Explicit Reciprocity Laws for Higher Local Fields. II

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Abstract. Using the previously constructed explicit reciprocity laws for the generalized Kummer pairing of an arbitrary (one-dimensional) formal group, in this article a special consideration is given to Lubin-Tate formal groups. In particular, this allows for a completely explicit description of the Kummer pairing in terms of multidimensional $p$-adic differentiation. The results obtained here constitute a generalization, to higher local fields, of the formulas of Artin-Hasse, Iwasawa, Kolyvagin and Wiles.

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1. Introduction

1.1. Background. For a prime $p > 2$, let $\zeta_p^n$ be a fixed $p^n$th root of unity and $L$ be the cyclotomic field $\mathbb{Q}_p(\zeta_p^n)$. The Hilbert symbol for $L$ $(,)_p^n : L^\times \times L^\times \to \langle \zeta_p^n \rangle$ is defined as $$(u, w)_p^n = \frac{\theta_L(u)(\sqrt[p^n]{w})}{\sqrt[p^n]{w}},$$ where $\theta_L : L^\times \to \text{Gal}(L^{ab}/L)$ is Artin’s local reciprocity map. Iwasawa [10] deduced explicit formulas for this pairing in terms of $p$-adic differentiation as follows

$$ (u, w)_p^n = \zeta_p^{\text{Tr}_{L/Q_p}(\psi(w) \log u)/p^n}, \quad \text{where } \psi(w) = -\zeta_p^n \frac{w}{w^{-1}} \left( \frac{d}{d\pi_n} \right), $$
and subject to the conditions $v_L(w - 1) > 0$ and $v_L(u - 1) > 2v_L(p)/(p - 1)$. Here $v_L$ is the discrete valuation of $L$, $\pi_n$ is the uniformizer $\zeta_p^n - 1$ and $dw/d\pi_n$ denotes $g'($\(\pi_n))$, for any power series $g(x) \in \mathbb{Z}_p[[X]]$ such that $w = g(\pi_n)$.

Iwasawa’s formula stemmed from the Artin-Hasse formula

$$ (u, \zeta_p^n)_p^n = \zeta_p^{\text{Tr}_{L/Q_p}(-\log u)/p^n}, $$
where $u$ is any principal unit in $L$.

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In this paper we will deduce analogous formulas of (1) for the Kummer pairing associated to a Lubin-Tate formal group, from a generalized Artin-Hasse formula for the Kummer pairing. The method we use to derive these formulas is inspired by that of Kolyvagin in [15]. As a consequence we obtain a generalization, to higher local fields, of the results of Artin-Hasse, Iwasawa and Wiles.

We point out that Zinoviev [29] and Kurihara [17] have similar formulas for the generalized Hilbert symbol.

1.2. Description of the results. Let \( K/\mathbb{Q}_p \) be a local field with ring of integers \( C = \mathcal{O}_K, k_K \) its residue field and \( q = |k_K| \). Fix a uniformizer \( \pi \) for \( K \). Let \( \Lambda_\pi \) be the subset of \( \mathcal{O}_K[[X]] \) consisting of the series \( f \) such that

1. \( f(X) \equiv \pi X \pmod{\deg 2} \).
2. \( f(X) \equiv X^q \pmod{\pi} \).

For a fixed \( f \in \Lambda_\pi \), let \( F = F_f \) be the Lubin-Tate formal group such that \( [\pi]F_f = f \) (cf. [2] VI §3.3 or [15] §7). The ring \( C \) is identified with the endomorphism ring of \( F \) as follows: \( C \to \text{End}(F) : a \to [a]_F(X) = aX + \cdots \). Let \( \kappa_n (\simeq C/\pi^n C) \) be the \( \pi^n \)th torsion group of \( F, \kappa = \varprojlim \kappa_n (\simeq C) \) be the Tate module and \( \kappa_\infty = \bigcup_{n \geq 1} \kappa_n \) (\( \simeq K/C \)). We will fix a generator \( e \) for \( \kappa \) and let \( e_n \) be the corresponding reduction to the group \( \kappa_n \).

In order to describe our formulas, let \( L \supset K \) be a (one-dimensional) local field containing the torsion group \( \kappa_n \) and with ring of integers \( \mathcal{O}_L \), and maximal ideal \( \mu_L \). Consider the \( d \)-dimensional local field of mixed characteristic \( \mathcal{L} = L \{ \{ T_1 \} \cdots \{ T_{d-1} \} \} \) (cf. §1.3). The ring of integers and maximal ideal of \( \mathcal{L} \) are, respectively,

\[ \mathcal{O}_L = \mathcal{O}_L \{ \{ T_1 \} \} \cdots \{ \{ T_{d-1} \} \} \quad \text{and} \quad \mu_L = \mu_L \{ \{ T_1 \} \} \cdots \{ \{ T_{d-1} \} \} . \]

We will denote by \( F(\mu_L) \) the set \( \mu_L \) endowed with the group structure of \( F \). For \( m \geq 1 \) we let \( L_m = L(\xi_m), \mathcal{L}_m = \mathcal{L}(\xi_m) \) and we fix a uniformizer \( \gamma_m \) for \( L_m \).

We define the Kummer pairing (cf. [8] §2.2)

\[ (,)_L, n : K_d(\mathcal{L}) \times F(\mu_L) \to \kappa_n \quad \text{by} \quad (\alpha, x) \to (\alpha, x)_L, n = \Upsilon_L(\alpha)(z) \odot F z, \]

where \( K_d(\mathcal{L}) \) is the \( d \)th Milnor \( K \)-group of \( \mathcal{L} \) (c.f. [8]§2.1.1), \( \Upsilon_L : K_d(\mathcal{L}) \to G_\mathbb{Q}^ab \) is Kato’s reciprocity map for \( \mathcal{L} \) (cf. [8] §2.1.4), \( f^{(n)}(z) = x \) and \( \odot F \) is the subtraction in the formal group \( F \).

The main result in this paper is the following (cf. Theorem 4.0.1 for the precise formulation).

Theorem. Let \( r \) be maximal and \( r' \) minimal such that \( K(\kappa_r) \subset L \cap K(\kappa_\infty) \subset K(\kappa_{r'}) \). Take \( s \geq \max\{r', n + r + \log_q(e(L/K_r))\} \); \( e(L/K_r) \) denotes the ramification index of \( L/K_r \). Then

\[ (N_{\mathcal{L}_s/L}(\alpha), x)_{\mathcal{L}_s, n} = \left[ T_{L_s/K}(QL_s(\alpha) I_{F}(x)) \right]_{F}(\xi_n), \]

where

\[ QL_s(\alpha) = \frac{T_1 \cdots T_{d-1}}{I_F(\xi_n)} \left[ \frac{\partial a_i}{\partial T_j} \right]_{1 \leq i, j \leq d}, \]

for all \( x \in F(\mu_L) \) and \( \alpha = \{a_1, \ldots, a_d\} \in \bigcap_{l \geq s} N_{\mathcal{L}_l/L}(K_d(\mathcal{L}_l)) \). Here \( N_{\mathcal{L}_s/L} \) and \( N_{\mathcal{L}_s/L} \) denote the norm on Milnor \( K \)-groups (c.f. [8] §2.2.2), \( I_F \) is the logarithm of the formal group, \( T_{L_s/K} \) denotes the generalize trace (cf. §1.3) and \( \partial a_i/\partial T_j \).
denotes the partial derivative of an element in $L_s$ with respect to the uniformizers $T_1,\ldots,T_{d-1},T_d = \gamma_s$ (cf. §1.3).

Just like in the work of Iwasawa, the formulas (3) will be deduced from the following generalized Artin-Hasse formula (cf. Proposition 3.0.1)

\begin{equation}
\left(\{T_1,\ldots,T_{d-1},u\},e_t\right)_{\mathcal{M},m} = T_{\mathcal{M}/S} \left(\log(u) - \frac{1}{\pi t}\right),
\end{equation}

where $\mathcal{M} = K(\kappa_t)\{\{T_1\}\cdots\{T_{d-1}\}\}$, $u \in V_{\mathcal{M},1} = \{u \in \mathcal{O}_\mathcal{M} : v_\mathcal{M}(u - 1) > v_\mathcal{M}(p)/(p - 1)\}$ and log is the usual logarithm.

This paper is organized as follows. In Section 2 we formally construct the derivations and logarithmic derivatives describing the Kummer pairing. In Section 3 we derived an Artin-Hasse type formula (4) for the generalized Kummer pairing. Finally, in Section 4 we derived the formula (3) from the Artin-Hasse formula (4).

The author would like to thank V. Kolyvagin for suggesting the problem treated in this article, for reading the manuscript and providing valuable comments and improvements.

1.3. Notation. We will fix a prime number $p > 2$. If $x$ is a real number then $\lfloor x \rfloor$ denotes the greatest integer $\leq x$.

We will denote by $D(M/L)$ the different of a finite extension of local fields $M/L$. If $R$ is a discrete valuation ring, the symbols $v_R$, $\mathcal{O}_R$, $\mu_R$ and $\pi_R$ will always denote the valuation, ring of integers, maximal ideal, and some fix uniformizer of $R$, respectively.

Let $L$ be a complete discrete valuation field and define

\[ L = \{\{T\}\} = \left\{ \sum_{i=-\infty}^{\infty} a_i T^i : a_i \in L, \inf_i v_L(a_i) > -\infty, \lim_{i \to -\infty} v_L(a_i) = +\infty \right\}. \]

Let $v_L(\sum a_i T^i) = \min_{i \in \mathbb{Z}} v_L(a_i)$, so $\mathcal{O}_L = \mathcal{O}_L\{\{T\}\}$ and $\mu_L = \mu_L\{\{T\}\}$. Observe that the residue field $k_L$ of $L$ is $k_L(\{T\})$, where $k_L$ is the residue field of $L$. Associated to $\mathcal{L}$ we have the map $c_{\mathcal{L}/L} : \mathcal{L} \to L$ such that $c_{\mathcal{L}/L}(\sum_{i=-\infty}^{\infty} a_i T^i) = a_0$.

We define $L = L\{\{T\}\} \cdots \{\{T_{d-1}\}\}$ and $c_{\mathcal{L}/L}$ inductively. Moreover, we define the generalization $T_{\mathcal{L}/K}$ to be the composition $T_{\mathcal{L}/K} = c_{\mathcal{L}/L}$. We also define $\frac{\partial}{\partial T_k} : \mathcal{O}_L \to \mathcal{O}_L$, $k = 1,\ldots,d-1$ to be the partial derivative of $\alpha \in \mathcal{O}_L$ with respect to its canonical Laurent expansion in $\mathcal{L}$. Moreover, if $L$ be a finite extension of $K$ we define $\frac{\partial}{\partial X_k} = \frac{\partial}{\partial X_k} : \mathcal{O}_L \to \mathcal{O}_L/D(L/K)\mathcal{O}_L$ by $\frac{\partial}{\partial T_k} = g^*(\pi_L)$, where $g(X) \in \mathcal{O}_L[X]$ is any polynomial such that $\alpha = g(\pi_L)$; here $\mathcal{L} = \tilde{L}\{\{T\}\} \cdots \{\{T_{d-1}\}\}$ and $\tilde{L}$ is the maximal subextension of $L$ unramified over $K$ (c.f. [8] Definition 4.1.2 for a full account on the derivations $\frac{\partial}{\partial T_k}$, $k = 1,\ldots,d$).

2. Canonical derivations and logarithmic derivatives

In this section we will give an explicit construction of the logarithmic derivative $QL_s$ in the formula (3). Then, in Section 4 we will show that (3) holds.

The following two propositions of Kolyvagin [15] will be needed in the construction of the logarithmic derivative and its properties; we state them here for easy reference.
Proposition 2.0.1. Let \( w_n = v'(e_n) \), \( n \geq 1 \). Then \( w_n \) generate \( \Omega_C(\mathcal{O}_\mathbf{K}) \), the
module of differentials of \( \mathcal{O}_\mathbf{K} \) with respect to \( C \), as \( \mathcal{O}_\mathbf{K} \)-modules. We also have
\[
(5) \quad w_n = \pi w_{n+1}.
\]
Proof. cf. [15] Proposition 7.9. \( \square \)

Let \( K_n = K(\kappa_n) \) and \( K_\infty = K(\kappa) \), where \( \kappa = \cup \kappa_n \). Fix a uniformizer \( \pi_n \) of \( K_n \)
and let
\[
\mathcal{K}_n = K_n\{\{T_1\}\} \cdots \{\{T_d-1\}\} \quad \text{and} \quad \mathcal{K} = K\{\{T_1\}\} \cdots \{\{T_d-1\}\}.
\]
The extension \( K_n/K \) is totally ramified and \( e_n \) is a uniformizer for \( K_n \). Therefore, we
will assume from now on that \( \pi_n = e_n \). Moreover, \( [K_n/K] = q^n - q^{n-1} \).

Let \( \tau : G_K \to \mathcal{O}_K^* \) be the galois representation induced by the action of \( G_K \)
on the Tate module \( \kappa = \lim \kappa_n (\simeq C) \) (cf. § 5.2, Equation (41) of [8] ). This induces
an embedding
\[
\tau_n : \text{Gal}(\mathcal{K}_n/K) \to (\mathcal{O}_K/\pi^n\mathcal{O}_K)^*,
\]
which turns out to be an isomorphism since \([\mathcal{K}_n/K]\) and \(|(\mathcal{O}_K/\pi^n\mathcal{O}_K)^*|\) are both
\( q^n - q^{n-1} \).

Proposition 2.0.2. \( w_n^g = \tau(g)w_n \) for all \( g \in \text{Gal}(\mathcal{K}/K) \).
Proof. cf. [15] § 7.2.3. \( \square \)

2.1. The canonical derivations \( Q_{\mathcal{M},s} \). Let \( M \) be a finite extension of \( K_s \), \( \pi_M \)
a uniformizer for \( M \) and \( \mathcal{M} = M\{\{T_1\}\} \cdots \{\{T_d-1\}\} \). Let \( \pi_1 \) be a uniformizer for
\( K_1 = K(\kappa_1) \). Let
\[
P_\mathcal{M} = (1/(\pi_1 D(M/K))) \mathcal{O}_\mathcal{M} = \{x \in \mathcal{M} : v_\mathcal{M}(x) \geq -v_\mathcal{M}(\pi_1 D(M/K))\}.
\]
We define a \( d \)-dimensional derivation
\[
Q_{\mathcal{M},s} : \mathcal{O}_\mathcal{M}^d \to P_\mathcal{M}/(\pi^s/\pi_1)P_\mathcal{M},
\]
over \( \mathcal{O}_K \), in the following way. Let \( b' \in \mathcal{O}_M \) such that \( w_s = b'd_M \), then \( b'\mathcal{O}_M = D(M/K_s) \). Let us put
\[
(6) \quad Q_{\mathcal{M},s}(T_1, \ldots, T_{d-1}, \pi_M) = \frac{T_1 \cdots T_{d-1}}{b'\pi^s}.
\]
Clearly, \( T_1 \cdots T_{d-1}/(b'\pi^s) \in P_\mathcal{M} \). By Proposition 4.1.5 (2) of [8] we have that
\( D(K_s/K) = \pi^s/\pi_1 \mathcal{O}_s \), then
\[
D(M/K) = D(M/K_s)D(K_s/K) = D(M/K_s)(\pi^s/\pi_1).
\]
Hence \( D(M/K)Q_{\mathcal{M},s}(T_1, \ldots, T_{d-1}, \pi_M) \in (\pi^s/\pi_1)P_\mathcal{M} \). Therefore, by [8] Proposition
4.2.3, \( Q_{\mathcal{M},s} \) defines a \( d \)-dimensional derivation as follows
\[
Q_{\mathcal{M},s}(a_1, \ldots, a_d) := \frac{T_1 \cdots T_{d-1}}{b'\pi^s} \det \frac{\partial a_i}{\partial T_j} \quad \text{for } 1 \leq i,j \leq d
\]
where \( a_1, \ldots, a_d \in \mathcal{O}_\mathcal{M} \). Note that the definition of \( Q_{\mathcal{M},s} \) is independent of the
choice of uniformizer \( \pi_M \) of \( M \).

In the following two propositions we will show some properties of the derivations
\( Q_{\mathcal{M},s} \) that will be needed later in the deduction of the formulas.
**Proposition 2.1.1.** Let $N$ be a finite field extension of $M$ such that $N \supset K_t$. Let $N = N\{\{T_1\}\} \cdots \{\{T_{d-1}\}\}$.

Suppose that $D(N/M)|\pi^{t-s}$. Then $(\pi^t/\pi_1)P_N$ is contained in $X = (\pi^s/\pi_1)P_MO_N$ and so $Q_{N,t}(mod\ X)$ is well-defined. Let $\varpi: \pi^t/\pi_1P_M \to \pi^t/\pi_1P_N/X$ be the injection induced from $P_M \subset P_N$. Then for $y \in \mathcal{O}_M$

$$\varpi(Q_{M,s}(y)) = Q_{N,t}(y) \pmod{X}.$$  

**Proof.** The Proposition 2.1.1 and its proof were suggested by Professor V. Kolyvagin. First,

$$\frac{\pi^tP_N}{\pi_1} = \frac{1}{\pi_1\pi_1D(N/K)} = \frac{1}{\pi_1}\frac{\pi_1D(N/M)D(M/K)}{\pi_1D(N/M)D(M/K)} = \frac{\pi^t}{\pi_1}\frac{\pi^{t-s}}{\pi_1}\frac{1}{\pi_1D(N/M)} \subset X$$

because $(\pi^{t-s}/D(N/M)) \subset \mathcal{O}_N$ by our assumption.

Let $a$ and $c$ in $\mathcal{O}_N$ and $b \in \mathcal{O}_M$ are such that

$$w_t = c\piN,\ w_s = b\piM \text{ and } d\piM = a\piN.$$  

Because $\pi^{t-s}w_t = w_s$ we have $\pi^{t-s}c\piN = b\piN = ba\piN$. So

$$\pi^{t-s}c = ba \pmod{D(N/M)},$$

Dividing this congruence by $\pi^tcb$ and taking into account that $c\mathcal{O}_N = D(N/K_t)$ and $b\mathcal{O}_M = D(M/K_s)$, we have

$$\frac{1}{\pi^t c} = a \pmod{Z},$$

where

$$Z = \frac{D(N/K)}{\pi^t D(N/K_t)D(M/K_s)} = \frac{\mathcal{O}_M}{\pi_1\pi_1D(M/K_s)} = \frac{\mathcal{O}_N}{\pi_1\pi_1D(M/K)} = X.$$  

Here we are using that $D(N/K) = D(N/K_t)D(K_t/K)$ and $D(K_t/K) = \pi^t/\pi_1$; similarly $D(M/K) = (\pi^s/\pi_1)D(M/K_s)$. Thus

$$Q_{N,t}(T_1, \ldots, T_{d-1}, \piM) = \frac{a}{\pi_1 c} = \frac{1}{\pi^s b} \pmod{X} = \varpi(Q_{M,s}(T_1, \ldots, T_{d-1}, \piM)).$$

This implies the corresponding equality for arbitrary $y \in \mathcal{O}_M$ since the right hand and left hand mappings are multidimensional derivations of $\mathcal{O}_M$ over $\mathcal{O}_K$ so they are determined by their value at $(T_1, \ldots, T_{d-1}, \piM)$ (cf. Proposition 4.2.3 of [8]). \hfill \Box

**Proposition 2.1.2.** $Q_{M,s}^g = \tau(g^{-1})Q_{M,s}$ for all $g \in Gal(\overline{K}/K)$. Here $Q_{M,s}^g$ is the $d$-dimensional derivation defined by $Q_{M,s}^g(a_1, \ldots, a_d) = [Q_{M,s}(a_1^{-1}, \ldots, a_d^{-1})]^g$.

**Proof.** Notice that it is enough to check that

$$Q_{M,s}^g(T_1, \ldots, T_{d-1}, \piM) = \tau(g^{-1})Q_{M,s}(T_1, \ldots, T_{d-1}, \piM),$$

by Equation (36) of Proposition 4.2.3.
Let \( b' \in \mathcal{O}_M \) such that \( w_s = b'd\pi_M \). Then \( w_s^g = (b')^g d\pi_M^g \) and, by Proposition 2.0.2, we have that \( w_s = \tau(g^{-1})(b')^g d\pi_M^g \). Since \( \pi_M^g \) is also a uniformizer for \( M \) and the definition of \( \mathcal{Q}_{M,s} \) is independent of the uniformizer, then

\[
\mathcal{Q}_{M,s}(T_1, \ldots, T_{d-1}, \pi_M^g) = \frac{1}{\tau(g^{-1})(b')^g \pi^g},
\]

but

\[
\frac{1}{\tau(g^{-1})(b')^g \pi^g} = \tau(g) \left( \frac{1}{b' \pi^g} \right)^g.
\]

This last expression is equal to \( \tau(g)^{\mathcal{Q}_{M,s}(T_1, \ldots, T_{d-1}, \pi_M)^g} \) by equation (6).

### 2.2. The canonical logarithmic derivatives \( \mathcal{Q}L_{M,s} \)

Let \( M \) be a finite extension of \( K_s \) and \( \mathcal{M} = M\{\{T_1\}\} \cdots \{\{T_{d-1}\}\} \).

**Definition 2.2.1.** From \( \mathcal{Q}_{M,s} \) we define the logarithmic derivative

\[
\mathcal{Q}L_{M,s} : K_d(\mathcal{M}) \to \frac{1}{\pi_M} \mathcal{P}_M
\]

by

\[
\mathcal{Q}L_{M,s}(u_1, \ldots, u_{d-1}, \pi_M) = \frac{\mathcal{Q}_{M,s}(u_1, \ldots, u_{d-1}, \pi_M)}{u_1 \cdots u_{d-1} \pi_M} \left( \text{mod} \frac{\pi^g}{\pi_M \mathcal{P}_M} \right),
\]

\[
\mathcal{Q}L_{M,s}(u_1, \ldots, u_d) = \frac{\mathcal{Q}_{M,s}(u_1, \ldots, u_d)}{u_1 \cdots u_d} \left( \text{mod} \frac{\pi^g}{\pi_M} \mathcal{P}_M \right),
\]

\[
\mathcal{Q}L_{M,s}(u_1, \ldots, u_{d-1}, u_{d}, \pi_M) = k \mathcal{Q}L_{M,s}(u_1, \ldots, u_{d}) + \mathcal{Q}L_{M,s}(u_1, \ldots, u_{d}), \quad k \in \mathbb{Z}
\]

\[
\mathcal{Q}L_{M,s}(a_1, \ldots, a_{d-1}) = 0, \quad \text{whenever } a_i = a_j \text{ for } i \neq j \text{ and } a_1, \ldots, a_{d-1} \in \mathcal{M}^*.
\]

where \( u_1, \ldots, u_d \) are in \( \mathcal{O}_M = \{ x \in \mathcal{O}_M : v_M(x) = 0 \} \).

Notice that the forth property says that \( \mathcal{Q}L_{M,s} \) is alternating, in particular it is skew-symmetric, i.e.,

\[
\mathcal{Q}L_{M,s}(a_1, \ldots, a_i, \ldots, a_j, \ldots, a_d) = -\mathcal{Q}L_{M,s}(a_1, \ldots, a_j, \ldots, a_i, \ldots, a_d),
\]

whenever \( i \neq j \).

Let us fix a finite extension \( L/K_n \). Let \( \gamma_m \) be a uniformizer for \( L_m = L(\kappa_m) \) and let \( L_m \) be the higher local field \( L_m\{\{T_1\}\} \cdots \{\{T_{d-1}\}\}, m \geq 1 \). From now on we will be using the following notation

\[
\begin{align*}
P_s &= P_{\mathcal{L},s}, \quad (s \geq 1) \\
Q_s &= Q_{\mathcal{L},s}, \quad Q_s L_{s} = Q_{\mathcal{L},s} \\
N_s &= N_{\mathcal{L},s}, \quad N_s = N_{\mathcal{L},s} \\
T_s &= T_{\mathcal{L},s}, \quad T_{s} = T_{\mathcal{L},s} \\
\end{align*}
\]

Notice that

\[
T_{s} / (1/\gamma_s) P_s \subset P_s \quad \text{and} \quad T_{s} / (1/\gamma_s) P_s \subset (1/\gamma_s) P_s.
\]

Indeed, from \( D(L \cap K) = D(L \cap L_s)D(L_s / K) \) we have that \( P_s = (1/D(L \cap L_s)) P_s \), thus

\[
T_{s} / (1/\gamma_s) P_s = T_{s} / (1/D(L \cap L_s)) P_s = P_s T_{s} / (1/D(L \cap L_s)) P_s \subset P_s, \quad \text{and} \quad T_{s} / (1/\gamma_s) P_s \subset T_{s} / (1/\gamma_s) P_s \subset P_s.
\]
Let \( r' \) be smallest positive integer such that \( L \cap K_r \subset K_{r'} \). Then

\[
L_s \cap K_r = (L \cap K_r)K_s = K_s \quad (s \geq r').
\]

**Proposition 2.2.1.** For any \( t \geq s \geq r' \) we have

\[
QL_s(N_{t/s}(a_1), a_2, \ldots, a_d) = \text{Tr}_{t/s}(QL_t(a_1, \ldots, a_d)) \quad (\in \frac{(1/\gamma_s)P_s}{(\pi^s/\pi_1\gamma_s)P_s})
\]

for \( a_1 \in L^*_t \) and \( a_2, \ldots, a_d \in L^*_s \).

**Proof of Proposition 2.2.1.** The proof follows the same ideas as in [15] Proposition 7.13. It will be enough to consider the case \( t = s + 1 \). From Proposition 2.1.2 it follows that

\[
QL_t\left( \prod_{g \in G(L_t/L_s)} a^g_1, a_2, \ldots, a_d \right) = \left( \sum \tau_t(g)g \right) QL_t(a_1, \ldots, a_d) = \left( \sum g \right) QL_t(a_1, \ldots, a_d) + \left( \sum (\tau_t(g) - 1)g \right) QL_t(a_1, \ldots, a_d).
\]

By Proposition 2.2.2 and Equation (10), we see that \( \sum (\tau_t(g) - 1)g \) takes \( \frac{\pi^s}{\pi_1}C \) to \( \frac{\pi^s}{\pi_1}P_sO_{L_t} \).

Then

\[
QL_s(N_{t/s}(a_1), a_2, \ldots, a_d) = QL_t(N_{t/s}(a_1), a_2, \ldots, a_d) \quad (\text{mod} \quad \frac{\pi^s}{\pi_1\gamma_s}P_s)
\]

\[
= \text{Tr}_{t/s}(QL_t(a_1, \ldots, a_d)),
\]

where the first equality follows from Proposition 2.1.1.

**Proposition 2.2.2.** Let \( M \) be a finite extension of \( K \) such that \( M \cap K_{\infty} = K_s \) and let \( N = M^{s+1} \). Put \( \mathcal{N} = N\{\{T_1\}\} \cdots \{\{T_{d-1}\}\} \) and \( \mathcal{M} = M\{\{T_1\}\} \cdots \{\{T_{d-1}\}\} \). Then

\[
\tau_{s+1}(G(N/M)) = \frac{1 + \pi^sC}{1 + \pi^{s+1}C} \subset (C/\pi^{s+1}C)^s
\]

where \( C = O_K \), and the element

\[
\sum_{g \in G(N/M)} (\tau_{s+1}(g) - 1)g
\]

takes \( \pi/(D(N/M))O_N \) to \( (\pi^{s+1}/\pi_1)O_N \). Also \( D(N/M)|\pi \).

**Proof.** This follows immediately from the fact that \( G(N/M) \cong G(N/M) \) and Proposition 5.12 in [15] and its proof.

\( \square \)
3. Generalized Artin-Hasse formulas

In this section we will give a version of Artin-Hasse formulas for the Kummer pairing over higher local fields. In the next section, we will use these results to deduce the main formulas.

We will assume that \( p \neq 2 \). Let \( \varrho \) is the ramification of index of \( K/Q_p \). Let \((k,t)\) be a pair of integers for which there exist a positive integer \( m \) such that \( t - k - 1 \geq m \varrho \geq k \), in other words: \((k,t)\) is an admissible pair as in Definition 2.3.1 of [8]. Also, let \( M = K_t \), \( M = \{\{T_1\} \cdots \{T_{d-1}\}\} \) and

\[
R_{M,1} = \{x \in M : v_M(x) \geq -v_M(D(M/Q)) - \left\lfloor \frac{v_M(p)}{p-1} \right\rfloor - 1\}.
\]

Since \( K_t/K \) is a totally ramified extension \( \pi_t = e_t \) will be a uniformizer for \( K_t \).

By Equation (44) of Definition 5.2.1 of [8] there exists a \( c_\beta = c_{\beta,i} \in R_{M,1}/\pi^k R_{M,1} \) such that

\[
\left(\{T_1, \ldots, T_{d-1}, u\}, e_t\right)_{M,K} = T_{M/K}\left(\log u c_\beta\right) \pmod{\pi^k C}.
\]

In the following proposition we will compute this constant \( c_\beta \) explicitly, in other words, we will show that

\[
c_\beta = -1/\pi^t \pmod{\pi^k R_{M,1}}.
\]

**Proposition 3.0.1.** For all \( u \in V_{M,1} = \{u \in O_M : v_M(u - 1) > v_M(p)/(p - 1)\} \) we have

\[
\left(\{T_1, \ldots, T_{d-1}, u\}, e_t\right)_{M,K} = T_{K_t/K}\left(c_{M/K_t}(\log u)\left(-\frac{1}{\pi^t}\right)\right) \pmod{\pi^k C},
\]

or equivalently,

\[
\left(\{T_1, \ldots, T_{d-1}, u\}, e_t\right)_{M,K} = T_{K_t/K}\left(c_{M/K_t}(\log u)\left(-\frac{1}{\pi^t}\right)\right)
\]

**Proof.** We start by observing that every \( u \in V_{M,1} \) can be expressed as

\[
\prod_{i = (i_1, \ldots, i_d) \in S, i_d \geq \left\lfloor v_M(p)/(p-1)\right\rfloor + 1} (1 + \theta; T_1^{i_1} \cdots T_{d-1}^{i_{d-1}} \pi_t^{i_d})
\]

where \( \theta; \in R, R \) is the group of \((q - 1)\)th roots of 1 in \( K_t^* \), and \( S \subset \mathbb{Z}^d \) is an admissible set (see Corollary from Section 1.4.3 of [7]). The convergence of this product is with respect to the Parshin topology of \( M^* \) (cf. [7] Chapter 1 §1.4.2.).

On the other hand, by Proposition 2.2.1 (6) of [8] the mapping

\[
M^* \rightarrow \kappa_k : u \rightarrow \left(\{u, T_2, \ldots, T_{d-1}\}, e_t\right)_{M,K}
\]

is continuous with respect to the Parshin topology \( M^* \). Then it is enough to check (14) for

\[
u = 1 + \theta T_1^{i_1} \cdots T_{d-1}^{i_{d-1}} \pi_t^{i_d}, \quad (\theta q^{-1} = 1).
\]

We are going to do this by cases:

Case 1) Suppose \((i_1, \ldots, i_{d-1}) \neq (0, \ldots, 0)\). Then the right hand side of (14) is zero since \( c_{M/K_t}(\log u) = 0 \). Let us show that the left hand side is also zero as well.

Suppose first \( i_r > 0 \) for some \( 1 \leq r \leq d-1 \). Consider \( N = K_t\{\{Y_1\}\} \cdots \{\{Y_{d-1}\}\} \) where \( Y_r = T_r^{i_r} \) and \( Y_m = T_m \), for \( m \neq r \). By lemma 3.0.1 below, \( M/N \) is a finite extension of degree \( i_r \) and \( N_M/M_N(T_r) = \pm Y_r \). Let also \( i'_r = 1 \) and \( i'_m = i_m \) for
$m \neq r$. To simplify the notation we will denote $T_1^{i_1} \cdots T_{d-1}^{i_{d-1}}$ and $Y_1^{i_1} \cdots Y_{d-1}^{i_{d-1}}$ by $T \bar{i}$ and $Y \bar{v}$, respectively. Therefore by [8] Proposition 2.2.1 (4)

\[
\left(\{T_1, \ldots, T_{d-1}, 1 + \theta T \bar{i}^{i_d}\}, e_t\right)_{\mathcal{M}, k} =
\]

\[
= \left(N_{\mathcal{M}/\mathcal{N}}(T_1, \ldots, T_{d-1}, 1 + \theta T \bar{i}^{i_d}), e_t\right)_{\mathcal{N}, k}
\]

\[
= \left(\{Y_1, T_1, \ldots, Y_{d-1}, 1 + \theta Y \bar{v} \pi_{i_d}^{i_d}\}, e_t\right)_{\mathcal{N}, k}
\]

\[
= \left(\{Y_1, \ldots, 1, \ldots, Y_{d-1}, 1 + \theta Y \bar{v} \pi_{i_d}^{i_d}\}, e_t\right)_{\mathcal{N}, k}
\]

\[
\oplus \left(\{Y_1, \ldots, Y_{d-1}, 1 + \theta Y \bar{v} \pi_{i_d}^{i_d}\}, e_t\right)_{\mathcal{N}, k}
\]

Since $p \neq 2$, \(\{Y_1, \ldots, 1, \ldots, Y_{d-1}, 1 + \theta Y \bar{v} \pi_{i_d}^{i_d}\}, e_t\)\right)_{\mathcal{N}, k} = 0. On the other hand, since $\theta^{q-1} = 1$ then

\[
\left(\{Y_1, \ldots, \bar{v}, \ldots, Y_{d-1}, 1 + \theta Y \bar{v} \pi_{i_d}^{i_d}\}, e_t\right)_{\mathcal{N}, k}
\]

\[
= [1/(q-1)] \left(\{Y_1, \ldots, \bar{v}, \ldots, Y_{d-1}, 1 + \theta Y \bar{v} \pi_{i_d}^{i_d}\}, e_t\right)_{\mathcal{N}, k} = 0
\]

and also \(\{Y_1, \ldots, \pi, \ldots, Y_{d-1}, 1 + \theta Y \bar{v} \pi_{i_d}^{i_d}\}, e_t\)\right)_{\mathcal{N}, k} = 0 by the norm series relation for Lubin-Tate formal groups (\(\{a_1, \ldots, \bar{v}, \ldots, a_d\}, X\))\right)_{\mathcal{N}, k} = 0; recalling that 

\[
\pi_t = e_t. Thus
\]

\[
\left(\{T_1, \ldots, T_{d-1}, 1 + \theta T \bar{i}^{i_d}\}, e_t\right)_{\mathcal{M}, k}
\]

\[
= \left(\{Y_1, \ldots, T, \ldots, Y_{d-1}, 1 + \theta Y \bar{v} \pi_{i_d}^{i_d}\}, e_t\right)_{\mathcal{N}, k}
\]

\[
= \left(\{Y_1, \ldots, \theta Y \bar{v} \pi_{i_d}^{i_d}, \ldots, Y_{d-1}, 1 + \theta Y \bar{v} \pi_{i_d}^{i_d}\}, e_t\right)_{\mathcal{N}, k}
\]

\[
= \left(\{Y_1, \ldots, -\theta Y \bar{v} \pi_{i_d}^{i_d}, \ldots, Y_{d-1}, 1 + \theta Y \bar{v} \pi_{i_d}^{i_d}\}, e_t\right)_{\mathcal{N}, k}
\]

The second equality follows from the fact that \(\{Y_1, \ldots, Y_m, \ldots, Y_{d-1}, 1 + \theta Y \bar{v} \pi_{i_d}^{i_d}\} \) is trivial, for $m \neq r$, in the Milnor K-group $K_d(\mathcal{M})$. Moreover, the last expression in the chain of equalities is again zero because \(\{Y_1, \ldots, -\theta Y \bar{v} \pi_{i_d}^{i_d}, \ldots, Y_{d-1}, 1 + \theta Y \bar{v} \pi_{i_d}^{i_d}\} \) is the zero element, by the Steinberg property, in the Milnor K-group $K_d(\mathcal{M})$.

Suppose now \(i_r < 0\). We take $T_r = T_{k}^{-i_r}$ instead and by lemma 3.0.1 we have $N_{\mathcal{M}/\mathcal{N}}(T_{k}^{-i_r}) = \pm T_{r}^{-i_r} = \pm Y_r$. Noticing that

\[
\left(\{T_1, \ldots, T_r, \ldots, T_{d-1}, 1 + \theta T \bar{i}^{i_d}\}, e_t\right)_{\mathcal{M}, k}
\]

\[
= - \left(\{T_1, \ldots, T_{r}^{-1}, \ldots, T_{d-1}, 1 + \theta T \bar{i}^{i_d}\}, e_t\right)_{\mathcal{M}, k}
\]

we can now apply the same argument as before to conclude that

\[
\left(\{T_1, \ldots, T_r, \ldots, T_{d-1}, 1 + \theta T \bar{i}^{i_d}\}, e_t\right)_{\mathcal{M}, k} = 0.
\]
Case 2) Suppose \((i_1, \ldots, i_{d-1}) = (0, \ldots, 0)\). That is, when \(u\) is an element of the one dimensional local field \(K_t\). In this case, we will show in lemma 3.0.2 below that the pairing \((\{T_1, \ldots, T_{d-1}, u\}, \epsilon_i)_{M,k}\) coincides with the pairing taking values in the one dimensional local field \(K_t\), namely \((u, \epsilon_i)_{K_t,k}\). Thus, by [15] section 7.3.1 and the fact that \(c_{M/K}(\log(u)) = \log(u)\) formula (14) follows. □

Lemma 3.0.1. Let \(M\) be a complete discrete valuation field and \(M = M\{\{T\}\}\). Put \(Y = T^j\) for \(j > 0\). Define \(N = M\{\{Y\}\}\). Then \(M/N\) is a finite extension of degree \(j\) and \(N_{M/N}(T) = \pm Y\).

Remark: Since \(M\) is a complete discrete valuation field the result immediately generalizes to the \(d\)-dimensional case, for if \(L = L\{\{T_1\}\} \cdots \{\{T_{d-1}\}\}\), then we can take \(M\) to be \(L\{\{T_1\}\} \cdots \{\{T_{d-2}\}\}\) and apply the result to \(M = M\{\{T_{d-1}\}\}\) and \(N = M\{\{Y_{d-1}\}\}\), where \(Y_{d-1} = T_{d-1}^d\).

Proof. We can assume that \(M\) contains \(\zeta_j\), a primitive \(j\)th root of unity of 1. Otherwise we can considering the diagram

\[
\begin{array}{ccc}
M(\zeta_j)\{\{T\}\} & & M(\zeta_j)\{\{Y\}\} \\
\downarrow & & \downarrow \\
M = M\{\{T\}\} & & M = M\{\{Y\}\} \\
\downarrow & & \\
N = M\{\{Y\}\}
\end{array}
\]

we see that \([M/N] = [M(\zeta_j)\{\{T\}\} : M(\zeta_j)\{\{Y\}\}]\) since \(M\{\{T\}\}\cap(M(\epsilon_i)\{\{T\}\}) = M\{\{Y\}\}\).

Note that \(Y\) has exact order \(j\) in \(N^*/(N^*)^j\) for if \(Y = \alpha^k\), \(\alpha \in N^*\), then \(0 = v_N(Y) = kv_N(\alpha)\), thus \(\alpha \in O_N\) and we can go to the residue field \(k_N\) of \(N\) where we have \(1 = v_{k_N}(Y) = kv_{k_N}(\alpha)\), which implies \(k = 1\). Then by Kummer theory (cf. [2] Chapter 3 Lemma 2) we have that the polynomial \(P(X) = X^j - Y \in O_N[X]\) is irreducible. Thus \([M/N] = j\), and \(N_{M/N}(T)\) is the product of the roots of the polynomial \(P(X)\). These roots are \(\zeta_j^k T\), \(k = 1, \ldots, j\). Thus

\[
N_{M/N}(T) = \prod_{k=1}^j \zeta_j^k T = \zeta_j^{\frac{j(j+1)}{2}} T^j = \begin{cases} T^j = Y, & \text{if } j \text{ is odd,} \\ -T^j = -Y, & \text{if } j \text{ is even.} \end{cases} = (-1)^{j+1} Y.
\]

Lemma 3.0.2. For a local field \(L/\mathbb{Q}_p\), let \(L = L\{\{T_1\}\} \cdots \{\{T_{d-1}\}\}\) and define the map

\[
(15) \quad f : L^* \to \text{Gal}(L^{ab}/L) \quad \text{by} \quad a \mapsto \gamma_L(\{T_1, \ldots, T_{d-1}, a\})|_{\text{Gal}(L^{ab}/L)},
\]

induced by the natural restriction \(\text{Gal}(L^{ab}/L) \to \text{Gal}(L^{ab}/L)\). Then \(f\) coincides with the Artin’s local reciprocity map for \(L\): \(\theta_L : L^* \to \text{Gal}(L^{ab}/L)\). Thus, for
Let \( L = K_t \) and \( M = K_t \{ \{ T_1 \} \} \cdots \{ \{ T_{d-1} \} \} \) we have
\[
\{ T_1, \ldots, T_{d-1}, u \}, \quad (u, e_t)_{M, k} = (u, e_t)_{K_t, k},
\]
for all \( u \in V_{L, 1} = \{ x \in L : v_L(x - 1) > v_L(p)/(p - 1) \} \). Here \((u, e_t)_{K_t, k}\) denotes the Kummer pairing of the (one-dimensional) local field \( K_t \).

**Proof.** It is enough to verify the two conditions of [2] Chapter 5 §2.8 Proposition 6. Let \( \mathcal{L}(d) \) be \( \mathcal{L} \), and \( \mathcal{L}(d-1) = k_{\mathcal{L}(t_1)} \cdots (t_{d-1}), \ldots, \mathcal{L}(0) = k_{\mathcal{L}} \) the chain of residue fields of \( \mathcal{L} \). Let \( \partial: K_m(\mathcal{L}(m)) \to K_{m-1}(\mathcal{L}(m-1)) \), \( m = 2, \ldots, d \), be the boundary map defined in Chapter 6, Section 6.4.1 of [7]. (See also [6] IX §2). Then by Theorem 3 of Section 10.5 of [7] the following diagram commutes:

\[
\begin{array}{ccc}
K_d(\mathcal{L}(d)) & \xrightarrow{\gamma(\mathcal{L}(d))} & \text{Gal}(\mathcal{L}^{ab}_{(d)}/\mathcal{L}(d)) \\
\downarrow \partial & & \downarrow \sigma \to \sigma \\
K_{d-1}(\mathcal{L}(d-1)) & \xrightarrow{\gamma(\mathcal{L}(d-1))} & \text{Gal}(\mathcal{L}^{ab}_{(d-1)}/\mathcal{L}(d-1)) \\
& \vdots & \vdots \\
& \downarrow \partial & \downarrow \sigma \to \sigma \\
\mathbb{Z} = K_0(\mathcal{L}(0)) & \xrightarrow{\gamma(\mathcal{L}(0))} & \text{Gal}(\mathcal{L}^{ab}_{(0)}/\mathcal{L}(0))
\end{array}
\]

Moreover, by Section 6.4.1 of [7] (See also [12] Section 1, Theorem 2) the composition of the vertical maps \( \partial \circ \cdots \circ \partial(\{ T_1, \ldots, T_{d-1}, a \}) \) coincide with the valuation \( v_L(a) \) for \( a \in L^* \). Therefore \( f: L^* \to \text{Gal}(L^{ab}/L) \to \text{Gal}(L^{un}/L) \) is the valuation map \( v_L: L^* \to \mathbb{Z} \). Thus condition (1) of [2] Chapter 5 §2.8 Proposition 6 is verified.

Suppose \( L'/L \) is a finite abelian extension and let \( \mathcal{L}' = L'\{ \{ T_1 \} \} \cdots \{ \{ T_{d-1} \} \} \). If \( a \in L^* \) is a norm from \( (L')^* \), namely \( a = N_{L'/L}(\alpha) \), then clearly \( \{ T_1, \ldots, T_{d-1}, a \} \) is a norm from \( K_d(\mathcal{L}') \), namely
\[
\{ T_1, \ldots, T_{d-1}, a \} = \{ T_1, \ldots, T_{d-1}, N_{L'/L}(\alpha) \} = N_{L'/L}(\{ T_1, \ldots, T_{d-1}, a \}).
\]

Then by (1) of Theorem 2.1.1 of [8] we have that \( \gamma_{\mathcal{L}}(T_1, \ldots, T_{d-1}, a) \) is trivial on \( \mathcal{L}' \) and so \( f(a) \) is trivial on \( L' \). Thus condition (2) of [2] Chapter 5 §2.8 Proposition 6 is verified. \( \square \)

### 4. Generalized Kolyvagin formulas

In this subsection we will provide a refinement of the formulas given for Theorem 5.3.1. of [8] to the case of a Lubin-Tate formal group \( F_f \).

Using the notation in (8) we define
\[
K_d(\mathcal{L}_s)' = \bigcap_{t \geq s} N_{t/s}(K_d(\mathcal{L}_t)).
\]

Also let \( r \) maximal, and \( r' \) minimal, such that
\[
K_r \subset L \cap K_\infty \subset K_{r'},
\]
Theorem 4.0.1. Take \( s \geq \max\{r', n + r + \log_q(e(L/K))\} \). Then \( \text{Tr}_s \) takes \((\pi^s/\pi_1\gamma_s)P_s \) to \( \pi^nR_L \) so that it induces a homomorphism

\[
\text{Tr}_s : \frac{1}{\gamma_t}P_s \frac{1}{\pi_1\gamma_s} \rightarrow L/\pi^nR_L
\]

and the following formula holds

\[
(\alpha, x)_{L,n} = \left[ \mathbb{T}_L/K \left( \text{Tr}_s \left( QL_s(\alpha) l_F(x) \right) \right) \right] F(e_n)
\]

all \( x \in F(\mu_L) \) and all \( \alpha \in K_d(L_s) \).

Proof. Let \( M = L_t \) and \( M = M\{\{T_1\}\} \cdots \{\{T_{d-1}\}\} \). The strategy of the proof is the following: First, we will show that \( QL_s \) coincides with the logarithmic derivative \( \mathcal{D} \mathcal{O} \mathcal{M},m \) from Theorem 5.3.1 of [8]; where \( t, k \) and \( m \) are as in the statement of that Theorem. Then, we will descend to the logarithmic derivative \( QL_s \), for \( s \geq \max\{r', n + r + \log_q(e(L/K))\} \), using the fact that \( \text{Tr}_s((\pi^s/\pi_1\gamma_s)P_s) \subset \pi^nR_L \); this fact will be shown at the end of the proof.

Let \( v \) be the normalized valuation \( v_M/v_M(p) \) and let \( R_{t,1} = R_{M,1} \). Since \( K_1/K \) is totally ramified and \( [K_1/K] = q - 1 \) then \( \pi_1^{q-1} \sim \gamma_t^{\epsilon(M)} \), so \( \pi_1 \) divides \( \gamma_t^{\epsilon(M)/(p-1)} \) and we have

\[
\frac{1}{\gamma_t}P_t = \frac{1}{\gamma_t\pi_1 D(L_t/K)}\mathcal{O}_{L_t} \subset \frac{1}{\gamma_t^{\epsilon(M)/(p-1)+1}} D(L_t/K)\mathcal{O}_{L_t} = R_{t,1}.
\]

If \( k < t \), then \( \pi_1^{k-t} \pi_t \gamma_t \), which implies \((\pi^k/\gamma_t)|((\pi^t/\pi_1))\) and so

\[
\pi_t^{k-t} \pi_t \gamma_t \subset \pi_t\gamma_t^{k}R_{t,1}.
\]

Consider \( t, k \) and \( m \) as in Theorem 5.3.1 of [8]. In particular, since \( t = 2k+q+1 \) then \( k < t \) and therefore we can look at the factorization of \( QL_t : (\mathcal{O}_M)^d \rightarrow P_t/((\pi^t/\pi_1))P_t \) into \( R_{t,1}/(\pi^k/\gamma_t)R_{t,1} \). This is a \( d \)-dimensional derivation such that

\[
QL_t(T_1, \ldots, T_{d-1}, e_t) = T_1 \cdots T_{d-1}/\pi^t l'(e_t) = -T_1 \cdots T_{d-1} c_\beta l'(e_t),
\]

by Proposition 3.0.1, where \( c_\beta \) is the constant from Equation 12. But from Lemma 5.2.1 and Proposition 5.2.3 of [8] we know that all the derivations satisfying such condition must coincide when reduced to \( R_{t,1}/(\pi^m/\gamma_t)R_{t,1} \). Therefore the reduction of \( QL_t \) to \( R_{t,1}/(\pi^m/\gamma_t)R_{t,1} \) coincides, after Theorem 5.3.1 of [8], with the derivation \( \mathcal{D} \mathcal{O} \mathcal{M},m \) from Definition 5.2.1 of [8]. This allows us to replace the logarithmic derivative \( \mathcal{D} \mathcal{O} \mathcal{M},m \) from Definition 5.2.1 of [8] by the logarithmic derivative

\[
QL_t : K_d(M) \rightarrow \frac{1}{\gamma_t}P_t \frac{1}{\pi_1\gamma_s}P_s
\]

defined in (7). Then for \( \epsilon \in K_d(L_t) \) we have

\[
(\text{Tr}_{M/L}(\psi_{M,m}^t(\epsilon))) = \text{Tr}_s(\epsilon) \mod \pi^nR_L.
\]

In particular, if we pick \( \epsilon \) of the form

\[
\{a_1, a_2, \ldots, a_d\} \in K_d(L_t) \quad \text{with} \quad a_2, \ldots, a_d \in L_s^*.
\]

then by Proposition 2.2.1 we know that for \( s \geq r' \) the following identity holds

\[
(\text{Tr}_{M/L}(\psi_{M,m}^t(\epsilon))) = \text{Tr}_{t/s}(\epsilon) \mod \pi_t^{s} \pi_1\gamma_s P_s.
\]
Suppose for the moment that \( \text{Tr}_s((\pi^s/\pi_1 \gamma_s) P_s) \subset \pi^n R_L \), then taking \( \text{Tr}_s \) we get
\[
\text{Tr}_s(Q_L t(e)) = \text{Tr}_s \left( \text{Tr}_{t/s}(Q_L t(e)) \right) = \text{Tr}_s \left( Q_L (N_{t/s}(\epsilon)) \right) \mod \pi^n R_L
\]
Therefore if \( \alpha \in K_d(L_i)' \) then in particular \( \alpha \in N_{t/s}(K_d(L_i)) \). According to Lemma 4.0.2 there exists an \( \epsilon \in K_d(L_i) \) of the form (18) such that \( N_{t/s}(\epsilon) = \alpha \) and thus, by (17) and Proposition 5.1.2. of [8], identity (16) follows.
Therefore it remains to prove that \( \text{Tr}_s((\pi^s/\pi_1 \gamma_s) P_s) \subset \pi^n R_L \) for \( s \geq n + r + \log_q(e(L/K)) \). Let \( x \in F(\mu_L) \). Then \( f(x) \equiv x^q \mod (\pi x) \) implies
\[
(19) \quad v(f(x)) \geq \min \{v(x^q), v(\pi x)\}.
\]
Let
\[
v(z) = \log_q \left( \frac{e(L/K)}{q-1} \right) + 1 = r + \log_q(e(L/K)) \cdot
\]
Then \( v(f(z)(x)) \geq \min \{v(x^{q-1}), v(\pi x)\} \). But
\[
v(x^{q-1}) = \pi^{q-1} v(x) = \frac{e(L/K)}{q-1} e(L/K) q \geq \frac{1}{q(q-1)} = \frac{v(\pi)}{q-1},
\]
so \( v(f(\pi)(x)) \geq v(\pi)/(q-1) \). Thus by equation (19) applied to \( x = f(\pi)(x) \) we have
\[
v(f(\pi)(x)) \geq \min \left\{ v \left( \left( f(\pi)(x) \right)^q \right), v(\pi f(\pi)(x)) \right\} \geq \left( 1 + \frac{1}{q-1} \right) v(\pi) \geq \frac{v(\pi)}{q-1}
\]
By Lemma 4.0.1 below, we have \( v(l_F(f(z)(x))) = v(f(z)(x)) \), and since \( v(\pi)/(q-1) = v(\pi_1) \), then
\[
v(\pi^{q-1} l_F(x)) \geq (1 + 1/(q-1)) v(\pi) = v(\pi) + v(\pi_1).
\]
This implies \((\pi^{q-1}/\pi_1) T_L \subset O_L \), where \( T_L = l_F(F(\mu_L)) \). Taking duals with respect to \( T_{\mu/L} \) we get
\[
\frac{\pi^{q-1}}{\pi_1} \frac{1}{D(L/K)} O_L \subset R_L.
\]
Since \( \text{Tr}_s(P_s) \subset P_L \), by equation (9), then \( \text{Tr}_s \) takes
\[
\frac{\pi^s}{\pi_1 \gamma_s} P_s \subset \frac{\pi^s}{\pi_1 \pi_L} P_s
\]
to
\[
\frac{\pi^s}{\pi_1 \pi_L} P_L = \frac{\pi^s}{\pi_1 \pi_L} \frac{1}{D(L/K)} O_L \subset \frac{\pi^s q^{q-1} + 1}{\pi_1 \pi_L} R_L
\]
Noticing now that \( q \neq 2 \), since \( p \neq 2 \), implies that \( \pi_1 \pi_L \pi \), because \( v_L(\pi/\pi_1 \pi_L) = e(L/L_1)(q-2) - 1 \). Thus, if \( s \geq n + s' \) then \( \text{Tr}_s \) takes \((\pi^s/\pi_1 \gamma_s) P_s \) to \( \pi^n R_L \) and we conclude the theorem.

\[\square\]

**Lemma 4.0.1.** For every element \( w \in F(\mu_L) \) such that
\[
v(w) > v(\pi)/(q-1),
\]
we have that
\[
v(l_F(w)) = v(w).
\]
Proof. The inequality \( v(w) > v(\pi)/q - 1 \) is equivalent to \( v(w^q) > v(\pi) + v(w) \), and since \( f(x) \equiv x^q \pmod{\pi x} \) this implies \( v(f(w)) = v(\pi w) \). Thus \( v(f^{(n)}(w)) = v(\pi^n w) \) and therefore we can take \( n \) large enough such that \( v(f^{(n)}(w)) > 1/(q - 1) \). By Proposition 6.2.3 of [8] we have that \( v(l_F(f^{(n)}(w))) = v(f^{(n)}(w)) \). Thus \( v(\pi^n l_F(w)) = v(\pi^n w) \), and this proves the lemma. \( \square \)

Lemma 4.0.2. If \( M/L \) is an abelian extension of \( d \)-dimensional local fields, then the group \( N_{M/L}(K_d(L)) \) is generated by all the elements of the form \( \{a_1, \ldots, a_d\} \in K_d(L) \) with some \( a_i \in N_{M/L}(M^*) \).

Proof. Let \( \mathcal{S} \) be the subgroup of \( K_d(L) \) generated by the elements of the form \( \{a_1, \ldots, a_d\} \in K_d(L) \) with some \( a_i \in N_{M/L}(M^*) \). Suppose \( a_i = N_{M/L}(b_i) \), \( b_i \in M^* \). Then by Proposition (4) of [8] we know that

\[
\{a_1, \ldots, N_{M/L}(b_i), \ldots, a_d\} = N_{M/L}(\{a_1, \ldots, b_i, \ldots, a_d\}).
\]

This implies

\[
\mathcal{S} \subset N_{M/L}(K_d(M)) \subset K_d(L).
\]

We know that the index of \( N_{M/L}(K_d(M)) \) in \( K_d(L) \) is \( r := [M : L] \); this follows, for example, from the main statement of higher class field theory (cf. [8] Theorem 2.1.1. (1)). On the other hand, it is clear that \( r \geq |K_d(L) : \mathcal{S}| \), since for \( \{a_1, \ldots, a_d\} \in K_d(L) \) we have

\[
\{a_1, \ldots, a_d\}^r = \{a_1^r, \ldots, a_d\} = \{N_{M/L}(a_1), \ldots, a_d\} \in \mathcal{S}
\]

Therefore we conclude that \( \mathcal{S} = N_{M/L}(K_d(M)) \). \( \square \)

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