MEAN-CVAR PORTFOLIO SELECTION MODEL WITH AMBIGUITY IN DISTRIBUTION AND ATTITUDE

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Abstract. In this paper, we develop α-robust (maxmin) models, where the Conditional Value-at-Risk (CVaR) is to be optimized under ambiguity in distribution, mean returns, and covariance matrix. Our models allow the investor to distinguish ambiguity and ambiguity attitude with different levels of ambiguity aversion. For the case when there is a risk-free asset and short-selling is allowed, we obtain the analytic solution for the α-robust CVaR optimization model subject to a minimum mean return constraint. Moreover, we also derive a closed-form portfolio rule for the α-robust mean-CVaR optimization problem in a market without the risk-less asset. The results obtained from solving the numerical example show that if an investor is more ambiguity-averse, his investment strategy will always be more conservative.

1. Introduction. Portfolio management is one of the fundamental problems in financial decision making. It is formulated as an optimization problem, where the risk measure is minimized through optimally allocating wealth into a number of available financial assets subject to an expected return constraint. In financial industry, Value-at-Risk (VaR), is the standard measure of risk. It reflects the maximum potential loss of an asset or a portfolio in a given period. Another measure of risk, Conditional Value-at-Risk (CVaR), has emerged as a popular alternative to replace VaR as the financial industry standard. It calculates the expected value of the loss exceeding VaR. In an investment problem, the returns of underlying assets in the financial market are random variables. It is normally assumed that their probability laws are available. However, in reality, there exist many factors which
are beyond the knowledge of most investors. This type of uncertainty is referred as ambiguity or Knightian uncertainty (see the seminal work of Ellsberg [7]). This motivates the study of the distributionally robust formulation of the risk measures which describes the worst-possible risk level over a set of potential distributions.

For distributionally robust optimization, instead of assuming that there is an underlying probability distribution that is known to the decision maker, the optimization is to be done subject to an ambiguity set of distributions. It is a generalization of classical robust optimization. It is well-known that the classical robust optimization can generate portfolios that are immune to noise and uncertainty in the parameters. However, such portfolios do not make use of any available information on distribution, such as moment information, and thus can be overly conservative (Cheng et al. [5]). For distributionally robust optimization, the probability distributions of uncertain parameters are assumed to belong to an ambiguity set (a family of distributions that share common properties), which is a key ingredient of any distributionally robust optimization model. Two types of ambiguity sets have been proposed: moment-based ambiguity sets and metric-based ambiguity sets. For moment-based ambiguity sets, it is assumed that all distributions in the distribution family satisfy certain moment constraints (Popescu [22]; Delage and Ye [6]; Kang and Li [15]). Metric-based ambiguity sets may contain all distributions that are sufficiently close to a reference distribution or most likely distribution with respect to a probability metric such as the ϕ-divergence (Bayraksan and Love [1]), and the Wasserstein metric (Esfahani and Kuhn [8]; Jiang and Guan [13]). We refer to Wiesemann et al. [25] for a survey of literatures on ambiguity models.

For the max-min expected utility framework (MEU) axiomatized by Gilboa and Schmeidler [11], it deals with extremely ambiguity-averse attitude by considering the worst-case scenario. This formulation does not distinguish between ambiguity and aversion to ambiguity. Experimental evidence shows that investors’ attitude to ambiguity is not systematically negative and they are not always extremely ambiguity averse (Heath and Tversky [12]; Fox and Tversky [9]). Heath and Tversky [12] show that people may even be ambiguity-loving in some circumstances if they are knowledgeable, experienced, or competent on the game (confident investors are willing to pay a considerable premium to bet on their own judgement rather than a chance event). In light of complex attitudes of decision makers towards ambiguity, Ghirardato et al. [10] axiomatize a model, termed as α-maxmin expected utility (α-MEU), for which it is possible in a certain sense to differentiate conceptually ambiguity from ambiguity attitude. The α-MEU model is

\[ (1 - \alpha) \sup_{P \in \mathcal{D}} \mathbb{E}_P[U(X)] + \alpha \inf_{P \in \mathcal{D}} \mathbb{E}_P[U(X)], \]

where \( \alpha \in [0, 1] \) is an index of ambiguity attitude, \( X \) is a random payoff, \( U \) is a general utility function, and \( \mathcal{D} \) is a set of prior probability measures. This approach considers a decision maker who evaluates each portfolio by a convex combination of its best and worst expected utilities over probabilities in \( \mathcal{D} \), using the weights \( 1 - \alpha \) and \( \alpha \) for the best and worst expected utilities, respectively. A key feature of α-MEU is that it differentiates the level of ambiguity aversion, specified by \( \alpha \), and the

\footnote{Later in this work, we restrict \( \alpha \in [\frac{1}{2}, 1] \). First, under this assumption, our α-robust CVaR optimization problems can be guaranteed to be convex, and the global optimal analytical strategies can be obtained. Second, Bossaerts et al. [4] show that there is a positive correlation between risk-aversion and ambiguity-aversion. In order to ensure consistency with the mean-CVaR framework which is risk-averse, we only discuss the behavior of ambiguity-averse investors.}
level of ambiguity, specified by the range of $D$. $\alpha$-MEU can better capture the ambiguity attitude (parameterized by $\alpha$) of the decision maker. In particular, $\alpha = 1, 0$ represent, respectively, extreme ambiguity-aversion and extreme ambiguity-loving attitudes, $\alpha = \frac{1}{2}$ represents the ambiguity-neutral attitude, and a higher value of $\alpha$ represents a more ambiguity-averse attitude. Recently, Li et al. [16] study the $\alpha$-robust mean-variance reinsurance-investment problem under the ambiguity framework (1). Li et al. [17] investigate the robust utility maximization problems taking into consideration of extreme ambiguity-loving and extreme ambiguity-aversion in a continuous time framework. Using parametric methods, Lotfi et al. [19] formulate tractable adjustable robust versions of the problem, where the VaR is computed using VaR portfolio optimization. It is then shown that the results obtained using this approach is robust with respect to both solution and structure without being too conservative.

Our work is motivated by the insight gained from the works of Li et al. [16] and Li et al. [17]. To avoid extremely ambiguity-averse attitude and the conservatism of optimal portfolio, similar to the $\alpha$-MEU criterion (Ghirardato et al. [10]), we propose a new mean-CVaR criterion, called $\alpha$-robust mean-CVaR criterion for the case where there is a risk-free asset and short-selling is allowed. Our model allows the investor to have different levels of ambiguity aversion, rather than having only the extremely ambiguity-averse attitude as in the existing literature (e.g., Paç and Pinar [21]; Liu et al. [18]). For the portfolio optimization problem without including a risk-less asset, we derive a closed-form solution. This result can be considered as a generalization of that obtained by Liu et al. [18] for the worst-case scenario. Our analytical results could help investors to manage an appropriate trade-off between robustness and less conservativeness of their investment strategies. It is clearly seen that if an investor is less ambiguity-averse, his investment strategy will be less conservative.

The developed robust mean-CVaR model in Paç and Pinar [21] is subject to ambiguity in terms of distribution and mean return, but not covariance matrix. On the other hand, our models consider the scenarios involving uncertainty in terms of mean, covariance and distribution. Although the axiomatic characterization of $\alpha$-MEU has been well studied, there has been little work on the financial investment problem for $\alpha$-MEU. Our models just take into account the worst case and best case of the set of distributions, which is the main feature of $\alpha$-MEU. For most of distributionally robust portfolio selection problems with unknown first and second order moments, such as Delage and Ye [6], no closed-form solution has been found (with the exception of e.g. Liu et al. [18]). The aim of this paper is to give an explicit solution to the optimal portfolio selection problem with short-selling being allowed, where a convex combination of the worst-case and best-case CVaR measures is minimized subject to a given level of ambiguity-aversion $\alpha$. A computationally simple analytical solution is obviously much more useful, especially for unsophisticated investors. To the best of our knowledge, there is no other study of $\alpha$-robust mean-CVaR portfolio selection problem allowing the investors to have different levels of ambiguity aversion. The existing worst-case CVaR optimization models (e.g., Paç and Pinar [21], Liu et al. [18]) with a given set of probability measures may be considered as a limiting case of our models with infinite level of ambiguity aversion.

The rest of this paper is set up as follows. In Section 2, the mathematical model of $\alpha$-robust CVaR in the presence of a risk-less asset over a set of distributions
with unknown first and second moments is formulated. We derive a closed-form optimal portfolio rule. For the optimal portfolio problem with no risk-less asset, we derive, in Section 3, the $\alpha$-maxmin optimal portfolio choice in closed form. This is a generalization of the result obtained by Liu et al. [18] for the case of extreme ambiguity-aversion. Some numerical examples are provided to support our theoretic results. Section 4 concludes the paper.

2. Optimal portfolios of $\alpha$-robust mean-CVaR optimization problem. We consider the problem of a MC (Mean-CVaR) investor operating in a market consisting of $n$ risky assets and a risk-less asset with return rate $r_f$, as treated in Pa¸c and Pinar [21]. The random return rates of risky assets are denoted by $\xi \in \mathbb{R}^n$. The corresponding loss function associated with decision variable $x \in \mathbb{R}^n$ of allocations to $n$ risky assets is denoted by

$$l(x, \xi) = - (\xi^T x + r_f(1 - e^T x)),$$

(2)

where $e$ denotes an $n$-vector of ones. Suppose that the random return vector $\xi$ has a probability distribution $p(\xi)$. Then, given a confidence level of $\beta \in (0, 1)$, VaR is defined as

$$\text{VaR}_\beta(x) = \inf \left\{ \zeta \in \mathbb{R} : \int_{\{\xi : l(x, \xi) \leq \zeta\}} p(\xi) d\xi \geq \beta \right\}. $$

The corresponding CVaR at level $1 - \beta$ with respect to the distribution $P$, which is defined as the expected value of the loss $l(x, \xi)$ exceeding VaR, can be expressed as

$$\text{CVaR}_{\beta}(x, P) = \min_{\zeta \in \mathbb{R}} \left\{ \zeta + \frac{1}{1 - \beta} \mathbb{E}_P [l(x, \xi) - \zeta]^+ \right\},$$

where $[a]^+ := \max\{a, 0\}$ for any $a \in \mathbb{R}$.

In light of ambiguity aversion and risk aversion, a significant body of research (see, for instance, Quaranta and Zaffaroni [23]; Zhu and Fukushima [26]; Ruan and Fukushima [24]; Pa¸c and Pinar [21]; Lotfi and Zenios [20]; Liu et al. [18]; Kang et al. [14]) has been done to study the robust CVaR of the form

$$\min_{x \in \mathbb{R}^n} \sup_{P \in D} \text{CVaR}_\beta(x, P),$$

(3)

s.t. $\inf_{P \in D} \mathbb{E}_P (-l(x, \xi)) \geq d$, where $D$ is a set of prior probability measures, $d$ is the minimum target mean return. We assume that the target mean return is larger than the risk-less return, i.e., $d > r_f$. Otherwise, it is optimal to invest all wealth into the risk-less asset. If the true model is the worst one, then this will be its solution. However, if the true model is the best one or something close to it, this will be a poor solution.

For the robust CVaR optimization problem (3), it only takes into account extreme ambiguity aversion. It is well-known that the worst case optimization is most conservative, while the best case optimization is, on the opposite, the least conservative. In this paper, we propose a class of more general functions, called $\alpha$-robust
\(\alpha\)-ROBUST MEAN-CVaR INVESTMENT STRATEGY

Mean-CVaR criterion, given by\(^3\)

\[
\min_{x \in \mathbb{R}^n} \{(1 - \alpha) \inf_{P \in \mathcal{D}} \text{CVaR}_\beta(x, P) + \alpha \sup_{P \in \mathcal{D}} \text{CVaR}_\beta(x, P)\},
\]

s.t. \((1 - \alpha) \sup_{P \in \mathcal{D}} \mathbb{E}_P(-l(x, \xi)) + \alpha \inf_{P \in \mathcal{D}} \mathbb{E}_P(-l(x, \xi)) \geq d,\]

(4)

where \(\alpha \in [0, 1]\) can be thought of as an ambiguity aversion index,\(^4\) which is a parameter that describes the relative weight put on pessimism versus optimism. A special case is for the level of ambiguity-aversion \(\alpha = 1\) (see, Remark 3), which leads to the worst case CVaR optimization problem as studied in Pac and Pinar [21]. Solution becomes more conservative as \(\alpha\) gets closer to one.

Remark 1. Similar to the \(\alpha\)-MEU criterion (Ghirardato et al. [10]), our objective is to minimize a convex combination of the worst-case and the best-case CVaR risk measures, where the worst-case and the best case are taken with respect to all distributions in the set \(\mathcal{D}\). A key feature of \(\alpha\)-robust mean-CVaR criterion (4) is that it differentiates the level of ambiguity aversion, specified by \(\alpha\), and the level of ambiguity, specified by the range of \(\mathcal{D}\). In the existing literature, a significant body of research on CVaR optimization problem considers only the extreme case of this criterion, i.e., \(\alpha = 1\).

In this paper, for greater computational tractability, we restrict ourselves to the following ambiguity set (used in Assumption 1 of Liu et al. [18]).

**Definition 2.1.** (Ambiguity in distribution). The random variable \(\xi \in \mathbb{R}^n\) assumes a distribution from the following set \(\mathcal{D}\)

\[
\mathcal{D} = \left\{ P \in \mathcal{M}_+ : \begin{array}{l}
(E_P(\xi) - \hat{\mu})^T \hat{\Sigma}^{-1}(E_P(\xi) - \hat{\mu}) \leq \gamma_1, \\
\text{Cov}_P(\xi) \preceq \gamma_2 \hat{\Sigma}, \text{Cov}_P(\xi) \succ 0
\end{array} \right\},
\]

where \(\mathcal{M}_+\) is the set of all probability measures on the measurable space \((\mathbb{R}^n, \mathcal{B})\) with the \(\sigma\)-algebra \(\mathcal{B}\) on \(\mathbb{R}^n\). \(\hat{\mu}\) and \(\hat{\Sigma}\) are estimated values of the mean vector and the covariance matrix of \(\xi\). Here, \(\gamma_1, \gamma_2 \in \mathbb{R}^+\) are two scale parameters controlling the size of the uncertainty set.

We assume that \(\hat{\mu}\) is not a multiple of \(e\), i.e., \(\hat{\mu}\) and \(e\) are linearly independent and \(\hat{\Sigma}\) is symmetric and positive definite. \(A \succeq B (A > B)\) implies that \(A - B\) is positive semidefinite (positive definite). In this paper, we choose this set as a starting point because the optimal strategy of mean-risk criterion can be expressed explicitly. This makes it more convenient for the later analysis on the behavior of investors with different levels of ambiguity aversion. For convenience, we define the following two sets, which will be used in subsequent proofs. The random variable \(\xi \in \mathbb{R}^n\) assumes a distribution from the following set \(\mathcal{F}(\hat{\mu}, \hat{\Sigma})\) with fixed \(\hat{\mu}\) and \(\hat{\Sigma}\), i.e.,

\[
\mathcal{F}(\hat{\mu}, \hat{\Sigma}) = \{ P \in \mathcal{M}_+ : \mathbb{E}_P(\xi) = \hat{\mu}, \text{Cov}_P(\xi) = \hat{\Sigma} \succ 0 \}.
\]

\(^3\)We will assume that at optimum the constraint is binding. This is the case when \(d\) is not smaller than the return for the minimal risk portfolio, i.e., the portfolio which solves the problem

\[
\min_{x \in \mathbb{R}^n} \{ (1 - \alpha) \inf_{P \in \mathcal{D}} \text{CVaR}_\beta(x, P) + \alpha \sup_{P \in \mathcal{D}} \text{CVaR}_\beta(x, P) \}
\]

without additional constraint.

\(^4\)There is no consensus whether \(\alpha\) should denote the weight attribute to the worst case or to the best case. Since \(\alpha\) is attributed to the worst case in our model, we also call this parameter the degree of pessimism.
Meanwhile, mean return \( \bar{\mu} \) and covariance matrix \( \bar{\Sigma} \) belong to the set
\[
U(\hat{\mu}, \hat{\Sigma}) = \left\{ (\bar{\mu}, \bar{\Sigma}) \in \mathbb{R}^n \times S^n : \begin{align*}
(\bar{\mu} - \hat{\mu})^T \bar{\Sigma}^{-1} (\bar{\mu} - \hat{\mu}) &\leq \gamma_1, \\
\bar{\Sigma} &\preceq \gamma_2 \hat{\Sigma}
\end{align*} \right\},
\]
where \( S^n \) denotes the space of the symmetric matrices of dimension \( n \). The uncertainty set \( U(\mu, \Sigma) \) accounts for information about the mean of random terms in the ellipsoidal set, and an upper bound \( \hat{\Sigma} \) on the covariance matrix of the random vector \( \xi \). \( \gamma_1 \) can be viewed as the squared radius of the confidence region of the mean vector. A higher value of \( \gamma_1 \) implies a higher level of ambiguity towards expected return. It is clear that
\[
D = \bigcup_{(\bar{\mu}, \bar{\Sigma}) \in U(\mu, \Sigma)} F(\bar{\mu}, \bar{\Sigma}). \tag{6}
\]

The following conclusion comes directly from Paç and Pinar [21]. It will be used in our subsequent proofs.

**Lemma 2.2.** Assume that \( l(x, \xi) = -(\xi^T x + r_f (1 - e^T x)) \) and that random vector \( \xi \in \mathbb{R}^n \), with mean \( \bar{\mu} \) and covariance \( \bar{\Sigma} \succ 0 \), follows a family of distributions \( F(\bar{\mu}, \bar{\Sigma}) \), which is defined by (5). Then we have
\[
\sup_{P \in F(\bar{\mu}, \bar{\Sigma})} \text{CVaR}_\beta(x, P) = -r_f - (\bar{\mu} - r_f e)^T x + \kappa \sqrt{x^T \Sigma x}, \tag{7}
\]
where \( \kappa = \sqrt{\frac{\beta}{1 - \beta}} \).

Using a well-known result in robust optimization (Ben-Tal and Nemirovski [2]), we have the following lemma.

**Lemma 2.3.** Define the ellipsoidal uncertainty set \( U_{\bar{\mu}} = \{ \bar{\mu} \in \mathbb{R}^n \mid (\bar{\mu} - \hat{\mu})^T \bar{\Sigma}^{-1} (\bar{\mu} - \hat{\mu}) \leq \gamma_1 \} \) for the mean return denoted by \( \bar{\mu} \), then
\[
\inf_{\bar{\mu} \in U_{\bar{\mu}}} \bar{\mu}^T x = \hat{\mu}^T x - \sqrt{\gamma_1} \sqrt{x^T \Sigma x},
\]
\[
\sup_{\bar{\mu} \in U_{\bar{\mu}}} \bar{\mu}^T x = \hat{\mu}^T x + \sqrt{\gamma_1} \sqrt{x^T \Sigma x}.
\]

**Proof.** It can be obtained directly by Ben-Tal and Nemirovski [2]. \( \square \)

**Proposition 1.** Suppose that the unknown distribution of the random variable \( \xi \in \mathbb{R}^n \) is within a family of distributions \( D \), Then, the \( \alpha \)-robust mean-CVaR model (4) is described by the following program
\[
\min_{x \in \mathbb{R}^n} -r_f - (\bar{\mu} - r_f e)^T x + [(2\alpha - 1) \sqrt{\gamma_1} + \alpha \kappa \sqrt{\gamma_2}] \sqrt{x^T \Sigma x},
\]
\[
s.t. (\bar{\mu} - r_f e)^T x - (2\alpha - 1) \sqrt{\gamma_1} \sqrt{x^T \Sigma x} \geq d - r_f. \tag{8}
\]

**Proof.** Based on the structure of the ambiguity set \( D \) and from Corollary 3.1 in Zhu and Shao [27], we have
\[
\inf_{P \in D} \text{CVaR}_\beta(x, P) = \inf_{(\bar{\mu}, \bar{\Sigma}) \in U(\mu, \Sigma)} \inf_{P \in F(\bar{\mu}, \bar{\Sigma})} \text{CVaR}_\beta(x, P) = \inf_{(\bar{\mu}, \bar{\Sigma}) \in U(\mu, \Sigma)} -r_f - (\bar{\mu} - r_f e)^T x
\]
\[
= -r_f - (\bar{\mu} - r_f e)^T x - \sqrt{\gamma_1} \sqrt{x^T \Sigma x}. \tag{9}
\]
The last equality comes directly from Lemma 2.3.
From Lemma 2.2 and Lemma 2.3, it follows that

$$\sup_{P \in \mathcal{D}} \text{CVaR}_\beta(x, P) = \sup_{(\hat{\mu}, \Sigma) \in \mathcal{U}(\hat{\mu}, \Sigma)} \sup_{P \in \mathcal{F}(\hat{\mu}, \Sigma)} \text{CVaR}_\beta(x, P)$$

$$= \sup_{(\hat{\mu}, \Sigma) \in \mathcal{U}(\hat{\mu}, \Sigma)} -r_f - (\mu - r_f e)^T x + \kappa \sqrt{x^T \Sigma x}$$

$$= \sup_{(\hat{\mu}, \Sigma) \in \mathcal{U}(\hat{\mu}, \Sigma)} -r_f - \inf_{\hat{\mu} \in \mathcal{U}_\mu} (\mu - r_f e)^T x + \kappa \sup_{\Sigma \in \mathcal{U}_\Sigma} \sqrt{x^T \Sigma x}$$

$$= \sup_{(\hat{\mu}, \Sigma) \in \mathcal{U}(\hat{\mu}, \Sigma)} -r_f - (\mu - r_f e)^T x + \sqrt{\gamma_1} \sqrt{x^T \Sigma x} + \kappa \sqrt{\gamma_2} \sqrt{x^T \Sigma x}, \quad (10)$$

where \( \kappa = \sqrt{\frac{\beta}{1-\beta}} \).

Substituting (9) and (10) into the objective function of (4), we obtain

$$-r_f - (\mu - r_f e)^T x + [(2\alpha - 1)\sqrt{\gamma_1} + \alpha \kappa \sqrt{\gamma_2}] \sqrt{x^T \Sigma x}. \quad (11)$$

Using the same transformation on the minimum mean return constraint, the constraint of problem (4) becomes

$$r_f + (\mu - r_f e)^T x - (2\alpha - 1)\sqrt{\gamma_1} \sqrt{x^T \Sigma x} \geq d. \quad (12)$$

Combining this with (11) leads to our desired result.

**Remark 2.** Problem (8) is convex for \( \frac{1}{2} \leq \alpha \leq 1 \) and hence it can be solved efficiently.

We are now in a position to present our first main result on the solution to the \( \alpha \)-robust mean-CVaR investment problem in the presence of risk-less asset under distribution ambiguity. Define the excess mean return vector estimate \( \hat{\mu} = \mu - r_f e \), and the optimal Sharpe ratio in the market \( H = \hat{\mu}^T \hat{\Sigma}^{-1} \hat{\mu} \) (\( H \) is positive because \( \hat{\Sigma} \) is positive definite and \( \hat{\mu} \) and \( e \) are linearly independent).

**Theorem 2.4.** (a). Under the assumption of Slater constraint qualification, i.e., there exists \( \bar{x} \in \mathbb{R}^n \) such that

$$r_f + (\mu - r_f e)^T \bar{x} - (2\alpha - 1)\sqrt{\gamma_1} \sqrt{\bar{x}^T \Sigma \bar{x}} > \rho.$$  

If \( (2\alpha - 1)\sqrt{\gamma_1} < \sqrt{H} \) and \( \beta > \frac{(\sqrt{H} - (2\alpha - 1)\sqrt{\gamma_1})^2}{(\sqrt{H} - (2\alpha - 1)\sqrt{\gamma_1})^2 + (\alpha \kappa)^2} \), then the optimal portfolio for problem (8) (i.e., the \( \alpha \)-robust mean-CVaR model (4)) is given by

$$x^* = \frac{d - r_f}{(\sqrt{H} - (2\alpha - 1)\sqrt{\gamma_1})\sqrt{H}} \hat{\Sigma}^{-1} \hat{\mu}. \quad (13)$$

(b). If \( (2\alpha - 1)\sqrt{\gamma_1} \geq \sqrt{H} \), then problem (8) is infeasible.

**Proof.** (a). The Lagrangian function of the optimization problem specified by (8) is

$$L(x, \omega) = -r_f - \hat{\mu}^T x + [(2\alpha - 1)\sqrt{\gamma_1} + \alpha \kappa \sqrt{\gamma_2}] \sqrt{x^T \Sigma x}$$

$$+ \omega(d - r_f - \hat{\mu}^T x + (2\alpha - 1)\sqrt{\gamma_1} \sqrt{x^T \Sigma x})$$

where \( \omega \) is a non-negative Lagrangian multiplier. The first-order conditions (these conditions are necessary and sufficient under Slater constraint qualification) give

$$-\hat{\mu} + [(2\alpha - 1)\sqrt{\gamma_1} + \alpha \kappa \sqrt{\gamma_2}] \frac{\Sigma x}{\sqrt{x^T \Sigma x}} - \omega \hat{\mu} + \omega (2\alpha - 1) \sqrt{\gamma_1} \frac{\Sigma x}{\sqrt{x^T \Sigma x}} = 0.$$
This yields the candidate solution
\[ x_c = \frac{\sigma(\omega + 1)}{(2\alpha - 1)\sqrt{\gamma_1}(\omega + 1) + \alpha \kappa \sqrt{\gamma_2}}. \] (14)

From the identity \( x_c^T \tilde{\Sigma} x_c = \sigma^2 \), we obtain the quadratic equation in \( \omega \)
\[ H - ((2\alpha - 1)\sqrt{\gamma_1})^2\omega^2 + 2H - 2((2\alpha - 1)\sqrt{\gamma_1})^2 - 2(2\alpha - 1)\sqrt{\gamma_1}\kappa \sqrt{\gamma_2} \omega + \omega + H - ((2\alpha - 1)\sqrt{\gamma_1})^2 - 2(2\alpha - 1)\sqrt{\gamma_1}\kappa \sqrt{\gamma_2} - \alpha^2 \kappa^2 \gamma_2 = 0. \]

The two roots are given by
\[ \omega_1 = -\frac{\alpha \kappa \sqrt{\gamma_1} + \sqrt{H} + (2\alpha - 1)\sqrt{\gamma_1}}{\sqrt{H} + (2\alpha - 1)\sqrt{\gamma_1}}, \quad \omega_2 = -\frac{-\alpha \kappa \sqrt{\gamma_1} + \sqrt{H} - (2\alpha - 1)\sqrt{\gamma_1}}{\sqrt{H} - (2\alpha - 1)\sqrt{\gamma_1}}. \]

Since the root \( \omega_1 \) is always non-positive, it is discarded. If \( \sqrt{H} > (2\alpha - 1)\sqrt{\gamma_1} \) and \( \alpha \kappa \sqrt{\gamma_1} > \sqrt{H} - (2\alpha - 1)\sqrt{\gamma_1} \), the root \( \omega_2 \) is positive. Returning to the conic constraint (12) which is assumed to be tight and substituting \( \omega_2 \) into the expression for \( x_c \), we obtain an equation in \( \sigma \):
\[ r_f + \sigma \sqrt{H} - (2\alpha - 1)\sqrt{\gamma_1} \sigma = d. \]

Simple algebraic manipulation gives
\[ \sigma = \frac{d - r_f}{\sqrt{H} - (2\alpha - 1)\sqrt{\gamma_1}}, \]
which is positive provided \( \sqrt{H} > (2\alpha - 1)\sqrt{\gamma_1} \). Substituting \( \sigma \) and \( \omega_2 \) obtained above into (14), we obtain our desired result.

(b). Assume that problem (8) is feasible. Then, there exists an \( x_0 \) such that
\[ (2\alpha - 1)\sqrt{\gamma_1} \sqrt{x_0^T \tilde{\Sigma} x_0} \leq (\hat{\mu} - r_f e)^T x_0 + (r_f - d). \]

Since \( r_f - d < 0 \), we get
\[ (2\alpha - 1)\sqrt{\gamma_1} \sqrt{x_0^T \tilde{\Sigma} x_0} < (\hat{\mu} - r_f e)^T x_0. \] (15)

On the other hand, if \( (2\alpha - 1)\sqrt{\gamma_1} \geq \sqrt{H} \), then we have
\[ \sqrt{H} \sqrt{x_0^T \tilde{\Sigma} x_0} \leq (2\alpha - 1)\sqrt{\gamma_1} \sqrt{x_0^T \tilde{\Sigma} x_0}. \] (16)

Using (15), (16) and the Cauchy-Schwartz inequality, we get
\[ (\hat{\mu} - r_f e)^T x_0 \leq \sqrt{H} \sqrt{x_0^T \tilde{\Sigma} x_0} < (\hat{\mu} - r_f e)^T x_0 \]
which is a contradiction. \( \square \)

In the following, we examine two special cases of Theorem 2.4: (i) \( \alpha = 1 \), which corresponds to the extreme ambiguity-aversion preference; and (ii) \( \alpha = \frac{1}{2} \), which corresponds to the ambiguity-neutral preference (equal contribution from the worst case and the best case optimisation).

**Corollary 1.** (1) For an extremely ambiguity-averse (AA) investor (\( \alpha = 1 \)), if \( \sqrt{\gamma_1} < \sqrt{H} \) and \( \beta > \frac{(\sqrt{H} - \sqrt{\gamma_1})^2}{(\sqrt{H} - \sqrt{\gamma_1})^2 + (\sqrt{\gamma_2})^2} \), the optimal investment strategy \( x^* \) is given by
\[ x^*_{AA} = \frac{d - r_f}{(\sqrt{H} - \sqrt{\gamma_1})} \tilde{\Sigma}^{-1} \hat{\mu}. \]

If \( \sqrt{\gamma_1} \geq \sqrt{H} \), then the problem is infeasible.
(2) For an ambiguity-neutral (AN) investor ($\alpha = \frac{1}{2}$), if $\beta > \frac{4H}{4H + \gamma_1}$, the optimal investment strategy $x^\ast$ is given by

$$x^\ast_{AN} = \frac{d - rf \hat{\Sigma}^{-1} \hat{\mu}}{H},$$

which is close to the classical mean-variance portfolio.\(^5\)

**Remark 3.** The $\alpha$-robust framework (4) with $\alpha = 1$ and $\gamma_2 = 1$ is related to the robust CVaR optimization considered in Paç and Pinar [21]. More specifically, when $\alpha = 1$ and $\gamma_2 = 1$, case (1) in Corollary 1 is exactly identical to Theorem 3 of Paç and Pinar [21].

Thanks to the closed-form expressions for the optimal investment strategy $x^\ast$ given in (13), we are now ready to derive the equation of the $\alpha$-robust mean-CVaR ($\alpha$-CVaR) efficient frontier. On this basis, we will further investigate the relationships between $x^\ast$ and the ambiguity-aversion parameter $\alpha$. The $\alpha$-robust CVaR efficient frontier is a straight line given by

$$d = b(\alpha) + k(\alpha) R$$

where $b(\alpha) = \frac{-\kappa \sqrt{\gamma_2}(\sqrt{H} + \sqrt{\gamma_1})}{\alpha \kappa \sqrt{\gamma_2} - \sqrt{H} + (2\alpha - 1) \sqrt{\gamma_1}}$ and $k(\alpha) = \frac{\sqrt{\gamma_2} - (2\alpha - 1) \sqrt{\gamma_1}}{\alpha \kappa \sqrt{\gamma_2} - \sqrt{H} + (2\alpha - 1) \sqrt{\gamma_1}}$ and $R = \alpha$-CVaR ($\alpha$-CVaR is a convex mixture between the worst-case and best-case values of CVaR risk measures).

We can see that the slope and intercept of the line are all $\alpha$-dependent functions, and different values of $\alpha$ corresponds to different effective frontiers. We only show that $k(\alpha)$ is decreasing in $\alpha$. The monotonicity of $b(\alpha)$ can be proved in the same way.

**Proposition 2.** For any fixed $\gamma_1 \geq 0$ and $\gamma_2 > 0$, $k(\alpha)$ and $b(\alpha)$ are decreasing functions for $\alpha \in \left[\frac{1}{2}, 1\right]$. i.e.,

$$\frac{dk(\alpha)}{d\alpha} = \frac{-\kappa \sqrt{\gamma_2}(\sqrt{H} + \sqrt{\gamma_1})}{(\alpha \kappa \sqrt{\gamma_2} - \sqrt{H} + (2\alpha - 1) \sqrt{\gamma_1})^2} < 0,$$

and

$$k(\alpha) - k(1) = \frac{(1 - \alpha) \kappa \sqrt{\gamma_2}(\sqrt{H} + \sqrt{\gamma_1})}{(\alpha \kappa \sqrt{\gamma_2} - \sqrt{H} + (2\alpha - 1) \sqrt{\gamma_1})(\kappa \sqrt{\gamma_2} - \sqrt{H} + \sqrt{\gamma_1})} \geq 0,$$

where $k(1) = \frac{\sqrt{\gamma_2} - \sqrt{H}}{\kappa \sqrt{\gamma_2} - \sqrt{H} + \sqrt{\gamma_1}}$.

The proof is straightforward. We provide a numerical example with $H = 0.4722$, $r_f = 1.01$, $\gamma_1 = 0.0001$, $\gamma_2 = 1.2$, $\beta = 0.95$ in Fig.1. The left panel of Fig.1 shows that $k(\alpha)$ and $b(\alpha)$ are decreasing in $\alpha$. From the right panel of Fig.1., we can see that the effective frontier becomes steeper as $\alpha$ decreases. That is to say, extreme ambiguity-aversion investors will adopt the extreme conservative investment strategy, and the decrease in $\alpha$ means that investors are less ambiguity averse and less conservative.

\(^5\)The mean-variance portfolio can be found by solving the following optimization problem

$$\min_{x \in \mathbb{R}^n} \{x^T \Sigma x \mid \hat{\mu}^T x + (1 - e^T x) r_f \geq d\}$$

where $d$ is a target mean return.
Figure 1. The left panel shows that $k(\alpha)$ and $b(\alpha)$ are decreasing in $\alpha$. The efficient frontier lines for $\alpha$-robust CVaR model are shown in the right panel. The steepest line (Dash-dot line, black) and flattest line (Solid line, blue) correspond to the cases $\alpha = 0.5$ and $\alpha = 1$, respectively. ($H = 0.4722$, $r_f = 1.01$, $\gamma_1 = 0.0001$, $\gamma_2 = 1.2$, $\beta = 0.95$)

3. Optimal portfolios of $\alpha$-maxmin mean-CVaR optimization problem.
In this section, as in Liu et al. [18], we consider a financial market consisting of $n$ risky assets whose random return rates are denoted by $\xi \in \mathbb{R}^n$. The corresponding loss function associated with decision variable $x \in \mathbb{R}^n$ of allocations to $n$ risky assets is given by\footnote{Similarly, we can also consider a financial market composed by a risk-less asset and risky assets in this Section. To conserve our space, we only consider the situation without a risk-less asset.}

$$l(x, \xi) = -\xi^T x. \quad (18)$$

Define mean-CVaR objective as follows

$$\rho_P(x) = (1 - \lambda)\mathbb{E}_P(l(x, \xi)) + \lambda \text{CVaR}_{\beta}(x, P)$$

where $\lambda \in [0, 1]$ is the level of risk aversion which represents a trade off between risk (CVaR metric) and return.

Similar to the previous section, we propose a less conservative variant of the portfolio selection problem, called $\alpha$-maxmin mean-CVaR criterion, where a convex combination of the worst-case and the best-case mean-CVaR objective function given by

$$\min_{x \in \mathcal{X}} \{(1 - \alpha) \inf_{P \in \mathcal{D}} \rho_P(x) + \alpha \sup_{P \in \mathcal{D}} \rho_P(x)\}, \quad (19)$$

is minimized. Here, $\alpha \in [0, 1]$ determines a trade-off between the worst case and the best case optimization problems and $\mathcal{X} \subseteq \mathbb{R}^n$ denotes the admissible set of portfolios.
Proposition 3. The worst-case and the best-case mean-CVaR functionals under the ambiguity set \( \mathcal{F}(\bar{\mu}, \bar{\Sigma}) \) are equivalent, respectively, to

\[
\sup_{P \in \mathcal{F}(\bar{\mu}, \bar{\Sigma})} \rho_P(x) = -\bar{\mu}^T x + \lambda \kappa \sqrt{x^T \Sigma x}, \tag{20}
\]

\[
\inf_{P \in \mathcal{F}(\bar{\mu}, \bar{\Sigma})} \rho_P(x) = -\bar{\mu}^T x, \tag{21}
\]

where \( \kappa = \sqrt{1 - \beta}. \)

Proof. From the definition of \( \mathcal{F}(\bar{\mu}, \bar{\Sigma}) \) and Lemma 2.2, we obtain

\[
\sup_{P \in \mathcal{F}(\bar{\mu}, \bar{\Sigma})} \rho_P(x) = \sup_{P \in \mathcal{F}(\bar{\mu}, \bar{\Sigma})} \left\{ (1 - \lambda)E_P(l(x, \xi)) + \lambda \text{CVaR}_\beta(x, P) \right\}
\]

\[
= (1 - \lambda)(-\bar{\mu}^T x + \lambda \sup_{P \in \mathcal{F}(\bar{\mu}, \bar{\Sigma})} \text{CVaR}_\beta(x, P))
\]

\[
= (1 - \lambda)(-\bar{\mu}^T x) + \lambda (-\bar{\mu}^T x + \kappa \sqrt{x^T \Sigma x})
\]

\[
= -\bar{\mu}^T x + \lambda \kappa \sqrt{x^T \Sigma x}.
\]

From Corollary 3.1 in Zhu and Shao (2018), it follows that

\[
\inf_{P \in \mathcal{F}(\bar{\mu}, \bar{\Sigma})} \rho_P(x) = \inf_{P \in \mathcal{F}(\bar{\mu}, \bar{\Sigma})} \left\{ (1 - \lambda)E_P(l(x, \xi)) + \lambda \text{CVaR}_\beta(x, P) \right\}
\]

\[
= (1 - \lambda)(-\bar{\mu}^T x) + \lambda (-\bar{\mu}^T x)
\]

\[
= -\bar{\mu}^T x.
\]

We now have the desired conclusion. \( \square \)

Proposition 4. Suppose that the unknown distribution of the random variable \( \xi \in \mathbb{R}^n \) is within a family of distributions \( \mathcal{D} \). Then, the \( \alpha \)-maxmin mean-CVaR model (19) is described by the following program

\[
\min_{x \in X} -\bar{\mu}^T x + \left[ (2\alpha - 1) \sqrt{\gamma_1} + \lambda \alpha \kappa \sqrt{\gamma_2} \right] \sqrt{x^T \Sigma x}. \tag{22}
\]

Proof. From the definition of \( \mathcal{F}(\bar{\mu}, \bar{\Sigma}) \) and \( \mathcal{U}(\bar{\mu}, \bar{\Sigma}) \) and Proposition 3, we have

\[
\sup_{P \in \mathcal{D}} \rho_P(x) = \sup_{(\bar{\mu}, \bar{\Sigma}) \in \mathcal{U}(\bar{\mu}, \bar{\Sigma})} \sup_{P \in \mathcal{F}(\bar{\mu}, \bar{\Sigma})} \rho_P(x)
\]

\[
= \sup_{(\bar{\mu}, \bar{\Sigma}) \in \mathcal{U}(\bar{\mu}, \bar{\Sigma})} -\bar{\mu}^T x + \lambda \kappa \sqrt{x^T \Sigma x}
\]

\[
= -\inf_{\bar{\mu} \in \mathcal{U}_\mu} \bar{\mu}^T x + \lambda \kappa \sup_{\Sigma \in \mathcal{U}_\Sigma} \sqrt{x^T \Sigma x}
\]

\[
= -\bar{\mu}^T x + \sqrt{\gamma_1} \sqrt{x^T \Sigma x} + \lambda \kappa \sqrt{\gamma_2} \sqrt{x^T \Sigma x}, \tag{23}
\]

where \( \mathcal{U}_\Sigma = \{ \Sigma \in S_+^n \mid \Sigma \preceq \gamma_2 \bar{\Sigma} \}. \)

Similarly, from Lemma 2.3, we have

\[
\inf_{P \in \mathcal{D}} \rho_P(x) = \inf_{(\bar{\mu}, \bar{\Sigma}) \in \mathcal{U}(\bar{\mu}, \bar{\Sigma})} \inf_{P \in \mathcal{F}(\bar{\mu}, \bar{\Sigma})} \rho_P(x)
\]

\[
= -\sup_{\bar{\mu} \in \mathcal{U}_\mu} \bar{\mu}^T x
\]

\[
= -\bar{\mu}^T x - \sqrt{\gamma_1} \sqrt{x^T \Sigma x}. \tag{24}
\]
Substituting (23) and (24) into (19) yields the desired result.

In the following, we set \( X = \{ x \in \mathbb{R}^n | e^T x = 1 \} \) unless otherwise stated, and define the following quantities:

\[
A = e^T \hat{\Sigma}^{-1} e, \quad B = \hat{\mu}^T \hat{\Sigma}^{-1} e, \quad C = \hat{\mu}^T \hat{\Sigma}^{-1} \hat{\mu}, \quad \Delta = AC - B^2,
\]

\[
\Omega = (2\alpha - 1) \sqrt{\gamma_1} + \lambda \kappa \sqrt{\gamma_2}.
\]

By Cauchy-Schwarz inequality, we can check that \( \Delta > 0 \).

**Remark 4.** The problem (22) is convex for \( \Omega \geq 0 \), i.e.,

\[
\frac{1}{2} - \frac{1}{2} \frac{\lambda \kappa \sqrt{\gamma_2}}{\sqrt{\gamma_1} + \lambda \kappa \sqrt{\gamma_2}} \leq \alpha \leq 1.
\]

If \( \Omega < 0 \), problem (22) is unbounded. Therefore, the assumption \( \Omega \geq 0 \) is realistic.

Solving this optimization problem leads to efficient frontiers that vary between the worst case and best case efficient frontiers depending on the value of \( \alpha \).

The following theorem gives the main results of this section.

**Theorem 3.1.** If \( \Omega > \sqrt{\frac{2A}{A}} \), then the optimal portfolio of the \( \alpha \)-maxmin mean-CVaR model (19) is given by

\[
x^* = \frac{1}{\sqrt{A\Omega^2 - \Delta}} \hat{\Sigma}^{-1}(\hat{\mu} - \frac{B}{A} e) + \frac{\hat{\Sigma}^{-1} e}{A}.
\]

If \( \Omega \leq \sqrt{\frac{2A}{A}} \), then problem (19) is unbounded.

**Proof.** Since \( X = \{ x \in \mathbb{R}^n | e^T x = 1 \} \), the Lagrangian function of the optimization problem given by (22) is

\[
L(x, \nu) = -\hat{\mu}^T x + \Omega \sqrt{x^T \hat{\Sigma} x} - \nu(1 - e^T x),
\]

where \( \nu \) is the Lagrangian multiplier for the budget constraint. Clearly, the first order conditions with respect to \( x \) is given by

\[-\hat{\mu} + \Omega \frac{\hat{\Sigma} x}{\sqrt{x^T \hat{\Sigma} x}} + \nu e = 0.
\]

Let \( \sigma = \sqrt{x^T \hat{\Sigma} x} \). Then

\[
x^* = \frac{\sigma}{\Omega} \hat{\Sigma}^{-1}(\hat{\mu} - \nu e).
\]

Using the definition of \( \sigma \), we obtain the following quadratic equation:

\[
A\nu^2 - 2B\nu + C - \Omega^2 = 0
\]

with the roots \( \nu = \frac{B \pm \sqrt{A^2 - A}}{A} \). Clearly, \( \Omega > \sqrt{\frac{2A}{A}} \).

From the budget constraint \( e^T x = 1 \), we have \( \sigma = \frac{\Omega}{B - \nu A} \). Since we need \( \sigma > 0 \), we require \( \nu < \frac{B}{A} \). Thus, we have \( \nu^* = \frac{B - \sqrt{A^2 - A}}{A} \) and we obtain \( \sigma = \frac{\Omega}{\sqrt{A^2 - A}} \).

Substituting \( \nu \) into the candidate solution \( x \) given by (26) yields

\[
x^* = \frac{1}{\sqrt{A\Omega^2 - \Delta}} \hat{\Sigma}^{-1}(\hat{\mu} - \frac{B - \sqrt{A\Omega^2 - \Delta}}{A} e).
\]

If \( \Omega \leq \sqrt{\frac{2A}{A}} \), then we have an infeasible Lagrange dual problem. Since the primal problem (22) is trivially feasible, it implies that the problem (22) is unbounded. \( \square \)
Markowitz suggest to determine an optimal tradeoff between the expected return and the risk of the portfolio. The optimal mean-variance (MV) portfolio can thus be found by solving the following deterministic convex quadratic program (mean-variance model)

$$\min_{x \in X} -\bar{\mu}^T x + \tau x^T \hat{\Sigma} x,$$

(27)

where $\tau$ is the risk-aversion coefficient and $X$ is the budget constraint. It is easy to find that the MV portfolio is

$$x^*_\text{mv} = \frac{1}{2\tau} \hat{\Sigma}^{-1}(\hat{\mu} - \frac{B}{A} e) + \frac{\hat{\Sigma}^{-1} e}{A}.$$  

Notice that the model (22) is quite similar to mean-variance model (27). Both models take into account the trade-off between portfolio return and portfolio risk. The main difference between two models is that portfolio variance in (27) is replaced by its standard deviation and the risk averse parameter $\tau$ is replaced by $\Omega$. Moreover, by comparing with the mean-variance portfolio, we can observe that the portfolio (25) is equivalent to a mean-variance portfolio $x^*_\text{mv}$ with $\tau$ being $\sqrt{\frac{A\Omega - \Delta}{A}}$. Therefore, the optimal portfolio of model (19) is mean-variance efficient.

We close this section by considering some special cases of Theorem 3.1.

**Corollary 2.** (1) For an extremely AA investor ($\alpha = 1$), if $\sqrt{\gamma_1} + \lambda\kappa\sqrt{\gamma_2} > \sqrt{\frac{\Delta}{A}}$, then the optimal investment strategy $x^*$ is given by

$$x^*_{\text{AA}} = \frac{1}{\sqrt{A(\sqrt{\gamma_1} + \lambda\kappa\sqrt{\gamma_2})^2 - \Delta}} \hat{\Sigma}^{-1}(\hat{\mu} - \frac{B}{A} e) + \frac{\hat{\Sigma}^{-1} e}{A}.$$  

(2) For an AN investor ($\alpha = \frac{1}{2}$), if $\lambda\kappa\sqrt{\gamma_2} > 2\sqrt{\frac{\Delta}{A}}$, then the optimal investment strategy $x^*$ is given by

$$x^*_{\text{AN}} = \frac{1}{\sqrt{A\lambda^2\kappa^2\gamma_2 - 4\Delta}} \hat{\Sigma}^{-1}(\hat{\mu} - \frac{B}{A} e) + \frac{\hat{\Sigma}^{-1} e}{A}.$$  

It corresponds to equal contribution from the worst and the best case optimisation problems.

**Remark 5.** The $\alpha$-maxmin framework (19) with $\alpha = 1$ is related to the robust CVaR optimization problem considered in Liu et al. [18]. More specifically, case (1) in Corollary 2 is identical to Theorem 2 of Liu et al. [18].

In the following, we use an example with three risky assets considered in Liu et al. [18] to compare the $\alpha$-maxmin mean-CVaR models. We set the two levels of ambiguity $\gamma_1 = 0.0001$ and $\gamma_2 = 1.2$ and the confidence level (used in calculation of CVaR) $\beta = 0.95$. The expected return and covariance matrix are given by

$$\hat{\mu} = \begin{pmatrix} 1.162 \\ 1.246 \\ 1.228 \end{pmatrix} \text{ and } \hat{\Sigma} = \begin{pmatrix} 0.0146 & 0.0187 & 0.0145 \\ 0.0187 & 0.0854 & 0.0104 \\ 0.0145 & 0.0104 & 0.0289 \end{pmatrix}.$$  

We check the effect of the ambiguity aversion parameter $\alpha$ in the $\alpha$-maxmin mean-CVaR model with unknown moments. For a fixed value of $\alpha$ ($\alpha = 1, 0.9, 0.8, 0.5$), we let the risk aversion parameter $\lambda$ vary in the range $(0, 1)$, and get two sequences of $\alpha$-maxmin portfolio return means and $\alpha$-maxmin portfolio CVaRs. We generate a range of efficient frontiers with different level of conservatism by changing the parameter $\alpha$ in $\alpha$-maxmin mean-CVaR optimization problem, and plot the curve of $\alpha$-maxmin portfolio return mean with respect to $\alpha$-maxmin portfolio CVaR in Fig.2. Furthermore, we set $\lambda = 0.5$ and fix all the other parameters at the same
values as above. Let $\alpha$ vary from 0.5 to 1, we plot the corresponding curves of the $\alpha$-maxmin portfolio return and $\alpha$-maxmin portfolio CVaR with respect to $\alpha$ in Fig. 3. We notice that when $\alpha$ is larger, $\alpha$-maxmin portfolio return is smaller and the $\alpha$-maxmin portfolio CVaR is larger, i.e., the investor is more ambiguity averse. The $\alpha$-maxmin mean-CVaR approach can help investors to construct portfolios with a balance between robustness and conservatism. In fact, using $\alpha$-maxmin mean-CVaR strategy with values of $\alpha$ closer to 1, we can incorporate estimation error into our models while keeping the resulting portfolios less conservative.

**Figure 2.** Efficient frontiers of the $\alpha$-maxmin mean-CVaR model with different parameter $\alpha$. The $\alpha$-maxmin portfolio CVaR in the x-axis ($\alpha$-maxmin portfolio return in the y-axis) is a convex mixture between the worst-case and best-case values of CVaR risk measures (expected return).

**Figure 3.** Effects of $\alpha$ (the level of ambiguity aversion) on the $\alpha$-maxmin portfolio return and $\alpha$-maxmin portfolio CVaR.
In Fig. 4, we examine the impact of ambiguity aversion parameter $\alpha$ on the efficient portfolios. We show the portfolio composition (weights) for each one of the assets we propose, using our $\alpha$-maxmin approach. Additionally, with the increase of $\alpha$ (vary from 0.5 to 1), the weight of asset 1 with the lowest return increases, while the proportion of assets 2 and 3 are always decreasing. Furthermore, to illustrate the variation of optimal portfolio strategies at different levels of ambiguity $\gamma_1$ and $\gamma_2$, we fix one of them and let the other change in a certain range. More specifically, we let $\gamma_1$ (for a given $\gamma_2 = 1.2$) and $\gamma_2$ (for a given $\gamma_1 = 0.0001$) varies in the intervals $(0.00005, 0.00015)$ and $(0.9, 1.5)$, respectively, and simultaneously set $\alpha = 0.8$. The portfolio compositions for different levels of $\gamma_1$ and $\gamma_2$ are depicted in Fig. 5a and Fig. 5b. Every vertical cut in the picture represents the portfolio composition for a given level of $\gamma_1$ or $\gamma_2$. Although it is known that errors in the mean estimates impact portfolio weights more significantly than do the errors in the covariance matrix (Best and Grauer [3]), from Fig. 5, we can see that our investment decisions are less sensitive to $\gamma_1$ than $\gamma_2$.

4. Conclusions. In this paper, the closed-form solutions of the $\alpha$-robust CVaR optimization problems have been presented under the condition of distribution ambiguity. Moment-based ambiguity set is used to model the uncertainty and short-selling is allowed. The $\alpha$-robust CVaR optimization models, with allowance to cater for investors with different level of ambiguity aversion, can help investors to construct portfolios with highly valuable quality of robustness with less conservativeness.

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Figure 5. The variations of optimal portfolio strategies under different levels of ambiguity $\gamma_1$ (for a given $\gamma_2 = 1.2$) and $\gamma_2$ (for a given $\gamma_1 = 0.0001$). The percentage allocation of assets 1-3 in the optimal allocation $x^*$ have been illustrated in different colors.

REFERENCES

[1] G. Bayraksan and D. K. Love, Data-driven stochastic programming using phi-divergences, in The Operations Research Revolution, INFORMS, 2015, 1–19.

[2] A. Ben-Tal and A. Nemirovski, Robust solutions of uncertain linear programs, Ops. Research Letters, 25 (1999), 1–13.

[3] M. J. Best and R. R. Grauer, Sensitivity analysis for mean-variance portfolio problems, Mgmt. Science, 37 (1991), 980–989.

[4] P. Bossaerts, P. Ghirardato, S. Guarnaschelli and W. R. Zame, Ambiguity in asset markets: Theory and experiment The Review of Finan. Studies, 23 (2010), 1325–1359.

[5] J. Cheng, R. Chen, H. Najm, A. Pinar, C. Safta and J. P. Watso, Distributionally robust optimization with principal component analysis, SIAM J. on Optimization, 28 (2018), 1817–1841.

[6] E. Delage and Y. Ye, Distributionally robust optimization under moment uncertainty with application to data-driven problems, Ops. Research, 58 (2010), 595–612.

[7] D. Ellsberg, Risk, ambiguity, and the savage axioms, The Quarterly Journal of Economics, 75 (1961), 643–669.

[8] P. M. Esfahani and D. Kuhn, Data-driven distributionally robust optimization using the Wasserstein metric: Performance guarantees and tractable reformulations, Math. Programming, 171 (2018), 115–166.

[9] C. R. Fox and A. Tversky, Ambiguity aversion and comparative ignorance, The Quarterly Journal of Economics, 110 (1995), 585–603.

[10] P. Ghirardato, F. Maccheroni and M. Marinacci, Differentiating ambiguity and ambiguity attitude, J. of Econ. Theory, 118 (2004), 133–173.

[11] I. Gilboa and D. Schmeidler, Maxmin expected utility with non-unique prior, J. of Math. Econ., 18 (1989), 141–153.

[12] C. Heath and A. Tversky, Preference and belief: Ambiguity and competence in choice under uncertainty, J. of Risk and Uncertainty, 4 (1991), 5–28.

[13] R. Jiang and Y. Guan, Data-driven chance constrained stochastic program, Math. Programming, 158 (2016), 291–327.

[14] Z. Kang, X. Li, Z. Li, and S. Zhu, Data-driven robust mean-CVaR portfolio selection under distribution ambiguity, Quant. Finan., 19 (2019), 105–121.
[15] Z. Kang and Z. Li, An exact solution to a robust portfolio choice problem with multiple risk measures under ambiguous distribution, *Math. Methods of Ops. Research*, 87 (2018), 169–195.

[16] B. Li, D. Li and D. Xiong, Alpha-robust mean-variance reinsurance-investment strategy, *J. of Econ. Dynamics and Control*, 70 (2016), 101–123.

[17] B. Li, L. Wang and D. Xiong, Robust utility maximization with extremely ambiguity-loving and ambiguity-aversion preferences, *Stochastics*, 90 (2018), 524–538.

[18] J. Liu, Z. Chen, A. Lisser and Z. Xu, Closed-Form optimal portfolios of distributionally robust mean-CVaR problems with unknown mean and variance, *Appl. Math. & Optimization*, 79 (2019), 671–693.

[19] S. Lotfi, M. Salahi and F. Mehrdoust, Adjusted robust mean-value-at-risk model: Less conservative robust portfolios, *Optimization and Engineering*, 18 (2017), 467–497.

[20] S. Lotfi and S. A. Zenios, Robust VaR and CVaR optimization under joint ambiguity in distributions, means, and covariances, *European J. of Oper. Research*, 269 (2018), 556–576.

[21] A. B. Paç and M. Ç. Pinar, Robust portfolio choice with CVaR and VaR under distribution and mean return ambiguity, *TOP*, 22 (2014), 875–891.

[22] I. Popescu, Robust mean-covariance solutions for stochastic optimization, *Ops. Research*, 55 (2007), 98–112.

[23] A. G. Quaranta and A. Zaffaroni, Robust optimization of Conditional Value-at-Risk and portfolio selection, *J. of Banking & Finance*, 32 (2008), 2046–2056.

[24] K. Ruan and M. Fukushima, Robust portfolio selection with a combined WCVar and factor model, *J. of Indust. & Mgmt. Optimization*, 8 (2012), 343–362.

[25] W. Wiesemann, D. Kuhn and M. Sim, Distributionally robust convex optimization, *Ops. Research*, 62 (2014), 1358–1376.

[26] S. Zhu and M. Fukushima, Worst-case conditional value-at-risk with application to robust portfolio management, *Ops. Research*, 57 (2009), 1155–1168.

[27] W. Zhu and H. Shao, Closed-form solutions for extremely-case distortion risk measures and applications to robust portfolio management, 2018. Available from: https://ssrn.com/abstract=3103458.

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