Generalized model for steady-state bifurcations without parameters in memristor-based oscillators with lines of equilibria

Ivan A. Korneev · Andrei V. Slepnev · Anna S. Zakharova · Tatiana E. Vadivasova · Vladimir V. Semenov

1 Introduction

The simultaneous and continuous dependence of the oscillatory dynamics both on parameter values and initial conditions is a frequent occurrence in various dynamical systems. In most cases, such peculiarities are associated with the properties of conservative oscillators. However, such behavior can be exhibited by non-linear dissipative oscillators with manifolds of equilibria. The manifolds of equilibria consist of non-isolated equilibrium points and take different forms: line of equilibria [9–11,22], surface of equilibria [16], circle [12,13] and square [13] equilibria, etc. A special class of such manifolds is represented by $m$-dimensional normally hyperbolic manifolds of equilibria characterized by $m$ pure imaginary eigenvalues or $m$ eigenvalues being equal to zero, whereas all the other eigenvalues have non-zero real parts. In the simplest case $m = 1$, these manifolds constitute a line of equilibria. The significant feature of systems with a line of equilibria is the occurrence of so-called bifurcations without parameters, i.e., the bifurcations corresponding to fixed parameters when the normal hyperbolicity condition is violated at some points of the line of equilibria [8–11,22,25,26]. Resultantly, the bifurcations without parameters can be observed by means numerical simulations of dynamical systems starting from varying initial conditions.
Memristor-based oscillators are widely represented in a variety of systems with a line of equilibria. One of the frequently considered models is the series RLC-circuit with negative resistance, where a flux-controlled memristor is connected in parallel with the capacitor. Bifurcation mechanisms of the periodic solution appearance in this system have been explored numerically and analytically for different kinds of nonlinearity [2,17,20,21,24,25,28]. Analytical solution of the model equations has enabled to recognize significant features of the supercritical [17,21] and subcritical [20] Andronov-Hopf bifurcation. In addition, the recently published research [20] has demonstrated how the saddle-node bifurcation of limit cycles observed in classical self-oscillators with hard self-oscillation excitation transforms in systems with a line of equilibria. Thus, all the bifurcations of limit cycles exhibited by self-oscillators with one degree of freedom have been described in terms of oscillators with a line of equilibria. The similarity between classical self-oscillators and the considered systems with a line of equilibria is emphasized by the fact that undamped periodic oscillations in systems with a line of equilibria can be synchronized by external periodic forcing [19].

In contrast to our previous publications focused on the self-oscillatory dynamics (see Refs. [17, 19-21,28]), in the current paper we show how classical bifurcations of steady states are manifested in systems with a line of equilibria. This kind of bifurcations is less studied in the context of oscillators with a line of equilibria in comparison with Andronov-Hopf-like bifurcations. In particular, only the transcritical bifurcation without parameter was studied before [26], while the other steady-state bifurcations are still unexplored phenomena which can be potentially observed in dynamical systems with a line of equilibria.

Three bifurcations are studied in the present research: the pitchfork bifurcation, the transcritical bifurcation and the saddle-node bifurcation of steady states. It is demonstrated that the bifurcations under study become bifurcations without parameters due to the memristor properties. Our purpose is to describe all the models and bifurcations in a unified manner. For this reason, all the considered systems belong to the same class of circuits, the RLC-circuit with feedback including the memristor. Thus, in this paper we propose a universal physical model for the steady-state bifurcations without parameters and describe new bifurcations on its example.

2 Model and methods

According to the idea proposed by Leon Chua [5], the memristor relates the transferred electrical charge, \( q(t) \), and the magnetic flux linkage, \( \phi(t) \), by means of the linear relationship \( dq = Wd\phi \), whence it follows that \( W = W(\phi) = \frac{dq}{d\phi} \). By this way, using the relationships \( d\phi = V_m dt \) and \( dq = I_m dt \) \( (V_m \) is the voltage across the memristor, \( I_m \) is the current passing through the memristor), the memristor current-voltage characteristic can be derived: \( I_m = W(\phi)V_m \). In such a case, \( W \) is the flux-controlled conductance (memductance) and depends on the entire past history of \( V_m(t) \):

\[
W(\phi) = \frac{dq}{d\phi} = q' \left( \int_{-\infty}^{t} V_m(t')dt' \right).
\]

It must be noted that the term ‘flux’ (or ‘flux linkage’) is used below only for denoting the memristor state variable being proportional to the integral \( \int_{-\infty}^{t} V_m(t')dt' \). The issues concerning the physical realizability of the postulated relationship \( dq = Wd\phi \) are not discussed in the current paper. Thus, the memristor is considered as a resistive component which conductance is dictated by the state variable which is not necessarily associated with magnetic phenomena. This approach reflects the conception of ‘memristive system’ [7] implying the mathematical definition, which does not concern a physical sense of the dynamical variables and their functional dependence. It allows for grouping a broad variety of elements of different nature identified by a continuous functional dependence of characteristics on previous states.

One of the simplest memristor model is Chua’s memristor described by the piecewise-linear dependence \( q(\phi) \) which takes the following form for the flux-controlled memristor:

\[
q(\phi) = \begin{cases} 
(a - b)\phi + b\phi, & \phi \geq \phi_*, \\
\alpha \phi, & |\phi| < \phi_*, \\
-(a - b)\phi + b\phi, & \phi \leq -\phi_* 
\end{cases}
\]

where \( \phi \) plays a role of the memristor state variable. Then the memristor conductance \( W(\phi) \) becomes:

\[
W(\phi) = \begin{cases} 
a, & |\phi| < \phi_*, \\
b, & |\phi| \geq \phi_* 
\end{cases}
\]

Nonlinearity (3) can be approximated by the hyperbolic tangent function:

\[
W(\phi) = \frac{b - a}{2} \tanh \left( k(\phi^2 - \phi_*) \right) + \frac{b + a}{2},
\]
where a parameter $k$ characterizes the sharpness of the transitions between two memristor’s states. It has been shown in [20] that changing the memristor conductance function to the smooth one does not qualitatively modify the memristor properties. The classical loop in the current-voltage characteristic of the memristor driven by the external periodic influence persists (see Fig. 2 in Ref. [20]). The memristors well-described by piecewise-smooth functions or by continuous ones such as tanh-function can be synthesized as semiconductor devices as well as by means of analog modeling methods. The second approach implies the development of an analog one-port circuit with memristive properties. One of the possible ways to create memristive circuits implies using transistor-based switches. For example, such circuit is illustrated and described in recent paper [27]. The memristor model including tanh-nonlinearity is not the only smooth memristor model. There is a number of smooth models describing various memristor properties [1,3,6,14,23,29,30].

Real memristive systems can 'forget' the state history over time, which results in finite correlation between current and previous states. That means the impact of the past states weakens with increase of the time distance between present and past states. For sufficiently long time distances, the previous state impact can be neglected and one can assume that a memristive system forgot that states. Such effects in memristors based on metal oxides are associated with the diffusion of charged particles [3,4,31] (however, the 'forgetting' can happen very slowly). One of the simplest form of the memristor state equation which implies the forgetting effect is the following:

$$\frac{dz}{dt} = g(x, z) = x - \delta z,$$

where $z$ plays a role of a memristor state variable, $x$ is an input signal, a parameter $\delta$ characterizes the forgetting effect strength.

Consider the circuit in Fig. 1a. It contains the operational-amplifier-based non-inverting memristive amplifier which output signal is $V_a = (1 + R_1 W(\varphi)) V_{in}$, where $\varphi$ varies according to memristor state equation (5) taken in the form $\frac{d\varphi}{dt} = g(V_{in}, \varphi)$. The second block is an analog divider producing the output voltage such that the resulting circuit response is $V_{out} = 1 + Z + R_1 W(\varphi)$, where $Z$ is a summing input voltage. The memristive circuit in Fig. 1a is included as the memristive block $M$ into the schematic circuit diagram in Fig. 1b.

The circuit in Fig. 1b represents the linear series RLC-circuit forced by the feedback signal $V_f(t)$ being an output signal of the complex nonlinear amplifier (the grey block in Fig. 1b). The amplifier has two inputs: the voltage drop across the resistor, $V_R = RI$ and the voltage across the capacitor, $V$. The amplifier input currents are assumed to be zero. After the voltage $V_R$ is reproduced by a differential amplifier, it becomes an input signal of the memristive circuit $M$ which output $V_M$ is comes to the analog multiplier and to the two-state switch. If the switch state is 1, the corresponding output signal is $V_s = V_M V$. Otherwise, the signal coming to the summing block $\sum$ is $V_s = V_M$. In addition, the amplifier contains the nonlinear block for the square or cubic transformation $mV^n$ of the input. Thus, the amplifier output signal is $V_f = V_s + V + mV^n$. The presented in Fig. 1b system is described by the following dynamical variables: $V$ is the voltage across
the capacitor $C$, $I$ is the current through the inductor $L$ and $\varphi$ is the magnetic flux linkage controlling the memristor. Using Kirchhoff’s laws, one obtains differential equations for the considered system in physical time $t'$:

\[
\begin{aligned}
&C \frac{dV}{dt'} = I, \\
&L \frac{dI}{dt'} = -RI + V_s(V, V_R, \varphi) + mV^n, \\
&\frac{d\varphi}{dt'} = RI - k\varphi,
\end{aligned}
\]

where the third equation reflects the memristor forgetting effect. In the dimensionless variables $x = V/V_0$, $Y = I/I_0$ and $z = \varphi/(L\varphi_0)$ with $V_0 = 1$ V, $I_0 = 1$ A, $\varphi_0 = 1$ sec × $V_0$ and the dimensionless time $t = [(V_0/I_0)L]t'$, Eqs. (6) can be rewritten in the following form:

\[
\begin{aligned}
&dx \over dt = \nu Y, \\
&\frac{dY}{dt} = -\gamma Y + f(x, Y, z) + \mu x^n, \\
&\frac{dz}{dt} = \gamma Y - \delta z,
\end{aligned}
\]

where $\nu = (L/C)(I_0/V_0)^2$ is a dimensionless parameter being numerically equal to $L/C$, $\gamma = R(I_0/V_0)$ is a dissipation factor, $f(x, Y, z)$ is a function reflecting the memristive properties of the block $M$ in Fig. 1d and represents a dimensionless analog of the signal $V_s$, $\mu$ is a dimensionless equivalent of the parameter $m$, $\delta = kL(I_0/V_0)$. The substitution $y = \gamma Y$ transforms model (7) into the finalized form:

\[
\begin{aligned}
&dx \over dt = \alpha y, \\
&\frac{1}{\gamma} \frac{dy}{dt} = -y + f(x, y, z) + \mu x^n, \\
&\frac{dz}{dt} = y - \delta z,
\end{aligned}
\]

where $\alpha = \nu/\gamma$. The function $f(x, y, z)$ contains the functional dependence on the memristor instantaneous state. In physical variables, the dependence is described as the expression for the memristor conductance given either by Exps. (3) or (4). In the context of model (8), the function $W_M(\varphi)$ corresponds to the equivalent dimensionless form $G_M(z)$. Two options for $G_M(z)$ are under consideration:

\[
G_M(z) = \begin{cases}
  a, & |z| < z_0, \\
  b, & |z| \geq z_0,
\end{cases}
\]

and

\[
G_M(z) = \frac{b - a}{2} \tanh \left( k(z^2 - z_0) \right) + \frac{b + a}{2}.
\]

Equations (8) represent a model written in a generalized form. It exhibits all the bifurcations without parameters discussed below. The occurrence of a concrete bifurcation depends on the properties of the term $f(x, y, z) + \mu x^n$ and of the memristor state equation. All the particular modifications of model (8) are explored both theoretically by using the linear stability analysis and numerically by means of integration methods. Numerical simulations are carried out by integration of the systems under study using the fourth-order Runge–Kutta method with the time step $\Delta t = 0.0001$ from different initial conditions.

### 3 Pitchfork bifurcation

Suppose that the switch in Fig. 1b operates in state 1 while the nonlinear transformation is cubic. Then the expression for the feedback voltage takes the form $V_t = (1 + Z + R_1 W(\varphi)) V + V + mV^3$, where the state variable varies according to the equation $d\varphi/dt = RI - k\varphi$ and $m = -1$. Adjusting the voltage $Z$ and the resistance $R_1$, one controls the coefficient $(1 + Z + R_1 W(\varphi))$ which can possess both negative and positive values. Then the physical model (see Eqs. (6)) corresponding to the presence of piecewise-smooth memristor (9) or its continuous analog (10) is transformed to the dimensionless form

\[
\begin{aligned}
&dx \over dt = \alpha y, \\
&\frac{1}{\gamma} \frac{dy}{dt} = -y + \beta(z)x - x^3, \\
&\frac{dz}{dt} = y - \delta z,
\end{aligned}
\]

for two options $\beta(z)$:

\[
\begin{cases}
  \beta_1, & |z| < 1, \\
  \beta_2, & |z| \geq 1,
\end{cases}
\]

and

\[
\beta(z) = \frac{\beta_2 - \beta_1}{2} \tanh \left( k(z^2 - 1) \right) + \frac{\beta_2 + \beta_1}{2}.
\]
All the trajectories in the subspace $z = \beta \delta = a$ represent a line of equilibria as a projection on the plane $y \in (-1; 1)$ (the green solid line in Fig. 2a). The right insets schematically show the manifolds of equilibria as a projection on the plane $(x, z)$: (c) Phase trajectories in system (11)-(13) for $\delta = 10^{-3}$ tracing motions to the stable steady state (green point) in the origin. 

Other system parameters are: $\alpha = \gamma = 1, \beta_1 = -1, \beta_2 = 1, k = 5$.

System (11) is studied for fixed parameters: $\alpha = \gamma = 1, \beta_1 = -1, \beta_2 = 1, k = 5$. First, the memristor forgetting effect is excluded from the consideration: $\delta = 0$.

System (11) with piecewise-smooth function (12) does not contain the variable $z$ in an explicit form. Then two cases $\beta = \beta_1$ and $\beta = \beta_2$ are considered independently. In case, $|z| < 1$ the system attractor is a manifold of steady states with the coordinates $x_1 = 0, y_1 = 0, z \in (-1; 1)$ (the green solid line in Fig. 2a). All the trajectories in the subspace $z \in (-1; 1)$ are attracted to the manifold of equilibria. Thus, the attractor represents a line of equilibria characterized by the eigenvalues $\lambda_1 = 0, \lambda_{2,3} = (-1 \pm i \sqrt{3})/2$ considered as stable fixed points since $\lambda_{2,3}$ have negative real part. Each point of the line of equilibria is neutrally stable in the OZ-axis direction. Hereinafter, using the terms ‘stable’ or ‘unstable’ point at the line of equilibria, we mean the behavior of trajectories (attraction or repelling) in the neighborhood of the equilibrium point, which is determined by the eigenvalues $\lambda_{2,3}$.

For two subspaces $|z| \geq 1$, there exist three lines of equilibria. The first one $(x_1 = 0, y_1 = 0, |z| \geq 1)$ represent a line of saddle-like equilibria (the green dashed line in Fig. 2a) with eigenvalues $\lambda_1 = 0, \lambda_{2,3} = (-1 \pm \sqrt{5})/2$. The second and third lines of equilibria (the red solid lines in Fig. 2a) are two coexisting attractors which consist of steady states $(x_{2,3} = \pm 1, y_{2,3} = 0, |z| \geq 1)$ with the corresponding eigenvalues $\lambda_1 = 0, \lambda_{2,3} = (-1 \pm i \sqrt{7})/2$. Thus, for $|z| \geq 1$ the central line of equilibria plays a role of an unstable fixed points between two attractors in classical bistable oscillators.

In summary, changing $z$ and keeping parameter values to be fixed, one implements the transition between the existence of a single attractor to the regime of bistability which consists in the coexistence of two stable lines of equilibria. The described bifurcation transition corresponds to the pitchfork bifurcation which occurs at $z = \pm 1$: the line of equilibria becomes unstable for $|z| \geq 1$ and two coexisting attractors appear at the moment of bifurcation. If system (11) involves tanh-function (13), the similarity with the classical pitchfork bifurcation is more evident (Fig. 2b): there exists a single line of equilibria in the central subspace, which loses stability when $z$ passes through the threshold value $z_\text{s} = \pm 1$. At the bifurcation moment, two stable lines of equilibria begin in the vicinity of the unstable line of equilibria.

For non-zero values of the parameter $\delta$, the lines of equilibria disappear (Fig. 2c). In such a case, the steady state $x = y = z = 0$ becomes a single attractor in the phase space. Then the transition to the bistability occurs in a classical way by varying parameter $\beta_1$: the pitchfork bifurcation occurs at $\beta_1 = 0$. However, for small values $\delta$ the phase trajectories describing motion to the stable fixed point can trace the lines of equilibria initially existed at $\delta = 0$ (see the trajectories in Fig. 2c).

4 Transcritical bifurcation

Suppose that the switch in Fig. 1b operates in state 1 while the nonlinear transformation is quadratic. Then the expression for the feedback voltage becomes $V_f = (1 + Z + R_l W(\varphi))V + V + mV^2$, where the state variable varies according the equation $d\varphi/dt = R_l - k\varphi$ and $m = -1$. Following the same procedures as in the previous section, one obtains the dimensionless model of the circuit:
\[
\begin{align*}
\frac{dx}{dt} &= \alpha y, \\
\frac{1}{\gamma} \frac{dy}{dt} &= -y + \beta(z)x - x^2, \\
\frac{dz}{dt} &= y - \delta z,
\end{align*}
\]  

where the function \( \beta(z) \) has two options: Exp. (12) or Exp. (13). System (14) is studied for fixed parameters: \( \alpha = \gamma = 1, \beta_1 = -1, \beta_2 = 1, k = 5 \).

First, system (14) is considered for piecewise-smooth function \( \beta(z) \) (function (12)) and \( \delta = 0 \). Two cases \( \beta = \beta_1 \) and \( \beta = \beta_2 \) are considered independently. In case \( |z| < 1 \) the system attractor is a manifold of steady states with the coordinates \( x_1 = 0, y_1 = 0, z \in (-1; 1) \) characterized by the eigenvalues \( \lambda_1 = 0, \lambda_{2,3} = (-1 \pm i \sqrt{3})/2 \) (the red solid line in Fig. 3a).

All the phase trajectories in the subspace \( z \in (-1; 1) \) are attracted to the line of equilibria \( x_1 = 0, y_1 = 0, z \in (-1; 1) \). In addition, there exists the second line of equilibria \( (x_2 = -1, y_2 = 0, z \in (-1; 1)) \) (the green dashed line in Fig. 3a) exhibiting the properties of a saddle fixed point (the corresponding eigenvalues are \( \lambda_1 = 0, \lambda_{2,3} = (-1 \pm \sqrt{5})/2 \)).

For two subspaces \( |z| \geq 1 \), there also exist two lines of equilibria. The first one \( (x_1 = 0, y_1 = 0, |z| \geq 1) \) represents a line of saddle-like equilibria (the green dashed lines in Fig. 3a) with eigenvalues \( \lambda_1 = 0, \lambda_{2,3} = (-1 \pm \sqrt{3})/2 \). The second line of equilibria (the red solid lines in Fig. 3a) is an attractor and consists of steady states \( (x_2 = 1, y_2 = 0, |z| \geq 1) \) with the corresponding eigenvalues \( \lambda_1 = 0, \lambda_{2,3} = (-1 \pm i \sqrt{3})/2 \).

In conclusion, changing \( z \) and keeping parameter values to be fixed one implements the bifurcation transition at \( z = \pm 1 \) when the central segment of the line of equilibria \( (x_1 = 0, y_1 = 0) \) changes the stability at the same moment with the coexisting lines of equilibria \( (x_2 = \beta, y_2 = 0) \). If system (14) involves tanh-function (13), the described transition is transformed into the transcritical bifurcation without parameters (Fig. 3b): two lines of equilibria \( (x_1 = 0, y_1 = 0, z \in (-\infty; \infty)) \) and \( (x_2 = \beta, y_2 = 0, z \in (-\infty; 0)) \) intersect at \( z = \pm 1 \) and exchange the stability.

For non-zero values of the parameter \( \delta \) the lines of equilibria disappear (Fig. 3c). In such a case, the steady state \( x = y = z = 0 \) becomes a single attractor in the phase space and the transcritical bifurcation can be induced only by varying the system parameter \( \beta_1 \). However, for small values \( \delta \) the phase trajectories describing motion to the stable fixed point can trace the attracting segments of the lines of equilibria initially existed at \( \delta = 0 \) (see the trajectories in Fig. 3c).

### 5 Saddle-node bifurcation

Suppose that the switch in Fig. 1b operates in state 2 while the nonlinear transformation is quadratic. Then the expression for the feedback voltage becomes \( V_t = (1 + Z + R_1 W(\varphi)) + V + m V^2 \), where the state variable varies according the equation \( d\varphi/dt = RI - k \varphi \) and \( m = -1 \). Then the dimensionless model of the circuit takes the form:

\[
\begin{align*}
\frac{dx}{dt} &= \alpha y, \\
\frac{1}{\gamma} \frac{dy}{dt} &= -y + \beta(z) - x^2, \\
\frac{dz}{dt} &= y - \delta z,
\end{align*}
\]  

![Fig. 3 Transcritical bifurcation without parameters in the phase space of system (14) for memristive nonlinearities (12) (panel a) and (13) (panel b) and the absence of the forgetting effect, \( \delta = 0 \). The attracting segments of the lines of equilibria are marked by solid red lines while the repelling ones are marked by green dashed lines. Phase trajectories are black arrowed curves. The right insets schematically show the manifolds of equilibria as a projection on the plane \((x, z)\); c Phase trajectories in system (14)–(13) for \( \delta = 10^{-3} \) tracing motions to the stable steady state (green point). Other system parameters are: \( \alpha = \gamma = 1, \beta_1 = -1, \beta_2 = 1, k = 5 \).](image)
where the function $\beta(z)$ has two options: Exp. (12) or Exp. (13). System (15) is studied for fixed parameters: $\alpha = \gamma = 1$, $\beta_1 = -1$, $\beta_2 = 1$, $k = 5$.

First, system (15) is considered for piecewise-smooth function $\beta(z)$ (function (12)) and $\delta = 0$. Then in case $|z| < 1$ the system has no equilibria while two lines of equilibria $(x_1, y_1, z_1) = (\pm \sqrt{2}, 0, 0)$, $|z| \geq 1$ exist for two subspaces $|z| \geq 1$ (Fig. 4a). The first line of equilibria $(x_1 = 1, y_1 = 0, |z| \geq 1)$ characterized by the eigenvalues $\lambda_1 = 0$, $\lambda_{2,3} = (-1 \pm i \sqrt{7})/2$ is the system attractor. The second line of equilibria $(x_1 = -1, y_1 = 0, |z| \geq 1)$ exhibits the saddle-equilibrium properties (the corresponding eigenvalues are $\lambda_1 = 0$, $\lambda_{2,3} = (-1 \pm 3)/2$).

In summary, changing $z$ and keeping parameter values to be fixed one implements the bifurcation transition at $z = \pm 1$ when two equilibria $(x = \pm 1, y = 0)$ appear: stable and unstable ones. If system (15) involves tanh-function (13), the described transition is transformed into the saddle-node bifurcation without parameters (Fig. 4b): two lines of equilibria $(x = \pm 1, y = 0, |z| \geq 1)$ appear at $z = \pm 1$. The first line of equilibria is an attractor while the second one is unstable.

For non-zero values of the parameter $\delta$ the lines of equilibria disappear. In such a case, the system has no equilibria. As demonstrated in Fig. 4c, the phase trajectory can trace the attracting lines of equilibria initially existed at $\delta = 0$, but after that the trajectories come to infinity.

**Conclusions**

The circuit model proposed in the current paper represents a universal dynamical system for the implementation of bifurcations without parameters related to basic bifurcations of steady states: the pitchfork, transcritical and saddle-node bifurcations. The demonstrated bifurcations are associated with the existence of lines of equilibria. The transcritical and saddle-node bifurcations imply the coexistence of a stable and unstable line of equilibria. The pitchfork bifurcation realizes the transition to the regime of bistability which consists in the coexistence of two lines of equilibria attracting the phase trajectories and an unstable line of equilibria between them.

The impact of the memristor forgetting effect is manifested in the contraction of limit sets along the OZ-axis. Then the resulting dynamics is determined by the limit sets crossing the plane $z = 0$ in the phase space. If any line of equilibria intersect the plane $z = 0$, it transforms into an isolated fixed point with the same stability properties. Resultantly, a continuous dependence of the oscillation characteristics on the initial condition $z_0$ disappears. A similar character of the forgetting effect action was reported in earlier publications [18–20].

The transcritical bifurcation without parameter was discussed before [26]. However, we are not aware of any models of dynamical systems exhibiting the pitchfork and saddle-node bifurcations without parameters. Apparently, the current paper is the first publication where the occurrence of the pitchfork and saddle-node bifurcations without parameters is analyzed.

The obtained results allow to summarize the materials addressing bifurcations without parameters and provide for comparative analysis with basic bifurcations of isolated limit sets. As a first step in this direction, we present table 1 which contains a brief overview on papers focused on bifurcations without parameters.
Table 1  Brief overview on literature addressing bifurcations without parameters

| Bifurcation of isolated limit sets | Reference to the analog without parameter |
|-----------------------------------|------------------------------------------|
| Supercritical Andronov-Hopf bifurcation | Refs. [2, 9–11, 15, 17, 21, 24, 28] |
| Subcritical Andronov-Hopf bifurcation | Ref. [20] |
| Saddle-node bifurcation of limit cycles | Ref. [20] |
| Pitchfork bifurcation | Current research |
| Transcritical bifurcation | Current research |
| Saddle-node bifurcation of fixed points | Current research |

Acknowledgements  V.V.S. and A.S.Z. acknowledge support by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation)—Projektnummer—163436311-SFB-910. V.V.S. also acknowledges support by the Russian Science Foundation (Project No. 22-72-00038).

Data availability  The data that support the findings of this study are available from the corresponding author upon reasonable request.

Declarations

Conflict of interest  The authors declare that they have no conflict of interest.

References

1. Ascoli, A., Senger, V., Tetzlaff, R.: Memristor model comparison. IEEE Circuits Syst. Mag. 13(2), 89–105 (2013)
2. Botta, V., Néspoli, C., Messias, M.: Mathematical analysis of a third-order memristor-based Chua’s oscillator. TEMATend. Mat. Apl. Comput. 12(2), 91–99 (2011)
3. Chang, T., Jo, S., Kim, K., Sheridan, P., Gaba, S., Lu, W.: Synaptic behaviors and modeling of a metal oxide memristive device. Appl. Phys. A 102(4), 857–863 (2011)
4. Chen, L., Li, C., Huang, T., Chen, Y., Wen, S., Qi, J.: A synapse memristor model with forgetting effect. Phys. Lett. A 377(45–48), 3260–3265 (2013)
5. Chua, L.: Memristor—the missing circuit element. IEEE Trans. Circuit Theory CT–18(5), 507–519 (1971)
6. Chua, L.: Resistance switching memories are memristors. Appl. Phys. A 102(4), 765–783 (2011)
7. Chua, L., Kang, S.: Memristive devices and systems. Proc. IEEE 64(2), 209–223 (1976)
8. Corinto, F., Forti, M., Chua, L.: Nonlinear Circuits and Systems with Memristors. Springer, New York (2020)
9. Fiedler, B., Liebscher, S.: Hopf bifurcation from lines of equilibria without parameters: ii. Systems of viscous hyperbolic balance laws. SIAM J. Math. Anal. 31(6), 1396–1404 (2000)
10. Fiedler, B., Liebscher, S., Alexander, J.: Generic Hopf bifurcation from lines of equilibria without parameters: I. Theory. J. Differ. Equ. 167(1), 16–35 (2000)
11. Fiedler, B., Liebscher, S., Alexander, J.: Generic Hopf bifurcation from lines of equilibria without parameters: iii. Binary oscillators. Int. J. Bifurc. Chaos 10(7), 1613–1621 (2000)
12. Gotthans, T., Petzela, J.: New class of chaotic systems with circular equilibrium. Nonlinear Dyn. 81(3), 1143–1149 (2015)
13. Gotthans, T., Sprott, J., Petzela, J.: Simple chaotic flow with circle and square equilibrium. Int. J. Bifurc. Chaos 26(8), 1650,137 (2016)
14. Guseinov, D., Matyushkin, I., Chernyaev, N., Mikhailov, A., Pershin, Y.: Capacitive effects can make memristors chaotic. Chaos Solitons Fractals 144, 110,699 (2021)
15. Itoh, M., Chua, L.: Memristor oscillators. Int. J. Bifurc Chaos 18(11), 3183–3206 (2008)
16. Jafari, S., Sprott, J., Pham, V.T., Volos, C., Li, C.: Simple chaotic 3d flows with surfaces of equilibria. Nonlinear Dyn. 86(2), 1349–1358 (2016)
17. Korneev, I., Semenov, V.: Andronov-Hopf bifurcation with and without parameter in a cubic memristor oscillator with a line of equilibria. Chaos 27(8), 081,104 (2017)
18. Korneev, I., Semenov, V., Slepnev, A., Vadivasova, T.: Complete synchronization of chaos in systems with nonlinear inertial coupling. Chaos Solitons Fractals 142, 110,459 (2021)
19. Korneev, I., Slepnev, A., Vadivasova, T., Semenov, V.: Forced synchronization of an oscillator with a line of equilibria. Eur. Phys. J. Spec. Top. 229(12), 2215–2224 (2020)
20. Korneev, I., Slepnev, A., Vadivasova, T., Semenov, V.: Sub-critical Andronov-Hopf scenario for systems with a line of equilibria. Chaos 31(7), 073,102 (2021)
21. Korneev, I., Vadivasova, T., Semenov, V.: Hard and soft excitation of oscillations in memristor-based oscillators with a line of equilibria. Nonlinear Dyn. 89(4), 2829–2843 (2017)
22. Liebscher, S.: Bifurcation without Parameters. Lectures Notes in Mathematics, vol. 2117. Springer, New York (2015)
23. Linn, E., Siemon, A., Waser, R., Menzel, S.: Applicability of well-established memristive models for simulations of resistive switching devices. IEEE Trans. Circuits Syst. I Regul. Pap. 61(8), 2402–2410 (2014)
24. Messias, M., Nespoli, C., Botta, V.: Hopf bifurcation from lines of equilibria without parameters in memristor oscillators. Int. J. Bifurc. Chaos 20(2), 437–450 (2010)
25. Riaza, R.: Manifolds of equilibria and bifurcations without parameters in memristive circuits. SIAM J. Appl. Math. 72(3), 877–896 (2012)
26. Riaza, R.: Transcritical bifurcation without parameters in memristive circuits. SIAM J. Appl. Math. 78(1), 395–417 (2018)

27. Semenov, V.: Synthesis of memristive one-port circuits with piecewise-smooth characteristics. ArXiv: 2203.12708 (2022)

28. Semenov, V., Korneev, I., Arinushkin, P., Strelkova, G., Vadivasova, T., Anishchenko, V.: Numerical and experimental studies of attractors in memristor-based Chua’s oscillator with a line of equilibria. Noise-induced effects. Eur. Phys. J. Spec. Top. 224(8), 1553–1561 (2015)

29. Singh, J., Raj, B.: An accurate and generic window function for nonlinear memristor models. J. Comput. Electron. 18(2), 640–647 (2019)

30. Tetzlaff, R. (ed.): Memristor and Memristive Systems. Springer-Verlag, New York (2014)

31. Zhou, E., Fang, L., Yang, B.: A general method to describe forgetting effect of memristors. Phys. Lett. A 383(11), 942–948 (2019)

Publisher’s Note  Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.