A Computational Study of Rotating Spiral Waves and Spatio-Temporal Transient Chaos in a Deterministic Three-Level Active System

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(Dated: March 10, 2005)

Spatio-temporal dynamics of a deterministic three-level cellular automaton (TLCA) of Zykov-Mikhailov type (Sov. Phys. – Doklady, 1986, Vol.31, No.1, P.51) is studied numerically. Evolution of spatial structures is investigated both for the original Zykov-Mikhailov model (which is applicable to e. g. Belousov-Zhabotinskii chemical reactions) and for proposed by us TLCA, which is a generalization of Zykov-Mikhailov model for the case of two-channel diffusion. Such the TLCA is a minimal model for an excitable medium of microwave phonon laser, called phaser (D. N. Makovetskii, Tech. Phys., 2004, Vol.49, No.2, P.224; cond-mat/0402640). The most interesting observed manifestations of TLCA dynamics are as follows: (a) spatio-temporal transient chaos in form of highly bottlenecked collective evolution of excitations by rotating spiral waves (RSWs) with variable topological charges; (b) competition of left-handed and right-handed RSWs with unexpected features, including self-induced alteration of integral effective topological charge; (c) transient chimera states, i. e. coexistence of regular and chaotic domains in TLCA patterns; (d) branching of TLCA states with different symmetry which may lead to full restoring of symmetry of imperfect starting pattern. Phenomena (a) and (c) are directly related to phaser dynamics features observed earlier in real experiments at liquid helium temperatures on corundum crystals doped by iron-group ions.

ACM classes: F.1.1 Models of Computation (Automata), I.6 Simulation and Modeling (I.6.3 Applications), J.2 Physical Sciences and Engineering (Chemistry, Physics)

PACS numbers: 05.65.+b, 07.05.Tp, 82.20.Wt

I. INTRODUCTION

During last years vortices and, in particular, rotating spiral waves (RSWs) are widely studied in excitable media \([13, 21, 67]\), lasers \([72, 76, 87, 88, 89]\), Bose-Einstein condensates \([26]\), saturable nonlinear media \([92]\), ferromagnetics \([31, 50]\), photoelectronic optical lattices \([96]\), open microwave billiards \([35]\), three-level spatial Lotka-Volterra’s type models \([104, 101]\) etc.

Evolution of such complex spatio-temporal structures, as patterns of RSWs, is now a subject of extensive computer modeling, especially for intrinsically unstable systems possessing hypersensitivity to initial and/or border conditions, to imperfections of active medium etc.

Choice of appropriate model is the crucial point for computer studying of these complex systems. Even fully deterministic model of continuous unstable system may demonstrate unpredictable behaviour due to exponential increasing of small deviation of initial conditions from their precise values \([11]\). This unpredictability cannot be eliminated because of finite precision of initial data, available for a digital computer, even if no round-off errors exist in certain “perfect” digital machine. Generally, instead of computing of the evolution of a continuous unstable system (in the framework of given model), a digital machine may compute an evolution of some composite system, which includes the computer itself.

Another serious trouble is the large dimension problem, which is typical for modeling of many-particle systems. Using of appropriate averaging is a good solution for systems near equilibrium, but for strongly nonequilibrium systems it often is not so.

A possible way to bypass these troubles consists in using of partially or fully discrete analogs of continuous models \([12, 13, 22, 28, 29, 32, 42, 68, 79, 80, 85, 98]\). Construction of such models is very nontrivial task, but adequate discrete model is much more suitable for numerical investigation of unstable and/or multiparticle systems, than, say, a model based on partial differential equations.

In this work, we study evolution of RSWs and some other spatio-temporal structures in discrete active (excitable) system consisting of locally interacting three-level particles (units) with finite times of relaxation. This active system is described by Zykov-Mikhailov (ZM) model, which was primarily introduced in 1986 for chemical excitable systems \([95]\) (see also \([42]\)).

A modification of ZM model is proposed by us to adapt it for emulation of some aspects of dynamics of microwave phonon lasers \([43, 65, 82]\) (called also phasers \([24, 27]\).
(43) which demonstrate strong instabilities, deterministic chaos and other manifestations of complexity [43, 44, 45, 52, 53]. The relaxation properties of active units (paramagnetic ions) in phasers are of the same type, as in class-B lasers [51], and there are experimental confirmations [18, 48, 52, 53] of common properties of phasers and this class of lasers.

In class-B lasers the inequality \( T_1 \gg T_{\text{field}} \gg T_2 \) [51] takes place, where \( T_{\text{field}} \) is the lifetime of photons; \( T_1 \) and \( T_2 \) are respectively the times of longitudinal and transverse relaxation of active units. The same inequality we have for phasers, if one replace photons by phonons (see [43, 48, 44, 52, 53]). From this point of view phasers are acoustic analogs of class-B lasers. Moreover, acoustic wavelengths in phasers is of the same order as electromagnetic ones in optical lasers, because sound velocity in phaser is about 5 orders less than light velocity in laser (and phasers usually operate at frequencies \( F = F_{\text{phaser}} \approx 10^{-5} F_{\text{laser}} \)).

In papers [74, 87, 89] class-B lasers were treated as a kind of excitable systems, that are widely studied in chemical kinetics, especially for the Belousov-Zhabotinskii (BZ) reaction and other similar chemical phenomena [4, 21, 33, 67, 91]. Some of differences and similarities between class-B lasers and excitable systems were examined in [70].

"The difference is that excitable systems in zero-dimensional case (with spatially homogeneous fields) display self-sustained oscillations of the field amplitude, whereas ... [class-B systems] display (damped) relaxation oscillations only. In the limit\(^1\) \( \gamma_{||} \to 0 \) the relaxation oscillations decay very slowly; thus class-B lasers may be considered quasi-excitable systems in this limit. In two spatial dimensions (unlike the zero-dimensional case) class-B lasers display self-sustained oscillations: They behave quite similarly to excitable systems. The non-stationarity of the vortices [57] is one indication of the similarity with the excitable systems." (70, page 1143).

Using pointed similarity and taking into account the analogy between phasers and class-B lasers, a phaser dynamics in two-dimension (2D) case may be emulated by ZM model [12, 58] (with some extensions of the last, concerning relaxational properties of active units). The ZM model is a cellular automaton (CA) [02], i.e. discrete mapping, which may be defined as follows. Let an active discretized medium \( \mathcal{M}_e \) has the form of rectangular 2D-lattice. Each cell of the lattice contains one cellular-automaton unit (CAU) with coordinates \( (i, j) \), where \( (\min(i) = 1) \land (\min(j) = 1) \). All the CAUs in the \( \mathcal{M}_e \) are identical, and they interact by the same set of rules (CA is homogeneous and isotropic in the von Neumann sense [62]). The upgrade of state \( S_{ij}^{(n)} \equiv S_{ij}(i, j) \) of each CAU is carried out synchronously at each step \( n \leq N \) during the cellular automaton evolution. The final step \( N \) may be either predefined or it will be searched during CA evolution as time (i.e. quantity of discrete steps) for reaching an attractor of this CA. The conditions of upgrade depends both on \( S_{ij}^{(n-1)} \) and on \( S_{ij'}^{(n-1)} \), where \( \{i', j'\} \) belongs to certain active neighborhood of CAU at site \( (i, j) \).

In order to formulate boundary conditions for a \( \mathcal{M}_e \) of finite size (i.e. when \( \max(i) = M_X, \max(j) = M_Y \)), a set \( \mathcal{M}_e \) of virtual cells with coordinates \( (i = 0) \lor (j = 0) \lor (i = M_X + 1) \lor (j = M_Y + 1) \) may be introduced.

The original ZM model has the single channel of diffusion [42, 58]. Such one-channel (1C) models are adequate for chemical reaction-diffusion systems [83]. In phaser active system, the multichannel diffusion of spin excitations is the typical case, because it proceeds via (near)-resonant dipole-dipole \( (d - d) \) magnetic interactions between paramagnetic ions [18]. For a three-level system, which is the simplest phaser system, there are 3 possible channels of resonant diffusion. In the case when \( d - d \) interactions are forbidden at one of three resonant frequencies of a three-level system, the asymmetric two-channel (2C) diffusion is realized under conditions of perfect refractoriness of the intermediate level — see Appendix A. Note that asymmetric diffusion is well known in biology (see, e.g., the book of D. A. Frank-Kamenetzky [102]). Recently a kind of asymmetric diffusion was proposed by N. Packard and R. Shaw [103] for a mechanical system.

Another important feature of phaser is extremely low level of unavoidable (quantum) noise in its active medium because \( I_{\text{spont}} \propto F^3 \), where \( I_{\text{spont}} \) is intensity of spontaneous emission. Accordingly, \( I_{\text{spont}}^{(\text{phaser})} \) is about 15 orders lower than \( I_{\text{spont}}^{(\text{laser})} \), because \( F_{\text{phaser}} / F_{\text{laser}} \approx 10^{-5} \). It gives the basis to consider the phaser system as close to deterministic, i.e. one need not include stochastic terms [85] in CA rules for phaser modeling.

The paper is organized as follows. In Section II we review deterministic CA generating vortices, giving main attention to those of CA, in which dynamically stable RSWs are possible. Section III is devoted to detailed description of ZM-like TLCA model used in our computer experiments. The results of computational study of RSWs and discussion are presented in Sections IV and V. Section VI contains some concluding remarks. In Appendices A and B the structure of energy levels and relaxation properties of the Ni\(^{2+} \) : Al\(_2\)O\(_3\) spin-system (used in real experiments with phaser) are discussed. Phenomena of inversion states collapse and critical slowing down for lumped (point-like) phaser models are described in Appendix C in order to compare them with slowing down phenomena in TLCA which is a minimal multiparticle model of an active (excitable) medium of phaser.

\(^1\) In [28], \( \gamma_{||} = T_{\text{field}} / T_1 \).
II. ROTATING SPIRAL WAVES IN CELLULAR AUTOMATA: A SHORT REVIEW

Probably the simplest cellular automaton which generates rotating (but nonspiral) objects is the Conway’s Game of Life (CGL) [11]. CGL is a two-level (or nonspiral) cellular automaton defined on square grid (\(\mathbb{N}^2\) in CGL is usually infinite). The state of each cell with coordinates \((i, j)\) at step \(n\) of evolution is described by very simple relationship \(G = \Phi_{ij}^{(n)}\). Here \(\Phi_{ij}^{(n)}\) is binary phase counter: \(\Phi_{ij}^{(n)} \in \{0, 1\}\) with such the rules of upgrading:

\[
\Phi_{ij}^{(n+1)} = \begin{cases} 
\Phi_{ij}^{(n)}, & \text{if } U_{ij}^{(n+1)} = 2; \\
1, & \text{if } U_{ij}^{(n+1)} = 3; \\
0, & \text{otherwise,}
\end{cases}
\]

(1)

and the pair of indexes \(\{i', j'\} \equiv \{i+a, j+b\}\) describes the active neighborhood for the cell \((i, j)\). For the classical CGL [11], this is the Moore neighborhood, i.e. octet of the nearest cells: \([|a| \leq 1] \land [|b| \leq 1] \land (\delta_{a0}\delta_{b0} \neq 1)\), where \(\delta_{xy}\) is the Kronecker symbol.

In CGL, rotating (and simultaneously moving) structures called gliders are dynamically stable if there are no another CGL objects in their vicinity. Gliders are of great interest from the mathematical point of view, especially because it is possible to construct universal computing machine on the basis of gliders and glider guns. On the other hand, CGL (at least in its classical two-level form) does not describe any known real-world vortices.

Much more realistic description of vorticity, including RSWs, is possible in framework of three-level cellular automata (TLCA) or many-level ones. The simplest TLCA which generates rotating spiral objects is the single-step relaxation (SSR) model, which was extensively investigated by L. V. Reshodko with co-authors at the beginning of 1970-th \([12, 28, 68]\). Later a similar CA was independently proposed and studied by J. Greenberg and S. Hastings (GH model) [28].

A. The single-step relaxations model

In SSR model, every CAU at each step of evolution \(n\) occupies one of the three discrete levels \(L_K \in \{L_1, L_{III}, L_{II}\}\). The order of levels in curly brackets corresponds to the order of CAU’s state advancing direction (by cycle): \(L_1 \rightarrow L_{III} \rightarrow L_{II} \rightarrow L_1 \rightarrow \cdots\). Here \(L_1\) is the ground (lowest) level, which is stable in the absence of nearest neighbors with \(K \neq I\); \(L_{III}\) is the excited (upper) level, and the \(L_{II}\) is the refractory (intermediate) level. This “arrangement” of levels may be described by apparently formal (from the mathematical point of view) inequality \(L_{III} > L_{II} > L_1\). But from the physical point of view such the inequality is well defined (see Appendix A for an example of the concrete physical system).

The essence of the SSR model is as follows [12, 28, 68]. Let us introduce the phase counters \(\phi_{ij}^{(n)} \in \{-1, 0, +1\}\). In the SSR model, there is such the correspondence between \(L_K\) and \(\phi_{ij}^{(n)}\):

\[
\begin{align*}
(L_K = L_1) & \Leftrightarrow (\phi_{ij}^{(n)} = 0); \\
(L_K = L_{III}) & \Leftrightarrow (\phi_{ij}^{(n)} = +1); \\
(L_K = L_{II}) & \Leftrightarrow (\phi_{ij}^{(n)} = -1).
\end{align*}
\]

(3)

The iterative process (which governs the whole SSR cellular automaton evolution) describes transitions of every individual CAU by mapping of phase counters \(\phi_{ij}^{(n)}\):

\[
\phi_{ij}^{(n+1)} = \phi_{ij}^{(n)} + \delta_{ij}^{(n+1)}(\phi_{ij}^{(n)}, \phi_{i+r,j+s}^{(n)}),
\]

(6)

where

\[
\delta_{ij}^{(n+1)}(\phi_{ij}^{(n)}) = \begin{cases} 
-1, & \text{if } \phi_{ij}^{(n)} = +1; \\
0, & \text{otherwise,}
\end{cases}
\]

(7)

and

\[
\delta_{ij}^{(n+1)}(\phi_{ij}^{(n)}, \phi_{i+r,j+s}^{(n)}) = \begin{cases} 
+1, & \text{if } (\phi_{ij}^{(n)} = 0) \land (\exists \phi_{i+r,j+s}^{(n)} = +1); \\
0, & \text{otherwise.}
\end{cases}
\]

(8)

Here the pair of indexes \(\{i + r, j + s\}\) describes the von Neumann neighborhood for the cell \((i, j)\). This neighborhood consists of non-diagonal nearest cells only: \(|r| = 1 \lor |s| = 1\) (the symbol \(\lor\) means exclusive OR).

The SSR cellular automaton is one of the simplest 2D discrete mappings, which models RSWs in excitable media. But the original SSR model has insufficient resources to model real physical or chemical excitable systems. In particular, transitions in simplest cellular automata of the SSR type are “instant”: the lifetimes of excited and refractory states are precisely equal to the single step of an iterative process.

Improving of the SSR model may be achieved by some different ways: increasing of level numbers (“colors”) and/or extending of the active neighborhood (see R. Fisch, J. Gravener, and D. Griffeath [22] for details). There are also qualitatively different models which are able to generate RSW, as the 2D Oono-Kohmoto CA, described in the following Subsection.

B. The 2D Oono-Kohmoto model

The well-known 1D Oono-Kohmoto model (Y. Oono and M. Kohmoto [63]) may be generalized for the case
of 2D active medium (see paper of G. G. Malinetskii and M. S. Shakaeva \[57\]). Let every CAU has three levels \(\{L_0, L_1, L_M\}\), which corresponds to three discrete values of certain global attribute \(\chi\) (e. g., concentration of a chemical agent): \(\chi_{ij}^{(n)} \in \{0, 1, M\}\), where \(M > 0\). The evolution of the Oono-Kohmoto 2D (OK2) cellular automaton is described by such the discrete mapping:

\[
\chi_{ij}^{(n+1)} = \begin{cases} 
1, & \text{if } \chi_{ij}^{(n)} \geq h_2; \\
0, & \text{if } h_1 \leq \chi_{ij}^{(n)} < h_2; \\
M, & \text{if } \chi_{ij}^{(n)} < h_1,
\end{cases}
\]

(9)

where

\[
\tilde{\chi}_{ij}^{(n)} = (\alpha/N) \sum_{i',j'} \chi_{i',j'}^{(n)} + (1 - \alpha)\chi_{ij}^{(n)}.
\]

(10)

Here \(h_1, h_2\) are lower and upper thresholds for \(\tilde{\chi}_{ij}^{(n)}\) (e. g. \(h_1 = 1/2; h_2 = 3/2\)), and \(\alpha\) characterizes the diffusion rate. The pair of indexes \(\{i',j'\} = \{i + p, j + q\}\) describes the active neighborhoods for the cell \((i, j)\); \(N\) is the coordination number (quantity of cells in the active neighborhood). This may be, e. g., the von Neumann neighborhood \((N = 4)\), the Moore neighborhood \((N = 8)\) etc.

G. G. Malinetskii and M. S. Shakaeva revealed RSWs, gliders and other interest objects in the OK2 cellular automaton \[57\]. But the OK2 model (as far as the SSR one) does not take into consideration relaxation processes. In the next Subsection we describe some CA models with multi-step relaxation (MSR) mechanisms.

C. Multi-step relaxation models of excitable media

Introducing of finite times of relaxation (i.e. delay of interlevel transition, which is independent of state of an active neighborhood) leads to an important improving of cellular-automata models of excitable media. Each of non-ground levels in an MSR model has some virtual “sublevels”, and relaxation of CAU may be formally described as subsequent jumps between these “sublevels”.

An extensive investigation of various three-level CA with MSR was fulfilled by L. V. Reshodko with co-authors in early 1970-th \[12, 13, 34, 65\] to model excitations in smooth muscle tissue. The starting point of their work was three-level Wiener-Rosenblueth (WR) model \[91\], elaborated in 1946 by N. Wiener and A. Rosenblueth for continuous media. In the original WR model \[91\], only the refractory level had finite relaxation time. The original WR model was modified by A. Rosenblueth in 1958 \[62\] with the purpose to take into consideration finite-time relaxation of excited level. The WR model was developed in \[94\] to model RSWs in inhomogeneous media only. But, as it was shown by O. Selfridge \[71\] and I. S. Balakhovskii \[10\], such the waves are possible in a parametrically homogeneous excitable medium too.

Another successful attempt to use of such the approach was undertaken by T. Toffoli and N. Margolus (their results, to our knowledge, were firstly published in 1987 in the book \[50\]). About one year earlier (in 1986) V. S. Zykov and A. S. Mikhailov had published paper \[98\], in which a simple and very clear TLCA was proposed to model excitable media with arbitrary relaxation times (both at excited and refractory levels), arbitrary factor of activator accumulation (at ground level) and arbitrary active neighborhood. In essence, the model of Zykov and Mikhailov \[98\] is a generalization of Bogach and Reshodko (BR) model \[12, 13\].

Birth, evolution, interaction and decay of RSWs and other spatio-temporal structures in active (excitable) media are very sensitive to mechanisms of diffusion. Both BR \[12, 13\] and ZM \[12, 98\] models are based on the 1C diffusion mechanism, which seems to be a good approximation for some chemical systems (see, e.g., \[83\]). Realistic models of several physical excitable system (class-B lasers \[76, 87, 89\], phasers \[43, 44, 48, 49, 52, 53\] etc.) need to take into consideration additional channels of diffusion, as it was discussed in Section II. In the next Section we present a detailed description of the MSR model of excitable system, based on ZM model and modified by us in order to introduce such an additional channel in TLCA. This modified ZM cellular automaton with 2C diffusion was used in the present paper as the basic model for computer studying of RSWs and transient spatio-temporal chaos in active (excitable) media.

III. THREE-LEVEL CELLULAR AUTOMATON WITH TWO-CHANNEL DIFFUSION
(A MODIFIED ZYKOV-MIKHAILOV MODEL)

A. States of cellular automaton units and branches of evolution operator in TLCA

Let \(M_e\) is a rectangular 2D lattice containing \(M_X \times M_Y\) three-level CAUs. The upgrade of state \(S_{ij}^{(n)}\) of each CAU is (as usually) carried out synchronously during the cellular automaton evolution. The excited (upper) level \(L_{III}\) has the time of relaxation \(\tau_e \geq 1\), and the refractory (intermediate) level \(L_{II}\) has the time of relaxation \(\tau_r \geq 1\) (as in the original ZM model \[98\] and in contrary to the SSR model, where the equality \(\tau_e = \tau_r = 1\) is embedded in). Both \(\tau_e\) and \(\tau_r\) are integer numbers.

In the TLCA model with 2C diffusion, the first channel of diffusion accelerates the transitions \(L_1 \rightarrow L_{III}\) for a given CAU, and the second channel of diffusion (which is absent in the ZM model) accelerates the transitions \(L_{III} \rightarrow L_{II}\). The complete description of CAU’s states \(S_{ij}^{(n)}\) in such the TLCA model includes one type of global attributes (the phase counters \(\psi_{ij}^{(n)}\)) and two types of partial attributes \(u_{ij}^{(n)}\) and \(z_{ij}^{(n)}\) for each individual CAU in \(M_e\). Full description of all CAU’s possible states is as
follows:

\[ S_{ij}^{(n)} (L_I) = (\varphi_{ij}^{(n)}, u_{ij}^{(n)}) ; \quad (11) \]

\[ S_{ij}^{(n)} (L_{III}) = \begin{cases} (\varphi_{ij}^{(n)}), & \text{if } n = 0, \\ (\varphi_{ij}^{(n)}, \varphi_{ij}^{(n)}), & \text{if } n \neq 0; \end{cases} \quad (12) \]

\[ S_{ij}^{(n)} (L_{II}) = (\varphi_{ij}^{(n)}). \quad (13) \]

In the framework of the TLCA model, the phase counters lie in the interval \( \varphi_{ij}^{(n)} \in [0, \tau_e + \tau_r] \). The following correspondences between \( \varphi_{ij}^{(n)} \) and \( L_K \) take place (by definition) for all the CAUs in \( \mathcal{M}_e \) at all steps \( n \) of evolution:

\[ (L_K = L_I) \Leftrightarrow (\varphi_{ij}^{(n)} = 0); \quad (14) \]

\[ (L_K = L_{III}) \Leftrightarrow (0 < \varphi_{ij}^{(n)} \leq \tau_e); \quad (15) \]

\[ (L_K = L_{II}) \Leftrightarrow (\tau_e < \varphi_{ij}^{(n)} \leq \tau_e + \tau_r). \quad (16) \]

These correspondences are of key significance for all MSR models (BR model [12, 13], ZM model [42, 98] etc.): there are only three discrete levels, and relaxation of each CAU is considered as intralevel transitions (or transitions between virtual sublevels).

The evolution of each individual CAU proceeds by sequential cyclic transitions \( L_K \rightarrow L_{K'} \) (where \( K \) and \( K' \in \{I, III, II\} \)), induced by the Kolmogorov evolution operator \( \hat{\Omega} \) [84]. In the TLCA model, the evolution operator \( \hat{\Omega} \) has three orthogonal branches \( \hat{\Omega}_I, \hat{\Omega}_{III} \) and \( \hat{\Omega}_{II} \), which we call ground, excited and refractory branches respectively. The choice of the branch at iteration \( n + 1 \) is dictated only by the global attribute of the CAU at step \( n \), namely:

\[ \varphi_{ij}^{(n+1)} = \hat{\Omega} \left( \varphi_{ij}^{(n)} \right) = \begin{cases} \hat{\Omega}_{I} \left( \varphi_{ij}^{(n)} \right), & \text{if } \varphi_{ij}^{(n)} = 0, \\ \hat{\Omega}_{III} \left( \varphi_{ij}^{(n)} \right), & \text{if } 0 < \varphi_{ij}^{(n)} \leq \tau_e, \\ \hat{\Omega}_{II} \left( \varphi_{ij}^{(n)} \right), & \text{if } \tau_e < \varphi_{ij}^{(n)} \leq \tau_e + \tau_r. \end{cases} \quad (17) \]

**B. Ground branch of the evolution operator**

At step \( n + 1 \), the branch \( \hat{\Omega}_I \) by definition fulfills operations only over those CAUs, which have \( L_K = L_I \) at step \( n \). These operations are precisely the same, as in ZM model [42, 98], namely:

\[ \varphi_{ij}^{(n+1)} = \hat{\Omega}_I \left( \varphi_{ij}^{(n)} \right) = \begin{cases} 0, & \text{if } (\varphi_{ij}^{(n)} = 0) \land (u_{ij}^{(n+1)} < h); \\ 1, & \text{if } (\varphi_{ij}^{(n)} = 0) \land (u_{ij}^{(n+1)} \geq h), \end{cases} \quad (18) \]

\[ u_{ij}^{(n+1)} = A_{ij}^{(n)} \left( S_{ij}^{(n)} \right) + D_{ij}^{(n)} \left( S_{i+p,j+q}^{(n)} \right) = g u_{ij}^{(n)} + \sum_{p,q} C(p,q) J_{i+p,j+q}^{(n)}, \quad (19) \]

where \( A_{ij}^{(n)} \) is the accumulating term for the \( u \)-agent, which is an analog of chemical activator [12, 98]; \( D_{ij}^{(n)} \) is the first-channel diffusion term; \( h \) is the threshold for the \( u \)-agent (\( h > 0 \)); \( g \) is the accumulation factor for the \( u \)-agent (\( g \in [0, 1] \)); \( C(p,q) \) is the active neighborhood of the CAU at site \((i,j)\); and the \( u \)-agent arrives to CAUs with \( L_K = L_I \) only from CAUs with \( L_K = L_{III} \) in \( C(p,q) \):

\[ J_{i+p,j+q}^{(n)} = \begin{cases} 1, & \text{if } (0 < \varphi_{i+p,j+q}^{(n)} \leq \tau_e); \\ 0, & \text{if } (\varphi_{i+p,j+q}^{(n)} > \tau_e) \lor (\varphi_{i+p,j+q}^{(n)} = 0). \end{cases} \quad (20) \]

The definition of the diffusion term \( D_{ij}^{(n)} \) in Eqn. [19] is very flexible. Apart of well-known neighborhoods of the Moore \( C(p,q) = C_M(p,q) \) and the von Neumann \( C(p,q) = C_N(p,q) \) types:

\[ C_M(p,q) = \begin{cases} 1, & \text{if } \left[ (|p| \leq 1) \land (|q| \leq 1) \right] \land (\delta_{p0} \delta_{q0} \neq 1); \\ 0, & \text{otherwise}, \end{cases} \quad (21) \]
\[ C_N(p, q) = \begin{cases} 1, & \text{if } \left[ (|r| = 1) \oplus (|s| = 1) \right]; \\ 0, & \text{otherwise}, \end{cases} \] (22)

one can easily define another neighborhoods, e.g., of the box type \( C_B(p, q) \) or of the diamond type \( C_D(p, q) \) (22) which are the straightforward generalization of the Moore \( C_M(p, q) \) and the von Neumann \( C_N(p, q) \) neighborhoods, etc.

On the other hand, the ZM definition \( J_i^{(n)} \) of the weight factors \( J_i^{(n)}(x, p, q) \) may be extended out of the binary set \{0,1\} to model various distance-dependent phenomena within, e.g., \( C_D(p, q), C_B(p, q) \).

In this work, however, we restricted ourselves to the Moore neighborhood and by weight factors of the form \( J_i^{(n)} \). Another types of neighborhoods, weight factors and some other modifications of the model will be studied in subsequent papers.

C. Excited branch of the evolution operator

At the same step \( n + 1 \), the branch \( \hat{\Omega}_{III} \) fulfills operations only over those CAUs, which have \( L_K = L_{III} \) at step \( n \):

\[ \varphi_{ij}^{(n+1)} = \hat{\Omega}_{III} \left( \varphi_{ij}^{(n)} \right) = \begin{cases} \varphi_{ij}^{(n)} + 1, & \text{if } \left[ (0 < \varphi_{ij}^{(n)} < \tau_e) \land (z_{ij}^{(n+1)} < f) \right] \lor (\varphi_{ij}^{(n)} = \tau_e); \\ \varphi_{ij}^{(n)} + 2, & \text{if } (0 < \varphi_{ij}^{(n)} < \tau_e) \land (z_{ij}^{(n+1)} \geq f). \end{cases} \] (23)

\[ z_{ij}^{(n+1)} = D_{ij}^{(n)} \left( S_{i+p,j+q}^{(n)} \right) = \sum_{p,q} C(p, q) Q_{i+p,j+q}^{(n)}, \] (24)

where \( D_{ij}^{(n)} \) is the second-channel diffusion term; \( f \) is the threshold for the \( z \)-agent (\( f > 0 \)), and we assume that \( z \)-agent arrives to excited CAU (\( L_K = L_{III} \)) only from those CAUs, which have \( L_K = L_I \) in \( C(p, q) \):

\[ Q_{i+p,j+q}^{(n)} = \begin{cases} 1, & \text{if } \varphi_{i+p,j+q}^{(n)} = 0; \\ 0, & \text{if } \varphi_{i+p,j+q}^{(n)} \neq 0. \end{cases} \] (25)

One can see from (24) that the \( z \)-agent does not accumulate during successive iterations. In other words, the branch \( \hat{\Omega}_{III} \) at step \( n + 1 \) produces "memoryless" values of partial attributes \( z_{ij}^{(n+1)} \) for CAUs having \( L_K = L_{III} \) at step \( n \) (in contrary to the \( u \)-agent for CAUs having \( L_K = L_I \) at step \( n \)). The \( z \)-agent may accelerate transitions from excited CAUs to refractory ones. This is important difference between 2C model and the original 1C model of ZM \( \{42, 98\} \); and 2C model qualitatively describes the real active media used in quantum acoustics (see Appendices \( A \) and \( B \)).

D. Refractory branch of the evolution operator

The branch \( \hat{\Omega}_{II} \) does not produce/change any partial attributes at all (because the intermediate level \( L_{II} \) is in the state of refractority). It fulfills only such the operations over CAUs, which have \( L_K = L_{II} \) at step \( n \):

\[ \varphi_{ij}^{(n+1)} = \hat{\Omega}_{II} \left( \varphi_{ij}^{(n)} \right) = \begin{cases} \varphi_{ij}^{(n)} + 1, & \text{if } \tau_e < \varphi_{ij}^{(n)} < \tau_e + \tau_r; \\ 0, & \text{if } \varphi_{ij}^{(n)} = \tau_e + \tau_r. \end{cases} \] (26)

Generally speaking, there are many examples of active media with weak refractority (when the unit at the intermediate level \( L_{II} \) is not absolutely isolated from its neighborhood units). But in this work we restrict ourselves by the case of perfect refractority \( \{20\} \), which is valid, e.g., for the phaser systems of the Ni\( ^2+ \):\ Al\( _2 \)O\( _3 \) type (as it was pointed out in Chapter \( II \) and is explained in Appendix \( A \)).

E. Reduction of the TLCA model to the ZM automaton

The second-channel diffusion gives the contribution to the TLCA dynamics if \( f \leq N \), i.e., if \( f \leq 4 \) for \( C = C_N \), \( f \leq 8 \) for \( C = C_M \) and so on.
At \((C = C_X) \land (f > 4)\) our TLCA model is of ZM-like (i.e. 1C) type, and at \((C = C_M) \land (f > 8)\) it becomes equivalent to the original ZM model \[12\ Q8\], which (with slight rearrangement of cases) is as follows:

\[
\varphi_{ij}^{(n+1)} = \begin{cases} \varphi_{ij}^{(n)} + 1, & \text{if } 0 < \varphi_{ij}^{(n)} < \tau_e + \tau_r; \\ 1, & \text{if } (\varphi_{ij}^{(n)} = 0) \land (u_{ij}^{(n+1)} \geq h); \\ 0, & \text{if } \left( (\varphi_{ij}^{(n)} = 0) \land (u_{ij}^{(n+1)} < h) \right) \lor (\varphi_{ij}^{(n)} = \tau_e + \tau_r), \end{cases}
\]

where \(u_{ij}^{(n)}\) is defined as in \[19\]–\[21\].

**F. Comparison with other models**

In our computer experiments we used such the conditions: \(J_{i+p,j+q}^{(n)} \in \{0;1\} \land (C(p,q) \in \{0;1\})\). Under these conditions and in particular (but very important) case of \(g \in \{0;1\}\), the ZM model \[12\ Q8\] becomes of fully integer kind. If all these conditions are supplemented by \(Q_{i+p,j+q}^{(n)} \in \{0;1\}\), the TLCA model \[11\ ]–\[20\] becomes fully integer too. Finally, at \(C = C_M, g = 0, h = 1, f > 8\) the TLCA model \[11\ ]–\[20\] is equivalent to the simplest variant of the ZM model \[12\ Q8\], which corresponds to a discrete form of the original WR model \[90\].

On the other hand, the TLCA model \[11\ ]–\[20\] (and, usually, the original ZM model \[12\ Q8\] may be considered as the generalization of the SSR model. This generalization is different from those of R. Fisch e. a. \[22\] in many aspects; first of them consists in using a concept of multilevel transitions in \[22\] instead of a concept of relaxation times, used in works \[12\ Q8\] and in the present work, formulae \[11\ ] to \[20\]. But in the limiting case of “instant” (i.e. one-step) transitions both TLCA model and the model of R. Fisch e. a. \[22\] reduce to the same original SSR model. For the TLCA model this is the case of \(C = C_X, f > 4, g = 0, h = 1, \tau_e = \tau_r = 1\). Under these conditions the system of equations \[11\ ]–\[20\] reduces to \[3\ ]–\[5\] if one uses the ansatz \((\phi_{ij}^{(n)} \in (-1,0,1)) \rightarrow (\phi_{ij}^{(n)} \in \{0,1,2\})\) in the SSR model. Such the ansatz is correct because the SSR model in the form of \[3\ ]–\[5\] is almost insensitive to the phase counter value for \(L_{II}\); the “-1” in the equations \[1\] and \[4\] may be changed to any \(X \not\in \{0,+1\}\).

At \((\tau_e > 1) \land (\tau_r > 1)\) our TLCA model is of MSR type, and relaxation of each individual CAU is described in the same manner as in various Reshodko’s MSR models of excitable media \[12\], but diffusion mechanism in TLCA model is algorithmized using more versatile way.

There is no direct correspondence between TLCA (ZM) model and the OK2 model introduced in \[57\], despite of definite similarities in spatio-temporal dynamics of these CA. Partial correspondence is between TLCA (ZM) model and one of variants of Toffoli-Margolus model, namely with its “monotonic-threshold” variant (see Chap. 9 in \[80\]). The TLCA model reduces to that variant of Toffoli-Margolus model if \(C = C_M, f > 8, g = 0, h \in \{2,3,4\}, \tau_e = 1, \tau_r = 2\).

**G. Geometry of active media, boundary conditions and transients in CA**

Pattern evolution is very sensitive to geometry of active medium and boundary conditions even in the framework of operation of the same CA. There are three types of geometry and boundary condition (GBC) combinations most commonly used in computer experiments with CA (see Table 1). Let us consider them in details.

- The GBC-1 type is defined as follows:

\[
S^{(n)}(i_v, j_v) = S^{(n)}(i^\Gamma, j^\Gamma_v),
\]

where

\[
i^\Gamma_v = \begin{cases} M_X & \text{if } i_v = 0; \\
1, & \text{if } i_v = M_X + 1; \\
i_v, & \text{otherwise,}
\end{cases}
\]

\[
j^\Gamma_v = \begin{cases} M_Y & \text{if } j_v = 0; \\
1, & \text{if } j_v = M_Y + 1; \\
j_v, & \text{otherwise.}
\end{cases}
\]

Here \((i_v, j_v) \in \mathbb{M}_v\), and the coordinates \((i^\Gamma_v, j^\Gamma_v)\) belong to the border set \(\mathbb{M}^* \subset \mathbb{M}_v\) of excitable area, i. e. \((i^\Gamma_v, j^\Gamma_v) = (i^\Gamma_v, j^\Gamma_v)\) if \((i = 1) \lor (j = 1) \lor (i = M_X) \lor (j = M_Y)\).

- The GBC-2 type can be easily defined by using an extended interval for the phase counter \(\varphi\). Let every cell \((i_v, j_v)\) in \(\mathbb{M}_v\) contains one unexcitable CAU. All these unexcitable CAUs have frozen \(\varphi^{(n)}(i_v, j_v)\) at all \(n \in [0, N]\) without reference to states of \((i^\Gamma_v, j^\Gamma_v)\), and

\[
\varphi^{(n)}(i_v, j_v) = \tau_e + \tau_r + \alpha,
\]

where \(\alpha > 0\), say \(\alpha = 1\) without loss of generality. These unexcitable CAUs in \(\mathbb{M}_v\) “absorb” excitations at the border of \(\mathbb{M}_v\) in contrary to the case of GBC-1, where excitations at the border \(\mathbb{M}_v\) are re-injected in active medium. In many cases this difference leads to qualitatively different behaviour of the whole CA. Note that by this way one may also define CA with inhomogeneous active medium (where some of CAUs are unexcitable not only in \(\mathbb{M}_v\), but in \(\mathbb{M}_e\) too), introducing an additional orthogonal branch \(\hat{\Theta}_0\) into the evolution operator \(\hat{\Theta}\). This branch is activated at \(\varphi^{(n)}(i_v, j_v) = \tau_e + \tau_r + 1\) only and is simply the
identity operator $\hat{\Omega}_0 = \hat{I}$, i.e.: $\varphi_{ij}^{(n+1)} = \hat{\Omega}_0 \varphi_{ij}^{(n)} = \varphi_{ij}^{(n)}$. And even more complex behaviour of such “impurity” CAUs may be defined using analogous approach (i.e. by introducing additional orthogonal branches in an evolution operator): there may be pacemakers [42] or another special units embedded in active medium and described by phase counter in extended areas $(\varphi > \tau_c + \tau_r) \vee (\varphi < 0)$.

- The GBC-3 type is, strictly speaking, a case of borderless system. But the starting pattern has, of course, bounded quantity of CAUs with $L_K \neq L_t$, located in bounded part of active medium. Hence, during the whole evolution ($1 \leq n \leq N$) the front of growing excited area will meet the unexcited (but excitable!) CAUs only. For the case GBC-3, there is neither absorption of excitations at boundaries (as for GBC-2), nor feedback by reinsertion of excitations in the active system (as for GBC-1).

From this point of view CA of GBC-3 type are “simpler”, than CA of GBC-1 and GBC-2 types. On the other hand, CA of GBC-3 type are potentially infinite discrete systems where true aperiodic (irregular, chaotic) motions are possible, in contrast to finite discrete systems, possessing, of course, only periodic trajectories in certain phase space after ending of transient stage [11].

On the other hand, there is also very important difference between GBC-2 and other two types of GBC. The system of GBC-2 type is the single from the three ones under consideration which interacts with the external world dynamically (besides of relaxation). Sure enough, GBC-1 and GBC-3 are connected to this outer world only by relaxation channels (the $\tau_c$ and $\tau_r$ are the measures of this connection). In contrary, a CA of the GBC-2 type interacts with surroundings through the real boundary, which is in fact absent in toroidal finite-size active medium of GBC-1 type and it is absent by definition for CA of GBC-3 type. Despite of elementary mode of such interaction (boundary simply "absorbs" the outside-directed flow of excitations from $j_c f_j^r$, a CA of the GBC-2 type may demonstrate very special behaviour. The main attention in this work is devoted just to TLCA of GBC-2 type as the most realistic model of phaser active system, where both the mechanisms of interaction (dynamical and relaxational) of an active medium with the outer world are essential. Dissipation in a phaser active medium (highly perfect single crystal at liquid helium temperatures) is caused by two main mechanisms: (a) dynamical, by coherent microwave phonon and photon emission directly through crystal boundaries and (b) relaxational, by thermal phonon emission. There are, of course, many more or less important differences between the TLCA model and the real phaser systems [24, 25, 14, 18, 11, 52, 53, 67, 82], but in any case autonomous phaser has only these two mechanisms of interaction with the outer world.

### H. Initial conditions

Patterns $P^{(n)}$ in CA are usually described in terms of levels (or “colors”) of CAUs, and initial conditions $P^{(0)}$ are formulated simply as matrix of CAUs levels $L_K$. But states of CAUs in such the automata as ZM or TLCA are fully defined not only by levels $L_K$ itself. There are global attributes which must be predefined before TLCA evolution is started. Some of parital attributes must be predefined too. These points are essential for reproducing of the results of computer experiments.

In our work such the initial conditions for the global attributes $\varphi_{ij}$ are used:

$$
\begin{aligned}
\varphi_{ij}^{(0)} &= 0, & \text{if } L_K^{(0)} &= L_t; \\
\varphi_{ij}^{(0)} &= 1, & \text{if } L_K^{(0)} &= L_{tt}; \\
\varphi_{ij}^{(0)} &= \tau_c + 1, & \text{if } L_K^{(0)} &= L_{tt},
\end{aligned}
$$

Initial conditions for $u_{ij}$ must be defined for ground-state CAU’s $(L_K^{(0)} = L_t)$ only. In this work, we suppose

$$
(u_{ij}^{(0)}) \text{ IFF } (\varphi_{ij}^{(0)}) = 0,
$$

where IFF means “if and only if”. Initial conditions for $z_{ij}$ are undefined for all $L_K^{(0)}$, because $z$-agent is not defined at $n = 0$, see Eq. (33). As the result, starting pattern $P^{(0)}$ may be defined (both for the ZM and TLCA models) as the matrix $\| \varphi_{ij}^{(0)} \|$ (where $\varphi_{ij}^{(0)} \in \{0,1,\tau_{e}+1\}$) with additional condition given by Eq. (33).

### IV. RESULTS OF COMPUTER EXPERIMENTS

Cellular automata are inherently based on irreducible algorithms. Generally speaking, there are no predictive procedures for such the systems. So, the best way to investigate cellular automaton (TLCA in particular) is to run it, because, as S. Wolfram pointed out, ”their own evolution is effectively the most efficient procedure for determining their future” (see [32], page 737).

Here we present results of our computer experiments both with original 1C automaton of ZM and with 2C automaton described in Section III. The main tool in these experiments was the program “Three-Level Cellular Automaton” (TLCA © 2004 S. D. Makovetskiy [50]), which is included now in the software package “Generalized Wiener-Rosenblueth Model” (GWR © 2001-2003 S. D. Makovetskiy [55]; see also [32]).

Computer experiments were fulfilled for wide ranges of control parameters (CP): $(1 \leq \tau_c \leq 50) \land (1 \leq \tau_r \leq 50)$.
(for all the cases); $0 < g < 1$ (for non-integer versions of TLCA and GWR); $g \in \{0; 1\}$ (for integer versions of TLCA and GWR); $2 \leq h \leq 100$ (for all the cases); $1 \leq f \leq 9$ (the case $f = 9$ corresponds to original ZM model — see Subsection III.E).

A discrete system with finite set of levels may have only two types of dynamically stable states at bounded lattice. The first of them is stationary state, and the second is periodic one. They may be called attractors by analogy with lumped dynamical systems (see e.g. [40, 93, 94]). For our cellular automaton, the first of such the attractors is spatially-uniform and time-independent state with $\varphi_{ij}^{(n)} = 0$, where $(i \in [1, M_X]) \land (j \in [1, M_Y])$, $n \geq n_C$. In other words, the only stationary state of TLCA is the state of full collapse of excitations at some step $n_C$ (by definition of excitable system). The second type of TLCA attractors includes many various periodically repeated states (RSWs is a typical but not the single case). In this case $\varphi_{ij}^{(n)} = \varphi_{ij}^{(n+T)}$ where $(i \in [1, M_X]) \land (j \in [1, M_Y])$, $n \geq n_P$; $n_P$ is the first step of motion at a periodic attractor; $T$ is the integer-number period of this motion, $T > 1$.

If starting patterns $P_{ID}^{(0)}$ (where $ID$ is the pattern identifier) are generated with random spatial distribution by levels, then the time intervals $n_C$ or $n_P$ may be considered as times of full ordering in the system. Such irregular transients are of special interest from the point of view of nonlinear dynamics of distributed systems [18, 90, 60, 61] because they may be bottlenecked by very slow, intermittent morphogenesis of spatial-temporal structures. In the next Subsection we study this collective relaxation both for cases of collapse and periodic final states of TLCA evolution.

### A. Bottlenecked collective relaxation in TLCA

Evolution of TLCA usually has several stages with very different characteristic times and qualitatively different spatio-temporal dynamics. An example of relatively simple evolution is shown at Figure 1 for starting pattern $P_A^{(0)} = P_A^{(0)}$, \(^2\) Here $f = 8$, but at $f = 9$ this starting pattern reaches precisely the same attractor. This is because our TLCA model at $f = 8$ is very close to ZM model. But even at $f = 8$ such coinciding of attractors is relatively rare (less than 10 percent for the CP set pointed at caption to Figure 1 and starting patterns of the type shown at Figure A1). In other words TLCA model even at $f = 8$ is not equal to ZM model. Much more typical case is gradual divergence of spatio-temporal structures for $f = 8$ and $f = 9$ during evolution (see Figure 2). And for $f < 8$ we have not observed any attractors coinciding with ones for $f = 9$.

Evolution of TLCA with $f < 8$ is more complicated, and transient time may rich giant values (millions of steps for the same $M_X = M_Y = 100$ as for patterns at Figure 1 and Figure 2) due to bottlenecked RSWs morphogenesis. Under certain conditions, the morphogenesis of RSWs proceeds by multiple irregular changings of effective topological charge\(^3\) $Q_T$, including reversing of $\text{sgn}(Q_T)$. For example, at $f = 4$ the system with starting pattern $P_C^{(0)} = P_C^{(0)}$ (see the $P_C^{(0)}$ at Figure A1) reaches its periodic attractor only at $n \approx 4 \cdot 10^6$. Seemingly perfect RSW with $Q_T = +2$ is forming at $n \approx 3 \cdot 10^5$, but it is obviously not an attractor. At $n \approx 4.5 \cdot 10^5$ the core of RSW becomes complicated and gradually evolves to state with $Q_T = +3$ (Figure 3, step $n = 500525$). During sequence of several metamorphoses, the system returns to $Q_T = +2$ (Figure 3, step $n = 1.35 \cdot 10^6$). And at $n = 1.63 \cdot 10^6$ one can see RSW with opposite sign ($Q_T = -2$). The further evolution continues this unpredictable scenario, ending at attractor with $Q_T = -3$ (steps $n = 3.97 \cdot 10^6$ to $n = 4.10^6$ at Figure 3). Besides of quantity and sign of $Q_T$, the frequency of spatial waves irregularly changes too — compare, e.g., patterns at $n = 2 \cdot 10^5$ and $n = 1.09 \cdot 10^6$ (or at $n = 1.94 \cdot 10^6$ and $n = 2.59 \cdot 10^6$ etc.).

It is interesting to compare such scenario of evolution of TLCA with a scenario of crystal growth due to spiral (screw) defects increasing. The results are just opposite: crystal growth is usually accelerated by many orders due to screw defects, but TLCA relaxation (transient to an attractor) is highly bottlenecked by RSWs.

There is a huge amount of periodic attractors for such TLCA, but the steady-state attractor is the single one. It is fully collapsed state of an excitable system, as it was pointed out earlier. On the other hand, there is very many starting patterns that evolve to this collapsed state (our TLCA is an irreversible system). An example of very slow transition to collapse is shown at Figure 4.

Bottlenecked collective relaxation shown at Figure 3 and Figure 4 is, of course, aperiodic. But it has some quasiperiodic features because observed at Figure 3 and Figure 4 spirals (or spiral “domains") possess more or less regular structure. Much more chaotic spatio-temporal behaviour one can observe varying CP of the TLCA. Such a complex, slow and unpredictable evolution of a simple TLCA system (with $f = 9$) is shown at Figure 5. This is a typical example of transient spatio-temporal chaos in collective relaxation of excitations, which we observed in TLCA.

Such highly bottlenecked and irregular transient dynamics (metaphorically, “turbulence") was observed numerically by J. P. Crutchfield and K. Kaneko [18] in an 1D model which is a kind of coupled map lattice (CML) model [32]. In contrary to CA and other discrete map-

---

\(^2\) Typical set of starting patterns with $M_X = M_Y = 100$, used in our numerical experiments, see Figure A1.

\(^3\) Effective topological charge is defined in the near vicinity of the vortex core.
ings, CML is not fully discrete system because of continuous spectrum of states of its elementary units. In other words, CML has not any well resolved levels. But both TLCA and CML demonstrate lethargic transients which are not consequences of usual critical slowing down (i.e. singularity of transient time at several combination of CP). Critical slowing down takes place near critical points only, and fine tuning is needed to reach the (very narrow) band of CP where this type of slowing-down is observable.

In phaser amplifier the observed phenomenon of critical slowing down may be described by simple lumped model of inversion states, see Appendix C. But in autonomous phaser generator we experimentally observed phenomena of spin-phonon interaction with very long, lethargic aperiodic transients (phonon “turbulence”) which have qualitatively another nature than usual critical slowing down.

There are some qualitatively different mechanisms of slowing down, for which such the tuning is not necessary. A well known mechanism of this type is the effect of self-organized criticality, which takes place when a dissipative system holds itself in critical state and no external tuning of CP is needed (models of sandpiles, earthquakes, forest fire etc.).

Some interesting scenarios of bottlenecks relaxation were proposed recently by K. Kaneko and his co-authors. In particular, self-organized bottleneck was revealed by H. Morita and K. Kaneko in transient processes for simple excited Hamiltonian system. Here “the critical state is spontaneously formed without continuous driving, once a part of a system is highly excited”. And in work of A. Awazu and K. Kaneko it was found that forming and evolution of transient dissipative structures lead to non-critical slowing down in closed chemical system (a modified Brusselator model).

Super-slow evolution of seemingly “frozen” spiral 2D structures was revealed and investigated by another group (C. Brito, I. S. Aranson, and H. Chaté) in 2D system modeled by complex Ginzburg-Landau equation. During last years some publications on slow evolving continuous systems has been appeared, particularly in the “Condensed Matter” and “Nonlinear Sciences” arXives.

Collective relaxation phenomena observed by us in TLCA are of similar kind despite of fully discrete nature of our model. Bottlenecked transition to final state of TLCA caused by RSWs may qualitatively explain why relaxation times of level-populations in dissipative three-level (or multilevel) paramagnetic systems differ by several orders from those predicted by one-particle spin-lattice relaxation theory (this old problem has not satisfactory solution up to now). On the other hand, freezing, quasi-freezing and slow evolution are now a subject of investigation of three-level systems in many alternative directions of modern nonlinear science, see e.g. work of F. Vazquez, P. L. Krapivsky, and S. Redner etc.

Hypersensitivity of the TLCA to changing of initial conditions (which will be discussed in the next Subsections of this work), instability of vortices and extremely slow, lethargic moving to a regular attractor are typical features of transient deterministic chaotic behaviour of TLCA. Birth, evolution, interaction and decay of RSWs in TLCA (including original ZM model), as it is obvious from Figures 1 to 5, is much more complex and unpredictable than it was expected at early days of investigation of such axiomatic CA models of excitable systems. In particular, collisions of RSWs with absorbing borders (in the GBC-2 case) violate conservation of Q_T (see Figure 3 and Figure 5). In the next Subsection we describe our numerical results concerning the problem of competition of left- and right-handed spatio-temporal structures, which is of interest not only from physical point of view.

B. Competition of left- and right-handed RSWs

A bounded solitary domain of excited CAUs having appropriate relaxation times and placed far from grid boundaries (or at unbounded grid) may evolve to RSWs if and only if there is an adjoined (but not a surrounding) domain of refractory CAUs. In this case RSWs appear by pair with opposite sgn(σ) domain of refractory CAUs. In particular, the internal tuning of CP is needed (models of sandpiles, earthquakes, forest fire etc.).

In the previous Subsection, it was already pointed out that, for some CP sets, an effective topological charge Q_T of RSW and integral Q_T = ∑_m Q_T(m) is obviously conserved (by infinity in time if grid is unbounded). If such solitary excited-refractory area is placed in a starting pattern near the grid boundary and the GBC-2 conditions take place, the single RSW appears and evolves and Q_T is, of course, not conserved in this case. These simplest and well known examples illustrate possibilities of coexistence and competition of one or two RSWs in excitable media, and possible scenarios of the RSW(s) evolution may be easily forecasted.

An evolution of complex patterns with multiple, irregularly appearing, chaotically-like drifting and colliding RSWs (or, possibly, another spatio-temporal structures possessing handedness) is in essence unpredictable without direct computing of the whole transient stage. In particular, some intuitive predictions about of Q_T at attractor may be fully erroneous, as it will be shown in this Subsection.

In the previous Subsection, it was already pointed out that, for some CP sets, an effective topological charge Q_T (more precisely, Q_T = ∑_m Q_T(m)) irregularly changes, including reversing of its sign during a system evolution. Extremely high complexity of almost all pre-attractor evolution (see e.g. Figure 5) does not permit to prognose even the near future of such TLCA. But sometimes it seems that long-term prediction is yet possible for another CP sets. In particular, Figure 6 demonstrate such “possibility” at least at pre-finish stage of transient process in TLCA with f = 3 (other CP are the same as for Figures 1 to 4) and with M_X = M_Y = 300. Indeed, beginning
from $n > 10^6$ the sign of the integral effective topological charge becomes equal to $\text{sgn} \left( Q_T^{\text{int}} \right) = -1$ (Figure 6, $n = 1.6 \cdot 10^5$) and stays unchanged up to winning of the single RSW in the spiral competitions. In other word, a “perfect” left-handed word is formed here.

Nevertheless, this pretty picture of “predictable” self-organization is fully destroyed at $n \approx 3.96 \cdot 10^5$ due to collision of the winner’s core with the grid boundary and subsequent full collapse of excitations at $n_C = 396982$. This last stage is not shown at Figure 6 because it is of the same type as collapse at Figures 4 and 5 – it is simply a gradual absorbing of non-spiral waves of excitation by boundaries of our GBC-2 system.

But much more unexpected, fully counter-intuitive TLCA dynamics may be illustrated by two snapshots (see Figures 7 and 8) of evolution of a system with the same CP set as for Figure 6, but with $M_X = M_Y = 900$. At Figure 7, one can see almost purely right-handed pattern at step $n = 1.48 \cdot 10^6$ of TLCA evolution. The only deviation from ideal right-handedness at Figures 7 is the rudimentary counter-clockwise “tail” of the clockwise spiral at top-center part of Figure 7 (note that this highly asymmetric spiral-antispiral pair is inside of the closed wavefront).

One may expect that the winner RSW in this system will be of the same handedness (by analogy with the Figure 6, where left-handed multi-spiral pattern was evolved to left-handed solitary RSW). But the subsequent evolution of the system is very surprising, see Figure 8. The winner RSW is shown here at step $n = 2 \cdot 10^6$, it rotates counter-clockwise ($Q_T = -1$), i. e. the system becomes left-handed despite of their long right-handed previous life. During further evolution ($n > 2 \cdot 10^6$, not shown here), the winner’s core is moving irregularly across the grid up to collision with the grid boundary, and full collapse of excitations takes place at $n_C = 2260964$ with final stage by scenario shown at Figures 4 and 5.

We will emphasize that the single left-handed RSW (Figure 8) is the result of evolution of the right-handed pattern of RSWs (Figure 7). A series of intermediate snapshots (not shown here due to their big size) at the time interval from $n = 1.48 \cdot 10^6$ to $n = 2 \cdot 10^6$ demonstrate a scenario of such reversing of the system handedness. An the cause of this reversing is not rudimentary counter-clockwise “tail” of the clockwise spiral at top-center part of Figure 7, because it quickly dies during “spiral war”. Moreover, at $n = 1.88 \cdot 10^6$ the single fully developed right-handed RSW remains at the whole grid, in the right-bottom corner of the grid. The second competing right-handed RSW is in a dangerous proximity from left border of the grid at the same $n = 1.88 \cdot 10^6$. And a small dislocation in spiral wavefront of the first RSW appears as the nucleus of future left-handed RSW.

At $n = 1.89 \cdot 10^6$, the two competing right-handed RSWs are still alive (both of them are near corners of grid), but the mentioned dislocation has already transformd to fully developed left-handed RSW. During the subsequent evolution, both right-handed RSWs are died, the remnant of wavefronts of right-handed structure is crowded out by giant lacuna, and the left-handed RSW becomes the winner in the world with almost perfect right-handed history. Finally, this solitary left-handed RSW must die too at $n_C \approx 2.26 \cdot 10^6$, as it was pointed already.

This intriguing story confirms high level of unpredictability of the system under consideration, and this subject will be discussed in details below (see Section V). And now we describe another unusual form of motions in our TLCA observed at different values $f = 5$ of the threshold for the

C. Chimera states - coexistence of regular and chaotic domains in TLCA

At the same CP $\tau_e, \tau_r, h, g$ as in Figures 1 to 4 and Figures 6 to 8, but with $f = 5$ we revealed unusual, long-living stages of CA evolution, during which coexistence of periodic and aperiodic spatio-temporal structures takes place in TLCA. A useful tool for investigation of such the phenomenon is generalized Poincaré cross-section of pattern sequence.

Ascending and descending generalized Poincaré cross-sections, used in this work, are defined as follows. Let $B(n)$ is an 1-bit (e. g. black = TRUE, white = FALSE) pattern and $\Delta n$ is a selection interval ($\Delta n > 1$). For each binary cell $B_{ij}$ in B, we fulfill the sequence of $k$ logical additions in the ascending time (starting from a step $n_1 \leq N - k \Delta n$):

$$B_{ij}^{(+)}(n_1, \Delta n, k) = \left[ \left[ B_{ij}(n_1) \lor B_{ij}(n_1 + \Delta n) \right] \lor \left[ B_{ij}(n_1 + 2\Delta n) \right] \lor \cdots \lor B_{ij}(n_1 + k\Delta n) \right],$$ (34)

and a sequence of $m$ logical additions in the descending time (starting from a step $n_2 > m\Delta n$):

$$B_{ij}^{(-)}(n_2, \Delta n, m) = \left[ \left[ B_{ij}(n_2) \lor B_{ij}(n_2 - \Delta n) \right] \lor \left[ B_{ij}(n_2 - 2\Delta n) \right] \lor \cdots \lor B_{ij}(n_2 - m\Delta n) \right].$$ (35)

Ascending and descending generalized Poincaré cross-sections ($P^{(+)}$ and $P^{(-)}$) are the patterns, which consist of $B_{ij}^{(+)}(n_1, \Delta n, k)$ and $B_{ij}^{(-)}(n_2, \Delta n, m)$ respectively:

$$P^{(+)}(n_1, \Delta n, k) = \| B_{ij}^{(+)}(n_1, \Delta n, k) \|$$ (36)

$$P^{(-)}(n_2, \Delta n, m) = \| B_{ij}^{(-)}(n_2, \Delta n, m) \|$$ (37)
Figure 9 demonstrates phenomenon of coexistence of long-living periodic and aperiodic spatio-temporal structures in TLCA by using of ascending and descending generalized Poincaré cross-sections (left and right columns of Figure 9 respectively; the central column at Figure 9 is usual sequence of TLCA patterns showed for clarity). We use term “transient chimera states” for such structures by analogy to term introduced by D. M. Abrams and S. H. Strogatz [3] for analogous non-transient phenomena revealed earlier by Y. Kuramoto and D. Battogtokh [4].

The series of ascending generalized Poincaré cross-sections \( \mathcal{P}^{(+)\,(n_1, \Delta n, k_p)} \) shows that spatial domain of aperiodic motions is bounded at the whole interval of selection. In other words, periodic domain is dynamically stable (beginning at list from 1.5 \( \cdot \) 10\(^5\)) for more than half of the grid area. And the series of descending generalized Poincaré cross-sections \( \mathcal{P}^{(-\,(n_2, \Delta n, m_p)} \) demonstrates growing of the regular domain, i.e. how the system reaches the attractor keeping previously occupied territory of local periodicity. This evolving coexistence of two qualitatively different (periodically and aperiodically oscillating) domains in high-dimensional discrete mapping we call transient chimera states.

For \( \tau_\epsilon = \tau_r = h = 50, g = 1 \) and \( f \neq 5 \) reaching of an attractor proceeds typically by metamorphoses of aperiodic waves to periodic ones over the whole grid, without coexistence of periodic and aperiodic domains. But for some alternative values of \( \tau_\epsilon, \tau_r, h, g \) and for \( f \neq 5 \) we observed chimera states again. Evolution of some of these states are different comparatively to ones shown at Figure 9. For example, small but strictly regular spatial islands in chaotic sea may appear; they live by long time and then disappear during that or those stages of TLCA evolution; and attractor may be reached after disappearing of all of these islands. In other words, emergence of regular domain(s) of CAUs (shown at Figure 9) is not the single scenario of an attractor reaching when spatially-periodic and spatially-chaotic states are coexisting.

Similar phenomena of coexistence of incongruous or even antagonistic spatio-temporal structures were discovered during last years in computer experiments with 1D arrays of identical coupled oscillators [2] [3] and 2D grids emulating blackouts in electric power systems [17]. The most essential difference between systems, which was used in [2] [17] [3] and in the present work, consists in the nature of their elementary units. In works [2] [3] and in overview [17] one deals with units, which by definition oscillate even without any coupling with their neighborhoods. In the TLCA model (as well as in all models of excitable media) each isolated elementary unit has the single stable state, namely steady ground state.

In other words, the TLCA or another system of this kind may be treated as oscillator if (and only if) there is more or less strong interaction between elementary units. And namely the whole TLCA is an oscillator in this case. Electric power generators or another monochromatic oscillators, of course, may work autonomously — with their own frequencies and phases. But a grid with noninteracting excitable CAUs at \( n > \tau_\epsilon + \tau_r \) is simply a collection of anabiotic cells. So the concepts of phase synchronization/desynchronization are unapplicable to individual CAUs by definition. But synchronization may take place between domains of CAUs (and this is not a trivial “synchronization” of e.g. pacemaker-like domains in excitable media with stable domain walls).

To conclude this Subsection, we mention that coexistence of static regular and disordered structures is well known in CGL (complex “still life” configurations of CAUs). E.g., some unbounded (GBC-3 conditions) two-level \( \{ L_G \in \{ L_0, L_1 \} \} \) CGL-like automata demonstrate unlimited growing of domain(s) with frozen CAU states (Turing structures), were static spatially-chaotic “cloud” is irregularly grided by static spatially-periodic “rays”.

The simplest CA of this type is the INKSPOT automaton [80], which may be defined similarly to CGL automaton (see Equs. [11] and [2]), using binary phase counter \( \Phi_{ij}^{(n)} \in \{0, 1\} \) and the following elementary upgrading rules:

\[
\Phi_{ij}^{(n+1)} = \begin{cases} 
\Phi_{ij}^{(n)}, & \text{if } U_{ij}^{(n+1)} \neq 3; \\
1, & \text{if } U_{ij}^{(n+1)} = 3,
\end{cases}
\]

where \( U_{ij}^{(n+1)} \) is defined by Eqn. [2] and the Moore neighborhood is used.

Only \( L_0 \rightarrow L_1 \) transitions are permitted in the INKSPOT automaton [38], but this restriction is not critical for the phenomenon of coexisting of regular and chaotic static domains. A similar behavior demonstrate, e.g., the RtC (“Rays through Clouds”) two-level automaton [40] defined by rules

\[
\Phi_{ij}^{(n+1)} = \begin{cases} 
\Phi_{ij}^{(n)}, & \text{if } U_{ij}^{(n+1)} = 2 \lor \left( U_{ij}^{(n+1)} \in \{4; 8\} \right); \\
1, & \text{if } U_{ij}^{(n+1)} = 3; \\
0, & \text{otherwise},
\end{cases}
\]

or, in more convenient form:

\[
\Phi_{ij}^{(n+1)} = \begin{cases} 
\Phi_{ij}^{(n)}, & \text{if } U_{ij}^{(n+1)} \notin \{0; 1; 3\}; \\
1, & \text{if } U_{ij}^{(n+1)} = 3; \\
0, & \text{if } U_{ij}^{(n+1)} \notin \{0; 1\},
\end{cases}
\]

It is clear from the equation [40] that both transitions \( L_0 \leftrightarrow L_1 \) are permitted for the RtC automaton, but emergent Turing structures of chimera type is still the main form of evolution of this CA (except of special starting patterns giving pure rays) [40].

D. Spatial symmetry breaking and restoring

Computer experiments with spatially random input leave out another important case of initialization of a
system by symmetric or almost symmetric starting pattern. In this Subsection we investigate TLCA evolution with such (almost) symmetric initial conditions.

For a perfectly symmetric input, the TLCA evolution will proceed by conserved-symmetry scenario. It directly follows from the TLCA definition as homogeneous and isotropic (in the von Neumann sense) Kolmogorov machine with deterministic rules. But how will the TLCA evolve if a perfect symmetry of input pattern is more or less distorted? Will symmetry be broken out or it may be restored? Or both these scenarios are possible due to multistability of TLCA? One cannot find answers to these questions a priori, using only general guidelines as Curie’s principle.

To find such the answers empirically, let us study TLCA evolution with strictly determined asymmetry of starting patterns (Figure 10).

Left-column starting pattern \( P_{ML}^{(0)} \) at Figure 10 is not fully symmetric (ten black cells at the top of the excited left part of \( P_{ML}^{(0)} \) are changed to ten white cells at the top of the excited right part \( P_{ML}^{(0)} \)). But the \( C_r \) symmetry is fully restored already at \( n = 48 \) (not shown at Figure 10), and at \( n = 10129 \) the system is already at periodic attractor (“Smile of Cheshire cat” — see Figure 10, left column, row \( d \)). This attractor has period \( T = 3592 \) and consists of two mirror-symmetric, counter-rotating RSW with \( |Q_T| = 3 \).

The right-column starting pattern \( P_{MR}^{(0)} \) at Figure 10 differs from the left-column starting pattern \( P_{ML}^{(0)} \) only by the single CAU with coordinates \( i = 66; j = 41 \) (cells are numbered beginning from the left bottom corner of the grid). But evolution of \( P_{MR} \) is qualitatively different from the \( P_{ML} \) case. At \( n = 500 \) (right column, row \( b \) at Figure 10) \( P_{MR} \) is fairly asymmetric. At \( n = 1256 \) one can see that \( P_{MR} \) takes form of a RSW with \( Q_T = +4 \) (right column, row \( c \) at Figure 10). But this RSW is unstable — the attractor for \( P_{MR} \) is RSW with \( Q_T = +3 \) and period \( T = 2308 \) (right column, row \( d \) at Figure 10).

This example shows that hypersensitivity of TLCA to changing of initial conditions may lead not only to full “breaking” of the slightly imperfect symmetry, but to restoring of such imperfect symmetry to the perfect one. Besides of \( C_r \) or \( C_h \) cases, a similar competition of symmetry “breaking” and restoring is possible for the cases of \( C_2 \), \( C_4 \) (both under GBC-1 or GBC-2 conditions). If starting patterns consists of multiple symmetric parts, then similar phenomena will appear at least under GBC-1 conditions (for perfect mosaic pattern and the GBC-1 conditions, a regular vortex grid is obviously possible).

Besides of these broken/restored symmetry phenomena, there are another group of effects, which are connected with competition of left/right spinning of RSWs.

We already described such phenomena in Subsection IV B for cases of random starting patterns. One can mistakenly conclude that unpredictability of direction of RSW spinning is the consequence of randomness of inputs in general. But in the next Subsection we will show that the main source of unpredictability of handedness in TLCA is hypersensitivity to initial conditions, when minimal possible change of starting pattern may reverse direction of RSW spinning (both at attractor and at transient stages of evolution).

### E. Hypersensitivity to initial condition and unpredictability of left- or right-handed forms of vorticity

Changing of a single CAU may lead to strongly expressed mutation of spatio-temporal structures in TLCA, as it was demonstrated in the previous Subsection. Such hypersensitivity is an inherent property of very wide class of nonlinear multicomponent systems with growing spatial disturbances and multiple attractors. Some more or less close analogs one can find in such well-known area of technology as electric power grids, which already has been mentioned within the framework of chimera states consideration in Subsection IV C (on cascading effects in power grids and other real-world and artificial multicomponent systems see also references therein).

Now we will investigate mutations which produce reversing of handedness of attractors reached during TLCA evolution in conditions of hypersensitivity to initial conditions. Let the configurations of ground-state CAUs in starting patterns at Figure 11 (\( P_{SL}^{(0)} \) at top of left column and \( P_{SR}^{(0)} \) at top of right column) are perfectly symmetric (in distinction to starting patterns \( P_{ML}^{(0)} \) and \( P_{MR}^{(0)} \) shown at Figure 10). So asymmetry in both \( P_{SL}^{(0)} \) and \( P_{SR}^{(0)} \) (Figure 11) is determined only by slightly differentiated populations of two non-ground levels.

Namely, left-column starting pattern \( P_{SL}^{(0)} \) at Figure 11 is not fully mirror-symmetric, because ten black cells at the top of the excited left part of \( P_{SL}^{(0)} \) are changed to ten gray cells at the top of the excited right part \( P_{SL}^{(0)} \). At \( n = 2010 \) pattern \( P_{SL}^{(0)} \) takes form of a RSW with \( Q_T = +4 \) (left column, row \( c \) at Figure 11). But this RSW is unstable — the attractor for \( P_{SR}^{(0)} \) is RSW with \( Q_T = +3 \) (left column, row \( d \) at Figure 11).

The right-column starting pattern \( P_{SR}^{(0)} \) at Figure 11 differs from the left-column starting pattern \( P_{SL}^{(0)} \) only by the single CAU with coordinates \( i = 66; j = 41 \) (similarly to that for Figure 10, but black cell at site \( 66; 41 \) is changed by gray one). Evolution of \( P_{SR}^{(0)} \) is qualitatively different from the \( P_{SL}^{(0)} \) case and leads to an attractor having another \(|Q_T|\) and opposite \( \text{sgn}(Q_T) \), namely to RSW with \( Q_T = -2 \) (right column, row \( d \) at Figure 11).

This last result illustrates that hypersensitivity of TLCA to changing of initial conditions is a cause of un-

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4 This is, of course, not a “Cheshire cat” of C. R. Tompkins (a two-level block predecessor, which is known in CGL).
predictability of the direction of a RSW spinning even for the robust vortices. Minimal possible disturbance of the initial conditions (by changing of the state of a single CAU in a starting pattern) trigger the sign of $Q_T$, not to speak of the $Q_T$ modulo. In other words, a result of self-organization of a dynamically stable world in TLCA cannot be forecasted even for almost identical initial conditions. Left-handed or right-handed life in the framework of the model under consideration is a result of pure accident despite of extremely simple, absolutely deterministic underlying algorithm.

In conclusion of this computer-experiment Section it will be marked out the following. Rich phenomenology of TLCA evolution, illustrated in this Section by several important examples concerning RSWs, demonstrates that this kind of deterministic mapping is one of the simplest, fully discrete, bounded (except the case GBC-3) system possessing complexity of very high degree. Brief list of observed complex phenomena includes:

- Transient spatio-temporal chaos (including long-term irregular evolution of complicated RSW structures);
- Competition of left-handed and right-handed RSWs with chaotic alterations of $Q_T^{(\text{integer})}$ and unexpected handedness of the winner RSW;
- Congruous or fully antagonistic spatio-temporal structures (chimerae);
- Amazing symmetry-connected properties of TLCA caused by hypersensitivity to initial condition, including full restoring of (initially broken) symmetry of evolving TLCA pattern. Reversing of $\text{sgn} \left( Q_T^{(\text{integer})} \right)$ with changing of $|Q_T^{(\text{integer})}|$ (mutations) caused by modification of a single CAU state only at the beginning of evolution.

In the Section V we discuss relationship between simplicity of the TLCA model and complexity of its behavior as well as some other issues concerning self-organization in CA.

V. DISCUSSION

A model of an active medium based on cyclic transitions in a three-level system was introduced for the first time by astrophysicist A. D. Thackerey in 1930’s. In essence, this was a model of resonatorless laser system, but proposed about 20 years before the beginning of the laser era (and about 10 years before work of N. Wiener and A. Rosenbluth on excitable systems and work of J. von Neumann on cellular automata).

Similarity of class-B lasers and excitable (chemical) systems was revealed and investigated by C. O. Weiss, K. Staliunas and their co-workers with using of CA or any other discrete mapping of this kind. Are algorithmically simple (but having phase space of huge dimension) CA models of class-B lasers and/or laser-like systems appropriate for at least qualitative study of these active devices? Are there advantages of CA in modeling of laser (phaser) dynamics at all?

There are many publications on CA and other algorithmically simple models of complex systems, which may be joined under the common title: “Complex worlds from simple rules?” (this is the real title of the shortest, to our knowledge, paper on this subject — see note of U. Yurtsever). Criticism of such publications is directed against simplicity of CAU operation, but it usually does not take into account emergent co-operation of large or very large quantity of CAUs in an intrinsically unstable system with random (or randomly perturbed) initial conditions.

It is obvious that discrete, spatially bounded deterministic system, e.g. CA (ZM or TLCA in particular) may have only two types of attractors, both of which are regular (as it was already stressed in Section IV). In contrary, continuous deterministic system may have a great variety of irregular attractors or even much more complicated spatio-temporal structures. Algorithmic complexity of models simulating continuous systems is usually higher (by many reasons, including purely technical). Introducing of stochastic terms in a (classic) model brings new sources of complexity. And quantum-mechanical models in most cases has the highest level of complexity. This hierarchy of complexity is built by the principle of algorithmic complexity. But algorithmic or Kolmogorov complexity is not the unique tool and it possibly is not an adequate tool in the area under consideration. So (in contrary to the U. Yurtsever point of view) “cellular automata are one of the more popular and distinctive classes of models of complex systems” (see page 20).

Results of the present work demonstrate complex, unpredictable behaviour of algorithmically simple system containing mesoscopic quantity of discrete elementary units (usually $10^{4}$ – $10^{6}$). We would like to stress that we deal with high-dimensional system, which cannot be reduced to any averaged, low-dimensional one. Distinctions between “standard” system having dimension of phase space $d_{\text{phase}} = 3$ and a system with $d_{\text{phase}} = 10^{4} – 10^{6}$ is, of course, very significant circumstance in modeling of active systems containing discrete interacting particles. Phaser system is one of such systems. Real phaser contains macroscopic quantity of particles ($d_{\text{phase}} > 10^{16} – 10^{17}$), but we hope that mesoscopic system is more appropriate for modeling of spatio-temporal chaotic (“turbulent”) motions.

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5 Besides of RSWs, there are several qualitatively another types of robust spatio-temporal structures in TLCA, and a lot of fragile ones. They will be discussed in separate publication.

6 Some intermediate cases are represented by discrete, spatially unbounded systems (e.g. CA of GBC-3 type, see Section III), where true irregular spatio-temporal structures may appear.
self-organization and slow self-detunings, inversion collapse, and other nonlinear phenomena in this multi-particle dissipative system.  

In particular, transient spatio-temporal chaos observed in TLCA may be collated with complex, “turbulent” behaviour of autonomous phaser generator. On the other hand, experimentally revealed coexistence of regular and irregular motions in ruby phaser may be interpreted as chimera states observed in TLCA as transient phenomenon (see Figure 9).  

Generally, TLCA model describes oscillatory collective states in system of non-oscillatory CAUs, including local self-synchronization of oscillations caused by diffusion of excitations, and global synchronization (or, alternatively, full collapse of excitations) as final state of the system evolution. In the first case a robust RSW or more complicated but strictly periodic in time structure is formed being a cyclic attractor of TLCA at several initial conditions — see e. g. Figure 3. This is an analog of limit cycle, i. e. regular attractor known in lumped dynamical systems. In the second case, when all non-ground CAU break out at finish, the full freezing of TLCA takes place — see e. g. Figure 4. This is an analog of point-like attractor in lumped dynamical systems. Both attractors may coexist for the same set of CP, as it is for the cases at Figure 3 and Figure 4. Such coexistence of final states is not the full analog of generalized multistability, which is well known for lumped dynamical systems, but branching of self-organized states in TLCA is evident.  

It is interesting to discuss shortly the question of degree of self-organization in the systems of CA type. It is obvious intuitively that final robust RSW in a TLCA (Figure 3) is a result of high-degree self-organization in TLCA. But it is not obvious intuitively that absolutely frozen final state (Figure 4) of TLCA is self-organized. At the same time, both routes to final states at these figures are very similar. This or that attractor is reached due to predefined (but deeply hidden) route dependent on combination of TLCA rules, control parameters and initial conditions. Slight changing of initial conditions frequently leads to another attractor, but the evolution itself may by qualitatively the same. If evolution is highly bottlenecked then attractor may not be reached at all during the whole time of numerical or real experiment.  

From this point of view a final state itself may be of minor interest, more important is a transient dynamics. The work of J. P. Crutchfield and K. Kaneko, which was already cited in Section, was titled: “Are Attractors Relevant to Turbulence?”. We may slightly reformulate this question in the context of our study: “Are Attractors Relevant to Transient Spatio-Temporal Chaos?”. The answer is “Yes” if an attractor may be reached for a reasonable time (in fully discrete and bounded system is always limited by quantity of all possible states of the system). But the answer is “No” if exceeds any possible duration of an experiment. In this case a system with transient spatio-temporal chaos cannot be distinguished from true chaotic system without additional testing.  

As a matter of fact, there are some intermediate classes of phenomena “at the edge between order and chaos” which may appear in bounded discrete deterministic system with large phase space. And self-organization scenario which includes super-slow, bottlenecked, chaotic-like stages is a signature of dominance of such an intermediate class of system dynamics in numerical experiments. It is important to define not only qualitative criteria of self-organization, but quantitative ones too. Really, having limited time and computer capacity, one cannot reach final self-organized state for a system with huge dimension of phase space. Irreducibility of CA or another discrete mappings does not permit direct forecasting of the system future without direct computation. So the computed part of transient process is the single source of available information of our fully deterministic but partially determined system.  

Criteria of self-organization introduced during last decades are of great interest. The brightest of them, absolutely counter-intuitive example is suggestion of Yu. L. Klimontovich, which may be formulated shortly as follows: Turbulence is more organized state than laminar motion. This suggestion is based on using of normalized entropy of nonequilibrium state. Paradoxical approach of Klimontovich is actually deep and have definite perspective for development in nonequilibrium thermodynamics. But it is inadequate for very wide class of (self-)organized complex system including high-dimensional system of CA type.  

Nonequilibrium thermodynamics is very capricious thing, it may give both excellent and inappropriate results for the same physical object under different conditions. Our previous experiments on phaser systems confirmed the need of careful analysis of applicability of thermodynamical approach to concrete cases. Very good qualitative and even quantitative interpretation of experiments on phaser amplification is contrasted to the case of phaser generation, where spin-temperature model sometimes gives very bad description of phenomena observed (and such models are failed to produce any heuristics of phaser generation). In an amplifier, there are no self-organized states at all, and saturated populations of levels admit averaged description (by spin temperature in rotating frame) — in contrary to some regimes of phaser generator. Using of exact high-dimensional CA models is, to our mind, more appropriate for investigation of self-organized multiparticle systems.  

Recently some interesting results were reported on quantitative investigation of self-organization in cyclic CA, which are a kind of multilevel CA with SSR (see Section). Cyclic CA are more expedient for chemical systems, which haven’t real discrete levels. Phasers...
(as far other active systems based on inversion of energy-level populations), in contrary to chemical excitable systems, have well resolved, discrete levels (usually three or four), so TLCA is an adequate model from this point of view. Moreover, our TLCA is of MSR type, so it is a flexible tool taking into account relaxation times for non-ground levels. Application of the approach, proposed in [72], to the TLCA may give a useful tool for direct quantitative study of self-organization in this cellular automaton and, possibly, for understanding of nonlinear phenomena observed in real physical experiments with Gigahertz phaser generator [13, 11, 52, 53] and similar active systems with negligible level of quantum noise.

And a bit about applicability of the TLCA to modeling of optical-range phonon lasers must be explained. Several successful attempts of laser-like generation of phonons at low-frequency side of Terahertz range were undertaken at the end of 1970-th [58], and now the renewed interest in this frequency range is coming into sight [77]. There are two main differences between 10-Gigahertz (usual microwaves) and, say, 100-Terahertz (or 1000-Terahertz) ranges: (i) about 12 (up to 15) orders difference in level of energies; (ii) sufficient distinction in distribution of level-population, which will be considered here for the case of three-level system.

In an excitable medium of a Gigahertz quantum device, all three working energy levels (ground, excited and refractory) in context of the present work) are populated even without inversion and even at liquid helium temperatures. Short pulse inversion almost instantly changes relative distribution of active centers with different states, but after the inversion pulse ending, this or that part of active centers remains in the refractory state. Examples of such distributions are shown at Figure A1.

In corundum (Al$_2$O$_3$) crystal matrix doped by nickel, each Ni$^{2+}$ is under action of static six-coordinated trigonal electric field (called crystal field [1, 6]) with the highest-symmetry axis $O_3$. This third-order axis coincides with the optical axis of corundum crystal matrix. In the Appendices A, B, we will use the Cartesian coordinate system with the applicata $O_3$ directed along the optical $z$.

The electric field of the trigonal symmetry splits the ground term $^3$F (see, e.g. [1, 6, 63]). This term consists of $(2\ell+1) \times (2S+1) = 21$ levels, which are unsplit for a Ni$^{2+}$ ion in free space (Figure A2). Here $\ell$ and $S$ are the orbital and the spin quantum numbers respectively: $\ell = 3$, $S = 1$.

In this approximation does not interact with spin (magnetic), because crystal (electric) field in this approximation does not interact with spin (magnetic) moment.

But this degeneracy is partially takes off by the relativistic spin-orbit interaction, described by operator $\lambda \mathbf{LS}$, where $\lambda$ is the constant of spin-orbit interaction; $\mathbf{L}$ and $\mathbf{S}$ are the operators of the orbital and the spin momenta respectively. As the result, spin triplet $^3A_2$ becomes splitted into low-lying spin doublet and the excited spin singlet (Figure A2). The spin singlet for the Ni$^{2+}$ : Al$_2$O$_3$ spin-system sits at $D_0/\hbar \approx 39.8$ GHz over the doublet $^1\Delta$, where $D_0$ is so called zero field (more rigorously, zero-magnetic-field) splitting of quantum energy levels.

This splitting in Ni$^{2+}$ : Al$_2$O$_3$ is about fourfold greater than $D_0$ in ruby (Cr$^{3+}$ : Al$_2$O$_3$) due to sufficient difference in $|\lambda|$ for Ni$^{2+}$ and Cr$^{3+}$ ions: $|\lambda|_{\text{Ni}^{2+}} = 335$ cm$^{-1}$ versus $|\lambda|_{\text{Cr}^{3+}} = 87$ cm$^{-1}$. Accordingly, spin-phonon interaction in Ni$^{2+}$ : Al$_2$O$_3$ is much stronger (and longitudinal paramagnetic relaxation is

**APPENDIX A: ENERGY LEVELS AND MICROWAVE-FREQUENCY TRANSITIONS IN Ni$^{2+}$ : Al$_2$O$_3$ SPIN SYSTEM**

Free $3d^8$-ion Ni$^{2+}$ has ground term $^3$F (see, e.g. [1, 6, 63]). This term consists of $(2\ell+1) \times (2S+1) = 21$ levels, which are unsplit for a Ni$^{2+}$ ion in free space (Figure A2). Here $\ell$ and $S$ are the orbital and the spin quantum numbers respectively: $\ell = 3$, $S = 1$.

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This splitting in Ni$^{2+}$ : Al$_2$O$_3$ is about fourfold greater than $D_0$ in ruby (Cr$^{3+}$ : Al$_2$O$_3$) due to sufficient difference in $|\lambda|$ for Ni$^{2+}$ and Cr$^{3+}$ ions: $|\lambda|_{\text{Ni}^{2+}} = 335$ cm$^{-1}$ versus $|\lambda|_{\text{Cr}^{3+}} = 87$ cm$^{-1}$. Accordingly, spin-phonon interaction in Ni$^{2+}$ : Al$_2$O$_3$ is much stronger (and longitudinal paramagnetic relaxation is
much faster) than in ruby — compare experimental data on tensor of spin-phonon interaction for Ni$^{2+}$ : Al$_2$O$_3$ and Cr$^{3+}$ : Al$_2$O$_3$. We want to underline close interconnection of static spin-lattice interaction (which defines zero-magnetic-field structure of the low-lying energy levels), dynamical spin-phonon processes (which respond for the resonant interaction of paramagnetic ions with microwave ultrasound) and longitudinal relaxation in such the diluted paramagnetics. The features of longitudinal relaxation of Nickel ions in Al$_2$O$_3$ will be consider in the Appendix B. Now we will continue description of the energy levels and microwave-frequency transitions in the Ni$^{2+}$ : Al$_2$O$_3$ paramagnetic system.

In nonzero static magnetic field $\mathbf{H}$, which is applied along $\mathbf{O}_3$, the doublet splits into pair of Zeeman levels with $M = \pm 1$, and the three-level spin system Ni$^{2+}$ : Al$_2$O$_3$ becomes fully split (Figure A2). The static magnetic field does not affect the level with $M = 0$. Here $\{M\} = \{-1, 0, +1\}$ is the set of the eigenvalues of the scalar operator $\hat{S}_z$ (this operator is the projection of the vectorial spin operator $\hat{S}$ onto applicata):

$$\hat{S}_z |\Psi_M\rangle = M |\Psi_M\rangle,$$  
(A1)

where $|\Psi_M\rangle$ are the wave functions (eigenfunctions of $\hat{S}_z$).

The energy levels $E(M)$ of the Ni$^{2+}$ : Al$_2$O$_3$ spin system for the case under consideration ($\mathbf{H} \parallel \mathbf{O}_3$) are as follows:

$$E(0) = (+2/3) D_0;$$  
(A2)

$$E(+1) = (−1/3) D_0 + g_\parallel \beta_B H;$$  
(A3)

$$E(−1) = (−1/3) D_0 − g_\parallel \beta_B H,$$  
(A4)

where $g_\parallel$ is the appropriate component of the effective g-factor; $\beta_B$ is the Bohr magneton; $H = |\mathbf{H}|$.

From the practical point of view (e. g. for using the Ni$^{2+}$ : Al$_2$O$_3$ as phaser active system) the most interest is the case of small magnetic fields: $H < D_0/h\gamma_\parallel$. In this case the following correspondence between Ni$^{2+}$ : Al$_2$O$_3$ levels $E(M)$ and TLCA levels $E_K$ (Section III) is suggested:

$$E(0) \Leftrightarrow L_{III};$$  
(A5)

$$E(+1) \Leftrightarrow L_{II};$$  
(A6)

$$E(−1) \Leftrightarrow L_I.$$  
(A7)

Possibility of such modeling of the Ni$^{2+}$ : Al$_2$O$_3$ active spin system by the 2C cellular automaton (SectionIII) is sustained by the obvious selection rules for transitions between $E(M)$:

$$|\Delta M| = \begin{cases} 1, & \text{for } (E(−1) \rightarrow E(0)) \wedge (E(0) \rightarrow E(+1)); \\ 2, & \text{for } (E(+1) \rightarrow E(−1)). \end{cases}$$  
(A8)

The main mechanism of diffusion of spin excitations in paramagnetic systems of the Ni$^{2+}$ : Al$_2$O$_3$ type is the magnetic dipole interaction between active units at resonance frequencies $\omega \approx \omega_{0} \pm \omega_{L}$. These interactions are permitted for $|\Delta M| = 1$, and they are forbidden for $|\Delta M| = 2$. So, there are only two channels for diffusion of spin excitations in physical system Ni$^{2+}$ : Al$_2$O$_3$ at $H \parallel \mathbf{O}_3$ — this is the case of the diffusion of excitations in the TLCA model (Section III). As the result we have such the correspondence between $E(M) \rightarrow E(M')$ and $L_K \rightarrow L_K'$:

$$(E(0) \rightarrow E(+1)) \Leftrightarrow (L_{III} \rightarrow L_{II});$$  
(A9)

$$(E(+1) \rightarrow E(−1)) \Leftrightarrow (L_{II} \rightarrow L_{I});$$  
(A10)

$$(E(−1) \rightarrow E(0)) \Leftrightarrow (L_{I} \rightarrow L_{III}).$$  
(A11)

For $H \parallel \mathbf{O}_3$ this close correspondence is, generally speaking, destroyed. The larger is angle $\alpha$ between $\mathbf{H}$ and $\mathbf{O}_3$, the more mixing of wave functions $|\Psi_M\rangle$ for each level takes place. But the case of $H \parallel \mathbf{O}_3$ (in practice $|\alpha| \lesssim 5^\circ$) is the single configuration where the phaser generation in the Ni$^{2+}$ : Al$_2$O$_3$ was experimentally realized [45]. An attempt of P. D. Peterson and E. H. Jacobson [65] to excite phaser generation in Ni$^{2+}$ : Al$_2$O$_3$ at $\alpha = 76^\circ$ was unsuccessful despite of large amplification of injected microwave ultrasound (some possible reasons of this “silence” of an active system Ni$^{2+}$ : Al$_2$O$_3$ in experiments of P. D. Peterson and E. H. Jacobson [65] were discussed in [45]. In this work we assume $H \parallel \mathbf{O}_3$, and the Ni$^{2+}$ : Al$_2$O$_3$ system is modeled by TLCA on the basis of $|\Psi_0\rangle \rightarrow |\Psi_1\rangle$.

APPENDIX B: TIMES OF LONGITUDINAL PARAMAGNETIC RELAXATION IN Ni$^{2+}$ : Al$_2$O$_3$ SPIN SYSTEM

Most of our early microwave experiments on spin-phonon interaction and phaser generation in active paramagnetic medium Ni$^{2+}$ : Al$_2$O$_3$ were fulfilled [45] at

$$H \ll D_0/hg_\parallel \beta_B,$$  
(B1)

i.e. under conditions:

$$F_{(+1,−1)} \ll (F_{(0,−1)}, F_{(0,+1)}, D_0/2\pi\hbar),$$  
(B2)

where $F_{(M,M')}$ are the frequencies of quantum transitions in our three-level spin-system:

$$F_{(M,M')} \equiv (E(M) − E(M'))/2\pi\hbar.$$  
(B3)

Typical values of CP in experiments [25, 43] were as follows: $H \approx 0.5$ kOe; $F_{(+1,−1)} = 3.0$ GHz, $F_{(0,−1)} \approx 41.3$ GHz, $F_{(0,+1)} \approx 38.3$ GHz. At first blush, such system may be described by TLCA with $\tau_r \ll \tau_\gamma$, because time of direct one-phonon longitudinal relaxation $T_1^{(direct)}$ (in the two-level approximation for each the transition) depends on $F_{(M,M')}$ by such the way [1, 2]:

$$T_1^{(direct)} \propto F_{(M,M')}^{-3} \tanh (\pi F_{(M,M')}/\theta).$$  
(B4)
Here $\theta$ is temperature of the crystal, $\varkappa$ is normalizing factor. We use the standard denotation $T_1$ for the longitudinal relaxation time in physical systems instead of $\tau$, which is used for relaxation time in the TLCA model system. In this case $\tau_r$ in the TLCA model (Section III) is an analog of $T_1(E_{(+1)})$, and $\tau_r$ is an analog of $T_1(E_{(0)})$ (this analogy is, of course, only qualitative, but it is clear from the physical point of view and expedient for numerical modeling of active systems).

For the case $F_{(M,M')} = F_{(+1, -1)} = 3$ GHz the inequality $(\varkappa F_{(M,M')}/\theta) \ll 1$ takes place at liquid helium temperatures ($\theta \leq 4.2$ K). So the time of longitudinal relaxation $T_1^{(\text{direct})}$ for the level $E_{(+1)}$ (which is the analog of the refractory level $L_1$ in TLCA — see Appendix A) at these temperatures must have such the form:

$$T_1^{(\text{direct})}(E_{(+1)}) \propto F_{(+1, -1)}^{-2} \theta^{-1}.$$  \hspace{1cm} (B5)

This two-level approach (which is often used in modeling of passive nonlinear systems) leads to wrong conclusion that relaxation of the level $E_{(0)}$ must be much faster than relaxation of the level $E_{(+1)}$, and that the correspondent TLCA with $\tau_e \ll \tau_r$ is a good model for the Ni$^{2+} : \text{Al}_2\text{O}_3$ physical system.

In a three-level (or, generally, multilevel) spin system, especially with large difference in splittings, there usually are several different mechanisms of longitudinal relaxation (direct, Orbach, Raman etc. [1, 2]):

$$1/T_1 = (1/T_1^{(\text{direct})}) +$$

$$+ (1/T_1^{(\text{Orbach})}) + (1/T_1^{(\text{Raman})}) + \ldots.$$  \hspace{1cm} (B6)

Measurements in Ni$^{2+} : \text{Al}_2\text{O}_3$ at low temperatures [25, 43] showed that the dominant mechanism of longitudinal relaxation for the spin doublet $M = \pm 1$ at $F_{(+1, -1)} \ll D_0$ is the Orbach two-phonon process [11, 12]:

$$T_1(E_{(+1)}) \approx T_1^{(\text{Orbach})}(E_{(+1)}),$$  \hspace{1cm} (B7)

with:

$$T_1^{(\text{Orbach})}(E_{(+1)}) \approx k_0 D_0^{-3} \exp(\varkappa D_0/\theta) \approx$$

$$\approx k_0 F_{(+0, +1)}^{-3} \exp(\varkappa F_{(+0, +1)}/\theta) \approx T_1(E_{(0)}),$$  \hspace{1cm} (B8)

where $k_0$ is normalizing factor. As the result, mentioned early inequality $\tau_e \ll \tau_r$ is generally incorrect for modeling of dynamics of the Ni$^{2+} : \text{Al}_2\text{O}_3$ active system (at least for $H \parallel \mathbf{O}_3$, $\theta = 1.8 - 4.2$ K, where the experiments on phaser generation [25, 43] were fulfilled). In the case under consideration, an approximate equality $\tau_r \approx \tau_r$ takes place instead of the above strong inequality, as it follows from Eqs. (B7) and (B8).

Additional changing of physical relaxation times at spin transitions of Ni$^{2+} : \text{Al}_2\text{O}_3$ is caused by interaction (cross-relaxation and some other mechanisms) of Ni$^{2+}$ ions with the Jahn-Teller 3d$^2$ ions Ni$^{3+}$ [43] in nickel-doped corundum. In strong six-coordinated crystal field the single unpaired 3d electron of Ni$^{3+}$ ion is at $e$-orbital, i.e. the ground state of Ni$^{3+}$ in corundum is $^3E$ (orbital doublet and spin doublet simultaneously) [7]. Non-zero ground state orbital momentum of the Ni$^{3+}$ ion in the Ni$^{3+} : \text{Al}_2\text{O}_3$ system (in contrast to zero ground-state orbital momentum for Ni$^{2+} : \text{Al}_2\text{O}_3$) is the cause of strong electron-phonon interaction and fast longitudinal relaxation of the Ni$^{3+}$.

The last type of ions always is present in nickel doped corundum, because ion Ni$^{3+}$ has the same charge as Al$^{3+}$, and after usual crystal growing the main part of nickel ions is in trivalent state. Reducing of Ni$^{3+}$ to Ni$^{2+}$ needs special technological operations, and such a reducing can not be full in principle.

Interaction of Ni$^{2+}$ with Ni$^{3+}$ in corundum strongly depends on magnetic resonance frequencies $F_{(M,M')} (\mathbf{H})$, this interaction is very sensitive to concentrations of both ions in corundum, etc. As the result, the ratios of relaxation times in Ni$^{2+} : \text{Al}_2\text{O}_3$ spin-system varies in very wide range, depending not only on experimental conditions, but on a crystal growing technology too. Due to these circumstances, the adequate TLCA model of the Ni$^{2+} : \text{Al}_2\text{O}_3$ active system must have possibilities for working with arbitrary ratios $\tau_e/\tau_r$.

APPENDIX C: COLLAPSE OF INVERSION STATES AND CRITICAL SLOWING-DOWN IN A THREE-LEVEL PHASER AMPLIFIER WITH BISTABLE PUMPING

Quantum amplifiers with bistable pumping were primarily investigated for the simplest case of weak signal [13, 14], when amplified field does not affect the active medium. The spin system of nonlinear three-level phaser amplifier (of a Ni$^{2+} : \text{Al}_2\text{O}_3$ type [25, 43, 50, 51]) is saturated by two microwave fields simultaneously.

The first of them is the electromagnetic pumping field with frequency $F_{\text{pump}} = F_{(+0, -1)}$ and normalized amplitude of magnetic component $Y$. This microwave magnetic component interacts with the spin system, exciting $|\Delta M| = 1$ transitions between the lower $E_{(-1)}$ and the upper $E_{(0)}$ spin levels.

The second field is microwave ultrasound (called also hypersound) with frequency $F_{\text{signal}} = F_{(+1, -1)}$ and normalized acoustic intensity $J$. This acoustic field interacts with the spin system, exciting $|\Delta M| = 2$ transitions between the lower $E_{(-1)}$ and the intermediate $E_{(+1)}$ spin levels. Such unusual selection rule ($|\Delta M| = 2$) is caused

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8 Relaxation times $\tau_e$ and $\tau_r$ are, of course, not the precise analogs of the correspondent physical relaxation times $T_1(E_{(0)})$ and $T_1(E_{(+1)})$ (the last are not additive etc.). Nevertheless, very close correspondence is obviously present here, and not only qualitative but quantitative interrelations between physical relaxation times may be used for defining of $\tau_e$ and $\tau_r$ in a TLCA model of physical (particularly, phaser) media.
by the quadratic on $\tilde{S}$ Hamiltonian of spin-phonon interaction \[1, 23, 24, 52\] for iron group ions in crystals.

Let us normalize the population differences $N(M, M') \equiv N(M) - N(M')$ at the spin transitions $E_M \rightarrow E_{M'}$ to their equilibrium values $N^\infty(M, M') \equiv N(M, M') |_{Y = 0, J = 0}$:

$$D \equiv \frac{N(-1, 0)}{N^\infty(-1, 0)}, \quad K \equiv \frac{N(+1, -1)}{N^\infty(+1, -1)}, \quad (C1)$$

where $0 < D < 1$ and $-1 \leq K < L - 1$ (note sign “minus” in the definition of $K$, Eq. (C1)). Here $K$ is the inversion ratio at the $E_{(+1)} \rightarrow E_{(-1)}$ spin transition, and the parameter $L$ characterizes properties of an active medium. Value $L$ depends on the ratio of pump and signal frequencies $F_{\text{pump}} / F_{\text{signal}}$, on the times of longitudinal relaxation etc. The third population difference $N(+1, 0)$ is univocally defined by $D$ and $K$, because $\sum_M N(M, M') = 1$.

For linear autonomous quantum amplifiers with weak signal ($J \ll 1$) standing-wave pumping becomes bistable \[55, 57\] if parameter of cooperativity $C$ is greater than its critical value $C_{\text{cr}}$ (where $C_{\text{cr}}$ is codimension-2 bifurcation point), and if $Y_1 \leq Y \leq Y_7$ ($Y_1$ and $Y_7$ are codimension-1 bifurcation points). For nonlinear autonomous quantum amplifiers bifurcation values $C_{\text{cr}}, Y_1$ and $Y_7$ are renormalized by an intense running-wave signal with constant amplitude \[54, 51\], and the collapse of inversion state becomes possible.

Phaser amplifier with modulated microwave acoustic signal is nonautonomous (at least one CP in our dynamic system becomes time dependent). But the equations of motions for nonautonomous dynamic system may be transformed to equivalent autonomous form in an extended phase space \[62\]. The only requirement for this transformation is regularity of modulation. In an extended phase space $(D, K, \zeta) \subset \mathbb{R}^3$ (where $\zeta = \omega_m t$, $\omega_m$ is the modulation frequency, and $t$ is usual, i. e. non-discretized time) the equations of motion for non autonomous phaser amplifier with bistable pump \[54, 51\] transforms to such the autonomous form \[54\]:

$$\begin{align*}
T_1 \frac{\partial \Delta}{\partial t} &= F(\Delta, \zeta/\omega_m, \Theta_A); \quad \Delta(0) = \Delta_0; \\
\frac{\partial \zeta}{\partial t} &= \omega_m; \quad \zeta(0) = \omega_m t_0.
\end{align*} \quad (C2)$$

Here $\Delta \equiv (D, K)$; $\Theta_A$ is time-independent vector of CP; $F \equiv (F_D, F_K)$. We suppose $T_1 \equiv T_1(E_{(0)}) \approx T_1(E_{(1)})$ (i. e. $T_1 \approx \tau_e$) by arguments of Appendix E and the inequality characterizing class-B active systems $T_1 \gg T_{\text{field}} \gg T_2$ takes place by definition \[31\] (see also Section I of the present work). Components of vector $F$ are as follows:

$$\begin{align*}
F_D &= 1 - D - \frac{Y^2 D}{(1 + 2CD)^2} + \frac{\bar{J}K}{4L}; \quad (C3) \\
F_K &= -1 - K + \frac{LY D}{(1 + 2CD)^2} - \bar{J}K, \quad (C4)
\end{align*}$$

where $\bar{J} = (1 + k_m \sin \zeta) J_0$; $k_m \zeta$ is the coefficient of modulation.

Eqs. (C2) to (C4) describe bistability, collapse of inversion states, critical slowing-down and other nonlinear processes in nonautonomous phaser amplifier (in framework of lumped, i.e. point-like three-level model). An example of critical slowing-down during collapse of inversion states is shown at Figure A3.

**APPENDIX D: LIST OF ABBREVIATIONS**

| Abbreviation | Meaning |
|--------------|---------|
| 1D, 2D, ... | One-Dimensional, Two-Dimensional, ... |
| 1C, 2C, ... | One-Channel, Two-Channel, ... |
| BZ | Belousov-Zhabotinskii |
| BR | Bogach-Reshodko |
| CA | Cellular Automaton |
| CAU | Cellular Automaton Unit |
| CP | Control Parameters |
| CGL | Conway’s Game of Life |
| CML | Coupled Map Lattice |
| GBC | Geometry and Boundary Conditions |
| GH | Greenberg-Hastings |
| GWR | Generalized Wiener-Rosenbluth (model) |
| MSR | Multi-Step Relaxation |
| OK2 | Oono-Kohmoto 2D (model) |
| Rrc | Rays through Clouds (automaton) |
| RSW | Rotating Spiral Wave |
| SSR | Single-Step Relaxation |
| TLCA | Three-Level Cellular Automaton |
| WR | Wiener-Rosenbluth |
| ZM | Zykov-Mikhailov |

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**FIGURE CAPTIONS**

to the paper of S. D. Makovetskiy and D. N. Makovetskii

“A Computational Study of Rotating Spiral Waves and Spatio-Temporal Transient Chaos in a Deterministic Three-Level Active System”
(figures see as separate PNG-files)

Figure 1: Typical stages of TLCA evolution. CP set is as follows: \(\tau_e = \tau_r = h = 50; g = 1; f = 8\). Starting pattern \(P^{(0)} = P^{(0)}_A\) has \(100 \times 100 = 10^4\) CAUs. Black, gray and white pixels denote excited, refractory and ground-state CAUs respectively. Numbers under patterns are equal to the steps \(n\) of evolution. At \(n = 3.66 \cdot 10^5\) the system is already at periodic attractor.

Figure 2: Divergence of spatio-temporal structures for \(f = 8\) (upper row) and \(f = 9\) (lower row) during TLCA evolution. Values of \(\tau_e, \tau_r, h, g\) are the same as in Figure 1, but for another starting pattern \(P^{(0)} = P^{(0)}_C\) (see Figure A1 for \(P^{(0)}_C\)). At \(n = 3 \cdot 10^5\) both systems are at their attractors. Only excited CAUs are shown at this figure (black pixels).

Figure 3: Slow evolution of TLCA and metamorphoses of RSW topology. Starting pattern \(P^{(0)} = P^{(0)}_C\). Values \(\tau_e, \tau_r, h, g\) are the same as in Figure 1 and Figure 2, but \(f = 4\). Only excited CAUs are shown at this figure (black pixels). The system reaches an attractor with \(Q_T = -3\) at \(n \approx 4 \cdot 10^6\).

Figure 4: Collapse of excitations for TLCA by slow evolution. Values \(\tau_e, \tau_r, h, g, f\) are the same as in Figure 3, but starting pattern is another: \(P^{(0)} = P^{(0)}_N\) (see the \(P^{(0)}_N\) at Figure A1). Transient time \(n_C = 3385479\) is the lifetime of nontrivial state of the whole cellular automaton with the pointed starting pattern.

Figure 5: Developed transient spatio-temporal chaos in TLCA. Set of CP is as follows: \(\tau_e = 9, \tau_r = 12, h = 25; g = 1; f = 9\) (ZM model). Starting pattern \(P^{(0)} = P^{(0)}_C\). Lifetime \(n_C = 2427063\). Black, gray and white pixels denote excited, refractory and ground-state CAUs respectively.

Figure 6: Pre-finish stage of transient process in TLCA with \(f = 3\). All RSWs have the same (by magnitude and sign) effective topological charge. Black, gray and white pixels denote excited, refractory and ground-state CAUs respectively. The perfect left-handed multi-RSW pattern is shown for \(n = 1.6 \cdot 10^5\). The CP set is as follows: \(\tau_e = \tau_r = h = 50; g = 1; f = 3\). This pattern is unstable. Multiple RSWs with \(Q_T = -1\) compete strongly after the left-handed pattern is formed, and “the winner takes all” at \(n > 3 \cdot 10^5\) — the single RSW with the same \(Q_T = -1\) occupies the whole active medium (not shown). But winner must die — excitation collapses fully at \(n_C = 396882\) due to collision of the winner’s core with boundary (not shown).

Figure 7: Reversing of sign of effective topological charge during TLCA evolution. Part 1: Almost purely right-handed pattern. Step \(n = 1.48 \cdot 10^6\) of evolution of starting pattern \(P^{(0)} = P^{(0)}_C\), with \(M_X = M_Y = 900\) (pattern \(P^{(0)}_{BC}\) itself is not shown here). The CP set is the same as at Figure 6. Only excited CAUs are shown by black pixels. Multiple RSW with \(Q_T = +1\) forms right-handed pattern. One may expect that winner will be right-handed too. But the subsequent evolution of the system is very surprising, see Figure 8.

Figure 8: Reversing of sign of effective topological charge during TLCA evolution. Part 2: The single left-handed RSW is the result of evolution of the right-handed pattern of RSWs. This is one of the snapshots (namely at step \(n = 2 \cdot 10^6\)) of subsequent evolution of the pattern shown at Figure 7. The winner RSW, shown here, has \(Q_T = -1\), i.e. the system becomes left-handed despite of their long right-handed previous life. During further evolution (\(n > 2 \cdot 10^6\), not shown here), the winner’s core is moving irregularly across the grid, it collides with the grid boundary, and full collapse of excitations takes place at \(n_C = 2260964\) with final stage by scenario shown at Figure 4.

Figure 9: Transient chimera states, i.e. coexistence of periodic and aperiodic spatio-temporal structures in TLCA (demonstrated by sequences of generalized Poincaré cross-sections). Starting pattern \(P^{(0)} = P^{(0)}_C\). Excited CAUs are black (TRUE), and non-excited CAUs (both refractive and ground-state ones) are white (FALSE). Parameters \(\tau_e, \tau_r, h, g\) are the same as in Figure 1, but \(f = 5\). Central column of the Figure — usual sequence of TLCA patterns, left column — forward-updated sequence of ascending Poincaré cross-sections \(P^{(+)}(n_1, \Delta n, k_p)\), right column — backward-updated sequence of descending Poincaré cross-sections \(P^{(-)}(n_2, \Delta n, m_q)\). Here \(n_1 = 154000; n_2 = 338000; \Delta n = 1000; k_{max} = m_{max} = 184\) (i.e. ending point for the whole sequence of cross-sections \(P^{(+)}\) is the starting point for \(P^{(-)}\) and vice versa). The TLCA at \(n_2 = 338000\) is already at the attractor.
Figure 10: **Symmetry restoring (left column) and “breaking” (right column) in TLCA.** Black, gray and white cells denote excited, refractory and ground-state CAUs respectively. Dimensions of the patterns are $M_X = 75$; $M_Y = 50$ (grid lines are shown for clarity). The CP set is as follows: $\tau_e = 5$; $\tau_r = 7$; $g = 0.3$; $h = 3$; $f = 9$. Denotations of the rows corresponds to the same steps of evolution for the left and the right starting patterns: $a \rightarrow n = 0$; $b \rightarrow n = 500$; $c \rightarrow n = 1256$; $d \rightarrow n = 10129$. Starting patterns differ by the single CAU with $i = 66$; $j = 41$ (cells are numbered beginning from the left bottom corner of the grid). This CAU in the left starting pattern $P_{ML}^{(0)}$ is at level $L_{III}$ (excited state, black cell). Analogous CAU in the right starting pattern $P_{MR}^{(0)}$ is at level $L_1$ (ground state, white cell).

Figure 11: **Hypersensitivity to initial condition as cause of unpredictability of left- or right-handed vorticity.** Dimensions of the patterns and the CP set are the same as at Figure 10, but starting patterns $P_{SL}^{(0)}$ (left column, row $a$) and $P_{SR}^{(0)}$ (right column, row $a$) are slightly changed comparatively to $P_{ML}^{(0)}$ and $P_{ML}^{(0)}$. Denotations of the rows corresponds to such the steps of evolution: $a \rightarrow n = 0$; $b \rightarrow n = 500$; $c \rightarrow n = 2010$; $d \rightarrow n = 10118$. Starting patterns $P_{SL}^{(0)}$ and $P_{SR}^{(0)}$ differ by the single CAU with the same coordinates $i = 66$; $j = 41$ as for Figure 10. This CAU in the left starting pattern $P_{SL}^{(0)}$ is at level $L_{III}$ (excited state, black cell). But analogous CAU in the right starting pattern $P_{SR}^{(0)}$ is at level $L_2$ (ground state, gray cell).

Figure A1: **Samples of starting patterns.** The population of excited CAUs (relative quantity of black pixels) is equal to 0.4 for all these patterns. This is slightly less than population of ground-state CAUs (white pixels), but more than twice greater than population of refractory CAUs (gray pixels). Resulting inversion of populations at $L_{III} \leftrightarrow L_1$ transition corresponds to an alternative variant of the phaser medium activation scheme (in the original scheme of the Ni$^{2+}$: Al$_2$O$_3$ phaser [25, 43], the analog of $L_{III} \leftrightarrow L_1$ transition was inverted).

Figure A2: **Energy levels of divalent nickel in corundum.** This is the simplest system by which microwave phonon laser (phaser) amplification, generation and inertial self-focusing were realized at Gigahertz-range frequencies. At non-zero static magnetic field, a collection of such active centers (interacting by $d-d$ mechanism) is a kind of excitable three-level system with two-channel diffusion of excitations. See Appendix A for explanation of the energy levels structure and underlying mechanisms of the levels splitting. For relaxation properties of this system at low temperatures see Appendix B.

Figure A3: **Critical slowing down and collapse of inversion states in a lumped three-level class-B active system.** Various-time oscillating routes to collapse are shown for the inversion ratio $K$, see Eqns. (C2) to (C4) in Appendix C. The invariant subset of CP (i.e. unchanged components of $\Theta_A$) is as follows: $C = 5$; $L = 2$; $Y = 6$; $J_0 = 0.15$; $k_{m,j} = 0.7$. The control parameter $\omega_m$ is varied relatively to relaxation time $T_1$. Parameter “n” enumerates dependencies $K(t)$ for different values of $\omega_mT_1$, namely: $(n = 1) \rightarrow (\omega_mT_1 = 0.1)$; $2 \rightarrow 0.15$; $3 \rightarrow 0.19$; $4 \rightarrow 0.199$; $5 \rightarrow 0.2$; $6 \rightarrow 0.204$; $7 \rightarrow 0.2005$; $8 \rightarrow 0.20053$. Initial conditions: $D(t = 0) = 0.01$; $K(t = 0) = 0.7$; $\zeta(t = 0) = 0$. 
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