Soliton molecules in Sharma-Tasso-Olver-Burgers equation

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Abstract

Soliton molecules have been experimentally discovered in optics and theoretically investigated for coupled systems. This paper is concerned with the formation of soliton molecules by the resonant mechanism for a noncoupled system, the Sharma-Tasso-Olver-Burgers (STOB) equation. In terms of introducing velocity resonance conditions, we derive the soliton (kink) molecules, half periodic kink (HPK) molecules and breathing soliton molecule of STOB equation. Meanwhile, the fission and fusion phenomena among kinks, kink molecules, HPKs and HPK molecules have been revealed. Moreover, we also discuss the central periodic kink solutions from the multiple solitary wave solutions.

Keywords: Soliton molecules, velocity resonance, half periodic kinks, fissions and fusions, Sharma-Tasso-Olver-Burgers equation

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1 Introduction

The Burgers system and the Sharma-Tasso-Olver (STO) equation are widely investigated from both mathematical and physical point of view \cite{1-6}. It is known that STO equation is an odd ordered equation of the Burgers hierarchy which has extensive application in physical and engineering fields, such as the plasma physics, fluid mechanics and statistical physics. Lots of approaches have been applied to study STO equation including first integral method\cite{7}, bi-Hamiltonian formulation method \cite{8}, the fractional sub-equation method \cite{9}, the sine-cosine method \cite{10} and q-functions approach \cite{11}. Furthermore, several transformations which involve Cole-Hopf transformation, fractional complex transform \cite{12}, Darboux transformation \cite{13} and Bäcklund transformation have been used to discuss the STO equation \cite{13,14}.

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Solitons have been experimentally found in plasma physics and optics, as well as in nonlinear science area including Bose-Einstein condensation and DNA mechanical waves [15, 16]. It is well-known that solitons have an important application in a variety of areas, such as atmospheric and ocean dynamics [17], optical fibers [18]-[22], photonic crystals [23] and plasmas [24]. Later on the soliton molecules which are the soliton bound states have attracted considerable attention. Soliton molecules have been observed in optical systems [25]-[29] and analyzed in Bose-Einstein condensates [30]. Various theoretical proposals to form soliton molecules have been established [31]-[33]. It is known that soliton molecules in coupled systems have been well discussed [34]. Recently, the breathing soliton molecules have been experimentally observed in a mode-locked fiber laser [28].

The resonant theory of solitons is applicable to a variety of integrable systems, such as the KP-(II) equation [35, 36] and Novikov-Veselov equation [37] in which Wronskian representation of the $\tau$-function have been employed. By restricting different resonant conditions on the solitons, it may derive many types resonant excitations. The resonant solutions have been generated by using linear superposition principle and the Bell polynomials for the bilinear equations [38]. By virtue of Hirota’s direct method and the Bäcklund transformation, the researchers study the fission and fusion of the solitary wave solution of the Burgers and STO equations [14]. Furthermore, the interaction between the lump soliton and the pair of resonance stripe solitons form a rogue wave solution [39]. The rational solutions to the modified Korteweg-de Vries equation have been analyzed by imposing the wave number resonance constraints [40]. Lately, by means of introducing velocity resonant mechanism, Lou investigated some types of soliton molecules in fluid systems [41] and nonlinear optical systems [42] involving the third-fifth order Korteweg de-Vries (KdV) equation, the KdV-Sawada-Kotera (KdVSK) equation, the KdV-Kaup-Kupershmidt (KdVKK) equation, the Hirota system and the potential modified KdV-sG system which derive kink-kink molecule, kink-antikink molecule, kink-breather molecule and breather-breather molecule etc. Although the structure and properties of the integrable systems have been widely investigated, much less is known about soliton molecules of the integrable systems. Our objective in this work is to build new types of molecules in a noncoupled system, i.e., the STOB equation by using velocity resonant conditions.

The article is structured as follows. Sec. II introduces the fission and fusion phenomenon of soliton solutions for the STOB equation. In Sec. III, we present some properties including the fission and fusion among the half periodic kink (HPK) and kink solutions. We also develop central periodic kink solutions for STOB equation. The fission and fusion among the soliton molecules, HPK molecules and kink molecules have been studied in Sec. IV. In addition, we derive a dissipative soliton molecule of multiple solitary solution for STOB equation. Section IV is devoted to summary and conclusion.
2 The fission and fusion in Sharma-Tasso-Olver-Burgers equation

Firstly, we give a brief introduction of the Sharma-Tasso-Olver-Burgers (STOB) equation which will be useful in what follows. We consider the nonlinear evolution equation STOB:

\[ u_t + \alpha (3u_x^2 + 3u^2u_x + 3uu_{xx} + u_{xxx}) + \beta (2uu_x + u_{xx}) = 0, \]

(1)

for which \( \alpha = 0 \) reduces to the Burgers equation \[43\], while \( \beta = 0 \) represents the STO equation \[44\].

With the aim of construction of the solutions for STOB equation, we introduce the truncated Painlevé expansion of the STOB equation

\[ u = \sum_{j=0}^{\alpha} u_j v^{j-\alpha}, \]

(2)

where \( u_j \) are functions of the derivatives of \( v \). Especially, substituting \( \alpha = 1 \) into (2), we obtain

\[ u = \frac{u_0}{v} + u_1, \]

(3)

where \( u_1 \) is an arbitrary solution of Eq.(1). (3) is a Bäcklund transformation of Eq.(1). By substituting (3) with \( u_1 = 0 \) into Eq.(1), we derive an equation involves the relation between \( u_0 \) and \( v \). Restricting the coefficient of \( v^{-4} \) being vanished, we have

\[ 3\alpha v^2 u_0^2 - \alpha u_0^3 v_x - 2\alpha u_0^3 v_3^2 = 0. \]

(4)

Solving the Eq.(4), we get

\[ u_0 = v_x, \quad u_0 = 2v_x. \]

(5)

Here we only consider the first case. Inserting \( u_0 = v_x, u_1 = 0 \) into (3), we have the following Cole-Hopf transformation

\[ u = \frac{v_x}{v}. \]

(6)

If we plug (6) back into (1) and solve the solution, we obtain

\[ v_t = -\alpha v_{xxx} - \beta v_{xx} - cv, \]

(7)

where \( c \) is a function of \( t \). Without loss of generality, we can simply take \( c = 0 \).

Firstly, let us concentrate on a multiple solitary wave solution which possesses the form

\[ v = 1 + \sum_{i=1}^{N} a_i e^{\omega_i t + k_i x + \xi_i} \]

(8)
with the wave numbers $k_i$, frequencies $\omega_i$ and original positions $\xi_i$. In (8), $a_i$ have been inserted for convenience though they can be taken as 1 without loss of generality.

It is straightforward to show that (8) is a solution of the STOB under the dispersion relations

$$\omega_i = -\alpha k_i^3 - \beta k_i^2, \quad i = 1, \ldots, N.$$  \hfill (9)

Inserting (8) with $N = 2, a_i = 1$ and (9) into (10), we obtain the two solitary wave solution

$$u = \frac{k_1 e^{-\alpha k_1^3 t - \beta k_1^2 t + k_1 x + \xi_1} + k_2 e^{-\alpha k_2^3 t - \beta k_2^2 t + k_2 x + \xi_2}}{1 + e^{-\alpha k_1^3 t - \beta k_1^2 t + k_1 x + \xi_1} + e^{-\alpha k_2^3 t - \beta k_2^2 t + k_2 x + \xi_2}}.$$  \hfill (10)

It is easy to obtain the potential field $z = u_x$ of (10)

$$z = \frac{k_1^2 e^{-\alpha k_1^3 t - \beta k_1^2 t + k_1 x + \xi_1} + k_2^2 e^{-\alpha k_2^3 t - \beta k_2^2 t + k_2 x + \xi_2}}{1 + e^{-\alpha k_1^3 t - \beta k_1^2 t + k_1 x + \xi_1} + e^{-\alpha k_2^3 t - \beta k_2^2 t + k_2 x + \xi_2}} \quad (1 + e^{-\alpha k_1^3 t - \beta k_1^2 t + k_1 x + \xi_1} + e^{-\alpha k_2^3 t - \beta k_2^2 t + k_2 x + \xi_2})^2.$$  \hfill (11)

Let us concentrate on the fission and fusion phenomenon for the two solitary wave solutions (10). Soliton fission is the splitting of a soliton to produce two or more solitons. More than two solitons can generate a soliton which is called soliton fusion. Let us present the limit expression for the potential field $z$.

For the soliton fusion case

(a) For $k_1 k_2 < 0$,

$$z \to \begin{cases} z_1 + z_2, & t \to -\infty, \\ z_3, & t \to +\infty, \end{cases}$$  \hfill (12)

where

$$z_1 = \frac{k_1^2}{4} \text{sech}^2 \left( \frac{1}{2} k_1 (x - (\alpha k_1^2 + \beta k_1) t + \xi_1) \right),$$  \hfill (13)

$$z_2 = \frac{k_2^2}{4} \text{sech}^2 \left( \frac{1}{2} k_2 (x - (\alpha k_2^2 + \beta k_2) t + \xi_2) \right),$$  \hfill (14)

$$z_3 = \frac{(k_1 - k_2)^2}{4} \text{sech}^2 \left[ \frac{1}{2} (k_1 - k_2) (x - (\alpha k_1^2 + \beta k_1 + k_1 k_2 + \beta (k_1 + k_2)) t) \right] + \xi_1 - \xi_2.$$  \hfill (15)

(b) For $k_1 k_2 > 0$,

$$z \to \begin{cases} z_2 + z_3, & t \to -\infty, \\ z_1, & t \to +\infty. \end{cases}$$  \hfill (16)
Concentrating on soliton fission, we have

(c) For \( k_1 k_2 < 0 \),

\[
z \to \begin{cases} 
z_1, & t \to -\infty, \\
z_2 + z_3, & t \to +\infty. 
\end{cases}
\]

(d) For \( k_1 k_2 > 0 \),

\[
z \to \begin{cases} 
z_3, & t \to -\infty, \\
z_1 + z_2, & t \to +\infty.
\end{cases}
\]

The single soliton solutions of (13) and (14) with \( k_1 k_2 < 0 \) are the exact solutions of STOB equation. Then two solitary waves occur interaction which makes two solitary waves fuse to a resonant solution by (15). Based on (12), it is obvious that before the fusion, the two single solitary waves lie on \( x = (\alpha k_1^2 + \beta k_1) t - \xi_1 \) and \( x = (\alpha k_2^2 + \beta k_2) t - \xi_2 \) with amplitudes \( \frac{1}{4} k_1^2 \) and \( \frac{1}{4} k_2^2 \) respectively. While after fusion, the resonant solitary wave changes its location to \( \alpha((k_1^2 + k_2^2 + k_1 k_2) + \beta(k_1 + k_2)) t - \xi_1 + \xi_2 \) with the amplitude \( \frac{1}{4}(k_1 - k_2)^2 \).

Fig. 1 is a plot of the fission phenomenon of the two solitary waves with parameter selections given by

\[
k_1 = 1, \ k_2 = -2, \ \alpha = \frac{2}{3}, \ \beta = 1, \ \xi_1 = -15, \ \xi_2 = 0.
\]

It should be noted that one resonant solitary wave fission to two single solitary wave at one time.

\[\text{Figure 1: Plot of two solitary wave fission of STOB equation with the parameter selections (19): (I) \ z; (II) \ u.}\]

\[\text{Fig. 2 is a plot of the fusion phenomenon of the two solitary waves with parameter selections}\]

\[
k_1 = 1, \ k_2 = -2, \ \alpha = 1, \ \beta = \frac{2}{3}, \ \xi_1 = 0, \ \xi_2 = 0.
\]

\[\text{From Fig. 2 we find two single solitary waves fusion to one resonant solitary wave at some time.}\]
Based on Fig. 1 and Fig. 2, we conclude the soliton's velocity, amplitude, width and wave shape changed after the collision. It is known that only fusion and no fission happen in the Burgers equation, while both fission and fusion occur for STO and STOB equations.

3 The half periodic kink solutions and their fissions and fusions

In this section, we turn to investigate some special structures and the properties of the solutions which are valid only for the STOB equation but not for the Burgers and STO equations.

From the expression of the two soliton solution, we know that if the wave number \( k_2 \) and the position parameter \( \xi_2 \) are taken as the complex conjugates of \( k_1 = k + i\kappa \) and \( \xi_1 = \eta + i\zeta \), the solution will become a singular kink (or named singular complexiton),

\[
\begin{align*}
u &= \frac{a \exp(\kappa x - \Omega t + \zeta) \cos(\kappa x - \Omega t + \zeta)}{1 + a \exp(\kappa x - \omega t + \eta) \cos(\kappa x - \Omega t + \zeta)},
\end{align*}
\]

with \( a = 2a_2 = 2a_1 \), \( \omega = \alpha k(k^2 - 3\kappa^2) + \beta(k^2 - \kappa^2) \), and \( \Omega = |\alpha(3k^2 - \kappa^2) + 2\beta k|\kappa \). The solution (21) is singular at the points \( 1 + a \exp(\kappa x - \omega t + \eta) \cos(\kappa x - \Omega t + \zeta) = 0 \).

In order to find nonsingular kink with periodic function, one more soliton should be added with the resonant conditions

\[
k_3 = k, \quad \omega_3 = -\alpha k^3_3 - \beta k^5_3 = -\omega.
\]

The resonant condition (22) leads to \( k = -\frac{\beta}{3\alpha} \). Thus, we get the following solution with both periodic and kink behaviors,

\[
\begin{align*}
u &= -\frac{1}{3\alpha} \frac{b \beta + a \beta \cos(\kappa x - \Omega t + \zeta) + 3a\alpha \kappa \sin(\kappa x - \Omega t + \zeta)}{\exp[\frac{\beta x}{3\alpha} + \frac{2\beta^3 t}{27a^2} - \eta]} + b + a \cos(\kappa x - \Omega t + \zeta)
\end{align*}
\]
with arbitrary real constants $a$, $b$, $\kappa$, $\eta$, $\zeta$ and $\Omega = -\frac{\kappa}{3\alpha}(3\alpha^2\kappa^2 + \beta^2)$.

It is clear that the solution (23) is analytic for the parameter condition $b > |a|$. From the expression (23), we know that the solution tends to a constant zero for the half plane $\frac{\partial u}{\partial x} + \frac{2\beta^3}{27\alpha^2} - \eta > 0$ and the solution will tend to a periodic wave

$$u \to -\frac{1}{3\alpha} \frac{b\beta + a\beta \cos(\kappa x - \Omega t + \zeta) + 3a\alpha \kappa \sin(\kappa x - \Omega t + \zeta)}{b + a \cos(\kappa x - \Omega t + \zeta)}, \quad \frac{\beta}{3\alpha} + \frac{2\beta^3}{27\alpha^2} - \eta \to -\infty$$

at the other half plane $\frac{\partial u}{\partial x} + \frac{2\beta^3}{27\alpha^2} - \eta < 0$. Thus, for simplicity later, we call the solution (23) as the half periodic kink (HPK).

Fig. 3 (I) indicates a special HPK structure for the STOB equation described by Eq. (23) with the parameter selections

$$a = -k = \frac{1}{15}, \quad \kappa = \frac{1}{10}, \quad \alpha = 5, \quad \beta = b = 1, \quad \eta = \zeta = 0.$$  \hspace{1cm} (24)

Figure 3: (I) HPK solution for STOB equation described by Eq. (23) with the parameter selections (24), (II) The HPK fissions to one usual kink and one HPK described by Eq. (25) with $N = 4$ and the parameter selections (26).

Now let us consider the fission and fusion phenomenon between the HPK solution and kink solution by adding more solitons. For $N$ soliton solutions, we have

$$u = \frac{\sum_{i=1}^{N} a_i k_i e^{k_i x - \alpha k_i^2 t - \beta k_i^2 t + \xi_i}}{1 + \sum_{i=1}^{N} a_i e^{k_i x - \alpha k_i^2 t - \beta k_i^2 t + \xi_i}}. \hspace{1cm} (25)$$

A usual kink may be split out from an HPK. Fig. 3 (II) displays the HPK soliton fissions to an HPK and a usual kink for STOB equation described by (25) with $N = 4$ and the parameter selections

$$a_1 = a_2 = 1, \quad a_3 = a_4 = \frac{1}{30}, \quad k_1 = \frac{1}{10}, \quad k_2 = -\frac{1}{15}, \quad k_3 = -\frac{1}{15} + \frac{i}{10}, \quad k_4 = -\frac{1}{15} - \frac{i}{10}, \quad \alpha = 5, \quad \beta = 1, \quad \xi_j = 0, \quad (j = 1, \ldots, 4). \hspace{1cm} (26)$$
Two HPKs may be fused to one usual kink. Fig. 4 demonstrates that two HPKs are fused to an ordinary kink described by (25) with \( N = 4 \) and the parameter selections

\[
\begin{align*}
a_1 &= 100, \quad a_2 = 1, \quad a_3 = a_4 = \frac{1}{20}, \quad k_1 = 1, \quad k_2 = \frac{1}{3}, \quad k_3 = 1 + \frac{i}{3}, \\
k_4 &= 1 - \frac{i}{3}, \quad \alpha = -1, \quad \beta = 1, \quad \xi_j = 0, \quad (j = 1, \ldots, 4).
\end{align*}
\]

Two HPKs are fused to an ordinary kink given by (26) with \( N = 4 \) and (27).

4 Kink molecules and Soliton molecules for the STOB equation

Soliton molecules are bound states of solitons. In this section, we shall present some types of molecules which contain soliton molecules, kink molecules, HPK molecules and breathing soliton molecule for STOB equation. The soliton molecules presented in this section exist only for the STOB equation but not the Burgers and STO equations.

To find possible soliton molecules for the STOB system (1), we should introduce the velocity resonant conditions. If \( i \)th and \( j \)th solitons constitute a soliton molecule, then the velocity resonant condition

\[
\frac{\omega_i}{\omega_j} = \frac{k_i}{k_j}, \quad i \neq j, \quad i, j = 1, \ldots, N
\]

should be satisfied. Using (9) and (28), the velocity resonant condition (28) becomes

\[
k_j = -k_i - \frac{\beta}{\alpha}.
\]

From the resonant condition (28), the two solitary wave solutions are bounded to generate a soliton molecule. Fig. 5 shows the plot of soliton molecule structure (10) of the STOB equation with the parameters being selected as

\[
k_1 = 1, \quad k_2 = -\frac{5}{2}, \quad \alpha = \frac{2}{3}, \quad \beta = 1, \quad \xi_1 = -15, \quad \xi_2 = 0.
\]
Figure 5: Soliton molecule structure for STOB equation expressed by (10) with the parameter selections (30) (I) \( z \); (II) \( u \).

By using the velocity condition (29) for some pairs of solitons for \( N \) soliton solutions, one may find fission and fusion phenomena among soliton molecules and solitons. For a multiple solitary wave solution (25) with \( N = 4 \), let us select the parameters as follows

\[
\begin{align*}
\alpha &= 5, \quad \beta = 1, \quad k_1 = \frac{1}{10}, \quad k_2 = -\frac{1}{10}, \quad k_3 = -\frac{3}{10}, \quad k_4 = -\frac{1}{10}, \\
\xi_1 &= -10, \quad \xi_2 = 10, \quad \xi_3 = \xi_4 = 0, \quad a_j = 1, \quad (j = 1, \ldots, 4).
\end{align*}
\]

(31)

Under the conditions (31), we have

\[
u = \frac{\frac{1}{10}e^{-\frac{1}{200}t + \frac{1}{10}x - 10} - \frac{1}{10}e^{-\frac{1}{200}t - \frac{1}{10}x + 10} - \frac{3}{10}e^{\frac{9}{200}t - \frac{1}{10}x} - \frac{1}{10}e^{-\frac{1}{200}t - \frac{1}{10}x}}{1 + e^{-\frac{1}{200}t + \frac{1}{10}x - 10} + e^{-\frac{1}{200}t - \frac{1}{10}x + 10} + e^{\frac{9}{200}t - \frac{1}{10}x} + e^{-\frac{1}{200}t - \frac{1}{10}x}}.
\]

(32)

Fig. 6 (I) is the plot of the four soliton molecule structure (32) with the parameter selections (31). Fig. 6 (II) indicates the density plot of the molecule structure \( z = u_x \) with \( u \) being given by (32). From Fig. 6 (II), one finds that the more detailed soliton interactions among four solitons expressed by (32) can be divided into two parts, a fission interaction is followed by a fusion procedure. In other words, one soliton (right soliton from the bottom of the figure) is firstly split to two solitons (named soliton A (left) and soliton B (right)). Then the soliton A and another soliton (left soliton from the bottom of the figure) are fused to one soliton which is combined with the soliton B and then they form a soliton molecule.

In the previous section, we present some interesting structure of the multiple solitary solutions of the STOB equation with paired conjugate complex parameters. Especially, the special new type of kink, HPK, solutions are found. Next we establish some different types of molecules from the multiple solitary wave solutions (25) by selecting the paired conjugate complex parameters.

For one paired complex conjugate wave numbers, we derive one simplest HPK molecule in the form

\[
\begin{align*}
\zeta &= -\frac{2\beta^3}{27\alpha}t - \frac{\beta}{3\alpha}x, \quad \eta = -\kappa x - \frac{a(3a^2\kappa^2 + 6a^3\kappa)}{3a}t \\
u &= \frac{1}{3a} \frac{6a_3\alpha \kappa \sin \eta - 2a_3 \beta \cos \eta - 2a_1 \beta \exp \zeta - a_2 \beta}{2a_3 \cos \eta + 2\sqrt{a_1} \cosh[\zeta + \ln(\sqrt{a_1})] + a_2}
\end{align*}
\]

(33)
Figure 6: (I) Two kinks are fused to one kink molecule for the field $u$ described by Eq. (32), (II) Density plot of the interaction procedure (two solitons are fused to one soliton molecule) for the field $z = u_x$ with $u$ being described by Eq. (32).

by fixing the parameters in $\alpha$ with $N = 4$ as

$$a_3 = a_4, \quad k_1 = \frac{2\beta}{3\alpha}, \quad k_2 = \frac{\beta}{3\alpha}, \quad k_3 = k_2 + i\kappa, \quad k_4 = k_2 - i\kappa, \quad \xi_j = 0, \quad (i = 1, \ldots, 4). \quad (34)$$

Substituting parameters

$$\alpha = 5, \quad \beta = 1, \quad a_1 = 1002, \quad a_2 = 1, \quad a_3 = 20, \quad \kappa = -\frac{1}{3}, \quad (35)$$

into Eq. (33), we have

$$u = \frac{1}{15} \exp \zeta + 20 \cos \eta + 100 \sin \eta + 501 \cosh \zeta + 20 \cos \eta + 501, \quad (36)$$

where $\zeta = -\frac{2}{675}t - \frac{1}{15}x, \quad \eta = \frac{28}{135}t + \frac{1}{3}x$.

Figure 7: (I) The structure of the HPK molecules described by (36), (II) Breathing dissipative solutions for $z$ of (36) with the parameters (35).
Fig. 7 (I) shows the HPK molecules determined by (36). Fig. 7 (II) is the breathing dissipative solutions for the field \( z = u_x \) while \( u \) being given by (36). The breathing dissipative solution can be regarded as the breathing kink-antikink molecule which has been observed in a modelocked fiber laser [29].

From the expression (33), we know that the parameter \( a_1 \) determines the distance between two parallel HPKs of the molecule. If we take \( a_1 = 1 \), two HPKs are overlapped and the molecule is reduced to some kinds of central periodic kinks (CPK). Here, we list two special types of CPKs.

Case (a). Substituting \( \alpha = 5, \beta = 1, a_1 = 2, a_2 = 1, a_3 = \frac{1}{10}, \kappa = -3 \) into (33), we obtain

\[
u = -\frac{1}{15} \frac{10 \exp \zeta + \cos \eta_1 + 45 \sin \eta_1 + 10}{10 \cosh \zeta + \cos \eta_1 + 10}, \tag{38}
\]

where \( \zeta = -\frac{2}{5^1} t - \frac{1}{15} x, \ \eta_1 = \frac{676}{5} t + 3 x. \) Case (b). If the parameters of (33) are given by

\[
\alpha = 5, \ \beta = 1, a_1 = 2, a_2 = 1, a_3 = \frac{1}{10}, \ \kappa = -\frac{3}{4}, \tag{39}
\]

we have \( \zeta = -\frac{2}{675} t - \frac{1}{15} x, \ \eta_2 = \frac{601}{320} t + \frac{2}{3} x, \) then we obtain

\[
u = -\frac{1}{60} \frac{40 \exp \zeta + 4 \cos \eta_2 + 45 \sin \eta_2 + 40}{10 \cosh \zeta + \cos \eta_2 + 10}. \tag{40}
\]

Figure 8: (I) The structure plot of the first type of CPK solution (38); (II) The structure plot of the second type of CPK solution (40).

The plot in Fig. 8 (I) and (II) displays the central periodic kink for (38) and (40), respectively. The central periodic kink (CPK) solution is composed of periodic solution in the center and a kink solution. Comparing Fig. 8 (I) with (II), we find the amplitude of Fig. (I) is higher than that of (II). From Fig. 3 and Fig. 8, we have analyzed the fission and fusion between HPK and kink.
solutions, and the CPK solutions have also been presented. However, the Burgers equation and STO equations do not have such kind of solutions. It is verified the STOB equation is a new systems possessing interesting and good structure and properties.

To find the fission and fusion properties related to HPK molecules, we take into account the solution of (25) for \( N = 5 \) with one paired complex conjugate parameters. Substituting \( \alpha = -1, \beta = 1, k_1 = \frac{2}{3}, k_2 = 1, k_3 = \frac{1}{3}, k_4 = k_3 + i, k_5 = k_3 - i, \)
\[
a_1 = \frac{1}{100}, \ a_2 = 1, \ a_3 = 100, \ a_4 = a_5 = 5, \ \xi_i = 0, \ (i = 1, \ldots, 5)
\]
into (25) with \( N = 5 \), we obtain
\[
u = \frac{2}{3} e^{2X} + 150e^x + 500(10 - 10 \sin Y + \cos Y)e^X
\]
\[
e^{2X} + 100(1 + e^x) + 1000(10 + \cos Y),
\]
where \( X = \frac{1}{3}x - \frac{2}{27}t, \ Y = x - \frac{4}{3}t. \)

Figure 9: (I) Absorption interaction expressed by (42): an HPK molecule is absorbed by a usual kink, (II) Escape interaction described by (44): an HPK molecule is split out from the usual kink.

Fig. 9 (I) reveals an HPK molecule and a usual kink are fused to an ordinary kink. In other words, a usual kink absorbs an HPK molecule. In the same way, an HPK molecule may be split out from a usual kink. By substituting
\[
\alpha = -1, \beta = 1, k_1 = \frac{2}{3}, k_2 = -\frac{1}{2}, k_3 = \frac{1}{3}, k_4 = k_3 + i, k_5 = k_3 - i,
\]
\[
a_1 = \frac{1}{100}, \ a_2 = 1, \ a_3 = 100, \ a_4 = a_5 = 5, \ \xi_i = 0, \ (i = 1, \ldots, 5),
\]
into (25) with \( N = 5 \), we get
\[
u = \frac{2}{3} e^{2X} - 75e^{-Z} + 500(10 - 10 \sin Y + \cos Y)e^X
\]
\[
e^{2X} + 100(1 + e^{-Z}) + 1000(10 + \cos Y),
\]
where $X = \frac{1}{3}x - \frac{2}{3}t$, $Y = x - \frac{4}{3}t$, $Z = \frac{1}{2}x + \frac{3}{8}t$.

Fig. 9 (II) indicates that a usual kink is split into an ordinary kink and an HPK molecule. In alternatively speaking, an HPK molecule is escaped from a usual kink.

In this section, we investigate the soliton (kink) molecules, HPK molecules, breathing dissipative solitons (breathing kink-antikink molecules) and the fissions and fusions between HPK and kink molecules from the multiple solitary wave solutions of the STOB equation by introducing velocity resonant mechanism.

5 Summary

By means of introducing the velocity resonant mechanism, we investigate the solitary wave solutions including soliton molecules, HPKs, HPK molecules and CPKs for the STOB equation. The fission and fusion phenomena have been analyzed not only among (kink) solitons but also among usual kinks, HPKs and HPK molecules. These resonant solitary waves have changed the corresponding amplitude, widths, velocity and wave shape after the interactions. It is proved that the Burgers equation only admit fusion phenomenon. However, the STOB equation possess both fission and fusion with imposing the resonant conditions. Furthermore, we establish some new types of solutions and molecules in STOB equation which Burgers and STO equation do not have. The fission and fusion of HPK and kink solutions of multiple solitary solutions have been discussed and CPK solutions have also been derived. Meanwhile, we investigate many kinds of molecules for STOB equation, such as soliton molecules, kink molecules, HPK molecules and the breathing soliton molecules.

The kink-kink molecules as shown in Fig. 6 II may be used to describe the well known layer dislocations in solid physics and the domain walls known in magnetic materials science. The breathing dissipative solitons (breathing kink-antikink molecule) as shown in Fig. 10 may be one of the candidate to describe the known observations in experiments [28]. How to discuss the structure and properties of molecules for other integrable systems deserves further study. The various types of molecules presented in this article should provide new insight into applications in physics.

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