Supplementary Materials for

Experimental test of quantum causal influences

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NOISE MODEL

In this section, we describe the noise model which best fits our experimental data, reported in the curves shown in Fig. 5 of the main text. In detail, we considered a noise model, including a fraction of white noise (the visibility $v$) and a fraction of coloured noise ($\lambda$), which is typical of SPDC sources [42].

White noise is the result of a partial isotropic depolarization of the state. Indicating the two-qubit ideal pure state as $|\psi\rangle\langle\psi|$, the state affected by white noise is given by

$$\rho_{\text{white}} = v' |\psi\rangle\langle\psi| + \frac{(1 - v')}{4} I \quad (1)$$

On the other hand, coloured noise appears as a decoherence-type term. Its action on the pure state $|\psi\rangle\langle\psi|$, quantified by the parameter $\lambda'$, is described as

$$\rho_{\text{col}} = \lambda' |\psi\rangle\langle\psi| + \frac{(1 - \lambda')}{2} (|\psi^+\rangle\langle\psi^+| + |\psi^-\rangle\langle\psi^-|). \quad (2)$$

with $|\psi^\pm\rangle = \frac{|00\rangle \pm |11\rangle}{\sqrt{2}}$.

Therefore, our overall noisy state is modelled combining these two contributions and then given by

$$\rho_{\text{noisy}} = v |\psi\rangle\langle\psi| + (1 - v) \left( \frac{\lambda}{2} (|\psi^+\rangle\langle\psi^+| + |\psi^-\rangle\langle\psi^-|) + \frac{(1 - \lambda)}{4} I \right), \quad (3)$$

where the visibility $v$ is such that $v = 1$ yields the pure state $|\psi\rangle\langle\psi|$, while $\lambda$ refers to the fraction of coloured noise with respect to the visibility.

The parameters $v$ and $\lambda$ that best model our experimental state are

$v = 0.81$,  
$\lambda = 0.93$.  

An additional source of experimental noise that we considered, in order to predict the expected values for the cACE and qACE reported in Fig. 5 of the main text, is the non-perfect phase $\delta$ introduced by the electro-optical device (Pockels cell, PC) employed in Bob’s measurement station.

For our experiment, we require that $\delta = \pi$, so that, when triggered by high voltage, the action of such a device is identical of that of a half-wave plate (HWP). Otherwise, it performs the identity operation. However, the largest effective phase inserted by the PC (for a voltage of 1350V) oscillated between 0.7$\pi$ and 0.8$\pi$.

Indeed, the $\delta$ value which best models our experimental data equals 0.802$\pi$ for the curve in Fig. 5a of the main text and 0.716$\pi$ for that in Fig. 5b.

Moreover, we also took into account minor noise sources such as the non-perfect rotation of the adopted HWPs and the non-perfect efficiencies of the detectors.

OPTIMIZATION AND IMPLEMENTATION OF THE OPERATORS

In this supplementary section, we report the optimal operators for Alice and Bob, corresponding to the measurement settings $MS_1$ and $MS_2$, given the degree of entanglement $\alpha$, to reproduce, respectively, the curve in Fig. 2a and Fig. 2b of the main text.
We define the operator respectively measured by Alice and Bob using the notation [37]

\[ M(\eta_x) := M^x = M_0^x - M_1^x, \quad N(\phi_a) := N^a = N_0^a - N_1^a, \quad \{ x, a \} \in \{ 0, 1 \}, \]

where, \( M^x \) and \( N^a \) are given by

\[ M^x = \cos(\eta_x)\sigma_x + \sin(\eta_x)\sigma_z, \]

and

\[ N^a = \cos(\phi_a)\sigma_z + \sin(\phi_a)\sigma_x. \]

Thus, \( q\text{ACE} \) can be expressed as

\[ q\text{ACE}_{A\rightarrow B} = \max_{\theta} \{ \text{Tr}[I \otimes (N_0^a - N_1^a)]|\psi\rangle\langle\psi| \} = \frac{1}{2}|\langle I \otimes (N(\phi_0) - N(\phi_1))\rangle|. \]

Without loss of generality, we can assume \( |\langle I \otimes (N(\phi_0) - N(\phi_1))\rangle| \geq 0 \). If this is not the case, one can always consider the scenario where Alice relabels the measurement outcomes and the subsequent derivations would follow accordingly.

Following Eq. (5) of the main text, the classical lower bound on the ACE, \( c\text{ACE}_{A\rightarrow B}^* \), is defined as

\[ c\text{ACE}_{A\rightarrow B}^* = 2\langle M_0^0 \otimes N_0^0 \rangle + \langle M_1^0 \otimes N_1^0 \rangle + \langle M_0^1 \otimes N_1^0 \rangle + \langle M_1^1 \otimes N_1^1 \rangle - 2. \]

This quantity can be rewritten, after some simplifications, as

\[ c\text{ACE}_{A\rightarrow B}^* = \frac{1}{4}(-3 + \langle I \otimes (N(\phi_0)) \rangle + \langle M(\theta_0) \otimes I \rangle - 2\langle I \otimes N(\phi_1) \rangle + f(\theta_0, \theta_1, \phi_0, \phi_1, \alpha)). \]

The expected quantum violation of the causal bound, namely the quantity \( c\text{ACE}_{A\rightarrow B}^* - q\text{ACE}_{A\rightarrow B} \), amounts to

\[ c\text{ACE}_{A\rightarrow B}^* - q\text{ACE}_{A\rightarrow B} = \frac{1}{4}(-3 - \langle I \otimes (N(\phi_0)) \rangle + \langle M(\eta_0) \otimes I \rangle + f(\eta_0, \eta_1, \phi_0, \phi_1, \alpha)). \]

It can be numerically shown that this quantity is maximal when imposing \( \phi_1 = -\phi_0 \) and \( \theta_1 = -\frac{\pi}{8} \). In this case, the following equality holds for function \( f(\eta_0, \eta_1, \phi_0, \phi_1, \alpha) = 3\cos(\eta_0)\cos(\phi_0) + \sin(2\alpha)\sin(\phi_0)(2 + \sin(\eta_0)) \). This choice of angles \( \{ \phi_0, \phi_1, \eta_1 \} \) characterizes the operators which reproduce the curve in Fig. 2a. Moreover, let us note that the assumption \( \phi_1 = -\phi_0 \) ensures that \( q\text{ACE} = 0 \). The parameter \( \eta_0 \) can be optimized analytically such that it maximizes the violation, yielding

\[ \theta_0 = \arccot \left( \frac{\cos(2\alpha) + 3\cos(\phi_0)}{\sin(2\alpha)\sin(\phi_0)} \right) \]

and \( \phi_0 = \arctan \left( \frac{2}{\sqrt{3} + 2} \right) \).

To reproduce the curve in Fig. 2b of the main text, we let the entanglement degree \( \alpha \) range in the interval \( \alpha \in [\pi/8, \pi/4] \), while the angles of Alice and Bob’s operators are given by

\[ \eta_0 = 3 \left( \alpha - \frac{\pi}{8} \right), \]

\[ \eta_1 = \pi, \]

\[ \phi_0 = 2 \left( \alpha - \frac{\pi}{8} \right), \]

\[ \phi_1 = \pi - 3 \left( \alpha - \frac{\pi}{8} \right). \]

In order to experimentally implement Bob’s operators for MS1, we adopted a PC followed by a fixed HWP. The latter is rotated of \( \frac{\pi}{4} \), so that, when the PC is not triggered and behaves like an identity operator, Bob performs the projective measurement onto the eigenstates of \( N^1 \), i.e. \( \cos(\phi_1)\sigma_z + \sin(\phi_1)\sigma_x \). Then, in order to switch to \( N^0 \), i.e. \( \cos(\phi_0)\sigma_z + \sin(\phi_0)\sigma_x \), with \( \phi_0 = -\phi_1 \), when the Pockels cell is triggered, we require the PC to be in its optical axis.
and to insert a π shift between the horizontal and vertical polarization states. Indeed, the Pockels cell has the Jones matrix given by

$$P(\delta) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\delta} \end{pmatrix},$$

which, when $\delta = \pi$, is equal to $\sigma_z$. Hence, the combined action of the PC and the fixed HWP amounts to

$$\sigma_z (\cos(\phi_1)\sigma_z + \sin(\phi_1)\sigma_x) \sigma_z = \cos(\phi_1)\sigma_z - \sin(\phi_1)\sigma_x = \cos(\phi_0)\sigma_z + \sin(\phi_0)\sigma_x = N^0.$$  \hspace{1cm} (14)

In order to experimentally implement Bob’s operators for MS2 (see Eq. (12)), we keep $\delta = \pi$ and the HWP fixed to $\frac{\phi}{2}$, but we change the polarization orthogonal states between which the PC inserts the phase, by rotating the PC of an angle $\theta$. In this way, the combined action of the rotated PC and the fixed HWP is

$$N(\theta) = R(\theta)P(\pi)R(-\theta)N^1 R(\theta)P^\dagger(\pi)R(-\theta) = R(\theta)\sigma_z R(-\theta)N^1 R(\theta)\sigma_z R(-\theta),$$

where $R(\eta)$ is the rotation matrix

$$R(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.$$  \hspace{1cm} (16)

At this point, in order to select the proper angle $\eta$ for each $\alpha$, the following system of equations is solved:

$$\begin{align*}
\text{Tr}[M_0^x \otimes N^0_0 \rho(\alpha)] &= \text{Tr}[M_0^x \otimes N_0(\theta) \rho(\alpha)], \\
\text{Tr}[M_0^y \otimes N^0_0 \rho(\alpha)] &= \text{Tr}[M_0^y \otimes N_0(\theta) \rho(\alpha)], \\
\text{Tr}[M_0^z \otimes N^0_0 \rho(\alpha)] &= \text{Tr}[M_0^z \otimes N_1(\theta) \rho(\alpha)], \\
\text{Tr}[M_1^x \otimes N^0_0 \rho(\alpha)] &= \text{Tr}[M_1^x \otimes N_0(\theta) \rho(\alpha)], \\
\text{Tr}[M_1^y \otimes N^0_0 \rho(\alpha)] &= \text{Tr}[M_1^y \otimes N_0(\theta) \rho(\alpha)], \\
\text{Tr}[M_1^z \otimes N^0_0 \rho(\alpha)] &= \text{Tr}[M_1^z \otimes N_1(\theta) \rho(\alpha)], \\
\text{Tr}[M_0^1 \otimes N^0_0 \rho(\alpha)] &= \text{Tr}[M_0^1 \otimes N_0(\theta) \rho(\alpha)], \\
\text{Tr}[M_1^1 \otimes N^0_0 \rho(\alpha)] &= \text{Tr}[M_1^1 \otimes N_0(\theta) \rho(\alpha)], \\
\text{Tr}[M_0^0 \otimes N^0_1 \rho(\alpha)] &= \text{Tr}[M_0^0 \otimes N_1(\theta) \rho(\alpha)], \\
\text{Tr}[M_1^0 \otimes N^0_1 \rho(\alpha)] &= \text{Tr}[M_1^0 \otimes N_1(\theta) \rho(\alpha)].
\end{align*}$$  \hspace{1cm} (17)

where $M^x$ and $N^\alpha$ are defined as in Eqs. (5)-(6) and $\theta_x$ and $\phi_\alpha$ are those in Eq. (12). Furthermore, $\rho(\alpha) = |\psi_\alpha\rangle\langle\psi_\alpha|$ and $|\psi_\alpha\rangle = \cos(\alpha)|00\rangle + \sin(\alpha)|11\rangle$. An equivalent way to implement $N(\theta)$ in Eq. (15), which is the one we choose, is to keep the PC in its optical axis and put it in between of two HWPs, rotated of $\frac{\phi}{2}$, so that the input polarization state is rotated of $\theta$ and, after the PC, it is rotated back (see the section Experimental data and details of this supplementary material and the Experimental setup section of the main text).

**EXPERIMENTAL QACE LOWER BOUNDS**

The curve in Fig. 1a of the main text displays a case where the expected qACE amounts to 0, for any degree of entanglement $\alpha$, considering a generated state $|\psi\rangle = \cos(\alpha)|00\rangle + \sin(\alpha)|11\rangle$ and the measurement operators constituting MS1. Hence, the lower bound of this quantity is trivially 0, given that, according to its definition, any average causal effect must belong to the interval $(0, 1)$. From an experimental point of view, this was confirmed by our apparatus, because the lower bounds of the qACE were negative for all of the tested $\alpha$. Therefore, in the experimental figure 5a, we set all of the negative lower bounds to 0. In the table below, we report the trivial experimental values that we experimentally obtained applying Eq. (10) of the main text. The table also reports the distance, in terms of standard deviations, from 0, to quantify the confidence, with which we can conclude that those values are lower than 0. Let us note that the same situation occurred also for the cACE lower bound corresponding to $\alpha = 0$, for MS1, and for the qACE lower bound corresponding to $\alpha = 0.785$, for MS2. The experimental values amounted, respectively, to $-0.01040 \pm 0.00010$ and to $-0.1573 \pm 0.0030$. 
Table S 1: Trivial experimental (negative) values of qACE lower bounds (MS1). In the first column on the left, we report the parameter $\alpha$, which specifies the generated state $|\cos(\alpha)00 + \sin(\alpha)11\rangle$, while, in the last on the right, we report the distance, in terms of standard deviations from 0, $D_{\sigma}$, to quantify the confidence with which we can conclude that those values are lower than 0.

| $\alpha$     | qACE$_{LB}$ | $\pm$  | $D_{\sigma}$ |
|--------------|-------------|--------|-------------|
| 0            | -0.35118    | 0.00055| 642.80      |
| 0.209        | -0.2776     | 0.0027 | 100.72      |
| 0.305        | -0.1823     | 0.0046 | 39.72       |
| 0.393        | -0.2472     | 0.0024 | 101.90      |
| 0.523        | -0.1870     | 0.0036 | 53.49       |
| 0.698        | -0.1211     | 0.0042 | 27.95       |
| 0.785        | -0.1782     | 0.0046 | 39.42       |

Table S 1: Experimental frequencies $p(a,b|x = 0)$ (observational data, MS1). The uncertainties were valuated by propagation, considering a Poissonian distribution for the coincidence counts. Then, $\alpha$ refers to twice the angle of which the pump waveplate ($H_{1}$, see Fig. 3 of the main text) is rotated, to obtain the state $|\cos(\alpha)00 + \sin(\alpha)11\rangle$.

| $\alpha$ | $p(0,0|0)$ | $\pm$  | $p(0,1|0)$ | $\pm$  | $p(1,0|0)$ | $\pm$  | $p(1,1|0)$ | $\pm$  |
|----------|------------|--------|------------|--------|------------|--------|------------|--------|
| 0.7854   | 0.5295     | 0.0043 | 0.02145    | 0.0071 | 0.1107     | 0.0017 | 0.3382     | 0.0032 |
| 0.6981   | 0.6316     | 0.0050 | 0.03304    | 0.0090 | 0.1314     | 0.0019 | 0.2040     | 0.0024 |
| 0.6109   | 0.5861     | 0.0046 | 0.05040    | 0.0011 | 0.1200     | 0.0018 | 0.2435     | 0.0027 |
| 0.5236   | 0.6163     | 0.0038 | 0.04565    | 0.0082 | 0.1176     | 0.0014 | 0.2205     | 0.0020 |
| 0.3927   | 0.6969     | 0.0048 | 0.10121    | 0.0014 | 0.04833    | 0.00098| 0.1536     | 0.0018 |
| 0.2094   | 0.9074     | 0.0050 | 0.03473    | 0.00072| 0.02314    | 0.00058| 0.03471    | 0.00072|
| 0        | 0.9940     | 0.0026 | 0.002553   | 0.00070| 0.00150    | 0.00012| 0.000070   | 0.000057|

Table S 2: Experimental frequencies $p(a, b|x = 0)$ (observational data, MS1). The uncertainties were valuated by propagation, considering a Poissonian distribution for the coincidence counts. Then, $\alpha$ refers to twice the angle of which the pump waveplate ($H_{1}$, see Fig. 3 of the main text) is rotated, to obtain the state $|\cos(\alpha)00 + \sin(\alpha)11\rangle$.
Table S 3: Experimental frequencies $p(a,b|x=1)$ (observational data, MS1). The corresponding uncertainties were valued by propagation, considering a Poissonian distribution for the coincidence counts. Then, $\alpha$ refers to twice the angle of which the pump waveplate ($H_1$, see Fig. 3 of the main text) is rotated, to obtain the state $\cos(\alpha)|00\rangle + \sin(\alpha)|11\rangle$.

| $\alpha$ | $p(0,0|0)$ | $\pm$ | $p(1,0|1)$ | $\pm$ | $p(0,1|1)$ | $\pm$ | $p(1,1|1)$ | $\pm$ |
|----------|------------|--------|------------|--------|------------|--------|------------|--------|
| 0.7854   | 0.1715     | 0.0021 | 0.3920     | 0.0034 | 0.1233     | 0.0017 | 0.3131     | 0.0030 |
| 0.6981   | 0.2504     | 0.0027 | 0.3228     | 0.0032 | 0.1040     | 0.0016 | 0.3229     | 0.0032 |
| 0.6109   | 0.2126     | 0.0023 | 0.3615     | 0.0031 | 0.1116     | 0.0016 | 0.3142     | 0.0029 |
| 0.5236   | 0.2439     | 0.0021 | 0.3366     | 0.0026 | 0.1261     | 0.0015 | 0.2933     | 0.0024 |
| 0.3927   | 0.2449     | 0.0024 | 0.2984     | 0.0027 | 0.2440     | 0.0024 | 0.2128     | 0.0022 |
| 0.2094   | 0.5226     | 0.0036 | 0.0787     | 0.0012 | 0.3051     | 0.0026 | 0.0936     | 0.0013 |
| 0        | 0.5507     | 0.0018 | 0.000873   | 0.000056 | 0.4477     | 0.0015 | 0.00725   | 0.00051 |

Table S 4: Experimental frequencies $p(b|do(a))$ (interventional data, MS1). The corresponding uncertainties were valued by propagation, considering a Poissonian distribution for the coincidence counts. Then, $\alpha$ refers to twice the angle of which the pump waveplate ($H_1$, see Fig. 3 of the main text) is rotated, to obtain the state $\cos(\alpha)|00\rangle + \sin(\alpha)|11\rangle$.

| $\alpha$ | $p(0|do(0))$ | $\pm$ | $p(1|do(0))$ | $\pm$ | $p(0|do(1))$ | $\pm$ | $p(1|do(1))$ | $\pm$ |
|----------|--------------|--------|------------|--------|------------|--------|------------|--------|
| 0.7854   | 0.5338       | 0.0012 | 0.4662     | 0.0011 | 0.5432     | 0.0014 | 0.4568     | 0.0013 |
| 0.6981   | 0.5152       | 0.0081 | 0.4848     | 0.0077 | 0.5187     | 0.0032 | 0.4813     | 0.0030 |
| 0.6109   | 0.5334       | 0.0032 | 0.4666     | 0.0030 | 0.5291     | 0.0050 | 0.4709     | 0.0046 |
| 0.5236   | 0.6057       | 0.0049 | 0.3943     | 0.0037 | 0.6113     | 0.0070 | 0.3887     | 0.0052 |
| 0.3927   | 0.6961       | 0.0098 | 0.3039     | 0.0057 | 0.7014     | 0.0057 | 0.2986     | 0.0032 |
| 0.2094   | 0.9132       | 0.0043 | 0.08676    | 0.00099 | 0.9096     | 0.0051 | 0.0904     | 0.0012 |
| 0        | 0.9985       | 0.0037 | 0.00151    | 0.00010 | 0.9985     | 0.0033 | 0.001484   | 0.000090 |

Table S 5: Rotation angles of the waveplates within the apparatus in Fig. 3. In detail, $\alpha$ refers to twice the angle of which the pump waveplate ($H_1$, see Fig. 3 of the main text) is rotated, to obtain the state $\cos(\alpha)|00\rangle + \sin(\alpha)|11\rangle$. Then, $\eta_0$ and $\eta_1$ corresponds to 4 times the angle of which Alice’s waveplate ($H_2$, see Fig. 3 of the main text) is rotated, to perform projection onto the eigenvectors of the following operators: $M_{(a,1)} = \cos(\eta_{(0,1)})\sigma_x + \sin(\eta_{(0,1)})\sigma_y$. Regarding Bob’s measurement station, $\phi_0$ and $\phi_1$ correspond to 4 times the angle of which Bob’s waveplate ($H_3$, see Fig. 3 of the main text) is rotated, to perform projection onto the eigenvectors of the following operators: $N_{(0,1)} = \cos(\phi_{(0,1)})\sigma_x + \sin(\phi_{(0,1)})\sigma_y$. Then, $\theta$ amounts to twice the angle of which the half waveplates before and after the Pockels cell are rotated ($H_4$ and $H_5$, see Fig. 3 of the main text), in order to select the two orthogonal polarization states, between which the fast electro-optical device inserts a $\pi$ phase shift. In detail, an angle $\theta$ selects those two states to be the following: $\cos(\theta)|0\rangle + \sin(\theta)|1\rangle$ and $-\sin(\theta)|0\rangle + \cos(\theta)|1\rangle$.

| $\alpha$ | $\eta_0$ | $\eta_1$ | $\phi_0$ | $\phi_1$ | $\theta$ |
|----------|----------|----------|----------|----------|---------|
| 0.7854   | 0.5186   | -1.5708  | 0.6591   | -0.6591  | 0       |
| 0.6981   | 0.5503   | -1.5708  | 0.6777   | -0.6777  | 0       |
| 0.6109   | 0.5571   | -1.5708  | 0.6773   | -0.6773  | 0       |
| 0.5236   | 0.6097   | -1.5708  | 0.6566   | -0.6566  | 0       |
| 0.3927   | 0.7639   | -1.5708  | 0.5835   | -0.5835  | 0       |
| 0.2094   | 1.2050   | -1.5708  | 0.3790   | -0.3790  | 0       |
| 0        | 0        | -1.5708  | 0        | 0        | 0       |
Table S 6: Experimental frequencies $p(a, b|x = 0)$ (observational data, MS2). The corresponding uncertainties were valuated by propagation, considering a Poissonian distribution for the coincidence counts. Then, $\alpha$ refers to twice the angle of which the pump waveplate ($H_1$, see Fig. 3 of the main text) is rotated, to obtain the state $\cos(\alpha)|00\rangle + \sin(\alpha)|11\rangle$.

| $\alpha$ | $p(0,0|00)$ | $\pm$ | $p(0,1|00)$ | $\pm$ | $p(1,0|00)$ | $\pm$ | $p(1,1|00)$ | $\pm$ |
|----------|-------------|-------|-------------|-------|-------------|-------|-------------|-------|
| 0.3927   | 0.8063      | 0.0039| 0.02604     | 0.00054| 0.1364      | 0.0013| 0.03120     | 0.00058|
| 0.5230   | 0.7339      | 0.0042| 0.02150     | 0.00053| 0.1947      | 0.0018| 0.04996     | 0.00083|
| 0.5934   | 0.6652      | 0.0059| 0.02340     | 0.00073| 0.2159      | 0.0026| 0.09555     | 0.0016 |
| 0.6630   | 0.6077      | 0.0016| 0.01880     | 0.00027| 0.21150     | 0.00090| 0.16197     | 0.00078|
| 0.7854   | 0.5064      | 0.0028| 0.04445     | 0.00068| 0.1210      | 0.0012| 0.3281      | 0.0021 |

Table S 7: Experimental frequencies $p(a, b|x = 1)$ (observational data, MS2). The corresponding uncertainties were valuated by propagation, considering a Poissonian distribution for the coincidence counts. Then, $\alpha$ refers to twice the angle of which the pump waveplate ($H_1$, see Fig. 3 of the main text) is rotated, to obtain the state $\cos(\alpha)|00\rangle + \sin(\alpha)|11\rangle$.

| $\alpha$ | $p(0,1|11)$ | $\pm$ | $p(0,1|11)$ | $\pm$ | $p(1,0|11)$ | $\pm$ | $p(1,1|11)$ | $\pm$ |
|----------|-------------|-------|-------------|-------|-------------|-------|-------------|-------|
| 0.3927   | 0.03175     | 0.00057| 0.1835      | 0.0015| 0.1243      | 0.0012| 0.6605      | 0.0033|
| 0.5230   | 0.05335     | 0.00080| 0.2987      | 0.0021| 0.06906     | 0.00090| 0.5769      | 0.0032|
| 0.5934   | 0.06262     | 0.00116| 0.3651      | 0.0034| 0.0820      | 0.0013| 0.4902      | 0.0043|
| 0.6630   | 0.08406     | 0.00052| 0.4080      | 0.0012| 0.07523     | 0.00049| 0.4327      | 0.0012|
| 0.7854   | 0.16069     | 0.00132| 0.3787      | 0.0022| 0.07653     | 0.00088| 0.3840      | 0.0022|

Table S 8: Experimental frequencies $p(b|do(a))$ (interventional data, MS2). The corresponding uncertainties were valuated by propagation, considering a Poissonian distribution for the coincidence counts. Then, $\alpha$ refers to twice the angle of which the pump waveplate ($H_1$, see Fig. 3 of the main text) is rotated, to obtain the state $\cos(\alpha)|00\rangle + \sin(\alpha)|11\rangle$.

| $\alpha$ | $p(0|do(0))$ | $\pm$ | $p(1|do(0))$ | $\pm$ | $p(0|do(1))$ | $\pm$ | $p(1|do(1))$ | $\pm$ |
|----------|-------------|-------|-------------|-------|-------------|-------|-------------|-------|
| 0.3927   | 0.8058      | 0.0055| 0.1942      | 0.0022| 0.3738      | 0.0045| 0.6262      | 0.0063|
| 0.5230   | 0.7300      | 0.0067| 0.2770      | 0.0034| 0.394       | 0.012 | 0.606       | 0.015 |
| 0.5934   | 0.6513      | 0.0109| 0.3487      | 0.0067| 0.4399      | 0.0071| 0.5501      | 0.0082|
| 0.6630   | 0.5922      | 0.0040| 0.4077      | 0.0031| 0.4394      | 0.0075| 0.5066      | 0.0076|
| 0.7854   | 0.5412      | 0.0046| 0.4857      | 0.0044| 0.511       | 0.017 | 0.489       | 0.016 |
Table S 9: Rotation angles of the waveplates within the apparatus in Fig. 3. In detail, $\alpha$ refers to twice the angle of which the pump waveplate ($H_1$, see Fig. 3 of the main text) is rotated, to obtain the state $\cos(\alpha)\ket{00} + \sin(\alpha)\ket{11}$. Then, $\eta_0$ and $\eta_1$ corresponds to 4 times the angle of which Alice’s waveplate ($H_2$, see Fig. 3 of the main text) is rotated, to perform projection onto the eigenvectors of the following operators: $M_{(0,1)} = \cos(\eta_{(0,1)})\sigma_z + \sin(\eta_{(0,1)})\sigma_x$. Regarding Bob’s measurement station, $\phi_0$ and $\phi_1$ correspond to 4 times the angle of which Bob’s waveplate ($H_3$, see Fig. 3 of the main text) is rotated, to perform projection onto the eigenvectors of the following operators: $N_{(0,1)} = \cos(\phi_{(0,1)})\sigma_z + \sin(\phi_{(0,1)})\sigma_x$. In the end, $\theta$ amounts to twice the angle of which the half waveplates before and after the Pockels cell are rotated ($H_4$ and $H_5$, see Fig. 3 of the main text), in order to select the two orthogonal polarization states, between which the fast electro-optical device inserts a $\pi$ phase shift. In particular, an angle $\theta$ selects those two states to be the following: $\cos(\theta)\ket{0} + \sin(\theta)\ket{1}$ and $-\sin(\theta)\ket{0} + \cos(\theta)\ket{1}$.

| $\alpha$ | $\eta_0$ | $\eta_1$ | $\phi_0$ | $\phi_1$ | $\theta$ |
|----------|----------|----------|----------|----------|----------|
| 0.3927   | 0        | $\pi$    | 0        | $\pi$    | 0.7854   |
| 0.5230   | 0.3909   | $\pi$    | 0.2606   | 2.7507   | 0.7527   |
| 0.5934   | 0.6021   | $\pi$    | 0.4014   | 2.5395   | 0.7352   |
| 0.6630   | 0.8109   | $\pi$    | 0.5406   | 2.3307   | 0.7178   |
| 0.7854   | 1.1781   | $\pi$    | 0.7854   | 1.9635   | 0.6872   |