2n-dimensional models with topological mass generation

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The 4-dimensional model with topological mass generation that has recently been presented by Dvali, Jackiw and Pi [G. Dvali, R. Jackiw, and S.-Y. Pi, Phys. Rev. Lett. 96, 081602 (2006), hep-th/0610228] is generalized to any even number of dimensions. As in the 4-dimensional model, the 2n-dimensional model describes a mass-generation phenomenon due to the presence of the chiral anomaly. In addition to this model, new 2n-dimensional models with topological mass generation are proposed, in which a Stückelberg-type mass term plays a crucial role in the mass generation. The mass generation of a pseudoscalar field such as the η’ meson is discussed within this framework.

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I. INTRODUCTION

Recently, Dvali, Jackiw and Pi have presented a novel 4-dimensional model [1] consisting of well-known topological entities: Chern-Pontryagin density $\mathcal{P}$ and Chern-Simons current $\mathcal{C}^\mu$, $\mathcal{P} = \partial_\mu \mathcal{C}^\mu$. This model can describe the mass-generation phenomenon in a 4-dimensional non-Abelian system without treating details of the underlying dynamics. Dvali et al. found the model as a partial, 4-dimensional generalization of the Schwinger model [2] reformulated in terms of the topological entities in 2 dimensions. The reformulated Schwinger model and the 4-dimensional model share the common mass-generation mechanism described in topological terms. Noting this, Dvali et al. stated that the present formulation offers a unified topological description of the mass-generation phenomena in seemingly unrelated systems.

In this paper, we first consider a straightforward 2n-dimensional generalization of the 4-dimensional model and demonstrate that the topological mass generation studied by Dvali et al. is present in any even number of dimensions. There, as in the 4-dimensional model, it is verified that the presence of the chiral anomaly is essential for generating mass. Next, we propose a new 2n-dimensional model with topological mass generation, in which a Stückelberg-type mass term gives rise to mass generation in a gauge invariant manner. In addition, we consider a hybrid of the 2n-dimensional models mentioned above, in which a mass is caused by both the Stückelberg-type mass term and the presence of the chiral anomaly. The hybrid model is applied, after a few modifications, to the mass generation of a pseudoscalar field such as the $\eta'$ meson.

In the process of deriving equations of motion in the 2n-dimensional models, it is necessary to know the variation of the Chern-Simons current in 2n dimensions. To find this, we adopt an elegant method developed on (2n + 1)-dimensional space.

This paper is organized as follows. Section 2 introduces the topological entities in 2n dimensions. Section 3 presents a straightforward 2n-dimensional generalization of the model found by Dvali et al. Section 4 proposes new 2n-dimensional models with a Stückelberg-type mass term. Section 5 contains a summary and discussion. The appendix is devoted to calculating the variation of the Chern-Simons current in 2n dimensions.

II. TOPOLOGICAL ENTITIES

Let $A$ be a (Hermitian) Yang-Mills connection on 2n-dimensional Minkowski space, $M^{2n}$, with local coordinates $(x^\mu)$. The connection $A$ is assumed to take values in a compact semisimple Lie algebra $\mathfrak{g}$, and hence $A$ can be expanded as $A = g_{\mu}^a T_a dx^\mu$. Here, $g$ is a coupling constant with mass dimension $(2 - n)$, $\{T_a\}$ are Hermitian basis of $\mathfrak{g}$ satisfying the commutation relations $[T_a, T_b] = i f_{abc} T_c$ and the normalization conditions $\text{Tr}(T_a T_b) = \delta_{ab}$. The curvature 2-form of $A$ is given by

$$F = dA - i A^2 = \frac{1}{2} g F^a_{\mu\nu} T_a dx^\mu dx^\nu,$$  \hspace{1cm} (1)

with $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^a_{bc} A^b_\mu A^c_\nu$. (Throughout this paper, the symbol $\wedge$ of the wedge product is omitted.)

Consider the Chern-Pontryagin 2n-form

$$P_{2n} = \text{Tr} F^n = \frac{1}{2^n} g^n h_{a_1 \cdots a_n} F^{a_1}_{\mu_1\mu_2} \cdots F^{a_n}_{\mu_{2n-1}\mu_{2n}} \wedge dx^{\mu_1} dx^{\mu_2} \cdots dx^{\mu_{2n-1}} dx^{\mu_{2n}},$$  \hspace{1cm} (2)

where $h_{a_1 \cdots a_n} \equiv \text{Tr}(T_{a_1} \cdots T_{a_n})$. The Bianchi identity $dF = i (AF - FA)$ guarantees $dP_{2n} = 0$. Then, in accordance with Poincaré's lemma, $P_{2n}$ is expressed at least locally as

$$P_{2n} = dC_{2n-1},$$  \hspace{1cm} (3)

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with the Chern-Simons $(2n-1)$-form \[ C_{2n-1}(A, F) \equiv n \int_0^1 dt \text{Tr}(AF_t^{n-1}), \] (4)
where $F_t \equiv IF - i(t^2 - t)A^2$.
We now introduce the Hodge $* \dagger$ operator defined by
\[ *(dx^{\mu_1} \cdots dx^{\mu_p}) = \frac{1}{(2n - p)!} \epsilon^{\mu_1 \cdots \mu_p \mu_{p+1} \cdots \mu_{2n}} dx^{\mu_{p+1}} \cdots dx^{\mu_{2n}}. \] (5)
The $*$ operator transforms $p$-forms into their dual $(2n-p)$-forms. For a $p$-form $\alpha_p = (p!)^{-1} \alpha_{\mu_1 \cdots \mu_p} dx^{\mu_1} \cdots dx^{\mu_p}$ on $M^{2n}$, it is verified that
\[ * \dagger \alpha_p = (-1)^{(2n-p)+1} \alpha_p, \] (6)
\[ \dagger * \alpha_p = (-1)^{(p-1)(2n-p)+1} \delta^\mu_{\mu_1 \cdots \mu_{p-1}} \times dx^{\mu_1} \cdots dx^{\mu_{p-1}}. \] (7)

Using (5), the Hodge $*$ operation of $P_{2n}$ is found to be
\[ P_{2n} \equiv * P_{2n} = \frac{1}{2^n} \eta^{\alpha_1 \cdots \alpha_n} \epsilon^{\mu_1 \mu_2 \cdots \mu_{2n-1} \mu_{2n}} \times F_{\mu_1 \mu_2} \cdots F_{\mu_{2n-1} \mu_{2n}}. \] (8)
The $0$-form $P_2$ is referred to as the Chern-Pontryagin density. Applying the $*$ operator to (8) and using the formulas (6) and (7), we have the dual form of (8):
\[ P_{2n} = \partial_\mu C_{2n}^\mu, \] (9)
where the $C_{2n}^\mu$ are the components of the $1$-form $C_{2n} \equiv \dagger C_{2n-1}$. This $1$-form, or simply $C_{2n}^\mu$, is referred to as the Chern-Simons current. The $P_{2n}$ and $C_{2n}^\mu$ are topological entities essential for constructing the $2n$-dimensional models with topological mass generation.

III. MASS GENERATION DUE TO CHIRAL ANOMALY

Now, we show that the mass-generation mechanism studied in Ref.1 works in any even number of dimensions. The Lagrangian that we adopt, $\mathcal{L}_{2n}$, is a $2n$-dimensional analogue of the Lagrangian for the 4-dimensional model:
\[ \mathcal{L}_{2n} = \frac{1}{2} p_{2n}^2 + \Lambda^2 (C_{2n}^\mu - \partial_\mu p_{2n}^\mu) (J_5^\mu - \partial_\rho q_{\rho \mu}). \] (10)

Here, $p_{2n}^{\mu}$ and $q_{\mu \nu}$ are antisymmetric tensor fields, $J_5^\mu$ is an axial vector current, and $\Lambda$ is a constant with mass dimension. (An overall dimensionful constant is omitted.)

Under the (infinitesimal) gauge transformation
\[ \delta_\omega A_\mu^a = D_\mu A_\omega^a, \] (11)
the Chern-Pontryagin density $P_{2n}$ remains invariant, while the Chern-Simons current $C_{2n}^\mu$ transforms as
\[ \delta_\omega C_{2n}^\mu = \partial_\mu U_{2n}^{\mu \nu}. \] (12)
Here, $U_{2n}^{\mu \nu}$ is an antisymmetric tensor that is a polynomial in $(A_\mu^a, F_{\mu \nu}^a, \omega^a)$ and linear in $\omega^a$. (For further details, see the appendix.) We impose the gauge transformation rule
\[ \delta_\omega p_{2n}^{\mu} = U_{2n}^{\mu \nu} p_{2n}^{\nu} \] (13)
on $p_{2n}^{\mu}$ so that the combination $C_{2n}^\mu - \partial_\mu p_{2n}^{\mu}$ can be gauge invariant; thereby the gauge invariance of $\mathcal{L}_{2n}$ can be secured. In this sense, $p_{2n}^{\mu}$ plays the role of the Stickelberg field. By contrast, $q_{\mu \nu}$ is assumed to be gauge invariant, $\delta_\omega q_{\mu \nu} = 0$, by considering the gauge invariance of $J_5^\mu$. As a result, $\mathcal{L}_{2n}$ remains invariant under the gauge transformation $\delta_\omega$. The field $p_{2n}^{\mu}$ is necessary for the gauge invariance of $\mathcal{L}_{2n}$, while $q_{\mu \nu}$ is necessary to avoid the integrability condition $\partial_\rho J_5^\rho = \partial_\nu J_5^\nu$.
As can be seen in the appendix, the variation of the Chern-Simons current $C_{2n}^\mu$ is given by (see \[ \Lambda \cdot \partial \delta_2 \])
\[ \delta C_{2n}^\mu = W_{2n,a}^{\mu \nu} \delta A_{\alpha}^{\mu} + \partial_\mu W_{2n}^{\nu \mu}, \] (14)
where
\[ W_{2n,a}^{\mu \nu} \equiv \frac{n}{2^{n-1}} \epsilon^{\alpha_1 \cdots \alpha_{n-1} \rho \nu} \epsilon^{\mu_1 \mu_2 \cdots \mu_{2n-3} \mu_{2n-2} \mu_{2n}} \times F_{\mu_1 \mu_2} \cdots F_{\mu_{n-1} \mu_n} \cdot F_{\mu_{n+1} \mu_{n+2}} \cdots F_{\mu_{2n-1} \mu_{2n}}. \] (15)
and $W_{2n}^{\mu \nu}$ is an antisymmetric tensor that is a polynomial in $(A_\mu^a, F_{\mu \nu}^a, \delta A_{\alpha}^a)$ and linear in $\delta A_{\alpha}^a$. Using (14), variation of the action $S_{2n} = \int \mathcal{L}_{2n} dx$ with respect to $A_\mu^a$ is readily calculated, yielding the equation of motion
\[ \{- \partial_\nu p_{2n} + \Lambda^2 (J_5^\nu - \partial_\rho q_{\rho \nu})\} W_{2n,a}^{\mu \nu} - \Lambda^2 \partial_\mu (J_5^\mu - \partial_\nu q_{\nu \mu}) \delta W_{2n}^{\mu \nu} = 0. \] (16)
Variation of $S_{2n}$ with respect to $p_{2n}^{\mu}$ and $q_{\mu \nu}$ yields the equations
\[ \partial_\mu (J_5^\mu - \partial_\nu q_{\nu \mu}) - (\mu \leftrightarrow \nu) = 0, \] (17)
\[ \partial^\mu (C_{2n}^\mu - \partial_\nu p_{2n}^{\nu}) - (\mu \leftrightarrow \nu) = 0. \] (18)
By virtue of (17), the second line of (16) vanishes. Also, we can strip away $W_{2n,a}^{\mu \nu}$ \[ \text{using the identity} \ W_{2n,a}^{\mu \nu} F_{\rho \nu} = 2 \delta^{\mu \nu} P_{2n}. \] As a result, provided $P_{2n} \neq 0$, (16) reduces to
\[ - \partial_\nu p_{2n} + \Lambda^2 (J_5^\nu - \partial_\rho q_{\rho \nu}) = 0. \] (19)
Taking the divergence of (19) and considering antisymmetry of $q_{\rho \nu}$, we have
\[ \partial^\mu J_5^\mu = -N P_{2n}, \] (20)
where $N$ is a dimensionless positive constant. Then, (20) becomes
\[ \partial^2 P_{2n} + N \Lambda^2 P_{2n} = 0. \] (22)
This shows that the pseudoscalar $\mathcal{P}_{2n}$ has acquired the mass $\sqrt{N}\Lambda$. It should be stressed that the mass $\sqrt{N}\Lambda$ is generated owing to the presence of the chiral anomaly. The topological mass generation studied by Dvali et al. \cite{i} is thus valid in any even number of dimensions.

IV. OTHER MODELS

Until now, we have merely considered a $2n$-dimensional generalization of the 4-dimensional model given in Ref. \cite{i}. In this section, we propose new $2n$-dimensional models with topological mass generation.

A. A Stückelberg-type model

With the topological entities $\mathcal{P}_{2n}$ and $\mathcal{C}_{2n}^\mu$ and the antisymmetric tensor field $p^{\mu\nu}$, we first propose a model governed by the Lagrangian

$$\tilde{\mathcal{L}}_{2n} = \frac{1}{2} \mathcal{P}_{2n}^2 - \frac{1}{2} m^2 (\mathcal{C}_{2n,\mu}^\nu - \partial_\mu p^{\mu\nu}) (\mathcal{C}_{2n,\nu}^\mu - \partial_\nu p_{\mu\nu}) ,$$

(23)

where $m$ is a constant with mass dimension. Obviously, $\tilde{\mathcal{L}}_{2n}$ is invariant under the gauge transformation $\delta_\nu$. Variation of the action $\tilde{\mathcal{S}}_{2n} = \int \tilde{\mathcal{L}}_{2n} dx$ with respect to $A_{0}^\mu$ gives, with the help of (14), the equation of motion

$$\{ - \partial_\mu \mathcal{P}_{2n} - m^2 (\mathcal{C}_{2n,\mu}^\nu - \partial_\nu p_{\mu\nu}) \} \mathcal{W}_{2n,a}^{\mu} + m^2 \partial_\mu (\mathcal{C}_{2n,\nu}^\mu - \partial_\nu p_{\mu\nu}) \frac{\delta \mathcal{Y}_{2n}^{\mu\nu}}{\delta A_{0}^\nu} = 0 .$$

(24)

Variation of $\tilde{\mathcal{S}}_{2n}$ with respect to $p^{\mu\nu}$ yields the equation

$$\partial_\mu (\mathcal{C}_{2n,\nu}^\mu - \partial_\nu p_{\mu\nu}) - (\mu \leftrightarrow \nu) = 0 .$$

(25)

By virtue of (25), the second line of (24) vanishes. Also, we can strip away $\mathcal{W}_{2n,a}^{\mu}$ in (24) in the same manner as what we used under (15). Consequently, provided $\mathcal{P}_{2n} \neq 0$, (24) reduces to

$$- \partial_\mu \mathcal{P}_{2n} - m^2 (\mathcal{C}_{2n,\mu}^\nu - \partial_\nu p_{\mu\nu}) = 0 .$$

(26)

Taking the divergence of (26), and noting (9) and antisymmetry of $p_{\mu\nu}$, we have

$$\partial^2 \mathcal{P}_{2n} + m^2 \mathcal{P}_{2n} = 0 .$$

(27)

This shows that the pseudoscalar $\mathcal{P}_{2n}$ has the mass $m$, which is immediately caused by the second term on the right-hand side of (24). Because this term provides a mass in a gauge invariant manner, it can be called the Stückelberg-type mass term of $\mathcal{P}_{2n}$. Accordingly, we refer to the present model as the Stückelberg-type model. The mass-generation mechanism in this model is obviously different from that in the model presented in section 3.

B. A hybrid model

Next, we propose a hybrid of the previous two models. The Lagrangian that we adopt to define the hybrid is

$$\mathcal{L}_{2n} = \frac{1}{2} \mathcal{P}_{2n}^2 - \frac{1}{2} m^2 (\mathcal{C}_{2n,\nu}^\mu - \partial_\mu p^{\mu\nu})(\mathcal{C}_{2n,\nu}^\mu - \partial_\nu p_{\mu\nu}) + \Lambda^2 (\mathcal{C}_{2n,\nu}^\mu - \partial_\mu p^{\mu\nu})(\mathcal{J}_{\nu}^\delta - \partial_\nu q_{\mu\nu}) .$$

(28)

This certainly inherits characteristics of the Lagrangians \cite{10} and \cite{23}. Variation of the action $\tilde{\mathcal{S}}_{2n} = \int \mathcal{L}_{2n} dx$ with respect to $A_{0}^\mu$ gives the equation of motion

$$\{ - \partial_\mu \mathcal{P}_{2n} - m^2 (\mathcal{C}_{2n,\mu}^\nu - \partial_\nu p_{\mu\nu}) \} \mathcal{W}_{2n,a}^{\mu} + \{ m^2 \partial_\mu (\mathcal{C}_{2n,\nu}^\mu - \partial_\nu p_{\mu\nu}) \} \frac{\delta \mathcal{Y}_{2n}^{\mu\nu}}{\delta A_{0}^\nu} - \Lambda^2 \partial_\mu (\mathcal{J}_{\nu}^\delta - \partial_\nu q_{\mu\nu}) \frac{\delta \mathcal{Y}_{2n}^{\mu\nu}}{\delta A_{0}^\nu} = 0 .$$

(29)

Variation of $\tilde{\mathcal{S}}_{2n}$ with respect to $p^{\mu\nu}$ and $q_{\mu\nu}$ yields the equations

$$m^2 \partial_\mu (\mathcal{C}_{2n,\nu}^\mu - \partial_\nu p_{\mu\nu}) - \Lambda^2 \partial_\mu (\mathcal{J}_{\nu}^\delta - \partial_\nu q_{\mu\nu}) - (\mu \leftrightarrow \nu) = 0 ,$$

(30)

$$\partial^\mu (\mathcal{C}_{2n,\nu}^\mu - \partial_\nu p_{\mu\nu}) - (\mu \leftrightarrow \nu) = 0 .$$

(31)

Combining (30) and (31) leads to (17). In the same procedure as was taken to derive (20) and (27) from (14) and (24), respectively, we obtain, from (29) and (30),

$$\partial^2 \mathcal{P}_{2n} + m^2 \mathcal{P}_{2n} - \Lambda^2 \partial_\mu (\mathcal{J}_{\nu}^\delta - \partial_\nu q_{\mu\nu}) = 0 .$$

(32)

When the chiral anomaly is presented, (21) holds and (32) becomes

$$\partial^2 \mathcal{P}_{2n} + (m^2 + N\Lambda^2) \mathcal{P}_{2n} = 0 .$$

(33)

This demonstrates that the pseudoscalar $\mathcal{P}_{2n}$ has the mass $\tilde{m} \equiv \sqrt{m^2 + N\Lambda^2}$. Obviously, the mass $\tilde{m}$ is caused by both the Stückelberg-type mass term and the presence of the chiral anomaly. The hybrid model can be reduced to either of the previous models depending on choices of the mass parameters $m$ and $\Lambda$.

V. SUMMARY AND DISCUSSION

The topological mass generation studied by Dvali et al. is valid in any even number of dimensions with no essential changes. That is, the $2n$-dimensional Chern-Pontryagin density $\mathcal{P}_{2n}$ acquires a mass owing to the presence of the chiral anomaly. Here, just as in the 4-dimensional model, the presence of the chiral anomaly is assumed without specifying its dynamical origin. To bring the $2n$-dimensional model close to a complete one, it will be necessary to investigate the underlying dynamics that leads to the mass generation due to the chiral anomaly.
By incorporating the St"uckelberg-type mass term into the Lagrangian \( \mathcal{L}_{2n} \), the 2\( n \)-dimensional model is extended to the hybrid model governed by the Lagrangian \( \mathcal{L}_4 \). The hybrid model becomes the St"uckelberg-type model in the absence of the chiral anomaly. Now we concentrate our discussion on the hybrid model, because it involves the other two models. In the case \( n = 1 \), the hybrid model reduces to the 2-dimensional massive Yang-Mills theory with a vector current.

In the case \( n \geq 2 \), the Lagrangian \( \mathcal{L}_{2n} \) consists of higher dimensional terms such as \( P^2_{2n} \). For this reason, \( \mathcal{L}_{2n} \) cannot be regarded as a fundamental Lagrangian; it should be viewed as an effective Lagrangian that is derived from a fundamental gauge theory. The hybrid model in the case \( n \geq 2 \) will be applied to a phenomenological description of mass-generation phenomena expected in the fundamental theory. In this connection, now we propose an application of the hybrid model to the mass generation of a pseudoscalar field.

As in Ref. [1], we consider the axial vector current of the form

\[
J^5_\mu = \sqrt{N} \Lambda^{-1} \partial_\mu \eta_0 , \tag{34}
\]

where \( \eta_0 \) is a pseudoscalar field. Adding an \( \eta_0 \) kinetic term to \( \mathcal{L}_4 \), and removing \( q_{\mu \nu} \) and a total derivative, we have the Lagrangian

\[
\mathcal{L}'_{2n} = \frac{1}{2} P^2_{2n} - \frac{1}{2} m^2 (C_{2n, \mu} - \partial_\mu \eta_0) (C_{2n, \nu} - \partial_\nu \eta_0) - \sqrt{N} \Lambda P_{2n} \eta_0 + \frac{1}{2} \partial_\mu \eta_0 \partial^\mu \eta_0 . \tag{35}
\]

This is gauge invariant and leads to the field equations

\[
- \partial_\mu P_{2n} - m^2 (C_{2n, \mu} - \partial_\mu \eta_0) + \sqrt{N} \Lambda \partial_\mu \eta_0 = 0 , \tag{36}
\]

\[
\partial^2 \eta_0 + \sqrt{N} \Lambda P_{2n} = 0 , \tag{37}
\]

and (31). Because the divergence of \( \mathcal{L} \) reproduces \( \mathcal{L}' \), the chiral anomaly is considered in the Lagrangian \( \mathcal{L}' \). Taking the divergence of (36) and using (9) and (37) yield (33). Hence, as before, \( P_{2n} \) acquires the mass \( \hat{m} \). Using (37), (33) can be written in terms of \( \eta_0 \):

\[
(\partial^2 + \hat{m}^2) \partial^2 \eta_0 = 0 . \tag{38}
\]

This equation implies that \( \eta_0 \) possesses both the massless and massive modes. Because the massive mode is recognized to be physical, it follows that \( \eta_0 \) can behave as a pseudoscalar field with the mass \( \hat{m} \). In this way, a mass of the field \( \eta_0 \) is generated.

The Lagrangian \( \mathcal{L}_{2n} \) in 4 dimensions, \( \mathcal{L}_4 \), is very similar to what Di Vecchia used for solving the U(1) problem in a simple model [2]. The similarity can be seen by identifying \( \hat{m} \) and \( m \) with the masses of the singlet and nonsinglet pseudoscalar-mesons, respectively. (The \( \eta' \) mass is evaluated by taking into account the mixing between the singlet meson \( \eta_0 \) and a nonsinglet meson.)

A remarkable difference between \( \mathcal{L}_4 \) and Di Vecchia's Lagrangian, \( \mathcal{L}_D \), is that whereas \( \mathcal{L}_D \) contains the mass term \( M = -\frac{1}{2} m^2 \eta_0^2 \), \( \mathcal{L}_4 \) does not contain it. Instead of \( M \), \( \mathcal{L}_4 \) contains the St"uckelberg-type mass term to provide the mass \( m \). Unlike \( M \), the St"uckelberg-type mass term does not break the symmetry under a constant shift of \( \eta_0 \). In spite of such a difference, the hybrid model should have a close connection with the effective Lagrangian approach to the U(1) problem [3, 4].

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APPENDIX: VARIATION OF THE CHERN-SIMONS CURRENT

In this appendix, we calculate the variation of the Chern-Simons current \( \mathcal{C}_2^{2n} \). For this purpose, we adopt a geometric method developed on the product space \( M^{2n} \times \mathbb{R} \), a direct product of 2\( n \)-dimensional Minkowski space \( M^{2n} \) and 1-dimensional real space \( \mathbb{R} \). The exterior derivative in \( M^{2n} \times \mathbb{R} \) takes the form

\[
d = d + \delta_y = \frac{\partial}{\partial x^\mu} dx^\mu + \frac{\partial}{\partial y} dy , \tag{A.1}
\]

where \( y \) denotes the coordinate of \( \mathbb{R} \). We now consider the following Yang-Mills connection defined on \( M^{2n} \times \mathbb{R} \):

\[
A = A + \Omega = g A^a_\mu T_a dx^\mu + g \omega^a T_a dy , \tag{A.2}
\]

where \( A \) is a 1-form that, at \( y = 0 \), agrees with the connection \( A \) that is already present in \( M^{2n} \). The components \( (A^a_\mu, \omega^a) \) of \( A \) are understood to be functions of \( (x^\mu, y) \). The curvature 2-form of \( A \) is defined in the manner same as (1):

\[
F = dA - i A^2 . \tag{A.3}
\]

Substituting (A.1) and (A.2) into (A.3) and noting the nilpotency \( dy dy = 0 \), we have

\[
F = F + \Xi , \tag{A.4}
\]

with \( \Xi \equiv \delta_y A + D \Omega \). Here, \( D \Omega \) is the exterior covariant derivative of \( \Omega \): \( D \Omega \equiv d \Omega - i (A \Omega + \Omega A) \). Obviously, \( \Xi \) can be expressed as \( \Xi = g \xi^a_\mu T_a dy dx^\mu \), with \( \xi^a_\mu \) being functions of \( (x^\mu, y) \). Now we write the definition of \( \Xi \) as

\[
\delta_y A = - D \Omega + \Xi . \tag{A.5}
\]

This expression can be read as a transformation rule of \( A \). In fact, the right-hand side is understood as the sum
of the (infinitesimal) gauge transformation with a parameter \( \Omega \) and the shift transformation with a parameter \( \Xi \). For the sake of convenience, we decompose into the sum of the two transformation rules:

\[
\delta_\Omega A = -D\Omega, \tag{A.6}
\]

\[
\delta_\Xi A = \Xi, \tag{A.7}
\]

in such a way that \( \delta_\mu A = \delta_\Omega A + \delta_\Xi A \). Accordingly, the exterior derivative \( d \) is expressed as

\[
d = d + \delta_\Omega + \delta_\Xi. \tag{A.8}
\]

The transformation rules (A.6) and (A.7) can be written in terms of the component fields as

\[
\delta_\omega A_\mu = D_\mu \omega^a, \tag{A.9}
\]

\[
\delta_\Xi A_\mu = \xi_\mu^a, \tag{A.10}
\]

with \( D_\mu \omega^a = \partial_\mu \omega^a + g f_{abc} A^a_\rho \partial^\rho \omega^c \). Here, \( \delta_\omega \) and \( \delta_\Xi \) are defined by \( \delta_\Omega = \delta_\omega dy \) and \( \delta_\Xi = \delta_\Xi dy \), respectively.

Replacing \((A, F)\) in formula (6) by \((A, F)\), we have an analogue of (6) valid in \( M^{2n} \times \mathbb{R} \):

\[
\text{Tr} F^n = dC_{2n-1}, \tag{A.11}
\]

where \( C_{2n-1} \equiv C_{2n-1}(A, F) \). The \((2n - 1)\)-form \( C_{2n-1} \) can be expanded in powers of \( dy \); by virtue of the nilpotency \( dy dy = 0 \), the expansion has only a finite number of expansion terms:

\[
C_{2n-1} = C_{2n-1}(A + \Omega, F + \Xi)
= C_{2n-1}(A, F) + U_{2n-1}(A, F, \Omega)
+ V_{2n-1}(A, F, \Xi). \tag{A.12}
\]

Here, \( U_{2n-1} \) is first order in \( \Omega \) and includes no \( \Xi \), while \( V_{2n-1} \) is first order in \( \Xi \) and includes no \( \Omega \). Concrete forms for \( U_{2n-1} \) and \( V_{2n-1} \) can be found from (6) and (A.12). Applying \( d \) to (A.12) gives

\[
dC_{2n-1} = dC_{2n-1} + dU_{2n-1} + dV_{2n-1}
+ \delta_\Omega C_{2n-1} + \delta_\Xi C_{2n-1}. \tag{A.13}
\]

Also, the following expansion is valid with (A.4):

\[
\text{Tr} F^n = \text{Tr} F^n + n \text{Tr}(F^{n-1} \Xi). \tag{A.14}
\]

Substituting (A.13) and (A.14) into (A.11) and decomposing the resultant with respect to \( \Omega \) and \( \Xi \), we have

\[
\delta_\Omega C_{2n-1} = -dU_{2n-1}, \tag{A.15}
\]

\[
\delta_\Xi C_{2n-1} = n \text{Tr}(F^{n-1} \Xi) - dV_{2n-1}. \tag{A.17}
\]

Equation (A.13) is identical to (3), (A.16) is the (infinitesimal) gauge transformation of \( C_{2n-1} \), and (A.17) is the shift transformation of \( C_{2n-1} \). In this way, the transformation rules of \( C_{2n-1} \) have together been derived.

We can write (A.16) and (A.17) as

\[
\delta_\omega C_{2n-1} = dU_{2n-2}, \tag{A.18}
\]

\[
\delta_\Xi C_{2n-1} = n \text{Tr}(F^{n-1} \xi) + dV_{2n-2}, \tag{A.19}
\]

with \( \xi \equiv g \xi_{\mu}^a T_a dx^\mu \). Here, \( U_{2n-2} \) and \( V_{2n-2} \) are \((2n - 1)\)-forms defined by \( U_{2n-1} = U_{2n-2} dy \) and \( V_{2n-1} = V_{2n-2} dy \), respectively. We hereafter treat (A.18) and (A.19) as transformation rules in \( M^{2n} \) by setting \( y = 0 \).

Applying the \( \ast \) operator to (A.13) and (A.19) and using the formulas (6) and (7) lead to the dual forms:

\[
\delta_\omega C^a_{2n} = \partial_n U^a_{\mu \nu}, \tag{A.20}
\]

\[
\delta_\Xi C^a_{2n} = \mathcal{W}^a_{2n,a} \xi^a_{\mu} + \partial_n \nu^a_{\nu}, \tag{A.21}
\]

where

\[
\mathcal{W}^a_{2n,a} = \frac{n}{2^{n-1}} g^{a a_{1} \cdots a_{n-1}} a_{c}^{\mu_{1} \

\mu_{2} \cdots \mu_{2n-3} \mu_{2n-2} a} \nu^a_{2n-1} \nu^a_{2n-2} \cdots 

\times F_{\mu_{1} \mu_{2}}^{a_{1}} \cdots F_{\mu_{2n-3} \mu_{2n-2}}^{a_{n-1}}, \tag{A.22}
\]

the \( U^a_{\mu \nu} \) are the components of the 2-form \( U_{2n} \equiv - \ast U_{2n-2} \), and the \( \nu^a_{2n, \nu} \) are the components of the 2-form \( V_{2n} \equiv - \ast V_{2n-2} \). Obviously, \( U^a_{\mu \nu} \), \( \nu^a_{2n, \nu} \), and \( \mathcal{W}^a_{2n,a} \) are antisymmetric tensors.

Because \( \xi^a_{\mu} \) are arbitrary functions of \( x^\mu \), the shift transformation (A.10) can be identified with the variation of \( A^a_{\mu} \). Replacing \( \xi^a_{\mu} \) by the variation \( \delta A^a_{\mu} \), we express (A.21) in the form of the variation of \( C^a_{2n} \):

\[
\delta C^a_{2n} = \mathcal{W}^a_{2n,a} \delta A^a_{\mu} + \partial_n \nu^a_{2n, \nu}, \tag{A.23}
\]

where \( \nu^a_{2n, \nu} \) here is linear in \( \delta A^a_{\mu} \). Thus, the variation of the Chern-Simons current has been obtained using a geometric method.

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