Rational Points of Bounded Height on Compactifications of Anisotropic Tori

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Abstract

We investigate the analytic properties of the zeta-function associated with heights on equivariant compactifications of anisotropic tori over number fields. This allows to verify conjectures about the distribution of rational points of bounded height.

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Introduction

In this paper we prove new results on the distribution of $K$-rational points of bounded height on algebraic varieties $X$ defined over a number field $K$ \[2, 9\].

Let $\mathcal{L} = (L, \| \cdot \|_v)$ be an ample metrized invertible sheaf on $X$ with a family $\{\| \cdot \|_v\}$ of $v$-adic metrics and $H_{\mathcal{L}} : X(K) \to \mathbb{R}_{\geq 0}$ the height function associated with $(L, \| \cdot \|_v)$. We are interested in analytical properties of the height zeta-function defined by the series

$$Z_{\mathcal{L}}(s) = \sum_{x \in X(K)} H_{\mathcal{L}}(x)^{-s}.$$

The investigation of $Z_{\mathcal{L}}(s)$ was initiated by Arakelov and Faltings \[1, 8\] who considered the case when $X = S^d(C)$ is $d$-th symmetric power of an algebraic curve $C$ of genus $g < d - 1$. It was shown in \[9\] that zeta-functions associated with heights on generalized flag varieties are Eisenstein series.

In this paper we consider the case when $X$ is an equivariant compactification of an anisotropic torus $T$. It is easy to show (cf. 1.3.6) that all $K$-rational points of $X$ must be contained in $T(K)$. Therefore

$$Z_{\mathcal{L}}(s) = \sum_{x \in T(K)} H_{\mathcal{L}}(x)^{-s}.$$

Our main idea for the computation of the zeta-function $Z_{\mathcal{L}}(s)$ is to use the group structure on $T(K)$ and to apply the Poisson formula in the following form:

Let $\mathcal{G}$ be a locally compact topological abelian group with a Haar measure $dx$, $\mathcal{H} \subset \mathcal{G}$ a discrete subgroup such that $\mathcal{G}/\mathcal{H}$ is compact, $F : \mathcal{H} \to \mathbb{R}$ a function on $\mathcal{H}$ which can be extended to an $L^1$-function on $\mathcal{G}$. Let $\text{vol}(\mathcal{G}/\mathcal{H})$ be the $dx$-volume of the fundamental domain of $\mathcal{H}$ in $\mathcal{G}$. Denote by $\hat{F}$ the Fourier transform of $F$ with respect to the Haar measure $dx$. Suppose that $\hat{F} \in L^1(\mathcal{H}^\perp)$. Then

$$\sum_{x \in \mathcal{H}} F(x) = \frac{1}{\text{vol}(\mathcal{G}/\mathcal{H})} \sum_{\chi \in \mathcal{H}^\perp} \hat{F}(\chi)$$

where $\mathcal{H}^\perp$ is the group of characters of $\mathcal{G}$ which are trivial on $\mathcal{H}$. 

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We will apply this formula to the case when $G = T(A_K)$ is the adele group of an anisotropic algebraic torus $T$, $H$ is the subgroup $T(K) \subset T(A_K)$ of all $K$-rational points of $T$ and $F(x, s) = H^{-s}_L(x)$ for some ample metrized line bundle $L$ on the compactification of $T$. The adelic method turns out to be very convenient, because the height function $H_L$ splits into the product of local Weil functions $H_{L,v}: T(K_v) \to \mathbb{R}_{\geq 0}$. This allows to extend $H_L^{-s}$ to a function on the whole adele group $G$. Moreover, the local Weil functions $H_{L,v}$ can be chosen to be $T(O_v)$-invariant. Therefore, the Fourier transform

$$\int_{T(A_K)} H_L(x) \chi(x) d\mu = \prod_{v \in \text{Val}(K)} \int_{T(K_v)} H_{L,v}(x) \chi_v(x) d\mu_v$$

equals zero unless the restriction of $\chi_v$ on $T(O_v)$ is trivial for all $v \in \text{Val}(K)$. The Fourier transform of $H_L(x)^{-s}$ can be calculated separately for each local factor $H_{L,v}(x)^{-s}$. The analytic properties of the Fourier transform of $H_L(x)^{-s}$ can be investigated by a method of Draxl [7].

In Section 1 we recall basic facts from theories of algebraic tori and toric varieties $P_{\Sigma}$ associated with fans $\Sigma$ over arbitrary fields.

In Sections 2, following ideas in [9], we define canonical families of $v$-adic metrics on all $T$-linearized invertible sheaves $L$ on $P_{\Sigma}$ simultaneously. This allows us to construct the complex height $H_{\Sigma}(x, \varphi)$ and the associated zeta-function $Z_{\Sigma}(\varphi)$ as a function of $\varphi$, where $\varphi$ represents an element of the complexified Picard group $\text{Pic}(P_{\Sigma}) := \text{Pic}(P_{\Sigma}) \otimes \mathbb{C}$. One obtains the one-parameter zeta-function $Z_L(s)$ via restriction of $Z_{\Sigma}(\varphi)$ to the complex line $s[L], s \in \mathbb{C}$.

In Section 3 we recall some facts about characteristic functions of convex cones.

In Section 4 we investigate the analytic properties of zeta-functions in order to obtain an asymptotic formula for the number of rational points of bounded height. As one of our main results, we prove the following refinement of the conjecture of Manin:

**Let $P_{\Sigma}$ be a smooth compactification of an anisotropic torus $T$ (notice that we do not need to assume that $P_{\Sigma}$ is a Fano variety). Let $r$ be the rank of $\text{Pic}(P_{\Sigma,K})$. Then there exist only a finite number $N(P_{\Sigma,K}^{-1}, B)$ of**
$K$-rational points $x \in T(K)$ having the anticanonical height $H_{K^{-1}}(x) \leq B$. Moreover,

$$N(P_\Sigma, K^{-1}, B) = \frac{\Theta(\Sigma, K)}{(r-1)!} \cdot B(\log B)^{r-1}(1 + o(1)), \quad B \to \infty,$$

where the constant $\Theta(\Sigma, K)$ depends on:

1. the cone of effective divisors $\Lambda_{\text{eff}}(\Sigma) \subset \text{Pic}(P_\Sigma)_\mathbb{R}$;
2. the Brauer group of $P_\Sigma$;
3. the Tamagawa number $\tau_K(P_\Sigma)$ associated with the metrized canonical sheaf on $P_\Sigma$ as defined by E. Peyre in \cite{15}.

We prove the Batyrev-Manin conjecture \cite{2} which describes the asymptotic for the number of $K$-rational points $x \in T(K)$ such that $H_L(x) \leq B$ in terms of $[\mathcal{L}]$ and the geometry of the cone of effective divisors $\Lambda_{\text{eff}}(\Sigma)$. Since for compactifications of anisotropic tori the cone of effective divisors $\Lambda_{\text{eff}}(\Sigma) \subset \text{Pic}(P_\Sigma)_\mathbb{R}$ is simplicial (cf. \ref{3.7}), our situation is very close to the case of generalized flag varieties considered in \cite{4, 9}.

We observe that our results provide first examples of asymptotics for the number of rational points of bounded height on unirational varieties of small dimension which are not rational (nonrational anisotropic tori in dimension 3 were constructed in \cite{11}). Another new phenomenon is the appearance of the Brauer group $\text{Br}(P_\Sigma)$ in the asymptotic formulas.

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1 Toric varieties over arbitrary fields

1.1 Algebraic tori

Let $K$ be an arbitrary field, $\overline{K}$ the algebraic closure of $K$, $G_m(\overline{K}) = \overline{K}^*$ the multiplicative group of $\overline{K}$.

Let $X$ be an arbitrary algebraic variety over $\overline{K}$. Let $E/K$ be a finite extension such that $X$ is defined over $E$. To stress this fact we sometimes
will denote $X$ also by $X_E$. The set of $E$-rational points of $X_E$ will be denoted by $X_E(E)$.

**Definition 1.1.1** A linear algebraic group $T$ over $K$ is called a $d$-dimensional algebraic torus if its base extension $T_K = T \times_{\text{Spec}(K)} \text{Spec}(K)$ is isomorphic to $(\mathbb{G}_m(K))^d$.

We notice that an isomorphism between $T$ and $(\mathbb{G}_m(K))^d$ is always defined over a finite Galois extension $E$ of $K$.

**Definition 1.1.2** Let $T$ be an algebraic torus over $K$. A finite Galois extension $E$ of $K$ such that $T_E = T \times_{\text{Spec}(K)} \text{Spec}(E)$ is isomorphic to $(\mathbb{G}_m(E))^d$ is called a splitting field of $T$.

**Definition 1.1.3** We denote by $\hat{T}_K = \text{Hom}(T, \overline{K}^*)$ the group of regular $\overline{K}$-rational characters of $T$. For any subfield $E \subset \overline{K}$ containing $K$, we denote by $\hat{T}_E$ the group of characters of $T$ defined over $E$.

There is well-known correspondence between Galois representations by integral matrices and algebraic tori [16, 23]:

**Theorem 1.1.4** Let $G = \text{Gal}(E/K)$ be the Galois group of the splitting field $E$ of a $d$-dimensional torus $T$. Then $\hat{T}$ is a free abelian group of rank $d$ with a structure of $G$-module defined by the natural representation

$$\rho : G \to \text{Aut}(\hat{T}) \cong \text{GL}(d, \mathbb{Z}).$$

Every $d$-dimensional integral representation of $G$ defines a $d$-dimensional algebraic torus over $K$ which splits over $E$. One obtains a one-to-one correspondence between $d$-dimensional algebraic tori over $K$ with the splitting field $E$ up to isomorphism, and $d$-dimensional integral representations of $G$ up to equivalence.

**Remark 1.1.5** The group $\hat{T}_K$ is a sublattice in $\hat{T} \cong \mathbb{Z}^d$ consisting of all $G$-invariant elements.
Definition 1.1.6 An algebraic torus $T$ over $K$ is called anisotropic if $\hat{T}_K$ has rank zero.

Example 1.1.7 Let $f(z) \in K[z]$ be a separable polynomial of degree $d$. Consider the $d$-dimensional $K$-algebra

$$A(f) = K[z]/(f(z)).$$

Then the multiplicative group $A^*(f)$ is a $d$-dimensional algebraic torus over $K$. This torus has the following properties:

(i) The rank of the group of characters of $A^*(f)$ is equal to the number of irreducible components of $\text{Spec}(A(f))$.

(ii) If $f(z)$ splits in linear factors over some finite Galois extension $E$ of $K$, then $A(f) \otimes_K E \cong E^n$, and $A^*(f) \otimes_K E \cong (E^*)^n$. Thus, $E$ is a splitting field of $A^*(f)$.

(iii) Since the classes of $1, z, \ldots, z^{d-1}$ in $A(f)$ give rise to a $K$-basis of the $d$-dimensional algebra $A(f)$, we can consider $A^*(f)$ as a commutative subgroup in $\text{GL}(d, K)$. Thus, the determinant of the matrix defines a regular $K$-character

$$\mathcal{N} : A^*(f) \to K^*.$$

We denote by $A_1^*(f)$ the $(d - 1)$-dimensional algebraic torus which is the kernel of $\mathcal{N}$.

(iv) The multiplicative group $K^*$ is a subgroup of $A^*(f)$ and the restriction of $\mathcal{N}$ to $K^*$ sends $x \in K^*$ to $x^d \in K^*$. The factor-group $A^*(f)/K^*$ is a $(d - 1)$-dimensional torus which is isogeneous to $A_1^*(f)$.

Example 1.1.8 Let $K'$ be a finite separable extension of $K$. By primitive element theorem, $K' \cong A(f)$ for some irreducible polynomial $f(z) \in K[z]$. Thus, we come to a particular case of the previous example. In this case, $\mathcal{N}$ is the norm $N_{K'/K}$, the algebraic torus $A^*(f)$ is usually denoted by

$$R_{K'/K}(\mathbb{G}_m),$$

and the torus $A_1^*(f)$ is usually denoted by

$$R_{1K'/K}(\mathbb{G}_m).$$

Since $\text{Spec}(K')$ is irreducible, $R_{1K'/K}(\mathbb{G}_m)$ and $R_{K'/K}(\mathbb{G}_m)/K^*$ are examples of anisotropic tori.
1.2 Compactifications of split tori

We recall standard facts about toric varieties over algebraically closed fields [4, 6, 15]. Let $M$ be a free abelian group of rank $d$ and $N = \text{Hom}(M, \mathbb{Z})$ the dual abelian group.

**Definition 1.2.1** A finite set $\Sigma$ consisting of convex rational polyhedral cones in $N_\mathbb{R} = N \otimes \mathbb{R}$ is called a **complete regular $d$-dimensional fan** if the following conditions are satisfied:

(i) every cone $\sigma \in \Sigma$ contains $0 \in N_\mathbb{R}$;
(ii) every face $\sigma'$ of a cone $\sigma \in \Sigma$ belongs to $\Sigma$;
(iii) the intersection of any two cones in $\Sigma$ is a face of both cones;
(iv) $N_\mathbb{R}$ is the union of cones from $\Sigma$;
(v) every cone $\sigma \in \Sigma$ is generated by a part of a $\mathbb{Z}$-basis of $N$.

We denote by $\Sigma(i)$ the set of all $i$-dimensional cones in $\Sigma$. For each cone $\sigma \in \Sigma$ we denote by $N_{\sigma, \mathbb{R}}$ the minimal linear subspace containing $\sigma$.

Every complete regular $d$-dimensional fan defines a smooth equivariant compactification $P_\Sigma$ of the split $d$-dimensional algebraic torus $T$. The variety $P_\Sigma$ has the following two geometric properties:

**Proposition 1.2.2** The toric variety $P_\Sigma$ is the union of split algebraic tori $T_\sigma$ ($\dim T_\sigma = d - \dim \sigma$):

$$P_\Sigma = \bigcup_{\sigma \in \Sigma} T_\sigma.$$  

For each $k$-dimensional cone $\sigma \in \Sigma(k)$, $T_\sigma$ is the kernel of a homomorphism $T \to (\mathbb{G}_m(\overline{K})^k)$ defined by a $\mathbb{Z}$-basis of the sublattice $N \cap N_{\sigma, \mathbb{R}} \subset N$.

Let $\check{\sigma}$ denote the cone in $M_\mathbb{R}$ which is dual to $\sigma$.

**Proposition 1.2.3** The toric variety $P_\Sigma$ has a $T$-invariant open covering by affine subsets $U_\sigma$:

$$P_\Sigma = \bigcup_{\sigma \in \Sigma} U_\sigma$$

where $U_\sigma = \text{Spec}(\overline{K})[M \cap \check{\sigma}]$. 

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Definition 1.2.4 A continuous function $\varphi : N_\mathbb{R} \to \mathbb{R}$ is called $\Sigma$-piecewise linear if the restriction $\varphi_\sigma$ of $\varphi$ to every cone $\sigma \in \Sigma$ is a linear function. It is called integral if $\varphi(N) \subset \mathbb{Z}$.

Definition 1.2.5 For any integral $\Sigma$-piecewise linear function $\varphi : N_\mathbb{R} \to \mathbb{R}$ and any cone $\sigma \in \Sigma(d)$, we denote by $m_{\sigma, \varphi}$ the restriction of $\varphi$ to $\sigma$ considered as an element in $M$. We put $m_{\sigma', \varphi} = m_{\sigma, \varphi}$ if $\sigma'$ is a face of a $d$-dimensional cone $\sigma \in \Sigma$.

Definition 1.2.6 For any integral $\Sigma$-piecewise linear function $\varphi : N_\mathbb{R} \to \mathbb{R}$, we define the invertible sheaf $L(\varphi)$ as the subsheaf of the constant sheaf of rational functions on $P_\Sigma$ generated over $U_\sigma$ by the element $-m_{\sigma, \varphi}$ considered as a character of $T \subset P_\Sigma$.

Remark 1.2.7 The $T$-action on the sheaf of rational functions restricts to the subsheaf $L(\varphi)$ so that we can consider $L(\varphi)$ as a $T$-linearized line bundle over $P_\Sigma$.

Denote by $e_1, \ldots, e_n$ the primitive integral generators of all 1-dimensional cones in $\Sigma$. Let $T_i$ ($i = 1, \ldots, n$) be the $(d - 1)$-dimensional torus orbit corresponding to the cone $\mathbb{R}_{\geq 0} e_i \in \Sigma$ and $D_i$ the Zariski closure of $T_i$ in $P_\Sigma$. Define $D(\Sigma) \cong \mathbb{Z}^n$ as the free abelian group of $T$-invariant Weil divisors on $P_\Sigma$ with the basis $D_1, \ldots, D_n$.

Proposition 1.2.8 The correspondence $\varphi \mapsto L(\varphi)$ gives rise to an isomorphism between the group of $T$-linearized line bundles on $P_\Sigma$ and the group $PL(\Sigma)$ of all $\Sigma$-piecewise linear integral functions on $N_\mathbb{R}$. There is the canonical isomorphism

$$PL(\Sigma) \cong D(\Sigma), \quad \varphi \mapsto (\varphi(e_1), \ldots, \varphi(e_n)).$$

The Picard group $\text{Pic}(P_\Sigma)$ is isomorphic to $PL(\Sigma)/M$ where elements of $M$ are considered as globally linear integral functions on $N_\mathbb{R}$, so that we have the exact sequence

$$0 \to M \to D(\Sigma) \to \text{Pic}(P_\Sigma) \to 0$$

(1)
Definition 1.2.9 Let $\Lambda_{\text{eff}}(\Sigma)$ be the cone in $\text{Pic}(P_\Sigma)$ generated by classes of effective divisors on $P_\Sigma$. Denote by $\Lambda^\ast_{\text{eff}}(\Sigma)$ the dual to $\Lambda_{\text{eff}}(\Sigma)$ cone.

Proposition 1.2.10 $\Lambda_{\text{eff}}(\Sigma)$ is generated by the classes $[D_1], \ldots, [D_n]$.

Proof. Any divisor $D$ on $P_\sigma$ is linearly equivalent to an integral linear combination of $D_1, \ldots, D_n$. Assume that $D = a_1D_1 + \cdots + a_nD_n$ is effective. Then there exists a rational function $f$ on $P_\Sigma$ having no poles and zeros on $T$ such that

$$(f) + D \geq 0. \quad (2)$$

We can assume that $f$ is character of $T$ defined by an element $m_f \in M$. Then the condition (2) is equivalent to

$$b_i = \langle m_f, e_i \rangle + a_i \geq 0, \quad i = 1, \ldots, n \quad (3)$$

Then $D' = b_1D_1 + \cdots + b_nD_n$ is linearly equivalent to $D$. So every effective class $[D]$ is a non-negative integral linear combination of $[D_1], \ldots, [D_n]$. $\square$

Proposition 1.2.11 Let $\varphi_\Sigma$ be the $\Sigma$-piecewise linear integral function such that $\varphi(e_1) = \cdots = \varphi(e_n) = 1$. Then $L(\varphi_\Sigma)$ is isomorphic to the $T$-linearized anticanonical line bundle on $P_\Sigma$.

Example 1.2.12 Projective spaces. Consider a $d$-dimensional fan $\Sigma$ whose 1-dimensional cones are generated by $d+1$ elements $e_1, \ldots, e_d, e_{d+1} = -(e_1 + \cdots + e_d)$, where $\{e_1, \ldots, e_d\}$ is a $\mathbb{Z}$-basis of $d$-dimensional lattice $N$, and $k$-dimensional cones in $\Sigma$ are generated by all possible $k$-element subsets in $\{e_1, \ldots, e_{d+1}\}$. Then the corresponding compactification $P_\Sigma$ of the $d$-dimensional split torus is $P^d$.

Remark 1.2.13 It is easy to see that the combinatorial construction of toric varieties $P_\Sigma$ immediately extends to arbitrary fields $E$; i.e., using a rational complete polyhedral fan $\Sigma$, one can define the toric variety $P_{\Sigma,E}$ as the equivariant compactification of the split torus $(G_m(E))^d$. 
1.3 Compactifications of nonsplit tori

Let $T$ be a $d$-dimensional algebraic torus over $K$ with a splitting field $E$ and $G = \text{Gal}(E/K)$. Denote by $M$ the lattice $\hat{T}$ and put $N = \text{Hom}(M, \mathbb{Z})$. Let $\rho^*$ be the integral representation of $G$ in $\text{GL}(N)$ which is dual to $\rho$. In order to construct a projective compactification of $T$ over $K$, we need a complete fan $\Sigma$ of cones having an additional combinatorial structure: an action of the Galois group $G$.

**Definition 1.3.1** A complete fan $\Sigma \subset \mathbb{N}_R$ is called $G$-invariant if for any $g \in G$ and for any $\sigma \in \Sigma$, one has $\rho^*(g)(\sigma) \in \Sigma$.

**Theorem 1.3.2** Let $\Sigma$ be a complete regular $G$-invariant fan in $\mathbb{N}_R$. Then there exists a complete algebraic variety $P_{\Sigma,K}$ over $K$ such that its base extension $P_{\Sigma,K} \otimes_{\text{Spec} K} \text{Spec} E$ is isomorphic to the toric variety $P_{\Sigma,E}$ defined over $E$ by $\Sigma$.

Let $\Sigma^G$ be the subset of all $G$-invariant cones $\sigma \in \Sigma$. Then

$$P_{\Sigma}(K) = \bigcup_{\sigma \in \Sigma^G} T_\sigma(K),$$

where $T_\sigma$ is the $(d - \dim \sigma)$-dimensional algebraic torus over $K$ corresponding to the restriction of the integral $G$-representation in $\text{GL}(M)$ to the sublattice $(\hat{\sigma} \cap -\hat{\sigma}) \cap M \subset M$.

Taking $G$-invariant elements in the short exact sequence

$$0 \to M \to D(\Sigma) \to \text{Pic}(P_{\Sigma,E}) \to 0$$

we obtain the exact sequence

$$0 \to M^G \to D(\Sigma)^G \to \text{Pic}(P_{\Sigma,E})^G \to H^1(G, M) \to 0 \quad (4)$$

**Proposition 1.3.3** The group $\text{Pic}(P_{\Sigma,E})^G$ is canonically isomorphic to the Picard group $\text{Pic}(P_{\Sigma,K})$. Moreover $H^1(G, M)$ is the Picard group of $T$. 

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Corollary 1.3.4 The correspondence $\phi \to L(\phi)$ induces an isomorphism between the group of $T$-linearized invertible sheaves on $P_{\Sigma,K}$ and the group $PL(\Sigma)^G$ of all $\Sigma$-piecewise linear integral $G$-invariant functions on $N_R$. An invertible sheaf $L$ on $P_{\Sigma,K}$ admits a $T$-linearization if and only if the restriction of $L$ on $T$ is trivial. In particular, some tensor power of $L$ always admits a $T$-linearization.

Corollary 1.3.5 Let $\Lambda_{\text{eff}}(\Sigma, K)$ be the cone of effective divisors of $P_{\Sigma,K}$. Then $\Lambda_{\text{eff}}(\Sigma, K)$ consists of $G$-invariant elements in $\Lambda_{\text{eff}}(\Sigma)$.

Corollary 1.3.6 Let $P_{\Sigma,K}$ be a compactification of an anisotropic torus $T$. Then all $K$-rational points of $P_{\Sigma,K}$ are contained in $T$ itself.

Proof. By [1.3.2], it is sufficient to prove that for an anisotropic torus $T$ defined by some Galois representation of $G$ in $GL(M)$, there is no $G$-invariant cone $\sigma$ of positive dimension in $\Sigma$.

Assume that a $k$-dimensional cone $\sigma$ with the generators $\{e_{i_1}, \ldots, e_{i_k}\}$ is $G$-invariant. Then $e_{i_1} + \cdots + e_{i_k}$ is a nonzero $G$-invariant integral vector in the interior of $\sigma$. Hence the sublattice $N^G$ of $G$-invariant elements in $N$ has positive rank. Thus $M^G \cong \hat{T}_K$ also has positive rank. Contradiction.

Proposition 1.3.7 Let $P_{\Sigma,K}$ be a compactification of an anisotropic torus $T$. Then the cone of effective divisors $\Lambda_{\text{eff}}(\Sigma, K)$ is simplicial. The rank of the Picard group $\text{Pic}(P_{\Sigma,K})$ equals to the number of $G$-orbits in $\Sigma(1)$.

Proof. Let $A_1(P_{\Sigma})$ be the group of 1-cycles on $P_{\Sigma,E}$ modulo numerical equivalence. We identify $A_1(P_{\Sigma})$ with the dual to $\text{Pic}(P_{\Sigma})$ group. Consider the dual cone $\Lambda_{\text{eff}}^*(\Sigma, K)$. Since $\Lambda_{\text{eff}}^*(\Sigma, K) = \Lambda_{\text{eff}}^*(\Sigma)^G$, by [1.2.10], $\Lambda_{\text{eff}}^*(\Sigma, K)$ consists of non-negative $G$-invariant $R$-linear relations among primitive generators of $\Sigma(1)$. Let

$$\Sigma(1) = \Sigma_1(1) \cup \ldots \cup \Sigma_l(1)$$

be the decomposition of $\Sigma(1)$ into a union of $G$-orbits. Then every $G$-invariant linear relation among the primitive generators $e_1, \ldots, e_n$ of the 1-dimensional cones has the form

$$\sum_{1 \leq i \leq l} \lambda_i \left( \sum_{\sigma_j \in \Sigma_i(1)} e_j \right) = 0 \quad (\sigma_j = R_{\geq 0} e_j).$$

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For every $i$ ($1 \leq i \leq l$), the sum

$$\sum_{\sigma_j \in \Sigma_i(1)} e_j$$

is a $G$-invariant element of the lattice $N$. Since $T$ is anisotropic, $N^G = 0$ and all sums $\sum_{\sigma_j \in \Sigma_i(1)} e_j$ must be equal to zero. These integral relations give rise to a $\mathbb{Z}$-basis $r_1, \ldots, r_l$ of the group of integral linear relations among $e_1, \ldots, e_n$. Thus $A_1(P_{\Sigma})^G_R$ is isomorphic to $\mathbb{Z}$ and the cone $\Lambda^*_{\text{eff}}(\Sigma, K)$ consists of nonnegative linear combinations of $r_1, \ldots, r_l$. So the cone $\Lambda^*_{\text{eff}}(\Sigma, K)$ is also an $l$-dimensional simplicial cone in $\text{Pic}(P_{\Sigma,K}) \otimes \mathbb{R}$.

Below we consider several examples of compactifications of anisotropic tori.

**Example 1.3.8** Consider a $d$-dimensional fan $\Sigma$ as in 1.2.12. It has a natural action of the symmetric group $S_{d+1}$. Let $E$ be a Galois extension of $K$ such that the Galois group $\text{Gal}(E/K)$ is a subgroup of $S_{d+1}$ (for instance, $E$ is a simple algebraic extension defined by an $K$-irreducible polynomial $f$). Then the action of $G$ on $\Sigma$ defines a $d$-dimensional toric variety $P_{\Sigma,K}$ which over $E$ is isomorphic to $d$-dimensional projective space; i.e. $P_{\Sigma,K}$ is a Severi-Brauer variety. In particular, if $E = K(f)$, then $P_{\Sigma,K}$ is a compactification of the $d$-dimensional anisotropic torus $R_{E/K}(G_m)/K^*$. Since $P_{\Sigma,K}$ contains infinitely many $K$-rational points, $P_{\Sigma,K}$ is in fact isomorphic to $P^d$ over $K$.

**Example 1.3.9** A complete fan $\Sigma$ is called *centrally symmetric* if it is invariant under the map $-Id$ of $N_R$.

Let $\Sigma$ be a centrally symmetric 4-dimensional fan and let $E$ be an extension of $K$ of degree 2. The $d$-dimensional torus $T$ corresponding to the integral representation of $\text{Gal}(E/K) \cong \mathbb{Z}/2\mathbb{Z}$ by $Id$ and $-Id$ is isomorphic to the anisotropic torus $(R_{E/K}^1)^d$. The $\mathbb{Z}/2\mathbb{Z}$-invariant fan $\Sigma$ defines the compactification $P_{\Sigma,K}$ of $(R_{E/K}^1)^d$.

**Example 1.3.10** Let $K'$ be a cubic extension of a number field $K$. We construct a smooth compactification of the 2-dimensional anisotropic $K$-torus $R_{K'/K}^1(G_m)$ as follows. Let $Y$ be the cubic surface in $P^3$ defined by the equation

$$N_{K'/K}(z_1, z_2, z_3) = z_0^3$$
where $N_{K'/K}(z_1, z_2, z_3)$ is the homogeneous cubic norm-form. Over the algebraic closure $\overline{K}$ it is isomorphic to the singular cubic surface $z_1z_2z_3 = z_0^3$. The 3 quadratic singular points $p_1, p_2, p_3 \in Y_{\overline{K}}$ are defined over a splitting field $E$ of $R_{K'/K}(\mathbb{G}_m)$ and the Galois group $G = \text{Gal} K'/K$ acts on $\{p_1, p_2, p_3\}$ by permutations. There exists a minimal simultaneous resolution $\psi : Y' \to Y$ of singularities which is defined over $K$. By contraction $\psi' : Y' \to X$ of the proper pull-back of three $(-1)$ curves which are preimages of lines passing through the singular points we obtain a Del Pezzo surface $X$ of anticanonical degree 6 which is a smooth compactification of the anisotropic torus $R_{K'/K}(\mathbb{G}_m)$.

Let $k$ be a finite field of characteristic $p$ containing $q = p^n$ elements. Any finite extension $k'$ of $k$ is a cyclic Galois extension and the group $G = \text{Gal}(k'/k)$ is generated by the Frobenius automorphism $\phi : z \to z^q$. By [1.1.4], any $d$-dimensional algebraic torus $T$ over $k$ splitting over $k'$ is uniquely defined by the conjugacy class in $\text{GL}(d, \mathbb{Z})$ of the integral matrix

\[ \Phi = \rho(\phi). \]

The characteristic polynomial of the matrix $\Phi$ gives the following formula obtained by T. Ono [16] for the number of $k$-rational points in $T$:

**Theorem 1.3.11** Let $T$ be a $d$-dimensional algebraic torus defined over a finite field $k$. In the above notations, one has the following formula for the number of $k$-rational points of $T$:

\[ \text{Card}[T(k)] = (-1)^d \det(\Phi - q \cdot \text{Id}). \]

**Proposition 1.3.12** Let $P_{\Sigma}$ be a toric variety over a finite field $k$ defined by a $\Phi$-invariant fan $\Sigma \subset \mathbb{N}_R$. For any $\Phi$-invariant cone $\sigma \in \Sigma^G$, let $M_{R, \sigma} = \sigma \cap (-\sigma)$ be the maximal linear subspace in the dual cone $\sigma^* \subset M_R$. Let $\Phi_{\sigma}$ be the restriction of $\Phi$ on $M_{R, \sigma}$. Then

\[ \text{Card}[P_{\Sigma}(k)] = \sum_{\sigma \in \Sigma^G} (-1)^{\dim \sigma} \det(\Phi_{\sigma} - q \cdot \text{Id}). \]
Proof. By 1.3.2,
\[ P_\Sigma(k) = \bigcup_{\sigma \in \Sigma^G} T_\sigma(k). \]
Observe that \( k' \) is a splitting field for every algebraic torus \( T_{\sigma,k} \) defined by
the \( \rho \)-action of \( \Phi_\sigma \). Now the statement follows from 1.3.11. \( \square \)

1.4 Algebraic tori over local and global fields

First we fix our notations. Let \( \text{Val}(K) \) be the set of all valuations of a global
field \( K \). For any \( v \in \text{Val}(K) \), we denote by \( K_v \) the completion of \( K \) with
respect to \( v \). Let \( v \) be a non-archimedian absolute valuation of a number
field \( K \) and \( E \) a finite Galois extension of \( K \). Let \( \mathcal{V} \) be an extension of \( v \)
and \( (v) \) to \( E \), \( E_{\mathcal{V}} \) the completion of \( E \) with respect to \( \mathcal{V} \). Then
\[ \text{Gal}(E_{\mathcal{V}}/K_v) \cong G_v \subset G, \]
where \( G_v \) is the decomposition subgroup of \( G \) and \( K_v \otimes_K E \cong \prod_{\mathcal{V}|v} E_{\mathcal{V}} \).
Let \( T \) be an algebraic torus over \( K \) with the splitting field \( E \). Denote by
\( T_{K_v} = T \otimes K_v \).

Definition 1.4.1 We denote the group of characters \( \hat{T}_{K_v} = M^{G_v} \) by \( M_v \) and
the dual group \( \text{Hom}(\hat{T}_{K_v}, \mathbb{Z}) = N^{G_v} \) by \( N_v \).

Let \( (K_v \otimes_K E)^* \) and \( E_{\mathcal{V}}^* \) be the multiplicative groups of \( K_v \otimes_K E \) and \( E_{\mathcal{V}} \)
respectively. One has
\[ T_{K_v} = \text{Hom}_G(\hat{T}, (K_v \otimes_K E)^*) = \text{Hom}_{G_v}(M, E_{\mathcal{V}}^*). \]
Denote by \( \mathcal{O}_{\mathcal{V}} \) the maximal compact subgroup in \( E_{\mathcal{V}}^* \). There is a short exact
sequence
\[ 1 \to \mathcal{O}_{\mathcal{V}} \to E_{\mathcal{V}}^* \to \mathbb{Z} \to 1, \quad b \to \text{ord } b |_{\mathcal{V}}. \]
Denote by \( T(\mathcal{O}_{\mathcal{V}}) \) the maximal compact subgroup in \( T(K_v) \). Applying the
functor \( \text{Hom}_{G_v}(M_v, *) \) to the short exact sequence above, we obtain the short
exact sequence
\[ 1 \to N_v \otimes \mathcal{O}_{\mathcal{V}} \to N_v \otimes E_{\mathcal{V}}^* \to N_v \to 1 \]
which induces an injective homomorphism
\[ \pi_v : T(K_v)/T(\mathcal{O}_{\mathcal{V}}) \hookrightarrow N_v = N^{G_v}. \]
Proposition 1.4.2  The homomorphism $\pi_v$ has finite cokernel. Moreover, $\pi_v$ is an isomorphism if $E$ is unramified in $v$.

Definition 1.4.3  Let $S$ be a finite subset of $\text{Val}(K)$ containing all archimedean and ramified non-archimedean valuations of $K$. We denote by $S_\infty$ the set of all archimedean valuations of $K$ and put $S_0 = S \setminus S_\infty$.

Now we assume that $v$ is an archimedean absolute valuation, i.e., $K_v$ is $\mathbb{R}$ or $\mathbb{C}$. It is known that any torus over $\mathbb{R}$ is isomorphic to the product of some copies of $\mathbb{C}^*$, $\mathbb{R}_1$, or $S^1 = \{z \in \mathbb{C} \mid z\bar{z} = 1\}$. The quotient $T(K_v)/T(\mathcal{O}_v)$ is isomorphic to the $\mathbb{R}$-linear space $N_v \otimes \mathbb{R}$. The homomorphism $T(K_v) \to T(K_v)/T(\mathcal{O}_v)$ is simply the logarithmic mapping onto the Lie algebra of $T(K_v)$. Hence, we obtain:

Proposition 1.4.4  For any archimedean absolute valuation $v$, the quotient $T(K_v)/T(\mathcal{O}_v)$ can be canonically identified with the real Lie algebra of $T(K_v)$ embedded in the $d$-dimensional $\mathbb{R}$-subspace $N_\mathbb{R}$.

Definition 1.4.5  Denote by $T(A_K)$ the adele group of $T$, i.e., the restricted topological product

$$
\prod_{v \in \text{Val}(K)} T(K_v)
$$

consisting of all elements $t = \{t_v\} \in \prod_{v \in \text{Val}(K)} T(K_v)$ such that $t_v \in T(\mathcal{O}_v)$ for almost all $v \in \text{Val}(K)$. Let

$$
T^1(A_K) = \{t \in T(A_K) \mid \prod_{v \in \text{Val}(K)} |m(t_v)|_v = 1, \text{ for all } m \in \hat{T}_K \subset M\}.
$$

We put also

$$
K_T = \prod_{v \in \text{Val}(K)} T(\mathcal{O}_v),
$$
Proposition 1.4.6  The groups $T(A_K)$, $T^1(A_K)$, $T(K)$, $K_T$ have the following properties which are generalizations of the corresponding properties of the adelization of $G_m(K)$:

(i) $T(A_K)/T^1(A_K) \cong \mathbb{R}^k$, where $k$ is the rank of $\hat{T}_K$;

(ii) $T^1(A_K)/T(K)$ is compact;

(iii) $T^1(A_K)/K_T \cdot T(K)$ is isomorphic to the direct product of a finite group $\text{cl}(T_K)$ (this is an analog of the idele-classes group $\text{Cl}(K)$) and a connected compact abelian topological group which dimension equals the rank $r'$ of the group of $\mathcal{O}_K$-units in $T(K)$ (this rank equals $r_1 + r_2 - 1$ for $G_m$);

(iv) $W(T) = K_T \cap T(K)$ is a finite group of all torsion elements in $T(K)$ (this is the analog of the group of roots of unity in $G_m(K)$).

The following theorem of A. Weil plays a fundamental role in the definition of adelic measures on algebraic varieties.

Theorem 1.4.7  Let $X$ be an $d$-dimensional smooth algebraic variety over a global field $K$. Denote by $\mathcal{K}$ the canonical sheaf on $X$ with a family of local metrics $\| \cdot \|_v$. Then these local metrics uniquely define natural $v$-adic measures $\omega_{\mathcal{K},v}$ on $X(K_v)$.

Let $U \subset X$ be a Zariski open subset of $X$. Then for almost all $v \in \text{Val}(K)$ one has

$$\int_{U(\mathcal{O}_v)} \omega_{\mathcal{K},v} = \frac{\text{Card}[U(k_v)]}{q_v^d},$$

where $k_v$ is the residue field of $K_v$ and $q_v = \text{Card}[k_v]$.

Remark 1.4.8  We notice that the structure sheaf $\mathcal{O}_X$ of any algebraic variety $X$ has a natural metrization defined by $v$-adic valuations of the field $K$. If $X = G$ is an algebraic group, then there exists a natural way to define a metrization of the canonical sheaf $\mathcal{K}$ on $G$ by choosing a $G$-invariant algebraic differential $d$-form $\Omega$. Such a form defines an isomorphism of $\mathcal{K}$ with the structure sheaf $\mathcal{O}_G$. We denote the corresponding local measure on $G(K_v)$ by $\omega_{\Omega,v}$.

Let $T$ be a $d$-dimensional torus over $K$ with a splitting field $E$. Take a $T$-invariant differential $d$-form $\Omega$ on $T$ (it is unique up to a constant from $K$). According to A. Weil (1.4.8), we obtain a family of local measures $\omega_{\Omega,v}$ on $T(K_v)$. 
Definition 1.4.9 [16] Let

\[ L_S(s, T; E/K) = \prod_{v \notin S} L_v(s, T; E/K) \]

be the Artin L-function corresponding to the representation

\[ \rho : G = \text{Gal}(E/K) \to \text{GL}(M) \]

and a finite set \( S \subset \text{Val}(K) \) containing all archimedean valuations and all non-archimedean valuations of \( K \) which are ramified in \( E \). By definition, \( L_v(s, T; E/K) \equiv 1 \) if \( v \in S \). The numbers

\[ c_v = L_v(1, T; E/K) = \frac{1}{\det(Id - q_v^{-1}\Phi_v)}, \quad v \notin S \]

are called canonical correcting factors for measures \( \omega_{\Omega,v} \) (\( \Phi_v \) is the \( \rho \)-image of a local Frobenius element in \( G \)).

By [1.3.11], one has

\[ c_v^{-1} = \int_{T(O_v)} \omega_{\Omega,v} = \frac{\text{Card}[T(k_v)]}{q_v^d}, \quad v \notin S. \]

Let \( d\mu_v = c_v \omega_{\Omega,v} \). We put \( c_v = 1 \) for \( v \in S \). Since

\[ \int_{T(O_v)} d\mu_v = 1 \]

for \( v \notin S \), the \( \{c_v\} \) defines the canonical measure

\[ \omega_{\Omega,S} = \prod_{v \in \text{Val}(K)} d\mu_v \]

on the adele group \( T(A_K) \). By the product formula, \( \omega_{\Omega,S} \) does not depend on the choice of \( \Omega \).

Let \( dx \) be the standard Lebesgue measure on \( T(A_K)/T^1(A_K) \cong R^k = M_R^G \). There exists a unique Haar measure \( \omega_{\Omega_1} \) on \( T^1(A_K) \) such that \( \omega_{\Omega_1} dx = \omega_{\Omega,S} \).
Definition 1.4.10 The Tamagawa number of $T_K$ is defined as
\[ \tau(T_K) = \frac{b_S(T_K)}{l_S(T_K)} \]
where
\[ b_S(T_K) = \int_{T^1(A_K)/T(K)} \omega_{1,S}^1, \]
\[ l_S(T_K) = \lim_{s \to 1} (s - 1)^k L_S(s, T; E/K). \]

Remark 1.4.11 Although the numbers $b_S(T_K)$ and $l_S(T_K)$ do depend on the choice of the finite subset $S \subset \text{Val}(K)$, the Tamagawa number $\tau(T_K)$ does not depend on $S$.

Theorem 1.4.12 [16, 17] The Tamagawa number $\tau(T)$ of $T$ does not depend on the choice of a splitting field $E$. It satisfies the following properties:
(i) $\tau(G_m(K)) = 1$;
(ii) $\tau(T \times T') = \tau(T) \cdot \tau(T')$ where $T'$ and $T$ are tori over $K$;
(iii) $\tau_K(R_{K'/K}(T)) = \tau_{K'}(T)$ for any torus $T$ over $K'/K$.
Moreover, $\tau(T)$ is the ratio of two positive integers
\[ h(T_K) = \text{Card}[H^1(G, M)] \]
and $i(T_K) = \text{Card}[\text{III}(T)]$ where
\[ \text{III}(T) = \text{Ker} [H^1(G, T(K)) \to \prod_v H^1(G_v, T(K_v))]; \]
in particular, $\tau(T_K)$ is a rational number.

Definition 1.4.13 Let $\overline{T(K)}$ be the closure of $T(K)$ in $\prod_v T(K_v)$ in the direct product topology. Define the obstruction group to weak approximation as
\[ A(T) = \prod_v T(K_v)/\overline{T(K)}. \]
Theorem 1.4.14 Let $P_\Sigma$ be a complete smooth toric variety over $K$. There is an exact sequence:

$$0 \to A(T) \to \text{Hom}(H^1(G, \text{Pic}(P_{\Sigma,E})), \mathbb{Q}/\mathbb{Z}) \to III(T) \to 0.$$ 

The group $H^1(G, \text{Pic}(P_{\Sigma,E}))$ is canonically isomorphic to $\text{Br}(P_{\Sigma,K})/\text{Br}(K)$, where $\text{Br}(P_{\Sigma,K}) = H^2_{\text{et}}(P_{\Sigma,K}, \mathbb{G}_m)$.

Corollary 1.4.15 Denote by $\beta(P_\Sigma)$ the cardinality of $H^1(G, \text{Pic}(P_{\Sigma,E}))$. Then

$$\text{Card}[A(T)] = \frac{\beta(P_\Sigma)}{i(T_K)}.$$ 

2 Heights and their Fourier transforms

2.1 Complexified local Weil functions and heights

A theory of heights on an algebraic variety $X$ defined over a number field $K$ is the unique functorial homomorphism from $\text{Pic}(X)$ to equivalence classes of functions $X(K) \to \mathbb{R}_{\geq 0}$ which on metrized line bundles $\mathcal{L}$ is given by the formula

$$H_{\mathcal{L}}(x) = \prod_v \|f(x)\|_v^{-1}$$

where $f$ is a rational section of $L$ not vanishing in $x \in X(K)$. Two functions are equivalent if they differ by a bounded on $X(K)$ function. For our purposes it will be convenient to extend these notions to the complexified Picard group $\text{Pic}(X) \otimes \mathbb{C}$.

Let $P_\Sigma$ be a compact toric variety over a global field $K$. We define a canonical compact covering of $P_{\Sigma}(K_v)$ by compact subsets $C_{\sigma,v} \subset U_\sigma(K_v)$. For this purpose we identify lattice elements $m \in M$ with characters of $T$ and define the compact subset $C_{\sigma,v} \subset U_\sigma(K_v)$ as follows

$$C_{\sigma,v} = \{x_v \in U_\sigma(K_v) \mid \|m(x_v)\|_v \leq 1 \text{ for all } m \in M^{G_v} \cap \sigma\}.$$
Proposition 2.1.1  The compact subsets $C_{\sigma,v}$ ($\sigma \in \Sigma$) form a compact covering of $P_\Sigma(K_v)$ such that for any two cones $\sigma, \sigma' \in \Sigma$ one has

$$C_{\sigma,v} \cap C_{\sigma',v} = C_{\sigma\cap\sigma',v}.$$  

Proof. The last property of the compact subsets $C_{\sigma,v}$ follows immediately from their definition. Since the $T(K_v)$-orbit of maximal dimension is dense in $P_\Sigma(K)$, it is sufficient to prove that the compacts $C_{\sigma,v}$ cover $T(K_v)$.

Let $x_v \in T(K_v)$. Denote by $\overline{x_v}$ the image of $x_v$ in $T(K_v)/T(O_v) \subset N_R$. By completeness of the fan $\Sigma$, the point $-\overline{x_v}$ is contained in some cone $\sigma \in \Sigma$. Hence $x_v \in C_{\sigma,v}$. $\square$

Now we define canonical metrizations of $T(K_v)$-linearized line bundles on $P_\Sigma(K_v)$.

Let $L(\varphi)$ be a line bundle on $P_\Sigma(K_v)$ corresponding to a $\Sigma$-piecewise linear integral $G_v$-invariant function $\varphi$ on $N_R$.

Proposition 2.1.2 Let $f$ be a rational section of $L(\varphi)$. We define the $v$-norm of $f$ at a point $x_v \in P_\Sigma(K_v)$ as

$$\|f(x_v)\|_v = \left| \frac{f(x_v)}{m_{\sigma,\varphi}(x_v)} \right|_v$$

where $\sigma$ is a cone in $\Sigma$ such that $x_v \in C_{\sigma,v}$ and $m_{\sigma,\varphi} \in M$ is the restriction of $\varphi$ on $\sigma$. Then this $v$-norm defines a $T(O_v)$-invariant $v$-adic metric on $L(\varphi)$.

Proof. The statement follows from the fact that

$$\left| \frac{m_{\sigma,\varphi}(x_v)}{m_{\sigma',\varphi}(x_v)} \right|_v$$

if $x_v \in C_{\sigma,v} \cap C_{\sigma',v}$. $\square$

A family of local metrics on all $T$-linearized line bundles on $P_\Sigma$ corresponding to $\Sigma$-piecewise linear $G$-invariant functions $\varphi \in PL(\Sigma)G$ uniquely determines a family of local Weil functions on $(P_\Sigma)$ corresponding to $T$-invariant divisors

$$D_\varphi = \varphi(e_1)D_1 + \cdots + \varphi(e_n)D_n.$$  

We extend these local Weil functions to the group of $T$-invariant Cartier divisors with complex coefficients as follows.
**Definition 2.1.3** A $T$-invariant $C$-Cartier divisor is a formal linear combination $D_s = s_1D_1 + \cdots + s_nD_n$, with $s = (s_1, \ldots, s_n) \in C^n$ or equivalently a complex piecewise linear function $\varphi$ in $PL(\Sigma)_G^C$ having the property $\varphi(e_i) = s_i$ ($i = 1, \ldots, n$).

**Definition 2.1.4** Let $\varphi \in PL(\Sigma)_G^C$. For any point $x_v \in T(K_v) \subset P_\Sigma(K_v)$, denote by $x_v$ the image of $x_v$ in $N_v$ (resp. $N_v \otimes \mathbb{R}$ for archimedian valuations), where $N_v$ is considered as a canonical lattice in the real space $N_\mathbb{R}$. Define the complexified local Weil function $H_{\Sigma,v}(x_v, \varphi)$ by the formula

$$H_{\Sigma,v}(x_v, \varphi) = e^{\varphi(x_v) \log q_v}$$

where $q_v$ is the cardinality of the residue field $k_v$ of $K_v$ if $v$ is non-archimedian and $\log q_v = 1$ if $v$ is archimedian.

**Proposition 2.1.5** The complexified local Weil function $H_{\Sigma,v}(x_v, \varphi)$ satisfies the following properties:

(i) If $s_i = \varphi(e_i) \in \mathbb{Z}^n$ ($i = 1, \ldots, n$), then $H_{\Sigma,v}(x_v, \varphi)$ is a classical local Weil function $H_{L(\varphi),v}(x_v)$ corresponding to a $T$-invariant Cartier divisor $D_s = s_1D_1 + \cdots + s_nD_n$ on $P_\Sigma$.

(ii) $H_{\Sigma,v}(x_v, \varphi)$ is $T(O_v)$-invariant.

(iii) $H_{\Sigma,v}(x_v, \varphi + \varphi') = H_{\Sigma,v}(x_v, \varphi)H_{\Sigma,v}(x_v, \varphi')$.

**Definition 2.1.6** Let $\varphi \in PL(\Sigma)_G^C$. We define the complexified height function on $P_\Sigma$ by

$$H_\Sigma(x, \varphi) = \prod_{v \in Val(K)} H_{\Sigma,v}(x, \varphi).$$

**Remark 2.1.7** Although all local factors $H_{\Sigma,v}(x, \varphi)$ of $H_\Sigma(x, \varphi)$ are functions on $PL(\Sigma)_G^C$, by the product formula, the global complex height function $H_\Sigma(x, \varphi)$ depends only on the class of $\varphi \in PL(\Sigma)_G^C$ modulo complex global linear $G$-invariant functions on $N_\mathbb{C}$, i.e., $H_\Sigma(x, \varphi)$ depends only on the class of $\varphi$ in $\text{Pic}(P_{\Sigma,K}) \otimes \mathbb{C}$. 22
Definition 2.1.8 We define the zeta-function of the complex height-function \( H_\Sigma(x, \varphi) \) as
\[
Z_\Sigma(\varphi) = \sum_{x \in T(K)} H_\Sigma(x, -\varphi).
\]

Remark 2.1.9 One can see that the series \( Z_\Sigma(\varphi) \) converges absolutely and uniformly in the domain \( \text{Re}(\varphi(e_j)) \gg 0 \) for all \( j \). Since \( H_\Sigma(x, \varphi) \) is the product of the local complex Weil functions \( H_{\Sigma,v}(x, \varphi) \) and \( H_{\Sigma,v}(x, \varphi) = 1 \) for almost all \( v \) (\( x \in T(K) \)), we can immediately extend \( H_\Sigma(x, \varphi) \) to a function on the adelic group \( T(A_K) \).

2.2 Fourier transforms of non-archimedean heights

Definition 2.2.1 Let \( \Sigma \) be a complete regular fan of cones in \( N \) whose 1-dimensional cones are generated by \( e_1, \ldots, e_n \). We establish a one-to-one correspondence between \( e_1, \ldots, e_n \) and \( n \) independent variables \( z_1, \ldots, z_n \). The Stanley-Reisner ring \( R(\Sigma) \) is defined as the factor of the polynomial ring \( A[z] = \mathbb{C}[z_1, \ldots, z_n] \) by the ideal \( I(\Sigma) \) generated by all monomials \( z_{i_1} \cdots z_{i_k} \) such that \( e_{i_1}, \ldots, e_{i_k} \) are not generators of a \( k \)-dimensional cone in \( \Sigma \).

Proposition 2.2.2 There is a natural identification between the elements of the lattice \( N \) and the monomial \( \mathbb{C} \)-basis of the ring \( R(\Sigma) \).

Proof. Every integral point \( x \in N \) belongs to the interior of a unique cone \( \sigma \in \Sigma \). Let \( e_{i_1}, \ldots, e_{i_k} \) be an integral basis of \( \sigma \). Then there exist positive integers \( a_1, \ldots, a_k \) such that
\[
\overline{\sigma} = a_1 e_{i_1} + \cdots + a_k e_{i_k}.
\]
Therefore, \( \overline{\sigma} \) defines the monomial \( m(\overline{\sigma}) = z_1^{a_1} \cdots z_k^{a_k} \).

By definition, \( m(\overline{\sigma}) \notin I(\Sigma) \). It is clear that \( I(\Sigma) \) has a monomial \( \mathbb{C} \)-basis. Hence, we have constructed a mapping \( \overline{\sigma} \mapsto m(\overline{\sigma}) \) from \( N \) to the monomial basis of \( R(\Sigma) \). It is easy to see that this mapping is bijective. \( \square \)

Now choose a valuation \( v \notin S \). Then we obtain a cyclic subgroup \( G_v = \langle \Phi_v \rangle \subset G \) generated by a lattice automorphism \( \Phi_v : N \to N \) representing...
the local Frobenius element at place \( v \). Then \( \Sigma(1) \) splits into a disjoint union of \( G_v \)-orbits

\[
\Sigma(1) = \Sigma_1(1) \cup \cdots \cup \Sigma_l(1).
\]

Let \( d_j \) be the length of the \( G_v \)-orbit \( \Sigma_j(1) \). One has

\[
\sum_{i=1}^l d_j = n.
\]

**Definition 2.2.3** Define the \( \mathbb{Z}^l_{\geq 0} \)-grading of the polynomial ring \( A[z] = C[z_1, \ldots, z_n] \) and the Stanley-Reisner ring \( R(\Sigma) \) by the decomposition of the set of variables \( \{z_1, \ldots, z_n\} \) into the disjoint union of \( l \) sets \( Z_1 \cup \cdots \cup Z_l \) which is induced by the decomposition of \( \Sigma(1) \) into \( G_v \)-orbits. The standard \( \mathbb{Z}_{\geq 0} \)-grading of the polynomial ring \( A[z] = C[z_1, \ldots, z_n] \) and the Stanley-Reisner ring \( R(\Sigma) \) will be called the total grading.

**Definition 2.2.4** We define the power series \( P(\Sigma, \Phi_v; t_1, \ldots, t_l) \) by the formula

\[
P(\Sigma, \Phi_v; t_1, \ldots, t_l) = \sum_{(i_1, \ldots, i_l) \in \mathbb{Z}^l_{\geq 0}} (\text{Tr} \Phi_v^{i_1 \cdots i_l}) t_1^{i_1} \cdots t_l^{i_l},
\]

where \( \Phi_v^{i_1 \cdots i_l} \) is the linear operator induced by \( \Phi_v \) on the homogeneous \((i_1, \ldots, i_l)\)-component of \( R(\Sigma) \).

**Proposition 2.2.5** One has

\[
P(\Sigma, \Phi_v; t_1, \ldots, t_l) = \frac{Q_\Sigma(t_1^{d_1}, \ldots, t_l^{d_l})}{(1 - t_1^{d_1}) \cdots (1 - t_l^{d_l})}
\]

where \( Q_\Sigma(t_1^{d_1}, \ldots, t_l^{d_l}) \) is a polynomial in \( t_1^{d_1}, \ldots, t_l^{d_l} \) having the total degree \( n \) such its all nonconstant monomials have the total degree at least 2.

**Proof.** Since \( \dim A[z] - \dim R(\Sigma) = n - d \), there exists the minimal \( \mathbb{Z}^l_{\geq 0} \)-graded free resolution

\[
0 \to F^{n-d} \to \cdots \to F^1 \to F^0 = A[z] \to R(\Sigma) \to 0
\]

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of the Stanley-Reisner ring \( R(\Sigma) \) considered as a module over the polynomial ring \( A[z] \). Let \( \Phi^{i_1,\ldots,i_l}_v \) be the linear operator on the homogeneous \((i_1,\ldots,i_l)\)-component of \( A[z] \) induced by the action of \( \Phi_v \) on \( z_1,\ldots,z_n \). Then

\[
\sum_{(i_1,\ldots,i_l)\in\mathbb{Z}_{\geq 0}} (\text{Tr} \Phi^{i_1,\ldots,i_l}_v) t_1^{i_1} \cdots t_r^{i_l} = \frac{1}{\prod_{j=1}^{l} (1 - t_j^{d_j})}.
\]

We notice that \( \text{Tr} \Phi^{i_1,\ldots,i_l}_v \) and \( \text{Tr} \Phi^{i_1,\ldots,i_l}_v \) can be nonzero only if the length \( d_k \) of the \( G_v \)-orbit \( \Sigma_k(1) \) divides \( i_k \) \((k = 1,\ldots,l)\). Therefore the polynomial \( Q(\Sigma) \) is defined by ranks and \( \mathbb{Z}_{\geq 0} \)-degrees of generators of the free \( A[z] \)-modules \( F^i \). Notice that every monomial in \( I(\Sigma) \) has the total degree at least 2, because every element \( e_i \in \{e_1,\ldots,e_n\} \) generates a 1-dimensional cone of \( \Sigma \). So all generators of \( F^i \) \((i \geq 1)\) have the total degree at least 2. Therefore, the polynomial \( Q_\Sigma \) has only monomials of the total degree at least 2. Since \( R(\Sigma) \) is a Gorenstein ring, \( F^{n-d} \) is a free \( A[z] \)-module of rank 1 with a generator of the degree \((d_1,\ldots,d_l)\). Therefore, \( Q \) has the total degree \( n = d_1 + \cdots + d_l \).

Let \( \chi \) be a topological character of \( T(A_K) \) such that its \( v \)-component \( \chi_v : T(K_v) \to S^1 \subset \mathbb{C}^\ast \) is trivial on \( T(O_v) \). For each \( j \in \{1,\ldots,l\} \), we denote by \( n_j \) the sum of \( d_j \) generators of all 1-dimensional cones of the \( G_v \)-orbit \( \Sigma_j(1) \). Then \( n_j \) is a \( G_v \)-invariant element of \( N \). By 1.4.2 \( n_j \) represents an element of \( T(K_v) \) modulo \( T(O_v) \). Therefore, \( \chi_v(n_j) \) is well defined.

**Proposition 2.2.6** Let \( v \notin S \). Denote by \( q_v \) the cardinality of the finite residue field \( k_v \) of \( K_v \). Then for any local topological character \( \chi_v \) of \( T(K_v) \), one has

\[
\hat{H}_{\Sigma,v}(\chi,-\varphi) = \int_{T(K_v)} H_{\Sigma,v}(x_v,-\varphi) \chi_v(x_v) d\mu_v =
\]

\[
\frac{Q_{\Sigma} \left( \chi_v(n_1)_{q_v(\chi_1)}, \ldots, \chi_v(n_l)_{q_v(\chi_l)} \right)}{(1 - \chi_v(n_1)_{q_v(\chi_1)}) \cdots (1 - \chi_v(n_l)_{q_v(\chi_l)})}
\]

if \( \chi_v \) is trivial on \( T(O_v) \), and

\[
\int_{T(K_v)} H_{\Sigma,v}(x_v,-\varphi) \chi_v(x_v) d\mu_v = 0
\]

otherwise.
Proof. Since the local Haar measure $\mu_v$ is $T(\mathcal{O}_v)$-invariant, one has

$$\int_{T(K_v)} H_{\Sigma,v}(x_v, -\varphi) \chi_v(x_v) d\mu_v = \sum_{\mathcal{F}_v \in T(K_v)/T(\mathcal{O}_v)} H_{\Sigma,v}(\mathcal{F}_v, -\varphi) \chi_v(\mathcal{F}_v) \int_{T(\mathcal{O}_v)} \chi_v d\mu_v$$

where $\mathcal{F}_v$ denotes the image of $x_v$ in $T(K_v)/T(\mathcal{O}_v) = N_v$. Notice that $\int_{T(\mathcal{O}_v)} \chi_v d\mu_v = 0$ if $\chi_v$ has nontrivial restriction on $T(\mathcal{O}_v)$. By 2.2.2, there exists a natural identification between $G_v$-invariant elements of $N$ and $G_v$-invariant monomials in $R(\Sigma)$. Since $\Phi_v$ acts by permutations on monomials in the homogeneous $(i_1, \ldots, i_l)$-component of $R(\Sigma)$, the number of $G_v$-invariant monomials in $R^{i_1, \ldots, i_l}(\Sigma)$ equals $\text{Tr} \Phi_v^{i_1, \ldots, i_l}$. Take a $G_v$-invariant element $\mathcal{F}_v \in N$ such that $m(\mathcal{F}_v) \in R^{i_1, \ldots, i_r}(\Sigma)$. Put $i_k = d_kb_k (k = 1, \ldots, l)$. Then

$$\varphi(\mathcal{F}_v) = b_1 \varphi(n_1) + \cdots + b_l \varphi(n_l)$$

and

$$\chi_v(\mathcal{F}_v) = \chi_{v}^{b_1}(n_1) \cdots \chi_{v}^{b_l}(n_l).$$

This implies the claimed formula. \qed

Let $A^*(\mathbf{P}_\Sigma) = \bigoplus_{i=0}^d A^i(\mathbf{P}_\Sigma)$ be the Chow ring of $\mathbf{P}_{\Sigma, E_v}$. The groups $A^i(\mathbf{P}_\Sigma)$ have natural $G_v$-action. Denote by $\Phi_v(i)$ the operator on $A^i(\mathbf{P}_\Sigma)$ induced by $\Phi_v$.

**Proposition 2.2.7** Denote by $1_v$ the trivial topological character of $T(K_v)$. Then the restriction of

$$\int_{T(K_v)} H_{\Sigma,v}(x_v, -\varphi) 1_v(x_v) d\mu_v$$

to the line $s_1 = \cdots = s_r = s$ is equal to

$$L_v(s, T; E/K) \left( \sum_{k=0}^d \frac{\text{Tr} \Phi_v(i)}{q_v^k} \right).$$

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Proof. The Chow ring is the quotient of $R(\Sigma)$ by a regular sequence $[4]$. This gives the $\mathbb{Z}_{\geq 0}$-graded (by the total degree) Koszul resolution having a $\Phi_v$-action:

$$0 \to \Lambda^d M \otimes R(\Sigma) \to \cdots \to \Lambda^1 M \otimes R(\Sigma) \to R(\Sigma) \to A^*(\mathbb{P}_\Sigma) \to 0.$$ 

We apply the trace operator to the $k$-homogeneous component of the Koszul complex, then we multiply the result by $1/q_v^{ks}$ and take the sum over $k \geq 0$. By 2.2.6, we have

$$L_v^{-1}(s, T; E/K) \cdot \int_{T(K_v)} H_{\Sigma,v}(x_v, -\varphi) 1_v(x_v) d\mu_v = \sum_{k=0}^d \frac{\text{Tr} \Phi_v(i)}{q_v^{ks}}$$

because

$$\sum_{k=0}^d \frac{(-1)^k}{q_v^{ks}} \text{Tr}(\Lambda^k \Phi_v) = \det(Id - q_v^{-s} \Phi_v) = L_v^{-1}(s, T; E/K).$$

\[\square\]

2.3 Fourier transforms of archimedian heights

**Proposition 2.3.1** Let $\chi_v(y) = e^{-2\pi i \langle x, y \rangle}$ be a topological character of $T(K_v)$ which is trivial on $T(\mathcal{O}_v)$. Then the Fourier transform $\hat{H}_{\Sigma,v}(\chi_v, -\varphi)$ of a local archimedian Weil function $H_{\Sigma,v}(x, -\varphi)$ is a rational function in $s_j = \varphi(e_j)$ for $\text{Re}(s_j) > 0$.

**Proof.** First we consider the case $K_v = \mathbb{C}$. Then $T(K_v)/T(\mathcal{O}_v) = N_\mathbb{R}$ and

$$\hat{H}_{\Sigma,v}(\chi_v(y), -\varphi) = \int_{T(\mathcal{O}_v)} e^{-\varphi(x) - 2\pi i \langle x, y \rangle} dx = \sum_{\sigma \in \Sigma(d)} \int_{\sigma} e^{-\varphi(x) - 2\pi i \langle x, y \rangle} dx$$

where $dx = d\mathbf{x}_v$ is the standard measure on $N_v \otimes \mathbb{R} \simeq \mathbb{R}^d$. On the other hand,

$$\int_{\sigma} e^{-\varphi(x) - 2\pi i \langle x, y \rangle} dx = \frac{1}{\prod_{e_j \in \sigma}(s_j + 2\pi i \langle x, y \rangle)}.$$ 

If $K_v = \mathbb{R}$, then the following simple statements allows to repeat the arguments:
Lemma 2.3.2 Let $\Sigma \subset N_R$ be a complete regular $G$-invariant fan of cones. Denote by $\Sigma^G \subset N^G_R$ the fan consisting of $\Sigma = \sigma \cap N^G_R$, $\sigma \in \Sigma$. Then $\Sigma^G$ is again a complete regular fan.

The proof of the following proposition was suggested to us by W. Hoffmann.

Proposition 2.3.3 Let $K \subset C^r$ be a compact such that $\text{Re}(s_j) > \delta$ for all $(s_1, \ldots, s_r) \in K$. Then there exists a constant $c(K, \Sigma)$ such that

$$|\hat{H}_{\Sigma, v}(y, -\varphi)| \leq c(K, \Sigma) \sum_{\sigma \in \Sigma(d)} \frac{1}{\prod e_k \in \sigma (1 + |\langle y, e_k \rangle|)^{1+1/d}}.$$

Proof. Let $f_1, \ldots, f_d$ be a basis of $M$. Put $x_i = \langle x, f_i \rangle$. We denote by $y_1, \ldots, y_d$ the coordinates of $y$ in the basis $f_1, \ldots, f_d$. Let $\varphi_i(x) = \frac{\partial}{\partial x_i} \varphi(x)$. $\varphi_i(x)$ has a constant value $\varphi_i, \sigma$ in the interior of a cone $\sigma \in \Sigma(d)$.

$$\hat{H}_{\Sigma, v}(y, -\varphi) = \int_{N_R} e^{-\varphi(x) - 2\pi i < y, x >} dx = \frac{1}{2\pi i y_j} \int_{N_R} \frac{\partial}{\partial x_j} (e^{-\varphi(x)} e^{-2\pi i < y, x >}) dx$$

$$= -\frac{1}{2\pi i y_j} \int_{N_R} \varphi_j(x) e^{-\varphi(x) - 2\pi i < y, x >} dx$$

$$= \frac{i}{2\pi y_j} \sum_{\sigma \in \Sigma(d)} \frac{\varphi_j, \sigma}{\prod e_k \in \sigma (s_k + 2\pi i < y, e_k >)}$$

Notice that $M_R$ is covered by $d$ domains:

$$V_j = \{ y = \sum_i y_i f_i \in M_R \mid |y_j| = \max_i |y_i| \}.$$

Let $\|y\|^2 = \sum_i y_i^2$. Then $\|y\| \leq \sqrt{d} |y_j|$ for $y \in V_j$. Then

$$|\hat{H}_{\Sigma, v}(y, -\varphi)| \leq \frac{\sqrt{d}}{\|y\|} \sum_{\sigma \in \Sigma(d)} \frac{1}{\prod e_k \in \sigma |s_k + 2\pi i < y, e_k >|}$$

for $y \in V_j$. Furthermore, we obtain

$$|\hat{H}_{\Sigma, v}(y, -\varphi)| \leq \frac{C'(\delta)}{1 + \|y\|} \sum_{\sigma \in \Sigma(d)} \frac{1}{\prod e_k \in \sigma (1 + |< y, e_k | >)}$$

using the following obvious statement:
Lemma 2.3.4 Assume that $\text{Re}(s) > \delta > 0$. Then there exists a positive constant $C(\delta)$ such that for all $t$ one has $C(\delta)(|s + 2\pi it|) \geq 1 + |t|$.

Since $|\langle y, e_k \rangle| \leq \|y\|\|e_k\|$, it follows that there exist constants $c_\sigma$ such that

$$c_\sigma(1 + |\langle y, e_k \rangle|) \geq \prod_{e_k \in \sigma} (1 + |\langle y, e_k \rangle|).$$

Finally, we obtain

$$|\hat{\mathcal{H}}_{\Sigma,v}(y, -\varphi)| \leq c_j(\delta, \Sigma) \cdot \sum_{\sigma \in \Sigma(d)} \frac{1}{\prod_{e_k \in \sigma}(1 + |\langle y, e_k \rangle|)^{1+1/d}}$$

for all $y \in V_j$. It remains to put $c(K, \Sigma) = \max_j c_j(\delta, \Sigma)$. \qed

Corollary 2.3.5 Let $g : M_C \to \mathbb{C}$ be a continuous function such that

$$|g(iy)| \leq \|y\|^\varepsilon, \quad \varepsilon < 1, \quad y \in M_R.$$

Then

$$\sum_{y \in O} g(iy)\hat{\mathcal{H}}_{\Sigma,v}(y, -\varphi)$$

is absolutely and uniformly convergent on $K$ for any function $g(iy)$ for any lattice $O \subset M_R$.

3 Characteristic functions of convex cones

Let $V$ be an $r$-dimensional real vector space, $V_C$ its complex scalar extension, $\Lambda \subset V$ a convex $r$-dimensional cone such that $\Lambda \cap -\Lambda = 0 \in V$. Denote by $\Lambda^o$ the interior of $\Lambda$, $\Lambda_C^o = \Lambda^o + iV$ the complex tube domain over $\Lambda^o$, by $V^*$ the dual space, by $\Lambda^* \subset V^*$ the dual to $\Lambda$ cone and by $dy$ a Haar measure on $V^*$.

Definition 3.0.6 The characteristic function of $\Lambda$ is defined as the integral

$$\mathcal{X}_\Lambda(dy, u) = \int_{\Lambda^*} e^{-(u, y)}dy,$$

where $u \in \Lambda_C$. 

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Remark 3.0.7 Characteristic functions of convex cones have been investigated in the theory of homogeneous cones by M. K"ocher, O.S. Rothaus, and E.B. Vinberg [10, 22, 19].

Remark 3.0.8 We will be interested in characteristic functions of convex cones $\Lambda$ in real spaces $V$ which have natural lattices $L \subset V$ of the maximal rank $r$. Let $L^*$ be the dual lattice in $V^*$, then we can normalize the Haar measure $dy$ on $V^*$ so that the volume of the fundamental domain $V^*/L^*$ equals 1. In this case the corresponding characteristic function will be denoted simply by $\mathcal{X}_\Lambda(u)$.

Proposition 3.0.9 [22] Let $u \in \Lambda^0 \subset V$ be an interior point of $\Lambda$. Denote by $\Lambda_u^*(t)$ the convex $(r-1)$-dimensional compact
\[
\{ y \in \Lambda^* \mid \langle u, y \rangle = t \}
\]
We define the $(r-1)$-dimensional measure $dy'_t$ on $\Lambda_u^*(t)$ in such a way that for any function $f : V \to \mathbb{R}$ with compact support one has
\[
\int_{V^*} f(y)dy = \int_{-\infty}^{+\infty} dt \left( \int_{\langle (u,y) = t \rangle} f(y)dy'_t \right).
\]
Then
\[
\mathcal{X}_\Lambda(u) = (r-1)! \int_{\Lambda_u^*(1)} dy'_1.
\]
The characteristic function $\mathcal{X}_\Lambda(u)$ has the following properties [19, 22]:

Proposition 3.0.10 (i) If $A$ is any invertible linear operator on $V$, then
\[
\mathcal{X}_\Lambda(Au) = \frac{\mathcal{X}_\Lambda(u)}{|\det A|};
\]
(ii) If $\Lambda^0 = \mathbb{R}^r_{\geq 0}$, $L = \mathbb{Z}^r \subset \mathbb{R}^r$, then
\[
\mathcal{X}_\Lambda(u) = (u_1 \cdots u_r)^{-1}, \text{ for } \text{Re}(u_i) > 0;
\]
(iii) If $z \in \Lambda^0$, then
\[
\lim_{z \to \partial \Lambda} \mathcal{X}_\Lambda(z) = \infty;
\]
(iv) $\mathcal{X}_\Lambda(u) \neq 0$ for all $u \in \Lambda^0_C$. 

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Proposition 3.0.11 If $\Lambda$ is an $r$-dimensional finitely generated polyhedral cone, then $\mathcal{X}_\Lambda(u)$ is a rational function of degree $-r$. In particular, $\mathcal{X}_\Lambda(u)$ has a meromorphic extension to the whole complex space $V_{\mathbb{C}}$.

Proof. It follows from Proposition 3.0.10(i) that $\mathcal{X}_\Lambda(\lambda u) = \lambda^{-r}\mathcal{X}_\Lambda(u)$. Hence $\mathcal{X}_\Lambda(u)$ has degree $-r$. In order to calculate $\mathcal{X}_\Lambda(u)$, we subdivide the dual cone $\Lambda^*$ into a union of simplicial subcones

$$\Lambda^* = \bigcup_j \Lambda_j^*.$$

Then $\Lambda$ is the intersection

$$\Lambda = \bigcap_j \Lambda_j.$$

For $\text{Re}(u) \in \bigcap_j \Lambda_j^o$, one has

$$\mathcal{X}_\Lambda(u) = \sum_j \mathcal{X}_{\Lambda_j}(u).$$

By Proposition 3.0.10(i),(ii), every function $\mathcal{X}_{\Lambda_j}(u)$ is rational. \qed

Definition 3.0.12 Let $X$ be a smooth proper algebraic variety. Denote by $\Lambda_{\text{eff}} \subset \text{Pic}(X)_{\mathbb{R}}$ the cone generated by classes of effective divisors on $X$. Assume that the anticanonical class $[K^{-1}] \in \text{Pic}(X)_{\mathbb{R}}$ is contained in the interior of $\Lambda_{\text{eff}}$. We define the constant $\alpha(X)$ by

$$\alpha(X) = \mathcal{X}_{\Lambda_{\text{eff}}}(K^{-1}).$$

Corollary 3.0.13 If $\Lambda_{\text{eff}}$ is a finitely generated polyhedral cone, then $\alpha(X)$ is a rational number.

Example 3.0.14 Let $P_{\Sigma,K}$ be a smooth compactification of an anisotropic torus $T_K$. By 1.3.7, $\Lambda_{\text{eff}} \subset \text{Pic}(P_{\Sigma,K}) \otimes \mathbb{R}$ is a simplicial cone. Using 3.0.10 and the exact sequence

$$0 \to PL(\Sigma)^G \to \text{Pic}(P_{\Sigma,K}) \to H^1(G, M) \to 0$$

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we obtain
\[ X_{\Lambda_{\text{eff}}} (u) = \frac{1}{h(T_K) u_1 \cdots u_r}, \]
where \( u = \varphi, \varphi(e_j) = u_j \ (j = 1, \ldots l) \). In particular,
\[ \alpha(P_\Sigma) = \frac{1}{h(T_K)}. \]

**Example 3.0.15** Consider an example of a non-simplicial cone of Mori \( \Lambda_{\text{eff}} \) in \( V = \text{Pic}(X)_R \) where \( X \) is a Del Pezzo surface of anticanonical degree 6. The cone \( \Lambda \) has 6 generators corresponding to exceptional curves of the first kind on \( X \). We can construct \( X \) as the blow up of 3 points \( p_1, p_2, p_3 \) in general position on \( \mathbb{P}^2 \). The exceptional curves are \( C_1, C_2, C_3, C_{12}, C_{13}, C_{23} \), where \( C_{ij} \) is the proper pullback of the line joining \( p_i \) and \( p_j \).

If \( u = u_1[C_1] + u_2[C_2] + u_3[C_3] + u_{12}[C_{12}] + u_{13}[C_{13}] + u_{23}[C_{23}] \in \Lambda^0_{\text{eff}}, \) then
\[ X_{\Lambda_{\text{eff}}} (u) = \frac{u_1 + u_2 + u_3 + u_{12} + u_{13} + u_{23}}{(u_1 + u_{23})(u_2 + u_{13})(u_3 + u_{12})(u_1 + u_2 + u_3)(u_{12} + u_{13} + u_{23})}, \]
and
\[ \alpha(X) = 1/12. \]

**Proposition 3.0.16** Assume that \( \Lambda \) is a finitely generated polyhedral cone and \( \Lambda \cap -\Lambda = 0 \). Let \( p_0 \) and \( p_1 \) be two points in \( E \) such that \( p_1 \notin \Lambda \) and \( p_0 \in \Lambda^0 \). Let \( t_0 \) be a positive real number such that \( t_0 p_0 + p_1 \in \partial \Lambda \). We define a meromorphic function in one complex variable \( t \) as
\[ Z(p_0, p_1, t) = X_\Lambda(t p_0 + p_1). \]

Let \( k \) be the codimension of the minimal face of \( \Lambda \) containing \( t_0 p_0 + p_1 \). Then the rational function \( Z(p_0, p_1, t) \) is analytic for \( \text{Re}(t) > t_0 \) and it has a pole of order \( k \) at \( t = t_0 \).

**Proof.** As in the proof of the previous statement, we can subdivide the dual cone \( \Lambda^* \) into simplicial subcones \( \Lambda_j^* \) such that \( t_0 p_0 + p_1 \in \partial \Lambda_1 \) and \( t_0 p_0 + p_1 \notin \partial \Lambda_j \ (j > 1) \). It suffices now to apply Proposition 3.0.11(i),(ii) to \( \Lambda_1 \).  \( \Box \)
Corollary 3.0.17 Assume that \( \Lambda \) is only locally polyhedral at the point \( t_0 p_0 + p_1 \) and \( k \) is the codimension of a minimal polyhedral face of \( \Lambda \) containing \( t_0 p_0 + p_1 \). Then

(i) \( Z(p_0, p_1, t) \) is an analytical function for \( \Re(t) > t_0 \).

(ii) \( Z(p_0, p_1, t) \) has meromorphic continuation to some neighbourhood of \( t_0 \).

(iii) \( Z(p_0, p_1, t) \) has a pole of order \( k \) at \( t = t_0 \).

4 Distribution of rational points

4.1 The method of Draxl

Let \( \Sigma \) be a \( G \)-invariant regular fan, \( \Sigma(1) = \Sigma_1(1) \cup \cdots \cup \Sigma_r(1) \) be the decomposition of \( \Sigma(1) \) into \( G \)-orbits. We choose a representative \( \sigma_j \) in each \( \Sigma_j(1) \) (\( j = 1, \ldots, r \)). Let \( e_j \) be the primitive integral generator of \( \sigma_j \), \( G_j \subset G \) be the stabilizer of \( e_j \). Denote by \( k_j \) the length of \( G \)-orbit of \( e_j \), and by \( K_j \subset E \) the subfield of \( G_j \)-fixed elements. Then \( k_j = [K_j : K] \) (\( j = 1, \ldots, r \)).

Consider the \( n \)-dimensional torus

\[
T' := \prod_{j=1}^{r} R_{K_j/K}(G_m).
\]

Notice that the group \( D(\Sigma) \) can be identified with the \( G \)-module \( \hat{T}_K' \). The homomorphism of \( G \)-modules \( M \rightarrow D(\Sigma) \) induces the homomorphism \( T' \rightarrow T \) and a map

\[
\gamma : \prod_{j=1}^{r} \mathbb{G}_m(A_{K_j})/\mathbb{G}_m(K_j) \rightarrow T(A_K)/T(K)
\]

We get a map of characters

\[
\gamma^* : (T(A_K)/T(K))^* \rightarrow \prod_{j=1}^{r} (\mathbb{G}_m(A_{K_j})/\mathbb{G}_m(K_j))^*.
\]

Remark 4.1.1 The kernel of \( \gamma^* \) is dual to the obstruction group to weak approximation \( A(T) \) defined above.
Let
\[ \chi : T(A_K) \to S^1 \subset \mathbb{C}^* \]
be a topological character which is trivial on \( T(K) \). Then \( \chi \circ \gamma \) defines Hecke characters of the idele groups
\[ \chi_j : G_m(A_{K_j}) \to S^1 \subset \mathbb{C}^*. \]
If \( \chi \) is trivial on \( K_T \), then all characters \( \chi_j \) \((j = 1, \ldots, r)\) are trivial on the maximal compact subgroups in \( G_m(A_{K_j}) \). We denote by \( L_{K_j}(s, \chi_j) \) the Hecke \( L \)-function corresponding to the character \( \chi_j \). The following statement is well-known:

**Theorem 4.1.2** The function \( L_{K_j}(s, \chi_j) \) is holomorphic in the whole plane unless \( \chi_j \) is trivial. In the later case, \( L_{K_j}(s, \chi_j) \) is holomorphic for \( \text{Re}(s) > 1 \) and has a meromorphic extension to the complex plane with a pole of order 1 at \( s = 1 \).

We come to the main statement which describes the analytical properties of the Fourier transform of height functions.

**Theorem 4.1.3** Define affine complex coordinates \( \{s_1, \ldots, s_r\} \) on the vector space \( PL(\Sigma)^G_C \) by \( s_j = \varphi(e_j) \) \((j = 1, \ldots, r)\). Then the Fourier transform \( \hat{H}_\Sigma(\chi, -\varphi) \) of the complex height function \( H_\Sigma(x, -\varphi) \) is always an analytic function for \( \text{Re}(s_j) > 1 \) \((1 \leq j \leq l)\), and
\[
\hat{H}_\Sigma(\chi, -\varphi) \prod_{i=1}^r L_{K_j}^{-1}(s_j, \chi_j)
\]
has an analytic extension to the domain \( \text{Re}(s_j) > 1/2 \) \((1 \leq j \leq r)\).

**Proof.** The idea of the proof is essentially due to Draxl [7]. We have the Euler product
\[
\hat{H}_\Sigma(\chi, -\varphi) = \prod_{v \in \text{Val}(K)} \hat{H}_{\Sigma,v}(\chi_v, -\varphi)
\]
In order to prove the above properties of \( \hat{H}_\Sigma(\chi, -\varphi) \), it is sufficient to investigate the product
\[
\hat{H}_{\Sigma,S}(\chi, -\varphi) = \prod_{v \notin S} \hat{H}_{\Sigma,v}(\chi_v, -\varphi).
\]
Choose a valuation \( v \not\in S \). Then we obtain a cyclic subgroup \( G_v = \langle \Phi_v \rangle \subset G \) generated by a lattice automorphism \( \Phi_v : N \to N \) representing the local Frobenius element at place \( v \). Let \( l \) be the number of \( G_v \)-orbits in \( \Sigma(1) \). By 2.2.6,

\[
\hat{H}_{\Sigma,S}(\chi, -\varphi) = \prod_{v \not\in S} P \left( \Sigma, \Phi_v; \frac{\chi_v(n_1)}{q_v^{\varphi(n_1)}}, \ldots, \frac{\chi_v(n_l)}{q_v^{\varphi(n_l)}} \right).
\]

By 2.2.5, we have

\[
\hat{H}_{\Sigma,v}(\chi, -\varphi) = \frac{Q_\Sigma \left( \frac{\chi_v(n_1)}{q_v^{\varphi(n_1)}}, \ldots, \frac{\chi_v(n_l)}{q_v^{\varphi(n_l)}} \right)}{(1 - \frac{\chi_v(n_1)}{q_v^{\varphi(n_1)}}) \cdots (1 - \frac{\chi_v(n_l)}{q_v^{\varphi(n_l)}})}.
\]

Moreover,

\[
\prod_{v \not\in S} Q_\Sigma \left( \frac{\chi_v(n_1)}{q_v^{\varphi(n_1)}}, \ldots, \frac{\chi_v(n_l)}{q_v^{\varphi(n_l)}} \right)
\]

is an absolutely convergent Euler product for \( \text{Re}(s_j) > 1/2 \) (\( j = 1, \ldots, l \)).

It remains to show the relation between

\[
\left(1 - \frac{\chi_v(n_1)}{q_v^{\varphi(n_1)}}\right)^{-1} \cdots \left(1 - \frac{\chi_v(n_l)}{q_v^{\varphi(n_l)}}\right)^{-1}
\]

and local factors of the product of the Hecke \( L \)-functions

\[
\prod_{j=1}^{r} L_{K_j}(s_j, \chi_j).
\]

For this purpose, we compare two decompositions of \( \Sigma(1) \) into the disjoint union of \( G_v \)-orbits and \( G \)-orbits. Notice that for every \( j \in \{1, \ldots, r\} \), the \( G \)-orbit \( \Sigma_j(1) \) decomposes into a disjoint union of \( G_v \)-orbits

\[
\Sigma_j(1) = \Sigma_{j,1}(1) \cup \cdots \cup \Sigma_{j,l_j}(1).
\]

Let \( d_{ji} \) be the length of the \( G_v \)-orbit \( \Sigma_{i,j}(1) \); i.e., we put \( \{d_{ji}\} = \{d_1, \ldots, d_l\} \). One has

\[
\sum_{i=1}^{l_j} d_{ji} = k_j.
\]
and
\[ l = \sum_{i=1}^{r} l_j. \]

On the other hand, \( l_j \) is the number of different valuations \( V_{ji1}, \ldots, V_{jlj} \in \text{Val}(K_j) \) over of \( v \in \text{Val}(K) \). Let \( k_v \) be the residue field of \( v \in \text{Val}(K) \), \( k_{V_{ji}} \) the residue field of \( V_{ji} \in \text{Val}(K_j) \). Then
\[ d_{ji} = [k_{V_{ji}} : k_v]. \]

We put also \( \{n_1, \ldots, n_l\} = \{n_{ji}\} \), where \( n_{ji} \) denotes the sum of \( d_{ji} \) generators of all 1-dimensional cones of the \( G_v \)-orbit \( \Sigma_{ji}(1) \). Therefore, \( \chi_v(n_{ji}) \) is the \( V_{ji} \)-adic component of the Hecke character \( \chi_j \). Hence
\[ \prod_{i=1}^{l_j} \left( 1 - \frac{\chi_v(n_{ji})}{q_v^{\varphi(n_{ji})}} \right)^{-1} \]
equals the product of the local factors
\[ \prod_{V_{ji}} \left( 1 - \frac{\chi_{V_{ji}}}{q_{V_{ji}}^{\varphi(n_{ji})}} \right)^{-1} \]
of the Hecke \( L \)-function \( L_{K_j}(s_j, \chi_j) \).

\section*{4.2 The meromorphic extension of \( Z_{\Sigma}(\varphi) \)}

\textbf{Theorem 4.2.1} For any \( \varepsilon > 0 \) there exists a \( \delta > 0 \) and a constant \( c(\varepsilon) \) such that
\[ |L_K(s, \chi)| \leq c(\varepsilon)(\text{Im}(s))^{\varepsilon} \text{ for } u = \text{Re}(s) > 1 - \delta \]
for every Hecke \( L \)-function \( L_K(s, \chi) \) with a nontrivial nonramified character \( \chi \).

\textbf{Proof.} We use the following standard statement based on the Phragmén-Lindelöf principle:

\textbf{Lemma 4.2.2} (\cite{21}, p.181) Let \( f(s) \) be a single valued analytic function in the strip \( u_1 \leq \text{Re}(s) \leq u_2 \) satisfying the conditions:
(i) \( |f(u + it)| < A_0 \exp(e^{C|t|}) \) for some real constants \( A_0 > 0 \) and \( 0 < C < \pi/(u_2 - u_1) \);

(ii) \( |f(u_1 + it)| \leq A_1|t|^{a_1}, \ |f(u_2 + it)| \leq A_2|t|^{a_2} \) for some constants \( a_1, a_2 \).

Then for all \( u_1 \leq u \leq u_2 \), we have the estimate

\[
|f(u + it)| \leq A_3|t|^{a(u)}
\]

where

\[
a(u) = a_1 \frac{u_2 - u}{u_2 - u_1} + a_2 \frac{u - u_1}{u_2 - u_1}.
\]

Choose a sufficiently small \( \delta_1 \). Then \( |L_K(s, \chi)| \) is bounded by \( A_2(\delta_1) = \zeta_K(1 + \delta_1) \) for \( \text{Re}(s) = 1 + \delta_1 \). Consider the functional equation \( L_K(s, \chi) = C(s)L_K(1 - s, \chi) \). Since \( \chi \) (as well as \( \chi \)) is unramified, the function \( C(s) \) depends only on the field \( K \). Using standard estimates for \( \Gamma \)-factors in \( C(s) \), we obtain \( |L_K(s, \chi)| < A_1(\text{Im}(s))^{a_1} \) for \( \text{Re}(s) = -\delta_1 \) and some sufficiently large explicit constants \( A_1(\delta_1), a_1(\delta_1) \).

We apply Lemma 4.2.2 to the \( L \)-function \( L_K(s, \chi) \) where \( u_1 = -\delta_1, u_2 = 1 + \delta_1 \), and \( a_2 = 0 \). Then for \( 1 - \delta < \text{Re}(s) < 1 + \delta_1 \), one has

\[
|L_K(s, \chi)| \leq A_3(\delta, \delta_1)(\text{Im}(s))^{a_1\delta + \delta_1 \delta_1}.
\]

It is possible to choose \( \delta \) and \( \delta_1 \) in such a way that

\[
a_1(\delta_1) \frac{\delta + \delta_1}{1 + 2\delta_1} < \varepsilon.
\]

\[\square\]

**Theorem 4.2.3** Let \( s_j = \varphi(e_j) \) \((j = 1, \ldots, r)\). Then the height zeta function \( Z_\Sigma(\varphi) \) is holomorphic for \( \text{Re}(s_j) > 1 \). There exists an analytic continuation of \( Z_\Sigma(\varphi) \) to the domain \( \text{Re}(s_j) > 1 - \delta \) such that the only singularities of \( Z_\Sigma(\varphi) \) in this domain are poles of order \( \leq 1 \) along the hyperplanes \( s_j = 1 \) \((j = 1, \ldots, r)\).

**Proof.** By the Poisson formula,

\[
Z_\Sigma(\varphi) = \frac{1}{\text{vol}(T^1(A_K)/T(K))} \sum_{\chi \in (T(A_K)/T(K))^*} \hat{H}_\Sigma(\chi, -\varphi).
\]
Since \( H_\Sigma(x, -\varphi) \) is \( K_T \)-invariant, we can assume that in the above formula \( \chi \) runs over the elements of the group \( \mathcal{P} \) consisting of characters of \( T(A_K) \) which are trivial on \( K_T \cdot T(K) \).

Let \( J \) be a subset of \( I = \{1, \ldots, r\} \). Denote by \( \mathcal{P}_J \) the subset of \( \mathcal{P} \) consisting of all characters \( \chi \in \mathcal{P} \) such that the corresponding Hecke character \( \chi_j \) is trivial if and only if \( j \in J \). Then

\[
Z_\Sigma(\varphi) = \sum_{J \subset I} Z_{\Sigma,J}(\varphi)
\]

where

\[
Z_{\Sigma,J}(\varphi) = \frac{1}{\text{vol}(T^1(A_K)/T(K))} \sum_{\chi \in \mathcal{P}_J} \hat{H}_{\Sigma}(\chi, -\varphi).
\]

Consider the logarithmic space

\[
N_{R,\infty} = \prod_{v \in \text{Val}_\infty(K)} T(K_v)/T(O_v)
\]

containing the full sublattice \( T(O_K)/W(T) \) of \( O_K \)-integral points of \( T(K) \) modulo torsion. Let \( \chi_\infty \) be the restriction of a character \( \chi \in \mathcal{P} \) to \( N_{R,\infty} \). Then \( \chi_\infty(x) = e^{2\pi i <x, y_\chi>} \) where \( y_\chi \) is an element of the dual logarithmic space

\[
M_{R,\infty} = \prod_{v \in \text{Val}_\infty(K)} \text{Hom}(T(K_v)/T(O_v), R).
\]

Moreover, \( y_\chi \) belongs to the dual lattice \( (T(O_K)/W(T))^* \subset M_{R,\infty} \).

Let \( u_1, \ldots, u_r \) be a basis of the lattice \( T(O_K)/W(T) \subset N_{R,\infty} \), \( f_1, \ldots, f_r \) the dual basis of the dual lattice \( (T(O_K)/W(T))^* \subset M_{R,\infty} \). We extend

\[
e^{2\pi i <x,f_1>}, \ldots, e^{2\pi i <x,f_r>}
\]

to some adelic characters \( \eta_1, \ldots, \eta_{r^r} \in \mathcal{P} \). Using [1,4,6] and the basis \( \eta_1, \ldots, \eta_{r^r} \), we can extend

\[
\chi_\infty = \prod_{k=1}^{r^r} e^{2\pi i a_k <x,f_k>}, \ a_k \in \mathbb{Z}
\]

to a character

\[
\tilde{\chi} = \prod_{k=1}^{r^r} \eta_{a_k} \in \mathcal{P}
\]
such that $\tilde{\chi}_\infty = \chi_\infty$ and $\chi \cdot \tilde{\chi}^{-1}$ is a character of the finite group $\text{cl}(T)$.

We fix a character of $\chi_c$ of $\text{cl}(T)$. Denote by $\mathcal{P}_{J,\chi_c}$ the set of all characters $\chi \in \mathcal{P}_J$ such that $\chi \cdot \tilde{\chi}^{-1} = \chi_c$. Then a character $\chi \in \mathcal{P}_{J,\chi_c}$ is uniquely defined by its archimedian component $\chi_\infty$.

By 4.1.3,

$$Q_\Sigma(\chi, -\varphi) = \prod_{v \notin S_\infty} \hat{H}_{\Sigma,v}(\chi_v, \varphi) \prod_{i=1}^r L_{K_j}^{-1}(s_j, \chi_j)$$

is absolutely convergent Euler product for $\Re(s_j) > 1 - \delta > 1/2$.

By 4.2.1,

$$\prod_{j \notin J} L_{K_j}(s_j, \chi_j) < C(\varepsilon) \prod_{j \notin J} \Im(s_j)^{\varepsilon}$$

$\Re(s_j) > 1 - \delta$.

We apply 2.3.3 to the archimedian Fourier transform

$$\hat{H}_{\Sigma,\infty}(\chi, \varphi) = \prod_{v \in S_\infty} \hat{H}_{\Sigma,v}(\chi_v, \varphi).$$

Then, by 2.3.3,

$$\sum_{\chi \in \mathcal{P}_{J,\chi_c}} \hat{H}(\chi, -\varphi) \prod_{j \in J} \zeta_{K_j}(s_j)^{-1}$$

is absolutely convergent for $\Re(s_j) > 1 - \delta$.

Therefore, we have obtained that

$$Z_{\Sigma,J}(\varphi) \prod_{j \in J} \zeta_{K_j}(s_j)^{-1}$$

is a holomorphic function for $\Re(s_j) > 1 - \delta$ and for any $J \subset I$.

It remains to notice, that in the considered domain $\prod_{j \in J} \zeta_{K_j}(s_j)$ has only poles of order 1 along hyperplanes $s_j = 1$.

$\square$

4.3 Rational points of bounded height

Recall the standard tauberian statement:

**Theorem 4.3.1** [4] Let $X$ be a countable set, $F : X \to \mathbb{R}_{>0}$ a real valued function. Assume that

$$Z_F(s) = \sum_{x \in X} F(x)^{-s}$$

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is absolutely convergent for \( \text{Re}(s) > a > 0 \) and has a representation
\[
Z_F(s) = (s - a)^{-r} g(s) + h(s)
\]
with \( g(s) \) and \( h(s) \) holomorphic for \( \text{Re}(s) \geq a, \ g(a) \neq 0, \ r \in \mathbb{N} \). Then for any \( B > 0 \) there exists only a finite number \( N(F, B) \) of elements \( x \in X \) such that \( F(x) \leq B \). Moreover,
\[
N(F, B) = \frac{g(a)}{a(b - 1)!} B^a (\log B)^{r-1} (1 + o(1)).
\]

Let \( \mathcal{L} = \mathcal{L}(\varphi_0) \) be a metrized invertible sheaf over a smooth compactification \( \mathbf{P}_\Sigma \) of an anisotropic torus \( T_K \) defined by a \( G \)-invariant fan \( \Sigma \). We denote by \( Z_{\Sigma, \mathcal{L}}(s) = Z_{\Sigma}(s\varphi_0) \) the restriction of \( Z_{\Sigma}(\varphi) \) to the line \( s[\mathcal{L}] \subset \text{Pic}(\mathbf{P}_\Sigma)_\mathbb{R} \).

Let \( a(\mathcal{L}) \) be the abscissa of convergence of \( Z_{\Sigma, \mathcal{L}}(s) \) and \( b(\mathcal{L}) \) the order of the pole of \( Z_{\Sigma, \mathcal{L}}(s) \) at \( s = a(\mathcal{L}) \). By 4.2.3,
\[
a(\mathcal{L}) \leq \min_{j=1}^r \frac{1}{\varphi(e_j)}.
\]

By 4.3.1, we obtain:

**Theorem 4.3.2** Assume that \( \varphi(e_j) > 0 \) for all \( j = 1, \ldots, r \); i.e., the class \([\mathcal{L}]\) is contained in the interior of the cone of effective divisors \( \Lambda_{\text{eff}}(\Sigma) \). Then there exists only finite number \( N(\mathbf{P}_\Sigma, \mathcal{L}, B) \) of \( K \)-rational points \( x \in T(K) \) having the \( \mathcal{L} \)-height \( H_{L}(x) \leq B \). Moreover,
\[
N(\mathbf{P}_\Sigma, \mathcal{L}, B) = B^{a(\mathcal{L})} \cdot (\log B)^{b(\mathcal{L})-1}(1 + o(1)), \ B \to \infty.
\]

The following statement implies Batyrev-Manin conjectures about the distribution of rational points of bounded \( \mathcal{L} \)-height for smooth compactifications of anisotropic tori:

**Theorem 4.3.3** The number \( a(\mathcal{L}) \) equals
\[
a(\mathcal{L}) = \inf \{ \lambda \mid \lambda[\mathcal{L}] + [\mathcal{K}] \in \Lambda_{\text{eff}}(\Sigma) \};
\]
i.e.,
\[
a(\mathcal{L}) = \min_{j=1}^r \frac{1}{\varphi(e_j)}.
\]
Moreover, \( b(\mathcal{L}) \) equals the codimension of the minimal face of \( \Lambda_{\text{eff}}(\Sigma) \) containing \( a(\mathcal{L})[\mathcal{L}] + [\mathcal{K}] \).
Proof. By [16], we can choose the finite set $S$ such that the natural homomorphism

$$\pi_S : T(K) \to \prod_{v \not\in S} T(K_v)/T(O_v) = \prod_{v \in S} N_v$$

is surjective. Denote by $T(O_S)$ the kernel of $\pi_S$ consisting of all $S$-units in $T(K)$. The group $T(O_S)/W(T)$ has the natural embedding in the finite-dimensional space

$$N_{S,R} = \prod_{v \in S} T(K_v)/T(O_v) \otimes R$$
as a full sublattice.

Let $\Delta$ be the fundamental domain of $T(O_S)/W(T)$ in $N_{S,R}$. For any $x \in T(K)$, denote by $\overline{x}_S$ the image of $x$ in $N_{S,R}$. Define $\phi(x)$ to be the element of $T(O_S)$ such that $\overline{x}_S - \phi(x) \in \Delta$. Thus, we have obtained the mapping

$$\phi_S : T(K) \to T(O_S).$$

Define the new height function $\tilde{H}_\Sigma(x, \varphi)$ on $T(K)$ by

$$\tilde{H}_\Sigma(x, \varphi) = H_\Sigma(\varphi, \phi_S(x)) \prod_{v \not\in S} H_{\Sigma,v}(x_v, \varphi).$$

Notice the following easy statement:

**Lemma 4.3.4** Choose a compact subset $K \subset \mathbb{C}^r$ such that $\text{Re}(s_j) > \delta$ ($j = 1, \ldots, r$) for $\varphi \in K$. Then there exist positive constants $C_1, C_2$ such that

$$0 < C_1 < \frac{\tilde{H}_\Sigma(x, \varphi)}{H_\Sigma(x, \varphi)} < C_2, \text{ for } \varphi \in K, \ x \in T(K).$$

Define $\tilde{Z}_\Sigma(\varphi)$ by

$$\tilde{Z}_\Sigma(\varphi) = \sum_{x \in T(K)} \tilde{H}_\Sigma(x, -\varphi).$$

Then $\tilde{Z}_\Sigma(\varphi)$ splits into the product

$$\tilde{Z}_\Sigma(\varphi) = \prod_{v \not\in S} \left( \sum_{z \in N_v} H_{\Sigma,v}(z, -\varphi) \right) \cdot \left( \sum_{u \in T(O)} H_\Sigma(u, -\varphi) \right).$$
By \([7]\), the Euler product
\[
\prod_{j=1}^r \zeta_{K_j}(s_j) \prod_{v \notin S} \left( \sum_{z \in \mathbb{N}_v} H_{\Sigma,v}(z, -\varphi) \right)
\]
is a holomorphic function without zeros for \(\text{Re}(s_j) > 1/2\).

On the other hand,
\[
\sum_{u \in T(\mathcal{O})} H_{\Sigma}(u, -\varphi)
\]
is an absolutely convergent series nonvanishing for \(\text{Re}(s_j) > 0\). Therefore, \(\tilde{Z}_{\Sigma}(\varphi)\) has a meromorphic extension to the domain \(\text{Re}(s_j) > 1/2\) where it has poles of order 1 along the hyperplanes \(s_j = 1\).

By \([4.3.4]\) and \([4.3.1]\), \(\tilde{Z}_{\Sigma}(\varphi)\) and \(Z_{\Sigma}(\varphi)\) must have the same poles in the domain \(\text{Re}(s_j) > 1 - \delta\). Therefore, \(Z_{\Sigma}(\varphi)\) has poles of order 1 along the hyperplanes \(s_j = 1\). By taking the restriction of \(Z_{\Sigma}(\varphi)\) to the line \(\varphi = s\varphi_0\), we obtain the statement. \(\square\)

### 4.4 The residue at \(s_j = 1\)

Recall the definition of the Tamagawa number of Fano varieties \([18]\). This definition immediately extends to arbitrary algebraic varieties \(X\) with a metrized canonical sheaf \(\mathcal{K}\).

Let \(x_1, \ldots, x_d\) be local analytic coordinates on \(X\). They define a homeomorphism \(f : U \to K_v^d\) in \(v\)-adic topology between an open subset \(U \subset X\) and \(f(U) \subset K_v^d\). Let \(dx_1 \cdots dx_d\) be the Haar measure on \(K_v^d\) normalized by the condition
\[
\int_{\mathcal{O}_v^d} dx_1 \cdots dx_d = \frac{1}{(\sqrt{\delta_v})^d}
\]
where \(\delta_v\) is the absolute different of \(K_v\). Denote by \(dx_1 \wedge \cdots \wedge dx_d\) the standard differential form on \(K_v^d\). Then \(g = f^*(dx_1 \wedge \cdots \wedge dx_d)\) is a local analytic section of the metrized canonical sheaf \(\mathcal{K}\). We define the local measure on \(U\) by
\[
\omega_{\mathcal{K},v} = f^*(\|g(f^{-1}(x))\|_v dx_1 \cdots dx_d).
\]
The adelic Tamagawa measure \(\omega_{\mathcal{K},S}\) is defined by
\[
\omega_{\mathcal{K},S} = \prod_{v \in \text{Val}(K)} \lambda_v^{-1} \omega_{\mathcal{K},v}
\]
where \(\lambda_v = L_v(1, \text{Pic}(X^\mathcal{K}); \overline{K}/K)\) if \(v \notin S\), \(\lambda_v = 1\) if \(v \in S\).
**Definition 4.4.1** Let \( \overline{X(K)} \) be the closure of \( X(K) \subset X(\mathbb{A}_K) \) in the direct product topology. Then the **Tamagawa number** of \( X \) is defined by

\[
\tau_K(X) = \lim_{s \to 1} (s - 1) L_S(s, \text{Pic}(\overline{X(K)}; \overline{K}/K)) \cdot \int_{\overline{X(K)}} \omega_{K,S}.
\]

**Proposition 4.4.2** Let \( K = \mathcal{L}(-\varphi_{\Sigma}) \) be the metrized canonical sheaf on a toric variety \( P_{\Sigma} \). Then the restriction of the \( v \)-adic measure \( \omega_{K,v} \) to \( T(K_v) \subset P_{\Sigma}(K_v) \) coincides with the measure

\[
H_{\Sigma,v}(x, -\varphi_{\Sigma}) \omega_{\Omega, v}.
\]

**Proof.** The rational differential \( d \)-form \( \Omega \) is a rational section of \( K \). By definition of the \( v \)-adic metric on \( \mathcal{L}(-\varphi_{\Sigma}) \), \( H_{\Sigma,v}(x, -\varphi_{\Sigma}) \) equals the norm \( \| \Omega \|_v \) of the \( T \)-invariant section \( \Omega \). This implies the statement. \( \square \)

**Proposition 4.4.3** One has

\[
\int_{T(K)} \omega_{K,S} = \int_{P_{\Sigma}(K)} \omega_{K,S}.
\]

**Proof.** Since \( P_{\Sigma}(K) \setminus T(K) \) is a subset of \( P_{T}(\mathbb{A}_K) \setminus T(\mathbb{A}_K) \), it is sufficient to prove that

\[
\int_{T(K_v)} \omega_{K,v} = \int_{P_{\Sigma}(K_v)} \omega_{K,v}.
\]

Since the measure \( \omega_{K,v} \) is \( T(\mathcal{O}_v) \)-invariant and the stabilizer in \( T(\mathcal{O}_v) \) of any point \( x \in P_{\Sigma}(K_v) \setminus T(K_v) \) is uncountable, the \( \omega_{K,v} \)-volume of \( P_{\Sigma}(K_v) \setminus T(K_v) \) is zero. \( \square \)

**Theorem 4.4.4** Let \( \Theta(\Sigma, K) \) be the the residue of the zeta-function \( Z_{\Sigma}(\varphi) \) at \( s_1 = \cdots = s_r = 1 \). Then

\[
\Theta(\Sigma, K) = \alpha(P_{\Sigma}) \beta(P_{\Sigma}) \tau_K(P_{\Sigma}).
\]

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Proof. By the Poisson formula,

$$Z\Sigma(\varphi) = \frac{1}{\text{vol}(T^1(A_K)/T(K))} \sum_{\chi \in (T(A_K)/T(K))^*} \hat{H}_\Sigma(\chi, -\varphi).$$

By [4.2.3], the residue of $Z\Sigma(\varphi)$ at $s_1 = \cdots = s_r = 1$ appears from $Z\Sigma, I(\varphi)$ containing only the terms $\hat{H}_\Sigma(\chi, -\varphi)$ such that $\chi_1, \ldots, \chi_r$ are trivial characters of $G_m(A_{K_j})/G_m(K_j)$ ($j = 1, \ldots, l$); i.e., $\chi$ is a character of the finite group $A(T)$. We apply again the Poisson formula to the finite sum

$$Z_{\Sigma, I}(\varphi) = \frac{1}{\text{vol}(T^1(A_K)/T(K))} \sum_{\chi \in (A(T))^*} \hat{H}_\Sigma(\chi, -\varphi).$$

Using [1.4.14, 1.4.15, 4.1.1], we have

$$Z_{\Sigma, I}(\varphi) = \frac{\beta(P_\Sigma)}{i(T_K) \cdot \text{vol}(T^1(A_K)/T(K))} \int_{T(K)} H_\Sigma(x, -\varphi) \omega_{1, S}^1.$$  

Notice that $\omega_{1, S}^1 = \omega_{\Omega, S}$ for anisotropic tori.

Now we assume that $\varphi(e_1) = \cdots = \varphi(e_r) = s$. Our purpose is to compute the constant

$$\Theta(\Sigma, K) = \lim_{s \to 1} (s - 1)^r Z_{\Sigma, I}(s \varphi_\Sigma).$$

By [1.4.10, 1.4.12],

$$\Theta(\Sigma, K) = \frac{\beta(P_\Sigma)}{h(T_K)} L_s^{-1}(1, T; E/K) \lim_{s \to 1} (s - 1)^r \int_{T(K)} H_\Sigma(x, -s \varphi_\Sigma) \omega_{\Omega, S}.$$  

Notice that $\overline{T(K)}$ contains $T(K_v)$ for $v \notin S$. Denote by $\overline{T(K)}_S$ the image of $\overline{T(K)}$ in $\prod_{v \notin S} T(K_v)$. By [2.2.7], we have

$$\int_{\overline{T(K)}} H_{\Sigma, v}(x, -s \varphi_\Sigma) \omega_{\Omega, S} =$$

$$= \prod_{v \notin S} \int_{T(K_v)} H_{\Sigma, v}(x, -s \varphi_\Sigma) d\mu_v \cdot \lim_{s \to 1} \prod_{v \notin S} H_\Sigma(x, -s \varphi_\Sigma) \omega_{\Omega, v} =$$

$$= L_s(s, T; E/K) \cdot \prod_{v \notin S} \left( \sum_{k=0}^d \frac{\text{Tr} \Phi_v(i)}{q_v^{k_s}} \right) \cdot \lim_{s \to 1} \prod_{v \notin S} H_{\Sigma, v}(x, -s \varphi_\Sigma) \omega_{\Omega, v}. 44
\[ L_S^{-1}(1, T; E/K)\omega_{\Omega,S} = \prod_{v \in \text{Val}(K)} \omega_{\Omega,v}. \]

By 2.2.5 and 2.2.6,

\[ L_S^{-1}(s, \text{Pic}(P_{\Sigma,E}); E/K) \prod_{v \notin S} \left( \sum_{k=0}^{d} \frac{\text{Tr} \Phi_v(i)}{q_v^k} \right) \]

has no singularity at \( s = 1 \). Moreover, by 4.4.2,

\[ \prod_{v \notin S} L_S^{-1}(1, \text{Pic}(P_{\Sigma,E}); E/K) \left( \sum_{k=0}^{d} \frac{\text{Tr} \Phi_v(i)}{q_v^k} \right) = \prod_{v \notin S} \int_{T(K_v)} \lambda_v^{-1} \omega_{K,v}. \]

Therefore

\[ \Theta(\Sigma, K) = \beta(P_{\Sigma}) \lim_{s \to 1} (s - 1)^r L_S(s, \text{Pic}(P_{\Sigma,E}); E/K) \int_{T(K)} \omega_{\Sigma,S}. \]

It remains to apply [4.4].

Using [4.3.1], we obtain:

**Corollary 4.4.5** Let \( T \) be an anisotropic torus and \( P_{\Sigma} \) its smooth compactification (notice that we do not need to assume that \( P_{\Sigma} \) is a Fano variety). Let \( r \) be the rank of \( \text{Pic}(P_{\Sigma,K}) \). Then the number \( N(P_{\Sigma}, \mathcal{K}^{-1}, B) \) of \( K \)-rational points \( x \in T(K) \) having the anticanonical height \( H_{\mathcal{K}^{-1}}(x) \leq B \) has the asymptotic

\[ N(P_{\Sigma}, \mathcal{K}^{-1}, B) = \frac{\Theta(\Sigma, K)}{(r - 1)!} \cdot B(\log B)^{r-1}(1 + o(1)), \quad B \to \infty. \]

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