DIRAC SPECTRUM IN ADJOINT QCD

D. TOUBLAN AND J.J.M. VERBAARSCHOT
Department of Physics and Astronomy,
University at Stony Brook,
Stony Brook, NY 11794, USA

In this lecture we discuss some exact results for the low-lying spectrum of the Dirac operator in adjoint QCD. In particular, we find an analytical expression for the slope of the average spectral density. These results are obtained by means of a generating function which is an extension of the QCD partition function with fermionic and bosonic ghost quarks. The low-energy limit of this generating function is completely determined by chiral (super-)symmetries. Our results for the slope of the average spectral density are consistent with the results for the scalar susceptibility which can be obtained from the usual chiral Lagrangian.

1 INTRODUCTION

Both from phenomenological arguments and lattice QCD simulations we know that chiral symmetry in QCD is spontaneously broken by the formation of a chiral condensate (this issue has been discussed in several recent reviews\cite{1,2,3,4,5}). However, a complete analytical understanding of the underlying mechanism of chiral symmetry breaking is not yet available. The situation is much better in Supersymmetric Gluodynamics. In this theory it can be shown analytically that the chiral condensate is non-vanishing\cite{6,7,8,9}. One important difference with QCD is that in this case the fermions are in the adjoint representation. In both cases the partition function can be viewed as the average of a fermion determinant. The chiral condensate, which is the mass derivative of the free energy, is thus directly related to the eigenvalues of the Dirac operator.

Generally, the Dirac spectrum cannot be obtained analytically. However, because the low-energy limit of theories with Goldstone bosons and a mass gap is uniquely determined by the pattern of spontaneous symmetry breaking, we expect that we will be able to derive analytical results for the low-lying Dirac spectrum. This program was initiated by Leutwyler and Smilga\cite{10} whose work resulted in sum-rules for the inverse Dirac eigenvalues. A complete analytical understanding of the low-lying Dirac spectrum came from the realization that it is described by a Random Matrix Theory with the global symmetries of QCD, also known as chiral Random Matrix Theory\cite{11,12}. This conjecture has been proved analytically starting from the low-energy limit of a generating function for the Dirac spectrum\cite{13,14}. In addition to the usual fermionic quarks, this QCD-like partition function contains fermionic and bosonic ghost quarks with a mass determined by the magnitude of the Dirac eigenvalues we are...
interested in. It was understood early on\textsuperscript{[3,4]} that the domain of validity of chiral Random Matrix Theory is determined by the mass scale for which the kinetic term of this low-energy effective theory can be neglected, i.e. when the wavelength of the corresponding Goldstone modes is much larger than the size of the box. In this domain, the thermodynamic limit can only be taken if the mass is decreased such that this condition remains satisfied. However, this is exactly what happens for the low-lying Dirac eigenvalues which scale as the inverse Euclidean volume. The recent work on the generating function for the Dirac spectrum\textsuperscript{[13,14,17]} made it possible to include the effect of the kinetic term and to study the Dirac spectrum in the physical domain with box size much larger than the Compton wavelength of the Goldstone modes. This allowed us to extract properties of the average spectral density at a scale that remains fixed in the thermodynamic limit. In particular, an analytical expression for its slope was found\textsuperscript{[13,17]}.

In this lecture we are interested in the QCD Dirac spectrum for fermions in the adjoint representation. We will calculate the slope of the Dirac spectral density in two different ways. First, via the scalar susceptibility which can be calculated by means of the usual chiral Lagrangian, and second, via the valence quark mass dependence of the chiral condensate. The first method was originally introduced by Smilga and Stern\textsuperscript{[18]} for the case of QCD with three or more colors and fundamental fermions. The second approach relies on the introduction of a generating function for the resolvent of the QCD Dirac operator as discussed above. We will find that both methods give the same results. One of the advantages of the second method, which proceeds by a direct calculation of the average spectral density of the Dirac operator, is its validity for $N_f = 1$.

In our convention, with an anti-Hermitian Dirac operator $D$, the eigenvalues are given by

$$D\phi_k = i\lambda_k\phi_k.$$  
\text{(1)}

The average spectral density is defined as

$$\rho(\lambda) = \langle \sum_k \delta(\lambda - \lambda_k) \rangle,$$  
\text{(2)}

where the average is over the ensemble of spectra. There are two important observables that are directly related to the Dirac spectrum, the chiral condensate and the scalar susceptibility. The chiral condensate is given by,

$$\Sigma \equiv \lim_{m \to 0} \lim_{V \to \infty} \frac{1}{V} \int \frac{\rho(\lambda)d\lambda}{i\lambda + m},$$  
\text{(3)}
and the scalar susceptibility can be written as,

\[ K = - \lim_{V \to \infty} \frac{1}{V} \int \frac{\rho(\lambda)d\lambda}{(i\lambda + m)^2} \] (4)

Here, and below, the Euclidean space time volume is denoted by \( V \). If the spectral density near zero can be expanded as

\[ \rho(\lambda) = \rho(0) + |\lambda|\rho'(0) + \frac{1}{2}\rho''(0)\lambda^2 \cdots, \] (5)

the chiral condensate is given by by the Banks-Casher formula\(^{19}\)

\[ \Sigma = \pi \rho(0) \frac{V}{V}, \] (6)

and the infrared singular part of the scalar susceptibility is given by

\[ K \sim \frac{\rho'(0)}{V} \log(\Lambda/m). \] (7)

The chiral condensate is obtained from the first term in (5). This term does not contribute to \( K \); the infrared singular part arises from the linear term in (5). All higher order terms in (5) can be neglected if the chiral limit is taken at fixed value of the cutoff \( \Lambda \) for the integration over \( \lambda \). The smallest Dirac eigenvalues thus provide us with important information about the vacuum properties of QCD.

2 QCD WITH ADJOINT FERMIONS

The Euclidean Dirac operator for quarks in the adjoint representation is given by

\[ D = \gamma_\mu(\partial_\mu \delta_{bc} + f_{abc}A_\mu^a), \] (8)

where the \( f_{abc} \) are the anti-symmetric structure constants of \( SU(N_c) \) and the \( \gamma_\mu \) are the Euclidean \( \gamma \)-matrices. Because the gauge fields \( A_\mu^a \) are real this Dirac operator satisfies the reality relation\(^{20}\)

\[ [iD, CK] = 0, \] (9)

where \( C \) is the charge conjugation matrix \((C^\dagger = \gamma_2\gamma_4)\) and \( K \) is the complex conjugation operator. Because

\[ (CK)^2 = -1, \] (10)
all Dirac eigenvalues are doubly degenerate with eigenfunctions given by $\phi$ and $CK\phi$. The linear independence of $\phi$ and $CK\phi$ follows from properties of the scalar product under anti-unitary transformations,

$$(CK\phi, \phi) = ((CK)^2\phi, CK\phi)^* = -(CK\phi, \phi),$$

so that $(CK\phi, \phi) = 0$. Another consequence of (9) is that it is always possible to find a basis for which the matrix elements of the Dirac operator are arranged into self-dual quaternions. In other words, the Dyson index of the Dirac operator in adjoint QCD has the value $\beta = 4$.

Because $\gamma_2\gamma_4(D + m)$ is anti-symmetric under transposition, the square root of the fermion determinant is given by its Pfaffian, and the Euclidean partition function for $N_f$ Majorana flavors can be written as

$$Z = \int DA \det^{1/2}(D + M)e^{-S_{YM}} = \int DA \prod_{f=1}^{N_f} d\lambda_f e^{\frac{d}{4} \lambda_f^4 C(D\delta_{fg} + M_{fg})\lambda^g} e^{-S_{YM}}.$$  

(12)

The mass matrix $M$ is symmetric under transposition. In the case of one massless flavor this theory is Supersymmetric Gluodynamics. Its properties have been investigated in great detail (an excellent introductory review of this topic is available). In particular, it has been shown that the gluino condensate is non-vanishing. Since the adjoint Dirac operator in the field of an instanton has $2N_c$ zero modes, this result cannot be understood in terms of explicit symmetry breaking by instantons as is the case for QCD with one massless fundamental flavor. A non-vanishing chiral condensate in Supersymmetric Gluodynamics thus suggests the existence of field configurations with winding number equal to $1/N_c$. Considerable progress was made in this direction in several recent articles. Some evidence for the existence of such configurations has been found in lattice QCD as well.

3 DIRAC SPECTRUM

In this section we review some properties of the QCD Dirac spectrum. Because of the axial $U_A(1)$ symmetry, $\{D, \gamma_5\} = 0$, all nonzero eigenvalues occur in pairs $\pm \lambda_k$ with eigenfunctions given by $\phi_k$ and $\gamma_5\phi_k$. If $\phi_k \sim \gamma_5\phi_k$, the corresponding eigenvalue is necessarily zero. This happens in the field of an instanton.

For eigenvalues much larger than $\Lambda_{QCD}$, we expect that the gauge fields do not significantly modify the Dirac spectrum so that its spectral density is given by a theory of noninteracting quarks,

$$\rho(\lambda) \sim V\lambda^3 \quad \text{for} \quad \lambda \to \infty.$$  

(13)
In the case of spontaneous broken chiral symmetry, the chiral condensate is nonzero if the thermodynamic limit is taken before the chiral limit. This can be understood in terms of the existence of a tower of roughly equally spaced eigenvalues (indicated by the wavy curve in Fig. 1) with the smallest nonzero eigenvalue at about one average spacing from zero,

\[ \lambda_{\text{min}} \approx \frac{1}{\rho(0)} = \frac{\pi}{\Sigma V}. \]  

(14)

Such accumulation of eigenvalues near zero does not occur in the free theory. It is only possible if the Dirac spectrum near zero is dominated by the interactions of the theory. Strong interactions give rise to repulsion of the eigenvalues which, viewed as positions of particles, condense into a Wigner crystal. This phenomenon has been studied in great detail in the context of Random Matrix Theory. As will be explained next, the smallest QCD Dirac eigenvalues are correlated according to chiral Random Matrix Theory (chRMT). An important energy scale in the Dirac spectrum is the Thouless energy. This is the quark mass scale \( m_c \), for which the Compton wavelength
of the corresponding Goldstone boson is equal to the length of the box, i.e.

$$\frac{m_c \Sigma}{F^2} = L^2.$$  \hspace{1cm} (15)

For $m \ll m_c$ the kinetic term of the Goldstone modes can be ignored. In this domain, all theories with the same pattern of chiral symmetry breaking and a mass gap are equivalent (several explicit examples have been constructed [28,29,30,31]). In particular, the Dirac spectrum below $m_c$ is given by a chiral Random Matrix Theory [13,14]. For adjoint fermions this is a random Dirac operator with quaternion real matrix elements with a probability distribution that includes the fermion determinant [12]. This has been confirmed by numerous lattice QCD simulations [32,33,34,35,36,37,38,39,40]; (a complete list of references can be found in a recent review [4]). The basis for the predictive power of Random Matrix Theory is universality: the fluctuation properties of the eigenvalues on the scale of the average level spacing are not sensitive to a wide class of large modifications of the probability distribution. In chiral Random Matrix Theory, this was first shown for QCD with three or more colors and fundamental fermions [42,43] and only more recently for the case of adjoint fermions [44,45,46,47,48].

Should we also expect an accumulation of small Dirac eigenvalues for one massless Majorana flavor? Let us consider the chiral condensate,

$$\langle \bar{\lambda} \lambda \rangle = \frac{1}{Z} \frac{1}{V} \left\langle \sum_k \frac{1}{i \lambda_k + m} \prod (i \lambda_k + m) \right\rangle.$$  \hspace{1cm} (16)

For $m \to 0$ the partition function is dominated by configurations with zero topological charge, whereas the numerator in (16) obtains its main contribution from the $\nu = 1/N_c \equiv \tilde{\nu}/N_c$ configurations for which the sum over eigenvalues in (16) can be approximated by the $\lambda_k = 0$ term. With a cancellation of the $1/m$ factor from the chiral condensate and a factor $m$ in the fermion determinant, we thus find

$$\langle \bar{\lambda} \lambda \rangle = \frac{1}{V} \frac{\langle \prod_k i \lambda_k \rangle_{\nu=1}}{\langle \prod_k i \lambda_k \rangle_{\nu=0}}.$$  \hspace{1cm} (17)

The average fermion determinants for $\tilde{\nu} = 0$ and $\tilde{\nu} = 1$ differ by a factor $1/\langle \bar{\lambda} \lambda \rangle V$. How can we understand this? Our explanation is that the zero eigenvalue repels all nonzero eigenvalues such that their average position (denoted by $\bar{\lambda}_k$) is shifted away from zero by exactly one half average eigenvalue spacing $\Delta \lambda$. We thus have

$$\prod (\bar{\lambda}_k^{\nu=1})^2 \approx \prod (\bar{\lambda}_k^{\nu=0} + \frac{1}{2} \Delta \lambda)^2 \approx \prod \bar{\lambda}_k^{\nu=0} \bar{\lambda}_{k+1}^{\nu=0} = \lambda_{\min} \prod (\bar{\lambda}_k^{\nu=1})^2.$$  \hspace{1cm} (18)
Explicit chiral symmetry breaking by instantons also requires the accumulation of small nonzero eigenvalues exactly as happens in the case of spontaneous breaking of chiral symmetry when the chiral condensate is given by the Banks-Casher formula \cite{19}. These results are in agreement with the analysis based on finite volume partition functions for $N_f = 1$ for which the contributions to the chiral condensate from the different topological sectors can be obtained analytically \cite{10,49}.

4 CHIRAL LAGRANGIAN AND SCALAR SUSCEPTIBILITY

The adjoint QCD partition function is invariant under

$$\lambda \equiv \begin{pmatrix} w \\ \bar{w} \end{pmatrix} \rightarrow \begin{pmatrix} U \\ U^{-1} \end{pmatrix} \begin{pmatrix} w \\ \bar{w} \end{pmatrix}, \quad (19)$$

where $U \in SU(N_f)$ (a $U_A(1)$ axial symmetry is broken by the anomaly). The gluino condensate is a color singlet with flavor structure given by \cite{50}

$$\langle \lambda^f C \lambda^g \rangle = \delta^{fg} \Sigma, \quad (20)$$

so that the $SU(N_f)$ flavor symmetry is broken to $O(N_f)$. The chiral Lagrangian corresponding to this pattern of chiral symmetry breaking can be constructed in the standard way. To lowest order in the momenta and the quark masses it is given by \cite{17}

$$Z(M) = \int_{U \in SU(N_f)/O(N_f)} dU e^{-\int d^4x \left[ \frac{1}{2} \text{Tr} \partial_\mu U \partial_\mu U^{-1} - \frac{1}{4} \text{Tr} (MU + M^\dagger U^{-1}) \right]}. \quad (21)$$

This partition function is invariant under the flavor transformations

$$U \rightarrow VUV^T, \quad M \rightarrow V^*MV^\dagger, \quad (22)$$

as required by the transformation properties of the QCD partition function. Such invariance arguments are very powerful and even allow us to construct the low energy limit of adjoint QCD at nonzero baryon density \cite{43,44}, but this topic will not be addressed today.

To calculate the scalar susceptibility it is convenient to introduce scalar sources in the mass matrix of the QCD partition function

$$M = m \delta^{fg} + s_a T^a, \quad (23)$$

where the $T_a$ are the symmetric generators of $SU(N_f)$. The scalar susceptibility is then given by the second derivative of the QCD partition function with

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respect to the scalar sources

\[ K_{ab} = \frac{1}{V} \partial_{s_a} \partial_{s_b} \bigg|_{s_a = 0, s_b = 0} \log Z(M), \]

\[ = -\frac{1}{V} \left\langle \text{Tr} \frac{T_a}{D + m} \frac{T_b}{D + m} \right\rangle = \frac{1}{2} \delta_{ab} K. \quad (24) \]

Since the scalar sources and the mass matrix have the same transformation properties in the QCD Lagrangian, they also enter in the same way in the chiral Lagrangian. To lowest order in chiral perturbation theory, we expand

\[ U = \exp[i 2\pi_a T^a / F] \]

to second order in the \( \pi \) fields,

\[ \frac{1}{2}(U + U^{-1}) = 1 - \frac{2}{F^2} \pi_k \pi_l T^k T^l, \quad (25) \]

so that the scalar susceptibility is given by

\[ K^{ab} = \frac{4\Sigma^2}{F^4} \text{Tr}(T^a T^k T^l) \text{Tr}(T^b T^m T^n) \left( \frac{1}{V} \int d^4 x d^4 y \pi_k(x) \pi_i(x) \pi_m(y) \pi_n(y) \right) \text{1-loop} \]

\[ = \frac{\Sigma^2}{16\pi^2 F^4} \text{Tr}(T^a \{ T^k, T^l \}) \text{Tr}(T^b \{ T^k, T^l \}) \log(\Lambda / m). \quad (26) \]

Carrying out the traces in flavor space we find

\[ K^{ab} = -\delta^{ab} \frac{\Sigma^2}{128\pi^2 F^4} \frac{(N_f - 2)(N_f + 4)}{N_f} \log(m / \Lambda). \quad (27) \]

The scalar susceptibility can also be calculated for QCD with fundamental fermions and two colors or for QCD with fundamental fermions and three of more colors with Goldstone manifold given by \( SU(2N_f) / Sp(2N_f) \) and \( SU(N_f) \), respectively. These three cases can be distinguished by the value of the Dyson index given by \( \beta = 4, \beta = 1 \) and \( \beta = 2 \), in this order. For Dyson index different from \( \beta = 4 \), the only change in (27) is the replacement \( (N_f + 4) / 4 \rightarrow (N_f + \beta) / \beta \). The case \( \beta = 2 \) was first analyzed by Smilga and Stern.

If the average spectral density has a simple linear expansion as given in (5), the slope of the average spectral density follows immediately from the result for the scalar susceptibility. It is given by

\[ \frac{\langle \rho'(0) \rangle}{V} = \frac{\Sigma^2 (N_f - 2)(N_f + \beta)}{16\pi^2 \beta N_f F^4}, \quad (28) \]

where we have included the dependence on the Dyson index. In the next section we will present a derivation for the slope that does not rely on this assumption [8], and moreover, is also valid for \( N_f = 1 \).
5 GENERATING FUNCTION FOR THE DIRAC SPECTRUM

In order to obtain a generating function for the Dirac spectrum one has to extend the QCD partition function with additional fermionic and bosonic ghost quarks.

\[ Z_{pq}(z,J) = \int DA \frac{\det(D + z + J)}{\det(D + z)} \det^{N_f/2}(D + m) e^{-S_M[A]} . \quad (29) \]

The resolvent is then given by

\[ \Sigma(z) = \frac{1}{V} \sum_k \frac{1}{z + i\lambda_k} = \frac{1}{V} \partial J|_{J=0} \log Z_{pq}(z,J). \quad (30) \]

The average spectral density follows from the discontinuity of the resolvent across the imaginary axis

\[ \rho(\lambda) = \frac{1}{2\pi} (\Sigma(i\lambda + \epsilon) - \Sigma(i\lambda - \epsilon)) = \frac{1}{2\pi} (\Sigma(i\lambda + \epsilon) + \Sigma(-i\lambda + \epsilon)), \quad (31) \]

where the second equality follows from the relation \( \Sigma(z) = -\Sigma(-z) \). Notice that \( \det^{-1/2}(D + z) \) cannot be written as a Gaussian integral. The minimal generating function thus requires the introduction of ghost determinants as in eq. (29), corresponding to one complex bosonic ghost quark and a pair of Majorana ghost quarks.

In the sector of fermionic quarks, the symmetry is broken by the formation of a chiral condensate with flavor structure as given in (20). The pattern of symmetry breaking is thus given by

\[ SU(N_f + 2) \rightarrow O(N_f + 2). \quad (32) \]

In the sector of bosonic quarks, the quadratic form in the action can be written as

\[ \begin{pmatrix} \phi_L^* & m \sigma_\mu d_\mu \\ \phi_R^* & \sigma_\mu d_\mu \end{pmatrix} = \begin{pmatrix} \phi_L^* & \sigma_\mu d_\mu \\ \sigma_2 \phi_R^* & \sigma_2 d_\mu \end{pmatrix} + \frac{1}{2} \begin{pmatrix} \phi_L^* & m \sigma_2 \\ \sigma_2 \phi_R^* & m \sigma_2 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} \phi_L^* & m \sigma_2 \\ \sigma_2 \phi_R^* & m \sigma_2 \end{pmatrix} \quad (33) \]
Here, $\sigma_\mu = (1, i\sigma_k)$ and $d_\mu = \partial_\mu + f_{abc}A^a_\mu$. The kinetic term is invariant under $U(2)$ transformations. However, the fields no longer occur in complex conjugated pairs after this transformation. But notice that the kinetic term is also invariant under the symmetry group $U^*(2, R)$, which does not affect the reality properties of the quadratic form. The chiral condensate given by the mass derivative of the partition function as well as the mass term in the partition function are only left invariant by the subgroup $Sp(2)$. The Goldstone manifold corresponding to the sector of bosonic quarks is thus given by $U^*(2)/Sp(2)$ and has only one degree of freedom.

In the sector of fermionic quarks we could extended the unitary symmetry to $Gl(N_f+2)$, but a noncompact symmetry group would lead to an effective partition function with an incorrect small mass expansion. The symmetry group of the sector of fermionic quarks should thus be $U(N_f+2)$. In addition, the generating function is invariant under supersymmetry transformations mixing fermions and bosons. The full symmetry group is thus given by the graded Lie group $Gl(N_f+2|2)$ which is broken spontaneously to the ortho-symplectic graded Lie group $Osp(2|N_f+2)$. The Goldstone manifold is then given by the maximum Riemannian submanifold of $Gl(N_f+2|2)/Osp(2|N_f+2)$ with fermionic sector given by $U(N_f+2)/O(N_f+2)$ and bosonic sector $U^*(2)/Sp(2)$.

We denote this manifold by $\hat{G}/H$.

The low-energy effective partition function is given by

$$Z_{\text{eff}} = \int_{U \in \hat{G}/H} dU e^{-\int d^4x L_{\text{eff}}}, \quad (34)$$

where

$$L_{\text{eff}} = \frac{F^2}{4} \text{Str}(\partial_\mu U \partial_\mu U^{-1}) - \sum_2 \text{Str}([M(U + U^{-1})] + m_0^2 \Phi_0^2 + \alpha \partial_\mu \Phi_0 \partial_\mu \Phi_0, \quad (35)$$

and $U = \exp(i2\Phi/F)$. The last two terms in (35) represent the mass term and the kinetic term of the super-$\eta^f$ flavor-singlet field $\Phi_0 = \text{Str}(\Phi)$. This partition function has the same transformation properties as the generating function (29) including the explicit breaking of an axial $Gl(1|1)$ symmetry. The mass matrix is given by $M = \text{diag}(m, \cdots, m, z + J, z + J, z, z)$, with $N_f$ masses equal to $m$. This partition function has both bosonic and fermionic Goldstone bosons with mass $M_{vv} = \sqrt{2z\Sigma}/F$, $M_{vz} = \sqrt{(m + z)\Sigma}/F$ and $M_{zz} = \sqrt{2m\Sigma}/F$. The generating function (29) was first introduced to study the quenched approximation in QCD. For symmetry class $\beta = 2$ a version of the effective partition function based on compact supergroups was first introduced by Bernard and
Golterman. It gives the correct perturbative expansion, but the nonperturbative integrations over $U$ are not reproduced correctly. Notice that a function is not determined by its asymptotic expansion.

To lowest order in chiral perturbation theory, the resolvent is simply given by tadpoles coming from the differentiation with respect to the source $J$. There are three different types of contributions corresponding to three different kinds of mesons that can be excited by the source $J$. First, there are tadpoles with bosonic mesons of mass $M_{vs}$ that do not mix with the super-$\eta'$, second, there are tadpoles with fermionic mesons of mass $M_{vv}$, and third there are tadpoles with bosonic mesons of mass $M_{vv}$ that mix with the super-$\eta'$. We thus find the following result for the resolvent:

$$
\Sigma(z) = \Sigma_0 \left[ 1 - \frac{1}{F^2} \left\{ \frac{N_f}{2} \Delta(M_{vs}^2) - \frac{1}{2} \Delta(M_{vv}^2) + G_{vv} \right\} \right],
$$

(36)

where the three terms in-between the braces correspond to the three different types of contributions discussed above, respectively. The trace of the propagator of the first two types of mesons is given by

$$
\Delta(M^2) = \frac{1}{V} \sum_p \frac{1}{p^2 + M^2} = \frac{1}{16\pi^2} M^2 \log \frac{M^2}{\Lambda^2},
$$

(37)

where $\Lambda$ is a momentum cutoff. The propagator of the third type of mesons is more complicated but it is known analytically. In the limit of $m_0 \to \infty$, the trace of this propagator simplifies to

$$
G_{vv} = \frac{1}{V} \sum_p \left[ \frac{1}{p^2 + M_{vv}^2} - \frac{1}{N_f} \frac{p^2 + M_{vv}^2}{(p^2 + M_{vv}^2)^2} \right].
$$

(38)

Using the explicit expressions for the trace of the propagators we find,

$$
\Sigma(z) = \Sigma \left[ 1 - \frac{16\pi^2 N_f F^4}{\Delta} \left\{ \frac{N_f}{2} (z + m) \log \frac{z + m}{2\mu} + \left( 2m + (N_f - 4)z \right) \log \frac{z}{\mu} \right\} \right],
$$

(39)

where $\mu = \Lambda^2 F^2 / 2\Sigma$. In the limit of $m \to 0$ this result simplifies to

$$
\Sigma(z) \approx \Sigma \left[ 1 - \frac{\Sigma(N_f - 2)(N_f + 4)}{32\pi^2 N_f F^4} \log \frac{z}{\mu} \right].
$$

(40)
From the discontinuity of the resolvent along the imaginary axis calculated using the second equation of (31), we find that the spectral density in the limit $m \to 0$ is given by

$$
\frac{\langle \rho(\lambda) \rangle}{V} = \frac{\Sigma}{\pi} \left[ 1 + \frac{(N_f - 2)(N_f + \beta)\Sigma}{16\pi\beta N_f F^4} |\lambda| \right],
$$

(41)

where the Dyson index is $\beta = 4$ in the case of adjoint QCD. We have also included the results for QCD with two colors and fundamental fermions ($\beta = 1$) and QCD with three or more colors and fundamental fermions ($\beta = 2$). These latter two cases can be derived along the same lines \cite{13, 17}. This result is in agreement with the slope obtained from the scalar susceptibility. It shows that the spectral density can be expanded in powers of $|\lambda|$. Finally, we wish to emphasize that the above derivation is also valid for $N_f = 1$. Results from instanton liquid simulations \cite{55, 56} are consistent with eq. (41).

As an alternative to the supersymmetric method, the mass dependence of the resolvent can be calculated by introducing $n$ flavors of fermionic ghost quarks with mass $z$ and take the limit $n \to 0$ at the end of the calculation. This so-called replica method was used to derive the low-energy limit of the quenched scalar susceptibility in lattice QCD with staggered fermions \cite{40}. A critical comparison of the supersymmetric calculation and the replica calculation was given in \cite{57}. However, we stress that, disregarding exceptional cases that the asymptotic series terminates \cite{58, 59}, only perturbative results \cite{60, 61} have been obtained by the replica method.

\section{CONCLUSIONS}

The QCD Dirac spectrum can be obtained from the discontinuity of the resolvent of the Dirac operator. Its generating function is given by the QCD partition function with additional bosonic and fermionic ghost quarks. Under the assumption of maximum breaking of the axial symmetry, the low-energy limit of this generating function can be written down on the basis of the global symmetries of the theory. The leading infrared singularity of the resolvent, which provides us with the slope of the average spectral density, is obtained from a simple one-loop calculation. This result, and results for two other patterns of chiral symmetry breaking, can be summarized in a single formula that depends in a natural way on the Dyson index of the symmetry class.

Our results for the spectral density are consistent with the infrared singularities of the scalar susceptibility which can be calculated by the usual chiral Lagrangian without relying on ghost quarks. Amazingly, the two calculations also agree for $N_f = 1$ when the scalar susceptibility cannot be calculated by...
chiral perturbation theory. Apparently, it is possible to perform an analytical continuation in $N_f$. Since, as an alternative to the supersymmetric generating function, the resolvent can also be calculated from an analytical calculation in the number of additional fermionic flavors, the agreement for $N_f = 1$ should not come as a surprise.

Lattice QCD with two colors and staggered (fundamental) fermions is in the same symmetry class as QCD with adjoint fermions. Our results have been extended to quenched lattice QCD and an impressive agreement between analytical and numerical results for the connected and disconnected scalar susceptibilities has been found. We are looking forward to a direct lattice calculation of the slope of the average spectral density as well.

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