RADIATION CONDITIONS FOR THE DIFFERENCE
SCHRÖDINGER OPERATORS

W. SHABAN AND B. VAINBERG

DEPT. OF MATHEMATICS
UNC AT CHARLOTTE
CHARLOTTE, NC, 28223

Abstract. The problem of determining a unique solution of the Schrödinger equation $(\Delta + q - \lambda) \psi = f$ on the lattice $\mathbb{Z}^d$ is considered, where $\Delta$ is the difference Laplacian and both $f$ and $q$ have finite supports. It is shown that there is an exceptional set $S_0$ of points on $Sp(\Delta) = [-2d, 2d]$ for which the limiting absorption principle fails, even for unperturbed operator $(q(x) = 0)$. This exceptional set consists of the points $\{\pm 4n\}$ when $d$ is even and $\{\pm 2(2n + 1)\}$ when $d$ is odd. For all values of $\lambda \in [-2d, 2d] \setminus S_0$, the radiation conditions are found which single out the same solutions of the problem as the ones determined by the limiting absorption principle. These solutions are combinations of several waves propagating with different frequencies, and the number of waves depends on the value of $\lambda$.

1991 AMS Classification: 35P25, 47B39
Key words: Schrödinger operator, lattice, limiting absorption principle, radiation conditions.

1. Introduction: We investigate the problem of determining a unique solution of the Schrödinger equation

$$(\Delta + q - \lambda) \psi = f,$$

(1)
on the lattice $\mathbb{Z}^d$, where both $f$ and $q$ are functions with bounded supports. Let $C_0(\mathbb{Z}^d)$ be the set of such functions.

In the above equation, $\Delta$ is the difference Laplacian in $\mathbb{Z}^d$ defined by

$$\Delta \psi (\xi) = \sum_{|\xi' - \xi| = 1} \psi (\xi'),$$

The work of B. Vainberg was partly supported by the NSF Grant DMS-9971592.
where $\xi = (\xi_1, \xi_2, ..., \xi_d) \in \mathbb{Z}^d$ and $\psi \in l^2(\mathbb{Z}^d)$. The Fourier transform of $\psi$ is defined by the formula

$$\hat{\psi}(k) = (2\pi)^{-d/2} \sum_{\xi \in \mathbb{Z}^d} \psi(\xi) e^{-ik\cdot\xi},$$

where $k = (k_1, k_2, ..., k_d) \in [-\pi, \pi]^d$, and $k \cdot \xi := \sum_{i=1}^d k_i \xi_i$. The Fourier transform of $\Delta$ is an operator of multiplication by $\phi(k)$:

$$\hat{\Delta}\hat{\psi}(k) = \hat{\Delta}\psi(k) = \phi(k) \hat{\psi}(k), \quad \phi(k) = 2\sum_{i=1}^d \cos k_i.$$

Thus, the operator $\Delta$ is self adjoint and its spectrum is a.c and coincides with the range of the function $\phi$, that is $Sp(\Delta) = [-2d, 2d]$. Hence, $Sp_{ess}(\Delta + q) = [-2d, 2d]$ since $q$ has a bounded support. As we shall see later, there is an important exceptional set $S_0$ of values of $\lambda$ on the interval $[-2d, 2d]:$

$$S_0 := \{ \pm 4n \text{ when } d \text{ is even, } \pm 2(2n + 1) \text{ when } d \text{ is odd, } n \in \mathbb{Z} \text{ and } 2n \leq d \}.$$

Let

$$S := Sp(\Delta) \setminus S_0 = [-2d, 2d] \setminus S_0.$$

There are two well known principles that are very natural from the point of view of physics and which allow one to single out the unique solution of the Schrödinger equation in $\mathbb{R}^d$ (see \cite{11}, \cite{12}). These principles are the limiting absorption principle and the Sommerfeld radiation conditions. It turns out that there is an essential difference in both the validity and the form of these principles when applied to the Schrödinger equation in $\mathbb{R}^d$ and on the lattice. In $\mathbb{R}^d$, the spectrum of the negative Laplacian is $Sp(-\Delta) = \{ \lambda \geq 0 \}$. The limiting absorption principle and the radiation conditions are valid for any $\lambda > 0$ (see \cite{11}, \cite{12}). In \cite{4}, \cite{5}, both of these principles are investigated for general equations on the lattice of the form

$$(A + q - \lambda)u = f, \quad q \in C_0(\mathbb{Z}^d),$$

where operator $A$ in the dual space (after the Fourier transform) is an operator of multiplication by a smooth, real valued $2\pi$–periodic function $a(k)$. In \cite{4}, \cite{5}, the limiting absorption principle is justified for values of $\lambda$ such that $\nabla a(k) \neq 0$ on the surface

$$\Gamma(\lambda) = \{ k : k \in T^d, \ a(k) = \lambda \},$$

and the radiation conditions are found when this surface is strictly convex. Here $T^d$ is the torus $\mathbb{R}^d/2\pi\mathbb{Z}^d$.

By applying results from \cite{4}, \cite{5} to the difference Schrödinger equation \cite{11}, one gets the limiting absorption principle when $\lambda \in S$, and the radiation
conditions when $\lambda$ belongs to the following two intervals on the continuous spectrum of the Laplacian:

$$2d - 4 < |\lambda| < 2d.$$  

The goal of this paper is to investigate the validity of the limiting absorption principle when $\lambda \in S_0$ and, more importantly, find the radiation conditions when $|\lambda| < 2d - 4$. Possibly, one of the most important observations in this paper consists of the fact that the fundamental solutions of the difference Schrödinger operator with $|\lambda| < 2d - 4$, singled out by the limiting absorption principle or by the radiation conditions, decay at infinity as fast as in the continuous case only for non-singular directions. The decay is much slower for singular directions. It is also important that the asymptotic behavior of the fundamental solutions at infinity is more complicated, and it changes dramatically when $\lambda$ passes through the points of the set $S_0$.

Let us recall that the limiting absorption principle enables us to obtain two solutions $\psi_\pm$ of (1), for any $\lambda \in S$, as a (pointwise) limit of $\psi_\eta(\xi) = (R_\eta f)(\xi)$ as $\eta \to \lambda \pm i0$. Here,

$$R_\eta = (\Delta + q - \eta)^{-1} : l^2(\mathbb{Z}^d) \to l^2(\mathbb{Z}^d), \quad \text{Im} \; \eta \neq 0.$$  

In order to clarify the situation with $\lambda \in S_0$, one can consider the simplest case $d = 2$ where the set $S_0$ consists of two end points $\lambda = \pm 4$ of the spectrum and the point $\lambda = 0$. It can be shown that the exponentially decaying at infinity fundamental solution $E_\eta(\xi)$ of the operator $\Delta - \eta$, $\xi \in \mathbb{Z}^2$, $\text{Im} \; \eta > 0$, has the following form as $\eta \to \pm i0$:

$$E_\eta(\xi) = \pm C(\xi) \ln |\eta| + \tilde{E}_\pm(\xi) + o(1), \quad C(\xi) = \frac{i}{8\pi} \left[ (-1)^{\xi_1} + (-1)^{\xi_2} \right],$$

Thus, $E_\eta$ does not have a (pointwise) limit as $\eta \to \pm i0$, and therefore the limiting absorption principle is not valid when $\lambda = 0$. Note that $C(\xi)$ satisfies the equation $\Delta C(\xi) = 0$, and therefore $\tilde{E}_\pm$ are fundamental solutions of the Laplacian. However, these fundamental solutions grow logarithmically at infinity, unlike the fundamental solutions obtained by the limiting absorption principle. The latter ones decay as $|\xi|^{-1/2}$ in the case of $d = 2$. The statements above and similar statements when $d > 2$ are not very difficult to prove, but the proofs are rather technical, and we do not include them in this paper.

The other way to single out unique solutions of (1) is by imposing some conditions at infinity called the Sommerfeld radiation conditions, or by requiring a special asymptotic behavior of the solution at infinity. We find the asymptotic behavior of solutions and the radiation conditions for the difference Schrödinger operators with arbitrary $\lambda \in S$. We show that, for any fixed $\lambda \in S$, the radiation conditions or the asymptotic behavior of
the solutions single out the same two solutions of (1) found by the limiting absorption principle. One of the two solutions corresponds to waves propagating to infinity and the other one corresponds to waves coming from infinity. These waves are not spherical as in the case of \( \mathbb{R}^d \). More importantly, in the lattice case, each solution of (1) is a combination of several waves propagating with different frequencies, and the number of waves depends on the value of \( \lambda \). (Only one wave exists if \( 2d - 4 < |\lambda| < 2d \)). For equations in \( \mathbb{R}^d \), several different waves appear in the case of nonisotropic elasticity equations with compactly supported right-hand sides and in the case of more general systems of equations of higher order (see [12]), but not for the Schrödinger equation. In fact, our approach to study the difference equation (1) follows the one used in [12] to study the general systems in \( \mathbb{R}^d \).

The form of the radiation conditions and the asymptotic behavior of the solutions of (1) depends dramatically on the shape of the surface \( \Gamma(\lambda) \),

\[
\Gamma(\lambda) = \{ k : k \in T^d, \phi(k) = \lambda \}, \quad \lambda \in [-2d, 2d].
\]

Here \( T^d = \mathbb{R}^d / 2\pi \mathbb{Z}^d \), function \( \phi \) is defined in (2). We fix the orientation of \( \Gamma(\lambda) \) by choosing the normal vector

\[
n := \nabla \phi(k).
\]

If \( 2d - 4 < |\lambda| < 2d \), then the surface \( \Gamma(\lambda) \) is strictly convex, and there is a unique point

\[
k = k(\omega, \lambda) \in \Gamma(\lambda)
\]

where the normal \( n \) to the surface \( \Gamma(\lambda) \) is parallel to and has the same direction as the unit vector \( \omega = \frac{\xi}{|\xi|} \). In the remaining part of \( S = Sp(\Delta) \setminus S_0 \) when \( |\lambda| < 2d - 4 \), the surface \( \Gamma(\lambda) \) is not convex and, for some \( \omega \in \Omega \), there exist more than one point at which the normal \( n \) to the surface is parallel to and has the same direction as \( \omega \) (see Fig 4.5 below). In addition, the curvature of the surface \( \Gamma(\lambda) \) is zero at some points.

We shall call a point \( \omega \in \Omega \) singular if there is a point \( k(\omega, \lambda) \) on the surface \( \Gamma(\lambda) \) at which the normal \( n \) to this surface is parallel to \( \omega \) and the total curvature (the product of principle curvatures) of the surface at \( k(\omega, \lambda) \) is zero. For a fixed \( \lambda \in S \), let \( \Omega_0 \) be the set of singular points \( \omega \) of \( \Omega \). The set \( \Omega \setminus \Omega_0 \) is an open subset of \( \Omega \) and consists of a finite number of connected components. We shall call these components non-singular domains. For any non-singular domain \( V \subset \Omega \), let \( m_V \) be the number of the points \( k(\omega, \lambda; s), 1 \leq s \leq m_V \), at which the normal \( n = \nabla \phi(k) \) to the surface \( \Gamma(\lambda) \) is parallel to and has the same direction as \( \omega \). Let \( \sigma = \sigma(V, s) \) be the difference between the number of positive and negative principle curvatures at the point \( k(\omega, \lambda; s) \). Since \( V \) is connected and the multi valued function \( \omega \to \{ k(\omega, \lambda; s) \} \), where \( \lambda \) is fixed, \( \omega \in V \), is smooth,
then \( m_V \) and \( \sigma \) do not depend on \( \omega \in V \). They depend only on the domain \( V \). One can show that \( m_V \leq 2^d d \). Let \( \mu(\omega, \lambda; s) = k(\omega, \lambda; s) \cdot \omega \) be the projection of the vector \( k(\omega, \lambda; s) \) on \( \omega \). We denote the cube \([-r, r]^d \) in \( \mathbb{Z}^d \) by \( B_r \).

We prove that, for any \( \lambda \in S \), equation (1) admits unique solutions \( \psi_+ \) and \( \psi_- \) such that for any integer \( R \geq 1 \),

\[
\frac{1}{R} \sum_{\xi \in B_{2R} \setminus B_R} |\psi_\pm(\xi)|^2 < C < \infty, \quad (4)
\]

and for any non-singular domain \( V \subset \Omega \), and \( \omega = \frac{\xi}{|\xi|} \in V \),

\[
\psi_\pm(\xi) = \sum_{s=1}^{m_V} e^{\pm i\mu(\omega, \lambda; s) \cdot \xi} a_\pm(\omega, \lambda; s) + O \left( \frac{1}{|\xi|^{\frac{d+1}{2}}} \right) \quad \text{as} \quad |\xi| \to \infty, \quad (5)
\]

where the coefficients \( a_\pm(\omega, \lambda; s) \) are smooth and the remainder can be estimated uniformly in \( \omega = \frac{\xi}{|\xi|} \) on any compact set \( Q \subset V \). Moreover, these solutions are equal to \( R_{\lambda \pm 0} f \). The same solutions can be singled out by the following radiation conditions at infinity: instead of (5), one can assume that \( \psi_\pm \) can be represented as a sum

\[
\psi_\pm(\xi) = \sum_{s=1}^{m_V} \psi_{s \pm}(\xi), \quad \omega = \frac{\xi}{|\xi|} \in V, \quad (6)
\]

where each term \( \psi_{s \pm}(\xi) \) satisfies the following conditions as \( |\xi| \to \infty \),

\[
\begin{cases}
\psi_{s \pm}(\xi) = O \left( \frac{1}{|\xi|^{\frac{d+1}{2}}} \right) \\
\psi_{s \pm}(\xi + e_j) = e^{\pm ik_j(\omega, \lambda; s)} \psi_{s \pm}(\xi) + O \left( \frac{1}{|\xi|^{\frac{d+1}{2}}} \right), \quad j = 1, \ldots, d,
\end{cases} \quad (7)
\]

with the remaining terms decaying uniformly in \( \omega = \frac{\xi}{|\xi|} \in Q \) for any compact set \( Q \subset V \). Here \( e_j = \begin{pmatrix} 0, \ldots, 0, 1, 0, \ldots, 0 \end{pmatrix} \) and \( k_j(\omega, \lambda; s) \) is the \( j \)-th coordinate of the point \( k(\omega, \lambda; s) \). Note that the second condition in (7) can be written in the following form which is more similar to the standard radiation condition in \( \mathbb{R}^d \):

\[
\frac{\partial \psi_{s \pm}}{\partial \xi_j}(\xi) = \left( e^{\pm ik_j(\omega, \lambda; s)} - 1 \right) \psi_{s \pm}(\xi) + O \left( |\xi|^{-\frac{d+1}{2}} \right),
\]

where

\[
\frac{\partial \psi}{\partial \xi_j}(\xi) := \psi(\xi + e_j) - \psi(\xi).
\]
Let us note again that $\Omega_0$ is empty when $2d - 4 < |\lambda| < 2d$. In this case, 
$$\psi_\pm = O \left( |\xi|^{\frac{1-d}{2}} \right), \quad |\xi| \to \infty,$$
and this estimate is uniform in $\omega \in \Omega$. Estimate (8) fails when $|\lambda| < 2d - 4$ and $\omega \in \Omega_0$.

The plan of the paper is as follows. In section 2, we discuss the limiting absorption principle and the radiation conditions for the unperturbed equation ($q = 0$). In section 3, we carry out the obtained results to the Schrödinger equation. The appendix contains the proofs of the lemmas on the properties of the surface $\Gamma(\lambda)$.

2. The Unperturbed Problem: We start with an investigation of the unperturbed problem (1):

$$(\Delta - \lambda) \psi = f, \quad \xi \in \mathbb{Z}^d,$$

which describes the propagation of waves in a homogeneous medium. When $\lambda \notin [-2d, 2d]$, the resolvent of the difference Laplacian is a bounded operator in $l^2(\mathbb{Z}^d)$, and it is given by the formula

$$(R_\lambda f)(\xi) = \frac{1}{(2\pi)^{d/2}} \int_{T^d} \hat{f}(k) e^{ik \cdot \xi} \phi(k) - \lambda \, dk; \quad \text{where } T^d = \mathbb{R}^d/2\pi \mathbb{Z}^d,$$ (10)

In (10) and in other similar formulas, we shall always identify $T^d$ with the cube $[-\pi, \pi]^d$ if $\lambda > 0$, and with the cube $[0, 2\pi]^d$ if $\lambda < 0$. If $\lambda \in S$, then $|\nabla \phi| \neq 0$ on the surface $\Gamma(\lambda)$. This allows to prove (see [4], [12]) the following statement.

**Theorem 1.** For any $\lambda \in S$, any $f \in C_0(\mathbb{Z}^d)$, and any fixed $\xi \in \mathbb{Z}^d$, the pointwise limits of $R_\eta f$ as $\eta \to \lambda \pm i0$ exist and are given by the following expression:

$$(R_{\lambda \pm i0} f)(\xi) = \frac{1}{(2\pi)^{d/2}} \int_{T^d} \frac{(1 - \chi(k)) \hat{f}(k) e^{ik \cdot \xi}}{\phi(k) - \lambda} \, dk$$

$$\pm \frac{1}{(2\pi)^{d/2}} \int_{[\rho - \lambda < \delta]} \chi(k) \hat{f}(k) e^{ik \cdot \xi} \chi(k) \, 1 \, |\nabla \phi| \, dsd\rho$$

$$\pm \frac{\pi i}{(2\pi)^{d/2}} \int_{\Gamma(\lambda)} \hat{f}(k) e^{ik \cdot \xi} \, ds,$$ (11)

where $\chi$ is an infinitely smooth function on $T^d$ such that

$$\chi(k) = \begin{cases} 
0 & \text{if } |\phi(k) - \lambda| > \delta \\
1 & \text{if } |\phi(k) - \lambda| \leq \delta/2,
\end{cases}$$
with \(0 < \delta < \text{dist}(\lambda, S_0)\), the surface \(\Gamma(\rho)\) is defined in (3), and \(ds\) is the surface element.

In order to describe the asymptotic behavior at infinity of the solutions to the equation (9) constructed in Theorem 1 we need to know geometrical properties of the surface \(\Gamma(\lambda)\). The following statement is proved in the appendix:

**Lemma 2.** When \(2d - 4 < \lambda < 2d \ (-2d < \lambda < -2d + 4)\), the surface \(\Gamma(\lambda)\) is located strictly inside the cube \([-\pi, \pi]^d\) (\([0, 2\pi]^d\), respectively), and it is smooth, convex and closed with the curvature \(K(k)\) not vanishing at any point.

Graphs of the surface \(\phi(k) = \lambda, k \in \mathbb{R}^d\), are shown in Figures 1-3 for \(\lambda = 0, 1, -1\), and \(d = 2\). The surface \(\Gamma(\lambda)\) can be obtained if these graphs are taken by modulo \(2\pi\mathbb{Z}^2\). As it was mentioned at the beginning of this section, we identify \(\Gamma(\lambda)\) with the graph of \(\phi(k) = \lambda\) in the cube \([-\pi, \pi]^d\) if \(\lambda > 0\), or in the cube \([0, 2\pi]^d\) if \(\lambda < 0\). Figures 4 and 5 below give the graphs of \(\Gamma(\lambda)\) for \(\lambda = 1\) and \(\lambda = 3\) respectively, and \(d = 3\).

We specify the orientation on the surface \(\Gamma(\lambda)\) by choosing the normal vector \(n\) to be

\[n = \nabla \phi(k) = -2 (\sin k_1, ..., \sin k_d).\]

When \(|\lambda| < 2d - 4\), the surface \(\Gamma(\lambda)\) is not convex and may have several normal vectors with the same direction. The following statement is also proved in the appendix:

**Lemma 3.** The number \(m\) of points of \(\Gamma(\lambda), \lambda \in S\), at which the normal \(n = \nabla \phi\) to \(\Gamma(\lambda)\) is parallel to and has the same direction as a fixed unit vector \(\omega\), and the total curvature of the surface at these points is non-zero, is such that \(m \leq 2^d d\).

(Note that the above inequality gives a very rough estimate. For example, \(m \leq 4\) when \(d = 3\).)

We denote by \(W_{\pm}\) the two classes of functions \(\psi_{\pm}(\xi)\) for which (4) and (5) hold. Let \(W'_{\pm}\) be two classes of functions \(\psi_{\pm}(\xi)\) that satisfy the estimate (4) and can be represented in the form (6), (7).

**Theorem 4.** For any \(f \in C_0(\mathbb{Z}^d)\) and for any \(\lambda \in S\), the equation

\[(\Delta - \lambda) \psi = f\]  \hspace{1cm} (12)

admits unique solutions in the classes \(W_{\pm}\) and unique solutions in \(W'_{\pm}\). The solutions in \(W_{\pm}\) and the corresponding ones in \(W'_{\pm}\) are the same and coincide with \(R_{\lambda \pm 0} f\).
Remark The amplitudes $a_+$ and $a_-$ in expansion (5) are equal to $a$ and $\overline{a}$ respectively, where $\overline{a}$ is the complex conjugate of $a$ and

$$a = a(\omega, \lambda; s) = \frac{\sqrt{2\pi f(k(\omega, \lambda; s))} e^{i(\sigma + 2) \pi/4}}{\sqrt{|K(\omega, \lambda; s)|} ||\nabla \phi(k(\omega, \lambda; s))||}. \quad (13)$$

Here $K(\omega, \lambda; s))$ is the total curvature of the surface $\Gamma(\lambda)$ at the point $k(\omega, \lambda; s)$, $\sigma = \sigma(V, s)$ is the difference between the number of positive and negative principle curvatures at $k(\omega, \lambda; s)$.

In order to prove this theorem, we need the following lemma.

Lemma 5. Let $\Gamma \subset T^d$ be a smooth surface defined by the equation

$$k_j = g(k'), \quad k' = (k_1, \ldots, k_{j-1}, k_{j+1}, \ldots, k_d) \in T^{d-1},$$
and let
\[ F(\xi) = (2\pi)^{-\frac{d}{2}} \int_{\Gamma} f(k) e^{ik \cdot \xi} dk', \quad \xi \in \mathbb{Z}^d, \quad f \in C^\infty(T^d). \]

Then for any \( R > 0 \),
\[ \sum_{\xi \in B_R} |F(\xi)|^2 \leq 2\pi (2R + 1) \int_{\Gamma} |f(k)|^2 dk'. \tag{14} \]

Proof. For each fixed \( \xi_j \), the function \( F \) is the Fourier transform of the function
\[ h(k') = \sqrt{2\pi} f(k) e^{ik_j \xi_j} |_{k_j = g(k')} . \]

Hence, from Parseval’s equality it follows that
\[ \sum_{\xi' \in \mathbb{Z}^{d-1}} |F(\xi', \xi_d)|^2 = \int_{T^{d-1}} |h(k')|^2 dk' = 2\pi \int_{\Gamma} |f(k)|^2 dk', \]
which immediately implies (14).

\[ \square \]

**Proof of Theorem** To prove the theorem, we show first that (5) implies (7), and therefore \( W_\pm \subset W'_\pm \). Then, we show that \( R_{\lambda \pm i0}f \) are the solutions of (9) which belong to \( W'_\pm \). After this, it suffices to prove only the uniqueness of the solutions in \( W'_\pm \).
Let $\psi_\pm$ satisfy (6). Then, obviously, $\psi_\pm$ can be represented in the form (6) with

$$\psi_{s\pm}(\xi) = \frac{e^{\pm i\mu(\omega,\lambda_0)|\xi|}}{|\xi|^{d-1}}a_\pm(\omega,\lambda; s) + O\left(|\xi|^{-\frac{d+1}{2}}\right), \quad |\xi| \to \infty. \quad (15)$$

Thus, the first relation of (7) holds, and

$$\psi_{s\pm}(\xi + e_j) = \frac{e^{\pm ik(\omega,\lambda_0,\xi+e_j)}}{|\xi + e_j|^{\frac{d-1}{2}}}a_\pm(\omega,\lambda; s) + O\left(|\xi|^{-\frac{d+1}{2}}\right) \quad \text{as} \quad |\xi| \to \infty,$$

where $\omega' = \frac{\xi + e_j}{|\xi + e_j|}$. By taking into account that

$$|\xi + e_j| = |\xi| \left(1 + O\left(|\xi|^{-1}\right)\right), \quad \omega' = \omega + O\left(|\xi|^{-1}\right),$$

and $\nabla_\omega k(\omega,\lambda)$ is orthogonal to $\omega$, we obtain that

$$\psi_{s\pm}(\xi + e_j) = \frac{e^{\pm i\mu(\omega,\lambda_0)|\xi+e_j|}}{|\xi + e_j|^{\frac{d-1}{2}}}a_\pm(\omega,\lambda; s) + O\left(|\xi|^{-\frac{d+1}{2}}\right) \quad \text{as} \quad |\xi| \to \infty,$$

which together with (15) immediately leads to the second part of (7). Thus, the asymptotic behavior (5) implies the radiation conditions (7), and the inclusion $W_\pm \subset W'_\pm$ is proved.

Next, we prove that $\psi_\pm = R_{\lambda \pm i0}f$ are solutions of (10) and $\psi_\pm \in W_\pm$. First, let us note that $R_{\eta}f$, defined in (11), satisfies (10) with $\lambda = \eta \notin [-2d,2d]$, and from Theorem 1 it follows that one can pass to the limit in the equation as $\eta \to \lambda \pm i0$. Thus, $\psi_\pm$ are solutions of (10). Let us show that $\psi_\pm$ satisfy (5).

The first integrand on the right-hand side of (11) is a smooth function because $\chi(k) = 1$ whenever $\phi(k) = \lambda$. By integrating by parts as many times as needed, we obtain that the first term on the right-hand side of (11) has order $O\left(|\xi|^{-\infty}\right)$ as $|\xi| \to \infty$. Thus, (11) implies

$$\psi_{\pm}(\xi) = \frac{1}{2\pi} \int_{|\rho - \lambda| < \delta} \frac{\Phi(\xi,\rho)}{\rho - \lambda} d\rho \pm i\frac{\Phi(\xi,\lambda)}{2} + O\left(|\xi|^{-\infty}\right), \quad |\xi| \to \infty. \quad (16)$$

where

$$\Phi(\xi,\rho) := (2\pi)^{-\frac{d}{2}} \int_{\Gamma(\rho)} \frac{\chi(k) \hat{f}(k) e^{ik\xi}}{|\nabla \phi|} ds. \quad (17)$$

We apply the stationary phase method to the integral (17) in order to get the asymptotic behavior of $\Phi(\xi,\rho)$ as $|\xi| \to \infty$. The asymptotic behavior of $\Phi(\xi,\rho)$ depends on points $k(\omega,\rho)$ on the surface $\Gamma(\rho)$ at which the normal $n = \nabla \phi(k)$ to the surface is parallel to $\omega = \frac{\xi}{|\xi|}$ (see [12], Theorem 9 of Chapter 1). If $\rho = \lambda$ and $\omega$ belongs to a nonsingular domain $V \subset \Omega$ then there are $m_V$ points $k(\omega,\lambda; s)$ on $\Gamma(\lambda)$ at which the normal $n = \nabla \phi(k)$ to the surface $\Gamma(\lambda)$ is parallel to and has the same direction as $\omega$. Due to the
symmetry of $\Gamma(\lambda)$, there are exactly $m_V$ points $k = -k(\omega, \lambda; s)$ where the directions of $n$ and $\omega$ are opposite. The total curvature of $\Gamma(\lambda)$ at points $\pm k(\omega, \lambda; s)$ is not vanishing. Then from (12), Theorem 9 of Chapter I) it follows that

$$\Phi(\xi, \lambda) = -i \sum_{s=1}^{m_V} \left( e^{i\mu(\omega, \lambda; s)|\xi|} a(\omega, \lambda; s) - e^{-i\mu(\omega, \lambda; s)|\xi|} a(\omega, \lambda; s) \right) |\xi|^{-\frac{d+1}{2}}$$

$$+ O \left( |\xi|^{-\frac{d+1}{2}} \right), \quad \omega = \frac{\xi}{|\xi|} \in V, \quad |\xi| \to \infty,$$

where $a(\omega, \lambda; s)$ is defined in (13) and the expansion is uniform with respect to $\omega$ in any compact subset of $V$.

Let us fix a compact $Q \subset V$. The surface $\Gamma(\rho)$ depends analytically on $\rho$ when $\lambda \in S$ and $|\rho - \lambda|$ is small enough. Thus, if $\omega = \frac{\xi}{|\xi|} \in Q$, $|\rho - \lambda| \leq \delta_1$ and $\delta_1$ is small enough, then: 1) there are exactly $m_V$ points $k(\omega, \rho; s)$ on $\Gamma(\rho)$ at which the normal $n = \nabla \phi(k)$ to the surface $\Gamma(\rho)$ is parallel to and has the same direction as $\omega$, 2) the points depend analytically on $\rho$ and 3) the total curvature of $\Gamma(\rho)$ at the points $\pm k(\omega, \rho; s)$ is not vanishing. Finally (see Remark 2 after Theorem 9 of Chapter I in [12]), expansion (18) is valid with $\lambda$ replaced by $\rho$ when $\omega \in Q$, $|\rho - \lambda| \leq \delta_1$:

$$\Phi(\xi, \rho) = -i \sum_{s=1}^{m_V} \left( e^{i\mu(\omega, \rho; s)|\xi|} a(\omega, \rho; s) - e^{-i\mu(\omega, \rho; s)|\xi|} a(\omega, \rho; s) \right) |\xi|^{-\frac{d+1}{2}}$$

$$+ O \left( |\xi|^{-\frac{d+1}{2}} \right),$$

and it is uniform and admits differentiation with respect to all arguments.

We shall choose the constant $\delta$ in (11) and (16) so small that $\delta < \delta_1$. Then (16), (19), and the following simple estimate

$$\int_{|\rho - \lambda| < \delta} \frac{F(\rho)}{\rho - \lambda} d\rho \leq 2\delta \max_{|\rho - \lambda| \leq \delta} |F'(\rho)|$$

which holds for any differentiable function $F$, imply that $|\xi|^{-\frac{d+1}{2}} \psi(\xi)$ is equal to

$$-i \int_{|\rho - \lambda| < \delta} \frac{1}{\rho - \lambda} \sum_{s=1}^{m_V} \left( e^{i\mu(\omega, \rho; s)|\xi|} a(\omega, \rho; s) - e^{-i\mu(\omega, \rho; s)|\xi|} a(\omega, \rho; s) \right) d\rho$$

$$\pm \frac{1}{2} \sum_{s=1}^{m_V} \left( e^{i\mu(\omega, \lambda; s)|\xi|} a(\omega, \lambda; s) - e^{-i\mu(\omega, \lambda; s)|\xi|} a(\omega, \lambda; s) \right) + O \left( |\xi|^{-1} \right).$$
when $\omega \in Q$, $|\xi| \to \infty$. Now from Lemma 5 in (\[12\], Chapter VII) on asymptotic behavior of the principal value integrals it follows that

$$\psi_{\pm}(\xi) = \frac{1}{2} \sum_{s=1}^{m_{V}} \gamma_{s} \left( e^{i\mu(\omega,\lambda;s)}|\xi| a(\omega, \lambda; s) + e^{-i\mu(\omega,\lambda;s)}|\xi| a(\omega, \lambda; s) \right) |\xi|^{-\frac{d-1}{2}}$$

$$\pm \frac{1}{2} \sum_{s=1}^{m_{V}} \left( e^{i\mu(\omega,\lambda;s)}|\xi| a(\omega, \lambda; s) - e^{-i\mu(\omega,\lambda;s)}|\xi| a(\omega, \lambda; s) \right) |\xi|^{-\frac{d-1}{2}}$$

$$+ O \left( |\xi|^{-\frac{d-1}{2}} \right), \quad \gamma_{s} = \text{sign} \frac{\partial}{\partial \rho} \left( \mu(\omega, \rho; s) \right) \bigg|_{\rho=\lambda}, \quad (20)$$

where $\omega \in Q$, $|\xi| \to \infty$. In order to find $\gamma_{s}$, we note that $\phi(k(\omega, \rho; s)) = \rho$, since $k(\omega, \rho; s) \in \Gamma(\rho)$. Thus,

$$\nabla \phi(k(\omega, \rho; s)) \cdot k_{\rho}(\omega, \rho; s) = 1. \quad (21)$$

On the other hand, the normal $n = \nabla \phi(k)$ to the surface $\Gamma(\rho)$ at $k(\omega, \rho; s)$ is parallel to $\omega$ with the same direction. Hence,

$$\nabla \phi(k(\omega, \rho; s)) \cdot c = c(\omega, \rho; s) > 0.$$ 

From here and (21), it follows that

$$\omega \cdot k_{\rho}(\omega, \rho; s) > 0.$$ 

The left-hand side of this inequality is equal to $\mu_{\rho}$, since $\mu(\omega, \rho; s) = k(\omega, \rho; s) \cdot \omega$. Thus $\gamma_{s} = +1$, and therefore (20) leads to (7).

Our next step is to prove that the functions $\psi_{\pm} = R_{\lambda \pm 0} f$ satisfy (4). We shall need the following estimate of the function $\Phi$ defined in (17):

$$\frac{1}{R} \sum_{\xi \in B_{R}} \left[ |\Phi(\xi, \rho)|^{2} + \left| \frac{\partial}{\partial \rho} \Phi(\xi, \rho) \right|^{2} \right] \leq C, \quad |\rho - \lambda| \leq \delta, \quad R \geq 1. \quad (22)$$

In order to prove (22), we cover $\Gamma(\lambda)$ by a finite number of balls $B^{n}$, $n = 1, \ldots, N$, such that the orthogonal projection of $\Gamma(\lambda) \cap B^{n}$ into one of the hyperplanes $k_{j} = 0$, $j = j(n)$, is smooth. Then, the same balls cover $\Gamma(\rho)$ with the same property of smoothness of the projections if $|\rho - \lambda| \leq \delta_{2}$ and $\delta_{2}$ is small enough. One can construct a partition of unity on $\bigcup_{|\rho - \lambda| \leq \delta} \Gamma(\rho)$ subordinate to the covering $\{B^{n}\}$, i.e., find $N$ functions $\alpha_{n}(k) \in C^{\infty}(T^{d})$ with support in $B^{n}$ such that $\sum_{n=1}^{N} \alpha_{n}(k) = 1$ on $\Gamma(\rho), \ |\rho - \lambda| \leq \delta_{2}$. Then $\Gamma(\rho), \ |\rho - \lambda| \leq \delta_{2},$ has the following form on the support of $\alpha_{n}$:

$$k_{j} = g(k', \rho), \quad k' = (k_{1}, \ldots, k_{j-1}, k_{j+1}, \ldots, k_{d}), \quad j = j(n).$$
We shall choose the constant \( \delta \) in (11) and (16) so small that \( \delta < \delta_2 \). Then we write the function \( \Phi \) in the form

\[
\Phi (\xi, \rho) = \sum_{n=1}^{N} \Phi_n (\xi, \rho), \quad \Phi_n (\xi, \rho) = (2\pi)^{1-d/2} \int_{\Gamma(\rho)} \frac{\alpha_n (k) \chi (k) \tilde{f} (k) e^{ik \cdot \xi}}{|\nabla \phi|} ds.
\]

(23)

Obviously,

\[
\Phi_n (\xi, \rho) = (2\pi)^{1-d/2} \int_{T^{d-1}} \left( \frac{\alpha_n (k) \chi (k) \tilde{f} (k)}{|\nabla \phi|} \sqrt{1 + |\nabla g|^2 e^{ik \cdot \xi}} \right) |_{k_j = g(k', \rho)} dk',
\]

(24)

where \( j = j(n) \).

Let us estimate each term in (23). Note that \( \hat{f} \) is analytic, since \( f \) has a bounded support. Hence, the integrand is infinitely smooth, and when differentiating (24) with respect to \( \rho \), the derivative can be passed under the integral sign. Thus, Lemma 5 implies that (22) holds for the functions \( \Phi_n \), and this proves (22) for \( \Phi \).

From (22), it follows that the second term on the right-hand side of (16) satisfies (4). Moreover, estimate (22) implies (4) also for the first term on the right-hand side of (16). Indeed, if \( F = \text{Re}\Phi \), then

\[
\int_{|\rho-\lambda| \leq \delta} \frac{F (\xi, \rho)}{\rho-\lambda} d\rho = \int_{|\rho-\lambda| \leq \delta} \frac{F (\xi, \rho) - F (\xi, \lambda)}{\rho-\lambda} d\rho = \int_{|\rho-\lambda| \leq \delta} F_\rho (\xi, \theta (\rho)) d\rho,
\]

where \( |\theta (\rho) - \lambda| \leq |\rho-\lambda| \leq \delta \), and therefore, for any \( R \geq 1 \),

\[
\sum_{\xi \in B_R} \left| \int_{|\rho-\lambda| \leq \delta} \frac{F (\xi, \rho)}{\rho-\lambda} d\rho \right|^2 \leq \sum_{\xi \in B_R} 2\delta \int_{|\rho-\lambda| \leq \delta} |F_\rho (\xi, \theta (\rho))|^2 d\rho \leq 4\delta^2 CR.
\]

Similar estimates are valid for \( \text{Im}\Phi \). This proves (4) for the first term on the right-hand side of (16). Since the remainder in (16) decays uniformly in \( \omega \in \Omega \), this proves that \( \psi_\pm = R_{\lambda \mp \rho} f \) satisfy (4). Since it was already proven that these functions satisfy (5), we have that \( \psi_\pm \in W_\pm \).

Finally, it remains to prove the uniqueness of the solution. So, we need to prove that if \( \psi_\pm \in W_\pm' \) are solutions of the homogeneous equation \( (\Delta - \lambda) \psi = 0 \), then \( \psi_\pm = 0 \). Let \( \xi^0 \) be an arbitrary point of \( \mathbb{Z}^d \), and let \( E_\pm \) be fundamental solutions of the difference operator \( \Delta - \lambda : \)

\[
(\Delta - \lambda) E_\pm = \delta (\xi), \quad E_\pm = R_{\lambda \mp \rho} (\xi).
\]

Let us apply the Green formula for the difference Laplacian to \( \psi_\pm (\xi) \) and \( E_\pm (\xi - \xi^0) \) in a cube \( B_r \) of \( \mathbb{Z}^d \) that contains the point \( \xi^0 \), \( B_r = [-r, r]^d \),
\( r \in \mathbb{N}, \)
\[
\sum_{\xi \in B_r} (\Delta \psi_\pm \cdot E_\pm - \psi_\pm \cdot \Delta E_\pm) (\xi) = \sum_{\xi \in \partial B_r} (\psi_\pm' E_\pm - \psi_\pm E_\pm') (\xi) \tag{25}
\]
where \( u' := u(\xi + e) \) and \( e \) is the outward unit normal to the boundary \( \partial B_r \) of the cube \( B_r \) at the point \( \xi \). If \( \xi \in \partial B_r \) is a point of intersection of several faces, then the right-hand side of (25) includes terms with normals to each face containing \( \xi \). In (25), one can replace \( \Delta \) by \( \Delta - \lambda \). Hence, the left-hand side of (25) is equal to \(-\psi(\xi_0)\).

Let us take the average of both sides of equation (25) over \( r \in (R, 2R], R \in \mathbb{N}, \)
\[
\psi_\pm(\xi_0) = \frac{1}{R} \sum_{\xi \in B_{2R} \setminus B_R} (\psi_\pm \cdot E_\pm - \psi_\pm' \cdot E_\pm) (\xi). \tag{26}
\]

The uniqueness follows if we prove that
\[
\frac{1}{R} \sum_{\xi \in B_{2R} \setminus B_R} (\psi_\pm \cdot E_\pm' - \psi_\pm' \cdot E_\pm) (\xi) \to 0 \quad \text{as} \quad R \to \infty. \tag{27}
\]

In order to prove (26), we need to improve estimate (4) for the solutions \( \psi_\pm = R_{\lambda \pm i0} f \) of equation (12). Let \( \Omega_0' \) be the following extension of the set \( \Omega_0 \) of singular points:
\[
\Omega_0' = \Omega_0 \cup \{ \omega \in \Omega; \ k_j(\omega, \lambda; s) = -k_j(\omega, \lambda; t) \text{ for some } j, s, t \}. \]

Let \( \Omega_\varepsilon \) be the \( \varepsilon \)-neighborhood on \( \Omega \) of the set \( \Omega_0' \):
\[
\Omega_\varepsilon = \{ \omega \in \Omega; \ \text{dist}(\omega, \Omega_0') < \varepsilon \},
\]
and let
\[
B(R, \varepsilon) = \{ \xi \in B_{2R} \setminus B_R; \ \omega = \frac{\xi}{|\xi|} \in \Omega_\varepsilon \}.
\]

We need to show that
\[
\frac{1}{R} \sum_{\xi \in B(R, \varepsilon)} \left| \psi_\pm(\xi) \right|^2 \to 0 \quad \text{as} \quad \varepsilon + \frac{1}{R} \to 0. \tag{28}
\]

In order to prove (27), we are going first to prove a similar estimate for the function \( \Phi \) given by (17) and for its derivative:
\[
\frac{1}{R} \sum_{\xi \in B(R, \varepsilon)} \left( |\Phi(\xi, \rho)|^2 + \left| \frac{\partial}{\partial \rho} \Phi(\xi, \rho) \right|^2 \right) \to 0 \quad \text{as} \quad \varepsilon + \frac{1}{R} + \delta \to 0. \tag{28}
\]
Relation (28) immediately leads to (27) if we take into account the fact that \( \psi_\pm \) in (11) and (16) do not depend on \( \delta \). Indeed, (27) can be proved using the same arguments as in the proof of estimate (4). One needs only to refer to (28) instead of (22) and to keep in mind that \( \delta \) is arbitrarily small.
In order to prove (28), we represent \( \Phi \) as before in the form (23). To investigate the asymptotic behavior of the integrals in (24) as \( |\xi| \to \infty \), we look for the stationary phase points of the integrands in (23) and (24) for each fixed \( \omega = \frac{\xi}{|\xi|} \) and \( \rho \) such that \( |\rho - \lambda| \leq \delta \). For the integral in (23), these are the points \( k = \pm k'(\omega, \rho; s) \in \Gamma(\rho) \) where the normal vector to \( \Gamma(\rho) \) is parallel to \( \omega = \frac{\xi}{|\xi|} \). Since the integral (24) is obtained by using local coordinates \( k' \) in (23), the stationary phase points in (24) are the projections \( \pm k'(\omega, \rho; s) \) of the points \( k(\omega, \rho; s) \) into the \( k' \)-hyperplane.

We represent (24) as a sum of two functions \( \Phi_n = \Phi_{n,1} + \Phi_{n,2} \) with additional factors \( h_\alpha \) and \( (1 - h_\alpha) \), respectively, in the integrand, where \( h_\alpha \) is the following function. Let

\[ K = \{ k \in T^d; \ k = \pm k(\omega, \lambda, s), \ \omega \in \Omega'_0 \}, \]

and let \( K' \) be the orthogonal projection of the set \( K \) into the torus \( T^{d-1} \) defined by the equation \( k_j = 0, j = j(n) \). Let \( K'_\alpha \) be the \( \alpha \)-neighborhood (in \( T^{d-1} \)) of the set \( K' \). Since the set \( K' \) is analytic,

\[ \text{meas} K'_\alpha \to 0 \quad \text{as} \ \alpha \to 0. \]

We choose \( h_\alpha = h_\alpha(k') \) in such a way that \( h_\alpha \in C^\infty(T^{d-1}), \ |h_\alpha| \leq 1, \ h_\alpha = 1 \) on \( K'_2 \), and \( h_\alpha = 0 \) outside of \( K' \). Then we apply Lemma 5 to \( \Phi_{n,1} \) and to \( \frac{\partial}{\partial \rho} \Phi_{n,1} \). In the integral representations of these two functions, the \( L^2 \)-norm of the integrands tends to zero as \( \alpha \to 0 \), due to the presence of the factor \( h_\alpha \). Hence, if \( |\rho - \lambda| < \delta' \) and \( \delta' \) is small enough then

\[
\frac{1}{R} \sum_{\xi \in B_{2R}} \left( |\Phi_{n,1}(\xi, \rho)|^2 + \left| \frac{\partial}{\partial \rho} \Phi_{n,1}(\xi, \rho) \right|^2 \right) \to 0 \quad \text{as} \ \alpha \to 0.
\]

Thus, for any \( \gamma > 0 \), there is an \( \alpha = \alpha(\gamma) \) such that, for any \( \varepsilon \),

\[
\frac{1}{R} \sum_{\xi \in B(R, \varepsilon)} \left( \left| \sum_n \Phi_{n,1}(\xi, \rho) \right|^2 + \left| \frac{\partial}{\partial \rho} \sum_n \Phi_{n,1}(\xi, \rho) \right|^2 \right) < \frac{\gamma}{4} \quad \text{as} \ \alpha < \alpha(\gamma).
\]

Now we apply the stationary phase method to functions \( \Phi_{n,2} \) which are given by the integrals (24) with an additional factor \( 1 - h_\alpha \) in the integrand. We take into account that the functions \( (\omega, \rho) \to k'(\omega, \rho, s) \) are continuous, and we choose \( \varepsilon = \varepsilon(\gamma) \) and \( \delta = \delta(\gamma) \) to be so small that the points \( \pm k'(\omega, \rho, s) \) belong to \( K'_\alpha(\gamma) \) when \( \omega \in \Omega_{\varepsilon(\gamma)} \), \( |\rho - \lambda| < \delta(\gamma) \). Since the stationary phase points are the points \( k' = \pm k'(\omega, \rho; s) \), and \( 1 - h_\alpha = 0 \) in a neighborhood of those points, we obtain that \( \Phi_{n,2} \) and \( \frac{\partial}{\partial \rho} \Phi_{n,2} \) are of order
\[ O(\|\xi\|^{-\infty}) \text{ as } \|\xi\| \to \infty, \text{ i.e. for any } m \text{ and some } C_m(\alpha) \]
\[
\left| \sum_n \Phi_{n,2}(\xi, \rho) \right|^2 + \left| \frac{\partial}{\partial \rho} \sum_n \Phi_{n,2}(\xi, \rho) \right|^2 \leq \frac{C_m(\alpha)}{\|\xi\|^m}, \quad \frac{\xi}{\|\xi\|} \in \Omega_{\varepsilon}, \quad |\rho - \lambda| \leq \delta.
\] (30)

The number of points in \( B(R, \varepsilon) \), \( R > 1 \), does not exceed \((2R + 1)^d < (3R)^d\). Thus, (30) with \( m = d \) implies estimate (29) for \( \sum \Phi_{n,2} \) if
\[
\omega = \frac{\xi}{\|\xi\|} \in \Omega_{\varepsilon}(\gamma), \quad |\rho - \lambda| < \delta(\gamma), \quad R > R(\gamma) = C_d(\alpha) 3^d \gamma/4.
\]

This estimate together with (29) proves (28). Thus, (27) is also proved.

Now we return to the proof of (26). In fact, it is enough to prove that
\[
\frac{1}{R} \sum_{\xi \in B_{2R} \setminus B_R} (\psi_{\pm} \cdot E_{\pm})(\xi) \to 0 \quad \text{as } R \to \infty,
\] (31)
since a similar relation for the second part in the sum (26) can be proved in the same way. Note that \( E_{\pm}(\xi - \xi^0) \in W_{\pm} \), since this inclusion was proved for \( R_{\lambda \pm i0} f \) if \( f \) has a bounded support, in particular, if \( f = \delta(\xi - \xi^0) \). By the same reason, (27) is valid for \( E_{\pm}(\xi - \xi^0) \):
\[
\frac{1}{R} \sum_{\xi \in B(R, \varepsilon)} \left| E_{\pm}(\xi - \xi^0) \right|^2 \to 0 \quad \text{as } \varepsilon + \frac{1}{R} \to 0.
\] (32)

The proof of (31) will be based on (32) and on the facts that \( \psi_{\pm}(\xi) \) and \( E_{\pm}(\xi - \xi^0) \) are in \( W'_{\pm} \).

Let \( D_{\varepsilon} = \Omega \setminus \Omega_{\varepsilon} \). Obviously, \( D_{\varepsilon} \) is a compact subset of \( \Omega \), and \( D_{\varepsilon} \) can be represented as a union of a finite number of compacts, \( D_{\varepsilon} = \bigcup_{i=1}^{m} D_{\varepsilon,i} \), such that each \( D_{\varepsilon,i} \) is contained in a non-singular domain \( V_i \subset \Omega \). We represent \( B_{2R} \setminus B_R \) as a union of \( B(R, \varepsilon) \) and sets
\[
B_i(R, \varepsilon) = \{ \xi \in B_{2R} \setminus B_R; \quad \omega = \frac{\xi}{\|\xi\|} \in D_{\varepsilon,i} \}, \quad i = 1, \ldots, m.
\]

Obviously,
\[
\left| \sum_{B(R, \varepsilon)} (\psi_{\pm} \cdot E_{\pm})(\xi) \right| \leq \left( \sum_{B(R, \varepsilon)} \left| \psi_{\pm}(\xi) \right|^2 \right)^{\frac{1}{2}} \left( \sum_{B(R, \varepsilon)} \left| E_{\pm}(\xi - \xi^0) \right|^2 \right)^{\frac{1}{2}}.
\] (33)
Note that the estimate (4) holds for $\psi'_\pm$ if it holds for $\psi_\pm$. From that estimate, (33) and (32) it follows that
\[
\left| \frac{1}{R} \sum_{\xi \in B(R,\varepsilon)} (\psi'_\pm \cdot E_\pm)(\xi) \right| \to 0 \quad \text{as} \quad \varepsilon + \frac{1}{R} \to 0. \tag{34}
\]

Hence, (31) will be proved if we show that for each $\varepsilon > 0$ and each $i = 1, \ldots, m$,
\[
\left| \frac{1}{R} \sum_{\xi \in B_i(R,\varepsilon)} (\psi'_\pm \cdot E_\pm)(\xi) \right| \to 0 \quad \text{as} \quad R \to \infty. \tag{35}
\]

Since $\psi_\pm$ satisfy (5) and (7), we have that $\psi'_\pm$ satisfy the same relations (see arguments in the second paragraph of the proof of Theorem 4). Hence,
\[
\psi'_\pm(\xi) = \sum_{s=1}^{m_{V_i}} \varphi_{s\pm}(\xi) \quad \text{and} \quad E_\pm(\xi - \xi^0) = \sum_{s=1}^{m_{V_i}} \zeta_{s\pm}(\xi), \quad \xi \in B_i(R,\varepsilon),
\]
where $\varphi_{s\pm}$ and $\zeta_{s\pm}$ satisfy the conditions (7). Thus, in order to prove (35) and to complete the proof of Theorem 4 it is enough to show that for each $\varepsilon > 0$ and each $i = 1, \ldots, m$, and $1 \leq s, t \leq m_{V_i}$,
\[
I := \frac{1}{R} \sum_{\xi \in B_i(R,\varepsilon)} (\varphi_{s\pm} \cdot \zeta_{t\pm})(\xi) \to 0 \quad \text{as} \quad R \to \infty. \tag{36}
\]

When $\omega \in D_{\varepsilon,i}$, there exists a constant $c(\varepsilon) > 0$ such that
\[
\left| e^{\pm i[k_j(\omega,\lambda,s)+k_j(\omega,\lambda,t)]} - 1 \right| > c(\varepsilon), \quad 1 \leq j \leq d. \tag{37}
\]

Now we fix $j = 1$ and recall that $e_1 = (1, 0, \ldots, 0)$. We represent the terms in (36) in the form
\[
\frac{e^{\pm i[k_1(\omega,\lambda,s)+k_1(\omega,\lambda,t)]}\varphi_{s\pm} \cdot \zeta_{t\pm}}{e^{\pm i[k_1(\omega,\lambda,s)+i k_1(\omega,\lambda,t)]} - 1} - \frac{\varphi_{s\pm} \cdot \zeta_{t\pm}}{e^{\pm i[k_1(\omega,\lambda,s)+k_1(\omega,\lambda,t)]} - 1},
\]
and then we use (7). This allows us to rewrite $I$ as $R \to \infty$ in the following form:
\[
I = \frac{1}{R} \sum_{\xi \in B_i(R,\varepsilon)} \left[ \frac{(\varphi_{s\pm} \cdot \zeta_{t\pm})(\xi + e_1)}{e^{\pm i[k_1(\omega,\lambda,s)+k_1(\omega,\lambda,t)]} - 1} - \frac{(\varphi_{s\pm} \cdot \zeta_{t\pm})(\xi)}{e^{\pm i[k_1(\omega,\lambda,s)+k_1(\omega,\lambda,t)]} - 1} \right] + o(1).
\]

Due to (37), we have
\[
\left| \frac{1}{e^{\pm i[k_1(\omega,\lambda,s)+k_1(\omega,\lambda,t)]} - 1} - \frac{1}{e^{\pm i[k_1(\omega',\lambda,s)+k_1(\omega',\lambda,t)]} - 1} \right| \leq \frac{C}{R},
\]
if \( \omega' = \frac{\xi + e_1}{|\xi + e_1|} \), \( \xi \in B_1(R, \varepsilon) \) and \( R \) is big enough. Thus,

\[
I = \frac{1}{R} \sum_{\xi \in B_i(R, \varepsilon)} [u(\xi + e_1) - u(\xi)] + o(1), \quad u(\xi) = \frac{(\varphi_{s\pm} \cdot \zeta_{i\pm})(\xi)}{e^{i[k_1(\omega, \lambda,s) + k_1(\omega, \lambda,t)]} - 1}.
\]  

(38)

Let \( C_\varepsilon \) be the cone in \( \mathbb{R}^d \), defined by the conditions \( \frac{\xi}{|\xi|} \in \partial \Omega_\varepsilon, |\xi| > 0 \). Then, (38) implies that

\[
|I| \leq \frac{1}{R} \sum_{\xi \in A_1} |u(\xi)| + \sum_{\xi \in A_2} |u(\xi)| + o(1),
\]

(39)

where \( R \to \infty \),

\[
A_1 = \{ \xi \in B_{2R+1} \setminus B_R, \quad \text{dist}(\xi, C_\varepsilon) \leq 1 \}
\]

and

\[
A_2 = \left\{ \xi \in \mathbb{Z}^d, \quad \text{dist}(\xi, \partial (B_{2R} \setminus B_R)) \leq 1, \quad \frac{\xi}{|\xi|} \in D_{\varepsilon,i} \right\}.
\]

Obviously, \( \frac{\xi}{|\xi|} \in D_{\varepsilon,i} \) if \( \xi \in A_1 \) and \( R \) is big enough. Hence, the first relation in (7) is valid for the functions \( \varphi_{s\pm} \) and \( \zeta_{i\pm} \) when \( \xi \in A_1 \cup A_2 \) and \( R \to \infty \). Thus,

\[
|u| = O\left(R^{1-d}\right), \quad \text{as } \xi \in A_1 \cup A_2, \quad R \to \infty.
\]

(40)

Since the number of points in \( A_1 \cup A_2 \) does not exceed \( CR^{d-1} \), (39) and (40) lead to (36).

3. The Equation with a Potential: In this section, we extend the results of the previous sections to the Schrödinger equation \((\Delta - \lambda + q) \psi = f\), where \( f \) and \( q \) are two functions from \( C_0(\mathbb{Z}^d) \) and \( q \) is real valued.

Let the support of \( q \) be contained in the cube \([-r, r]^d\), for some positive integer \( r \). Also, let \( N = (2r + 1)^d \) be the upper bound on the number of points of the lattice in the support of \( q \).

**Theorem 6.** The spectrum of \( \Delta + q \), outside the interval \([-2d, 2d]\), consists of at most \( N \) real eigenvalues less than \(-2d\) and at most \( N \) real eigenvalues greater than \(2d\).

**Proof.** This statement is a direct consequence (see [6], Theorem 13bis of Chapter I) of the facts that \( Sp(\Delta) = [-2d, 2d] \) and that the rank of the operator of multiplication by \( q \) does not exceed \( N \). \( \square \)

**Lemma 7.** Let \( q \) and \( f \) be any two functions from \( C_0(\mathbb{Z}^d) \). The relation 

\[
\psi = R_\lambda \varphi, \quad \text{when } \text{Im} \lambda \neq 0 \quad (\text{resp. } \psi_{\pm} = R_{\lambda \pm i0} \varphi, \quad \text{when } \lambda \in S = [-2d, 2d] \setminus S_0),
\]

(41)
gives a one-to-one correspondence between the solutions \( \psi \in l^2(\mathbb{Z}^d) \) (resp. \( \psi_\pm \in W_\pm \)) of the equation
\[
(\Delta - \lambda + q) \psi = f, \quad \text{Im} \lambda \neq 0 \quad (\text{resp. } \lambda \in S),
\]
and the solutions \( \varphi \in C_0(\mathbb{Z}^d) \) of
\[
(I + T_\lambda) \varphi = f, \quad (\text{resp. } (I + T_{\lambda \pm i0}) \varphi = f),
\]
where \( T_\lambda = qR_\lambda, \) (resp. \( T_{\lambda \pm i0} = qR_{\lambda \pm i0} \)).

Proof. Let \( \psi \in l^2(\mathbb{Z}^d) \) be a solution of
\[
(\Delta - \lambda + q) \psi = f, \quad \text{Im} \lambda \neq 0. \tag{41}
\]
Then, \( (\Delta - \lambda) \psi = f - q\psi, \) and \( \psi = R_\lambda \varphi \) with \( \varphi = f - q\psi \in C_0(\mathbb{Z}^d) \) and \( R_\lambda \) given by (10). By substituting this value of \( \psi \) into (41), we get
\[
\varphi + qR_\lambda \varphi = f, \quad \text{Im} \lambda \neq 0. \tag{42}
\]

Conversely, let \( \varphi \in C_0(\mathbb{Z}^d) \) be a solution of (42). Then, \( \psi = R_\lambda \varphi \in l^2(\mathbb{Z}^d) \) and satisfies the relation \( (\Delta - \lambda) \psi = \varphi \). From this relation and (42), it follows that (41) holds. Thus, the statement of this lemma for \( \text{Im} \lambda \neq 0 \) is proved.

In the case when \( \lambda \in S \), the statement can be proved in a similar way. \( \square \)

Theorem 8. Any solution of the homogeneous equation
\[
(\Delta - \lambda + q) \psi = 0, \quad \lambda \in S \in [-2d, 2d] \setminus S_0,
\]
that belongs to one of the classes \( W_\pm \), belongs also to the other one.

Proof. To be specific, let \( \psi \in W_+ \). We apply the Green formula (25) to \( \psi \) and \( \overline{\psi} \), the complex conjugate of \( \psi \), in a cube \( B_r = [-r, r]^d \subset \mathbb{Z}^d \), where \( r \) is a positive integer. Thus,
\[
\sum_{\xi \in B_r} \left( \Delta \psi \cdot \overline{\psi} - \psi \cdot \Delta \overline{\psi} \right) (\xi) = \sum_{\xi \in \partial B_r} \left( \psi \overline{\psi} - \psi \overline{\psi} \right) (\xi).
\]

Since both \( \psi \) and \( \overline{\psi} \) are solutions of the homogeneous Schrödinger equation, the left-hand side of the Green formula is equal to zero. By taking the average with respect to \( r \in (R, 2R] \), we get
\[
\frac{1}{R} \sum_{\xi \in B_{2R} \setminus B_R} \left( \psi \overline{\psi} - \psi \overline{\psi} \right) (\xi) = 0. \tag{43}
\]

We are going to use some arguments from the proof of Theorem 4 where the expression similar to the left-hand side of (43) was studied. The difference is that one of the factors in (43) contains the complex conjugation, which is not present in (23). That is why we need to change slightly the
definition of the set $\Omega'_0 \subset \Omega$ in order to be able to refer to the proof of Theorem 4. We denote by $\Omega'_0$ the following set:

$$\Omega'_0 = \Omega_0 \cup \{ \omega \in \Omega; k_j(\omega, \lambda; s) = k_j(\omega, \lambda; t) \text{ for some } j, s, t \text{ with } s \neq t \}.$$ 

We preserve the same definition of the sets $B(R, \varepsilon)$ and $B_i(R, \varepsilon)$ introduced in the proof of Theorem 4, but with the new set $\Omega'_0$ in these definitions.

We split the sum (43) in two parts: over $\xi \in B(R, \varepsilon)$ and over $\xi \in \bigcup_i B_i(R, \varepsilon)$. On the first set,

$$\frac{1}{R} \sum_{\xi \in B(R, \varepsilon)} \left( \psi' \bar{\psi} - \psi \bar{\psi}' \right) (\xi) \to 0 \quad \text{as } \varepsilon + \frac{1}{R} \to 0. \quad (44)$$

Indeed, from Theorem 4 it follows that $\psi = R_{\lambda+0}(-q\psi)$, and therefore (27) is valid for $\psi$. Now (44) follows similarly to (34).

Note that the expansion (5) is valid for $\psi$:

$$\psi(\xi) = \sum_{s=1}^{m_i} e^{i\mu(\omega, \lambda; s)|\xi|} \frac{a(\omega, \lambda; s)}{|\xi|^{d-1}} + O\left( \frac{1}{|\xi|^\frac{d+1}{2}} \right) \quad \text{as } \xi \in V_i, \quad |\xi| \to \infty,$$

where $m_i = m_{V_i}$. As it was mentioned in the proof of Theorem 4, the same expansion is valid for $\psi'$. Thus, as $R \to \infty$,

$$\frac{1}{R} \sum_{\xi \in B_i(R, \varepsilon)} \left( \psi' \bar{\psi} - \psi \bar{\psi}' \right) (\xi) = \frac{1}{R} \sum_{1 \leq s \neq t \leq m_i} \sum_{B_i(R, \varepsilon)} \left( \psi_s \bar{\psi}_s - \psi_t \bar{\psi}_t \right) (\xi)$$

$$+ \frac{1}{R} \sum_{s=1}^{m_i} \sum_{B_i(R, \varepsilon)} \frac{1}{|\xi|^{d-1}} |a(\omega, \lambda; s)|^2 + o(1), \quad (46)$$

where $\psi_s$ are the terms under the summation sign in (45). The first sum on the right-hand side of (46) (which includes the functions $\psi_s, \psi_t, s \neq t$) tends to zero as $R \to \infty$. This can be proved by repeating the arguments which led to (33). From here, (43), (44) and (46), it follows that for any $i$,

$$\frac{1}{R} \sum_{s=1}^{m_i} \sum_{\xi \in B_i(R, \varepsilon)} \frac{1}{|\xi|^{d-1}} |a(\omega, \lambda; s)|^2 \to 0 \quad \text{as } \varepsilon + \frac{1}{R} \to 0.$$

This is possible only if all $a(\omega, \lambda; s)$ are zeros. In this case, $\psi$ belongs to both classes $W_\pm$.

**Theorem 9.** For any $\lambda \in S$, and for any $q \in C_0(\mathbb{Z}^d)$, the homogeneous equation $(\Delta - \lambda + q) \psi = 0$ has only trivial solutions in the classes $W_\pm$.

**Proof.** Let $\lambda \in S$ and

$$\Gamma_\varepsilon(\lambda) = \{ k \in \mathbb{C}^d / \mod 2\pi \mathbb{Z}^d, \varphi(k) = \lambda \},$$
i.e., $\Gamma_c(\lambda)$ consists of points $k \in \mathbb{C}^d$ such that $\varphi(k) = \lambda$ and with any two points $k^{(1)}, k^{(2)}$ identified if $(k^{(1)} - k^{(2)})/2\pi \in \mathbb{Z}^d$. This is an analytic $(d-1)$-dimensional complex manifold without singularities because

$$\nabla \varphi = -2(\sin k_1, \ldots, \sin k_d) \neq 0 \quad \text{on} \quad \Gamma_c(\lambda).$$

(47)

It is not difficult to check that $\Gamma_c(\lambda)$ is connected and its intersection with the real space $\mathbb{R}^d$ is $\Gamma(\lambda)$ given by (3). Since $\Gamma_c(\lambda)$ is connected and smooth, it is irreducible (see section 1 of [3] or section 2 of [8]).

Let $\psi$ be a solution of $(\Delta - \lambda + q)\psi = 0$. Then $(\Delta - \lambda)\psi = f$, where $f = -q\psi \in C_0(\mathbb{Z}^d)$. According to Theorem 8, $\psi$ belongs to both classes $W_{\pm}$, and therefore the coefficients $a_{\pm}$ in expression (5) for $\psi$ are zeros. From the remark to Theorem 4 it follows that this can only happen if $\hat{f}(k) = 0$ on the set $\Gamma(\lambda)$. Note that $\hat{f}(k)$ is invariant under the transformation $k \rightarrow k + 2\pi \xi$, $\xi \in \mathbb{Z}^d$. Since $\Gamma(\lambda)$ (which is $\Gamma_c(\lambda) \cap \mathbb{R}^d$) is a $(d-1)$-dimensional real manifold, $\Gamma_c(\lambda)$ is irreducible and $\hat{f}(k) = 0$ on $\Gamma(\lambda)$, we have that $\hat{f}(k) = 0$ on $\Gamma_c(\lambda)$ (see [8], Corollary 2.7). From here and (47), it follows that $\hat{f}(k)$ is an entire function of $k \in \mathbb{C}^d$.

Let $z_j = e^{ik_j}$ be the conformal transformation which maps the strip $|\text{Re} k_j| < \pi$ into the complex $z_j$-plane without the origin. With the new variables, $\hat{f}$ takes the form

$$\hat{f}(k) = \sum_{\xi \in \text{supp}(f)} f(\xi) e^{-ik\cdot\xi} = \sum_{\xi \in \text{supp}(f)} f(\xi) z^{-\xi}, \quad \text{where} \quad z^\xi = \prod_{j=1}^d z_j^{\xi_j}.$$ 

Obviously, this function can be written as

$$\hat{f}(k) = F(z) \prod_{j=1}^d z_j^{-\alpha_j},$$

where $F(z)$ is a polynomial of $z$ and the $\alpha_j$ are positive integers. Similarly,

$$\phi(k) - \lambda = \sum_{i=1}^d \left(z_i + \frac{1}{z_i}\right) - \lambda = Q(z) \prod_{i=1}^d z_i^{-1},$$

where

$$Q(z) = \left(\sum_{i=1}^d z_i\right) \prod_{i=1}^d z_i + \sum_{j=1}^d \left(\prod_{i \neq j}^d z_i\right) - \lambda \prod_{i=1}^d z_i.$$
Then,
\[ \frac{\hat{f}(k)}{\phi(k) - \lambda} = \frac{F(z)}{Q(z)} \prod_{i=1}^{d} z_i^{1-\alpha_i}. \tag{48} \]

The right-hand side of (48) is analytic for all complex \( z \) except possibly on the hyperplanes \( z_j = 0, j = 1, \ldots, d \) (where the transform: \( k \to z \) is not defined). Thus, the same is true for \( \frac{F(z)}{Q(z)} \). But \( Q(z) \) is not equal to zero if \( z_j = 0 \) for some \( j_0 \), and \( z_j \neq 0 \) for \( j \neq j_0 \). Hence, \( \frac{F(z)}{Q(z)} \) is analytic except possibly on some set of complex dimension \( d - 2 \) (the intersection of two hyperplanes). Therefore, \( \frac{F(z)}{Q(z)} \) is an entire function (see [7], Corollary 7.3.2).

Since \( F(z) \) and \( Q(z) \) are both polynomials and \( \frac{F(z)}{Q(z)} \) is an entire function, then \( \frac{F(z)}{Q(z)} \) is a polynomial (see [10], appendix, a corollary to The Hilbert Nullstellensatz). Hence from (48), it follows that \( \hat{\psi}(k) = \frac{\hat{f}(k)}{\phi(k) - \lambda} \) is a trigonometric polynomial, which implies that \( \psi(\xi) \) has bounded support. The only solution of the homogeneous equation with a bounded support is \( \psi \equiv 0 \). \( \square \)

Let us recall that we denote the resolvent \( (\Delta - \lambda + q)^{-1} \) by \( \mathcal{R}_\lambda \) if \( q \neq 0 \) and denote it by \( R_\lambda \) if \( q = 0 \).

**Theorem 10.** For any \( f \in C_0(\mathbb{Z}^d), \) any \( q \in C_0(\mathbb{Z}^d), \) and any \( \lambda \in S, \) the limits \( \psi_\pm \) of \( \psi_n = \mathcal{R}_n f \) as \( n \to \lambda \pm i0 \) exist. Moreover, for each \( \lambda \in S, \) the equation
\[ (\Delta - \lambda + q) \psi = f \tag{49} \]
adopts unique solutions in \( W_+ \) and in \( W_- \), and these solutions are \( \psi_\pm = \mathcal{R}_{\lambda \pm i0}f \).

**Proof.** The homogeneous equation \( (\Delta - \eta + q) \psi = 0 \) has only trivial solutions in \( l^2(\mathbb{Z}^d) \) if \( \text{Im} \eta \neq 0 \). Therefore, by Lemma 7, \( I + T_\eta \) has only a trivial kernel in this case. Thus, if \( \text{Im} \eta \neq 0 \), \( I + T_\eta \) is invertible since \( T_\eta \) is a finite-dimensional operator.

Similarly, from Theorem 9 and Lemma 7 it follows that \( I + T_{\lambda \pm i0} \) is invertible if \( \lambda \in S \). Theorem 11 implies that \( T_\eta = qR_\eta \to qR_{\lambda \pm i0} = T_{\lambda \pm i0} \) as \( \eta \to \lambda \pm i0 \) if \( \lambda \in S \). Thus, for \( \lambda \in S, \)
\[ (I + T_\eta)^{-1} \to (I + T_{\lambda \pm i0})^{-1} \quad \text{as} \quad \text{Im} \eta \neq 0, \quad \eta \to \lambda \pm i0. \tag{50} \]

Moreover, we have that \( \mathcal{R}_\eta = R_\eta (I + T_\eta)^{-1} \). By applying the limiting absorption principle to \( R_\eta \) and using (50), we get, for \( \lambda \in S, \) that
\[ \mathcal{R}_\eta \to R_{\lambda \pm i0} (I + T_{\lambda \pm i0})^{-1} \quad \text{as} \quad \text{Im} \eta \neq 0, \quad \eta \to \lambda \pm i0, \tag{51} \]
and this concludes the proof of the first statement of the theorem.
From (51), it also follows that $R_{\lambda \pm i 0} f = R_{\lambda \pm i 0} \varphi$, where $\varphi = (I + T_{\lambda \pm i 0})^{-1} f \in C_0 (Z^d)$. Thus, the solutions $\psi_+ = R_{\lambda \pm i 0} f$ of equation (49) belongs to $W_\pm$ due to Theorem 4. The uniqueness of solutions in $W_\pm$ is proved in Theorem 9. □

5. Appendix:
Proof of Lemma 2. We shall assume that $\lambda > 0$ since the case $\lambda < 0$ can be studied similarly. At any point $k \in \Gamma(\lambda)$, the curvature of the surface is equal to

$$K(k) = \frac{\sum_{j=1}^{d} \sin^2 k_j \prod_{m=1}^{d} \cos k_m}{\left( \sum_{j=1}^{d} \sin^2 k_j \right)^{\frac{d+1}{2}}}, \quad k = (k_1, \ldots, k_d).$$

For the sake of simplicity of formulas we shall provide the proof of the Lemma only in the cases $d = 2$ and $d = 3$. Let

$$s_i = \sum_{j=1}^{i} \cos k_j \quad \text{and} \quad p_i = \prod_{j=1}^{i} \cos k_j.$$

When $d = 2$, the equation of the surface $\Gamma(\lambda)$ is

$$\cos k_1 + \cos k_2 = \frac{\lambda}{2}, \quad \text{where} \quad (k_1, k_2) \in [-\pi, \pi]^2.$$

The total curvature $K(k)$ of this surface at any point $k = (k_1, k_2)$ is equal to

$$K(k) = \frac{\cos k_1 \sin^2 k_2 + \sin^2 k_1 \cos k_2}{\left( \sin^2 k_1 + \sin^2 k_2 \right)^{\frac{3}{2}}} = \frac{s_2 \left( 1 - p_2 \right)}{\left( \sin^2 k_1 + \sin^2 k_2 \right)^{\frac{3}{2}}} > 0,$$

since $\lambda \in (0, 4)$, $s_2 = \lambda/2 > 0$, and $0 \leq |p_2| < 1$. The convexity of $\Gamma(\lambda)$ follows from the facts that $K(k) > 0$ and that $\Gamma(\lambda)$ is located strictly inside the square $[-\pi, \pi]^2$ (inside $[0, 2\pi]^2$ if $\lambda < 0$).

When $d = 3$, the spectrum of the difference Laplacian is $Sp(\Delta) = [-6, 6]$, and the total curvature $K(k)$ of the surface $\Gamma(\lambda)$ at any point $k$ is equal to

$$K(k) = \frac{\cos k_1 \cos k_2 \sin^2 k_3 + \cos k_1 \cos k_3 \sin^2 k_2 + \cos k_2 \cos k_3 \sin^2 k_1}{\left( \sin^2 k_1 + \sin^2 k_2 + \sin^2 k_3 \right)^{\frac{3}{2}}}.$$

(52)

If $\cos k_j > 0$ for $1 \leq j \leq 3$, then $K(k) > 0$ due to (52). Else, only one of the three cosines is negative because otherwise $s_3 < 1$, and that contradicts the assumption of the Lemma that $\lambda \in (2, 6)$ if $\lambda > 0$. Without
loss of generality, let \( \cos k_3 < 0 \). It is easy to show that the curvature can be written in the following form:

\[
K(k) = \frac{p_3}{(\sin^2 k_1 + \sin^2 k_2 + \sin^2 k_3)^2} \left( \frac{1}{\cos k_3} + \frac{s_2}{p_2} - s_3 \right).
\]  

(53)

Since \( s_3 > 1 \),

\[
\cos k_3 > 1 - s_2.
\]

(54)

Moreover,

\[
p_2 \geq s_2 - 1,
\]

(55)
due to the fact that \( \cos k_{1,2} \leq 1 \). By combining the inequality \( s_3 > 1 \) with (54) and (55), we get that the last factor on the right-hand side of (53) is negative:

\[
\frac{1}{\cos k_3} + \frac{s_2}{p_2} - s_3 < \frac{1}{1 - s_2} + \frac{s_2}{s_2 - 1} - 1 = 0.
\]

After taking into account that \( p_3 \) is negative, (53) implies that \( K(k) > 0 \).

The convexity of \( \Gamma(\lambda) \) follows from the fact that \( \Gamma(\lambda) \) is located strictly inside the cube \([-\pi, \pi]^3\).

**Proof of Lemma 3.** Let \( \omega = (\omega_1, \omega_2, ..., \omega_d) \) be a unit vector and \( k = (k_1, k_2, ..., k_d) \) be a point of \( \Gamma(\lambda) \) that satisfies the assumptions of the Lemma. Hence,

\[
\sum_{j=1}^{d} \cos k_j = \frac{\lambda}{2} \quad \text{and} \quad \sin k_j = -\kappa \omega_j, \quad j = 1, ..., d,
\]

where \( \kappa \) is a positive constant. Thus, \( \sum_{j=1}^{d} \left( \pm \sqrt{1 - \kappa^2 \omega_j^2} \right) = \frac{\lambda}{2} \). Since \( \kappa > 0 \), the number \( m \) does not exceed the number of solutions of the equations

\[
\sum_{j=1}^{d} \left( \pm \sqrt{1 - x \omega_j^2} \right) = \frac{\lambda}{2}.
\]

Without loss of generality, we consider only \( 0 \leq \lambda \leq 2d \) and \( 0 \leq \omega_1 \leq \omega_2 \leq ... \leq \omega_d \). Let \( f(x) \) denote one of the following functions:

\[
f(x) = \sum_{j=1}^{d} \left( \pm \sqrt{1 - x \omega_j^2} \right) - \frac{\lambda}{2}.
\]

(56)

Since there are \( 2^d \) different functions \( f(x) \), \( m \leq m_0 2^d \), where \( m_0 \) is the maximal number of zeros that any individual function \( f(x) \) can admit.
We are going to prove the following assertion: For each \( i = 0, 1, \ldots, d \), there exists a function

\[
f_i(x) = \sum_{j=1}^{d-i} a_j (1 - xb_j)^{\frac{1}{2} - i} + c, \quad x < \frac{1}{bd-i},
\]

such that \( 0 < b_1 < \ldots < b_{d-i} \) and \( m_0 \leq m_i + i \), where \( m_i \) is the number of zeros of the function \( f_i(x) \). Let us prove this assertion by induction.

For \( i = 0 \), one only needs to show that the function \( f(x) \), defined in (56), can be written in the form (57) with \( i = 0 \). If \( 0 < \omega_1 < \ldots < \omega_d \), then \( f(x) \) already has the form (57) with \( a_j = \pm 1, b_j = \omega_j^2 \), and \( c = -\omega^2 \). Else if \( \omega_1 = \omega_2 = \ldots = \omega_k = 0 \), then we drop the first \( k \) terms in (56) and choose \( c = \sum_{j=1}^{k} (\pm 1) - \frac{1}{2} \). Also, we combine the terms in (56) with equal \( \omega_j \)'s. Last, in order to write (56) in the form (57), it is left to add several terms to (56), with small positive \( b_j \) and zero coefficient \( a_j \), to keep the number of terms in the sum (56) equal to \( d \).

Now, suppose that the assertion holds for \( i = n \). Let us prove it for \( i = n + 1 \). Obviously,

\[
f_n'(x) = \sum_{j=1}^{d-n} a_j \left( n - \frac{1}{2} \right) b_j (1 - xb_j)^{-(n+\frac{1}{2})}, \quad x < \frac{1}{bd-n}, \tag{58}
\]

and \( m_n \leq m_n' + 1 \), where \( m_n' \) is the number of zeros of the function \( f_n'(x) \). On the other hand, \( m_0 \leq m_n + n \) because the assertion holds for \( i = n \). Thus,

\[
m_0 \leq m_n' + n + 1. \tag{59}
\]

From (58), it follows that

\[
\frac{1}{(n - \frac{1}{2}) (1 - xb_{d-n})^{-(n+\frac{1}{2})}} f_n'(x) = \sum_{j=1}^{d-(n+1)} a_j b_j \left( 1 - xb_j \right)^{-(n+\frac{1}{2})} \left( \frac{1}{1 - xb_{d-n}} \right)^{-(n+\frac{1}{2})} + a_d b_d, \quad x < \frac{1}{bd-n}.
\]

With the one-to-one change of variable \( x = \frac{1}{z} + \frac{1}{bd-n} \), we get

\[
\frac{(-z)^{-(n+\frac{1}{2})}}{(n - \frac{1}{2})} f_n'(x(z)) = \sum_{j=1}^{d-(n+1)} a_j b_j^{\frac{1}{2} - n} \left( 1 - z \left( \frac{1}{b_j} - \frac{1}{b_{d-n}} \right) \right)^{-(n+\frac{1}{2})} + a_d b_d^{\frac{1}{2} - n}, \quad z < 0.
\]

where \( z < 0 \). Let \( f_{n+1}(z) \) be the function in the left-hand side of this equation. The number of negative zeros of this function is equal to \( m_n' \).

Certainly, the function \( f_{n+1}(z) \) will have the form (57) if one changes the index of summation in (50): \( j \rightarrow d-n-j \). Moreover, this function is defined
for $z < \frac{1}{b_i} - \frac{1}{b_{d-n}}$, and the number $m_{n+1}$ of its zeros on this bigger interval is not less than the number $m'_n$ of its negative zeros. This, together with (59), proves the assertion for $i = n + 1$. Hence by induction, the assertion is true for any $i = 0, \ldots, d$.

If $a_j \neq 0$ for any $j = 1, \ldots, d$, in particular when the $\omega_j$'s are distinct and positive, the function $f_d(x)$ is constant and $f_d(x) = a_1b_1^{-d+\frac{3}{2}} \neq 0$. In this case, $m_d = 0$ and $m_0 \leq m_d + d = d$. Based on the construction of the functions $f_i(x)$, one can check that if some of the $a_j$'s are zero, one of the functions $f_i(x)$ is a non-zero constant. For this $i$, $m_0 \leq m_i + i = i \leq d$. So, $m_0 \leq d$, and $m \leq m_02^d \leq 2^dd$.

REFERENCES

[1] F. V. Atkinson: *Discrete and continuous boundary problems*, Academic Press, New York and London 1964.

[2] E. D. Bloch: *A first course in geometric topology and differential geometry*, Birkhäuser, Boston 1997.

[3] E. M. Chirka: *Complex analytic sets*, Encyclopedia of mathematical sciences, vol. 7, Several complex variables I, Springer-Verlag, Berlin 1990.

[4] M. S. Eskina: The scattering problem for partial-difference equations, in *Mathematical Physics*, Naukova Dumka, Kiev, 1967 (in Russian), 248-273.

[5] M. S. Eskina: The direct and the inverse scattering problem for a partial-difference equation, *Soviet Math. Doklady*, v.7, no.1 (1966), 193-197

[6] I. M. Glazman: *Direct methods of qualitative spectral analysis of singular differential operators*, *Israel Program for scientific translation*, Jerusalem 1965 and by Daniel Davey and Co., New York 1966.

[7] S. G. Krantz: *Function theory of several complex variables*, John Wiley & Sons Inc., 1982.

[8] P. Kuchment and B. Vainberg: On absence of embedded eigenvalues for Schrödinger operators with perturbed periodic potentials, *Comm. PDE*, 25, no. 9-10 (2000), 1809 - 1826.

[9] S. Molchanov: *Lectures on random media*, *Lecture notes in mathematics*, vol. 1581, Springer Verlag, Berlin 1994.

[10] I. R. Shafarevich: *Basic algebraic geometry: Varieties in projective space*, vol. 1, Springer-Verlag, Berlin Heidelberg and New York 1944.

[11] A. N. Tikhonov and A. A. Samarskii: *Equations of mathematical physics*, Pergamon Press Ltd, Canada 1963.

[12] B. R. Vainberg: *Asymptotic methods in equations of mathematical physics*, Gordon and Breach, New York 1989.