ERGODIC UNITARILY INVARIANT MEASURES ON THE SPACE OF INFINITE HERMITIAN MATRICES

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Abstract

Let $H$ be the space of all Hermitian matrices of infinite order and $U(\infty)$ be the inductive limit of the chain $U(1) \subset U(2) \subset \ldots$ of compact unitary groups. The group $U(\infty)$ operates on the space $H$ by conjugations, and our aim is to classify the ergodic $U(\infty)$-invariant probability measures on $H$ by making use of a general asymptotic approach proposed in Vershik’s note [V]. The problem is reduced to studying the limit behavior of orbital integrals of the form

$$\int_{B \in \Omega_n} e^{i \text{tr}(AB)} M_n(dB),$$

where $A$ is a fixed $\infty \times \infty$ Hermitian matrix with finitely many nonzero entries, $\Omega_n$ is a $U(n)$-orbit in the space of $n \times n$ Hermitian matrices, $M_n$ is the normalized $U(n)$-invariant measure on the orbit $\Omega_n$, and $n \to \infty$.

We also present a detailed proof of an ergodic theorem for inductive limits of compact groups that has been announced in [V].

There is a remarkable link between our subject and Schoenberg’s [S2] theory of totally positive functions, and our approach leads to a new proof of Schoenberg’s [S2] main theorem, originally proved by function-theoretic methods.

On the other hand, our results have a representation-theoretic interpretation, because the ergodic $U(\infty)$-invariant measures on $H$ determine irreducible unitary spherical representations of an infinite-dimensional Cartan motion group.

The present paper is closely connected with a series of articles by S. V. Kerov and the authors on the asymptotic representation theory of “big” groups, but it can be read independently.

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§0. Introduction

1. Ergodic measures. The description of ergodic invariant measures for group actions is a traditional problem of ergodic theory. It is well known that it is not always that this problem can be solved in a satisfactory way. It is a surprising fact that for certain infinite-dimensional (or “big”\(^1\)) groups, there exist nice actions whose ergodic measures can be completely described. The simplest examples are given by the classical theorems due to B. de Finetti and I. J. Schoenberg.

We recall that, by de Finetti’s theorem, the ergodic measures on a product space \(X^\infty\), invariant under the group of permutations of the copies of \(X\) (the infinite symmetric group), are exactly the product measures with identical factors. Schoenberg’s theorem \([S1]\) states that the ergodic \(O(\infty)\)-invariant measures on the space \(\mathbb{R}^\infty\) are exactly the Gaussian product measures (by \(O(\infty)\) we denote the inductive limit group \(\lim_{\rightarrow} O(n)\)).\(^2\)

These classical examples and a number of more complicated ones were discussed in the note \([V]\) by one of the authors. In that note, a general “ergodic method” for inductive limits of compact groups was proposed. This method was further developed in the papers \([VK1, VK2, KV]\) by Vershik and Kerov. The aim of the present paper is to give a detailed exposition of the method on a model example considered in \([V]\).

Let \(H(n)\) denote the space of \(n \times n\) complex Hermitian matrices, and let \(H = \lim H(n)\) be the space of all infinite Hermitian matrices. Let \(U(\infty) = \lim U(n)\) be the group of infinite unitary matrices \(u = [u_{ij}]\) such that \(u_{ij} = \delta_{ij}\) when \(i + j\) is large enough. The group \(U(\infty)\) operates on the space \(H\) by conjugations, and we are interested in the class \(\mathcal{M}\) of all ergodic \(U(\infty)\)-invariant Borel probability measures on \(H\).

Further, let \(H(\infty) = \lim H(n)\) be the space of \(\infty \times \infty\) Hermitian matrices with finitely many nonzero entries. The spaces \(H(\infty)\) and \(H\) are in a natural duality, and any measure on \(H\) is uniquely determined by its characteristic function (Fourier transform), which is a function on \(H(\infty)\).

Classification Theorem. The characteristic functions of the measures \(M \in \mathcal{M}\) are exactly those of the form

\[
f(A) = e^{i\gamma_1 \text{tr } A - \gamma_2 \frac{\text{tr}(A^2)}{2}} \det \left( \prod_{k=1}^{\infty} \frac{e^{-ix_k A}}{1 - i x_k A} \right), \quad A \in H(\infty),
\]

where \(\gamma_1, \gamma_2, x_1, x_2, \ldots\), the parameters of the measure \(M\), are real numbers such that \(\gamma_2 \geq 0\) and \(\sum x_k^2 < \infty\).

The Classification Theorem implies that any measure \(M \in \mathcal{M}\) can be written as the convolution product of a Gaussian measure and a (finite or countable) family of

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\(^1\)The term “big groups” was suggested by one of us in \([V]\). This term has no rigorous definition, it can mean “infinite-dimensional” or “out of the class of locally compact groups” or something like that. For instance, the group \(S(\infty) := \lim S(n)\), the inductive limit of finite symmetric groups, is a discrete, hence a locally compact group, but its properties are very similar to that of the infinite-dimensional group \(U(\infty) := \lim U(n)\), so we prefer to rank \(S(\infty)\) among the “big” groups.

\(^2\)About this theorem, see also Berg–Christensen–Ressel \([BCR]\).
non-Gaussian “elementary” ergodic measures. The latter are essentially supported by rank one matrices and are related to Wishart distributions, well-known in multivariate statistical analysis. Let us emphasize that in this theorem, the structure of the answer turns out to be much more complicated than in de Finetti’s and Schoenberg’s [S1] theorems: up to trivial exceptions, the ergodic measures $M \in \mathcal{M}$ are neither product nor Gaussian measures.

Note that formula (0.1) can be rewritten in the following form:

$$f(A) = \prod_{a \in \text{Spec}(A)} F(a), \quad A \in H(\infty),$$

(0.2)

where $\text{Spec}(A)$ is the collection of eigenvalues of $A$ (taken with their multiplicities) and

$$F(a) = e^{i\gamma_1 a - \gamma_2 a^2/2} \prod_{k=1}^{\infty} \frac{e^{-ix_k a}}{1 - ix_k a}, \quad a \in \mathbb{R}.$$  

(0.3)

The function $F(a)$ has a simple meaning. Let us regard the matrices $B \in H$ as random matrix variables defined on the probability space $(H, \mathcal{M})$. Then $F(a)$ is the characteristic function for the distribution of any diagonal entry $B_{ii}$, $i = 1, 2, \ldots$.  

As shown by Pickrell [Pi2], the Classification Theorem can be derived from a deep function-theoretic result due to Schoenberg [S2]. In the present paper, we prove the Classification Theorem in an entirely different way: our method consists in studying how an ergodic measure $M \in \mathcal{M}$ is approximated by finite-dimensional “orbital measures”. (By an orbital measure we mean the $U(n)$-invariant probability measure supported by a $U(n)$-orbit $\Omega_n \subset H(n)$, where $n = 1, 2, \ldots$.)

We would like to emphasize that such an approach not only leads to the description (0.1) of ergodic measures, but also provides us with additional information about them. Namely, we find a characterization of those sequences $\{M_n\}$ of orbital measures that weakly converge, as $n \to \infty$, to an ergodic measure $M \in \mathcal{M}$, so we can understand how $M$ “grows up” from orbital measures. In particular, we can relate the parameters $\gamma_1, \gamma_2, x_1, x_2, \ldots$ to the asymptotic behavior of the eigenvalues of large Hermitian matrices.

A characterization of the convergent sequences $M_n$ is the main result of the paper—Theorem 4.1. The statement is too long to be reproduced here in detail, but roughly speaking, the picture is as follows. Given $n = 1, 2, \ldots$, pick any matrix from the orbit $\Omega_n$ supporting $M_n$ and represent the collection of the eigenvalues of that matrix as an $n$-point configuration on the real line. Then this configuration, being contracted with scaling factor $1/n$, must converge, as $n \to \infty$, to a (countable) point configuration $\{x_1, x_2, \ldots\} \subset \mathbb{R}$. (Here the $x_k$’s coincide with the parameters appearing in (0.1); in fact, there are also additional restrictions on the growth of the orbits related to the parameters $\gamma_1$ and $\gamma_2$.)

2. **Relationships with representations and totally positive functions.**

We are interested in the measures of class $\mathcal{M}$ for several reasons:

1) (The initial motivation of [V].) The action of the group $U(\infty)$ on the space $H$ is a natural example of a “big” group action. As compared with the action of
the group $O(\infty)$ on the space $\mathbb{R}^\infty$ (Schoenberg’s case [S1]) or the action of $U(\infty)$ on $\mathbb{C}^\infty$, this example is of the next level of complexity. So, from the viewpoint of ergodic theory, it is interesting to find ergodic measures and to compare the result with the one in Schoenberg’s case.

2) The ergodic measures of class $\mathcal{M}$ determine some irreducible unitary representations of a certain infinite-dimensional group $G(\infty)$.

3) These measures are closely related to Schoenberg’s totally positive functions.

4) They are also related to harmonic analysis on the group $U(\infty)$ (cf. [KOV]).

We shall briefly comment on items 2) and 3).

The group $G(\infty)$ is a model example of an infinite-dimensional Cartan motion group. It can be defined as the inductive limit $G(\infty) = \lim_{\to} G(n)$, where $G(n)$ stands for the semidirect product $U(n) \rtimes H(n)$. There is a natural bijective correspondence between ergodic $U(\infty)$-invariant probability measures on $H$ and irreducible unitary representations of the group $G(\infty)$ that are spherical with respect to the subgroup $U(\infty) \subset G(\infty)$. Under this correspondence, the spherical functions of $G(\infty)$, when restricted to $H(\infty) := \lim_{\to} H(n)$, coincide with the characteristic functions of the ergodic measures.

It should be noted that all the results and constructions concerning the ergodic measures of class $\mathcal{M}$ can be translated into the language of representation theory. In particular, weak convergence of measures turns into uniform compact convergence of spherical functions, and the problem that is solved in the present paper is a particular case of the following general problem in representation theory of “big” (inductive limit) groups: Given a (irreducible) unitary representation of an inductive limit group $\mathcal{G} = \lim_{\to} \mathcal{G}(n)$, study its approximation by (irreducible) unitary representations of the growing subgroups $\mathcal{G}(n)$ as $n \to \infty$.

(About various asymptotic results in representation theory of “big” groups, see the series of papers by the authors and S. V. Kerov, [VK1–VK3, Ke, KV, O1–O7].)

A remarkable feature of representation theory of “big” groups is its connection with analytic problems of total positivity theory (see Thoma [T], Boyer [Bo], Vershik and Kerov [VK1, VK2], Voiculescu [Vo1]; for a systematic exposition of total positivity theory, see Karlin’s fundamental monograph [K]).

To describe this connection, we need the definition of totally positive (TP) functions: these are real-valued functions $\varphi(t)$ on $\mathbb{R}$ such that for any $n$ and any choice of real numbers $t_1 < \cdots < t_n$ and $s_1 < \cdots < s_n$, the determinant of the $n \times n$ matrix $[\varphi(t_i - s_j)]$ is nonnegative. Similarly, one also defines two-sided and one-sided TP sequences: these are TP functions defined on the 1-dimensional lattice $\mathbb{Z} \subset \mathbb{R}$ or its nonnegative part $\mathbb{Z}_+$, respectively.

It turns out that the Fourier transform of TP functions $\varphi$ on $\mathbb{R}$ leads exactly to functions $F$ of the form (0.3): this claim is Schoenberg’s main theorem in [S2].

Thus, there is a correspondence $M \leftrightarrow \varphi$ between ergodic measures $M \in \mathcal{M}$ and TP functions $\varphi$ (normalized by the condition $\int \varphi(t) \, dt = 1$) that can be stated as follows: $\varphi(t) \, dt$ coincides with the distribution of (any) diagonal entry $B_{ii}$, where $B$ stands for a random Hermitian matrix distributed according to $M$.

Further, there exists a similar correspondence between characters of the group $U(\infty)$ or of the infinite symmetric group $S(\infty)$ and two-sided or one-sided TP sequences, respectively.

This correspondence can be used in two directions:

On the one hand, old theorems in total positivity theory, obtained by analytic tools in [ASW, E1, E2, S2, T, K], can be applied to representation-theoretic
problems (classification of spherical functions or characters). Such an approach is adopted by Thoma [T], Boyer [Bo], and Pickrell [Pi2].

But on the other hand, if we can classify spherical functions or characters independently, then we can prove some theorems on TP functions or TP sequences in a new way.

It is the second approach that is adopted in Vershik and Kerov’s papers [VK1, VK2], and also in the present paper: as a corollary of our main result, we obtain a new derivation of Schoenberg’s classification for TP functions on the real line.

Also note a recent paper by Okounkov [Ok], where a different (direct representation-theoretic) method is used; the result of [Ok] gives yet another way to classify one-sided TP sequences.

3. The method. Now let us describe our techniques in more detail.

The starting point of our approach is a general approximation theorem for ergodic measures, see Theorem 3.2 below\(^4\). When applied to our concrete situation, it implies that any ergodic measure \( M \in \mathcal{M} \) can be approximated by a sequence \( \{M_n\} \) of orbital measures (see Theorem 3.3 below). Then the main problem is to understand what sequences \( \{M_n\} \) are weakly convergent, as \( n \to \infty \), and what are their limits.

An attempt to do this was made in [V], but the solution proposed there was incomplete, because of a gap in the calculations. Nevertheless, the method itself was correct, and a refinement of the calculations leads to the right result.

To study the weak convergence of orbital measures \( M_n \), we deal with their characteristic functions \( f_n \),

\[
f_n(A) = \int_{B \in \Omega_n} e^{i \text{tr}(AB)} M_n(dB), \quad A \in H(n),
\]

where \( \Omega_n \subset H(n) \) is the \( U(n) \)-orbit carrying \( M_n \). We calculate the Taylor decomposition at the origin of a characteristic function \( f_n \) and then analyze the asymptotics of its Taylor coefficients as \( n \to \infty \). Note that these Taylor coefficients are nothing but the moments of the measure \( M_n \). (In fact, due to the symmetry of \( f_n \), it is convenient to rewrite its Taylor decomposition as a series of Schur polynomials.) A non evident fact is that weak convergence of orbital measures can always be controlled by the moments. Such a phenomenon was first discovered in [VK2]; there it was applied to an allied problem: classifying characters of the group \( U(\infty) \).

Note that we are able to generalize our results to the spaces of real symmetric and quaternionic Hermitian matrices. Instead of Schur polynomials, we must then use Jack symmetric polynomials. As a further generalization, one could consider all matrix spaces of the form \( H = \lim \rightarrow H(n) \), where \( H(n) \) ranges over one of the 10 series of classical symmetric spaces of Euclidean type, the parameter \( n \) being the rank of the space (see [O5, Pi1] and especially [Pi2] for a discussion of these spaces \( H \)). Characteristic functions of orbital measures on the spaces \( H(n) \) are sometimes called generalized Bessel functions; for 3 of the 10 spaces \( H \), they can be expressed in terms of elementary functions, and for the 7 other spaces, these are certain multidimensional special functions. The problem consists in studying their limiting behavior as \( n \to \infty \). We conjecture that our approach can be transferred to all the spaces \( H \).

\(^4\)There is also another version of this theorem that deals with spherical functions, see Theorem 3.5 below.
4. **Contents.** The present paper is organized as follows.

In §1, we introduce ergodic measures and explain their relationship with spherical unitary representations.

In §2, we formulate the so-called Multiplicativity Theorem for characteristic functions of ergodic measures (Theorem 2.1). This important result states that an $U(\infty)$-invariant probability measure $M$ on $H$ is ergodic if and only if its characteristic function is multiplicative in a certain sense; it follows that the set of ergodic measures is closed under convolution. Then we state the classification result (Theorem 2.9), which shows that any ergodic measure is a convolution product of certain “elementary” measures. We also discuss a number of corollaries of these theorems.

Section 3 is devoted to a proof of a general approximation theorem that has been announced in [V] (Theorem 3.2 below). This result may be viewed as an ergodic theorem for actions of general inductive limits of compact groups.

In §4, we state the main result of the paper, Theorem 4.1, and outline its proof.

Section 5 contains a preliminary result, needed for the proof of Theorem 4.1. There we decompose the characteristic function of an orbital measure into a series of Schur symmetric functions.\(^5\)

In §6, we prove Theorem 4.1.

In §7, following Pickrell’s arguments in [Pi2], we establish the equivalence of two classification problems: that of ergodic invariant measures $M \in \mathcal{M}$ and that of totally positive functions $\varphi$ on $\mathbb{R}$. Note that our own contribution to the results of this section is very modest. Here we only aimed to clarify some technical details of the correspondence $M \leftrightarrow \varphi$ and to explain how our main result implies Schoenberg’s classification.\(^6\)

In the final §8, we explain a connection between our main theorem and the main result of the remarkable work [CS] by Curry and Schoenberg.

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§1. **Ergodic measures and spherical representations**

Let $H(n)$ denote the real vector space formed by complex Hermitian $n \times n$ matrices.

\(^5\) It would be interesting to find analogues of this result for all generalized Bessel functions.

\(^6\) Note, however, that we are not completely satisfied by the way we obtain Schoenberg’s theorem. We think there should exist a more conceptual derivation of this result by approximation methods. A similar remark can be made à propos TP sequences as well.
matrices, \( n = 1, 2, \ldots \). There is a natural embedding
\[ H(n) \to H(n + 1), \quad A \mapsto \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \]
and we denote by \( H(\infty) \) the corresponding inductive limit space \( \lim H(n) \). Then \( H(\infty) \) is identified with the space of infinite Hermitian matrices with a finite number of nonzero entries. We equip \( H(\infty) \) with the inductive limit topology. In particular, a function \( f : H(\infty) \to \mathbb{C} \) is continuous if its restriction to \( H(n) \) is continuous for any \( n \).

Let \( H \) stand for the space of all infinite Hermitian matrices. For \( A \in H \) and \( n = 1, 2, \ldots \), we denote by \( \theta_n(A) \in H(n) \) the upper left \( n \times n \) corner of \( A \). Using the projections \( \theta_n : H \to H(n), n = 1, 2, \ldots \), we may identify \( H \) with the projective limit space \( \lim \leftarrow H(n) \). We equip \( H \) with the corresponding projective limit topology. In particular, \( H \) is a Borel space.

There is a natural pairing \( H(\infty) \times H \to \mathbb{R}, \quad (A, B) \mapsto \text{tr}(AB). \)

Using it, we may regard \( H \) as the algebraic dual space of \( H(\infty) \).

Note that \( H \) can be identified, in an obvious manner, with the space \( \mathbb{R}^\infty = \mathbb{R} \times \mathbb{R} \times \cdots \). Under this identification, \( H(\infty) \subset H \) turns into \( \mathbb{R}_0^\infty := \bigcup_{n \geq 1} \mathbb{R}^n \), and the pairing defined above becomes the standard pairing between \( \mathbb{R}_0^\infty \) and \( \mathbb{R}^\infty \).

Given a Borel probability measure \( M \) on \( H \), we define its Fourier transform, or characteristic function, as the following function on \( H(\infty) \):
\[ f(A) = f_M(A) = \int_H e^{i\text{tr}(AB)} M(dB). \quad (1.1) \]

We need the following statement:

**Proposition 1.1.** The Fourier transform (1.1) establishes a bijective correspondence between Borel probability measures \( M \) on \( H \) and continuous positive definite normalized functions \( f \) on \( H(\infty) \).

Here “normalized” means that \( f(0) = 1 \).

**Proof.** This is an immediate corollary of Kolmogorov’s consistency theorem and Bochner’s theorem. Indeed, let us identify \( (H(\infty), H) \) with \( (\mathbb{R}_0^\infty, \mathbb{R}^\infty) \). Then Kolmogorov’s theorem implies that Borel probability measures on \( \mathbb{R}^\infty \) are just the projective limits of probability measures on the \( \mathbb{R}^n \)’s, and Bochner’s theorem allows us to restate this fact in terms of characteristic functions. \( \square \)

Let \( U(n) \) be the group of unitary \( n \times n \) matrices, \( n = 1, 2, \ldots \). For any \( n \), we embed \( U(n) \) into \( U(n + 1) \) using the mapping \( u \mapsto \begin{bmatrix} u & 0 \\ 0 & 1 \end{bmatrix} \). Let \( U(\infty) = \lim U(n) \) denote the corresponding inductive limit group. We regard \( U(\infty) \) as the group of infinite unitary matrices \( u = [u_{ij}]_{i,j=1}^\infty \) with a finite number of entries \( u_{ij} \neq \delta_{ij} \). The group \( U(\infty) \) acts by conjugations both on \( H(\infty) \) and \( H \), and the pairing between these two spaces is clearly \( U(\infty) \)-invariant.

Let us recall some basic facts concerning ergodic measures.
**Definition 1.2.** Let $\mathcal{X}$ be a Borel space, let $\mathcal{G}$ be a group of Borel transformations of $\mathcal{X}$, and let $M$ be a $\mathcal{G}$-invariant probability Borel measure on $\mathcal{X}$.

(i) A Borel subset $Y \subset \mathcal{X}$ is said to be $\mathcal{G}$-invariant mod 0 if $M((gY) \triangle Y) = 0$ for any $g \in \mathcal{G}$.

(ii) The measure $M$ is said to be ergodic with respect to $\mathcal{G}$ if the $M$-volume of any $\mathcal{G}$-invariant mod 0 Borel subset $Y \subset \mathcal{X}$ is equal either to 0 or 1.

**Proposition 1.3.** The following conditions on a Borel probability measure $M$ on $\mathcal{X}$ are equivalent:

(i) $M$ is ergodic;

(ii) $M$ is an extreme point of the convex set formed by all Borel probability measures on $\mathcal{X}$;

(iii) the subspace of $\mathcal{G}$-invariant vectors in the Hilbert space $L^2(\mathcal{X}, M)$ is exhausted by constant functions.

**Proof.** See, e.g., Phelps [Ph, Proposition 10.4]. □

For compact group actions on locally compact spaces, ergodic measures coincide with invariant measures supported by the orbits, so that for such actions, classifying orbits and classifying ergodic measures are equivalent problems. However, for “big” groups like $U(\infty)$, these two problems are quite different: the first one seems to be out of reach, while the second one has, in certain cases, a nice solution. According to a general principle of ergodic theory, ergodic measures may be viewed as true substitutes of orbits.

Let $M$ denote the set of all ergodic $U(\infty)$-invariant Borel probability measures $M$ on $H$, and let $\mathcal{F}$ denote the set of the corresponding characteristic functions $f = f_M$.

The following claim, which follows from Propositions 1.1 and 1.3, gives a useful characterization of the class $\mathcal{F}$:

**Proposition 1.4.** The functions $f \in \mathcal{F}$ are exactly the extreme points of the convex set formed by all continuous $U(\infty)$-invariant positive definite normalized functions on $H(\infty)$.

Let $G(n) = U(n) \ltimes H(n)$ be the semidirect product of $U(n)$ and the additive group of the vector space $H(n)$. The elements $g \in G(n)$ are the pairs $(u, A) \in U(n) \times H(n)$ with the multiplication rule

$$(u, A) \cdot (v, B) = (uv, v^{-1}Av + B).$$

In the same way, we define the group $G(\infty) = U(\infty) \ltimes H(\infty)$. The group $G(\infty)$ can be also viewed as the inductive limit group $\varinjlim G(n)$, and we equip it with the inductive limit topology. Using the embeddings

$$u \mapsto (u, 0) \quad \text{and} \quad A \mapsto (1, A), \quad u \in U(\infty), \quad A \in H(\infty),$$

we may identify $U(\infty)$ and $H(\infty)$ with the corresponding subgroups in $G(\infty)$.

Let $T$ be a unitary representation of $G(\infty)$ in a Hilbert space $\mathcal{H}(T)$ (we tacitly assume $T$ is continuous with respect to the inductive limit topology of the group $G(\infty)$). Then $T$ is said to be spherical if it is irreducible and the subspace $\mathcal{H}(T)^{U(\infty)}$ of $U(\infty)$-invariants in $\mathcal{H}(T)$ is nonzero.
Suppose $T$ is spherical. It is then known that the space $\mathcal{H}(T)^{U(\infty)}$ is one-dimensional (see Olshanski [O5, Theorem 23.6]). A vector $h \in \mathcal{H}(T)^{U(\infty)}$ of norm 1 is called a spherical vector of $T$, and the corresponding matrix element

$$\varphi_T(g) = (T(g)h, h), \quad g \in G(\infty),$$

is called the spherical function of $T$. Since $\varphi_T$ does not change when $h$ is multiplied by a complex number of absolute value 1, $\varphi_T$ is an invariant of $T$; moreover, it uniquely determines $T$. Further, as $\varphi_T$ is bi-invariant with respect to the subgroup $U(\infty) \subset G(\infty)$, it is uniquely determined by its restriction $\varphi_T|H(\infty)$ to $H(\infty)$, which is a continuous $U(\infty)$-invariant function.

**Proposition 1.5.** There is a natural bijective correspondence $T \leftrightarrow M$ between (equivalence classes of) spherical representations $T$ of the group $G(\infty)$ and ergodic measures $M \in \mathcal{M}$. Under that correspondence, the functions $\varphi_T|H(\infty)$ coincide with the characteristic functions $f_M$. Given $M \in \mathcal{M}$, the corresponding representation $T$ can be realized in the Hilbert space $L^2(H, M)$ as follows:

$$(T(u)\Psi)(B) = \Psi(u^{-1}Bu), \quad u \in U(\infty) \subset G(\infty),$$

$$(T(A)\Psi)(B) = e^{i\text{tr}(AB)}\Psi(B), \quad A \in H(\infty) \subset G(\infty),$$

where $\Psi \in L^2(H, M)$ and $B \in H$. Finally, in this realization, the spherical vector is the constant function $\Psi_0(B) \equiv 1$.

**Proof.** Indeed, it is enough to remark that the spherical functions, when restricted to $H(\infty)$, are just the extreme $U(\infty)$-invariant continuous positive definite functions, normalized at $0 \in H(\infty)$, and then apply Proposition 1.4. □

**§2. The Classification Theorem**

Denote by $D(\infty)$ the subspace of diagonal matrices in $H(\infty)$. An element of $D(\infty)$ will be written as $\text{diag}(a_1, a_2, \ldots)$, where $a_1, a_2, \ldots \in \mathbb{R}$ and $a_k = 0$ for $k$ large enough. Since any matrix in $H(\infty)$ can be diagonalized under the action of the group $U(\infty)$, a $U(\infty)$-invariant function $f$ on $H(\infty)$ is uniquely determined by $f|D(\infty)$, the restriction of $f$ to $D(\infty)$.

Assume $f$ is an $U(\infty)$-invariant function on $H(\infty)$, $f(0) = 1$. Let us say that $f$ is multiplicative if

$$f(\text{diag}(a_1, a_2, \ldots)) = F(a_1)F(a_2) \cdots, \quad (2.1)$$

where $F$ is a function on $\mathbb{R}$ such that $F(0) = 1$. In other words,

$$f(A) = F(a_1)F(a_2) \cdots, \quad A \in H(\infty),$$

(2.2)

where $a_1, a_2, \ldots$ are the eigenvalues of $A$. Note that $F(a) = f(\text{diag}(a, 0, 0, \ldots))$, i.e., $F = f|H(1)$.

**Theorem 2.1** (Multiplicativity Theorem). *Let $f$ be a continuous $U(\infty)$-invariant positive definite function on $H(\infty)$, $f(0) = 1$. Then $f$ is extreme (i.e., $f \in \mathcal{F}$) if and only if $f$ is multiplicative.*

**Proof.** See, e.g., Olshanski [O5, Theorem 23.8]. There the Multiplicativity Theorem is stated for allied groups (like $GL(n, \mathbb{C})$) but the proof is exactly the same. □
Not that there are many theorems of this type and a lot of different methods to prove them: see Thoma [T], Ismagilov [I1] and [I2] (Ismagilov’s method is also explained in Olshanski [O4, section 2.5]), Nessonov [N], Olshanski [O1], Pickrell [Pi1], Vershik–Kerov [VK4], Voiculescu [Vo1] and [Vo2], Stratila–Voiculescu [SV].

Theorem 2.1 implies that a function \( f \in \mathcal{F} \) is uniquely determined by the function \( F := f|H(1) \), which is a continuous function in one real variable. Note that
\[
F(a) = F_M(a) = \int_H e^{iaB_{11}} M(dB), \quad a \in \mathbb{R}, \ B \in H, \tag{2.3}
\]
where \( M \in \mathcal{M} \) corresponds to \( f \).

Definition 2.2. Let \( \mathcal{F}_1 = \{ F \} \) denote the class of all functions on \( \mathbb{R} \) of the form \( F = f|H(1) \) where \( f \) ranges over \( \mathcal{F} \). In other words, a function \( F \) on \( \mathbb{R} \) belongs to \( \mathcal{F}_1 \) if and only if \( F \) is continuous, \( F(0) = 1 \), and the corresponding function (2.2) on \( H(\infty) \) is positive definite.

Clearly, the classification of the measures \( M \in \mathcal{M} \) is reduced to that of the functions \( F \in \mathcal{F}_1 \).

Theorem 2.1 has a number of corollaries.

**Corollary 2.3.** The class \( \mathcal{F}_1 \) is stable under pointwise multiplication. \( \square \)

This implies that the class \( \mathcal{F} \) is also stable under multiplication, and the class \( \mathcal{M} \) is stable under convolution.

**Corollary 2.4.** If a sequence \( F_1, F_2, \cdots \in \mathcal{F}_1 \) pointwise converges to a continuous function \( F \) on \( \mathbb{R} \), then \( F \in \mathcal{F}_1 \). \( \square \)

**Corollary 2.5.** For a real \( \gamma \), the Dirac measure concentrated at \( \gamma \cdot 1 \in H \) belongs to \( \mathcal{M} \). The corresponding function \( F \in \mathcal{F}_1 \) is
\[
F(a) = e^{i\gamma a}, \quad a \in \mathbb{R}. \tag{2.4}
\]

**Proof.** Since scalar matrices are \( U(\infty) \)-invariant, our measure is invariant. It is clearly ergodic. It is evident that the corresponding function from \( \mathcal{F}_1 \) is given by (2.4). \( \square \)

**Corollary 2.6.** Given \( \gamma \geq 0 \), let \( M \) be the Gaussian distribution on \( H \) such that, for a matrix \( B \in H \), the diagonal entries \( B_{ii} \) and the off-diagonal entries \( \text{Re}B_{ij}, \text{Im}B_{ij}, i < j \), are independent Gaussian variables with mean 0 and variance \( \gamma \). Then \( M \in \mathcal{M} \), and the corresponding function \( F \in \mathcal{F}_1 \) is
\[
F(a) = e^{-\gamma a^2/2}, \quad a \in \mathbb{R}. \tag{2.5}
\]

**Proof.** It is easily verified that \( M \) is invariant and its characteristic function satisfies (2.1), where \( F \) is given by (2.5). \( \square \)
Corollary 2.7. Let \( \omega \) denote the Gaussian measure on \( \mathbb{C} \) with density given by \( \pi^{-1} \exp(-|z|^2) \), \( z \in \mathbb{C} \). Given \( y \in \mathbb{R} \), let \( M \) be the image of the Gaussian product measure \( \omega^{\otimes \infty} \) on \( \mathbb{C}^\infty \) under the following Borel mapping \( \mathbb{C}^\infty \rightarrow H \):

\[
\mathbb{C}^\infty \ni \xi \mapsto y(-1 + \xi^* \xi) = B \in H.
\]

(Here \( \xi = (\xi_1, \xi_2, \ldots) \in \mathbb{C}^\infty \) is regarded as a row vector, so that the \((i, j)\)-entry of the matrix \( B \) is equal to \( y(-1 + \bar{\xi}_i \xi_j) \), \( i, j = 1, 2, \ldots \)). Then \( M \in \mathcal{M} \), and the corresponding function \( F \in \mathcal{F}_1 \) is given by

\[
F(a) = \frac{e^{-iya}}{1 - iya}, \quad a \in \mathbb{R}.
\]

Proof. The invariance of \( M \) is obvious. A direct calculation shows that the characteristic function of \( M \) satisfies (2.1) with \( F \) given by (2.7). \( \Box \)

We shall call the measures defined in Corollaries 2.5–2.7 the elementary ergodic measures.

Note that we could omit the term \(-1\) in (2.6) and then the numerator in (2.7) would disappear. However, due to this term, the mean of \( M \) is equal to zero and the function (2.7) has the property

\[
F(a) = \frac{e^{-iya}}{1 - iya} = 1 - \frac{3}{2} a^2 y^2 + O(y^3) \quad \text{as } y \to 0,
\]

i.e., the term of degree 1 in \( y \) vanishes. This is important for the following construction.

Proposition 2.8. The class \( \mathcal{F}_1 \) contains all functions of the form

\[
F_{\gamma_1, \gamma_2, x}(a) = e^{i\gamma_1 a - \gamma_2 a^2/2} \prod_{k} \frac{e^{-ix_k a}}{1 - ix_k a},
\]

where \( \gamma_1 \in \mathbb{R} \) and \( \gamma_2 > 0 \) are arbitrary constants, and \( x = (x_1, x_2, \ldots) \) is a sequence of real numbers such that \( \sum x_k^2 < \infty \).

Proof. Suppose first the sequence \( x \) is finite. Then (2.9) is a finite product of functions that belong to the class \( \mathcal{F}_1 \) due to Corollaries 2.5, 2.6, and 2.7. By Corollary 2.2, their product belongs to \( \mathcal{F}_1 \) also. Finally, if the sequence \( x \) is infinite, then the assumption \( \sum x_k^2 < \infty \) and the estimate (2.8) imply that the product in (2.9) is convergent for any \( a \in \mathbb{R} \); moreover, the result is a continuous function. Then Corollary 2.4 implies that this function belongs to \( \mathcal{F}_1 \). \( \Box \)

Comments. 1) The order of the \( x_k \)'s is unessential, so that \( x = (x_k) \) is rather a point configuration (or multiset) than a sequence.

2) If \( \sum |x_k| < \infty \) then (2.9) may be rewritten as

\[
F_{\gamma_1, \gamma_2, x}(a) = e^{i\gamma_1 a - \gamma_2 a^2/2} \prod_{k} \frac{1}{1 - ix_k a}, \quad \gamma_1 := \gamma_1 - \sum_{k} x_k.
\]

3) The function (2.9) admits a holomorphic continuation to the horizontal strip \( \{ z \in \mathbb{C} \mid | \Im z | < \varepsilon \} \), where \( \varepsilon^{-1} = \sup \{|x_k| \} \).

4) The parameters \( \gamma_1, \gamma_2, x \) are uniquely determined by the function \( F_{\gamma_1, \gamma_2, x} \).

5) The characteristic function (2.2) corresponding to the function (2.9) can be written as

\[
f(A) = e^{i\gamma_1 \text{tr} A - \gamma_2 \text{tr}(A^2)/2} \det \left( \prod_{k} \frac{e^{-ix_k A}}{1 - ix_k A} \right), \quad A \in H(\infty).
\]
Theorem 2.9 (Classification Theorem). The class $\mathcal{F}_1$ is exhausted by the functions of the form (2.9). Thus, the characteristic functions of the ergodic $U(\infty)$-invariant Borel probability measures on $H$ are just the functions of the form (2.11), where $\gamma_1, \gamma_2, x_1, x_2, \ldots$ are real parameters such that $\gamma_2 \geq 0$ and $\sum x_k^2 < \infty$.

This result gives a description of ergodic measures $M \in \mathcal{M}$: any such $M$ is a convolution of the elementary ergodic measures constructed in Corollaries 2.5–2.7. The proof will be given in §4: we shall derive Theorem 2.9 from a more general result, Theorem 4.1.

Note that the elementary ergodic measures of Corollaries 2.5 and 2.6 are infinitely divisible with respect to convolution, whereas those of Corollary 2.7 are not.

Remark 2.10. The construction of Corollary 2.7 can be generalized as follows. Fix $k = 1, 2, \ldots$, consider the space $\mathbb{C}^{k \times \infty}$ of all complex matrices $\Xi$ with $k$ rows and infinitely many columns, and equip it with the Gaussian product measure

$$\omega^{k \times \infty} := \omega^\otimes \infty \otimes \cdots \otimes \omega^\otimes \infty. \quad (2.12)$$

Let $z, x_1, \ldots, x_k$ be any real numbers, and let $X = \text{diag}(x_1, \ldots, x_k)$ denote the diagonal matrix of order $k$ with diagonal entries $x_1, \ldots, x_k$. Consider the mapping

$$\mathbb{C}^{k \times \infty} \ni \Xi \mapsto z \cdot 1 + \Xi^* X \Xi = B \in H. \quad (2.13)$$

Then the image of the Gaussian measure (2.12) under the mapping (2.13) is an ergodic measure $M_{z;x_1,\ldots,x_k}$. Its parameters are

$$\gamma_1 = z + x_1 + \cdots + x_k, \quad \gamma_2 = 0, \quad x = (x_1, \ldots, x_k, 0, 0, \ldots). \quad (2.14)$$

$M_{z;x_1,\ldots,x_k}$, $k = 1, 2, \ldots$, form a weakly dense subset of $\mathcal{M}$. Further, let $H_{\leq k}$ denote the closed subspace of $H$ formed by matrices of rank $\leq k$. One can prove that the measures $M_{0;x_1,\ldots,x_k}$ are just those measures $M \in \mathcal{M}$ that are supported by $H_{\leq k}$. Finally, note that the measures $M_{0;x_1,\ldots,x_k}$ with $x_1 = \cdots = x_k$ are infinite-dimensional analogs of the well-known Wishart distributions (see e.g., Muirhead [Mu]).

Remark 2.11. Let $H_+(n) \subset H(n)$ and $H_+ \subset H$ denote the subsets of nonnegative definite matrices. Then $H_+$ coincides with $\lim H_+(n)$ and is a closed cone in $H$. One can prove that a measure $M \in \mathcal{M}$ is supported by $H_+$ if and only if its parameters satisfy the conditions

$$\gamma_2 = 0, \quad x_1 \geq 0, \quad x_2 \geq 0, \quad \ldots, \quad \sum x_k \leq \gamma_1 < \infty. \quad (2.17)$$

Remark 2.12. Let $M \in \mathcal{M}$ and let $F \in \mathcal{F}_1$ be the corresponding function (2.9). We may regard $(H, M)$ as a probability space and the matrix elements $B_{ij}$ of a matrix $B \in H$ as random variables. Let $\mu$ denote the distribution of the real random variable $B_{11}$; then $F$ is the characteristic function of $\mu$. We know that $M$ is completely determined by $\mu$. Let us describe the structure of $\mu$. It follows from (2.9) that $\mu$ is the convolution of a (not necessarily centered) normal distribution with a family of distributions possessing characteristic functions of the form (2.7). It is easily verified that if $y > 0$, then the distribution with characteristic function (2.7) is supported by the half-line $t \geq -y$ and has density

$$t \mapsto y^{-1}e^{-y^{-1}(t+y)}, \quad t \geq -y. \quad (2.18)$$
This is the shifted exponential distribution with variance $y^2$ and mean 0. If the parameter $y$ is negative, it suffices to replace $t$ by $-t$. Thus, $\mu$ is the convolution of a normal distribution with a family of modified exponential distributions.

**Remark 2.13.** Let $M \in \mathcal{M}$ be an ergodic measure for which not all the parameters $x_k$ vanish. Then the characteristic function $f(A)$, see (2.11), cannot be factorized into a product of factors each of which depends on a single matrix element $A_{ij}$ only. This means that $M$ is not a product measure, so that the matrix elements $B_{pq}$, as random variables defined on $(H,M)$, are not independent on the whole (in the particular case of the measures $M_{z;x_1,\ldots,x_k}$, this is seen from their construction via the mapping (2.13)). However, certain matrix elements are independent. For instance, it is evident that the diagonal elements are independent. More generally, for any $(p_1,q_1),\ldots,(p_m,q_m)$ such that $i \neq j$ implies $p_i \neq p_j$, $q_i \neq q_j$, $p_i \neq q_j$, the matrix elements $B_{p_1q_1},\ldots,B_{p_mq_m}$ are independent. This claim can be deduced from (2.2).

According to Proposition 1.5, Theorem 2.9 also gives a complete description of spherical representations of the group $G(\infty)$. We shall now discuss the possibility of extending spherical representations to some topological completions of the inductive group $G(\infty)$.

Let us regard $U(\infty)$ as a group of unitary operators in the complex coordinate Hilbert space $\ell_2$, and let $\overline{U}(\infty) \supset U(\infty)$ stand for the group of all unitary operators in $\ell_2$. Note that $\overline{U}(\infty)$ is a topological group with respect to the weak operator topology (which coincides on unitary operators with the strong operator topology). Further, let $H(\infty)_1 \supset H(\infty)$ (respectively, $H(\infty)_2 \supset H(\infty)$) denote the space of the trace class (respectively, Hilbert–Schmidt) Hermitian operators in $\ell_2$, equipped with the topology defined by the trace norm $\| \cdot \|_1$ (respectively, by the Hilbert–Schmidt norm $\| \cdot \|_2$).

One can check that the actions

$$\overline{U}(\infty) \times H(\infty)_1 \rightarrow H(\infty)_1, \quad \overline{U}(\infty) \times H(\infty)_2 \rightarrow H(\infty)_2,$$

where $(u,A) \mapsto uAu^{-1}$, are continuous (cf. Shale [Sha]). Thus we may form the semidirect products

$$G(\infty)_1 = \overline{U}(\infty) \ltimes H(\infty)_1, \quad G(\infty)_2 = \overline{U}(\infty) \ltimes H(\infty)_2,$$

which are topological groups with respect to the corresponding product topologies. Note that the inductive limit group $G(\infty) = U(\infty) \ltimes H(\infty)$ is contained as a dense subgroup both in $G(\infty)_1$ and $G(\infty)_2$.

**Corollary 2.14.** (i) Any spherical representation of the group $G(\infty)$ admits an extension to a continuous representation of the topological group $G(\infty)_1$.

(ii) It can be continued to $G(\infty)_2$ if and only if the parameter $\gamma_1$ in (2.11) vanishes.

**Proof.** (i) Indeed, for any function $F$ of the form (2.9), the corresponding function $f$ on $H(\infty)$, which is defined by (2.11), can be extended to a continuous function on $H(\infty)_1$. Then the latter function can be extended to a continuous $\overline{U}(\infty)$-bi-invariant function on the group $G(\infty)_1$, and our claim follows.

(ii) We argue as in (i), using the fact that $f$ is continuous with respect to the Hilbert–Schmidt norm if and only if $\gamma_1 = 0$. □
§3. APPROXIMATION OF ERGODIC MEASURES BY ORBITAL MEASURES

Let $\mathcal{X}$ be a separable metric space and $C(\mathcal{X})$ be the Banach space of bounded continuous functions on $\mathcal{X}$. By a measure on $\mathcal{X}$ we shall always mean a Borel probability measure. Recall that a sequence $\{\nu_n\}$ of measures on $\mathcal{X}$ is said to be weakly convergent to a measure $\nu$ (notation: $\nu_n \Rightarrow \nu$) if $\langle \psi, \nu_n \rangle \to \langle \psi, \nu \rangle$ as $n \to \infty$ for any $\psi \in C(\mathcal{X})$, see, e.g., [Bi, Pa1].

Proposition 3.1. There exists a countable set $\Psi \subset C(\mathcal{X})$ of functions with the following property: a sequence of measures $\nu_1, \nu_2, \ldots$ weakly converges to a measure $\nu$ provided convergence occurs for any function $\psi \in \Psi$.

Proof. See, e.g., [Pa1, Chapter II, Theorem 6.6]. □

Let $K(1) \subset K(2) \subset \ldots$ be an ascending chain of compact groups, and let $K = \lim\limits_{\rightarrow} K(n)$ be the corresponding inductive limit topological group. We assume there is a jointly continuous action $(u, x) \mapsto u \cdot x$ of the group $K$ on the space $\mathcal{X}$. Let $m_n$ denote the normalized Haar measure on $K(n)$. Given a point $x \in \mathcal{X}$, let $m_n(x)$ denote the image of $m_n$ under the mapping $u \mapsto u \cdot x$, where $u$ ranges over $K(n)$, i.e., $m_n(x)$ is the unique $K(n)$-invariant probability measure supported by the orbit $K(n) \cdot x$.

Theorem 3.2 (Vershik [V, Theorem 1]). Let $\nu$ be an ergodic $K$-invariant Borel probability measure on $\mathcal{X}$. Then there exists a point $x \in \mathcal{X}$ such that $m_n(x) \Rightarrow \nu$ as $n \to \infty$. Moreover, the set of all points $x \in \mathcal{X}$ with this property is of full measure with respect to $\nu$.

Proof. Given $\psi \in C(\mathcal{X})$, put

$$\psi_n(x) = \langle \psi, m_n(x) \rangle = \int_{K(n)} \psi(u \cdot x) m_n(du), \quad x \in \mathcal{X}, \quad (3.1)$$

and let $\overline{\psi}$ denote the constant function

$$\overline{\psi}(x) \equiv \langle \psi, \nu \rangle \cdot 1 = \left( \int_{\mathcal{X}} \psi(x) \nu(dx) \right) \cdot 1. \quad (3.2)$$

To prove the theorem, it is enough to check that $\psi_n \to \overline{\psi}$ almost everywhere for any $\psi \in C(\mathcal{X})$. Indeed, it will follow that, given an arbitrary countable family of functions $\Psi \subset C(\mathcal{X})$, there exists a subset $\mathcal{X}'$ of full measure such that

$$\langle \psi, m_n(x) \rangle \to \langle \psi, \nu \rangle \quad \text{for any } \psi \in \Psi \text{ and any } x \in \mathcal{X}'. \quad (3.3)$$

Taking as $\Psi$ a subset from Proposition 3.1, we shall obtain $m_n(x) \Rightarrow \nu$ for all $x \in \mathcal{X}'$.

Now we proceed to check that, given $\psi \in C(\mathcal{X})$, we have $\psi_n \to \overline{\psi}$ almost everywhere.

Remark that the $\psi_n$ are bounded Borel functions. Indeed, given $x \in \mathcal{X}$ and $n$, the function $u \mapsto \psi(u \cdot x)$ is continuous on $K(n)$, so that the integral in (3.1) can be approximated by its Riemannian integral sums. It follows that $\psi_n$ is a pointwise limit of continuous functions, so it is a Borel function. Its boundedness is immediate.
Now it is enough to verify the following two claims:

**Claim 1.** \( \psi_n \to \overline{\psi} \) in the metric of \( L^2(\mathcal{X}, \nu) \).

**Claim 2.** For almost all points \( x \in \mathcal{X} \), the following limit exists

\[
\psi_{\infty}(x) := \lim_{n \to \infty} \psi_n(x).
\]

Indeed, since the \( \psi_n \)'s are uniformly bounded, Claim 2 will imply that \( \psi_n \to \psi_{\infty} \) in \( L^2(\mathcal{X}, \nu) \), whence \( \psi_{\infty} = \overline{\psi} \) almost everywhere, so that \( \psi_n \to \overline{\psi} \) almost everywhere.

To verify Claim 1, consider the natural unitary representation of the group \( \mathcal{K} \) in the Hilbert space \( L^2(\mathcal{X}, \nu) \). Let \( P_n \) denote the orthoprojection in this space onto the subspace of \( \mathcal{K}(n) \)-invariant vectors, \( n = 1, 2, \ldots \). Then \( P_n \) strongly converges, as \( n \to \infty \), to \( P \), the orthoprojection onto the subspace of \( \mathcal{K} \)-invariant vectors. In particular, \( P_n \psi \to P \psi \) in the metric of \( L^2(\mathcal{X}, \nu) \). Since the measure \( \nu \) is ergodic, the only \( \mathcal{K} \)-invariant vectors are the constants (this is the only place when we use the assumption that \( \nu \) is ergodic). Therefore, \( P \psi \) is a constant function, which clearly equals \( \overline{\psi} \). Finally, \( P_n \psi = \psi_n \), so that \( \psi_n \to \overline{\psi} \) in \( L^2(\mathcal{X}, \nu) \).

Now we shall prove Claim 2 by a method similar to the one used in the proof of Birkhoff–Khinchine’s individual ergodic theorem (see, e.g., Parthasarathy [Pa2, §49]).

Without loss of generality one may assume that \( \psi \) is real-valued. For \( N = 1, 2, \ldots \), put

\[
E_N = E_N(\psi) = \left\{ x \in \mathcal{X} \mid \sup_{1 \leq n \leq N} \psi_n(x) > 0 \right\},
\]

\[
E_{\infty} = E_{\infty}(\psi) = \left\{ x \in \mathcal{X} \mid \sup_{1 \leq n \leq \infty} \psi_n(x) > 0 \right\} = \bigcup_{N=1}^{\infty} E_N(\psi).
\]

Note that each \( E_N \) is a Borel subset, whence \( E_{\infty} \) also is a Borel subset.

The following claim is an analog of the maximal ergodic theorem.

**Claim 3.** We have

\[
\int_{E_{\infty}} \psi(x) \nu(dx) \geq 0.
\]

Indeed, since \( (E_N) \) is a monotone family of sets, it is enough to check that

\[
\int_{E_N} \psi(x) \nu(dx) \geq 0, \quad N = 1, 2, \ldots
\]

We can write each \( E_N \) as a disjoint union of Borel subsets,

\[
E_N = E_{1N} \cup \cdots \cup E_{NN},
\]

where

\[
E_{mN} = \{ x \in \mathcal{X} \mid \psi_m(x) > 0, \psi_i(x) \leq 0 \text{ for } m + 1 \leq i \leq N \}.
\]

Then it is enough to show that

\[
\int_{E_{mN}} \psi(x) \nu(dx) \geq 0, \quad 1 \leq m \leq N.
\]
To do this, we remark that $E_{m,N}$ is invariant relative to the action of $K(m)$. Since $\nu$ is an invariant measure, it follows that

$$\int_{E_{m,N}} \psi(x) \nu(dx) = \int_{E_{m,N}} \psi(u \cdot x) \nu(dx), \quad u \in K(m). \quad (3.11)$$

Since the function $(u, x) \to \psi(u \cdot x)$ is continuous on $K(m) \times E_{m,N}$, we may integrate the right-hand side of (3.11) over $K(m)$ (with respect to the Haar measure) and then interchange the integrals over $K(m)$ and over $E_{m,N}$. This yields

$$\int_{E_{m,N}} \psi(x) \nu(dx) = \int_{E_{m,N}} \psi_m(x) \nu(dx). \quad (3.12)$$

By definition (3.9), $\psi_m$ is positive on $E_{m,N}$, so that (3.12) is nonnegative.

Thus, we have checked Claim 3.

Further, for arbitrary real $a < b$, put

$$X_{ab} = \{x \in X \mid \lim_{n} \psi_n(x) < a < b < \lim_{n} \psi_n(x)\}. \quad (3.13)$$

This is a Borel subset. Let us establish the double inequality

$$a \nu(X_{ab}) \geq \int_{X_{ab}} \psi(x) \nu(dx) \geq b \nu(X_{ab}), \quad (3.14)$$

which will imply $\nu(X_{ab}) = 0$.

Indeed, $X_{ab}$ is a $K$-invariant Borel subset of $X$. Let us replace $X$ by $X_{ab}$ and apply Claim 3 to the functions

$$\psi' := (\psi - b)|_{X_{ab}} \quad \text{and} \quad \psi'' := (a - \psi)|_{X_{ab}}. \quad (3.15)$$

By definition (3.13) of $X_{ab}$,

$$E_{\infty}(\psi') = E_{\infty}(\psi'') = X_{ab}, \quad (3.16)$$

whence

$$\int_{X_{ab}} \psi'(x) \nu(dx) \geq 0, \quad \int_{X_{ab}} \psi''(x) \nu(dx) \geq 0, \quad (3.17)$$

which is equivalent to (3.14).

Finally, applying (3.14) to various couples of rational numbers $a < b$, we see that

$$\lim_{n} \psi_n(x) = \lim_{n} \psi_n(x) \quad (3.18)$$

almost everywhere. This completes the proof of Claim 2 and of the theorem. \[ \square \]

We shall apply Theorem 3.2 to $X = H$ and $K = U(\infty)$. Note that the assumptions of Theorem 3.2 are satisfied. Indeed, $H$ is homeomorphic to a separable metric space, because it is essentially a copy of $\mathbb{R}^{\infty}$, and, further, the action $U(\infty) \times H \to H$ is jointly continuous.

For $n = 1, 2, \ldots$, denote by $M_n$ the set of $U(n)$-invariant probability measures that are supported by the $U(n)$-orbits in the space $H(n)$. These measures will be called orbital measures. Since $H(n)$ is contained in $H$, we may view orbital measures as measures on the space $H$.\[ 17 \]
**Theorem 3.3.** For any ergodic measure $M \in \mathcal{M}$, there exists a sequence $\{M_n \in \mathcal{M}_n\}$ of orbital measures such that $M_n \Rightarrow M$ as $n \to \infty$.

**Proof.** We shall write $m_n(B)$ instead of $m_n(x)$; here $B$ is a matrix from $H$. By the first claim of Theorem 3.2, there exists $B \in H$ such that $m_n(B) \Rightarrow M$ as $n \to \infty$. Consider the projections $\theta_n$: $H \to H(n)$ defined at the beginning of §1 and remark that $\theta_n(m_n(B))$ coincides with the orbital measure in $H(n)$ corresponding to the matrix $\theta_n(B)$. Let us take this orbital measure as $M_n$. If $k \leq n$, then $m_n(B)$ and $M_n$ have the same image under $\theta_k$; fixing $k$ and letting $n \to \infty$, we see that the measures $\theta_k(M_n)$ on $H(k)$ weakly converge to the measure $\theta_k(M)$. Now identify $H$ with $\mathbb{R}^\infty$ and recall that on $\mathbb{R}^\infty$ weak convergence of probability measures is equivalent to weak convergence of their finite-dimensional projections (see, e.g., [Bi, chapter 1, §3]). Applying this to our sequence $\{M_n\}$, we see that $M_n \Rightarrow M$. □

It is convenient to analyze weak convergence of measures in terms of characteristic functions. Assume $\nu, \nu_1, \nu_2, \ldots$ are Borel probability measures on $H$ and $f, f_1, f_2, \ldots$ denote their characteristic functions. We shall need the following simple claim, which again is essentially a well-known fact about the space $\mathbb{R}^\infty$.

**Proposition 3.4.** Weak convergence $\nu_n \Rightarrow \nu$ on $H$ is equivalent to uniform convergence $f_n \to f$ on compact subsets in $H(\infty)$. □

(Note that any compact subset in $H(\infty)$ is always contained in $H(n)$ for sufficiently large $n$.)

**Proof.** We again pass to finite-dimensional projections and then use the fact that weak convergence of probability measures on $\mathbb{R}^n$ is equivalent to uniform convergence, on compact sets, of their characteristic functions. □

To obtain Theorem 3.3, we could use, instead of Theorem 3.2, another general result, where we have to specialize $\mathcal{G} = G(\infty)$, $\mathcal{K} = U(\infty)$:

**Theorem 3.5** (Olshanski [O3, Theorem 2.5] and [O5, Theorem 22.10]). Let $\mathcal{G} = \lim_n \mathcal{G}(n)$ be an inductive limit of separable locally compact groups and $\mathcal{K}$ be a subgroup of $\mathcal{G}$. Let $\mathcal{K}(n) = \mathcal{K} \cap \mathcal{G}(n)$, so that $\mathcal{K} = \lim_n \mathcal{K}(n)$.

Then any extreme $\mathcal{K}$-bi-invariant continuous positive definite function $f$ on $\mathcal{G}$, normalized at unity, can be approached, uniformly on compact sets of the group $\mathcal{G}$, by a sequence $\{f_n\}$ of extreme $\mathcal{K}(n)$-bi-invariant continuous positive definite normalized functions on the subgroups $\mathcal{G}(n)$. □

Note, however, that Theorem 3.2, due to its second claim, provides us with more detailed information on the approximation process than Theorem 3.5.

**Remark 3.6.** Note that Claim 2 in the proof of Theorem 3.2 can also be deduced from Doob’s theorem on convergence of (reversed) martingales (see [D, Chapter VII, Theorem 4.2]). Indeed, denote by $\mathcal{B}_n$ the $\sigma$-algebra of all $\mathcal{K}(n)$-invariant Borel subsets of the space $\mathcal{X}$. We have $\mathcal{B}_1 \supseteq \mathcal{B}_2 \supseteq \ldots$, so that, by Doob’s theorem, as $n \to \infty$, the conditional expectation $E(\psi|\mathcal{B}_n)$ of the bounded Borel function $\psi$ converges almost everywhere, with respect to $\nu$, to a function, which is the conditional expectation $E(\psi|\mathcal{B}_\infty)$, where $\mathcal{B}_\infty = \bigcap \mathcal{B}_n$. To derive Claim 2, we only need to show that $E(\psi|\mathcal{B}_n) = \psi_n$ almost everywhere with respect to $\nu$, $n = 1, 2, \ldots$. By definition of conditional expectation, this means that $\psi_n$ is $\mathcal{B}_n$-measurable and
the following condition holds:

$$A \in \mathcal{B}_n \implies \int_A \psi(x) \nu(dx) = \int_A \psi_n(x) \nu(dx). \quad (3.19)$$

Since \(\psi_n\) is a \(K(n)\)-invariant Borel function, it is \(\mathcal{B}_n\)-measurable. Further, since the action of \(K\) on \(\mathcal{X}\) is jointly continuous, the function \((u, x) \mapsto \psi(u \cdot x)\) is continuous on \(K \times \mathcal{X}\), hence is a Borel function on \(K(n) \times \mathcal{A}\). By Fubini’s theorem,

$$\int_{K(n)} \left( \int_A \psi(u \cdot x) \nu(dx) \right) m_n(du) = \int_A \left( \int_{K(n)} \psi(u \cdot x) m_n(du) \right) \nu(dx). \quad (3.20)$$

The left-hand side of (3.20) is equal to \(\int_A \psi(x) \nu(dx)\), because \(A\) and \(\nu\) are \(K(n)\)-invariant, whereas the right-hand side is equal to \(\int_A \psi_n(x) \nu(dx)\) by the definition of \(\psi_n\).

§4. **Main Theorem**

We shall deal with a sequence \(\{M_n \in \mathcal{M}_n\}, n = 1, 2, \ldots\), of orbital measures. By \(f_n\) we denote the characteristic function of \(M_n\); recall that

$$f_n(A) = \int_{\Omega_n} e^{i \text{tr}(AB)} M_n(dB), \quad A \in H(n), \quad (4.1)$$

where \(\Omega_n\) stands for the \(U(n)\)-orbit that carries \(M_n\). Let \(\Lambda(n) = (\lambda_1(n), \ldots, \lambda_n(n))\) be the common spectrum of all the matrices \(B \in \Omega_n\). Then \(\Omega_n\) may be specified as the orbit containing the diagonal matrix \(\text{diag} \Lambda(n)\) with diagonal entries \((\lambda_1(n), \ldots, \lambda_n(n))\). The eigenvalues \(\lambda_1(n), \ldots, \lambda_n(n)\) may be arranged in any order; it will be convenient for us to separate the positive and the negative eigenvalues and to regard \(\Lambda(n)\) as a double sequence formed by positive and negative eigenvalues, respectively, written in decreasing order of their absolute values:

$$\Lambda(n) = (\Lambda'(n), \Lambda''(n)), \quad (4.2)$$

where

$$\Lambda'(n) = (\lambda_1'(n) \geq \lambda_2'(n) \geq \cdots \geq 0),$$

$$\Lambda''(n) = (\lambda_1''(n) \leq \lambda_2''(n) \leq \cdots \leq 0). \quad (4.3)$$

The possible zero values may be included either in \(\Lambda'(n)\) or in \(\Lambda''(n)\); in fact, we prefer to view both \(\Lambda'(n)\) and \(\Lambda''(n)\) as infinite sequences with a finite number of nonzero terms.

**Theorem 4.1** (Main Theorem). Let \(\{M_n \in \mathcal{M}_n\}\) be an infinite sequence of orbital measures defined by a sequence \(\{\Omega_n \subset H(n)\}\) of \(U(n)\)-orbits. For \(n = 1, 2, \ldots\), pick a matrix \(B_n\) from the orbit \(\Omega_n\) and write the collection \(\Lambda(n)\) of its eigenvalues as a double sequence (4.2). (Note that \(\Lambda(n)\) does not depend on the choice of \(B_n\).)

(i) Suppose that the following limits exist:

$$x'_k = \lim_{n \to \infty} \frac{\lambda_k'(n)}{n} \geq 0, \quad x''_k = \lim_{n \to \infty} \frac{\lambda_k''(n)}{n} \leq 0, \quad k = 1, 2, \ldots, \quad (4.4)$$

$$\gamma_1 = \lim_{n \to \infty} \frac{1}{n} \sum_k (\lambda_k'(n) + \lambda_k''(n)) = \lim_{n \to \infty} \frac{1}{n} \text{tr} B_n, \quad (4.5)$$

$$\tilde{\gamma}_2 = \lim_{n \to \infty} \frac{1}{n^2} \sum_k (\lambda_k'(n)^2 + \lambda_k''(n)^2) = \lim_{n \to \infty} \frac{1}{n^2} \text{tr}(B_n^2). \quad (4.6)$$
Then the measures $M_n$ weakly converge to an ergodic measure $M \in \mathcal{M}$ with the multiplicative characteristic function $f$ defined by

$$f(A) = \prod_{a \in \text{Spec}(A)} F(a), \quad A \in H(\infty), \quad (4.7)$$

where $\text{Spec}(A)$ stands for the collection $\{a_1, a_2, \ldots, 0, 0, \ldots\}$ of the eigenvalues of $A$ and

$$F(a) = e^{i\gamma_1 a - \gamma_2 a^2/2} \prod_k \frac{e^{-ix'_k a}}{1 - ix'_k a} \prod_k \frac{e^{-ix''_k a}}{1 - ix''_k a}, \quad a \in \mathbb{R}; \quad (4.8)$$

here $\gamma_1$ is given by (4.5), the parameters $x'_k \geq 0$ and $x''_k \leq 0$ are given by (4.4), and, finally,

$$\gamma_2 = \tilde{\gamma}_2 - \sum_k ((x'_k)^2 + (x''_k)^2), \quad (4.9)$$

where $\tilde{\gamma}_2$ is given by (4.6).

(ii) Conversely, if the measures $M_n$ weakly converge to a probability measure on $H$, then the limits (4.4)–(4.6) do exist.

Comment. It follows from the definition of the parameters $x'_k$, $x''_k$, and $\tilde{\gamma}_2$ that $\sum((x'_k)^2 + (x''_k)^2) \leq \tilde{\gamma}_2$, so that $\gamma_2 \geq 0$.

**Corollary 4.2.** Let $\{M_n \in \mathcal{M}_n\}$ be a sequence of orbital measures that weakly converges to a Borel probability measure $M$ on $H$. Then $M \in \mathcal{M}$. \[ \square \]

**Derivation of Theorem 2.9 from Theorem 4.1.** Let $F \in \mathcal{F}_1$ and let $M \in \mathcal{M}$ be the corresponding ergodic measure. We must show that $F$ is of the form (2.9). By Theorem 3.3, there exists a sequence $\{M_n \in \mathcal{M}_n\}$ that weakly converges to $M$. Next, by claim (ii) of Theorem 4.1, the limits (4.4)–(4.6) exist. Finally, by claim (i) of Theorem 4.1, $F$ is of the form (4.8) that coincides with (2.9) up to a reordering of the points in $x = (x_1, x_2, \ldots)$ only; we recall (see Comment 1 to Proposition 2.8) that we may order the parameters $x_1, x_2, \ldots$ in (2.9) in any way. \[ \square \]

**Outline of the proof of Theorem 4.1.** In §5, we establish a preliminary result – we expand the orbital integral (4.1) into a series of Schur polynomials. The proof of the theorem is given in §6; it is divided into three steps.

In Step 1, we check that under assumptions (4.4)–(4.6),

$$f_n(\text{diag}(a, 0, 0, \ldots)) \to F_{\gamma_1, \gamma_2, x}(a), \quad a \in \mathbb{R}, \quad (4.10)$$

where $\text{diag}(\cdots)$ stands for a diagonal matrix and $F_{\gamma_1, \gamma_2, x}$ is given by (4.8) or, that is the same, by (2.9).

In Step 2, we generalize this to arbitrary diagonal matrices:

$$f_n(\text{diag}(a_1, \ldots, a_k, 0, 0, \ldots)) \to \prod_{i=1}^{k} F_{\gamma_1, \gamma_2, x}(a_i), \quad (a_1, \ldots, a_k) \in \mathbb{R}^k. \quad (4.11)$$

This proves claim (i).

Finally, in Step 3, using a simple trick, we show that conditions (4.4)–(4.6) are indeed necessary (claim (ii)).
Let us emphasize that to prove Theorem 2.9 only, one could avoid Step 2. However, we need this step to characterize the convergent sequences \( \{M_n\} \).

**Remark 4.3.** To check (4.10) or, more generally, (4.11), we consider the Taylor series decomposition at zero for the left-hand side and show that its coefficients tend, as \( n \to \infty \), to the corresponding Taylor coefficients for the right-hand side. Note that these coefficients are nothing but the moments of the measures.

Moreover, it follows from claim (ii) and the proof of (i) that whenever a sequence \( \{M_n \in \mathcal{M}_n\} \) weakly converges to a probability measure \( M \), the moments of \( M_n \) must tend to the moments of \( M \). Thus, in our situation, weak convergence \( M_n \to M \) is always controlled by moments—a fact that is not at all evident a priori.

Such a “moment method” also works in allied classification problems, related to characters of \( U(\infty) \) (see Vershik–Kerov [VK2]) and spherical functions of \( GL(\infty, \mathbb{C}) \) (see Nessonov [N]). We conjecture that it can be used for all families of classical symmetric spaces. (About spherical functions on infinite-dimensional symmetric spaces, see Olshanski˘ı [O5] and Pickrell [Pi1].)

**Remark 4.4.** Assume that in the spectrum \( \Lambda(n) = (\lambda_1(n), \ldots, \lambda_n(n)) \) there are at most \( k \) nonzero eigenvalues, where \( k \) does not depend on \( n \). Then we can prove claim (i) of Theorem 4.1 directly, i.e., without using moments. Indeed, let

\[
\lim_{n \to \infty} \frac{\lambda_i(n)}{n} = x_i, \quad 1 \leq i \leq n, \quad \lambda_i(n) = 0, \quad i > k. \tag{4.12}
\]

Then the measure \( M_n \) can be viewed as the image of the normalized Haar measure of the group \( U(n) \) under the mapping

\[
U(n) \ni u \mapsto uu^* \text{ diag}(\lambda_1(n), \ldots, \lambda_k(n), 0, \ldots, 0) = B \in H(n) \subset H. \tag{4.13}
\]

The matrix \( B \) can be rewritten as follows:

\[
B = (\Xi(n))^* X(n) \Xi(n), \tag{4.14}
\]

where \( \Xi(n) \) denotes the \( k \times n \) matrix formed by the first \( k \) rows of the matrix \( u \in U(n) \) multiplied by the scalar \( \sqrt{n} \), and

\[
X(n) = \text{ diag} \left( \frac{\lambda_1(n)}{n}, \ldots, \frac{\lambda_k(n)}{n} \right). \tag{4.15}
\]

Now let \( n \to \infty \). Then \( X(n) \to X : = \text{ diag}(x_1, \ldots, x_k) \), because of (4.12). Further, fix \( m = 1, 2, \ldots \) and regard the \( k \times m \) matrix formed by the first \( m \) columns of \( \Xi(n) \) as a random matrix variable (with respect to the Haar measure of \( U(n) \)). It is well-known that the limit distribution of this matrix is given by the Gaussian product measure \( \omega^{k \times m} \) (product of \( km \) copies of \( \omega \), cf. (2.12)), where \( \omega \) stands for the Gaussian measure on \( \mathbb{C} \) specified in Corollary 2.7. This fact is proved, e.g., in Olshanski [O5, Lemma 5.3]. It follows that the measures \( M_n \) weakly converge, as \( n \to \infty \), to the measure \( M_0; x_1, \ldots, x_k \in \mathcal{M} \) defined in Remark 2.10.

**Remark 4.5.** In general, for a convergent sequence of orbital measures, the eigenvalues in the spectrum \( \Lambda(n) \) must grow linearly in \( n \) as \( n \to \infty \). But if the limiting ergodic measure is Gaussian (that is, \( x \equiv 0 \)), then the order of growth of the eigenvalues becomes equal to \( \sqrt{n} \). Example: let

\[
\Lambda(n) = (\sqrt{\gamma n}, \ldots, \sqrt{\gamma n}, -\sqrt{\gamma n}, \ldots, -\sqrt{\gamma n}),
\]

then the limiting measure is the Gaussian measure \( M_{0, \gamma, 0} \).
§5. Expanding spherical functions into series of Schur polynomials

In this section, we fix \( n = 1, 2, \ldots \). Let \( (a_1, \ldots, a_n) \in \mathbb{C}^n \), \( (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n \), and suppose
\[
A = \text{diag}(a_1, \ldots, a_n), \quad \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)
\]
are the corresponding diagonal matrices. We fix \( \Lambda \) and deal with the \( U(n) \)-orbit \( \Omega \subset H(n) \) passing through \( \Lambda \). Let \( f_\Lambda \) stand for the characteristic function of the invariant probability measure supported by \( \Omega \). Then
\[
f_\Lambda(A) = \int_{U(n)} e^{i \text{tr}(Au \Lambda u^{-1})} du,
\]
where \( du \) is the normalized Haar measure on \( U(n) \).

Since \( f_\Lambda \) is an entire function of \( (a_1, \ldots, a_n) \in \mathbb{C}^n \), it admits an everywhere convergent Taylor series expansion. But since \( f_\Lambda(a_1, \ldots, a_n) \) is also symmetric in \( a_1, \ldots, a_n \), it is more convenient to rewrite this Taylor series as a series of Schur polynomials \( s_\mu \),
\[
f_\Lambda(A) = \sum_\mu c_\mu s_\mu(a_1, \ldots, a_n),
\]
where \( \mu \) ranges over the set of all Young diagrams with at most \( n \) rows. (About Schur polynomials, see, e.g., Macdonald [M].)

We shall use some standard notation concerning Young diagrams: \( \mu \vdash m \) means that \( \mu \) is a partition of \( m \) (i.e., \( m \) equals \( |\mu| \), the number of boxes in \( \mu \)), \( \ell(\mu) \) is the number of (nonzero) rows in \( \mu \), \( (p, q) \in \mu \) denotes the box of \( \mu \) lying on the intersection of \( p \)th row and \( q \)th column, \( \dim \mu \) is the dimension of the irreducible representation of the symmetric group \( S(m) \), \( m = |\mu| \), that corresponds to the diagram \( \mu \).

**Theorem 5.1.** The coefficients in (5.3) are given by the following formula:
\[
c_\mu = \prod_{(p,q) \in \mu} \frac{1}{n + q - p} \cdot s_\mu(i \lambda_1, \ldots, i \lambda_n) = i^{|\mu|} \prod_{(p,q) \in \mu} \frac{1}{n + q - p} \cdot s_\mu(\lambda_1, \ldots, \lambda_n).
\]

**Proof.** Step 1. Let \( \pi \) be an irreducible representation of \( U(n) \), \( \text{Dim} \pi \) be its dimension, and \( \chi \) be its normalized character:
\[
\chi(g) = \frac{\text{tr} \pi(g)}{\text{Dim} \pi}, \quad g \in U(n).
\]
It is well known that \( \chi \) satisfies the following functional equation:
\[
\int_{U(n)} \chi(guhu^{-1}) du = \chi(g) \chi(h), \quad g, h \in U(n).
\]

Now let us take as \( \pi \) the irreducible polynomial representation with highest weight \( \mu = (\mu_1, \ldots, \mu_n) \). Then
\[
\chi(g) = \frac{s_\mu(\text{Spec}(g))}{\text{Dim}_n \mu}, \quad g \in U(n),
\]

(5.6)
where, given an $n \times n$ matrix $g$, Spec$(g)$ stands for the collection $(z_1, \ldots, z_n)$ of its eigenvalues, and Dim$_n \mu$ denotes the dimension of the representation $\pi$ of the group $U(n)$.

Substituting (5.6) into (5.5), we obtain

$$\int_{U(n)} s_\mu(\text{Spec}(guhu^{-1})) \, du = \frac{1}{\text{Dim}_n \mu} s_\mu(\text{Spec}(g)) s_\mu(\text{Spec}(h)).$$  (5.7)

By analytic continuation, this formula holds for arbitrary $n \times n$ complex matrices $g$ and $h$.

**Step 2.** Let us come back to the orbital integral (5.2), where $A$ and $\Lambda$ are given by (5.1). We may write

$$e^{i \text{tr}(Au\Lambda u^{-1})} = \sum_{m \geq 0} \frac{1}{m!} p_1^m(\text{Spec}(iAu\Lambda u^{-1})),$$  (5.8)

where $p_1(z_1, \ldots, z_n) = z_1 + \cdots + z_n$ is the first power sum.

Recall the well-known identity (see, e.g., Macdonald [M, chapter I, (7.8)]):

$$p_1^m(z_1, \ldots, z_n) = \sum_{\mu \vdash m, \ell(\mu) \leq n} \dim \mu \cdot s_\mu(z_1, \ldots, z_n).$$  (5.9)

Using it, we may rewrite (5.8) as follows:

$$e^{i \text{tr}(Au\Lambda u^{-1})} = \sum_{m \geq 0} \sum_{\mu \vdash m, \ell(\mu) \leq n} \frac{\dim \mu}{m!} s_\mu(\text{Spec}(iAu\Lambda u^{-1})).$$  (5.10)

Integrating both sides of (5.10) over $u \in U(n)$ and applying the functional equation (5.7) with $g = A$, $h = i\Lambda$, we obtain

$$\int_{U(n)} e^{i \text{tr}(Au\Lambda u^{-1})} \, du = \sum_{m \geq 0} \sum_{\mu \vdash m, \ell(\mu) \leq n} \frac{\dim \mu}{m! \text{Dim}_n \mu} s_\mu(a_1, \ldots, a_n) s_\mu(i\lambda_1, \ldots, i\lambda_n).$$  (5.11)

**Step 3.** For a box $(p,q) \in \mu$, let $h(p,q)$ denote the corresponding hook length. Recall the well-known formulas

$$\dim \mu = m! \prod_{(p,q) \in \mu} \frac{1}{h(p,q)}, \quad \text{Dim}_n \mu = \prod_{(p,q) \in \mu} \frac{n+q-p}{h(p,q)},$$

see, e.g., [M, Chapter I, §3, Example 4]. It follows

$$\frac{\dim \mu}{m! \text{Dim}_n \mu} = \prod_{(p,q) \in \mu} \frac{1}{n+q-p}.$$

Substituting this into (5.11), we obtain (5.4). □

(After work on this paper was completed, we learned that the argument presented above was used much earlier by James, see his survey paper [J, (60)].)
Corollary 5.2. The orbital integral (5.2) is given by the following explicit formula:

\[ f_{\lambda}(A) = \int_{U(n)} e^{i \text{tr}(AuA^{-1})} \, du = \frac{(n-1)! \cdots 0! \det [e^{i a_j \lambda_k}]_{j,k=1}^n}{V(a_1, \ldots, a_n) V(i \lambda_1, \ldots, i \lambda_n)}. \] (5.12)

Here \( V(\cdots) \) is the Vandermonde determinant,

\[ V(z_1, \ldots, z_n) = \prod_{1 \leq j < k \leq n} (z_j - z_k), \]

and the coordinates \( a_1, \ldots, a_n \), as well as \( \lambda_1, \ldots, \lambda_n \), are assumed to be pairwise distinct.

**Proof.** The statement of Theorem 5.1 may be rewritten as follows:

\[ f_{\lambda}(A) = (n-1)! \cdots 0! \sum_{\ell(\mu) \leq n} s_{\mu}(a_1, \ldots, a_n) s_{\mu}(i \lambda_1, \ldots, i \lambda_n) \frac{(\mu_1 + n - 1)! (\mu_2 + n - 2)! \cdots \mu_n!}{(\mu_1 + n - 1)! (\mu_2 + n - 2)! \cdots \mu_n!}. \]

Using the determinant formula for Schur polynomials, we obtain

\[
\frac{V(a_1, \ldots, a_n) V(i \lambda_1, \ldots, i \lambda_n)}{(n-1)! \cdots 0!} f_{\lambda}(A) \\
= \sum_{\mu} \sum_{\ell(\mu) \leq n} \frac{\det [a_{j+k-n-k}^{\mu}]_{j,k=1}^n \cdot \det [(i \lambda_j)^{\mu_k+n-k}]_{j,k=1}^n}{(\mu_1 + n - 1)! (\mu_2 + n - 2)! \cdots \mu_n!} \\
= \sum_{m_1 > \cdots > m_n \geq 0} \frac{\det [a_{j+k-n-k}^{m_k}]_{j,k=1}^n \cdot \det [(i \lambda_j)^{m_k}]_{j,k=1}^n}{m_1! m_2! \cdots m_n!}.
\]

Since the numerator in the latter expression is symmetric in \( m_1, \ldots, m_n \) and vanishes if some of these numbers are equal, we may drop the assumption \( m_1 > \cdots > m_n \). Then we obtain

\[
\frac{V(a_1, \ldots, a_n) V(i \lambda_1, \ldots, i \lambda_n)}{(n-1)! \cdots 0!} f_{\lambda}(A) \\
= \frac{1}{n!} \sum_{m_1, \ldots, m_n \geq 0} \frac{\det [a_{j+k-n-k}^{m_k}]_{j,k=1}^n \cdot \det [(i \lambda_j)^{m_k}]_{j,k=1}^n}{m_1! m_2! \cdots m_n!} \\
= \frac{1}{n!} \sum_{m_1, \ldots, m_n \geq 0} \frac{1}{m_1! \cdots m_n!} \sum_{\sigma, \tau} \text{sgn}(\sigma) \text{sgn}(\tau) (a_{\sigma_1} \cdot i \lambda_{\tau_1})^{m_1} \cdots (a_{\sigma_n} \cdot i \lambda_{\tau_n})^{m_n},
\]

where \( \sigma = (\sigma_1, \ldots, \sigma_n) \) and \( \tau = (\tau_1, \ldots, \tau_n) \) range over all the permutations of \( 1, \ldots, n \). Changing the order of the summation, we obtain

\[
\frac{1}{n!} \sum_{\sigma, \tau} \text{sgn}(\sigma) \text{sgn}(\tau) e^{i a_{\sigma_1} \lambda_{\tau_1}} \cdots e^{i a_{\sigma_n} \lambda_{\tau_n}} = \det [e^{i a_{j_k} \lambda_k}]_{j,k=1}^n,
\]

which completes the proof. \( \square \)

**Remark 5.3.** Formula (5.12) is well known and can be proved in a number of different ways. For instance, it can be obtained from the Gelfand–Naimark calculation of spherical functions on \( GL(n, \mathbb{C}) \) (see Gelfand–Naimark [GN]) by a passage to the limit. Another way consists in using the radial parts of invariant differential operators. Yet another method can be found in [BGV, Section 7.5] (we are grateful to Michel Duflo for the latter reference). Note also that writing the above calculations in reverse order, we can deduce Theorem 5.1 from formula (5.12).
Corollary 5.4. Let us substitute $A = \text{diag}(a,0,\ldots,0)$ into (5.2). Then

$$f_\Lambda(\text{diag}(a,0,\ldots,0)) = \int_{\text{U}(n)} e^{ia(Au^{-1})\Lambda} \, du = \sum_{m \geq 0} \frac{h_m(i\lambda_1,\ldots,i\lambda_n)}{n(n+1)\cdots(n+m-1)} a^m,$$

where $h_m$ is the $m$th complete symmetric function.

Proof. Indeed, note that $s_\mu(a,0,\ldots,0)$ vanishes unless $\mu = (m)$, where $m = 0,1,\ldots$. Thus the summation is really taken over the diagrams $\mu = (m)$ only. For these diagrams, $s_\mu$ reduces to $h_m$ and the product $\prod_{(p,q) \in \mu} (n+q-p)$ turns into $n(n+1)\cdots(n+m-1)$, so that we obtain (5.13). □

Remark 5.5. The expansion (5.3) of theorem 5.1 may be written in the following form, which emphasizes the symmetry between $A$ and $\Lambda$,

$$f_\Lambda(A) = \sum_{\mu} \frac{1}{i^{\|\mu\|}} \sum_{(p,q) \in \mu} \frac{1}{n+q-p} s_\mu(\lambda_1,\ldots,\lambda_n) s_\mu(a_1,\ldots,a_n).$$

The symmetry can already be seen from (5.2).

§ 6. PROOF OF THEOREM 4.1

We start with the following simple claim.7

Proposition 6.1. Let $f,f_1,f_2,\ldots$ be analytic functions on $\mathbb{R}^k$ satisfying the following conditions:

(i) $f_1,f_2,\ldots$ are positive definite and normalized at the origin;

(ii) the Taylor coefficients of $f_n$ at the origin tend, as $n \to \infty$, to the corresponding Taylor coefficients of $f$;

(iii) The Taylor decomposition of $f_n$ converges absolutely and uniformly on $n$ in a neighborhood of the origin that does not depend on $n$.

Then $f_n \to f$ uniformly on compact subsets of $\mathbb{R}^k$.

Proof. This is a standard exercise. It is clear that $f_n \to f$ in a neighborhood of the origin. Consider the probability measures $\nu_1,\nu_2,\ldots$ on $\mathbb{R}^k$ that correspond to the functions $f_1,f_2,\ldots$ by Bochner’s theorem. By Paul Lévy’s classical continuity theorem (see, e.g., Shiryaev’s textbook [Shi, chapter III, §3]), the measures $\nu_n$ weakly converge to a probability measure $\nu$. Let $\tilde{f}$ stand for the characteristic function of $\nu$. Then $f_n \to \tilde{f}$ uniformly on compact sets of $\mathbb{R}^k$. Further, (ii) and (iii) imply that $\tilde{f}$ is analytic in a neighborhood of the origin, and this implies that $\tilde{f}$ is analytic on the whole space $\mathbb{R}^k$. Therefore, $\tilde{f} = f$ and our claim follows. □

We proceed to prove the theorem. The notation of §4 is maintained.

Step 1. Let us fix a sequence $\{\Lambda(n)\}$ such that the limits (4.4), (4.5), and (4.6) exist. Let us abbreviate

$$f_n(a) = f_n(\text{diag}(a,0,0,\ldots)), \quad a \in \mathbb{R}.$$
Note that \( f_n(a) \) is the characteristic function of the probability measure \( \nu_n := M_n^{(1)} \), the image of the measure \( M_n \) under the projection \( \theta_1: H(n) \to H(1) = \mathbb{R} \).

The purpose of this step is to prove that
\[
\lim_{n \to \infty} f_n(a) = F_{\gamma_1, \gamma_2, x}(a), \quad a \in \mathbb{R},
\]
uniformly on bounded sets in \( \mathbb{R} \). (Recall that \( \gamma_1 \) is given by (4.5), \( \gamma_2 \) is given by (4.6) and (4.9), and \( x = (x'_1, x'_2, \ldots; x''_1, x''_2, \ldots) \), where the latter parameters are defined by (4.4). The function \( F_{\gamma_1, \gamma_2, x} \) is given by (4.8) or, that is the same, by (2.9).)

To do this, let us expand both the sides of (6.2) into Taylor series:
\[
f_n(a) = \sum_{m \geq 0} c_m^{(n)} a^m, \quad \quad \quad (6.3)
\]
\[
F_{\gamma_1, \gamma_2, x}(a) = \sum_{m \geq 0} c_m^{(\infty)} a^m. \quad \quad \quad (6.4)
\]

By Proposition 6.1, it is enough to verify the following two claims: First,
\[
\lim_{n \to \infty} c_m^{(n)} = c_m^{(\infty)}, \quad m = 0, 1, 2, \ldots \quad (6.5)
\]
Second, the series (6.3) converges absolutely and uniformly on \( n \) in a sufficiently small neighborhood of the origin, i.e.,
\[
|c_m^{(n)}| \leq C_1 C_2^m, \quad m = 0, 1, 2, \ldots, \quad (6.6)
\]
where the constants \( C_1 > 0, C_2 > 0 \) do not depend on \( n \).

By (5.13), we have
\[
c_m^{(n)} = \frac{h_m(i\lambda_1(n), \ldots, i\lambda_n(n))}{n(n+1) \cdots (n+m-1)}
\]
\[
= \frac{n^m}{n(n+1) \cdots (n+m-1)} h_m \left( \frac{i\lambda_1(n)}{n}, \ldots, \frac{i\lambda_n(n)}{n} \right). \quad (6.7)
\]

It is clear that in both claims, (6.5) and (6.6), we may replace \( c_m^{(n)} \) by
\[
\tilde{c}_m^{(n)} := h_m \left( \frac{i\lambda_1(n)}{n}, \ldots, \frac{i\lambda_n(n)}{n} \right). \quad (6.8)
\]

Further, instead of the series \( \sum c_m^{(n)} a^m \) and \( \sum c_m^{(\infty)} a^m \), it is more convenient to deal with their logarithms \( \ln(\sum \tilde{c}_m^{(n)} a^m) \) and \( \ln(\sum c_m^{(\infty)} a^m) \), respectively.

Now recall a well-known identity from the theory of symmetric functions,
\[
\sum_{m \geq 0} h_m(\cdot) a^m = \exp \left( \sum_{m \geq 1} p_m(\cdot) \frac{a^m}{m} \right), \quad (6.9)
\]
where \( p_m(\cdot) \) are the power sum symmetric functions (see, e.g., [M]). It follows from (6.8) and (6.9) that
\[
\ln \left( \sum_{m \geq 0} \tilde{c}_m^{(n)} a^m \right) = \sum_{m \geq 1} p_m \left( \frac{i\lambda_1(n)}{n}, \ldots, \frac{i\lambda_n(n)}{n} \right) \frac{a^m}{m}. \quad (6.10)
\]
On the other hand, by definition (2.9) of the function \( F_{\gamma_1, \gamma_2, x} \),
\[
\ln \left( \sum_{m \geq 0} e_m(\infty) a^m \right) = i\gamma_1 a - \frac{1}{2} \gamma_2 a^2 + \sum_{m \geq 2} p_m(ix) \frac{a^m}{m},
\]
where
\[
p_m(ix) = i^m p_m(x) = i^m \sum_{k=1}^{\infty} ((x_k')^m + (x_k'')^m).
\]

Note that the sum in the right-hand side of (6.12) is convergent, because, due to assumption (4.6), we have \( p_2(x) < \infty \) (see the Comment to Theorem 4.1).

Thus, our first claim reduces to the existence of the limits
\[
\lim_{n \to \infty} p_1 \left( \frac{i\lambda_1(n)}{n}, \ldots, \frac{i\lambda_n(n)}{n} \right) = i\gamma_1, \quad (6.13)
\]
\[
\lim_{n \to \infty} p_2 \left( \frac{i\lambda_1(n)}{n}, \ldots, \frac{i\lambda_n(n)}{n} \right) = -\gamma_2 + p_2(ix), \quad (6.14)
\]
\[
\lim_{n \to \infty} p_m \left( \frac{i\lambda_1(n)}{n}, \ldots, \frac{i\lambda_n(n)}{n} \right) = p_m(ix), \quad m \geq 3, (6.15)
\]

and our second claim reduces to an estimate of the form
\[
\left| p_m \left( \frac{i\lambda_1(n)}{n}, \ldots, \frac{i\lambda_n(n)}{n} \right) \right| \leq C_1'(C_2')^m, \quad m \geq 3, (6.16)
\]
where \( C_1' > 0, C_2' > 0 \) are some constants not depending on \( n \).

Clearly, (6.13) is just the assumption (4.5). Further, (6.14) immediately follows from (4.6) and (4.9). Indeed, by (4.6),
\[
\lim_{n \to \infty} p_2 \left( \frac{i\lambda_1(n)}{n}, \ldots, \frac{i\lambda_n(n)}{n} \right) = -\lim_{n \to \infty} p_2 \left( \frac{\lambda_1(n)}{n}, \ldots, \frac{\lambda_n(n)}{n} \right) = -\tilde{\gamma}_2.
\]

Now, by (4.9),
\[
-\tilde{\gamma}_2 = -\gamma_2 - \sum_k ((x_k')^2 + (x_k'')^2) = -\gamma_2 + p_2(ix).
\]

Let us verify (6.15). Using the notation (4.2), (4.3), we have
\[
p_m \left( \frac{i\lambda_1(n)}{n}, \ldots, \frac{i\lambda_n(n)}{n} \right) = \sum_{r \geq 1} \left( \frac{i\lambda_r(n)}{n} \right)^m + \sum_{r \geq 1} \left( \frac{i\lambda_r''(n)}{n} \right)^m, \quad (6.17)
\]
\[
p_m(ix) = \sum_{r \geq 1} (ix_r')^m + \sum_{r \geq 1} (ix_r'')^m. \quad (6.18)
\]

To deduce (6.15) from (4.4), it suffices to show that both sums in the right-hand side of (6.17) converge absolutely and uniformly on \( n \). Let us examine the first
sum (for the second one the reasoning is just the same). Since $m \geq 3$ and $\lambda'_1(n) \geq \lambda'_2(n) \geq \cdots$, we have for $N = 1, 2, \ldots$

$$\left| \sum_{r \geq N} \left( \frac{i\lambda'_r(n)}{n} \right)^m \right| = \sum_{r \geq N} \left( \frac{\lambda'_r(n)}{n} \right)^m \leq \left( \frac{\lambda'_N(n)}{n} \right)^{m-2} \sum_{r \geq N} \left( \frac{\lambda'_r(n)}{n} \right)^2 \leq \frac{\lambda'_N(n)}{n} p_2 \left( \frac{\lambda_1(n)}{n}, \ldots, \frac{\lambda_n(n)}{n} \right). \quad (6.19)$$

By (4.6), $p_2(\lambda_1(n)/n, \ldots, \lambda_n(n)/n)$ remains bounded as $n \to \infty$. Finally, since $x'_N \to 0$ as $N \to \infty$ and since $\lambda'_N(n)/n \to x'_N$ as $n \to \infty$, the value of $\lambda'_N(n)/n$ may be made arbitrarily small provided first $N$ and then $n$ are chosen large enough.

Let us verify (6.16). For $m \geq 3$

$$\left| p_m \left( \frac{i\lambda_1(n)}{n}, \ldots, \frac{i\lambda_n(n)}{n} \right) \right| \leq p_2 \left( \frac{\lambda_1(n)}{n}, \ldots, \frac{\lambda_n(n)}{n} \right) \cdot \sup_{1 \leq r \leq n} \left( \frac{\lambda_r(n)}{n} \right)^{m-2} \leq p_2 \left( \frac{\lambda_1(n)}{n}, \ldots, \frac{\lambda_n(n)}{n} \right) p_2 \left( \frac{\lambda_1(n)}{n}, \ldots, \frac{\lambda_n(n)}{n} \right)^{(m-2)/2}. \quad (6.20)$$

Since $p_2(\lambda_1(n)/n, \ldots, \lambda_n(n)/n)$ remains bounded as $n \to \infty$, we obtain (6.16).

This completes Step 1.

**Step 2.** The purpose of this step is to prove that under the same assumptions as in Step 1, we have a more general result: for any fixed $k = 1, 2, \ldots$,

$$\lim_{n \to \infty} f_n(\text{diag} (a_1, \ldots, a_k, 0, \ldots, 0)) = \prod_{p=1}^{k} F_{\gamma_1, \gamma_2, x}(a_p) \quad (6.21)$$

uniformly on bounded subsets in $\mathbb{R}^k$.

Let us abbreviate

$$f_n(a_1, \ldots, a_k) = f_n(\text{diag} (a_1, \ldots, a_k, 0, \ldots, 0)) \quad (6.22)$$

and note that $f_n(a_1, \ldots, a_k)$ is again the characteristic function of some probability measure $\nu_n$ on $\mathbb{R}^k$. Namely, $\nu_n$ is the radial part of the measure $M_n^{(k)} = \theta_k(M_n)$ on $H(k)$ with respect to the projection

$$H(k) \ni A \mapsto \text{Spec} (A) \in \mathbb{R}^k. \quad (6.23)$$

Our reasoning will be similar to that of Step 1. We expand both sides of (6.21) into multidimensional Taylor series. However, since these are symmetric functions of $(a_1, \ldots, a_k)$, we prefer to rewrite the Taylor series as series of Schur polynomials $s_\mu(a_1, \ldots, a_k)$, where $\mu$ ranges over the set of all Young diagrams with $\ell(\cdot) \leq k$:

$$f_n(a_1, \ldots, a_k) = \sum_{\mu} c_\mu^{(n)} s_\mu(a_1, \ldots, a_k), \quad (6.24)$$

$$\prod_{p=1}^{k} F_{\gamma_1, \gamma_2, x}(a_p) = \sum_{\mu} c_\mu^{(\infty)} s_\mu(a_1, \ldots, a_k). \quad (6.25)$$
As in Step 1, applying Proposition 6.1, we reduce our problem to verifying the following two claims:

First,

$$\lim_{n \to \infty} c^{(n)}_\mu = c^{(\infty)}_\mu \quad \text{for any } \mu \text{ with } \ell(\mu) \leq k,$$

(6.26)

and, second,

$$|c^{(n)}_\mu| \leq C_1 C_2^{[\mu]},$$

(6.27)

where $C_1 > 0$, $C_2 > 0$ are some constants not depending on $n$.

By (5.4),

$$c^{(n)}_\mu = \prod_{(p,q) \in \mu} \frac{1}{n + q - p} \cdot s_\mu(i\lambda_1(n), \ldots, i\lambda_n(n))$$

$$= \prod_{(p,q) \in \mu} \frac{n}{n + q - p} \cdot s_\mu\left(\frac{i\lambda_1(n)}{n}, \ldots, \frac{i\lambda_n(n)}{n}\right).$$

(6.28)

Thus, in both claims, (6.26) and (6.27), we may replace $c^{(n)}_\mu$ by

$$\tilde{c}^{(n)}_\mu := s_\mu\left(\frac{i\lambda_1(n)}{n}, \ldots, \frac{i\lambda_n(n)}{n}\right).$$

(6.29)

Let us prove (6.26) with $c^{(n)}_\mu$ replaced by $\tilde{c}^{(n)}_\mu$.

Recall the Jacobi–Trudi identity expressing the Schur functions in terms of the complete symmetric functions (see, e.g., [M, Chapter I, (3.4)]):

$$s_\mu = \det [h_{\mu_i-i+j}]^k_{i,j=1}, \quad \ell(\mu) \leq k.$$  

(6.30)

It follows from (6.8), (6.29), and (6.30) that

$$\tilde{c}^{(n)}_\mu = \det [\tilde{c}^{(n)}_{\mu_i-i+j}]^k_{i,j=1}. $$

(6.31)

By Step 1, $\tilde{c}^{(n)}_m \to c^{(\infty)}_m$ as $n \to \infty$, so that

$$\lim_{n \to \infty} \tilde{c}^{(n)}_\mu = \det [c^{(\infty)}_{\mu_i-i+j}]^k_{i,j=1}. $$

(6.32)

Further, it is well known that for an arbitrary formal series $\sum_{m \geq 0} c_m a^m$ with $c_0 = 1$, we have

$$\prod_{p=1}^k \left(\sum_{m=0}^\infty c_m a_p^m\right) = \sum_\mu \det [c^{(\infty)}_{\mu_i-i+j}]^k_{i,j=1} s_\mu(a_1, \ldots, a_k).$$

(6.33)

Applying (6.33) to the left-hand side of (6.25), we conclude that

$$c^{(\infty)}_\mu = \det [c^{(\infty)}_{\mu_i-i+j}]^k_{i,j=1} = \lim_{n \to \infty} \tilde{c}^{(n)}_\mu.$$  

(6.34)

Thus, we have verified the first claim. As for the second one, an estimate of type (6.27) for $\tilde{c}^{(n)}_\mu$ follows at once from (6.31) and from the estimate (6.6) for $\tilde{c}^{(n)}_m$, proved in Step 1.
This completes Step 2.

It follows that \( f_n \to f \) (where \( f \) is given by (4.7)) uniformly on compact subsets of \( H(\infty) \). Clearly, \( f \) is a continuous positive definite normalized function on \( H(\infty) \). By Proposition 1.1, it is the characteristic function of a \( U(\infty) \)-invariant Borel probability measure \( M \) on the space \( H \), invariant under the action of \( U(\infty) \). Then, by Proposition 3.4, \( M \), weakly converges to \( M \). Since \( f \) is multiplicative, \( M \) is ergodic. Thus, we have verified claim (i) of Theorem 4.1.

**Step 3.** Let us fix a sequence \( \{\Lambda(n)\} \), where \( \Lambda(n) = (\lambda_1(n), \ldots, \lambda_n(n)) \), and let \( f_n(a), a \in \mathbb{R} \), be defined as in Step 1. Let us assume that
\[
\lim_{n \to \infty} f_n(a) = f(a), \quad a \in \mathbb{R},
\]
uniformly on bounded subsets in \( \mathbb{R} \), where \( f(a) \) is a function on \( \mathbb{R} \) (of course, \( f \) is automatically continuous). We shall prove that then \( \{\Lambda(n)\} \) must satisfy the assumptions (4.4)–(4.6) of Theorem 4.1(i).

Suppose first that
\[
\sup_n \left\{ p_2 \left( \frac{\lambda_1(n)}{n}, \ldots, \frac{\lambda_n(n)}{n} \right) + \left( p_1 \left( \frac{\lambda_1(n)}{n}, \ldots, \frac{\lambda_n(n)}{n} \right) \right)^2 \right\} < \infty.
\]
Then, given an infinite subset \( N \subseteq \{1, 2, \ldots\} \), there exists a possibly smaller infinite subset \( N' \subseteq N \) such that the limits (4.4), (4.5), and (4.6) exist provided \( n \) goes to infinity inside \( N' \). Then, by Step 1, \( f_n \to F_{\gamma_1, \gamma_2, x} \), as \( n \to \infty \) inside \( N' \), so that \( F_{\gamma_1, \gamma_2, x} = f \). For any other \( N \) and \( N' \), the parameters \( \gamma_1, \gamma_2, \) and \( x \) will be the same, because they are uniquely determined by the function itself, see Comment 4 after Proposition 2.8. It follows that the limits (4.4)–(4.6) exist as \( n \) ranges over the set of all natural numbers.

Suppose now that (6.36) does not hold. We shall show that this leads to a contradiction with the initial assumption (6.35).

Indeed, since the expression \( \{\ldots\} \) in (6.36) is a homogeneous function of \( \Lambda(n) \), we can choose an infinite subset \( N \subseteq \{1, 2, \ldots\} \) and a sequence of positive numbers \( \{\varepsilon_n \mid n \in N\} \) such that \( \lim_{n \in N} \varepsilon_n = 0 \) and
\[
\lim_{n \in N} \left\{ p_2 \left( \varepsilon_n \frac{\lambda_1(n)}{n}, \ldots, \varepsilon_n \frac{\lambda_n(n)}{n} \right) + \left( p_1 \left( \varepsilon_n \frac{\lambda_1(n)}{n}, \ldots, \varepsilon_n \frac{\lambda_n(n)}{n} \right) \right)^2 \right\} = 1.
\]
Then, replacing \( n \) by a smaller infinite subset \( N' \), we can arrange so that for the sequence \( \{\varepsilon_n \Lambda(n)\} \), the limits (4.4)–(4.6) will exist provided \( n \) goes to infinity inside \( N' \). Moreover, at least one of the corresponding parameters \( \gamma_1, \gamma_2 \) will be nonzero.

Note that the effect of multiplying \( \Lambda(n) \) by \( \varepsilon_n \) is the same as that of multiplying \( a \) by \( \varepsilon_n \). Thus, by Step 1,
\[
\lim_{n \in N'} f_n(\varepsilon_n a) = F_{\gamma_1, \gamma_2, x}, \quad a \in \mathbb{R}.
\]
Since at least one of the parameters \( \gamma_1, \gamma_2 \) of the function \( F_{\gamma_1, \gamma_2, x} \) is nonzero, it follows from the definition of this function that it is not equal identically to 1. But since \( F_{\gamma_1, \gamma_2, x} \) is analytic, the same is true in an arbitrarily small neighborhood of the point \( a = 0 \). Then, comparing (6.38) with (6.35), we arrive at a contradiction.

Thus, we have verified claim (ii) of Theorem 4.1. □
Remark 6.2. Note that the estimates (6.6) and (6.16) are not necessary to assert the convergence of the functions $f_n$. Indeed, Proposition 6.1 may be replaced by the following stronger claim:

Let $f_1, f_2, \ldots$ and $f$ be smooth positive definite functions on $\mathbb{R}^k$, normalized at the origin. Expand them into Taylor series at the origin and assume that each Taylor coefficient of $f_n$ tends, as $n \to \infty$, to the corresponding coefficient of $f$. Finally, assume that the moment problem defined by the coefficients of $f$ has a unique solution (the latter condition is satisfied, e.g., if $f$ is analytic).

Then the sequence $(f_n)$ converges to $f$ uniformly on compact subsets of $\mathbb{R}^n$.

By virtue of this claim, the verification of the uniform convergence of the Taylor expansions may be omitted.

§7. Total positivity

Definition 7.1. Let $\varphi(t)$ be a real nonnegative measurable function on $\mathbb{R}$. Then $\varphi$ is said to be a totally positive function if for $n = 1, 2, \ldots$

$$\det[\varphi(t_i - s_j)]_{i,j=1}^n \geq 0 \quad \text{for any } t_1 < \cdots < t_n \text{ and } s_1 < \cdots < s_n.$$  \hspace{1cm} (7.1)

It will be convenient for us to include in the definition the following additional assumption: $\varphi$ is summable and $\int \varphi(t) \, dt = 1$, i.e., $\varphi(t) \, dt$ is a probability measure on $\mathbb{R}$. (In [S2], functions satisfying both conditions are called Pólya frequency functions, the second condition is in fact not restrictive, see [S2, lemma 4].)

Proposition 7.2 (Schoenberg [S2, p. 341, Lemma 5]). The set of totally positive functions is stable under convolution.

Proof. For two summable functions $\varphi$ and $\psi$, the convolution $\varphi * \psi$ is correctly defined and the following formula is readily verified:

$$\det[(\varphi * \psi)(t_i - s_j)]_{i,j=1}^n = \frac{1}{n!} \int_{\mathbb{R}^n} \det[\varphi(t_i - u_k)]_{i,k=1}^n \cdot \det[\psi(u_k - s_j)]_{k,j=1}^n \, du_1 \cdots du_n. \hspace{1cm} (7.2)$$

Now suppose $\varphi$ and $\psi$ are totally positive and let $t_1 < \cdots < t_n$, $s_1 < \cdots < s_n$. Then the integrand in (7.2) is nonnegative for all (pairwise distinct) $u_1, \ldots, u_n$, because both determinants have the same sign, equal to that of $\prod_{k \geq l}(u_k - u_l)$. □

Proposition 7.3 (Schoenberg [S2, p. 335 and p. 343]). The densities of the normal and exponential distributions,

$$\psi_\gamma(t) = \frac{1}{\sqrt{2\pi} \gamma} e^{-t^2/(2\gamma)}, \quad \gamma > 0,$$  \hspace{1cm} (7.3)

and

$$\varphi_y(t) = \begin{cases} y^{-1} e^{-y^{-1}t}, & t \geq 0, \\ 0, & t < 0, \end{cases} \quad y > 0,$$  \hspace{1cm} (7.4)

are totally positive. □

Note that the result remains true after the shift $t \mapsto t + \text{const}$ of the argument or the change of sign $t \mapsto -t$.

Note also that $\{\psi_\gamma\}$ and $\{\varphi_y\}$ are one-parametric semigroups with respect to the convolution product.
**Theorem 7.4** (Schoenberg’s theorem on totally positive functions, see Schoenberg [S2], Karlin [K]). The Fourier transforms \( \hat{\varphi} \) of totally positive functions \( \varphi \) are just the functions \( F_{\gamma_1, \gamma_2, x} \), defined in (2.9), where at least one of the parameters \( \gamma_2, x_1, x_2, \ldots \) is nonzero. \( \square \)

This fundamental result shows that a totally positive function \( \varphi \) is a convolution product \( \varphi_0 * \varphi_1 * \varphi_2 * \cdots \), where \( \varphi_0(t) \, dt \) is a normal distribution and \( \varphi_k(t) \, dt, k = 1, 2, \ldots \), are, up to transformations \( t \mapsto \pm t + \text{const} \), exponential distributions.

We have to exclude \( \gamma_2 = x_1 = x_2 = \cdots = 0 \), because the inverse Fourier transform of the corresponding function \( F \) is a Dirac measure.

Comparing Theorem 2.9 and Theorem 7.4, we obtain the following correspondence \( M \leftrightarrow \varphi \) between the ergodic measures \( M \in \mathcal{M} \) (except the Dirac measures on scalar matrices) and the totally positive functions \( \varphi \) on the real line:

\[
\theta_1(M)(dt) = \varphi(t) \, dt
\]

(recall that the mapping \( \theta_1 \) assigns to a matrix \( B \in \mathcal{H} \) its matrix element \( B_{11} \in \mathbb{R} \)). In other words, this means that the distribution of the random variable \( B_{11} \) with respect to the probability distribution \( M \) on the matrices \( B \in \mathcal{H} \) is given by the density \( \varphi \).

The easy part of Theorem 7.4 consists in verifying the fact that \( F_{\gamma_1, \gamma_2, x} = \hat{\varphi} \) with a totally positive \( \varphi \). This is done by making use of Proposition 7.2 and Proposition 7.3 and an evident passage to the limit (cf. Proposition 2.8).

The hard part of Theorem 7.4 is to prove that the Fourier transform \( \hat{\varphi} \) of any totally positive function \( \varphi \) is of the form (2.9). Our purpose is to show that this claim is equivalent to Theorem 2.9.

**Definition 7.5** (Karlin [K, p. 49]). A real smooth nonnegative function \( \varphi(t) \) on \( \mathbb{R}, \int \varphi(t) \, dt = 1 \), is called extended totally positive if

\[
\det[\varphi^{(t-1)}(v_j)]_{i,j=1}^n \geq 0, \quad n = 1, 2, \ldots, v_1 > \cdots > v_n. \tag{7.5}
\]

**Proposition 7.6.** (i) Any smooth totally positive function \( \varphi \) is extended totally positive.

(ii) Conversely, if \( \varphi \) is an extended totally positive function, then the function \( \varphi * \psi_\gamma \), where \( \psi_\gamma \) was defined by (7.3), is totally positive for any \( \gamma > 0 \).

**Proof.** (i) By definition of total positivity, for any pairwise distinct real \( t_1, \ldots, t_n \) and any \( s_1 < \cdots < s_n \),

\[
\prod_{p>q}(t_p - t_q)^{-1} \cdot \det[\varphi(t_i - s_j)]_{i,j=1}^n \geq 0. \tag{7.6}
\]

Putting \( s_1 = -v_1, \ldots, s_n = -v_n \) and letting \( t_1, \ldots, t_n \to 0 \) in (7.6), we obtain (7.5).

(ii) By Theorem 2.1 in Karlin [K, p. 50], if \( \varphi \) verifies strict inequalities in (7.5), then it is totally positive. So it suffices to prove that if we replace \( \varphi \) by \( \varphi * \psi_\gamma \), then the inequalities in (7.5) become strict.

For \( n = 1 \) this is evident, because \( \varphi \) is nonnegative and not identically equal to zero whereas \( \psi_\gamma \) is strictly positive. For \( n > 1 \) this argument is generalized as follows.
First, remark that the functions $\varphi, \varphi', \varphi'', \ldots$ are linearly independent. Indeed, if this is not true, then $\varphi$ satisfies a linear differential equation with constant coefficients, whence $|\varphi(t)| \to \infty$ as $t \to \infty$ or $t \to -\infty$. But this contradicts the assumption $\varphi \in L^1(\mathbb{R})$.

Next, substitute $\psi = \psi_\gamma$ into formula (7.2) and repeat the argument used in the proof of (i). Then we obtain, for any $s_1 < \cdots < s_n$,

$$
\det \left[ (\varphi \ast \psi_\gamma)^{(i-1)}(-s_j) \right]_{i,j=1}^n
= \frac{1}{n!} \int_{\mathbb{R}^n} \det \left[ \varphi^{(i-1)}(-u_k) \right]_{i,k=1}^n \det \left[ \psi_\gamma(u_k - s_j) \right]_{k,j=1}^n \, du_1 \cdots du_n. \quad (7.7)
$$

Arguing just as in the proof of Proposition 7.2, we see that the integrand in (7.7) is nonnegative.

Finally, as noted in Schoenberg [S2, p. 336], the second determinant in the integrand is nonzero provided $u_1, \ldots, u_n$ are pairwise distinct. On the other hand, since $\varphi, \varphi', \ldots$ are linearly independent, the first determinant in the integrand does not vanish for certain $(u_1, \ldots, u_n) \in \mathbb{R}^n$, hence on an open subset of $\mathbb{R}^n$. We conclude that the integrand is everywhere nonnegative and strictly positive on an open subset, so that the integral (7.7) is strictly positive. □

Note that Proposition 7.6 corresponds to a part of Pickrell’s proof in [Pi2, p. 154–155]. There it is claimed that an analytic extended totally positive function is totally positive; however, the arguments are too sketchy and seem to be incomplete. To avoid this difficulty, we modified the claim somewhat and used a trick suggested by Boyer’s paper [Bo, p. 218].

Note also that not all totally positive functions are smooth: for instance, the function (7.4) is not smooth at 0. Thus, the class of extended totally positive functions, as defined above, does not coincide with the class of totally positive functions (although the two classes are very close, as is seen from Proposition 7.6). For this reason, to use property (7.5), we must first smooth totally positive functions.

The next theorem is Pickrell’s main calculation in [Pi2, pp. 154–155]. It is simple but instructive. For completeness and for reader’s convenience, we give the proof (which is presented here in slightly more detail than in [Pi2]).

**Theorem 7.7** (Pickrell [Pi2, pp. 154–155]). Let $\varphi$ be a smooth nonnegative function on $\mathbb{R}$, $\int \varphi(t) \, dt = 1$, and let $F = \hat{\varphi}$ be its Fourier transform. Then $F$ belongs to the class $\mathcal{F}_1$ (see definition (2.2)) if and only if $\varphi$ is extended totally positive.

**Proof.** Given $n = 1, 2, \ldots$ and $A \in H(n)$, denote by $a_1, \ldots, a_n$ the eigenvalues of $A$ and put

$$
f_n(A) = F(a_1) \cdots F(a_n);
$$

this is a continuous $U(n)$-invariant function on $H(n)$. Let us fix $n$ and show that positive definiteness of $f_n$ is equivalent to condition (7.5).

By Bochner’s theorem, $f_n$ is positive definite if and only if its inverse Fourier transform is a measure. This condition is equivalent to the following one: for any function $\Psi \geq 0$ from the Schwartz space $\mathcal{S}(H(n))$,

$$
\langle f_n, \Psi \rangle := \int_{H(n)} f_n(A) \overline{\Psi(A)} \, dA \geq 0. \quad (7.8)
$$
Since $f_n$ is $U(n)$-invariant, one may assume $\Psi$ is $U(n)$-invariant too.

Let $D(n)$ denote the subspace of diagonal matrices in $H(n)$. We identify $D(n)$ with $\mathbb{R}^n$ and write elements of $D(n)$ as diag$(a_1, \ldots, a_n)$ where $(a_1, \ldots, a_n) \in \mathbb{R}^n$. It is well known and easily verified that the radial part of the Lebesgue measure on $H(n)$ with respect to the action of $U(n)$ is the measure

$$\text{const } V^2(a_1, \ldots, a_n) \, da_1 \cdots da_n, \quad \text{const } > 0$$

on $D(n)$, where

$$V(a_1, \ldots, a_n) = \prod_{p<q} (a_p - a_q).$$

It follows that

$$\langle f_n, \hat{\Psi} \rangle = \text{const} \int_{D(n)} V(a_1, \ldots, a_n) \hat{\varphi}(a_1) \cdots \hat{\varphi}(a_n) \times V(a_1, \ldots, a_n) \hat{\Psi}(\text{diag}(a_1, \ldots, a_n)) \, da_1 \cdots da_n. \quad (7.10)$$

Put

$$\theta(t_1, \ldots, t_n) = V(t_1, \ldots, t_n) \Psi(\text{diag}(t_1, \ldots, t_n)). \quad (7.11)$$

We shall show that the integral (7.10) is equal, up to a positive factor, to

$$\int_{\mathbb{R}^n} \det [\varphi^{(j-1)}(t_k)]_{j,k=1}^n \theta(t_1, \ldots, t_n) \, dt_1 \cdots dt_n. \quad (7.12)$$

Indeed, the function $V(a_1, \ldots, a_n) \hat{\varphi}(a_1) \cdots \hat{\varphi}(a_n)$ is the Fourier transform of

$$V\left( i \frac{\partial}{\partial t_1}, \ldots, i \frac{\partial}{\partial t_n} \right) \cdot \varphi(t_1) \cdots \varphi(t_n) = i^{(n-1)/2} \det [\varphi^{(j-1)}(t_k)]_{j,k=1}^n. \quad (7.13)$$

On the other hand, using the $U(n)$-invariance of $\Psi$, we have

$$V(a_1, \ldots, a_n) \hat{\Psi}(\text{diag}(a_1, \ldots, a_n))$$

$$= V(a_1, \ldots, a_n) \int_{T \in H(n)} e^{i \text{tr}(\text{diag}(a_1, \ldots, a_n)T)} \Psi(T) \, dT$$

$$= V(a_1, \ldots, a_n) \int_{H(n)} \left( \int_{U(n)} e^{i \text{tr}(\text{diag}(a_1, \ldots, a_n)uT u^{-1})} \, du \right) \Psi(T) \, dT. \quad (7.14)$$

Using formula (5.12) for the interior integral and again applying formula (7.8) for the radial part of the Lebesgue measure, we see that (7.14) is equal, up to a positive factor, to

$$(-i)^{(n-1)/2} \int_{\mathbb{R}^n} \det [e^{ia_j t_k}]_{j,k=1}^n \cdot \theta(t_1, \ldots, t_n) \, dt_1 \cdots dt_n, \quad (7.15)$$

where $\theta$ was defined in (7.11). Developing the determinant and using the fact that $\theta(t_1, \ldots, t_n)$ is antisymmetric with respect to permutations of $t_1, \ldots, t_n$, we conclude that (7.15) is equal to

$$n! (-i)^{n(n-1)/2} \hat{\theta}(a_1, \ldots, a_n). \quad (7.16)$$
Now (7.13) and (7.16) imply that the integral (7.10) is equal, up to a positive factor, to
\[ (-1)^{n(n-1)/2} \int_{\mathbb{R}^n} \det [\varphi^{(n-j)}(t_k)]_{j,k=1}^n \cdot \theta(t_1, \ldots, t_n) \, dt_1 \cdots dt_n = \int_{\mathbb{R}^n} \det [\varphi^{(j-1)}(t_k)]_{j,k=1}^n \cdot \theta(t_1, \ldots, t_n) \, dt_1 \cdots dt_n. \] (7.17)

Thus, we have verified (7.12).

Since the integrand in (7.12) is symmetric with respect to permutations of \( t_1, \ldots, t_n \), we see that (7.12) is equal, up to a positive factor, to
\[ \int_{t_1 > \cdots > t_n} \det [\varphi^{(j-1)}(t_k)]_{j,k=1}^n \cdot \theta(t_1, \ldots, t_n) \, dt_1 \cdots dt_n. \] (7.18)

Recall that \( \theta \) is given by (7.11), where \( \Psi \) is an \( U(n) \)-invariant nonnegative function from the Schwartz space, and remark that \( V(t_1, \ldots, t_n) > 0 \) in the domain \( t_1 > \cdots > t_n \). It follows that (7.18) is nonnegative for any such \( \Psi \) if and only if \( \varphi \) is extended totally positive. □

**Corollary 7.8.** Theorem 2.9 (classification of ergodic measures \( M \in M \) or of functions \( F \in \mathcal{F}_1 \)) and Schoenberg’s Classification Theorem 7.4 can be derived one from another.

**Proof.** Let us show that Theorem 2.9 implies Theorem 7.4. Let \( \varphi \) be a totally positive function. We must prove that the Fourier transform \( F = \hat{\varphi} \) is of the form (2.9), where at least one of the parameters \( \gamma_2, x_1, x_2, \ldots \) is nonzero.

First suppose \( \varphi \) is smooth. Then, by Proposition 7.6 (i), \( \varphi \) is extended totally positive. Next, by Theorem 7.7, \( F \in \mathcal{F}_1 \), and, finally, by Theorem 2.9, \( F \) is of the form (2.9).

The general case can be reduced to that of a smooth \( \varphi \) as follows. We again use the Gaussian totally positive function \( \psi_\gamma \) (see (7.3)) to smooth \( \varphi \). Then, for any \( \gamma > 0 \), we have a smooth (even analytic) totally positive function \( \varphi \ast \psi_\gamma \). This implies that \( F(a) e^{-\gamma a^2/2} \) is of the form (2.9) for any \( \gamma > 0 \), whence \( F \) itself is of this form.

It should be added that \( F \) cannot be equal to \( F_{\gamma_1,0,0} \), because we know that the inverse Fourier transform of \( F \) is a function and not a Dirac measure. So at least one of the parameters \( \gamma_2, x_1, x_2, \ldots \) is nonzero.

The inverse implication is verified similarly, by making use of Proposition 7.6 (ii). □

Thus, our proof of Theorem 2.9 leads to a new proof of Schoenberg’s Theorem 7.4.

§8. **Totally positive functions as limits of splines**

After reading the preliminary version [OV] of the present paper, Andrei Okounkov remarked that the one-dimensional projections of orbital measures coincide with the so-called fundamental splines (= B-splines) whose limits were studied in an important paper by Curry and Schoenberg [CS]. The purpose of this section is to briefly discuss the relationship between the results of [CS] and our results.

We start by stating some classical facts used in [CS].
Fix real numbers $t_1 < \cdots < t_n$, called the knots $(n \geq 3)$. There exists a (unique) function $M_{n-1}(t) = M_{n-1}(t; t_1, \ldots, t_n)$ on $\mathbb{R}$ such that:

(i) On each open interval determined by adjacent knots, $M_{n-1}(t)$ is a polynomial of degree $n - 2$.

(ii) $M_{n-1}(t)$ vanishes when $t < t_1$ or $t > t_n$.

(iii) $M_{n-1}(t)$ has $n - 3$ continuous derivatives at each knot.

(iv) $\int M_{n-1}(t) \, dt = 1$.

In [CS], the function $M_{n-1}(t)$ is called the fundamental spline (with knots $t_1, \ldots, t_n$). Another term, used in the modern literature, is B-spline.

The fundamental spline $M_{n-1}(t)$ is given by the following explicit formula:

$$M_{n-1}(t; t_1, \ldots, t_n) = (n - 1) \sum_{k=1}^{n} \frac{(\max(t_k - t, 0))^{n-2}}{\prod_{i \neq k}(t_k - t_i)}. \quad (8.1)$$

Let $\sigma_{n-1}$ denote the standard $(n-1)$-dimensional simplex,

$$\sigma_{n-1} = \{(p_1, \ldots, p_n) \mid 0 \leq p_1, \ldots, p_n \leq 1, p_1 + \cdots + p_n = 1\} \subset \mathbb{R}^n, \quad (8.2)$$

and let $\xi$ denote the affine functional on the simplex taking values $t_1, \ldots, t_n$ at its vertices. Then $M_{n-1}(t)$ coincides with the density of the image under $\xi$ of the Lebesgue measure on $\sigma_{n-1}$, so normalized that the volume of $\sigma_{n-1}$ is equal to 1. This implies, in particular, that $M_{n-1}(t)$ is nonnegative, so that $M_{n-1}(t) \, dt$ is a probability measure on $\mathbb{R}$.

Using a passage to the limit, one easily extends the definition of $M_{n-1}(t)$ to the case when some of the knots coincide.

For further properties of the functions $M_{n-1}(t)$ and for proofs of the facts mentioned above, see [CS] or, e.g., Babenko’s textbook [Ba].

Now we are in a position to state the main result of Curry and Schoenberg [CS, Theorem 6]:

**Theorem 8.1.** Consider the class of probability measures on $\mathbb{R}$ which can be obtained as weak limits of measures of the form $M_{n-1}(t; t_1, \ldots, t_n) \, dt$, where $n \to \infty$ and the knots $t_1, \ldots, t_n$ depend on $n$. Then the characteristic functions of the measures of this class are exactly those given by formula (4.8).

We shall briefly describe the method of proof used in [CS]. Given $t_1, \ldots, t_n$, put

$$F_n(a) = \int_{-\infty}^{\infty} \left(1 - \frac{iat}{n}\right)^{-n} M_{n-1}(t; t_1, \ldots, t_n) \, dt. \quad (8.3)$$

Since

$$(1 - \frac{iat}{n})^{-n} = e^{iat} \left(1 + O\left(\frac{1}{n}\right)\right), \quad (8.4)$$

$F_n(a)$ may be viewed as the ‘approximate Fourier transform’ of $M_{n-1}(t)$. Its advantage with respect to the ordinary Fourier image of $M_{n-1}(t)$ is that it is given by a very simple expression, namely

$$F_n(a) = \prod_{k=1}^{n} \left(1 - \frac{iat_k}{n}\right)^{-1}. \quad (8.5)$$
Now note that $F_n(ia)^{-1}$ is a polynomial in $a$ with only real zeros, and use a well-known theorem, due to Laguerre and Pólya, which describes the class of entire functions that can be approximated by polynomials with real zeros: up to change of a variable $a \mapsto ia$, these are exactly the reciprocals to functions of type (4.8), see Hirschman–Widder [HW].

Note that this result on entire functions also plays an important role in Schoenberg’s classification of totally positive functions (Theorem 7.4 above).

Comparing Theorem 8.1 with Theorem 7.4, we conclude that a probability measure on $\mathbb{R}$ (distinct from a Dirac measure) can be approximated by a sequence of fundamental splines with growing number $n$ of knots if and only if it is given by a totally positive density.

The following fact, remarked by Andrei Okounkov, is crucial for our discussion.

**Proposition 8.2** (A. Yu. Okounkov). Let $\lambda_1 \leq \ldots \leq \lambda_n$ be real numbers and $n \geq 3$. Consider the $U(n)$-orbit in $H(n)$ passing through the diagonal matrix $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$ and denote by $\mu(dB)$ the corresponding orbital measure.

Then the image of $\mu$ under the projection $H(n) \ni B \mapsto B_{11} \in \mathbb{R}$ coincides with the fundamental spline $M_{n-1}(t; \lambda_1, \ldots, \lambda_n) dt$.

**Proof.** For $u \in U(n)$, we have

$$(u\Lambda u^{-1})_{11} = \sum_{k=1}^{n} u_{1k} \lambda_k (u^{-1})_{k1} = \sum_{k=1}^{n} |u_{1k}|^2 \lambda_k. \quad (8.6)$$

When the matrix $u$ ranges over $U(n)$, its first row

$$z = (z_1, \ldots, z_n) := (u_{11}, \ldots, u_{1n}) \quad (8.7)$$

ranges over the unit sphere $S^{2n-1} \subset \mathbb{C}^n$, and under the mapping $u \mapsto z$, the orbital measure $\mu$ projects onto the normalized invariant measure on $S^{2n-1}$, which may be written as

$$\text{const} \frac{d(\text{Re} z_1) d(\text{Im} z_1) \cdots d(\text{Re} z_n) d(\text{Im} z_n)}{d(|z_1|^2 + \cdots + |z_n|^2 - 1)}. \quad (8.8)$$

Further, under the mapping

$$(z_1, \ldots, z_n) \mapsto (|z_1|^2, \ldots, |z_n|^2) = (p_1, \ldots, p_n) \quad (8.9)$$

of the sphere onto the simplex $\sigma_{n-1}$, the measure (8.8) projects onto the measure

$$\text{const} \frac{dp_1 \cdots dp_n}{d(p_1 + \cdots + p_n - 1)}. \quad (8.10)$$

which coincides with the normalized Lebesgue measure on the simplex.

Finally, the right-hand side of (8.6), which may be rewritten as $\sum_{k=1}^{n} p_k \lambda_k$, is just the value at the point $(p_1, \ldots, p_n) \in \sigma_{n-1}$ of the linear functional $\xi: \mathbb{R}^n \to \mathbb{R}$ taking values $\lambda_1, \ldots, \lambda_n$ at the vertices of the simplex. By a property of the spline function $M_{n-1}(t)$ mentioned above, we conclude that the image of the orbital measure $\mu$ under the mapping $B \mapsto B_{11}$ is equal to $M_{n-1}(t; \lambda_1, \ldots, \lambda_n) dt$. □

By virtue of Proposition 8.2, the one-dimensional projections of orbital measures admit a very nice analytic interpretation in terms of splines, and Curry–Schoenberg’s result described in Theorem 8.1 turns out to be almost equivalent to
the ‘one-dimensional part’ of our Theorem 4.1, that is, to the results of Steps 1 and 3 in §6. The difference in the statements is that Curry and Schoenberg do not obtain necessary and sufficient conditions on the knots under which a sequence of fundamental splines would be weakly convergent; they are only interested in describing the limiting functions.

Theorems 7.4 and 8.1 together imply that totally positive functions are exactly those functions which may be approximated by fundamental splines with a growing number of knots. This fact seems to be highly nontrivial, because the fundamental splines themselves are not totally positive. We hope that the chain of relations traced in the present paper furnishes a certain explanation of this phenomenon.

Remark 8.3. Recall that the Dirichlet distribution $D(\theta_1, \ldots, \theta_n)$ with parameters $\theta_1 > 0, \ldots, \theta_n > 0$ is defined as the probability measure on the $(n-1)$-dimensional simplex (8.2) whose density with respect to the Lebesgue measure is given by

$$\text{const} p_1^{\theta_1-1} \cdots p_n^{\theta_n-1}, \quad (8.11)$$

see Kingman [Ki, Section 9.1]. Now in Proposition 8.2 let us replace the space $H(n)$ of $n \times n$ Hermitian matrices by the space of $n \times n$ real symmetric (respectively, quaternion Hermitian) matrices. Then the one-dimensional projections of orbital measures coincide with various one-dimensional projections of the Dirichlet distribution $D(1/2, \ldots, 1/2)$ (respectively, of the Dirichlet distribution $D(2, \ldots, 2)$).

More generally, one can consider one-dimensional projections of the Dirichlet distribution $D(\theta, \ldots, \theta)$ with arbitrary parameter $\theta > 0$. If $\theta = 1, 2, 3, \ldots$ then one-dimensional projections of this distribution are the fundamental splines with multiple knots: the multiplicity of each knot is equal to $\theta$. For general $\theta > 0$ there is no such interpretation. However, for any $\theta$, using the ‘moment method’, one can still obtain an analog of Theorem 8.1. The limiting measures will have a characteristic function of the following form (cf. (4.8)):

$$F(a) = e^{i\gamma_1 a - \gamma_2 a^2/2} \prod_k e^{-i\theta x_k^0 a} \prod_k e^{-i\theta x_k^0 a}/(1 - ix_k^0 a)^\theta, \quad a \in \mathbb{R},$$

where the parameters are the same as in (4.8).

Finally, note that the parameter $\alpha = \theta^{-1}$ exactly corresponds to the parameter that appears in the theory of Jack’s symmetric functions (see Stanley [Sta] or the 2nd edition (1995) of Macdonald’s book [M]). In particular, the $m$th moment of a one-dimensional projection of the Dirichlet distribution $D(\theta, \ldots, \theta)$ is equal, up to a scalar factor, to $P_m(t_1, \ldots, t_n; \theta^{-1})$, where $t_1, \ldots, t_n$ stand for the parameters of the projection (i.e., the values of the corresponding affine functional $\xi$ at the vertices of the simplex) and $P_m(\cdot; \alpha)$, $m = 1, 2, \ldots$, are one-row Jack’s symmetric functions with parameter $\alpha$.

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