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Quantum group perturbative formalism for affine Toda models

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Abstract. In the case of non-compact space we consider the quantum affine Toda systems in the operator approach. We introduce quantum analogues of the systems of equations as well as elements of quantum connections. For the solution of the quantum affine Toda systems we use the light-cone as well as Yang-Feldman formalisms of quantization. We then propose solutions to the quantum affine Toda equations based on the quantum-group approach.

1. Introduction
Two-dimensional integrable field theories always attract much attention [1–8]. Such systems appear to be a very important tool in understanding of main non-perturbative aspects of mathematical physics. The importance of two-dimensional integrable models consists also in their beautiful algebraic structure. Consideration of algebraic foundations of classical and quantum two-dimensional exactly solvable models leads also to interesting results either in the theory of classical (infinite dimensional) Lie algebras and groups and in the theory of q-deformed structures. For conformal and affine Toda models in classical and quantum regions [1–4,9,12–19] we consider the most important properties of two-dimensional exactly solvable models by applying methods based on deformed universal enveloping algebras of Lie algebras. Toda models have applications in many fields of applied mathematics and mathematical physics [2,20].

2. Affine Toda systems in the classical region
Let $\mathcal{M}$ be two-dimensional manifold $\mathbb{R}^2$ with standard coordinates $z^\pm = t \pm x$ and derivatives $\partial_{\pm}$. Let $\mathfrak{g}$ be a rank $r$ simple Lie algebra with simple roots $\alpha_i$, $i = 1, \ldots, r$. Let $\hat{\mathfrak{g}}$ be an affine Lie algebra corresponding to $\mathfrak{g}$ endowed with $\mathbb{Z}$-grading, and $\hat{\mathcal{G}}$ be corresponding infinite-dimensional Lie group. In the principle grading, the subspace $\hat{\mathfrak{g}}_0$ in the decomposition $\hat{\mathfrak{g}} = \bigoplus_{m \in \mathbb{Z}} \hat{\mathfrak{g}}_m$ is abelian. The affine Toda system fields $\phi = \sum_{i=1}^r h_i \phi_i$ (where $h_i$ denotes Cartan elements of the Lie algebra $\hat{\mathfrak{g}}$) satisfying equations

$$
\partial_+ \partial_- \phi + \frac{4\eta^2}{\beta} \left( \sum_{i=1}^r m_i \frac{\alpha_i}{\alpha_i^2} e^{\beta \alpha_i \phi} - \frac{\psi}{\psi^2} e^{-\beta \psi \phi} \right) = 0,
$$

(1)

where $\alpha_i$, $i = 1, \ldots, r$, are simple roots of $\mathfrak{g}$, and $\psi = -\alpha_0$, is the highest root

$$
\frac{\psi}{\psi^2} = \sum_{i=1}^r m_i \frac{\alpha_i}{\alpha_i^2},
$$

(2)
which determines \( m_i \). Here \( \eta \) denotes a real valued inverse length constant and \( \beta \) is a complex interaction constant. The equations (1) appear from the zero curvature condition on a Lax pair, related to the Lie algebra \( \hat{g} \) in the principal grading. Note that the equations of the affine Toda systems with pure imaginary \( \beta \) \((\beta \in \mathbb{C} \setminus \mathbb{R})\) possess solitonic solutions while when \( \beta \in \mathbb{C} \cap \mathbb{R} \) such solutions are missing. The coefficients in the equations (1) have been chosen in such a way that \( \phi = 0 \) gives a solution.

The formal general solution to the equations (1) was obtained in [9]

\[
e^{-\beta \lambda_j \cdot \phi} = (\lambda_j)_{j=0}^{\pm} \frac{\mu_0^{-1} \mu_+^{-1} \mu_- | \lambda_j \rangle}{\mu_0 | \lambda_j \rangle},
\]

where where \( \lambda_j \), \( 0 \leq j \leq r \) are \( r + 1 \) fundamental weights of \( \hat{g} \), and \( | \lambda_j \rangle \), the corresponding highest–weight states for the fundamental representations. These have as levels the integers \( m_i \) – labels on Dynkin diagram defined by (2). Here we use the irreducible highest-weight representations, which are generated by the action of arbitrary products of the negative step operators on a state called the highest-weight state. This state is annihilated by the step operators corresponding to positive roots. The representation is characterised by the action of the Cartan subalgebra on this weight, which must be an eigenvector under this action (for details see Appendix A in ref. [9]).

Holomorphic and anti-holomorphic \( \gamma_{\pm} (z^\pm) : \mathcal{M} \longrightarrow \hat{G}_0 \) and \( \mu_{\pm} (z^\pm) : \mathcal{M} \longrightarrow \hat{G}_\pm \) map points of the manifold \( \mathcal{M} \) to the subgroups \( \hat{G}_0 \) and \( \hat{G}_\pm \) of the infinite-dimensional group \( \hat{G} \). The general theory of infinite-dimensional Lie algebras can be found in [10,11].

The sine–Gordon equation

\[
\partial_+ \partial_- \phi + \frac{4 \eta^2}{\beta} \left( \frac{\alpha_1}{\alpha_1^2} e^{\beta \alpha_1 \cdot \phi} - \frac{\psi \dot{\psi}}{\psi^2} e^{-\beta \cdot \phi} \right) = 0,
\]

is a particular case of affine Toda systems connected with the Lie algebra \( \hat{g} = \mathfrak{sl}_2 \) in the principal and homogeneous gradings. In both gradings the general solution to the equation (4) has the same form (3) but it is assumed that the group elements are certain exponentials of the Lie algebra elements associated to the corresponding grading. The mappings \( \mu_{\pm} (z^\pm) \) satisfy the conditions

\[
\partial_{\pm} \mu_{\pm} = \mu_{\pm} \kappa_{\pm},
\]

where

\[
\kappa_{\pm} (z^\pm) = \pm \eta \sum_{i=0}^{r} \Psi_{\pm i}^0 x_{\pm i},
\]

\( (x_{\pm i}, i = 0, ..., r, \) are Chevalley generators of \( \hat{g} \),

\[
\Psi_{\pm i}^0 = \sqrt{m_i} e^{\beta \sum_{i=0}^{r} k_{ij} \phi_{\pm j}^0}.
\]

Here \( k_{ij} \) are elements of the Cartan matrix, and \( \phi_{\pm i}^0 \) are free fields satisfying \( \partial_{\pm} \partial_{\pm} \phi_{\pm i}^0 = 0 \), for \( \phi_{\pm i}^0 = \sum_{i=1}^{r} h_i \phi_{\pm i}^0 \). The elements (6) can be represented in the form

\[
\kappa_{\pm} (z^\pm) = \pm \eta \gamma_{\pm}^1 \hat{E}_{\pm 1} \gamma_{\pm}^{\pm 1},
\]

\[
\hat{E}_{\pm 1} = \sum_{i=0}^{r} \sqrt{m_i} x_{\pm i}.
\]

The mappings \( \gamma_{\pm} (z^\pm) : \mathcal{M} \longrightarrow \hat{G}_0 \) in (3) have the form

\[
\gamma_{\pm} = e^{-\beta \sum_{i=0}^{r} \phi_{\pm i}^0 h_i}.
\]
3. Quantization in the light cone formalism: affine Toda systems

In this paper we quantize the conformal and affine Toda systems in the case of non-compact space (plane) is the formalism of the light-cone quantization. It was applied in [24, 25] to the conformal Toda systems. In this section we apply the formalism of the light-cone quantization of affine Toda models in the particular case of the sine–Gordon equation. Let \( \Phi(z^+, z^-) \) be the Heisenberg fields satisfying the canonical the commutation relations. We construct the quantum sine–Gordon equation and quantum Lax pair on the basis of the affine Lie algebra \( \mathfrak{sl}_2 \). Let’s use the loop affinization [10] of the Lie algebra \( \mathfrak{sl}_2 \) with a parameter \( \lambda \). Let \( \Phi = \sqrt{2} \phi \) and \( \psi = \frac{\sqrt{2}}{2} \beta \phi \). Denote

\[
\rho = \frac{1}{2} \beta \phi.
\]

Then, following [23], one can introduce the quantum Lax operators. The zero curvature condition applied to the Lax pair results in the quantum sine–Gordon equation.

Introduce the operators

\[
\hat{\omega}_+ = \hbar \partial_+ \rho + \alpha (x_+ + \lambda x_-) + \zeta \Omega, \quad \hat{\omega}_- = \alpha \sigma \left( : e^{\beta \phi} : x_- + \lambda^{-1} : e^{-\beta \phi} : x_+ \right),
\]

(12)

where \( \hbar, x_+, x_- \) are generators of the Lie algebra \( \mathfrak{sl}_2 \), and \( \alpha, \zeta, \sigma \) are constants while \( \Omega = \frac{1}{4} h^2 \).

In (12) the normal ordering is defined in the light-cone \( I^\pm = \{ z^\pm = t/2, z^\mp \geq t/2 \} \), with respect to the associated Fock space oscillator operators

\[
a_j(p) = \int_{I^\pm} dz^- e^{i\vec{p} \cdot \vec{x}} \hat{\partial}_- \phi_i + \int_{I^-} dz^+ e^{i\vec{p} \cdot \vec{x}} \hat{\partial}_+ \phi_i,
\]

where \( a \hat{\partial}_\pm b = a(\partial_\pm b) - (\partial_\pm a)b \).

If we apply the zero curvature condition for the operators \( \hat{\omega}_\pm \)

\[
[\partial_+ + \hat{\omega}_+, \partial_- + \hat{\omega}_-] = 0,
\]

(13)

we obtain

\[
\begin{align*}
\alpha \sigma \left( \partial_+ \left( : e^{\beta \phi} : \right) x_- + \lambda^{-1} \partial_+ \left( : e^{-\beta \phi} : \right) x_+ \right) - \partial_+ \partial_- \rho h \\
+ \alpha \sigma \left( \partial_+ \partial_- \rho h, : e^{\beta \phi} : x_+ \right) + \alpha \sigma \lambda^{-1} \left( \partial_+ \partial_- \rho h, : e^{-\beta \phi} : x_+ \right) \\
+ \alpha^2 \sigma \left( : e^{\beta \phi} : - : e^{-\beta \phi} : \right) h - \alpha \sigma \zeta : e^{\beta \phi} : \{ h, x_- \} \\
+ \alpha \sigma \lambda^{-1} \zeta : e^{-\beta \phi} : \{ h, x_+ \} = 0.
\end{align*}
\]

(14)

The last equality is equivalent to the following two:

\[
\begin{align*}
\alpha \sigma \partial_+ \left( : e^{\beta \phi} : \right) x_- - \partial_+ \partial_- \rho h + \alpha \sigma \left( \partial_+ \partial_- \rho h, : e^{\beta \phi} : x_- \right) \\
+ \alpha^2 \sigma \left( : e^{\beta \phi} : - : e^{-\beta \phi} : \right) h - \alpha \sigma \zeta : e^{\beta \phi} : \{ h, x_- \} = 0,
\end{align*}
\]

(15)

\[
\begin{align*}
\partial_+ \left( : e^{-\beta \phi} : \right) x_+ + \left[ \partial_+ \partial_- \rho h, : e^{\beta \phi} : x_+ \right] + \zeta : e^{-\beta \phi} : \{ h, x_+ \} = 0.
\end{align*}
\]

(15)

Here we use the property of \( \Omega \):

\[
\Omega, x_\pm = \pm \{ h, x_\pm \},
\]

(16)

and the identity

\[
[a A, b B] = \frac{1}{2} \left( [a, b] \{ A, B \} + \{ a, b \} [A, B] \right),
\]

(17)
in which the operators $A$, $B$ commute with $a$, $b$ (in our case $a$, $b$ are operators, depending on the fields and $A$, $B$ are Lie algebra elements of $\mathfrak{sl}_2$). Using the commutation relations it follows that the commutator $\partial_+ \Phi_i$ and $\Phi_i$ commute at coinciding points with $\Phi_i$, and thus

$$\partial_+ e^{\beta \Phi_i} = \frac{1}{2} \left\{ \partial_+ \beta \Phi_i, e^{\beta \Phi_i} \right\}.$$  \hfill (18)

The same is true for the normal ordered exponent

$$\partial_+ : e^{\beta \Phi_i} := \frac{1}{2} \left\{ \partial_+ \beta \Phi_i, : e^{\beta \Phi_i} : \right\}.$$  \hfill (19)

From (19) it follows that

$$\partial_+ : e^{\pm \beta \phi} := \pm \left\{ \partial_+ \rho, : e^{\pm \beta \phi} : \right\}.$$  \hfill (20)

Besides this, using the commutation relations we conclude that

$$\left[ \partial_+ \psi(z_+, z^-), : e^{\pm \beta \Phi_j(z^+, z^-)} : \right] = \mp \frac{1}{2} i \hbar \beta^2 \delta_{ij} \delta(z^+ - \bar{z}^+) : e^{\pm \beta \Phi_j(z^+, z^-)} :,$$  \hfill (21)

from which it follows that

$$\left[ \partial_+ \rho(z_+, z^-), : e^{\pm \beta \phi(z^+, z^-)} : \right] = \mp \frac{1}{2} i \hbar \beta^2 \delta(z^+ - \bar{z}^+) : e^{\pm \beta \phi(z^+, z^-)} :.$$  \hfill (22)

The comparison of summands in (15), (19) and the formulae (22) determines the choice of the constant $\varsigma$ in (12)

$$\varsigma = - \frac{i \hbar \beta^2}{4} \delta(0),$$  \hfill (23)

(the infinite constant $\varsigma$ vanishes in the final expressions). From (15) we obtain the quantum sine–Gordon equation.

$$\partial_+ \partial_- \phi = \frac{2 \alpha^2 \sigma}{\beta} \left( : e^{\beta \phi} : - : e^{-\beta \phi} : \right).$$  \hfill (24)

Since $\alpha$ is an arbitrary constant then it is possible to let $\alpha^2 = \frac{2 \eta}{\beta}$ so that coefficients in the quantum equation (24) coincide with the coefficients of the sine–Gordon equation in the classical region (4).

4. **Group-theoretical solution**

In this section we construct solutions to the equation (24). Let $g$ be $\tilde{G}$ group element. The zero curvature condition (13) implies the gradient form

$$\partial_+ g^{-1} = g^{-1} \partial_+, \quad \partial_- g = - \partial_- g.$$  \hfill (25)

Therefore

$$\partial_-(\lambda_i) \ = \ - \alpha \sigma \langle \lambda_i \rangle \left( : e^{\beta \phi} : f_1 + : e^{-\beta \phi} : f_0 \right) g \ = \ 0,$$  \hfill (26)

where $f_1 = x_-$, $f_0 = \lambda^{-1} x_+$, are the lowering operators and $\langle \lambda_i \rangle$, $i = 0, 1$, are the highest weight vectors of the $i$-th fundamental weight of the Lie algebra $\mathfrak{sl}_2$ representation. Note that

$$\partial_+ \left( g^{-1} : e^{-\rho \phi} : \right) = \left( \partial_+ g^{-1} \right) : e^{-\rho \phi} : + g^{-1} \partial_+ \left( : e^{-\rho \phi} : \right) \ = \ g^{-1} \partial_+ \omega_+ : e^{-\rho \phi} : + g^{-1} \left[ \partial_+ \rho h_i : e^{-\rho \phi} : \right] \ = \ g^{-1} (\omega_+ - \partial_+ \rho h) : e^{-\rho \phi} : + \frac{1}{2} g^{-1} \left[ \partial_+ \rho h_i : e^{-\rho \phi} : \right] \ = \ g^{-1} \omega_- (\partial_+ \rho h - \varsigma \Omega) : e^{-\rho \phi} : = g^{-1} (\alpha (e_0 + e_1)) : e^{-\rho \phi} :.$$  \hfill (27)
where \( e_1 = x_+ \), \( e_0 = \lambda x_- \) are the raising operators of the Lie algebra \( \widehat{sl}_2 \). Therefore

\[
\partial_+ \left( g^{-1} : e^{-\rho} : |\lambda_i\rangle \right) = g^{-1} \alpha (e_0 + e_1) : e^{-\rho} : |\lambda_i\rangle = 0, \tag{28}
\]

\( i = 0, 1 \). We conclude that the right hand side of the expression

\[
: e^{-\rho(z^+, z^-)} : = \frac{\langle \lambda_1 | g(z^+, z^-) g^{-1}(z^+, z^-) : e^{-\rho(z^+, z^-)} : |\lambda_1\rangle}{\langle \lambda_0 | g(z^+, z^-) g^{-1}(z^+, z^-) |\lambda_0\rangle}, \tag{29}
\]

does not depend on \( \tilde{z}^\pm \) and equals to \( : e^{-\rho(z^+, z^-)} : \) when \( \tilde{z}^\pm = z^\pm \). In the classical limit, the quantum expression (29) returns to the general solution to the affine Toda system of equations (3). Thus, at the point \( P = (z^+, z^-) \) inside the light-cone and two points on the branches \( I^\pm \) we have an expression for the Heisenberg operator.

We compute then the group elements in (29). Since the right hand side of the expression does not depend on \( \tilde{z}^\pm \), with no loss of generality, one can set \( \tilde{z}^+ = \tilde{z}^- = 0 \). From (25) it follows that

\[
g^{-1}(0, z^-) = g^{-1}(0, 0) \tilde{T} \exp \int_{(0, 0)}^{(0, z^-)} \phi dz^-.
\]

Thus, on the basis of (27) we obtain

\[
\partial_+ \left( : e^{\rho} : g \right) = -\alpha \left( : e^{\rho} : g \right) (x_+ + \lambda x_-) g. \tag{31}
\]

From the commutation relations of the Lie algebra \( \widehat{sl}_2 \) we obtain the identity:

\[
e^{\rho} x_\pm = x_\pm e^{\pm 2 \rho} e^{\rho}.
\]

Nevertheless, exponents of the operator \( \rho \) in (32) are not normal ordered and do not have a finite action on the Fock space \( \mathcal{F}(I^\pm) \). Let us replace the operators of the fields in (32) by the regularized expressions and finally let us take \( p_i \to \infty \), \( i = 1, 2 \). Using the commutation relation we obtain the following identities

\[
e^{\pm \rho} = e^{\pm \rho} : e^{\frac{i}{2} \hbar \beta^2 \Delta_{\text{reg}}(0)} \Omega, \tag{33}
\]

\[
e^{3 \phi} = e^{3 \phi} : e^{\hbar \beta^2 \Delta_{\text{reg}}(0)}, \tag{34}
\]

Then we have

\[
: e^{\rho} : x_+ = u^{-1} x_+ u \left( : e^{2 \rho} : \right) e^{\frac{i}{2} \hbar \beta^2 \Delta_{\text{reg}}(0)},
\]

\[
\lambda : e^{\rho} : \lambda x_- = u^{-1} \lambda x_- u \left( : e^{-2 \rho} : \right) e^{\frac{i}{2} \hbar \beta^2 \Delta_{\text{reg}}(0)}, \tag{35}
\]

where

\[
u = e^{\frac{i}{2} \hbar \beta^2 \Delta_{\text{reg}}(0)} \Omega. \tag{36}
\]

Therefore

\[
\partial_+ \left( u : e^{\rho} : g \right) = \tilde{B}_+ \cdot \left( u : e^{\rho} : g \right), \tag{37}
\]

where

\[
\tilde{B}_+ = -\alpha \left( x_+ : e^{2 \rho} : + \lambda x_- : e^{-2 \rho} : \right) e^{\hbar \beta^2 \Delta_{\text{reg}}(0)}. \tag{38}
\]
The equation (37) can be integrated to obtain

\[ u : e^{\rho h} : g(z^+, 0) = \left( T \exp \int_{(0,0)}^{(z^+,0)} \hat{B}_+ dz_1^+ \right) \cdot u : e^{\rho(0,0)h} : g(0,0). \]  

(39)

As a result, we are able to rewrite (29) as

\[ : e^{-\rho(z^+, z^-)} : = \frac{\langle \lambda_1 | G : e^{-\rho(0,z^-)h} : | \lambda_1 \rangle}{\langle \lambda_0 | G | \lambda_0 \rangle}, \]  

(40)

where

\[ G = : e^{-\rho(z^+, 0)h} : u^{-1} \left( T \exp \int_{(0,0)}^{(z^+,0)} \hat{B}_+ dz_1^+ \right) u : e^{\rho(0,0)h} : \left( T \exp \int_{(0,0)}^{(0,z^-)} \hat{\omega}_- dz_1^- \right). \]  

(41)

Note that this expression formally coincides (when the normal ordering is not taken into consideration) with the expression for the general solution to the classical affine Toda system obtained according to the Leznov–Saveliev method [1, 9].

5. Yang–Feldman approach

The Yang–Feldman perturbative method is a way to compute field operators in a quantum theory in the expansion over free fields [1, 30]. In this section we recall, following [21, 22] results of computation of orders in the expansion of the Heisenberg operators satisfying the quantum conformal and affine Toda equations over field operators in absence of interactions.

Let us start with the quantum conformal Toda system of equations. Let \( \varphi \), \( \alpha = 1, ..., r \), be field operators satisfying the quantum analogues of the conformal Toda system. The asymptotic fields \( \varphi_{\pm \alpha}^0 \), \( \alpha = 1, ..., r \), satisfy the canonical commutation relations

\[ [\varphi_{\pm \alpha}^0(z^\pm), \varphi_{\pm \beta}^0(\bar{z}^\pm)] = -\frac{i\hbar}{4}(k_{\alpha \beta})^{-1}w_\beta^{-1}\epsilon(z^\pm - \bar{z}^\pm), \quad [\varphi_{\pm \alpha}^0(z^\pm), \varphi_{\mp \beta}^0(\bar{z}^\mp)] = 0, \]  

(42)

(here \( \epsilon \) is the standard sign function). Then let

\[ u_{\pm \alpha}^0 = \sum_{\beta=1}^{r} k_{\alpha \beta} \varphi_{\pm \beta}^0, \quad \varphi_{\pm \alpha}^0 = \sum_{\beta=1}^{r} (k_{\alpha \beta})^{-1} u_{\pm \beta}^0, \]  

(43)

where \( k_{\alpha \beta} \) is the Cartan matrix element corresponding to the Lie algebra \( g \), and \( w_\alpha \) is a symmetrizer (a diagonal matrix) satisfying the relation

\[ w_\alpha k_{\alpha \beta} = w_\beta k_{\beta \alpha}, \]  

(44)

\( \alpha, \beta = 1, ..., r \). Then, correspondingly

\[ [\psi_{\pm \alpha}^0(z^\pm), \psi_{\pm \beta}^0(\bar{z}^\pm)] = -\frac{i\hbar}{4}(\hat{k}_{\alpha \beta})^{-1}w_\beta^{-1}\epsilon(z^\pm - \bar{z}^\pm), \]  

\[ [\psi_{\pm \alpha}^0(z^\pm), \varphi_{\mp \beta}^0(\bar{z}^\mp)] = -\frac{i\hbar}{4 w_\beta} \delta_{\alpha \beta} \epsilon(z^\pm - \bar{z}^\pm), \quad \hat{k}_{\alpha \beta} = k_{\alpha \beta} w_\beta^{-1}. \]  

(45)
In the Yang–Feldman formalism [1,21], the $n$-th order of the exponential Heisenberg operator is given by the expression

$$
(e^{-\varphi_\alpha})_{(n)} = \frac{1}{(i\hbar)^n} \int_{-\infty}^{+\infty} dz_1...dz_n \theta(t - t_1)...\theta(t_{n-1} - t_n) \left[ e^{-\varphi_0^0}, V_{11}, ..., V_{nn} \right], \tag{46}
$$

where $\theta(z - z) = \theta(z^+ - z^+)$ and $\theta(z^- - z^-)$ is the ordinary step-function and

$$
V_{ij} = V(z_i^+, z_j^-) = \sum_{\alpha=1}^r 2w_\alpha e^{\psi_0^0(z_i^+)}. e^{\psi_0^0(z_j^-)}. \tag{47}
$$

In (46) we use the notation

$$[A, B, ..., C] = [[[A, B],], ..., C]. \tag{48}$$

The expression (46) can be rewritten in the form

$$
(e^{-\varphi_\alpha})_{(n)} = \frac{1}{(i\hbar)^n} \int_{-\infty}^{+\infty} dz_1...dz_n \theta(z - z_1)...\theta(z_{n-1} - z_n) \sum_{P(k_1,...,k_n)} \left[ e^{-\varphi_0^0}, V_{1k_1}, ..., V_{nk_n} \right], \tag{49}
$$

where $P(k_1,...,k_n)$ denotes a permutation of indexes $(k_1,...,k_n)$. The operator of the field exponent in the first thee orders of the Yang–Feldman procedure (49) has the form

$$
\begin{align*}
(e^{-\varphi_\alpha})_{(0)} &= e^{-\varphi_0^0}, \\
(e^{-\varphi_\alpha})_{(1)} &= e^{-\varphi_0^0} \frac{2w_\alpha}{i\hbar} \left[ 1 - e^{-\frac{ik}{x_{\alpha\gamma}}} \right] \Phi^+_{\alpha\gamma} \Phi^-_{\alpha\gamma}, \\
(e^{-\varphi_\alpha})_{(2)} &= e^{-\varphi_0^0} \frac{2w_\alpha}{i\hbar} \left[ 1 - e^{-\frac{ik}{x_{\alpha\gamma}}} \right] \sum_{\gamma \neq \alpha} \frac{2w_\alpha}{i\hbar} \left[ 1 - e^{-\frac{ik}{x_{\gamma\alpha}}} \right] \Phi^+_{\alpha\gamma} \Phi^-_{\gamma\alpha}, \tag{50}
\end{align*}
$$

where $\alpha = 1, ..., r$, and

$$
\Phi^\pm_{\alpha_1, ..., \alpha_m}(z^+) = \int_{-\infty}^{z^+} dz_1^+ e^{\psi_0^0(z_1^+)} \int_{-\infty}^{z_2^+} dz_2^+ e^{\psi_0^0(z_2^+)} ... \int_{-\infty}^{z_{m-1}^+} dz_{m-1}^+ e^{\psi_0^0(z_{m-1}^+)}. \tag{51}
$$

Thus,

$$
\Phi^\pm_{\alpha}(z^+) = \int_{-\infty}^{z^+} dz_1^+ e^{\psi_0^0(z_1^+)}, \tag{52}
$$

$$
\Phi^\pm_{\alpha_1,\beta}(z^+) = \int_{-\infty}^{z^+} dz_1^+ e^{\psi_0^0(z_1^+)} \int_{-\infty}^{z_2^+} dz_2^+ e^{\psi_0^0(z_2^+)}. \tag{53}
$$

Note that in the quantum conformal Toda system case the series (49) is finite, i.e., terminates after certain order. The expression for $n$-th order of expression for the solution (49) can be found in [21].

The Yang–Feldman procedure is also applicable to the affine Toda system of equations. In that case it is convenient to use the loop realization of the affine Lie algebra $\mathfrak{g}$. As in the
case of conformal Toda system, the Heisenberg operators of the field $\varphi_\alpha$ satisfy the canonical commutation relations (42) as well as to the quantum equations, and coincide in the form with (1). The classical fields in (1) are formally substituted by the Heisenberg operators. Nevertheless, instead of the perturbation operator (47), one has to use another operator. For example, in a partial case the solution of the system of the affine Toda equations, the sine–Gordon model, we have

$$V_{ij} \equiv V(z^+_i, z^-_j) = 2w \left( e^{\psi^0_+ (z^+_i)} e^{\psi^0_- (z^-_j)} - e^{-\psi^0_+ (z^+_i)} e^{-\psi^0_- (z^-_j)} \right),$$  \hspace{1cm} (54)$$

(here $w$ is a constant) which results in

$$(e^{-\varphi})(0) = e^{-\varphi^0},$$  \hspace{1cm} (55)$$

$$(e^{-\varphi})(1) = e^{-\varphi^0} \frac{2w}{i\hbar} \left( \left(1 - e^{-i\frac{\eta\varphi}{\kappa}} \right) \Phi_+^+ \Phi^-_+ - \left(1 - e^{i\frac{\eta\varphi}{\kappa}} \right) \Phi^-_+ \Phi^-_+ \right),$$  \hspace{1cm} (56)$$

where we introduce

$$\Phi^+_\pm (z^+) = \int_{-\infty}^{z^+} dz^+ e^{\pm \psi^0(z^+_i)}, \quad \Phi^-_\pm (z^-) = \int_{-\infty}^{z^-} dz^- e^{\pm \psi^0(z^-_j)},$$  \hspace{1cm} (57)$$

and $\psi^0$ is defined in the expression (43). Then the $n$-the order of the exponent of the field operator is given by the expression (46) with the interaction operator (54). Note that in the case of the affine Toda system we obtain an infinite series of the expansion of the Heisenberg operators. Nevertheless, as in the case of conformal Toda systems, one can find the general formula for the exponent of the Heisenberg operator at any order.

6. Quantum group solutions for Toda systems

Originally, a relation between the quantization of the conformal Toda system of equations and quantum groups was found in [22]. Indeed, consider the Toda fields $\varphi_i$, $i = 1, ..., r$, as Heisenberg operators satisfying a certain quantum analogue of the conformal Toda system of equations in which the classical fields are replaced by field operators. Let us apply, as in [21] (see also [1]), the perturbative Yang–Feldman procedure (see the previous section). We arrive at the exact expressions for the exponents of the Heisenberg operators $\varphi_i$ in the form of finite series over free field operators $\varphi^0_i$, satisfying the canonical commutation relations. In this section we show that the formal expressions constructed on the basis of classical solutions in the Leznov–Saveliev approach, though containing quantum group generators, coincide with the perturbative solutions for Toda systems of equations obtained in frames of the Yang–Feldman procedure.

Let us take the general solution to the conformal Toda system of equations and formally replace the group elements in the right and side by elements of the quantum universal enveloping algebra $U_q(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$, on the basis of which the system of equations was constructed. Let us replace also the highest weight vectors $|\lambda_i\rangle$ of the fundamental highest weight representation by the highest weight vectors $|\lambda^0_i\rangle_q$ of the fundamental representation of the quantum universal enveloping algebra $U_q(\mathfrak{g})$ of $\mathfrak{g}$, and the mappings $q\kappa_{\pm}$ by the quantum mappings $q\kappa_{\pm} : \mathcal{M} \longrightarrow U_q(\mathfrak{g})$

$$q\kappa_{\pm}(z^\pm) = \pm \eta \sum_{i=1}^{r} \Psi^0_{\pm i} q x_{\pm i},$$  \hspace{1cm} (58)$$

$$\Psi^0_{\pm i} = e^{i \sum_{j=1}^{r} k_{ij} \varphi^0_{\pm j}},$$  \hspace{1cm} (59)$$
where $U_q(\mathfrak{g})$.

The generators of the quantum universal enveloping algebra satisfy the deformed commutation relations in the Jimbo–Drinfeld form [26, 27]:

$$
[q h_i , q h_j ] = 0, \quad [q h_i , q x_{\pm j} ] = \pm 2 k_{ji} q x_{\pm j},
$$

$$
[q x_{\pm i} , q x_{\pm j} ] = \delta_{ij} \frac{q^{h_i} - q^{-h_i}}{q_i - q_i^{-1}},
$$

where $q_i = e^{i \hbar}$, $\hbar$ is the Planck constant, and $d_i$ = adjoint prime integers such that $dk$ is symmetric matrix. Then one obtains [22] the following formal expression in the Leznov–Saveliev form

$$
e^{-\check{\varphi}} = q \langle \lambda_1^q \rangle q M_+^{-1} q M_- \langle \lambda_0^q \rangle q,
$$

(63)

where $q M_\pm = e^{-\sum_{i=1}^r q h_i \varphi^\pm_{\pm i}}$, $q \mu_\pm$ are quantum group $U_q(\mathfrak{g})$ elements, and $h_i$, $i = 1, \ldots, r$, Cartan elements of $U_q(\mathfrak{g})$. Here $|\lambda_j^q\rangle$, $1 \leq j \leq r$ are elements of the finite dimensional highest vector representation [22] for $U_q(\mathfrak{g})$ with the conditions $q h_i |\lambda_j^q\rangle = \delta_{ij} |\lambda_j^q\rangle$, $q x_{\pm i} |\lambda_j^q\rangle = 0$, for some $l_j$. The remaining basis elements are determined by the action of the lowering operators $x_{-i}$ on the highest vectors $|\lambda_j^q\rangle$ [22].

The group elements $q \mu_\pm$ satisfy the condition

$$
\partial_\pm q \mu_\pm = q \mu_\pm q \kappa_\pm.
$$

(64)

At the same time, in the classical case, due to the properties of the fundamental highest weight representations of $U_q(\mathfrak{g})$, the number of summands in the series of the expansion of (63) is finite and exactly equals to the dimension of the $i$-th fundamental representation of $U_q(\mathfrak{g})$. It is easy to find, for example, the first three orders in the expansion (63):

$$
(e^{-\varphi^i})_{(0)} = q \langle \lambda_1^q \rangle e^{-\sum_{j=1}^r q h_j (\varphi^0_{+j} + \varphi^0_{-j})} |\lambda_0^q\rangle q = e^{-\check{\varphi}^0},
$$

(65)

$$
(e^{-\varphi^i})_{(1)} = -e^{-\check{\varphi}^0} \sum_{\theta = 1}^r \Phi^+_{\theta} \Phi^-_{\theta} \cdot q \langle \lambda_1^q \rangle q x_{+\theta} q x_{-p} |\lambda_0^q\rangle q e^{-\varphi^0_{-i}},
$$

(66)

$$
(e^{-\varphi^i})_{(2)} = -e^{-\check{\varphi}^0} \sum_{\theta, \tau = 1}^r q \langle \lambda_1^q \rangle q x_{+\theta} q x_{+\tau} q x_{-p} q x_{-q} |\lambda_1^q\rangle q \Phi^+_{\theta \tau} \Phi^-_{pq} e^{-\varphi^0_{-i}},
$$

(67)

Here

$$
\Phi^-_{pq} = \int_{\mathbb{R}} dz_1^{-} \int_{\mathbb{R}} dz_2^{-} \theta(z_2^{-} - z_1^{-}) e^{\gamma \psi \bar{\psi}} e^{\gamma \bar{\psi} \psi} \ldots = e^{-\frac{\hbar}{4} \bar{\Phi} \Phi},
$$

(68)

$$
i = 1, \ldots, r, \text{ and } \bar{\Phi}_{pq}, w_i \text{ were determined in (44) and (45). The operators } \Phi^+_{\theta}, \Phi^-_{\theta} \text{ are the same as in (52), (53). The expression for the } n \text{-th order in the expansion (63) can be found in [22]. One can see that the first three orders of (65) – (67) coincide with (50). The same is true}.$$
for any order. Thus as it was shown in [22], the perturbative expressions found in the Yang–Feldman approach for the quantum conformal Toda systems of equations coincides with the quantum group expressions containing generators of the quantum universal enveloping algebra $U_q(\mathfrak{g})$ instead of ordinary group elements.

By analogy with the conformal Toda system case, we construct in this paper quantum group solutions in the approach similar to [22] for affine Toda systems. In the general solution formula (3) for the classical system of equations for the affine Toda one can replace the group elements and vectors in the $\mathfrak{g}$ Lie algebra representations by the elements and vectors of the quantum group $U_q(\mathfrak{g})$ representations. Then we obtain

$$e^{-\varphi_i} = \frac{q^\langle \lambda^q_i | q^M^{-1}_+ \cdot q^M_- | \lambda^q_i \rangle_q}{q^\langle \lambda^q_i | q^M_- \cdot q^M_+ | \lambda^q_i \rangle_q}, \quad (69)$$

where $q^M_\pm = e^{-\sum_{i=0}^r q^{h_i} \mu_i^\pm} \cdot q^\mu_\pm$, $i = 1, ..., r$. When in the finite dimensional case the series representing the solutions (63) terminates and it is easy to compute it order by order, in the case of the affine Toda system of equations the nominator and denominator as well as the solution itself (69) are infinite series. The summands of series in the nominator and denominator are corresponding orders of the action of generators of the quantum universal enveloping algebra of corresponding Lie algebra on the highest vectors of $i$-th and 0-th fundamental representations. Nevertheless, one can consider the ratio of the numerator and denominator series and use a formula for the $n$-th order for such a series [31]. Then one can find the general formula for $(e^{-\varphi_i})^{(n)}$. It turns out that in the case of the affine Toda system of equations the expressions, obtained on the basis of the Yang–Feldman procedure coincide with the quantum group expressions.

7. Comparison of the approaches

We have to mention that in the quantum case the analogy with well-developed group-theoretical Leznov–Saveliev method [1] is not completely fulfilled. First of all it remains unclear how to introduce Lax pairs for the quantum analogues of the classical exactly solvable systems containing elements of the quantum universal enveloping algebra $U_q(\mathfrak{g})$. As mentioned before the quantum flat connection elements (12) introduced in [23, 24] contain both Heisenberg field operators satisfying the quantum equations (24) and elements of affine Lie algebras. At the same time, in corresponding solutions we used group elements for ordinary groups parameterized by Heisenberg field operators. This is possibly the reason why we had to introduce an infinite constant $\zeta$ in the Lax pair (let us mention that $\zeta$ is absent in the final expressions for solutions). Nevertheless, as it was shown in [25], even if we use such pairs of Lax operators, a quantum group structure of the theory shows up. Moreover, an algebraic origin of the summands containing $\Omega$ remains unclear (see (12)). It is possible that the Lax operator pair (12) appears as a result of a gauge transformation of a symmetric version of Lax operator pairs $A_\pm = \sum_\alpha (u_{\pm\alpha} h_\alpha + f_{\pm\alpha} x_{\pm\alpha})$

(here $u_{\pm\alpha}$, $f_{\pm\alpha}$ are operators). Nevertheless, such a transformation is not known. We hope that in the future one could construct on the base of elements of the quantum universal enveloping algebras suitable elements of quantum connections not containing auxiliary infinite constants. Then the zero curvature condition would lead to the quantum conformal and affine Toda equations.

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