Fair Division Minimizing Inequality

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Abstract

Behavioural economists have shown that people are often averse to inequality and will make choices to avoid unequal outcomes. In this paper, we consider how to allocate indivisible goods fairly so as to minimize inequality. We consider how this interacts with axiomatic properties such as envy-freeness, Pareto efficiency and strategy-proofness. We also consider the computational complexity of computing allocations minimizing inequality. Unfortunately, this is computationally intractable in general so we consider several tractable greedy online mechanisms that minimize inequality. Finally, we run experiments to explore the performance of these methods.

Introduction

In resource allocation, one of the most frequently used normative measures of fairness is envy-freeness (no agent envies another’s allocation). Unfortunately, when the resources are indivisible, envy-free allocations may not exist. In addition, computing an envy-free allocation when it exists is computationally intractable. Another desirable property in resource allocation is Pareto efficiency. In contrast to envy-free allocations, Pareto efficient allocations always exist and can be computed quickly. However, Pareto efficient allocations may not be very fair (e.g. giving all items to a single agent is Pareto efficient). We consider here whether minimizing the inequality between agents offers an alternative to envy-freeness and Pareto efficiency for the fair division of indivisible items. A number of different measures of inequality have been proposed in economics (e.g. Gini, 1912; Atkinson, 1970; Hoover, 1936). We focus on the Gini index as it has been commonly used in many other settings. However, it would be interesting to consider other measures such as the Atkinson, and Hoover (aka Robin Hood) indices.

Our results: We start our paper with a motivating example. We consider three normative inequality measures for fair division: the Gini index, the subjective Gini index and the envy index. These three indices measure the quality of allocations and mechanisms between perfect equitability and envy-freeness. Unlike envy-free allocations which may not exist, allocations that minimize these three measures always exist. We study the relationship between the Gini, subjective Gini and envy indices and envy-freeness, Pareto efficiency and strategy-proofness. For example, we show that there are fair division problems when none of the envy-free allocations minimizes the inequality indices. We further study the complexity of computing allocations minimizing each of these indices. Unfortunately, most of these computational problems are intractable. For this reason, we propose three tractable online mechanisms that allocate each item in a given sequence thus minimizing the three inequality indices without the knowledge of the future items in the sequence. We finally run experiments with these online mechanisms.

Formal background

We consider a fair division problem with $n$ agents and $m$ indivisible items. Each agent has some private cardinal utility $u_i(o_j) \in \mathbb{Q}^{\geq 0}$ for each item $o_j$ but can submit a public cardinal bid $v_i(o_j) \in \mathbb{Q}^{\geq 0}$ for each item $o_j$. An instance of a fair division problem thus has (1) agents $a_1, \ldots, a_n$, (2) indivisible items $o_1, \ldots, o_m$, and (3) a bid matrix $(v_i(o_j))_{n \times m}$. Let $A$ be an allocation of items to agents. We write $A_i$ for the bundle of items allocated to agent $i$, and $u_i(B)$ for the utility to agent $a_i$ of the items in the bundle $B$. We assume additive utilities. That is, $u_i(B) = \sum_{o_j \in B} u_i(o_j)$. In economics, incomes and wealth are additive for the population. Also, in a food bank, donated products are additive for the bank. Additivity offers an elegant compromise between simplicity and expressivity in our model as well as in many other theoretical models (e.g. Bevia, 1998; Brams et al., 2003; Chevaleyre et al., 2008; de Keijzer et al., 2009; Lesca and Permy, 2010).

We consider welfare, fairness and efficiency notions. The utilitarian welfare of $A$ is equal to $\sum_{i \in [1,n]} u_i(A_i)$. The egalitarian welfare of $A$ is equal to $\min_{i \in [1,n]} u_i(A_i)$. An allocation $A$ is envy-free iff $u_i(A_i) \geq u_i(A_j)$ for every $i,j$. An allocation $A$ is Pareto efficient iff there is no other allocation $A'$ such that $\forall i : u_i(A'_i) \geq u_i(A_i)$ and $\exists k : u_k(A'_k) > u_k(A_k)$. We further consider only responsive mechanisms that compute an allocation of items to agents based on their positive bids. A desirable property of mechanisms is that they cannot be manipulated. A mechanism is strategy-proof if, for each instance, an agent cannot increase their utility by misreporting their bids. We are interested in properties of the actual ex post outcomes returned by mechanisms.
One of the most frequently used measures of inequality is the Gini index. It is commonly used to measure inequality in income or wealth. The Gini index satisfies a number of desirable properties such as anonymity, scale independence, population independence, and the transfer principle (inequality reduces when we take from the rich and give to the poor). We will use it here to measure inequality between agents in the utility of the items allocated to them. More precisely, the Gini index of an allocation equals half of the relative mean absolute difference in utilities of the agents.

\[
\text{Gini} = \frac{\sum_{i=1}^{n} \sum_{j=1}^{n} |u_i(A_i) - u_j(A_j)|}{2 \sum_{i=1}^{n} \sum_{j=1}^{n} u_i(A_i)} = \frac{\sum_{i=1}^{n} \sum_{j=1}^{n} |u_i(A_i) - u_j(A_j)|}{2n \sum_{i=1}^{n} u_i(A_i)}
\]

The Gini index lies in the interval [0,1], taking the value 0 when all \( n \) agents get the same utility, and 1 \(-\frac{1}{n}\) when all but one agent get zero utility. In a plot of the cumulative distribution, the Gini index measures the area that lies between the line of equality (i.e. all \( n \) agents get the same utility) and the Lorenz curve [Endriss, 2013].

A motivating example

A simple example provides some motivation. Suppose Alice, Bob and Carol arrive at the car hire office and are offered to rent a Renault, a Skoda, or a Toyota car. Alice knows that Skoda’s share their mechanicals with VW, and likes reliable German cars, so she prefers the Skoda most. Bob is torn between the Skoda and the more unusual Renault. And Carole loves quirky cars, so has a strong preference for the Renault. She is also an environmentalist, so dislikes VW and has a strong preference against the Skoda. Their precise utilities for the different cars are given in the following table. Who gets what car?

|       | Renault | Skoda | Toyota |
|-------|---------|-------|--------|
| Alice | 1       | 8     | 3      |
| Bob   | 8       | 7     | 1      |
| Carol | 18      | 1     | 8      |

There is no envy-free allocation. Bob and Carol both most prefer the Renault and only one of them can get it. The allocation with the least amount of envy (either of one person for another or in total) allocates the Renault to Carol, the Skoda to Bob and the Toyota to Alice. This is also the optimal allocation from a welfare perspective with both the maximum utilitarian and egalitarian welfare. However, Alice might not consider this allocation fair as she gets less than half the utility of Bob or Carol, as well as less than half the utility of her most preferred car, whilst Carol gets her most preferred car and Bob gets a car with value close to his maximum utility.

We might decide instead that it is fairer to choose from amongst those allocations which minimize the inequality between Alice, Bob and Carol. For instance, allocating the Renault to Bob, the Skoda to Alice and the Toyota to Carole is such an allocation. Everyone gives their car the same 8 units of utility. This allocation is Pareto efficient and has a Gini index of zero, the minimum possible. In this allocation, only Carol envies Bob, but since she gets as much utility for her car as both Alice and Bob get for their cars, this might be acceptable.

Note that there is another allocation that minimizes inequality. Allocating the Renault to Alice, the Skoda to Carol and the Toyota to Bob gives everyone the same unit of utility. This also has a Gini index of zero. However, everyone now has their least preferred car, and everyone envies everyone else. Moreover, this allocation is not Pareto efficient and has the minimal welfare possible, both from the utilitarian and egalitarian perspective.

To sum up, this example suggests that whilst the Gini index can help in choosing between allocations, we cannot minimize inequality alone. Amongst allocations that minimize inequality, we might look to maximize welfare, minimize envy, etc. Minimizing inequality does, however, have an advantage over envy-freeness as a primary measure of fairness. An allocation of indivisible items minimizing inequality always exists whilst an envy-free allocation may not.

The subjective Gini index

As remarked earlier, the Gini index is typically used to measure inequality in income and wealth distribution. However, we are concerned here with the distribution of indivisible items not money, and importantly agents can have different subjective utilities for these items. For example, the utility you get for an item is not necessarily the same as the utility I get for it.

Should it increase the “inequality” of an allocation that someone else gets an item they value when you have little or even no value for it? To return to our motivating example, suppose Alice gets the Renault, Bob gets the Toyota, and Carol gets the Skoda. Everyone gets 1 unit of utility so this allocation has a Gini index of zero. But from everyone’s subjective perspective, this is not a very equitable allocation of items. For instance, from Alice’s perspective, rather than the 1 unit of utility she gets, she would get 8 units of utility for Carol’s car and 3 for Bob’s. And from Bob’s perspective, rather than the 1 unit of utility he gets, he would get 8 units of utility for Alice’s car and 7 for Carol’s.

We propose the subjective Gini index to take such differences into consideration. We modify the definition of the Gini index to sum the difference in utility an agent has for its allocation of the same agent has for the allocation of items to other agents.

\[
\text{subjective Gini} = \frac{\sum_{i=1}^{n} \sum_{j=1}^{n} |u_i(A_i) - u_i(A_j)|}{2 \sum_{i=1}^{n} \sum_{j=1}^{n} u_i(A_j)}
\]

Like the Gini index, the subjective Gini index is between [0,1] taking the value 0 when each agent gives the same utility to each bundle of items, and 1 \(-\frac{1}{n}\) when one agent gets all items. Returning again to our motivating example, the allocation in which each agent gets 1 unit of utility has a Gini index of 0 but a subjective Gini index of 23/55 (\(=0.41818181818\)). The allocation in which each agent gets 8 units of utility might be more preferred as it has a lower subjective Gini index of 37/110 (\(=0.33636363636\)).
The envy index

Minimizing the subjective Gini index will find allocations which divide the items into bundles so that each bundle has similar utility for each agent. This reminds us of a fairness concept such as the maximin share when each agent’s utility should be at least as high as the agent can guarantee by dividing the items into as many bundles as there are players and receiving their least desirable bundle [Budish, 2011].

On the plus side, an allocation which minimizes the subjective Gini index always exists, unlike maximin fair shares [Procaccia and Wang, 2014]. On the negative side, such an allocation may not be envy-free. To overcome this, we propose also an envy index whose definition is closely related to that of the subjective Gini index. This new index is focused on the amount of envy in an allocation. Minimizing this index will return an envy-free allocation when it exists.

$$\text{envy} = \frac{\sum_{i=1}^{n} \sum_{j=1}^{n} \max\{0, u_i(A_j) - u_i(A_i)\}}{\sum_{i=1}^{n} \sum_{j=1}^{n} u_i(A_j)}$$

The envy index is between [0, 1] taking the value 0 when the allocation is envy-free, and tending towards 1 as we increase the number of agents and allocate all items to just one agent. It is easy to see that the envy index is never greater (and sometimes smaller) than the subjective Gini index. Returning to our motivating example, the unique allocation minimizing the envy with index of 6/110 (=0.054545454545) allocates the Renault to Carol, the Skoda to Bob and the Toyota to Alice. As we noted, this is also the optimal allocation from a welfare perspective with both the maximum utilitarian and egalitarian welfare.

Relationship to envy-freeness

We consider how these indices relate to a fairness concept such as envy-freeness. Suppose that an envy-free allocation exists. Clearly, such an allocation minimizes the envy index. On the other hand, envy-free allocations may not minimize the Gini or subjective Gini indices.

**Theorem 1** There exist problems with envy-free allocations on which no envy-free allocation minimizes the Gini or subjective Gini index.

**Proof.** Consider 2 agents and 2 items. Suppose the first agent gives the first item a utility of 1 and the second a utility of 2, whilst the second agent gives utilities of 3 and 1 respectively. The only envy free allocation gives the first item to the second agent and the second item to the first agent. However, the unique allocation that minimizes the Gini index gives the first item to the first agent and the second item to the second agent. In this allocation, both agents envy each other.

Consider 3 agents and 3 items. Suppose the first agent has a utility of 9, 1 and 5 for the items respectively, the second agent has a utility of 5, 9 and 1 respectively, and the third agent has a utility of 1, 5, and 9 respectively. Then the unique envy-free allocation gives each agent their most valued item. However, the unique allocation that minimizes the subjective Gini index gives each agent their second most preferred item, i.e. the one they value with utility of 5.

The examples in the proof of Theorem 1 critically depend on the agents not sharing common utilities for items. When utilities are common, there is no incompatibility between envy-freeness and minimizing the Gini or subjective Gini indices. If an allocation is envy-free and agents have common utilities, then every agent must get the same utility for every bundle of items.

**Observation 1** With common utilities, an allocation is envy-free iff the Gini and subjective Gini indices are zero.

Relationship to Pareto efficiency

Another fundamental notion in fair division is Pareto efficiency. We would prefer allocations where no agent can improve their outcome without making others worse off. Pareto efficiency is not necessarily compatible with minimizing inequality. The first example in the proof of Theorem 1 shows that Pareto efficiency and the Gini index are incompatible. This should perhaps not be surprising as other fairness properties are also incompatible with Pareto efficiency. For example, an allocation that is envy free may not necessarily be Pareto efficient. Moreover, each envy-free allocation can be Pareto dominated only by allocations that are not envy-free [de Keijzer et al., 2009]. It follows quickly that minimizing the envy index is not compatible with Pareto efficiency. We can show that the same is true for the subjective Gini index.

**Theorem 2** There exist problems on which no Pareto efficient allocation minimizes the subjective Gini index.

**Proof.** Consider 2 agents and 4 items. Suppose the first agent gives items $o_1, o_3$ a utility of 1, item $o_2$ a utility of $2 - \epsilon$ and $o_4$ a utility of $\epsilon$, whilst the second agent gives utilities of $2 - \epsilon$, 1, $\epsilon$, 1 to $o_1, o_2, o_3, o_4$ respectively. Then the only allocation minimizing the subjective Gini index allocates $o_1, o_3$ to the first agent, and $o_2, o_4$ to the second agent. However, the only Pareto efficient allocation swaps items $o_1, o_2$, giving $o_3$ to the second agent, and $o_2$ to the first agent.

Again, with common utilities, there is no incompatibility between Pareto efficiency and minimizing the Gini, subjective Gini and envy indices. This follows because each allocation, including those that minimize these indices, is Pareto efficient.

**Observation 2** With common utilities, any allocation minimizing the Gini, subjective Gini or envy index is Pareto efficient.

We can measure the trade-off between Pareto efficiency and minimizing one of these indices. The egalitarian/utilitarian price of an index for a given welfare is the ratio between the best welfare of any Pareto efficient allocation and the worst welfare of an allocation minimizing the index.

**Theorem 3** The utilitarian and egalitarian prices of the Gini and subjective Gini indices are unbounded.

**Proof.** Consider 2 agents, 2 items and let $\epsilon < \frac{1}{2}$. Suppose the first agent gives item $o_1$ a utility of $\epsilon$ and $o_2$ a utility of $1 - \epsilon$, whilst the second agent gives utilities of $2 - \epsilon$ and $\epsilon$ respectively. Then the Pareto efficient outcome with the best utilitarian and egalitarian welfare allocates $o_1$ to the second
agent, and $o_2$ to the first agent. However, the only allocation that minimizes the Gini index does the reverse. The egalitarian price of the Gini index is then $\frac{1}{1-\epsilon}$, which is unbounded as $\epsilon$ goes to zero. The utilitarian price is $\frac{3-2\epsilon}{2\epsilon}$ which is unbounded as $\epsilon$ goes to zero. The same example demonstrates that the utilitarian and egalitarian price of the subjective Gini index are also unbounded. 

For the envy index, we have examples where the utilitarian price grows as the number $n$ of agents. We conjecture that this may also be an upper bound. For the egalitarian price, we can show that the price is unbounded.

**Theorem 4** The egalitarian price of the envy index is unbounded.

**Proof.** Consider 3 agents, and 3 items. Suppose the first agent gives a utility of 1 to each item, and both the second and third agents give utilities of 8, 4, and 4 respectively to the 3 items. The Pareto efficient outcome with the best egalitarian welfare allocates the item with utility 8 to the second or third agent, and each of the remaining items to one of the other agents. This has an egalitarian welfare of 1 unit. However, the allocation that minimizes the envy index gives the item with utility 8 to the second agent, both the other items to the third agent, or vice versa. As the first agent gets no items, this has an egalitarian welfare of zero units. Hence, the egalitarian price of the envy index is unbounded. 

**Relationship to strategy proofness**

If we use a mechanism that minimizes one of these indices, agents have an incentive to declare false utilities. Again, this should not be too surprising. We often need to choose between fairness and strategy-proofness. For example, the random priority is strategy-proof but it can return allocations which are not envy-free [Bogomolnaia and Moulin, 2001].

**Theorem 5** A mechanism which minimizes the Gini, subjective Gini or envy index is not strategy proof.

**Proof.** For the Gini index, consider the first example from proof of Theorem[1] If agents sincerely report their utilities, the first agent gets $o_1$ and the second agent gets $o_2$. If the first agent misreports their utilities as 1/2 and 3 respectively, the agents swap items, and both agents are better off. Similarily if the second agent misreports their utilities as 2 and 1/2 respectively, the agents swap items, and both agents are better off.

For the subjective Gini index, consider 2 agents and 4 items. Let the first agent have utilities $u_{11} = 1, u_{12} = 3/2, u_{13} = 1, u_{14} = 1/2$ whereas the second agent have utilities $u_{21} = 3/2, u_{22} = 1, u_{23} = 1/2, u_{24} = 1$. Suppose sincere play. The mechanism that minimizes the subjective Gini index gives to each agent both items for which they have utility 1, or both items for which they have utility 3/2 and 1/2. The expected utility of each agent is then 2. Suppose next that the first agent reports utilities 1, 3/2, 0, 0 respectively. The mechanism now gives the first and second items to the first agent and the third and fourth items to the second agent. The utility of the first agent increases to 5/2.

For the envy index, we can use the same instance as for the subjective Gini index.

**Computational complexity**

In this section, we turn our attention to computational properties of the Gini, subjective Gini and envy indices. Computing envy-free allocations is NP-hard even with just 2 agents, and common utilities [Bouveret and Lang, 2008]. It immediately follows that finding an allocation minimizing the envy index is NP-hard. The proof from [Schneckenburger et al., 2017] showing that minimizing the Atkinson index is NP-hard can be reused to prove that finding an allocation that minimizes the Gini or subjective Gini index is NP-hard.

One way to deal with this intractability is to use algorithms that are fast enough for small values of $n$ or $m$. Another way is to identify some tractable cases. For example, with $n$ agents and $n$ items, minimizing the subjective Gini or envy index is polynomial. Each envy-free allocation (whenever it exists) minimizes the envy index. Each envy-free allocation with common utilities (whenever it exists) minimizes the subjective Gini index. Interestingly, minimizing the Gini index is also polynomial in this case. For each utility value $u$, consider the instance in which only the utilities equal to $u$ are left. Each envy-free allocation in this instance minimizes the Gini index. Computing allocations minimizing the indices in this setting with $n$ agents and $n$ items takes $O(n^{5/2})$ time [Hopcroft and Karp, 1973].

**Online mechanisms**

Another approach to deal with the intractability of computing allocations that minimize inequality or envy is to use greedy online mechanisms. These will often return an allocation with little inequality or envy, even if there is no guarantee that it is minimal. Online mechanisms are also applicable when the allocation problem is itself online [Aleksandrov and Walsh, 2017a; Mehta, 2013; Mattei et al., 2017]. We consider three online randomized mechanisms. These mechanisms can be applied to an offline problem by picking an (perhaps random) order of the items. WLOG, let $o = (o_1, \ldots, o_n)$ be such an order. Each mechanism computes a set of agents feasible for each next $o_j$ in $o$ given an allocation $A_{j-1}$ of $o_1$ to $o_{j-1}$. A feasible agent then receives $o_j$ with probability that is uniform with respect to the other feasible agents.

- **GINI:** this decides that $a_i$ is feasible for $o_j$ if $v_i(o_j) > 0$ and $A_{j-1} \cup \{(a_i, o_j)\}$ minimizes the Gini index
- **SUBJECTIVE GINI:** this decides that $a_i$ is feasible for $o_j$ if $v_i(o_j) > 0$ and $A_{j-1} \cup \{(a_i, o_j)\}$ minimizes the subjective Gini index
- **ENVY:** this decides that $a_i$ is feasible for $o_j$ if $v_i(o_j) > 0$ and $A_{j-1} \cup \{(a_i, o_j)\}$ minimizes the envy index

A powerful technique to study online mechanisms is competitive analysis [Sleator and Tarjan, 1985]. This has recently been applied to online fair division [Aleksandrov and Walsh, 2017b]. Competitive analysis identifies the loss in efficiency due to the data arriving in an online fashion. An online mechanism $M$ is $c$-competitive for a given welfare $w$ if there exists a constant $b$ such that, whatever the order $o$ of items, $w(OPT) \leq c \cdot w(M, o) + b$ holds where $w(M, o)$ is the welfare of $M$ on $o$ and $w(OPT)$ is the optimal offline welfare.
A mechanism that is $c$-competitive has a ratio $c$. Most of the ratios of our mechanisms are unbounded. For example, we can use the instance from the proof of Theorem 10 in [Aleksandrov et al., 2015] and show that both the utilitarian and egalitarian ratios of SUBJECTIVE GINI are unbounded. We next prove similar results for GINI and ENVY.

**Theorem 6** The utilitarian and egalitarian competitive ratios of GINI are unbounded.

**Proof.** For GINI, consider the online fair division of items $o_1, o_2$ to agents $a_1, o_2$. Let the first item have a utility 1 for $o_1$ and $\epsilon$ for $o_2$ whilst the second agent have a utility $\epsilon$ for $o_1$ and 1 for $o_2$ where $\epsilon > 0$. The mechanism allocates $o_1$ to $o_2$ and $o_2$ to $o_1$ and thus returns utilitarian and egalitarian welfares of $2\epsilon$ and $\epsilon$. The optimal offline allocation allocates $o_2$ to $o_2$ and $o_1$ to $o_1$ and thus returns utilitarian and egalitarian welfares of 2 and 1. The competitive ratios are equal to $\frac{1}{2}$ which goes to $\frac{1}{\epsilon}$ as $\epsilon$ goes to zero. ∗

**Theorem 7** The utilitarian competitive ratio of ENVY is at least $\frac{n}{2}$ whilst its egalitarian competitive ratio is unbounded.

**Proof.** For the utilitarian ratio, consider $n$ agents and $n$ items. Let the first agent have utility $n$ for each item, and each other agent have utility 1 for each item. Then ENVY will allocate the first item to the first agent, and then each subsequent item to a new agent. The utilitarian welfare of this allocation is $2n - 1$. The optimal utilitarian welfare is $n^2$.

For the egalitarian ratio, consider the online fair division of items $o_1, o_2$ to agents $a_1, a_2$. Let the first agent have a utility 1 for each item whilst the second agent have a utility $\epsilon$ for $o_1$ and 0 for $o_2$ where $\epsilon > 0$. The mechanism allocates both items to the first agent, and thus returns an egalitarian welfare of 0. The optimal offline allocation gives to each agent an item they like, and returns egalitarian welfare of $\epsilon$. The egalitarian ratio is $\infty$. ∗

We can also measure the price of anarchy of these online mechanisms. The price of anarchy is closely related to the competitive ratio but now supposing agents act strategically. Koutsoupias and Papadimitriou, 1999 [Aleksandrov et al., 2015]. The price of anarchy of an online mechanism for a given welfare is the ratio between the best welfare of an allocation when agents are sincere and the worst welfare of an allocation when agents are strategic. Interestingly, the price of anarchy of each of our online mechanisms is at least to $n$. We conjecture that this may also be their upper bound.

**Theorem 8** The utilitarian and egalitarian prices of anarchy of GINI, SUBJECTIVE GINI and ENVY are at least $n$.

**Proof.** Consider an instance with $n$ agents and $n$ items. For $i \in \{1, \ldots, n\}$, let $a_i$ has utility of 1 for $o_i$, and utility of $\epsilon > 0$ for each other item. The optimal offline allocation gives to each $a_i$ their most valued item. The utilitarian and egalitarian welfares of this allocation are $n$ and 1 respectively.

We start with GINI. At round 1, this mechanism gives the first item to one of the agents who likes with it $\epsilon$. The first agent then has an incentive to report $\epsilon$ for this item simply because they do not know what items will arrive next. By a similar argument, at round 2, the optimal play for the second agent is to bid $\epsilon$, and so on for each other round. At the end of the allocation, each agent gets expected utility of $\frac{1}{n} + \frac{n-1}{n} \epsilon$. The utilitarian and egalitarian welfares of this strategic allocation go to 1 and $\frac{1}{n}$ respectively as $\epsilon$ goes to zero. The prices are consequently at least $n$.

We next consider SUBJECTIVE GINI. The sincere play is optimal for each agent with this mechanism because they get each item with probability $\frac{1}{n}$. The welfares go to 1 and $\frac{1}{n}$ respectively as $\epsilon$ goes to zero. The prices are at least $n$.

We finally consider ENVY. This mechanism tends to allocate each item to agents with the highest utility for this item. By similar arguments as for GINI, we conclude that the optimal play of each agent is to bid 1 for each item. Each agent thus gets expected utility of $\frac{1}{n} + \frac{(n-1)}{n} \epsilon$. ∗

Despite the fact that these mechanisms are not competitive supposing agents act sincerely, they become more competitive supposing agents act strategically. Moreover, each of these mechanisms does as well as any other online mechanism at minimizing their respective index. An online mechanism $M_1$ is ex post optimal for a given index iff, for each other online mechanism $M_2$, each online problem and each allocation $A_2$ returned by $M_2$, there exist an allocation $A_1$ returned by $M_1$ such that the index of $A_2$ is at least as the index of $A_1$. We show ex post optimality only for GINI. The proof for the other two mechanisms can similarly be done by analogy.

**Theorem 9** The GINI mechanism is ex post optimal for the Gini index.

**Proof.** Suppose that GINI is not optimal. Hence, there is another online mechanism $M$, an online problem and an allocation $A_M$ such that the index of $A_M$ is strictly lower than the minimum index of an allocation $A_{\text{Gini}}$ returned by GINI. This means that there is a round $j \in \{1, \ldots, n\}$ at which $A_M$ and $A_{\text{Gini}}$ differ for item $o_j$, but coincide for items $o_1$ to $o_{j-1}$. Let $A_{j-1}$ denote the allocation of $o_1$ to $o_{j-1}$ in $A_M$ and $A_{\text{Gini}}$. Without loss of generality, let $M$ allocate $o_j$ to $a_1$ whereas GINI allocate it to $o_2$ given $A_{j-1}$. We have that the Gini index of $A_{j-1} \cup \{(o_1, a_1)\}$ with $M$ is lower than the Gini index of this allocation with GINI. Hence, GINI does not minimize this index given $A_{j-1}$. This is a contradiction with the definition of GINI. ∗

Finally, Theorems 6 and 7 suggest that, in the worst-case, these online mechanisms have performance that cannot be bounded, whereas the Theorem 8 suggests that no other online mechanism can do better.

**Experiments**

We ran an experiment to see how these online mechanisms would perform in practice. We generated 100 instances of $n = 5$ agents, $m \in \{10, 20, 30, 40, 50, 60, 70, 80, 90, 100\}$ items and integer utilities drawn uniformly at random from $\{0, 1, \ldots, m\}$. For each combination of $n$ and $m$, we computed the Gini index, the subjective Gini index, the envy index, the egalitarian welfare and the utilitarian welfare of 100 000 sampled allocations returned by GINI, SUBJECTIVE GINI and ENVY. We report in our graphs only the average results because their standard deviations were less than 1% of them. We further omit our results for the subjective Gini index for reasons of space.
In the first graph, GINI achieves the lowest value of the Gini index for each number of items. For example, the Gini value of GINI is nearly 50% lower than the Gini values of SUBJECTIVE GINI and ENVY for 100 items. This gap actually remains almost the same for any number of items in our experiment. Unfortunately, GINI fails to minimize envy. In the second graph, we could clearly see that ENVY outperforms GINI. In fact, ENVY achieves an envy index of almost 0 for 100 items. Interestingly, SUBJECTIVE GINI tends to favor envy-freeness to equitability. Moreover, the performance of GINI diverges from envy-freeness and converges to perfect equitability with more items. Perhaps, we observe this as GINI tends to allocate items to agents with low utilities. In contrast, SUBJECTIVE GINI and ENVY tend to allocate items to agents with great utilities. They thus tend to minimize simultaneously both the envy and inequality.

We next report our results for the utilitarian and egalitarian ratios. The utilitarian/egalitarian ratio is the ratio between the utilitarian/egalitarian welfare returned by an online mechanism and the optimal offline utilitarian/egalitarian welfare.

From a utilitarian perspective (the first graph), ENVY outperforms the other two mechanisms for each number of items. For example, this mechanism achieves a utilitarian ratio close to 0.7 for 100 items. This value is nearly 16% higher than the ratio of SUBJECTIVE GINI and 100% higher than the ratio of GINI for 100 items. From an egalitarian perspective (the second graph), again ENVY outperforms SUBJECTIVE GINI and GINI, followed closely by SUBJECTIVE GINI. Interestingly, for each number of items, ENVY not only minimizes the envy but also maximizes the egalitarian welfare. For 100 items, its egalitarian ratio is nearly 0.95. This value is nearly 82% higher than the value of GINI for 100 items. For both welfares, the performance of SUBJECTIVE GINI is close to the performance of ENVY.

Finally, our experimental results indicate that envy-freeness, equitability and welfare efficiency may be achievable in practice.

Related work
Endriss has formulated the task of reducing inequality as a combinatorial optimisation problem [Endriss, 2013]. In particular, he studied the problem of deciding if there exists an inequality reducing improvement such as a Pigou-Dalton or Lorenz transfer. The complexity of such decision problems depends on the language used to represent the (possibly non-additive) utilities. He also provided a modular mixed integer programming formulation that returns an allocation to minimize inequality measures such as the Gini and Hoover indices when utilities are specified with the XOR-language. Schneckenburger, Dorn and Endriss [Schneckenburger et al., 2017] consider allocating indivisible goods to minimize inequality as measured by the Atkinson index. They demonstrated that a sequence of local deals would converge on a globally optimal allocation with the minimum Atkinson index possible, but that the number of agents and items involved in such deals could not be bounded. For the Gini index, they conjectured that such convergence would be very challenging if not impossible to achieve.

By comparison, we show that computing allocations with small inequalities might be fast in practice. Moreover, none of these works relates to other axiomatic properties. For example, [Aziz et al., 2015] studied a taxonomy of fairness concepts related to envy-freeness and proportionality. However, there are fair division problems in which even the weakest of these concepts may not exist, whereas allocations minimizing our indices always exist. Moreover, the Gini index is characterized in [Sanchez-Perez et al., 2012]. The subjective Gini and envy indices are inspired by two measures of envy that are analyzed in [Bosmans and ¨Ozt¨urk, 2018]. However, the idea of measuring envy was first proposed in [Feldman and Kirman, 1974].

Conclusions
We defined three new indices that measure the quality of allocations: the Gini, subjective Gini and envy indices. The first two indices measure inequality within an allocation, whilst the third index measures the amount of envy. Each index could be used as a second order criterion in choosing between allocations. For example, we could choose the Pareto efficient allocation with the the least value of an index. Unlike envy-free allocations which may not exist, allocations that minimize these three indices always exist. We studied the relationship of these indices with envy-freeness, Pareto efficiency and strategy-proofness. We further studied the complexity of computing allocations minimizing each of these indices. Unfortunately, most of these computational problems are intractable. For this reason, we proposed three tractable online mechanisms that greedily minimize these three indices. Experiments showed that, even for modest sized problems, we may be able to efficiently compute allocations with limited inequality or envy as well as with reasonably high values of the egalitarian and utilitarian welfares.
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