How to study correlation functions in fluctuating Bose liquids using interference experiments

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Interference experiments with independent condensates provide a powerful tool for analyzing correlation functions. Scaling of the average fringe contrast with the system size is determined by the two-point correlation function and can be used to study the Luttinger liquid liquid behavior in one-dimensional systems and to observe the Kosterlitz-Thouless transition in two-dimensional quasicompressible condensates. Additionally, higher moments of the fringe contrast can be used to determine the higher order correlation functions. In this article we focus on interference experiments with one-dimensional Bose liquids and show that methods of conformal field theory can be applied to calculate the full quantum distribution function of the fringe contrast.

Correlation functions provide a convenient language for characterizing quantum states. First-order coherence in optics, which underlies classical Young’s interference experiments, is mathematically equivalent to the factorizability of the first order correlation function \( G^{(1)}(x_1, x_2) = \langle E^-(x_1) E^+(x_2) \rangle \), where \( E^\pm \) are the electric field components which vary as \( e^{\pm i \omega t} \). Photon bunching in Hanbury-Brown and Twiss experiments can be understood using the second order correlation function \( G^{(2)}(\tau) = \langle E^-(t) E^-(t+\tau) E^+(t+\tau) E^+(t) \rangle \). The same correlation function describes photon antibunching in quantum mechanical states of light such as number states of photons\textsuperscript{1}. Analogously, in condensed matter physics many common experiments can be understood as probes of the appropriate two point correlation functions. For example, conductivity measurements (dc or finite frequency) correspond to current-current correlation functions\textsuperscript{2}, angle resolved photoemission\textsuperscript{3, 4} and tunneling experiments\textsuperscript{5} probe single particle Green’s functions, X-Ray and neutron scattering experiments can be related to charge and spin correlation functions\textsuperscript{6}.

An important question for current experiments with ultracold atoms is finding new methods of characterizing strongly correlated many-body states, such as systems near Feshbach resonances, rotating condensates, atoms in optical lattices, low-dimensional systems (see ref. \textsuperscript{7} for a review). The existing experimental toolbox includes Bragg scattering, which measures the dynamic structure factor\textsuperscript{8} (i.e. the imaginary part of the density-density correlation function), and the RF spectroscopy (see e.g. Refs. \textsuperscript{9, 10}), which is essentially equivalent to finite frequency conductivity measurements in solid state systems. These techniques measure different types of two point correlation functions. An interesting question to ask is whether one can do experiments with systems of cold atoms that would measure higher order correlation functions. In two recent papers\textsuperscript{11, 12} we argued that this can be done using interference experiments with independent condensates (see also Ref. \textsuperscript{13}). In this article we will review these ideas while focusing on a specific case of one-dimensional condensates.

We mention in passing that another interesting technique, which is unique to systems of cold atoms, is the time of flight experiments (see e.g. Ref. \textsuperscript{14}). When expansion of atoms released from the trap is ballistic, the momentum distribution of atoms inside the original system gets mapped into the density distribution after the expansion. Measurements of quantum noise after the expansion have been used to study Hanbury-Brown-Twiss correlations for bosons and fermions in optical lattices\textsuperscript{15, 16, 17} and pairing correlations on the molecular side of the Feshbach resonance\textsuperscript{18}. The correlation function measured in such experiments is a non-local operator from the point of view of the original system but should provide “smoking gun” signatures of several many-body phases including antiferromagnetically ordered Mott states and paired states of fermions\textsuperscript{19}.

The interference experiments we consider are shown in Fig. \textsuperscript{1}. Two independent quasi-condensates are allowed to expand in the transverse direction. After sufficient expansion, the atom distribution is measured using the imaging beam. Everywhere in this paper we consider the two condensates to be identical, although our analysis can be generalized to the case of different condensates. In the set-up we discuss, the measuring beam is orientated along the axis of the original cloud. This beam integrates over local interference patterns and the fringes that we see on a detector are a result of averaging over the entire imaging length. In the presence of thermal or quantum fluctuations, interference patterns are not in phase at different points and the resulting interference fringes have a reduced contrast. Such reduction carries information about fluctuations in the original clouds. The main topic of this article is a non-trivial information that we can extract by analyzing such reduced contrast. We point out that an alternative set-up, which is also possible with current experiments, is to measure not along the system axis but from the side. This provides a local picture of interference fringes rather than the integrated one. A convenient way to analyze such data would be to integrate interference patterns numerically and study how the average contrast changes with the system size (see discussion below). So the main advantage of the latter set-up is a dramatic reduction in the number
of measurements needed to determine the scaling of interference fringes. Conceptually, however, analysis is identical to the case that we discuss in this paper. We assume that before the expansion, atoms are confined to the lowest transverse channels of their respective traps and that the optical imaging length $L$ (which is smaller than the size of the system in the axial direction) is much larger than the coherence length of the condensates. This allows us to use an effective Luttinger liquid description of the interacting bosons\[20].

The quantum observable corresponding to the interference amplitude of the two condensates \[11\] is given by $A_Q = \int_0^L dz a_1^\dagger(z) a_2^\dagger(z) a_1(z) a_2(z)$. Here $a_1$ and $a_2$ are the bosonic operators in the two systems before the expansion, and the integrals are taken along the condensates. $L$ is the imaging length that is in general smaller than the full condensate length. If the two condensates are decoupled from each other, the expectation value of $\langle A_Q \rangle$ vanishes. This does not mean that $|A_Q|$ is zero in each individual measurement but rather shows that the phase of $A_Q$ is random. Hence the position of interference fringes is completely random from shot to shot. The quantum mechanical expectation value is defined as a result of averaging over many experimental runs. Superimposed interference patterns with random phases wash each other out and interference fringes averaged over many shots disappear\[21\]. However, what we are interested in is the amplitude of the fringes in an individual measurement. Thus we should consider an observable that does not involve the random phase of $A_Q$. We take $|A_Q|^2 = \int_0^L \int_0^L dz_1 dz_2 a_1^\dagger(z_1) a_1(z_2) a_2^\dagger(z_2) a_2(z_1)$ (1)

To simplify calculations we consider the limit of a large system when we are allowed to take the normal ordered expression\[22\]. Taking the expectation value of the last equation produces a nontrivial result since on the right hand side operators that correspond to different clouds decouple. Assuming that the condensates are identical gives $\langle |A_Q|^2 \rangle = L^2 \int_0^L dz \langle a^\dagger(z) a(0) \rangle^2$. (2)

Thus the average intensity of the interference pattern depends on the two point correlation function along the individual one-dimensional condensates. In this regime the long distance correlations decay as a power law within the imaging length $L$, $\langle a^\dagger(z) a(0) \rangle \sim z^{-1/2K}$, where $K$ is the Luttinger parameter. For bosons with a repulsive short-range potential, $K$ ranges between 1 and $\infty$, with $K = 1$ corresponding to strong interactions, or “impenetrable” bosons, while $K \rightarrow \infty$ for non-interacting bosons. Using the power law correlations in (2), we find that the interference intensity scales as a non-trivial power of the imaging length $\langle |A_Q|^2 \rangle = L^{2-1/K}$. This property can be used as a sensitive probe of the Luttinger parameter. In the non-interacting limit ($K \rightarrow \infty$), the scaling is linear, $\bar{A}_Q \equiv \sqrt{\langle |A_Q|^2 \rangle} \sim L$, as expected for a fully coherent system. Interestingly, in the impenetrable limit, $\bar{A}_Q \sim \sqrt{L}$, as for short range exponentially decaying correlations. One may ask whether it makes sense to discuss zero temperature limit when experiments are always done at finite temperature. It is important to realize that experiments are done in systems of finite size. At finite temperatures and at long distances, correlations always decay exponentially. However, long distances means distances larger than the correlation length $\xi_T \sim v_s/T$, where $v_s$ is the sound velocity. As the temperature is lowered, $\xi_T$ increases and at some point becomes larger than the system size. At this point (and at all lower temperatures), correlations within the system size are those of the zero temperature system. It may be useful to make one remark regarding experimental constrains for observing quantum Luttinger liquid behavior with ultracold atoms. A common
parameter used to characterize the strength of interactions between particles is the ratio of the interaction energy to the kinetic energy $\gamma = mg/h^2n$. Here $m$ is the mass of atoms, $g$ is the interaction strength, and $n$ is the one-dimensional density. To observe strongly interacting Bose liquid we want $\gamma$ to be large, which may be achieved by reducing the density. However we also need to satisfy the condition $\xi_T \gg L$. If we use Bogoliubov expression for $v_s$ we find $L^{-1}\sqrt{mg/m} \gg T$. Hence for a smaller density it takes a lower temperature to reach the quantum limit. In experiments one needs to find the balance between having the interesting regime of strong interactions and reaching the quantum limit.

We point out that analysis presented above for one dimensional systems can be easily generalized to interference between planar condensates at finite temperatures[11]. The power law decay of correlations, and thus the scaling of the interference intensity with imaging size is related in this case to the superfluid stiffness of the condensates. This was used by Hadzibabic et al [23] to observe the universal jump of superfluid stiffness across the Kosterlitz-Thouless transition in pancake condensates.

So far we have discussed the average intensity of the interference pattern and showed that it contains information about two-point correlation functions. It is important to realize that interference experiments correspond to a classical measurement of a quantum mechanical state. Hence they contain an intrinsic quantum mechanical noise and the result of each individual measurement will be different from the average value. Generalization of the argument that led to eq. (2) shows that higher moments of the distribution function of interference amplitudes correspond to high-order correlation functions. Hence the knowledge of the entire distribution function reveals global properties of the system that depend on very high order non-local correlation functions.

From equation (11) one finds that for two one-dimensional condensates higher moments of the interference fringe amplitude are given by [11]

$$\langle |A_Q|^{2n} \rangle = A_0^{2n}Z_{2n}, \text{ where } A_0 = \sqrt{C \rho \xi_h} L^{1-1/K},$$

where $C$ is a constant of order unity, $\rho$ is the particle density in each condensate, $\xi_h$ is the short range cutoff equal to the healing length. The coefficients $Z_{2n}$ in Eq. (3) are given by [24]:

$$Z_{2n}(K) = \int_0^{2\pi} \cdots \int_0^{2\pi} \prod_{i=1}^n du_i dv_i \frac{1}{2\pi} \left( \prod_{i<j} 2 \sin \left( \frac{u_i - u_j}{2} \right) \prod_{i<k<l} 2 \sin \left( \frac{u_i - u_l}{2} \right) \right)^{1/K}.$$

The coefficients $Z_{2n}$ originally appeared in the grand canonical partition function of a neutral two-component Coulomb gas on a circle

$$Z(K, x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!^2} Z_{2n}(K).$$

Here $x$ is the fugacity of Coulomb charges and $Z_{2n}$ describes contributions from configurations with $2n$ charges (i.e. canonical partition functions). The partition function (5) with $K > 1$ describes several problems in statistical physics (see Ref [25] and references therein). It is also related to the problems of an impurity in a Luttinger liquid [12, 26] and Kondo effect [27].

When describing interference experiments it is convenient to define the normalized amplitude of interference fringes $\alpha = |A_Q|^{2}/A_0^{2}$. From (3) we find that $\langle \alpha^{2n} \rangle = Z_{2n}$, so by performing experiments that measure the distribution function $W(\alpha)$, we get direct access to the partition function (3). We point out that $W(\alpha)$ can be used to compute all moments of $|A_Q|^2$, and therefore contains information about high order correlation functions of the interacting Bose liquids.

From the Taylor expansion of the modified Bessel function we find

$$Z(K, x) = \int_0^{\infty} W(\alpha) I_0(2x\sqrt{\alpha}) d\alpha.$$  \hspace{1cm} (6)

Inverting Eq. (6) allows one to express the probability $W(\alpha)$ through the partition function $Z(K, x)$. Noting that $I_0(ix) = J_0(x)$ and using the completeness relation for Bessel functions, $\int_0^\infty J_0(\lambda x)J_0(\lambda y)|x|d\lambda = \delta(|x| - |y|)$, we obtain

$$W(\alpha) = 2 \int_0^{\infty} Z(K, ix)I_0(2x\sqrt{\alpha})dx.$$  \hspace{1cm} (7)

The remaining problem is to calculate the partition function $Z(K, ix)$. In this paper we will use a method based on the studies of the integrable structure of conformal field theories [28]. In particular, it was shown that the vacuum
expectation value of Baxter’s $Q$ operator (central to the integrable structure of the models\cite{29}), coincides with the grand partition function of interest:

$$Q^{\text{vac}}(\lambda) = Z(K, -ix),$$

where $x$ is related to the spectral parameter $\lambda$, $x = \pi \lambda / \sin(\pi/2K)$. It was conjectured in Refs.\cite{30,31} that the vacuum expectation value $Q^{\text{vac}}(\lambda)$ is proportional to the spectral determinant of the single particle Schrödinger equation

$$-\partial_x^2 \Psi(x) + \left(x^{4K-2} - \frac{1}{4x^2}\right) \Psi(x) = E \Psi(x),$$

So, $Q^{\text{vac}}(\lambda) = D(\rho \lambda^2)$, where $\rho = (4K)^{2-1/K} |\Gamma(1-1/(2K))|^2$, $D(E)$ is the spectral determinant defined as $D(E) = \prod_{n=1}^{\infty} (1 - E/E_n)$, and $E_n$ are the eigenvalues of (9). Thus, we have

$$Z(K, ix) = \prod_{n=1}^{\infty} \left(1 - \frac{\rho \lambda^2}{E_n}\right).$$

The distribution function $W(\alpha)$ is shown in Fig. 2 for several values of $K$. For $K$ close to 1 (Tonks-Girardeau limit), $W(\alpha)$ is a wide Poissonian function, which gradually narrows as $K$ increases, finally becoming a narrow $\delta$-function at $K \to \infty$ (limit of noninteracting bosons). Note that the distribution function remains asymmetric for arbitrarily large $K$. In fact we find that $W(\hat{\alpha} - 1)$, where $\hat{\alpha} = \alpha/\langle \alpha \rangle = |A|^2/\langle |A|^2 \rangle$, tends to a universal scaling form, parameterized by a single number characterizing the width of the distribution: $\delta \equiv \sqrt{\langle \alpha^2 \rangle - 1} \approx \pi/\sqrt{6}K$. We conjecture that the limiting form of $W(\hat{\alpha})$ is the Gumbel distribution function\cite{12}. The appearance of the Gumbel distribution in this problem is not surprising. This distribution was introduced to describe rare events, such as earthquakes or stock market crashes, which act in one direction. For example, earthquakes destroy property but do not create it. In the limit of large $K$, bosonic quasicondensates have small fluctuations and exhibit good interference patterns in most cases. But occasionally there are strong fluctuations which lead to an appreciable decrease of the contrast. This gives rise to a strong asymmetry of the resulting distribution function.

![FIG. 2: Evolution of the distribution function $W(\alpha)$ for different values of $K$. At larger values of $K$ the function $W$ tends to the delta-function (see the text).](image)

The construction of\cite{28} is based on the representation space of the Virasoro algebra for the set of central charges satisfying $c = 1 - 6(\sqrt{2K} - 1/\sqrt{2K})^2$ and the highest weight $\Delta = (c - 1)/24$ which gives $c \leq -2$ for $K \geq 1$. For all $K > 1$ the central charge of the Virasoro algebra is negative. Theories with negative central charges appear in different contexts in statistical mechanics, stochastic growth models, 2D quantum gravity, models of 2D turbulence and even high-energy QCD. The experimentally measured distribution function $W$ can be inverted (using the Bessel functions completeness relation) to obtain the $Q$-operator which in turn can be used to reconstruct the transfer matrices of the above mentioned models with negative $c$. Therefore, the interference of condensates provides a possible way to explore the interesting physics of various models ranging from statistical to high energy physics.

To summarize, in this paper we discussed interference experiments between two independent one-dimensional quasi-condensates. We showed that quantum phase fluctuations act to suppress the average interference contrast and to
FIG. 3: Limit of large $K$. Scaled distribution function $\tilde{\delta} \omega W ((\tilde{\alpha} - 1)/\tilde{\delta} \tilde{\alpha})$, where $\tilde{\alpha} = \alpha/\langle \alpha \rangle = |A|^2/\langle |A|^2 \rangle$ and $\delta \tilde{\alpha}$ is the width of the distribution. The function $W$ is multiplied by $\delta \tilde{\alpha}$ to preserve the total probability, which must be equal to unity. The dashed and dotted lines correspond to different values of $K$. The solid line corresponds to the conjectured Gumbel distribution.

induce fluctuations in this quantity from one experimental run to the next. The average interference contrast depends on the two point correlation. It therefore scales as a non trivial power of the imaging length related to the power law decay of the correlations. The distribution function characterizing the shot to shot fluctuations of the contrast contains information on high order correlation functions. We computed the distribution function of the amplitude of interference fringes using the relation of this problem to the partition function of the logarithmic Coulomb gas on a circle and to the properties of $Q$ operators of conformal field theories with negative central charges. We showed that the distribution function of fringe amplitudes is related to non-trivial physical properties of a variety of interesting statistical and field-theoretical models.

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We point out that Eq. (4) corresponds to periodic boundary condition. This may give a small quantitative change but will not affect the qualitative picture of the evolution of the distribution function with $K$.

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