Extensible Black Hole Embeddings
for Apparently Forbidden Periodicities

Aharon Davidson* and Uzi Paz**

Physics Department, Ben-Gurion University of the Negev, Beer-Sheva 84105, Israel

Abstract

Imposing extensibility on Kasner-Fronsdal black hole local isometric embedding is equivalent to removing conic singularities in Kruskal representation. Allowing for globally non-trivial (living in $M_5 \times S^1$) embeddings, parameterized by $k$, extensibility can be achieved for apparently forbidden frequencies $\omega_1(k) \leq \omega(k) \leq \omega_2(k)$. The Hawking-Gibbons limit, $\omega_{1,2}(0) = \frac{1}{4M}$ for Schwarzschild geometry, is respected. The corresponding Kruskal sheets are viewed as slices in some Kaluza-Klein background. Euclidean $k$ discreteness, dictated by imaginary time periodicity, is correlated with twistor flux quantization.

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Local isometric embedding \([1]\) of a curved \(d\)-dimensional manifold, within some parent \(D\)-dimensional flat space-time, has been traditionally invoked to classify \([2]\) the variety of general relativity solutions. The embedding is fully characterized by its induced metric, twistor Yang-Mills gauge field and the extrinsic curvature which, on consistency grounds, are subject to Gauss, Codazzi, and Ricci equations. Interesting attempts \((i)\) to interpret the embedding functions as alternative canonical variables for gravity \([3]\), \((ii)\) to relate the normal space symmetries with the electro/nuclear interactions \([4]\), and \((iii)\) to view our world as confined to a \((3+1)\)-dimensional membrane by some potential well \([5]\), are worth recalling. Depending on the differentiable nature of the embedding functions, \(D \leq \frac{1}{2}d(d+1)\) for analytic embeddings \([6]\), whereas \(D \leq \frac{1}{2}d(d + 3)\) if only integrability is required \([7]\). Of particular interest for us, however, are the \((3 + 1)\)-dimensional radially symmetric solutions; they fall categorically into the \(D = 6\) embedding class. The latter fact was already known to Kasner \([8]\) who presented a \((4 + 2)\)-dimensional embedding, non-causal and non-extendible though, of the exterior Schwarzschild geometry. However, only the Fronsdal \((5 + 1)\)-dimensional embedding \([9]\) has the advantage of being one-to-one correlated with the Kruskal-Szekeres analysis \([10]\). As far as black holes geometries are concerned, the removal of apparent horizon singularities in the Lorentzian Kruskal representation, which is fully equivalent to discharging conic singularities in the Euclidean world by means of Hawking-Gibbons \([11]\) periodicity, can be translated into imposing extendibility in the Kasner-Fronsdal approach. In this paper, following a pedagogical introduction, we present extensible Schwarzschild embeddings for apparently forbidden periodicities.

To be more specific, but keeping a certain amount of generality, consider the radially symmetric 4-metric

\[
ds^2 = \frac{1}{B(r)} \left( -A(r) dt^2 + dr^2 + r^2 d\Omega \right),
\]

(1)

where \(d\Omega \equiv d\vartheta^2 + \sin^2 \vartheta d\varphi^2\). For this metric to describe a black hole, \(\sqrt{AB}\) must well behave near the critical radius where \(A(r_h) = 0\). Indeed, invoking a set of Kruskal coordinates, namely \(v = C(r) \sinh \omega t\) and \(u = C(r) \cosh \omega t\), one is led to the compelling requirement
that $A \exp(-2\omega \int \frac{dr}{\sqrt{AB}})$ must reach a non-zero finite value at the horizon. This is to assure a singularity-free scale for the light-cone combination ($-dv^2 + du^2$). In turn, the parameter $\omega$ gets fixed

$$\omega = \frac{1}{2} \sqrt{\frac{B}{A}} \frac{dA}{dr} \bigg|_{r=r_h}. \tag{2}$$

The would be imaginary time periodicity $\frac{2\pi}{\omega}$ is the Hawking-Gibbons [11] key to Bekenstein-Hawking [12] black hole thermodynamics.

Alternatively, one may consider the embedding of the above 4–metric in $M_6$ with flat metric

$$ds^2 = -dy_0^2 + \sum_{n=1}^{5} dy_n^2. \tag{3}$$

Apart from the usual assignments $y_1 = r \cos \vartheta$, $y_2 = r \sin \vartheta \cos \varphi$, and $y_3 = r \sin \vartheta \sin \varphi$, which define the radial marker, one further introduces

$$y_0 = f(r) \sinh \omega t, \quad y_4 = f(r) \cosh \omega t, \quad y_5 = g(r). \tag{4}$$

These are supposed to cover the (say) $\frac{y_0}{y_4} \leq 1$ section of the 4-manifold characterized by $y_4^2 - y_0^2 = \frac{A}{\omega^2}$. After some algebra we arrive at

$$\left( \frac{dg}{dr} \right)^2 = -1 + \frac{1}{B} - \frac{1}{4\omega^2 A} \left( \frac{dA}{dr} \right)^2, \tag{5}$$

noticing that it is precisely that $\omega$ given by eq.(2) for which $\frac{dg}{dr}$ remains finite at the horizon. This establishes the correspondence between the Kruskal and the Fronsdal schemes.

For pedagogical reasons, let us focus attention on the Schwarzschild geometry, specified by $A(r) = B(r) = 1 - \frac{2m}{r}$. The crucial point then is the interplay of the two zeroes of $\sqrt{AB} \frac{dg}{dr}$. One infers that,
(i) For $\omega \geq \frac{1}{4m}$, the embedding does not cover the interior strip $\left( \frac{m}{2\omega^2} \right)^{\frac{1}{3}} < r < 2m$, whereas
(ii) For $\omega \leq \frac{1}{4m}$, the embedding does not cover the exterior strip $2m < r < \left( \frac{m}{2\omega^2} \right)^{\frac{1}{3}}$.

The conclusion being that Fronsdal extendibility requires

\[ \left( 1 - \frac{2m}{r} \right) \left( 1 - \frac{m}{2\omega^2 r^3} \right) \geq 0 \]  

(6)

for all $0 < r < \infty$, and thus can be achieved only provided $\omega = \frac{1}{4m}$, recognized as the Kruskal value. Appreciating the role played by inequality (6), we note that its direction would have been harmfully reversed had we considered a time-like $y_5$.

We now claim that eq.(5) is in fact not the most general ansatz capable of giving birth to the radially symmetric (and hence time independent) metric (1). Introducing a new parameter $k$, one can still have

\[ y_0 = f(r) \sinh[\omega t + k\psi(r)], \]
\[ y_1 = f(r) \cosh[\omega t + k\psi(r)], \]
\[ y_5 = kt + g(r), \]  

(7)

with $k = 0$ serving as the Fronsdal limit. We now state, skipping the proof due to length limitations (to be published elsewhere), that the embedding in hand is accompanied by the $SO(2)$ twistor vector field of the two normal directions

\[ A_t(r) = -k\omega \sqrt{\frac{B}{A}}. \]  

(8)

The latter vanishes at the Fronsdal limit, and appears as a harmless pure gauge configuration for a long list of general relativity solutions (including Schwarzschild and Reissner-Nordstrom). Nonetheless, for $k \neq 0$, it may leave non-trivial global fingerprints upon Euclidization, once imaginary time is identified with a certain period.

The functions involved in the embedding procedure are:
\[ f^2 = \frac{1}{\omega^2} \left( 1 + k^2 - \frac{2m}{r} \right), \]
\[ \frac{dg}{dr} = \frac{1}{\left( 1 - \frac{2m}{r} \right)^{\sqrt{k^2 + (1 - \frac{2m}{r})(1 - \frac{m}{2\omega^2 r^3})}}} \]
\[ \frac{d\psi}{dr} = \frac{1}{\omega f^2} \frac{dg}{dr}, \]

(9)

where \( \sqrt{k^2 + (1 - \frac{2m}{r})(1 - \frac{m}{2\omega^2 r^3})} \). The first thing to notice is that, unlike in the Fronsdal limit, both \( g(r) \) and \( \psi(r) \) are now singular (logarithmic singularity) at \( r = 2m \). This is not necessarily a problem, however, as one has still the option of performing a gauge transformation (that is, a general coordinate transformation), and blame it for the singularity induced. Radial symmetry (and hence time independence by virtue of Birkhoff theorem) still allows for \( t \to t + \Lambda(r) \). Such a shift in \( t \) is equivalent to a redefinition of \( g(r) \) and \( \psi(r) \), namely \( g \to g + k\Lambda \) and \( \psi \to \psi + \frac{\omega}{k} \Lambda \), under which only the combination \( g - \frac{k^2}{\omega} \psi \) would not change. The fact that such a ‘gauge-invariant’ combination exhibits no singularity at \( r = 2m \), as can be seen from \( \frac{d}{dr}(\psi - \frac{\omega}{k^2}g) = -\frac{\sqrt{\omega k^2 f^2}}{\omega k^2 f^2} \), comes with no surprise.

Two tenable gauges offer their services:

(i) Using \( \frac{d\Lambda_1}{dr} = -\frac{\sqrt{k}}{k \left( 1 - \frac{2m}{r} \right)} \), one is led to the convenient choice \( g_1 = 0 \) accompanied by \( \psi_1 = \psi - \frac{\omega}{k^2} g \), and

(ii) Using \( \frac{d\Lambda_2}{dr} = -\frac{k}{\omega^2 f^2 \left( 1 - \frac{2m}{r} \right)} \), on the other hand, paying attention to the extra \( f^2 \) in the denominator, one obtains \( g_2 = g - \frac{k^2}{\omega} \psi \) on the expense of \( \psi_2 = 0 \).

Both gauges give rise to non-diagonal Schwarzschild variants of the Eddington-Finkelstein type.

Witnessing the smooth behavior of the embedding functions near the horizon \( r_h = 2m \), we turn attention now to the apparently critical radius \( r_c = \frac{2m}{1 + k^2} \). This is where the matching of the two sections \( \left| \frac{y_0}{y_1} \right| \leq 1 \) and \( \left| \frac{y_0}{y_4} \right| \geq 1 \) of the manifold characterized by \( y_4^2 - y_0^2 = \frac{1}{\omega^2} \left( 1 + k^2 - \frac{2m}{r} \right) \) is supposed to take place. A closer inspection of the two gauges a priori permissible is thus in order. The first gauge offers us the advantage that
\( f \cosh k\psi_1 \) and \( f \sinh k\psi_1 \), the amplitudes of \( \sinh \omega t \) and \( \cosh \omega t \), are perfectly regular at \( r = r_c \). To be more specific, \( f \sim \sqrt{\epsilon} \) whereas \( k\psi_1 \sim -\frac{1}{2} \ln \epsilon \) as \( \epsilon \equiv r - r_c \to 0 \). The second gauge offers us nothing but a major drawback: namely, \( g_2 \) gets (logarithmically) singular at \( r = r_c \). The gauge choice is thus clear.

For the embedding to cover any given region of the 4–dim manifold, the \( \sqrt{\epsilon} \) function involved must stay real in that region. Consequently, generalizing eq.\((\text{6})\) in a very simple manner, \( k \)-extendibility requires

\[
\left( 1 - \frac{2m}{r} \right) \left( 1 - \frac{m}{2\omega^2 r^3} \right) + k^2 \geq 0
\]

(10)

to hold for all \( 0 < r < \infty \). We now argue that if \( \omega \leq \omega_1(k) \), the embedding does not cover some (exterior) strip, whereas \( \omega \geq \omega_2(k) \) leaves another (interior) strip without coverage. While fully respecting the \( k = 0 \) Hawking-Gibbons limit, the door is widely open now for apparently forbidden black hole frequencies in the range \( \omega_1(k) \leq \omega \leq \omega_2(k) \). The allowed region in the \((\omega,k)\)-plane is depicted in the enclosed Figure. \( \omega_{1,2}(k) \) are the two roots of

\[
\left( 1 - \frac{2m}{a} \right) \left( 1 - \frac{m}{2\omega^2 a^3} \right) + k^2 = 0,
\]

(11)

where the radius \( a(m,\omega) \), for which \( \left( 1 - \frac{2m}{r} \right) \left( 1 - \frac{m}{2\omega^2 r^3} \right) \) is minimal, is given by

\[
a(m,\omega) = \frac{m}{\omega^\frac{2}{3}} \left[ \left( \sqrt{1 + \frac{1}{64m^2 \omega^2}} + 1 \right)^{\frac{1}{3}} \right] - \left( \sqrt{1 + \frac{1}{64m^2 \omega^2}} - 1 \right)^{\frac{1}{3}}.
\]

(12)

In particular, for large \( k \), the extremal \( \omega_{1,2}(k) \) behave like

\[
\omega_1(k) \approx \frac{3\sqrt{3}}{64mk}.
\]

(13)

\[
\omega_2(k) \approx \frac{4k^3}{3\sqrt{3}m}.
\]

(14)

In search of proper ‘dispersion relations’, the naive candidate family is of course \( \omega(\xi k) \), conveniently parameterized by the continuous parameter \(-1 \leq \xi \leq 1\), and given as the solution of
\[
\left(1 - \frac{2m}{a}\right) \left(1 - \frac{m}{2\omega^2 a^3}\right) + \xi^2 k^2 = 0. \tag{15}
\]

Near the Hawking-Gibbons limit, that is for small enough \(k\), one derives
\[
\omega(\xi k) \simeq \frac{1}{4m} \left(1 + \sqrt{3}\xi k\right). \tag{16}
\]

The global aspects of the \(k\)-embedding are next. The fundamental role played by the hyperbolic functions of \(\omega t\) in the Kruskal and the Fronsdal schemes is very much established by now. But here, the situation appears to be a bit more complicated, due to the fact that a linear function of \(t\), namely \(y_5 = kt\) (using the preferred \(\Lambda_1\)-gauge), is floating around as well. We first infer, recalling that the argument of the hyperbolic functions is \(\omega t + k\psi_1(r)\), that when going Euclidean, \(t \to i\tau\) must be accompanied by \(k \to -i\kappa\) (and also by \(\xi \to i\zeta\)). In turn, and this is a central point, \(y_5 \to y_5\) without a change in signature. A potential problem then arises: Imaginary time periodicity is violated in principle, unless of course the fifth dimension acts cooperatively. In other words, if \(\tau\)-periodicity is important to us (and we believe quantum mechanics is rather important), \(y_5\) better be a closed coordinate, and this must be the case at the Lorentzian level as well. This is why the embedding space-time must have the topology of \(M_5 \times S_1\) (to be contrasted with Fronsdal’s \(M_6\)), thereby establishing the arena for the linear function of \(\tau\) to play its non-trivial global role. We remark in passing that the Schwarzschild metric is not a plane-wave metric, and hence its global \((1 + N)\)-embedding in not Penrose restricted [4].

Now, the \(\tau\)-periodicity of \(\frac{2\pi}{\omega}\) must be in accord with the underlying topology, but this can only be discretely satisfied, leading to
\[
\kappa_n = n\omega R, \tag{17}
\]
with \(R\) denoting the radius of the fifth dimension. The latter quantization condition has a rather interesting 6-dimensional interpretation. Recalling the attached twistor vector potential [8], and noticing that
\[
\int A_\mu dx^\mu \to \kappa \omega \int d\tau = 2\pi \kappa,
\]

one realizes that (17) is nothing but magnetic flux quantization in disguise (in Kaluza-Klein-like units, with \( \frac{1}{2\pi \omega R} \) serving as the twist electric charge).

To complete the correspondence between the Kruskal removal of conic singularities and the Fronsdal extendibility, we are after the so-called \( k \)–generalization of the original Kruskal scheme. Let our starting point be the 5-geometry

\[
ds_5^2 = dx_5^2 + ds_4^2,
\]

where \( x_5 \) is a compactified (a la Kaluza-Klein) fifth dimension, and the 4-metric takes the form

\[
ds_4^2 \equiv - \left( 1 + k^2 - \frac{2m}{r} \right) dt^2 + \left[ 1 + \frac{m^2}{(1 + k^2 - \frac{2m}{r}) \omega^2 r^4} \right] dr^2 + r^2 d\Omega.
\]

The above carefully designed 4–metric exhibits a major feature. Namely, as can be verified by means of eq.(2), this metric is Kruskalizable for any arbitrary prescribed \( \omega \). There is no mystery about this; \( ds_4^2 \) has the familiar Fronsdal embedding in \( M_5 \), that is \( ds_4^2 = -dy_0^2 + \sum_{n=1}^{4} dy_n^2 \) (with \( n = 5 \) notably excluded). Using momentarily Euclidean language, where \( t \to i\tau \) (and \( k \to -i\kappa \)), we deal with a torus specified by its periodicities \( \Delta x_5 = 2\pi R \) and \( \Delta \tau = \frac{2\pi}{\omega} \).

Consider now a class of 4–dimensional manifolds which reside within the given 5–dimensional space-time, and proceed in steps:

(i) By arbitrarily assigning \( x_5(x^\mu) \), one induces the 4–metric \( ds_4^2 + \left( \frac{\partial x_5}{\partial x^\mu} dx_\mu \right)^2 \). This will generically kill all parent periodicities.

(ii) Symmetry-wise, one can do better by choosing \( x_5(t,r) = at + b(r) \). This way, and here we switch again to the Euclidean framework (with \( a \to -i\alpha \)), one may still recover closed lines on the torus. In particular, if \( \alpha_n = n\omega R \), the \( \Delta \tau = \frac{2\pi}{\omega} \) periodicity stays alive.

(iii) If it so happens that the induced metric is locally Schwarzschild, we are done.
The prescription is then clear. Given the seed metric (20), first shift
\[ t \rightarrow \tilde{t} + \frac{k}{\omega} \psi(r), \]                                               (21)
and then cut out the simple Kaluza-Klein slices
\[ x_5(\tilde{t}, r) = k\tilde{t} + g(r). \]                                      (22)

To import no apparent singularities this way, however, it is advisable to make contact with our preferred $\Lambda_1$-gauge. The restrictions imposed on $\omega$ originate then from the $\sqrt{\cdot}$ function which enters $\psi_1(r)$. This completes the presentation of the generalized Kruskal scheme.

In this paper, although restraining ourselves to solely deal with the geometrical and the topological aspects, we have serendipitously challenged the fundamental formula which governs the quantum theory of black holes. The Hawking-Gibbons formula, say $\omega(0) = \frac{1}{4m}$ for the prototype Schwarzschild black hole, is viewed as the tip of an iceberg, the $k \rightarrow 0$ limit of a full class of dispersion relations $\omega(\xi k)$. On the technical side we have first shown that, for globally non-trivial (living in $M_5 \times S^1$) $k$-embeddings, Kasner-Fronsdal extendibility can be achieved for apparently forbidden frequencies. Then, motivated by the fact that extendibility actually means the removal of conic singularities in the (Euclidean) Kruskal language, we have derived the corresponding Kruskal-like sheets as slices in some Kaluza-Klein background. We are partially aware of the potential impact the present work may have on black hole physics. In fact, preliminary results already suggest a discrete (reflecting the twistor flux quantization) quantum family of classically degenerate black holes.

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FIGURES

FIG. 1. The extendibility allowed region $\omega_1(k) \leq \omega \leq \omega_2(k)$. Notice the Hawking-Gibbons limit $\omega_{1,2}(0) = \frac{1}{4M}$. 
\[ \omega_1(k) \approx \frac{3\sqrt{3}}{64mk} \]

\[ \omega_2(k) \approx \frac{4k^3}{3\sqrt{3}m} \]

\[ \omega_{1,2}(k) \approx \frac{1}{4m}(1 \pm \sqrt{3}k) \]