CONTRAADJUSTED MODULES, CONTRAMODULES, AND REDUCED COTORSION MODULES

LEONID POSITSELSKI

Abstract. This paper is devoted to the more elementary aspects of the contramodule story, and can be viewed as an extended introduction to the more technically complicated [27]. Reduced cotorsion abelian groups form an abelian category, which is in some sense covariantly dual to the category of torsion abelian groups. An abelian group is reduced cotorsion if and only if it is isomorphic to a product of \( p \)-contramodule abelian groups over prime numbers \( p \). Any \( p \)-contraadjusted abelian group is \( p \)-adically complete, and any \( p \)-adically separated and complete group is a \( p \)-contramodule, but the converse assertions are not true. In some form, these results hold for modules over arbitrary commutative rings, while other formulations are applicable to modules over one-dimensional Noetherian rings.

Contents

Introduction 2
1. Orthogonality Classes as Abelian Categories 4
2. \( s \)-Contraadjustedness, \( s \)-Contramoduleness, and \( s \)-Completeness 6
3. \( s \)-Power Infinite Summation Operations 11
4. \([s, t]\)-Power Infinite Summation Operations 14
5. \([s_1, \ldots, s_m]\)-Contraadjustedness, \( I \)-Contramoduleness, and \( I \)-Adic Completeness 17
6. \( s \)-Torsion Modules and the Functor \( \Gamma_s \), \( s \)-Contramodules and the Functor \( \Delta_s \) 22
7. \( I \)-Torsion Modules and the Functor \( \Gamma_I \), \( I \)-Contramodules and the Functor \( \Delta_I \) 31
8. Covers, Envelopes, and Cotorsion Theories 37
9. Contramodules and Relatively Cotorsion Modules 42
10. Flat, Projective, and Free Contramodules 45
11. Free Contramodules and Flat Cotorsion Modules 50
12. Cotorsion and Contraadjusted Abelian Groups 54
13. Noetherian Rings of Krull Dimension 1 61
References 70
An abelian group $C$ is said to be cotorsion if for any torsion-free abelian group $F$ one has $\text{Ext}^1_\mathbb{Z}(F,C) = 0$. Considering the short exact sequence $F \to \mathbb{Q} \otimes \mathbb{Z} F \to \mathbb{Q}/\mathbb{Z} \otimes \mathbb{Z} F$ and having in mind that the category of abelian groups has homological dimension 1, one easily concludes that $C$ is cotorsion if and only if $\text{Ext}^1_\mathbb{Z}(\mathbb{Q},C) = 0$.

An abelian group is said to be reduced if it has no divisible subgroups. Clearly, any abelian group $B$ has a unique maximal divisible subgroup $B_{\text{div}} \subset B$; the quotient group $B_{\text{red}} = B/B_{\text{div}}$ is the maximal reduced quotient group of $B$. The short exact sequence $B_{\text{div}} \to B \to B_{\text{red}}$ is (noncanonically) split. For any element $b \in B_{\text{div}}$, one can construct an abelian group homomorphism $\mathbb{Q} \to B$ taking 1 to $b$. So the subgroup $B_{\text{div}} \subset B$ can be computed as the image of the restriction map $\text{Hom}_\mathbb{Z}(\mathbb{Q},B) \to B$ induced by the embedding $\mathbb{Z} \to \mathbb{Q}$. In particular, an abelian group $B$ is reduced if and only if $\text{Hom}_\mathbb{Z}(\mathbb{Q},B) = 0$.

For any abelian category $A$ and any object $U$ of projective dimension $\leq 1$ in $A$, the full subcategory $C \subset A$ consisting of all the objects $C$ satisfying $\text{Hom}_A(U,C) = 0 = \text{Ext}^1_A(U,C)$ is closed under the kernels and cokernels of morphisms between its objects, and also extensions and infinite products, taken in $A$. It follows that the category $C$ is abelian and its embedding functor $C \to A$ is exact.

We have explained that the category of reduced cotorsion abelian groups is abelian. One can try to read this assertion between the lines of the exposition in the book [8, §54]. The feeling that this fact deserves to be emphasized was one of the motivating impulses for writing this paper. In fact, the term “co-torsion group” first appeared in the paper [13], where it meant what we now call “reduced cotorsion”.

Notice that the category of torsion abelian groups, consisting of all the groups $T$ satisfying $\mathbb{Q} \otimes \mathbb{Z} T = 0$, has even stronger properties: it contains all the subobjects and quotients of its objects, in addition to extensions (and infinite direct sums) in the category of abelian groups $\text{Ab}$; so it is a Serre subcategory. That is because the group $\mathbb{Q}$ has flat dimension 0, but projective dimension 1.

Just as the category of torsion abelian groups, the category of reduced cotorsion abelian groups splits into a Cartesian product of categories indexed by the prime numbers. In fact, the functor right adjoint to the embedding of the full subcategory of torsion abelian groups (i.e., the maximal torsion subgroup functor) can be computed as $\text{Tor}^1_\mathbb{Z}(\mathbb{Q}/\mathbb{Z},-)$, and the functor left adjoint to the embedding of the full subcategory of reduced cotorsion abelian groups into $\text{Ab}$ can be computed as $\text{Ext}^1_\mathbb{Z}(\mathbb{Q}/\mathbb{Z},-)$ (cf. [20, Theorem 7.1]). Decomposing $\mathbb{Q}/\mathbb{Z}$ into the direct sum of its $p$-primary components $\bigoplus_p \mathbb{Q}_p/\mathbb{Z}_p$, one concludes that an abelian group $C$ is reduced cotorsion if and only if it can be (always uniquely) presented as a product of abelian groups of the form $\text{Ext}^1_\mathbb{Z}(\mathbb{Q}_p/\mathbb{Z}_p, A)$, where $A \in \text{Ab}$. Moreover, one can take $A = C$.

Abelian groups of the form $\text{Ext}^1_\mathbb{Z}(\mathbb{Q}_p/\mathbb{Z}_p, A)$ form an abelian category which we call the category of $p$-contramodule abelian groups. It can be described as the full subcategory of all abelian groups $C$ satisfying $\text{Hom}_\mathbb{Z}(\mathbb{Z}[p^{-1}],C) = 0 = \text{Ext}^1_\mathbb{Z}(\mathbb{Z}[p^{-1}],C)$. Actually, both the abelian categories of $p$-contramodule and reduced cotorsion abelian
groups lie in the intersection of two natural classes of abelian categories (so there are two independent ways to explain why they are abelian). Besides their above description as the Ext\(^{0,1}\)-orthogonality classes, they can be also defined as the categories of contramodules over the topological rings of \(p\)-adic integers \(\mathbb{Z}_p\) and finite integral adèles \(\prod_p \mathbb{Z}_p\), respectively. These are modules with infinite summation operations, as introduced in [23, Remark A.3] and [24], and overviewed in [26].

An abelian group \(C\) is called \(p\)-contraadjusted if \(\text{Ext}_1^p(\mathbb{Z}[p^{-1}], C) = 0\). Similarly one defines \(s\)-contraadjusted abelian groups for any natural number \(s\); and a group is called contraadjusted if it is \(s\)-contraadjusted for all \(s \geq 2\). Contraadjusted modules were introduced in connection with contraherent cosheaves [25] and further studied in [33, 34]. For any \(p\)-contraadjusted abelian group \(C\), the natural map into the \(p\)-adic completion \(C \rightarrow \varprojlim_n C/p^n C\) is surjective. When this map is an isomorphism, the group \(C\) is even a \(p\)-contramodule. The converse implications to both these assertions fail in general, but they are true for \(p\)-torsion-free abelian groups.

Most of the results mentioned above extend to modules over an arbitrary commutative ring \(R\) with a fixed element \(s \in R\), or even with a fixed ideal \(I \subset R\). The results based on the decomposition of the quotient module \(Q/R\), where \(Q\) is the ring/field of fractions of \(R\), into a direct sum over the maximal ideals, are the main exception. These require \(R\) to be a Noetherian ring of Krull dimension 1.

The original motivation for our study of the cotorsion and contraadjusted modules comes from algebraic geometry. Given a commutative ring \(R\), the category of contraherent cosheaves over the affine scheme Spec \(R\) is equivalent to the category of contraadjusted \(R\)-modules, while the category of locally cotorsion contraherent cosheaves on Spec \(R\) is equivalent to the category of cotorsion \(R\)-modules [25]. One would like to have a supply of examples and easily computable special (e. g., low-dimensional) cases, just for developing an intuition of what the contraherent cosheaves are. From this geometric point of view, cotorsion and contraadjusted abelian groups are nothing but the (global cosections of) contraherent cosheaves over Spec \(\mathbb{Z}\).

The experience seems to teach that the locally cotorsion contraherent cosheaves behave well on Noetherian schemes, while for schemes of more general nature there are not enough of these, and one wishes to consider arbitrary (i. e., locally contraadjusted) contraherent cosheaves. Hence the importance of cotorsion modules over Noetherian rings and contraadjusted modules over both Noetherian and non-Noetherian rings, from our geometric standpoint. Flat cotorsion modules, corresponding geometrically to projective locally cotorsion contraherent cosheaves, over Noetherian rings were classified by Enochs [7] (see [25, Section 5.1] for the nonaffine case). Arbitrary cotorsion modules over Noetherian rings of Krull dimension 1 are described (and contraadjusted modules discussed) in the present paper, based on the approaches originated by Nunke [20] and Slávik–Trlifaj [33, 34].

Concerning the answers that we seem to obtain, the following vague analogy may be illuminating. Quasi-coherent sheaves over Spec \(\mathbb{Z}\) are the same thing as abelian groups, and coherent sheaves correspond to finitely generated abelian groups. Arbitrary abelian groups are hopelessly complicated, but the finitely generated ones (or,
say, finitely generated modules over a Dedekind domain) can be classified. In fact, they are described as finite direct sums, where one summand is a free group (respectively, a projective module) and the remaining ones are torsion modules indexed by the prime numbers (sitting at the closed points of the spectrum).

Likewise, contraadjusted abelian groups may be too complicated to describe, but one can say a lot about cotorsion groups or cotorsion modules over Dedekind domains. In fact, the latter are described as the infinite products of a divisible group (= injective module) and contramodules sitting at the prime numbers (closed points). So one obtains a geometric picture of locally cotorsion contraherent cosheaves over smooth curves bearing some vague similarity to the classification of coherent sheaves over such curves. For Noetherian rings of Krull dimension 1 (= singular curves) there is essentially the same description of reduced cotorsion modules as the products of contramodules (sitting at the points), while an injective direct summand turns into a more complicated (noninjective) divisible submodule.

Acknowledgement. This paper grew out of my visits to Prague for a conference in September and then for a longer stay in November–December 2015. I would like to express my gratitude to Jan Trlifaj and Alexander Slávik for stimulating discussions. In particular, Alexander’s presentations influenced the present work, and he told me about the reference [20]. I learned from Jan about the $R$-topology and the references [8, §54–55] and [9]. A large part of this paper is but an extended writeup of my last talk in Prague in December.

Subsequently in Israel, I had a conversation with Joseph Bernstein, and I want to thank him for pointing out that abelian categories are rare and interesting. Last but not least, I am grateful to Vladimir Hinich and the Israeli Academy of Sciences for warm hospitality in offering me a postdoctoral position at the University of Haifa, with excellent working conditions and financial support. I also want to thank the anonymous referee for a number of helpful remarks and suggestions. The author was supported by the ISF grant # 446/15 in Israel and by the Grant Agency of the Czech Republic under the grant P201/12/G028 in Prague.

1. Orthogonality Classes as Abelian Categories

All rings and modules in this paper are unital.

Let $R$ be an associative ring. We denote by $R\text{-mod}$ and $\text{mod}-R$ the categories of (arbitrarily large) left and right $R$-modules, respectively.

Given an abelian category $A$, a full subcategory $C \subseteq A$ is said to be closed under subobjects (respectively, quotient objects) if every subobject (resp., quotient object) of an object of the class $C$ in the category $A$ also belongs to $C$. A full subcategory closed under subobjects, quotients, and extensions is called a Serre subcategory. For such subcategories $C \subseteq A$, an abelian quotient category $A/C$ is defined.

A full subcategory $C \subseteq A$ is said to be closed under the kernels (respectively, cokernels), if for every morphism $f : C \to D$ in $A$ between two objects $C, D \in C$ the kernel $\text{ker}_A(f)$ (resp., the cokernel $\text{coker}_A(f)$) of the morphism $f$ computed in the
category $A$ belongs to $C$. A full subcategory $C$ closed under the kernels, cokernels, and finite direct sums in an abelian category $A$ is also an abelian category with an exact embedding functor $C \to A$.

**Theorem 1.1.** (a) Let $U$ be a projective left $R$-module. Then the full subcategory $C \subseteq R\mod$ formed by all the left $R$-modules $C$ for which $\text{Hom}_R(U,C) = 0$ is closed under subobjects, quotient objects, extensions, and infinite products in $R\mod$. In particular, $C \subseteq R\mod$ is a Serre subcategory.

(b) Let $U$ be a flat right $R$-module. Then the full subcategory $T \subseteq R\mod$ formed by all the left $R$-modules $T$ for which $U \otimes_R T = 0$ is closed under subobjects, quotient objects, extensions, and infinite direct sums in $R\mod$. In particular, $T \subseteq R\mod$ is a Serre subcategory.

**Proof.** More generally, for any abelian categories $A$ and $B$ and any exact functor $F: A \to B$ the full subcategory $S \subseteq A$ formed by all the objects $S \in A$ for which $F(S) = 0$ is a Serre subcategory in $A$. \hfill \square

The following result can be found in [10, Proposition 1.1].

**Theorem 1.2.** (a) Let $U$ be a left $R$-module of projective dimension $\leq 1$. Then the full subcategory $C \subseteq R\mod$ formed by all the left $R$-modules $C$ for which $\text{Hom}_R(U,C) = 0 = \text{Ext}_R^1(U,C)$ is closed under the kernels, cokernels, extensions, and infinite products in $R\mod$. In particular, $C$ is an abelian category and the embedding functor $C \to R\mod$ is exact.

(b) Let $U$ be a right $R$-module of flat dimension $\leq 1$. Then the full subcategory $T \subseteq R\mod$ formed by all the left $R$-modules $T$ for which $U \otimes_R T = 0 = \text{Tor}_1^R(U,T)$ is closed under the kernels, cokernels, extensions, and infinite direct sums in $R\mod$. In particular, $T$ is an abelian category and the embedding functor $T \to R\mod$ is exact.

**Proof.** Generally, let $A$, $B$ be abelian categories and $(F^0,F^1)$ be a cohomological $\delta$-functor (of cohomological dimension $\leq 1$) from $A$ to $B$, that is a pair of functors $F^0, F^1: A \to B$ together with a 6-term exact sequence

$$
0 \to F^0(X) \to F^0(Y) \to F^0(Z) \to F^1(X) \to F^1(Y) \to F^1(Z) \to 0
$$

in $B$ defined functorially for any short exact sequence $0 \to X \to Y \to Z \to 0$ in $A$. Then the full subcategory $C \subseteq A$ formed by all the objects $C$ for which $F^0(C) = 0 = F^1(C)$ is closed under the kernels, cokernels, and extensions in $A$.

Indeed, the class of all object $X \in A$ such that $F^0(X) = 0$ is closed under subobjects and extensions, while the class of all $Z \in A$ such that $F^1(Z) = 0$ is closed under extensions and quotients. Now let $f: C \to D$ be a morphism between two objects $C, D \in C$ and let $E$ be the image of $f$. Then $F^0(E) = 0$, as $E$ is a subobject in $D$; and $F^1(E) = 0$, because $E$ is a quotient of $C$. Finally, let $K$ and $L$ denote the kernel and cokernel of the morphism $f$. Considering the long exact sequence of functor $F^\ast$ for the short exact sequences $0 \to K \to C \to E \to 0$ and
0 → E → D → L → 0 in the category A, one concludes that the objects K and L belong to C. \qed

**Remark 1.3.** It is clear from the proof of Theorem 1.2 that a module U of projective (respectively, flat) dimension ≤ 1 in its formulation can be replaced by a two-term complex \( U^{-1} → U^0 \) of projective (resp., flat) \( R \)-modules. In other words, a module can be replaced with an object of the derived category of modules with the similar restriction on the projective/flat dimension. Then it is claimed that the full subcategory of all modules \( C \) for which \( \text{Hom}_R(u, C) \) is an isomorphism (resp., all modules \( T \) for which \( u ⊗_R T \) is an isomorphism) is closed under the kernels, cokernels, and extensions in \( R\text{-mod} \); so it is an abelian category. One just applies the above argument to the \( \delta \)-functor \((F^0, F^1)\) with \( F^0(X) = \ker \text{Hom}_R(u, X) \) and \( F^1(X) = \text{coker} \text{Hom}_R(u, X) \).

Similarly, if \( V^0 → V^1 \) is a two-term complex of injective left \( R \)-modules, then the left \( R \)-modules \( T \) for which \( \text{Hom}_R(T, v) \) is an isomorphism form a full subcategory closed under the kernels, cokernels, extensions, and infinite direct sums in \( R\text{-mod} \), hence an abelian category. (Cf. the discussion of infinite systems of nonhomogeneous linear equations in the next section.)

2. \( s \)-Contraadjustedness, \( s \)-Contramoduleness, and \( s \)-Completeness

Let \( R \) be a commutative ring and \( s \in R \) be a fixed element. In this section, we will consider \( R \)-modules \( C \) with certain conditions imposed on the action of \( s \) in \( C \). The action of the rest of \( R \) will be less important, and almost just as well we could be talking about abelian groups \( C \) endowed with an endomorphism \( s : C → C \).

An \( R \)-module \( C \) is called \( s \)-torsion-free if the operator \( s : C → C \) is injective. An \( R \)-module \( C \) is called \( s \)-divisible if the map \( s : C → C \) is surjective.

We denote by \( R[s^{-1}] \) the localization of \( R \) with respect to the multiplicative system \( \{1, s, s^2, \ldots, s^n, \ldots\} \subset R \). An \( R \)-module \( C \) is said to be \( s \)-contraadjusted if \( \text{Ext}^1_R(R[s^{-1}], C) = 0 \). An \( R \)-module \( C \) is called an \( s \)-contramodule (or an \( s \)-contraadjusted \( R \)-module) if \( \text{Hom}_R(R[s^{-1}], C) = 0 = \text{Ext}^1_R(R[s^{-1}], C) \).

Clearly, one has \( \text{Hom}_R(R[s^{-1}], C) = 0 \) if and only if there are no nonzero \( s \)-divisible \( R \)-submodules in \( C \), or which is equivalent, if there are no nonzero abelian subgroups \( D \subset C \) for which \( sD = D \). The following lemma [24, Lemma B.7.1], [34, Lemma 5.1] explains what does the condition \( \text{Ext}^1_R(R[s^{-1}], C) = 0 \) mean.

**Lemma 2.1.** (a) An \( R \)-module \( C \) is \( s \)-contraadjusted if and only if for any sequence of elements \( a_0, a_1, a_2, \ldots \in C \) the infinite system of nonhomogeneous linear equations

\[ b_n - sb_{n+1} = a_n, \quad n ≥ 0 \]

has a solution \( b_0, b_1, b_2, \ldots \in C \).

(b) An \( R \)-module \( C \) is an \( s \)-contramodule if and only if for any sequence of elements \( a_0, a_1, a_2, \ldots \in C \) the infinite system of nonhomogeneous linear equations (1) has a unique solution \( b_0, b_1, b_2, \ldots \in C \).
Proof. Computing $R[s^{-1}] \simeq \lim_{n \geq 0} R$ (with all the maps $R \to R$ being the multiplication with $s$) using the telescope construction for $\mathbb{Z}_{\geq 0}$-indexed inductive limits, one obtains a free resolution of the form

$$0 \longrightarrow \bigoplus_{n=0}^{\infty} Rf_n \longrightarrow \bigoplus_{n=0}^{\infty} R\epsilon_n \longrightarrow R[s^{-1}] \longrightarrow 0,$$

with the differential taking the basis vector $f_n$ to $e_n - s\epsilon_{n+1}$, for the $R$-module $R[s^{-1}]$. Applying the functor $\text{Hom}_R(-, C)$, one computes the $R$-modules $\text{Hom}_R(R[s^{-1}], C)$ and $\text{Ext}^1_R(R[s^{-1}], C)$ as, respectively, the kernel and the cokernel of the map

$$\prod_{n=0}^{\infty} C \longrightarrow \prod_{n=0}^{\infty} C$$

taking a sequence $(b_n)_{n=0}^{\infty}$ to the sequence $(a_n = b_n - sb_{n+1})_{n=0}^{\infty}$. □

**Lemma 2.2.** The class of $s$-contraadjusted $R$-modules is closed under quotients, extensions, and infinite products in $R\text{-mod}$.

**Proof.** See the proof of Theorem 1.2. Alternatively, the closedness with respect to quotients can be easily deduced from Lemma 2.1(a). □

An $R$-module $C$ is said to be $s$-adically complete (or simply $s$-complete) if the natural map from it to its $s$-adic completion

$$\lambda_{s,C} \colon C \longrightarrow \lim_{\leftarrow n \geq 1} C/s^nC$$

is surjective. The $R$-module $C$ is called $s$-adically separated (or s-separated) if the map $\lambda_{s,C}$ is injective. Clearly, the kernel of $\lambda_{s,C}$ is the submodule $\bigcap_n s^n C \subset C$.

The following result can be found in [33, Lemma 4.4] and [34, Lemma 5.4].

**Theorem 2.3.** (a) Any $s$-contraadjusted $R$-module is $s$-complete. (Hence, in particular, any $s$-contramodule $R$-module is $s$-complete.)

(b) Any $s$-torsion-free $s$-complete $R$-module is $s$-contraadjusted.

**Proof.** By the definition, $s$-completeness of $C$ means that for any sequence of elements $c_n \in C$, $n \geq 1$ such that $c_{n+1} - c_n \in s^n C$ for all $n \geq 1$ there exists an element $b \in C$ for which $b - c_n \in s^n C$ for all $n \geq 1$. Set $a_0 = c_1$; and pick elements $a_n \in C$ such that $c_{n+1} = c_n + s^n a_n$ for all $n \geq 1$. Then we have $c_n = a_0 + sa_1 + \cdots + s^{n-1}a_{n-1}$ for $n \geq 1$. This shows that an $R$-module $C$ is $s$-complete if and only if for every sequence of elements $a_n \in C$, $n \geq 0$ there exists an element $b \in C$ such that

$$b - a_0 - sa_1 - \cdots - s^{n-1}a_{n-1} \in s^n C \quad \text{for all } n \geq 1.$$ Set $b_0 = b$. Now the latter condition on $C$ can be expressed as the solvability of the system of linear equations

$$b_0 - s^{n+1}b_{n+1} = a_0 + sa_1 + \cdots + s^n a_n, \quad n \geq 0$$
in $b_0$, $b_1$, $b_2$, $\ldots \in C$ for all $a_0$, $a_1$, $a_2$, $\ldots \in C$. Finally, we can rewrite this system of equations equivalently as

$$(2) \quad s^n(b_n - sb_{n+1}) = s^n a_n, \quad n \geq 0,$$

because one always has

$$b_0 - s^{n+1}b_{n+1} = (b_0 - sb_1) + s(b_1 - sb_2) + \cdots + s^n(b_n - sb_{n+1}).$$
We have shown that an $R$-module $C$ is $s$-complete if and only if the system of nonhomogeneous linear equations (2) is solvable in $(b_n \in C)_{n=0}^\infty$ for every sequence $(a_n \in C)_{n=0}^\infty$. Now it is obvious that the system of equations (1) implies (2), and the converse implication holds whenever $sc = 0$ implies $c = 0$ for $c \in C$. This proves both the assertions (a) and (b).

A generalization of Theorem 2.3(a) to the $I$-adic completions for finitely generated nonprincipal ideals $I$ will be given below in Theorem 5.6. A precise criterion of contraadjustedness extending the one provided by Theorem 2.3(a-b) to the non-$s$-torsion-free module case can be found in Corollary 6.10(b) (see also Remark 6.11).

**Theorem 2.4.** (a) Any $s$-separated $s$-complete $R$-module is an $s$-contramodule.

(b) Any $s$-torsion-free $s$-contramodule $R$-module is $s$-separated (and $s$-complete).

**Proof.** From the proof of Theorem 2.3 one concludes that an $R$-module $C$ is $s$-separated if and only if for any sequence $a_0, a_1, a_2, \ldots \in C$ the element $b_0 \in C$ is uniquely determined by the system of equations (2). In other words, for any two solutions $(b'_n \in C)_{n=0}^\infty$ and $(b''_n \in C)_{n=0}^\infty$ of the system (2) with the given sequence $(a_n \in C)_{n=0}^\infty$, one should have $b'_0 = b''_0$. We will say that “a solution of the system (2) is weakly unique” if this condition is satisfied.

Assuming that this is the case for all sequences $(a_n \in C)_{n=0}^\infty$, one can see that a solution of the system of equations (1) is unique (in the conventional sense of the word) in the $R$-module $C$. Indeed, for every $n \geq 0$ the system (1) implies

$$s^i(b_{n+i} - sb_{n+i+1}) = s^ia_{n+i}$$

for all $i \geq 0$,

which uniquely determines the element $b_n$.

Now assume that a solution of (2) exists and is weakly unique for all sequences $(a_n \in C)_{n=0}^\infty$. In this case, in order to solve the system (1), one for every $n \geq 0$ solves the auxiliary equation system

$$s^i(d_i^{(n)} - sd_{i+1}^{(n)}) = s^ia_{n+i}, \quad i \geq 0,$$

in $d_i^{(n)} \in C$, $n, i \geq 0$; and puts $b_n = d_0^{(n)}$. Then, for every $n \geq 0$, the sequence $e_i^{(n)} = sd_i^{(n+1)} + a_{n+i}$ provides another solution of the system (3), because

$$s^i(e_i^{(n)} - se_{i+1}^{(n)}) = s^i(sd_i^{(n+1)} + a_{n+i} - s^2d_{i+1}^{(n+1)} + sa_{n+i+1})$$

$$= ss^i(d_i^{(n+1)} - sd_{i+1}^{(n+1)}) + s^i(a_{n+i} - sa_{n+i+1})$$

$$= ss^i(a_{n+i+1} + s^i(a_{n+i} - sa_{n+i+1}) = s^ia_{n+i}.$$  

Hence, due to the weak uniqueness condition on the solutions of (2), we have

$$b_n = d_0^{(n)} = e_0^{(n)} = sd_0^{(n+1)} + a_n = sb_{n+1} + a_n$$

and a solution of the equations (1) is obtained. This proves part (a).

To check (b), it remains to recall that in an $s$-torsion-free $R$-module $C$ the systems (1) and (2) are equivalent, so (existence and) uniqueness of solutions of (1) implies (existence and) uniqueness of solutions of (2). □
Remark 2.5. The assertion of Theorem 2.4(b) is well-known; see Corollary 10.3(b) below for a generalization to Noetherian rings. Theorem 2.4(a) is even more standard, and easier proved with the standard methods. It suffices to say that the class of $s$-contramodule $R$-modules is closed under kernels and infinite products in $R$–mod by Theorem 1.2(a); hence it is also closed under projective limits of all diagrams. Any $R$-module $D$ for which $s^nD = 0$ for some $n \geq 1$ is an $s$-contramodule (since the action of $s$ in $\text{Ext}_R^*(R[s^{-1}], D)$ is then simultaneously invertible and nilpotent); thus the $R$-module $\lim_{\leftarrow n} C/s^nC$ is an $s$-contramodule, too (cf. Lemma 5.7 below). The more complicated argument above is presented for comparison with the proof of Theorem 2.3, and also in order to illustrate the workings of the technique of infinite systems of nonhomogeneous linear equations.

One can sum up the results of Theorems 2.3–2.4 in the following diagram:

\[
\begin{array}{ccc}
\text{s-complete} & \uparrow & \downarrow \text{if s-torsion-free} \\
\text{s-contraadjusted} & \uparrow & \\
\text{s-contramodule} & \uparrow & \downarrow \text{if s-torsion-free} \\
\text{s-separated and s-complete} & \\
\end{array}
\]

Remark 2.6. The two “inner” properties in the above diagram are defined in terms of (the existence and/or uniqueness of solutions of) the equation system (1), while the two “outer” properties translate into (the existence and/or weak uniqueness of solutions of) the equation system (2). The system (2) does not look good, though, and appears to be nothing more than an unsuccessful naïve attempt to arrive at (1). This is supposed to teach us that the $s$-completeness and $s$-separatedness are not the right conditions to consider unless $s$-torsion-freeness, or some weaker form of it guaranteeing essential equivalence of (1) and (2), is assumed.

Indeed, it is known [32, Example 2.5] that the class of $s$-separated and $s$-complete $R$-modules does not have good homological properties (of the kind listed in Theorem 1.2(a) for the class of $s$-contramodules): it is not closed under cokernels (even under the cokernels of injective morphisms) in $R$–mod, nor is it closed under extensions. The category of $s$-separated and $s$-complete $R$-module is not abelian. Similarly, the counterexample below shows that the class of $s$-complete $R$-modules is not closed under extensions (cf. Lemma 2.2), even though it is closed under quotients.

Concerning the latter assertion, given a short exact sequence of $R$-modules $0 \rightarrow K \rightarrow L \rightarrow M \rightarrow 0$, the exact sequence $0 \rightarrow K/(K \cap s^nL) \rightarrow L/s^nL \rightarrow M/s^nM \rightarrow 0$ with surjective maps between the modules in the projective system $K/(K \cap s^nL)$ yields surjectivity of the map $\lim_{\leftarrow n} L/s^nL \rightarrow \lim_{\leftarrow n} M/s^nM$, so $s$-completeness of $L$ implies $s$-completeness of $M$. 

Examples 2.7. (1) Let us start with reproducing the now-classical counterexample of a nonseparated contramodule (see [26, Section 1.5] and the references therein). Set $R = \mathbb{Z}$ and choose a prime number $p$. Let $C$ denote the subgroup in abelian group $\prod_{n=0}^{\infty} \mathbb{Z}_p$ consisting of all sequences of $p$-adic integers $u_0, u_1, u_2, \ldots, u_n, \ldots$ converging to zero in the topology of $\mathbb{Z}_p$. Let $D \subset C$ denote the group of all sequences of $p$-adic integers of the form $v_0, pv_1, p^2v_2, \ldots$, where $v_n \in \mathbb{Z}_p$; and let $E \subset D$ be the subgroup of all sequences $u_n = p^n v_n$ such that $v_n \to 0$ in $\mathbb{Z}_p$ as $n \to \infty$.

All the three groups $C, D, E$ are $p$-contramodules; in fact, they are $p$-separated and $p$-complete. Hence the quotient group $C/E$ is a $p$-contramodule, too. However, it is not $p$-separated; in fact, one has $\bigcap_n p^n(C/E) = D/E \subset C/E$, so $\lim_{\to n}(C/E)/p^n(C/E) = C/D$. Furthermore, there is a short exact sequence $0 \to D/E \to C/E \to C/D \to 0$, where the groups $D/E \simeq (\prod_{n=0}^{\infty} \mathbb{Z}_p)/C$ and $C/D \subset \prod_{n=0}^{\infty} \mathbb{Z}/p^n\mathbb{Z}$ are $p$-separated and $p$-complete, but the group $C/E$ is not [32].

Computing the cokernel of the embedding $E \to C$ in the category of $p$-separated and $p$-complete abelian groups $\text{Ab}_{p,\text{secmp}}$, one obtains the group $C/D$ and the kernel of the morphism $C \to C/D$ is $D$, while the kernel of $E \to C$ is, of course, zero, and the cokernel of $0 \to E \to C$ is $E$. Thus the category $\text{Ab}_{p,\text{secmp}}$ is not abelian.

(2) Now let us demonstrate an example of a $p$-complete abelian group that is not $p$-contraadjusted. Choose a bijection $(\phi, \psi): \mathbb{Z}_{\geq 0} \simeq \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$, and assign to every eventually vanishing sequence of $p$-adic integers $w_0, w_1, w_2, \ldots, w_i, \ldots$ the sequence $w_{\phi(0)}, pw_{\phi(1)}, p^2w_{\phi(2)}, \ldots, p^nw_{\phi(n)}, \ldots$. This construction provides a homomorphism $\bigoplus_{n=0}^{\infty} \mathbb{Z}_p \to D$ whose composition with the projection $D \to D/E$ is still an injective map $\bigoplus_{n=0}^{\infty} \mathbb{Z}_p \to D/E$, because $w_{\phi(n)} \to 0$ as $n \to \infty$ implies $w_i = 0$ for all $i$. Denote the image of this embedding by $A \subset D/E$. Consider the surjective homomorphism $A \to \mathbb{Q}_p$ taking $(w_i)_{i=0}^{\infty}$ to $\sum_{i=0}^{\infty} w_i p^i$, and extend it to an abelian group homomorphism $f: C/E \to \mathbb{Q}_p$ in an arbitrary way. Set $F = \ker(f) \subset C/E$, and denote by $G$ the kernel of the composition of $f$ with the projection $C \to C/E$.

Then from the exact sequence $0 \to F \to C/E \to \mathbb{Q}_p \to 0$ one concludes that $F/p^nF \simeq (C/E)/p^n(C/E)$, so $\lim_{\to n} F/p^nF = C/D$. The map $F \to C/D$ is surjective, because the map $D \to C/G \simeq \mathbb{Q}_p$ is. Hence the group $F$ is $p$-complete. Similarly, from the short exact sequence $0 \to G \to C \to \mathbb{Q}_p \to 0$ we get $G/p^nG \simeq C/p^nC$, so $\lim_{\to n} G/p^nG = C$ and $G$ is not $p$-complete. Finally, we have a short exact sequence $0 \to D \to G \to F \to 0$. Both the groups $D$ and $F$ are $p$-complete, and $D$ is also $p$-separated. Thus $F$ is not $p$-contraadjusted (for otherwise $G$ would have to be $p$-contraadjusted, too, but it is not even $p$-complete).

Remark 2.8. In addition to Theorems 2.3–2.4, it is instructive to compare the two versions of the separatedness property. An $R$-module is $s$-separated if and only if it has no elements divisible by an arbitrary power of $s$. This means weak uniqueness of solutions of the equation system (2). An $R$-module $C$ satisfies $\text{Hom}_R(R[s^{-1}], C) = 0$ if and only if it has no $s$-divisible submodules. This is equivalent to uniqueness of solutions of the equation system (1). The former condition is “naïve” and the latter one is its “well-behaved version”, in that the class of all $R$-modules without $s$-divisible
submodules is closed under extensions (see the proof of Theorem 1.2), while the class of $s$-separated $R$-modules is not (as the above Example 2.7 (1) demonstrates). For $s$-torsion-free modules, the two conditions are equivalent.

3. $s$-Power Infinite Summation Operations

Let $s$ be a formal symbol. We will say that an abelian group $C$ is endowed with an $s$-power infinite summation operation if for every sequence of elements $a_0, a_1, a_2, \ldots \in C$ an element denoted formally by

$$\sum_{n=0}^{\infty} s^n a_n \in C$$

is specified. The axioms of additivity

$$\sum_{n=0}^{\infty} s^n (a_n + b_n) = \sum_{n=0}^{\infty} s^n a_n + \sum_{n=0}^{\infty} s^n b_n$$

for any $(a_n, b_n \in C)_{n=0}^{\infty}$, contraunitality

$$\sum_{n=0}^{\infty} s^n a_n = a_0$$

when $a_1 = a_2 = a_3 = \cdots = 0$ in $C$, and contraassociativity

$$\sum_{i=0}^{\infty} s^i \left( \sum_{j=0}^{\infty} s^j a_{ij} \right) = \sum_{n=0}^{\infty} s^n \left( \sum_{i+j=n} a_{ij} \right)$$

for any $(a_{ij} \in C)_{i,j=0}^{\infty}$ have to be satisfied (cf. [26, Sections 1.3–1.4]).

Given an abelian group $C$ endowed with an $s$-power infinite summation operation, one defines an additive operator (abelian group endomorphism) $s: C \to C$ by the rule

$$sa = \sum_{i=0}^{\infty} s^i a_i,$$

where $a_1 = a$ and $a_i = 0$ for $i \neq 1$.

It follows from the contraassociativity axiom that the powers (iterated compositions) $s^n = s \circ \cdots \circ s$ of the endomorphism $s$ can be obtained as

$$s^n a = \sum_{i=0}^{\infty} s^i a_i,$$

where $a_n = a$ and $a_i = 0$ for $i \neq n$.

Examples 3.1. (1) For any abelian group $V$, the group of formal power series $V[[z]]$ is naturally endowed with a $z$-power infinite summation operation. The group of $p$-adic integers $\mathbb{Z}_p$ is naturally endowed with a $p$-power infinite summation operation. More generally, for any set $X$ the group $\prod_{x \in X} \mathbb{Z}_p$ is endowed with a $p$-power infinite summation operation. The subgroup $C = \mathbb{Z}_p[[\mathbb{Z}_{\geq 0}]] \subset \prod_{n=0}^{\infty} \mathbb{Z}_p$ of all sequences of $p$-adic integers converging to zero in the topology of $\mathbb{Z}_p$ is preserved by the $p$-power infinite summation operation in $\prod_{n=0}^{\infty} \mathbb{Z}_p$, hence also endowed with a $p$-power infinite summation operation. This generalizes to the case of the subgroup $\mathbb{Z}_p[[X]] \subset \prod_{x \in X} \mathbb{Z}_p$ consisting of all the families of elements $u_x \in \mathbb{Z}_p$, $x \in X$ such that for every $n \geq 0$ one has $u_x \in p^n \mathbb{Z}_p$ for all but a finite number of indices $x \in X$. In all these cases, the infinite sum can be computed as the limit of finite partial sums in the $s$-adic (i.e., $z$-adic or $p$-adic, resp.) topology of the group in question.
(2) For any two abelian groups $C$ and $D$ with $s$-power infinite summation operations and a group homomorphism $f: C \to D$ preserving the infinite summation operations, the groups $\ker(f)$ and $\coker(f)$ inherit the $s$-power infinite summation operations of $C$ and $D$. It follows that the category of abelian groups with $s$-power infinite summation operations is abelian. Furthermore, for any family of groups $C_{\alpha}$ with $s$-power infinite summation operations, there is a natural infinite summation operation on the infinite product $\prod_{\alpha} C_{\alpha}$.

(3) The construction of Example 2.7 (1) allows to show that there exists an abelian group $B$ with an $s$-power infinite summation operation and a sequence of elements $b_0, b_1, b_2, \ldots \in B$ such that $\sum_{n=0}^{\infty} s^n b_n \neq 0$ in $B$, but $s^n b_n = 0$ for every $n \geq 0$. Indeed, set $B = C/E$, and let $b_n = c_n + E$ be the coset of the sequence $c_n = (u_0, u_1, u_2, \ldots)$ with $u_0 = 0$, $u_1 = 0$, $u_2 = 0$, $u_3 = 0$, $u_4 = 0$, $u_5 = 0$, $u_6 = 0$, $u_7 = 0$. Then $p^c c_n \in E$, but $\sum_{n=0}^{\infty} p^n c_n \notin E$, because the sequence $u_0 = 0$, $u_1 = 1$, $u_2 = p$, $u_3 = p^2$, $u_4 = p^3$, $u_5 = p^4$ does not have the form $u_n = p^n v_n$ with $v_n \to 0$ in $\mathbb{Z}_p$ as $n \to \infty$. This counterexample shows that the $s$-power infinite summation operations, generally speaking, cannot be interpreted as any kind of limit of finite partial sums.

**Lemma 3.2.** Let $C$ be an abelian group endowed with an $s$-power infinite summation operation and $D \subset C$ be a subgroup for which $sD = D$. Then $D = 0$.

**Proof.** Let $a_0, a_1, a_2, \ldots$ be a sequence of elements in $C$ satisfying $a_n = sa_{n+1}$ for every $n \geq 0$. Our aim is to show that $a_n = 0$ for all $n \geq 0$. The idea is to consider the element $\sum_{n=0}^{\infty} s^n a_n$ and perform the transformations

$$\sum_{n=0}^{\infty} s^n a_n = \sum_{n=0}^{\infty} s^n sa_{n+1} = \sum_{n=0}^{\infty} s^{n+1} a_{n+1} = \sum_{n=1}^{\infty} s^n a_n,$$

implying $a_0 = 0$ (which is clearly sufficient). To do it more rigorously, consider the double-indexed array of elements $a_{ij} = a_{i+1}$ for $i \geq 0$, $j = 1$ and $a_{ij} = 0$ for other values of $i$, $j$, and apply the contraassociativity axiom,

$$\sum_{i=0}^{\infty} s^i a_i = \sum_{i=0}^{\infty} s^i \left( \sum_{j=0}^{\infty} s^j a_{ij} \right) = \sum_{n=0}^{\infty} s^n \left( \sum_{i+j=n} a_{ij} \right) = \sum_{n=0}^{\infty} s^n a'_n,$$

where $a'_0 = 0$ and $a'_n = a_n$ for $n \geq 1$. Using the additivity and contraunitality axioms allows to deduce the desired equation $a_0 = 0$. \hfill $\square$

**Theorem 3.3.** (a) An $s$-power infinite summation operation on an abelian group $C$ is uniquely determined by the endomorphism $s: C \to C$. In other words, given an abelian group $C$ with an additive operator $s: C \to C$, there exists at most one $s$-power infinite summation operation structure on $C$ restricting to the prescribed action of the operator $s$ in $C$.

(b) Given two abelian groups $C$ and $D$ endowed with $s$-power infinite summation operations, an abelian group homomorphism $f: C \to D$ preserves the $s$-power infinite summation operations if and only if it commutes with the endomorphisms $s$ on $C$ and $D$, i.e., $fs = sf$.

(c) An endomorphism $s: C \to C$ of an abelian group $C$ can be extended to an $s$-power infinite summation operation on $C$ if and only if one has $\text{Hom}_{\mathbb{Z}[s]}(\mathbb{Z}[s, s^{-1}], C) = 0 = \text{Ext}^1_{\mathbb{Z}[s]}(\mathbb{Z}[s, s^{-1}], C)$. 

12
In other words, the theorem says that the category of abelian groups with \( s \)-power infinite summation operations is equivalent (if one wishes, even isomorphic) to the category of \( s \)-contramodule \( \mathbb{Z}[s] \)-modules (cf. [24, Lemma B.5.1]).

**Proof.** Let \( C \) be an abelian group endowed with an \( s \)-power infinite summation operation. We already know from Lemma 3.2 that \( \text{Hom}_{\mathbb{Z}[s]}([\mathbb{Z}[s, s^{-1}], C] = 0 \); let us check that \( \text{Ext}_{\mathbb{Z}[s]}^{1}([\mathbb{Z}[s, s^{-1}], C] = 0 \). According to Lemma 2.1(a), we have to check that for any sequence of elements \( a_{0}, a_{1}, a_{2}, \ldots \in C \) the system of equations (1) can be solved in \( C \). Put

\[
(4) \quad b_{n} = \sum_{i=0}^{\infty} s^{i} a_{n+i}.
\]

We claim that (1) is satisfied. Indeed,

\[
b_{n} - s b_{n+1} = \sum_{i=0}^{\infty} s^{i} a_{n+i} - s \sum_{i=0}^{\infty} s^{i} a_{n+1+i} = \sum_{i=0}^{\infty} s^{i} a_{n+i} - \sum_{i=0}^{\infty} s^{i+1} a_{n+i+1} = \sum_{i=0}^{\infty} s^{i} a_{n+i} - \sum_{i=1}^{\infty} s^{i} a_{n+i} = a_{n}.
\]

To make a rigorous argument out of this calculation, one can, similarly to the above proof of Lemma 3.2, apply the contraassociativity axiom to the array of elements \( a_{ij} = a_{n+j+1} \) for \( i = 1, j \geq 0 \) and \( a_{ij} = 0 \) for other values of \( i \) and \( j \).

Conversely, let \( C \) be an \( s \)-contramodule \( \mathbb{Z}[s] \)-module; so the system of equations (1) is uniquely solvable in \( C \) for any sequence of elements \( (a_{n} \in C)_{n=0}^{\infty} \). Given such a sequence, solve the system (1) and put

\[
(5) \quad \sum_{n=0}^{\infty} s^{n} a_{n} = b_{0}.
\]

The additivity and contraunitality axioms being pretty straightforward, let us check the contraassociativity axiom for the infinite summation operation so defined. For this purpose, one has to compute the sum over the area \( i + j \geq n \), \( i, j \geq 0 \) as the sum of the sums over the rows. Let us start with solving the equations

\[
c_{i,m} - s c_{i,m+1} = a_{i,m}, \quad c_{i,m} \in C, \quad i, m \geq 0;
\]

so, according to our definition, \( c_{i,m} = \sum_{j=0}^{\infty} s^{j} a_{i,m+j} \) and, in particular, \( c_{i,0} = \sum_{j=0}^{\infty} s^{j} a_{i,j} \). Furthermore, solve the equations

\[
d_{n} - s d_{n+1} = c_{n,0}, \quad d_{n} \in C, \quad n \geq 0;
\]

so \( d_{n} = \sum_{i=0}^{\infty} s^{i} c_{n+i,0} \) and, in particular, \( d_{0} = \sum_{i=0}^{\infty} s^{i}(\sum_{j=0}^{\infty} s^{j} a_{ij}) \). Finally, set \( e_{n} = c_{0,n} + c_{1,n-1} + \cdots + c_{n-1,1} + d_{n} \); this is our sum over the area \( i + j \geq n \), \( i, j \geq 0 \). In particular, by the definition, we have \( e_{0} = d_{0} \). On the other hand,

\[
e_{n} - e_{n+1}
\]

\[
= (c_{0,n} - s c_{0,n+1}) + (c_{1,n-1} - s c_{1,n}) + \cdots + (c_{n-1,1} - s c_{n-1,2}) - s c_{n,1} + (d_{n} - s d_{n+1})
\]

\[
= a_{0,n} + a_{1,n-1} + \cdots + a_{n-1,1} - s c_{n,1} + a_{n,0}
\]

\[
= a_{0,n} + a_{1,n-1} + \cdots + a_{n-1,1} + a_{n,0},
\]

hence \( e_{0} = \sum_{n=0}^{\infty} s^{n}(\sum_{i+j=n} a_{ij}) \) and we are done.
We have shown that the infinite system of nonhomogeneous linear equations (1) is solvable by (4) in any abelian group $C$ with an $s$-power infinite summation operation. Moreover, the system (1) is uniquely solvable in $C$, as the related system of homogeneous linear equations has no nonzero solutions according to (the proof of) Lemma 3.2. Furthermore, solving the equations (1) allows to recover the $s$-power infinite summation operation in $C$ by the rule (5).

Therefore, the infinite summation operation structure with the prescribed map $s: C \rightarrow C$ is unique when it exists. Furthermore, any abelian group homomorphism $f: C \rightarrow D$ commuting with the operators $s$ takes solutions of the system (1) in $C$ to similar solutions in $D$, and consequently preserves the infinite summation operations. All the assertions of the theorem are now proved. □

4. $[s, t]$-Power Infinite Summation Operations

Let us now have two formal symbols $s$ and $t$. We say that an abelian group $C$ is endowed with an $[s, t]$-power infinite summation operation if for every array of elements $a_{mn} \in C$, $m, n \geq 0$, an element denoted formally by

$$\sum_{n=0}^{\infty} s^m t^n a_{mn} \in C$$

is defined. The axioms of additivity

$$\sum_{m,n=0}^{\infty} s^m t^n (a_{mn} + b_{mn}) = \sum_{m,n=0}^{\infty} s^m t^n a_{mn} + \sum_{m,n=0}^{\infty} s^m t^n b_{mn} \quad \text{for any } (a_{mn}, b_{mn} \in C)_{m,n=0}^{\infty},$$

contraunitality

$$\sum_{m,n=0}^{\infty} s^m t^n a_{mn} = a_{00} \quad \text{whenever } a_{mn} = 0 \text{ in } C \text{ for all } (m, n) \neq (0, 0),$$

and contraassociativity

$$\sum_{i,j=0}^{\infty} s^i t^j \left( \sum_{k,l=0}^{\infty} s^k t^l a_{ij,kl} \right) = \sum_{m,n=0}^{\infty} s^m t^n \left( \sum_{j+k=m} a_{ij,kl} \right) \quad \text{for any } (a_{ij,kl} \in C)_{i,j,k,l=0}^{\infty}$$

are imposed.

**Theorem 4.1.** The category of abelian groups $C$ endowed with an $[s, t]$-power infinite summation operation is isomorphic to the category of abelian groups $C$ endowed with a pair of commuting endomorphisms $s$, $t: C \rightarrow C$, $st = ts$, such that both the $s$-power and the $t$-power infinite summation operations exist in $C$.

In other words, an abelian group $C$ with an $[s, t]$-power infinite summation operation is the same thing as a $\mathbb{Z}[s, t]$-module satisfying

$$\text{Hom}_{\mathbb{Z}[s, t]}(\mathbb{Z}[s, s^{-1}, t] \oplus \mathbb{Z}[s, t, t^{-1}], P) = 0 = \text{Ext}^1_{\mathbb{Z}[s, t]}(\mathbb{Z}[s, s^{-1}, t] \oplus \mathbb{Z}[s, t, t^{-1}], P)$$
(cf. [24, Lemma B.6.1 and Theorem B.1.1]).

**Proof.** Given an abelian group $C$ with an $[s, t]$-power infinite summation operation, one defines an $s$-power infinite infinite summation operation and a $t$-power infinite summation operation on $C$ by the obvious rules

$$
\sum_{m=0}^{\infty} s^m a_m = \sum_{n=0}^{\infty} s^m t^n a_{m,n} \quad \text{if } a_{m,0} = a_m \text{ and } a_{m,n} = 0 \text{ for } n > 0,
$$

$$
\sum_{n=0}^{\infty} t^n a_n = \sum_{m=0}^{\infty} s^m t^n a_{m,n} \quad \text{if } a_{0,n} = a_n \text{ and } a_{m,n} = 0 \text{ for } m > 0.
$$

(6)

Specializing further to arrays $(a_{m,n})_{m,n=0}^{\infty}$ with the only nonzero component $a_{1,0}$ or $a_{0,1}$, one obtains the additive operators $s: C \to C$ and $t: C \to C$ (as explained in Section 3). Applying the contraassociativity axiom to the arrays $(a_{ij,kl})$ with the only nonzero component $a_{1,0,1,0}$ or $a_{0,1,1,0}$, one proves that $st = ts$.

Conversely, suppose that $s$-power infinite summation operations and $t$-power infinite summation operations are defined in $C$, and the operators $s$ and $t$ in $C$ commute. Then one can define the $[s, t]$-power infinite summation operation on $C$ as

$$
\sum_{m,n=0}^{\infty} s^m t^n a_{m,n} = \sum_{m=0}^{\infty} s^m \left( \sum_{n=0}^{\infty} t^n a_{m,n} \right).
$$

(7)

This obviously satisfies the additivity and contraunitality axioms. Checking the contraassociativity axiom,

$$
\sum_{i,j=0}^{\infty} s^i t^j \left( \sum_{k,l=0}^{\infty} s^k t^l a_{ij,kl} \right) = \sum_{i=0}^{\infty} s^i \left( \sum_{j=0}^{\infty} t^j \left( \sum_{k=0}^{\infty} s^k \left( \sum_{l=0}^{\infty} t^l a_{ij,kl} \right) \right) \right)
$$

$$
= \sum_{i=0}^{\infty} s^i \left( \sum_{k=0}^{\infty} s^k \left( \sum_{j=0}^{\infty} t^j \left( \sum_{l=0}^{\infty} t^l a_{ij,kl} \right) \right) \right)
$$

$$
= \sum_{m=0}^{\infty} s^m \left( \sum_{i+k=m}^{\infty} \left( \sum_{n=0}^{\infty} t^n \left( \sum_{j+l=n}^{\infty} a_{ij,kl} \right) \right) \right)
$$

$$
= \sum_{m,n=0}^{\infty} s^m t^n \left( \sum_{i+k=m}^{\infty} a_{ij,kl} \right),
$$

reduces to showing that the two infinite summation operations commute with each other, that is

$$
\sum_{j=0}^{\infty} t^j \left( \sum_{k=0}^{\infty} s^k a_{jk} \right) = \sum_{k=0}^{\infty} s^k \left( \sum_{j=0}^{\infty} t^j a_{jk} \right) \quad \text{for any } (a_{jk} \in C)_{j,k=0}^{\infty}.
$$

(8)

To prove (8), one can do a computation with infinite systems of nonhomogeneous linear equations similar to the one in the proof of Theorem 3.3, based on considering the sums over the areas $j \geq n$, $k \geq m$. Alternatively, there is the following conceptual argument based on the assertion of Theorem 3.3(b). The $t$-power infinite summation is a group homomorphism

$$
\sum_t = \left( (a_j)_{j=0}^{\infty} \mapsto \sum_{j=0}^{\infty} t^j a_j \right): \prod_{j=0}^{\infty} C \to C.
$$

There is a natural (termwise) $s$-power infinite summation operation in the group $\prod_{j=0}^{\infty} C$ (see Example 3.1 (2)). The equation (8) says that $\sum_t$ is a morphism of groups with $s$-power infinite summation operations. According to Theorem 3.3(b),
it suffices to show that \( \sum_t \) commutes with the operators \( s \) in \( \prod_{j=0}^{\infty} C \) and \( C \), which means the equation

\[
\sum_{j=0}^{\infty} t^j(sa_j) = s \sum_{j=0}^{\infty} t^j a_j \quad \text{for any } (a_j \in C)_{j=0}^{\infty}.
\]

Now, the equation (9) is also equivalent to the assertion that \( s : C \to C \) is a morphism of groups with \( t \)-power infinite summation operations. Applying Theorem 3.3(b) again, we finally conclude that this follows from \( st = ts \).

We have constructed functors in both directions between the two categories in question; it remains to check that their compositions are the identity functors. Here the essential part is to check that, for an abelian group \( C \) with an \([s, t]\)-power infinite summation operation and its restrictions to an \( s \)-power and a \( t \)-power infinite summation operations provided by (6), the equation (7) holds. For this purpose, it suffices to apply the contraassociativity axiom to the array \((a_{ij,kl})_{i,j,k,l=0}^{\infty} \) with \( a_{m,0,0,n} = a_{mn} \) for all \( m, n \geq 0 \) and \( a_{ij,kl} = 0 \) when \( j > 0 \) or \( k > 0 \).

The following lemma, extending Lemma 3.2 to the case of two variables, belongs to the class of results known as the “contramodule Nakayama lemma”. A number of other versions can be found in the literature [22, Corollary 0.3], [23, Lemma A.2.1], [24, Lemma 1.3.1], [25, Lemma D.1.2], [29, Lemma 6.14].

**Lemma 4.2.** Let \( C \) be an abelian group endowed with an \([s, t]\)-power infinite summation operation and \( D \subset C \) a subgroup such that \( sD + tD = D \). Then \( D = 0 \).

**Proof.** Let \( a_0 \in D \) be an element. Choose a pair of elements \( b_0' \) and \( b_0'' \in D \) such that \( a_0 = sb_0' + tb_0'' \). Set \( a_{1,0} = b_0' \) and \( a_{0,1} = b_0'' \). Choose two pairs of elements \( b_{1,0}', b_{1,0}'', b_{0,1}', b_{0,1}'' \in D \) such that \( a_{1,0} = sb_{1,0}' + tb_{1,0}'' \) and \( a_{0,1} = sb_{0,1}' + tb_{0,1}'' \). Set \( a_{2,0} = b_{1,0}' \), \( a_{1,1} = b_{0,1}' + b_{0,1}'' \), and \( a_{0,2} = b_{0,1}' \), etc. Proceeding by induction in \( n \geq 0 \), we choose for every \( i, j \geq 0, \ i + j = n \), a pair of elements \( b_{ij}' \) and \( b_{ij}'' \in D \) such that

\[
a_{ij} = sb_{ij}' + tb_{ij}''.
\]

Then we set \( b_{n+1,-1}' = b_{-1,n+1}' = 0 \) and

\[
a_{i,j} = b_{i-1,j}' + b_{i,j-1}' \quad \text{for all } i, j \geq 0, \ i + j = n + 1.
\]

Now we have

\[
\sum_{i,j=0}^{\infty} s^i t^j a_{ij} = \sum_{i,j=0}^{\infty} s^i t^j (sb_{ij}' + tb_{ij}'') = \sum_{i \geq 1, j \geq 0} s^i t^j b_{i-1,j}' + \sum_{i \geq 0, j \geq 1} s^i t^j b_{i,j-1}'' = \sum_{i \geq 0, j \geq 0} s^i t^j (b_{i-1,j}' + b_{i,j-1}'') = \sum_{i \geq 0, j \geq 0} s^i t^j a_{ij},
\]

hence \( a_0 = 0 \). \( \square \)

**Remark 4.3.** Similarly to the above exposition in this section, one can define the notion of an \([s_1, \ldots, s_m]\)-power infinite summation operation in an abelian group \( C \) for any \( m \geq 1 \). Then one can prove that the category of abelian groups with
[s_1,\ldots,s_m]-power infinite summation operations is isomorphic to the category of
abelian groups $C$ with $m$ pairwise commuting endomorphisms $s_j: C \to C$, $1 \leq j \leq m$ such that for every $j$ there exists an $s_j$-power infinite summation operation structure on $C$ restricting to the prescribed endomorphism $s_j$. Furthermore, if $C$ is an abelian group with an $[s_1,\ldots,s_m]$-power infinite summation operation and $D \subset C$ a subgroup for which $D \subset s_1D + \cdots + s_mD$, then $D = 0$.

5. $[s_1,\ldots,s_m]$-Contraadjustedness, $I$-Contramoduleness, and $I$-Adic Completeness

One of the aims of this paper is to discuss the following theorem.

**Theorem 5.1.** Let $R$ be a commutative ring, $I \subset R$ an ideal generated by some elements $s_j \in R$, and $C$ an $R$-module. Assume that $C$ is an $s_j$-contramodule for every $j$. Then $C$ is an $s$-contramodule for every $s \in I$.

**Remark 5.2.** Notice that the analogue of the assertion of Theorem 5.1 is *not* true for $s$-contraadjusted modules. E. g., any $R$-module is 1-contraadjusted, but it does not have to be $s$-contraadjusted for other elements $s \in R$. Or, to give another example, the abelian group $\mathbb{Z}[[\frac{1}{2},\frac{1}{3}]]$ is 2-contraadjusted and 3-contraadjusted, but not 5-contraadjusted (as one can readily check using Theorem 2.3).

Two proofs of Theorem 5.1 are given in this paper (cf. [27, Section 2]). One of them, based on the technique of infinite summation operations developed in Sections 3–4, is presented immediately below. The other one is postponed to Sections 6–7, because it uses an explicit construction of the functor $\Delta$ left adjoint to the embedding of the category of contramodules into $R\text{-mod}$, which will be introduced there.

Yet another proof can be found in [38, Theorem 5 and Lemma 7 (1)]).

**First proof of Theorem 5.1.** Given a commutative ring $R$ and an $R$-module $C$, denote by $I_C$ the set of all elements $s \in R$ for which $C$ is an $s$-contramodule. We will show that $I_C$ is an ideal in $R$.

**Lemma 5.3.** Let $C$ be an abelian group and $r, s: C \to C$ be two commuting endomorphisms of $C$. Assume that $C$ admits an $s$-power infinite summation operation. Then an $(rs)$-power infinite summation operation also exists in $C$.

**Proof.** Set

$$\sum_{n=0}^{\infty} (rs)^n a_n = \sum_{n=0}^{\infty} s^n (r^n a_n)$$

for any $(a_n \in C)_{n=0}^{\infty}$.

One will have to use Theorem 3.3(b) (the commutativity of $r$ with the $s$-power infinite summation operation) in order to check the contraassociativity axiom for the $(rs)$-power infinite summation operation so defined. \qed
Lemma 5.4. Let $C$ be an abelian group and $s, t : C \to C$ be two commuting endomorphisms of it. Assume that $C$ admits an $s$-power infinite summation operation and a $t$-power infinite summation operation. Then $C$ also admits an $(s + t)$-power infinite summation operation.

Proof. According to Theorem 4.1, there is an $[s, t]$-power infinite summation operation in $C$. So we can put
\[
\sum_{n=0}^{\infty} (s + t)^n a_n = \sum_{i,j=0}^{\infty} s^i t^j \left( \binom{i+j}{i} a_{i+j} \right) \quad \text{for any } (a_n \in C)^{\infty}_{n=0}.
\]
We leave it to the reader to check the axioms. □

In view of Theorem 3.3(c), it follows from Lemmas 5.3–5.4 that $I_C$ is an ideal in $R$. The first proof of Theorem 5.1 is finished. □

Remark 5.5. Notice that the property of an $R$-module $C$ to be an $s$-contramodule for some element $s \in R$ does not depend on the $R$-module structure on $C$, but only on the abelian group $C$ with the endomorphism $s$, as it is clear from Lemma 2.1. So we can choose the ring $R$ as we find convenient. Furthermore, for any $R$-module $C$ the ideal $I_C \subset R$ is a radical ideal in $R$, that is, for any $s \in R$ and $n \geq 1$ such that $s^n \in I_C$ one has $s \in I_C$. In other words, if $s : C \to C$ is an endomorphism of an abelian group $C$ such that $C$ admits an $s^n$-power infinite summation operation, then $C$ also admits an $s$-power infinite summation operation. Indeed, the two related localizations of the ring $R$ coincide, $R[s^{-1}] = R[(s^n)^{-1}]$, so the $s$-contramodule and $s^n$-contramodule properties are equivalent.

Let $R$ be a commutative ring and $I$ an ideal in $R$. We denote the $I$-adic completion functor by
\[
C \mapsto \Lambda_I(C) = \lim_{\leftarrow n \geq 1} C/I^n C.
\]
An $R$-module $C$ is called $I$-adically complete if the natural map
\[
\lambda_{I,C} : C \longrightarrow \Lambda_I(C)
\]
is surjective. The $R$-module $C$ is called $I$-adically separated if the map $\lambda_{I,C}$ is injective, i.e., if the intersection $\ker(\lambda_{I,C}) = \bigcap_{n \geq 1} I^n C$ vanishes.

The following result generalizes Theorem 2.3(a) (by replacing a principal ideal with a finitely generated one) and improves upon [38, Theorem 10] (by removing the irrelevant separatedness assumption).

Theorem 5.6. Let $R$ be a commutative ring and $I \subset R$ be the ideal generated by a finite set of elements $s_1, \ldots, s_m \in R$. Assume that an $R$-module $C$ is $s_j$-contradjusted for every $j = 1, \ldots, m$. Then the $R$-module $C$ is $I$-adically complete.

Proof. The idea is that $I$-adic completeness can be thought of in the language of $[s_1, \ldots, s_m]$-power infinite sums and such infinite sums obtained in terms of solutions of the equations (1), even if those solutions are not unique and the infinite sums accordingly only ambiguously defined. Specifically, let $(c_n \in C)^{\infty}_{n=1}$ be a sequence
of elements satisfying \( c_{n+1} - c_n \in I^nC \) for all \( n \geq 1 \). Then there exist elements \( a_{n_1, \ldots, n_m} \in C \), \( n_1, \ldots, n_m \geq 0 \), \( n_1 + \cdots + n_m \geq 1 \) such that

\[
c_{n+1} - c_n = \sum_{n_1 \geq 0, \ldots, n_m \geq 0} s_1^{n_1} \cdots s_m^{n_m} a_{n_1, \ldots, n_m} \quad \text{for all } n \geq 1.
\]

Set \( a_{0, \ldots, 0} = c_1 \). Applying Lemma 2.1(a) to the element \( s_m \in R \) and the \( R \)-module \( C \), for every \( n_1, \ldots, n_m \geq 0 \) choose elements \( b_{n_1, \ldots, n_m-1;k}^{(1)} \in C \), \( k \geq 0 \) such that

\[
b_{n_1, \ldots, n_m-1;k}^{(1)} - b_{n_1, \ldots, n_m-1;k+1}^{(1)} = a_{n_1, \ldots, n_m-1,k} \quad \text{for all } n_1, \ldots, n_m-1 \geq 0, k \geq 0.
\]

Proceeding by decreasing induction in \( j \leq m-1 \) and applying Lemma 2.1(a) to \( s_j \in R \) and the module \( C \), for every \( n_1, \ldots, n_{j-1} \geq 0 \) choose elements \( b_{n_1, \ldots, n_{j-1};k}^{(m-j+1)} \in C \), \( k \geq 0 \) such that

\[
b_{n_1, \ldots, n_{j-1};k}^{(m-j+1)} - b_{n_1, \ldots, n_{j-1};k+1}^{(m-j+1)} = b_{n_1, \ldots, n_{j-1},k;0}^{(m-j)} \quad \text{for all } n_1, \ldots, n_{j-1} \geq 0, k \geq 0.
\]

Eventually, for \( j = 2 \) we will obtain

\[
b_{n_1;k}^{(m-1)} - b_{n_1;k+1}^{(m-1)} = b_{n_1;k;0}^{(m-2)} \quad \text{for all } n_1 \geq 0, k \geq 0,
\]

and for \( j = 1 \),

\[
b_{k}^{(m)} - b_{k+1}^{(m)} = b_{k;0}^{(m-1)} \quad \text{for all } k \geq 0.
\]

Set \( b = b_0^{(m)} \). We claim that \( b - c_n \in I^nC \) for all \( n \geq 1 \); so the element \( b \in C \) is a preimage of the element \( c \in \varprojlim C/I^nC \) represented by \( (c_n)_{n \geq 1} \) under the natural map \( \lambda_{I,C} : C \to \varprojlim C/I^nC \). Indeed,

\[
b_0^{(m)} = b_{0;0}^{(m-1)} + b_1^{(m-1)} = b_{0;0}^{(m-1)} + s_1 b_2^{(m-1)} + \cdots = \sum_{n_1=0}^{n-1} s_1^{n_1} b_{n_1;0}^{(m-1)} + b_n^{(m)} = \cdots
\]

\[
= \sum_{n_1+\cdots+n_{m-1}<n} s_1^{n_1+\cdots+n_{m-1}} b_{n_1,\ldots, n_{m-1};0}^{(1)} + s^n \sum_{n_1=0}^{n-1} b_{n_1;0}^{(m-1)} + b_n^{(m)} = \cdots
\]

\[
= \sum_{n_1+\cdots+n_{m-1}<n} s_1^{n_1+\cdots+n_{m-1}} a_{n_1,\ldots, n_{m-1}} + s^n \sum_{j=1}^{m} \left( \sum_{n_1+\cdots+n_{m-j}+k=n} b_{n_1,\ldots, n_{m-j};k}^{(j)} \right)
\]

\[
= c_n + s^n \sum_{j=1}^{m} \left( \sum_{n_1+\cdots+n_{m-j}+k=n} b_{n_1,\ldots, n_{m-j};k}^{(j)} \right).
\]

To explain the last step without long formulas, one can argue as follows. Reducing solutions of the equation systems (1) in \( C \) modulo \( I^nC \), one obtains solutions of the same equation systems (1) in \( C/I^nC \). For any \( s \in I \), the equation systems (1) are uniquely solvable in \( C/I^nC \) (e. g., because \( C/I^nC \) is \( s \)-separated and \( s \)-complete, so
Lemma 5.7. Let the solutions of (1) in \( C/I \) can be expressed in the form (4) (where the sum is actually finite, as the action of \( s \) is nilpotent). Therefore, we have

\[
b_0^{(m)} = \sum_{n_1=0}^{n-1} s^{n_1} b_{n_1;0}^{(m-1)} = \sum_{n_1,n_2=0}^{n-1} s^{n_1+n_2} b_{n_1,n_2;0}^{(m-2)} = \ldots
\]

\[
= \sum_{n_1,\ldots,n_m-1=0}^{n-1} s^{n_1+\ldots+n_m-1} b_{n_1,\ldots,n_m-1;0}^{(1)} = \sum_{n_1,\ldots,n_m=0}^{n-1} s^{n_1+\ldots+n_m} a_{n_1,\ldots,n_m}
\]

\[
\equiv c_n \pmod{I^n C}.
\]

For a stronger version of Theorem 5.6, see Remark 7.6 below.

Lemma 5.7. Let \( R \) be a commutative ring, \( I \subset R \) an ideal, and \( C \) an \( R \)-module. Then the \( R \)-module \( \Lambda_I(C) = \lim_{\leftarrow n \geq 1} C/I^n C \) is an \( s \)-contramodule for every element \( s \in I \). In particular, any \( I \)-adically separated and complete \( R \)-module is an \( s \)-contramodule.

Proof. Any \( R \)-module in which \( s \) acts nilpotently is an \( s \)-contramodule, and the projective limit of any diagram of \( s \)-contramodule \( R \)-modules is an \( s \)-contramodule \( R \)-module (see Remark 2.5). Alternatively, one can define an \( s \)-power infinite summation operation in \( \lim_{\leftarrow n} C/I^n C \) (see Theorem 3.3(c)) as the limit of finite partial sums in the topology of the projective limit (of discrete groups \( C/I^n C \)) on \( \lim_{\leftarrow n} C/I^n C \), where the kernels of the projection maps \( \lim_{\leftarrow n} C/I^n C \to C/I^n C \) form a base of neighborhoods of zero.

Let us denote the full subcategory of \( I \)-adically separated and complete \( R \)-modules by \( R\text{-mod}_{I,\text{secmp}} \subset R\text{-mod} \).

Theorem 5.8. Let \( R \) be a commutative ring and \( I \subset R \) a finitely generated ideal. Then the functor of \( I \)-adic completion \( \Lambda_I : R\text{-mod} \to R\text{-mod}_{I,\text{secmp}} \) is left adjoint to the fully faithful embedding functor \( R\text{-mod}_{I,\text{secmp}} \to R\text{-mod} \).

Proof. The formulation of the theorem tacitly includes the claim that the functor \( \Lambda_I \) takes values in \( R\text{-mod}_{I,\text{secmp}} \), i.e., the \( I \)-adic completion of any \( R \)-module is \( I \)-adically separated and complete. This is the result of [37, Corollary 3.6], which is not true without the assumption that \( I \) is finitely generated [37, Example 1.8]. The point is that the projective limit topology is, by the definition, always complete on \( \Lambda_I(C) = \lim_{\leftarrow n} C/I^n C \), but the \( I \)-adic topology on \( \Lambda_I(C) \) can differ from the projective limit topology (see [37, Example 1.13 and the preceding discussion]).

The two topologies are the same when the ideal \( I \) is finitely generated. A more precise claim, which is of key importance here, is that the submodule \( I^n \Lambda_I(C) \) coincides with the kernel \( K_n \) of the natural projection \( \Lambda_I(C) \to C/I^n C \). One obviously has \( I^n \Lambda_I(C) \subset K_n \), so \( \Lambda_I(C) \) is always \( I \)-adically separated. Furthermore, when \( I \) is finitely generated, combining the results of Lemma 5.7 and Theorem 5.6 proves that \( \Lambda_I(C) \) is \( I \)-adically complete. Then one can apply [37, Theorem 1.5] in order to deduce the assertion that \( K_n = I^n \Lambda_I(C) \).
An explicit proof of the equation \( K_n = I^n \Lambda_I(C) \) based on the infinite summation operation technique would look as follows. Let \( c_i \in C, i \geq 1 \) be a sequence of elements such that \( c_{i+1} - c_i \in I^nC \) for \( i \geq 1 \) and \( c_0 = 0 \) for \( i \leq n \). Let \( s_1, \ldots, s_m \) be some set of generators of the ideal \( I \). Arguing as in the proof of Theorem 5.6, there are elements \( a_{i_1, \ldots, i_m} \in C, \ i_1, \ldots, i_m \geq 0 \) such that \( a_0, \ldots, 0 = c_1 \) and

\[
c_{i+1} - c_i = \sum_{i_1+\cdots+i_m=i} s_1^{i_1} \cdots s_m^{i_m} a_{i_1, \ldots, i_m} \quad \text{for all } i \geq 1.
\]

In addition, we can assume that \( a_{i_1, \ldots, i_m} = 0 \) whenever \( i_1 + \cdots + i_m < n \). Following Lemma 5.7 and Remark 4.3, or just using directly the limits of finite partial sums in the projective limit topology on \( \lim_{\leftarrow} C/I^nC \), we have an \([s_1, \ldots, s_m]\)-power infinite summation operation in \( \Lambda_I(C) \). Hence the element \( b \in \lim_{\leftarrow} C/I^nC \) represented by the sequence \((c_i)_{i=1}^\infty\) can be expressed as

\[
b = \sum_{i_1, \ldots, i_m=0}^\infty s_1^{i_1} \cdots s_m^{i_m} a_{i_1, \ldots, i_m},
\]

where the images of the elements \( a_{i_1, \ldots, i_m} \) in \( \lim_{\leftarrow} C/I^nC \) are denoted for simplicity also by \( a_{i_1, \ldots, i_m} \). Finally, it is a standard exercise in combinatorics to rewrite this infinite sum as a finite sum of \([s_1, \ldots, s_j]\)-power infinite sums, \( 1 \leq j \leq m \), each of which is divisible by a certain monomial of degree \( n \) in \( s_1, \ldots, s_m \).

Now we can show that the two functors are adjoint. Let \( D = \lim_{\leftarrow} D/I^nD \) be an \( I \)-adically separated and complete \( R \)-module, and let \( C \) be an arbitrary \( R \)-module. Then \( R \)-module homomorphisms \( C \to D \) correspond bijectively to morphisms of projective systems of \( R \)-modules \( (C/I^nC)_{n \geq 1} \to (D/I^nD)_{n \geq 1} \), since any \( R \)-module homomorphism \( C \to D/I^nD \) factorizes through the surjection \( C \to C/I^nC \) in a unique way. In particular, \( R \)-module homomorphisms \( \Lambda_I(C) \to D \) correspond bijectively to morphisms of projective systems \( (\Lambda_I(C)/I^n\Lambda_I(C))_n \to (D/I^nD)_n \). But we have just seen that \( \Lambda_I(C)/I^n\Lambda_I(C) = \Lambda_I(C)/K_n = C/I^nC \). Hence the desired bijection \( \text{Hom}_R(\Lambda_I(C), D) = \text{Hom}_R(C, D) \). \( \square \)

**Remark 5.9.** The category \( R\text{-mod}_{I, \text{secmp}} \) is “naïve” and not well-behaved (see Example 2.7 (1)); and so the functor \( \Lambda_I \) is “naïve” and not well-behaved outside of the classical situation of finitely-generated modules over Noetherian rings (when the Artin–Rees lemma is applicable). In fact, for \( R = \mathbb{Z} \) and \( I = p\mathbb{Z} \), applying the \( p \)-adic completion functor \( \Lambda_p \) to the embedding \( D/E \to C/E \) from Example 2.7 (1) produces the zero map \( D/E \to C/D \), while applying the functor \( \Lambda_p \) to the short exact sequence \( 0 \to E \to C \to C/E \to 0 \) one obtains the sequence \( 0 \to E \to C \to C/D \to 0 \), which is not exact at the middle term [37, Example 3.20]. So, as pointed out in [37, Proposition 1.2], preserving surjections seems to be the only good property of the functor \( \Lambda_I \) in general. When the ideal \( I \) is finitely generated, one can also say that \( \Lambda_I \) preserves infinite products and is a reflector onto its image (hence idempotent); but that is about it.

The explanation is that the functor \( \Lambda_I \) is constructed by composing the right exact functor of reduction modulo \( I^n \) with the left exact functor of projective limit.
Composing functors that are exact from different sides is not generally a good procedure. Composing the derived functors between derived categories (and then taking the degree 0 cohomology if needed) is advisable instead. In the situation at hand, this would mean applying the functor $\Lambda I$ to a flat resolution of a given $R$-module [21], which works well for many, but not all, finitely generated ideals $I$ (cf. the counterexample in [27, Example 2.6]). The “well-behaved version” of the functor $\Lambda I$, denoted by $\Delta I$, will be discussed in the next two sections.

6. $s$-Torsion Modules and the Functor $\Gamma_s$
$s$-Contramodules and the Functor $\Delta_s$

The fundamental idea of covariant duality between torsion modules and contramodules, which lurked beneath the surface of our exposition before, becomes explicitly utilized in this and the next section. Let $R$ be a commutative ring and $s \in R$ an element. An $R$-module $M$ is said to be $s$-torsion if for each element $x \in M$ there exists an integer $n \geq 1$ such that $s^nx = 0$ in $M$. Equivalently, one can say that an $R$-module $M$ is $s$-torsion if and only if $R[s^{-1}] \otimes_R M = 0$.

The following lemma provides one of the simplest illustrations of the torsion module-contramodule duality.

**Lemma 6.1.** (a) Let $N$ and $M$ be $R$-modules. Then the tensor product $R$-module $N \otimes_R M$ is $s$-torsion whenever either $N$, or $M$ is $s$-torsion.

(b) Let $M$ and $C$ be $R$-modules. Then the $\text{Hom}_R$-module $\text{Hom}_R(M,C)$ is an $s$-contramodule whenever either $M$ is $s$-torsion, or $C$ is an $s$-contramodule.

**Proof.** Part (a): one has

$$R[s^{-1}] \otimes_R (N \otimes_R M) = (R[s^{-1}] \otimes R N) \otimes_R M,$$

hence $R[s^{-1}] \otimes_R N = 0$ implies $R[s^{-1}] \otimes_R (N \otimes_R M) = 0$.

Part (b): denoting the derived functor of $R$-module homomorphisms, viewed as a functor between the derived categories of $R$-modules, by $\mathbb{R}\text{Hom}_R$, one has

$$\text{Hom}_R(R[s^{-1}], \text{Hom}_R(M,C)) = H^0(\mathbb{R}\text{Hom}_R(R[s^{-1}], \mathbb{R}\text{Hom}_R(M,C))),$$

$$\text{Ext}^1_R(R[s^{-1}], \text{Hom}_R(M,C)) \subseteq H^1(\mathbb{R}\text{Hom}_R(R[s^{-1}], \mathbb{R}\text{Hom}_R(M,C))).$$

Hence from

$$\mathbb{R}\text{Hom}_R(R[s^{-1}], \mathbb{R}\text{Hom}_R(M,C)) = \mathbb{R}\text{Hom}_R(R[s^{-1}] \otimes_R M, C) = \mathbb{R}\text{Hom}_R(M, \mathbb{R}\text{Hom}_R(R[s^{-1}], C))$$

we can conclude that either $R[s^{-1}] \otimes_R M = 0$ or $\text{Ext}^*_R(R[s^{-1}], \text{Hom}_R(M,C)) = 0$ is sufficient to imply $\text{Ext}^*_R(R[s^{-1}], \text{Hom}_R(M,C)) = 0$.

Alternatively, one can define an $s$-power infinite summation operation on the $R$-module $\text{Hom}_R(M,C)$ by the rules

$$(\sum_{n=0}^{\infty} s^n f_n)(x) = \sum_{n=0}^{\infty} f_n(s^n x)$$
for all $f_n \in \text{Hom}_R(M, C)$, $x \in M$ if $M$ is $s$-torsion (so the sum in the right-hand side is finite, because $s^n x = 0$ for large enough $n$), or
\[
(\sum_{n=0}^{\infty} s^n f_n)(x) = \sum_{n=0}^{\infty} s^n (f_n(x))
\]
if $C$ is an $s$-contramodule (so the sum in the right-hand side refers to the $s$-power infinite summation operation on $C$).

The next lemma is a generalization of the previous one that will be useful in Section 13. In its context, the derived tensor product $N^\bullet \otimes_R^L M^\bullet$ of unbounded complexes is computed, as usually in the unbounded derived category of modules, using homotopy flat resolutions, while the $R$-modules of homomorphisms in the unbounded derived category $\text{Hom}_{D(R\text{-mod})}(M^\bullet, C^\bullet[n]) = H^n(\mathbb{R}\text{Hom}_R(M^\bullet, C^\bullet))$ can be computed using homotopy projective or homotopy injective resolutions.

**Lemma 6.2.** (a) Let $N^\bullet$ and $M^\bullet$ be two complexes of $R$-modules such that either the modules $H^n(N)$ are $s$-torsion for all $n \in \mathbb{Z}$, or the modules $H^n(M)$ are $s$-torsion for all $n \in \mathbb{Z}$. Then the derived tensor product modules $H^n(N^\bullet \otimes_R^L M^\bullet)$ are $s$-torsion for all $n \in \mathbb{Z}$.

(b) Let $M^\bullet$ and $C^\bullet$ be two complexes of $R$-modules such that either the modules $H^n(M)$ are $s$-torsion for all $n \in \mathbb{Z}$, or the modules $H^n(C)$ are $s$-contramodules for all $n \in \mathbb{Z}$. Then the $R$-modules $\text{Hom}_{D(R\text{-mod})}(M^\bullet, C^\bullet[n])$ are $s$-contramodules for all $n \in \mathbb{Z}$.

**Proof.** Part (a): the $R$-module $R[s^{-1}]$ is flat, so given a complex of $R$-modules $L^\bullet$, the $R$-modules $H^n(L^\bullet)$ are $s$-torsion for all $n \in \mathbb{Z}$ if and only if the complex $R[s^{-1}] \otimes_R L^\bullet$ is acyclic. Now one has
\[
R[s^{-1}] \otimes_R (N^\bullet \otimes_R^L M^\bullet) = (R[s^{-1}] \otimes_R N^\bullet) \otimes_R^L M^\bullet,
\]
hence acyclicity of the complex $R[s^{-1}] \otimes_R N^\bullet$ implies acyclicity of the complex $R[s^{-1}] \otimes_R (N^\bullet \otimes_R^L M^\bullet)$.

Part (b): the $R$-module $R[s^{-1}]$ has projective dimension at most 1, so for any complex of $R$-modules $B^\bullet$ there are short exact sequences
\[
0 \longrightarrow \text{Ext}_R^1(R[s^{-1}], H^{n-1}(B^\bullet)) \longrightarrow H^n(\mathbb{R}\text{Hom}_R(R[s^{-1}], B^\bullet)) \longrightarrow \text{Hom}_R(R[s^{-1}], H^n(B^\bullet)) \longrightarrow 0.
\]
Therefore, the $R$-modules $H^n(B^\bullet)$ are $s$-contramodules for all $n \in \mathbb{Z}$ if and only if $\mathbb{R}\text{Hom}_R(R[s^{-1}], B^\bullet) = 0$ in $D(R\text{-mod})$. Now one has
\[
\mathbb{R}\text{Hom}_R(R[s^{-1}], \mathbb{R}\text{Hom}_R(M^\bullet, C^\bullet)) = \mathbb{R}\text{Hom}_R(R[s^{-1}] \otimes_R M^\bullet, C^\bullet)
= \mathbb{R}\text{Hom}_R(M^\bullet, \mathbb{R}\text{Hom}_R(R[s^{-1}], C^\bullet)),
\]
hence acyclicity of one of the complexes $R[s^{-1}] \otimes_R M^\bullet$ or $\mathbb{R}\text{Hom}_R(R[s^{-1}], C^\bullet)$ implies acyclicity of the complex $\mathbb{R}\text{Hom}_R(R[s^{-1}], \mathbb{R}\text{Hom}_R(M^\bullet, C^\bullet))$.

Alternatively, for bounded complexes $N^\bullet$, $M^\bullet$, and $C^\bullet$ one can deduce Lemma 6.2 from Lemma 6.1. For example, let us explain the first case in part (b). Since the class of $s$-contramodule $R$-modules is closed the kernels, cokernels, and extensions,
the question reduces to proving that the $R$-modules $\text{Ext}_R^n(M, C)$ are $s$-contramodules for every $s$-torsion $R$-module $M$ and every $R$-module $C$. Here it suffices to choose an injective $R$-module resolution for $C$ and apply Lemma 6.1(b).

Let $R\text{-mod}_{s\text{-tors}} \subset R\text{-mod}$ denote the full subcategory of $s$-torsion $R$-modules in $R\text{-mod}$. We recall from Theorem 1.1(b) that $R\text{-mod}_{s\text{-tors}}$ is an abelian category, and, in fact, even a Serre subcategory in $R\text{-mod}$. Denote by $\Gamma_s(M) \subset M$ the maximal $s$-torsion submodule of an $R$-module $M$. Then the functor $\Gamma_s$ is right adjoint to the fully faithful embedding functor $R\text{-mod}_{s\text{-tors}} \hookrightarrow R\text{-mod}$.

We start from computing the functor $\Gamma_s$ and then proceed to construct the dual-analogous functor $\Delta_s$.

**Lemma 6.3.** For any $R$-module $M$, the following $R$-modules are naturally isomorphic to each other:

(i) the $R$-module $\Gamma_s(M)$;

(ii) the kernel of the $R$-module morphism

$$\psi^M_s : \bigoplus_{n \geq 0} M \longrightarrow \bigoplus_{n \geq 1} M$$

taking an eventually vanishing sequence $x_0, x_1, x_2, \ldots \in M$ to the sequence

$$(y_1, y_2, y_3, \ldots) = \psi^M_s (x_0, x_1, x_2, \ldots), \quad y_n = x_n - sx_{n-1}, \quad n \geq 1;$$

(iii) assuming that $s$ is not a zero-divisor in $R$, the $R$-module $\text{Tor}^R_1(R[s^{-1}]/R, M)$.

**Proof.** (i) $\simeq$ (ii): The equations $y_n = 0, n \geq 1$ mean that $x_n = sx_{n-1}$ for $n \geq 1$, that is $x_n = s^n x_0$. The submodule formed by all the elements $x_0 \in M$ for which the sequence $s^n x_0 \in M, \quad n \geq 1$ is eventually vanishing coincides with $\Gamma_s(M) \subset M$.

(ii) $\simeq$ (iii): When $s$ is a nonzero-divisor in $R$, the $R$-module $R[s^{-1}]/R$ has a two-term flat left resolution $R \longrightarrow R[s^{-1}]$. Hence the $R$-module $\text{Tor}^R_1(R[s^{-1}]/R, M)$ is computed as the kernel of the morphism $(R \rightarrow R[s^{-1}]) \otimes_R M = (M \rightarrow R[s^{-1}]) \otimes_R M$, which obviously coincides with $\Gamma_s(M)$.

(iii) $\simeq$ (iii): When $s$ is a nonzero-divisor, the two-term complex

$$(10) \quad \psi^R_s : \bigoplus_{n \geq 0} R \longrightarrow \bigoplus_{n \geq 1} R$$

is a projective left resolution of the $R$-module $R[s^{-1}]/R$ (cf. the proof of Lemma 2.1). Therefore, the $R$-module $\text{Tor}^R_1(R[s^{-1}]/R, M)$ is computed as the kernel of the morphism $\psi^R_s \otimes_R M = \psi^M_s$. \qed

For any commutative ring $R$ and element $s \in R$, we denote by $R\text{-mod}_{s\text{-ctra}} \subset R\text{-mod}$ the full subcategory of $s$-contramodule $R$-modules. According to Theorem 1.2(a), the category $R\text{-mod}_{s\text{-ctra}}$ is abelian and the embedding functor $R\text{-mod}_{s\text{-ctra}} \longrightarrow R\text{-mod}$ is exact.

**Theorem 6.4.** For any $R$-module $C$, the following $R$-modules are naturally isomorphic to each other:
(i) the cokernel of the $R$-module morphism

\[ \phi^s_C : \prod_{n \geq 1} C \longrightarrow \prod_{n \geq 0} C \]

taking a sequence $c_1, c_2, c_3, \ldots \in C$, $c_0 = 0$ to the sequence

\[ (b_0, b_1, b_2, \ldots) = \phi^s_C(c_1, c_2, c_3, \ldots), \quad b_n = c_n - sc_{n+1}, \quad n \geq 0; \]

(ii) the cokernel of the endomorphism of the $R$-module $C[[z]]$ of formal power series in one variable $z$ with coefficients in $C$

\[ (z - s) : C[[z]] \longrightarrow C[[z]] \]

which is the difference of the endomorphism of multiplication by $z$ (coming from the formal power series structure) and the endomorphism $s$ (induced by the endomorphism $s$ on the coefficient module $C$);

(iii) assuming that $s$ is not a zero-divisor in $R$, the $R$-module $\Ext^1_R(R[s^{-1}]/R, C)$.

Denote the $R$-module produced by either of the constructions (i)–(iii) by $\Delta_s(C)$. Then the functor $\Delta_s : R\text{-mod} \longrightarrow R\text{-mod}_{s\text{-contra}}$ is left adjoint to the fully faithful embedding functor $R\text{-mod}_{s\text{-contra}} \longrightarrow R\text{-mod}$.

**Proof.** (i) $\simeq$ (ii): The obvious isomorphisms $\prod_{n \geq 1} C \simeq C[[z]] \simeq \prod_{n \geq 0} C$ identify the morphism $\phi^s_C$ with the morphism $z - s$.

(i) $\simeq$ (iii): When $s$ is a nonzero-divisor, the $R$-module $\Ext^1_R(R[s^{-1}]/R, C)$ can be computed as the cokernel of the morphism $\Hom_R(\psi^R_s, C)$ (see (10)), which is readily identified with the morphism $\phi^s_C$.

Before proving that the two functors are adjoint (cf. [27, the proof of Proposition 2.1]), we have to check that $\Delta_s$ takes values in $R\text{-mod}_{s\text{-contra}}$, i.e., the $R$-module $\Delta_s(C)$ is an $s$-contramodule for every $R$-module $C$. This claim only depends on the action of $s$ and not on the rest of the $R$-module structures on our modules, so we can assume that $R = \mathbb{Z}[s]$. Furthermore, the functor $\Delta_s$ is right exact by construction (as the cokernel of a morphism is a right exact functor and the functor of infinite product of abelian groups is exact). So it can be computed using an initial fragment of a free $\mathbb{Z}[s]$-module resolution of the module $C$.

In other words, the functor $\Delta_s$ preserves cokernels. Let us present the $\mathbb{Z}[s]$-module $C$ as the cokernel of a morphism of free $\mathbb{Z}[s]$-modules $u : A'[s] \longrightarrow A''[s]$, where $A'$ and $A''$ are some (free) abelian groups. Then $\Delta_s(C)$ is the cokernel of the morphism $\Delta_s(u) : \Delta_s(A'[s]) \longrightarrow \Delta_s(A''[s])$. Since the class of $s$-contramodules is closed under the cokernels, it suffices to check that the $\mathbb{Z}[s]$-module $\Delta_s(A[s])$ is an $s$-contramodule for any abelian group $A$.

This is accomplished by an explicit computation; and the construction (ii) seems to be the most convenient one. We leave it to the reader to compute that the quotient module $A[[s]]/(z - s)$ is naturally isomorphic to the $\mathbb{Z}[s]$-module of formal power series $A[[s]]$; so $\Delta_s(A[s]) = A[[s]]$. That is clearly an $s$-contramodule.

Now let $D$ be an $s$-contramodule $R$-module and $C$ an arbitrary $R$-module. By the definition, the adjunction morphism $\delta_{s,C} : C \longrightarrow \Delta_s(C)$ is induced by the embedding of $C$ into $\prod_{n \geq 0} C$ as the $(n = 0)$-indexed factor of the product. To any $R$-module
morphism \( g: \Delta_s(C) \to D \) one assigns the composition \( g\delta_{s,C}: C \to \Delta_s(C) \to D \). We have to check that this construction defines an isomorphism of the Hom modules \( \text{Hom}_R(\Delta_s(C), D) \cong \text{Hom}_R(C, D) \).

Let \( f: C \to D \) be an \( R \)-module morphism; we need to show that it factorizes through the morphism \( \delta_{s,C} \). Define a morphism \( g': \prod_{n \geq 0} C \to D \) by the rule

\[
g'(b_0, b_1, b_2, \ldots) = \sum_{n=0}^{\infty} s^n f(b_n).
\]

Then for any sequence \((c_n \in C)_{n \geq 1}, \ c_0 = 0\) we have

\[
g'(\phi^s_C((c_n)_{n=1}^{\infty})) = g'((c_n - sc_{n+1})_{n=0}^{\infty}) = \sum_{n=0}^{\infty} s^n (f(c_n - sc_{n+1})) = \sum_{n=0}^{\infty} s^n f(c_n) - \sum_{n=0}^{\infty} s^n (sf(c_{n+1})) = f(c_0) = 0,
\]

so the morphism \( g' \) factorizes through the cokernel of \( \phi^s_C \), and induces a morphism \( g: \Delta_s(C) \to D \). Using the contraunitality axiom for infinite summation operations, one can easily check that \( g\delta_{s,C} = f \).

Finally, let \( h: \Delta_s(C) \to D \) be an \( R \)-module morphism for which \( h\delta_{s,C} = 0 \). Denote the composition of \( h \) with the projection \( \prod_{n=0}^{\infty} C \to \Delta_s(C) \) by \( h': \prod_{n=0}^{\infty} C \to D \). Given any sequence \((b_n \in C)_{n=0}^{\infty}, \) set

\[
d_n = h'((b_{n+1})_{n=0}^{\infty}) = h'(b_n, b_{n+1}, b_{n+2}, \ldots) \in D.
\]

Then we have

\[
d_n - sd_{n+1} = h'(b_n - sb_{n+1}, b_{n+1} - sb_{n+2}, b_{n+2} - sb_{n+3}, \ldots) = h'(-sb_{n+1}, b_{n+1} - sb_{n+2}, b_{n+2} - sb_{n+3}, \ldots) = h'((\phi^s_C(b_{n+1}, b_{n+2}, b_{n+3}, \ldots)) = 0
\]

for every \( n \geq 0 \), since \( h'(b_0, 0, 0, \ldots) = 0 \) for any \( b \in C \) by assumption. Since \( D \) has no nonzero \( s \)-divisible submodules, we conclude that \( d_n = 0 \) for all \( n \geq 0 \), and in particular \( d_0 = 0 \). Thus \( h'((b_n)_{n=0}^{\infty}) = 0 \) for any sequence \((b_n \in C)_{n=0}^{\infty} \) and \( h = 0 \). \( \square \)

**Remark 6.5.** The two-term complex (10) plays a central role in the constructions of Lemma 6.3 and Theorem 6.4. Let us denote it by \( T^\bullet(R; s) \) and place in the cohomological degrees 0 and 1, so that \( T^0(R; s) = \bigoplus_{n=0}^{\infty} R \) and \( T^1(R; s) = \bigoplus_{n=1}^{\infty} R \). The complex \( T^\bullet(R; s)[1] \) is quasi-isomorphic to the two-term complex \( R \to R[s^{-1}] \) (where the term \( R \) sits in the cohomological degree \(-1\) and the term \( R[s^{-1}] \) sits in the cohomological degree \( 0 \)). The quasi-isomorphism is provided by the map taking an eventually vanishing sequence \( x_0, x_1, x_2, \ldots \in R \) to the element \( x_0 \in R \) and an eventually vanishing sequence \( y_1, y_2, y_3, \ldots \in R \) to the element \(-\sum_{n=1}^{\infty} y_n/s^n \in R[s^{-1}] \). Hence the assumption that \( s \) is not a zero-divisor in \( R \) can be removed from the constructions of Lemma 6.3(iii) and Theorem 6.4(iii) by saying that \( \Gamma_s(M) = H^{-1}((R \to R[s^{-1}]) \otimes_R M) \) and \( \Delta_s(C) = \text{Hom}_{\text{R-mod}}((R \to R[s^{-1}]), C[1]) \) for any commutative ring \( R \), an element \( s \in R \), and \( R \)-modules \( M \) and \( C \).

**Remark 6.6.** The following observations, suggested to the author by the anonymous referee, shed some additional light on Lemma 6.3 and Theorem 6.4. Pick a
derived category symbol \( \star = \mathbb{b}, +, - \), or \( \emptyset \), and consider the related (bounded or unbounded) derived category of \( R \)-modules \( \mathcal{D}^*(\mathcal{R-mod}) \). The restriction-of-scalars functor \( \mathcal{D}^*(\mathcal{R}s^{-1})\mathcal{-mod}) \rightarrow \mathcal{D}^*(\mathcal{R-mod}) \) is a fully faithful embedding whose essential image is the full subcategory in \( \mathcal{D}^*(\mathcal{R-mod}) \) formed by all the complexes in whose cohomology modules \( s \) acts by automorphism. The functor \( \mathcal{D}^*(\mathcal{R}s^{-1})\mathcal{-mod}) \rightarrow \mathcal{D}^*(\mathcal{R-mod}) \) has adjoints on both sides, the left adjoint being the extension of scalars \( \mathcal{M}^\star \mapsto \mathcal{R}[s^{-1}] \otimes_{\mathcal{R}} \mathcal{M}^\star \) and the right adjoint being the coextension of scalars \( \mathcal{C}^\star \mapsto \mathcal{R} \text{Hom}_\mathcal{R}(\mathcal{R}[s^{-1}], \mathcal{C}^\star) \). The kernel of this extension-of-scalars functor is the full subcategory \( \mathcal{D}^*_{s-tors}(\mathcal{R-mod}) \subset \mathcal{D}^*(\mathcal{R-mod}) \) of complexes of \( R \)-modules with \( s \)-torsion cohomology modules, while the kernel of the coextension of scalars is the full subcategory \( \mathcal{D}^*_{s-terra}(\mathcal{R-mod}) \subset \mathcal{D}^*(\mathcal{R-mod}) \) of complexes of \( R \)-modules with \( s \)-contramodule cohomology modules (cf. the proof of Lemma 6.2).

The embedding functor \( \mathcal{D}^*_{s-tors}(\mathcal{R-mod}) \rightarrow \mathcal{D}^*(\mathcal{R-mod}) \) has a right adjoint, which can be computed as the functor \((\mathcal{R} \rightarrow \mathcal{R}[s^{-1}][{-1}] \otimes_{\mathcal{R}} \mathcal{R})\), while the embedding functor \( \mathcal{D}^*_{s-terra}(\mathcal{R-mod}) \rightarrow \mathcal{D}^*(\mathcal{R-mod}) \) has a left adjoint, which is computed as the functor \( \text{Hom}_\mathcal{R}(\mathcal{T}^\star(\mathcal{R}; \mathcal{s}), -) \) (see [27, Section 3] or [28, Section 4]). The functor \( \Gamma_s \) is the composition \( R^\mathcal{mod} \rightarrow \mathcal{D}^*(\mathcal{R-mod}) \rightarrow \mathcal{D}^*_{s-tors}(\mathcal{R-mod}) \rightarrow \mathcal{R} \text{mod}_{s-tors} \) of the functor \((\mathcal{R} \rightarrow \mathcal{R}[s^{-1}][{-1}] \otimes_{\mathcal{R}} \mathcal{R})\) with the embedding \( \mathcal{R} \text{mod} \rightarrow \mathcal{D}^*(\mathcal{R-mod}) \) and the degree-zero cohomology functor \( \mathcal{D}^*_{s-tors}(\mathcal{R-mod}) \rightarrow \mathcal{R} \text{mod}_{s-tors} \), while the functor \( \Delta_s \) is the similar composition \( R^\mathcal{mod} \rightarrow \mathcal{D}^*(\mathcal{R-mod}) \rightarrow \mathcal{D}^*_{s-terra}(\mathcal{R-mod}) \rightarrow \mathcal{R} \text{mod}_{s-terra} \) of the functor \( \text{Hom}_\mathcal{R}(\mathcal{T}^\star(\mathcal{R}; \mathcal{s}), -) \) with the degree-zero cohomology functor \( \mathcal{D}^*_{s-terra}(\mathcal{R-mod}) \rightarrow \mathcal{R} \text{mod}_{s-terra} \).

It follows that both the triangulated categories \( \mathcal{D}^*_{s-tors}(\mathcal{R-mod}) \) and \( \mathcal{D}^*_{s-terra}(\mathcal{R-mod}) \) are equivalent to the quotient category \( \mathcal{D}^*(\mathcal{R-mod})/\mathcal{D}^*_{\mathcal{R}s^{-1}}\mathcal{-mod}) \). The mutually inverse equivalences between \( \mathcal{D}^*_{s-tors}(\mathcal{R-mod}) \) and \( \mathcal{D}^*_{s-terra}(\mathcal{R-mod}) \) are given by the restrictions of the functors \((\mathcal{R} \rightarrow \mathcal{R}[s^{-1}][{-1}] \otimes_{\mathcal{R}} \mathcal{R})\) and \( \text{Hom}_\mathcal{R}(\mathcal{T}^\star(\mathcal{R}; \mathcal{s}), -) \). The resulting two \( t \)-structures on the triangulated category \( \mathcal{D}^*_{s-tors}(\mathcal{R-mod}) \simeq \mathcal{D}^*_{s-terra}(\mathcal{R-mod}) \) are connected by tilting with respect to torsion pairs (in the sense of an appropriate generalization of [12, Section 1.2]). The torsion pair in the abelian category \( \mathcal{R} \text{mod}_{s-tors} \) consists of the classes of \( s \)-divisible \( s \)-torsion \( R \)-modules and \( s \)-reduced \( s \)-torsion \( R \)-modules. The torsion pair in the abelian category \( \mathcal{R} \text{mod}_{s-terra} \) consists of the classes of \( s \)-special \( s \)-contramodule \( R \)-modules and \( s \)-torsion-free \( s \)-contramodule \( R \)-modules [28, Section 5].

Given a commutative ring \( \mathcal{R} \), an element \( t \in \mathcal{R} \), and an \( R \)-module \( \mathcal{M} \), we denote by \( t\mathcal{M} \subset \mathcal{M} \) the submodule of all elements annihilated by \( t \) in \( \mathcal{M} \).

For any element \( s \) in a commutative ring \( \mathcal{R} \), one can assign to every \( R \)-module \( \mathcal{C} \) two projective systems of \( R \)-modules. One of them is formed by the \( R \)-modules \( \mathcal{C}/s^n\mathcal{C} \) and the natural surjective morphisms between them. The other one consists of the \( R \)-modules \( s^n\mathcal{C} \) and the multiplication maps \( s: s^{n+1}\mathcal{C} \rightarrow s^n\mathcal{C} \).

Let us denote by \( \Lambda_s \) the \( s \)-completion functor \( \mathcal{C} \mapsto \Lambda_s(\mathcal{C}) = \lim_{n \geq 1} \mathcal{C}/s^n\mathcal{C} \). The following lemma provides a comparison between the functors \( \Lambda_s \) and \( \Delta_s \).
Lemma 6.7. Let $R$ be a commutative ring and $s \in R$ be an element. Then for any $R$-module $C$ there is a natural short exact sequence of $R$-modules

$$0 \longrightarrow \lim_{\leftarrow n \geq 1} s^n C \longrightarrow \Delta_s(C) \longrightarrow \Lambda_s(C) \longrightarrow 0.$$ 

Proof. Denote by $T_n^*(R; s)$, $n \geq 1$ the subcomplex

$$\bigoplus_{i=0}^{n-1} R \longrightarrow \bigoplus_{i=1}^{n} R$$

of the complex (10). As in Remark 6.5, $T_n^*(R; s)$ is viewed as a complex concentrated in the cohomological degrees 0 and 1. The complex $T_n^*(R; s)$ is homotopy equivalent to the two-term complex of $R$-modules $R \xrightarrow{s^n} R$. The complexes $\text{Hom}_R(T_n^*(R; s), C)$ form a projective system with termwise surjective morphisms of complexes, hence there is a short exact sequence of homology modules

$$0 \longrightarrow \lim_{\leftarrow n} H_1(\text{Hom}_R(T_n^*(R; s), C)) \longrightarrow H_0(\lim_{\leftarrow n} \text{Hom}_R(T_n^*(R; s), C)) \longrightarrow \lim_{\leftarrow n} H_0(\text{Hom}_R(T_n^*(R; s), C)) \longrightarrow 0.$$ 

Furthermore, we have

$$\lim_{\leftarrow n \geq 1} \text{Hom}_R(T_n^*(R; s), C) = \text{Hom}_R(T^*(R; s), C).$$

It remains to recall that $H_0(\text{Hom}_R(T^*(R; s), C)) = \Delta_s(C)$ by Theorem 6.4(i), while $H_1(\text{Hom}_R(T_n^*(R; s), C)) = H_1(\text{Hom}_R((R \xrightarrow{s^n} R), C)) = s^n C$ and $H_0(\text{Hom}_R(T_n^*(R; s), C)) = H_0(\text{Hom}_R((R \xrightarrow{s^n} R), C)) = C/s^n C$. (See the proof of Lemma 7.5 below for a more detailed discussion.)

Corollary 6.8. Let $R$ be a commutative ring and $s \in R$ an element. Then

(a) for any $R$-module $C$, there is a natural surjective $R$-module morphism $\Delta_s(C) \longrightarrow \Lambda_s(C)$;

(b) for any $s$-torsion-free $R$-module $C$, the morphism $\Delta_s(C) \longrightarrow \Lambda_s(C)$ is an isomorphism.

Proof. Both assertions follow immediately from Lemma 6.7. Alternatively, one can deduce part (a) from Theorem 2.3(a) and part (b) from Theorem 2.3(b), as we will now explain.

Part (a): by Theorem 2.4(a), the category of $s$-separated and $s$-complete $R$-modules $R\text{-mod}_{s,\text{secmp}}$ is contained in the category of $s$-contramodule $R$-modules $R\text{-mod}_{s,\text{ctra}}$, which in turn is contained in the category $R\text{-mod}$. By Theorem 5.8, the functor $\Lambda_s$ is left adjoint to the embedding functor $R\text{-mod}_{s,\text{secmp}} \longrightarrow R\text{-mod}$, while by Theorem 6.4, the functor $\Delta_s$ is left adjoint to the embedding functor $R\text{-mod}_{s,\text{ctra}} \longrightarrow R\text{-mod}$. It follows that there is a natural transformation $\Delta_s \longrightarrow \Lambda_s$ forming a commutative triangle with the adjunction morphisms

$$C \longrightarrow \Delta_s(C) \longrightarrow \Lambda_s(C)$$

for any $R$-module $C$. Furthermore, we have $\Lambda_s(\Delta_s(C)) = \Lambda_s(C)$. Since $\Delta_s(C)$ is an $s$-contramodule, it follows from Theorem 2.3(a) that $\Delta_s(C)$ is $s$-complete, hence the morphism $\Delta_s(C) \longrightarrow \Lambda_s(C)$ is surjective.
Part (b): in view of the proof of part (a), it suffices to check that the $R$-module $\Delta_s(C)$ is $s$-torsion-free for every $s$-torsion-free $R$-module $C$. According to Remark 6.5, we have $\Delta_s(B) = \text{Hom}_{\mathcal{D}}((R \to R[s^{-1}]), B[1])$ for any $R$-module $B$. Hence from the short exact sequence $0 \to C \to C \to C/sC \to 0$ we obtain a long exact sequence

$$\cdots \to \text{Hom}_{\mathcal{D}}((R \to R[s^{-1}]), C/sC) \to \Delta_s(C) \to \Delta_s(C/sC) \to 0.$$ 

It remains to notice that $\text{Hom}_{\mathcal{D}}((R \to R[s^{-1}]), D) = \text{Hom}_R(R[s^{-1}]/R, D)$ for any $R$-module $D$, and the right-hand side vanishes when $sD = 0$. \hfill \(
\)

**Remark 6.9.** One can also show that the natural morphism $\Delta_s(C) \to \Lambda_s(C)$ is an isomorphism whenever the $R$-module $\Lambda_s(C)$ is $s$-torsion-free. Indeed, both $\Delta_s(C)$ and $\Lambda_s(C)$ are $s$-contramodules, so the kernel $K$ of the morphism in question is an $s$-contramodule, too. Furthermore, the morphism $\Delta_s(C)/s\Delta_s(C) \to \Lambda_s(C)/s\Lambda_s(C)$ is always an isomorphism, because for any $R$-module $M$ with $sM = 0$ one has

$$\text{Hom}_R(\Delta_s(C)/s\Delta_s(C), M) = \text{Hom}_R(\Delta_s(C), M) = \text{Hom}_R(C, M)$$

and

$$\text{Hom}_R(\Lambda_s(C)/s\Lambda_s(C), M) = \text{Hom}_R(\Lambda_s(C), M) = \text{Hom}_R(C, M).$$

Assuming that $\Lambda_s(C)$ is $s$-torsion-free, from the exact sequence $0 \to K \to \Delta_s(C) \to \Lambda_s(C) \to 0$ we get an exact sequence $0 \to K/sK \to \Delta_s(C)/s\Delta_s(C) \to \Lambda_s(C)/s\Lambda_s(C) \to 0$. Hence $K = sK$ and it follows that $K = 0$. (See Lemma 10.1 below for a generalization.)

More generally, an $R$-module $C$ is said to have bounded $s$-torsion if there exists $m \geq 1$ such that $s^n c = 0$ implies $s^{m+n} c = 0$ for all $n \geq 1$ and $c \in C$. One can observe that $\lim_{n \to -\infty} s^n C = 0$ whenever the $s$-torsion in $C$ is bounded. By Lemma 6.7, the natural morphism $\Delta_s(C) \to \Lambda_s(C)$ is an isomorphism in this case. Hence it follows that the morphism $\Delta_s(C) \to \Lambda_s(C)$ is an isomorphism whenever the $R$-module $\Delta_s(C)$ has bounded $s$-torsion.

Conversely, assume that the $R$-module $\Lambda_s(C)$ has bounded $s$-torsion. Then, for every $n \geq 1$, from the exact sequence $0 \to K \to \Delta_s(C) \to \Lambda_s(C) \to 0$ we get an exact sequence $s^n\Lambda_s(C) \to K/s^n K \to \Delta_s(C)/s^n \Delta_s(C) \to \Lambda_s(C)/s^n \Lambda_s(C) \to 0$, where $tM$ denotes the submodule annihilated by an element $t \in R$ in an $R$-module $M$. For any $n \geq k$, we have a morphism of such exact sequences with the maps $M/s^n M \to M/s^k M$ being the natural surjections and the map $s^n \Lambda_s(C) \to s^k \Lambda_s(C)$ being the multiplication with $s^{n-k}$. Since the map $\Delta_s(C)/s^n \Delta_s(C) \to \Lambda_s(C)/s^n \Lambda_s(C)$ is an isomorphism, the map $s^n \Lambda_s(C) \to K/s^n K$ is surjective. So is the map $K/s^n K \to K/s^k K$. Now if the $s$-torsion in $\Lambda_s(C)$ is bounded, then the map $s^n \Lambda_s(C) \to \Lambda_s(C)$ vanishes for $n$ large enough, and it follows from the commutativity of the diagram that $K/s^k K = 0$. Hence $K = 0$.

Now we deduce a corollary providing a “non-naïve” version of the contraadjustedness criterion from Theorem 2.3.
Corollary 6.10. Let $R$ be a commutative ring, $s \in R$ an element, and $C$ an $R$-module. Then

(a) an $R$-module $C$ has no $s$-divisible submodules if and only if the adjunction morphism $C \to \Delta_s(C)$ is injective;
(b) an $R$-module $C$ is $s$-contraadjusted if and only if the adjunction morphism $C \to \Delta_s(C)$ is surjective.

Proof. First let us consider the case when $s$ is not a zero-divisor in $R$. Then from the short exact sequence

$$0 \longrightarrow R \longrightarrow R[s^{-1}] \longrightarrow R[s^{-1}]/R \longrightarrow 0$$

we obtain the long exact sequence of Ext groups

$$0 \longrightarrow \text{Hom}_R(R[s^{-1}]/R, C) \longrightarrow \text{Hom}_R(R[s^{-1}], C) \longrightarrow C \longrightarrow \text{Ext}^1(R[s^{-1}]/R, C) \longrightarrow \text{Ext}^1(R[s^{-1}], C) \longrightarrow 0$$

Recalling that $\text{Ext}^1_R(R[s^{-1}]/R, C) = \Delta_s(C)$ by Theorem 6.4(iii), we obtain both the assertions (a) and (b). In the general case, it suffices to notice that the properties in question do not depend on the $R$-module structure on $C$, but only on the action of the element $s$; so a ring $R$ can be replaced with the polynomial ring $\mathbb{Z}[s]$.

Alternatively, the argument in the general case can be similar, except that one has to use the cone in lieu of the cokernel. Denote by $(R \to R[s^{-1}]) \in \mathcal{D}^b(R\text{-mod})$ the derived category object represented by the two-term complex $R \longrightarrow R[s^{-1}]$, which is quasi-isomorphic to the complex (10). Placing the complex $R \longrightarrow R[s^{-1}]$ in the cohomological degrees $-1$ and $0$, one has a distinguished triangle

$$R \longrightarrow R[s^{-1}] \longrightarrow (R \to R[s^{-1}]) \longrightarrow R[1]$$

in $\mathcal{D}^b(R\text{-mod})$. Hence the related long exact sequence of triangulated Hom

$$\cdots \longrightarrow \text{Hom}_{\mathcal{D}^b(R\text{-mod})}(R[s^{-1}], C) \longrightarrow \text{Hom}_{\mathcal{D}^b(R\text{-mod})}(R, C) \longrightarrow \text{Hom}_{\mathcal{D}^b(R\text{-mod})}((R \to R[s^{-1}]), C[1]) \longrightarrow \cdots,$n

with $\text{Hom}_{\mathcal{D}^b(R\text{-mod})}(R[s^{-1}], C) = \text{Hom}_R(R[s^{-1}], C)$, $\text{Hom}_{\mathcal{D}^b(R\text{-mod})}(R, C) = \text{Hom}_R(R, C) = C'$, $\text{Hom}_{\mathcal{D}^b(R\text{-mod})}(R[s^{-1}], C[1]) = \text{Ext}^1_R(R[s^{-1}], C)$, and $\text{Hom}_{\mathcal{D}^b(R\text{-mod})}(R, C[1]) = \text{Ext}^1_R(R, C) = 0$. Finally, notice the isomorphism

$$\text{Hom}_{\mathcal{D}^b(R\text{-mod})}((R \to R[s^{-1}]), C[1]) = \Delta_s(C)$$

from Remark 6.5. Now exactness of the long sequence implies both (a) and (b). □

Remark 6.11. Combining Corollaries 6.8 and 6.10(b), one can obtain a new proof of Theorem 2.3. Similarly, Theorem 2.4(b) follows from Corollary 6.8.

More generally, the morphism $\Delta_s(C) \longrightarrow \Delta_s(C)$ is an isomorphism when the $s$-torsion in $C$ is bounded (see Remark 6.9). This allows to weaken the “$s$-torsion-free” assumption to “bounded $s$-torsion” in the assertions of Theorems 2.3(b) and 2.4(b).
Similarly, it follows from Corollary 6.10(a) that any \( R \)-module with bounded \( s \)-torsion and without \( s \)-divisible submodules is \( s \)-separated (cf. Remark 2.8).

7. \( I \)-Torsion Modules and the Functor \( \Gamma_I \),
\( I \)-Contramodules and the Functor \( \Delta_I \)

Let \( I \) be an ideal in a commutative ring \( R \). An \( R \)-module \( M \) is said to be \( I \)-torsion if it is \( s \)-torsion for every \( s \in I \). Clearly, it suffices to check the latter condition for some set of generators of the ideal \( I \): if \( I \) is generated by some elements \( s_j \) and \( M \) is \( s_j \)-torsion for every \( j \), then \( M \) is \( I \)-torsion. The dual-analogous contramodule version of this observation is Theorem 5.1 (to be proved again below in this section).

Let \( R \)-mod\(_{I\text{-tors}} \subset R\text{-mod} \) denote the full subcategory of \( I \)-torsion \( R \)-modules. Denote by \( \Gamma_I(M) \) the maximal \( I \)-torsion submodule of an \( R \)-module \( M \). Then the functor \( \Gamma_I \) is right adjoint to the fully faithful embedding \( R\text{-mod}_{I\text{-tors}} \to \to R\text{-mod} \).

As the category \( R\text{-mod}_{I\text{-tors}} \) is abelian and its embedding functor \( R\text{-mod}_{I\text{-tors}} \to \to R\text{-mod} \) is exact, the functor \( \Gamma_I \) (viewed either as a functor \( R\text{-mod} \to \to R\text{-mod}_{I\text{-tors}} \) or as a functor \( R\text{-mod} \to \to R\text{-mod} \)) is left exact. It also preserves infinite direct sums; and the full subcategory \( R\text{-mod}_{I\text{-tors}} \subset R\text{-mod} \) is closed under subobjects, quotients, extensions, and infinite direct sums in \( R\text{-mod} \).

We recall the notation \( T^\bullet(R; s) \) for the complex (10) (see Remark 6.5). Furthermore, we set

\[
T^\bullet(R; s_1, \ldots, s_m) = T^\bullet(R; s_1) \otimes_R \cdots \otimes_R T^\bullet(R; s_m).
\]

Hence \( T^\bullet(R; s_1, \ldots, s_m) \) is a complex of countably-generated free \( R \)-modules concentrated in the cohomological degrees \( 0, \ldots, m \). Since the complex \( T^\bullet(R; s_1) \) is quasi-isomorphic to the two-term complex \( R \to \to R[s_1^{-1}] \), the complex \( T^\bullet(R; s_1, \ldots, s_m) \) is quasi-isomorphic to the complex

\[
\tilde{\mathcal{C}}^\sim(R; s_1, \ldots, s_m) = (R \to \to R[s_1^{-1}], \ldots, R \to \to R[s_m^{-1}]),
\]

which looks explicitly as

\[
R \to \to \bigoplus_{j=1}^m R[s_j^{-1}] \to \to \bigoplus_{j' < j''} R[s_{j'}^{-1}, s_{j''}^{-1}] \to \to \cdots \to \to R[s_1^{-1}, \ldots, s_m^{-1}].
\]

For the reasons discussed below in Remark 7.4, we call the complex \( \tilde{\mathcal{C}}^\sim(R; s_1, \ldots, s_m) \) the \textit{augmented \( \check{\text{C}} \)ech complex} of the ring \( R \) with the elements \( s_1, \ldots, s_m \). This is a complex of flat (in fact, even very flat, see \cite{25, 34} for the definition) \( R \)-modules concentrated in the cohomological degrees \( 0, \ldots, m \).

**Lemma 7.1.** For any \( R \)-module \( M \), the following \( R \)-modules are naturally isomorphic to each other:

(i) the maximal \( I \)-torsion submodule \( \Gamma_I(M) \);
(ii) the maximal \( s_1 \)-, \( s_2 \)-, \ldots, and \( s_m \)-torsion submodule \( \Gamma_{s_m} \cdots \Gamma_{s_2} \Gamma_{s_1}(M) \);
(iii) the cohomology module \( H^0(T^\bullet(R; s_1, \ldots, s_m) \otimes_R M) \).
Proof. (i) and (ii) are clearly the same submodule in \( M \).

(i) \( \simeq \) (iii): The complex \( T^\bullet(R; s_1, \ldots, s_m) \otimes_R M \) is quasi-isomorphic to the complex \( \hat{C}^\bullet(R; s_1, \ldots, s_m) \otimes_R M \), which starts as \( M \to \bigoplus_{j=1}^m M[s_j^{-1}] \). The kernel of the latter morphism coincides with \( \Gamma_i(M) \).

(ii) \( \simeq \) (iii): For any complex of \( R \)-modules \( K^\bullet \) concentrated in the cohomological degrees \( \geq 0 \) and any complex of flat \( R \)-modules \( T^\bullet \) concentrated in the cohomological degrees \( \geq 0 \), one has

\[
H^0(T^\bullet \otimes_R K^\bullet) \simeq H^0(T^\bullet \otimes_R H^0(K^\bullet)).
\]

By Lemma 6.3 (i)\( \simeq \) (ii), we have \( H^0(T^\bullet(R; s_j) \otimes_R M) \simeq \Gamma_{s_j}(M) \). Hence

\[
H^0(T^\bullet(R; s_1, s_2) \otimes_R M) \simeq H^0(T^\bullet(R; s_2) \otimes_R (T^\bullet(R; s_1) \otimes_R M)) \simeq H^0(T^\bullet(R; s_2) \otimes_R H^0(T^\bullet(R; s_1) \otimes_R M)) \simeq H^0(T^\bullet(R; s_2) \otimes_R \Gamma_{s_1}(M)) \simeq \Gamma_{s_2} \Gamma_{s_1}(M),
\]

e tc. The argument finishes by induction on \( j \).

Let \( I \subset R \) be the ideal generated by a finite set of elements \( s_1, \ldots, s_m \). As our aim is to prove Theorem 5.1 rather than just use it, let us introduce the temporary notation \( R-\text{mod}_{[s_1, \ldots, s_m]-\text{ctra}} \subset R-\text{mod} \) for the full subcategory of all \( R \)-modules that are \( s_j \)-contramodules for every \( 1 \leq j \leq m \). (When our proof is finished, we will switch to the permanent notation \( R-\text{mod}_{[s_1, \ldots, s_m]-\text{ctra}} = R-\text{mod}_{I-\text{ctra}} \).

According to Theorem 1.2(a), the category \( R-\text{mod}_{[s_1, \ldots, s_m]-\text{ctra}} \) is abelian and its embedding functor \( R-\text{mod}_{[s_1, \ldots, s_m]-\text{ctra}} \to R-\text{mod} \) is exact. The full subcategory \( R-\text{mod}_{[s_1, \ldots, s_m]-\text{ctra}} \) is also closed under infinite products in \( R-\text{mod} \).

**Theorem 7.2.** For any \( R \)-module \( C \), the following \( R \)-modules are naturally isomorphic to each other:

(i) the module \( \Delta_{s_m} \cdots \Delta_{s_2} \Delta_{s_1}(C) \);

(ii) the quotient module

\[
C[[z_1, \ldots, z_m]]/\sum_{j=1}^m (z_j - s_j)C[[z_1, \ldots, z_m]]
\]

of the module of formal power series \( C[[z_1, \ldots, z_m]] \) in \( n \) variables \( z_1, \ldots, z_m \) with coefficients in \( C \) by the sum of the images of the operators

\[
z_1 - s_1, \ z_2 - s_2, \ldots, z_m - s_m : C[[z_1, \ldots, z_m]] \to C[[z_1, \ldots, z_m]];
\]

(iii) the homology module \( H_0(\text{Hom}_R(T^\bullet(R; s_1, \ldots, s_m), C)) \).

Denote the \( R \)-module produced by either of the constructions (i)–(iii) by \( \Delta_{s_1, \ldots, s_m}(C) \). Then the functor \( \Delta_{s_1, \ldots, s_m} : R-\text{mod} \to R-\text{mod}_{[s_1, \ldots, s_m]-\text{ctra}} \) is left adjoint to the fully faithful embedding functor \( R-\text{mod}_{[s_1, \ldots, s_m]-\text{ctra}} \to R-\text{mod} \).

**Proof.** (i) \( \simeq \) (ii): The natural isomorphism is easily constructed using the observation that the functor assigning to an abelian group \( A \) the group \( A[[z]] \) is exact and, in particular, preserves cokernels.
(i) \(\simeq\) (iii): For any complex of \(R\)-modules \(K_\bullet\) concentrated in the homological degrees \(\geq 0\) and any complex of projective \(R\)-modules \(T_\bullet\) concentrated in the cohomological degrees \(\geq 0\), one has
\[
H_0(\text{Hom}_R(T_\bullet, K_\bullet)) \simeq H_0(\text{Hom}_R(T_\bullet, H_0(K_\bullet))).
\]

Hence
\[
H_0(\text{Hom}_R(T_\bullet(R; s_1, s_2), C)) \simeq H_0(\text{Hom}_R(T_\bullet(R; s_2), \text{Hom}_R(T_\bullet(R; s_1), C)))
\]
\[
\simeq H_0(\text{Hom}_R(T_\bullet(R; s_2), H_0(\text{Hom}_R(T_\bullet(R; s_1), C))))
\]
\[
\simeq H_0(\text{Hom}_R(T_\bullet(R; s_2), \Delta_{s_1}(C))) \simeq \Delta_{s_2}\Delta_{s_1}(C),
\]
etc.

It remains to show that the functor \(\Delta_{s_1,\ldots,s_m}\) is left adjoint to the embedding functor \(R\text{-mod}_{[s_1,\ldots,s_m]\text{-ctra}} \rightarrow R\text{-mod}\) (cf. [27, proof of Proposition 2.1]). The key observation is that, for any two elements \(s\) and \(t\in R\), the functor \(\Delta_t\) takes \(s\)-contramodules to \(s\)-contramodules. Indeed, the class of \(s\)-contramodules is closed under the infinite products and cokernels in \(R\text{-mod}\), hence the cokernel of the morphism (11) is an \(s\)-contramodule whenever the \(R\)-module \(C\) is. It follows that, for any \(R\)-module \(C\), the \(R\)-module \(\Delta_{s_m}\cdots\Delta_{s_2}\Delta_{s_1}(C)\) is an \(s_j\)-contramodule for every \(1 \leq j \leq m\).

Now for an arbitrary \(R\)-module \(C\) and an \(R\)-module \(D\) from the subcategory \(R\text{-mod}_{[s_1,\ldots,s_m]\text{-ctra}} \subset R\text{-mod}\) one has
\[
\text{Hom}_R(\Delta_{m}\cdots\Delta_2\Delta_1(C), D) \simeq \text{Hom}_R(\Delta_{m-1}\cdots\Delta_2\Delta_1(C), D)
\]
\[
\simeq \cdots \simeq \text{Hom}_R(\Delta_2\Delta_1(C), D) \simeq \text{Hom}_R(\Delta_1(C), D) \simeq \text{Hom}_R(C, D)
\]
due to the adjointness properties of the functors \(\Delta_{s_j}\). \(\square\)

For any ideal \(I\subset R\), one denotes by \(\sqrt{I}\subset R\) the radical of the ideal \(I\), i.e., the ideal formed by all the elements \(s\in R\) for which there exists \(n \geq 1\) such that \(s^n\in I\). The following result is due to Porta, Shaul, and Yekutieli [21, Theorem 6.1].

**Theorem 7.3.** Let \(s_1,\ldots,s_m\) and \(t_1,\ldots,t_k\) be two finite sets of elements in a commutative ring \(R\). Denote by \(I = (s_1,\ldots,s_m)\) and \(J = (t_1,\ldots,t_k)\subset R\) the ideals generated by the first and the second set, respectively. Suppose that \(\sqrt{I} = \sqrt{J}\) in \(R\). Then the two complexes of \(R\)-modules \(T_\bullet(R; s_1,\ldots,s_m)\) and \(T_\bullet(R; t_1,\ldots,t_k)\) are homotopy equivalent.

**Sketch of proof** [21]. According to Lemma 7.1 (i)\(\simeq\)(iii), for any \(R\)-module \(M\) we have
\[
H^0(T_\bullet(R; s_1,\ldots,s_m) \otimes_R M) \simeq \Gamma_I(M) \simeq H^0(T_\bullet(R; t_1,\ldots,t_k) \otimes_R M).
\]

Therefore, the cohomology of the complexes \(T_\bullet(R; s_1,\ldots,s_m) \otimes_R M\) and \(T_\bullet(R; t_1,\ldots,t_k) \otimes_R M\) form cohomological \(\delta\)-functors of the argument \(M\in R\text{-mod}\) taking values in \(R\text{-mod}\). A sequence of elements \(s_1,\ldots,s_m\) in a commutative ring \(R\) is called \emph{weakly proregular}, if the cohomology \(H^*(T_\bullet(R; s_1,\ldots,s_m) \otimes_R M)\) computes the right
derived functor $\mathbb{R}\Gamma_I^*(M)$ of the functor $\Gamma_I : R^{\text{mod}} \to R^{\text{mod}}$. In other words, it means that one should have

$$H^q(T^*(R; s_1, \ldots, s_m) \otimes_R K) = 0 \text{ for all } q > 0$$

when $K$ is an injective $R$-module. (After the theorem will have been proved, it will follow that the weak prerreularity is a property of the ideal $I$ and even $\sqrt{J}$ rather than of the generating sequence of elements.)

The fact that the two ideals $\sqrt{I}$ and $\sqrt{J}$ coincide in $R$ can be expressed as a finite system of equations on a finite set of elements involved (including the elements $s_j$, $t_i$ and the elements used as the coefficients of the expressions of powers of $t_i$ and vice versa). Denote by $R' \subset R$ the subring generated by this finite set of elements over $\mathbb{Z}$. Then we have

$$T^*(R; s_1, \ldots, s_m) = R \otimes_{R'} T^*(R'; s_1, \ldots, s_m),$$

$$T^*(R; t_1, \ldots, t_k) = R \otimes_{R'} T^*(R'; t_1, \ldots, t_k).$$

Hence it suffices to prove that the complexes of $R'$-modules $T^*(R'; s_1, \ldots, s_m)$ and $T^*(R'; t_1, \ldots, t_k)$ are homotopy equivalent. This reduces the assertion to be proved to a Noetherian ring $R$.

According to [21, Theorem 4.34], any finite sequence of elements in a Noetherian ring is weakly prerreular. Hence both complexes $T^*(R; s_1, \ldots, s_m)$ and $T^*(R; t_1, \ldots, t_k)$ compute the derived functor $\mathbb{R}\Gamma_I(R)$ for the $R$-module $R$. Now let $K^*$ be an injective $R$-module resolution of the module $R$; then we have a chain of quasi-isomorphisms

$$T^*(R; s_1, \ldots, s_m) \longrightarrow T^*(K^*; s_1, \ldots, s_m) \longrightarrow \Gamma_I(K^*)$$

$$T^*(R; t_1, \ldots, t_k) \longrightarrow T^*(K^*; t_1, \ldots, t_k) \longrightarrow \Gamma_I(K^*)$$

connecting $T^*(R; s_1, \ldots, s_m)$ with $T^*(R; t_1, \ldots, t_k)$ (where the bicomplexes are presumed to have been replaced with their total complexes). Finally, two finite complexes of free $R$-modules are homotopy equivalent whenever they are quasi-isomorphic. □

Second proof of Theorem 5.1. Clearly, it suffices to consider the case of a finitely generated ideal. Let $R$ be a commutative ring, and let $s_1, \ldots, s_m$ and $t_1, \ldots, t_m$ be two sequences of elements in $R$ generating ideals $I$ and $J$ such that $\sqrt{I} = \sqrt{J}$. Then, by Theorem 7.2, the full subcategories $R^{\text{mod}}[s_1, \ldots, s_m]_{\text{cta}}$ and $R^{\text{mod}}[t_1, \ldots, t_k]_{\text{cta}} \subset R^{\text{mod}}$ are the essential images of their reflector functors $\Delta_{s_1, \ldots, s_m}$ and $\Delta_{t_1, \ldots, t_k}$. Furthermore, for any $R$-module $C$, the $R$-modules $\Delta_{s_1, \ldots, s_m}(C)$ and $\Delta_{t_1, \ldots, t_k}(C)$ can be computed as the homology modules

$$H_0(\text{Hom}_R(T^*(R; s_1, \ldots, s_m), C)) \quad \text{and} \quad H_0(\text{Hom}_R(T^*(R; t_1, \ldots, t_k), C)).$$

Finally, according to Theorem 7.3, the two complexes of $R$-modules $T^*(R; s_1, \ldots, s_m)$ and $T^*(R; t_1, \ldots, t_k)$ are homotopy equivalent. Thus the functors $\Delta_{s_1, \ldots, s_m}$ and $\Delta_{t_1, \ldots, t_k}$ are isomorphic, and it follows that the two subcategories $R^{\text{mod}}[s_1, \ldots, s_m]_{\text{cta}}$ and $R^{\text{mod}}[t_1, \ldots, t_k]_{\text{cta}}$ in $R^{\text{mod}}$ coincide. □

Now we can set $R^{\text{mod}}_{I, \text{cta}} = R^{\text{mod}}[s_1, \ldots, s_m]_{\text{cta}}$ and $\Delta_I = \Delta_{s_1, \ldots, s_m}$.  

34
**Remark 7.4.** One can avoid the reduction to Noetherian rings in the proof of Theorem 7.3 using the following geometric argument instead. Let $X = \text{Spec } R$ denote the affine scheme with the ring of functions $R$. Then the category of quasi-coherent sheaves on $X$ is equivalent to the category of $R$-modules; let us denote by $\mathcal{M}$ the quasi-coherent sheaf corresponding to the $R$-module $M$. For any element $s \in R$, we have the related principal affine open subscheme $U_s \subset X$. Given a sequence $s_1, \ldots, s_m \in R$ generating an ideal $I \subset R$, consider the open subscheme $U = \bigcup_{j=1}^m U_{s_j} \subset X$. Then the open subscheme $U \subset X$ depends only on the radical $\sqrt{I} \subset R$ of the ideal $I$ and not on the generating sequence $s_1, \ldots, s_m$ itself.

The complex of $R$-modules $T^\bullet(R; s_1, \ldots, s_m)$ is quasi-isomorphic to the complex $\check{C}^\sim(R; s_1, \ldots, s_m)$ (13). Let $\check{C}(R; s_1, \ldots, s_m)$ denote the nonaugmented version of the same complex

$$\bigoplus_{j=1}^m R[s_j^{-1}] \longrightarrow \bigoplus_{j'<j} R[s_j^{-1}, s_j^{-1}] \longrightarrow \cdots \longrightarrow R[s_1^{-1}, \ldots, s_m^{-1}].$$

Then $\check{C}(R; s_1, \ldots, s_m) \otimes_R M$ is the Čech complex computing the quasi-coherent sheaf cohomology $H^\bullet(U, \mathcal{M}_U)$ in terms of the affine covering $U = \bigcup_{j=1}^m U_{s_j}$, while the complex $\check{C}^\sim(R; s_1, \ldots, s_m) \otimes_R M$ computes the cohomology of the cone of the restriction morphism between (the complexes representing) $M = H^\bullet(X, \mathcal{M})$ and $H^\bullet(U, \mathcal{M}_U)$. It follows that the cohomology of both the complexes $\check{C}(R; s_1, \ldots, s_m) \otimes_R M$ and $\check{C}^\sim(R; s_1, \ldots, s_m) \otimes_R M$ depend only on the open subscheme $U \subset X$ and not on the generating sequence of the ideal. Thus, given our two sequences $s_1, \ldots, s_m$ and $t_1, \ldots, t_k$, the complexes $T^\bullet(R; s_1, \ldots, s_m)$ and $T^\bullet(R; t_1, \ldots, t_k)$ are connected by a chain of quasi-isomorphisms

$$T^\bullet(R; s_1, \ldots, s_m) \leftarrow T^\bullet(R; s_1, \ldots, s_m, t_1, \ldots, t_k) \longrightarrow T^\bullet(R; t_1, \ldots, t_k).$$

Weak prregularity of ideals in Noetherian rings is explained, in the same geometric terms, by the facts that injective quasi-coherent sheaves on Noetherian schemes are flasque and their restrictions to open subschemes remain injective [14, §II.7]. Hence, denoting by $K^\bullet$ the complex of quasi-coherent sheaves on $X$ corresponding to an injective $R$-module resolution $K^\bullet$ of a given module $M$, one can compute $H^\bullet(U, \mathcal{M}_U)$ as the cohomology of the complex of sections $H^\bullet(K^\bullet(U))$. Furthermore, the morphism of complexes $K^\bullet(X) \rightarrow K^\bullet(U)$ is surjective, so one can replace the cone with the kernel, which leads one to the complex $\Gamma_I(K^\bullet)$ [27, Section 1].

This discussion is meant to suggest that, generally speaking, it is not a good idea to use the underived global sections of the restrictions of injective quasi-coherent sheaves to open subschemes in lieu of the cohomology of quasi-coherent sheaves on such open subschemes. Therefore, outside of the weakly prregular case, it is the derived functor $\mathbb{R}\Gamma_I$ that appears to be “naive”, and the cohomology of the complex $T^\bullet(R; s_1, \ldots, s_m) \otimes_R M$ or $\check{C}^\sim(R; s_1, \ldots, s_m) \otimes_R M$ is its “well-behaved replacement”. Similarly, the homology of the complex $\text{Hom}_R(T^\bullet(R; s_1, \ldots, s_m), C)$ is preferable to the derived functor $\mathbb{L}\Delta_I$ outside of the weakly prregular case [27, Section 3], while in the weakly prregular case they coincide [27, Lemma 2.7].
Recall the notation $T_1^\bullet(R; s) \subset T^\bullet(R; s)$, $n \geq 1$, from the proof of Lemma 6.7. The complex $T_1^\bullet(R; s)$ is quasi-isomorphic to the two-term complex $R \to R$. Set [21, Section 5]

$$T^\bullet(R; s_1, \ldots, s_m) = T_1^\bullet(R; s_1) \otimes_R \cdots \otimes_R T^\bullet(R; s_m).$$

Then $T_1^\bullet(R; s_1, \ldots, s_m)$ is a subcomplex in $T^\bullet(R; s_1, \ldots, s_m)$ and one has

$$T^\bullet(R; s_1, \ldots, s_m) = \lim_{\to n \geq 1} T_1^\bullet(R; s_1, \ldots, s_m).$$

**Lemma 7.5.** Let $R$ be a commutative ring and $I \subset R$ be an ideal generated by a sequence of elements $s_1, \ldots, s_m \in R$. Then for any $R$-module $C$ there is a natural short exact sequence of $R$-modules

$$0 \to \lim_{\leftarrow n \geq 1} H_1(\text{Hom}_R(T_1^\bullet(R; s_1, \ldots, s_m), C)) \to \Delta_I(C) \to \Lambda_I(C) \to 0.$$

**Proof.** The complexes $\text{Hom}_R(T_1^\bullet(R; s_1, \ldots, s_m), C)$ form a projective system of complexes and termwise surjective morphisms between them indexed by the integers $n \geq 1$. Furthermore, we have $H_0(\text{Hom}_R(T_1^\bullet(R; s_1, \ldots, s_m), C)) \simeq C/(s_1^n, \ldots, s_m^n)C$, where $(s_1^n, \ldots, s_m^n)$ denotes the ideal in $R$ generated by the elements $s_1^n, \ldots, s_m^n \in R$. Hence

$$\lim_{\to n} H_0(\text{Hom}_R(T_1^\bullet(R; s_1, \ldots, s_m), C)) \simeq \Lambda_I(C).$$

On the other hand, we have

$$\lim_{\to n} \text{Hom}_R(T_1^\bullet(R; s_1, \ldots, s_m), C) = \text{Hom}_R(T^\bullet(R; s_1, \ldots, s_m), C),$$

so

$$H_0(\lim_{\to n} \text{Hom}_R(T_1^\bullet(R; s_1, \ldots, s_m), C)) \simeq \Delta_I(C).$$

Finally, since the projective limit functor $\lim_{\to n \geq 1}$ has cohomological dimension 1, for any projective system of complexes of $R$-modules and termwise surjective morphisms between them $K_1^\bullet \leftarrow K_2^\bullet \leftarrow K_3^\bullet \leftarrow \cdots$, the hypercohomology spectral sequence reduces to “universal coefficient” short exact sequences

$$0 \to \lim_{\leftarrow n \geq 1} H^{q-1}(K_n^\bullet) \to H^q(\lim_{\leftarrow n \geq 1} K_n^\bullet) \to \lim_{\leftarrow n \geq 1} H^q(K_n^\bullet) \to 0,$$

where $\lim_{\leftarrow n \geq 1}$ denotes the (first) derived functor of projective limit. \hfill \Box

**Remark 7.6.** As a stronger version of Theorem 5.6, one can prove that the adjunction morphism $C \to \Delta_I(C)$ is surjective whenever the $R$-module $C$ is $s_j$-contraadjusted for every $j = 1, \ldots, m$. Indeed, by Corollary 6.10(b) the morphism $C \to \Delta_{s_1}(C)$ is surjective. Since the class of $s$-contraadjusted $R$-modules is closed under quotients for any $s \in R$, it follows that the $s_1$-contramodule $R$-module $\Delta_{s_1}(C)$ is $s_j$-contraadjusted for every $j = 2, \ldots, m$. Applying Corollary 6.10(b) again, we see that the morphism $\Delta_{s_1}(C) \to \Delta_{s_2}\Delta_{s_1}(C)$ is surjective, etc. Hence the morphism $C \to \Delta_{s_m} \cdots \Delta_{s_1}(C)$ is surjective. (Cf. [25, Section C.2].)

**Remark 7.7.** Given an arbitrary (not necessarily finitely generated) ideal $I$ in a commutative ring $R$, one denotes by $R-\text{mod}_{I\text{-tr}}$ the full subcategory in $R-\text{mod}$ consisting of all the $R$-modules that are $s$-contramodules for every $s \in I$. In this generality, one can show, using category-theoretic techniques, that the embedding functor
$R\text{-mod}_{I\text{-tra}} \to R\text{-mod}$ has a left adjoint functor $\Delta_I: R\text{-mod} \to R\text{-mod}_{I\text{-tra}}$ [29, Examples 4.1 (2–3)].

8. Covers, Envelopes, and Cotorsion Theories

This section consists almost entirely of the definitions. Its aim is to supply preliminary material for the remaining part of the paper, including the more general theoretical discussion in Sections 9–11 and, most importantly, the examples considered in Sections 12–13. All the proofs in this section are omitted and replaced with references to the book [36] and the papers [31, 5, 3, 6, 35, 29].

Let $R$ be an associative ring. Given two full subcategories $F \subset R\text{-mod}$, one denotes by $F^\perp$ the full subcategory $F^\perp = \{ C \in R\text{-mod} | \text{Ext}^1_R(F,C) = 0 \text{ for all } F \in F \}$.

A cotorsion theory (or cotorsion pair) in $R\text{-mod}$ is a pair of full subcategories $F, C \subset R\text{-mod}$ such that $F^\perp = C$ and $C^\perp = F$. Given a class of modules $S \subset R\text{-mod}$, one construct a cotorsion theory by setting $C = S^\perp$ and $F = C^\perp$. The cotorsion theory $(F, C) = (\perp(S^\perp), S^\perp)$ is said to be generated by $S$.

Example 8.1. Let $R\text{-mod}_{fl} \subset R\text{-mod}$ denote the class of all flat left $R$-modules. A left $R$-module $C$ is said to be cotorsion if it belongs to $R\text{-mod}_{fl}^\perp$, i.e., one has $\text{Ext}^1_R(F,C) = 0$ for every flat left $R$-module $F$. The class of all cotorsion left $R$-modules is denoted by $R\text{-mod}_{cot} \subset R\text{-mod}$. The pair of full subcategories $(R\text{-mod}_{fl}, R\text{-mod}_{cot})$ is called the flat cotorsion pair/theory in the category of left $R$-modules. This is the classical example of a cotorsion theory.

Notice that one actually needs to prove that $(R\text{-mod}_{fl}, R\text{-mod}_{cot})$ is a cotorsion theory, i.e., that any left $R$-module belonging to $R\text{-mod}_{cot}^\perp$ is flat. This is the result of [36, Lemma 3.4.1], which can be also obtained from Theorems 8.5–8.6 below.

One says that a cotorsion theory $(F, C)$ has enough projectives if every $R$-module $M$ can be included in a short exact sequence

$$0 \to C' \to F \to M \to 0, \quad F \in F, \ C' \in C,$$

and that $(F, C)$ has enough injectives if every for every $R$-module there exists a short exact sequence

$$0 \to M \to C \to F' \to 0, \quad F' \in F, \ C \in C.$$

Examples of exact sequences (14–15) in the flat cotorsion theory will be provided in Sections 12 and 13.

Remark 8.2. For any cotorsion theory $(F, C)$ in $R\text{-mod}$, the full subcategories $F$ and $C \subset R\text{-mod}$ are closed under extensions in $R\text{-mod}$. Therefore, they inherit the exact category structures from the abelian category $R\text{-mod}$ [4]. The intersection $F \cap C$
coincides with the class of injective objects in the exact category $F$ and with the class of projective objects in the exact category $C$.

If an $R$-module $M$ in a short exact sequence (14) belongs to $C$, then the $R$-module $F$, being an extension of $M \in C$ and $C' \in C$, belongs to $F \cap C$. So, when a cotorsion theory $(F, C)$ has enough projectives, the exact category $C$ has enough projective objects. Building up short exact sequences (14), one can then construct a projective resolution of a given object $M$ in the exact category $C$.

Similarly, if an $R$-module $M$ in a short exact sequence (15) belongs to $F$, then the $R$-module $C$ belongs to $F \cap C$. So if a cotorsion theory $(F, C)$ has enough injectives then the exact category $F$ has enough injective objects. Building up short exact sequences (15), one can construct an injective resolution of a given object $M \in F$.

In particular, Theorem 8.6 below implies that there are enough injective objects in the exact category of flat left $R$-modules and enough projective objects in the exact category of cotorsion left $R$-modules. Similarly, it will follow from Corollary 8.14 that the exact category of contraadjusted modules over a commutative ring has enough projective objects. These are important observations for the theory of contraherent cosheaves [25, Section 4.4].

The following lemma is due to Salce [31] (see also [11, Lemma 5.20], [5, second paragraph of the proof of Theorem 10], or [25, Lemma 1.1.3]).

**Lemma 8.3.** A cotorsion theory in $R$–$\text{mod}$ has enough projectives if and only if it has enough injectives. □

A cotorsion theory having enough projectives/injectives is called complete.

Given a class of modules $S \subset R$–$\text{mod}$, one says that an $R$-module $F$ is a transfinite iterated extension (in the sense of the inductive limit) of modules from $S$ if there exists an ordinal $\alpha$ and an increasing chain of submodules $F_i \subset F$ indexed by the ordinals $i \leq \alpha$ such that $F_0 = 0$, $F_\alpha = F$, $F_i = \bigcup_{j<i} F_j$ for all limit ordinals $i \leq \alpha$, and $F_{i+1}/F_i \in S$ for all $i < \alpha$. The class of all modules representable as transfinite iterated extensions of modules from $S$ is denoted by $\text{filt}(S)$.

The next result is known as the Eklof lemma [5, Lemma 1].

**Lemma 8.4.** For any class of modules $S \subset R$–$\text{mod}$, one has $S^\perp = \text{filt}(S)^\perp$. □

The following important existence theorem is due to Eklof and Trlifaj [5, Theorem 10].

**Theorem 8.5.** (a) Any cotorsion theory $(F, C)$ generated by a set (rather than a proper class) of modules $S \subset R$–$\text{mod}$ is complete.

(b) If the $R$-module $R$ belongs to $S$, then the class $F$ consists precisely of all the direct summands of the modules belonging to $\text{filt}(S)$. □

A class of modules $F \subset R$–$\text{mod}$ is called deconstructible if there exists a set of modules $S \subset R$–$\text{mod}$ such that $F = \text{filt}(S)$. The following corollary of Theorem 8.5 is due to Enochs [3, Proposition 2].

**Theorem 8.6.** For any associative ring $R$, the class of all flat $R$-modules is deconstructible. Therefore, the flat cotorsion theory in $R$–$\text{mod}$ is complete. □
According to Lemma 8.4, in any cotorsion theory in $R\text{-mod}$ the class $F$ is closed under transfinitely iterated extensions in the sense of the inductive limit; this includes finitely iterated extensions and infinite direct sums. The class $C$ is closed under extensions and the infinite products (and, in fact, even under transfinitely iterated extensions in the sense of the projective limit [5, Proposition 18], [29, Lemma 4.5], cf. Lemma 9.7 below). Both the classes $F$ and $C$ are closed under direct summands.

**Lemma 8.7.** Let $(F, C)$ be a cotorsion theory in $R\text{-mod}$. Then the following four conditions are equivalent:

1. $\text{Ext}_R^2(F, C) = 0$ for all $F \in F$ and $C \in C$;
2. $\text{Ext}_R^n(F, C) = 0$ for all $F \in F$, $C \in C$, and $n \geq 2$;
3. the class $F$ is closed under the kernels of surjective morphisms;
4. the class $C$ is closed under the cokernels of injective morphisms.

**Proof.** See [35, Lemma 6.17] (cf. Lemma 9.4 below). □

A cotorsion theory in $R\text{-mod}$ is said to be hereditary if it satisfies the equivalent conditions of Lemma 8.7. In particular, the flat cotorsion theory is hereditary, because the condition (iii) clearly holds.

Let $F \subset R\text{-mod}$ be a class of $R$-modules. An $R$-module morphism $f: F \rightarrow M$ with $F \in F$ is called an $F$-precover of an $R$-module $M$ if any morphism $f': F' \rightarrow M$ with $F' \in F$ factorizes through $f$ (that is for any such $f'$ there exists a morphism $u: F' \rightarrow F$ such that $f' = fu$). A special $F$-precover of an $R$-module $M$ is a surjective morphism $f: F \rightarrow M$ with $F \in F$ and $\text{ker}(f) \in F^\perp$; in other words, it is a morphism that can be included into a short exact sequence like (14).

An $F$-precover $f: F \rightarrow M$ is called an $F$-cover of the $R$-module $M$ if for any endomorphism $u: F \rightarrow F$ the equation $fu = f$ implies that $u$ is an automorphism of $F$, i. e., $u$ is invertible. Clearly, an $F$-cover of a given module $M$, if it exists, is unique up to a (nonunique) isomorphism.

**Lemma 8.8.** (a) Any special $F$-precover is an $F$-precover.

(b) If the class $F$ is closed under extensions in $R\text{-mod}$, then the kernel of any $F$-cover belongs to $F^\perp$. In particular, any surjective $F$-cover is special.

(c) Assume that an $R$-module $M$ admits an $F$-cover. In this case, an $F$-precover $f: F \rightarrow M$ is an $F$-cover if and only if the $R$-module $F$ has no nonzero direct summands contained in $\text{ker}(f)$.

**Proof.** Part (a) is [36, Proposition 2.1.3]. Part (b) is known as Wakamatsu’s lemma; this is [36, Lemma 2.1.1]. Part (c) is [36, Corollary 1.2.8]. □

Dually, let $C \subset R\text{-mod}$ be another class of $R$-modules. An $R$-module morphism $g: M \rightarrow C$ with $C \in C$ is called a $C$-preenvelope of $M$ if any morphism $g': M \rightarrow C'$ with $C' \in C$ factorizes through $g$ (that is for any $g'$ there exists a morphism $u: C \rightarrow C'$ such that $g' = ug$). A special $C$-preenvelope of an $R$-module $M$ is an injective morphism $g: M \rightarrow C$ with $C \in C$ and $\text{coker}(f) \in C^\perp$; in other words, it is a morphism that can be included in a short exact sequence like (15).
A \( C \)-preenvelope \( g : M \to C \) is called a \( C \)-envelope of the \( R \)-module \( M \) if for any endomorphism \( u : C \to C \) the equation \( ug = g \) implies that \( u \) is an automorphism of \( C \). A \( C \)-envelope of a given module \( M \), if it exists, is unique up to a (nonunique) isomorphism.

**Lemma 8.9.** (a) Any special \( C \)-preenvelope is a \( C \)-preenvelope.
(b) If the class \( C \) is closed under extensions in \( R \text{-mod} \), then the cokernel of any \( C \)-envelope belongs to \( {}^\perp C \). In particular, any injective \( C \)-envelope is special.
(c) Assume that an \( R \)-module \( M \) admits a \( C \)-envelope. In this case, a \( C \)-preenvelope \( g : M \to C \) is a \( C \)-envelope if and only if the \( R \)-module \( C \) has no proper direct summands containing \( \text{im} \, g \).

**Proof.** Part (a) is [36, Proposition 2.1.4]. Part (b) is Wakamatsu’s lemma [36, Lemma 2.1.2]. Part (c) is [36, Corollary 1.2.3].

**Lemma 8.10.** (a) Let \( F \subset R \text{-mod} \) be a full subcategory closed under finite direct sums. Then the direct sum of any two \( F \)-precovers is an \( F \)-precover, the direct sum of any two special \( F \)-precovers is a special \( F \)-precover, and the direct sum of any two \( F \)-covers is an \( F \)-cover.
(b) Let \( C \subset R \text{-mod} \) be a full subcategory closed under finite direct sums. Then the direct sum of any two \( C \)-preenvelopes is a \( C \)-envelope, the direct sum of any two special \( C \)-preenvelopes is a special \( C \)-preenvelope, and the direct sum of any two \( C \)-envelopes is a \( C \)-envelope.

**Proof.** Part (a): the assertions concerning precovers and special precovers are straightforward. Concerning the covers, let \( M_1 \) and \( M_2 \) be two \( R \)-modules and \( f_i : F_i \to M_i \) be their \( F \)-covers. We have to show that \( f_1 \oplus f_2 : F_1 \oplus F_2 \to M_1 \oplus M_2 \) is an \( F \)-cover. This is obvious when \( \text{Hom}_R(F_1, F_2) = 0 \) or \( \text{Hom}_R(F_2, F_1) = 0 \), because endomorphisms of \( F_1 \oplus F_2 \) are then represented by triangular matrices, and a triangular matrix with invertible diagonal entries is invertible.

In the general case, one observes that any \( 2 \times 2 \) matrix with an invertible diagonal entry can be naturally decomposed into the product of an upper triangular and a lower triangular matrices (by a kind of Gaussian elimination). The original matrix is invertible whenever the diagonal entries of its two triangular factors are. In the situation at hand, the latter is guaranteed by the condition that the two original morphisms \( f_i \) are covers [36, Remark 1.4.2]. The proof of part (b) is similar.

**Theorem 8.11.** (a) Let \( F \subset R \text{-mod} \) be a full subcategory closed under filtered inductive limits. Suppose that an \( R \)-module \( M \) admits an \( F \)-precover. Then \( M \) also admits an \( F \)-cover.
(b) Let \( F \subset R \text{-mod} \) be a full subcategory closed under extensions and filtered inductive limits. Set \( C = F^\perp \). Suppose that an \( R \)-module \( M \) admits a special \( C \)-preenvelope with the cokernel belonging to \( F \). Then \( M \) also admits a \( C \)-envelope.

**Proof.** These two results are due to Enochs. Part (a) is [36, Theorem 2.2.8] or [6, Theorem 1.2]. Part (b) is easily obtained from [36, Theorem 2.2.6]. For generalizations, see [29, Theorem 2.7 or Corollary 4.17] in the case of part (a), and [29, Corollary 4.18 or Remark 4.19] in the case of part (b).
The following theorem is due to El Bashir [6, Theorem 3.2] (for a generalization, see [29, Theorem 2.5 and Proposition 2.6]).

**Theorem 8.12.** Let \( F \subset R\text{-mod} \) be a full subcategory closed under direct sums and filtered inductive limits. Suppose that there is a set (not a proper class) of modules \( S \subset F \) such that every module from \( F \) is a filtered inductive limit of modules from \( S \). Then every \( R \)-module has an \( F \)-cover. □

**Corollary 8.13.** (a) For any associative ring \( R \), any left \( R \)-module has a flat cover.

(b) For any associative ring \( R \), any left \( R \)-module has a cotorsion envelope.

**Proof.** Since the class of all flat \( R \)-modules \( F = R\text{-mod}_{fl} \) is closed under filtered inductive limits and consists precisely of the filtered inductive limits of finitely generated free modules, Theorem 8.12 applies, proving part (a). In view Lemma 8.8(b) and Lemma 8.3, this provides another proof of completeness of the flat cotorsion theory [3, Section 3]. Conversely, one can deduce part (a) from Theorem 8.6 using Lemma 8.8(a) and Theorem 8.11(a). Part (b) of the corollary follows from completeness of the flat cotorsion theory by means of Theorem 8.11(b), since the class of all flat left \( R \)-modules is closed under filtered inductive limits. □

Now we return to our usual setting of a commutative ring \( R \). An \( R \)-module \( C \) is said to be **contraadjusted** if it is \( s \)-contraadjusted for every \( s \in R \). An \( R \)-module \( F \) is called **very flat** if it is a direct summand of a transfinitely iterated extension (in the sense of the inductive limit) of \( R \)-modules of the form \( R[s^{-1}] \), \( s \in R \). We denote the class of all contraadjusted \( R \)-modules by \( R\text{-mod}_{ctaa} \) and the class of all very flat \( R \)-modules by \( R\text{-mod}_{vfl} \subset R\text{-mod} \).

**Corollary 8.14.** For any commutative ring \( R \), the pair of full subcategories \((R\text{-mod}_{vfl}, R\text{-mod}_{ctaa})\) is a hereditary complete cotorsion theory in \( R\text{-mod} \).

**Proof.** This is the result of [25, Section 1.1]. The assertion that \((R\text{-mod}_{vfl}, R\text{-mod}_{ctaa})\) is a complete cotorsion theory is provided by Theorem 8.5 applied to the set of \( R \)-modules \( S = \{ R[s^{-1}] \mid s \in R \} \). Furthermore, any cotorsion theory generated by a class of modules of projective dimension \( \leq 1 \) is hereditary. □

The pair of full subcategories \((R\text{-mod}_{vfl}, R\text{-mod}_{ctaa})\) in \( R\text{-mod} \) is called the **very flat cotorsion theory**. According to Corollary 8.14, any \( R \)-module has a special very flat precover and a special contraadjusted preenvelope. On the other hand, it is proved in the paper [34] that for any Noetherian commutative ring \( R \) with infinite spectrum there exist an \( R \)-module having no very flat cover and an \( R \)-module having no contraadjusted envelope.

For a Noetherian domain \( R \) with finite spectrum, it is shown in [34, Lemma 2.13] that the very flat cotorsion theory in \( R\text{-mod} \) coincides with the flat cotorsion theory, \( R\text{-mod}_{vfl} = R\text{-mod}_{fl} \) and \( R\text{-mod}_{ctaa} = R\text{-mod}_{cot} \). In Section 13 below, we will extend this result to all Noetherian commutative rings with finite spectrum. On the other hand, for a von Neumann regular commutative ring \( R \), every very flat \( R \)-module is projective and every \( R \)-module is contraadjusted [34, Example 2.9].
9. Contramodules and Relatively Cotorsion Modules

According to Example 8.1 and Lemma 8.7, a left module \( C \) over an associative ring \( A \) is said to be cotorsion if for every flat left \( A \)-module \( F \) one has \( \text{Ext}^1_A(F, C) = 0 \), or equivalently, for every flat left \( A \)-module \( F \) one has \( \text{Ext}^q_A(F, C) = 0 \) for all \( q \geq 1 \). The class of cotorsion left \( A \)-modules is closed under the cokernels of injective morphisms, extensions, and infinite products (see Lemma 9.4 below for a discussion in the somewhat greater generality of relatively cotorsion modules).

**Lemma 9.1.** For any homomorphism of associative rings \( A \rightarrow A' \), any cotorsion \( A' \)-module is also a cotorsion \( A \)-module (in the induced \( A \)-module structure).

**Proof.** For any flat left \( A \)-module \( F \), there is a natural isomorphism \( \text{Ext}^q_A(F, C) \cong \text{Ext}^q_{A'}(A' \otimes_A F, C) \) for all \( q \geq 0 \); and the \( A' \)-module \( A' \otimes_A F \) is flat. \( \square \)

Let \( R \) be a commutative ring and \( I \subset R \) be an ideal. An \( R \)-module \( C \) is said to be an \( I \)-contramodule (or an \( I \)-contramodule \( R \)-module) if \( C \) is an \( s \)-contramodule for every \( s \in I \) (cf. Theorem 5.1). The full subcategory of \( I \)-contramodule \( R \)-modules is denoted by \( R \text{-mod}_{I,\text{ctra}} \subset R \text{-mod} \).

**Remark 9.2.** Any \( I \)-contramodule \( R \)-module is actually a module over the localization \( (1 + I)^{-1}R \) of the ring \( R \) with respect to the multiplicative set \( (1 + I) = \{1 + s \mid s \in I\} \). Indeed, for any \( s \in I \), the inverse map to the action of \( 1 - s \) in an \( I \)-contramodule \( R \)-module \( C \) can be constructed as

\[
(1 - s)^{-1}(c) = \sum_{n=0}^{\infty} s^n c.
\]

In particular, if \( m \) is a maximal ideal in a commutative ring \( R \), then any \( m \)-contramodule \( R \)-module is actually a module over the local ring \( R_m = (R \setminus m)^{-1}R \).

When \( I \) is an ideal in a Noetherian ring \( R \), one can show that the category of \( I \)-contramodule \( R \)-modules is isomorphic to the category of \( \Lambda_I(R) \)-modules [24, Theorem B.1.1].

The aim of this section is to prove the following theorem (cf. [24, Proposition B.10.1] and [25, Proposition 1.3.7(a)]).

**Theorem 9.3.** Let \( m \) be a maximal ideal in a Noetherian ring \( R \). Then every \( m \)-contramodule \( R \)-module is a cotorsion \( R \)-module.

More generally, let \( A \rightarrow A' \) be a homomorphism of associative rings. Let us call a left \( A \)-module \( C \) cotorsion relative to \( A' \) (or \( A\mid A' \)-cotorsion) if \( \text{Ext}^1_A(F, C) = 0 \) for every flat \( A \)-module \( F \) such that the left \( A' \)-module \( A' \otimes_A F \) is projective.

**Lemma 9.4.** (a) For any \( A\mid A' \)-cotorsion left \( A \)-module \( C \) and any flat left \( A \)-module \( F \) such that the left \( A' \)-module \( A' \otimes_A F \) is projective one has \( \text{Ext}^q_A(F, C) = 0 \) for all \( q \geq 1 \).

(b) The class of all \( A\mid A' \)-cotorsion left \( A \)-modules is closed under the cokernels of injective morphisms, extensions, and infinite products in \( A \text{-mod} \).
Proof. Part (a): arguing by induction in \( q \geq 1 \), choose a surjective homomorphism \( P \twoheadrightarrow F \) onto the \( A \)-module \( F \) from a projective left \( A \)-module \( P \). Denoting the kernel of this \( A \)-module morphism by \( G \), we have a short exact sequence of flat left \( A \)-modules \( 0 \twoheadrightarrow G \twoheadrightarrow P \twoheadrightarrow F \twoheadrightarrow 0 \) and a short exact sequence of left \( A' \)-modules \( 0 \twoheadrightarrow A' \otimes_A G \twoheadrightarrow A' \otimes_A P \twoheadrightarrow A' \otimes_A F \twoheadrightarrow 0 \). Since the \( A' \)-modules \( A' \otimes_A P \) and \( A' \otimes_A F \) are projective, the \( A' \)-module \( A' \otimes_A G \) is projective, too. On the other hand, we have \( \text{Ext}_A^q(F,C) \cong \text{Ext}_{A'}^{q-1}(G,C) \) for \( q \geq 2 \), and now the right-hand side of this isomorphism vanishes by the induction assumption.

Part (b): we will only check closedness under the cokernels of injective morphisms. Let \( 0 \twoheadrightarrow C \twoheadrightarrow D \twoheadrightarrow E \twoheadrightarrow 0 \) be a short exact sequence of left \( A \)-modules; suppose that the modules \( C \) and \( D \) are \( A|A' \)-cotorsion. Let \( F \) be a flat left \( A \)-module such that \( A' \otimes_A F \) is a projective left \( A' \)-module. Applying part (a) for \( q = 2 \), we conclude from the exact sequence \( \text{Ext}_A^1(F,D) \twoheadrightarrow \text{Ext}_A^1(F,E) \twoheadrightarrow \text{Ext}_A^2(F,C) \) that \( \text{Ext}_A^1(F,E) = 0 \). So \( E \) is an \( A|A' \)-cotorsion left \( A \)-module, too. \( \square \)

We will obtain Theorem 9.3 as a particular case of the following result, which extends the assertion of [24, Proposition B.10.1] to non-Noetherian rings.

**Theorem 9.5.** Let \( R \) be a commutative ring and \( I \subset R \) a finitely generated ideal. Then every \( I \)-contramodule \( R \)-module is \( R/(R/I) \)-cotorsion (i.e., cotorsion relative to the quotient ring \( R/I \)).

The proof of Theorem 9.5 occupies the rest of this section. We start with the following lemma, which can be found in [24, Lemma B.10.2], but the argument is standard and goes back, at least, to the famous [1, Theorem P].

**Lemma 9.6.** Let \( A \) be an associative ring and \( J \subset A \) an ideal such that \( J^n = 0 \) for a certain \( n \geq 1 \). Suppose that \( P \) is a flat left \( A \)-module such that \( P/JP \) is a projective left \( A/J \)-module. Then the \( A \)-module \( P \) is projective.

Similarly, if the \( P \) is a flat \( A \)-module and \( P/JP \) is a free \( A/J \)-module, then \( P \) is a free \( A \)-module.

Proof. Let \( G \) be a free \( A \)-module such that \( P/JP \) is a direct summand of \( G/JG \). Then there exists an idempotent endomorphism \( e \) of the \( A/J \)-module \( G/JG \) such that the \( A/J \)-module \( P/JP \) is isomorphic to \( e(G/JG) \). The functor \( A/J \otimes_A - \) taking \( G \) to \( G/JG \) provides an associative ring homomorphism \( \pi \colon \text{Hom}_A(G,G) \twoheadrightarrow \text{Hom}_{A/J}(G/JG,G/JG) \). Since the \( A \)-module \( G \) is projective, the homomorphism \( \pi \) is surjective. Its kernel \( I = \ker \pi = \text{Hom}_A(G,JG) \subset \text{Hom}_A(G,G) \) satisfies \( I^n = 0 \); so one can lift idempotents modulo \( I \). Let \( f \in \text{Hom}_A(G,G) \) be an idempotent endomorphism for which \( \pi(f) = e \). The projective \( A \)-module \( fG \) is endowed with a natural homomorphism onto the \( A/J \)-module \( fG/J(fG) \cong e(G/JG) \cong P/JP \), which can lifted to an \( A \)-module morphism \( l \colon fG \twoheadrightarrow P \).

We claim that \( l \) is an isomorphism. Indeed, let \( K \) and \( L \) denote its kernel and cokernel, respectively. Then \( L/JL = 0 \), since the morphism \( fG \twoheadrightarrow P/JP \) was surjective. Given that \( J^n = 0 \), it follows that \( L = 0 \). Now we have a short exact sequence of \( A \)-modules \( 0 \twoheadrightarrow K \twoheadrightarrow fG \twoheadrightarrow P \twoheadrightarrow 0 \). The \( A \)-module \( P \) being flat
by assumption, the sequence remains exact after taking the tensor product with $A/J$ over $A$; so the short sequence $0 \rightarrow K/JK \rightarrow fG/J(fG) \rightarrow P/JP \rightarrow 0$ is also exact. We have shown that $K/JK = 0$ and it follows that $K = 0$.

To prove the second assertion, it suffices to say that when $e = 1$ one can choose $f = 1$. 

The following lemma is a particular case of [24, Lemma B.10.3].

**Lemma 9.7.** Let $A$ be an associative ring, $F$ a left $A$-module, and $C_1 \leftarrow C_2 \leftarrow C_3 \leftarrow \cdots$ a projective system of left $A$-modules. Assume that the induced maps $\text{Hom}_A(F,C_{n+1}) \rightarrow \text{Hom}_A(F,C_n)$ are surjective for all $n \geq 1$ and $\text{Ext}^1_A(F,C_n) = 0$ for all $n \geq 1$. Then $\text{Ext}^1_A(F, \lim \leftarrow_{n\geq 1} C_n) = 0$.

**Proof.** This is almost the dual version of (the ordinal $\omega$ particular case of) the Eklof lemma (Lemma 8.4 above; cf. [5, Proposition 18] and [29, Lemma 4.5]). Suppose we are given a short exact sequence of left $A$-modules

$$0 \rightarrow \lim \leftarrow_{n\geq 1} C_n \xrightarrow{i} M \xrightarrow{p} F \rightarrow 0.$$ 

The push-forwards with respect to the projection morphisms $\lim \leftarrow_{m\geq 1} C_m \rightarrow C_n$ provide a projective system of short exact sequences

$$0 \rightarrow C_n \xrightarrow{i_n} M_n \xrightarrow{p_n} F \rightarrow 0.$$ 

By assumption, all the short exact sequences in this projective system split. We only have to show that one can choose the splittings $t_n: M_n \rightarrow C_n$, $t_n i_n = \text{id}_{C_n}$ in a compatible way, as then composing with the morphisms $M \rightarrow M_n$ and passing to the projective limit will provide the desired splitting $M \rightarrow \lim \leftarrow_{n\geq 1} C_n$.

Given a splitting $t_n: M_n \rightarrow C_n$ and a splitting $t'_{n+1}: M_{n+1} \rightarrow C_{n+1}$, the difference between the two compositions $M_{n+1} \rightarrow M_n \rightarrow C_n$ and $M_{n+1} \rightarrow C_{n+1} \rightarrow C_n$ is a morphism $M_{n+1} \rightarrow C_n$ that vanishes in the composition with $i_{n+1}: C_{n+1} \rightarrow M_{n+1}$, and therefore factorizes through $p_{n+1}: M_{n+1} \rightarrow F$. We have obtained a morphism $f: F \rightarrow C_n$. By assumption, it can be lifted to a morphism $g: F \rightarrow C_{n+1}$. Adding the composition $M_{n+1} \rightarrow F \rightarrow C_{n+1}$ to the splitting $t'_{n+1}$, that is replacing $t'_{n+1}$ by $t_{n+1} = t'_{n+1} + gp_{n+1}$, provides a splitting $t_{n+1}$ compatible with $t_n$. Now one can proceed by induction in $n$. \hfill \Box

**Corollary 9.8.** Let $I$ be a finitely generated ideal in a commutative ring $R$. Then every $I$-adically separated and complete $R$-module is $R/(R/I)$-cotorsion.

**Proof.** Let $F$ be a flat $R$-module such that the $R/I$-module $F/IF$ is projective. Applying Lemma 9.6 to the ring $A = R/I^n$ with the ideal $J = I/I^n$ and the $A$-module $F/I^n F$, we conclude that the $R/I^n$-module $F/I^n F$ is projective.

Let $C$ be an $I$-adically complete and separated $R$-module; so $C = \lim \leftarrow_{n\geq 1} C/I^n C$. It suffices to check that the conditions of Lemma 9.7 are satisfied for the ring $A = R$, the module $F$, and the projective system of modules $(C_n = C/I^n C)_{n=1}^\infty$. Since $F$ is a flat $R$-module, we have

$$\text{Ext}^1_R(F,C_n) = \text{Ext}^1_R(I^n F, C_n) = 0.$$ 

44
Finally, any morphism $F \to C_n$ factorizes through the surjection $F \to F/I^{n+1}F$, providing a morphism $F/I^{n+1}F \to C_n$, which can be lifted to a morphism $F/I^{n+1}F \to C_{n+1}$.

**Lemma 9.9.** Let $C_1 \leftarrow C_2 \leftarrow C_3 \leftarrow \cdots$ be a projective system of $I$-adically separated and complete $R$-modules. Then the projective limit $\lim_{n \geq 1} C_n$ and the derived projective limit $\check{\lim}_{n \geq 1} C_n$ are $R/(R/I)$-cotorsion $R$-modules.

**Proof.** The $R$-modules $\lim_{n \geq 1} C_n$ and $\check{\lim}_{n \geq 1} C_n$ are computed as, respectively, the kernel and the cokernel of the morphism $f = \text{id} - \text{shift}: \prod_{n=1}^{\infty} C_n \rightarrow \prod_{n=1}^{\infty} C_n$.

Since the class of $I$-contramodule $R$-modules is closed under the kernels, cokernels, and infinite products in $R\text{-mod}$, both $\lim_{n \geq 1} C_n$ and $\check{\lim}_{n \geq 1} C_n$ are $I$-contramodules. Furthermore, the $R$-module $\lim_{n \geq 1} C_n$ is $I$-adically separated as a submodule of the $I$-adically separated module $\prod_{n} C_n$. By Corollary 9.8, we can conclude that $\lim_{n \geq 1} C_n$ and $\prod_{n} C_n$ are $R/(R/I)$-cotorsion $R$-modules.

Now we have short exact sequences of $R$-modules

$$0 \longrightarrow \lim_{n \geq 1} C_n \longrightarrow \prod_{n=1}^{\infty} C_n \longrightarrow \text{im } f \longrightarrow 0$$

$$0 \longrightarrow \text{im } f \longrightarrow \prod_{n=1}^{\infty} C_n \longrightarrow \check{\lim}_{n \geq 1} C_n \longrightarrow 0.$$ 

According to Lemma 9.4, the class of $R/(R/I)$-cotorsion $R$-modules is closed under the cokernels of injective morphisms. Thus $\text{im } f$ and $\check{\lim}_{n \geq 1} C_n$ are also $R/(R/I)$-cotorsion.

**Proof of Theorem 9.5.** We will show that the $R$-module $\Delta_I(C)$ is $R/(R/I)$-cotorsion for every $R$-module $C$; since one has $\Delta_I(C) = C$ for any $I$-contramodule $R$-module $C$, this is sufficient. The argument is based on the exact sequence from Lemma 7.5. The elements $s^n_1, \ldots, s^n_m \in R$ act in the complex $T^*_n(R; s_1, \ldots, s_m)$ by contractible endomorphisms, so all the homology modules of the complex $\text{Hom}_R(T^*_n(R; s_1, \ldots, s_m), C)$ are annihilated by each of these elements. Hence $H_n(\text{Hom}_R(T^*_n(R; s_1, \ldots, s_m), C))$ are $I$-adically separated and complete $R$-modules. By Lemma 9.9, both the leftmost and the rightmost terms of our short exact sequence are $R/(R/I)$-cotorsion $R$-modules; and it follows that the middle term is $R/(R/I)$-cotorsion, too.

---

**10. Flat, Projective, and Free Contramodules**

The aim of this section and the next one is to discuss Enochs’ classification of flat cotorsion modules over Noetherian rings [7] and explain the connection with free contramodules over Noetherian local rings, as stated in [25, Theorem 1.3.8].

We start with the following application of the contramodule Nakayama lemma.
Lemma 10.1. Let $R$ be a commutative ring, $I \subset R$ a finitely generated ideal, $C$ an $R$-module, and $\Lambda_I(C)$ its $I$-adic completion. Assume that $\text{Tor}_1^R(R/I, \Lambda_I(C)) = 0$. Then the natural morphism $\Delta_I(C) \to \Lambda_I(C)$ is an isomorphism.

Proof. The morphism $f : \Delta_I(C) \to \Lambda_I(C)$ is surjective by Lemma 7.5 (the proof of Corollary 6.8(a) is also applicable, and shows additionally that $\Lambda_I(C) = \Lambda_I(\Delta_I(C))$). Its kernel $K = \ker f$ is an $I$-contramodule, because both $\Delta_I(C)$ and $\Lambda_I(C)$ are. Furthermore, the maps

$$C/I^nC \to \Delta_I(C)/I^n\Delta_I(C) \to \Lambda_I(C)/I^n\Lambda_I(C)$$

are isomorphisms for any $R$-module $C$ and any $n \geq 1$ (see the proof of Theorem 5.8 or the argument in Remark 6.9). Now from the short exact sequence $0 \to K \to \Delta_I(C) \to \Lambda_I(C) \to 0$ we obtain the long exact sequence

$$\cdots \to \text{Tor}_1^R(R/I, \Lambda_I(C)) \to K/IK \to \Delta_I(C)/I\Delta_I(C) \to \Lambda_I(C)/I\Lambda_I(C) \to 0,$$

and the vanishing of $\text{Tor}_1^R(R/I, \Lambda_I(C))$ implies the vanishing of $K/IK$. Finally, it remains to apply the Nakayama Lemma 4.2, extended to any finite number of variables as mentioned in Remark 4.3, in order to conclude that $K = IK$ implies $K = 0$ for an $I$-contramodule $R$-module $K$. □

The following lemma is specific to Noetherian rings.

Lemma 10.2. Let $R$ be a Noetherian commutative ring, $I \subset R$ an ideal, and $C$ an $I$-adically separated and complete $R$-module. Assume that the $R/I^n$-module $C/I^nC$ is flat for every $n \geq 1$. Then the $R$-module $C$ is flat.

Proof. It suffices to show that the functor $M \mapsto C \otimes_R M$ is exact on the category of finitely generated $R$-modules $M$. We consider two functors on the category of finitely generated $R$-modules:

$$M \mapsto C \otimes_R M \quad \text{and} \quad M \mapsto \Lambda_I(C \otimes_R M) = \lim_{n \to 1} (C/I^nC \otimes_{R/I^n} M/I^nM).$$

First let us check that the functor $M \mapsto \Lambda_I(C \otimes_R M)$ is exact. Indeed, let $0 \to K \to L \to M \to 0$ be a short exact sequence of finitely generated $R$-modules. Then there are short exact sequences

$$(16) \quad 0 \to K/(K \cap I^nL) \to L/I^nL \to M/I^nM \to 0, \quad n \geq 1.$$

According to the Artin–Rees lemma, there exists $m \geq 0$ such that $K \cap I^nL = I^{n-m}(K \cap I^mL)$ for all $n \geq m$. Consequently, one has

$$I^nK \subset K \cap I^nL \subset I^{n-m}K \quad \text{for all} \quad n \geq m,$$

and therefore

$$\lim_{n \to 1} (C \otimes_R K/(K \cap I^nL)) \simeq \lim_{n \to 1} (C \otimes_R K/I^nK).$$
Furthermore, the sequence (16) is a short exact sequence of $R/I^n$-modules, so flatness of the $R/I^n$-module $C/I^nC$ implies exactness of the sequence

$$0 \longrightarrow C \otimes_R K/(K \cap I^nL) \longrightarrow C \otimes_R L/I^nL \longrightarrow C \otimes_R M/I^nM \longrightarrow 0.$$  

The sequences (17) form a projective system of short exact sequences and termwise surjective morphisms between them, so the passage to the projective limit over $n \geq 1$ preserves exactness of (17). We have constructed the desired short exact sequence

$$0 \longrightarrow \lim_{\rightarrow n \geq 1} (C \otimes_R K/I^nK) \longrightarrow \lim_{\rightarrow n \geq 1} (C \otimes_R L/I^nL) \longrightarrow \lim_{\rightarrow n \geq 1} (C \otimes_R M/I^nM) \longrightarrow 0.$$  

Now we have a morphism of functors

$$C \otimes_R M \longrightarrow \Lambda_I(C \otimes_R M)$$

of the argument $M$ running over the abelian category of finitely generated $R$-modules. We claim that this is an isomorphism of functors. The functor in the left-hand side is right exact (preserves cokernels), while the functor in the right-hand side is exact, as we have just shown. When $M$ is a finitely generated free $R$-module, the map (18) is an isomorphism, because the map $C \longrightarrow \Lambda_I(C)$ is. In the general case, one can present a finitely generated $R$-module $M$ as the cokernel of a morphism of finitely generated free $R$-modules $f: G \longrightarrow F$, and then

$$C \otimes_R \text{coker} f = \text{coker}(C \otimes_R f) = \Lambda_I(C \otimes_R f) = \Lambda_I(C \otimes \text{coker} f)$$

because $C \otimes_R f = \Lambda_I(C \otimes_R f)$.  

Finally, since the two functors are isomorphic and the functor $M \longmapsto \Lambda_I(C \otimes_R M)$ is exact, the functor $M \longmapsto C \otimes_R M$ is exact, too.  

As a corollary, we obtain the following result [24, Lemma B.9.2], many versions and generalizations of which are known by now. See [25, Proposition C.5.4] for the noncommutative Noetherian case; or [25, Corollary D.1.7] and [29, Corollary 6.15] for far-reaching generalizations to contramodules over topological rings.

**Corollary 10.3.** Let $R$ be a Noetherian commutative ring, $I \subset R$ an ideal, and $C$ an $I$-contramodule $R$-module. Then

(a) $C$ is a flat $R$-module if and only if $C/I^nC$ is a flat $R/I^n$-module for every $n \geq 1$;

(b) whenever either of the two conditions in (a) is satisfied, $C$ is $I$-adically separated.

**Proof.** Clearly, if $C$ is a flat $R$-module then $C/I^nC$ is a flat $R/I^n$-module. Conversely, assume that $C/I^nC$ is a flat $R/I^n$-module for every $n$. Then the $R$-module $\Lambda_I(C)$ satisfies the assumptions of Lemma 10.2, since $\Lambda_I(C)/I^n\Lambda_I(C) = C/I^nC$. Hence $\Lambda_I(C)$ is a flat $R$-module. By Lemma 10.1, it follows that $C = \Delta_I(C) \longrightarrow \Lambda_I(C)$ is an isomorphism. Therefore, the $R$-module $C$ is flat and $I$-adically separated.  

**Corollary 10.4.** Let $R$ be a Noetherian commutative ring, $I \subset R$ an ideal, and $F$ a flat $R$-module. Then

(a) the natural morphism $\Delta_I(F) \longrightarrow \Lambda_I(F)$ is an isomorphism;
(b) the $R$-module $\Delta_I(F) = \Lambda_I(F)$ is flat.

Proof. Since $\Delta_I(F)/I^n\Delta_I(F) = F/I^nF$ is a flat $R/I^n$-module for every $n \geq 1$, the assertions follow from Corollary 10.3 applied to the $R$-module $C = \Delta_I(F)$. □

A generalization of the assertion of Corollary 10.4(a) to the case of a commutative ring with a weakly proregular finitely generated ideal can be found in [27, Lemma 2.5]. A partial generalization of the assertion (b) is briefly discussed in [27, Remark 5.6].

Let $R$ be a commutative ring and $I \subset R$ an ideal. For any $R$-module $M$ and a set $X$, we denote by $M^{(X)}$ the direct sum of $X$ copies of $M$ and by $M^X$ the direct product of $X$ copies of $M$. The $R$-module $\Delta_I(R^{(X)})$ is called the free $I$-contramodule $R$-module generated by $X$, because for any $I$-contramodule $R$-module $C$ one has

$$\text{Hom}_R(\Delta_I(R^{(X)}), C) \cong \text{Hom}_R(R^{(X)}, C) \cong C^X.$$ 

Free $I$-contramodule $R$-modules are projective objects in the abelian category $R\text{-mod}_{I,\text{ctra}}$, and every $I$-contramodule $R$-module $C$ is a quotient object of some free $I$-contramodule $R$-module (e.g., one can take $X = C$). It follows that an $I$-contramodule $R$-module is a projective object in $R\text{-mod}_{I,\text{ctra}}$ if and only if it is a direct summand of some free $I$-contramodule $R$-module.

The following result can be found in [24, Corollary B.8.2] (see also [37, Theorem 3.4] and [22, Corollary 1.8]).

**Theorem 10.5.** Let $R$ be a Noetherian commutative ring, $I \subset R$ an ideal, and $C$ an $I$-contramodule $R$-module. Then the following conditions are equivalent:

(i) $C$ is a projective $I$-contramodule $R$-module;
(ii) $C/I^nC$ is a projective $R/I^n$-module for every $n \geq 1$;
(iii) $C/I^nC$ is a flat $R/I^n$-module for every $n \geq 1$ and $C/IC$ is a projective $R/I$-module;
(iv) $C$ is a flat $R$-module and $C/IC$ is a projective $R/I$-module.

Furthermore, any projective $I$-contramodule $R$-module is $I$-adically separated.

Proof. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\iff$ (iv) are obvious, while (iii) $\Rightarrow$ (ii) is provided by Lemma 9.6 and (iii) $\Rightarrow$ (iv) by Corollary 10.3(a). Any one of the conditions (iii)–(iv) implies that $C$ is $I$-adically separated by Corollary 10.3(b).

It remains to prove (iv) $\Rightarrow$ (i). Assuming (iv), by Theorem 9.5 we have $\text{Ext}_R^1(C, D) = 0$ for any $I$-contramodule $R$-module $D$. Hence any short exact sequence of $I$-contramodule $R$-modules $0 \rightarrow D \rightarrow E \rightarrow C$ splits.

Alternatively, one can use an idempotent-lifting argument similar to the proof of Lemma 9.6 in order to show that (iv) implies (i). Let $G = R^{(X)}$ be a free $R$-module such that $C/IC$ is a direct summand of $G/IG$. Then there is an idempotent endomorphism $e : G/IG \rightarrow G/IG$ such that the $R$-module $C/IC$ is isomorphic to $e(G/IG)$. Proceeding by induction in $n$, one lifts the idempotent element $e \in \text{Hom}_R(G/IG, G/IG)$ to a compatible sequence of idempotent elements $e_n \in \text{Hom}_R(G/I^nG, G/I^nG)$. Passing to the projective limit, we obtain an idempotent endomorphism $f : \Lambda_I(G) \rightarrow \Lambda_I(G)$. By Corollary 10.4(a), $\Lambda_I(G) = \Delta_I(G)$ is a free
$I$-contramodule $R$-module. Set $F = \text{fA}_I(G)$; then $F$ is a projective $I$-contramodule $R$-module. Therefore, the isomorphism $F/IF \simeq e(G/IG) \simeq C/IC$ can be lifted to an $R$-module homomorphism $l: F \rightarrow C$.

Let $K$ and $L$ denote the kernel and cokernel of $l$. Then $L/IL = \text{coker}(F/IF \rightarrow C/IC) = 0$, so by Lemma 4.2 with Remark 4.3, we have $L = 0$ and the morphism $l$ is surjective. The $R$-module $C$ is flat by assumption, so from the exact sequence $0 \rightarrow K \rightarrow F \rightarrow C \rightarrow 0$ we obtain the exact sequence $0 \rightarrow K/IK \rightarrow F/IF \rightarrow C/IC \rightarrow 0$. Hence $K/IK = 0$. Applying the Nakayama Lemma 4.2 again, we conclude that $K = 0$. □

Free $I$-contramodule $R$-modules can be characterized in a way similar to the above characterization of the projective ones.

**Theorem 10.6.** Let $R$ be a Noetherian commutative ring, $I \subset R$ an ideal, and $C$ an $I$-contramodule $R$-module. Then the following conditions are equivalent:

(i) $C$ is a free $I$-contramodule $R$-module;
(ii) $C/I^nC$ is a free $R/I^n$-module for every $n \geq 1$;
(iii) $C/I^nC$ is a flat $R/I^n$-module for every $n \geq 1$ and $C/IC$ is a free $R/I$-module;
(iv) $C$ is a flat $R$-module and $C/IC$ is a free $R/I$-module.

**Proof.** The implications (i) $\implies$ (ii) $\implies$ (iii) $\iff$ (iv) are obvious, while (iii) $\implies$ (ii) is provided by Lemma 9.6 (the second assertion) and (iii) $\implies$ (iv) by Corollary 10.3(a). Finally, the second (longer) proof of Theorem 10.5(iv) $\implies$ (i) above is also a proof of (iv) $\implies$ (i) in the present theorem (take $e_n = 1$ and $f = 1$ when $e = 1$). □

A version of the next result can be found in [37, Corollary 4.5]; for a generalization, see [24, Lemma 1.3.2].

**Corollary 10.7.** Let $m$ be a maximal ideal in a Noetherian commutative ring $R$. Then the classes of projective $m$-contramodule $R$-modules and free $m$-contramodule $R$-modules coincide.

**Proof.** Compare Theorem 10.5(iv) and Theorem 10.6(iv). □

Let $R$ be a commutative ring and $I \subset R$ be a finitely generated ideal such that the free $I$-contramodule $R$-modules are $I$-adically separated: e. g., $R$ is Noetherian (Corollary 10.4(a) or the last assertion of Theorem 10.5), or $I$ is weakly proregular ([27, Lemma 2.5]). Then the free $I$-contramodule $R$-module generated by a set $X$ can be computed as

$$\Delta_I(R^{(X)}) = \Lambda_I(R^{(X)}) = \varprojlim_{n \geq 1} (R/I^n)^{(X)} = \Lambda_I(R)[[X]],$$

where $\Lambda_I(R)[[X]]$ denotes the $R$-module of all families of elements $u_x \in \Lambda_I(R)$, $x \in X$ converging to 0 in the $I$-adic (= projective limit) topology of $\Lambda_I(R)$. In other words, the $R$-module $\Lambda_I(R)[[X]]$ consists of all the maps $X \rightarrow \Lambda_I(R)$, $x \mapsto u_x$ such that for every $n \geq 1$ the set of all $x \in X$ for which

$$u_x \notin I^n\Lambda_I(R) = \ker(\Lambda_I(R) \rightarrow R/I^n)$$

is finite (see the proof of Theorem 5.8; cf. Example 3.1 (1)).
11. Free Contramodules and Flat Cotorion Modules

Throughout this section, \( R \) is a Noetherian commutative ring. The following lemma is a dual version of Corollary 10.4(b).

**Lemma 11.1.** Let \( I \) be an ideal in \( R \) and \( K \) an injective \( R \)-module. Then the maximal \( I \)-torsion submodule \( \Gamma_I(K) \) of \( K \) is also an injective \( R \)-module.

**Proof.** It suffices to check that for any finitely generated \( R \)-module \( M \) and a submodule \( N \subset M \) any \( R \)-module morphism \( g: N \to \Gamma_I(K) \) can be extended to a morphism \( f: M \to \Gamma_I(K) \). Since the \( R \)-module \( N \) is finitely generated, there exists \( n \geq 1 \) such that \( g \) annihilates \( I^nM \). By the Artin–Rees lemma, there exists \( m \geq 1 \) such that \( N \cap I^m \subset I^n \). Then we have \( N/(N \cap I^mM) \subset M/I^mM \) and the morphism \( g \) factorizes through the surjection \( N \to N/(N \cap I^mM) \), providing an \( R \)-module morphism \( g': N/(N \cap I^mM) \to \Gamma_I(K) \subset K \). Since \( K \) is an injective \( R \)-module, the morphism \( g' \) can be extended to an \( R \)-module morphism \( f': M/I^mM \to K \). Obviously, the image of \( f' \) is contained in \( \Gamma_I(K) \subset K \). \( \square \)

It follows from Lemma 11.1 that every injective object of the category \( R \text{-mod}_{I\text{-tors}} \) is at the same time an injective \( R \)-module (i.e., an injective object in \( R \text{-mod} \)).

For any \( R \)-module \( M \), we denote by \( E_R(M) \supset M \) an injective envelope of the \( R \)-module \( M \) (cf. the definition of a \( C \)-envelope for a class of objects \( C \subset R \text{-mod} \) in Section 8). It follows from Lemma 11.1 that whenever \( M \) is an \( I \)-torsion \( R \)-module for some ideal \( I \subset R \), the \( R \)-module \( E_R(M) \) is also \( I \)-torsion.

On the other hand, suppose that \( M \) is an \( s \)-torsion-free for each element \( s \) from a certain multiplicative subset \( S \subset R \). Then one has \( E_R(S^{-1}M) = E_R(M) \), because \( M \subset S^{-1}M \) and any nonzero submodule of \( S^{-1}M \) has a nonzero intersection with \( M \). For any \( R \)-module \( N \), any endomorphism of the \( R \)-module \( E_R(N) \) restricting to an automorphism of \( N \) is an automorphism of \( E_R(N) \). Applying this observation to \( N = S^{-1}M \), we conclude that the elements of \( S \) act by automorphisms of \( E_R(M) \), that is \( E_R(M) \) is an \( (S^{-1}R) \)-module.

Furthermore, every injective \( (S^{-1}R) \)-module is an injective \( R \)-module, since \( S^{-1}R \) is a flat \( R \)-module. It follows that

\[ E_R(M) = E_R(S^{-1}M) = E_{S^{-1}R}(S^{-1}M). \]

In particular, let \( p \subset R \) be a prime ideal. Denote by

\[ k_R(p) = ((R/p) \mathord\setminus 0)^{-1}(R/p) = ((R \mathord\setminus p)^{-1}R)/(((R \mathord\setminus p)^{-1}p) \]

the residue field of \( p \). Then the injective \( R \)-module \( E_R(R/p) = E_R(k_R(p)) \) is \( p \)-torsion. It is also a module over the local ring \( R_p = (R \mathord\setminus p)^{-1}R \), and in fact, an injective \( R_p \)-module isomorphic to \( E_{R_p}(k_R(p)) \). In particular, if \( R \) is an integral domain, then \( E_R(R) = k_R(0) \) is the field of fractions of \( R \).

**Lemma 11.2.** Let \( p \) and \( q \subset R \) be two prime ideals. Then

\[ \text{Hom}_R(E_R(R/p), E_R(R/q)) = 0 \quad \text{if} \quad p \nsubseteq q. \]
Proof. Let $s \in p$ be an element not belonging to $q$. Then the $R$-module $E_R(R/p)$ is $s$-torsion, while the action of $s$ in $E_R(R/q)$ is invertible, so there are no nonzero $s$-torsion elements in $E_R(R/q)$.

The next result is sometimes called the Matlis duality [16, Corollary 4.3] (not to be confused with the covariant Matlis category equivalence of [18, Section 3], [9, Section VIII.2]). We will only need the simplest finite-length version.

Lemma 11.3. Let $m \subset R$ be a maximal ideal. Then the functor

$$M \mapsto \text{Hom}_R(M, E_R(R/m))$$

is an involutive auto-anti-equivalence of the category of finitely generated $m$-torsion $R$-modules. In other words, the $R$-module $\text{Hom}_R(M, E_R(R/m))$ is a finitely generated $m$-torsion $R$-module for every finitely generated $m$-torsion $R$-module $M$, and the natural morphism

$$M \longrightarrow \text{Hom}_R(\text{Hom}_R(M, E_R(R/m)), E_R(R/m))$$

is an isomorphism.

Proof. Set $E = E_R(R/m)$. The quotient module $R/m = k_R(m)$ is a field, and the $R$-module $\text{Hom}_R(R/m, E)$ is a vector space over this field. This vector space is one-dimensional, since, by the definition of an injective envelope, $R/m$ is an $R$-submodule in $E$ and any nonzero $R$-submodule in $E$ has a nonzero intersection with this particular submodule. Hence $\text{Hom}_R(\text{Hom}_R(R/m, E), E)$ is a one-dimensional $R/m$-vector space, too. Since there exists an injective morphism $R/m \longrightarrow E$, the natural map $R/m \longrightarrow \text{Hom}_R(\text{Hom}_R(R/m, E), E)$ is injective, and therefore an isomorphism.

Now an $R$-module is finitely generated $m$-torsion if and only if it is a finitely iterated extension of copies of the $R$-module $R/m$. The functor $\text{Hom}_R(\_ , E)$ is exact, so it takes extensions to extensions. This proves that the functor $\text{Hom}_R(\_ , E)$ takes finitely generated $m$-torsion $R$-modules to finitely generated $m$-torsion $R$-modules. Finally, if $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ is a short exact sequence of $R$-modules such that $L \longrightarrow \text{Hom}_R(\text{Hom}_R(L, E), E)$ and $N \longrightarrow \text{Hom}_R(\text{Hom}_R(N, E), E)$ are isomorphisms, then $M \longrightarrow \text{Hom}_R(\text{Hom}_R(M, E), E)$ is an isomorphism, too.

The following lemma is a dual version of Corollary 10.7.

Lemma 11.4. Let $m \subset R$ be a maximal ideal. Then the injective objects of the category of $m$-torsion $R$-modules are precisely the direct sums of copies of the $R$-module $E_R(R/m)$.

Proof. Direct sums of copies of $E = E_R(R/m)$ are injective $R$-modules, because the class of injective left modules over a left Noetherian ring is closed under infinite direct sums. Conversely, for any $m$-torsion $R$-module $M$ denote by $mM$ the submodule of elements annihilated by $m$ in $M$. The argument is based on the observation that $mM = 0$ implies $M = 0$ (cf. [26, Lemma 2.1(a)]). Let $K$ be an injective $m$-torsion $R$-module. Choose a basis indexed by a set $X$ in the $R/m$-vector space $mK$, consider the direct sum $E^{(X)}$ of $X$ copies of $E$, and extend the isomorphism $mK \simeq mE^{(X)}$ to
an $R$-module morphism $f: K \rightarrow E^{(X)}$. Then $\ker f = 0$, since $K$ is $m$-torsion and $\ker f \subset K$ does not intersect $mK$. Since $K$ is injective, it follows that it is a direct summand in $E^{(X)}$; hence $m \coker f = 0$ and $\coker f = 0$. □

The following classification theorem is due to Matlis [16].

**Theorem 11.5.** An $R$-module $K$ is injective if and only if it is isomorphic to an $R$-module of the form

$$\bigoplus_{p \in \text{Spec } R} E_R(R/p)^{(X_p)},$$

where $p \mapsto X_p$ is a correspondence assigning some set $X_p$ to every prime ideal $p \subset R$. The cardinality of every set $X_p$ is uniquely determined by the $R$-module $K$. □

Recall that we can use the notation $\Lambda_p(R_p) = \lim_{\leftarrow n \geq 1} R_p/p^nR_p$ for the completion of the local ring $R_p = (R \setminus p)^{-1}R$ of a prime ideal $p \in \text{Spec } R$. For a maximal ideal $m \subset R$, we may write simply $\Lambda_m(R) = \Lambda_m(R_m)$.

The following classical result can be viewed as a simple version of the covariant Matlis category equivalence. At the same time, it is one of the simplest manifestations of the underived co-contra correspondence phenomenon [26, Sections 1.2 and 3.6], [27, Proposition 1.5.1].

**Theorem 11.6.** Let $m \subset R$ be a maximal ideal. Then the pair of functors

$$M \mapsto \text{Hom}_R(E_R(R/m), M) \quad \text{and} \quad C \mapsto E_R(R/m) \otimes_R C$$

provides an equivalence between the additive category of injective $m$-torsion $R$-modules $M$ and the additive category of projective $m$-contramodule $R$-modules $C$.

**Proof.** As in the previous proof, we set $E = E_R(R/m)$. Obviously, $M \mapsto \text{Hom}_R(E, M)$ and $C \mapsto E \otimes_R C$ form a pair of adjoint functors from the category of $R$-modules to itself. According to Lemma 11.4, injective $m$-torsion $R$-modules are precisely the $R$-modules of the form $E^{(X)}$, where $X$ is a set. According to Corollary 10.7, projective $m$-contramodule $R$-modules are precisely the $R$-modules $\Lambda_m(R)[[X]] = \Lambda_m(R^{(X)})$, where $X$ is again an arbitrary set.

For every integer $n \geq 1$, denote by $nE \subset E$ the submodule of all elements annihilated by $m^n$ in $E$. Then $nE \cong \text{Hom}_R(R/m^n, E)$ is a finitely generated $m$-torsion $R$-module by Lemma 11.3, and $E = \lim_{\leftarrow n \geq 1} nE$. Now we have

$$E \otimes_R \Lambda_m(R)[[X]] = \lim_{\leftarrow n \geq 1} nE \otimes_R \Lambda_m(R)[[X]]$$

$$= \lim_{\leftarrow n \geq 1} nE \otimes_R m^n \left( \Lambda_m(R)[[X]]/m^n\Lambda_m(R)[[X]] \right)$$

$$= \lim_{\leftarrow n \geq 1} nE \otimes_R m^n \left( R/m^n \right)^{(X)} = \lim_{\leftarrow n \geq 1} \left( nE \right)^{(X)} = E^{(X)}.$$ 

Conversely,

$$\text{Hom}_R(E, E^{(X)}) = \lim_{\leftarrow n \geq 1} \text{Hom}_R(m^n, nE^{(X)}) = \lim_{\leftarrow n \geq 1} \text{Hom}_R(m^n, nE^{(X)})$$

$$= \lim_{\leftarrow n \geq 1} \text{Hom}_R(m^n, R/m^n)^{(X)} = \lim_{\leftarrow n \geq 1} \left( R/m^n \right)^{(X)} = \Lambda_m(R)[[X]],$$

and

$$\text{Hom}_R(E, C) = \lim_{\leftarrow n \geq 1} \text{Hom}_R(m^n, C) = \lim_{\leftarrow n \geq 1} \left( R/m^n \right)^{(X)} = \Lambda_m(R)[[X]].$$
since \( \text{Hom}_R(nE, nE) = \text{Hom}_R(R/m^n, R/m^n) \) by Lemma 11.3. 

The next lemma is a dual version of Lemma 11.2.

**Lemma 11.7.** Let \( q \subset R \) be a prime ideal. Denote by \( P \subset \text{Spec} \) the set of all prime ideals \( p \subset R \) such that \( p \nsubseteq q \). Let \( X_p, \ p \in P \) and \( X_q \) be some sets. Then

\[
\text{Hom}_R \left( \prod_{p \in P} \Lambda_p(R)[[X_p]], \Lambda_q(R)[[X_q]] \right) = 0.
\]

**Proof.** More generally, let \( M_p \) be arbitrary \( R_p \)-modules and \( C_q \) a \( q \)-contramodule \( R \)-module. Then we claim that

\[
\text{Hom}_R \left( \prod_{p \in P} M_p, C_q \right) = 0.
\]

Indeed, let \( s_1, \ldots, s_m \) be a finite set of generators of the ideal \( q \). For every \( j = 1, \ldots, m \), denote by \( P_j \subset \text{Spec} \) the set of all prime ideals \( p \subset R \) not containing \( s_j \). Then \( P = \bigcup_{j=1}^m P_j \), hence \( \prod_{p \in P} M_p \) is a direct summand of \( \bigoplus_{j=1}^m \prod_{p \in P_j} M_p \), and it suffices to show that for every \( 1 \leq j \leq m \) one has \( \text{Hom}_R \left( \prod_{p \in P_j} M_p, C_q \right) = 0 \).

Now, the action of \( s_j \) is invertible in \( M_p \) for every \( p \in P_j \), hence also in \( \prod_{p \in P_j} M_p \); while \( C_q \) is an \( s_j \)-contramodule, so it contains no \( s_j \)-divisible submodules. (Cf. [25, Lemma 5.1.2(a)].) 

The following version of Lemma 11.7 will be useful in the sequel.

**Lemma 11.8.** Let \( I \subset R \) be an ideal. Denote by \( P \subset \text{Spec} \) the set of all maximal ideals \( m \subset R \) such that \( I \nsubseteq m \). Let \( C_m, \ m \in P \) and \( C_I \) an \( I \)-contramodule \( R \)-module. Then

\[
\text{Hom}_R \left( \prod_{m \in P} C_m, C_I \right) = 0.
\]

**Proof.** Following the argument in the proof of Lemma 11.7, we only have to check that \( C_m \) is a module over the local ring \( R_m \) for every maximal ideal \( m \in P \). This is explained in Remark 9.2. 

The next lemma is a classical result.

**Lemma 11.9.** Let \( N \) and \( K \) be \( R \)-modules. Then

(a) if \( K \) is an injective \( R \)-module, then \( \text{Hom}_R(N, K) \) is a cotorsion \( R \)-module;

(b) if both \( N \) and \( K \) are injective \( R \)-modules, then \( \text{Hom}_R(N, K) \) is a flat cotorsion \( R \)-module.

**Proof.** Part (a): for any left \( R \)-module \( F \), one has \( \text{Ext}_R^q(F, \text{Hom}_R(N, K)) \simeq \text{Hom}_R(\text{Tor}_R^q(N, F), K) \), and the right-hand side vanishes for \( q > 0 \) when \( F \) is flat. Part (b): for any finitely generated \( R \)-module \( M \), one has \( M \otimes_R \text{Hom}_R(N, K) \simeq \text{Hom}_R(\text{Hom}_R(M, N), K) \), and the functor in the right-hand side is exact. 

Finally, we come to the following classification theorem due to Enochs [7] (see also [25, Theorem 1.3.8]).
Theorem 11.10. An $R$-module $C$ is flat and cotorsion if and only if it is isomorphic to an $R$-module of the form
\[(20) \prod_{p \in \text{Spec } R} \Lambda_p(R_p)[[X_p]],\]
where $p \mapsto X_p$ is a correspondence assigning some set $X_p$ to every prime ideal $p \subset R$. The cardinality of the set of generators $X_p$ of a free $(R_p p)$-contramodule $R_p$-module $\Lambda_p(R_p)[[X_p]]$ is uniquely determined by the $R$-module $C$.

Brief sketch of proof. $R_p p$ is the maximal ideal of a local ring $R_p$, so by Theorem 9.3 all $R_p p$-contramodule $R_p$-modules are cotorsion $R_p$-modules, and by Lemma 9.1 they are also cotorsion $R$-modules. By Corollary 10.4(b) or Theorem 10.6 $(i) \iff (iv)$, all free $(R_p p)$-contramodule $R_p$-modules are flat $R_p$-modules, hence also flat $R$-modules (as $R_p$ is a flat $R$-module). Alternatively, by (the proof of) Theorem 11.6 one has
\[
\Lambda_p(R_p)[[X]] \simeq \text{Hom}_R(E_R(R/p), E_R(R/p)^{(X)}),
\]
and the right-hand side is a flat cotorsion $R$-module by Lemma 11.9.

Conversely, Enochs proves in [7] that every flat cotorsion $R$-module is a direct summand of an $R$-module of the form $\text{Hom}_R(E, E')$, where $E$ and $E'$ are injective $R$-modules. He then proceeds to compute the $R$-module $\text{Hom}_R(E, E')$ using the classification of injective $R$-modules (19), and shows that it has the form (20). Finally, one can use, e. g., Lemma 11.7 together with Corollary 10.7 in order to check that direct summands of the form (20) also have the form (20), and that the factors in the direct product (20) can be recovered from the product. (Cf. [25, Theorem 5.1.1].)

Remark 11.11. Matlis’ classification of injective modules over Noetherian rings was used by Hartshorne in his theory of injective quasi-coherent sheaves over locally Noetherian schemes [14, § II.7]. Analogously, there is a theory of projective locally cotorsion contraherent cosheaves over locally Noetherian schemes [25, Section 5.1] based on Enochs’ classification of flat cotorsion modules over Noetherian rings.

12. Cotorsion and Contraadjusted Abelian Groups

We start with presenting several examples of flat covers and cotorsion envelopes in the category $\text{Ab} = \mathbb{Z}$–mod, before proceeding to describe cotorsion abelian groups and discuss contraadjusted abelian groups. First of all, we recall that an abelian group is flat if and only if it is torsion-free.

Example 12.1. Let $m \geq 2$ be a natural number. Then in the short exact sequence
\[0 \longrightarrow \bigoplus_{p|m} \mathbb{Z}_p \longrightarrow \bigoplus_{p|m} \mathbb{Z}_p \longrightarrow \mathbb{Z}/m\mathbb{Z} \longrightarrow 0\]
the middle term is a flat $\mathbb{Z}$-module, and the leftmost term is a cotorsion $\mathbb{Z}$-module. So it is a short exact sequence of the type (14) for the group $\mathbb{Z}/m\mathbb{Z}$ (with respect to the flat cotorsion theory in $\mathbb{Z}$–mod). Here the direct sums are taken over the finite set.
of all the prime numbers $p$ dividing $m$, and the leftmost arrow acts by multiplication with $m$ on the groups of $p$-adic integers $\mathbb{Z}_p$.

Indeed, the group $\bigoplus_{p|m} \mathbb{Z}_p$ is torsion-free, so it is a flat $\mathbb{Z}$-module. To show that the groups $\mathbb{Z}_p$ are cotorsion, one can apply Theorem 9.3. Alternatively, it suffices to notice that $\mathbb{Z}_p = \text{Hom}_\mathbb{Z}(\mathbb{Q}_p/\mathbb{Z}_p, \mathbb{Q}_p/\mathbb{Z}_p)$, where the group $\mathbb{Q}_p/\mathbb{Z}_p$ is injective (because it is divisible, see below), and use Lemma 11.9(a).

Moreover, the map $f : \bigoplus_{p|m} \mathbb{Z}_p \to \mathbb{Z}/m\mathbb{Z}$ is a flat cover of the cyclic group $\mathbb{Z}/m\mathbb{Z}$. Indeed, we have seen that $\mathbb{Z}_p$ is a flat precover. So it remains to check that any endomorphism $u : \bigoplus_{p|m} \mathbb{Z}_p \to \bigoplus_{p|m} \mathbb{Z}_p$ for which $fu = f$ is an automorphism.

One easily computes that $\text{Hom}_\mathbb{Z}(\mathbb{Z}_p, \mathbb{Z}_q) = 0$ for $p \neq q$ (e.g., because $\mathbb{Z}_p$ is $q$-divisible and there are no $q$-divisible subgroups in $\mathbb{Z}_q$) and $\text{Hom}_\mathbb{Z}(\mathbb{Z}_p, \mathbb{Z}_p) = \mathbb{Z}_p$ (e.g., because $\mathbb{Z}_p = \Delta_p(\mathbb{Z}) = \Lambda_p(\mathbb{Z})$, so $\text{Hom}_\mathbb{Z}(\mathbb{Z}_p, \mathbb{Z}_p) \to \text{Hom}_\mathbb{Z}(\mathbb{Z}, \mathbb{Z}_p)$ is an isomorphism). Thus $\text{Hom}_\mathbb{Z}(\bigoplus_{p|m} \mathbb{Z}_p, \bigoplus_{p|m} \mathbb{Z}_p) = \bigoplus_{p|m} \mathbb{Z}_p$. Now, $fu = f$ for $u \in \bigoplus_{p|m} \mathbb{Z}_p$ means that $u - 1$ is divisible by $m$ in $\bigoplus_{p|m} \mathbb{Z}_p$; hence $u \equiv 1 \pmod{p}$ for every $p$ dividing $m$ and $u$ is invertible in $\bigoplus_{p|m} \mathbb{Z}_p$.

**Example 12.2.** Let $p$ be a prime number. Then in the short exact sequence

$$0 \longrightarrow \mathbb{Z}_p \longrightarrow \mathbb{Q}_p \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow 0,$$

the middle term $\mathbb{Q}_p$ is a flat $\mathbb{Z}$-module, and the leftmost term $\mathbb{Z}_p$ is a cotorsion $\mathbb{Z}$-module (as we have seen in Example 12.1). So it is a short exact sequence of the type (14) for the group $\mathbb{Q}_p/\mathbb{Z}_p = \mathbb{Z}[p^{-1}]/\mathbb{Z}$.

Moreover, the map $f : \mathbb{Q}_p \to \mathbb{Q}_p/\mathbb{Z}_p$ is a flat cover of the group $\mathbb{Q}_p/\mathbb{Z}_p$. Indeed, by Corollary 8.13(a), a flat cover of $\mathbb{Q}_p/\mathbb{Z}_p$ exists in $\text{Ab}$. Applying Lemma 8.8(a,c), we conclude that it suffices to check that $\mathbb{Q}_p$ has no nonzero direct summands contained in $\ker(f)$. However, such a direct summand would be also a direct summand in $\ker(f) = \mathbb{Z}_p$, and $\mathbb{Z}_p$ has no nontrivial direct summands, because there are nontrivial idempotents in the ring $\mathbb{Z}_p = \text{Hom}_\mathbb{Z}(\mathbb{Z}_p, \mathbb{Z}_p)$. It remains to point out that $\ker(f)$ itself is not a direct summand in $\mathbb{Q}_p$.

**Example 12.3.** Consider the short exact sequence

$$0 \longrightarrow \prod_p \mathbb{Z}_p \longrightarrow \prod'_p \mathbb{Q}_p \longrightarrow \bigoplus_p \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow 0,$$

where the product in the leftmost term and the direct sum in the rightmost term are taken over all the prime numbers $p$. The middle term is the “restricted product” of the groups/fields of $p$-adic rationals $\mathbb{Q}_p$: by the definition, it consists of all the collections $(a_p \in \mathbb{Q}_p)_p$ such that $a_p \in \mathbb{Z}_p$ for all but a finite subset of the primes $p$.

Alternatively, the middle term can be defined as the tensor product $\mathbb{Q} \otimes_{\mathbb{Z}} \prod_p \mathbb{Z}_p$. This is what is called “the ring of finite adèle” in the algebraic number theory.

Clearly, the middle term $\prod'_p \mathbb{Q}_p$ is a flat $\mathbb{Z}$-module. The leftmost term $\prod_p \mathbb{Z}_p$ is a cotorsion $\mathbb{Z}$-module, since we already know that every factor $\mathbb{Z}_p$ is cotorsion and the class of cotorsion modules is closed under infinite products. So our sequence is
short exact sequence of the type (14) for the group $\bigoplus_p \mathbb{Q}_p / \mathbb{Z}_p = \mathbb{Q} / \mathbb{Z}$ (with respect to the flat cotorsion theory in $\mathbb{Z}$-mod).

Moreover, the map $f: \prod_p' \mathbb{Q}_p \rightarrow \mathbb{Q} / \mathbb{Z}$ is a flat cover of the group $\mathbb{Q} / \mathbb{Z}$. Indeed, let $u: \prod_p' \mathbb{Q}_p \rightarrow \prod_p' \mathbb{Q}_p$ be a group homomorphism such that $fu = f$. Then $f(u - 1) = 0$, hence the morphism $(u - 1): \prod_p' \mathbb{Q}_p \rightarrow \prod_p' \mathbb{Q}_p$ factorizes through the embedding $\prod_p \mathbb{Z}_p \rightarrow \prod_p' \mathbb{Q}_p$. However, any group homomorphism from the $\mathbb{Q}$-vector space $\prod_p' \mathbb{Q}_p$ into the group $\prod_p \mathbb{Z}_p$ vanishes, as there are no divisible subgroups in $\prod_p \mathbb{Z}_p$. We have shown that $u = 1$.

Example 12.4. In the short exact sequence

$$0 \rightarrow \mathbb{Z} \rightarrow \prod_p \mathbb{Z}_p \rightarrow \prod_p \mathbb{Z}_p / \mathbb{Z} \rightarrow 0$$

the middle term $\prod_p \mathbb{Z}_p$ is a cotorsion $\mathbb{Z}$-module, and the rightmost term $(\prod_p \mathbb{Z}_p) / \mathbb{Z}$ is a flat $\mathbb{Z}$-module. In fact, the rightmost term is a $\mathbb{Q}$-vector space isomorphic to $(\prod_p' \mathbb{Q}_p) / \mathbb{Q}$. So this is a short exact sequence of the type (15) for the group $\mathbb{Z}$.

Moreover, the map $g: \mathbb{Z} \rightarrow \prod_p \mathbb{Z}_p$ is a cotorsion envelope of the group $\mathbb{Z}$. Indeed, we have seen that it is a special preenvelope, hence (by Lemma 8.9(a)) a preenvelope. According to Lemma 11.7, we have

$$\text{Hom}_{\mathbb{Z}}(\prod_p \mathbb{Z}_p, \prod_p \mathbb{Z}_p) = \prod_p \text{Hom}_{\mathbb{Z}}(\mathbb{Z}_p, \mathbb{Z}_p) = \prod_p \mathbb{Z}_p = \text{Hom}_{\mathbb{Z}}(\mathbb{Z}, \prod_p \mathbb{Z}_p).$$

Thus the equation $ug = g$ for an endomorphism $u: \prod_p \mathbb{Z}_p \rightarrow \prod_p \mathbb{Z}_p$ implies $u = 1$.

Example 12.5. All finite abelian groups are cotorsion, e. g., by Lemma 11.9(a). Moreover, by Lemma 9.6 all flat modules over an Artinian commutative ring $R$ are projective. Hence all $R$-modules are cotorsion. Applying Lemma 9.1, we conclude that all the $\mathbb{Z}/m\mathbb{Z}$-modules are cotorsion over $\mathbb{Z}$, that is every abelian group annihilated by some integer $m \geq 2$ is cotorsion. Hence so are all the infinite products of such abelian groups.

On the other hand, the groups $\bigoplus_{n=0}^{\infty} \mathbb{Z}_p$ and $\bigoplus_{n=1}^{\infty} \mathbb{Z}/p^n\mathbb{Z}$ are not cotorsion. We will see below in Examples 12.12 what their cotorsion envelopes are.

An abelian group $B$ is said to be divisible if it is $s$-divisible for every $0 \neq s \in \mathbb{Z}$. Any abelian group $B$ has a unique maximal divisible subgroup $B_{\text{div}}$, which can be constructed as the sum of all divisible subgroups in $B$.

An abelian group $B$ is said to be reduced if it has no nonzero divisible subgroups, i. e., $B_{\text{div}} = 0$. For any abelian group $B$, the quotient group $B_{\text{red}} = B / B_{\text{div}}$ is reduced. It is the (unique) maximal reduced quotient group of $B$.

An abelian group is an injective object in $\mathbb{Z}$-mod if and only if it is divisible. Hence the natural short exact sequence

$$0 \rightarrow B_{\text{div}} \rightarrow B \rightarrow B_{\text{red}} \rightarrow 0$$

is always (noncanonically) split.
Lemma 12.6. For any abelian group $B$, the subgroup $B_{\text{div}} \subset B$ is equal to the image of the map $\text{Hom}_\mathbb{Z}(\mathbb{Q}, B) \longrightarrow \text{Hom}_\mathbb{Z}(\mathbb{Z}, B) = B$. An abelian group $B$ is reduced if and only if $\text{Hom}_\mathbb{Z}(\mathbb{Q}, B) = 0$.

Proof. Essentially, the claim is that for every element $b \in B_{\text{div}}$ there is a homomorphism $f: \mathbb{Q} \longrightarrow B$ such that $b = f(1)$. There are several ways to explain why this is true, the simplest of them being, because $\mathbb{Z}$ is a countable integral domain (for another approach, see Theorem 13.8(a) below). E. g., one can choose an element $b_1 \in B_{\text{div}}$ such that $b = 2 \cdot b_1$, then an element $b_2 \in B_{\text{div}}$ such that $b_1 = 2 \cdot 3 \cdot b_2$, an element $b_3 \in B_{\text{div}}$ such that $b_2 = 2 \cdot 3 \cdot 5 \cdot b_3$, etc. After all such choices have been made, one obtains the desired homomorphism from the group

$$\mathbb{Q} = \lim\limits_{\longrightarrow}(\mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{2 \cdot 3} \mathbb{Z} \xrightarrow{2 \cdot 3 \cdot 5} \mathbb{Z} \longrightarrow \cdots)$$

into $B$. □

The classification of divisible abelian groups is provided by Theorem 11.5: an abelian group $B$ is divisible if and only if it is isomorphic to a group of the form

$$\mathbb{Q}^{(X)} \oplus \bigoplus_p (\mathbb{Q}_p / \mathbb{Z}_p)^{(X_p)},$$

where $X$ and $X_p$ are some sets (whose cardinalities are uniquely determined by the group $B$) and the direct sum is taken over all the prime numbers $p$.

For any abelian group $C$ and an integer $s \in \mathbb{Z}$, we consider the group $\Delta_s(C)$ and the adjunction morphism $\delta_{s,C}: C \longrightarrow \Delta_s(C)$ constructed in Theorem 6.4.

Lemma 12.7. For any abelian group $C$, one has

(a) $\Delta_0(C) = C$ and $\delta_{0,C} = \text{id}_C$;
(b) $\Delta_1(C) = 0$;
(c) for any integer $s \neq 0$, there is a natural isomorphism

$$\Delta_s(C) \simeq \bigoplus_{p|s} \Delta_{p}(C),$$

where the direct sum is taken over all the prime numbers $p$ dividing $s$. The components of the map $\delta_{s,C}$ with respect to this direct sum decomposition are equal to $\delta_{p,C}$.

Proof. Part (a): for any commutative ring $R$ and the element $s = 0$, one has $R[s^{-1}] = 0$, hence every $R$-module is a 0-contramodule and the reflector $\Delta_0: R\text{-mod} \longrightarrow R\text{-mod}_{0\text{-ctra}}$ is the identity functor. Part (b): for any commutative ring $R$ and the element $s = 1$, one has $R[s^{-1}] = R$, hence the only 1-contramodule $R$-module is the zero module.

Part (c): according to Theorem 6.4(iii), we have

$$\Delta_s(C) = \text{Ext}^1_{\mathbb{Z}}(\mathbb{Z}[s^{-1}]/\mathbb{Z}, C).$$

The assertion now follows from the isomorphism $\mathbb{Z}[s^{-1}]/\mathbb{Z} = \bigoplus_{p|s} \mathbb{Z}[p^{-1}]/\mathbb{Z}$. □

For any abelian group $C$, we consider the natural group homomorphism

$$\delta_{Z,C} = (\delta_{p,C})_p : C \longrightarrow \prod_p \Delta_p(C),$$

57
where the product is taken over all the prime numbers $p$. The following theorem goes back to Nunke [20, Theorem 7.1].

**Theorem 12.8.** For any abelian group $C$, one has

(a) $\ker(\delta_{Z,C}) = C_{\text{div}}$;
(b) $C$ is reduced if and only if the map $\delta_{Z,C}$ is injective;
(c) $\text{coker}(\delta_{Z,C})$ is a $\mathbb{Q}$-vector space;
(d) $C$ is cotorsion if and only if the map $\delta_{Z,C}$ is surjective;
(e) if $C$ is reduced, then the morphism $\delta_{Z,C}$ is a cotorsion envelope of $C$.

**Proof.** From the short exact sequence $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow 0$ together with the isomorphisms $\mathbb{Q}/\mathbb{Z} = \bigoplus_p \mathbb{Z}[p^{-1}]/\mathbb{Z}$ and $\Delta_p(C) = \text{Ext}_1^\mathbb{Z}(\mathbb{Z}[p^{-1}]/\mathbb{Z},C)$, we obtain the long exact sequence

$$0 \rightarrow \text{Hom}_\mathbb{Z}(\mathbb{Q}/\mathbb{Z}, C) \rightarrow \text{Hom}_\mathbb{Z}(\mathbb{Q}, C)$$

$$\rightarrow C \rightarrow \prod_p \Delta_p(C) \rightarrow \text{Ext}_1^\mathbb{Z}(\mathbb{Q}, C) \rightarrow 0.$$

Hence the kernel of $\delta_{Z,C}$ is equal to the image of the map $\text{Hom}_\mathbb{Z}(\mathbb{Q}, C) \rightarrow C$, which coincides with $C_{\text{div}}$ by Lemma 12.6. This proves part (a), and part (b) follows trivially. Furthermore, the group $\text{coker}(\delta_{Z,C})$ is identified with the group $\text{Ext}_1^\mathbb{Z}(\mathbb{Q}, C)$, where the field $\mathbb{Q}$ acts via its action in the first argument. This proves part (c).

If the group $C$ is cotorsion, then $\text{Ext}_1^\mathbb{Z}(\mathbb{Q}, C) = 0$, so the map $\delta_{Z,C}$ is surjective. Conversely, if $\delta_{Z,C}$ is surjective, then we have a short exact sequence

$$0 \rightarrow C_{\text{div}} \rightarrow C \rightarrow \prod_p \Delta_p(C) \rightarrow 0.$$  

(21)

Now the groups $\Delta_p(C)$ are always cotorsion by Theorem 9.3, the group $C_{\text{div}}$ is cotorsion since it is injective, and it follows that $C$ is cotorsion, because the class of cotorsion modules is closed under infinite products and extensions.

Alternatively, if $\delta_{Z,C}$ is surjective, then $\text{Ext}_1^\mathbb{Z}(\mathbb{Q}, C) = 0$. Given a torsion-free abelian group $F$, one considers the short exact sequence $0 \rightarrow F \rightarrow \mathbb{Q} \otimes_\mathbb{Z} F \rightarrow \mathbb{Q}/\mathbb{Z} \otimes_\mathbb{Z} F \rightarrow 0$. Since the category of abelian groups has homological dimension 1, i.e., $\text{Ext}_2^\mathbb{Z}(A,B) = 0$ for any abelian groups $A$ and $B$, from the corresponding long exact sequence we see that the map

$$\text{Ext}_1^\mathbb{Z}(\mathbb{Q} \otimes_\mathbb{Z} F, C) \rightarrow \text{Ext}_1^\mathbb{Z}(F, C)$$

is surjective for any group $C$. Since the group $\mathbb{Q} \otimes_\mathbb{Z} F$ is a $\mathbb{Q}$-vector space, that is a direct sum of copies of $\mathbb{Q}$, we conclude that $\text{Ext}_1^\mathbb{Z}(\mathbb{Q}, C) = 0$ implies $\text{Ext}_1^\mathbb{Z}(F, C) = 0$. This proves part (d).

When the group $C$ is reduced, we have a short exact sequence

$$0 \rightarrow C \rightarrow \prod_p \Delta_p(C) \rightarrow \text{Ext}_1^\mathbb{Z}(\mathbb{Q}, C) \rightarrow 0,$$

(22)

where the middle term $\prod_p \Delta_p(C)$ is cotorsion and the rightmost term $\text{Ext}_1^\mathbb{Z}(\mathbb{Q}, C)$ is flat. So the morphism $\delta_{Z,C}$ is a special cotorsion preenvelope, hence a cotorsion preenvelope, of the group $C$. To check that it is an envelope, one computes
\[ \text{Hom}_Z(\prod_p \Delta_p(C), \prod_p \Delta_p(C)) = \prod_p \text{Hom}_Z(\Delta_p(C), \Delta_p(C)) = \prod_p \text{Hom}_Z(C, \prod_p \Delta_p(C)) = \text{Hom}_Z(C, \prod_p \Delta_p(C)) \]

by Lemma 11.8 (cf. Example 12.4). Thus the equation \( u\delta_{Z,C} = \delta_{Z,C} \) for an endomorphism \( u: \prod_p \Delta_p(C) \to \prod_p \Delta_p(C) \) implies \( u = 1 \), proving (e). \( \square \)

**Corollary 12.9.** Choosing a splitting \( t_C: C \to C_{\text{div}} \) of the embedding \( C_{\text{div}} \to C \), one can construct a cotorsion envelope of an arbitrary abelian group \( C \) as the map
\[
(t_C, \delta_{Z,C}): C \longrightarrow C_{\text{div}} \oplus \prod_p \Delta_p(C).
\]

**Proof.** Clearly, the map \((t_C, \delta_{Z,C})\) is injective with the cokernel isomorphic to \( \text{Ext}^1_Z(\mathbb{Q}, C) \); so it is a special cotorsion preenvelope of the group \( C \). To check that it is an envelope, one can use the computation of \( \text{Hom}_Z(\prod_p \Delta_p(C), \prod_p \Delta_p(C)) \) in the above proof together with the observation that \( \text{Hom}_Z(C_{\text{div}}, \prod_p \Delta_p(C)) = 0 \).

So endomorphisms \( u \) of the group \( C_{\text{div}} \oplus \prod_p \Delta_p(C) \) are represented by triangular matrices with the entries \( u_{dd}: C_{\text{div}} \to C_{\text{div}}, u_{cc}: \prod_p \Delta_p(C) \to \prod_p \Delta_p(C), \) and \( u_{dc}: \prod_p \Delta_p(C) \to C_{\text{div}} \). The equation \( u(t_C, \delta_{Z,C}) = (t_C, \delta_{Z,C}) \) implies \( u_{cc} = 1 \) and \( u_{dd} = 1 \); and a triangular matrix with invertible diagonal entries is invertible.

More generally, the direct sum of any two cotorsion envelopes is a cotorsion envelope (see Lemma 8.10). \( \square \)

**Corollary 12.10.** An abelian group \( C \) is cotorsion if and only if it is isomorphic to a group of the form
\[
D \oplus \prod_p C_p,
\]
where the product is taken over all the prime numbers, \( D \) is a divisible group, and \( C_p \) are \( p \)-contramodule abelian groups. The groups \( D \) and \( C_p \) are uniquely determined by the group \( C \).

**Proof.** Follows from the short exact sequence (21) (which is even functorial, so the groups \( C_p \) and \( D \) are functors of a cotorsion group \( C \)) together with Theorem 9.3 and the arguments in the (first) proof of Theorem 12.8(d). \( \square \)

Let \( C \subset \text{Ab} \) denote the full subcategory of reduced cotorsion abelian groups.

**Corollary 12.11.** (a) The full subcategory \( C \) is closed under the kernels, cokernels, extensions, and infinite products in \( \text{Ab} \). In particular, \( C \) is an abelian category and its embedding \( C \hookrightarrow \text{Ab} \) is an exact functor.

(b) The functor \((C_p)_p \hookrightarrow \prod_p C_p\) establishes an equivalence between the Cartesian product of the abelian categories \( Z-\text{mod}_{p\text{-ctra}} \) of \( p \)-contramodule abelian groups, taken over all the prime numbers \( p \), and the category \( C \).

**Proof.** Part (a): by Lemma 12.6, an abelian group \( C \) is reduced if and only if \( \text{Hom}_Z(\mathbb{Q}, C) = 0 \). We have also seen that an abelian group \( C \) is cotorsion if and only if \( \text{Ext}^1_Z(\mathbb{Q}, C) = 0 \). Hence the assertions of part (a) follow from Theorem 1.2(a).
Part (b) claims that an abelian group $C$ is reduced cotorsion if and only if it can be presented in the form $C = \prod_p C_p$ with $C_p \in \mathbb{Z}\mod_{p\text{-ctra}}$, and

$$\text{Hom}_\mathbb{Z}(\prod_p B_p, \prod_p C_p) = \prod_p \text{Hom}_\mathbb{Z}(B_p, C_p)$$

for any $B_p$ and $C_p \in \mathbb{Z}\mod_{p\text{-ctra}}$. The former assertion is provided by Corollary 12.10, and the latter one by Lemma 11.8. □

Examples 12.12. (1) Fix a prime number $p$, and let $B_\alpha$ be a family of $p$-contra-module abelian groups indexed by some set of indices $\{\alpha\}$. Then the group

$$B = \bigoplus_\alpha B_\alpha$$

is not a $p$-contra-module in general, but it is $q$-divisible for all the primes $q \neq p$ and has no $p$-divisible subgroups. Hence one has $B_{\text{div}} = 0$ and $\Delta_q(B) = 0$ for all $q \neq p$. Applying Theorem 12.8(e), we conclude that

$$\delta_{p,B}: B \rightarrow \Delta_p(B)$$

is a cotorsion envelope of $B$.

The functor $\Delta_p$, being a left adjoint functor to the embedding $\mathbb{Z}\mod_{p\text{-ctra}} \rightarrow \mathbb{Z}\mod$, preserves categorical coproducts. Hence it takes the coproduct of abelian groups $B = \bigoplus_\alpha B_\alpha$ to the coproduct of the objects $B_\alpha = \Delta_p(B_\alpha)$ taken in the category of $p$-contra-module abelian groups $\mathbb{Z}\mod_{p\text{-ctra}}$. We have explained that the group $\Delta_p(B)$ is the coproduct of the $p$-contra-module abelian groups $B_\alpha$ taken in the category of $p$-contra-module abelian groups.

(2) Consider the particular case when all the groups $B_\alpha$ are copies of $\mathbb{Z}_p$. As it was explained at the end of Section 10, we have $\Delta_p(\mathbb{Z}_p^{(X)}) = \mathbb{Z}_p[[X]]$. So $\mathbb{Z}_p^{(X)} \rightarrow \mathbb{Z}_p[[X]]$ is a cotorsion envelope of the group $\mathbb{Z}_p^{(X)}$.

In particular, $\Delta_p(\bigoplus_{n=0}^\infty \mathbb{Z}_p)$ is the group $C$ of all the sequences of $p$-adic integers converging to zero in the topology of $\mathbb{Z}_p$ from Example 2.7 (1). So $\bigoplus_{n=0}^\infty \mathbb{Z}_p \rightarrow C$ is a cotorsion envelope of the group $\bigoplus_{n=0}^\infty \mathbb{Z}_p$.

(3) Now let us consider the group $B = \bigoplus_{n=1}^\infty \mathbb{Z}/p^n\mathbb{Z}$. It is explained in [26, Section 1.5] that the coproduct of the groups $\mathbb{Z}/p^n\mathbb{Z}$, $n \geq 1$ in the category $\mathbb{Z}\mod_{p\text{-ctra}}$ is the non-$p$-separated $p$-contra-module abelian group $C/E$ from Example 2.7 (1). So we have $\Delta_p(B) = C/E$, and $B \rightarrow C/E$ is a cotorsion envelope of the group $B$.

Remark 12.13. It is probably impossible to describe cotorsion modules over commutative rings much more complicated than $\mathbb{Z}$. The intuition seems to be that torsion-free abelian groups are “many”, so cotorsion groups are relatively “few”. But over more complicated rings, flat modules are “few”, so cotorsion modules must be “many”. Nevertheless, we will see in the next section that much of the theory of this section can be extended to Noetherian rings of Krull dimension 1.

It remains to say a few words about contraadjusted abelian groups. Here we follow [33, Section 4] and [34, Example 5.2].
Clearly, an abelian group $C$ is contraadjusted if and only if the group $C_{\text{red}}$ is contraadjusted. The group $C_{\text{red}}$ is a subgroup in the product $\prod_p \Delta_p(C)$ over all the prime numbers $p$. According to Corollary 6.10(b), a group $C$ is $s$-contraadjusted if and only if the map $\delta_{s,C}: C \rightarrow \Delta_s(C)$ is surjective for every $s \in \mathbb{Z}$. By Lemma 12.7(c), we conclude that an abelian group $C$ is contraadjusted if and only if the composition

$$C_{\text{red}} \rightarrow \prod_p \Delta_p(C) \rightarrow \bigoplus_{p \in P} \Delta_p(C)$$

is surjective for any finite set of prime numbers $P$.

Examples 12.14. (1) The groups $\bigoplus_p \mathbb{Z}_p$ and $\bigoplus_p \mathbb{Z}/p\mathbb{Z}$, where the direct sums are taken over all (or any infinite subset of) the prime numbers $p$, are contraadjusted, but not cotorsion.

(2) According to [25, Lemma 1.7.3 or Theorem 1.7.6] (see also [34, Lemma 2.3(ii)]), for any multiplicative set $S \subset \mathbb{Z}$ containing infinitely many prime numbers, the group $S^{-1}\mathbb{Z}$ is flat, but not very flat.

Remark 12.15. We are not aware of any explicit nontrivial example of a short exact sequence (14) or (15) in the very flat cotorsion theory in the category of abelian groups. Neither do we know anything about how the very flat contraadjusted abelian groups might look like (cf. Theorem 11.10).

13. Noetherian Rings of Krull Dimension 1

In this section, $R$ is a Noetherian ring of Krull dimension $\leq 1$. This means that every prime ideal in $R$ is either minimal or maximal. As in any Noetherian ring, the set of all minimal prime ideals in $R$ is finite.

It is possible, however, that some prime ideals in $R$ are simultaneously minimal and maximal. Such prime ideals correspond to isolated points of the topological space $\text{Spec } R$, i.e., to Artinian ring direct summands of the ring $R$. In order to apply the theory developed in this section, one has to mark such prime ideals in $R$ for being considered either on par with the other minimal ideals, or on par with the other maximal ideals. In other words, we presume that the set of all prime ideals in $R$ has been divided into two disjoint subsets,

$$\text{Spec } R = P_0 \sqcup P_1,$$

where all the prime ideals $q \in P_0$ are minimal and all the prime ideals $p \in P_1$ are maximal. Given a prime ideal that is both minimal and maximal, one has to decide whether to put it into $P_0$ or into $P_1$, but not into both.

Generally speaking, the set of all zero-divisors in a Noetherian ring is the union of the associated primes of the zero ideal. All the minimal primes belong to this set of associated primes $\text{Ass}_R(0)$, and there may be also a finite set of nonminimal prime ideals belonging to $\text{Ass}_R(0)$. We denote the intersection $P_1 \cap \text{Ass}_R(0)$ by $P_a \subset P_1$. In
particular, all the minimal prime ideals belonging to \( P_1 \) are in \( P_a \). So the set of all zero-divisors in \( R \) is
\[
\bigcup_{q \in P_0} q \cup \bigcup_{p \in P_a} p,
\]
while the nilradical of \( R \) is
\[
\bigcap_{q \in P_0} q \cap \bigcap_{p \in P_a} p
\]
(where it, of course, suffices to intersect the minimal prime ideals of \( R \)).

Let \( S \subset R \) denote the multiplicative set
\[
S = R \setminus \bigcup_{q \in P_0} q.
\]
All nonzero-divisors in \( R \) always belong to \( S \); conversely, all the elements of \( S \) are nonzero-divisors if and only if \( P_a = \emptyset \). When \( P_0 \) is the set of all minimal prime ideals in \( R \), one can describe \( S \) as the set of all elements in \( R \) whose images are nonzero-divisors in the quotient ring of \( R \) by its nilradical.

For every prime ideal \( m \in P_1 \), there exists \( s \in S \) such that \( s \in m \). So prime ideals of the ring \( S^{-1}R \) correspond bijectively to the prime ideals \( q \in P_0 \) in the ring \( R \). Hence the ring \( S^{-1}R \) is zero-dimensional Noetherian, and consequently Artinian. When \( S \) consists of nonzero-divisors in \( R \), or in other words, when the map \( R \rightarrow S^{-1}R \) is injective, one says that \( S^{-1}R \) is an Artinian classical ring of fractions of \( R \).

For every element \( s \in S \), prime ideals of the quotient ring \( R/(s) \) correspond bijectively to those prime ideals \( m \subset R \) that contain \( s \). All such prime ideals in \( R \) belong to \( P_1 \) (in particular, they are maximal), so the ring \( R/(s) \) is Artinian, too. We denote the finite set of all prime ideals in \( R \) containing \( s \) by \( P(s) \subset P_1 \).

As usually, for any \( R \)-module \( M \) and a prime ideal \( p \subset R \), we denote by \( M_p \) the localization \( (R \setminus p)^{-1}M = R_p \otimes_R M \) of the module \( M \) at the prime ideal \( p \). Similarly, we use the notation \( S^{-1}M = S^{-1}R \otimes_R M \) for the localization with respect to \( S \). We start with the following decomposition lemma (cf. [19, Theorem 3.1]).

**Lemma 13.1.** Let \( M \) be an \( R \)-module such that \( S^{-1}M = 0 \). Then the map \( M \rightarrow \prod_{p \in P_1} M_p \) is injective with the image coinciding with \( \bigoplus_{p \in P_1} M_p \subset \prod_{p \in P_1} M_p \), so we have a natural isomorphism
\[
M \cong \bigoplus_{p \in P_1} M_p.
\]

The \( R \)-module \( M_p \) is \( p \)-torsion (in the sense of the definition in Section 7).

**Proof.** By assumption, for every element \( m \in M \) there exists \( s \in S \) such that \( sm = 0 \). We have a finite set \( P(s) \subset P_1 \) of all prime ideals \( p \subset R \) such that \( s \in p \). For every \( p \in P_1 \setminus P(s) \), the image of the element \( m \) is zero in \( M_p \). Hence the image of the map \( M \rightarrow \prod_{p \in P_1} M_p \) is contained in \( \bigoplus_{p \in P_1} M_p \).

Furthermore, if \( m \neq 0 \) then there exists a maximal ideal \( p \in P_1 \) containing the annihilator ideal \( \text{Ann}_R(m) \subset R \) of the element \( m \). The image of the element \( m \) in \( M_p \) is nonzero. Hence the map \( M \rightarrow \bigoplus_{p \in P_1} M_p \) is injective.

In order to check that this map is surjective, it suffices to consider the case of a finitely generated module \( M \). Then the annihilator ideal \( I \) of \( M \) has a nonempty intersection with \( S \). Hence \( I \) is not contained in any prime ideal \( q \in P_0 \), and the
prime ideals of the ring $R/I$ correspond to a finite subset of prime ideals $P(I) \subseteq P_1$ in $R$. The ring $R/I$ is Artinian, so it is a finite direct sum of the Artinian local rings indexed by the maximal ideals of $R/I$, that is $R/I = \bigoplus_{p \in P_1} R_p/R_p I$. Thus every $R/I$-module $M$ is the direct sum of the $R_p/R_p I$-modules $M_p$. Finally, we have $S^{-1}M_p = 0$, hence for every $m \in M_p$ there exists $s \in S$ such that $sm = 0$. Clearly, $s \in p$ if $m \neq 0$. The quotient ring $R_p/R_ps$ is an Artinian local ring with the maximal ideal $R_p p/R_p s$, so every module over it is $p$-torsion.

Denote by $K^* = K^*_R$ the two-term complex of $R$-modules $R \longrightarrow S^{-1}R$ with the term $R$ placed in the cohomological degree $-1$ and the term $S^{-1}R$ in the cohomological degree $0$. When $S^{-1}R$ is an Artinian classical ring of fractions of $R$, one can use the quotient module $(S^{-1}R)/R$ in lieu of the complex $K^*$.

The following theorem is a key technical result.

**Theorem 13.2.** The complex $K^*$ is naturally isomorphic to the direct sum $\bigoplus_{p \in P_1} K^*_p$ in the derived category of complexes of $R$-modules $D^b(R\text{-}mod)$.

**Proof.** Let $H \subset R$ denote the kernel of the map $R \longrightarrow S^{-1}R$. Then $S^{-1}H = 0$ and $H$ is a finitely generated $R$-module, so $H$ is isomorphic to the direct sum of its localizations $H_p$ over some finite set of prime ideals $p \in P_1$ in $R$. In fact, for a prime ideal $p \in P_1$ one has $H_p \neq 0$ if and only if $p \in P_a = \text{Ass}_R(0) \cap P_1$, so

$$H = \bigoplus_{p \in P_a} H_p.$$ 

Besides, let $G$ denote the cokernel of the map $R \longrightarrow S^{-1}R$. Then, of course, we have $S^{-1}G = 0$. The assertion of the theorem now reduces to the next lemma. \[\square\]

**Lemma 13.3.** Let $f : M \longrightarrow N$ be a two-term complex of $R$-modules such that $S^{-1}\ker(f) = 0 = S^{-1}\operatorname{coker}(f)$ and $\ker(f)_p = 0$ for all but a finite set of ideals $p \in P_1$. Then the complex $M \longrightarrow N$ is naturally isomorphic to the complex $\bigoplus_{p \in P_1} M_p \longrightarrow \bigoplus_{p \in P_1} N_p$ in the derived category $D^b(R\text{-}mod)$.

**Proof.** Denote by $N^+$ the fibered product of the pair of morphisms $\prod_p N_p \longrightarrow \prod_p \operatorname{ker}(f_p)$ and $\bigoplus_p \operatorname{coker}(f_p) \longrightarrow \prod_p \operatorname{coker}(f_p)$. In other words, $N^+ \subset \prod_p N_p$ consists of all the collections of elements $(n_p \in N_p)_{p \in P_1}$ such that $n_p$ belongs to the image of the morphism $M_p \longrightarrow N_p$ for all but a finite subset of the ideals $p \in P_1$. Then we have a pair of morphisms of two-term complexes of $R$-modules

$$(M \rightarrow N) \longrightarrow (\prod_{p \in P_1} M_p \rightarrow N^+) \leftarrow (\bigoplus_{p \in P_1} M_p \rightarrow \bigoplus_{p \in P_1} N_p).$$

Here a natural map $N \longrightarrow N^+$ exists, because the image of the composition $N \longrightarrow \operatorname{coker}(f) \longrightarrow \prod_p \operatorname{coker}(f_p)$ is contained in $\bigoplus_p \operatorname{coker}(f_p)$.

We claim that both these morphisms of complexes are quasi-isomorphisms. Indeed, the kernel of the morphism $\prod_p M_p \longrightarrow N^+$ is equal to $\prod_p \operatorname{ker}(f_p)$, which coincides with $\bigoplus_p \operatorname{ker}(f_p)$ by assumption. On the other hand, the cokernel of the morphism $\prod_p M_p \longrightarrow N^+$ is equal to $\bigoplus_p \operatorname{coker}(f_p)$ by construction. \[\square\]
Remark 13.4. A much more general version of Theorem 13.2 can be obtained by combining Lemma 13.1 with the result of [28, Theorem 6.6(a)]. In fact, any complex $C^\bullet \in \mathcal{D}(R{-}\text{mod})$ for which the complex $S^{-1}C^\bullet$ is acyclic is naturally isomorphic to the direct sum $\bigoplus_{p \in P_1} C_p^\bullet$ as an object of $\mathcal{D}(R{-}\text{mod})$. Indeed, according to [28, Theorem 6.6(a) and Remark 6.8], the full subcategory $\mathcal{D}_{S{-}\text{tors}}(R{-}\text{mod})$ of complexes with $S$-torsion cohomology modules in $\mathcal{D}(R{-}\text{mod})$ is equivalent to the derived category $\mathcal{D}(R{-}\text{mod}_{S{-}\text{tors}})$ of the abelian category of $S$-torsion $R$-modules (where an $R$-module $M$ is said to be $S$-torsion if $S^{-1}M = 0$). This holds for any multiplicative subset $S$ in a Noetherian commutative ring $R$, because the $S$-torsion in $R$ is always bounded in the Noetherian case. Now, in the situation at hand, the abelian category of $S$-torsion $R$-modules is equivalent to the Cartesian product of the categories of $p$-torsion $R_p$-modules over the prime ideals $p \in P_1$, the equivalence being provided by the direct sum decomposition of Lemma 13.1. This argument was suggested to the author by the anonymous referee.

Lemma 13.5. Suppose that $R$ is a semilocal ring. Then

(a) there exists an element $s \in S$ such that $s \in m$ for every maximal ideal $m \in P_1$;

(b) the radical $\sqrt{(s)}$ of the principal ideal generated by $s$ in $R$ is equal to the intersection $\bigcap_{m \in P_1} m$;

(c) one has $S^{-1}R = R[s^{-1}]$.

Proof. Part (a): as we have already mentioned, by prime avoidance for every maximal ideal $m \in P_1$ there exists an element $s_m \in S \cap m$. Take $s = \prod_{m \in P_1} s_m$. Part (b): the radical $\sqrt{(s)}$ is the intersection of all the prime ideals $p \in \text{Spec } R$ containing $s$. Now one has $s \notin q$ for every $q \in P_0$ and $s \in m$ for every $m \in P_1$, so the assertion follows. Part (c): for every $t \in S$, we have $\sqrt{(s)} \subset \sqrt{(t)}$. Hence there exists $n \geq 1$ for which $s^n \in (t)$. Thus inverting $s$ in $R$ also gets $t$ inverted. □

Corollary 13.6. For every element $s \in S$, the complex $R \longrightarrow R[s^{-1}]$ is naturally isomorphic to the direct sum $\bigoplus_{m \in P(s)} K^\bullet_m$ in the derived category $\mathcal{D}^b(R{-}\text{mod})$.

Proof. The conditions of Lemma 13.3 are clearly satisfied for the complex $R \longrightarrow R[s^{-1}]$, so we have a natural isomorphism

$$(R \longrightarrow R[s^{-1}]) \simeq \bigoplus_{m \in P_1} (R_m \longrightarrow R_m[s^{-1}]).$$

in $\mathcal{D}^b(R{-}\text{mod})$. Now the complex $R_m \longrightarrow R_m[s^{-1}]$ is acyclic when $s \notin m$, so we are reduced to a direct sum over the finite set of ideals $m \in P(s)$. Finally, for $m \in P(s)$ we have $R_m[s^{-1}] = S^{-1}R_m$ by Lemma 13.5(c) applied to the local ring $R_m$; so the complex $R_m \longrightarrow R_m[s^{-1}]$ is isomorphic to $K^\bullet_m$. □

A finite complex of $R$-modules $M^\bullet$ is said to have projective dimension $\leq d$ if one has $\text{Hom}_{\mathcal{D}^b(R{-}\text{mod})}(M^\bullet, N[n]) = 0$ for all $R$-modules $N$ and all the integers $n > d$. We will denote the projective dimension of a finite complex $M^\bullet$ by $\text{pd}_R M^\bullet$. For a nonacyclic complex $M^\bullet$, one has $\text{pd}_R M^\bullet \in \mathbb{Z} \cup \{\infty\}$; and the projective dimension of an acyclic complex is equal to $-\infty$. 64
The next result is an extension of the classical theory of Matlis domains \cite[Section 8]{Mat}, \cite[Section IV.4]{Eis} to Noetherian rings of Krull dimension 1.

**Corollary 13.7.** One has

(a) \( \text{pd}_R(S^{-1}R) \leq 1 \);
(b) \( \text{pd}_R K^\bullet \leq 1 \).

**Proof.** Part (a) follows from part (b), because \( \text{pd}_R R = 0 \). To prove part (b), in view of Theorem 13.2, it suffices to show that the projective dimension of the complex of \( R \)-modules \( K^\bullet \) does not exceed 1 for every ideal \( m \in P_1 \). Choose an element \( s \in S \cap m \). By Corollary 13.6, the complex \( K^\bullet \) is a direct summand of the complex \( R \rightarrow R[s^{-1}] \) as an object of the derived category \( D^b(R\text{-mod}) \). Finally, the projective dimension of \( R \rightarrow R[s^{-1}] \) does not exceed 1, e. g., because \( \text{pd}_R R = 0 \) and \( \text{pd}_R R[s^{-1}] \leq 1 \) (cf. the discussion of the complex \( T^\bullet(R; s) \) in Sections 6–7).

Let \( A \) be a commutative ring and \( T \subset A \) a multiplicative set. An \( A \)-module \( B \) is called \( T \)-divisible if it is \( t \)-divisible for every \( t \in T \). The class of \( T \)-divisible \( A \)-modules is closed under the passages to quotient objects, extensions, infinite direct sums, and infinite products. An \( A \)-module \( B \) is said to be \( T \)-reduced if it has no \( T \)-divisible submodules. The class of \( T \)-reduced \( A \)-modules is closed under subobjects, extensions, infinite direct sums, and infinite products. Clearly, every \( A \)-module \( B \) has a unique maximal \( T \)-divisible submodule \( B_{\text{div}} \), equal to the sum of all the \( T \)-divisible submodules in \( B \). The quotient module \( B_{\text{red}} = B/B_{\text{div}} \) is \( T \)-reduced; it is the (unique) maximal \( T \)-reduced quotient module of \( B \).

We will be interested in \( S \)-divisible \( R \)-modules. The notation \( B_{\text{div}} \) and \( B_{\text{red}} \) for an \( R \)-module \( B \) will stand for the maximal \( S \)-divisible submodule and the maximal \( S \)-reduced quotient module of \( B \), respectively. Notice that for every maximal ideal \( m \in P_1 \), every \( m \)-contramodule \( R \)-module is \( S \)-reduced (because there exists \( s \in S \cap m \), and \( m \)-contramodules contain no \( s \)-divisible submodules).

Part (a) of the following theorem is a version of the classical theory of \( h \)-divisible modules \cite[Section VII.2]{Mat} for Noetherian rings of Krull dimension 1.

**Theorem 13.8.** (a) Every \( S \)-divisible \( R \)-module is a quotient module of an \( (S^{-1}R) \)-module.

(b) Every \( S \)-divisible \( R \)-module is cotorsion.

**Proof.** Let \( B \) be an \( S \)-divisible \( R \)-module. There is a natural distinguished triangle

\[
R \rightarrow S^{-1}R \rightarrow K^\bullet \rightarrow R[1]
\]

in the derived category \( D^b(R\text{-mod}) \). Applying the functor \( \text{Hom}_{D^b(R\text{-mod})}(-, B[*]) \), we get a fragment of the long exact sequence

\[
\cdots \rightarrow 0 \rightarrow \text{Hom}_{D^b(R\text{-mod})}(K^\bullet, B) \rightarrow \text{Hom}_R(S^{-1}R, B) \rightarrow B \rightarrow \text{Hom}_{D^b(R\text{-mod})}(K^\bullet, B[1]) \rightarrow \cdots
\]

According to Theorem 13.2, we have \( K^\bullet \simeq \bigoplus_{m \in P_1} K^\bullet_m \) in \( D^b(R\text{-mod}) \). By the last assertion of Lemma 13.1, the cohomology modules \( H^{-1}(K^\bullet_m) \) and \( H^0(K^\bullet_m) \) are...
m-torsion. Applying Lemma 6.2, we conclude that for every $n \in \mathbb{Z}$ the $R$-module $\text{Hom}_{\text{D}^b(R-\text{mod})}(K^\bullet, B[n])$ is a product of $m$-contramodule $R$-modules over the maximal ideals $m \in P_1$.

In particular, the $R$-module $\text{Hom}_{\text{D}^b(R-\text{mod})}(K^\bullet, B[1])$ is $S$-reduced. Since $B$ is $S$-divisible, it follows that the morphism $B \to \text{Hom}_{\text{D}^b(R-\text{mod})}(K^\bullet, B[1])$ vanishes and the morphism $\text{Hom}_R(S^{-1}R, B) \to B$ is surjective. We have proved part (a).

Furthermore, every $(S^{-1}R)$-module is cotorsion, because $S^{-1}R$ is an Artinian ring. By Lemma 9.1, every $(S^{-1}R)$-module is also a cotorsion $R$-module. By Theorem 9.3, the $R$-module $\text{Hom}_{\text{D}^b(R-\text{mod})}(K^\bullet, B)$ is cotorsion, too. Since the class of cotorsion $R$-modules is closed under the cokernels of injective morphisms, it follows that the $R$-module $B$ is cotorsion. □

Let us call an $R$-module $C$ weakly cotorsion (or Matlis cotorsion [18]) if $\text{Ext}^1_R(S^{-1}R, C) = 0$. Since the $R$-module $S^{-1}R$ is flat, every cotorsion $R$-module is weakly cotorsion. Part (b) of the next theorem provides the inverse implication, extending to Noetherian rings of Krull dimension 1 the theory of almost perfect domains of Bazzoni and Salce [2, Section 4].

**Theorem 13.9.** (a) The class of weakly cotorsion $R$-modules is closed under quotients (that is any quotient $R$-module of a weakly cotorsion $R$-module is weakly cotorsion).

(b) The classes of cotorsion $R$-modules and weakly cotorsion $R$-modules coincide.

(c) An $R$-module $C$ is cotorsion if and only if the $R$-module $C_{\text{red}}$ is cotorsion.

**Proof.** Part (a) follows immediately from Corollary 13.7(a). To prove part (b), let $C$ be a weakly cotorsion $R$-module. Consider another fragment of the long exact sequence from the proof of Theorem 13.8

$$\cdots \to \text{Hom}_R(S^{-1}R, C) \to C \to \text{Hom}_{\text{D}^b(R-\text{mod})}(K^\bullet, C[1]) \to \text{Ext}^1_R(S^{-1}R, C) \to \cdots$$

Since $\text{Ext}^1_R(S^{-1}R, C) = 0$, the $R$-module $C$ is an extension of the $R$-module $\text{Hom}_{\text{D}^b(R-\text{mod})}(K^\bullet, C)$, which is a product of $m$-contramodule $R$-modules over the maximal ideals $m \in P_1$, and a quotient module of an $(S^{-1}R)$-module $\text{Hom}_R(S^{-1}R, C)$. By Theorem 9.3, every product of $m$-contramodule $R$-modules is cotorsion. By Theorem 13.8(b), every quotient $R$-module of an $(S^{-1}R)$-module is cotorsion (being clearly $S$-divisible). The assertion follows.

Part (c) holds because the $R$-module $C_{\text{div}}$ is always cotorsion by Theorem 13.8(b), and the class of cotorsion $R$-modules is closed under extensions and the cokernels of injective morphisms. □

**Remark 13.10.** By the famous result of Raynaud–Gruson [30, Corollaire II.3.3.2], the projective dimension of any flat module over a Noetherian commutative ring $R$ does not exceed the Krull dimension of $R$. This covers the result of our Corollary 13.7(a), and also implies that the class of cotorsion $R$-modules is closed under quotients. We prefer not to use the difficult result of [30] here, but rather to
have a self-contained exposition in our generality. In fact, we have proved in The-
orem 13.9(a-b) that the class of cotorsion $R$-modules is closed under quotients, and
one can easily deduce from this the assertion that the projective dimension of every
flat $R$-module does not exceed 1. So we have obtained an independent proof of the
Krull dimension 1 case of the Raynaud–Gruson theorem with our methods.

On the other hand, the assertions of Theorem 13.8(a-b), Theorem 13.9(b), and
Corollaries 13.12–13.13 below do not seem to follow from known results.

Now we can deduce a corollary promised at the end of Section 8.

**Corollary 13.11.** Let $R$ be a Noetherian ring with finite spectrum. Then the classes
of cotorsion $R$-modules and contraadjusted $R$-modules coincide. The classes of flat
$R$-modules and very flat $R$-modules coincide.

**Proof.** As explained in [15, Theorem 144] (cf. [34, Lemma 2.10]), any Noetherian ring
with finite spectrum has Krull dimension $\leq 1$. Hence Theorem 13.9(b) applies, and it
suffices to prove that the $R$-module $S^{-1}R$ is very flat. The ring $R$ being semilocal, it
remains to use Lemma 13.5(a,c) in order to show that there exists an element $s \in S$
such that $S^{-1}R = R[s^{-1}]$. □

**Corollary 13.12.** An $R$-module $C$ is cotorsion if and only if it can be included into
a short exact sequence

$$0 \to D \to C \to \prod_{m \in P_1} C_m \to 0$$

where $D$ is an $S$-divisible $R$-module and $C_m$ are $m$-contramodule $R$-modules. Both
the short exact sequence and the direct product decomposition of the rightmost term
are uniquely defined and depend functorially on a cotorsion $R$-module $C$.

**Proof.** We have already seen in the proof of Theorem 13.9 that every cotorsion
$R$-module can be included into a such exact sequence. The exact sequence is unique,
because products of $m$-cotorsion $R$-modules are $S$-reduced, so $D = C_{\text{div}}$. The direct
product decomposition is unique by Lemma 11.8. □

We denote by $C_R \subset R\text{-mod}$ the full subcategory of $S$-reduced cotorsion $R$-modules.

**Corollary 13.13.** (a) The full subcategory $C_R$ is closed under the kernels, cokernels,
extensions, and infinite products in $R\text{-mod}$. In particular, $C_R$ is an abelian category
and its embedding $C_R \to R\text{-mod}$ is an exact functor.

(b) The functor $(C_m)_{m \in P_1} \mapsto \prod_m C_m$ establishes an equivalence between the Carte-
sian product of the abelian categories $R\text{-mod}_{m\text{-ctra}}$ of $m$-contramodule $R$-modules,
taken over the maximal ideals $m \in P_1$ of the ring $R$, and the category $C_R$.

**Proof.** Part (a): by Theorem 13.8(a), an $R$-module $C$ is reduced if and only if
$\text{Hom}_R(S^{-1}R, C) = 0$. By Theorem 13.9(b), an $R$-module $C$ is cotorsion if and only
if $\text{Hom}_R(S^{-1}R, C) = 0$. It remains to apply Theorem 1.2(a). Part (b) follows from
Corollary 13.12; see the proof of Corollary 12.11(b) for a discussion. □
Lemma 13.14. Let \( m \) be a maximal ideal in a Noetherian commutative ring \( A \) and let \( M \) be an \( m \)-torsion \( A \)-module. Then for every \( A \)-module \( C \) and all \( n \geq 0 \) the natural maps

\[
\text{Ext}_A^n(M, C) \rightarrow \text{Ext}_A^n(M, C_m) \rightarrow \text{Ext}_A^n(M, C_m)
\]

are isomorphisms.

Proof. First of all, \( M \) is an \( A_m \)-module, so the rightmost \( \text{Ext} \) module is well-defined. All the \( \text{Ext} \) modules in question can be viewed as the derived functors of the functor \( \text{Hom} \) with respect to its second argument with the first argument \( M \) fixed. Notice that the localization functor \( C \mapsto C_m \) takes injective \( A \)-modules to injective \( A_m \)-modules, which are also injective \( A \)-modules. Hence it suffices to consider the case \( n = 0 \).

Now the map \( \text{Hom}_A(M, C_m) \rightarrow \text{Hom}_{A_m}(M, C_m) \) is clearly an isomorphism, and the map \( \text{Hom}_A(M, C) \rightarrow \text{Hom}_A(M, C_m) \) is an isomorphism because the map \( \Gamma_m(C) \rightarrow \Gamma_m(C_m) \) is. To check the latter claim, one applies the functor of localization at \( m \) to the short exact sequence \( 0 \rightarrow \Gamma_m(C) \rightarrow C \rightarrow C/\Gamma_m(C) \rightarrow 0 \) and notices that \( \Gamma_m(C) \) is an \( A_m \)-module, while the \( A \)-module \( C/\Gamma_m(C) \) and the \( A_m \)-module \( (C/\Gamma_m(C))_m \) are \( m \)-torsion-free.

□

Theorem 13.15. For any maximal ideal \( m \in P_1 \), there is an isomorphism of functors

\[
(23) \quad \text{Hom}_{D^b(R \text{-mod})}(K_m^*, -[1]) \simeq \Delta_m: R \text{-mod} \longrightarrow R \text{-mod}_{m \text{-contra}}.
\]

For every \( R \)-module \( C \), the product of the isomorphisms \( \text{Hom}_{D^b(R \text{-mod})}(K_m^*, C[1]) \simeq \Delta_m(C) \) over all \( m \in P_1 \) together with the isomorphism \( \bigoplus_{m \in P_1} K_m^* \simeq K^* \) transform the morphism \( C = \text{Hom}_R(R, C) \rightarrow \text{Hom}_{D^b(R \text{-mod})}(K^*, C[1]) \) into the morphism \( C \rightarrow \prod_m \Delta_m(C) \) whose components are the adjunction morphisms \( \delta_{m,C}: C \rightarrow \Delta_m(C) \).

Proof. By Remark 9.2, the category of \( m \)-contra-module \( R \)-modules is a full subcategory of the category of \( R_m \)-modules, which is a full subcategory of the category of all \( R \)-modules,

\[
R \text{-mod}_{m \text{-contra}} \subset R_m \text{-mod} \subset R \text{-mod}.
\]

The localization functor \( C \mapsto C_m \) is left adjoint to the embedding \( R_m \text{-mod} \rightarrow R \text{-mod} \). Furthermore, according to Lemma 13.5 applied to the local ring \( R_m \), there exists an element \( s \in S \) such that \( S^{-1}R_m = R_m[s^{-1}] \) and the radical of the ideal \( (s) \subset R_m \) is equal to \( R_m m \). So the complex \( R_m \rightarrow R_m[s^{-1}] \) is isomorphic to \( K_m^* \). In view of Remark 5.5 (or the second proof of Theorem 5.1 in Section 7), the full subcategories of \( s \)-contra-module \( R_m \)-modules and \( (R_m m) \)-contra-module \( R_m \)-modules in \( R_m \text{-mod} \) coincide. Therefore, the functor left adjoint to the embedding \( R \text{-mod}_{m \text{-contra}} = R_m \text{-mod}_{(R_m m) \text{-contra}} \rightarrow R_m \text{-mod} \) can be computed as

\[
D \mapsto \text{Hom}_{D^b(R_m \text{-mod})}(K_m^*, D[1]), \quad D \in R_m \text{-mod}
\]

(see Theorem 6.4(iii) and Remark 6.5). It follows that the functor \( \Delta_m \) left adjoint to the composition of the two embeddings of categories can be obtained as the composition

\[
C \mapsto \text{Hom}_{D^b(R_m \text{-mod})}(K_m^*, C_m[1]), \quad C \in R \text{-mod},
\]

\[ 
\]
and the adjunction morphism is the composition
\[ C \to C_m \to \text{Hom}_{D^b(Rm\text{-mod})}(K^\bullet_m, C_m[1]). \]

Now, by Lemma 13.14 we have
\[ \text{Hom}_{D^b(R\text{-mod})}(K^\bullet_m, C_m[1]) \simeq \text{Hom}_{D^b(Rm\text{-mod})}(K^\bullet_m, C_m[1]), \]
which provides the desired isomorphism of functors (23). We leave it to the reader to finish the proof of the second assertion of the theorem. □

In view of the result of Theorem 13.15, the long exact sequence from the proofs of Theorems 13.8–13.9 takes the form
\[ \cdots \to \text{Hom}_R(S^{-1}R, C) \to C \to \prod_{m \in P_1} \Delta_m(C) \to \text{Ext}^1_R(S^{-1}R, C) \to 0 \]
for every \( R \)-module \( C \). The image of the morphism \( \text{Hom}_R(S^{-1}R, C) \to C \) is the submodule \( C_{\text{div}} \subset C \). So every \( S \)-reduced \( R \)-module \( C \) is naturally a submodule in the product \( \prod_{m \in P_1} \Delta_m(C) \).

Recall that a ring is called \textit{reduced} if it contains no nonzero nilpotent elements. In our setting, if the ring \( R \) is reduced, then \( S^{-1}R \) is a reduced Artinian ring, that is a direct sum of a finite number of fields.

\textbf{Corollary 13.16.} If the ring \( R \) is reduced and the \( R \)-module \( C \) is \( S \)-reduced, then \((\delta_{m,C})_{m \in P_1}: C \to \prod_{m \in P_1} \Delta_m(C)\) is a cotorsion envelope of the \( R \)-module \( C \).

\textit{Proof.} In the assumptions of the corollary, the morphism \( \delta_{S,C} = (\delta_{m,C})_{m \in P_1} \) is injective and its cokernel \( \text{Ext}^1_R(S^{-1}R, C) \) is an \((S^{-1}R)\)-module. Furthermore, every \((S^{-1}R)\)-module is a flat (and even projective) \((S^{-1}R)\)-module and a flat \( R \)-module. Hence the morphism \( \delta_{S,C} \) is a special cotorsion preenvelope. To prove that it is an envelope, argue as in the proof of Theorem 12.8(e): using Lemma 11.8, compute
\[ \text{Hom}_R(\prod_m \Delta_m(C), \prod_m \Delta_m(C)) = \prod_m \text{Hom}_R(\Delta_m(C), \Delta_m(C)) = \prod_m \text{Hom}_R(C, \Delta_m(C)) \]
and conclude that the equation \( u\delta_{S,C} = \delta_{S,C} \) for an endomorphism \( u: \prod_m \Delta_m(C) \to \prod_m \Delta_m(C) \) implies \( u = 1 \). □

\textbf{Remark 13.17.} It would be interesting to know how to construct cotorsion envelopes of nonreduced modules over Noetherian domains of Krull dimension 1. The construction of Corollary 12.9 requires the embedding \( C_{\text{div}} \to C \) to be split, so it only seems to work for Dedekind domains.

Now we return to our usual setting of an arbitrary Noetherian ring \( R \) of Krull dimension 1. Our aim is to characterize \( s \)-contraadjusted \( R \)-modules.

\textbf{Corollary 13.18.} For any element \( s \in S \) and every \( R \)-module \( C \), there is a natural isomorphism
\[ \Delta_s(C) \simeq \bigoplus_{m \in P(s)} \Delta_m(C), \]
where the direct sum is taken over the set \( P(s) \subset P_1 \) of all prime ideals \( m \subset R \) containing \( s \). The components of the map \( \delta_{s,C}: C \to \Delta_s(C) \) with respect to this direct sum decomposition are equal to \( \delta_{m,C} \).

**Proof.** Follows from Corollary 13.6 and Theorem 13.15.

**Corollary 13.19.** Let \( C \) be an \( R \)-module and \( s \in S \) be an element. Then

(a) an \( R \)-module \( C \) is \( s \)-contraadjusted if and only if the \( R \)-module \( C_{\text{red}} \) is \( s \)-contraadjusted;

(b) an \( R \)-module \( C \) is \( s \)-contraadjusted if and only if the map \( C \to \bigoplus_{m \in P(s)} \Delta_m(C) \) is surjective.

**Proof.** Part (b) follows from Corollaries 6.10 and 13.18. Part (a) holds, because the \( R \)-module \( C_{\text{div}} \) is \( s \)-divisible, hence \( \Delta_s(C_{\text{div}}) = 0 \) and \( \Delta_s(C) = \Delta_s(C_{\text{red}}) \), so the map \( C \to \Delta_s(C) \) is surjective if and only if the map \( C_{\text{red}} \to \Delta_s(C_{\text{red}}) \) is.

**Corollary 13.20.** An \( R \)-module \( C \) is \( s \)-contraadjusted for every element \( s \in S \) if and only if for every finite subset \( P \subset P_1 \) the map \( C \to \bigoplus_{m \in P} \Delta_m(C) \) is surjective.

**Proof.** Follows from Corollary 13.19(b), because for every finite subset \( P \subset P_1 \) there exists an element \( s \in S \) such that \( P \subset P(s) \).

**Remark 13.21.** For a Noetherian domain \( R \) of Krull dimension 1, the result of Corollary 13.20 says that an \( R \)-module \( C \) is contraadjusted if and only if for every finite set of maximal ideals \( P \subset P_1 \) in \( R \) the map \( C \to \bigoplus_{m \in P} \Delta_m(C) \) is surjective. Indeed, in this case \( S = R \setminus \{0\} \), and every \( R \)-module is \( 0 \)-contraadjusted. This extends the characterization of contraadjusted modules over Dedekind domains provided by [33, Corollary 4.13] to Noetherian domains of Krull dimension 1. It would be interesting to extend this kind of characterization of contraadjusted modules further to Noetherian rings of Krull dimension 1, but we do not know how to do it.

**References**

[1] H. Bass. Finitistic dimension and a homological generalization of semi-primary rings. *Transactions of the American Math. Society* 95, p. 466–488, 1960.

[2] S. Bazzoni, L. Salce. Strongly flat covers. *Journ. of the London Math. Society* 66, #2, p. 276–294, 2002.

[3] L. Bican, R. El Bashir, E. Enochs. All modules have flat covers. *Bulletin of the London Math. Society* 33, #4, p. 385–390, 2001.

[4] T. Bühler. Exact categories. *Expositiones Math.* 28, #1, p. 1–69, 2010. arXiv:0811.1480 [math.HO]

[5] P. C. Eklof, J. Trlifaj. How to make Ext vanish. *Bulletin of the London Math. Society* 33, #1, p. 41–51, 2001.

[6] R. El Bashir. Covers and directed colimits. *Algebras and Representation Theory* 9, #5, p. 423–430, 2006.

[7] E. Enochs. Flat covers and flat cotorsion modules. *Proceedings of the American Math. Society* 92, #2, p. 179–184, 1984.

[8] L. Fuchs. Infinite abelian groups, vol. I. Academic Press, New York–San Francisco–London, 1970.
[9] L. Fuchs, L. Salce. Modules over non-Noetherian domains. Mathematical Surveys and Monographs 84, American Math. Society, 2001.

[10] W. Geigle, H. Lenzing. Perpendicular categories with applications to representations and sheaves. Journ. of Algebra 144, #2, p. 273–343, 1991.

[11] R. Göbel, J. Trlifaj. Approximations and endomorphism algebras of modules. Second Revised and Extended Edition. De Gruyter Expositions in Mathematics 41, De Gruyter, Berlin–Boston, 2012.

[12] D. Happel, I. Reiten, S. O. Smalø. Tilting in abelian categories and quasitilted algebras. Memoirs of the American Math. Society 120, #575, 1996.

[13] D. K. Harrison. Infinite abelian groups and homological methods. Annals of Math. 69, #2, p. 366–391, 1959.

[14] R. Hartshorne. Residues and duality. Lecture Notes in Math. 20, Springer, 1966.

[15] I. Kaplansky. Commutative rings. Revised edition. University of Chicago Press, 1974.

[16] E. Matlis. Injective modules over Noetherian rings. Pacific Journ. of Math. 8, #3, p. 511–528, 1958.

[17] E. Matlis. Divisible modules. Proceedings of the American Math. Society 11, #3, p. 385–391, 1960.

[18] E. Matlis. Cotorsion modules. Memoirs of the American Math. Society 49, 1964.

[19] E. Matlis. Decomposable modules. Transactions of the American Math. Society 125, #1, p. 147–179, 1966.

[20] R. J. Nunke. Modules of extensions over Dedekind rings. Illinois Journ. of Math. 3, #2, p. 222–241, 1959.

[21] M. Porta, L. Shaul, A. Yekutieli. On the homology of completion and torsion. Algebras and Representation Theory 17, #1, p. 31–67, 2014. arXiv:1010.4386 [math.AC]. Erratum in Algebras and Representation Theory 18, #5, p. 1401–1405, 2015. arXiv:1506.07765 [math.AC]

[22] M. Porta, L. Shaul, A. Yekutieli. Cohomologically cofinite complexes. Communications in Algebra 43, #2, p. 597–615, 2015. arXiv:1208.4064 [math.AC]

[23] L. Positselski. Homological algebra of semimodules and semicontramodules: Semi-infinite homological algebra of associative algebraic structures. Appendix C in collaboration with D. Rumynin; Appendix D in collaboration with S. Arkhipov. Monografie Matematyczne vol. 70, Birkhäuser/Springer Basel, 2010. xxiv+349 pp. arXiv:0708.3398 [math.CT]

[24] L. Positselski. Weakly curved $A_{\infty}$-algebras over a topological local ring. Electronic preprint arXiv:1202.2697 [math.CT].

[25] L. Positselski. Contraherent cosheaves. Electronic preprint arXiv:1209.2995 [math.CT].

[26] L. Positselski. Contramodules. Electronic preprint arXiv:1503.00991 [math.CT].

[27] L. Positselski. Dedualizing complexes and MGM duality. Journ. of Pure and Appl. Algebra 220, #12, p. 3866–3909, 2016. arXiv:1503.05523 [math.CT]

[28] L. Positselski. Triangulated Matlis equivalence. Electronic preprint arXiv:1605.08018 [math.CT], to appear in Journ. of Algebra and its Appl.

[29] L. Positselski. J. Rosický. Covers, envelopes, and cotorsion theories in locally presentable abelian categories and contramodule categories. Journ. of Algebra 483, p. 83–128, 2017. arXiv:1512.08119 [math.CT]

[30] M. Raynaud, L. Gruson. Critères de platitude et de projectivité: Techniques de “platification” d’un module. Inventiones Math. 13, #1–2, p. 1–89, 1971.

[31] L. Salce. Cotorsion theories for abelian groups. Symposia Math. XXIII, Academic Press, London–New York, 1979, p. 11–32.

[32] A.-M. Simon. Approximations of complete modules by complete big Cohen–Macaulay modules over a Cohen–Macaulay local ring. Algebras and Representation Theory 12, #2–5, p. 385–400, 2009.

[33] A. Slávík. Classes of modules arising in contemporary algebraic geometry. Master Thesis, Department of Algebra, Faculty of Mathematics and Physics, Charles University, Prague, 2015.
A. Slávik, J. Trlifaj. Very flat, locally very flat, and contraadjusted modules. *Journ. of Pure and Appl. Algebra* **220**, #12, p. 3910–3926, 2016. arXiv:1601.00783 [math.AC]

J. Šťovíček. Exact model categories, approximation theory, and cohomology of quasi-coherent sheaves. In: Advances in Representation Theory of Algebras, EMS Series of Congress Reports, European Math. Society, Zurich, 2013, 297–367. arXiv:1301.5206 [math.CT]

J. Xu. Flat covers of modules. *Lecture Notes in Math.* **1634**, Springer, 1996.

A. Yekutieli. On flatness and completion for infinitely generated modules over noetherian rings. *Communications in Algebra* **39**, #11, p. 4221–4245, 2011. arXiv:0902.4378 [math.AC]

A. Yekutieli. A separated cohomologically complete module is complete. *Communications in Algebra* **43**, #2, p. 616–622, 2015. arXiv:1312.2714 [math.AC]

Department of Mathematics, Faculty of Natural Sciences, University of Haifa, Mount Carmel, Haifa 31905, Israel; and Laboratory of Algebraic Geometry, National Research University Higher School of Economics, Moscow 119048; and Sector of Algebra and Number Theory, Institute for Information Transmission Problems, Moscow 127051, Russia

E-mail address: posic@mccme.ru