Sustainable Usability Distribution Mechanisms under Multi-Attribute Sports Management Schemes

En-Cheng Chi 1 and Yu-Hsien Liao 2,*

1 Office of Physical Activities, National Pingtung University, Pingtung County 90003, Taiwan; g731007@mail.nptu.edu.tw
2 Department of Applied Mathematics, National Pingtung University, Pingtung County 900391, Taiwan
* Correspondence: twincos@mail.nptu.edu.tw; Tel.: +886-958-631-010

Abstract: Recently, game-theoretical methods have been adopted to analyze the reasonability of usability distribution mechanisms. On the other hand, sustainability has become a major conception among many fields by focusing on various influences that arose from environmental change, including usability distribution under multi-attribute sports management schemes. In many real-world situations, however, performers and its energy levels (strategies) should be essential factors simultaneously. Based on maximal-usability among energy level (strategy) vectors, we define an output, its efficacious extension and normalization to analyze usability distribution mechanisms under multi-attribute sports management schemes. We also adopt axiomatic processes to present the reasonability for these outputs. By considering reduced scheme and excess mapping, we adopt alternative formulation to offer dynamic processes for the efficacious extension and the normalization, respectively.

Keywords: multi-attribute sports management scheme; reduced scheme; excess mapping; dynamic process

1. Introduction

Due to the constant renovation of the trend of real-world environmental change, distribution mechanisms of a combination of different theoretical fields, including sports usability distribution, have become a major conception in the context of sustainability. In the framework of sports usability distribution, the use of several notions could promote the distribution efficiency no matter for improvement of distribution methods or sports management techniques of usability. On the other hand, the axiomatic outcomes of game-theoretical distribution mechanisms could be always adopted to analyze various interaction relationships and related models among agents and coalitions by applying mathematical outcomes. In addition to theoretical analysis, game-theoretical outcomes also have been applied to offer optimal outcomes or equilibrium situations for many real-world models. Based on the demands for accounting, economics, management sciences and even sports sciences, some more power outputs are introduced as adequate mappings from weights and decision quota to power. The most extensively applied one is the Banzhaf–Coleman output. The Banzhaf–Coleman output, named after Banzhaf [1], is a power output proposed by the probability of changing a result of a vote where voting interests are not necessarily distributed equally among the voters. Plainly speaking, the Banzhaf–Coleman output is a distribution notion that collects each performer’s average marginal contribution from all participated coalitions. Several related outcomes may be found in, e.g., Banzhaf [1], Owen [2], Dubey and Shapley [3], Moulin [4], Lehrer [5], Haller [6] and van den Brink and van der Laan [7].

Under traditional schemes, each performer is either entirely participated or not participated at all in participation with other performers, while under a multi-choice scheme, each performer could be consented to take finite many different energy levels. It is known
that outputs on multi-choice schemes could be performed under various issues such as economics, management sciences, sports sciences and so on. Hwang and Liao [8,9] and Liao [10] proposed several extensions of the max-reduced scheme and the complement-reduced scheme to characterize several core concepts in the context of multi-choice schemes. Later, Hwang and Liao [11] also considered an extended self-reduced scheme to analyze a multi-choice generalization of the Shapley value.

In different topics, from biomedical engineering, sciences to environment and the management sciences, performers confront an increasing demand to focus on multiple objectives effectively in its operational procedures. Related conditions include analyzing allocation tradeoffs, selecting optimal strategy or course designs or an arbitrary other situation where you need an effective rule with tradeoffs among several objectives. Under various cases, these real-world effective conditions might be modeled as a mathematical multi-attribute optimization status. The rules of such conditions require appropriate techniques to present optimal outcomes that—unlike traditional notions or viewpoints—apply several properties of the objectives into account.

Under the axiomatic processes for outputs under cooperative schemes, consonance is a crucial property of useful outputs. The notion behind this type of consonance is as follows: for a given scheme, performers might develop prospects of the scheme and may be willing to consent the computation of its remunerations to be based upon these prospects. The output concept is consonant if it gives the same remunerations to performers in the original scheme as it does to performers of the imaginary reduced scheme. Thus, consonance is a requisite of the inner “robustness” of compromises. In addition to axiomatic characterizations, dynamic processes can be represented that lead the performers to that output, commencing from an arbitrary efficacious remuneration vector. The main base of a dynamic notion was raised from Stearns [12].

The above-mentioned existing-outcomes generate one motivation:
• whether the power (usability) outputs could be defined by simultaneously considering multi-choice behavior and multi-attribute situations.

Different from the contexts of traditional schemes and multi-choice schemes, we focus on the framework of multi-attribute multi-choice schemes throughout this paper.

1. In Section 2.1, the multi-consideration Banzhaf–Coleman output, the multi-consideration efficient Banzhaf–Coleman output and the multi-consideration normalized Banzhaf–Coleman output are further defined by adopting maximal-usability among multi-choice level vectors on multi-attribute multi-choice schemes.
2. In Section 2.1, some motivating and practical examples are provided to present related applications for sports management.
3. By applying an extended reduction, we propose some axiomatic processes to present the reasonability for these outputs in Section 3. In order to analyze the dynamic processes of these outputs, we adopt alternative formulations for these outputs in terms of excess mappings.
4. In Section 4, specific reductions and excess mapping are applied to present that these outputs can be reached by performers who commence from an arbitrary efficacious remuneration vector.

2. Preliminaries

2.1. Definitions and Notations

Let $UV$ be the universe of performers and $P \subseteq UV$ be a set of performers. Let $e = (e_p)_{p \in P}$ with $e_p \in \mathbb{N}$ be the vector that shows the amount of energy levels for each performer, at which it can operate. For $p \in UV$, we define $E_p = \{0, 1, \ldots, e_p\}$ to be the level repository of performer $p$, where $0$ means not acting, and $E_p^+ = E_p \setminus \{0\}$. For $P \subseteq UV$, $P \neq \emptyset$, let $E^P = \prod_{p \in P} E_p$ be the product set of the level repositories for performers in $P$. For every $K \subseteq P$, we define $\sigma^K \in E^P$ as the vector with $\sigma^K_p = 1$ if $p \in K$, and $\sigma^K_p = 0$ if $p \in P \setminus K$. Denote $0_P$ to be the zero vector in $\mathbb{R}^P$. For $m \in \mathbb{N}$, let $\Gamma_m = \{1, \ldots, m\}$. 
A multi-choice scheme is denoted by \((P,e,h)\), where \(P \neq \emptyset\) is a finite collection of performers, \(e = \{e_p\}_{p \in P} \in E^P\) is the vector that appears the amount of energy levels for each performer and \(h : E^P \rightarrow \mathbb{R}\) is a map which apportions to each \(\omega = \{\omega_p\}_{p \in P} \in E^P\) the usability that the performers can accept when each performer \(p\) takes energy level \(\omega_p \in E_p\) with \(h(0) = 0\). A scheme \((P,e,h)\) will be denoted by the map \(h\) if there is no confusion. Given a multi-choice scheme \((P,e,h)\) and \(\omega \in E^P\), we denote \((P,\omega,h)\) to be the multi-choice subscheme defined by restricting \(h\) to \(\{\chi \in E^P \mid \chi_p \leq \omega_p \forall p \in P\}\). A multi-attribute multi-choice scheme is a triple \((P,e,H^m)\), where \(m \in \mathbb{N}\), \(H^m = \{h^t\}_{t \in \Gamma}\) and \((P,e,h^t)\) is a multi-choice scheme for every \(t \in \Gamma_m\). Let \(\Delta\) be the collection of all multi-attribute multi-choice schemes.

Given \((P,e,h) \in \Delta\) and \(\omega \in \mathbb{R}^P\), we define that \(S(\omega) = \{p \in P \mid \omega_p \neq 0\}\) and \(\omega_K\) to be the restriction of \(\omega\) at \(K\) for each \(K \subseteq P\). Further, we define \(h^t_s(K) = \max_{\omega \in E^P} \{h(\omega)|S(\omega) = K\}\) to be the maximal-usability (Here we apply bounded multi-choice schemes, defined as \((P,e,h)\) such that, there exists \(E_h \in \mathbb{R}\) such that \(h(\omega) \leq E_h\) for every \(\omega \in E^P\). We apply it to guarantee that \(h_s(K)\) is well-defined.) among all vectors \(\omega\) with \(S(\omega) = K\). A remuneration vector of \((P,e,H^m)\) is a vector \(x = (x_t)_{t \in \Gamma}\) and \(x^t = (x^t_p)_{p \in P} \in \mathbb{R}^P\), where \(x^t_p\) denotes the remuneration to performer \(p\) in \((P,e,h^t)\) for every \(t \in \Gamma_m\) and for every \(p \in P\). A remuneration vector \(x\) of \((P,e,H^m)\) is multi-attribute efficacious if \(\sum_{p \in P} x_p = h^t_s(P)\) for every \(t \in \Gamma_m\). The collection of all multi-attribute efficacious vectors of \((P,e,H^m)\) is denoted by \(ME(P,e,H^m)\).

An output is a map \(\rho\) apportioning to each \((P,e,H^m) \in \Delta\), an element

\[\rho(P,e,H^m) = (\rho^t(P,e,H^m))_{t \in \Gamma_m}\]

where \(\rho^t(P,e,H^m) = (\rho^t_p(P,e,H^m))_{p \in P} \in \mathbb{R}^P\) and \(\rho^t_p(P,e,H^m)\) is the remuneration of the performer \(p\) apportioned by \(\rho\) in \((P,e,h^t)\).

Next, we introduce three outputs under the multi-attribute multi-choice situation.

**Definition 1.** The multi-consideration Banzhaf–Coleman output (MBCO), \(\Theta\), is defined to be for every \((P,e,H^m) \in \Delta\), for every \(t \in \Gamma_m\) and for every \(p \in P\),

\[\Theta^t_p(P,e,H^m) = \frac{1}{2^{|P|-1}} \sum_{\substack{S \subseteq P \setminus \{p\}}} \left[h^t_s(S) - h^t_s(S \setminus \{p\})\right].\]

Under the output \(\Theta\), all performers receive an average marginal contribution of maximal-usability in each \(S \subseteq P\).

An output \(\rho\) conforms multi-attribute efficacy (MEIY) if for every \((P,e,H^m) \in \Delta\) and for every \(t \in \Gamma_m\), \(\sum_{p \in P} \rho^t_p(P,e,H^m) = h^t_s(P)\). MEIY presents that all performers completely allocate all the usability. Clearly, the MBCO violates MEIY. In the following, we consider an efficacious extension and an normalization.

**Definition 2.**

- The multi-consideration efficacious Banzhaf–Coleman output (MEBCO), \(\overline{\Theta}\), is defined for every \((P,e,H^m) \in \Delta\), for every \(t \in \Gamma_m\) and for every \(p \in P\),

\[\overline{\Theta}^t_p(P,e,H^m) = \Theta^t_p(P,e,H^m) + \frac{1}{|P|} \cdot \left[h^t_s(P) - \sum_{k \in P} \Theta^t_k(P,e,H^m)\right].\]

Under the output \(\overline{\Theta}\), all performers first receive its MBCO, and further allocate the remaining usability equally.
The multi-consideration normalized Banzhaf–Coleman output (MNBCO), $\overline{\Phi}_p$, is defined for every $(P, e, H^m) \in \Delta^*$, for every $t \in \Gamma_m$ and for every $p \in P$,

$$\overline{\Phi}_p(P, e, H^m) = \frac{h^t_k(P)}{\sum_{k \in P} \Theta^t_k(P, e, H^m)} \cdot \Theta^t_p(P, e, H^m),$$

where $\Delta^* = \{(P, e, H^m) \in \Delta \mid \sum_{p \in P} \Theta^t_k(P, e, H^m) \neq 0 \text{ for every } t \in \Gamma_m\}$. Under the notion of $\overline{\Phi}$, all performers allocate the maximal-usability of the grand coalition proportionally by applying the MBCO of all performers.

**Lemma 1.** The MEBCO and the MNBCO conform MEIY on $\Delta$ and $\Delta^*$, respectively.

**Proof.** For every $(P, e, H^m) \in \Delta$ and for every $t \in \Gamma_m$,

$$\sum_{p \in P} \overline{\Phi}_p(P, e, H^m) = \sum_{p \in P} \left[ \Theta^t_p(P, e, H^m) + \frac{1}{|P|} \cdot \left[ h^t_k(P) - \sum_{k \in P} \Theta^t_k(P, e, H^m) \right] \right]$$

$$= \sum_{p \in P} \Theta^t_p(P, e, H^m) + \frac{|P|}{|P|} \cdot \left[ h^t_k(P) - \sum_{k \in P} \Theta^t_k(P, e, H^m) \right]$$

$$= \sum_{p \in P} \Theta^t_p(P, e, H^m) + h^t_k(P) - \sum_{k \in P} \Theta^t_k(P, e, H^m)$$

$$= h^t_k(P).$$

Thus, the MEBCO conforms MEIY on $\Delta$. For every $(P, e, H^m) \in \Delta^*$ and for every $t \in \Gamma_m$,

$$\sum_{p \in P} \overline{\Phi}_p(P, e, H^m) = \sum_{p \in P} \left[ \frac{h^t_k(P)}{\sum_{k \in P} \Theta^t_k(P, e, H^m)} \cdot \Theta^t_p(P, e, H^m) \right]$$

$$= \frac{h^t_k(P)}{\sum_{k \in P} \Theta^t_k(P, e, H^m)} \cdot \left[ \sum_{p \in P} \Theta^t_p(P, e, H^m) \right]$$

$$= h^t_k(P).$$

Thus, the MNBCO conforms MEIY on $\Delta^*$.

2.2. Motivating and Practical Examples

As we mentioned in the introduction, each performer might be allowed to participate with different levels in real situations respectively. On the other hand, multi-attribute analysis is a notion of multiple criterion analysis that is concerned with situations involving simultaneously more than one objective to be optimized. Multi-attribute analysis has been adopted in many issues, including biomedical engineering, economics, politics, sports management sciences, logistics, where efficacious decisions need to be adopted in the presence of trade-offs among several objectives. For instance, minimizing cost while maximizing comfort while marketing a central air conditioning system, and maximizing efficacy whilst minimizing energy consumption and emission of pollutants are examples of multi-attribute efficacious problems involving respectively two and three objectives. Under various cases, there might be more than three objectives. Hence, we focus on the framework of multi-attribute multi-choice schemes throughout this paper. The advantages of our methods are that these outputs of the multi-attribute multi-choice scheme always exist and result in a kind of global outcome for a specific performer by summarizing the result under all its energy levels.

Here we provide a brief motivating example of multi-choice schemes in the setting of “sports management”. Let $P = \{1, 2, \ldots, p\}$ be a set of all performers of a sports management system $(P, e, h)$. The function $h$ could be treated as a usability function which assigns to each level vector $a = (a_p)_{p \in P} \in E^P$ the worth that the performers can obtain when each performer $p$ participates at operation strategy $a_p \in E_p$ in $(P, e, h)$. Modeled in this way, the sports management system $(P, e, h)$ could be considered as a multi-choice
scheme, with $h$ being each characteristic function and $E_p$ being the set of all operation strategies of the performer $p$.

In the following, we also provide a practical application of power distribution in a sports association, such as NBA, MLB and so on. Let $P = \{1, 2, \ldots, p\}$ be a set of all performers of a sports association. In the sports association, all the performers are elected by voting or recommendation by sports parties (or teams). All performers have the power to propose, discuss, establish and veto all bills (or rules). They dedicate different levels of attention and participation to different bills depending on their academic expertise and the public opinion they represent. The level of involvement is also closely associated with the alliance strategy formed for the interests of different sports parties. For the aforementioned reasons, strategies adopted by each performer of the parliament show distinct levels of participation and certain amounts of ambiguity. The function $h$ could be treated as a power function which assigns to each level vector $a = (a_p)_{p \in P} \in E^P$ the power that the performers can dedicate when each performer $p$ participates at operation strategy $a_p \in E_p$.

Modeled in this way, the sports association operational system $(P, e, h)$ could be considered as a multi-choice scheme, with $h$ being each characteristic function and $E_p$ being the set of all operation strategies of the performer $p$. To evaluate the influence of each performer in the sports association, using the power indicators we proposed, we first assess the influence each sports association performer has arisen over previous bill meetings based on various strategies, which are the outputs mentioned in Definitions 1 and 2.

Here we provide an application with real data as follows. Let $(P, e, H^m) \in \Delta$ with $P = \{i, j, k\}$, $m = 2$ and $e = (2, 1, 1)$. Thus, $(P, e, H^m) = ((P, e, h^1), (P, e, h^2))$. Further, let $h^1(2, 1, 1) = 5, h^1(1, 1, 1) = 7, h^1(2, 1, 0) = 3, h^1(2, 0, 1) = 2, h^1(2, 0, 0) = 9$, $h^1(1, 1, 0) = 3, h^1(1, 0, 1) = −4, h^1(0, 1, 1) = 5, h^1(1, 0, 0) = −1, h^1(0, 1, 0) = 2, h^1(0, 0, 1) = −3, h^2(2, 1, 1) = 9, h^2(1, 1, 1) = 5, h^2(2, 1, 0) = 7, h^2(2, 0, 1) = 3, h^2(2, 0, 0) = 2, h^2(1, 1, 0) = 9, h^2(1, 0, 1) = 3, h^2(0, 1, 1) = −4, h^2(1, 0, 0) = 5, h^2(0, 1, 0) = −2, h^2(0, 0, 1) = 5$ and $h^2(0, 0, 0) = 0 = h^2(0, 0, 0)$. Therefore, we have that $h^1_i(\{i, j, k\}) = 7, h^1_i(\{i, j\}) = 3, h^1_i(\{i, k\}) = 2, h^1_i(\{j, k\}) = 5, h^1_i(\{i\}) = 9, h^1_i(\{j\}) = 2, h^1_i(\{k\}) = −3, h^2_i(\{i, j, k\}) = 9, h^2_i(\{i, j\}) = 9, h^2_i(\{i, k\}) = 3, h^2_i(\{j, k\}) = −4, h^2_i(\{i\}) = 5, h^2_i(\{j\}) = −2, h^2_i(\{k\}) = 5$ and $h^2_i(\emptyset) = 0 = h^2_i(\emptyset)$. By Definitions 1 and 2,

\[
\begin{align*}
\Theta^1_1(P, e, H^m) &= \frac{18}{4}, \quad \Theta^1_2(P, e, H^m) = \frac{15}{4}, \quad \Theta^1_3(P, e, H^m) = \frac{7}{4}, \\
\Theta^2_1(P, e, H^m) &= \frac{55}{12}, \quad \Theta^2_2(P, e, H^m) = \frac{49}{12}, \quad \Theta^2_3(P, e, H^m) = \frac{13}{12}, \\
\Phi^1_1(P, e, H^m) &= \frac{144}{31}, \quad \Phi^1_2(P, e, H^m) = \frac{120}{31}, \quad \Phi^1_3(P, e, H^m) = \frac{-16}{31}, \\
\Phi^2_1(P, e, H^m) &= \frac{27}{4}, \quad \Phi^2_2(P, e, H^m) = 0, \quad \Phi^2_3(P, e, H^m) = \frac{3}{4}, \\
\Phi^3_1(P, e, H^m) &= \frac{31}{8}, \quad \Phi^3_2(P, e, H^m) = 1, \quad \Phi^3_3(P, e, H^m) = \frac{1}{4}. 
\end{align*}
\]

In the following sections, we would like to demonstrate that the MBCO, the MECBO and the MNBCO could present "optimal distribution mechanisms" among all performers, in the sense that this organization can receive remuneration from each combination of operational levels of all performers under multi-choice behavior and multi-attribute situations.

3. Axiomatic Outcomes

Here, we demonstrate that there exists corresponding reduced schemes that could be applied to axiomatize the MBCO, the MECBO and the MNBCO.
Let $\rho$ be an output, $(P, e, H^m) \in \Delta$ and $S \subseteq P$. The 1-reduced scheme $(S, e_S, H_{S, \rho}^{1,m})$ is defined by $H_{S, \rho}^{1,m} = (h_{S, \rho}^{1,l})_{t \in T_m}$ and

$$h_{S, \rho}^{1,l}(\omega) = \begin{cases} 0, & \omega = 0_S, \\ \frac{1}{2^{\rho_S}} \sum_{Q \subseteq P \setminus S} \left[ \max_{\tau \in E^Q} \{ h(\omega, \tau, 0_{P \setminus (S \cup Q)} \} | S(\tau) = Q \} \right. \\ \left. - \sum_{p \in Q} \rho_p(P, e, h) \right], & \text{otherwise} \end{cases}$$

for every $\omega \in E^S$. An output $\rho$ conforms 1-consonance (1CSE) if $\rho_p'(S, e_S, H_{S, \rho}^{1,m}) = \rho_p'(P, e, H^m)$ for every $(P, e, H^m) \in \Delta$, for every $S \subseteq P$ with $|S| = 2$, for every $t \in T_m$ and for every $p \in S$. Further, $\rho$ conforms 1-criterion for schemes (1CS) if $\rho(P, e, H^m) = \Theta(P, e, H^m)$ for every $(P, e, H^m) \in \Delta$ with $|P| \leq 2$. In the following, we characterize the MBCO by adopting 1CSE and 1CS.

**Lemma 2.**

1. The MBCO conforms 1CSE on $\Delta$.
2. On $\Delta$, the MBCO is the only output conforming 1CS and 1CSE.

**Proof.** To demonstrate result 1, let $(P, e, H^m) \in \Delta$ and $S \subseteq P$. The proof is finished if $|P| = 1$. Suppose that $|P| \geq 2$ and $S = \{p, q\}$ for some $p, q \in P$, for every $t \in T_m$ and for every $p \in S$,

$$\Theta_p'(S, e_S, H_{S, \rho}^{1,m}) = \frac{1}{2^{\rho_S}} \cdot \sum_{T \subseteq T_m} \sum_{p \in T} \left[ (h_{S, \rho}^{1,l})_s(T) - (h_{S, \rho}^{1,l})_s(T \setminus \{p\}) \right]$$

$$= \frac{1}{2^{\rho_S}} \cdot \sum_{T \subseteq T_m} \sum_{p \in T} \left[ h^1(T \cup Q) - h^1(T \setminus \{p\} \cup Q) \right]$$

$$= \sum_{K \subseteq P \setminus S} \sum_{p \in K} \left[ h^1(K) - h^1(K \setminus \{p\}) \right]$$

$$= \Theta_p'(P, e, H^m).$$

Hence, $\Theta$ conforms 1CSE.

By result 1, $\Theta$ conforms 1CSE on $\Delta$. Clearly, $\Theta$ conforms 1CS on $\Delta$. To demonstrate uniqueness of result 2, suppose that $\rho$ conforms 1CS and 1CSE. Let $(P, e, h) \in \Delta$. If $|P| \leq 2$, then $\rho(P, e, h) = \Theta(P, e, h)$ by 1CS. The condition $|P| > 2$: Let $p \in P$ and $t \in T_m$, and let $S \subseteq P$ with $|S| = 2$ and $p \in S$. Therefore,

$$\rho_p'(P, e, H^m) = \rho_p'(S, e_S, H_{S, \rho}^{1,m}) = \Theta_p'(S, e_S, H_{S, \rho}^{1,m})$$

$$= \frac{1}{2^{\rho_S}} \cdot \sum_{T \subseteq T_m} \sum_{p \in T} \left[ (h_{S, \rho}^{1,l})_s(T) - (h_{S, \rho}^{1,l})_s(T \setminus \{p\}) \right]$$

$$= \frac{1}{2^{\rho_S}} \cdot \sum_{T \subseteq T_m} \sum_{p \in T} \left[ h^1(T \cup Q) - h^1((T \setminus \{p\}) \cup Q) \right]$$

$$= \sum_{K \subseteq P \setminus S} \sum_{p \in K} \left[ h^1(K) - h^1(K \setminus \{p\}) \right]$$

$$= \Theta_p'(P, e, H^m).$$

Thus, $\rho(P, e, H^m) = \Theta(P, e, H^m)$. □

In the following we take some examples to manifest that each of the properties applied in Lemma 2 are independent of the rest of properties.
Example 1. Consider an output $\rho$ for every $(P, e, H^n) \in \Delta$, for every $t \in \Gamma_m$ and for every $p \in P$, $\rho_p(P, e, H^n) = 0$. Clearly, $\rho$ conforms 1CSE, but it does not conform 1CS.

Example 2. Consider an output $\rho$ for every $(P, e, H^n) \in \Delta$, for every $t \in \Gamma_m$ and for every $p \in P$,

$$\rho_p^t(P, e, H^m) = \begin{cases} \Theta_p^t(P, e, H^m) & \text{if } |P| \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

On $\Delta$, $\rho$ conforms 1CS, but it does not conform 1CSE.

It is easy to demonstrate that the output $\overline{\Theta}$ and $\overline{\Theta}$ violates 1CSE. In the following, we consider the 2-reduced scheme. Let $\rho$ be an output, $(P, e, H^n) \in \Delta$ and $S \subseteq P$. The 2-reduced scheme $(S, e_S, H_{S, \rho}^{2m})$ is defined by $H_{S, \rho}^{2m} = (h_{S, \rho}^{2j})_{t \in \Gamma_m}$ and

$$h_{S, \rho}^{2j}(\omega) = \begin{cases} 0 & , \omega = 0_S, \\ \max_{t \in E_{P, S}} \{h(\omega, \tau) | S(\tau) = P \setminus S\} - \sum_{p \in P \setminus S} \rho_p(P, e, h) , S(\omega) = S, \\ \frac{1}{2^{S^2}} \sum_{Q \subseteq P, S} \left[ \max_{t \in E_{Q}} \{h(\omega, \tau, 0_{P \setminus (S \cup Q)} | S(\tau) = Q\} - \sum_{p \in Q} \rho_p(P, e, h) \right] , \text{otherwise} \end{cases}$$

for every $\omega \in E^S$. An output $\rho$ conforms 2-consonance (2CSE) if $\rho_p^t(S, e_S, H_{S, \rho}^{2m}) = \rho_p^t(P, e, H^n)$ for every $(P, e, H^n) \in \Delta$, for every $S \subseteq P$ with $|S| = 2$, for every $t \in \Gamma_m$ and for every $p \in S$. Furthermore, $\rho$ conforms 2-criterion for schemes (2CS) if $\rho(P, e, H^n) = \overline{\Theta}(P, e, H^n)$ for every $(P, e, H^n) \in \Delta$ with $|P| \leq 2$.

Clearly, $(S, e_S, H_{S, \rho}^{2m})$ does not exist if $\sum_{p \in S} \Theta_p^t(P, e, H^n) = 0$. Thus, we consider the 3-consonance as follows. An output $\rho$ conforms 3-consonance (3CSE) if $(S, e_S, H_{S, \rho}^{2m}) \in \Delta^*$, for some $(P, e, H^n) \in \Delta$ and for some $S \subseteq P$ with $|S| = 2$, it holds that $\rho_p^t(S, e_S, H_{S, \rho}^{2m}) = \rho_p^t(P, e, H^n)$ for every $t \in \Gamma_m$ and for every $p \in S$. Further, $\rho$ conforms 3-criterion for schemes (3CS) if $\rho(P, e, H^n) = \overline{\Theta}(P, e, H^n)$ for every $(P, e, H^n) \in \Delta$ with $|P| \leq 2$.

Subsequently, we characterize the MEBCO and the MNBCO by respectively adopting 2CSE, 3CSE, 2CS and 3CS. In order to establish consonance of the MEBCO and the MNBCO, it will be useful to introduce alternative formulation for the MEBCO and the MNBCO in terms of excess. Let $(P, e, H^n) \in \Delta$, $S \subseteq P$ and $x$ be a remuneration vector in $(P, e, H^n)$. Define that $x(S) = \sum_{p \in S} x_p$ for every $t \in \Gamma_m$. The excess of a coalition $S \subseteq P$ at $x$ is considered to be

$$\text{Ex}(S, H^n, x) = (\text{Ex}(S, h^t, x^t))_{t \in \Gamma_m} \text{ and } \text{Ex}(S, h^t, x^t) = h^t(S) - x^t(S).$$

The value $\text{Ex}(S, h^t, x^t)$ can be regarded as the complaint of coalition $S$ when all performers in $S$ receive its remunerations from $x^t$ in $(P, e, h^t)$.

Lemma 3. For every $(P, e, H^n) \in \Delta$, for every $x \in \text{ME}(P, e, H^n)$, for every $p, q \in P$ and for every $t \in \Gamma_m$,

$$\sum_{S \subseteq P \setminus \{p, q\}} \text{Ex}(S \cup \{p\}, h^t, x^t) + \sum_{S \subseteq P \setminus \{p, q\}} \text{Ex}(S \cup \{q\}, h^t, x^t) \iff x = \overline{\Theta}(P, e, H^n).$$
Proof. Let \((P, e, H^m)\) \(\in \Delta\) and \(x \in ME(P, e, H^m)\). For every \(t \in \Gamma_m\) and for every \(p, q \in P\),

\[
\sum_{S \subseteq P \setminus \{p,q\}} Ex(S \cup \{p\}, h^t, x^t) \not\equiv \sum_{S \subseteq P \setminus \{p,q\}} Ex(S \cup \{q\}, h^t, x^t)
\]

\[
\iff \sum_{S \subseteq P \setminus \{p,q\}} \left[ h^t_i(S \cup \{i\}) - x^t_i(S \cup \{i\}) \right] = \sum_{S \subseteq P \setminus \{p,q\}} \left[ h^t_i(S \cup \{j\}) - x^t_i(S \cup \{j\}) \right]
\]

\[
\iff \left[ \sum_{S \subseteq P \setminus \{p,q\}} h^t_i(S \cup \{i\}) \right] - 2^{|P| - 2} \cdot x^t_p = \left[ \sum_{S \subseteq P \setminus \{p,q\}} h^t_i(S \cup \{j\}) \right] - 2^{|P| - 2} \cdot x^t_q
\]

\[
x^t_p - x^t_q = \frac{1}{2^{|P| - 2}} \cdot \sum_{S \subseteq P \setminus \{p,q\}} \left[ h^t_i(S \cup \{i\}) - h^t_i(S \cup \{j\}) \right].
\]

By definition of \(\overline{\Theta}\),

\[
\overline{\Theta}_p(P, e, H^m) - \overline{\Theta}_q(P, e, H^m) = \frac{1}{2^{|P| - 2}} \cdot \sum_{S \subseteq P \setminus \{p,q\}} \left[ h^t_i(S \cup \{p\}) - h^t_i(S \cup \{q\}) \right].
\]

By Equations (2) and (3), for every \(p, q \in P\),

\[
x^t_p - x^t_q = \overline{\Theta}_p(P, e, H^m) - \overline{\Theta}_q(P, e, H^m).
\]

Hence,

\[
\sum_{q \in P} \left[ x^t_p - x^t_q \right] = \sum_{q \in P} \left[ \overline{\Theta}_p(P, e, H^m) - \overline{\Theta}_q(P, e, H^m) \right].
\]

That is, \(|P| \cdot x^t_p - \sum_{q \in P} x^t_q = |P| \cdot \overline{\Theta}_p(P, e, H^m) - \sum_{q \in P} \overline{\Theta}_q(P, e, H^m)\). Since \(x \in ME(P, e, H^m)\) and \(\overline{\Theta}\) conforms MEY, \(|P| \cdot x^t_p - h^t_i(P) = |P| \cdot \overline{\Theta}_p(P, e, H^m) - h^t_i(P)\). Hence, \(x^t_p = \overline{\Theta}_p(P, e, H^m)\) for every \(t \in \Gamma_m\) and for every \(p \in P\), i.e., \(x = \overline{\Theta}(P, e, H^m)\). \(\square\)

Remark 1. Clearly,

\[
\sum_{S \subseteq P \setminus \{p,q\}} Ex(S \setminus \{p\}, H^m, \Theta(P, e, H^m)) = \sum_{S \subseteq P \setminus \{p,q\}} Ex(S \setminus \{q\}, H^m, \Theta(P, e, H^m))
\]

for every \((P, e, H^m) \in \Delta\) and for every \(p, q \in P\).

Theorem 1.

1. The MEBCO conforms 2CSE on \(\Delta\).
2. If \(\rho\) conforms 2CS and 2CSE, then it also conforms MEY.
3. On \(\Delta\), the MEBCO is the only output conforming 2CS and 2CSE.

Proof. To demonstrate result 1, let \((P, e, H^m) \in \Delta\) and \(S \subseteq P\). The proof is finished if \(|P| = 1\). Suppose that \(|P| \geq 2, x = \Theta(P, e, H^m)\) and \(S = \{p, q\}\) for some \(p, q \in P\). For every \(t \in \Gamma_m\) and for every \(l \in S\),

\[
Ex(\{l\}, h^t_{S \setminus \{l\}}, x^t_l) = h^t_{S \setminus \{l\}}(\{l\}) - x^t_l
\]

\[
= \left( \frac{1}{2^{|P|}} \cdot \sum_{Q \subseteq P \setminus S} \left[ h^t_i(\{l\} \cup Q) - \sum_{k \in Q} x^t_k \right] \right) - x^t_l
\]

\[
= \left( \frac{1}{2^{|P|}} \cdot \sum_{Q \subseteq P \setminus S} \left[ h^t_i(\{l\} \cup Q) - \sum_{k \in \{l\} \cup Q} x^t_k \right] \right)
\]

\[
= \left( \frac{1}{2^{|P|}} \cdot \sum_{Q \subseteq P \setminus \{p,q\}} Ex(Q \cup \{l\}, h^t_l) \right).
\]
Since $\mathcal{\overline{O}}$ conforms MEIY, $x^*_S \in \text{Ex}(S, e_S, h^2_{S,x})$ by definition of 2-reduced scheme. Further, by Lemma 3 and Equation (4),

$$\text{Ex}(\{p\}, h^2_{S,x}, x^*_S) = \frac{1}{2^{|P|}} \cdot \sum_{Q \subseteq P \setminus \{p\}} \text{Ex}(Q \cup \{p\}, h^t, x^t) \quad \text{(by Equation (4))}$$

$$= \frac{1}{2^{|P|}} \cdot \sum_{Q \subseteq P \setminus \{p\}} \text{Ex}(Q \cup \{q\}, h^t, x^t) \quad \text{(by Lemma 3)}$$

$$= \text{Ex}(\{q\}, h^2_{S,x}, x^*_S). \quad \text{(by Equation (4))}$$

Therefore, $x_S = \mathcal{\overline{O}}(S, e_S, h^2_{S,x})$. That is, $\mathcal{\overline{O}}$ conforms 2CSE.

To demonstrate result 2, suppose $\rho$ conforms 2CS and 2CSE. Let $(P, e, H^m) \in \Delta$ and $t \in \Gamma_m$. By 2CS, $\rho$ conforms MEIY if $|P| \leq 2$. The condition $|P| > 2$: Suppose, on the contrary, that there exists $(P, e, H^m) \in \Delta$ such that $\sum_{p \in P} \rho^e_P(P, e, H^m) \neq h^t(P)$. This presents that there exists $p \in P$ and $q \in P$ such that $[h^t_P(P) - \sum_{k \in P \setminus \{p, q\}} \rho^e_k(P, e, H^m)] \neq [\rho^e_P(P, e, H^m) + \rho^e_q(P, e, H^m)]$. By 2CSE, $\rho$ conforms MEIY for two-person schemes and this contradicts with

$$\rho^e_P(P, e, H^m) + \rho^e_q(P, e, H^m) = \rho^e_P(\{p, q\}, e, H^m) + \rho^e_q(\{p, q\}, e, H^m) = h^t(P) - \sum_{k \in P \setminus \{p, q\}} \rho^e_k(P, e, H^m).$$

Hence, $\rho$ conforms MEIY.

To demonstrate result 3, $\mathcal{\overline{O}}$ conforms 2CSE by result 1. Clearly, $\mathcal{\overline{O}}$ conforms 2CS. To demonstrate uniqueness, suppose $\rho$ conforms 2CS and 2CSE; hence, by result 2, $\rho$ also conforms MEIY. Let $(P, e, H^m) \in \Delta$. By 2CS, $\rho(P, e, H^m) = \mathcal{\overline{O}}(P, e, H^m)$ if $|P| \leq 2$. The condition $|P| > 2$: Let $p \in P, t \in \Gamma_m$ and $S = \{p, q\}$ for some $q \in P \setminus \{p\}$. Then,

$$\rho^e_P(P, e, H^m) - \mathcal{\overline{O}}^e_P(P, e, H^m) = \rho^e_P(S, e_S, h^2_{S,\rho}) - \mathcal{\overline{O}}^e_P(S, e_S, h^2_{S,\rho}) \quad \text{(by 2CSE of $\rho$, $\mathcal{\overline{O}}$)}$$

$$= \frac{1}{2^{|P|}} \cdot \sum_{Q \subseteq P \setminus \{p, q\}} \left[ h^t(S) - h^t(Q) \right] \left[ h^2_{S,\rho}(S) - h^2_{S,\rho}(Q) \right] \quad \text{by 2CS of $\rho$, $\mathcal{\overline{O}}$}$$

$$= \frac{1}{2} \cdot \left[ (h^2_{S,\rho})_*(\{p\}) - (h^2_{S,\rho})_*\{q\} \right]. \quad \text{(5)}$$

By definitions of $h^2_{S,\rho}$ and $h^2_{S,\mathcal{\overline{O}}}$,

$$(h^2_{S,\rho})_*(\{p\}) - (h^2_{S,\rho})_*\{q\} = \frac{1}{2^{|P| - 1}} \cdot \sum_{Q \subseteq P \setminus \{p, q\}} \left[ h^t(S) - h^t(Q) \right] \quad \text{(6)}$$

By Equation (6), Equation (5) becomes

$$\rho^e_P(P, e, H^m) - \mathcal{\overline{O}}^e_P(P, e, H^m) = \frac{1}{2} \cdot \left[ (h^2_{S,\rho})_*(S) - (h^2_{S,\rho})_*(Q) \right]$$

$$= \frac{1}{2} \cdot \left( \rho^e_P(P, e, H^m) + \rho^e_q(P, e, H^m) - \mathcal{\overline{O}}^e_P(P, e, H^m) + \mathcal{\overline{O}}^e_q(P, e, H^m) \right).$$

That is, for every $p, q \in P$,

$$\rho^e_P(P, e, H^m) - \rho^e_q(P, e, H^m) = \mathcal{\overline{O}}^e_P(P, e, H^m) - \mathcal{\overline{O}}^e_q(P, e, H^m).$$

By MEIY of $\rho$ and $\mathcal{\overline{O}}$,

$$|P| \cdot \rho^e_P(P, e, H^m) - h^t(P) = |P| \cdot \mathcal{\overline{O}}^e_P(P, e, H^m) - h^t(P).$$
Thus, $\rho^t_P(P, e, H^m) = \Theta^t_P(P, e, H^m)$ for every $p \in P$, i.e., $\rho(P, e, H^m) = \Theta(P, e, H^m)$. \hfill \square

In the following we take some examples to show that each of the properties applied in Theorem 1 are independent of the rest of properties.

**Example 3.** Consider an output $\rho$ for every $(P, e, H^m) \in \Delta$, for every $t \in \Gamma_m$ and for every $p \in P$, $\rho^t_P(P, e, H^m) = 0$. Clearly, $\rho$ conforms 2CSE, but it does not conform 2CS.

**Example 4.** Consider an output $\rho$ for every $(P, e, H^m) \in \Delta$, for every $t \in \Gamma_m$ and for every $p \in P$,

$$
\rho^t_P(P, e, H^m) = \begin{cases} 
\Theta^t_P(P, e, H^m) & \text{if } |P| \leq 2, \\
0 & \text{otherwise.}
\end{cases}
$$

On $\Delta$, $\rho$ conforms 2CS, but it does not conform 2CSE.

**Lemma 4.** Let $(P, e, H^m) \in \Delta^*$ and $x \in ME(P, e, H^m)$. Then, for every $p, q \in P$ and $t \in \Gamma_m$,

$$
\sum_{S \subseteq P \setminus \{p, q\}} \text{Ex}(S \cup \{p\}, h^t, \frac{x^t}{a^t}) = \sum_{S \subseteq P \setminus \{p, q\}} \text{Ex}(S \cup \{q\}, h^t, \frac{x^t}{a^t})
$$

where $a^t = \sum_{k \in P} h^t(k, P, e, H^m)$.

**Proof.** Let $(P, e, H^m) \in \Delta^*$ and $x \in ME(P, e, H^m)$. For every $p, q \in P$ and $t \in \Gamma_m$,

$$
\sum_{S \subseteq P \setminus \{p, q\}} \text{Ex}(S \cup \{p\}, h^t, \frac{x^t}{a^t}) = \sum_{S \subseteq P \setminus \{p, q\}} \text{Ex}(S \cup \{q\}, h^t, \frac{x^t}{a^t})
$$

where $a^t = \sum_{k \in P} h^t(k, P, e, H^m)$.

By definition of $\Phi$,

$$
\Phi^t_P(P, e, H^m) - \Phi^t_q(P, e, H^m) = \sum_{k \in P \setminus \{p, q\}} \text{Ex}(S \cup \{p\}, h^t, \frac{x^t}{a^t}) - \sum_{k \in P \setminus \{p, q\}} \text{Ex}(S \cup \{q\}, h^t, \frac{x^t}{a^t})
$$

By Equations (7) and (8), for every pair $\{p, q\} \subseteq P$ and $t \in \Gamma_m$,

$$
x^t_p - x^t_q = \Phi^t_P(P, e, H^m) - \Phi^t_q(P, e, H^m).
$$

Hence,

$$
\sum_{q \neq p} x^t_p - x^t_q = \sum_{q \neq p} [\Phi^t_P(P, e, H^m) - \Phi^t_q(P, e, H^m)].
$$

That is, $(|P| - 1) \cdot x^t_p - \sum_{q \neq p} x^t_q = (|P| - 1) \cdot \Phi^t_P(P, e, H^m) - \sum_{q \neq p} \Phi^t_q(P, e, H^m)$. Since $x \in ME(P, e, H^m)$ and $\Phi$ conforms MEIY, $|P| \cdot x^t_p - \Phi^t_P(P, e, H^m) = |P| \cdot \Phi^t_P(P, e, H^m) - \Phi^t_P(P, e, H^m)$. Hence, $x^t_p = \Phi^t_P(P, e, H^m)$ for every $(P, e, H^m) \in \Delta^*$, for every $p \in P$ and for every $t \in \Gamma_m$. \hfill \square

**Theorem 2.**

1. The MNBCO conforms 3CSE on $\Delta^*$.
2. If $\rho$ conforms 3CS and 3CSE, then it also conforms MEIY.
3. On $\Delta^*$, the MNBCO is the only output conforming 3CS and 3CSE.

**Proof.** The proofs of outcomes 1 and 2 are similar to the proofs of outcomes 1 and 2 of Theorem 1, so we omit them. Next, we demonstrate result 3. By result 1, the MNBCO conforms 3CS. Clearly, the MNBCO conforms 3CSE and 3CS on $\Delta^*$. Let $(P, e, H^m) \in \Delta^*$. The proof could be finished by induction on $|P|$. By 3CS, it is trivial that $\rho(P, e, H^m) = \Theta(P, e, H^m)$ if $|P| \leq 2$. Suppose that it holds if $|P| \leq l - 1$, $l \geq 3$. The condition $|P| = l$: Let $p, q \in P$ with $p \neq q$ and $t \in \Gamma_m$. By Definition 2, $\Phi^i_k(P, e, H^m) = \sum_{h \not\in \rho(P, e, H^m)} \phi_k^i(P, e, H^m)$

for every $k \in P$. Assume that $\omega^i_k = \sum_{h \not\in \rho(P, e, H^m)} \phi_k^i(P, e, H^m)$ for every $k \in P$ and for every $t \in \Gamma_m$.

Therefore,

$$\rho^i_p(P, e, H^m) = \rho_{P \setminus \{q\}}^i(P, e, H^m) = \sum_{h \in \rho(P, e, H^m)} \phi_k^i(P, e, H^m) = \sum_{h \not\in \rho(P, e, H^m)} \phi_k^i(P, e, H^m)$$

(by 3CSE of $\rho$)

$$= \Phi^i_k(P \setminus \{q\}, e, H^m)$$

(by 3CS of $\rho$)

$$= \Phi^i_k(P \setminus \{q\}, e, H^m) + \Phi^i_k(P, e, H^m)$$

(by Equation (8))

$$= \Phi^i_k(P, e, H^m).$$

By Equation (9),

$$\rho^i_p(P, e, H^m) \cdot [1 - \omega^i_q] = [h^i_p(P) - \rho^i_q(P, e, H^m)] \cdot \omega^i_q$$

$$\implies \sum_{p \in P} \rho^i_p(P, e, H^m) \cdot [1 - \omega^i_q] = [h^i_p(P) - \rho^i_q(P, e, H^m)] \cdot \sum_{p \in P} \omega^i_q$$

$$\implies h^i_p(P) \cdot [1 - \omega^i_q] = [h^i_p(P) - \rho^i_q(P, e, H^m)] \cdot 1$$ (by MEIY of $\rho$)

$$\implies h^i_p(P) - h^i_p(P) \cdot \omega^i_q = h^i_p(P) - \rho^i_q(P, e, H^m)$$

$$\implies \Phi^i_q(P, e, H^m) = \rho^i_q(P, e, H^m).$$

The proof is finished. □

In the following we take some examples to manifest that each of the properties applied in Theorem 1 are independent of the rest of the properties.

**Example 5.** Consider an output $\rho$ by for every $(P, e, H^m) \in \Delta^*$, for every $t \in \Gamma_m$ and for every $p \in P$, $\rho^i_p(P, e, H^m) = 0$. On $\Delta^*$, $\rho$ conforms 3CS, but it does not conform 3CS.

**Example 6.** Consider an output $\rho$ by for every $(P, e, H^m) \in \Delta^*$, for every $t \in \Gamma_m$ and for every $p \in P$,

$$\rho^i_p(P, e, H^m) = \begin{cases} \Phi^i_k(P, e, H^m) & \text{if } |P| \leq 2, \\ 0 & \text{otherwise}. \end{cases}$$

On $\Delta^*$, $\rho$ conforms 3CS, but it does not conform 3CSE.

4. Dynamic Processes

Here, we adopt excess mappings and reductions to offer dynamic outcomes for the MEBCO and the MNBCO.

In order to present the dynamic processes of the MEBCO and the MNBCO, we firstly define switch mappings by means of excess mappings. The switch mappings are based on the notion that each performer shortens the complaint relating to its own and others’
non-participation, and uses these regulations to revise the original remuneration. In the following, we firstly introduce the dynamic processes for the MEBCO.

**Definition 3.** Let \((P,e,H^m) \in \Delta \) and \(p \in P\). The switch mapping is considered to be \(f = (f^t)_{t \in \Gamma_m}\), where \(f^t = (f^t_p)_{p \in P}\) and \(f^t_p : ME(P,e,H^m) \to \mathbb{R}\) is defined by

\[
f^t_p(x) = x_p^t + \sum_{q \in P \setminus \{p\}} \frac{1}{2^{|P|-2}} \sum_{Q \subseteq P \setminus \{p,q\}} \left[ Ex(Q \setminus \{q\}, h^t, x^t) - Ex(Q \setminus \{p\}, h^t, x^t) \right]
\]

Define \([x]^0 = x\), \([x]^1 = f([x]^0)\), \ldots, \([x]^n = f([x]^{n-1})\) for every \(n \in \mathbb{N}\).

**Lemma 5.** \(f(x) \in ME(P,e,H^m)\) for every \((P,e,H^m) \in \Delta\) and for every \(x \in ME(P,e,H^m)\).

**Proof.** Let \((P,e,H^m) \in \Delta\), \(t \in \Gamma_m\), \(p,q \in P\) and \(x \in ME(P,e,H^m)\). Similar to Equation (2),

\[
\sum_{q \in P \setminus \{p\}} \frac{1}{2^{|P|-2}} \sum_{Q \subseteq P \setminus \{p,q\}} \left[ Ex(Q \setminus \{q\}, h^t, x^t) - Ex(Q \setminus \{p\}, h^t, x^t) \right]
\]

By definition of \(\Theta\),

\[
\Theta^p(P,e,H^m) - \Theta^q(P,e,H^m) = \frac{1}{2^{|P|-2}} \sum_{Q \subseteq P \setminus \{p,q\}} \left[ h^t(Q \setminus \{q\}) - h^t(Q \setminus \{p\}) \right].
\]

By Equations (10) and (11),

\[
\sum_{q \in P \setminus \{p\}} \frac{1}{2^{|P|-2}} \sum_{Q \subseteq P \setminus \{p,q\}} \left[ Ex(Q \setminus \{q\}, h^t, x^t) - Ex(Q \setminus \{p\}, h^t, x^t) \right]
\]

Moreover,

\[
\sum_{p \in P} |P| \cdot \left( \Theta^p(P,e,H^m) - x_p^t \right)
\]

By MEIY of \(\Theta\), \(x \in ME(P,e,H^m)\)

\[
|P| \cdot \left( \sum_{p \in P} \Theta^p(P,e,H^m) - \sum_{p \in P} x_p^t \right)
\]

By MEIY of \(\Theta\), \(x \in ME(P,e,H^m)\)

\[
0.
\]
Therefore, we have that
\[
\sum_{p \in P} f_p^j(x) = \sum_{p \in P} \left[ x_p^t + \alpha \sum_{q \in P \setminus \{p\}} \frac{1}{2^{P \setminus \{p\}}} \sum_{Q \subseteq P \setminus \{p\}} \left[ \mathbb{E}(Q \setminus \{q\}, h^i, x^t) - \mathbb{E}(Q \setminus \{p\}, h^i, x^t) \right] \right]
\]
\[
= \sum_{p \in P} \left[ x_p^t + \alpha \sum_{q \in P \setminus \{p\}} \sum_{\tilde{Q} \subseteq P \setminus \{q\}} \frac{1}{2^{P \setminus \{q\}}} \sum_{T \subseteq P \setminus \{q\}} \left[ \mathbb{E}(T \setminus \{q\}, h^i, x^t) - \mathbb{E}(T \setminus \{p\}, h^i, x^t) \right] \right]
\]
\[
= h^i_t(p) + 0 \quad (\text{by Equation (13) and } x \in \text{ME}(P, e, H^m))
\]

Hence, \( f(x) \in \text{ME}(P, e, H^m) \) if \( x \in \text{ME}(P, e, H^m) \).

**Theorem 3.** Let \( (P, e, H^m) \in \Delta \). If \( 0 < \alpha < \frac{2}{|T^p|} \), then \( \{[x]^n\}_{n=1}^{\infty} \) converges to \( \Theta(P, e, H^m) \) for each \( x \in \text{ME}(P, e, H^m) \).

**Proof.** Let \( (P, e, H^m) \in \Delta \), \( t \in \Gamma_m \), \( p \in P \) and \( x \in \text{ME}(P, e, H^m) \). By Equation (12) and the definition of \( f \),
\[
f_p^j(x) - x^t_p = \alpha \sum_{q \in P \setminus \{p\}} \frac{1}{2^{P \setminus \{p\}}} \sum_{Q \subseteq P \setminus \{p,q\}} \left[ \mathbb{E}(Q \setminus \{q\}, h^i, x^t) - \mathbb{E}(Q \setminus \{p\}, h^i, x^t) \right]
\]
\[
= \alpha \cdot |P| \cdot \left( \Theta_p(P, e, H^m) - x^t_p \right).
\]

Hence,
\[
\Theta_p(P, e, H^m) - f_p^j(x) = \Theta_p(P, e, H^m) - x^t_p + x^t_p - f_p^j(x)
\]
\[
= \Theta_p(P, e, H^m) - x^t_p - \alpha \cdot |P| \cdot (\Theta_p(P, e, H^m) - x^t_p)
\]
\[
= \left( 1 - \alpha \cdot |P| \right) \left[ \Theta_p(P, e, H^m) - x^t_p \right].
\]

Therefore, for every \( n \in \mathbb{N} \),
\[
\Theta(P, e, H^m) - [x]^n = \left( 1 - \alpha \cdot |P| \right)^n \left[ \Theta(P, e, H^m) - x \right].
\]

If \( 0 < \alpha < \frac{2}{|T^p|} \), then \(-1 < \left( 1 - \alpha \cdot |P| \right) < 1 \) and \( \{[x]^n\}_{n=1}^{\infty} \) converges to \( \Theta(P, e, H^m) \).

Similar to the work of Maschler and Owen [13], a dynamic outcome could be provided under reductions as follows.

**Definition 4.** Let \( \rho \) be an output, \( (P, e, H^m) \in \Delta \), \( S \subseteq P \) and \( x \in \text{ME}(P, e, H^m) \). The \( (x, \rho) \)-reduced scheme \( (S, \rho, H^m_{\rho,S,x}) \) is given by \( H^m_{\rho,S,x} = (h^i_{\rho,S,x})_{t \in \Gamma_m} \) for every \( T \subseteq S \),
\[
h^i_{\rho,S,x}(x) = \begin{cases} 
\max_{\tau \in \mathbb{E}^{F \rightarrow S}} \{ h(\omega, \tau)|S(\tau) = P \setminus S \} - \sum_{p \in P \setminus S} x^t_p, & S(\omega) = S, \\
\Theta^j_{\rho,S,p}(\omega), & \text{otherwise}.
\end{cases}
\]

Similar to the work of Maschler and Owen [13], we also introduce related switch mapping as follows. The R-switch mapping is \( g = (g^t)_{t \in \Gamma_m} \), where \( g^t = (g^t_{\rho,p})_{p \in P} \) and \( g^t_{\rho,p} : \text{ME}(P, e, H^m) \rightarrow \mathbb{R} \) is defined by
\[
g^t_{\rho,p}(x) = x^t_p + \alpha \sum_{k \in P \setminus \{p\}} \left( \Theta^j_{\rho,\{p,k\}}(\omega) - x^t_p \right).
\]

Define \( \theta^0 = x \), \( \theta^1 = g([\theta]^0) \), \ldots, \( \theta^n = g([\theta]^{n-1}) \) for every \( n \in \mathbb{N} \).

**Lemma 6.** \( g(x) \in \text{ME}(P, e, H^m) \) for every \( (P, e, H^m) \in \Delta \) and for every \( x \in \text{ME}(P, e, H^m) \).
Proof. Let \((P, e, H^m) \in \Delta, t \in \Gamma_m, p, k \in P\) and \(x \in ME(P, e, H^m)\). Let \(S = \{p, k\}\), by MEIY of \(\Theta\) and Definition 4,
\[
\Theta_p^t(S, e_S, H_{S,x}^m) + \Theta_k^t(S, e_S, H_{S,x}^m) = x_p^t + x_k^t.
\]
By 2CSE and 2CS of \(\Theta\),
\[
\Theta_p^t(S, e_S, H_{S,x}^m) - \Theta_k^t(S, e_S, H_{S,x}^m) = (h_{p,1}^t)_{\{p\}} - (h_{k,1}^t)_{\{k\}} = \Theta_p^t(S, e_S, H_{S,x}^m) - \Theta_k^t(S, e_S, H_{S,x}^m) = \Theta_p^t(P, e, H^m) - \Theta_k^t(P, e, H^m).
\]
Therefore,
\[
2 \cdot \left[\Theta_p^t(S, e_S, H_{S,x}^m) - x_p^t\right] = \Theta_p^t(P, e, H^m) - \Theta_k^t(P, e, H^m) - x_p^t + x_k^t. \tag{14}
\]
By definition of \(g\) and Equation (14),
\[
g_p^t(x) = x_p^t + \frac{|p| - a}{2} \left[ \sum_{k \in P \setminus \{p\}} \Theta_p^t(P, e, H^m) - \sum_{k \in P \setminus \{p\}} x_k^t \right] = x_p^t + \frac{|p| - a}{2} \left[ \Theta_p^t(P, e, H^m) - (|P| - 1)x_p^t \right] - \left( h_p^t(P) - x_p^t \right).
\]
Therefore, we have that
\[
\sum_{p \in P} g_p^t(x) = \sum_{p \in P} \left[ x_p^t + \frac{|p| - a}{2} \cdot \Theta_p^t(P, e, H^m) - x_p^t \right] = \sum_{p \in P} x_p^t + \frac{|P| - a}{2} \cdot \left[ \Theta_p^t(P, e, H^m) - \sum_{p \in P} x_p^t \right] = h_p^t(P) + \frac{|P| - a}{2} \cdot \left[ h_p^t(P) - h_p^t(P) \right] = h_p^t(P).
\]
Thus, \(g(x) \in ME(P, e, H^m)\) for every \(x \in ME(P, e, H^m)\). □

Theorem 4. Let \((P, e, H^m) \in \Delta\). If \(0 < \alpha < \frac{1}{|P|}\), then \(|\theta|^n_{n=1}\) converges to \(\Theta(P, e, H^m)\) for each \(x \in ME(P, e, H^m)\).

Proof. Let \((P, e, H^m) \in \Delta, t \in \Gamma_m\) and \(x \in ME(P, e, H^m)\). By Equation (15), \(g_p^t(x) = x_p^t + \frac{|P| - a}{2} \cdot \Theta_p^t(P, e, H^m) - x_p^t\) for every \(p \in P\). Therefore,
\[
(1 - \frac{|P| - a}{2}) \cdot \Theta_p^t(P, e, H^m) - x_p^t = \Theta_p^t(P, e, H^m) - g_p^t(x).
\]
Therefore, for every \(n \in \mathbb{N}\),
\[
\Theta(P, e, H^m) - |\theta|^n = \left(1 - \frac{|P| - a}{2}\right)^n \Theta(P, e, H^m) - x.
\]
If $0 < \alpha < \frac{4}{|\nabla|}$, then $-1 < \left(1 - \frac{|\nabla|\alpha}{2}\right) < 1$ and \(\{[\theta]^n\}_{n=1}^{\infty}\) converges to \(\Theta(P,e,h)\) for every \((P,e,H^m) \in \Delta\), for every \(t \in \Gamma_m\) and for every \(p \in P\). \(\square\)

Subsequently, we present the dynamic outcomes for the MNBCO.

**Definition 5.**

- Let \((P,e,H^m) \in \Delta^*\) and \(p \in P\). The N-switch mapping is considered to be \(f^N = (f^{N_J})_{t \in \Gamma_m}\), where \(f^{N_J} = (f^{N_J}_{p})_{p \in P}\) and \(f^{N_J}_{p} : ME(P,e,H^m) \rightarrow \mathbb{R}\) is considered by
  \[
  f^{N_J}_{p}(x) = \sum_{q \in P \setminus \{p\}} \frac{\lambda^*(\{q\})}{\sum_{l \in P \setminus \{p\}} \lambda^*(\{l\})} \left(Ex(S \cup \{q\},H^m) - Ex(S \cup \{p\},H^m)\right),
  \]
  where \(\lambda^* = \frac{h^p\{P,P\}}{\sum_{l \in P \setminus \{p\}} h^p\{l\}}\). Let \([\lambda]^0 = x, [\lambda]^1 = f^{N_J}([\lambda]^0),\ldots, [\lambda]^n = f^{N_J}([\lambda]^{n-1})\) for every \(n \in \mathbb{N}\).

- The N-R-switch mapping is \(g^N = (g^{N_J})_{t \in \Gamma_m}\), where \(g^{N_J} = (g^{N_J}_{p})_{p \in P}\) and \(g^{N_J}_{p} : ME(P,e,H^m) \rightarrow \mathbb{R}\) is considered by
  \[
  g^{N_J}_{p}(x) = x_p^+ \sum_{k \in P \setminus \{p\}} \left(h^p_{\{k\}} \Phi^p_{\{p,k\},x} - x_p^i\right).
  \]
  Let \([\chi]^0 = x, [\chi]^1 = g^{N_J}([\chi]^0),\ldots, [\chi]^n = g^{N_J}([\chi]^{n-1})\) for every \(n \in \mathbb{N}\).

**Theorem 5.**

1. \(f^N(x) \in ME(P,e,H^m)\) and \(g^N(x) \in ME(P,e,H^m)\) for every \((P,e,H^m) \in \Delta^*\) and for every \(x \in ME(P,e,H^m)\).
2. Let \((P,e,H^m) \in \Delta^*\). If \(0 < \alpha < \frac{2}{|\nabla|}\), then \(\{[\lambda]^n\}_{n=1}^{\infty}\) converges to \(\Theta(P,e,H^m)\) for each \(x \in ME(P,e,H^m)\).
3. Let \((P,e,H^m) \in \Delta^*\). If \(0 < \alpha < \frac{4}{|\nabla|}\), then \(\{[\chi]^n\}_{n=1}^{\infty}\) converges to \(\Theta(P,e,H^m)\) for each \(x \in ME(P,e,H^m)\).

**Proof.** The proofs of this theorem are similar to Lemmas 5, 6 and Theorems 3 and 4. \(\square\)

5. Conclusions

In this article, we investigate the multi-consideration Banzhaf–Coleman output, the multi-consideration efficacious Banzhaf–Coleman output and the multi-consideration normalized Banzhaf–Coleman output. Based on the reduced scheme, several axiomatic outcomes for these outputs are proposed to analyze the reasonability. By adopting reductions and excess mappings, we also present alternative formulations and related dynamic outcomes for these outputs. One should compare related existing outcomes with the outcomes proposed in this article.

- The multi-consideration Banzhaf–Coleman output, the multi-consideration efficacious Banzhaf–Coleman output and the multi-consideration normalized Banzhaf–Coleman output are presented initially in the contexts of traditional schemes and multi-attribute multi-choice schemes.
- The notion of our switch mappings in Definitions 3–5 and related dynamic outcomes are based on that of Maschler and Owen’s [13] dynamic outcomes for the Shapley value [14]. The major difference is that our switch mappings in Definitions 3 and 5 are based on “excess mappings”, and the Maschler and Owen [13] switch mapping is based on “reduced schemes”.

The outcomes proposed in this article raise one motivation:

- Whether there exist other generalizations and related outcomes for some more outputs in the context of multi-attribute multi-choice schemes.
To our knowledge, these issues are still open questions.

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