GROUPOIDS AND HERMITIAN BANACH ∗-ALGEBRAS

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ABSTRACT. We study when the twisted groupoid Banach ∗-algebra $L^1(G, \sigma)$ is Hermitian. In particular, we prove that Hermitian groupoids satisfy the weak containment property. Furthermore, we find that for $L^1(G, \sigma)$ to be Hermitian it is sufficient that $L^1(\mathcal{G}_x)$ is Hermitian. Moreover, if $\mathcal{G}$ is ample, we find necessary conditions for $L^1(\mathcal{G}, \sigma)$ to be Hermitian in terms of the fibers $\mathcal{G}_x$.

1. INTRODUCTION

Due to the seminal paper [18], Hermitian Banach ∗-subalgebras of $C^*$-algebras have been intimately linked to Wiener’s lemma and the notion of spectral invariance. As such, they appear in a variety of areas of mathematics, such as approximation theory, time-frequency analysis and signal processing and noncommutative geometry, see e.g. [10, 19, 20, 21, 22].

We say that a locally compact group $G$ is Hermitian if the convolution algebra $L^1(G)$ is a Hermitian Banach ∗-algebra. In the realm of Hermitian locally compact groups the literature is very extensive (see for example [31, 32]). The class of Hermitian locally compact groups includes compact extensions of nilpotent groups [28] and famously also all compactly generated groups of polynomial growth [27]. In [36] it was proved that a Hermitian locally compact group must be amenable. In another, but related context, it was proved in [25] that if $\mathcal{A}$ is a Hermitian Banach ∗-algebra and $\Gamma$ is a compact group acting on $\mathcal{A}$ by ∗-automorphisms, the generalized $L^1$-algebra $L^1(\Gamma, \mathcal{A})$ is Hermitian.

One of the main motivations for this article is the use of spectral invariance of noncommutative tori in time-frequency analysis as in e.g. [22]. Noncommutative tori can be viewed as the twisted group $C^*$-algebra $C^*(\mathbb{Z}^{2d}, \sigma)$, where $\sigma$ is the Heisenberg 2-cocycle. However, this can generalized to the group $C^*$-algebra $C^*(\Delta, \sigma)$ where $\Delta$ is a closed subgroup of the time-frequency plane $G \times \hat{G}$, where $G$ is a locally compact abelian group. This extension has been used to construct finitely generated modules over noncommutative solenoids [16, 24]. Spectral invariance of $L^1(\Delta, \sigma)$ was studied in [4]. Pushing forward this generalization, given a quasicrystal $\Lambda \subseteq \mathbb{R}^{2d}$ one can construct a twisted groupoid $C^*$-algebra $C^*(\mathcal{G}_\Lambda, \sigma)$ where $\mathcal{G}_\Lambda$ is an étale groupoid. Finitely generated projective modules over $C^*(\mathcal{G}_\Lambda, \sigma)$ were constructed in [24].

This paper together with [6] is a starting point for a project that aims to extend the tools of operator algebras applied to Gabor analysis for lattices in the time-frequency plane (see for example in [5, 16, 29, 35]) to the more general context of quasicrystals [13, 23]. In particular, in this paper we focus our attention on when...
the Banach $\ast$-algebra $L^1(\mathcal{G}, \sigma)$ associated to a locally compact groupoid $\mathcal{G}$ with a twist $\sigma$ is Hermitian.

The paper is structured as follows. First, in Section 2 we give the necessary background and results about groupoids and Banach $\ast$-algebras.

In Section 3 we prove the main result of the paper, namely, if $L^1(\mathcal{G}, \sigma)$ is Hermitian, then the full and the reduced twisted groupoid $C^\ast$-algebras coincide (also known as the weak containment property). When $\mathcal{G}$ is a group, this is equivalent to amenability. However, in the case of locally compact groupoids the situation is more subtle.

Then, in Section 4 we give prove that for ample groupoids a necessary condition for $L^1(\mathcal{G}, \sigma)$ to be Hermitian is that the "fibers" $L^1(\mathcal{G}_x, \sigma_x)$ are Hermitian for every $x \in \mathcal{G}^{(0)}$. Using this we give a simple example of an amenable étale groupoid such that $L^1(\mathcal{G}, \sigma)$ is not Hermitian.

Finally, in Section 5 we give sufficient conditions for $L^1(\mathcal{G}, \sigma)$ to be Hermitian. We construct the twisted groupoid $\mathcal{G}_\sigma$, and show that $L^1(\mathcal{G}, \sigma)$ is Hermitian if the "untwisted" groupoid Banach $\ast$-algebra $L^1(\mathcal{G}_\sigma)$ is Hermitian. In particular, we prove that if $\Gamma$ is a compact group or a locally compact abelian group acting by homeomorphisms on a locally compact Hausdorff space $X$, then $L^1(X \rtimes \Gamma, \sigma)$ is Hermitian for every group $2$-cocycle $\sigma$ of $\Gamma$, where $X \rtimes \Gamma$ is the transformation groupoid.

2. Preliminaries

2.1. Hermitian Banach $\ast$-algebras. If $\mathcal{A}$ is a unital Banach algebra, we denote by $\mathcal{A}^{-1}$ the set of invertible elements of $\mathcal{A}$. Given a unital Banach algebra $\mathcal{A}$ we denote by

$$\text{Sp}_\mathcal{A}(a) = \{ \lambda \in \mathbb{C} : \lambda 1_{\mathcal{A}} - a \notin \mathcal{A}^{-1} \},$$

the spectrum of $a$ in $\mathcal{A}$, and by

$$r_\mathcal{A}(a) := \sup \{ |\lambda| : \lambda \in \text{Sp}_\mathcal{A}(a) \} = \lim_{n \to \infty} \| f^n \|^1/n$$

the spectral radius of $a$. If $\mathcal{A}$ is not unital, given $a \in \mathcal{A}$ we define $\text{Sp}_\mathcal{A}(a) := \text{Sp}_{\mathcal{A}^+}(a)$ and $r_\mathcal{A}(a) = r_{\mathcal{A}^+}(a)$, where $\mathcal{A}^+$ is the minimal unitization of $\mathcal{A}$.

Now let $\mathcal{A}$ be a Banach $\ast$-algebra. Then $a \in \mathcal{A}$ is Hermitian if $a^\ast = a$. If $\mathcal{S}$ is a subset of $\mathcal{A}$, we denote by $\mathcal{S}_h = \{ a \in \mathcal{S} : a^\ast = a \}$ the set of Hermitian elements in $\mathcal{S}$.

**Definition 2.1.** A Banach $\ast$-algebra $\mathcal{A}$ is Hermitian if $\text{Sp}_\mathcal{A}(a) \subseteq \mathbb{R}$ for every $a \in \mathcal{A}_h$. $\mathcal{A}$ is called symmetric if $\text{Sp}_\mathcal{A}(aa^\ast) \subseteq [0, \infty)$ for every $a \in \mathcal{A}$.

**Remark 2.2.** By the celebrated Shirali-Ford theorem, a Banach $\ast$-algebra is Hermitian if and only if it is symmetric.
A \textit{*-representation} of a Banach *-algebra $A$ is a *-homomorphism $\pi : A \to B(\mathcal{H})$, where $B(\mathcal{H})$ denotes the bounded linear operators on a Hilbert space $\mathcal{H}$. We say $A$ is \textit{reduced} if $A_R = \{a \in A : \pi(a) = 0 \text{ for every } *-\text{representation } \pi \text{ of } A\} = \{0\}$. All Banach *-algebras we consider in the sequel will be \textit{reduced}. The \textit{enveloping} $C^*$-\textit{algebra} of a reduced Banach *-algebra $A$ is the completion of $A$ with respect to the norm

$$\|a\| := \sup\{\|\pi(a)\| : \pi : A \to B(\mathcal{H}) \text{ is a } *-\text{representation}\}$$

for every $a \in A$, and it is denoted by $C^*(A)$. The enveloping $C^*$-algebra of a Banach *-algebra always exists [32, Section 10.1].

2.2. Invariant spectral radius.

\textbf{Definition 2.3.} We say that $A \subseteq B$ is a \textit{nested pair} of reduced Banach *-algebras if $A$ and $B$ are reduced Banach *-algebras and $A$ embeds continuously into $B$ as a dense *-subalgebra. A \textit{nested triple} of reduced Banach *-algebras is defined similarly.

\textbf{Definition 2.4.} Let $A \subseteq B$ be a nested pair of reduced Banach *-algebras and $S$ a (not necessarily closed) *-subalgebra of $A$. We say $S_h$ has \textit{invariant spectral radius} in $(A, B)$ if

$$r_A(a) = r_B(a),$$

for every $a \in S_h$. If $A_h$ has invariant spectral radius in $(A, B)$, we say that $A_h$ has \textit{invariant spectral radius} in $B$. Moreover, we say $S$ is a spectrally invariant subalgebra of $(A, B)$ if

$$\text{Sp}_A(a) = \text{Sp}_B(a),$$

for every $a \in S$. If $A$ is a spectrally invariant subalgebra of $(A, B)$, we say $A$ is spectrally invariant in $B$.

Clearly, if $S$ is a spectrally invariant subalgebra of $(A, B)$, then $S_h$ has invariant spectral radius in $(A, B)$. The Barnes-Hulanicki Theorem [7] provides a partial converse when $B$ is a $C^*$-algebra.

\textbf{Theorem 2.5.} Let $A$ be a Banach *-algebra, $S$ a *-subalgebra of $A$, and $\pi : A \to B(\mathcal{H})$ a faithful *-representation. If $A$ is unital, we assume that $\pi(1_A) = 1_{B(\mathcal{H})}$. If

$$r_A(a) = \|\pi(a)\|,$$

for all $a \in S_h$, then

$$\text{Sp}_A(a) = \text{Sp}_B(\mathcal{H})(\pi(a))$$

for every $a \in S$.

In [36, Proposition 2.7] they prove a very useful property related to invariant spectral radius.

\textbf{Proposition 2.6.} Let $A \subseteq B$ be a nested pair of reduced Banach *-algebras, and let $S$ be a dense *-subalgebra of $A$. Suppose that $S_h$ has invariant spectral radius in $(A, B)$. Then $A$ and $B$ have the same $C^*$-envelope.
2.3. Hermitian and quasi-hermitian Banach *-algebras. Let \( A \) be a reduced Banach *-algebra. Then the following statements are equivalent:

1. \( A \) is Hermitian,
2. \( A \) is symmetric,
3. \( A \) is spectrally invariant in \( C^*(A) \),
4. \( r_A(a) = r_{C^*(A)}(a) \) for every \( a \in A \).

(see [36] Lemma 2.8 and [26] p. 340).

Definition 2.7. A dense *-subalgebra \( S \) of a Banach *-algebra \( A \) is called quasi-Hermitian in \( A \) if \( \operatorname{Sp}_A(a) \subseteq \mathbb{R} \) for every \( a \in S_h \). We say \( S \) is quasi-symmetric in \( A \) if \( \operatorname{Sp}_A(a^*a) \subseteq [0, \infty) \) for every \( a \in S \).

Let \( S \) be a dense *-subalgebra of a Banach *-algebra \( A \). If \( A \) is Hermitian, then \( S \) is automatically quasi-Hermitian in \( A \). The converse is not true in general, but holds whenever \( A \) is commutative [36 Proposition 2.10].

2.4. Spectral interpolation.

Definition 2.8. Suppose \( A \subseteq B \subseteq C \) is a nested triple of reduced Banach *-algebras and \( S \) is a dense *-subalgebra of \( A \). We say that \( (A, B, C) \) is a spectral interpolation triple relative to \( S \) if there exists \( \theta \in (0, 1) \) such that

\[
\operatorname{r}_B(a) \leq \operatorname{r}_A(a)^{1-\theta} \operatorname{r}_C(a)^{\theta}
\]

for every \( a \in S_h \).

A nice property of spectral interpolation triples of reduced Banach *-algebras was proved in [36 Proposition 3.4].

Proposition 2.9. If \( (A, B, C) \) is a spectral interpolation triple of reduced Banach *-algebras relative to a quasi-Hermitian dense *-subalgebra \( S \) of \( A \), then \( S_h \) has invariant spectral radius in \( (B, C) \). In particular, \( C^*(B) = C^*(C) \) and

\[
\operatorname{r}_B(a) = \operatorname{r}_C(a),
\]

for every \( a \in S_h \).

2.5. Groupoids, twists and associated algebras. We now introduce the Banach *-algebras and \( C^* \)-algebras associated to a groupoid as described in [33]. Given a topological groupoid \( G \) we will denote by \( G^{(0)} \) its unit space and write \( r, s : G \to G^{(0)} \) for the continuous range and source maps, respectively. We will also denote by \( G^{(2)} = \{(\alpha, \beta) \in G \times G : s(\alpha) = r(\beta)\} \) the set of composable elements. \( G^{(2)} \) inherits the subspace topology from \( G \times G \). Given \( x \in G^{(0)} \) we define \( G_x := \{ \gamma \in G : s(\gamma) = x \} \) and \( G^x := \{ \gamma \in G : r(\gamma) = x \} \).

Let \( \lambda = \{ \lambda_x \}_{x \in G^{(0)}} \) be a Haar system for \( G \), where \( \lambda_x \) are measures with support \( G_x \) such that

1. for every \( f \in C_c(G) \), the function \( x \mapsto \int_{G_x} f(\gamma)d\lambda_x(\gamma) \) is continuous,
2. for every \( \eta \in G \) and \( f \in C_c(G) \) we have that

\[
\int_{G_{s(\eta)}} f(\gamma\eta)d\lambda_{r(\eta)}(\gamma) = \int_{G_{s(\eta)}} f(\gamma)d\lambda_{s(\eta)}(\gamma).
\]
If $\mathcal{G}$ is a locally compact groupoid, then there always exist Haar systems for $\mathcal{G}$.

A groupoid $\mathcal{G}$ is called \textit{étale} if the range map, and hence also the source map, is a local homeomorphism. In this case the sets $\mathcal{G}^x$ and $\mathcal{G}_x$ are discrete sets for every $x \in \mathcal{G}^{(0)}$, and the Haar system consists of counting measures. A subset $B$ of an étale groupoid $\mathcal{G}$ is called a \textit{bisection} if there is an open set $U \subseteq \mathcal{G}$ containing $B$ such that $r: U \to r(U)$ and $s: U \to s(U)$ are homeomorphisms onto open subsets of $\mathcal{G}^{(0)}$. Second-countable étale groupoids have countable bases consisting of open bisections.

The \textit{orbit} of $x \in \mathcal{G}^{(0)}$ is defined to be $\text{Orb}_\mathcal{G}(x) = \{r(\gamma) : \gamma \in \mathcal{G}_x\}$. The \textit{isotropy group} of $x$ is given by $\mathcal{G}_x^x := \mathcal{G}^x \cap \mathcal{G}_x = \{\gamma \in \mathcal{G} : s(\gamma) = r(\gamma) = x\}$, and the \textit{isotropy subgroupoid} of $\mathcal{G}$ is the subgroupoid $\text{Iso}(\mathcal{G}) := \bigcup_{x \in \mathcal{G}^{(0)}} \mathcal{G}_x^x$ with the relative topology from $\mathcal{G}$.

We will consider groupoid twists where the twist is implemented by a normalized continuous 2-cocycle. To be more precise, let $\mathcal{G}$ be a locally compact groupoid. A \textit{normalized continuous 2-cocycle} is then a continuous map $\sigma: \mathcal{G}^{(2)} \to \mathbb{T}$ satisfying
\begin{equation}
\sigma(r(\gamma), \gamma) = 1 = \sigma(\gamma, s(\gamma))
\end{equation}
for all $\gamma \in \mathcal{G}$, and
\begin{equation}
\sigma(\alpha, \beta)\sigma(\alpha\beta, \gamma) = \sigma(\beta, \gamma)\sigma(\alpha, \beta\gamma)
\end{equation}
whenever $(\alpha, \beta), (\beta, \gamma) \in \mathcal{G}^{(2)}$. The set of normalized continuous 2-cocycles on $\mathcal{G}$ will be denoted $Z^2(\mathcal{G}, \mathbb{T})$. Note that this is not the most general notion of a twist of a groupoid (see [37, Chapter 5]).

Let $\mathcal{G}$ be a locally compact groupoid with Haar system $\lambda$. We will define the $\sigma$-twisted convolution algebra $C_c(\mathcal{G}, \sigma)$ as follows: As a set it is just
\[ C_c(\mathcal{G}, \sigma) = \{f : \mathcal{G} \to \mathbb{C} : f \text{ is continuous with compact support}\}, \]
but equipped with $\sigma$-twisted convolution product
\begin{equation}
(f \ast_\sigma g)(\gamma) = \int_{\mathcal{G}_s(\gamma)} f(\gamma\mu^{-1})g(\mu)\sigma(\gamma\mu^{-1}, \mu)\,d\lambda_s(\gamma)(\mu), \quad f, g \in C_c(\mathcal{G}, \sigma), \gamma \in \mathcal{G},
\end{equation}
and $\sigma$-twisted involution
\begin{equation}
f^*\sigma(\gamma) = \overline{\sigma(\gamma^{-1}, \gamma)f(\gamma^{-1})}, \quad f \in C_c(\mathcal{G}, \sigma), \gamma \in \mathcal{G}.
\end{equation}
We complete $C_c(\mathcal{G}, \sigma)$ in the "fiberwise 1-norm", also known as the $I$-norm, given by
\begin{equation}
\|f\|_I = \sup_{x \in \mathcal{G}^{(0)}} \left\{ \max \left\{ \int_{\mathcal{G}_x} |f(\gamma)|\,d\lambda_x(\gamma), \int_{\mathcal{G}_x} |f(\gamma^{-1})|\,d\lambda_x(\gamma) \right\} \right\}
\end{equation}
\begin{equation}
= \sup_{x \in \mathcal{G}^{(0)}} \left\{ \max \left\{ \int_{\mathcal{G}_x} |f(\gamma)|\,d\lambda_x(\gamma), \int_{\mathcal{G}_x} |f^*\sigma(\gamma)|\,d\lambda_x(\gamma) \right\} \right\}
\end{equation}
for $f \in C_c(\mathcal{G}, \sigma)$. Denote by $L^1(\mathcal{G}, \sigma, \lambda)$ (or $L^1(\mathcal{G}, \sigma)$ when there is no ambiguity on the Haar system) the completion of $C_c(\mathcal{G}, \sigma)$ with respect to the $I$-norm. If $\mathcal{G}$ is an étale groupoid, then we denote it by $\ell^1(\mathcal{G}, \sigma)$. 
The (full) twisted groupoid $C^*$-algebra $C^*_r(\mathcal{G}, \sigma, \lambda)$ (or $C^*(\mathcal{G}, \sigma)$ when there is no ambiguity on the Haar system) is the completion of $C_c(\mathcal{G}, \sigma)$ in the norm
\[
\|f\| := \sup\{\|\pi(f)\| : \pi \text{ is an } I\text{-norm bounded } *\text{-representation of } C_c(\mathcal{G}, \sigma)\},
\]
for $f \in C_c(\mathcal{G}, \sigma)$. It was observed in [13] Lemma 3.3.19 that if $\mathcal{G}$ is a locally compact Hausdorff étale groupoid, then every *-representation of $C_c(\mathcal{G}, \sigma)$ is bounded by the $I$-norm. In addition, if $\mathcal{G}$ is a transformation groupoid (see Example 2.12), every *-representation is bounded by the $I$-norm. In these cases, since we are completing with respect to a supremum over *-representations, $C^*(\mathcal{G}, \sigma)$ is just the $C^*$-envelope of $L^1(\mathcal{G}, \sigma)$.

Now we will construct a faithful *-representation of $L^1(\mathcal{G}, \sigma)$ called the $\sigma$-twisted left regular representation. In particular, we have that $L^1(\mathcal{G}, \sigma)$ is reduced. The completion of the image of $L^1(\mathcal{G}, \sigma)$ under the $\sigma$-twisted left regular representation is called the $\sigma$-twisted reduced groupoid $C^*$-algebra of $\mathcal{G}$ and will be denoted $C^*_r(\mathcal{G}, \sigma, \lambda)$ (or $C^*_r(\mathcal{G}, \sigma)$ when there is no ambiguity on the Haar system). Let $x \in \mathcal{G}^{(0)}$. Then there is a representation $L^2_x : C_c(\mathcal{G}, \sigma) \to B(L^2(\mathcal{G}_x))$ (here $L^2(\mathcal{G}_x) = L^2(\mathcal{G}_x, \lambda_x)$) which is given by
\[
L^2_x(f)\xi(\gamma) = \int_{\mathcal{G}_x} \sigma(\gamma\mu^{-1}, \mu)f(\gamma\mu^{-1})\xi(\mu)\,d\lambda_{\mu}(\gamma)(\mu)
\]
for $f \in C_c(\mathcal{G}, \sigma)$, $\xi \in L^2(\mathcal{G}_x)$ and $\gamma \in \mathcal{G}_x$.

We then obtain a faithful $I$-norm bounded *-representation of $C_c(\mathcal{G}, \sigma)$ given by
\[
\bigoplus_{x \in \mathcal{G}^{(0)}} L^2_x : C_c(\mathcal{G}, \sigma) \to \bigoplus_{x \in \mathcal{G}^{(0)}} B(L^2(\mathcal{G}_x)) \subset B\left( \bigoplus_{x \in \mathcal{G}^{(0)}} L^2(\mathcal{G}_x) \right).
\]

$C^*_r(\mathcal{G}, \sigma)$ is then the completion of $C_c(\mathcal{G}, \sigma)$ with respect of the image of $C_c(\mathcal{G}, \sigma)$ under the $\sigma$-twisted left regular representation, so given $f \in C_c(\mathcal{G}, \sigma)$
\[
\|f\|_{r, 2} := \sup_{x \in \mathcal{G}^{(0)}} \{\|L^2_x(f)\|_{B(L^2(\mathcal{G}_x))}\}.
\]

As the representation is $I$-norm bounded, it extends to a *-representation of $L^1(\mathcal{G}, \sigma)$, and $C^*_r(\mathcal{G}, \sigma)$ is also the $C^*$-completion of $L^1(\mathcal{G}, \sigma)$ in the extended *-representation. Moreover, since $C^*(\mathcal{G}, \sigma)$ is the completion of $C_c(\mathcal{G}, \sigma)$ with respect to the supremum of the $I$-bounded norms, the identity map on $C_c(\mathcal{G}, \sigma)$ extends to a (surjective) *-homomorphism $\pi : C^*(\mathcal{G}, \sigma) \to C^*_r(\mathcal{G}, \sigma)$.

**Definition 2.10.** Let $\mathcal{G}$ be a locally compact groupoid with Haar system $\lambda$ and let $\sigma \in Z^2(\mathcal{G}, \mathbb{T})$. We say that $\mathcal{G}$ twisted by $\sigma$ has the weak containment property when the natural map $\pi : C^*(\mathcal{G}, \sigma) \to C^*_r(\mathcal{G}, \sigma)$ is an isomorphism.

If $\mathcal{G}$ is an amenable groupoid with Haar measure $\lambda$ [2], we have that $C^*_r(\mathcal{G}, \sigma, \lambda) = C^*(\mathcal{G}, \sigma, \lambda)$ for every $\sigma \in Z^2(\mathcal{G}, \mathbb{T})$ [2 Proposition 6.1.8], and hence $\mathcal{G}$ twisted by $\sigma$ has the weak containment property for every $\sigma \in Z^2(\mathcal{G}, \mathbb{T})$. In [33] it was proved that amenability is not equivalent to having the weak containment property.

**Remark 2.11.** It was shown in [33] that the full and reduced $C^*$-algebras don’t depend, up to Morita equivalence, on the Haar system. Suppose $\lambda, \lambda'$ are two...
Haar systems for a locally compact groupoid $G$, let $\sigma \in Z^2(G, \mathbb{T})$, and suppose $C^*_r(G, \sigma, \lambda) = C^*_r(G, \sigma, \lambda)$. Then $C^*_r(G, \sigma, \lambda)$ and $C^*_r(G, \sigma, \lambda')$ are Morita equivalent, as are $C^*_r(G, \sigma, \lambda)$ and $C^*_r(G, \sigma, \lambda')$. However, it is not known to the authors if this also implies $C^*_r(G, \sigma, \lambda') = C^*_r(G, \sigma, \lambda')$, that is, if weak containment is independent of the Haar system.

**Example 2.12.** Let $\Gamma$ be a locally compact group with left Haar measure $\lambda$ and with unit $e$, and let us consider an action of $\Gamma$ by homeomorphisms on a locally compact Hausdorff space $X$. Then we define the transformation groupoid $X \rtimes \Gamma$ as the set $X \times \Gamma$ with the product topology, such that

\[
s(x, \gamma) = (x, e), \quad r(x, \gamma) = (\gamma \cdot x, e) \quad \text{and} \quad (\gamma \cdot x, \gamma_2)(x, \gamma_1) = (x, \gamma_2 \gamma_1).
\]

Then $X \rtimes \Gamma$ is a locally compact groupoid. Moreover, if $X$ and $\Gamma$ are both second-countable, then so is $X \rtimes \Gamma$. One defines the Haar system $\{\delta_x \times \lambda\}_{x \in G^{(0)}}$ where $\delta_x$ is the Dirac measure, and in this case, given $f \in C_c(X \rtimes \Gamma)$ we have that

\[
\|f\|_I = \sup \left\{ \max \left\{ \int \|f(x, \gamma)\| d\lambda(\gamma), \int \|f(\gamma \cdot x, \gamma^{-1})\| d\lambda(\gamma) \right\} : x \in G^{(0)} \right\} = \sup \left\{ \int \|f(x, \gamma)\| d\lambda(\gamma) : x \in G^{(0)} \right\}.
\]

Now let $\sigma \in Z^2(\Gamma, \mathbb{T})$. Then we can extend $\sigma$ to a 2-cocycle of $X \rtimes \Gamma$, also denoted by $\sigma$, by defining

\[
\sigma((x_1, \gamma_1), (x_2, \gamma_2)) := \sigma(\gamma_1, \gamma_2),
\]

for all $x_1, x_2 \in X$ and $\gamma_1, \gamma_2 \in \Gamma$ for which $((x_1, \gamma_1), (x_2, \gamma_2)) \in (X \rtimes \Gamma)^{(2)}$. Then $C^*(L^1(X \rtimes \Gamma, \sigma)) \cong C^*(X \rtimes \Gamma, \sigma) \cong C_0(X) \rtimes_{\sigma} \Gamma$

is the full twisted crossed product $C^*$-algebra, and $C^*_r(X \rtimes \Gamma, \sigma) \cong C_0(X) \rtimes_{\sigma} \Gamma$

is the reduced twisted crossed product $C^*$-algebra.

**3. Quasi-Hermitian Groupoids Have the Weak Containment Property.**

In this section we prove the main result of the paper, namely that if $L^1(G, \sigma, \lambda)$ is Hermitian, then $C^*_r(G, \sigma, \lambda) = C^*_r(G, \sigma, \lambda)$. As a consequence, we also prove that if $L^1(G, \sigma, \lambda)$ is Hermitian, then $L^1(G, \sigma, \lambda)$ is spectrally invariant in $C^*_r(G, \sigma, \lambda)$. The general strategy is to follow [36, Section 4], but not all the steps trivially extend to our situation.

**Definition 3.1.** Let $G$ be a locally compact groupoid with Haar system $\lambda$ and let $\sigma \in Z^2(G, \mathbb{T})$. We say that $G$ is $\sigma$-quasi-Hermitian (resp. $\sigma$-quasi-symmetric) if $C_c(G, \sigma)$ is quasi-Hermitian (resp. quasi-symmetric) in $L^1(G, \sigma, \lambda)$.

**Proposition 3.2.** Let $G$ be a locally compact groupoid with Haar system $\lambda$ and $\sigma \in Z^2(G, \mathbb{T})$. If $G$ is $\sigma$-quasi-symmetric, then $G$ is $\sigma$-quasi-Hermitian.
Lemma 3.5. Let $G$ be a second-countable locally compact groupoid with Haar system $\lambda$, $\sigma \in Z^2(G, \mathbb{T})$ and $1 \leq p \leq \infty$. The reduced groupoid $L^p$-Banach algebra, denoted by $F^p(G, \sigma, \lambda)$, is the completion of $C_c(G, \sigma)$ with respect to the norm
\[
\|f\|_{r,p} := \sup_{x \in G(0)} \{\|L^p_x(f)\|_{B(B(L^p(G_x)))}\}
\]
for all $f \in C_c(G, \sigma)$. We will denote $F^p(G, \sigma, \lambda)$ by $F^p(G, \sigma)$ when there is no ambiguity on the Haar system $\lambda$.

Let $1 \leq p \leq q \leq \infty$ with $1 = \frac{1}{p} + \frac{1}{q}$. Then, given any $x \in G(0)$, there is a duality relation $(L^p(G_x))^* \cong L^q(G_x)$ given by
\[
\langle \xi, \zeta \rangle := \int_{G_x} \xi(\gamma)\overline{\zeta(\gamma)} \, d\lambda_x,
\]
for $\xi \in L^p(G_x)$ and $\zeta \in L^q(G_x)$.

Lemma 3.5. Let $G$ be a second-countable locally compact groupoid with Haar system $\lambda$, let $\sigma \in Z^2(G, \mathbb{T})$, and let $1 \leq p, q \leq \infty$ be such that $1 = \frac{1}{p} + \frac{1}{q}$. Then
\[
\langle L^p_x(f^*)\xi, \zeta \rangle = \langle \xi, L^q_x(f)\zeta \rangle,
\]
for every $x \in G(0)$, $f \in L^1(G, \sigma, \lambda)$, $\xi \in L^p(G_x, \lambda_x)$ and $\zeta \in L^q(G_x, \lambda_x)$. 

Proof. The proof of [15, Proposition 4.1] adapts trivially to give us the following result. Let $f \in C_c(G, \sigma)_h$, then by assumption $Sp_{L^1(G, \sigma)}(f * \sigma f^*) \subseteq [0, \infty)$. Therefore
\[
\{\lambda^2 : \lambda \in Sp_{L^1(G, \sigma)}(f) \subseteq Sp_{L^1(G, \sigma)}(f * \sigma f) = Sp_{L^1(G, \sigma)}(f * \sigma f^*) \subseteq [0, \infty)\}.
\]
Hence $Sp_{L^1(G, \sigma)}(f) \subseteq \mathbb{R}$ for every $f \in C_c(G, \sigma)_h$. \qed
Proof. Fix \( x \in \mathcal{G}^{(0)}, f \in L^1(\mathcal{G}, \sigma), \xi \in L^p(\mathcal{G}_x) \) and \( \zeta \in L^q(\mathcal{G}_x) \). Then

\[
\langle L^\sigma_x(f^{*\sigma})(\xi, \zeta) = \int_{\mathcal{G}_x} \left( \int_{\mathcal{G}_x} \sigma(\gamma \mu^{-1}, \mu) f^{*\sigma}(\gamma \mu^{-1}) \xi(\mu) d\lambda_x(\mu) \right) \bar{\zeta}(\gamma) d\lambda_x(\gamma)
\]

\[
= \int_{\mathcal{G}_x} \left( \int_{\mathcal{G}_x} \sigma(\gamma \mu^{-1}, \mu) \sigma((\gamma \mu^{-1})^{-1}, \gamma \mu^{-1}) f((\gamma \mu^{-1})^{-1}) \xi(\mu) d\lambda_x(\mu) \right) \bar{\zeta}(\gamma) d\lambda_x(\gamma)
\]

\[
= \int_{\mathcal{G}_x} \xi(\mu) \left( \int_{\mathcal{G}_x} \sigma(\gamma \mu^{-1}, \mu) \sigma((\gamma \mu^{-1})^{-1}, \gamma \mu^{-1}) f(\mu \gamma^{-1}) \zeta(\gamma) d\lambda_x(\gamma) \right) d\lambda_x(\mu)
\]

\[
= \int_{\mathcal{G}_x} \xi(\mu) \int_{\mathcal{G}_x} \sigma(\mu \gamma^{-1}, \gamma) f(\mu \gamma^{-1}) \zeta(\gamma) d\lambda_x(\gamma) \right) d\lambda_x(\mu)
\]

\[
= \int_{\mathcal{G}_x} \xi(\mu) L^p_x(f) \zeta(\mu) d\lambda_x(\mu) = \langle \xi, L^q_x(f) \zeta \rangle.
\]

Here we used that \( \sigma(\gamma \mu^{-1}, \mu) \sigma((\gamma \mu^{-1})^{-1}, \gamma \mu^{-1}) = \sigma(\mu \gamma^{-1}, \gamma), \) or equivalently, that

\[
\sigma(\mu \gamma^{-1}, \gamma \mu^{-1}) = \sigma(\mu \gamma^{-1}, \gamma) \sigma(\gamma \mu^{-1}, \mu)
\]

for every \( \mu, \gamma \in \mathcal{G}_x \). We get this from (2.2) using \( \mu \gamma^{-1}, (\mu \gamma^{-1})^{-1}, \mu \) instead of \( \alpha, \beta, \gamma \), and then applying (2.1).

\[
\square
\]

**Proposition 3.6.** Let \( \mathcal{G} \) be a second-countable locally compact groupoid with Haar system \( \lambda \) and \( \sigma \in Z^2(\mathcal{G}, \mathbb{T}) \), and suppose \( 1 \leq p, q \leq \infty \) with \( 1 = \frac{1}{p} + \frac{1}{q} \). The algebra \( L^1(\mathcal{G}, \sigma, \lambda) \) is a normed \(*\)-algebra with norm

\[
\|f\|_{p,q} := \max\{\|f\|_{r,p}, \|f\|_{r,q}\},
\]

for \( f \in C_c(\mathcal{G}, \sigma) \), with the convolution and involution in \( L^1(\mathcal{G}, \sigma, \lambda) \).

**Proof.** We will adapt the proof of [36] Proposition 4.2 to our setting. Given \( f \in C_c(\mathcal{G}, \sigma) \) we have that

\[
\|f\|_{X^p} = \max\left\{ \sup_{x \in \mathcal{G}^{(0)}} \{\|L^\sigma_x(f)\|\}, \sup_{x \in \mathcal{G}^{(0)}} \{\|L^\sigma_x(f)\|\} \right\}
\]

\[
= \sup_{x \in \mathcal{G}^{(0)}} \{\|L^\sigma_x(f)\|, \|L^\sigma_x(f)\|\} \leq \|f\|_1,
\]

by (3.11), so \( \|f\|_{p,q} \) is well defined. It is easy to check that \( (L^1(\mathcal{G}, \sigma, \lambda), \| \cdot \|_{p,q}) \) is a normed algebra. We only have to prove that the involution is an isometry with respect to \( \| \cdot \|_{p,q} \). Let \( f \in L^1(\mathcal{G}, \sigma) \). Given \( x \in \mathcal{G}^{(0)} \), by Lemma 3.3 we have that

\[
\|L^\sigma_x(f^{*\sigma})\|_{B(L^p(\mathcal{G}_x))} = \sup\{\|L^\sigma_x(f^{*\sigma})(\xi, \zeta)\|_{p} : \|\xi\|_p \leq 1, \|\zeta\|_q \leq 1\}
\]

\[
= \sup\{\|\xi, L^\sigma_x(f)(\zeta)\|_{p} : \|\xi\|_p \leq 1, \|\zeta\|_q \leq 1\}
\]

\[
= \|L^\sigma_x(f)\|_{B(L^q(\mathcal{G}_x))}.
\]

Similarly, switching \( p \) and \( q \) we obtain that

\[
\|L^\sigma_x(f^{*\sigma})\|_{B(L^q(\mathcal{G}_x))} = \|L^\sigma_x(f)\|_{B(L^p(\mathcal{G}_x))},
\]

and therefore \( \|f^{*\sigma}\|_{p,q} = \|f\|_{p,q} \), as desired.

\[
\square
\]
**Definition 3.7.** Let $G$ be a second-countable locally compact groupoid with Haar system $\lambda$ and $\sigma \in Z^2(G, \mathbb{T})$, and let $1 \leq p \leq \infty$. The Banach $*$-algebra $F^p_\sharp(G, \sigma, \lambda)$ ($F^p_\sharp(G, \sigma)$ when there is no ambiguity on the Haar system) is defined to be the completion of $L^1(G, \sigma, \lambda)$ with respect to the norm $\| \cdot \|_{p,\sigma}^\sharp$.

Given $1 \leq p, q \leq \infty$ with $1 = \frac{1}{p} + \frac{1}{q}$, the Banach $*$-algebras $F^p(G, \sigma, \lambda)$ and $F^q_\sharp(G, \sigma, \lambda)$ are isometrically isomorphic. Moreover, $F^2_\sharp(G, \sigma, \lambda) = C^*_r(G, \sigma, \lambda)$.

The following result is a combination of [9, Theorem 5.1.1] and [14, Section 10.1].

**Lemma 3.8.** Let $G$ be a second-countable locally compact groupoid with Haar system $\lambda$, let $x \in G^{(0)}$, and let $1 \leq p_1 \leq p_2 \leq p_3 \leq \infty$. Suppose that $T$ is a bounded operator defined on $L^{p_1}(G_x, \lambda_x) \cap L^{p_3}(G_x, \lambda_x)$ such that it extends continuously to bounded operators on both $L^{p_1}(G_x, \lambda_x)$ and $L^{p_3}(G_x, \lambda_x)$: Then $T$ extends continuously on $L^{p_2}(G_x, \lambda_x)$. Furthermore, if $M_i$ is the norm of the extension of $T$ on $L^{p_i}(G_x, \lambda_x)$ for $i = 1, 2, 3$, then

$$M_2 \leq M_1^{1-\theta} M_3^\theta,$$

for $0 < \theta < 1$ satisfying

$$\frac{1}{p_2} = \frac{1 - \theta}{p_1} + \frac{\theta}{p_3}.$$ 

**Proposition 3.9.** Let $G$ be a second-countable locally compact groupoid with Haar system $\lambda$ and $\sigma \in Z^2(G, \mathbb{T})$, and suppose $1 < p < 2$. Then

$$(L^1(G, \sigma, \lambda), F^p_\sharp(G, \sigma, \lambda), C^*_r(G, \sigma, \lambda))$$

is a spectral interpolation triple of reduced Banach $*$-algebras relative to $L^1(G, \sigma, \lambda)$.

**Proof.** Let $q \in (2, \infty)$ such that $1 = \frac{1}{p} + \frac{1}{q}$, and let

$$\theta = \frac{2p - 2}{p} \in (0, 1),$$

and hence

$$\frac{1}{p} = \frac{1 - \theta}{1} + \frac{\theta}{2}.$$ 

Then, for $x \in G^{(0)}$ and $f \in L^1(G, \sigma)$, using Lemma 3.8 and (2.5) we have that

$$\|L^p_x(f)\|_{B(L^p(G_x))} \leq \|L^1_x(f)\|^{1-\theta}_{B(L^1(G_x))}\|L^2_x(f)\|^{\theta}_{B(L^2(G_x))}$$

$$\leq \|f\|_{L^{1,1}}^{1-\theta}\|f\|_{L^{2,2}}^\theta \leq \|f\|_{L^{1,1}}^{1-\theta}\|f\|_{L^{2,2}}^\theta,$$

and therefore $\|f\|_{r,p} \leq \|f\|_{L^{1,1}}^{1-\theta}\|f\|_{L^{2,2}}^\theta$. On the other hand,

$$\frac{1}{q} = 1 - \frac{1}{p} = 1 - \left(\frac{1 - \theta}{1} + \frac{\theta}{2}\right)$$

$$= 1 - (1 - \theta) - \theta \left(1 - \frac{1}{2}\right)$$

$$= 0 + \frac{\theta}{2} = \frac{1 - \theta}{\infty} + \frac{\theta}{2}.$$
so we can apply the same above argument to show that \( \|f\|_{r,q} \leq \|f\|_{r,2}^{1-\theta} \|f\|_{r,2}^\theta \).

Hence,

\[
(3.2) \quad \|f\|_{z,p} \leq \|f\|_{t}^{1-\theta} \|f\|_{r,2}^\theta ,
\]

for every \( f \in L^1(\mathcal{G}, \sigma) \).

Therefore given \( f \in L^1(\mathcal{G}, \sigma) \), we have that

\[
\|f^n\|_{z,p} \leq \|f^n\|_{t}^{1-\theta} \|f^n\|_{r,2}^\theta ,
\]

for every \( n \in \mathbb{N} \). Then taking the limit as \( n \to \infty \) we have that

\[
(3.3) \quad r_{F_z^p(\mathcal{G}, \sigma)}(f) \leq r_{L^1(\mathcal{G}, \sigma)}(f)^{1-\theta} r_{C^*_r(\mathcal{G}, \sigma)}(f)\theta ,
\]

for every \( f \in L^1(\mathcal{G}, \sigma) \).

To finish the proof we only need to prove that \((L^1(\mathcal{G}, \sigma), F_z^p(\mathcal{G}, \sigma), C^*_r(\mathcal{G}, \sigma))\) is a nested triple of reduced Banach \(*\)-algebras.

Let \( \theta \in (0, 1) \) be such that

\[
\frac{1}{2} = \frac{1-\theta}{p} + \frac{\theta}{q}.
\]

Then for \( f \in L^1(\mathcal{G}, \sigma) \),

\[
\|f\|_{z,p} = \sup_{x\in\mathcal{G}^{(0)}} \{ \|L^n(f)\|_{B(L^p(\mathcal{G}_x))}, \|L^n(f)\|_{B(L^q(\mathcal{G}_x))} \}
\geq \sup_{x\in\mathcal{G}^{(0)}} \{ \|L^n(f)\|_{B(L^p(\mathcal{G}_x))}^{1-\theta} \|L^n(f)\|_{B(L^q(\mathcal{G}_x))}^{\theta} \}
\geq \sup_{x\in\mathcal{G}^{(0)}} \{ \|L^n(f)\|_{B(L^2(\mathcal{G}_x))} \} = \|f\|_{r,2} ,
\]

by Lemma \([3.8]\) Therefore the identity map on \( L^1(\mathcal{G}, \sigma) \) extends to a contractive \(*\)-homomorphism

\[
\pi_{p,2} : F_z^p(\mathcal{G}, \sigma) \to C^*_r(\mathcal{G}, \sigma) .
\]

Now we want to see that \( \pi_{p,2} \) is injective. Let \( f \in \ker(\pi_{p,2}) \). Then there exists a sequence \( \{f_n\}_{n=1}^{\infty} \) in \( L^1(\mathcal{G}, \sigma) \) such that \( \lim_{n} f_n = f \) in \( F_z^p(\mathcal{G}, \sigma) \). Then given \( x \in \mathcal{G}^{(0)} \) and \( \xi \in C_r(\mathcal{G}_x) \) we have that

\[
\lim_{n \to \infty} L_x^{\alpha-2}(f_n)\xi = 0 \quad \text{in } L^2(\mathcal{G}_x),
\]

that is

\[
0 = \lim_{n \to \infty} L_x^{\alpha-2}(f_n)\xi\|_{L^2(\mathcal{G}_x)}
\]

\[
= \left( \int_{\mathcal{G}_x} \lim_{n \to \infty} \left| L_x^{\alpha-2}(f_n)\xi(\gamma) \right|^2 d\lambda_x(\gamma) \right)^{1/2}
\]

\[
= \left( \int_{\mathcal{G}_x} \lim_{n \to \infty} \int_{\mathcal{G}_x} \left| \sigma(\gamma\mu^{-1}, \mu) f_n(\gamma\mu^{-1})\xi(\mu) \right|^2 d\lambda_x(\mu) \right)^{1/2} ,
\]

which forces

\[
(3.4) \quad \lim_{n \to \infty} \int_{\mathcal{G}_x} \left| \sigma(\gamma\mu^{-1}, \mu) f(\gamma\mu^{-1})\xi(\mu) \right| d\lambda_x(\mu) = 0 .
\]
Now observe that the map 
\( \Psi : L^1(\mathcal{G}, \sigma) \to B \left( \bigoplus_{x \in \mathcal{G}(0)} (L^p(\mathcal{G}_x) \oplus L^q(\mathcal{G}_x)) \right) \),
defined by \( f \mapsto \bigoplus_{x \in \mathcal{G}(0)} (L_x^s \sigma \rho(x)(f) \oplus L_x^s \sigma \rho(x)^*(f)) \), extends isometrically to a map 
\( \Psi : F^p_\mathcal{G}(\mathcal{G}, \sigma) \to B \left( \bigoplus_{x \in \mathcal{G}(0)} (L^p(\mathcal{G}_x) \oplus L^q(\mathcal{G}_x)) \right) \).

Now fix \( x \in \mathcal{G}(0) \) and \( i \in \{ p, q \} \). Then given \( \xi \in C_c(\mathcal{G}_x) \) we have that 
\( \Psi(f)\xi = \lim_{n \to \infty} \Psi(f_n)\xi \),
but 
\[
\| \Psi(f)\xi \|_{L^1(\mathcal{G}_x)} = \| \lim_{n \to \infty} \Psi(f_n)\xi \|_{L^1(\mathcal{G}_x)} \\
= \| \lim_{n \to \infty} L_x^{\sigma,i}(f_n)\xi \|_{L^1(\mathcal{G}_x)} \\
= \lim_{n \to \infty} \left( \int_{\mathcal{G}_x} \left( \int_{\mathcal{G}_x} \sigma(\gamma \mu^{-1}, \mu) f_n(\gamma \mu^{-1}) \xi(\mu) \right) d\lambda_x(\mu) \right)^{1/i} d\lambda_x(\gamma) \\
= 0
\]
because of (3.3). Thus, \( \Psi(f) = 0 \) and since \( \Psi \) is isometric we have that \( f = 0 \). Hence \( \pi_{n, 2} \) is injective.

Now, by (3.2) and the fact that the regular representation is \( I \)-norm bounded, we get
\[
\| f \|_{L^p} \leq \| f \|_{L^1}^{1-\theta} \| f \|_{L^2}^\theta \leq \| f \|_{L^1(\mathcal{G}, \sigma)}^{1-\theta} \| f \|_{L^2(\mathcal{G}, \sigma)}^\theta = \| f \|_I,
\]
for every \( f \in L^1(\mathcal{G}, \sigma) \). Hence, the identity map on \( L^1(\mathcal{G}, \sigma) \) extends to a contraction
\( \Phi_{p, I} : L^1(\mathcal{G}, \sigma) \to F^p_\mathcal{G}(\mathcal{G}, \sigma) \).

Observe that then \((\pi_{n, 2} \circ \Phi_{p, I}) : L^1(\mathcal{G}, \sigma) \to C^*_\mathcal{G}(\mathcal{G}, \sigma)\) is the regular representation of \( L^1(\mathcal{G}, \sigma) \), which is injective. It follows that \( \Phi_{p, I} : L^1(\mathcal{G}, \sigma) \to F^p_\mathcal{G}(\mathcal{G}, \sigma) \) is injective too.

**Proposition 3.10.** Let \( \mathcal{G} \) be a second-countable locally compact groupoid with Haar system \( \lambda \) and \( \sigma \in \mathcal{Z}^2(\mathcal{G}, \mathbb{T}) \). Then the following statements are equivalent:

1. \( \mathcal{G} \) is \( \sigma \)-quasi-symmetric,
2. \( \mathcal{G} \) is \( \sigma \)-quasi-Hermitian,
3. \( r_{L^1(\mathcal{G}, \sigma, \lambda)}(f) = r_{C^*_\mathcal{G}(\mathcal{G}, \sigma, \lambda)}(f) \) for every \( f \in C_c(\mathcal{G}, \sigma) \),
4. \( \text{Sp}_{L^1(\mathcal{G}, \sigma, \lambda)}(f) = \text{Sp}_{C^*_\mathcal{G}(\mathcal{G}, \sigma, \lambda)}(f) \) for every \( f \in C_c(\mathcal{G}, \sigma) \).

**Proof.** (1) \( \Rightarrow \) (2) was proved in Proposition 3.2. (3) \( \Rightarrow \) (4) is proved in Theorem 2.5 and (4) \( \Rightarrow \) (1) is clear. So we only need to prove (2) \( \Rightarrow \) (3).

Suppose that \( \mathcal{G} \) is \( \sigma \)-quasi-Hermitian and \( 1 < p < 2 \). Then
\[
(L^1(\mathcal{G}, \sigma), F^p_\mathcal{G}(\mathcal{G}, \sigma), C^*_\mathcal{G}(\mathcal{G}, \sigma))
\]
is a spectral interpolation triple of reduced Banach *-algebras relative to \( L^1(\mathcal{G}, \sigma) \) by Proposition 3.9. Hence, by Proposition 2.9 we have that
\[
r_{F^p_\mathcal{G}(\mathcal{G}, \sigma)}(f) = r_{C^*_\mathcal{G}(\mathcal{G}, \sigma)}(f),
\]
for every \( f \in C_c(\mathcal{G}, \sigma) \).
Now fix \( f \in C_c(G, \sigma)_h \). Then the sets \( U = \text{Supp}(f) \) and \( s(U) \) are compact sets. Replacing \( U \) by \( U \cup U^{-1} \) we can assume that \( r(U) = s(U) \), and that given any \( x \in s(U) \), the map \( G_x \cap U \to G^x \cap U \) given by \( \gamma \mapsto \gamma^{-1} \) is a bijection. Then since \( G \) is locally compact, using a partition of the unit, there exists a function \( g_1 \in C_c(G, \sigma) \) such that \( g_1(\gamma) = 1 \) for every \( \gamma \in U \). Then since the map \( x \mapsto \int_{G_x} g_1(\gamma) \, d\lambda_x(\gamma) \) is continuous we have that

\[
K := \sup \left\{ \int_{G_x} g_1(\gamma) \, d\lambda_x(\gamma) : x \in s(U) \right\} < \infty.
\]

Observe that given \( x \in s(U) \) we have that

\[
\int_{G_x} g_1(\gamma) \, d\lambda_x(\gamma) = \lambda_x(G_x \cap U) \leq K.
\]

Now given \( n \in \mathbb{N} \), we denote by \( f^n \) the \( n \)’th convolution power \( f \ast_1 \cdots \ast_1 f \). Then we have that

\[
\text{Supp}(f^n) \subseteq U^{(n)} = \{ \gamma_1 \cdots \gamma_n : \gamma_i \in U \text{ such that } r(\gamma_i) = s(\gamma_{i+1}) \text{ for } i = 1, \ldots, n-1 \}.
\]

By continuity of the groupoid product \( U^{(n)} \) is a compact subset of \( G \). Now given \( x \in s(U^{(n)}) \) we define \( U_x^{(n)} := U^{(n)} \cap G_x \), so

\[
\lambda_x(U_x^{(n)}) = \int_{U_x^{(n-1)}} \lambda_x(U_{r(\gamma)} \gamma) \, d\lambda_x(\gamma) \leq \int_{U_x^{(n-1)}} \lambda_x(U_{r(\gamma)} \gamma) \, d\lambda_x(\gamma)
\]

\[
\leq \int_{U_x^{(n-1)}} K \, d\lambda_x(\gamma) = K \lambda_x(U_x^{(n-1)})
\]

\[
K \lambda_x(U_x^{(n-2)}) \leq \cdots \leq K^{n-1} \lambda_x(G_x \cap U)
\]

\[
\leq K^n,
\]

by using the invariance of the Haar measures.

Now, fix \( n \in \mathbb{N} \) and \( x \in s(U^{(n)}) \). Let \( 1_{U_x^{(n)}} \in L^1(G_x) \) be the characteristic function on \( U_x^{(n)} \). Then we have that

\[
\|f^n\|_{L^1(G_x)} = \|f^n \cdot 1_{U_x^{(n)}}\|_{L^1(G_x)}
\]

\[
\leq \|f^n\|_{L^p(G_x)} \|1_{U_x^{(n)}}\|_{L^q(G_x)} \quad (\text{where } 1 = \frac{1}{p} + \frac{1}{q})
\]

\[
\leq \|L_p^\sigma(f^{n-1})\|_{B(L^q(G_x))} \|f\|_{L^p(G_x)}\|G_x\|^{-\frac{n}{q}} K \frac{q}{p}
\]

\[
\leq \|f^{n-1}\|_{\frac{p}{q},p} \|f\|_{L^p(G_x)}\|G_x\|^{-\frac{n}{q}} K \frac{q}{p}.
\]

In a similar way we have that

\[
\|(f^\ast \sigma)\|_{1,1} \leq \|(f^\ast \sigma)^{n-1}\|_{\frac{p}{q},p} \|f^n\|_{L^p(G_x)} K \frac{q}{p} = \|f^{n-1}\|_{\frac{p}{q},p} \|f^n\|_{L^p(G_x)} K \frac{q}{p}.
\]

Now using (3.5) but replacing \( g_1 \) with \( |f|^p \) and \( |f^\ast \sigma|^p \), we have that

\[
\sup_{x \in \mathcal{G}} \left\{ \max\{\|f\|_{L^p(G_x)}, \|f^n\|_{L^p(G_x)}\} \right\} = C < \infty,
\]
and then
\[
\|f^n\|_I = \sup_{x \in G(0)} \{\max\{|f^n_{|G_x}|_{L^1(G_x)}, |(f^*)^n_{|G_x}|_{L^1(G_x)}\}\}
\]
\[
\leq \|f^{n-1}\|_{L^p} \sup_{x \in G(0)} \{\max\{|f_{|G_x}|_{L^p(G_x)}, |(f^*)_{|G_x}|_{L^p(G_x)}\}\} K^{\frac{n}{p}}
\]
\[
\leq \|f^{n-1}\|_{L^p} C K^{\frac{n}{p}}.
\]
Therefore,
\[
\|f^n\|_I^\frac{1}{n} \leq \|f^{n-1}\|_{L^p}^\frac{1}{n} C^{\frac{1}{n}} K^{\frac{1}{n}},
\]
and then when \(n \to \infty\), we have that
\[
r_{L^1(G,\sigma)}(f) \leq r_{F^\ast_1}(G,\sigma)(f) K^\frac{1}{2} = r_{C^\ast_1}(G,\sigma)(f) K^\frac{1}{2}.
\]
Then, taking the limit for \(p \to 1^+\), we have that \(q \to \infty\), so \(r_{L^1(G,\sigma)}(f) \leq r_{C^\ast_1}(G,\sigma)(f)\).
But we always have that \(r_{C^\ast_1}(G,\sigma)(f) \leq r_{L^1(G,\sigma)}(f)\), and hence
\[
r_{C^\ast_1}(G,\sigma)(f) = r_{L^1(G,\sigma)}(f).
\]
\[
\square
\]

**Theorem 3.11.** Let \(G\) be a second-countable locally compact groupoid with Haar system \(\lambda\) and \(\sigma \in Z^2(G,\mathbb{T})\). If \(G\) is \(\sigma\)-quasi-Hermitian, then \(C^\ast_1(G,\sigma)\) is the \(C^\ast\)-envelope of \(L^1(G,\sigma)\). In particular, \(G\) with Haar system \(\lambda\) and the twist \(\sigma\) has the weak containment property.

**Proof.** By Proposition 2.7, if \(C_\ast(G,\sigma)\) is quasi-Hermitian in \(L^1(G,\sigma,\lambda)\), then for every \(f \in C_\ast(G,\sigma)_h\) we have \(r_{C^\ast_1(G,\sigma)}(f) = r_{L^1(G,\sigma)}(f)\). Therefore by Proposition 2.10, we have that \(C^\ast_1(G,\sigma,\lambda)\) is the \(C^\ast\)-envelope of \(L^1(G,\sigma,\lambda)\). But this means that the reduced norm is the maximal norm, and so \(C^\ast_1(G,\sigma,\lambda) = C^\ast(G,\sigma,\lambda)\).

Finally, we address the problem of spectral invariance. Recall that spectra in non-unital algebras are defined in terms of the spectra in their minimal unitizations. Since a Banach *-algebra \(A\) is Hermitian if and only if its minimal unitization is Hermitian [34, Theorem (4.7.9)], we obtain the following corollary.

**Corollary 3.12.** Let \(G\) be a second-countable locally compact groupoid with Haar system \(\lambda\) and \(\sigma \in Z^2(G,\mathbb{T})\). Then \(L^1(G,\sigma,\lambda)\) is Hermitian if and only if \(L^1(G,\sigma,\lambda)\) is spectrally invariant in \(C^\ast_1(G,\sigma,\lambda)\).

**Proof.** Suppose that \(L^1(G,\sigma)\) is Hermitian. Then \(C^\ast(L^1(G,\sigma)) = C^\ast_1(G,\sigma)\) by Theorem 3.11. But as \(L^1(G,\sigma)\) is Hermitian, it must be spectrally invariant in its enveloping \(C^\ast\)-algebra, hence it is spectrally invariant in \(C^\ast_1(G,\sigma)\). The converse implication is trivial.

\[
\square
\]


4. Non-Hermitian ample amenable groupoids

In the previous section we proved that if $G$ is a second-countable locally compact quasi-Hermitian groupoid, then $G$ satisfies the weak containment property, i.e. $C^*_r(G) \cong C^*_r(G)$. In the case $G$ is a group, this translates to the fact that quasi-Hermitian groups are amenable. In the case of groupoids the situation is more subtle, since the weak containment property is not equivalent to $G$ being amenable. There are examples of non-amenable groupoids with the weak containment property in [11, 39]. In this section we will see that when $G$ is an ample groupoid, an obstruction to $G$ being quasi-Hermitian is that the isotropy groups are not quasi-Hermitian. We can then easily construct an amenable ample groupoid that is not Hermitian.

**Definition 4.1.** A locally compact Hausdorff étale groupoid is called *ample* if it has a basis consisting of open and compact bisections.

Let $G$ be an ample groupoid and let $\sigma \in Z^2(G, \mathbb{T})$. Then, given $x \in G^{(0)}$, the restriction of $\sigma$ to the isotropy group $G^x_x$ is a group 2-cocycle. We denote this restricted 2-cocycle by $\sigma_x$.

**Proposition 4.2.** Let $G$ be a second-countable ample groupoid and $\sigma \in Z^2(G, \mathbb{T})$, with $G^{(0)}$ compact. Suppose that for every $\gamma \in \text{Iso}(G)$ there exists a clopen bisection $U \subseteq G$ such that $\gamma \in U$ and $r(U) = s(U) = G^{(0)}$. If $G$ is $\sigma$-quasi-Hermitian, then $G^x_x$ is a $\sigma_x$-quasi-Hermitian groupoid for every $x \in G^{(0)}$.

**Proof.** Let us suppose that there exists $x \in G^{(0)}$ such that $G^x_x$ is not $\sigma_x$-quasi-Hermitian. Then there exists $f \in C_c(G^x_x, \sigma_x)$ such that $\text{Sp}_{\ell^1(G^x_x, \sigma_x)}(f) \not\subset \mathbb{R}$. Observe that $f$ must be of the form

$$f = \sum_{i=1}^m \lambda_i \delta_{\gamma_i} + \sum_{i=1}^m \lambda_i \sigma(\gamma_i^{-1}, \gamma_i) \delta_{\gamma_i^{-1}},$$

where $\lambda_i \in \mathbb{C}$ and $\gamma_i \in G^x_x$. Let $\lambda \in \text{Sp}_{\ell^1(G^x_x, \sigma_x)}(f) \setminus \mathbb{R}$. By assumption, for every $\gamma \in G^x_x$ there exists a bisection $U_\gamma \subseteq G$ such that $\gamma \in U_\gamma$ and $r(U_\gamma) = s(U_\gamma) = G^{(0)}$. Then $\hat{f} = \sum_{i=1}^m \lambda_i 1_{U_{\gamma_i}} + \sum_{i=1}^m \lambda_i \sigma(\gamma_i^{-1}, \gamma_i) 1_{U_{\gamma_i}} \in C_c(G, \sigma)_h$. We claim that $\lambda \in \text{Sp}_{\ell^1(G, \sigma)}(\hat{f})$, that is $\lambda 1_{G^{(0)}} - \hat{f}$ is not invertible in $\ell^1(G, \sigma)$. Suppose that $\lambda 1_{G^{(0)}} - \hat{f}$ is invertible in $\ell^1(G, \sigma)$, so there exists $\hat{g} \in \ell^1(G, \sigma)$ such that $(\lambda 1_{G^{(0)}} - \hat{f}) \ast_\sigma \hat{g} = 1_{G^{(0)}}$. Let $\{\hat{g}_n\}_{n=1}^\infty$ be a sequence in $C_c(G, \sigma)$ such that $\hat{g}_n \to \hat{g}$ in $\ell^1(G, \sigma)$, and hence $(\lambda 1_{G^{(0)}} - \hat{f}) \ast_\sigma \hat{g}_n \to 1_{G^{(0)}}$ in $\ell^1(G, \sigma)$. In particular, $((\lambda 1_{G^{(0)}} - \hat{f}) \ast_\sigma \hat{g}_n)(\gamma) \to 1$ and $((\lambda 1_{G^{(0)}} - \hat{f}) \ast_\sigma \hat{g}_n)(\gamma) \to 0$ for every $\gamma \in G^x_x \setminus \{x\}$. Let $\hat{g}_n = \sum_{j=1}^{l_n} \beta_{j,n} 1_{V_{j,n}}$, where the $V_{j,n}$’s are compact open bisections. Then

$$((\lambda 1_{G^{(0)}} - \hat{f}) \ast_\sigma \hat{g}_n) = (\lambda 1_{G^{(0)}} - \hat{f}) \ast_\sigma (\sum_{j=1}^{l_n} \beta_{j,n} 1_{V_{j,n}})$$

$$= \sum_{j=1}^{l_n} \lambda \beta_{j,n} 1_{V_{j,n}} - \sum_{j=1}^{l_n} \beta_{j,n} (\hat{f} \ast_\sigma 1_{V_{j,n}}).$$
Therefore, defining \( \eta_{j,n} := xV_{j,n}x \in G_x^x \) and \( g_n = \sum_{j=1}^{l_n} \beta_{j,n}\delta_{\eta_{j,n}} \in \ell^1(G_x^x, \sigma_x) \), we have that the sequence \( \{g_n\}_{n=1}^{\infty} \) converges in \( \ell^1(G_x^x, \sigma_x) \) because \( \{\hat{g}_n\}_{n=1}^{\infty} \) converges in \( \ell^1(G, \sigma) \), and

\[
(\lambda - f) *_{\sigma_x} g_n \rightarrow (\lambda - f) *_{\sigma_x} g = 1.
\]

Therefore \( \lambda \notin \text{Sp}_1(G_x^x, \sigma_x)(f) \), a contradiction.

\[\Box\]

Remark 4.3. It was observed in [39] Lemma 4.9 that the condition that for every \( \gamma \in \text{Iso}(G) \) there exists a clopen bisection \( U \subset G \) such that \( \gamma \in U \) and \( r(U) = s(U) = G^{(0)} \) is satisfied if \( |\text{Orb}_{G}(x)| \geq 2 \) for every \( x \in G^{(0)} \).

Example 4.4. Willett constructed in [39] a second-countable locally compact ample groupoid \( G \) with \( G^{(0)} \) compact that satisfies the weak-containment property. \( G \) is a group bundle so it clearly satisfies the assumptions in Proposition 4.2. Moreover, \( G \) has an isotropy group isomorphic to the free non-abelian group with two generators, which is not quasi-Hermitian by [36] Corollary 4.8. Therefore, by Proposition 4.2 we have that \( G \) is not quasi-Hermitian.

Example 4.5. Let \( \Gamma \) be a countable discrete group with unit \( e \), and let us consider an action of \( \Gamma \) on a second-countable compact Hausdorff space \( X \). Then \( X \rtimes \Gamma \) is a second-countable locally compact Hausdorff étale groupoid. Let us suppose that \( \Gamma \) contains a free semigroup on two generators \( z, t \). Given \( \gamma \in \Gamma \) we define the bisection \( U_\gamma = (X, \gamma) \) of \( X \rtimes \Gamma \). Observe that given \( \gamma, \gamma' \in \Gamma \) we have that \( U_\gamma U_{\gamma'} = U_{\gamma\gamma'} \). Let us consider

\[ f = a_01_{U_e} + a_11_{U_z} + a_21_{U_t} = C_c(X \rtimes \Gamma), \]

where \( a_0, a_1, a_2 \in \mathbb{C} \) satisfy \( |a_0| = |a_1| = |a_2| = \frac{1}{3} \) and

\[ \sup\{|a_0 + a_1x + a_2x^2| : x \in \mathbb{T}\} < 1. \]

Observe that \( 1_{U_e} \) is the unit of \( \ell^1(X \rtimes \Gamma) \). We then have

\[ r_{\ell^1(X \rtimes \Gamma)}(f) < 1, \]

by using the spectral mapping theorem and maximum modulus principle, and since \( f \) is normal, i.e. \( ff^* = f^*f \), we obtain

\[ \|f\|_{C^*_\Gamma(X \rtimes \Gamma)} = r_{C^*_\Gamma(X \rtimes \Gamma)}(f) < 1. \]

Now, since \( 1_{U_t} \) is a unitary in \( C^*(X \rtimes \Gamma) \) it follows that

\[ \|f *_{\Gamma} 1_{U_t}\| = \left\|a_01_{U_t} + a_11_{U_{zt}} + a_21_{U_{zt}}\right\|. \]

Then since \( t, z \) generate a free non-abelian group we have that

\[ (a_01_{U_t} + a_11_{U_{zt}} + a_21_{U_{zt}})^n \]

has \( 3^n \) linearly independent terms. Hence, since \( U_\gamma \cap U_{\gamma'} = \emptyset \) if and only if \( \gamma \neq \gamma' \), we have that

\[ \|(a_01_{U_t} + a_11_{U_{zt}} + a_21_{U_{zt}})^n\|_{\ell^1(X \rtimes \Gamma)} = 1 \]

for every \( n \in \mathbb{N} \), and so \( r_{\ell^1(X \rtimes \Gamma)}(a_01_{U_t} + a_11_{U_{zt}} + a_21_{U_{zt}}) = 1 \). Thus we have that

\[ \text{Sp}_{\ell^1(X \rtimes \Gamma)}(a_01_{U_t} + a_11_{U_{zt}} + a_21_{U_{zt}}) \neq \text{Sp}_{C^*_\Gamma(X \rtimes \Gamma)}(a_01_{U_t} + a_11_{U_{zt}} + a_21_{U_{zt}}). \]
and hence by Proposition 5.10 we have that \( X \rtimes \Gamma \) is not quasi-Hermitian.

Then, let \( F_2 \) be the non-abelian free group with two generators. It is known that there exists a locally compact space \( X \) and an amenable action of \( F_2 \) on \( X \) (see for example \[38\]). Hence, \( X \rtimes F_2 \) is an amenable groupoid but not quasi-Hermitian.

### 5. Hermitian Twisted Groupoid Banach \(*\)-algebras

Let \( G \) be a second-countable locally compact groupoid with Haar system \( \lambda \) and let \( \sigma \in Z^2(G, \mathbb{T}) \). In this section we give a sufficient condition for the Banach \(*\)-algebra \( L^1(G, \sigma, \lambda) \) to be Hermitian. As a consequence we are able to give conditions for twisted transformation groupoids so that the associated twisted transformation groupoid Banach \(*\)-algebras are Hermitian.

**Definition 5.1.** Given a second-countable locally compact groupoid \( G \) with Haar system \( \lambda \) and \( \sigma \in Z^2(G, \mathbb{T}) \), we define the twisted groupoid \( G_{\sigma} \) to be the groupoid \( G \times \mathbb{T} \) with product topology and operations defined by

\[
(\gamma_1, z_1) \cdot (\gamma_2, z_2) = (\gamma_1 \gamma_2, z_1 z_2 \overline{\sigma(\gamma_1, \gamma_2)}) \quad \text{if} \quad (\gamma_1, \gamma_2) \in G^{(2)},
\]

and

\[
(\gamma, z)^{-1} = (\gamma^{-1}, \overline{z} \sigma(\gamma, \gamma^{-1})).
\]

The Haar system of \( G_{\sigma} \) is the one given by \( \lambda \times \eta = \{\lambda_x \times \eta\}_{x \in G^{(0)}} \), where \( \eta \) is the normalized Lebesgue measure on \( \mathbb{T} \).

**Proposition 5.2.** Let \( G \) be a second-countable locally compact groupoid with Haar system \( \lambda \) and \( \sigma \in Z^2(G, \mathbb{T}) \). The map

\[
j : L^1(G, \sigma, \lambda) \rightarrow L^1(\mathcal{G}_\sigma, \lambda \times \eta),
\]

given by \( j(f)(\gamma, z) = zf(\gamma) \) is an isometric \(*\)-homomorphism.

**Proof.** First we prove that \( j \) is a \(*\)-homomorphism. Fix \( f, g \in L^1(G, \sigma) \), \( \gamma \in G \) and \( z \in \mathbb{T} \). Then we have that

\[
j(f \ast_\sigma g)(\gamma, z) = z(f \ast_\sigma g)(\gamma) = z \int_{G_{\sigma}(\gamma)} \sigma(\gamma \mu^{-1}, \mu) f(\gamma \mu^{-1}) g(\mu) d\lambda_{\sigma(\gamma)}(\mu)
\]

\[
= \int_{G_{\sigma}(\gamma)} z \sigma(\gamma, \mu^{-1}) \sigma(\mu, \mu^{-1}) f(\gamma \mu^{-1}) g(\mu) d\lambda_{\sigma(\gamma)}(\mu) dt
\]

\[
= \int_{G_{\sigma}(\gamma)} \int_{\mathbb{T}} \sigma(\gamma, \mu^{-1}) \sigma(\mu, \mu^{-1}) f(\gamma \mu^{-1}) g(\mu) d\lambda_{\sigma(\gamma)}(\mu) dt
\]

\[
= \int_{G_{\sigma}(\gamma)} \int_{\mathbb{T}} \overline{\sigma(\gamma, \mu^{-1})} \overline{\mu} \sigma(\mu, \mu^{-1}) f(\gamma \mu^{-1}) g(\mu) d\lambda_{\sigma(\gamma)}(\mu) dt
\]

\[
= \int_{G_{\sigma}(\gamma)} \int_{\mathbb{T}} j(f)(\gamma \mu^{-1}) \overline{\sigma(\gamma, \mu^{-1})} \overline{\mu} \sigma(\mu, \mu^{-1}) g(\mu) d\lambda_{\sigma(\gamma)}(\mu) dt
\]

\[
= \int_{G_{\sigma}(\gamma)} \int_{\mathbb{T}} j(f)((\gamma, z)(\mu, t)^{-1}) j(g)(\mu, t) d\lambda_{\sigma(\gamma)}(\mu) dt
\]

\[
= (j(f) \ast j(g))(\gamma, z),
\]
where, at the third equality, we have used \( \sigma(\gamma \mu^{-1}, \mu) \sigma(\gamma, \mu^{-1}) = \sigma(\mu^{-1}, \mu) \). This identity follows from (2.2) by using \( \gamma, \mu^{-1}, \mu \) instead of \( \alpha, \beta, \gamma \) and then applying (2.1). We then use that \( \sigma(\mu^{-1}, \mu) = \sigma(\mu, \mu^{-1}) \), which follows from (2.2) using \( \mu, \mu^{-1}, \mu \) instead of \( \alpha, \beta, \gamma \) and then applying (2.1).

Moreover,

\[
\int \left| f(\gamma) \right| d\lambda_x(\gamma) = \int \left| \int T \left[ |f(\gamma)| d\lambda_x(\gamma) \right] dz \right| = \int T \left[ \int |f(\gamma)| d\lambda_x(\gamma) \right] dz
\]

and

\[
\int \left| f(\gamma^{-1}) \right| d\lambda_x(\gamma) = \int \left| \int T \left[ |f(\gamma^{-1})| d\lambda_x(\gamma) \right] dz \right| = \int T \left[ \int |f(\gamma^{-1})| d\lambda_x(\gamma) \right] dz.
\]

Finally,

\[
j(f^*) (\gamma, z) = zf^{*} (\gamma) = z \sigma(\gamma, \gamma^{-1}) f(\gamma^{-1})
\]

\[
= j(f)(\gamma^{-1}, \sigma(\gamma, \gamma^{-1}))
\]

\[
= j(f)((\gamma, z)^{-1})
\]

\[
= j(f)^* (\gamma, z).
\]

Therefore,

\[
\|f\|_I = \sup_{x \in T} \max \left\{ \int \left| f(\gamma) \right| d\lambda_x(\gamma), \int \left| f(\gamma^{-1}) \right| d\lambda_x(\gamma) \right\}
\]

\[
= \sup_{x \in T} \max \left\{ \int T |f(\gamma, z)| d\lambda_x(\gamma) dz, \int T |j(f)((\gamma, z)^{-1})| d\lambda_x(\gamma) dz \right\}
\]

\[
= \|j(f)\|_I.
\]

Proposition 5.3. Let \( \mathcal{G} \) be a second-countable locally compact groupoid with Haar system \( \lambda \) and \( \sigma \in \mathcal{Z}^2(\mathcal{G}, T) \). Suppose that \( L^1(\mathcal{G}_{\sigma}) \) is Hermitian. Then \( L^1(\mathcal{G}, \sigma) \) is Hermitian.

Proof. By Proposition 5.2 \( L^1(\mathcal{G}, \sigma) \) is a closed Banach *-subalgebra of \( L^1(\mathcal{G}_{\sigma}) \). Then by [10, Proposition 7.10] \( L^1(\mathcal{G}, \sigma) \) is Hermitian.
Example 5.4. Let $X$ be a second-countable locally compact Hausdorff space, and let $\Gamma$ be a second-countable locally compact group acting on $X$ by homeomorphisms. Let $G = X \rtimes \Gamma$ be the transformation groupoid, which is locally compact. Recall that $G$ is étale if and only if $\Gamma$ is discrete. Let $\sigma \in Z^2(\Gamma, \mathbb{T})$, and extend $\sigma$ to a 2-cocycle of $X \rtimes \Gamma$ as shown in Example 2.13. We define $\Gamma_\sigma$ to be the group that is $\Gamma \times \mathbb{T}$ with the product topology, and

$$(\gamma_1, z_1)(\gamma_2, z_2) = (\gamma_1 \gamma_2, z_1 z_2 \bar{\sigma}(\gamma_1, \gamma_2)),$$

for every $z_1, z_2 \in \mathbb{T}$ and $\gamma_1, \gamma_2 \in \Gamma$. Now if we define the action of $\Gamma_\sigma$ on $X$ by

$$(\gamma, z) \cdot x := \gamma \cdot x,$$

then the transformation groupoid $X \rtimes \Gamma_\sigma$ is isomorphic to $(X \rtimes \Gamma)_\sigma$.

Let $X$ be a second-countable locally compact Hausdorff space, and let $\Gamma$ be a second-countable locally compact group with modular function $\Delta$, acting on $X$ by homeomorphisms. Further, let $\sigma \in Z^2(\Gamma, \mathbb{T})$, and let $C_0(X)$ be the Banach $\ast$-algebra of continuous functions on $X$ that vanish at infinity with the supremum norm $\| \cdot \|_\infty$. Then $\Gamma$ acts on $C_0(X)$ by $\gamma \cdot f(x) = f(\gamma^{-1} \cdot x)$ for every $f \in C_0(X)$ and $\gamma \in \Gamma$. Let us define the generalized $L^1$-algebra $L^1(\Gamma, C_0(X), \sigma)$ to be the completion of

$$C_c(\Gamma, C_0(X), \sigma) = \{ f : \Gamma \to C_0(X) : f \text{ is continuous with compact support} \}$$

with respect to the norm

$$\| f \| := \int_\Gamma \| f(\gamma) \|_\infty d\lambda(\gamma),$$

where $\lambda$ is a left Haar measure of $\Gamma$. Then $L^1(\Gamma, C_0(X), \sigma)$ becomes a Banach $\ast$-algebra with the operations

$$(f \ast_\sigma g)(\gamma)(x) = \int_\Gamma \sigma(\gamma \mu^{-1}, \mu) f(\gamma \mu^{-1}) (\mu \cdot x) g(\mu)(x) d\lambda(\mu),$$

and

$$(f^{\ast \sigma})(\gamma)(x) = \Delta(\gamma^{-1}) \bar{\sigma}(\gamma, \gamma^{-1}) f(\gamma^{-1})(\gamma \cdot x),$$

for every $f, g \in L^1(\Gamma, C_0(X), \sigma)$, $\gamma \in \Gamma$ and $x \in X$.

If $\sigma$ is the trivial twist, we denote $L^1(\Gamma, C_0(X), \sigma)$ by $L^1(\Gamma, C_0(X))$. The $C^*$-envelope of $L^1(\Gamma, C_0(X), \sigma)$ is the twisted crossed product $C^*$-algebra $C_0(X) \rtimes^\sigma \Gamma$.

Lemma 5.5. Let $X$ be a second-countable locally compact Hausdorff space, and let $\Gamma$ be a second-countable locally compact unimodular group ($\Delta \equiv 1$) acting on $X$ by homeomorphisms. Let $\sigma \in Z^2(\Gamma, \mathbb{T})$. Then there exists a surjective $\ast$-homomorphism $\Phi : L^1(\Gamma, C_0(X), \sigma) \to L^1(X \rtimes \Gamma, \sigma)$. Consequently, if $L^1(\Gamma, C_0(X), \sigma)$ is Hermitian, then so is $L^1(X \rtimes \Gamma, \sigma)$.

Proof. First observe that $C_c(\Gamma, C_c(X), \sigma)$ is a dense $\ast$-subalgebra of $L^1(\Gamma, C_0(X), \sigma)$. The map $\Phi : C_c(\Gamma, C_c(X), \sigma) \to C_c(X \rtimes \Gamma, \sigma)$ given by $\Phi(f)(x, \gamma) = \Phi(f)(\gamma)(x)$ defines a $\ast$-homomorphism. Indeed, given $f, g \in C_c(\Gamma, C_c(X), \sigma)$, $\gamma \in \Gamma$ and $x \in X$
\[\Phi(f \ast g)(x, \gamma) = (f \ast g)(\gamma)(x)\]
\[= \int_{\Gamma} \sigma(\gamma \mu^{-1}, \mu) f(\gamma \mu^{-1})(\mu \cdot x) g(\mu)(x) \, d\lambda(\mu)\]
\[= \int_{\Gamma} \sigma(\gamma \mu^{-1}, \mu) \Phi(f)(\mu \cdot x, \gamma \mu^{-1}) \Phi(g)(x, \mu) \, d\lambda(\mu)\]
\[= (\Phi(f) \ast \Phi(g))(x, \gamma),\]

and
\[\Phi(f^{*\ast})(x, \gamma) = f^{*\ast}(\gamma)(x) = \overline{\sigma(\gamma, \gamma^{-1}) f(\gamma^{-1})(\gamma \cdot x)}\]
\[= \overline{\sigma(\gamma, \gamma^{-1}) \Phi(f)(\gamma \cdot x, \gamma^{-1})} = \Phi(f)^{*\ast}(x, \gamma).\]

Observe that clearly \(\Phi\) is a bijection. Now given \(f \in C_c(\Gamma, C_c(X), \sigma)\) we have that
\[\|\Phi(f)\|_1 = \sup \left\{ \int_{\Gamma} |\Phi(f)(x, \gamma)| \, d\lambda(\gamma) : x \in X \right\}\]
\[= \sup \left\{ \int_{\Gamma} |f(\gamma)(x)| \, d\lambda(\gamma) : x \in X \right\}\]
\[\leq \int_{\Gamma} \sup \{ |f(\gamma)(x)| : x \in X \} \, d\lambda(\gamma)\]
\[= \int_{\Gamma} \|f(\gamma)\|_\infty \, d\lambda(\gamma) = \|f\|_i.\]

Thus, \(\|\Phi(f)\|_1 \leq \|f\|\) for every \(f \in C_c(\Gamma, C_c(X), \sigma)\), and so \(\Phi\) extends to a continuous surjective \(*\)-homomorphism \(\Phi : L^1(\Gamma, C_0(X), \sigma) \to L^1(X \rtimes \Gamma, \sigma)\).

Finally, the last statement follows by the fact that quotients of Hermitian \(*\)-algebras are Hermitian [32, Theorem 10.4.4]. \(\square\)

**Corollary 5.6.** Let \(X\) be a second-countable locally compact Hausdorff space, and let \(\Gamma\) be a second-countable compact group or a locally compact abelian group acting on \(X\) by homeomorphisms. Let \(\sigma \in Z^2(\Gamma, \mathbb{T})\). Then \(L^1(X \rtimes \Gamma, \sigma)\) is Hermitian, and, in particular, \(L^1(X \rtimes \Gamma, \sigma)\) is spectrally invariant in \(C^*_r(X \rtimes \Gamma, \sigma)\).

**Proof.** By Example 5.4 we have that \((X \rtimes \Gamma)_\sigma \cong X \rtimes \Gamma_\sigma\). Then by Proposition 5.3 \(L^1(X \rtimes \Gamma, \sigma)\) is Hermitian if \(L^1(X \rtimes \Gamma_\sigma)\) is Hermitian. By Lemma 5.5 \(L^1(X \rtimes \Gamma_\sigma)\) is Hermitian if \(L^1(\Gamma_\sigma, C_0(X))\) is Hermitian. Now if \(\Gamma\) is compact, then \(\Gamma_\sigma\) is compact too, and if \(\Gamma\) is abelian, then \(\Gamma_\sigma\) is nilpotent, and hence \(L^1(\Gamma_\sigma, C_0(X))\) is Hermitian [8, pg. 1285]. So in both cases \(L^1(X \rtimes \Gamma, \sigma)\) is Hermitian, and it then follows by Theorem 3.11 that \(X \rtimes \Gamma\) has the weak containment property with respect to \(\sigma\). Finally, by Corollary 3.12 it follows that \(L^1(X \rtimes \Gamma, \sigma)\) is spectrally invariant in \(C^*_r(X \rtimes \Gamma, \sigma)\). \(\square\)

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