\dot{x}\). Such dynamics are crucial for the subsequent design, since the available structural parameters can be adjusted to match the behavior of \(x\) with the reference \(r_x\). We now illustrate how the zero dynamics are computed via an input-output linearizing control law.
Taking advantage of the explicit expressions of $M, C\dot{x}, G,$ and $B,$ and inverting the inertia matrix, we can rewrite system (8) as follows:

$$
\begin{align*}
\ddot{x} &= f_x(p_m, p_s, x, \dot{x}, y, \dot{y}) + g_x(p_m, x, y)u \\
\dot{y} &= f_y(p_m, p_s, x, \dot{x}, y, \dot{y}) + g_y(p_m, x, y)u
\end{align*}
$$

where:

$$
\begin{align*}
&f_x = -M_{22}(C_1 + G_1) + M_{12}(C_2 + G_2) \\
&\quad + M_{11}M_{22} \dot{x}^2 \\
&g_x = \frac{M_{22}B_1 - M_{22}B_2}{M_{11}M_{22} - M_{12}^2} \\
&f_y = \frac{M_{12}(C_1 + G_1) - M_{11}(C_2 + G_2)}{M_{11}M_{22} - M_{12}^2} \\
&g_y = -\frac{M_{12}B_1 - M_{11}B_2}{M_{11}M_{22} - M_{12}^2}.
\end{align*}
$$

For the feasibility of the proposed approach, consider the following Assumption.

**Assumption 2:** For all $p_m \in \mathcal{P}_m$ and all $\chi \in \mathcal{N},$ it holds $g_y(p_m, \chi) \neq 0.$

This way, we can apply to system (12) a torque of the form

$$
u = -\frac{f_y(p_m, p_s, x, \dot{x}, y, \dot{y}) + g_y(p_m, x, y)\nu}{g_y(p_m, x, y)} + v,
$$

for a generic input $v,$ which can be employed in the perspective of Remark 1 to define a controller for asymptotic tracking or orbital stabilization. We can rewrite system (12) as

$$
\begin{align*}
\ddot{x} &= f_x(p_m, p_s, x, \dot{x}, y, \dot{y}) \\
&\quad - \frac{g_x(p_m, x, y)}{g_y(p_m, x, y)}\left[ f_y(p_m, p_s, x, \dot{x}, y, \dot{y}) + v \right] \\
\dot{y} &= v,
\end{align*}
$$

therefore the zero dynamics are computed by imposing $y = \bar{r}_y$ and $v = 0,$ for all $t,$ leading to:

$$
\begin{align*}
\ddot{x} &= f_x(p_m, p_s, x, \dot{x}, \bar{r}_y, 0) - \frac{g_x(p_m, x, \bar{r}_y)}{g_y(p_m, x, \bar{r}_y)}\left[ f_y(p_m, p_s, x, \dot{x}, \bar{r}_y, 0) + \nu \right].
\end{align*}
$$

Let $\bar{M}, \bar{G}$ and $\bar{B}$ be the respective maps evaluated in $y = \bar{r}_y$:

$$
\begin{align*}
\bar{M}(p_m, x) &= M(p_m, x, \bar{r}_y), \\
\bar{G}(p_m, p_s, x) &= G(p_m, p_s, x, \bar{r}_y), \\
\bar{B}(x) &= B(x, \bar{r}_y),
\end{align*}
$$

and let $\bar{C}$ be $C$ computed for $y = \bar{r}_y$ and for $\dot{y} = 0,$ i.e.,

$$
\bar{C}(p_m, x, \dot{x}) = C(p_m, x, \dot{x}, \bar{r}_y, 0).
$$

Notice that, having set $\dot{y} = 0,$ (10) can be simplified as follows:

$$
\begin{align*}
\bar{C}_1(p_m, x, \dot{x}) &= C_1(p_m, x, \dot{x}, \bar{r}_y, 0) = \frac{1}{2} \frac{\partial M_{11}}{\partial x} \bigg|_{y=\bar{r}_y} \dot{x}^2, \\
\bar{C}_2(p_m, x, \dot{x}) &= C_2(p_m, x, \dot{x}, \bar{r}_y, 0) = \left( \frac{\partial M_{12}}{\partial x} - \frac{1}{2} \frac{\partial M_{11}}{\partial y} \right) \bigg|_{y=\bar{r}_y} \dot{x}^2.
\end{align*}
$$

Due to (13), system (16) is equivalent to:

$$
(M_{11}\bar{B}_2 - M_{12}\bar{B}_1)\ddot{x} = \bar{B}_1(\bar{C}_2 + \bar{G}_2) - \bar{B}_2(\bar{C}_1 + \bar{G}_1),
$$

while the residual forcing input torque $u_x(p_m, p_s, x, \dot{x})$ can be expressed as:

$$
u_x = -\frac{f_y(p_m, p_s, x, \dot{x}, \bar{r}_y, 0)}{g_y(p_m, x, \bar{r}_y)} = \frac{M_{12}(\bar{C}_1 + \bar{G}_1) - M_{11}(\bar{C}_2 + \bar{G}_2)}{M_{11}B_1 - M_{12}B_2}.
$$

Finally, combining (18) and (19), we can rewrite the zero dynamics as follows:

$$
\alpha(p_m, x)\ddot{x} + \beta(p_m, x)\dot{x}^2 + \gamma(p_m, p_s, x) = 0,
$$

where

$$
\begin{align*}
\alpha(p_m, x) &= M_{11}(p_m, x)\bar{B}_2(x) - M_{12}(p_m, x)\bar{B}_1(x), \\
\beta(p_m, x) &= -\bar{B}_1(x) \left( \frac{\partial M_{12}}{\partial x} - \frac{1}{2} \frac{\partial M_{11}}{\partial y} \right) \bigg|_{y=\bar{r}_y} + \frac{1}{2} \bar{B}_2(x) \left( \frac{\partial M_{11}}{\partial x} \right) \bigg|_{y=\bar{r}_y}, \\
\gamma(p_m, p_s, x) &= -\bar{B}_1(x)\bar{G}_2(p_m, p_s, x) + \bar{B}_2(x)\bar{G}_1(p_m, p_s, x).
\end{align*}
$$

We make some relevant observations concerning the structure of the zero dynamics (21).

- The differential equation (21) is a second order nonlinear dynamics whose stability properties are not guaranteed a priori. In Section 3A we show how it can be ensured that, by proper selection of the mass distribution parameters $p_m,$ the resulting trajectories are periodic.

- Rewrite system (21) as $\ddot{x} = \psi(x, \dot{x}).$ By direct inspection of (21), we see that $\psi(x, \dot{x}) = \psi(x, -\dot{x}).$ Consider $\{x, z\} = \{x, -\dot{x}\}$ and note that $dx/d(-t) = z,$ $dz/d(-t) = \psi(x, z).$ It follows that for any solution of system (21) $(\bar{x}(t), \dot{x}(t))$ such that $\dot{x}(t_0) = 0,$ for some $t_0 \in \mathbb{R},$ the forward and backward evolutions are related through $(\bar{x}(t_0 + t), \dot{x}(t_0 + t)) = (\bar{x}(t_0 - t), -\dot{x}(t_0 - t)),$ i.e., the orbit is symmetric w.r.t. the $x$ axis. This result implies that the target orbit associated with $(\bar{x}(t), \dot{x}(t))$ must satisfy the same symmetry condition to guarantee that $r(t)$ is compatible with the zero dynamics.

**Assumption 3:** For all $t_0 \in \mathbb{R}$ such that $\dot{x}(t_0) = 0,$ it holds:

$$
\begin{align*}
\begin{pmatrix} r_x(t_0 + t) \\ \dot{r}_x(t_0 + t) \end{pmatrix} &= \begin{pmatrix} r_x(t_0 - t) \\ -\dot{r}_x(t_0 - t) \end{pmatrix}, \quad \forall t \in \mathbb{R}.
\end{align*}
$$

- Finally, if we exploit the explicit value of $\bar{G}_1$ and $\bar{G}_2$ in (11), the following relation between $\gamma(\cdot)$ and $S(\cdot)$ arises:

$$
\gamma(p_m, p_s, x) = \zeta_S(x)S(p_s, \eta(x, \bar{r}_y)) + \zeta_U(p_m, x),
$$

where

$$
\begin{align*}
\zeta_S(x) &= \bar{B}_2 \left( \frac{\partial q_1}{\partial x} \right) \bigg|_{y=\bar{r}_y} - \bar{B}_1 \left( \frac{\partial q_1}{\partial y} \right) \bigg|_{y=\bar{r}_y}, \\
\zeta_U(p_m, x) &= \bar{B}_2 \left( \frac{\partial U}{\partial x} \right) \bigg|_{y=\bar{r}_y} - \bar{B}_1 \left( \frac{\partial U}{\partial y} \right) \bigg|_{y=\bar{r}_y}.
\end{align*}
$$

Eq. (24) indicates that system (21) can be shaped via an appropriate selection of the spring $S(\cdot).$
III. TWO-STEP SOLUTION

The approach that we propose is based on dividing the problem into two sub-problems that we solve sequentially:

1) Firstly, we optimize the parameter vector \( p_m \) to minimize the RMS value of the linearizing torque along a given trajectory \( (x(t), \dot{x}(t)) \). In particular, we constrain \( (x(t), \dot{x}(t)) \), solution of the zero dynamics \((21)\) with initial conditions \( (r_x(0), \dot{r}_x(0)) \), to be periodic and equal to \( (r_x(t), \dot{r}_x(t)) \). To do so, we suppose that \( \gamma \) in \((22)\) can be shaped arbitrarily as any desired map \( \hat{\gamma}(p_m, r_x) \), related to an ideal spring characteristic \( \sigma^*(\cdot) \). This optimization procedure and the explicit computation of \( \hat{\gamma}(\cdot) \) and \( \sigma^*(\cdot) \) are described in detail in subsection III-A.

2) Secondly, we choose a convenient class of functions \( S \), parametrized through \( p_s \), to approximate a generic spring characteristic. This way, we can optimize \( p_s \) so that the resulting \( S(\cdot) \) approximates \( \sigma^*(\cdot) \) as much as possible. Notably, our approach follows the intuition that a higher approximation of \( S(\cdot) \) is needed for perfect tracking of \( \sigma^*(\gamma(r_x, \tilde{r}_y)) \) by \( \gamma(p_m, r_x) \).

Before formalizing the optimization problem, we need to ensure that the generated trajectory is periodic and remains periodic in a neighborhood \( N \) of \( r \). We exploit \((13)\), where it is shown that if the zero dynamics \((21)\), linearized around an equilibrium point, presents a center, then the resulting trajectories near that equilibrium point are periodic. We begin by imposing the following feasibility assumption.

Assumption 5: There exists a compact set \( \bar{P}_m \subset P_m \) such that for all \( p_m \in \bar{P}_m \), there exists a point \( r_{x0}(p_m) \in [r_{x_{\min}}, r_{x_{\max}}] \), satisfying \( (r_{x0}(p_m), \tilde{r}_y) \in N \), and such that \( \hat{\gamma}(p_m, r_{x0}(p_m)) = 0 \).

This way, we are able to guarantee the existence of at least an equilibrium point for the zero dynamics. From the continuity of functions \( \alpha, \beta, \gamma, \), the hypotheses of \((13)\) Theorem 3\) hold, thus we ensure that the zero dynamics, linearized around \( (r_{x0}, 0) \), have a center if \( \Omega(p_m) > 0 \), where:

\[
\Omega(p_m) = \frac{\partial}{\partial r_x} \left( \hat{\gamma}(p_m, r_x) \right) \bigg|_{r_x = r_{x0}(p_m)}.
\]

Therefore, to guarantee the presence of periodic orbits near \( r_{x0} \), we impose \( \Omega(p_m) > 0 \) as an inequality constraint in the optimization problem. The last requirement is the existence of at least a \( p_m \) guaranteeing \( \Omega(p_m) > 0 \).

Assumption 6: There exists \( \bar{p}_m \in \bar{P}_m \) such that, for an equilibrium of the form \( r_{x0}(\bar{p}_m) \), it holds \( \Omega(\bar{p}_m) > 0 \).

We can thus formalize the optimization problem as follows:

\[
\min_{p_m \in \bar{P}_m} \sqrt{T_r} \int_0^T \nu(p_m, r_x(t), \dot{r}_x(t)) \, dt
\]

sub. to: \((26), (28), (29), \Omega(p_m) > 0\).

For the numerical solution of \((31)\) we used a genetic algorithm due to the non-convex structure of the problem. In fact, a gradient descent algorithm might converge to a local minimum potentially far away from the global one. Concerning the implementation of \((31)\), we decided to discretize the problem, i.e., we sampled the reference trajectory in \( N \) points equally spaced in time (with sample time \( T_s = T_r / N \)) and we imposed the system dynamics to coincide with the reference only for the discrete trajectory samples, i.e., \((x(iT_s), \dot{x}(iT_s), y(iT_s), \tilde{y}(iT_s)) \) = \((r_x(iT_s), \dot{r}_x(iT_s), \tilde{r}_y, 0) \) with \( i \in \{0, 1, \ldots, N - 1\} \). The resulting discretized problem
is the following:

$$\min_{p_m \in \mathcal{P}_m} \left\{ \frac{1}{N} \sum_{i=0}^{N-1} u^2(p_m, r_x(i T_s), \dot{r}_x(i T_s)) \right\}$$

subject to: \(26\), \(28\), \(29\), \(\Omega_s(p_m) > 0\).

In this case, the constraints \(26\), \(28\), \(29\) are not evaluated for any \(t \in [0, T]\) but only in correspondence of the discrete-time samples \(i T_s\). Moreover, for the inequality constraint, a finite difference approximation of the exact derivative defined in \(30\) has been introduced:

$$\Omega_s(p_m) = \frac{1}{r_{x1} - r_{x0}} \left( \frac{\dot{\gamma}(p_m, r_{x1})}{\alpha(p_m, r_{x1})} - \frac{\dot{\gamma}(p_m, r_{x0})}{\alpha(p_m, r_{x0})} \right),$$

where \(r_{x1} = r_x(i T_s)\) and \(r_{x0} = r_x((i + 1) T_s)\), for some \(i \in \{0, \ldots, N - 1\}\) such that:

$$\frac{\dot{\gamma}(p_m, r_{x1})}{\alpha(p_m, r_{x1})} - \frac{\dot{\gamma}(p_m, r_{x0})}{\alpha(p_m, r_{x0})} < 0.$$ (34)

**B. Nonlinear stiffness approximation**

Consider the ideal spring characteristic, \(\sigma^*(\cdot)\), obtained in the previous subsection. If we were able to realize exactly such characteristic, then the term \(\gamma\) of the zero dynamics \(27\) would match the desired profile, so that \(26\) would hold true (at the discrete samples of the reference trajectory), forcing \((x(t), \dot{x}(t))\) to be (almost) equal to \((r_x(t), \dot{r}_x(t))\). The purpose of this subsection is twofold: firstly, we present a strategy to parameterize through a vector \(p_s\) the characteristic of a nonlinear spring; then, we set up an optimization problem that searches the best parameters, \(\hat{p}_s\), that approximate \(\sigma^*(\cdot)\), for all \(\theta \in [\theta_{\min}, \theta_{\max}]\), where

$$\theta_{\min} = \min_{\tau \in [0, T_s]} \eta(r_x(\tau), \dot{r}_y), \quad \theta_{\max} = \max_{\tau \in [0, T_s]} \eta(r_x(\tau), \dot{r}_y).$$

(35)

There are several works in the literature related to the design of nonlinear springs (see [10] and references therein). In this context, particularly interesting are the approaches found in [10] and [11], where a piecewise linear stiffness is used to fit the desired nonlinear characteristic. As a consequence, it is possible to physically implement the computed stiffness through a combination of multiple linear springs. This procedure has been employed, e.g., in [10] to compensate the weight of an inverted pendulum in a wide interval of angular configurations.

Taking advantage of these results, we propose to approximate \(\sigma^*(\cdot)\) with a piecewise linear stiffness. To parameterize such a profile, we suppose that the overall characteristic is given by the sum of different elementary components. In particular, we start the approximation with a linear spring with stiffness \(k_0\) and rest angle \(\theta_0\). Then, we increase the complexity of the characteristic by introducing pairs of positive-negative sub-springs. By positive sub-spring, we intend a spring that gives no contribution until \(\theta\) reaches a specific threshold \(\theta^+\). For \(\theta \geq \theta^+\), the positive sub-spring has a linear characteristic of slope \(k^+\). Conversely, a negative sub-spring is a linear spring (with stiffness \(k^-\)) for \(\theta \leq \theta^-\), for a given threshold \(\theta^-\). In view of the possibility of realizing springs with negative stiffness [14], we allow \(k^+\) and \(k^-\) to be either positive or negative. On the other hand, we impose \(k_0 \geq 0\) for the linear spring.

The overall characteristic is thus obtained through the combination of one linear spring and \(n\) pairs of positive-negative sub-springs. Notice that increasing \(n\) improves the characteristic approximation, even though it increases the complexity in the actual realization of the spring. Based on the described modeling strategy, the ensuing analytical representation of the spring characteristic is:

$$S(p_s, \theta) = S_0(k_0, \theta_0, \theta) + \sum_{j=1}^{n} S_j^+(k_j^+, \theta_j^+, \theta) + S_j^-(k_j^-, \theta_j^-, \theta)$$

(36)

where

$$S_0(k_0, \theta_0, \theta) = k_0 (\theta - \theta_0);$$

$$S_j^+(k_j^+, \theta_j^+, \theta) = \begin{cases} 0, & \text{if } \theta < \theta_j^+; \\ k_j^+ (\theta - \theta_j^+), & \text{if } \theta \geq \theta_j^+, \quad j \in \{1, \ldots, n\}, \\ S_j^-(k_j^-, \theta_j^-, \theta) = \begin{cases} k_j^- (\theta - \theta_j^-), & \text{if } \theta \leq \theta_j^-; \\ 0, & \text{if } \theta > \theta_j^- \quad j \in \{1, \ldots, n\}. \end{cases} \end{cases}$$

(37)

We can thus introduce an optimization algorithm aiming to approximate \(S^*(\theta)\). We collect all the parameters determining the spring characteristic in vector \(p_s\):

$$p_s = (k_0 \theta_0 k_1^+ \theta_1^+ k_1^- \theta_1^- \ldots k_n^+ \theta_n^+ k_n^- \theta_n^-)^T \in \mathbb{R}^d$$

(38)

where \(d = 2 + 4n\), since every pair of positive-negative sub-springs introduces four parameters \((k^+, \theta^+, k^-, \theta^-)\), while the linear spring is characterized by two parameters \((k_0, \theta_0)\). The domain of \(p_s\) is \(\mathcal{P}_s = [0, \bar{k}] \times [\theta_{\min}, \theta_{\max}] \times \{[-\bar{k}, \bar{k}] \times [\theta_{\min}, \theta_{\max}]\}^{2n}\), where \(\bar{k}\) is the maximum slope (in absolute value) admissible for the linear characteristic of any spring or sub-spring, while \(\theta_{\min}\) and \(\theta_{\max}\) are defined in (35). Therefore, we look for an optimal parameter configuration, \(\hat{p}_s\), minimizing the mismatch between the actual spring characteristic and the ideal one, both evaluated in \(N\) samples of angle \(\theta = \eta(r_x(i T_s), \dot{r}_y)\) corresponding to the \(N\) samples of \(r_x\) imposed in the mass distribution optimization problem:

$$\min_{p_s \in \mathcal{P}_s} \frac{1}{N} \sum_{i=0}^{N-1} \left[ S(p_s, \eta(r_x(i T_s), \dot{r}_y)) - \sigma^*(\eta(r_x(i T_s), \dot{r}_y)) \right]^2.$$ (39)

As for the mass distribution optimization, to account for the non-convex cost function, we employed the genetic algorithm to solve the problem.

**Remark 2:** In this second optimization problem, we have not imposed any condition regarding the properties of the equilibrium point of the resulting zero dynamics. In fact, we assumed implicitly that the obtained \(\gamma(\hat{p}_m, \hat{p}_s, r_x)\) is sufficiently close to \(\hat{\gamma}(\hat{p}_m, r_x)\). Hence, both the equilibrium of the zero dynamics
and the trajectories near such equilibrium are expected to have similar properties w.r.t. the ideal ones. If this were not the case, we could always increase the accuracy in the spring characteristic approximation by increasing the number of sub-springs, so that a better approximation of \( \tilde{\gamma}(\tilde{p}_m, r_x) \) is obtained.

### C. Overall problem solution

We can sum up our solution to Problem 1. Firstly, from the mass distribution optimization we obtain the optimal \( \tilde{p}_m \) and the ideal spring characteristic \( \sigma^*(\cdot) \). Secondly, choosing the number \( d = 2 + 4n \) of parameters used to model the spring profile, we run the spring parameter optimization determining \( \tilde{p}_s \). This second step can be repeated more than once, with different values of \( n \), i.e., with a different degree of accuracy in the approximation of \( \sigma^*(\cdot) \). Finally, the input torque that completes the solution of the problem is the following:

\[
\bar{u}(\tilde{p}_m, \tilde{p}_s, q, \dot{q}) = -\frac{f_y(\tilde{p}_m, \tilde{p}_s, x(q), \dot{x}(q, \dot{q}), y(q), \dot{y}(q, \dot{q}))}{g_y(\tilde{p}_m, x(q), y(q))} \tag{40}
\]

with \( f_y \) and \( g_y \) defined in (13), \( (x \ y)^\top = h(q), (\dot{x} \ y)^\top = \frac{\partial h}{\partial q}(q)\dot{q} \) and \( h \) as in (3). Recalling Remark 1, we highlight that this input action does not stabilize the system on the reference trajectory. Instead, \( \bar{u} \) is the feedback-linearizing input torque that keeps the system on the reference \( r \) in perfect tracking conditions.

### IV. Numerical results

In this section, we present the results of our optimization strategy applied to the mechanism of Fig. 1. The parameters of the links are summarized in Table I. In particular, the values of the masses and the moments of inertia (computed w.r.t. the center of mass of each link) refer to the links without the additional masses. Such additional masses were allowed to be at most equal to 0.1kg. The domain considered in the mass distribution optimization problem is given by the following box: \( \mathcal{P}_m = [0, 0.1]^2 \times [-0.05, 0.05]^2 \). For the domain \( \mathcal{P}_s \), described in Section III-B, we chose \( \bar{k} = 10\text{Nm/rad} \), while \( \theta_{\text{min}} \) and \( \theta_{\text{max}} \) have been computed as in (3) according to the assigned reference trajectory.

Concerning the solution of problems (22) and (19), we ran the optimization on an AMD Ryzen Threadripper 2950X 16-core processor with 3.5GHz base clock frequency and 4.4GHz maximum boost frequency. In both optimization steps, we fixed \( N = 1000 \) and we employed the genetic algorithm available in Matlab, \textit{ga}.

Two different horizontal reference trajectories \( r_x(t) \) have been considered for testing our two-step solution. Both these trajectories satisfy Assumption 3 and have a period \( T_1 \) equal to 0.5s. In both cases, the end-effector height \( \tilde{r}_y \) is set to 0.15m, while the position of joint \( \mathcal{J}_1 \) is fixed to \((-0.19, 0.15)\)m in the \((e_x, e_y)\) frame.

The first horizontal trajectory taken into account has the profiles of \( r_x(t) \) and \( \dot{r}_x(t) \) shown in red respectively in Figs. 2(a) and 2(b), while the red plot of Fig. 2(c) represents the target orbit associated with \((r_x(t), \dot{r}_x(t))\). The forward motion of this trajectory (and conversely, the backward phase) consists of an initial low-speed phase, followed by a second high-speed phase. The optimized mass distribution parameters are found

| Length [m] | Total mass [kg] | Moment of inertia [kg m^2] |
|------------|----------------|-----------------------------|
| \( l_1 \) | 0.080 \( m_1 \) | 0.071 \( J_1 \) | \( 0.747 \times 10^{-4} \) |
| \( l_2 \) | 0.235 \( m_2 \) | 0.195 \( J_2 \) | \( 10.413 \times 10^{-4} \) |
| \( l_3 \) | 0.052 \( m_3 \) | 0.049 \( J_3 \) | \( 0.345 \times 10^{-4} \) |
| \( l_4 \) | 0.135 \( m_4 \) | 0.115 \( J_4 \) | \( 2.430 \times 10^{-4} \) |

Figure 2. Periodic motion of the zero dynamics for the first reference trajectory. (a): reference \( r_x \) (red) and zero dynamics evolution \( \dot{x} \) (blue), \( n = 1 \). (b): reference \( \dot{r}_x \) (red) and zero dynamics evolution \( \dot{x} \) (blue), \( n = 1 \). (c): reference \( r_x \) (red) and zero dynamics evolution \( \dot{x} \) (blue), \( n = 3 \). (f): target orbit (red) and the one generated by the zero dynamics (blue), \( n = 3 \).

Figure 3. Spring optimization results for the first reference trajectory. (a): \( \sigma^*(\theta) \) (red) and \( S(\tilde{p}_x, \theta) \) comparison with \( n \in \{1, 2, 3\} \). (b): \( \sigma^*(\theta) - S(\tilde{p}_x, \theta) \) with \( n \in \{1, 2, 3\} \).

Figure 4. Periodic motion of the zero dynamics for the second reference trajectory.

Table I: PARAMETERS OF THE MECHANISM LINKS

| Length [m] | Total mass [kg] | Moment of inertia [kg m^2] |
|------------|----------------|-----------------------------|
| \( l_1 \) | 0.080 \( m_1 \) | 0.071 \( J_1 \) | \( 0.747 \times 10^{-4} \) |
| \( l_2 \) | 0.235 \( m_2 \) | 0.195 \( J_2 \) | \( 10.413 \times 10^{-4} \) |
| \( l_3 \) | 0.052 \( m_3 \) | 0.049 \( J_3 \) | \( 0.345 \times 10^{-4} \) |
| \( l_4 \) | 0.135 \( m_4 \) | 0.115 \( J_4 \) | \( 2.430 \times 10^{-4} \) |
\[
\begin{align*}
\sigma^* (\theta) &= (0.100\text{kg}, 0.062\text{kg}, -0.050\text{m}, 0.022\text{m})^\top, \\
\text{in} \ \bar{p}_m &= (0.100\text{kg}, 0.062\text{kg}, -0.050\text{m}, 0.022\text{m})^\top, \\
\end{align*}
\]
providing an ideal input with RMS value equal to 0.143Nm. Notice that, without any additional mass, the RMS value of the input would be equal to 0.193Nm. Therefore, a reduction of 25.8% in the RMS value of the input is obtained due to the mass distribution optimization. The ideal spring characteristic \(\sigma^* (\theta)\), resulting from the first optimization procedure, is approached with 1, 2, and 3 couples of sub-springs, i.e., \(n \in \{1, 2, 3\}\). Fig. 3(a) shows the comparison between \(\sigma^* (\theta)\), red, and \(S(\bar{p}_s, \theta)\) obtained with the three different choices of \(n\) (color associations: orange corresponds to \(n = 1\), green corresponds to \(n = 2\), blue corresponds to \(n = 3\)). As expected, Fig. 3(b) confirms that a higher number of sub-springs improves the approximation, i.e., the mismatch between \(\sigma^* (\theta)\) and \(S(\bar{p}_s, \theta)\) is reduced. Figs. 3(a), 3(b), 3(c), and Figs. 3(d), 3(e), 3(f) illustrate the zero dynamics behavior (in blue), compared with the reference (in red), obtained respectively in the case of \(n = 1\) and \(n = 3\). Notably, it is possible to appreciate a significantly improved approximation of the target orbit as a consequence of the more accurate approximation of \(\sigma^* (\theta)\).

The second horizontal reference trajectory that we considered has a trapezoidal velocity profile (with appropriate smoothing to satisfy the requirement \(r_x \in C^2\)). The optimized mass distribution parameters are found in \(\bar{p}_m = (0.100\text{kg}, 0.044\text{kg}, -0.050\text{m}, 0.017\text{m})^\top\) and provide a reduction of 25.9% w.r.t. the RMS value of the input obtained in case of \(m_{A_3} = m_{A_4} = 0\text{kg}\) (i.e., \(0.188\text{Nm}\) instead of \(0.187\text{Nm}\)). In this case, we approximated \(\sigma^* (\theta)\) with only one pair of sub-springs, obtaining a very satisfactory zero dynamics matching. Fig. 4 reports the profiles of \(S(\bar{p}_s, \theta)\) and \((\dot{x}(t), \ddot{x}(t))\), compared with \(\sigma^* (\theta)\) and \((r_x(t), \dot{r}_x(t))\).

V. CONCLUSIONS

We presented a control-oriented structural design strategy that generates an optimal periodic motion for a 2-DOF underactuated mechanism. The proposed solution is based on the co-design of an input-output linearizing torque and some system’s structural parameters. In particular, we formalized a two-step optimization strategy. Firstly, we minimized the RMS value of the input by selecting the mass distribution and obtaining a nonlinear stiffness that imposes the desired periodic behavior for the system’s zero dynamics. Afterward, we approximated the nonlinear spring through a piecewise linear characteristic for a simpler physical implementation of the system. Future works will be dedicated to integrating the two optimization steps and extending the approach to more general underactuated mechanisms.

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