Trajectory analysis of the propagation of Rossby waves on the Earth’s δ-surface

SONG Jian,a XUE Hai-Leb and DA Chao-Jiuc

aCollege of Sciences, Inner Mongolia University of Technology, Hohhot, China; bState Key Laboratory of Severe Weather, Chinese Academy of Meteorological Sciences, Beijing, China; cSchool of Mathematics and Computer Science Institute, Northwest University for Nationalities, Lanzhou, China

ABSTRACT

The propagation of Rossby waves on the Earth’s δ-surface is discussed in a reference frame propagation with the zonal phase speed of the wave. In this reference frame, trajectories and streamlines coincide; the propagation of Rossby waves has a different interpretation than in a reference frame fixed on the ground; and the mechanism for the propagation of Rossby waves depends not only on the different positions of the wave relative to the basic current, but also the variation of β with latitude (called the δ-effect).

1. Introduction

Rossby waves are a fundamental wave motion in large-scale atmospheric and oceanic dynamics, affecting weather and climate. The propagation of Rossby waves is related to the variation in the Coriolis force with latitude, the so-called β-effect. There have been many studies on the subject of Rossby waves. Plzaman (1968) and Hoskings, McIntyre, and Robertson (1985) explained the Rossby wave propagation mechanism. Holton and Hakim (2012) showed the mechanism of Rossby wave propagation in a barotropic atmosphere. Cai and Huang (2013) used the mechanical-Coriolis oscillation mechanism to show how the β-effect and topographic-effect induce Rossby waves. Song and Yang (2009, 2010) and Song et al. (2009, 2017) have studied solitary Rossby waves in barotropic fluids and stratified fluids using the Wenzel-Kramers-Brillouin theory. In the barotropic and baroclinic model, Schneider (2015) analyzed the mechanism for the westward propagation of Rossby waves on the β-plane. Yang (1987, 1988) introduced the δ-surface approximation of the Earth’s surface by considering the second derivative of the Coriolis parameter with respect to latitude (the variation of β with the latitudinal δ-effect), revealing Rossby wave packet structural vacillation to exist in some basic currents or topographies on the Earth’s δ-surface.

The object of this paper is, depending on potential vorticity conservation to describe Rossby wave propagation when modeling on the Earth’s δ-surface, a reference frame propagation with the zonal phase speed of the wave is used to analyze the physical mechanism of Rossby wave propagation. The results show that the variation in β with the latitudinal δ-effect has an important influence on the propagation of Rossby waves.

2. Barotropic Rossby waves on the Earth’s δ-surface

Following the shallow-water formulation of Holton (1979), Pedlosky (1979), and Gill (1982), the barotropic vorticity equation is

$$\frac{D\zeta + f}{Dt} = 0,$$

where $Dh/\partial t$ stands for the material derivative, $\zeta, f$ are the relative vorticity and the planetary vorticity (the Coriolis parameter), respectively.

The Earth’s δ-surface approximation is the same as employed in Yang (1987, 1988):
\[ f = f_0 + \beta (y - y_0) - \frac{1}{2} \delta (y - y_0)^2, \]

where the second derivative of the Coriolis parameter with respect to latitude has been included; \( f_0 \) is a constant midlatitude (at \( y_0 \)) Coriolis parameter; and \( \beta \) and \( \delta \) are constants, defined as follows:

\[ \beta = \frac{df}{dy} \bigg|_{y = y_0} \tag{3} \]

\[ \delta = \frac{d^2 f}{dy^2} \bigg|_{y = y_0}. \tag{4} \]

Here, \( f_0, \beta, \) and \( \delta \) can be taken as positive at the midlatitudes. Yang (1987, 1988) showed the main difference between the Earth’s \( \delta \)-surface approximation and the Earth’s \( \beta \)-plane approximation. The main difference is: on the \( \beta \)-plane the surface of the Earth is considered to be a plane whereas on the \( \delta \)-surface; the surface of the Earth is considered to be a quadratic surface.

The \( \delta \)-surface approximation is more realistic than the \( \beta \)-plane approximation, especially at higher latitudes (Liu and Tan 1992; Luo 1995). With \( d \) representing the change along a trajectory, Equation (1) can be reduced to

\[ d\zeta + df = 0. \tag{5} \]

Then, Equation (2) gives

\[ df = [\beta - \delta (y - y_0)] dy. \tag{6} \]

and it follows that

\[ d\zeta = -[\beta - \delta (y - y_0)] dy. \tag{7} \]

Using Equation (7), we draw the following main conclusions:

(1) For a northward displacement, \( dy > 0 \): if \( \beta - \delta (y - y_0) > 0 \), from Equation (7) the relative vorticity change is anticyclonic, \( d\zeta < 0 \); if \( \beta - \delta (y - y_0) < 0 \), the relative change vorticity change is cyclonic, \( d\zeta > 0 \).

(2) For a southward displacement, \( dy < 0 \): if \( \beta - \delta (y - y_0) > 0 \), from Equation (7) the relative vorticity change is cyclonic, \( d\zeta > 0 \); if \( \beta - \delta (y - y_0) < 0 \), the relative change vorticity change is anticyclonic, \( d\zeta < 0 \).

(3) If the \( \delta \)-effect is neglected, the conclusion of the relative vorticity coincides with the literature (Schneider 2015).

The material derivative of Equation (1) can be obtained as

\[ \frac{D_h}{Dt} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y}. \tag{8} \]

For a barotropic atmospheric model, the horizontal velocity \((u, v)\) is nondivergent and given in terms of the streamfunction \( \psi \), as

\[ u = -\frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}. \tag{9} \]

The relative vorticity is

\[ \zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \nabla^2 \psi, \tag{10} \]

and the streamfunction is (Holton and Hakim 2012)

\[ \psi = -\bar{u} \psi + \psi'(x, y, t), \tag{11} \]

where \( \bar{u} \) is a constant basic-state zonal flow and \( \psi'(x, y, t) \) is a perturbation streamfunction. The linearized vorticity, Equation (1), on the Earth’s \( \delta \)-surface is

\[ (\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x}) \nabla^2 \psi' + [\beta - \delta (y - y_0)] \frac{\partial \psi'}{\partial x} = 0. \tag{12} \]

The solutions of Equation (12) are the same as employed in Holton and Hakim (2012):

\[ \psi' = \psi_0 \cos [k(x - ct) + ly], \tag{13} \]

where \( k \) and \( l \) are the local wave numbers in the zonal and meridional direction, respectively, and \( c \) is the phase speed of the barotropic Rossby waves.

Substituting Equation (13) into Equation (12), the dispersion relation can be obtained by

\[ c = \bar{u} - \frac{\beta - \delta (y - y_0)}{K^2} = \bar{u} - \frac{\beta}{K^2} + \frac{\delta (y - y_0)}{K^2}, \tag{14} \]

where \( K^2 = k^2 + l^2 \) is the full wavenumber. From

\[ \bar{u} - c = \frac{\beta - \delta (y - y_0)}{K^2}, \]

the Rossby waves propagate westward, but the \( \delta \)-surface effect prevents the Rossby waves from propagating westward. Thus, the Rossby waves propagate westward relative to the basic state flow. Furthermore, the Rossby wave phase speed depends on the zonal, the horizontal, and the \( \delta \)-effect.

(4) In the spherical coordinate system, the \( \beta \)-plane approximation is the same as employed in Liu and Liu (2011), Pedlosky (1979), and Grimshaw (1975):

\[ f = f_0 + \beta (\phi - \phi_0), \tag{15} \]

where \( f_0 = 2\Omega \sin \phi_0 \) and \( \beta = 2\Omega \cos \phi_0 \) are the Coriolis parameter and the Rossby parameter at the central latitude \( \phi_0 \), respectively, and \( \Omega \) is the angular speed of rotation to the Earth. In the dispersion relation of the spherical coordinate, the Rossby parameter \( \beta \) only plays the role of the spherical effect (Deng et al. 2004; Li and Li 2012; Gu 2016). In the spherical coordinate and local Cartesian coordinate system, the classical \( \beta \)-plane approximation
shows that the Coriolis parameter $f$ is a linear function of latitude. It is convenient to use the Cartesian coordinate; on the $\delta$-surface, the surface of the Earth is the Coriolis parameter as a quadratic function of latitude. Equation (14) shows that the Earth $\delta$-surface approximation is more realistic than the $\delta$-plane approximation, and the resulting equation, Equation (14), expresses the Earth’s curvature is not neglected in the Cartesian coordinate.

Transforming to the wave reference frame on the Earth’s $\delta$-surface, the coordinates in the wave frame are the same as employed in Schneider (2015):

$$(\tilde{x}, \tilde{y}) = (x - ct, y).$$

(16)
The streamfunction $\tilde{\psi}$ is

$$\tilde{\psi} = (c - \tilde{u})y + \psi'(\tilde{x}, y, t_0),$$

(17)
where $t_0$ is a constant, and the streamfunction $\tilde{\psi}$ differs from the streamfunction in the ground reference frame, Equation (11), such as in Schneider (2015). Equation (12) is given by in the wave frame

$$(\tilde{u} - c) \frac{\partial}{\partial \tilde{x}} \nabla^2 \psi' + [\beta - \delta(y - y_0)] \frac{\partial \psi'}{\partial \tilde{x}} = 0,$$

(18)
where

$$\nabla^2 \psi' = \frac{\partial^2 \psi'}{\partial \tilde{x}^2} + \frac{\partial^2 \psi'}{\partial y^2}$$

is the Laplace operator in the wave frame. The description of the conservation of absolute vorticity along trajectories parallels that of Schneider (2015) on the Earth’s $\delta$-surface:

$$d\tilde{\zeta} = -[\beta - \delta(y - y_0)]dy.$$  

(19)
In the wave reference frame, $d\tilde{\psi} = 0$. Along a trajectory and, following Equation (17), we obtain

$$dy = \frac{d\tilde{\psi}(\tilde{x}, y)}{\tilde{u} - c}$$

(20)
Here, $\tilde{u} - c \neq 0$ and, using Equations (14) and (20) in the wave frame, the relative vorticity is

$$d\tilde{\zeta} = -[\beta - \delta(y - y_0)]dy = \frac{\beta - \delta(y - y_0)}{\tilde{u} - c} d\psi' = -K^2 d\psi'$$

(21)
Equation (21) can be written as

$$d\tilde{\zeta} = -K^2 (\tilde{u} - c) dy.$$

(22)
Equation (21) or Equation (22) shows the kinematic relationship: Under $dy > 0$ (northward displacement) conditions, if $\beta - \delta(y - y_0)$ is positive, and thus $d\tilde{\zeta}$ is negative the parcels approach from the west, but the $\delta$-effect slows the parcels’ westward movement, and this may be of great importance near the Earth’s poles.

Where $\beta = 0$, the relative vorticity $d\tilde{\zeta} = \delta(y - y_0)dy$, and thus $d\tilde{\zeta}$ is positive, the parcels approach from the east. If $\beta - \delta(y - y_0)$ is negative, in this case, from the kinematic relationship, Equation (22), the parcels approach from the east, and the $\delta$-effect strengthens its eastward movement. Analysis of southward displacement ($dy < 0$) gives a similar result.

### 3. Baroclinic Rossby waves on the Earth’s $\delta$-surface

The baroclinic quasigeostrophic potential vorticity equation (Holton and Hakim 2012) is

$$\frac{D_p q}{Dt} = 0.$$  

(23)
Here,

$$q = \zeta + f + \frac{f^2}{\rho_0 N^2} \frac{\partial}{\partial z} (\rho_0 \frac{\partial \psi}{\partial z}),$$

(24)
where the streamfunction $\psi$ satisfies Equation (9); the definitions of $z = -H \ln(p/p_i)$, $H = RT_s/g$, $p_s$, $T_s$, $R$, $\rho_0$, and $N^2$ parallel those in Holton and Hakim (2012); and the planetary vorticity $f$ satisfies Equation (2). Assuming a small-amplitude disturbance on a constant zonal-mean flow, the streamfunction for a wave mode takes the form (Schneider 2015)

$$\psi = -\tilde{u}y + \psi'(x, y, t).$$

(25)
Substituting Equation (25) into Equation (23), the linearized version of Equation (23) is

$$\left(\frac{\partial}{\partial t} + \tilde{u} \frac{\partial}{\partial x}\right)[\nabla^2 \psi' + \frac{f_0^2}{\rho_0 N^2} \frac{\partial}{\partial z} (\rho_0 \frac{\partial \psi}{\partial z})] + [\beta - \delta(y - y_0)] \frac{\partial \psi'}{\partial \tilde{x}} = 0.$$  

(26)

The baroclinic Rossby wave solution to Equation (26) is the same as employed in Holton and Hakim (2012) and Schneider (2015):

$$\psi' = \psi_0 e^{iH} \cos[k(x - ct) + ly + mz + \varphi].$$

(27)
Here, the factor $e^{iH}$ (which is proportional to $\rho_0^{-1/2}$) is included to simplify the equation for the vertical dependence, and $m$ is the vertical wavenumber. With the substitution of Equation (27), the baroclinic Rossby wave dispersion relation on the Earth’s $\delta$-surface can be obtained:

$$c = \tilde{u} - \frac{\beta - \delta(y - y_0)}{K^2}$$

(28)
where the full wavenumber $K^2 = k^2 + l^2 + m^2 + \lambda^{-2}$, $m^2 = \frac{f_0^2}{\rho_0} m^2$, and $\lambda^{-2} = \frac{f_0^2}{\rho_0} \lambda^2$. 


From the dispersion relation, Equation (28), if
\[ \bar{u} - c = \frac{\beta - \delta (y - y_0)}{K^2} > 0, \]
the baroclinic Rossby waves propagate westward with respect to the basic state flow, but the \( \delta \)-effect suppress the Rossby waves propagating westward. If
\[ \bar{u} - c = \frac{\beta - \delta (y - y_0)}{K^2} < 0, \]
the \( \delta \)-effect promotes the eastward propagation of the baroclinic Rossby waves.

Analagous to the barotropic Rossby waves in the wave frame,
\[ dy = \frac{d\psi'(\bar{x}, y, z)}{\bar{u} - c} \]
(29)
The relative vorticity is proportional to the streamfunction
\[ d\bar{q}' = -[\beta - \delta (y - y_0)]dy = -K^2 (\bar{u} - c)dy, \]
(30)
where \( \bar{q}' \) is the sum of the perturbation relative vorticity with the \( \delta \)-effect.

In the wave frame, from Equation (30), under \( dy > 0 \) (northward displacement): if \( \bar{u} - c > 0 \) (with \( \beta - \delta (y - y_0) > 0 \)), parcels must approach from the west, but the \( \delta \)-effect slows the parcels’ westward movement; and if \( \bar{u} - c < 0 \) (with \( \beta - \delta (y - y_0) < 0 \)), parcels must approach from the east, and the \( \delta \)-effect promotes the parcels’ eastward movement. Analysis under southward displacement (\( dy > 0 \)) gives a similar result.

4. Conclusions
From the above discussion, we can draw the following main conclusions: The conservation equation of potential vorticity based on the \( \delta \)-effect of Yang (1987, 1988) and Liu and Tan (1992) of Rossby waves cannot be ignored, especially in high-latitude regions, showing that the curvature of Earth is of great importance in the study of geophysical fluid dynamics. The propagation trajectory of barotropic and baroclinic Rossby waves is studied in a reference frame. Consequently, the \( \delta \)-effect plays an important role in the propagation of barotropic and baroclinic Rossby waves. This effect increases the speed of the Rossby wave moving eastward, which weakens the westward propagation of the Rossby wave.

Disclosure statement
No potential conflict of interest was reported by the authors.

Funding
This study was supported by the National Natural Science Foundation of China [grant numbers 11362012, 11562014, 41465002, and 41765004] and the High School Science Research Project of the Inner Mongolia Autonomous Region [grant number NJZY16096], the Natural Science Foundation of Inner Mongolia[grant number 2018LH04005].

References
Cai, M., and B. H. Huang. 2013. “A New Look at the Physics of Rossby Waves: A mechanical-Coriolis Oscillation.” Journal of the Atmospheric Sciences 70 (1): 303–316. doi:10.1175/JAS-D-12-0941.1.
Deng, L. T., S. S. Liu, X. D. Xu, and Z. T. Fu. 2004. “The Effect of Rossby Parameter in Vortex Rossby Waves(In Chinese).” Journal of Tropical Meteorology 20 (5): 483–492.
Gill, A. E. 1982. Atmosphere-Ocean Dynamic, 189–243. New York: Academic Press.
Grimshaw, R. H. J. 1975. “A Note the \( \beta \)-plane Approximation.” Tellus 27 (4): 351–357. doi:10.3402/tellusa.v27i4.9982.
Gu, X. Z. 2016. “Research on the Rossby Wave on Propagation and Dispersion in the Westerlies and Easterlies, Part II: Analysis and Diagnosis of Deducing Spherical Rossby Wave Equation(In Chinese).” Journal of Tropical Meteorology 32 (5): 622–633.
Holton, J. R. 1979. An Introduction to Dynamic Meteorology, 79–98. 2nd ed. New York: Academic Press.
Holton, J. R., and G. J. Hakim. 2012. An Introduction to Dynamic Meteorology, 407–434. 5th ed. New York: Academic Press.
Hoskins, B. J., M. E. McIntyre, and A. W. Robertson. 1985. “On the Use and Significance of Isentropic Potential Vorticity Maps.” Quarterly Journal of the Royal Meteorological Society 111 (470): 877–946. doi:10.1002/qj.49711147002.
Li, Y. J., and J. P. Li. 2012. “Propagation of Planetary Waves in the Horizontal Non-Uniform Basic Flow(In Chinese).” Chinese Journal of Geophysics 55 (2): 361–371.
Liu, S. K., and S. D. Liu. 2011. Atmospheric Dynamics(In Chinese), 14–23. 2nd ed. Peking: Peking University Press.
Liu, S. K., and B. K. Tan. 1992. “Rossby Waves with the Change of \( \beta \) (In Chinese).” Applied Mathematics Mechanics 13 (1): 35–44.
Luo, D. H. 1995. “Solitary Rossby Waves with the Beta Parameter and Dipole Blocking(In Chinese).” Quarterly Journal of Applied Meteorology 6 (2): 220–227.
Pedlosky, J. 1979. Geophysical Fluid Dynamics, 57–130. 2nd ed. New York: Springer-Verlag.
Platzman, G. W. 1968. “The Rossby Wave.” Quarterly Journal of the Royal Meteorological Society 94 (401): 225–248. doi:10.1002/qj.49709440102.
Schneider, E. K. 2015. “Trajectory Analysis of Mechanism for Westward Propagation of Rossby Waves.” Journal of the Atmospheric Sciences 72 (5): 2178–2182. doi:10.1175/JAS-D-14-0242.1.
Song, J., Q. S. Liu, R. T. Cen, and L. G. Yang. 2017. “Rossby Waves with the Change of Beta and the Influence of Topography in a Two-Layer Fluid.” Journal of Mathematics 37 (4): 751–760.
Song, J., and L. G. Yang. 2009. “Modified KdV Equation for Solitary Rossby Waves with \( \beta \) Effect in Barotropic Fluids.” Chinese Physics B 18 (7): 2873–2877. doi:10.1088/1674-1056/18/7/042.
Song, J., L. G. Yang, C. J. Da, and H. Q. Zhang. 2009. “mKdV Equation for the Amplitude of Solitary Rossby Waves in Stratified Shear Flows with a Zonal Shear Flow.” *Atmospheric and Oceanic Science Letters* 2 (1): 18–23. doi:10.1080/16742834.2009.11446771.

Song, J., and L. G. Yang. 2010. “mKdV Equation for Solitary Rossby Waves with Linear Topography Effect in Barotropic Fluids.” *Plateau Meteorology* 29 (5): 1137–1141.

Yang, H. J. 1987. “Evolution of a Rossby Wave Packet in Barotropic Flow with Asymmetric Basic Current, Topography and δ-effect.” *Journal of the Atmospheric Sciences* 44 (6): 2267–2276. doi:10.1175/1520-0469(1987)044<2267:EOARWP>2.0.CO;2.

Yang, H. J. 1988. “Global Behavior of the Evolution of a Rossby Wave Packet in Barotropic Flow on the Earth’s δ-surface.” *Journal of the Atmospheric Sciences* 45 (2): 133–146.