Thermal Stability Of Charged Rotating Quantum Black Holes

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Criteria for thermal stability of charged rotating black holes of any dimension are derived, for horizon areas that are large relative to the Planck area (in these dimensions). The derivation uses results of loop quantum gravity and equilibrium statistical mechanics of the Grand Canonical ensemble. There is no explicit use of classical spacetime geometry at all in this analysis. The only assumption is that the mass of the black hole is a function of its horizon area, charge and angular momentum. Our stability criteria are then tested in detail against specific classical black holes in spacetime dimensions 4 and 5, whose metrics provide us with explicit relations for the dependence of the mass on the charge and angular momentum of the black holes. This enables us to predict which of these black holes are expected to be thermally unstable under Hawking radiation.

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I. INTRODUCTION

It is well-known from semiclassical analysis that nonextremal, asymptotically flat black holes are thermally unstable due to decay under Hawking radiation, leading to their specific heat being negative [1]. This interesting fact has motivated the study of thermal stability of black holes, from a perspective that relies on a definite proposal for quantum spacetime (like Loop Quantum Gravity, [2, 3]) rather than on semiclassical assumptions. A consistent understanding of the issue of quantum black hole entropy has been obtained through Loop Quantum Gravity [4, 7], where not only has the Bekenstein-Hawking area law been retrieved for macroscopic (astrophysical) black holes, but a whole slew of corrections to it, due to quantum spacetime fluctuations have been derived as well [5-11], with the leading correction being logarithmic in area with the coefficient $-\frac{3}{2}$.

The implications of this quantum perspective, on the thermal stability of black holes from decay due to Hawking radiation, has therefore been an important aspect of black hole thermodynamics beyond semiclassical analysis, and also somewhat beyond the strictly equilibrium configurations that Isolated Horizons represent. Classically a black hole in general relativity is characterized by its mass ($M$), charge ($Q$) and angular momentum ($J$). Intuitively, therefore, we expect that thermal behaviour of black holes will depend on all of these parameters. For a given classical metric characterizing a black hole, the mass can be derived explicitly to be a function of the other ‘hairs’, viz., the charge and angular momentum. However, the quantum spacetime perspective frees us from having to use classical formulae for this functional dependence of the mass. Instead, the assumption is simply this: the mass is a monotonically increasing function of the horizon area, along with the charge and angular momentum.

The simplest case of vanishing charge and angular momentum has been investigated longer than a decade ago [12 - 14]. This has been generalized, via the idea of thermal holography [15, 16], and the saddle point approximation to evaluate the canonical partition function corresponding to the horizon, retaining Gaussian thermal fluctuations. The consequence is a general criterion of thermal stability as an inequality connecting area derivatives of the mass and the microcanonical entropy. This inequality is nontrivial only when the microcanonical entropy has corrections (of a particular algebraic sign) beyond the area law, as is the case for the loop quantum gravity calculation of the microcanonical entropy [7]. This body of work has been generalized more recently [17] to include black holes with charge. The generalized stability criterion indeed ‘predicts’ the thermal instability of asymptotically flat Reissner-Nordstrom black holes contrasted with the thermal stability of anti-de Sitter Reissner-Nordstrom black holes (for a range of cosmological constants).

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In this paper, this approach is generalized to quantum black holes carrying both charge and angular momentum, thus covering the entire gamut of general relativistic black holes. Further, several other types of black holes in diverse spacetime dimensions are also considered as examples where the ensuing thermal stability criteria are applicable, either in a positive or in a negative sense. One thus has a substantive list of predictions for black hole spacetimes which, even though formal constructs at the moment, may have elements of reality awaiting discovery, and therefore of prospective interest from a phenomenological perspective. On the basis of the thermal stability criteria derived here, one is thus able to predict quantitatively the decay properties of such black holes under Hawking radiation. 

The paper is organized as follows: In section 2, the idea of thermal holography, along with the concept of (holographic) mass associated with horizon of a black hole is briefly reviewed and the grand canonical entropy of charged rotating large quantum black hole is determined. In section 3, the criterion for thermal stability of such black holes is determined by using saddle point approximation to evaluate the horizon partition function for Gaussian thermal fluctuations around thermal equilibrium. In the next section, this stability criterion is used to test on various explicit classical black holes, with the objective of predicting their behaviour under decay due to Hawking radiation. We end in section 5 with a brief summary and outlook.

II. THERMAL HOLOGRAPHY

In this section, we present a generalization of the thermal holography for non-rotating electrically charged quantum radiant horizons discussed in [17], to the situation when the horizon has both charge and angular momentum. Such a generalization completes the task set out in [12] and [15] to include all possible general relativistic hair in consideration of thermal stability of the horizon under Hawking radiation. To make this section self-contained, some overlap with ref. [17] is inevitable.

A. Mass Associated With horizon

Black holes at equilibrium are represented by isolated horizons, which are internal boundaries of spacetime. Hamiltonian evolution of this spacetime gives the first law associated with isolated horizon, which is given as,

\[ \delta E_t^h = \frac{\kappa_t}{8\pi} \delta A_h + \Phi_t \delta Q_h + \Omega_t \delta J_h \]  

where, \( E_t^h \) is the energy function associated with the horizon, \( \kappa_t \), \( \Phi_t \) and \( \Omega_t \) are respectively the surface gravity, electric potential and angular velocity of the horizon; \( Q_h, A_h \) and \( J_h \) are respectively the charge, area and angular momentum of the horizon. The label ‘\( t \)’ denotes the particular time evolution field (\( t^\mu \)) associated with the spatial hypersurface chosen. This hypersurface foliates the horizon. \( E_t^h \) is assumed here to be a function of \( A_h, Q_h \) and \( J_h \).

The advantage of the isolated (and also the radiant or dynamical) horizon description is that one can associate with it a mass \( M_t^h \), related to the ADM energy of the spacetime through the relation

\[ E_{t,ADM}^h = M_t^h + E_{t,rad}^h \]  

where, \( E_{t,rad}^h \) is the energy associated with spacetime between the horizon and asymptopia. For stationary spacetimes, the existence of a global timelike killing vector requires that \( E_{t,rad}^h = 0 \), implying that the ADM energy, measured at asymptopia, is the only definition of the energy of such a spacetime. An isolated horizon, in contrast, does not require stationarity, and therefore admits \( E_{t,rad}^h \neq 0 \), and hence a mass defined locally on the horizon.

B. Quantum Geometry

The Hilbert space of a generic quantum spacetime is given as, \( \mathcal{H} = \mathcal{H}_b \otimes \mathcal{H}_v \), where \( b(v) \) denotes the boundary (bulk) space. A generic quantum state is thus given as

\[ |\Psi\rangle = \sum_{b,v} C_{b,v} |\chi_b\rangle \otimes |\psi_v\rangle \]  

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Now, the full Hamiltonian operator ($\hat{H}$), operating on $\mathcal{H}$ is given by

$$\hat{H}\Psi = (\hat{H}_b \otimes I_v + I_b \otimes \hat{H}_v)\Psi$$

(4)

where, respectively, $I_b(I_v)$ are identity operators on $\mathcal{H}_b(\mathcal{H}_v)$ and $\hat{H}_b(\hat{H}_v)$ are the Hamiltonian operators on $\mathcal{H}_b(\mathcal{H}_v)$.

The first class constraints are realized on Hilbert space as annihilation constraints on physical states. The bulk Hamiltonian operator thus annihilates bulk physical states

$$\hat{H}_v|\psi_v\rangle = 0$$

(5)

The bulk quantum spacetime is assumed to be free of electric charge and angular momentum, so that eqn. (3) is augmented by the relation

$$[\hat{H}_v - \Phi \hat{Q}_v - \Omega \hat{J}_v]|\psi_v\rangle = 0.$$  

(6)

C. Grand Canonical Partition Function

Consider the black hole immersed in a heat bath, at some (inverse) temperature $\beta$, with which it can exchange energy, charge and angular momentum. The grand canonical partition function of the black hole is given as,

$$Z_G = Tr(exp(-\beta \hat{H} + \beta \Phi \hat{Q} + \beta \Omega \hat{J}))$$

(7)

where the trace is taken over all states. This definition, together with eqn.s (3) and (6) yield

$$Z_G = \sum_{b,v} |C_{b,v}|^2 \langle \psi_v | \psi_v \rangle \langle \chi_b | exp(-\beta \hat{H} + \beta \Phi \hat{Q} + \beta \Omega \hat{J}) | \chi_b \rangle$$

$$= \sum_b |C_b|^2 \langle \chi_b | exp(-\beta \hat{H} + \beta \Phi \hat{Q} + \beta \Omega \hat{J}) | \chi_b \rangle ,$$

(8)

assuming that the bulk states are normalized. The partition function thus turns out to be completely determined by the boundary states ($Z_{Gb}$), i.e.,

$$Z = Z_{Gb} = Tr_b exp(-\beta \hat{H} + \beta \Phi \hat{Q} + \beta \Omega \hat{J})$$

$$= \sum_{k,l,m} g(k,l,m) \exp(-\beta (E(A_k, Q_l, J_m) - \Phi Q_l - \Omega J_m)) .$$

(9)

Where $g(k,l,m)$ is the degeneracy corresponding to energy $E(A_k, Q_l, J_m)$ and $k,l,m$ are the quantum numbers corresponding to area, charge and angular momentum respectively. Here, the spectrum of the boundary Hamiltonian operator is assumed to be a function of area, charge and angular momentum of the boundary, considered here to be the horizon. Following [17], it is further assumed that these ‘quantum hairs’ all have a discrete spectrum. In the semiclassical limit of quantum isolated horizons of macroscopic area, $A_h > l_p^2$, they all have large eigenvalues i.e. $(k,l,m >> 1)$, so that, application of the Poisson resummation formula [12] gives

$$Z_G = \int dx \ dy \ dz \ g(A(x), Q(y), J(z)) \ \exp(-\beta (E(A(x), Q(y), J(z)) - \Phi Q(y) - \Omega J(z)))$$

(10)

where $x, y, z$ are respectively the continuum limit of $k, l, m$ respectively.

Following [17], we now assume that the semiclassical spectrum of the area, charge and angular momentum are linear in their arguments, so that a change of variables gives, with constant Jacobian, the result

$$Z_G = \int dA \ dQ \ dJ \ \exp[S(A) - \beta (E(A, Q, J) - \Phi Q - \Omega J)] ,$$

(11)

where, following [18], the microcanonical entropy of the horizon is defined by $exp S(A) \equiv \frac{g(A, Q, J(z))}{4\pi^{3/2} l_p^3}$. 
III. STABILITY AGAINST GAUSSIAN FLUCTUATIONS

A. Saddle Point Approximation

The equilibrium configuration of black hole is given by the saddle point $\bar{A}, \bar{Q}, \bar{J}$ in the three dimensional space of integration over area, charge and angular momentum. This configuration is identified with an isolated horizon, as already mentioned. The idea now is to examine the grand canonical partition function for fluctuations $a = (\bar{A} - A), q = (\bar{Q} - Q), j = (\bar{J} - J)$ around the saddle point, in order to determine the stability of the equilibrium isolated horizon under Hawking radiation. We restrict our attention to Gaussian fluctuations. Taylor expanding eqn (11) about the saddle point, yields

$$Z_G = \exp[S(\bar{A}) - \beta M(\bar{A}, \bar{Q}, \bar{J}) + \beta \Phi \bar{Q} + \beta \Omega \bar{J}]$$

$$\times \int da \ dq \ dj \ \exp(-\frac{\beta}{2}(S_{AA} - \frac{S_{AA}}{\beta})a^2 + (M_{QQ})q^2 + (2M_{AQ})aq)$$

$$+ (M_{JJ})^2 + (2M_{AJ})aj + (2M_{QJ})qj$$,

where $M(\bar{A}, \bar{Q}, \bar{J})$ is the mass of equilibrium isolated horizon. Here $M_{AQ} \equiv \partial^2 M/\partial A \partial Q|_{(\bar{A}, \bar{Q}, \bar{J})}$ etc.

Observe that all observables of Loop Quantum Gravity used here are self-adjoint operators over the boundary Hilbert space, and hence their eigenvalues are real. It suffices therefore to restrict integrations over the spectra of these operators to the real axes.

Now, in the Saddle point approximation the coefficients of terms linear in $a, q, j$ vanish by definition of the saddle point. These imply that, at saddle point

$$\beta = \frac{S_A}{M_A}, \quad M_Q = \Phi, M_J = \Omega$$

B. Stability Criteria

Convergence of the integral (12) implies that the Hessian matrix $(H)$ has to be positive definite, where

$$H = \begin{pmatrix}
\beta M_{AA}(\bar{A}, \bar{Q}, \bar{J}) - S_{AA}(\bar{A}, \bar{Q}, \bar{J}) & \beta M_{AQ}(\bar{A}, \bar{Q}, \bar{J}) & \beta M_{AJ}(\bar{A}, \bar{Q}, \bar{J}) \\
\beta M_{AQ}(\bar{A}, \bar{Q}, \bar{J}) & \beta M_{QQ}(\bar{A}, \bar{Q}, \bar{J}) & \beta M_{QJ}(\bar{A}, \bar{Q}, \bar{J}) \\
\beta M_{AJ}(\bar{A}, \bar{Q}, \bar{J}) & \beta M_{QJ}(\bar{A}, \bar{Q}, \bar{J}) & \beta M_{JJ}(\bar{A}, \bar{Q}, \bar{J})
\end{pmatrix}$$

The necessary and sufficient conditions for a real symmetric square matrix to be positive definite is: 'determinants all principal square submatrices, and the determinant of the full matrix, are positive.' This condition leads to the following 'stability criteria':

$$\beta M_{AA}(\bar{A}, \bar{Q}, \bar{J}) - S_{AA}(\bar{A}, \bar{Q}, \bar{J}) > 0$$

$$\beta M_{QQ}(\bar{A}, \bar{Q}, \bar{J}) > 0$$

$$\beta M_{JJ}(\bar{A}, \bar{Q}, \bar{J}) > 0$$

$$M_{QQ}(\bar{A}, \bar{Q}, \bar{J})M_{JJ}(\bar{A}, \bar{Q}, \bar{J}) - (M_{QJ}(\bar{A}, \bar{Q}, \bar{J}))^2 > 0$$

$$M_{JJ}(\bar{A}, \bar{Q}, \bar{J})(\beta M_{AA}(\bar{A}, \bar{Q}, \bar{J}) - S_{AA}(\bar{A}, \bar{Q}, \bar{J})) - \beta (M_{AJ}(\bar{A}, \bar{Q}, \bar{J}))^2 > 0$$

$$M_{QQ}(\bar{A}, \bar{Q}, \bar{J})(\beta M_{AA}(\bar{A}, \bar{Q}, \bar{J}) - S_{AA}(\bar{A}, \bar{Q}, \bar{J})) - \beta (M_{AQ}(\bar{A}, \bar{Q}, \bar{J}))^2 > 0$$

$$[(\beta M_{AA}(\bar{A}, \bar{Q}, \bar{J}) - S_{AA}(\bar{A}, \bar{Q}, \bar{J}))(M_{QQ}(\bar{A}, \bar{Q}, \bar{J})M_{JJ}(\bar{A}, \bar{Q}, \bar{J}) - (M_{QJ}(\bar{A}, \bar{Q}, \bar{J}))^2)$$

$$-\beta M_{AQ}(\bar{A}, \bar{Q}, \bar{J})(M_{AQ}(\bar{A}, \bar{Q}, \bar{J})M_{JQ}(\bar{A}, \bar{Q}, \bar{J}) - M_{QJ}(\bar{A}, \bar{Q}, \bar{J})M_{AJ}(\bar{A}, \bar{Q}, \bar{J}))$$

$$+ \beta M_{AJ}(\bar{A}, \bar{Q}, \bar{J})(M_{AQ}(\bar{A}, \bar{Q}, \bar{J})M_{QJ}(\bar{A}, \bar{Q}, \bar{J}) - M_{QQ}(\bar{A}, \bar{Q}, \bar{J})M_{AJ}(\bar{A}, \bar{Q}, \bar{J}))] > 0$$

Of course, (inverse) temperature $\beta$ is assumed to be positive for a stable configuration. What is new is the requirement that this temperature must increase with horizon area, inherent in the positivity of the quantity $(\beta M_{AA} - S_{AA})$ which appears in several of the stability criteria. If this is violated, as for example in case of the standard Schwarzschild black hole, thermal instability is inevitable.

The convexity property of the entropy follows from the condition of convergence of partition function under gaussian fluctuations [12, 18, 21]. The thermal stability is related to the convexity property of entropy.
Hence, the above conditions are correctly the conditions for thermal stability. For chargeless, non-rotating horizons, eqn. [15] reproduces the thermal stability criterion and condition of positive specific heat given in [13], as expected. For charged, non-rotating black holes, eqns. [15], [16] and [20] describe the stability, in perfect agreement with [17], while [15], [17] and [19] describe the thermal stability criteria for uncharged rotating radiant horizons. The new thing for black holes with both charge and angular momentum is that not only does the specific heat has to be positive for stability, but the charge and the angular momentum play important roles as well.

As claimed in the Introduction, the thermal stability criteria above are derived by the application of standard statistical mechanical formalism to a quantum horizon characterized by various observables having discrete eigenvalue spectra. Thus, no aspect of classical geometry enters the derivation of these criteria. Given the classical metrics specifying various classical black hole spacetimes, the mass can be obtained as an explicit function of the area, charge and angular momentum of the horizon. It is then possible, on the basis of our stability criteria, to predict which classical black holes will radiate away to extinction, and which ones might find some stability, and for what range of in parameters. This is what is attempted in the next section.

### IV. PREDICTING THERMAL STABILITY OF CLASSICAL BLACK HOLES

Notice that in the stability criteria derived in the last section, first and second order derivatives of the microcanonical entropy of the horizon at equilibrium play a crucial role, in making some of the criteria non-trivial. Thus, corrections to the microcanonical entropy beyond the Bekenstein-Hawking area law, arising due to quantum spacetime fluctuations, are very significant, because without these, some of the stability criteria might lose their essential physical content. It has been shown that [6] the microcanonical entropy for macroscopic isolated horizons has the form

$$S = S_{BH} - \frac{3}{2} \log S_{BH} + O(S_{BH}^{-1})$$  \hspace{1cm} (22)

$$S_{BH} = \frac{A_B}{4A_P}, \quad A_P \to \text{Planck area}.$$  \hspace{1cm} (23)

The corrections beyond $S_{BH}$ in eqn. (22) are responsible for the non-triviality of the thermal stability criteria discussed above.

#### A. Kerr-Newman Black Hole

The Kerr-Newman metric of asymptotically flat Black Hole is given in Boyer-Lindquist coordinates as

$$ds^2 = -\frac{\Delta}{\Sigma} (dt - a \sin^2 \theta \, d\phi)^2 + \frac{\sin^2 \theta}{\Sigma} ((r^2 + a^2) d\phi - adt)^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2$$  \hspace{1cm} (24)

where, $\Delta = r^2 - 2Mr + a^2 + Q^2$, \quad $\Sigma = r^2 + a^2 \cos^2 \theta$, \quad $a = \frac{J}{M}$

The generalized Smarr formula for the Kerr-Newman Black Hole is given as [22]

$$M^2 = \frac{A}{16\pi} + \frac{\pi}{A} (4J^2 + Q^4) + \frac{Q^2}{2}$$  \hspace{1cm} (25)

One can now perform implicit differentiation of the logarithm of eqn. (25) to determine the derivatives of $M(A, Q, J)$ with respect to its arguments, appearing in the stability criteria [15]-[21]. Since the stability criteria hold for macroscopic horizons of large area ($A >> A_P$), it is sufficient to evaluate these derivatives for $A >> \sqrt{4J^2 + Q^2}$. Alternatively, one can approximate the square root of eqn. (25) for large areas and then evaluate relevant derivatives for substitution into the stability criteria. Both procedures agree to leading order terms in area.

Retaining the leading order term in horizon area, eqns. [18], [22] and [23] and give the inverse temperature ($\beta$) as

$$\beta = \frac{2\pi^{1/2}A^{1/2}}{A_P}.$$  \hspace{1cm} (26)

So, $\beta$ is positive and is an essential criterion for thermodynamic stability.
The thermal stability of Kerr-Newman black holes can now be ‘predicted’ on the basis of explicit verification as to whether the inequalities (15)-(21) are all satisfied. It is straightforward to verify that inequalities (15)-(21) in fact do not hold for the explicit mass-area relation (24) for the Kerr-Newman black hole, implying that this black hole is thermally unstable. The black hole either radiates away or accretes, indefinitely, depending on its ambient thermal conditions. This is exactly the sort of behaviour that is known for non-rotating asymptotically flat black holes like the Schwarzschild [14, 15] or the Reissner-Nordstrom spacetimes [17]. The usual Kerr spacetime follows the same pattern as the Reissner-Nordstrom, exhibiting thermal instability for a generic range of parameters. Our analysis in this paper rounds up the story for asymptotic flat general relativistic black holes.

B. AdS Black Holes

The thermal instability discerned in the last subsection for the standard asymptotically flat general relativistic black hole spacetimes raises the question as to whether the asymptotically Anti-de Sitter versions of these spacetimes are thermally more stable, for some region of their parameter space, as has been noticed decades ago by Hawking and Page [20] within a semiclassical approach. In contrast, the quantum geometry underpinning of our analysis in the previous sections is independent of specific black hole metrics, giving us very general criteria for thermal stability. The mass-horizon area functional dependence derived from classical metrics of specific black holes therefore permits predictions of stability behaviour of specific black holes under Hawking radiation. In this sense, classical geometry provides us with fiducials for verification of validity of the stability criteria derived earlier. This motivates their application to AdS black holes in this subsection.

The AdS Kerr-Newman black hole is given in Boyer-Lindquist coordinates as

$$ds^2 = -\frac{\Delta}{\rho^2}(dt - \frac{a \sin^2 \theta}{\Sigma} d\phi)^2 + \frac{\Delta \sin^2 \theta}{\rho^2} (\rho^2 + a^2 \sin^2 \theta d\phi - a dt)^2 + \frac{\rho^2}{\Delta r} dr^2 + \frac{\rho^2}{\Delta \theta} d\theta^2$$

where, $\Sigma = 1 - \frac{a^2}{r^2}$. $\Delta_r = (r^2 + a^2)(1 + \frac{e^2}{r^2}) - 2 Mr + Q^2$, $\Delta_\theta = 1 - \frac{a^2 \cos^2 \theta}{r^2}$, $\rho^2 = r^2 + a^2 \cos^2 \theta$, $a = \frac{J}{M}$.

The generalized Smarr formula for the AdS Kerr-Newman Black Hole is given as [22]

$$M^2 = \frac{A}{16\pi} + \frac{\pi}{4}(4J^2 + Q^4)^{1/2} + \frac{Q^2}{2} + \frac{a^2}{r^2} + \frac{A}{8\pi l^2} (Q^2 + \frac{A}{4\pi} + \frac{A^2}{32\pi^2 l^2})$$

(28)

where the cosmological constant ($\Lambda$) is defined in terms of a cosmic length parameter as $\Lambda = -1/l^2$.

As before, our interest is in astrophysical (macroscopic) charged, rotating black holes whose horizon area exceeds by far both the cosmic ‘area’ $l^2$ and the Planck area. Once again, taking the logarithm of eqn. (28), and obtaining the derivatives of $M(A, Q, J)$ with respect to its arguments by implicit differentiation, and evaluating these derivatives for large area $A >> l^2$ and also $A^2 >> 4J^2 + Q^4$, one can substitute them in the stability criteria to examine the thermal stability of the black hole. Alternatively, one can approximate (28) as follows

$$M \approx A^{3/2} \frac{\Lambda}{16\pi^{3/2} l^2} + A^{1/2} \frac{\Lambda^{1/2}}{4\pi l^{1/2}} + \pi^{1/2} Q^2 \frac{\Lambda^{1/2}}{A^{1/2}} + 8\pi^{3/2} J^2 \frac{\Lambda^{3/2}}{A^{3/2}}$$

(29)

and then evaluate appropriate derivatives before substituting them into the stability criteria. Once again, to leading order in area, both approaches agree. In other words, because of the requirement of large area, and the functional form of $M^2$, whether one approximates the exact derivatives for large area, or evaluates derivatives directly from the approximate mass formula (29), before substitution into the stability criteria, it seems not to matter.

Retaining the leading terms in the horizon area, as before, Eqns. (13), (22) and (28) and give the inverse temperature ($\beta$) as

$$\beta = \frac{8\pi^{3/2} l^2}{3A_P A^{1/2}} - \frac{16\pi^{3/2} l^2}{A^{3/2}} - \frac{32\pi^{5/2} l^4}{9A_P A^{3/2}}$$

(30)

Since we are dealing with macroscopic black holes with a large event horizon area and hence the second term in the expression of $\beta$ is highly suppressed. So, $\beta$ is positive only when $A > l^2$ i.e. radius of horizon is greater than cosmic length. So, $\beta$ is positive and is an essential criterion for thermodynamic stability. One also verifies that unlike the asymptotically flat case, for anti-de Sitter black holes, the horizon temperature does increase with horizon area!
To complete the test for thermal stability, conditions (15) - (21) have to be checked. On detailed algebraic calculation, it turns out that all stability conditions, are satisfied, so long as the magnitude of the cosmological constant exceeds the square of the inverse horizon area. Thus, the AdS Kerr-Newman black hole turns out to be thermodynamically stable for a sufficiently large negative asymptotic curvature.

A physical understanding of this phenomenon emerges at the semiclassical level along the lines of [20], where, the presence of Cauchy horizons in asymptotically AdS spacetimes is emphasized, requiring incoming boundary conditions for a specification of the full dynamics. In our quantal approach, though, such an intuitive understanding is not yet available.

C. Asymptotically Flat String Theoretic Black Hole

Here we consider the low energy effective field theory describing heterotic string theory, which describes a black hole carrying finite amount of charge and angular momentum [23]. In low energy limit the effective four dimensional theory contains gravity, maxwell field, dilaton field and antisymmetric gauge field. The solution of the metric turns out be a black hole whose charge, mass and angular momentum are determined by various fundamental parameters of the theory [23]. The classical metric for such a black hole is

\[
\begin{align*}
\text{ds}^2 &= -\frac{\rho^2 + a^2 \cos^2 \theta - 2 m \rho}{\rho^2 + a^2 \cos^2 \theta + 2 m \rho \sin^2 \frac{\alpha}{2}} dt^2 + \frac{\rho^2 + a^2 \cos^2 \theta + 2 m \rho \sin^2 \frac{\alpha}{2}}{\rho^2 + a^2 - 2 m} d\rho^2 \\
&+ (\rho^2 + a^2 \cos^2 \theta + 2 m \rho \sin^2 \frac{\alpha}{2}) d\theta^2 - \frac{4 m \rho a \cosh^2 \frac{\alpha}{2} \sin^2 \theta}{\rho^2 + a^2 \cos^2 \theta + 2 m \rho \sin^2 \frac{\alpha}{2}} dt d\phi \\
&+ \left\{ (\rho^2 + a^2)(\rho^2 + a^2 \cos^2 \theta) + 2 m \rho a^2 \sin^2 \theta + 4 m \rho (\rho^2 + a^2) \sin^2 \frac{\alpha}{2} + 4 m^2 \rho^2 \sin^4 \frac{\alpha}{2} \right\} \sin^2 \theta d\rho d\phi
\end{align*}
\]

This metric describes a black hole solution with mass \(M\), charge \(Q\), and angular momentum \(J\) given by

\[
M = m \frac{\alpha}{2} (1 + \cosh \alpha), \quad Q = m \sqrt{2} \sinh \alpha, \quad J = m a \frac{\alpha}{2} (1 + \cosh \alpha)
\]

It will be more convenient to express \(m\), \(a\) and \(\alpha\) in terms of the independent physical parameters \(M\), \(J\) and \(Q\) by inverting the relations given in (32). We get

\[
m = M - \frac{Q^2}{2M}, \quad \sinh \alpha = \frac{2\sqrt{2}QM}{2M^2 - Q^2}, \quad a = \frac{J}{M}
\]

The area of the horizon turns out to be

\[
A = 8\pi M \left( M - \frac{Q^2}{2M} + \sqrt{(M - \frac{Q^2}{2M})^2 - \frac{J^2}{M^2}} \right)
\]

which gives the mass of the black hole \(M\) as

\[
M^2 = \frac{A}{16\pi} + \frac{Q^2}{2} + \frac{4\pi J^2}{A}
\]

In the large area limit i.e. in the limit \(A >> J\) and \(A >> Q^2\), it is obvious that the expression (35) above for the squared mass is very close to that for the asymptotically flat Kerr-Newman black hole given in eqn. (25). Without any calculation, therefore, we can conclude that this black hole must be thermally unstable under Hawking radiation (or accretion, depending upon the ambient conditions), since the Hawking temperature must, exactly as for the Kerr-Newman case, decrease with area.

D. Five Dimensional Asymptotically Flat Dilatonic Black Hole With Rotation

Here we consider the dilaton field coupled to gravity in presence of the Maxwell field in five dimension [24]. Such solutions are derived from the standard four dimensional Kerr solution of Einstein’s equation, by
constructing the five dimensional product space obtained by tensoring the Kerr spacetime with $R$. Boosting the Kerr solution along the real line thus gives a rotating charged black hole in a five dimensional Lorentzian spacetime. Let, $m$, $a$ be the mass and rotation parameter of the original Kerr solution and $v$ is the velocity of the boost in the extra direction.

The resulting metric is as,

$$ds^2 = -\frac{1-Z}{B}dt^2 - \frac{2aZ \sin^2 \theta}{B\sqrt{1-v^2}}dt d\phi + \left[ B(r^2 + a^2) + a^2 \sin^2 \theta \frac{Z}{B} \right] \sin^2 \theta d\phi^2 + B \frac{\Sigma}{\Delta_0} dr^2 + B\Sigma d\theta^2$$

where $B = \sqrt{1 + v^2}$, $Z = \frac{2mr}{\Sigma}$, $\Delta_0 = r^2 + a^2 - 2mr$, $\Sigma = r^2 + a^2 \cos^2 \theta$. The dilaton field is given by $\phi = (-\sqrt{3}/2) \log B$.

This gives the mass($M$), charge($Q$) and angular momentum ($J$) of the black hole as,

$$M = m \left( 1 + \frac{v^2}{2(1-v^2)} \right)$$

$$Q = \frac{mv}{1-v^2}$$

$$J = \frac{ma}{\sqrt{1-v^2}}$$

One can solve eqns. (37) and (38) for the boost velocity $v$ in terms of these parameters

$$v = \sqrt{2 + \left( \frac{M}{Q} \right)^2 - \frac{M}{Q}}$$

Similarly, eqns. (37), (38) and (40) can be inverted to yield $m$ as a function of these parameters

$$m = \frac{3M}{2} - \frac{\sqrt{M^2 + 2Q^2}}{2}.$$  

The area of the black hole $A$ is then given as

$$A = 8\pi \left[ C + \sqrt{C^2 - J^2} \right]$$

where,

$$C = \frac{m^2}{\sqrt{1-v^2}}$$

$$= \frac{9M^2Q}{4} - \frac{3M^2Q}{2} \left( \sqrt{1 + \frac{2Q^2}{M^2}} + \frac{M^2Q + 2Q^3}{4} \right)$$

$$\sqrt{2M^2 \left( \sqrt{1 + \frac{2Q^2}{M^2}} - 2M^2 - Q^2 \right)}$$

While eqn. (37) expresses the mass of the black hole as a function of the parameters $v$ and $m$, eliminating the latter in terms of the parameters $A$, $Q$, $J$ is a complicated algebraic task, involving the inversion of eqn. (43). An easier approach is to rederive the stability criteria from the Grand Canonical Partition Function and evaluate the saddle point integrals over the variables $m$, $v$, $a$, taking the appropriate Jacobian into account.

Assume,

$$\cosh \eta = \frac{1}{\sqrt{1-v^2}}, \quad \cos \mu = \frac{a}{m}$$

Eqns. (37), (38), (39), (42) and (44) together imply

$$M = m \left( 1 + \frac{\sinh^2 \eta}{2} \right), \quad J = m^2 \cos \mu \cosh \eta$$

$$Q = \frac{m}{2} \sinh 2\eta, \quad A = 8\pi m^2 \cosh \eta \left( 1 + \sin \mu \right)$$
Eqn. (45) implies that large area \( (A) \) means large value of \( 'm' \) and \( \cosh \eta \).

Now, we will calculate the Grand Canonical Partition function \( (Z_G) \) in terms of the new variables \( (m, \eta, \mu) \). Let us assume that the saddle point be at \( (\bar{m}, \bar{\eta}, \bar{\mu}) \) and define the fluctuations around this point as
\[
m = (m - \bar{m}), \quad \mu = (\mu - \bar{\mu}), \quad \eta = (\eta - \bar{\eta})
\]

Eqns. (12), (22), (23), (45), and (46) together give
\[
Z_G = \exp \left\{ X(\bar{m}, \bar{\mu}, \bar{\eta}) \right\} \times \int \mathrm{d}m \, \mathrm{d}\mu \, \mathrm{d}\eta \exp \left\{ \frac{1}{2} \left[ (X_{mm})m^2 + (X_{\mu\mu})\mu^2 + (X_{\eta\eta})\eta^2 + (2X_{m\mu})m\mu \right] + \left( 2X_{m\eta}m\eta + (2X_{\eta\mu})\eta\mu \right) \right\}
\]

Where,
\[
X(\bar{m}, \bar{\mu}, \bar{\eta}) = \log(T) + S - \beta M + \beta \Psi Q + \beta \Omega J
= \log(m) + \frac{1}{2} \log \cosh \eta - \frac{1}{2} \log(1 + \sin \mu) + \log(3 \cosh^2 \eta - 1)
+ 2\pi m^2 \cosh \eta (1 + \sin \mu) - \beta m \left( 1 + \frac{\sin^2 \eta}{2} \right) + \beta \Psi \frac{m}{2} \sinh 2\eta
+ \beta \Omega m^2 \cos \mu \cosh \eta
\]

Here, \( T \) is Jacobian due to change of variables from \( (A, Q, J) \) to \( (m, \mu, \eta) \) and is given as,
\[
T = 8\pi m^4 (1 + \sin \mu) (3 \cosh^2 \eta - 1) \cosh^2 \eta
\]

Saddle point approximation method implies \( X_m = 0 = X_\mu = X_\eta \) at the saddle point and hence in the large area limit \( X_\mu = 0 \) and eqn. (48) together give
\[
\beta \Omega = \frac{2\pi}{\tan \mu}
\]

In large area limit, eqns. (48) and (50) give \( X_{mm} = 4\pi \cosh \eta (1 + cosec \mu) \) and is non-negative. This implies that \( Z_G \) will diverge and macroscopically large dilatonic black holes in asymptotically flat spacetime must be thermally unstable with respect to Hawking radiation, over their entire parameter space.

V. SUMMARY AND DISCUSSION

In the following table, we summarize the work of previous sections, for a clearer perspective.

| Type Of Black Hole | Whether Stable | Region of stability                      |
|--------------------|----------------|------------------------------------------|
| Asymptotically Flat Black Holes | Unstable | For all values of parameters |
| ADS Black Holes    | Stable        | Radius of Horizon is greater than Cosmic length |
| String Theoretic Black Holes | Unstable | For all values of parameters |
| Dilaton Black Holes without Cosmological Constant | Unstable | For all values of parameters |

We reiterate that our analysis is quite independent of specific classical spacetime geometries, relying as it does on quantum aspects of spacetime. The construction of the partition function used standard formulations of equilibrium statistical mechanics augmented by results from canonical Quantum Gravity, with extra inputs regarding the behaviour of the microcanonical entropy as a function of area beyond the Bekenstein-Hawking area law, as for instance derived from Loop Quantum Gravity. However, we emphasize that the results are more general than being restricted to any specific proposal for quantum spacetime geometry, requiring only certain functional dependences on horizon area and other parameters of statistical mechanical quantities like entropy. It also stands to reason that our stability criteria are useful for predicting the thermal behaviour vis-a-vis Hawking radiation for specific astrophysical black holes.

It is also noteworthy that the approach is useful for making predictions on the thermal stability of black holes in Lorentzian spacetimes with arbitrary number of spatial dimensions. It can also be generalized to black holes with arbitrary ‘hairs’ (charges) - either quantum or classical.
There are however, subtleties of a statistical mechanical nature which have not be addressed in this paper. The most important of these is the nature of the thermal instability discerned by us. While there are indications that the instability in most cases can be associated with some sort of phase transition [13], the very general approach here has not yet been applied to discuss the full range of thermal behaviour exhibited specifically for AdS Schwarzschild black holes, for instance, as discussed in detail in [20]. Crucially, there are ‘phases’ discussed in that paper which have not been fully explored via our more ‘quantum geometry’ approach, as distinct from the semiclassical approach employed in [20]. We hope to return to these important issues in a future publication.

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