THE LIMITS OF SOLUTIONS OF A LINEAR DELAY INTEGRAL EQUATION

KAZUKI HIMOTO AND HIDEAKI MATSUNAGA*

Department of Mathematical Sciences, Osaka Prefecture University
Sakai 599-8531, Japan

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Abstract. In this paper we classify the limits of solutions of a linear integral equation with finite delay. In particular, if the solution tends to a point or a periodic orbit, we establish the explicit expressions depending on given initial functions by using analysis of characteristic roots and the formal adjoint theory. Our results also present a necessary and sufficient condition for the exponential stability of the equation.

1. Introduction. In the last half century, qualitative theories of Volterra integral equations and Volterra integro-differential equations have undergone developments rapidly. It has been strongly promoted by many applications that these theories have found in physics, engineering, and biology. For the general background of Volterra integral equations, one can refer to some books [2, 3, 5].

We consider a linear integral equation with finite delay

\[ x(t) = A \int_{t-h}^{t} x(s)ds, \quad t \geq 0 \]  

with the initial condition

\[ x(\xi) = \varphi(\xi), \quad -h \leq \xi \leq 0, \]

where \( A \) is a real \( m \times m \) constant matrix, \( h \) is a positive constant, and the initial function \( \varphi \) belonging to \( L^1([-h, 0], \mathbb{R}^m) \) is given arbitrarily. Recently, integral equations with finite or infinite delay are studied by several authors; one can refer to [1, 4, 6, 7, 8, 9] and the references therein. The purpose of this paper is to completely classify the limits of solutions of (1.1) in terms of eigenvalues of \( A \) and the delay parameter \( h \), and to present asymptotic formulae of the solutions. More precisely, if the solution of (1.1) tends to a point or a periodic orbit, we establish the explicit expressions depending on the initial function \( \varphi \).

Our research is motivated by the following stability result (Theorem A) and asymptotic formula of solutions (Theorem B) for the equation of convolution type

\[ x(t) = \int_{t-h}^{t} G(t-s)x(s)ds, \quad t \geq 0, \]

where the kernel \( G \) is a measurable real \( m \times m \) matrix valued function on \([0, h]\).

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* Corresponding author: Hideaki Matsunaga.
Theorem A. ([9, Proposition 2]) Suppose that
\[ \sup\{\|G(t)\| \mid 0 \leq t \leq h\} h < 1. \]
Then the zero solution of (1.3) is exponentially stable, that is, every solution of (1.3) converges to 0 exponentially as \( t \to \infty \).

Theorem B. ([8, Example 4.1]) Suppose that \( m = 1 \) and \( G(t) \) is a nonnegative function satisfying \( \int_0^h G(t)dt = 1 \). Let \( x(t; \varphi) \) be a solution of (1.3) with (1.2) where \( \varphi \in L^1([-h,0], \mathbb{R}) \). Then
\[ \lim_{t \to \infty} x(t; \varphi) = \frac{1}{\gamma} \int_{-h}^0 G(-\xi) \int_{\xi}^0 \phi(\tau)d\tau d\xi, \]
where \( \gamma = \int_0^h tG(t)dt \).

Judging from these theorems above, we have planned the classification of the limits of solutions of (1.1). By the transformation \( x(t) = Py(t) \) with a nonsingular matrix \( P \), Eq. (1.1) can be written as
\[ y(t) = P^{-1}AP \int_{t-h}^t y(s)ds, \quad t \geq 0. \]
Consequently, we may assume without loss of generality that the matrix \( A \) is the Jordan canonical form. Moreover, throughout this paper, we treat the essential case where \( A \) is the \( 2 \times 2 \) matrix given in the following three cases:

(I) \( A = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \), (II) \( A = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix} \), (III) \( A = a \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \),

where \( a_1, a_2, a \) and \( \theta \) are real numbers and \( 0 < |\theta| \leq \pi/2 \). In fact, one can easily extend our results to higher dimensional cases.

Our main results are stated as follows.

**Theorem 1.1.** Suppose that the matrix \( A \) is of the form (I). Let \( x(t; \varphi) \) be a solution of (1.1) with (1.2) where \( \varphi \in L^1([-h,0], \mathbb{R}^2) \). Then the following statements hold:

(i) If \( a_1h < 1 \) and \( a_2h < 1 \), then \( x(t; \varphi) \) converges to 0 exponentially as \( t \to \infty \).

(ii) If \( a_1h = 1 \) and \( a_2h < 1 \), then \( x(t; \varphi) \) converges to \( \text{col}(c_1^{(1)}, 0) \) exponentially as \( t \to \infty \).

(iii) If \( a_1h < 1 \) and \( a_2h = 1 \), then \( x(t; \varphi) \) converges to \( \text{col}(0, c_1^{(2)}) \) exponentially as \( t \to \infty \).

(iv) If \( a_1h = a_2h = 1 \), then \( x(t; \varphi) \) converges to \( c_1 \) exponentially as \( t \to \infty \).

(v) If \( a_1h > 1 \) or \( a_2h > 1 \), there exists an unbounded solution of (1.1). Here \( c_1^{(1)} = c_1^{(1)}(\varphi) \), \( c_1^{(2)} = c_1^{(2)}(\varphi) \) and \( c_1 = c_1(\varphi) \) are given by
\[ c_1 = \begin{pmatrix} c_1^{(1)} \\ c_1^{(2)} \end{pmatrix} = \frac{2}{h^2} \int_{-h}^0 \int_{\xi}^0 \varphi(\tau)d\tau d\xi. \]

**Theorem 1.2.** Suppose that the matrix \( A \) is of the form (II). Let \( x(t; \varphi) \) be a solution of (1.1) with (1.2) where \( \varphi \in L^1([-h,0], \mathbb{R}^2) \). Then the following statements hold:

(i) If \( ah < 1 \), then \( x(t; \varphi) \) converges to 0 exponentially as \( t \to \infty \).

(ii) If \( ah \geq 1 \), there exists an unbounded solution of (1.1).
**Theorem 1.3.** Suppose that the matrix $A$ is of the form (III). Let $x(t; \varphi)$ be a solution of (1.1) with (1.2) where $\varphi \in L^1([-h, 0], \mathbb{R}^2)$. Then the following statements hold:

(i) If $(|\theta| - \pi)/\sin \theta < ah < \theta/\sin \theta$, then $x(t; \varphi)$ converges to 0 exponentially as $t \to \infty$.

(ii) If $ah = \theta/\sin \theta$ or $ah = (|\theta| - \pi)/\sin \theta$, then $|x(t; \varphi) - \tilde{\Phi}_3(t)c_3|$ converges to 0 exponentially as $t \to \infty$.

(iii) If $ah > \theta/\sin \theta$ or $ah < (|\theta| - \pi)/\sin \theta$, there exists an unbounded solution of (1.1).

Here $\tilde{\Phi}_3(t)$ and $c_3 = c_3(\varphi)$ are given by

$$
\tilde{\Phi}_3(t) = \begin{pmatrix} \cos \omega^* t & -\sin \omega^* t \\ \sin \omega^* t & \cos \omega^* t \end{pmatrix}, \quad \omega^* = \begin{cases} \frac{2|\theta|}{h} \text{sgn}(\theta), & ah = \frac{\theta}{\sin \theta}, \\
\frac{2(|\theta| - \pi)}{h} \text{sgn}(\theta), & ah = \frac{|\theta| - \pi}{\sin \theta}, \end{cases}
$$

$$
c_3 = \left(-\int_{-h}^{0} \xi \tilde{\Phi}_3(\xi) d\xi\right)^{-1} \int_{-h}^{0} \int_{\xi}^{0} \tilde{\Phi}_3(\xi - \tau) \varphi(\tau) d\tau d\xi.
$$

By virtue of Theorems 1.1, 1.2 and 1.3, one can immediately obtain the following stability criterion for (1.1) (Corollary 1.1) and asymptotic formula of solutions of (1.1) with $m = 1$ (Corollary 1.2), which may be considered as some improvements of Theorems A and B for (1.3), respectively, in the case where $G(t)$ is constant.

**Corollary 1.1.** Let $a_j e^{i \theta_j}$ ($j = 1, 2, \ldots, m$) be the eigenvalues of the real $m \times m$ matrix $A$, where $a_j$ and $\theta_j$ (possibly 0) are real numbers and $|\theta_j| \leq \pi/2$. Then the zero solution of (1.1) is exponentially stable if and only if

$$
\begin{cases}
a_j h < 1, & \theta_j = 0, \\
\frac{|\theta_j| - \pi}{\sin |\theta_j|} < a_j h < \frac{\theta_j}{\sin \theta_j}, & \theta_j \neq 0 \quad \text{for } j = 1, 2, \ldots, m.
\end{cases}
$$

**Corollary 1.2.** Suppose that $m = 1$, and $A = a$ is a real number. Let $x(t; \varphi)$ be a solution of (1.1) with (1.2) where $\varphi \in L^1([-h, 0], \mathbb{R})$. Then the following statements hold:

(i) If $ah < 1$, then $x(t; \varphi)$ converges to 0 exponentially as $t \to \infty$.

(ii) If $ah = 1$, then $x(t; \varphi)$ converges to $c$ exponentially as $t \to \infty$.

(iii) If $ah > 1$, there exists an unbounded solution of (1.1).

Here $c = c(\varphi)$ is given by

$$
c = \frac{2}{h^2} \int_{-h}^{0} \int_{\xi}^{0} \varphi(\tau) d\tau d\xi.
$$

Note that, to the best of our knowledge, these corollaries above are fundamental but unknown results for linear integral equations with finite delay.

This paper is outlined as follows. In section 2, we summarize preparatory results on linear integral equations to establish an explicit asymptotic formula of solutions of (1.1). In section 3, we study the distributions of the characteristic roots of (1.1) in detail, which is an important role in our research. Finally in section 4, we give proofs of main results.
2. Preliminaries. In this section we will introduce the decomposition theory and the formal adjoint theory for linear autonomous integral equations with delay developed in [7, 8].

Let \( \rho \) be a fixed positive constant and let \( \mathbb{R}^- \) be the set of nonpositive real numbers. Denote by \( X \) the function space defined by

\[
X = \{ \varphi : \mathbb{R}^- \to \mathbb{C}^m \mid \varphi(\xi)e^{\rho \xi} \text{ is integrable on } \mathbb{R}^- \}
\]

with norm \( \| \varphi \| = \int_{-\infty}^{0} |\varphi(\xi)|e^{\rho \xi}d\xi \) for \( \varphi \in X \). For any function \( x : (\mathbb{R}^- \to \mathbb{C}^m) \) and \( t < b \), we define a function \( x_t : \mathbb{R}^- \to \mathbb{C}^m \) by \( x_t(\xi) = x(t + \xi) \) for \( \xi \in \mathbb{R}^- \).

Let us consider the linear integral equation with infinite delay

\[
x(t) = \int_{-\infty}^{t} K(t-s)x(s)ds, \tag{2.1}
\]

where the kernel \( K \) is a measurable \( m \times m \) matrix valued function on \([0, \infty)\) with complex components satisfying the conditions

\[
\int_{0}^{\infty} \|K(t)\|e^{\rho t}dt < \infty \quad \text{and} \quad \text{ess sup} \{\|K(t)\|e^{\rho t} \mid t \geq 0\} < \infty.
\]

Eq. (2.1) can be formulated as an autonomous equation on \( X \) of the form

\[
x(t) = L(x_t), \tag{2.2}
\]

where \( L \) is a bounded linear operator defined by \( L(\varphi) = \int_{-\infty}^{0} K(-\xi)\varphi(\xi)d\xi \) for \( \varphi \in X \). For any \( \varphi \in X \), a function \( x : (\mathbb{R}^- \to \mathbb{C}^m) \), \( b > 0 \), is called a solution of (2.2) (or (2.1)) with the initial function \( \varphi \) if \( x \) satisfies the following conditions:

\[
x(\xi) = \varphi(\xi) \quad \text{for} \quad \xi \in \mathbb{R}^-, \quad x \text{ is locally integrable on } [0, b], \quad \text{and} \quad x(t) = L(x_t) \quad \text{for} \quad t \in (0, b).
\]

It is known by [7, Proposition 3] that there exists a unique global solution \( x : \mathbb{R} \to \mathbb{C}^m \) of (2.2) such that \( x_0 = \varphi \) on \( \mathbb{R}^- \), which is called the solution of (2.2) through \((0, \varphi)\), and denoted by \( x(t; \varphi) \).

For any \( t \geq 0 \) and \( \varphi \in X \), the solution operator \( T(t) : X \to X \) is defined by the relation \( T(t)\varphi = x_t(\cdot; \varphi) \). Since the family \( \{T(t)\}_{t \geq 0} \) is a strongly continuous semigroup of linear operators on \( X \), the generator \( A_0 \) of \( \{T(t)\}_{t \geq 0} \) is characterized as follows:

\[
\mathcal{D}(A_0) = \{ \varphi \in X \mid \varphi(\xi) = \tilde{\varphi}(\xi) \text{ a.e.} \xi \in \mathbb{R}^- \text{ for some } \tilde{\varphi} \in \tilde{X} \},
\]

\[
A_0\varphi = (d/d\xi)\tilde{\varphi}, \quad \varphi \in \mathcal{D}(A_0),
\]

where \( \tilde{X} = \{ \tilde{\varphi} \in X \mid \tilde{\varphi} \text{ is locally absolutely continuous on } \mathbb{R}^-, \ (d/d\xi)\tilde{\varphi} \in X \text{ and } \tilde{\varphi}(0) = L(\tilde{\varphi}) \} \).

Denote by \( \sigma(A_0) \), \( P_\rho(A_0) \) and \( \text{ess}(A_0) \) the spectrum, the point spectrum and the essential spectrum of \( A_0 \), respectively. Let \( \mathbb{C}_{-\rho} = \{ z \in \mathbb{C} \mid \text{Re} z > -\rho \} \), and introduce the characteristic equation for (2.1) defined by

\[
\text{det} \Delta(z) = 0, \quad \Delta(z) = E_m - \int_{0}^{\infty} K(t)e^{-zt}dt, \tag{2.3}
\]

where \( E_m \) is the \( m \times m \) unit matrix. We call \( \lambda \in \mathbb{C}_{-\rho} \) such that \( \text{det} \Delta(\lambda) = 0 \) a characteristic root of (2.1). Then the spectra of \( A_0 \) are characterized as

\[
\sigma(A_0) \cap \mathbb{C}_{-\rho} = P_\rho(A_0) \cap \mathbb{C}_{-\rho} = \{ \lambda \in \mathbb{C}_{-\rho} \mid \text{det} \Delta(\lambda) = 0 \}
\]

and \( \sup\{\text{Re} \lambda \mid \lambda \in \text{ess}(A_0)\} \leq -\rho \).
Let $\Sigma^\text{cu} = \{ \lambda \in \sigma(A_0) \mid \text{Re}\lambda \geq 0 \}$ and $\Sigma^* = \sigma(A_0) \setminus \Sigma^\text{cu}$. Then $\Sigma^\text{cu} \cap \text{ess}(A_0) = \emptyset$ and $\Sigma^\text{cu}$ is a finite set, and hence $X$ is decomposed as a direct sum
\[ X = X^\text{cu} \oplus X^*, \]
where $X^\text{cu}$ and $X^*$ are closed subspaces satisfying the following properties:
\[ \dim X^\text{cu} < \infty, \quad \sigma(A_0|_{X^\text{cu}}) = \Sigma^\text{cu} \quad \text{and} \quad \sigma(A_0|_{X^* \cap D(A_0)}) = \Sigma^*. \]
Denote by $\Pi^\text{cu}$ the projection from $X$ onto $X^\text{cu}$. Then we have the following result on the asymptotic behavior of solutions of (2.1).

**Proposition 2.1.** ([8, Corollary 2.1]) The following statements hold:

(i) If $\Sigma^\text{cu} = \emptyset$, then the zero solution of (2.1) is exponentially stable.

(ii) If $\Sigma^\text{cu} \neq \emptyset$, then the solution $x(t; \varphi)$ of (2.1) tends to $x_*(t) = x(t; \Pi^\text{cu}\varphi)$ exponentially as $t \to \infty$. More precisely, there exist constants $K_1 \geq 1$ and $\beta > 0$ such that
\[ |x(t; \varphi) - x_*(t)| \leq K_1 e^{-\beta t} \|\varphi\|, \quad t > 0, \quad \varphi \in X. \]

Let $\lambda_0$ be a characteristic root of (2.1), and let $\mathcal{M}_\lambda(A_0) = \bigcup_{k \geq 1} \mathcal{N}((A_0 - \lambda_0 I)^k)$. $\mathcal{M}_\lambda$ is called the generalized eigenspace of $A_0$ for $\lambda_0$. If there exists the smallest positive integer $p$ such that $\mathcal{N}((A_0 - \lambda_0 I)^p) = \mathcal{N}((A_0 - \lambda_0 I)^{p+1})$, then $p$ is called the ascent of $\lambda_0$, where $\mathcal{N}((A_0 - \lambda_0 I)^p)$ denotes the null space of the operator $(A_0 - \lambda_0 I)^p$. The following result on the ascent of $\lambda_0$ and the dimension of $\mathcal{M}_\lambda(A_0)$ is satisfied.

**Proposition 2.2.** ([8, Proposition 2.4 and Corollary 3.1]) Let $\lambda_0 \in \mathbb{C}_{-\rho}$ and $\det \Delta(\lambda_0) = 0$, and let $q$ be the order of $\lambda_0$ as a zero of $\det \Delta(z)$. Then the ascent of $\lambda_0$ does not exceed $q$, and $\dim \mathcal{M}_{\lambda_0} = \dim \mathcal{N}_1 = q$ holds. In particular, $X$ is decomposed as
\[ X = \mathcal{R}((A_0 - \lambda_0 I)^q) \oplus \mathcal{N}((A_0 - \lambda_0 I)^q), \]
where $\mathcal{R}((A_0 - \lambda_0 I)^q)$ denotes the range of the operator $(A_0 - \lambda_0 I)^q$.

Next we introduce the formal adjoint theory for (2.1) to give the explicit form of $x(t; \Pi^\text{cu}\varphi)$. Let $\mathbb{R}^+$ be the set of nonnegative real numbers and let $\mathbb{C}^{m*}$ be the space of all $m$-dimensional row vectors. Consider the function space $X^\sharp$ defined by
\[ X^\sharp = \{ \psi : \mathbb{R}^+ \to \mathbb{C}^{m*} \mid \psi(s)e^{-\rho s} \text{ is integrable on } \mathbb{R}^+ \} \]
with the norm $\|\psi\| = \int_0^\infty |\psi(s)|e^{-\rho s}ds$ for $\psi \in X^\sharp$. Also we set
\[ D(A_0^\sharp) = \{ \psi \in X^\sharp \mid \psi(s) = \tilde{\psi}(s) \text{ a.e. } s \in \mathbb{R}^+ \text{ for some } \tilde{\psi} \in \tilde{X}^\sharp \}, \]
\[ A_0^\sharp \psi = -(d/ds)\tilde{\psi}, \quad \psi \in D(A_0^\sharp), \]
where $\tilde{X}^\sharp = \{ \tilde{\psi} \in X^\sharp \mid \tilde{\psi} \text{ is locally absolutely continuous on } \mathbb{R}^+, \ (d/ds)\tilde{\psi} \in X^\sharp \text{ and } \tilde{\psi}(0) = \int_0^\infty \tilde{\psi}(s)K(s)ds \}$.

We call the operator $A_0^\sharp$ the formal adjoint operator of $A_0$. Furthermore, let us consider the bilinear form $\langle \cdot, \cdot \rangle$ on $X^\sharp \times X$ defined by
\[ \langle \psi, \varphi \rangle = \int_{-\infty}^0 \int_0^\infty \psi(\tau-\xi)K(-\xi)\varphi(\tau)d\tau d\xi, \quad \varphi \in X, \ \psi \in X^\sharp. \tag{2.4} \]
Then for $\varphi \in D(A_0)$ and $\psi \in D(A_0^\sharp)$, the dual relation $\langle \psi, A_0\varphi \rangle = \langle A_0^\sharp \psi, \varphi \rangle$ is satisfied. For $\lambda \in \mathbb{C}_{-\rho}$ and $k \in \mathbb{N}$, the function $\varphi$ belongs to $\mathcal{R}((A_0 - \lambda I)^k)$ if and only if $\langle \psi, \varphi \rangle = 0$ for all $\psi \in \mathcal{N}((A_0^\sharp - \lambda I)^k)$.
For $\lambda \in \mathbb{C}_{-\rho}$ and $k \in \mathbb{N}$, we introduce functions $w_k(\lambda) : \mathbb{R}^- \to \mathbb{C}$, $w_k^q(\lambda) : \mathbb{R}^+ \to \mathbb{C}$ and a $(km) \times (km)$ matrix $D_k(\lambda)$ defined by

$$
[w_k(\lambda)](\xi) = \frac{\xi^{k-1}}{(k-1)!} e^{\lambda \xi}, \quad \xi \in \mathbb{R}^-,
$$

$$
[w_k^q(\lambda)](s) = [w_k(\lambda)](-s), \quad s \in \mathbb{R}^+.
$$

$$
D_k(\lambda) = \begin{pmatrix}
\Delta(\lambda) & \Delta'(\lambda) & \cdots & \Delta^{(k-1)}(\lambda)/(k-1)!
0 & \Delta(\lambda) & \cdots & \Delta^{(k-2)}(\lambda)/(k-2)!
\vdots & \vdots & \ddots & \vdots
0 & \cdots & 0 & \Delta(\lambda)
\end{pmatrix},
$$

respectively, where $\Delta^{(n)}(z) = (d^n/dz^n) \Delta(z)$ for $n \in \mathbb{N}$. The null spaces $\mathcal{N}((A_0 - \lambda I)^k)$ and $\mathcal{N}((A_0^d - \lambda I)^k)$ are characterized as follows.

**Proposition 2.3.** ([8, Propositions 3.1 and 3.4]) Let $\lambda \in \mathbb{C}_{-\rho}$ and $k \in \mathbb{N}$. Then the following statements hold:

(i) $\varphi \in \mathcal{N}((A_0 - \lambda I)^k)$ if and only if it is written as $\varphi = \sum_{j=1}^{k} w_j(\lambda) \eta_j$, where $\eta_1, \ldots, \eta_k \in \mathbb{C}^m$ with the relation $D_k(\lambda) \text{col}(\eta_1, \ldots, \eta_k) = \text{col}(0, \ldots, 0)$.

(ii) $\psi \in \mathcal{N}((A_0^d - \lambda I)^k)$ if and only if it is written as $\psi = \sum_{j=1}^{k} w_j^q(\lambda) \zeta_{k+1-j}$, where $\zeta_1, \ldots, \zeta_k \in \mathbb{C}^{m*}$ with the relation $(\zeta_1, \ldots, \zeta_k) D_k(\lambda) = (0, \ldots, 0)$.

Let $\Sigma^{cu} = \{\lambda_1, \ldots, \lambda_r\}$, and denote by $p_i$ the ascent of $\lambda_i$ for each $i = 1, \ldots, r$. It follows from Proposition 2.2 that

$$
X^{cu} = \mathcal{N}((A_0 - \lambda_1 I)^{p_1}) \oplus \cdots \oplus \mathcal{N}((A_0 - \lambda_r I)^{p_r}).
$$

We consider the subspace $X^{cu}$ of $X$ as well as the subspace $\mathcal{N}^2$ of $X^2$ defined by

$$
\mathcal{N}^2 = \mathcal{N}((A_0^d - \lambda_1 I)^{p_1}) \oplus \cdots \oplus \mathcal{N}((A_0^d - \lambda_r I)^{p_r}).
$$

Let $\{\varphi_1, \ldots, \varphi_d\}$ be a basis for $X^{cu}$, and set $\Phi = (\varphi_1, \ldots, \varphi_d)$. Similarly, let $\{\psi_1, \ldots, \psi_d\}$ be a basis for $\mathcal{N}^2$, and set $\Psi = \text{col}(\psi_1, \ldots, \psi_d)$. We call $\Phi$ and $\Psi$ a *basis vector* for $X^{cu}$ and $\mathcal{N}^2$, respectively. Since each $\varphi_j$ in $\Phi = (\varphi_1, \ldots, \varphi_d)$ belongs to the space $X^{cu}$, it is expressed as the form

$$
\varphi_j = \sum_{i=1}^{r} \sum_{k=1}^{p_i} w_k(\lambda_i) \eta_{k,i,j}
$$

by Proposition 2.3, where $\eta_{k,i,j} \in \mathbb{C}^m$ satisfies $D_{p_i}(\lambda_j) \text{col}(\eta_{1,i,j}, \ldots, \eta_{p_i,i,j}) = \text{col}(0, \ldots, 0)$. For this $\varphi_j$, we define

$$
\hat{\varphi}_j = \sum_{i=1}^{r} \sum_{k=1}^{p_i} \hat{w}_k(\lambda_i) \eta_{k,i,j}, \quad [\hat{w}_k(\lambda)](t) = \frac{t^{k-1}}{(k-1)!} e^{\lambda t}, \quad t \in \mathbb{R}.
$$

Clearly, $\hat{\varphi}_j(\xi) = \varphi_j(\xi)$ for $\xi \in \mathbb{R}^-$ by the definition of $\hat{w}_k(\lambda)$, and we call $\hat{\varphi}_j$ the canonical prolongation of $\varphi_j$. Also, set $\tilde{\Phi} = (\hat{\varphi}_1, \ldots, \hat{\varphi}_d)$. Then $\tilde{\Phi}(\xi) = \Phi(\xi)$ for $\xi \in \mathbb{R}^-$, and we call $\tilde{\Phi}$ the canonical prolongation of the basis vector $\Phi$. Furthermore, let $\langle \Psi, \Phi \rangle$ be the $d \times d$ matrix $((\langle \psi_i, \varphi_j \rangle))_{i,j = 1,2, \ldots, d}$, and denote by $\langle \Psi, \Phi \rangle$ the column vector $\text{col}(\langle \psi_1, \varphi_1 \rangle, \ldots, \langle \psi_d, \varphi_d \rangle)$ for $\varphi \in X$. The following result presents the explicit representation form of the projection $\Pi^{cu} : X \to X^{cu}$ and an asymptotic formula for solutions of (2.1).
Proposition 2.4. ([8, Theorems 3.1 and 3.2]) Let $\Phi$ and $\Psi$ be a basis vector for $X^c$ and $N'\Sigma$, respectively. Let $\hat{\Phi}$ be the canonical prolongation of $\Phi$ defined on $\mathbb{R}$. Then the matrix $\langle \langle \Psi, \hat{\Phi} \rangle \rangle$ is nonsingular, and the projection $\Pi^c$ is given by
\[
\Pi^c \varphi = \hat{\Phi} \langle \langle \Psi, \hat{\Phi} \rangle \rangle^{-1} \langle \langle \Psi, \varphi \rangle \rangle, \quad \varphi \in X.
\]
Moreover, the solution $x_s(t)$ of (2.1) through $(0, \Pi^c \varphi)$ is expressed as
\[
x_s(t) = \hat{\Phi}(t) \langle \langle \Psi, \hat{\Phi} \rangle \rangle^{-1} \langle \langle \Psi, \varphi \rangle \rangle, \quad t > 0.
\]
In particular, for any $\varphi \in X$, the solution $x(t; \varphi)$ of (2.1) satisfies the relation
\[
\lim_{t \to \infty} |x(t; \varphi) - \hat{\Phi}(t) \langle \langle \Psi, \hat{\Phi} \rangle \rangle^{-1} \langle \langle \Psi, \varphi \rangle \rangle| = 0 \quad \text{(exponentially)}.
\]

3. Analysis of characteristic roots. In this section we will investigate the characteristic roots of (1.1). Since Eq. (1.1) can be regarded as Eq. (2.1) with $K(t) = \begin{cases} A, & 0 \leq t \leq h, \\ O_m, & h < t < \infty, \end{cases}$ where $O_m$ is the $m \times m$ zero matrix, the characteristic equation for (1.1) with $m = 2$ becomes
\[
\det \Delta(z) = 0, \quad \Delta(z) = E_2 - A \int_0^h e^{-zt} dt. \quad (3.1)
\]
Then we obtain the following results on the distributions of the roots of (3.1).

Theorem 3.1. Suppose that the matrix $A$ is of the form (I) with $a_1 \neq 0$ or $a_2 \neq 0$. Then the following statements hold:

(i) If $a_1 h < 1$ and $a_2 h < 1$, then all the roots of (3.1) have negative real parts.

(ii) If $a_1 h = 1 > a_2 h$ or $a_1 h < 1 = a_2 h$, then (3.1) has a simple root $0$ and the remaining roots have negative real parts.

(iii) If $a_1 h = a_2 h = 1$, then (3.1) has a double root $0$ and the remaining roots have negative real parts.

(iv) If $a_1 h > 1$ or $a_2 h > 1$, there exists a root of (3.1) with a positive real part.

Theorem 3.2. Suppose that the matrix $A$ is of the form (II) with $a \neq 0$. Then the following statements hold:

(i) If $ah < 1$, then all the roots of (3.1) have negative real parts.

(ii) If $ah = 1$, then (3.1) has a double root $0$ and the remaining roots have negative real parts.

(iii) If $ah > 1$, there exists a root of (3.1) with a positive real part.

Theorem 3.3. Suppose that the matrix $A$ is of the form (III) with $a \neq 0$. Let $\omega_0 = 2\theta/h$ and $\omega_{-1} = 2(|\theta| - \pi)/h$. Then the following statements hold:

(i) If $(|\theta| - \pi)/\sin \theta < ah < \theta/\sin \theta$, then all the roots of (3.1) have negative real parts.

(ii) If $ah = \theta/\sin \theta$, then (3.1) has a pair of simple roots $\pm i\omega_0$ and the remaining roots have negative real parts.

(iii) If $ah = (|\theta| - \pi)/\sin \theta$, then (3.1) has a pair of simple roots $\pm i\omega_{-1}$ and the remaining roots have negative real parts.

(iv) If $ah > \theta/\sin \theta$ or $ah < (|\theta| - \pi)/\sin \theta$, there exists a root of (3.1) with a positive real part.
First, we will treat the roots of (3.1) when the matrix \( A \) has real eigenvalues. For simplicity, let
\[
f(z; a) = 1 - a \int_0^h e^{-zt} dt.
\]
If the matrix \( A \) is of the form (I), we have
\[
\det \Delta(z) = \begin{vmatrix} 1 - a_1 \int_0^h e^{-zt} dt & 0 \\ 0 & 1 - a_2 \int_0^h e^{-zt} dt \end{vmatrix} = f(z; a_1)f(z; a_2).
\]
Also if the matrix \( A \) is of the form (II), we get
\[
\det \Delta(z) = \begin{vmatrix} 1 - a \int_0^h e^{-zt} dt & - \int_0^h e^{-zt} dt \\ 0 & 1 - a \int_0^h e^{-zt} dt \end{vmatrix} = \{f(z; a)\}^2.
\]
Hence, to prove Theorems 3.1 and 3.2, it suffices to verify the following proposition.

**Proposition 3.1.** For \( a \neq 0 \) the following statements hold:

(i) If \( ah < 1 \), then all the roots of \( f(z; a) = 0 \) have negative real parts.

(ii) If \( ah = 1 \), then \( f(z; a) = 0 \) has a simple root 0 and the remaining roots have negative real parts.

(iii) If \( ah > 1 \), there exists a root of \( f(z; a) = 0 \) with a positive real part.

To prove Proposition 3.1, we will prepare some lemmas. Note that since \( f(z; a) \) is an analytic function of \( z \) and \( a \), one can regard the root \( z = z(a) \) of \( f(z; a) = 0 \) as a continuous function of \( a \).

**Lemma 3.1.** If \( 0 < |a|h < 1 \), there exist no roots of \( f(z; a) = 0 \) in the right half open domain of the complex plane.

**Proof.** Let \( 0 < |a|h < 1 \). Assume that there exists a root \( \lambda \) of \( f(\lambda; a) = 0 \) such that \( \text{Re} \lambda > 0 \). Then \( 1 = a \int_0^h e^{-\lambda t} dt \) and
\[
1 = \left| a \int_0^h e^{-\lambda t} dt \right| \leq |a| \int_0^h e^{-(\text{Re}\lambda)t} dt \leq |a|h,
\]
which contradicts \( |a|h < 1 \).

**Lemma 3.2.** \( f(z; a) = 0 \) has the root 0 if and only if \( a = 1/h \). Here the root 0 is simple.

**Proof.** The relation \( f(0; a) = 1 - ah \) shows that the former assertion holds. Since
\[
f'(z; a) = -a \frac{d}{dz} \left( \int_0^h e^{-zt} dt \right) = a \int_0^h te^{-zt} dt,
\]
we obtain \( f'(0; 1/h) = h/2 \neq 0 \), which implies that the root 0 is simple.

**Lemma 3.3.** There exist no purely imaginary roots of \( f(z; a) = 0 \) except for 0.

**Proof.** Assume that there exists a root \( i\omega \) of \( f(z; a) = 0 \) with \( \omega \neq 0 \). Then
\[
f(i\omega; a) = 1 - a \int_0^h e^{-i\omega t} dt = 1 + \frac{a}{i\omega} (e^{-i\omega h} - 1) = 0,
\]
which yields
\[
1 = \frac{a}{i\omega} (e^{-i\omega h} - 1) = \frac{a}{\omega} \{\sin \omega h + i(\cos \omega h - 1)\},
\]
for some \( \omega \neq 0 \).
namely, $\sin \omega h = \omega/a$ and $\cos \omega h = 1$. Therefore we must have $\omega/a = 0$, which contradicts $\omega \neq 0$. \hfill \Box

**Lemma 3.4.** The root $0$ moves in the right half open domain of the complex plane as $a$ increases from $1/h$.

**Proof.** By taking the derivative of $z$ with respect to $a$, we have

$$-\int_0^h e^{-zt} dt - a \frac{d}{da} \left( \int_0^h e^{-zt} dt \right) \frac{dz}{da} = 0,$$

that is,

$$\frac{dz}{da} = -\frac{1}{a^2} \int_0^h te^{-zt} dt = \frac{1}{a^2} \int_0^h te^{-zt} dt.$$

Thus we get $\text{Re}((dz/da)_{z=0}) = 2/(ah)^2 > 0$, which leads to the assertion of the lemma. \hfill \Box

Now we are in a position to prove Proposition 3.1.

**Proof of Proposition 3.1.** Let $a \neq 0$. There are three cases to consider.

(i) $ah < 1$. Lemma 3.1 shows if $|a| < 1/h$, there exist no roots of $f(z; a) = 0$ which have positive real parts. Also, Lemmas 3.2 and 3.3 show if $a \neq 1/h$, there exist no purely imaginary roots of $f(z; a) = 0$. By virtue of the continuity of the roots with respect to $a$, we conclude that if $a < 1/h$, all the roots of $f(z; a) = 0$ have negative real parts.

(ii) $ah = 1$. Lemmas 3.2 and 3.3 and Proposition 3.1 (i) imply the conclusion, as required.

(iii) $ah > 1$. Let $\lambda_0(a)$ be the branch of the root of $f(z; a) = 0$ satisfying $\lambda_0(1/h) = 0$. Lemma 3.4 shows $\text{Re} \lambda_0(a) > 0$ for all sufficiently small $a - 1/h > 0$. Since $\lambda_0(a)$ cannot cross the imaginary axis from right to left as $a$ increases from $1/h$, one can obtain $\text{Re} \lambda_0(a) > 0$ for all $a > 1/h$. \hfill \Box

Next, we will treat the roots of (3.1) when the matrix $A$ has complex eigenvalues. For brevity, let

$$g(z; a) = 1 - ae^{i\theta} \int_0^h e^{-zt} dt.$$

If the matrix $A$ is of the form (III), we have

$$\det \Delta(z) = \begin{vmatrix} 1 - a \cos \theta \int_0^h e^{-zt} dt & a \sin \theta \int_0^h e^{-zt} dt \\ -a \sin \theta \int_0^h e^{-zt} dt & 1 - a \cos \theta \int_0^h e^{-zt} dt \end{vmatrix}$$

$$= \left( 1 - a \cos |\theta| \int_0^h e^{-zt} dt \right)^2 - \left( ia \sin |\theta| \int_0^h e^{-zt} dt \right)^2$$

$$= \left( 1 - ae^{i|\theta|} \int_0^h e^{-zt} dt \right) \left( 1 - ae^{-i|\theta|} \int_0^h e^{-zt} dt \right) = g(z; a)\bar{g}(\bar{z}; a),$$

where $\bar{z}$ denotes the complex conjugate of any complex $z$. Note that if $\lambda$ is a root of $\det \Delta(z) = 0$, then $\bar{\lambda}$ is also a root of $\det \Delta(z) = 0$. Indeed, $g(\lambda; a) = 0$ implies $g(\bar{\lambda}; a) = \bar{g}(\bar{\lambda}; a) = 0$. Consequently, to prove Theorem 3.3, it suffices to verify the following proposition.

**Proposition 3.2.** Let $\omega_0 = 2|\theta|/h$ and $\omega_{-1} = 2(|\theta| - \pi)/h$. Then for $a \neq 0$ the following statements hold:
Conversely, if \( a \), then \( g(z; a) = 0 \) has a simple root \( i\omega \) and the remaining roots have negative real parts.

(iii) If \( a = (|\theta| - \pi)/\sin |\theta| \), then \( g(z; a) = 0 \) has a simple root \( i\omega_{-1} \) and the remaining roots have negative real parts.

(iv) If \( a > \theta/\sin \theta \) or \( a < (|\theta| - \pi)/\sin |\theta| \), there exists a root of \( g(z; a) = 0 \) with a positive real part.

**Lemma 3.5.** If \( 0 < |a|h < 1 \), there exist no roots of \( g(z; a) = 0 \) in the right half open domain of the complex plane.

The lemma can be verified by the same argument as in the proof of Lemma 3.1; so we omit the proof.

**Lemma 3.6.** Let \( i\omega \) be a root of \( g(z; a) = 0 \) where \( \omega \) is a real number. Then the values of \( \omega \) and \( a \) are expressed as

\[
\omega = \omega_n := \frac{2(|\theta| + n\pi)}{h}, \quad a = a_n := \frac{\omega_n}{2\sin |\theta|}, \quad n \in \mathbb{Z}.
\]

Conversely, if \( a = a_n \) for \( n \in \mathbb{Z} \), then \( i\omega_n \) is a simple root of \( g(z; a) = 0 \).

**Proof.** We first notice that 0 is not a root of \( g(z; a) = 0 \) because of \( g(0; a) = 1 - ahe^{i|\theta|} \neq 0 \) under \( \theta \neq 0 \). If \( g(i\omega; a) = 0 \) for \( \omega \in \mathbb{R} \setminus \{0\} \), then

\[
g(i\omega; a) = 1 - ae^{i|\theta|} \int_0^h e^{-i\omega t} dt = 1 + \frac{ae^{i|\theta|}}{i\omega}(e^{-i\omega h} - 1) = 0,
\]

namely,

\[
sin(\omega h - |\theta|) + \sin |\theta| = 2 \sin \frac{\omega h}{2} \cos \frac{\omega h - 2|\theta|}{2} = \frac{\omega}{a}
\]

(3.2)

and

\[
cos(\omega h - |\theta|) - \cos |\theta| = -2 \sin \frac{\omega h}{2} \sin \frac{\omega h - 2|\theta|}{2} = 0.
\]

(3.3)

From \( \sin(\omega h/2) \neq 0 \), we have \( \sin((\omega h - 2|\theta|)/2) = 0 \) by (3.3), which together with (3.2) yields

\[
\omega = \frac{2(|\theta| + n\pi)}{h} = \omega_n, \quad a = \frac{\omega_n}{2\sin |\theta|} = a_n, \quad n \in \mathbb{Z}.
\]

Conversely, if \( a = a_n \) for \( n \in \mathbb{Z} \), we see that

\[
g(i\omega_n; a_n) = 1 + \frac{a_ne^{i|\theta|}}{i\omega_n}(e^{-i\omega_n h} - 1) = 1 + \frac{e^{-i|\theta|} - e^{i|\theta|}}{2i \sin |\theta|} = 0,
\]

which implies that \( i\omega_n \) is a root of \( g(z; a) = 0 \). Moreover it follows that

\[
g'(z; a) = -ae^{i|\theta|} \frac{d}{dz} \left( \int_0^h e^{-zt} dt \right) = ae^{i|\theta|} \int_0^h te^{-zt} dt
\]

\[
= -ae^{i|\theta|}he^{-z\omega} + ae^{i|\theta|} \int_0^h e^{-zt} dt = -ahe^{-(zh - i|\theta|)} + 1 - g(z; a)
\]

and hence

\[
g'(i\omega_n; a_n) = \frac{1 - ahe^{-i(\omega_n h - |\theta|)}}{i\omega_n} = \frac{1 - ahe^{-i|\theta|}}{i\omega_n} \neq 0,
\]

which yields that the root \( i\omega_n \) is simple. \( \square \)
Remark 3.1. The definition of $a_n$ leads to the relation
\[ \cdots < a_{-2} < a_{-1} = \frac{|\theta| - \pi}{h \sin |\theta|} < - \frac{1}{h} < 0 < \frac{1}{h} < \frac{\theta}{h \sin \theta} = a_0 < a_1 < \cdots. \tag{3.4} \]

Lemma 3.7. All the roots of $g(z; a) = 0$ on the imaginary axis move in the right half open domain as $|a|$ increases.

Proof. It suffices to verify the relation $\text{sgn}\{\text{Re}(dz/da)_{z=i\omega_n}\} = \text{sgn}(a)$. By taking the derivative of $z$ with respect to $a$ on $g(z; a) = 0$, we have
\[ -e^{i\theta} \int_0^h e^{-zt} dt - ae^{i\theta} \frac{d}{dz} \left( \int_0^h e^{-zt} dt \right) \frac{dz}{da} = 0, \]
that is,
\[ \frac{dz}{da} = \frac{e^{i\theta} \int_0^h e^{-zt} dt}{ae^{i\theta} \int_0^h te^{-zt} dt} = \frac{1/a}{\frac{1}{a} - ah e^{-i|\theta|-zh}} = \frac{z}{a(1 - ah e^{-i|\theta|-zh})}. \]

Therefore it follows that
\[ \frac{dz}{da} \bigg|_{z=i\omega_n} = \frac{i\omega_n}{a(1 - ah e^{-i|\theta|-zh})} = \frac{2i \sin |\theta|}{1 - ah e^{-i|\theta|}} = \frac{2 \sin |\theta|}{1 - ah \cos |\theta| + iah \sin |\theta|} = \frac{2ah \sin^2 |\theta| + 2i \sin |\theta|(1 - ah \cos |\theta|)}{(1 - ah \cos |\theta|)^2 + (ah \sin |\theta|)^2}, \]
which implies
\[ \text{sgn}\left\{\text{Re}\left( \frac{dz}{da} \bigg|_{z=i\omega_n} \right) \right\} = \text{sgn}\left\{ \frac{2ah \sin^2 |\theta|}{(1 - ah \cos |\theta|)^2 + (ah \sin |\theta|)^2} \right\} = \text{sgn}(a), \]
as required. \vspace{0.5cm}

Now we are ready to prove Proposition 3.2.

Proof of Proposition 3.2. Let $a \neq 0$. There are four cases to consider.

(i) $a_{-1} < a < a_0$. Lemma 3.5 shows that if $|a| < 1/h$, there exist no roots of $g(z; a) = 0$ which have positive real parts. Also, Lemma 3.6 shows that if $a \neq a_n$ for $n \in \mathbb{Z}$, there exist no purely imaginary roots of $g(z; a) = 0$. By virtue of relation (3.2) and the continuity of the roots with respect to $a$, we conclude that if $0 < a < a_0$ or $a_{-1} < a < 0$, all the roots of $g(z; a) = 0$ have negative real parts.

(ii) $a = a_0$. Lemma 3.6, relation (3.4) and Proposition 3.2 (i) yield the conclusion, as required.

(iii) $a = a_{-1}$. Lemma 3.6, relation (3.4) and Proposition 3.2 (i) lead to the conclusion, as required.

(iv) $a > a_0$ or $a < a_{-1}$. Let $\lambda_1(a)$ and $\lambda_2(a)$ be the branches of the roots of $g(z; a) = 0$ satisfying $\lambda_1(a_0) = i\omega_0$ and $\lambda_2(a_{-1}) = i\omega_{-1}$, respectively. Lemma 3.7 shows $\text{Re} \lambda_1(a) > 0$ for all sufficiently small $a - a_0 > 0$ and $\text{Re} \lambda_2(a) > 0$ for all sufficiently small $a_{-1} - a > 0$. Since $\lambda_1(a)$ cannot cross the imaginary axis from right to left as $a$ increases from $a_0$ by Lemma 3.7, one can obtain $\text{Re} \lambda_1(a) > 0$ for all $a > a_0$. Similarly, since $\lambda_2(a)$ cannot cross the imaginary axis from right to left as $a$ decreases from $a_{-1}$ by Lemma 3.7, one can obtain $\text{Re} \lambda_2(a) > 0$ for all $a < a_{-1}$. \vspace{0.5cm}
4. **Proofs of main results.** To prove Theorems 1.1, 1.2 and 1.3, we regard Eq. (1.1) as Eq. (2.1) with

\[
K(t) = \begin{cases} 
A, & 0 \leq t \leq h, \\
O_2, & h < t < \infty.
\end{cases}
\]

Since \( K(t) \) satisfies the bounded condition by \( K(t) = O_2 \) on \((h, \infty)\), we can treat the phase space \( X \) (and \( X^2 \)) for any \( \rho > 0 \). Recall that \( \Sigma^{cu} \) coincides with the set \( \{ \lambda \in \mathbb{C} \mid \det \Delta(\lambda) = 0, \Re \lambda \geq 0 \} \). The bilinear form (2.4) is reduced to

\[
\langle \langle \psi, \varphi \rangle \rangle = \int_{-h}^{0} \int_{\xi}^{0} \psi(\tau - \xi)A\varphi(\tau)d\tau d\xi, \quad \varphi \in X, \ \psi \in X^2. \quad (4.1)
\]

**Proof of Theorem 1.1.** Suppose that the matrix \( A \) is of the form (i). Let \( x(t; \varphi) \) be the solution of (1.1) with (1.2) where \( \varphi \in L^1([-h, 0], \mathbb{R}^2) \).

(i) \( a_1h < 1 \) and \( a_2h < 1 \). Clearly, if \( a_1 = a_2 = 0 \), we have \( x(t; \varphi) = 0 \) for all \( t > 0 \). If \( a_1 \neq 0 \) or \( a_2 \neq 0 \), Theorem 3.1 (i) shows \( \Sigma^{cu} = \emptyset \), which together with Proposition 2.2 (i) implies that \( x(t; \varphi) \) converges to 0 exponentially as \( t \to \infty \).

(ii) \( a_1h = 1 \) and \( a_2h < 1 \). Theorem 3.1 (ii) shows \( \Sigma^{cu} = \{ 0 \} \) and the root 0 is simple, which yield that the ascent of 0 is equal to 1 by Proposition 2.2. Thus we get \( X^{cu} = \mathcal{N}(A_0) \). Notice that

\[
D_1(0) = \Delta(0) = \begin{pmatrix} 0 & 0 \\ 0 & 1 - a_2h \end{pmatrix}, \quad [w_1(0)](\xi) = 1, \quad \xi \in \mathbb{R}^-.
\]

From Proposition 2.3 (i), we see that \( \mathcal{N}(A_0) = \{ [w_1(0)](\xi) \eta \mid D_1(0)\eta = 0 \} = \text{span} \{ \text{col} \ (1 \ 0) \} \). Similarly, from Proposition 2.3 (ii), we have \( \mathcal{N}^2 = \mathcal{N}(A_0^2) = \text{span} \{ (1 \ 0) \} \).

Therefore basis vectors \( \Phi \) for \( X^{cu} \) and \( \Psi \) for \( \mathcal{N}^2 \) are expressed as

\[
\Phi(\xi) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \xi \in \mathbb{R}^-, \quad \Psi(s) = (1\ 0), \quad s \in \mathbb{R}^+.
\]

This, together with (4.1), yields that

\[
\langle \langle \Psi, \Phi \rangle \rangle = \int_{-h}^{0} \int_{\xi}^{0} a_1 d\tau d\xi = \frac{h}{2}.
\]

\[
\langle \langle \Psi, \varphi \rangle \rangle = \int_{-h}^{0} \int_{\xi}^{0} (a_1 \ 0)\varphi(\tau)d\tau d\xi = (1/h \ 0) \int_{-h}^{0} \int_{\xi}^{0} \varphi(\tau)d\tau d\xi.
\]

Let \( \Phi(t)\langle \langle \Psi, \Phi \rangle \rangle^{-1} \langle \langle \Psi, \varphi \rangle \rangle = \frac{2}{h^2} \int_{-h}^{0} \int_{\xi}^{0} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \varphi(\tau)d\tau d\xi := \begin{pmatrix} c_1(t) \\ 0 \end{pmatrix} \]

and \( x(t; \varphi) \) converges to \( \text{col} \ (c_1(t), 0) \) exponentially as \( t \to \infty \).

(iii) \( a_1h < 1 \) and \( a_2h = 1 \). The proof can be carried out by almost the same argument as in the proof of Theorem 1.1 (ii); so we omit the proof.

(iv) \( a_1h = a_2h = 1 \). Theorem 3.1 (iii) shows \( \Sigma^{cu} = \{ 0 \} \) and the root 0 is double, which yields that the ascent of 0 is equal to 1 or 2 by Proposition 2.2. Hence we get \( X^{cu} = \mathcal{N}(A_0) \) or \( X^{cu} = \mathcal{N}(A_0^2) \). Also, by Proposition 2.2, we know that
Proof of Theorem 1.2. Suppose that the matrix \( \dim X^\text{cu} = \dim M_0(A_0) = 2 \). Notice that \([w_1(0)](\xi) = 1, [w_2(0)](\xi) = \xi \) for \( \xi \in \mathbb{R}^+ \) and

\[
D_1(0) = \Delta(0) = O_2, \quad D_2(0) = \begin{pmatrix} \Delta(0) & \Delta'(0) \\ O_2 & O_2 \end{pmatrix} = \begin{pmatrix} O_2 & (h/2)E_2 \\ O_2 & O_2 \end{pmatrix}.
\]

From Proposition 2.3 (i), we find that

\[
\mathcal{N}(A_0) = \left\{ [w_1(0)](\xi) \eta \mid D_1(0) \eta = 0 \right\} = \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\},
\]

\[
\mathcal{N}(A_0^2) = \left\{ [w_1(0)](\xi)\eta_1 + [w_2(0)](\xi)\eta_2 \mid D_2(0) \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = 0 \right\} = \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\},
\]

which lead to \( X^\text{cu} = \mathcal{N}(A_0) \). Similarly, from Proposition 2.3 (ii), we have \( \mathcal{N}^2 = \mathcal{N}(A_0^2) = \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \). Therefore basis vectors \( \Phi \) for \( X^\text{cu} \) and \( \Psi \) for \( \mathcal{N}^2 \) are expressed as

\[
\Phi(\xi) = E_2, \quad \xi \in \mathbb{R}^-, \quad \Psi(s) = E_2, \quad s \in \mathbb{R}^+.
\]

This, together with (4.1), yields that

\[
\langle \Psi, \Phi \rangle = \int_{-h}^{0} \int_{-h}^{0} A d\tau d\xi = \frac{h}{2} E_2,
\]

\[
\langle \Psi, \varphi \rangle = \int_{-h}^{0} \int_{-h}^{0} A \varphi(\tau) d\tau d\xi = \frac{1}{h} \int_{-h}^{0} \int_{-h}^{0} \varphi(\tau) d\tau d\xi.
\]

Let \( \hat{\Phi} \) be the canonical prolongation of \( \Phi \), that is, \( \hat{\Phi}(t) = E_2 \) for \( t \in \mathbb{R} \). By applying Proposition 2.4, we therefore obtain

\[
\hat{\Phi}(t) \langle \Psi, \Phi \rangle^{-1} \langle \Psi, \varphi \rangle = \frac{2}{h^2} \int_{-h}^{0} \int_{-h}^{0} \varphi(\tau) d\tau d\xi := c_1
\]

and \( x(t; \varphi) \) converges to \( c_1 \) exponentially as \( t \to \infty \).

(v) \( a h > 1 \) or \( a^2 h > 1 \). Theorem 3.1 (iv) shows there exists a root of (3.1) with a positive real part, which implies that there exists an unbounded solution of (1.1) with an initial function in \( X^\text{cu} \).

Proof of Theorem 1.2. Suppose that the matrix \( A \) is of the form (II). Let \( x(t; \varphi) \) be the solution of (1.1) with (1.2) where \( \varphi \in L^1([-h, 0], \mathbb{R}^2) \).

(i) \( a h < 1 \). Clearly, if \( a = 0 \), we have \( x(t; \varphi) = 0 \) for all \( t > 0 \). If \( a \neq 0 \), Theorem 3.2 (i) shows \( \Sigma^\text{cu} = \emptyset \), which together with Proposition 2.1 (i) implies that \( x(t; \varphi) \) converges to 0 exponentially as \( t \to \infty \).

(ii) \( a h \geq 1 \). If \( a h > 1 \), Theorem 3.2 (iii) shows there exists a root of (3.1) with a positive real part, which implies that there exists an unbounded solution of (1.1) with an initial function in \( X^\text{cu} \).

If \( a h = 1 \), Theorem 3.1 (ii) shows \( \Sigma^\text{cu} = \{0\} \) and the root 0 is double, which yields that the ascent of 0 is equal to 1 or 2 by Proposition 2.2. Thus we get \( X^\text{cu} = \mathcal{N}(A_0) \) or \( X^\text{cu} = \mathcal{N}(A_0^2) \). Also, by Proposition 2.2, we know that \( \dim X^\text{cu} = \dim M_0(A_0) = 2 \). Notice that \([w_1(0)](\xi) = 1, [w_2(0)](\xi) = \xi \) for \( \xi \in \mathbb{R}^- \) and

\[
D_1(0) = \Delta(0) = \begin{pmatrix} 0 & -h \\ 0 & 0 \end{pmatrix}, \quad D_2(0) = \begin{pmatrix} \Delta(0) & \Delta'(0) \\ O_2 & O_2 \end{pmatrix} = \begin{pmatrix} 0 & -h & h/2 & h^2/2 \\ 0 & 0 & 0 & h/2 \\ 0 & 0 & 0 & -h \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\]
From Proposition 2.3 (i), it is easy to see that
\[ \mathcal{N}(A_0) = \text{span}\{\varphi_1\}, \quad \mathcal{N}(A_0^2) = \text{span}\{\varphi_1, \varphi_2\}, \]
where \( \varphi_1(\xi) = \cos(\xi) \) and \( \varphi_2(\xi) = \sin(\xi) \) for \( \xi \in \mathbb{R}^- \). Since \( \varphi_2 \notin \mathcal{N}(A_0) \), we get \( X^{cu} = \mathcal{N}(A_0^2) \). Similarly, from Proposition 2.3 (ii), we have \( \mathcal{N}^\sharp = \mathcal{N}(A_0^2)^2 = \text{span}\{\psi_1, \psi_2\} \), where \( \psi_1(s) = (1/2 - s) \) and \( \psi_2(s) = (0, 1) \) for \( s \in \mathbb{R}^+ \). Therefore basis vectors \( \Phi \) for \( X^{cu} \) and \( \Psi \) for \( \mathcal{N}^\sharp \) are expressed as
\[ \Phi(\xi) = \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix}, \quad \xi \in \mathbb{R}^- \]
\[ \Psi(s) = \begin{pmatrix} 1/2 & -s \\ 0 & 1 \end{pmatrix}, \quad s \in \mathbb{R}^+. \]

Set \( c_2 = \left\langle \langle \Psi, \Phi \rangle \right\rangle^{-1} \left\langle \langle \Psi, \varphi \rangle \right\rangle \). Then, by (4.1), an easy calculation yields that
\[ c_2 = \frac{2}{\hbar^2} \int_0^h \int_{-\hbar}^0 \begin{pmatrix} 1 & 4h/3 - 2(\tau - \xi) \\ 0 & 2 \end{pmatrix} \varphi(\tau)d\tau d\xi := \begin{pmatrix} c_2^{(1)} \\ c_2^{(2)} \end{pmatrix}. \]

Let \( \tilde{\Phi} \) be the canonical prolongation of \( \Phi \). By applying Proposition 2.4, we therefore conclude that \( |x(t; \varphi) - \tilde{\Phi}(t)c_2| \) tends to 0 exponentially as \( t \to \infty \), where
\[ \tilde{\Phi}(t)c_2 = \begin{pmatrix} 1 & t \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} c_2^{(1)} \\ c_2^{(2)} \end{pmatrix} = \begin{pmatrix} c_2^{(1)} + c_2^{(2)}t \\ c_2^{(2)}/2 \end{pmatrix}. \]

In particular, if we take \( \varphi \) with \( c_2^{(2)}(\varphi) \neq 0 \), the solution \( x(t; \varphi) \) is unbounded, as required. \qed

**Proof of Theorem 1.3.** Suppose that the matrix \( A \) is of the form (III). Let \( x(t; \varphi) \) be the solution of (1.1) with (1.2) where \( \varphi \in L^1([-h, 0], \mathbb{R}^2) \).

(i) \( (|\theta| - \pi)/\sin|\theta| < ah < \theta/\sin \theta \). Clearly, if \( a = 0 \), we have \( x(t; \varphi) = 0 \) for all \( t > 0 \). If \( a \neq 0 \), Theorem 3.3 (i) shows \( \Sigma^{cu} = \emptyset \), which together with Proposition 2.1 (i) implies that \( x(t; \varphi) \) converges to 0 exponentially as \( t \to \infty \).

(ii) \( ah = \theta/\sin \theta \) or \( ah = (|\theta| - \pi)/\sin|\theta| \). We will only consider the case \( ah = \theta/\sin \theta \). The other case can be dealt with similarly. Theorem 3.3 (ii) shows \( \Sigma^{cu} = \{i\omega_0, -i\omega_0\} \) and the roots \( \pm i\omega_0 \) are simple, which yield that the ascents of \( i\omega_0 \) and \( -i\omega_0 \) are equal to 1 by Proposition 2.2. Hence we get
\[ X^{cu} = \mathcal{N}(A_0 - i\omega_0 I) \oplus \mathcal{N}(A_0 + i\omega_0 I), \]
\[ \mathcal{N}^\sharp = \mathcal{N}(A_0^2 - i\omega_0 I) \oplus \mathcal{N}(A_0^2 + i\omega_0 I). \]

Taking into account that \( g(i\omega_0; a) = 0 \), we have
\[ D_1(i\omega_0) = \Delta(i\omega_0) \]
\[ = E_2 - a \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \int_0^h e^{-i\omega_0 t} dt = E_2 - e^{-i|\theta|} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \]
\[ = e^{-i|\theta|} \begin{pmatrix} e^{i|\theta|} & -\sin \theta \\ -\sin \theta & e^{i|\theta|} \end{pmatrix} \sin |\theta| = ie^{-i|\theta|} \sin |\theta| \begin{pmatrix} 1 \\ \text{sgn}(\theta)i \end{pmatrix}. \]

Similarly
\[ D_1(-i\omega_0) = \Delta(-i\omega_0) = -ie^{i|\theta|} \sin |\theta| \begin{pmatrix} 1 \\ -\text{sgn}(\theta)i \end{pmatrix}. \]
Since \([w_1(\pm i\omega_0)](\xi) = e^{\pm i\omega_0 \xi} \) for \(\xi \in \mathbb{R}^-\) and \([w_1^\ast(\pm i\omega_0)](s) = e^{\mp i\omega_0 s} \) for \(s \in \mathbb{R}^+\), it follows from Proposition 2.3 that

\[
\mathcal{N}(A_0 - i\omega_0 I) = \text{span} \left\{ \hat{\varphi}_1(\xi) := e^{i\omega_0 \xi} \left( \frac{1}{-\text{sgn}(\theta)i} \right) \right\}, \quad \xi \in \mathbb{R}^-,
\]

\[
\mathcal{N}(A_0 + i\omega_0 I) = \text{span} \left\{ \hat{\varphi}_2(\xi) := e^{-i\omega_0 \xi} \left( \frac{1}{\text{sgn}(\theta)i} \right) \right\}, \quad \xi \in \mathbb{R}^-,
\]

\[
\mathcal{N}(A_0^2 - i\omega_0 I) = \text{span} \left\{ \hat{\psi}_1(s) := e^{-i\omega_0 s} (1 - \text{sgn}(\theta)i) \right\}, \quad s \in \mathbb{R}^+,
\]

\[
\mathcal{N}(A_0^2 + i\omega_0 I) = \text{span} \left\{ \hat{\psi}_2(s) := e^{i\omega_0 s} (1 - \text{sgn}(\theta)i) \right\}, \quad s \in \mathbb{R}^+.
\]

We define new bases for the eigenspaces of \(A_0\) and \(A_0^2\) for \(\pm i\omega_0\) as

\[
\varphi_1(\xi) = \frac{\hat{\varphi}_1(\xi) + \hat{\varphi}_2(\xi)}{2} = \left( \frac{\cos \omega_0 \xi}{-\sin \omega_0 \xi \cdot \text{sgn}(\theta)} \right), \quad \xi \in \mathbb{R}^-,
\]

\[
\varphi_2(\xi) = \frac{-\hat{\varphi}_1(\xi) + \hat{\varphi}_2(\xi)}{2\iota \text{sgn}(\theta)} = \left( -\sin \omega_0 \xi \cdot \text{sgn}(\theta) \right), \quad \xi \in \mathbb{R}^-,
\]

\[
\psi_1(s) = \frac{\hat{\psi}_1(s) + \hat{\psi}_2(s)}{2\iota \text{sgn}(\theta)} = \left( \cos \omega_0 s \cdot \frac{\sin \omega_0 s \cdot \text{sgn}(\theta)}{\cos \omega_0 s} \right), \quad s \in \mathbb{R}^+,
\]

\[
\psi_2(s) = \frac{-\hat{\psi}_1(s) + \hat{\psi}_2(s)}{2\iota \text{sgn}(\theta)} = \left( -\sin \omega_0 s \cdot \frac{\text{sgn}(\theta)}{\cos \omega_0 s} \right), \quad s \in \mathbb{R}^+.
\]

Therefore basis vectors \(\Phi\) for \(X^{cu}\) and \(\Psi\) for \(\mathcal{N}^2\) are expressed as

\[
\Phi(\xi) = (\varphi_1(\xi) \varphi_2(\xi)), \quad \xi \in \mathbb{R}^-,
\]

\[
\Psi(s) = (\psi_1(s) \psi_2(s)), \quad s \in \mathbb{R}^+.
\]

This, together with (4.1), yields that

\[
\langle \Psi, \Phi \rangle = \int_{-h}^0 \int_{-h}^0 \Psi(\tau - \xi) A\Phi(\tau) d\tau d\xi = -A \int_{-h}^0 \xi \Phi(\xi) d\xi,
\]

\[
\langle \Psi, \varphi \rangle = \int_{-h}^0 \int_{-h}^0 \Psi(\tau - \xi) A\varphi(\tau) d\tau d\xi = A \int_{-h}^0 \int_{-h}^0 \Phi(\xi - \tau) \varphi(\tau) d\tau d\xi
\]

because the relation \(\Psi(s) = \Phi(-s)\) for \(s \in \mathbb{R}^+\) and the commutativity of \(A\) and \(\Phi\). Set \(c_3 = \langle \Psi, \Phi \rangle^{-1} \langle \Psi, \varphi \rangle\). Then the above relations yield that

\[
c_3 = \left( -\int_{-h}^0 \xi \Phi(\xi) d\xi \right)^{-1} \int_{-h}^0 \int_{-h}^0 \Phi(\xi - \tau) \varphi(\tau) d\tau d\xi.
\]

Let \(\hat{\Phi}\) be the canonical prolongation of \(\Phi\). By applying Proposition 2.4, we therefore conclude that \(|x(t; \varphi) - \hat{\Phi}(t)c_3|\) tends to 0 exponentially as \(t \to \infty\).

(iii) \(ah > \theta / \sin \theta\) or \(ah < (|\theta| - \pi) / \sin |\theta|\). Theorem 3.3 (iii) shows there exists a root of (3.1) with a positive real part, which implies that there exists an unbounded solution of (1.1) with an initial function in \(X^{cu}\). □

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*E-mail address: kazuki.k_himoto@yahoo.co.jp*

*E-mail address: hideaki@ms.osakafu-u.ac.jp*