Intrinsic curvature and topology of shadow in Kerr spacetime

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From the viewpoint of differential geometry and topology, we investigate the characterization of the shadows in a Kerr spacetime. Two new quantities, the length of the shadow boundary and the local curvature radius are introduced. Each shadow can be uniquely determined by these two quantities. For the black hole case, the result shows that we can constrain the black hole spin and the angular coordinate of the observer only by measuring the minimum and maximum of the curvature radius. While for the naked singularity case, we adopt the length parameter and the maximum of the curvature radius. This technique is completely independent of the coordinate system and the location of the shadow, and is expected to uniquely determine the parameters of the spacetime. Moreover, we propose a topological covariant quantity to measure and distinguish different topological structures of the shadows.

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\textbf{Introduction.}—Black holes are amongst the most fascinating objects in the universe. Recently, for the first time, the twin Laser Interferometer Gravitational-wave Observatory (LIGO) detectors directly measured gravitational waves due to black hole collisions, arousing great interest in gravitational waves and providing the most direct evidence for the existence of both black holes and binary black hole systems. From this data the masses and spins of these colliding black holes can be determined. Furthermore, it is generally believed that there exists a super-massive black hole at the center of each galaxy. Such objects will cast a two-dimensional shadow from light emitted by objects behind it, and whose particular form may reveal not only black hole properties but perhaps even new tests of gravitational theory. Very recently, the Event Horizon Telescope has been able to observe the shadow casted by the super-massive black hole in the center of our galaxy at very high angular resolution, and its shape is expected to be determined with the new imaging techniques.

From a theoretical point of view, the boundary of the shadow corresponds to the photon capture sphere as seen by a distant observer. Conversely a computation of this latter structure via null geodesics determines the black hole shadow. In order to fit astronomical observations, several observables were constructed using special points on the shadow boundary in the celestial coordinates. Various related investigations have been carried out by many groups. Although the velocity of the observer can yield aberrational effects, these were found to be negligible for observing the shadow of Sgr A* and Sgr B*. A formalism for describing the boundary of the shadow, independent of the location its center, has been proposed, and several distortion parameters introduced. This method is expected to give a possible precise detection of the black hole shadow.

How to precisely describe the boundary of the black hole shadow is therefore crucial for measuring black hole parameters with astronomical observations. It is known that the boundary of the shadow casted by a black hole is a one-dimensional closed curve. However, other dark objects, in particular naked singularities, have shadow boundaries that are open curves. Fortunately differential geometry provides a natural way to precisely describe the intrinsic properties of closed and open curves (i.e., surfaces and curves). Here we employ such methods to study and compare these two kinds of shadows, introducing covariant quantities to measure the topological structure of a shadow. Our approach can therefore not only distinguish different black hole and naked singularity backgrounds but can also reveal any potential topological phase transition between these systems.

\textbf{Set up.}—The pattern of a shadow is characterized by its boundary, a one-dimensional curve. By determining the nature of the curve, we will obtain the basic properties of a black hole or naked singularity. From the viewpoint of differential geometry, a curve has two intrinsic quantities: its length \(\lambda\) and its local curvature radius \(R\). A sketch is provided in Fig.\textsuperscript{1} where \(\lambda\) parameterizes the curve. Using \((\lambda_0, R)\), a curve will be uniquely determined without introducing any coordinate system. Note that if the curvature radius at point B is defined to be positive, then the one at point C will be negative.

\textbf{Black hole shadow.}—Here we consider a simple case in which all light sources are located at infinity and distributed uniformly in all directions. The observer is likewise located at infinity. The rotating dark object has inclination angle \(\theta_0\), defined by the angle between its rotation axis and the observer’s line of sight. Then the
are parameterized by the radius of the circular unstable
the boundary of the shadow, the parameters
\( \xi \) Carter constant of the null geodesic of energy
celestial coordinates describing its shadow are \([20, 21]\)

FIG. 1: Sketch picture. The curve starting at point A is
parameterized by the line length parameter \( \lambda \) and local
curvature radius \( R(\lambda) \). The circle denotes the curvature circle
with radius \( R(\lambda) \) at point B.

\[ \alpha = -\xi \csc \theta_0, \quad \beta = \pm \sqrt{\eta + a^2 \cos^2 \theta_0 - \xi^2 \cot^2 \theta_0}, \]

where \( a \) is the black hole spin, and \( \xi = Lc/E \) and \( \eta = Q/E^2 \) are two conserved parameters for the photon orbit
associated respectively with the angular momentum and
Carter constant of the null geodesic of energy \( E \). On
the boundary of the shadow, the parameters \( \xi \) and \( \eta \)
are parameterized by the radius of the circular unstable

![Shadows for the Kerr black hole (upper row) and Kerr naked singularity (lower row). The spin \( a \) increases from left to right.](image)

FIG. 2: Shadows for the Kerr black hole (upper row) and Kerr naked singularity (lower row). The spin \( a \) increases from left to right.

\[
R(\lambda_s - \lambda) = R(\lambda).
\]

Moreover, for a black hole shadow, one has
\[ R(\lambda + \lambda_s) = R(\lambda), \]

with \( \lambda_s \) being the length of the boundary of the shadow.

Consider an observer located at the equatorial plane \( \theta_0 = \pi/2 \) of the black hole spacetime background. On
this plane, \( r_0 \in (r_A, r_B) \) with
\[ r_{A, B} = 2M \left( 1 + \cos \left( \frac{2}{3} \arccos \left( \frac{a}{M} \right) \right) \right), \]

respectively corresponding to direct and retrograde or-
bits. The celestial coordinates reduce to \( \alpha = -\xi(r_0), \)
\( \beta = \pm \sqrt{\eta(r_0)} \). For the Schwarzschild black hole, \( a = 0 \)
yielding \( r_0 = 3M \) and \( \alpha^2 + \beta^2 = (3\sqrt{3}M)^2 \), which indicates that the shadow of the Schwarzschild black hole is
a standard circle with radius \( 3\sqrt{3}M \). For \( a \neq 0 \), taking
this parameterization, we can calculate the length \( \lambda_s \) of
the shadow boundary with
\[ \lambda_s = 2 \int_{r_A}^{r_B} \sqrt{(\partial_{r_0} \alpha)^2 + (\partial_{r_0} \beta)^2} \, dr_0. \]

The factor ‘2’ comes from the \( \mathbb{Z}_2 \) symmetry. A simple calculation shows that \( \lambda_s \) decreases with the black hole
spin \( a \). However, the difference is very small. For example,
when \( a/M = 0, 0.99 \), and 1, we have \( \lambda_s/M = 6\pi\sqrt{3}, \)
31.2636, and 16\( \sqrt{3} \), respectively. The relative deviation
between \( a/M = 0 \) and 0.99 is about 4.3%.

Alternatively, the local curvature radius at each point
can also be parameterized by \( r_0 \). Taking nearby points
\( r_0 \pm \epsilon \), these three points on the boundary of shadow can
be used to uniquely construct a circle. As \( \epsilon \to 0 \), the
radius of the circle will exactly coincide with the local
curvature radius at the point corresponding to \( r_0 \). Fortu-
nately it has an analytic form
\[ R = \frac{8\sqrt{M}r_0(r_0^3 - 3Mr_0^2 + 3M^2r_0 - Ma^2)}{3(r_0 - M)^3}. \]

We plot the curvature radius \( R \) as a function of the
length parameter \( \lambda \) in Fig. 3 for different values of the

![The local curvature radius \( R \) as a function of the line
length parameter \( \lambda \). Near the well, the spin \( a/M = 0, 0.01, 0.5, 0.6, 0.7, 0.8, 0.9, 0.93, 0.96, 0.99 \) from top to bottom,
with the horizontal black line being the Schwarzschild case.](image)
black hole spin $a$ when point A (shown in Fig. 2) is chosen as the starting point. Since our treatment is independent of the coordinate system, the starting point can be chosen arbitrarily, with the pattern of the curvature radius $R$ unchanged. For $a/M = 0$, we can see that the curvature radius $R = 3\sqrt{3}M$, which means that the black hole shadow is just a standard circle. For nonvanishing spin $a/M \neq 0$, the curvature radius has two maxima at $\lambda = 0$ and $\lambda$, and two minima at some $\lambda$ satisfying $\partial_{\lambda} R = 0$, due to the $Z_2$ symmetry. Reflecting about the symmetric point, the curvature radius will have only one maximum and one minimum for each fixed spin $a$. Moreover, the minimum value of $R$ decreases and shifts to smaller $\lambda$ with increasing spin, approaching its (approximate) minimum value $2.7M$ for an extremal black hole with $a/M = 1$. Conversely the maximum value of $R$ increases with the black hole spin and diverges for the extremal black hole. In general each black hole shadow is characterized by maximal and minimal values of $R$, providing a way to measure black hole spin whilst ignoring the length of the shadow boundary.

For the observer located at $\theta_0 \neq \pi/2$ the situation is similar. The celestial coordinates can also be parameterized by the radius of the circular unstable photon orbit, i.e., $\alpha = \alpha(r_0)$ and $\beta = \beta(r_0)$, but $r_0$ is no longer in the range $(r_A, r_B)$ given in Eq. (7), but instead is determined by solving $\beta(r_{A,B}) = 0$. The local curvature radius can also be expressed in terms of $r_0$.

\[
R = \frac{64M^{1/2}(r_0^3 - a^2r_0 \cos^2 \theta_0)^{3/2}}{(r_0 - M)^3 [3(8r_0^3 - a^3 - 8a^2r_0^2) - 4a^3(6r_0^2 + a^2) \cos(2\theta_0) - a^4 \cos(4\theta_0)]},
\]

(Naked singularity shadow)—When the spin of the Kerr black hole is beyond its extremal value, i.e.,

\[
|a| > M,
\]

the system becomes a naked singularity. Although naked singularities have generally been thought to be unstable \[22\]—indeed, the cosmic censorship conjecture forbids them \[23\]—recent work has shown that when scalar fields or other fields are included, stability can be restored \[24\], and the cosmic censorship conjecture will be violated \[25\]. In particular, it has recently been claimed (using an analytical treatment employing the Teukolsky equation) that Kerr naked singularities are stable under a variety of boundary conditions \[26\]. Thus it is of great interest to study the naked singularity shadow.

As mentioned above, there exists a dramatic change of the shadow shape and topology for the naked singularity. The boundary closed curve will open, and will be a one-dimensional dark arc. In realistic observations it inclines to form a two-dimensional dark “lunate” shadow; observation of such shapes would provide evidence for the existence of naked singularities in our universe.

For this case, we can also calculate the curvature radius $R$ and boundary length $\lambda_s$. After a detailed examination, we find that the curvature radius $R$ increases with the length parameter $\lambda$. So for each naked singularity shadow, $R$ has one minimum at point A or C and one maximum at point B (shown in Fig. 2). Thus, similar to the black hole shadow, we can measure $a$ and $\theta_0$ by observing the minimum and maximum of the curvature radius. However, the minimum curvature radius corresponding to the starting point A or C is very hard to calculate. We therefore make use of the length $\lambda_s$ of the shadow, as well as the maximum of $R$ to determine the shadow. In Fig. 5 we list the contours of $R_{\text{max}}$ and $\lambda_s$.

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**FIG. 4:** Contour lines of $R_{\text{min}}$ and $R_{\text{max}}$ in the $\theta_0-a/M$ plane for a Kerr black hole shadow. $R_{\text{max}}/M$ is described by the red solid lines with fixed values 4.9, 5. 5.1, 5.15, 5.18, 5.19, 5.2, 5.3, 5.5, 5.8, 6.5, 8, 10, 14 from bottom to top, whereas $R_{\text{min}}/M$ is described by the blue dashed lines with fixed values 3.2, 4.0, 4.3, 4.5, 4.6, 4.7, 4.8, 4.9, 5.0, 5.1, 5.15, 5.18 from right to left.

\[
R = \frac{64M^{1/2}(r_0^3 - a^2r_0 \cos^2 \theta_0)^{3/2}}{(r_0 - M)^3 [3(8r_0^3 - a^3 - 8a^2r_0^2) - 4a^3(6r_0^2 + a^2) \cos(2\theta_0) - a^4 \cos(4\theta_0)]}.
\]
Interestingly, if \( \theta_0 \) is fixed, we can determine the spin \( a \) of the naked singularity, which is very different from the black hole case. Note that a measurement only of \( \lambda_s \) cannot do that. One can accurately obtain the spin \( a \) and inclination angle \( \theta_0 \) by measuring both \( R_{\text{max}} \) and \( \lambda_s \).

**Topology.**—While the black hole and naked singularity have distinct shadows that can be distinguished by observation, we are also interested in seeking some topological covariant quantities to describe their difference. The boundary of each shadow is in a one-dimensional curve, and cannot cross itself. Motivated by the Gauss-Bonnet-Chern theorem we introduce the topological covariant quantity

\[
\delta = \frac{1}{2\pi} \left( \int \frac{d\lambda}{R(\lambda)} + \sum_i \theta_i \right),
\]

where \( \lambda \) and \( R(\lambda) \) are the line length parameter and the local curvature radius we introduced above. The first term measures the smooth part of the shadow boundary and the parameter \( \theta_i \) denotes the \( i \)-th angle of the boundary. Since the shadow boundary of the Kerr black hole is smooth, \( \sum_i \theta_i = 0 \). Cases for which \( \theta_i \) is non-vanishing are related to the existence of stable fundamental photon orbits; examples include scalar hair black holes [13], rotating Konoplya-Zhidenko black holes, or a compact object with magnetic dipole [27, 28]. However, for the Kerr spacetime, all the fundamental photon orbits are unstable, and so the second term in Eq. (12) vanishes.

We exhibit in Fig. 6 the topological quantity \( \delta \) for various fixed \( \theta_0 = 30^\circ, 45^\circ, 60^\circ, 90^\circ \). For the black hole \( (a \leq M) \), \( \delta \) keeps a constant value 1, whereas for the naked singularity \( (a > M) \), \( \delta \) deviates from unity, rapidly decreasing with increasing spin \( a \). For \( a > M \) there is a naked singularity, and the shadow experiences a structural change from a two-dimensional dark region to a one-dimensional dark lunate. As the dimension of the structure changes, this is a topological change and so the quantity \( \delta \) can be regarded as a topological covariant quantity whose value is indicative of such topological change. Noting that \( R(\lambda) \) might be negative (at point C in Fig. 1), \( \delta \) can also be used to measure the structural change of the shadow of a scalar hair black hole [13], as well as for more ‘square’ or ‘hammer-like’ shadows. For multiple disconnected shadows [12, 29], our topological quantity gives

\[
\delta = n,
\]

where \( n \) is the number of disconnected shadows. So different values of \( \delta \) are indicative of different topological structures of the shadows. Note that in Ref. [30], the authors have discussed the the topological structure of the past and future trapped null geodesics, however which is quite different from our case.

**Summary.**—Making use of concepts from differential geometry we have constructed a novel approach for determining the shadow cast by a rotating black hole and its naked singularity counterpart. Our approach depends only on intrinsic properties of the shadow, and is independent of the coordinate system and the location of the shadow.

We found that both the spin \( a \) and viewing angle \( \theta_0 \) can be accurately determined by measuring the maximum and minimum of the local curvature radius \( R(\lambda) \) of the shadow boundary of the black hole. The minimal value of \( R \) is difficult to obtain for the naked singularity, but this can be dealt with by measuring the length of the lunate shadow and the maximum of \( R \) to determine \( a \) and \( \theta_0 \). A full description of the shadow entails measuring the local curvature radius at each point of its boundary.

We have also introduced a topological covariant quantity, \( \delta \), that characterizes the topological structure of the shadow. This quantity distinguishes between naked singularities and black holes, and is sensitive to shadow connectivity. We expect that our technique will have considerable advantage in fitting theoretical models to the
distinguish or rule out this feature, particularly when
understanding the black hole that generates it. So any evidence of a nonzero value provides significant
information about the black hole that generates it. This can provide us with an additional tool to precisely
distinguish GR from alternative gravity theories. Moreover, it is conceivable that improved resolution could better
distinguish or rule out this feature, particularly when
lensing rings around the shadow are taken into account. These are easily observed due to their high brightness,
and the inner one is extremely close to the edge of the shadow. It is therefore natural to identify the inner lensing
ring as the boundary of the shadow: in combination with the curvature radius, one may determine the corres-
ponding black hole parameters.

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