Response of the Fractional Damped Oscillator to a Driving Delta Pulse

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Abstract. Exact solutions for a fractional damped oscillator that is acted by a Dirac delta pulse are reported. It is found that the response of the oscillator to such a pulse can be manipulated by either the damping parameter or the fractional order of the corresponding dynamical law.

1. Introduction

The study of physical systems usually implies the solving of either differential or partial–differential equations of integer order. For instance, relaxation and oscillation processes deal respectively with differential equations of first and second order. However, intermediate phenomena between oscillation and relaxation may be modeled by differential equations of order \( \alpha \in \mathbb{R} \), where \( \alpha \) can take fractional values between 1 and 2, see e.g. [1, 2]. Systems associated to such phenomena are called fractional oscillators and their properties can be investigated within the mathematical framework of fractional calculus [3–9] (other interesting fractional systems appear in viscoelasticity [10] and in the transition between diffusive and wave phenomena [11–13]). Of particular interest, a fractional damped oscillator is also affected by a force that is proportional to the fractional position–derivative of \( \alpha \)th order [14].

In this communication we report exact solutions for a fractional damped oscillator that is subjected to an external force modeled by the Dirac delta distribution [15]. The model is based on the results for fractional oscillators in oscillatory external forces [16], and represents an example of the exactly solvable fractional oscillators studied in [17, 18]. On the other hand, it is well known that, among other applications, the \( \delta \)-distribution is useful to model either the charge density produced by a point charge [19], the response of electric circuits to very high voltages applied in short intervals of time [20], or the strike of a hammer in a mass attached to a spring [21]. Our interest is to use the Dirac delta function to model a sudden pulse of force that is acted on a fractional damped oscillator.

The paper is structured as follows. In Section 2 we briefly revisit the results reported in [18] for fractional damped oscillators subjected to driving forces \( f(t) \). In Section 2.1 we specialize to the case of a driving force proportional to the Dirac delta distribution. The analysis of the results is included in Section 2.2. Finally, we give some concluding remarks in Section 3.
2. Fractional damped oscillator in external forces

To describe the time-evolution of a fractional damped oscillator with natural frequency $\omega_0$ that is affected by an external time-dependent force $f(t)$ we use the fractional linear differential equation

\[ (D^\alpha D^\alpha + 2\beta D^\alpha + \omega_0^{2\alpha} - \Omega^2) x(t) = f(t), \quad \alpha \in (1/2, 1), \]

where $\beta$ is a damping constant expressed in units of frequency. Hereafter $D^\alpha$ stands for the fractional derivative operator defined in the Caputo sense. For the sake of consistency in units we impose $[f(t)] = [L][T]^{-2\alpha}$, where $[L]$ and $[T]$ stand for the units of length and time respectively.

For $\beta \neq \omega_0$ the straightforward calculation [18] (see also [22]) gives the solution

\[ x(t) = \frac{[t^\alpha-1(E_{a,a}(-\Omega_-t^\alpha) - E_{a,a}(-\Omega_+t^\alpha))]*f(t)}{2\sqrt{\beta^2 - \omega_0^{2\alpha}}}, \]

with $E_{a,b}(z)$ the Mittag-Leffler function [23], and $\Omega_{\pm} = \beta \pm \sqrt{\beta^2 - \omega_0^{2\alpha}}$. The symbol $*$ stands for the Laplace convolution operator. On the other hand, for $\beta = \omega_0$ it may be shown [18] that the solution acquires the form

\[ x(t) = \frac{[t^2\alpha-1 E_{a,2\alpha}(-\omega_0^a t^\alpha)]*f(t)}{2\sqrt{\beta^2 - \omega_0^{2\alpha}}}, \]

where $E_{a,b}^\gamma(z)$ is the Prabhakar function [24].

2.1. Fractional damped oscillator driven by a Delta pulse

To investigate the form in which the fractional damped oscillator responds to a very strong impulse of instantaneous duration at $t = 0$ let us make $f(t) = f_0\delta(t)$. For $\beta \neq \omega_0$ the solution (2) is therefore given by

\[ x(t) = \frac{f_0 t^\alpha-1[ E_{a,a}(-\Omega_-t^\alpha) - E_{a,a}(-\Omega_+t^\alpha)]}{2\sqrt{\beta^2 - \omega_0^{2\alpha}}}, \]

The above expression has been already reported in [22], we have included it in the present work for the sake of completeness. In turn, for $\beta = \omega_0$ the convolution (3) leads to the solution

\[ x(t) = f_0 t^2\alpha-1 E_{a,2\alpha+1}(-\omega_0^a t^\alpha). \]

- Models for (conventional) damped oscillators driven by a delta pulse are obtained from (4)-(5) after calculating the limit $\alpha \to 1$. To be concrete, for $\beta < \omega_0$ the solution (4) gives

\[ \lim_{\alpha \to 1} x(t) = \frac{f_0 e^{-\beta t}}{\sqrt{\omega_0^2 - \beta^2}} \sin \left( \sqrt{\omega_0^2 - \beta^2} t \right). \]

In turn, for $\beta > \omega_0$ we obtain

\[ \lim_{\alpha \to 1} x(t) = \frac{f_0 e^{-\beta t}}{\sqrt{\beta^2 - \omega_0^2}} \sinh \left( \sqrt{\beta^2 - \omega_0^2} t \right). \]

On the other hand, for $\beta = \omega_0$ the solution (3) yields

\[ \lim_{\alpha \to 1} x(t) = f_0 t e^{-\omega t}. \]

- Fractional oscillators driven by a delta pulse are also obtained from (4)-(5), but this time calculating the limit $\beta \to 0$. In particular, from (4) one gets

\[ \lim_{\beta \to 0} x(t) = f_0 t^{2\alpha-1} E_{2\alpha,2\alpha}(-\omega_0^{2\alpha} t). \]

Some insights concerning the latter case have been already given in [25], the results of which are recovered from (9) by the changes $2\alpha \to \alpha$ and $f_0 \to A$. 

2
2.2. Analysis of results

The response solution \( x(t) \) for some combinations of \( \alpha \) and \( \beta \) is depicted in Figure 1. At \( t = 0 \) the system is stimulated with a very strong and instantaneous driving force, then it leaves its position of equilibrium. The system reaches a maximum distance from its rest position in a very short interval of time, and then it returns to the initial position. The shorter the value of \( \alpha \) the stronger significant is the response of the system and the faster is the coming back to the initial position. On the other hand, the response of the system is less significant as the value of \( \beta \) increases. The latter is clear by paying attention to the maximum of the curves depicted in the figure.

![Figure 1](image)

(a) \( \alpha = 0.9, \beta = 1 \)
(b) \( \alpha = 0.9, \beta = 2 \)
(c) \( \alpha = 0.9, \beta = 3 \)
(d) \( \alpha = 0.7, \beta = 1 \)
(e) \( \alpha = 0.7, \beta = 2 \)
(f) \( \alpha = 0.7, \beta = 3 \)

Figure 1. Response of the fractional damped oscillator to a delta driving pulse (\( f_0 = 1 \) and \( \omega_0 = 2 \)).

3. Concluding remarks

We have analyzed the response of the fractional damped oscillator to sudden pulse of force that is modeled by a Dirac delta distribution. In general, once applied the delta pulse the position of the system reaches a maximum from its point of equilibrium and then comes back to its initial position. The maximum displacement and celerity of return depend on the parameters \( \alpha \) and \( \beta \). The maximum increases with \( \alpha \) but it decreases as \( \beta \) goes larger. We have assumed that the delta pulse strikes the system at \( x = x_0 = 0 \) and \( t = t_0 = 0 \), from the left (\( x < 0 \)). The situation when the pulse comes from the right (\( x > 0 \)) is quite similar to the one shown in Figure 1, but in the opposite direction. A more elaborated model includes a succession of delta pulses at the times \( t_1, t_2, \ldots, t_N \), from which a periodic series may be obtained with \( t_j - t_{j-1} = T \) and \( N \to \infty \). Results in this direction are in progress and will be reported elsewhere.

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