Abstract. - We present some generalizations of a recently proposed alternative approach to nonabelian gauge theories based on the causal Epstein-Glaser method in perturbative quantum field theory. Nonabelian gauge invariance is defined by a simple commutator relation in every order of perturbation theory separately, using only the linear (abelian) BRS-transformations of the asymptotic fields. This condition is sufficient for the unitarity of the S-matrix in the physical subspace. We derive the most general specific coupling compatible with the conditions of nonabelian gauge invariance and normalizability. We explicitly show that the quadrilinear terms, the four-gluon-coupling and the four-ghost-coupling, are generated by our linear condition of nonabelian gauge invariance. Moreover, we work out the required generalizations for linear gauges.

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1. Introduction

The causal Epstein-Glaser formalism [1] is an alternative approach to (perturbative) quantum field theory. The $S$-matrix is constructed in the well-defined Fock space of free asymptotic fields in the form of a formal power series

$$S(g) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int d^4x_1...d^4x_n T_n(x_1, ..., x_n)g(x_1)...g(x_n),$$  \hspace{1cm} (1.1)

where $g(x)$ is a tempered test function which switches the interaction. Only well-defined free field operators occur in the whole construction. Thus, one does not need the Haag-Ruelle (LSZ-) formalism. However, interacting field operators can be perturbatively constructed in an additional step as certain functional derivatives of the $S$-matrix [2].

The central objects are the $n$-point distributions $T_n$. They may be viewed as mathematically well-defined time-ordered products.

The defining equations of the theory in the causal formalism are the fundamental (anti-) commutation relations of the free field operators, their dynamical equations and the specific coupling of the theory $T_{n=1}$. The $n$-point distributions $T_n$ in (1.1) are then constructed inductively from the given first order $T_{n=1}$. In fact, Epstein and Glaser [1] present an explicit inductive construction of the general perturbation series in the sense of (1.1) which is compatible with causality and Poincare invariance. It is the physical condition of causality which makes a direct construction of the renormalized (finite) perturbation series possible - without any intermediate modification of the theory and without introduction of any new mathematical concept. The whole perturbative $S$-matrix is already determined by the conditions of causality, Poincare invariance and the specific coupling of the theory $T_{n=1}$ except for a number of free constants which have to be fixed by further physical conditions.

In the causal formalism the purely technical details which are essential for explicit calculations are separated from the simple physical structure of the theory:

The well-known ultraviolet (UV-)problem is reduced to a conceptionally simple and mathematically well-defined problem, namely the splitting of an operator-valued distribution with causal support into a distribution with retarded and a distribution with advanced support. The physical infrared (IR-)problem is naturally separated by the adiabatic switching of the $S$-matrix $S(g)$ with a tempered test function $g$ and does only arise if one considers the adiabatic limit $g \to 1$ in (1.1).

As a consequence, general properties of the perturbative quantum field theory
like normalizability, gauge invariance, unitarity or the infrared behaviour of the theory can be separately analyzed and inductively proven.

With this background the causal analysis of the abelian gauge theory in the four dimensional Minkowski space was worked out ([3,4,5]). The causal Epstein-Glaser construction of a nonabelian gauge theory in four(3+1) dimensional space-time was done for the Yang-Mills theory with Faddeev-Popov coupling and with fermionic matter fields in the Feynman gauge [6,7]:

A new definition of nonabelian gauge invariance is given as a simple commutator condition in every order of perturbation theory separately. For this purpose one only needs the concept of the linear (abelian) BRS-transformations of the free asymptotic field operators. The condition implies unitarity of the $S$-matrix in the physical subspace, i.e. decoupling of the unphysical degrees of freedom. Moreover, one can derive the well-known Slavnov-Taylor identities. 

It is remarkable that the content of nonabelian gauge invariance in perturbation theory can be completely discussed in the well-defined Fock space of free asymptotic fields.

Furthermore, one can imagine, that the analysis of nonabelian gauge invariance in perturbation theory is simplified considering the facts that in the usual Lagrange formalism the BRS-transformations of the interacting fields [8] - the central objects in this approach - relate basic fields with composite fields and that the latter transformations connect different orders of perturbation theory.

The Epstein-Glaser method is also well-suited to analyze genuine massive nonabelian gauge theories [9]. The causal formalism allows for a comprehensive and simplified analysis of such models [10].

The paper is organized as follows: In section 2 we present the Epstein-Glaser construction of the Faddeev-Popov theory with fermionic matter fields in four(3+1) dimensional space-time. In section 3 we generalize these results to the most general gauge invariant theory. In section 4 we discuss general $\xi$–gauges. In Appendix A we prove Lemma 3.1. In Appendix B, we give a brief introduction into the causal Epstein-Glaser formalism.
2. A Linear Condition of Nonabelian Gauge Invariance

In this section we present the causal construction of the (massless) Yang-Mills theory with Faddeev-Popov coupling and with fermionic matter fields in four (3+1)-dimensional space-time. We state the main theorems. For proofs we refer to [6,7].

The corresponding specific coupling in the Feynman gauge is

\[ T_1 = igf^{abc}(\frac{1}{2} : A_{\mu a}A_{\nu b}F_{\nu c}^\mu : - : A_{\mu a}u_b\partial^\mu \tilde{u}_c :) + i\frac{g}{2} : \bar{\psi}_\alpha(\lambda_a)_{\alpha\beta}\gamma^\mu\psi_\beta : A_{\mu a}. \] (2.1)

All the field operators herein are well-defined free fields and these are the only quantities appearing in the whole theory. Double dots denote their normal ordering.

The second term, the gluon-ghost-coupling, is naturally introduced by our linear condition of nonabelian gauge invariance (see equations (2.14),(2.17) below). The specific coupling \( T_{n=1}(x) \) of the theory does contain no quadrilinear term proportional to \( g^2 \). Such terms are automatically generated in second order by our gauge invariance condition (see equation (2.17) below) as we explicitly show in section 3.

\( A_{\mu a}(x) \) are the (free) gauge potentials satisfying the commutation relations (Feynman gauge)

\[ [A_{\mu a}^{(-)}(x), A_{\nu b}^{(+)}(y)] = i\delta_{ab}g^{\mu\nu}D_0^{(+)}(x-y), \] (2.3)

where \( A^{(\pm)} \) are the emission and absorption parts of \( A \) and \( D_0^{(\pm)} \) are the zero-mass Pauli-Jordan distributions. \( u_a(x) \) and \( \tilde{u}_a(x) \) are the free massless fermionic ghost fields fulfilling the anti-commutation relations

\[ \{ u_a^{(\pm)}(x), \tilde{u}_b^{(\mp)}(y) \} = -i\delta_{ab}D_0^{(\mp)}(x-y). \] (2.4)

The gauge fields are minimally coupled to spinor fields \( \psi_\alpha, \bar{\psi}_\beta \). The latter satisfy the anti-commutation relation

\[ \{ \psi_\alpha^{(-)}(x), \bar{\psi}_\beta^{(+)}(y) \} = \delta_{\alpha\beta}\frac{1}{i}S^{(+)}(x-y) \] (2.5)

where \( S^{(+)} = (i\gamma^\mu\partial^\mu + m) = D_0^{(+)} \). \( f_{abc} \) denotes the usual antisymmetric structure constants of the gauge group, say \( SU(N) \); \((-i/2)\lambda_a\) are the generators of the
fundamental representation of the Lie algebra of the gauge group. The time-
dependence of $A, u$ and $\tilde{u}, \psi$ and $\bar{\psi}$ is given by the wave equations

$$\Box A^\mu_a(x) = 0, \quad \Box u_a(x) = 0, \quad \Box \tilde{u}_a(x) = 0,$$  \hspace{1cm} (2.6)

respectively by the free Dirac equation

$$i\gamma_\mu \partial^\mu \psi_\alpha(x) = M_{\alpha\beta} \psi_\beta(x)$$  \hspace{1cm} (2.7)

with a real and diagonal mass matrix $M_{\alpha\beta}$ which satisfies $\lambda_\alpha M = M \lambda_\alpha$. For a simple gauge group like $SU(N)$ Schur's lemma implies that $M_{\alpha\beta}$ is a multiple of the unit matrix (2.5). In the following we omit colour indices for the spinor fields in order to simplify the notation. We define

$$F^{\mu\nu}_a \overset{\text{def}}{=} \partial^\mu A^\nu_a - \partial^\nu A^\mu_a, \quad j^\mu_a = : \bar{\psi} \gamma^\mu \lambda_a \psi :.$$  \hspace{1cm} (2.8)

According to (2.7), $j^\mu_a$ is a conserved current: $\partial_\mu j^\mu_a = 0$.

Now one considers the linear (abelian!) BRS transformations $[8,11]$ of the free asymptotic field operators. The generator of the abelian operator transformations is the charge

$$Q \overset{\text{def}}{=} \int d^3 x \left( \partial^\nu A^\nu_a - u^\nu_a \right), \quad Q^2 = 0,$$  \hspace{1cm} (2.9)

with the (anti-)commutation relations

$$[Q, A^\mu_a]_- = i \partial_\mu u_a, \quad \{ Q, u_a \}_- = -i \partial_\nu A^\nu_a, \quad \{ Q, u_a \}_+ = 0,$$

\begin{align*}
[Q, \psi]_- = 0, & \quad [Q, \bar{\psi}]_- = 0, & \quad [Q, F^a_{\mu\nu}]_- = 0.
\end{align*}  \hspace{1cm} (2.10)

In addition to the charge $Q$, one defines the ghost charge

$$Q_c := i \int d^3 x : (\bar{u} \partial^\mu u ) :$$  \hspace{1cm} (2.11)

In the algebra, generated by the fundamental field operators, one introduces a gradation by the ghost number $G(\hat{A})$ which is given on the homogenous elements by

$$[Q_c, \hat{A}] = -G(\hat{A}) \cdot \hat{A}.$$  \hspace{1cm} (2.12)

One can define an anti-derivation $d_Q$ in the graded algebra by

$$d_Q \hat{A} := Q \hat{A} - (e^{i \pi Q_c} \hat{A} e^{-i \pi Q_c}) Q$$  \hspace{1cm} (2.13)
The anti-derivation $d_Q$ is obviously homogenous of degree (-1) and satisfies $d_Q^2 = 0$.

Nonabelian gauge invariance in the causal approach means that the commutator of the specific coupling (2.1) with the charge $Q$ is a divergence (in the sense of vector analysis):

$$[Q, T_{n=1}] = i \partial_\nu [ig f_{abc} (A^\mu_a u_b F^{\mu \nu}_c : - \frac{1}{2} : u_a u_b \partial_\nu \tilde{u}_c : ) + ig j^\nu_a u_a] \overset{\text{def}}{=} i \partial_\nu T_1^\nu$$  \hspace{1cm} (2.14)

The second term in (2.1) (the gluon-ghost-coupling) is essential for that $d_Q T_{n=1}$ can be written as a divergence. Note this different compensation of terms in the invariance equation (2.14) compared with the invariance of the Yang-Mills Lagrangean under the full BRS-transformations of the interacting fields in the conventional formalism where the gauge boson part ($TrFF$) alone is BRS invariant.

The representation of $[Q, T_{n=1}]$ as a divergence is not unique in general. The most general (so-called) $Q$-vertex $\tilde{T}_1^\nu$ with the same mass dimension and ghost number as $T_1^\nu$ in (2.14) is the following:

$$[Q, T_1] = i \partial_\nu [T_1^\nu + \gamma B_1^\nu] \overset{\text{def}}{=} i \partial_\nu \tilde{T}_1^\nu$$

with $B_1^\nu = ig f_{abc} \partial_\mu (u_a A^\mu_b A^\nu_c :), \quad \partial_\nu B_1^\nu = 0, \quad \gamma \in \mathbb{C}$ free.  \hspace{1cm} (2.15)

The choice of $\gamma$ has just practical reasons and has no physical consequences.

In addition, $T_{n=1}$ in (2.1) is also anti-gauge invariant in the sense that

$$[\bar{Q}, T_{n=1}] \quad \text{(where} \quad \bar{Q} := \int d^3 x (\partial_\nu A^\nu_a \tilde{\partial}_0 \tilde{u}_a) \quad \text{with} \quad \bar{Q}^2 = 0)$$

is also a divergence:

$$[\bar{Q}, T_{n=1}] = i \partial_\nu [ig f_{abc} (\tilde{u}_a A^\nu_b F^{\nu \kappa}_c : - \frac{1}{2} : A^\nu_a \partial_\nu \tilde{u}_c : ) + ig j^\nu_a \tilde{u}_a] \overset{\text{def}}{=} i \partial_\nu [\tilde{T}_1^\nu + \beta \tilde{B}_1^\nu].$$  \hspace{1cm} (2.16)

Having defined nonabelian gauge invariance in the first order of perturbation theory by the relation (2.14), the condition of nonabelian operator gauge invariance in the causal approach is similarly expressed in every order of perturbation theory separately by a simple commutator relation of the n-point distributions $T_n$ with the charge $\bar{Q}$, the generator of the free operator gauge transformations:
Theorem 2.1 (Nonabelian Gauge Invariance) In the Faddeev-Popov theory in (3+1)-dimensional space-time with the defining equations (2.1-2.7), the following linear condition holds in every order of perturbation theory:

\[
[Q, T_n(x_1, ..., x_n)] = d_Q T_n(x_1, ..., x_n) = i \sum_{l=1}^{n} \partial_{\mu} T^\mu_{n/l}(x_1, ..., x_n),
\]

(2.17)

where \(T^\nu_{n/l}(x_1, ..., x_n)\) are n-point distributions of an extended theory whose first order S-matrix is equal to

\[
S_1(g_0, g_1) \overset{\text{def}}{=} \int d^4 x [T_1(x)g_0(x) + T^\nu_1(x)g^\nu_1(x)].
\]

(2.18)

g_1 = (g^\nu_1)_{\nu=0,1,2,3} \in (\mathcal{S}(\mathbb{R}^4))^4 must be an anti-commuting C-number field. The higher orders are determined by the usual inductive Epstein Glaser construction up to local normalization terms. The \(T^\nu_{n/l}\) are the n-point distributions of the extended theory with one \(Q\)-vertex at \(x_l\), all other \(n-1\) vertices are ordinary Yang-Mills vertices (2.1).

The simple linear operator condition (2.17) involving only well-defined asymptotic field operators expresses the full content of the nonabelian gauge structure of the quantized theory in perturbation theory, namely the Slavnov-Taylor identities and the unitarity of the S-matrix in the physical subspace, i.e. the decoupling of the unphysical degrees of freedom in the theory. It can be proven by induction on the order \(n\) of perturbation theory following the causal construction of \(T_n\) and \(T^\nu_{n/l}\) [6,7]: The linear operator condition is expressed by a set of identities between C-number distributions analogously to the Slavnov-Taylor identities. The different types of these identities are derived and proven by suitable normalization. All symmetries of the theory, in particular the global \(SU(N)\)-symmetry and charge conjugation invariance, are needed. But in order to express the operator gauge invariance condition (2.17) in a set of identities between C-number distributions, one has to work out the explicit form of the divergence in the operator gauge invariance condition. Moreover, one has to distinguish the operator and its derivative, which implies the relative largeness of the set of identities to be proven separately.

In [12] we present a direct algebraic analysis of the operator gauge invariance condition without using the identities between C-number distributions.

Normalizability of the theory is the second important property. In the causal approach the question of the normalizability of a quantum field theory does not require proving its finiteness. Ultraviolet Divergences do not appear at all in our
approach. The problem of normalizability means that we have to show that the number of the finite constants to be fixed by physical conditions stays the same in all orders of perturbation theory. This means that finitely many normalization conditions are sufficient to determine the S-matrix completely. In the causal approach, the question of normalizability is totally separated from the analysis of gauge invariance.

The concept of the singular order of distributions (see Appendix B) is a rigorous definition of the usual power-counting degree [4]. The singular order $\omega$ depends on the external field operators only so that there are only finitely many cases with nonnegative $\omega$, i.e. with free normalization terms. Therefore, the following theorem establishes the normalizability of the Yang-Mills theory.

**Theorem 2.2 (Normalizability)** In the theory defined by (2.1-2.7) the singular order $\omega$ of a distribution with $b$ external gluons, $g_u$ external ghost operators, $g_{\tilde{u}}$ anti-ghost operators, $d$ derivatives on these external operators and $f$ quark or anti-quark pairs, is given by the following simple expression:

$$\omega \leq 4 - b - g_u - g_{\tilde{u}} - d - 3f$$

(2.19)

This expression is obviously independent of the order $n$ of perturbation theory.

The proof is simply based on rigorous power counting arguments and does not require any analysis of combinatorial or topological properties of Feynman graphs. The crucial inputs of the proof are the following properties of the theory:

(a) The specific coupling $T_1$ (2.1) has a mass dimension smaller or equal than four and

(b) the singular order of the (anti-) commutator distributions in (2.3),(2.4) and (2.5) are smaller than zero.

The most important and most subtle property of the S-matrix $S(g)$ is its unitarity in the physical subspace of the Fock space. The subtlety comes from the well-known fact that (because of the gauge structure) the gauge boson sector of the Fock space contains more elements than are physically distinguishable.

As is well-known, the realization of the gauge boson field on a positive definite Hilbert space $F$ is not possible in a manifestly Lorentz covariant way: The zeroth component of the gauge boson field must be skew-hermitean, in contrast to the hermitean spatial components:

$$A_0 = -A_0^+, \quad A_j = A_j^+ \quad j = 1, 2, 3$$

(2.20)

where ‘$+$’ denotes the hermitean conjugation with regard to the positive definite
scalar product of the Fock space

\[ < \cdot | \cdot > : F \times F \longrightarrow C^+ \]  \hspace{1cm} (2.21)

In addition, one introduces a sesquilinear form in \( F \) (an indefinite metric) defined by a metric tensor \( \eta_A^+ = \eta_A^{-1} = \eta_A \)

\[ < \cdot | \eta_A \cdot > : F \times F \longrightarrow C \]  \hspace{1cm} (2.22)

In the gauge boson sector of \( F \), it is given by \( \eta_A = (-1)^{N_{A0}} \), where \( N_{A0} \) is the particle number operator of the scalar gauge bosons. The corresponding conjugation \( 'k' \) for any operator \( \hat{O} \) is given by

\[ \hat{O}^k = \eta \hat{O}^+ \eta. \]  \hspace{1cm} (2.23)

One finds that the gauge boson field is pseudo-hermitean \( A^k = A^\mu. \) One defines a sesquilinear form in the ghost sector of \( F \) with \( \bar{u}^k = u^k \) and \( \bar{\tilde{u}}^k = -\tilde{u}. \) The specific coupling is then pseudo-hermitean with regard to the sesquilinear form: \( T^k_1 = -T_1 = \tilde{T}_1 \) This holds for all n-point distributions \( T_n(x) \) by induction [7]:

**Theorem 2.3. (Pseudo-Unitarity)**

\[ T^k_n(x) = \tilde{T}_n(x) \hspace{1cm} \forall n \]  \hspace{1cm} (2.24a)

This implies the following statement about the formal power series:

\[ S^k(g) = S^{-1}(g)) \]  \hspace{1cm} (2.24b)

The \( \tilde{T}_n(x) \) are the n-point distributions of the inverse \( S^{-1} \)-matrix (see Appendix B).

Let \( N \) be the particle number operator of the unphysical particles: the scalar and longitudinal vector bosons, and the ghosts. It is a positive self-adjoint operator with discrete spectrum \( n = 0, 1, 2, 3, \ldots \) The operator \( Q \) (2.9) manifestly does not change the number of unphysical particles. This means that \( N \) commutes with \( Q \). Hence the eigenspaces of the operator \( N \) for fixed \( n \), \( Eig(N, n) \), are invariant under \( Q \) and \( Q \) commutes with the corresponding projection operators. The nullspace \( \text{Ker}N \) is the physical subspace \( F_\perp \) of transversal gauge bosons. \( F_\perp \) is a subspace of \( \text{Ker}Q \). We state the definitions:

\[ \text{Ker}Q := \{ \alpha \in F \mid Q\alpha = 0 \} \]  \hspace{1cm} (2.25)
\[ F_\perp := \{ \alpha \in F \mid Q\alpha = 0 \land N\alpha = 0 \} \quad (2.26) \]

\[ F_0 := \text{Ker}Q \cap (\oplus_{n>0} \text{Eig}(N,n)) \quad (2.27) \]

We call the corresponding projection operators \( P, P_\perp \) and \( P_0 \). One shows [7]

\[ \text{Ker}Q = F_\perp \oplus F_0 \quad \text{and} \quad F_0 = QF = \text{Range}Q. \quad (2.28) \]

According to (2.24), we have pseudo-unitarity. The physical unitarity means the corresponding perturbative relation for the restriction of the S-matrix to the physical subspace,

\[ P_\perp S(g)P_\perp = S_\perp(g) \quad (2.29) \]

Its inverse is given by

\[ (P_\perp S(g)P_\perp)^{-1} = \sum_n \frac{1}{n!} \int d^4x_1 \ldots \int d^4x_n \quad \tilde{T}_n^{P_\perp}(x_1, \ldots, x_n)g(x_1) \ldots g(x_n) \quad (2.30) \]

where the n-point distributions are equal to the following sum over subsets of \( X = \{ x_1, \ldots, x_n \} \)

\[ \tilde{T}_n^{P_\perp}(X) = \sum_{r=1}^{n} (-1)^r \sum_{P_r} P_\perp T_n(1)P_\perp \ldots P_\perp T_r(X_r)P_\perp. \quad (2.31) \]

**Theorem 2.4. (Physical Unitarity)**

\[ \tilde{T}_n^{P_\perp} = P_\perp T_n^+ P_\perp + \text{div} \quad \forall n \quad (2.32a) \]

where \( \text{div} \) denotes terms of divergence form as in the condition of gauge invariance (2.17). (2.32a) implies the following statement about a formal power series:

\[ (S_\perp)(g)^{-1} = S_\perp^+(g) + \text{div}(g) \quad \text{where} \quad S_\perp = P_\perp SP_\perp \quad (2.32b) \]

The straightforward inductive proof can be found in [7, Chapter 7]. Therein physical unitarity is shown as a direct consequence of the operator gauge invariance condition (2.17) and of the nilpotency of the operator \( Q \).

We stress that in the causal approach the physical infrared problem is naturally separated by adiabatic switching of the S-matrix by a tempered testfunction \( g \) and also absent before the limit \( g \rightarrow 1 \) is taken. So all examinations regarding gauge invariance and unitarity are mathematically well-defined.
3. The Most General Gauge Invariant Coupling

Starting from the Faddeev-Popov coupling $T_1 (2.1)$ and leaving out the unproblematic coupling to matter fields, we search for the most general gauge invariant coupling $T_1^g$, ie.

\[ (A) \quad dQ T_1^g = \text{div} \quad (3.1) \]

**Lemma 3.1.** The most general gauge invariant coupling $T_1^g$, $dQ T_1^g = \text{div}$, which is also invariant under the special Lorentz group $L^\mu_\lambda$ (B) and under the structure group $G$ (C), which has ghost number zero - $G(T_1^g) = 0$ - (D) and has maximal mass dimension 4 (E) can be written as

\[
T_1^g = \alpha_1 \left[ : u_a \tilde{u}_b : \delta_{ab} + \frac{1}{2} : A_\mu^a A^\mu_b : \delta_{ab} \right] + \\
+ \alpha_2 : F^a_{\mu\nu} F^\mu_\nu : \delta_{ab} + \alpha_3 \varepsilon_{\mu\nu\rho\lambda} : F^a_{\rho\lambda} F^\mu_\nu : \delta_{ab} + \\
+ \alpha_4 dQ L_4 + \partial_\mu \sum_{i=5}^9 \alpha_i L^\mu_i + \\
+ \frac{i}{2} g f_{abc} : A^a_\mu A^b_\nu F^\nu_\mu : - i g f_{abc} : A^a_\mu u_b \partial^\mu \tilde{u}_c : + \\
+ \beta_1 dQ K_1 + \beta_2 \partial_\mu K^\mu_1 + \\
+ \beta_3 dQ K_3 + \beta_4 \partial_\mu K^\mu_3 + \beta_5 \partial_\mu K^\mu_5 \quad (3.2a.) \\
+ \beta_6 dQ L^\mu_4 + \partial_\mu \sum_{i=5}^9 \beta_i L^\mu_i + \\
+ \beta_7 dQ K^\mu_4 + \beta_8 \partial_\mu K^\mu_4 + \beta_9 \partial_\mu K^\mu_5 \\
\quad (3.2b.c.) \\
+ \beta_1 dQ K_1 + \beta_2 \partial_\mu K^\mu_1 + \\
+ \beta_3 dQ K_3 + \beta_4 \partial_\mu K^\mu_3 + \beta_5 \partial_\mu K^\mu_5 \\
\quad (3.2d.e.) \\
+ \beta_6 dQ L^\mu_4 + \partial_\mu \sum_{i=5}^9 \beta_i L^\mu_i + \\
+ \beta_7 dQ K^\mu_4 + \beta_8 \partial_\mu K^\mu_4 + \beta_9 \partial_\mu K^\mu_5 \\
\quad (3.2f.g.) \\
+ \beta_1 dQ K_1 + \beta_2 \partial_\mu K^\mu_1 + \\
+ \beta_3 dQ K_3 + \beta_4 \partial_\mu K^\mu_3 + \beta_5 \partial_\mu K^\mu_5 \\
\quad (3.2h.i.) \\
+ \beta_6 dQ L^\mu_4 + \partial_\mu \sum_{i=5}^9 \beta_i L^\mu_i + \\
+ \beta_7 dQ K^\mu_4 + \beta_8 \partial_\mu K^\mu_4 + \beta_9 \partial_\mu K^\mu_5 \\
\quad (3.2j.k.l.)
\]

where

\[
L_1 = i : \tilde{u}_a \partial_\mu A^\mu_a : \delta_{ab}; \quad L^\mu_1 = : u_a \partial^\mu \tilde{u}_b : \delta_{ab}; \\
L^\mu_2 = : \partial^\mu u_a \tilde{u}_b : \delta_{ab}; \quad L^\mu_7 = : A^a_\mu \partial^\nu A^\nu_b : \delta_{ab}; \\
L^\mu_9 = : A^a_\mu \partial_\nu A^\nu_b : \delta_{ab}; \\
K_1 = g f_{abc} : u_a \tilde{u}_b \tilde{u}_c ; \\
K^\mu_2 = i g f_{abc} : A^a_\mu u_b \tilde{u}_c ; \\
K_3 = g d_{abc} : A^a_\mu A^b_\nu \tilde{u}_c ; \\
K^\mu_4 = i g d_{abc} : A^a_\mu A^b_\nu A^\nu_c ; \\
K^\mu_5 = i g d_{abc} : A^a_\mu u_b \tilde{u}_c ;
\]

The proof of Lemma 3.1. is given in Appendix A. We make some remarks on this result:
• In Appendix A it is explicitly shown that $d_Q T_{n=1}^g$ has the following representation as a divergence:

$$d_Q T_{n=1}^g = \partial_\mu \left( T_{1,g}^\mu + B_{1,g}^\mu \right)$$

(3.3)

where

$$T_{1,g}^\mu = \alpha_1 : u_a A_b^\mu : \delta_{ab} + \frac{1}{i} \sum_{i=5}^g \alpha_i d_Q L_i^\mu +$$

(3.3a.b.)

$$+ig f_{abc} \left( : A_c^\mu u_b (\partial^\mu A_c^\nu - \partial^\nu A_c^\mu) : -\frac{1}{2} (u_a u_b \partial^\mu \bar{u}_c : ) \right) +$$

(3.3c.d.e.)

$$+\beta_2 \frac{1}{i} d_Q K_2^\mu + \beta_4 \frac{1}{i} d_Q K_4^\mu + \beta_5 \frac{1}{i} d_Q K_5^\mu$$

(3.3f.g.h.)

$$B_{1,g}^\mu = \gamma_1 i g f_{abc} \partial_\nu ( : u_a A_b^\mu A_c^\nu : ), \quad \partial_\mu B_{1,g}^\mu = 0$$

(3.4)

• Most free constants in $T_{1,g}^g$ (3.1) correspond to pure divergences (terms proportional to $\alpha_5, \ldots, \alpha_9, \beta_2, \beta_4, \beta_5$, also $\alpha_3$) or to cocycles with respect to the antiderivation $d_Q$ (terms proportional to $\alpha_4, \beta_1, \beta_3$).

Besides the Faddeev-Popov-coupling $T_{1,g}^g$ (2.1) and the latter terms which are of course automatically gauge invariant in the sense of (A), we have the quadratic terms proportional $\alpha_1$ and $\alpha_2$. The $\alpha_1$-term would generate masses of the gauge bosons and of the ghost fields. Note that the operator gauge invariance condition fixes the relation between the mass term of the gauge bosons and the ghosts uniquely. It means that if one introduces masses into the theory perturbatively, the masses are already fixed by condition (A).

The $\alpha_2$-term is the usual kinetic term of the gauge boson. The quadratic $\alpha_3$-term is a divergence, but it is ruled out by the condition of the invariance under the discrete symmetries (see below).

In the causal formalism, the information of such quadratic terms are already contained in the fundamental (anti-)commutation relations and the dynamical equations of the operators. Therefore we set all quadratic terms in $T_{1,g}^g$ to zero.

• The condition (E) - together with the corresponding condition for the fundamental (anti-)commutation relations - is sufficient for the normalizability of the theory (see section 2).

• In addition, we pose the condition of invariance under the discrete symmetry transformations, parity $P$, charge conjugation $C$ and time inversion $T$, on $T_{1,g}^g$:

In [7, Chapter 5] we established the following (anti-)unitary representations $U_i$
of the discrete symmetry transformations in the Fock space which leave invariant the defining equations of the theory in the causal formalism:

\[ U_P A_\mu^a(x) U_p^{-1} = A_\mu^a(x_p), \]  
\[ U_P u_a(x) U_p^{-1} = u_a(x_p), \quad U_P \tilde{u}_a(x) U_p^{-1} = \tilde{u}_a(x_p) \]  
\[ U_T A_\mu^a(x) U_T^{-1} = -U^{ab} A_\mu^b(x_T), \]  
\[ U_T u_a(x) U_T^{-1} = -U_{ab} u^b(x_T), \quad U_T \tilde{u}_a(x) U_T^{-1} = -U_{ab} \tilde{u}^b(x_T) \]  
\[ U_c A_\mu^a(x) U_c^{-1} = U_{ab} A_\mu^b(x), \]  
\[ U_c u_a(x) U_c^{-1} = U_{ab} u_b(x), \quad U_c \tilde{u}_a(x) U_c^{-1} = U_{ab} \tilde{u}_b(x) \]

\( U_{ab} \) is defined by the equation \( \lambda_a = U_{ab} \lambda_b = -\bar{\lambda} \) where \( \lambda_a \) are the fundamental representation of the \( SU(N) \)-generators.

The condition of invariance under the discrete symmetry transformations posed on \( T_1^q \),

\[ U_c T_1^q(x) U_c^{-1} = T_1^q, \quad U_P T_1^q(x) U_p = T_1^q(x_p), \quad U_T T_1^q(x) U_T = T_1^q(x_T) \]  
\[ \text{(F)} \]  
leads to

\[ \alpha_3 = 0, \quad \beta_3 = 0, \quad \beta_4 = 0, \quad \beta_5 = 0 \]  
\[ \text{(3.9)} \]

For the \( \alpha_3 \)-term one should keep in mind that

\[ \partial_{x \mu} = \partial_{x^\mu}, \quad \partial_{x^\mu} = -\partial_{x \mu}, \quad \epsilon_{\mu
u\kappa\lambda} = -\epsilon^{\mu
u\kappa\lambda} \]

in four \((3+1)\) dimensional space-time; for the \( \beta \)-terms note

\[ f' = f, \quad d' = -d, \quad \delta' = \delta. \]

The invariance under the discrete symmetry transformations posed on general \( T_n \)-distributions and its compatibility with pseudo-unitarity can be inductively proven [7, Chapter 5].

- The condition of anti-gauge invariance

\[ d_Q T_1^q = div \]  
\[ \text{(G)} \]  

does not give any further restriction: This condition posed on \( T_1^q \) in (3.1) only leads to \( \beta_3 = 0 \). But this already results from condition \( \text{(F)} \) (see 3.9).
• As explicitly shown in Appendix A, all Lorentz invariant \((B)\), G-invariant \((C)\) terms with ghost number zero \((D)\) and with four normal ordered operators which would be compatible with normalizability \((E)\) are ruled out by the gauge invariance condition \((A)\). But the well-known quadrilinear terms, the four-gluon- and the four-ghost-vertex are automatically generated in second order of perturbation theory by our gauge invariance condition as we will show in the following:

We study whether gauge invariance, defined in first order by the equation \(d_Q T_{n=1}^g = [Q, T_{n=1}^g] = div \quad (A)\), can be maintained in second order,

\[
d_Q T_{n=2} = [Q, T_{n=2}] = div \quad (3.11)
\]

considering the tree-contributions only. There exists an effective method to reach this goal [13]. According to Epstein-Glaser method we construct the causal commutator

\[
D_{n=2}(x, y) = [T_1^g(x), T_1^g(y)],
\]

whose gauge invariance is a direct consequence of \((A)\):

\[
d_Q D_{n=2}(x, y) = [Q, [T_1^g(x), T_1^g(y)]] =
\]

\[
= i\partial_v^\nu ( [T_{1,g}^\nu(x), T_{1}^g(y)]) + i\partial_y^\nu ( [T_{1}^g(x), T_{1,g}^\nu(y)]) \quad (3.12)
\]

The first term is a divergence with regard to \(x\) and the second with regard to \(y\). In fact, the second term is obtained from the first by interchanging \(x\) and \(y\) and multiplying it by \((-1)\). The question is whether the same (divergence form) is true for the commutator \([Q, R_2(x, y)]\) obtained by causal splitting of \((3.12)\). Since this commutator agrees with \((3.12)\) on \(\{(x - y)^2 \geq 0, x^0 - y^0 > 0\}\), gauge invariance can only be spoiled by local terms with support \(x = y\). But such terms do arise in the process of distribution splitting of \((3.12)\), so they can be probably removed.

Note that in the splitting of \([Q, D_2]\), we have to split only those numerical distributions which also appear in \(D_2\) because the commutation does not affect the numerical distributions in \(D_2\). It only changes the field operators without disturbing normal ordering. With the same convention of normalization in the splitting of these numerical distributions, we can calculate \([Q, R_2]\) directly by splitting \([Q, D_2]\). This procedure has the advantage that it preserves the divergence structure and shows immediately where gauge invariance may break down.

Considering the commutator \([T_{1,g}^\nu(x), T_{1}^g(y)]\), the splitting of \((3.12)\) must be performed as follows: We carry out the derivative \(\partial^\nu_v\) and then we uniquely split the causal \(D\)-distributions in each term according to the formula

\[
D(x - y) = D^{ret}(x - y) - D^{av}(x - y). \quad (3.13)
\]
Then we have to examine whether the resulting retarded distribution $R_2$ is again a divergence, that means, whether the derivative $\partial_\nu D^\mu(x-y) = \delta(x-y)$ in contrast to $\Box D(x-y) = 0$. This is the only mechanism to spoil gauge invariance in the tree-contribution. Note that it is only necessary to analyze the first term in (3.12) since the splitting of the second commutator in (3.12) leads to local terms with the same sign as in the first term, so that no compensation is possible.

Now it is an easy job to pick up the possible local terms in the splitting solution $d_Q R_{n=2|\text{tree}}$ of $d_Q D_{n=2|\text{tree}}$. There are the following mechanism to get the $\Box$-operator:

(a) There is a second derivative $\partial_\nu \partial^\nu$ in the fermionic coupling $T^\nu_{1/g}(x)$ or
(b) there is an operator $\partial_\nu A^\nu(y)$ in the specific coupling $T^\nu_1(y)$, which can be contracted with an operator $A_\nu(x)$ in $T^\nu_{1/g}(x)$.

Using the formulae (3.1) and (3.3) and taking into account the additional constraints by the discrete symmetry condition (3.9), we get the following list of local terms with four normal ordered operators generated by the terms in $T^\sigma_1$ and $T^\nu_{1/g}$ with three normal ordered operators. In the brackets, we state the corresponding term in the commutator $\partial_\nu \left[ T^\nu_{1/g}, T^\sigma_1 \right]$ which leads to the specialized local term:

| $A_{n_1}$ | $A_{n_2}$ | $A_{n_3}$ | $A_{n_4}$ | $A_{n_5}$ | $A_{n_6}$ | $A_{n_7}$ | $A_{n_8}$ | $A_{n_9}$ | $A_{n_{10}}$ |
|---|---|---|---|---|---|---|---|---|---|
| $-g^2 f_{abc} f_{a'c'} : \partial_\nu u_b A^\mu_\mu \delta(x-y) A^\nu_\mu A_\mu : \left( \partial_\nu [3.3.c), (3.1.f) ]_\nu \right)$ | $-\frac{1}{2} g^2 f_{b'c'} f_{c'ac} : u_b u_c \delta(x-y) A^\nu_\mu \partial^\mu \tilde{u}_{c'} : \left( \partial_\nu [3.3.c), (3.1.g) ]_\nu \right)$ | $\frac{1}{2} g^2 f_{bb'c'} f_{cac'} : u_b u_c \delta(x-y) A^\nu_\mu \partial^\mu \tilde{u}_{c'} : \left( \partial_\nu [3.3.c), (3.1.g) ]_\nu \right)$ | $\beta g^2 f_{a'b'} f_{a'c'} : \partial_\nu [ A^\mu_\mu u_b ] \delta(x-y) u_b \tilde{u}_{c'} : \left( \partial_\nu [3.3.d), (3.1.i) ]_\nu \right)$ | $(\beta g^2)^2 f_{abc} f_{a'c'} : u_b \tilde{u}_{c'} \delta(x-y) u_b \partial_\sigma A^\nu_\sigma : \left( \partial_\nu [3.3.f), (3.1.i) ]_\nu \right)$ | $-\beta g^2 f_{abc} f_{a'c'} : \partial_\nu [ A^\mu_\mu u_b ] \delta(x-y) u_b \tilde{u}_{c'} : \left( \partial_\nu [3.3.f), (3.1.g) ]_\nu \right)$ | $2\beta g^2 f_{abc} f_{a'c'} : \partial_\nu [ A^\mu_\mu u_b ] \delta(x-y) u_b \tilde{u}_{c'} : \left( \partial_\nu [3.3.c), (3.1.h) ]_\nu \right)$ | $-2\beta g^2 f_{abc} f_{a'c'} : \partial_\nu [ A^\mu_\mu u_b ] \delta(x-y) u_b \tilde{u}_{c'} : \left( \partial_\nu [3.3.d), (3.1.h) ]_\nu \right)$ | $-\beta g^2 f_{abc} f_{a'c'} : u_b u_b \delta(x-y) \tilde{u}_{c'} \partial_\nu A^\nu_\sigma : \left( \partial_\nu [3.3.e), (3.1.h) ]_\nu \right)$ | $4\beta g^2 f_{abc} f_{a'c'} : u_b u_b \delta(x-y) \tilde{u}_{c'} \partial_\nu A^\nu_\sigma : \left( \partial_\nu [3.3.f), (3.1.h) ]_\nu \right)$ |
One easily verifies that $A_{n_7} + A_{n_8} = 0$, $A_{n_2} + A_{n_3} = 0$ and $A_{n_4} + A_{n_6} = 0$. So we have four local terms left in the commutator:

$$[Q, R_{n=2}] = div + 2(A_{n_1} + A_{n_5} + A_{n_9} + A_{n_{10}})$$ (3.16)

The factor 2 represents the fact that there is another anomaly contribution of the second commutator in (3.12). Using the Jacobi identity, we arrive at

$$A_{n_1} + A_{n_5} + A_{n_9} + A_{n_{10}} = g^2 f_{abc} f_{a'b'c'} : \partial \nu u_a A^\nu_b \delta(x - y) A^\mu_{a'} A^{\nu}_{b'} : +$$ $$(\beta_2)^2 + 2\beta_1 - 4\beta_1 \beta_2) g^2 f_{b'a'c'} f_{b'c'c} : u_{b'} u_b \delta(x - y) \bar{\nu}_c \partial \kappa A^\kappa_{a'}$$ (3.17)

On the other hand, we have three terms in $R_{n=2}$ with singular order $\omega \geq 0$. This means that the general splitting solution $\tilde{R}_{n=2}$ contains the following three unfixed local normalization terms with free constants $C_1, C_2, C_3$

$$N_{0_1} = -i C_1 g^2 f_{abc} f_{a'b'c'} : A^a_\mu A^b_\nu \delta(x - y) A^\mu_{a'} A^{\nu}_{b'}$$
$$N_{0_2} = -i C_2 g^2 f_{b'a'c'} f_{b'c'c} : u_{b'} u_b \delta(x - y) \bar{\nu}_c \partial \kappa A^\kappa_{a'}$$
$$N_{0_3} = -i C_3 g^2 f_{abc} f_{a'b'c} : A^a_\mu A^\mu_{a'} \delta(x - y) u_b \bar{\nu}_{b'}$$ (3.18)

The question whether there exist a gauge invariant splitting solution $\tilde{R}_{n=2}|_{tree, 4}$ is then equivalent to the solvability of the following anomaly equation:

$$d_Q (N_{0_1} + N_{0_2} + N_{0_3}) - 2(A_{n_1} + A_{n_5} + A_{n_9} + A_{n_{10}}) \equiv div$$ (3.19)

Calculating the commutators of the normalization terms, one directly shows that one can fulfill this equation for all $\beta_1$ and $\beta_2$. We have the following unique solution of the anomaly equation:

$$C_1 = \frac{1}{2}; \quad C_2 = (\beta_2)^2 + 2\beta_1 - 4\beta_1 \beta_2); \quad C_3 = 0$$ (3.20)

So we arrive at the following lemma:
Lemma 3.2: The operator gauge invariance condition,

\[ dQ_{T_{n=2}|_{tree,A}} = \text{div}, \]

uniquely fixes the normalization of \( T_{n=2}|_{tree,A} \) and naturally introduces a four gluon coupling and a four ghost coupling in \( T_{n=2} \):

\[ N_{0_1} = -\frac{1}{2}g^2 i f_{abc} f_{a'b'c'} : A_{\mu}^a A_{\mu}^b \delta(x-y) A_{\nu}^c A_{\nu}^{b'} : \quad (3.21) \]

\[ N_{0_2} = -g^2 \left( (\beta_2)^2 + 2\beta_1 - \beta_1 \beta_2 \right) i f_{bc'd'} f_{'c'e'\nu} : u_{b'} u_{b} \delta(x-y) \tilde{u}_c \tilde{u}_{a'} : \]

Remark:
One can directly compare equations (3.1), (3.9), (3.21) and (3.22) with the most general Lagrangian (written in terms of interacting field operators) which is invariant under the full BRS-transformations of the interacting fields and fulfills reasonable certain additional conditions like (B)-(G). For example see [14, formula (3.13)]:

Taking into account the conventions in [14] and setting the parameters \( \lambda \) and \( \alpha \) in formula (3.13) of [14] to:

\[ \lambda = 1 (\text{Feynman gauge}), \quad \alpha = \beta_2 = 2\beta_1 \quad (3.22) \]

one verifies in a straightforward comparison that in the perturbative analysis the interaction terms of this Lagrangean, the \( g^- \) and also the \( g^2 \) terms, agree with the terms in \( T_{g1}^g \) and the local normalization terms in \( T_{g2}^g \) (3.21) (Note the \( \frac{1}{n!} \) factor in (1.1) ).

Once again one can realize that the linear operator condition of nonabelian gauge invariance (2.17) is sufficient to derive the whole content of nonabelian gauge symmetry in perturbation theory.

The theorems (2.1) - (2.4) about nonabelian gauge invariance, normalizability, pseudo-unitarity and physical unitarity hold also in the generalized theory defined by \( T_{g1}^g \). The proof of normalizability can be taken over without any changes, taking into account that the two crucial inputs are not changed: the maximal mass dimension of the specific coupling and the singular order of the (anti-) commutation distributions .
The algebraic analysis of the nonabelian gauge invariance condition [12] also covers the generalized theory defined by $T_1^g$. The condition of pseudo-hermiticity of $T_1^g$

$$(T_1^g)^k = -T_1^g = \tilde{T}_1^g$$  \hspace{1cm} (3.23)

poses some additional restrictions on the parameter set:

$$\text{Re} \alpha_i = 0, \quad \text{Im} \beta_i = 0$$  \hspace{1cm} (3.24)

The proof of the pseudo-unitarity and of the physical unitarity can then be taken over without any changes, as well.

**Section 4 Linear $\xi$- Gauges**

The analysis so far has been carried out in the Feynman gauge. In this section, we discuss the required generalization to other gauge fixings. In this context, we focus on the so-called linear $\xi$-gauges.

The defining equations of the theory are modified accordingly:

- The wave equation of the gauge boson field in the Feynman gauge

$$\Box A_\mu = 0 \quad (\xi = 1)$$  \hspace{1cm} (4.1)

generalizes to

$$\Box A_\mu - \frac{(\xi - 1)}{\xi} \partial_\mu \partial_\nu A^\nu = 0$$  \hspace{1cm} (4.2)

- The commutation relation of the gauge boson field in the Feynman gauge is

$$[A_\mu(x), A_\nu(y)] = ig_{\mu\nu} D_0(x - y) \quad (\xi = 1)$$  \hspace{1cm} (4.3)

The Pauli-Jordan distribution $D_0$ is the only one of the well-known two linear independent Lorentz invariant solutions of the wave equations which is causal, that means $D_0(x) = 0$ for $x^2 < 0$. The right side of (4.3) thus represents the general Lorentz invariant and causal ansatz which is compatible with the manifest normalizability of the theory - the singular order of $D_0$ is smaller than zero. We search for the corresponding general ansatz in the case of general $\xi$.

Here one should keep in mind that (4.2) implies the dipole equation

$$\Box^2 A_\mu(x) = 0$$  \hspace{1cm} (4.4)
If $F$ is a causal and Lorentz invariant distribution fulfilling the dipole equation, we have $\Box F(x) = D_0(x)$ because $\text{supp} \, \Box F(x) \subseteq \text{supp} \, F(x)$. We arrive at the following general ansatz for the commutator distribution in a linear $\xi$-gauge compatible with causality (I), Lorentz invariance (II), and equation (4.4) (III).

$$[A_\mu(x), A_\nu(y)] = ig_{\mu\nu}D(x - y) + i\alpha\partial_\mu\partial_\nu D(x - y) + i\beta g_{\mu\nu}E(x - y) + i\gamma\partial_\mu\partial_\nu E(x - y)$$

(4.5)

where $\alpha, \beta, \gamma$ are free constants and $E(x)$ the well-known dipole distribution

$$E(x) = \int sgn(p_0)\delta'(p^2)e^{-ipx}d^4p$$

(4.6)

with the properties

$$\Box^2 E(x) = 0, \quad \Box E(x) = D_0, \quad E(x) = 0 \quad \text{for} \quad x^2 < 0$$

Note that the positive and negative frequency parts of $\hat{E}(p) \sim \Theta(\pm p_0)\delta'(p^2)$ are not uniquely defined because of the indeterminacy of the product $\Theta(\pm p_0)\delta'(p^2)$. However, the product $sgn(p_0)\delta'(p^2)$ is a well-defined distribution as an odd homogeneous tempered distribution (see [15] for details). This point indicates a well-known difficulty in a general $\xi$-gauge [15,16] which we will discuss in a forthcoming note.

The additional condition of manifest normalizability (IV) leads to $\alpha = 0$ in (4.5). Furthermore, the required compatibility with equation (4.2) (V) imply $\beta = 0$ and $\gamma = (\xi - 1)$. So the second defining equation of the theory in a general linear $\xi$-gauge reads as follows

$$[A_\mu(x), A_\nu(y)] = ig_{\mu\nu}D_0(x - y) + (\xi - 1)i\partial_\mu\partial_\nu E(x - y)$$

(4.7)

• Generalizing the formula (2.9), we now define the generator of the linear gauge transformations

$$Q := \int d^3x \quad \frac{\partial_\nu A_\mu^\nu(x)}{\xi} \overset{\leftrightarrow}{\partial_0} u(x)$$

(4.8)

Note that we still have the crucial property: $Q^2 = 0$ for all $\xi$. (4.8) implies for the corresponding antiderivation $d_Q$: 
\[
d_Q A_\mu^a = i \partial_\mu u_a; \quad d_Q \tilde{u}_a(x) = -i \frac{\partial_\nu A_\nu^a}{\xi};
\]
\[
d_Q \partial_\mu \tilde{u}_a = -i \partial_\mu \partial_\nu A_\nu^1 \frac{1}{\xi} = i \partial_\nu F^{\nu\mu}; \quad d_Q u_a = 0. \tag{4.9}
\]

Now we further pursue the standard procedure in the causal formalism and construct the most general gauge invariant specific coupling in a general $\xi-$gauge.

**Lemma 4.1:** The most general gauge invariant coupling $T_1^g$ for a general linear $\xi$-gauge, \(d_Q T_1^g = div\), which fulfills the conditions (B)-(G) agree with the result for $\xi = 1$ (Feynman gauge) (see formulae (3.1/3.9)) with one minor change in the $\alpha_1$-term:

\[
T_1^g = \alpha_1 \left[ \xi : u_a \tilde{u}_b : \delta_{ab} + \frac{1}{2} : A_\mu^a A_\nu^b : \delta_{ab} \right] + \tag{4.10}
\]
\[
\alpha_2 : F^{a}_{\mu\nu} F^{a}_{\mu\nu} : \delta_{ab} + \alpha_4 d_Q L_4 + \partial_\mu \sum_{i=5}^{9} \alpha_i L_\mu^i + 
\]
\[
\frac{i}{2} g f_{abc} : A_\mu^a A_\nu^b F^{\nu\mu}_c : -i g f_{abc} : A_\mu^a u_b \partial_\mu \tilde{u}_c : + 
\]
\[
+ \beta_1 d_Q K_1 + \beta_2 \partial_\mu K_\mu^2
\]

where

\[
L_4 = i : \tilde{u}_a \partial_\lambda A_\lambda^b : \delta_{ab}; \quad L_5^\mu = : u_a \partial_\mu \tilde{u}_b : \delta_{ab};
\]
\[
L_6^\mu = : \partial_\mu u_a \tilde{u}_b : \delta_{ab}; \quad L_7^\mu = : A_\mu^a \partial_\nu A_\nu^b : \delta_{ab};
\]
\[
L_9^\mu = : A^\mu_a \partial_\nu A_\nu^b : \delta_{ab};
\]
\[
K_1 = g f_{abc} : u_a \tilde{u}_b \tilde{u}_c ; ; \quad K_2^{\mu} = i g f_{abc} : A_\mu^a u_b \tilde{u}_c ; ;
\]

The proof is straightforward and analogous to the corresponding one in section 3. The explicit representation of $d_Q T_1^g$ as a divergence reads

\[
d_Q T_1^g = \partial_\mu \left( T_1^\mu_{1,g} + B_1^\mu_{1,g} \right) \tag{4.11}
\]

where

\[
T_1^\mu_{1,g} = \alpha_1 : u_a A_\mu^a : \delta_{ab} + \frac{1}{2} \sum_{i=5}^{9} \alpha_i d_Q L_\mu^i + 
\]
\[
ig f_{abc} \left( : A_\mu^a u_b (\partial_\mu A_\nu^c - \partial_\nu A_\mu^c) : - \frac{1}{2} (u_a u_b \partial_\mu \tilde{u}_c :) \right) + \beta_2 \frac{1}{7} d_Q K_2^{\mu}.
\]
\[ B_{1,g}^\mu = \gamma_1 i g f_{a b c} \partial_\nu (\cdot u_a A_\nu^a A_\xi^c) , \quad \partial_\mu B_{1,g}^\mu = 0 \]

The \( \alpha_1 \)-term again would generate masses of the gauge bosons and of the ghost fields. Note that the operator gauge invariance condition fixes the relation between the mass term of the gauge bosons and the ghosts uniquely. This relation now depends on the parameter \( \xi \). Again we leave out the quadratic terms in the following.

Again the gauge invariance condition (A) rules out all terms with four normal ordered operators. But the quadrilinear terms are automatically generated in second order of perturbation theory by the operator gauge invariance condition: The condition \( dq T_{n=2}^{1214} = \text{div} \) fixes all free normalization terms in second order of perturbation theory.

In fact, using the procedure of section 3 we can calculate the local terms in the splitting solution of the commutator \( \partial_\nu [T_{1/g}^\nu, T_1^\mu] \) in a general \( \xi \)-gauge. Because of the second term in the commutation relation (4.7) there are additional mechanism to get a local term. Also note that there are additional \( \frac{1}{\xi} \) factors in \( T_{n=1}^\mu \) and \( T_{n=1,g}^\mu \) (namely in \( \beta_1 dq K_1 \) and in \( \beta_2 \frac{1}{\xi} dq K_2^\mu \)). As a consequence, we get the following changes compared with the case \( \xi = 1 \) (see (3.15):

\[
A_{n_1}^\xi = A_{n_1}^{\xi_1}, A_{n_2}^{\xi_2} = A_{n_2}^{\xi_3}, A_{n_3}^{\xi_4} = A_{n_3}^{\xi_5}, A_{n_4}^{\xi_6} = \xi A_{n_4}^{\xi_7}, A_{n_5}^{\xi_8} = A_{n_5}^{\xi_9}, \quad \text{for } \xi = 1, 2, 3, \ldots, \quad (4.12)
\]

Moreover, we have an additional local term in \( \left( \partial_\nu [3.3.c], 3.1.i \right) \):

\[
A_{n_{11}}^{\xi_7} = -\beta_2 g^2 (\xi - 1) f_{abc} f_{a'c} \partial_\mu (A_\mu^a u_b) \delta(x - y) u_{b'c} \quad (4.13)
\]

One easily verifies the cancellations:

\[
A_{n_7}^{\xi_1} + A_{n_8}^{\xi_2} = 0, \quad A_{n_2}^{\xi_3} + A_{n_3}^{\xi_4} = 0
\]

\[
A_{n_4}^{\xi_5} + A_{n_6}^{\xi_6} + A_{n_11}^{\xi_8} = 0. \quad (4.14)
\]

We have four local terms left in the commutator again:

\[
A_{n_1}^{\xi_1} + A_{n_5}^{\xi_3} + A_{n_9}^{\xi_7} + A_{n_{10}}^{\xi_9} = g^2 f_{abc} f_{a'c} \partial_\nu (u_a A_\nu^a A_\xi^c) \delta(x - y) a' A_\mu^\alpha A_\nu^\beta +
\]

\[
-((\beta_2)^2 + 2\beta_1 (\xi)^{-1} - 4\beta_1 \beta_2 (\xi)^{-1}) g^2 f_{ba'c} f_{b'c} \partial_\nu (u_a u_b \delta(x - y)) u_{c} \partial_\mu A_\alpha^c \quad (4.15)
\]

As in the \( \xi = 1 \) case we have three unfixed local normalization terms \( N_{01}, N_{02}, N_{03} \) with free constants \( C_1, C_2, C_3 \) (see (3.18)). The question whether there exist a
gauge invariant 2-point distribution $\tilde{T}_{n=2|_{\text{tree,4}}}$ is equivalent to the solvability of the following anomaly equation:

$$d_Q(N_0_1 + N_0_2 + N_0_3) - 2(A_{n_1} + A_{n_5} + A_{n_9} + A_{n_9}) \overset{!}{=} \text{div}$$  \hspace{1cm} (4.16)

For any $\beta_1$ and $\beta_2$ we can find the following (unique) solution:

$$C_1 = \frac{1}{2}; \quad C_2 = ((\beta_2)^2 + 2\beta_1(\xi)^{-1} - 4\beta_1\beta_2(\xi)^{-1}); \quad C_3 = 0$$  \hspace{1cm} (4.17)

So we have also in a general $\xi$-gauge:

**Lemma 4.2:** The operator gauge invariance condition,

$$d_QT_{n=2|_{\text{tree,4}}} = \text{div}$$

uniquely fixes the normalization of $T_{n=2|_{\text{tree,4}}}$ and naturally introduces a four gluon coupling and a four ghost coupling in $T_{n=2}$:

$$N_0_1 = -\frac{1}{2}g^2if_{abc}f_{a'b'c'}A_{\mu}^aA_{\nu}^b\delta(x-y)A_{\mu'}^aA_{\nu'}^b$$  \hspace{1cm} (4.18)

$$N_0_2 = -g^2((\beta_2)^2 + 2\beta_1(\xi)^{-1} - 4\beta_1\beta_2(\xi)^{-1})if_{bc'd'}f_{b'cc'}: u_{b'}u_{b}\delta(x-y)\tilde{u}_{c}\tilde{u}_{c'} :$$

The comparison with the Lagrange formalism [14] leads to the same conclusion as in section 3: We consider the most general Lagrangian in a general $\xi$-gauge (written in terms of interacting field operators) which is invariant under the full BRS-transformations of the interacting fields and fulfills reasonable certain additional conditions like (B)-(G); for example see again formula (3.13) in [14]. In perturbative analysis the interaction terms of the Lagrangian, the $g-$ and also the $g^2-$terms, agree with the terms in $T_{1}^g$ (4.10) and the local normalization terms in $T_{2}^g$ (4.18). For an explicit comparison one has to set the parameters $\lambda$ and $\alpha$ in formula (3.13) of [14] to:

$$\lambda = \xi \quad \alpha = \beta_2 = 2\beta_1(\xi)^{-1}$$  \hspace{1cm} (4.19)

The theorems about gauge invariance, normalizability, pseudo-unitarity and physical unitarity hold also in the theory with a general $\xi$-gauge. Again, the proofs
can be taken over with minor changes from the $\xi = 1$ case. The $\xi$-independence of the $S$-matrix elements is not yet discussed here. Finally, a short remark on other gauge fixings is in order. The generalization to nonlinear gauge fixings does not represent a principal difficulty since one can introduce the nonlinear terms in the specific coupling. As far as the Coulomb gauge is concerned, Strocchi and Wightmann have shown that a theory of the free electromagnetic field that maintains the Maxwell equations as operator identities must use a vector potential being nonlocal and Lorentz-variant [17]. This is precisely what happens in a gauge theory fixed in the Coulomb gauge. The specific coupling in the Coulomb gauge contains a nonlocal term. Moreover, the fundamental commutator also includes a non-causal part. (This missing microcausality is of course unproblematic because the commutator of the field strength includes only causal distributions.) It is obvious, that in a formalism where causality appears as the decisive inductive construction element, it is difficult to analyze theories in which this very property is not manifest in their fundamental equations.

Conclusions

We have derived the main features of nonabelian gauge theories in four (3+1) dimensional space time using the causal Epstein-Glaser method. In this formalism, the technical details concerning the well-known UV- and IR-problem in quantum field theory are separated and reduced to mathematically well-defined problems, namely the causal splitting and the adiabatic switching of operator-valued distributions. We have shown that the whole analysis of nonabelian gauge symmetry can be done in the well-defined Fock space of free asymptotic fields; the LSZ-formalism is not used in our construction. Nonabelian gauge symmetry is introduced by an operator condition in every order of perturbation theory separately. The approach allows for a simplified analysis of the different properties of nonabelian gauge theories.

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Appendix A

We give a sketch proof of Lemma 3.1:
Because of condition (E), according to which the maximal mass dimension is four, only terms with two, three or four operators in $T_i^q$ are possible.

- The most general Lorentz invariant, $SU(N)$ invariant local normalordered local operator-valued distribution with ghost number zero and maximal mass dimension four with two basic operators can be written ($\delta_i \in \mathbb{C}$ free)

\[
T_1|_2 = \delta_1 : u_a \tilde{u}_b : \delta_{ab} + \delta_2 : A^\mu_a A^\mu_b : \delta_{ab} \\
+ \delta_3 : \partial_\mu u_a \partial^\mu \tilde{u}_b : \delta_{ab} + \delta_4 : \partial_\mu A^\mu_a \partial_\kappa A^\kappa_b : \delta_{ab} \\
+ \delta_5 : \partial_\mu A^\mu_a \partial^\rho A^\rho_b : \delta_{ab} + \delta_6 : \partial_\mu A^\mu_a \partial^\rho A^\rho_b : \delta_{ab} \\
+ \delta_7 : \partial_\mu A^\mu_a \partial^\nu F^{\nu\kappa}_b : \delta_{ab} + \delta_8 \epsilon_{\mu
u\kappa\lambda} : F^\mu\nu F^\kappa\lambda : \delta_{ab}
\]

The most general $SU(N)$ invariant normalordered local operator-valued distribution with two fundamental operators, ghost number $(-1)$ and maximal mass dimension four, which is covariant under the adjoint representation of the Lorentz group, is the following: ($\gamma_i \in \mathbb{C}$ free)

\[
T'^1_1|_2 = \gamma_1 : u_a A^\nu_a : \delta_{ab} + \gamma_2 : \partial_\nu u_a \partial_\kappa A^\kappa_b : \delta_{ab} \\
+ \gamma_3 : \partial_\nu u_a \partial^\nu A^\kappa_b : \delta_{ab} + \gamma_4 : \partial_\kappa u_a \partial^\nu A^\nu_b : \delta_{ab} \\
+ \gamma_5 : u_a \partial_\kappa F^{\nu\kappa}_b : \delta_{ab}
\]

We search for restrictions on the free parameters $\delta_i \in \mathbb{C}$ in (A.1.) by the equation

\[
d_Q T_1(\delta_i)|_2 = i \partial_\nu T'^{\nu}_1(\gamma_i)|_2.
\]

(A.3) leads to the following relations between the two parameter-sets $\gamma_i$ and $\delta_i$:

\[
\delta_1 = 2\delta_2 = \gamma_1; \quad 2(\delta_5 + \delta_6) = \gamma_3 + \gamma_5; \quad \delta_3 + \delta_7 = \gamma_2 + \gamma_4 + \gamma_5
\]

(A.4)

Thus, we have only one restriction on the set ($\delta_i$), namely $\delta_1 = 2\delta_2$. It means, that if one introduces masses into the theory perturbatively, then gauge invariance already fixes the relation between the mass of the ghost and the gauge boson
at first order.

Using (A.4) and making a simple reparametrization one easily arrives at the two-operator terms in (3.1) and (3.3).

- The corresponding ansatz for three basic operators are

\[ (\alpha_i, \beta_i, \varepsilon_i, \mu_i \in \mathbb{C} \text{ for SU(2)} \] free note that we have already fixed the coefficient of the three-gluon coupling; \( f_{abc} \) and \( d_{abc} \) are the well-known totally antisymmetric and symmetric invariant tensors. These are the only two independent invariant tensors of rank three for \( SU(N) \), \( N > 2 \); for \( SU(2) \) we have to set \( d = 0 \):)

\[
T_1|_3 = ig f_{abc} \left\{ A^\alpha_A A^\beta_B \partial^\gamma A^\varepsilon_C : +\varepsilon_1 : A^\alpha_A u_b \partial^\nu \tilde{u}_c : +\varepsilon_2 : \partial_\lambda A^\alpha_A u_b \tilde{u}_c + \varepsilon_3 : A^\lambda_A \partial_\lambda u_b \tilde{u}_c : \right\} \\
+ ig d_{abc} \left\{ \frac{\beta_1}{2} : A^\alpha_A A^\beta_B (\partial_\nu A^\varepsilon_C - \partial^\mu A^\varepsilon_C) : +\beta_2 : A^\alpha_A A^\beta_B \partial_\nu A^\varepsilon_C : +\beta_3 : A^\alpha_A u_b \partial^\mu \tilde{u}_c : \\
+ \beta_4 : A^\alpha_A \partial^\mu u_b \tilde{u}_c : +\beta_5 : \partial_\mu A^\alpha_A u_b \tilde{u}_c : \right\}
\]

\[
T_1'|_3 = ig f_{abc} \left\{ \mu_1 : A^\alpha_A u_b \partial^\nu A^\mu_C : +\mu_2 : A^\alpha_A u_b \partial^\mu A^\nu_C : \\
+ \mu_3 : u_a u_b \partial^\nu \tilde{u}_c : +\mu_4 : \partial_\nu u_a u_b \tilde{u}_c : \\
+ \mu_5 : A^\alpha_A u_b \partial^\nu A^\varepsilon_C : +\mu_6 : \partial^\nu u_a A^\beta_B A^\varepsilon_C : \right\} \\
+ ig d_{abc} \left\{ \alpha_1 : u_a A^\beta_B \partial_\mu A^\varepsilon_C : +\alpha_2 : u_a A^\beta_B \partial^\mu A^\varepsilon_C : \\
+ \alpha_3 : u_a A^\beta_B \partial^\nu A^\mu_C : +\alpha_4 : \partial_\nu u_a A^\beta_B A^\mu_C : \\
+ \alpha_5 : \partial^\nu u_a A^\beta_B A^\mu_C : +\alpha_6 : \partial_\nu u_a u_b \tilde{u}_c : \right\}
\]

(A.5)

The condition of gauge invariance:

\[
d_Q T_1(\epsilon_i, \beta_i)|_3 = i \partial_\nu T_1'|_3(\mu_i, \alpha_i)|_3 \tag{A.6}
\]

leads to the following restrictions on the parameter sets:

\[
\alpha_1 = \beta_3, \quad \alpha_1 = \beta_5, \quad \alpha_2 = 0, \quad \alpha_3 = 0, \quad \alpha_4 = \beta_1, \quad \alpha_5 = \frac{\beta_1}{2}
\]

\[
\alpha_6 = \beta_3, \quad \alpha_1 + \alpha_4 = 2\beta_2 + \beta_4
\]

\[
\mu_1 = 1, \quad \mu_2 = \mu_6 - 1, \quad \mu_3 = -\frac{1}{2},
\]

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\( \mu_4 = 1 + \epsilon_1, \quad \mu_5 = \epsilon_1 - \mu_2, \quad \epsilon_3 = \epsilon_1 + 1 \)

So we have the following restrictions on \( T_{1/3} \) (A.5):

\[
\beta_2 = \frac{1}{2}(\beta_3 + \beta_1 - \beta_4), \quad \beta_5 = \beta_3, \quad \beta_1, \beta_3, \beta_4 \text{ still free} \quad (A.8)
\]

\[ \epsilon_3 = \epsilon_1 + 1, \quad \epsilon_1, \epsilon_2 \text{ still free} \]

With the help of a simple reparametrization we arrive at the representation of the three-operator terms in (3.1) and (3.3).

- The corresponding ansatz for four normalordered operators is (For \( SU(N), N > 3 \) we have 9 invariant tensors 
\( \delta_{ab}\delta_{cd}, \delta_{ad}\delta_{bc}, \delta_{ac}\delta_{bd}, d_{abc}d_{cde}, d_{ade}d_{cbe}, d_{abe}f_{cde}, d_{ade}f_{bce}, d_{ace}f_{dce} \). Note that the following ansatz is also valid for \( SU(3) \), where we have only eight instead of nine invariant tensors of rank four due to the well-known relation \( \delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} = 3(d_{ijm}d_{kem} + d_{skm}d_{jim} + d_{ilm}d_{jkm}) \). The case \( SU(2) \) is straightforward: One has to set \( d = 0 \):)

\[
T^\nu_4 = (\zeta_1 d_{ace}d_{bde} + \phi_1 \delta_{ac}\delta_{bd}) \ g : u_a u_b \bar{u}_c \bar{u}_d : + \\
+ (\zeta_2 \delta_{ab}d_{cde} + \phi_2 \delta_{ab}\delta_{cd}) \ g : A^\mu_\nu A^\alpha_\beta A^\nu_\mu A^\beta_\alpha : + \\
+ (\zeta_3 d_{ace}d_{bde} + \phi_3 \delta_{ac}\delta_{bd}) \ g : A^\mu_\nu A^\alpha_\beta A^\nu_\mu A^\beta_\alpha : + \\
+ (\zeta_4 d_{abe}d_{cde} + \phi_4 \delta_{ab}\delta_{cd}) \ g : u_a u_b A^\nu_\mu A^\mu_\nu u_c \bar{u}_d : + \\
+ (\zeta_5 d_{ace}d_{bde} + \phi_5 \delta_{ac}\delta_{bd}) \ g : u_a \bar{u}_b A^\nu_\mu A^\mu_\nu u_c \bar{u}_d : + \\
+ (\kappa_1 d_{abe}f_{cde} + \kappa_2 d_{ace}f_{bde}) \ g : A^\mu_\nu A^\alpha_\beta u_c \bar{u}_d : + \\
+ (\kappa_3 d_{ade}f_{bce}) \ g : A^\mu_\nu A^\alpha_\beta u_c \bar{u}_d : + \\
+ (\kappa_4 d_{ace}f_{bde}) \ g : u_a u_b \bar{u}_c \bar{u}_d :
\]

\[
T^\nu_4 = (\lambda_1 d_{ace}d_{bde} + \mu_1 \delta_{ac}\delta_{bd}) \ g : u_a u_b \bar{u}_c A^\nu_\mu : + \\
+ (\lambda_2 d_{ace}d_{bde} + \mu_2 \delta_{ab}\delta_{cd}) \ g : u_a A^\nu_\mu A^\mu_\nu A^\beta_\alpha : + \\
+ (\lambda_3 d_{ace}d_{bde} + \mu_3 \delta_{ac}\delta_{bd}) \ g : u_a A^\nu_\mu A^\mu_\nu A^\beta_\alpha : + \\
+ (\nu_1 d_{ace}d_{bde} + \nu_2 f_{ace}d_{bde}) \ g : u_a u_b \bar{u}_c A^\nu_\mu : + \\
+ (\nu_3 d_{abe}f_{cde}) \ g : u_a u_b \bar{u}_c A^\nu_\mu : + \\
+ (\nu_4 d_{ace}d_{bde}) \ g : u_a A^\nu_\mu A^\mu_\nu A^\beta_\alpha : + \\
+ (\nu_5 d_{ace}d_{bde}) \ g : u_a A^\nu_\mu A^\mu_\nu A^\beta_\alpha : + \\
+ (\nu_6 d_{ace}d_{bde}) \ g : u_a A^\nu_\mu A^\mu_\nu A^\beta_\alpha :
\]
\[(\zeta_i, \vartheta_i, \kappa_i, \lambda_i, \mu_i, \nu_i \in \mathbb{C} \text{ free constants}).\]

The equation
\[d_\alpha T_1 (\zeta_i, \vartheta_i, \kappa_i) \big|_4 = i \partial_\nu T_1^{\nu} (\lambda_i, \mu_i, \nu_i) \big|_4\] (A.10)
has no nontrivial solution. As a consequence, there are no quadrilinear terms in the specific coupling \(T_1\).

## Appendix B

In this appendix we state some main features of the Epstein-Glasers method in quantum field theory (for details see [1,4]):

Epstein and Glaser followed the Bogoliubov’s formalism in order to keep apart the different difficulties encountered in perturbative quantum field theory. In contrast to the usual Lagrangean approach, Epstein and Glaser construct the perturbative scattering matrix \(S(g)\) directly in the well-defined Fock space of free fields \(F\). In order to obtain the explicit form of the \(S\)-matrix, they use certain physical conditions. Besides Poincaré invariance the condition of causality plays the most important role:

- If the support of \(g_1\epsilon S\) in Minkowski space is earlier than the support of \(g_2\epsilon S\) in some Lorentz frame (\(\text{suppg}_1 < \text{suppg}_2\)), then the \(S\)-matrix fulfills the following functional equation:
  \[S(g_1 + g_2) = S(g_2) \cdot S(g_1) \quad \text{[Causality (I)]} \] (B.1)

Note that we use a slightly different condition of causality compared with the one used by Epstein and Glaser.
- \(U(a, \Lambda)\) shall be the usual representation of the Poincaré group \(P^+_\uparrow\) in the Fock space \(F\). The condition of **Poincare invariance** of the \(S\)-matrix can be expressed as follows:
  \[U(a, 1)S(g)U(a, 1)^{-1} = S(g_a) \quad \forall a \in \mathbb{R}^4\]
  \[U(0, \Lambda)S(g)U(0, \Lambda)^{-1} = S(g_\Lambda) \quad \forall \Lambda \in L^\uparrow_\uparrow\]
  [Translational invariance (II)] (B.2)
  [Lorentz Invariance (III)] (B.3)
• Epstein and Glaser search for the most general Poincare invariant solution of
the functional equation for the S-matrix of the following form (formal power series
in $g \in \mathcal{S}$)

$$S(g) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int d^4 x_1 \cdots d^4 x_n T_n(x_1, \ldots, x_n)g(x_1) \cdots g(x_n)$$

$$\overset{\text{def}}{=} 1 + T. \quad \text{[Perturbative Ansatz (IV)]} \quad (B.4)$$

if the **Specific Coupling** of the theory $T_{n=1}$ ($V$) is given. The $T_n$ are operator-valued n-point distributions.

Epstein and Glaser show that the whole perturbative S-matrix in the sense of a
formal power series (IV) is already determined by the conditions of causality (I),
translational invariance (II) and the specific coupling of the theory (V) except for
a number of finite (!) free constants which have to be fixed by further physical
conditions. The main steps of their inductive construction are the following:

• Analogously to (B.4), Epstein and Glaser express the inverse S-matrix also by
a formal power series:

$$S(g)^{-1} = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int d^4 x_1 \cdots d^4 x_n \tilde{T}_n(x_1, \ldots, x_n)g(x_1) \cdots g(x_n)$$

$$= (1 + T)^{-1} = 1 + \sum_{n=1}^{\infty} (-T)^n. \quad (B.5)$$

Since by definition $\tilde{T}(x_1, \ldots, x_n)$ and also $T_n(x_1, \ldots, x_n)$ are symmetric in $x_1, \ldots, x_n,$
it is convenient to use a set-theoretical notation $X = x_1, \ldots, x_n.$ The distributions
$\tilde{T}$ can be computed by formal inversion of (B.4):

$$\tilde{T}_n(X) = \sum_{r=1}^{n} (-)^r \sum_{P_r} T_{n_1}(X_1) \cdots T_{n_r}(X_r), \quad (B.6)$$

where the second sum runs over all partitions $P_r$ of $X$ into $r$ disjoint subsets

$$X = X_1 \cup \ldots \cup X_r, \quad X_j \neq \emptyset, \quad |X_j| = n_j. \quad (B.7)$$

We stress the fact that all products of distributions are well-defined because the
arguments are disjoint sets of points so that the products are tensor products of
distributions.

• Epstein and Glaser translate the conditions imposed on $S(g)$ into the corre-
sponding perturbative conditions on the n-point distributions $T_n(x_1, \ldots, x_n)$ and
$\tilde{T}_n(x_1, \ldots, x_n) \quad n \in \mathbb{N},$ according to the Bogoliubov’s approach.
Now Epstein and Glaser introduce the retarded and the advanced n-point distributions:

\[ R_n(x_1, \ldots, x_n) = T_n(x_1, \ldots, x_n) + R'_n \quad \text{where} \quad R'_n = \sum_{P_2} T_{n-n_1}(Y, x_n) \tilde{T}_{n_1}(X) \]

\[ A_n(x_1, \ldots, x_n) = T_n(x_1, \ldots, x_n) + A'_n \quad \text{where} \quad A'_n = \sum_{P_2} \tilde{T}_{n_1}(X) T_{n-n_1}(Y, x_n). \]

The sum runs over all partitions \( P_2 : \{x_1, \ldots, x_{n-1}\} = X \cup Y, \ X \neq \emptyset \) into disjoint subsets with \( |X| = n_1 \geq 1, \ |Y| \leq n - 2 \).

Both sums, \( R'_n \) and \( A'_n \), contain \( T_j \)'s with \( j \leq n - 1 \) only and are therefore known quantities in the inductive step from \( n - 1 \) to \( n \) - in contrast to \( T_n \).

Note that the last argument \( x_n \) is marked as the reference point for the support of \( R_n \) and \( A_n \).

The following proposition is a consequence of the causality condition (I):

**Proposition B.1**

\[ \text{supp} R_m(x_1, \ldots, x_m) \subseteq \Gamma^+_{m-1}(x_m), \quad m < n \]

\[ \text{supp} A_m(x_1, \ldots, x_m) \subseteq \Gamma^-_{m-1}(x_m), \quad m < n \]

where \( \Gamma^+_{m-1} (\Gamma^-_{m-1}) \) is in the (m-1)-dimensional closed forward (backward) cone

\[ \Gamma^+_{m-1}(x_m) = \{(x_1, \ldots, x_{m-1}) \mid (x_j - x_m)^2 \geq 0, \ x_j^0 \geq x_n^0, \forall j\}. \]

In the difference

\[ D_n(x_1, \ldots, x_n) \overset{\text{def}}{=} R_n - A_n = R'_n - A'_n \]

the unknown n-point distribution \( T_n \) cancels. Hence this quantity is also known in the inductive step. It should be added that \( D_n \) has a causal support:

**Proposition B.2**

\[ \text{supp} D_n \subseteq \Gamma^+_{n-1}(x_n) \cup \Gamma^-_{n-1}(x_n) \]

This crucial support property is preserved in the inductive step. It directly results from causality.

- Given the aforegoing facts, the following inductive construction of the n-point distribution \( T_n \) becomes possible:

Starting off with the known \( T_m(x_1, \ldots, x_n), m \leq n - 1 \), one computes \( A'_n, R'_n \) and \( D_n = R'_n - A'_n \). With regard to the supports, one can decompose \( D_n \) in the following way:

\[ D_n(x_1, \ldots, x_n) = R_n(x_1, \ldots, x_n) - A_n(x_1, \ldots, x_n) \]

\[ \text{(B.13)} \]
Then $T_n'$ is given by

$$T_n' = R_n - R_n' = A_n - A_n'$$  \hspace{1cm} (B.14)

One can verify that the $T_n'$ satisfy the perturbative versions of conditions [1].

Because of the marked $x_n$-variable, we finally symmetrize:

$$T_n(x_1, \ldots, x_n) = \sum_{\pi} \frac{1}{n!} T_n'(x_{\pi_1}, \ldots, x_{\pi_n})$$  \hspace{1cm} (B.15)

The only nontrivial step in the construction is the splitting of the operator-valued distribution $D_n$ with support in $\Gamma^+ \cup \Gamma^-$ into a distribution $R_n$ with support in $\Gamma^+$ and a distribution $A_n$ with support in $\Gamma^-$. In causal perturbation theory the usual renormalization program is reduced to this conceptually simple and mathematically well-defined problem:

- Let there be an operator-valued tempered distribution $D$ with causal support:

$$D \in S'((\mathbb{R}^m)^n), \quad \text{supp} \, D \subset \Gamma^+(x_n) \cup \Gamma^-(x_n)$$  \hspace{1cm} (B.16)

The question is whether it is possible to find a pair $(R, A)$ of tempered distributions on $\mathbb{R}^{4n}$ with the following characteristics:

- $R, A \in S'((\mathbb{R}^m)^n)$ \hspace{1cm} (A)
- $R \subset \Gamma^+(x_n), \quad A \subset \Gamma^-(x_n)$ \hspace{1cm} (B)
- $R - A = D$ \hspace{1cm} (C)

Because of $\Gamma^+_{n-1}(x_n) \cap \Gamma^-_{n-1}(x_n) = \{(x_n, \ldots, x_n)\}$, it is obvious that the behaviour of the distribution at $x = (x_n, \ldots, x_n)$ is crucial for the splitting problem. One has to classify the singularities of distributions in this region. This can be carried out with the help of the singular order of distributions which is a rigorous definition of the usual power-counting degree. We finally state the main definitions:

We assume $d(x)$ to be a tempered distribution in $S'((\mathbb{R}^m), m = 4(n-1)$.

**Definition B.1** The distribution $d(x) \in S'((\mathbb{R}^m)$ has quasi-asymptotics $d_0(x)$ at $x = 0$, with regard to a positive continuous function $\rho(\delta), \delta > 0$ if the limit

$$\lim_{\delta \to 0} \rho(\delta) \delta^m d(\delta x) = d_0(x) \neq 0$$  \hspace{1cm} (B.18)

exists in $S'((\mathbb{R}^m)$.

By scaling transformation it follows that

$$\lim_{\delta \to 0} \frac{\rho(a\delta)}{\rho(\delta)} = a^\omega$$  \hspace{1cm} (B.19)
with some real $\omega$. $\rho$ is called power-counting function.

**Definition B.2** The distribution $d(x) \in \mathcal{S}'(\mathbb{R}^m)$ is called singular of order $\omega$ at $x = 0$, if it has a quasi-asymptotics $d_0(x)$ at $x = 0$ with power-counting function $\rho(\delta)$ satisfying (B.19).

Note that this definition differs from the one introduced by Epstein and Glaser [1]. The latter definition is hampered by the fact that the corresponding definitions in the $x$-space and $p$-space are not completely equivalent. Our definition does not have this defect.

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