Entanglement and Bell Inequalities.
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Abstract. The entangled quantum states play a key role in quantum information. The association of the quantum state vector with each individual physical system in an attributive way is a source of many paradoxes and inconsistencies. The paradoxes are avoided if the purely statistical interpretation (SI) of the quantum state vector is adopted. According to the SI the quantum theory (QT) does not provide any deterministic prediction for any individual experimental result obtained for a free physical system, for a trapped ion or for a quantum dot. In this article it is shown that if the SI is used then, contrary to the general belief, the QT does not predict for the ideal spin singlet state perfect anti-correlation of the coincidence counts for the distant detectors. Subsequently the various proofs of the Bell’s theorem are reanalyzed and in particular the importance and the implications of the use of the unique probability space in these proofs are elucidated. The use of the unique probability space is shown to be equivalent to the use of the joint probability distributions for the non commuting observables. The experimental violation of the Bell’s inequalities proves that the naive realistic particle like spatio-temporal description of the various quantum mechanical experiments is impossible. Of course it does not give any argument for the action at the distance and it does not provide the proof of the completeness of the QM. The fact that the quantum state vector is not an attribute of a single quantum system and that the quantum observables are contextual has to be taken properly into account in any implementation of the quantum computing device.

Keywords: Entanglement, Bell’s inequalities, quantum information, quantum computing, EPR correlations, quantum cryptography

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0.1 Introduction

The long range non classical correlations characterizing the entangled quantum states are at the base of the quantum computer project [42, 32, 26], state teleportation and quantum cryptography [24, 9, 17]. The mathematical structure and possible time evolutions of the quantum states have been studied and a considerable progress has been achieved [29, 4, 47]. The entanglement witnesses have been constructed which may help to distinguish between different entangled states in the experiment [40, 18]. Quantum states and quantum process tomography have been studied and experimentally implemented [41, 42, 13, 30, 31]. In
spite of this incontestable progress of quantum information in some papers the state vectors (qubits) are treated as the attributes of the individual quantum system which can be manipulated and modified quasi-instantaneously. One may also occasionally find the picture of the Schrödinger cat and hear a story of the twin point-like particles communicating at the distance with faster than light signals. It seems that the abstract, statistical and contextual character of the quantum description of the Nature is sometimes forgotten. Besides it is usually assumed that a single measurement reduces instantaneously the state vector of a physical system.

The problems related to the quantum theory of the measurement and a notion of the state vector reduction have been for decades a subject of discussions between people interested in the foundations of the quantum theory (QT) and still there is no unanimity. The most consistent seems to us a point of view of the followers of so-called purely statistical interpretation (SI) of QT which evolved from the interpretation advocated for the first time by Einstein.[23,22]. According to SI the pure state vector $\Psi$ or the density matrix $\rho$ describes only the statistical properties of an ensemble of a similarly prepared systems. For the trapped ions and the quantum dots it describes the statistical properties of the repeated measurements on the same ion or the quantum dot after the same initial preparation. The statistical interpretation was extensively discussed by Ballentine [14]. In his already classic textbook of the quantum mechanics based on the SI we may read [15]: "Once acquired, the habit of considering an individual particle to have its own wave function is hard to break. Even though it has been demonstrated to be strictly incorrect, it is surprising how seldom it leads to a serious error." In the SI the state vector reduction is a passage from the description of the whole ensemble to the description of the sub-ensemble obtained from the initial ensemble by so-called non-destructive measurements. The important additional arguments in favour of the SI have been recently given by Allaverdyan, Balian and Nieuwenhuizen[48].

Since most of the predictions of the QT are of statistical nature a famous EPR question [23] might be asked whether and in what sense the QT provides a complete description of the individual physical system. In fact the SI leaves in principle a place for the introduction of the supplementary parameters (called often hidden variables) which would determine the behavior of each particular physical system during the experiment. Several theories with supplementary parameters (TSP) have been discussed [7]. The most influential was the paper by Bell[8], who analyzed a large family of TSP so-called local or realistic hidden variable theories (LRHV) and showed that their predictions must violate, for some configurations of the experimental set-up, the quantum mechanical predictions for spin polarization correlations experiments (SPCE) dealing with pairs of electrons or photons produced in a singlet state. Bell’s argument was put into experimentally verifiable form, by Clauser, Horne, Shimony and Holt[19]. Several experiments in particular those by Aspect et al. [5,6] confirmed the predictions of QM. The general conclusion summarized in the excellent review by Clauser and Shimony [21] was that if one wants to understand the experimental data "either one must totally abandon the realistic philosophy of most
working scientists or dramatically revise our concept of space time ” which encouraged unwillingly speculations about a spooky action on a distance.

It was shown by many authors that assumptions made in LRHV were more restrictive and questionable that they seemed to be and the Bell’s inequalities may be violated not only by quantum experiments but also by macroscopic ones[1,3, 35-38]. The recent experiments seemed to close the remaining loop holes [45,43 ] but the violation of CHSH may not be consider neither as a proof of the completeness of QM nor the indication of the faster than light communication [44, 29, 2, 30, 39, 25] . The extensive discussion of the concept of probability were given by Khrennikov[49] and Holevo[50]. The role of the contextuality and the remaining loopholes in Bell’s proof were recently underlined by Khrennikov and Volovich [51-53].

In this short paper we want to refine and complement some of our old arguments and forgotten ideas [35-38] hoping that it could shed some light on the problems we face in the quantum information.

The paper is organized as follows in the section 2 we reanalyze in view of the SI the properties of the entangled idealized spin singlet state. In particular we show that there is no prediction for the perfect correlations of the counts for the far away detectors and no EPR-Bohm paradox. Let us underline that this lack of perfect correlations is of more deep nature than the lack of perfect correlations in all real experiments which is attributed to the decoherence, experimental systematic and statistical errors and efficiency of detectors.\[21,30,25\]. In section 3 we analyze some proofs of Bell and CHSH inequalities clearly demonstrating that the use of the unique probability space is equivalent to the use of the joint probability distributions for the noncommuting observables or to the assumption that all random variables corresponding to physical observables studied are completely independent thus uncorrelated. Let us note that most of the proofs of the recent generalizations of the CHSH inequalities to the n qubits are usually done assuming the factorization of the expectation functions thus the statistical independence of the corresponding random variables.

0.2 A singlet state.

Let us state the essential points of the EPR-Bohm reasoning using the notation and phrasing from the reference [15].

The singlet spin state vector for the system of two particles has the form

\[ \Psi_0 = (|+\rangle \otimes |-\rangle - |-\rangle \otimes |+\rangle) \sqrt{1/2} \]  

where the single particle vectors \(|+\rangle\) and \(|-\rangle\) denote ”spin up” and ”spin down” with respect to some coordinate system.

a) Even if the orbital state is not stationary, the interactions do not involve spin and so the spin states will not change.

b) The particles are allowed to separate , and when they are well beyond the range of the interaction we can measure the z component of spin of the particle #1.
c) Because the total spin is zero, we can predict with certainty, and without any way disturbing the second particle, that the z component of spin of particle #2 must have the opposite value. Thus the values of $\sigma_z^{(2)}$ is an element of reality, according to EPR criterion.

d) But the singlet state is invariant under rotation and it has the same form (1) in term of "spin up" and "spin down" if the directions "up" and "down" are referred to any other axis. Thus following EPR we may argue that the values of $\sigma_x^{(2)}$, $\sigma_y^{(2)}$ and any number of other spin components are also elements of the reality for the particle #2.

What is wrong with this argument? In a) all possible decoherence due to the interaction with the environment is neglected. In b) by saying that the particles had a time to separate we assume a mental image of two point-like particles which are produced and which after some time become separated and free. Even if we assume that the points a) and b) are correct then the point c) is wrong and it will be proven below using the SI. We do not see any particular couple of the particles and we do not follow its space time evolution. We record only the clicks on the far away coincidence counters. To be able to deduce the value of a particular spin projection for the particle #2 from the measurement made on the particle #1 we should have had for each experiment (A,B) a different experimental design (impossible to realize) giving us much more information about each couple of the particles than we have in a simple coincidence experiment. Similar arguments were given by Bohr [12,11] in his neither well understood nor frequently read answer to the original EPR paper.

We interpret a click as a detection of the particle which passed by a polarization filter and which was registered by a detector. According to SI only the ensemble of these particles is described by the one particle state vector $|+\rangle$ or $|-\rangle$ with respect to the axis determined by the filter. Let us note that if c) is not correct than d) does not follow and there is no EPR paradox. According to SI a state $\Psi_0$ allows only to find the statistical correlations observed in a long run of the various experiments with different couples (A,B) of the spin polarization analyzers, characterized by macroscopic direction vectors A and B. Since the angle between A and B is a continuous variable the QT gives us the probability density functions not the probabilities. Let’s go back to the mathematical formalism of the QT.

Let $\sigma_a = \sigma \circ a$ denote the component of the Pauli spin operator in the direction of the unit vector a, and $\sigma_b = \sigma \circ b$ denote the component of the Pauli spin operator in the direction of the unit vector b. If we "measure" the spin of the particle #1 along the direction a and the spin of particle #2 along the direction b, the results will be correlated, and for the singlet state the correlation is

$$\langle \Psi_0 | \sigma_a \otimes \sigma_b | \Psi_0 \rangle = -\cos \theta_{ab}$$

(2)

where $\theta_{ab}$ is the angle between the directions a and b.

Each spin polarization correlation experiment (A,B) is defined by two macroscopic orientation vectors A and B being some average orientation vectors of the analyzers. An analyzer A is defined by a probability distribution $d\rho_A(a)$, where
a are the microscopic direction vectors, \( a \in O_A = \{ a \in S^{(2)}; |1 - a \cdot A| \leq \varepsilon_A \} \).

Similarly a polarizer B is defined by \( d\rho_B(b) \). The probability \( p(A,B) \) that a particle #1 is detected by the analyzer A and a particle #2, correlated with the particle #1 is detected by a analyzer B could be given by

\[
p(A,B) = \eta(A) \eta(B) \int_{O_A} \int_{O_B} p_{12}(a,b) d\rho_A(a) d\rho_B(b)
\]  

where \( p_{12}(a,b) \) is a probability density function given by QM:

\[
p_{12}(a,b) = \frac{1}{2} \sin^2(\theta_{ab}/2)
\]

and \( \eta \)'s are some factors related to the efficiency of the detectors. Similarly the predicted correlation function \( E(A,B) \) to be compared with the experimental data is given by

\[
E(A,B) = \eta(A) \eta(B) \int_{O_A} \int_{O_B} -\cos\theta_{ab} d\rho_A(a) d\rho_B(b)
\]  

We see that the observable value of the spin projection characterizes only the whole beam of the "particles" which passed through a given analyzer A. Nearly 100% of the "particles" of this beam would pass by the subsequent identical analyzer A, but we have no prediction concerning any individual "particle" from the beam. and we have no strict spin anti-correlations between the members of each pair.

Let us now discuss the various proofs of the Bell’s inequalities.

### 0.3 Bell’s Theorem

For any random experiment we may find a non unique mathematical probabilistic model describing it. Given a probabilistic model there exist in general several random experiments which can be described by the model. To obtain the consistency of the probabilistic model with the experiment a particular experimental design and a protocol have to be adopted. It was clearly demonstrated by Bertrand [10] and discussed by us [38, 39].

To each random experiment we associate a random variable \( X \), a probability space \( S \) and a probability density function \( f_X(x) \) for all \( x \in S \).

If \( X \) is a discrete random variable \( \sum_x f_X(x) = 1 \) and \( P(X=x) = f_X(x) \) If \( X \) is a continuous random variable \( \int_S f_X(x) dx = 1 \) and

\[
P(a \leq X \leq b) = \int_a^b f_X(x) dx
\]  

where \( P(a \leq X \leq b) \) is a probability of finding a value of \( X \) included between \( a \) and \( b \). Note that \( P(X=x) = 0 \) for all \( x \in S \).

If in a random experiment we can measure simultaneously values of \( k \)-random variables \( X_1, \ldots, X_k \) we describe the experiment by a \( k \)-dimensional random variable \( X = (X_1, \ldots, X_k) \) a common probability space \( S \) and some joint probability density function \( f_{X_1,X_2,\ldots,X_k}(x_1, \ldots, x_k) \). From the joint probability density
function we can obtain various conditional probabilities and by integration over k-1 variables we obtain k marginal probability density functions \( f_{X_i}(x_i) \) describing \( k \) different random experiments each performed to measure only one random variable \( X_i \) and neglecting all the others. In this case we say that \( f_{X_i}(x_i) \) were obtained by conditionalization from a unique probability space \( S \). In general if the random variables \( X_i \) are dependent (correlated)

\[
f_{X_1, X_2, \ldots, X_k}(x_1, x_2, \ldots, x_k) \neq f_{X_1}(x_1) f_{X_2}(x_2) \cdots f_{X_k}(x_k)
\]

As we found in the preceding section each spin polarization correlation experiment \((A, B)\) is defined by two macroscopic orientation vectors \( A \) and \( B \) and the coincidence probabilities are given by (3) and the correlation functions are given by (4). It is impossible to perform different experiments \((A, B)\) simultaneously on the same couple of the particles therefore it does not seem possible to use a unique probability space \( S \) and to obtain, by conditionalization, the probabilities \( p(A, B) \) for all such experiments. This is why that it is not so strange that Bell’s inequalities proven using a common probability space do not agree with the predictions of QT.

Let us now analyze a model used by Clauser and Horne [20] to prove their inequalities:

\[
p(A, B) = \int_{\Lambda} p_1(\lambda, A) p_2(\lambda, B) d\rho(\lambda) \tag{7}
\]

where \( p_1(\lambda, A) \) and \( p_2(\lambda, B) \) are the probabilities of detecting component 1 and component 2 respectively, given the state \( \lambda \) of the composite system.

We see from (7) that a state \( \lambda \) is determined by all the values of strictly correlated spin projections of two components for all possible orientations of the polarizers \( A \) and \( B \). The polarizers are not perfect therefore the detection probabilities have been introduced. Therefore it is assumed in the model that even before the detection each component has well defined spin projection in all directions. The model is using a single probability space \( \Lambda \) and obtains the predictions on the probabilities \( p(A, B) \) measured in different experiments by conditionalization. As we told the same assumption was used in all other proofs of Bell’s theorem. Explicit description of states \( \lambda \) by the values of spin projections is also clearly seen in Wigner’s proof[46]. As we told the experiments \((A, B)\) are mutually exclusive so there is no justification for using such models.

If we try to prove the Bell’s inequalities by comparing only the experimental runs of different experiments we can not do it without some additional and questionable assumptions.

Let us simplify the argument we gave in[38]. We want to estimate a value of the spin expectation function \( E(A, B) \) for an experiment \((A, B)\). We have to perform several runs of the length \( N \) and find the value of the empirical spin expectation function \( r_N(A, B) \) for each run and after to estimate \( E(A, B) \) by
averaging over various runs. Let us associate with each member of a pair a spin function $s_1(x)$ or $s_2(x)$, taking the values 1 or -1, on the unit sphere $S^{(2)}$ (representing the orientation vectors of various polarizers). We assume also that $s_1(x) = - s_2(x) = s(x)$ for all vectors $x \in S^{(2)}$. We saw in equation (3) that the macroscopic directions $A$ and $B$ were not sharp therefore in each particular run we might have different direction vectors $(a, b)$ representing them. If for the simplicity we neglect this possibility, we get:

$$r_N(A, B) = -\frac{1}{N} \sum_i s_i(A)s_i(B)$$

(8)

where $N$ functions $s_i$ are drawn from some uncountable set of spin functions $F_0$.

If we consider a particular run of the same length from the experiment $(A, C)$ we get

$$r_N(A, C) = -\frac{1}{N} \sum_j s'_j(A)s'_j(C)$$

(9)

where $N$ functions $s'_j$ are drawn from the same uncountable set of spin functions $F_0$.

A probability that we have the same sets of spin functions in both experimental runs is equal to zero. Therefore in general we have completely distinct sets of functions in (8) and (9) and we are unable to prove the Bell’s theorem by using $r_N(A, B) - r_N(A, C)$. If we used the same sets of spin functions in the runs from the different experiments then we could replace (9) by (10)

$$r_N(A, C) = -\frac{1}{N} \sum_i s_i(A)s_i(C)$$

(10)

and we could easily reproduce the Bell’s proof finding his inequalities in the standard form or in the form given for the first time in the reference [15]:

$$|E(A, B) - E(A, B')| + |E(A', B') + E(A', B)| \leq 2$$

(11)

One could still have some doubts concerning the above argument for the sharp directions of the polarizers. (the samples are not the same but on the long run everything should average out, etc.) However if the directions of the polarizers are not sharp our random experiment is not only a random sampling from some unique population of the spin functions followed by their exact evaluation.

In the subquantal description of the experiment $(A, B)$ we have 3 populations: population of couples of correlated spin functions, microscopic directions of the polarizer $A$ and microscopic directions of the polarizer $B$. The sampling from these three populations produce the effective samples of the experimental data which are sets of couples of the numbers $\pm 1$ corresponding to a draw from these populations and the evaluation of the spin functions. Therefore if we change the experiment into $(C, D)$ the results may not be represented by conditionalization from some unique probability space common for $(A, B)$ and $(C, D)$. The smearing of the polarization directions is important in the impossibility of the rigorous proof of the Bell inequalities in this type of subquantal description of the phenomenon.
When the validity of the inequality is tested (11) one should estimate properly all the quantities and include the correct error bars [25].

Let us also notice that the act of passage of the i-th particle through a given analyzer A depends in a complicated way on its interaction with this polarizer. Therefore we should not consider a spin function as describing a state of a particle independent of its interaction with A. The spin functions $s_i$ in the (8) and (9) resume the interactions of the subsequent particles with the polarizers in a particular experiment. Therefore if we want to be rigorous we should replace (8) by (12).

$$r_N(A,B) = -\frac{1}{N} \sum_i s_i(A) s_i(B)$$

where $a_i \in O_A$ and $b_i \in O_B$. If we use the formula (12) there is no possibility of proving Bell’s theorem. Using this formula we can always obtain the results consistent with the equation (3). The formula (12) visualizes the contextual character of the observables.

In a trivial but artificial way a common probability space $S$ could be used in a case if we had four independent experiments described by four independent random variables $X_1, X'_1, X_2, X'_2$ and their probability density functions. If all possible values of these variables had the absolute value smaller or equal to 1 a proof of Bell’s inequalities would be extremely easy. In such a case the "spin" expectation function $E(X_1, X_2)$ is a product of expectation values of $X_1$ and $X_2$:

$$E(X_1, X_2) = \langle X_1 \rangle \langle X_2 \rangle$$

and we immediately get

$$|\langle X_1 \rangle \langle X_2 \rangle - \langle X_1 \rangle \langle X'_2 \rangle| + |\langle X'_1 \rangle \langle X'_2 \rangle + \langle X'_1 \rangle \langle X_2 \rangle| \leq$$

$$\leq |\langle X_2 \rangle - \langle X'_2 \rangle| + |\langle X_2 \rangle + \langle X'_2 \rangle| \leq 2$$

which is the exactly the inequality (11)

Of course if we assume the independence there are no correlations. The statistical independence is related to the separability of the statistical operator used recently by Krüger in his proofs of Bell’s inequalities in [33].

In the similar way the quantum correlations are neglected in the cryptographic proof by Herbert [29] reviewed by Ballentine [15]. The source of the singlet state is represented as a generator of the two correlated signals. If the two detectors (A, B) are aligned in the same directions the two messages (the strings of +1 and -1 are identical). If the detector B is rotated by an angle $\theta$ it is assumed that the rate of disagreement between the two $d(\theta)$ is due only to the change in the orientation of B and does not depend on the orientation of the spatially separated detector A. This assumptions leads to the inequality $d(2\theta) \leq 2d(\theta)$ which does not agree with the predictions of QM. Let us note that quantum mechanical correlations are the correlations between the counts of the distant detectors obtained by the coincidence technique and they are never perfect. The messages, string of the bits, are not send by the source they are only created by the coincidence technique after the results of the measurements for
each pair of the analyzers (A,B) are recorded. Therefore in each experiment the rate of the disagreement depends on the directions of both macroscopic devices not only on the one of them. Before the measurement there is no message. In Herbert’s approach the rate of disagreement is treated like a measure of the random errors of reading some preexisting incoming message which depends on the rotation of only one of the analyzers from its initial position. The subtle quantum mechanical statistical correlations between counts of A and B are simply ignored and the contextual character of the quantum observables is neglected.

0.4 Conclusions

The violation of the Bell inequalities requires neither the abandon of the Einsteinian separability nor the abandon of the realistic point of view according to which external reality is assumed to exist and to have definite properties. The properties of the reality are however not attributive but contextual. Without doubt in the SPCE a source is producing the pulses of some real physical field. These pulses are interacting with far away analyzers and produce the correlated clicks of the detectors. The interference and the diffraction of light has been successfully explained by the wave picture of Huygens and Maxwell in the classical physics. The violation of the Bell inequalities forces us to abandon naive realistic models according to which the source is producing a stream of couples of point like particles flying to the detectors, the couples having well defined individuality and the properties possessed in the attributive way. The subquantal intuitive picture, if it did exist, it would have to be of a completely different nature. This subquantal picture is however not needed. Quantum theory with its statistical interpretation provides the algorithms allowing to explain the results of the experiments in the microworld without providing any spatio-temporal description of the physical phenomena involved.

The lack of the deterministic predictions for the individual measurements and the SI interpretation of the quantum state vectors have implications for the quantum information. There is no problem with the implementation of quantum cryptography since transmission of the secret key can be realized successfully with the use of the short pulses of polarized light or with gaussian-modulated coherent states [32] instead of using the single photons.

The fact that the quantum state vector is not an attribute of a single quantum system requires more caution in the problems related to the implementation of the quantum computing devices [42,32,26]. A more detailed discussion of the contextual character of quantum observables [11,34,39] and its implication for the quantum computing will be given in the following paper.

0.5 References

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