NUMERICAL SOLUTION OF FRACTIONAL STURM–LIOUVILLE EQUATION IN INTEGRAL FORM

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Abstract

In this paper a fractional differential equation of the Euler–Lagrange / Sturm–Liouville type is considered. The fractional equation with derivatives of order $\alpha \in (0, 1]$ in the finite time interval is transformed to the integral form. Next the numerical scheme is presented. In the final part of this paper examples of numerical solutions of this equation are shown. The convergence of the proposed method on the basis of numerical results is also discussed.

MSC 2010: Primary 26A33: Secondary 34A08, 65L10

Key Words and Phrases: fractional Euler–Lagrange equation, fractional Sturm–Liouville equation, fractional integral equation, numerical solution

1. Introduction

The fractional differential equations, both ordinary and partial ones, are very useful tools for modelling many phenomena in physics, mechanics, control theory, biochemistry, bioengineering and economics [10, 12, 21, 22, 24, 38]. Therefore, the theory of fractional differential equations is an area that has developed extensively over the last decades. In the monographs [13, 14, 15, 30, 31] one can find a review of methods of solving fractional differential equations.

In recent years, subtopic of the theory of fractional differential equations gains importance: it concerns the variational principles for functionals involving fractional derivatives. These principles lead to equations known in the literature as the fractional Euler–Lagrange equations. The equations of this type were derived when fractional integration by parts rule [13] has been applied.

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pp. xxx–xxx
This approach was initiated by Riewe in [36], where he used non-integer order derivatives to describe nonconservative systems in mechanics. Next Klimek [15] and Agrawal [1] noticed that such equations can be investigated in the sequential approach. A fractional Hamiltonian formalism for the combined fractional calculus of variations was introduced in [27]. In the work [29] Green theorem for generalized partial fractional derivatives was proved. Other applications of fractional variational principles are presented in [2, 19, 25, 26, 28].

Recently, the fractional Sturm–Liouville problems were formulated by Klimek and Agrawal in [18] and Rivero et al. in [37]. Authors in these papers considered several types of the fractional Sturm–Liouville equations and they investigated the eigenvalues and eigenfunctions properties of the fractional Sturm–Liouville operators.

Unfortunately, the fractional Euler–Lagrange / Sturm–Liouville equations contain the composition of the left- and right-sided derivatives. It is an additional drawback for computation of an exact solution (even with simple Lagrangian, see [5, 16, 17]). Consequently, numerous studies have been devoted to numerical schemes for the fractional equations (see [4, 6, 8, 23, 39]). For numerical methods in the fractional calculus of variations we refer the reader to [33, 34, 35].

In our previous works [7, 8, 9] we proposed numerical scheme on the basis of a finite difference method of solution for a special case of the problem, namely the fractional oscillator equation. In this paper we propose a numerical solution of the fractional Sturm–Liouville equation. We investigate a new integral form of this equation and a numerical method of solution of considered equation in conjunction with analysis of a rate of convergence. Another integral form of the fractional Euler-Lagrange equations (containing the Caputo derivatives) has been recently considered in [20].

2. Statement of the problem and definitions

We consider the fractional differential equation with derivatives of order \( \alpha \in (0, 1] \) in the finite time interval \( t \in [0, b] \), for parameter \( \lambda \in \mathbb{R} \) and variable potential determined by function \( q(t) \)

\[
^C D_0^\alpha - D_0^\alpha \cdot f(t) + (\lambda + q(t)) \ f(t) = 0, \tag{2.1}
\]

where \( f(t) \in AC[0, b] \) is an unknown function (absolutely continuous on \([0, b]\), satisfying boundary conditions

\[
f(0) = 0, \quad f(b) = L. \tag{2.2}
\]
According [13, 30, 31] we recall the definitions of the left- and right-sided Riemann-Liouville fractional derivative operators for \( \alpha \in (0, 1) \)

\[
D_{0+}^\alpha f(t) := D I_{0+}^{1-\alpha} f(t) \quad (2.3)
\]

\[
D_{b-}^\alpha f(t) := -D I_{b-}^{1-\alpha} f(t) \quad (2.4)
\]

where \( D \) is operator of the first order derivative and operators \( I_{0+}^\alpha \) and \( I_{b-}^\alpha \) are respectively the left- and right-sided fractional integrals of order \( \alpha > 0 \) defined by

\[
I_{0+}^\alpha f(t) := \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau \quad (t > 0) \quad (2.5)
\]

\[
I_{b-}^\alpha f(t) := \frac{1}{\Gamma(\alpha)} \int_t^b \frac{f(\tau)}{(\tau-t)^{1-\alpha}} d\tau \quad (t < b), \quad (2.6)
\]

whereas operators \( C D_{0+}^\alpha \) and \( C D_{b-}^\alpha \) represent the left- and right-sided Caputo fractional derivatives, respectively. Between both definitions occur the following relationships [13] (only valid for \( \alpha \in (0, 1) \))

\[
C D_{0+}^\alpha f(t) := D_{0+}^\alpha f(t) - \frac{t^{-\alpha}}{\Gamma(1-\alpha)} f(0) \quad (2.7)
\]

\[
C D_{b-}^\alpha f(t) := D_{b-}^\alpha f(t) - \frac{(b-t)^{-\alpha}}{\Gamma(1-\alpha)} f(b). \quad (2.8)
\]

In the further part of this paper we will use the following composition rules of fractional operators (for \( \alpha \in (0, 1) \)) [13]

\[
I_{0+}^\alpha \; C D_{0+}^\alpha f(t) = f(t) - f(0) \quad (2.9)
\]

\[
I_{b-}^\alpha \; C D_{b-}^\alpha f(t) = f(t) - f(b) \quad (2.10)
\]

and the fractional integral of a constant \( C \)

\[
I_{0+}^\alpha C = C \frac{t^\alpha}{\Gamma(1+\alpha)}, \quad I_{b-}^\alpha C = C \frac{(b-t)^\alpha}{\Gamma(1+\alpha)} \quad (2.11)
\]

In particular, when \( \alpha = 1 \), then \( C D_{b-}^1 D_{0+}^1 = -D^2 \) and for \( q(t) = 0 \) Eq. \((2.1)\) becomes

\[
-D^2 f(t) + \lambda f(t) = 0 \quad (2.12)
\]

and for \( \lambda < 0 \) (the oscillatory character of solutions of the equation) its analytical solution satisfying boundary conditions \((2.2)\) is of the form

\[
f(t) = L \frac{\sin \left( \sqrt{-\lambda} t \right)}{\sin \left( \sqrt{-\lambda} b \right)}, \quad \lambda \neq -\left( \frac{k\pi}{b} \right)^2, \quad k \in \mathbb{Z}. \quad (2.13)
\]

The formula for the analytical solution of Eq. \((2.1)\) for \( \alpha \in (0, 1) \) is rather involved. For example, the analytical solution for \( q(t) = 0 \) which
contains the series of fractional integrals is presented in [8, 16, 17]. This makes it difficult to carry out any operations on them. There is a problem in calculations of the values of \( f \). The analytical solution of Eq. (2.1) can be expressed by elementary functions only in the special case (see details in further description).

### 3. Transformation of fractional equation to the integral form

**Proposition 3.1.** The equivalent integral form of Eq. (2.1) with boundary conditions (2.2) is given as

\[
f(t) + I_{0+}^{\alpha} I_{b-}^{\alpha} (\lambda + q(t)) f(t) - \left( \frac{t}{b} \right)^{\alpha} I_{0+}^{\alpha} I_{b-}^{\alpha} (\lambda + q(t)) f(t)|_{t=b} = \frac{L}{b^\alpha} t^\alpha.
\] (3.1)

**Proof.** By using the fractional integral operators \( I_{0+}^{\alpha} I_{b-}^{\alpha} \) acting on Eq. (2.1), we obtain

\[
I_{0+}^{\alpha} I_{b-}^{\alpha} f(t) + I_{0+}^{\alpha} I_{b-}^{\alpha} (\lambda + q(t)) f(t) = 0.
\] (3.2)

Next we use the composition rule of operators \( I_{0+}^{\alpha} C D_{b-}^{\alpha} \) (see Eq. (2.9)) in Eq. (3.2), thus we get

\[
I_{0+}^{\alpha} (D_{0+}^{\alpha} f(t) - D_{0+}^{\alpha} f(t)|_{t=b}) + I_{0+}^{\alpha} I_{b-}^{\alpha} (\lambda + q(t)) f(t) = 0
\] (3.3)

The above equation contains an unknown value \( D_{0+}^{\alpha} f(t)|_{t=b} \) and for the unknown function \( f(t) \) this value is treated here as a constant.

Using again the composition rule of operators \( I_{0+}^{\alpha} D_{0+}^{\alpha} \) (see Eq. (2.9)) and the fact that if \( f(0) = 0 \), then \( D_{0+}^{\alpha} f(t) = C D_{0+}^{\alpha} f(t) \) in Eq. (2.7), hence \( I_{0+}^{\alpha} D_{0+}^{\alpha} f(t) = f(t) \) and the fractional integral of a constant (2.11), we obtain the following form of equation

\[
f(t) - D_{0+}^{\alpha} f(t)|_{t=b} \frac{t^{\alpha}}{\Gamma(\alpha + 1)} + I_{0+}^{\alpha} I_{b-}^{\alpha} (\lambda + q(t)) f(t) = 0.
\] (3.4)

In order to determine the value \( D_{0+}^{\alpha} f(t)|_{t=b} \) we substitute the value \( t = b \) into Eq. (3.4)

\[
f(b) - D_{0+}^{\alpha} f(t)|_{t=b} \frac{b^{\alpha}}{\Gamma(\alpha + 1)} + I_{0+}^{\alpha} I_{b-}^{\alpha} (\lambda + q(t)) f(t)|_{t=b} = 0
\] (3.5)

and obtain

\[
D_{0+}^{\alpha} f(t)|_{t=b} = \frac{\Gamma(\alpha + 1)}{b^{\alpha}} \left( f(b) + I_{0+}^{\alpha} I_{b-}^{\alpha} (\lambda + q(t)) f(t)|_{t=b} \right).
\] (3.6)
Next we substitute the right-hand side of (3.6) into Eq. (3.4) and get the integral form of Eq. (2.1)

\[ f(t) + I_0^\alpha I_b^\alpha (\lambda + q(t)) f(t) - \left( \frac{t}{b} \right)^\alpha I_0^\alpha I_b^\alpha (\lambda + q(t)) f(t) \bigg|_{t=b} = \left( \frac{t}{b} \right)^\alpha f(b). \]  

Taking into account the boundary conditions (2.2) we obtain Eq. (3.1).

**Remark 3.1.** One can easily check that \( I_0^\alpha I_b^\alpha (\lambda + q(t)) f(t) \bigg|_{t=0} = 0 \) (on the basis of definitions (2.5) and (2.6)). If we put \( t = 0 \) and \( t = b \) into Eq. (3.1) we can confirm that this equation fulfills the boundary conditions (2.2). In particular, for \( \lambda + q(t) = 0 \) Eq. (3.1) simplifies to the form

\[ f(t) = L \frac{t^\alpha}{b^\alpha}. \]  

**4. Numerical algorithm**

In this section we present a numerical scheme for Eq. (3.1). We introduce the homogeneous grid of nodes (with the constant time step \( \Delta t = b/n \), where \( n+1 \) is a number of nodes): \( 0 = t_0 < t_1 < ... < t_i < t_{i+1} < ... < t_n = b \), and \( t_i = i \Delta t \). In order to simplify notation we will introduce a new function \( \phi(t) \equiv (\lambda + q(t)) f(t) \) and we denote the values of functions \( f(t) \), \( q(t) \) and \( \phi(t) \) at the node \( t_i \) by \( f_i = f(t_i) \), \( q_i = q(t_i) \) and \( \phi_i = \phi(t_i) = (\lambda + q_i) f_i \).

Now we determine the numerical schemes of integration [11, 30, 32] for both fractional integral operators occurring in Eq. (3.1).

At node \( t_0 \) we have \( I_0^\alpha \phi(t) \bigg|_{t=t_0} = 0 \). Discrete form of the integral operator [25] at nodes \( t_i \) for \( i = 1, 2, ..., n \) is approximated by the formula

\[ I_0^\alpha \phi(t) \bigg|_{t=t_i} = \frac{1}{\Gamma(\alpha)} \int_0^{t_i} \frac{\phi(\tau)}{(t_i - \tau)^{1-\alpha}} d\tau = \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} \frac{\phi(\tau)}{(t_i - \tau)^{1-\alpha}} d\tau \]

\[ \approx \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{i-1} \phi_j + \phi_{j+1} \frac{1}{2} \int_{j \Delta t}^{(j+1)\Delta t} \frac{1}{(i \Delta t - \tau)^{1-\alpha}} d\tau \]

\[ = \frac{(\Delta t)^\alpha}{2\Gamma(\alpha + 1)} \sum_{j=0}^{i-1} \phi_j + \phi_{j+1} ((i-j)^\alpha - (i-j-1)^\alpha) \]

\[ = \sum_{j=0}^{i} \phi_j w_{i,j}, \]  

(4.1)
where the coefficients \( w_{i,j} \) (also including the case for \( i = 0 \)) are as follows

\[
    w_{i,j} = \frac{(\Delta t)^\alpha}{2\Gamma(\alpha + 1)} \begin{cases} 
        0 & \text{for } i = 0 \text{ and } j = 0 \\
        \tau^\alpha - (i-1)^\alpha & \text{for } i > 0 \text{ and } j = 0 \\
        (i-j+1)^\alpha - (i-j-1)^\alpha & \text{for } i > 0 \text{ and } 0 < j < i \\
        1 & \text{for } i > 0 \text{ and } j = i
    \end{cases}
\]

We determine discrete form of the fractional integral operator (2.6) in a similar way. This operator at node \( t_n \) is equal to \( I^\alpha_{b-} \phi(t)|_{t=t_n} = 0 \), whereas at nodes \( t_i \), \( i = 0,1,\ldots,n-1 \), the discrete values are determined by the formula

\[
    I^\alpha_{b-} \phi(t)|_{t=t_i} = \frac{1}{\Gamma(\alpha)} \int^{t_n}_{t_i} \frac{\phi(\tau)}{\tau^1-\alpha} d\tau = \frac{1}{\Gamma(\alpha)} \sum_{j=i}^{n-1} \int^{t_{j+1}}_{t_j} \frac{\phi(\tau)}{\tau^1-\alpha} d\tau 
\]

\[
    \approx \frac{1}{\Gamma(\alpha)} \sum_{j=i}^{n-1} \frac{\phi_j + \phi_{j+1}}{2} \int_{j}^{(j+1)\Delta t} \frac{\phi(\tau)}{\tau^1-\alpha} d\tau = \frac{1}{\Gamma(\alpha)} \sum_{j=1}^{n} (\phi_j + \phi_{j+1}) ((j-i)^\alpha - (j-i-1)^\alpha)
\]

\[
    = \sum_{j=1}^{n} \phi_j v_{i,j}, \quad (4.3)
\]

where the coefficients \( v_{i,j} \) (together with the case for \( i = n \)) have the form

\[
    v_{i,j} = \frac{(\Delta t)^\alpha}{2\Gamma(\alpha + 1)} \begin{cases} 
        0 & \text{for } i = n \text{ and } j = n \\
        (n-i)^\alpha - (n-i-1)^\alpha & \text{for } i < n \text{ and } j = n \\
        (j-i+1)^\alpha - (j-i-1)^\alpha & \text{for } i < n \text{ and } i < j < n \\
        1 & \text{for } i < n \text{ and } j = i
    \end{cases}
\]

(4.4)

The discrete form of the composition of both operators \( I^{\alpha}_{0+} I^{\alpha}_{b-} \phi(t) \) at nodes \( t = t_i \) for \( i = 0,1,\ldots,n \), has the following form

\[
    I^{\alpha}_{0+} I^{\alpha}_{b-} \phi(t)|_{t=t_i} \approx \sum_{j=0}^{i} \sum_{k=j}^{n} w_{i,j} \phi_k v_{j,k}, \quad (4.5)
\]

or

\[
    I^{\alpha}_{0+} I^{\alpha}_{b-} (\lambda + q(t)) f(t)|_{t=t_i} \approx \sum_{j=0}^{i} \sum_{k=j}^{n} (\lambda + q_k) f_k v_{j,k}. \quad (4.6)
\]

One can note that at the node \( t_0 \) we have \( I^{\alpha}_{0+} I^{\alpha}_{b-} \phi(t)|_{t=t_0} = 0 \).

Now we present the discrete form of the integral equation (3.1). The solution one can write in the form of the system of \( n+1 \) linear equations.
For every grid node $t_i$, $i = 0, 1, ..., n$, we write the following equation
\[
f_i + \sum_{j=0}^{i} \sum_{k=j}^{n} (\lambda + q_k) f_{k,j} - \left( \frac{i}{n} \right)^{\alpha} \sum_{j=0}^{n} \sum_{k=j}^{n} (\lambda + q_k) f_{k,v,j,k} = \left( \frac{i}{n} \right)^{\alpha} L.
\]

(4.7)

Analysing the above system of equations, created equations for node indexes $i = 0$ and $i = n$ one can reduce to the forms $f_0 = 0$ and $f_n = L$, respectively. In this way the obtained system of linear equations can be solved numerically.

5. Simulation results and numerical error analysis

In this section we present the results of calculation obtained by our numerical approach to the fractional Sturm–Liouville equation (2.1). In order to numerically solve the system of equations (4.7) we used the LUP decomposition method [32]. We present several examples of calculations for different values of parameters $\alpha$, $\lambda$ and forms of function $q(t)$. In all these examples we assumed: $b = 1$ and $L = 1$ in the right boundary condition. For all presented graphs of functions (Figures 1-3), in the calculations we assume that the time domain $t \in [0, 1]$ has been divided into $n = 2048$ subintervals.

5.1. Example of results

Figures 1 and 2 show the example graphs of solutions of Eq. (2.1) for $q(t) = 0$ (the case of the fractional oscillator equation). Figure 1 presents solutions for $\lambda \in \{-3, -10, -20, -25\}$ and variable values of parameter $\alpha$. We can see how changes of parameter $\alpha$ influence on the frequency of oscillations in comparison to the classical oscillator equation ($\alpha = 1$). Whereas in the Figure 3 we show the influence of parameter $\lambda \in \{-5, -7.5, -10, -12.5\}$ at the constant values of $\alpha = 0.6$ and $\alpha = 0.8$ on the solution.

In the last Figure 3 we present solutions of Eq. (2.1) for different form of function $q(t) \neq 0$ (the case of the fractional Sturm–Liouville equation). We show the influence of parameters $\alpha$, $\lambda$ and forms of function $q(t)$ on the solution.

5.2. Error analysis

Next we analyse errors and convergence of the numerical scheme (4.7) for $q(t) = 0$, any $\lambda$ and $\alpha \in (0, 1]$. When analytical solution is not available, the rate of convergence $p = p_i(\Delta t, \alpha, \lambda)$ at nodes $t_i$, for fixed parameters $\alpha$, $\lambda$.
Figure 1. Numerical solution of Eq. (2.1) for different parameters $\alpha, q(t) = 0, b = 1, L = 1$ and $\lambda = -3$ (left/top), $\lambda = -10$ (right/top), $\lambda = -20$ (left/bottom) and $\lambda = -25$ (right/bottom)

Figure 2. Numerical solution of Eq. (2.1) for $\lambda \in \{-5, -7.5, -10, -12.5\}$, $b = 1, L = 1, q(t) = 0$ and $\alpha = 0.6$ (left-side), $\alpha = 0.8$ (right-side)

$\lambda$ and variable values of $\Delta t$, can be determined from the following formula
Figure 3. Numerical solution of Eq. (2.1) for different parameters $\lambda$, forms of function $q(t)$ and $\alpha = 1$ (left-side), $\alpha = 0.5$ (right-side). (see proposition [3])

$$R_i^{(\Delta t, \alpha, \lambda)} = f_i^{(\Delta t, \alpha, \lambda)} - f_i^{(2\Delta t, \alpha, \lambda)} \quad \text{(5.1)}$$

We thus have

$$p_i(\Delta t, \alpha, \lambda) = \log_2 \frac{f_i^{(\Delta t, \alpha, \lambda)} - f_i^{(2\Delta t, \alpha, \lambda)}}{f_i^{(\Delta t/2, \alpha, \lambda)} - f_i^{(\Delta t, \alpha, \lambda)}}. \quad \text{(5.2)}$$

We present numerical values at three selected nodes and rate of convergence for $\alpha \in \{0.3, 0.5, 0.7\}$ and $\lambda = -3$, $q(t) = 0$ in Table 1. In Table 2 the numerical values at three selected nodes and rates of convergence for $\alpha = 0.6$, $\lambda \in \{-5, -7.5, -10\}$ and $q(t) = 0$ are shown. The values from Tables 1 and 2 are also shown in plots - Figures 1 and 2, respectively.

Analysing the values in the above tables, one can observe that the rate of convergence $p$ is dependent on the fractional order $\alpha$ and does not depend on parameter $\lambda$. The rate of convergence $p$ is close to $1 + \alpha$. 
Table 1. Numerical values of $f$ at nodes $t_i, i \in \{n/4, n/2, 3n/4\}$ and rates of convergence $p$ for parameters $\lambda = -3$, $q(t) = 0, b = 1, L = 1$

| $\alpha$ | $\Delta t = 1/n$ | $t = 0.25$ | $t = 0.5$ | $t = 0.75$ |
|----------|-----------------|------------|------------|------------|
|          | $f_{n/4}$      | $p$        | $f_{n/2}$  | $p$        | $f_{3n/4}$ | $p$     |
| 0.3      | 1/256           | 1.53966755 | -7.0003222 | -6.5071282 | -5.6011461 | 1.20    |
|          | 1/512           | 1.58297521 | -6.1759387 | 1.12       | -5.6084351 | 1.12    |
|          | 1/1024          | 1.60340765 | -6.2570276 | 1.19       | -5.6011461 | 1.20    |
|          | 1/2048          | 1.61245514 | -6.2925534 | 1.24       | -5.6415701 | 1.24    |
|          | 1/4096          | 1.61632922 | -6.3074929 | 1.26       | -5.6587385 | 1.26    |
|          | 1/8192          | 1.61795763 | -6.3137773 | -5.6659021 | -5.6659021 | -5.6659021 |
| 0.5      | 1/256           | 3.73516937 | -4.7305234 | -3.5745003 | -3.5745003 | -3.5745003 |
|          | 1/512           | 3.72842313 | -4.7221146 | -3.5691099 | -3.5691099 | -3.5691099 |
|          | 1/1024          | 3.72583618 | -4.7189072 | -3.5670826 | -3.5670826 | -3.5670826 |
|          | 1/2048          | 3.72487655 | -4.7177105 | -3.5663310 | -3.5663310 | -3.5663310 |
|          | 1/4096          | 3.72451075 | -4.7172708 | -3.5660453 | -3.5660453 | -3.5660453 |
|          | 1/8192          | 3.72438081 | -4.7171110 | -3.5659415 | -3.5659415 | -3.5659415 |
| 0.7      | 1/256           | 0.94678077 | -1.4024198 | -1.4276444 | -1.4276444 | -1.4276444 |
|          | 1/512           | 0.94671534 | -1.4023161 | -1.4275495 | -1.4275495 | -1.4275495 |
|          | 1/1024          | 0.94669252 | -1.4022808 | -1.4275185 | -1.4275185 | -1.4275185 |
|          | 1/2048          | 0.94668482 | -1.4022690 | -1.4275082 | -1.4275082 | -1.4275082 |
|          | 1/4096          | 0.94668228 | -1.4022523 | -1.4275043 | -1.4275043 | -1.4275043 |
|          | 1/8192          | 0.94668146 | -1.4022400 | -1.4275038 | -1.4275038 | -1.4275038 |

6. Conclusions

In this paper the fractional Sturm–Liouville equation with derivatives of order $\alpha \in (0, 1]$ in the finite time interval $t \in [0, b]$ is considered. This equation was transformed using the composition rules for fractional integrals and derivatives to the integral form. Next the discrete form of the integral equation was presented as the system of linear algebraic equations. The obtained system of equations was solved numerically. The equation was solved for derivatives of different orders $\alpha$, different values of parameter $\lambda$ and different forms of function $q(t)$. The presented results showed the influence these values on the character (i.e. the occurrence of oscillations) of the solution. One can note that for $q(t) = 0$ the oscillations occur only for $\lambda < 0$. The number of oscillations increases when the value of order $\alpha$ decreases for fixed value of parameter $\lambda$. Similarly, for fixed order $\alpha$, the number of oscillations increases when the value of parameter $\lambda$ decreases. In order to ensure stability of the computation, the convergence study of the numerical scheme was conducted. The rate of convergence $p$ was estimated to be close to $1 + \alpha$. 
Table 2. Numerical values of $f$ at nodes $t_i$, $i \in \{n/4, n/2, 3n/4\}$ and rates of convergence $p$ for parameters $\alpha = 0.6$, $q(t) = 0$, $b = 1$, $L = 1$

| $\lambda$ | $\Delta t = 1/n$ | $f_{n/4}$ | $p$ | $f_{n/2}$ | $p$ | $f_{3n/4}$ | $p$ |
|---|---|---|---|---|---|---|---|
| -5 | 1/256 | -2.16736188 | - | -2.58746851 | - | -0.79882549 | - |
| | 1/512 | -2.16934247 | 1.46 | -2.59034981 | 1.44 | -0.80084057 | 1.45 |
| | 1/1024 | -2.17006290 | 1.50 | -2.59140830 | 1.49 | -0.80157662 | 1.49 |
| | 1/2048 | -2.17031750 | 1.53 | -2.59178487 | 1.52 | -0.80183747 | 1.52 |
| | 1/4096 | -2.17040578 | 1.55 | -2.59191604 | 1.54 | -0.80192808 | 1.54 |
| | 1/8192 | -2.17043598 | - | -2.59196106 | - | -0.80195912 | - |
| -7.5 | 1/256 | -1.51542247 | - | 0.06057150 | - | 1.83620282 | - |
| | 1/512 | -1.51337627 | 1.46 | 0.05880539 | 1.40 | 1.8322891 | 1.45 |
| | 1/1024 | -1.51263480 | 1.50 | 0.05815391 | 1.46 | 1.8307807 | 1.50 |
| | 1/2048 | -1.51237316 | 1.53 | 0.05792141 | 1.50 | 1.8302679 | 1.53 |
| | 1/4096 | -1.51228251 | 1.54 | 0.05784024 | 1.53 | 1.8300898 | 1.55 |
| | 1/8192 | -1.51225151 | - | 0.05781234 | - | 1.8300288 | - |
| -10 | 1/256 | 1.58290914 | - | -1.66780189 | - | -1.87108209 | - |
| | 1/512 | 1.58736775 | 1.42 | -1.67135428 | 1.46 | -1.8751726 | 1.42 |
| | 1/1024 | 1.58904541 | 1.48 | -1.67236756 | 1.51 | -1.88265567 | 1.48 |
| | 1/2048 | 1.58965039 | 1.51 | -1.67341544 | 1.53 | -1.88377958 | 1.52 |
| | 1/4096 | 1.58986289 | 1.54 | -1.67331027 | 1.56 | -1.88417251 | 1.54 |
| | 1/8192 | 1.58993623 | - | -1.67336695 | - | -1.88430769 | - |

Acknowledgements

This work was supported by the Czestochowa University of Technology Grant Number BS/MN 1-105-302/13/P.

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Received: June 18, 2013