Non-local properties of a symmetric two-qubit system

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Non-local properties of symmetric two-qubit states are quantified in terms of a complete set of entanglement invariants. We prove that negative values of some of the invariants are signatures of quantum entanglement. This leads us to identify sufficient conditions for non-separability in terms of entanglement invariants. Non-local properties of two-qubit states extracted from (i) Dicke state (ii) state generated by one-axis twisting Hamiltonian, and (iii) one-dimensional Ising chain with nearest neighbour interaction are analyzed in terms of the invariants characterizing them.

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I. INTRODUCTION

Quantum entanglement is one of the most striking properties of quantum theory and it plays a crucial role in the rapidly developing areas such as quantum computation and quantum communication, quantum teleportation, and quantum cryptography. Entanglement reflects itself through non-local correlations among the subsystems of a quantum system. These non-local properties remain unaltered by local manipulations on the subsystems and provide a characterization of quantum entanglement. Two quantum states \( \rho_1 \) and \( \rho_2 \) are said to be equally entangled if they are related to each other through local unitary operations, which merely imply a choice of bases in the spaces of the subsystems. The non-local properties associated with a quantum state can be represented in terms of a complete set of local invariants characterizing quantum entanglement. Such two-qubit systems (N=2) restrict them to a complete set of 18 polynomial invariants has been identified in the case of pure three qubit states. Linden et. al. have outlined a general prescription to identify the invariants associated with a multi particle system. In this paper we will focus on entanglement invariants of symmetric two-qubit systems, which exhibit exchange symmetry. Such two-qubit systems (N=2) restrict themselves to a space spanned by the symmetric angular momentum states \( |S, M\rangle \) with \( S = \frac{N}{2} = 1 \), \( M = -1, 0, 1 \). We have organized this paper as follows: In Sec. III we identify that the number of invariants required to characterize a symmetric two-qubit system reduces from 18 (proposed by Makhlin) to 6. Moreover, we consider a specific case of symmetric two-qubit system, which is realized in several physically interesting examples like, even and odd spin states, Kitagawa - Ueda state generated by one-axis twisting Hamiltonian, and steady state of two-level atoms in a squeezed bath etc. For this special case of symmetric two-qubit system, we show that a subset of three independent invariants is sufficient to characterize the non-local properties completely. Sec. IV discusses separable symmetric two-qubit states, for which some of the invariants necessarily assume positive values. Using this result, we propose criteria, which provide a characterization of non-separability (entanglement) in symmetric two-qubit states. In Sec. V we discuss physical examples of symmetric two-qubits extracted from N qubit systems like (i) Dicke state, (ii) state generated by one-axis twisting Hamiltonian and (iii) state extracted from a one-dimensional Ising chain with nearest neighbour interaction. Sec. VI contains a brief summary of our results.

II. ENTANGLEMENT INVARIANTS

Density matrix of an arbitrary two-qubit state in the Hilbert-Schmidt space \( \mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2 \) is given by

\[
\rho = \frac{1}{4} \left( I \otimes I + s^i \cdot (\vec{\sigma} \otimes I) + (I \otimes \vec{\sigma}) \cdot \vec{r} + \sum_{i,j=1}^{3} t_{ij} (\sigma_i \otimes \sigma_j) \right),
\]

where \( I \) denotes the \( 2 \times 2 \) unit matrix; \( \sigma_i, \ i = 1, 2, 3 \) are the standard Pauli matrices; \( s^i = Tr [\rho (\vec{\sigma} \otimes I)] \) and \( \vec{r} = Tr [\rho (I \otimes \vec{\sigma})] \) denote average spins of the qubits; \( t_{ij} = Tr [\rho (\sigma_i \otimes \sigma_j)] \) are the elements of the real \( 3 \times 3 \) matrix \( T \) corresponding to two-qubit correlations. The set of 15 state parameters \( s_i, r_i, t_{ij} \) transform under local unitary operations \( U_1 \otimes U_2 \) on the qubits as follows:

\[
s_i' = \sum_{j=1}^{3} O_{ij}^{(1)} s_j, \quad r_i' = \sum_{j=1}^{3} O_{ij}^{(2)} r_j
\]

\[
t_{ij}' = \sum_{k,l=1}^{3} O_{ik}^{(1)} O_{jl}^{(2)} t_{kl} \quad \text{or} \quad T' = O^{(1)} T O^{(2)}, \quad (2)
\]
where $O(i) \in SO(3, R)$ are the $3 \times 3$ rotation matrices, uniquely corresponding to the $2 \times 2$ unitary matrices $U_i \in SU(2)$. Quantum entanglement of a given two-qubit state remains unaltered under local unitary transformations $U_1 \otimes U_2$ and has been quantitatively captured in terms of a complete set of 18 invariants [6] given below:

\[ I_1 = \det T, \quad I_2 = \text{Tr} (T^d T), \quad I_3 = \text{Tr} (T^d T^2), \]
\[ I_4 = s^1 s, \quad I_5 = s^1 T T^d s, \quad I_6 = s^1 (T T^d)^2 s, \]
\[ I_7 = r^1 r, \quad I_8 = r^1 T T^d r, \quad I_9 = r^1 (T T^d)^2 r, \]
\[ I_{10} = \epsilon_{ijk} s_i (T T^d)^j s_k, \quad I_{11} = \epsilon_{ijk} r_i (T T^d)^j r_k, \quad I_{12} = s^1 T T^d r, \]
\[ I_{13} = s^1 T T^d T, \quad I_{14} = s^1 t_{ij} t_{jm} t_{kn}, \quad I_{15} = \epsilon_{ijk} (T T^d)^j r_k, \]
\[ I_{16} = \epsilon_{ijk} (T T^d)^j r_k, \quad I_{17} = \epsilon_{ijk} (T T^d)^j r_k, \quad I_{18} = \epsilon_{ijk} s_i (T T^d)^j r_k. \]

It is easy to verify that all the $I_k$, $k = 1, 2, \ldots, 18$, are invariant, when the state parameters $s_i, r_i, t_{ij}$ transform - under local unitary transformations of the two-qubit state $\rho$ - as shown in Eq. 2. Makhlin [6] has given an explicit procedure to find local unitary operations that transform any equivalent density matrices to a specific form, uniquely determined by the set of 18 invariants given in Eq. 3. For example, $I_1 - I_3$ determine the diagonal form $T^d$ of the correlation matrix $T$ (achieved through local operation $T^d = O(1) T O(2)$); the invariants $I_4 - I_6$ and $I_7 - I_9$ specify the absolute values of the six components of the average spin $s$ and $r$ of the qubits; other invariants $I_{10} - I_{18}$ determine the signs of these components. It is thus shown [6] that the two-qubit states $\rho_1$ and $\rho_2$ are locally equivalent iff the invariants $I_1 - I_{18}$ are identical for these states. Further, any quantitative measure of entanglement should be a function of these invariants.

We turn our focus to symmetric two-qubit systems, which respect exchange symmetry viz.,

\[ \Pi_{12} \rho(\text{sym}) \Pi_{12}^{-1} = \rho(\text{sym}), \]

$\Pi_{12}$ being the permutation operator. Quantum states of symmetric two-qubit $N = 2$ systems get confined to a three dimensional subspace \{ $\frac{1}{\sqrt{2}}, M; M = \pm 1, 0$ \} of the Hilbert space $C^2 \otimes C^2$. The state parameters of a symmetric two-qubit system satisfy the following constraints due to exchange symmetry [6]:

\[ r_i = s_i, \quad t_{ij} = t_{ji}, \quad \text{Tr} (T) = 1, \]

and thus 8 real state parameters viz., $r_i$ and the elements $t_{ij}$ of the real symmetric correlation matrix $T$, which has unit trace, characterize a symmetric two-qubit system. For a two-qubit state obeying exchange symmetry it is easy to see that $I_4 = I_7, I_5 = I_8, I_6 = I_9, I_{10} = I_{11}, I_{15} = I_{16}$ and $I_{17} = I_{18}$, reducing the number of invariants to 9. Moreover, since the matrix $T$ is real symmetric with $\text{Tr}(T) = 1$, the invariants $I_1$ and $I_2$ alone suffice to determine the diagonal form of the two-qubit correlation matrix $T$ (i.e., $I_1 = t_1 t_2 t_3$ and $I_2 = t_1^2 + t_2^2 + t_3^2$ determine the eigenvalues $t_1, t_2, t_3$ of the real symmetric two-qubit correlation matrix $T$ in the case of symmetric qubits). In other words, for a symmetric two-qubit system, the subset ($I_1, I_2, I_3$) of three invariants reduces to ($I_1, I_2$). Along with $I_1, I_2$, the invariants

\[ I_4 = s_1^2 + s_2^2 + s_3^2, \]
\[ I_{12} = s_1^2 t_1 + s_2^2 t_2 + s_3^2 t_3, \]
\[ I_{14} = 2 (s_1^2 t_2 t_3 + s_2^2 t_1 t_3 + s_3^2 t_1 t_2), \]

(5)

allow for the determination of $s_1^2, s_2^2$ and $s_3^2$, thus fixing the absolute values of the components of qubit orientation vector $\vec{s}$. One more invariant,

\[ I_{10} = (t_1^2 (t_3^2 - t_2^2) + t_2^2 (t_1^2 - t_3^2) + t_3^2 (t_2^2 - t_1^2)) s_1 s_2 s_3, \]

(6)

fixes the relative signs of $s_1^2, s_2^2$ and $s_3^2$. Therefore the subset ($I_1, I_2, I_4, I_{10}, I_{12}$ and $I_{14}$) of six invariants gives a complete characterization of the non-local properties of symmetric two-qubit states.

Many physically interesting cases of symmetric two-qubit states like for e.g., even and odd spin states [11], Kitagawa - Ueda state generated by one-axis twisting Hamiltonian [12], qubits in a one-dimensional Ising chain [13], steady state of two-level atoms in a squeezed bath [14], exhibit a particularly simple structure

\[ \rho(\text{sym}) = \begin{pmatrix} a & 0 & 0 & b \\ 0 & c & 0 & 0 \\ 0 & c & 0 & 0 \\ b^* & 0 & 0 & d \end{pmatrix}, \]

(7)

of the density matrix (in the standard two-qubit basis $|01, 02, 10, 12, 11, 02, 12, 12, 12, 12\rangle$ and it will be interesting to analyze the non-local properties of such systems through local invariants. The specific structure $\rho(\text{sym})$ given by Eq. 7 of the two-qubit density matrix further reduces the number of parameters essential for the problem. Entanglement invariants associated with the symmetric two-qubit system, with a simple structure $\rho(\text{sym})$ given by Eq. 7 for the state, may now be identified through a simple calculation to be

\[ I_1 = (4 c^2 - 4 |b|^2) (1 - 4 c), \]
\[ I_2 = (2 c^2 + 2 |b|^2) (1 - 2 |b|^2) + (1 - 4 c)^2, \]
\[ I_4 = (a - d)^2, \quad I_{10} = 0, \]
\[ I_{12} = (a - d)^2 (1 - 4 c), I_{14} = 8 (a - d)^2 (c^2 - |b|^2). \]

(8)

In this special case we can express the invariants $I_1$ and $I_2$ in terms of $I_4, I_{12}$ and $I_{14}$ (provided $I_4 \neq 0$):

\[ I_1 = I_{14} I_{12} I_{12} / 2 I_4^2, \quad I_2 = (I_4 - I_{12} I_{12} - I_4 I_{14} + I_{12}^2) / I_4^2. \]

(9)
If \( I_4 = 0 \), the set containing six invariants reduces to the subset of non-zero invariants \( (I_1, I_2) \) leading to the identification that the set \( (I_4, I_{12}, I_{14}) \) (or \( (I_1, I_2) \) if \( I_4 = 0 \)) characterizes the non-local properties of symmetric two-qubit states having a specific structure \( \varrho^{(\text{sym})} \) given by Eq. \( 4 \) for the density matrix.

III. CHARACTERIZATION OF SEPARABLE SYMMETRIC TWO-QUBIT STATES

A separable symmetric two-qubit density matrix is an arbitrary convex combination of direct product of identical single qubit states \( \rho_w = \frac{1}{2} \left( I + \sum_{i=1}^{3} \sigma_i s_{wi} \right) \) and is given by

\[
\rho^{(\text{sym-sep})} = \sum_w p_w \rho_w \otimes \rho_w; \quad \sum_w p_w = 1. \tag{10}
\]

Separable symmetric system is a classically correlated system, which can be prepared through classical communications between two parties. One of the important goals of quantum information theory has been to identify and characterize separability. We look for such identifying criteria for separability, in terms of entanglement invariants, in the following theorem:

Theorem: The invariants, \( I_{14}, I_{12} \) and a combination \( I_{12} - I_4^2 \) of the invariants, necessarily assume positive values for a symmetric separable two-qubit state, which has a non-zero value for the invariant \( I_4 \).

Proof: First we note that in a separable symmetric system the correlation matrix elements are given by

\[
t_{ij} = \sum_w p_w s_{wi} s_{wj}, \quad \text{and the components of the average spin of the qubits are given by } s_i = \sum_w p_w s_{wi}. \]

We therefore obtain,

(i) \[
I_{12} = \frac{1}{2} t^* T t = \sum_{i,j=1}^{3} t_{ij} s_i s_j
= \sum_w p_w \left( \sum_{i=1}^{3} s_{wi} s_i \right) \left( \sum_{j=1}^{3} s_{wj} s_j \right)
= \sum_w p_w (\vec{s}_w \cdot \vec{s})^2 \geq 0;
\]

(ii) \[
I_{14} = \epsilon_{ijk} \epsilon_{lmn} s_i s_l t_{jm} t_{kn}
= \sum_{w,w'} p_w p_{w'} (\vec{s}_w \times \vec{s}_{w'})^2 \geq 0;
\]

(iii) \[
I_{12} - I_4^2 = \sum_w p_w (\vec{s}_w \cdot \vec{s})^2 - \left( \sum_w p_w (\vec{s}_w \cdot \vec{s}) \right)^2
= \left( (\vec{s}_w \cdot \vec{s})^2 \right) - \left( (\vec{s}_w \cdot \vec{s}) \right)^2 \geq 0, \tag{11}
\]

for a separable symmetric two-qubit system, proving the above theorem.

We conclude that non-positive values of \( I_{12}, I_{14} \) or \( I_{12} - I_4^2 \) serve as a signature of entanglement and hence provide sufficient conditions for non-separability of the quantum state. It would be interesting to explore how these constraints on the invariants, get related to the other well established criteria of entanglement. For two qubits states, it is well known that Peres’s PPT (positivity of partial transpose) criterion is both necessary and sufficient for separability. We now proceed to show that, in the case of symmetric states given by Eq. \( 7 \), there exists a simple connection between the Peres’s PPT criterion and the non-separability constraints (see Eq. \( 11 \)) on the invariants.

It is easy to identify the eigenvalues of the partially transposed density matrix \( \varrho^{(\text{sym})} \) of Eq. \( 7 \):

\[
\lambda_{1,2} = \frac{1}{2} \left( (a + d) \mp \sqrt{(a - d)^2 + 4c^2} \right),
\]

\[
\lambda_{3,4} = c \mp |b|
\]

of which \( \lambda_1 \) and \( \lambda_3 \) can assume negative values. Now, (i) for \( \lambda_1 < 0 \), we must have \((a + d)^2 < (a - d)^2 + 4c^2\), which on using \( \text{Tr}(\varrho^{(\text{sym})}) = a + d + 2c = 1 \), reduces to the inequality \((1 - 4c) < (a - d)^2\). Obviously (see Eq. \( 8 \)),

\[
I_{12} - I_4^2 = (a - d)^2 \left( 1 - 4c \right) - (a - d)^2, \tag{12}
\]

associated with the state \( \varrho^{(\text{sym})} \), is negative when \( \lambda_1 < 0 \). Further, (ii) we may express the invariants \( I_{14} \) (using Eq. \( 5 \)) corresponding to the state \( \varrho^{(\text{sym})} \), in terms of the eigenvalue \( \lambda_3 \) as

\[
I_{14} = 8 (a - d)^2 (c + |b|) \lambda_3, \tag{13}
\]

from which it follows immediately that, \( I_{14} < 0 \) if \( \lambda_3 < 0 \). We have thus established an equivalence between the PPT criterion and the constraints on the invariants for two qubit symmetric state of the form \( \varrho^{(\text{sym})} \). In other words, the nonseparability conditions \( I_{12} - I_4^2, I_{14} < 0 \), are both necessary and sufficient for a class of two-qubit symmetric states given by Eq. \( 7 \). However, we leave open the question whether nonseparability constraints \( I_{12} - I_4^2, I_{12} < 0, I_{14} < 0 \) on the invariants, serve as both necessary and sufficient for an arbitrary symmetric two qubit system.

IV. EXAMPLES OF SYMMETRIC TWO-QUBIT STATES

We consider some physical examples of two-qubit states, where random pairs of qubits are extracted from a symmetric state of \( N \) qubits.

A. Two-qubit system drawn from a Dicke State

Density matrix of a random pair of qubits picked from a symmetric Dicke state \( |S = \frac{N}{2}, M\rangle, -\frac{N}{2} \leq M \leq \frac{N}{2} \)
\( \frac{N}{4} \) of \( N \) qubits has the simple structure given by Eq. (7) with
\[
a = \frac{(N + 2M)(N - 2M)}{4N(N - 1)}, \quad b = 0, \quad c = \frac{N^2 - 4M^2}{4N(N - 1)}
\]
and \( d = 1 - a - 2c \). The invariants \( I_4, I_{12}, I_{14} \) and \( I_{12} - I_4^2 \) associated with this system are given by
\[
I_4 = \left( \frac{2M}{N} \right)^2, \quad I_{12} = I_4 \left( \frac{4M^2 - N}{N(N - 1)} \right),
\]
\[
I_{14} = 8I_4 \left( \frac{N^2 - 4M^2}{4N(N - 1)} \right)^2,
\]
\[
I_{12} - I_4^2 = I_4 \left( \frac{4M^2 - N^2}{N^2(N - 1)} \right)
\]
and it is easy to identify that \( I_{14} \geq 0 \), but \( I_{12} - I_4^2 \) is negative, implying that the system is indeed non-separable (except for the states with \( M = \pm \frac{N}{2} \)). However it is interesting to identify that as \( N \to \infty \), \( I_{12} - I_4^2 \) approaches zero, revealing that the system tends to be separable in this limit.

### B. Two-qubit state extracted from non-linear one-axis twisting Hamiltonian

Kitagawa and Ueda [12] had proposed a non-linear Hamiltonian \( H = \chi S_x^2 \), where \( S_x \) is the \( x \)-component of collective angular momentum of a \( N \) two level system, to generate multi-atom spin squeezed states. A random pair of two-qubits of such a spin squeezed state has the density operator [13] with
\[
a = \frac{1}{8} (3 + \cos^{(N-2)}(2\chi t) - 4 \cos^{(N-1)}(\chi t)), \quad \text{Im } b = \frac{1}{2} \cos^{(N-1)}(\chi t) \sin(\chi t), \quad c = \frac{1}{8} (1 - \cos^{(N-2)}(2\chi t)) = -\text{Re } b
\]
and the corresponding invariants \( (I_4, I_{12}, I_{14}) \) are given by,
\[
I_4 = \cos^{2(N-1)}(\chi t), \quad I_{12} = \frac{I_4}{2} \left( 1 + \cos^{(N-2)}(2\chi t) \right),
\]
\[
I_{14} = -2I_4 \cos^{2(N-2)}(\chi t) \sin^2(\chi t)
\]
Note that \( I_{14} \) is manifestly negative, highlighting the non-local nature of the state.

### C. State extracted from 1-d Ising chain of N-qubits

A one dimensional chain of \( N \) qubits with nearest neighbour interaction is characterized by the Ising model Hamiltonian \( H = \frac{\chi}{4} \sum_{i=1}^{N} \sigma_i x \sigma_{i+1} x \), where \( \chi \) denotes the interaction strength. The Hamiltonian evolution of \( N \) qubits, taken in a all-down initial state, results in a two-qubit density matrix [13] with the state parameters,
\[
a = \frac{4(N-1)(1 + \cos^2(\chi t)) - \sin^2(\chi t)}{8(N-1)}, \quad b = -\frac{\sin(\chi t)(\sin(\chi t) + 4i)}{8(N-1)}, \quad c = \frac{\sin^2(\chi t)}{8(N-1)}
\]
and the local entanglement invariants are given by,
\[
I_4 = \cos^4 \left( \frac{\chi t}{2} \right), \quad I_{12} = I_4 \left( 1 - \frac{\sin^2(\chi t)}{2(N-1)} \right), \quad I_{14} = -2I_4 \sin^2(\chi t) \left( \frac{1}{(N-1)^2} \right)
\]
in this model the local invariant \( I_{14} \) assumes negative value indicating that the system is entangled.

### V. CONCLUSIONS

We have shown that a subset \( (I_1, I_2, I_4, I_{10}, I_{12}, I_{14}) \), of a more general set of 18 invariants proposed by Makhlin [7], is sufficient to characterize the non-local properties of a symmetric two-qubit system. Invariants \( I_{12}, I_{14} \) and \( I_{12} - I_4^2 \) of separable symmetric two-qubit states are shown to be non-negative. We have proposed sufficient conditions for identifying entanglement in symmetric two-qubit states, when the qubits have a non-zero value for the average spin. Moreover these conditions on the invariants are shown to be necessary and sufficient for a class of symmetric two-qubit states. We have calculated the invariants of some physical examples of two-qubit states picked from \( N \) qubit states like, (i) symmetric Dicke state, (ii) state generated by one-axis twisting Hamiltonian and (iii) 1-d Ising model of \( N \) qubits. We have explicitly demonstrated the non-separability of such states through our characterization in terms of local invariants.
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[9] The squared collective angular momentum operator satisfies the condition $S_x^2 + S_y^2 + S_z^2 = S(S + 1) = \frac{N(N + 1)}{4}$ for symmetric states. More explicitly we have, $\frac{1}{4}(\sigma_1 + \sigma_2)^2 = \frac{N(N + 1)}{4} = 2$, which in turn leads to condition $\text{Tr}(T) = 1$ on the correlation matrix of the two-qubit symmetric states.

[10] Note that if the product $s_1 s_2 s_3$ is positive, and say, $(+,+,+)$ are the signs of each components, identical local rotations about the axes $i = 1, 2, 3$ on the qubits, by an angle $180^\circ$ affect only the signs of the components and lead to the alternative possibilities $(+,-,-), (-,+,-)$ or $(-,-,+)$, all of which have positive sign for the product $s_1 s_2 s_3$. Similarly, if the product $s_1 s_2 s_3$ is negative, and say, has $(-,-,-)$ signs for each components, these identical local rotations through an angle $180^\circ$ change the signs of the components to $(−,+,−)$, $(+,−,+)$ or $+(+,−)$, all of which have negative sign for the product $s_1 s_2 s_3$ and the absolute sign of the product $s_1 s_2 s_3$ is a local invariant.

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