A uniform contraction principle for bounded Apollonian embeddings.

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Abstract

Let \( \hat{H} = H \cup \{\infty\} \) denote the standard one-point completion of a real Hilbert space \( H \). Given any non-trivial proper sub-set \( U \subset \hat{H} \) one may define the so-called ‘Apollonian’ metric \( d_U \) on \( U \). When \( U \subset V \subset \hat{H} \) are nested proper subsets we show that their associated Apollonian metrics satisfy the following uniform contraction principle: Let \( \Delta = \text{diam}_V(U) \in [0, +\infty] \) be the diameter of the smaller subsets with respect to the large. Then for every \( x, y \in U \) we have

\[
d_V(x, y) \leq \tanh \frac{\Delta}{4} d_U(x, y).
\]

In dimension one, this contraction principle was established by Birkhoff [Bir57] for the Hilbert metric of finite segments on \( \mathbb{RP}^1 \). In dimension two it was shown by Dubois in [Dub09] for subsets of the Riemann sphere \( \hat{\mathbb{C}} \sim \hat{\mathbb{R}}^2 \). It is new in the generality stated here.

1 Introduction and results

There are striking similarities between the projective group for the real or complex projective lines and the conformal group of the one-point completion of a real Hilbert space of dimension at least 3. In the first case, the group consists of Möbius maps of the form \( z \mapsto \frac{az+b}{cz+d} \) and in the second it is generated by linear isometries, homotheties and the inversion, corresponding to Möbius transformations supplemented with a complex conjugation. In both cases one needs at least 4 points to define a group invariant quantity, i.e. the cross-ratio. Fixing a subset \( U \) whose complement contains at least 2 points, the logarithm of cross-ratios may then be used to construct a (semi-)metric on \( U \). On the interval \( I = (-1, 1) \), there is a unique (up to a constant) distance invariant under Möbius transformations preserving \( I \). This is precisely the restriction of the Poincaré metric \( 2|dz|(1 - |z|^2)^{-1} \) on the unit disk in the complex plane. In the case of Hilbert spaces of higher dimensions one may derive the so-called ‘Apollonian metric’ (see below). This latter metric was first introduced for \( \hat{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\} \) by Barbilian [Bar34] and later rediscovered by Beardon [Bea98].

From a dynamical point of view it is of interest to know how a subset \( U \) metrically embeds into a larger subset \( V \) with respect to the associated metrics \( d_U \) and \( d_V \) (see below for more precise

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exchanges the origin and $\infty$ is generated by the set of isometries, homotheties (both fixing $\{\infty\}$ which is finite dimensional. The space $(\hat{H}, \hat{d})$ is a complete metric space of diameter one with respect to the metric:

$$
\hat{d}(x, y) = \frac{\|x - y\|}{\sqrt{1 + \langle x, x \rangle} \sqrt{1 + \langle y, y \rangle}}, \quad \hat{d}(\infty, y) = \frac{1}{\sqrt{1 + \langle y, y \rangle}}.
$$

(1.1)

**Definition 1.1** Given four points $\{x_1, x_2, u_1, u_2\} \in \hat{H}$ such that $\{x_1, x_2\}$ and $\{u_1, u_2\}$ are disjoint we define their cross-ratio to be:

$$[x_1, x_2; u_1, u_2] = \frac{\|x_2 - u_1\| \|x_1 - u_2\|}{\|x_1 - u_1\| \|x_2 - u_2\|}.
$$

(1.2)

Here, $\|\cdot\|$ denotes the Hilbert norm in $H$ and we adapt usual conventions for dealing with the point at $\infty$. When $U \subset \hat{H}$ is a proper subset (by proper we mean that $U$ and $U^c$ are both non-empty) one defines the Apollonian (semi, pseudo-)distance between points $x_1, x_2 \in U$:

$$d_U(x_1, x_2) = \sup_{u_1, u_2 \in U^c} \log[x_1, x_2; u_1, u_2] \in [0, +\infty]
$$

(1.3)

We denote by $GM(\hat{H})$ the general conformal group which acts continuously upon $(\hat{H}, \hat{d})$ and is generated by the set of isometries, homotheties (both fixing $\infty$) and the inversion (which exchanges the origin and $\infty$):

$$I(x) = \frac{x}{\langle x, x \rangle}, \quad I(0) = \infty \quad \text{and} \quad I(\infty) = 0.
$$

(1.4)

When $\dim H \geq 3$ the Liouville theorem (see e.g. [Nev60]) shows that any conformal map is in $GM(\hat{H})$. In dimension 1 or 2, it is the Möbius group (supplemented with complex conjugation in the 2 dimensional case). That $d_U$ is $GM(\hat{H})$ invariant is trivial for isometries and homotheties and in the case of inversions it follows easily from the formula $\|I(x) - I(y)\| = \|x - y\| / \|x\| \|y\|$ (with some care taken with respect to the point at infinity). From the cross-ratio identity $[x, z; u, v] = [x, y; u, v] [y, z; u, v]$ and taking sup in the right order one also sees that $d_U$ verifies the triangular inequality. When $U^c$ has non-empty interior $d_U$ is a genuine metric, but in the general case it need not distinguish points. We refer to e.g. [Bea98, Chapter 3] and [Has04] for further details on the geometry of this metric. Our main result is the following:
Theorem 1.2 [Main Theorem] Let $U \subset V \subset \hat{H}$ be non-empty proper subsets with $d_U$ and $d_V$ being the associated Apollonian metrics. Let $\Delta = \sup \{d_V(u_1, u_2) \mid u_1, u_2 \in U\}$ be the diameter of the smaller subset within the larger. Then for every $x_1, x_2 \in U$:

$$d_V(x_1, x_2) \leq \left( \tanh \frac{\Delta}{4} \right) d_U(x_1, x_2).$$

(1.5)

If $\text{diam}_V(U) < +\infty$, the embedding $i : (U, d_V) \hookrightarrow (U, d_U)$ is a uniform contraction.

Proof: We will base our proof upon Birkhoff’s inequality [Bir57] for cross-ratios on the projective real line. It is, in fact, a special case of our main theorem when $n = 1$. We will use it in the following version: Let $K = (a_1, a_2)$ be a non-empty open sub-interval of $J = (0, +\infty)$. The Hilbert distance of $s_1, s_2 \in K$ relative to $K$ and $J$ are given by:

$$d_K(s_1, s_2) = \left| \log \frac{s_1 s_2}{a_1 a_2} \right| \quad \text{and} \quad d_J(s_1, s_2) = \left| \log \frac{s_2}{s_1} \right|.$$

The quantity $\Delta = \text{diam}_J(K) = \log \frac{a_2}{a_1} \in (0, +\infty)$ measures the diameter of $K$ for the $J$-metric. Birkhoff [Bir57, p.220] showed the fundamental inequality:

$$d_J(s_1, s_2) \leq \left( \tanh \frac{\Delta}{4} \right) d_K(s_1, s_2), \quad \forall \ s_1, s_2 \in K.$$

(1.6)

Proof of (1.6): It suffices to show this for $s_1$ and $s_2$ infinitesimally close. So we differentiate with respect to $s_2$ at $s_2 = s_1 = s \in (a_1, a_2)$ and search for the optimal value of $\theta > 0$ so that for every $a_1 < s < a_2$: $\frac{1}{s} \leq \theta \frac{a_1 - a_2}{(s - a_1)(a_2 - s)}$, or equivalently

$$\theta \geq \inf_{a_1 < s < a_2} \frac{(s - a_1)(a_2 - s)}{s(a_2 - a_1)}.$$  

(1.7)

The minimum value is at $s = \sqrt{a_1 a_2}$ and equals $\theta_{\text{min}} = \frac{\sqrt{a_2} - \sqrt{a_1}}{\sqrt{a_2} + \sqrt{a_1}} = \tanh \frac{\log(a_2/a_1)}{4}$ which is therefore the desired contraction constant.

Now, returning to the general case let $x_1, x_2 \in U$ be distinct points. We have $d_V(x_1, x_2) \leq d_U(x_1, x_2)$ since the sup in the latter case is over a larger set. So we may assume that $\Delta = \text{diam}_V(U) < +\infty$ and also that $0 < d_V(x_1, x_2) \leq d_U(x_1, x_2) < +\infty$ (or else the statement is trivial). Let $\epsilon > 0$ and pick $v_1, v_2 \in V^c$ so that $d_V(x_1, x_2) \leq (1 + \epsilon) \log [x_1, x_2; v_1, v_2]$. To simplify calculations, we choose a transformation in $GM(\hat{H})$ which maps $v_1$ to zero and $v_2$ to infinity. We recall that this preserves cross-ratios. By a slight abuse of notation we still write $x_1, x_2$ for the images in $\hat{H}$ of the corresponding points. We have then $0 < d_V(x_1, x_2) \leq (1 + \epsilon) \log \frac{\|x_2\|}{\|x_1\|}$ so in particular, $\|x_1\| < \|x_2\|$. When $u_1, u_2 \in U$ we have in these new coordinates, $\log \frac{\|u_2\|}{\|u_1\|} = \|\log[u_1, u_2, 0, \infty]\| \leq d_V(u_1, u_2) \leq \Delta < +\infty$. In other words, $U$ is bounded away from the origin and infinity.
Consider now the formula for the distance of \( x_1, x_2 \) relative to \( U \). It splits into a sum of two suprema (this splitting is one of the deeper reasons why the Apollonian metric is easy to handle):

\[
d_{U}(x_1, x_2) = \sup_{u_1 \in \mathcal{U}^c} \log \frac{\|x_2 - u_1\|}{\|x_1 - u_1\|} + \sup_{u_2 \in \mathcal{U}^c} \log \frac{\|x_1 - u_2\|}{\|x_2 - u_2\|}.
\]

The suprema of these two terms are denoted \( \alpha_1 \) and \( \alpha_2 \). They are both finite. We define the Apollonian ball

\[
B_1 = B_{\alpha_1}(x_1, x_2) = \left\{ u \in \mathcal{H} : \frac{\|x_1 - u\|}{\|x_2 - u\|} < \alpha_1 \right\} \subset U
\]

and similarly for the ball \( B_2 = B_{\alpha_2}(x_2, x_1) \subset U \) (see Figure 1).

A priori \( B_1 \) is a generalized open ball containing \( x_1 \) but as \( U \) is bounded \( B_1 \) must be an open ball in the usual bounded sense (and \( \alpha_1 \) must be greater than one). Now let \( t_1 x_1 \) (with \( 0 < t_1 < 1 \)) be the unique intersection of the segment \( \{ t x_1 : 0 \leq t \leq 1 \} \) and the sphere \( \partial B_{u_1}(x_1, x_2) \). Similarly, let \( t_2 x_2 \) (with \( 1 < t_2 < +\infty \)) be the unique intersection between the segment \( \{ t x_2 : 1 \leq t \leq +\infty \} \) and \( \partial B_{u_2}(x_2, x_1) \) (see Figure 1). Then \( t_1 x_1, t_2 x_2 \in \text{Cl} U \) and \( \|t_1 x_1\| < \|x_1\| < \|x_2\| < \|t_2 x_2\| \). From the way we defined \( t_1 \) and \( t_2 \) we have the following lower bound

\[
d_{U}(x_1, x_2) = \alpha_1 + \alpha_2 = \log \frac{\|x_2 - t_1 x_1\|}{\|x_1 - t_1 x_1\|} \times \frac{\|x_1 - t_2 x_2\|}{\|x_2 - t_2 x_2\|} \geq \log \frac{\|x_2 - t_1 x_1\|}{\|x_1 - t_1 x_1\|} \times \frac{\|x_1 - t_2 x_2\|}{\|x_2 - t_2 x_2\|} \geq \log \frac{\|x_1\| - \|t_1 x_1\|}{\|x_2\| - \|t_1 x_1\|} \times \frac{\|t_2 x_2\| - \|x_1\|}{\|t_2 x_2\| - \|x_2\|}. \tag{1.8}
\]

The last expression is the cross-ratio of the four (ordered) points on the positive real line \( 0 < \|t_1 x_1\| < \|x_1\| < \|x_2\| < \|t_2 x_2\| < +\infty \). Let us write \( J = (0, \infty), \ K = (\|t_1 x_1\|, \|t_2 x_2\|) \) and \( s_1 = \|x_1\|, \ s_2 = \|x_2\| \). By our construction \( \text{diam}_J(K) = \log \frac{\|t_2 x_2\|}{\|t_1 x_1\|} \leq d_{V}(t_1 x_1, t_2 x_2) \leq \text{diam}_V(U) = \Delta \), where we used that \( t_1 x_1, t_2 x_2 \in \text{Cl} U \) and that \( v_1 = 0, v_2 = +\infty \in V \). Also \( d_{K}(\|x_1\|, \|x_2\|) \leq d_{U}(x_1, x_2) \) by the above bound (1.8). So using Birkhoff’s inequality we
get
\[ d_V(x_1, x_2)(1 + \epsilon)^{-1} \leq d_J(s_1, s_2) \leq \left( \tanh \frac{\text{diam}_J(K)}{4} \right) d_K(s_1, s_2) \leq \left( \tanh \frac{\Delta}{4} \right) d_U(x_1, x_2), \]
and since \( \epsilon > 0 \) was arbitrary we see that
\[ d_V(x_1, x_2) \leq \left( \tanh \frac{\Delta}{4} \right) d_U(x_1, x_2), \]
which is what we aimed to show. \[\blacksquare\]

2 Some applications

In the one dimensional case, the result of Birkhoff \cite{Bir57} has a vast variety of applications related to Perron-Frobenius type of results and the presence of spectral gaps of real operators contracting a real convex cone, see e.g. \cite{Bal00}. In the case of complex operators similar spectral gap results were obtained first in \cite{Rug10} and then simplified in \cite{Dub09} using a complex Hilbert metric and the 2-dimensional version of the UCP for the Apollonian metric. We discuss in the following some possible applications in the case of arbitrary dimension.

**Corollary 2.1** Let \( U \subset V \) and \( \Delta \) be as in the Main theorem and write \( \Gamma(V, U) = \{ \gamma \in GM(\hat{H}) : \gamma(V) \subset U \} \) for the elements of the conformal group that map \( V \) into \( U \). Then for every \( \gamma \in \Gamma(V, U) \) we have \( \gamma^{-1} \in \Gamma(U^c, V^c) \) and the mappings \( \gamma : (V, d_V) \to (V, d_V) \) and \( \gamma^{-1} : (U^c, d_{U^c}) \to (U^c, d_{U^c}) \) are \( \left( \tanh \frac{\Delta}{4} \right) \)-Lipschitz.

Proof: \( \gamma \in \Gamma(U, V) \) preserves cross-ratios, and \( \gamma(V) \subset U \) so writing \( \theta = \tanh \Delta/4 \) we have for \( v_1, v_2 \in V \):
\[ d_V(\gamma(v_1), \gamma(v_2)) \leq \theta d_U(\gamma(v_1), \gamma(v_2)) \leq \theta d_{\gamma(V)}(\gamma(v_1), \gamma(v_2)) \leq \theta d_V(v_1, v_2). \]
The inverse map is bijective so it maps \( U^c \) into \( V^c \). We have the same bound for its contraction rate since
\[ \text{diam}_V(U) = \text{diam}_{U^c}(V^c) = \sup_{v_1, v_2 \in V} \sup_{u_1, u_2 \in U^c} \log[u_1, u_2; v_1, v_2]. \] \[\blacksquare\]

**Corollary 2.2** In finite dimension when \( \text{Cl} U \subset \text{Int} V \) for the topology of \( (\hat{H}, \hat{d}) \), then from compactness we see that \( \text{diam}_V(U) < +\infty \) so the embedding \( (U, d_U) \to (V, d_V) \) is a strict Lipschitz contraction.

**Lemma 2.3** Suppose that \( U \subset B(x_0, R), R < \infty \). Then
\[ \| u_1 - u_2 \| \leq \frac{R}{2} d_U(u_1, u_2), \forall u_1, u_2 \in U. \] \[(2.9)\]
Suppose that \( U \subset V \) and that \( r = \text{dist}(U, V^c) = \sup_{u \in U, w \in V^c} \| u - w \| > 0 \). Then
\[ d_V(u_1, u_2) \leq \frac{2}{r} \| u_1 - u_2 \|, \forall u_1, u_2 \in U. \] \[(2.10)\]
Proof: When \( x \in B(x_0, R) \) and \( h \) is small we get from a straight-forward calculation:

\[
d_B(x, x + h) = \frac{2R}{R^2 - \|x - x_0\|^2}\|h\| + o(h).
\]

Thus, \( ds = \frac{2R}{R^2 - \|x - x_0\|^2}\|dx\| \geq \frac{2}{R}\|dx\| \) and \( \|v_1 - v_2\| \leq d_B(v_1, v_2) \leq d_V(v_1, v_2) \) (since \( V \subset B \)).

When \( B(u_1, r), B(u_2, r) \subset V \) then for \( w \in V^c \):

\[
\|u_2 - w\| \leq 1 + \frac{\|u_2 - u_1\|}{r} \quad \text{and} \quad d_V(u_1, u_2) \leq 2\log \left(1 + \frac{\|u_2 - u_1\|}{r}\right) \leq \frac{2}{r}\|u_2 - u_1\|. \quad \square
\]

**Theorem 2.4**  Let \( U \subset V \) be non-empty proper subsets of \( (\tilde{H}, \tilde{d}) \) such that \( Cl \ V \neq \tilde{H} \) and \( \Delta = \text{diam}_V(U) < +\infty \). Let \( \gamma_1, \ldots, \gamma_k \in \Gamma(V, U) \) and write

\[
\Lambda \equiv \Lambda(\gamma_1, \ldots, \gamma_k) = \bigcap_{n \geq 1} Cl \bigcup_{1 \leq i_1, \ldots, i_n \leq k} \gamma_{i_1} \circ \cdots \circ \gamma_{i_n}(V)
\]

for the associated limit set. Then \( \Lambda \) is compact and has Hausdorff and Box dimensions not greater than \( -\log k / \log \tanh \frac{\Delta}{r} \).

Proof: Pick \( q \in \tilde{H} \setminus Cl \ V \) and map \( q \) to infinity by an inversion in \( q \). In the new coordinates \( V \) is bounded so by the previous Lemma, Hilbert distances are bounded by Apollonian distances. At level \( n \geq 1 \) each set in the finite union has diameter not greater than \( r = \Delta(\tanh \frac{\Delta}{r})^{-n} \) which becomes arbitrarily small as \( n \to \infty \). There are \( N_r = k^n \) elements in the union. As \( \Lambda \) is closed and has finite covers of arbitrarily small diameters it is compact and we have the bound

\[
\dim_H(\Lambda) \leq \limsup_n \frac{\log N_r}{\log 1/r} = \frac{\log k}{\log \tanh \frac{\Delta}{r}}. \quad \square
\]

When the images \( Cl(\gamma_i(V)), 1 \leq i \leq k \) are pairwise disjoint the Hausdorff dimension may also be obtained from a Bowen-like formula as in [Rug08] or [MU98]. We omit the details. Note that we do not assume here that \( H \) is finite dimensional.

**Remark 2.5** In finite dimension \( d \geq 2 \) the Apollonian metric for an open ball \( V = B(0, R) \) is the same as the hyperbolic metric for the ball, i.e. \( ds = 2r/(r^2 - \|x\|^2)\|dx\| \). In this case it is well-known that if \( \gamma \) maps \( V \) inside \( V \) and \( \gamma(V) \) has bounded diameter then \( \gamma \) is a uniform contraction.

Other metrics may be constructed from the Apollonian metric (cf. [Has04]). Let \( v \in H^*, \|h\| \leq \|v\|/4 \) and write \( x = \langle h, v \rangle / \langle v, v \rangle \) \([-1/4, 1/4]\] and \( \|h\|^2 = \|v\|^2(x^2 + y^2) \). Calculus shows that \( \frac{1}{2}\log((1 + x^2 + y^2) - x) \leq x^2 + y^2 \) when \( x \geq -1/4 \). Therefore,

\[
\left| \log \frac{\|v + h\|}{\|v\|} - \langle I(v), h \rangle \right| = \left| \log \frac{\|v + h\|}{\|v\|} - \frac{\langle v, h \rangle}{\langle v, v \rangle} \right| \leq \frac{\langle h, h \rangle}{\langle v, v \rangle}.
\]

We assume in the following that \( U \) is open. Let \( x \in U \) and set \( r = \inf_{u \in U^c} d(x, u) > 0 \). When \( u_1, u_2 \in U^c \) and \( \|h\| \leq r/4 \) we get:

\[
\left| \log [x, x + h; u_1, u_2] - \langle I(x - u_1) - I(x - u_2), h \rangle \right| \leq 2\|h\|^2/r^2.
\]

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It follows that the following limit exists and define a Finsler (pseudo-) norm on the tangent space of \( U \):

\[
p_{U,x}(h) \equiv \lim_{t \to 0} \frac{1}{t} d_U(x + th, x) = \sup_{u_1,u_2 \in U^c} |\langle I(x - u_1) - I(x - u_2), h \rangle|.
\] (2.11)

It is only a pseudo-norm when \( U^c \) is contained in a generalized ball, since in that case \( p_{U,x} \) may vanish in some directions. If \( \gamma : [0,1] \to U \) is a continuous path then we may define its (pseudo-) length to be

\[
\ell(\gamma) \equiv \limsup_{\delta \to 0} \sum_{k=0}^{n} d_U(\gamma(t_{k+1}), \gamma(t_k)),
\]

where \( 0 = t_0 < t_1 < \cdots < t_n = 1 \) and \( t_{k+1} - t_k < \delta \). Then

\[
d_V^{\text{in}}(x,y) = \inf\{\ell(\gamma) : \gamma \in C([0,1], U), \gamma(0) = x, \gamma(1) = y\} \tag{2.12}
\]
defines a (pseudo-)metric which in \([Has04]\) was coined the Apollonian inner metric. When \( \gamma \) is piecewise \( C^1 \) we have \( \ell(\gamma) = \int_0^1 p_{U,x}(\dot{\gamma}(t)) \, dt \). Another possibility is to maximize (2.11) over directions. This leads to a conformal Riemannian metric \( ds = g_U(x)dx \) with

\[
g_U(x) = \sup_{\|h\| = 1} p_{U,x}(h) = \sup_{u_1,u_2 \in U^c} \frac{\|u_1 - u_2\|}{\|x - u_1\| \|x - u_2\|}. \tag{2.13}
\]

An advantage of this metric is perhaps that it distinguishes points when \( U^c \) contains at least two points. It is easy to see that \( g_U(x) \) is continuous (as we assumed \( U \) to be open). We write \( d_V^{\text{Rie}}(x,y) \) for the Riemannian distance of \( x \) and \( y \) with respect to this metric.

**Corollary 2.6** Let \( U \subset V \subset \hat{H} \) (with \( \text{Cl} \, V \neq \hat{H} \)) be non-empty proper subsets and let \( \Delta = \sup_{u_1,u_2 \in U} d_V(u_1,u_2) \) be the diameter of the smaller subset within the larger with respect to the Apollonian metric. Then for every \( x,y \in U \):

\[
p_{V,x}(h) \leq \left( \tanh \frac{\Delta}{4} \right) p_{U,x}(h), \quad h \in E,\tag{2.14}
\]

\[
d_V^{\text{in}}(x,y) \leq \left( \tanh \frac{\Delta}{4} \right) d_U^{\text{in}}(x,y),\tag{2.15}
\]

\[
d_V^{\text{Rie}}(x,y) \leq \left( \tanh \frac{\Delta}{4} \right) d_U^{\text{Rie}}(x,y).\tag{2.16}
\]

Proof: For \( x, x+th \in U \) we have by the Main Theorem \( \frac{1}{t} d_U(x, x+th) \leq \tanh \frac{\Delta}{4} d_U(x, x+th) \). The first inequality follows. The second follows by taking limits in the right order. For the Riemannian metric one has

\[
g_V(x) \leq \sup_{\|h\| = 1} p_{U,x}(h) \leq \sup_{\|h\| = 1} \left( \tanh \frac{\Delta}{4} \right) p_U(x) = \left( \tanh \frac{\Delta}{4} \right) g_U(x)
\]

which yields the last inequality. □
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