REMARKS ON LOCAL LIE ALGEBRAS
OF PAIRS OF FUNCTIONS

JOSEF JANYŠKA

Abstract. We study local Lie algebras of pairs of functions which
generate infinitesimal symmetries of almost-cosymplectic-contact
structures of odd dimensional manifolds.

Keywords: Almost-cosymplectic-contact structure; almost-coPoisson-
Jacobi structure; infinitesimal symmetry; Lie algebra.

Mathematics Subject Classification 2010: 53C15, 53B99, 17B66.

1. Introduction

It is very well known, [10], that (1-dimensional) local Lie algebras of
functions on a $(2n + 1)$-dimensional manifold $M$ are in one-to-one cor-
respondence with Jacobi structures on $M$ given by a skew symmetric
2-vector field $\Lambda$ and a vector field $E$ such that

$$[E, \Lambda] = 0, \quad [\Lambda, \Lambda] = -2 E \wedge \Lambda,$$

where $[,]$ is the Schouten-Nijenhuis bracket (see, for example, [16]) of
skew symmetric multi-vector fields.

Then the Jacobi bracket

$$[f, h] = \{f, h\} - f E.h + h E.f,$$

where $\{f, h\} = \Lambda(df, dh)$ is the Poisson bracket, defines on the sheaf of
functions $C^\infty(M)$ the structure of a local (Jacobi) Lie algebra.

Let us assume the subsheaf $C^\infty_E(M)$ of functions constant on integral
curves of the vector field $E$, i.e. such that $E.f = 0$ (conserved functions
in the terminology of [9]). Then the restriction of the above Jacobi
bracket defines the Lie subalgebra $(C^\infty_E(M); \{,\}) \subset (C^\infty(M); [,])$ of
conserved functions. Really, if $f, h \in C^\infty_E(M)$ then

$$E.\{f, h\} = \{E.f, h\} + \{f, E.h\} = 0.$$

Moreover, the Hamilton-Jacobi lift

$$X_f = df^2 - f E$$

Supported by the grant GA ČR 14–02476S.
of a function $f \in C_\infty^\infty(E)$ is an infinitesimal symmetry of $(E, \Lambda)$, i.e. $L_X f = [X, f] = 0$ and $L_X \Lambda = [X, \Lambda] = 0$. In what follows we shall use the notation $\Lambda^\sharp(\alpha) := \alpha^\sharp = i_\alpha \Lambda$ for any 1-form $\alpha$.

If the pair $(E, \Lambda)$ is regular (transitive in the terminology of [10]), i.e. $E \wedge \Lambda^n \not\equiv 0$, then there exist the unique 1-form $\omega$ such that $i_E \omega = 1$ and $i_\omega \Lambda = 0$. Moreover, $\omega$ is a contact form, i.e. $\omega \wedge d\omega^n \not\equiv 0$ and $E$ is the Reeb vector field of the contact structure $(\omega, \Omega = d\omega)$, i.e. $i_E \Omega = 0$. The pairs $(\omega, \Omega)$ and $(E, \Lambda)$ are said to be mutually dual. It is easy to see that the Hamilton-Jacobi lift of a conserved function is an infinitesimal symmetry of the contact pair $(\omega, \Omega)$, i.e. $L_X f \omega = 0$ and $L_X f \Omega = 0$.

The Hamilton-Jacobi lift of conserved functions is a Lie algebra homomorphism from the Lie algebra $(\mathcal{C}_\infty^\infty(E), \{,\})$ to the Lie algebra $(\mathcal{L}(\omega, \Omega), [\, ,\,])$ of infinitesimal symmetries of the contact structure $(\omega, \Omega)$ (respective to the Lie algebra $(\mathcal{L}(E, \Lambda), [\, ,\,])$ of infinitesimal symmetries of the Jacobi structure $(E, \Lambda)$). Here $(\mathcal{X}(M), [\, ,\,])$ is the Lie algebra of vector fields.

In [8] dual contact and Jacobi structures were generalized in the following sense. An almost-cosymplectic-contact (regular) structure (pair) is given by a pair $(\omega, \Omega)$ such that $d\Omega = 0$, $\omega \wedge \Omega^n \not\equiv 0$.

According to [12] there exists a unique dual almost-coPoisson-Jacobi structure (pair) given by the pair $(E, \Lambda)$ such that

\[
\Omega^b : T^*M \to TM \quad \text{is given by} \quad \Omega^b(X) := X^b = i_X \Omega.
\]

Then (see [8])

\[
[E, \Lambda] = -E \wedge \Lambda^\sharp(L_E \omega), \quad [\Lambda, \Lambda] = 2 E \wedge (\Lambda^\sharp \otimes \Lambda^\sharp)(d\omega).
\]

**Remark 1.1.** The almost-cosymplectic-contact pair and the dual almost-coPoisson-Jacobi pair generalize not only the contact and the dual Jacobi structures but also cosymplectic structures. Really, if $d\omega = 0$ we obtain a cosymplectic pair (see, for example, [11]). The corresponding dual pair is coPoisson pair, [8], given by the pair $(E, \Lambda)$ such that $[E, \Lambda] = 0$, $[\Lambda, \Lambda] = 0$.

**Remark 1.2.** For cosymplectic and contact structures we have distinguished forms $\omega, \Omega$ and vector fields $E, \Lambda$. But for an almost-cosymplectic-contact structure we have distinguished forms $\omega, L_E \omega, \Omega, d\omega$ and distinguished vector fields $E, (L_E \omega)^\sharp, \Lambda, (\Lambda^\sharp \otimes \Lambda^\sharp)(d\omega)$.

In what follows we assume an odd dimensional manifold $M$ with a regular almost-cosymplectic-contact structure $(\omega, \Omega)$. We assume the
dual (regular) almost-coPoisson-Jacobi structure \((E, \Lambda)\). Then we have \(\text{Ker}(\omega) = \text{Im}(\Lambda^2)\) and \(\text{Ker}(E) = \text{Im}(\Omega^p)\) and we have the splitting

\[
TM = \text{Im}(\Lambda^2) \oplus \langle E \rangle, \quad T^*M = \text{Im}(\Omega^p) \oplus \langle \omega \rangle,
\]

i.e. any vector field \(X\) and any 1-form \(\beta\) can be decomposed as

\[
(1.1) \quad X = X_{(\alpha, h)} = \alpha^\sharp + h E, \quad \beta = \beta_{(Y, f)} = Y^\flat + f \omega,
\]

where \(h, f \in C^\infty(M)\), \(\alpha\) be a 1-form and \(Y\) be a vector field. Moreover, \(h = \omega(X_{(\alpha, h)})\) and \(f = \beta_{(Y, f)}(E)\). Let us note that the splitting (1.1) is not defined uniquely, really \(X_{(\alpha_1, h_1)} = X_{(\alpha_2, h_2)}\) if and only if \(\alpha_1^\sharp = \alpha_2^\sharp\) and \(h_1 = h_2\), i.e. \(\alpha_1^\sharp - \alpha_2^\sharp = 0\) that means that \(\alpha_1 - \alpha_2 \in \langle \omega \rangle\). Similarly \(\beta_{(Y_1, f_1)} = \beta_{(Y_2, f_2)}\) if and only if \(Y_1 - Y_2 \in \langle E \rangle\) and \(f_1 = f_2\).

The projections \(p_2 : TM \to \langle E \rangle\) and \(p_1 : TM \to \text{Im}(\Lambda^2) = \text{Ker}(\omega)\) are given by \(X \mapsto \omega(X) E\) and \(X \mapsto X - \omega(X) E\). Equivalently, the projections \(q_2 : T^*M \to \langle \omega \rangle\) and \(q_1 : T^*M \to \text{Im}(\Omega^p) = \text{Ker}(E)\) are given by \(\beta \mapsto \beta(E) \omega\) and \(\beta \mapsto \beta - \beta(E) \omega\). Moreover, \(\Lambda^\sharp \circ \Omega^p = p_1\) and \(\Omega^p \circ \Lambda^\sharp = q_1\).

**Remark 1.3.** Let us consider the pair \((\omega, F)\) where \(F = \Omega + d\omega\) is a closed 2–form. Then the pair \((\omega, F)\) is not generally an almost-cosymplectic-contact pair because it may not to be regular. We shall use the form \(F\) later in Theorem 3.1. \(\square\)

In [3] we have studied infinitesimal symmetries of tensor fields \(\omega, \Omega, E, \Lambda\) generating the almost-cosymplectic-contact and the dual almost-coPoisson-Jacobi structures. Such symmetries are vector fields of the type \(X_{(\alpha, h)} = \alpha^\sharp + h E\), where \(\alpha\) and \(h\) meet certain conditions and the fact that they generate infinitesimal symmetries of a tensor field defines a Lie algebra structure on some subsheaf of \(\Omega^1(M) \times C^\infty(M)\) of generators \((\alpha, h)\) of infinitesimal symmetries.

In this paper we shall study the situation where the 1-form \(\alpha\) is locally exact, i.e. \(\alpha = df\) for \(f \in C^\infty(M)\). This leads to local Lie algebras of pairs of functions generating infinitesimal symmetries of some tensor fields. Such Lie algebras are 2-dimensional local Lie algebras in the sense of [10].

Generally, we can define local Lie algebra structure in \(C^\infty(M) \times C^\infty(M)\) by a (double) bracket \(\{\{f_1, h_1\}; (f_2, h_2)\}\) in \(C^\infty(M) \times C^\infty(M)\) satisfying the following conditions:

1. it defines a Lie algebra structure in \(C^\infty(M) \times C^\infty(M)\) over \(\mathbb{R}\),
2. \(\{\{f_1, h_1\}; (f_2, h_2)\}\) is continuous in \(f_i\) and \(h_i, i = 1, 2,\)
3. \(\text{supp}(\{\{f_1, h_1\}; (f_2, h_2)\}) \subset \text{supp}(f_1, h_1) \cap \text{supp}(f_2, h_2)\) for each \((f_1, h_1)\) and \((f_2, h_2)\), where \(\text{supp}(f_i, h_i) = \text{supp}(f_i) \cap \text{supp}(h_i)\).
As an example we can consider two Jacobi pairs \((E_i, \Lambda_i), i = 1, 2\), and the corresponding Jacobi brackets \([, ]_i\). Then the bracket
\[
\left\langle (f_1, h_1); (f_2, h_2) \right\rangle = ([f_1, f_2]; [h_1, h_2])
\]
defines the Lie algebra structure in \(C^\infty(M) \times C^\infty(M)\). As far as we know, the classification of local Lie algebras of pairs of functions is not known. In the paper we shall describe several subsheafs of \(C^\infty(M) \times C^\infty(M)\) with a Lie algebra structure given by the fact that pairs of functions generate infinitesimal symmetries of basic tensor fields \(\omega, \Omega, E, \Lambda\) given by the almost-cosymplectic-contact and the dual almost-coPoisson-Jacobi structures.

All manifolds and mappings are assumed to be smooth.

2. Local Lie algebras of generators of infinitesimal symmetries

In this section we shall study local Lie algebras of pairs of functions which generate infinitesimal symmetries of basic tensor fields.

For two functions \(f, h \in C^\infty(M)\) we define their pre-Hamiltonian lift to vector fields on \(M\) by
\[
X_{(f,h)} = df^2 + hE.
\]

Lemma 2.1. Let \((f_i, h_i) \in C^\infty(M) \times C^\infty(M), i = 1, 2\), be two pairs of functions on \(M\). Then
\[
[X_{(f_1, h_1)}, X_{(f_2, h_2)}] = (d\{f_1, f_2\} + (E.f_1)(L_{df_2} + h_2 L_E)\omega
\]
\[
- (E.f_2)(L_{df_1} + h_1 L_E)\omega - h_2 d(E.f_1) + h_1 d(E.f_2)) E
\]
\[
+ \{f_1, h_2\} - \{f_2, h_1\} - d\omega(df_1^2, df_2^2)
\]
\[
+ h_1(E.h_2 + \Lambda(L_E\omega, df_2)) - h_2(E.h_1 + \Lambda(L_E\omega, df_1)) E.
\]

Proof. We have (see [8])
\[
[E, df^2] = (d(E.f) - (E.f)(L_E\omega))^2 + \Lambda(L_E\omega, df) E,
\]
\[
[df^2, dh^2] = (d\Lambda(df, dh) + (E.f)(i_{dh^2}d\omega)
\]
\[
- (E.h)(i_{df^2}d\omega))^2 - d\omega(df^2, dh^2) E
\]
which implies (2.2). \qed

It is easy to see that the vector field (2.2) is not generally the pre-Hamiltonian lift of a pair of functions. It is so in the case when the projection \(p_1 : TM \to \text{Ker}(\omega)\) of (2.2) is the \(\Lambda^2\)-lift of the differential of a function. We shall describe several examples of subsheafs of \(C^\infty(M) \times C^\infty(M)\) such that the Lie bracket of two pre-Hamiltonian
lifts of pairs of functions from the subsheaf is the pre-Hamiltonian lift of a pair from the subsheaf. All such subsheafs are given by local generators of infinitesimal symmetries of basic tensor fields.

2.1. **Infinitesimal symmetries of** $\omega$ **and** $\Omega$ **generated by pairs of functions.**

**Theorem 2.2.** The pre-Hamiltonian lift (2.1) is an infinitesimal symmetry of $\omega$ if and only if

\[ i_{df_1}d\omega + h_i E d\omega + dh = 0. \]  

**Proof.** From $i_E\omega = 1$ and $i_{df_i}\omega = 0$ we get

\[ L_{X_{(f,h)}}\omega = i_{df_i}d\omega + h_i E d\omega + dh \]

which proves Theorem 2.2. 

Let us denote by $\text{LGen}(\omega) \subset C^\infty(M) \times C^\infty(M)$ the subsheaf of pairs of functions $(f, h)$ on $M$ which generate (locally) infinitesimal symmetries of $\omega$, i.e. satisfy the condition (2.5).

**Theorem 2.3.** The Lie bracket of the pre-Hamiltonian lifts of two pairs $(f_i, h_i) \in \text{LGen}(\omega)$, $i = 1, 2$, is the pre-Hamiltonian lift of a pair of functions from $\text{LGen}(\omega)$.

**Proof.** If (2.5) is satisfied then $dh_i = -(i_{df_i} + h_i i_E) d\omega = -(L_{df_i} + h_i L_E)\omega$ and $E. h_i + \Lambda(L_E\omega, df_i) = 0$ which follows by evaluating (2.5) on $E$. Then we can rewrite (2.2) as

\[ [X_{(f_1, h_1)}, X_{(f_2, h_2)}] = (d(\{f_1, f_2\} - h_2 (E.f_1) + h_1 (E.f_2))) + \Lambda(\{f_1, h_2\} - \{f_2, h_1\} - d\omega(df_1^z, df_2^z)) \]

which is the pre-Hamiltonian lift of the pair

\[ \{f_1, f_2\} - h_2 (E.f_1) + h_1 (E.f_2); \]

\[ \{f_1, h_2\} - \{f_2, h_1\} - d\omega(df_1^z, df_2^z) \].

The pair (2.7) is in $\text{LGen}(\omega)$ which follows from the fact that, according to Theorem 2.2, the pre-Hamiltonian lifts (2.1) of pairs from $\text{LGen}(\omega)$ are infinitesimal symmetries of $\omega$. Then from $L_{[X,Y]} = L_X L_Y - L_Y L_X$ the Lie bracket (2.6) of two pre-Hamiltonian lifts of pairs from $\text{LGen}(\omega)$ is an infinitesimal symmetry of $\omega$ and, by Theorem 2.2, the pair (2.7) has to satisfy the condition (2.5), i.e. it is in $\text{LGen}(\omega)$.

As a consequence we obtain the Lie bracket

\[ [(f_1, h_1); (f_2, h_2)] = (\{f_1, f_2\} - h_2 (E.f_1) + h_1 (E.f_2); \]

\[ \{f_1, h_2\} - \{f_2, h_1\} - d\omega(df_1^z, df_2^z) \]
which defines the local Lie algebra structure on $\text{LGen}(\omega)$. Moreover, the pre-Hamiltonian lift (2.1) is the Lie algebra homomorphism from the local Lie algebra $(\text{LGen}(\omega); [,])$ to the Lie algebra $(\mathcal{L}(\omega); [ , ] ) \subset (\mathcal{X}(\text{M}); [ , ] )$ of infinitesimal symmetries of $\omega$.

Now, we shall define a Lie algebra structure on the subsheaf $C^\infty_E(\text{M}) \times C^\infty(\text{M})$ of pairs $(f, h)$ of functions, where $f$ is conserved.

**Theorem 2.4.** The Lie bracket of the pre-Hamiltonian lifts of two pairs $(f_i, h_i) \in C^\infty_E(\text{M}) \times C^\infty(\text{M})$, $i = 1, 2$, is the pre-Hamiltonian lift of a pair of functions from $C^\infty_E(\text{M}) \times C^\infty(\text{M})$.

**Proof.** For $E \cdot f_i = 0$ the vector field (2.2) has the expression

$$[X_{(f_1, h_1)}, X_{(f_2, h_2)}] = (df_1, df_2) + \{f_1, h_2\} - \{f_2, h_1\} - d\omega(df_1^2, df_2^2) + h_1(E \cdot h_2 + \Lambda(L_E \omega, df_2)) - h_2(E \cdot h_1 + \Lambda(L_E \omega, df_1))$$

This vector field is the pre-Hamiltonian lift (2.1) of the pair

$$\{f_1, h_2\} - \{f_2, h_1\} - d\omega(df_1^2, df_2^2) + h_1(E \cdot h_2 + \Lambda(L_E \omega, df_2)) - h_2(E \cdot h_1 + \Lambda(L_E \omega, df_1))$$

Moreover, the above pair is in $C^\infty_E(\text{M}) \times C^\infty(\text{M})$. Really,

$$E \cdot \{f_1, f_2\} = \{E \cdot f_1, f_2\} + \{f_1, E \cdot f_2\} + i_{(E, \Lambda)}(df_1 \wedge df_2) = -i_{E \Lambda(L_E \omega)}(df_1 \wedge df_2) = 0$$

which proves Theorem 2.4. □

As a consequence, by observing $\Lambda(L_E \omega, df) = (L_E \omega)^{\sharp} \cdot f$, we obtained the Lie bracket

$$\llbracket (f_1, h_1); (f_2, h_2) \rrbracket = \llbracket f_1, f_2 \rrbracket;$$

$$\{f_1, h_2\} - \{f_2, h_1\} - d\omega(df_1^2, df_2^2) + h_1(E \cdot h_2 + (L_E \omega)^{\sharp} \cdot f_2) - h_2(E \cdot h_1 + (L_E \omega)^{\sharp} \cdot f_1)$$

of pairs of functions from $C^\infty_E(\text{M}) \times C^\infty(\text{M})$ which defines the local Lie algebra structure on $C^\infty_E(\text{M}) \times C^\infty(\text{M})$.

In [3] it was proved that all infinitesimal symmetries of $\Omega$ are vector fields $X_{(\alpha, h)} = \alpha^{\sharp} + h \cdot E$, where $\alpha$ is a closed 1-form such that $\alpha(E) = 0$. Hence, locally, $\alpha = df$ for a function $f \in C^\infty_E(\text{M})$ and any infinitesimal symmetry of $\Omega$ is locally the pre-Hamiltonian lift of a pair of functions from $C^\infty_E(\text{M}) \times C^\infty(\text{M})$. So the Lie algebra of local generators of infinitesimal symmetries of $\Omega$ is $(\text{LGen}(\Omega), [ , ] ) \equiv (C^\infty_E(\text{M}) \times C^\infty(\text{M}), [ , ] )$ with the Lie bracket (2.9). The pre-Hamiltonian lift (2.1) is the Lie
algebra homomorphism from the local Lie algebra \((\text{LGen}(\Omega); [\cdot, \cdot])\) to the Lie algebra \((\mathcal{L}(\Omega); [\cdot, \cdot]) \subset (\mathcal{X}(M); [\cdot, \cdot])\) of infinitesimal symmetries of \(\Omega\).

**Theorem 2.5.** A vector field \(X(f,h)\) is an infinitesimal symmetry of the almost-cosymplectic-contact structure \((\omega, \Omega)\) if and only if \(f \in C_\infty^\omega(M)\) and the condition \((2.5)\) is satisfied.

**Proof.** It follows from Theorems 2.2 and 2.4. \(\Box\)

**Lemma 2.6.** A vector field \(X(f,h)\) is an infinitesimal symmetry of \((\omega, \Omega)\) if and only if the following conditions are satisfied

1. \(E \cdot f = i_E df = 0\),
2. \(i_E dh + i_E i_{df} d\omega = E \cdot h + (L_E \omega)^2 \cdot f = 0\),
3. \(d\omega(df^2, \beta^2) + h d\omega(E, \beta^2) + dh(\beta^2) = 0\) for any 1-form \(\beta\), especially, if we put \(\beta = dg\) for a \(g \in C_\infty(M)\), we get \(\{g, h\} = d\omega(df^2, df^2) + h d\omega(dg^2, E)\).

**Proof.** It is a consequence of Theorem 2.5 and Theorem 2.2 where we have evaluated the 1-form on the left hand side of \((2.5)\) on \(E\) (which gives the condition \((2)\)) and on \(\beta^2\) for any 1-form \(\beta\) (which gives the condition \((3)\)). \(\Box\)

We denote the sheaf of pairs of functions which locally generate infinitesimal symmetries of the almost-cosymplectic-contact structure \((\omega, \Omega)\) as \(\text{LGen}(\omega, \Omega) = \text{LGen}(\omega) \cap \text{LGen}(\Omega)\). The brackets \((2.8)\) and \((2.9)\) restricted for generators of infinitesimal symmetries of \((\omega, \Omega)\) give the equivalent expressions of the bracket

\[
\left[ (f_1, h_1); (f_2, h_2) \right] =
\]

\[
= (\{f_1, f_2\}; \{f_1, h_2\} - \{f_2, h_1\} - d\omega(df^2_1, df^2_2)) \]

\[
= (\{f_1, f_2\}; d\omega(df^2_1, df^2_2) + h_2 (L_E \omega)^2 . f_1 - h_1 (L_E \omega)^2 . f_2) \]

\[
= (\{f_1, f_2\}; d\omega(df^2_1, df^2_2) + h_1 E . h_2 - h_2 E . h_1) \]

which defines the Lie algebra structure on \(\text{LGen}(\omega, \Omega)\).

**Corollary 2.7.** An infinitesimal symmetry of the cosymplectic structure \((\omega, \Omega)\) is of local type \(X(f,h) = df^2 + h E\), where \(f \in C_\infty^\omega(M)\) and \(h\) is a constant.

Then the bracket \((2.10)\) is reduced to

\[
\left[ (f_1, h_1); (f_2, h_2) \right] = (\{f_1, f_2\}, 0).
\]

I.e. we obtain the subalgebra \((\text{LGen}_{\text{cos}}(\omega, \Omega); [\cdot, \cdot]) = (C_\infty^\omega(M); \{\cdot, \cdot\}) \oplus (\mathbb{R}; [\cdot, \cdot])\) of local generators of infinitesimal symmetries of the cosymplectic structure. Here \([\cdot, \cdot]\) is the trivial Lie bracket in \(\mathbb{R}\).
Proof. It follows from \( d\omega = 0 \), then, from (2.5), \( dh = 0 \).

**Corollary 2.8.** Any infinitesimal symmetry of the contact structure \((\omega, \Omega = d\omega)\) is of local type

\[
X_{(f,-f)} = d f^\sharp - f E ,
\]

where \( f \in C^\infty_{E}(M) \).

Then the bracket (2.10) is reduced to

\[
\left[ \left[ (f_1, -f_1); (f_2, -f_2) \right] = (\{f_1, f_2\}, -\{f_1, f_2\}) \right].
\]

I.e. we obtain the subalgebra \((\mathbb{LGen}_{con}(\omega); [\ ; \]) \subset (\mathbb{LGen}(\omega, \Omega); [\ ; \])\) of local generators of infinitesimal symmetries of the contact structure. Moreover, \((\mathbb{LGen}_{con}(\omega); [\ ; \]) \equiv (C^\infty_{E}(M), \{\ ; \})\).

Proof. It follows from \( d\omega = \Omega, i_E \Omega = 0 \) and \( i_d f^\sharp \Omega = df \). Then, from (2.5), \( df = -dh \). □

2.2. Infinitesimal symmetries of \( E \) and \( \Lambda \) generated by pairs of functions.

**Theorem 2.9.** The pre-Hamiltonian lift (2.1) is an infinitesimal symmetry of the Reeb vector field \( E \) if and only if

\[
\begin{align*}
(2.11) & \quad (d(E.f) - (E.f)L_E\omega)^\sharp = 0 , \\
(2.12) & \quad (E.h) + (L_E\omega)^\sharp . f = 0 .
\end{align*}
\]

Proof. Let us assume that the pre-Hamiltonian lift of a pair \((f, h)\) is an infinitesimal symmetry of the Reeb vector field. Then from (2.3)

\[
0 = [X_{(f,h)}, E] = -(d(E.f) - (E.f)L_E\omega)^\sharp - ((E.h) + (L_E\omega)^\sharp . f) E
\]

which proves Theorem 2.9. □

If we have two pairs \((f_i, h_i)\) generating infinitesimal symmetries of \( E \) then the Lie bracket of the corresponding pre-Hamiltonian lifts is

\[
(2.13) \quad [X_{(f_1,h_1)}, X_{(f_2,h_2)}] = \left( d\{f_1, f_2\} + (E.f_1) L_{d f_2^\sharp} \omega - (E.f_2) L_{d f_1^\sharp} \omega \right)^\sharp
\]

\[
+ \left( \{f_1, h_2\} - \{f_2, h_1\} - d\omega(df_1^\sharp, df_2^\sharp) \right) . E.
\]

The above vector field (2.13) is not generally the pre-Hamiltonian lift of a pair of functions. So, the sheaf of local generators of infinitesimal symmetries of \( E \) is not a Lie algebra.

**Theorem 2.10.** The pre-Hamiltonian lift (2.1) is an infinitesimal symmetry of \( \Lambda \), i.e. \( L_{X_{(f,h)}} \Lambda = [X_{(f,h)}, \Lambda] = 0 \), if and only if the following condition is satisfied

\[
(2.14) \quad [df^\sharp, \Lambda] - E \wedge (dh + h L_E \omega)^\sharp = 0.
\]
Proof. We have
\[ L_{X(f,h)} \Lambda = [df^2, \Lambda] + [h E, \Lambda]. \]

Theorem 2.10 follows from
\[ [h E, \Lambda] = h [E, \Lambda] - E \wedge dh^\sharp = -E \wedge (dh + h L_{E \omega})^\sharp. \]

Lemma 2.11. A vector field \( X(f,h) \) is an infinitesimal symmetry of \( \Lambda \) if and only if the following conditions
\begin{align*}
(2.15) \quad & E \cdot f = 0, \\
(2.16) \quad & d\omega (df^\sharp, \beta^\sharp) + h d\omega (E, \beta^\sharp) + dh (\beta^\sharp) = 0,
\end{align*}
are satisfied for any 1-form \( \beta \).

Proof. It is sufficient to evaluate the 2-vector field on the left hand side of (2.14) on \( \omega, \beta \) and \( \beta, \gamma \), where \( \beta, \gamma \) are closed 1-forms. We get
\[ i_{[df^\sharp, \Lambda]-E \wedge (dh + h L_{E \omega})^\sharp} (\beta \wedge \gamma) = -\Lambda (i_{df^\sharp} d\omega + h L_{E \omega} + dh, \beta) \]
which vanishes if and only if (2.16) is satisfied.

On the other hand
\[ i_{[df^\sharp, \Lambda]-E \wedge (dh + h L_{E \omega})^\sharp} (\beta \wedge \gamma) = \Lambda (df, d\Lambda(\beta, \gamma)) + \Lambda (\beta, d\Lambda(\gamma, df)) + \Lambda (\gamma, d\Lambda(df, \beta)) - \beta (E) \Lambda (h L_{E \omega} + dh, \gamma) + \gamma (E) \Lambda (h L_{E \omega} + dh, \beta) \]
which, by using (2.16), can be rewritten as
\[ i_{[df^\sharp, \Lambda]-E \wedge (dh + h L_{E \omega})^\sharp} (\beta \wedge \gamma) = -\frac{1}{2} i_{[\Lambda, \Lambda]} (df \wedge \beta \wedge \gamma) + \beta (E) \Lambda (i_{df^\sharp} d\omega, \gamma) - \gamma (E) \Lambda (i_{df^\sharp} d\omega, \beta) - i_{E \wedge (\Lambda^\sharp \otimes \Lambda^\sharp)} (d\omega) (df \wedge \beta \wedge \gamma) + \beta (E) \Lambda (i_{df^\sharp} d\omega, \gamma) - \gamma (E) \Lambda (i_{df^\sharp} d\omega, \beta) - df (E) d\omega (\beta^\sharp, \gamma^\sharp) = -(E \cdot f) d\omega (\beta^\sharp, \gamma^\sharp) \]
which vanishes if and only if (2.15) is satisfied.

On the other hand if (2.15) and (2.16) are satisfied, then the 2-vector field \( L_{X(f,h)} \Lambda \) is the zero 2-vector field.

So, local generators of infinitesimal symmetries of \( \Lambda \) are from \( LGen(\Omega) \) and generates also infinitesimal symmetries of \( \Omega \), i.e. \( LGen(\Lambda) \subset LGen(\Omega) \) and the Lie algebra of local generators of infinitesimal symmetries of \( \Lambda \) is the subalgebra \( (LGen(\Lambda); [\; , \; ]) \subset (LGen(\Omega); [\; , \; ]) \) with the bracket
\[ [(f_1, h_1); (f_2, h_2)] = \]
\[ = \{(f_1, f_2); d\omega (df_1^\sharp, df_2^\sharp) + h_1 E \cdot h_2 - h_2 E \cdot h_1 \} \]
which we obtain from the bracket (2.9) restricted for functions satisfying (2.16).

**Corollary 2.12.** The Lie algebra of local generators of infinitesimal symmetries of the almost-coPoisson-Jacobi pair \((E, \Lambda)\) coincides with the local Lie algebra of generators of infinitesimal symmetries of \((\omega, \Omega)\).

**Proof.** From Theorem 2.9 and Lemma 2.11 the pre-Hamiltonian lift \((2.1)\) is an infinitesimal symmetry of the pair \((E, \Lambda)\) if and only if the conditions (1), (2) and (3) of Lemma 2.6 are satisfied, i.e. if and only if it is an infinitesimal symmetry of the pair \((\omega, \Omega)\). □

**Remark 2.1.** Let us consider subsheaves of \(C^\infty(M) \times C^\infty(M)\) given by conditions (1), (2) and (3) of Lemma 2.6. The subsheaf \(LGen(\Omega)\) is given by the condition (1), the subsheaf \(LGen(\omega)\) is given by the conditions (2) and (3) and the subsheaf \(LGen(\Lambda)\) is given by the conditions (1) and (3). If we assume the subsheaf given by conditions (1) and (2) then, from Theorem 2.9 it is the subsheaf \(LGen(E, \Omega)\) of local generators of infinitesimal symmetries of the Reeb vector field \(E\) and \(\Omega\). The corresponding bracket will be

\[ [ (f_1, h_1); (f_2, h_2) ] = (\{ f_1, f_2 \}; \{ f_1, h_2 \} - \{ f_2, h_1 \} - d\omega(df_1^\sharp, df_2^\sharp)) . \]

So we obtain the following local Lie algebras of generators of infinitesimal symmetries

| LGen(\Omega), [, ] | (1), \(E.f = 0\) |
|---------------------|------------------|
| LGen(\omega), [, ] | (2), \(E.h + (L_E\omega)^\sharp . f = 0\) |
|                     | (3), \(d\omega(df_1^\sharp, \beta^\sharp) + h d\omega(E, \beta^\sharp) + dh(\beta^\sharp) = 0\) |
| LGen(\Lambda), [, ] | (1) and (3) |
| LGen(E, \Omega), [, ] | (1) and (2) |
| LGen(\omega, \Omega), [, ] | (1), (2) and (3) |

\(\equiv (LGen(E, \Lambda), [, ]\) | (1), (2) and (3) |

**Remark 2.2.** In the Lie algebra \((LGen(\Omega); [, ]\) we have the Abelian subalgebra formed by pairs of constant functions \(K(M) = \mathbb{R} \times \mathbb{R} \subset LGen(\Omega)\). The centralizer of the Lie subalgebra \(K(M)\) in \(LGen(\Omega)\) is the Lie subalgebra \((LGen(E, \Omega); [, ]\) \subset (LGen(\Omega); [, ])\) of generators of infinitesimal symmetries of \(E\) and \(\Omega\).

Really, let \((c, k) \in K(M)\) and \((f, h) \in LGen(\Omega)\) such that

\[ (0; 0) = [ (c, k); (f, h) ] = (0; -k(E.f + (L_E\omega)^\sharp . f)) . \]
Then, from Theorem 2.9 it follows that the pair \((f, h)\) generates an infinitesimal symmetry of \(E\). □

2.3. **Multiplicative algebra** \((\text{LGen}(\Omega), \cdot)\). In Section 2.1 we have defined the local Lie algebra structure on \(\text{LGen}(\Omega)\). We define the multiplication in \(\text{LGen}(\Omega)\) by
\[
(f_1, h_1)(f_2, h_2) = (f_1 f_2, f_1 h_2 + f_2 h_1)
\]
which defines on \(\text{LGen}(\Omega)\) the structure of an associative commutative algebra with the unit \((1, 0)\). Really it is easy to see that
\[
(f_1, h_1)(f_2, h_2) = (f_2, h_2)(f_1, h_1),
\]
\[
((f_1, h_1)(f_2, h_2))(f_3, h_3) = (f_1, h_1)((f_2, h_2)(f_3, h_3)),
\]
\[
(1, 0)(f, h) = (f, h).
\]

**Lemma 2.13.** We have
\[
X_{(f_1, f_2, f_1 h_2 + f_2 h_1)} = f_1 X_{(f_2, h_2)} + f_2 X_{(f_1, h_1)}.
\]

**Proof.** We have
\[
X_{(f_1, f_2, f_1 h_2 + f_2 h_1)} = d(f_1 f_2)^2 + (f_1 h_2 + f_2 h_1) E
\]
\[
= f_1 (d f_2^2 + h_2 E) + f_2 (d f_1^2 + h_1 E)
\]
\[
= f_1 X_{(f_2, h_2)} + f_2 X_{(f_1, h_1)}. 
\]
□

From (2.9) it is easy to see that the bidifferential operator \(D\) on \(\text{LGen}(\Omega)\) given by
\[
D_{(f_1, h_1)}(f_2, h_2) = \left( (f_1, h_1); (f_2, h_2) \right)
\]
is of order 1.

From the Jacobi identity we get
\[
D_{[(f_1, h_1); (f_2, h_2)]}(f_3, h_3) = \left( D_{(f_1, h_1)}D_{(f_2, h_2)} - D_{(f_2, h_2)}D_{(f_1, h_1)} \right)(f_3, h_3)
\]
and
\[
D_{(f_1, h_1)} \left( (f_2, h_2); (f_3, h_3) \right) =
\]
\[
= \left[ D_{(f_1, h_1)}(f_2, h_2); (f_3, h_3) \right] + \left[ (f_2, h_2); D_{(f_1, h_1)}(f_3, h_3) \right],
\]
i.e., \(D_{(f,h)}\) is a derivation on \((\text{LGen}(\Omega); [, ]\)).

**Theorem 2.14.** The 1st order differential operator
\[
D_{(f,h)} : \text{LGen}(\Omega) \rightarrow \text{LGen}(\Omega)
\]
is a derivation on \((\text{LGen}(\Omega), \cdot)\), i.e. for \((f_i, h_i) \in \text{LGen}(\Omega), \ i = 1, 2, 3,\)

we have

\[
D_{(f_1, h_1)}((f_2, h_2)(f_3, h_3)) = (f_2, h_2)D_{(f_1, h_1)}(f_3, h_3) + (f_3, h_3)D_{(f_1, h_1)}(f_2, h_2).
\]

Proof. From (2.9) we can prove

\[
[(f_1, h_1); (f_2, h_2)(f_3, h_3)] = (f_2, h_2) [(f_1, h_1); (f_3, h_3)] + (f_3, h_3) [(f_1, h_1); (f_2, h_2)]
\]

which proves Theorem 2.14.

2.4. Lie derivation of pairs of functions. We define the Lie derivation of pairs of functions \((f, h) \in C^\infty(M) \times C^\infty(M)\) given by a vector field \(X\) on \(M\) by \(L_X(f, h) = (L_X f, L_X h) = (X.f, X.h)\). Generally \(L_X\) is not an operator on \(\text{LGen}(\Omega)\).

Lemma 2.15. Let \(X\) be a vector field on \(M\) such that \(L_X E = [X, E] = 0\). Then for \((f, h) \in \text{LGen}(\Omega)\) the Lie derivation \(L_X(f, h) \in \text{LGen}(\Omega)\).

Proof. We have to prove that \(L_X f \in C^\infty_E(M)\). But from

\[
0 = L_{[X,E]} f = L_X L_E f - L_E L_X f = -E.(L_X f),
\]

hence \(L_X(f, h) \in \text{LGen}(\Omega)\).

Lemma 2.16. If a vector field \(X\) is an infinitesimal symmetry of \(E\) then \(L_X\) is a derivation on \((\text{LGen}(\Omega), \cdot)\), i.e.

\[
L_X \left( (f_1, h_1)(f_2, h_2) \right) = \left( L_X f_1, L_X h_1 \right) (f_2, h_2) + (f_1, h_1)(L_X f_2, L_X h_2).
\]

Proof. By Lemma 2.15 \(L_X ((f_1, h_1)(f_2, h_2)) \in \text{LGen}(\Omega)\) and

\[
L_X ((f_1, h_1)(f_2, h_2)) = L_X (f_1 f_2, f_1 h_2 + f_2 h_1)
= ((L_X f_1) f_2 + f_1 (L_X f_2);
(L_X f_1) h_2 + f_1 (L_X h_2) + (L_X f_2) h_1 + f_2 (L_X h_1))
= (L_X f_1, L_X h_1)(f_2, h_2) + (f_1, h_1)(L_X f_2, L_X h_2).
\]

On the other hand it is easy to see that if \(X\) is an infinitesimal symmetry of \(E\), then \(L_X\) is not a derivation on the Lie algebra \((\text{LGen}(\Omega), [\cdot, \cdot])\). But we have

Theorem 2.17. Let \(X\) be an infinitesimal symmetry of the almost-coPoisson-Jacobi structure \((E, \Lambda)\), then \(X\) is a Lie derivation on the Lie algebra \((\text{LGen}(\omega, \Omega); [\cdot, \cdot])\).
Proof. $L_XE = 0$ and $L_X\Lambda = 0$ imply

$$L_X\{f_1, f_2\} = \{L_X f_1, f_2\} + \{f_1, L_X f_2\},$$

$LE L_X \omega = 0$, $LE L_X f = L_X LE f$ and $L_X d\omega = 0$.

We have to prove that $(L_X f, L_X h) \in LGen(\omega, \Omega)$ for any $(f, h) \in LGen(\omega, \Omega)$. First, from Lemma 2.15, $L_X f \in C_\infty^E(M)$. Further, we have to prove conditions (2) and (3) of Lemma 2.6 for the pair of functions $(L_X f, L_X h)$. The condition (2) can be expressed as

$$0 = dh(E) + \Lambda(LE \omega, df).$$

If we apply $L_X$ on the above identity we get, from $L_X df = dL_X f$,

$$0 = (L_X dh)(E) + dh(L_X E) + (L_X \Lambda)(LE \omega, df)$$
$$+ \Lambda(L_X LE \omega, df) + \Lambda(LE \omega, L_X df)$$
$$= (dL_X h)(E) + \Lambda(LE \omega, dL_X f)$$

which is the condition (2) for the pair $(L_X f, L_X h)$.

Further, applying $L_X$ on (3) we get

$$0 = (L_X d\omega)(df^2, \beta^2) + d\omega(L_X df^2, \beta^2) + d\omega(df^2, L_X \beta^2)$$
$$+ (L_X h) d\omega(E, \beta^2) + h d\omega(L_X E, \beta^2) + h d\omega(E, L_X \beta^2)$$
$$+ (L_X dh)(\beta^2) + dh(L_X \beta^2)$$
$$= d\omega(d(L_X f)^2, \beta^2) + (L_X h) d\omega(E, \beta^2) + (dL_X h)(\beta^2)$$

which follows from $L_X df^2 = d(L_X f)^2$ (see [4], Lemma 2.15), and the condition (3) for the pair $(L_X f, L_X h)$ is satisfied. Hence $(L_X f, L_X h) \in LGen(\omega, \Omega)$.

Finally, we assume the bracket (2.10) in the form

$$\left[ (f_1, h_1); (f_2, h_2) \right] = \{\{f_1, f_2\}; d\omega(df_1^2, df_2^2) + h_1 E.h_2 - h_2 E.h_1\}. $$

Then

$$L_X \left[ (f_1, h_1); (f_2, h_2) \right] = \{\{L_X f_1, f_2\} + \{f_1, L_X f_2\};$$
$$d\omega(L_X df_1^2, df_2^2) + d\omega(df_1^2, L_X df_2^2)$$
$$+ L_X h_1 LE h_2 + h_1 L_X LE h_2 - L_X h_2 LE h_1 - h_2 L_X LE h_1\}. $$
On the other hand
\[
\left[ (L_X f_1, L_X h_1); (f_2, h_2) \right] + \left[ (f_1, h_1); (L_X f_2, L_X h_2) \right] = \\
\left( \{L_X f_1, f_2\}; d\omega(d(L_X f_1)^\sharp, df_2^\sharp) \\
+ L_X h_1 L_E h_2 - h_2 L_E L_X h_1 \right) \\
+ \left( \{f_1, L_X f_2\}; d\omega(df_1^\sharp, d(L_X f_2)^\sharp) \\
+ h_1 L_E L_X h_2 - L_X h_2 L_E h_1 \right)
\]
and from \( L_X df^\sharp = d(L_X f)^\sharp \) we get
\[
L_X \left[ (f_1, h_1); (f_2, h_2) \right] = \\
\left[ (L_X f_1, L_X h_1); (f_2, h_2) \right] + \left[ (f_1, h_1); (L_X f_2, L_X h_2) \right],
\]
i.e. \( L_X \) is a derivation on the Lie algebra \( \mathfrak{LGen}(\omega, \Omega); [\, , \, ] \).

**Remark 2.3.** We have
\[
(2.17) \quad \left[ (f_1, h_1); (f_2, h_2) \right] = \frac{1}{2}(L_X(\{f_1, h_1\})(f_2, h_2) - L_X(\{f_2, h_2\})(f_1, h_1)).
\]
Really
\[
L_X(\{f_1, h_1\})(f_2, h_2) - L_X(\{f_2, h_2\})(f_1, h_1) = \\
= (2 \{f_1, f_2\}; \{f_1, h_2\} - \{f_2, h_1\} + h_1 E.h_2 - h_2 E.h_1)
\]
and from (2) and (3) of Lemma 2.6 we have
\[
\{f_1, h_2\} - \{f_2, h_1\} = 2 d\omega(df_1^\sharp, df_2^\sharp) + h_1 E.h_2 - h_2 E.h_1
\]
which implies (2.17).

### 3. Lie algebroid and infinitesimal symmetries

Let us assume a closed 2–form \( F \) on \( M \) and the vector bundle \( E = T M \oplus_M \mathbb{R} \to M \). Then sections of \( E \) are pairs \((X, f)\) of vector fields on \( M \) and functions on \( M \). We define a bracket of sections of \( E \) by
\[
(3.1) \quad \left[ (X_1, f_1); (X_2, f_2) \right]_F = \left( [X_1, X_2]; X_1.f_2 - X_2.f_1 + F(X_1, X_2) \right).
\]
This bracket defines an \( F \)-Lie algebroid structure (see, for instance, [13]) on \( E \) with the anchor given by the projection on the first component.

Really, the bracket (3.1) is antisymmetric and from the closure of \( F \) the Jacobi identity is satisfied. Moreover, for any \( h \in C^\infty(M) \) we get
\[
\left[ (X_1, f_1); h (X_2, f_2) \right]_F = \left[ (X_1, f_1); (h X_2, h f_2) \right]_F \\
= h \left[ (X_1, f_1); (X_2, f_2) \right]_F + (X_1, h) (X_2, f_2)
\]
and the Leibniz-type formula is satisfied.
Now, let us assume the sheaf mapping from the local Lie algebra \( \mathsf{LGen}(\omega, \Omega) ; [ , ] \) to sections of the \( F \)-Lie algebroid given by pairs of vector fields on \( M \) and functions on \( M \) given by

\[
(3.2) \quad s : (f, h) \mapsto (X_{(f, h)}, f - h).
\]

**Theorem 3.1.** For the closed 2-form

\[
F = \Omega + d\omega
\]

the sheaf mapping

\[
s : \mathsf{LGen}(\omega, \Omega) \to \mathcal{X}(M) \times C^\infty(M)
\]

given by \( \text{(3.2)} \) is a Lie algebra morphism.

**Proof.** For \((f_i, h_i) \in \mathsf{LGen}(\omega, \Omega), i = 1, 2, \) we have

\[
[(f_1, h_1); (f_2, h_2)] = \left( \{f_1, f_2\}; \{f_1, h_2\} - \{f_2, h_1\} - d\omega(df_1^\ast, df_2^\ast) \right)
\]

\[
\mapsto \left( d\{f_1, f_2\}^\ast + \{f_1, h_2\} - \{f_2, h_1\} - d\omega(df_1^\ast, df_2^\ast) \right) E;
\]

\[
\{f_1, f_2\} - \{f_1, h_2\} + \{f_2, h_1\} + d\omega(df_1^\ast, df_2^\ast).
\]

On the other hand we have

\[
[(s(f_1, h_1); s(f_2, h_2)]_F =
\]

\[
= \left( d\{f_1, f_2\}^\ast + \{f_1, h_2\} - \{f_2, h_1\} - d\omega(df_1^\ast, df_2^\ast) \right) E;
\]

\[
2 \{f_1, f_2\} - \{f_1, h_2\} + \{f_2, h_1\}
\]

\[
- h_1 E. h_2 + h_2 E. h_1 + F(df_1^\ast + h_1 E, df_2^\ast + h_2 E).
\]

The first parts of the above pairs are equal. The second parts are equal if and only if

\[
0 = \{f_1, f_2\} - d\omega(df_1^\ast, df_2^\ast) - h_1 E. h_2 + h_2 E. h_1 + F(df_1^\ast + h_1 E, df_2^\ast + h_2 E)
\]

which can be rewritten, by using \( \Omega(df_1^\ast, df_2^\ast) = -\Lambda(df_1, df_2) = -\{f_1, f_2\} \) (see \( \text{[8]} \)), as

\[
0 = -h_1 (E. h_2 - F(E, df_2^\ast)) + h_2 (E. h_1 - F(E, df_1^\ast))
\]

\[
+ (F - d\omega - \Omega)(df_1^\ast, df_2^\ast).
\]

From \( \Omega(E, df_i^\ast) = 0, E. h_i - d\omega(E, df_i^\ast) = 0 \) (see Lemma 2.6) it is equivalent with

\[
0 = (F - d\omega - \Omega)(X_{(f_1, h_1)}, X_{(f_2, h_2)})
\]

which is satisfied for \( F = \Omega + d\omega. \) \( \square \)
Lemma 3.2. Let us consider the $F$-Lie algebroid given by the closed 2-form $F = \Omega + d\omega$. Let $(X_i, \tilde{f}_i) \in \mathcal{X}(M) \times \mathcal{C}^\infty(M)$, $i = 1, 2$, be pairs such that $X_i$ are infinitesimal symmetries of $(\omega, \Omega)$ and $E.\tilde{f}_i = -E.(\omega(X_i))$. Then the bracket $[,]_F$ of these pairs satisfy the same conditions.

Proof. $[X_1, X_2]$ is an infinitesimal symmetry of $(\omega, \Omega)$ so it is sufficient to prove that
\[
E.(X_1.\tilde{f}_2 - X_2.\tilde{f}_1 + F(X_1, X_2)) = -E.([X_1, X_2])
\]
which can be rewritten as
\[
X_1.(E.\tilde{f}_2) - X_2.(E.\tilde{f}_1) + E.(F(X_1, X_2)) =
\]
\[
= -E.(X_1.(\omega(X_2)) - X_2.(\omega(X_1) - d\omega(X_1, X_2))).
\]
From the condition $E.\tilde{f}_i = -E.(\omega(X_i))$ this equation is satisfied if and only if
\[
E.(F(X_1, X_2)) = E.(d\omega(X_1, X_2))
\]
which is satisfied for $F = \Omega + d\omega$ because of $L_E\Omega = 0$ and $L_EX_i = [E, X_i] = 0$. \hfill \Box

Theorem 3.3. Let us consider the $F$-Lie algebroid given by the closed 2-form $F = \Omega + d\omega$. Let $(X_i, \tilde{f}_i) \in \mathcal{X}(M) \times \mathcal{C}^\infty(M)$, $i = 1, 2$, be pairs such that $X_i$ are infinitesimal symmetries of $(\omega, \Omega)$ and $E.\tilde{f}_i = -E.(\omega(X_i))$. Then the sheaf mapping
\[
r : \mathcal{X}(M) \times C^\infty(M) \to C^\infty(M) \times C^\infty(M)
\]
given by
\[
(3.3) \quad r : (X, \tilde{f}) \mapsto (\omega(X) + \tilde{f}, \omega(X))
\]
has values in $\text{LGen}(\omega, \Omega)$ and it is a Lie algebra morphism inverse to $s$.

Proof. First we have to prove that the mapping (3.3) has values in $\text{LGen}(\omega, \Omega)$. Let us assume that $X$ is an infinitesimal symmetry of $(\omega, \Omega)$. By Theorem 2.5 \[X = df^\sharp + h E,\] where $E.f = 0$ and the condition (2.5) is satisfied. It is easy to see that $h = \omega(X)$. Then
\[
f = \omega(X) + \tilde{f}
\]
is a conserved function. So we have
\[
(3.4) \quad X = d(\omega(X) + \tilde{f})^\sharp + \omega(X) E
\]
and the pair $(\omega(X) + \tilde{f}, \omega(X))$ is in $\text{LGen}(\omega, \Omega)$. 


Now, we have

\[
\begin{align*}
\left[ (X_1, \tilde{f}_1); (X_2, \tilde{f}_2) \right]_F \mapsto & \mapsto \\
& (\omega([X_1, X_2]) + X_1 \cdot \tilde{f}_2 - X_2 \cdot \tilde{f}_1 + \Omega(X_1, X_2) + d\omega(X_1, X_2); \\
& \omega([X_1, X_2])) .
\end{align*}
\]

On the other hand

\[
\begin{align*}
\left[ r(X_1, \tilde{f}_1); r(X_2, \tilde{f}_2) \right] = & \left\{ \omega(X_1) + \tilde{f}_1, \omega(X_2) + \tilde{f}_2 \right\} ; \\
& \{ \omega(X_1) + \tilde{f}_1, \omega(X_2) \} - \{ \omega(X_2) + \tilde{f}_2, \omega(X_1) \} \\
& - d\omega(d(\omega(X_1) + \tilde{f}_1)^\sharp, \omega(X_2) + \tilde{f}_2)^\sharp) .
\end{align*}
\]

These expressions are equal if and only if the following two equations are satisfied

\begin{align*}
(3.5) & \quad \omega([X_1, X_2]) + X_1 \cdot \tilde{f}_2 - X_2 \cdot \tilde{f}_1 + \Omega(X_1, X_2) + d\omega(X_1, X_2) = \\
& \left\{ \omega(X_1) + \tilde{f}_1, \omega(X_2) + \tilde{f}_2 \right\} ,
\end{align*}

\begin{align*}
(3.6) & \quad \omega([X_1, X_2]) = \left\{ \omega(X_1) + \tilde{f}_1, \omega(X_2) \right\} - \left\{ \omega(X_2) + \tilde{f}_2, \omega(X_1) \right\} \\
& - d\omega(d(\omega(X_1) + \tilde{f}_1)^\sharp, \omega(X_2) + \tilde{f}_2)^\sharp) .
\end{align*}

By using \([3.4]\) and \((\Lambda^\sharp \otimes \Lambda^\sharp)(\Omega) = -\Lambda\) we can rewrite the left hand side of \([3.5]\) as

\[
X_1(\omega(X_2) + \tilde{f}_2) - X_2(\omega(X_1) + \tilde{f}_1) + \Omega(X_1, X_2) = \\
= i d(\omega(X_1) + \tilde{f}_1)^\sharp + \omega(X_1) E d(\omega(X_2) + \tilde{f}_2) - i d(\omega(X_2) + \tilde{f}_2)^\sharp + \omega(X_2) E d(\omega(X_1) + \tilde{f}_1) \\
+ \Omega(d(\omega(X_1) + \tilde{f}_1)^\sharp, \omega(X_2) + \tilde{f}_2)^\sharp) = \\
= \left\{ \omega(X_1) + \tilde{f}_1, \omega(X_2) + \tilde{f}_2 \right\} - \left\{ \omega(X_2) + \tilde{f}_2, \omega(X_1) + \tilde{f}_1 \right\} \\
- \left\{ \omega(X_1) + \tilde{f}_1, \omega(X_2) + \tilde{f}_2 \right\} = \\
= \left\{ \omega(X_1) + \tilde{f}_1, \omega(X_2) + \tilde{f}_2 \right\} .
\]
Similarly, by using $i_E d(\omega(X)) = d\omega(E, X)$, we can rewrite the left hand side of (3.6) as

$$X_1.(\omega(X_2)) - X_2.(\omega(X_1)) - d\omega(X_1, X_2) =$$

$$= i_{d(\omega(X_1) + \tilde{f}_1) + \omega(X_1) E} d(\omega(X_2)) - i_{d(\omega(X_2) + \tilde{f}_2) + \omega(X_2) E} d(\omega(X_1))$$

$$- d\omega(X_1, X_2) =$$

$$= \{\omega(X_1) + \tilde{f}_1, \omega(X_2)\} - \{\omega(X_2) + \tilde{f}_2, \omega(X_1)\}$$

$$- d\omega(X_1 - \omega(X_1) E, X_2 - \omega(X_2) E)$$

$$= \{\omega(X_1) + \tilde{f}_1, \omega(X_2)\} - \{\omega(X_2) + \tilde{f}_2, \omega(X_1)\}$$

$$- d\omega(d(\omega(X_1) + \tilde{f}_1)^\sharp, d(\omega(X_2) + \tilde{f}_2)^\sharp)$$

which proves that (3.3) is a Lie algebra morphism.

Finally, it is easy to see that $(r \circ s)(f, h) = (f, h)$ for all $(f, h) \in \text{LGen}(\omega, \Omega)$.

4. Examples

In this section we recall results obtained for structures of the classical phase space. These results were the motivation of the paper.

We assume classical space-time to be an oriented and time oriented 4–dimensional manifold $E$ equipped with a Lorentzian metric $g$, with signature $(1, 3)$ [7]. We denote by $(x^\lambda) = (x^0, x^i)$, $\lambda = 0, 1, 2, 3$, local coordinates on $E$ such that $\partial_0$ is time-like and $\partial_i$ are space-like. A motion is defined to be a 1–dimensional time-like submanifold of space-time. We define the classical (Einsteinian) phase space to be the open subspace $J_1 E \subset J_1(E, 1)$ consisting of all 1–jets (1st order contact elements) of motions. So elements of $J_1 E$ are classes of non-parametrized curves which have in a point $x \in E$ the same tangent line lying inside the light cone. $\pi_1^1 : J_1 E \to E$ is a fibred manifold but not an affine bundle! We have the induced coordinate chart $(x^\lambda, x^0_0)$.

The metric $g$ gives naturally the unscaled horizontal time form

$$\tilde{\tau} : J_1 E \to T^* E, \quad \tilde{\tau} = \tilde{\tau}_\lambda dx^\lambda.$$

4.1. Infinitesimal symmetries of the gravitational contact structure. The pair $(-\tilde{\tau}, \Omega^\theta)$, where

$$\Omega^\theta = -d\tilde{\tau} : J_1 E \to \bigwedge^2 T^* J_1 E,$$

is the contact (gravitational) regular structure on $J_1 E$. The dual Jacobi structure is given by a pair $(-\tilde{\gamma}^\theta, \Lambda^\theta)$, where $\tilde{\gamma}^\theta$ and $\Lambda^\theta$ are naturally given by the metric field (for details see [7]).
By Corollary 2.8 infinitesimal symmetries of the gravitational contact phase structure are Hamilton-Jacobi lifts of conserved functions, i.e.

they are of the type $X = d\tilde{f}^\sharp + \tilde{f}^\gamma g$, where $\gamma g. f = 0$ and, moreover, $f = \tilde{\tau}(X) = \tilde{\tau}(\hat{X})$. Here $\hat{X} = T\pi_0^1(X) : J_1E \to TE$ is a generalized vector field in the terminology of [14]. So infinitesimal symmetries are of the type

\begin{equation}
X = d(\tilde{\tau}(\hat{X}))^\sharp + \tilde{\tau}(\hat{X})^\gamma g,
\end{equation}

where the following conditions are satisfied

1. (Projectability) The Hamilton-Jacobi lift (4.1) projects on $X$.
2. (Conservation) $\gamma g. (\tilde{\tau}(\hat{X})) = 0$.

The following results were proved in [3].

**Lemma 4.1.** A symmetric $k$-vector field $\hat{K}$, $k \geq 1$, on $E$ admits the generalized vector field satisfying the projectability condition. Such generalized vector fields are given by

$X[\hat{K}] = k\underbrace{\tilde{\tau} \ldots \tilde{\tau}}_{(k-1)\text{-times}} \hat{K} - (k-1)\hat{K}(\tilde{\tau}, \ldots, \tilde{\tau}) \hat{\kappa} : J_1E \to TE$,

where $\hat{\kappa} = T\pi_0^1(\gamma g)$. Then we obtain the induced phase function

$\tilde{\tau}(X[\hat{K}]) = k\hat{K} = \hat{K}(\tilde{\tau}, \ldots, \tilde{\tau}) = k^{\lambda_1 \ldots \lambda_k} \tilde{\tau}_{\lambda_1} \ldots \tilde{\tau}_{\lambda_k}$.

\[\square\]

**Lemma 4.2.** Let $\hat{K}$ be a space-time function. Then $\gamma g. \hat{K} = 0$ if and only if $\hat{K}$ is a constant. The phase function $\hat{K}(\tilde{\tau})$, $k \geq 1$, is conserved with respect to the gravitational Reeb vector field, i.e. $\tilde{\tau}.(\hat{K}(\tilde{\tau})) = 0$, if and only if $\hat{K}$ is a Killing $k$-vector field.

\[\square\]

**Theorem 4.3.** The Hamilton-Jacobi lift of a phase function

\begin{equation}
K = \hat{K} + \sum_{k \geq 1} \hat{K}(\tilde{\tau})
\end{equation}

is an infinitesimal symmetry of the gravitational contact phase structure $(\tilde{\tau}, \Omega^g)$ if and only if $\hat{K}$ is a constant and $\hat{K}$, $k \geq 1$, are Killing $k$-vector fields. Moreover, for $k = 1$ the corresponding infinitesimal symmetry coincides with the jet flow lift $J_1\hat{K}$ and is projectable on space-time. For $k \geq 2$ the corresponding infinitesimal symmetry is hidden.
Remark 4.1. It is very well known that Killing multivector fields generates on $\mathcal{T}^*E$ functions constant of motion (functions constant on lifts of geodesic curves) (see, for instance, [15]). In [4] it was proved that if we consider the mapping $-\hat{\tau} : J_1 E \to \mathcal{T}^*E$, then a conserved phase function of type (4.2) is obtained as the pull-back $K = -\hat{\tau}^*(\tilde{K})$ of a function $\tilde{K}$ constant of motion. □

4.2. Infinitesimal symmetries of the total almost-cosymplectic-contact phase structure. Let us assume an electromagnetic (Maxwell) field which is a closed 2–form $\hat{F} : E \to \wedge^2 \mathcal{T}^*E$. Then we can consider the total phase 2–form

$$\Omega^j = \Omega^\theta + \Omega^\xi = -d\hat{\tau} + \frac{1}{2} \hat{F}$$

and the pair $(-\hat{\tau}, \Omega^j)$ turns out to be an almost-cosymplectic-contact structure of the phase space, i.e. $\Omega^j$ is closed and $\hat{\tau} \wedge \Omega^j \wedge \Omega^j \wedge \Omega^j$ is a volume form.

The dual almost-coPoisson-Jacobi pair is then given by the total Reeb vector field $\hat{\gamma}^j = \hat{\gamma}^g + \hat{\gamma}^e$ and the total phase 2-vector $\Lambda^j = \Lambda^g + \Lambda^e$, where $\hat{\gamma}^e$ and $\Lambda^e$ are $E$-vertical given by $g$ and $\hat{F}$, see [7].

In [4] it was proved that all phase infinitesimal symmetries of the total phase structure are vector fields of the type

$$X = d(\hat{\tau}(X) + \hat{f})^j + \hat{\tau}(X) \hat{\gamma}^j$$

where $X$ is a generalized vector field and $\hat{f} \in C^\infty(E)$ such that:

1) $df = X \cdot \hat{F}$;
2) (Projectability) The vector field $X$ projects on $\hat{X}$;
3) (Conservation) $\hat{\gamma}^j.(\hat{\tau}(X) + \hat{f}) = 0$.

The projectability condition is the same as in the case of the contact gravitational structure which follows from the fact that the fields $\hat{\gamma}^e$ and $\Lambda^e$ are $E$-vertical. So it is sufficient to describe conditions that the function (4.2), where $\hat{f} = K$, is conserved.

Theorem 4.4. [4] A phase function (4.2) is conserved, i.e. $\hat{\gamma}^j.K = 0$, if and only if

$$g^\rho_\lambda \partial_\rho \hat{K} + \hat{K}_\rho \hat{F}^\rho_\lambda = 0,$$

$$\nabla^{(\lambda_1 \ldots \lambda_{k+1})} K_{\lambda_1 \ldots \lambda_{k+1}} + (k + 1) \hat{K}^\rho_{(\lambda_1 \ldots \lambda_k} \hat{F}^{\lambda_{k+1})}_\rho = 0,$$

for $k \geq 1$. □
Corollary 4.5. Let us assume a special phase function $K = K_0 + K_1(\hat{\tau})$. Then the conditions (4.3) and (4.4) are reduced to
\[
\partial_\rho K_0 - K_1^\sigma \hat{F}_{\sigma \rho} = 0, \quad \nabla^{(\lambda_1} K_1^{\lambda_2)} = 0
\]
and we obtain the result of \([9]\), i.e. $K$ is a Killing vector field and $K_0$ and $K_1$ are related by the formula $dK = K_1 \lrcorner \hat{F}$. Moreover, the corresponding infinitesimal symmetry is the jet flow lift $J_1 K$ which projects on $K$. □

Remark 4.2. Let us assume a phase function $K = K(\hat{\tau})$, $k \geq 2$. Then the condition (4.4) gives
\[
\nabla^{(\lambda_1} K_1^{\lambda_2} \cdots \lambda_{k+1)} = 0, \quad K_1^\rho(\lambda_1 \cdots \lambda_{k-1} \hat{F}^{\lambda_k})_\rho = 0
\]
and we obtain that $K$ is a Killing–Maxwell $k$-vector field. But the corresponding lift has to satisfy also the condition 1) which is of the form $X[K] \lrcorner \hat{F} = 0$. This condition implies $\hat{F} \lrcorner \hat{F} = 0$, which implies $\hat{F} \equiv 0$ and the structure is reduced to the gravitational case. This implies that there are no non-projectable (hidden) infinitesimal symmetries generated by Killing–Maxwell $k$-vector fields for $k \geq 2$.

So all infinitesimal symmetries of $(-\hat{\tau}, \Omega)$ are projectable and can be generated by pairs $(X, \hat{f})$ of Killing vector fields and spacetime functions such that $d\hat{f} = X \lrcorner \hat{F}$. Such pairs are sections of the Lie algebroid $TE \oplus \mathbb{R} \to E$ with the bracket $[;]_\hat{F}$ (see Section 3 and \([9]\)). The sections of the Lie algebroid described in Section 3 are obtained as the 1-jet flow lifts of $X$ and the pull-backs of $\hat{f}$. □

References

[1] M. de Leon, G.M. Tuynman: A universal model for cosymplectic manifolds, Journal of Geometry and Physics 20 (1996), 77–86, DOI: 10.1016/0393-0440(96)00047-2.

[2] J. Janyška: Special phase functions and phase infinitesimal symmetries in classical general relativity, AIP Conf. Proc. 1460, XX Internat. Fall Workshop on Geometry and Physics, 135–140, DOI: 10.1063/1.4733369.

[3] J. Janyška: Hidden symmetries of the gravitational contact structure of the classical phase space of general relativistic test particle, Arch. Math. (Brno) 50(5) (2014), 297–316, DOI: 10.5817/AM2014-5-297.

[4] J. Janyška: Relations between constants of motion and conserved functions, Arch. Math.(Brno) 51 (2015), 297–313, DOI: 10.5817/AM2015-5-297.

[5] J. Janyška: On Lie algebras of generators of infinitesimal symmetries of almost-cosymplectic-contact structures, to appear in Arch. Math. (Brno), arXiv: 1610.06714.
[6] J. Janyška, M. Modugno: *Graded Lie algebra of Hermitian tangent valued forms*, J. Math. Pures Appl. 85 (2006), 687–697, DOI: 10.1016/j.matpur.2005.11.004.

[7] J. Janyška, M. Modugno: *Geometric Structures of the Classical General Relativistic Phase Space*, Int. J. Geom. Methods Mod. Phys. 5 (2008), 699–754, DOI: 10.1142/S021988780800303X.

[8] J. Janyška, M. Modugno: *Generalized geometrical structures of odd dimensional manifolds*, J. Math. Pures Appl. (9) 91 (2009), 211–232, DOI: 10.1016/j.matpur.2008.09.007.

[9] J. Janyška, R. Vitolo: *On the characterization of infinitesimal symmetries of the relativistic phase space*, J. Phys. A: Math. Theor. 45 (2012), 485205, DOI: 10.1088/1751-8113/45/48/485205.

[10] A. A. Kirillov: *Local Lie algebras*, Russian Math. Surveys 31 (1976), 55–76.

[11] I. Kolář, P. W. Michor, J. Slovák: *Natural Operations in Differential Geometry*, Springer 1993.

[12] A. Lichnerowicz: *Les variétés de Jacobi et leurs algèbres de Lie associées*, J. Math. Pures et appl. 57 (1978), 453–488.

[13] K. Mackenzie: *General Theory of Lie Groupoids and Lie Algebroids*, London Mathematical Society, Lecture Note Series 213, Cambridge University Press 2005.

[14] P. Olver: *Applications of Lie groups to differential equations*, Graduate Texts in Mathematics 107, Springer 1986.

[15] P. Sommers: *On Killing tensors and constant of motions*, J. Math. Phys. 14 (1973), 787-790.

[16] I. Vaisman: *Lectures on the Geometry of Poisson Manifolds*, Birkhäuser Verlag, Basel-Boston-Berlin 1994.

Department of Mathematics and Statistics, Masaryk University
Kotlářská 2, 611 37 Brno, Czech Republic
E-mail: janyska@math.muni.cz