ON THE AUTOMORPHISM GROUP OF A CLOSED $G_2$-STRUCTURE

FABIO PODESTÀ AND ALBERTO RAFFERO

Abstract. We study the automorphism group of a compact 7-manifold $M$ endowed with a closed non-parallel $G_2$-structure, showing that its identity component is abelian with dimension bounded by $\min\{6, b_2(M)\}$. This implies the non-existence of compact homogeneous manifolds endowed with an invariant closed non-parallel $G_2$-structure. We also discuss some relevant examples.

1. Introduction

A seven-dimensional smooth manifold $M$ admits a $G_2$-structure if the structure group of its frame bundle can be reduced to the exceptional Lie group $G_2 \subset \text{SO}(7)$. Such a reduction is characterized by the existence of a global 3-form $\varphi \in \Omega^3(M)$ satisfying a suitable non-degeneracy condition and giving rise to a Riemannian metric $g_\varphi$ and to a volume form $dV_\varphi$ on $M$ via the identity

$$g_\varphi(X, Y) dV_\varphi = \frac{1}{6} \iota_X \varphi \wedge \iota_Y \varphi \wedge \varphi,$$

for all $X, Y \in \mathfrak{x}(M)$ (see e.g. [1, 11]).

By [9], the intrinsic torsion of a $G_2$-structure $\varphi$ can be identified with the covariant derivative $\nabla^g_\varphi \varphi$, and it vanishes identically if and only if both $d\varphi = 0$ and $d^* \varphi = 0$, $* \varphi$ being the Hodge operator defined by $g_\varphi$ and $dV_\varphi$. On a compact manifold, this last fact is equivalent to $\Delta_\varphi \varphi = 0$, where $\Delta_\varphi = d^2 d + dd^*$ is the Hodge Laplacian of $g_\varphi$. A $G_2$-structure $\varphi$ satisfying any of these conditions is said to be parallel and its associated Riemannian metric $g_\varphi$ has holonomy contained in $G_2$. Consequently, $g_\varphi$ is Ricci-flat and the automorphism group $\text{Aut}(M, \varphi) := \{f \in \text{Diff}(M) | f^* \varphi = \varphi\}$ of $(M, \varphi)$ is finite when $M$ is compact and $\text{Hol}(g_\varphi) = G_2$.

Parallel $G_2$-structures play a central role in the construction of compact manifolds with holonomy $G_2$, and known methods to achieve this result involve closed $G_2$-structures, i.e., those whose defining 3-form $\varphi$ satisfies $d\varphi = 0$ (see [11 2 5 12 14 17]).

Most of the known examples of 7-manifolds admitting closed $G_2$-structures consist of simply connected Lie groups endowed with a left-invariant closed $G_2$-form $\varphi$ [4 7 8 10 15]. Compact locally homogeneous examples can be obtained considering the quotient of such groups by a co-compact discrete subgroup, whenever this exists. Further non-homogeneous closed $G_2$-structures on the 7-torus can be constructed starting from the symplectic half-flat $SU(3)$-structure on $T^6$ described in [6 Ex. 5.1] (see Example 2.4 for details).

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Up to now, the existence of compact homogeneous 7-manifolds admitting an invariant closed non-parallel G\(_2\)-structure is still not known (cf. \[15\] Question 3.1 and \[16, 21\]). Moreover, among the G\(_2\)-manifolds acted on by a cohomogeneity one simple group of automorphisms studied in \[3\] no compact examples admitting a closed G\(_2\)-structure occur.

In this short note, we investigate the properties of the automorphism group Aut(M, \(\varphi\)) of a compact 7-manifold M endowed with a closed non-parallel G\(_2\)-structure \(\varphi\). Our main results are contained in Theorem 2.1, where we show that the identity component Aut(M, \(\varphi\))^0 is necessarily abelian with dimension bounded by \(\min\{6, b_2(M)\}\). In particular, this answers negatively \[15\] Question 3.1 and explains why compact examples cannot occur in \[3\]. Moreover, we also prove some interesting properties of the automorphism group action. Finally, we describe some relevant examples.

Similar results hold for compact symplectic half-flat 6-manifolds, and they will appear in a forthcoming paper.

2. The automorphism group

Let M be a seven-dimensional manifold endowed with a closed G\(_2\)-structure \(\varphi\), and consider its automorphism group

\[
\text{Aut}(M, \varphi) := \{ f \in \text{Diff}(M) \mid f^* \varphi = f \}.
\]

Notice that Aut(M, \(\varphi\)) is a closed Lie subgroup of Iso(M, \(g_\varphi\)), and that the Lie algebra of its identity component \(G := \text{Aut}(M, \varphi)^0\) is

\[
\mathfrak{g} = \{ X \in \mathfrak{X}(M) \mid \mathcal{L}_X \varphi = 0 \}.
\]

In particular, every \(X \in \mathfrak{g}\) is a Killing vector field for the metric \(g_\varphi\) (cf. \[17\] Lemma 9.3).

When M is compact, the Lie group Aut(M, \(\varphi\)) \(\subset\) Iso(M, \(g_\varphi\)) is also compact, and we can show the following.

**Theorem 2.1.** Let M be a compact seven-dimensional manifold endowed with a closed non-parallel G\(_2\)-structure \(\varphi\). Then, there exists an injective map

\[
F : \mathfrak{g} \to \mathcal{H}^2(M), \quad X \mapsto \iota_X \varphi,
\]

where \(\mathcal{H}^2(M)\) is the space of \(\Delta_\varphi\)-harmonic 2-forms. As a consequence, the following properties hold:

1) \(\dim(\mathfrak{g}) \leq b_2(M)\);
2) \(\mathfrak{g}\) is abelian with \(\dim(\mathfrak{g}) \leq 6\);
3) for every \(p \in M\), the isotropy subalgebra \(\mathfrak{g}_p\) has dimension \(\dim(\mathfrak{g}_p) \leq 2\), with equality only when \(\dim(\mathfrak{g}) = 2, 3\);
4) the G-action is free when \(\dim(\mathfrak{g}) \geq 5\).

**Proof.** Let \(X \in \mathfrak{g}\). Then, \(0 = \mathcal{L}_X \varphi = d(\iota_X \varphi), \) as \(\varphi\) is closed. We claim that \(\iota_X \varphi\) is co-closed (see also \[17\] Lemma 9.3]. Indeed, by \[13\] Prop. A.3] we have

\[
\iota_X \varphi \wedge \varphi = -2 \ast_\varphi (\iota_X \varphi),
\]

from which it follows that

\[
0 = d(\iota_X \varphi \wedge \varphi) = -2d \ast_\varphi (\iota_X \varphi).
\]
Consequently, the 2-form $\iota_X \varphi$ is $\Delta \varphi$-harmonic and $F$ is the restriction of the injective map $Z \mapsto \iota_Z \varphi$ to $\mathfrak{g}$. From this follows.

As for 2), we begin observing that $\mathcal{L}_Y (\iota_X \varphi) = 0$ for all $X, Y \in \mathfrak{g}$, since every Killing field on a compact manifold preserves every harmonic form. Hence, we have

$$0 = \mathcal{L}_Y (\iota_X \varphi) = \iota_{[Y,X]} \varphi + \iota_X (\mathcal{L}_Y \varphi) = \iota_{[Y,X]} \varphi.$$  

This proves that $\mathfrak{g}$ is abelian, the map $Z \mapsto \iota_Z \varphi$ being injective. Now, $G$ is compact abelian and it acts effectively on the compact manifold $M$. Therefore, the principal isotropy is trivial and $\dim(\mathfrak{g}) \leq 7$. When $\dim(\mathfrak{g}) = 7$, $M$ can be identified with the 7-torus $\mathbb{T}^7$ endowed with a left-invariant metric, which is automatically flat. Hence, if $\varphi$ is closed non-parallel, then $\dim(\mathfrak{g}) \leq 6$.

In order to prove 3), we fix a point $p$ of $M$ and we observe that the image of the isotropy representation $\rho : G_p \to \text{O}(7)$ is conjugated into $G_2$. Since $G_2$ has rank two and $G_p$ is abelian, the dimension of $\mathfrak{g}_p$ is at most two. If $\dim(\mathfrak{g}_p) = 2$, then the image of $\rho$ is conjugate to a maximal torus of $G_2$ and its fixed point set in $T_p M$ is one-dimensional. As $T_p (G \cdot p) \subseteq (T_p M)^{G_p}$, the dimension of the orbit $G \cdot p$ is at most one, which implies that $\dim(\mathfrak{g})$ is either two or three.

The last assertion is equivalent to proving that $G_p$ is trivial for every $p \in M$ whenever $\dim(\mathfrak{g}) \geq 5$. In this case, $\dim(\mathfrak{g}_p) \leq 1$ by 3). Assuming that the dimension is precisely one, then the dimension of the orbit $G \cdot p$ is at least four. This means that the $G_p$-fixed point set in $T_p M$ is at least four-dimensional. On the other hand, the fixed point set of a closed one-parameter subgroup of $G_2$ is at most three-dimensional. This gives a contradiction. □

The following corollary answers negatively a question posed by Lauret in [15].

**Corollary 2.2.** There are no compact homogeneous 7-manifolds endowed with an invariant closed non-parallel $G_2$-structure.

**Proof.** The assertion follows immediately from point 2) of Theorem 2.1 □

In contrast to the last result, it is possible to exhibit non-compact homogeneous examples. Consider for instance a six-dimensional non-compact homogeneous space $H/K$ endowed with an invariant symplectic half-flat SU(3)-structure, namely an SU(3)-structure $(\omega, \psi)$ such that $d\omega = 0$ and $d\psi = 0$ (see [20] for the classification of such spaces when $H$ is semisimple and for more information on symplectic half-flat structures). If $(\omega, \psi)$ is not torsion-free, i.e., if $d(J\psi) \neq 0$, then the non-compact homogeneous space $(H \times S^1)/K$ admits an invariant closed non-parallel $G_2$-structure defined by the 3-form

$$\varphi := \omega \wedge ds + \psi,$$

where $ds$ denotes the global 1-form on $S^1$.

**Remark 2.3.** In [3], the authors investigated $G_2$-manifolds acted on by a cohomogeneity one simple group of automorphisms. Theorem 2.1 explains why compact examples in the case of closed non-parallel $G_2$-structures do not occur.

The next example shows that $G$ can be non-trivial, that the upper bound on its dimension given in 2) can be attained, and that 4) is only a sufficient condition.
Example 2.4. In [3], the authors constructed a symplectic half-flat SU(3)-structure $(\omega, \psi)$ on the 6-torus $\mathbb{T}^6$ as follows. Let $(x^1, \ldots, x^6)$ be the standard coordinates on $\mathbb{R}^6$, and let $a(x^1), b(x^2)$ and $c(x^3)$ be three smooth functions on $\mathbb{R}^6$ such that

$$
\lambda_1 := b(x^2) - c(x^3), \quad \lambda_2 := c(x^3) - a(x^1), \quad \lambda_3 := a(x^1) - b(x^2),
$$

are $\mathbb{Z}^6$-periodic and non-constant. Then, the following pair of $\mathbb{Z}^6$-invariant differential forms on $\mathbb{R}^6$ induces an SU(3)-structure on $\mathbb{T}^6 = \mathbb{R}^6/\mathbb{Z}^6$:

$$
\omega = dx^{14} + dx^{25} + dx^{36},
$$
$$
\psi = -e^{\lambda_3} dx^{126} + e^{\lambda_2} dx^{135} - e^{\lambda_1} dx^{234} + dx^{456},
$$

where $dx^{ijk\ldots}$ is a shorthand for the wedge product $dx^i \wedge dx^j \wedge dx^k \wedge \cdots$. It is immediate to check that both $\omega$ and $\psi$ are closed and that $d(J\psi) \neq 0$ whenever at least one of the functions $a(x^1), b(x^2), c(x^3)$ is not identically zero. Thus, the pair $(\omega, \psi)$ defines a symplectic half-flat SU(3)-structure on the 6-torus. The automorphism group of $(\mathbb{T}^6, \omega, \psi)$ is $\mathbb{T}^3$ when $a(x^1) b(x^2) c(x^3) \neq 0$, while it becomes $\mathbb{T}^4 (\mathbb{T}^5)$ when one (two) of them vanishes identically.

Now, we can consider the closed $G_2$-structure on $\mathbb{T}^7 = \mathbb{T}^6 \times S^1$ defined by the 3-form $\varphi = \omega \wedge ds + \psi$. Depending on the vanishing of none, one or two of the functions $a(x^1), b(x^2), c(x^3)$, $\varphi$ is a closed non-parallel $G_2$-structure and the automorphism group of $(\mathbb{T}^7, \varphi)$ is $\mathbb{T}^4$, $\mathbb{T}^5$ or $\mathbb{T}^6$, respectively.

Finally, we observe that there exist examples where the upper bound on the dimension of $g$ given in [4] is more restrictive than the upper bound given in [2].

Example 2.5. In [4], the authors obtained the classification of seven-dimensional nilpotent Lie algebras admitting closed $G_2$-structures. An inspection of all possible cases shows that the Lie algebras whose second Betti number is lower than seven are those appearing in Table 1.

| nilpotent Lie algebra $\mathfrak{n}$ | $b_2(\mathfrak{n})$ |
|-------------------------------------|-------------------|
| $(0, 0, e^{12}, e^{13}, e^{23}, e^{15} + e^{24}, e^{16} + e^{34})$ | 3 |
| $(0, 0, e^{12}, e^{13}, e^{23}, e^{15} + e^{24}, e^{16} + e^{34} + e^{25})$ | 3 |
| $(0, 0, e^{12}, 0, e^{13} + e^{24}, e^{14}, e^{46} + e^{34} + e^{15} + e^{23})$ | 5 |
| $(0, 0, e^{12}, 0, e^{13}, e^{24} + e^{23}, e^{25} + e^{34} + e^{15} + e^{16} - 3e^{26})$ | 6 |

Table 1.

Let $\mathfrak{n}$ be one of the Lie algebras in Table 1 and consider a closed non-parallel $G_2$-structure $\varphi$ on it. Then, left multiplication extends $\varphi$ to a left-invariant $G_2$-structure of the same type on the simply connected nilpotent Lie group $N$ corresponding to $\mathfrak{n}$. Moreover, as the structure constants of $\mathfrak{n}$ are integers, there exists a co-compact discrete subgroup $\Gamma \subset N$ giving rise to a compact nilmanifold $\Gamma \backslash N$ [18]. The left-invariant 3-form $\varphi$ on $N$ passes to the quotient defining an invariant closed non-parallel $G_2$-structure on $\Gamma \backslash N$. By Nomizu Theorem [19], the de Rham cohomology group $H^2_{dR}(\Gamma \backslash N)$ is isomorphic to the cohomology group $H^2(\mathfrak{n}^*)$ of the Chevalley-Eilenberg complex of $\mathfrak{n}$. Hence, $b_2(\Gamma \backslash N) = b_2(\mathfrak{n})$. 

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Dipartimento di Matematica e Informatica “U. Dini”, Università degli Studi di Firenze, Viale Morgagni 67/a, 50134 Firenze, Italy
E-mail address: podesta@math.unifi.it, alberto.raffero@unifi.it