Null-orbit reflexive operators

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Abstract

We introduce and study the notion of null-orbit reflexivity, which is a slight perturbation of the notion of orbit-reflexivity. Positive results for orbit reflexivity and the recent notion of $C$-orbit reflexivity both extend to null-orbit reflexivity. Of the two known examples of operators that are not orbit-reflexive, one is null-orbit reflexive and the other is not. The class of null-orbit reflexive operators includes the classes of hyponormal, algebraic, compact, strictly block-upper (lower) triangular operators, and operators whose spectral radius is not 1. We also prove that every polynomially bounded operator on a Hilbert space is both orbit-reflexive and null-orbit reflexive.

1 Introduction

In a recent paper [11] the authors and M. McHugh introduced a new notion of reflexivity for operators, $C$-orbit reflexivity as well as its linear-algebraic analogue. This notion is related to the notion of orbit reflexivity [5]. Examples of Hilbert space operators that are not orbit reflexive can be found in two very remarkable papers; the first example was given by S. Grivaux and M. Roginskaya [1], and the second, much simpler, example was given by V. Müller and J. Vršovský [11].

Although even in finite-dimensions there is an ample supply of operators that are not $C$-orbit reflexive, it was easy to show that operators that are strictly block-upper(or lower)-triangular are $C$-orbit reflexive. This fact combined with the example of a non-orbit-reflexive operator in [11], led us naturally to a new version of orbit reflexivity, null-orbit reflexivity, that includes all of the previously-proved orbit-reflexive operators but excludes the counterexample in [11].
Suppose $T$ is a linear transformation on a vector space. We define the null-orbit of $T$ as

$$\text{nullOrb}(T) = \{0, 1, T, T^2, \ldots\}.$$

The orbit of $T$ is $\text{Orb}(T) = \{1, T, T^2, \ldots\}$. We define $\text{nullOrb}_0(T)$ to be the set of all linear transformations $S$ such that for every vector $x$

$$Sx \in \text{null-Orb}(T)x$$

and we say that $T$ is algebraically null-orbit reflexive if

$$\text{nullOrb}_0(T) = \text{nullOrb}(T).$$

If $T$ is a bounded operator on a Banach space, we define $\text{nullOrb}_0(T)$ to be the set of all operators $S$ such that, for every vector $x$

$$Sx \in [\text{nullOrb}(T)x]^-,$$

and we say that $T$ is null-orbit reflexive if $\text{nullOrb}_0(T)$ is the strong-operator closure of $\text{nullOrb}(T)$. Orbit reflexivity is defined as in the above definition replacing $\text{nullOrb}(T)$ with $\text{Orb}(T)$. The slight change in definitions causes drastic changes in the two notions.

In this paper we extend all of the positive known results for orbit reflexivity to null-orbit reflexivity, and we show that most of the positive results for $C^*$-orbit reflexivity extend to null orbit reflexivity. Moreover, for the example in [11] of a Hilbert space operator $T$, that is not orbit reflexive, we show that $T$ is null-orbit reflexive. In the example in [11] of a Hilbert space operator that is not orbit reflexive, the proof shows that the operator is also not null-orbit reflexive.

We first prove a number of results in the purely algebraic case, and we use these to prove several results for operators on a normed space or a Hilbert space. We next extend the results of [8] and [11] to the null-orbit reflexivity case. We finish with a new result that every polynomially bounded operator on a Hilbert space is both orbit-reflexive and null-orbit reflexive.

Suppose $X$ is a normed space and $A$ is an algebra of (bounded linear) operators on $X$. A (closed linear) subspace $M$ of $X$ is $A$-invariant if $A(M) \subseteq M$ for every $A \in A$. We let $\text{Lat}A$ denote the set of all invariant subspaces for $A$, and we let $\text{AlgLat}A$ denote the algebra of all operators that leave invariant every $A$-invariant subspace. The algebra $A$ is reflexive if $A = \text{AlgLat}A$. If the algebra $A$ contains the identity operator $I$, then $S \in \text{AlgLat}A$ if and only if, for every $x \in X$, $Sx$ is in the closure of $Ax$. This characterization works equally well for a linear subspace $S$ of $B(X)$ (the set of all operators on $X$), i.e., we define $\text{ref}S$ to be the set of all operators $A$ such that, for every $x \in X$, we have $Ax$ is in the closure of $Sx$, and we say that $S$ is reflexive if $S = \text{ref}S$. If we let $T$ be a single operator and let $S = \text{Orb}(T) = \{T^n : n \geq 0\}$, we apply the same process to obtain the notion of orbit reflexivity. (Note that in this case $S$ is not a linear space.) We define $\text{OrbRef}(T)$ to be the set of all operators $A$ such that, for every vector $x$, we have $Ax$ is in the closure of $\text{Orb}(T)x$. We say that $T$ is orbit reflexive if $\text{OrbRef}(T)$ is the closure of $\text{Orb}(T)$ in the strong operator topology (SOT).
2 Algebraic Results

Throughout this section $\mathbb{F}$ will denote an arbitrary field, $X$ will denote a vector space over $\mathbb{F}$, and $L(X)$ will denote the algebra of all linear transformations on $X$.

A transformation $T \in L(X)$ is locally nilpotent if $X = \cup_{n \geq 1} \ker(T^n)$. More generally, $T$ is locally algebraic if, for each $x \in X$, there is a nonzero polynomial $p_x \in \mathbb{F}[t]$ such that $p_x(T)x = 0$. If $p_x(t)$ is chosen to be monic with minimal degree, we call $p_x$ a local polynomial for $T$ at $x$.

**Theorem 1** Every locally nilpotent linear transformation on a vector space $X$ over field $\mathbb{F}$ is algebraically null-orbit reflexive. Moreover, if $S \in \text{nullOrbRef}_0(T)$, $x \in X$, and $Sx = T^k x \neq 0$, then $S = T^k$.

**Proof.** We know from [3, Theorem 1] that $T$ is algebraically $\mathbb{F}$-orbit reflexive. Thus if $S \in \text{nullOrbRef}_0(T)$ and $S \neq 0$, then there is an $x \in X$ and an integer $n \geq 0$ such that $Sx = T^n x \neq 0$, and it follows from [3, Theorem 1] that $S = T^n$.

For infinite fields the next theorem reduces the problem of algebraic null-orbit reflexivity to the case of locally algebraic transformations. A key ingredient in the proof is an algebraic reflexivity result from [2] that says if $\mathbb{F}$ is infinite and $T \in L(X)$ is not locally algebraic, then, whenever $S \in L(X)$ and for every $x \in X$ there is a polynomial $p_x$ such that $Sx = p_x(T)x$, we must have $S = p(T)$ for some polynomial $p$.

**Theorem 2** Suppose $X$ is a vector space over an infinite field $\mathbb{F}$, and suppose $T \in L(X)$ is not locally algebraic. Then $T$ is algebraically null-orbit reflexive.

**Proof.** Suppose $S \in \text{nullOrbRef}_0(T)$. Then $Sx \in \text{nullOrb}(T)x$ for every $x \in X$. It follows from [2] that $T$ is algebraically reflexive, so we know there is a polynomial $p \in \mathbb{F}[t]$ such that $S = p(T)$. Since $T$ is not locally algebraic, there is a vector $e \in X$ such that for every nonzero polynomial $q \in \mathbb{F}[t]$, we have $q(T)e \neq 0$. Since $S \in \text{nullOrbRef}_0(T)$, we know that there is an $n \geq 0$ such that $Se = T^n e$. Hence $p(t) = t^n$, and thus $S \in \text{nullOrb}(T)$. ■

**Remark 3** If there is an $A \in \text{OrbRef}_0(T)$ such that $AT \neq TA$, then, since $\text{OrbRef}_0(T) \subseteq \text{nullOrbRef}_0(T)$, it follows that $T$ is not algebraically null-orbit reflexive. Similarly, if $T$ acts on a Banach space, and there is an $A \in \text{OrbRef}(T)$ such that $AT \neq TA$, then $T$ is not null-orbit reflexive. Hence the Hilbert space operator constructed by S. Grivaux and M. Roginskaya [1] is not null-orbit reflexive.

The preceding remark naturally leads to a pair of questions.
Question 1. If \( S \in \text{nullOrbRef}_0(T) \) and \( ST = TS \), must \( S \in \text{nullOrb}(T) \)?

Question 2. If \( T \) acts on a Hilbert space, \( S \in \text{nullOrbRef}(T) \) and \( ST = TS \), must \( S \) be in the strong-operator closure of \( \text{nullOrb}(T) \)? What is the answer if we assume that \( S \) is in the double commutant of \( \{T\} \)?

Note that the example of V. Müller and J. Vršovský [11, Example 1], where \( S = 0 \in \text{OrbRef}(T) \setminus \text{Orb}(T)^{-SOT} \) shows that the analog of Question 2 for orbit reflexivity has a negative answer. We will see later (Corollary 15) that their example is null-orbit reflexive, so it has no bearing on Question 2. In [11] an example is given of an operator on \( \ell^1 \) that is reflexive but not orbit reflexive. In view of Theorem 2.8 and Proposition 3.1 in [4], it seems feasible that the operator \( T \) in Example 1 of [11] Example 1] is reflexive. We know that \( \text{AlgLat}T \subseteq \{T\}^\prime \) and that if \( S \in \text{AlgLat}T \), then there is a sequence \( \{a_n\}_{n \geq 0} \) such that, for every vector \( x \), \( Sx \sim \sum_{n=0}^{\infty} a_n T^n \) in the sense of [4].

Question 3. Is the operator in Example 1 of [11 Example 1] is reflexive?

The proof of Theorem 2 shows that if \( T \) is algebraically \( F \)-orbit reflexive (reflexive) and \( F\cdot \text{Orb}(T) \) (\( \{p(T) : p \in F[t]\} \)) has a separating vector, then \( T \) is algebraically null-orbit reflexive. This immediately gives us the following (see [3, Theorem 3]).

**Theorem 4** Suppose \( X \) is a finite-dimensional vector space over a field \( F \) not isomorphic to \( \mathbb{Z}/p\mathbb{Z} \) for some prime \( p \). Then every linear transformation on \( X \) whose minimal polynomial splits over \( F \) is algebraically null-orbit reflexive.

**Corollary 5** If \( X \) is a finite-dimensional vector space over an algebraically closed field \( F \), then every linear transformation on \( X \) is algebraically null-orbit reflexive.

Recall from ring theory that if \( R \) is a principal ideal domain, \( M \) is an \( R \)-module, \( 0 \neq r \in R \) and \( rM = \{0\} \), then \( M \) is a direct sum of cyclic \( R \)-modules; Applying this fact to \( R = F[t] \), we get that any algebraic linear transformation on a vector space is a direct sum of transformations on finite-dimensional subspaces, and therefore has a Jordan form when the minimal polynomial splits over \( F \). (See [4] for details.) This gives us the following corollary.

**Corollary 6** Suppose \( X \) is a vector space over a field \( F \) not isomorphic to \( \mathbb{Z}/p\mathbb{Z} \) for some prime \( p \). Then every algebraic linear transformation on \( X \) whose minimal polynomial splits over \( F \) is algebraically null-orbit reflexive.
3 Null-orbit reflexivity

The following result was proved in [5, Proposition 3].

**Lemma 7** Suppose $\mathcal{N}$ is a commuting family of normal operators on a Hilbert space $X$ and $A \in B(X)$ satisfies, for every $x \in X$, $Ax \in (\mathcal{N}x)^{-}$. Then $A$ is in the SOT-closure of $\mathcal{N}$.

If in the preceding lemma we let $\mathcal{N} = \{0, 1, T, T^2, \ldots\}$, we obtain the following.

**Proposition 8** Every normal operator on a Hilbert space is null-orbit reflexive.

The next two results are consequences of Theorem 1.

**Theorem 9** Suppose $T$ is a bounded linear operator on a real or complex normed space $X$ such that $\bigcup_{n=1}^{\infty} \ker(T^n)$ is dense in $X$. Then $T$ is null-orbit reflexive and nullOrb $(T)$ is SOT-closed. Moreover, if $S \in \mathrm{nullOrbRef} (T)$, $x \in \bigcup_{n=1}^{\infty} \ker (T^n)$, $k \geq 0$, and $Sx = T^k x 
eq 0$, then $S = T^k$.

**Proof.** Suppose $S \in \mathrm{nullOrbRef} (T)$, and let $M = \bigcup_{n=1}^{\infty} \ker (T^n)$. It is clear that $S(M) \subseteq M$ and $T(M) \subseteq M$ and $S|_M \in \mathrm{nullOrbRef}_0 (T|_M)$. But $T|_M$ is locally nilpotent, and if $x \in M$ and $T^nx = 0$, then

$$\mathrm{nullOrb}(T)x = \{0\} \cup \{x, Tx, \ldots, T^{n-1}x\}$$

is norm closed. Hence, $\mathrm{nullOrbRef} (T|M) = \mathrm{nullOrbRef}_0 (T|M)$, which, by Theorem 1, is $\mathrm{nullOrb}(T|M)$. Hence there is an $A \in \mathrm{nullOrb}(T)$ such that $S|M = A|M$. However, $M$ is dense in $X$, so $S = A \in \mathrm{nullOrb}(T)$. □

The preceding theorem implies a stronger version of itself.

**Corollary 10** Suppose $X$ is a real or complex normed space, and there is a decreasingly directed family $\{X_\lambda : \lambda \in \Lambda\}$ of $T$-invariant closed linear subspaces such that

1. for every $\lambda \in \Lambda$, $\bigcup_{n=0}^{\infty} (T^n)^{-1}(X_\lambda)$ is dense in $X$, and
2. $\bigcap_{\lambda \in \Lambda} X_\lambda = \{0\}$.

Then $T$ is null-orbit reflexive and $\mathrm{nullOrbRef} (T) = \mathrm{nullOrb}(T)$.

**Proof.** Suppose $S \in \mathrm{nullOrbRef} (T)$ and $S \neq 0$. Choose $e \in X$ such that $Se \neq 0$. It follows from (2) that both (1) and (2) remain true if we consider only those $X_\lambda$ that contain neither $e$ nor $Se$. Since $T(X_\lambda) \subseteq X_\lambda$, $\hat{T}_\lambda (x + X_\lambda) = Tx + X_\lambda$ defines a bounded linear operator $\hat{T}_\lambda$ on $X/X_\lambda$. Condition (1) implies that
$\bigcup_{n=1}^{\infty} \ker \left( \hat{T}_n \right)$ is dense in $X/X\lambda$; whence, by Theorem 9 \( \hat{T}_\lambda \) is null-orbit reflexive. However, \( S \in \text{nullOrbRef} \left( T \right) \) implies that \( S \left( X\lambda \right) \subseteq X\lambda \), so \( \hat{S}_\lambda (x + X\lambda) = Sx + X\lambda \) defines an operator on $X/X\lambda$ such that \( \hat{S}_\lambda \in \text{nullOrbRef} \left( \hat{T}_\lambda \right) \). Hence, by Theorem 9 there is a unique nonnegative integer \( n_\lambda \) such that \( \hat{S}_\lambda = \hat{T}_\lambda^{n_\lambda} \).

Suppose \( \eta \in \Lambda \). Since the \( X\lambda \)'s are decreasingly directed, there is a \( \sigma \in \Lambda \) such that \( X\sigma \subseteq X\lambda \cap X\eta \). Applying the same arguments we used on \( X\lambda \), there is a unique integer \( m \geq 0 \) such that \( \hat{S}_\sigma = \hat{T}_\sigma^{n_\sigma} \). However, it follows from (1) that there is a vector \( x \in \left[ \bigcup_{n=0}^{\infty} (T^n)^{-1} (X\sigma) \right] \setminus X\lambda \). Then there is an \( n \) such that \( T^n x \in X\sigma \subseteq X\lambda \) and thus \( \hat{T}^n (x + X\lambda) = 0 \) but \( x + X\lambda \neq 0 \). However, \( Sx - T^n x \in X\sigma \subseteq X\lambda \), so \( \hat{S}_\lambda (x + X\lambda) = \hat{T}_\lambda^{n_\lambda} (x + X\lambda) \), which implies that \( n_\sigma = n_\lambda \). Similarly, \( n_\sigma = n_\lambda \). Hence there is an integer \( n \geq 0 \) such that, for every \( \lambda \in \Lambda \), \( n_\lambda = n \). Hence, for every \( x \in X \) and every \( \lambda \in \Lambda \),

\[
Sx - T^n x \in X\lambda,
\]

which, by (2), implies \( S = T^n \). 

The following corollary applies to operators that have a strictly upper-triangular operator matrix with respect to some direct sum decomposition.

**Corollary 11** If a normed space \( X \) over \( \mathbb{F} \in \{ \mathbb{R}, \mathbb{C} \} \) is a direct sum of spaces \( \{ X_n : n \in \mathbb{N} \} \) such that \( T \left( X_1 \right) = \{ 0 \} \), and for every \( n > 1 \),

\[
T \left( X_n \right) \subseteq \left( \bigoplus_{k<n} X_k \right)^-, \n\]

then \( T \) is null-orbit reflexive and \( \text{nullOrbRef} \left( T \right) = \text{nullOrb} \left( T \right) \).

The preceding corollary has some familiar special cases.

**Corollary 12** If \( T \) is an operator-weighted (unilateral, bilateral, or backwards) shift or if \( T \) is a direct sum of nilpotent operators on a real or complex normed space \( X \), then \( T \) is null-orbit reflexive.

**Theorem 13** Suppose \( X \) is a normed space over \( \mathbb{F} \in \{ \mathbb{R}, \mathbb{C} \} \), \( T \in B \left( X \right) \) and \( \bigcap_{n=1}^{\infty} T^n \left( X \right)^- = \{ 0 \} \). Then \( T \) is null-orbit reflexive and \( \text{nullOrbRef} \left( T \right) = \text{nullOrb} \left( T \right) \). Moreover, if \( S \in \text{nullOrbRef} \left( T \right), x \in X, \) and \( 0 \neq Sx = T^k x \), then \( S = T^k \).
Proof. We will first show that $T$ is algebraically null-orbit reflexive. If $M$ is a finite-dimensional invariant subspace for $T$ and $T|M$ is not nilpotent, then there is a nonzero $T$-invariant subspace $N$ of $M$ such that $\ker(T|N) = 0$. Thus $T(N) = N \neq 0$, which violates $\cap_{n=1}^\infty T^n(X)^- = \{0\}$. Thus, either $T$ is not locally algebraic or $T$ is locally nilpotent. In these cases it follows either from Theorem 2 or Theorem 1 that $T$ is indeed algebraically null-orbit reflexive.

Furthermore, the hypothesis on $T$ implies, for each $x \in X$, that

$$\cap_{n=1}^\infty \{ T^n x : k \geq N \}^- = \{0\},$$

so $\text{nullOrb}(T)x$ is closed in $X$. Thus $\text{nullOrbRef}(T) = \text{nullOrbRef}_0(T) = \text{nullOrb}(T)$. For the last statement suppose $x \in X$, and $k, n \geq 0$ are integers, and

$$0 \neq Sx = T^n x = T^k x.$$

Suppose $k < n$. Then $M = sp\{ x, T x, \ldots, T^{n-1} x \}$ is a nonzero finite-dimensional invariant subspace for $T$ with $\dim M \leq n$. Since $T^n x \neq 0$, we know $T|M$ is not nilpotent, which, as remarked earlier, contradicts $\cap_{n=1}^\infty T^n(X)^- = \{0\}$. 

This theorem also implies a stronger version of itself.

Corollary 14 Suppose $X$ is a real or complex normed space, $T \in B(X)$, and there is an increasingly directed family $\{ X_\lambda : \lambda \in \Lambda \}$ of $T$-invariant linear subspaces such that

1. for every $\lambda \in \Lambda$, $\cap_{n=1}^\infty \overline{T^n(X_\lambda)} = \{0\}$, and
2. $\cup_{\lambda \in \Lambda} X_\lambda$ is dense in $X$.

Then $T$ is null-orbit reflexive, and $\text{nullOrbRef}(T) = \text{nullOrb}(T)$. Moreover, if $S \in \text{nullOrbRef}(T)$, $x \in X$, and $0 \neq Sx = T^k x$, then $S = T^k$.

Proof. Suppose $0 \neq S \in \text{nullOrbRef}(T)$. It follows from (2) that there is a $\lambda_0 \in \Lambda$ and an $f \in X_{\lambda_0}$ such that $0 \neq Sf$. However, we must have $S(X_{\lambda_0}) \subseteq X_{\lambda_0}$, and $S|X_{\lambda_0} \in \text{nullOrbRef}(T|X_{\lambda_0}) = \text{nullOrb}(T|X_{\lambda_0})$ (by (1) and the preceding theorem). Thus there is an integer $k \geq 0$ such that

$$S|X_{\lambda_0} = T^k|X_{\lambda_0}.$$ 

The same $k$ must work for any $X_\lambda$ that contains $X_{\lambda_0}$. It follows from the fact that the family is increasingly directed and (2) that $S = T^k$. 

If $T$ is the operator constructed in [11] that is not orbit reflexive, it is easy to show that $\cap_{n \geq 0} T^n(X)^- = \{0\}$. 

Corollary 15 The non orbit reflexive operator constructed in [11] is null-orbit reflexive.
Irving Kaplansky [6] (see also [7], [8], [10]) proved that a (bounded linear) operator on a Banach space is locally algebraic if and only if it is algebraic. This immediately gives us the following result from Theorem 2.

**Proposition 16** Suppose $X$ is a real or complex Banach space and $T \in B(X)$ is not algebraic. Then $T$ is algebraically null-orbit reflexive.

The results in the paper of [11] also extend to the null-orbit case. If $T$ is an operator on a Banach space, then $r(T)$ denotes the spectral radius of $T$, i.e.,

$$r(T) = \max \{|\lambda| : \lambda \in \sigma(T)\}.$$

**Lemma 17** If $X$ is a normed space, $T \in B(X)$ and

$$E = \{x \in X : \text{nullOrb}(T)x \text{ is norm closed}\}$$

is not contained in a countable union of nowhere dense subsets of $X$, then $T$ is null-orbit reflexive and nullOrbRef(T) = nullOrb(T). (Note that $E$ contains all $x \in X$ such that $T^n x \to 0$ weakly or $\|T^n x\| \to \infty$.)

**Proof.** If $S \in \text{nullOrbRef}(T)$, then $E \subseteq \bigcup_{A \in \text{nullOrb}(T)} \ker(S - A)$, so there is an $A \in \text{nullOrb}(T)$ such that $\ker(S - A)$ has nonempty interior, which means that $S = A$. ■

**Corollary 18** If $X$ is a Banach space, $T \in B(X)$ and $r(T) < 1$, then $T$ is null-orbit reflexive.

**Proof.** It follows that $\|T^n\| \to 0$, and thus the set $E$ in Lemma 17 is all of $X$.

The proof of the following theorem is almost exactly the same as the proof of Theorem 7 in [11].

**Theorem 19** If $X$ is a Banach space and $T \in B(X)$ and $\sum_{n=1}^{\infty} \frac{1}{\|T^n\|} < \infty$, then $T$ is null-orbit reflexive. If $X$ is a Hilbert space and $\sum_{n=1}^{\infty} \frac{1}{\|T^n\|^2} < \infty$, then $T$ is null-orbit reflexive. In particular, if $r(T) \neq 1$, then $T$ is null-orbit reflexive.

**Corollary 20** The set of null-orbit reflexive operators on a Banach space $X$ is norm dense in $B(X)$. 

8
Theorem 21  If $X$ is a Hilbert space and $T \in B(X)$ and is polynomially bounded, then $T$ is null-orbit reflexive and orbit reflexive.

Proof. We prove the null-orbit reflexivity; the orbit reflexivity is proved in a similar fashion. Suppose $T$ is polynomially bounded. It was proved by W. Mlak [2] that $T$ is similar to the direct sum of a unitary operator $U$ and an operator $A$ with a weakly continuous $H^\infty$ functional calculus. In particular, $A^n \to 0$ in the weak operator topology. We can assume $T = U \oplus A$. We can also assume that the $A$ summand is present; otherwise, $T$ is null-orbit reflexive by Proposition [3]. Since $A^n \to 0$ in WOT, it follows from Lemma [17] that $\text{nullOrbRef} (A) = \text{nullOrb} (A)$. Hence we can assume that the $U$ summand is also present. Suppose $S \in \text{nullOrbRef} (T)$. Then we can write $S = B \oplus C$. Hence $C \in \text{nullOrb} (A)$. We also know that $B \in \text{nullOrbRef} (U)$.

Case 1. $C = 0$, and $B \neq 0$. For a fixed $x_0$ with $Bx_0 \neq 0$ and any $y$ there is a sequence $\{n_k\}$ of nonnegative integers such that $\|T^{n_k} (x_0 \oplus y) - Bx_0 \oplus 0\| \to 0$. In particular, $\|A^{n_k}y\| \to 0$. However, $A^n \to 0$ WOT implies there is an $M > 0$ such that $\|A^n\| < M$ for all $n \geq 0$. We want to show $\|A^n y\| \to 0$. Suppose $\varepsilon > 0$. Then there is an $n_k$ such that $\|A^{n_k}y\| < \varepsilon/M$. If $n \geq n_k$, then

$$\|A^n y\| \leq \|A^{n-n_k}\| \|A_{n_k} y\| < M (\varepsilon/M) = \varepsilon.$$

We now know that $A^n \to 0$ in the strong operator topology.

Now suppose $m \geq 0$ and $A^m \neq 0$. Choose $y_0$ such that $A^m y_0 \neq 0$. For any $x$, there is a sequence $\{n_k\}$ of integers such that $T^{n_k} (x \oplus y_0) \to S (x \oplus y_0)$, and it follows that eventually $n_k > m$. Thus, for every $x$ we have $Bx \in \{U^n x : n > m\}$, so it follows from Lemma [17] that $B \in \{U^n : n > m\}^{\text{SOT}}$. It now follows that there is a net $\{n_\lambda\}$ of positive integers such that $T^{n_\lambda} \to S$ in the strong operator topology.

Case 2. $C \neq 0$. Since $C \in \text{nullOrb} (A)$, there is an integer $s \geq 0$ such that $C = A^s \neq 0$. Since $A^n \to 0$ in the WOT, it follows that $\text{Ker} (A^k - 1) = 0$ for $k > 0$. Thus if $A^n y = A^m y$ with $n < m$, then $(A^{m-n} - 1) A^s y = 0$, which implies that $A^nx = 0$ and therefore $A^m y = 0$. Choose $y_1$ so that $A^s y_1 \neq 0$. It follows that if $\{n_k\}$ is a sequence of nonnegative integers and $A^{n_k} y_1 \to A^s y_1$, then $n_k$ must eventually become $s$. By considering vectors of the form $x \oplus y_1$, we see that $B = U^s$, and therefore $S = T^s$.

Since the only remaining case is $S = 0 \in \text{nullOrb} (T)$, the proof is complete.

Corollary 22  If $T$ is a Hilbert space operator and $\|T\| \leq 1$, then $T$ is null-orbit reflexive.

Corollary 23  If $T$ is a Hilbert space operator with $\|T\| = r(T)$ (e.g., $T$ is hyponormal), then $T$ is null-orbit reflexive.
The following lemma is a consequence of Theorem 19.

Lemma 24 Suppose $X$ is a Hilbert space, $T \in B(X)$, $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. If $\ker(T - \lambda) \neq \ker(T - \lambda)^2$, then $T$ is null orbit reflexive.

Proof. Suppose $\|x\| = 1$ and $(T - \lambda)^2 x = 0$ and $(T - \lambda) x \neq 0$. It follows that $\|T^n x\| = \|\lambda + (T - \lambda)\|^n \|x\| = \|\lambda^n x + n(T - \lambda) x\| \geq n \|(T - \lambda) x\| - \|x\| \geq \varepsilon n$ for some $\varepsilon > 0$ and for sufficiently large $n$. Thus $\sum 1/\|T^n\|^2 < \infty$, which, by Theorem 19 implies $T$ is null-orbit reflexive. □

Theorem 25 Suppose $X$ is a Hilbert space, $T \in B(X)$, $r(T) = 1$ and no point in $E = \sigma(T) \cap \{z \in \mathbb{C}: |z| = 1\}$ is a limit point of the spectrum of $T$. If the restriction of $T$ to the spectral subspace $M_E$ for the clopen subset $E$ of $\sigma(T)$ is an algebraic operator, then $T$ is null-orbit reflexive. In particular, every compact operator, or algebraic operator on a Hilbert space is null-orbit reflexive. Hence every operator on a finite-dimensional space is null-orbit reflexive.

Proof. It follows from Lemma 24 that we need only consider the case when $\ker(T - \lambda) = \ker(T - \lambda)^2$ for every $\lambda \in E$. This implies that the restriction of $T$ to $M_E$ is similar to a unitary operator, and since the restriction of $T$ to $M_{\sigma(T) \setminus E}$ has spectral radius less than 1, we see that $T$ is similar to a contraction. Hence, by Theorem 24 $T$ is null-orbit reflexive. If $T$ is compact or algebraic and $r(T) = 1$, then the first part of this theorem applies. If $r(T) \neq 1$, then $T$ is null-orbit reflexive by Theorem 19. □

We conclude with another question.

Question 4. Is every power bounded Hilbert space operator orbit reflexive or null-orbit reflexive?

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