Regular second-order perturbations of binary black holes in the extreme mass ratio regime

Carlos O Lousto and Hiroyuki Nakano

Center for Computational Relativity and Gravitation, and School of Mathematical Sciences, Rochester Institute of Technology, Rochester, NY 14623, USA

E-mail: colsma@rit.edu and hxnsma@rit.edu

Received 24 April 2008, in final form 29 September 2008
Published 16 December 2008
Online at stacks.iop.org/CQG/26/015007

Abstract
In order to derive the precise gravitational waveforms from extreme mass ratio inspirals (EMRI), we develop a formulation for the second-order metric perturbations produced by a point particle moving in the Schwarzschild spacetime. The second-order waveforms satisfy a wave equation with an effective source build-up from the $T_{\mu\nu}$ and products of the first-order perturbations and its derivatives. We have explicitly regularized this source at the horizon and at spatial infinity. We show that the effective source does not contain squares of Dirac’s delta and that perturbations are regular at the particle location. We introduce an asymptotically flat gauge for the $\ell = 0$ and $\ell = 1$ modes to compute the (leading radiative) second-order $\ell = 2$ waveforms in the head-on collision case.

PACS numbers: 04.25.Nx, 04.70.Bw

1. Introduction

The past 20 years have witnessed a steady increase in the interest in gravitational waves from astrophysical sources particularly driven by the design and construction of laser interferometric detectors, both ground and space based. Alongside these experimental developments, theoretical progress has also been steady. We are now in conditions to predict the gravitational radiation from the astrophysical scenarios more likely to produce the strongest signals, i.e. the merging binary black holes.

The two main astrophysical scenarios involving binary black holes are, first, galactic binaries with black holes having comparable masses (a few tens of solar masses). They are, for instance, the by-product of supernova explosions plus subsequent accretion. The second scenario involves supermassive black holes (of several million solar masses) residing in the center of an active galaxy. They attract stars in the inner nuclei toward unstable orbits with a subsequent plunge generating observable gravitational radiation. This scenario clearly
involves extreme mass ratio collisions. Let us also mention that a less common event, but most energetic, is the close collision of galaxies and consequently of the supermassive black holes in their respective cores.

From the theoretical point of view one advantage of dealing with binary black holes is that one can treat the problem of generation of radiation in terms of only its gravitational field, ignoring the (small) effects of matter around the binary system. A second important feature is that the equations of general relativity scale with the total mass of the system. In this way, one can choose the dimensions of the systems \textit{a posteriori}, i.e. after solving for the scale-free problem (see [1] for the case of three black holes). It is then convenient to characterize the binary black hole systems in terms of the mass ratio of its components (besides the individual spins, orbital parameters and spatial orientation with respect to the observer).

In binary black hole systems most of the generation of gravitational radiation takes place during the final few orbits before the merger. In the case of comparable masses, this stage involves highly nonlinear interactions among black holes and can only be described by directly numerically solving the full general relativity field equations. Until recently this represented an insurmountable task that held the field for nearly 30 years, but during the year 2005 two successful approaches [2–4] have led to stable codes that allow us to simulate binary black holes in supercomputers. These breakthroughs in numerical relativity have led to numerous studies in the last year, including the last few quasi-circular orbits [5, 6], the effect of eccentricity [7], and spin–orbit coupling leading to a change in the merger time [8], corotation [9], and spin-flip and precession [10]. Finally, unequal mass black holes have been studied in [11–14] reaching a minimum mass ratio of nearly 1:4. But simulations with mass ratios up to 1:10 are currently underway. All these simulations indicate the dominance of the $\ell = 2$ mode in the radiation to infinity. Generic binary simulations, i.e. unequal masses and spins have first been reported in [15]; these simulations lead to the shocking discovery that merging spinning black holes can acquire recoil velocities of up to 4000 km s$^{-1}$ [16]. Even multi-black-hole spacetimes can now be evolved numerically [17, 18].

In the extreme mass ratio regime the smaller hole orbiting the larger one is described as a perturbation. In this approach, the smaller hole is mathematically described by a Dirac’s delta particle and the spacetime is no longer empty but has a non-vanishing energy–momentum tensor at the location of the particle. The simplicity of this approach is appealing, but the problem notably gets complicated when self-force effects are taken into account to compute the corrections to the background geodesic motion. A consistent approach to this problem was first laid down over 10 years ago by Mino, Sasaki and Tanaka [19] and soon after confirmed by Quinn and Wald [20] by providing a regularization procedure. (See [21, 22] for a detailed review and [23] for a practical regularization method.) Self-force corrections can be considered second-order effects on the mass ratio of the holes, the natural perturbation parameter of the system. To consistently compute the gravitational waveforms, energy, angular and linear momentum radiated to infinity (and onto the horizon) we need to solve the second-order perturbations problem for the gravitational field. The formalism to study second- (and higher) order perturbations of rotating black holes with sources was extensively discussed in [24] based on the Newman–Penrose approach to curvature perturbations.

The first explicit computation of the gravitational self-force was done for the head-on collision of two black holes in [25, 26]. This allows us to complete the second-order program applied to the head-on collision of black holes. The first-order self-force is important, but its effects can be isolated at the second perturbative order. Therefore, we do not deal here with the self-force explicitly, but rather analyze the regularization of the second-order calculation. Note that this step is also crucial in order to close the gap between the full numerical simulations with comparable masses and the perturbative approach for extreme mass ratios.
Second-order calculations are required to derive precise gravitational waveforms to be used as templates for gravitational wave data analysis. In general, these second-order computations have to be done by numerical means. It is hence important to derive a well-behaved second-order effective source for practical calculations. As a first step, we deal here only with the $\ell = 2$ second-order perturbations because this gives the leading correction to gravitational radiation.

The second-order approach was pioneered by Tomita \cite{27, 28}, and vacuum perturbations in the Schwarzschild background were studied by Gleiser \textit{et al} \cite{29–32}. There are also studies on second-order quasi-normal modes \cite{33, 34}, and the second-order analysis was also extended to cosmology \cite{35–39}.

This paper is organized as follows. We consider second-order metric perturbations and the equations they satisfy, i.e., the perturbed Hilbert–Einstein equations of general relativity in section 2. We study in detail only the $\ell = 2$ second-order perturbation here. The convergence of the second-order metric perturbations in the harmonics expansion is discussed in section 3. Throughout the paper, we use the Regge–Wheeler–Zerilli formalism \cite{40, 41}. This formalism in our notation is summarized in section 4. In section 5, we discuss the first-order metric perturbations in the case of a particle falling radially into a Schwarzschild black hole. To calculate the second-order source, the first-order $\ell = 0$ and $\ell = 1$ modes are given in an appropriate gauge in the appendix. In section 6, we derive the regularized second-order effective source where the singular behavior is removed analytically. In section 7, we summarize the results of this paper and discuss some remaining open problems. Throughout this paper, we use units in which $c = G = 1$.

2. Second-order metric perturbations

We consider second-order metric perturbations which arise from a point particle with mass $\mu$ on black hole backgrounds with mass $M$,

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}^{(1)} + h_{\mu\nu}^{(2)},$$

(1)

with expansion parameter $\mu/M$ corresponding to the mass ratio of the holes. Here, $g_{\mu\nu}$ is the background metric, and superscripts $(i)$ ($i = 1, 2$) denote the perturbative order, i.e., $h_{\mu\nu}^{(1)}$ and $h_{\mu\nu}^{(2)}$ are called the first- and second-order metric perturbations, respectively. In the perturbative calculation, we raise and lower all tensor indices with the background metric. The Hilbert–Einstein tensor up to the second perturbative order is given by

$$G_{\mu\nu}[\tilde{g}_{\mu\nu}] = G_{\mu\nu}^{(1)}[h_{\mu\nu}^{(1)}] + G_{\mu\nu}^{(1)}[h_{\mu\nu}^{(2)}] + G_{\mu\nu}^{(2)}[h_{\mu\nu}^{(1)}, h_{\mu\nu}^{(1)}],$$

(2)

where we have omitted the spacetime indices $\mu$ and $\nu$ of the metric perturbations, $h_{\mu\nu}^{(1)}$ and $h_{\mu\nu}^{(2)}$, and ignored $O((\mu/M)^3)$ and higher order terms. $G_{\mu\nu}^{(1)}$ is the well-known linearized Hilbert–Einstein tensor,

$$G_{\mu\nu}^{(1)}[H] = -\frac{1}{2} H_{\mu\nu;\alpha}^{\cdot\alpha} + H_{\alpha(\mu;\nu)}^{\cdot\alpha} - R_{\alpha\mu\beta\nu} H^{\alpha\beta} - \frac{1}{2} H_{\alpha;\mu\nu}^{\cdot\alpha} - \frac{1}{2} g_{\mu\nu} \left( H^{\cdot\alpha} \cdot_{\alpha} \cdot_{\lambda} - H_{\cdot\alpha}^{\alpha} \cdot_{\lambda} \right).$$

(3)

Here, $H_{\mu\nu}$ denotes $h_{\mu\nu}^{(1)}$ or $h_{\mu\nu}^{(2)}$, and semicolon ‘;’ in the index denotes the covariant derivative with respect to the background metric. $G_{\mu\nu}^{(2)}$ consists of quadratic terms in the first-order perturbations,

$$G_{\mu\nu}^{(2)}[h_{\mu\nu}^{(1)}, h_{\mu\nu}^{(1)}] \equiv R_{\mu\nu}^{(2)}[h_{\mu\nu}^{(1)}, h_{\mu\nu}^{(1)}] - \frac{1}{2} g_{\mu\nu} R_{\alpha\mu\beta\nu}^{(2)}[h_{\mu\nu}^{(1)}, h_{\mu\nu}^{(1)}];$$

$$R_{\mu\nu}^{(2)}[h_{\mu\nu}^{(1)}, h_{\mu\nu}^{(1)}] \equiv \frac{1}{2} h_{\mu\nu,\alpha}^{(1)} h_{\alpha\beta}^{(1)} + \frac{1}{2} h_{\mu\nu,\alpha}^{(1)} h_{\alpha\beta}^{(1)} - 2 h_{\mu\nu,\alpha}^{(1)} h_{\alpha\beta}^{(1)}$$

$$- \frac{1}{2} \left( h_{\alpha\beta}^{(1)} \cdot_{\beta} - \frac{1}{2} h_{\beta}^{(1)} \cdot_{\beta} \right) (2 h_{\alpha(\mu;\nu)}^{(1)} - h_{\mu\nu}^{(1)}) + \frac{1}{2} h_{\mu\nu,\beta}^{(1)} h_{\alpha}^{(1)} \cdot_{\alpha} - \frac{1}{2} h_{\mu\nu,\beta}^{(1)} h_{\alpha}^{(1)} \cdot_{\alpha}.$$  

(4)

$$\ell$$
On the other hand, the stress–energy tensor includes two parts:

\[ T_{\mu\nu} = T_{\mu\nu}^{(1)} + T_{\mu\nu}^{(2)}. \]  

(5)

The first-order stress–energy tensor \( T_{\mu\nu}^{(1)} \) which is the one of a point particle moving along a background geodesic is given by

\[ T_{\mu\nu}^{(1)} = \mu \int_{-\infty}^{+\infty} \delta^{(4)}(x - z(\tau)) \frac{dz^\mu}{d\tau} \frac{dz^\nu}{d\tau} d\tau, \]  

(6)

where

\[ z^\mu = [T(\tau), R(\tau), \Theta(\tau), \Phi(\tau)], \]  

(7)

for the particle’s orbit. \( T_{\mu\nu}^{(2)} \) denotes the effect of the self-force as derived by the MiSaTaQuWa formalism [19, 20]. (See [42].) Although this self-force effect is important, we do not treat this stress–energy tensor explicitly in this paper. This is because this effect does not derive any singular behavior and we can evaluate the contribution to gravitational radiation by the same method of the first-order calculation. Thus, we discuss only effects due to the nonlinearity of the Einstein equation in [42].

With the above expansion of the Hilbert–Einstein and stress–energy tensors, we may solve the following equation for the first perturbative order:

\[ G_{\mu\nu}^{(1)} h^{(1)} = 8\pi T_{\mu\nu}^{(1)}. \]  

(8)

And for the second perturbative order, we have the following equation:

\[ G_{\mu\nu}^{(1)} h^{(2)} = 8\pi T_{\mu\nu}^{(2)} - G_{\mu\nu}^{(2)} h^{(1)}. \]  

(9)

Once the first-order metric perturbations \( h^{(1)} \) (and the self-force) are obtained, we may solve (9) with a second-order source that can be considered as an effective stress–energy tensor. Systematically expanding the Hilbert–Einstein equations, one can obtain the perturbative equations order by order [24, 43–46].

3. Harmonics of gravitational radiation

In this paper, we discuss only the \( \ell = 2 \) mode of the second perturbative order in the harmonics expansion, since this gives the leading contribution to gravitational radiation. There remains a question about the convergence of the second-order metric perturbations, though. Based on the works by Rosenthal [47–49], we discuss this problem.

First, we consider a second-order wave equation which is given by

\[ \Box h^{(2)} = S_h^{(2)}, \]  

(10)

where \( \Box \) is the wave operator and the second-order source \( S_h^{(2)} \) is derived from the first-order metric perturbations with local behavior around the particle location as

\[ h^{(1)} \sim O(\epsilon^{-1}), \]  

(11)

where the spatial separation \( \epsilon = |x - x_0| \) with a particle location \( x_0 \). In this section, we are not on the particle’s worldline but rather take a limit to the particle location. We need second derivatives to compute \( S_h^{(2)} \), then the local behavior becomes

\[ S_h^{(2)} \sim O(\epsilon^{-4}). \]  

(12)

For the above source, if we solve the wave equation by using a usual four-dimensional Green’s function method, this solution diverges everywhere [47]. Therefore, to obtain the finite solution, we need to remove the \( O(\epsilon^{-4}) \) and \( O(\epsilon^{-3}) \) terms from \( S_h^{(2)} \). Here, we note that we
can remove the $O(\epsilon^{-3})$ terms by using a regular gauge transformation [48]. Since we consider the second-order gravitational wave at infinity which is gauge invariant, the $O(\epsilon^{-3})$ terms do not contribute. In practice, we use the gauge-invariant wavefunction in our calculation. Thus, we may discuss only the problematic $O(\epsilon^{-4})$ terms.

Here, Rosenthal has already shown the most singular part of the second-order metric perturbations as

\[ h^{(2)}_s \sim O(\epsilon^{-2}). \]  

This is a peculiar solution of

\[ \Box h^{(2)}_s \sim O(\epsilon^{-4}). \]  

(13)

Using this solution, we rewrite the second-order wave solution as

\[ h^{(2)}_r = h^{(2)}_s - \Box^{-1}(S^{(2)} - O(\epsilon^{-4})) \sim \Box^{-1}(O(\epsilon^{-2})). \]  

(14)

The right-hand side of the last line is finite after the integration by using the retarded Green function as $\Box^{-1}$. The final result is

\[ h^{(2)} = h^{(2)}_s + h^{(2)}_r. \]  

(16)

is the physical second-order gravitational perturbations [48]. Thus, if we construct a second-order gauge invariant $\psi^{(2)}$ from the above metric perturbations, this is given by

\[ \psi^{(2)} = \psi^{(2)}_s + \psi^{(2)}_r, \]  

(17)

where $\psi^{(2)}_s$ and $\psi^{(2)}_r$ are derived from $h^{(2)}_s$ and $h^{(2)}_r$, respectively. As a result, the apparent divergence derived from the first consideration is only a gauge contribution. It should be noted that the divergence at the location of the particle remains.

Next, we consider the expansion in terms of tensor harmonics of the second-order gauge invariant wavefunction $\psi^{(2)}$ (which need not be the same as the Regge–Wheeler or Zerilli function)

\[ \Box_{\ell m} \psi^{(2)}_{\ell m} = S^{(2)}_{\ell m}. \]  

(18)

In our situation, we solve the above equation by numerical integration. Formally, we can write the solution as

\[ \psi^{(2)}_{\ell m} = \Box^{-1}_{\ell m} S^{(2)}_{\ell m}. \]  

(19)

This $\Box^{-1}_{\ell m}$ means a numerical integration under an appropriate boundary condition. We may also use the retarded Green function. Since the solution $\psi^{(2)}_{\ell m}$ is gauge invariant, this summation over the $(\ell, m)$ modes coincides with $\psi^{(2)}$ in (17) by definition. Hence, the summation of $\psi^{(2)}_{\ell m}$ over modes has a finite value, except for the location of the particle. Conversely, we can carry out the harmonics expansion where the harmonics coefficient $\psi^{(2)}_{\ell m}$ is finite everywhere. In particular, the asymptotic behavior for large $r$, where we need to compute gravitational radiation, is well defined.

From the above fact, the second-order source $S^{(2)}_{\ell m}$ is finite except for the particle’s location because this is derived by, at most, the second-order derivative in (18). Also note that the singularity at the particle’s location is integrable in the gauge invariant formulation.

4. Tensor harmonics expansion in the Regge–Wheeler–Zerilli formalism

In this paper, we consider the Schwarzschild background,

\[ ds^2 = - \left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(\sin^2 \theta d\phi^2) \]  

(20)

in Boyer–Lindquist coordinates.
of the first perturbative order is as follows. First, we expand metric perturbations, i.e., the first-order Hilbert–Einstein equation given in (8). The treatment before considering the second perturbative order, it is necessary to discuss the first-order

5. First-order perturbations in the Regge–Wheeler gauge

The Regge–Wheeler–Zerilli formalism [40, 41] is used to discuss the metric perturbations. The basic formalism has been given in Zerilli’s paper [41], and it has been summarized in the time domain in [50, 51]. Although we have similar notation to that of Zerilli’s paper [41], there are some differences. We establish our notation and summarize the Regge–Wheeler–Zerilli formalism in the time domain here.

For the first- and second-order metric perturbations, and stress–energy tensors, we expand $h^{(i)}_{μν}$ and $T^{(i)}_{μν}$ ($i = 1, 2$) in tensor harmonics,

$$
T^{(i)} = \sum_{ℓm} \left[ A^{(i)}_{0ℓm} a_{0ℓm} + A^{(i)}_{1ℓm} a_{1ℓm} + A^{(i)}_{2ℓm} a_{2ℓm} + A^{(i)}_{3ℓm} b_{0ℓm} + A^{(i)}_{4ℓm} b_{1ℓm} + A^{(i)}_{5ℓm} b_{2ℓm} + A^{(i)}_{6ℓm} b_{3ℓm} \right],
$$

(22)

where $a_{0ℓm}$, $a_{1ℓm}$, $a_{2ℓm}$, $b_{0ℓm}$, $b_{1ℓm}$, $b_{2ℓm}$, and $A^{(i)}_{0ℓm}$, $A^{(i)}_{1ℓm}$, $A^{(i)}_{2ℓm}$, $A^{(i)}_{3ℓm}$, $A^{(i)}_{4ℓm}$, $A^{(i)}_{5ℓm}$, $A^{(i)}_{6ℓm}$ are tensor harmonics defined by (3.2–11) in [34], and $H^{(i)}_{0ℓm}$, $H^{(i)}_{1ℓm}$, $H^{(i)}_{2ℓm}$, $H^{(i)}_{3ℓm}$, $H^{(i)}_{4ℓm}$, $H^{(i)}_{5ℓm}$, $H^{(i)}_{6ℓm}$ denote the coefficients of the tensor harmonics.

The tensor harmonics can be classified into even and odd parities from the above expressions. Even parity modes are defined by the parity $(-1)^ℓ$ under the transformation $(θ, φ) → (π − θ, φ + π)$, while odd parity modes are by the parity $(-1)^{ℓ+1}$. Using the orthogonality of the above tensor harmonics, we can derive the coefficient of the corresponding tensor harmonics expansion. For example,

$$
A^{(i)}_{0ℓm}(t, r) = \int T^{(i)} · a^{*}_{0ℓm} \, dΩ = \int δ^{\muα} δ^{νβ} T^{(i)}_{μν} a^{*}_{0ℓmμα} \, dΩ.
$$

(23)

where $*^*$ denotes the complex conjugate, $dΩ = \sin θ \, dθ \, dφ$ and $δ^{\muα}$ has the component,

$$
δ^{1α} = \text{diag}(1, 1, 1/r^2, 1/(r^2 \sin^2 θ)).
$$

5. First-order perturbations in the Regge–Wheeler gauge

Before considering the second perturbative order, it is necessary to discuss the first-order metric perturbations, i.e., the first-order Hilbert–Einstein equation given in (8). The treatment of the first perturbative order is as follows. First, we expand $h^{(1)}_{μν}$ and $T^{(1)}_{μν}$ in ten tensor harmonics components, given in (21) and (22). We then obtain the linearized field equations for each harmonic mode. Here, for example, for the even parity modes which have the even parity behavior, we may consider the Zerilli equation. Finally, imposing the Regge–Wheeler (RW) gauge conditions:

$$
h^{(e)(1)RW}_{0ℓm} = h^{(e)(1)RW}_{1ℓm} = T^{(1)RW}_{μν} = 0,
$$

(24)

where the suffix RW stands for the RW gauge, we obtain the first-order metric perturbations.
5.1. Geodesic motion and the first-order stress–energy tensor

The linearized Hilbert–Einstein equation is given in (8). Here, we need to specify that the stress–energy tensor on the right-hand side of the above equation be the one of a point particle moving along a geodesic,

\[ T^{(1)\mu\nu} = \mu U^0 \int_{-\infty}^{+\infty} \delta(4)(x - z(\tau)) \frac{dz^\mu}{d\tau} \frac{dz^\nu}{d\tau} d\tau \]

\[ = \mu U^0 \frac{dz^\mu}{d\tau} \frac{dz^\nu}{d\tau} \frac{d}{dr} \frac{d}{dr} \frac{d}{dr} \delta(\cos \Theta - \cos \Theta(t)) \delta(\phi - \Phi(t)), \]

where we have used the following notation,

\[ z^\mu = z^\mu(\tau) = \{T(\tau), R(\tau), \Theta(\tau), \Phi(\tau)\}, \quad U^0 = \frac{dT(\tau)}{d\tau}, \]

for the particle’s orbit. This stress–energy tensor is expressed in terms of the tensor harmonics as given in table 1 for the even parity modes. In this table, the angular functions \( X_{lm} \) and \( W_{lm} \) are given by

\[ X_{lm} = 2 \frac{\partial}{\partial \phi} \left( \frac{\partial}{\partial \theta} \cot \theta \right) Y_{lm}, \]

\[ W_{lm} = \left( \frac{\partial^2}{\partial \theta^2} \cot \theta \frac{\partial}{\partial \theta} - \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right) Y_{lm}. \]

In this paper, we consider a particle falling radially into a Schwarzschild black hole as the first-order source. Assuming \( \Theta(t) = \Phi(t) = 0 \), the equation of motion of the particle is given as

\[ \left( \frac{dR(t)}{dt} \right)^2 = - \left( 1 - \frac{2M}{R(t)} \right)^3 \frac{1}{E^2} \left( 1 - \frac{2M}{R(t)} \right)^2, \]

where \( R(t) \) is the location of the particle and the energy \( E \) is written by

\[ E = \left( 1 - \frac{2M}{R(t)} \right) \frac{dT(t)}{d\tau}. \]
We will also use
\[ \frac{d^2 R(t)}{dt^2} = -\frac{3}{E^2} \left( 1 - \frac{2M}{R(t)} \right)^2 \frac{M}{R(t)^2} + 2 \left( 1 - \frac{2M}{R(t)} \right) \frac{M}{R(t)^2}. \] (31)
to simplify equations. The tensor harmonics coefficients of the first-order stress–energy tensor which are given in table 1 are
\[
\mathcal{A}^{(1)}_{lm}(t, r) = \mu \frac{ER(t)}{R(t) - 2M} \left( \frac{dR(t)}{dt} \right)^2 \frac{1}{(r - 2M)^2} \delta(r - R(t)) Y^*_{lm} (0, 0),
\]
\[
\mathcal{A}^{(1)}_{0m}(t, r) = \mu \frac{ER(t)}{R(t) - 2M} \frac{(r - 2M)^2}{r^4} \delta(r - R(t)) Y^*_{lm} (0, 0),
\]
\[
\mathcal{A}^{(1)}_{1m}(t, r) = \sqrt{2} \mu \frac{ER(t)}{R(t) - 2M} \frac{dR(t)}{dt} \frac{1}{r^2} \delta(r - R(t)) Y^*_{lm} (0, 0).
\] (32)
The remaining coefficients are zero. Because of the symmetry of the problem, we have only to consider the even parity modes.

5.2. First-order metric perturbations ($\ell \geq 2$)

Substituting (21) and (22) into (8), we obtain the linearized Hilbert–Einstein equation for each harmonic mode. Here, we use the RW gauge condition, $h^{(1)}_{2lm} = 0$ for the odd part and $h^{(1)}_{0lm} = h^{(1)}_{1lm} = G^{(1)}_{lm} = 0$ for the even part. From these ten linearized equations, we derive the Regge–Wheeler–Zerilli equations and construct the metric perturbations from the Regge–Wheeler–Zerilli functions in the RW gauge. In the following, we focus on the even parity modes. For the $\ell = 0$ and 1 modes, we will need a different treatment as described in section 5.4 and section 5.5. (See also [52, 53].)

We introduce the following function for $\ell \geq 2$ modes:
\[
\psi^{\text{even}}_{lm}(t, r) = \frac{2r}{\ell(\ell + 1)} \left[ K^{(1)}_{lm}(t, r) + 2 \frac{(r - 2M)}{r^2 + r \ell - 2r + 6M} \left( H^{(1)}_{2lm}(t, r) - r \frac{\partial}{\partial r} K^{(1)}_{lm}(t, r) \right) \right].
\] (33)
This function is related to Zerilli’s even parity function $\psi^{Z, \text{even}}_{lm}$ as
\[
\psi^{Z, \text{even}}_{lm}(t, r) = \frac{2}{r^2 + r \ell - 2r + 6M} \left( r^2 \frac{\partial}{\partial t} K^{(1)}_{lm}(t, r) - (r - 2M) H^{(1)}_{lm}(t, r) \right)
= \frac{\partial}{\partial t} \psi^{\text{even}}_{lm}(t, r) - \frac{16\sqrt{2} \pi i r^2 (r - 2M)}{\ell(\ell + 1)(r^2 + r \ell - 2r + 6M)} \mathcal{A}^{(1)}_{lm}(t, r).
\] (34)
The function $\psi^{\text{even}}_{lm}$ obeys the Zerilli equation,
\[
\left[ -\frac{\partial^2}{\partial \ell^2} + \frac{\partial^2}{\partial r^{\ast 2}} - V^{\text{even}}(r) \right] \psi^{\text{even}}_{lm}(t, r) = S^{\text{even}}_{lm}(t, r),
\] (35)
where $r^* = r + 2M \ln(r/2M - 1)$ and
\[
V^{\text{even}}(r) = \frac{r - 2M}{r^4 (r^2 + r \ell - 2r + 6M)^2} \left( \ell(\ell + 1)(\ell + 2)^2(\ell - 1)^2 r^3 + 6M(\ell + 2)^2(\ell - 1)r + 72M^3 \right).
\] (36)
and the source term is given by

\[
S^{\text{even}}_{\ell m}(t, r) = \frac{16\pi (r - 2M)^2 (r\ell^2 + r\ell - 4r + 2M)}{(\ell^2 + r\ell - 2r + 6M)r} A^{(1)}_{\ell m}(t, r) \\
- \frac{16\sqrt{2}\pi (r - 2M)}{\sqrt{(\ell + 1)(\ell - 1)(\ell + 2)}} f^{(1)}_{\ell m}(t, r) + \frac{32\pi (r - 2M)^2 \sqrt{2}}{(r\ell^2 + r\ell - 2r + 6M)\sqrt{\ell(\ell + 1)}} b^{(1)}_{\ell m}(t, r) \\
- \frac{32\pi (r - 2M)^3}{(r\ell^2 + r\ell - 2r + 6M)\ell(\ell + 1)} \frac{\partial}{\partial r} f^{(1)}_{\ell m}(t, r) - [16\pi r (\ell^4 + 2r^2 \ell^3 - 5r^2 \ell^2 \\
+ 16r \ell^2 M - 6r^2 \ell + 16r \ell^2 M + 8r^2 - 68r M^2 + 108M^2)/(\ell(\ell + 1)\ell(\ell + 2)r\ell^2 + r\ell \\
- 2r + 6M)^2)]A^{(1)}_{\ell m(0m)}(t, r) + \frac{32\pi (r - 2M)^2 r^2}{(r\ell^2 + r\ell - 2r + 6M)\ell(\ell + 1)} \frac{\partial}{\partial r} f^{(1)}_{\ell m(0m)}(t, r) \\
+ \frac{32\sqrt{2}\pi (r - 2M)^2}{(r\ell^2 + r\ell - 2r + 6M)\ell(\ell + 1)} g^{(1)}_{\ell m(0m)}(t, r).
\]

(37)

Here \( T^{(1,v)}_{\mu\nu} = 0 \) have been used to simplify the source term. Using the function \( \psi^{(\text{even})}_{\ell m} \), the four coefficients for the metric perturbations in the RW gauge are expressed as

\[
K^{(1,\text{RW})}_{\ell m}(t, r) = \frac{1}{2} (L^4 r^2 + 2r^2 \ell^3 - r^2 \ell^2 + 6r \ell^2 M - 2r^2 \ell + 6r \ell M \\
- 12r M + 2M^2)/(r(\ell^2 + r\ell - 2r + 6M)r^2)]\psi^{(\text{even})}_{\ell m}(t, r) \\
+ (r - 2M) \frac{\partial}{\partial r} \psi^{(\text{even})}_{\ell m}(t, r) - \frac{32\pi r^3}{\ell(\ell + 1)(r\ell^2 + r\ell - 2r + 6M)} A^{(1)}_{\ell m(0m)}(t, r),
\]

\[
H^{(1,\text{RW})}_{2m}(t, r) = \frac{1}{2} \frac{(r - 2M)}{r - 2M} \frac{\partial}{\partial r} K^{(1,\text{RW})}_{\ell m}(t, r) + \frac{r}{\ell(\ell + 1)(r\ell^2 + r\ell - 2r + 6M)} A^{(1)}_{\ell m(0m)}(t, r),
\]

(38)

Here, \( H^{(1,\text{RW})}_{2m}(t, r) \) + 16 \( \frac{\pi r^2 \sqrt{2}}{\sqrt{(\ell + 1)(\ell - 1)(\ell + 2)}} f^{(1)}_{\ell m}(t, r) \\
- \frac{1}{4} \frac{\partial}{\partial r} \psi^{(\text{even})}_{\ell m}(t, r),
\]

\[
H^{(1,\text{RW})}_{1m}(t, r) = H^{(1,\text{RW})}_{2m}(t, r) + \frac{2}{(r - 2M)} \frac{(r - 3M)}{r - 2M} \frac{\partial}{\partial t} K^{(1,\text{RW})}_{\ell m}(t, r) + \frac{2}{\ell(\ell + 1)} \frac{\partial^2}{\partial t \partial \ell} K^{(1,\text{RW})}_{\ell m}(t, r) \\
- 2\frac{r}{\ell(\ell + 1)} \frac{\partial}{\partial t} H^{(1,\text{RW})}_{2m}(t, r) + \frac{8\pi \sqrt{2} r^2}{\ell(\ell + 1)} A^{(1)}_{\ell m(0m)}(t, r).
\]

Here, the metric perturbations in the RW gauge are \( C^0 \) (continuous across the particle) in the head-on collision case. One can see this as follows. First, using the following linearized Hilbert–Einstein equations for each harmonics mode,

\[
H^{(1)}_{0m(0)} - H^{(1)}_{2m(0)} = \frac{8\pi r^2 f^{(1)}_{\ell m}}{2} \frac{1}{\sqrt{(\ell + 1)(\ell - 1)(\ell + 2)/2}}.
\]

(39)

and \( f^{(1)}_{\ell m} = 0 \), we obtain \( H^{(1,\text{RW})}_{2m(0)} = H^{(1,\text{RW})}_{0m(0)} \). Then, removing \( H^{(1,\text{RW})}_{1m(0)} \) from two linearized Hilbert–Einstein equations,

\[
\frac{\partial}{\partial t} \left[ (1 - \frac{2M}{r}) H^{(1,\text{RW})}_{1m(t)} \right] - \frac{\partial}{\partial r} \left( H^{(1,\text{RW})}_{2m(t)} + K^{(1,\text{RW})}_{\ell m} \right) = \frac{8\pi i r}{\sqrt{(\ell + 1)/2}} f^{(1)}_{0m},
\]

(40)

\[
- \frac{\partial H^{(1,\text{RW})}_{1m(t)}}{\partial t} + (1 - \frac{2M}{r}) \frac{\partial}{\partial r} \left( H^{(1,\text{RW})}_{0m(t)} - K^{(1,\text{RW})}_{\ell m(t)} + \frac{2M}{r^2} H^{(1,\text{RW})}_{0m(t)} \right) + \frac{1}{r} \left( 1 - \frac{M}{r} \right) \left( H^{(1,\text{RW})}_{2m(t)} - H^{(1,\text{RW})}_{0m(t)} \right) = \frac{8\pi (r - 2M)}{\sqrt{(\ell + 1)/2}} B^{(1)}_{\ell m},
\]

(41)
we obtain the relation
\[
\left[ -\frac{\partial^2}{\partial t^2} + \left( 1 - \frac{2M}{r} \right) \frac{\partial^2}{\partial r^2} \right] H_{2m}^{(1)\text{RW}}(t, r) \equiv \left[ -\frac{\partial^2}{\partial t^2} + \left( 1 - \frac{2M}{r} \right) \frac{\partial^2}{\partial r^2} \right] K_{2m}^{(1)\text{RW}}(t, r) \\
+ \text{(first differential terms of } H_{2m}^{(1)\text{RW}} \text{ and } K_{2m}^{(1)\text{RW}}),
\]
where we have used \(B_{0m}^{(1)} = B_{1m}^{(1)} = 0\). Therefore, we find that \(H_{2m}^{(1)\text{RW}}\) and \(K_{2m}^{(1)\text{RW}}\) have the same differential behavior. Here, we note that the wavefunction in the second-order source term.

In the next subsection, we treat only the \(\ell = 2\) mode which is the leading contribution in the first-order perturbations. In this \(\ell = 2\) mode, we may consider only the \(m = 0\) mode because of \(Y_{2m}(0, 0) = 0\) for \(m \neq 0\) in (32).

### 5.3. \(\ell = 2, m = 0\) mode

We focus here only on the \(\ell = 2, m = 0\) mode. For this mode, the Zerilli equation in (35) becomes
\[
\left[ -\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial r^2} - \frac{6}{r^2} \left( 4r^2 + 4r^2M + 6rM^2 + 3M^3 \right) \right] \psi^{\text{even}}_{20}(t, r) \\
= -8\pi \frac{\mu (R(t))^2 - 2R(t)E^2 M + 6R(t)M + M^2 (R(t) - 2M)}{E^2 r^2 (2r + 3M)^2} \psi^{\text{even}}_{20}(r, r) \\
\times Y_{20}(0, 0) \delta(r - R(t)) + \frac{8\pi \mu (R(t) - 2M)^3}{3 E^2 r^2 (2r + 3M)} Y_{20}(0, 0) \frac{d}{dr} \delta(r - R(t)),
\]
where we have used the formula
\[
F(r) \frac{d}{dr} \delta(r - R) = F(R) \frac{d}{dr} \delta(r - R) - \frac{d}{dr} F'(r) \bigg|_{r=R} \delta(r - R),
\]
to simplify the source term.

Here, we decompose the wavefunction in the following form,
\[
\psi^{\text{even}}_{20}(t, r) = \Psi^{\text{out}}_{20}(t, r)\delta(r - R(t)) + \Psi^{\text{in}}_{20}(t, r)\theta(R(t) - R) \\
= \Psi^{\text{out}}_{20}(t, r)\theta(r - R(t)) + \Psi^{\text{out}}_{20}(t, r);
\]
\[
\Psi^{\text{out}}_{20}(t, r) = \Psi^{\text{out}}_{20}(t, r) - \Psi^{\text{in}}_{20}(t, r), \quad \Psi^{\text{out}}_{20}(t, r) = \Psi^{\text{in}}_{20}(t, r),
\]
where \(\Psi^{\text{out}}_{20}\) and \(\Psi^{\text{in}}_{20}\), i.e., also \(\Psi^{\text{out}}\) and \(\Psi^{\text{in}}\), are homogeneous solutions to the Zerilli equation. Using the fact that the first-order metric perturbations are \(C^0\), we can derive the following six quantities,
\[
\psi^{\text{out}}_{20}(t, r) \bigg|_{r=R(t)}, \quad \frac{\partial}{\partial r} \psi^{\text{out}}_{20}(t, r) \bigg|_{r=R(t)}, \quad \frac{\partial}{\partial t} \psi^{\text{out}}_{20}(t, r) \bigg|_{r=R(t)}, \quad \frac{\partial}{\partial r} \psi^{\text{out}}_{20}(t, r) \bigg|_{r=R(t)}, \quad \frac{\partial}{\partial t} \psi^{\text{out}}_{20}(t, r) \bigg|_{r=R(t)}, \quad \frac{\partial}{\partial r} \psi^{\text{out}}_{20}(t, r) \bigg|_{r=R(t)}.
\]
Here, first we take the derivatives, and then set \(r = R(t)\). These quantities allow us to calculate the coefficients of the \(\delta\) terms in the second-order source.
Next, we consider the $\ell = 0$ perturbation which is present only in even parity. The metric perturbations and the gauge transformation are given as

$$h_{00}^{(1)} = \left(1 - \frac{2M}{r}\right) H_{100}^{(1)}(t, r) a_{00} - i\sqrt{2} H_{100}^{(1)}(t, r) a_{100} + \left(1 - \frac{2M}{r}\right)^{-1} H_{200}^{(1)}(t, r) a_{00} + \sqrt{2} K_{00}^{(1)}(t, r) g_{00}.$$  

(47)

$$x^\mu \rightarrow x^\mu + \xi_{\mu}^{(1)}(x^\alpha), \quad \xi_{\mu}^{(1)} = \left\{ V_0^{(1)}(t, r) Y_{00}(\theta, \phi), V_1^{(1)}(t, r) Y_{00}(\theta, \phi), 0, 0 \right\}.$$  

(48)

respectively. Here, $V_0^{(1)}$ and $V_1^{(1)}$ are the harmonics coefficients which denote two degrees of gauge freedom in the $\ell = 0$ mode. The metric perturbations transform under the above gauge transformation from the $G$ gauge to the $G'$ gauge as

$$H_{000}^{(1)G'}(t, r) = H_{000}^{(1)G}(t, r) + \frac{2}{r} \frac{\partial}{\partial t} V_0^{(1)G\rightarrow G'}(t, r), \quad V_0^{(1)G\rightarrow G'}(t, r) = \frac{2M}{r(r - 2M)} V_1^{(1)G\rightarrow G'}(t, r),$$  

(49)

$$H_{100}^{(1)G'}(t, r) = H_{100}^{(1)G}(t, r) - \frac{r}{2M} \frac{\partial}{\partial r} V_0^{(1)G\rightarrow G'}(t, r), \quad V_0^{(1)G\rightarrow G'}(t, r) = \frac{2M}{r(r - 2M)} V_1^{(1)G\rightarrow G'}(t, r),$$  

(50)

$$H_{200}^{(1)G'}(t, r) = H_{200}^{(1)G}(t, r) - \frac{2}{r} \frac{\partial}{\partial r} V_1^{(1)G\rightarrow G'}(t, r), \quad V_1^{(1)G\rightarrow G'}(t, r) = \frac{2M}{r(r - 2M)} V_1^{(1)G\rightarrow G'}(t, r),$$  

(51)

$$K_{00}^{(1)G'}(t, r) = K_{00}^{(1)G}(t, r) + \frac{2}{r} V_1^{(1)G\rightarrow G'}(t, r).$$  

(52)

Here, we can choose $V_0^{(1)G\rightarrow G'}$ and $V_1^{(1)G\rightarrow G'}$ so that $H_{100}^{(1)Z} = K_{00}^{(1)Z} = 0$ where the suffix $Z$ stands for the Zerilli gauge [41]. In this gauge, we obtain a

$$H_{000}^{(1)Z}(t, r) = 8\pi \mu E \left(\frac{3}{r^4} \frac{1}{r - 2M} \frac{1}{R(t) - 2M} - \frac{R(t)^2}{(R(t) - 2M)^3} \left(\frac{dR(t)}{dt}\right)^2\right) \times Y_{00}^{*}(0, 0) \theta(r - R(t)).$$  

(53)

$$H_{000}^{(1)Z}(t, r) = 8\pi \mu E \left(\frac{3}{r^4} \frac{1}{r - 2M} \frac{1}{(R(t) - 2M)^3} \left(\frac{dR(t)}{dt}\right)^2\right) \times Y_{00}^{*}(0, 0) \theta(r - R(t)).$$  

(54)

It is difficult however to construct the second-order source from the above metric perturbations, since these are not $C^0$. We instead consider a new (singular) gauge transformation, chosen to make the metric perturbations $C^0$.

We consider the following gauge transformation. We call this the $C$ gauge, where the first-order metric perturbations are $C^0$ at the particle location

$$V_0^{(1)Z\rightarrow C}(t, r) = \frac{2\pi \mu Y_{00}^{*}(0, 0)}{3E} \left(r - 2M(r + 2M)(r^2 + 4M^2)\right) \text{INT}(t) - \frac{2\pi \mu E Y_{00}^{*}(0, 0)}{3}$$

$$\times \frac{r - 2M}{R(t) - 2M} \frac{dR(t)}{dt} \left(-r^2 R(t)^3 - R(t)^2 r^3 - r^4 R(t) + 10r^2 M R(t)^2 + 10M R(t)^3 + 8Mr^4 + 10rM R(t) - 16rM^2 R(t)^3 - 16M^2 R(t)^2 - 16M^2 r^2 - R(t)^4 r + 10R(t)^5 - 62R(t)^4 M \right) + 72R(t)^3 M^2 \theta(r - R(t)).$$  

(55)
\[ V_1^{(1)Z\rightarrow C}(t, r) = 4\pi\mu E Y_{00}^0(0, 0) \frac{(r - 2M)(r - R(t))R(t)^6}{r^6(R(t) - 2M)^2}\theta(r - R(t)); \]  

(56)

\[ \text{INT}(t) = \int \left\{ (-54M^2 + 39R(t)M - 50ME^2R(t) + 12E^2R(t)^2) - 6R(t)^2 + 16M^2E^2/(R(t) - 2M)^3 \right\} \text{d}t \]

\[ = 6\left(-1 + 2E^2\right) \frac{E}{\sqrt{1 - E^2}} \arctan \left\{ \left( \frac{M}{1 - E^2} - R(t) \right) \left[ R(t) \left( \frac{2M}{1 - E^2} - R(t) \right) \right]^{-1/2} \right\} \]

\[ + 12E^2 \ln \left\{ \left( \frac{4M^2}{1 - E^2} + \frac{2R(t)M}{1 - E^2} - 4R(t)M \right) \right\} \frac{4ME}{\sqrt{1 - E^2}} \left( \frac{R(t)}{2M - 2M} \right)^{1/2} \frac{R(t)}{2M} \}

\[ + \frac{(13R(t)^2 + 48M^2 - 56R(t)M)E^2R(t) - R(t) + 2M} {2M} \frac{1}{\sqrt{R(t)}} \].

(57)

We then obtain the \( \ell = 0 \) mode of the metric perturbations in this \( C \) gauge, \( H_{000}^{(1)C} \), \( H_{100}^{(1)C} \), \( H_{200}^{(1)C} \) and \( K_{00}^{(1)C} \) by using (49), (50), (51) and (52). Note that all of the above metric perturbations are \( C^0 \) at the particle location and go to zero at \( r = \infty \) and the horizon. Using the metric perturbations in the \( C \) gauge, we have discussed the second-order source in [54]. But, with this method, we cannot obtain the second-order gravitational wave at infinity because this is not an asymptotic flat gauge.

We therefore must consider another treatment. The metric perturbations in the \( C \) gauge have the following asymptotic behavior for large \( r \):

\[ H_{000}^{(1)C}(t, r) = \left( 8\frac{\mu\pi E}{r} + 16\frac{\mu\pi EM}{r^2} \right) Y_{00}^0(0, 0) + O(r^{-3}), \quad H_{100}^{(1)C}(t, r) = O(r^{-4}); \]

\[ H_{200}^{(1)C}(t, r) = \left( 8\frac{\mu\pi E}{r} + 16\frac{\mu\pi EM}{r^2} \right) Y_{00}^0(0, 0) + O(r^{-3}), \quad K_{00}^{(1)C}(t, r) = O(r^{-5}). \]

(58)

Hence, the metric up to the first perturbative order in the system becomes

\[ ds^2 = -\left(1 - \frac{2M}{r} \right) \left(1 - H_{000}^{(1)C}(t, r)Y_{00}(\theta, \phi) \right) dt^2 + \frac{H_{100}^{(1)C}(t, r)Y_{00}(\theta, \phi) dt \ dr}{1 - \frac{2M}{r}} \]

\[ + \left(1 - \frac{2M}{r} \right)^{-1} \left(1 + H_{200}^{(1)C}(t, r)Y_{00}(\theta, \phi) \right) \ dr^2 + r^2 \left(1 + K_{00}^{(1)C}(t, r)Y_{00}(\theta, \phi) \right) d\Omega^2 \]

\[ \sim -\left(1 - \frac{2M + 2\mu E}{r} \right) dt^2 + \frac{1}{r} \left(1 - \frac{2M + 2\mu E}{r} \right)^{-1} dr^2 + r^2 d\Omega^2. \]

(59)

We thus find that this perturbation is related to the mass increase of the system.

From the above analysis, we define the total mass \( M_{\text{tot}} = M + \mu E \) as that of the system.

This means that the particle’s mass is absorbed in the background Schwarzschild mass. Then, the first-order \( \ell = 0 \) renormalized metric perturbations become

\[ H_{000}^{(1)N}(t, r) = H_{000}^{(1)C}(t, r) - H_{000}^{(1)M}(t, r), \]

\[ H_{100}^{(1)N}(t, r) = H_{100}^{(1)C}(t, r) - H_{100}^{(1)M}(t, r), \]

\[ H_{200}^{(1)N}(t, r) = H_{200}^{(1)C}(t, r), \]

\[ K_{00}^{(1)N}(t, r) = K_{00}^{(1)C}(t, r). \]

(60)
where we have labeled these renormalized metric perturbations by $N$, and

$$H^{(1)M}_{000}(t, r) = H^{(1)M}_{200}(t, r) = \frac{8}{r^2} \frac{\mu E}{r - 2M} Y^e_{00}(0, 0). \quad (61)$$

Now for the asymptotic behavior for large $r$, we have

$$H^{(1)N}_{000}(t, r) = O(r^{-4}), \quad H^{(1)N}_{200}(t, r) = O(r^{-5}). \quad (62)$$

To derive the second-order source, we will use the coefficients of the first-order metric perturbations labeled by $N$ in (60).

5.5. $\ell = 1, m = 0$ mode (dipole, even parity perturbation)

Although the $\ell = 1$ even parity mode in the vacuum regions can be completely eliminated in the center-of-mass coordinate system, the metric perturbation is not globally pure gauge [53]. We therefore have to include the $\ell = 1$ mode contributions. For this mode, the metric perturbations are given by

$$h^{(1)}_{10} = \left(1 - \frac{2M}{r}\right) H^{(1)}_{010}(t, r) a_{10} + i \sqrt{2} H^{(1)}_{110}(t, r) a_{110} + \left(1 - \frac{2M}{r}\right)^{-1} H^{(1)}_{210}(t, r) a_{10}
$$
$$- \frac{2i}{r} H^{(1)}_{010}(t, r) b_{10} + \frac{2}{r} H^{(1)}_{110}(t, r) b_{110} + \sqrt{2} K^{(1)}_{10}(t, r) g_{10}. \quad (63)$$

The generator of the gauge transformation can be written as

$$x^\mu \to x^\mu + \xi^{(1)}_{\ell=1} (x^\mu) = \left\{ \begin{array}{l}
V^{(1)}_0(t, r) Y_{10}(\theta, \phi), V^{(1)}_1(t, r) Y_{11}(\theta, \phi), V^{(1)}_2(t, r) \frac{\partial Y_{10}(\theta, \phi)}{\sin^2 \theta}
\end{array} \right\}. \quad (64)$$

Here, $V^{(1)}_0$, $V^{(1)}_1$, and $V^{(1)}_2$ are coefficients which denote the three degrees of gauge freedom in the $\ell = 1$ mode. The metric perturbations transform under the above gauge transformation from a $G$ gauge to a $G'$ gauge as

$$H^{(1)G}_{010}(t, r) = H^{(1)G}_{010}(t, r) + 2 \frac{\partial}{\partial t} V^{(1)G\to G'}_0(t, r) + 2 \frac{M}{r (r - 2M)} V^{(1)G\to G'}_1(t, r), \quad (65)$$

$$H^{(1)G}_{110}(t, r) = H^{(1)G}_{110}(t, r) + \frac{r - 2M}{r} \frac{\partial}{\partial r} V^{(1)G\to G'}_0(t, r) - \frac{r}{r - 2M} \frac{\partial}{\partial t} V^{(1)G\to G'}_1(t, r), \quad (66)$$

$$H^{(1)G}_{210}(t, r) = H^{(1)G}_{210}(t, r) - \frac{2}{r} \frac{\partial}{\partial r} V^{(1)G\to G'}_1(t, r) + 2 \frac{M}{r (r - 2M)} V^{(1)G\to G'}_1(t, r), \quad (67)$$

$$K^{(1)G}_{10}(t, r) = K^{(1)G}_{10}(t, r) - \frac{2}{r} V^{(1)G\to G'}_1(t, r) + 2 V^{(1)G\to G'}_2(t, r), \quad (68)$$

$$h^{(1)G}_{010}(t, r) = h^{(1)G}_{010}(t, r) + \frac{r - 2M}{r} V^{(1)G\to G'}_0(t, r) - r \frac{\partial}{\partial t} V^{(1)G\to G'}_2(t, r), \quad (69)$$

$$h^{(1)G}_{110}(t, r) = h^{(1)G}_{110}(t, r) - \frac{r - 2M}{r} V^{(1)G\to G'}_1(t, r) - r^2 \frac{\partial}{\partial r} V^{(1)G\to G'}_2(t, r). \quad (70)$$
When we choose \( V_0^{(0)G\rightarrow G'}, V_1^{(0)G\rightarrow G'} \) and \( V_2^{(0)G\rightarrow G'} \) so that \( h_{010}^{(0)Z} = h_{110}^{(0)Z} = K_{10}^{(1)Z} = 0 \),
where the suffix \( Z \) stands for the Zerilli gauge [41], we obtain

\[
H_{010}^{(1)Z}(t, r) = \frac{8\pi \mu E}{3M(r-2M)^2} \left( r^3 \frac{d^2 R(t)}{dt^2} + M(R(t) - 2M) \right) Y_{10}^*(0, 0) \theta(r - R(t)),
\]

(71)

\[
H_{110}^{(1)Z}(t, r) = -\frac{8\pi \mu E r}{(r-2M)^2} \frac{dR(t)}{dt} Y_{10}^*(0, 0) \theta(r - R(t)),
\]

(72)

\[
H_{210}^{(1)Z}(t, r) = \frac{8\pi \mu E}{(r-2M)^2} (R(t) - 2M) Y_{10}^*(0, 0) \theta(r - R(t)).
\]

(73)

Here, we considered the same treatment as for the \( \ell = 0 \) mode. In order to obtain the \( C^0 \) metric perturbations at the particle location, we use the following gauge transformation,

\[
V_0^{(1)Z\rightarrow C}(t, r) = 4\pi \mu E Y_{10}^*(0, 0) \frac{R(t)^6(r - R(t))}{(R(t) - 2M)^2 r^2} \frac{dR(t)}{dt} \theta(r - R(t))
\]

\[-\frac{4}{3} \pi \mu E Y_{10}^*(0, 0) \frac{(r - 2M)^2(r^3 + 8r^2 M + 40M^2 r + 160M^3)}{r^3 M} \frac{dR(t)}{dt},
\]

(74)

\[
V_1^{(1)Z\rightarrow C}(t, r) = 4\pi \mu E Y_{10}^*(0, 0) \frac{R(t)^4(R(t) - 2M)}{(R(t) - 2M)^2 r^4} \theta(r - R(t))
\]

\[-\frac{4}{3} \pi \mu E Y_{10}^*(0, 0) \frac{(r - 2M)^2(r^2 + 6r M + 24M^2)}{r^4 M},
\]

(75)

\[
V_2^{(1)Z\rightarrow C}(t, r) = -\frac{4}{3} \pi \mu E Y_{10}^*(0, 0) \frac{(r - 2M)^2(r^2 + 6r M + 24M^2)}{r^3 M},
\]

(76)

where we have used the part of the step function to make the metric perturbations \( C^0 \), and the homogeneous transformation to obtain well-behaved metric perturbations at \( r = 2M \) and \( r \to \infty \). The latter transformation is derived by a series expansion with respect to \( M \) for the
gauge transformation to the center of mass of the system. Then, the metric perturbations in the \( C \) gauge are calculated by using (65), (66), (67), (68), (69) and (70), and have the following asymptotic behavior for large \( r \):

\[
H_{010}^{(1)C}(t, r) = O(r^{-3}), \quad H_{110}^{(1)C}(t, r) = O(r^{-3}),
\]

\[
H_{210}^{(1)C}(t, r) = O(r^{-4}), \quad K_{10}^{(1)C}(t, r) = O(r^{-4}),
\]

(77)

6. Second-order perturbations in the Regge–Wheeler gauge

Since the leading radiative modes asymptotically far from the system are dominated by the \( \ell = 2 \) mode we will restrict ourselves to this second-order correction to the radiation. As discussed in [31, 32] in an asymptotically flat gauge the radiative behavior is ‘gauge invariant’ and the near zone of the source terms can be changed by different gauge choices without affecting the computation of radiation at infinity. We make use of this freedom to compute the leading second-order corrections by considering only the \( (\ell = 2) \cdot (\ell = 2) \), \( (\ell = 0) \cdot (\ell = 2) \) and \( (\ell = 1) \cdot (\ell = 1) \) couplings.
6.1. The second-order Zerilli equation

Since the first-order metric perturbations contain only $m = 0$ even parity modes, we can discuss the second-order metric perturbations via the Zerilli equation. We will choose the RW gauge condition. Here, we use a wavefunction for the second perturbative order,

$$\chi^{Z}_{20}(t, r) = \frac{1}{2r + 3M} \left( r^2 \frac{\partial}{\partial t} K^{(2)RW}_{20}(t, r) - (r - 2M) H^{(2)RW}_{120}(t, r) \right).$$  \hspace{1cm} (78)

This is the same definition as in (34) for the first perturbative order. Here, we have considered the contribution from the $\ell = 0, 1$ and $2$ modes of the first perturbative order to the $\ell = 2$ mode of the second perturbative order since this gives the leading contribution to gravitational radiation. This Zerilli function satisfies the equation

$$\frac{\partial^2}{\partial t^2} \chi^{Z}_{20}(t, r) = \left[ - \frac{\partial^2}{\partial t^2} + \frac{\partial}{\partial r} - 6 \frac{(r - 2M)(4r^3 + 4r^2M + 6rM^2 + 3M^3)}{r^4(2r + 3M)^2} \right] \chi^{Z}_{20}(t, r) = S_{20}^{Z}(t, r);$$ \hspace{1cm} (79)

$$S_{20}^{Z}(t, r) = \frac{8\pi \sqrt{3} (r - 2M)^2}{3(2r + 3M)} \frac{\partial}{\partial t} \chi^{Z}_{20}(t, r) + \frac{8\pi (r - 2M)^2}{2r + 3M} \frac{\partial A_{20}^{(2)}(t, r)}{\partial t}$$

$$- \frac{8\sqrt{3}\pi (r - 2M)}{3} \partial \chi^{Z}_{20}(t, r) - \frac{4\sqrt{3}\pi (r - 2M)^2}{2r + 3M} \frac{\partial A_{20}^{(2)}(t, r)}{\partial r}$$

$$- \frac{8\sqrt{3}\pi (r - 2M)(5r - M)}{r (2r + 3M)} A_{120}^{(2)}(t, r) - \frac{8\sqrt{3}\pi (r - 2M)^2}{3(2r + 3M)} \frac{\partial B_{020}^{(2)}(t, r)}{\partial r} + \frac{32\sqrt{3}\pi (3M^2 + r^2)(r - 2M)}{3r(2r + 3M)^2} B_{020}^{(2)}(t, r).$$ \hspace{1cm} (80)

The functions $B^{(2)}_{20}$ etc are derived from the effective stress–energy tensor on the left-hand side of (9),

$$T^{(2, eff)}_{\mu\nu} = T^{(2)}_{\mu\nu} - \frac{1}{8\pi} G^{(2)}_{\mu\nu}[h^{(1)}, h^{(1)}],$$ \hspace{1cm} (81)

by the same tensor harmonics expansion as for the first perturbative order. (We do not discuss $T^{(2)}_{\mu\nu}$ here.)

The second-order metric perturbations from $\chi^{Z}_{20}$ in the RW gauge are given by

$$\frac{\partial}{\partial t} K^{(2)RW}_{20}(t, r) = \frac{6(r^2 + rM + M^2)}{(2r + 3M)r^2} \chi^{Z}_{20}(t, r) + \frac{(r - 2M)}{r} \frac{\partial}{\partial r} \chi^{Z}_{20}(t, r)$$

$$+ \frac{4\sqrt{3}\pi r (r - 2M)}{2r + 3M} A_{120}^{(2)}(t, r) + \frac{8\sqrt{3}\pi (r - 2M)}{3(2r + 3M)} B_{020}^{(2)}(t, r),$$ \hspace{1cm} (82)

$$\frac{\partial}{\partial t} H^{(2)RW}_{120}(t, r) = \frac{r^2}{\partial r} \frac{\partial}{\partial t} K^{(2)RW}_{20}(t, r) + \frac{3M}{r^2} \frac{\partial}{\partial r} \chi^{Z}_{20}(t, r) - \frac{(2r + 3M)}{r} \frac{\partial}{\partial r} \chi^{Z}_{20}(t, r)$$

$$- \frac{8\sqrt{3}}{3} i\pi B_{020}^{(2)}(t, r),$$ \hspace{1cm} (83)

$$H^{(2)RW}_{020}(t, r) = H^{(2)RW}_{120}(t, r) + \frac{8\sqrt{3}}{3} i\pi F^{(2)}_{20}(t, r) r^2.$$ \hspace{1cm} (84)

Since the RW gauge is not asymptotically flat, we need to derive the second-order metric perturbations in an asymptotic flat (AF) gauge to obtain the second-order gravitational wave at spatial infinity. This is discussed in the appendix.
In the second-order source, given in (80), we may wonder if there is any \( \delta^2 \) term which prevents us from making the second-order calculation. The answer is ‘no’. This is because, in the head-on collision case, the first-order metric perturbations in the RW gauge are \( C^0 : G_{\mu\nu}^{(2)}(h^{(1)}, h^{(1)}) \) includes second derivatives and we need one more derivative to construct the second-order source of (80). Here, \( (h^{(1)})^2 \) is \( C^0 \times C^0 \) and its third derivative yields \( C^0 \times \delta^2 \) and \( \theta \times \delta^2 \) as the most singular terms. Note also that there are no \( \delta^2 \) terms coming from \( T_{\mu\nu}^{(2)} \) [42], which we ignored in this paper. Using the result of [25, 26], we can include the contributions of \( T_{\mu\nu}^{(2)} \).

In the following subsection, we derive the second-order source of the Zerilli equation in (80). The summary is given here. From (80), we obtain the second-order source as

\[
S_{20}^{\ell m}(t, r) = (2, 2)S_{20}^{\ell m}(t, r) + (0, 2)S_{20}^{\ell m}(t, r) + (1, 1)S_{20}^{\ell m}(t, r),
\]

where \( (2, 2)S_{20}^{\ell m}, (0, 2)S_{20}^{\ell m} \) and \( (1, 1)S_{20}^{\ell m} \) are the contributions from the \( (\ell = 2) \cdot (\ell = 2), (\ell = 0) \cdot (\ell = 2) \) and \( (\ell = 1) \cdot (\ell = 1) \) couplings, respectively. Note that while the above source term is locally integrable near the particle’s location, some terms diverge as \( r \to \infty \) or \( 2M \).

This is not readily suitable for numerical calculations. We then consider some regularization for the asymptotic behavior. (See e.g. [31,1].) In order to obtain a second-order source which behaves well everywhere, we define a regularized Zerilli function by

\[
\tilde{\chi}_{20}^Z(t, r) = \chi_{20}^Z(t, r) - \chi_{20}^{reg, (2, 2)} - \chi_{20}^{reg, (0, 2)}.
\]

Here, we should note that we do not need the regularization for the \( (\ell = 1) \cdot (\ell = 1) \) contribution, because the first-order metric perturbations for the \( \ell = 1 \) mode are well behaved in the whole region\(^1\). The best suited equation to solve the Zerilli equation numerically is then

\[
\tilde{\chi}_{20}^{even}(t, r) = S_{20}^{reg}(t, r),
\]

where the regular source \( S_{20}^{reg} \) is given by

\[
S_{20}^{reg}(t, r) = (2, 2)S_{20}^{reg}(t, r) - \tilde{\chi}_{20}^{even, (2, 2)}(t, r) + \chi_{20}^{reg, (0, 2)} + (1, 1)S_{20}^{reg}(t, r).
\]

In the following, we focus on the contribution from \( (\ell = 2) \cdot (\ell = 2) \) and \( (\ell = 0) \cdot (\ell = 2) \).

6.2. The regularized second-order source from \( (\ell = 2) \cdot (\ell = 2) \)

When we consider the asymptotic behavior of the second-order source for large \( r \), we use the retarded solution of the first-order wavefunction \( \psi_{20}^{even} \) with the retarded time \( t - r^* \), which we expand in inverse powers of \( r \). The wavefunction becomes

\[
\psi_{20}^{even}(t, r) = F(t - r^*) + \frac{3}{r} F(t - r^*) + O(r^{-2}),
\]

where we have introduced a function \( F \) which satisfies the asymptotic expansion of the Zerilli equation (43) for large \( r \) and \( F(x) = dF(x)/dx \). Equation (89) can be obtained by substituting the Zerilli equation.

On the other hand, the wavefunction is expanded near the horizon as

\[
\psi_{20}^{even}(t, r) = F_{20}(t + r^*) + \frac{1}{4M} F_{20}(t + r^*) + \frac{27(r - 2M)}{56M^2} F_{20}(t + r^*) + O((r - 2M)^2),
\]

where we have also introduced a function \( F_{20} \) which satisfies the asymptotic expansion of the Zerilli equation (43) near the horizon, and has the variable \( (t + r^*) \). Using these expansions (89) and (90), we derive a second-order source which is regular everywhere.

\(^1\) \( (1, 1)S_{20}^{reg} \) behaves as \( O(r^{-6}) \) for large \( r \), and vanishes as \( O(r - 2M)^3 \) at \( r = 2M \).
In order to obtain a well-behaved source for large values of \( r \), we define a regularization function by

\[
\chi^{\text{reg},(2,2)}_{20}(t, r) = \frac{\sqrt{5}}{7\sqrt{\pi}} \frac{r^2}{2r + 3M} \left( \frac{\partial}{\partial t} K^{(1)\text{RW}}_{20}(t, r) \right) K^{(1)\text{RW}}_{20}(t, r).
\]

(91)

Note that the regularization function is not unique. Therefore, this affects the formal expression of the second-order gravitational wave as in (A.27). However, for any specific computation, there is no ambiguity in the physical final results.

Using the above regularization function, the regularized second-order source \( Z^{\text{reg}}_{20} \) from the \((\ell = 2) \cdot (\ell = 2)\) coupling is obtained in the following form:

\[
Z^{\text{reg}}_{20}(t, r) = \left( Z^{(2,2)\text{reg}}_{20}(t, r) - \frac{2}{\sqrt{2\pi}} \chi^{\text{reg},(2,2)}_{20}(t, r) \right)
\]

\[
= G^{(2,2)} \left( Z_{20}^{\text{out}}(t, r) \theta(r - R(t)) + G^{(2,2)} S_{20}^{\text{in}}(t, r) \theta(R(t) - r)ight)
\]

\[
+ G^{(2,2)} S_{20}^{\text{ef}}(t, r) \frac{d}{dr} \delta(r - R(t)),
\]

(92)

where the superscripts \( G \) denote the source terms which are derived from \( G^{(2,2)}[h^{(1)}, h^{(1)}] \). For the factors of the step functions in the above equation, \( G^{(2,2)} S_{00}^{\text{out}} \) behaves as \( O(r^{-2}) \) for large \( r \), and \( G^{(2,2)} S_{20}^{\text{in}} \) vanishes as \( O((r - 2M)^4) \) at \( r = 2M \).

6.3. The regularized second-order source from \((\ell = 0) \cdot (\ell = 2)\)

The second-order source derived from the \((\ell = 0) \cdot (\ell = 2)\) coupling needs regularization of its behavior near the horizon. Note that when we choose another gauge for the first-order \( \ell = 0 \) mode, some regularization is also needed for the behavior for large \( r \). To this end, we use the regularization function,

\[
\chi^{\text{reg},(0,2)}_{20}(t, r) = \frac{1061r - 2728M}{5488\sqrt{\pi}} r \frac{H^{(1)N}_{200}(t, r) H^{(1)\text{RW}}_{220}(t, r)}{H^{(1)N}_{200}(t, r) \frac{d}{dr} H^{(1)\text{RW}}_{220}(t, r)}
\]

\[
- \frac{107}{2744\sqrt{\pi}} \frac{(r - 2M) M}{r} \frac{H^{(1)N}_{200}(t, r) \frac{d}{dr} H^{(1)\text{RW}}_{220}(t, r)}{H^{(1)N}_{200}(t, r)}
\]

\[
+ \frac{1}{2744\sqrt{\pi}} M (583r - 756M) \frac{H^{(1)N}_{200}(t, r) \frac{d}{dr} H^{(1)\text{RW}}_{220}(t, r)}{H^{(1)N}_{200}(t, r)}.
\]

(93)

Then, we obtain the second-order regularized source from the \((\ell = 0) \cdot (\ell = 2)\) coupling

\[
S^{(2,2)\text{reg}}_{20}(t, r) = \left( S^{(2,2)\text{reg}}_{20}(t, r) - \frac{2}{\sqrt{2\pi}} \chi^{\text{reg},(2,2)}_{20}(t, r) \right)
\]

\[
= G^{(2,2)} \left( S^{\text{out}}_{00}(t, r) \theta(r - R(t)) + G^{(2,2)} S^{\text{in}}_{20}(t, r) \theta(R(t) - r)ight)
\]

\[
+ G^{(2,2)} S^{\text{ef}}_{20}(t, r) \frac{d}{dr} \delta(r - R(t)),
\]

(94)

In the above equation, \( G^{(2,2)} S^{\text{out}}_{00} \) behaves as \( O(r^{-2}) \) for large \( r \), and \( G^{(2,2)} S^{\text{in}}_{20} \) vanishes as \( O((r - 2M)^4) \) at \( r = 2M \).

7. Summary and discussion

In this paper, we have discussed the problem of binary black hole second-order perturbations in the extreme mass ratio limit. Our analysis applies to head-on collisions, but can be extended to arbitrary orbits if worked in the Lorenz gauge [55] since our approach relies basically on the continuity of the metric coefficients at the particle location. The first-order perturbations of two black holes starting from rest at a finite distance have been solved in the frequency [56]...
and time [57] domains. Then the corrected trajectories (from the background geodesics) via the computation of the self-force have been computed in [25, 26].

Here, we completed the program by obtaining the regularized source for the second-order Zerilli equation in the case of a particle falling radially into a Schwarzschild black hole. This is given by (88) with (92) and (94). Using this second-order source, one would be able to compute the second-order contributions to gravitational radiation by numerical integration of the wave equation (87) and compare to full numerical simulations. A key property of the effective source term we build up is its regularity at the horizon and particle location, and the appropriate fall-off at large distances \( r \) from the source. The derivation of the radiative second-order gravitational waveform from the second-order Zerilli function is discussed in the appendix.

To prove that there are no \( \delta^2 \) terms in the second-order source we have used the fact that the first-order metric perturbations in the RW gauge are \( C^0 \). In the general orbit case (including circular orbits), although the first-order metric perturbations are not \( C^0 \) in the RW gauge, it is possible to build up a well-behaved second-order source term following the discussions in section 3. On the other hand, the first-order metric perturbations are \( C^0 \) in the Lorenz gauge [55]. Therefore, it seems that this later gauge choice favors the study of the second-order perturbations for generic orbits (as well as for the self-force computation).

We have shown that the \( \ell = 2 \) mode of the second-order effective source can be regularized at the particle location as well as at the horizon and radial infinity. Even if assuming (as one would expect) that the same can be done for all higher \( \ell \)-modes, the sum over all \( \ell \) could be non-converging at the particle location. The resolution of this issue can be reached by a proper choice of (an asymptotically flat) gauge that removes the non-convergent part of each \( \ell \)-mode at the particle location (and the region close to the holes) leaving unchanged the asymptotically large \( r \) region. Since we know that in this asymptotic (radiation) region the waveforms are strongly dominated by the \( \ell = 2 \) mode (from the results of full numerical simulations, for instance), our computation, based on a global \( \ell = 2 \) effective source is justified as providing a good approximation to the second-order radiative corrections.

To be fully consistent in the second perturbative order, we have to include the self-force contribution \( T^{(2)}_{\mu \nu} \) which is derived from the deviation from the background geodesic motion. The self-force for a head-on collision has been calculated in [25, 26], and in a circular orbit around a Schwarzschild black hole in [58–60], but has not yet been obtained in the general case.

Acknowledgments

We would like to thank K Ioka, K Nakamura and T Tanaka for useful discussions. This work is supported by JSPS for Research Abroad (HN) and by NSF grants PHY-0722315, PHY-0701566, PHY-0714388, and PHY-0722703, and from a NASA grant 07-ATFP07-0158.

Appendix. Second-order gauge transformation

In this appendix, we deal with first- and second-order gauge transformations. In order to obtain the second-order waveform, it is necessary to derive the second-order metric perturbations in an asymptotic flat (AF) gauge. The Regge–Wheeler–Zerilli formalism that we have employed in the RW gauge is not asymptotically flat. Therefore, we will focus on the gauge transformation from the RW gauge to an AF gauge. We also need to discuss the first-order gauge transformation to an AF gauge simultaneously.
Here, we consider the following gauge transformation [43],

\[ x_{\text{RW}}^\mu \rightarrow x_{\text{AF}}^\mu = x_{\text{RW}}^\mu + \xi^{(1)\mu} (x^\nu) + \frac{1}{2} \left( \xi^{(2)\mu} (x^\nu) + \xi^{(1)\nu} \xi^{(1)\mu} (x^\nu) \right), \quad (A.1) \]

where comma ‘,’ in the index indicates the partial derivative with respect to the background coordinates, and \( \xi^{(1)\mu} \) and \( \xi^{(2)\mu} \) are generators of the first- and second-order gauge transformations, respectively. Then, the metric perturbations change as

\[ h_{\text{RW}}^{(1)\mu\nu} \rightarrow h_{\text{AF}}^{(1)\mu\nu} = h_{\text{RW}}^{(1)\mu\nu} - \mathcal{L}_{\xi^{(1)}} g_{\mu\nu}, \]

\[ h_{\text{RW}}^{(2)\mu\nu} \rightarrow h_{\text{AF}}^{(2)\mu\nu} = h_{\text{RW}}^{(2)\mu\nu} - \frac{1}{2} \mathcal{L}_{\xi^{(2)}} g_{\mu\nu} + \frac{1}{2} \mathcal{L}_{\xi^{(1)}} \mathcal{L}_{\xi^{(1)}} h_{\text{RW}}^{(1)\mu\nu}, \quad (A.2, A.3) \]

where \( \mathcal{L}_{\xi^{(\ell)}} \) denotes the Lie derivative. Next, we discuss the \( \ell = 2 \) \( \cdot (\ell = 2) \) and \( \ell = 0 \) \( \cdot (\ell = 2) \) parts separately, and it is not necessary to consider the \( \ell = 1 \) \( \cdot (\ell = 1) \) part because the first-order \( \ell = 1 \) mode has already been in the AF gauge and we do not need any regularization function for this part.

### A.1. First-order \( \ell = 2 \) mode and second-order \( \ell = 2 \) \( \cdot (\ell = 2) \) part

In this paper, we have used only the even parity mode; therefore, a generator of the gauge transformation for \( \ell = 2, m = 0 \) modes can be written as

\[ \xi^{(i)\mu}_{\ell = 2} = \left\{ V_0^{(i)} (t, r) Y_{20} (\theta, \phi), V_1^{(i)} (t, r) Y_{20} (\theta, \phi), \right. \\
\[ \left. \times V_2^{(i)} (t, r) \frac{\partial Y_{20} (\theta, \phi)}{\sin^2 \theta} \right\}, \quad (A.4) \]

where \( i = 1 \) and \( 2 \) denote the first and second perturbative orders, respectively. Here, \( V_0^{(i)} \), \( V_1^{(i)} \) and \( V_2^{(i)} \) are the harmonics coefficients which denote three degrees of gauge freedom for each perturbative order.

The gauge transformation of the metric perturbations is explicitly given as follows: for the first-order metric perturbations, we find

\[ H_{20}^{(1)\text{AF}} (t, r) = H_{20}^{(1)\text{RW}} (t, r) + 2 \frac{\partial}{\partial t} V_0^{(1)} (t, r) + 2 M \frac{1}{r (r - 2M)} V_1^{(1)} (t, r), \]

\[ H_{120}^{(1)\text{AF}} (t, r) = H_{120}^{(1)\text{RW}} (t, r) + (r - 2M) \frac{\partial}{\partial r} V_0^{(1)} (t, r) - r \frac{\partial}{\partial t} V_1^{(1)} (t, r), \]

\[ H_{220}^{(1)\text{AF}} (t, r) = H_{220}^{(1)\text{RW}} (t, r) - 2 \frac{\partial}{\partial r} V_1^{(1)} (t, r) + 2 M \frac{1}{r (r - 2M)} V_1^{(1)} (t, r), \]

\[ K_{20}^{(1)\text{AF}} (t, r) = K_{20}^{(1)\text{RW}} (t, r) - \frac{2}{r} V_1^{(1)} (t, r), \]

\[ h_{020}^{(1)\text{AF}} (t, r) = \frac{(r - 2M)}{r} V_0^{(1)} (t, r) - r \frac{\partial}{\partial t} V_1^{(1)} (t, r), \]

\[ h_{120}^{(1)\text{AF}} (t, r) = - \frac{r}{r - 2M} V_1^{(1)} (t, r) - r \frac{\partial}{\partial t} V_2^{(1)} (t, r), \]

\[ G_{20}^{(1)\text{AF}} (t, r) = - 2 V_1^{(1)} (t, r). \]

For the second-order metric perturbations, we can calculate the gauge transformation straightforwardly, but we obtain very long expressions. For example, they can be written formally as

\[ K_{20}^{(2)\text{AF}} (t, r) = K_{20}^{(2)\text{RW}} (t, r) - \frac{1}{r} V_1^{(2)} (t, r) + \delta K_{20}^{(2)} (t, r), \quad (A.6) \]
perturbations in the RW gauge. The asymptotic expansion of the wavefunction
given, from (38), as follows:

\[ \psi_{20}(t, r) = \frac{r^2}{2(r - 2M)} V_1(t, r) + \frac{r}{2} \frac{\partial}{\partial r} V_2(t, r) + \delta h_{120}^{(2)(e)}(t, r). \]  
(A.7)

\[ G_{20}^{(2)(AF)}(t, r) = -V_2^{(1)}(t, r) + \delta G_{20}^{(2)(AF)}(t, r), \]  
(A.8)

where \( \delta K_{20}^{(2)} \), \( \delta h_{120}^{(2)(e)} \) and \( \delta G_{20}^{(2)} \) are defined by the tensor harmonics expansion of the last two terms on the right-hand side of (A.3). This includes only quadratic terms of the first-order wavefunction.

First, we consider the asymptotic behavior on the \( \ell = 2 \) mode of the first-order metric perturbations in the RW gauge. The asymptotic expansion of the wavefunction \( \psi_{20}^{\text{even}} \) is given by

\[ \psi_{20}^{\text{even}}(t, r) = \frac{1}{3} \frac{d^2}{dT_r} F(T_r) + \left( \frac{d}{dT_r} F(T_r) \right) r^{-1} + \left( F(T_r) - M \frac{d}{dT_r} F(T_r) \right) r^{-2} + O(r^{-3}), \]  
(A.9)

where we have introduced a function \( F \) which satisfies the asymptotic expansion of the Zerilli equation (43) for large \( r \) and has the variable \( (t - r_*) \), and we have introduced \( T_r = t - r_*(r) \) for simplicity. In the following calculation, we need only the leading order contribution with respect to the above large \( r \) expansion. Then, the coefficients of the metric perturbations are given, from (38), as follows:

\[ H_{020}^{(1)(RW)}(t, r) = H_{220}^{(1)(RW)}(t, r) = \frac{1}{3} \left( \frac{d^4}{dT_r^4} F(T_r) \right) r + O(r^0), \]

\[ H_{120}^{(1)(RW)}(t, r) = -\frac{1}{3} \left( \frac{d^3}{dT_r^3} F(T_r) \right) r + O(r^0), \]  
(A.10)

\[ K_{20}^{(1)(RW)}(t, r) = -\frac{1}{3} \frac{d^3}{dT_r^3} F(T_r) + O(r^{-1}). \]

On the other hand, the metric perturbations in an AF gauge should behave as

\[ H_{020}^{(1)(AF)}(t, r) = H_{120}^{(1)(AF)}(t, r) = h_{220}^{(1)(1)(AF)}(t, r) = 0, \quad H_{220}^{(1)(AF)}(t, r) = O(r^{-3}), \]

\[ h_{120}^{(e)(1)(AF)}(t, r) = O(r^{-1}), \quad K_{20}^{(1)(AF)}(t, r) = O(r^{-1}), \quad G_{20}^{(1)(AF)}(t, r) = O(r^{-1}). \]  
(A.11)

This asymptotic behavior will also be considered for the second-order calculation. We find the following gauge transformation to go to the AF gauge:

\[ V_0^{(1)}(t, r) = -\frac{1}{6} \left( \frac{d^3}{dT_r^3} F(T_r) \right) r + O(r^0), \]

\[ V_1^{(1)}(t, r) = \frac{1}{6} \left( \frac{d^3}{dT_r^3} F(T_r) \right) r + O(r^0), \]  
(A.12)

\[ V_2^{(1)}(t, r) = -\frac{1}{6} \left( \frac{d^2}{dT_r^2} F(T_r) \right) r^{-1} + O(r^{-2}). \]

The above results are calculated iteratively for large \( r \) expansion. Since the transverse-traceless tensor harmonics for the even parity part is \( f_{lm}^{(2)} \) in (21), the coefficient of the metric perturbations related to the gravitational wave is \( G_{lm}^{(1)(AF)} \). This becomes

\[ G_{20}^{(1)(AF)}(t, r) = \frac{1}{3} \frac{d^2}{dT_r^2} F(T_r) + O(r^{-2}) \]

\[ = \frac{1}{r} \psi_{20}^{\text{even}}(t, r) + O(r^{-2}) \]  
(A.13)

with the use of (A.9).
Next, we discuss the second perturbative order. When we treat the second-order metric perturbations from the \((\ell = 2) \cdot (\ell = 2)\) coupling, in practice, we calculate \(\tilde{\chi}_{20}^Z\) numerically instead of \(\chi_{20}^Z\), where \(\chi_{20}^Z\) has been considered in (86) as

\[
\tilde{\chi}_{20}^Z(t, r) = \chi_{20}^Z(t, r) - \chi_{20}^{\text{reg.}(2,2)}(t, r) - \chi_{20}^{\text{reg.}(0,2)}(t, r),
\]  

(A.14)

where \(\chi_{20}^{\text{reg.}(0,2)}\) is the \((\ell = 0) \cdot (\ell = 2)\) contribution, to be discussed in the following subsection. Here, to derive the gravitational wave amplitude for the second perturbative order, we also need to obtain the coefficient \(G_{20}^{(2)AF}\) in an AF gauge as in the first-order case.

The asymptotic expansion of \(\tilde{\chi}_{20}^Z\) is

\[
\tilde{\chi}_{20}^Z(t, r) = \frac{d^3}{3 dT_r^3} F_2(T_r) + O(r^{-1}),
\]  

(A.15)

where we have introduced a function \(F_2\) which satisfies the asymptotic expansion of the Zerilli equation (79) for large \(r\) and has the variable \((t - r)\). Here, we may consider only the leading order with respect to large \(r\) expansion in the same manner as the first-order calculation. The \(\tilde{\chi}_{20}^Z\) contribution to the waveform is derived by the same method as that for the first perturbative order.

First, we obtain \(\frac{\partial K_{20}^{(2)RW}}{\partial t}\) in the RW gauge from (82) as

\[
\frac{\partial}{\partial t} K_{20}^{(2)RW}(t, r) = - \frac{d^3}{3 dT_r^3} F_2(T_r) + \frac{\sqrt{5}}{18 \sqrt{\pi}} \left( \frac{d^4}{dT_r^4} F(T_r) \right) \frac{d^3}{dT_r^3} F(T_r) + O(r^{-1}).
\]  

(A.16)

where the second term on the right-hand side of the above equation arises from the regularization function \(\chi_{20}^{\text{reg.}(2,2)}\) and the \(A_{20}^{(1)}\) and \(B_{20}^{(2)}\) terms in (82). Integrating the above equation for \(K_{20}^{(2)RW}\)

\[
K_{20}^{(2)RW}(t, r) = - \frac{d^3}{3 dT_r^3} F_2(T_r) + \frac{\sqrt{5}}{36 \sqrt{\pi}} \left( \frac{d^3}{dT_r^3} F(T_r) \right)^2 + O(r^{-1}).
\]  

(A.17)

The second-order gauge transformation of \(K_{20}^{(2)}\) is given by

\[
K_{20}^{(2)AF}(t, r) = K_{20}^{(2)RW}(t, r) - \frac{1}{r} V_{1}^{(2)}(t, r) + \frac{\delta K_{20}^{(2)}}{\partial t}(t, r),
\]  

(A.18)

where \(\delta K_{20}^{(2)}\) is defined by the tensor harmonics expansion of \((1/2)\mathcal{L}_{\ell = 2}^2 \delta g_{\mu\nu} - \mathcal{L}_{\ell = 2}^0 \delta h_{\text{RW}}^{(1)}\) in (A.3) and derived as

\[
\delta K_{20}^{(2)}(t, r) = \frac{\sqrt{5}}{252 \sqrt{\pi}} \left[ \frac{d^3}{dT_r^3} F(T_r) \right] \frac{d^2}{dT_r^2} F(T_r) - 2 \left( \frac{d^3}{dT_r^3} F(T_r) \right)^2 + O(r^{-1}).
\]  

(A.19)

From the above results and the AF gauge condition for \(K_{20}^{(2)}\) in (A.11), in order to remove the \(O(r^0)\) terms, the second-order gauge transformation \(V_{1}^{(2)}\) is

\[
V_{1}^{(2)}(t, r) = - \frac{1}{3} \left( \frac{d^3}{dT_r^3} F_2(T_r) \right) r
\]

\[
+ \frac{\sqrt{5}}{252 \sqrt{\pi}} \left[ 5 \left( \frac{d^3}{dT_r^3} F(T_r) \right)^2 + \left( \frac{d^3}{dT_r^3} F(T_r) \right) \frac{d^2}{dT_r^2} F(T_r) \right] r + O(r^0).
\]  

(A.20)

Here, we note that it is sufficient to consider the leading order with respect to large \(r\) to derive the second-order waveform.

Next, we consider to derive \(V_{1}^{(2)}\) from the condition of \(h_{120}^{(2)AF}\). The second-order gauge transformation is given by

\[
h_{120}^{(2)AF}(t, r) = - \frac{r}{2(r - 2M)} V_{1}^{(2)}(t, r) - \frac{r^2}{2} \frac{\delta}{\delta r} V_{2}^{(2)}(t, r) + \frac{\delta h_{120}^{(2)}}{\partial t}(t, r).
\]  

(A.21)
\( \delta h_{120}^{(2)} \) is defined by the tensor harmonics expansion of \( (1/2) \mathcal{L}_{\xi^{(1)}} g_{\mu\nu} - \mathcal{L}_{\xi^{(1)}} h_{RW\mu\nu} \) in equation (A.3). By considering the asymptotic expansion, we obtain

\[
\delta h_{120}^{(2)} (t, r) = \frac{\sqrt{5}}{504 \sqrt{\pi}} \left[ \left( \frac{d^4}{dt^4} F(T_r) \right)^2 + \left( \frac{d^3}{dt^3} F(T_r) \right)^2 \right] r + O(r^0). \tag{A.22}
\]

Then, \( V_2^{(2)} \) is calculated from the above value and the result for \( V_1^{(2)} \) in (A.20) with the AF gauge condition in equation (A.11) as

\[
\frac{\partial}{\partial r} V_2^{(2)} (t, r) = \frac{1}{3} \frac{\partial}{\partial t} G_{20} (t, r)^{-1} - \frac{\sqrt{5}}{63 \sqrt{\pi}} \left( \frac{d^3}{dt^3} F(T_r) \right)^2 r^{-1} + O(r^{-2})
\]

\[
= - \frac{\partial}{\partial t} V_2^{(2)} (t, r) + O(r^{-2}). \tag{A.23}
\]

In the last line, we have used the definition of the retarded time, \( T_r = t - r_+ (r) \).

From the above results, we can consider the metric perturbations related to the gravitational wave amplitude, i.e., \( G_{20}^{(2)AF} \). The gauge transformation is given by

\[
G_{20}^{(2)AF} (t, r) = - V_2^{(1)} (t, r) + \delta G_{20}^{(2)} (t, r). \tag{A.24}
\]

Here, \( \delta G_{20}^{(2)} \) is defined by the tensor harmonics expansion of \( (1/2) \mathcal{L}_{\xi^{(1)}} g_{\mu\nu} - \mathcal{L}_{\xi^{(1)}} h_{RW\mu\nu} \) in equation (A.3), and found to be

\[
\delta G_{20}^{(2)} (t, r) = \frac{\sqrt{5}}{126 \sqrt{\pi}} \left( \frac{d^2}{dt^2} F(T_r) \right) \left( \frac{d^3}{dt^3} F(T_r) \right) r^{-1} + O(r^{-2}). \tag{A.25}
\]

Inserting (A.23) and (A.25) into (A.24), we obtain

\[
\frac{\partial}{\partial t} G_{20}^{(2)} (t, r) = \frac{1}{3} \frac{\partial}{\partial t} G_{20} (t, r)^{-1}
\]

\[
+ \frac{\sqrt{5}}{126 \sqrt{\pi}} \left[ - \left( \frac{d^2}{dt^2} F(T_r) \right)^2 + \left( \frac{d^3}{dt^3} F(T_r) \right) \frac{d^2}{dt^2} F(T_r) \right] r^{-1} + O(r^{-2}). \tag{A.26}
\]

The gravitational waveform is now (by the use of \( \psi_{20}^{even} \) and \( \tilde{\chi}_{20}^{2} \))

\[
\frac{\partial}{\partial t} G_{20}^{(2)} (t, r) = \frac{1}{r} \tilde{\chi}_{20}^{(2)} (t, r)
\]

\[
+ \frac{\sqrt{5}}{14 \sqrt{\pi}} \frac{1}{r} \left[ - \left( \frac{\partial}{\partial t} \psi_{20}^{even} (t, r) \right)^2 + \psi_{20}^{even} (t, r) \frac{\partial^2}{\partial t^2} \psi_{20}^{even} (t, r) \right] + O(r^{-2}). \tag{A.27}
\]

If we consider higher order corrections with respect to the \( 1/r \) expansion, we can show that all metric components satisfy the asymptotic flat gauge condition in (A.11).

### A.2. Second-order (\( \ell = 0 \)) \( \cdot (\ell = 2) \) part

We have already discussed the first-order perturbations for the \( \ell = 0 \) mode in section 5.4. In this paper, we use the first-order \( \ell = 0 \) metric perturbation given by (60). This satisfies the AF gauge condition in (A.11). Therefore, it is not necessary to consider the first-order gauge transformation of the \( \ell = 0 \) mode in (A.2) and (A.3), and we will focus on the second perturbative order related to the \( (\ell = 0) \cdot (\ell = 2) \) coupling.

We discuss the gravitational wave amplitude for the second perturbative order which arises from the \( (\ell = 0) \cdot (\ell = 2) \) coupling by using the same method as in the case of the
\((\ell = 2) \cdot (\ell = 2)\) part. Note that we have already discussed the contribution from \(\tilde{\chi}^{Z}_{20}\) and \(\chi^{\text{reg.}(2,2)}_{20}\) in the second-order wavefunction of \((86)\). In the following, we consider \(\chi^{\text{reg.}(0,2)}_{20}\) and the contribution of the last term on the right-hand side of \((A.3)\), i.e., \(-\mathcal{L}_{\xi^{(1)}}h^{(1)}_{\text{RW}\mu\nu}\) where \(\xi^{(1)}\) is the generator of the gauge transformations for \(\ell = 2\), and we use \((60)\) as the \(\ell = 0\) mode of \(h^{(1)}_{\text{RW}\mu\nu}\).

The regularization function \(\chi^{\text{reg.}(0,2)}_{20}\) is given in \((93)\). This function behaves \(O(r^{-4})\) for large \(r\), therefore we expect that \(\chi^{\text{reg.}(0,2)}_{20}\) does not contribute to the second-order gravitational waveform at infinity.

In the same way as for the \((\ell = 2) \cdot (\ell = 2)\) part, we consider the gauge transformation of \(K_{20}^{(2)}\) in \((A.18)\). Here, we obtain

\[
K_{20}^{(2)\text{RW}}(t, r) + \delta K_{20}^{(2)}(t, r) = O(r^{-3}),
\]

where \(K_{20}^{(2)\text{RW}}\) and \(\delta K_{20}^{(2)}\) arise from \(\chi^{\text{reg.}(0,2)}_{20}\) and the tensor harmonics expansion of \(-\mathcal{L}_{\xi^{(1)}}h^{(1)}_{\text{RW}\mu\nu}\) in \((A.3)\), respectively. This is already asymptotically flat, therefore we do not need any further gauge transformation, \(V_{1}^{(2)}\) for the \((\ell = 0) \cdot (\ell = 2)\) part.

\[
V_{1}^{(2)}(t, r) = O(r^{-5}),
\]

In \((A.21)\), we have

\[
\delta h_{120}^{(2)(e)}(t, r) = O(r^{-3}),
\]

where \(\delta h_{120}^{(2)(e)}\) is defined by the tensor harmonics expansion of \(-\mathcal{L}_{\xi^{(1)}}h^{(1)}_{\text{RW}\mu\nu}\) in \((A.3)\). From this equation, we also conclude

\[
\frac{\partial}{\partial r} V_{2}^{(2)}(t, r) = O(r^{-4}).
\]

Hence, there is no contribution from the \((\ell = 0) \cdot (\ell = 2)\) coupling to the second-order gravitational wave, except for \(\tilde{\chi}^{Z}_{20}\). This means that if we obtain \(\tilde{\chi}^{Z}_{20}\) in the numerical calculation, we can obtain the gravitational wave amplitude for the second perturbative order by using \((A.27)\).

**References**

[1] Lousto C O and Nakano H 2008 *Class. Quantum Grav.* 25 195019 (arXiv:0710.5542)
[2] Pretorius F 2005 *Phys. Rev. Lett.* 95 121101
[3] Campanelli M, Lousto C O, Marronetti P and Zlochower Y 2006 *Phys. Rev. Lett.* 96 111101
[4] Baker J G, Centrella J, Choi D I, Koppitz M and van Meter J 2006 *Phys. Rev. Lett.* 96 111102
[5] Campanelli M, Lousto C O and Zlochower Y 2006 *Phys. Rev. D* 73 061501
[6] Baker J G, Centrella J, Choi D I, Koppitz M and van Meter J 2006 *Phys. Rev. D* 73 104002
[7] Pretorius F 2006 *Class. Quantum Grav.* 23 S529
[8] Campanelli M, Lousto C O and Zlochower Y 2006 *Phys. Rev. D* 74 041501
[9] Campanelli M, Lousto C O and Zlochower Y 2006 *Phys. Rev. D* 74 084023
[10] Campanelli M, Lousto C O, Zlochower Y, Krishnan B and Merritt D 2007 *Phys. Rev. D* 75 064030
[11] Campanelli M 2005 *Class. Quantum Grav.* 22 S387
[12] Herrmann F, Shoemaker D and Laguna P 2007 *Class. Quantum Grav.* 24 S33 (arXiv:gr-qc/0601026)
[13] Baker J G, Centrella J, Choi D I, Koppitz M, van Meter J R and Miller M C 2006 *Astrophys. J.* 653 L93
[14] Gonzalez J A, Sperhake U, Bruegmann B, Hannam M and Husa S 2007 *Phys. Rev. Lett.* 98 091101
[15] Campanelli M, Lousto C O, Zlochower Y and Merritt D 2007 *Astrophys. J.* 659 L5
[16] Campanelli M, Lousto C O, Zlochower Y and Merritt D 2007 *Phys. Rev. Lett.* 98 231102
[17] Campanelli M, Lousto C O and Zlochower Y 2008 *Phys. Rev. D* 77 101501 (arXiv:0711.0879)
[18] Lousto C O and Zlochower Y 2008 *Phys. Rev. D* 77 024034
[19] Mino Y, Sasaki M and Tanaka T 1997 *Phys. Rev. D* 55 3457
[20] Quinn T C and Wald R M 1997 *Phys. Rev. D* 56 3381
