Bi-invariant metric on contact diffeomorphisms group

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Abstract

We show the existence of a weak bi-invariant symmetric nondegenerate 2-form on the contact diffeomorphisms group $\mathcal{D}_\theta$ of a contact Riemannian manifold $(M, g, \theta)$ and study its properties. We describe the Euler’s equation on a Lie algebra of group $\mathcal{D}_\theta$ and calculate the sectional curvature of $\mathcal{D}_\theta$. In a case $\dim M = 3$ connection between the bi-invariant metric on $\mathcal{D}_\theta$ and the bi-invariant metric on volume-preserving diffeomorphisms group $\mathcal{D}_\mu$ of $M^3$ is discover.

1 Contact transformations group and bi-invariant metric

Let $M$ be a smooth (of class $C^\infty$) compact orientable manifold of dimension $n = 2m + 1$ without boundary. The manifold $M$ is said to be contact if a 1-form $\theta$ with the following property is given on it: the $(2m + 1)$-form $\theta \wedge (d\theta)^m$ does not vanish everywhere on $M$. Such a form $\theta$ is said to be contact. A vector field $\xi$ is said to be characteristic if it has the properties

$$\theta(\xi) = 1, \quad d\theta(\xi, \cdot) = 0.$$ 

A contact distribution $E$ on $M$ is defined as the kernel of the form $\theta$, $E = \text{Ker} \theta$. Clearly, $TM = E \oplus \mathbb{R}\xi$. Let $\Gamma(TM)$ be the space of smooth vector fields on $M$ and $\Gamma(E)$ be the space of smooth vector fields on $M$ belonging to a distribution $E$.

A Riemannian structure $g$ on $M$ is said to be associated with $\theta$ if there exists a tensor field $\varphi$ of type $(1,1)$ on $M$ such that for any vector fields $X$ and $Y$ on $M$,

\begin{enumerate}
  \item $g(X, \xi) = \theta(X)$;
  \item $\varphi^2 = -I + \theta \otimes \xi$;
  \item $d\theta(X, Y) = g(X, \varphi Y)$,
\end{enumerate}

where $\varphi$ is considered as a morphism $\varphi : TM \to TM$ and $I$ is the identity morphism.

Let’s note some additional properties [7]. For any vector fields $X$ and $Y$ on $M$,

\begin{enumerate}
  \item $g(\xi, \xi) = 1$;
  \item the distribution $E$ is orthogonal to the field $\xi$;
  \item $\varphi(\xi) = 0$, $\varphi(E) = E$;
  \item $\varphi$ is skew-symmetric and $(\varphi|_E)^2 = -I_E$;
  \item $d\theta(\varphi X, \varphi Y) = d\theta(X, Y)$;
  \item $g(X, Y) = \theta(X)\theta(Y) + d\theta(\varphi X, Y)$.
\end{enumerate}

A mapping $\eta : M \to M$ is said to be contact if it preserves the contact structure $\theta = 0$, i.e., if $\eta^*\theta = h\theta$, where $h$ is a function of $M$. Although the algebra of infinitesimal contact transformations is usually defined as a subalgebra of the algebra of vector fields:

$$\Gamma_\theta(TM) = \{ V \in \Gamma(TM); \quad L_V \theta = f \theta, \text{ for a certain function } f \},$$

we define $\widetilde{\Gamma}_\theta$ as a subalgebra of $C^\infty(M) \oplus \Gamma(TM)$:

$$\widetilde{\Gamma}_\theta = \{ (f, V) \in C^\infty(M) \oplus \Gamma(TM); \quad f \theta + dV \theta + i_V d\theta = 0 \}, \quad (1.1)$$

where the Lie algebra structure on $C^\infty(M) \oplus \Gamma(TM)$ is defined as follows:

$$[(f, U), (g, V)] = (U(g) - V(f), [U, V]). \quad (1.2)$$
It is easy to verify that \( \widetilde{\Gamma}_\theta \) is indeed a subalgebra of \( C^\infty(M) \oplus \Gamma(TM) \).

**Remark 1.** The algebra \( C^\infty(M) \oplus \Gamma(TM) \) defined above is the Lie algebra of the semidirect product \( C^\infty_*(M) \ast D \) of the group \( C^\infty_*(M) \) of positive smooth functions on \( M \) and the diffeomorphisms group \( D \). The group operation is given by

\[
(a, \eta) \ast (b, \zeta) = (a(b \circ \eta), \eta \circ \zeta).
\]

It is shown in [19, Theorem 4.5.1] that \( C^\infty_*(M) \ast D \) is strongly an ILH-Lie group. The coordinate mapping can be given by

\[
\psi : C^\infty(M) \oplus \Gamma(TM) \to C^\infty_*(M) \oplus D, \quad (f,V) \mapsto (e^f, E(V)),
\]

where \( V \mapsto E(V) \) is the coordinate mapping on the diffeomorphisms group defined in [9].

There exists a one-to-one linear correspondence between \( \widetilde{\Gamma}_\theta \) and the function space \( C^\infty(M) \). Indeed, let \((f,U) \in \widetilde{\Gamma}_\theta\); then \( L_U \theta + f \theta = 0 \). Decompose the field \( U \) in ac-
cordance with the decomposition \( TM = \mathbb{R} \xi \oplus E \): \( U = h \xi + v \). Then

\[
L_U \theta + f \theta = dh + i_v d\theta + f \theta = 0.
\]

Hence \( \text{grad} h - \varphi(v) + f \xi = 0 \). We obtain from the latter relation that

\[
f = -\xi(h), \quad \varphi(\text{grad} h) + v = 0, \quad v = -\varphi(\text{grad} h).
\]

Therefore,

\[
(f, U) = (-\xi(h), h \xi - \varphi(\text{grad} h)).
\]

The latter formula defines the correspondence of \( \widetilde{\Gamma}_\theta \) with the function space \( C^\infty(M) \): \( (f, U) \leftrightarrow h = \theta(U) \).

**Theorem 1.1** (see [19, Theorem 8.3.6]). The group

\[
\widetilde{\mathcal{D}}_\theta = \{(f, \eta) \in C^\infty_*(M) \ast D; \ f \eta^* \theta = \theta \}
\]

is a strong, closed ILH-subgroup of the group \( C^\infty_*(M) \ast D \) with the Lie algebra \( \widetilde{\Gamma}_\theta \).

A contact form \( \theta \) is said to be **regular** [7] if the vector field \( \xi \) induces a free action of the unit circle \( S^1 \) on the manifold \( M \). In this case, the quotient manifold \( N = M/S^1 \) is defined; the 2-form \( d\theta \) is lowered to \( N \) and defines a symplectic structure \( \omega \) on \( N \). If \( \pi : M \to N \) is the natural projection, then \( \pi^* \omega = d\theta \).

On \( M \) we fix the **associated metric** \( g \) and the **affinor** \( \varphi \) corresponding to it. Assume that the contact structure \( \theta \) on \( M \) is regular.

The transformation \( \eta : M \to M \) is said to be **exactly contact**, or it is called **quantomorphism** if it preserves the contact form \( \theta \): \( \eta^* \theta = \theta \). Let \( \mathcal{D}_\theta \) be the connected component of the group of all exact contact transformations of the manifold \( M \):

\[
\mathcal{D}_\theta = \{ \eta \in \mathcal{D}_0, \ \eta^* \theta = \theta \}.
\]

For any \( \eta \in \mathcal{D}_\theta \) we have:

\[
\text{Ad}_{\eta} \xi = \xi.
\]

Omori showed in [19] in the case of a regular contact manifold \((M, \theta)\) that the group \( \mathcal{D}_\theta \) is an ILH-Lie group. In proving this, he used the fact that the group \( \mathcal{D}_\theta \) consists of diffeomorphisms commuting with the free action of the compact group \( S^1 \) on \( M \).
Theorem 1.2 (see [19]). The group $\mathcal{D}_\theta$ is a strong ILH-subgroup of the group $\widetilde{\mathcal{D}}_\theta$. The Lie algebra of the group $\mathcal{D}_\theta$ consists of contact vector fields $X$ on $M$, i.e., those fields for which $L_X \theta = 0$.

Let $T_e \mathcal{D}_\theta = \{X \in \Gamma(TM); \ L_X \theta = 0\}$ be the Lie algebra of the group $\mathcal{D}_\theta$. If $X \in T_e \mathcal{D}_\theta$ is a contact vector field, then the function $f = \theta(X)$ is called the contact Hamiltonian of the field $X$, and the field $X$ itself is usually denoted by $X_f$. The condition $L_X \theta = 0$ immediately implies $i_X d\theta = -df$. Therefore, the function $f$ is constant on the trajectories of the characteristic vector field $\xi$. Using the latter relation, it is easy to show that the contact vector field $X_f$ has the form

$$X_f = f\xi - \varphi \, \text{grad} f. \quad (1.5)$$

Let us define the Lagrange bracket $[f, g]$ of contact Hamiltonians $f$ and $g$ by

$$[f, g] = X_f(g). \quad (1.6)$$

Then the following relation holds for the Lie bracket of contact vector fields and the Lagrange bracket:

$$[X_f, X_g] = X_{[f,g]}, \quad (1.7)$$

The group $\mathcal{D}_\theta$ has the following natural right-invariant weak Riemannian structure: if $X, Y \in T_e \mathcal{D}_\theta$, then

$$(X, Y)_e = \int_M g(X(x), Y(x)) d\mu(x), \quad (1.8)$$

where $\mu = \theta \wedge (d\theta)^n$ and at integration the form $\mu$ we denote as $d\mu$.

Introduce one more inner product on $T_e \mathcal{D}_\theta$:

$$\langle X_f, X_h \rangle_e = \int_M f h \, d\mu. \quad (1.9)$$

The relation between the natural weak Riemannian structure (1.8) and (1.9) is expressed by the formula

$$\langle X_f, X_h \rangle_e = \langle X_{f + \Delta f}, X_h \rangle_e, \quad (1.10)$$

where $\Delta = -\text{div} \circ \text{grad}$ is the Laplacian. Indeed, we have

$$\langle X_f, X_h \rangle_e = \int_M g(X_f, Y_h) d\mu = \int_M g(f \xi - \varphi \, \text{grad} f, h \xi - \varphi \, \text{grad} h) d\mu =$$

$$= \int_M f h \, d\mu + \int_M g(\varphi \, \text{grad} f, \varphi \, \text{grad} h) d\mu = \int_M f h d\mu + \int_M g(\text{grad} f, \text{grad} h) d\mu =$$

$$= \int_M f h d\mu + \int_M -\text{div}(\text{grad} f) \ h \, d\mu = \int_M (f + \Delta f) h \, d\mu.$$

Theorem 1.3. The inner product (1.3) on the Lie algebra $T_e \mathcal{D}_\theta$ of the group $\mathcal{D}_\theta$ defines a bi-invariant weak Riemannian structure on the group $\mathcal{D}_\theta$.

Proof. It is easy to see that it is invariant under the adjoint action of the group $\mathcal{D}_\theta$ on $T_e \mathcal{D}_\theta$. Indeed, if $X_f = f\xi - \varphi \, \text{grad} f$, then

$$\text{Ad}_\eta X_f = (f \circ \eta^{-1})\xi - \text{Ad}_\eta(\varphi \, \text{grad} f) = X_{f \circ \eta^{-1}}. \quad (1.11)$$

Moreover, $\eta^*(\mu) = \eta^*(\theta \wedge (d\theta)^n) = \theta \wedge (d\theta)^n = \mu$. Therefore,

$$\langle \text{Ad}_\eta(X_f), \text{Ad}_\eta(X_g) \rangle_e = \langle X_{f \circ \eta^{-1}}, X_{g \circ \eta^{-1}} \rangle_e = \int_M (f \circ \eta^{-1})(g \circ \eta^{-1}) \, d\mu =$$

$$= \int_M f \cdot g \ \eta^*(d\mu) = \int_M f \cdot g \, d\mu = \langle X_f, X_g \rangle_e.$$

Therefore, (1.9) defines a bi-invariant weak Riemannian structure on the group $\mathcal{D}_\theta$. \qed
2 Euler equation on the Lie algebra $T_c D_\theta$

Let $\mathfrak{g}$ be a semisimple, finite-dimensional Lie algebra, and let $H$ be a certain function on $\mathfrak{g}$. In [16], it was shown that the extension of the Euler equation on the Lie algebra of the group $SO(n, \mathbb{R})$ of motions of an $n$-dimensional rigid body to the case of the general semisimple Lie algebra $\mathfrak{g}$ is an equation of the form

$$\frac{d}{dt} X = [X, \text{grad} H(X)],$$

where $X \in \mathfrak{g}$ and the gradient of the Hamiltonian function $H$ is calculated with respect to the invariant Killing–Cartan inner product on $\mathfrak{g}$.

As $\mathfrak{g}$, let us consider the Lie algebra $T_c D_\theta$ of contact vector fields on the regular contact Riemannian manifold $M$. On $T_c D_\theta$, we have the invariant nondegenerate form (1.9) and the function (kinetic energy)

$$T(X_f) = \frac{1}{2} (X_f, X_f)_e = \frac{1}{2} \int_M g(X_f, X_f) d\mu, \quad X_f \in T_c D_\theta.$$

The function $T$ can be written as follows in terms of the inner product (1.9):

$$T(X_f) = \frac{1}{2} (X_f, X_f)_e = \frac{1}{2} \langle X_f + \triangle f, X_f \rangle_e.$$

Perform the Legendre transform $Y_h = X_f + \triangle f$; then

$$T(X_f) = T(Y_{(1+\triangle)^{-1} h}) = \frac{1}{2} \langle Y_h, Y_{(1+\triangle)^{-1} h} \rangle_e.$$

Consider the Hamiltonian function $H(Y_h) = \frac{1}{2} \langle Y_h, Y_{(1+\triangle)^{-1} h} \rangle_e$ on the Lie algebra $T_c D_\theta$. The gradient of the function $H$ with respect to the invariant inner product (1.9) is easily calculated:

$$\text{grad} H(Y_h) = Y_{(1+\triangle)^{-1} h}.$$

As in the finite-dimensional case, let us write the Euler equation on the Lie algebra $T_c D_\theta$:

$$\frac{d}{dt} Y_h = [Y_h, Y_{(1+\triangle)^{-1} h}] = Y_{[h, (1+\triangle)^{-1} h]},$$

$$\frac{d}{dt} h = [h, (1 + \triangle)^{-1} h].$$

Since $h = f + \triangle f$ and $[f, f] = 0$, it follows that

$$\frac{\partial (f + \triangle f)}{\partial t} = [\triangle f, f],$$

On the Lie algebra $T_c D_\theta$, Eq. (2.2) or (2.4) has the following two quadratic first integrals:

$$m(Y_h) = \langle Y_h, Y_h \rangle_e = (X_{(1+\triangle)f}, X_f)_e = \int_M g(X_{(1+\triangle)f}, X_f) d\mu,$$

$$H(Y_h) = T(X_f) = \frac{1}{2} \int_M g(X_f, X_f) d\mu,$$

where $Y_h = X_{(1+\triangle)f}$. The first of them $m(Y_h)$ is naturally called the kinetic moment. The invariance of the function $m$ follows from the invariance of the inner product (1.9) on $T_c D_\theta$. The second integral $H(Y_h)$ is the kinetic energy.
Since $H(X_f) = \frac{1}{2} \langle X_f, X_{(1+\Delta)f} \rangle_e$, the operator $1 + \Delta : T_e \mathcal{D}\theta \to T_e \mathcal{D}\theta$ is the inertia operator of our mechanical system $(T_e \mathcal{D}\theta, H)$. The eigenvectors $V_i$ of the operator $1 + \Delta$ are naturally called (analogously to the rigid body motion) the \textit{axes of inertia}, and the eigenvalues $\lambda_i$ of $1 + \Delta$ are called the \textit{moments of inertia} with respect to the axes $V_i$.

The Euler equation (2.2) can be written in the form

$$\frac{\partial X_{(1+\Delta)f}}{\partial t} = -L_{X_f}X_{(1+\Delta)f},$$

where $L_{X_f}$ is the Lie derivative. Therefore [15], the vector field $X_{(1+\Delta)f}(t)$ is transported by the flow $\eta_t$ of the field $X_f$:

$$X_{(1+\Delta)f}(t) = d\eta_t(X_{(1+\Delta)f}(0) \circ \eta_t^{-1}) = Ad_{\eta_t}(X_{(1+\Delta)f_0}) = X_{((1+\Delta)f_0)\circ \eta_t^{-1}},$$

where $X_f(0) = X_{f_0}$ is the initial velocity field. Hamiltonian function $(1 + \Delta)f(t, x)$ is transported by the flow $\eta_t$:

$$(1 + \Delta)f(t, x) = ((1 + \Delta)f_0)(\eta_t^{-1}(x)).$$

This immediately implies the following theorem.

\textbf{Theorem 2.1.} Let $f = f(t, x)$ be a solution of the Euler equation (2.4), and let $\eta_t$ be the flow on $M$ generated by the vector field $X_f$. Then the following quantities are independent of time $t$:

$$T = \frac{1}{2} (X_f, X_f)_e = \frac{1}{2} \int_M f(1 + \Delta)f \, d\mu, \quad (2.7)$$

$$I_k = \int_M ((1 + \Delta)f)^k \, d\mu. \quad (2.8)$$

The curve $Y_h(t) = X_{(1+\Delta)f}(t) = Ad_{\eta_t}(X_{(1+\Delta)f_0}) = Ad_{\eta_t}(Y_{f_0})$ on the Lie algebra $T_e \mathcal{D}\theta$ that is a solution of the Euler equation (2.2) lies on the orbit $\mathcal{O}(Y_{f_0}) = \{Ad_{\eta}Y_{f_0}; \ \eta \in \mathcal{D}\theta\}$ of the coadjoint action of the group $\mathcal{D}\theta$.

Since the kinetic moment $m(X)$ is preserved under the motion, the orbit $\mathcal{O}(Y_{f_0})$ lies on the "pseudo-sphere"

$$S = \{Y_h \in T_e \mathcal{D}\theta; \ \langle Y_h, Y_h \rangle_e = r\},$$

where $r = \langle Y_{f_0}, Y_{f_0} \rangle_e$. It is natural to expect that the critical points of the function $H(Y_h)$ on the orbit $\mathcal{O}(Y_{f_0})$ are stationary motions. The orbit $\mathcal{O}(Y_h(0))$ is the image of the smooth mapping

$$\mathcal{D}\theta \to T_e \mathcal{D}\theta, \ \eta \to Ad_{\eta}Y_{f_0}.$$ 

Therefore, the tangent space $T_Y \mathcal{O}(Y_{f_0})$ to the orbit at a point $Y$ is

$$T_Y \mathcal{O}(Y_{f_0}) = \{[W, Y]; \ W \in T_e \mathcal{D}\theta\}.$$ 

A point $Y \in \mathcal{O}(Y_{f_0})$ is critical for the function $H(Y_h)$ if grad$H(Y_h)$ is orthogonal to the space $T_Y \mathcal{O}(Y_{f_0})$. Since grad$H(Y_h) = Y_{(1+\Delta)^{-1}h}$, the latter condition is equivalent to

$$\langle Y_{(1+\Delta)^{-1}h}, [W, Y] \rangle_e = 0 \ \text{for any} \ W \in T_e \mathcal{D}\theta(M).$$

The invariance of $\langle \cdot, \cdot \rangle_e$ implies

$$\langle [Y, Y_{(1+\Delta)^{-1}h}], W \rangle_e = 0 \ \text{for any} \ W \in T_e \mathcal{D}\theta(M).$$
From the nondegeneracy of the form $\langle \cdot, \cdot \rangle_e$ on $T_e \mathcal{D}_\theta(M)$, we obtain that a point $Y$ on the orbit $\mathcal{O}(Y_h(0))$ is a critical point of the function $H$ iff

$$[Y, Y_{(1+\Delta)^{-1}h}] = 0.$$ 

It follows from the Euler equation $\frac{d}{dt}Y_h = [Y_h, Y_{(1+\Delta)^{-1}h}]$ that a field $Y_h$ is an equilibrium state of our system (i.e., a stationary motion) iff $Y_h$ is a critical point of the kinetic energy $H(Y_h)$ on the orbit $\mathcal{O}(Y_{h_0})$.

### 3 Curvature of the group $\mathcal{D}_\theta$

The sectional curvature of the group $\mathcal{D}_\theta$ with respect to the bi-invariant metric (1.9) in the direction to a 2-plane $\sigma$ given by an orthonormal pair of contact vector fields $X_f, X_h \in T_e \mathcal{D}_\theta$ is expressed by the formula

$$K_\sigma = \frac{1}{4} \int_M [f, h]^2 d\mu.$$ 

Thus, the group $\mathcal{D}_\theta$ is of nonnegative sectional curvature, and $K_\sigma = 0$ iff the contact Hamiltonians $f$ and $h$ commutes, $[f, h] = 0$.

The sectional curvature of the natural right-invariant Riemannian structure (1.8) in the direction to a 2-plane $\sigma$ given by an orthonormal pair of contact vector fields $X, Y \in T_e \mathcal{D}_\theta$ can be found by the general formula [23]

$$K_\sigma = -\frac{3}{4} ([X, Y], [X, Y])_e - \frac{1}{2} ([X, [X, Y]], Y)_e - \frac{1}{2} ([Y, [Y, X]], X)_e -$$

$$- (P_e(\nabla_X X), P_e(\nabla_Y Y)) + \frac{1}{4} (P_e(\nabla_X Y + \nabla_Y X), P_e(\nabla_X Y + \nabla_Y X)).$$

where the operator $P_e$ must be considered as the orthogonal projection of the space $\Gamma(TM)$ on the space $T_e \mathcal{D}_\theta$ of contact vector fields on $M$ in accordance with the orthogonal decomposition $\Gamma(TM) = T_e \mathcal{D}_\theta \oplus (T_e \mathcal{D}_\theta)^\perp$.

**Lemma 3.1.** For any contact vector fields $X = X_f$ and $Y = X_h$ on $M$, the contact Hamiltonian $s$ of the vector field $P_e(\nabla_X Y) = X_s$ is expressed through $f$ and $h$ in the following way:

$$Ds = \frac{1}{2} (D[f, h] + [f, Dh] + [h, Df]),$$

where $D = 1 + \Delta$, $\Delta$ is the Laplacian, and $[f, g]$ is the Lagrange bracket.

**Proof.** It was demonstrated in [9] that, given the weak right-invariant Riemannian structure (1.8) on $\mathcal{D}_\theta$, there exists a Riemannian connection whose covariant derivative $\tilde{\nabla}$ at the identity $e \in \mathcal{D}_\theta$ is given by the formula

$$\tilde{\nabla}_{X} Y)_e = P_e(\nabla_{X_e} Y_e),$$

where $\nabla$ is the covariant derivative of the Riemannian connection on $M$ and $X(\eta) = X_e \circ \eta$, $Y(\eta) = Y_e \circ \eta$ are right-invariant vector fields on $\mathcal{D}_\theta$, $X_e, Y_e \in T_e \mathcal{D}_\theta$. For determining $P_e(\nabla_{X_e} Y_e) = (\tilde{\nabla}_X Y)_e$, we use the six-term formula

$$2(\tilde{\nabla}_XY, Z)_e = X(Y, Z) + Y(Z, X) - Z(X, Y) + (Z, [X, Y])_e + (Y, [Z, X])_e - (X, [Y, Z])_e,$$
where \( X = X_f \), \( Y = X_h \) and \( Z = X_g \) are regarded as right-invariant vector fields on \( \mathcal{D}_g \). Taking the right invariance of the weak Riemannian structure (1.8) into account, we obtain \( X(Y, Z) = Y(Z, X) = Z(X, Y) = 0 \). Using the bi-invariant scalar product (1.3), we obtain

\[
2(\nabla_X Y, Z)_e = (X_g, [X_f, X_h])_e + (X_h, [X_g, X_f])_e - (X_f, [X_h, X_g])_e =
\]

\[
= (X_g, X_{[f,h]})_e + (X_h, X_{[g,f]})_e - (X_f, X_{[h,g]})_e = (X_g, X_{D[f,h]})_e + (X_h, X_{D[g,f]})_e - (X_f, X_{D[h,g]})_e =
\]

\[
= (X_g, X_{D[f,h]})_e + ([X_f, X_{D[h]})_e + ([X_h, X_{D[f]})_e = (X_{D[f,h]} + [X_{D[f,h]} + X_{D[g,f]}]_e).
\]

On the other hand, we have

\[
2(\nabla_X Y, Z)_e = 2(P_e(\nabla_X Y), Z)_e = 2(X_s, X_g)_e = 2(X_{Ds}, X_g)_e.
\]

\[
\square
\]

**Corollary 3.2.** Let \( X = X_f \) and \( Y = X_h \). If \( X_s = P_e(\nabla_X Y) \) and \( X_q = P_e(\nabla_X Y + \nabla_Y X) \), then

\[
(1 + \Delta)s = [f, \Delta f], \quad (1 + \Delta)q = [f, \Delta h] - [\Delta f, h]. \tag{3.3}
\]

From (3.1) and Lemma 3.1 immediately implies the following theorem.

**Theorem 3.3.** The sectional curvature of the group \( \mathcal{D}_g \) with respect to metric (1.8) in the direction of a 2-plane \( \sigma \subset T_e \mathcal{D}_g \) composed of an orthonormal pair of contact vector fields \( X_f, X_h \in T_e \mathcal{D}_g \) is expressed by the formula

\[
K_\sigma = \frac{1}{4} \int_M [f, h]^2 d\mu - \frac{3}{4} \int_M [f, h] \Delta [f, h] d\mu +
\]

\[
+ \frac{1}{2} \int_M [f, h] ([\Delta f, h] + [f, \Delta h]) d\mu - \int_M [f, \Delta f] D^{-1}([h, \Delta h]) d\mu +
\]

\[
+ \frac{1}{4} \int_M ([f, \Delta h] - [\Delta f, h]) D^{-1}([f, \Delta h] - [\Delta f, h]) d\mu, \tag{3.4}
\]

where \( D = 1 + \Delta \), \( \Delta = -\text{div} \circ \text{grad} \) is the Laplacian, and \([f, h]\) is the Lagrange bracket of functions on a contact manifold.

In the case where, as the contact Hamiltonians \( f \) and \( h \), we choose eigenfunctions of the Laplace operator, \( \Delta f = \alpha f \) and \( \Delta h = \beta h \), formula (3.4) is considerably simpler:

\[
K_\sigma = -\frac{3}{4} \int_M [f, h] \Delta [f, g] d\mu + \frac{1 + 2(\alpha + \beta)}{4} \int_M [f, h]^2 d\mu + \frac{(\alpha - \beta)^2}{4} \int_M [f, h] D^{-1}([f, h]) d\mu. \tag{3.5}
\]

If the structural constants defined from

\[
[f, h] = e_i f_i,
\]

where \( f_i \) is the orthonormal system of eigenfunctions of the Laplace operator with eigenvalues \( \alpha_i \), of the Lie algebra \( T_e \mathcal{D}_g \) of contact vector fields are known, then the formula for the sectional curvatures becomes

\[
K_\sigma = -\frac{1}{(1 + \alpha)(1 + \beta)} \left( -\frac{3}{4} \sum_{i > 0} \alpha_i (e_i^f)^2 + \frac{1 + 2(\alpha + \beta)}{4} \sum_{i > 0} (e_i^f)^2 + \frac{(\alpha - \beta)^2}{4} \sum_{i > 0} \frac{(e_i^f)^2}{1 + \alpha_i} \right), \tag{3.6}
\]
here, it is assumed that the $L^2$-norms of the functions $f$ and $h$ are equal to $1$.

Let us show that the calculation of the sectional curvatures of the group $D_\theta$ by formula (3.6) reduces to the search for the structural constants of the Lie algebra of Hamiltonian vector fields on the symplectic manifold $N = M/S^1$. Recall that the symplectic structure on $N$ is defined by the 2-form $\omega$ uniquely found from the relation $\pi^*\omega = d\theta$, where $\pi$ is the projection of $M$ on $N = M/S^1$.

An exact contact transformation $\eta : M \to M$ preserves the differential forms $\theta$ and $d\theta$. Therefore, the diffeomorphism $\eta$ preserves the characteristic vector field $\xi$ and commutes with the action of $S^1$ on $M$. This implies that the contact diffeomorphism $\eta$ defines a diffeomorphism $\eta$ of the manifold $N$ such that $\pi \circ \eta = \eta \circ \pi$. From the relation $\eta^*d\theta = d\theta$, we obtain $\eta^*\omega = \omega$, i.e., $\eta$ is a symplectic diffeomorphism of the manifold $N$. Therefore, we have obtained the homomorphism

$$ p : D_\theta \to D_\omega(N), \quad p(\eta) = \eta. $$

The image $p(D_{\theta,0})$ of the connected component $D_{\theta,0}$ of the identity is (see [19]) a closed ILH-Lie subgroup $G$ of exact symplectic transformations. Recall that the Lie algebra $\mathcal{H} = T_eG$ of this group consists of globally Hamiltonian vector fields on $N$.

The sequence of groups

$$ 1 \to S^1 \to D_{\theta,0} \to G \to 1, \quad (3.7) $$

is exact. In the corresponding exact sequence of algebras

$$ 0 \to \mathbb{R} \to T_eD_\theta \to \mathcal{H} \to 0, \quad (3.8) $$

the kernel of the differential $dp$ is the one-dimensional space of vector fields proportional to the field $\xi$: $\text{Ker} dp = \mathbb{R}\xi$.

In $T_eD_\theta$, let us find the orthogonal complement to the kernel of the differential $dp$ with respect to the inner product (1.8). The contact Hamiltonian of the characteristic vector field $\xi$ is identically equal to 1. Let $X_f$ be any other contact vector field; then

$$ (\xi, X_f) = \int_M (1 + \Delta(1))f \, d\mu = \int_M f \, d\mu = 0. $$

Thus, the contact vector field $X_f$ lies in the orthogonal complement to the vector field $\xi$ iff

$$ \int_M f \, d\mu = 0. $$

As we already noted, a contact vector field on $M$ has the form

$$ X_f = f\xi - \varphi \text{grad}f, $$

where $f$ is a smooth function on $M$ that is constant on the trajectories of the vector field $\xi$. Therefore, it defines the function $F$ on $N$ by $f = F \circ \pi$. Under the projection $\pi : M \to N$, the contact vector field $X_f$ is mapped into the Hamiltonian vector field $X_F$, $X_F = d\pi(X_f)$. In this case, the Lie bracket $[X_f, X_h]$ passes to the Lie bracket $[X_F, X_H]$ of the corresponding Hamiltonian vector fields $X_F$, $X_H$ on $N$ [15]. Therefore, the Poisson bracket $\{F, H\}$ on $N$ corresponds to the Lagrange bracket $[f, h] : [f, h] = \{F, H\} \circ \pi$.

Note that if $X_f$ lies in the orthogonal complement to $\mathbb{R}\xi$ in $T_eD_\theta$, then the Hamiltonian $F$ on $N$ corresponding to $f$ has the property $\int_M F \, d\mu = 0$. 
It follows from the above arguments that the structural constants \( c^i_{fh} \) of the Lie algebra \( T_eD_\theta \),
\[
[f, h] = c^i_{fh} f_i, \quad X_f, X_h \in (\mathbb{R}\xi)^\perp,
\]
coincide with the structural constants \( C^i_{FH} \) of the algebra \( \mathcal{H} \),
\[
\{F, H\} = C^i_{FH} F_i.
\]

**Conclusion.** The structural constants \( C^i_{FH} \) of the algebra \( \mathcal{H} \) of Hamiltonian vector fields on \( N \) can be used in formula \( (3.6) \) instead of the structural constants \( c^i_{fh} \) of the Lie algebra of contact vector fields on \( M \) for calculating the sectional curvatures of the group \( D_\theta \) in directions \( \sigma \) orthogonal to the field \( \xi \).

**Remark 2.** Since the contact vector fields \( X_f \) commute with the characteristic vector field \( \xi \), i.e., \([\xi, X_f] = 0\), it follows that the other structural constants of \( T_eD_\theta \) of the form \( c^i_{\xi h} \) vanish. Therefore, we have the following conclusion:
the sectional curvature of the group \( D_\theta \) in directions \( \sigma \) containing the characteristic vector field \( \xi \) vanishes.

**Remark 3.** Assume that the associated metric \( g \) on \( M \) is \( K \)-contact. This means that the metric tensor \( g \) is invariant with respect to the action of \( S^1 \) on \( M \), \( L_{\xi}g = 0 \). Then on \( N = M/S^1 \), we can define the Riemannian structure \( \mathfrak{g} \) with respect to which the projection \( \pi : M \to N = M/S^1 \) is a Riemannian submersion. If the contact Hamiltonian \( f \) on \( M \) is an eigenfunction of the Laplace operator, \( \Delta f = \lambda f \), then the corresponding Hamiltonian \( F \) on \( N \) is also an eigenfunction of the Laplace operator on \( N \), \( \Delta F = \lambda F \). In the general case, this is not true. Therefore, the use of the structural constants \( C^i_{FH} \) of the algebra \( \mathcal{H}(N) \) for calculating the sectional curvatures of the group \( D_\theta \) can be especially effectively used in the case of the \( K \)-contact structure. Therefore, we can calculate the sectional curvatures of the group \( D_\theta \) of contact transformations of the three-dimensional unit sphere \( S^3 \) in the basis corresponding to the Legendre functions \( Y_{lm}(z, \varphi) \) on the sphere \( S^2 \). The corresponding bundle \( S^3 \to S^2 \) is the Hopf bundle in this case.

# 4 Case of a three-dimensional manifold \( M^3 \)

Let \( M \) be a three-dimensional contact manifold. In this case, the contact metric structure on \( M \) also has the following properties in addition to properties (1)–(9) given in the beginning of this paper:

- (10) \( \theta \wedge d\theta = \mu \) is the Riemannian volume element on \( M \);
- (11) \( d\theta(\varphi X, X) = 1 \) for any unit vector field \( X \) belonging to the distribution \( E \);
- (12) the triple of vectors \( (\varphi X, X, \xi) \), \( X \in E_x \) composes a positively oriented basis of the space \( T_xM \);
- (13) \( \varphi X = X \times \xi \), where \( \times \) is the vector product on the three-dimensional Riemannian manifold \( M \);
- (14) \( *\theta = d\theta, *d\theta = \theta \), where \( * \) is the Hodge operator.

The exact contact transformation \( \eta \) of the manifold \( M \) preserves the forms \( \theta \) and \( d\theta \) and hence preserves the volume element \( \mu = \theta \wedge d\theta \). Therefore, the group \( D_\theta \) of exact contact diffeomorphisms is a subgroup of the group \( D_\mu(M^3) \) of diffeomorphisms preserving the volume element \( \mu \).

Let \( D_\mu(M^3) \) be the group of exact diffeomorphisms preserving the volume element \( \mu \). The Lie algebra \( T_eD_\mu(M^3) \) consists of vector fields \( X \) of a type \( X = \text{rot}Y \). We will prove
that if the contact structure \((M^3, \theta, \xi, g)\) is \(K\)-contact, then the group \(D_\theta\) is a subgroup in \(D_{\mu, \theta}(M^3)\).

**Lemma 4.1.** The characteristic vector field \(\xi\) is an eigenvector of the operator \(\text{rot}\),

\[
\text{rot} \xi = \xi. \tag{4.1}
\]

**Proof.** Indeed, let \(\omega_X = g^{-1}X\) be the differential 1-form corresponding to the vector field \(X\) on \(M\) by the metric tensor \(g\). Then \(\text{rot}X\) is found from the relation \(\omega_{\text{rot}X} = *d\omega_X\), where \(*\) is the Hodge operator. For the characteristic vector field, we have \(\omega_\xi = \theta\). Therefore,

\[
\omega_{\text{rot} \xi} = *d\omega_\xi = *d\theta = \theta = \omega_\xi.
\]

\[\square\]

For the \(K\)-contact structure, the metric tensor \(g\) and the affinor \(\varphi\) are invariant with respect to the action of the group \(S^1\) generated by the field \(\xi\): \(L_\xi g = 0, L_\xi \varphi = 0\), on \(M\). As is known, the contact vector fields commute with the field \(\xi, [\xi, X_f] = 0\). Taking into account that \(X_f = f \xi - \varphi \text{grad} f\) and \(\xi(f) = 0\), we immediately obtain from the latter relation that \([\xi, \varphi \text{grad} f] = 0\). In the case of the \(K\)-contact structure, it is easy to show that the gradients of the contact Hamiltonians \(f\) also commute with \(\xi\):

\([\xi, \text{grad} f] = 0\).

Indeed, since \(\text{grad} f \in \Gamma(E)\), it follows: \([\xi, \text{grad} f] = [\xi, -\varphi^2 \text{grad} f] = -L_\xi \varphi(\varphi \text{grad} f) - \varphi(L_\xi (\varphi \text{grad} f)) = -\varphi(L_\xi (\varphi \text{grad} f)) = -\varphi[\xi, \varphi \text{grad} f] = 0\).

In what follows, we assume that the contact metric structure \((M^3, \theta, \xi, g)\) is \(K\)-contact.

**Lemma 4.2.** If \(X_f = f \xi - \varphi \text{grad} f\) is a contact vector field on \(M^3\), then

\[
\text{rot}X_f = (f - \Delta f) \xi + \varphi \text{grad} f,
\]

where \(\Delta = -\text{div} \circ \text{grad}\) is the Laplacian. In particular,

\[
\text{rot}f \xi = f \xi + \varphi \text{grad} f, \quad \text{rot}(\varphi \text{grad} f) = \Delta f \xi.
\]

**Proof.**

\[
\text{rot}X_f = \text{rot}(f \xi + \xi \times \text{grad} f) = \text{grad} f \times \xi + f \text{rot} \xi + [\text{grad} f, \xi] + \xi \text{div}(\text{grad} f) - \text{grad} f \text{div} \xi = \\
= \varphi \text{grad} f + f \xi + (-\Delta f) \xi = (f - \Delta f) \xi + \varphi \text{grad} f.
\]

Here, we have used the relations \(\varphi X = X \times \xi, \text{div} \xi = 0, [\xi, \text{grad} f] = 0\) and \(\text{rot}(X \times Y) = [Y, X] + X \text{div} Y - Y \text{div} X\). 

In the case of the \(K\)-contact metric, the operator \(\text{rot}^{-1}\) can be explicitly calculated on the contact vector fields.

**Lemma 4.3.** If the contact vector field \(X_f = f \xi - \varphi \text{grad} f \in T_eD_\theta(M^3)\) lies in the orthogonal complement to the vector field \(\xi\) with respect to the inner product \((\ref{1.8})\), that is \(\int_M f \, d\mu_g = 0\), then

\[
\text{rot}^{-1}X_f = -f \xi + 2 \varphi \text{grad}(\Delta^{-1} f), \tag{4.3}
\]

where \(h = \Delta^{-1} f\) there is a function, such that \(\Delta h = f\), \(\xi(h) = 0\) and \(\int_M h \, d\mu_g = 0\).
Proof. A direct verification of the relation \( \text{rot}(\text{rot}^{-1}X_f) = X_f \):

\[
\text{rot}(\text{rot}^{-1}X_f) = \text{rot}(-f \xi + 2 \text{grad}(\Delta^{-1}f) \times \xi) =
\]

\[
= -\text{grad} f \times \xi - f \text{rot} \xi + 2 [\xi, \text{grad}(\Delta^{-1}f)] + 2 \text{grad}(\Delta^{-1}f) \text{div} \xi - 2 \xi \text{div}(\text{grad} (\Delta^{-1}f)) =
\]

\[
= -\varphi \text{grad} f - f \xi + 2 \xi (\Delta)(\Delta^{-1}f) = -\varphi \text{grad} f - f \xi + 2 \xi f = f \xi - \varphi \text{grad} f = X_f.
\]

Here, we have used the relations: \( \varphi X = X \times \xi, \text{div} \xi = 0, [\xi, \text{grad} f] = 0 \) and \( \text{rot}(X \times Y) = [Y, X] + X \text{div} Y - Y \text{div} X \). Note that the vector field \(-f \xi + 2 \varphi \text{grad}(\Delta^{-1}f)\) is divergence-free. \(\square\)

**Corollary 4.4.** If the contact vector field \( X_f = f \xi - \varphi \text{grad} f \in T_e \mathcal{D}_\theta(M^3) \) lies in the orthogonal complement to the vector field \( \xi \), that is \( \int_M f \text{d} \mu_g = 0 \), then

\[
X_f = \text{rot}(-f \xi + 2 \varphi \text{grad}(\Delta^{-1}f))). \quad (4.4)
\]

From the lemma 4.1 and corollary 4.4 follows:

**Corollary 4.5.** If the contact structure \((M^3, \theta, \xi, g)\) is \( K \)-contact, then the group \( \mathcal{D}_\theta \) is a subgroup in \( \mathcal{D}_{\mu \theta}(M^3) \).

On the group \( \mathcal{D}_{\mu \theta}(M^3) \) of exact diffeomorphisms preserving the volume element \( \mu \) there exists the bi-invariant weak Riemannian structure [21]:

\[
\langle X, Y \rangle^{D_\mu}_{e} = \int_M g \left( \text{rot}^{-1}X, Y \right) \text{d} \mu, \quad X, Y \in T_e \mathcal{D}_\mu(M^3). \quad (4.5)
\]

Therefore the group \( \mathcal{D}_\theta \subset \mathcal{D}_{\mu \theta}(M^3) \) inherits one more bi-invariant form

\[
\langle X_f, X_h \rangle^{D_\mu}_{e} = \int_M g \left( \text{rot}^{-1}X_f, X_h \right) \text{d} \mu(x). \quad (4.6)
\]

**Theorem 4.6.** If the contact metric structure on \( M^3 \) is \( K \)-contact, then the bi-invariant inner products \((4.6)\) and \((1.9)\) on \( \mathcal{D}_\theta(M^3) \) are connected by following relations:

If contact vector fields \( X_f, X_h \in T_e \mathcal{D}_\theta(M^3) \) lies in the orthogonal complement to the vector field \( \xi \) with respect to the inner product \((1.8)\), that is \( \int_M f \text{d} \mu_g = \int_M h \text{d} \mu_g = 0 \), then

\[
\langle X_f, X_h \rangle^{D_\mu}_{e} = -3 \langle X_f, X_h \rangle^{D_\theta}_{e}. \quad (4.7)
\]

If \( X_f = \xi \) then for all \( X_h \in T_e \mathcal{D}_\theta(M^3) \),

\[
\langle \xi, X_h \rangle^{D_\mu}_{e} = \langle \xi, X_h \rangle^{D_\theta}_{e}. \quad (4.8)
\]

In particular,

\[
\langle \xi, \xi \rangle^{D_\mu}_{e} = \langle \xi, \xi \rangle^{D_\theta}_{e} = \int_M \text{d} \mu,
\]

and if \( \int_M h \text{d} \mu_g = 0 \), then \( \langle \xi, X_h \rangle^{D_\mu}_{e} = \langle \xi, X_h \rangle^{D_\theta}_{e} = 0 \).

Proof. Indeed, if the contact vector fields \( X_f, X_h \in T_e \mathcal{D}_\theta(M^3) \) lies in the orthogonal complement to the vector field \( \xi \) with respect to the inner product \((1.8)\), then

\[
\langle X_f, X_h \rangle^{D_\mu}_{e} = (\text{rot}^{-1}X_f, X_h)_{e} = (-f \xi + 2 \varphi \text{grad}(\Delta^{-1}f), h \xi - \varphi \text{grad} h)_{e} =
\]

\[
= -(f \xi, h \xi)_{e} - 2 (\varphi \text{grad}(\Delta^{-1}f), \varphi \text{grad} h)_{e} - \int_M f h \text{d} \mu - 2 \int_M g(\text{grad}(\Delta^{-1}f), \text{grad} h) \text{d} \mu =
\]

\[
= -\int_M f h \text{d} \mu + 2 \int_M \text{div}(\text{grad}(\Delta^{-1}f)) h \text{d} \mu = -\int_M f h \text{d} \mu - 2 \int_M f h \text{d} \mu = -3 \langle X_f, X_h \rangle^{D_\theta}_{e}. \]

If $X_f = \xi$, then
\[
\langle \xi, X_h \rangle^D = (\text{rot}^{-1}\xi, X_h)_e = (\xi, h\xi - \varphi \text{grad} h)_e = \int_M h \, d\mu = \langle \xi, X_h \rangle^D.
\]
\[
\langle \xi, \xi \rangle^D_e = \int_M d\mu.
\]

References

[1] Arnold V. Sur la geometrie differentielle des groupes de Lie de dimenzi on infinite et ses applications a l’hidrodynamique des fluides parfaits. Ann. Institut Fourier. 1966, Vol. 16, No. 1, P. 319-361.

[2] Arnold V. I. Variational principle for three-dimensional stationary flows of the ideal fluid. Prikl. Mat. Mekh., 1965, Vol. 29, No. 5, P. 846|851.

[3] Arnold V.I. Mathematical Methods of Classical Mechanics. Springer. 1989.

[4] Arnold V.I. and Khesin B. Topological Methods in Hydrodynamics. Springer Verlag, New York, 1998.

[5] Atiyah M.F., Patodi V.K., Singer I.M. Spectral asymmetry and Riemannian Geometry. I. Math. Proc. Camb. Phil. Soc. 1975, Vol. 77, P. 43-69.

[6] Atiyah M.F., Patodi V.K., Singer I.M. Spectral asymmetry and Riemannian Geometry. II. Math. Proc. Camb. Phil. Soc. 1975, Vol. 78, 405-432.

[7] Blair D.E. Riemannian Geometry of Contact and Symplectic Manifolds. Progress in Mathematics, Vol. 203. Birkhauser, 2010.

[8] Ebin D., Marsden J. Groups of diffeomorphisms and the motion of an incompressible fluid. Ann. of Math. 1970, Vol. 92, No. 1, P. 102-163.

[9] Ebin D.G., Preston S. C. Riemannian geometry on the quantomorphism group. arXiv:1302.5075 [math.DG], 36 p.

[10] Etnyre J., Ghrist R. Contact Topology and Hydrodynamics. arXiv:dg-ga/9708011, 155 p.

[11] Etnyre J., Ghrist R. Contact topology and hydrodynamics II: solid tori. arXiv:math/9907112 [math.SG], 14 p.

[12] Etnyre J., Ghrist R. Contact topology and hydrodynamics III: knotted flowlines. arXiv:math-ph/9906021, 17 p.

[13] Etnyre J., Ghrist R. An index for closed orbits in Beltrami fields. arXiv:math/0101095 [math.DS], 14 p.

[14] Gromov M. Partial Differential Relations. Springer, 1986.

[15] Kobayashi S. and Nomizu K. Foundations of Differential Geometry, Vol. 1, 2. Inter-science Publ. 1963.
[16] Mishchenko A.S. and Fomenko A.T. Euler equations on finite-dimensional Lie groups. Izv. Akad. Nauk SSSR, Ser. Mat. 1978, Vol. 42, No. 2, 396-415.

[17] Omori H. On the group of diffeomorphisms on a compact manifold. Proc. Symp. Pure Math., vol. 15, Amer. Math. Soc. 1970. P. 167-183.

[18] Omori H. On smooth extension theorems. J. Math. Soc. Japan. 1972, Vol. 24, No. 3, P. 405-432.

[19] Omori H. Infinite dimensional Lie transformations groups. Lect. Notes Math., vol. 427, 1974.

[20] Smolentsev N. K. On a certain weak Riemannian structure on the diffeomorphism group. Izv. Vyssh. Ucheb. Zaved., Ser. Mat. 1979, Vol. 5, 78-80.

[21] Smolentsev N. K. Bi-invariant metric on the diffeomorphism group of a three-dimensional manifold. Sib. Mat. Zh. 1983, Vol. 24, No. 1, 152-159. [arXiv:1401.7415v1 [math.DG], 2014].

[22] Smolentsev N. K. Bi-invariant metrics on the symplectic diffeomorphism group and the equation $\frac{\partial}{\partial t} \Delta F = \{\Delta F, F\}$. Sib. Mat. Zh. 1986, Vol. 27, No. 1, 150156.

[23] Smolentsev N. K. Curvature of the classical diffeomorphism groups. Sib. Mat. Zh. 1994, Vol. 35, No. 1, 169176.

[24] Serrin J. Mathematical Principles of Classical Fluid Mechanics. Springer, 1959.