Decomposing planar graphs into graphs with degree restrictions

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Abstract

Given a graph G, a decomposition of G is a partition of its edges. A graph is \((d, h)\)-decomposable if its edge set can be partitioned into a \(d\)-degenerate graph and a graph with maximum degree at most \(h\). For \(d \leq 4\), we are interested in the minimum integer \(h_d\) such that every planar graph is \((d, h_d)\)-decomposable. It was known that \(h_1 \leq 4\), \(h_2 \leq 8\), and \(h_1 = \infty\). This paper proves that \(h_4 = 1\), \(h_3 = 2\), and \(4 \leq h_2 \leq 6\).

Key words

graph decomposition, planar graph

1 | INTRODUCTION

We consider only finite simple graphs. Given a graph \(G\), a decomposition of \(G\) is a collection of spanning subgraphs \(H_1, \ldots, H_t\) such that each edge of \(G\) is an edge of \(H_i\) for exactly one \(i \in \{1, \ldots, t\}\). In other words, \(E(H_1), \ldots, E(H_t)\) is a partition of \(E(G)\).

A graph is \(d\)-degenerate if every subgraph has a vertex of degree at most \(d\). Given nonnegative integers \(d\) and \(h\), a \((d, h)\)-decom and an acyclic orientation of a graph \(G\) is a decomposition \(H_1, H_2\) of \(G\) such that \(H_1\) is \(d\)-degenerate and \(H_2\) has maximum degree at most \(h\). We say \(G\) is \((d, h)\)-decomposable if there exists a \((d, h)\)-decomposition of \(G\). This paper studies \((d, h)\)-decomposability of planar graphs.

Decomposing a graph into subgraphs with a simpler structure is a fundamental problem in graph theory. The classical Nash-Williams Arboricity Theorem [14] (see also [13,18]) gives a necessary and sufficient condition under which a graph can be decomposed into \(k\) forests. The
Nine-Dragon Tree Conjecture [12], confirmed by Jiang and Yang [10], gives a sharp density condition under which a graph can be decomposed into $k$ forests with one of them having bounded maximum degree. The page number of a graph $G$ is the minimum $k$ such that $G$ can be decomposed into $k$ planar graphs. A proper edge-coloring of $G$ is a decomposition of $G$ into matchings. The problem of decomposing a graph into star forests, linear forests, graphs of bounded maximum degree, and so forth is studied extensively in the literature [1,5,21,20,17].

The concept of a $(d, h)$-decomposition has not been explicitly defined in the literature (as far as we know). However, such decompositions arise naturally in the study of many problems. For example, it was proved in [8] that if a graph $G$ is $(1, h)$-decomposable, then $G$ has game chromatic number $\chi_g(G)$ at most $h + 4$. It was shown in [8] that outerplanar graphs are $(1, 3)$-decomposable, and hence have game chromatic number at most 7. A result in [22] implies that planar graphs are $(2, 8)$-decomposable, and such a decomposition (with additional structural constraints) was used to show that planar graphs have game chromatic number at most 19 (the current best-known upper bound for the game chromatic number of planar graphs is 17 [23]). A similar decomposition was used to derive an upper bound on the game chromatic number of a graph $G$ embeddable on an orientable surface of genus $g \geq 1$, namely, $\chi_g(G) \leq \frac{1}{2}(3\sqrt{1 + 48g} + 23)$. It is known that if $G$ decomposes into $H_1, H_2, \ldots, H_k$, then the spectral radius of $G$ is bounded by the summation of the spectral radius of each $H_i$, that is, $\rho(G) \leq \rho(H_1) + \rho(H_2) + \cdots + \rho(H_k)$ [19,3]. The current best-known upper bounds on the spectral radius of a planar graph (namely, $\rho(G) \leq \sqrt{8\Delta - 16} + 3.47$) was obtained by Dvořák and Mohar [3] by applying the result that every planar graph $G$ decomposes into $H_1, H_2$ where $H_1$ has an orientation of maximum out-degree 2 and $H_2$ has maximum degree at most 4.

In this paper, we are interested in the minimum integer $h_d$ such that every planar graph is $(d, h_d)$-decomposable. Since every planar graph is 5-degenerate, the problem is interesting only for $d \leq 4$. As observed above, the result in [22] implies that every planar graph is $(2, 8)$-decomposable, that is, $h_2 \leq 8$. A result in [6] implies that every planar graph is $(3, 4)$-decomposable, that is, $h_3 \leq 4$. In this paper, we prove the following results:

Theorem 1.1. Every planar graph is $(4, 1)$-decomposable.

Theorem 1.2. Every planar graph is $(3, 2)$-decomposable.

Since a planar graph of minimum degree 5 is neither $(3, 1)$-decomposable nor $(4, 0)$-decomposable, we conclude that $h_4 = 1$ and $h_3 = 2$. Determining the exact value of $h_2$ turns out to be more difficult. We were able to improve the best-known bounds in the literature to obtain the following two results.

Theorem 1.3. Every planar graph is $(2, 6)$-decomposable.

Proposition 1.4. Not all planar graphs are $(2, 3)$-decomposable.

As a consequence of Theorem 1.3 and Proposition 1.4, we have $4 \leq h_2 \leq 6$. As determining $h_2$ is an intriguing open problem, any improvement on the aforementioned bounds would be exciting.

Note that for every integer $h$, the complete bipartite graph with two vertices in one part and $2h + 2$ vertices in the other part is not $(1, h)$-decomposable. Thus $h_1 = \infty$. 
A graph \( G \) is \( h \)-defective \( k \)-choosable if for every \( k \)-list assignment \( L \) of \( G \), there is an \( L \)-coloring of \( G \) in which each vertex \( v \) has at most \( h \) neighbors colored the same color as \( v \). The concept of \( h \)-defective \( k \)-paintability is an online version of \( h \)-defective \( k \)-choosability, defined through a two-person game (see [9] for its definition), and \( h \)-defective \( k \)-DP-colourability is a generalization of \( h \)-defective \( k \)-choosability (see [11] for its definition). We remark that \((d, h)\)-decomposable graphs are easily seen to be \( h \)-defective \((d + 1)\)-choosable, \( h \)-defective \((d + 1)\)-paintable, as well as \( h \)-defective \((d + 1)\)-DP-colorable. On the other hand, \((d, h)\)-decomposable seems to be considerably stronger than \( h \)-defective \((d + 1)\)-choosability and \( h \)-defective \((d + 1)\)-paintability. Cushing and Kierstead [2] proved that every planar graph is \( 1 \)-defective 4-choosable. This result was strengthened recently by Grytczuk and Zhu [7] who proved that every planar graph is \( 1 \)-defective 4-paintable. As observed above, planar graphs with minimum degree 5 are not \( 3 \)\,(1)-decomposable. Eaton and Hull [4], and independently Škrekovski [15] proved that every planar graph is \( 2 \)-defective 3-choosable. Gutowski, Han, Krawczyk, and Zhu [9] showed that there are planar graphs that are not \( 2 \)-defective 3-paintable, but every planar graph is \( 3 \)-defective 3-paintable. We show in this paper that not every planar graph is \((2, 3)\)-decomposable.

To obtain Theorems 1.2 and 1.3, we utilize an approach used by Thomassen in his ingenious proof of 5-choosability of planar graphs [16], but in a much more technical fashion. Namely, we prove by induction stronger statements regarding near triangulations where we introduce various conditions on each vertex depending on its type. The technical statement used to derive Theorem 1.3 is more intriguing and the proof is also more complicated. See Sections 2 and 3 for those statements and proofs. The proof of Theorem 1.1 (given in Section 4) uses the standard discharging method.

We end this section with some definitions and notations. A vertex ordering \( \sigma \) of \( G \) is \( d \)-degenerate if every vertex has at most \( d \) earlier neighbors in the ordering \( \sigma \). Note that a graph \( G \) is \( d \)-degenerate if and only if it has a \( d \)-degenerate ordering. For \( S \subseteq V(G) \) and a vertex ordering \( \sigma \) of \( G \), let \( \sigma − S \) denote a subordering of \( \sigma \) obtained by deleting the vertices in \( S \). For a digraph \( D \) and a vertex \( v \in V(D) \), we denote by \( \deg^{+}_{D}(v) \) (resp., \( \deg^{-}_{D}(v) \)) and \( N^{+}_{D}(v) \) (resp., \( N^{-}_{D}(v) \)) the out-degree of \( v \) in \( D \) (resp., the in-degree of \( v \) in \( D \)) and the set of out-neighbors of \( v \) in \( D \) (resp., the set of in-neighbors of \( v \) in \( D \)), and we denote by \( \Delta^{+}(D) \) the maximum out-degree of \( D \). We note that a graph \( G \) is \( d \)-degenerate if and only if it has an acyclic orientation whose maximum out-degree is at most \( d \). Therefore, when we prove Theorems 1.2 and 1.3, we find a pair \((D, H)\), where \( H \) is a subgraph of \( G \) with \( \Delta(H) \leq h \) and \( D \) is an acyclic orientation of \( G − E(H) \) with \( \Delta^{+}(D) \leq d \).

Let \( G \) be a plane graph. A plane subgraph of \( G \) is a subgraph of \( G \) whose plane embedding is inherited. We say \( G \) is a near triangulation if \( G \) is a 2-connected plane graph and every face of \( G \) except the outer face is a triangle. Note that the outer face of a near plane triangulation \( G \) is a cycle since \( G \) is 2-connected. A boundary vertex and a boundary edge of \( G \) are a vertex and an edge, respectively, on the boundary cycle of \( G \). For a boundary edge \( uv, v \) is called a boundary neighbor of \( u \).

An arc, which is a directed edge, is represented by an ordered pair of vertices; namely, \( uv \) is an (undirected) edge whereas \((u, v)\) is an arc from \( u \) to \( v \). For a graph \( G \) and a set \( E \) of unordered pairs on \( V(G) \), let \( G + E \) (resp., \( G − E \)) denote the graph obtained from \( G \) by adding (resp., deleting) the elements of \( E \) to (resp., from) the edge set of \( G \). If \(|E| = 1\), say \( E = \{ww'\}\), then denote \( G + E \) (resp., \( G − E \)) by \( G + ww' \) (resp., \( G − ww' \)). For a digraph \( D \) and a set \( A \) of ordered pairs on \( V(D) \), define \( D + A, D − A, D + (w, w') \), and \( D − (w, w') \) similarly. Moreover, for a digraph \( D \) and vertices \( x, y \in V(D) \), let \( D − xy \) denote the subdigraph
We often drop the parentheses to improve the readability. For instance, for a digraph \( D \) and sets \( A_1, A_2, A_3 \) of ordered pairs on \( V(D) \), both \( D - A_1 + A_2 + A_3 \) and \( D - A_1 + (A_2 + A_3) \) denote \((D - A_1) + A_2 + A_3\). For two (di)graphs \( G_1 \) and \( G_2 \), let \( G_1 \cup G_2 \) be the (di)graph such that \( V(G_1 \cup G_2) = V(G_1) \cup V(G_2) \) and \( E(G_1 \cup G_2) = E(G_1) \cup E(G_2) \).

\[ D - \{(x, y), (y, x)\}. \] We often drop the parentheses to improve the readability. For instance, for a digraph \( D \) and sets \( A_1, A_2, A_3 \) of ordered pairs on \( V(D) \), both \( D - A_1 + A_2 + A_3 \) and \( D - A_1 + (A_2 + A_3) \) denote \((D - A_1) + A_2 + A_3\). For two (di)graphs \( G_1 \) and \( G_2 \), let \( G_1 \cup G_2 \) be the (di)graph such that \( V(G_1 \cup G_2) = V(G_1) \cup V(G_2) \) and \( E(G_1 \cup G_2) = E(G_1) \cup E(G_2) \).

\[ 2 \quad \text{PROOF OF (3, 2)-DECOMPOSABILITY} \]

Note that for a near triangulation and a boundary edge \( xy \), there always exists a boundary vertex \( z \) distinct from \( x, y \) that is not incident with a chord of the boundary cycle. Instead of proving Theorem 1.2 directly, we prove the following more technical result.

**Theorem 2.1.** Let \( G \) be a near triangulation, \( xy \) be a boundary edge of \( G \), and \( z \) be a boundary vertex other than \( x, y \) that is not incident with a chord of the boundary cycle. When neither \( x \) nor \( y \) is a boundary neighbor of \( z \), let \( z' \) be a boundary neighbor of \( z \). Then there exist a subgraph \( H \) and an acyclic orientation \( D \) of \( G - E(H) \) satisfying the following:

(i) For every interior vertex \( w \), \( \deg_D^+(w) \leq 3 \) and \( \deg_H(w) \leq 2 \).

(ii) For every boundary vertex \( w \), \( \deg_D^+(w) \leq 2 \) and \( \deg_H(w) \leq 2 \). Moreover, if \( w \neq z' \), then \( \deg_D^+(w) + \deg_H(w) \leq 3 \).

(iii) \( \deg_D^+(y) = \deg_H(x) = \deg_H(y) = 0 \), \( N_D^+(x) = \{y\} \), and \( \deg_D^+(z) + \deg_H(z) \leq 2 \).

Let us call such \((D, H)\) a \((3, 2)\)-decomposition of \( G \) with respect to \((x, y, z)\) or \((x, y, z, z')\).

**Proof:** We use induction on \( |V(G)| \). If \( |V(G)| = 3 \), then \( G = K_3 \). Let \( D \) be a digraph with arcs \((x, y), (z, x)\), and \((z, y)\), and \( H \) be the empty graph. Then \((D, H)\) is a \((3, 2)\)-decomposition of \( G \) with respect to \((x, y, z)\). Suppose \( |V(G)| \geq 4 \). Let \( C \) be the boundary cycle of \( G \).

**Case 1.** \( C = (x, y, z) \) is a triangle.

Let \( G' = G - z \). Note that \( G' \) is a near triangulation and let \( C' \) be the boundary cycle of \( G' \). Let \( w \in N_G(z) \setminus \{x, y\} \) such that \( w \) is not incident with a chord of \( C' \). See Figure 1. By the induction hypothesis, there is a \((3, 2)\)-decomposition \((D', H')\) of \( G' \) with respect to \((x, y, w, w') \) or \((x, y, w)\) depending on the existence of \( w' \). Then \((D, H)\), where \( D = D' + \{(z, y), (z, x)\} + \{(u, z) | u \in V(C') \setminus \{x, y\}\} \) and \( H = H' \), satisfies Conditions (i)–(iii).

**Case 2.** \( C \) has a chord \( uv \).

**Case 2-1.** There is a chord \( uv \) of \( C \) such that \( x, y, z \in V(G_i) \) for some \( i \in \{1, 2\} \), where \( G_1 \) and \( G_2 \) are the plane subgraphs of \( G \) separated by \( uv \).

Let \( C_i \) be the boundary cycle of \( G_i \). Without loss of generality, assume \( x, y, z \in V(G_1) \). See the first figure of Figure 2. Choose the chord \( uv \) so that \( G_2 \) is minimum, so \( C_2 \) has no chord. Note that \( z \notin \{u, v\} \), since \( z \) is not incident with a chord of \( C \). Therefore, \( z' \in V(G_1) \) if neither \( x \) nor \( y \) is a boundary neighbor of \( z \) in \( G \). By the induction hypothesis, there is a \((3, 2)\)-decomposition \((D_1, H_1)\) of \( G_1 \) with respect to \((x, y, z, z')\) or
(x, y, z) depending on the existence of \( z' \). Let \( z'' \) be a boundary neighbor of \( v \) in \( G_2 \) other than \( u \). By the induction hypothesis, there is a \((3, 2)\)-decomposition \((D_2, H_2)\) of \( G_2 \) with respect to \((u, v, z'')\). Note that \( z'' \) is not incident with a chord of \( C_2 \) since it has no chord. Let \( D = D_1 + (D_2 - uv) \) and \( H = H_1 + (H_2 - uv) \). Since \( N_{H_1}^+(u) = \{v\}, \deg_{D_2}^+(v) = \deg_{H_2}^+(u) = \deg_{H_2}^+(v) = 0 \) by Condition (iii) for \((D_2, H_2)\), it follows that \( D \) is acyclic and Conditions (i)–(iii) are easily verified.

**Case 2.2.** For every chord \( uv \) of \( C \), \( x, y \in V(G_1) \) and \( z \in V(G_2) \setminus V(G_1) \), where \( G_1 \) and \( G_2 \) are the plane subgraphs of \( G \) separated by \( uv \). See the second figure of Figure 2.

Let \( C_1 \) be the boundary cycle of \( G_1 \). Choose the chord \( uv \) so that \( G_1 \) is minimum, so \( C_1 \) has no chord. By the induction hypothesis, there is a \((3, 2)\)-decomposition \((D_1, H_1)\) of \( G_1 \) with respect to \((x, y, w)\) where \( w \) is a boundary vertex of \( G_1 \) so that \( wx \) is a boundary edge of \( G_1 \). Note that \( w \) is not incident with a chord of \( C_1 \).

If either \( zw \) or \( zv \) is a boundary edge of \( G \), then there exists a \((3, 2)\)-decomposition \((D_2, H_2)\) of \( G_2 \) with respect to \((u, v, z)\) by the induction hypothesis. If \( z \) is neither adjacent to \( u \) nor \( v \), then \( z' \in V(G_2) \setminus \{u, v\} \), so let \((D_2, H_2)\) be a \((3, 2)\)-decomposition of \( G_2 \) with respect to \((u, v, z, z')\). Let \( D = D_1 + (D_2 - uv) \) and \( H = H_1 + (H_2 - uv) \). Since \( N_{H_2}^+(u) = \{v\}, \deg_{D_2}^+(v) = \deg_{H_2}^+(u) = \deg_{H_2}^+(v) = 0 \) by Condition (iii) for \((D_2, H_2)\), it follows that \( D \) is acyclic and Conditions (i)–(iii) are also easily verified.

**Case 3.** \( C \) is not a triangle and has no chord.

Let \( zw \) be the boundary edge of \( G \) where \( w \not\in \{x, y, z'\} \), and let \( w^* \) be the other boundary neighbor of \( z \) in \( G \). Note that \( w^* \in \{x, y, z'\} \). For simplicity, let \( U = N_G(z) \setminus \{w, w^*\} \). Let \( G' = G - z \). Note that \( G' \) is a near triangulation, and let \( C' \) be the boundary cycle of \( G' \). Let \( w' \) be the interior vertex of \( G \) which is a boundary neighbor of \( w \) in \( G' \). (Such \( w' \) exists, since \( G \) has no chord and so \( \deg_G(z) \geq 3 \).)
Case 3-1. $C'$ has no chord at the vertex $w$.

We find a $(3, 2)$-decomposition $(D', H')$ of $G'$ with respect to $(x, y, w, w')$ (if $y$ or $x$ is a boundary neighbor of $w$ in $G$, then we do not consider $w'$) by the induction hypothesis. Note that $\deg_{DH}(w) + \deg_{DH}(w') \leq 2$. Let $\tilde{D} = D' + \{(u, z) | u \in U\}$ for simplicity.

Suppose $w \in \{x, y\}$. See the first figure of Figure 3. If $\deg_{DH}(w) \leq 1$, then let $D = \tilde{D} + (z, w)$ and $H = H' + zw$. If $\deg_{DH}(w) = 2$, then let $D = \tilde{D} + \{(w, z), (z, w')\}$ and $H = H'$. Since $\deg_{DH}(y) = 0$ and $N_{DH}(x) = \{y\}$, it follows that $D$ is acyclic. Moreover, $\deg_{DH}(w) = 2$ implies $\deg_{DH}(w') = 0$, so Conditions (i)–(iii) are verified.

Suppose $w = z'$. See the second figure of Figure 3. We divide into four cases according to $\deg_{DH}(w)$ and $\deg_{DH}(z')$. Note that $\deg_{DH}(w) = 2$ implies $\deg_{DH}(w') = 0$.

- If $\deg_{DH}(w) \leq 1$ and $\deg_{DH}^+(z') \leq 1$, then let $D = \tilde{D} + \{(z', z)\}$ and $H = H' + zw$.
- If $\deg_{DH}(w) = 2$ and $\deg_{DH}^+(z') \leq 1$, then let $D = \tilde{D} + \{(z', z), (w, z)\}$ and $H = H'$.
- If $\deg_{DH}(w) \leq 1$ and $\deg_{DH}^+(z') = 2$, then let $D = \tilde{D}$ and $H = H' + [zw, zz']$.
- If $\deg_{DH}(w) = 2$ and $\deg_{DH}^+(z') = 2$, then let $D = \tilde{D} + \{(w, z)\}$ and $H = H' + zz'$.

Clearly, the resulting digraph $D$ is acyclic. It is also easy to check Conditions (i) and (iii). By Condition (ii) for $(D', H')$, we have $\deg_{DH}(z') + \deg_{DH}(z) \leq 3$, so Condition (ii) is also satisfied.

Case 3-2. $C'$ has a chord $wv$.

Since there is no chord of $C$ by the case assumption, $v \in N_G(z) \setminus \{w, w'\}$. Then $G - \{z, w, v\}$ has two components $V_1$ and $V_2$. Let $G_i = G[V_i \cup \{z, w, v\}]$ for each $i$, and assume $x, y \in V(G_1)$. Note that each $G_i$ is a near triangulation. See the last figure of Figure 3. By the induction hypothesis, there is a $(3, 2)$-decomposition $(D_1, H_1)$ of $G_1$ with respect to $(x, y, z, z')$ (if either $zv$ or $zv$ is a boundary edge of $G_1$ (or $G$), then we do not consider $z'$). By the induction hypothesis, there is a $(3, 2)$-decomposition $(D_2, H_2)$ of $G_2$ with respect to $(w, v, z)$. Let $D = D_1 + (D_2 - \{zw, vz, wv\})$ and $H = H_1 + H_2$.  

**Figure 3** Illustrations for Case 3

**Figure 4** Illustrations for Case 2
By Condition (iii) for \((D_2, H_2), (s, t)\) is an arc of \(D_2\), for every edge \(st\) of \(G\) joining an interior vertex \(s\) of \(G_2\) and a boundary vertex \(t\) of \(G_2\). Hence, \(D\) is acyclic and Conditions (i)–(iii) are easily verified. 

\[ \text{□} \]

3 PROOF OF (2, 6)-DECOMPOSABILITY

Assume \(G\) is a near triangulation, \(xy\) is a boundary edge of \(G\), and \(w\) is a boundary vertex of \(G\). Recall that \(w\) is a boundary neighbor of \(x\) if \(xw\) is a boundary edge of \(G\). We denote by \(b_{G,xy}(w)\) the number of vertices in \([x, y]\) that are boundary neighbors of \(w\). In particular, \(b_{G,xy}(x) = b_{G,xy}(y) = 1\). Note that \(b_{G,xy}(w) = 2\) if and only if the boundary cycle of \(G\) is a triangle and \(w\) is the boundary vertex other than \(x\) and \(y\). In other words, if the boundary cycle of \(G\) is not a triangle and \(w\) is a boundary neighbor of \(x\) or \(y\), then \(b_{G,xy}(w) = 1\). If there is no confusion, then we use \(b(w)\) to denote \(b_{G,xy}(w)\). Instead of proving Theorem 1.3 directly, we prove the following more technical result, which is easily seen to imply Theorem 1.3.

**Theorem 3.1.** Let \(G\) be a near triangulation, \(xy\) be a boundary edge of \(G\), and \(z\) be a boundary vertex of \(G\) other than \(x\) and \(y\). Then there exist a subgraph \(H\) and an acyclic orientation \(D\) of \(G - E(H)\) satisfying the following:

(i) For every interior vertex \(w\), \(\deg_D^+(w) \leq 2\) and \(\deg_H(w) \leq 6\).

(ii) For every boundary vertex \(w\), \(\deg_D^+(w) \leq 1\) and \(\deg_H(w) \leq 5 - b(w)\).

(iii) \(\deg_D^+(y) = \deg_H(y) = 0\), \(N_D^+(x) = \{y\}\), and \(\deg_H(x) \leq 1\). If \(\deg_H(x) = 1\), then there exists a boundary vertex \(s\) adjacent to both \(x\) and \(y\), and \(sx \in E(H)\).

(iv) \(\deg_H(z) \leq 4 - b(z)\). If equality holds, then \(\deg_H(w) \leq 4 - b(w)\) for every boundary neighbor \(w\) of \(z\).

(v) For the boundary neighbors \(z'\) and \(z''\) of \(z\), \(\deg_H(z) + \deg_H(z') + \deg_H(z'') \leq 12 - b(z') - b(z'')\).

Let us call such \((D, H)\) a \((2, 6)\)-decomposition of \(G\) with respect to \((x, y, z)\).

**Remark 1.** In a \((2, 6)\)-decomposition \((D, H)\) with respect to \((x, y, z)\), the following holds:

(a) If there exists a boundary vertex \(s\) adjacent to both \(x\) and \(y\), then \(sx \in E(H)\) and \((s, y)\) is an arc of \(D\) by (ii) and (iii). Therefore, \(\deg_H(x) = 1\) is equivalent to the existence of a boundary vertex that is a common neighbor of \(x\) and \(y\).

(b) By (iii), \(\deg_H(x) \leq 1 < 3 = 4 - b(x)\) and \(\deg_H(y) = 0 < 3 = 4 - b(y)\). Hence, for a boundary neighbor \(z'\) of \(z\), if \(z' \in [x, y]\), then the inequality in the second part of (iv) holds for \(z'\). Moreover, in this case, (ii) and (iii) imply (v) because \(\deg_H(z) \leq 5 - b(z) \leq 4\), \(\deg_H(z') < 4 - b(z')\), and \(\deg_H(z'') \leq 5 - b(z'')\). Therefore, if the boundary cycle of \(G\) is a triangle or a quadrangle, then we can ignore (v) because either \(x\) or \(y\) is a boundary neighbor of \(z\).

The following lemma shows that \(G\) has a \((2, 6)\)-decomposition with respect to \((x, y, z)\) if and only if it has one with respect to \((y, x, z)\).
**Lemma 3.2.** Let $G$ be a near triangulation, $xy$ be a boundary edge of $G$, and $z$ be a boundary vertex of $G$ other than $x$ and $y$. If $(D, H)$ is a $(2, 6)$-decomposition of $G$ with respect to $(x, y, z)$, then there is a $(2, 6)$-decomposition of $G$ with respect to $(y, x, z)$.

**Proof:** Let $(D, H)$ be a $(2, 6)$-decomposition of $G$ with respect to $(x, y, z)$. If $\deg_{H}(x) = 0$, then let $D' = D - (x, y) + (y, x)$ and $H' = H$. If $\deg_{H}(x) = 1$, then by Condition (iii), there exists a common boundary neighbor $s$ of $x$ and $y$, and $sx \in E(H)$. Let $D' = D - \{(x, y), (s, y)\} + \{(s, x), (y, x)\}$ and $H' = (H + sy) - sx$. Then $(D', H')$ is a $(2, 6)$-decomposition of $G$ with respect to $(y, x, z)$.

**Proof of Theorem 3.1.** We use induction on $|V(G)|$. If $|V(G)| = 3$, then $G = K_{3}$. Let $D$ be a digraph with two arcs $(x, y)$ and $(z, y)$, and $H$ be a graph with one edge $xz$. Then $(D, H)$ is a $(2, 6)$-decomposition of $G$ with respect to $(x, y, z)$. Suppose $|V(G)| \geq 4$. Let $C$ be the boundary cycle of $G$, and let $z'$ and $z''$ be the boundary neighbors of $z$. For simplicity, we denote $b_{G, xy}(w)$ by $b(w)$.

**Case 1.** $C = (x, y, z)$ is a triangle.

Let $G' = G - z$. Since $G$ contains at least four vertices, $G'$ is a near triangulation. Let $C'$ be the boundary cycle of $G'$, and let $w$ be a boundary vertex of $G'$ other than $x$ and $y$. See Figure 1. By the induction hypothesis, there is a $(2, 6)$-decomposition $(D', H')$ of $G'$ with respect to $(x, y, w)$.

If $\deg_{H}(x) = 0$, then let $D = D' + \{(u, z) | u \in V(C') \setminus \{x, y\}\} + (z, y)$ and $H = H' + xz$. If $\deg_{H}(x) = 1$, then there exists a boundary vertex $s$ in $G'$ adjacent to both $x$ and $y$, and $sx \in E(H')$ by Condition (iii), so let $D = D' + \{(s, x), (z, y)\} + \{(u, z) | u \in V(C') \setminus \{x, y, s\}\}$ and $H = (H' - sx) + \{sz, xz\}$.

In both cases, we can easily check Conditions (i)–(iii), so Condition (v) holds by Remark 1(b). Since $b(z) = 2$ and $\deg_{H}(z) \leq 2$, Condition (iv) holds by Remark 1(b). Thus $(D, H)$ is a $(2, 6)$-decomposition of $G$ with respect to $(x, y, z)$.

**Case 2.** $C$ has a chord $uv$ that either separates $xy$ and $z$ or is incident with $x$, $y$, or $z$.

Let $G_{1}$ and $G_{2}$ be the plane subgraphs of $G$ separated by $uv$. Namely, $G_{1} = G[V_{1}]$, $G_{2} = G[V_{2}]$, where $V_{1} \cup V_{2} = V(G)$ and $V_{1} \cap V_{2} = \{u, v\}$. Then each $G_{i}$ is a near triangulation. Let $C_{i}$ be the boundary cycle of $G_{i}$. Without loss of generality, assume $x, y \in V(G_{1})$. We divide the proof into three subcases: (1) $z \notin V(G_{1})$, (2) $z \in \{u, v\}$, and (3) $z \in V(G_{1}) \setminus \{u, v\}$. In each case, we will find a $(2, 6)$-decomposition $(D_{1}, H_{1})$ of $G_{1}$ with respect to $(x, y, w')$ for some $w' \in \{z, u, v\}$, and a $(2, 6)$-decomposition $(D_{2}, H_{2})$ of $G_{2}$ with respect to $(u, v, w')$ or $(v, u, w')$ for some vertex $w'$. Let $D = D_{1} \cup (D_{2} - uv)$ and $H = H_{1} \cup H_{2}$. It is clear that $D$ is acyclic since both $D_{1}$ and $D_{2}$ are acyclic and there are no arcs from $V(G_{1})$ to $V(G_{2}) \setminus \{u, v\}$.

For simplicity, denote $b_{G_{1}, xy}(w)$ and $b_{G_{2}, uv}(w)$ by $b_{1}(w)$ and $b_{2}(w)$, respectively. For $w \in V(G_{1}) \setminus \{u, v\}$, we have $\deg_{D}^{+}(w) = \deg_{D_{1}}^{+}(w)$, $\deg_{H}(w) = \deg_{H_{1}}(w)$, and if $w$ is a boundary vertex of $G$, then $b_{1}(w) \geq b(w)$. For $w \in \{u, v\}$, we have $\deg_{D}^{+}(w) = \deg_{D_{1}}^{+}(w)$, $\deg_{H}(w) = \deg_{H_{1}}(w) + \deg_{H_{2}}(w)$, and $b_{1}(w) \geq b(w)$. Hence, Condition (i) immediately holds, and Conditions (ii)–(v) hold except those regarding the degrees in $H$ involving $u$ or $v$. Therefore, we only need to check the following conditions to show that $(D, H)$ is a $(2, 6)$-decomposition of $G$ with respect to $(x, y, z)$.
• Condition (ii) holds for \( u \) and \( v \).
• Condition (iii) holds for every vertex in \( \{x, y\} \cap \{u, v\} \).
• Condition (iv) holds when \( \{z', z'', z''\} \cap \{u, v\} \neq \emptyset \).
• Condition (v) holds when \( \{z, z', z''\} \cap \{u, v\} \neq \emptyset \).

Case 2-1. \( z \notin V(G_1) \).
We may assume \( v \notin \{x, y\} \). See the first figure of Figure 4. By Lemma 3.2, we may assume \( x \neq u \). Note that \( z \notin \{u, v\} \) and \( z', z'' \in V(C_2) \). Let \((D_1, H_1)\) be a \((2, 6)\)-decomposition of \( G_1 \) with respect to \((x, y, v)\), and \((D_2, H_2)\) be a \((2, 6)\)-decomposition of \( G_2 \) with respect to \((v, u, z)\).

Condition (ii) holds for \( u \) and \( v \), since

\[
\deg_{H_1}(u) \leq \deg_{H_1}(u) + \deg_{H_1}(u) \leq (5 - b_1(u)) + 0 = 5 - b(u),
\]

\[
\deg_{H_1}(v) \leq \deg_{H_1}(v) + \deg_{H_1}(v) \leq (4 - b_1(v)) + 1 \leq 5 - b(v).
\]

If \( \{x, y\} \cap \{u, v\} \neq \emptyset \), then \( u = y \), and so \( \deg_D^+(y) = \deg_D^+(y) = 0 \) and \( \deg_{H_1}(y) = \deg_{H_1}(y) = 0 + 0 = 0 \). Thus Condition (iii) holds for every vertex in \( \{x, y\} \cap \{u, v\} \).

For Condition (iv), note that \( z \notin \{u, v\} \) by the case assumption. Suppose that \( \deg_{H_1}(z) = 4 - b(z) \) and \( \{z', z''\} \cap \{u, v\} \neq \emptyset \). Since \( \deg_{H_1}(z) = \deg_{H_1}(z) \leq 4 - b_2(z) \leq 4 - b(z) \), we conclude that \( b_2(z) = b(z) \) and \( \deg_{H_1}(z) = 4 - b_2(z) \). Since \( b_2(z) = b(z) \), either \( b(z) = b_2(z) = 0 \) or \( b(z) = b_2(z) = 1 \). If \( b(z) = b_2(z) = 0 \), then \( \{z', z''\} \cap \{u, v\} = \emptyset \), a contradiction. So \( b(z) = b_2(z) = 1 \).

Then \( u = y, y \) is a boundary neighbor of \( z \), and \( v \) is not a boundary neighbor of \( z \). We may assume that \( z'' \notin V(G_1) \), we have the following:

\[
\deg_{H_1}(z') \leq \deg_{H_1}(z') + \deg_{H_1}(z') \leq 4 - b(z'),
\]

\[
\deg_{H_1}(z'') \leq \deg_{H_1}(z'') \leq 4 - b_2(z'') \leq 4 - b(z'').
\]

Thus Condition (iv) holds when \( \{z, z', z''\} \cap \{u, v\} \neq \emptyset \).

By Condition (ii), \( \deg_{H_1}(z') \leq 5 - b(z') \) and \( \deg_{H_1}(z'') \leq 5 - b(z'') \). If \( \{z', z''\} = \{u, v\} \), then \( \deg_{H_2}(z) \leq 4 - b_2(z) = 2 \), so \( \deg_{H_1}(z) + \deg_{H_1}(z') + \deg_{H_1}(z'') \leq 2 + (5 - b(z')) + (5 - b(z'')) = 12 - b(z') - b(z'') \). If \( z' \in \{u, v\} \) and \( z'' \notin \{u, v\} \), then \( \deg_{H_1}(z) = \deg_{H_1}(z) \leq 4 - b_2(z) = 3 \). If \( \deg_{H_2}(z) = 3 \), then \( \deg_{H_1}(z') = \deg_{H_1}(z') \leq 4 - b_2(z'') \) by Condition (iv), and so \( \deg_{H_1}(z) + \deg_{H_1}(z') \leq 7 - b_2(z'') \leq 7 - b(z'') \). So we have

\[
\deg_{H_1}(z) + \deg_{H_1}(z') + \deg_{H_1}(z'') \leq (7 - b(z'')) + (5 - b(z')) = 12 - b(z') - b(z''),
\]

which implies Condition (v). Thus Condition (v) holds when \( \{z, z', z''\} \cap \{u, v\} \neq \emptyset \).

Case 2-2. \( z \in \{u, v\} \).
We may assume \( v = z \). Let \( z' \in V(G_1) \) and \( z'' \in V(G_2) \). See the second figure of Figure 4. Let \((D_1, H_1)\) be a \((2, 6)\)-decomposition of \( G_1 \) with respect to \((x, y, z)\). If \( u = y \) or \( \deg_{H_1}(u) = 5 - b_1(u) \), then let \((D_2, H_2)\) be a \((2, 6)\)-decomposition of \( G_2 \) with respect to \((v, u, z')\).
Otherwise, that is, if \( u \neq y \) and \( \deg_{H_1}(u) \leq 4 - b_1(u) \), then let \((D_2, H_2)\) be a \((2, 6)\)-decomposition of \( G_2 \) with respect to \((u, v, z')\). Note that \( b_1(v) = b(v) + 1 \) if \( u = y \), and \( \deg_{H_2}(v) \leq 3 - b_1(v) \) if \( \deg_{H_1}(u) = 5 - b_1(u) \) by Condition (iv).

Condition (ii) holds for \( u \) and \( v \), since

\[
\deg_{H}(u) \leq \deg_{H_1}(u) + \deg_{H_2}(u)
\]

\[
\leq \begin{cases}
0 + 0 \leq 4 - b(u) & \text{if } u = y, \\
(5 - b_1(u)) + 0 = 5 - b(u) & \text{if } \deg_{H_1}(u) = 5 - b_1(u), \\
(4 - b_1(u)) + 1 = 5 - b(u) & \text{otherwise},
\end{cases}
\]

\[
\deg_{H}(v) \leq \deg_{H_1}(v) + \deg_{H_2}(v)
\]

\[
\leq \begin{cases}
(4 - b_1(v)) + 1 = 4 - b(v) & \text{if } u = y, \\
(3 - b_1(v)) + 1 \leq 4 - b(v) & \text{if } \deg_{H_1}(u) = 5 - b_1(u), \\
(4 - b_1(v)) + 0 \leq 4 - b(v) & \text{otherwise}.
\end{cases}
\]

To check Condition (iii) we consider the case that \( \{x, y\} \cap \{u, v\} \neq \emptyset \). Then \( u = y \). So \( \deg_{D_1}(y) = \deg_{D_1}(v) = 0 \) and \( \deg_{H}(y) = \deg_{H_1}(y) + \deg_{H_2}(y) = 0 + 0 = 0 \). Thus Condition (iii) holds for every vertex in \( \{x, y\} \cap \{u, v\} \).

To check Condition (iv), note that \( \deg_{H}(z) \leq 4 - b(z) \) since \( z = v \) and

\[
\deg_{H}(z) \leq \deg_{H_1}(z) + \deg_{H_2}(z)
\]

\[
\leq \begin{cases}
4 - b_1(z) + 1 \leq 4 - b(z) & \text{if } u = y, \\
(3 - b_1(z)) + 1 = 4 - b(z) & \text{if } \deg_{H_1}(u) = 5 - b_1(u), \\
(4 - b_1(z)) + 0 = 4 - b(z) & \text{otherwise}.
\end{cases}
\]

Also, if \( \deg_{H}(z) = 4 - b(z) \), then either

\[
\deg_{H_1}(z) = 3 - b_1(z) \quad \text{and} \quad \deg_{H_1}(u) = 5 - b_1(u)
\]

or

\[
\deg_{H_1}(z) = 4 - b_1(z).
\]

For the first case, we have

\[
\deg_{H_1}(z') \leq (12 - b_1(z') - b_1(u)) - \deg_{H_1}(z) - \deg_{H_1}(u)
\]

\[
= (12 - b_1(z') - b_1(u)) - (3 - b_1(z)) - (5 - b_1(u))
\]

\[
= 4 - b_1(z') + b_1(z).
\]

Here, \( b_1(z) = 0 \) since otherwise, \( z' = x \), which by the above inequality, \( \deg_{H}(x) = 5 - b_1(x) \), a contradiction. Thus \( \deg_{H_1}(z') \leq 4 - b_1(z') \). Also, for the second case, we have \( \deg_{H_1}(z') \leq 4 - b_1(z') \). Thus, in each of the cases, we know \( \deg_{H}(z') = \deg_{H_1}(z') \leq 4 - b(z') \). Thus Condition (iv) holds.
Note that $\deg_H(z'') = \deg_{H_2}(z'') \leq 4 - b_2(z'') < 4 - b(z'')$. Thus

$$\deg_H(z) + \deg_H(z') + \deg_H(z'') < 12 - b(z') - b(z''),$$

and Condition (v) holds.

**Case 2-3.** $z \in V(G_1) \setminus \{u, v\}$.

By the case assumption, the chord $uv$ is incident with either $x$ or $y$. By Lemma 3.2, we may assume $y = u$. See the last figure of Figure 4. Note that $z', z'' \in V(C_1)$. Let $(D_1, H_1)$ be a $(2, 6)$-decomposition of $G_1$ with respect to $(x, y, z)$, and let $(D_2, H_2)$ be a $(2, 6)$-decomposition with respect to $(v, u, z^*)$, where $z^* \in V(C_2) \setminus \{u, v\}$. Note that Condition (iii) clearly holds by definition. Conditions (ii), (iv), and (v) hold since $b_1(v) = b(v) + 1$ and $\deg_{H_1}(v) \leq \deg_{H_1}(v) + 1$.

**Case 3.** Neither Case 1 nor Case 2 applies, in other words, $C$ has at least four vertices and for every chord $uv$ of $C$, the vertices $x, y, z$ lie in the same component of $G - \{u, v\}$.

**Case 3-1.** $z$ is a boundary neighbor of either $x$ or $y$.

By Lemma 3.2, we may assume $yz \in E(C)$. Since $C$ has at least four vertices, we may assume $z' \notin \{x, y\}$. See the first figure of Figure 5. Let $G' = G - z$, and let $P$ be the boundary path of $G'$ from $y$ to $z'$ not containing $x$. Let $(D', H')$ be a $(2, 6)$-decomposition of $G'$ with respect to $(x, y, z')$. For simplicity, let $X = V(P) \setminus \{y, z'\}$.

If $\deg_{H_1}(x) = 0$, then let $D = D' + (z, y) + \{(u, z)\}u \in X\}$ and $H = H' + zz'$. It is easy to observe that $(D, H)$ is a $(2, 6)$-decomposition of $G$ with respect to $(x, y, z)$. Suppose $\deg_{H_1}(x) = 1$. Then by Condition (iii) for $(D', H')$, there exists a boundary vertex $s$ of $G'$ adjacent to both $x$ and $y$, and $sx \in E(H')$. See the second figure of Figure 5. Since $G$ has no chord incident with either $x$ or $y$, $s \in X$. Let $D = D' + \{(s, x), (z, y)\} + \{(u, z)\}u \in X\setminus\{s\}\}$ and $H = (H' - sx) + \{zz', sz\}$. Then $(D, H)$ is a $(2, 6)$-decomposition of $G$ with respect to $(x, y, z)$.

**Case 3-2.** Neither $x$ nor $y$ is a boundary neighbor of $z$.

Then $z', z''$ are different from $x, y$. By Lemma 3.2, we may assume $x, y, z', z, z'$ is the clockwise ordering on $C$. See Figure 6. Let $p_1$ be the boundary neighbor of $y$ other than $x$. Let $P$ be the clockwise subpath of $C$ joining $p_1$ and $z$. Note that by the case assumption, $|V(P)| \geq 2$. Let $G'$ be the block of $G - V(P)$ containing $x, y, z'$ (such a block exists since there is no chord separating $\{x, y\}$ and $z$ in $G$), and let $C'$ be the boundary cycle of $G'$. Let $Q$ be the clockwise subpath of $C'$ joining $y$ and $z'$.

**Claim 3.3.** Every two adjacent vertices $q$ and $q'$ on $Q$ have a common neighbor in $V(P)$.
Proof. Since $G$ is a near triangulation, $q$ and $q'$ have a common neighbor $w$ in $V(G) \setminus V(G')$. If $w$ is not on $P$, then $qq'$ cannot be a boundary edge of $G - V(P)$, which is a contradiction. \qed

Since there is no chord incident with $y$ in $G$, $|V(Q)| \geq 3$. By Claim 3.3, every vertex of $Q$ has a neighbor in $V(P)$. If every vertex of $Q$ has exactly one neighbor in $V(P)$, then $P[z] = \emptyset$, which is a contradiction. Let $q_y = 0$ and $q_{q,k}$ be the vertices of $Q$ in the order from $y$ to $z'$ that are adjacent to at least two vertices in $V(P)$. Note that $q_k \neq z'$ since no chords of $G$ separate $(x, y)$, and $z$. Let $q_{k+1} = z'$. By Claim 3.3, for $i \in \{1, 2, \ldots, k+1\}$, let $p_i \in V(P)$ be the vertex adjacent to both $q_{i-1}$ and $q_i$. Note that $P$ is a path from $p_1$ to $p_{k+1} = z$, and $yp_1 \in E(G)$.

For $j \in \{0, 1, \ldots, k\}$, let $Q_j$ be the subpath of $Q$ from $q_j$ to $q_{j+1}$. For $i \in \{1, \ldots, k\}$, let $P_i$ be the subpath of $P$ from $p_i$ to $p_{i+1}$. Let $C_i$ be the cycle consisting of $P_i$ and the vertex $q_i$, and let $G_i$ be the maximal plane subgraph of $G$ with boundary cycle $C_i$. Let $(D_i, H_i)$ be a $(2, 6)$-decomposition of $G_i$ with respect to $(p_{i+1}, q_i, p_i)$. Then, clearly $(p_{i+1}, q_i, p_i, q_i)$ are arcs of $D_i$. Modify $D_i$ and $H_i$ by reversing the orientation of $pq_{i+1}$ in $D_i$, removing $(p_{i+1}, q_i)$ from $D_i$, and adding $p_{i+1}q_i$ to $H_i$. Then for $i \in \{1, \ldots, k\}$,

$$\deg_{D_i}(q_i) = \deg_{H_i}(q_i) = 1, \quad \deg_{D_i}(p_i) = \deg_{D_i}(p_{i+1}) = 0, \quad \deg_{H_i}(p_i) \leq 3, \quad \deg_{H_i}(p_{i+1}) \leq 2,$$

and $D_i$ is still acyclic because $\deg_{D_i}(p_i) = 0$. Let $(D', H')$ be a $(2, 6)$-decomposition of $G'$ with respect to $(x, y, z')$. Let

$$D = D' \cup \left( \bigcup_{i=1}^k D_i \right) + \{(p_1, y)\} + \{(q, p_{i+1})|q \in V(Q_i)\setminus\{q_i, q_{i+1}\}, i \in \{0, 1, \ldots, k\}\},$$

$$H = H' \cup \left( \bigcup_{i=1}^k H_i \right) + zz'.$$

Suppose $\deg_{H_i}(x) = 0$. Then $x$ and $y$ have no common boundary neighbor in $G'$ by Remark 1(a). We claim that $(D, H)$ is a $(2, 6)$-decomposition of $G$ with respect to $(x, y, z)$. Clearly, $D$ is acyclic and has maximum out-degree at most 2, and $H$ has maximum degree at most 6. So, Condition (i) holds.

From the definition of $(D, H)$, $N^+_D(x) = \{y\}$ and $\deg^+_D(y) = \deg_H(x) = \deg_H(y) = 0$, so Condition (iii) holds.

It is easy to check that
(\*1) \( \deg_H^+(p_i) = 1, \deg_H^+(p_i) = \deg_{D_{i+1}}^+(p_i) + \deg_{D_i}^+(p_i) = 0 \) for \( i \in \{2, \ldots, k\} \),

(\*2) \( \deg_{EH}(p_i) = \deg_{H_i}(p_i) \leq 3 < 5 - b(p_i) \),

\( \deg_{EH}(p_i) = \deg_{H_{i-1}}(p_i) + \deg_{EH}(p_i) \leq 2 + 3 = 5 - b(p_i) \) for \( i \in \{2, \ldots, k\} \),

\( \deg_E(z) = 0, \deg_H(z) \leq 1 + \deg_{H_k}(z) \leq 1 + 2 = 3 = 3 - b(z) \), and

\( \deg_D^+(z') = \deg_{D_i}^+(z') \leq 1, \deg_{EH}^+(z') = \deg_{EH}^+(z') + 1 \leq 5 - b(z') \).

We also have \( \deg_{EH}(z'') \leq 4 - b(z'') \) since if \( z'' \neq p_k \), then \( \deg_{EH}(z'') \leq 5 - b_{G_k,p_{k+1}q_k}(z'') = 4 = 4 - b(z'') \) and if \( z'' = p_k \), then the boundary cycle of \( G_k \) is a triangle, and so \( \deg_{H_k}(p_k) \leq 4 - b_{G_k,p_{k+1}q_k}(p_k) = 2 \), which implies that

\[
\deg_{EH}^+(z'') = \deg_{EH}^+(p_k) \leq \begin{cases} 2 + 2 = 4 = 4 - b(z'') & \text{if } k > 1, \\ 1 + 2 = 3 = 4 - b(z'') & \text{if } k = 1. \end{cases}
\]

So Conditions (ii) and (iv) hold.

It remains to check Condition (v). As shown above, whether \( z'' \) is equal to \( p_k \) or not, we have \( \deg_{EH}^+(z'') \leq 4 - b(z'') \). Therefore

\[
\deg_{EH}^+(z) + \deg_{EH}^+(z') + \deg_{EH}^+(z'') \leq 3 + (5 - b(z')) + (4 - b(z'')) = 12 - b(z') - b(z'').
\]

Now we assume \( \deg_{EH}^+(x) = 1 \). Then there exists a boundary vertex \( s \) adjacent to both \( x \) and \( y \) in \( G' \), \( xs \in E(H') \subseteq E(H) \), and \( y \) is the unique out-neighbor of \( s \) in \( D \). By the case assumption, \( s \) belongs to \( V' \setminus \{y, z'\} \).

We modify \( D \) and \( H \) as follows: Delete \( sx \) from \( H \) and then add arc \((s, x)\) to \( D \). For the smallest index \( i \) such that \( s \in V(Q_i) \), if \( i \geq 1 \), then modify \( D \) by reversing the orientation of \((s, p_{i+1})\) in \( D \), and if \( i = 0 \), then modify \( D \) and \( H \) by deleting arc \((s, p_{i+1})\) from \( D \) and then adding the edge \( sp_{i+1} \) to \( H \).

Note that the above modification increases either \( \deg_{EH}^+(p_i) \) or \( \deg_{EH}^+(p_i) \) for some \( i \geq 2 \) by 1, and preserves \( D \) to be acyclic so that \((D, H)\) satisfies Conditions (i)–(v) by (\*1) and (\*2).

Therefore \((D, H)\) is a \((2, 6)\)-decomposition of \( G \) with respect to \((x, y, z)\).

We finish this section by proving the following, which implies Proposition 1.4.

**Proposition 3.4.** Let \( G \) be a plane triangulation on at least 11 vertices. If \( G' \) is the plane graph obtained from \( G \) by adding a new vertex \( v_f \) to every face \( f \) of \( G \) and adding all edges between \( v_f \) and the vertices of \( f \), then \( G' \) is not \((2, 3)\)-decomposable.

**Proof.** Let \( n = |V(G)| \). Since \( G \) is a triangulation, \( G \) has \( 2n - 4 \) faces and \( 3n - 6 \) edges. Suppose to the contrary that \( G' \) is \((2, 3)\)-decomposable. Let \((D, H)\) be a \((2, 3)\)-decomposition of \( G' \) that maximizes \(|E(H) \cap (E(G') \setminus E(G))|\). Let \( \sigma \) be a 2-degenerate ordering of \( D \).

We claim that for every face \( f \) of \( G \), \( \deg_{EH}(v_f) \geq 1 \). Suppose \( \deg_{EH}(v_f) = 0 \) for some face \( f \) of \( G \). Let \( v_1, v_2, v_3 \) be the vertices of \( G \) incident with \( f \). Then \( \deg_G(v_f) = 3 \), so some \( v_f \) comes later than \( v_f \) in \( \sigma \). We may assume \( v_1 \) is the last in \( \sigma \) among \( \{v_f, v_1, v_2, v_3\} \). Since \( \sigma \) is a 2-degenerate ordering, either \( v_1v_2 \) or \( v_1v_3 \) is in \( H \), say \( v_1v_2 \in E(H) \). Let \( D' = D - v_f v_1 + v_1v_3 \) and \( H' = H - v_1v_2 + v_f v_1 \). Then \((D', H')\) is a \((2, 3)\)-decomposition of \( G' \), which is a contradiction to the maximality of
$|E(H) \cap (E(G') \setminus E(G))|$. Therefore $\deg_{E}(v_f) \geq 1$ for every face $f$ of $G$, and thus $|E(H) \setminus E(G)| \geq |F(G)| = 2n - 4$.

In $\sum_{v \in V(G)} \deg_{E}(v)$, an edge in $E(H) \cap E(G)$ is counted twice and an edge in $E(H) \setminus E(G)$ is counted once. Hence, together with the fact that $\Delta(H) \leq 3$,

$$3n \geq \sum_{v \in V(G)} \deg_{E}(v) \geq 2|E(H) \cap E(G)| + |E(H) \setminus E(G)|$$

$$\geq 2|E(H) \cap E(G)| + (2n - 4).$$

From the fact that $|E(D) \cap E(G)| \leq 2n - 3$, we have $E(H) \cap E(G)| \geq (3n - 6) - (2n - 3) = n - 3$, so $3n \geq 2(n - 3) + 2n - 4 = 4n - 10$, which is a contradiction since $n \geq 11$.  

\[\square\]

### 4 PROOF OF (4, 1)-DECOMPOSABILITY

A $d$-vertex, a $d^+$-vertex, and a $d^-$-vertex are a vertex of degree $d$, at least $d$, and at most $d$, respectively. A $d$-neighbor is a neighbor that is a $d$-vertex. A $d^+$-neighbor and a $d^-$-neighbor are defined analogously. Note that even though a matching is a collection of edges, we sometimes refer to it as a subgraph with maximum degree one.

Let $G$ be a minimum counterexample to Theorem 1.1 with respect to the number of vertices. We may assume that $G$ is a triangulation, and fix an embedding of $G$. The following lemma reveals some reducible configurations of $G$.

**Lemma 4.1.** The following structures cannot appear in $G$:

(i) A $4^-$-vertex.

(ii) Two adjacent 5-vertices.

(iii) A 5-vertex with three consecutive $6^-$-neighbors.

(iv) A 5-vertex with two 7-neighbors and three $6^-$-neighbors.

(v) A 7-vertex with three consecutive $6^-$-neighbors where two of them are 5-vertices.

**Proof.** In all cases, we will obtain a (4, 1)-decomposition of $G$, which is a contradiction.

(i) Suppose to the contrary that there is a $4^-$-vertex $v$. By the minimality of $G$, $G - v$ has a $(4, 1)$-decomposition $(D', M')$ with a 4-degenerate ordering $\sigma'$ of $D'$. Let $M = M'$, and let $D$ be the graph from $D'$ by adding all edges incident to $v$. Clearly, $M$ is a matching and the ordering obtained by appending $v$ to $\sigma'$ is a 4-degenerate ordering of $D$, so $D$ is 4-degenerate.

(ii) Suppose to the contrary that there are two adjacent 5-vertices $u$ and $v$. By the minimality of $G$, $G - \{u, v\}$ has a $(4, 1)$-decomposition $(D', M')$ with a 4-degenerate ordering $\sigma'$ of $D'$. Let $M = M' \cup \{uv\}$ and let $D = G - M$. Clearly, $M$ is a matching and the ordering obtained by appending $u, v$ to $\sigma'$ is a 4-degenerate ordering of $D$, so $D$ is 4-degenerate.

(iii) Suppose to the contrary that there is a 5-vertex $v$ with three $6^-$-neighbors $u_1, u_2, u_3$, and $u_1u_2, u_2u_3 \in E(G)$. By the minimality of $G$, $G - \{v, u_1, u_2, u_3\}$ has a $(4, 1)$-decomposition $(D', M')$ with a 4-degenerate ordering $\sigma'$ of $D'$. Let $M = M' \cup \{vu_1, u_2u_3\}$ and let
$D = G - M$. Clearly, $M$ is a matching, and the ordering obtained by appending $u_3, u_1, u_2, v$ to $\sigma'$ is a 4-degenerate ordering of $D$, so $D$ is 4-degenerate.

(iv) Suppose to the contrary that there is a 5-vertex $v$ with three 6°-neighbors and two 7-neighbors. Let $N_G(v) = \{u_1, u_2, u_3, u_4, u_5\}$ where $u_i u_5 \in E(G)$ and $u_i u_{i+1} \in E(G)$ for $i \in \{1, 2, 3, 4\}$.

By (ii) and (iii), we may assume that $u_1, u_2, u_4$ are the 6-vertices, and $u_3$ and $u_5$ are the 7-vertices. By the minimality of $G$, $G - N_G[v]$ has a $(4, 1)$-decomposition $(D', M')$ with a 4-degenerate ordering $\sigma'$ of $D'$. Let $M = M' \cup \{vu_5, u_1 u_2, u_3 u_4\}$ and let $D = G - M$. Clearly, $M$ is a matching, and the ordering obtained by appending $u_5, u_1, u_3, u_2, u_4, v$ to $\sigma'$ is a 4-degenerate ordering of $D$, so $D$ is 4-degenerate.

(v) Suppose to the contrary that there is a 7-vertex $v$ with three consecutive neighbors $u_1, u_2, u_3$ where two of them are 5-vertices. By (ii), $u_1, u_3$ are 5-vertices and $u_2$ is a 6-vertex. By the minimality of $G$, $G - \{v, u_1, u_2, u_3\}$ has a $(4, 1)$-decomposition $(D', M')$ with a 4-degenerate ordering $\sigma'$ of $D'$. Let $M = M' \cup \{vu_1, u_2 u_3\}$ and let $D = G - M$. Clearly, $M$ is a matching, and the ordering obtained by appending $v, u_2, u_3, u_1$ to $\sigma'$ is a 4-degenerate ordering of $D$, so $D$ is 4-degenerate.

We use the discharging method to reach the final contradiction, to conclude that the minimum counterexample $G$ could not have existed. By Euler’s formula, recall that

$$\sum_{v \in V(G)} (\deg_G(v) - 6) + \sum_{f \in F(G)} (2\deg_G(f) - 6) = -12.$$ 

Since $\deg_G(f) \geq 3$ for every face $f \in F(G)$, we know

$$\sum_{v \in V(G)} (\deg_G(v) - 6) \leq -12.$$ 

Let the initial charge of each vertex $v$ be $\deg_G(v) - 6$, and note that the initial charge sum is negative. We will reach a contradiction by showing that the final charge at each vertex is nonnegative after the discharging rules, which preserves the charge sum. The following is our one discharging rule:

[R] Each 6°-vertex $v$ sends charge $(\deg_G(v) - 6)/d_5(v)$ to each of its 5-neighbors, where $d_5(v)$ is the number of 5-neighbors of $v$.

By Lemma 4.1 (ii), $d_5(v) \leq \left\lfloor \frac{\deg_G(v)}{2} \right\rfloor$. Thus, an 8-vertex and a 7-vertex send charge at least $\frac{1}{2}$ and at least $\frac{1}{3}$, respectively, to each 5-neighbor.

By the rule [R], the final charge of a 6°-vertex is nonnegative. By Lemma 4.1 (i), it remains to check 5-vertices. Take a 5-vertex $v$, and let $N_G(v) = \{u_1, u_2, u_3, u_4, u_5\}$ where $u_i u_5 \in E(G)$ and $u_i u_{i+1} \in E(G)$ for $i \in \{1, 2, 3, 4\}$. If $v$ has at least two 8°-neighbors, then the final charge of $v$ is nonnegative. If $v$ has no 8°-neighbors, then by Lemma 4.1(iii) and (iv), it has at least three 7-neighbors, and the final charge of $v$ is nonnegative.

Assume $v$ has exactly one 8°-neighbor $u_5$. If $v$ has at least two 7°-neighbors other than $u_5$, then it has nonnegative final charge. If $v$ has no 7°-neighbor other than $u_5$, then this is a contradiction to Lemma 4.1(iii). Thus, $v$ has exactly one 7-neighbor, so it has three 6°-neighbors. By Lemma 4.1(iii),
we may assume that \( u_3 \) is the 7-neighbor and \( u_1, u_2, u_4 \) are the 6-neighbors. By Lemma 4.1(ii) and (v), the 7-vertex \( u_3 \) has at most two 5-neighbors. Thus \( u_3 \) sends charge at least \( \frac{1}{2} \) to \( v \) by the rule [R]. Since \( u_3 \) sends charge at least \( \frac{1}{2} \) to \( v \) by the rule [R], the final charge of \( v \) is nonnegative.

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