ON THE CHOW RING OF SOME LAGRANGIAN FIBRATIONS

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ABSTRACT. Let $X$ be a hyperkähler variety admitting a Lagrangian fibration. Beauville’s “splitting property” conjecture predicts that fibres of the Lagrangian fibration should have a particular behaviour in the Chow ring of $X$. We study this conjectural behaviour for two very classical examples of Lagrangian fibrations.

1. INTRODUCTION

For a smooth projective variety $X$ over $\mathbb{C}$, let $A^i(X) := CH^i(X)_\mathbb{Q}$ denote the Chow groups (i.e. the groups of codimension $i$ algebraic cycles on $X$ with $\mathbb{Q}$-coefficients, modulo rational equivalence).

The domain of algebraic cycles is an alluring treasure trove for anyone looking for open problems [5], [19], [20], [32], [47], [33]. Inside this treasure trove, one niche of particular interest is occupied by hyperkähler varieties (i.e. projective irreducible holomorphic symplectic manifolds [2], [1]). For these varieties, recent years have seen an intense amount of new constructions and significant progress in the understanding of their Chow groups [3], [45], [48], [43], [38], [42], [35], [36], [10], [26], [27], [40], [15], [14], [30], [50]. Much of this progress has centered around the following conjecture:

Conjecture 1.1 (Beauville, Voisin [3], [43]). Let $X$ be a hyperkähler variety. Let $D^*(X) \subset A^*(X)$ denote the $\mathbb{Q}$-subalgebra generated by divisors and Chern classes of $X$. Then the cycle class maps induce injections

$$D^i(X) \hookrightarrow H^{2i}(X, \mathbb{Q}) \quad \forall i.$$  

(For some cases where Conjecture 1.1 is satisfied, cf. [3], [43], [4], [36], [10], [42], [51], [14], [23], [12].)

The “motivation” underlying Conjecture 1.1 is that for a hyperkähler variety $X$, the Chow ring $A^*(X)$ is expected to have a bigrading $A^*_{[i]}(X)$, where the piece $A^*_{[i]}(X)$ corresponds to the graded $\text{Gr}_F A^*(X)$ for the conjectural Bloch–Beilinson filtration. In particular, it is expected that the subring $A^*_{[0]}(X)$ injects into cohomology, and that $D^*(X) \subset A^*_{[0]}(X)$.

In addition to divisors and Chern classes, what other cycles should be in the subring $A^*_{[0]}(X)$ (assuming this subring exists) ? A conjecture of Voisin provides more candidate members:
Conjecture 1.2 (Voisin [48]). Let $X$ be a hyperkähler variety of dimension $n = 2m$. Let $Z \subset X$ be a codimension $i$ subvariety swept out by $i$-dimensional constant cycle subvarieties. There exists a subring $A^*_i(X) \subset A^*(X)$ injecting into cohomology, containing $D^*(X)$ and

$$Z \in A^*_i(X).$$

A constant cycle subvariety is a closed subvariety $T \subset X$ such that the image of the natural map $A^0(T) \to A^n(X)$ has dimension 1. In particular, Conjecture 1.2 stipulates that Lagrangian constant cycle subvarieties (i.e., constant cycle subvarieties of dimension $m$) should lie in $A^m_i(X)$. Some results towards Conjecture 1.2 can be found in [45], [27], [11], [12].

Amongst hyperkähler varieties, of particular interest are those admitting a Lagrangian fibration (i.e. a proper surjective morphism $\pi : X \to B$ with connected fibers and $0 < \dim B < \dim X$; in this case the general fiber of $\pi$ is an abelian variety that is Lagrangian with respect to the symplectic form on $X$ [29]. In dimension 2, a Lagrangian fibration is an elliptic K3 surface [1]).

As explained in [21], Conjecture 1.2 plus the Bloch–Beilinson conjectures lead in particular to the following:

Conjecture 1.3. Let $X$ be a hyperkähler variety of dimension 4. Assume that $X$ admits a Lagrangian fibration with general fibre $A$. Then

$$\text{Im}(A^2(X) \to A^4(X)) = \mathbb{Q}[c_4(T_X)].$$

(For a more general conjecture, which is more awkward to state, cf. [21, Conjecture 1.3].)

The goal of this note is to study the conjectural injectivity property (as outlined by Conjectures 1.1, 1.2 and 1.3) for some classical examples of Lagrangian fibrations.

The first result is as follows:

Theorem (=Theorem 4.1). Let $X$ be a hyperkähler fourfold, and assume that $X$ admits a Lagrangian fibration $\pi$ which is a compactified Jacobian of a family of curves. Let $A$ be a general fibre of $\pi$. Then

$$\text{Im}(A^2(X) \to A^4(X)) = \mathbb{Q}[c_4(T_X)].$$

To prove Theorem 4.1 thanks to Markushevich [28] one is reduced to fibrations arising from hyperplane sections of a genus 2 K3 surface (such fibrations are cited as examples of Mukai flops in the introduction of Mukai’s beautiful paper [31, Example 0.6]). Then, we exploit the existence of a multiplicative Chow–Künneth decomposition [38], combined with results concerning the Franchetta property for families of Hilbert powers of low degree K3 surfaces [11].

The second result is about the six-dimensional Lagrangian fibration $\pi : J_1 \to \mathbb{P}^3$, where $J_1$ is the compactified Jacobian of genus 3 curves arising as hyperplane sections of a general quartic K3 surface $S$. This is another example in the introduction of Mukai’s foundational paper [31, Example 0.8], where it is shown that the flop of $J_1$ along a certain codimension 2 subvariety $P \subset J_1$ is isomorphic to a moduli space of sheaves on $S$.

Footnote: For background on Lagrangian fibrations, cf. the foundational [29] as well as the recent [7], [18] and the references given there.
Theorem (=Theorem 5.2). Let $h_1 \in A^1(J_1)$ be the polarization class, let $h_2 \in A^1(J_1)$ be $\pi^*(d)$ where $d \subset \mathbb{P}^3$ is a hyperplane class, and let $P \subset J_1$ be as above. Let $R^*(J_1)$ be the $\mathbb{Q}$-subalgebra

$$R^*(J_1) := \langle h_1, h_2, P, c_j(T_{J_1}) \rangle \subset A^*(J_1).$$

The cycle class map induces an injection

$$R^*(J_1) \hookrightarrow H^*(J_1, \mathbb{Q}).$$

In particular, Conjecture 1.1 is true for the very general sixfold $J_1$. Theorem 5.2 is also in agreement with Conjecture 1.2, because $P$ (being a $\mathbb{P}^2$-bundle over $S$) has codimension 2 and is swept out by constant cycle surfaces. In proving Theorem 5.2, we rely on (a sharpening of) a recent result of Bülles [8], combined with results on the Franchetta property from [11].

For Lagrangian fibrations of higher dimension (such as the tenfolds of [24] or [49]), the argument of the present note quickly runs into problems: this is because the Franchetta property is not known outside of a few selected cases (e.g., for the tenfolds of [49], in view of [25] one would need the Franchetta property for the 5th relative power of cubic fourfolds; this is currently unknown and perhaps not even true).

Conventions. In this note, the word variety will refer to a reduced irreducible scheme of finite type over $\mathbb{C}$. For a smooth variety $X$, we will denote by $A^j(X)$ the Chow group of codimension $j$ cycles on $X$ with $\mathbb{Q}$-coefficients. The notation $A^j_{\text{hom}}(X)$ will be used to indicate the subgroups of homologically trivial cycles.

For a morphism between smooth varieties $f : X \to Y$, we will write $\Gamma_f \in A^*(X \times Y)$ for the graph of $f$, and $^t \Gamma_f \in A^*(Y \times X)$ for the transpose correspondence.

The contravariant category of Chow motives (i.e., pure motives with respect to rational equivalence as in [37], [33]) will be denoted $M_{\text{rat}}$.

## 2. Preliminaries

### 2.1. Bülles’ result revisited.

The following theorem is a slight sharpening of a result of Bülles [8]:

**Theorem 2.1.** Let $S$ be a projective K3 surface or an abelian surface, and $\alpha \in Br(S)$ a Brauer class. Let $M$ be a smooth projective moduli space of Gieseker stable $\alpha$-twisted sheaves on $S$, of dimension $\dim M = 2m$. There is an inclusion as direct summand

$$h(M) \hookrightarrow \bigoplus_{i=1}^r h(S^{k_i})(\ell_i) \quad \text{in } M_{\text{rat}},$$

where $\ell_i \in \mathbb{Z}$ and $1 \leq k_i \leq m$.

**Proof.** We follow Bülles’ proof, with a slight twist to get a better bound on the integers $k_i$ (in [8, Theorem 0.1], the $k_i$ are bounded by $2m$). Let

$$[\text{Ext}_n^\alpha] := \sum_i (-1)^i [\text{Ext}_n^\alpha(\mathcal{E}, \mathcal{F})] \in K_0(M \times M)$$

and
be as in [8] Proof of Theorem 0.1. Then (as explained in loc. cit.) a result of Markman’s gives the equality
\begin{equation}
\Delta_M = c_{2m}(-[\text{Ext}^1_n]) \quad \text{in } A^{2m}(M \times M).
\end{equation}

As in loc. cit., we consider the two-sided ideal in the ring of correspondences
\[ I := \bigcup_{k \geq 1} I_k \subset A^*(M \times M), \]
where \( I_k \) is defined as
\[ I_k := \langle \beta \circ \alpha \mid \alpha \in A^*(M \times S^\ell), \beta \in A^*(S^\ell \times M), 1 \leq \ell \leq k \rangle. \]
Bülles shows [8] Proof of Theorem 0.1 that \( I \) is closed under intersection product, and more precisely that \( I_k \cdot I_\ell \subset I_{k+\ell} \). In addition, let us state a lemma:

**Lemma 2.2.** Let \( \gamma \in I_k \) (for some \( k \geq 1 \), and \( \delta \in A^1(M \times M) \). Then
\[ \gamma \cdot \delta \in I_k. \]

**Proof.** Since the irregularity \( q(M) = 0 \), every divisor \( \delta \in A^1(M \times M) \) can be written as a sum of pullbacks \( (p_i)^*(D_i) \) under the two projections \( p_i : M \times M \to M \). We may thus suppose \( \delta \) is of the form \( D \times M \) or \( M \times D \), where \( D \subset M \) is an irreducible reduced divisor. Let \( \iota : D \to M \) denote the inclusion morphism. We have
\[ \gamma \cdot \delta = \gamma \cdot (D \times M) = (\iota \times \text{id})_*(\iota \times \text{id})^*(\gamma) = (\Gamma_\iota \times \Delta_M)_*(\Gamma_\iota \times \Delta_M)^*(\gamma) \]
\[ = \Gamma_\iota \circ \iota \Gamma_\iota \circ \gamma \quad \text{in } A^*(M \times M), \]
where the last equality is by virtue of Lieberman’s lemma [41 Lemma 3.3], [33 Proposition 2.1.3]. Similarly, in case \( \delta \) is of the form \( M \times D \), we find that
\[ \gamma \cdot \delta = \gamma \cdot (M \times D) = \gamma \circ \Gamma_\iota \circ \iota \Gamma_\iota \quad \text{in } A^*(M \times M). \]
In both cases, it follows that \( \gamma \in I_k \) implies that also \( \gamma \cdot \delta \in I_k \). \( \square \)

Let us write \( c_n := c_n(-[\text{Ext}^1_n]) \in A^n(M \times M) \). As shown by Bülles, we have
\begin{equation}
c_n = (-1)^{n-1}(n-1)! \text{ch}_{m+n} + p(c_1, \ldots, c_{n-1}) \quad \text{in } A^n(M \times M) \quad \forall n \geq 1.
\end{equation}
Here \( \text{ch}_{m+n} \) denotes the degree \( m \) part of the Chern character \( \text{ch}(-[\text{Ext}^1_n]) \in A^*(M \times M) \), and \( p \) is some weighted homogeneous polynomial of degree \( n \). We have \( \text{ch}_{n} \in I_1 \) for all \( n \geq 1 \). In particular, for \( n = 2 \) we find that
\[ c_2 = \frac{1}{2}c_1^2 - \text{ch}_2 \quad \text{in } A^2(M \times M). \]
The class \( c_1 = \text{ch}_1 \) is in \( I_1 \) and so (using lemma 2.2) \( c_1^2 \) is also in \( I_1 \). It follows that
\[ c_2 \in I_1. \]
Likewise, \( c_3 \) can be expressed in terms of \( \text{ch}_3 \in I_1 \) and \( c_1^3 \in I_1 \) and \( c_1 \cdot c_2 \in I_1 \), and so (again using lemma 2.2) we see that
\[ c_3 \in I_1. \]
We now make the claim that
\[ p(c_1, \ldots, c_{n-1}) \in I_{\frac{2}{2}} \]
for any weighted homogeneous polynomial \( p \) of degree \( n \geq 2 \). Let us prove this claim by induction. From what we have just checked, it is clear that the claim is true for \( n = 2, 3 \). Let us now suppose \( n \geq 4 \). The polynomial \( p \) can be decomposed
\[ p(c_1, \ldots, c_{n-1}) = \lambda c_1 \cdot c_{n-1} + \mu c_1^2 c_{n-2} + \nu c_2 c_{n-2} + c_1^2 q(c_1, \ldots, c_{n-3}) + c_2 r(c_1, \ldots, c_{n-3}) \],
where \( \lambda, \mu, \nu \in \mathbb{Q} \) and \( q \) and \( r \) are weighted homogeneous polynomials of degree \( n - 2 \). By the induction hypothesis combined with (2), we know that \( c_1 \cdot c_{n-1} \) and \( c_1^2 c_{n-2} \) are in \( I_{\frac{2}{1}} \); using lemma 2.2 this implies that the terms \( c_1 \cdot c_{n-1} \) and \( c_1^2 c_{n-2} \) are in \( I_{\frac{2}{1}} \). By the induction hypothesis, \( c_{n-2} \in I_{\frac{n-2}{2}} \); since \( c_2 \in I_1 \) this gives \( c_2 c_{n-2} \in I_{\frac{n-2}{2}} \). Again by the induction hypothesis, the polynomials \( q \) and \( r \) are in \( I_{\frac{n-2}{2}} \). Using lemma 2.2 it follows that \( c_1^2 q(c_1, \ldots, c_{n-3}) \in I_{\frac{n-2}{2}} \). Using the fact that \( c_2 \in I_1 \), it follows that \( c_2 r(c_1, \ldots, c_{n-3}) \in I_{\frac{n-2}{2}} \). Altogether, this proves the claim (3).

Claim (3), combined with relation (2) and the fact that \( ch_n \in I_1 \), implies that
\[ c_n \in I_{\frac{n}{2}} \quad \forall n \geq 2 \, . \]
In view of equality (1), it follows that
\[ \Delta_m \in I_m \cap A^{2m}(M \times M) \, , \]
which proves the theorem.

\begin{proof}

\end{proof}

**Remark 2.3.** As noted by Bülls [8, Remark 2.1], Theorem 2.1 is also valid for moduli spaces of \( \sigma \)-stable objects on a K3 surface or abelian surface, where \( \sigma \) is a generic stability condition. For instance, Theorem 2.1 applies to Ouchi’s eightfolds [34] and to the Laza–Saccà–Voisin tenfolds [24].

In [12], we use Theorem 2.1 to prove the generalized Franchetta conjecture for Lehn–Lehn–Sorger–van Straten eightfolds.

**2.2. The Franchetta property.**

**Definition 2.4.** Let \( \pi : X \to B \) be a smooth projective family of varieties, and let us write \( X_b := \pi^{-1}(b) \) for a fibre. We say that the family \( \pi : X \to B \) has the Franchetta property if for any \( \Gamma \in A^2(X) \) there is equivalence
\[ \Gamma|_{X_b} = 0 \quad \text{in} \quad H^8(X_b) \quad \text{for} \quad b \in B \quad \text{very general} \quad \iff \quad \Gamma|_{X_b} = 0 \quad \text{in} \quad A^8(X_b) \quad \text{for} \quad b \in B \quad \text{very general} \, . \]

**Remark 2.5.** In view of [47, Lemma 3.2], the vanishing \( \Gamma|_{X_b} = 0 \) in \( A^8(X_b) \) for \( b \in B \) very general is equivalent to the vanishing \( \Gamma|_{X_b} = 0 \) in \( A^8(X_b) \) for all \( b \in B \).

**Notation 2.6.** Let \( \mathbb{P} \) denote weighted projective space \( \mathbb{P}(1^3, 3) \). Let \( S_{g_2} \to B_{g_2} \) denote the universal family of K3 surfaces of genus 2, where
\[ B_{g_2} \subset \mathbb{P} H^0(\mathbb{P}, \mathcal{O}_2(6)) \]
is the Zariski open parametrizing smooth sections.
Let $S_{g3} \to B_{g3}$ denote the universal family of K3 surfaces of genus 3, where

$$B_{g3} \subset \mathbb{P}H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(4))$$

is the Zariski open parametrizing smooth sections.

**Notation 2.7.** For any family $X \to B$ and $m \in \mathbb{N}$, we write $X^{m/B} := X \times_B \cdots \times_B X$ for the $m$-fold fibre product.

**Theorem 2.8 ([11]).** The families $S_{g2}^{m/B_{g2}} \to B_{g2}$, $m \leq 3$ and $S_{g3}^{m/B_{g3}} \to B_{g3}$, $m \leq 5$ have the Franchetta property.

**Proof.** This is (part of) [11, Theorem 1.5].

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### 3. Multiairative Chow–Künneth Decomposition

**Definition 3.1 (Murre [32]).** Let $X$ be a smooth projective variety of dimension $n$. We say that $X$ has a **CK decomposition** if there exists a decomposition of the diagonal

$$\Delta_X = \pi_X^0 + \pi_X^1 + \cdots + \pi_X^{2n}$$

in $A^n(X \times X)$, such that the $\pi_X^i$ are mutually orthogonal idempotents and $(\pi_X^i)_*H^*(X) = H^i(X)$. Given a CK decomposition for $X$, we set

$$A^i_{(j)}(X) := (\pi_X^{2i-j})_*A^i(X).$$

The CK decomposition is said to be **self-dual** if

$$\pi_X^i = t\pi_X^{2n-i}$$

in $A^n(X \times X) \quad \forall i$.

(Here $t\pi$ denotes the transpose of a cycle $\pi$.)

(NB: “CK decomposition” is short-hand for “Chow–Künneth decomposition”.)

**Remark 3.2.** The existence of a Chow–Künneth decomposition for any smooth projective variety is part of Murre’s conjectures [32, [33]. It is expected that for any $X$ with a CK decomposition, one has

$$A^i_{(j)}(X) \equiv 0 \quad \text{for } j < 0, \quad A^i_{(0)}(X) \cap A^i_{num}(X) \equiv 0.$$

These are Murre’s conjectures B and D, respectively.

**Definition 3.3 (Definition 8.1 in [38]).** Let $X$ be a smooth projective variety of dimension $n$. Let $\Delta_X^{sm} \in A^{2n}(X \times X \times X)$ be the class of the small diagonal

$$\Delta_X^{sm} := \{ (x, x, x) : x \in X \} \subset X \times X \times X.$$

A CK decomposition $\{ \pi_X^i \}$ of $X$ is **multiplicative** if it satisfies

$$\pi_X^k \circ \Delta_X^{sm} \circ (\pi_X^i \otimes \pi_X^j) = 0$$

in $A^{2n}(X \times X \times X) \quad \text{for all } i + j \neq k$.

In that case,

$$A^i_{(j)}(X) := (\pi_X^{2i-j})_*A^i(X)$$

defines a bigraded ring structure on the Chow ring; that is, the intersection product has the property that

$$\text{Im} \left( A^i_{(j)}(X) \otimes A^{i'}_{(j')} \to A^{i+i'}(X) \right) \subseteq A^{i+i'}_{(j+j')}(X).$$
Remark 3.4. The property of having an MCK decomposition is severely restrictive, and is closely related to Beauville’s “splitting property” conjecture [3]. Examples of varieties admitting an MCK decomposition include hyperelliptic curves, K3 surfaces, abelian varieties, cubic hypersurfaces. For more ample discussion and more examples, we refer to [38, Chapter 8], as well as [42, [39], [15], [16], [22], [11].

There are the following useful general results:

Theorem 3.5 (Shen–Vial [38]). Let \( X \) be a hyperkähler fourfold that is birational to a Hilbert square \( S^{[2]} \) where \( S \) is a K3 surface. Then \( X \) has an MCK decomposition.

Proof. The statement for \( S^{[2]} \) is [38, Theorem 13.4] (a more general result is [42, Theorem 1]). The statement for \( X \) then follows by applying the result of Rieß [35] (as duly noted in [42, Introduction]). \(\square\)

Proposition 3.6 (Shen–Vial [38]). Let \( M, N \) be smooth projective varieties that have an MCK decomposition. Then the product \( M \times N \) has an MCK decomposition.

Proof. This is [38, Theorem 8.6], which shows more precisely that the product CK decomposition 
\[
\pi^{k+\ell}_{M \times N} := \sum_{k+\ell = i} \pi^k_M \times \pi^\ell_N \in A^{\dim M + \dim N}((M \times N) \times (M \times N))
\]
is multiplicative. \(\square\)

Theorem 3.7 (Shen–Vial [39]). Let \( M \) be a smooth projective variety, and let \( f : \tilde{M} \to M \) be the blow–up with center a smooth closed subvariety \( N \subset M \). Assume that

1. \( M \) and \( N \) have a self-dual MCK decomposition;
2. the Chern classes of the normal bundle \( N_{N/M} \) are in \( A^*_0(N) \);
3. the graph of the inclusion morphism \( N \to M \) is in \( A^*_0(N \times M) \).

Then \( \tilde{M} \) has a self-dual MCK decomposition, and
\[
f^*A^*_0(M) \subset A^*_0(\tilde{M}) \quad \text{and} \quad f_*A^*_0(\tilde{M}) \subset A^*_0(M) .
\]

Proof. This is [39, Proposition 2.4]. \(\square\)

4. Examples in Dimension 4

Theorem 4.1. Let \( X \) be a hyperkähler fourfold, and assume that \( X \) admits a Lagrangian fibration \( \pi : X \to B \) which is a compactified Jacobian of a family of curves. Let \( A \) be a general fibre of \( \pi \). Then
\[
\text{Im}(A^2(X) \to A^4(X)) = \mathbb{Q}[c_4(T_X)] .
\]

Proof. Thanks to a result of Markushevich [28, Theorem 1.1], we know that \( B \cong \mathbb{P}^2 \) and \( X \cong J_0 \), where \( J_0 \) is the compactified Jacobian of the genus 2 curves arising as hyperplane sections of a
genus 2 K3 surface $S$. The fibration $\pi: J_0 \to \mathbb{P}^2$ occurs in [31, Example 0.6], where it is shown that there is a birational map

$$J_0 \dashrightarrow S^{[2]}$$

which is a Mukai flop. Precisely, $S^{[2]}$ contains a subvariety $P \cong \mathbb{P}^2$ (defined as the pairs of points in $S$ that are in the same fibre of the double cover $S \to \mathbb{P}^2$). There are birational transformations $J_0 \leftarrow \tilde{J}_0 \dashrightarrow S^{[2]}$, where $s$ is the blow-up with center $P \subset S^{[2]}$, and $r$ is the blow-down of the exceptional divisor of $s$ onto a closed subvariety $P' \subset J_0$.

The theorem will follow by combining the following 4 claims:

**Claim 4.2.** The variety $\tilde{J}_0$ has an MCK decomposition, and this induces a splitting $A^2(\tilde{J}_0) = A^{2(0)}(\tilde{J}_0) + A^{2(2)}(\tilde{J}_0)$.

**Claim 4.3.** Let $A$ be a fibre of the Lagrangian fibration $\pi: J_0 \to \mathbb{P}^2$. Then $r^*(A) \in A^{2(0)}(\tilde{J}_0)$.

**Claim 4.4.** Let $A$ be a general fibre of $\pi: J_0 \to \mathbb{P}^2$. The map

$$A^{2(2)}(\tilde{J}_0) \xrightarrow{r^*(A)} A^4(\tilde{J}_0)$$

is zero.

**Claim 4.5.** One has

$$r^*c_4(T_{J_0}) \in A^{4(0)}(\tilde{J}_0).$$

Let us show that these claims imply the theorem; Since $A^2(\tilde{J}_0) = A^{2(2)}(\tilde{J}_0) + A^{2(0)}(\tilde{J}_0)$, we have

$$\text{Im}(A^2(\tilde{J}_0) \xrightarrow{r^*(A)} A^4(\tilde{J}_0)) = A^{2(2)}(\tilde{J}_0) \cdot r^*(A) + A^{2(0)}(\tilde{J}_0) \cdot r^*(A).$$

Using Claim [4.4] this reduces to

$$\text{Im}(A^2(\tilde{J}_0) \xrightarrow{r^*(A)} A^4(\tilde{J}_0)) = A^{2(0)}(\tilde{J}_0) \cdot r^*(A).$$

Using Claim [4.3] we see that $A^{2(0)}(\tilde{J}_0) \cdot r^*(A)$ is contained in $A^{4(0)}(\tilde{J}_0) \cong \mathbb{Q}$. Claim [4.5] plus the fact that $c_4(T_{J_0})$ has strictly positive degree, then implies that

$$A^2(\tilde{J}_0) \cdot r^*(A) \in \mathbb{Q}[r^*c_4(T_{J_0})].$$

Pushing forward to $J_0$, this gives an inclusion

$$A^2(J_0) \cdot A \subset \mathbb{Q}[c_4(T_{J_0})] \subset A^4(J_0).$$

Since the left-hand side is one-dimensional (the intersection of $A$ with 2 ample divisors has strictly positive degree), this inclusion is an equality, proving the theorem.

It remains to prove the claims. To prove Claim [4.2] we use Theorem [3.7] with $M = S^{[2]}$ and $N = P \cong \mathbb{P}^2$. Points (1) and (2) are clearly satisfied. For point (3), we note that $S^{[2]}$ and $P$ have a “universal MCK decomposition”, i.e. there exist

$$\pi_{S^{[2]} \times_B P}^i \in A^6(S^{[2]} \times_B P), \ i = 0, \ldots, 12,$$
such that for each \( b \in B \) the restriction

\[
\pi_{(S_b)^2 \times P_b}^i := \pi_{(S_b)^2 \times P_b}^i \big|_b \in A^6((S_b)^2 \times P_b)
\]
defines an MCK decomposition for \((S_b)^2 \times P_b\). Let \( \iota : P \to (S_b)^2 \) denote the inclusion morphism, and \( \iota_b : P_b \to (S_b)^2 \) the restriction to a fibre. For any \( k \neq 0 \), we have that

\[
(\pi_k^{(S_b)^2 \times P_b})_*(\Gamma_{\iota_b}) = \left( (\pi_k^{(S_b)^2 \times P_b})_*(\Gamma_{\iota_b}) \right) \big|_b \in A^4((S_b)^2 \times P_b)
\]
is homologically trivial. Theorem 2.8 then implies that it is rationally trivial, and so

\[
\text{We have now checked that the conditions of Theorem 3.7 are satisfied, and so } \tilde{J}_0 \text{ has an MCK decomposition. The “blow-up” isomorphism } A^2(\tilde{J}_0) \cong A^2(S^{[2]} \times P) \oplus A^1(P) \text{ is homogeneous with respect to the lower grading. Since } A^2(\tilde{J}_0) = 0 \text{ for } j \in \{0, 2\} \text{ and } A^1(P) = A^1(P), \text{ this shows the second part of claim 4.2.}
\]

Claim 4.2 is elementary: writing \( A = \pi^{-1}(x) \) where \( x \in \mathbb{P}^2 \), we see that \( r^*(A) = r^* \pi^*(x) \) in \( A^2(\tilde{J}_0) \) is an intersection of divisors, which proves the claim. To prove the other 2 claims, we consider things familywise. That is, we let \( S \to B \) denote the universal family of genus 2 K3 surfaces as in notation 2.6, and we write \( S^{[2]} \to B \) for the universal family of Hilbert squares of genus 2 K3 surfaces. There are morphisms of \( B \)-schemes

\[
\mathcal{J} \xleftarrow{r} \tilde{J} \xrightarrow{s} S^{[2]},
\]
such that restriction to a fibre gives the Mukai flop mentioned above. The morphism \( r \) is the blow-up with center \( \mathcal{P} \) (which is a \( \mathbb{P}^2 \)-bundle over \( B \)), and the morphism \( s \) is the blow-up with center \( \mathcal{P} \) (which is again a \( \mathbb{P}^2 \)-bundle over \( B \)). We now establish the following result:

**Proposition 4.6.** Let \( \tilde{J} \to B \) be as above. The families \( \tilde{J} \to B \) and \( \tilde{J} \times_B S \to B \) have the Franchetta property.

**Proof.** For the first family, one notes that there is a commutative diagram

\[
\begin{array}{ccc}
A^i(\tilde{J}) & \to & A^i(S^{[2]}) \oplus A^{i-1}(\mathcal{P}) \\
\downarrow & & \downarrow \\
A^i(\tilde{J}_b) & \cong & A^i((S_b)^2) \oplus A^{i-1}(P_b).
\end{array}
\]
(Here, we write \( \tilde{J}_b, S_b, P_b \) for the fibre over \( b \in B \) of the family \( \tilde{J} \) resp. \( S \) resp. \( \mathcal{P} \).) The family \( S^{[2]} \to B \) has the Franchetta property (Theorem 2.8), and \( P_b \cong \mathbb{P}^2 \) so the family \( \mathcal{P} \to B \) trivially has the Franchetta property. This settles the Franchetta property for \( \tilde{J} \to B \).
For the second family, there is a similar commutative diagram
\[
A^i(\tilde{J} \times_B S) \rightarrow A^i(S^{[2]} \times_B S) \oplus A^{i-1}(P \times_B S)
\]
\[
\downarrow \quad \sim \quad \downarrow
\]
\[
A^i(\tilde{J}_b \times S_b) \rightarrow A^i((S_b)^{[2]} \times S_b) \oplus A^{i-1}(P_b \times S_b)
\].

The family \(S^{[2]} \times S \rightarrow B\) has the Franchetta property (Theorem 2.3, or more exactly [11, Theorem 1.5]), and so does the family \(P \times_B S \rightarrow B\) (using the projective bundle formula, one reduces to \(S \rightarrow B\)). This settles the Franchetta property for \(\tilde{J} \times_B S \rightarrow B\). \[\square\]

Let us now prove the two remaining claims. We will rely on the existence of an MCK decomposition that is generically defined for the family \(\tilde{J} \rightarrow B\), in the following sense:

**Lemma 4.7.** Let \(\tilde{J} \rightarrow B\) be as above. There exist \(\pi^i_{\tilde{J}} \in A^4(\tilde{J} \times_B \tilde{J})\), \(i = 0, \ldots, 8\), such that for each \(b \in B\) the restriction \(\pi^i_{\tilde{J}_b} := \pi^i_{\tilde{J}}|_b \in A^4(\tilde{J}_b \times \tilde{J}_b)\) defines an MCK decomposition for \(\tilde{J}_b\).

**Proof.** The Hilbert squares \((S_b)^{[2]}\) have an MCK decomposition that exists universally (this is just because the “distinguished 0-cycle” of [4] exists universally). Looking at the argument of Theorem 3.7 (i.e. the proof of [39, Proposition 2.4], one sees that the induced MCK decomposition for the blow-up \(\tilde{J}_b\) exists universally as well. \[\square\]

To prove Claim 4.5, we observe that \((r_b)^*c_4(T_{\tilde{J}_b}) = (r^*c_4(T_{\tilde{J}/B}))|_b \in A^4(\tilde{J}_b)\) is universally defined. This forces \((r_b)^*c_4(T_{\tilde{J}_b})\) to lie in \(A^4_{(0)}(\tilde{J}_b)\): for any \(k \neq 8\), we have that
\[
(\pi^k_{\tilde{J}_b})_*(\pi^k_{\tilde{J}})_*(r_b)^*c_4(T_{\tilde{J}_b})) = \left( (\pi^k_{\tilde{J}})_*(r^*c_4(T_{\tilde{J}/B}))\right)|_b \in A^4(\tilde{J}_b)
\]
is homologically trivial, for all \(b \in B\). In view of Proposition 4.6, this implies
\[
(\pi^k_{\tilde{J}_b})_*(r_b)^*c_4(T_{\tilde{J}_b})) = 0 \quad \text{in} \quad A^4(\tilde{J}_b) \quad \forall b \in B, \quad \forall k \neq 8,
\]
proving claim 4.5.

To prove Claim 4.4, let \(A \subset \mathcal{J}\) denote a general fibre of \(\pi: \mathcal{J} \rightarrow \cup_{b \in B}|O_{J_b}(1)|\), and let \(\tilde{A}\) denote a general fibre of \(\pi \circ r\). Let us write \(\tau: \tilde{A} \rightarrow \tilde{J}\) for the inclusion. We are interested in the correspondence
\[
\Gamma_b := \tau_{\tilde{A}} \circ \Gamma_{\tilde{J}_b} \circ \pi^2_{\tilde{J}_b} \in A^6(\tilde{J}_b \times \tilde{J}_b),
\]
which by construction is such that
\[
(\Gamma_b)_*A^2(\tilde{J}_b) = A^2_{(2)}(\tilde{J}_b) \cdot (r_b)^*(A_b).
\]
The correspondence $\Gamma_b$ is universally defined, i.e. there exists $\Gamma \in A^6(\tilde{\mathcal{J}} \times_B \tilde{\mathcal{J}})$ such that

$$\Gamma_b = \Gamma|_b \in A^6(\tilde{J}_b \times \tilde{J}_b) \quad \forall b \in B.$$ 

Since $A_b \subset J_b$ is Lagrangian, the cup product of $A_b$ with $H^{2,0}(J_b)$ is zero. By a standard Hodge theory argument, this means that the cup product of $A_b$ with the transcendental cohomology $H^2_{tr}(J_b)$ is also zero. Since $H^2_{tr}$ is a birational invariant, the same holds on $\tilde{J}_b$, and so the map

$$H^2(\tilde{J}_b) \xrightarrow{\Gamma_b} H^6(\tilde{J}_b)$$

is the same as the map

$$N^1H^2(\tilde{J}_b) \xrightarrow{\sim} N^3H^6(\tilde{J}_b)$$

(where $N^iH^j(\cdot)$ denotes the algebraic classes in cohomology). It follows that there exist (for each $b \in B$) a finite union of curves $C_b \subset \tilde{J}_b$ and a cycle $\gamma_b$ supported on $C_b \times C_b$ such that

$$\Gamma_b = \gamma_b \text{ in } H^{12}(\tilde{J}_b \times \tilde{J}_b).$$

(Indeed, for $C_b$ one can take a basis of $N^3H^6(\tilde{J}_b)$, and add curves forming a dual basis to $N^1H^2(\tilde{J}_b)$.) Using Voisin’s Hilbert schemes argument as in [44, Proposition 3.7], these fibre-wise data can be spread out, i.e. there exist a finite union of codimension 3 closed subvarieties $C \subset \tilde{\mathcal{J}}$ and a cycle $\gamma$ supported on $C \times_B C \subset \tilde{\mathcal{J}} \times_B \tilde{\mathcal{J}}$ with the property that

$$(\Gamma - \gamma)|_b = 0 \quad \text{in } H^{12}(\tilde{J}_b \times \tilde{J}_b) \quad \forall b \in B.$$

At this point, we need another lemma:

**Lemma 4.8.** Set-up as above. There exist relative correspondences

$$\Theta_1, \Theta_2 \in A^4(S \times_B \tilde{\mathcal{J}}), \quad \Xi_1, \Xi_2 \in A^2((\tilde{\mathcal{J}} \times_B S)$$

such that for each $b \in B$, the composition

$$A^2(\tilde{J}_b) \xrightarrow{(\Xi_1|_b, \Xi_2|_b)} A^2(S_b) \oplus A^2(S_b) \xrightarrow{((\Theta_1 + \Theta_2)|_b)} A^2(\tilde{J}_b)$$

is the identity.

**Proof.** By virtue of Theorem 3.7, the isomorphism

$$A^2(\tilde{J}_b) \cong A^2((S_b)^{[2]} \oplus A^1(P_b)$$

respects the bigrading. Since $A^1_{(2)}(P_b) = 0$, it follows that

$$A^2_{(2)}(\tilde{J}_b) \xrightarrow{(s_b)^*} A^2_{(2)}((S_b)^{[2]}) \xrightarrow{(s_b)^*} A^2(\tilde{J}_b)$$

is the identity. Let $\Psi_b \in A^4((S_b)^{[2]} \times (S_b)^2)$ be the correspondence such that $(\Psi_b)^*(\Psi_b)_* = id$ on $A^2_{(2)}((S_b)^{[2]})$. This $\Psi_b$ is obviously the restriction of a relative correspondence $\Psi$ (cf. for instance [21 Proof of Corollary 3.4]). The argument of [21 Proposition 2.15] gives that $A^2_{(2)}((S_b)^{[2]}$ factors (via universally defined correspondences) over $A^2(S_b) \oplus A^2(S_b)$. Composing with $\Psi \circ \Gamma_s$ and its transpose, we obtain the required relative correspondences.  \[\Box\]
Let us now return to the relative correspondence \( \Gamma - \gamma \in A^6(\mathcal{F} \times_B \mathcal{F}) \) constructed above. We define the compositions
\[
\Gamma_i := (\Gamma - \gamma) \circ \Theta_i \in A^6(S \times_B \mathcal{F}) \quad (i = 1, 2).
\]
In view of (4), these correspondences are fibrewise homologically trivial:
\[
(\Gamma_i)|_b = 0 \quad \text{in} \quad H^{12}(S_b \times \mathcal{F}) \quad \forall b \in B \quad (i = 1, 2).
\]
Applying proposition 4.6, it follows that they are fibrewise rationally trivial:
\[
(\Gamma_i)|_b = 0 \quad \text{in} \quad A^6(S_b \times \mathcal{F}) \quad \forall b \in B \quad (i = 1, 2).
\]
But then a fortiori
\[
(\Gamma_i)|_b \circ (\Xi_i)|_b = (\Gamma - \gamma)|_b \circ (\Theta_i)|_b \circ (\Xi_i)|_b = 0 \quad \text{in} \quad A^6(\mathcal{F} \times \mathcal{F}) \quad \forall b \in B \quad (i = 1, 2).
\]
Taking the sum, this implies the fibrewise vanishing
\[
(\Gamma - \gamma)|_b \circ (\Theta_1 \circ \Xi_1 + \Theta_2 \circ \Xi_2)|_b = 0 \quad \text{in} \quad A^6(\mathcal{F} \times \mathcal{F}) \quad \forall b \in B.
\]
In view of Lemma 4.8, we find that
\[
((\Gamma - \gamma)|_b)_\ast : A^2(\mathcal{F}) \rightarrow A^4(\mathcal{F}) \quad \forall b \in B.
\]
But the correspondence \( \gamma|_b \) does not act on \( A^2(\mathcal{F}) \) for dimension reasons, and so
\[
(\Gamma)|_b \ast = 0 : A^2(\mathcal{F}) \rightarrow A^4(\mathcal{F}) \quad \forall b \in B.
\]
Since (by construction) \( \Gamma|_b = \Gamma_b \) acts on \( A^2(\mathcal{F}) \) as multiplication by \( A_b \), this proves claim 4.4.

**Remark 4.9.** The fourfold \( J_0 \), being birational to \( S^{[2]} \), has an MCK decomposition (theorem 3.5). In proving theorem 4.1 it would be more natural to use this MCK decomposition of \( J_0 \), rather than the one of \( \mathcal{F} \). However, when trying to do this one runs into the following problem: it is not clear whether the MCK decomposition of \( J_0 \) is universal (in the sense of lemma 4.7); for this one would need to know that the correspondence \( Z \) constructed in [35] is universally defined. (On a related note, it is not clear whether the map \( (r_b)_\ast : A^*(\mathcal{F}) \rightarrow A^*(\mathcal{F}) \) respects the bigrading coming from the two MCK decompositions, i.e. I have not been able to prove that \( r_b \) is “of pure grade 0” in the sense of [39].)

**5. Examples in dimension 6**

**Theorem 5.1** (Mukai [31]). Let \( S \subset \mathbb{P}^3 \) be a quartic K3 surface, and assume that every element in \( |\mathcal{O}_S(1)| \) is irreducible. Let \( \pi : J_1 \rightarrow |\mathcal{O}_S(1)| \cong \mathbb{P}^3 \) be the component of the compactified Picard scheme that parametrizes torsion free degree 1 line bundles \( \xi \) on curves \( C \in |\mathcal{O}_S(1)| \).

The subset \( P \subset J_1 \) parametrizing line bundles \( \xi \) such that \( H^0(C, \xi) \neq 0 \) has the structure of a \( \mathbb{P}^2 \)-bundle over \( S \). The flop of \( P \subset J_1 \) is isomorphic to the moduli space \( M_v(S) \), where \( v \) is the Mukai vector \( v = (3, \mathcal{O}_S(-1), 0) \).

**Proof.** This is [31] Example 0.8. \( \square \)
Theorem 5.2. Let \( J_i \) and \( P \) be as in theorem 5.1. Let \( h_1 \in A^1(J_1) \) be the polarization class, and let \( h_2 := \pi^*(d) \in A^1(J_1) \) where \( d \subset \mathbb{P}^3 \) is a hyperplane class. Let \( R^*(J_1) \) be the \( \mathbb{Q} \)-subalgebra
\[
R^*(J_1) := \langle h_1, h_2, P, c_j(T_{J_1}) \rangle \subset A^*(J_1).
\]

The cycle class map induces an injection
\[
R^*(J_1) \hookrightarrow H^*(J_1, \mathbb{Q}).
\]

Proof. Let \( J \to B \) be the universal family of sixfolds \( J_i \) as in theorem 5.1 (here \( B \) is some open in the parameter space \( B_{g3} \) of notation 2.6). We will prove that \( J \to B \) has the Franchetta property. Since the classes defining the subring \( R^*(J_1) \) are universally defined (i.e., they are restrictions of classes in \( A^*(J_1) \)), this settles the theorem.

We claim that there exist morphisms of \( B \)-schemes
\[
J \xleftarrow{J} \tilde{J} \xrightarrow{s} M,
\]
where \( M \to B \) is the universal moduli space with Mukai vector \( v = (3, \mathcal{O}_S(-1), 0) \), and \( r: \tilde{J} \to J \) is the blow-up of \( P \) (the relative version of \( P \)) and \( s: \tilde{J} \to M \) is the blow-up of \( \mathcal{P}' \) (the relative version of the dual \( \mathbb{P}^2 \)-bundle \( P' \subset M_B \)). To ascertain that \( M \) and \( s \) exist as claimed, one may reason as follows: \( \mathcal{P} \subset \mathcal{J} \) can obviously be defined and has the structure of a \( \mathbb{P}^2 \)-bundle over \( S \). Let \( r: \tilde{J} \to J \) be the blow-up with center \( \mathcal{P} \), and let \( \mathcal{E} \subset \tilde{J} \) denote the exceptional divisor of \( r \). This \( \mathcal{E} \) maps to \( \mathcal{P}' \), which is the dual \( \mathbb{P}^2 \)-bundle over \( S \). The Nakano–Fujiki criterion for the existence of a blow-down [17], as used by Mukai [31]. Proof of Theorem 0.7 needs that the normal bundle of \( \tilde{J} \subset \tilde{J} \) restricts to the tautological bundle of the fibres \( F_p \) of \( \mathcal{E} \to \mathcal{P}' \). Since \( \mathcal{N}_{\mathcal{E}/\tilde{J}}|_{F_p} = \mathcal{N}_{E_b/F_p} \), and the criterion is satisfied fibrewise, this is OK. That is, thanks to Nakano–Fujiki we conclude that there exists a blow-down \( s: \tilde{J} \to M \) with \( M \) smooth and \( s(\mathcal{E}) = \mathcal{P}' \), as claimed.

To prove the Franchetta property for \( J \), it suffices to prove the Franchetta property for \( \tilde{J} \). On the other hand, the morphism \( s \) is the blow-up with center \( \mathcal{P}' \), and \( \mathcal{P}' \) is a \( \mathbb{P}^2 \)-bundle over \( S \). The formulae for Chow groups of blow-ups and projective bundles give a commutative diagram
\[
\begin{align*}
A^i(\tilde{J}) & \to A^i(M) \oplus A^{i-1}(S) \oplus A^{i-2}(S) \oplus A^{i-3}(S) \\
\downarrow & \downarrow \\
A^i(\tilde{J}_b) & \xrightarrow{\cong} A^i(M_b) \oplus A^{i-1}(S_b) \oplus A^{i-2}(S_b) \oplus A^{i-3}(S_b).
\end{align*}
\]
(Here, we write \( \tilde{J}_b, M_b, S_b \) for the fibre over \( b \in B \) of the family \( \tilde{J} \) resp. \( M \) resp. \( S \).) Since we already know the Franchetta property holds for \( S \to B \), the Franchetta property for \( \tilde{J} \to B \) follows from that for \( M \to B \). Thus, the following result settles the proof of theorem 5.2.

Proposition 5.3. The family \( M \to B \) has the Franchetta property.

To prove proposition 5.3, we use Bülles’ result (Theorem 2.1) to reduce to the family \( S^{3/B} := S \times_B S \times_B S \). That is, Theorem 2.1 tells us that for every \( b \in B \) there exist correspondences
\[
\Gamma_1^b, \ldots, \Gamma_r^b \in A^*(M_b \times (S_b)^{k_j}), \quad \Psi_1^b, \ldots, \Psi_r^b \in A^*((S_b)^{k_j} \times M_b).
\]
with the property that

$$
\Delta_{M_b} = \sum_{j=1}^{r} \Psi_j^b \circ \Gamma_j^b \quad \text{in} \quad A^6(M_b \times M_b).
$$

Using a Hilbert schemes argument as in [44, Proposition 3.7], these fibrewise data can be spread out over the family, i.e. there exist relative correspondences

$$
\Gamma_1, \ldots, \Gamma_r \in A^*(\mathcal{M} \times_B S^{k_j/B}), \quad \Psi_1, \ldots, \Psi_r \in A^*(S^{k_j/B} \times_B \mathcal{M})
$$

with the property that

$$
(5) \quad \Delta_{M_b} = \sum_{j=1}^{r} (\Psi_j \circ \Gamma_j)|_{M_b \times M_b} \quad \text{in} \quad A^6(M_b \times M_b) \quad \forall b \in B.
$$

(Alternatively, instead of invoking a Hilbert schemes argument, one may observe that the cycles in [8] are universal expressions in the Chern classes of a quasi-universal object, and thus naturally can be constructed in families. This is the same argument as [12, Proof of Theorem 3.1].)

Now, given a cycle \( \Gamma \in A^*(\mathcal{M}) \) which is homologically trivial on the very general fibre, the element

$$
(\Gamma_1 \circ \Gamma, \ldots, \Gamma_r \circ \Gamma) \in A^*(S^{k_1/B}) \oplus \cdots \oplus A^*(S^{k_r/B})
$$

will also be homologically trivial on the very general fibre. The families \( S^{k_j/B} \) have the Franchetta property by Theorem 2.8 (note that \( k_j \leq 3 \)), and so it follows that

$$
(\Gamma_1 \circ \Gamma, \ldots, \Gamma_r \circ \Gamma)|_b = (0, \ldots, 0) \quad \text{in} \quad A^*(S^{k_1}) \oplus \cdots \oplus A^*(S^{k_r}),
$$

for \( b \in B \) very general. But then, in view of (5), it follows that also

$$
\Gamma|_b = \sum_{j=1}^{r} (\Psi_j \circ \Gamma_j \circ \Gamma)|_b = 0 \quad \text{in} \quad A^*(\mathcal{M}),
$$

for \( b \in B \) very general, i.e. \( \mathcal{M} \to B \) has the Franchetta property.

This proves the proposition, and hence the theorem.

\[ \square \]

**Remark 5.4.** A general fibre \( A \) of the Lagrangian fibration \( J_1 \to \mathbb{P}^3 \) has \( A = h_2^3 \) in \( A^3(J_1) \) (because a point \( p \in \mathbb{P}^3 \) has \( p = d^3 \) in \( A^3(\mathbb{P}^3) \), with \( d \) the hyperplane class). As such, \( A \) is in the subalgebra \( R^*(J_1) \) of Theorem 5.2.

**Remark 5.5.** It would be interesting to prove something like Theorem 4.1 for the sixfolds \( J_1 \) of theorem 5.2. To do this, it would suffice to have an MCK decomposition for \( J_1 \) that is universal, and with the property that \( A^2_{(2)}(J_1) \) comes from \( A^2(S) \).

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