Abstract—Recent developments in quaternion-valued widely linear processing have illustrated that the exploitation of complete second-order statistics requires consideration of both the covariance and complementary covariance matrices. Such matrices have a tremendous amount of structure, and their decomposition is a powerful tool in a variety of applications, however, this has proven rather difficult, owing to the non-commutative nature of the quaternion product. To this end, we introduce novel techniques for a simultaneous decomposition of the covariance and complementary covariance matrices in the quaternion domain, whereby the quaternion version of the Takagi factorisation is explored to diagonalise symmetric quaternion-valued matrices. This gives new insight into the quaternion uncorrelating transform (QUT) and forms a basis for the proposed quaternion approximate uncorrelating transform (QAUT) which simultaneously diagonalises all four covariance matrices associated with improper quaternion signals. The effectiveness of the proposed uncorrelating transforms is validated by simulations on synthetic and real-world quaternion-valued signals.

Index Terms—Quaternion matrix diagonalisation, complementary covariance, pseudo-covariance, widely linear processing.

I. INTRODUCTION

Advances in sensing technology have enabled recording from 3-D and 4-D data sources, such as measurements from seismometers [1], ultrasonic anemometers [2], and inertial body sensors [3]. It is convenient to express these measurements as vectors in the $\mathbb{R}^3$ and $\mathbb{R}^4$ fields of reals, however, vector algebra is not a division algebra and is inadequate when modelling orientation and rotation [4]. Quaternions have advantages in representing 3-D and 4-D data, owing to their division algebra, a generic extension of the real and complex algebras. Quaternions also account for data, owing to their division algebra, a generic extension of the real and complex algebras. Quaternions also account for meaningful interpretation for real-world applications in the fields of navigation, communication, and image processing [5], [6]. A recent resurgence in research on quaternion signal processing spans the areas of neural networks [7], adaptive filtering [8], independent component analysis (ICA) [9], and Fourier transforms [10].

Diagonalisable matrices play a fundamental role in engineering applications, whereby the related uncorrelating transforms greatly simplify the analysis of complex problems, both in terms of enabling a solution and providing a methodological framework (e.g. performance bounds). Computational advantages associated with matrix diagonalisations include a reduction in the size of the parameter space from $\mathbb{N}^2$ to $\mathbb{N}$, while the statistical advantages include the possibility to match source properties (e.g. orthogonality) in blind source separation. In addition, the joint diagonalisation of matrices in some blind source separation applications has a close link with the canonical decomposition problem for tensors [11], [12]. The real linear algebra is already a mature area, however, covariance matrices in widely linear signal processing [13] in the complex and quaternion division algebras and their structures [14] have recently received plenty of attention.

In the context of complex-valued signal processing, for a complex random vector $z$, both of the covariance, $C = E\{zz^H\}$, and the pseudo-covariance, $P = E\{zz^T\}$, are necessary to capture complete second-order statistical information [15]. This is achieved by a joint diagonalisation of the Hermitian covariance matrix and the symmetric pseudo-covariance matrix, which is performed respectively via the eigenvalue decomposition (EVD) and the Takagi factorisation. While such decorrelations exist in their general forms in mathematics [16], De Lathauwer and De Moor [13] and Eriksson and Koivunen [14] have given them a practical value through an important engineering contribution called the strong uncorrelating transform (SUT), together with its extension, the generalised uncorrelating transform [17]. Our own recent contributions in this case include a computationally efficient approximate uncorrelating transform (AUT) [18], while to preserve the integrity of the original bivariate sources in the decorrelation process, we have proposed the correlation preserving transform [19].

While real-valued and complex-valued matrices are well understood [16], the quaternion-valued matrix algebra is still under development [20], [21]. The recent theory of quaternion matrix derivatives provides a systematic framework for the calculation of derivatives of quaternion matrix functions with respect to quaternion matrix variables [22]. Advances in structural quaternion matrix decompositions include tools to diagonalise quaternion matrices [23], [24], while in the context of quaternion widely linear processing, it was shown that for a quaternion random vector $x$, the diagonalisation of the Hermitian covariance matrix $C_x = E\{xx^H\}$ can be performed straightforwardly using the quaternion EVD, whereas
the diagonalisation of the complementary covariance matrices (\(C_{x^i} = E\{xx^iH\}\), \(C_{xj} = E\{xx^jH\}\), \(C_{xk} = E\{xx^kH\}\)) can be computed via the quaternion singular value decomposition (SVD) [21]. To jointly diagonalise the covariance and complementary covariance matrices, the SUT has been extended from the complex domain to the quaternion domain [25]. It is important to notice that the relationship between the quaternion covariance matrices in the widely linear model is governed by [26]

\[
P_x = \frac{1}{2}(C_{x^i} + C_{x^j} + C_{x^k} - C_x)
\]

which demonstrates that the diagonalisation of symmetric matrices, such as the pseudo-covariance matrix, \(P_x = E\{xx^T\}\), requires solutions to the following issues: 1) there is no closed-form solution[3] to perform a simultaneous diagonalisation of the covariance matrix and the three complementary covariance matrices, 2) given the dimensionality of the problem, a simple approximate diagonalising transform which would apply to both the covariance and pseudo-covariance matrices is desired. In this paper, we therefore set out to propose solutions to these open problems, and support the analysis with illustrative examples.

The rest of this paper is organised as follows. Section II provides an overview of quaternion algebra. Insights into quaternion covariance matrices are provided in Section III. Section IV proposes novel techniques for the diagonalisation of symmetric quaternion matrices and a simultaneous diagonalisation of two quaternion covariance matrices. Section V proposes an approximate simultaneous diagonalisation of the four quaternion covariance matrices. Simulation results are given in Section VII and Section VII concludes the paper. Throughout the paper, we use boldface capital letters to denote matrices, \(\mathbf{A}\), boldface lowercase letters for vectors, \(\mathbf{a}\), and standard letters for scalar quantities, \(a\). Superscripts \((\cdot)^T, (\cdot)^*\) and \((\cdot)^H\) denote the transpose, conjugate, and Hermitian (i.e. transpose and conjugate), respectively, \(I\) the identity matrix, and \(E(\cdot)\) the statistical expectation operator.

II. QUATERNION ALGEBRA

The quaternion domain \(\mathbb{H}\) is a 4-D vector space over the field of reals, spanned by the basis \(\{1, i, j, k\}\). A quaternion vector \(\mathbf{x} \in \mathbb{H}^{L \times 1}\) consists of a real part \(\Re[\cdot]\) and an imaginary part \(\Im[\cdot]\) which comprises \(i-, j-,\) and \(k-\) imaginary components, so that

\[
\mathbf{x} = \Re[\mathbf{x}] + \Im[\mathbf{x}]
\]

\[
= \Re[\mathbf{x}] + \Im_i[\mathbf{x}] i + \Im_j[\mathbf{x}] j + \Im_k[\mathbf{x}] k
\]

\[
= x_a + x_i i + x_j j + x_k k
\]

where \(\Re[\mathbf{x}] = x_a, \Im_i[\mathbf{x}] = x_i, \Im_j[\mathbf{x}] = x_j, \Im_k[\mathbf{x}] = x_k\) are \(L \times 1\) real vectors, and \(i, j, k\) are orthogonal imaginary units with properties

\[
i j = - j i = k, \quad j k = - k j = i, \quad k i = - i k = j
\]

\[i^2 = j^2 = k^2 = i j k = -1\]

A quaternion variable \(x\) is called a pure quaternion if it satisfies \(\Im[x] = 0\). The modulus of a quaternion variable \(x \in \mathbb{H}\) is defined as

\[
|x| = \sqrt{x_a^2 + x_i^2 + x_j^2 + x_k^2}
\]

A quaternion variable \(x\) is called a unit quaternion if it satisfies \(|x| = 1\). The product of two quaternions \(x, y \in \mathbb{H}\) are defined as

\[
x y = \Re[x]\Re[y] - \Im[x] \cdot \Im[y] + \Re[x] \Im[y] + \Re[y] \Im[x] + \Im[x] \times \Im[y]
\]

where the symbol ‘\(\cdot\)’ denotes the scalar product and ‘\(\times\)’ the vector product. Notice that the presence of the vector product in the above expression causes the non-commutativity of the quaternion product, that is, \(xy \neq yx\).

Another important notion in the quaternion domain is the so-called “quaternion involution” [28], which defines a self-inverse mapping analogous to the complex conjugate. The general involution of the quaternion vector \(x\) is defined as \(x^\alpha = -\alpha x_\alpha\), and represents the rotation of the vector part of \(x\) by \(\pi\) about a unit pure quaternion \(\alpha\). The special cases of involutions about the \(i, j\) and \(k\) imaginary axes are given by [28]

\[
x^i = -i x_i = x_a + x_k j - x_j k - x_d i
\]

\[
x^j = -j x_j = x_a - x_k i + x_d j - x_d k
\]

\[
x^k = -k x_k = x_a - x_i k + x_d j + x_d k
\]

This set of involutions, together with the original quaternion, \(x\), forms the most frequently used basis for augmented quaternion statistics [26], [29]. While an involution represents a rotation along a single unit axis, the quaternion conjugate operator \((\cdot)^*\) rotates the quaternion along all three imaginary axes, and is given by

\[
x^* = \Re[x] - \Im[x] = x_a - x_i i - x_j j - x_k k
\]

A useful algebraic operation which conjugates only one imaginary component is given by

\[
x^{i*} = (x^i)^* = (x^i)^* = x_a - x_i i + x_j j + x_k k
\]

\[
x^{j*} = (x^j)^* = (x^j)^* = x_a + x_k i + x_d j - x_d k
\]

\[
x^{k*} = (x^k)^* = (x^k)^* = x_a + x_i k + x_d j - x_d k
\]

Since the only difference between \(x\) and \(x^{\alpha*}\), for \(\alpha \in \{i, j, k\}\), is the sign of the \(\alpha\)-imaginary component, a quaternion vector \(x\) is called \((\cdot)^{\alpha*}\) invariant if \(\alpha x_\alpha = 0\). For example, if \(x\) has a vanishing \(i\)-imaginary part, then \(x^{i*} = x\). For rigour, the structure and general properties of quaternion matrices are given in Appendix A.

III. QUATERNION COVARIANCE MATRICES: STRUCTURAL INSIGHTS

The recent success of complex augmented statistics is largely due to the simplicity and physical meaningfulness of the statistical descriptors in the form of the Hermitian standard covariance \(C_x = E\{xx^H\}\) and the pseudo-covariance \(P_x = E\{xx^T\}\). In particular, the pseudo-covariance enables us to account for the improperness (channel power imbalance or correlation) of complex variables, while the symmetry of \(P_x\) implies that its decomposition can be computed using the
Takagi factorisation as \( P_x = Q \Sigma Q^T \), where \( Q \) is a complex unitary matrix and \( \Sigma \) a real non-negative diagonal matrix \([16]\).

However, the pseudo-covariance of a quaternion random vector \( \mathbf{x} = [x_1, x_2, \ldots, x_L]^T \) does not exhibit symmetry, owing to the non-commutativity of the quaternion product, whereby \( x_m x_n \neq x_n x_m \) \((m \neq n)\), yield an asymmetric matrix

\[
P_x = E\{\mathbf{xx}^T\} = \begin{bmatrix}
E\{x_1x_1\} & E\{x_1x_2\} & \cdots & E\{x_1x_L\} \\
E\{x_2x_1\} & E\{x_2x_2\} & \cdots & E\{x_2x_L\} \\
\vdots & \vdots & \ddots & \vdots \\
E\{x_Lx_1\} & E\{x_Lx_2\} & \cdots & E\{x_Lx_L\}
\end{bmatrix}
\]

(7)

The quaternion involution basis in \([3]\) is at the core of the recently proposed widely linear processing \([29, 30]\), which provides theoretical and practical performance gains over traditional strictly linear processing \([31]\). For example, the widely linear minimum mean square error (MMSE) estimation of the quaternion signal \( y \) in terms of the observation \( \mathbf{x} \) is performed as

\[
\hat{y} = E\{y|x, x^*, x_j^j, x^k\}
\]

which is analogous to the complex widely linear MMSE estimation \([13]\), given by

\[
\hat{y} = E\{y|x, x^*\}
\]

Then, the involution basis for the quaternion widely linear processing provides a useful description of quaternion second-order statistics, represented by the standard covariance matrix and the \( r-, j-, \) and \( k- \) complementary covariance matrices, which are given by \([26]\)

\[
C_{\alpha^*} = E\{\mathbf{xx}^{\alpha H}\} = \begin{bmatrix}
E\{x_1x_1^{\alpha*}\} & E\{x_1x_2^{\alpha*}\} & \cdots & E\{x_1x_L^{\alpha*}\} \\
E\{x_2x_1^{\alpha*}\} & E\{x_2x_2^{\alpha*}\} & \cdots & E\{x_2x_L^{\alpha*}\} \\
\vdots & \vdots & \ddots & \vdots \\
E\{x_Lx_1^{\alpha*}\} & E\{x_Lx_2^{\alpha*}\} & \cdots & E\{x_Lx_L^{\alpha*}\}
\end{bmatrix}
\]

(9)

where \( \alpha \in \{r, j, k\} \). The \( \alpha \)-complementary covariance matrix is \( \alpha \)-Hermitian, that is, \( C_{\alpha} = (C_{\alpha^*})^{\alpha H} \), which stems from the fact that its diagonal entries are \( \cdot^{\alpha*} \) invariant, whereas its off-diagonal entries are governed by the relationship \( C_{\alpha^*}[m, n] = (C_{\alpha^*}[n, m])^{\alpha*} \). The relationship between the pseudo-covariance and the three complementary covariances is given in \([1]\).

**Remark 1.** The knowledge of both the covariance matrix and the three complementary covariance matrices is necessary to ensure the utilisation of complete second-order statistical information in the quaternion domain.

**A. Properness of quaternions**

The notion of non-circularity (improperness) is unique to division algebras. While non-circularity refers to probability distributions which are not rotation-invariant, a proper complex random vector \( z = z_r + z_i \) \((z_r, z_i \in \mathbb{R}^{L \times 1})\) has a vanishing pseudo-covariance \( E\{zz^T\} = 0 \). In other words, its real and imaginary parts are uncorrelated and have equal variance, that is, \( E\{z_r z_r^*\} = 0 \) and \( E\{z_i z_i^*\} = E\{z_r z_r^*\} \).

Similarly, improperness in the quaternion domain is characterised by the degree of correlation and/or power imbalance between imaginary components relative to the real component. The additional degrees of freedom in the quaternion domain allow for two types of properness: \( \mathbb{H} \)-properness and \( \mathbb{C}^\alpha \)-properness \([32]\).

**Definition 1 (\( \mathbb{H} \)-properness).** A quaternion random vector \( \mathbf{x} \) is \( \mathbb{H} \)-proper if it is uncorrelated with its vector involutions, \( \mathbf{x}^r, \mathbf{x}^j, \) and \( \mathbf{x}^k \), so that

\[
C_{\mathbf{x}^r} = E\{\mathbf{xx}^{rH}\} = 0 \quad C_{\mathbf{x}^j} = E\{\mathbf{xx}^{jH}\} = 0 \quad C_{\mathbf{x}^k} = E\{\mathbf{xx}^{kH}\} = 0
\]

(10)

**Definition 2 (\( \mathbb{C}^\alpha \)-properness).** A quaternion random vector \( \mathbf{x} \) is \( \mathbb{C}^\alpha \)-improper with respect to \( \alpha = i, j \) or \( k \) if it is correlated only with the involution \( \mathbf{x}^\alpha \), so that all the complementary covariances except for \( C_{\mathbf{x}^\alpha} \) vanish.

Appendix B shows that a \( \mathbb{C}^\alpha \)-improper quaternion random vector can be generated from two proper complex random vectors.

**IV. DIAGANISATION OF QUATERNION COVARIANCE MATRICES**

Prior to introducing the diagonalisation of the quaternion pseudo-covariance matrix and a simultaneous diagonalisation of two quaternion covariance matrices, the following two observations establish results essential for subsequent analyses.

**Observation 1.** The EVD applied to the Hermitian quaternion covariance matrix, \( C_{\mathbf{x}^*} \), gives \( C_{\mathbf{x}^*} = Q^H \Lambda \mathbf{x} Q \), where \( Q \) is a quaternion unitary matrix and \( \Lambda_{\mathbf{x}} \) a real-valued diagonal matrix for which the eigenvalues are the singular values of \( C_{\mathbf{x}^*} \).\([27]\)

**Observation 2.** The \( \alpha \)-complementary covariance matrix, \( C_{\mathbf{x}^*}, \alpha \in \{i, j, k\} \), can be factorised as \( C_{\mathbf{x}^*} = Q_\alpha^H \Lambda_{\mathbf{x}} Q_\alpha \), where \( Q_\alpha \) is a quaternion unitary matrix and \( \Lambda_{\mathbf{x}} \) is a real-valued non-negative diagonal matrix for which the diagonal entries are the singular values of \( C_{\mathbf{x}^*} \)\([27]\).

Observation 1 provides an algebraic tool to diagonalise the quaternion covariance matrix. Observation 2 enables the diagonalisation of quaternion complementary covariance matrices, and degenerates into the Takagi factorisation of complex symmetric matrices if the quaternion vector \( \mathbf{x} \) has two vanishing imaginary parts \([34]\).

**A. Diagonalisation of symmetric matrices**

The tools for the decomposition of quaternion pseudo-covariance matrices are still in their infancy; this is because 1) the tools for the analysis of the symmetric matrices in \( \mathbb{C} \) cannot be readily generalised to \( \mathbb{H} \), and 2) it is required to simultaneously diagonalise all the three complementary covariance matrices, a task with prohibitively many degrees of freedom. We shall start our analysis with a tool for the factorisation of quaternion symmetric matrices.

**Proposition 1:** A symmetric matrix \( \mathbf{A} \in \mathbb{H} \) which satisfies \((\mathbf{AA}^*)^* = \mathbf{A} \mathbf{A} \) admits the factorisation \( \mathbf{A} = \mathbf{U} \Lambda \mathbf{U}^T \).

\(^3\)It is called \( \mathbb{C}^\alpha \)-improper in some literature, but we tend to think that \( \mathbb{C}^\alpha \)-improper is more intuitive.
Therefore, we arrive at

\[ C = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \]

because both diagonal, then

\[ \text{Since} \]

\[ \text{Proof:} \]

\[ \text{is} \]

\[ \alpha \]

\[ \text{Proposition 2:} \]

\[ \text{Quaternion uncorrelating transform} \]

\[ \text{x} \]

\[ \text{and} \]

\[ \lambda \]

\[ \text{A} \]

\[ \text{is identical to the Takagi factorisation of complex symmetric matrices} \]

\[ \text{A} = \text{U} \text{U}^T. \]

A consequence of Proposition 1 is that the Takagi factorisation of quaternion symmetric matrices is possible if \((\text{A}^*)^* = \text{A}^*\text{A}\) holds. However, in general, the non-commutativity of the quaternion product means that \((\text{A}^*)^* \neq \text{A}^*\text{A}\). Appendix C shows that a \(2 \times 2\) symmetric matrix \(\text{A}\) satisfies the condition \((\text{A}^*)^* = \text{A}^*\text{A}\) if either its diagonal or off-diagonal elements are real-valued.

**B. Simultaneous diagonalisation of two covariance matrices**

We next proceed to introduce a simultaneous diagonalisation of the covariance and complementary covariance matrices. The finding that a non-singular transform is sufficient to diagonalise both \(C_x\) and \(C_{x^o}\) will form a basis for the quaternion uncorrelating transform [25].

**Proposition 2:** There exists a unitary matrix \(\text{U}\) such that \(U^H C_{x^o} U = U^H C_{x^o} U\) are both diagonal only if \(C_{x^o} C_{x^o}^o\) is \(\alpha\)-Hermitian, that is, \(C_{x^o} C_{x^o}^o = (C_{x^o} C_{x^o}^o)^{\alpha H}\).

**Proof:** Since \(U^H C_x U = \text{A}_x\) and \(U^H C_{x^o} U = \text{A}_x\) are both diagonal, then \(C_x = U \text{A}_x U^H\) and \(C_{x^o} = U \text{A}_x U^H\), whereby

\[ C_x C_{x^o} = U \text{A}_x U^H U \text{A}_x U^H = U \text{A}_x U^H U \text{A}_x U^H = C_x C_{x^o} = C_x^o C_x. \]

Because \(C_x\) is Hermitian and \(C_{x^o}\) is \(\alpha\)-Hermitian, we can readily obtain

\[ C_{x^o} C_x^o = C_x^o C_x = (C_x C_{x^o})^{\alpha H}. \]

Therefore, we arrive at \(C_x C_{x^o} = (C_x C_{x^o})^{\alpha H}\).

In general, \(C_x C_{x^o}\) is not \(\alpha\)-Hermitian, so there is no unitary transform that can simultaneously diagonalise the covariance matrices. However, inspired by the complex SUT, we propose a non-unitary transform for the simultaneous diagonalisation of \(C_x\) and \(C_{x^o}\). On the basis of Observations 1 and 2, \(C_x\) and \(C_{x^o}\) can be diagonalised as

\[ C_x = \text{V} \text{A}_x \text{V}^H \]

\[ C_{x^o} = U \text{A}_x U^H \]

where \(U\) is a quaternion unitary matrix. Define a whitening transform \(D = \text{D} \text{A}_x^{-\frac{1}{2}} \text{V}^H\), and denote \(s = \text{D} x\), to obtain the covariance matrix of \(s\) as

\[ C_s = \text{D} C_x \text{D}^H = \text{D} \text{A}_x^{-\frac{1}{2}} \text{V}^H \text{A}_x \text{V}^H \text{D} \text{A}_x^{-\frac{1}{2}} \text{V}^H = I. \]

The Takagi factorisation is a special case of the complex/quaternion SVD when \(A = A^T\).

Corollary 4.5.18 (c) in [16] is the corresponding result in the complex domain.

Based on Observation 2, the \(\alpha\)-complementary covariance matrix of \(s\) can be factorised as \(C_{s^o} = \text{W} \text{A}_x \text{W}^\alpha\), where \(\text{W}\) is a quaternion unitary matrix, and \(\text{A}\) is a real diagonal matrix. Now, the non-singular uncorrelating transform \(Q = \text{W}^H \text{D}\), and its application in the form \(y = \text{Q} x\) yields the simultaneous diagonalisation

\[ C_y = \text{W}^H C_s \text{W} = \text{W}^H \text{I} \text{W} = I \]

\[ C_{y^o} = \text{W}^H C_{s^o} \text{W}^\alpha = \text{W}^H \text{W} \text{A}_x \text{W}^\alpha \text{W}^H \text{W}^\alpha = \text{A}_x. \]

Denoting \(\text{A}_x\) by \(\text{A}_x\), we refer to the transform \(Q\) as the *quaternion uncorrelating transform* (QUT), which is summarised in Proposition 3 and Algorithm [11].

**Proposition 3 (QUT):** For a random quaternion vector \(x\) with finite second-order statistics, there exists a quaternion non-singular matrix \(Q\) for which \(y = \text{Q} x\) performs the diagonalisation of the following covariance matrices:

\[ C_y = I \quad C_{y^o} = \text{A}_x, \quad \text{where} \quad \alpha \in \{i, j, k\}. \]

**Algorithm 1. Quaternion Uncorrelating Transform (QUT).**

1. Compute the EVD of the covariance matrix \(C_x = E\{xx^H\} = \text{V} \Lambda \text{V}^H\).
2. Compute the whitening matrix \(\text{D} = \text{V} \Lambda^{-\frac{1}{2}} \text{V}^H\).
3. Calculate the \(\alpha\)-complementary covariance of the whitened data, \(s = \text{D} x\), as \(C_{s^o} = E\{ss^H\}\).
4. Compute the factorisation of the complementary covariance matrix, \(C_{s^o} = \text{W} \text{A}_x \text{W}^\alpha\), where \(\text{W}\) is a quaternion unitary matrix and \(\text{A}_x\) a real diagonal matrix.
5. The QUT matrix is then \(Q = \text{W}^H \text{D}\).

Recall that for a \(C^\alpha\)-improper vector \(x\), only the \(\alpha\)-complementary covariance exists while the other two complementary covariances vanish. Thus, the QUT of \(C^\alpha\)-improper signals, for example, for the case \(\alpha = \kappa\), satisfies

\[ C_y = I \quad C_{y^o} = \text{A}_i \quad C_{y^j} = \text{A}_j \quad C_{y^k} = \text{A}_\kappa \]

where \(\text{A}_i\) is a diagonal matrix and \(\text{A}_i = 0\). In other words, for a \(C^\alpha\)-improper quaternion vector, the QUT can be regarded as the quaternion version of SUT [13], [14]. For a general improper vector, however, the remaining two complementary covariance matrices are still not diagonalised by the QUT. In fact, so far there is no closed-form solution for general improper vectors. To circumvent the problem, we shall next introduce an approximate way to simultaneously diagonalise all four covariance matrices.

**V. SIMULTANEOUS DIAGONALISATION OF FOUR COVARIANCE MATRICES**

Following the approach in Section [V], we next extend our proposed approximate uncorrelating transform (AUT) [18] from the complex domain to the quaternion domain.

**A. Univariate QAUT**

Consider first the univariate quaternion random vector \(x\). Our aim is to find a unitary transform \(y = \text{A} x\) which jointly diagonalises the covariance matrix, \(C_y = E\{yy^H\} = \text{V} \Lambda \text{V}^H\).
\[ \Phi C_x \Phi^H, \] and the three complementary covariance matrices, \( C_{y^\alpha} = A \{ y y^\alpha \} = \Phi C_x \Phi^H, \alpha \in \{ i, j, k \}. \] Notice that if \( C_x C_x^H = C_{x^r} C_{x^r}^H, \) the matrix product

\[ C_y C_y^H = \Phi C_x C_x^H \Phi^H \]

is equal to

\[ C_y C_y^H = \Phi C_x C_x^H \Phi^H \]

Following the complex AUT [13], for \( N \) samples of \( x, N > L, \) the quadratic forms \( C_x C_x^H \) and \( C_{x^r} C_{x^r}^H, \) are estimated from the data as

\[ C_x C_x^H = \frac{1}{N} \sum_{n=1}^{N} x(t+n_1) x^H(t+n_2) \]

\[ = \frac{1}{N} \sum_{n=1}^{N} \sum_{n_2=1}^{N} x(t+n_1) x^H(t+n_2) \]

\[ C_{x^r} C_{x^r}^H = \frac{1}{N} \sum_{n=1}^{N} x(t+n_1) x^H(t+n_2) \]

\[ \approx \frac{1}{N^2} \sum_{n=1}^{N} \sum_{n_2=1}^{N} x(t+n_1) x^H(t+n_2) \]

Upon applying the approximation

\[ \sum_{n=1}^{N} x(t+n_1) x^H(t+n_2) \approx \sum_{n=1}^{N} x(t+n_1) x^H(t+n_2) \]

we obtain \( C_x C_x^H \approx C_{x^r} C_{x^r}^H, \) and hence \( C_y C_y^H \approx C_{y^r} C_{y^r}^H, \) which implies that if \( C_y \) is diagonal, so too is \( C_{y^r}, \) and vice versa. Furthermore, if

\[ \sum_{n=1}^{N} x(t+n_1) x^H(t+n_2) \in \mathbb{R} \]

then (11) holds for all \( \alpha \in \{ i, j, k \}, \) and thus the simultaneous diagonalisation applies to all of the three complementary covariance matrices, producing the following quaternion approximate uncorrelating transform (QAUT).

**Proposition 4 (QAUT):** If any of \( C_x, C_{x^r}, C_{x^i}, \) and \( C_{x^r} \) is diagonalised by the unitary matrix \( \Phi, \) the similarity of the structures of the covariance matrices facilitates the approximate diagonalisation of the other three covariance matrices by \( \Phi. \) From (1), the pseudo-covariance matrix is thus also approximately diagonalised.

The transformation matrix \( \Phi \) can take the form of \( Q \) in Observation 1 or \( Q_\alpha \) in Observation 2. For example, implementing the transform \( Q \) obtained from the EVD of \( C_x \)

\[ C_x = Q^H \Lambda_x Q \]

\[ C_{x^r} \approx Q^H \Lambda_x Q^r \]

\[ C_{x^i} \approx Q^H \Lambda_x Q^i \]

\[ C_{x^r} \approx Q^H \Lambda_x Q^r \]

**B. Principle of QAUT**

The QAUT is based on the approximation (12). To examine the validity of this approximation for highly improper quaternion signals, assume that the four components of the quaternion variable \( x_t \) are perfectly correlated and consider the Cayley-Dickinson construction [35]

\[ x_t = x_a + x_b i + x_c j + x_d k = c_1 + c_2 j \]

where \( c_1 = x_a + x_b i \) and \( c_2 = x_c + x_d i \) are complex variables defined in the plane spanned by \( \{ 1, i \}. \) This gives

\[ x_t^* (n_1) x_t (n_2) = c_1^* (n_1) c_1 (n_2) + c_2 (n_1) c_2^* (n_2) + \]

\[ c_1^* (n_1) c_2 (n_2) - c_2 (n_1) c_1^* (n_2) \]

\[ j \]

and the approximation (12) holds given the assumption

\[ x_t^* (n_1) x_t (n_2) = c_1^* (n_1) c_1 (n_2) + c_2 (n_1) c_2^* (n_2) \]

\[ + E \{ c_1^* (n_1) c_2 (n_2) - c_2 (n_1) c_1^* (n_2) \} \]

\[ j \]

in which

\[ E \{ c_1^* (n_1) c_1 (n_2) \} = E \{ x_a (n_1) x_a (n_2) + x_b (n_1) x_b (n_2) \}

\[ + E \{ x_a (n_1) x_b (n_2) - x_a (n_2) x_b (n_1) \} j \]

\[ \approx E \{ x_a (n_1) x_a (n_2) + x_b (n_1) x_b (n_2) \}

\[ + E \{ x_b (n_1) x_b (n_2) - x_b (n_2) x_b (n_1) \} j \]

\[ = E \{ x_a (n_1) x_a (n_2) + x_b (n_1) x_b (n_2) \} \in \mathbb{R} \]

(16)

and likewise for \( E \{ c_2 (n_1) c_2^* (n_2) \} \in \mathbb{R}. \) Moreover, if \( c_1 \) and \( c_2 \) are highly correlated, they can be expressed as \( c_2(t) = k c_1(t) + g(t), \) where \( k \) and \( g(t) \) are complex valued, to give

\[ E \{ c_1^* (n_1) c_2 (n_2) - c_2 (n_1) c_1^* (n_2) \} \approx E \{ c_1^* (n_1) [k c_1 (n_2) + g(n_2)] - [k c_1 (n_1) + g(n_1)] c_1^* (n_2) \} \]

\[ = 2 k E \{ c_1^* (n_1) c_1 (n_2) \} + E \{ g(n_2) \} E \{ c_1^* (n_1) \} - E \{ g(n_1) \} E \{ c_1^* (n_2) \} \approx 0 \]

(17)

Therefore,

\[ E \{ x_t^* (n_1) x_t (n_2) \} \in \mathbb{R} \]

(18)

and the approximation (12) holds given the assumption

\[ E \{ x_t^* (n_1) x_t (n_2) \} \approx \sum_{n=1}^{N} x_t^* (n_1) x_t (n_2) \]

From the above analysis, the QAUT holds exactly when the four components of the quaternion variable are perfectly correlated. For general quaternion signals, however, these components are never perfectly correlated. This causes the approximation error which can be analysed through the correlation coefficients

\[ \rho_{ab} = \frac{\text{cov}[x_a, x_b]}{\sigma_{x_a} \sigma_{x_b}}, \quad \rho_{ac} = \frac{\text{cov}[x_a, x_c]}{\sigma_{x_a} \sigma_{x_c}}, \quad \rho_{ad} = \frac{\text{cov}[x_a, x_d]}{\sigma_{x_a} \sigma_{x_d}} \]

\[ \rho_{bc} = \frac{\text{cov}[x_b, x_c]}{\sigma_{x_b} \sigma_{x_c}}, \quad \rho_{bd} = \frac{\text{cov}[x_b, x_d]}{\sigma_{x_b} \sigma_{x_d}}, \quad \rho_{cd} = \frac{\text{cov}[x_c, x_d]}{\sigma_{x_c} \sigma_{x_d}} \]

whose values range from -1 to 1, while the symbol ‘cov’ denotes the covariance and \( \sigma \) the standard deviation. From
\[ \mathcal{J} [x_t^* (n_1) x_l (n_2)] = [x_a (n_1) x_b (n_2) - x_a (n_2) x_b (n_1) + x_c (n_2) x_d (n_1) - x_c (n_1) x_d (n_2)]^t + [x_a (n_1) x_c (n_2) - x_a (n_2) x_c (n_1) + x_b (n_1) x_c (n_2) - x_b (n_2) x_c (n_1)] j + [x_a (n_1) x_d (n_2) - x_a (n_2) x_d (n_1) + x_b (n_1) x_d (n_2) - x_b (n_2) x_d (n_1)] k \]  

(20)

the diagonalisation condition (11), a natural measure of the diagonalisation error for \( C_{\alpha} \) is

\[ x_t^* (n_1) x_l (n_2) - [x_t^* (n_1) x_l (n_2)]^2 = 2 \mathcal{J} [x_t^* (n_1) x_l (n_2)] \gamma \]  

(19)

for distinct \( \alpha, \beta, \gamma \in \{ 1, j, k \} \)

in which \( \mathcal{J} [x_t^* (n_1) x_l (n_2)] \) is given by (20). Similarly to the complex AUT in (13), Equation (20) can be expressed as in (21) where \( \xi_{ab}, \xi_{cd}, \xi_{ac}, \xi_{bd}, \xi_{ad}, \xi_{bc} \) have real values determined by the statistics of \( x_t \).

It is obvious that the diagonalisation error in (21) decreases with an increase in correlation between the components of quaternion data. Owing to the cumulative nature of the error in (12), the approximation error also increases with the data segment length \( L \).

Remark 2. The univariate and multivariate QAUT become exact for quaternion signals with special second-order statistical properties, such as \( \mathbb{H} \)-proper signals, the signals with fully correlated components, and the signals for which the four components are uncorrelated and their covariance matrices have the same eigenvectors. This can be verified by exploring the matrix structure (29).

C. Multivariate QAUT

To extend the univariate QAUT to the multivariate case, consider an \( M \)-variate quaternion data matrix \( X = [x_1, \ldots, x_M]^T \), where the \( m \)-th column random vector \( x_m = [x_{m,1}, \ldots, x_{m,L}]^T (L > M) \) represents \( L \) samples of the \( m \)-th variate, which is assumed to have highly correlated components. The covariance and \( \alpha \)-covariance matrices are then given by \( C_X = E\{XX^H\} \) and \( C_{\alpha} = E\{XX^{\alpha H}\} \), and the aim is to find a unitary transform

\[ Y = \Phi X \]  

(22)

which jointly diagonalises the covariance matrix, \( C_Y = E\{YY^H\} = \Phi C_X \Phi^H \), and the three complementary covariance matrices, \( C_{\alpha} = E\{YY^{\alpha H}\} = \Phi C_{\alpha} \Phi^{\alpha H}, \alpha \in \{1, j, k\} \). Notice that, if \( C_X C_X^H = C_X C_X^{\alpha H} \), then the matrix product

\[ C_Y C_Y^H = \Phi C_X C_X^H \Phi^H \]

is equal to

\[ C_{\alpha} C_{\alpha}^H = \Phi C_{\alpha} C_{\alpha}^H \Phi^H \]

For \( N \) samples of \( X \), the quadratic forms \( C_X C_X^H \) and \( C_{\alpha} C_{\alpha}^H \) can be estimated from the data as

\[ C_X C_X^H = \frac{1}{N} \sum_{t=1}^{N} X(t+n_1) X^H(t+n_1) \]

(23)

\[ C_{\alpha} C_{\alpha}^H = \frac{1}{N} \sum_{t=1}^{N} X(t+n_1) X^{\alpha H}(t+n_1) \]

(24)

Following the analysis in (14)-(17), we can prove that

\[ E\{x_m^*(n_1)x_{m,l}(n_2)\} \in \mathbb{R}, \text{ where } l_1, l_2 \in \{1, \ldots, L\}, \text{ so that } E\{x_m^*(n_1)x_{m,l}^T(n_2)\} \in \mathbb{R}^{L \times L} \]

Under this condition, we obtain \( C_X C_X^H \approx C_{\alpha} C_{\alpha}^H \), so that \( C_Y C_Y^H = C_{\alpha} C_{\alpha}^H \), which implies that if \( C_Y \) is diagonal, so too is \( C_{\alpha} \), and vice versa. This means that the four covariance matrices of multivariate quaternion data can be simultaneously diagonalised via the QAUT, such as in (13).

Similarly to the univariate case, the diagonalisation error of the multivariate QAUT decreases with an increase in the correlation between the components of quaternion data, and increases with the number of variates, \( M \).

VI. SIMULATIONS

The effectiveness of the proposed QUT and QAUT techniques is illustrated via simulations on both synthetic and real-world quaternion-valued data.

A. Performance of QUT

In the first experiment, the QUT was applied for the decorrelation of multivariate \( \mathbb{C}^{\ast} \)-improper quaternion signals. First, three uncorrelated \( \mathbb{C}^{\ast} \)-improper quaternion signals were generated through the approach introduced in Appendix B and subsequently mixed using a \( 3 \times 3 \) matrix for which the elements were drawn from a standard normal distribution. The mixed signals \( x_1, x_2, x_3 \) were uncorrelated in terms of the \( \nu \)- and \( \gamma \)-complementary covariances and correlated in terms of the \( \kappa \)-complementary covariance, with the correlation coefficients 0.69 (between \( x_1 \) and \( x_2 \)), 0.34 (between \( x_1 \) and \( x_3 \)), and
0.91 (between \( x_2 \) and \( x_3 \)). Fig. 1 shows 3-D scatter plots of the original signals, \( x_1, x_2 \) and \( x_3 \), illustrating their high correlation, as indicated by elliptical shapes of the scatter plots at an angle to the coordinate axes. The QUT was used to decorrelate \( x_1, x_2 \) and \( x_3 \) into signals \( y_1, y_2 \) and \( y_3 \). Fig. 2 shows the much more circular nature of the decorrelated signals \( y_1, y_2 \) and \( y_3 \). The simulation results confirmed that the correlation coefficients between any two of \( y_1, y_2 \) and \( y_3 \) were zero, which indicates their successful decorrelation using the QUT.

### B. Performance of QAUT for synthetic signals

To assess the performance of QAUT, the squared diagonal error of the complementary covariance matrix \( C_{xx} \) was measured by a power ratio of the off-diagonal, \( \lambda_{\alpha,ij} \), versus diagonal, \( \lambda_{\alpha,ii} \), elements of the approximately diagonal matrix \( \Lambda_\alpha \), given by

\[
\varepsilon_\alpha^2 = \frac{E\left(\left|\lambda_{\alpha,ij}\right|^2\right)}{E\left(\lambda_{\alpha,ii}^2\right)} \times 100\%, \quad i \neq j
\]

Synthetic quaternion signals with a varying degree of correlation between the real, \( i, j \) and \( \kappa \) components were considered. For simplicity, we assumed the six correlation coefficients of the four components to be equally distributed in \((0, 1]\) and denoted by \( \rho \), where \( \rho = 1 \) indicates full correlation and \( \rho = 0 \) no correlation.

In the univariate simulations, a synthetic quaternion data vector with a varying \( \rho \) was generated from a quaternion white Gaussian signal. Then, the performance of the univariate QAUT was assessed comprehensively against the degree of correlation present in the data components, \( \rho \in [0.5, 1] \), and the length of data segment, \( L \in [5, 20] \). Conforming with the analysis, Fig. 3 shows that the diagonalisation error increased as \( \rho \) or \( L \) increased. The performance was excellent, for example, for a moderate correlation degree, with \( \rho = 0.5 \), and a long data segment, \( L = 20 \), the squared error in the diagonalisation was less than 1%, while for \( \rho = 1 \), the error was negligible.

In the multivariate simulations, multiple channels of quaternion signals with a varying \( \rho \) were generated and then mixed using a matrix for which the elements were drawn from a standard normal distribution, to generate multivariate quaternion data. The performance of the multivariate QAUT was assessed against the degree of correlation present in the data components, \( \rho \in [0.5, 1] \), and the number of variates, \( M \in [5, 20] \). Fig. 4 shows that the diagonalisation error increased as \( \rho \) or \( M \) increased. Similarly to the univariate case, for a moderate correlation degree, with \( \rho = 0.5 \), and a large number of data variates, \( M = 20 \), the squared error in the diagonalisation was less than 1%, while for \( \rho = 1 \), the error was negligible.

### C. Performance of QAUT for real-world signals

The QAUT was also applied to 4-D real-world electroencephalogram (EEG) signals, four adjacent channels of the multichannel real-valued EEG signal were combined into a quaternion-valued signal. The channels measuring adjacent brain regions exhibited strong correlation with each other while the channels measuring distant brain regions were relatively weakly correlated. We tested the QAUT on such quaternion EEG signals with low, medium and high correlations between the four dimensions, with the data segment length \( L \) varying between 5 and 20. Fig. 5 illustrates that the diagonalisation error increased with either an increase in \( L \) or a decrease in the degree of correlation present in the data. The squared diagonal error was less than 0.2%, which indicates that the QAUT may be used for various analyses in brain computer interface.

### D. QAUT for rank reduction

In many practical applications the acquired data are highly correlated or even collinear, with redundant elements that do not contribute to the accuracy of processing but require excessive computational resources and may affect stability of algorithms. It is therefore desirable to use the minimum number of variables to describe the information within a data set; this can be achieved through a low-rank matrix approximation. For data requiring rank reduction, the assumption that the components of variables are highly correlated is sensible, that is, the approximation conditions for the QAUT hold. The multivariate QAUT in (22) can be therefore rearranged to decompose the data matrix \( X \) as a sum of uncorrelated rank one vector outer products:

\[
X = \Phi^H Y = \sum_{m=1}^{M} \phi_m^H y_m
\]

where \( \phi_m \) and \( y_m \) are rows in the matrices \( \Phi \) and \( Y \) respectively, and the transformed variables, \( y_m \), are monotonically non-increasing in variance. The use of variables for which the variance is above a defined threshold and the disposal of the remaining variables, which account for noise, provides the reduced rank approximation of \( X \):

\[
\hat{X} = \sum_{m=1}^{P} \phi_m^H y_m, \quad \text{where} \quad P < M
\]

which is illustrated in Fig. 6. This is analogous to the real domain where the SVD provides a change of coordinates into uncorrelated variables for which the variance is monotonically non-increasing.

To illustrate the potential of the QAUT in rank reduction for quaternion data, a 2-D quaternion process, \( x_1, x_2 \in \mathbb{H} \) was generated where \( x_1 \) and \( x_2 \) are highly correlated as shown in Fig. 7 which plots the component relationship in the real, \( i, j \) and \( \kappa \) planes. The QAUT decorrelated the original data into signals \( y_1 \) and \( y_2 \). Fig. 8 shows significant variation only in the \( y_2 \) axis while the \( y_1 \) variation corresponds to noise, indicating that the true dimensionality of the process is one rather than two.

Rank reduction is commonly used in chemometrics where many sets of measurements are required to examine a process and the important information may not be directly observable. For example, Near Infrared Spectra (NIR) are used to examine
the chemical content of a data sample and are often highly correlated or collinear. We applied the QAUT to NIR obtained from 80 samples of corn using three different spectrometers [37]. A pure quaternion variable is formed for each corn sample by combining its spectra from the three spectrometers. The QAUT diagonalised the data covariance matrices successfully to show that one quaternion variable was sufficient to express most of the information, as the largest diagonal element was 39 compared to 0.01 for the next largest. Fig. 9 displays the logarithm of the ratio of variance explained by each variable, compared to the total variance in the data. The insert plot shows the cumulative ratio of variance explained by including further variables for the original and transformed data. Observe that the transformed data explains the same information as the original with much fewer variables, thus verifying the QAUT.

E. QAUT for imbalance detection in three-phase power systems

As shown in Section V, when the components of data are highly correlated the QAUT diagonalises all complementary covariance matrices to a small error, however, this error increases for lower correlations. We can exploit this data dependence of the QAUT and the structure of the complementary covariance matrices to detect imbalance in a three-phase power system. In general, the instantaneous voltages of a three-phase power system are given by [38]

\[
\begin{align*}
    v_1(n) &= A_1 \sin(2\pi f DTn + \varphi_1) \\
    v_2(n) &= A_2 \sin(2\pi f DTn + \varphi_2) \\
    v_3(n) &= A_3 \sin(2\pi f DTn + \varphi_3)
\end{align*}
\]

where \( A_1, A_2, A_3 \) are the instantaneous amplitudes, \( \varphi_1, \varphi_2, \varphi_3 \) the instantaneous phases, \( f \) the system frequency, and \( DT \) the sample interval. In a balanced three-phase system, \( A_1 = A_2 = A_3, \varphi_2 = \varphi_1 - \frac{2\pi}{3}, \varphi_3 = \varphi_2 - \frac{2\pi}{3} \), and as a result the correlation between each component is equivalent. The power grid is usually designed to operate optimally under balanced conditions; however, faults in the power system can cause unbalanced operating conditions that propagate through the network, threatening its stability. Therefore, it is important in fault detection and mitigation applications to identify the incidences when the power grid is operating in an unbalanced fashion.

Pure quaternions have been used to deal with data recordings from all three phases simultaneously as \( x(n) = v_1(n) + v_2(n)j + v_3(n)k \) [38]. Organising the recorded data into a vector, \( x = [v_1 + v_2 + v_3] \), where \( v_1, v_2 \) and \( v_3 \) are column vectors consisting of records of \( v_1, v_2 \) and \( v_3 \), we can apply the univariate QAUT as an indicator for power imbalance by considering the diagonalisation error of the complementary covariance matrices for the pure quaternion vector \( x \), for which the overall diagonalisation error expressed in (20) becomes

\[
\begin{align*}
    J[x^*_1(n_1)x_1(n_2)] &= |v_2(n_2)v_3(n_2) - v_1(n_2)v_3(n_1)|i \\
    &+ |v_1(n_1)v_3(n_2) - v_1(n_2)v_3(n_1)|j \\
    &+ |v_1(n_2)v_2(n_1) - v_1(n_1)v_2(n_2)|k
\end{align*}
\]

When the system is balanced, the covariance \( E\{v_pv_q^T\} \) is equivalent for all phases \( p, q \in \{1, 2, 3\} \), so the three imaginary parts of \( x^*_1(n_1)x_1(n_2) \) in (24) have equal absolute value, leading to equivalent diagonalisation errors for the three complementary covariance matrices. On the other hand, when the system becomes unbalanced, the correlation degrees between the lines are different, and hence the three imaginary parts \( x^*_1(n_1)x_1(n_2) \) differ, resulting in different diagonalisation errors for three complementary covariance matrices. This allows the QAUT to be used in an imbalance detection application. To this end, we applied the QAUT to a sliding window of 0.04 seconds of a three-phase power system for which a fault occurred at 0.25s lasting for 0.25s before the system returned to a balanced state. During this fault, \( v_1 \) decreased significantly in amplitude whereas \( v_2 \) and \( v_3 \) increased in amplitude (\( v_2 > v_3 \)).

Fig. 10 shows that the diagonalisation error for \( C_x \) and \( C_\kappa \) increased while the error for \( C_\chi \) decreased and therefore analysing the difference between the errors is a suitable detection method for the imbalance. This can be explained via (24) and (19): the decrease in the amplitude of \( v_1 \) and the increase in the amplitudes of \( v_2 \) and \( v_3 \) induce an increased \( \alpha \)-imaginary part and decreased \( \beta \)-imaginary and \( \kappa \)-imaginary parts in \( x^*_1(n_1)x_1(n_2) \), so that the error for \( C_\chi \) declines and the errors for \( C_\kappa \) and \( C_\kappa \) rise.

VII. Conclusion

Novel matrix factorisation techniques for widely linear quaternion algebra have been introduced. The diagonalisation of symmetric quaternion matrices has been addressed, as the Takagi factorisation of complex symmetric matrices cannot be readily extended to the quaternion domain. We have shown that the symmetry is not sufficient for a quaternion matrix \( A \) to be diagonalisable, and instead the condition \( (AA^*)^T = A^*A \) must also hold, which in general is not the case. The simultaneous diagonalisation of two quaternion covariance matrices has then been introduced as an analogue to the SUT in the complex domain. It has also been shown that for improper quaternion data, a typical case in quaternion signal processing applications, a single EVD of the covariance matrix is sufficient to diagonalise approximately the three complementary matrices. The analysis of the so introduced QAUT has shown its excellent practical usefulness for highly improper quaternion signals. Simulation studies on synthetic and real-world signals support the proposed QUT and QAUT techniques.

APPENDIX A

PROPERTIES OF QUATERNION MATRICES

For two general quaternion matrices, \( A \) and \( B \), the following properties hold [21]:

\begin{enumerate}
    \item \( A^{*T} = (A^*)^T = (A^T)^* \) and \( A^{\alpha*} = (A^*)^\alpha = (A^\alpha)^* \),
    \item \( A^{\alpha^T} = (A^\alpha)^T = (A^T)^\alpha \) and \( A^H = (A^\alpha)^H = (A^H)^\alpha \),
    \item \( (A^\alpha)^\beta = (A^\beta)^\alpha \) for distinct \( \alpha, \beta, \gamma \in \{i, j, k\} \),
    \item \( (AB)^* \neq A^*B^* \),
    \item \( (AB)^T \neq B^TA^T \).
\end{enumerate}
\[ (A B)^{H} = B^{H} A^{H}, \]

\[ (A B)^{\alpha} = A^{\alpha} B^{\alpha}, \]

\[ (A B)^{\alpha H} = B^{\alpha H} A^{\alpha H}. \]

**APPENDIX B**

\[ \mathbb{C}^{n} \text{-improper quaternion random vectors} \]

For distinct \( \alpha, \beta, \gamma \in \{ 1, \beta, \gamma \} \), the Cayley-Dickson construction allows for the complex representation of a quaternion random vector \( x \), as

\[ x = z_{1} + z_{2} \beta \]

where \( z_{1} \) and \( z_{2} \) are complex random vectors defined in the plane spanned by \( \{ 1, \alpha \} \). The quaternion random vector \( x \) is \( \mathbb{C}^{n} \text{-improper} \) if and only if both of the following two conditions are fulfilled [32):

1) \( z_{1} \) and \( z_{2} \) are proper complex vectors, which is achieved when their real and imaginary parts are uncorrelated and with the same variance,

2) \( z_{1} \) and \( z_{2} \) are with unequal variance or correlated with each other.

**APPENDIX C**

**Symmetric quaternion matrices**

Consider a \( 2 \times 2 \) symmetric quaternion matrix \( A \), whereby

\[ A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \]

\[ A^{*} = \begin{bmatrix} a_{11}^{*} & a_{12}^{*} \\ a_{21}^{*} & a_{22}^{*} \end{bmatrix} \]

\[ A A^{*} = \begin{bmatrix} a_{11}^{2} + a_{12}^{2} & a_{11} a_{12}^{*} + a_{12} a_{11}^{*} \\ a_{21} a_{12}^{*} + a_{22} a_{11}^{*} & a_{21}^{2} + a_{22}^{2} \end{bmatrix} \]

By considering the symmetry \( A = A^{T} \), the matrices \((A A^{*})^{*}\) and \( A A^{*} \) are structured as

\[ (A A^{*})^{*} = \begin{bmatrix} |a_{11}|^{2} + |a_{12}|^{2} & a_{11} a_{12}^{*} + a_{12} a_{11}^{*} \\ a_{11} a_{12}^{*} + a_{12} a_{11}^{*} & |a_{21}|^{2} + |a_{22}|^{2} \end{bmatrix} \]

\[ A A^{*} = \begin{bmatrix} |a_{11}|^{2} + |a_{12}|^{2} & a_{11} a_{12}^{*} + a_{12} a_{11}^{*} \\ a_{11} a_{12}^{*} + a_{12} a_{11}^{*} & |a_{21}|^{2} + |a_{22}|^{2} \end{bmatrix} \]

Note that \((A A^{*})^{*} = A A^{*}\) if their off-diagonal elements are commutative, implying that either the diagonal or off-diagonal elements of \( A \) are real-valued.

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Fig. 1: Scatter plots, indicating the high correlation degrees, of the three correlated $\mathbb{C}^\kappa$-improper quaternion signals, $x_1$, $x_2$ and $x_3$.

Fig. 2: Scatter plots, indicating the low correlation degrees, of the three $\mathbb{C}^\kappa$-improper quaternion signals, $y_1$, $y_2$ and $y_3$, decorrelated via the QUT.
Fig. 3: Squared diagonal error [in %] against the correlation degree and the data segment length for synthetic univariate data.

Fig. 4: Squared diagonal error [in %] against the correlation degree and the number of variates for synthetic multivariate data.
Fig. 5: Squared diagonal error [in %] against the correlation degree and the data segment length for EEG data.
\[ X = \mathbf{y}_1 + \mathbf{y}_2 + \cdots + \mathbf{y}_r + \cdots + \mathbf{y}_w \]

\[ \hat{X} \]

Fig. 6: Rank reduction using the QAUT.
Fig. 7: Scatter plots showing high correlation between the original $x_1$ and $x_2$ variables.

Fig. 8: Scatter plots showing no correlation between the transformed $y_1$ and $y_2$ variables.

Fig. 9: The logarithm of the ratio of the variance explained by each variable to the total variance. *Insert diagram:* The cumulative ratio of variance against the number of variables for the original and decorrelated corn data.

Fig. 10: Three-phase power signal and imbalance detection.