Quantum Driven Dissipative Parametric Oscillator in a Blackbody Radiation Field

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We consider the general open system problem of a charged quantum oscillator confined in a harmonic trap, whose frequency can be arbitrarily modulated in time, that interacts with both an incoherent quantized (blackbody) radiation field and with an arbitrary coherent laser field. We assume that the oscillator is initially in thermodynamic equilibrium with its environment, a non-factorized initial density matrix of the system and the environment, and that at $t = 0$ the modulation of the frequency, the coupling to the incoherent and the coherent radiation are switched on. The subsequent dynamics, induced by the presence of the blackbody radiation, the laser field and the frequency modulation, is studied in the framework of the influence functional approach.

This approach allows incorporating, in \textit{analytic closed formulae}, the non-Markovian character of the oscillator-environment interaction at any temperature as well the non-Markovian character of the blackbody radiation and its zero-point fluctuations. Expressions for the time evolution of the covariance matrix elements of the quantum fluctuations and the reduced density-operator are obtained.

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I. INTRODUCTION

Since the seminal work of Magalinskiĭ [1], Feynman and Vernon [2], Ullersma [3], and Caldeira-Leggett [4], the theory of open quantum systems has been source of great interest and object of continuous development, refinement and application (cf. Refs. 5–9 and references therein). This theory has, for example, provided a solid conceptual framework to explain fundamental phenomena such as the quantum-classical transition [10], violations of the second law of the thermodynamics [11], the survival of quantum features at high temperature [12, 13], and has found applications in several fields in physics and chemistry [5–9].

Despite the beauty and power of this theory, a study of the dynamics of a particular system can be very cumbersome due to the sheer complexity of correctly incorporating the various time, energy and coupling scales. In order to circumvent this problem various approximations, such as the weak-coupling-to-the-bath, Markovian, high temperature, or the initial factorizing condition is usually invoked [5–9, 14]. However, the development of modern experimental techniques for preparing and manipulating physical and chemical systems has reached the regime where such approximations are questionable. This fact has encouraged the development of techniques for more consistently treating and analyzing open quantum systems (cf. Ref. 15).

Despite the refinement in technique, no approach is completely approximation-free [5–9, 15], and therefore some physical features of the dissipative dynamics are often absent in these descriptions [5–7, 15, 16].

The breakdown of these common approximations is expected to occur in the low temperature regime [17] and/or in the presence of initial correlations between the system and the environment [14] and, in particular, in the case of driven non-equilibrium quantum systems [12, 18–20]. The ubiquitous presence of this situation (an open quantum systems under the presence of time dependent fields in, e.g., coherent control scenarios of chemical systems [21] and assorted physical systems [15, 20, 22]), motivates a formal and detailed treatment of such dynamics.

In this paper, we derive formal \textit{exact} results for the non-Markovian dynamics of a prototypical system, including the presence of initial correlations between the system and the bath, and the possibility of arbitrary rapidly oscillating driving forces. The method can be applied in the low temperature and strong coupling-to-the-bath regimes. The particular system studied here is a charged quantum oscillator confined in a harmonic trap that is initially in thermodynamic equilibrium with its environment (non-factorized initial conditions [14]). For $t > 0$, we start varying the frequency of the harmonic trap and couple the oscillator, via the dipole, with an incoherent quantized blackbody radiation field and with an arbitrary coherent laser field. By means of the influence functional approach [2, 4, 14, 23], we derive analytic closed expressions for the non-Markovian time evolution of the covariance matrix elements of the quantum fluctuations valid at any temperature, any system-environment coupling strength and incorporating the zero-point fluctuations of the radiation.

This robust and general model allows us to address many different physical problems in generic open quantum systems. The results derived here can be used to study, for example:

\begin{itemize}
  \item \textit{i} The incoherent excitation of open quantum systems: In particular, our results allow us to model situations when a molecular system such as retinal or a photosynthetic light-harvesting complex, equilibrated with
it surroundings, is excited by either coherent (coherent laser pulses) or incoherent (sunlight or moonlight) light sources, a subject of great significance in the chemical physics community [24–27]. One such application of this method is given in Ref. 22.

ii Environmentally-assisted one-photon phase control: That weak field one-photon phase control is not possible for certain isolated quantum systems is a known result (cf. Refs. 21, 28, 29 and references therein). Recently, it was suggested that the coupling to the environment could allow, in principle, for the coherent manipulation of quantum systems in such cases [28, 29]. However, it is still unclear what physical mechanisms are behind this process, mainly because a formal study of this situation involves the presence of ultrafast field-induced modulations in open quantum systems at low temperature, a situation where non-Markovian processes cannot be disregarded and where approximations such as the rotating-wave fail [19]. Moreover, in this case of environmentally-assisted control, the presence of initial correlations between the system and the bath is vital. A treatment of this problem using the method developed herein is to be provided in Ref. 30.

iii Optimal-control-based cooling of quantum nano-resonators by means of parametric driving. On the basis of a numerical protocol, it was recently shown that nano-mechanical resonators can be cooled down by the delicate interplay of dissipative and driving process [20]. Being based on an optimal-control protocol, the possibility of parametrically driving the frequency of the resonator with arbitrary rapidly oscillating fields, as we consider here, is a key element in this cooling process. As in the previous case, this scenario is well beyond standard approximation schemes, but can be immediately cast as a particular case of our general model, with the great advantage of having an analytic formulae for the system dynamics.

iv The establishment of a quantum limit on non-Markovian time scales. In thermodynamical equilibrium, quantum features survive in the limit \( h\omega_0/k_{\text{B}}T > 1 \), where \( h\omega_0 \) is a typical energy scale of the system and \( T \) the temperature. According to Ref. 12, quantum features can persist for higher temperatures under non-equilibrium situations. However, results in Ref. 12 are based on the Markovian approximation, so the derivation a quantum limit consistent with the presence of non-Markovian effects is desirable. This problem can be addressed using our general approach and this study is currently in progress.

v Non-Markovian thermodynamics. For quantum systems, it is possible to have very fast control of heat and entropy due to anomalies induced by the non-Markovian character of the relaxation [11]. Our general results, can be immediately applied to study, e.g., heat transport between two non-Markovian reservoirs at the quantum regime.

This provides a sample list of problems that can be readily examined using the exact solution, derived below. It is the variety of challenging problems that can be addressed that motivates this paper, the derivation of an all-in-one versatile model that can treat a host of problems and which can be generalized to consider additional phenomena.

\[ \hat{H}_S = \frac{1}{2m} p^2 + \frac{1}{2} m \omega(t)^2 q^2 \]  
\[ \hat{H}_{TB} = \sum_{j} \left[ \frac{\hat{p}_j^2}{2m_j} + m_j \omega_j^2 \hat{q}_j^2 \right] \]  
\[ \hat{H}_{S-TB} = -\hat{q} \sum_{j} c_j \hat{q}_j + \hat{q}^2 \sum_{j} \frac{c_j^2}{2m_j \omega_j^2} \]

with \( \hat{p} \) and \( \hat{q} \) the canonically conjugate momentum and position of the oscillator (an analogous notation describes the bath modes), \( m \) the mass of the quantum oscillator and \( \omega_j(t)^2 = \omega_0^2 + \omega_j^2(t) \) the parametrically modulated frequency. This frequency comprises two components: \( \omega_0 \), a constant frequency and an arbitrary time-dependent-frequency \( \omega_j(t) \). The magnitude of the interaction between the system and the bath is determined by the coupling constants \( c_j \).

In the presence of the blackbody radiation the Hamiltonian \( \hat{H}_0 = \hat{H}_S + \hat{H}_{S-TB} + \hat{H}_{TB} \) needs to be augmented to include the interaction with the field as well as the Hamiltonian of the field modes,

\[ \hat{H}_{BB} = \sum_{k,s} \hbar c \left( \hat{a}_{k,s} + \frac{1}{2} \right). \]

Assuming that the charged oscillator interacts weakly with each mode in the field, we can adopt a dipole-dipole type interaction, giving the overall Hamiltonian

\[ \hat{H} = \frac{1}{2m} \left( \hat{p} - \frac{e}{c} \hat{A} \right)^2 + \frac{1}{2} m \omega(t)^2 \hat{q}^2 + \hat{H}_{S-TB} + \hat{H}_{TB} + \hat{H}_{BB}, \]

where \( e/c \) is the coupling constant to the radiation, \( \hat{a}_{k,s} \).
and \( \hat{a}_{k,s} \) are the annihilation and creation operators of the field mode of momentum \( k \) and polarization \( s \). The vector potential is given by

\[
\hat{A} = \sum_{k,s} \left( \frac{\hbar c}{kV} \right)^{1/2} f_k \hat{e}_{k,s} \left( \hat{a}_{k,s} + \hat{a}_{k,s}^\dagger \right),
\]

where \( \hat{e} \) is the polarization unit vector, \( V \) is the volume of the auxiliary cavity containing the field modes and \( f_k \) is the electron form-factor (Fourier transform of the charge distribution) that incorporates the electron structure \cite{31, 32}. We have assumed, with no loss of generality, that the form factor and polarization vector are real. Note that by virtue of Eq. (4), Eq. (5) already contains zero-point or vacuum fluctuations.

Equation (5) can be generalized to include an additional term \(-\hat{q}E_L(t)\), which allows for the possible manipulation of the charged oscillator, via dipole coupling, by means of the electric field \( E_L(t) \) of a pulsed or continuous laser field.

Since Eq. (5) includes the diamagnetic term \( \hat{A}^2 \), it is not suitable for a path integration calculation, which is why it is usually omitted \cite{33, 34}. However, the contribution of this term is relevant for the derivation of the partition function of the oscillator in the presence of the blackbody radiation (cf. the discussion in \cite{35, 36}). In our case, this term can be introduced by means of the Power-Zienau’s transformation \cite{37} (see also Ref. 38 for the original version and Ref. 39 and references therein for a short historical review on this transformation). \( \hat{T} = \exp \left\{ \frac{i}{\hbar} \hat{z} \cdot \hat{A} \right\} \), which transforms \( \hat{p} \to \hat{p} + \frac{\hbar}{i} \hat{z}, \hat{q} \to \hat{q}, \hat{p}_{k,s} \to \hat{p}_{k,s} + \frac{\hbar}{i} m_k \omega_k \hat{q}, \hat{q}_{k,s} \to \hat{q}_{k,s} \), where we have defined \cite{31, 32}

\[
\hat{a}_{k,s} = (m_k \omega_k \hat{q}_{k,s} + i \hat{p}_{k,s}) / \sqrt{2m_k \hbar \omega_k}, \tag{7}
\]

with \( m_k = 4\pi e^2 f_k^2 / (\omega_k V) \). The corresponding total Hamiltonian reads

\[
\hat{H} = \frac{1}{2m} \hat{p}^2 + \frac{1}{2m} \omega(t)^2 \hat{q}^2 - \hat{q}E_L(t) + \sum_j \left( \frac{\hbar^2}{2m_j} \hat{p}_j^2 + \frac{m_j \omega_j^2}{2} \left( \hat{q}_j - \frac{c_j}{m_j \omega_j^2} \hat{q} \right)^2 \right) + \sum_{k,s} \left( \frac{\hbar^2}{2m_k} \hat{p}_{k,s}^2 + m_k \omega_k \hat{q}_{k,s}^2 \right) + \frac{1}{2} \sum_{k,s} \frac{m_k \omega_k^2}{2} \hat{q}_{k,s}^2. \tag{8}
\]

Here the oscillator is seen to be coupled to the momentum coordinate \( p_{k,s} \).

Recalling that the electric field of the blackbody radiation is given by

\[
\hat{E} = -\frac{\partial \hat{A}}{\partial t} = i \sum_{k,s} \left( \frac{\hbar c^3}{kV} \right)^{1/2} f_k \hat{e}_{k,s} \left( \hat{a}_{k,s} - \hat{a}_{k,s}^\dagger \right), \tag{9}
\]

and examining Eq. (7), we see that in Eq. (8), the position of the oscillator is coupled to the electric field of the blackbody radiation. From an open-quantum-systems perspective, this implies that any statistical behavior induced by the blackbody radiation (as seen elsewhere \cite{22, 31, 32}) is dictated by the stochastic fluctuations of the electric field.

### III. INITIAL DENSITY MATRIX

After characterizing the Hamiltonian of the total system, the next step in the description of the dynamics is the determination of the initial state, taken to be the equilibrated state of \( S+TB \). By denoting the coordinates of \( S+TB \) as \( \hat{Q} = \{ \hat{q}, \hat{Q} \} \), the matrix elements of the initial density operator of the system \( S \) plus the environment TB can be calculated as \cite{14}

\[
\langle \hat{Q}' | \hat{\rho}_0 | \hat{Q}' \rangle = Z_{\beta_{TB}}^{-1} \int \mathcal{D} \hat{Q} \exp \left( -\frac{1}{\hbar} S^E[\hat{Q}] \right), \tag{10}
\]

where the integral is over all paths \( \hat{Q}(\tau), 0 \leq \tau \leq \hbar \beta_{TB} \) with \( \hat{Q}(0) = \hat{Q}' \) and \( \hat{Q}(\hbar \beta_{TB}) = \hat{Q}'' \). The bar stands for the trajectories \( \hat{Q}(t) \) in imaginary time \( t \to -\hbar \beta \), with \( \beta_{TB} = 1/(\hbar k_B T_{TB}) \). In the imaginary-time path integral described by Eq. (10), \( S^E[\hat{Q}] \) denotes the Euclidean action of the system \( S \) as \( \hat{q}^2 + \hat{Q}^2 \) and defined as \( Z_{\beta_{TB}} \int \mathcal{D} \hat{Q} \exp \left( -\frac{1}{\hbar} S^E[\hat{Q}] \right) \), obtained by introducing a global minus sign in the potential energy \cite{14, 40}. Since \( \hat{\rho}_0 \) denotes the density operator at \( t = 0 \), we assume that \( \omega(0) = 0 \) and \( E_L(0) = 0 \) in Eq. (8). The matrix elements of the initial total density operator of interest are

\[
\langle \hat{q}_{k,s}' | \hat{\rho}_0 | \hat{q}_{k,s}' \rangle = \langle \hat{q}_{k,s}^\dagger | \hat{\rho}_0 | \hat{q}_{k,s} \rangle \langle \hat{q}_{k,s}' | \hat{\rho}_0 | \hat{q}_{k,s}' \rangle, \tag{11}
\]

where \( \hat{\rho}_0 \) denotes the equilibrium density operator of the radiation only, at temperature \( T_{BB} \) and defined as in Eq. (10) using the Euclidean action \( S^E_{BB}[\hat{q}_{k,s}] \) of the bare radiation, \( \hat{\rho}_0 \) is the thermal density operator of system and bath introduced in Eq. (10), and the blackbody-radiation-mode-coordinates \( \hat{q}_{k,s} \) are defined in Eq. (7). When a system \( S \) is in contact with two thermal baths, what is usual in the literature (cf. Refs. 22, 30) is that the initial state of the total system is assumed to be factorized, in our case this is equivalent to take \( \hat{\rho}(0) = \hat{\rho}_{\beta_{BB}} \otimes \hat{\rho}_{\beta_{TB}} \otimes \hat{\rho}_S \), being \( \hat{\rho}_{\beta_{TB}} \) and \( \hat{\rho}_S \) the density operator of the thermal bath TB and the system \( S \), respectively. Here we deal with a more complex situation because we take into account the initial correlations between the system \( S \) and the thermal bath TB.

In general, one would also like to study the system \( S \) prepared in a state other than the equilibrium state, e.g., in a coherent state or in a squeezed state. According to Refs. 14, 41, one could prepare a different ini-
tial state by allowing the operators $\hat{A}_n, \hat{A}'_n$ acting only in the system Hilbert space of $S$ to generate an initial non-equilibrium density operator of system and bath $\hat{\rho}_S = \sum_n (\hat{A}_n \otimes 1_{TB}) \hat{\rho}_B (1_{TB} \otimes \hat{A}'_n)$, where $1_{TB}$ denotes the unit operator in the Hilbert space of the bath alone.

In the position representation, the matrix elements of $\hat{\rho}_S$ are given by

$$\langle Q'_+ | \hat{\rho}_S | Q'_- \rangle = \int d\bar{Q}' d\bar{Q}'' \lambda(q'_+, q''_-, q'_-, q'') \times \delta(Q'_+ - \bar{Q}'') \delta(Q'_- - \bar{Q}'),$$

where the propagating function $\lambda(q'_+, q''_-, q'_-, q'') = \sum_n (q'_+ | \hat{A}_n | q''_-) / (q' + q'_- | \hat{A}'_n | q''_-)$ characterizes the action of these operators. The delta functions indicate that the imaginary-time paths for the bath degrees of freedom are continuously connected to the real-time paths describing the time evolution of the initial state $|1_S\rangle$. The thermal initial state in Eq. (10) can be recovered by setting the auxiliary operators $\hat{A}_n$ and $\hat{A}'_n$ to $1_S$, which yields $\lambda(q'_+, q''_-, q'_-, q'') = \delta(q'_+ - q''_-) \delta(q'_- - q''_-)$. As distinct from $Q'$ and $Q''$, which are the endpoints of a single imaginary-time-trajectory, $Q'_+$ and $Q'_-$ denote the initial condition for two real-time-trajectories “+” and “−” (see below).

For the case of unitary evolution these can be identified as the forward and backward trajectories associated to the unitary time-evolution-operator and its adjoint, respectively. Additional details about different initial preparations can be found in Refs. 14, 41.

IV. DYNAMICS OF THE SYSTEM

In this section we solve for the time evolution of the initial density matrix [Eq. (11)] under the action of the Hamiltonian [Eq. (8)], using the Feynman and Vernon influence functional approach [2, 23]. For this problem we require a mixture of the influence functional for factorizing initial conditions [2, 4, 18, 23] in order to consider the effect of the radiation, and for non-factorizing initial conditions [14, 41] in order to correctly characterize the equilibrium state between the oscillator and the thermal bath at $t = 0$.

It is worth mentioning that the standard path integral calculations are performed for position-position couplings [2, 4, 5, 41] [cf. the term $\hat{q} \hat{q}_j$ in Eq. (8)], and for momentum-momentum coupling [4]. The oscillator-radiation coupling is of position-momentum type, $\hat{q} \hat{p}_{k,s}$ [cf. Eq. (8)]. By means of a set of unitary transformations [32], one could invert the role of the momentum and position operators of the field [see Eq. 7], with the caveat that this generates an additional term proportional to the initial position of the oscillator (see below). However, since the system described in Eq. (8) is still linear, the path integral calculation can also be carried out analytically for the position-momentum coupling. This is the approach followed below.

A. Derivation of the propagating function and the influence functional

The time evolution of the system S is described by the reduced density operator $\hat{\rho}_S(t) = \text{tr}_{TB,BB} \hat{\rho}(t)$. Following Ref. 14, we obtain that the matrix elements of $\hat{\rho}_S(t)$ are given by

$$\langle q''_+ | \hat{\rho}_S(t) | q''_- \rangle = \int dq'_+ dq''_+ dq''_- \lambda(q'_+, q''_-, q''_-) \times J(q''_+, q''_-; t, q'_+, q''_-, 0; q''_-, q''),$$

where $J(q''_+, q''_-, t; q'_+, q''_-, 0; q''_-, q'')$ is the propagating function of the system density matrix which can be expressed in terms of the functional phase $\Sigma[q'_+, q''_-, 0]$ by means of the three-fold path integral expression

$$J(q''_+, q''_-, t; q'_+, q''_-, 0; q''_-, q'') =$$

$$\frac{1}{Z} \int_{q''_+} \int_{q''_-} \int_{\bar{q}'} \exp \left( \frac{i}{\hbar} \Sigma[q'_+, q''_-, 0] \right),$$

where $Z$ normalizes $J(q''_+, q''_-, t; q'_+, q''_-, 0; q''_-, q'')$ to $\delta(q''_+ - q'_+) \delta(q''_- - q''_-) \delta(q''_+ - q''_-) / (\hat{\rho}_B, S | q'')$ at $t = 0$, being $\hat{\rho}_B, S = \text{tr}_{TB} (\hat{\rho}_B)$. The real time path-integrals over $q'_+$ and $q''_-$ are carried out subject to the endpoints $q'_+(0) = q''_+$, $q'_+(t) = q''_+$, $q''_-(0) = q'_-$ and $q''_-(t) = q''_-$, while the imaginary time path integral are over $\bar{q}(0) = \bar{q}$ and $\bar{q}(t h_{TB}) = \bar{q}'$. Recall that the imaginary time path-integral allows for the calculation of the equilibrated density operator of $S + \text{TB}$ and the influence of their initial correlations in the subsequent time evolution.

After tracing over the degree of freedom of TB and BB, and after defining $q_+ = (r + x)/2$ and $q_- = r - x$, we have that the functional phase $\Sigma[x, r, \bar{q}]$ is given by
The last term in the second line of Eq. (15) accounts for the equilibrium density operator of \( S \) in the presence of the thermal bath TB while the second term containing \( \bar{q}(\tau)x(s) \) is responsible of the the effect of initial correlations between the environment and the system on the parametric harmonic potential, while the fourth term is responsible for the evolution induced by the laser field \( E_L \). As previously noted, we take \( \omega_T(0) = 0 \) and \( E_L(0) = 0 \). The last term in the second line of Eq. (15) arises from the incoherent excitation induced by the position-position coupling to the thermal bath TB. Since the coupling to the blackbody radiation field BB is of a different nature, position-momentum coupling, this transient term proportional to the initial position \( r' \) is not present. However, if one changes the role of position and momentum, as discussed above, this transient term enters implicitly. The terms in the third line constitute the exponent of the influence functional of the Feynman-Vernon theory under the action of the thermal bath TB and blackbody radiation BB.

The additional time integration in the last line of Eq. (15) over \( u \) accounts for the non-local time (non-Markovian) evolution of the density operator. Although the temporal non-locality is determined by the various kernels in a cumbersome way, we can identify two kinds of non-Markovian contributions: one from the dissipative part and determined by the non-local character of \( \eta_{TB} \) and \( \eta_{BB} \), and a second determined by the thermal fluctuations described (see below) by the kernels \( K_{TB}^r \) and \( K_{BB} \). The presence of the latter is not determined by the presence of the former, i.e., in the limit of local dissipative Ohmic kernels, \( \eta_{TB}(s) \sim \delta(s) \), the non-local character of the thermal fluctuations is still present; it only vanishes in the high temperature regime [5, 42].

\[
\begin{align*}
\Sigma[x, r, \bar{q}] &= \frac{h\beta_{TB}}{i} \int_0^t d\tau \left[ \frac{m}{2} \dot{r}^2 + \frac{1}{2} m \omega_0^2 \bar{q}^2 + \frac{1}{2} \int_0^{h\beta_{TB}} d\sigma k_{TB}(\tau - \sigma)\bar{q}(\tau)\bar{q}(\sigma) \right] + \int_0^t d\tau \int_0^t ds K_{TB}^r(s - i\tau)\bar{q}(\tau)x(s) \\
&+ \int_0^t ds \left\{ m\dot{x}(s)\dot{r}(s) - m\omega_0^2 \gamma(s) x(s) - m\omega_T(t)^2 r(s)x(s) + E_L(t)x(s) - r'\eta_{TB}(s)x(s) \right\} \\
&- \int_0^t ds \left\{ \int_0^t du [\eta_{TB}(s) + \eta_{BB}(s)] x(s)\dot{r}(u) - \frac{i}{2} \int_0^t du [K_{TB}^{\eta}(s-u) + K_{BB}(s-u)] x(s)\dot{u} \right\}.
\end{align*}
\]

Note that \( r' = q'_s + \frac{1}{2} q''_s \), \( r'' = q''_s + \frac{1}{2} q''_s \) and analogously for \( x'' \) and \( x' \). The various kernels entering into Eq. (15) are defined in the next section.

### B. Kernels in the functional action

The quantities introduced in the effective action \( \Sigma[x, r, q] \) are defined in terms of the bath spectral density \( J_{TB} \) and the blackbody-radiation spectral density \( J_{BB} \). These spectral densities are determined [4, 14, 31, 32] from the parameters of the bath modes and the coupling constants by means of

\[
J_{TB}(\omega) = \frac{\pi}{2} \sum_{j=1}^{\infty} \frac{c_j^2}{m_j \omega_j^2} \delta(\omega - \omega_j),
\]

\[
J_{BB}(\omega) = \frac{\pi}{2} \sum_{k,s} m_k \omega_k^3 \delta(\omega - \omega_k).
\]

Assuming that the thermal bath is dense in the frequency of the modes [4], it is customary to describe the spectral density in Eq. (16) by, e.g., assuming the Ohmic model

\[
J_{TB}(\omega) = m\gamma_{TB}\omega^3 \Omega_{TB}^2 / (\Omega_{TB}^2 + \omega^2),
\]

where \( \gamma_{TB} \) is the coupling constant to the bath TB and \( \Omega_{TB} \) is a cutoff parameter related to the inverse of the bath memory time. In contrast with the thermal bath case, no assumption on the functional form of the spectral density of the blackbody is needed in the continuous limit [31, 32, 34, 43].

The transversality condition implies that only two of the three components of \( k \) contribute to the coupling [32], giving a global factor of two-thirds for the spectral density in Eq. (17). In the continuous limit, \( \sum_k \rightarrow \frac{1}{(2\pi)^3} \int d\mathbf{k} \), the spectral density for the blackbody radiation is

\[
J_{BB}(\omega) = M\tau_{BB}\omega^3 \Omega_{BB}^2 / (\Omega_{BB}^2 + \omega^2),
\]

where \( M = m + M_{TB} \Omega_{BB} \) is the renormalized mass, \( \tau_{BB} = 2e^2/3mc^3 \) and \( \Omega_{BB} \) is a frequency cutoff. This path-integral-based expression coincides completely with the seminal results in Refs. 31, 32, 43 using the quantum Langevin formalism. It also coincides with the result derived in Ref. 34 using the standard path integral approach. However, we need to note that in Ref. 34, the system, an electron, is interacting with its own radiation;
here, by difference, we consider the system as being irradiated by an external blackbody radiation such as sunlight or moonlight for $t > 0$. This is precisely what allows us to separate the initial density operator of the system and the radiation. This natural emerging functional form of $J_{BB}(\omega)$ reveals, from a statistical viewpoint, the intrinsic non-Markovian character of the radiation [31, 32, 34, 43]. This fact implies that the two point correlation function, $(\langle \mathcal{E}(t') \mathcal{E}(t''\rangle )_{BB}$, of the electric field in Eq. (9) is not delta correlated. From an optics point of view [44] this means that the blackbody radiation is coherent, although the coherence time is very short, $\sim 1.3$ fs at $T_{BB} = 5900$ K (cf. Refs. 22 and Chap. 13 in Ref. 45).

Once we have condensed the relevant information for the thermal bath and the radiation field in the spectral densities Eqs. (18) and (19), we are in the position to define the various functions entering into the functional actions $\Sigma[x, r, \tilde{q}]$.

The kernels $K_{TB}(s - ir)$ and $k_{TB}(\tau)$ are given by [14]

\[ K_{TB}(s - ir) = K_{TB}^{re}(s - ir) + iK_{TB}^{im}(s - ir), \]

\[ k_{TB}(\tau) = \frac{m}{\hbar \beta_{TB}} \sum_{n = -\infty}^{\infty} \zeta_n(0) \exp(i\nu_n \tau), \]

with the Matsubara frequencies $\nu_n = 2\pi n / \hbar \beta_{TB}$. The kernel $K_{TB}(s - ir)$ contains the information of the thermal fluctuations due to the bath and its influence in the loss of coherence as well as the in the decay of correlations between the system and the bath. The kernel $k_{TB}(\tau)$ contains the influence of the bath on the thermal equilibrium state of the system $S$. The real and imaginary parts of $K_{TB}(s - ir)$ are given by

\[ K_{TB}^{re}(s - ir) = \int_0^\infty \frac{d\omega}{\pi} J_{TB}(\omega) \frac{\cosh[\omega(\frac{1}{2} \hbar \beta_{TB}) - \tau]}{\sinh(\frac{1}{2} \omega \hbar \beta_{TB})} \cos(\omega s), \]

\[ = \frac{m}{\hbar \beta_{TB}} \sum_{n = -\infty}^{\infty} g_n(s) \exp(i\nu_n \tau), \]

\[ K_{TB}^{im}(s - ir) = -\int_0^\infty \frac{d\omega}{\pi} J_{TB}(\omega) \frac{\sinh[\omega(\frac{1}{2} \hbar \beta_{TB}) - \tau]}{\sinh(\frac{1}{2} \omega \hbar \beta_{TB})} \sin(\omega s), \]

\[ = \frac{m}{\hbar \beta_{TB}} \sum_{n = -\infty}^{\infty} j_n(s) \exp(i\nu_n \tau). \]

while $\zeta_n(s) = \frac{1}{m} \int_0^\infty \frac{d\omega}{\pi} \frac{J_{TB}(\omega)}{\omega} \frac{2\nu_n^2}{\omega^2 + \nu_n^2} \cos(\omega s)$ or in terms of the damping kernel $\gamma_{TB}(s)$ [14], $\zeta_n(s) = \frac{1}{\gamma_{TB}(s)} \int_0^\infty du \gamma_{TB}(u)[\exp(-|\nu_s(s + u)|) + \exp(-|\nu_s(s - u)|)].$

The functions $g_n(s)$ and $f_n(s)$ can be expressed in terms of the damping kernels

\[ \gamma_{TB, BB}(s) = \frac{2}{m} \int_0^\infty \frac{d\omega}{\pi} \frac{J_{TB, BB}(\omega)}{\omega} \cos(\omega s), \]

and $\zeta_n(s)$ as $g_n(s) = \gamma_{TB}(s) - \zeta_n(s)$ and $f_n = -\frac{1}{m} \frac{d}{ds} g_n(s)$. The spectral density in Eq. (18) generates the damping kernel $\gamma(s) = \gamma_{TB} \Omega_{TB} \exp(-\Omega_{TB}|s|)$. In the limit when the cutoff frequency $\Omega_{TB}$ tends to infinity, $\gamma(s) \rightarrow 2\gamma_{TB}\delta(s)$, which corresponds to Markovian Ohmic dissipation. The spectral density in Eq. (19) generates $\gamma_{BB}(s) = \tau_{BB}\Omega_{BB}^2 [2\delta(s) - \Omega_{BB} \exp(-\Omega_{BB}|s|)]$. Note that there is a fundamental limitation to the use of Eq. (19). That is, in the limit $\Omega_{BB} \rightarrow \infty$, we get the surprising result that $\gamma_{BB}(s) = 0$, i.e. no relaxation [46, 47]. This corresponds to the point-electron limit $[fe = \Omega_{BB}^2/(\Omega_{BB}^2 + \omega^2) = 1$ in Eq. (6)] and is unphysical because even for the electron, $\Omega_{BB}$ remains finite, although large. According to Refs. 46, 47, there is a natural upper value given by $\Omega_{BB} = \tau_{BB}^{-1}$, which corresponds to two-thirds of the time for a photon to traverse the classical electron radius ($r_c = 2.818 \times 10^{-15}$m). Beyond this natural limit, causality is violated [46] and the bare mass $m$ takes negative values [46].

Finally, the kernel $K_{BB}(s)$ is given by

\[ K_{BB}(s) = \int_0^\infty \frac{d\omega}{\pi} J_{BB}(\omega) \coth\left(\frac{\omega \beta_{BB}}{2}\right) \cos(\omega s). \]

This kernel is responsible for the decoherence due to thermal fluctuations induced by the blackbody radiation, while the kernels $\eta_{TB}(s) = m\gamma_{TB}(s)$ and $\eta_{BB}(s) = m\gamma_{BB}(s)$ in Eq. (15) induce the relaxation process.

C. Explicit calculation of the propagating function

The explicit calculation of the propagating function demands evaluating the path integral in Eq. (14). Since, the system is linear, the path integral can be performed by evaluating the action in Eq. (15) along its stationary trajectories and condensing the effect of the fluctuations in a global time dependent factor [5, 14]. The extremum of the action for imaginary time is given by

\[ m\tilde{q} - m\omega_0^2 \tilde{q} - \int_0^{\hbar \beta_{TB}} d\sigma k_{TB}(\tau - \sigma) \tilde{q}(\sigma) = -i \int_0^t ds K_{TB}^{re}(s - ir)x(s), \]

where we can see how the dynamics in real time, represented by $x(s)$, drives the system-bath correlations, by driving the imaginary time path $q(\tau)$ in a non-local way.
For real time, the action is stationary along

\[ m\dot{r} + m\omega(t)^2 r - E_L(t) + \frac{d}{ds} \int_0^s du \eta(s-u)r(u) \]

\[ = r'\eta_{BB}(s) + i \int_0^t du K(s-u)x(u) + \int_0^{\hbar\beta_{TB}} d\tau K_{TB}^{re}(s-i\tau)q(\tau), \]

\[ m\ddot{x} + m\omega(t)^2 x - \frac{d}{ds} \int_0^s du \eta(s-u)x(u) = 0, \]

(27)

where we have defined

\[ \eta(s) = \eta_{TB}(s) + \eta_{BB}(s), \quad K(s) = K_{TB}^{re}(s) + K_{BB}(s). \]

The term \( r'\eta_{BB}(s) \) appears here as a consequence of the sudden turn-on of the blackbody radiation. Since, we assume that the parametric driving, as well as the laser field and the blackbody radiation act after \( t = 0 \), the equilibrium state of our system coincides with the one derived in Ref. 14, so we need to focus only on the evaluation of the real part of the action.

The real part of the action is stationary along the solution to the equation of motion

\[ m\ddot{r} + m\omega(t)^2 r + m\omega(t)^2 r = \bar{E}_L(s) + i \int_0^t du R(s-u)x(u), \]

\[ m\ddot{x} - m\omega(t)^2 x + m\omega(t)^2 x = 0, \]

(28)

where we have defined \( \bar{E}_L(s) = E_L(s) + r'\eta_{BB}(s) + m\bar{x}C_1(s) - i\bar{x}C_2(s) \) with \( \bar{r} = \bar{q} + \bar{q}'/2 \) and \( \bar{x} = \bar{q} - \bar{q}' \). Additionally, we have defined \( \eta(s) = m\gamma(s) \) and

\[ R(s,u) = R_{TB}(s,u) + K(s-u)/m \]

with

\[ R_{TB}(s,u) = -A_{TB} C_1(s) C_1(u) \]

\[ + \frac{1}{\hbar\beta_{TB}} \sum_{n=-\infty}^{\infty} u_n [g_n(s)g_n(u) - f_n(s)f_n(u)], \]

(31)

\[ C_1(s) = \frac{1}{\hbar\beta_{TB} \Lambda_{TB}} \sum_{n=-\infty}^{\infty} u_n g_n(s), \]

(32)

\[ C_2(s) = \frac{1}{\hbar\beta_{TB}} \sum_{n=-\infty}^{\infty} u_n \nu_n f_n(s), \]

(33)

where \( \Lambda_{TB} = \frac{1}{\hbar\beta_{TB}} \sum_{n=-\infty}^{\infty} u_n \) can be related to the second moment of the position of the system at equilibrium, \( \langle q^2 \rangle_{\text{equil.}} = (\hbar/m)\Lambda_{TB}, \) and \( u_n = (\omega_n^2 + \nu_n^2 + \zeta_n)^{-1}. \)

Since, for a harmonic potential, the functional action \( \Sigma[x, \dot{r}, \dot{q}] \) can be evaluated using only the real part of the trajectories \( r(s) \) and \( x(s) \) [14, 41, 48, 49], we need to solve only for the real part of Eq. (29) and (30). Due to the linear character of (29), the solution to the homogeneous part can be written as

\[ r(s) = r_0 \frac{\phi_1(s)}{\phi_1(t)} + r' \left( \frac{\phi_2(s) - \phi_2(t)}{\phi_1(t)} \right) \phi_1(s), \]

(34)

where \( \phi_1(s) \) is the fundamental solution for \( r(0) = 0 \) and \( \dot{r}(0) = 1 \), while \( \phi_2(s) \) is the fundamental solution for \( r(0) = 1 \) and \( \dot{r}(0) = 0 \). Thus, for Eqs. (29) and (30) we have

\[ r_{re}(s) = r' \phi_1(s) \frac{\phi_1(t)}{\phi_1(t)} + r' \left( \frac{\phi_2(s) - \phi_2(t)}{\phi_1(t)} \phi_1(s) \right) \]

\[ + \frac{1}{m} \int_0^s du \phi_1(s-u) \dot{F}(u) - \frac{1}{m} \frac{\phi_1(s)}{\phi_1(t)} \int_0^t du \phi_1(t-u) \dot{F}(u). \]

(35)

Since \( \dot{F}'(s) \) contains the term induced by the sudden coupling to the radiation, \( r'\eta_{BB}(s) \), we can see that \( r(s) \) is driven by this sudden turn-on. Following a similar procedure for \( x(s) \), we get

\[ x(s) = x' \frac{\varphi_1(s)}{\varphi_1(t)} + x' \left( \varphi_2(s) - \varphi_2(t) \right) \frac{\varphi_1(s)}{\varphi_1(t)} \]

(36)

where \( \varphi_1(s) \) is the fundamental solution for \( x(0) = 0 \) and \( \dot{x}(0) = 1 \), while \( \varphi_2(s) \) is the fundamental solution for \( x(0) = 1 \) and \( \dot{x}(0) = 0 \). For the particular case of no frequency modulation, \( \phi_1(s), \phi_2(s), \varphi_1(s) \) and \( \varphi_2(s) \) can be derived from standard Laplace techniques [14]. For Markovian dissipation, \( \Omega_{TB} \to \infty \) in Eq. (18), and harmonic modulation of the frequency, \( \phi_1(s), \phi_2(s), \varphi_1(s) \) and \( \varphi_2(s) \) are related to the Mathieu functions [12, 18]. For more general cases, these functions must be calculated numerically. However, the functional form of Eqs. (35) and (36) is very convenient of the subsequent analytical calculations. For further convenience we define

\[ v_1(t, s) = \varphi_2(s) - \frac{\varphi_2(t)}{\varphi_1(t)} \varphi_1(s), \quad v_2(t, s) = \frac{\varphi_1(s)}{\varphi_1(t)}, \]

(37)

\[ u_1(t, s) = \phi_2(s) - \frac{\phi_2(t)}{\phi_1(t)} \phi_1(s), \quad u_2(t, s) = \frac{\phi_1(s)}{\phi_1(t)} \]

(38)

The influence functional in Eq. (14) can now be rewritten as

\[ J(r'', x'', \tau'; r', x', \tau; \bar{r}, \bar{x}) = \frac{1}{N(t)} \exp \left( \frac{1}{\hbar} \Sigma[r, x, \tau, \bar{r}, \bar{x}] \right), \]

(39)
where $N(t)$ is a normalization factor given by $N(t) = \frac{2\pi \hbar}{\Delta\Omega_{TB}} |\tilde{u}_2(t, 0)| (2\pi \hbar \frac{1}{T^2})^{1/2}$. After evaluating Eq. (15) along $r^s(s)$ and $x(s)$, we get

$$
\Sigma[r'', x'', r', x', \bar{r}, \bar{x}] = 
\text{im} \left( \frac{1}{2\Delta T_{TB}^2} + \frac{\Omega_{TB}}{2} \bar{x}^2 \right) + m [x''r'' \tilde{u}_2(t, t) + x'r' \tilde{u}_1(t, 0)] - m [x'r'' \tilde{u}_2(t, 0) - x''r' \tilde{u}_1(t, t)]
+ m \int_0^t ds \ [x'v_1(t, s) + x''v_2(t, s)] \left[ \frac{1}{m} E_L(s) + r' \gamma_{BB}(s) + r C_1(s) - i\bar{x}C_2(s) \right]
+ \frac{i}{2} m x' x' \int_0^t ds \int_0^t du R(s, u)v_1(t, s)v_1(t, u) + \frac{i}{2} m x' x'' \int_0^t ds \int_0^t du R(s, u)v_1(t, s)v_2(t, u)
+ \frac{i}{2} m x'' x' \int_0^t ds \int_0^t du R(s, u)v_2(t, s)v_1(t, u) + \frac{i}{2} m x'' x'' \int_0^t ds \int_0^t du R(s, u)v_2(t, s)v_2(t, u),
$$

where $\Omega_{TB} = \frac{1}{\hbar T_{TB}} \sum_{n=-\infty}^{\infty} u_n(\omega_0 + \zeta_n)$ can be related to the second moment of the momentum, $(q^2)_{\text{equl}} = h m \Omega_{TB}$, at equilibrium with TB. The first term in Eq. (40) containing $\Delta T_{TB}$ and $\Omega_{TB}$ can be associated to the thermal equilibrium state influenced by the presence of the thermal bath TB. These results provide the general expression for the influence functional.

**D. Limiting cases**

The general result in Eq. (39) includes, and agrees with, several limiting cases. These include:

i. In absence of blackbody radiation and with no parametric modulation of the frequency, Eq. (39) reduces to the result in Ref. 14.

ii. In absence of the thermal bath and no parametric modulation of the frequency, Eq. (39) is the formal path integral equivalent of Refs. 31, 32 (based on the quantum Langevin equation formalism).

iii. In absence of the thermal bath and with no parametric modulation of the frequency, Eq. (39) is the formal path integral equivalent of Refs. 31, 32 to the result in Refs. 50, 51.

$$
A(t) = \frac{m}{\hbar} \begin{pmatrix}
R_{12}(t) & -i\bar{u}_2(t, t) & -C_2(t) + R_{12}(t) & -i\bar{u}_1(t, t) - i\bar{C}_1(t) & \Omega_{TB} - 2C_2^+(t) + R_{11}(t) & -i\bar{u}_1(t, 0) - i\bar{C}_1^+(t) \\
-i\bar{u}_2(t, t) & 0 & -C_2(t) + R_{12}(t) & -i\bar{u}_1(t, t) - i\bar{C}_1(t) & 0 & 1/\Delta T_{TB} \\
-R_{12}(t) & i\bar{u}_2(t, 0) & i\bar{u}_2(t, 0) & 0 & 0 & 0 \\
-C_2(t) + R_{12}(t) & i\bar{u}_2(t, 0) & i\bar{u}_2(t, 0) & 0 & 0 & 0 \\
-i\bar{u}_1(t, t) - i\bar{C}_1(t) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
$$

being

$$
C^+_j(t) = \int_0^t ds C_j(s) v_1(t, s), \quad C^-_j(t) = \int_0^t ds C_j(s) v_2(t, s).
$$

iv. In absence of blackbody radiation, for no laser field, harmonic modulation of the frequency, Eq. (39) reduces to Ref. 18 (see also Ref. 12).

v. In absence of blackbody radiation, for no laser field, no modulation of the frequency and for factorized initial conditions, Eq. (39) reduces to Refs. 4, 52 (see also Ref. 49 for a description in terms of the Wigner function and Ref. 53 for a master equation approach).

**E. Explicit form of the propagating function**

For an initial thermal state, i.e. $\lambda(q^+_+, q^0_-, q^0_-, q^-) = \delta(q^+_+ - q^0_+) \delta(q^0_- - q^-)$ in Eq. (12) and correspondingly $\bar{r} = r'$ and $\bar{x} = x'$, the influence functional in Eq. (39) with the function phase given in Eq. (40) can be written in the very compact form

$$
J(r'', x'', t; r', x', 0; r', x') = \frac{1}{N(t)} \times \exp \left\{ -\frac{1}{2} x^T A(t) x + E_L^+(s) x' + E_L^-(s) x'' \right\}
$$

where $x^T A(t) x = x_f^T A_f(t) x_f + x_i^T A_i(t) x_i + 2B x$, where $x = (x'', x', r', x')$, $x_f = (x'', r')$, $x_i = (x', r')$, $A_f = (A_{13}x'' + A_{23}r'')$, $A_i = (A_{14}x'' + A_{24}r'')$ with the time dependent matrix $A(t)$ given by

The $C^+_j(t)$'s functions account for the influence of the initial correlations between the system and the bath on
the system dynamics. \( \hat{C}_+^+(t) \) is obtained by replacing \( C_1(t) \rightarrow C_1(t) + \gamma_{\text{BH}}(t) \) in the definition of \( C_+^+(t) \) in Eq. (43) and contains the effect of the turn on of the interaction with the blackbody radiation. The effects of the laser field \( E_L \) on the dynamics are encoded in

\[
E_L^+(t) = \int_0^t ds E_L(s)v_1(t, s), \quad E_L^-(t) = \int_0^t ds E_L(s)v_2(t, s).
\]

The decoherence dynamics is governed by the \( R_{ij}(t) \) functions given by

\[
R_{ij}(t) = \int_0^t \int_0^t ds du R(s, u)v_i(t, s)v_j(t, s).
\]

Once we have completely characterized the functional form of the propagating function, we proceed in the next sections to derive explicit expressions for the covariance matrix elements of the quantum fluctuations and the time evolution of the reduced density-operator.

V. TIME EVOLUTION OF THE COVARIANCE MATRIX ELEMENTS

Due to the linear nature of the system under consideration, the dynamics as well as the statistical properties can be characterized in terms of the first and second moments \[52\], defined as \( \langle f(q(t)) \rangle = \int dr'' f(r'')\rho_\beta(r'', x'') = 0, t \), or in terms of the propagating function in Eq. (41) by,

\[
\langle f(q(t)) \rangle = \int dr'' dr' dx' f(r'')J(r'', x'' = 0, r', x', x').
\]

So, we can now make use of the explicit form of the propagating function in Eq. (41). Note that the propagating function in Eq. (41) also contains the initial thermal state \( \rho_\beta \). These moments define the variances or dispersion relations

\[
\sigma_{qq}(t) = \langle q^2(t) \rangle - \langle q(t) \rangle^2,
\]

\[
\sigma_{qp}(t) = \frac{1}{2} \langle q(t)p(t) + p(t)q(t) \rangle - \langle q(t)p(t) \rangle,
\]

\[
\sigma_{pp}(t) = \langle p^2(t) \rangle - \langle p(t) \rangle^2,
\]

which will be used in order to express explicitly the time evolution of the density matrix in Eq. (56).

The first moments are determined by

\[
\langle q(t) \rangle = \frac{1}{m \tilde{u}_2(t, 0)} \int_0^t ds v_1(t, s)E_L(s)
\]

\[
\langle p(t) \rangle = \int_0^t ds \left[ v_1(t, s) \frac{\tilde{u}_2(t, t)}{\tilde{u}_2(t, 0)} + v_2(t, s) \right] E_L(s)
\]

where the overdot denotes the derivative with respect to \( s \), i.e., \( \dot{u}_2(t, 0) = \partial u_2(t, s)/\partial s \, |_{s=0} \).

The second moments are given by

\[
\langle q^2(t) \rangle = \langle q(t) \rangle^2 + \frac{E^2}{m^2 \tilde{u}_2^2(t, 0)} \left[ M_{11}(t) - \frac{\hbar}{m} M_{12}(t)^2 \right],
\]

\[
\frac{1}{2} \langle pq + qp \rangle = m \langle q^2(t) \rangle \dot{u}_2(t, t) + \frac{\hbar}{m \tilde{u}_2(t, 0)} \left[ R_{12}(t) - C_2(t) \right]
\]

\[
+ \frac{i \hbar^2 \Lambda_{\text{TB}}}{m \tilde{u}_2(t, 0)} \left[ \hat{C}_1^-(t) + \hat{u}_1(t, t) \right] M_{12}(t)
\]

\[
+ \langle q(t) \rangle \int_0^t ds v_2(t, s)E_L(s),
\]

\[
\langle p^2(t) \rangle = \hbar m R_{22}(t) + \hbar m \Lambda_{\text{TB}} \left[ \hat{C}_1^-(t) + \hat{u}_1(t, t) \right]^2
\]

\[
- m^2 \dot{u}_2^2(t, t) \langle q^2(t) \rangle + m \tilde{u}_2(t, 0) \langle pq + qp \rangle
\]

\[
+ \left[ \int_0^t ds v_2(t, s)E_L(s) \right]^2,
\]

where

\[
M = \frac{m}{\hbar} \left( \Omega_{\text{TB}} - 2C_2^+(t) + R_{11} - i \left[ \hat{u}_1(t, 0) + \hat{C}_+^+(t) \right] \right).
\]

In the absence of the parametric driving, the blackbody radiation and the laser field, Eqs. (52)-(54) are time-independent and coincide with the expressions (6.62)-(6.64) in Ref. 14, i.e., \( \langle q^2(t) \rangle = (\hbar/m) \Lambda_{\text{TB}} \), \( \langle pq + qp \rangle = 0 \) and \( \langle p^2(t) \rangle = \hbar m \Omega_{\text{TB}} \).

VI. TIME EVOLUTION OF THE REDUCED DENSITY-OPERATOR

If one is interested in the reduced-density operator itself, it can be written in terms of the second moments as

\[
\langle r'' | \hat{\rho}_\beta(t) | x'' \rangle = (2\pi \sigma_{qq}(t))^{-1/2} \exp \left[ -\frac{1}{2\sigma_{qq}(t)} [r'' - \langle q(t) \rangle]^2 \right] - \frac{1}{2\hbar^2} \left( \frac{\sigma_{pp}(t) - \sigma_{pq}(t)^2}{\sigma_{qq}(t)} \right) \frac{x''}{2} \)

\[
+ \frac{i}{\hbar} \left\{ \langle p(t) \rangle + \frac{\sigma_{pq}(t)^2}{\sigma_{qq}(t)} (r'' - \langle q(t) \rangle) \right\} x'.
\]

In summary, to obtain the time evolution of the reduced density operator, we proceed as follows:

i One first specifies the spectral density \( J_{\text{TB}}(\omega) \) to obtain the function describing the modulating force in Eq. (8). This permits us to obtain the fundamental
solutions $\phi_{1,2}(s)$ and $\varphi_{1,2}(s)$ in Eq. (35) and Eq. (36), respectively.

ii One then calculates the kernels $K_{TB}(s)$, $k(\tau)$ and $K_{BH}(s)$ defined in Eqs. (20), (21) and (25).

iii With the fundamental solutions obtained, and all the kernels calculated, we calculate the auxiliary functions $v_{1,2}(t, s)$ and $v_{1,2}(t, s)$ given in Eqs. (37) and (37), and subsequently the functions $C^2_{1,2}(t)$, $E^2(t)$ and $R_{ij}(t)$ defined by Eqs. (43), (44) and (45), respectively.

iv One then calculates the first and second moments given in Eqs. (50–54), and subsequently the dispersion relations in Eqs. (47–49) and system dynamics via Eq. (56).

This brief prescription concludes our completely formal and approximation-free treatment. Note that we have successfully applied the method to a number of cases, some of which are reported elsewhere [22, 30].

VII. CONCLUDING REMARKS

As discussed in the introduction, the results derived here can be used to study a wide variety of problems, e.g., the incoherent [22] or coherent [30] excitation of open quantum systems in order to provide physical insight into the role of coherences detected in photosynthetic light-harvesting complexes (for a review in the subject see Ref. 15). In doing so, we need to translate the propagating function in Eq. (41) into the energy basis in order to identify the incoherent/coherent nature of the excitation.

Our results allow us to directly study the possibility of environmentally assisted one-photon phase control [28, 29] provided by the fact that the initial equilibrium density matrix deviates from the canonical distribution. In this respect, we eliminate the incoherent radiation and the frequency modulation contributions and focus on how the phase information encoded in $E_{ij}(t)$ can be used to manipulate the populations of the oscillator [30].

Additionally, the analytic closed expression could be useful in understanding the delicate balance between dissipation and driving under non-Markovian evolution that has been pointed out in Ref. 20 in the context of optimal control theory and cooling of nano-mechanical resonators. In particular, the optimal cooling protocol addressed in Ref. 20 by means of numerical techniques, can be analyzed in great detail from the second moments derived in Eqs. (52–54) and the theory of variational calculus.

In Ref. 12, it was established that the usual quantum limit, $\hbar\omega/k_B T > 1$, needs to be reformulated for out-of-equilibrium systems. However, in that work a Markovian Ohmic spectral density, $\Omega_{TB} \to \infty$ in Eq. (18), was used, disregarding in this way the dynamics during non-Markovian time scales. The physical system considered in Ref. 12 consisted of two identical harmonic oscillators with time-depend coupling, in the normal mode description we get two independent parametric oscillators. So, under the same circumstances considered in Ref. 12, the results derived here allow us to explore the limit for the presence of quantum features in non-Markovian-driven-open-quantum systems, which is of great importance in, e.g., quantum statistical mechanics or control theory.

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