Associative Subalgebras of the Norton-Sakuma Algebras

A. Castillo-Ramirez

Imperial College London, Department of Mathematics.
South Kensington Campus, London, SW7 2AZ.
Email address: ac1209@imperial.ac.uk

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Abstract

A Majorana representation of a transposition group is a non-associative commutative real algebra that satisfies some of the properties of the Griess algebra. The term was introduced by A. A. Ivanov in 2009 inspired by Sakuma’s theorem, which establishes that the Majorana representations of the dihedral groups are the so-called Norton-Sakuma algebras. Since these algebras classify the isomorphism types of any algebra generated by two Majorana axes, they have become the fundamental building blocks in the construction of Majorana representations. In the present paper, we revisit Mayer and Neutsch’s theorem on associative subalgebras of the Griess algebra in the context of Majorana theory, and we apply this result to determine all the maximal associative subalgebras of the Norton-Sakuma algebras.

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1 Majorana Representations

The largest of the sporadic simple groups, the Monster group \( M \), was constructed for the first time by Griess in [G82] as a group of automorphisms of a 196,884-dimensional commutative nonassociative algebra \( V_3 \). This construction was later simplified by Conway [C84], who, among various things, associated to every 2A-involution \( t \in M \) an idempotent \( \psi(t) \in V_3 \) called a 2A-axis.

Frenkel, Lepowsky and Meurman [FLM] constructed a Vertex Operator Algebra (VOA) \( V \), called the Moonshine module, such that \( \text{Aut}(V) = M \) and the weight 2 subspace of \( V \) coincides with \( V_3 \). It was this construction of the Monster the one which eventually allowed Richard Borcherds to prove the famous Moonshine conjecture.

Let \( \psi(t) \) and \( \psi(g) \) be distinct 2A-axes of \( V_3 \). In [N96], Norton showed that the possible isomorphism types of the subalgebra of \( V_3 \) generated by \( \psi(t) \) and \( \psi(g) \) may be labeled by

\[
2A, 2B, 3A, 3C, 4A, 4B, 5A \text{ and } 6A,
\]

as they are completely determined by the conjugacy class in \( M \) of the product \( tg \). In the context of a generalized Griess subalgebra of a VOA, Sakuma [Sa07] showed that the possible isomorphism types of an algebra generated by two distinct Ising vectors coincides with the isomorphism types described by Norton.

Inspired in Sakuma’s theorem, Ivanov [I09] defined the concept of a Majorana representation of a transposition group; we will introduce some notation before stating this definition.

Let \( V = (V, \cdot, (,)) \) be a commutative real algebra with inner product. For \( S \subseteq V \), denote by \( \langle (S) \rangle \) the smallest subalgebra of \( V \) containing \( S \). Define the adjoint transformation of \( v \in V \) as the linear map

\[
ad_v(u) = v \cdot u \text{ for } u \in V.
\]

We say that \( \mu \in \mathbb{R} \) is an eigenvalue of \( v \in V \) if \( \mu \) is an eigenvalue of the transformation \( ad_v \). In this situation, denote by \( V^{(v)}_\mu \) the \( \mu \)-eigenspace of \( ad_v \).

**Definition 1.** A pair \((G,T)\) is called a transposition group if \( G \) is a finite group and \( T \) is a \( G \)-stable set of involutions of \( G \) such that \( \langle T \rangle = G \).

**Definition 2.** Let \((G,T)\) be a transposition group, \( V = (V, \cdot, (,)) \) a commutative non-associative real algebra with inner product, \( \varphi : G \to \text{GL}(V) \) a linear representation of \( G \) such that \( \varphi(G) \leq \text{Aut}(V) \) and \( \psi : T \to V \setminus \{0\} \) an injective map such that

\[
\psi(t^g) = \psi(t)^{\varphi(g)} \text{ for all } t \in T, g \in G.
\]

The quintuple

\[
(G, T, V, \varphi, \psi)
\]

is called a Majorana representation of \((G,T)\) if the following \textbf{M1-M8} axioms are satisfied:

**M1** The inner product associates with the algebra product in the sense that

\[
(u, v \cdot w) = (u \cdot v, w) \text{ for all } u, v, w \in V.
\]

\(2\)
For every \( u, v \in V \), the Norton inequality holds:
\[
(u \cdot u, v \cdot v) \geq (u \cdot v, u \cdot v).
\]

The elements of \( \psi(T) \) are idempotents of length 1.

The elements of \( \psi(T) \) are semisimple with spectrum contained in
\[
\{ 0, 1/2, 1/2 \}.
\]

For any \( a \in \psi(T) \), 1 is a simple eigenvalue of \( a \).

For any \( a \in \psi(T) \), the endomorphism \( \tau(a) \) of \( V \) defined by
\[
u \tau(a) := (-1)^{a^2} v, \text{ for } v \in V_{\mu}, \mu \in \text{Sp},
\]
preserves the algebra product of \( V \).

For any \( a \in \psi(T) \), the endomorphism \( \sigma(a) \) of \( C_V(\tau(a)) \) defined by
\[
u \sigma(a) := (-1)^{a^2} v, \text{ for } v \in V_{\mu}, \mu \in \text{Sp} \setminus \{ 1/2 \},
\]
preserves the algebra product of \( C_V(\tau(a)) \).

For any \( t \in T \), we have that
\[
\tau(\psi(t)) = \varphi(t).
\]

The idempotents in \( \psi(T) \) are called Majorana axes while the automorphisms of \( \varphi(T) \) are called Majorana involutions. In order to simplify notation, denote by \( a_t \), the Majorana axis corresponding to \( t \in T \). In the language of VOAs, the Majorana involutions may be seen as restrictions of the Miyamoto involutions [Mi96]. When \( \varphi \) and \( \psi \) are clear in the context, we often simply say that \( V \) is a Majorana representation of \( (G,T) \).

When \( G \) is a subgroup of \( \mathbb{M} \) and \( T \subseteq 2A \), we say that a Majorana representation \( V \) of \( (G,T) \) is based on an embedding in the Monster if
\[
V = \langle \langle a_t \in T \rangle \rangle \leq V_\mathbb{M}.
\]

Sakuma’s theorem implies that the Majorana representations of the dihedral groups coincide with the subalgebras of \( V_\mathbb{M} \) described by Norton; hence, these algebras have been named the Norton-Sakuma algebras. Besides this, Majorana representations of various transposition groups have been described in [IPSS10], [IS12], [Iv11b], [Iv11a] and [IS].

Meyer and Neutsch [MN93] showed the existence of a maximal 48-dimensional subalgebra of \( V_\mathbb{M} \) generated by 2A-axes by proving that every associative subalgebra of \( V_\mathbb{M} \) is generated by a set of pairwise orthogonal idempotents. Furthermore, they conjectured that 48 was the largest possible dimension of an associative subalgebra of \( V_\mathbb{M} \). This conjecture was proved later by Miyamoto [Mi96].

In this paper, we discuss Meyer and Neutsch’s results in the context of an abstract Majorana representation. Furthermore, we find all the maximal associative subalgebras of the Norton-Sakuma using the information about idempotents obtained in [CR13]. In particular, we prove that 3 the largest dimension of an associative subalgebra of a Norton-Sakuma algebra and that the algebra of type 6A has 45 non-trivial associative subalgebras.
2 Associative subalgebras of Majorana representations

Throughout this section, we assume that $V$ is a Majorana representation of $(G,T)$ with identity $\text{id} \in V$. We will not use explicitly axioms M6–M8 of Definition 2, but we will replace M2 by a stronger statement:

**M2’** The Norton inequality holds for every $u,v \in V$, with equality precisely when the adjoint transformations $\text{ad}_u$ and $\text{ad}_v$ commute.

It is shown in Section 15 of [C84] that axiom M2’ holds in $V_M$. The importance of this stronger axiom in our discussion lies in the following proposition:

**Lemma 2.1.** Let $x \in V$ be an idempotent. Then the eigenspaces $V_0^{(x)}$ and $V_1^{(x)}$ are subalgebras of $V$.

**Proof.** Let $\mu \in \{0,1\}$ and $y_1, y_2 \in V_\mu^{(x)}$. Observe that, for $i \in \{1,2\}$,

$$(x \cdot y_i, y_i) = (\mu y_i, y_i) = (\mu y_i, \mu y_i) = (x \cdot y_i, x \cdot y_i).$$

Therefore, M2’ implies that $\text{ad}_x$ and $\text{ad}_{y_i}$ commute, and so

$$x \cdot (y_1 \cdot y_2) = y_1 \cdot (x \cdot y_2) = \mu (y_1 \cdot y_2).$$

This shows that $(y_1 \cdot y_2) \in V_\mu^{(x)}$. \qed

If $v \in V$, define the length of $v$ by the non-negative real number $l(v) := (v,v)$. The following are further useful elementary results.

**Lemma 2.2.** Let $\{x_i \in V : 1 \leq i \leq k\}$ be a finite set of idempotents. Let $\lambda_i \in \mathbb{R}$ and suppose that

$$x = \sum_{i=1}^{k} \lambda_i x_i,$$

is also an idempotent. Then, we have that

$$l(x) = \sum_{i=1}^{k} \lambda_i l(x_i).$$

**Proof.** The lemma follows by axiom M1 and the linearity of the inner product:

$$l(x) = (x,x) = (x \cdot x, \text{id}) = (x, \text{id}) = \sum_{i=1}^{k} \lambda_i (x_i, \text{id}) = \sum_{i=1}^{k} \lambda_i l(x_i).$$ \qed

**Lemma 2.3.** Let $x,y \in V$ be idempotents. The following are equivalent:

(i) $(x,y) = 0$.

(ii) $x \cdot y = 0$.

(iii) $x + y$ is idempotent.
Proof. If \((x, y) = 0\), Norton inequality implies that

\[(x \cdot y, x \cdot y) \leq (x \cdot x, y \cdot y) = (x, y) = 0.\]

Hence \(x \cdot y = 0\) by the positive definiteness of the inner product. It is clear that statement (ii) implies (iii). Finally, Lemma 2.2 shows that (iii) implies (i).

We say that two idempotents are orthogonal if they satisfy the equivalent statements (i) - (iii) of Lemma 2.3. For \(v \in V\), define \(d(v) := \dim(V_0^{(v)})\).

Lemma 2.4. Let \(x \in V\) be a non-zero non-identity idempotent. The following statements hold:

(i) \(1\) and \(0\) are eigenvalues of \(x\).

(ii) For every \(g \in \text{Aut}(V)\), the spectrum of \(x^g\) is equal to the spectrum of \(x\).

(iii) If \(v \in V\) is a \(\lambda\)-eigenvector of \(x\), then \(v\) is a \((1 - \lambda)\)-eigenvector of \(\text{id} - x\).

(iv) If \(V_0^{(x)}\) has finitely many idempotents, then \(x\) is orthogonal to at most \(2^{d(x)}\) idempotents.

Proof. Part (i) follows because \(x\) and \(\text{id} - x\) are 1- and 0-eigenvectors of \(\text{ad}_x\), respectively. Part (ii) follows since, for any \(g \in \text{Aut}(V)\), \(v\) is an eigenvector of \(\text{ad}_x\) if and only if \(v^g\) is an eigenvector of \(\text{ad}_x^g\). Parts (iii) is trivial while part (iv) follows by Lemma 2.1 and Bézout’s theorem.

When \(x \in V\) is a non-zero non-identity idempotent, it is clear that \(\{x, \text{id} - x\}\) is the orthogonal basis of a 2-dimensional associative subalgebra

\[V_x := \langle\langle x, \text{id} - x\rangle\rangle \leq V.\]

Define \(V_0 := \langle\langle 0\rangle\rangle\) and \(V_{\text{id}} := \langle\langle \text{id}\rangle\rangle\).

Definition 3. We say that an associative subalgebra \(U\) of \(V\) is trivial associative if \(U = V_x\) for some idempotent \(x \in V\).

An idempotent is decomposable if it may be expressed as a sum of at least two nonzero idempotents; otherwise, we say the idempotent is indecomposable.

The following results were obtained in [MN93] in the context of the Griess algebra; however, it should be noted that they hold as well in the context of a Majorana representation with identity that satisfies axiom M2’.

Proposition 2.5 (Meyer, Neutsch). An idempotent \(x\) of \(V\) is indecomposable if and only if 1 is a simple eigenvalue of \(x\).

Corollary 2.6. The Majorana axes of \(V\) are indecomposable.

Theorem 2.7 (Meyer, Neutsch). Let \(U\) be a subalgebra of \(V\). The following statements hold:

(i) \(U\) is associative if and only if \(U\) has an orthogonal basis of idempotents.

(ii) \(U\) is maximal associative if and only if \(\text{id} \in U\) and the idempotents in the orthogonal basis of \(U\) are indecomposable.
Corollary 2.8. A trivial associative subalgebra $V_x \leq V$ is maximal associative if and only if 0 and 1 are simple eigenvalues of $x \in V$.

Proof. The result follows by Lemma 2.4 (iii) and Theorem 2.7 (ii).

In view of Theorem 2.7, it is relevant to study the eigenspaces $V_0^{(x)}$ and $V_1^{(x)}$, where $x \in V$ is an idempotent.

The following lemmas will be useful in our discussion about the associative subalgebras of the Norton-Sakuma algebras.

Lemma 2.9. Suppose that, for every idempotent $x \in V$, the space $V_0^{(x)}$ has finitely many idempotents and $d(x) \leq 2$. Then every associative subalgebra of $V$ is at most three-dimensional.

Proof. If $\{x, y, z, w\}$ is a set of four pairwise idempotents of $V$, then $x$ is orthogonal to 7 idempotents: $y, z, w, y + z, y + w, z + w$ and $y + z + w$. This contradicts Lemma 2.4 (iv).

Lemma 2.10. Let $x \in V$ be an idempotent and suppose that $V_0^{(x)}$ has finitely many idempotents. The following statements hold:

(i) If $d(x) = 1$, then $x$ is not contained in the orthogonal basis of any three-dimensional associative subalgebra of $V$.

(ii) If $d(x) \geq 2$, then $x$ is in the orthogonal basis of at most $2^{d(x) - 1} - 1$ three-dimensional maximal associative subalgebras of $V$.

Proof. Part (i) is trivial. Let $d(x) \geq 2$ and suppose that $\{x, y, z\}$ is the orthogonal basis of a maximal associative subalgebra of $V$, where $y, z \in V_0^{(x)}$ are idempotents. Note that, if there is an idempotent $w \in V_0^{(x)}$ such that $\{x, y, w\}$ is the orthogonal basis of a maximal associative subalgebra, then Theorem 2.7 (ii) implies that $x + y + z = \text{id} = x + y + w$.

so $z = w$. This shows that the three-dimensional maximal associative subalgebras of $V$ with $x$ in their orthogonal basis correspond to disjoint two-sets of non-zero idempotents of $V_0^{(x)}$. By Bézout’s theorem, $V_0^{(x)}$ has at most $2^{d(x)}$ idempotents. Therefore, there are at most $\frac{2^{d(x)} - 2}{2}$ disjoint two-sets of non-zero idempotents in $V_0^{(x)}$.

3 The Norton-Sakuma Algebras

Sakuma’s theorem [Sa07] states that the product of any two Majorana involutions is at most 6 and that there are at most eight possibilities for the isomorphism type of an algebra generated by two Majorana axes. The following version of this theorem was established in [IPSS10].

Theorem 3.1. Let $(G, T, V, \varphi, \psi)$ be a Majorana representation. Let $t, g \in T$, $t \neq g$, and define $\rho := \varphi(tg)$. For $i \in \mathbb{Z}$, let $a_{2i} := \psi(t^i \rho)$. Then the subalgebra $\langle (a_t, a_g) \rangle$ of $V$ is isomorphic to a Norton-Sakuma algebra of type $NX$, as described in Table 1, where $N = |\rho|, X \in \{A, B, C\}$.
| Type | Basis | Products |
|------|-------|----------|
| 2A   | $a_t$, $a_g$, $a_\rho$ | $a_t \cdot a_g = \frac{1}{2\pi}(a_t + a_g - a_\rho)$, $a_t \cdot a_\rho = \frac{1}{2\pi}(a_t + a_\rho - a_g)$, $(a_t, a_g) = (a_t, a_\rho) = (a_g, a_\rho) = \frac{1}{2\pi}$ |
| 2B   | $a_t$, $a_g$ | $a_t \cdot a_g = 0$, $(a_t, a_g) = 0$ |
| 3A   | $a_t$, $a_g$, $a_{g-1}$, $u_\rho$ | $a_t \cdot a_g = \frac{1}{2\pi}(2a_t + 2a_g + a_{g-1}) - \frac{3\pi}{2\pi}u_\rho$, $a_t \cdot u_\rho = \frac{1}{2\pi}(2a_t - a_g - a_{g-1}) + \frac{\pi}{2\pi}u_\rho$, $(a_t, a_g) = (a_t, u_\rho) = \frac{1}{2\pi}$, $(u_\rho, u_\rho) = \frac{3\pi}{5}$ |
| 3C   | $a_t$, $a_g$, $a_{g-1}$ | $a_t \cdot a_g = \frac{1}{2\pi}(a_t + a_g - a_{g-1})$, $(a_t, a_g) = \frac{1}{2\pi}$ |
| 4A   | $a_t$, $a_g$, $a_{g-1}$, $a_{g2}$, $v_\rho$ | $a_t \cdot a_g = \frac{1}{2\pi}(3a_t + 3a_g + a_{g2} + a_{g-1}) - 3v_\rho$, $a_t \cdot v_\rho = \frac{1}{2\pi}(5a_t - 2a_g - a_{g2} - 2a_{g-1} + 3v_\rho)$, $v_\rho \cdot v_\rho = v_\rho$, $a_t \cdot a_{g2} = 0$, $(a_t, a_g) = \frac{1}{2\pi}$, $(a_t, a_{g2}) = 0$, $(a_t, v_\rho) = \frac{1}{2\pi}$, $(v_\rho, v_\rho) = 2$ |
| 4B   | $a_t$, $a_g$, $a_{g-1}$, $a_{g2}$, $a_\rho^2$ | $a_t \cdot a_g = \frac{1}{2\pi}(a_t + a_g - a_{g-1} - a_{g2} + a_\rho^2)$, $a_t \cdot a_{g2} = \frac{1}{2\pi}(a_t + a_{g2} - a_\rho^2)$, $(a_t, a_g) = \frac{1}{2\pi}$, $(a_t, a_{g2}) = (a_t, a_\rho^2) = \frac{1}{2\pi}$ |
| 5A   | $a_t$, $a_g$, $a_{g-1}$, $a_{g2}$, $a_{g-2}$, $w_\rho$ | $a_t \cdot a_g = \frac{1}{2\pi}(3a_t + 3a_g - a_{g2} - a_{g-1} - a_{g-2}) + w_\rho$, $a_t \cdot a_{g2} = \frac{1}{2\pi}(3a_t + 3a_{g2} - a_{g1} - a_{g-1} - a_{g-2}) - w_\rho$, $a_t \cdot w_\rho = \frac{7}{2\pi}(a_g + a_{g-1} - a_{g2} - a_{g-3}) + \frac{7}{2\pi}w_\rho$, $w_\rho \cdot w_\rho = \frac{5\pi}{2\pi}(a_{g-2} + a_{g-1} + a_t + a_g + a_{g2})$, $(a_t, a_g) = \frac{1}{2\pi}$, $(a_t, w_\rho) = 0$, $(w_\rho, w_\rho) = \frac{3\pi}{5}$ |
| 6A   | $a_t$, $a_g$, $a_{g-1}$, $a_{g2}$, $a_{g-2}$, $a_{g3}$, $a_\rho^3$, $u_\rho^2$ | $a_t \cdot a_g = \frac{1}{2\pi}(a_t + a_g - a_{g-2} - a_{g-3} - a_{g2} - a_{g3} + a_\rho^3) + \frac{3\pi}{2\pi}u_\rho^2$, $a_t \cdot a_{g2} = \frac{1}{2\pi}(2a_t + 2a_{g2} + a_{g-2}) - \frac{3\pi}{2\pi}u_\rho^2$, $a_t \cdot a_{g3} = \frac{1}{2\pi}(a_t + a_{g3} - a_\rho^3)$, $a_\rho^3 \cdot u_\rho^2 = 0$, $(a_t, a_g) = \frac{5}{2\pi}$, $(a_t, a_{g2}) = \frac{1}{2\pi}$, $(a_t, a_{g3}) = \frac{1}{2\pi}$, $(a_\rho^3, u_\rho^2) = 0$ |

Table 1: Norton-Sakuma algebras.
The scaling of the products of Table 1 coincides with the one used in \[\text{IPSS10}\]. The missing products of basis vectors may be obtained using the symmetries of the algebras and their mutual inclusions:

\[
2A \leftrightarrow 4B, \quad 2B \leftrightarrow 4A, \quad 2A \leftrightarrow 6A, \quad 3A \leftrightarrow 6A.
\]

For the rest of the paper, denote by \(V_{NX}\) the Norton-Sakuma algebra of type \(NX\). Each of these algebras has an identity \(\text{id}_{NX}\), which is given explicitly in Table 2 of \[\text{CR13}\]. Denote the automorphism group of \(V_{NX}\) by \(\text{Aut}(NX)\). The following result is Theorem 4.1 in \[\text{CR13}\].

**Proposition 3.2.** For \(2 \leq N \leq 6\) and \(X \in \{A, B, C\}\), let \((D, T, V_{NX}, \varphi, \psi)\) be the Majorana representation of the dihedral group of order \(2N\), where \(V_{NX} = \langle \langle a_t, a_g \rangle \rangle, \ t, g \in T, \) is a Norton-Sakuma algebra of type \(NX\). The following statements hold:

1. The algebra \(V_{2A}\) has exactly 8 idempotents and \(\text{Aut}(2A) = \text{Sym} \{a_t, a_g, a_{tg}\} \cong S_3\).
2. The algebra \(V_{3A}\) has exactly 16 idempotents and \(\text{Aut}(3A) = \varphi(D) \cong S_3\).
3. The algebra \(V_{3C}\) has exactly 8 idempotents and \(\text{Aut}(3C) = \varphi(D) \cong S_3\).
4. The algebra \(V_{4A}\) has an infinite family of idempotents plus 18 extra idempotents. In this case, \(\text{Aut}(4A) = \langle \varphi(t), \phi_{4A} \rangle \cong D_8\), with \(\phi_{4A} = (a_t, a_g) (a_{g-1}, a_{g_2})\).
5. The algebra \(V_{4B}\) has exactly 32 idempotents and \(\text{Aut}(4B) = \langle \varphi(t), \phi_{4B} \rangle \cong D_8\), with \(\phi_{4B} = (a_t, a_g) (a_{g-1}, a_{g_2})\).
6. The algebra \(V_{5A}\) has exactly 44 idempotents and \(\text{Aut}(5A) = \langle \varphi(t), \varphi(g), \phi_{5A} \rangle\), with \(\phi_{5A} = (a_g, a_{g_2}, a_{g_{-1}}, a_{g_{-2}})\), is isomorphic to the Frobenius group of order 20.
7. The algebra \(V_{6A}\) has exactly 208 idempotents and \(\text{Aut}(6A) = \langle \varphi(t), \phi_{6A} \rangle \cong D_{12}\), with \(\phi_{6A} = (a_t, a_g) (a_{g_{-1}}, a_{g_2}) (a_{g_{-2}}, a_{g_3})\).

The next result was verified by direct computations in \[\text{MAP}\], using the idempotents found in \[\text{CR13}\].

**Lemma 3.3.** Every idempotent of every Norton-Sakuma algebra is semisimple.

In view of Lemma 3.3, the spectra of the idempotents of \(V_{NX}\) gives essential information about the associative subalgebras of \(V_{NX}\). Although we only require the multiplicities of the eigenvalues 0 and 1, we believe that the full spectrum of each idempotent may be of general interest.
### 3.1 Norton-Sakuma Algebras of Small Dimensions

Table 2 contains the spectra, given as multisets, of the non-trivial non-identity idempotents of $V_{2A}$, $V_{3A}$ and $V_{3C}$. Because of Lemma 2.4 (ii), we organize these spectra in terms of $\text{Aut}(NX)$-orbits of idempotents. The idempotent $y_{3A}$ that appears in Table 2 is defined by

$$y_{3A} := \frac{2}{9}(4a_t + 4a_g + a_{g-1}) - \frac{1}{4}u_\rho.$$  

**Lemma 3.4.** For $NX \in \{2A, 3A, 3C\}$, the following statements hold:

(i) The identity of $V_{NX}$ is the unique decomposable idempotent of $V_{NX}$

(ii) There are no non-trivial associative subalgebras of $V_{NX}$.

**Proof.** This is a direct consequence of Table 2, Proposition 2.5 and Corollary 2.8.

| Norton-Sakuma algebra of type 2A | Orbit | Size | Spectrum | Orbit | Size | Spectrum |
|----------------------------------|-------|------|----------|-------|------|----------|
| $[a_t]$                          | 3     | $\{0, 1, \frac{1}{4}\}$ | $[\text{id}_{2A} - a_t]$ | 3     | $\{0, 1, \frac{3}{4}\}$ |

| Norton-Sakuma algebra of type 3A | Orbit | Size | Spectrum | Orbit | Size | Spectrum |
|----------------------------------|-------|------|----------|-------|------|----------|
| $[a_t]$                          | 3     | $\{0, 1, \frac{1}{4}, \frac{1}{32}\}$ | $[\text{id}_{3A} - a_t]$ | 3     | $\{0, 1, \frac{3}{4}, \frac{31}{32}\}$ |
| $[u_\rho]$                       | 1     | $\{0, 1, \frac{1}{4}, \frac{1}{2}\}$ | $[\text{id}_{3A} - u_\rho]$ | 1     | $\{0, 1, \frac{2}{3}, \frac{2}{3}\}$ |
| $[y_{3A}]$                       | 3     | $\{0, 1, \frac{1}{8}, \frac{13}{16}\}$ | $[\text{id}_{3A} - y_{3A}]$ | 3     | $\{0, 1, \frac{2}{3}, \frac{4}{3}, \frac{4}{16}\}$ |

| Norton-Sakuma algebra of type 3C | Orbit | Size | Spectrum | Orbit | Size | Spectrum |
|----------------------------------|-------|------|----------|-------|------|----------|
| $[a_t]$                          | 3     | $\{0, 1, \frac{1}{32}\}$ | $[\text{id}_{3C} - a_t]$ | 3     | $\{0, 1, \frac{31}{32}\}$ |

Table 2: Spectra of the idempotents of $V_{2A}$, $V_{3A}$ and $V_{3C}$.

### 3.2 The Norton-Sakuma Algebra of Type 4A

The Norton-Sakuma algebra $V = V_{4A}$ has an infinite family of idempotents of length 2. In particular, for any $\lambda \in [-\frac{2}{3}, 1]$,

$$y_{4A}^{(1)}(\lambda) := f(\lambda)(a_t + a_{g2}) + \overline{f(\lambda)}(a_g + a_{g-1}) + \lambda u_\rho$$

is an idempotent, where

$$f(\lambda) := \frac{1}{2} (1 - \lambda) - \frac{1}{6} \sqrt{-15\lambda^2 + 6\lambda + 9},$$
and $\bar{f}(\lambda)$ is the conjugate of $f(\lambda)$ in $\mathbb{Q}(\sqrt{-15\lambda^2 + 6\lambda + 9})$. The spectrum of $y_{4A}^{(1)}(\lambda)$ is

$$\{0, 1, \frac{1}{2}, h(\lambda), \bar{h}(\lambda)\}$$

where

$$h(\lambda) := \frac{1}{2^2}(17 - 5\lambda - 5\sqrt{-15\lambda^2 + 6\lambda + 9}).$$

**Lemma 3.5.** The following statements hold:

(i) For any $\lambda \in [-\frac{3}{5}, 1]$, $\lambda \neq 0, \frac{2}{5}$, the algebra $V_{y_{4A}^{(1)}(\lambda)}$ is maximal associative.

(ii) The idempotent $y := y_{4A}^{(1)}(\frac{2}{5})$ is indecomposable with $d(y) = 2$.

**Proof.** The equation $h(\lambda) = 1$ has no solutions while $\bar{h}(\lambda) = 1$ has the unique solution $\lambda = 0$. On the other hand, the equation $h(\lambda) = 0$ has the unique solution $\lambda = \frac{2}{5}$ while $\bar{h}(\lambda) = 0$ has no solutions. Therefore, 0 and 1 are simple eigenvalues of $y_{4A}^{(1)}(\lambda)$, for any $\lambda \in [-\frac{3}{5}, 1]$, $\lambda \neq 0, \frac{2}{5}$. The result follows by Corollary 2.8.

The previous lemma implies that, for every idempotent $x \in V_{4A}$, the algebra $V_{x}^{(0)}$ has finitely many idempotents. Observe that

$$y = \text{id}_{4A} - a_t - a_{g_2}.$$

Table 3 contains the spectra, given as multisets, of the rest of the non-zero non-identity idempotents of $V_{4A}$, where

$$y_{4A}^{(2)} := \frac{2}{7} \left(2 - \sqrt{2}\right) (a_t + a_g) + \frac{2}{7} \left(2 + \sqrt{2}\right) (a_{g-1} + a_{g_2}) - \frac{2}{7} v_p.$$

| Orbit  | Size | Spectrum                                      | Orbit  | Size | Spectrum                                      |
|--------|------|-----------------------------------------------|--------|------|-----------------------------------------------|
| $[a_t]$ | 4    | $\{0, 0, 1, \frac{1}{7}, \frac{1}{2}\}$        | $[\text{id}_{4A} - a_t]$ | 4    | $\{0, 1, \frac{3}{7}, \frac{31}{56}\}$      |
| $[v_p]$ | 1    | $\{0, 1, \frac{1}{7}, \frac{3}{8}, \frac{3}{7}\}$ | $[\text{id}_{4A} - v_p]$ | 1    | $\{0, 1, \frac{1}{2}, \frac{5}{8}, \frac{5}{8}\}$ |
| $[y_{4A}]^{(2)}$ | 4    | $\{0, 1, \frac{1}{7}, \frac{3}{8}, \frac{3}{7}\}$ | $[\text{id}_{4A} - y_{4A}]^{(2)}$ | 4    | $\{0, 1, \frac{11}{14}, \frac{9}{14}, \frac{9}{14}\}$ |

Table 3: Spectra of the idempotents of $V_{4A}$.

Since $d(x) \leq 2$ for every idempotent $x \in V_{4A}$, Lemma 2.9 implies that every associative subalgebra of $V_{4A}$ is at most three-dimensional.

**Lemma 3.6.** The subalgebras of $V_{4A}$,

$$\langle\langle a_t, a_{g_2}, \text{id}_{4A} - a_t - a_{g_2}\rangle\rangle$$

and

$$\langle\langle a_g, a_{g-1}, \text{id}_{4A} - a_g - a_{g-1}\rangle\rangle,$$

are maximal associative.

**Proof.** The idempotents generating any of the above subalgebras are clearly pairwise orthogonal and their sum is $\text{id}_{4A}$. Moreover, they are indecomposable by Table 3 and Proposition 2.5. The result follows by Theorem 2.7.
Lemma 3.7. The Norton-Sakuma algebra of type $4A$ has infinitely many maximal associative subalgebras. However, it has only two non-trivial maximal associative subalgebras.

Proof. The first part of this lemma follows by Lemma 3.5. By Table 3 and Lemma 3.3 only the idempotents in the orbits $[a_4]$ and $[id_4A - a_t - a_g]$ have a two-dimensional 0-eigenspace, so they are the only idempotents with non-maximal trivial subalgebra. By Lemma 2.10 each one of these idempotents is contained in the orthogonal basis of at most one three-dimensional maximal associative subalgebra of $V_4A$. Lemma 3.6 describes such algebras, so the result follows.

3.3 The Norton-Sakuma Algebra of Type $4B$

The Norton-Sakuma algebra $V = V_4B$ contains a subalgebra of type $2A$ with basis $\{a_t, a_g, a_\rho\}$. Let $id_2A$ be the identity of this subalgebra. We begin this section by proving the following result.

Lemma 3.8. Every associative subalgebra of $V_4B$ is at most three-dimensional.

Proof. Suppose $U$ is an associative subalgebra of $V_6A$ of dimension $k \geq 4$. Let $\{x_i : 1 \leq i \leq k\}$ be the orthogonal basis of idempotents of $U$. Without loss of generality, we may assume $U$ is maximal associative. By Theorem 2.7, $id_4B \in U$ and $\sum_{i=1}^{k} x_i = id_4B$. By Lemma 2.10 we have that

$$\sum_{i=1}^{k} l(x_i) = l(id_4B) = \frac{19}{5}. \quad (1)$$

The orthogonal basis of $U$ contains at most one idempotent of length 1, since there is no pair of orthogonal idempotents of length 1 in $V_4B$. The non-zero idempotents with the smallest length different from 1 are $[id_2A - a_t] \cup [id_2A - a_\rho^2]$; these idempotents have length $\frac{7}{5}$. Therefore,

$$\sum_{i=1}^{k} l(x_i) \geq 1 + 3 \cdot \frac{7}{5} = \frac{26}{5} > \frac{19}{5},$$

which contradicts (1).

Table 4 gives the spectra of the non-zero non-identity idempotents of $V_4B$, where

$$y_{4B} := \frac{2^2}{11} (1 + \sqrt{2}) (a_t + a_g) + \frac{2^2}{11} (1 - \sqrt{2}) (a_g + a_{g^2}) + \frac{5}{11} a_{\rho^2}.$$

Lemma 3.9. The subalgebras of $V_4B$,

$$U_{4B}^{(1)} := \langle \langle a_t, id_2A - a_t, id_2A \rangle \rangle,$$

$$U_{4B}^{(2)} := \langle \langle a_{\rho^2}, id_2A - a_{\rho^2}, id_2A \rangle \rangle,$$

are maximal associative.
Table 4: Spectra of the idempotents of $V_{4B}$.

| Orbit                | Size | Spectrum       |
|----------------------|------|----------------|
| $[a_t]$              | 4    | $\{0, 1, \frac{1}{2}, \frac{3}{4}\}$ |
| $[\text{id}_{4B} - a_t]$ | 4    | $\{0, 1, \frac{1}{2}, \frac{3}{4}\}$ |
| $[a_{\rho^2}]$      | 1    | $\{0, 0, \frac{1}{2}, \frac{1}{2}\}$ |
| $[\text{id}_{4B} - a_{\rho^2}]$ | 1    | $\{0, 1, \frac{1}{2}, \frac{1}{2}\}$ |
| $[\text{id}_{2A}]$  | 2    | $\{0, 1, 1, \frac{1}{2}\}$ |
| $[\text{id}_{2A} - a_{\rho^2}]$ | 2    | $\{0, 0, 1, \frac{3}{4}\}$ |
| $[\text{id}_{2A} - a_t]$ | 4    | $\{0, 0, 1, \frac{3}{4}\}$ |
| $[\text{id}_{4B} - \text{id}_{2A} + a_t]$ | 4    | $\{0, 1, \frac{3}{4}, \frac{3}{2}\}$ |
| $[y_{4B}]$          | 4    | $\{0, 1, \frac{10}{11}, \frac{21}{22}, \frac{9}{22}\}$ |
| $[\text{id}_{4B} - y_{4B}]$ | 4    | $\{0, 1, \frac{10}{11}, \frac{21}{22}, \frac{13}{22}\}$ |

**Proof.** Since 

$$(\text{id}_{2A}, \text{id}_{2A}^{\phi_{4B}}) = (a_{\rho^2}, \text{id}_{2A}^{\phi_{4B}}) = 1$$ and 

$$(a_t, \text{id}_{2A} - a_t) = (a_{\rho^2}, \text{id}_{2A}^{\phi_{4B}} - a_{\rho^2}) = (\text{id}_{2A} - a_t, \text{id}_{2A}^{\phi_{4B}} - a_{\rho^2}) = 0,$$

we have that 

$$(a_t, \text{id}_{2A} - a_t) = (a_{\rho^2}, \text{id}_{2A}^{\phi_{4B}} - a_{\rho^2}) = (\text{id}_{2A} - a_t, \text{id}_{2A}^{\phi_{4B}} - a_{\rho^2}) = 0,$$

and 

$$(a_{\rho^2}, \text{id}_{2A} - a_{\rho^2}) = (a_{\rho^2}, \text{id}_{2A}^{\phi_{4B}} - a_{\rho^2}) = (\text{id}_{2A} - a_{\rho^2}, \text{id}_{2A}^{\phi_{4B}} - a_{\rho^2}) = 0.$$ 

As the relation 

$$\text{id}_{4B} = \text{id}_{2A} + \text{id}_{2A}^{\phi_{4B}} - a_{\rho^2}$$

holds, we have that 

$$\text{id}_{4B} = a_t + (\text{id}_{2A} - a_t) + (\text{id}_{2A}^{\phi_{4B}} - a_{\rho^2}),$$

$$\text{id}_{4B} = a_{\rho^2} + (\text{id}_{2A} - a_{\rho^2}) + (\text{id}_{2A}^{\phi_{4B}} - a_{\rho^2}).$$

By Table 4 and Proposition 2.5, the idempotents generating $U_{4B}^{(1)}$ and $U_{4B}^{(2)}$ are indecomposable, so the result follows by Theorem 2.7.

If $U$ is a subalgebra of any Majorana representation $V$, we introduce the notation 

$$[U] := \{ U^g : g \in \text{Aut}(V) \}.$$ 

**Lemma 3.10.** The Norton-Sakuma algebra of type $4B$ has exactly 9 maximal associative subalgebras; 4 of these subalgebras are trivial associative while 5 are three-dimensional.
Proof. The 4 trivial maximal associative subalgebras are the ones in the orbit $[V_{yA}]$. If $U^{(1)}_{4B}$ and $U^{(2)}_{4B}$ are the subalgebras of Lemma 3.9, the orbits $[U^{(1)}_{4B}]$ and $[U^{(2)}_{4B}]$ contain 4 and 1 maximal associative subalgebras respectively. We will show that there are no more maximal associative subalgebras. By Lemma 3.8, there are no four-dimensional associative subalgebras of $V_{4B}$. Let $N_x$ be the number of associative algebras in $\bigcup U^{(1)}_{4B}$ and $\bigcup U^{(2)}_{4B}$. The following table gives the values of $d(x)$ and $N_x$ for the orbit representatives of idempotents of $V_{4B}$ with $d(x) \geq 2$:

| Idempotent $x$ | $d(x)$ | $N_x$ | Idempotent $x$ | $d(x)$ | $N_x$ |
|----------------|--------|------|----------------|--------|------|
| $a_t$          | 2      | 1    | $\text{id}_{2A} - a_{\rho^2}$ | 3      | 3    |
| $a_{\rho^2}$   | 2      | 1    | $\text{id}_{2A} - a_t$       | 2      | 1    |

If $M_x$ is the number of three-dimensional maximal associative subalgebras of $V_{4B}$ where the idempotent $x$ is contained, Lemma 2.10 shows that $N_x \leq M_x \leq 2d(x) - 1$, whenever $d_x \geq 2$. However, the above table shows that $N_x = 2d(x) - 1$, so $N_x = M_x$. Therefore, there are no more three-dimensional associative subalgebras of $V_{4B}$ besides the ones in $[U^{(1)}_{4B}] \cup [U^{(2)}_{4B}]$.

3.4 The Norton-Sakuma Algebra of Type $5A$

Table 5 gives the spectra of the idempotents of $V_{5A}$, where

$$y_{5A}^{(1)} := \frac{24}{35} \left( a_t + a_g + a_{g^{-1}} + a_{g_2} + a_{g^{-2}} \right) + \frac{211}{179} \sqrt{5}w_{\rho},$$

$$y_{5A}^{(2)} := \frac{1}{5} \left( \frac{3}{14} a_t + \alpha (a_g + a_{g^{-1}}) + \alpha \beta (a_{g_2} + a_{g^{-2}}) - \frac{27}{4} \sqrt{5}w_{\rho} \right),$$

$$y_{5A}^{(3)} := \frac{22}{5} \left( \frac{2}{7} a_t + \beta (a_g + a_{g^{-1}}) + \beta (a_{g_2} + a_{g^{-2}}) - \frac{384}{35} \sqrt{5}w_{\rho} \right),$$

where $\alpha = \frac{4}{7} + \sqrt{5}$, $\beta = \frac{22}{7} - \frac{1}{7} \sqrt{5}$, and $\alpha, \beta$ are their conjugates in $\mathbb{Q}(\sqrt{5})$.

Lemma 3.11. The subalgebra of $V_{5A}$,

$$\langle (a_t, y_{5A}^{(2)}, (y_{5A}^{(2)})^{\phi_{\alpha}}) \rangle,$$

is maximal associative.

Proof. The result follows by Table 5, Proposition 2.5, and Theorem 2.7, since

$$(a_t, y_{5A}^{(2)}) = (a_t, (y_{5A}^{(2)})^{\phi_{\alpha}}) = (y_{5A}^{(2)}, (y_{5A}^{(2)})^{\phi_{\alpha}}) = 0$$

and

$$\text{id}_{5A} = a_t + y_{5A}^{(2)} + (y_{5A}^{(2)})^{\phi_{\alpha}}.$$
Proposition 3.12. The Norton-Sakuma algebra of type 5A has exactly 11 maximal associative subalgebras; 6 of these algebras are trivial while 5 are three-dimensional.

Proof. By Table 5 and Corollary 2.9 the trivial maximal associative subalgebras of $V_{5A}$ are exactly the ones in the orbits $[V_{5A}^i]$ for $i = 1$ and $i = 3$, which have sizes 1 and 5 respectively. Using the action of $\text{Aut}(5A)$, Lemma 3.11 defines 5 three-dimensional maximal associative subalgebras of $V_{5A}$. The result follows using Lemma 2.9 and Lemma 2.10.

| Orbit     | Size | Spectrum            |
|-----------|------|---------------------|
| $[a_i]$   | 5    | $\{0, 1, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\}$ |
| $[id_{5A} - a_i]$ | 5    | $\{0, 1, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\}$ |
| $[b_{5A}^{(1)}]$ | 2    | $\{0, 1, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\}$ |
| $[b_{5A}^{(3)}]$ | 10   | $\{0, 1, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\}$ |
| $[y_{5A}^{(2)}]$ | 10   | $\{0, 1, \frac{59}{64}, \frac{7}{8}, \frac{7}{8}\}$ |
| $[id_{5A} - y_{5A}^{(2)}]$ | 10   | $\{0, 1, \frac{59}{64}, \frac{7}{8}, \frac{7}{8}\}$ |

Table 5: Spectra of the idempotents of $V_{5A}$.

3.5 The Norton-Sakuma Algebra of Type 6A

The following list of idempotents of $V_{6A}$ was obtained in [CR13]:

$$y_{6A}^{(1)} := \frac{1}{21} (2^4(a_g + a_{g-1} + a_{g_2} + a_{g-2}) + 2^2(a_t + a_{g_3}) + 12a_{\rho^3} - 9u_{\rho^2}),$$

$$y_{6A}^{(2)} := \frac{1}{36} (36a_t + 2^5(a_g + a_{g-1}) + 2^3a_{g_3} - 3^2u_{\rho^2}),$$

$$y_{6A}^{(3)} := \frac{1}{252} (48(a_t + 2^2a_{g_2} + 2^2a_{g_2}) - 8(a_{g_3} + 2^2a_g + 2^2a_{g_2}) + 144a_{\rho^3} - 45u_{\rho^2}),$$

$$y_{6A}^{(4)} := \frac{1}{216} (2^4[\alpha(a_t + a_g) + \rho(a_{g-2} + a_{g_3}) - (a_{g-1} + a_{g_2})] + 36a_{\rho^3} + 45u_{\rho^2}),$$

$$y_{6A}^{(5)} := \frac{1}{1080} (80[\sigma(a_t + a_g) + \alpha(a_{g-2} + a_{g_3})] + 784(a_{g-1} + a_{g_2}) - 36a_{\rho^3} + 225u_{\rho^2}),$$

$$y_{6A}^{(6)} := \frac{1}{66} (2^4[\beta(a_t + a_g) + \beta(a_{g-2} + a_{g_3})] - 2^3(a_{g-1} + a_{g_2}) + 6a_{\rho^3} + 45u_{\rho^2}),$$

where $\alpha = 5 + 2\sqrt{3}$, $\beta = 1 + \sqrt{3}$, and $\sigma, \rho$ are their conjugates in $\mathbb{Q}(\sqrt{3})$.

Moreover, it was shown that there is an idempotent

$$y_{6A}^{(7)} := \gamma_1a_t + \gamma_2a_g + \gamma_3a_{g-1} + \gamma_4a_{g_2} + \gamma_5a_{g_3} + \gamma_6a_{g_2} + \gamma_7a_{\rho^3} + \gamma_8a_{\rho^2},$$

with

$\gamma_1 \in B_r (0.118600343195)$, $\gamma_2 \in B_r (0.116899056660)$,

$\gamma_3 \in B_r (0.672945208716)$, $\gamma_4 \in B_r (0.89196849266)$,

$\gamma_5 \in B_r (0.034809133018)$, $\gamma_6 \in B_r (0.960846592395)$,

$\gamma_7 \in B_r (-0.258738375363)$, $\gamma_8 \in B_r (-0.226937866453)$,
for \( r = 10^{-10} \), and an idempotent

\[
y_{6A}^{(8)} := \delta_1 a_t + \delta_2 a_g + \delta_3 a_{g-1} + \delta_4 a_{g2} + \delta_5 a_{g-2} + \delta_6 a_{g3} + \delta_7 a_{g3} + \delta_8 a_{g2},
\]

with

\[
\begin{align*}
\delta_1 & \in B_r (0.753376146443), & \delta_2 & \in B_r (-0.031896831434), \\
\delta_3 & \in B_r (-0.153112021089), & \delta_4 & \in B_r (0.729547069626), \\
\delta_5 & \in B_r (0.11069245253), & \delta_6 & \in B_r (0.844782757936), \\
\delta_7 & \in B_r (0.620071135272), & \delta_8 & \in B_r (-0.121749860276).
\end{align*}
\]

| Orbit | Size | Spectrum |
|-------|------|----------|
| \([a_t]\) | 6 | \{0, 0, 0, 1, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\} |
| \([a_{g3}]\) | 1 | \{0, 0, 0, 0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\} |
| \([u_{g3}]\) | 1 | \{0, 0, 0, 1, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\} |
| \([a_t + u_{g3}]\) | 1 | \{0, 1, 1, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\} |
| \([id_{2A}]\) | 3 | \{0, 1, 1, 1, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}\} |
| \([id_{2A} - a_t]\) | 6 | \{0, 0, 1, \frac{2}{3}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\} |
| \([id_{2A} - a_{g3}]\) | 3 | \{0, 0, 0, 1, \frac{2}{3}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\} |
| \([id_{3A}]\) | 2 | \{0, 1, 1, 1, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}\} |
| \([id_{3A} - a_t]\) | 6 | \{0, 0, 1, \frac{2}{3}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\} |
| \([id_{3A} - u_{g3}]\) | 2 | \{0, 0, 1, \frac{2}{3}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\} |
| \([y_{3A}]\) | 6 | \{0, 0, 0, 1, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\} |
| \([id_{3A} - y_{3A}]\) | 6 | \{0, 0, 1, \frac{2}{3}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\} |
| \([y_{6A}^{(1)}]\) | 3 | \{0, 1, 1, 1, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}\} |
| \([y_{6A}^{(1)} - a_{g3}]\) | 3 | \{0, 0, 1, \frac{2}{3}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\} |
| \([y_{6A}^{(2)}]\) | 6 | \{0, 1, 1, \frac{2}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\} |
| \([y_{6A}^{(3)}]\) | 6 | \{0, 0, 1, \frac{2}{3}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\} |
| \([y_{6A}^{(4)}]\) | 6 | \{0, 0, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\} |
| \([y_{6A}^{(5)}]\) | 6 | \{0, 1, 1, \frac{2}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\} |
| \([y_{6A}^{(6)}]\) | 6 | \{0, 1, \frac{2}{3}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\} |
| \([y_{6A}^{(7)}]\) | 12 | \{0, 1, \lambda_i : 1 \leq i \leq 6\} |
| \([y_{6A}^{(8)}]\) | 12 | \{0, 1, \mu_i : 1 \leq i \leq 6\} |

Table 6: Spectra of the idempotents of \( V_{6A} \).
Lemma 3.13. Every associative subalgebra of $V_{6A}$ is at most 3-dimensional.

Proof. The result follows by a similar argument as the one used in Lemma 3.8 since $\frac{7}{5}$ is the smallest length greater that 1 of an idempotent in $V_{6A}$ and $l(id_{6A}) = \frac{51}{10}$.

The spectra of half of the non-trivial non-zero idempotents of $V_{6A}$ is given in Table 6. The missing idempotents are $id_{6A} - x$, where of $x$ is an idempotent of the table. In this case, $id_{2A}$ is the identity of the subalgebra of type $2A$ with basis $\{a_t, a_{g_2}, a_{\rho^3}\}$ while $id_{3A}$ is the identity of the subalgebra of type $3A$ with basis $\{a_t, a_{g_2}, a_{g_{-2}}, u_{\rho^2}\}$. Using the approximation of the coordinates of $y_{6A}^{(7)}$ and $y_{6A}^{(7)}$, it may be shown that $0 \preceq \lambda_i, \mu_i \preceq 1$ for every $i$.

Lemma 3.14. The subalgebras of $V_{6A}$ given in Table 7 are maximal associative.

Proof. This follows by Table 6, Proposition 2.5 and Theorem 2.7.

| Associative subalgebra | Orbit size |
|------------------------|------------|
| $\langle (a_{\rho^3}, u_{\rho^2}, id_{6A} - a_{\rho^3} - u_{\rho^2}) \rangle$ | 1 |
| $\langle (a_{\rho^3}, id_{6A} - id_{2A}, id_{2A} - a_{\rho^3}) \rangle$ | 3 |
| $\langle (a_{\rho^3}, y_{6A}^{(1)} - a_{\rho^3}, id_{6A} - y_{6A}^{(1)}) \rangle$ | 3 |
| $\langle (u_{\rho^2}, id_{3A} - u_{\rho^2}, id_{6A} - id_{3A}) \rangle$ | 2 |
| $\langle (a_t, id_{2A} - a_t, id_{6A} - id_{2A}) \rangle$ | 6 |
| $\langle (a_t, id_{3A} - a_t, id_{6A} - id_{3A}) \rangle$ | 6 |
| $\langle (a_t, id_{6A} - y_{6A}^{(2)}), (y_{3A})^{(6A)} \rangle$ | 6 |
| $\langle (y_{3A}, id_{3A} - y_{3A}, id_{6A} - id_{3A}) \rangle$ | 6 |
| $\langle (y_{6A}^{(3)}, (y_{3A})^{(6A)}, id_{6A} - y_{6A}^{(1)}) \rangle$ | 6 |
| $\langle (y_{6A}^{(4)}, id_{6A} - y_{6A}^{(5)}, id_{2A}^{(2)} - a_{\rho^3}) \rangle$ | 6 |

Table 7: Non-trivial maximal associative subalgebras of $V_{6A}$.

Lemma 3.15. The Norton-Sakuma algebra of type $6A$ has exactly 75 maximal associative subalgebras; 30 of these algebras are trivial while 45 are three-dimensional.

Proof. By Table 6 and Corollary 2.8, the trivial maximal associative subalgebras of $V_{6A}$ are contained in the orbits $[V_{6A}^{(i)}]$ for $i = 6, 7, 8$, of sizes 6, 12 and 12, respectively. Using the action of Aut$(6A)$, Table 6 defines 45 non-trivial maximal associative subalgebras of $V_{6A}$. By Lemma 3.13, there are no four-dimensional associative subalgebras of $V_{6A}$. Now we show that there are no more three-dimensional associative subalgebras in $V_{6A}$. For each idempotent $x \in V_{6A}$, let $N_x$ be the number of three-dimensional associative subalgebras
Idempotent $x \quad d(x) \quad N_x \quad Idempotent x \quad d(x) \quad N_x$

| $a_t$     | 3    | 3   | $id_{3A} - u_{p^2}$ | 2    | 1   |
| $a_{p^3}$ | 4    | 7   | $y_{3A}$             | 3    | 3   |
| $u_{p^2}$ | 3    | 3   | $id_{3A} - y_{3A}$   | 2    | 1   |
| $id_{6A} - a_{p^3} - u_{p^2}$ | 2    | 1   | $id_{6A} - y_{6A}^{(1)}$ | 3    | 3   |
| $id_{6A} - id_{2A}$ | 3    | 3   | $y_{6A}^{(1)} - a_{p^2}$ | 2    | 1   |
| $id_{2A} - a_t$ | 2    | 1   | $id_{6A} - y_{6A}^{(2)}$ | 2    | 1   |
| $id_{6A} - id_{2A} + a_{p^3}$ | 3    | 3   | $y_{6A}^{(3)}$        | 2    | 1   |
| $id_{6A} - id_{3A}$ | 4    | 7   | $y_{6A}^{(4)}$        | 2    | 1   |
| $id_{3A} - a_t$ | 2    | 1   | $id_{(6A)} - y_{6A}^{(5)}$ | 2    | 1   |

Table 8: Values of $d(x)$ and $N_x$ for idempotents $x \in V_{6A}$.

4 Conclusions

The following theorem summarises the main results of this paper:

**Theorem 4.1.** Consider the Norton-Sakuma algebras $V_{NX}$. The following statements hold:

(i) Every idempotent of every Norton-Sakuma algebra is semisimple.

(ii) There are no four-dimensional associative subalgebras in any of the Norton-Sakuma algebras.

(iii) There are no non-trivial associative subalgebras of $V_{2A}$, $V_{3A}$ and $V_{3C}$.

(iv) There are exactly 2 non-trivial maximal associative subalgebras of $V_{4A}$.

(v) There are exactly 5 non-trivial maximal associative subalgebras of $V_{4B}$.

(vi) There are exactly 5 non-trivial maximal associative subalgebras of $V_{5A}$.

(vii) There are exactly 45 non-trivial maximal associative subalgebras of $V_{6A}$.
References

[CR13] A. Castillo-Ramirez, ‘Idempotents of the Norton-Sakuma algebras’, J. Group Theory, 16.3 (2013) 419-444.

[C84] J.H. Conway, ‘A simple construction for the Fischer-Griess Monster group’, Invent. Math., 79, (1984) 513–540.

[FLM] I. Frenkel, J. Lepowsky and A. Meurman, ‘Vertex operator algebras and the Monster’, Academic Press, 134, Pure Appl. Math., Boston (1988).

[G82] R.L Griess, ‘The Friendly Gigant’, Invent. Math., 69 (1982) 1–102.

[Iv09] A. A. Ivanov, The Monster Group and Majorana Involutions, Cambridge Univ. Press, Cambridge, Cambridge Tracts in Mathematics 176 (2009).

[IPSS10] A. A. Ivanov, D. V. Pasechnik, Á. Seress and S. Shpectorov, ‘Majorana representations of the symmetric group of degree 4’, J. Algebra 324 (2010) 2432-2463.

[Iv11a] A. A. Ivanov, ‘On Majorana Representations of $A_6$ and $A_7$’, Comm. Math. Phys. 307 (2011) 1-16.

[Iv11b] A. A. Ivanov, ‘Majorana Representations of $A_6$ involving 3C-algebras’, Bull. Math. Sci. 1 (2011) 356-378.

[IS12] A. A. Ivanov and Á. Seress, ‘Majorana Representations of $A_5$’, Math. Z. 272 (2012) 269-295.

[IS] A. A. Ivanov and S. Shpectorov, ‘Majorana Representations of $L_3(2)$’, Adv. Geom., 14 (2012) 717-738.

[MAP] Maplesotf, Maple 16.00 - The Essential Tool for Mathematics and Modelling, Licensed to: Imperial College Centre for Computing Services (2012).

[MN93] W. Meyer and W. Neutsch, ‘Associative subalgebras of the Griess algebra’, J. Algebra, 158 (1993) 1-17.

[Mi96] M. Miyamoto, ‘Griess algebras and conformal vectors in vertex operator algebras’, J. Algebra, 179 (1996) 523-548.

[N96] S. P. Norton, ‘The Monster algebra: some new formulae’, in Moonshine, the Monster and Related Topics, Contemp. Math. 193, AMS, Providence, RI, (1996) 297-306.

[Sa07] S. Sakuma, ‘6-Transposition property of $\tau$-involutions of Vertex Operator Algebras’, Int. Math. Res. Not. rnm030 (2007) 19.