RESONANCES AND LOCALIZATION
IN MULTI-PARTICLE DISORDERED SYSTEMS

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Abstract. This is a complement to our earlier work [4] where a new eigenvalue concentration bound for multi-particle disordered quantum lattice systems was obtained. Here we show that the new bound leads to a simplified proof of multi-particle spectral and dynamical localization.

1. Introduction. The model and the motivation for this paper

We study multi-particle quantum systems in a disordered environment, usually referred to as Anderson-type models. Consider an $N$-particle tight-binding Hamiltonian $H_{V,U}(\omega)$ in the Hilbert space $\ell^2(\mathbb{Z}^d)$,

$$H_{V,U} = \Delta + gV + U = \sum_{j=1}^{N} \left( \Delta^{(j)} + gV_{x_j, \omega} \right) + U, \quad (1.1)$$

where $\Delta$ is the nearest-neighbor lattice Laplacian,

$$\Delta^{(j)} \Psi(x) \equiv \Delta^{(j)} \Psi(x_1, \ldots, x_N) = \sum_{y \in \mathbb{Z}^d: |y|=1} \Psi(x_1, \ldots, x_j + y, \ldots, x_N),$$

$V : \mathbb{Z}^d \times \Omega \to \mathbb{R}$ is a random field relative to a probability space $(\Omega, F, P)$, $g > 0$ is a constant measuring the "amplitude" of the potential $V$, and $U$ is the multiplication operator by a function $U(x)$ which we assume bounded, but not necessarily symmetric. In addition, we assume that $U$ has a finite range $r_0 < \infty$. In the case of a two-body interaction generated by an interaction potential $U^{(2)} : \mathbb{Z}^d \to \mathbb{R}$,

$$U(x_1, \ldots, x_N) = \sum_{i<j} U^{(2)}(|x_i - x_j|),$$

this hypothesis is equivalent to the following condition: $\text{supp} U^{(2)} \subset [0, r_0]$. The assumptions on the random field $V$ are described in Sect. 1.3. Unless otherwise specified, the boldface symbols denote objects relative to multi-particle systems.

Given any finite cube $C_L(u) := \{ x \in \mathbb{Z}^d : \|x - u\| \leq L \}$, we will consider a finite-volume approximation of the Hamiltonian $H$

$$H_{C_L(u)} = H |_{\ell^2(C_L(u))}$$

with Dirichlet boundary conditions on $\partial C_L(u)$ acting in the finite-dimensional Hilbert space $\ell^2(C_L(u))$. In [5] the following "two-volume" version of the Wegner bound was established for pairs of two-particle operators $H_{C_L(u)}, H_{C_L(u')}$, such that $L \geq L'$ and $\text{dist}(C_L(u), C_L(u')) \geq 8L$: if $\nu$ is the continuity modulus of the marginal distribution function $F_V$, then

$$\mathbb{P} \left\{ \text{dist}(\sigma(H_{C_L(u)}), \sigma(H_{C_L(u')})) \leq \epsilon \right\} \leq (2L + 1)^{2d} (2L' + 1)^d \nu(2\epsilon). \quad (W2)$$

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1.1. More efficient eigenvalue concentration bounds. In \cite{3} the MPMSA was used to prove spectral localization (i.e., exponential decay of eigenfunctions) in the strong disorder regime. Aizenman and Warzel \cite{2} used the FMM to prove directly dynamical localization (hence, spectral localization) in various regions in the parameter space, including strong disorder, “extreme” energies and weak interactions.

However, due to a highly correlated nature of the potential energy in multi-particle systems, it was difficult to obtain optimal decay bounds for eigenfunctions in terms of some norm in \( \mathbb{Z}^{Nd} \). This difficulty had been analyzed by Aizenman and Warzel \cite{2} and served as the motivation for a more efficient eigenvalue concentration bound proven in \cite{4} (cf. Thm. 1.1 below). Here we prove spectral and dynamical localization for multi-particle systems with the potential \( V(x; \omega) \) satisfying the hypotheses of Thm. 1.1, for \(|g|\) large enough \( \|g\| \) (strong disorder regime).

We would like to emphasize that the novelty of the present paper consists in the observation that the proof of dynamical (and spectral) localization for multi-particle systems, based upon Thm. 1.1, can be obtained by a modification of the single-particle version, once the key MSA bounds are established for the multi-particle system in question. In the author’s opinion, this opens a way to numerous extensions of existing “single-particle” techniques to disordered systems with interaction.

1.2. Basic geometrical definitions. Consider the lattice \( (\mathbb{Z}^d)^N \cong \mathbb{Z}^{Nd}, N > 1 \). Vectors \( x = (x_1, \ldots, x_N) \in \mathbb{Z}^{Nd} \) will be identified with \( N \)-particle configurations in \( \mathbb{Z}^d \).

We use below the max-norm \( \| \cdot \|_\infty \) on \( \mathbb{R}^{nd} \supset \mathbb{Z}^{nd} \), \( n \geq 1 \):

\[
\|x\|_\infty = \|(x_1, \ldots, x_n)\|_\infty := \max_{i \in [1,n]} \max_{j \in [1,d]} |x^{(i)}_j|.
\]

This norm canonically induces the notion of diameter for subsets of \( \mathbb{R}^{nd} \) and \( \mathbb{Z}^{nd} \), denoted below as “diam”. We denote by \( \mathbb{D} \) the “principal diagonal” in \( (\mathbb{Z}^d)^N \):

\[
\mathbb{D} = \{ x \in \mathbb{Z}^{Nd} : x = (x, \ldots, x), x \in \mathbb{Z}^d \}.
\]

We will often use the standard notation \( [[a, b]] := [a, b] \cap \mathbb{Z} \).

Taking into account the symmetry of the potential energy \( V(x_1; \omega) + \cdots + V(x_N; \omega) \), it is natural to introduce also the “symmetrized” distance

\[
d_S(x, y) := \min_{\tau \in \mathbb{D}_N} \|x - y\|.
\]

Given a cube \( C_L(u) = \{ x : \|x - u\| \leq L \} \subset \mathbb{Z}^{Nd} \), we define three kinds of its “boundaries”:

\[
\partial^- C_L(u) = \{ x : \|x - u\| = L \},
\]

\[
\partial^+ C_L(u) = \{ x : \|x - u\| = L + 1 \},
\]

\[
\partial C_L(u) = \{ (x, x') : x \in \partial^- C_L(u), x' \in \partial^+ C_L(u), \|x - x'\| = 1 \}.
\]

1.3. The main result on multi-particle eigenvalue concentration. Introduce the following notations. Given a parallelepiped \( Q \subset \mathbb{Z}^d \), we denote by \( \xi_Q(\omega) \) the sample mean of the random field \( V \) over \( Q \),

\[
\xi_Q(\omega) = \frac{1}{|Q|} \sum_{x \in Q} V(x, \omega)
\]

\(^1\)An adaptation of our method to the case of weak disorder, at “extreme” energies, is the subject of a forthcoming manuscript by T. Ekanga \cite{10}.
and introduce the ”fluctuations” of $V$ relative to the sample mean,

$$\eta_x = V(x, \omega) - \xi_Q(\omega), \ x \in Q.$$ 

We denote by $\mathcal{F}_{V,Q}$ the sigma-algebra generated by $\{\eta_x, x \in Q; V(y, \cdot), y \notin Q\}$, and by $F_\xi(\cdot | \mathcal{F}_{V,Q})$ the conditional distribution function of $\xi_Q$ given $\mathcal{F}_{V,Q}$:

$$F_\xi(s | \mathcal{F}_{V,Q}) := \mathbb{P} \{ \xi_Q \leq s | \mathcal{F}_{V,Q} \}.$$

We will assume that the random field $V$ satisfies the following condition:

\textbf{(CM($\nu$))}: For any $R \geq 0$ there exists a function $\nu_R : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ vanishing at 0 and such that $\forall Q \subset \mathbb{Z}^d$ with $\text{diam}(Q) \leq R$ the conditional distribution function $F_\xi(\cdot | \mathcal{F}_{V,Q})$ satisfies

$$\forall t, s \in \mathbb{R}, \ \text{ess sup} |F_\xi(t | \mathcal{F}_{V,Q}) - F_\xi(s | \mathcal{F}_{V,Q})| \leq \nu_R(|t - s|). \quad (1.2)$$

This condition may prove useless for applications if $\nu_R(s) \downarrow 0$ too slowly as $s \downarrow 0$. For this reason, we assume below, in addition to (1.2), that

$$\nu_R(t) \leq \text{Const } R^A \ln^{-B} |t|^{-1}, \ |t| < 1, \quad (1.3)$$

for some $A < \infty$ and sufficiently large $B$. Alternatively, a stronger condition of Hölder- or Lipshitz-continuity can be used: for some $A < \infty$ and $b > 0$,

$$\nu_R(t) \leq \text{Const } R^A |t|^b, \ |t| < 1. \quad (1.4)$$

Note that in the particular case of a Gaussian IID field $V$ with zero mean and unit variance, $\xi_Q$ is a Gaussian random variable with variance $|Q|^{-1}$, independent of the ”fluctuations” $\eta_x$, so that its probability density exists and is bounded:

$$p_{\xi_Q}(s) = |Q|^{1/2} (2\pi)^{-1/2} e^{-\frac{|Q|^2 s^2}{2}} \leq |Q|^{1/2} (2\pi)^{-1/2},$$

although $\|p_{\xi_Q}\|_\infty$ grows with $|Q|$, and so does the continuity modulus of $F_{\xi_Q}$.

\textbf{Theorem 1.1} (Cf. [4]). Let $V : \mathbb{Z}^d \times \Omega \rightarrow \mathbb{R}$ be a random field satisfying \textbf{(CM($\nu$))}. Then for any pair of $N$-particle operators $H_{C_L(u',\cdot)}$, $H_{C_{L''}(u'',\cdot)}$, $0 \leq L', L'' \leq L$, satisfying $d_{s}(u',u'') > 2NL$, and any $s > 0$ the following bound holds:

$$\mathbb{P} \{ \text{dist}(\sigma(H_{C_{L'}(u')}), \sigma(H_{C_{L''}(u'')})) \leq s \} \leq |C_{L'}(u')| \cdot |C_{L''}(u'')| |\nu_L(2s)|. \quad (1.5)$$

An adaptation of our method to a class of bounded potentials $V(\cdot, \omega)$, for which \textbf{(CM($\nu$))} does not hold, is the subject of a forthcoming manuscript by M. Gaume [12].

2. Multi-particle MSA

2.1. Decay of resolvents in finite cubes: the key bound.

\textbf{Definition 2.1.} Given a sample $H(\omega)$ of an $N$-particle Hamiltonian of the form \textbf{[11]}, a cube $C_L(u)$ is said $(E,m)$-non-singular ($(E,m)$-NS), with $E \in \mathbb{R}$ and $m > 0$, if

$$\max_{x : \|x-u\| \leq L^{1/n}} \max_{y \in \partial - C_L(u)} |(H_{C_L(u)} - E)^{-1}(x,y)| \leq e^{-\gamma(m,L,N)},$$

where

$$\gamma(m,L,n) := mL(1 + L^{-1/4})^{N-n+1}, \ 1 \leq n \leq N.$$ 

Otherwise, it is called $(E,m)$-singular ($(E,m)$-S).
Traditionally, the $k$-dependence is put in the decay exponent, $m = m_k$, which is re-calculated recursively. We prefer to keep fixed the decay parameter $m$, while the actual decay bound depends on $L_k$ and on $N$ explicitly, through the function $\gamma$.

The main result of the multi-particle MSA, used for the derivation of spectral and dynamical localization, can be formulated as follows:

**Lemma 2.1.** There exists $g^* < \infty$ and positive functions $m^*(g), p^*(g)$ defined for $|g| \geq g^*$ such that

1. $m^*(g), p^*(g) \to +\infty$ as $|g| \to \infty$;
(2) the property (DS.1, 0, N) holds for the operators $H^{(N)}(\omega) = \Delta^{(N)} + gV(\omega) + U$ with $m = m^*(g)$ and $p = p^*(g)$.

This statement can be proven for $N$-particle Hamiltonians in the same way as for $N = 1$; see, e.g., [8, 11]. In fact, the non-random interaction $U$ (and/or a non-random component of $V$) can simply be ignored in the proof. Note for a reader not familiar with the conventional MSA techniques that $m^*(g) \sim O(\ln|g|)$, for $|g| \gg 1$. The asymptotic behavior of $p^*(g)$ depends upon the regularity of the marginal distribution function $F_{\omega}$ of the random field $V$.

2.5. "Radial descent" bound for resolvents. In this subsection we state an analytic result which does not rely upon single- or multi-particle structure or random nature of the potential energy of the Hamiltonian $H$. All constants involved depend only upon the dimension of the lattice. In order to emphasize this, we change here our notations and do not use boldface symbols. Since $V$ is arbitrary here, in applications to the multi-particle systems we can assume that $V$ contains also an inter-particle interaction energy. In a certain sense, Lemma 2.2 below encapsulates an argument going back to [8,11] and used since then in many papers.

**Definition 2.2.** A bounded function $f : \Lambda \to \mathbb{C}$ on a subset $\Lambda \subset \mathbb{Z}^n$, $n \geq 1$, is called $(\ell, q, S, c)$-subharmonic, with $\ell \in \mathbb{N}$, $q > 0$, $S \subset \Lambda$, $c \geq 1$, if for any $x \in \Lambda \setminus S$ such that $C_{\ell}(x) \subset \Lambda$, one has

$$|f(x)| \leq q \max_{\|y-x\| = \ell} |f(y)|,$$

while for $x \in S$

$$|f(x)| \leq q \max_{y \in \Lambda : \ell \leq \|y-x\| \leq (1+c)\ell} |f(y)|.$$  

**Lemma 2.2.** Consider an $(\ell, q, S, c)$-subharmonic function $f$ on $C_{\ell}(u) \subset \mathbb{Z}^n$. Suppose that the $c\ell$-neighborhood of the set $S$ can be covered by a collection $A$ of annuli

$$A_i = C_{b_i}(u) \setminus C_{a_i}(u),$$

of total width $W_A$. Then for any $r \in [W_A + \ell, L - W_A + \ell]$

$$\max_{x \in C_{b_i}(u)} |f(x)| \leq q^{(L-r-W(A))/\ell-1} \max_{y \in C_{\ell}(u)} |f(y)|.$$  

In particular,

$$|f(0)| \leq q^{(L-W(A))/\ell-1} \max_{y \in C_{\ell}(u)} |f(y)|.$$  

The proof, based on a "reverse induction" in $r$ ("radial descent") is given in Sect. 5. A direct application of this bound to the resolvents, making use of the GRI, leads to the following statement.

**Lemma 2.3.** Consider a lattice Schrödinger operator $\Delta + V(x)$ in a cube $C_{L_{k+1}}(u) \subset \mathbb{Z}^n$, $n \geq 1$. Fix an energy $E \in I$ and suppose that the $cL_k$-neighborhood of all $(E, m)$-singular cubes of radius $L_k$ inside $C_{L_{k+1}}(u)$ can be covered by a collection $A$ of annuli $A_i = C_{b_i}(u) \setminus C_{a_i}(u)$ of total width $W_A$. Suppose also that $C_{L_{k+1}}(u)$ does not contain any $E$-resonant cube of radius $L \geq L_k$ (including itself). Then, for any fixed value of the constant $\bar{c}$ and $L_0 \geq L_0^*(c, n)$ large enough, the cube $C_{L_{k+1}}(u)$ is $(E, m)$-NS.

We will see below (cf. Sect. 2.7) that, in application to MPMSA, with sufficiently high probability a cube of radius $L_{k+1}$ does not contain more than 4 singular cubes.
of radius \( L_k \) which are pairwise \( 2NL_k \)-distant, in which case all singular cubes can be covered by a collection of annuli of total width \( O(L_k) \).

We stress that Lemma 2.3 is purely "deterministic" and does not rely upon single- or multi-particle structure of the potential.

2.6. Localization bounds for decomposable systems. We need a simple property of quantum systems decomposed in a union of perfectly non-interacting subsystems. Loosely speaking, it says that if two distant subsystems \( u' \in \mathbb{Z}^n \), \( u'' \in \mathbb{Z}^{n''} \) are "localized" and do not interact, then their union \( u = (u', u'') \) is also "localized".

Definition 2.3.

(1) Let \( n' \in \{1, \ldots, N - 1\} \), \( k \geq 0 \) and \( u' = (u_1, \ldots, u_{n'}) \in \mathbb{Z}^{n'd} \). Given a bounded interval \( I \subset \mathbb{R} \) and \( m > 0 \), the \( n' \)-particle cube \( C_{L_k}^{(n')}(u') \) is said \((m, I)\)-tunneling \( ((m, I)\)-T, for short) if \( \exists E \in I \) and there are \( 2NL_{k-1} \)-distant \( n \)-particle cubes \( C_{L_{k-1}}^{(n)}(v_1), C_{L_{k-1}}^{(n)}(v_2) \subset C_{L_k}^{(n')}(u') \) which are \((E, m)\)-S.

(2) An \( N \)-particle cube \( C_{L_k}^{(N)}(u) \) is said \((m, I)\)-partially tunneling \( ((m, I)\)-PT) if, for some permutation \( \tau \in \mathfrak{S}_N \) (acting on the components of the vectors \( u = (u_1, \ldots, u_N) \)) and some \( n', n'' \geq 1 \), it admits a representation

\[
C_{L_k}^{(n)}(\tau(u)) = C_{L_{k-1}}^{(n')}(u') \times C_{L_{k-1}}^{(n'')}(u'')
\]

\( u' = (u_1, \ldots, u_{n'}) \), \( u'' = (u_{n'+1}, \ldots, u_N) \), where either \( C_{L_{k-1}}^{(n')}(u') \) or \( C_{L_{k-1}}^{(n'')}(u'') \) is \((m, I)\)-T. Otherwise, it is said \((m, I)\)-NPT.

Lemma 2.4 (Cf. Lemma 3 in [7]). Fix an interval \( I \subset \mathbb{R} \) and an energy \( E \in I \). Consider an \( N \)-particle cube \( C_{L_k}^{(N)}(u) = C_{L_k}^{(n')}(u') \times C_{L_k}^{(n'')}(u'') \), with \( n', n'' \geq 1 \), and a sample of the potential \( V(\cdot ; \omega) \) such that

(a) \( \rho \left( \Pi C_{L_k}^{(n')}(u'), \Pi C_{L_k}^{(n'')}(u'') \right) > 2L_k + r_0 \); \( (r_0 = \text{the range of interaction}) \)

(b) \( C_{L_k}^{(n)}(u) \) is \( E \)-non-resonant;

(c) \( C_{L_{k-1}}^{(n)}(u') \) and \( C_{L_{k-1}}^{(n'')}(u'') \) are \((m, I)\)-NT;

(d) the inductive assumptions \( \{(DS.I, k', n), k' \geq 0, n \leq N - 1\} \) hold true.

Then the cube \( C_{L_k}^{(N)}(u) \) is \((E, m)\)-non-singular.

Lemma 2.5 (Cf. Lemma 5 in [7]). If \( L_0 \) is large enough, then

\[
\mathbb{P} \{ C_{L_k}(u) \text{ is } (m, I)\text{-PT} \} \leq \frac{1}{2} L_{\frac{k}{2}}^{-p2^{N-(N-1)+1}+2Nd\alpha}. \tag{2.7}
\]

Indeed, the "tunneling" property says that one of the projection cubes \( C_{L_k}^{(n)}(u') \) contains at least two distant singular cubes of radius \( L_{k-1} \), so that Lemma 2.4 follows from the inductive assumptions \( \{(DS.I, k', n), k' \geq 0, 1 \leq n \leq N - 1\} \) combined with a simple bound on the number of pairs of points in \( C_{L_k}^{(n)}(u') \) by \( \frac{1}{2} |C_{L_k}^{(n)}(u')|^2 \).

Introducing the following random variables, depending upon the samples \( V(\cdot ; \omega) : \)

\[
K^{PI}(C_{L_{k+1}}(u), I) = \text{the maximal number of } 2NL_k \text{-distant PI cubes of radius } L_k \text{ in } C_{L_{k+1}}(u) \text{ which are simultaneously } (E, m)\text{-S, with } E \in I, \]

\[
K^{FI}(C_{L_{k+1}}(u), I) = \text{the maximal number of } 2NL_k \text{-distant FI cubes of radius } L_k \text{ in } C_{L_{k+1}}(u) \text{ which are simultaneously } (E, m)\text{-S, with } E \in I.
\]
The maximal collections of resonant and distant FI (or PI) cubes are, of course, not uniquely defined, but their (maximal) cardinality is well-defined, since there is only a finite number of choices for cubes $C_{L_k}(v(1)), \ldots, C_{L_k}(v(n))$ inside $C_{L_{k+1}}(u)$.

**Corollary 1.** If $p > Nda/(2 - \alpha)$ and $L_0^p(2 - \alpha) - Nda \geq 2$, then
\[
P \{ K^{PI}(C_{L_{k+1}}(u), I) \geq 2 \} \leq \frac{1}{2} L_k^{-2p+2Nda} \leq \frac{1}{8} L_k^{-2p}. \tag{2.8}
\]

2.7 Pairs of non-decomposable (FI) cubes.

**Lemma 2.6 (Cf. [7]).** If $p > 2Nda/(2 - \alpha)$, then
\[
P \{ K^{PI}(C_{L_{k+1}}(u), I) \geq 4 \} \leq \frac{1}{4!} L_k^{-4p \alpha^{-1} + 4Nd} \leq \frac{1}{4} L_k^{-2p}. \tag{2.9}
\]

For the proof, it suffices to notice that distant FI cubes give rise to independent samples of the potential $V$.

As was mentioned before, we see that with sufficiently high probability $K^{FI} + K^{PI} < 3 + 1 = 4$, and Lemma 2.3 can be used to prove non-singularity of a non-resonant cube of radius $L_{k+1}$ with at most $K^{FI} + K^{PI} \leq 4$ pairwise $2NL_k$-distant singular cubes of radius $L_k$ inside it. Indeed, $O(L_k)$-neighborhood of all such cubes can be covered by a collection of annuli of total width $O(L_k)$.

Note that, with $\alpha = 3/2$, it suffices to require that $p > 6Nd$ and $L_0 \geq 2$; then the hypotheses on $p$ in Corollary 2 and in Lemma 2.6 are automatically satisfied.

**Corollary 2.** Assume that the bound (DS.I,k,N) (Eqn (2.2)) holds for some $k \geq 0$. Then the bound of the form (DS.I,k+1,N) also holds for all $2NL_{k+1}$-distant pairs of FI cubes.

**Proof.** Consider two $2NL_{k+1}$-distant FI cubes $C_{L_{k+1}}(x), C_{L_{k+1}}(y)$. Both of them are $(E, m)$-S for some $E \in I$ only if at least one of the following events occurs:

1. for some $E \in I$, both $C_{L_{k+1}}(x)$ and $C_{L_{k+1}}(y)$ are $E$-R;
2. $K^{FI}(C_{L_{k+1}}(x), I) \geq 4$ or $K^{FI}(C_{L_{k+1}}(x), I) \geq 2$;
3. $K^{FI}(C_{L_{k+1}}(y), I) \geq 4$ or $K^{FI}(C_{L_{k+1}}(y), I) \geq 2$.

By Thm. 1.1 the probability of the event (1) can be bounded by $\frac{1}{2} L_k^{-2p+1}$ (actually, by any power of $L_k^{-1}$, provided that 1.3 is fulfilled with $B > 0$ large enough). In the case where 1.4 holds, one obtains even a stronger bound by $e^{-L_{k+1}^\beta}$, $\beta > 0$. Next, it follows from Corollary 2 and Corollary 1 that the probability of the event (2) (as well of the event (3)) is bounded by
\[
\frac{1}{8} L_k^{-2p+1} \leq L_k^{-2p}. \tag{2.10}
\]

Therefore, the cubes $C_{L_{k+1}}(x)$ and $C_{L_{k+1}}(y)$ are simultaneously $(E, m)$-singular, for some $E \in I$, with probability \( \leq (\frac{1}{4} + \frac{1}{4}) L_k^{-2p} < L_k^{-2p} \), as required. \( \square \)

2.8 Pairs of decomposable (PI) cubes and mixed pairs.

**Lemma 2.7.** Assume that the bound (DS.I,k,N) (Eqn (2.2)) holds for some $k \geq 0$. Then the bound of the form (DS.I,k+1,N) also holds for all $2NL_{k+1}$-distant pairs of cubes $C_{L_{k+1}}(x)$ and $C_{L_{k+1}}(y)$, of which at least one is “partially interactive”.

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values of $k$ must $(E,m)$-S. Indeed, assume otherwise. Then there exist an infinite number of values of $k$ (hence, arbitrarily large values of $L_k$) such that $C_{L_k}(0)$ is $(E,m)$-NS.

Proof. Without loss of generality, assume that $C_{L_{k+1}}(y)$ is PI. The cubes $C_{L_{k+1}}(x)$ and $C_{L_{k+1}}(y)$ are simultaneously $(E,m)$-singular, for some $E \in I$, only if at least one of the following events occurs:

1. for some $E \in I$, both $C_{L_{k+1}}(x)$ and $C_{L_{k+1}}(y)$ are $E$-R;
2. $C_{L_{k+1}}(x)$ is FI, and $K^{FI}(C_{L_{k+1}}(x), I) \geq 4$ or $K^{PI}(C_{L_{k+1}}(x), I) \geq 2$;
3. the PI cube $C_{L_{k+1}}(y)$ is partially tunneling.

Naturally, the option (2) is to be considered only for a mixed pair of cubes: if both of them are PI, it suffices to analyze only the cube $C_{L_{k+1}}(y)$.

The probabilities of the events (1) and (2) can be estimated exactly as in the proof of Corollary 2 so that their sum does not exceed $\frac{2}{4}L_{k+1}^{-2p}$. By inductive assumptions $\{(DS, I_k, N), k \geq 0\}$ on systems with $n \leq N - 1$ particles, the event (3) has probability bounded by $\frac{1}{4}L_{k+1}^{-2p}$. Combining these estimates, the lemma follows. \hfill $\square$

Now the key bound of the MPMSA, $(DS, I_k, N)$, is established for all $k \geq 0$.

3. From the MSA bounds to multi-particle localization

It has been known since more than twenty years that the principal MSA bounds – in the single-particle theory – imply the spectral localization; see the original papers \cite{8,11}. In fact, as was pointed out in \cite{6,7}, the main argument used in such a derivation is not specific to single- or multi-particle structure of the random potential. Note for a reader familiar with the traditional, single-particle version of Lemma 5.1 given below, that the main argument used in the single-particle context applies to multi-particle systems, with one minor technical modification: instead of disjoint pairs of cubes, one should consider $2NL_k$-distant pairs of cubes at each scale $L_k$.

In essence, owing to the new eigenvalue concentration bound given by Thm.\,1.1, the derivation of 2-particle spectral localization from MSA bounds of the form $(DS, I_k, N)$ described in \cite{6} extends – in a fairly simple way – to any $N \geq 2$.

Lemma 3.1 (Cf. \cite{7}). Fix an interval $I \subset \mathbb{R}$ and suppose that the bound $(DS, I_k, N)$ (cf. Eqn. (2.2)) holds for all $k \geq 0$. Then with probability one, the Hamiltonian $H^{(N)}(\omega)$ has pure point spectrum in $I$, and all its eigenfunctions $\Psi_n(\omega)$ with eigenvalues $E_n \in I$ decay exponentially fast at infinity:

$$\exists C_n(\omega) : \forall x \in \mathbb{Z}^N \quad |\Psi_n(x; \omega)| \leq C_n(\omega)e^{-m|x|}.$$ 

Proof. As it is well-known, spectrally-a.e. generalized eigenfunction $\Psi$ of a lattice Schrödinger operator $H$ is polynomially bounded,

$$|\Psi(x)| \leq Const \|x\|^A, \quad A < \infty,$$

so it suffices to show that every polynomially bounded solution of the equation $H\Psi = E\Psi$ with $E \in I$ is exponentially decaying at infinity.

If $\Psi \neq 0$, then there is a point $u \in \mathbb{Z}^N$ where $\Psi(u) \neq 0$. We start the analysis of the decay properties of $\Psi$ by finding the smallest integer $k_0 \geq 0$ such that the cube $C_{L_{k_0}}(0)$ contains all points $\tau(u)$, $\tau \in \mathbb{Z}^N$, obtained by permutations of the components $u_j$ of the vector $u = (u_1, \ldots, u_N)$.

The first observation is that for some $k_1 \geq k_0$ and all $k \geq k_1$ the cubes $C_{L_k}(0)$ must $(E,m)$-S. Indeed, assume otherwise. Then there exist an infinite number of values of $k$ (hence, arbitrarily large values of $L_k$) such that $C_{L_k}(0)$ is $(E,m)$-NS.
that the function $\Psi$. Moreover, all cubes of radius $L$ of them is $(E, m)$-S cubes of radius $L_k$ outside $C_{L_{k+1}}(0)$ are also $(E, m)$-NS. This implies that the function $\Psi$ is $(L_k, q)$-subharmonic in the cube $C_R(x)$, with

$$q = e^{-m(L_k + O(\ln L_k))} \leq e^{-m(L_k + \frac{3}{4}L_k^{3/4})},$$

if $L_k$ is large enough, and

$$R := \|x\| - (L_k + 2NL_k), \quad \|x\| \geq L_k^{9/8} \gg L_k.$$ 

It is easy to see that $R/\|x\| = 1 - O(L_k^{-1/8})$, and applying Lemma 2.3, we obtain, for $L_k$ large enough,

$$-\frac{\ln |\Psi(x)|}{\|x\|} \geq m \left( \frac{L_k + \frac{3}{4}L_k^{3/4}}{L_k} \cdot \left(1 - O\left(L_k^{-1/8}\right)\right) \right) \geq m,$$

as required.

Recall that the validity of the hypothesis of Thm. 4.1 is established in Sect. 2.3. We come, therefore, to our main result on the $N$-particle spectral localization:

**Theorem 3.1.** Consider the random operators $H^{(N)}(\omega)$ of the form $\hat{H}$. Suppose that $U$ is bounded and $V$ satisfies CM($\nu$). Then there exists $g^* < \infty$ such that if $|g| \geq g^*$, then with probability one the operator $H(\omega)$ has pure point spectrum, and all its eigenfunctions $\Psi_n(\omega)$ with eigenvalues $E_n \in I$ decay exponentially fast at infinity:

$$\exists C_n(\omega) : \forall x \in \mathbb{Z}^d \quad |\Psi_n(x; \omega)| \leq C_n(\omega) e^{-m\|x\|}.$$
Remark 3.1. While the statement of Thm. 3.1 is similar to that of the main result of [7], a detailed analysis shows that the actual bound on the random constants $C_n(\omega)$ obtained in [7] depends upon the position of the localization center $x_n$ for the corresponding eigenfunction $\Psi_n(\omega)$; Thm. 4.1 rules out such a dependence.

4. Strong dynamical localization

To prove the dynamical localization, in addition to the assumption [13] on the random potential $V(\cdot; \omega)$ we need the following hypothesis: for any finite interval $I \subset \mathbb{R}$

$$\exists \kappa = \kappa(I, N, d) < \infty : \mathbb{P} \{ \text{tr} (P_I(\mathbf{H}_{\mathcal{C}_L(u)}(\omega))) > C L^{\kappa N d} \} \leq L^{-B'},$$

(4.1)

where $P_I(\cdot)$ is the spectral projection on $I$ and $B' > 0$ will be required below to be sufficiently large, depending on other parameters. It is readily seen that the trace of $P_I(\mathbf{H}_{\mathcal{C}_L(u)})$ grows not faster than linearly in $|\mathcal{C}_L(u)|$, if the random potential $V$ is bounded from below. Since we would like to allow also Gaussian potentials for which $\mathbf{CM}(\nu)$ becomes most simple, we allow a faster rate of growth. Observe that

$$\text{tr} (\mathbf{H}_{\mathcal{C}_L(u)}(\omega)) = \text{tr} (\Delta_{\mathcal{C}_L(u)} + U_{\mathcal{C}_L(u)}) + g \text{tr} (V_{\mathcal{C}_L(u)}(\omega)),$$

and

$$\text{tr} (V_{\mathcal{C}_L(u)}(\omega)) = \sum_{x \in \mathcal{C}_L(u)} \sum_{j=1}^N V(x_j; \omega).$$

Therefore, for a large class of marginal distributions including Gaussian ones, the required bound follows from standard results for the sums of IID random variables.

4.1. Strong dynamical localization. Here we follow closely the scheme which is well-described in [13]; see also [9, 13]. Specifically, Propositions 1–6 below correspond to assertions at Steps 1–6 from Sect. 3.4 in [13].

Introduce the operator $\mathbf{X}$ of multiplication by max-norm in $l^2(\mathbb{Z}^{N d})$:

$$(\mathbf{X}\Phi)(x) := \| x \| \Phi(x), \ x \in \mathbb{Z}^{N d}. $$

**Theorem 4.1.** Assume that the property $(\mathbf{DS.I, k, N})$ holds true for a given $N > 1$ and for all $k \geq 0$, with $p > (3 N d \alpha + \alpha \kappa)/2$, $s > 0$. Then the random operators $\mathbf{H}^{(N)}(\omega)$ feature strong dynamical localization in the energy interval $I$: for any finite subset $K \subset \mathbb{Z}^{N d}$ and any bounded measurable function $\eta$ with supp $\eta \subset I$,

$$\mathbb{E} \left[ \| \mathbf{X}^s \eta(\mathbf{H}(\omega)) 1_K \| \right] < \infty.$$

Note that the condition $p > 2 N d \alpha/(2 - \alpha)$ was required to prove $(\mathbf{DS.I, k, N})$.

Since the validity of the hypothesis of Thm. 4.1 is established in Sect. 2.3, it implies the $N$-particle dynamical localization for the Hamiltonians $\mathbf{H}(\omega)$:

**Theorem 4.2.** Assume that the random field $V : \mathbb{Z}^d \times \Omega \to \mathbb{R}$ satisfies the condition $(\mathbf{CM}(\nu)$ and $[\mathbf{I, N})$ with $B' > 2p - N d \alpha$. Assume also that the initial scale estimate $(\mathbf{DS.I, 0, N})$ is fulfilled for some interval $I \subset \mathbb{R}$ with $p > (3 N d \alpha + \alpha \kappa)/2$, $s > 0$. Then the random operators $\mathbf{H}^{(N)}(\omega)$ feature strong dynamical localization in the energy interval $I$: for any finite subset $K \subset \mathbb{Z}^{N d}$ and any $\eta \in L^\infty(\mathbb{R})$ with supp $\eta \subset I$,

$$\mathbb{E} \left[ \| \mathbf{X}^s \eta(\mathbf{H}(\omega)) 1_K \| \right] < \infty.$$

In particular, $\mathbf{H}^{(N)}(\omega)$ features complete dynamical localization with any given value of $s > 0$, if $g$ is large enough: $|g| \geq g^*(s)$, $g^*(s) \leq \infty$.

Recall that [3] is not required for random potentials bounded from below.
Now we will describe the strategy of the proof of Thm. 4.1. Our main goal here is to show that the proof, making use of the simpler and more general eigenvalue concentration bound (3.5), is very close to that used in the single-particle context.

4.1.1. "Bad" and "good" events. Fix $s > 0$. We will always assume that

$$p > \max\{2Nd\alpha/(2-\alpha), (3Nd\alpha+\alpha s)/2\}$$

(4.2)

and set $b = b(p, N, d, \alpha) := 2p - 2Nd\alpha$. For each $j \geq 1$ consider the events

$$S_j = \{\omega : \exists E \in I \exists y, z \in C_{4(N+1)L_{j+1}^b}(0) \text{ such that } d_{S}(y, z) > 2NL_j \text{ and } C_{L_j}(y), C_{L_j}(z) \text{ are } (m, E)\text{-singular}\}.$$

and (in the case where the random potential $V$ is not bounded from below)

$$T_j = \{\omega : \text{tr}(P_1(H\mathbf{C}_{L_{j+2}^b}(u))) > CL_j^{Nd}\}.$$

Further, for $k \geq 1$ denote

$$\Omega_k^{(bad)} = \bigcup_{j \geq k} (S_j \cup T_j)$$

and consider the annuli

$$M_k = C_{4(N+1)L_{k+1}^b}(0) \setminus C_{4(N+1)L_k}(0).$$

Proposition 1. Under the assumption 4.1 with $B' > 2p - 2Nd\alpha$

$$\forall k \geq 1 \quad \mathbb{P}\{\Omega_k^{(bad)}\} \leq c(\alpha, d, p, N)L_k^{-(2p-2Nd\alpha)}.$$

Proof. The number of pairs $y, z \in C_{4(N+1)L_{j+1}^b}(0)$ figuring in the definition of the event $S_j$ is bounded by

$$\frac{1}{2} (4(N+1)L_{j+1})^{2Nd} = C(N, d)L_j^{2Nd\alpha},$$

and (DS.I,j,N) says that

$$\mathbb{P}\{C_{L_j}(y), C_{L_j}(z) \text{ are } (m, E)\text{-singular}\} \leq L_j^{-2p},$$

while for the event $T_j$ we have the bound (4.1). Now we require that the exponent $B'$ be large enough, so that $\mathbb{P}\{T_j\} \leq L_j^{-2p+2Nd\alpha}$ and

$$\mathbb{P}\{\Omega_k^{(bad)}\} \leq \sum_{j \geq k} (\mathbb{P}\{S_j\} + \mathbb{P}\{T_j\}) \leq \sum_{j \geq k} Const L_j^{2Nd\alpha} \cdot L_j^{-2p}$$

$$= L_k^{-b} \left[1 + \sum_{j > k} L_k^b L_k^{-(2p-2Nd\alpha)}\right] \leq Const L_k^{-(2p-2Nd\alpha)}. \quad (4.3)$$

4.1.2. Centers of localization. By Thm. 3.1 there exists a subset $\Omega_1 \subset \Omega$ with $\mathbb{P}\{\Omega_1\} = 1$ such that for any $\omega \in \Omega_1$ the spectrum of $H^{(N)}(\omega)$ in $I$ is pure point. Fix $\omega \in \Omega_1$ and let $\Phi_n(\omega)$ be a normalized eigenfunction of $H(\omega)$, with eigenvalue $E_n(\omega) \in I$. We call a center of localization for $\Phi_n$ every point $x_n(\omega) \in \mathbb{Z}^{Nd}$ such that

$$|\Phi_n(x_n(\omega))| = \max_{y \in \mathbb{Z}^{Nd}} |\Phi_n(y)|. \quad (4.4)$$

Since $\Phi_n \in L^2(\mathbb{Z}^{Nd})$, such centers always exist and, due to the normalization $\|\Phi_n\| = 1$, the number of centers of localization $x_{n,a}$ for a given $n$ must be finite.

Proposition 2. There exists $k_0$ such that for all $\omega \in \Omega_1$ and $k \geq k_0$, if one of the centers of localization $x_{n,a}$ for an eigenfunction $\Psi_n$ with eigenvalue $E_n \in I$ belongs to a cube $C_{L_k}(x)$, then the cube $C_{L_{k+1}^b}(x)$ is $(m, E_n)$-S.
Proposition 4. There exists \( j_0 = j_0(m, \alpha, d) \) large enough such that for \( j \geq j_0, j \geq k \) and \( x_{n,1} \in C_{L_{j+1}}(0) \)

\[
\|(1 - 1_{C_{4(L_{j+1})}}(0)) \Phi_n\| < \frac{1}{4}.
\]

Proof. Using the annuli \( M_k \), we can write

\[
\|(1 - 1_{C_{4(L_{j+1})}}(0)) \Phi_n\|^2 = \sum_{i \geq j+2} \|1_{M_i} \Phi_n\|^2 = \sum_{i \geq j+2} \sum_{y \in M_i} |\Phi_n(y)|^2
\]

Fix \( i \geq j + 2 \). The cube \( C_{L_{i-1}}(0) \supset C_{L_{j+1}}(0) \) contains the center of localization \( x_{n,1} \), so, by Prop. 2, \( C_{L_i}(y) \) is \((E_{n,m})\)-singular. By construction of the event \( \Omega_k^{(good)} \), \( k \leq j \), the cube \( C_{L_i}(y) \) must be \((E_{n,m})\)-NS. In turn, this implies by the GRI for eigenfunctions that \( |\Phi_n(y)|^2 \leq e^{-2mL_i} \). Now the claim follows from a polynomial (in \( L_i \)) bound on the number of terms in the sum \( \sum_{y \in M_i} \).

Proposition 5. There exists \( c_4 = c_4(m, d, \kappa) \) such that for \( \omega \in \Omega_k^{(good)} \) and \( j \geq k \) the following bound holds:

\[
\# \{ n : x_n \in C_{L_{j+1}}(0) \} \leq c_4 L_{j+1}^{\alpha d}.
\]  

Proof. Observe that we have

\[
\sum_{x_n,1 \in C_{L_{j+1}}(0)} (1_{C_{L_{j+2}}(0)} P_I(H) 1_{C_{L_{j+2}}(0)} \Phi_n, \Phi_n) \leq \text{tr} (1_{C_{L_{j+2}}(0)} P_I(H)).
\]

It suffices to show that each term in the LHS is bigger than \( \frac{1}{2} \). Indeed, by Prop. 3

\[
\begin{align*}
(1_{C_{L_{j+2}}(0)} P_I(H) 1_{C_{L_{j+2}}(0)} \Phi_n, \Phi_n) &= (1_{C_{L_{j+2}}(0)} P_I(H) \Phi_n, \Phi_n) - (1_{C_{L_{j+2}}(0)} P_I(H)(1 - 1_{C_{L_{j+2}}(0)}) \Phi_n, \Phi_n) \\
&\geq (1_{C_{L_{j+2}}(0)} \Phi_n, \Phi_n) - \frac{1}{4} = (\Phi_n, \Phi_n) - \frac{1}{4}
\end{align*}
\]

\( \geq \frac{1}{2}. \)

4.1.3. Eigenfunction correlator bounds.

Proposition 5. There exists an integer \( k_1 = k_1(\kappa, L_0) \) such that for all \( k \geq k_1 \), \( \omega \in \Omega_k^{(good)} \) and \( x \in M_k \)

\[
|\eta(H(\omega))(x,0)| \leq e^{-mL_{k-1}/2}\|\eta\|_{\infty}.
\]  

(4.7)
Proof. Without loss of generality, assume that $\|\eta\|_{\infty} \neq 0$:

$$\|\eta\|_{\infty}^{-1} \eta(H(\omega))(x, 0) \leq \sum_{n, E_n \in I} \left| \Phi_n(x) \right| \| \Phi_n(0) \|$$

$$\leq \sum_{n, E_n \in I} \sum_{x_n, 1 \in M_k(x)} \left| \Phi_n(x) \right| \| \Phi_n(0) \| + \sum_{j > k} \sum_{n, E_n \in I} \left| \Phi_n(x) \right| \| \Phi_n(0) \|. (4.8)$$

For $k \geq k_0$ and $L_k$ large enough, the first sum in the RHS can be bounded as follows:

$$\sum_{n, E_n \in I} \left| \Phi_n(x) \right| \| \Phi_n(0) \| \leq \text{const} L_{k+1}^{\alpha N_d} e^{-mL_k} \leq \frac{1}{2} e^{-mL_k/2}, (4.9)$$

since one of the cubes $C_{L_k}(x)$, $C_{L_k}(0)$ must be $(E_n, m)$-NS; indeed, these cubes are $2NL_k$-distant. Next, fix any $j \geq k + 1$ and consider the sum with $x_n, 1 \in M_k(0)$. The cubes $C_{L_j}(0)$ and $C_{L_j}(x_n, 1)$ are $2NL_j$-distant and by Prop. 2 for $k$ (hence, $j$) large enough the cube $C_{L_j}(x_n, 1)$ is $(E_n, m)$-S, so for $\omega \in \Omega_{(k)}^{\text{good}}$ the cube $C_{L_j}(0)$ must be $(E_n, m)$-NS. Therefore,

$$\left| \Phi_n(x) \right| \| \Phi_n(0) \| \leq \text{const} e^{-mL_j}. (4.10)$$

Now the claim follows from (4.9) and (4.10).

**Proposition 6.** Fix $k_1$ as in Prop. 2. Then for any $k \geq k_1$ and $x \in M_k$,

$$\mathbb{E} \left[ \| 1_{C_{L_k}(x)} \eta(H(\omega)) 1_{C_{L_k}(0)} \| \right] \leq \| \eta \|_{\infty} \left( CL_k^{-2p+2N\alpha} + e^{-mL_k/2} \right). (4.11)$$

**Proof.** Using Prop. 3 and Prop. 1 we can write

$$\mathbb{E} \left[ \| 1_{C_{L_k}(x)} \eta(H(\omega)) 1_{C_{L_k}(0)} \| \right] = \mathbb{E} \left[ 1_{\Omega_{(k)}^{\text{bad}}} \left\| 1_{C_{L_k}(x)} \eta(H(\omega)) 1_{C_{L_k}(0)} \| \right\| \right] + \mathbb{E} \left[ 1_{\Omega_{(k)}^{\text{good}}} \left\| 1_{C_{L_k}(x)} \eta(H(\omega)) 1_{C_{L_k}(0)} \| \right\| \right] \leq \| \eta \|_{\infty} \left( C L_k^{-2p+2N\alpha} + e^{-mL_k/2} \right).$$

4.1.4. **Conclusion.** Fix a set $K \subset \mathbb{Z}^{Nd}$ and find $k \geq k_1$ such that $K \subset C_{L_k}(0)$. Then

$$\mathbb{E} \left[ \| X^* \eta(H(\omega)) 1_K \| \right] \leq c N_d L_k^s + \sum_{j \geq k} \mathbb{E} \left[ \| X^* 1_{M_j} \eta(H(\omega)) 1_K \| \right] \leq c(k) + \sum_{j \geq k} c N_d L_{j+1}^s \sum_{w \in M_j} \mathbb{E} \left[ \left\| 1_{C_{L_k}(w)} \eta(H(\omega)) 1_{C_{L_k}(0)} \| \right\| \right] \leq C \left[ 1 + \sum_{j \geq k} L_j^{\alpha s} L_j^{N\alpha} \left( L_j^{-2p+2N\alpha} + e^{-mL_j/2} \right) \right] < \infty,$$

since $2p - 3N\alpha - \alpha s > 0$, and $L_j = (L_0)^{\alpha j}$ grow fast enough.

This completes the proof of Thm. 4.1.
5. **Appendix. Proof of the Radial Descent bound (Lemma 2.2)**

**Proof.** It suffices to consider non-negative functions; otherwise, we replace \( f \) by \( |f| \).

Let a function \( f : C_L(0) \to \mathbb{R}_+ \) be \((\ell, q, S, c)-\)subharmonic. Introduce "spheres"

\[
S'_r = \{ x : \| x \| = r \} \quad \text{and the sets}
\]

\[
S' = \{ x : S_{\| x \|} \cap S \neq \emptyset \}, \quad S'' = \bigcup_{r:S \cap S \neq \emptyset} \bigcup_{j=0}^{\ell} S_{r+j}
\]

and also

\[
\mathcal{R} = \{ r \geq 0 : C_r(0) \subset C_L(0) \setminus S'' \}.
\]

Note that if \( \| x \| = r \in \mathcal{R} \), then

\[
f(x) \leq q \max_{y: \| y-x \| \leq \ell} f(y) \leq q \max_{y: \| y \| \leq r+\ell} f(y).
\]

Moreover, for any \( u \) with \( \| u \| \in [r-c\ell, r] \) the definition of the sets \( \mathcal{R} \) and \( S'' \) implies that \( u \notin S \), so that

\[
f(u) \leq q \max_{y: \| y-u \| \leq \ell} f(y) \leq q \max_{y: \| y \| \leq \| u \| + \ell} f(y)
\]

Further, for any \( u \) with \( \| u \| < r-c\ell \) there exists a value \( R \in \{ \ell, (1+c)\ell \} \) such that

\[
f(u) \leq q \max_{y: \| y-u \| \leq R} f(y) \leq q \max_{y: \| y \| \leq (r-c\ell)+(1+c)\ell} f(y)
\]

Combining (5.1) with (5.2), we conclude that for any \( r \in \mathcal{R} \)

\[
\max_{u \in C_r(0)} f(u) \leq q \max_{y \in C_{r+\ell}(0)} f(y).
\]

Construct a sequence of points \( \{ r_n \geq 0, 0 \leq n \leq M \} \) by recursion:

\[
r_0 = L; \quad r_n = \max\{ r \in \mathcal{R} : r \leq r_{n-1} - \ell \}, 1 \leq n \leq M,
\]

with some \( M = M(\mathcal{R}) \). Set, formally, \( r_{M+1} = 0 \). We see that, as long as \( r_n \geq 0 \), either \( r_n = r_{n-1} - \ell \), or \( J_n := [r_n, r_{n-1} - \ell] \subset \mathcal{R}^c \). The total length of all non-empty intervals \( J_n \) is bounded by the total width \( W(S'') \) of annuli covering \( S'' \). Now we can write

\[
L = \sum_{n=0}^{M} (r_{n-1} - r_n)
\]

\[
= (r_M - r_{M+1}) + \sum_{n:J_n \neq \emptyset} (r_{n-1} - r_n) + \ell \#\{ n : r_n = r_{n-1} - \ell \}
\]

\[
\leq \ell + W(S'') + \ell \#\{ n : r_n = r_{n-1} - \ell \}
\]

yielding

\[
M \geq \#\{ n : r_n = r_{n-1} - \ell \} \geq \frac{L - W(S'')}{\ell} - 1 \geq \frac{L - W_A}{\ell} - 1.
\]

Further, \( W(S'') \leq W_A \), since the annuli \( A_i \in A \), by assumption, cover the \( c\ell \)-neighborhood of the set \( S \). Therefore,

\[
f(0) \leq q^M \| f \|_{\infty}.
\]

\[3 R = (1+c)\ell \] if \( u \in S \), and \( R = \ell \), otherwise.
More generally, if we stop the construction of the sequence \( \{r_n\} \) at \( n = M' \) as soon as \( r_{M'+1} < r \), for some given \( r > W_A + \ell \geq W(S'') + \ell \), then we obtain
\[
f(r) \leq q^{M'} \|f\|_\infty, \quad M' \geq \frac{L - r - W_A}{\ell} - 1.
\]
This completes the proof of Lemma 2.2.\( \square \)

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