Cosmological solutions in five-dimensional minimal supergravity

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Abstract

We construct new non-stationary cosmological solutions to five-dimensional minimal supergravity that do not have any tri-holomorphic \( U(1) \) isometries. Our new solutions, in part, contain some of the previously constructed solutions to the minimal supergravity. The \( c \)-function of solutions shows monotonic increasing/decreasing behavior in time, in agreement with the expected behavior of the \( c \)-function in spacetimes with a positive cosmological constant.

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1. Introduction

In the strong coupling limit of brane systems, many horizonless three-charge brane configurations undergo a geometric transition and become smooth horizonless geometries with black hole or black ring charges. These charges come completely from fluxes wrapping on non-trivial cycles. The three-charge black hole (ring) systems are dual to the states of corresponding CFTs: in favor of the idea that non-fundamental black hole (ring) systems effectively arise as a result of many horizonless configurations [1]. In the heart of 11-dimensional three-charged supergravity solutions, there is a four-dimensional hyper-Kähler metric (which is equivalent to a metric with self-dual curvature in four dimensions) that guarantees the solutions preserve some supersymmetries [2]. The five-dimensional sub-spacetime of the 11-dimensional three-charged metric together with the Maxwell field makes the bosonic sector of five-dimensional minimal supergravity equivalent to the Einstein–Maxwell–Chern–Simons theory. The Einstein–Maxwell–(Dilaton–(Axion)) or -(Chern–Simons) theories in different dimensionalities have been extensively explored from many different directions. The black hole solutions have been considered in [3] as well as solitonic and gravitational instantons, dyonic and pp-wave solutions in [4], supergravity solutions in [5], brane worlds and cosmology in [6] and NUT and Bolt solutions, Liouville potential, rotating solutions and string theory.
extensions of Einstein–Maxwell fields in [7]. In five dimensions, unlike the four dimensions that only horizon topology is 2-sphere, we can have different more interesting horizon topologies such as black holes with horizon topology of 3-sphere [8], black rings with horizon topology of 2-sphere × circle [9], black saturn: a spherical black hole surrounded by a black ring [10], black lens which the horizon geometry is a lens space \( L(p, q) \) [11]. All allowed horizon topologies have been classified in [12]. In [13, 14], the authors consider hyper-Kähler Atiyah–Hitchin and Einstein–hyper-Kähler triaxial Bianchi type IX base spaces to construct five-dimensional supergravity solutions that only have rotational \( U(1) \) isometries. The complete solutions are regular around the critical surface of base spaces. The solutions in [14] are quite remarkable because the Einstein–hyper-Kähler Bianchi type IX geometry (that includes hyper-Kähler Atiyah–Hitchin as a special case) does not have any tri-holomorphic \( U(1) \) isometry. Hence, the solutions could be used to study interesting physical processes such as merger of two Breckenridge–Myers–Peet–Vafa black holes [15] or the geometric transition of a three-charge supertube of arbitrary shape, which do not respect any tri-holomorphic \( U(1) \) symmetry. We should emphasize that, in general, constructing solutions with non-tri-holomorphic \( U(1) \) isometries is a rather complicated, tedious and challenging task. To our knowledge, for classical black holes and black rings, only two solutions exist [16].

In this paper, we use Bianchi type IX space as the base space to construct five-dimensional cosmological solutions to minimal supergravity with a positive cosmological constant. The solutions enjoy generic non-tri-holomorphic \( U(1) \) isometries. The idea behind this paper is the first step to search for and construct black hole solutions in the presence of positive cosmological constant on Bianchi type IX space.

In fact, in [17], the authors constructed multi-black hole solutions of Einstein–Maxwell theory in spacetimes with a positive cosmological constant. The solutions describe an arbitrary number of charged black holes that are in motion with respect to each other. Moreover, the five- and higher-dimensional black hole solutions with positive cosmological constant were found in [18] and [19]. Specially in [19], the Eguchi–Hanson-based black hole solutions are in a contracting phase derived by the cosmological constant; hence, the solutions can describe coalescence of black holes in asymptotically de Sitter (dS) spacetimes. The Eguchi–Hanson space (as well as Atiyah–Hitchin space) is a special case of Bianchi type IX Einstein-Kähler space (see appendix C); hence this supports our idea to search for black hole solutions based on Bianchi type IX space, in spacetimes with the positive cosmological constant.

The outline of this paper is as follows. In section 2, we give a brief review of self-dual Bianchi type IX space and five-dimensional minimal supergravity with cosmological constant. In section 3, we present the class of cosmological non-stationary supergravity solutions over Bianchi type IX space and discuss the asymptotics of the solutions as well as the behavior of the \( c \)-function for the solutions. We conclude in section 4 with a summary of our solutions and possible future research directions as well as three appendices.

2. The Bianchi type IX space and minimal supergravity

The Bianchi type IX metric is locally given by the following metric with an \( SU(2) \) or \( SO(3) \) isometry group [20]:

\[
\text{d}s^2 = e^{2A(\zeta) + B(\zeta) + C(\zeta)} \text{d}\zeta^2 + e^{2A(\zeta)} \sigma_1^2 + e^{2B(\zeta)} \sigma_2^2 + e^{2C(\zeta)} \sigma_3^2 ,
\]

(2.1)

where \( \sigma_i \)'s are Maurer–Cartan one-forms. Integrating the Einstein equations (appendix A) as well as self-duality of the curvature implies

\[
\frac{\text{d}A}{\text{d}\zeta} = \frac{1}{2} [ e^{2B} + e^{2C} - e^{2A} ] - \alpha_1 e^{B+C} ,
\]

(2.2)
\[
\frac{dB}{d\zeta} = \frac{1}{2} \{e^{2C} + e^{2A} - e^{2B}\} - \alpha_2 e^{A+C},
\]

\[
\frac{dC}{d\zeta} = \frac{1}{2} \{e^{2A} + e^{2B} - e^{2C}\} - \alpha_3 e^{A+B},
\]

where \(\alpha_i, i = 1, 2, 3\), are integration constants obeying \(\alpha_i \alpha_j = \varepsilon_{ijk} \alpha_k\).

The first set of solutions corresponds to \((\alpha_1, \alpha_2, \alpha_3) = (1, 1, 1)\) that yields the Atiyah–Hitchin metric (appendix B). The Atiyah–Hitchin metric (and its ambipolar extension) was considered extensively in [13] for the construction of supergravity/Einstein–Maxwell–Chern–Simons solutions. The second set of solutions corresponds to \((\alpha_1, \alpha_2, \alpha_3) = (0, 0, 0)\).

Changing the coordinate from \(\zeta\) to \(r\) given by (C.8) (appendix C), we find the metric of triaxial Bianchi type IX space as

\[
ds^2 = \frac{dr^2}{\sqrt{F(r)}} + \frac{r^2}{4} \sqrt{F(r)} \left( \frac{\sigma_1^2}{1 - \alpha_1^4} + \frac{\sigma_2^2}{1 - \alpha_2^4} + \frac{\sigma_3^2}{1 - \alpha_3^4} \right),
\]

where

\[
F(r) = \prod_{i=1}^{3} \left( 1 - \alpha_i^4 \right),
\]

and \(\alpha_1, \alpha_2\) and \(\alpha_3\) are three parameters that, without loss of generality, are chosen such that \(0 = \alpha_1 \leq \alpha_2 = 2kc \leq \alpha_3 = 2c\). We note that coordinate \(r\) must be greater than or equal to \(\alpha_3\). Here \(0 \leq k \leq 1\) is the square root of the modulus of different types of Jacobian elliptic functions (appendix C) and \(c > 0\). If \(k > 1\), all we need is just to interchange the two and three directions. To embed the Bianchi space into five-dimensional minimal supergravity with cosmological constant, we take the metric ansatz as

\[
ds^2 = -H(t, \zeta)^{-2} dt^2 + H(t, \zeta) ds^2_4,
\]

and the only non-vanishing component of the gauge field as

\[
A_t = \frac{\eta \sqrt{3}}{2} \frac{1}{H(t, \zeta)},
\]

where \(\eta = \pm 1\). The five-dimensional minimal supergravity with a positive cosmological constant is described by the action

\[
S = \frac{1}{16\pi} \int d^5x \sqrt{-g} \left( R - 4\Lambda - F_{\mu\nu} F^{\mu\nu} - \frac{2}{3\sqrt{3}} \epsilon^{\mu\nu\rho\sigma\xi} F_{\mu\nu} F_{\rho\sigma} A_\xi \right),
\]

where \(R\) and \(F_{\mu\nu}\) are the five-dimensional Ricci scalar and Maxwell field. The Einstein and Maxwell equations are

\[
R_{\mu\nu} - \left( \frac{1}{2} R - 2\Lambda \right) g_{\mu\nu} = 2 \left( F_{\mu\lambda} F^{\lambda\nu} - \frac{1}{2} g_{\mu\nu} F^2 \right),
\]

\[
F^{\mu\nu}_{;\xi} = \frac{2}{3\sqrt{3}} \epsilon^{\mu\nu\rho\sigma\xi} F_{\rho\sigma} F_{\xi},
\]

respectively.

### 3. Minimal supergravity solutions with cosmological constant

We find the solutions to supergravity equations (2.10) and (2.11) assuming that the five-dimensional metric, the embedded four-dimensional space and gauge field are given by (2.7).
(2.1) and (2.8) respectively. All gravitational equations of motion are satisfied provided $H(t, \zeta)$ is a solution to the differential equations

$$\frac{3}{2}H(t, \zeta) \frac{d^2H(t, \zeta)}{dt^2} - 3 \left( \frac{\partial H(t, \zeta)}{\partial t} \right)^2 H^2(t, \zeta) + 4 \Lambda H^2(t, \zeta) = 0, \quad (3.1)$$

$$3 \left( \frac{\partial H(t, \zeta)}{\partial t} \right)^2 + 3 \frac{\partial^2 H(t, \zeta)}{\partial t^2} H(t, \zeta) - 4 \Lambda = 0, \quad (3.2)$$

$$\frac{\partial^2 H(t, \zeta)}{\partial \zeta \partial t} = 0. \quad (3.3)$$

Taking $H(t, \zeta) = 2 \epsilon \sqrt{\frac{\Lambda}{3}} t + m \zeta + \gamma$ where $\epsilon = \pm 1$ and $m, \gamma$ are two constants of integration, all the equations of motion are satisfied; hence, we obtain the supergravity metric as

$$ds^2 = -\frac{dt^2}{\left(2 \epsilon \sqrt{\frac{\Lambda}{3}} t + m \zeta + \gamma\right)^2} + \left(2 \epsilon \sqrt{\frac{\Lambda}{3}} t + m \zeta + \gamma\right)$$

$$\times \left(e^{2A(\zeta)} e^{B(\zeta)} e^{C(\zeta)} \right) d\zeta^2 + e^{2A(\zeta)} \sigma_1^2 + e^{2B(\zeta)} \sigma_2^2 + e^{2C(\zeta)} \sigma_3^2. \quad (3.4)$$

The metric functions $A(\zeta)$, $B(\zeta)$ and $C(\zeta)$ are given by equations (C.1), (C.2) and (C.3), respectively, and their exponentials plotted in figure 1, where we set $c = 1$ and $k = 1/2$.

We would prefer to find the analytic solutions to supergravity equations of motion to embed triaxial Bianchi type IX space (2.5) in (2.7). However, we find too unlikely to get the analytic solutions despite that metric (2.5) looks simpler in structure than (2.1).

We note in figure 1, the coordinate $\zeta$ in (2.1) varies as $\zeta_0/2 \leq \zeta \leq \zeta_0$ covers the range of $0 \leq r < \infty$ where $\sin(\zeta = \zeta_0, 1/4) = 0$ and $\zeta_0 = 3.192448 + O(10^{-7})$ (see figure 2). In general for any $c$ and $0 < k < 1$, the coordinate $\zeta$ should be chosen as $\zeta_{m, c, k}/2 \leq \zeta \leq \zeta_{m, c, k}$ where $\zeta_{m, c, k}$ is the $m$th positive root of $\sin(c^2 \zeta, k^2)$.

For both $\epsilon = \pm 1$, we choose $m, \gamma > 0$ (the mass of solutions is proportional to parameter $m$ that appears in metric (3.4)). The choice makes solution (3.4) with $\epsilon = +1$ regular everywhere for $t \geq 0$. Where $\epsilon = -1$, the solutions are still regular everywhere for $t \leq 0$.
The radial coordinate \( r \geq 2c \) in (2.5) as a function of coordinate \( \zeta \) in (2.1) where we set \( c = 1 \) and \( k = 1/2 \).

The Ricci scalar of solution (3.4) is given by

\[
R = \frac{20G(t, \zeta) - 27/2m^2 e^{-2(A(\zeta) + B(\zeta) + C(\zeta))}}{H^3(t, \zeta)},
\]

(3.5)

where

\[
G(t, \zeta) = 27\Lambda y^2 m\zeta + 9\Lambda m^3\zeta^3 + 36\Lambda^2 t^2 m\zeta + 27\Lambda y m^2\zeta^2 + 36\Lambda^2 t^2 y + 9\Lambda y^3 + \epsilon(8\sqrt{3}\Lambda^{5/2} t^3 + 18\sqrt{3}\Lambda^{3/2} t m^2\zeta^2 + 18\sqrt{3}\Lambda^{3/2} t y^2 + 36\sqrt{3}\Lambda^{3/2} t y m\zeta),
\]

and

\[
H(t, \zeta) = 2\sqrt{3}e\Lambda^{1/2} t + 3y + 3m\zeta.
\]

(3.6)

For the highest possible value of \( \zeta \) where \( \zeta \to \zeta_0 \), metric (3.4) reduces to

\[
\text{ds}^2 = -dT^2 + e^{2\epsilon\sqrt{A/T}} \text{ds}_{\mathbb{R}^4}^2,
\]

(3.8)

where \( \text{ds}_{\mathbb{R}^4}^2 \) is given by (A.5) and \( T = \frac{\ln(2e\epsilon\sqrt{A/T} \text{harm}(\omega \gamma))}{2\epsilon\sqrt{A/T}} \). The equal time hypersurfaces in (3.8) are flat \( \mathbb{R}^4 \). These hypersurfaces have an infinitely large size at \( T \to -\infty \) (where \( \epsilon = -1 \)), which decreases to a minimum value of 1 as \( T \to 0 \). Hence, the five-dimensional spacetime (3.8) with \( \epsilon = -1 \) shows a collapsing patch of five-dimensional dS spacetime. This contracting patch of solutions (3.4) at future infinity implies that black hole solutions based on Bianchi type IX space can describe the coalescence of black holes in asymptotically dS spacetimes. Indeed, the Ricci scalar (3.5) of (3.8) is equal to \( \frac{20\Lambda}{\epsilon} \), that is exactly the Ricci scalar of five-dimensional dS spacetime [21]. The flat hypersurfaces \( \mathbb{R}^4 \) have the smallest size at \( T = 0 \) (where \( \epsilon = +1 \)), which increases to an infinitely large size as \( T \to \infty \). Hence, spacetime (3.8) with \( \epsilon = +1 \) shows an expanding patch of five-dimensional dS spacetime. So in the limit of \( \zeta \to \zeta_0 \), spacetime (3.4) asymptotically reduces to the expanding/collapsing patches of five-dimensional dS spacetime [21].

In the other extreme case where \( \zeta = \zeta_0/2 + \hat{\zeta} \) with \( \hat{\zeta} \to 0^+ \), solution (3.4) transforms to

\[
\text{ds}^2 = -dT^2 + e^{2\epsilon\sqrt{A/T}} \left\{ d\hat{\zeta}^2 + \sigma_1^2 + \sigma_2^2 + \frac{1}{\lambda^2} \sigma_3^2 \right\},
\]

(3.9)

where \( T = \frac{\ln(2e\epsilon\sqrt{A/T} \text{harm}(\omega \gamma))}{2\epsilon\sqrt{A/T}} \) and we denote \( e^{2A(\zeta_0/2)} = e^{2B(\zeta_0/2)} = e^{-2C(\zeta_0/2)} \) by \( \lambda \to 0^+ \).

Metric (3.9) clearly shows a bolt structure at \( \hat{\zeta} = 0 \) in which both Ricci scalar and Kretschman invariant diverge as \( \frac{1}{\hat{\zeta}} \) and \( \frac{1}{\hat{\zeta}^2} \), respectively.
Finally, we consider the geometry of solutions (3.4) where $\epsilon = +1$ at future infinity. The metric in the limit of large values of $t$ approaches

$$ds^2 = -dt^2 + \mu e^{\mu t} ds_B^2,$$

where $T = \frac{\ln t}{\mu}$, $\mu = 2 \sqrt{\frac{A}{3}}$ and $ds_B^2$ is given by (2.1). The equal time hypersurfaces of (3.10) are Bianchi spaces. Metric (3.10) shows the big bang patch which covers half of the $dS$ spacetime from a big bang at the past horizon to the Bianchi hypersurfaces at future infinity. The other half of the $dS$ spacetime is covered by the big crunch patch that includes the Bianchi hypersurfaces at past infinity to a big crunch at future horizon. This contracting patch of solutions (3.4) at future infinity implies that black hole solutions based on Bianchi type IX space can describe the coalescence of black holes in asymptotically $dS$ spacetimes.

As is well known, there is a natural correspondence between phenomena occurring near the boundary (or in the deep interior) of asymptotically $dS$ (or $AdS$) spacetime and UV (IR) physics in the dual CFT. Hence, any solutions in asymptotically (locally) $dS$ spacetimes lead to interpretation in terms of renormalization group flows and the associated generalized $dS$ $c$-theorem. In any contracting patch of $dS$ spacetimes, the renormalization group flows toward the infrared and in any expanding patch, it flows toward the ultraviolet. The $c$-function for representation of the $dS$ metric with a wide variety of boundary geometries involving direct products of flat space, the sphere and hyperbolic space was studied in [22]. For our five-dimensional exact cosmological solution (3.4), the $c$-function is $c = (G_{\mu \nu} n^\mu n^\nu)^{-2}$ where $n^\mu$ is the unit vector along the time direction. The $c$-function should show an increasing (decreasing) behavior versus time for any expanding (contracting) patch of the spacetime. The explicit expression for $c$-function is given by

$$c^{-2/3}(t, \zeta) = \frac{3}{2} \left( \frac{\partial H(t, \zeta)}{dt} \right)^2 + 3 e^{-2(A(t) + B(t) + C(t))} \left( \frac{\partial H(t, \zeta)}{\partial \zeta} \right)^2 - 2 H(t, \zeta) \frac{\partial^2 H(t, \zeta)}{\partial \zeta^2},$$

which shows that the $c$-function depends on time as well as $\zeta$. We consider three different values for $\zeta$ and evaluate the corresponding $c$-function as function of time. First, we consider $\zeta = \zeta_0/2 \approx 1.60$ that is the smallest value of $\zeta$. Second, we set $\zeta = 2.50$ and finally

![Figure 3. The $c$-function/$10^3$ versus time (for $\epsilon = +1$) at three different $\zeta$-fixed slices, where we set $c = 1$ and $k = 1/2$. The up and down solid curves are $c$-functions at $\zeta = 3.19$ and 1.60 slices, respectively. The dashed curve shows the $c$-function for slice at $\zeta = 2.50.$](image-url)
we consider $\zeta = \zeta_0 \simeq 3.19$. In figure 3, we plot the $c$-functions for these three different fixed values of $\zeta$ where we set $\epsilon = +1$. The plots show explicitly the expansion of $\zeta$-fixed slices, in perfect agreement with what we concluded after equation (3.8). Moreover, setting $\epsilon = -1$ yields the decreasing $c$-functions of figure 4, again in agreement with our previous conclusions.

4. Conclusions

In this paper, we have constructed an exact cosmological class of non-stationary solutions to five-dimensional minimal supergravity, based on Bianchi type IX Einstein–hyper-Kähler space. The Bianchi type IX Einstein–hyper-Kähler space does not have any tri-holomorphic $U(1)$ isometries; hence, the solutions could be used to study the physical processes that do not respect any tri-holomorphic Abelian symmetries. The solutions are regular everywhere except on the location of bolt in four-dimensional Bianchi type IX base space. We note the $c$-function for our solutions depends on cosmological time as well as radial coordinate of Bianchi type IX base space. For any fixed value of coordinate $\zeta$, the $c$-function has monotonically increasing (or decreasing) behavior with time depending on $\epsilon = +1$ (or $\epsilon = -1$). The behavior of the $c$-function is in perfect agreement with asymptotic reduction of solutions to expanding (or collapsing) patches of five-dimensional dS spacetimes. We also note that the geometry of solutions at future infinity contains a contracting patch. This implies that black hole solutions based on Bianchi type IX space can describe the coalescence of black holes in asymptotically dS spacetimes. We leave the black hole solutions based on Bianchi type IX space as well as the thermodynamics of solutions and application of dS/CFT correspondence to the solutions presented in this paper, for a future paper.

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Appendix A

The Maurer–Cartan one-forms in (2.1) are given respectively by

\[ \sigma_1 = d\psi + \cos \theta \, d\phi, \]  
\[ \sigma_2 = -\sin \psi \, d\theta + \cos \psi \sin \theta \, d\phi, \]  
\[ \sigma_3 = \cos \psi \, d\theta + \sin \psi \sin \theta \, d\phi, \]

and satisfy

\[ d\sigma_i = \frac{1}{2} \varepsilon_{ijk} \sigma_j \wedge \sigma_k. \]

We note that the metric on the \( \mathbb{R}^4 \) (with a radial coordinate \( R \) and Euler angles \( (\theta, \phi, \psi) \) on an \( S^3 \)) could be written in terms of Maurer–Cartan one-forms via

\[ ds^2 = dR^2 + \frac{R^2}{4}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2), \]

with \( \sigma_1^2 + \sigma_2^2 + \sigma_3^2 \) is the standard metric of \( S^2 \) with unit radius; \( 4(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) \) gives the same for \( S^3 \).

The Bianchi type IX metric (2.1) satisfies Einstein equations provided

\[ \frac{d^2 A}{d\zeta^2} = \frac{1}{2} \{ e^{4A} - (e^{2B} - e^{2C})^2 \}, \]  
\[ \frac{d^2 B}{d\zeta^2} = \frac{1}{2} \{ e^{4B} - (e^{2C} - e^{2A})^2 \}, \]  
\[ \frac{d^2 C}{d\zeta^2} = \frac{1}{2} \{ e^{4C} - (e^{2A} - e^{2B})^2 \}, \]

as well as

\[ \frac{dA}{d\zeta} \frac{dB}{d\zeta} + \frac{dB}{d\zeta} \frac{dC}{d\zeta} + \frac{dC}{d\zeta} \frac{dA}{d\zeta} = \frac{1}{2} \{ e^{2(A+B)} + e^{2(B+C)} + e^{2(C+A)} \} - \frac{1}{4} \{ e^{4A} + e^{4B} + e^{4C} \}. \]

Integrating equations (A.6), (A.7), (A.8) and (A.9), we get equations (2.2), (2.3) and (2.4) where \( \alpha_i \)'s are three integration constants.

Appendix B

The first set of solutions to equations (2.2), (2.3) and (2.4) corresponds to \( (\alpha_1, \alpha_2, \alpha_3) = (1, 1, 1) \). The solutions describe the Atiyah–Hitchin metric [23] in the general form of (2.1).

The metric functions are

\[ e^{2A(\zeta)} = \frac{2}{\pi} \frac{\partial_2 \partial_3 \partial_4}{\partial_2 \partial_3 \partial_4}, \]  
\[ e^{2B(\zeta)} = \frac{2}{\pi} \frac{\partial_2 \partial_3 \partial_4}{\partial_2 \partial_3 \partial_4}, \]  
\[ e^{2C(\zeta)} = \frac{2}{\pi} \frac{\partial_2 \partial_3 \partial_4}{\partial_2 \partial_3 \partial_4}. \]
where the \( \vartheta \)'s are Jacobi theta functions with complex modulus \( i\xi \). The Jacobi theta functions \( \vartheta \) are given by \( \vartheta_i(\tau) = \vartheta_i(0|\tau) \), where we have used the Jacobi–Erdélyi notation \( \vartheta_i(\tau) = \vartheta_i[0]_i(\tau) \), \( \vartheta_i(\tau) = \vartheta_i[0]_i(\tau) \), \( \vartheta_i(\tau) = \vartheta_i[0]_i(\tau) \) and \( \vartheta_i(\tau) = \vartheta_i[0]_i(\tau) \).

The Jacobi theta functions with characteristics \( \vartheta_i[0]_i(\tau) \) are defined by the following series:

\[
\vartheta_i[0]_i(\tau) = \sum_{n \in \mathbb{Z}} e^{i\pi(n - 1/2)\tau (n - 1/2 - \tau)}.
\]

(B.4)

where \( a \) and \( b \) are two real numbers.

The other possible values for the integration constants \( \alpha_1, \alpha_2 \) and \( \alpha_3 \) are \((1, -1, -1), (-1, 1, -1) \) and \((-1, -1, 1) \). However, in all these three cases, a redefinition of metric functions change the solutions into the Atiyah–Hitchin metric with metric functions (B.1), (B.2) and (B.3).

Appendix C

The second set of solutions to equations (2.2), (2.3) and (2.4) corresponds to \((\alpha_1, \alpha_2, \alpha_3) = (0, 0, 0) \). We find that the differential equations (2.2), (2.3) and (2.4) can be solved exactly and the solutions are given by [14]

\[
A(\xi) = \frac{1}{2} \ln \left( c^2 \frac{\cn(c^2 \xi, k^2) \dn(c^2 \xi, k^2)}{\sn(-c^2 \xi, k^2)} \right),
\]

(C.1)

\[
B(\xi) = \frac{1}{2} \ln \left( c^2 \frac{\sn(c^2 \xi, k^2) \dn(c^2 \xi, k^2)}{\cn(c^2 \xi, k^2)} \right),
\]

(C.2)

\[
C(\xi) = \frac{1}{2} \ln \left( c^2 \frac{\dn(c^2 \xi, k^2)}{\cn(c^2 \xi, k^2) \dn(-c^2 \xi, k^2)} \right),
\]

(C.3)

where \( \sn(z, k), \cn(z, k) \) and \( \dn(z, k) \), the standard Jacobian elliptic \( SN \), \( CN \) and \( DN \) functions, are related to \( \am(z, k) \), the Jacobian elliptic \( AM \) function, by

\[
\sn(z, k) = \sin(\am(z, k)),
\]

(C.4)

\[
\cn(z, k) = \cos(\am(z, k)),
\]

(C.5)

\[
\dn(z, k) = \sqrt{1 - k^2 \sn^2(z, k)}.
\]

(C.6)

The Jacobian elliptic \( AM \) function is the inverse of the trigonometric form of the elliptic integral of the first kind, which means \( \am(f(\sin \phi, k), k) = \phi \), where \( f(\psi, k) \), the elliptic integral of the first kind, is defined by

\[
f(\psi, k) = \int_0^{\sin^{-1}(\psi)} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}.
\]

(7)

We change the coordinate \( \zeta \) in metric (2.1) to the coordinate \( r \) by [14]

\[
r = \frac{2c}{\sqrt{\sn(c^2 \zeta, k^2)}}.
\]

(8)

The metric then changes from (2.1) into (2.5) with one metric function (2.6) only. The latter form of metric is more convenient to be considered in special cases, as we consider here. In a special case of \( k = 0 \), where \( a_1 \) and \( a_2 \) coincide, we get the metric

\[
ds^2 = \frac{dr^2}{h(r)} + \frac{r^2}{4} (d\psi^2 + \sin^2 \theta \, d\phi^2) + \frac{r^2}{4h(r)} (d\phi + \cos \theta \, d\phi)^2.
\]

(9)
which is the Eguchi–Hanson type I metric with \( h(r) = \left(1 - \frac{(2c)^4}{r^4}\right)^{1/2} \). The Eguchi–Hanson type I metric can also be written as [24]

\[
\text{ds}^2 = \tilde{f}(r)^2 \, \text{dr}^2 + \frac{r^2}{4} \tilde{g}(r) \left[ \text{d}\theta^2 + \sin^2 \theta \, \text{d}\phi^2 \right] + \frac{r^2}{4} \left( \text{d}\psi + \cos \theta \, \text{d}\phi \right)^2, 
\]

(C.10)

where

\[
\tilde{f}(r) = \frac{1}{2} \left( 1 + \frac{1}{\sqrt{1 - a^4 r^4}} \right),
\]

\[
\tilde{g}(r) = \sqrt{\frac{1}{2} \left( 1 + \sqrt{1 - a^4 r^4} \right)}.
\]

(C.11)

In the other extreme case where \( k = 1, a_2 \) is equal to \( a_3 \) and we obtain the Eguchi–Hanson type II metric

\[
\text{ds}^2 = \frac{dr^2}{h^2(r)} + \frac{r^2}{4} h^2(r) \sigma_1^2 + \frac{r^2}{4} \left( \sigma_2^2 + \sigma_3^2 \right),
\]

(C.12)

which is of the same form of the well-known Eguchi–Hanson metric

\[
\text{ds}^2 = \frac{r^2}{4g(r)} \left[ \text{d}\psi + \cos(\theta) \, \text{d}\phi \right]^2 + g(r) \, \text{dr}^2 + \frac{r^2}{4} \left( \text{d}\theta^2 + \sin^2(\theta) \, \text{d}\phi^2 \right),
\]

(C.13)

by making the substitution \( 2c = a \) and \( h(r) = \frac{1}{\sqrt{h^2(r)}} \) in (C.12). We note that only for special values of \( k = 0 \) and \( k = 1 \), metric (2.5) admits a tri-holomorphic \( U(1) \) isometry, hence could be put into the Gibbons–Hawking form. In both special cases of \( k = 0 \) and \( k = 1 \), the five-dimensional supergravity solutions can be constructed simply by four harmonic functions on the base space. The case with \( k = 1 \) was considered explicitly in [25] and [26], where the authors constructed five-dimensional supersymmetric black ring solutions as well as 11-dimensional solutions on the four- and six-dimensional hyper-Kähler Eguchi–Hanson type II base spaces, respectively. Their solutions have the same two angular momentum components and the asymptotic structure on time slices is locally Euclidean. The circle-direction of the black ring is along the equator on a two-sphere bolt on the base space. The case with \( k = 0 \) gives a separable five-dimensional metric for Eguchi–Hanson type I manifold with a time direction. The most general five-dimensional supergravity solutions with parameter \( k \) varies as \( 0 < k \leq 1 \) were studied in [14].

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