Relativistic kinetic dispersion theory of linear parallel waves in magnetized plasmas with isotropic thermal distributions

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Abstract. New relativistic dispersion laws are obtained for electromagnetic waves parallel propagating in an isotropic thermal plasma embedded in an ambient magnetic field. The relativistic Maxwellian distribution is considered to describe both the susceptible growing and the damped waves, by analytical continuation on the whole complex frequency plane. Classical results correspond to the formal limit of the infinite speed of light, and are quite different from the relativistically correct dispersion relation obtained here. The relativistic solutions then exhibit different behaviours as illustrated for low-frequency waves in finite $\beta$ plasmas. For an isothermal plasma typically encountered in galactic cosmic rays, a visible discrepancy is observed in the high limit of non-relativistic ion temperature. More prominent relativistic differences are obtained for pair plasma conditions corresponding either to a pulsar magnetosphere or to an active galactic nuclei engine which are currently created in the laboratory. The relativistic effects are found to be more efficient in a non-isothermal plasma leading to an important correction for the low-frequency dispersion and even at low non-relativistic temperatures.

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1. Introduction

The diversity of plasma-wave effects has established a growing interest in the most important challenges of plasma research and technology, such as the analysis of space plasma emissions, or the management of fusion plasma.

In collisionless states the most powerful formalism describing plasma-wave phenomena is developed by using kinetic theory. It becomes vital on length or time scales smaller than those of binary collisions of plasma particles. The relativistic kinetic theory essentially removes some unphysical results of the non-relativistic theory (e.g. Landau or cyclotron damping of superluminal waves in equilibrium thermal plasmas).

We continue here the series of investigations of electromagnetic waves in hot magnetized plasma, with a special interest in the relativistic effects associated with relativistic particle mass variation and which occur at large but still non-relativistic thermal velocities. Only the relativistically correct dispersion relations reveal these relativistic effects and some of them are analysed in this paper by comparison with previous results from non-relativistic theory.

Starting from the relativistically correct set of Vlasov–Maxwell equations, we find in section 2 the general dispersion relation of transverse waves parallel propagating in magnetized plasmas of arbitrary composition and arbitrary distribution function. In section 3 the dispersion relation is reduced to the case of magnetized and thermally isotropic plasma. The relativistic Maxwellian distribution is introduced in section 4 leading to a new expression of the dispersion relation which describes the transverse waves for any plasma temperature. The new relativistically correct dispersion relation corresponding to the limit of non-relativistic plasma temperatures is obtained in section 5 in terms of the Fried and Conte plasma dispersion function and for
the whole complex frequency plane. Re-emphasizing the generality of the relativistic dispersion relation even for non-relativistic temperatures, we show in section 6, that only in the strictly unphysical formal limit of an infinitely large speed of light, $c \to \infty$, the new relativistic dispersion relation reduces to the standard non-relativistic form. Their dependence on frequency and wavenumber is markedly different, and we illustrate in section 7, by comparison, their low-frequency solutions for some concludent examples of finite $\beta \neq 0$ plasmas. In the low-frequency regime the ion contribution dominates and we obtain a new relativistically correct dispersion relation dependent on the ion temperature ($T_i$) and on the isothermal factor $\theta = T_e / T_i \neq 1$ (ratio of electron and ion temperatures). The main results of our investigation are summarized in the last section.

Comparing with standard solutions [10, 11], the relativistic dispersion curves are significantly lower towards the highest limit of non-relativistic temperature, which is an expected behaviour due to the relativistic decrease of cyclotron frequency with increasing plasma particle thermal velocity. These differences between classical and relativistic dispersion curves become more prominent in pair plasmas where the mass contrast of plasma components is reduced to unity ($m^+/m^- = 1$), and in non-isothermal electron–ion plasmas with a high isothermal factor, $\theta \gg 1$, where the relativistic effects occur at lower non-relativistic (ion) temperatures.

2. Derivation of the dispersion relations

2.1. Basic equations

From the dielectric tensor for general gyrophase-averaged one-particle phase space distribution function $F_a(p_{||}, p_{\perp})$ of species $a$ in an ordered magnetic field we obtain for the dispersion relation of parallel propagating left-handed (LH) circularly polarized waves (e.g. [12], p 229ff)

$$\Lambda_{\text{LH}}^+(k, \omega) = 1 - N^2 + \frac{\pi}{\omega} \sum_a \omega_{p,a}^2 \int_{-\infty}^{\infty} dp_{||} \int_0^\infty dp_{\perp} \frac{p_{\perp}}{\gamma(\omega - kv_{||}) - \Omega_{a,0}}$$

$$\times \left[ p_{\perp} \frac{\partial F_a}{\partial p_{||}} + k v_{\perp} \frac{\partial F_a}{\omega} \left( p_{\perp} \frac{\partial F_a}{\partial p_{||}} - p_{||} \frac{\partial F_a}{\partial p_{\perp}} \right) \right] = 0,$$

where $\omega_{p,a}$ denotes the plasma frequency of particles of type $a$, $k = k_{||}$ and the index of refraction $N^2 = c^2 k^2 / \omega^2$. For positive frequencies $\omega > 0$, (1) describes LH circularly polarized waves whereas negative frequencies $\omega < 0$ refer to right-handed (RH) circularly polarized waves.

Because we consider an infinitely extended physical system we choose the wave number $k$ to be real but allow, in general, that the Laplace transform in time gives rise to complex frequencies $\omega = \omega_R + i\Gamma$, implying also a complex index of refraction. In the above derivation it has been assumed that the imaginary part of the complex frequency is sufficiently positive, $\Gamma > 0$. The index ‘+’ in (1) indicates this condition.

We immediately notice the symmetry conditions

$$\Lambda_{\text{LH}}^+(−k, \omega) = \Lambda_{\text{LH}}^+(k, \omega),$$

so that the wave mode properties are the same irrespective of the wave propagation direction. Without loss of generality we therefore may choose the wavenumber $k$ to be positive.

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2.2. Limit of infinite speed of light $c \to \infty$

For completeness and later comparison with the fully relativistic form of the dispersion relation we note the limit of an infinitely large speed of light $c \to \infty$ so that simply $p_\parallel \simeq m_a v_\parallel$ and $p_\perp \simeq m_a v_\perp$, where we have set

$$\lim_{c \to \infty} \gamma = \lim_{c \to \infty} \sqrt{1 + \frac{p_\parallel^2 + p_\perp^2}{m_a c^2}} = 1.$$  \hfill (3)

With $\int d^3 p\, F_a(\vec{p}) = \int d^3 v\, f_a(\vec{v})$ the dispersion relation (1) becomes

$$\Lambda_{\text{LH},\infty}^+(k, \omega) = 1 - N^2 + \sum_a \frac{\pi \omega_{p,a}^2 (m_a c)^3}{k^2 c^2 z^2} \int_1^{+\sqrt{E^2 - 1}} dE \int_{-\sqrt{E^2 - 1}}^{+\sqrt{E^2 - 1}} dy \frac{E^2 - 1 - y^2}{y - E z + x_a}$$

$$\times \left[ \frac{\partial F_a(E, y)}{\partial y} + z \frac{\partial F_a(E, y)}{\partial E} \right] = 0,$$  \hfill (5)

where we have introduced the inverse index of refraction

$$z = \frac{\omega}{k c}$$  \hfill (6)

and the dimensionless non-relativistic gyrofrequency

$$x_a = \frac{\Omega_{a,0}}{k c}.$$  \hfill (7)

2.3. Relativistic case

In the general relativistic case it is convenient to transform to the new variables of integration [13], $y \equiv p_\parallel / (m_a c)$ and $E \equiv \sqrt{1 + (p_\parallel^2 + p_\perp^2)/(m_a c^2)}$, which yields for the dispersion relation (1)

$$\Lambda_{\text{LH}}^+(k, \omega) = 1 - N^2 + \frac{\pi}{k^2 c^2 z^2} \sum_a \omega_{p,a}^2 (m_a c)^3 \int_1^{+\sqrt{E^2 - 1}} dE \int_{-\sqrt{E^2 - 1}}^{+\sqrt{E^2 - 1}} dy \frac{E^2 - 1 - y^2}{y - E z + x_a}$$

$$\times \left[ \frac{\partial F_a(E, y)}{\partial y} + z \frac{\partial F_a(E, y)}{\partial E} \right] = 0,$$  \hfill (5)

3. Isotropic equilibrium plasma

Throughout this work we consider isotropic equilibrium plasma distribution functions $F_a(E, y) = F_a(E)$ or $f_a(v_\perp, v_\parallel) = f_a(v)$, where $v = \sqrt{v_\perp^2 + v_\parallel^2}$. The Maxwell operator $\Lambda^+(k, \omega)$ from (5) then simplify to (only for LH mode)

$$\Lambda_{\text{LH}}^+(k, \omega) = 1 - \frac{1}{z^2} - \frac{\pi}{k^2 c^2 z^2} \sum_a \omega_{p,a}^2 (m_a c)^3 \int_1^{+\sqrt{E^2 - 1}} dE \frac{\partial F_a(E)}{\partial E} L(E, z, x_a),$$  \hfill (8)

with the integral

$$L(e, z, x_a) \equiv \int_{-\sqrt{E^2 - 1}}^{+\sqrt{E^2 - 1}} dy \frac{E^2 - 1 - y^2}{y - E z + x_a} = 0.$$  \hfill (9)
3.1. Reduction of the integral $L(E, z, x_a)$

With the substitution $y + x_a = E s$ the integral (9) becomes

$$L = \int_{\sqrt{1-E^2+x_a/E}^-}^{\sqrt{1-E^2+x_a/E}^+} ds \frac{E^2 - 1 - (E^2 s^2 - 2x_a E s + x_a^2)}{s-z} = (E^2 - 1 - x_a^2) I_1 + 2x_a E I_2 - E^2 I_3$$

with the integrals

$$I_1(E) = \int_{\sqrt{1-E^2+x_a/E}^-}^{\sqrt{1-E^2+x_a/E}^+} ds \frac{1}{s-z} = \int_{0}^{\sqrt{1-E^2-x_a/E}^-} \frac{ds}{s-z} - \int_{0}^{\sqrt{1-E^2-x_a/E}^+} \frac{ds}{s+z}$$

$$I_2(E) = \int_{\sqrt{1-E^2+x_a/E}^-}^{\sqrt{1-E^2+x_a/E}^+} ds \frac{s}{s-z} = \int_{\sqrt{1-E^2-x_a/E}^-}^{\sqrt{1-E^2-x_a/E}^+} ds \frac{s-z+z}{s-z} = z I_1(E) + 2 \sqrt{1-E^2}$$

and

$$I_3(E) = \int_{\sqrt{1-E^2+x_a/E}^-}^{\sqrt{1-E^2+x_a/E}^+} ds \frac{s^2}{s-z} = \int_{\sqrt{1-E^2-x_a/E}^-}^{\sqrt{1-E^2-x_a/E}^+} ds \frac{s^2 - z^2 + z^2}{s-z}$$

$$= z^2 I_1(E) + \int_{\sqrt{1-E^2-x_a/E}^-}^{\sqrt{1-E^2-x_a/E}^+} ds (s+z) = z^2 I_1(E) + 2 \left[z + \frac{x_a}{e}\right] \sqrt{1-E^2}.$$}

Collecting terms, equation (10) becomes

$$L = [E^2(1-z^2) + 2x_aE - (1+x_a^2)] I_1(E) + 2(x_a - Ez) \sqrt{E^2 - 1}.$$ (14)

3.2. Dispersion relation

Inserting (14) into the dispersion relation (8) readily yields

$$\Lambda_{1H}^+(k, \omega) = 1 - \frac{1}{z^2} - \frac{2 \pi}{k^2 c^2 z} \sum_a \omega_{p,a}^2 (m_a c)^3 \int_{1}^{\infty} \frac{dE}{\sqrt{E^2 - 1}(x_a - Ez)} \frac{\partial F_a(E)}{\partial E}$$

$$- \frac{\pi}{k^2 c^2 z} \sum_a \omega_{p,a}^2 (m_a c)^3 T = 0$$ (15)

with

$$T \equiv \int_{1}^{\infty} dE \left[ (1-z^2) E^2 \frac{dF_a(E)}{dE} + 2x_a z \frac{dF_a(E)}{dE} - (1+x_a^2) \frac{dF_a(E)}{dE} \right] I_1(E)$$

$$- \int_{1}^{\infty} dE \left[ (1-z^2) \frac{du_a}{dE} + 2x_a z \frac{du_a}{dE} - (1+x_a^2) \frac{du_a}{dE} \right] I_1(E),$$ (16)

where we used the function $u_a(E)$ defined such that [13]

$$\frac{du_a}{dE} = -E^n \frac{df_a}{dE}.$$ (17)
Partial integration of (16) yields
\[
\begin{align*}
T &= \int_{\infty}^{1} dE[(1 - z^2)u_2(E) + 2x_au_1(E) - (1 + x_a^2)u_0(E)] \frac{\partial I_1(E)}{\partial E} \\
+ I_1(1) &\left[(1 - z^2)u_2(1) + 2x_au_1(1) - (1 + x_a^2)u_0(1)\right] \\
= \int_{1}^{\infty} dE[(1 - z^2)u_2(E) + 2x_au_1(E) - (1 + x_a^2)u_0(E)] \frac{\partial I_1(E)}{\partial E} 
\end{align*}
\]
(18)
because according to (11)
\[
I_1(1) = \int_{x_a}^{s} \frac{ds}{s - z} = 0. 
\] (19)
Moreover, from (11) we obtain
\[
\frac{\partial I_1(E)}{\partial E} = \frac{1}{E^2 - 1} \left( \frac{E^2 - 1}{E^2 - 1 + x_aE^{-1} - z} - \frac{E^2 - 1}{E^2 - 1 - x_aE^{-1} - z} \right) \\
= \frac{2(z - x_aE)}{\sqrt{E^2 - 1} \left[ E^2 - 1 - (x_a - zE)^2 \right]} 
\] (20)
so that
\[
T = 2 \int_{1}^{\infty} dE[(1 - z^2)u_2(E) + 2x_au_1(E) - (1 + x_a^2)u_0(E)] \\
\times \frac{z - x_aE}{\sqrt{E^2 - 1} \left[ E^2 - 1 - (x_a - zE)^2 \right]} 
\] (21)
The dispersion relation (15) then reads
\[
0 = \Lambda_{\text{th}}^{+}(k, \omega) = 1 - \frac{1}{z^2} - \frac{2\pi k^2c^2}{\omega_p^2} \sum a \omega_{p,a}^2 (m_ac)^3 \left( \int_{1}^{\infty} dE \sqrt{E^2 - 1(x_a - Ez)} \frac{\partial F_a(E)}{\partial E} \right. \\
- \int_{1}^{\infty} dE[(1 - z^2)u_2(E) + 2x_au_1(E) - (1 + x_a^2)u_0(E)] \\
\times \frac{z - x_aE}{\sqrt{E^2 - 1} \left[ E^2 - 1 - (x_a - zE)^2 \right]} \right). 
\] (22)

4. Thermal equilibrium distribution

We now consider in detail the Maxwellian distribution function
\[
F_a(E) = C_a e^{-\mu_aE}, \quad C_a = \frac{\mu_a}{4\pi (m_ac)^3 K_2(\mu_a)}, 
\] (23)
where the dimensionless parameter \( \mu_a = m_ac^2/(k_BT_0) \) characterizes the plasma temperature. The normalization factor \( C_a \) is determined by the integral \( \int d^3p F_a(E) = 1 \).

According to (17) we then obtain for the three functions
\[
\begin{align*}
u_0(E) &= -C_a e^{-\mu_aE}, \quad v_1(E) &= -C_a e^{-\mu_aE} \left[ E + \frac{1}{\mu_a} \right], \\
u_2(E) &= -C_a e^{-\mu_aE} \left[ E^2 + \frac{2E}{\mu_a} + \frac{1}{\mu_a^2} \right]. 
\end{align*}
\] (24)
Using (23), (24) and the two integrals

\[ \int_{1}^{\infty} dE \sqrt{E^2 - 1} e^{-\mu E} = \frac{K_1(\mu)}{\mu}, \quad \int_{1}^{\infty} dE \sqrt{E^2 - 1} e^{-\mu E} = \frac{K_2(\mu)}{\mu}, \]  

(25)

the dispersion relation (22) becomes in this case

\[ \Lambda_{LH}^{+}(k, \omega) = 1 - \frac{1}{z^2} - \frac{1}{2k^2c^2z} \sum_{a} \omega_{p,a}^2 \mu_a \left[ \frac{z - x_a}{K_1(\mu_a)K_2(\mu_a)} \right] + \frac{1}{2k^2c^2z} \sum_{a} \omega_{p,a}^2 \mu_a \left[ \frac{z - x_a}{K_1(\mu_a)K_2(\mu_a)} \right] \]

\[ \times \int_{1}^{\infty} dE \sqrt{E^2 - 1} e^{-\mu_a E} \]

\[ \times [(1 - z^2) \left( E^2 + 2 \frac{E}{\mu_a} + \frac{2}{\mu_a^2} \right) + 2x_a z \left( E + \frac{1}{\mu_a} \right) - (1 + x_a^2)] = 0. \]

(26)

With the factorization

\[ (1 - z^2)(E^2 + 2E/\mu_a + 2/\mu_a^2) + 2x_a z(E + 1/\mu_a) - (1 + x_a^2) \]

\[ E^2 - 1 - (x_a - zE)^2 \]

\[ = \frac{E^2(1 - z^2) + 2x_a zE - (1 + x_a^2) + (2/\mu_a)(E + 1/\mu_a)(1 - z^2) + 2x_a z/\mu_a}{E^2(1 - z^2) + 2x_a zE - (1 + x_a^2)} \]

\[ = 1 + \frac{2}{\mu_a} \frac{(E + 1/\mu_a)(1 - z^2) + x_a z}{E^2 - 1 - (x_a - zE)^2}, \]

(27)

the two integrals

\[ \int_{1}^{\infty} dE \frac{e^{-\mu E}}{\sqrt{E^2 - 1}} = K_0(\mu), \quad \int_{1}^{\infty} dE \frac{e^{-\mu E}}{\sqrt{E^2 - 1}} = K_1(\mu), \]

(28)

and the Bessel function relation

\[ K_0(\mu) = K_2(\mu) - \frac{2K_1(\mu)}{\mu} \]

(29)

the dispersion relation (26) simplifies to

\[ 0 = \Lambda_{LH}^{+}(k, \omega) = 1 - \frac{1}{z^2} - \frac{1}{k^2c^2z} \sum_{a} \omega_{p,a}^2 \mu_a \left[ \frac{z - x_a}{K_1(\mu_a)K_2(\mu_a)} \right] + \frac{1}{k^2c^2z} \sum_{a} \omega_{p,a}^2 \mu_a \left[ \frac{z - x_a}{K_1(\mu_a)K_2(\mu_a)} \right] \]

\[ \times \int_{1}^{\infty} dE \frac{(z - x_a E)e^{-\mu_a E}}{\sqrt{E^2 - 1} \left[ E^2 - 1 - (x_a - zE)^2 \right]} \left[ (1 - z^2) \left( E + \frac{1}{\mu_a} \right) + x_a z \right] \]

(30)

This is the relativistically correct form of the dispersion relation of transverse oscillations in a magnetized homogeneous Maxwellian plasma of arbitrary composition.
5. Non-relativistic thermal plasmas

We now consider the limit of non-relativistic plasma temperatures $\mu_a \gg 1$. To lowest order in $\mu_a^{-1} \ll 1$

$$\frac{K_1(\mu_a)}{K_2(\mu_a)} \approx 1, \quad K_2(x) \approx \sqrt{\frac{\pi}{2x}} e^{-x}$$

(31)

In this limit the dispersion relation (30) becomes

$$\Lambda_{1H}^+(k, \omega) = 1 - \frac{1}{z^2} - \frac{\sum_a \omega_{p,a}^2}{k^2 c^2} + \sqrt{\frac{2}{\pi k^2 c^2 z}} \sum_a \omega_{p,a}^2 \mu_a^{1/2} J = 0$$

(32)

with

$$J \equiv \int_1^\infty \frac{dE}{\sqrt{E^2 - 1}} \frac{(z - x_a E) [(1 - z^2)(E + 1/\mu_a) + x_a z]}{E^2 - 1 - (x_a - z E)^2}.$$ 

(33)

Substituting $E = \sqrt{1 + s^2}$ yields

$$J = \int_0^\infty ds \frac{\exp(-\mu_a s^2/2)}{\sqrt{1 + s^2}} \frac{(z - x_a \sqrt{1 + s^2})}{s^2 - (x_a - z \sqrt{1 + s^2})^2} \times \left[ 1 - \frac{1}{\mu_a} \right] + x_a z.$$ 

(34)

Because of the exponential function the main contribution to the integral for large values of $\mu_a \gg 1$ comes from small values of $s \ll 1$, so that we may approximate $\sqrt{1 + s^2} \approx s^2/2$ yielding

$$J \approx (z - x_a) \int_0^\infty ds \frac{\exp(-\mu_a s^2/2)}{s^2 - (z - x_a)^2 (1 - z^2 + x_a z)} = (z - x_a) \sqrt{\frac{\mu_a}{2}} \int_0^\infty dt \frac{\exp(-t^2)}{t^2 - f^2}$$

$$= (z - x_a) \sqrt{\frac{\mu_a}{2}} \frac{1}{2 f} \int_{-\infty}^\infty dt \frac{\exp(-t^2)}{t - f} = \frac{\pi^{1/2} \mu_a^{1/2}}{2^{1/2}} \frac{z - x_a}{f} Z^+(f),$$

(35)

where we have used the plasma dispersion function of Fried and Conte [20] of the argument

$$f_a \equiv \frac{\mu_a(z - x_a)^2}{2(1 - z^2 + x_a z)} = \frac{\omega - \Omega_{a,0}}{v_{th,a} k} \left( 1 - \frac{\omega(\omega - \Omega_{a,0})}{k^2 c^2} \right)^{-1/2},$$

(36)

where $v_{th,a} = \sqrt{2 k_B T_a/m_a}$ denotes the thermal velocity. The dispersion relation (32) then reads

$$\Lambda_{1H}^+(k, \omega) = 1 - \frac{1}{z^2} - \frac{\sum_a \omega_{p,a}^2}{k^2 c^2} + \frac{1}{2 k^2 c^2 z} \sum_a \omega_{p,a}^2 \mu_a \frac{z - x_a}{f_a} Z^+(f_a)$$

$$= 1 - \frac{1}{z^2} - \frac{\sum_a \omega_{p,a}^2}{k^2 c^2} + \frac{1}{2 k^2 c^2 z} \sum_a \omega_{p,a}^2 \mu_a \sqrt{\frac{\mu_a}{2}} \sqrt{1 - z^2 + x_a z} Z^+(f_a)$$

$$= 1 - \frac{1}{z^2} - \frac{\sum_a \omega_{p,a}^2}{k^2 c^2} + \sum_a \frac{\omega_{p,a}^2}{\omega k_{th,a}} \sqrt{1 - \frac{\omega(\omega - \Omega_{a,0})}{k^2 c^2}}$$

$$\times Z^+ \left[ \frac{\omega - \Omega_{a,0}}{v_{th,a} k} \left( 1 - \frac{\omega(\omega - \Omega_{a,0})}{k^2 c^2} \right)^{-1/2} \right] = 0.$$ 

(37)
Equation (37) is the relativistically correct dispersion relation of parallel propagating transverse waves in a magnetized thermal plasma of non-relativistic temperature.

Using the well-known analytic continuation of the plasma dispersion relation into the negative imaginary frequency plane ($\Gamma < 0$) (e.g. Roos [14])

\[
Z^-(f) = \frac{1}{\pi^{1/2}} \int_{-\infty}^{\infty} \frac{e^{-x^2}}{x - f} + 2i\sqrt{\pi}e^{-f^2}, \quad \Im(f) < 0,
\]

(38)

the damped waves are described now by

\[
\Lambda_{LH}^{-}(k, \omega) = \Lambda_{LH}^{+}(k, \omega) + 2i\sqrt{\pi} \sum_a \frac{\omega_{p,a}^2}{\omega_k v_{th,a}} \left[ 1 - \frac{\omega(\omega - \Omega_{a,0})}{k^2 c^2} \right]^{1/2} \\
\times \exp \left[ -\left( \frac{\omega - \Omega_{a,0}}{v_{th,a} k} \right)^2 \right] \left( 1 - \frac{\omega(\omega - \Omega_{a,0})}{k^2 c^2} \right) = 0,
\]

(39)

where $\Lambda_{LH}^{+}(k, \omega)$ is given by (37).

6. Limit of infinite speed of light $c \to \infty$

In this section, we consider the transverse dispersion relations in the formal limit of an infinitely large speed of light $c \to \infty$.

From equation (36) we obtain

\[
f_{a,\infty} = \lim_{c \to \infty} f_a = \frac{\omega - \Omega_{a,0}}{v_{th,a} k}
\]

(40)

and dispersion relation (37), in this limit becomes

\[
\Lambda_{LH,\infty}^{+}(k, \omega) \simeq 1 - \frac{1}{z^2} + \sum_a \frac{\omega_{p,a}^2}{\omega_k v_{th,a}} Z^+ \left( \frac{\omega - \Omega_{a,0}}{v_{th,a} k} \right) = 0,
\]

(41)

which agrees with the standard result (equation (25.5) from the textbook of Brambilla [15]) of the non-relativistic dispersion theory.

The classical dispersion relation for damped waves is similar to (39)

\[
\Lambda_{LH,\infty}^{-}(k, \omega) = \Lambda_{LH,\infty}^{+}(k, \omega) + i \frac{2\sqrt{\pi}}{k \omega} \sum_a \frac{\omega_{p,a}^2}{v_{th,a}} \exp \left[ -\frac{(\omega - \Omega_{a,0})^2}{k^2 v_{th,a}^2} \right] = 0.
\]

(42)

We note here that the main difference between relativistically correct form (37) and the standard form (41) consists of the relativistic factor

\[
f_c = \sqrt{1 - \frac{\omega(\omega - \Omega_{a,0})}{k^2 c^2}}, \quad \lim_{c \to \infty} f_c = 1.
\]

(43)
7. Low-frequency waves in finite $\beta$ plasmas

Because wave generation is probably dominated by the heavier charges (protons, ions) in most astrophysically relevant situations [11], we will show here what is the relativistic correction introduced by the relativistically correct dispersion relation (39) in the range of low-frequency waves, $|\omega| < \Omega_{i,0} < |\Omega_e,0|$. This is the case of magnetohydrodynamic (MHD) waves with subluminal phase velocities ($\omega \ll kc < \sqrt{k^2c^2 + \Omega_{i,0}^2}$) and well described by asymptotic approximation ($f_a > 1$ in (36)) of plasma dispersion function in the dispersion relations (37) and (39).

$$\Lambda_{LH}^{-1}(k, \omega) = \Lambda_{LH}(k, \omega) + \frac{i 2\sqrt{\pi}}{k\omega} \sum_a \frac{\omega^2_{p,a}}{v_{th,a}} e^{-(\omega - \Omega_{a,0})^2/k^2v_{th,a}^2} = 0 \quad (44)$$

with

$$\Lambda_{LH}^{+}(k, \omega) = 1 - \frac{k^2c^2}{\omega^2} - \sum_a \frac{\omega^2_{p,a}}{k^2c^2} \left( \frac{k^2c^2}{\omega(\omega - \Omega_{a,0})} + \frac{k^4c^2v_{th,a}^2}{2\omega(\omega - \Omega_{a,0})^3} \left( 1 - \frac{\omega(\omega - \Omega_{a,0})}{k^2c^2} \right)^2 \right). \quad (45)$$

Assuming an isothermal factor $\theta = T_e/T_i < m_i/m_e$ and $\Omega_{i,0}/kv_{th,i} > 1$ the sum terms from (45) can be processed as follows:

$$- \sum_a \frac{\omega^2_{p,a}}{\omega(\omega - \Omega_{a,0})} \simeq \sum_a \frac{\omega^2_{p,a}}{\omega \Omega_{a,0}} \left( 1 + \frac{\omega}{\Omega_{a,0}} \right) \simeq \sum_a \frac{\omega^2_{p,a}}{\Omega_{a,0}^2} \simeq \frac{c^2}{v_A^2}, \quad (46)$$

$$- \sum_a \frac{\omega^2_{p,a} k^2v_{th,a}^2}{2\omega(\omega - \Omega_{a,0})^3} \left( 1 - \frac{\omega(\omega - \Omega_{a,0})}{k^2c^2} \right)^2 \simeq \frac{k^2c^2v_{th,i}^2}{2\omega \Omega_{i,0} v_A^2} + \frac{v_{th,i}^2}{v_A^2} (1 + \theta) \quad (47)$$

We collect now the last (46), (47) to find in (44) (with (45)):

$$\left\{ 1 + \frac{v_A^2}{c^2} [1 + (1 + \theta) \beta] \right\} \Omega^2 + K^2(2\Omega - 1) + \frac{i\sqrt{\pi}}{8} \frac{\beta^{3/2}}{\sqrt{K}} \exp \left( -\frac{\beta}{16K^2} \right) = 0, \quad (48)$$

where $\beta = (8\pi P)/B^2 = v_{th,i}^2/v_A^2$ is the so-called plasma beta (ratio of thermal to magnetic energy density), $v_A = \gamma \Omega_{i,0}/\omega_{p,i}$ is Alfvén velocity. The dimensionless symbols $\Omega = \Omega_i + i\Omega_e = \omega\beta/4\Omega_{i,0}$ and $K = kc\beta/4\omega_{p,i}$ correspond to wave frequency and wave number, respectively, and have been chosen to generalize the analysis and implicit the solutions for any species of ions.

Similar dispersion relations were obtained in the classical non-relativistic approach, first by Foote and Kulsrud ([10] equation (18)) omitting the last term representing (ion) cyclotron damping in (48), and later, by Achterberg ([11] equation (4)) who took into account only a low ion cyclotron damping, twice as low as our last term in (48). Furthermore, in both non-relativistic dispersion relations first term was only $(1 + v_A^2/c^2)\Omega^2$, but now in the first term of (48), $[(1 + v_A^2/c^2) + C_c]\Omega^2$ we have obtained the correction relativistic factor:

$$C_c = \frac{v_A^2}{c^2} \beta (1 + \theta) = \frac{v_{th,i}^2}{c^2} (1 + \theta) = \frac{2}{\mu_i} (1 + \theta), \quad (49)$$
which depends on the both temperatures of plasma components and therefore it can become important (irrespective to unity), even for low non-relativistic plasma temperatures \(\mu_i > 1\). Obviously, equation (48) is of general practical use because it admits relative simple approximative analytical solutions, but we will analyse the exact numerical solutions, and discuss the cases which stress the importance of the relativistic correction. Involving the isothermal factor \(\theta = T_e/T_i \neq 1\) in (49) leads to an interesting situation also discussed at the end.

From the second term of (48), we observe that low-frequency (Alfvén) waves (i.e. \(2 \Omega \ll 1\)) are non-dispersive, being described by the approximative relation

\[
\frac{\Omega}{K} = \left[1 + \frac{v_A^2}{c^2} (1 + \theta) \beta\right]^{-1/2},
\]

where we have made use of the fact that \(v_A/c = \Omega_{i,0}/\omega_{pi} \ll 1\) in conventional plasmas. And explicating the dimensionless symbols \(\Omega\) and \(K\) in (50) we find the new relativistic Alfvén velocity

\[
v_A^c = \frac{v_A}{\sqrt{1 + (2/\mu_i)(1 + \theta)}},
\]

which has been similarly defined by Gedalin [16]. In the limit of a non-relativistic approach \(\lim_{c \to \infty} v_A^c = v_A\).

### 7.1. Isothermal plasmas, \(\theta = 1\)

In the case of isothermal plasma, \(\theta = 1\), the correction term (49) is usually negligible, \(C = 4/\mu_i \to 0\), for non-relativistic temperatures, i.e. \(\mu_i = (m_i/m_e)\mu_e \gg \mu_e \gg 1\). Otherwise, it could reach significant values for lower ion thermal factor, \(\mu_i \gtrsim 4\), but the electrons would be ultrarelativistic \((\mu_e \ll 1)\) and our assumption of very large values of thermal factor, \(\mu_a \gg 1\), for any species, \(a = e, i\), will fail. However, for low-frequency oscillations only the ions are able to follow them dynamically, and the dispersion relation (48) with the relativistic correction (49), \(C_c \approx 4/\mu_i\), could be more useful than previous standard results, in the limit of near-relativistic ions \((\mu_i \gtrsim 4)\).

The scattering of cosmic rays by low-frequency MHD plasma waves guarantees the near isotropy of cosmic rays [17], and because the interstellar magnetic turbulence [18] is much less than the mean total magnetic field in the Galaxy, the application of quasi-linear theory to cosmic ray transport and acceleration is well justified. Therefore in figure 1 we compare numerically the relativistic solutions of (48) with those provided by equation (4) of Achterberg [11] for some typical conditions of space plasma immersed by galactic (or solar [19]) cosmic rays [17]: a relative dilute plasma, with density \(n \sim 10^{-9} \, \text{cm}^{-3}\), but with very high temperatures, \(T_{\text{max}} > 10 \, \text{MeV}\), and with an ambient magnetic field, \(B_0 \sim 10^{-6} \, \text{G}\), so that Alfvén velocity is sufficiently high \(v_A \simeq 0.3c\).

In order to obtain only the differences provided by the relativistic correction from the first term of (48), we replaced the third term of Achterberg’s equation with the third term from (48) which removes the limitation of weak damping approximation.

The numerical solutions for each of the four cases presented in figure 1 have been obtained keeping the same value as those mentioned before, for plasma density and for the static magnetic field, so that \(v_A^2/c^2 = 2/(\beta\mu_i) = 0.08 = \text{constant}\). Only the plasma temperature increases (from...
Figure 1. Each panel illustrates by comparison the dispersion curves (upper positive plane, $\Omega_r$ versus $K$) and the damping curves (lower negative plane, $\Omega_i$ versus $K$) provided by relativistic (······) and classical theory (——). These curves practically coincide at very low (non-relativistic) temperature, as shown in the first panel (1) $\mu_i \geq 100$, $\beta = 0.25$, but they are markedly different at high (near relativistic temperatures), as shown in the next three panels (2) $\mu_i = 20$, $\beta = 1.25$; (3) $\mu_i = 10$, $\beta = 2.5$; (4) $\mu_i = 5$, $\beta = 5$.

the first to the last panel) strongly affecting the wave dispersion which exhibits a significant changing with a deep minimum before the asymptotic approaching of the ion cyclotron frequency. This thermal effect was discussed also by Achatz et al (1993), for a similar hot plasma (relativistic electrons and non-relativistic protons) with $\beta \gtrsim 1$, and it is clearly associated to the LH wave frequencies very closed to ion gyrofrequency and consequently with a significant cyclotron damping which tends to suppress the propagation in thermal plasmas. Otherwise, the relativistic solutions are less affected than those from classical theory.

Alfvén velocity attains very high values in this case, $v_A \simeq 0.3c$, and provides therefore another important contribution in the first term of (48), by the ratio $v_A^2/c^2 \lesssim 1$ which is often neglected in high beta plasma (see the considerations of Achterberg [11]).

Towards the highest limit of non-relativistic ion temperature ($\mu_i \lesssim 1$), the relativistic dispersion and damping curves are slightly lower than classical solutions (excepting the regions of a heavily damped regime). This is a natural consequence of the relativistic decreasing of cyclotron frequency with increasing plasma particle thermal velocity.

7.2. Pair plasmas, $m_{e^-} = m_{e^+}$

The neutral pair plasma at equilibrium is a particular case of isothermal plasma ($\theta = 1$) with two symmetric species, $m_{e^-} = m_{e^+}$, as one expects in the pulsar polar magnetosphere [21]. In
In order to develop diagnostic tools for the pair plasma created in the laboratory [22], Iwamoto [23] provided a non-relativistic kinetic analysis for the collective modes in low temperature electron–positron plasmas (with or without a static spatially uniform magnetic field). Furthermore, Zank and Greaves [24] argued the new interest for astrophysical low temperature pair plasma: ‘electron–positron plasmas radiate very effectively by cyclotron emission and must, therefore, cool eventually’. They also extended the linear analysis of Iwamoto [23] to nonlinear modes including two stream instability and solitary waves in non-relativistic pair plasma. In both of these papers it was shown rigorously that time scales for pair annihilation are usually many orders of magnitude larger than the plasma period, so that long time scale plasma physics experiments should be possible. This is the case of very low, $\mu > 5 \times 10^3$, or higher energies, $\mu < 50$, and only for non-relativistic energies in the range of $\mu = 5 \times 10^2–5 \times 10^3$ direct annihilation dominates the loss process. Thus, the investigation of low-frequency waves in pair plasmas towards the high limit of non-relativistic temperatures, i.e. $1 < \mu < 50$, is fully motivated.

For parallel propagation, we agree with the previous outlines [23, 24] that (i) the dispersion relation for the LH circularly polarized wave is the same as that for the RH circularly polarized wave; (ii) Alfvén waves exist as in the case of electron–ion plasma, and resonate with positrons for LH wave polarization or with electrons for RH wave polarization; (iii) the whistler mode does not exist (in contrast with ion–electron plasma). But in the derivation of the exact transverse dispersion relations they minimized the thermal effects by considering $k v_{th,e} \ll |\omega \pm \Omega_{e,0}|$, or even the cold pair plasma limit $T \to 0$.

Now we are able to use dispersion relation (48) which includes thermal effects of a warm pair plasma, and where all the parameters should be expressed in terms of the pair plasma parameter $\alpha = 2 \omega_{p,e}/\Omega_{e,0} = 2c/v_A$ and thermal factor $\mu = \mu_{e^-} = \mu_{e^+}$. Then $\beta = 2\alpha^2/\mu$ and also (49) becomes

$$C^p = C_0^p + \frac{4}{\mu} = \frac{1}{\alpha^2} + \frac{4}{\mu}. \quad (52)$$

In conventional conditions the pair plasma parameter is small, $\alpha \lesssim 1$, and the characteristic velocity of Alfvén waves derived non-relativistically [25] as

$$v_A^p = \frac{c}{\sqrt{1 + \alpha^2}} = \frac{v_A}{\sqrt{2 + v_A^2/c^2}}. \quad (53)$$

differs from the usual Alfvén velocity in an electron–ion plasma, $v_A = B_0/[4\pi n(m_i + m_e)]^{1/2} \approx B_0/[4\pi n m_i]^{1/2}$. For example in the limit of $v_A/c = \Omega_{e,0}/\omega_{p,e} \ll 1$ in a pair plasma, one obtains from (53): $v_A^p = B_0/[8\pi n m_i]^{1/2} = v_A/\sqrt{2}$ where we thus emphasize the factor $2^{-1/2}$. But here, as a consequence of the relativistic treatment, we can use again the limit of Alfvén waves in (48) to obtain the new relativistically correct form of Alfvén velocity in a thermal pair plasma

$$v_A^{p,c} = c \left[ 1 + \alpha^2 \left( 1 + \frac{4}{\mu} \right) \right]^{-1/2}, \quad (54)$$
Figure 2. The low-frequency dispersion curves (upper panels, (1) and (3) with \( \Omega_r \) versus \( K \)) and the damping curves (lower panels, (2) and (4) with \( \Omega_i \) versus \( K \)) for pair plasmas with parameter \( \alpha \simeq 1 \). The left panels (1) and (2), and the right panels (3) and (4) correspond to near relativistic temperatures with \( \mu = 5 \) and \( \mu = 20 \), respectively.

with \( \lim_{v_c \to \infty} v_p = v_A \). We distinguish three situations here:

1. \( \alpha < 1 \) for which (49) becomes \( C \simeq 1/\alpha^2 \);
2. \( \alpha \sim 1 \) for which (49) becomes \( C \simeq 2 + 4/\mu \);
3. \( \alpha > 1 \) or \( \alpha^2 \gg 1 \), for which (49) becomes \( C \simeq 1 + 4/\mu \).

The first case of \( \alpha < 1 \) corresponds to a pulsar magnetosphere plasma [21], where the quasi-stationary magnetic field can reach very high values (\( \sim 10^{12} \) G). In this case, the gyrofrequencies are so high that damping rates are negligible, and the relativistic and standard dispersion curves do not behave significantly different at non-relativistic temperatures.

The second case (\( \alpha \sim 1 \)) is similar to the last one (\( \alpha > 1 \)), and both of them could be specific for laboratory-created pair plasmas [22] or for quasi-equilibrium pair plasma in AGN engine, after an initial phase of rapid cooling by synchrotron and inverse Compton energy losses [26]. In figure 2 we compare numerically only for the second case of \( \alpha \sim 1 \), the relativistic (dotted line) and the standard solution (solid line) obtained from (48) with or without correction factor (\( C_c \)), respectively.

7.3. Non-isothermal plasmas, \( \theta \neq 1 \)

In a non-isothermal electron–ion plasma the range of validity of the relativistically correct dispersion relation (48) extends to lower ion temperatures for a high isothermal factor \( \theta > 1 \), or to a fully non-relativistic plasma (low-temperature electrons and near-relativistic protons) when \( \theta < 1 \).
In most astrophysical plasmas the ions are limited to low non-relativistic temperatures and because of their small rest mass, the electrons will have to be relativistic [11]. Such a non-isothermal plasma model has been extensively used for detailed investigations of particles acceleration in plasma wave turbulence (see [27] and references therein).

The realistic model of solar flare plasmas is non-isothermal, favourable to very strong variations in the temperature of plasma components, $T_e \neq T_i$. For example, in the majority of solar flares the energy partition favours electrons but there is a significant fraction of flares where more energy resides in protons than in electrons [19, 28]. Some estimations on the values of isothermal factor, $\theta$, have been communicated by Miller et al [28] and their extreme values were extended also by the latest observations [19, 27] to $\theta_{\min} = 10^{-4}$, and $\theta_{\max} = 10^4$.

If we consider first the case of very small $\theta \ll 1$, it can well describe the non-relativistic temperature plasmas where the protons are more energetic than the electrons, and for which the relativistic correction term (49) becomes $C_c \simeq 2/\mu_i$. This situation is then very closed to the case of isothermal plasmas (where $C_c \simeq 4/\mu_i$) and therefore it adds only small changes in the relativistic dispersion (48) comparing with standard solutions [11], as we have shown in figure 1. But here the electrons could be considered to have a lower (even non-relativistic) temperature because $\theta = T_e/T_i \ll 1$, and thus, this case constitutes a substantial support for our covariant dispersion relation (48) which provides a safe extension of the standard results to fully non-relativistic non-isothermal plasmas.

In the other limit of $\theta \gg 1$, the electrons are usually ultrarelativistic because the ion (proton) temperature in solar flares is supposed to be limited to $T_i > 1$ MeV [27, 28], and the relativistic correction becomes $C_c \simeq (2 \theta)/\mu_i$. If we accept again that for low-frequency waves, the non-relativistic protons dominate dynamically the neutralizing electronic background, then the relativistic dispersion relation (48) provides solutions more different than standard solutions, and which are illustrated in figure 3 by dotted and solid lines, respectively. Considering proton temperature $k_B T_p \simeq 2.5$ MeV ($\mu_p \simeq 400$) and a high isothermal factor, $\theta = 300$ we have chosen to compare the relativistic and non-relativistic solutions in four situations characterized by different ambient magnetic field which decreases, and plasma $\beta$ will increase (for a plasma density $n = \text{constant}$), from $\beta = 0.1$ in panel (1) to $\beta = 200$ in panel (4).

Some suggestive similarities with figure 1 can be followed in figure 3: dispersion curves exhibit a deeper minimum just before asymptotic approaching of the ion cyclotron frequency (where cyclotron damping rates reach considerable values), and prominent differences are obtained between relativistic and non-relativistic solutions for high plasma-$\beta \gg 1$. Otherwise, in figure 1 we have recorded small differences between relativistic and non-relativistic solutions, and only at the limit of near relativistic ion temperatures ($\mu_i = 1–20$). But in figure 3, as we have mentioned before for a non-isothermal plasma with $\theta > 1$, the relativistic effects occur for lower ion (proton) temperatures ($\mu_p = 400$) and even for large plasma-$\beta$ (see the panels (2)–(4) where the damping becomes sufficiently high).

8. Summary

We have formulated the relativistic kinetic theory of linear parallel plasma waves in a magnetized hot plasma with an isotropic distribution and arbitrary composition. Using the relativistically correct set of Vlasov–Maxwell equations, we have derived the dispersion relation for plasmas of arbitrary composition and arbitrary isotropic distribution functions. The dispersion relation
Figure 3. Each panel compares the (low frequency) dispersion curves (upper positive plane, $\Omega_r$ versus $K$) and the damping curves (lower negative plane, $\Omega_i$ versus $K$) provided by relativistic (―) and classical theory (——) for an electron–proton plasma with $\mu_p = 400$, $\theta = 300$. A considerable difference between covariant and non-covariant solutions is observed for each of the situations: (1) $\beta = 0.1$; (2) $\beta = 2$; (3) $\beta = 20$; (4) $\beta = 200$.

describes the linearized response of the system to the initial perturbation and thus defines all existing linear parallel transverse plasma modes in the system. We then restrict our analysis to the case of an equilibrium plasma with relativistic Maxwellian distribution characterized by the temperature parameter $\mu_a$. Equation (30) is the relativistic dispersion relation of transverse oscillations parallel propagating in a magnetized relativistic Maxwellian plasma of arbitrary composition.

Investigating the limit of non-relativistic thermal plasmas ($\mu_a \gg 1$) we have derived, apparently for the first time, the relativistically correct dispersion relation (37) in terms of the plasma dispersion function. There is a significant difference between the classical and the relativistic forms mainly introduced by the relativistic factor (43). We have demonstrated that only in the strictly unphysical formal limit of an infinitely large speed of light $c \to \infty$ the dispersion relations reduce to the standard non-relativistic dispersion relations.

In order to illustrate these differences we have limited our investigation to low-frequency range (MHD waves) where we have derived a new relativistic form of dispersion relation which includes now a relativistic correction factor given by relation (49). This is directly dependent on the ion temperature ($\sim \mu_i^{-1}$), and on the isothermal factor $\theta = T_e/T_i \neq 1$. Using the relativistic correction (49), we have defined a new relativistic Alfvén speed in (51).

Consequently, the relativistically correct solutions should be markedly different from the non-relativistic solutions that we have shown numerically for isothermal or non-isothermal plasmas. In an isothermal electron–ion plasma, we have assumed that only the ions are able...
to respond dynamically to low-frequency perturbations. And considering the typical conditions of galactic cosmic rays with high temperatures, low magnetic field and very low density, we have obtained small but visible differences between relativistic and non-relativistic solutions to the limit of high but still non-relativistic ion temperatures ($1 < \mu_i < 20$). Furthermore, we have particularized to the case of electron–positron plasma where the differences becomes more prominent at lower plasma temperatures because the mass contrast is reduced to unity ($m^+/m^- = 1$). But the most important contribution of the new relativistic solutions has been obtained for a non-isothermal electron–ion plasma. First, the results obtained for a plasma with small isothermal factor are very close to those for isothermal plasmas, but when $\theta = T_e/T_i \ll 1$ we are able to consider lower temperature (even non-relativistic) of the electrons and plasma becomes fully non-relativistic as we have assumed. This case provides then an important justification for a relativistically correct theory of plasma waves at low non-relativistic temperatures, and it will be investigated in detail in our future work. In the other limit of a strong factor $\theta = T_e/T_i \gg 1$, we have considered again only the ion contribution to the low-frequency oscillations, and the relativistic effects are found to be more efficient and even at lower ion temperatures.

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