Statistical Errors in the Measurement of Particle Thresholds

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Simple rules of thumb are derived for the precision with which s-wave and p-wave thresholds can be determined by a series of equally spaced cross section measurements near threshold. Backgrounds and beam spread are ignored.

An important question for a linear collider is the precision with which particle thresholds can be measured. The thresholds in question may have either s-wave or p-wave behavior, that is, the cross section may rise as \( p_1 \) or \( p_3 \). Measurements of the production cross section near the threshold determine particle masses quite well, but the question is how well. The result must depend on the number of events registered, \( N \), and the interval over which the data are taken. The beamstrahlung will degrade the nearly monochromatic beam, but the effective beam energy spectrum still has a delta function at the nominal energy. The 50% or so of the beam that has its energy reduced by 5% or so simply becomes a source of a diffuse background. Because the highest energy component of the beam occurs in the delta function, the sharp threshold in the s-wave survives. For our purposes we ignore the beamstrahlung altogether. This means that our estimates are too optimistic. We also ignore the effect of the possible widths of the produced particles. Detailed threshold measurements might even measure these widths \[1\].

Here we suppose the energy is set at evenly spaced discrete values and a fixed amount of luminosity is delivered. Let us suppose that at the \( i \)th energy value the expected number of events given the cross section with some trial values for the parameters and the delivered luminosity is \( n_i^{\text{exp}} \). If the measured number of events is \( n_i^{\text{obs}} \), then probability of this set of measurements with these parameters is

\[
P = \prod_i e^{-n_i^{\text{exp}}} \frac{n_i^{\text{exp}}}{n_i^{\text{obs}}} \tag{1}
\]

Let us suppose that we accumulate an integrated luminosity \( L \) at each of \( N + 1 \) energies, \( E_0 + (i/N)\Delta E, \ i = 0, 1, ... N \). We suppose that the threshold occurs between \( E_0 \) and \( E_1 \). We can take the log-likelihood function \( \mathcal{L} = \ln P + \text{constant} \) to be a function of the number of observed events, \( n_i^{\text{obs}} \), and the number of expected events, \( n_i^{\text{exp}} \), at each energy:

\[
\mathcal{L} = \sum_{i=1}^{N} n_i^{\text{obs}} \ln(n_i^{\text{exp}}/n_i^{\text{obs}}) - n_i^{\text{exp}} + n_i^{\text{obs}} \tag{2}
\]

where we have added the \( n_i^{\text{obs}} \) term so that \( \mathcal{L} \) vanishes when \( n_i^{\text{obs}} = n_i^{\text{exp}} \). The values of \( n_i^{\text{exp}} \) depend on the parameters describing the cross section.

Now the expected uncertainty in a parameter is given by

\[
\sigma_{\alpha}^{-2} = -\frac{\partial^2 \mathcal{L}}{\partial \alpha^2} = \sum_{i=1}^{N} \frac{1}{n_i^{\text{exp}}} \left( \frac{\partial n_i^{\text{exp}}}{\partial \alpha} \right)^2 \tag{3}
\]

evaluated at the true value of \( \alpha \).

Here we shall be interested solely in the threshold for the process. The cross section can be written

\[
\sigma = \sigma_0 ((E - 2m)/E_{\text{thr}})^\nu \tag{4}
\]

where \( 2m = E_{\text{thr}} \) is the trial value of the threshold. However, in order not to mix the unknown value of \( E_{\text{thr}} \) with the parameter \( m \), we replace \( E_{\text{thr}} \) in the denominator by \( E_0 \), where \( E_0 \) is the lowest energy setting, the one just below threshold. Thus \( E_0 \) differs from \( E_{\text{thr}} \) typically by less than a percent.
We write \( \overline{m} = E_0 + \lambda \Delta E \) where \( \overline{m} \) is the true mass and where \( 0 < \lambda < (1/N) \). By hypothesis, the first non-zero contribution is for \( i = 1 \).

\[
\sigma_m^{-2} = \sum_{i=1}^{N} \frac{4L \sigma_0 \nu^2}{E_0^2} \left( \frac{E_i - 2 \overline{m}}{E_0} \right)^{\nu - 2} = \frac{4L \sigma_0 \nu^2}{E_0^2} \epsilon^{\nu - 2} \sum_{i=1}^{N} \left( \frac{i}{N} - \lambda \right)^{\nu - 2}
\]

where \( \epsilon = \Delta E/E_0 \).

In the p-wave case \( \nu = 3/2 \) and we have

\[
\sigma_m^{-2} = \frac{9L \sigma_0}{E_0^2} \epsilon^{-1/2} N^{1/2} \sum_{i=1}^{N} (i - \lambda N)^{-1/2} = \frac{18L \sigma(\Delta E)N}{\Delta E^2} f_p(\lambda N, N)
\]

where we noted that \( \sigma_0 \epsilon^{3/2} = \sigma(\Delta E) \) is the cross section at the upper end of the \( \Delta E \) interval and where

\[
f_p(z, N) = \frac{1}{2 \sqrt{N}} \sum_{i=1}^{N} (i - z)^{-1/2}
\]

See Fig. 1. Thus we find

\[
\sigma_m = \frac{\Delta E}{\sqrt{18N L \sigma(\Delta E) f_p(\lambda N, N)}}
\]

Martyn and Blair [1] give an example of \( e^{-} e^{+} \rightarrow \tilde{\mu}_R \tilde{\mu}_R \), with \( \Delta E = 10 \text{ GeV} \), \( \sigma(\Delta E) = 10 \text{ fb} \), \( N = 10 \), \( L = 10 \text{ fb}^{-1} \). With these values we find

\[
\sigma_m = 0.0745 \text{ GeV}/\sqrt{f_p(\lambda N, N)}
\]

For \( \lambda = 0 \), this gives 0.084 GeV, while Martyn and Blair give 0.09 GeV.

In the s-wave case

\[
\sigma_m^{-2} = \frac{L \sigma_0}{E_0^2} \epsilon^{-3/2} \sum_{i=1}^{N} \left( \frac{i}{N} - \lambda \right)^{-3/2}
\]

We define

\[
f_s(z, N) = \frac{1}{\zeta(3/2)} \sum_{i=1}^{N} (i - z)^{-3/2}
\]

where the zeta function is defined by \( \zeta(s) = \sum_{n=1}^{\infty} n^{-s} \) so that \( f_s(0, \infty) = 1 \) See Fig. 1. We then find

\[
\sigma_m^{-2} = \frac{L \sigma_0 N}{\Delta E^2} \epsilon^{1/2} N^{1/2} \zeta(3/2) f_s(\lambda N, N)
\]

FIG. 1: The functions \( f_p(z, N) \) and \( f_s(z, N) \) for two values of \( N \)
TABLE I: Comparison of Eq.(13) and toy Monte Carlo study with $N = 10$ data points for an s-wave threshold near 255 GeV, with $\sigma_0 = 800$ fb and $10 \text{ fb}^{-1}$ at each point, corresponding to the study of Martyn and Blair [1] for $e^- e^+ \rightarrow \chi^- \chi_1^+$. Martyn and Blair give an uncertainty of 0.04 GeV for a threshold of 255.0 GeV. As the threshold is placed closer and closer to the first data point, 256 GeV, the resolution steadily improves. The conservative estimate is obtained by taking $\lambda N = 0$.

| Threshold (GeV) | $\lambda N$ | $f_s(\lambda N, N)$ | $\sigma_m(\text{GeV})$ | $\sigma_m(\text{GeV})$ |
|-----------------|-------------|----------------------|------------------------|------------------------|
| 255.0           | 0.0         | 0.77                 | 0.030                  | 0.030                  |
| 255.1           | 0.1         | 0.85                 | 0.030                  | 0.029                  |
| 255.5           | 0.5         | 1.59                 | 0.022                  | 0.021                  |
| 255.7           | 0.7         | 2.91                 | 0.016                  | 0.015                  |
| 255.9           | 0.9         | 12.8                 | 0.007                  | 0.007                  |

and

$$
\sigma_m = \frac{\Delta E}{\sqrt{N L \sigma(\Delta E)}} N^{-1/4} [\zeta(3/2) f_s(\lambda N, N)]^{-1/2}
$$

(13)

where we noted that $\sigma_0 \epsilon^{1/2} = \sigma(\Delta E)$ is the cross section at the upper end of the $\Delta E$ interval.

For the example given by Martyn and Blair [1], $L = 10 \text{ fb}^{-1}$, $\sigma(\Delta E) = 160$ fb, $N = 10$, we find

$$
\sigma = 0.0276[f_s(\lambda N, N)]^{-1/2}
$$

(14)

A comparison of a toy Monte Carlo study and Eq.(13) is shown in the Table.

The functions $f_s$ and $f_p$ reflect the possible fortuitousness of our scan. If we get a cross section measurement just above threshold, we get a much smaller error. The conservative errors are obtained by setting $\lambda = 0$ in these relations. We see that since the total luminosity used in the measurement is $NL$, the p-wave measurement doesn’t depend directly on the choice of $N$, while the s-wave measurement benefits weakly from increasing $N$.

With the approximations

$$
f_s(0, N) = 1 - \frac{0.77}{\sqrt{N}}
$$

$$
f_p(0, N) = 1 - \frac{0.72}{\sqrt{N}}
$$

(15)

we have the conservative estimates

$$
\sigma_p(m) = \frac{\Delta E}{\sqrt{18 NL \sigma(\Delta E)}} [1 + \frac{0.36}{\sqrt{N}}]
$$

$$
\sigma_s(m) = \frac{\Delta EN^{-1/4}}{\sqrt{2.612 NL \sigma(\Delta E)}} [1 + \frac{0.38}{\sqrt{N}}]
$$

(16)

[1] H.-U. Martyn and G. A. Blair, hep-ph/9910416.
[2] See, for example, G. A. Blair, in these proceedings.