Automorphisms of Set Families and of Families of Cliques in an Interval Graph in FPT Time

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Abstract

We consider the following problem closely related to graph isomorphism. In a simplified version, the task is to compute the automorphism group of a given set family (or a hypergraph), that is, the group of all automorphisms of the given sets which are compatible with some permutation of their elements. In a general setting, the set family in question is a collection of cliques (called marked cliques) of a given interval graph, and the task is to compute the group of all permutations of the cliques which result from some automorphism of the underlying interval graph. This problem is obviously at least as hard as the graph isomorphism (GI-hard) already in the simplified version – consider the set family of edges of a graph, and we give an FPT-time algorithm parameterized by the maximum number of sets in the family which are incomparable by inclusion (its antichain size).

To our best knowledge, the general version of the problem has not been formulated in the literature so far. The problem has been inspired by the research of special cases of the isomorphism problem of chordal graphs; namely, the simplified set-family version is the core of our FPT algorithm for the isomorphism of so-called \( S_d \)-graphs [MFCS 2021], and the general version extends and improves a cumbersome technical step in our FPT algorithm for the isomorphism of chordal graphs of bounded leafage [WALCOM 2022]. The new algorithm combines two classical tools – PQ-trees of interval graphs and Babai’s tower-of-groups, in a nontrivial way.

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1 Introduction

The graph isomorphism problem is to determine whether the two given graphs are isomorphic, denoted by \( G \cong H \); i.e., to decide whether there exists a bijection between the vertex sets, \( f : V(G) \rightarrow V(H) \), such that \( f \) preserves the edges, \( \{u, v\} \in E(G) \iff \{f(u), f(v)\} \in E(H) \) for all \( u, v \in V(G) \). All isomorphisms of a graph \( G \) onto itself form a permutation group called the automorphism group of \( G \).

Graph isomorphism is in a sense a quite special problem in computer science; on one hand, under some widely-believed complexity-theoretic assumptions, it can be shown that graph isomorphism is not an NP-hard problem, while on the other hand, a polynomial-time algorithm for graph isomorphism is still elusive (and not everybody expects existence of such algorithm). It has actually defined its own complexity class \( GI \) of the problems which are reducible in polynomial time to graph isomorphism. The current state of the art is a quasi-polynomial algorithm of Babai [5]. Nevertheless, the problem has been shown to be solvable efficiently for various natural graph classes such as trees, planar and interval graphs [1][2][6].

For a set family \( \mathcal{X} \subseteq 2^M \), the automorphism group of \( \mathcal{X} \) is the group of all permutations \( \sigma \) of \( \mathcal{X} \) for which there exists a permutation \( \pi \) of \( M \) such that \( \sigma(X) = \pi(X) \) holds for all \( X \in \mathcal{X} \). If there are no restrictions on the considered permutations of the ground set \( M \), the condition on \( \sigma \) can be simply translated as that \( \sigma \) preserves the cardinalities of all intersections of
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sets from $\mathcal{X}$ (cf. Lemma 8). Note that this problem is at least as hard as computing the automorphism group of a graph (that is, GI-hard); simply take the edge set of the graph as the considered family $\mathcal{X}$. In our approach, we focus on such instances in which the maximum number of inclusion-incomparable sets in $\mathcal{X}$, i.e., the antichain size of $\mathcal{X}$, is a parameter.

In an extended view of the problem, we have a permutation group $\Gamma$ over the ground set $M$, and we ask to compute the group $\Delta$ of all permutations $\sigma$ of $\mathcal{X}$ for which there exists $\pi \in \Gamma$ such that $\sigma(X) = \pi(X)$ holds for all $X \in \mathcal{X}$. We call such $\Delta$ the action of $\Gamma$ on $\mathcal{X}$. As the example of $\mathcal{X}$ being the edge set of a graph shows, the computation of $\Delta$ is nontrivial even if we can efficiently compute with the group $\Gamma$. We are here interested in the case that $\Gamma$ is the automorphism group of some graph (then $\mathcal{X}$ are subsets of vertices of this graph, hereafter called marked sets) and, in particular, of an interval graph for which the automorphism group $\Gamma$ can be easily computed [5, 7].

Problem definition and main result

A graph $G$ is an interval graph if the vertex set of $G$ can be mapped into some set of intervals on the real line such that two vertices of $G$ are adjacent if and only if the corresponding intervals intersect.

Problem 1 (Action of the marking-preserving automorphism group of an interval graph).

The problem AUTOMMARKEDINT($G; \mathcal{A}^1, \ldots, \mathcal{A}^m$) is defined as follows:

Input An interval graph $G$, and families $\mathcal{A}^1, \ldots, \mathcal{A}^m$ of nonempty subsets of $V(G)$ such that every set $A \in \mathcal{A}^i$ where $i \in \{1, \ldots, m\}$ induces a clique of $G$. The sets $A \in \mathcal{A}^1 \cup \ldots \cup \mathcal{A}^m$ are called the marked sets of $G$, and specially, $A \in \mathcal{A}^i$ is called a marked set of color $i$.

Task Compute the group $\Gamma$ of such permutations of $\mathcal{A} := \mathcal{A}^1 \cup \ldots \cup \mathcal{A}^m$ that are the actions of the $(\mathcal{A}^1, \ldots, \mathcal{A}^m)$-preserving automorphisms of $G$. In other words, a permutation $\tau$ of $\mathcal{A}$ belongs to $\Gamma$, if and only if there exists an automorphism $\sigma$ of $G$ such that, for every $i \in \{1, \ldots, m\}$ and all $A \in \mathcal{A}^i$, we have $\tau(A) \in \mathcal{A}^i$ and $\sigma(A) = g(A)$.

To stay on the more general side, we consider set families in the multiset setting, meaning that the same set $A$ may occur in a family $\mathcal{A}^i$ multiple times. In regard of the above stated condition, a permutation $\tau$ of $\mathcal{A}$ then obviously has to map $A \in \mathcal{A}$ to a set $\tau(A) = A' \in \mathcal{A}$ such that the multiplicities of $A$ and of $A'$ in each of $\mathcal{A}^1, \ldots, \mathcal{A}^m$ are the same.

Regarding computational complexity, we remark that by ‘computing a group’ we mean to output a set of its generators which is at most polynomially large by [3] (while the permutation group itself is often exponentially large compared to the ground set). It is also important to mention what is the input size of an AUTOMMARKEDINT($G; \mathcal{A}^1, \ldots, \mathcal{A}^m$) instance. If $G$ is an $n$-vertex graph, the number of sets in $\mathcal{A}$ may be up to exponential in $n$. However, considering our parameter $a$ equal to the maximum antichain size of $\mathcal{A}$, we easily get that there are at most $an$ distinct sets in $\mathcal{A}$ (at most $a$ of each cardinality between 1 and $n$). Hence we can always upper-bound the input size by $(|V(G)| + |E(G)| + \sum_{A \in \mathcal{A}} |A|) \in O(a \cdot |V(G)|^2)$.

Theorem 2. The problem AUTOMMARKEDINT($G; \mathcal{A}^1, \ldots, \mathcal{A}^m$) is solvable in FPT-time with respect to the parameter $a$ which is the maximum antichain size of $\mathcal{A} := \mathcal{A}^1 \cup \ldots \cup \mathcal{A}^m$.

In the course of proving Theorem 2, we combine classical PQ-trees for capturing the internal structure of interval graphs [6, 7] (Section 2), and a colored extension of the algorithm for computing the automorphism group of a set family of bounded antichain size from 2 which is based on another classical tool – Babai’s tower-of-groups procedure [11] (Section 3). The full proof is finished in Section 4.
Motivation of the problem

The problem given by Definition 1 is not artificial, but naturally follows from our recent research of special cases of the isomorphism problem of chordal graphs. In detail, the simplified set-family version of it (with no colors and no underlying graph) is implicitly used and solved in the core of an FPT algorithm for the isomorphism of so-called $S_{d}$-graphs [2]. A complicated extension of the algorithm of [2] was used recently in [3] to solve the isomorphism problem of chordal graphs of bounded leafage also in FPT. Subsequently to that, we have formulated and solved the problem of Definition 1, which is among other things a handy replacement and rigorous improvement over the cumbersome technical part of the algorithm [3].

To informally explain the mentioned use of the algorithm of Theorem 2, we summarize that [3] defines and proves a canonical (i.e., isomorphism-invariant) decomposition procedure of a chordal graph into a collection of interval graphs over clique-cutsets. The condition of bounded leafage of the graph implies that these clique-cutsets in every component of the decomposition have bounded-size antichains by inclusion. Since isomorphism of interval graphs (the decomposed components) is well-understood (Section 2), it then remains to efficiently handle the actions of automorphisms of the components on their clique-cutsets to finish the algorithm for isomorphism of chordal graphs of bounded leafage (see [3] for more details).

Even more informally, the previous can be illustrated by a picture in Figure 1 in which the interval graph represents a linear part of the clique-tree representation of a chordal graph, and the clique-cutsets separate the part from other “branching” parts of the clique-tree.

2 Interval Graphs and PQ-trees

A clique in a graph is a set of its vertices which are pairwise adjacent. A clique is maximal if it cannot be extended by adding another vertex. Interval graphs have linearly many maximal
cliques which can easily be listed in linear time using the simplicial-vertex elimination procedure (a simplicial vertex is one whose neighbors induce a clique).

To recognize and test the isomorphism of interval graphs, Booth and Lueker [6] invented PQ-trees (see Figure 2), ordered rooted trees which have the maximal cliques of an interval graph in its leaves, and every internal node is either one of the following:

- A P-node: the order of its children can be permuted arbitrarily.
- A Q-node: the order of its children can be reversed (but not changed otherwise).

The above permissible reorderings at P- and Q-nodes are called equivalence transformations of a PQ-tree, and the following is well-known:

▶ **Theorem 3** (Booth and Lueker [6]). For every interval graph $G$, one can in linear time construct a PQ-tree $T$, such that the following hold:

- This PQ-tree $T$ of $G$ is unique up to equivalence transformations.
- Every possible interval representation of $G$ corresponds to a PQ-tree $T'$ of $G$ (i.e., one equivalent to $T$). The correspondence is that the linear order of the maximal cliques in the representation of $G$ is the same as the one given by the linear order of the leaves of $T'$.

In particular, the latter point means that every automorphism of $G$ can be represented as an equivalence transformation of a PQ-tree of $G$ (though, not the other way round without further information associated with the tree). One may go further this way. We say that an assignment of PQ-trees to interval graphs is canonical if, whenever we take isomorphic graphs $G \cong G'$ and their canonical PQ-trees $T$ and $T'$, then $T$ and $T'$ are isomorphic respecting the order of the trees (one may say “the same”). The following fact is crucial for us:

▶ **Corollary 4** (Colbourn and Booth [7], noted already in [6]). For every interval graph $G$, one can in linear time compute the automorphism group of $G$ and a PQ-tree of $G$ which is canonical.

The definition of a PQ-tree (of an interval graph) explicitly refers only to the maximal cliques of $G$ and not directly to its vertices, but it will be useful to clearly understand the relation of PQ-tree nodes to the particular vertices of $G$. Every node $p$ of a PQ-tree $T$ of $G$ can be associated with a subgraph of $G$ formed by the union of all cliques of the descendant leaves of $p$ – this subgraph is said to belong to $p$. Then, for a node $p$, we define the inner vertices assigned to $p$ as those vertices of $G$ which belong to $p$ and, if $p$ is not a leaf, they belong to at least two child nodes of $p$, but they do not belong to any node which is not an ancestor or a descendant of $p$. (See the illustration in Figure 2 and further in Figure 4.)

Note that every vertex of $G$ is an inner vertex of precisely one node of its PQ-tree $T$. Moreover, by the ‘consecutive-ones’ property of a PQ-tree, the following holds: if a vertex $v$
of \(G\) is an inner vertex of a node \(p\) of \(T\), and \(v\) also belongs to a son \(p_1\) of \(p\), then \(v\) belongs to every descendant of \(p_1\) in \(T\). This illustrates the unique nature of the node \(p\) (to which \(v\) is an inner vertex) for the vertex \(v\) within the PQ-tree \(T\).

In the case of a P-node, the inner vertices assigned to \(p\) belong to all child nodes of \(p\), but this is generally not true for Q-nodes. We thus additionally define the ranking of inner vertices; this is trivial for P-nodes (all inner vertices of the same rank). For a Q-node \(q\) of \(T\), we index the sons of \(q\) from left to right in a palindromic way (i.e., as 1, 2, 3, 2, 1 or 1, 2, 3, 3, 2, 1 depending on parity), which is invariant upon reversal. The rank of every inner vertex \(w\) assigned to \(q\) is then the multiset of indices of the sons of \(q\) that \(w\) belongs to (obviously, this must be a consecutive section of the index sequence). Observe that since two inner vertices of the same node and of the same rank are in the same collection of maximal cliques of \(G\), they are mutually symmetric in the automorphism group of \(G\).

3 Automorphisms of Set Families of Bounded Antichain Size

In this section, we give the main technical tool of this paper—a procedure efficiently computing the automorphism group of a set family under the assumption of bounded antichain size, which builds on ideas used already in our past paper \([2]\). Here we formulate those ideas in an extended form as the standalone result in Theorem \([7]\). Again, to stay on the more general side, we consider set families in the multiset setting, meaning that the same set may occur in a family multiple times (but this does not pose any additional difficulties in the coming arguments besides having to observe the multiplicity as a label on a set).

Note that the problem of computing the automorphism group of a colored set family is actually a special case of the problem \(\text{AUTOMMARKEDINT}(G; A^1, \ldots, A^m)\), where \(G\) is a clique, but, at the same time, we are going to prove in the next Section 4 that the \(\text{AUTOMMARKEDINT}\) reduces to the problem solved here.

Definition 5 (Automorphism group of a colored set family). The problem \(\text{AUTOMSET}(X; U^1, \ldots, U^m)\) is defined as follows:

Input: For a finite ground set \(X\), a finite set family \(U \subseteq 2^X\) partitioned into \(m \geq 1\) color classes \(U = U^1 \cup \ldots \cup U^m\) (allowing the same set to occur in a family multiple times).

Task: Compute the group \(\Gamma\) of the color-preserving permutations of \(U\) which come from a permutation of the ground set \(X\). Precisely, a permutation \(\tau\) of \(U\) belongs to \(\Gamma\), if and only if there exists a permutation \(\sigma\) of \(X\) such that, for every \(i \in \{1, \ldots, m\}\) and all \(B \in U^i\), we have \(\tau(B) \in U^i\) and \(\tau(B) = \sigma(B)\) (this trivially gives \(|B| = |\sigma(B)|\)). Note that the latter condition also immediately implies that the (possible) multiplicities of \(B\) and of \(\sigma(B)\) in each family \(U^i\) are equal.

The problem \(\text{AUTOMSIMPLESET}(X; U^1, \ldots, U^m)\) is the same as \(\text{AUTOMSET}(X; U^1, \ldots, U^m)\) with the following condition on the input: For every set \(B \in U\), there is exactly one index \(i \in \{1, \ldots, m\}\) such that \(B \in U^i\), and the multiplicity of \(B\) in \(U^i\) equals one.

We again, as with Theorem \([2]\) above, estimate the input size here by the same simple argument. If the maximum antichain size in the family \(U = U^1 \cup \ldots \cup U^m \subseteq 2^X\) equals \(a\), then the input size of an instance of \(\text{AUTOMSET}(X; U^1, \ldots, U^m)\) is at most \(\sum_{U \in \mathcal{U}} |U| \in O(a \cdot |X|^2)\). (Possible multiplicities of sets in the family \(U\) are negligible in this regard since they are encoded as integer labels of the multiple sets.)

We start with a simple observation that will simplify the next Theorem \([4]\).

Proposition 6. The problem \(\text{AUTOMSET}(X; V^1, \ldots, V^m)\) reduces in linear time to the problem \(\text{AUTOMSIMPLESET}(X; U^1, \ldots, U^m)\) for suitable \(U^1, \ldots, U^m\).
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**Proof.** For every set $B \in \mathcal{V} = \mathcal{V}^1 \cup \ldots \cup \mathcal{V}^m$, we record the integer vector $m_B := (b_1, \ldots, b_n)$ where $b_i$ is the multiplicity of $B$ in the family $\mathcal{V}^i$. As noted already in Definition 5, any permutation $\tau$ in the solution of $\text{AUTOMSET}(\mathcal{X}; \mathcal{V}^1, \ldots, \mathcal{V}^m)$ must preserve this multiplicity vector; $m_B = m_{\tau(B)}$. Let $\mathcal{U}$ be the simplification of the (multi)family $\mathcal{V}$, i.e., without repetition of multiple member sets. We hence define a partition $(\mathcal{U}^1, \ldots, \mathcal{U}^m)$ as the partition of $\mathcal{U}$ given by the equality of the vectors $m_B$. Then $\tau$ projects to a permutation in $\text{AUTOMSIMPLESET}(\mathcal{X}; \mathcal{U}^1, \ldots, \mathcal{U}^m)$.

Conversely, for any permutation $\sigma$ in the solution of $\text{AUTOMSIMPLESET}(\mathcal{X}; \mathcal{U}^1, \ldots, \mathcal{U}^m)$, we expand $\sigma$ to permutations $\tau$ of $\mathcal{X}$ as follows. Every functional assignment $B \mapsto \sigma(B)$ is lifted, for each $i \in \{1, \ldots, m\}$, to all bijections of the set (possibly empty) of multiple copies of $B$ in $\mathcal{V}^i$ to the set of multiple copies of $\sigma(B)$ in $\mathcal{V}^i$. Then every such $\tau$ belongs to the solution group of $\text{AUTOMSET}(\mathcal{X}; \mathcal{U}^1, \ldots, \mathcal{U}^m)$.

On the other hand, we remark that in the problem $\text{AUTOMSIMPLESET}(\mathcal{X}; \mathcal{U}^1, \ldots, \mathcal{U}^m)$, the number $m$ of colors is not bounded, and so there is (likely) no easy way to reduce this problem to the uncolored case.

**Theorem 7.** The problem $\text{AUTOMSET}(\mathcal{X}; \mathcal{U}^1, \ldots, \mathcal{U}^m)$ is solvable in FPT-time with respect to the parameter $m$ which is the maximum antichain size of $\mathcal{U}$.

The rest of the section is devoted to the proof of Theorem 7 which, in view of Proposition 6, is sufficient to prove for the problem $\text{AUTOMSIMPLESET}(\mathcal{X}; \mathcal{U}^1, \ldots, \mathcal{U}^m)$.

**Cardinality Venn diagrams.** First of all, we review what Definition 5 specifically the words “there exists a permutation $\sigma$ of $X$”, mean for us. Since it is not much efficient to deal with the many permutations of $X$, we now show a simple fact that it will be enough to observe certain cardinalities to decide the existence of such permutation $\sigma$ as required in the definition. For a set family $\mathcal{U}$, we call a cardinality Venn diagram of $\mathcal{U}$ the integer vector $(\ell_{U_{\mathcal{U}}} : \emptyset \neq U_{\mathcal{U}} \subseteq \mathcal{U})$ such that $\ell_{U_{\mathcal{U}}} := |L_{U_{\mathcal{U}}}|$ where $L_{U_{\mathcal{U}}} = \bigcap_{A \subseteq U_{\mathcal{U}}} A \setminus \bigcup_{B \subseteq U_{\mathcal{U}}} B$. That is, informally, we record the cardinality of every internal cell of the Venn diagram of $\mathcal{U}$. See an illustration in Figure 3.

For $U_{\mathcal{U}} \subseteq \mathcal{U}$, let naturally $\varrho(U_{\mathcal{U}}) = \{ \varrho(B) : B \in U_{\mathcal{U}} \}$. We have easily got:

**Lemma 8 (cf.).** Let $\varrho$ be a permutation of $\mathcal{U}$ over $X$. There exists a permutation $\sigma$ of $X$ such that, for every $B \in \mathcal{U}$, we have $\varrho(B) = \sigma(B)$, if and only if the cardinality Venn diagrams of $\mathcal{U}$ and of $\varrho(\mathcal{U})$ are the same (equal), meaning that $\ell_{U_{\mathcal{U}}} = \ell_{\varrho(U_{\mathcal{U}})}$ for all $\emptyset \neq U_{\mathcal{U}} \subseteq \mathcal{U}$.

Furthermore, for $\mathcal{U}' \subseteq \mathcal{U}$ such that $\varrho(\mathcal{U}') = \mathcal{U}'$, one can in $O(|X| + \sum_{U \in \mathcal{U}} |U|)$ time test the condition as above, i.e., whether the equalities $\ell_{U_{\mathcal{U}}} = \ell_{\varrho(U_{\mathcal{U}})}$ hold for all $\emptyset \neq U_{\mathcal{U}} \subseteq \mathcal{U}'$.

**Proof.** $\Rightarrow$ Suppose that there exists such a permutation $\sigma$ of $X$. Then, for all $\emptyset \neq U_{\mathcal{U}} \subseteq \mathcal{U}$, every element of $L_{U_{\mathcal{U}}}$ is mapped by $\sigma$ into $L_{\varrho(U_{\mathcal{U}})}$, and so the claim follows since $\sigma$ is a permutation.

$\Leftarrow$ For any $\emptyset \neq U_{\mathcal{U}} \subseteq \mathcal{U}$, we have that $|L_{U_{\mathcal{U}}}| = |L_{\varrho(U_{\mathcal{U}})}|$ which implies an existence of a bijection from $L_{U_{\mathcal{U}}}$ to $L_{\varrho(U_{\mathcal{U}})}$. Since $L_{U_{\mathcal{U}}} \cap L_{\varrho(U_{\mathcal{U}})} = \emptyset$ for $U_{\mathcal{U}} \neq \mathcal{U}$, the composition of these bijections is sound and results in a permutation $\sigma$ of $X$. Picking any $B \in \mathcal{U}$, we have $B = \bigcup \{ L_{U_{\mathcal{U}}} : \emptyset \neq U_{\mathcal{U}} \subseteq \mathcal{U} \}$, and hence $\sigma(B) = \bigcup \{ \sigma(L_{U_{\mathcal{U}}} : \emptyset \neq U_{\mathcal{U}} \subseteq \mathcal{U} \} = \bigcup \{ L_{U_{\mathcal{U}}} : \emptyset \neq U_{\mathcal{U}} \subseteq \mathcal{U} \} = \varrho(B)$.

As for testing the condition $\ell_{U_{\mathcal{U}}} = \ell_{\varrho(U_{\mathcal{U}})}$ hold for all $\emptyset \neq U_{\mathcal{U}} \subseteq \mathcal{U}'$, we loop through all elements $x \in X$, and for each $x$ we record in $O(n)$ time to which of the sets in $\mathcal{U}'$ this $x$ belongs to. Summing the obtained records at the end precisely gives the $O(|X|)$ nonzero.
Figure 3 An illustration of the concept of a cardinality Venn diagram, and of Lemma 8. We have 4 sets $A, B, C, D$, and the ground set elements are depicted with the dots. The numbers in the cells of the diagram are the cardinalities of these cells. One can check, using the depicted cardinalities, that there exists a permutation of the ground set which permutes our sets $(A, B, C, D)$ into $(D, B, C, A)$, but there is no such permutation permuting $(A, B, C, D)$ into $(A, C, B, D)$.

Our strategy for the proof of Theorem 7 is as follows.

- Let, for $n = |X|$ and every $j \in \{1, \ldots, m\}$, $\mathcal{U}^j = \bigcup_{c=1}^n \mathcal{U}_c^j$ be a partition of $\mathcal{U}^j$ into subfamilies of sets of cardinality $c \in \{1, \ldots, n\}$. We make the initial group $\Gamma''$ (of permutations of $\mathcal{U} = \bigcup_{j=1}^m \mathcal{U}^j$) as the direct product of the symmetric groups (those of all permutations) on all nonempty families $\mathcal{U}_c^j$ over $j \in \{1, \ldots, m\}$ and $c \in \{1, \ldots, n\}$.

- We compute the subgroup $\Gamma' \subseteq \Gamma''$ of those permutations which fulfill the condition $\ell_{\mathcal{U}', \mathcal{U}_c^j} = \ell_{\mathcal{U}', \mathcal{U}_c^j\varrho}$ for all $\emptyset \neq \mathcal{U}_c^j \subseteq \mathcal{U}'$ of Lemma 8. Then $\Gamma'$ will be, by Lemma 8, the solution to the $\text{AUTOMSIMPLESET}(X; \mathcal{U}^1, \ldots, \mathcal{U}^m)$ problem. At this point it will be quite important that each of the subfamilies $\mathcal{U}_c^j$ (as an antichain) has bounded cardinality.

The latter point, however, is not an easy task, and we will employ classical Babai’s tower-of-groups procedure to “gradually refine” $\Gamma'$ into $\Gamma$, ensuring that the conditions of Lemma 8 hold, in a sense, for more and more combinations of the subfamilies $\mathcal{U}_c^j$ until all are satisfied.

Before getting into the group-computing tools, we introduce one more technical result which will be crucial in the gradual refinement of $\Gamma'$ into $\Gamma$. In the setting of Lemma 8, we say that $\mathcal{U}' \subseteq \mathcal{U}$ is $\text{Venn-good}$ with $\varrho$ if $\ell_{\mathcal{U}', \mathcal{U}_c^j} = \ell_{\mathcal{U}', \mathcal{U}_c^j\varrho}$ holds true for all $\emptyset \neq \mathcal{U}_c^j \subseteq \mathcal{U}'$, and we call such $\mathcal{U}_c^j$ a witness (of $\mathcal{U}'$ not being Venn-good) if $\ell_{\mathcal{U}', \mathcal{U}_c^j} \neq \ell_{\mathcal{U}', \mathcal{U}_c^j\varrho}$.

Lemma 9 ([2]). Let $\varrho$ be a permutation of a set family $\mathcal{U}$, and $\mathcal{U}' \subseteq \mathcal{U}$ be such that $\varrho(\mathcal{U}') = \mathcal{U}'$. If $\mathcal{U}'$ is not $\text{Venn-good}$ with $\varrho$, then there exist $\mathcal{U}_2, \mathcal{U}_3 \subseteq \mathcal{U}'$ such that $|\mathcal{U}_2| \leq 2$ or $\mathcal{U}_2$ is an antichain in the inclusion, $\emptyset \neq \mathcal{U}_3 \subseteq \mathcal{U}_2$ and $\ell_{\mathcal{U}_2, \mathcal{U}_3} \neq \ell_{\mathcal{U}_2, \mathcal{U}_3\varrho}$ (not Venn-good).

Proof. Choose $\mathcal{U}_2 \subseteq \mathcal{U}'$ such that $\mathcal{U}_2$ is not $\text{Venn-good}$ with $\varrho$ and it is minimal such by inclusion, and assume (for a contradiction) that there are $\mathcal{A}_1, \mathcal{A}_2 \in \mathcal{U}_2$ such that $\mathcal{A}_1 \subseteq \mathcal{A}_2$. If $\varrho(\mathcal{A}_1) \subseteq \varrho(\mathcal{A}_2)$, then already $\mathcal{U}_2 := \{\mathcal{A}_1, \mathcal{A}_2\}$ is not $\text{Venn-good}$ (with a witness $\{\mathcal{A}_1\}$), and so let $\varrho(\mathcal{A}_1) \subset \varrho(\mathcal{A}_2)$. Let $\mathcal{U}_3$ be a witness of $\mathcal{U}_2$ not being $\text{Venn-good}$, and for $j = 2, 3$ denote: $\mathcal{V}_0^j := \mathcal{U}_j \setminus \{\mathcal{A}_1, \mathcal{A}_2\}$, $\mathcal{V}_1^j := (\mathcal{U}_j \cup \{\mathcal{A}_1\}) \setminus \{\mathcal{A}_2\}$, $\mathcal{V}_2^j := (\mathcal{U}_j \cup \{\mathcal{A}_2\}) \setminus \{\mathcal{A}_1\}$, $\mathcal{V}_3^j := \mathcal{U}_j \cup \{\mathcal{A}_1, \mathcal{A}_2\}$. \hfill $\Box$
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Since $A_1 \subseteq A_2$ and $\varrho(A_1) \subseteq \varrho(A_2)$ (and so $I_{U_2,v_3^2} = \emptyset$), we easily derive
\[
\ell_{U_2,v_3^2} = 0 = \ell_{\varrho(U_2),\varrho(v_3^2)},
\]
and since, by our minimality assumption, all three subfamilies $V_2^0$, $V_3^1$ and $V_3^2$ are Venn-good,
\[
\ell_{U_2,v_3^2} = \ell_{U_2 \setminus \{A_1\},v_3^2} = \ell_{V_3^2,v_3^2} = \ell_{\varrho(v_3^2),\varrho(v_3^2)} = \ell_{\varrho(U_2 \setminus \{A_1\}),\varrho(v_3^2)} = \ell_{\varrho(U_2),\varrho(v_3^2)},
\]
\[
\ell_{U_2,v_3^1} = \ell_{U_2 \setminus \{A_2\},v_3^1} = \ell_{V_3^1,v_3^1} = \ell_{\varrho(v_3^1),\varrho(v_3^1)} = \ell_{\varrho(U_2 \setminus \{A_2\}),\varrho(v_3^1)} = \ell_{\varrho(U_2),\varrho(v_3^1)}.
\]
Then, using the previous equalities, and the trivial observation $\ell_{V_2^2,v_3^2} = \ell_{U_2,v_3^2} + \ell_{U_2,v_3^1} + \ell_{U_2,v_3^1} + \ell_{U_2,v_3^1}$ with the analogous equality under $\varrho$, we conclude
\[
\ell_{U_2,v_3^2} = \ell_{V_2^0,v_3^0} - \ell_{U_2,v_3^1} - 0 - \ell_{U_2,v_3^1} = \ell_{\varrho(v_2^0),\varrho(v_3^0)} - \ell_{\varrho(U_2),\varrho(v_3^1)} - \ell_{\varrho(U_2),\varrho(v_3^2)} = \ell_{\varrho(U_2),\varrho(v_3^2)}.
\]
However, $U_2 \in \{V_3^0, V_3^1, V_3^2, V_3^3\}$, and so one of the latter four derived equalities contradicts the assumption that $U_2$ witnessed $U_2$ not being Venn-good with $\varrho$. ▶

Since we deal with set families of bounded-size antichains, Lemma 9 essentially tells us that either a family is Venn-good, or it contains a “small” subfamily not being Venn-good.

Babai’s tower-of-groups [4]. The famous classical paper of Babai [4] is not just a standalone algorithm, but more an outline of how to efficiently compute a subgroup $\Gamma$ of a given group $\Gamma'$ in a setting in which the conditions defining the subgroup $\Gamma$ can be stepwise refined in sufficiently small steps. Here, by “computing a group” we mean to output a set of its generators which is at most polynomially large by [8]. Note that this task, in general, cannot be done directly by processing all members (even though we have an efficient membership test at hand) of the group $\Gamma'$ which can be exponentially large.

Babai’s “tower-of-groups” approach informally works as follows: we iteratively compute a chain of subgroups $\Gamma = \Gamma_0 \supseteq \Gamma_1 \supseteq \ldots \supseteq \Gamma_h = \Gamma$, where each $\Gamma_{i+1}$ consists of those members of $\Gamma_i$ which satisfy a suitably chosen additional condition, until $\Gamma_h = \Gamma$ satisfies all the defining conditions, and so it is the desired outcome. The important ingredient which makes this procedure work efficiently is that the ratio of orders (sizes) of consequent groups $\Gamma_i$ and $\Gamma_{i+1}$ in the chain is always bounded. The number $h$ of steps should also not be too large. In this setting, each of the refinement steps can be done using another classical result:

Theorem 10 (Furst, Hopcroft and Luks [8], Cor. 1]). Let $\Pi$ be a permutation group given by its generators, and $\Pi_1$ be any subgroup of $\Pi$ such that one can test in polynomial time whether $\pi \in \Pi_1$ for any $\pi \in \Pi$ (membership test). If the ratio $|\Pi|/|\Pi_1|$ is bounded by a function of a parameter $d$, then a set of generators of $\Pi_1$ can be computed in FPT-time (with respect to $d$).

The collected ingredients show a clear road to computing the subgroup $\Gamma \subseteq \Gamma'$ from the above outlined proof strategy for Theorem 10. In every intermediate step of a chain $\Gamma = \Gamma_0 \supseteq \Gamma_1 \supseteq \ldots \supseteq \Gamma_h = \Gamma$, we assume that the whole family $U$ is not Venn-good (with some generator of $\Gamma_1$), and we use Lemma 9 to argue that there exists a small subfamily $U_2$ of $U$ which is not Venn-good with some generator of $\Gamma_{i-1}$. Such $U_2$ can be found efficiently with a little trick (note that a brute-force approach would not give an FPT-time algorithm here), and then we define and compute the next group $\Gamma_i$ as that of permutations for which $U_2$ is Venn-good with Theorem 10. In this way, we eventually arrive at the group $\Gamma$ such that $U$ is Venn-good with every generator of $\Gamma$. The details follow.
Algorithm 1 One (i-th) step of the computation of the subgroup $\Gamma \subseteq \Gamma'$

Require: a set family $\mathcal{U}$ of maximum antichain size $a \in \mathbb{N}$, a partition $\mathcal{W} \subseteq 2^\mathcal{U}$ of $\mathcal{U}$ into parts of size $\leq a$, and a group $\Gamma_{i-1}$ (via a generator set) of permutations of $\mathcal{U}$ which set-wise stabilizes every part in $\mathcal{W}$.

Ensure: either a certificate that $\mathcal{U}$ is Venn-good with every generator of $\Gamma_{i-1}$; or a subfamily $\mathcal{W} \subseteq \Gamma_{i-1}$ (via a generator set) such that, for $\mathcal{T} \subseteq \mathcal{U}$ which is the union of some at most $a$ parts of $\mathcal{W}$, $\mathcal{T}$ is Venn-good precisely with every member of $\Gamma_{i}$ (and not with members of $\Gamma_{i-1} \setminus \Gamma_{i}$).

1: Let $\mathcal{W} = \{\mathcal{W}_1, \mathcal{W}_2, \ldots, \mathcal{W}_k\}$
2: $\mathcal{T} \leftarrow \emptyset$
3: repeat for $p := 1, 2, \ldots, a$:
   4:    repeat for $q := 1, 2, \ldots, k + 1$:
      5:      if $q > k$ then return “$\mathcal{U}$ is Venn-good with all generators of $\Gamma_{i-1}$”;
      6:      $\mathcal{T}_1 \leftarrow (\mathcal{W}_1 \cup \mathcal{W}_2 \cup \cdots \cup \mathcal{W}_q) \cup \mathcal{T}$;
      7:      until $\mathcal{T}_1$ is not Venn-good (cf. Lemma 8) with some generator of $\Gamma_{i-1}$;
      8:      $\mathcal{T}_p \leftarrow \mathcal{W}_{1:p} \cup \mathcal{W}_{2:p} \cup \cdots \cup \mathcal{W}_{q:p}$;
      9:    until $\mathcal{T}_p = 1$ or $p = a$;
   10: Call the algorithm of Theorem 10 to compute the subgroup $\Gamma_i \subseteq \Gamma_{i-1}$, such that the membership test of $\varrho \in \Gamma_i$ checks whether $\mathcal{T}$ is Venn-good with $\varrho$ (again Lemma 8);
11: return $\Gamma_i$.

Proof of Theorem 7. Recall that, for an instance of AUTOMSIMPLESET($X; \mathcal{U}^1, \ldots, \mathcal{U}^m$) over an $n$-element set $X$ and for every $j \in \{1, \ldots, m\}$, we have defined a refined partition $\mathcal{U}^j = \bigcup_{c=1}^{n} \mathcal{U}^j_c$ where $\mathcal{U}^j_c$ consists of the sets from $\mathcal{U}^j$ of cardinality $c$. Since the maximum antichain size of $\mathcal{U}$ is $a$, we have that each $\mathcal{U}^j_c$ contains at most $a$ distinct sets. For simplicity, we may assume $a \geq 2$ since the case of $a = 1$ is trivial. Let $\mathcal{W} := \{\mathcal{U}^j_c : 1 \leq j \leq m, 1 \leq c \leq n, \mathcal{U}^j_c \neq \emptyset\}$ be a system of all these nonempty families.

To solve the given instance, we start with the group $\Gamma' = \Gamma_0$ of permutations of $\mathcal{U}$ which is the direct product of the symmetric groups on all parts of $\mathcal{W}$. For $i := 1, 2, \ldots$, we iteratively call Algorithm 1 with $\mathcal{U}$, $\mathcal{W}$ and $\Gamma_{i-1}$, until the outcome (in $h$-th step) is that $\mathcal{U}$ is Venn-good with all generators of $\Gamma_h$. Note that the latter immediately gives that $\mathcal{U}$ is Venn-good with all permutations in $\Gamma_h$. By the assurance of Algorithm 1, $\Gamma = \Gamma_h$ contains all permutations for which $\mathcal{U}$ is Venn-good and indeed is the desired solution of AUTOMSIMPLESET($X; \mathcal{U}^1, \ldots, \mathcal{U}^m$).

So, it remains to finish two things; analyze and prove one call to Algorithm 1 and prove that the number $h$ of steps is finite and not “too large”. We start with the former. If $\mathcal{U}$ is Venn-good for every generator of $\Gamma_{i-1}$, then we find this already in the first iteration of $p = 1$, on line 10. Hence we may further assume that $\mathcal{U}$ is not Venn-good. By Lemma 8 there exists a subfamily $\mathcal{U}_2 \subseteq \mathcal{U}$ which is also not Venn-good, and $|\mathcal{U}_2| \leq \max(a, 2) < a$ since $\mathcal{U}_2$ is an antichain. Note that $\mathcal{U}_2$ thus intersects at most $a$ parts of $\mathcal{W}$, which implies that there exist at most $a$ parts of $\mathcal{W}$ whose union is not Venn-good.

Let $k \geq j'_1 > j'_2 > \cdots > j'_r \geq 1$ be an index sequence of length $r \leq a$ such that the subfamily $\mathcal{W}_{j'_1} \cup \mathcal{W}_{j'_2} \cup \cdots \cup \mathcal{W}_{j'_r}$ is not Venn-good with some generator of $\Gamma_{i-1}$, and the vector $(j'_1, j'_2, \ldots, j'_r)$ is lexicographically minimal of these properties. Then one can straightforwardly verify that $(j'_1, j'_2, \ldots, j'_r)$ is a prefix of (or equal to) the vector $(j_1, j_2, \ldots, j_p)$ computed by Algorithm 1. Consequently, the collection $\mathcal{T}$, when leaving the cycle on line 10, is not
Venn-good with some generator of $\Gamma_{i-1}$. We look at the subset $\Gamma_i \subseteq \Gamma_{i-1}$ defined as on line 11. In particular, $\Gamma_i \subseteq \Gamma_{i-1}$. The important point is that, as $\Gamma_{i-1}$ set-wise stabilizes every part of $\mathcal{V}$, $\Gamma_i$ is closed under composition of permutations, and so it forms a subgroup as expected by the algorithm.

Next, we verify the fulfillment of the assumptions of Theorem 10. Generators of $\Gamma_{i-1} = \Pi$ have been given to Algorithm 1. The ratio $|\Pi|/|\Pi_1|$, where $\Pi_1 = \Gamma_i$ in our case, can be bounded as follows (despite we do not know $\Gamma_i$ yet): by standard algebraic arguments, $|\Gamma_{i-1}|/|\Gamma_i|$ equals the number of distinct cosets of the subgroup $\Gamma_i$ in $\Gamma_{i-1}$. If we consider two automorphisms $\alpha, \beta \in \Gamma_{i-1}$ which are equal when restricted to $\mathcal{T}$ (recall that it set-wise stabilize $\mathcal{T}$), then the automorphism $\alpha^{-1}\beta$ determines a permutation of $\mathcal{U}$ which is identical on $\mathcal{T}$ (so it is Venn-good with $\alpha^{-1}\beta$), and hence $\alpha^{-1}\beta \in \Gamma_i$. The latter means that $\alpha$ and $\beta$ belong to the same coset of $\Gamma_i$, and consequently, the number of distinct cosets is at most the number of distinct subpermutations on $\mathcal{T}$ possibly induced by $\Gamma_{i-1}$, that is at most $(\alpha\beta)^n$ (at most the symmetric group on each of at most $a$ parts of $\mathcal{V}$ which form $\mathcal{T}$). Therefore, we can finish one iteration on line 11 in FPT-time with respect to $d = a$ by Theorem 10.

Lastly, we estimate the number of steps $h$ in the refinement process $\mathcal{T}' = \mathcal{T}_0 \supseteq \mathcal{T}_1 \supseteq \ldots \supseteq \mathcal{T}_h = \mathcal{T}$. By Lagrange’s group theorem, $|\Gamma_i|$ divides $|\Gamma_{i-1}|$, and since $\Gamma_{i-1} \neq \Gamma_i$, we have $|\Gamma_i| \leq \frac{1}{2}|\Gamma_{i-1}|$. Hence the number of strict refinement steps in our chain of subgroups is $h \leq \log_2 |\Gamma_i|$. Since trivially $|\Gamma_i| \leq (\alpha\beta)^n$ (where $n = |\mathcal{X}|$), we get $h = O(na \log a)$. Therefore, the overall computation of the resulting group $\Gamma$ of the problem instance of AUTOMSIMPLESET$(X; \mathcal{U}_1, \ldots, \mathcal{U}_m)$ is finished in FPT-time with respect to the parameter $a$, the maximum antichain size of the set family $\mathcal{U}$. We remark that the only steps in our algorithm which require FPT-time (i.e., are possibly not of polynomial-time) are the calls to the algorithm of Theorem 10.

4 Handling PQ-trees of marked interval graphs

In this section, we provide the proof – an FPT algorithm, for Theorem 2. For this purpose, we revisit the PQ-trees of Section 2 from the point of view of marked interval graphs of Definition 1. While efficient handling of PQ-trees with an arbitrary marking seems basically infeasible, we deal with the special assumption of bounded antichains which appears very helpful within PQ-trees. Informally, we can say that the marked sets under this assumptions affect only very small part of the whole PQ-tree of our marked graph $G$.

Recall that we have got an interval graph $G$, families $\mathcal{A}_1, \ldots, \mathcal{A}_m$ of nonempty (marked) subsets of $V(G)$ such that every set $A \in \mathcal{A}_i$ where $i \in \{1, \ldots, m\}$ induces a clique of $G$, and $\mathcal{A} := \mathcal{A}_1 \cup \ldots \cup \mathcal{A}_m$. Let the maximum antichain size in $\mathcal{A}$ be $a$. Let $T$ be a PQ-tree of $G$, and recall what are inner vertices of $G$ assigned to nodes of $T$ and their rank from Section 2. We call a node $p$ of $T$ clean if the inner vertices assigned to $p$ are disjoint from $\bigcup \mathcal{A}$ (that is, not belonging to any marked set). The subtree rooted at $p$ is then clean if $p$ and all descendants of $p$ in $T$ are clean. It can be shown that the number of “incomparable” non-clean subtrees is bounded by the antichain size, but we skip the partial details since we actually prove much more in Lemma 11. See an illustration in Figure 4.

Our answer to handling PQ-trees with marked sets is to precompute the complete isomorphism types of all clean subtrees in $T$, and then to introduce new marked sets (in addition to $\mathcal{A}$) which encode the (few) non-clean subtrees of $T$. Importantly, the latter can be done without increasing the maximum antichain size of the marked sets. Altogether, this is formulated in detail as follows:

$\blacktriangleright$ **Lemma 11.** Assume an instance of the problem AUTOMMARKEDINT$(G; \mathcal{A}_1, \ldots, \mathcal{A}_m),$
where the maximum antichain size of $\mathcal{A} := \mathcal{A}^1 \cup \ldots \cup \mathcal{A}^m$ equals $a$. Then there are families $\mathcal{B}^1 \cup \ldots \cup \mathcal{B}^k = \mathcal{A}$ of subsets of $V(G)$ where $(\mathcal{B}^1, \ldots, \mathcal{B}^k)$ is a partition of $\mathcal{A}$ refining $(\mathcal{A}^1, \ldots, \mathcal{A}^m)$, and another family $\mathcal{C} \subseteq 2^V(G)$ of nonempty subsets of vertices where $\mathcal{C}$ is partitioned into families $(\mathcal{C}^1, \ldots, \mathcal{C}^\ell)$, such that the following holds. If a group $\Gamma$ is the solution to the problem \textsc{AutomSet}(V(G); $\mathcal{B}^1, \ldots, \mathcal{B}^k$, $\mathcal{C}^1, \ldots, \mathcal{C}^\ell$), then $\Gamma$ restricted to $\mathcal{A}$ is the sought solution of \textsc{AutomMarkedInt}(G; $\mathcal{A}^1, \ldots, \mathcal{A}^m$). Furthermore, the maximum antichain size in $\mathcal{A} \cup \mathcal{C}^1 \cup \ldots \cup \mathcal{C}^\ell$ is also $a$, and the families $\mathcal{B}^1, \ldots, \mathcal{B}^k$ and $\mathcal{C}^1, \ldots, \mathcal{C}^\ell$ can be computed in linear time.

Proof. Let $T$ be a PQ-tree of our interval graph $G$. We first show how to refine the partition $(\mathcal{A}^1, \ldots, \mathcal{A}^m)$ of $\mathcal{A}$, using an auxiliary annotation assigned to the sets of $\mathcal{A}$. Observe that since every set $A \in \mathcal{A}$ induces a clique in $G$, the set $A$ cannot intersect the inner vertices of two nodes of $T$ which are incomparable in the tree order. This holds since $A$ must be a subset of some maximal clique of $G$, and so if $A$ intersects some inner vertex assigned to a node $q$, the max clique containing $A$ must be among the descendant leaves of $q$ in $T$ (it may be useful to imagine this fact in an actual interval representation of $G$).

Consequently, the nodes with assigned inner vertices from $A \in \mathcal{A}$ lie on a root-to-leaf
Therefore, with the implied presentation of the (unlabelled) graph is an automorphism, rooted tree, but also the order of every Q-node up to reversal. As noted above, since maximal cliques of \(G\) which, in turn, gives an automorphism \(\beta'\) of the PQ-tree \(T\). Note that in this context, naturally, an automorphism of a PQ-tree not only preserves the underlying rooted tree, but also the order of every Q-node up to reversal. As noted above, since \(\beta\) is an automorphism, \(\alpha\) must preserve the annotations of the sets of \(A\), i.e., the partition \((B^1,\ldots,B^k)\). Since clean subtrees are also preserved by \(\beta'\), the restriction of \(\beta'\) gives an
annotation-preserving automorphism of the tree $T'$ by the definition of a canonical PQ-tree. 
Hence $\beta'$ induces a permutation of $C$ respecting the partition $(C_1, \ldots, C\ell)$, which composed 
with $\alpha$ on $A$ gives a unique permutation $\gamma \in \Gamma$.

Conversely, let $\gamma \in \Gamma$ be a permutation on $A \cup C$ respecting both partitions $(B^1, \ldots, B^\ell)$ 
and $(C_1, \ldots, C\ell)$. In particular, every set $C_q \in C^i \subseteq C$, where $q \in V(T')$ and $1 \leq i \leq \ell$, 
is mapped into $C_r = \gamma(C_q) \in C^i$ where $r \in V(T')$. Henceforth, $\gamma$ induces (as $q \mapsto r'$) an 
annotation-preserving permutation $\beta_0$ of $V(T')$. Since we have, for any two nodes $p, p' \in V(T')$, that $p$ is an ancestor of $p'$ if and only if $C_p \supseteq C_{p'}$, and $\gamma$ preserves the inclusion 
relation, we conclude that $\beta_0$ is an annotation-preserving automorphism of $T'$. Since our 
annotation at every node of $T'$ includes the canonical PQ-tree of the clean subtrees, and the 
ordering of the non-clean subtrees under Q-nodes, $\beta_0$ extends to a permutation $\beta_1$ of whole 
$V(T)$ which hence is an automorphism of the whole PQ-tree $T$. This gives an underlying 
automorphism $\beta$ of the graph $G$. Since, moreover, $\gamma$ preserves the annotations of the sets in $A$, the automorphism $\beta$ can be chosen such that it agrees with the permutation $\gamma$ on $A$. We 
have got the restriction of $\gamma$ to $A$ in the group $\Delta$.

It remains to analyze the runtime of the described reduction, i.e., of the computation of the 
families $B^1, \ldots, B^\ell$ and $C_1, \ldots, C\ell$. For that we first compute a PQ-tree $T$ of our interval 
graph $G$ in linear time by Theorem 3. From $T$, we straightforwardly compute the annotations 
of the sets in $A$, as specified above. Then we easily in linear time identify all clean subtrees 
of $T$, and hence get the tree $T'$. The annotations of the nodes of $T'$ are computed again in 
linear time using Corollary 4 applied to their clean subtrees, and since the considered clean 
subtrees in this computation are pairwise disjoint, the overall runtime is linear in the size of $G$. 
Knowing the annotations, we then easily output the families $B^1, \ldots, B^\ell$ and $C_1, \ldots, C\ell$. □

Proof of Theorem 2. We first apply Lemma 11 in this way, we transform an instance of 
$\text{AUTOMMARKEDINT}(G; A^1, \ldots, A^m)$ into an instance of $\text{AUTOMSET}(V(G); B^1, \ldots, B^\ell,$ 
$C_1, \ldots, C\ell)$ of the same parameter value $a$. Then, we use Theorem 7 to solve the latter 
in FPT time with respect to $a$, and straightforwardly output the corresponding restricted 
solution of $\text{AUTOMMARKEDINT}(G; A^1, \ldots, A^m)$. □

5 Conclusions

We have introduced the problem $\text{AUTOMMARKEDINT}(G; A^1, \ldots, A^m)$ which is a clean and 
rigorous new general formulation of an algorithmic task previously used in isomorphism 
algorithms for special classes of chordal graphs [2, 4]. We believe that this self-contained 
exposition of the solution of $\text{AUTOMMARKEDINT}(G; A^1, \ldots, A^m)$ can be interesting and 
useful on its own, not only as a minor technical tool. For instance, we think it can be useful 
development of an isomorphism algorithm for so-called $H$-graphs (these are intersection 
graphs of connected subgraphs in a suitable subdivision of the base graph $H$, and they 
naturally generalize interval graphs and chordal graphs of bounded leafage) in the case that 
$H$ contains one cycle (while for $H$ containing more than one cycle the problem is known to 
be GI-complete).

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