An a posteriori analysis of $C^0$ interior penalty methods for the obstacle problem of clamped Kirchhoff plates

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**Recommended Citation**

Brenner, S., Gedicke, J., Sung, L., & Zhang, Y. (2017). An a posteriori analysis of $C^0$ interior penalty methods for the obstacle problem of clamped Kirchhoff plates. *SIAM Journal on Numerical Analysis, 55* (1), 87-108. [https://doi.org/10.1137/15M1039444](https://doi.org/10.1137/10M1039444)
AN A POSTERIORI ANALYSIS OF C\(^0\) INTERIOR PENALTY METHODS FOR THE OBSTACLE PROBLEM OF CLAMPED KIRCHHOFF PLATES

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Abstract. We develop an a posteriori analysis of C\(^0\) interior penalty methods for the displacement obstacle problem of clamped Kirchhoff plates. We show that a residual based error estimator originally designed for C\(^0\) interior penalty methods for the boundary value problem of clamped Kirchhoff plates can also be used for the obstacle problem. We obtain reliability and efficiency estimates for the error estimator and introduce an adaptive algorithm based on this error estimator. Numerical results indicate that the performance of the adaptive algorithm is optimal for both quadratic and cubic C\(^0\) interior penalty methods.

1. Introduction

Let \( \Omega \subset \mathbb{R}^2 \) be a bounded polygonal domain, \( f \in L_2(\Omega) \), \( \psi \in C(\overline{\Omega}) \cap C^2(\Omega) \) and \( \psi < 0 \) on \( \partial\Omega \). The displacement obstacle problem for the clamped Kirchhoff plate is to find

\[
\begin{align*}
(1.1) \quad u &= \arg\min_{v \in K} \left[ \frac{1}{2} a(v,v) - (f,v) \right] \\
(1.2) \quad a(w,v) &= \int_{\Omega} D^2w : D^2v \, dx = \int_{\Omega} \sum_{i,j=1}^{2} \left( \frac{\partial^2 w}{\partial x_i \partial x_j} \right) \left( \frac{\partial^2 v}{\partial x_i \partial x_j} \right) \, dx, \quad (f,v) = \int_{\Omega} f v \, dx \\
(1.3) \quad K &= \{ v \in H^2_0(\Omega) : v \geq \psi \text{ in } \Omega \}.
\end{align*}
\]

The unique solution \( u \in K \) of (1.1)–(1.3) is characterized by the variational inequality

\[
\begin{align*}
(1.4) \quad a(u,v-u) &\geq (f,v-u) \quad \forall v \in K,
\end{align*}
\]

which can be written in the following equivalent complementarity form:

\[
\int_{\Omega} (u - \psi) \, d\lambda = 0,
\]
where the Lagrange multiplier \( \lambda \) is the nonnegative Borel measure defined by

\[
(1.5) \quad a(u, v) = (f, v) + \int_{\Omega} v \, d\lambda \quad \forall \, v \in H^2_0(\Omega).
\]

**Remark 1.1.** Since \( u > \psi \) near \( \partial\Omega \), the support of \( \lambda \) is disjoint from \( \partial\Omega \) because of (1.4).

**Remark 1.2.** We can treat \( \lambda \) as a member of \( H^{-2}(\Omega) = [H^2_0(\Omega)]' \) such that

\[
\langle \lambda, v \rangle = \int_{\Omega} v \, d\lambda \quad \forall \, v \in H^2_0(\Omega).
\]

\( C^0 \) interior penalty methods \([23, 13, 11, 9, 8, 26]\) form a natural hierarchy of discontinuous Galerkin methods that are proven to be effective for fourth order elliptic boundary value problems. The goal of this paper is to develop an \textit{a posteriori} error analysis of \( C^0 \) interior penalty methods for the obstacle problem defined by (1.1)–(1.3). While there is a substantial literature on the \textit{a posteriori} error analysis of finite element methods for second order obstacle problems (cf. \([30, 19, 37, 34, 2, 35, 36, 7, 6, 27, 28, 18]\) and the references therein), as far as we know this is the first paper on the \textit{a posteriori} error analysis for the displacement obstacle problem of Kirchhoff plates. We note that there is a fundamental difference between second order and fourth order obstacle problems, namely that the Lagrange multipliers for the fourth order discrete obstacle problems can be represented naturally as sums of Dirac point measures (cf. Section 2), which leads to a simpler \textit{a posteriori} error analysis (cf. Section 4 and Section 5).

The rest of the paper is organized as follows. We recall the \( C^0 \) interior penalty methods in Section 2 and analyze a mesh-dependent boundary value problem in Section 3 that plays an important role in the \textit{a posteriori} error analysis carried out in Section 4 and Section 5. An adaptive algorithm motivated by the \textit{a posteriori} error analysis is introduced in Section 6 and we report results of several numerical experiments in Section 7. We end the paper with some concluding remarks in Section 8.

### 2. \( C^0 \) Interior Penalty Methods

Let \( \mathcal{T}_h \) be a triangulation of \( \Omega \), \( V_h \) be the set of the vertices of \( \mathcal{T}_h \), \( E_h \) be the set of the edges of \( \mathcal{T}_h \), and \( V_h \subset H^1_0(\Omega) \) be the \( P_k \) Lagrange finite element space \((k \geq 2)\) associated with \( \mathcal{T}_h \). The discrete problem for the \( C^0 \) interior penalty method \([14, 15]\) is to find

\[
(2.1) \quad u_h = \arg\min_{v \in K_h} \left[ \frac{1}{2} a_h(v, v) - (f, v) \right],
\]

where \( K_h = \{ v \in V_h : v(p) \geq \psi(p) \quad \text{for all} \, p \in V_h \} \),

\[
a_h(w, v) = \sum_{T \in \mathcal{T}_h} \int_T D^2 w : D^2 v \, dx + \sum_{e \in E_h} \int_e \left( \left\{ \frac{\partial^2 w}{\partial n^2} \right\} \left[ \frac{\partial v}{\partial n} \right] + \left\{ \frac{\partial^2 v}{\partial n^2} \right\} \left[ \frac{\partial w}{\partial n} \right] \right) \, ds
\]

\[
+ \sum_{e \in E_h} \sigma |e| \int_e \left[ \frac{\partial w}{\partial n} \right] \left[ \frac{\partial v}{\partial n} \right] \, ds,
\]
\( \{ \cdot \} \) denotes the average across an edge, \( [\cdot] \) denotes the jump across an edge, \(|e|\) is the length of the edge \( e \), and \( \sigma \geq 1 \) is a penalty parameter large enough so that \( a_h(\cdot, \cdot) \) is positive-definite on \( V_h \). Details for the notation and the choice of \( \sigma \) can be found in [13, 31].

The unique solution \( u_h \in K_h \) of (2.1) is characterized by the variational inequality
\[
a_h(u_h, v - u_h) \geq (f, v - u_h) \quad \forall v \in K_h,
\]
which can be expressed in the following equivalent complementarity form:
\[
\sum_{p \in V_h} \lambda_h(p) (u_h(p) - \psi(p)) = 0,
\]
where the Lagrange multipliers \( \lambda_h(p) \) are defined by
\[
a_h(u_h, v) = (f, v) + \sum_{p \in V_h} \lambda_h(p) v(p) \quad \forall v \in V_h
\]
and satisfy
\[
\lambda_h(p) \geq 0 \quad \forall p \in V_h.
\]

We also use \( \lambda_h \) to denote the measure \( \sum_{p \in V_h} \lambda_h(p) \delta_p \), where \( \delta_p \) is the Dirac point measure at \( p \). The equation (2.3) can therefore be written as
\[
a_h(u_h, v) = (f, v) + \int_\Omega v \, d\lambda_h \quad \forall v \in V_h.
\]

**Remark 2.1.** For second order obstacle problems, the discrete Lagrange multiplier cannot be extended to \( H^{-1}(\Omega) \) as a sum of Dirac point measures since such measures do not belong to \( H^{-1}(\Omega) \). Consequently there are different choices for extending the discrete Lagrange multiplier to \( H^{-1}(\Omega) \) [37, 34, 35]. The fact that the Lagrange multiplier for the discrete fourth order obstacle problem can be expressed naturally as a sum of Dirac point measures leads to the simple \textit{a posteriori} error analysis in Section 4 and Section 5.

**Remark 2.2.** We can also treat \( \lambda_h \) as a member of \( H^{-2}(\Omega) = [H^2_0(\Omega)]' \) such that
\[
\langle \lambda_h, v \rangle = \int_\Omega v \, d\lambda_h = \sum_{p \in V_h} \lambda_h(p) v(p) \quad \forall v \in H^2_0(\Omega).
\]

Let the mesh-dependent norm \( \| \cdot \|_h \) be defined by
\[
\|v\|_h^2 = \sum_{T \in T_h} |v|_{H^2(T)}^2 + \sum_{e \in E_h} \frac{\sigma}{|e|} \|[\![\partial v / \partial n]\!]|_{L^2(e)}^2.
\]

Note that
\[
\|v\|_h = |v|_{H^2(\Omega)} \quad \forall v \in H^2_0(\Omega)
\]

The following \textit{a priori} error estimate is known [15, 14]:
\[
\|u - u_h\|_h \leq C h^\alpha,
\]
where the index of elliptic regularity $\alpha \in (\frac{1}{2}, 1]$ is determined by the interior angles of $\Omega$ and can be taken to be 1 if $\Omega$ is convex.

Our goal is to develop \textit{a posteriori} error estimates for $\|u - u_h\|_h$.

Two useful tools for the analysis of $C^0$ interior penalty methods are the nodal interpolation operator $\Pi_h : H^2_0(\Omega) \rightarrow V_h$ and an enriching operator $E_h : V_h \rightarrow W_h \subset H^2_0(\Omega)$, where $W_h$ is the Hsieh-Clough-Tocher macro finite element space [20].

The operator $E_h$ is defined by averaging (cf. [8, Section 4.1]) and hence

\begin{equation}
(E_h u_h)(p) = u_h(p) \quad \text{for all } p \in V_h.
\end{equation}

The following estimate can be found in the proof of [8, Lemma 1].

\begin{equation}
\eta_h = \sum_{T \in T_h} \eta^2_T \leq C \sum_{e \in \tilde{E}_T} \frac{1}{|e|} \|\frac{\partial v}{\partial n}\|_{L^2(e)}^2 \quad \forall T \in T_h,
\end{equation}

where $\tilde{E}_T$ is the set of the edges of $T$ emanating from the vertices of $T$, and the positive constant $C$ depends only on $k$ and the shape regularity of $T_h$.

From (2.10) and standard inverse estimates [21, 12], we also have

\begin{equation}
\eta_h \leq C \sum_{e \in \tilde{E}_T} \frac{1}{|e|} \|\frac{\partial v}{\partial n}\|_{L^2(e)}^2 \quad \forall T \in T_h,
\end{equation}

\begin{equation}
\sum_{T \in T_h} \|v - E_h v\|_{H^2(T)}^2 \leq C \sum_{e \in \tilde{E}_T} \frac{1}{|e|} \|\frac{\partial v}{\partial n}\|_{L^2(e)}^2 \quad \forall v \in V_h,
\end{equation}

\begin{equation}
\|v - E_h v\|_h^2 \leq C \sum_{e \in \tilde{E}_T} \frac{\sigma}{|e|} \|\frac{\partial v}{\partial n}\|_{L^2(e)}^2 \quad \forall v \in V_h,
\end{equation}

where the positive constant $C$ depends only on $k$ and the shape regularity of $T_h$.

3. A Mesh-Dependent Boundary Value Problem

Let $z_h \in H^2_0(\Omega)$ be defined by

\begin{equation}
a(z_h, v) = (f, v) + \int_{\Omega} v \, d\lambda_h = (f, v) + \sum_{p \in V_h} \lambda_h(p) v(p) \quad \forall v \in H^2_0(\Omega).
\end{equation}

Then $u_h$ is the approximate solution of (3.1) obtained by the $C^0$ interior penalty method.

Remark 3.1. The idea of considering such mesh-dependent boundary value problems was introduced in [5] for second order obstacle problems.

A residual based error estimator [10, 8] for $u_h$ (as an approximate solution of (3.1)) is given by

\begin{equation}
\eta_h = \left( \sum_{e \in \tilde{E}_h} \eta^2_{e,1} + \sum_{e \in \tilde{E}_h} \eta^2_{e,2} + \sum_{T \in T_h} \eta^2_T \right)^{\frac{1}{2}},
\end{equation}
where $\mathcal{E}_h^i$ is the set of the edges of $T_h$ interior to $\Omega$.

$$\eta_{e,1} = \frac{\sigma}{|e|^2} \left\| \left[ \frac{\partial u_h}{\partial n} \right] \right\|_{L^2(e)},$$

$$\eta_{e,2} = |e|^{\frac{1}{2}} \left\| \left[ \frac{\partial^2 u_h}{\partial n^2} \right] \right\|_{L^2(e)},$$

$$\eta_{e,3} = |e|^{\frac{3}{2}} \left\| \left[ \frac{\partial^3 u_h}{\partial n^3} \right] \right\|_{L^2(e)},$$

$$\eta_T = h_T^2 \left\| f - \Delta^2 u_h \right\|_{L^2(T)}.$$

The following result will play an important role in the \textit{a posteriori} error analysis of the obstacle problem. Note that its proof is made simple by the representation of the discrete Lagrange multiplier $\lambda_h$ as a sum of Dirac point measures supported at the vertices of $T_h$, which allows the analysis in [8] to be used here.

\textbf{Lemma 3.2.} There exists a positive constant $C$, depending only on $k$ and the shape regularity of $T_h$, such that

$$\| z_h - u_h \|_h \leq C \eta_h.$$  

\textbf{Proof.} We have an obvious estimate

$$\sum_{e \in \mathcal{E}_h} \left\| \left[ \frac{\partial(z_h - u_h)}{\partial n} \right] \right\|^2_{L^2(e)} = \sum_{e \in \mathcal{E}_h} \left\| \left[ \frac{\partial u_h}{\partial n} \right] \right\|^2_{L^2(e)} \leq \sum_{e \in \mathcal{E}_h} \eta_{e,1}^2,$$

and it only remains to estimate $\sum_{T \in \mathcal{T}_h} |z_h - u_h|^2_{H^2(T)}$.

Let $E_h : V_h \rightarrow H^2_0(\Omega)$ be the enriching operator. It follows from (2.12) and (3.3) that

$$\sum_{T \in \mathcal{T}_h} |z_h - u_h|^2_{H^2(T)} \leq 2 \sum_{T \in \mathcal{T}_h} \left[ |z_h - E_h u_h|^2_{H^2(T)} + |u_h - E_h u_h|^2_{H^2(T)} \right]$$

$$\leq 2 |z_h - E_h u_h|^2_{H^2(\Omega)} + C \sum_{e \in \mathcal{E}_h} \eta_{e,1}^2,$$

and, by duality,

$$|z_h - E_h u_h|_{H^2(\Omega)} = \sup_{\phi \in H^2_0(\Omega) \backslash \{0\}} \frac{a(z_h - E_h u_h, \phi)}{|\phi|_{H^2(\Omega)}}.$$  

In view of (2.3) and (3.1), the numerator on the right-hand side of (3.10) becomes

$$a(z_h - E_h u_h, \phi) = \sum_{T \in \mathcal{T}_h} \int_T D^2(z_h - E_h u_h) : D^2 \phi \, dx$$

$$= (f, \phi) + \sum_{p \in \mathcal{V}_h} \lambda_h(p) \phi(p)$$

$$+ \sum_{T \in \mathcal{T}_h} \int_T D^2(u_h - E_h u_h) : D^2 \phi \, dx - \sum_{T \in \mathcal{T}_h} \int_T D^2 u_h : D^2(\phi - \Pi_h \phi) \, dx.$$
which is precisely the equation [8, (7.9)].

(4.1)

\[ \frac{\|u\|_h}{\|\lambda - \lambda_h\|_{H^{-2}(\Omega)}} \leq C \eta_h + \sqrt{\int_\Omega (\psi - E_h u_h)^+ d\lambda}. \]

\[ |u - E_h u_h|_{H^2(\Omega)} = a(u - E_h u_h, u - E_h u_h) \]

(4.2)

\[ = a(u - z_h, u - E_h u_h) + a(z_h - E_h u_h, u - E_h u_h), \]

and, in view of (2.7), (2.13), (3.3) and Lemma 3.2 the second term on the right-hand side of (4.2) is bounded by

\[ a(z_h - E_h u_h, u - E_h u_h) \leq \|z_h - E_h u_h\|_{H^2(\Omega)}|u - E_h u_h|_{H^2(\Omega)} \]

(4.3)

\[ \leq C \eta_h \|u - E_h u_h\|_{H^2(\Omega)}. \]
By (1.3)–(1.5), (2.2), (2.4), (2.9) and (3.1), the first term on the right-hand side of (4.2) can be bounded as follows:

\[
a(u - z_h, u - E_h u_h) = \int_{\Omega} (u - E_h u_h) \, d\lambda - \sum_{p \in \mathcal{V}_h} \lambda_h(p) (u(p) - (E_h u_h)(p))
\]

\[
= \int_{\Omega} (\psi - E_h u_h) \, d\lambda - \sum_{p \in \mathcal{V}_h} \lambda_h(p) (u(p) - \psi(p)) \leq \int_{\Omega} (\psi - E_h u_h)^+ \, d\lambda.
\]

It follows from (2.7) and (4.2)–(4.4) that

\[
\|u - E_h u_h\|_h \leq C \eta_h + \sqrt{\int_{\Omega} (\psi - E_h u_h)^+ \, d\lambda},
\]

which together with (2.13) implies

\[
\|u - u_h\|_h \leq C \eta_h + \sqrt{\int_{\Omega} (\psi - E_h u_h)^+ \, d\lambda}.
\]

In order to estimate \(\|\lambda - \lambda_h\|_{H^{-2}(\Omega)}\), we observe that (1.5), (2.7) and (3.1) imply

\[
\|\lambda - \lambda_h\|_{H^{-2}(\Omega)} = \sup_{v \in H^2_0(\Omega)} \int_{\Omega} v \, d(\lambda - \lambda_h) \frac{|v|}{|H^2(\Omega)|} = \sup_{v \in H^2_0(\Omega)} \frac{a(u - z_h, v)}{|v|_{H^2(\Omega)}} = |u - z_h|_{H^2(\Omega)} \leq \|u - u_h\|_h + \|z_h - u_h\|_h.
\]

The estimate for \(\|\lambda - \lambda_h\|_{H^{-2}(\Omega)}\) then follows from Lemma 3.2 and (4.5).

We can also remove the inconvenient \(E_h\) in the estimate (4.1).

**Theorem 4.2.** There exists a positive constant \(C\), depending only on \(k\) and the shape regularity of \(T_h\), such that

\[
\|u - u_h\|_h + \|\lambda - \lambda_h\|_{H^{-2}(\Omega)} \leq C \left( \eta_h + |\lambda|^{1/2} \sqrt{\max_{T \in T_h} \sum_{e \in \tilde{E}_T} |e|^{-1/2} \|\partial u_h/\partial n\|_{L^2(e)}} \right)
\]

\[
+ |\lambda|^{1/2} \|\psi - u_h\|_{L^\infty(\Omega)}^+ \|\lambda\|_2^2,
\]

where \(\tilde{E}_T\) is the set of the edges in \(T_h\) that emanate from the vertices of \(T\).

**Proof.** We have

\[
\int_{\Omega} (\psi - E_h u_h)^+ \, d\lambda \leq \|\psi - u_h\|_{L^\infty(\Omega)}^+ \|u_h - E_h u_h\|_{L^\infty(\Omega)} |\lambda|,
\]
and, by (2.11),
\[(4.9)\]
\[\|u_h - E_h u_h\|_{L_\infty(\Omega)} \leq C \max_{T \in T_h} h_T \sum_{e \in \tilde{E}} |e|^{-1/2} \|\partial u_h / \partial n\|_{L_2(e)}.
\]

The estimate (4.7) follows from (4.1), (4.8), and (4.9).

Remark 4.3. The estimate (4.7) is not a genuine a posteriori error estimate since |λ| is not known. But it is useful for monitoring the asymptotic convergence of adaptive algorithms (cf. Lemma 6.1 and Lemma 6.2).

Remark 4.4. Under the stronger assumption ψ ∈ C^2(Ω) on the obstacle function, one can also obtain a genuine a posteriori error estimate by replacing |λ| with a computable bound.

Indeed, for any w ∈ K, we have
\[\frac{1}{2} |u|_{H^2(\Omega)}^2 \leq \frac{1}{2} |w|_{H^2(\Omega)}^2 - (f, w) + (f, u) \leq \frac{1}{2} |w|_{H^2(\Omega)}^2 - (f, w) + C \|f\|_{L_2(\Omega)}^2 + \frac{1}{4} |u|_{H^2(\Omega)}^2,
\]
by a Poincaré-Friedrichs inequality \[33\] and the arithmetic-geometric means inequality, and hence
\[(4.10)\]
\[|u|_{H^2(\Omega)}^2 \leq 2|w|_{H^2(\Omega)}^2 - 4(f, w) + C \|f\|_{L_2(\Omega)}^2,
\]
where C is a computable positive constant. Combining (4.10) with the Sobolev embedding (cf. \[1\]) \(H^2(\Omega) \hookrightarrow C^{0,\gamma}(\Omega)\) for any \(\gamma < 1\), we see that there is a computable \(\delta > 0\) such that \(u(x) > \psi(x)\) if the distance from \(x\) to \(\partial \Omega\) is < \(\delta\). Therefore there is a computable \(\phi \in C_c^\infty(\Omega)\) such that \(\phi = 1\) on the support of \(\lambda\).

We then have, in view of (1.5) and (4.10),
\[|\lambda| = a(u, \phi) - (f, \phi) \leq |u|_{H^2(\Omega)}^2 |\phi|_{H^2(\Omega)} + \|f\|_{L_2(\Omega)}^2 \|\phi\|_{L_2(\Omega)} \leq C,
\]
where the positive constant \(C\) is computable.

5. Efficiency Estimates for the Obstacle Problem

Let the local data oscillation \(\text{Osc}(f; T)\) be defined by
\[\text{Osc}(f; T) = h_T^2 \|f - \bar{f}_T\|_{L_2(T)},
\]
where \(\bar{f}_T\) is the \(L_2\) projection of \(f\) in the polynomial space \(P_j(T)\) with \(j = \max(k - 4, 0)\).

The global data oscillation is then given by
\[\text{Osc}(f; T_h) = \left( \sum_{T \in T_h} \text{Osc}(f; T)^2 \right)^{1/2}.
\]

Theorem 5.1. There exists a positive constant \(C\), depending only on the shape regularity of \(T^h\), such that
\[\eta_{e,1} \leq \frac{\sigma}{|e|^{1/2}} \|\partial (u - u_h)/\partial n\|_{L_2(e)} \quad \forall e \in E_h,
\]
\[
\eta_{e,2} \leq C \left[ \sum_{T \in \mathcal{T}_e} [\|u - u_h\|_{H^2(T)} + \text{Osc}(f; T)] + \|\lambda - \lambda_h\|_{H^{-2}(\Omega_e)} \right] \quad \forall e \in \mathcal{E}_h,
\]

\[
\eta_{e,3} \leq C \left[ \sum_{T \in \mathcal{T}_e} [\|u - u_h\|_{H^2(T)} + \text{Osc}(f; T)] + \|\lambda - \lambda_h\|_{H^{-2}(\Omega_e)} \vphantom{\sum_{T \in \mathcal{T}_e}} \right]
+ \frac{1}{|e|} \|\partial(u - u_h)/\partial n\|_{L^2(e)}^2 \quad \forall e \in \mathcal{E}_h,
\]

\[
\eta_T \leq C \left( \|u - u_h\|_{H^2(T)} + \text{Osc}(f; T) + \|\lambda - \lambda_h\|_{H^{-2}(T)} \right) \quad \forall T \in \mathcal{T}_h,
\]

where \( \mathcal{T}_e \) is the set of the two triangles that share the edge \( e \) and \( \Omega_e \) is the interior of \( \bigcup_{T \in \mathcal{T}_e} \bar{T} \).

**Proof.** The estimate for \( \eta_{e,1} \) is obvious. The other estimates are obtained by modifying the arguments in [8, Section 5.3].

In the proof of the estimate [8, (5.17)] (with \( v = u_h \)), we replace the relation

\[
\int_T (\bar{f}_T - \Delta^2 u_h)z \, dx = \int_T D^2(u - u_h) : D^2z \, dx + \int_T (\bar{f}_T - f)z \, dx
\]

by

\[
(5.1) \quad \int_T (\bar{f}_T - \Delta^2 u_h)z \, dx = \int_T D^2(u - u_h) : D^2z \, dx + \int_T (\bar{f}_T - f)z \, dx - \int_T z \, d(\lambda - \lambda_h)
\]

to obtain the estimate

\[
\int_T (\bar{f}_T - \Delta^2 u_h)z \leq C \left( h_T^{-2}\|u - u_h\|_{H^2(T)} + \|f - \bar{f}_T\|_{L^2(T)} + h_T^{-2}\|\lambda - \lambda_h\|_{H^{-2}(T)} \right) \|z\|_{L^2(T)}
\]

which then leads to the estimate for \( \eta_T \). Note that (5.1) holds because the bubble function \( z \) vanishes at the vertices of \( \mathcal{T}_h \).

In the proof of the estimate [8, (5.26)] (with \( v = u_h \)), we replace the relation

\[
\sum_{T \in \mathcal{T}_e} \left( - \int_T D^2 u_h : D^2(\xi_1 \xi_2) \, dx + \int_T (\Delta^2 u_h)(\xi_1 \xi_2) \, dx \right)
\]

\[
= \sum_{T \in \mathcal{T}_e} \int_T D^2(u - u_h) : D^2(\xi_1 \xi_2) \, dx - \sum_{T \in \mathcal{T}_e} \int_T (f - \Delta^2 u_h)(\xi_1 \xi_2) \, dx
\]

that appears in [8, (5.24)] by

\[
\sum_{T \in \mathcal{T}_e} \left( - \int_T D^2 u_h : D^2(\xi_1 \xi_2) \, dx + \int_T (\Delta^2 u_h)(\xi_1 \xi_2) \, dx \right)
\]

\[
= \sum_{T \in \mathcal{T}_e} \int_T D^2(u - u_h) : D^2(\xi_1 \xi_2) \, dx - \sum_{T \in \mathcal{T}_e} \int_T (f - \Delta^2 u_h)(\xi_1 \xi_2) \, dx
\]

\[
- \int_{\Omega_e} (\xi_1 \xi_2) \, d(\lambda - \lambda_h)
\]

(5.2)
to obtain the estimate
\[
\sum_{T \in T_h} \left( - \int_T D^2 u_h : D^2 (\zeta_1 \zeta_2) \, dx + \int_T (\Delta^2 u_h) (\zeta_1 \zeta_2) \, dx \right)
\leq C \left[ \sum_{T \in T_h} \left( h_T^{-2} |u - u_h|_{H^2(T)} + \| f - \Delta^2 u_h \|_{L^2(T)} \right) + h_T^{-2} \| \lambda - \lambda_h \|_{H^{-2}(\Omega_e)} \right] \| \zeta_1 \zeta_2 \|_{L^2(\Omega_e)},
\]
which then leads to the estimate for \( \eta_{e,2} \). Note that (5.2) holds because the bubble function \( \zeta_1 \zeta_2 \) vanishes at the vertices of \( T_h \).

Finally, in the proof of the estimate [8, (5.32)] (with \( v = u_h \)), we replace the relation
\[
\sum_{T \in T_h} \left( \int_T D^2 u_h : D^2 (\zeta_2 \zeta_3) \, dx - \int_T (\Delta^2 u_h) (\zeta_2 \zeta_3) \, dx \right)
= \sum_{T \in T_h} \int_T D^2 (u_h - u) : D^2 (\zeta_2 \zeta_3) \, dx + \sum_{T \in T_2} \int_T (f - \Delta^2 u_h) (\zeta_2 \zeta_3) \, dx
\]
that appears in [8, (5.30)] by
\[
\sum_{T \in T_h} \left( \int_T D^2 u_h : D^2 (\zeta_2 \zeta_3) \, dx - \int_T (\Delta^2 u_h) (\zeta_2 \zeta_3) \, dx \right)
= \sum_{T \in T_h} \int_T D^2 (u_h - u) : D^2 (\zeta_2 \zeta_3) \, dx + \sum_{T \in T_2} \int_T (f - \Delta^2 u_h) (\zeta_2 \zeta_3) \, dx
+ \int_{\Omega_e} (\zeta_2 \zeta_3) \, d(\lambda - \lambda_h)
\]
(5.3)
to obtain the estimate
\[
\sum_{T \in T_h} \left( \int_T D^2 u_h : D^2 (\zeta_2 \zeta_3) \, dx - \int_T (\Delta^2 u_h) (\zeta_2 \zeta_3) \, dx \right)
\leq C \left[ \sum_{T \in T_h} \left( h_T^{-2} |u - u_h|_{H^2(T)} + \| f - \Delta^2 u_h \|_{L^2(T)} \right) + h_T^{-2} \| \lambda - \lambda_h \|_{H^{-2}(\Omega_e)} \right] \| \zeta_2 \zeta_3 \|_{L^2(\Omega_e)},
\]
which then leads to the estimate for \( \eta_{e,3} \). Again (5.3) holds because the bubble function \( \zeta_2 \zeta_3 \) vanishes at the vertices of \( T_h \).

We can also prove a global efficiency result under the following assumption:

The triangles (resp. interior edges) of \( T_h \) can be divided into \( n \) disjoint groups \( 5.4 \) so that the ratio of the diameters of any two triangles (resp. interior edges) in the same group is bounded above by a constant \( \tau \geq 1 \).

**Theorem 5.2.** Under assumption \( 5.4 \), there exists a positive constant \( C \) depending only on \( \tau, k \) and the shape regularity of \( T_h \) such that
\[
\eta_h \leq C \left( \sqrt{\tau} \| u - u_h \|_h + \sqrt{n} \| \lambda - \lambda_h \|_{H^{-2}(\Omega)} + \text{Osc}(f; T_h) \right).
\]
(5.5)
Proof. We have a trivial estimate
\[
\sum_{T \in \mathcal{T}} \eta_{e,T}^2 \leq C \sum_{e \in \mathcal{E}} \sigma_e^2 \left\| \left\| \frac{\partial(u - u_h)}{\partial n} \right\| \right\|_{L^2(e)}^2.
\]

For the estimate involving \( \eta_T \), we first write \( \mathcal{T}_h \) as the disjoint union \( \mathcal{T}_{h,1} \cup \cdots \cup \mathcal{T}_{h,n} \) so that the ratio of the diameters of any two triangles in \( \mathcal{T}_{h,j} \) is bounded by \( \tau \). For \( 1 \leq j \leq n \), the subdomain \( \Omega_j \) is the interior of \( \cup_{T \in \mathcal{T}_{h,j}} \bar{T} \).

For any \( T \in \mathcal{T}_{h,j} \), let \( z_T \) be the bubble function in [8, Section 5.3.2] associated with \( T \) and we define \( z_j = \sum_{T \in \mathcal{T}_{h,j}} z_T \in H^2_0(\Omega_j) \). It follows from [8 (5.16)], (5.1) and a standard inverse estimate that
\[
\| \bar{f}_T - \Delta^2 u_h \|^2_{L^2(T)} \leq C \int_T (\bar{f}_T - \Delta^2 u_h) z_T \, dx
\]
\[
\leq C \left( [h_T^{-2}] \| u - u_h \|_{H^2(T)} + \| f - \bar{f}_T \|_{L^2(T)} \right) \| z_T \|_{L^2(T)} - \int_{\Omega_j} z_T \, d(\lambda - \lambda_h)
\]
and hence
\[
\sum_{T \in \mathcal{T}_{h,j}} \| \bar{f}_T - \Delta^2 u_h \|^2_{L^2(T)} \leq C \left( \sum_{T \in \mathcal{T}_{h,j}} [h_T^{-2}] \| u - u_h \|_{H^2(T)} + \| f - \bar{f}_T \|_{L^2(T)} \right) \| z_T \|_{L^2(T)}
\]
\[
- \int_{\Omega_j} z_T \, d(\lambda - \lambda_h)
\]
\[
\leq C \left[ \left( \sum_{T \in \mathcal{T}_{h,j}} [h_T^{-4}] \| u - u_h \|_{H^2(T)}^2 + \| f - \bar{f}_T \|_{L^2(T)}^2 \right)^\frac{1}{2} \left( \sum_{T \in \mathcal{T}_{h,j}} \| z_T \|_{L^2(T)}^2 \right)^\frac{1}{2} + \| \lambda - \lambda_h \|_{H^{-2}(\Omega_j)} \right] \left( \sum_{T \in \mathcal{T}_{h,j}} h_T^{-4} \| z_T \|_{L^2(T)}^2 \right)^\frac{1}{2}
\]
by a standard inverse estimate.

Therefore we have
\[
(5.6) \quad \sum_{T \in \mathcal{T}_{h,j}} h_T^4 \| \bar{f}_T - \Delta^2 u_h \|^2_{L^2(T)} \leq C \left( \sum_{T \in \mathcal{T}_{h,j}} [h_T^4] \| f - \bar{f}_T \|_{L^2(T)}^2 + \| u - u_h \|_{H^2(T)}^2 \right)
\]
\[
+ \| \lambda - \lambda_h \|_{H^{-2}(\Omega_j)}^2
\]
because (cf. [8 (5.16)])
\[
\| z_T \|_{L^2(T)} \approx \| \bar{f}_T - \Delta^2 u_h \|_{L^2(T)}
\]
and the diameters \( h_T \) are comparable for \( T \in \mathcal{T}_{h,j} \).

It follows from (5.6) that
\[
\sum_{T \in \mathcal{T}_h} \eta_T^2 = \sum_{T \in \mathcal{T}_h} h_T^4 \| f - \Delta^2 u_h \|^2_{L^2(T)}
\]
\[
\leq \sum_{j=1}^{n} \sum_{T \in T_{h,j}} 2h_T^4 \left[ \| \bar{f}_T - \Delta^2 u_h \|_{L_2(T)} + \| f - \bar{f}_T \|_{L_2(T)} \right]^2 \\
\leq C \sum_{j=1}^{n} \left( \sum_{T \in T_{h,j}} h_T^4 \| f - \bar{f}_T \|_{L_2(T)}^2 + \| u - u_h \|_{H^2(T)}^2 \right) \\
\quad + \| \lambda - \lambda_h \|_{H^{-2}(\Omega_j)}^2 \\
\leq C \left( \text{Osc}(f; T_h)^2 + \sum_{T \in T_h} |u - u_h|_{H^2(T)}^2 + n \| \lambda - \lambda_h \|_{H^{-2}(\Omega)}^2 \right),
\]

where we have also used the trivial estimate \( \| \lambda - \lambda_h \|_{H^{-2}(\Omega_j)} \leq \| \lambda - \lambda_h \|_{H^{-2}(\Omega)}. \)

The estimates for \( \eta_{e,2} \) and \( \eta_{e,3} \) can be established by using (5.2), (5.3) and results in [8, Sections 5.3.3 and 5.3.4]. Their derivations are similar to the derivation for \( \eta_T \) and hence are omitted. \( \square \)

6. An Adaptive Algorithm

In view of the efficiency estimates in Section 5, we will use \( \eta_h \) from (3.2) as the error indicator in the adaptive loop

\textbf{Solve} \longrightarrow \textbf{Estimate} \longrightarrow \textbf{Mark} \longrightarrow \textbf{Refine}

to define an adaptive algorithm for the \( C^0 \) interior penalty methods for (1.1)–(1.3).

In the step \textbf{Solve}, we compute the solution of the discrete obstacle problem (2.1) by a primal-dual active set method [3, 29]. In the step \textbf{Estimate}, we compute \( \eta_{e,1}, \eta_{e,2}, \eta_{e,3} \) and \( \eta_T \) defined in (3.3)–(3.6). In the step \textbf{Mark}, we use the Dörfler marking strategy [22] to mark a minimum number of triangles and edges whose contributions exceed \( \theta \eta_h \) for some \( \theta \in (0, 1) \).

In the step \textbf{Refine}, we refine the marked triangles and edges followed by a closure algorithm that preserves the conformity of the triangulation.

In the adaptive setting the subscript \( h \) will be replaced by the subscript \( \ell \), where \( \ell = 0, 1, \ldots \) denotes the level of refinements. The adaptive algorithm generates a sequence of triangulations \( T_\ell \) of \( \Omega \), a sequence of solutions \( u_\ell \in V_\ell \) of the discrete obstacle problems and a sequence of error indicators \( \eta_\ell \).

According to Theorem 4.2, we can use the following result to monitor the asymptotic convergence rate of the adaptive algorithm.

**Lemma 6.1.** Suppose \( \eta_\ell = O(N_\ell^{-\gamma}) \), where \( N_\ell \) is the number of degrees of freedom (dof) at the refinement level \( \ell \). Then we have

\[
\| u - u_\ell \|_\ell + \| \lambda - \lambda_\ell \|_{H^{-2}(\Omega)} = O(N_\ell^{-\gamma})
\]

provided that

\[
Q_{\ell,1} = \sqrt{\max_{T \in T_\ell} h_T \sum_{e \in \partial T} |e|^{-1/2} \| \partial u \|_{L_2(e)} = O(N_\ell^{-\gamma}),
\]

where

\[
\sqrt{\sum_{T \in T_\ell} h_T \sum_{e \in \partial T} |e|^{-1/2} \| \partial u \|_{L_2(e)}} = O(N_\ell^{-\gamma}),
\]

and

\[
\text{Osc}(f; T_h) = O(h_T^{-1/2} \| \partial f \|_{L_2(T)}).
\]
(6.3) \[ Q_{\ell,2} = \| (\psi - u_{\ell})^+ \|_{L_\infty(\Omega)}^{\frac{1}{2}} = O(N^{-\gamma}_\ell). \]

In particular, the estimate (6.1) holds if \( Q_{\ell,1} \) and \( Q_{\ell,2} \) are dominated by \( \eta_\ell \).

Note that \( \| \lambda - \lambda_h \|_{H^{-2}(\Omega)} \) is not computable. However we can test the convergence of \( \| \lambda - \lambda_h \|_{H^{-2}(\Omega)} \) indirectly as follows. Let \( \phi \in C_0^\infty(\Omega) \) be equal to 1 on the supports of \( \lambda \) and the \( \lambda_\ell \)'s. Then we have

(6.4) \[ |\lambda| - |\lambda_\ell| = \int_{\Omega} \phi d(\lambda - \lambda_\ell) \leq |\phi|_{H^2(\Omega)} \| \lambda - \lambda_\ell \|_{H^{-2}(\Omega)}, \]

which implies

(6.5) \[ |\lambda_\ell| - |\lambda_{\ell+1}| = (|\lambda_\ell| - |\lambda|) + (|\lambda| - |\lambda_{\ell+1}|) \leq |\phi|_{H^2(\Omega)} (\| \lambda - \lambda_\ell \|_{H^{-2}(\Omega)} + \| \lambda - \lambda_{\ell+1} \|_{H^{-2}(\Omega)}). \]

Let \( \Lambda_\ell \) be defined by

(6.6) \[ \Lambda_\ell = |||\lambda_\ell| - |\lambda_{\ell+1}|||. \]

The following result is an immediate consequence of Lemma 6.1 and (6.5).

**Lemma 6.2.** Suppose \( \eta_\ell = O(N^{-\gamma}_\ell) \), where \( N_\ell \) is the number of dof at the refinement level \( \ell \). Then we have

\[ \Lambda_\ell = O(N^{-\gamma}_\ell) \]

provided that (6.2) and (6.3) are valid.

**Remark 6.3.** In view of (6.4), we can also replace \( |\lambda| \) by \( |\lambda_\ell| \) in (6.7) to obtain a true a posteriori error estimate that is asymptotically reliable under the assumptions of Lemma 6.1.

### 7. Numerical Experiments

In this section we report numerical results that demonstrate the estimate (6.7) and illustrate the performance of the adaptive algorithm for quadratic and cubic \( C^0 \) interior penalty methods. We choose the penalty parameter \( \sigma \) to be 6 (resp. 18) for the quadratic (resp. cubic) \( C^0 \) interior penalty method. We also take \( \theta \) to be 0.5 in the Dörfler marking strategy.

We will consider three examples. The first one concerns a problem on the unit square with known exact solution. The second one is about a problem on a \( L \)-shaped domain with a two dimensional coincidence set (where \( u = \psi \)) that has a fairly smooth boundary. The third example is also about a problem on a \( L \)-shaped domain but with a coincidence set that is one dimensional. For the second and third examples where the exact solution is not known, we estimate the error \( \| u - u_\ell \|_\ell \) by using a reference solution computed on the mesh obtained by a uniform refinement of the last mesh generated by the refinement procedure.

In each of the experiment for the adaptive algorithm, we will present figures that display the convergence histories for \( \| u - u_\ell \|_\ell \) and \( \eta_\ell \), and for the quantities \( Q_{\ell,1} \) and \( Q_{\ell,2} \) defined in (6.2) and (6.3). We also present tables that contain numerical results for the quantity \( \Lambda_\ell \) defined in (6.6) and examples of adaptively generated meshes.
7.1. Example 1. In this example we consider an obstacle problem on the unit square $\Omega = (-0.5, 0.5)^2$ from Example 1 with $f = 0$, $\psi = 1 - |x|^2$ and nonhomogeneous boundary conditions, whose exact solution is given by

$$u(x) = \begin{cases} C_1|x|^2 \ln(|x|) + C_2|x|^2 + C_3 \ln(|x|) + C_4 & r_0 < |x| \\ 1 - |x|^2 & |x| \leq r_0 \end{cases}$$

where $r_0 \approx 0.18134453$, $C_1 \approx 0.52504063$, $C_2 \approx -0.62860905$, $C_3 \approx 0.017266401$ and $C_4 \approx 1.0467463$.

For this example the coincidence set is the disc centered at the origin with radius $r_0$ whose boundary is the free boundary, and we have $|\lambda| = 8\pi C_1 \approx 13.1957$.

Due to the nonhomogeneous boundary conditions, we modify the discrete obstacle problem (cf. [14]) to find

$$u_h = \arg\min_{v \in K_h} \left[ \frac{1}{2} a_h(v, v) - F(v) \right],$$

where $K_h = \{ v \in V_h : v - \Pi_h u \in H^1_0(\Omega), v(p) \geq \psi(p) \quad \forall p \in V_h \}$,

$$F(v) = (f, v) + \sum_{e \in \mathcal{E}_h^b} \int_e \left( \left[ \frac{\partial^2 v}{\partial n^2} \right] + \frac{\sigma}{|e|} \left[ \frac{\partial v}{\partial n} \right] \right) \left[ \frac{\partial u}{\partial n} \right] ds,$$

and $\mathcal{E}_h^b$ is the set of the edges of $T_h$ that are on the boundary of $\Omega$. We also modify the residual based error estimator:

$$\eta_h = \left( \sum_{e \in \mathcal{E}_{h,1}} \eta_{e,1}^2 + \sum_{e \in \mathcal{E}_{h,2}} \eta_{e,2}^2 + \sum_{e \in \mathcal{E}_{h,3}} \eta_{e,3}^2 + \sum_{T \in T_h} \eta_T^2 + \sum_{e \in \mathcal{E}_h^b} \sigma^2 |e|^{-1} \| \partial(u_h - u) / \partial n \|_{L_2(e)}^2 \right)^{\frac{1}{2}}.$$

In the first experiment we solve the discrete problem with the $P_2$ element on uniform meshes and compute the quantity

$$Q_h = C \left( \eta_h + |\lambda|^{\frac{1}{2}} \sqrt{\max_{T \in T_h} h_T} \sum_{e \in \mathcal{E}_T} |e|^{-1/2} \| \partial u_h / \partial n \|_{L_2(e)} \right)$$

$$+ |\lambda|^{\frac{1}{2}} \| (\psi - u_h) + \|_{L_\infty(\Omega)}$$

that appears on the right-hand side of (4.7), with $C = 0.32$ and $|\lambda| = 13.196$. The results for $\|u - u_h\|_h/Q_h$ (cf. Table 7.1) clearly demonstrate the estimate (4.7).

| $h$ | $\|u - u_h\|_h/Q_h$ |
|-----|-----------------|
| 2^{-1} | 0.93 |
| 2^{-2} | 0.94 |
| 2^{-3} | 0.96 |
| 2^{-4} | 0.97 |
| 2^{-5} | 0.97 |
| 2^{-6} | 0.97 |
| 2^{-7} | 0.98 |
| 2^{-8} | 0.98 |
| 2^{-9} | 0.99 |
| 2^{-10} | 1.04 |

Table 7.1. Numerical results for the estimate (4.7).

In the second experiment we solve the discrete obstacle problem with the cubic element on uniform and adaptive meshes. We observe optimal (resp. suboptimal) convergence rate for adaptive (resp. uniform) meshes in Figure 7.1(a) and also the reliability of $\eta_e$. Furthermore
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the optimal $O(N_{\ell}^{-1})$ convergence rate of $\|u-u_{\ell}\|_{\ell}$ is justified by Figure 7.1(b) and Lemma 6.1.

According to Lemma 6.2 and Figure 7.1(b), the magnitude of $\Lambda_{\ell}$ should be $O(N_{\ell}^{-1})$. This is confirmed by the results in Table 7.2 where $N_{\ell}$ increases from $N_{0} = 49$ to $N_{20} = 231328$.

$$
\begin{array}{cccccccccccc}
\ell & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\Lambda_{\ell}N_{\ell} & 155 & 250 & 60.5 & 214 & 164 & 99.6 & 101 & 75.8 & 21.8 & 31.9 & 17.9 \\
\ell & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 \\
\Lambda_{\ell}N_{\ell} & 12.2 & 1.48 & 23.8 & 5.37 & 0.403 & 4.65 & 20.5 & 51.1 & 259 & 730 \\
\end{array}
$$

Table 7.2. $\Lambda_{\ell}N_{\ell}$ for the adaptive cubic $C^0$ interior penalty method for Example 1

An adaptive mesh with roughly 3000 nodes is depicted in Figure 7.2 and strong refinement near the free boundary is observed.
7.2. Example 2. In this example we consider the obstacle problem from [14, Example 4] for a clamped plate occupying the $L$-shaped domain $\Omega = (-0.5, 0.5)^2 \setminus [0, 0.5]^2$ with $f = 0$ and $\psi(x) = 1 - \left[\frac{(x_1 + 1/4)^2}{0.2^2} + \frac{x_2^2}{0.35^2}\right]$. The coincidence set for this problem is presented in Figure 7.3(a).

**Figure 7.3.** $L$-shaped domain for Example 2: (a) Coincidence set for the obstacle problem (b) Adaptive mesh with $\approx 3000$ nodes for the $P_2$ element (c) Adaptive mesh with $\approx 5000$ nodes for the $P_3$ element

In the first experiment we solve the discrete obstacle problem with the $P_2$ element on uniform and adaptive meshes. Optimal (resp. suboptimal) convergence rate for adaptive (resp. uniform) meshes and the reliability of $\eta_\ell$ are observed in Figure 7.4(a), and the $O(N_\ell^{-1/2})$ convergence rate of $\|u - u_\ell\|_\ell$ is justified by Figure 7.4(b) and Lemma 6.1.

**Figure 7.4.** Convergence histories for the quadratic $C^0$ interior penalty method for Example 2: (a) $\|u - u_\ell\|_\ell$ and $\eta_\ell$, (b) $\eta_\ell$, $Q_{\ell,1}$ and $Q_{\ell,2}$

The $O(N_\ell^{-1/2})$ bound for $\Lambda_\ell$ predicted by Lemma 6.2 and Figure 7.4(b) is observed in Table 7.3, where $N_\ell$ increases from 65 to 827483.
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| $\ell$ | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  |
|--------|----|----|----|----|----|----|----|----|----|
| $\Lambda_{\ell}N_{\ell}^{1/2}$ | 2715 | 391 | 637 | 756 | 1454 | 654 | 613 | 467 | 411 |
| $\ell$ | 9  | 10 | 11 | 12 | 13 | 14 | 15 | 16 |    |
| $\Lambda_{\ell}N_{\ell}^{1/2}$ | 360 | 149 | 255 | 105 | 144 | 70 | 72 | 52 |    |

Table 7.3. $\Lambda_{\ell}N_{\ell}^{1/2}$ for the adaptive quadratic $C^0$ interior penalty method for Example 2

An adaptive mesh with roughly 3000 nodes is displayed in Figure 7.3(b), where we observe a strong refinement near the reentrant corner. In contrast the refinement near the free boundary is mild. This is due to the fact that away from the reentrant corner the solution belongs to $H^3$ (cf. [24, 4]) and we are using the $P_2$ element.

In the second experiment we solve the obstacle problem with the $P_3$ element on uniform and adaptive meshes. We observe optimal (resp. suboptimal) convergence rate for adaptive (resp. uniform) meshes in Figure 7.5(a) and that $\eta_{\ell}$ is reliable in both cases. Moreover the $O(N_{\ell}^{-1})$ convergence rate of $\|u - u_{\ell}\|_{\ell}$ is justified by Figure 7.5(b) and Lemma 6.1.

![Figure 7.5](image)

**Figure 7.5.** Convergence histories for the cubic $C^0$ interior penalty method for Example 2: (a) $\|u - u_{\ell}\|_{\ell}$ and $\eta_{\ell}$, (b) $\eta_{\ell}$, $Q_{\ell,1}$ and $Q_{\ell,2}$

The results for $\Lambda_{\ell}$ are reported in Table 7.4, where the $O(N_{\ell}^{-1})$ bound for $\Lambda_{\ell}$ predicted by Lemma 6.2 and Figure 7.5(b) can be observed. Note that there are large oscillations at the beginning before the coincidence has been captured by the adaptive mesh. Here $N_{\ell}$ increases from $N_0 = 133$ to $N_{22} = 358792$.

An adaptive mesh with roughly 5000 nodes is displayed in Figure 7.3(c), where we observe strong refinement near both the reentrant corner and the free boundary.

7.3. **Example 3.** In this example we consider the obstacle problem on the $L$-shaped domain $\Omega = (-0.5, 0.5)^2 \setminus [0, 0.5]^2$ with

$$\psi(x) = -[\sin(2\pi(x_1 + 0.5)(x_2 + 0.5))\sin(4\pi(x_1 - 0.5)(x_2 - 0.5))] - 0.35$$
and
\[
f(x) = \begin{cases} 
10^3 \left( \frac{1}{2} e^{(x_1 + 0.25)^2 + (x_2 + 0.25)^2} \right) & x_1 \leq 0, x_2 > 0 \\
0 & x_1 \leq 0, x_2 \leq 0 \\
10^3 \left( \frac{1}{2} + \left[ (x_1 - 0.25)^2 + (x_2 + 0.25)^2 \right]^{3/2} \right) & x_1 \geq 0, x_2 \leq 0 
\end{cases}.
\]

For this example, the coincidence set is one dimensional (cf. Figure 7.6(a)).

| $\ell$ | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  |
|--------|----|----|----|----|----|----|----|----|
| $\Lambda_\ell N_\ell$ | 15398 | 16877 | 2893 | 1035 | 11806 | 8925 | 15493 | 5993 |
| $\ell$ | 8  | 9  | 10 | 11 | 12 | 13 | 14 | 15 |
| $\Lambda_\ell N_\ell$ | 1048 | 5162 | 3544 | 3271 | 1362 | 119 | 580 | 778 |
| $\ell$ | 16 | 17 | 18 | 19 | 20 | 20 | 21 | 22 |
| $\Lambda_\ell N_\ell$ | 77  | 96 | 68 | 92 | 147 | 116 | 754 | 885 |

Table 7.4. $\Lambda_\ell N_\ell$ for the adaptive cubic $C^0$ interior penalty method for Example 2
Figure 7.7. Convergence histories for the quadratic $C^0$ interior penalty method for Example 3: (a) $\|u - u_\ell\|_\ell$ and $\eta_\ell$, (b) $\eta_\ell$, $Q_{\ell,1}$ and $Q_{\ell,2}$

| $\ell$ | $\Lambda_\ell N^{1/2}_\ell$ |
|-------|-----------------|
| 0     | 151             |
| 1     | 501             |
| 2     | 92              |
| 3     | 201             |
| 4     | 120             |
| 5     | 419             |
| 6     | 201             |
| 7     | 198            |
| 8     | 34 | 76  |
| 9     | 40             |
| 10    | 36             |

Table 7.5. $\Lambda_\ell N^{1/2}_\ell$ for the adaptive quadratic $C^0$ interior penalty method for Example 3

Figure 7.8. Convergence histories for the cubic $C^0$ interior penalty method for Example 3: (a) $\|u - u_\ell\|_\ell$ and $\eta_\ell$, (b) $\eta_\ell$, $Q_{\ell,1}$ and $Q_{\ell,2}$
The results in Table 7.6 agrees with the $O(N^{-1})$ bound for $\Lambda_\ell$ predicted by Lemma 6.2 and Figure 7.8(b). Here $N_\ell$ increases from $N_0 = 133$ to $N_{15} = 88699$.

| $\ell$ | $\Lambda_\ell N_\ell$ |
|--------|-----------------|
| 0      | 11724           |
| 1      | 1782            |
| 2      | 1842            |
| 3      | 32888           |
| 4      | 1046            |
| 5      | 6439            |
| 6      | 2974            |
| 7      | 2588            |

| $\ell$ | $\Lambda_\ell N_\ell$ |
|--------|-----------------|
| 8      | 2781            |
| 9      | 25657           |
| 10     | 2215            |
| 11     | 3805            |
| 12     | 5177            |
| 13     | 2030            |
| 14     | 1092            |
| 15     | 2355            |

Table 7.6. $\Lambda_\ell N_\ell$ for the adaptive cubic $C^0$ interior penalty method for Example 3

An adaptive mesh with 12841 dof is depicted in Figure 7.6(c), where we observe strong refinement around the reentrant corner and the coincidence set.

8. Conclusions

We have developed a simple \textit{a posteriori} error analysis of $C^0$ interior penalty methods for the displacement obstacle problem of clamped Kirchhoff plates by taking advantage of the fact that the Lagrange multiplier for the discrete problem can be represented naturally as the sum of Dirac point measures supported at the vertices of the triangulation. Numerical results indicate that the adaptive algorithm based on a standard \textit{a posteriori} error estimator originally developed for boundary value problems also performs optimally for quadratic and cubic $C^0$ interior penalty methods for obstacle problems. However the theoretical justification of convergence and optimality for adaptive $C^0$ interior penalty methods remains open even in the case when the obstacle is absent.

The results in this paper can be extended to the displacement obstacle problem of the biharmonic equation with the boundary conditions of simply supported plates or the Cahn-Hilliard type. In the case where $\Omega$ is convex, such problems are related to distributed elliptic optimal control problems with pointwise state constraints \[32\,25\,16\,17\] and can also be considered in three dimensional domains. Adaptive finite element methods for these problems based on the approach in this paper are ongoing projects.

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