State-dependent Topological Invariants and Anomalous Bulk-Boundary Correspondence in non-Hermitian Topological Systems

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The breakdown of the bulk-boundary correspondence in non-Hermitian (NH) topological systems is an open, controversial issue. In this paper, to resolve this issue, we ask the following question: Can a (global) topological invariant completely describe the topological properties of a NH system as its Hermitian counterpart? Our answer is no. One cannot use a global topological invariant (including non-Bloch topological invariant) to accurately characterize the topological properties of the NH systems. Instead, there exist a new type of topological invariants that are absent in its Hermitian counterpart – the state-dependent topological invariants. With the help of the state-dependent topological invariants, we develop a new topological theory for NH topological system beyond the general knowledge for usual Hermitian systems and obtain an exact formulation of the bulk-boundary correspondence, including state-dependent phase diagram, state-dependent phase transition and anomalous transport properties (spontaneous topological current). Therefore, these results will help people to understand the exotic topological properties of various non-Hermitian systems.

Topological systems, including topological insulators and topological superconductors have become the forefront of research in condensed matter physics for many years. These gapped topological system are always characterized by certain (global) topological invariants and have intrinsic topological properties that are robust and immune to perturbations. For two quantum phases with different topological invariants, one cannot deform the ground states from one quantum phase to the other without closing the energy gap. On the other hand, non-Hermitian (NH) topological systems have been intensively studied in both theory and experiments. The topological properties of NH systems show quite different properties as their Hermitian counterparts. Recently, within the generalization of Altland-Zirnbauer (AZ) theory, the classification of NH systems with topological bands is characterized by different symmetry-protected topological invariants.

An open issue is the breakdown the bulk-boundary correspondence (BBC) in NH systems that has recently become a subject of active and controversial discussion. Due to the existence of NH skin effect, the conventional approach of predicting boundary states from bulk topological invariants for periodic systems does not provide a conclusive physical picture. According to Ref. 22, it was known that it is non-Bloch topological invariant that characterizes the topological properties of the NH topological systems. However, the non-Bloch topological invariants cannot predict the existence of the (singular) defective edge state (an edge state on the ends of an one-dimensional (1D) topological system with NH coalescence).

Hence, to complete solve the open issue of the breakdown the bulk-boundary correspondence in NH systems, we develop a new theory for non-Hermitian topological system by proposing the state-dependent topological invariants. We point out that it is the state-dependent topological invariants rather than a global state-independent topological invariant that characterize the non-Hermitian topological phases. With the help of effective edge Hamiltonian, we show spontaneous EP phenomenon together with topological Hermitian-NH transition for a given edge state. In addition, due to the unbalance of the state-dependent topological invariants for the edge states on chemical potential there exists spontaneous topological current for 2D non-Hermitian Chern insulator.

State-dependent topological invariants for 1D NH topological insulator: Firstly, we take 1D nonreciprocal Su-Schrieffer-Heeger (SSH) model as an example to introduce the state-dependent topological invariants and provide a new description of bulk-boundary correspondence for 1D NH topological system.

The Bloch Hamiltonian for a nonreciprocal SSH model under periodic boundary condition (PBC) is $H_{PBC} = (t_1 + t_2 \cos k) \sigma_x + (t_2 \sin k + i \gamma) \sigma_y + \varepsilon \sigma_z$ where $t_1$ and $t_2$ describe the intra-cell and inter-cell hopping strengths, respectively. $\varepsilon$ is the staggered potential and $\gamma$ describes the unequal intra-cell hoppings. $\sigma_i$‘s are the Pauli matrices acting on the (A or B) sublattice subspace. In this paper, we set $t_2 = 1$. It was known that due to the NH skin effect the bulk spectrum of the system becomes that of a NH Hamiltonian $H_{OBC}$ with open boundary condition (OBC). As a result, the effective bulk Hamiltonian turns into $H_{OBC}(k) = (\tilde{t}_1 + t_2 \cos k) \sigma_x + (t_2 \sin k) \sigma_y + \varepsilon \sigma_z$ where the effective hopping parameters become $\tilde{t}_1 = \sqrt{(t_1 - \gamma)(t_1 + \gamma)}$, and $\tilde{t}_2 = t_2$. Here, $S_{NHP}$ is a similar-transformation, i.e., $|\psi(k)\rangle \rightarrow |\tilde{\psi}(k)\rangle = |\psi(k - i\tilde{\gamma})\rangle = S_{NHP} |\psi(k)\rangle |n\rangle \rightarrow |\tilde{n}\rangle = e^{-\tilde{\gamma}(n-1)} |n\rangle$ ($n$ denotes the cell number) with $e^{\tilde{\gamma}} = \sqrt{\frac{t_1 - \gamma}{t_1 + \gamma}}$.

To completely characterize the edge states, we in-
roduce the state-dependent topological invariants \( \{ \bar{v}_L, \bar{v}_R \} = \{ \bar{v}_L, \xi = L, R \} \) where \( \bar{v}_L \) and \( \bar{v}_R \) are topological invariants for the edge states at left and right, respectively. \( \{ \bar{v}_L, \bar{v}_R \} \) are combination of Bloch topological invariants from \( H_{\text{PBC}} \) and non-Bloch topological invariants from \( H_{\text{OBC}} \), i.e.,

\[
\bar{v}_L = \bar{w} \cdot v_L, \quad \bar{v}_R = \bar{w} \cdot v_R \quad (1)
\]

where \( v_L = \frac{1}{2\pi} \int dk \cdot \partial_k \varphi_+ \) and \( v_R = \frac{1}{2\pi} \int dk \cdot \partial_k \varphi_- \) are the Bloch winding number that are defined from the Hamiltonian under PBC \( H_{\text{PBC}} \). \( \varphi_{\pm} = \text{Arg}(h_{\pm}) \) and \( h_{\pm} \) is described by \( h_{\pm} = h_0 \pm it_0 \) \( (h_x = t_1 + t_2 \cos k, h_y = t_2 \sin k + i\gamma) \); \( \bar{w} = \frac{1}{2\pi} \int _{-\pi}^{\pi} \partial \varphi(k)dk \) is the non-Bloch topological invariant that is defined from the Hamiltonian under OBC \( H_{\text{OBC}} \), where \( \varphi(k) = \tan ^{-1}(h_y/h_x) \) and \( h_x = t_1 + t_2 \cos k, h_y = t_2 \sin k \).

The state-dependent topological invariants \( \{ \bar{v}_\xi, \xi = L, R \} \) become a complete description of BBC for 1D NH topological systems: There exist \( |\bar{v}_L| \) edge states at left end and \( |\bar{v}_R| \) edge states at right end. The state-dependent topological invariants are not applied to the case of \( \varepsilon = 0 \). The non-Hermitian SSH model with \( \varepsilon = 0 \) is very special and unstable to arbitrary perturbation breaking chiral symmetry. We will discuss the case of \( \varepsilon = 0 \) in supplementary materials in detail.

As a result, for the 1D NH SSH model, we have a state-dependent phase diagram with four phases (See Fig.1(a)): phase I, phase II, phase III, phase IV. There exist two kinds of topological phase transitions: the state-independent topological transition at \( |\bar{v}_L| = |\bar{v}_R| \) is characterized by the changing of non-Bloch topological invariant \( \bar{w} \) for \( H_{\text{OBC}} \) from a trivial phase with \( \bar{v}_L = \bar{v}_R = 0 \) (or \( \bar{w} = 0 \)) to topological phase with \( \bar{v}_L \neq 0 \) or \( \bar{v}_R \neq 0 \) (or \( \bar{w} = 1 \)); the other at \( t_1 = \pm \gamma = \pm 1 \) is state-dependent that is characterized by the changing of Bloch topological invariant \( v_L \) or \( v_R \) for \( H_{\text{PBC}} \) from a topological phase with the edge states \( (\bar{v}_L, \bar{v}_R) = 1 \), and \( \bar{w} = 1 \) to another without them \( \{ \bar{v}_L, \bar{v}_R = 0, \bar{w} = 1 \} \).

To verify the validity of the state-dependent topological invariants \( \{ \bar{v}_\xi, \xi = L, R \} \) and explore the corresponding topological transitions for the 1D NH topological insulators, we write down the effective Hamiltonian for the edge states \( H_{\text{edge}} = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} \) where \( h_{IJ} = \langle b^I | \hat{H}_{\text{NH}} | b^J \rangle \), \( I, J = 1, 2 \), \( ( b_1^1 b_2^1 b_1^2 b_2^2 ) \) are the basis under biorthogonal set of the edge states at left/right ends. For the NH SSH model with \( \gamma \neq 0 \), the effective edge Hamiltonian is obtained as

\[
H_{\text{edge}} = \Delta^+ \tau^+ + \Delta^- \tau^- + \varepsilon \tau^z \quad (2)
\]

where \( \Delta^+ = \Delta e^{N\phi_0} \) or \( \Delta^- = \Delta e^{-N\phi_0} \) of \( H_{\text{edge}} \), there exists the competition between the exponential decay of \( N \) from energy tunneling \( \Delta \sim e^{-N\phi_{0}} \) and the exponential increase with \( N \) from NH similarity transformation \( e^{N\phi_0} \). Therefore, in thermodynamic limit (or \( N \to \infty \)) there exist two phases: one is Hermitian phase with \( |\Delta e^{N\phi_0}| \to 0 \), the other is NH phase with \( |\Delta e^{N\phi_0}| \to \infty \). In the Hermitian phase with \( |\Delta e^{N\phi_0}| \to 0 \), the effective edge Hamiltonian is reduced into \( \hat{H}_{\text{edge}} \to \varepsilon \cdot \tau^z \). Now, the effect from the NH similarity transformation is irrelevant; On the other hand, in the NH phase with \( |\Delta e^{N\phi_0}| \to \infty \), the effective edge Hamiltonian is reduced into \( \hat{H}_{\text{edge}} \to e^{-N\phi_0} \tau^+ \) or \( e^{-N\phi_0} \tau^- \). Now, the effect from NH similarity transformation dominates and becomes relevant. Although the total energy splitting \( E_+ - E_- = 2\sqrt{\varepsilon^2 + \Delta^2} \) is finite, the system is at exceptional points (EPs) and we have singular defective edge state with NH coalescence. For this reason, we call it spontaneous EP phenomenon.

At \( |\Delta e^{N\phi_0}| = 1 \), the topological transition between Hermitian phase and NH phase with spontaneous EP occurs. We call the state-dependent topological transition to be topological Hermitian – NH transition. Fig.1(c) and
Fig. 1(d) are the numerical results for \( \Delta^+ = \bar{\Delta}_e^{-Nq_0} \) or \( \Delta^- = \bar{\Delta}_e^{Nq_0} \), in which \( \ln \frac{\Delta^+}{\Delta^-} = 0 \) or \( \frac{4\gamma^2}{\tau^2} e^{\pm 2q_0} = 1 \) denotes the topological Hermitian–NH transition. This condition for topological Hermitian–NH transition is just \( t_1 \pm \gamma = \pm 1 \) from phase with the edge states \( \langle \bar{v}_L \cdot \bar{v}_R \rangle = 1 \), and \( \bar{v}_L = 1 \) to another without them \( \langle \bar{v}_L \cdot \bar{v}_R \rangle = 0 \), and \( \bar{v}_L = 1 \). To verify this type of topological transitions and show the defectiveness of the edge states, we calculate the BBC ratio, \( \gamma_{BBC} = 1 - \langle \psi_e | \psi_e \rangle \), a quantity that characterizes number anomaly of the edge states. If \( \gamma_{BBC} \) is 1, there exists the usual BBC with \( \bar{v}_L = 1 \) and \( \bar{v}_R = 1 \); if \( \gamma_{BBC} \) is smaller than 1, there exists defective edge states with \( \bar{v}_L = 1 \) and \( \bar{v}_R = 0 \) or \( \bar{v}_L = 0 \) and \( \bar{v}_R = 1 \). In Fig. 1(b), we show the numerical and analytic results that are consistent with each other.

State-dependent topological invariants and topological spontaneous current for 2D NH topological insulators:

Next, we consider a lattice model of 2D NH Chern insulator - the 2D NH spin-orbital coupling model.[3] The Bloch Hamiltonian under PBC is \( H_B(k_x, k_y) = (\sin k_x)\sigma_x + (\sin k_y + i\gamma)\sigma_y + (m + \cos k_x + \cos k_y)\sigma_z \), where \( \sigma_{x,y,z} \) are Pauli matrices. The NH parameters \( \gamma \) appear as “imaginary Zeeman field”. Suppose that the cylinder has periodic-boundary condition along \( x \)-direction and open boundary condition along \( y \)-direction, by doing similar transformation \( \tilde{S}_{NHP} \), \( \langle \psi(k) \rangle \rightarrow \langle \psi(k) \rangle = \langle \psi(k - iq_0) \rangle = \tilde{S}_{NHP} \langle \psi(k) \rangle \), we have the non-Bloch “cylinder Hamiltonian”, \( H_{NHP}(k_x, k_y) = H_{NHP}(k_x \rightarrow k_x, k_y \rightarrow k_y - iq_0, \text{in which } e^{-2q_0(k_x)} = \frac{m + \cos k_x + \gamma}{m + \cos k_x - \gamma} \).

Then, we define the state-dependent topological invariants for edge states in the 2D NH spin-orbital coupling model. For OBC along \( y \)-direction and PBC along \( x \)-direction, the topological system exhibits modes localized on the edges and the wave vectors \( k_x = k = \frac{2\pi n}{L_x} \) \((n = 1, 2, ..., N_x)\) are good quantum numbers. The state-dependent topological invariants are defined as

\[
\langle \bar{v}_{\xi,k}, k \in T^1, k \neq 0, \xi = L, R \rangle \tag{3}
\]

where \( \bar{v}_{\xi,L} = C_y \cdot v_{k,L} \) are topological invariants for all edge state at left end with wave vector \( k \) and \( \bar{v}_{\xi,R} = C_y \cdot v_{k,R} \) are topological invariants for all edge state at right end with wave vector \( k \), respectively. Here, \( C_y = \frac{1}{2\pi} \int dk_x dk_y \epsilon^{ij}(\partial_i u_L(k_x, k_y)|\partial_j u_R(k_x, k_y)) \) is the non-Bloch Chern number that is defined from the Hamiltonian \( H_{NHP}(k_x, k_y) \) \((u_R(k_x, k_y) \rightarrow \langle u_R(k_x, k_y) \rangle \) denotes the bulk state of biorthogonal set under OBC; \( v_{L,k} = \frac{1}{2\pi} \int dk_y \partial_y \psi^+(k_y, k) \) and \( v_{R,k} = \frac{1}{2\pi} \int dk_y \partial_y \psi^-(k_y, k) \) are the Bloch winding number that are defined from the Hamiltonian \( H_{NHP}(k_x, k_y) \). \( h_{\pm} \) is described by \( h_x = h_x \pm ih_y \) \((h_x = (m + \cos k) + \cos k, h_y = \sin k + i\gamma) \) and \( \varphi_\pm = \text{Arg}(h_\pm) \).

Therefore, the state-dependent topological invariants \( \langle \bar{v}_{\xi,k}, k \in T^1, k \neq 0, \xi = L, R \rangle \) becomes a complete description of BBC for 2D NH topological systems: There exist \( \langle \bar{v}_{\xi,k} \rangle \) edge states with wave vector \( k \) at left end and \( \langle \bar{v}_{\xi,k} \rangle \) edge states with wave vector \( k \) at right end. The state-dependent topological invariants are not applied to the case of \( k = 0 \) and we will discuss the case of \( k = 0 \) in supplementary materials in detail. As a result, for the 2D NH topological system with fixed parameter, we have a state-dependent phase diagram via \( k \) and \( m \). See Fig. 2(a).

To verify the validity of the state-dependent topological invariants, we obtain the effective edge Hamiltonian for the NH spin-orbital coupling model.

Firstly, the effective edge Hamiltonian of edge states for Hermitian 2D Chern insulator with \( \gamma = 0 \) is obtained as \( \mathcal{H}_{\text{edge}} = \tau^z \xi_k + \tau^z \Delta_k \), where \( \xi_k = \text{sin } k \) is the dispersion of the edge states of semi-infinite system and \( \Delta_k = ((\cos k + m)^2 - 1) \cdot (\cos k + m)N_y \) is tunneling strength. As a result, the energy levels are \( E_k = \pm \sqrt{(\sin k)^2 + (\Delta_k)^2} \). In thermodynamic limit \( N_y \rightarrow \infty, \Delta_k \rightarrow 0 \), we have \( E_k \rightarrow \pm \text{sin } k \) or \( E_k \rightarrow -\text{sin } k \) on left/right edge and \( E_k \rightarrow -\text{sin } k \) on right/left edge.

When considering the NH skin effect, we do an additional NH similarity transformation \( U_{\text{edge}} = \begin{pmatrix} 1 & 0 \\ 0 & e^{-\gamma q_0(k)N_y} \end{pmatrix} \) on the effective edge Hamiltonian \( \mathcal{H}_{\text{edge}} \), i.e., \( \tau^z \rightarrow U_{\text{edge}}^{-1} \tau^z U_{\text{edge}} = \tau^z \cosh(q_0(k)N_y) - i\tau^y \sinh(q_0(k)N_y) \). As a result, the effective edge Hamiltonian turns into

\[
\mathcal{H}_{\text{edge}} = \Delta^+_k \tau^z + \Delta^-_k \tau^z + \tau^z \text{sin } k \tag{4}
\]
where $\Delta^+_k = \Delta_k e^{-N_{\gamma} k} \Delta_k = \Delta_k e^{N_{\gamma} k}$ and $\Delta_k = ((\cos k + m)^2 - \gamma^2) - 1/2$. Under $U_{edge}$, the energy levels become $E_{\pm}(k) = \pm \sqrt{\sin^2 k + (\Delta_k)^2}$.

With the help of the effective edge Hamiltonian $H_{edge}$, we show spontaneous EP phenomenon and topological Hermitian-NH transition for a given edge state with wave vector $k$. At $\ln \Delta_k = 0$ topological Hermitian-NH transition occurs that is just the condition of $|((\cos k + m)^2 - \gamma^2) - 1/2| = 1$ from phase with the edge states ($|\tilde{\psi}_{L,k} \cdot \tilde{\psi}_{R,k}| = 1$, and $C_y = 1$) to another without them ($\tilde{\psi}_{L,k} \cdot \tilde{\psi}_{R,k} = 0$, and $C_y = 1$). Fig. 2(c) and Fig. 2(d) are the numerical results for $\Delta^+_k$ or $\Delta^-_k$, in which $\ln \Delta^+_k = 0$ denotes the critical points of the topological transitions. We can also use $\Gamma_{BNC}$ to characterize the topological properties of the edge states with wave vector $k$, $\Gamma_{BNC} = 1 - |(|\psi_{k,+}|/|\psi_{k,-}|)/2$. See the results in Fig. 2(b).

In addition, we point out that there exists physics consequence of unbalance of the state-dependent topological invariants for the edge states on chemical potential – the spontaneous topological current for 2D NH Chern insulator.

For the system with open boundary condition (OPC), in general, the chemical potentials at the ends of system may be different, i.e., $\mu_L$ and $\mu_R$, respectively. We assume that the chemical potentials locate inside the energy gap of the bulk states. So, the transport of the system mainly comes from the edge states and we can apply the Landauer-Buttiker formalism on the transport of edge states. According to the Landauer-Buttiker formalism, the (Hall) current is defined as $I_H = -e v_{eff} n$. In thermodynamic limit, the effective velocity of the charge carriers is $v_{eff} = \frac{\partial E_k}{\partial k}$, where $E_k \sim \pm \sin k$. The density of the charge carriers is

$$n = n_L - n_R = \mu_L D(\mu_L) - \mu_R D(\mu_R)$$

$$\sim (\mu_L \nu_{L,k} (E) - \mu_R \nu_{R,k} (E)) D(E)$$

where $D(\mu_L) = \nu_{L,k} (E) D(E)$ and $D(\mu_R) = \nu_{R,k} (E) D(E)$, $\nu_{L,k} (E)$ and $\nu_{R,k} (E)$ are the state-dependent topological invariants for the edge states at left and right sides, respectively. The wave vectors $k_L$ and $k_R$ are obtained by calculating the following equations, $\sin k_L = \mu_L$ and $\sin k_R = \mu_R$, respectively. As a result, considering $\mu_L/k = -e V_{L,R}$, we derive the current for the NH 2D topological insulator,

$$I_H = \frac{e^2}{\hbar} C_y (\frac{\nu_{L,k} (E)}{\nu_{R,k} (E)}) = \sigma_0 (\nu_{L,k} (E) - \nu_{R,k} (E))$$

where $\sigma_0 = e^2 / \hbar$ is unit of quantized Hall conductance.

In phase III and phase IV of Fig. 2(a), when there doesn’t exist an external transverse electric field $\mu_L = \mu_R = \mu$, due to the imbalance of the state-dependent topological invariants for the edge states on chemical potential, the electric current still exists, i.e., $I_H = \sigma_0 (\tilde{\nu}_{L,k} - \tilde{\nu}_{R,k}) \neq 0$. Because the current is proportional to the unbalance of the state-dependent topological invariant ($\tilde{\nu}_{L,k} - \tilde{\nu}_{R,k}$), the spontaneous current is topological!

*State-dependent Topological Invariants for NH d-dimensional topological insulators:* Finally, we generalize the theory of state-dependent topological invariants to NH d-dimensional topological insulators.

A d-dimensional topological system can be approximatively described by continuum model. Assuming PBCs in all directions, we consider the following Hamiltonian for NH d-dimensional topological insulators as $H_{BBC}(k) = \sum_{i=1}^{d} k_i + (m - \frac{1}{2} \sum_i k_i^2) \Gamma_{d+1} + i \gamma \Gamma_j$, where $\Gamma_{d+1}$ denote the gamma matrices that satisfy $\{\Gamma_{d+1}, \Gamma_j\} = 2 \delta_{d+1,j}$ and $m$ is the real mass parameter. $\gamma$ denotes the strength of the NH term. With OBCs in the j-th direction and PBCs in all other directions, due to the residue translation symmetry, $\mathbf{k}$ denotes the vector of all momenta except $k_j$. Under a similar transformation $S_{NH} = \text{diag} \{1, \alpha, \cdots, \alpha, N_j \} \otimes \{1 + i(1 - \alpha) \Gamma_{d+1}\}$, the Hamiltonian is transformed to $H_{OBC}(k) = \sum_{j \neq d} k_i \Gamma_j + [\sqrt{M_k^2 - \gamma^2 \Gamma_{d+1}^2}]$, where $\alpha = \sqrt{(M_k + \gamma)/(M_k + \gamma)}$, $M_k = -\frac{1}{2} \sum_{j \neq d} k_i^2$, and $N_j$ stands for the number of layers in the j-th direction.

Then, to completely characterize the topological properties, we define the state-dependent topological invariants for edge states in the d-D NH topological model.

With OBCs in the j-th direction and PBCs in all other directions, due to the residue translation symmetry, there exist edge states on left/right edges with wave vector $\mathbf{k}$. The state-dependent topological invariants for edge states are defined as

$${\tilde{v}}_{\xi,\mathbf{k}}, \mathbf{k} \in T^{d-1}, \mathbf{k} \neq 0, \text{ and } \xi = L, R$$

where $\mathbf{v}_{L,k} = \mathbf{v}_{L,\mathbf{k}}$ for edge state at left end with wave vector $\mathbf{k}$ and $\mathbf{v}_{R,\mathbf{k}} = \mathbf{v}_{R,\mathbf{k}}$ for edge state at right end with wave vector $\mathbf{k}$. Here $\mathbf{v}_{\mathbf{j}}$ is the d-dimensional non-Bloch topological invariant that is defined from the Hamiltonian $H_{OBC}(k)$. $\mathbf{v}_{L,k}$ and $\mathbf{v}_{R,k}$ are the one-dimensional winding number that are defined from the Hamiltonian $H_{PBC}(k)$. $\mathbf{v}_{L,k}$ and $\mathbf{v}_{R,k}$ are the Bloch winding number that are defined from the Hamiltonian under closed boundary condition. $h_{\pm} = h_x \pm iy (h_x = \sum_{i=1}^{d} \frac{1}{2} k_i^2, h_y = k_j + i \gamma)$ and $\phi = \text{Arg}(h_{\pm})$. The topological Hermitian–NH transition occurs at $m - \sum_{j \neq d} k_i^2 = \pm \gamma \pm 1$.

Therefore, the state-dependent topological invariants $\{\tilde{v}_{\xi,\mathbf{k}}, \mathbf{k} \in T^{d-1}, \mathbf{k} \neq 0, \text{ and } \xi = L, R\}$ become a complete description of BBC for d-D NH topological systems:

There exist $\tilde{v}_{L,\mathbf{k}}$ edge states with wave vector $\mathbf{k}$ at left end and $\tilde{v}_{L,\mathbf{k}}$ edge states with wave vector $\mathbf{k}$ at right end. The state-dependent topological invariants are not
applied to the case of \( \mathbf{k} = 0 \) and we will discuss the case of \( \mathbf{k} = 0 \) in supplementary materials in detail.

**Conclusion and discussion:** In the end, we draw a brief conclusion. The theory of state-dependent topological invariants for NH topological insulators is developed. The key point of our theory is that each edge state is characterized by state-dependent topological invariants \( \{ \tilde{v}_{L,k}, \tilde{v}_{R,k} \} \) rather than a global (state-independent) non-Bloch topological invariant \( w_j \).

To completely characterize the edge states in a NH topological systems, one need to calculate all state-dependent topological invariants. With the help of effective edge Hamiltonian \( \mathcal{H}_{edge} \), we derive the state-dependent phase diagram and show spontaneous EP phenomenon together with (state-dependent) topological Hermitian-NH transition for a given edge state with given wave vector \( \tilde{\mathbf{k}} \).

In addition, there exists spontaneous topological current \( I_H \) for 2D NH Chern insulator that is proportional to the unbalance of the state-dependent topological invariants \( \tilde{v}_{L,k_L} - \tilde{v}_{R,k_R} \) for the edge states on chemical potential \( \mu_L = \mu_R = \mu \). In future, the theory of state-dependent topological invariants can be applied to other types of topological systems, including topological superconductors, higher order topological states, even the topological semi-metals.

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