The propagation of a polarized Gaussian beam in a smoothly inhomogeneous isotropic medium

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Abstract

We present a description of the evolution of a polarized Gaussian beam in a smoothly inhomogeneous isotropic medium in frame of the eikonal-based complex geometrical optics which describes the phase front and the cross section of the Gaussian beam using the quadratic expansion of the complex-valued eikonal. The linear complex-valued eikonal components are introduced to describe the influence of the spin-orbit interaction and the deformation of a polarized Gaussian beam on the propagation firstly in this paper. In an inhomogeneous medium, the interaction between the polarization and the rotation deformation of the light beam is presented besides the spin-orbit interaction, it corresponds to the spin-intrinsic orbital angular momentum interaction and makes the correction for the spin Hall effect of a polarized Gaussian light beam.

1. Introduction

The propagation of a polarized light has been investigated in various physical environment\cite{1-16}. In the process of the propagation, there are two important observable effects, the spin Hall effect\cite{1-12} and the Berry phase\cite{17} which describe the splitting of the propagation trajectory of different polarized light and the rotation of the polarization ellipse\cite{18,19,20} respectively. The two phenomena are caused by the spin-orbit interaction of photons, which describes the interaction between the polarization and the extrinsic orbital angular momentum of light\cite{1,4,10}.

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The eikonal-based complex geometrical optics[21,22,23,24] and the solution of a parabolic-type wave equation have given the description of the Gaussian beam propagation in an inhomogeneous medium, and the evolution of the Gaussian beam reduces to the solution of the Riccati type ordinary differential equations. One knows that the quadratic expansion complex-valued eikonal[22,23] is introduced to describe the phase front and the cross section of the Gaussian beam in frame of the eikonal-based complex geometrical optics. There is the deformation of the Gaussian which is due to the medium inhomogeneity[22,23]. we have known there is the spin-orbit interaction for a polarized light beam, but whether there is the interaction between the polarization and the deformation of the light beam? In this paper, this problem is investigated using the eikonal-based complex geometrical optics.

In frame of the eikonal-based complex geometrical optics, One knows that there are no the linear components in the complex-valued eikonal[23] when the Gaussian beam propagates in the homogeneous isotropic medium, but the inhomogeneous isotropic medium can be considered as the weakly anisotropic medium for different polarized light in the first-order geometrical optics approximation[3]. In order to describe the perturbation in the propagation, the linear complex-valued eikonal components are introduced in this paper. We obtain the result that the propagation of a polarized Gaussian beam in an inhomogeneous isotropic medium is affected by the spin-orbit interaction and the rotation deformation of the beam. The spin Hall effect of a polarized light beam[25] describes the transverse shift of the beam as a whole and is determined by the spin-orbit interaction. The rotation deformation of the light beam produces an equivalent intrinsic orbital angular momentum[26,27,28], there must be the interaction between the polarization and the rotation deformation of the light beam. This interaction makes the correction for the spin Hall effect of a polarized light beam.

The paper is organized as follows. First, the eikonal equation and amplitude transport equation which include the first-order geometrical optics approximation correction are derived. Then, the linear complex-valued eikonal components are introduced to describe the influence of the medium’s inhomogeneity on the propagation of a polarized beam within the paraxial approximation. And the differential equations describing the evolution of a polarized Gaussian beam are obtained. Finally, we reexamine the spin Hall effect of a polarized Gaussian beam and obtain result that both the spin-orbit interaction and the deformation of a polarized Gaussian affect the propagation trajectory of the light beam.
2. The eikonal equation in first-order geometrical optics approximation

Let’s consider the propagation of a monochromatic linearly polarized light in a smoothly inhomogeneous isotropic medium with dielectric permittivity $\varepsilon(\mathbf{r})$. There is a small geometrical optics parameter $\mu_{GO}$[21,29],

$$
\mu_{GO} = \frac{\lambda}{L} \ll 1,
$$

(1)

where $L \sim |\nabla \varepsilon/\varepsilon|^{-1}$ is the characteristic scale of the medium inhomogeneity, $\lambda = 2\pi/k_0 = 2\pi c/\omega$ is the wavelength in vacuum, $\omega$ is the angular frequency. Next, the eikonal equation will be derived in the first-order approximation in $\mu_{GO}$.

Maxwell equations have the form

$$
\nabla \times (\nabla \times \mathbf{E}) + \frac{\varepsilon}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0,
$$

(2)

then

$$
\nabla^2 \mathbf{E} - \frac{\varepsilon}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} - \nabla (\nabla \cdot \mathbf{E}) = 0.
$$

(3)

The last term in the left-hand side of Eq.(3) corresponds to the spin-orbit interaction of electromagnetic waves[1,10]. In geometrical optics, the propagation of the electromagnetic wave can be described by the ray trajectory in the short-wavelength limit. A ray-accompanying frame $(\eta_1, \eta_2, \tau)$ is introduced with the unit vectors $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{l})$, $\tau$ is the parameter along the ray, which is connected with the ray arc length $s$ by the relation $d\tau = ds/\varepsilon_c$. $\varepsilon_c = \varepsilon(\mathbf{r}_c)$, $\mathbf{r}_c = (0, 0, \tau)$ is the radius vector for the central ray in zero-order approximation in $\mu_{GO}$. The Lamé coefficients of the coordinate frame $(\eta_1, \eta_2, \tau)$ are[30,31]

$$
h_1 = h_2 = 1, \quad h = h_\tau = \sqrt{\varepsilon} \left[ 1 - \frac{\eta \cdot \nabla_\perp \varepsilon}{2\varepsilon} \right] |_{\mathbf{r} = \mathbf{r}_c},
$$

(4)

where

$$
\nabla = \mathbf{e}_1 \frac{\partial}{\partial \eta_1} + \mathbf{e}_2 \frac{\partial}{\partial \eta_2} + \mathbf{l} \frac{1}{h} \frac{\partial}{\partial \tau}, \quad \nabla_\perp = \mathbf{e}_1 \frac{\partial}{\partial \eta_1} + \mathbf{e}_2 \frac{\partial}{\partial \eta_2}.
$$

(5)

The electric field of the electromagnetic wave in the ray coordinates is $\mathbf{E} = E_1 \mathbf{e}_1 + E_2 \mathbf{e}_2 + E_\parallel \mathbf{l} = \mathbf{E}_\perp + E_\parallel \mathbf{l}$, $E_\parallel \ll E_\perp$. Keeping the terms up to the $\mu_{GO}$,
the wave equation for the transverse electric field $E_\perp$ can be written in the form

$$\nabla^2 E_\perp + k_0^2 \varepsilon E_\perp + (\nabla \ln \varepsilon \times \nabla) \times E_\perp = 0,$$

(6)

the third term in the left-hand side of Eq.(6) corresponds to the spin-orbit interaction, it originates from the third term in the left-hand side of Eq.(3). One knows a linearly polarized electromagnetic wave is a superposition of right-hand and left-hand circularly polarized waves, $E_\perp = E_+ e_+ + E_- e_-,$

$$e^\pm = \frac{1}{\sqrt{2}} (e_1 \pm i e_2), \quad E^\pm = \frac{1}{\sqrt{2}} (E_1 \mp i E_2).$$

(7)

Because of

$$l \times e^\sigma = -i \sigma e^\sigma,$$

(8)

we obtain the wave equation from Eq.(6),

$$k_0^{-2} \nabla^2 E^\sigma + \varepsilon E^\sigma - i k_0^{-2} \sigma 1 \cdot (\nabla \ln \varepsilon \times \nabla) E^\sigma = 0,$$

(9)

where $\sigma = \pm 1$ denote the wave helicity of right and left circular polarizations. One can takes $E^\sigma$ as wave function of photons and introduces the momentum operator of photons $-ik_0^{-1} \nabla,$ Eq.(9) can be considered as the schrodinger-type equation of photons, the Hamiltonian of photons[1,4,10] which include the spin-orbit interaction is obtained. Then the equation of motion for photons[1-4,6,10] can be derived using the canonical equations, which describes the spin Hall effect of photons.

The eikonal equation which includes the spin-orbit interaction also can be derived from the wave equation (9). From Eq.(7), $E^\sigma$ can be written in the following form

$$E^\sigma = A(r) \exp[i k_0 \psi(r)] \exp(-i \sigma \frac{\pi}{4}).$$

(10)

Substituting Eq.(10) into Eq.(9), the eikonal equation and amplitude transport equation are obtained in the first-order geometrical optics approximation,

$$(\nabla \psi)^2 = \varepsilon + \sigma \left( \frac{\nabla \ln \varepsilon}{k_0} \times \nabla \psi \right) \cdot 1,$$

(11)

$$2 \nabla \psi \cdot \nabla A + A \nabla^2 \psi = \sigma \left( \frac{\nabla \ln \varepsilon}{k_0} \times \nabla A \right) \cdot 1.$$

(12)
The second term in the right-hand side of Eq.(11) and Eq.(12) are the spin-orbit correction which corresponds to the first-order approximation in $\mu_{GO}$. The eikonal equation (11) is same as the equation of motion derived from the photon’s Hamiltonian, and it also describes the spin Hall effect of photons. The amplitude transport equation (12) describes the variation of the amplitude along the ray. If we take $A(\mathbf{r}) = A(\tau)$, the right-hand side of Eq.(12) disappear, then $\nabla \cdot (A^2 \nabla \psi) = 0$, this form means the conservation of the energy flux along the ray. Next, the propagation of a polarized Gaussian beam is investigated based on Eq.(11) in the paraxial approximation.

3. The evolution of a polarized Gaussian beam in an inhomogeneous isotropic medium

In order to describe the propagation of the Gaussian beam in the paraxial approximation, two small parameters are introduced,

$$\mu_1 = \frac{\lambda}{w}, \quad \mu_2 = \frac{w}{L},$$

(13)

where $w$ is the characteristic beam width. $\mu_1$ and $\mu_2$ describe the angle of the beam diffraction widening and the influence of the medium inhomogeneity on the diffraction respectively[23]. Three parameters, $\mu_1$, $\mu_2$ and $\mu_{GO}$ follow relation

$$\mu_{GO} \ll \mu_1, \mu_2.$$  

(14)

The propagation of the Gaussian beam in an inhomogeneous medium has been investigated with an accuracy of $\mu_1^2$ and $\mu_2^2$[23,30,31], we will keep the order in $\mu_{GO}\mu_1^2$ and $\mu_{GO}\mu_2^2$ because of the spin-orbit interaction.

In the paraxial approximation, the eikonal can be written as

$$\psi = \psi_c(\tau) + \delta \psi(\eta_1, \eta_2, \tau),$$

(15)

where $\psi_c(\tau)$ is the eikonal on the central ray and $\delta \psi$ is a small deviation in the paraxial approximation. $\delta \psi$ can be written as the following form[23]

$$\delta \psi(\eta_1, \eta_2, \tau) = \frac{\eta_1^2}{2} S_{11}(\tau) + \frac{\eta_2^2}{2} S_{22}(\tau) + \eta_1 \eta_2 S_{12}(\tau) + \eta_1 S_1(\tau) + \eta_2 S_2(\tau) + S_0(\tau),$$

(16)

where $S_{ij}$ ($i = 1, 2$ and $S_{12} = S_{21}$) are the quadratic complex-valued term in $\eta_i \eta_j$. The real and imaginary components of $S_{ij}$ describe the shape of
the phase front and the cross section of a polarized Gaussian beam [21-23]. The linear complex-valued term, $S_i$, describe the deviation of the center of the beam. One knows that the deviation of the center of the beam is due to the spin-orbit interaction in the first-order geometrical optics approximation. $S_0(\tau)$ is a small additional phase in the paraxial approximation. Substituting Eq.(15) and Eq.(16) into the eikonal equation (11),

$$
(\frac{\partial \psi}{\partial \eta_1})^2 + (\frac{\partial \psi}{\partial \eta_2})^2 + \frac{1}{\hbar^2} \left( \frac{\partial \psi}{\partial \tau} \right)^2 = \varepsilon + \frac{\sigma}{k_0} \left( \frac{\partial \ln \varepsilon}{\partial \eta_1} \frac{\partial \psi}{\partial \eta_2} - \frac{\partial \ln \varepsilon}{\partial \eta_2} \frac{\partial \psi}{\partial \eta_1} \right).
$$

(17)

The dielectric permittivity $\varepsilon$ can be expanded in a Taylor series,

$$
\varepsilon = \varepsilon_c + \mathbf{\eta} \cdot \nabla_\perp \varepsilon_c + \frac{1}{2} (\mathbf{\eta} \cdot \nabla_\perp)^2 \varepsilon_c,
$$

(18)

where $\mathbf{\eta} = \eta_1 \mathbf{e}_1 + \eta_2 \mathbf{e}_2$, $\nabla_\perp = \partial / \partial \mathbf{\eta}$, $\nabla_\perp \varepsilon_c \equiv (\nabla \varepsilon)|_{r=r_c}$. From Eq.(17) and Eq.(18), the eikonal equation has the form

$$
\hbar^2 \left[ (\frac{\partial \psi}{\partial \eta_1})^2 + (\frac{\partial \psi}{\partial \eta_2})^2 \right] + \left[ \frac{\partial \psi_c}{\partial \tau} + \frac{\partial \delta \psi_c}{\partial \tau} \right]^2 = \hbar^2 \varepsilon + \hbar^2 \frac{\sigma}{k_0} \left( \frac{\partial \ln \varepsilon_c}{\partial \eta_1} \frac{\partial \psi}{\partial \eta_2} - \frac{\partial \ln \varepsilon_c}{\partial \eta_2} \frac{\partial \psi}{\partial \eta_1} \right).
$$

(19)

In zero-order approximation, the ray equation is obtained

$$
\frac{\partial \psi_c}{\partial \tau} = \varepsilon_c.
$$

(20)

This equation describes the evolution of the trajectory of the center of the beam without the influence of the spin-orbit interaction [29,32].

From Eq.(19) and Eq.(20), we have

$$
\hbar^2 \left[ (\frac{\partial \psi}{\partial \eta_1})^2 + (\frac{\partial \psi}{\partial \eta_2})^2 \right] + \left[ \frac{2 \partial \psi_c}{\partial \tau} \frac{\partial \delta \psi_c}{\partial \tau} + \left( \frac{\partial \delta \psi_c}{\partial \tau} \right)^2 \right] = \frac{\varepsilon_c}{2} (\mathbf{\eta} \cdot \nabla_\perp)^2 \varepsilon_c - \frac{3}{4} (\mathbf{\eta} \cdot \nabla_\perp \varepsilon_c)^2
$$

$$
+ \hbar^2 \frac{\sigma}{k_0} \left( \frac{\partial \ln \varepsilon_c}{\partial \eta_1} \frac{\partial \psi}{\partial \eta_2} - \frac{\partial \ln \varepsilon_c}{\partial \eta_2} \frac{\partial \psi}{\partial \eta_1} \right)^2.
$$

(21)

Eq.(21) can be expanded in $\eta$, the differential equations for the complex
parameters $S_{ij}$ and $S_i$ are obtained keeping up to the $\mu_{GO}\mu_i^2$ and $\mu_{GO}\mu_2^2$,

\[
\frac{\partial S_{11}}{\partial \tau} + (S_{11}^2 + S_{12}^2) = \alpha_{11}, \tag{22}
\]

\[
\frac{\partial S_{22}}{\partial \tau} + (S_{12}^2 + S_{22}^2) = \alpha_{22}, \tag{23}
\]

\[
\frac{\partial S_{12}}{\partial \tau} + S_{12}(S_{11} + S_{22}) = \alpha_{12}, \tag{24}
\]

\[
\frac{\partial S_1}{\partial \tau} + (S_1S_{11} + S_2S_{12}) - \frac{\sigma}{2k_0} \left( \frac{\partial \ln \varepsilon_c}{\partial \eta_1} S_{12} - \frac{\partial \ln \varepsilon_c}{\partial \eta_2} S_{11} \right) = 0, \tag{25}
\]

\[
\frac{\partial S_2}{\partial \tau} + (S_1S_{12} + S_2S_{22}) - \frac{\sigma}{2k_0} \left( \frac{\partial \ln \varepsilon_c}{\partial \eta_1} S_{22} - \frac{\partial \ln \varepsilon_c}{\partial \eta_2} S_{12} \right) = 0, \tag{26}
\]

\[
\frac{\partial S_0}{\partial \tau} + \frac{1}{2}(S_1^2 + S_2^2) - \frac{\sigma}{2k_0} \left( \frac{\partial \ln \varepsilon_c}{\partial \eta_1} S_2 - \frac{\partial \ln \varepsilon_c}{\partial \eta_2} S_1 \right) = 0, \tag{27}
\]

where

\[
\alpha_{ij} = \frac{1}{2} \frac{\partial^2 \varepsilon_c}{\partial \eta_i \partial \eta_j} - \frac{3}{4\varepsilon_c} \frac{\partial \varepsilon_c}{\partial \eta_i} \frac{\partial \varepsilon_c}{\partial \eta_j}. \tag{28}
\]

Eqs.(22)-(27) describe the evolution of a polarized Gaussian beam in the smoothly inhomogeneous isotropic medium, the parameters $\alpha_{ij}$ describe the influence of the medium inhomogeneity on the propagation of the beam. These equations can be divided into three groups. The first group is Eqs.(22)-(24) which describe the evolution of the shape of the phase front and the cross section of a polarized Gaussian beam[23]. This group of equations has been obtained in [23] and is independent of the other equations. The second group is Eq.(25) and Eq.(26) which describe the transverse shift of the center of a polarized Gaussian beam. The third group is Eq.(27) which describes the evolution of the additional phase due to the deformation of the beam.

We have known that Eqs.(22)-(24) keep the terms in $\mu_1^2$ and Eqs.(25)-(26) in $\mu_{GO}\mu_2^2$. The spin Hall effect of a polarized Gaussian beam[25] describes the transverse shift of the beam as a whole, and the transverse shift is accord with the spin Hall effect of photons in the geometrical optics. In next section, the transverse shift will be reinvestigated, the interaction between the polarization and the rotation deformation of a polarized Gaussian beam will be presented besides the spin-orbit interaction.
4. The correction of the spin Hall effect of a polarized Gaussian beam

The complex-valued eikonal components can be written as \( S(\tau) = S^R(\tau) + iS^I(\tau) \), \( S^R(\tau) \) and \( S^I(\tau) \) are the real and imaginary components. Eq.(10) has the following form

\[
\exp[ik_0\psi(r)] = \exp[ik_0\psi_c]\exp\left[ik_0\left(\frac{\eta_1^2}{2}S_{11} + \frac{\eta_2^2}{2}S_{22} + \eta_1\eta_2S_{12} + \eta_1S_1 + \eta_2S_2\right)\right]
\]

\[
= \exp[ik_0\psi_c]\exp\left[ik_0\left(\frac{\eta_1^2}{2}S^R_{11} + \frac{\eta_2^2}{2}S^R_{22} + \eta_1\eta_2S^R_{12} + \eta_1S^R_1 + \eta_2S^R_2\right)\right]
\]

\[
\exp\left[-k_0\left(\frac{\eta_1^2}{2}S^I_{11} + \frac{\eta_2^2}{2}S^I_{22} + \eta_1\eta_2S^I_{12} + \eta_1S^I_1 + \eta_2S^I_2\right)\right]
\]

\[
= \exp[ik_0\psi_c]\exp\left[ik_0\left(\frac{S^R_{11}}{2}(\eta_1 + d^R_1)^2 + \frac{S^R_{22}}{2}(\eta_2 + d^R_2)^2 + \eta_1\eta_2S^R_{12} - \frac{S^R_{11}}{2}(d^R_1)^2 - \frac{S^R_{22}}{2}(d^R_2)^2\right)\right]
\]

\[
\exp\left[-k_0\left(\frac{S^I_{11}}{2}(\eta_1 + d^I_1)^2 + \frac{S^I_{22}}{2}(\eta_2 + d^I_2)^2 + \eta_1\eta_2S^I_{12} - \frac{S^I_{11}}{2}(d^I_1)^2 - \frac{S^I_{22}}{2}(d^I_2)^2\right)\right]
\]

where \( d_1 \) and \( d_2 \) are

\[
d^R_i = \frac{S^R_i}{S^R_{11}}, \quad d^I_i = \frac{S^I_i}{S^I_{11}}.
\]  

(30)

\( d^R_i \) and \( d^I_i \) are the transverse shift of the center of the wave phase front and the cross section[33]. From Eq.(29), one knows that the transverse shift[25] don’t change the shape of the phase front and the cross section of the beam, but the deformation of the phase front and the cross section can affect the transverse shift of the light beam. This transverse shift represents the spin Hall effect of a polarized light beam[25].

By virtue of Eqs.(22-26) and Eq.(30), the evolution equations of the transverse shift are obtained,

\[
\frac{\partial d^I_i}{\partial \tau} = \frac{1}{S^I_{11}} \frac{\partial S^I_{1}}{\partial \tau} - \frac{S^I_{1}}{(S^I_{11})^2} \frac{\partial S^I_{11}}{\partial \tau}
\]

\[
= -\frac{\sigma}{2k_0} \frac{\partial \ln \varepsilon_c}{\partial \eta_2} + \frac{\sigma}{2k_0} \frac{\partial \ln \varepsilon_c}{\partial \eta_1} \left(\frac{S^I_{12}}{S^I_{11}}\right) + O^I_i,
\]  

(31)
where

\[ O'_1 = \frac{1}{(S^l_{11})^2}[(S^l_{12}S^R_{11} - S^R_{11}S^l_{12})S^l_{11} + (S^l_{12}S^R_{11} - S^R_{11}S^l_{12})S^l_{12} + (S^l_{12}S^l_{11} - S^l_{11}S^l_{12})S^R_{12}]. \]  

(32)

Because of \( d'_1 = S^l_{11}/S^l_{11} \), Eq.(31) is written as

\[ \frac{\partial d'_1}{\partial \tau} = -\frac{\sigma}{2k_0} \frac{\partial \ln \varepsilon_c}{\partial \eta_2} + \frac{\sigma}{2k_0} \frac{\partial \ln \varepsilon_c}{\partial \eta_1} \left( \frac{S^l_{12}}{S^l_{11}} \right) d'_1 + O'_1. \]  

(33)

In the same way, we have

\[ \frac{\partial d'_1}{\partial \tau} = -\frac{\sigma}{2k_0} \frac{\partial \ln \varepsilon_c}{\partial \eta_2} + \frac{\sigma}{2k_0} \frac{\partial \ln \varepsilon_c}{\partial \eta_1} \left( \frac{S^l_{12}}{S^l_{11}} \right) d'_1 + O'_1, \]  

(34)

\[ \frac{\partial d'_1}{\partial \tau} = -\frac{\sigma}{2k_0} \frac{\partial \ln \varepsilon_c}{\partial \eta_2} + \frac{\sigma}{2k_0} \frac{\partial \ln \varepsilon_c}{\partial \eta_1} \left( \frac{S^l_{12}}{S^l_{11}} \right) d'_1 + O'_1, \]  

(35)

\[ \frac{\partial d'_1}{\partial \tau} = -\frac{\sigma}{2k_0} \frac{\partial \ln \varepsilon_c}{\partial \eta_2} + \frac{\sigma}{2k_0} \frac{\partial \ln \varepsilon_c}{\partial \eta_1} \left( \frac{S^l_{12}}{S^l_{11}} \right) d'_1 + O'_1, \]  

(36)

where

\[ O'_2 = \frac{1}{(S^R_{22})^2}[(S^l_{21}S^R_{22} - S^R_{22}S^l_{21})S^l_{21} + (S^l_{21}S^R_{22} - S^R_{22}S^l_{21})S^l_{22} + (S^l_{21}S^l_{22} - S^l_{22}S^l_{21})S^R_{22}], \]  

(37)

\[ O^R_1 = \frac{1}{(S^R_{11})^2}[(S^l_{12}S^R_{11} - S^R_{11}S^l_{12})S^l_{11} + (S^l_{12}S^R_{11} - S^R_{11}S^l_{12})S^l_{12} + ((S^R_{12})^2 - (S^l_{12})^2)S^R_{11}], \]  

(38)

\[ O^R_2 = \frac{1}{(S^R_{22})^2}[(S^l_{22}S^R_{22} - S^R_{22}S^l_{22})S^l_{22} + (S^l_{22}S^R_{22} - S^R_{22}S^l_{22})S^l_{22} + ((S^R_{22})^2 - (S^l_{22})^2)S^R_{22}], \]  

(39)

From Eqs.(33)-(36), the transverse shift in the plane \((\eta_1, \eta_2)\) are

\[ d^R = \frac{\sigma}{2k_0} \int (1 \times \nabla \ln \varepsilon_c) d\tau + \frac{\sigma}{2k_0} \int \left( \frac{\partial \ln \varepsilon_c d^R_{11}}{\partial \eta_1} - \frac{\partial \ln \varepsilon_c d^R_{12}}{\partial \eta_2} \right) S^R_{12} d\tau + O^R, \]  

(40)
\[ d' = \frac{\sigma}{2k_0} \int (1 \times \nabla \ln \varepsilon_c) d\tau + \frac{\sigma}{2k_0} \int \left( \frac{\partial \ln \varepsilon_c}{\partial \eta_1} \frac{d_1'}{S_1'} - \frac{\partial \ln \varepsilon_c}{\partial \eta_2} \frac{d_2'}{S_2'} \right) S_{12'} d\tau + O', \]

where \( O^R \) and \( O^I \) are

\[ O^R = \int (O_{1}'e_1 + O_{2}'e_2) d\tau, \]

\[ O^I = \int (O_{1}^R e_1 + O_{2}^R e_2) d\tau. \]

In Eq.(40) and Eq.(41), \( d = d_1e_1 + d_2e_2, d_i = d_ie_i \). From Eq.(40) and Eq.(41), the transverse shift of a polarized light beam is different from photons. In the right-hand side of Eq.(40) and Eq.(41), the first term corresponds to the transverse shift due to the spin-orbit interaction, it describes the influence of the interaction between the polarization and the trajectory of the propagation of the light beam. The second term is closely related to the quadratic terms \( S_{12} \). The term \( S_{12} \) describes the rotation of the cross section of the beam in the plane \((\eta_1, \eta_2)\)[23,30,31]. The rotation of the beam produces an additional equivalent intrinsic orbital angular momentum[26,27,28], so the second term corresponds to the spin-intrinsic orbital angular momentum interaction[34]. The third term describes the influence of the deformation on the transverse shift. One has known that the propagation of a polarized light(or photons) is influenced by the spin-orbit interaction, but a polarized light beam is different from photons because of the spatial energy distribution of the light beam. This result means that it is not enough if we only consider the influence of the spin-orbit interaction for the propagation of a polarized light beam in an inhomogeneous isotropic medium, the rotation deformation of the beam also is important. This is the main result in this paper.

The imaginary components of \( S_{ij} \) describe the amplitude(energy) spatial distribution of the Gaussian beam. From Eq.(40) and Eq.(41), the splitting of the center of gravity for the linear polarized Gaussian beam is obtained

\[ \delta d' = (d')^+ - (d')^- = \frac{1}{k_0} \int (1 \times \nabla \ln \varepsilon_c) d\tau + \frac{1}{k_0} \int \left( \frac{\partial \ln \varepsilon_c}{\partial \eta_1} \frac{d_1'}{S_1'} - \frac{\partial \ln \varepsilon_c}{\partial \eta_2} \frac{d_2'}{S_2'} \right) S_{12'} d\tau. \]

Here, the first term describes the splitting due to the spin-orbit interaction, the second term corresponds to the splitting due to the rotation deformation of the Gaussian beam. This is a significant difference between the light beam
and photons.

Although the analytic solutions of the differential equations (22)-(27) cannot be obtained, we give the numerical simulation when the linear polarized Gaussian beam propagates along the helical ray in the cylindrical symmetry medium [23]. Fig.1 shows the evolution of the splitting for the center (central ray) of the linear polarized Gaussian beam and the propagation trajectory of photons. Because of the deformation of a polarized Gaussian beam, the relative shift between the right-hand and left-hand circularly polarized Gaussian beam is slightly different from photons. Although the difference of the relative shift is very small, one also should pay attention to the spin-intrinsic orbital angular momentum interaction because of the deformation of a polarized light beam.

5. Conclusion

The propagation of a polarized Gaussian beam in an inhomogeneous isotropic medium has been presented using the description of the eikonal-based complex geometrical optics in this paper. The linear complex-valued terms are introduced to describe the perturbation of the center of a polarized Gaussian beam due to the inhomogeneity of medium and the deformation of a polarized light beam. Because the rotation deformation of a polarized Gaussian beam produces an additional equivalent intrinsic orbital angular momentum, the interaction between the polarization and the rotation deformation of a polarized light beam is presented. This interaction can be considered as the spin-intrinsic orbital angular momentum and makes the correction for the spin Hall effect of a polarized light beam.
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Figure 1: Numerical result for the splitting of the linear polarized Gaussian beam and photons which both propagate along the helical ray in the cylindrical symmetrical inhomogeneous medium. The red line and the blue line describe the evolution of the splitting of the trajectory of the beam and photons respectively. The parameters are as follows: the angle between the tangent to the helical ray and the cylinder axis, the dielectric permittivity $\varepsilon = \varepsilon_0 - \rho^2/L^2$, $\varepsilon_0 = 1$, $L = 200\lambda$, the initial beam waist radius $w_0 = 10\lambda$, the initial phase front is flat $r = \infty$. 

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