Internal transport barriers in some Hamiltonian systems modeling the magnetic lines dynamics in tokamak

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Abstract: The formation of internal transport barriers (ITB) in tokamaks has been experimentally associated with the region of low magnetic shear and the presence of main rational magnetic surfaces. We study the ITB from a mathematical point of view in a Hamiltonian model (the rev-tokamap model) that describes the magnetic field lines dynamics in reversed shear tokamaks. The non-twist properties of the map that generates the system are exploited in order to explain analytically the existence of ITB surrounding the shearless curve (in the low magnetic shear zone) and the robustness of invariant circles (corresponding to closed magnetic surfaces) situated in its proximity. We describe the location of the ITB for various q-profiles and perturbations of the (ideal) integrable system. We use an analytical method to estimate the maximum radius of magnetic field lines confinement, hence the position of the barrier, and we explain the destruction of invariant circles when the amplitude of the perturbation increases. The reconnection phenomena are also related with ITB and an explanation for the (experimentally observed) enlargement of ITB when the minimal value of the q-profile becomes close but less than a main rational is obtained.

1 Introduction
The tokamaks are toroidal chambers in which charged particles are confined by a magnetic field obtained by the superposition of two basic components acting in the direction of the major, respectively minor curvatures of the torus.
We are particularly interested in the study of the magnetic internal transport barriers (ITB), i.e. regions which separate two stochastic zones (one situated near the magnetic axis of the tokamak and the other situated near the tokamak’s wall). The ITB cannot be crossed by the magnetic field lines, i.e. the magnetic transport through the ITB is suppressed and the magnetic field lines passing through the central stochastic zone are confined. The existence of a magnetic transport barrier is crucial for the plasma confinement because it prevents the outside radial motion of charged particles (charged...
particles guiding centers follow magnetic field lines). A magnetic transport barrier can be directly observed in experiments, since the consequences of its existence for particles’ transport can be seen from the shape of the density or temperature radial profile on which a locally strong decrease indicates the position of an ITB. On the other hand, the fact that transport barriers for particles are determined by the magnetic field configuration (through its q-profile) has been reported from experiments in Tore Supra, JT-60U, JET, TFTR and other tokamaks [1], [2]. We focus on two important features:
- transport barriers are generally obtained in presence of a reversed magnetic shear, i.e. when the q-profile has a local maximum near the magnetic axis and a local minimum at a normalized radius between 0.3 and 0.4. The transport barrier appears in the negative shear region or in the low shear region (near the points where q has the minimum value). In this zone a reduced heat diffusivity is observed [1]; in [3] two spatially separated transport barriers are presented, located in both the positive and negative shear regions.
- zones with reduced transport can be noticed near the points where q has low order rational values (q=1, q=3/2, q=2, q=2.5, q=3) [1], [4].

In order to study the magnetic transport barriers from a mathematical point of view, the theory of dynamical systems can be used. It is well known that the magnetic field lines equations define a Hamiltonian system [5], [6] which is generically non-integrable. It can be studied using the Poincaré map or other maps associated with a poloidal section [7], [8], [9], [10].

We will analyze the magnetic configuration in tokamaks with reversed magnetic shear using the rev-tokamap model [8]. The non-twist properties of the systems are exploited in order to give an analytical explanation for the previous experimental observations.

The paper is organized as follows: Section 2 is dedicated to the presentation of the rev-tokamap model. The study of the magnetic transport barrier using the non-twist property is presented in Section 3. In Section 4 the transport barriers are described and located for various q-profiles and perturbations and a scenario for the destruction of invariant circles is proposed. Conclusions are drawn in Section 5.

2. The Rev-Tokamap model
The toroidal coordinates \((r, \theta, \zeta)\) are natural in order to describe the configuration of the magnetic field lines because the tokamaks are toroidal devices. \(\zeta\) is the toroidal angle and \((r, \theta)\) are the polar coordinates in a (circular) poloidal cross-section having the radius “a”. However, because \(\psi\) and \(\theta\) are the canonically conjugated variables in the Hamiltonian description of the magnetic field lines, the toroidal flux \(\psi = r^2 / 2\) is used instead of \(r\).

The intersection of the magnetic field line which starts from \((\theta, \psi)\) with a poloidal cross-section \(S : \zeta = cst\) after one toroidal turn will be denoted by \((\bar{\theta}, \bar{\psi})\).

The rev-tokamap [8] is \(T_k : (0.1) \times R_+ \rightarrow (0.1) \times R_+ , T_k (\theta, \psi) = (\bar{\theta}, \bar{\psi})\) defined by

\[
\bar{\theta} = \left( \theta + W(\bar{\psi}) - \frac{K}{4\pi^2} \cdot \cos(2\pi\theta) \cdot \frac{1}{1+\bar{\psi}^2} \right) \quad \text{(mod1)}, \quad \bar{\psi} = \psi - \frac{K}{2\pi} \cdot \sin(2\pi\theta) \cdot \frac{\bar{\psi}}{1+\bar{\psi}}. \tag{1}
\]

The winding function \(W(\psi) = w\left(1 - A(C\psi - 1)^2\right)\) is a quadratic function having a maximum value \(w\) and the limit values \(w_0 = W(0)\) and \(w_1 = W(1)\). The coefficients \(A, C\) are related with \(w, w_0, w_1\) by the formulas \(A = \frac{w-w_0}{w}, \quad C = 1 + \sqrt{\frac{w-w_0}{w-w_1}}\).

It is an area-preserving map compatible with the toroidal geometry (if \(\psi = 0\) then \(\bar{\psi} = 0\) and if \(\bar{\psi} > 0\) then \(\bar{\psi} > 0\)). \(W(\psi)\) is called the winding function, \(K\) is the stochasticity parameter describing
the strength of the magnetic perturbation. The function \( q = 1/W \) is called the safety factor or simply the q-profile and \( s = d \ln q / d \ln \psi \) is the shear profile. The shear is positive when \( q \) is an increasing function and it is negative if \( q \) is a decreasing function of \( \psi \).

The rev-tokamap is a non-twist map because the equation \( \frac{\partial \hat{\theta}}{\partial \psi} = 0 \) has solutions in \( S^1 \times R_+ \).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{phase_portraits.png}
\caption{The phase portrait of the rev-tokamap corresponding to \( w = 0.67,\ w_0 = 0.3333,\ w_1 = 0.1667 \) and a) \( K = 3.5 \), b) \( K = 4.5 \), c) \( K = 5 \).}
\end{figure}

Some phase portraits of the rev-tokamap corresponding to a fixed \( q \)-profile and various values of \( K \) are presented in figure 1. A regular zone that separates two chaotic regions can be noticed. It is the magnetic transport barrier, the object of our study.

In order to describe the properties of \( T_k \) the following fundamental curves are especially useful.

The non-twist curve of \( T_k \) is the set of points where the twist property is violated. Its equation is

\[ C_{nt} : \frac{\partial \hat{\theta}}{\partial \psi} (\theta, \psi) = 0. \]

The regular curve of \( T_k \) is \( C_{reg} : W^+(\psi) = 0 \). The folding curve \( C_f : \frac{\partial \hat{\theta}}{\partial \theta} (\theta, \psi) = 0 \).

The shearless curve, denoted by \( C_{sh} \), is the closure of the orbit having a rotation number which is a local extremum of all rotation numbers of orbits in the map.

In the unperturbed system the regular curve, the non-twist curve and the shearless curve coincide because \( \frac{\partial \hat{\theta}}{\partial \psi} (\theta, \psi) = W^+(\psi) = W'(\psi) \). They have the equation \( C_{nt} = C_{reg} = C_{sh} : \psi = 1/C \).

In the perturbed case \( (K \neq 0) \) these curves are different and a natural question rises on how to relate the dynamical properties of the system to each of them. These curves are presented in figure 2a for the rev-tokamap corresponding to \( K = 5,\ w = 0.67,\ w_0 = 0.3333 \) and \( w_1 = 0.1667 \).

The critical twist curve is a rotational circle that is not invariant under the action of \( T_k \). The closure of all orbits starting from \( C_{nt} \) is called the non-twist annulus \( (NTA) \).

The \( NTA \) is a \( T_k \)-invariant set that splits the phase space in two (invariant) regions on which the function \( T_k \) acts as a twist map with opposite twist properties. The map \( T_k \) is a negative twist map in the region bounded by the axis \( \psi = 0 \) and \( NTA \). In the unbounded region limited by \( NTA \) in the lower part, \( T_k \) has positive twist properties. \( NTA \) is the collection of all points that have non-twist dynamics, hence all non-twist phenomena (reconnection, existence of shearless curve) occur within it.

In figure 2b we present the phase portrait of the rev-tokamap corresponding to \( K = 4,\ w = 0.67,\ w_0 = 0.3333,\ w_1 = 0.1667 \). The non-twist annulus and the twist regions can also be remarked in this figure.
Figure 2 a) the fundamental curves b) the twist regions and the non-twist annulus.

The main cause of the destruction of invariant circles in non-twist maps is the fact that some circles \( \psi = \psi_0 = \text{cst} \) are folded by \( T_K \), i.e. their images through \( T_K \) are not graphs of some functions depending on the angle \( \theta \).

The image circles have indeed vertical tangents in the points of the folding curve \( C_f \). The folding curve does not intersect the critical twist circle \( C_{nt} \). It has an unbounded component, denoted by \( C_f^- \), in the region of negative twist. In this region it is the graph of a real function defined on the interval \((1/4,3/4)\) having the asymptotic lines \( \theta = 1/4 \) and \( \theta = 3/4 \). In the positive twist region the folding curve is empty for low perturbations and it is a closed curve for large perturbations. The bounded component of \( C_f \) will be denoted by \( C_f^+ \).

3 The study of the internal transport barrier using the non-twist properties of the system

The union of the invariant circles separating the negative twist chaotic zone (situated near the magnetic axis) from the positive twist chaotic zone (situated in the peripheral zone of the tokamak) is an internal transport barrier (\( ITB \)) because it cannot be crossed by the magnetic field lines.

The \( ITB \) can be numerically located by direct computation or using carefully sophisticated theories.

In order to study in a rigorous manner the \( ITB \), its relation with \( C_{reg} \) can be used.

An annulus on which the dynamics of \( T_K \) is regular appears in the region where \( T_K \) is almost integrable because it is close to an integrable map.

The region where \( T_K \) is closest (in \( C^0 \) topology) to a rigid rotation (which is an integrable map) is an annulus, called the regular annulus, which intersects the curve \( C_{reg} \).

Computer experiments point out that \( C_{nt} \) and \( C_{reg} \) are very close, even for large values of \( K \) (the case \( K = 5 \) can be visualized in figure 2a, the maximal values of \( K \) is \( 2\pi \) ) so that the regular annulus and the non-twist annulus almost coincide. This explains the robustness of the invariant circles in the non-twist annulus. It is obvious that the regular annulus is contained in \( ITB \).

The non-twist annulus contains the shearless, hence the shearless curve is also contained in \( ITB \) as long as it exists.

The \( ITB \) may also contain invariant circles situated in the positive or negative twist regions, so that the regular annulus is situated in the central part of \( ITB \) and its influence on the trajectories of charged particles is visible in experiments.
The curve $C_{reg}$ can be also used to determine the threshold for break-up of the last invariant circle: if there are not points of $C_{reg}$ with orbits densely filling a rotational circle, it results that the ITB is broken.

The destruction of invariant circles can be explained using the folding curve. Using the cone-crossing criterion [11], which essentially describes the twist dynamics, it can be proved that $C_{f^+}$ is not crossed by invariant circles. The same criterion is used to study the breaking-up of invariant circles in the negative twist zone. The intersection of $C_f^-$ with the line $\theta = 0.5$ is denoted by $(0.5, \psi^*)$.

As long as $C_f^-$ is contained in the negative region, no invariant circle passes through the points situated above $(0.5, \psi^*)$. In this case we obtain an analytical estimation of the radius of confinement: $r^* = \sqrt{2 \cdot \psi^*}$. This is the main analytical result of the present work.

4 ITB for various q-profiles and perturbations

The unperturbed system ($K = 0$ in (1)) is integrable. There are no chaotic zones and the ITB can be identified to the whole phase space, the regular and the non-twist annulus reduce to a circle and the shearless curve is the same circle. By increasing the stochasticity parameter $K$, chaotic regions are formed both in the positive twist and in the negative twist region, the ITB becomes smaller and smaller until it disappears. In the same time the non-twist annulus becomes larger.

When the stochasticity parameter is increased the components $C_{f^-}$ and $C_{f^+}$ of the folding curve approach the non-twist annulus from both sides. The (twist) invariant circles intersecting them are destroyed both in the positive and in the negative twist regions. The dynamics on NTA is close to integrable as long as the folding curve is far enough from it. When the folding curve enters NTA, the destruction of the invariant circles continues from both sides but it is no more related to the cone crossing criterion.

For a fixed winding function ($w = 0.67$, $w_0 = 0.3333$, $w_1 = 0.1667$) and various $K \in [0.5, 4.5]$ the positions of the upper boundary of ITB (numerically computed), of the boundaries of NTA (analytically determined) and of $\psi^*$ (analytically determined) are presented in figure 3a.

![Figure 3](image)

Figure 3 The position of the upper boundary of ITB, of the boundaries of NTA and of $\psi^*$

a) for the same winding function ($w = 0.67$, $w_0 = 0.3333$, $w_1 = 0.1667$) and various $K$, b) for $K = 2.5$ and various translated winding function.
For the winding function $W_0$ determined by $w = 0.67$, $w_0 = 0.3333$, $w_1 = 0.1667$ the component $C_f^+$ exists for $K > 4.137$. For $K^* = 3.923916$ the point $(0.5, \psi^*) = (0.5, 0.633728371)$ is on the common boundary of NTA and ITB. For $K < K^*$ the orbits of NTA are regular and the non-twist annulus is contained in the transport barrier, but for $K > K^*$ in the peripheral part of NTA some chaotic orbits are formed, the ITB enters NTA, becomes thinner and thinner and it is destroyed for $K_d = 6.2043$.

A similar analysis can be made for a fixed $K$ and various winding functions. We studied the case when the winding functions are translations of $W_0$ and $K = 2.5$. The position of $\psi^*$ and of the boundaries of NTA on the line $\theta = 0.5$ are presented in figure 3b. In this case the point $(0.5, \psi^*)$ is outside NTA in all the analyzed situations. A significant enlargement of NTA can be noticed when the maximal value of $W$, namely $w$, is close and larger than a main rational (in this situation the minimal value of $q$ is close and smaller than a main rational). It can be analytically explained using reconnection arguments (an extensive study of reconnectin in the rev-tokamap model was performed in [12]): before reconnection the twin island chains, previously located in the positive respectively negative twist zone, enter NTA because the reconnection is a specific non-twist phenomenon and the enlargement of NTA is produced.

This phenomenon induces an enlargement of ITB and generates a visible effect on the transport of charged particles. It can be related to the experimental observations.

5 Conclusions
Using the non-twist properties of the rev-tokamap system (which models the magnetic field lines configuration in a reversed shear tokamak) we obtained analytical explanation for some experimental observations concerning the transport barriers. The existence of the ITB surrounding the shearless curve was proved using the fact that the rev-tokamap is almost integrable inside the non-twist annulus and the enlargement of ITB was explained using reconnection arguments. An analytical explanation for the destruction of invariant circles was also proposed and we obtained an estimation for the confinement radius.

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