Dark energy as a spatial continuity condition

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ABSTRACT

Context. Observational evidence of dark energy that makes the Universe nearly flat at the present epoch is very strong.

Aims. We study the link between spatial continuity and dark energy.

Methods. We assume that comoving space is a compact 3-manifold of constant curvature, described by a homogeneous Friedman-Lemaître-Robertson-Walker metric. We assume that spatial continuity cannot be violated, i.e. that the global topology of the comoving section of the Universe cannot change during post-quantum epochs.

Results. We find that if the Universe was flat and compact during early epochs, then the presently low values of the radiation and matter densities imply that dark energy was created as a spatial continuity effect. Moreover, if the Universe is compact, then $\Omega_{\text{tot}} \equiv 1$ is dynamically stable, where $\Omega_{\text{tot}}$ is the total density parameter in units of the critical density.

Conclusions. Dark energy was observationally detected as a geometrical phenomenon. It is difficult to imagine a simpler explanation for dark energy than spatial continuity, finiteness and homogeneity.

Key words. Cosmology: theory – cosmological parameters – large-scale structure of Universe – early Universe

1. Introduction

The flatness problem (e.g. Sect. 3.1, Liddle 1999, and references therein) has long been a challenge for modern cosmology. Given the baryon density provided by galaxies as a lower limit to the total matter-energy density $\Omega_{\text{tot}}$ [expressed in units of the critical density $\rho_{\text{crit}} \equiv 3H^2/(8\pi G)$, where $H$ is the Hubble parameter and $G$ is the gravitational constant], why is the Universe within an order of magnitude or two of the critical density, which is the density at which the Universe is flat? Inflationary scenarios provide one possible answer to both this problem and to some other cosmological problems. However, during the 1990’s, observational evidence using many different observational strategies (Fukugita et al. 1990; Fort et al. 1997; Chiba & Yoshii 1997; Perlmutter et al. 1999) established the existence of “dark energy”, which together with matter density makes the Universe nearly flat (total density within about ten percent of the critical density) at the present epoch. More recent evidence constrains the total density to within about one percent of the critical density (Spergel et al. 2007; Reichardt et al. 2008). This is the new flatness problem.

Inflationary scenarios may well provide the correct explanation of why the density of the Universe is presently within a few orders of magnitude of the critical density. However, for primordial inflation to lead to a present-day inflationary epoch that takes over from matter domination just as the matter density is dropping below the critical density constitutes a new fine-tuning problem. Many hypotheses have been raised in order to provide an explanation for dark energy (e.g., Caldwell & Kamionkowski 2009) but the simplest possible explanation seems to have been overlooked.

The Friedman-Lemaître-Robertson-Walker models of the Universe implicitly assume that the comoving spatial section of the Universe is a 3-manifold. This assumption does not constitute “new physics”. It could better be described as “implicit physics”. Does this assumption have any consequences for dark energy?

In Sect. 2 our assumptions are presented. In Sect. 3 the consequences are examined. Discussion and conclusions are presented in Sect. 4 and Sect. 5.

2. Method

2.1. The dark energy/flatness problem

First consider the local properties of Friedman-Lemaître-Robertson-Walker (FLRW) models. Here, “local” refers to the fact that the Einstein equations describe the properties in the limit towards any arbitrary point $p$ in comoving space (averaged on sufficiently “large” scales), i.e. properties that apply at every $p$ in $M$, where the comoving space is a constant curvature Riemannian 3-manifold $M$. Let us write the Friedman equation as

$$H^2 = H_0^2 [\Omega_{\text{r}0}a^{-4} + \Omega_{\text{m}0}a^{-3} + \Omega_{\text{k}0}a^{-2} + \Omega_{\Lambda 0}a^{-3-3\omega}]$$ (1)

where the present values of matter-energy densities are written relative to the critical density $\rho_{\text{crit}}$ with the subscript “0” to indicate the present epoch $[a(t_0) \equiv a_0 \equiv 1]$, i.e. the radiation energy density is $\Omega_{\text{r}0}$, the non-relativistic matter density is $\Omega_{\text{m}0}$, dark energy is $\Omega_{\Lambda 0}$, and the curvature “density parameter” $\Omega_{\text{k}0}$ is the present day value of curvature density parameter $\Omega_{\text{k}}$ defined

$$\Omega_{\text{k}}(a) \equiv 1 - \Omega_{\text{r}}(a) - \Omega_{\text{m}}(a) - \Omega_{\Lambda}(a)$$ (2)
Fig. 1. Density parameters relative to the critical density from well into the radiation-dominated epoch at \( a = 10^{-10} \) to the present, showing \( \Omega_{\text{tot}} \) (solid line), \( \Omega_r \) (dotted line), \( \Omega_m \) (dotted-dashed line), \( \Omega_{\Lambda} \) (dashed line), where \( \Omega_{m0} = 4.95 \times 10^{-5}, \Omega_{m0} = 0.3, \Omega_{\Lambda0} = 0.715, w = -1 \). See Eqs (3) and (1).

at an arbitrary scale factor \( a \) (cf. Eqs (13.3), (13.4), Peebles 1993). We can write the evolution of these components in terms of their present-epoch values, allowing \( w \equiv p/\rho \) to parameterise a more general dark energy model than that of a cosmological constant:

\[
\begin{align*}
\Omega_r &= \frac{\Omega_{r0} a^{-4}}{(H/H_0)^2} \\
\Omega_m &= \frac{\Omega_{m0} a^{-3}}{(H/H_0)^2} \\
\Omega_k &= \frac{(1 - \Omega_{m0} - \Omega_{r0} - \Omega_{\Lambda0}) a^{-2}}{(H/H_0)^2} \\
\Omega_{\Lambda} &= \frac{\Omega_{\Lambda0} a^{-3-3w}}{(H/H_0)^2}. \quad (3)
\end{align*}
\]

The curvature radius (real for positive curvature, imaginary for negative curvature) can be written

\[
R_c^2 = \left( \frac{c}{H} \right)^2 \frac{1}{(-\Omega_k)}. \quad (4)
\]

Figure 1 shows the evolution of the radiation density, the matter density and dark energy with \( w = -1 \), i.e. a cosmological constant, from well before the Universe became matter-dominated to the present. The cosmological constant at \( a = 10^{-10} \) had to be \( \Omega_{\Lambda} \approx 1.82 \times 10^{-37} \) at that epoch in order to have just the right value in order to make the Universe close to flat today. The matter density at that epoch was \( \Omega_m(10^{-10}) \approx 6.42 \times 10^{-7} \). How was it possible for the physical content of the Universe at \( a = 10^{-10} \) to “know” that dark energy, which at the time was 30 orders of magnitude less dense than matter, should be within a few percent of the required values in order that it would start dominating the Universe just when matter density would drop significantly below unity?

This is the new fine-tuning problem, which is not solved by inflationary scenarios. The specific numbers cited here can be modified slightly by including a neutrino density or other estimates of \( \Omega_m, \Omega_{m0} \) and/or \( \Omega_{\Lambda0} \) consistent with present observational constraints, but the fine-tuning problem is only altered slightly. Inflationary scenarios are normally thought to finish much earlier than \( a = 10^{-10} \), so the required “conspiracy” between \( \Omega_m \) and \( \Omega_{\Lambda} \) is even worse than what is shown in Fig. 1. For example, if we ignore the matter-dominated epoch in order to make an order-of-magnitude calculation and write \( a \sim (t/t_0)^{1/2} \), then \( (H/H_0) \sim (t/t_0)^{-1} \) and a post-inflationary epoch of \( t \sim 10^{-30} \) s gives \( \Omega_{\Lambda} \sim \Omega_{\Lambda0}(H/H_0)^2 \sim \Omega_{\Lambda0}(t/t_0)^{-2} \sim 10^{-95} \). How was it possible that a dark energy density nearly 10 orders of magnitude smaller than the critical density was “waiting” in preparation for a future epoch when it would take over from matter in order to flatten the Universe at just the right moment?

### 2.2. The assumption that comoving space is a 3-manifold

The FLRW models of the Universe implicitly assume that the comoving spatial section of the Universe is a 3-manifold. An additional assumption that is often made implicitly, without any discussion nor theoretical or observational evidence, is that this 3-manifold is topologically trivial (has a trivial \( \pi_1 \) homotopy group). However, this assumption is arbitrary. The covering space of comoving space for an FLRW model is either \( \mathbb{H}^3 \), \( \mathbb{R}^3 \), or \( S^3 \) for negative, zero, or positive curvature respectively. The different possible fundamental groups of holonomy transformations \( \Gamma \) for these three covering spaces give the FLRW constant curvature Riemannian 3-manifolds of interest in cosmology, i.e. \( \mathbb{H}^3/\Gamma \), \( \mathbb{R}^3/\Gamma \), or \( S^3/\Gamma \) respectively for the three signs of the curvature. The groups \( \Gamma \) are, in general, non-isomorphic for different curvatures.

The arbitrary nature of this trivial topology assumption is first known to have been raised by Karl Schwarzschild (Schwarzschild 1900, 1998). It has been analysed most intensively during the last decade and a half (see Lachieze-Rey & Luminet 1995; Luminet 1998; Starkman 1998; Luminet & Roukema 1999; Blankoeil & Roukema 2000, and references therein) and different observational strategies for measuring cosmic topology (the topology of the 3-manifold of comoving space) have been developed and classified (e.g., Uzan et al. 1999; Luminet & Roukema 1999; Roukema 2002; Rebouças & Gomero 2004).

Some theoretical work that might constitute the basis for deciding which 3-manifold should be favoured by a theory of quantum cosmology has been carried out (Masafumi 1996; Anderson et al. 2004). A recent heuristic result is that of the dynamical effect of cosmic topology in the presence of density perturbations. A residual weak limit gravitational effect in the presence of a density perturbation selects well-proportioned spaces in general, and the Poincaré dodecahedral space \( S^3/I^* \) in particular, to be special in the sense of being “better balanced” (Roukema et al. 2007; Roukema & Rózanski 2009). Recent empirical analyses have mostly focussed on the Wilkinson Microwave Anisotropy Probe (WMAP) all-sky maps of the cosmic microwave background. The results are presently inconclusive, ranging from excluding detectable cosmic topology (Cornsilk et al. 2004; Key et al. 2007), preferring a simply connected infinite flat space (Niarcho & Jaffe 2007; Lew & Roukema 2008), preferring the Poincaré dodecahedral space over simply connected infinite flat space (Luminet et al. 2003; Aurich et al. 2005a, 2005b; Gundermann 2005;
Cailierie et al. 2007; Roukema et al. 2008a, 2008b), or preferring the regular, flat torus $T^3$ over simply connected infinite flat space (Spergel et al. 2003; Aurich et al. 2007; Aurich 2008; Aurich et al. 2008, 2009).

It is clear that even if cosmic microwave background maps from Planck Surveyor give evidence against both the Poincaré space and 3-torus models, there is at present no strong theoretical or empirical argument requiring comoving space to be either simply connected or multiply connected.

### 2.3. The assumption of spatial continuity

Now consider the assumption that the global topology of the comoving section of the Universe cannot change during post-quantum epochs. Could an infinite $\mathbb{R}^3$ comoving space suddenly become finite $T^3$? Or vice versa? Or could space evolve from one finite space to another? For example, would the following be physically reasonable during an epoch well past the quantum epoch? Changes corresponding to “cutting” vast square (comoving) gigaparsec surfaces of the Universe and then “sticking” them back together in a new way, in order to transform comoving space from a flat space whose fundamental domain is (for example) the hexagonal prism (before cutting) to $T^3$ (after “sticking” back together the cut surfaces), would have had to have occurred. Or could a $T^3$ comoving space be cut apart, stretched and curved slightly and then stuck back together to make one of the spherical spaces $S^3/\Gamma$, where $\Gamma$ is a group of holonomy transformations of $S^3$? At the quantum epoch, a superposition of various 3-manifold states whose probabilities depend on operators might be reasonable, but not at post-quantum epochs.

In principle, during post-quantum epochs for a comoving observer, black holes modify the topology of comoving space. However, astrophysically realistic black holes are relatively tiny and isolated from one another from the point of view of “average” comoving space on scales of hundreds of megaparsecs and above. Moreover, they form mostly at very recent epochs. Structure formation in the standard cosmological model does not give any hint that black holes could lead to modification of the topology of the comoving spatial section of the Universe. Therefore, for the purposes of this work, let us cut out a neighbourhood of radius 100 Schwarzschild radii around every black hole (whether stellar, intermediate or supermassive) and replace it by an homogeneous solid ball.

Hence, let us make the physically reasonable assumption that global topology change at post-quantum epochs is physically impossible. In other words, we assume comoving spatial continuity.

### 2.4. Compact comoving space

Objects of infinite size or mass are generally disliked in physics. Comoving space itself might be an exception to the rule. However, finite, continuous space without any boundaries provides a more conventional physical model than infinite space (Levin et al. 1998). For this reason, we focus on compact (finite) 3-manifolds (except for Sect. 3.1.2). These include multiply connected flat and hyperbolic 3-manifolds, and spherical 3-manifolds of any topology.

### 2.5. Intuitive tools for thinking about 3-manifolds

See Lachièze-Rey & Luminet (1995); Luminet (1998); Starkman (1998); Luminet & Roukema (1999); Blanlœil & Roukema (2000) for general reviews about 3-manifolds of interest to cosmology. Here, we note in particular that there are at least three different ways of thinking about a constant curvature 2-manifold, of which two are relatively easy to use for 3-manifolds. We focus on the two dimensional flat torus $T^2$ as an illustrative example.

(i) $T^2$ can be thought of embedded in $\mathbb{R}^3$, i.e. as “the surface of a doughnut with a hole in the middle”, but given its own intrinsic (flat) metric rather than the metric of $\mathbb{R}^3$. The continuity of $T^2$ is obvious, but the constant curvature of its metric is less obvious.

(ii) $T^2$ can be thought of as a fundamental domain, i.e. a square (if it is a regular $T^2$ model) with identified edges. The constant (zero) curvature is now obvious, but continuity may seem less obvious.

(iii) $T^2$ can be thought of as the covering space or apparent space $\mathbb{R}^2$, which contains (infinitely) many copies of $T^2$. The continuity and constant zero curvature are now obvious, at the cost of the existence of multiple images of any single physical point of space.

For 3-manifolds, method (i) would require intuition in at least $\mathbb{R}^4$ (and at most $\mathbb{R}^7$; Whitney 1936). For understanding the physical consequences of comoving space being a 3-manifold, it is generally useful to switch between methods (ii), thinking of a 3-manifold as a fundamental domain (polyhedron) with faces identified in a certain way, and (iii) thinking of the covering space ($\mathbb{H}^3$, $\mathbb{R}^3$ or $S^3$) tiled by identically shaped (in a metric sense) copies of the fundamental domain. The 3-manifold itself is, of course, physically identical no matter which method is used for modelling it cognitively.

### 3. Results

#### 3.1. Zero curvature during post-quantum epochs

#### 3.1.1. $T^3$

First consider the flat case. In particular, let us start with $T^3 = \mathbb{R}^3/\mathbb{Z}^3$. That is, let us suppose that at some post-quantum epoch, comoving space was $T^3$. For example, this was probably determined by the end of the quantum epoch. Suppose also that there is no present-epoch dark energy, i.e. $\Omega_\Lambda = 0$.

Now suppose that at much more recent epochs, the various components of matter-energy density dropped in density so that the density came to be dominated by the matter density alone, i.e. we arrive at the observed value $\Omega_{m_0} \approx 0.3$. By the Friedman equation, this implies that the curvature became significantly negative. Hence, our assumption of no topology change implies that comoving space evolved from $T^3 = \mathbb{R}^3/\mathbb{Z}^3$ to $\mathbb{H}^3/\mathbb{Z}^3$.

However, $\mathbb{H}^3/\mathbb{Z}^3$ as a constant curvature 3-manifold does not exist (see e.g. Best 1971). An elementary proof is given in Appendix A. Either comoving space became highly inhomogeneous on the largest scales possible, or comoving space remained homogeneous and remained $\mathbb{R}^3/\mathbb{Z}^3$. Since we assume an FLRW model, we ignore the former possibility and are forced to accept the latter.
How is it possible for space to have remained homogeneous $\mathbb{R}^3/\mathbb{Z}^3$ even though the matter density dropped to $\Omega_{m0} \approx 0.3$? The obvious explanation is the constant of integration allowed by solving the Einstein equations. A cosmological constant, or more generally, dark energy $\Omega_\Lambda$, allows the equations to find a solution that conserves spatial continuity and homogeneity even though the evolution of matter-energy density alone in the absence of dark energy would require a violation of either homogeneity or spatial continuity. In other words, if comoving space is finite, continuous and close to homogeneous and retains these three conditions as it evolves, then dark energy is a stretching effect necessarily induced by these conditions. Hereafter, we refer to this as “continuity dark energy”.

This is somewhat analogous to solving a general physical equation with a given set of boundary conditions. Multiply connected space is sometimes described as having “periodic boundary conditions”. However, this is a somewhat misleading method of thinking about the nature of multiply connected space, since there is no physical repetition of space-time events, only an effect similar to mirroring. Rather than referring to “continuity dark energy” as a “boundary condition effect”, it would be more physically useful to refer to it as a “spatial continuity effect”.

This can be rewritten as follows. If the Universe was $T^3$ when it exited the quantum epoch, then since that epoch, it has at different epochs been in one of the following three states:

1. **highly inhomogeneous**
2. homogeneous and “supported” as flat space by one or more forms of “bottom-up” matter-energy density (e.g. photons, or dark matter particles and baryons, or more exotic particles or fields at earlier epochs); or
3. homogeneous and flattened by “top-down” dark energy which compensated exactly for “bottom-up” forms of matter-energy density that were insufficient to “support” flatness.

Since $S^3/\mathbb{Z}^3$ does not exist either (the fundamental groups of spherical constant curvature 3-manifolds are finite, but $\mathbb{Z}$ is infinite), this reasoning applies even if some exotic form of matter-energy density would (in the absence of dark energy) try to increase the density to super-critical during certain epochs. Similar arguments apply for other flat compact spaces than $T^3$ (e.g., Sect. 6.2 Lachièze-Rey & Luminet 1995).

A useful way to think of this is in terms of the (comoving) fundamental domain. Provided we have the assumptions of homogeneity, spatial continuity and compactness, the curvature can change from one negative value to another or one positive value to another, but the sign (negative, zero, or positive) cannot change. To change the sign of curvature, at least one of these assumptions would have to be violated.

3.1.2. Stability of $\Omega_{\text{tot}} = 1$

It is often stated (e.g. Sect. 3.1, Liddle 1999) that $\Omega_{\text{tot}} = 1$ is an unstable point for FLRW models. This statement is partially true if the topology of comoving space is trivial. If comoving space is $\mathbb{R}^3$, then it can smoothly evolve to $\mathbb{R}^3$, or vice versa, without violating spatial continuity nor requiring large inhomogeneities to be created. Given the Friedman equation and the standard behaviour expected of matter-density components, it is not easy to change from a negatively curved space to a perfectly flat space or vice-versa. However, spatial continuity does not prevent this. Exotic fields could in principle allow this evolution without violating either homogeneity or continuity.

On the other hand, evolving from $\mathbb{R}^3$ to $S^3$ would require infinite comoving space to be suddenly wrapped up into a finite hypersphere using an extremely inhomogeneous transformation (radially inhomogeneous with respect to a given “centre”), with the addition of a new point at the antipode, or the placing of a black hole there. The opposite transformation would be equally unappealing. In other words, even with the assumption of trivial topology, the continuity requirement prevents evolution between $\Omega_{\text{tot}} \leq 1$ and $\Omega_{\text{tot}} > 1$. In this sense, $\Omega_{\text{tot}}$ might be described as semi-stable or asymmetrically stable.

Now consider compact flat space. For simplicity, consider $T^3$. Now suppose that, for example, density perturbations shift the average total density to slightly higher or lower than the critical density at a given cosmological time. This would *not* cause the Universe to shift further and further away from $\Omega_{\text{tot}} = 1$, since continuity and homogeneity would force dark energy to (on average) compensate for the slight (temporary?) curvature. Perturbations or changes in the dominant matter-energy components of the Universe cannot change $T^3 = \mathbb{R}^3/\mathbb{Z}^3$ into either $\mathbb{R}^3/\mathbb{Z}^3$ or $S^3/\mathbb{Z}^3$, since neither of the latter two spaces exist. Similar arguments apply for the other compact, flat spaces.

Hence, if the Universe is compact, then $\Omega_{\text{tot}} = 1$ is *dynamically stable*. Compact comoving flat space cannot evolve away from $\Omega_{\text{tot}} = 1$ without violating either homogeneity or spatial continuity. The disagreement between this result and previous statements in the literature is that previous work did not consider the fundamentally distinct nature of 3-manifolds of different curvatures. The stabi-
ity of $\Omega_{\text{tot}}$ for the different cases is shown schematically in 

This link between cosmic topology and dynamics should not be confused with the residual gravity effect (Roukema et al. 2007; Roukema & Rőzanski 2009). These constitute two very different mechanisms linking topology and dynamics.

### 3.2. Positive curvature during post-quantum epochs

Now consider the positively curved case, where at some post-quantum epoch, comoving space was $S^3/\Gamma$. The choice of $\Gamma$ is unlikely to play a role for the “creation” of dark energy, so let us adopt the Poincaré dodecahedral space $S^3/I^*$ as suggested empirically for the present epoch by the WMAP data, and theoretically by the residual gravity effect. Is it possible for this model to evolve by the present epoch to the present observed value $\Omega_m \approx 0.3$ in the absence of dark energy, i.e. with $\Omega_\Lambda = 0$? In the absence of adding a further exotic dark energy component, the Friedman equation already prevents evolution from positive curvature to negative curvature.

However, if some exotic component were expected to “push” space from positive to negative curvature, then this would imply that comoving space evolved from $S^3/I^*$ to $\mathbb{R}^3/I^*$ to $\mathbb{H}^3/I^*$ as the curvature changed. Our assumption of no topology change prevents this; the binary icosahedral group $I^*$ is a finite group, but the fundamental group $\Gamma$ of $\mathbb{R}^3/I^*$ or $\mathbb{H}^3/I^*$ must be infinite. Clearly, the constant curvature 3-manifolds $\mathbb{R}^3/I^*$ and $\mathbb{H}^3/I^*$ do not exist.

Nevertheless, the addition of an exotic dark energy component that in turn requires continuity dark energy would constitute a more complicated model rather than a simpler model. The present dark energy would be a continuity effect but a new primordial dark energy component would be required. At least for the simplest models of the Universe, dark energy does not arise as a consequence of continuity in a positively curved model.

### 3.3. Negative curvature during post-quantum epochs

One optimal member of the set of 3-manifolds is the Weeks space (Weeks 1985; Fagundes 1993), which minimises the volume of negatively curved constant curvature 3-manifolds for a fixed curvature radius. Suppose that the quantum epoch selected the Weeks space or another hyperbolic compact 3-manifold. Since the present estimate of the matter-energy density in the absence of dark energy is $\Omega_{\text{tot}} \approx \Omega_m \approx 0.3$, the evolution from primordial epochs to the present would not require any violation of homogeneity nor spatial continuity, since the evolution would be from $\mathbb{H}^3/I$ during a primordial epoch to $\mathbb{H}^3/I^*$ at the present for the same group of holonomy transformations $\Gamma$, with only an evolution in the curvature radius.

Similarly to the positive curvature case, in order for spatial continuity to induce dark energy if space is $\mathbb{H}^3/I^*$, an additional component of the matter-energy density that would tend to force the total density from below critical to above critical would have to exist. In other words, for “continuity dark energy” to exist in a negatively curved space, an additional, as yet unknown form of matter-energy density would also need to exist.

Figure 3. Schematic diagram showing how geometry can induce a physical effect with no requirement for “new physics”. A fluid of constant density flows at constant speed $v$ through a pipe of initially constant cross-sectional area $A$ through to a bottleneck of new constant cross-sectional area $A/4$. Its new speed is $4v$. The change in speed can be attributed either to the onset of a dark energy epoch, or to geometrical change together with the assumptions of spatial continuity, finiteness and homogeneity.

Again, this would constitute a more complicated model rather than a simpler model. By Occam’s razor, compactness, homogeneity, spatial continuity and the observed dark energy favour a flat, compact model in order for dark energy to arise as a spatial continuity effect without requiring the addition of any “new physics”.

### 4. Discussion

#### 4.1. A physical analogy: fluid flow of an homogeneous, incompressible material

Figure 3 shows a loose analogy for dark energy as a spatial continuity condition. The flow of a fluid composed of an homogeneous incompressible material from a length of pipe with a wide cross-section to a length of pipe with a narrow cross-section necessarily causes the speed of the fluid to increase in proportion to the change in cross-sectional area. The apparent creation of “dark energy” causing the fluid to accelerate is purely a continuity effect. No new model of particle physics nor braneworld model is required to explain the acceleration of fluid in a pipe that narrows.

This analogy is not perfect. Instead of comoving space changing its shape, it retains its shape, while the matter density drops below what is required by the Einstein equations in order to conserve the sign of the curvature. Nevertheless, the analogy may help illustrate why spatial continuity provides a simple physical understanding of what we refer to as dark energy.

#### 4.2. Exact flatness versus approximate flatness

The surface of the Earth is not an exact 2-sphere, and the orbits of the planets in our Solar System are not exact circles. It is equally unrealistic for the Universe to have had a curvature exactly equal to zero everywhere, in the mathematical sense of exactness. Density perturbations certainly exist today, and at small enough scales, particles do not consist of a genuinely uniform fluid.

On the other hand, it is quite possible that topological evolution during the quantum epoch led to the 3-manifold of comoving space being $T^3$. The residual gravity effect se-
lects the Poincaré dodecahedral space $S^3/I_*$ as a better balanced space than $T^3$ (Roukema & Rožański 2009), but for the sake of argument, let us suppose that the unknown theory of the quantum epoch led to comoving space being $T^3$. In this case, how do we reconcile the nature of $T^3$ as an idealised mathematical object being perfectly flat and a more realistic physical model which is very nearly flat, but contains some tiny perturbations and most likely has an “average” curvature (depending on the way that the average is calculated) which is “slightly” different from zero? In fact, there is no contradiction. From a topological point of view, $T^3$ with density perturbations is a mathematically valid 3-manifold. What happens when we apply the Friedman equation? To the best of our experimental knowledge, the Einstein equations provide an excellent approximation to reality. However, interpreting the Friedman equation assuming perfect homogeneity and a “slightly” non-zero curvature would imply that the curvature evolves further and further away from zero. Yet, as noted above, this is not possible without violating either homogeneity or spatial continuity. Retaining spatial continuity and approximate homogeneity solves the dilemma: a tiny amount of stretching with the properties of dark energy would result in order to keep the average curvature close to zero.

So the difference between idealised perfect flatness and approximate flatness is only a semantic problem that occurs if curvature is discussed without reference to topology.

If comoving space is $T^3$ and close to homogeneous as assumed in the FLRW models, then in this sense it “is flat” and will always stay “flat” unless a global topology change occurs. However, the particular values of the average curvature at a given epoch may differ from zero and will depend on the averaging method and on the details of how information on the spatial continuity requirement is spread throughout comoving space, i.e. on the details of how continuity dark energy responds to the changes in matter-energy density. So the same space could be considered “flat in a topological sense” while being “non-flat” in terms of $\Omega_{\text{tot}}$.

On the other hand, if comoving space “is approximately flat” in the sense that it is a compact, non-zero curvature 3-manifold, i.e. $S^3/I$ or $\mathbb{H}^3/I$ with a small average curvature at a given epoch, then this is physically very different from being “topologically flat”. No exotic form of matter-energy density, including an inflaton, can make it become “topologically flat” or switch it to a 3-manifold of the opposite curvature, unless we drop the assumption of spatial continuity.

4.3. The present-epoch fine-tuning problem (coincidence problem)

Continuity dark energy does not require any conspiracy. If we consider the $T^3$ model, then the “knowledge” encoded at $a = 10^{-10}$ was the “knowledge” that comoving space is $T^3$ and that any average deviations from perfect flatness “needed” to be compensated. The precise way in which this occurs (e.g. a field theory) is unknown, but it must occur unless we wish to drop approximate homogeneity or spatial continuity. The real evolution of $\Omega_\Lambda$ with $a$ might be quite different, but it would have to evolve in a way that space remains “approximately” flat. The physical way in which “approximately” is encoded in comoving space should determine the way in which $\Omega_\Lambda$ really evolved with $a$.

4.4. Possible consequences for galaxy formation

The Friedmann equation together with spatial continuity do not constrain in what way continuity dark energy information is communicated throughout comoving space. They only require that this occurs. Presumably, the information is to some degree represented everywhere locally, and changes in the information are transmitted at (or below) the speed of the space-time conversion constant $c$.

Did the stretching process occur uniformly? Continuity dark energy could reasonably have occurred inhomogeneously on large-scale-structure scales, i.e. at about 100 comoving megaparsecs and below, provided that the (global) continuity of comoving space and approximate homogeneity conditions were satisfied. As density perturbations collapsed into the cosmic web, especially filaments and massive clusters at the knots where the filaments intersect, it would seem reasonable that the stretching occurred more in the voids and less along filaments and at knots, since higher density regions are more tightly bound than lower density regions. This may provide the solution to the “void phenomenon” (Peebles 2001), without requiring any modification of gravity. Other puzzles in galaxy formation (e.g. Perivolaropoulos 2008) for the concordance model (Ostriker & Steinhardt 1995) could also potentially be resolved by continuity dark energy and/or offer clues for distinguishing continuity dark energy from the exotic dark energy models required in a simply connected, infinite flat space or in curved spaces.

4.5. Consequences for primordial inflationary scenarios

For a flat, compact universe, since dark energy as a continuity effect explains why the Universe is exactly flat in the 3-manifold sense, it also explains why the Universe is “close” to flat today, i.e. it resolves the order-of-magnitude flatness problem. However, the other arguments in favour of primordial inflation scenarios (the horizon problem, the magnetic monopoles problem) remain unchanged, so primordial inflation is certainly not excluded.

Moreover, in a flat compact space, the continuity argument also applies during primordial (post-quantum) epochs. If the dominant matter-energy density at an early epoch entered a phase where in the absence of dark energy the curvature would have become negative, then space must have been stretched unless continuity or homogeneity were violated. The question of the nature of the inflaton then becomes a search for the dominant matter-energy components prior to the primordial inflationary epoch, since the inflaton itself would just be a spatial continuity effect. It is also interesting that Linde (2004) has argued that under certain conditions, a compact hyperbolic or flat universe would be preferred in inflationary scenarios, i.e. in agreement with the spatial continuity requirement in the case of flat models, though not in the case of non-flat models.

5. Conclusion

The key assumptions required here are that

(i) comoving space was a compact 3-manifold after exiting the quantum epoch;
(ii) during post-quantum epochs, the Universe cannot have been ripped apart and pasted back together again in a different way (spatial continuity); and
Comoving spatial continuity provides an explanation for the present epoch cosmological constant without requiring any “new physics”, provided that the 3-manifold of comoving space is a compact, flat 3-manifold, such as \( T^3 \) [see Sect. 6.2, Lachi`eze-Rey & Luminet (1995), for other compact, flat 3-manifolds]. The same effect would shift the search for an “inflaton” to the search for a pre-inflationary matter-energy density that in the absence of the continuity requirement would cause the Universe to become hyperbolic.

This effect does not require multiple connectedness to be observable today by direct methods. However, the missing fluctuations problem in COsmic Background Explorer (COBE) and WMAP maps of the cosmic microwave background has become stronger with the release of the five-year COBE and WMAP maps of the cosmic microwave background (e.g. Copi et al. 2007, 2008) and would be simply explained by living in an observably compact space. Several analyses of the WMAP data favour a Poincaré dodecahedral space \((S^3/T^3)\) over the infinite, simply connected flat model (Luminet et al. 2003; Aurich et al. 2005a, 2005b; Gundermann 2005; Caillerie et al. 2007; Roukema et al. 2008a, 2008b). On the other hand, recent work has shown that a \( T^3 \) model (Spergel et al. 2003; Aurich et al. 2007; Aurich et al. 2008, 2009) also provides a better fit to the empirical constraints than the infinite, simply connected flat model. Given that dark energy is explained as a continuity effect in a compact flat space without requiring any “new physics”, Occam’s razor suggests that a \( T^3 \) model of comoving side length about 12h^{-1} \, \text{Gpc} (e.g. Aurich et al. 2009) and fundamental directions in the directions listed in Table 1 of Aurich (2008) may be our present best model of comoving space.

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References

Anderson, M., Carlip, S., Ratcliffe, J. G., Surya, S., & Tschantz, S. T. 2004, ClassQuartGra, 21, 729, [arXiv:gr-qc/0310002]
Aurich, R. 2008, ClassQuartGra, 25, 225017, [arXiv:0803.2130]
Aurich, R., Janzer, H. S., Lustig, S., & Steiner, F. 2008, Classical and Quantum Gravity, 25, 125006, [arXiv:0708.1420]
Aurich, R., Lustig, S., & Steiner, F. 2005a, ClassQuartGra, 22, 3443, [arXiv:astro-ph/0504650]
Aurich, R., Lustig, S., & Steiner, F. 2005b, ClassQuartGra, 22, 2061, [arXiv:astro-ph/0412569]
Aurich, R., Lustig, S., & Steiner, F. 2009, ArXiv e-prints, [arXiv:0903.3133]

Appendix A: Does the constant curvature 3-manifold \( \mathbb{H}^3/\mathbb{Z}^3 \) exist?

Suppose that the constant curvature 3-manifold \( \mathbb{H}^3/\mathbb{Z}^3 \) exists. By symmetry, let us represent it using \( \mathbb{R}^3 \) as a coordi-
Consider the cube in \( \mathbb{R}^3 \) composed of \((2n+1)^3\) copies of the fundamental domain, centred at the origin, as shown in Fig. 4. The surface area of this cube consists of \(6(2n+1)^2\) faces of the fundamental domain. Since every copy of the fundamental domain has exactly the same shape, in the metric sense, this surface area is

\[ A_{\text{big}} = 6(2n+1)^2A, \quad \text{(A.1)} \]

where \(A\) is the surface area of one face of the fundamental domain of \(\mathbb{H}^3/\mathbb{Z}^3\). Since space is hyperbolic, \(A > L^2\) and \(L^2/A\) are fixed values, independent of \(n\).

However, every one of the \(6(2n+1)\) faces of copies of the fundamental domain on the surface of this large cube is at a distance of at least \((n+1/2)L\) from the origin. Hence, the surface area of the large cube must be

\[ A'_{\text{big}} > 4\pi\{R \sinh[(n+1/2)L/R]\}^2. \quad \text{(A.2)} \]

Since \(A, L\) and \(R\) are fixed values, a large value of \(N\) can be found such that

\[ n > N \Rightarrow A'_{\text{big}}(n) > A_{\text{big}}(n). \quad \text{(A.3)} \]

However, \(A'_{\text{big}} = A_{\text{big}}\) by definition. Hence, \(\mathbb{H}^3/\mathbb{Z}^3\) cannot exist.