RIBET’S CONJECTURE FOR EISENSTEIN MAXIMAL IDEALS

DEBARGHA BANERJEE, NARASIMHA KUMAR, AND DIPRAMIT MAJUMDAR

ABSTRACT. According to Ogg’s conjecture (Mazur’s Theorem), cuspidal subgroup coincides with rational torsion points of the Jacobian variety of modular curves of the form $X_0(N)$ for a prime number $N$. There is a recent interest to generalize the conjecture for arbitrary $N$ by Ribet, Ohta and Yoo. In this direction, Ribet conjectured [28, p. 360] that all the Eisenstein maximal ideals are “cuspidal”. Hwajong Yoo proved the conjecture (under certain hypothesis) provided that those ideals are rational. In this article, we show that (under certain hypothesis), Ribet’s conjecture is true for non-rational Eisenstein maximal ideals.

CONTENTS
1. Introduction 1
2. Eisenstein series for $\Gamma_0(N)$ 2
3. Divisor associated to Eisenstein Series $E_{\psi,M,L}$ 3
4. Cuspidal subgroup associated to non-rational Eisenstein series 4
5. Classification of non-rational Eisenstein maximal ideals 10
6. Ribet’s conjecture for non-rational Eisenstein series 21
7. Numerical examples 25
8. Appendix: Computation of boundary of Eisenstein series 29

1. INTRODUCTION

For any $N \in \mathbb{N}$, the congruence subgroup $\Gamma_0(N)$ acts on the complex upper half plane $\mathbb{H}$ and consider the Riemann surface $Y_0(N) := \Gamma_0(N) \backslash \mathbb{H}$. Let $X_0(N)$ be the compactified modular curve obtained by adding a finite set of cusps $\partial(X_0(N))$. Let $X_0(N) / \mathbb{Q}$ be the Shimura’s canonical model of the Riemann surface $X_0(N)$ over $\mathbb{Q}$. Let $J_0(N) := \text{Jac}(X_0(N)) / \mathbb{Q}$ be the Jacobian abelian variety associated to the modular curve of the form $X_0(N)$ [8, Definition 6.1.1] and $C_{\Gamma_0(N)}$ be the cuspidal subgroup inside $J_0(N)$ obtained by the equivalence classes of divisors of degree zero supported at $\partial(X_0(N))$.

According to a conjecture of Ogg [13, Conjecture 2] rational torsion subgroups of the Jacobian variety $\text{Jac}(X_0(N))$ co-incide with the cuspidal subgroups if $N$ is a prime. The Ogg’s conjecture has been proved in a landmark paper by Barry Mazur [12]. There is a recent interest to generalize the work to arbitrary $N \in \mathbb{N}$ by series of papers by Masami Ohta [14], [15], Hwajong Yoo [28] for square free $N \in \mathbb{N}$. Yuan Ren [16] computed the same by reducing it to integers of the form $N = DC \in \mathbb{N}$ such that $D$ is square free and $C \mid D$ but for $\mathbb{Q}$-rational points of Jacobian variety of modular curves.

Consider the Hecke algebra $T$ acting on $X_0(N)$ and hence on $J_0(N)$ [8, §6.3, p. 230] consisting of all Hecke operators $T_l$ for all $l$ (including $l \mid N$). Let $m$ be a maximal ideal of $\mathbb{T}$ with residue characteristic

2010 Mathematics Subject Classification. Primary: 11F67, Secondary: 11F11, 11F30, 11F70, 11F80.

Key words and phrases. Eisenstein series, Modular symbols, Special values of $L$-functions.
l and denote by $\rho_m : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}_2(\mathbb{T}/m)$ the two dimensional semi-simple Galois representation associated to $m$. Upto isomorphism of Galois representations, these are completely determined by the following two properties \cite[p. 2430]{34}:

1. The Galois representation $\rho_m$ is unramified outside $lN$.
2. For all $p \nmid lN$, the characteristic polynomial of $\rho_m(\text{Frob}_p)$ is

$$X^2 - (T_p \mod m)X + p = 0.$$ 

We call $m$ to be an Eisenstein maximal ideal if $\rho_m$ is reducible. In order to generalize Ogg’s conjecture, we need to study the Eisenstein maximal ideals and Ribet conjectured that if $m$ is a Eisenstein maximal ideal then $C_{\Gamma_0(N)}[m] \neq 0$ \cite{28}. Hwajong Yoo proved the conjecture under the assumption that $N$ square free \cite{28}. If the level $N$ is not squarefree, the study of Eisenstein maximal ideals become very complicated. We restrict ourself to study of Eisenstein maximal ideal for $p$-good levels.

**Definition 1.** ($p$-good) Let $p$ be an odd prime. A natural number $N$ is called $p$-good if it is of the form $N = p^2N'$, where $N'$ is a square-free integer co-prime to $p$ such that all the prime factors of $N'$ are congruent to $\pm 1$ modulo $p$.

Note that, if $N$ is a square-free natural number which is co-prime to 9, then $9N$ is 3-good. If the level $N$ is $p$-good, we classify all non-rational Eisenstein ideal provided it’s residual characteristic is co-prime to $6p$ (see Corollary \cite{29}) and in that situation we show that Ribet’s conjecture is true.

**Theorem 2.** (Ribet’s conjecture) Let $N$ be a $p$-good integer for an odd prime $p$. For a Eisenstein maximal ideal $m$ of the Hecke algebra $\mathbb{T}(N)$, if the residual characteristic of $m$ is co-prime to $6p$, then we have

$$C_{\Gamma_0(N)}[m] \neq 0.$$ 

Eisenstein maximal ideals for a non-squarefree level are quite complicated. Following Yoo, we classify them as rational and non-rational Eisenstein maximal ideals.

**Definition 3.** (Rational and non-rational Eisenstein maximal ideals) We call an Eisenstein maximal ideal to be rational if $\rho_m \simeq 1 \bigoplus \chi_l$, here $\chi_l$ denotes the mod $\ell$ cyclotomic character. Otherwise, we call the Eisenstein maximal ideal to be non-rational.

Yoo proved that for rational Eisenstein maximal ideal with odd residue charactereristic $l$ with $l^2 \nmid N$ for general $N$ the Ribet’s conjecture is true \cite{30}. Thus in order to prove Theorem 2 it is enough to study non-rational Eisenstein maximal ideals for $p$-good levels. We prove that Ribet’s conjecture is true for non-rational Eisenstein maximal ideals for $p$-good level as long as the residual characteristic is co-prime to $6p$.

On the other hand, Faltings-Jordan \cite[p. 37, Conjecture 3.22]{10} conjectured that the every Eisenstein maximal ideal should correspond to a characteristic zero Eisenstein series. As a consequence of the classification of Eisenstein maximal ideals (cf. Corollary 29), we prove the conjecture under certain assumption on level and for non-rational Eisenstein maximal ideals. This conjecture is already known for rational Eisenstein maximal ideals by the series of work of Hwajong Yoo.

To prove Ribet’s conjecture, we found several examples of congruences between cusp forms and Eisenstein series (cf. \cite{27}). In particular in Example 39, we found examples of so called Eisenstein congruences when the residue field is not a prime field. In these numerical examples, we write down the Eisenstein maximal ideals explicitly. Interested readers may wish to write down other examples by following the recipe of this section. To the best of knowledge of the authors, these examples were never explicitly written down in the literature.

The main novelty in our work lies in the use of the cuspidal subgroup $C_E$ associated to any Eisenstein series $E \in E_2(\Gamma_0(N))$. In \cite{22}, Glenn Stevens computed the order of the subgroup $C_E$ of the
Theorem 4. Let $\chi$ be a primitive Dirichlet character associated to the cuspidal subgroup $E \subset E_2(\Gamma_0(N))$ for the congruence subgroup $\Gamma_0(N)$ from the order of the cuspidal subgroup for $\Gamma_1(N)$. Using this one can compute the order of the cuspidal subgroup up to the order of intersection of the said group with the Shimura subgroup. Using the above mentioned theorem of Stevens, we can only compute the order of certain component [22, Example 4.9, page 542] of the cuspidal group associated to the Eisenstein series rather than the whole group because of the dependence on the Shimura subgroups. To understand rational Eisenstein maximal ideals Yuan Ren also studied these subgroups [16].

In the present paper, we modify the argument of [22, Theorem 1.3] and prove a variant of the above theorem [cf. Theorem 1.7] that holds for the congruence subgroup $\Gamma_0(N)$ for any $N \in \mathbb{N}$. As a consequence, we can compute the order of cuspidal subgroup associated to an Eisenstein series $E \in E_2(\Gamma_0(N))$ without dependence on the order of the Shimura subgroups.

We now consider the subgroup $C_{\Gamma_0(N)}(E)$ inside the cuspidal group $C_{\Gamma_0(N)}$ as defined in [22, p. 523].

Let $\phi$ be a non-trivial primitive Dirichlet character of conductor $f > 1$. Let $N$ be a fixed integer such that $f^2 \mid N$. Let $\chi$ be the primitive Dirichlet character associated to $\varphi^2$ of conductor $n$. For a primitive Dirichlet character $\chi$ of conductor $m$, let $\tau(\chi)$ denote the Gauss sum of the character $\chi$ and $B_2(\chi) = B_2,\chi(0) := m \sum_{a=0}^{m-1} \chi(a)((\frac{a}{m})^2 - \frac{a}{m} + \frac{1}{2}) = -2L(\chi, -1)$ the generalised Bernoulli number associated to $\chi$ [22, p. 520].

Let $M$ and $L$ be integers such that $f^2 ML \mid N$ and $(fM, L) = 1$. Let $T_1 = \prod_{l | M} l^t_1 l_{M,l}$ and $T_2 = \prod_{q \mid L} q = q_1 \cdots q_s$. Let $T_{2,\phi} = \prod_{q \mid S_{\phi}} q$, where $q \in S_{\phi}$ if $q \mid T_2$ and $\phi(q) = \pm 1$. Let

$$\beta_{\Gamma_0(N), \phi, M, L} = \frac{f^2 T_1 \phi(T_{2,\phi})}{4N} \left( \prod_{p \mid f} p^{\nu_p(M)+\delta_p} \right) \frac{\tau(\varphi^{-1})\tau(\xi^{-1})}{\tau(\xi^{-1})} B_2(\xi^{-1}) \prod_{p | T_1} \left( 1 - \frac{\xi(p)}{p^2} \right) \in \mathbb{Q}(\zeta_f, \varphi),$$

where $\phi(T_{2,\phi}) = \prod_{q \in S_{\phi}} (q - 1)$ and $\delta_p = 1$ if $\nu_p(M) = 0$ and $\nu_p \left( \frac{\xi}{\chi} \right) \geq 1$, otherwise $\delta_p = 0$.

**Theorem 4.** Let $\phi$ be a (fixed) non-trivial primitive Dirichlet character of conductor $f$. Let $N$ be a (fixed) integer of the form $N = f^2(\prod_{p \mid f} p^{n_p})N'$, where $n_p \geq 0$ integer and $N'$ be a squarefree integer co-prime to $f$. Let $M$ and $L$ be integers such that $f^2 ML \mid N$ and $(fM, L) = 1$. Moreover, if for a prime divisor $p$ of $f$, $n_p > 1$, then $\nu_p(M) = n_p$. Consider the non-rational Eisenstein series $E_{\varphi, M, L} \in E_2(\Gamma_0(N))$ [Definition 7]. The cuspidal subgroup associated to $E_{\varphi, M, L}$ is supported at cusps which are $\mathbb{Q}(\zeta_f)$-rational, hence $C_{\Gamma_0(N)}(E_{\varphi, M, L})$ is a subgroup of $J_0(N)(\mathbb{Q}(\zeta_f, \varphi))$ and moreover it’s order is given by

$$|C_{\Gamma_0(N)}(E_{\varphi, M, L})| = |\mathbb{Z}[\zeta_f, \varphi]/Num(\beta_{\Gamma_0(N), \phi, M, L})|,$$

here $\beta_{\Gamma_0(N), \phi, M, L} := fT_1 \beta_{\Gamma_0(N), \phi, M, L}$ and $Num(\beta_{\Gamma_0(N), \phi, M, L})$ denotes the intersection of the fractional ideal $(\beta_{\Gamma_0(N), \phi, M, L})$ and $\mathbb{Z}[\zeta_f, \varphi]$.

Lemma 20 shows that $12 \beta_{\Gamma_0(N), \phi, M, L} \in \mathbb{Z}[\zeta_f, \varphi]$. Our Theorem 4 is a consequence of Theorem 4. We have not checked but an analogue of Theorem 4 should help us to prove Ribet’s conjecture for other congruence subgroups.

1.1. **Acknowledgements.** It is a pleasure to acknowledge several e-mail communication, advice and remark of Professor Ken Ribet. The first author was partially supported by the SERB grant MTR/2017/000357 and CRG/2020/000223. The second author was partially supported by the SERB grant MTR/2019/000137. The third author was partially supported by IIT Madras ERP project RF/2021/0658/MA/RFER/008794 and MHRD SPARC grant number 445.

2. **Eisenstein series for $\Gamma_0(N)$**

Let $A_{N,2}$ be the set of triples $(\varphi, t)$ such that $\varphi$ is a primitive Dirichlet characters of conductor $f$ and $t \mid N$ be such that $1 < t f^2 \mid N$. Recall the following theorem [8, p. 133, Theorem 4.6.2]:

Theorem 5. The set \( \{ E^{\varphi^{-1}, t}_2 : (\varphi, t) \in A_{N, 2} \} \) is a basis of \( E_2(\Gamma_0(N)) \).

Following the terminology of Yoo, there are two types of Eisenstein series as defined below.

1. Rational Eisenstein series with parameter \( t \in \mathbb{N} \): These are defined by \( t \in \mathbb{N} \) with \( 1 < t \mid N \). These Eisenstein series are in bijection with the set \( E^{11,11,t}_2 \) [8, p. 133].

2. Non-rational Eisenstein series with parameter \( (\varphi, t) \): These are the Eisenstein series \( E^{\varphi^{-1}, t}_2 \) determined by a character \( \varphi \) of conductor \( f \) with \( 1 < f \mid N \) and an integer \( t \) such that \( tf^2 \mid N \) [8, p. 133].

The Eisenstein series \( E^{11,11,t}_2 \) of [8, p. 133] are not Hecke eigenforms. Hence, Hwajong Yoo constructed a different set of Eisenstein series. For each \( t \), Yoo writes \( t = ML \) and produce several Eisenstein series out of it by applying ordinary and critical refinement on the normalized series \( E \) [cf. §2.2.1].

Note that Stevens [22, p. 523, Equation 1.5] uses the series \( \varphi \) and \( t \) to define \( \chi \) for the subgroups \( \Gamma(\chi) \) of degree \( \chi \) and conductor \( \chi \) of conductor \( m \) with \( (m, N) = 1 \), we define the twisted \( L \)-function as the analytic continuation of the series

\[
L(\varphi, \chi, s) := \sum_{n=1}^{\infty} a_n(E) \chi(n) \frac{n^s}{N^s}.
\]

The above series is convergent in a half plane \( \text{Re}(s) > 2 \). If \( \chi \) is trivial, we simply write it as \( L(E, s) \). Note that Stevens [22, p. 523, Equation 1.5] uses the series \( \chi^{-1}(N)N^s \sum_{n=1}^{\infty} \frac{a_n(E) \chi(n)}{n^s} \) to define \( L(E, \chi, s) \) for the subgroups \( \Gamma(N) \) and \( \Gamma_0(N), \Gamma_1(N) \). In [22], Stevens first consider the Eisenstein series of level \( \Gamma(N) \) and hence here \( a_n(E) \) corresponds to the coefficient of \( q^n \) in \( \chi^{-1}(N)N^s \). In loc. cit., the Eisenstein series for the subgroups \( \Gamma_0(N) \) of level \( \Gamma_0(N) \) and \( \Gamma_1(N) \) are obtained as a suitable linear combination of Eisenstein series for \( \Gamma(N) \) and \( \langle \frac{1}{2}, \frac{1}{2} \rangle \) is the smallest parabolic matrix in \( \Gamma(N) \). However, we choose to follow [23] and start directly with Eisenstein series for \( \Gamma_0(N) \). By [23, page 850, Remark 2.4], Fourier expansion of \( E_\varphi \) at \( \infty \) is given by

\[
E_\varphi(z) := E_{\varphi, \varphi^{-1}}(z) = \sum_{c=1}^{\infty} \sum_{b=1}^{\infty} \varphi(c) \varphi^{-1}(b) b q^b = \sum_{n=1}^{\infty} b_n q^n,
\]

where \( q = e^{2\pi iz} \) and \( b_n = \sum_{bc=n} \varphi(c) \varphi^{-1}(b) b \) for all \( n \geq 1 \).

Proposition 6. (1) The \( L \)-function \( L(E_\varphi, s) \) associated with the Eisenstein series \( E_\varphi := E_{\varphi, \varphi^{-1}} \) is given by:

\[
L(E_\varphi, s) = L(\varphi, s) L(\varphi^{-1}, s-1).
\]

More generally, for a Dirichlet character \( \chi \) whose conductor is co-prime to \( f \) we have,

\[
L(E_\varphi, \chi, s) = L(\chi \varphi, s) L(\chi \varphi^{-1}, s-1).
\]
(2) The action of the Hecke operators are then given by

\[ T_l(E_\varphi) = \begin{cases} 
(l\varphi^{-1}(l) + \varphi(l))E_\varphi & \text{if } l \nmid f, \\
0 & \text{if } l \mid f.
\end{cases} \]

**Proof.** First part of (1) follows proposition [23, page 850, Proposition 2.3]. From [23, page 850, Remark 2.4] we can see that \( a_n(E_\varphi)\chi(n) = \chi(n) \sum_{bc=n} \varphi(c)\varphi^{-1}(b)b = \sum_{bc=n} \chi(c)\varphi(c)\chi(b)\varphi^{-1}(b)b = a_n(E_{\chi \varphi, \chi \varphi^{-1}}) \) and the result follows from [23, Proposition 2.3(b)].

The proof of part 2 is similar to the proof of [23, page 852, Lemma 2.7] which we sketch below.

Thus the action of \( T_\ell \) on \( E_\varphi \) is given by

\[ T_\ell(E_\varphi) = \begin{cases} 
\sum_{n=1}^{\infty} b_{\ell n}q^n + \sum_{n=1}^{\infty} \ell b_n q^{\ell n}, \text{ for } \ell \nmid f, \\
\sum_{n=1}^{\infty} b_{\ell n}q^n, \text{ for } \ell \mid f.
\end{cases} \]

Now suppose that \( \ell \nmid f \). Then, proceeding as in [23, page 853] we get

\[ b_{\ell n} = \sum_{bc=\ell n} \varphi(c)\varphi^{-1}(b)b = \varphi(\ell) \sum_{bc=n, \ell \mid b} \varphi^{-1}(b)\varphi(c) + \ell \varphi^{-1}(\ell) \sum_{b'c=n, \ell \nmid b'} \varphi^{-1}(b')\varphi(c)b', \]

\[ \ell b_n = \ell \sum_{bc=n} \varphi^{-1}(b)\varphi(c)b = \varphi(\ell) \sum_{bc=n} \varphi^{-1}(b)\varphi(c) + \ell \varphi^{-1}(\ell) \sum_{b'c=n, \ell \nmid b'} \varphi^{-1}(b')\varphi(c)b'. \]

Also note that, for \( \ell \mid f, b_{\ell n} = 0 \). The result follows. \( \square \)

2.2.1. **Ordinary and critical refinement of \( E_\varphi \).** Recall the ordinary and critical refinement of Eisenstein series following [1, §1.3] and [2].

Note that, the Eisenstein series \( E_\varphi \) as defined above is a new form of level \( \Gamma_0(f^2) \) and \( f \nmid l \), it has \( T_l \) eigenvalue \( \varphi(l) + \ell \varphi^{-1}(l) \), the polynomial

\[ X^2 - (\varphi(l) + \ell \varphi^{-1}(l))X + \ell = 0 \]

has two roots \( \alpha := \varphi(l) \) and \( \beta := \ell \varphi^{-1}(l) \), note that \( \nu_l(\alpha) = 0 \) and \( \nu_l(\beta) = 1 = 2 - 1 \).

Consider the matrix \( \gamma_l = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \). For \( g \in M_2(\Gamma_0(f^2)) \), define

\[ [l]_\varphi^+ g(z) := g(z) - \varphi(l)g(lz) 
\]

\[ = g \begin{pmatrix} 0 & \alpha \\ 1 & \beta \end{pmatrix}(z); \]

and

\[ [l]_\varphi^- g(z) := g(z) - l\varphi^{-1}(l)g(lz) 
\]

\[ = g \begin{pmatrix} 0 & 1 \\ \alpha^{-1} & 1 \end{pmatrix}(z). \]

Note that, for \( l \nmid f \), \([l]_\varphi^+ E_\varphi \) is an Eisenstein series of level \( \Gamma_0(f^2l) \), it is eigenform for \( U_l \) operator and \( U_l([l]_\varphi^+ E_\varphi) = l\varphi^{-1}(l)[l]_\varphi^+ E_\varphi \), hence \([l]_\varphi^+ \) acts as critical refinement on \( E_\varphi \). Similarly, \([l]_\varphi^- E_\varphi \) is an Eisenstein series of level \( \Gamma_0(f^2l) \), it is eigenform for \( U_l \) operator and \( U_l([l]_\varphi^- E_\varphi) = \varphi(l)[l]_\varphi^- E_\varphi \), hence \([l]_\varphi^- \) acts as ordinary refinement on \( E_\varphi \). Note that, \([l]_\varphi^\pm \) acts as identity for \( l \mid f \).

Write \( t = ML \), where the integers \( M \) and \( L \) satisfies \( f^2ML \mid N \) and \( (fM, L) = 1 \). For such a choice of \( M \) and \( L \) define \( T_1 = \prod_{l \mid M, l \nmid t} l = l_1 \cdots l_m \) and \( T_2 = \prod_{q \mid L} q = q_1 \cdots q_s \). Hence \( T_1 \) and \( T_2 \) are square-free co-prime integers such that \( f \nmid T_1T_2 \). Define

\[ E_{\varphi, T_1, T_2} := \prod_{l \mid T_1} [l]_\varphi^+ \circ \prod_{q \mid T_2} [q]_\varphi^- E_\varphi. \]
Note that, we can rewrite
\[ E_{\varphi,T_1,T_2} = E_{\varphi}^{[\text{crit} \{ l \mid T_2 \}, \{ \text{ord}_q l \} \mid T_2]}, \]
and hence \( E_{\varphi,T_1,T_2} \) is an Eisenstein series of level \( \Gamma_0(f^2 T_1 T_2) \). Note that, \( E_{\varphi,T_1,T_2} \) is an eigenform for all Hecke operators with eigenvalues given by
\[ T_r(E_{\varphi,T_1,T_2}) = (\varphi(r) + r \varphi^{-1}(r))E_{\varphi,T_1,T_2} \text{ for } r \mid f T_1 T_2, \]
\[ U_l(E_{\varphi,T_1,T_2}) = l \varphi^{-1}(l)E_{\varphi,T_1,T_2} \text{ for } l \mid T_1, \]
\[ U_q(E_{\varphi,T_1,T_2}) = \varphi(q)E_{\varphi,T_1,T_2} \text{ for } q \mid T_2, \]
\[ U_p(E_{\varphi,T_1,T_2}) = U_p(E_{\varphi}) = 0 \text{ for } p \mid f. \]

**Definition 7.** For a matrix \( \gamma_{ML_{T_1T_2}} = \left( \begin{array}{cc} M & 0 \\ L & 1 \end{array} \right) \), define
\[ E_{\varphi,M,L}(z) := E_{\varphi,T_1,T_2} \mid \gamma_{ML_{T_1T_2}}(z) = \frac{ML}{T_1 T_2} E_{\varphi,T_1,T_2} \left( \frac{ML}{T_1 T_2} z \right). \]
Then \( E_{\varphi,M,L} \) is an Eisenstein series of level \( \Gamma_0(f^2 ML) \). We view \( E_{\varphi,M,L} \) as an Eisenstein series of \( \Gamma_0(N) \) via inclusion of \( \Gamma_0(N) \) in \( \Gamma_0(f^2 ML) \).

It is now clear that the set \( \{ E_{\varphi,M,L} \} \) generates the set \( \{ E_{\varphi,\varphi^{-1},ML} \} \) as we vary \( \varphi, M, L \) with \( f^2 ML \mid N \) and \( (L,f M) = 1 \). Hence they generate all non-rational Eisenstein series of \( \Gamma_0(N) \).

We further remark that if \( N = p^2 N' \) with \( N' \) squarefree and \( (p,N') = 1 \), then \( \{ E_{\varphi,M,L} \} \) is an eigenbasis of all non-rational Eisenstein series of \( \Gamma_0(N) \) as we vary \( \varphi \) among the characters of conductor \( p \) and \( M \) and \( L \) such that \( ML = N' \).

**Proposition 8.** Let \( E_{\varphi,M,L} \in E_2(\Gamma_0(N)) \) be the Eisenstein series as in definition 7 and \( \chi \) be a Dirichlet character whose conductor is coprime to \( N \). The twisted \( L \)-function \( L(E_{\varphi,M,L}, \chi, s) \) is given by
\[ L(E_{\varphi,M,L}, \chi, s) = \chi \left( \frac{ML}{T_1 T_2} \right) \left( \frac{ML}{T_1 T_2} \right)^{1-s} \left( \prod_{l \mid T_1} (1 - \chi(l) \varphi(l) l^{-s}) \right) \left( \prod_{q \mid T_2} (1 - \chi(q) \varphi^{-1}(q) q^{1-s}) \right) L(\chi \varphi, s) L(\chi \varphi^{-1}, s-1). \]

**Proof.** Recall that \( L(f \mid \gamma_d, \chi, s) = \chi(d) d^{-s} L(f, \chi, s) \) for any modular form \( f \), where \( \gamma_d = \left( \begin{array}{cc} d & 0 \\ 0 & 1 \end{array} \right) \). This essentially follows from the fact if the Fourier expansion of \( f(z) = \sum_{k=0} a_k q^k \) then the Fourier expansion of \( f \mid \gamma_d (z) = \sum_{n \geq 0} b_n q^n \), where \( b_n = \left\{ \begin{array}{ll} a_{dk} & \text{if } n = dk, \\ 0 & \text{otherwise.} \end{array} \right. \)

We obtain,
\[ L([l]^+ E_{\varphi}, \chi, s) = L(E_{\varphi}, \chi, s) - \frac{\varphi(l)}{l} L(E_{\varphi} \mid \gamma_l, \chi, s) = (1 - \chi(l) \varphi(l) l^{-s}) L(E_{\varphi}, \chi, s), \]
and
\[ L([q]^+ E_{\varphi}, \chi, s) = L(E_{\varphi}, \chi, s) - \varphi^{-1}(q) L(E_{\varphi} \mid \gamma_q, \chi, s) = (1 - \chi(q) \varphi^{-1}(q) q^{1-s}) L(E_{\varphi}, \chi, s). \]

Using the above two formulas we get,
\[ L(E_{\varphi,T_1,T_2}, \chi, s) = \left( \prod_{l \mid T_1} (1 - \chi(l) \varphi(l) l^{-s}) \right) \left( \prod_{q \mid T_2} (1 - \chi(q) \varphi^{-1}(q) q^{1-s}) \right) L(E_{\varphi}, \chi, s). \]
Result follows from the fact that, \( E_{\varphi,M,L} = E_{\varphi,T_1,T_2} \mid \gamma_{ML_{T_1T_2}} \) and \( L(E_{\varphi}, \chi, s) = L(\chi \varphi, s) L(\chi \varphi^{-1}, s-1). \) \( \square \)
3. DIVISOR ASSOCIATED TO EISENSTEIN SERIES $E_{e,M,L}$

Let $E_2(\Gamma_0(N); \mathbb{C})$ be the vector space of all Eisenstein series for $\Gamma_0(N)$. The Eisenstein series $E \in E_2(\Gamma_0(N); \mathbb{C})$ corresponds to the divisor

$$\delta_{\Gamma_0(N)}(E) = \sum_{x \in \text{Cusps}(\Gamma_0(N))} r_{\Gamma_0(N), E}(x) \{x\}.$$ 

By [22, Theorem 1.3(a)], we can compute the coefficient $r_{\Gamma_0(N), E}(x)$ in terms of ramification index and the constant term of the Eisenstein series at the cusp $x$.

Following Stevens [22] we recall some basic facts about cusps for the congruence subgroup $\Gamma_0(N)$. Let $\partial(X(N))$ (respectively $\partial(X_0(N))$) be the set of cusps for the congruence subgroup $\Gamma(N)$ (respectively $\Gamma_0(N)$). For the congruence subgroup $\Gamma(N)$, we follow Shimura’s notation and denote by $[a \ b \ \Gamma(N)] \in \partial(X(N))$ the cusp represented by $\frac{a}{b} \in \mathbb{Q}$ with $\alpha, \beta \neq 0$ and $\alpha \equiv a \ (\text{mod } N)$ and $\beta \equiv b \ (\text{mod } N)$. The group $\text{GL}_2(\mathbb{Z}/N\mathbb{Z})$ acts on $\partial(X(N))$ by matrix multiplication. The natural map $\Gamma_0(N) \to \text{GL}_2(\mathbb{Z}/N\mathbb{Z})$ determines an action of $\Gamma_0(N)$ on $\partial(X(N))$.

By $[a \ b \ \Gamma(N)]$ we denote a cusp in $\partial(X_\Gamma)$ for the congruence subgroup $\Gamma$. The ramification indices of the cusp $x = \frac{a}{b} \in \partial(X_\Gamma)$ is denoted by $e_\Gamma(x)$.

The cusps of $\Gamma_0(N)$ are determined by the $\Gamma_0(N)$ orbits of $[a \ b]_{\Gamma(N)}$ with the relations:

1. $[a \ b]_{\Gamma_0(N)} = [a' \ b']_{\Gamma_0(N)}$ if $a \equiv a' \ (\text{mod } N)$, $b \equiv b' \ (\text{mod } N)$.
2. $[a \ b]_{\Gamma_0(N)} = [a' \ b]_{\Gamma_0(N)}$ if $a \equiv a' \ (\text{mod } b)$.
3. $[ra \ b]_{\Gamma_0(N)} = [a \ rb]_{\Gamma_0(N)}$ for all $r \in \mathbb{Z}$ with $(r, N) = 1$.

The integer $d = (b, N)$ depends only on the cusp $[a \ b]_{\Gamma_0(N)}$. Following Stevens’ we’ll refer $d$ as the divisor of $[a \ b]_{\Gamma_0(N)}$. According to loc. cit., $e_{\Gamma_0(N)}([a \ b]_{\Gamma_0(N)}) = \frac{N}{d}$ with $t = \gcd(d, \frac{N}{d})$. Moreover a cusp $[a \ b]_{\Gamma_0(N)}$ of divisor $d$ is $Q(\zeta_d)$-rational [22, eqn. (4.9)].

Let $\mathcal{D}_{\Gamma_0(N)}$ be the subgroup of divisors supported on the cusps $\partial(X_0(N))$ and for $d | N$, $\mathcal{D}_{\Gamma_0(N), d}$ be the subgroup of divisors supported on the cusps with divisor $d$. The group $\text{GL}_2(\mathbb{Z}/N\mathbb{Z})$ acts on the set of cusps $\partial(X_0(N))$ and hence on $\mathcal{D}_{\Gamma_0(N)}$. Hence the torus $T(N) = \{[a \ 0 \ 0 \ d] \in \text{GL}_2(\mathbb{Z}/N\mathbb{Z})\}$ acts on $\mathcal{D}_{\Gamma_0(N)}$. By [22, p. 539], the action of $T(N)$ on $\mathcal{D}_{\Gamma_0(N)}$ preserves the subspace $\mathcal{D}_{\Gamma_0(N), d}$.

By [22, p. 100], we have a decomposition

$$\mathcal{D}_{\Gamma_0(N)} = \bigoplus_{\psi : T(N) \to \mathbb{C}^\times} \mathcal{D}_{\Gamma_0(N), \psi}$$

where $\mathcal{D}_{\Gamma_0(N), \psi}$ (respectively $\mathcal{D}_{\Gamma_0(N), d, \psi}$) is the subspace of $\mathcal{D}_{\Gamma_0(N)}$ (respectively $\mathcal{D}_{\Gamma_0(N), d}$) on which $T(N)$ acts by $\psi$. By [22, p. 540], we have

$$\mathcal{D}_{\Gamma_0(N), \psi} = \sum_{d | N} \mathcal{D}_{\Gamma_0(N), d, \psi}.$$
Notice that as a group \( T(N) = (\mathbb{Z}/N\mathbb{Z})^\times \times (\mathbb{Z}/N\mathbb{Z})^\times \) and hence the Pontryagin dual \( \hat{T}(N) = (\mathbb{Z}/N\mathbb{Z})^\times \times (\mathbb{Z}/N\mathbb{Z})^\times \). For any (but fixed) character \( \psi : T(N) \to C^\times \), there are two primitive Dirichlet characters \( \epsilon_1, \epsilon_2 \) of conductors \( N_1, N_2 \) divisors of \( N \) such that

\[
\psi\left( \begin{pmatrix} r & 0 \\ s & 1 \end{pmatrix} \right) = \epsilon_1^{-1}(s)\epsilon_2(r)
\]

for \( r, s \in (\mathbb{Z}/N\mathbb{Z})^\times \).

We now recall the following Proposition from loc. cit.:

**Proposition 9.** [22 Proposition 4.5]

1. The divisor group \( \mathcal{D}_{T(N), \psi} \neq 0 \) if and only if \( N \) divides \( \psi \) and \( \epsilon_1 = \epsilon_2^{-1} \).
2. If divisor group \( \mathcal{D}_{T(N), \psi} \) is non-zero, then it is generated as a \( \mathbb{Z}[\psi] \) module by

\[
\mathcal{D}_{T(N), \psi} := \sum_{a, b} \epsilon_1(b)\epsilon_2^{-1}(a) \begin{bmatrix} a \\ db \end{bmatrix} \in \mathcal{D}_{T(N), \psi}.
\]

The following theorem determines the divisor associated to the non-rational Eisenstein series \( E_{f, M, L} \in M_2(\Gamma_0(N)) \). The general outline of the proof is similar to the proof of [22 Proposition 4.7]. We give detail of the proof in the Appendix as the proof is computationally involved.

Let \( N \) be a fixed integer and \( \varphi \) be a primitive Dirichlet character of conductor \( f > 1 \) such that \( f^2 \mid N \). Let \( \xi \) be the primitive Dirichlet character associated to \( \varphi^2 \) of conductor \( n \).

By \( \tau(\varphi^{-1}) \) (resp. \( \tau(\xi^{-1}) \) ) we denote the Gauss sum of the character \( \varphi^{-1} \) (resp. \( \xi^{-1} \) ) and \( B_2(\xi^{-1}) = -2L(-1, \xi^{-1}) \).

Let \( M \) and \( L \) be integers such that \( f^2M \mid N \) and \( (fM, L) = 1 \). Let \( T_1 = \prod_{q \mid M} q = l_1 \cdots l_m \) and \( T_2 = \prod_{q \mid L} q = q_1 \cdots q_s \). Let \( T_2, \varphi = \prod_{q \in S_\varphi} q \) where \( q \in S_\varphi \) if \( q \mid T_2 \) and \( \varphi(q) = \pm 1 \). Let

\[
\beta_{T_2, \varphi} = \frac{f^3T_1\phi(T_2, \varphi)}{4\nu_1} \prod_{p | f} p^\nu_p(M + \delta_p) \frac{\tau(\varphi^{-1})}{\tau(\xi^{-1})} B_2(\xi^{-1}) \prod_{p | fT_1} \left( 1 - \frac{\xi(p)}{p^2} \right) \in \mathbb{Q}(\zeta_f, \varphi),
\]

where \( \phi(T_2, \varphi) = \prod_{q \in S_\varphi} (q - 1) \) and \( \delta_p = 1 \) if \( \nu_p(M) = 0 \) and \( \nu_p(N) \geq 1 \), otherwise \( \delta_p = 0 \).

**Theorem 10.**

\[
\delta_{T_2, \varphi}(E_{f, M, L}) = \beta_{T_2, \varphi}D_{T_2, \varphi}(E_{f, M, L}),
\]

where

\[
D_{T_2, \varphi}(E_{f, M, L}) = \sum_{i_1=0}^{\nu_{q_1}(M)-1} \cdots \sum_{i_m=0}^{\nu_{q_m}(M)-1} \sum_{j_1=0}^{\nu_{\nu_q(N)}(N)} \cdots \sum_{j_s=0}^{\nu_{\nu_q(N)}(N)} \sum_{k_1=0}^{\nu_{\nu_q(N)}(N)} \cdots \sum_{k_r=0}^{\nu_{\nu_q(N)}(N)} \frac{\alpha_{i_1, j_1, k_1}}{t_1} \cdots \frac{\alpha_{i_m, j_s, k_r}}{t_m} \phi(\varphi),
\]

Here \( l_i \)'s are primes which divides \( M \) but co-prime to \( f \), \( q_i \)'s are primes which divides \( L \) and \( \{t_1, \cdots, t_r\} \) is the full set of prime divisors of \( \frac{N}{l_1 \cdots l_m q_1 \cdots q_s} \) which are co-prime to \( fM \) and \( \alpha_{t_i, t_j, t_k} \)'s are as in Lemma [27]. We remark that the divisor \( D_{T_2, \varphi}(E_{f, M, L}) \) is in \( \text{Div}^0(X_0(N))(\mathbb{Q}(\zeta_f, \varphi)) \).
We obtain this by applying formulas from Remark \[ \text{Lemma } 47 \] two obtain our result. 

Recall $\delta_{\Gamma_0(f^2)}(E_\varphi) = \beta_\varphi D_{\Gamma_0(f^2),f}(\varphi)$, where $\beta_\varphi = \frac{f^2 \cdot (\varphi^{-1})}{4n \cdot (\varphi^{-1})} B_2(\xi^{-1}) \prod_{p \mid f} \left( 1 - \frac{\xi(q)}{q^\varphi} \right)$.

First, recall that we construct $E_{\varphi,T_1,T_2}$ from $E_\varphi$ by applying $[l]^+$ for $l \mid T_1$ and $[q]^-$ for $q \mid T_2$. Thus, we obtain $\delta_{\Gamma_0(f^2T_1,T_2)}(E_{\varphi,T_1,T_2})$ by applying $[l]^+$ for $l \mid T_1$ and $[q]^-$ for $q \mid T_2$ to $\delta_{\Gamma_0(f^2)}(E_{\varphi}) = \beta_\varphi D_{\Gamma_0(f^2),f}(\varphi)$.

We see that $\delta_{\Gamma_0(f^2T_1,T_2)}(E_{\varphi,T_1,T_2}) = \beta_1 D_1$, where $\beta_1 = \beta_\varphi \prod_{l \mid T_1} l(1 - \frac{\xi(l)}{l^2}) \prod_{q \in S_\varphi} (q - 1)$ and

$$D_1 = \sum_{j_1=0}^{\nu_1(M)_1-1} \cdots \sum_{j_m=0}^{\nu_m(L)_m} \beta_{j_1,j_1} \cdots \beta_{j_m,j_m} D_{\Gamma_0(f^2T_1,T_2),f,q_1^{j_1} \cdots q_m^{j_m}}(\varphi).$$

We obtain this by applying formulas from (1)(a) and (2)(a) of Lemma \[ \text{Lemma } 47 \].

Recall that we construct $E_{\varphi,M,L}$ from $E_{\varphi,T_1,T_2}$ by slashing with $\gamma_{M,L}^{\frac{N}{M}} = \prod_{p \mid M} \left[ \gamma_p \right]_{p^\varphi(M,L)}$. Thus, for each prime divisor $p$ of $\frac{M}{T_1T_2}$, we apply $\pi_p^+ \nu_p(\frac{M}{T_1T_2})$ times to $\delta_{\Gamma_0(f^2T_1,T_2)}(E_{\varphi,T_1,T_2})$ to obtain $\delta_{\Gamma_0(f^2ML)}(E_{\varphi,M,L})$.

Applying formulae Lemma [(1)(b), (2)(b) and (3)(a)]\[ \text{Lemma } 47 \] we see $\delta_{\Gamma_0(f^2ML)}(E_{\varphi,M,L}) = \beta_2 D_2$, where $\beta_2 = \beta_1(\prod_{p \mid f} p^{\nu_\varphi(M)})$ and

$$D_2 = \sum_{j_1=0}^{\nu_1(M)_1-1} \cdots \sum_{j_m=0}^{\nu_m(L)_m} \alpha_{j_1} \cdots \alpha_{j_m} \cdot \beta_{j_1,j_1} \cdots \beta_{j_m,j_m} \cdot \delta_{\Gamma_0(f^2ML),f,q_1^{j_1} \cdots q_m^{j_m}}(\varphi).$$

Finally, we see $E_{\varphi,M,L}$ as an Eisenstein seies of level $\Gamma_0(N)$ by the inclusion of $\Gamma_0(N)$ into $\Gamma_0(f^2ML)$. Thus for each prime divisor $p$ of $\frac{N}{M}$, we apply $\pi_p^+ \nu_p(\frac{N}{M})$ times to $\delta_{\Gamma_0(f^2ML)}(E_{\varphi,M,L})$ to obtain $\delta_{\Gamma_0(N)}(E_{\varphi,M,L})$. Thus we obtain the final result by applying formulas from (1)(c), (2)(c), (3)(b) and (4) of Lemma \[ \text{Lemma } 47 \].

\[ \square \]

Remark 11. Suppose that $\varphi$ is a non-trivial primitive Dirichlet character of conductor $f$ and $N$ is a (fixed) integer of the form $N = f^2(\prod_{p \mid f} p^{\nu_p})N'$, where $n_p \geq 0$ integer and $N'$ be a squarefree integer co-prime to $f$. Let $M$ and $L$ be integers such that $f^2ML \mid N$ and $(fM,L) = 1$. Moreover, if for a prime divisor $p$ of $f$, $n_p > 1$, then $\nu_p(M) = n_p$. Since the cusps of $\Gamma_0(N)$ of divisor $d$ are $\mathbb{Q}(\zeta_d)$-rational ($t = (d, N)$), it follows that $D_{\Gamma_0(N),M,L}(\varphi)$ is supported at the cusps which are $\mathbb{Q}(\zeta_f)$-rational. Thus in this case $D_{\Gamma_0(N),M,L}(\varphi) \in \text{Div}^0(X_0(N)(\mathbb{Q}(\zeta_f, \varphi)))$.

Remark 12. For convenience, we write down the divisor $D_{\Gamma_0(N),M,L}(\varphi) = \text{Div}^0(X_0(N)(\mathbb{Q}(\zeta_p, \varphi)))$ for an integer $N = p^2N'$ which is $p$-good. Let $N' = q_1 \cdots q_s$. Since $q_i \equiv \pm 1 \pmod{p}$, it follows that $S_\varphi = \{ q_1, \ldots, q_s \}$ and hence $\beta_{q_1,0} = 1$ and $\beta_{q_1,1} = -\varphi(q_1) \in \{ \pm 1 \}$ and $\beta_{\varphi,0,1} = 1$ and $\beta_{\varphi,1,1} = -\varphi(q_1) \in \{ \pm 1 \}$ and

$$D_{\Gamma_0(N),M,L}(\varphi) = \sum_{j_1=0}^{d} \cdots \sum_{j_s=0}^{d} \sum_{j_1=0}^{\beta_{q_1,j_1} \cdots \beta_{q_s,j_s} D_{\Gamma_0(N),p^2q_1^{j_1} \cdots q_s^{j_s}}(\varphi) = \sum_{p \mid d} c_\varphi D_{\Gamma_0(N),d}(\varphi),$$
here \( c_d(\varphi) = (-1)^n \prod_{i=1}^s \varphi(q_i) \in \{\pm 1\} \). It follows from the above expression that the cusps appearing in \( D_{\Gamma_0(N),M,N'}(\varphi) \) are \( \mathbb{Q}(\zeta_p) \)-rational.

4. CUSPIDAL SUBGROUP ASSOCIATED TO non-rational Eisenstein series

4.1. Cuspidal subgroup \( C_{\Gamma_0(N)}(E) \) associated to the Eisenstein series \( E \). Let \( J_0(N)(K) := \text{Pic}^0(X_0(N))(K) \) (the Jacobian variety of the modular curve) be the equivalence classes of degree 0 divisors of \( X_0(N)(K) \) and \( C_{\Gamma_0(N)}(K) \) be the subgroup of \( J_0(N)(K) \) consisting of equivalence classes of degree 0 divisors of \( X_0(N)(K) \) supported on the \( K \)-rational cusps of \( X_0(N)(K) \). Let \( E \in E_2(\Gamma_0(N)) \) and \( \delta_{\Gamma_0(N)}(E) = \lambda D \) be the associated degree 0 divisor of \( E \) supported at the cusps of \( X_0(N) \) for some \( \lambda \in \mathbb{C} \). Then the cuspidal subgroup associated to \( E \) is the subgroup generated by the equivalence class of \( D \) in \( J_0(N)(\mathbb{C}) \).

For remaining of the section, unless otherwise mentioned, \( \varphi \) denotes a (fixed) non-trivial primitive Dirichlet character of conductor \( f \) and \( N \) denotes a (fixed) integer of the form \( N = f^2(\prod_{p \mid f} p^{n_p})N' \), where \( n_p \geq 0 \) integer and \( N' \) be a squarefree integer co-prime to \( f \). Let \( M \) and \( L \) be integers such that \( f^2ML \mid N \). Moreover, if for a prime divisor \( p \) of \( f \), \( n_p > 1 \), then \( \nu_p(M) = n_p \). In particular, \( M = T_1(\prod_{p \mid f} p^{n_p}) \) with \( 0 \leq m_p \leq n_p \) (with \( m_p = n_p \) if \( n_p > 1 \)) and \( L = T_2 \), where \( T_1 \) and \( T_2 \) are two (co-prime) divisors of \( N' \). Note that, this condition on \( N' \) implies that the cusps which support \( D_{\Gamma_0(N),M,L}(\varphi) \) are \( \mathbb{Q}(\zeta_f) \)-rational, or in other words \( D_{\Gamma_0(N),M,L}(\varphi) \in \text{Div}^0(X_0(N))(\mathbb{Q}(\zeta_f, \varphi)) \).

**Definition 13.** (Cuspidal subgroups associated to \( E_{\varphi,M,L} \)) Let \( \varphi \) be a non-trivial primitive Dirichlet character of conductor \( f \). Let \( N \) be a integer of the form \( N = f^2(\prod_{p \mid f} p^{n_p})N' \), where \( n_p \geq 0 \) integer and \( N' \) be a squarefree integer co-prime to \( f \). Let \( M \) and \( L \) be integers such that \( f^2ML \mid N \) and \( (fM, L) = 1 \). Moreover, if for a prime divisor \( p \) of \( f \), \( n_p > 1 \), then \( \nu_p(M) = n_p \). The cuspidal subgroup associated to \( E_{\varphi,M,L} \), denoted by \( C_{\Gamma_0(N)}(E_{\varphi,M,L}) \) is the cyclic subgroup of \( J_0(N)(\mathbb{Q}(\zeta_f, \varphi)) \) generated by the equivalence class of the divisor \( D_{\Gamma_0(N),M,L}(\varphi) \). Here \( D_{\Gamma_0(N),M,L}(\varphi) \in \text{Div}^0(X_0(N))(\mathbb{Q}(\zeta_f, \varphi)) \) is the divisor as in Theorem [10].

4.2. The group \( A_{\Gamma_0(N)}(E) \) of Stevens. In the section, we recall the basic properties of the periods of the Eisenstein series for the congruence subgroup \( \Gamma_0(N) \). Denote by \( R_{\Gamma_0(N)}(E) \) the finitely generated \( \mathbb{Z} \)-submodule of \( \mathbb{C} \) generated by the co-efficients of \( \delta_{\Gamma_0(N)}(E) \), the divisor associated to the Eisenstein series \( E \).

**Definition 14.** (Periods of Eisenstein series) Fix a point \( z_0 \in \mathbb{H} \) and let \( c(\gamma) \) be the geodesic in \( Y_0(N) \) joining \( z_0 \) and \( \gamma(z_0) \). The integral

\[
\pi_E(\gamma) = \int_{c(\gamma)} E(z) \, dz
\]

is the period of the Eisenstein series \( E \).

The group \( P_{\Gamma_0(N)}(E) \) is the free abelian group generated by \( \pi_E(\gamma) \) with \( \gamma \in \Gamma_0(N) \) and \( R_{\Gamma_0(N)}(E) \subset P_{\Gamma_0(N)}(E) \). The details about the group \( P_{\Gamma_0(N)}(E) \) can be found in [16, Page 3]. For every Eisenstein series \( E \in E_2(\Gamma_0(N)) \), we have a homomorphism \( \pi_E : \Gamma_0(N) \to \mathbb{C} \).

**Definition 15.** For any Eisenstein series \( E \in E_2(\Gamma_0(N); \mathbb{C}) \), define the abelian group associated to \( E \):

\[
A_{\Gamma_0(N)}(E) := \frac{P_{\Gamma_0(N)}(E)}{R_{\Gamma_0(N)}(E)}.
\]
By [22, Proposition 1.1] \(A_{\Gamma_0(N)}(E)\) is a finite abelian group and by [22, Theorem 1.2] there is a pairing \(C_{\Gamma_0(N)}(E) \times A_{\Gamma_0(N)}(E) \to \mathbb{Q}/\mathbb{Z}\). As a consequence, we have \(|C_{\Gamma_0(N)}(E)| = |A_{\Gamma_0(N)}(E)|\). We generalise Stevens [22, Theorem 1.3(b)] to the case of \(\Gamma_0(N)\) and use that to compute cardinality of \(A_{\Gamma_0(N)}(E_{\varphi,M,L})\), which in turn gives us cardinality of \(C_{\Gamma_0(N)}(E_{\varphi,M,L})\).

**Definition 16.** Following Stevens [22, Equation (1.8)], we consider the set \(S\) consisting of primes \(m\) such that \(m \equiv 3 \pmod{4}\) and \(S\) intersects all arithmetic progressions of the form \(-1 + Nkr\) with \(r \in \mathbb{Z}\). Let \(\chi_S\) be the set of all Dirichlet character \(\chi\) such that \(\chi\) is not quadratic and its conductor \(m_{\chi} \in S\). We denote by \(\chi_S^+\) (resp. \(\chi_S^-\)) then set of even (resp. odd) characters of \(\chi_S\). By \(\mathbb{Z}[\chi_S]\) we denote the ring generated over \(\mathbb{Z}\) by the values of characters \(\chi \in \chi_S\).

For a nontrivial primitive Dirichlet character \(\chi\) of conductor \(t\) with \((t, N) = 1\) and \(t \equiv -1 \pmod{N^2}\), Stevens [22, Equation (2.1, 2.2)] considers the special values:

\[
\Lambda(\chi) := \sum_{a=0}^{t-1} \chi^{-1}(Na)\{0, \frac{Na}{t}\}_{\Gamma_0(N)} \in H_1(X_0(N) , \mathbb{Z}[\chi]),
\]

where \(\{0, \frac{Na}{t}\}_{\Gamma_0(N)} \in H_1(X_0(N) , \mathbb{Z})\) is the homology class represented by oriented geodesic in \(\Phi\) joining 0 and \(\frac{Na}{t}\). Moreover, if \(t = m^n\) for some \(m \in S\), then

\[
\Lambda_{\pm}(\chi) := \sum_{a=0}^{t-1} \chi^{-1}(Na)\{0, \frac{Na}{t}\}_{\Gamma_0(N)},
\]

where \(\chi_m(-) = \left( \frac{a}{m} \right)\) the quadratic residue modulo \(m\). Finally, for a \(\mathbb{Z}[\chi]\) module \(A\) and \(\Phi : H_1(X_0(N) , \mathbb{Z}) \to A\) a cohomology class on \(X_0(N)\), he considers twisted \(L\)-vales [22, Equation (2.4)]

\[
\Lambda_{\pm}(\Phi, \chi) = (\Phi \otimes 1)(\Lambda_{\pm}(\chi)).
\]

We now give a proof of Theorem [17] that in turn is the generalization of [22, Theorem 2.1, p. 525] for the congruence subgroup \(\Gamma_0(N)\).

**Theorem 17.** Let \(S\) be a set of prime defined as in Definition [16] and \(A \subseteq \mathbb{C}\) be a \(\mathbb{Z}[\chi_S]\)-module. Given a cohomology class \(\Phi : H_1(X_0(N) , \mathbb{Z}) \to A\) such that \(\Lambda_{\pm}(\Phi, \chi) = 0\) for all Dirichlet character \(\chi \in \chi_S^+\), we have \(\Phi = 0\).

**Proof.** Choose a prime ideal \(\mathfrak{P} \subseteq \mathbb{Z}[\chi_S]\) and consider the localization \(A_{\mathfrak{P}}\) of \(A\) at \(\mathfrak{P}\). Consider now the cohomology class \(\Phi_{\mathfrak{P}}\) obtained by composing \(\Phi\) with the inclusion \(A \to A_{\mathfrak{P}}\).

We define \(\Phi_{\mathfrak{P}} : \Gamma_0(N) \to A_{\mathfrak{P}}\) by defining \(\Phi_{\mathfrak{P}}(\gamma) = \Phi_{\mathfrak{P}}(\{x, \gamma x\})\). We show \(\Phi_{\mathfrak{P}} = 0\) in the following 4 steps.

**Step 1:** If \(m \in S\) and \(m \equiv -1 \pmod{N^2}\) and \(\mathfrak{P} \mid \frac{m-1}{2}\), then \(\Phi_{\mathfrak{P}}(\{0, \frac{Na}{m}\}_{\Gamma_0(N)}) = 0\) whenever \((m, Na) = 1\).

We remark that the proof of this step is essentially same as in Step 1 of [22, Theorem 2.1].

**Step 2:**

In this step, we show that \(\Phi_{\mathfrak{P}}\) vanishes on the subgroup:

\[
\Gamma'(N) = \left\{ \begin{pmatrix} a & bN \\ cN & d \end{pmatrix} \in SL_2(\mathbb{Z}) | a, d \equiv \pm 1 \pmod{N^2} \right\}.
\]

The proof of this step is same as in Step 2 of [22, Theorem 2.1].
Step 3: Let 
\[ \Gamma'_0(N) = \left\{ \begin{pmatrix} a & bN \\ cN & d \end{pmatrix} \right\} \subset \Gamma_0(N), \]
then \( \Phi_\mathfrak{p} \) vanishes on \( \Gamma'_0(N) \).

Since we have \( (a, N) = (d, N) = 1 \), which is equivalent to saying \( (a, N^2) = (d, N^2) = 1 \), we see that, 
\[ a^{\varphi(N^2)/2} = d^{\varphi(N^2)/2} = \pm 1 \pmod{N^2}, \]
here \( \varphi \) denotes the Euler’s totient function. Let \( \gamma = \begin{pmatrix} a & bN \\ cN & d \end{pmatrix} \equiv \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \pmod{N} \in \Gamma'_0(N) \). Then
\[ \gamma^{\varphi(N^2)/2} = \begin{pmatrix} a^{\varphi(N^2)/2} & 0 \\ 0 & d^{\varphi(N^2)/2} \end{pmatrix} \pmod{N}. \]
Since \( a^{\varphi(N^2)/2} = d^{\varphi(N^2)/2} = \pm 1 \pmod{N^2} \), we see that \( \gamma^{\varphi(N^2)/2} \in \Gamma'(N) \). Thus \( \Phi_\mathfrak{p}(\gamma^{\varphi(N^2)/2}) = 0 \) by Step 2.

Step 4: In this step, we show that \( \Phi_\mathfrak{p} = 0 \).

Let \( A \in \Gamma_0(N) \) be the matrix \( A = \begin{pmatrix} a & b \\ cN & d \end{pmatrix} \), now consider a parabolic matrix \( B = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \), here \( x \) is chosen such that \( ax \equiv b \pmod{N} \). Clearly, we then have \( AB^{-1} \in \Gamma'_0(N) \). By Step 3, \( \Phi_\mathfrak{p}(AB^{-1}) = 0 \) and by a well-know theorem due to Manin [11], we have \( \Phi_\mathfrak{p}(B) = 0 \). Hence, we conclude that \( \Phi_\mathfrak{p}(A) = \Phi_\mathfrak{p}(AB^{-1}) + \Phi_\mathfrak{p}(B) = 0 \).

Since the homomorphism \( \Gamma_0(N) \to H_1(X_0(N), \mathbb{Z}) \) is surjective, we see \( \Phi_\mathfrak{p} = 0 \).

Since \( \Phi_\mathfrak{p} = 0 \) for all primes \( \mathfrak{p} \), we get \( \Phi = 0 \).

As a consequence, we prove the following Theorem that is a direct generalization of [22, Th 1.3, p. 524]. We first recall some notations from [22].

Let \( E \) be an Eisenstein series for \( \Gamma_0(N) \) and \( m \in S \) (see definition 16) a prime. Let \( \chi_m \) denotes the quadratic character residue modulo \( m \) and \( \chi \in \chi_S \) (see definition 16) a Dirichlet character. The twisted special values of \( L \) function of \( E \) are defined as follows:

\[ \Lambda(E, \chi, 1) := \frac{\tau(\chi^{-1})}{2\pi i} L(E, \chi, 1) \quad \text{and} \quad \Lambda_{\pm}(E, \chi, 1) = \frac{1}{2}[\Lambda(E, \chi, 1) \pm \Lambda(E, \chi m, 1)]. \]

Theorem 18. Let \( M \subseteq \mathbb{C} \) be a finitely generated \( \mathbb{Z} \)-submodule. Let \( S \) be a set of prime as in definition 16. Then \( P_{\Gamma_0(N)}(E) \subseteq M \) if and only if
(1) \( R_{\Gamma_0(N)}(E) \subseteq M \);
(2) for every \( \chi \in \chi_S \) (see Definition 16),
\[ \Lambda_{\pm}(E, \chi, 1) \in M[\chi, \frac{1}{q\chi}]. \]
Proof. Suppose first that \( P_{\Gamma_0(N)}(E) \subset M \). Since \( R_{\Gamma_0(N)}(E) \subset P_{\Gamma_0(N)}(E) \), we have \( R_{\Gamma_0(N)}(E) \subset M \). That \( \Lambda_+(E, \chi, 1) \in M[\chi, \chi^{-1}] \) for any \( \chi \in \mathcal{X}_S \) follows from [22, Lemma 2.2 (1)].

The proof that \( R_{\Gamma_0(N)}(E) \subset M \) and \( \Lambda_+(E, \chi, 1) \in M[\chi, \chi^{-1}] \) for all \( \chi \in \mathcal{X}_S \) implies \( P_{\Gamma_0(N)}(E) \subset M \) is exactly same as in the proof of [22, Th. 1.3(b), p. 528], except we use Theorem 17 instead of [22, Th 2.1].

In the following proposition we calculate the twisted special values of \( E_{\varphi,m,L} \).

Proposition 19. Let \( \varphi \) be a non-trivial Dirichlet character of conductor \( f \), \( M \) and \( L \) are integers with \( (f M, L) = 1 \) and \( f^2 ML \mid N \). Let \( \chi \) be a primitive Dirichlet character of conductor \( m_\chi \) and \( (m_\chi, N) = 1 \). Then,

1) \[
\Lambda (E_{\varphi,m,L}, \chi, 1) = \frac{\varphi(m_\chi)}{2\pi i} \chi \left( \frac{ML}{T_1 T_2} \right) \prod_{l \mid T_1} \left( 1 - \frac{\varphi(l)}{l} \right) \prod_{q \mid T_2} (1 - \chi \varphi^{-1}(q)) B_1(\chi^{-1} \varphi^{-1}) B_1(\chi \varphi^{-1}).
\]

2) Moreover if \( \chi \in \mathcal{X}_S^{-\varphi(1)} \) (see definition 16), then
\[
\Lambda_\pm (E_{\varphi,m,L}, \chi, 1) = \frac{\varphi^{-1}(m_\chi) \tau(\varphi)}{f} \chi \left( \frac{ML}{T_1 T_2} \right) \prod_{l \mid T_1} \left( 1 - \frac{\varphi(l)}{l} \right) \prod_{q \mid T_2} (1 - \chi \varphi^{-1}(q)) B_1(\chi^{-1} \varphi^{-1}) B_1(\chi \varphi^{-1}).
\]

Proof. From Proposition 8 we have
\[
L(E_{\varphi,m,L}, \chi, s) = \chi \left( \frac{ML}{T_1 T_2} \right) \left( \prod_{l \mid T_1} (1 - \chi(l) \varphi(l) l^{-s}) \prod_{q \mid T_2} (1 - \chi(q) \varphi^{-1}(q) q^{-s}) \right) L(\chi, s) L(\chi^{-1}, s). \]

By [22, p. 544], we have
\[
L(\varphi, \chi, 1) = \frac{\pi i}{\tau(\varphi^{-1} \chi^{-1})} L(\varphi^{-1} \chi^{-1}, 0); \quad L(\varphi, 0) = -B_1(\varphi) \quad \text{and} \quad \frac{\tau(\chi^{-1})}{\tau(\chi^{-1} \varphi^{-1})} = \frac{\chi(f) \cdot \varphi(m_\chi)}{\tau(\varphi^{-1})}.
\]

We obtain
\[
\Lambda (E_{\varphi,m,L}, \chi, 1) := \frac{\tau(\chi^{-1})}{2\pi i} L(E_{\varphi,m,L}, \chi, 1) = \frac{\tau(\chi^{-1})}{2\pi i} \chi \left( \frac{ML}{T_1 T_2} \right) \left( \prod_{l \mid T_1} (1 - \chi(l) \varphi(l) l^{-1}) \prod_{q \mid T_2} (1 - \chi(q) \varphi^{-1}(q)) L(\chi, 1) L(\chi^{-1}, 0). \right.
\]

Note that, \( \chi \in \mathcal{X}_S^{-\varphi^{-1}} \) implies \( m \equiv 3 \pmod{4} \) a prime, as a consequence, we obtain \( \chi_m(-1) = -1 \). Also \( \chi \in \mathcal{X}_S^{-\varphi^{-1}} \) implies \( \varphi \chi(-1) = -1 \). Combining these two observations, we conclude that \( \varphi^{-1} \chi^{-1} \chi_m^{-1} \) is an even character, hence \( B_1(\varphi^{-1} \chi^{-1} \chi_m^{-1}) = 0 \), from part (1) of proposition, it follows that \( \Lambda(E_{\varphi,m,L}, \chi \chi_m, 1) = 0 \). Since \( \varphi(-1) \tau(\varphi) \tau(\varphi^{-1}) = f \), the statement of part (2) of the proposition follows.

We now prove the main theorem of this paper. Again the proof follows in the same way as [22, p. 542]. We first recall the notation. Let \( N \) be a fixed integer and \( \varphi \) be a primitive Dirichlet character of conductor \( f > 1 \) such that \( f^2 \mid N \). \( M \) and \( L \) are integers such that \( (f M, L) = 1 \) and \( f^2 ML \mid N \). Let
Thus from Theorem 18, we obtain we deduce that any primitive Dirichlet character \( \chi \) and function of \( \chi \) given by \( B_1(\chi) = -L(\chi, 0), B_2(\chi) = -2L(\chi, -1) \). We consider the quantity

\[
\beta_{\Gamma_0(N), \varphi, M, L} = \frac{f^3 T_1 \phi(T_2, \varphi)}{4N} (\prod_{p \mid M} p^{r_p(M) + \delta_p}) \frac{\tau(\varphi^{-1})}{\tau(\xi^{-1})} B_2(\xi^{-1}) \prod_{p \mid f T_1} \left( 1 - \frac{\xi(p)}{p^2} \right) \in \mathbb{Q}(\zeta_f, \varphi),
\]

where \( \phi(T_2, \varphi) = \prod_{q \in S_{\varphi}} (q - 1) \) and \( \delta_p = 1 \) if \( \nu_p(M) = 0 \) and \( \nu_p(f) \geq 1 \), otherwise \( \delta_p = 0 \).

**Proof of Theorem 4.** Recall that from Theorem 10 we have

\[
\delta_{\Gamma_0(N)}(E_{\varphi, M, L}) = \beta_{\Gamma_0(N), \varphi, M, L} D_{\Gamma_0(N), \varphi, M, L}(\varphi).
\]

Let \( T := f T_1 \) and \( \tilde{\beta}_{\Gamma_0(N), \varphi, M, L} = T \beta_{\Gamma_0(N), \varphi, M, L} \).

Since \( R_{\Gamma_0(N)}(E_{\varphi, M, L}) \) is by definition the free \( \mathbb{Z} \)-module generated by the co-efficients of \( \delta_{\Gamma_0(N)}(E_{\varphi, M, L}) \), we deduce that \( R_{\Gamma_0(N)}(E_{\varphi, M, L}) = \frac{\tilde{\beta}_{\Gamma_0(N), \varphi, M, L}}{T} \mathbb{Z}[\zeta_f, \varphi] \).

By Proposition 19 for \( \chi \in \mathcal{X}_S^{-\varphi(-1)} \) we have

\[
T \Lambda_{\pm}(E_{\varphi, M, L}, \chi, 1) = \varphi(-m_\chi) \tau(\varphi) \chi \left( \frac{FML}{T_1 T_2} \right) \prod_{l \mid T_1} (1 - \chi(p(l))) \prod_{q \mid T_2} (1 - \chi(q^{-1}(q))) \frac{B_1(\chi^{-1}, \varphi^{-1})}{2} \frac{B_1(\chi \varphi^{-1})}{2}.
\]

By [22, Theorem 4.2(b)], we have

\[
\frac{B_1(\chi^{-1}, \varphi)}{2}, \frac{B_1(\chi \varphi^{-1})}{2} \in \mathbb{Z}[\varphi, \chi, \frac{1}{m_\chi}];
\]

hence

\[
(4.1) \quad \Lambda_{\pm}(E_{\varphi, M, L}, \chi, 1) \in \frac{1}{T} \mathbb{Z}[\zeta_f, \varphi, \chi, \frac{1}{m_\chi}].
\]

Thus from Theorem 18 we obtain

\[
(4.2) \quad \mathcal{P}_{\Gamma_0(N)}(E_{\varphi, M, L}) \subseteq \frac{1}{T} \mathbb{Z}[\zeta_f, \varphi] + \frac{\tilde{\beta}_{\Gamma_0(N), \varphi, M, L}}{T} \mathbb{Z}[\zeta_f, \varphi].
\]

We now show that the inclusion (4.2) is equality. For simplification of notation let \( P := \mathcal{P}_{\Gamma_0(N)}(E_{\varphi, M, L}) \) and \( M := \frac{1}{T} \mathbb{Z}[\zeta_f, \varphi] + \frac{\tilde{\beta}_{\Gamma_0(N), \varphi, M, L}}{T} \mathbb{Z}[\zeta_f, \varphi] \). For any prime \( \mathfrak{p} \mid T \), we’ll show \( P_{\mathfrak{p}} = M_{\mathfrak{p}} \).

To do that, given a prime \( \mathfrak{p} \mid T \), we choose a character \( \chi \in \mathcal{X}_S \) in a suitable way as below:

(1) For any choice of plus or minus, we can choose infinitely many Dirichlet character \( \chi \in \mathcal{X}_S^{-\varphi} \) such that \( \mathfrak{p} \mid (l_i - \chi(l_i)) \) for all \( l_i \mid T_1 \) and \( \mathfrak{p} \mid (1 - \chi(q_j)) \) for all \( q_j \mid T_2 \). This essentially follows as in the proof of [22, Theorem 4.2(a)]. For convenience of the reader, we give a sketch of the proof. Let \( t \) be a prime such that \( \mathfrak{p} \mid t \). Since \( t \)th roots of unity are distinct modulo \( \mathfrak{p} \), we see that for each \( l_i \), \( \mathfrak{p} \mid (l_i - \zeta_{t}^{a_i} \varphi(l_i)) \) for at most one choice of \( a_i \). Similarly, for each \( q_j \), \( \mathfrak{p} \mid (1 - \zeta_{t}^{b_j} \varphi(q_j)) \) for at most one choice of \( b_j \). Let \( a_1, \ldots, a_m \) be the integers such that \( \mathfrak{p} \mid (l_i - \zeta_{t}^{a_i} \varphi(l_i)) \) and \( b_1, \ldots, b_s \) be the integers such that \( \mathfrak{p} \mid (1 - \zeta_{t}^{b_j} \varphi(q_j)) \). Choose a prime \( t \in S \) large enough such that \( \mathfrak{p} \mid t \). Then there is a Dirichlet character \( \chi \in \mathcal{X}_S^{-\varphi} \) of conductor \( t \) such that \( \chi(l_i) \neq \zeta_{t}^{a_i} \) for \( i \in \{1, \ldots, m\} \) and \( \chi(q_j) \neq \zeta_{t}^{b_j} \) for \( j \in \{1, \ldots, s\} \). The result follows as there are infinitely many choices of \( t \in S \).
we conclude that:

very far from being integral.

Lemma 20.

Thus, given a prime \( \mathfrak{P} \), there exists a character \( \chi \in \mathcal{X}_S^{\varphi(-1)} \) such that \( \mathfrak{P} \nmid (l_i - \chi(l_i) \varphi(l_i)) \) for all \( l_i \mid T_i \), \( \mathfrak{P} \nmid (1 - \chi(q_j) \varphi(q_j)) \) for all \( q_j \mid T_2 \) and both \( \frac{1}{2} B_1(\varphi^{-1} \chi) \) and \( \frac{1}{2} B_1(\varphi^{-1} \chi^{-1}) \) are \( \mathfrak{P} \) unit. For this choice of \( \chi \), we see that \( \Lambda_\pm(E_{\varphi, M, L}, \chi, 1) \) is a \( \mathfrak{P} \) unit. As a consequence we see that \( P_{\mathfrak{P}} = M_{\mathfrak{P}} \). Hence

\[
P_{T_\varphi(N)}(E_{\varphi, M, L}) = \frac{1}{T} Z[\zeta_f, \varphi] + \frac{\tilde{\beta}_{T_\varphi(N), \varphi, M, L}}{T} Z[\zeta_f, \varphi].
\]

Recall that \( P_{T_\varphi(N)}(E_{\varphi, M, L}) \) is a finite generated \( \mathbb{Z} \) (PID)-module and hence so is \( R_{T_\varphi(N)}(E_{\varphi, M, L}) \).

Since the quotient module is a finite abelian group so they are of same rank.

\[
\begin{aligned}
A_{T_\varphi(N)}(E_{\varphi, M, L}) &= P_{T_\varphi(N)}(E_{\varphi, M, L})/R_{T_\varphi(N)}(E_{\varphi, M, L}) \\
&= \frac{1}{T} Z[\zeta_f, \varphi] + \frac{\tilde{\beta}_{T_\varphi(N), \varphi, M, L}}{T} Z[\zeta_f, \varphi] \\
&\cong \frac{1}{T} Z[\zeta_f, \varphi]/(\beta_{T_\varphi(N), \varphi, M, L}) \cap \frac{1}{T} Z[\zeta_f, \varphi] \\
&\cong Z[\zeta_f, \varphi]/\text{Num}(\tilde{\beta}_{T_\varphi(N), \varphi, M, L}),
\end{aligned}
\]

here \( (\tilde{\beta}_{T_\varphi(N), \varphi, M, L}) \) denotes the fractional ideal generated by \( \tilde{\beta}_{T_\varphi(N), \varphi, M, L} \) and \( \text{Num}(\tilde{\beta}_{T_\varphi(N), \varphi, M, L}) \) is the ideal \( (\tilde{\beta}_{T_\varphi(N), \varphi, M, L}) \cap Z[\zeta_f, \varphi] \). From the perfect pairing between \( C_{T_\varphi(N)}(E_{\varphi, M, L}) \) and \( A_{T_\varphi(N)}(E_{\varphi, M, L}) \) we conclude that:

\[
|C_{T_\varphi(N)}(E_{\varphi, M, L})| = |A_{T_\varphi(N)}(E_{\varphi, M, L})| = |Z[\zeta_f, \varphi]/\text{Num}(\tilde{\beta}_{T_\varphi(N), \varphi, M, L})|.
\]

\( \square \)

Although \( \tilde{\beta}_{T_\varphi(N), \varphi, M, L} \) need not be an element of \( Z[\zeta_f, \varphi] \), the following lemma shows that it is not very far from being integral.

Lemma 20. \( 12 \tilde{\beta}_{T_\varphi(N), \varphi, M, L} \in Z[\zeta_f, \varphi] \).

Proof.

\[
\tilde{\beta}_{T_\varphi(N), \varphi, M, L} = \frac{f^4 T_2^2 \phi(T_2, \varphi)}{4n^2} \left( \prod_{p|f} p^{\nu_p(M) + \delta_p} \frac{\tau(p^{-1})}{\tau(\xi^{-1})} B_2(\xi^{-1}) \prod_{p|T_1} \left(1 - \frac{\xi(p)}{p^2}\right) \right)
\]

\[
= \frac{f^4 \phi(T_2, \varphi)}{4n^2} \left( \prod_{p|f} p^{\nu_p(M) + \delta_p} \frac{\tau(p^{-1})}{\tau(\xi^{-1})} \tau(\xi) B_2(\xi^{-1}) \prod_{p|f} \left(1 - \frac{\xi(p)}{p^2}\right) \prod_{p|T_2} \left(p^2 - \xi(p)\right) \right).
\]

Note that if \( p \mid n \), then \( \xi(p) = 0 \), thus \( \prod_{p|f} \left(1 - \frac{\xi(p)}{p^2}\right) = \prod_{p|f, p|n} \left(1 - \frac{\xi(p)}{p^2}\right) \). Hence

\[
\tilde{\beta}_{T_\varphi(N), \varphi, M, L} = \frac{f^4 \phi(T_2, \varphi)}{4n^2} \left( \prod_{p|f} p^{\nu_p(M) + \delta_p} \tau(p^{-1}) \tau(\xi) \prod_{p|T_2} \left(p^2 - \xi(p)\right) \right).
\]

Note that the Gauss sums are elements of \( Z[\zeta_f, \varphi] \). Observe that \( n^2 \prod_{p|f, p|n} p^2 \) divides \( f^2 \), hence to complete the proof we need to understand \( B_2(\xi^{-1}) \).
We divide the proof in two cases.

**Case I:**
First we consider the case when \( \varphi \) is a quadratic character. Then \( \xi = 1 \) and \( B_2(\xi^{-1}) = \frac{1}{6} \).

\[
\tilde{\beta}_{\Gamma_0(N), \varphi, M, L} = \frac{f^4 \phi(T_2, \varphi)}{24 \prod_{p \mid f} p^2} \left( \prod_{p \mid f} p^{\nu(M) + \delta_p} \right) \tau(\varphi^{-1}) \prod_{p \mid f} (p^2 - 1)
\]

Since \( f > 2 \), it follows that \( \frac{f}{\prod_{p \mid f} p} \cdot \frac{1}{24} \prod_{p \mid f} (p - 1)p(p + 1) \) is an integer. As a consequence, we see that \( \tilde{\beta}_{\Gamma_0(N), \varphi, M, L} \in \mathbb{Z}[\zeta_f, \varphi] \).

**Case II:**
Now we consider the case \( \varphi^2 \neq 1 \) (this implies \( f \geq 5 \)). In this situation \( \xi \) is an primitive Dirichlet character of conductor \( n > 1 \). Note that

\[
B_2(\xi^{-1}) = n \sum_{i=1}^{n-1} \xi^{-1}(i) (\frac{i}{n} - \frac{i}{n} + \frac{1}{6}) = \frac{1}{6n} \sum_{i=1}^{n-1} \xi^{-1}(i)(6i^2 - 6in + n^2).
\]

As a consequence, we obtain

\[
\tilde{\beta}_{\Gamma_0(N), \varphi, M, L} = \frac{1}{24n^3 \prod_{p \mid f, p \mid n} p^2} \left( f \phi(T_2, \varphi) \left( \sum_{i=1}^{n-1} \xi^{-1}(i)(6i^2 - 6in + n^2) \right) \tau(\varphi^{-1}) \tau(\xi) \prod_{p \mid f} p^{\nu(M) + \delta_p} \prod_{p \mid fT_1, p \mid n} (p^2 - \xi(p)) \right).
\]

Then it follows that \( 12 \tilde{\beta}_{\Gamma_0(N), \varphi, M, L} \in \mathbb{Z}[\zeta_f, \varphi] \) (since either \( f \) is even, or if \( f \) is odd, then

\[
\sum_{i=1}^{n-1} \xi^{-1}(i)(6i^2 - 6in + n^2) = 2 \sum_{i=1}^{(n-1)/2} \xi^{-1}(i)(6i^2 - 6in + n^2)) \]

5. **Classification of non-rational Eisenstein maximal ideals**

Let \( p \) be an odd prime and \( N = p^2N' \), where \( N' \) is a square-free integer which is co-prime to \( p \). Recall that, if all the prime divisors of \( N' \) are congruent to \( \pm 1 \) modulo \( p \), then \( N \) is \( p \)-good.

Denote by \( M_2(\Gamma_0(N), \mathbb{C}) \) the complex vector space of classical modular forms as in \( \mathbb{Z} \). Let \( M_2(\Gamma_0(N), \mathbb{Z}) \) denote the set of elements \( f \in M_2(\Gamma_0(N), \mathbb{C}) \) whose Fourier coefficients at the cusp \( \infty \) are in \( \mathbb{Z} \). By \( \mathbb{Z} \), Cor. 12.3.12, Proposition 12.4.1, \( M_2(\Gamma_0(N), \mathbb{Z}) \) is stable under the action of Hecke operators and contains a basis of \( M_2(\Gamma_0(N), \mathbb{C}) \). For any \( \mathbb{Z} \) algebra \( S \), we define, \( M_2(\Gamma_0(N), S) = M_2(\Gamma_0(N), \mathbb{Z}) \otimes S \) and we denote by \( S_2(\Gamma_0(N), S) \) the corresponding space of cusp forms (the forms for which constant terms are zero).

For any integer \( n \), the Hecke operators \( T_n \) act on the the space \( S_2(\Gamma_0(N), \mathbb{Z}) \). Let \( T(N) \subseteq \mbox{End}_{\mathbb{Z}}(S_2(\Gamma_0(N), \mathbb{Z})) \) be Hecke algebra, i.e., \( \mathbb{Z} \)-algebra generated by \( T_n \) for all \( n \in \mathbb{N} \). It follows that \( T := T(N) \) is a free \( \mathbb{Z} \)-module of rank \( d \), where \( d \) is the dimension of the complex vector space \( S_2(\Gamma_0(N), \mathbb{C}) \).

Note that for a maximal ideal \( m \) of \( T \), the contraction \( m \cap \mathbb{Z} = l\mathbb{Z} \) is a non zero prime ideal of \( \mathbb{Z} \). Hence, \( l \in m \) and \( T/m \) can be identified with a finite field of characteristic \( l \), say \( \kappa(m) \). By abuse of notation, we call \( l \) the characteristic of \( m \).
Following [17, Proposition 5.1], we associate Galois representation to any arbitrary maximal ideals \( m \) of \( \mathbb{T} \). According to loc. cit., there exist an unique continuous semi-simple Galois representation

\[
\rho_m : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}_2(\mathbb{T}/m)
\]

unramified outside \( lN \) such that \( \text{Tr}(\rho_m(\text{Frob}_p)) \equiv T_p \pmod{m} \) and \( \det(\rho_m(\text{Frob}_p)) \equiv p \pmod{m} \) for all primes \( p \nmid lN \).

Recall that, this proposition in turn follows from [3, Theorem 2.1]. Using Cheboterov density theorem, we see that \( \det(\rho_m) = \overline{\chi} \), here \( \overline{\chi} \) denotes the mod \( l \) cyclotomic character.

We choose an isomorphism \( \psi' : \mathbb{T}/m \to \kappa(m) \). Using the isomorphism \( \psi' \), we can view \( \rho_m \) as a representation \( \rho'_m : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}_2(\kappa(m)) \). Note that for a different choice of isomorphism \( \psi'' : \mathbb{T}/m \to \kappa(m) \), we obtain a different representation \( \rho''_m : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}_2(\kappa(m)) \). Observe that the representations \( \rho'_m \) and \( \rho''_m \) are Galois \( \text{Gal}(\kappa(m)/\mathbb{F}_l) \) conjugate. Thus the representation \( \rho'_m \) is unique up to Galois \( \text{Gal}(\kappa(m)/\mathbb{F}_l) \) conjugate.

**Definition 21.** (Cuspform associated to maximal ideal \( m \)) We have a natural ring homomorphism

\[
T_m : \mathbb{T} \to \frac{\mathbb{T}}{m} \cong \kappa(m).
\]

and there is a non-degenerate bilinear pairing [17, Eqn. (10), Pg. 465]:

\[
S_2(\Gamma_0(N), \kappa(m)) \times \text{Hom}(\mathbb{T}, \kappa(m)) \to \kappa(m).
\]

Hence the ring homomorphism \( T_m \) give rise to an unique cusp form (depending on the chosen isomorphism between \( \mathbb{T}/m \) and \( \kappa(m) \)) \( f_m := \sum_{n \geq 1} T_m(T_n)q^n \in S_2(\Gamma_0(N), \kappa(m)) \). We call \( f_m \) the cusp form associated to the Eisenstein maximal ideal \( m \). Different choice of isomorphisms between \( \mathbb{T}/m \) and \( \kappa(m) \) will give rise to \( \text{Gal}(\kappa(m)/\mathbb{F}_l) \)-conjugate cuspforms in \( S_2(\Gamma_0(N), \kappa(m)) \). The mod \( l \) modular form \( f_m \) may lift to several modular forms of different weights, levels and nebentypus [3, Theorem 2.1].

Let us fix an isomorphism \( \psi' : \mathbb{T}/m \to \kappa(m) \) and by abuse of notation, let us denote \( \rho'_m \) by \( \rho_m \). Recall that a maximal ideal \( m \) in \( \mathbb{T} \) is called Eisenstein maximal ideal if the Galois representation \( \rho_m : G_Q \to \text{GL}_2(\kappa(m)) \) associated to \( m \) is reducible. Since it is semi-simple, there exist continuous characters \( \vec{\alpha}_m, \vec{\beta}_m : G_Q \to \kappa(m)^\times \) such that

\[
\rho_m \simeq \vec{\alpha}_m \bigoplus \vec{\beta}_m.
\]

We classify the Eisenstein maximal ideals of \( \mathbb{T} \) in the following proposition. This is a generalisation of [19, Lemma 0.1]. For any prime \( p \), let \( G_p \) be the decomposition group at the prime \( p \).

**Proposition 22.** Let \( p \) be an odd prime and \( N = p^2N' \), where \( N' \) is a square-free integer which is coprime to \( p \). Let \( m \) be a maximal ideal of \( \mathbb{T} = \mathbb{T}(N) \). Assume that \( l \nmid 6p \), here \( l \) denotes the residual characteristic of \( m \). Then \( m \) is Eisenstein if and only if there exist there exists a unique character \( \tau_m : G_Q \to \kappa(m)^\times \) unramified outside \( p \) such that

\[
\rho_m \simeq \tau_m \oplus (\overline{\tau}_m)^{-1} \chi_l.
\]

**Proof.** If \( \overline{\tau} : G_Q \to \kappa(m)^\times \) is a continuous character, by [20, Prop 21.6.3, p.223], then \( \overline{\tau} = \overline{\theta} \chi_i \) for some integer \( i \) and some character \( \overline{\theta} \) unramified at \( l \). Since \( \rho_m = \vec{\alpha}_m \bigoplus \vec{\beta}_m \), for some continuous character \( \vec{\alpha}_m, \vec{\beta}_m : G_Q \to \kappa(m)^\times \), it follows that there are unramified (at \( l \)) characters \( \tau_1, \tau_2 : G_Q \to \kappa(m)^\times \) and some integer \( i \) and \( j \) such that

\[
\rho_m \simeq \begin{cases} \tau_1 \chi_i & \text{if } \overline{\tau}_1 \chi_i \neq 1 \\ 0 & \text{if } \overline{\tau}_1 \chi_i = 1 \end{cases}.
\]
Since $\chi_l = \det(\rho_m) = \tau_1 \tau_2 \chi_l^{i+j}$, it follows that $i + j \equiv 1 \pmod{l - 1}$ and $\tau_1 = (\tau_2)^{-1}$. Let $\tau_m := \tau_2$. Note that $\tau_m$ is unramified at $l$.

Let $f_m \in S_2(\Gamma_0(N), \kappa(m))$ be the cusp form associated to the Eisenstein maximal ideal $m$ (Definition 21). Let $\rho_{f_m}$ be the unique two dimensional semisimple odd Galois representation (continuous) $\rho_{f_m} : G_Q \to \text{GL}_2(\overline{\mathbb{F}_l})$ associated to $f_m$. Observe that $\rho_{f_m}$ and $\rho_m$ have same characteristic polynomial at $\text{Frob}_r$ for all prime $r \nmid lN = lp^2N'$. By Brauer-Nesbitt theorem, for any prime $q$, we have $(\rho_{f_m})^{a_q} \mid G_q \simeq \rho_m \mid G_q$. Recall that $N'$ is squarefree, it follows that $\rho_{f_m}$ is semistable outside $lp$ [12, Proposition 14.1, p. 113]. It follows that $\tau_m \chi_l$ and $(\tau_m)^{-1} \chi_l$ are unramified outside $pl$. By applying [9, Theorem 2.5] and noting that $(\rho_{f_m})^{a_q} \mid G_q \simeq \rho_m \mid G_q$, we deduce that $\tau_m$ is unramified outside $p$ as $p \neq l$.

Let $a_l$ denotes the $T_l$ eigenvalue of $f_m$. By our assumption, $l \geq 5$ and $l \nmid 6p$. If $l \nmid N'$, then as $\rho_{f_m} \mid G_q$ is reducible, we get that $a_l = 0$, as otherwise it will contradict Fontaine’s theorem [9, Theorem 2.6]. Hence from Deligne’s theorem [9, Theorem 2.5] we get $$(\rho_{f_m})_{|\text{Frob}_l} = \left(\frac{\lambda(a_l^{-1}) \chi_l}{\lambda(a_l)}\right),$$
where $\lambda(x)$ denotes the unramified character which sends $\text{Frob}_l$ to $x$. On the other hand if $l \mid N'$, we cannot directly apply [9, Theorem 2.5, Theorem 2.6]. We use the following trick of Ribet as described in [19]. By [18, Theorem 2.1, Step 4] we can replace mod $l$ cusp form $f_m \in S_2(\Gamma_0(N), \kappa(m))$ by some cusp form $g_m \in S_{l+1}(\Gamma_0(N/l), \kappa(m))$ as $f_m$ and $g_m$ have same mod $l$ Galois representation. Now as $l \nmid N/l$, using [9, Theorem 2.5, Theorem 2.6], we obtain
$$\rho_{g_m}|_{\text{Frob}_l} = \left(\frac{\lambda(a_l^{-1}) \chi_l}{\lambda(a_l)}\right).$$
Note that $\chi_l = \chi_l$. As a consequence, in either case we may assume that $\overline{\tau}_m = \tau_m^{-1} \chi_l$ and $\overline{\rho}_m = \tau_m$. □

Remark 23. For a different choice of isomorphism between $\mathbb{F}/m$ and $\kappa(m)$, we’ll obtain a different character $\overline{\tau}_m : G_Q \to \kappa(m)^\times$, but $\tau_m$ and $\overline{\tau}_m$ will be Galois $\text{Gal}(\kappa(m)/\mathbb{F})$ conjugate.

Corollary 24. Let $m$ be an Eisenstein maximal ideal of $T = \mathbb{F}(N)$. Assume that the residual characteristic $l$ of $m$ does not divide $6p$. Then for any prime $r \nmid N$, we have
$$T_r \equiv \tau_m(r) + r(\overline{\tau}_m)^{-1}(r) \pmod{\rho_m}.$$

Proof. The proof is essentially contained in Proposition 22.

If $r \nmid lN$, the result follows from the fact that $\rho_{f_m}$ and $\rho_m$ have same trace at $\text{Frob}_r$.

For $r = l \nmid 6p$, the proof follows from the proof of Proposition 22 which gives us $T_l \pmod{\rho_m} = a_l = \tau_m(l) = \tau_m(l) + (\overline{\tau}_m)^{-1}(l)$, as shown in in [19, Lemma 1.1]. □

As in [20, Proposition 21.6.3, p. 223-224], we can consider $\tau_m$ as a Dirichlet character $(\mathbb{Z}/p\mathbb{Z})^\times \to \kappa(m)^\times$.

Definition 25 (Character associated to Eisenstein maximal ideals). Let $p$ be an odd prime and $N = p^2N'$, where $N'$ is a square-free integer which is coprime to $p$. Let $m$ be a maximal ideal of $T = \mathbb{F}(N)$. Assume that the residual characteristic of $m$ is co-prime to $6p$. For a fixed choice of isomorphism between $\mathbb{F}/m$ and $\kappa(m)$, the character $\tau_m$ is called the Dirichlet character associated to the Eisenstein maximal ideal
m. 

Observe that a Eisenstein maximal ideal m is rational if the associated Dirichlet character \( \tau_m \) is trivial and is non-rational if the associated Dirichlet character \( \tau_m \) is non-trivial.

Let \( m \) be a non-rational Eisenstein maximal ideal of level \( N = p^2 N' \) (\( N' \) is square-free, co-prime to \( p \)) and residual characteristic of \( m \) is \( l \nmid 6p \). We fix an isomorphism \( \psi' : \mathbb{T}/m \to \kappa(m) \). Let \( \tau_m \) be the (non-trivial) Dirichlet character of conductor \( p \) associated to \( m \) as in definition \( \text{25} \). For a prime \( r \nmid N \), let \( f_r(X) \in \mathbb{F}_l[X] \) be the minimal polynomial of \( \tau_m(r) + r(\tau_m)^{-1}(r) \in \kappa(m) \) over \( \mathbb{F}_l \). Note that the polynomial \( f_r(X) \) does not depend on the choice of the isomorphism \( \psi' \). Let \( f_r(X) \in \mathbb{Z}[X] \) be a monic (necessarily irreducible) polynomial such that \( f_r(X) \equiv f_r(X) \mod l \).

**Corollary 26.** Let \( m \) be a non-rational Eisenstein maximal ideal of \( N = p^2 N' \) (\( N' \) is square-free, co-prime to \( p \)) and residual characteristic of \( m \) is \( l \nmid 6p \). Let 

\[
I_m(N) = (f_r(T_r), r \text{ primes such that } r \nmid N)
\]

Then \( m \supset (l, I_m(N)) \).

**Proof.** The ideal \((l, I_m(N))\) is independent of the choice of \( f_r(X) \) as any lift of \( f_r(X) \) satisfies \( f_r(X) \equiv f_r(X) \mod l \). We have a map,

\[
\mathbb{T}(N) \to \mathbb{F}(N)/m \cong \kappa(m) \text{ sending } T_r \mapsto \tau_m(r) + r(\tau_m)^{-1}(r) \text{ for primes } r \nmid N.
\]

It is obvious that \((l, I_m(N))\) is in the kernel of the map (for any choice isomorphism between \( \mathbb{T}/m \) and \( \kappa(m) \)) and hence \( m \supset (l, I_m(N)) \). \( \square \)

For rest of the section we assume that \( N \) is \( p \)-good, that is, there exists an odd prime \( p \) and a square-free integer \( N' \) whose all the prime divisors of are congruent to \( \pm 1 \mod 6p \) and \( N = p^2 N' \). We now describe non-rational Eisenstein maximal ideals of level \( N \) whose residual characteristic \( l \) is co-prime to \( 6p \).

First we recall definition of \( q \)-old and \( q \)-new Eisenstein ideal [\text{30}, Section 2]. For any prime \( q \mid N \), there are two degeneracy coverings \( \alpha_q(N/q), \beta_q(N/q) : X_0(N) \to X_0(N/q) \) given by \( \alpha_q(N)(z) = z \mod \Gamma_0(N/q) \) and \( \beta_q(N)(z) = qz \mod \Gamma_0(N/q) \). Let \( z \in H \cup \mathbb{P}^1(q) \) and we identify \( X_0(N) \) (resp. \( X_0(N/q) \)) with \( \Gamma_0(N) \backslash H \cup \mathbb{P}^1(q) \) (resp. \( \Gamma_0(N/q) \backslash H \cup \mathbb{P}^1(q) \)). This two degeneracy maps in turn define two maps \( \alpha_q(N/q)\ast, \beta_q(N/q)\ast : J_0(N/q) \to J_0(N) \).

Consider the map \( \gamma_q(N)\ast : J_0(N/q) \times J_0(N/q) \to J_0(N) \) given by \( \gamma_q(N)(x,y) = \alpha_q(N/q)\ast(x) + \beta_q(N/q)\ast(y) \). Image of \( \gamma_q(N)\ast \) is called the \( q \)-old subvariety of \( J_0(N) \), denoted by \( J_0(N)q^{\text{old}} \). The quotient of \( J_0(N) \) by \( J_0(N)q^{\text{old}} \) is called the \( q \)-new subvariety of \( J_0(N) \), denoted by \( J_0(N)q^{\text{new}} \).

The map \( \gamma_q(N)\ast \) is Hecke-equivariant. The image of \( \mathbb{T}(N) \) in \( \text{End}(J_0(N)q^{\text{old}}) \) (resp. \( \text{End}(J_0(N)q^{\text{new}}) \)) is denoted by \( \mathbb{T}(N)q^{\text{old}} \) (resp. \( \mathbb{T}(N)q^{\text{new}} \)). A maximal ideal of \( \mathbb{T}(N) \) is called \( q \)-old (resp. \( q \)-new) if its image in \( \mathbb{T}(N)q^{\text{old}} \) (resp. \( \mathbb{T}(N)q^{\text{new}} \)) is still maximal. Thus, any maximal ideal \( m \) of \( \mathbb{T}(N) \) is either \( q \)-old or \( q \)-new or both.

Now let \( N \) be a \( p \)-good integer for an odd prime \( p \). For a prime \( q \mid N \), let \( U_q \) (resp. \( \tau_q \)) denote the Hecke operator at \( q \) of level \( N \) (resp. \( q \)). Then from [\text{30}, Eqn (2.2), Eqn (2.5)] it follows that

\[
U_p = 0 \in \mathbb{T}^{p^{\text{new}}}, U_p^2 - \tau_p U_p \in \mathbb{T}^{p^{\text{old}}} \text{ and } U_p, w_p = 0 \in \mathbb{T}^{p^{\text{new}}}, U_p^2 - \tau_p U_p + p = 0 \in \mathbb{T}^{p^{\text{old}}}, \text{ here } p_i \mid N/p^2 \text{ a prime and } w_p, \text{ denotes the Atkin-Lehner involution.}
\]

**Lemma 27.** Let \( N = p^2 \) be \( p \)-good and \( m \) a non-rational Eisenstein maximal ideal of \( \mathbb{T} \) with residual characteristic co-prime to \( 6p \). Then \( m \) is \( p \)-new and hence \( U_p \in m \).
Proof. If \( m \) is \( p \)-new, then from (5.1), it follows that \( U_p \in m \) as \( U_p = 0 \in \mathbb{T}^{p-new} \).

Suppose that \( m \) is \( p \)-old, then there exists a maximal ideal \( n \) in \( \mathbb{T}(N/p) \) corresponding to \( m \) [30, Lemma 2.2], [17, Section 7]. Since \( N/p \) is square-free and \( n \) is an Eisenstein maximal ideal of \( \mathbb{T}(N/p) \), from [29, Proposition 2.1] it follows that \( n \) is rational Eisenstein ideal, that is \( \rho_n = 1 \oplus \mathfrak{f}_1 \). Contradiction. \( \square \)

Observe that as \( N = p^2N' \) is \( p \)-good, for any prime \( p_i \mid N' \), we have \( \tau_m(p_i) = \pm 1 \in \mathbb{F}_l \) as \( p_i = \pm 1 \pmod{p} \) and conductor of \( \tau_m \) is \( p \). We define \( \epsilon_m(p_i) = 1 \) if \( \tau_m(p_i) = 1 \in \mathbb{F}_l \) and we define \( \epsilon_m(p_i) = -1 \) if \( \tau_m(p_i) = -1 \in \mathbb{F}_l \).

**Lemma 28.** Let \( N = p^2N' \) be \( p \)-good and let \( p_i \mid N' \). Let \( m \) be a non-rational Eisenstein ideal of \( \mathbb{T} \) with residual characteristic co-prime to \( 6p \). Then either \( U_{p_i} - \epsilon_m(p_i) \in m \) or \( U_{p_i} - \epsilon_m(p_i)^{-1} \in m \).

**Proof.** We proceed as in [28, Lemma 2.1].

First, suppose that \( m \) is \( p_i \)-old. Let \( R \) be the common subring of \( \mathbb{T}(N/p_i) \) and \( \mathbb{T}(N)^{p_i-old} \). Let \( n \) be the corresponding maximal ideal of \( \mathbb{T}(N/p_i) \) to \( m \). Let \( \tau_{p_i} \) denote the \( p_i \)-th Hecke operator in \( \mathbb{T}(N/p_i) \). Then

\[
\mathbb{T}(N/p_i) = R[\tau_{p_i}], \quad \mathbb{T}(N)^{p_i-old} = R[U_{p_i}] \quad \text{and} \quad \mathbb{T}(N)/m \cong \mathbb{T}(N/p_i)/n.
\]

Note that \( n \) is an Eisenstein ideal of \( \mathbb{T}(N/p_i) \), it follows that \( I_m(N/p) \subset n \). Note that \( p_i \nmid N/p_i \) and \( p_i \equiv \pm 1 \pmod{l} \), hence \( \tau_{m}(p_i) - p_i(\tau_{m}^{-1}(p_i)) \in \mathbb{F}_l \). It follows that \( \tau_{p_i} - \epsilon_m(p_i) - p_i(\epsilon_m(p_i))^{-1} \in n \). From (5.1), we have \( U_{p_i}^2 - \tau_{p_i}U_{p_i} + p_i = 0 \). As a consequence, we see that in the ring \( \mathbb{T}(N)/m \cong \mathbb{T}(N/p_i)/n \) we have \( U_{p_i}^2 - (\tau_{m}(p_i) + p_i(\tau_{m}^{-1}(p_i)))U_{p_i} + p_i = (U_{p_i} - \tau_{m}(p_i))(U_{p_i} - p_i(\tau_{m}^{-1}(p_i))) = 0 \).

If \( l = p_i \mid N' \) and \( l \nmid 6p \), then from the proof of corollary 24, it follows that \( U_{p_i} \equiv \tau_m(p_i) \pmod{m} \). Hence for the remaining proof we assume that \( l \nmid 6N \).

Now suppose that \( m \) is \( p_i \)-new. As \( U_{p_i} + w_{p_i} = 0 \in \mathbb{T}^{p-new} \), it follows that \( U_{p_i} - a_{p_i} \in m \), where \( a_{p_i} \in \{1, -1\} \). We want to show

\[
a_{p_i} \equiv \tau_{m}(p_i) \quad \text{or} \quad p_i(\tau_{m}^{-1}(p_i)) \pmod{m}.
\]

The congruence is obvious if \( a_{p_i} = \epsilon_m(p_i) \). We need to show the above mentioned congruence holds even if \( a_{p_i} = -\epsilon_m(p_i) \). From now on, we assume that \( a_{p_i} = -\epsilon_m(p_i) \).

Since the residual characteristic \( l \) of \( m \) is odd and \( l \neq p_i \) and \( p_i \mid N \), from [4, Theorem 3.1(e)], it follows that \( (\rho_f \mid _{G_{p_i}})^{s,a} = \lambda(a_{p_i}) \mathfrak{f}_l \otimes \lambda(a_{p_i}) \), here \( \lambda(a_{p_i}) \) is the unramified quadratic character which sends \( \text{Frob}_{p_i} \) to \( a_{p_i} \). Since \( \rho_m = \tau_{m} \otimes (\tau_{m}^{-1} \mathfrak{f}_l) \) and \( a_{p_i} \neq \tau_m(p_i) \), it follows that

\[
\tau_m(p_i) = -a_{p_i} \equiv p_i a_{p_i} \pmod{m} \quad \Rightarrow \quad p_i \equiv -1 \pmod{m} \quad \Rightarrow \quad p_i \tau_m^{-1}(p_i) = p_i a_{p_i} \equiv a_{p_i} \pmod{m}.
\]

**Corollary 29.** (Classification of Eisenstein maximal ideals) Let \( N = p^2N' \) be \( p \)-good for an odd prime \( p \) and \( m \) be a non-rational Eisenstein ideal of \( \mathbb{T}(N) \) whose residual characteristic is co-prime to \( 6p \). Then \( m \) contains ideal

\[
I_{m,M}(N) = (I_m(N), U_{p_i}, U_s - s(\epsilon_m(s))^{-1}) \quad \text{for primes} \quad s \mid M, U_q - \epsilon_m(q) \quad \text{for primes} \quad q \mid \frac{N'}{M}
\]

for some divisor \( M \) of \( N' \). Hence \( m = (l, I_{m,M}(N)) \) and \( \kappa(m) \cong \mathbb{T}/m \cong \mathbb{F}_l[\mathfrak{f}_m] \).

**Proof.** The result follows from Corollary 26, Lemma 27 and Lemma 28. \( \square \)
Remark 30. Note that for any prime \( q \mid \frac{N'}{m} \) such that \( q \) is congruent to 1 modulo the residual characteristic of \( m \), we have \( I_{m,M}(N) = I_{m,Mq}(N) \). Thus from now on we assume that \( \frac{N'}{m} \) does not have a prime divisor which is congruent to 1 modulo the residual characteristic of \( m \).

6. Ribet’s conjecture for non-rational Eisenstein series

For this section, let us assume that \( N \) is \( p \)-good for some odd prime \( p \) and \( m \) a non-rational Eisenstein maximal ideal of \( \mathbb{T}(N) \) whose residual characteristic is co-prime to \( 6p \). Let us fix an isomorphism \( \psi' \) between \( T/m \) and \( \kappa(m) \) and \( \rho_m : \mathbb{C}_0 \to \text{GL}_2(\kappa(m)) \) be the representation associated to \( m \) and let \( \tau_m : (\mathbb{Z}/p\mathbb{Z})^\times \to \kappa(m)^\times \) be the non-trivial Dirichlet character of conductor \( p \) associated to \( m \). Let \( \alpha \) be a generator of \( (\mathbb{Z}/p\mathbb{Z})^\times \) and \( k \geq 2 \) be the order of \( \tau_m \) and \( \tau_m(a) = \alpha \). Since \( k \) is the order of the character, it follows that \( k \mid |\kappa(m)| - 1 \) and \( k \mid (p - 1) \), in particular, \( k \) is co-prime to \( p \) and \( l \). Let \( \zeta_k \in \mathbb{C} \) denotes a fixed primitive \( kl \)th root of unity. Since both \( \alpha \) and \( \zeta_k \) satisfies the \( kl \)th cyclotomic polynomial, it follows that there exists a prime \( L \) in \( \mathbb{Z}[\zeta_k] \) lying above \( l \) and an isomorphism \( \mathbb{Z}[\zeta_k]/(L) \to \mathbb{F}_l[[\tau]] \) which sends \( \zeta_k \mapsto \alpha \). By \( \epsilon_m \) we denote the character \( \epsilon_m : (\mathbb{Z}/p\mathbb{Z})^\times \to \mathbb{C}^\times \) given by \( \epsilon_m(a) = \zeta_k \). It follows that \( \epsilon_m \) is an unramified Dirichlet character such that for all primes \( r \neq p \), \( \epsilon_m(Frob_r) \equiv \tau_m(Frob_r) \pmod{L} \). Note that, instead of choosing a \( k \)-th complex root of unity \( \zeta_k \), we may choose a \( kl \)-th complex root of unity \( \zeta_{kl} \) and proceed similarly.

Definition 31 (Eisenstein series associated to Eisenstein maximal ideal). Let \( N \) be \( p \)-good integer for some odd prime \( p \) and \( m \) be a non-rational Eisenstein ideal of \( \mathbb{T}(N) \) whose residual characteristic is co-prime to \( 6p \). Let us fix an isomorphism \( \psi' \) between \( T/m \) and \( \kappa(m) \) and let \( \tau_m : (\mathbb{Z}/p\mathbb{Z})^\times \to \kappa(m)^\times \) be the Dirichlet character associated to \( m \) (w.r.t. \( \psi' \)). Let \( \epsilon_m : (\mathbb{Z}/p\mathbb{Z})^\times \to \mathbb{C}^\times \) be the lift of \( \tau_m \) as described above.

By Corollary 29 there exists a divisor \( M \) of \( N' = N/p^2 \) such that \( M \supseteq I_{m,M}(N) \). We choose largest such \( M \).

By Eisenstein series associated to \( m \) (w.r.t. \( \psi' \)), we understand the non-rational Eisenstein series \( E_{\epsilon_m,M,\frac{N'}{m}} \in E_2(\Gamma_0(N)) \). We denote the same by \( E_m \).

Remark 32. Note that if we choose a different isomorphism \( \psi' : T/m \to \kappa(m) \), then \( \tau_m \) gets replaced by a \( \text{Gal}(\kappa(m)/\mathbb{F}_l) \) conjugate character. As a consequence \( \epsilon_m \) is then also replaced by a \( \text{Gal}(\mathbb{Q}(\zeta_k)/\mathbb{Q}) \) conjugate character. We see that the associated Eisenstein series is unique up to \( \text{Gal}(\mathbb{Q}(\zeta_k)/\mathbb{Q}) \) conjugates.

For remaining of the section we use the following notations. Let \( K = \mathbb{Q}(\zeta_p) \), \( F = \mathbb{Q}(\epsilon_m) := \mathbb{Q}(\zeta_k) \) and \( L = EK \) be the cyclotomic number fields and \( \mathcal{O}_K = \mathbb{Z}[\zeta_p] \), \( \mathcal{O}_F := S = \mathbb{Z}[\epsilon_m] \) and \( \mathcal{O}_L := R = \mathbb{Z}[\zeta_k, \epsilon_m] \) be the corresponding ring of integers. We denote by \( \Omega_S(N) := \Omega(N) \otimes_{\mathbb{Z}} S \) and \( \Omega_L(N) := \Omega(N) \otimes_{\mathbb{Z}} R \).

Observe that we have a map \( \psi : \Omega_S(N) \to \mathbb{Z}[\epsilon_m] \to \mathbb{F}_l[[\tau]] \) where the first map is the evaluation map given by \( T_r \to \epsilon_m(r) + r^{-1} \epsilon_m^{-1}(r) \) for primes \( r \nmid N, U_p \to 0, U_s \to s \epsilon_m^{-1}(s) \) for primes \( s \mid M \) and \( U_q \to \epsilon_m(q) \) for primes \( q \mid \frac{N'}{M} \) and the second map is going modulo \( L \) given by \( \zeta_k \mapsto \alpha \). It follows that \( Ker(\psi) = (L, I_m,M) \) where \( I_m,M \) is the ideal \( I_m,M = (T_r - \epsilon_m(r) - r^{-1} \epsilon_m^{-1}(r) \) for primes \( r \nmid N, U_p, U_s - s \epsilon_m^{-1}(s) \) for primes \( s \mid M, U_q - \epsilon_m(q) \) for primes \( q \mid \frac{N'}{M} \) and moreover \( \Omega_S(N)/Ker(\psi) \cong \Omega[[\tau]] \cong \Omega(N)/\mathfrak{m} \). It follows that \( Ker(\psi) \supseteq \mathfrak{m} \mathcal{S}_N(N) \) and \( I_m,M \supseteq I_m,M \mathcal{S}_N(N) \).

Lemma 33. Let \( N \) be \( p \)-good integer for some odd prime \( p \) and \( m \) be a non-rational Eisenstein ideal of \( \mathbb{T}(N) \) whose residual characteristic is co-prime to \( 6p \) with associated Dirichlet character \( \tau_m \). There exists a non-zero
ideal $J$ of $S = \mathbb{Z}[\epsilon_m]$ such that

$$S/J \cong \mathbb{T}_S(N)/\mathbb{I}_{m,M}.$$

In particular, $|\mathbb{T}_S(N)/\mathbb{I}_{m,M}| = N_{F/\mathbb{Q}}(J)$.

Proof. We have a natural surjective map

$$\kappa : S = \mathbb{Z}[\epsilon_m] \to \mathbb{T}_S(N)/\mathbb{I}_{m,M} \text{ given by } z \mapsto z \pmod{\mathbb{I}_{m,M}}.$$  

Suppose $\mathbb{T}_S(N)/\mathbb{I}_{m,M} \cong \mathbb{Z}[\epsilon_m]$, then there exists a weight 2 cusp form over $\mathbb{Z}[\epsilon_m] \subset \mathbb{C}$ with $T_r$ eigenvalue $a_r(f) = \epsilon_m(r) + r\epsilon_m^{-1}(r)$ for all primes $r \nmid N$ and hence in particular for some prime $r \geq 7$. By Ramanujan bound, $|a_r(f)| \leq 4r$. Writing $\epsilon_m(r) = x + iy$ with $x^2 + y^2 = 1$, the Ramanujan bound gives us

$$1 + r^2 + 2r(2x^2 - 1) \leq 4r \text{ or equivalently } (1-r)^2 \leq 4r(1-x^2) \leq 4r.$$ 

Note that as $r \geq 7$, $(1-r)^2 > 4r$, hence the cuspform will not satisfy Ramanujan bound. As a consequence, we obtain that $|\mathbb{T}_S(N)/\mathbb{I}_{m,M}| \cong \mathbb{Z}[\epsilon_m]/J$ for some non-zero ideal $J$. □

Since $\mathbb{Z}[\zeta_p]$ is a flat $\mathbb{Z}$-module, by tensoring the exact sequence $0 \to J \to S \to \mathbb{T}_S(N)/\mathbb{I}_{m,M} \to 0$ by $\mathbb{Z}[\zeta_p]$, we obtain $R/JR \cong \mathbb{T}_R(N)/\mathbb{I}_{m,M}\mathbb{T}_R(N)$, here $R = \mathbb{Z}[\zeta_p, \epsilon_m]$. Hence $|\mathbb{T}_R(N)/\mathbb{I}_{m,M}\mathbb{T}_R(N)| = N_{L/\mathbb{Q}}(JR)$. We will relate this to the cardinality of the finite cyclic group $C_{\Gamma_0(N)}(E_m)$.

For a prime $q$, let

$$|(\mathbb{T}_R(N)/\mathbb{I}_{m,M}\mathbb{T}_R(N))[q^{\infty}]| = q^{\alpha(q)} \text{ and } |C_{\Gamma_0(N)}(E_m))[q^{\infty}]| = q^{\beta(q)}.$$  

We will show that $\alpha(q) = \beta(q)$ if $q \nmid 6p$.

Lemma 34. Let $N$ be $p$-good integer for some odd prime $p$ and $m$ be a non-rational Eisenstein ideal of $\Gamma(N)$ whose residual characteristic is co-prime to $6p$. The ideal $\mathbb{I}_{m,M}$ annihilates the cuspidal subgroup $C_{\Gamma_0(N)}(E_m)$.

Proof. Recall that the $C_{\Gamma_0(N)}(E_m)$ is generated by the divisor $D_{\Gamma_0(N),M}/\mathbb{I}_{m,M} \in \text{Div}^0(X_0(N),R)$ (definition[12] and remark[13]). Since $\delta_{\Gamma_0(N)}(E_m) = \lambda D_{\Gamma_0(N),M}/\mathbb{I}_{m,M}$ and since for all prime $q$, $\delta_{\Gamma_0(N)}(T_q E_m) = T_q \delta_{\Gamma_0(N)}(E_m)$, the result follows.

Since $\mathbb{I}_{m,M} R$ annihilates $C_{\Gamma_0(N)}(E_m) \cong \mathbb{Z}/t\mathbb{Z}$ (say), we see that $\mathbb{Z}/t\mathbb{Z} \cong \text{End}(C_{\Gamma_0(N)}(E_m))$ is a $\mathbb{T}_R(N)/\mathbb{I}_{m,M}\mathbb{T}_R(N)$ module. Thus $t \in JR$, which implies

$$t = |C_{\Gamma_0(N)}(E_m)| \leq |R/JR| = |\mathbb{T}_R(N)/\mathbb{I}_{m,M} R|.$$

We conclude that $\alpha(q) \geq \beta(q)$ for all $q$.

Theorem 35. Let $N$ be $p$-good integer for some odd prime $p$ and $m$ be a non-rational Eisenstein ideal of $\Gamma(N)$ whose residual characteristic is co-prime to $6p$. For every prime $q \nmid 6p$, we have

$$|(\mathbb{T}_R(N)/\mathbb{I}_{m,M}\mathbb{T}_R(N))[q^{\infty}]| = |C_{\Gamma_0(N)}(E_m))[q^{\infty}]|.$$  

Proof. To show $\alpha(q) = \beta(q)$, it is enough to show $\alpha(q) \leq \beta(q)$. This is obvious if $\alpha(q) = 0$. From now on we assume that $\alpha(q) \geq 1$ or equivalently $q \mid N_{L/\mathbb{Q}}(JR)$, hence $J$ is an ideal of $S$ as in lemma[33]. Using the unique factorisation of ideals in Dedekind domain $R$, we write $JR = J_q J_{q'}$, where prime factorisation of $J_q$ (resp. $J_q$) does not contain any prime above $q$ (resp. contains primes only above $q$). Consider the ideal $\mathbb{I} = (J_q, \mathbb{I}_{m,M}) \subset \mathbb{T}_R(N)$. Note that $\mathbb{T}_R(N)/\mathbb{I} \cong (R/JR)/J_q(R/JR) \cong R/J_q R$. Thus, $|R/J_q R| = |\mathbb{T}_R(N)/\mathbb{I}| = |(\mathbb{T}_R(N)/\mathbb{I}_{m,M}\mathbb{T}_R(N))[q^{\infty}]| = q^{\alpha(q)}$. 


Consider the cuspform $F \in S_2(\Gamma_0(N), \mathbb{T}_R(N)/\mathcal{I}) = S_2(\Gamma_0(N), R/J_q)$ whose Fourier expansion at the cusp $i\infty = \left[ \frac{1}{N} \right]_{\Gamma_0(N)}$ is given by

$$F = \sum_{n \geq 1} (T_n \mod \mathcal{I}) q^n.$$

From the divisor associated to the Eisenstein series $E_m$ it is obvious that the constant term of $E_m$ at the cusp $i\infty = \left[ \frac{1}{N} \right]_{\Gamma_0(N)}$ is 0 and hence it follows that the Fourier expansion of $E_m$ has coefficients in $R$, that is $E_m \in M_2(\Gamma_0(N), R)$. We now consider the modular form

$$G = F - E_m \mod J_q \in M_2(\Gamma_0(N), R/J_q).$$

Note that $q$-expansion of $G$ at the cusp $i\infty$ is 0. Therefore $G$ is identically 0 in the connected component of $C$ of $X_0(N)_{/R}$ containing the cusp $i\infty = \left[ \frac{1}{N} \right]_{\Gamma_0(N)}$.

For any natural number $N \in \mathbb{N}$, denote by $\lambda_0(N)$ the model of $X_0(N)$ over $\mathbb{Z}[\frac{1}{N}]$.

First we assume $q \nmid 6N$. In this situation, $\lambda_0(N)_{/R}$ is connected. It follows that the constant Fourier coefficient of $E_m \mod J_q'$ at any cusp of $X_0(N)$ is 0 and hence it follows that the Fourier expansion of $E_m$ has coefficients in $R$, that is $E_m \in M_2(\Gamma_0(N), R)$.

Next we assume that $q \geq 5$ be a prime such that $q \mid N$, then $N = qR$ with $(q, R) = 1$. The scheme $\lambda_0(N)_{/R} = \lambda_0(R)_{/\mathcal{I}} \cup \lambda_0(R)_{/\mathcal{I}}$ (cf. [5, Theorem 6.9, p. 286]). Let us denote one of the component containing $i\infty = \left[ \frac{1}{N} \right]_{\Gamma_0(N)}$ (respectively 0) to be $\lambda_0(R)_{/\mathcal{I}}$ (respectively $\lambda_0(R)_{/\mathcal{I}}$). Note that the constant Fourier coefficient of $E_m \mod J_q'$ at any cusp in the connected component of $i\infty$ of $X_0(N)$ is 0 and hence it follows that the Fourier expansion of $E_m$ has coefficients in $R$, that is $E_m \in M_2(\Gamma_0(N), R)$. We conclude that the constant Fourier coefficient of $E_m \mod J_q$ at the cusp $\left[ \frac{1}{P} \right]_{\Gamma_0(N)}$ is 0 in both of these cases.

Since $\delta_{\Gamma_0(N)}(E_m) = \sum_{x \in \text{Cusps}(\Gamma_0(N))} e_{\Gamma_0(N)}(x) \cdot a_0(E \mid \theta的应用 \{x\}) = \beta_{\Gamma_0(N), \epsilon, M, \frac{N'}{N}} D_{\Gamma_0(N), M, \frac{N'}{N}}(\phi)$, comparing the coefficient at the cusp $\left[ \frac{1}{P} \right]_{\Gamma_0(N)}$ we obtain

$$a_0(E_{\epsilon, M, \frac{N'}{N}} \mid \left[ \frac{1}{P} \right]_{\Gamma_0(N)}) = \frac{\beta_{\Gamma_0(N), \epsilon, M, \frac{N'}{N}}}{e_{\Gamma_0(N)}(\left[ \frac{1}{P} \right]_{\Gamma_0(N)})} = \frac{\beta_{\Gamma_0(N), \epsilon, M, \frac{N'}{N}}}{N'}.$$
Since the constant Fourier coefficient of $E_m \pmod{J_q}$ at the cusp $\frac{1}{p} \Gamma_0(N)$ is 0, we see that $\text{Num}(a_0(E_m | \Gamma_0(N))$ is a non-rational Eisenstein ideal of $\mathbb{T}(N)$. By corollary \[29\] $m = (l, I_{m,M})$. Now $\mathfrak{F}[\mathfrak{T}] \cong \mathfrak{T}(N)/m \cong \mathfrak{F}(\mathfrak{S}(N))/\mathfrak{L}, I_{m,M} \cong (S/J)/\mathfrak{L}(S/J)$. Thus $l \mid |(S/J) | = N_{F/Q}(J)$ and hence $l \mid |(S/J) | = \mathfrak{T}(N)/I_{m,M} \mathfrak{T}(N)$. From Theorem \[35\] it follows that $l \mid |C_{\Gamma_0(N)}(E_m)|$. Since $I_{m,M}$ annihilates $C_{\Gamma_0(N)}(E_m)$ and $I_{m,M} \supset I_{m,M} R$, it follows that $C_{\Gamma_0(N)}(I_{m,M} \supset C_{\Gamma_0(N)}|I_{m,M} | \supset C_{\Gamma_0(N)}(E_m)$. Now $C_{\Gamma_0(N)}[m] = C_{\Gamma_0(N)}[(l, I_{m,M}) \supset C_{\Gamma_0(N)}(E_m)] \neq \{0\}$ as the cardinality of the last group is divisible by $l$.

Given a $p$-good integer $N$, from Theorem 1 it follows that residual characteristic a non-rational Eisenstein ideal $m$ of $\mathfrak{T}(N)$ must divide $6p \cdot |C_{\Gamma_0(N)}(E_m)|$. Given a $p$ good integer $N$, we determine which primes (other than $2, 3, p$) may appear as residual characteristic of some non-rational Eisenstein ideal of $\mathfrak{T}(N)$. To do this we define the following two subsets of rational primes as follows. Let $S_1(N)$ denote the set of rational primes such that

$$S_1(N) = \{ r \text{ prime, } r \mid (q^2 - 1) \text{ for some prime } q \mid N \}$$

and let $S_2(N) = \cup_{\varphi} S_2(N, \varphi)$, here the union is over the set of non-trivial Dirichlet characters of conductor $p$ and $S_2(N, \varphi)$ denote the set of rational primes such that

$$S_2(N, \varphi) = \{ s \text{ prime, } s \mid N_{q(\varphi)/q}(6pB_2(\xi^{-1})) \}$$

where $\xi$ is the primitive Dirichlet character associated to $\varphi^2$. Note that $6pB_2(\xi^{-1}) \in \mathbb{Z}[\varphi]$.

**Proposition 36.** Let $N$ be an integer which is $p$-good for an odd prime $p$ and let $m$ be a non-rational Eisenstein maximal ideal of $\mathfrak{T}(N)$. The residual characteristic of $m$ belongs in the set $\{2, 3, p\} \cup S_1(N) \cup S_2(N)$.

**Proof.** If the residual characteristic divides $6p$, then nothing to prove. From now we assume that $m$ is a non-rational Eisenstein ideal of $\mathfrak{T}(N)$ whose residual characteristic $l$ is co-prime to $6p$. From the proof of Theorem 1 it follows that $E_m = E_{\varphi,M, \frac{N}{M}}$ for some non-trivial Dirichlet character $\varphi$ of conductor $p$ and some integer $M | N'$. Moreover, the residual characteristic $l$ of $m$ divides $|C_{\Gamma_0(N)}(E_m)| = |\mathbb{Z}[\varphi, \varphi]/(\text{Num}(\tilde{\beta}_{\Gamma_0(N), \varphi, M, \frac{N}{M}))|$. Recall that if $\varphi$ is the quadratic character, then

$$\tilde{\beta}_{\Gamma_0(N), \varphi, M, \frac{N}{M}} = \frac{p_2 \phi(N')}{24} \sqrt{\frac{-p}{p}} \prod_{l | p M} (l^2 - 1) = \frac{p_2 \phi(N')}{24} \sqrt{\frac{-p}{p}} \prod_{l | p M} (l^2 - 1)$$
We conclude that

\[ \hat{\beta}_{\Gamma_0(N), \varphi, M, \frac{N'}{M}} = \frac{p^2 \phi(N/M)}{4} B_2(\xi^{-1}) \tau(\varphi^{-1}) \tau(\xi) \prod_{l | M} (l^2 - 1) = \frac{p}{24} \tau(\varphi^{-1}) \tau(\xi) \left( 6pB_2(\xi^{-1}) \phi(N/M) \prod_{l | M} (p^2 - 1) \right). \]

Since \( l \nmid 6p \), in the case \( \varphi \) is quadratic character, \( l \mid |Z[\zeta_p, \varphi]/(Num(\hat{\beta}_{\Gamma_0(N), \varphi, M, \frac{N'}{M}}))| \) and only if \( l \mid |Z[\zeta_p, \varphi]/(\phi(N/M) \prod_{p \mid M} (p^2 - 1))| \). The last statement implies \( l \in S_1(N) \). Similarly, in the case that \( \varphi \) is not a quadratic character, then \( l \mid |Z[\zeta_p, \varphi]/(Num(\hat{\beta}_{\Gamma_0(N), \varphi, M, \frac{N'}{M}}))| \) if and only if

\[ l \mid (6pB_2(\xi^{-1}) \phi(N/M) \prod_{p \mid M} (p^2 - 1)). \]

The last statement implies that \( l \in S_1(N) \cup S_2(N) \).

7. Numerical examples

Given a \( p \)-good integer \( N \), from proposition 36 it follows that the possible residual characteristics belong to \{2, 3, p\} \( \cup \) \( S_1(N) \cup S_2(N) \). From proof of Theorem 1 and Theorem 35 it follows that if the residual characteristic \( l \) of \( m \) is co-prime to \( 6p \), then there exists (at least one) Eisenstein series \( E_{\varphi, M, \frac{N'}{M}} \in E_2(\Gamma_0(N)) \) for some non-trivial Dirichlet character \( \varphi \) of conductor \( p \) and some divisor \( M \mid N' \) and moreover in this situation there is a cuspform \( f \in S_2(\Gamma_0(N)) \) such that

\[ E_{\varphi, M, \frac{N'}{M}} \equiv f \pmod{\ell} \]

for some prime \( \ell \mid l \).

Given a \( p \)-good integer \( N \), we compute the \( \hat{\beta}_{\Gamma_0(N), \varphi, M, \frac{N'}{M}} \) for various \( E_{\varphi, M, \frac{N'}{M}} \in E_2(\Gamma_0(N)) \). If \( m \) is an Eisenstein maximal ideal whose residual characteristic is co-prime to \( 6p \), then we can determine possible residual characteristic of \( m \) by looking at the prime divisors of \( |C_{\Gamma_0(N)}(E_{\varphi, M, \frac{N'}{M}})| \).

Finally, for a prime divisor \( q \nmid 6p \) of \( |C_{\Gamma_0(N)}(E_{\varphi, M, \frac{N'}{M}})| \), we look for congruence between a cuspform \( f \in S_2(\Gamma_0(N)) \) and the specific Eisenstein series \( E_{\varphi, M, \frac{N'}{M}} \) above a prime over \( q \). If such a congruence exists then there exists a non-rational Eisenstein ideal of residual characteristic \( q \) whose associated Eisenstein series is \( E_{\varphi, M, \frac{N'}{M}} \). On the other hand, if no such congruence exists, then if \( m \) is a non-rational Eisenstein ideal whose associated Eisenstein series is \( E_{\varphi, M, \frac{N'}{M}} \), then the residual characteristic of \( m \) must divide \( 6p \). The examples below are computed using \([25]\) and \([24]\).

Example 37. Let us take \( N = 121 = 11^2 \) and \( \varphi \) to be the quadratic character of conductor 11 (given by \( \varphi(2) = -1 \)). Then \( \tau(\varphi^{-1}) = \sqrt{-11}, \xi = 1, n = 1, \tau(1) = 1, B_2(1) = \frac{1}{6} \). We obtain

\[ \hat{\beta}_{\Gamma_0(N), \varphi} = \frac{11^4}{4} \cdot \sqrt{-11} \cdot \frac{1}{6} (1 - \frac{1}{11^2}) = 5 \cdot 11^2 \sqrt{-11}. \]

We conclude that 5 is a possible residual characteristic of an Eisenstein ideal \( m \) for which the corresponding Eisenstein series is \( E_{\varphi} \).

Recall that the \( q \) expansion of \( E_{\varphi} = \sum_{n \geq 1} a_n q^n \), where \( a_n = \sum_{bc = n} \varphi(c)\varphi^{-1}(b) \), we obtain

\[ E_{\varphi} = \sum_{n=1}^{\infty} \varphi(n)(\sum_{d|n} d)q^n = q - 3q^2 + 4q^3 + 7q^4 + 6q^5 - 12q^6 - 8q^7 - 15q^8 + 13q^9 - 18q^{10} + 28q^{12} + O(q^{13}). \]
Let \( f \in S^\text{new}_2(121) \) be a newform with LMFDB label 121.2.a.d whose \( q \)-expansion is given by
\[
q + 2q^2 - q^3 + 2q^4 + q^5 - 2q^6 + 2q^7 - 2q^9 + 2q^{10} - 2q^{12} + O(q^{13}).
\]
We have,
\[
E_\varphi(z) \equiv f(z) \pmod{5}.
\]
We conclude that the corresponding non-rational Eisenstein maximal ideal of characteristic 5 is
\[
m = \langle 5, U_{11}, \{ T_r - 1 - r \} \text{ primes } r \equiv 1, 3, 4, 5, 9 \pmod{11}, \{ T_s + 1 + s \} \text{ primes } s \equiv 2, 6, 7, 8, 10 \pmod{11} \rangle.
\]
In this situation \( \mathbb{F} / m \cong \mathbb{F}_5 \).

**Example 38.** Let \( N = 121 = 11^2 \) and \( \varphi \) be the Dirichlet character of conductor 11 given by \( \varphi(2) = \zeta_{10} \).
Note that the order of the character is 10. Then \( \xi = \varphi^2 \) is a primitive Dirichlet character of conductor 11 and \( \xi^{-1} = \varphi^8 \). We have \( B_2(\xi^{-1}) = \frac{1}{11} \left( 61 - 59\zeta_5 - 23\zeta_5^2 - 47\zeta_5^3 + 13\zeta_5^4 \right) \). Using SAGE, we see that \( 5 \mid N_{\xi(\zeta_5)}/\xi(61 - 59\zeta_5 - 23\zeta_5^2 - 47\zeta_5^3 + 13\zeta_5^4) \).
\[
\tilde{\beta}_{\Gamma_0(N),\varphi} = \frac{11^4}{4 \cdot 11^2} \cdot \tau(\varphi^{-1}) = \frac{1}{33}(61 - 59\zeta_5 - 23\zeta_5^2 - 47\zeta_5^3 + 13\zeta_5^4) = \frac{11}{12} \cdot \tau(\varphi^{-1} \cdot 61 - 59\zeta_5 - 23\zeta_5^2 - 47\zeta_5^3 + 13\zeta_5^4).
\]
Note that \( \tau(\varphi^{-1}) \in \mathbb{Z}[\zeta_{11}, \zeta_{10}] \) and hence 5 is a possible residual characteristic of a Eisenstein ideal \( m \) for which the corresponding Eisenstein series is \( E_\varphi \). Since the order of \( \varphi = 10 \) (a multiple of 5), it follows that order of \( \tau_m \) should be 2, that is, it is the quadratic character from \( \mathbb{Z}/11\mathbb{Z} \lhd \mathbb{F}_5^\times \).
The Fourier expansion of \( E_\varphi = \sum_{n=1}^{\infty} \varphi(n) (\sum_{d|n} \varphi^8(d) d) q^n \) is given by
\[
q + \zeta_{10}(1 + 2\zeta_5^4) q^2 - \zeta_{10}^3 (1 + 3\zeta_5^2) q^3 + \zeta_5 (1 + 2\zeta_5^4 + 4\zeta_5^3) q^4 + \zeta_5^2 (1 + 5\zeta_5) q^5 - \zeta_5^3 (1 + 2\zeta_5^4 + 3\zeta_5^3 + 6\zeta_5) q^6 + O(q^7).
\]
Let \( \lambda = (1 - \zeta_5) \) be the prime above 5 in \( \mathbb{Q}(\zeta_5) = \mathbb{Q}(\zeta_{10}) \). We have,
\[
E_\varphi(z) \equiv f(z) \pmod{\lambda},
\]
here \( f(z) \) denotes the newform of \( \Gamma_0(121) \) whose LMFDB label is 121.2.a.d. We conclude that the corresponding non-rational Eisenstein maximal ideal of characteristic 5 is
\[
m = \langle 5, U_{11}, \{ T_r - 1 - r \} \text{ primes } r \equiv 1, 3, 4, 5, 9 \pmod{11}, \{ T_s + 1 + s \} \text{ primes } s \equiv 2, 6, 7, 8, 10 \pmod{11} \rangle.
\]
In this situation \( \mathbb{F} / m \cong \mathbb{F}_5 \).

**Example 39.** Let us take \( N = 725 = 5^2 \cdot 29 \). There are 3 non-trivial Dirichlet characters of conductor 5, namely \( \{ \varphi, \varphi^2, \varphi^3 \} \), where \( \varphi \) is given by \( \varphi(2) = i \). Obviously, \( \varphi^{-1} = \varphi^3 \) and \( \varphi^2 \) is the quadratic Dirichlet character of conductor 5. Note that \( \tau(\varphi^2) = \sqrt{5}, \quad B_2(\varphi^2) = 5 \sum_{a=1}^{4} \varphi^2(a)(\frac{a^2}{25} - \frac{a}{5} + \frac{1}{6}) = \frac{1}{47}, \quad \tau(\varphi) = \sum_{a=1}^{4} \varphi(a)\zeta_5^a = \zeta_5 (1 + \zeta_5) (1 + (1 + i) \zeta_5^4 + \zeta_5^2) = i\sqrt{-15 + 20i} \) and \( \tau(\varphi^3) = i\sqrt{-15 - 20i} \). Note that, in \( \mathbb{Q}(i, \zeta_5) \), the only prime divisors of \( \tau(\varphi) \) and \( \tau(\varphi^3) \) are primes lying above 5 in \( \mathbb{Q}(i, \zeta_5) \).
Our space of non-rational Eisenstein series is 6 dimensional given by \( E_{\varphi,29}^\text{ord} = E_{\varphi,1,29}, E_{\varphi,29}^\text{crit} = E_{\varphi,29,1}, E_{\varphi,129}^\text{ord} = E_{\varphi,1,29}, E_{\varphi,129}^\text{crit} = E_{\varphi,1,29} \) and \( E_{\varphi,29}^\text{crit} = E_{\varphi,29,1} \). Using Theorem 1, we obtain
\[
\tilde{\beta}_{\Gamma_0(N),\varphi^2,129} = \frac{5^4 \cdot 28}{4} \cdot \sqrt{5} \cdot \frac{1}{6} \cdot (1 - \frac{1}{25}) = 22 \cdot 5^2 \sqrt{5} \cdot 7,
\]
\[
\tilde{\beta}_{\Gamma_0(N),\varphi^2,29,1} = \frac{5^4 \cdot 29^2}{4} \cdot \sqrt{5} \cdot \frac{1}{6} \cdot (1 - \frac{1}{25}) = 22 \cdot 3 \cdot 5^3 \sqrt{5} \cdot 7,
\]
\[ \beta_{\Gamma_0(N), \varphi^{3}, 29} = \frac{5^4 \cdot 29^{2}}{4 \cdot 5} \cdot \frac{i \sqrt{15 - 20i}}{1} \cdot \frac{4}{5} \left( i \cdot 2^2 \cdot (5 \sqrt{5} \cdot \sqrt{15 - 20i}) \cdot 7, \right) \]

\[ \beta_{\Gamma_0(N), \varphi^{2}, 29} = \frac{5^4 \cdot 29^{2}}{4 \cdot 5} \cdot \frac{i \sqrt{15 - 20i}}{1} \cdot \frac{4}{5} \left( i \cdot 2^2 \cdot (5 \sqrt{5} \cdot \sqrt{15 - 20i}) \cdot 7, \right) \]

\[ \beta_{\Gamma_0(N), \varphi^{3}, 29} = \frac{5^4 \cdot 29^{2}}{4 \cdot 5} \cdot \frac{i \sqrt{15 + 20i}}{1} \cdot \frac{4}{5} \left( i \cdot 2^2 \cdot (5 \sqrt{5} \cdot \sqrt{15 + 20i}) \cdot 7, \right) \]

We conclude that 7 is the only prime (other than 2, 3 and 5) that can appear as the residual characteristic of non-rational Eisenstein ideals of level 725. Using the Fourier expansion of \( E_{\varphi^2} \), we see that

\[ E_{\varphi^2}(z) = \sum_{n=1}^{\infty} \varphi^2(n)(\sum_{d|n} d)q^n \]

\[ = q - 3q^2 - 4q^3 + 7q^4 + 12q^5 - 8q^7 - 15q^8 + 13q^9 + 12q^{10} - 28q^{12} - 14q^{13} + 24q^{14} + O(q^{16}). \]

Consider the newform \( f(z) \) of level \( \Gamma_0(725) \) with LMFDB label 725.2.a.b with coefficients in \( \mathbb{Q}(\sqrt{2}) \) whose Fourier expansion \( f(q) \) is given by (here \( \beta = \sqrt{2} \))

\[ q + (1 + \beta)q^2 + (-1 - \beta)q^3 + (1 + 2\beta)q^4 + (-3 - 2\beta)q^6 + 2\beta^7 + (3 + \beta)q^8 + 2\beta^9 + (1 - \beta)q^{11} + (5 - 3\beta)q^{12} + (1 + 2\beta)q^{13} + (4 + 2\beta)q^{14} + O(q^{16}). \]

Consider the prime \( \varphi = 3 - \sqrt{2} \) in \( \mathbb{Z}[\sqrt{2}] \) lying above 7. We observe that

\[ E_{\varphi^2}^{ord_{29}}(z) = E_{\varphi^2}(z) - 29E_{\varphi^2}(29z) \equiv E_{\varphi^2}(z) - E_{\varphi^2}(29z) = E_{\varphi^2}^{crit_{29}}(z) \equiv f(q) \pmod{\varphi}. \]

We conclude that the non-rational Eisenstein maximal ideal of characteristic 7 corresponding to both \( E_{\varphi^2}^{ord_{29}} \) and \( E_{\varphi^2}^{crit_{29}} \) is given by

\[ m = \langle 7, U_5, U_{29} - 1, \{ T_r - 1 - r \} \; \text{primes} \; r \equiv 1, 2 \pmod{5} \rangle, \{ T_s + 1 + s \} \; \text{primes} \; s \equiv 2, 3 \pmod{5}. \]

In this situation, \( \mathbb{T}/m \cong \mathbb{F}_7 \).

Now consider the newform \( g(z) \) of level \( \Gamma_0(725) \) with LMFDB label 725.2.a.l with coefficient in the number field \( K \) (defined by the polynomial \( a^3 - 13a^2 + 41a^2 - 1 \)) with Fourier expansion

\[ g(q) = q + \beta_1 q^2 + (\beta_1 - \beta_4)q^3 + (2 + \beta_2)q^4 + (2 - \beta_3)q^6 + (\beta_1 + \beta_4 + \beta_3)q^7 + (2\beta_1 + \beta_4 + \beta_3)q^8 + (2 - \beta_2 - \beta_3)q^9 + O(q^{10}), \]

here \( \beta_1 \) is a root of \( a^6 - 13a^4 + 41a^2 - 1, \beta_2 = \beta_1^2 - 4, \beta_3 = (\beta_1^2 + 8\beta_1^2 + 5)/2, \beta_4 = (\beta_1^2 + 12\beta_1^2 + 35\beta_1)/2 \) and \( \beta_5 = (\beta_1^2 + 14\beta_1^2 + 47\beta_1)/2. \)

Now the Fourier expansion of \( E_{\varphi} \) and \( E_{\varphi^3} \) are given by

\[ E_{\varphi}(z) = \sum_{n=1}^{\infty} \varphi(n)(\sum_{d|n} d)q^n, E_{\varphi^3}(z) = \sum_{n=1}^{\infty} \varphi^3(n)(\sum_{d|n} d)q^n. \]

Note that \( E_{\varphi} \) and \( E_{\varphi^3} \) have Fourier coefficient in the field \( \mathbb{Q}(i) \), moreover \( E_{\varphi} \) and \( E_{\varphi^3} \) are Galois \( \text{Gal}(\mathbb{Q}(i)/\mathbb{Q}) \)-conjugate. Using SAGE, we see that in \( K(i) \) the ideal (7) factors as \( p_1^2 p_2^2 p_3 p_4 \), with \( p_1 = ((\frac{1}{4}\beta_1 + \frac{1}{4}\beta_2 + 6\beta_3^2 - 2\beta_2^2 + \frac{27}{2}\beta_1 + \frac{13}{4})i + (\frac{1}{2}\beta_1^2 - \frac{2}{7}\beta_2^2 - \frac{13}{2}\beta_1^2 - \frac{27}{4}\beta_1 + \frac{1}{2})) \) and \( p_2 = ((\frac{1}{2}\beta_1 + \frac{1}{2}\beta_1^2 - 6\beta_1^3 - 2\beta_1^2 + \frac{27}{2}\beta_1 + \frac{13}{4})i + (\frac{1}{2}\beta_1^2 - \frac{2}{7}\beta_2^2 - \frac{13}{2}\beta_1^2 + \frac{27}{4}\beta_1 + \frac{1}{2})) \). We have,

\[ E_{\varphi}^{ord_{29}}(z) \equiv E_{\varphi}^{crit_{29}} \equiv g(z) \pmod{p_1}, \]
We obtain
\[ E^{\text{ord}_{29}}_{\phi^3}(z) \equiv E^{\text{crit}_{29}}_{\phi^3} \equiv g(z) \pmod{p_2}. \]

We conclude that the non-rational Eisenstein maximal ideal of characteristic 7 corresponding to \( E^{\text{ord}_{29}}_{\phi}, E^{\text{crit}_{29}}_{\phi}, E^{\text{ord}_{29}}_{\phi^3}(z) \) and \( E^{\text{crit}_{29}}_{\phi^3}(z) \) is given by
\[ m = \langle 7, U_5, U_{29} + 1, \{ T_r - 1 - r \} \text{ primes } r \equiv 1 \pmod{3}, \{ T_s + 1 + s \} \text{ primes } s \equiv 2 \pmod{3} \rangle. \]

In this situation \( \mathbb{T}/m \cong \mathbb{F}_{49}. \)

**Example 40.** Let \( N = 234 = 3^2 \cdot 2 \cdot 13 \) and \( \phi \) be the non-trivial Dirichlet character of conductor 3 given by \( \phi(2) = -1. \) There are 4 non-rational Eisenstein series of weight 2 and level \( \Gamma_0(N) \), namely \( E^{\text{ord}_{2}, \text{ord}_{13}}, E^{\text{ord}_{2}, \text{crit}_{13}}, E^{\text{crit}_{2}, \text{ord}_{13}} \) and \( E^{\text{crit}_{2}, \text{crit}_{13}}. \) We have,
\[ E_{\phi}(z) = \sum_{n=1}^{\infty} \frac{\phi(n)}{n} q^n = q - 3q^2 + 7q^4 - 6q^5 + 8q^7 - 15q^8 + 18q^{10} - 12q^{11} + 14q^{13} - 24q^{14} + O(q^{16}). \]

Then,
\[ E^{\text{crit}_{13}}_{\phi}(z) = E_{\phi}(z) - E_{\phi}(13z) = q - 3q^2 + 7q^4 - 6q^5 + 8q^7 - 15q^8 + 18q^{10} - 12q^{11} + 13q^{13} - 24q^{14} + O(q^{16}), \]
\[ E^{\text{ord}_{13}}_{\phi}(z) = E_{\phi}(z) - 13E_{\phi}(13z) = q - 3q^2 + 7q^4 - 6q^5 + 8q^7 - 15q^8 + 18q^{10} - 12q^{11} + q^{13} - 24q^{14} + O(q^{16}). \]

We obtain
\[ E^{\text{ord}_{2}, \text{crit}_{13}}_{\phi} = E^{\text{crit}_{13}}_{\phi}(z) + 2E^{\text{crit}_{13}}_{\phi}(2z) = q - 2q^2 + 4q^4 - 6q^5 + 8q^7 - 15q^8 - 6q^{10} - 12q^{11} + 13q^{13} - 8q^{14} + O(q^{16}), \]
\[ E^{\text{crit}_{2}, \text{crit}_{13}}_{\phi} = E^{\text{crit}_{13}}_{\phi}(z) + E^{\text{crit}_{13}}_{\phi}(2z) = q - 2q^2 + 4q^4 - 6q^5 + 8q^7 - 15q^8 - 12q^{10} - 12q^{11} + 13q^{13} - 16q^{14} + O(q^{16}), \]
\[ E^{\text{ord}_{2}, \text{ord}_{13}}_{\phi} = E^{\text{ord}_{13}}_{\phi}(z) + 2E^{\text{ord}_{13}}_{\phi}(2z) = q - 2q^2 + 4q^4 - 6q^5 + 8q^7 - 15q^8 - 12q^{10} - 12q^{11} + 13q^{13} - 16q^{14} + O(q^{16}). \]

Now,
\[ \tilde{\beta}_{1,26} = 36 \sqrt{-3}, \tilde{\beta}_{2,13} = 108 \sqrt{-3}, \tilde{\beta}_{26,1} = 7 \cdot 216 \sqrt{-3}, \tilde{\beta}_{13,2} = 7 \cdot 72 \sqrt{-3}. \]

Thus 7 can not be residual characteristic of Eisenstein maximal ideal whose associated Eisenstein series is either \( E^{\text{ord}_{2}, \text{ord}_{13}} \) or \( E^{\text{crit}_{2}, \text{ord}_{13}}. \) On the other hand 7 can be residual characteristic of Eisenstein maximal ideal whose associated Eisenstein series is either \( E^{\text{ord}_{2}, \text{crit}_{13}} \) or \( E^{\text{crit}_{2}, \text{crit}_{13}}. \)

There are 5 newforms of weight 2 and level \( \Gamma_0(234) \) (namely LMFDB label 234.2.a.a - 234.2.a.e) all of which have rational co-efficient. We do not expect any congruence (modulo 7) between \( E^{\text{ord}_{2}, \text{ord}_{13}} \) (resp. \( E^{\text{crit}_{2}, \text{ord}_{13}} \)) and the newforms of weight 2 and level \( \Gamma_0(234) \), which can be verified by looking at the respective Fourier coefficient.

On the other hand, \( E^{\text{ord}_{2}, \text{crit}_{13}} \) is congruent to 234.2.a.b modulo 7. We conclude that the corresponding non-rational Eisenstein maximal ideal of characteristic 7 is
\[ m = \langle 7, U_3, U_2 + 1, U_{13} - 1 \{ T_r - 1 - r \} \text{ primes } r \equiv 1 \pmod{3}, \{ T_s + 1 + s \} \text{ primes } s \equiv 2 \pmod{3} \rangle. \]

In this situation \( \mathbb{T}/m \cong \mathbb{F}_7. \)

Finally, from the Fourier expansion, it follows that the Eisenstein series \( E^{\text{crit}_{2}, \text{crit}_{13}} \) is not congruent to any of the above mentioned newforms modulo 7. We conclude that the Eisenstein maximal ideals \( m \) which corresponds to \( E^{\text{crit}_{2}, \text{crit}_{13}} \) must have residual characteristic (dividing \( 6p \), hence) 2 or 3. In fact,
Thus, as in proof of Proposition 4.7(d) [22, Page 545], we conclude that $E_{\varphi, M} \left( \frac{z}{d} \right)$ is congruent to a weight 2 newform $f(z)$ (with rational coefficient) of level $\Gamma_0(N)$ modulo a prime above $q$. Let $f_E$ denote the isogeny class of the elliptic curves (defined over $\mathbb{Q}$) associated to the modular form $f$. Then the isogeny class $f_E$ has $q$-isogeny. In fact, that seems to be the only prime degree isogeny in that class.

Based on these observations we ask the following Conjecture:

**Conjecture 42.** Is it necessarily true that $q$ is the only prime degree isogeny in the isogeny class $f_E$?

## 8. Appendix: Computation of boundary of Eisenstein series

In the appendix, we prove our computation regarding ramification indices.

**Proposition 43.** Let $\beta_\varphi = \frac{\pi^{3/2} \tau(\varphi^{-1})}{2 \pi} B_2(\zeta^{-1}) \prod_{p|f} (1 - \xi(p)p^{-2}) \in \mathbb{Q}(\zeta_f, \varphi)$, here $\xi$ is the primitive Dirichlet character associated to $\varphi^2$ of conductor $n$, $\tau(\varphi^{-1})$ (resp. $\tau(\xi^{-1})$) denotes the Gauss sum of the character $\varphi^{-1}$ (resp. $\xi^{-1}$) and $B_2(\zeta^{-1})$ denotes generalised Bernoulli number associated to $\xi^{-1}$. The divisor associated to the Eisenstein series $E_{\varphi}$ is given by

$$
\delta_{\Gamma_0(f^2)}(E_{\varphi}) = \beta_\varphi D_{\Gamma_0(f^2), f}(\varphi),
$$

$$
D_{\Gamma_0(f^2), f}(\varphi) = \sum_{\varphi(ab)} \left[ \frac{\varphi}{\mathfrak{f}_b} \right]_{\Gamma_0(f^2)} \text{ are as in Proposition 9}.
$$

Note that the cusps $\left[ \frac{\varphi}{\mathfrak{f}_b} \right]_{\Gamma_0(f^2)}$ are not rational cusps; these cusps are defined over the field $\mathbb{Q}(\zeta_f)$ and hence the divisor $D_{\Gamma_0(f^2), f}(\varphi) \in \text{Div}^0(X_0(N)(\mathbb{Q}(\zeta_f, \varphi))$.

**Proof.** From [22, Page 540], we note that if $\psi(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}) = \zeta_f^{-1}(s) \zeta_f(2r)$ then the action of $T_l$ in $\mathfrak{D}_{\Gamma_0(N)}(\psi)$ is given by $\psi(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}) = \zeta_f^{-1}(s) \zeta_f(2r)$. Hence, $D_{\Gamma_0(f^2)}(\varphi) \in \mathfrak{D}_{\Gamma_0(f^2), f}(\psi)$.

From proposition 9 part (2), we conclude that $\delta_{\Gamma_0(f^2)}(E_{\varphi}) = c(f) \cdot D_{\Gamma_0(f^2), f}(\varphi)$. On the other hand, from [22, Theorem 1.3(a)] recall that

$$
\delta_{\Gamma_0(N)}(E) = \sum_{x \in \text{Cusps}(\Gamma_0(N))} \mathfrak{r}_{\Gamma_0(N), E}(x) \{x\}
$$

where $\mathfrak{r}_{\Gamma_0(N), E}(x) = \mathfrak{r}_{\Gamma_0(N)}(x) \cdot a_0(E | \theta_x)$.

and $\theta_x \in \text{SL}_2(\mathbb{Z})$ is chosen arbitrarily so that $\theta_x \cdot \infty \in \text{Cusps}(\Gamma_0(N))$. Thus, as in proof of Proposition 4.7(d) [22, Page 545], we conclude that $\beta_\varphi := c(f) = r_{\Gamma_0(f^2), E_{\varphi}}(\begin{bmatrix} 1 \\ f \end{bmatrix}_{\Gamma_0(f^2)}) = e_{\Gamma_0(f^2)}(\begin{bmatrix} 1 \\ f \end{bmatrix}_{\Gamma_0(f^2)})a_0(E_{\varphi} | \begin{bmatrix} 1 \\ f \end{bmatrix}_{\Gamma_0(f^2)})$. Note that $e_{\Gamma_0(f^2)}(\begin{bmatrix} 1 \\ f \end{bmatrix}_{\Gamma_0(f^2)}) = f^2/d$, where $d = (f, f^2) = f$ and

$$
t = (f, f) = f \text{ and hence } e_{\Gamma_0(f^2)}(\begin{bmatrix} 1 \\ f \end{bmatrix}_{\Gamma_0(f^2)}) = 1.
$$
Hence,

\[(8.1)\]
\[
\delta \Gamma_0(f^2)(E_\varphi) = a_0(E_\varphi | \begin{bmatrix} 1 & 0 \\ f & 1 \end{bmatrix}) \cdot D \Gamma_0(f^2), f(\varphi).
\]

Comparing coefficients of equation \[8.1] at the cusp \[\begin{bmatrix} 1 \\ f \end{bmatrix} \Gamma_0(f^2)\] we find,

\[a_0(E_\varphi | \begin{bmatrix} 1 & 0 \\ af & 1 \end{bmatrix}) = \varphi(a)a_0(E_\varphi | \begin{bmatrix} 1 & 0 \\ f & 1 \end{bmatrix}).\]

Thus, following Stevens [22, Equation (5.5)], we define \[E' = \sum_{a=0}^{f-1} \varphi^{-1}(a)E_\varphi | \begin{bmatrix} 1 & 0 \\ af & 1 \end{bmatrix}\] and note that \[a_0(E_\varphi | \begin{bmatrix} 1 & 0 \\ f & 1 \end{bmatrix}) = \frac{1}{|\varepsilon(f)|}a_0(E')\]. Finally, following Stevens [22, c.f page 545-547] we compute \[a_0(E') = -L(0, E')\] as follows: Define

\[E_1 = E_\varphi | \begin{bmatrix} 1 & 0 \\ 0 & f \end{bmatrix}, E_2 = E_1 | R(\varphi), E_3 = E_2 | \begin{bmatrix} 0 & -1 \\ n & 0 \end{bmatrix}, E' = \varphi(-1)E_3 | \begin{bmatrix} \frac{f^2}{n} & 0 \\ 0 & 1 \end{bmatrix},\]

where \(n\) is the conductor for \(\varepsilon\), the primitive Dirichlet character associated to \(\varphi^2\) and for a primitive Dirichlet character \(\chi\) of conductor \(m\), \(R(\chi)\) is defined as \[R(\chi) = \sum_{a=0}^{m-1} \chi^{-1}(a) \begin{bmatrix} 1 & \frac{a}{m} \\ 0 & 1 \end{bmatrix}\]. Note that

\[L(E_1, s) = f^sL(E_\varphi | \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} f & 0 \\ 1 & 1 \end{bmatrix}, s) = L(E_\varphi | \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, s).\]

Recall that for any \(f \in E_2(\Gamma_0(N))\) the completed \(L\)-function of \(f\) is defined as \[D(f, s) := i\Gamma(s)(2\pi)^{-s}L(f, s)\] and \(D(f, s)\) satisfies functional equation [22, Lemma 5.1 (a)] \[D(f | \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, s) = -D(f, 2-s).\]

Hence,

\[D(E_1, s) = D(E_\varphi | \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, s) = -D(E_\varphi, 2-s),\]

which implies

\[L(E_1, s) = -\frac{\Gamma(2-s)(2\pi)^{2s-2}}{\Gamma(s)}L(E_\varphi, 2-s) = -\frac{\Gamma(2-s)(2\pi)^{2s-2}}{\Gamma(s)}L(\varphi, 2-s)L(\varphi^{-1}, 1 - s).\]

Note that for a primitive Dirichlet Character \(\chi\) of conductor \(d\), we have completed \(L\)-function

\[\Lambda(\chi, s) = \left(\frac{d}{\pi}\right)^{-\frac{s}{2}}\Gamma\left(s + \frac{\delta}{2}\right)L(\chi, s),\]

where \(\delta = \frac{1-\chi(-1)}{2}\). The completed function \(\Lambda(\chi, s)\) satisfies functional equation \(\Lambda(\chi^{-1}, 1 - s) = \frac{i\sqrt{d}}{\tau(\chi)}\Lambda(\chi, s)\), in other words, \(L(\chi, s)\) satisfies

\[L(\chi, s) = \frac{\Gamma\left(\frac{1-\tau(\chi)}{2}\right)}{\Gamma\left(s + \frac{\delta}{2}\right)}\left(\frac{d}{\pi}\right)^{-\frac{s}{2}}\tau(\chi)i^s\sqrt{d}L(\chi^{-1}, 1 - s).\]
Hence
\[ L(E_1, s) = -\frac{\Gamma(2-s)(2\pi)^{2s-2}}{\Gamma(s)} \left( \frac{\Gamma(s+\delta)\Gamma(s+\delta-1)}{\Gamma(\frac{s+\delta}{2})\Gamma(\frac{1-s+\delta}{2})} \left( \frac{f}{\pi} \right)^{2s-2} \frac{\tau(\varphi^1)}{i^{3-2s}f} \right) L(\varphi, s) L(\varphi^{-1}, s-1). \]

Following Stevens [22, Lemma 5.1 (b)],
\[ L(E_2, s) = \tau(\varphi^{-1}) L(E_1, \varphi, s) \]
\[ = -\tau(\varphi^{-1}) \frac{\Gamma(2-s)(2\pi)^{2s-2}}{\Gamma(s)} \left( \frac{\Gamma(s+\delta)\Gamma(s+\delta-1)}{\Gamma(\frac{s+\delta}{2})\Gamma(\frac{1-s+\delta}{2})} \left( \frac{f}{\pi} \right)^{2s-2} \frac{\tau(\varphi^1)}{i^{3-2s}f} \right) L(\varphi^2, s) L(\varphi\varphi^{-1}, s-1). \]

We obtain,
\[ D(E_2, s) = -\Gamma(2-s)(2\pi)^{s-2} \tau(\varphi^{-1}) \left( \frac{\Gamma(s+\delta)\Gamma(s+\delta-1)}{\Gamma(\frac{s+\delta}{2})\Gamma(\frac{1-s+\delta}{2})} \left( \frac{f}{\pi} \right)^{2s-2} \frac{\tau(\varphi^1)}{i^{2s-2}f} \right) L(\varphi^2, s) L(\varphi\varphi^{-1}, s-1). \]

Note that for an imprimitive Dirichlet character \( \chi \) of modulus \( N \), we have \( L(\chi^*, s) = L(\chi^*, s) \prod_{p|N} (1 - \chi^*(p)p^{-s}) \), where \( \chi^* \) is the primitive Dirichlet character associated to \( \chi \). Thus
\[ D(E_2, s) = -\Gamma(2-s)(2\pi)^{s-2} \tau(\varphi^{-1}) \left( \frac{\Gamma(s+\delta)\Gamma(s+\delta-1)}{\Gamma(\frac{s+\delta}{2})\Gamma(\frac{1-s+\delta}{2})} \left( \frac{f}{\pi} \right)^{2s-2} \frac{\tau(\varphi^1)}{i^{2s-2}f} \right) \left\{ \prod_{p|f} (1 - \xi(p)p^{-s})(1 - p^{1-s}) \right\} L(\xi, s) \zeta(s-1), \]
where \( \xi \) is the primitive character associated to \( \varphi^2 \) (of conductor \( n \)).

Now,
\[ D(E_3, s) = n^{1-s} D(E_2 \mid \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, s) = -n^{1-s} D(E_2, 2-s) \]
\[ = n^{1-s} \Gamma(s)(2\pi)^{-s} \left( \frac{\Gamma(\frac{s+\delta}{2})\Gamma(\frac{1-s+\delta}{2})}{\Gamma(\frac{s+\delta}{2})\Gamma(\frac{1-s+\delta}{2})} \left( \frac{f}{\pi} \right)^{2s-2} \frac{\tau(\varphi^1)}{i^{2s-2}f} \right) \left\{ \prod_{p|f} (1 - \xi(p)p^{-s})(1 - p^{1-s}) \right\} L(\xi, 2-s) \zeta(1-s) \]
\[ = \frac{\Gamma(s)(2\pi)^{-s} f^{1-2s}}{n^{1-s}} \tau(\varphi^1)^2 \tau(\xi) \left\{ \prod_{p|f} (1 - \xi(p)p^{-s})(1 - p^{1-s}) \right\} L(\xi^{-1}, s-1) \zeta(s). \]

For the last equality we have used functional equation of \( \xi \) and \( \zeta \). We obtain
\[ L(E_3, s) = \frac{D(E_3, s)}{i\Gamma(s)(2\pi)^{-s}} = -\frac{f^{1-2s}}{n} \tau(\varphi^1)^2 \tau(\xi) \left\{ \prod_{p|f} (1 - \xi(p)p^{-s})(1 - p^{1-s}) \right\} L(\xi^{-1}, s-1) \zeta(s). \]

Finally,
\[ L(E', s) = \varphi(-1) \left( \frac{f^2}{n} \right)^{1-s} L(E_3, s) = -\varphi(-1) \frac{f^{3-4s}}{n^{2-s}} \tau(\varphi^1)^2 \tau(\xi) \left\{ \prod_{p|f} (1 - \xi(p)p^{-s})(1 - p^{1-s}) \right\} L(\xi^{-1}, s-1) \zeta(s). \]

Hence
\[ \beta_{\varphi} = -\frac{L(E', 0)}{[(E/FZ)^*]} = \varphi(-1) \frac{f^{2}}{4n^{2}} \tau(\varphi^1)^2 \tau(\xi) L(\xi^{-1}, 1) \zeta(0) \prod_{p|f} (1 - \xi(p)p^{-2})(1 - p^{-1}) \]
\[ = \frac{\varphi(-1)f^2}{4n^2} \tau(\varphi^1)^2 \tau(\xi) B_2(\xi^{-1}) \prod_{p|f} (1 - \xi(p)p^{-2}). \]
For the last equality we have used $\zeta(0) = -\frac{1}{2}$ and $L(\chi, -1) = -B_2(\chi)/2$. The final formula for $\beta_\varphi$ is obtained by using $\tau(\varphi)\tau(\varphi^{-1}) = \varphi(-1)f$ and $\tau(\xi)\tau(\xi^{-1}) = \xi(-1)n = n$. □

Following Stevens we proceed with induction on the level of the congruence subgroup. The following propositions are analog of Stevens [22, Lemma 5.2]. Let $A$ be a natural number such that $f^2 \mid A$ and fix $l$ a prime number. We denote by $\pi_{\{l\}} : X_0(Al) \to X_0(A)$ the map induced from injection of $\Gamma_0(Al) \hookrightarrow \Gamma_0(A)$ and $\pi_l : X_0(Al) \to X_0(A)$ the map induced from $\gamma \mapsto \begin{bmatrix} l & 0 \\ 0 & 1 \end{bmatrix} \gamma \begin{bmatrix} l^{-1} & 0 \\ 0 & 1 \end{bmatrix}$.

For a map $f : X \to Y$ between Riemann surface such that $f(x) = y$, let $e(f; x, y)$ be the ramification index of the cusp $x$ for the map $f$.

**Proposition 44.** We have,

(1) 

$$
\pi_{\{l\}}^* (D_{\Gamma_0(A), d}(\varphi)) = \begin{cases} 
\left(\frac{1}{d}\right) D_{\Gamma_0(A), d}(\varphi) & \text{if } l \mid \frac{A}{d}, \\
\left(\frac{1}{d}\right) D_{\Gamma_0(A), d}(\varphi) + \varphi(l)e(\pi_{\{l\}}, \left[ \begin{array}{c} 1 \\ d \end{array} \right]_{\Gamma_0(A)}, \left[ \begin{array}{c} l \\ dl \end{array} \right]_{\Gamma_0(A)} \Gamma_0(A) ) & \text{if } l \nmid \frac{A}{d}.
\end{cases}
$$

(2) 

$$
\pi_l^* (D_{\Gamma_0(A), d}(\varphi)) = \begin{cases} 
\left(\frac{1}{d}\right) D_{\Gamma_0(A), d}(\varphi) & \text{if } l \mid d, \\
\left(\frac{1}{d}\right) D_{\Gamma_0(A), d}(\varphi) + \varphi(l)e(\pi_l, \left[ \begin{array}{c} 1 \\ dl \end{array} \right]_{\Gamma_0(A)}, \left[ \begin{array}{c} l \\ dl \end{array} \right]_{\Gamma_0(A)} \Gamma_0(A) ) & \text{if } l \nmid d.
\end{cases}
$$

**Proof.** We give detailed proof of case (1). Proof of case (2) is similar and is left to the reader. For convenience of notation, we denote by $\Gamma = \Gamma_0(A)$ and $\Gamma' = \Gamma_0(Al)$ and $\pi = \pi_{\{l\}}$. Recall that

$$
D_{\Gamma, d}(\varphi) = \sum_{(db, A) = d,(a,d) = 1} \varphi(ab) \left[ \begin{array}{c} a \\ db \end{array} \right]_{\Gamma}.
$$

We’ll omit $(a, d) = 1$ from the notation. Note that,

$$
\pi^* (D_{\Gamma, d}(\varphi)) = \sum_{(db, A) = d} \varphi(ab) \pi^* \left( \left[ \begin{array}{c} a \\ db \end{array} \right]_{\Gamma} \right)
$$

$$
= \sum_{(db, A) = d} \varphi(ab) \left( \sum_{x_{\Gamma'} \in \pi^{-1} \{ \left[ a \right]_{\Gamma'} \}} e(\pi, x_{\Gamma'}, \left[ \begin{array}{c} a \\ db \end{array} \right]_{\Gamma} \Gamma' ) \right).
$$
Proposition 45. Let \( d \) be a divisor of \( A \) such that \( (d, A) = d \). For an integer \( x \), by \( \nu_l(x) \) we denote the highest power of \( l \) that divides \( x \).

(1) The ramification index of \( \pi_l \) are given as follows.

(a) For any prime \( l \), we have

\[
e(\pi_l), \left[ \frac{a}{d} \right]_{\Gamma^l}, \left[ \frac{1}{d} \right]_{\Gamma} = \begin{cases} l & \text{if } \nu_l(d) \leq \nu_l\left(\frac{A}{d}\right), \\ 1 & \text{otherwise.} \end{cases}
\]
(b) For any prime $l \nmid \frac{A}{d}$, we have
\[ e(\pi(l), \begin{bmatrix} 1 \\ d \end{bmatrix}_\Gamma, \begin{bmatrix} 1 \\ l \end{bmatrix}_\Gamma) = 1 \]

(2) The ramification index of $\pi_1$ are given as follows.
(a) For any prime $l$, we have
\[ e(\pi(l), \begin{bmatrix} 1 \\ d \end{bmatrix}_\Gamma, \begin{bmatrix} 1 \\ d \end{bmatrix}_\Gamma) = \begin{cases} l & \text{if } \nu_l\left(\frac{A}{d}\right) \leq \nu_l(d), \\ 1 & \text{otherwise.} \end{cases} \]

(b) For any prime $l \nmid d$, we have
\[ e(\pi(l), \begin{bmatrix} 1 \\ l \end{bmatrix}_\Gamma, \begin{bmatrix} 1 \\ l \end{bmatrix}_\Gamma) = 1 \]

Proof. Proof of part (1) follows easily from the fact,
\[ e(\pi(l), \begin{bmatrix} a \\ db \end{bmatrix}_\Gamma, \begin{bmatrix} a \\ db \end{bmatrix}_\Gamma) = \frac{e_{\Gamma_0(A)}\left(\begin{bmatrix} a \\ db \end{bmatrix}_\Gamma \right)}{\Gamma_0(A)} \]
The computation is left to the reader.

Now we give a proof of part (2). Consider the subsets $V, V' \subset \mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})$ around the cusps $x$ and $y$ respectively and consider the corresponding neighborhood $U, \tilde{U}$ on the modular curves $X_0(A)$ and $X_0(A)$ respectively and $U', \tilde{U}'$ are the neighbourhood of the point $0$ of the unit disc. We have a diagram:

Observe that $\pi_1$ is achieved by the matrix $\tau_1 = \begin{bmatrix} \sqrt{7} & 0 \\ 0 & 1 \end{bmatrix} \in \text{SL}_2(\mathbb{R})$. For a cusp $s$, let $\delta_s \in \text{SL}_2(\mathbb{Z})$ be such that $\delta_s(s) = \infty$ and $h_s$ be the width of the cusp $s$. Consider the wrapping map $\rho_s(z) = e^{\frac{2\pi i}{h_s}z}$ and hence $\rho_s^{-1}(z) = \frac{h_s}{2\pi i} \log(z)$.

Note that $\delta_{\frac{1}{7}} = \left(\begin{smallmatrix} 1 \\ -d \end{smallmatrix}\right) \in \text{SL}_2(\mathbb{Z})$ (respectively $\delta_{\frac{1}{7}} = \left(\begin{smallmatrix} 1 \\ -d \end{smallmatrix}\right) \in \text{SL}_2(\mathbb{Z})$) is such that $\delta_{\frac{1}{7}}(\frac{1}{d}) = \infty$ and $\delta_{\frac{1}{7}}\left(\frac{1}{d}\right) = \infty$. We note that $\delta_{\frac{1}{7}} \circ \tau_1 \circ \delta_{\frac{1}{7}}^{-1} = \tau_1$.

In particular, let $h_{\frac{1}{7}}$ (respectively $h_{\frac{1}{7}}$) be the width of the cusp $\begin{bmatrix} 1 \\ d \end{bmatrix}_\Gamma$ (respectively $\begin{bmatrix} 1 \\ dl \end{bmatrix}_\Gamma$) for the modular curves $X_0(A)$ (respectively $X_0(Al)$). We now obtain the ramification index by the
following computation:
\[
\pi_l^0(z) = \tilde{\phi} \circ \tau_l \circ \pi_l \circ (\phi \circ \tau_l)^{-1}(z) = \rho_{\frac{l}{t}} \circ \delta_{\frac{l}{t}} \circ \tau_l \circ (\rho_{\frac{l}{t}} \circ \delta_{\frac{l}{t}})^{-1}(z) = \rho_{\frac{l}{t}} \circ \tau_l \circ (\rho_{\frac{l}{t}})^{-1}(z) = l_{\frac{h_{\frac{l}{t}}}{t}} \log(z) = z \frac{h_{\frac{l}{t}}}{t}.
\]

Note that \( h_{\frac{l}{t}} = \frac{A}{dl} \), where \( t = (d, \frac{A}{dl}) \) and \( h_{\frac{l}{t}} = \frac{Al}{dl} = \frac{A}{dl} \), where \( t'' = (dl, \frac{A}{dl}) \). Hence the ramification index is \( \frac{h_{\frac{l}{t}}}{t''} \). Writing \( d = l''k_1 \) and \( \frac{A}{dl} = l''k_2 \) with \( l \parallel k_1k_2 \), we see \( t = t'' \) if \( \nu_l(\frac{A}{dl}) \leq \nu_l(d) \) and \( t'' = lt \) otherwise. This concludes the proof of part (a).

We now show that \( e(\pi_l, \begin{bmatrix} 1 \\ d \end{bmatrix}, \begin{bmatrix} l \\ d \end{bmatrix}) = 1 \) if \( l \parallel d \). First observe that the width of the cusps \( \begin{bmatrix} 1 \\ d \end{bmatrix} \) and \( \begin{bmatrix} l \\ d \end{bmatrix} \) are given by \( h_{\frac{l}{t}} = \frac{Al}{dl} \) and \( h_{\frac{l}{t}} = \frac{A}{dl} \) and hence \( \frac{h_{\frac{l}{t}}}{h_{\frac{l}{t}}} = l \). Since \( l \parallel d \), we can find integers \( m \) and \( n \) such that \( ml + dn = 1 \). Consider the matrix \( \delta_{\frac{l}{t}} = (\frac{m}{-dl}) \in \text{SL}_2(\mathbb{Z}) \), we have \( \delta_{\frac{l}{t}} \circ \tau_l \circ \delta_{\frac{l}{t}}^{-1}(z) = \left( \begin{smallmatrix} \frac{m}{-dl} & 0 \\ 0 & \sqrt{l} \end{smallmatrix} \right) \circ (\begin{bmatrix} 1 \\ 0 \end{bmatrix} \circ (\frac{1}{0} 1)). \)

Hence,
\[
\pi_l^0(z) = \rho_{\frac{l}{t}} \circ \delta_{\frac{l}{t}} \circ \tau_l \circ (\rho_{\frac{l}{t}} \circ \delta_{\frac{l}{t}})^{-1}(z) = \rho_{\frac{l}{t}} \circ \left( \begin{smallmatrix} \frac{m}{-dl} & 0 \\ 0 & \sqrt{l} \end{smallmatrix} \right) \circ (\begin{bmatrix} 1 \\ 0 \end{bmatrix} \circ (\frac{1}{0} 1)) \circ (\rho_{\frac{l}{t}})^{-1}(z) = e^{\frac{h_{\frac{l}{t}}}{t}(\log(z) + n)} = e^n z.
\]

Combining Propositions 44, 45, we obtain the following result.

**Proposition 46.** For \( d^2 \mid A \) with \( (d, A) = d \), we have

(1)
\[
\pi_l^0(D_{\Gamma_0(A),d}(\varphi)) = \begin{cases} 
ID_{\Gamma_0(A),d}(\varphi) + \varphi(l)ID_{\Gamma_0(A),d}(\varphi) & \text{if } l \parallel \frac{A}{dl}, l \parallel d, \\
ID_{\Gamma_0(A),d}(\varphi) + \varphi(l)ID_{\Gamma_0(A),d}(\varphi) & \text{if } l \parallel \frac{A}{dl}, l \parallel d, \\
ID_{\Gamma_0(A),d}(\varphi) & \text{if } l \parallel \frac{A}{dl}, \nu_l(d) \leq \nu_l(\frac{A}{dl}), \\
ID_{\Gamma_0(A),d}(\varphi) & \text{if } l \parallel \frac{A}{dl}, \nu_l(d) > \nu_l(\frac{A}{dl}). 
\end{cases}
\]

(2)
\[
\pi_l^0(D_{\Gamma_0(A),d}(\varphi)) = \begin{cases} 
ID_{\Gamma_0(A),d}(\varphi) + \varphi(l)ID_{\Gamma_0(A),d}(\varphi) & \text{if } l \parallel \frac{A}{dl}, d \parallel \frac{A}{dl}, \\
ID_{\Gamma_0(A),d}(\varphi) & \text{if } l \parallel \frac{A}{dl}, d \parallel \frac{A}{dl}, \\
id_{\Gamma_0(A),d}(\varphi) + \varphi(l)ID_{\Gamma_0(A),d}(\varphi) & \text{if } l \parallel d, l \parallel \frac{A}{dl}, \\
ID_{\Gamma_0(A),d}(\varphi) & \text{if } l \parallel d, d \parallel \frac{A}{dl}, \nu_l(\frac{A}{dl}) \leq \nu_l(d), \\
ID_{\Gamma_0(A),d}(\varphi) & \text{if } l \parallel d, d \parallel \frac{A}{dl}, \nu_l(\frac{A}{dl}) > \nu_l(d).
\end{cases}
\]
Since the map \( \pi_t : X_0(Al) \to X_0(A) \) is given by \( z \to tz \), hence the pull-back map on Picard group \( \delta_{\Gamma_0(Al)}(E) = \delta_{\Gamma_0(Al)}(E \mid \gamma_t) (z) \).

Similarly,

\[ \pi_t^* \left( \delta_{\Gamma_0(A)}(E) \right) = \delta_{\Gamma_0(A)}(E) \mid_{\gamma_t} (z) \]

Applying Proposition \( 46 \), we get the following lemma which is analogue of Stevens \( 22 \), Lemma 5.3.

**Lemma 47.** Let \( M \) and \( L \) be integers such that \( f^2 ML \mid N \) and \( (fM, L) = 1 \). Let \( T_1 = \prod_{l \mid ML} l = l_1 \cdots l_m \) and \( T_2 = \prod_{q \mid qL} q = q_1 \cdots q_n \). Let \( p \) (resp. \( l \), resp. \( q \), resp. \( t \)) denotes a prime divisors of \( f \) (resp. \( T_1 \), resp. \( T_2 \), resp. \( N \) which does not appear in \( fT_1T_2 \)). By \( [y] \), we understand greatest integer less than or equals to \( y \). For an integer \( x \), by \( \nu_l(x) \) we denote the highest power of \( l \) that divides \( x \).

1. Suppose \( l \nmid A \) and \( B \mid A \). Then
   
   (a) \[ [l] + D_{\Gamma_0(A), B}(\varphi) = l(1 - \frac{\xi(l)}{l^2})D_{\Gamma_0(Al), B}(\varphi). \]

   (b) For \( 1 \leq n < \nu_l(M) \),
   
   \[ \pi_t^* \left( \sum_{i=0}^{n-1} \alpha_l^{n,i}D_{\Gamma_0(AL^n), B}(\varphi) \right) = \sum_{i=0}^{n} \alpha_l^{n+1,i}D_{\Gamma_0(AL^{n+1}), B}(\varphi), \]
   
   where \( \alpha_{l,0} = 1 \) and
   
   \[ \alpha_l^{n+1,i} = \begin{cases} \varphi(l)\alpha_l^{n,0} & \text{for } i = 0, \\ \alpha_l^{n,i-1} & \text{for } 1 \leq i \leq \frac{n+1}{2}, \\ l\alpha_l^{n,i-1} & \text{for } \frac{n+1}{2} + 1 \leq i \leq n. \end{cases} \]

   (c) For \( n \geq \nu_l(M) \) and \( 0 \leq i \leq \nu_l(M) - 1 \),
   
   \[ \pi_t^* \left( \alpha_l^{n,i}D_{\Gamma_0(AL^n), B}(\varphi) \right) = \alpha_l^{n+1,i}D_{\Gamma_0(AL^{n+1}), B}(\varphi), \]
   
   where
   
   \[ \alpha_l^{n+1,i} = \begin{cases} l\alpha_l^{n,i} & \text{for } 0 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor, \\ \alpha_l^{n,i} & \text{otherwise}. \end{cases} \]

2. Suppose \( q \mid A, B \mid A \). Let \( S_\varphi = \{ q \mid \varphi(q) = \pm 1 \} \).
   
   (a) \[ [q] - D_{\Gamma_0(A), B}(\varphi) = \begin{cases} (q - 1)(\beta_{q,0}D_{\Gamma_0(Aq), B}(\varphi) + \beta_{q,1}D_{\Gamma_0(Aq), Bq}(\varphi)) & \text{if } q \in S_\varphi, \\ \beta_{q,0}D_{\Gamma_0(Aq), B}(\varphi) + \beta_{q,1}D_{\Gamma_0(Aq), Bq}(\varphi) & \text{if } q \notin S_\varphi, \end{cases} \]
   
   where
   
   \[ \beta_{q,0} = \begin{cases} 1 & \text{if } q \in S_\varphi, \\ q - 1 & \text{if } q \notin S_\varphi \end{cases}, \quad \text{and} \quad \beta_{q,1} = \begin{cases} -\varphi(q) & \text{if } q \in S_\varphi, \\ \varphi(q) - q\varphi^{-1}(q) & \text{if } q \notin S_\varphi. \end{cases} \]

   (b) For \( 1 \leq n < \nu_q(L) \),
   
   \[ \pi_q^* \left( \sum_{i=0}^{n} \beta_q^{n,i}D_{\Gamma_0(Aq^n), Bq}(\varphi) \right) = \sum_{i=0}^{n+1} \beta_q^{n+1,i}D_{\Gamma_0(Aq^{n+1}), Bq}(\varphi), \]
where

$$\beta_{q^{n+1},i} = \begin{cases} \varphi(q)\beta_{q^n,0} & \text{for } i = 0, \\ \beta_{q^n,i-1} & \text{for } 1 \leq i \leq \left[\frac{n+1}{2}\right], \\ q\beta_{q^n,i-1} & \text{for } \left[\frac{n+1}{2}\right] + 1 \leq i \leq n+1. \end{cases}$$

(c) For $n \geq \nu_q(L)$,

$$\pi^*_q \left( \sum_{i=0}^{n} \beta_{q^n,i} D_{\Gamma_0(Aq^n),Bq} (\varphi) \right) = \sum_{i=0}^{n+1} \beta_{q^{n+1},i} D_{\Gamma_0(Aq^{n+1}),Bq} (\varphi),$$

where

$$\beta_{q^{n+1},i} = \begin{cases} q\beta_{q^n,i} & \text{for } 0 \leq i \leq \left[\frac{n}{2}\right], \\ \beta_{q^n,i} & \text{for } \left[\frac{n}{2}\right] + 1 \leq i \leq n, \\ \varphi(q)\beta_{q^n,n} & \text{for } i = n+1. \end{cases}$$

(3) Let $p \nmid A$ and $B \mid A$.

(a) For $0 \leq n < \nu_p(M)$,

$$\pi^*_p \left( D_{\Gamma_0(f^2Ap^n),fBp^n} (\varphi) \right) = pD_{\Gamma_0(f^2Ap^n+1),fBp^{n+1}} (\varphi).$$

(b) For $n \geq \nu_p(M)$,

$$\pi^*_p \left( D_{\Gamma_0(f^2Ap^n),fBp^n} (\varphi) \right) = p\delta_p D_{\Gamma_0(f^2Ap^n+1),fBp^{n+1}} (\varphi),$$

where $\delta_p = 1$ if $n = 0$ and $\delta_p = 0$ otherwise. Note $\delta_p = 1$ implies $\nu_p(M) = 0$.

(4) Let $t$ be a prime such that $t \nmid A$ and $B \mid A$. Let $\gamma_{1,0} = 1$ and for $n \geq 0$,

$$\pi_{(l)}^* \left( \sum_{i=0}^{n} \gamma_{l^n,i} D_{\Gamma_0(At^n),B^l} (\varphi) \right) = \sum_{i=0}^{n+1} \gamma_{l^{n+1},i} D_{\Gamma_0(A^l(t+n+1)),B^l} (\varphi),$$

where

$$\gamma_{l^{n+1},i} = \begin{cases} t\gamma_{l^n,i} & \text{for } 0 \leq i \leq \left[\frac{n}{2}\right], \\ \gamma_{l^n,i} & \text{for } \left[\frac{n}{2}\right] + 1 \leq i \leq n, \\ \varphi(l)\gamma_{l^n,n} & \text{for } i = n+1. \end{cases}$$

Proof. All these formulas are easy consequence of Proposition 46. We only show part (1) for illustration purpose. Remaining calculations are similar (also slightly easier). For (1)(a), note that,

$$[l]^+ D_{\Gamma_0(A),B} (\varphi) = \pi_{(l)}^* D_{\Gamma_0(A),B} (\varphi) - \frac{\varphi(l)}{l} \pi_{(l)}^* D_{\Gamma_0(A),B} (\varphi).$$

The result follows from the observation that since $l \nmid A$, $l \nmid \frac{A}{A}$ and $l \nmid B$ and applying proposition 46.
For (1)(b), we note,
\[\pi_l^* \left( \sum_{i=0}^{n-1} \alpha_{l^n,i} D_{\Gamma_0(At^n), B_l}(\phi) \right) = \pi_l^* (\alpha_{l^n,0} D_{\Gamma_0(At^n), B}(\phi)) + \sum_{i=1}^{n-1} \alpha_{l^n,i} D_{\Gamma_0(At^n), B_l}(\phi)\]
\[= \alpha_{l^n,0} \pi_l^* D_{\Gamma_0(At^n), B}(\phi) + \sum_{i=1}^{\left\lfloor \frac{n-1}{2} \right\rfloor-1} \alpha_{l^n,i} \pi_l^* D_{\Gamma_0(At^n), B_l}(\phi) + \sum_{i=\left\lceil \frac{n-1}{2} \right\rceil}^{n-1} \alpha_{l^n,i} \pi_l^* D_{\Gamma_0(At^n), B_l}(\phi).\]

The result follows from Proposition\[46\] in the first term \(l \mid d, l \mid \frac{d}{q}\), in the second term \(l \mid d, \nu_l(\frac{d}{q}) > \nu_l(d)\) and in the last term \(l \mid d, \nu_l(\frac{d}{q}) \leq \nu_l(d)\). Finally, for (1)(c), from Prop\[46\] we obtain
\[\pi_l^* \alpha_{l^n,0} D_{\Gamma_0(At^n), B_l}(\phi) = \begin{cases} l\alpha_{l^n,i} D_{\Gamma_0(At^{n+1}), B_l}(\phi) & \text{if } 0 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor, \\ \alpha_{l^n,i} D_{\Gamma_0(At^{n+1}), B_l}(\phi) & \text{otherwise.} \end{cases}\]

□

REFERENCES

[1] J. Bellaïche, Critical p-adic L-functions, Invent. Math. 189 (2012), no. 1, 1–60.
[2] J. Bellaïche and S. Dasgupta, The p-adic L-functions of evil Eisenstein series, Compos. Math. 151 (2015), no. 6, 999–1040.
[3] N. Billerey and R. Menares, On the modularity of reducible mod l Galois representations, Math. Res. Lett. 23 (2016), no. 1, 15–41.
[4] H. Darmon, F. Diamond, and R. Taylor, Elliptic curves, modular forms & Fermat’s last theorem (Hong Kong, 1993), Int. Press, Cambridge, MA (1997).
[5] P. Deligne and M. Rapoport, Les schémas de modules de courbes elliptiques, in Modular functions of one variable, II (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972), 143–316. Lecture Notes in Math., Vol. 349, Springer, Berlin (1973).
[6] P. Deligne and J.-P. Serre, Formes modulaires de poids 1, Ann. Sci. École Norm. Sup. (4) 7 (1974) 507–530.
[7] F. Diamond and J. Im, Modular forms and modular curves, in Seminar on Fermat’s Last Theorem (Toronto, ON, 1993–1994), Vol. 17 of CMS Conf. Proc., 39–133, Amer. Math. Soc., Providence, RI (1995).
[8] F. Diamond and J. Shurman, A first course in modular forms, Vol. 228 of Graduate Texts in Mathematics, Springer-Verlag, New York (2005).
[9] B. Edixhoven, The weight in Serre’s conjectures on modular forms, Invent. Math. 109 (1992), no. 3, 563–594.
[10] G. Faltings and B. W. Jordan, Crystalline cohomology and GL(2, Q), Israel J. Math. 90 (1995), no. 1-3, 1–66.
[11] J. I. Manin, Parabolic points and zeta functions of modular curves, Izv. Akad. Nauk SSSR Ser. Mat. 36 (1972) 19–66.
[12] B. Mazur, Modular curves and the Eisenstein ideal, Inst. Hautes Études Sci. Publ. Math. (1977), no. 47, 33–186 (1978).
[13] A. P. Ogg, Diophantine equations and modular forms, Bull. Amer. Math. Soc. 81 (1975) 14–27.
[14] M. Ohta, Eisenstein ideals and the rational torsion subgroups of modular Jacobian varieties, J. Math. Soc. Japan 65 (2013), no. 3, 733–772.
[15] ———, Eisenstein ideals and the rational torsion subgroups of modular Jacobian varieties II, Tokyo J. Math. 37 (2014), no. 2, 273–318.
[16] Y. Ren, Rational torsion subgroups of modular Jacobian varieties, J. Number Theory 190 (2018) 169–186.
[17] K. A. Ribet, On modular representations of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ arising from modular forms, Invent. Math. 100 (1990), no. 2, 431–476.
[18] ———, Report on mod $\ell$ representations of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, in Motives (Seattle, WA, 1991), Vol. 55 of Proc. Sympos. Pure Math., 639–676, Amer. Math. Soc., Providence, RI (1994).
[19] ———, Eisenstein primes for $J_0(pq)$, Unpublished (2008) 1–8.
[20] K. A. Ribet and W. A. Stein, Lectures on Serre’s conjectures, in Arithmetic algebraic geometry (Park City, UT, 1999), Vol. 9 of IAS/Park City Math. Ser., 143–232, Amer. Math. Soc., Providence, RI (2001).
[21] G. Stevens, Arithmetic on modular curves, Vol. 20 of Progress in Mathematics, Birkhäuser Boston, Inc., Boston, MA (1982), ISBN 3-7643-3088-0.
[22] ———, The cuspidal group and special values of $L$-functions, Trans. Amer. Math. Soc. 291 (1985), no. 2, 519–550.
[23] S.-L. Tang, Congruences between modular forms, cyclic isogenies of modular elliptic curves and integrality of $p$-adic $L$-functions, Trans. Amer. Math. Soc. 349 (1997), no. 2, 837–856.
[24] The LMFDB Collaboration, The $L$-functions and modular forms database, http://www.lmfdb.org (2021). [Online; accessed 01 November 2021].
[25] The Sage Developers, SageMath, the Sage Mathematics Software System (Version 9.3.4) (2021). https://www.sagemath.org.
[26] L. C. Washington, Introduction to cyclotomic fields, Vol. 83 of Graduate Texts in Mathematics, Springer-Verlag, New York, second edition (1997), ISBN 0-387-94762-0.
[27] H. Yoo, The index of an Eisenstein ideal and multiplicity one, Math. Z. 282 (2016), no. 3-4, 1097–1116.
[28] ———, On Eisenstein ideals and the cuspidal group of $J_0(N)$, Israel J. Math. 214 (2016), no. 1, 359–377.
[29] ———, Non-optimal levels of a reducible mod $\ell$ modular representation, Trans. Amer. Math. Soc. 371 (2019), no. 6, 3805–3830.
[30] ———, On rational Eisenstein primes and the rational cuspidal groups of modular Jacobian varieties, Trans. Amer. Math. Soc. 372 (2019), no. 4, 2429–2466.

INDIAN INSTITUTE OF SCIENCE OF EDUCATION AND RESEARCH, PUNE, INDIA

INDIAN INSTITUTE OF TECHNOLOGY HYDERABAD, INDIA

INDIAN INSTITUTE OF TECHNOLOGY MADRAS, INDIA