Dragged surfaces. On the accretion tori in the ergoregion

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Abstract
We discuss the conditions for the existence of extended matter configurations orbiting in the ergoregion or close to the outer ergosurface of the Kerr black hole ("dragged" configurations). The corotating tori under consideration are perfect fluid configurations with barotropic equation of state, orbiting on the equatorial plane of the central Kerr black hole. The possibility of magnetized tori with a toroidal magnetic field is also discussed. Indications on the attractors where dragged tori can be observed are provided with the analysis of the fluid characteristics and geometrical features, relevant for the tori stability and their observations. QPOs emissions from the inner edges of the dragged tori are also discussed. We argue that the smaller dragged tori could be subjected to a characteristic instability, effect of the frame-dragging. This possibility is thoroughly explored. This can finally lead to the destruction of the torus (disk exfoliation) which can combine with accretion and processes present in the regions very close to the black hole horizon. Tori are characterized according to the central attractor dimensionless spin. These structures can be observed orbiting black holes with dimensionless spin $a > 0.9897 M$.

Key words: Black hole physics—Hydrodynamics—Accretion, accretion disks—Galaxies: active—galaxies: jets

1 Introduction
Although restricted by the typical assumptions of the simplified models, geometrically thick (stationary) disks provide a striking good approximation of several aspects of accretion in more complex dynamical models, some of these features include indication on tori location, the disk elongation on their symmetry plane, the inner edge of quiescent and accreting disks, the tori thickness, the maximum height, and the critical pressure points in the disks. In this article we explore accreting toroidal matter orbiting a central Kerr black hole (BH), within the hydrodynamic model introduced and detailed in a series of works (Paczyński&Wiita 1980; Paczyński 1980; Kozłowski et al. 1978; Abramowicz et al. 1996; Fishbone&Moncrief 1976; Frank et al. 2002; Pugliese et al. 2013; Pugliese&Montani 2015).

This is a fully general relativistic model of an opaque and super-Eddington, pressure supported disk, cooled by advection based on the Boyer theory of the equilibrium and rigidity in general relativity (Boyer 1965). In the Boyer model, many features of the disk dynamics and morphology like the disk thickness, the disk stretching on the equatorial plane and the disk location, are predominantly constrained by the geometric properties of spacetime via an effective potential function regulating the pressure gradients in the Euler equation. The effective potential function contains two main components: a dynamical term, related to the orbiting matter by

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means of the fluid angular momentum, here assumed constant along the disk, and a geometrical one related to the properties of the spacetime background. The boundary of any stationary, barotropic, perfect fluid torus is determined by the equipotential surface, which are the surfaces of constant pressure, defined by the gradient of a scalar function, i.e., the effective potential. This property holds if, for barotropic fluids, the relativistic frequency $\Omega$ turns to be function of the fluid angular momentum $\ell$ only or $\Omega = \Omega(\ell)$ (von Zeipel conditions) (Kozłowski et al. 1978; Abramowicz 1971; Chakrabarti 1991; Chakrabarti 1990). Accretion onto the attractor is driven through the vicinity of the cusp (corresponding to the inner edge of the cusped toroid) in the self crossed (cusped) closed configurations from a violation of the hydrostatic equilibrium, i.e., due to Paczyński mechanism (Kozłowski et al. 1978; Paczyński&Wiita 1980; Paczyński 1980). This mechanism has been proved to be also an important stabilizing mechanism against the thermal and viscous instabilities locally, and against the so called Papaloizou&Pringle instability (PP1) globally (Blaes 1987; Abramowicz&Fragile 2013; Paczyński 1980; Paczyński&Wiita 1980; Kozłowski et al. 1978). During the evolution of dynamical processes, the functional form of the angular momentum and entropy distribution depends on the initial conditions of the system and on the details of the dissipative processes.

We focus on tori orbiting in regions close to the BH static limit and in the BH ergoregion, which is the region between the outer ergosurface and the (outer) event horizon, defining the partially contained and dragged surfaces. (The outer ergosurface is also called stationary limit surface or static limit, for the absence of any (timelike) static observers inside the region). A partially contained torus is centered (location of maximum pressure point) in the outer region, but a part of the torus inner region, which is bounded from the below by the inner edge and from the above by the torus center, orbits in the ergoregion.

Thick accretion disks are usually associated with very compact objects like black holes and super massive BHs (SMBHs). Directly involved in the equilibrium phases of the attractors, they are very likely the base of the jet formation and dynamics, the Active Galactic Nuclei (AGN) and Gamma Ray Bursts (GRBs) emissions. In this context the dynamics inside the ergoregion is extremely relevant in Astrophysics: accreting matter can support, for example, jets of matter or radiation originated inside the ergoregion (Meier 2012; Gariel et al. 2013), which can empower jets with powerful Poynting flux, as from ergospheric disks studied in Punsly (2007).

The processes occurring inside the BH ergoregion are essential for understanding the central engine mechanism of many emissions (Meier 2012; Frolov&Zelnikov 2011). The mechanism, by which energy from compact spinning objects is extracted, is of great astrophysical interest: BH rotational energy can be extracted through the classical Penrose process, and indeed the super-radiance as seen by asymptotic observers. Black hole thermodynamics has been grounded on the observation that the BH energy can actually be extracted from the black hole, and ergoregion is the background geometrical engine of many processes of energy extraction from the Kerr BH. The ergosurface is also the geometric basic of super-radiant scattering of incident waves which is the wave analog of the Penrose process, based on the fact that the particles energy, within the ergoregion, as measured by an observer at infinity, can also be negative (BH super-radiance appears to distinguish bosonic waves from the fermions in the super-radiant amplification).

The rotational energy can be extracted from the source, lowering its angular momentum. The Penrose energy extraction, and the super-radiance are classical phenomena due to the frame-dragging of the spinning spacetime. Another possibility is the extraction of energy from a rotating black hole through the Blandford-Znajek mechanism (Blandford&Znajek 1977; Pei et al. 2016; Komissarov 2009; Lasota et al. 2014; Penrose 1969) and magnetic Penrose process or radiative Penrose process(Kološ et al. 2021; Tursunov et al. 2020). The Hawking radiation instead arises from the vacuum fluctuations in the regions close to the BH horizon, it is the spontaneous emission of thermal radiation which is created in the vacuum regions surrounding a BH, and it may eventually lead to a decrease of the BH mass (Penrose&Floyd 1971; Hawking 1976; Hawking 1971; Bekenstein 1975).

In this article we characterize the orbiting extended matter configurations orbiting in the ergoregion of a Kerr BH, introducing the concept of dragged tori and partially contended tori made by corotating perfect fluids thick tori. We discuss possible instabilities by exploring the effects of geometry frame-dragging on the disk exfoliation and evaluating the pressure gradients which are the main factor for the tori formation and formation of a possible atmosphere of (free) particles swarm. A possible outcome of the instabilities of the smallest tori could be an emission of swarm of particles and photons, enhanced accretion, or the torus destruction. Dragged and partially contained tori are also subjected to processes typical of the regions very close to the black hole horizon as, for 1 The presence of negative energy particles is also a distinctive feature of the ergoregion in weakly rotating naked singularities (WNS) differentiating these from the BHs. (The superirradiance and ergoregion instability is a further characteristic differentiating BHs from NSs.). The static limit can act as a semi-permeable membrane separating the spacetime region, filled with negative energy particles from the external one, filled with positive. The WNSs ergoregion is characterized by the existence of zero and negative energy states circular orbits (stable circular geodesics with negative energy)(Stuchlík 1980; Stuchlík&Schée 2012; Charbulák&Stuchlík 2018; Stuchlík et al 2018). The presence of this special matter in an “antigravity” sphere, possibly filled with negative energy matter formed according to the Penrose process, and bounded by orbits with zero angular momentum, could have an important role in the source evolution(Hawking&Ellis 1973; Penrose&Floyd 1971).
example, the Runaway Instability. To evaluate this situation, we assess the torus verticality, the dragged torus geometrical thickness, the influence of the dragging frame on the disk thickness, and the location of the tori extreme of pressure with the respect to the static limit. We proceed to the exploration of the tori characteristics in dependence on the central attractor dimensionless spin, providing eventually indications on the attractors where dragged tori can be observed. Some extreme configurations in the ergoregion are also described, as proto-jets which are open cusped General Relativistic Hydrodynamic (GRHD) configurations, limiting solutions for the thick disks models, and the possibility of orbiting agglomerates of toroids composed by an aggregate of multi toroids orbiting in the ergoregion.

In the second part of this work, dragged and partially contained tori are studied in relation to the quasi-periodic oscillations (QPOs) in emission from the tori inner edges. We relate also the dragged tori in the ergoregion and the maximum extractable rotational energy $\xi$ from the BH horizon, constituting the state prior the total extraction of the energy from the BH, following the approach introduced in Daly (2009), which is quite independent from the details of the specific process of energy extraction, and based on the definition of BH irreducible mass function, rotational energy and using the BH classical thermodynamical law. (The energy extraction through the accretion process, the rotational energy converted into radiation corresponds to the binding energy of the fluid). For large part of this analysis we shall consider perfect fluids defined by a barotropic equation of state; however, we also investigate the possibility that the frame dragging could differentiate different polytropics. In this context we provide an estimation of the flux thickness, mass-flux and enthalpy-flux for dragged and partially contained thick disks with a fixed polytropic fluid.

The article is organized as follows: In Sec. (2) we introduce the thick accretion model and the fluid effective potential for the toroidal configurations in a Kerr spacetime background. Dragged surfaces and partially contended tori are introduced in Sec. (3.1). In Sec. (3.1.1) the tori verticality is studies and disk exfoliation is introduced. Dragged disks thickness is the focus of Sec. (3.1.2) where the influence of the dragging frame on the disk thickness is investigated. In Sec. (3.1.3) we explore the process of tori exfoliation. Extreme configurations as orbiting agglomerates of toroids and proto-jets are briefly considered in Sec. (3.1.4). Characteristic frequencies of the oscillations of the toroids are studied in Sec. (3.2) and in Sec. (3.2.1) where we discuss the origin of the QPOs emission in relations to the toroids in the ergoregion. Analysis of possible polytropic fluids for dragged and partially contained tori is in Sec. (3.3). Discussion on some aspects of tori energetics for these tori is in Sec. (3.3.1). Conclusions are Sec. (4). In Appendix (1) there are some notes on stationary magnetized tori in the ergoregion.

2 Fluid configuration on the Kerr spacetime

We consider a perfect fluid orbiting in the Kerr spacetime background, where the metric tensor can be written in Boyer-Lindquist (BL) coordinates \( \{t, r, \theta, \phi\} \) as follows

\[
ds^2 = -\alpha^2 dt^2 + \frac{A\sigma}{\rho^2} (d\phi - \omega_z dt)^2 + \frac{\rho^2}{\Delta} d\theta^2, \quad \sigma \equiv \sin^2 \theta, \quad A \equiv (r^2 + a^2)^2 - a^2 \Delta \sigma
\]

where \( \alpha = \sqrt{(\Delta \rho^2/A)} \) and \( \omega_z = 2aM \rho/A \) are the lapse function and the frequency of the zero angular momentum fiducial observer (ZAMOS) (Pugliese&Quevedo 2018), whose four velocity is \( a^\mu = (1/\alpha, 0, 0, \omega_z/\alpha) \), being orthogonal to the surface of constant \( t \). Here \( M \) is a mass parameter and the specific angular momentum is given as \( a = J/M \), where \( J \) is the total angular momentum of the gravitational source and \( \rho^2 \equiv r^2 + a^2 \cos^2 \theta \), \( \Delta \equiv r^2 - 2Mr + a^2 \). In the following it will be also convenient to introduce the quantity \( \sigma \equiv \sin^2 \theta \). We will consider the Kerr black hole (BH) case defined by \( a \in [0, M] \), the extreme black hole source \( a = M \), and the non-rotating limiting case \( a = 0 \) of the Schwarzschild metric. The horizons \( r_- < r_+ \) and the static (or stationary) limit \( r^+_\ast \) are respectively

\[
r_\pm \equiv M \pm \sqrt{M^2 - a^2}; \quad r^+_\ast \equiv M + \sqrt{M^2 - a^2 \cos^2 \theta}
\]

where \( r_+ < r^+_\ast \) on the planes \( \theta \neq 0 \) and it is \( r^+_\ast = 2M \) i on the equatorial plane \( \theta = \pi/2 \). In the region \( r \in [r_+, r^+_\ast] \equiv \Sigma^+_\ast \) (ergoregion or outer ergoregion), there is \( g_{rt} > 0 \) (time component of the metric tensor) and \( t \)-Boyer-Lindquist coordinate becomes spacelike, this fact implies that a static observer cannot exist inside the ergoregion. The radius \( r_+ \) is the ergosurface, or, precisely, outer ergosurface. In this work we investigate toroidal configurations of perfect fluid orbiting a Kerr SMBH, it will be therefore convenient to consider first the properties of the test particle circular motion. Since the metric is independent of \( \phi \) and \( t \), the covariant component of a particle four–momentum, \( p_\phi \) and \( p_t \), are conserved along the geodesics or\(^2\)

\(^2\) We adopt the geometrical units \( c = 1 = G \) and the \((-+++)\) signature, Latin indices run in \( \{0, 1, 2, 3\} \). The four-velocity satifsy \( u^a u_a = -1 \). The radius \( r \) has unit of mass \([M]\), and the angular momentum units of \([M]^2\), the velocities \([u^r] = [u^\theta] = 1 \) and \([u^\phi] = [u^t] = [M]^{-1} \) with \([a^r/a^\phi] = [M]^{-1} \) and
Kerr metric is symmetric under reflection through the equatorial plane 
there is 
$$u^r = \frac{E g_{\phi \phi} + g_{\phi r} L}{A_T}, \quad u^\phi = -\frac{E g_{\phi \phi} + g_{\phi r} L}{A_T}, \quad (A_T \equiv g_{\phi \phi} - g_{\phi r} g_{\phi r}).$$

The momentum $p^a = \mu u^a$ of the particle with mass $\mu$ and four-velocity $u^a$ can be normalized so that $g_{ab} u^b u^b = -k$, where $k = 0, -1, 1$ for null, spacelike and timelike curves, respectively. Quantities $(E, L)$ are constants of motion for circular test particles geodesics where $\xi = \partial_t$ is the Killing field representing the stationarity of the Kerr geometry and $\xi_\phi = \partial_\phi$ is the axial Killing field; the vector $\xi_t$ is spacelike in the ergoregion. In general, we may interpret $E$, for timelike geodesics, as representing the total energy of the test particle coming from radial infinity, as measured by a static observer at infinity, and $L$ as the axial component of the angular momentum of the particle. The normalization condition on the four-velocity can be solved for the energy $E$. For circular motion there is $u^r = 0$. Considering the motion on the fixed equatorial plane $\sigma = 1$, no motion is in the $\theta$ angular direction and $u^\theta = 0$ (the Kerr metric is symmetric under reflection through the equatorial plane $\theta = \pi/2$). Within these assumptions we obtain the effective potential $V_{\text{eff}}(\alpha; L, r) \equiv E\pm/\mu|_{u^\alpha=0}$. Circular orbits are therefore described by 
$$\dot{r} = 0, \quad V_{\text{eff}} = E/\mu, \quad \partial V_{\text{eff}}/\partial r = 0.$$

Some notable radii regulate the particle dynamics, namely the marginal circular orbit for timelike particles, $r^\pm_{\text{mbo}}$ which is also a photon orbit, the marginal bounded orbit at $r^\pm_{\text{mbo}}$, and the marginal stable circular orbit at $r^\pm_{\text{mso}}$ with angular momentum and energy $(\mp L_\pm, E_\pm)$ respectively, where $(\pm)$ is for counterrotating or corotating orbits with respect to the attractor. It is convenient to introduce also the relativistic angular frequency $\Omega$ and the specific angular momentum $\ell$ as follows 
$$\Omega \equiv \frac{u^\phi}{u^t} = -\frac{E g_{\phi t} + g_{\phi r} L}{E g_{\phi \phi} + g_{\phi r} L} = -\frac{g_{t \phi} + g_{\phi r} \ell}{g_{\phi \phi} + g_{\phi r} \ell}^\prime,$$

$$\ell \equiv \frac{L}{E} = \frac{u^\phi}{u_t} = \frac{u^b g_{\phi b} + g_{\phi r} u^r}{g_{t r} u^t + g_{\phi r} u^\phi} = \frac{g_{t \phi} + g_{\phi r} \Omega}{g_{t t} + g_{\phi r} \Omega}.$$

We consider a one-specie particle perfect fluid (simple fluid), where 
$$T_{ab} = (\rho + p)u_a u_b + p g_{ab},$$

is the fluid energy momentum tensor, $\rho$ and $p$ are the total energy density and pressure, respectively, as measured by an observer moving with the fluid. For the symmetries of the problem, we always assume $\partial_\phi Q = 0$ and $\partial_\sigma Q = 0$, being $Q$ a generic spacetime tensor (we can refer to this assumption as the condition of ideal hydrodynamics of equilibrium). The timelike flow vector field $u^a$ denotes now the fluid four-velocity. The motion of the fluid is described by the continuity equation and the Euler equation respectively: 
$$u^a \nabla_a Q + (p + \rho) \nabla^a u_a = 0, \quad (p + \rho) u^a \nabla_a u^c + h^{bc} \nabla_b p = 0,$$

where the projection tensor $h_{ab} = g_{ab} + u_a u_b$ (from the conservation equation $\nabla_a T^{ab} = 0$ along the tensor $(u^a, h_{ab})$ defined by the fluid, where $\nabla_a g_{bc} = 0$). We investigate in particular the case of a fluid circular configuration, defined by the constraint $u^r = 0$. Tori are centered on the central BH, orbiting on its equatorial plane which is also the tori symmetry plane—see for example Figs (1). On the fixed plane $\sigma = 1$, no motion is assumed in the $\theta$ angular direction, which means $u^\theta = 0$. We assume moreover a barotropic equation of state $p = p(\rho)$. The continuity equation is identically satisfied as consequence of these conditions. From the Euler equation (8) we obtain 
$$W \equiv \ln V_{\text{eff}}(\ell), \quad W \equiv \frac{L}{E},$$

where $W$ is the Paczynski-Witta (P-W) potential, function $V_{\text{eff}}(r)$ is the torus (fluid) effective potential, reflecting the background Kerr geometry and the centrifugal effects through the fluid specific angular momentum $\ell$. The fluid equilibrium is therefore 
$$[u_\sigma/u_t] = [M].$$

For the seek of convenience, we always consider the dimensionless energy and effective potential $|V_{\text{eff}}| = 1$ and an angular momentum per unit of mass $[L]/[M] = [M]$. We can relate the fluid effective potential $V_{\text{eff}}(\alpha; \ell, r)$ with the test particles circular orbits potential $V_{\text{eff}}^\pm (\alpha; L, r)$ as $u^2 = V_{\text{eff}}^\pm (\alpha; L, r) = A_T A^{-1} = -A_{r r}^{-1} (g_{t r} u^t + g_{\phi r} u^\phi)^2 = E^2 A_T A_{\phi \phi}^{-1}$ where $A_{r r}^{-1} \equiv g_{t r} (u^t)^2 + 2 g_{t r} u^t u^\phi + g_{\phi r} (u^\phi)^2$ and $A_{\phi \phi}^{-1} \equiv E^2 g_{\phi \phi} + 2 E g_{\phi r} L(\ell) + g_{t r} L(\ell)^2$ and where $A \equiv L^2 g_{t t} + 2 g_{t r} g_{\phi r} + g_{\phi \phi} = \ell^2 (L^2 - L^2 - L^2), \quad \omega \equiv \ell^2 (g_{t t} + 2 g_{t r} g_{\phi r} + g_{\phi \phi})$, here $\omega = 1/\ell$, and $A$ is related to the norm of stationary observers whose frequency is $\omega^2$ and $A$ is defined in Eq. (4). Light surfaces have $\omega = L^2 - L^2 = 0$. Note, that the effective potential of the fluids and the particles, as defined here in the hydrostatic model, is related to the constant energy $E$ of the test particle motion and $u_i$. These quantities could be negative in the ergoregion. The presence of (geodetical) circular motion with $E < 0$ or $E > 0$ in the ergoregion is a well known feature of the Kerr naked singularity solutions which has also zamos, zero angular momentum, $L = 0$ observers, which has not equivalent in the ergoregion of the BH and it is a distinctive feature of certain naked singularities with "small" spin–mass ratios. This characteristic has been attributed to the repulsive gravity effects of the NSs. We also note that this feature
regulated by the balance of the gravitational and pressure terms versus centrifugal factors arising due to the fluid rotation and the curvature effects of the Kerr background, encoded in the effective potential function $V_{eff}$. Here $\mu = (r, \theta)$ and, for large part of this analysis, we shall consider mainly the radial ($r$) gradients of pressure. Assuming the fluid is characterized by the specific angular momentum $\ell = \text{constant}$ across the torus (see also Lei et al. (2008); Abramowicz&Fragile (2013)), we consider, from Eq. (9) the critical points of the pressure in the torus as the extremes of the effective potentials considering the solutions is capable to define an effective ergoregion in the field of Reissner-Norström geometry (electrically charged and static metric). In the case we consider here circular orbit has $E > 0$ and therefore effective potential is always positive. (The energy $E$ may be negative as measured at infinity, locally it is always positive.) (Stuchlík&Schäfer 2013; Stuchlík& Schäfer 2012) Here we specify that $V^{2}_{eff} < 0$ (not well defined) in some circumstances inside the ergoregion, it is related to the normalization conditions on the 4-momentum which is stationary (i.e. the associated fluid four momentum has $u^\mu = u^r + \omega u^\phi$, where $\omega = \Omega_{\ell}$ is the relativistic angular velocity which satisfies to the von Zeipel conditions) therefore well defined in the ergoregion. There is $V_{eff}(r)^2 < 0$ for $r = r_+^\text{c}$ and $\ell / \ell_c \in [0, \ell_c \ell / M \sqrt{2}]$ with $r / M \in [1, 2]$ with $a / \ell < \sqrt{27 M - 7}, \sqrt{(7 - 33 M^2) / (2 M)}]$ which is a photon orbit in the ergoregion.
\( V_{\text{eff}} = K = \text{constant} \), introducing therefore the couple of tori parameters \((\ell, K)\). The toroidal surface are equipressure surfaces, therefore equipotential surfaces with \( V_{\text{eff}} = K = \text{constant} \), for a given \( \ell = \text{constant} \) (Abramowicz&Fragile 2013; Pugliese&Montani 2015; Pugliese&Stuchlík 2015; Pugliese et al. 2013) and for example Abramowicz (1971); Abramowicz (1985); Abramowicz et al. (1996); Abramowicz (2006); Adamek&Stuchlík (2013); Fishbone&Moncrief (1976); Font&Daigne (2002); Frank et al. (2002); Komissarov (2006); Lei et al. (2008); Montero et al. (2007); Pugliese&Montani (2013); Pugliese&Montani (2018); Pugliese&Stuchlík (2016); Pugliese&Stuchlík (2017); Pugliese&Stuchlík (2018a); Pugliese&Stuchlík (2018b); Pugliese&Stuchlík (2019); Pugliese&Stuchlík (2021); Zanotti&Pugliese (2015).

According to Eq. (9), tori centers (maximum pressure points) and cusps (minimum pressure points), are the critical points of \( V_{\text{eff}}(r) \) as function of the radius \( r \). The maximum of the hydrostatic pressure corresponds to the minimum of the effective potential \( V_{\text{eff}} \), and it is the \textit{torus center} \( r_{\text{center}} \). The instability points of the tori (P-W mechanics), are located at the minima of the pressure, correspondent to the maximum of \( V_{\text{eff}} \) as functions of \( r \)–see Figs (1).

More generally, equation \( \partial_r V_{\text{eff}} = 0 \) can be solved for the fluid specific angular momentum obtaining the function \( \ell(r) \) "leading function". We also introduce the function \( K(r) \) ("energy function") which, as function \( \ell(r) \), is also dependent on the BH dimensionless spin, and are defined as:

\[
\ell^\pm(r) : \partial_r V_{\text{eff}} = 0, \quad K^\pm(r) \equiv V_{\text{eff}}(r, \ell^\pm(r)),
\]

for counterrotating (+) and corotating (−) fluids respectively. Functions \( \ell(r) \) and \( K(r) \) are showed in connection with the tori surfaces in Figs (1). Functions \( (\ell(r), K(r)) \) define the hydrostatic pressure critical points \( r_{\text{crit}} \) in the torus, centers of maximum pressure \( r_{\text{center}} = r_{\text{crit}} > r_{\text{meso}}^\pm \) or cusps \( r_{\text{cusp}} = r_{\text{crit}} < r_{\text{meso}}^\pm \). Curves \( K^\pm(r) \) locate the tori centers and provide information on torus elongation and density and, for a torus accreting onto the central BH, determine the inner, \( r_{\text{inner}} \), and outer, \( r_{\text{outer}} \), torus edges, where the elongation on the equatorial plane is \( \lambda \equiv r_{\text{outer}} - r_{\text{inner}} \)–Figs (1) and Figs (2). The couple of constant parameters \((\ell, K)\) uniquely identifies each toroidal surface and these can be directly reduced to a single parameter \( \ell \), in presence of a cusp. (The critical points of the pressure are determined by the \( \ell \) parameter only, therefore in the case of cusped tori, the cusp fixes uniquely also the \( K \)-parameter, coincident with the maximum value of the torus effective potential–Figs (1)).

More specifically, solutions of the problem \( V_{\text{eff}} = K = \text{constant} \) lead to three classes of configurations corresponding to –C–cross sections of the closed surfaces (equilibrium quiescent torus); \( C_x \)–cross sections of the closed cusped surfaces (accreting torus)– \( J_x \)–cross sections of the open cusped surfaces, generally associated to proto-jet configurations.

The closed, not cusped, \( C \) surfaces are associated to stationary equilibrium (quiescent) toroidal configurations. For the cusped and closed equipotential surfaces, \( C_x \), the accretion onto the central black hole can occur through the cusp \( r_x \) of the equipotential surface. In this situation the outflow of matter through the cusp occurs as the torus surface exceeds the critical equipotential surface (having a cusp), leading to a mechanical non-equilibrium process, i.e. an instability in the balance of the gravitational and inertial forces and the pressure gradients, where matter inflows into the central black hole (a violation of the hydrostatic equilibrium known as Paczyński mechanism). In the case considered here the accreting flow “starts” across a “Lagrange point” (Roche lobe overflow, due to Paczynski accretion mechanics (Paczynski 1980; Paczynski & Wiita 1980). This is the cusp of the orbiting toroidal surface, which is an important aspect of thick disks since its presence also stabilizes the tori against several instabilities (thermal and viscous local instabilities, and globally against the Papaloizou-Pringle instability-PPI and it could be possibly connected to QPOs emission.) (Abramowicz&Fragile 2013).

Therefore, in this accretion model we shortly indicate the cusp of the self-crossed closed toroidal surface as the “inner edge of accreting torus”. Finally, the open equipotential surfaces, which are briefly considered in Sec. (3.1.4), have been associated to the formation of proto-jets (Pugliese&Stuchlík 2016; Pugliese&Stuchlík 2018c; Pugliese&Stuchlík 2018b; Pugliese&Stuchlík 2019).

The minimum points of the hydrostatic pressure are constrained by the geodesic structure of the Kerr geometry. Alongside the geodesic structure of the Kerr spacetime, we include radii \( r_{\Lambda}^\pm \) solution of \( \partial_r \ell = 0 \), and radii \( r_{\Theta}^{\pm} \) and \( r_{\phi}^{\pm} \) or more generally

This fully general relativistic model of pressure supported torus, traces back to the Boyer theory of the equilibrium and rigidity in general relativit, i.e. the analytic theory of equilibrium configurations of rotating perfect fluids (Boyer 1965; Frank et al. 2002). Each toroid is governed by the General Relativity hydrodynamic condition of equilibrium configurations of rotating perfect fluids applied to toroidal surfaces. These correspond also to the surfaces of constant density \( \rho \), specific angular momentum \( \ell \), and constant relativistic angular frequency \( \Omega \), where \( \Omega = \Omega(\ell) \) as a consequence of the von Zeipel theorem (Chakrabarti 1990; Chakrabarti 1991; Zanotti&Pugliese 2015). In this model the entropy is constant along the flow and the rotation law \( \ell = \ell(\Omega) \) is independent of the equation of state (Abramowicz&Fragile 2013). In these tori in fact the functional form of the angular momentum and entropy distribution during the evolution of dynamical processes, could be considered as dependent only on the initial conditions of the system see Abramowicz&Fragile (2013). In this situation the surfaces of constant enthalpy, pressure and density coincide with the surfaces of constant effective potential \( V_{\text{eff}} \), fixed by the integration parameter \( K \) related to the boundary conditions on the integration of the Euler equation. The torus surfaces are surfaces of constant pressure or \( \Sigma_i = \text{constant for} \ i \in (p, p, \ell, \Omega) \), (Frank et al. 2002), where it is indeed \( \Omega = \Omega(\ell) \) and \( \Sigma_{ij} = \Sigma_i \) for \( i, j \in (p, p, \ell, \Omega) \).
Fig. 2. Notable radii of the geodetic structure are represented as function of the BH dimensionless spin. \( r_\gamma \) is the photon orbit, \( r_{\text{mbo}} \) the marginally bounded orbit, \( r_{\text{mso}} \) marginally stable orbit. Radius \( r_\ell := \partial^2 \ell(r) = 0 \) is dashed purple line. Black region is the BH at \( r < r_+ \) (outer horizon). Center of quiescent \( C_3 \) configuration with specific momentum in \( \ell > \ell_\gamma \), are in \( r > r_{\ell}^+ \). Spins \( A^{+\epsilon} := \{a_\gamma, a_{\text{mbo}}, a_{\text{mso}}, a_{b_{\text{mbo}}}, a_{b_\gamma} \} \) of Eqs (13) are dashed lines, see discussion in Sec. (3.1). Upper left panel shows the entire range \( a \in [0, M] \), while other panels show the ranges \( \text{Range } A_1 \equiv [a_\gamma, a_{\text{mbo}}] \); \( \text{Range } A_2 \equiv [a_{\text{mbo}}, a_{\text{mso}}] \); \( \text{Range } A_3 \equiv [a_{\text{mso}}, a_{b_{\text{mbo}}} \); \( \text{Range } A_4 \equiv [a_{b_{\text{mbo}}, a_{b_\gamma}} \); \( \text{Range } A_5 \equiv [a_{b_\gamma}, M] \).

Fig. 3. Existence condition for accreting tori an double accretion (left panel), location of notable radii of the geodetic structure (right panel). Black region is \( r < r_+ \), \( r_+ \) is the outer horizon as function of the BH spin \( a/M \). Left panel: description of double accretion \( C_- < C_+ \) (inner corotating cusped closed torus and outer counter-rotating closed cusped torus) location of tori centers \( r_\pm \) (dotted strips) and accretion points \( r_\pm \) (shaded regions) for corotating \((-\)) and counterrotating \((+)\) fluids. \( r_{\text{mbo}} \) is the marginally bounded orbits \( r_{\text{mbo}} := \ell'(r_{\text{mbo}}) = \ell(r) \), \( r_\gamma \) is the photon orbit, \( r_{\text{mso}} \) is the marginally stable orbit. Black lines are \( A_7^{(+\epsilon)} := \{a_{\text{mso}}, a_{\text{mbo}}, a_{b_{\text{mbo}}, a_\gamma, a_{b_\gamma}} \} \) defined in Figs (2). Right panel: gray region is the outer ergosurface \( r_+ = 2M \) dotted lines are the spins \( A_7^+ \).

\[
\begin{align*}
(r_{b_{\text{mbo}}}^\pm, r_{b_\gamma}^\pm, r_M^\pm), & \text{ regulating the maximum points of the pressure, and defined as the solutions of the following equations} \\
& \ell^\pm(r_{b_{\text{mbo}}}^\pm) = \ell^\pm(r_{b_\gamma}^\pm) = \ell^\pm(r_M^\pm) = \ell^\pm(r_\ell^\pm) = \ell^\pm(r_\gamma^\pm), \quad \text{and} \\
r_{b_{\text{mbo}}}^\pm : \ell^\pm(r_{b_{\text{mbo}}}^\pm) = \ell^\pm(r_M^\pm) = \ell^\pm(r_\ell^\pm) & \quad \text{where} \\
r_\gamma^\pm < r_{b_{\text{mbo}}}^\pm < r_{b_{\text{mso}}}^\pm < r_{b_{\text{mbo}}}^\pm < r_\gamma^\pm,
\end{align*}
\]

respectively for the counterrotating \( (+) \) and corotating \( (-) \) orbits—Fig. (3). Radii \( \{r_{b_{\text{mso}}}^\pm, r_{b_{\text{mbo}}}^\pm, r_\gamma^\pm, r_M^\pm, r_\ell^\pm, r_\gamma^\pm\} \) constitute the extended geodetic structure of the spacetime. Location of radii \( \{r_M^\pm, r_\ell^\pm\} \) with respect to the radii Eqs (12) depend on the BH spin.

We summarize the constraints, defining ranges of fluids specific angular momentum \( (L_1, L_2, L_3) \) defining the closed quiescent, cusped or open configurations and the tori location (cusp and tori centers) with respect to the central BH:
- Tori ($C_1, C_x$) For $\ell \in L_1 \equiv [\ell_{mso}, \ell_{mbo}]$ there are quiescent and cusped tori, with topologies ($C_1, C_x$); accretion point is $r_x \in [r_{mbo}, r_{mso}]$ and center with maximum pressure is $r_{center} \in [r_{mso}, r_{mbo}]$.

- Tori ($C_2, J_x$) For $\ell \in L_2 \equiv [\ell_{mbo}, \ell_j]$ there are quiescent tori and proto-jets, with topologies ($C_2, J_x$); unstable point is on $r_j \in [r_j, r_{mbo}]$ and center with maximum pressure $r_{center} \in [r_{mbo}, r_j]$.

- Tori $C_3$ For $\ell \in L_3, \ell \geq \mp \ell_3$, there are quiescent tori $C_3$ with center $r_{center} > r_b^b$, holding respectively for corotating and countercorotating tori.

Let $(K_{center}, K_x)$ be function $K(r)$ evaluated at $r_{center}$ and $r_x$ respectively. Closed, not cusped configurations $C$ are for $K \in [K_{center}, K_x]$ where $K_x \in \{K_x, 1\}$ according to the momentum $\ell \in L_1$ or $\ell \in \{L_2, L_3\}$ respectively. Cusped configurations $C_x$ are for $K = K_x$—see Figs (3). As we study the configurations inside the ergoregion, we consider mainly corotating fluids, therefore notation $(-)$ to indicate the corotating tori will be generally omitted. For any quantity $Q$ evaluated on a general radius $r_*$, we adopt notation $Q_x \equiv Q(r_*)$. We also adopt the notation $Q_x$ or $Q^x$ for any quantity $Q$ relative to the cusped torus $C_x$. Where appropriate, to ease the reading of complex expressions, we will use the geometric units where $r \rightarrow r/M$ and $a \rightarrow a/M$.

3 In the outer ergoregion

Here we characterize the orbiting tori in the outer ergoregion or close to the static limit. In section (3.1) we introduce the concept of dragged surfaces and partially contained surfaces which are analyzed in Sec. (3.1.1) particularly with the investigation of the the tori verticality and introducing the disk exfoliation for dragged tori. The influence of the dragging frame on the disk thickness is the focus of Sec. (3.1.2), while in Sec. (3.1.3) we deepen the discussion on the tori exfoliation. We conclude this section in Sec. (3.1.4), with some notes on limiting configurations as the possibility of multi-tori (aggregates of multi-toroids) in the ergoregion, the proto-jets and $C_2$ tori.

3.1 Dragged surfaces

We explore two types of closed configurations in $\Sigma_+^+$:

1. The closed quiescent configurations $C$;
2. The closed tori $C_x$ with cusps $r_x$, having specific angular momentum $\ell \in L_1$.

In Sec. (3.1.4) we consider also the proto-jets, open toroidal configurations, in the ergoregion.

As discussed in Pugliese&Montani (2015), we can consider an evolutive parameter following the configurations evolution, from the phase of disk formation, assuming a simplified two-phase accretion tori evolution, to the accretion phase. Naturally one can consider the parameter $\ell$ evolution with a negative "time-gradient", where $\ell \geq \ell_{mso}$. Decreasing in region $\ell \in [\ell_{mso}, \ell_{mbo}]$, with the consequent evolution of $K$ to $K_x$ and to $K_x = [K_x, 1]$ with an "over-critical" toroid where there is formation of an accretion throat, corresponding to the torus cusp enlargement and associated to matter flowing—Figs (1). Cusped tori $C_x$ are considered limiting configurations for the accretion phases. The disk may also be subsequently stabilized (in quiescent tori) to a sequence of interrupted phases of super-Eddington accretion as often considered in models of SMBHs evolution (Volonteri et al. 2007; Volonteri 2007; Volonteri 2010; Volonteri et al. 2003; Li 2012; Oka et al. 2017; Kawakatu&Ohsuga 2011; Allen et al. 2006), occurring, for example, in composite systems of tori aggregates as the eRAD (Pugliese&Stuchlik 2015; Pugliese&Stuchlik 2017; Pugliese&Stuchlik 2019). A further possibility consists in the formation of an accretion throat, for over critical tori, with $K > K_x$, and formation of a fat torus with accretion material very close to the horizon.

We focus specifically on the following two types of configurations:

1. Dragged surfaces Dragged surfaces are closed quiescent, $C$, or closed cusped $C_x$ tori entirely contained in the ergoregion. This definition considers the tori radial dimension, i.e., we assume the torus inner, $r_{inner}$, and outer edge $r_{outer}$, be both located in the range $[r_m, r_+^+]$. This condition however does not imply the torus entire containment in $\Sigma_+^+$ at any plane $\theta \neq \pi/2$. Considering the disk vertical dimension, tori defined by constant fixed $K$ may cross the ergosurface at a plane $\theta \neq \pi/2$, including the tori geometrical maximum which is related to the maximum pressure point. We focus the analysis of the disks verticality in $\Sigma_+^+$ considering the ergosurface crossing at any $\theta$ in Figs (14,15,16,17,18) and Sec. (3.1.1).

2. Partially contained configurations The partially contained tori are characterized by the conditions $r_{inner} < r_+^+$ and $r_{outer} > r_+^+$. This case has the following two sub-cases, which can also be associated to different phases of the tori evolution inside the ergoregion: (a) In the first sub-class of partially contained tori, the disk inner part, region $[r_{inner}, r_{center}]$, is fully contained
in the ergoregion, which may include also a section the torus outer part, \(r_{\text{center}}, r_{\text{outer}}\). (b) In the second sub-class of tori, only a section of the torus inner part is contained in the ergoregion (that is a region \(r_{\text{inner}}, r_{\ast}\) with \(r_{\ast} < r_{\text{center}}\). This condition implies that the projection on the equatorial plane of the maximum geometrical point \(r_{\text{max}} > r_{\text{center}}\) is out of the ergoregion. (In the following we consider, when not otherwise stated, \(r_{\text{max}}\) the projection of the geometrical maximum of the surface on the equatorial plane.).

The torus stability in \(\Sigma^{+}\) is an important issue to be assessed, that can involve a series of hypothesis on the dragged and partially contained surfaces formation. Tori evolution may develop from a configuration formed outside the ergoregion, then evolving in partially contained and from partially contained to dragged surfaces, featuring, therefore, disk formation in three stages. This hypothesis can be grounded on the assessment of appropriate details of disk formation, as growing from accreting matter from the embedding, and conditions on the accretion process, for example in the case where the central SMBH is a part of a binary system having an accreting companion star. A second possibility consists in the formation of a torus in the ergoregion crossing the ergosurface from the outer part of the torus. (Note that there is \(r_{\text{inner}} < r_{\text{center}} < r_{\text{max}} < r_{\text{outer}}\), therefore there is a first phase when \(r_{\text{inner}}\) crosses the ergosurface, implying also the torus crossing the ergosurface on planes different from the equatorial plane).

Finally, a third hypothesis features a disk formed inside the ergoregion, growing in the ergoregion and accreting into the central attractor. The problem of disk approaching and crossing the static limit or the formation of the closed quiescent torus across the static limit is also faced by the analysis of the disk vertical dimension, rephrasing the problem for a torus crossing the ergoregion on different planes. Following the torus growing, and the accreting disk stretching on the equatorial plane, the torus outer edge may cross the static limit, leading the initial dragged surface to a subsequent partially contained configuration. It may be possible that the dragged configurations could appear as seeds, formed in-loci, on a range centered on the orbit \(r_{\text{center}} \in [r_{\ast}, r^{+}_{\ast}]\). This situations can be also largely affected by the details of fluids characteristics as the equations of state or differentiated in more complicated frames as the GRMHD models, we consider this issue briefly in Sec. (3.3) and Appendix (1). (However as we note from the Euler equation in the GRHD model, the “pressure-forces” are entirely regulated by the fluid effective potential gradient, and possibly the fluid relativistic angular velocity and the gradient of the specific angular momentum for model with \(\ell \neq \text{constant}\).)

The disks embedding is another relevant aspect of the dragged configurations, i.e., tori could be formed following accretion in a binary system or they can result also as the consequence, for instance, of the Bardeen–Petterson effect on an originally misaligned torus, broken due to the frame dragging and other factors, as the fluids viscosity, in an inner corotating torus and an outer torus which may also be counter-rotating (Bardeen & Petterson 1975; Nealon et al. 2015). A second possibility can also consider the Bardeen-Petterson effect on an original, partially contained, torus-(Nealon et al. 2015; Martin et al. 2014; King & Nixon 2018; Nixon et al. 2012a; Nixon et al. 2012b; Nealon et al. 2015; Lodato & Pringle 2006; Scheuurl & Feier 1996; King et al. 2005). Vertical stresses in the disks, and the polar gradient of pressure, can combine with the Lense-Thirring effect. Consequently, a relevant issue of these processes for the stability of the partially contained configurations, is whether the inner or outer part of the disk is in the ergoregion and, the case of crossing of the outer ergosurface on planes different from the equatorial plane, which may affect the torus (global) stability when stretching towards the horizon during accretion. For this reason we consider more in details the torus inner and outer edges location with respect to the static limit.

---

**The inner edge** The inner edge of tori with specific angular momentum in \(L_{1}\) satisfies the relation \(r_{\text{inner}} \geq r_{\ast} \in [r_{\text{mbo}}, r_{\text{maso}}]\) with \(r_{\text{center}} \in [r_{\text{maso}}, r_{\text{mbo}}^{a}]\). For any dragged or partially contained configuration, we concentrate on the problem of penetration of the inner edge of the tori in the outer ergoregion, assuming firstly \(\ell \in [\ell_{\text{maso}}, \ell_{\text{mbo}}]\), and considering cusped tori \(C_{\ast}\). A necessary but not sufficient condition for partially contained or dragged surfaces, is the occurrence of \(r_{\text{mbo}} \leq 2M\), distinguishing therefore two classes of BH attractors, which we consider more in details in Figs (2). The value \(r_{\text{maso}} = 2M\) is a limiting condition for configurations approaching the ergoregion, being very close to the outer ergosurface. Condition \(r_{\text{maso}} = 2M\) is a sufficient but not necessary condition for a partially contained torus (a section of the torus inner part being contained). It is clear that the necessary but not sufficient condition for the torus to be centered in the ergoregion, i.e. the maximum pressure point would be inside the ergoregion, is that \(r_{\text{maso}} \in \Sigma^{+}_{e}\). A sufficient but not necessary condition for the torus center to be included in the ergoregion is \(r_{\text{mbo}} \in \Sigma^{+}_{e}\). The analysis of the limiting situation \(r_{\text{center}} = r^{+}_{e}\) is faced in Figs (4) solving the equivalent problem \(\ell^{+}_{e} \in [\ell_{\text{maso}}, \ell_{\text{mbo}}]\) for a given spacetime \(a/M\), where \(\ell^{+}_{e} \equiv (r^{+}_{e})^{\ell}\). (We note that the center of maximum pressure and the minimum pressure point in the torus are determined by the specific angular momentum only, while the geometrical maximum is regulated by the parameter \(K\) once fixed \(\ell\)). The torus cusp \(r_{\text{cusp}}\) may also be in the near-horizon region for very high spins \(a/M\) of the central SMBH (partially contained for \(a > a_{c}\), and entirely contained for \(a > a_{c}^{b}\)). (For \(r_{\text{cusp}}\) we intend the minimum points of the pressure \(r_{\text{cusp}} \in [r_{\gamma}, r_{\text{maso}}]\), which can be a cusp of the closed torus.
of the critical points of the pressure, \( K \), of toroidal solutions, with one dragged surface with an outer corotating torus, considering the distance between the critical points. The function \( K \), the torus density, the torus thickness and other characteristics related to the tori energetics, as cusp luminosity and accretion rates.

This issue is indeed relevant for the stability of these configurations as the relativistic Roche lobe overflow at the cusp of the equipotential surfaces is also the primary driving mechanism against the thermal and viscous instabilities locally, and against the so called Papaloizou–Pringle instability globally (Blaes 1987). For a discussion on the relation between Papaloizou-Pringle (PP) modes in the tori, the Papaloizou-Pringle Instability (PPI), a global, hydrodynamic, non-axi-symmetric instability and the Magneto-Rotational Instability (MRI) modes see Pugliese & Montani (2018); Bugli et al. (2018).

The maximum elongation of the dragged torus is at most \( \lambda \approx 2M \). The surfaces with very large specific angular momentum \( \ell \), \( \ell \geq \ell_\gamma r^+ \), tend to be stable against the cusp formation, leading to quiescent closed configurations. An interesting question is whether these configurations, stabilized by a large centrifugal component (the cusp absence), are dragged or partially contained toroidal surfaces rather than tori with lower specific angular momentum characterized with the cusp formation. These configurations can also grow very huge, with the center of maximum pressure in the torus at \( r > r^+_\gamma \geq 9M \) for large spin \( a \gg a_\gamma \).

In Figs 3 we show the regions of tori parameters for the existence of the closed tori, featuring also the possibility of double cusped configurations for BHs with high-spin, while in Figs (5) curves \( \ell = \ell_\gamma = \ell(\ell^+_{\text{outer}} = 2M) \) are shown, fixing the location of the critical points of the pressure, \( r_{\text{center}} \) and \( r_{\text{cusp}} = (r_\gamma, r_\gamma) \). We investigate the distance between the two tori in the aggregates of toroidal solutions, with one dragged surface with an outer corotating torus, considering the distance between the critical points an the centers. We use the leading function \( \ell(\ell) \), distribution of the tori specific angular momentum. We also evaluated the energy function, related to tori densities and tori energetics. The function \( K(r) \) is related to an independent tori parameter \( K \) which regulates the torus elongation \( \lambda \) on its symmetry plane and the emergence of hydro-dynamical instability. It is also associated to the torus density, the torus thickness and other characteristics related to the tori energetics, as cusp luminosity and accretion rates. The function \( K(r) \) gives the critical values \( \lambda_{\text{crit}} = \{K_{\text{cusp}}, K_{\text{center}}\} \) of the tori with different specific angular momentum. The study of \( \ell(\ell) \) and \( K(\lambda) \) functions is also important to set constrains for the tori collision. Function \( K(r) \) provides the distribution of the effective potential values corresponding to the points of maximum and minimum of the density (and of the HD pressure),

\[
\begin{align*}
\ell_{\text{cusp}} &= r_\gamma \in [mbo, mso] \quad \text{or the cusp of a proto-jet} \quad r_{\text{cusp}} = r_\gamma \in [r_\gamma, mso].
\end{align*}
\]

- **The outer edge** If \( mso < 2M \), the condition \( r_{\text{outer}} \leq 2M \) is possible for some \( \ell \) and \( K \) values. The condition \( r_{\text{mbo}} < r^+_{\gamma} \) is necessary but not sufficient condition for the outer edge of a \( C_2 \) configuration to be included in the ergoregion. The inner part of such tori can be in part \( \Sigma^\_+ \). The inner part of a \( C_1 \) configuration must be in the ergoregion. The condition \( r^+_\gamma < r^+_{\gamma} \) is necessary but not sufficient condition for the outer edge of a \( C_3 \) configuration to be included in the ergoregion. The inner part of such tori can be in part \( \Sigma^\_+ \). The inner part of a \( C_2 \) configuration must be in the ergoregion. Analysis of the outer edge location with the respect to the static limit is in Figs (10) and (9).

Panels show different ranges of the BH dimensionless spin \( a/M \).
regulating the location of the points of the minimum of pressure in the torus with respect to the static limit, and

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A regulating the location of the centers of maximum pressure with respect to the static limit. Spins $K = \text{torus center}$ and eventually momentum $\ell = K$ for different spacetimes and angular momentum, The ”energy function”, defined as $V_{\text{eff}}(\ell(r), r)$, is therefore a function of $r$ with parameter $\alpha/M$, and it parameterizes each torus with equal angular momentum $\ell$ with the $K$-parameter governing the torus center and eventually $K_{\text{max}}$ at its cusp. Function $r_p(r) : K(r_p) = K(r)$ identifies a pair of tori ($T_1, T_2$) with different angular momentum $\ell$, with $K_{\text{center}}(T_1) = K_{\text{center}}(T_2) = K(r)$ where $r_p = r = \text{constant}$, and $r_{\text{ext}}$ is $r_{\text{ext}}$ where $K \in K_{\text{max}} \in [K_{\text{max}}, 1]$, or $r_{\text{cusp}}$ where $K \geq 1$—Figs (2). For fixed $\ell = \text{constant}$, the solution of $\ell = \ell(r)$ gives a torus center $r_{\text{center}}$, the maximum density and HD pressure points; tori with different $\ell \in [\ell_{\text{min}}, \ell_{\text{max}}]$ can have also equal minimum pressure (and density)-points (different inner edges, only one torus cusp). In general, tori characterized by same value of specific angular momentum $\ell$ have not same geometrical maximum, which depends on the $K$ parameter. The $K$ parameter sets the disk ”verticality”, defining the solution of $\partial_y V_{\text{eff}}(x, y) = 0$, where $V_{\text{eff}}(x, y)$ is the fluid effective potential on any plane $\sigma$ in Cartesian coordinates (there is $(x = r \cos \theta, y = r \sin \theta)$, on the equatorial plane there is $y \equiv r$).

In Figs (2), we explore five classes of the Kerr geometries defined according to the circular geodesics related to the ergosphere boundary that give rise to the characteristics values of the BH spin $A^+ = \{\alpha, a_{\text{mbo}}, a_{\text{mso}}, a_{\text{rb}, b}^+, a_{\text{rb}, b}^-, \gamma \}$. The geodetic radii used in the definition and related spins are given as

$$
\begin{align*}
\frac{\alpha}{M} &\equiv \frac{1}{\sqrt{2}} = 0.7071 : r_\gamma = r_\gamma^+ , \\
\frac{a_{\text{mbo}}}{M} &\equiv 2 (\sqrt{2} - 1) \approx 0.828 : r_{\text{mbo}} = r_\ell^+ , \\
\frac{a_{\text{mso}}}{M} &\equiv \frac{2}{3} \approx 0.9428 : r_{\text{mso}} = r_\ell^+ ,
\end{align*}
$$

regulating the location of the points of the minimum of pressure in the torus with respect to the static limit, and

$$
\begin{align*}
\frac{a_{\text{rb}, b}}{M} &\equiv 0.9987 \approx 1 : r_{\text{rb}, b} = r_\ell^+ , \\
\frac{\gamma}{M} &\equiv 0.9943 M : r_\ell^+ = r_\ell^+,
\end{align*}
$$

regulating the location of the centers of maximum pressure with respect to the static limit. Spins $A^+$ are represented in Figs (4).
In terms of the fluid specific angular momentum, spins of the set $A^+_1$ can also be defined as $a_{mso}: \ell^+_x = \ell_{mso}$, where $(a_{mbo}, a_{mso})$ satisfy $\ell^+_x = \ell_{mbo}$, and for the spins $(a, a^+_x)$ there is $\ell^+_x = \ell_{mso} - Figs (4)$ and (19). Some configurations orbiting in the geometries defined in $A^+_1$ are shown in Figs (14).

We study the conditions for the tori have outer edge $r_{outer}$ in $\Sigma^+_x$, in the different geometries of the set $A^+_1$ in Figs (6). A similar analysis for the outer edge of cusped surfaces $C_x$ is in Figs (7), showing the conditions in terms of torus specific angular momentum, and in Figs (8) where the conditions are expressed as functions of the disks inner edge $r_{inner}$ location. The crossing of outer edge of the disk with the outer ergosurface is considered in Figs (4).

Below we consider the dragged and partially contained tori in the five classes of geometries delimited by spins $A^+_1$:

- **Range $A^+_1 \equiv [a_{mbo}, a_{mso}] \in A^+_1$**: In the geometries of this range, with increasing dimensionless spin $a/M$, the radius $r_x$, cusp of the $C_x$ torus with momentum $\ell \in L_1$, approaches the static limit. An inner Roche lobe of orbiting matter appears close to the horizon. The closed configurations with $\ell \in L_3$, have center in $r > r^+_x$ which can also be, for small BH spins, very far from the outer ergosurface. These tori can be very huge. The inner edges of $C_1$ tori are out of the ergoregion, but approaching the outer ergosurface at distance that can be less then $\approx 0.5M$. The inner edge of the quiescent $C_2$ torus may be included in the ergoregion, the cusp of the proto-jet can be in the ergoregion, but the inner part of the disk ($[r_{inner}, r_{center}]$) is partially contained in the ergoregion as the center is at $r > r^+_{mbo}$.

- **Range $A^+_2 \equiv [a_{mbo}, a_{mso}] \in A^+_1$**: The center of the $C_1$ configurations is out the ergoregion. The cusp $r_x$ of the cusped closed torus $C_x$ is inside the ergoregion, therefore these tori must be partially contained with their inner part. Quiescent $C_1$ tori can be partially contained in $\Sigma^+_x$ and they cannot evolve into dragged tori. The quiescent $C_2$ and $C_3$ tori have center out of the ergoregion. The inner edge of $C_2$ can be contained in the ergoregion and the cusp of proto-jet must be inside the ergoregion. These tori can be huge, with elongation of the disk inner part of $\approx 4M$ for $C_2$ to $\approx 6M$.

- **Range $A^+_3 \equiv [a_{mso}, a_{mbo}] \in A^+_1$**: In these BH spacetimes, the center of configurations $C_x$ and $C_1$ can be in the ergoregion, while the center of $C_2$ and $C_3$ tori cannot be in the ergoregion. The cusp $r_x$ of $C_x$ tori must be in $\Sigma^+_x$, therefore the inner part of the cusped surface must be partially contained in the ergoregion. Configurations $C_1$ can be dragged.

- **Range $A^+_4 \equiv [a_{mbo}, a_{mso}] \in A^+_1$**: In the geometries of this class, the center of $C_1$ configurations must be contained inside the ergoregion. Center of $C_2$ configurations approaches the static limit.

- **Range $A^+_5 \equiv [a_{mbo}, M] \in A^+_1$**: The center of $C_1$ and $C_2$ tori must be in the ergoregion, while the $C_3$ tori centers can be in the ergoregion. Both $C_2$ and $C_3$ can be dragged and must be partially contained. Location of the inner edge, center and outer edge of the disk is shown in Figs (6) for different spins. Maximum elongation and location of the inner edge, outer edge and center of the disks are represented in Figs (7).

Results of equations $r : \ell(r) = \ell^+_x$ are shown in Figs (4) and Figs (19), where there is the study of the outer edge approaching the outer ergosurface. In Figs (12) we show the pressure gradients in the ergoregion $\Sigma^+_x$ according to Eq. (9), for different planes and spins.

**Notes on the torus inner edge, the static limit and tori topology**

From Figs (9)–right panel, we see that condition $r_{inner} = 2M$ holds only for tori orbiting BHs with sufficiently large spin-mass ratios $(a \geq a_{mbo})$, where $\ell$ and $K$ are progressively smaller with increasing $a/M$. Figs (9) detail a complicated situation where there are two classes, (1) and (2), of $C_x$, $C_2$ and $C_3$ configurations (defined according to the specific angular momentum classes there in Sec. (2)) such that $C_3 (1) < C_2 (1) < C_x (1) < C_x (2) < C_2 (2) < C_3 (2)$, where the ordering relation expresses the ordering relation between the central BH attractor dimensionless spins of the geometries where the configurations are orbiting. Therefore, for sufficiently small spins, $a < a_{mbo}$, solutions $r : \ell = \ell (r^+_x)$ define quiescent tori $C_3 (1)$ which are generally very huge. Whether the quiescent tori $C_3 (1)$, orbiting small spin attractors, can have inner edges approaching, or even crossing, the static limit, remains to be clarified. It is clear that as for the cases of $C_2$ tori, the limiting condition is $V_{eff} (r^+_x) = 1$ (or more precisely $V_{eff} (r^*_x) = 1$ where $r^*_x \leq r^+_x$) whose properties are described in Figs (11,13).

Solutions of the problem $V_{eff} = 1$ are

$$r^+_K \equiv \frac{1}{\sqrt{2}} \left( \ell^2 \pm \sqrt{\ell^4 - 16\ell^2 + 32\ell - 16a^2} \right)$$

alternatively

$$a_{K1} \equiv \ell - \sqrt{\frac{\ell^2 - 2r}{\sqrt{2}}}$$

and

$$\ell^+_K \equiv \frac{\sqrt{2\sqrt{r^2 - 2a}}}{\sqrt{r - 2}}$$

see Figs (13). In the case of torus $C_2$, we consider the limiting case of the proto-jet cusp on the static limit and, therefore, the quiescent disks could be at a relatively large distance from the static limit. For the cusped configurations, the cusp is the lower
Fig. 6. Tori edges and pressure extremes are shown as functions of the fluid specific angular momentum $\ell$. Purple line is the couple of radii $r_{\text{center}} > r_{\text{cusp}}$ of the center of maximum pressure point and minimum pressure point in the disk respectively. Number sets the values of $K$ parameter of the disks. $r_{\text{inner}}$ is the inner edge of the disk, $r_{\text{outer}}$ is the outer edge of the disk. $\ell$ is the fluid specific angular momentum. Spins $A^\gamma \equiv \{a_{mso}, a_{mbo}, a_{\gamma}, a_{\gamma}^b\}$ are defined in Figs (2).

Fig. 7. Plot of the outer edge $r_{\text{outer}}$, inner edge and the center $r_{\text{center}}$ of the cusped tori $C_x$ as functions of the fluid specific angular momentum, for spins $A^\gamma \equiv \{a_{mso}, a_{mbo}, a_{\gamma}, a_{\gamma}^b\}$ defined in Figs (2). $\ell \in [\ell_{mso}, \ell_{mbo}]$ is the fluid specific angular momentum, $mso$ is for marginally stable orbit, $mbo$ is for marginally bounded orbit, the outer ergosurface on the equatorial plane is $r_{\gamma}^e = 2M$ (gray line).
shown as function of the inner edge $r$. However, the disks might have the inner edges lower than the tori on $r$ as there is elongation for small $\epsilon$. Therefore, including also range $A_0 \equiv a < a_{\gamma}$, there is:

- For $A_0 : a < a_{\gamma}$, there are no tori with inner edge on $r^+_C$ as follows from the analysis of Figs (8,7,9,10). Whereas there is $r^+_{\gamma} > r^+_C$, therefore a proto-jet cusp can approach the static limit. The analysis of tori $C_3$ and $C_2$ with inner edge close to the static limit (or for $C_2$ on the static limit) is regulated by the solution of the problem $V_{\text{eff}}(r) \leq 1$ for these tori (note tori $C_2$ cannot have inner edge on the static limit).

- For $A_1 : [a_{mbo}, a_{mso}]$ there is $r_{mbo} > r^+_C$. The cusp of $C_1$ tori is always out of the ergoregion, approaching the static limit, where can be however a proto-jets cusp. Therefore, there is $r^+_C \in [r^+_{\gamma}, r_{mbo}]$, and to prove that $r^+_C$ can be the inner edge of a quiescent torus $C_2$, we need to assure if $V_{\text{eff}}(r^+_C) < 1$ for $\ell \in \mathbb{L}_2$ in these geometries, which is done in Figs (11,12) showing that this is not the case. Tori $C_2$ are defined in two sets of configurations with centers in $r_{\text{center}} \in [r^+_{\gamma}, r_M]$, and in $r_{\text{center}} \in [r_M, r_{mbo}^{b}]$ respectively.

- For $A_2 : [a_{mso}, a_{mso}]$ there are tori $C_{1}(1)$ whose cusp can be on the static limit as there is $r^+_C \in [r_{mbo}, r_{mso}]$; therefore the outer ergosurface can be the inner edge of $C_1$, $C_2$ and $C_3$ tori–see for details Figs (7,8,22).

- For $A_3 : [a_{mso}, a_{mso}]$ tori $C_{3}(2)$ have maximum pressure point $r_{\text{center}} = r^+_C$, while the cusp has to be inside the ergoregion. In this case there is $r^+_C \in [r_{mbo}, r_{mso}]$, therefore radius $r^+_C$ can coincide with the center of $C_1$ tori, and can be the inner edge of $C_1$, $C_2$ and $C_3$ tori. See Figs (7,8,22) for the analysis of the torus outer edge coincidence with the static limit.

- For $A_4 : [a_{mso}, a_{mso}]$ there are tori $C_2$ centered on $r^+_C$. The static limit can be the center and the inner edge of the $C_2$ tori, and the inner edge of $C_3$ tori (the analysis of the outer edge coincidence with the static limit is in Figs (7,8,22)).

- For $A_5 : [a_{mso}, M]$ the static limit can be also the inner edge or the center of $C_3$ tori (the analysis of the outer edge coincidence with the static limit has been done in Figs (7,8,22)).
**Fig. 9.** Analysis of tori center and inner edge with respect to the stationary limit. Left upper panel: Curves show the parameters $K^2 > K_{mso}^2$ and $\ell > \ell_{mso}$ ranges for the (quiescent and cusped) tori inner edge be coincident with the outer ergosurface $r_{inner} = r_x$, for different BH dimensionless spin $a/M$ signed along the curves. Right upper panel, and bottom panels: solutions $r: \ell = \ell(r^+_x)$ (black curves) where $r^+_x = 2M$ is the outer ergosurface. For cusped surfaces, the curves locate the cusp of the configuration whose center is in $r_{center} = r^+_x > r_{mso}$ or, vice versa, curves locate the center of the torus whose cusp is in $r_x = r^+_x < r_{mso}$, according to the different ranges of $r/M$ and BH spins $a/M$. Location of center $r_{center}$ and cusp $r_x$ is signed on the black curve. Gray region is $r < r_\gamma$, the radius $r_\gamma$ is the marginally circular orbit and photon orbit where $r_{b\gamma} > r_\gamma$: $\ell(r_\gamma) = \ell(r)$ (orange curves). Marginally bounded orbit $r_{mbo}$ and radius $r^+_{mbo}: \ell(r_{mbo}) = \ell(r)$ are the blue curves. Light-blue curve is the marginally stable orbit $r_{mso}$. Radius $r_M: \partial^2_\ell \ell = 0$ is the red curve. Spins $\mathbf{A}^+ \equiv \{a_{mbo}, a_{mso}, a_\gamma, a_{b\gamma}, a_{mso}\}$ are represented. We show the regions where there can be quiescent tori with specific angular momentum $\ell \in L_3$ (i.e., tori $C_1$), quiescent tori $C_2$ and proto-jets $J_3$ (having specific angular momentum $\ell \in L_2$) and cusped tori $C_3$ having center or cusp located on $r^+_x$ or regulated by the relation $r: \ell = \ell(r^+_x)$. Panels shows different ranges of spins.

**Fig. 10.** Solutions $r_{inner} = r^+_x$, inner edge of the torus coincident with the outer ergosurface on the equatorial plane, in terms of the parameter $K^2 > K_{mso}^2$ for different BH dimensionless spin $a/M$ signed close to the curve and for specific angular momentum $\ell \in \{\ell_{mbo}, \ell_{mso}\}$ (for tori $C_1$ and $C_3$)-left panel; for $\ell \in [\ell_{mbo}, \ell_\gamma]$ (for tori $C_2$ and protojets $J_3$)-center panel and $\ell > \ell_\gamma$ (for tori $C_3$)-right panel.
Fig. 11. Analysis of solutions $V_{eff} = 1$ on the ergosurface, $r^+ = 2M$, on the equatorial plane. Upper left panel: solution of $K(r) = 1$ for different radii signed on the curves close to the outer ergoregion. Dashed lines are the spins $a_{mbo}$ and $a_{mso}$. Central panel: plots of $K(r) = V_{eff}(\ell(r), r)$ as function of the radius $r/M$, for different spins of the set $A^+_{\gamma}$ signed in figures. Each point for the curves sets the value of the $K$ parameter on the minimum and maximum of the fluid effective potential, maximum and minimum of the pressure in the disk respectively. The limiting Schwarzschild spacetime $a = 0$ is the dotted-dashed curve, the extreme Kerr spacetime with spin $a = M$ is the dashed curve. The curve $K_{mso}(r) \equiv K(r_{mso}(a))$ is shown. Left panel: radius $r(\ell,a) : K(r) = 1$ is shown, each radius at $r > r_{mso}$, black curve can be a torus center. Gray region is region $K(r) < 1$. The fluid specific angular momentum $\ell_{\gamma} = \ell(r_{\gamma}), \ell_{mso} = \ell(r_{mso}), \ell_{mbo} = \ell(r_{mbo})$ are shown and dashed lines are the spins $\{a_{mso}, a_{\gamma}, a_{mso}\}$. Below panels: solutions $\ell : V_{eff} = 1$, black region is $r < r_{+}$, where $r_{+}$ the outer horizon Colored curves show $r_{+} < r_{mbo} < r_{mso} < r_{b} < r_{b+}$.

Fig. 12. Analysis of the pressure gradients, in the ergoregion $\Sigma^+_{\gamma}$ regulated by Eq. (9). 3D plots show the function $\partial_r W$ where $W$ is the P-W potential and $* = r$ (light gray surface for $\theta = \pi/2$, black surface is $\theta = \pi/9$), $* = \theta$ (green surface for $\theta = \pi/9$), for tori orbiting BHs with spins $a = 0.9999M$ and $a = a_{mbo}$. Plane $\partial_r W = \partial_r p = 0$ is shown as red surface. $\ell > \ell_{mso}$ is the fluid specific angular momentum and $r \in [r_{+}(a), r_{+}(a, \theta)] = \Sigma^+_{\gamma}$.
Fig. 13. Plots of radii $r_{\pm}^{\pm1} : V_{\text{eff}} = 1$ of Eq. (14) as functions of the fluids specific angular momentum $\ell$. Marginally bounded orbit $r_{\text{mbo}}$, marginally stable orbit $r_{\text{mso}}$ and marginally circular orbit $r_\gamma$ with the BH horizon $r_+$ and the outer ergosurface $r_{+}^{\pm} = 2M$ on the equatorial plane are plotted. There is $a_\gamma > a_{\text{mbo}} > a_{\text{mso}}$, see Eqs (13). Purple curve is solution $r : \ell = \ell(r)$ for the critical points (cusps) and center of maximum pressure in the tori. Vertical lines are $\ell_{\text{mso}} < \ell_{\text{mbo}} < \ell_\gamma$.

Fig. 14. Dragged tori in the five regions of Figs (8) for critical $C_x$ and quiescent configurations $C$ in the ergoregion or partially contained in the ergoregion or proximate to the outer ergosurface. Spins $A_\pm^{\pm} \equiv \{a_{\text{mbo}}, a_{\text{mbo}}, a_{\gamma}, a_{\gamma}, a_{\text{mso}}\}$ are defined in Eqs (13) and Figs (2). Black region is the BH $r < r_+$, gray region is the outer ergosurface, equipotential and equipressure surfaces are shown. $r_x$ is the cusp of the critical configurations $C_x$ chosen considering Figs (8) for the purple surface. Blue curves set the extremes of the pressure and density in the disks: the maximum pressure inside the disk from the center of the configurations to the geometrical maximum for different $K$ surfaces, and the inner minimum point of the pressures from the cusp. $\ell$ is the specific fluid angular momentum.
3.1.1 The tori verticality and disk exfoliation

Relevant aspects of the toroidal geometrically thick disks centered on the BH equatorial plane, are regulated by the dependence of the tori characteristics on the radial dimension, as the radial pressure gradients, which ultimately fixes the tori rotational law \( \ell(r) \). For the tori orbiting in the ergoregion it is necessary to explore the vertical pressure gradients inside the disk. The disk verticality is in fact a significant factor for the accretion disks also in presence of the magnetic fields, and in many models of tori oscillation, as for the analysis of QPOs from accretion disks, which we face here in Sec. (3.2.1). The disk verticality in this model is regulated by the pressure \( x \)-gradient and the curve of the extremes of pressure, including the points \( r_x \), the disks center \( r_{\text{center}} \), which is a maximum pressure point, and the minimum in the accretion flow at radius \( r_x < r_\ell \) (for the overcritical, \( C_1 \), configurations \( K > K_\ell \)) and the disk geometrical maximum—Figs (14). The curve relates the geometrical maxima and the extremes of the HD pressure, solution of \( \ell : \partial_y V_{\text{eff}}(x,y) = 0 \) (there is \( \partial_y V_{\text{eff}} = \partial_x V_{\text{eff}} \)). This curve depends on the BH dimensionless spin \( a/M \) only.

The curves, solution of equation \( \partial_y V_{\text{eff}} = 0 \), are bounded in a range analyzed in Figs (32) where we show that the zeros and the maximal extensions of the range of definition for the solutions \( \ell = \text{constant} \) for the vertical gradient of the pressure are limited in a bounded region of the plane \( x-y \). More precisely, in Figs (15) we note how the vertical pressure gradient is limited by a surface \( y(x) \) dependent on the spin only. The extension of this range is largely unchanged by the BH spin as clarified in Figs (32).

The boundary curve is

\[
y\ell \equiv \sqrt{2(2M^2 - a^2) + 2M} \quad \text{or, equivalently,} \quad a_\ell \equiv \sqrt{y \left( 2M - \frac{y}{2} \right)},
\]

represented in Figs (32), where the maximum \( y_\ell \leq 4M \) is shown. It is evident that increasing the BH spin, the curves \( \ell = \text{constant} \) are confined in the ergoregion, the density of curves in the region increases approaching the horizon. For the limiting case of a static Schwarzschild BH, the boundary radius corresponds to \( y_\ell = r_{\text{mbo}} = 4M \) (while the specific angular momentum \( \ell \) and the energy function \( K \) on this curve decreases with the spin, which can be seen analyzing the function \( \ell(r) \) and the function \( K(r) \)).

To ensure the disk is entirely contained in \( \Sigma^+_{\ell} \), at a general plane \( \theta \) we need to consider the condition \( \partial_y V_{\text{eff}}(x,y) = 0 \). In fact, the necessary but not sufficient condition to be satisfied for the outer part of the disk to be in the ergoregion (and also for the inclusion of the torus geometrical maximum) is \( r_{\text{center}} \in \Sigma^+_{\ell} \), while this is a sufficient condition for the inner part of the disk to be included in the ergoregion. It is clear from Figs (15) that the curve of geometrical maxima and minima depends on \( y(r) \) (on the equatorial plane) but also on the parameter \( K > K_{\text{center}}(x_{\text{max}}(y) = x_{\text{max}}(K)) \). There are two branches of the curve, one from \( r \leq r_x \) and one \( r \geq r_{\text{center}} \). The torus approaches the BH poles rotational axes, when the torus geometrical maximum crosses the outer ergosurface. Figs (16) show the results of the analysis of the conditions for tori crossing the outer ergosurface considering the line of geometrical maxima. Radius \( r_{\text{max}} \) is a limiting radius, being a maximum of the curve, as one value of the fluid specific angular momentum \( \ell \) corresponds to two radii \( r_1 < r_2 : \ell(r_1) = \ell(r_2) \). Solutions \( x/M \) (vertical axis) crossing of geometrical maximum line with the ergosurface as function of \( r > r_\ell \) (outer horizon) are plotted (as solution of \( \partial_y V_{\text{eff}} = 0 \)). The analysis refers to critical (cusped) configurations. The smaller (larger) \( x_\ell^+ \) (the intersection of the toroidal surface with the static limit on \( \theta \neq \pi/2 \)) is and the closest to the equatorial plane (rotational axis) of the BH is the crossing.
We argue that for the small surfaces crossing the outer ergosurface or in the ergoregion, the partially contained or dragged tori might undergo a form of instability, which could be also coupled with the usual internal processes, of the disk, induced by the geometry frame dragging, occurring in regions of the disks also far from the center of maximum pressure, leading to a process of disk “exfoliation” (“peeling”). We consider this process more in details below. In Figs (16) we show the conditions to be satisfied for the geometrical maximum of the cusped surfaces intersecting the outer ergosurface. In Figs (17) we approach the analysis of the torus exfoliation studying the solutions of the problem of the disks crossing the outer ergosurface on $\theta \neq \pi/2$, by its geometrical maximum (or the minimum) according to the analysis of Figs (15). We show the solutions $x/M$ (vertical axis), crossing of the line of geometrical maxima, defined by $\partial_{y}V_{eff} = 0$, with the ergosurface for fluid specific angular momentum $\ell > \ell_{mso}$, in the geometries of the set $A_{+}^{\ell} = \{a_{mbo}, a_{mbo}, a_{r}, a_{b}^{b}, a_{mso}\}$. The maximum heigh of the tori defined in the ergoregion is rather small.

We note, in accordance with the analysis of Figs (16), (17), (18), that for the geometries with spins $a = a_{mbo}$, $a = a_{mbo}$ and $a = a_{\ell}^{b}$, there is a solution of this problem for cusped tori which have specific angular momentum with $\ell \in \mathbf{L}$. At larger momentum, $\ell > \ell_{r}$, there is $r_{center} \geq r_{r}^{b}$, and the torus can be also very large. The results of the analysis of the torus inner edge are in Figs (7). In the restricted range of tori parameters where cusped tori are, only geometries of a particular range of dimensionless BH spin $a/M$ admit tori with center of maximum pressure $r_{center}$ in the ergoregion, which is necessary but not sufficient condition for the intersection of the geometrical maxima with the ergosurface–see Figs (8). An analogue situation is in Figs (18) where is shown the crossing of the critical toroidal configurations (any point) with the outer ergosurface. Notable radii and values of momenta are represented in Figs (19). Figs (20) describe solutions of the problem $f(r) = f(r_{g})$, for two radii $r \neq r_{g}$ and for $f \in \{\ell(r), K(r)\}$ at different spins of the set $A_{+}^{\ell}$. For $f = \ell$, solutions are the radii $(r_{g}, r_{g})$, which are the center $r > r_{mso}$ or the critical point of the tori as $r_{\times}$ or $r_{j}$, where $r_{\times} < r_{mso}$. Therefore the analysis show the distance between maximum and minimum pressure points in the cusped disk, as well as the location of the points. The curves shown Figs (20) are bounded in the region delimited by $r_{g}$ and the radius $r : \ell = \ell(a_{\pm})$ where $a_{\pm} \equiv \sqrt{r(2M - r)}$ is the horizon curve in the plane $a - r$. The solution of the problem for $f = K$ distinguishes the points
Fig. 17. Study of the vertical penetration of the outer ergosurface of the disk. Plot of solutions $x/M$ (vertical axis, here is $x = r \cos \theta$, $y = r \sin \theta$), on the equatorial plane there is $y \equiv r$ crossing of the line of geometrical maxima $\partial_x V_{eff} = 0$ on the ergosurface for $\ell > \ell_{mso}$, where $\ell$ is the specific angular momentum of the fluid. mbo is for marginally bounded orbit, mso is for marginally stable orbit, $r_c$ is the corotating photon orbit. $r_\ell$ is the torus elongation on the equatorial plane—Figs (8) for Figs (7).

Different spins $A^+_a \equiv \{a_{mbo}, a_{mbo}, a_\gamma, a_\gamma, a_{mso}\}$ are represented as in Figs (14). Momenta $\ell_{mso}, \ell_{mbo}$ and $\ell_\gamma$ are black lines. For $a_{mbo}, a_{mso}$ and $a_\gamma$, a zoom in the region $\ell < \ell_\gamma$ is shown. Cusped tori are for $\ell \in [\ell_{mso}, \ell_{mbo}]$.

$(r, r_\gamma)$ such that $K_{center}(\ell^*) = K_{crit}(\ell^*)$ for two tori with specific angular momenta $\ell^* \neq \ell^*$ respectively—see Pugliese & Stuchlik (2015).

### 3.1.2 Dragged disks thickness: influence of the dragging frame on the disk thickness

In Figs (15) and Figs (39) we show the results of the analysis of the disk verticality in terms of the polar gradient of the pressure, considering the lines of extremes of the HD pressure, which provide also the surfaces geometric maximum, obtaining clear indication of the maximum vertical extension of the torus in $\Sigma^\dagger_a$. The disk thickness is an essential parameter in the study tori oscillation processes, in the regulation of the instability processes and in the accretion. The analysis of Figs (21) clarifies the situation for the BH geometries with spins in $a \in A^+_a$. In the definition of $S$, $x_{max}$ is the geometrical maximum of the cusped toroidal surface, obtainable from the solution of $\partial_b V_{eff} = 0$ and defined by the condition $V_{eff} = K_\ell$ (the energy parameter $K$ at the cusp $r_\ell$), which is also studied in Figs (14), $\lambda \equiv r_{outer} - r_\ell$ is the torus elongation on the equatorial plane—Figs (8), Figs (7). Because we restricted our analysis on the cusped tori, we assumed $r \in [r_{mbo}, r_{mso}]$: for $r \equiv r_{mbo}$ there is the limiting case of a proto-jet, for $r = r_{mso}$ there is the limiting case of one ring torus located on the marginally stable orbit. It is evident how only tori for $a_\in \{a_{mbo}, a_{mso}, a_\gamma\}$ of the set $A^+_a$ have $r_{\ell} < r^\dagger_a$.

For larger $r_\ell \lesssim r_{mso}$, tori can be also very small, while for smaller $r_\ell \gtrsim r_{mbo}$, tori, including partially contained tori in the ergoregion, are larger. For radii $r_\ell$ very close to the marginally bounded orbit, for any spacetime considered, there is a turning point in the tori thickness, i.e., at larger $r$ there is $S_{\ell_\gamma} < S_{a_{mbo}} < S_{\ell_{mso}} < S_{a_{mbo}} < S_{a_{mbo}} < S_{a_{mbo}} < 1$. As value $S = 1$ is the limit for geometrically thick disks prevally tori under this condition are small tori. However, for cusp close to $r_{mbo}$, for all the spacetimes there is a region where $S_{a_{mbo}} < S_{a_{mbo}} < S_{a_{mbo}} < S_{a_{mbo}} < S_{a_{mbo}} < S_{a_{mbo}} < 1$ (right range) and a second region (left range at $r_\ell$ closer to $r_{mbo}$) where

$1 < S_{a_{mbo}} < S_{a_{mbo}} < S_{a_{mbo}} < S_{a_{mbo}} < S_{a_{mbo}} < S_{a_{mbo}} < S_{a_{mbo}}$. 


Fig. 18. Study of the tori crossing the outer ergosurface. Different spins \( A^\pm = \{a_{mbo}, a_{b\ mbo}, a_\gamma, a_{b\ \gamma}, a_{mso}\} \) are represented as in Figs (14). Plot of solutions \( x/M = x^\pm /M \in [0, r_+ \} \) the outer ergosurface (vertical axis, there is \( x = r \cos \theta, y = r \sin \theta \), on the equatorial plane there is \( y \equiv r \) crossing the torus surface \( V_{eff} = K_x \) on the ergosurface as function of \( r > r_+ \) (outer Killing horizon). \( \ell \) is the fluid specific angular momentum. We consider the critical configurations, evaluating \( V_{eff} \) on \( \ell(r) \), setting centers or cusps \( V_{eff}(\ell(r)) = K_x(r) \). \( r_{mbo} \) is the marginally bounded orbit, \( r_{mso} \) is the marginally stable orbit, \( r_\gamma \) is the corotating photon orbit. Therefore for \( r < r_{max} \) sets a cusp, while for \( r > r_{max} \) is a center. Center of (quiescent) tori with \( \ell > \ell_\gamma \) is at \( r > r_{b\ \gamma} \), center of (accreting) tori with \( \ell \in [\ell_{mso}, \ell_{mbo}] \) with \( r_{center} \in [r_{mso}, r_{mbo}] \), while tori with \( \ell \in [\ell_{mbo}, \ell_\gamma] \) associated to quiescent tori and proto-jets have center in \( r_{center} \in [r_{mbo}, r_{b\ \gamma}] \).

Fig. 19. Notable momenta and radii. Black region is the region \( r < r_+ \), where \( r_+ \) is the outer horizon. \( \ell \) is the specific angular momentum of the fluid. mbo is for marginally bounded orbit, mso is for marginally stable orbit, \( r_\gamma \) is the corotating photon orbit. Gray region is region \( \ell < \ell_{max} \) where no accretion tori can be defined. Dashed blue line is the curve \( \partial^2 \ell = 0 \), the curve \( r : \ell = \ell^2 \equiv (r^2)^2 \) is also represented. Central panel shows specific angular momentum of the fluid \( \ell \) for different spins \( A^\pm = \{a_{mbo}, a_{b\ mbo}, a_\gamma, a_{b\ \gamma}, a_{mso}\} \) are represented as in Figs (14), and solution of \( \partial_\theta \ell = 0 \).
Fig. 20. Solution of $r : \ell(r) = \ell(r_g)$ (left panel) and $r : K(r) = K(r_g)$ (right panel) as function of the radius $r_g/M$, for the spins $A^+ \equiv \{a_{mbo}, a_{mbo}^b, a_γ, a_γ^b, a_{mso}\}$—see Figs (14). $r_+$ is the outer BH horizon, yellow curve is $\ell(a_\pm)$, where $a_\pm \in [0, M]$ is the horizon curve in the plane $a - r$. The static Schwarzschild case is the green curve for $a = 0$. Dashed lines are the outer ergosurface $r_+^\epsilon = 2M$.

Fig. 21. Study of the tori geometrical thickness $S \equiv 2x_{\text{max}}/\lambda$ of the cusped tori in the ergoregion (dragged surfaces) or close to the outer the ergosurface. BH geometries with $A^+ \equiv a \in \{a_{mbo}, a_{mbo}^b, a_γ, a_γ^b, a_{mso}\}$ are considered—see also Figs (14). On the equatorial plane ($x = 0$) the outer ergosurface is $y = r_+^\epsilon = 2M$. $x_{\text{max}}$ is the geometrical maximum of the toroidal surface, solution of $\partial_y V_{eff} = 0$ and $V_{eff} = K_x$ (the energy parameter $K$ at the cusp $r_x$) (see also Figs (14)); $\lambda \equiv r_{\text{outer}} - r_x$ is the torus elongation on the equatorial plane—see Figs (8), (7). (There is $(x = r \cos \theta, y = r \sin \theta)$, on the equatorial plane there is $y \equiv r$). The geometrical thickness is represented as function of $\epsilon$, a parameter regulating the location of the critical radius (cusp) $r_{\text{times}} \in [r_{\text{mbo}}, r_{\text{mso}}]$: $r_x \equiv r_{\text{mso}} - (r_{\text{mso}} - r_{\text{mbo}})/\epsilon$. For $\epsilon = 1$ there is $r_x = r_{\text{mbo}}$, the limiting case of a proto-jet, for $\epsilon \to +\infty$ there is $r = r_{\text{mso}}$, the limiting case of one ring torus. Below panels show the range of existence of the radius $r(\epsilon)$ for selected spacetime in relation to the presence of cusp in the ergoregion $r_+^\epsilon = 2M$ (outer ergosurface). It is evident how only tori orbing BHs with spins $a \in \{a_\gamma^b, a_{mbo}^b, a_{mso}\}$ of the considered set have $r_x < r_+^\epsilon$. Upper panels: thickness as function of $\epsilon$. Central panel shows a zoom in the region $\epsilon \leq 1.30$. Right panel: tori for selected spacetimes according to the color range of the first panel. Smallest tori are for $\epsilon = 5.8$ middle plain color tori are for $\epsilon = 1.8$, larger dashed tori are at $\epsilon = 1.18$. 
3.1.3 Tori exfoliation

The Lense-Thirring effect of the Kerr geometry is expected to hugely affect partially contained and dragged tori (also if viscosity and radiation enter the play). The small tori can undergo a process induced by the dragging effects in the ergoregion, at different planes resulting in (almost vertical) fractiousness of the tori surface, which can combine with the accretion processes onto the central BH attractor, leading to a swarm of (initially corotating) particles and eventually photons, which can constitute a tori atmosphere. According to the initial data on the torus, they can escape in the outer region, remain trapped in the region \( r < r_{\text{inner}} \), impacting back to the tori surface and environment (analogously to a Poynting–Robertson effect), and eventually be captured by the central BH (disk exfoliation). This process, relevant for the smaller disks, may also occur in combination with the Bardeen&Petterson effect in tori with a slight inclination with respect to the equatorial plane. In this regard it is necessary to reassess the "pressure forces" in the and stability of the dragged tori (we also note there are no ZAMOs geodetics in the BH ergoregion). On the other hand, the pressure gradients are also the main factor to be evaluated for the exfoliation and the consequent formation of a possible atmosphere of (free) particles swarm. In this context it can be also important the role of external large scale magnetic field—see for example Kološ et al. (2021); Stuchlík&Kološ (2016); Kovár et al. (2016).

Here we do not focus on the particulars of the processes leading to the formation, reflection, or emission of particles and photons, but on the evaluation of the effectiveness of this process in the considered hydrodynamic models, giving thus a groundwork for the analysis of more complex systems. This process may also give rise to effective dissipative forces (or inducing internal turbulence), especially in small tori with a small inner elongation \( \lambda = r_{\text{center}} - r_{\text{inner}} \) (distance between the maximum pressure point and the inner edges), and crossing the ergosurface at some plane \( \theta \), as considered for example in Figs (14), characteristic of the BHs geometries with spins \( a > a_c \). Clearly, these particles and photons interact with the BH environment, for example in the presence of a magnetosphere, with surrounding magnetic fields and other accreting materials. However, it is necessary to understand the characteristic time scales for the eventual destruction of the disk, which would depend on the mass (regulated by the \( K \) parameter) and on the disk and BH specific momentum. In the context considered here, this phenomenon involves disks with \( r_{\text{inner}} \leq 2M \), which can combine with jet and proto-jets emission, regulated by the limiting light surfaces provided by the solutions of \( \mathcal{L} \cdot \mathcal{L} = 0 \)--Figs (39). A particularly interesting situation would occur when \( r_{\text{center}} \leq 2M \) as this condition, where the inner part of the disk is dragged and the outer part of the disk is fully contained or partially contained in the ergoregion, may imply the geometrical maximum of the disk approaches the outer ergosurface. Therefore in the process of torus exfoliation we can distinguish the following two cases:

(I) In the first case there is \( r_{\text{center}} \leq r_{\text{max}} \leq r_{s}^+ = 2M \), where \( r_{\text{max}} \) is the projection on the equatorial plane of the geometrical maximum of the torus. This condition is sufficient for partially contained tori with the inner region, \( [r_{\text{inner}}, r_{\text{center}}] \), in \( \Sigma^± \). The first inequality, \( r_{\text{center}} < r_{\text{max}} \), is always satisfied, while condition \( r_{\text{center}} = r_{\text{max}} \) occurs in the limit of \( K \approx K_{\text{center}} \).

(II) In the second case there is \( r_{\text{max}} \geq r_{s}^+ \). We note that even with \( r_{\text{max}} \in \Sigma^± \) the geometrical maximum of the torus can be outside the ergoregion.

We also consider the following three cases for the toroidal configurations:

(i) \( K > K_{\text{crit}} \) correspondent to overcritical tori with an accretion throat (there is \( K \in [K_{\text{crit}}, 1] \));

(ii) \( K = K_{\text{crit}} \), correspondent to a cusped torus;

(iii) \( K < K_{\text{crit}} \) associated to a quiescent torus (there is \( K \in [K_{\text{center}}, K_{\text{crit}}] \))

where \( K_{\text{crit}} \) is the value of the \( K \) parameter at the maximum of the fluid effective potential (minimum of the pressure). Tori in these three cases also have equal constant fluid specific angular momentum. Cusped tori are considered in the Figs (15)–first line-center panel. In some cases the toroidal configuration crosses the outer ergosurface on a point \( z_a \in [r_{\text{inner}}, r_{\text{max}}] \) or for external \( z_s \in [z_{\text{max}}, r_{\text{outer}}] \) (where \( z \) is here the crossing that occurs on a point at \( \theta \neq 0 \)) along the toroid surface. These cases are characterized by two different scenarios concerning accreting matter colliding with the embedding tori materials. The small dimensions of the tori (see Figs (21)) allow to perform a first analysis in the hypothesis of free test particle from the tori surface. We shall assume free particles with initial conditions dictated by the toroidal configurations studied in Figs (15) from four regions of the configurations crossing the static limit, which are the set \( R_{\text{free}} = (\text{I. the inner edge, 2. the accretion throat, 3. the outer edge, 4. the geometrical maximum}) \). In this first analysis we do not consider the details of the different accretion mechanisms which may also involve plasmas particles and magnetic fields.\(^6\) Particles models considered here inherit the initial data from the tori construction (values of \( \ell \) and

\(^6\) For the analysis of timelike orbits, integration can be done equivalently on two sets of equations. Considering derivation with respect to proper time \( \tau \), the
Therefore particles leave the tori from different points of the toroidal surfaces (data set $R_{\text{free}}$). From the definition of $E$ and $L$, we find $\phi'(0) = u^\phi(0)$ expressed in terms of $(\ell, E)$ using definition $L = \ell E$, having $\ell$ settled according to the torus model and $E$ for the timelike particles coincident to parameter $K$ of the torus. Initial data $t'(0) = u'(0)$ is managed according to the equations set up solving the constrain at the initial time. (Note $\ell$ and $K$ are torus "global" parameter in the sense that are constant on each toroidal surface). Particles are initially circularly orbiting on an orbit defined by the location $r(0)$ and $\theta(0)$ on the torus surface (there is no need to define explicitly $\phi(0)$) according with $R_{\text{free}}$, therefore $\theta'(0) = 0$ and $r'(0) = 0$.

Therefore, according to the torus and the particle model, we fix $\theta_0$, $r_0$ in $R_{\text{free}}$, for simplicity we refer to these points as "emission points", with initial data $Q_0 \equiv Q(0)$, where $Q_0$ is for any quantity evaluated at initial time $\tau = 0$. We consider the toroidal models of Figs (23). Using the Kerr geometry symmetries, there are two emission points obtainable in accordance with the reflection symmetries with respect to the BH rotation axis (coincident with the torus rotation axis) for the emission points on the equatorial plane, and four positions for emission points on planes different from the equatorial, found considering also the symmetry for reflection with respect to the disk equatorial plane (coincident with the BH equatorial plane, note there is no need to fix $\phi_0$). In the orbits representations of Figs (24) and (25) for convenience we used one or two emission points. Furthermore we considered the entire trajectory in the internal area of the disk to enlighten the trajectories impacting the disk, on its inner edge, as this back-reaction from the emission to the inner regions of the tori resolved eventually in photons and dust (or plasma) colliding with the toroid matter, being a characterizing factor of partially contained or dragged toroids. Particle leaving the surface can evolve in a swarm of particles created in an atmosphere wind around the torus surfaces. Or eventually, they can part from the torus, for example from the outer edge of the outer region crossing ergosurface as a wind of particles and photons, or part of the emission could impact on the torus surface if emitted from the inner region (note the model considered here is opaque, cooled with advection and with super Eddington luminosity).

In this analysis we should take into account the size of partially contained and dragged tori, studied in Figs (21), the torus maximum height and the location of the tori outer edge.

The torus maximum dimensions are constrained largely by the BH dimensionless spin. The reduced dimensions of the tori mean that the dragged surfaces are similar to globules where exfoliation should take start. Another important factor in this process is the back-reaction effect through tori self gravity for the huge tori, combined with accretion and runaway instability. The analysis of PPI or MRI for these tori should be also taken into consideration (Gammie 2004).

Runaway instability (RI), however, is an important process for thick disks orbiting SMBHs, particularly for partially contained and dragged surfaces. In this type of instability, the eventual rotational law $\ell \neq \text{constant}$, defining different tori model, can be significant. Neglecting the torus self-gravity, and using stationary models, RI can be studied with a fully relativistic hydrodynamics analysis. The RI features a sequence of varying mass BH phases following the mass transfer from the torus during the accretion and the consequent new status of the torus. The torus can become stable or unstable, depending on several factors as the mass ratio of the disk and the BH, if the BH is spinning, and most importantly the location of torus inner edge and cusp with respect to the central BH at the initial stage of the process. We expect therefore the RI be significant for dragged surfaces considered here and especially the partially contained tori, having the cusp close to the BH, but being large enough to interfere with RI with the eventual mass-transfer slowing-down and the torus exfoliation characterizing the smallest tori in $\Sigma_0^\nu$. The RI is generally a very fast process enduring on a dynamical timescale of a few orbital periods. In this process the geometry changes consequently to the increase of the BH mass for accretion from the disk. It is then clear that there can be also a BH spin variation due to rotational energy extraction, via the Blandford–Znajek mechanism, where the amount of extractable energy depends on the disk mass and can be ejected in jets or winds observable for example in GRBs. (Some considerations on the extraction of rotational energy are in Sec. (3.3)). The accretion disk therefore adapts to the new situation leading possibly to a series of steady states. The torus inner edge plays a key role in this process, the disk filling its Roche lobe can transfer mass through the cusp. Following the geometry change, the cusp can stretch towards the horizon, slowing the mass transfer, which can eventually stabilize the toroid, or otherwise the cusp can move outwardly, into the constraint equation (normalization condition) can be included in the set of four equations and integrated numerically. The three equations for the particles coordinates can be numerically integrated solving the constraint for $t'(\tau)$, according to the initial data. Note we numerically integrated the geodesic equations $u^a \nabla_a u^b = 0$ assuming zero pressure terms in Eq. (8) and $u^a$ the particles four-velocity with normalization condition $(u_a u^a) = -1$. To fix the initial data of Table (1) from the toroidal models of Figs (23), particles inherit aspects of the torus configuration reflected in the particle motion initial data, that is we express $(\theta', r')$ (at the initial time) from the $(E, L)$ definition of Eq. (3) (expressed in terms of $t'$ and $\phi'$), with $L = \ell E$, where $\ell$ is from the torus model detailed in Figs (23). As we consider timelike particles, parameter $E$ coincides with the $K$-parameter of the torus (this can also be tested by applying the constraint to the initial data, as done in the case of photon particles, used also for example for the cases where, for some particles models, there is $E \neq K$). We adapted the constraint to the case of photons, and $(E, L)$ parameters cannot be provided from the tori model as for timelike particles, depending on the particular reflection and emission model. We used $\phi'$ and $t'$, in the initial data, expressed in terms of the $(E, L)$ parameters and, similarly to the timelike particles cases, with $\ell = L/E$ adapted to the torus, the normalization condition on the initial surface is used to recover the parameter $E$. 
toroid interior, increasing the mass transfer, eventually the toroid could be "absorbed" by the central BH in a very fast process. The disk inner region, \([r_{\text{inner}}, r_{\text{center}}]\), regulates the two cases, therefore this analysis singles out the relevant tori.

In \(\Sigma^+\), we consider quiescent tori with \(r_{\text{inner}} > r_x\), cusped tori \(r_{\text{cusp}}\) and over critical tori with an accretion throat whose minima are given by the pressure extremes curve. Naturally we can evaluate the force term to which the particles of the different points of the toroidal surface are subjected by the pressure gradients in Euler’s law, in points other than the cusp where the free force hypothesis is guaranteed—Figs (22). It is expectable that the closer the inner edge is to the BH horizon, the more significant is the RI, which is therefore significant particularly for BH with high spins. In this case then, we note that tori with outer edge \(r_{\text{outer}} = r^+_x\), represented in Figs (22), have inner edge closer to the \(r_{\text{mso}}\) and therefore far from the horizon (Abramowicz et al. 1983; Abramowicz et al. 1998; Korobkin et al. 2013; Font&Daigne 2002).

Analysis of Figs (24) and Figs (25), considers the BH geometry with spin \(a_y\), studying tori with specific angular momentum \(\ell \in L_1\) and parameter \(K \in \{K_{\text{mso}}, 1\}\). This first study of the particles can be implemented point by point on the surface. In Figs (23) the blue line sets the center and the geometrical maximum points of all the tori at \(\ell = \ell_c\) and the minima of the accretion throats. Therefore, all the considered tori have at least the inner part of the disk contained in the ergoregion. Figs (22) show the results of the analysis of the torus outer edge crossing the outer ergosurface \(r_{\text{outer}} = r^+_x = 2M\) on the equatorial plane, for cusped tori \(C_1\), with specific angular momentum \(\ell \in L_1\) and parameter \(K \in \{K_{\text{mso}}, 1\}\). Tori having these characteristics can be observed only orbiting BH with spin \(a > a_{\text{mso}}\).

Increasing the BH spin, the range of values of the parameters \(\ell K^2\), for quiescent or cusped tori \(C_1\), decreases according to the fact that the tori are corotating. Relation \(\ell = \ell(K^2)\) can be considered almost linear. The analysis shows a larger range of values for \(\ell K^2\) for BH with spin in a range centered on the value \(a_{\text{mso}}^b\). The peculiarity of the geometries of this range is also shown in right panel, with the results for cusped tori. Note that cusped tori have equatorial elongation \(\lambda_x\) larger then the quiescent tori with equal specific angular momentum (a different case occurs for the overcritical tori having an accretion throat). The extended geodetic structure of this geometry is studied in Figs (2). Inner, outer edge and center of the tori of Figs (6) are shown as functions of the specific angular momentum. An alternative study of the outer edge, center and cusps of the tori \(C_x\) is in Figs (7). The analysis is repeated in Figs (8) as function of the cusp radius \(r_x\). A prospect of the possible toroidal surfaces orbiting in these geometries is in Figs (15). Analysis of the disks verticality is in Figs (16), Figs (17) and Figs (19). Distance cusps—centers is considered in Figs (20), the von Zeipel surfaces are Figs (32).

Tori \(C_x\) in Figs (22)—right panel have outer edge at \(r^\alpha_{\text{outer}} = r^+_x = 2M\). Curves show different characteristics of the cusps and center of the tori in a region \(P_{\text{mbo}}\), defined by spins \(a_{\text{mbo}}^b\) and \(a_y^b\), around the radius \(r = 1.6M\) and shown in Figs (22). The tori cusp at variation of the BH spin remains relatively close to the limiting radius \(r_{\text{mso}}\), increasing the distance in \(P_{\text{mbo}}\), being small tori. Tori are smaller and closer to the ergosurface for spins close to \(a_{\text{mso}}\), and are relatively small for the larger spins \(a \approx M\) where the cusp approaches the radius \(r_{\text{mbo}}\), maximum distance is reached in \(P_{\text{mbo}}\). The cusp decreases increasing the BH spin, approaching \(r_{\text{mbo}}\). Therefore increasing the BH spin the tori elongation \(\lambda_x\) increases.

However, the tori stability and tori exfoliation depend on their inner region, whose radial dimension is \(\lambda_{\text{inner}} \equiv r_{\text{center}} - r_x\) (the
outer region is \( \lambda_{\text{outer}} \equiv r_{\text{outer}} - r_{\text{center}} = 2M - r_{\text{center}} = \lambda - \lambda_{\text{inner}} \). We observe that there is always \( \lambda_{\text{inner}} \ll \lambda_{\text{outer}} \), where \( \lambda_{\text{inner}} \) is maximal in \( P_{\text{mbo}} \).

Tori studied in Figs (23) orbit around a BH with spin \( a_{\text{mbo}}^b \) and they are defined by the following four models with fixed \( \ell = \text{constant} \):

- (1): Torus \( T_2 \), yellow curve in Figs (23), is defined by parameter \( K = K_2 \equiv K_x + (1 - K_x) / 2 > K_x \) (the torus is characterized by the presence of an accretion throat), and it is partially contained in \( \Sigma^+ \) (with the torus inner part).
- (2): Torus \( T_M \), blue surface in Figs (23), is a partially contained torus, characterized by an accretion throat defined by parameter \( K_M = 0.765671 > K_x \). The torus geometrical maximum crosses the outer ergosurface.
- (3): Torus \( T_{10-4} \), green-dashed surface of Figs (23), is defined by \( K = K_{10-4} = K_x + 5 \times 10^{-4} > K_x \), it is a partially contained torus characterized by a small accretion throat, crossing the outer ergosurface on a plane \( \theta \neq \pi / 2 \).
- (4): Torus \( T_x \), black surface of Figs (23), is the cusped torus with \( K = K_x \), which is a dragged surface, with the outer edge coincident with the outer ergosurface \( r_\lambda^+ = 2M \) on the equatorial plane.

We set the initial data \( \{ \tau_0, \theta_0, \phi'(0), E, K, r_0, y_0, x_0 \} \) for the integration defining six particles models\(^7\) listed in Table (1) in Figs (23).

### Table 1
Timelike particle models initial data \( \{ \tau_0, \theta_0, \phi'(0), E, K, y_0, x_0 \} \) related to tori models \( \{ T_x, T_M, T_2, T_{10-4} \} \) of Figs (23), dimensionless units have been used. (') is intended a derivation with respect to the proper time.

| Torus | Description | Data |
|-------|-------------|------|
| 1. Torus \( T_M \). | Particles are initially located on the geometric maximum \( r_{\text{max}} = r_\lambda^+ \) on a plane \( \theta \neq \pi / 2 \). | \( \{ \phi'(0) = 0.83, \tau(0) = 1.96, \theta(0) = 1.289, E = 0.766, y(0) = 1.884, x(0) = 0.545 \} \) |
| 2. Torus \( T_2 \). | Particles leave the crossing point of the torus with the ergosurface. | \( \{ \phi'(0) = 1.257, \tau(0) = 1.851, \theta(0) = 1.015, E = 0.872, y(0) = 1.5722, x(0) = 0.9773 \} \) |
| 3. From the torus \( T_{10-4} \). | Particles leave the crossing point of the torus surface with the outer ergosurface. | \( \{ \phi'(0) = 0.750, \tau(0) = 1.98999, \theta(0) = 1.52575, E = K_{10-4} = 0.744353, y(0) = 1.57217, x(0) = 0.0900212 \} \) |
| 4. From the torus \( T_x \). | Particles leave a point close to the cusp \( r_\lambda \) of the torus \( T_x \). | \( \{ \phi'(0) = 5.0598, \tau(0) = 1.23872, \theta(0) = \pi / 2, E = K_x = 0.744 \} \) |
| 5. From the torus \( T_x \). | Particles are initially located on the torus outer edge, coincident with the outer ergosurface. | \( \{ \phi'(0) = 0.748, \tau(0) = 1.99992, \theta(0) = \pi / 2, E = K_x = 0.743853 \} \) |
| 6. From the torus \( T_x \), particles leave a point \( r_0 \). | \( \{ \phi'(0) = 6.505, \tau(0) = 1.2, \theta(0) = \pi / 2, E = K_x = 0.743853 \} \) |

We consider particular toroidal surfaces at different \( K \) from the set of Figs (14)–(upper line central panel). The analysis is a first simple attempt to consider the swarm of free particles and photons which is grounded on the reduced dimension of the tori in the ergoregion and the model of zero-pressure fluid at the tori edges—see Figs 21). We have used same approximation in Sec. (3.2.1) for the analysis of the oscillatory models of the torus using the characteristic Keplerian frequencies to discuss the adaptability of QPOs models on these structures. In general the test particles hypothesis is of astrophysical significance, also in the case of a tori atmosphere. In this situation we can also combine in characteristics processes of these tori as the Poynting–Robertson effects for partially included and dragged surfaces\(^8\) (Igumenschkev 2008; Bini et al. 2011; Bini et al. 2015; Lee 2001; Ballantyne&Everett 2005; Zanni et al. 2007c; Pudritz et al. 2007; De Falco et al. 2020).

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\(^7\) Note in the formulation of toroidal models of Figs (23) and initial data in Table (1) it is not necessary to integrate the equation for the torus surface. We have numerically integrated the condition \( V_{\alpha \beta} = K^2 \) of Eq. (9), fixing the two tori parameters \( (\ell, K) \), according to the tori topology and using the rotational law \( \ell(r) \) and the function \( K(r) \) to fix the parameters values. On the other hand, the inner and outer edges of the cusped tori, the torus center (maximum pressure point inside the disk), the geometrical maxima of the torus surface are in fact directly obtainable, these radii are given in Pugliese&Montani (2015) and for example in Pugliese&Stuchlik (2018a); Pugliese&Stuchlik (2019)—see also (Abramowicz&Fragile 2013).

\(^8\) The radiation field carrying out energy and momentum interacts back with the accretion disk plasma. The radiation field induces a radiation pressure which combines with a radiation drag force (which is the Poynting-Robertson effect), and the mass transfer during accretion. We should also note that the tori considered here are opaque and super-Eddington. The radiation drag can act as dissipative force for the orbiting matter, removing energy and angular momentum. The Poynting-Robertson effect can remove angular momentum and energy from small-sized test particles. Test particle radial motion is therefore affected by the Poynting-Robertson effect and radiation pressure and the accreting mass can lose angular momentum combining to accretion. It should be noted that Poynting flux (in presence of magnetic field) carries away from the BH energy and angular momentum, combining in the BH evolution with the accretion.) There can be therefore (radiatively and thermally) outflows and inflow followed eventually also by a change into the disk structure, considering outgoing or ingoing radial photon flux.
We show some results considering light-like particles from the torus surface and with fixed initial $\ell$ parameters, using the constraint and the definition of $(E, L)$. We used Eq. (4) for the photon to obtain initial data $u^{\ell}(0)$, determined by conditions on tori parameter and assuming the circular motion symmetries preserved, where there is $u^{\ell}(0) = 0$ and $u^{\phi}(0) = 0$, determined by parameter $\ell$ and $E$, the first fixed by the torus, while the second is fixed for the timelike particle by the $K$-parameter of the torus. Eq. (4) does not differ for photon and timelike particles, defined by $E$ and $L$ with the definition of Killing fields. We considered $L = \ell E$, using for timelike particles parameters $\ell$ and $E$. For the photons $\ell$, the initial data $u^{\ell}(0)$ has been fixed using the normalization condition used also in the set of equation to be integrated. We also considered the metric symmetries $(t, \phi) \rightarrow (-t, -\phi)$ for the two solutions in $u^{\ell}$. Details of particles model 1, 2, and 3, of Table (1) are shown in Figs (25). Models 4, 5, and 6, considering particles on the torus and BH equatorial plane are shown Figs (24).

As exemplified by the cases studied in this example, particles follow three classes of trajectories: particles can leave the ergoregion; particles can remain trapped in the inner region (leading to collision with torus inner edge), or be absorbed by the BH. Particle motion is shown in Figs (26), (27). Particle models of Table (1) have been considered for the initial locations and the tori models. The disk exfoliation is based on the free particles hypotheses, grounded on low pressure gradients, Lense-Thirring effect and small tori sizes, (a further open question concerns the process time scale for the total destruction of the dragged torus). The example of Figs (26), (27) has therefore to be understood as indicative, and the analysis should be framed considering the emission-reflection processes from the disk surface.

### 3.1.4 Extreme configurations: multi-tori, proto-jets and $C_2$ tori

On the sidelines of this analysis we also consider the proto-jets existence in $\Sigma_c^+$. An issue that can affect stability of tori orbiting in $\Sigma_c^+$ is the possibility that several orbiting toroidal configurations may be formed in aggregates of toroids orbiting the central BH, occurring, for example, in composed systems of tori agglomerates as in the eRAD framework (Pugliese&Stuchlik 2015; Pugliese&Stuchlik 2017), developed as aggregates of axisymmetric, corotating and counter-rotating toroidal configurations, coplanar and centered on the equatorial plane of the central Kerr attractor in AGNs-Figs (28). Figs (28) show for the BH spacetime with $a = a_c^b$, the case of three closed configurations, the inner cusped torus contained in the ergoregion as in Figs (14). The outer critical torus is counterrotating, the middle of the triplet is quiescent and corotating. This is a case of double accretion onto the central BH with a middle screening torus.

It is clear that each toroidal component of the aggregate of partially contained or dragged surfaces is constrained according to the analysis of Secs (3.1). Especially tori orbiting around faster rotating BH can rise to collision. These tori would be extremely small and at close distance (i.e. small displacement $\lambda_{(1,2)} \equiv r^{(2)}_{inner} - r^{(1)}_{outer}$ where $r^{(2)}_{inner}$ is the inner edge of the outer torus while $r^{(1)}_{outer}$ is the outer edge of the inner torus). There is $r_{M} > r_c^+$, and this property suggests that the dragged tori distribution (density of tori centers in a region around the central BH) has no extreme as function of $r/M$ for any spin $a/M$, if the agglomerate does

![Fig. 23. Equipressure surfaces for GRHD tori are shown at different $K$ (the energy parameter). Black region is the central BH with spin $a = a_m^b$, as in Figs (14)–(upper line central panel). Gray region is the outer ergosurface. Vertical dashed lines are $r_{mbo}$ the marginally bounded orbit, $r_{max}$, the marginally stable orbit, $r_c$, the corotating photon orbit on the equatorial plane. $\ell$ is the fluid specific angular momentum. $r_\times$ is the torus cusp. $K_\times \equiv K_\times + (1 - K_\times)/2$ and $K_{10^{-4}} \equiv K_\times + 5 \times 10^{-4}$ and $K_M = 0.765671$ (blue line torus whose geometrical maximum is on the outer ergosurface) see also analysis in Figs (28). Blue curves set the extremes of the pressure and density in the disks: the maximum pressure inside the disk from the center of the configurations to the geometrical maximum for different $K$ surfaces, and the inner minimum point of the pressures from the cusp. The cusped torus has outer edge in $r = 2M$ the outer ergosurface one the equatorial plane. Orbits of fluid particles and photons in this system is shown in Figs (24) and Figs (25), (26), (27) respectively.](image-url)
Fig. 24. Timelike particles analysis of configurations of Figs (23) on the equatorial plane \( \theta = \pi/2 \). Torus \( T_x \) represented in Figs (23), and described in Sec. (3.1.3) is considered. There is \( \{ x = r \sin \theta \cos \phi, y = r \sin \theta \sin \phi, z = r \cos \theta \} \). Black center is the BH \( r < r_+ \), gray region is the outer ergoregion \( r \in [r_+, r^{+}_e] \), where \( r_+ \) is the BH event horizon and \( r^{+}_e \) is the outer ergosurface. Yellow surface is the torus. The central BH has spin \( a_\text{spin} \), the torus specific angular momentum \( \ell \) is signed in figure. Complete particles trajectories are shown. Considering Table (1) upper line panels show details of particles model 4, central line panels show refer to particles model 5, the bottom line panels explore particles model 6. Left panels show motion on the BH equatorial plane, center panels show particles coordinates angles \( \theta \) (red curve), \( \phi \) (purple curve) and radius \( r \) (green curve).
Fig. 25. BH spacetime with spin $a^b_m$, torus specific angular momentum $\ell$ is signed in figure. Figures show examples of timelike particles leaving the torus from a point of the equipressure surface. Tori models $T_2$, $T_1$ and $T_{10}$ represented in Figs (23) and described in Sec. (3.1.3). Particles leave a crossing point $z^*$ of the toroidal surface with the ergoregion (clearly for the problem symmetry there are four "equivalent" points at fixed radius $r_0$ and four $\theta$s) on plane different from the equatorial plane, $r_{\text{max}}$ is the geometrical maximum of the surface in the considered model coincident with a point of the outer ergosurface. Complete particles trajectories are shown. Considering Table (1) upper line panels show details of particles model 1., central line panels show refer to particles model 2., the bottom line panels explore particles model 3. Gray region is the outer ergosurface, black region is the central BH orange surface is the torus. Right panel: equatorial plane is at $z = 0$, center panel: angles $\theta$ (red curve), $\phi$ (purple curve) and radius $r$ (green curve). Note that in the bottom right panel the projection on the equatorial plane of the motion is shown, as $\theta_0 \approx \pi/2$. (There is $\{x = r \sin \theta \cos \phi, y = r \sin \theta \sin \phi, z = r \cos \theta\}$.)
Fig. 26. Light-like particles analysis of configurations of Figs (23). Black center is the BH $r < r_+$, gray region is the outer ergoregion $r \in [r_+, r^*_{+}]$. (There is $\{x = r \sin \theta \cos \phi, y = r \sin \theta \sin \phi, z = r \cos \theta\}$). Yellow surface is the torus. BH spacetime has spin $a^b_{\text{mbo}}$. The torus specific angular momentum $\ell$ signed in figure. Upper line panels: photons leave radius $r_0 = r_{\max} = r_{+} + \epsilon$ (i.e. the torus geometrical maximum, coincident with the outer ergosurface $r^*_{+}$). Second and third line panels: the initial radius $r_0$ is the the crossing points of the torus surface with the outer ergosurface. Third line photon leaves the torus outer edge coincident with $r^*_{+} = 2M$. Right panels show particles coordinates $\theta$ (red curve), $\phi$ (purple curve) and radius $r$ (green curve).

Fig. 27. Light-like particles analysis of configurations of Figs (23). Particles from cusped torus $T_{\times}$, with cusps $r_{\times}$. There is $\{x = r \sin \theta \cos \phi, y = r \sin \theta \sin \phi, z = r \cos \theta\}$. Bottom line left and center panels: particles leave a point $r_{+} = 1.2M$ in the inner region. Bottom right panel, particles leave the torus outer edge $r_{\text{outer}} = r^*_{+}$. Upper right panel shows particles coordinates $\theta$ (red curve), $\phi$ (purple curve) and radius $r$ (green curve). Black center is the BH $r < r_+$ with spin $a^b_{\text{mbo}}$. Gray region is the outer ergoregion $r \in [r_+, r^*_{+}]$. Yellow surface is the torus. Torus specific angular momentum is $\ell = 2.107$. 
not vary with the spin or radius. However, this may not be the case for partially contained configurations because of the presence of the radius $r_M \in [r_{mbo}, r_b]$ in the stability range for the centers of $C_2$ tori. Curve $r_M(a)$ represents an “accumulation” point for the tori centers. Radius $r_M$ would be a center for the configuration with specific angular momentum in the range $L_2$, where proto-jets are possible. Correspondingly, there is the radius $r_M^+ \in [r_\gamma, r_mbo]$ for the proto-jets cusps in the geometries $a \in [0, a_M^1]$, where $a_M^1 = 0.934313M : \ell(r_M) = \ell(r_\gamma)$, and $\ell(r_M^+) = \ell(r_\gamma)$; in this spin range there is $r_M^+ < r_\gamma$, and $r_M > r_b$ for larger spin. Radius $r_M$ is a center for tori in $L_2$ correspondent to unstable points $r_M^-$ see Figs (29) and Figs (19)- Figs (2). On the other hand, there is a BH geometry with $\ell_M = \ell(r_j^+)$ where $\ell_M \in [\ell_{mbo}, \ell_\gamma]$, in the range $[a_\gamma, a_{mbo}]$. Then $r_{cusp} = r_j$ and the quiescent torus $C_2$ inner edge is $r_{inner} > r_j \in [r_\gamma, r_mbo]$, which can be in ergoregion also for relatively small BH spin, opening an interesting possibility for jet emission and the role of the ergoregion, which would be suppressed at large spin. The possibility of existence of configurations $L_2$ with the solutions of $\partial^2_\ell \ell = 0$, points of accumulation of pressure $r_M$, in the ergoregion, and therefore of inner edge of $C_2$, has been investigated in Figs (30). It is clear that this possibility is regulated by the spins $a_{\ell_2} = 0.93431 : r_M^+ = r_\gamma$ and $a_{\ell M 2} = 0.7667 : r_M^+ = r_j^+$. In $a \in [a_{\ell M 2}, a_{\ell M 2}]$ the critical points $r_j$ lie in the ergoregion.

3.2 Characteristic frequencies

The investigation of the disk verticality in the ergoregion includes the exploration of the characteristic frequencies affecting the torus. The analysis in Figs (31) shows the vertical frequency $\omega_z$ in the ergoregion considering therefore the frame dragging for very high spinning BH, as clear by the presence of extreme of $\omega_z$ as function of the spin $a$ and as function of $r$. Concerning more generally the tori dynamical oscillations it should be said that the spectrum modes are not completely studied. The axisymmetric, incompressible modes related to global oscillations comprise the epicyclic frequencies. The Keplerian, radial and vertical angular frequencies take the form (Stuchlík et al. 2013)
We apply then and the results of state. This property of the rotational law is linked to scale-times of the main physical processes involved in the disks, the accretion mechanism for transporting angular momentum in the disk, and the turbulence emergence dependent the magnetic fields of the so called Boyer condition within the conditions of the von Zeipel results reduces to an integrability condition on the Euler equations. In the magnetized plasma fluids(Zanotti&Pugliese 2015). Von Zeipel surfaces are shown in Figs (32). Essentially the application of the so called Boyer condition within the conditions of the von Zeipel results reduces to an integrability condition on the Euler equations. In the case of a barotropic fluid, the differential equation (8) can be integrated reducing to a gradient of a scalar, which is

\[ \omega = \text{velocity equations.} \]

In the case of a barotropic fluid, the differential equation (8) can be integrated reducing to a gradient of a scalar, which is...
Possible if and only if $\ell = \ell(\Omega)$. (Dragged tori are far from the quasi-spherical conditions). In general in these models of accretion disks the angular momentum of matter in the disks is considered to be sufficiently high for the centrifugal force to be a predominant component of the four forces regulating the disks’ balance (centrifugal, gravitational, pressure and magnetic forces, and eventually dissipative effects). This holds for situations where the gravitational background is generated by a SMBHs shaping morphology and a great part of dynamics on (micro and macroscopic scale of) the disks. In general accretion disks, there must be an extended region where there is $\pm \ell_+ > \pm L_+$ in the same orbital region (explicitly including counterrotating fluids on Kerr background). This limiting condition is assumed to hold for a general accretion torus with a general angular momentum distribution. The Bondi quasi-spherical accretion constitutes a situation when the condition $|\ell| > |L|$ is not fulfilled. In the Bondi quasi spherical accretion, the fluid angular momentum is everywhere smaller than the Keplerian one and therefore dynamically unimportant. However, the models under examinations here are based on a full GR onset for each RAD toroid, where in fact there exists an extended region where the fluids angular momentum in the torus is larger or equal in magnitude than the Keplerian (test particle) angular momentum.

Frequencies $\{\omega_K, \omega_r, \omega_z\}$ of Eq. (16) are then combined the lower and higher frequency peaks in the particles models of the QPOs emission.

3.2.1 On the origin of the QPOs emission and the ergoregion

We discuss the possibility presented in different literature that the origin of the QPOs (Quasi-periodic oscillations) emission could be attributable to and immediately traceable to the oscillatory phenomena reducible to the test particles, considering the case of particles in the ergoregion (Stuchlík et al. 2013). This hypotheses is plausible in these models of dragged and partially contained tori as they can be considered small tori, relatively close to the maximum pressure radius constituting the torus center. The small geometric thickness of the disks is investigated in Figs (21), although the thickness parameter considered for the construction on QPOs emission model based on the tori geometrical thickness has been discussed in Török et al. (2016a); Straub&Sramkova (2009).

QPOs are observed in the X-ray brightness at low (Hz) and high (kHz) frequencies. Observed QPO frequencies are associated in many models with frequencies of orbital and oscillatory motion of the particles from accretion disk matter. The peaks of high frequencies turn to be near the orbital frequency of the marginally stable orbit extreme, limit for the inner edge of thin disks. The upper $\omega_U$ and lower $\omega_L$ frequencies of the peak are in the particles geodesic models expressed in terms of the radial and vertical
oscillations and the azimuthal frequency. Particular attention is given to recognize the emergence of the twin HF QPOs with resonant frequency ratios $\omega_U/\omega_L = 3 : 2$. The observed ratio of the twin peak frequencies $\approx 3 : 2$ and other resonances characterized by ratios as $(2 : 1, 3 : 1, 5 : 2)$. could explain observed QPOs frequencies (with same $3 : 2$ ratio), considering the combinational frequencies, occurring in the inner parts of accretion flow around a black hole ( assumed radiating spots in thin marginally bounded orbit, $r_{mbo}$ is the marginally stable orbit. The cusped tori are associated with the cusp point of the minimum null-pressure. In this respect the torus cusp could be possibly connected to QPOs emission. The individual QPO models frequencies considered here are functions of three fundamental frequencies of perturbed circular geodesic motion and are:

RE: \( (\omega_L = \omega_r, \quad \omega_U = \omega_z) \), \quad \text{Kepl:} \quad (\omega_L = \omega_r, \quad \omega_U = \omega_K) \), \quad (17)

RP: \( (\omega_L = \omega_K - \omega_r, \quad \omega_U = \omega_K) \), \quad \text{RP1:} \quad (\omega_L = \omega_K - \omega_r, \quad \omega_U = \omega_z) \)

RP2: \( (\omega_L = \omega_K - \omega_r, \quad \omega_U = 2\omega_K - \omega_z) \);

TP: \( (\omega_L = \omega_z - \omega_r, \quad \omega_U = \omega_z) \), \quad \text{TP1:} \quad (\omega_L = \omega_z - \omega_r, \quad \omega_U = \omega_K) \)

TD: \( (\omega_U = \omega_K, \quad \omega_U = \omega_K + \omega_r) \), \quad \text{WD:} \quad (\omega_L = 2(\omega_K - \omega_r), \quad \omega_U = 2\omega_K - \omega_r) \)

(RE) and (Kepl) models are investigated in Figs (36). Frequencies on the static limit are in Figs (35). The radial frequency is well defined, $\omega^r > 0$, for $r > r_{mso}$, where it vanishes, determining orbits stability for radial oscillations around the point of the perturbed particle. We consider this problem and various oscillation models as in Eqs (17) in the ergoregion, where $r_{mbo} \in [r_+ , r^*]$ for $a \geq a_{mso}$, including geometry $a^b_{mbo}$. In Figs (34) we show the models in the BH geometry $a^b_{mbo}$ in comparison with different resonant ratios, in Figs (33) we show the models in the BH geometry $a^b_{mso}$. Each frequency model we study is borrowed from a specific context from which they are derived including slender tori and hot spot models (assuming radiating spots in thin

![Fig. 33. Plot of frequencies $\omega_U$ (blue) $\omega_L$ (black) and the ratio $\omega_U/\omega_L$ (red curve) as functions of the radius $r/M \in [r_+ , r^*]$ in the ergoregion, in the BH spacetime with spin $a^b_{mbo}$ see also Figs (14). Where the outer ergosurface on the equatorial plane is $r^*_+ = 2M$. Oscillation models $\{ (WD), (TD), (TP), (TP1), (RP), (RP1), (RP2), (RE) \}$ of Eqs (17) are studied. Resonant frequency ratios $R_1 = 2 : 1, R_2 = 3 : 1, R_3 = 3 : 2, R_4 = 4 : 3, R_5 = 5 : 4$ (dashed lines) are also shown. Black lines are $r_+ < r_0 < r_{mbo} < r_{mso}$, where $r_0$ is the marginally circular orbit, $r_{mbo}$ is the marginally stable orbit. Each frequency model we study is borrowed from a specific context from which they are derived including slender tori and hot spot models (assuming radiating spots in thin marginally bounded orbit, $r_{mbo}$ is the marginally stable orbit.]

Fig. 34. Plot of frequencies $\omega_U$ (blue curve) $\omega_L$ (black curve) and the ratio $\omega_U/\omega_L$ (red curve) as functions of the radius $r/M \in [r_+,r_m]$ in the ergoregion, where the outer ergosurface on the equatorial plane is $r^+ = 2M$. The BH spacetime has spin $a_{mb}$—see also Figs (14). Oscillation models $\{(\text{WD}), (\text{TD}), (\text{TP}), (\text{TP1}), (\text{RP}), (\text{RP1}), (\text{RP2}), (\text{RE})\}$ of Eqs (17). Resonant frequency ratios $R_1 = 2:1$, $R_2 = 3:1$, $R_3 = 3:2$, $R_4 = 4:3$, $R_5 = 5:4$ (dashed lines) are also shown. Black lines are $r_+ < r_\gamma < r_{mb} < r_{mso}$, where $r_\gamma$ is the marginally circular orbit, $r_{mb}$ is the marginally bounded orbit, $r_{mso}$ is the marginally stable orbit.

accretion disks). Therefore frequencies used in the fit of the models $\{(\text{WD}), (\text{TD}), (\text{TP}), (\text{TP1}), (\text{RP}), (\text{RP1}), (\text{RP2}), (\text{RE})\}$ are therefore expressions of particle frequencies that can demonstrate Keplerian and epicyclic frequencies: (WD) is for "warped disk" model introduced for oscillatory modes in a warped accretion disk. (TD) is for tidal disruption model (related to tidal disruption of accreting inhomogeneities) (Kostic et al. 2009). TP and TP1 are for total precession models. (RP), (RP1) and (RP2) are for relativistic-precession models, attributing the HF QPOs to modes of blobs in the inner parts of the accretion disk. (RE) is for "epicyclic resonance" model dealing with radial and vertical epicyclic oscillations, and the (Kepl) for "Keplerian resonance" model assuming a resonance between the orbital Keplerian motion and the radial epicyclic oscillations (Stuchlík et al. 2017; Stuchlík & Kološ 2016; Stuchlík et al. 2007b; Török & Stuchlík 2005; Kotrllová et al. 2017; Török et al. 2016b; Šramková et al. 2015; Stuchlík et al. 2013; Török et al. 2011; Stuchlík et al. 2011; Kotrllová et al. 2008; Stuchlík et al. 2007a).

3.3 Tori from polytropic fluids and the ergoregion

In this section we specify the polytropic equation of state (EoS) for dragged tori. An interesting issue to be investigated is whether the frame dragging of the Kerr spacetime may differentiate dragged and partially contained tori or configurations close to the outer ergosurface with different equations of state, for example if the polytropics can affect differently dragged tori. We consider a polytropic equation of state, assuming the pressure $p$ be a function of the matter density $\rho$: $p = \tilde{k}\rho^\gamma$, where $\tilde{k} > 0$ and $\gamma$ is the polytropic index. In Pugliese & Montani (2015) different polytropic EoS for tori were studied, distinguishing classes in relation with the tori parameters ranges and the locations, here we re-focus on this concept in relations with tori in the ergoregion. It has been shown that there is a specific classification of eligible geometric polytropes and a specific class of polytropes is characterized by a discrete range of values for the index $\gamma$, (Pugliese & Montani 2018; Frank et al. 2002; Pugliese & Montani 2015; Stuchlík et al. 2007a).
Fig. 35. The \((\text{RE})\) and \((\text{Kepl})\) QPO models frequencies of Eqs (17) evaluated on the static limit \(r^+_e = 2M\) on the equatorial plane are plotted as functions of the BH spin \(a/M\). Black lines are spins \(A^+_+ = \{a_mbo, a_b^{mbo}, a_\gamma, a_b^{\gamma}, a_mso\}\) — see also Figs (14). Dashed lines are resonant frequency ratios \(R1 = 2 : 1, R2 = 3 : 1, R3 = 3 : 2, R4 = 4 : 3, R5 = 5 : 4\). Plot of frequencies \(\omega_U, \omega_L\), and the ratio \(\omega_U/\omega_L\). Right panel is a zoom in the region \(a/M \in [0.95, 1]\). Frequencies ratios are \(\omega_k/\omega_r\) (red curve) \((\text{Kepl})\) QPO model, \(\omega_z/\omega_r\) (purple curve) \((\text{RE})\) model and \(\omega_k/\omega_z\) (orange curve). Frequencies are \(\omega_K\) (black), \(\omega_r\) (green) and \(\omega_z\) (blue).

Fig. 36. Plot of frequencies \(\omega_U, \omega_L\), and the ratio \(\omega_U/\omega_L\), as functions of the radius \(r/M \in [r_+^-, r_+^+]\) in the ergoregion, where the outer ergosurface on the equatorial plane is \(r_+^+ = 2M\). Resonant frequency ratios \(R1 = 2 : 1, R2 = 3 : 1, R3 = 3 : 2, R4 = 4 : 3, R5 = 5 : 4\) (dashed lines) are also shown. Black lines are \(r_+^- < r_\gamma < r_{mbo} < r_{mso}\), where \(r_+^\gamma\) is the marginally circular orbit, \(r_{mbo}\) is the marginally bounded orbit, \(r_{mso}\) is the marginally stable orbit. Spins \(A^+_+ = \{a_mbo, a_b^{mbo}, a_\gamma, a_b^{\gamma}, a_mso\}\) are represented—see also Figs (14). Frequencies are \(\omega_K\) (black), \(\omega_r\) (green) and \(\omega_z\) (blue) of Eqs (16) as functions of \(r/M\)—see also Figs (31). Frequencies ratios are \(\omega_k/\omega_r\) (red curve) \((\text{Kepl})\) QPO model, \(\omega_z/\omega_r\) (purple curve) \((\text{RE})\) model and \(\omega_k/\omega_z\) (orange curve).
(2009; Stuchlík et al. 2020). For the matter density \( \rho \) there is
\[
\bar{\eta}_r \equiv \left[ \frac{1}{k} \left( V_{e,ff}^{\frac{\gamma+1}{n}} - 1 \right) \right]^{\frac{1}{\gamma+1}} \quad \text{for} \quad \gamma \neq 1, \quad \bar{\eta}_r \equiv \frac{V_{e,ff}^{\frac{1}{n+1}}}{1 + k} \quad \text{for} \quad \gamma = 1, 
\]
where \( \bar{\eta}_r \equiv \tilde{k}^{1/(\gamma-1)} \bar{\eta}_r \), which is independent from \( \bar{k} \). We consider here \( \gamma \neq 1 \). The following two cases occur:

1. There is \( C > 0 \), which is for \( \gamma \neq 1 \), where \( V_{e,ff}^2 \geq 1 \) for \( \gamma \in [0,1] \) and \( V_{e,ff}^2 \in [0,1] \) for \( \gamma > 1 \).
2. There is \( C < 0 \), which is for \( \gamma \neq 1 \), within the condition \( V_{e,ff}^2 \in [0,1] \) for \( \gamma \in [0,1] \), and the condition \( V_{e,ff}^2 > 1 \) for \( \gamma > 1 \). The polytropic index satisfies the condition \( \gamma = 1/(2n) + 1 \) where \( n \in \mathbb{Z} \) and \( n \geq 1 \), and \( n \leq -1 \).

(Note, the case 1. includes polytropes with index \( \gamma = 4/3 \).)

### 3.3.1 Polytropic fluids and tori energetics

We consider tori with polytropic fluids: \( p = \kappa \rho^{1+1/n} \), providing an estimation of the mass-flux, enthalpy-flux (evaluating also the temperature parameter), and the flux thickness based on geometric consideration only on the thick disks in the ergoregion or close to ergoregion—see Abramowicz & Fragile (2013); Abramowicz (1985). In details, these quantities are listed in Table (2). Parameters

| Quantities \( O(r_x, r_s, n) \equiv q(n, \bar{k}) (W_x - W_s)^{d(n)} \) | Quantities \( P \equiv \frac{O(r_x, r_s, n) n \bar{k}}{\bar{\omega}_K (r_x)} \) |
| --- | --- |
| Enthalpy-flux \( = D(n, \bar{k}) (W_x - W_s)^{n+3/2} \) | torus-accretion-rate \( \dot{m} = \frac{M}{\bar{M}_{Ed}} \) |
| Mass-flux \( = C(n, \bar{k}) (W_x - W_s)^{n+1/2} \) | Mass-accretion-rate \( \dot{M}_x = \bar{A}(n, \bar{k}, r_x) \frac{(W_x - W_s)^{n+1}}{\bar{\omega}_K (r_x)} \) |
| Cusp-luminosity \( \bar{L}_x = B(n, \bar{k}) \bar{\omega}_K (r_x) (W_x - W_s)^{n+2} \) | Cusp-luminosity \( \bar{L}_x = B(n, \bar{k}) \bar{\omega}_K (r_x) (W_x - W_s)^{n+2} \) |

Table 2. Quantities \( O \) and \( \bar{N} \). \( \bar{L}_x \) stands for the fraction of energy produced inside the flow and not swallowed through the surface but by the central BH. Efficiency \( \eta \equiv L/\bar{L}/c^2 \), \( L \) representing the total luminosity, \( \bar{L} \) the total accretion rate where, for a stationary flow, \( M = \dot{M}_x \), \( W = \ln V_{e,ff} \) is the potential of Eq. (9), \( \bar{L} \) is the Keplerian (relativistic) angular frequency, \( W_x \geq W_x \) is the value of the equipotential surface, which is taken with respect to the asymptotic value, \( W_x = \ln K_{\text{max}} \) is the function at the cusp (inner edge of the accreting torus), \( D(n, \bar{k}), C(n, \bar{k}), \bar{A}, \bar{B} \) are functions of the polytropic index and the polytropic constant.

\((\bar{k}, n)\) within the constraints \( q(n, \bar{k}) = q = \text{constant} \), fix a polytropic-family. Then \( \kappa = n + 1 \), with \( \gamma = 1/n + 1 \) being the polytropic index. We use the fact that the pressure forces are vanishing at the edges of the accreting torus. These quantities can be express in the general form \( O(r_x, r_s, n) \). The \( O(r_x, r_s, n) \) depends on the location of the inner edge of the accreting torus, being determined only by the angular momentum, and on the radius \( r_x \) which is related to the thickness of the matter flow: \( K_x \) corresponds to the accreting point \( r_x \), while \( r_s \) is directly associated to the accreting flux thickness, where \( K_s \in [K_x, 1] \). In \( O(r_x, r_s, n) \), \( q(n, \bar{k}) \) and \( d(n) \) are different functions of the polytropic index \( \gamma = 1 + 1/n \) and of the polytropic constant \( \bar{k} \). The mass flow rate through the cusp (mass loss, accretion rates) \( \dot{M}_x \), and the cusp luminosity \( \bar{L}_x \) (and the accretion efficiency \( \eta \)), measuring the rate of the thermal-energy carried at the cusp could be expressed as \( P \) quantities, or alternately \( \bar{N} \) quantities, as in Table (2). \( \bar{N} \) and \( O \) quantities of Table (2) are evaluated for corotating tori in Figs (37) providing a simple way to evaluate the trends of these functions of the BH spin and the cusp location. The quantities at \( r_s < r_x \) are evaluated according to \( K_x \equiv K(r_x) = K_x + (1 - K_x)/\epsilon \), where \( \epsilon = 2 \) in Figs (37). The behaviour of the curves at different spins as functions of \( r_{crit} \) (therefore as functions of the specific fluid momentum) is qualitatively independent from variation of \( \epsilon \) (an increase of \( \epsilon \) corresponds to a lowering of \( K \) parameter over the critical maximum value and lowering of the matter level towards the cusp is associated to similar qualitative behaviour of the curves and decrease in magnitude). We note that clearly the choice of \( K \) corresponds to the choice of a point on the curves \( \partial V_{e,ff} = 0 \) at \( y < r_{crit} = r_x \) as in Figs (14). An analysis of the \( K_{crit} \) and location of the critical points for different BH spins in relation to the possibility of multi-orbiting tori around the central BH, consisting by both corotating or counterrotating fluids, can be found in Figs (5). We can note that there is an increase of the spin corresponding to a decreases of the range \( r_x \); on the contrary, the curves increase due to increasing the distance from the BH of the disk cusp which corresponds to the increase of the specific angular momentum (at fixed BH spin \( a/M \)). Despite the reduced dimensions of the dragged configurations it is clear that \( \bar{N} \) and \( O \) are quite large inside the ergoregion \( r_x \in [r_+ + r_0^2] \).

Considering Figs (2) we can see that the analysis of cusped configurations in Figs (37) is in agreement with the existence of the cusp in the ergoregion at \( a_{\text{crit}}^k, a_{\text{crit}}^{lho} \), and \( a_{\text{min}}^k \). This analysis, however, does not take into account both the deviations present from the perfectly stationary situation, and the fate of the flow in free falling after the cusp. We can analyze this case, taking into account that
Fig. 37. Evaluation of $N$- and $O$-quantities of Table (2) for corotating tori, versus the cusp location $r_{c} / M$, for the spins $A_{\pm}^{n} \equiv \{a_{mbo}, a_{mbbo}, a_{\gamma}, a_{b\gamma}, a_{mso}\}$—see Figs (14). The quantities at $r_{s} < r_{c}$ are evaluated according to $K^{*} \equiv K(r_{s}) = K_{x} + (1 - K_{x}) / \epsilon$ where $\kappa = n + 1 = 4$, with $\gamma = 1 / n + 1$ is the polytropic index and $\epsilon = 2$. $r_{s}^{*} = 2M$ is the outer ergosurface.

Fig. 38. Plane $r / M$ and $\xi$, where $\xi$ is the maximum extractable rotational energy parameter. Gray curves are the marginally bounded orbit, $r_{mbo}$, marginally stable orbit $r_{mso}$, and $r_{mco}$, marginally circular orbit. Left panel: Curves $\ell(\xi; r) =$constant (on a vertical line of the plane each curve describes one torus). Right panel: curves $K_{cr} =$constant on the equatorial plane. Spins $A_{\pm}^{n} \equiv \{a_{mbo}, a_{mbbo}, a_{\gamma}, a_{b\gamma}, a_{mso}\}$ are plotted as black lines—see Figs (14). The plot region corresponds in fact to the situation of minimum hydrostatic pressure. The extractable rotational energy $\xi$. In Figs (38) the analysis has been developed showing explicitly the relations between the dragged toroidal surfaces characteristics in the ergoregion and the maximum extractable rotational energy $\xi$ from the BH horizon, through the analysis of the leading function $\ell(r)$ and the energy function $K(r)$. The plots show the situation one might expect in a phase prior the total extraction of the energy from the BH. We start by considering the spin function $a(\xi)$ (Pugliese&Stuchlík 2021):

$$a(\xi) = 2 \sqrt{-(\xi - 2)(\xi - 1)^{2}\xi},$$

relating the dimensionless BH spin $a / M$ to the dimensionless ratio $\xi$, (total released rotational energy versus BH mass measured by an observer at infinity). We are assuming a process ending with the total extraction of the rotational energy of the central Kerr BH. Considering $M(0)$ and $J(0)$ the mass and angular momentum of the initial state of the BH, the upper limit for of the energy extraction from a stationary process bringing the BH at the state $(1)$ is $M(0) - M_{irr}(M(0), J(0))$. The BH angular momentum in the new state $(1)$ is zero. All the quantities therefore are evaluated at the state $(0)$ prior the process, which implies that all the quantities evaluated here inform on the status of the BH-accretion disk system at its stationary state $(0)$ prior the energy extraction.

that region corresponds in fact to the situation of minimum hydrostatic pressure.

The extractable rotational energy $\xi$.

In Figs (38) the analysis has been developed showing explicitly the relations between the dragged toroidal surfaces characteristics in the ergoregion and the maximum extractable rotational energy $\xi$ from the BH horizon, through the analysis of the leading function $\ell(r)$ and the energy function $K(r)$. The plots show the situation one might expect in a phase prior the total extraction of the energy from the BH. We start by considering the spin function $a(\xi)$ (Pugliese&Stuchlík 2021):

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Considering the dimensionless rotational energy as $\xi = 1 - M_{irr} / M$ ($M_{irr}$ is the irreducible mass), with quantities evaluated at the stationary state prior the process, we obtain the restricted range $\xi \in [0, \xi_{0}]$, where $\xi_{0} \equiv \frac{1}{2} \left(2 - \sqrt{2}\right)$ limiting therefore the energy extracted to a superior of $\approx 29\%$ of the mass $M$, where at the state $(0)$ (prior the extraction) there is an extreme Kerr BH
BHs that the thickness of tori is larger for larger BHs partially included tori in the ergoregion. Plots show details on the location of the inner edge and center of maximum pressure, in spacetime. The larger extractable rotational energy is related to the BHs, the maximum extractable rotational energy, occurring for an extreme Kerr BH.

In relations to BHs, the thickness of tori is larger for larger BHs, partially included tori in the ergoregion. Plots show details on the location of the inner edge and center of maximum pressure, in spacetime. The larger extractable rotational energy is related to the BHs, the maximum extractable rotational energy, occurring for an extreme Kerr BH.

4 Conclusions

In this work we investigated orbiting extended matter configurations in the ergoregion of a spinning black hole, introducing in Sec. (3.1) the concept of dragged tori. To this end, we considered corotating perfect fluids tori, constituting geometrically thick accretion disks, using the disk model detailed in Sec. (2) for tori orbiting a central Kerr BH.

It is argued that the smaller dragged tori could be subjected to a characteristic instability, disk exfoliation, consequence of the geometry frame-dragging. We discussed this hypothesis assessing the tori features for the occurrence of this phenomena, the possible outcomes as particle and photon emission, or in an enhanced accretion. This process is expected to lead to the destruction of the torus in combination with accretion, emission and other processes typical of the regions very close to the black hole horizon as, for example, the Runway Instability. An important parameter for the tori exfoliation is the torus verticality, given by the evaluation of the effective potential is function of Cartesian coordinates $(x, y)$ and we can note the role of the marginally stable orbits—see also Figs (32). In Sec. (1) we include some notes on the stationary magnetized tori in the ergoregion, considering the GRHD tori with a toroidal magnetic field according to the Komissarov solution (Komissarov 2006).

8 In relations to BHs of the set $A_x^+$, there is $\xi_{mbbo} = 0.244$, $\xi_{mbbo} = 0.117$, $\xi_0 = 0.256$, $\xi_0 = 0.0761$, $\xi_{max} = 0.183503$, $\xi_{max} = \frac{1}{2} \left( 2 - \sqrt{2} \right)$, where $\xi_{max}$ is the maximum extractable rotational energy, occurring for an extreme Kerr BH.
cording to the central attractor dimensionless spin. We individuated five classes of geometries, as in Figs (2), defined according to the BHs spin–mass ratios, bounded by the spins $A_{\pm} \equiv \{a_{R}, a_{mbo}, a_{mso}, a_{mho}, b_{\gamma}\}$, and regulating the constraints on the location of the extremes of the pressure in the tori–Figs (14),(4) and (19). We studied the toroids in each of these classes. Configurations orbiting in the geometries defined in $A_{\pm}$ are shown in Figs (14). The conditions for the tori outer edge $r_{outer}$ to be in $\Sigma_{\pm}$, in the different geometries of the set $A_{\pm}$, are shown in Figs (6), for cusped tori are in Figs (7), and in Figs (8) where the conditions are expressed in functions of the disks inner edge, $r_{inner}$, location. We provided in Figs (7,8,9,10) indications on the torus inner region elongation, $\lambda_{inner} \equiv r_{center} - r_{inner}$, the most active and significant part of the tori for this analysis (large tori inner elongation are expected for small BH spin). While results on the torus outer edge coincidence with the static limit are in Figs (7,8,22).

On the other hand, the pressure gradients are also the main factor to be evaluated for the exfoliation and the consequent formation of a possible atmosphere of free-particle swarm. In order to asses the reliability of an eventual process of disk exfoliation we studied in Sec. (3.1.1) the pressure gradients in the disks, results are in Figs (13), (14),(15),(32),(16),(17),(18),(7). In Figs (15) and(39) we show the results of the analysis of the disk verticality in terms of the polar gradients of the pressure, considering the lines of extremes of the HD pressure, providing also the surfaces of geometric maximum, obtaining therefore a clear indication of the maximum vertical extension of the torus in the ergoregion. Figs (21) summarize the results on the dragged tori geometrical thickness, distinguishing faster spinning and slower spinning black holes, and the situations where the inner Roche lobe of the tori is thicker then the outer Roche lobe for the dragged cusped tori. In Sec. (3.1.3) we explored the process of tori exfoliation considering five tori models, six particle models from four regions of the configurations crossing the static limit. We considered quiescent (inert) tori (topologically regularly surfaces) and cusped tori, characterized by the HD instability driven by a Paczyński mechanism. The accretion occurs through the torus cusp (from the outer Roche lobe of the disk ). On the sidelines of this analysis we also considered in Sec. (3.1.4) the proto-jets existence in $\Sigma_{\pm}$, and extreme configurations as agglomerates of toroids orbiting the ergoregion constituting aggregates of corotating tori which can collide .

Dragged and partially contained tori have been also studied in relation to the QPOs emissions from the tori inner edges. More precisely, the characteristic frequencies of the toroids are studied in Sec. (3.2) and in Sec. (3.2.1) we discuss the origin of the QPOs emission from toroids in the ergoregion. We focused on the resonant frequency ratios $R1 = 2 : 1$, $R2 = 3 : 1$, $R3 = 3 : 2$, $R4 = 4 : 3$, $R5 = 5 : 4$ and frequencies used to the fit of the models \{(WD), (TD), (TP), (TP1), (RP), (RP1), (RP2), (RE), (Kepl)\} are expressions of particle geodesic frequencies of Eqs (17) related to dragged or partially contained tori (with torus inner region contained in the ergoregion). Some results are in Figs (34)– (36).

In Sec. (3.3) we specified the polytropic equation of state for dragged and partially contained tori, investigating how the frame dragging of the Kerr spacetime could differentiate dragged and partially contained tori with different values of the polytropic index. Discussion on some aspects of tori energetics for these tori is in Sec. (3.3.1). We provided an estimation of the flux thickness, mass-flux and enthalpy-flux basing on geometric considerations for dragged and partially contained thick disks. These toroids can be observed orbiting black holes with dimensionless spin $a > 0.9897 M$.

In Sec. (3.3.1), we also discussed the relations between the dragged tori in the ergoregion and the maximum extractable rotational energy $\xi$ from the BH horizon, showing the situation of the state prior the total extraction of the energy from the BH, considering the approach introduced in Daly (2009) focused on the definition of SMBH (irreducible) mass function and the definition of rotational energy. Finally, in Appendix (1), we included in the GRHD model developed in Sec. (2) a toroidal magnetic field a la’ Komissarov (Komissarov 2006), developing the analysis on the stationary magnetized tori in the ergoregion. As mentioned in Sec. (3.1.3), disk exfoliation could combine with others tori characteristics processes as the Poynting–Robertson effects for partially included and dragged surfaces. Runaway instability is another major process for dragged thick disks orbiting SMBHs. We intend to deepen the relation with the Bardeen–Petterson effect, for tori having a slight inclination with respect to the equatorial plane, in regards to the possible origin on an inner corotating dragged or partially contained tori.

Data Availability

No new data were generated or analysed in support of this research.

Appendix 1 Notes on stationary magnetized tori in the ergoregion: the toroidal magnetic field

The study of the magnetized surfaces with toroidal magnetic field is an interesting issue of the toroidal configurations we consider here. The presence of a magnetic field with a relevant toroidal component can be related to the disk differential rotation, viewed as a generating mechanism of the magnetic field (Komissarov 2006; Montero et al. 2007). The Komissarov solution is a well known
and widely used magnetic field solution based on the construction of a toroidal magnetic field considering the set of results known as von Zeipel theorem—(Komissarov 2006). Here we consider the possibility that such field could be defined in the ergoregion. We reconsider briefly the onset developed in Pugliese&Montani (2013); Pugliese&Montani (2018). We also refer to Pugliese&Montani (2013); Hamersky&Karas (2013); Karas et al. (2014); Pugliese&Montani (2018); Adamek&Stuchlík (2013) where this solution is dealt in detail in the context of accretion disks. We consider an infinitely conductive plasma, with $F_{ab}u^a = 0$ where $F_{ab}$ is the Faraday tensor and $u^a$ is the fluid four-velocity and $B^a$ is the magnetic field with $\partial_t B^a = 0$ and $B^r = B^\phi = 0$.

The Euler equation for this system can be exactly integrated for the background spacetime of Schwarzschild and Kerr BHs with a magnetic field $B = B^\phi$ and magnetic pressure $p_B$

$$B^\phi = \sqrt{\frac{2nq}{\mathcal{A}}} \quad \text{where } \quad p_B = \mathcal{M}A_T^{-1} \varpi^4 \quad \text{or alternatively} \quad B^\phi = \sqrt{2M\varpi q A_T^{-2} V_{eff}(\ell)},$$

where $A \equiv \ell^2 g_{tt} + 2\ell g_{t\phi} + g_{\phi\phi}$, $A_T \equiv g_{\phi\phi}^2 - g_{tt} g_{\phi\phi}$,

$$\text{(A1)}$$

where $\varpi$ is the fluid enthalpy, $q$ and $\mathcal{M}$ are constant-Komissarov (2006); Montero et al. (2007). (A barotropic equation of state is assumed). Eq. (9) has been used in second term of equation A1. According to our set-up we introduce a deformed (magnetized) Paczyński potential function $\bar{W}$ and the Euler equation (9) becomes:

$$\partial_\mu \bar{W} = \partial_\mu [\ln V_{eff} + G] \quad \text{where } \quad \bar{W} \equiv G(r, \theta) + \ln(V_{eff}) = K,$$

where for $a \neq 0$: $G(r, \theta) = S(AV_{eff}^2)^{q-1} = A_T Q^{-1}$, and $S = \frac{q\mathcal{M}\omega^q - 1}{q-1}$.

$$\text{(A3)}$$

(A4) We therefore consider the equation for the function $\bar{W}$ and we introduce the function

$$\bar{V}_{eff}^2 \equiv V_{eff}^2 e^{2S(AV_{eff}^2)^{q-1}}.$$  

$$\text{(A5)}$$

The toroidal surfaces are obtained from the equipotential surfaces, for the potential $\bar{V}_{eff}$ which, for $S = 0$, reduces to the effective potential $V_{eff}$ for the non-magnetized case in Eq. (9). The tori are regulated by the modified rotational law $\bar{\omega}(r) : \partial_\phi \bar{V}_{eff} = 0$, for counterrotating and corotating magnetized fluids respectively- where there is $\lim_{\mathcal{M} \to 0} \bar{\ell}^2 = \lim_{q \to 1} \bar{\ell}^2 = \ell^\pm$. We rephrase the problem introducing an adapted function $S_{crit}(r; \ell, q)$, whose values $S_{crit}(r; \ell, q)$ =constant, provide the parameter $S$ defining the torus. Re-considering the equation for the hydrostatic pressure critical points, solution $S_{crit}(r; \ell, q)$ represents the values of $S$ as a function of $r$, for which critical points of the function $\bar{V}_{eff}$ exist:

$$S_{crit} \equiv -S_Q \frac{a^2(a - \ell)^2 + 2r^2(a - \ell)(a - 2\ell) - 4r(a - \ell) - \ell^2r^3 + r^4}{2r(\ell - 1)[r(a^2 - \ell^2) + 2(a - \ell)^2 + r^3]}, \quad \left(\text{where } \quad S_Q \equiv \frac{\Delta - Q}{Q}\right)$$

$$\text{(A6)}$$

(with $r \to r/M$ and $a \to a/M$ and $Q = q - 1$), giving the tori centers and, eventually the tori cusps. (A negative solution for $S_{crit} < 0$ may appear for $q > 1$). We are interested to the conditions for the magnetic field, defined in Eq. (A1), is well defined in the ergoregion at any plane $\theta = \text{constant}$. It is simple to see that the magnetic pressure, within the condition assumed on the enthalpy, is always positive. We note that, although the magnetic field is independent from the effective potential function $V_{eff}$ it depends implicitly on the $K$ parameter. In Figs (40) we considered the field $B$ in relation to tori present in the ergoregion. The definition of the magnetic field is delineated by the light-surfaces, solutions of $\mathcal{L} \cdot \mathcal{L} = 0$, which provide solutions in the ergoregion—Figs (39). (The photon circular orbit on the equatorial plane $r_\gamma$, is a particular relevant surface). The field is not well defined approaching the horizon $r_+$ There is $A = \ell^2 g_{tt} + 2\ell g_{t\phi} + g_{\phi\phi} = \ell^2 (\mathcal{L} \cdot \mathcal{L}) = \ell^2 (g_{tt} + 2\omega g_{t\phi} + \omega^2 g_{\phi\phi})$ where $\omega \equiv 1/\ell$, and the problem is reduced to find the solutions of the equations for the frequencies $\omega_\ell$ of the stationary observers. Light surfaces have frequencies $\omega_\ell$. Spacelike solutions are given for $\omega_\ell < \omega_-$ and $\omega_\ell > \omega_+$ (on all the planes $\theta$). On the ergosurface the condition is $\omega_\ell > \omega_+$ as $\omega_\ell = 0$.

References

Abramowicz M. A. 1971, Acta. Astron., 21, 81
Abramowicz, M. A. 1985, Astronomical Society of Japan, 37, 4, 727-734
Abramowicz, M. A., Calvani, M. & Nobili, L. 1983, Nature, 302, 597–599
Abramowicz M. A. & Fragile P. C. 2013, Living Rev. Rel., 16, 1
Abramowicz M. A., Karas V., Lanza A. 1998, A&A, 331, 1143
Abramowicz M.A., Lanza A., Percival M.J. 1996, Astrophys. J., 479, 179
Adamek K. & Stuchlík Z. 2013, Class. Quantum Grav. 30, 205007
Allen, S. W., Dunn, R.J.H., Fabian, A.C., et al. 2006, MNRAS, 1, 372, 21
Baltzlyne D. R. & Everett J. E. 2005, Astrophys. J. 626, 364-372
Bardeen J. M. & Petterson J.A. 1975, Astrophys. J., 195, L65.
Bekenstein J.D. 1975, Phys. Rev. D, 12, 3077
Bini D., Geradico A., Jantzen R. T., Semerák O. and Stella L. 2011, Class. Quantum Grav. 28, 035008
Fig. 40. Magnetized tori in the ergoregion. Upper panel: Black region is the BH $r < r_+$, gray region is the outer ergosurface. For spacetime $\alpha = a^b_\gamma$, right panel shows the tori surfaces at constant $K$ for a fixed value of the specific momentum $\ell$, the magnetic parameters $S$ and $q$, $r_\gamma$ is the photon orbits and $r_m^b: \ell(r_\gamma) = \ell(r)$. Right panel shows the magnetic pressure $p_B =$constant (pink) and the magnetic field $B =$constant (black) of Eqs (A1), for parameters $M = 1$. Dotted-dashed arrows indicate the increasing values of the magnetic pressure (pink) and of the magnetic field (black). For the analysis of the spacetime $\alpha = a^b_\gamma$, properties see Figs (2)-(6)-(7)-(8)-(15)-(32)-(39). Below left panel: photon orbital frequencies $\omega_\pm$ versus $r/M$ for the spacetime $\alpha = a^b_\gamma$, stationary observers exist for $\omega \in [\omega_-, \omega_+]$, where $\omega_+ = \omega_-$ on the horizon $r_+$. $r_m^b$ is the marginally stable orbit, $r_{mbo}$ is the marginally bounded orbit. Right panel: BH spacetime with spin $a^b_\gamma$, limits on the specific angular momentum $\ell < 1/\omega_+$ and $\ell > 1/\omega_-$ for the magnetic field in the ergoregion. Dashed horizontal lines are $\ell(r_\gamma) > \ell(r_{mbo}) > \ell(r_{mso})$.

Bini D., Geralico A., Jantzen R. T., Semerák O. 2015 , MNRAS, 446, 2317-2329
Blaes O. M. 1987, MNRAS, 227, 975-992
Blandford R. D., Znajek R. L. 1977, MNRAS, 179, 433
Boyer R. H. 1956, Proc. Cambridge Phil. Soc., 61, 527
Bugli, M., Guilet, J., Müller, E., et al. 2018, MNRAS, 475, 108
Chakrabarti, S. K. 1990, MNRAS , 245, 747
Chakrabarti S. K., 1991, MNRAS, 250, 7
Charbulák D., Stuchlík Z., 2018, EPJC, 78, 879.
Daly R. A. 2009, Astrophys. J., 691–L72-L76
DeFalco V., Bakala P., and Falanga M. 2020, Phys. Rev. D 101, 124031
Fishbone, L. G., Moncrief, V. 1976 Astrophys. J., 207, 962
Font, J. A. & Daigne, F. 2002, MNRAS, 334, 383
Frank J., King A., Raine D. 2002, Accretion Power in Astrophysics, (Cambridge University Press, Cambridge 2002)
Frolov V. P., Zelnikov A. 2011, Introduction to Black Hole Physics, (Oxford University Press, Oxford 2011)
Gammie C. F. 2004, Astrophys. J., 614:309–313
Gariel J., Marcilhacy G. and Santos N. O. 2013, Astrophys. J. 774, 109
Hamersky J.&Karas V. 2013, A&A, 555, A32
Hawking S.W. 1976, Comm. Math. Phys. 43, 199 (1975) Erratum - ibidem 46, 206 (1976)
Hawking S. W. & Ellis G. F. R. 1973, The large scale structure of spacetime, (Cambridge University Press, Cambridge, 1973)
Igumenshchev I. V. 2008, Astrophys. J., 677, 317
Karas V., Kopacek O., Runneriath D. & Hamersky J. 2014, Acta Polytech., 54, 6, 398
Kawakatu, N., Ohsuga, K. 2011, MNRAS, 417, 4, 2562-2570
King A. R., Lubow S. H., Ogilvie G. I., Pringle J. E. 2005, MNRAS, 363, 49–56
King A. and Nixon C. 2018, Astrophys. J., 857, 1, L7
Kološ M., Tursunov A., Stuchlík Z. 2021, PhRvD, 103, 034018
Komissarov S. S. 2006, MNRAS, 368, 993
Komissarov S. S. 2009, J. Korean Phys. Soc., 54, 2503
Korobkin, O., Abdikamalov, E., Stergioulas, N., et al. 2013, MNRAS, 431, 1, 354
Kostic, U., Cadez, A., Calvani, M., Gomboc, A. 2009, Astron. Astrophysic, 496, 30
Kotrová A., Stuchlík Z.&Török G. 2008, Class. Quantum Grav., 25, 225016
Kotrová A., Šrámková E., Török G. 2017, et al, A&A, 607, A69
