A lemma on a total function defined over the Baker–Gill–Solovay set of polynomial Turing machines.∗

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Abstract

We prove here a lemma that connects some properties of the so–called “counterexample function” to the \( P = NP \) conjecture over the Baker–Gill–Solovay set of polynomial Turing machines to the behavior of the same function “at large,” over the set of all polynomial Turing machines.

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1 Introduction

We deal here with a property of the Baker–Gill–Solovay (BGS) set \(^1\) of polynomial Turing machines. The BGS set is a kind of ‘representation set’ for polynomial machines in the following sense: for every computable function \(f\) with a polynomial algorithm (a polynomial Turing machine), there is one such polynomial algorithm for \(f\) in the BGS set, and there are only polynomial algorithms in BGS.

Actually there are infinitely many polynomial algorithms for each \(f\) in BGS, but not every polynomial algorithm for \(f\) will be in BGS.

The BGS set was conceived to rigorously formulate the \(P < NP\) question (see below). Since the set of all polynomial Turing machines isn’t recursive in the set of all Turing machines, it is certainly easier to deal with a set that contains copies of representatives of all polynomial algorithms and which moreover is a recursive set.

On our main result

As we see below, \(P < NP\) asserts that a given recursive function (noted \(f_{-G}\) in what follows) is total over the BGS set. That function has an extension (better said, a kind of copy) as a relative recursive function on the set of all polynomial Turing machines. So, it is of interest to relate what happens over the BGS set to what happens outside it.

We are interested in Peano Arithmetic (PA), which we take to be consistent. We deal with PA–provably total recursive functions. Recall that:

**Definition 1.1** \(F\) is PA–provably total recursive if,

1. \(F\) has an explicitly given Gödel number \(e\).
2. \(PA \vdash \forall x \exists z T(e, x, z)\).

\((T\) is Kleene’s predicate.) This means that we explicitly have a program for \(F\) (it is given by \(e\)) and that there is a proof in PA that every computation of \(F\) will eventually stop.

Our question in this paper is: suppose that the counterexample function is total over BGS. If it is PA–provably total, what happens outside BGS, over the (nonrecursive) set of all polynomial Turing machines?

We give here a partial answer to that question.

The BGS set

The BGS set is constructed as follows:

- A polynomial clock \(C_{(a, b)}\) is a total Turing machine that behaves as follows: for binary input \(x\) of length \(|x|\) it computes \(|x|^a + b\), \(a, b\) positive integers, and stops the operation of the coupled machine \(M_m(x)\) after \(|x|^a + b\) cycles, if it hasn’t stopped yet.


- Form all pairs \((M_m, C_{(a,b)})\) of a Turing machine \(M_m\) and a polynomial clock \(C_{(a,b)}\).

- The pairs \((M_m, C_{(a,b)})\) form the BGS set. If \(\langle \ldots, \ldots \rangle\) is the usual 1–1 and onto pairing function, we order BGS according to \(\langle m, \langle a, b \rangle \rangle\).

The BGS index is the triple \(\langle m, a, b \rangle\), \(m, a, b\) ranging over the whole of \(\omega\).

- There are several primitive recursive procedures to embed all such pairs into the set \(\mathcal{M}\) of all Turing machines. We suppose that one of them has been chosen and kept fixed. The pairs \((M_m, C_{(a,b)})\) or their recursive, embedded images, form the BGS set. (For our purposes it is indifferent whether we deal with the BGS pairs or with their image in \(\mathcal{M}\), the set of all Turing machines, via that p.r. embedding previously agreed upon, but we will consider here the BGS set as an entity separated from \(\mathcal{M}\).)

As mentioned above, one immediately sees that for every poly machine there is a pair machine–clock in BGS that corresponds to it (actually, infinitely many such pairs), and given an arbitrary pair, there is a corresponding poly machine.

\[\square\]

**Rigorous formulation of \(P < NP\) for the Satisfiability Problem**

**Remark 1.2** We consider the case \(3\) of SAT, the Satisfiability Problem for Boolean expressions in conjunctive normal form (cnf).

- Let \(x\) be a Boolean expression in cnf, adequately coded as a binary string of length \(|x|\). Let \(P_n\) be a polynomial machine of BGS index \(n = \langle m, a, b \rangle\).

- Given a binary string \(y\) of truth–values for the \(|y|\) Boolean variables of \(x\), there is a polynomial procedure (a polynomial Turing machine which we note \(V\)) that tests whether \(y\) satisfies \(x\), that is, say, \(V(\langle x, y \rangle) = 1\) if and only if \(y\) satisfies \(x\); and is 0 otherwise.

  (For the sake of completeness, we add that \(V(0, 0) = 1\), that is, the empty string is satisfied by the empty string. The empty string as a string of truth values makes true the empty string, seen as a string of propositional variables; we agree that for \(x > 0\), no such \(x\) is satisfied by 0.)

- We formulate the predicate:

\[G^*(m, x) \iff_{\text{Def}} \exists y (P_m(x) = y \land V(x, y) = 1).\]

\(G^*(m, x)\) is intuitively understood as “polynomial machine of BGS index \(m\) correctly guesses about Boolean cnf expression \(x\),” or, even more explicitly, “machine \(m\) inputs \(x\) and outputs a line of truth values that satisfies \(x\).”

- We can also write: \(G^*(m, x) \iff [V(x, P_m(x)) = 1]\).
• Form the pair \( z = \langle x, y \rangle \), and let \( \pi_i, i = 1, 2 \), be the usual (polynomial) projection functions. Recall that \( V \) is a polynomial machine that inputs a pair \( \langle x, y \rangle \). Then we can consider the predicate:

\[
- G(m, z) \iff V(z) = 1 \land V(\langle \pi_1 z, P_m(\pi_1 z) \rangle) = 0,
\]

or

\[
- G(m, z) \iff V(z) = 1 \land -G^*(m, \pi_1 z).
\]

• \(-G(m, z)\) can be intuitively understood as follows: polynomial machine \( P_m \) doesn’t accept the pair \( z \) if and only if \( z \) is such that \( \pi_1 z = x \) is satisfiable, but the output of \( P_m \) over \( x = \pi_1 z \) doesn’t satisfy \( x \). □

Then we can define:

Definition 1.3 \( P < NP \iff \forall m \exists z - G(m, z) \), where \( m \) ranges over the BGS set. □

Notice that \( P < NP \) is a \( \Pi^0_2 \) sentence. Also:

Definition 1.4 \( f_{-G} = \mu_x [-G(m, x)] \) is the counterexample function. □

2 Main result

Remark 2.1 We suppose here that the counterexample function \( f_{-G} \) is total over BGS. This means that we suppose that \( P < NP \) holds, for the sake of our argument.

Our query then is: if it is so, what do we need in order to have that \( f_{-G} \) be \( \text{PA–provably total recursive} \)? □

Write \( f^*_{-G} \) for the ‘complete’ counterexample function, that is the one which is defined over the (nonrecursive) set \( \mathcal{P} \subset \mathcal{M} \) of all poly Turing machines.

Let \( T \) be the exponential algorithm (truth–table computation) that settles \( \text{SAT} \). We have agreed that no instance \( > 0 \) is satisfied by 0.

Remark 2.2 An \( \text{A–quasi–trivial Turing machine with cutoff value } k \) is a Turing machine that equals some other total machine \( A \) up to instance \( k \), and then outputs 0 for every instance \( x > k \). □

Define:

Definition 2.3 A \( \text{T–quasi–trivial machine } T^k \) is a quasi–trivial machine that equals \( T \) up to instance \( k \), and then outputs 0 for every instance \( x > 0 \). □

Definition 2.4 A recursive subset \( B \) contained in the set of all Turing machines is \( \text{PA–recursive} \) iff its explicitly given characteristic function \( c_B \) is \( \text{PA–provably total recursive} \). □
**Remark 2.5** If $B$ as above is PA–provably recursive, then we can also explicitly obtain a PA–provably recursive function $c_B : B \to \omega$ that is 1–1 and onto. Thus we can code (via $c_B$) the machines in $B$ by the natural numbers.

Given a PA–provably total recursive function $F$ which is defined over all Turing machines coded by $\omega$, we can adequately define the restriction $F|_B$ for $B$ PA–provably total recursive, and see that $F|_B$ is PA–provably total over the set $B$ coded by $c_B$. □

**Remark 2.6**

Recall that, for $m$ the Gödel number of a polynomial machine, and for the corresponding BGS index $N(m)$,

$$f^*_G(m) = f_G(N(m)).$$

The map $m \mapsto N(m)$ isn’t in general recursive (it isn’t even a function in the general case, as there are infinitely many $N(m)$ that correspond to each $m$). However we use below recursive versions of it which also turn out to be functions. □

We now state our main result:

**Lemma 2.7** If the counterexample function $f_G$ is PA–provably total over BGS, then for any restriction of $f^*_G$ over a PA–recursive subset $B$ of $T$–quasi–trivial machines, $f^*_G|_B$ is PA–provably total recursive. □

**Proof of the lemma:** Keep in mind the correspondence

$$f^*_G(m) = f_G(N(m))$$

given in Remark 2.5. Pick up an arbitrary PA–recursive subset $B$ of the Turing machines which only contains $T$–quasi–trivial machines. Let’s embed it into BGS as follows:

- **Clocks that bound $B$.** Machines in $B$ are as follows ($m$ is the machine’s Gödel number):
  $$T^{k(m)}_m(x) = T(x), x \leq k(m),$$
  $$T^{k(m)}_m(x) = 0, x > k(m).$$

- Put $b_m = \max\{\text{operation time of } T(x), x \leq k(m)\} + 1$.

- Then clock $C_{(2,b_m)}$ bounds the operation of $T^{k(m)}_m$ without interrupting it.

- We thus form $B' \subset BGS$, whose elements are the pairs $(T^{k(m)}_m, C_{(2,b_m)})$.

  This gives us the recursive map (see Remark 2.6):

  $$B \subset \mathcal{P} \to B' \subset BGS,$$

  $$m \mapsto N(m) = \langle m, 2, b_m \rangle.$$
• Crucial step. Now we know that $f_{\neg G}$ is PA-provably recursive over BGS. For each pair $(T_m^{k(m)}, C_{(2,b_m)})$, and if the BGS index $N(m) = \langle m, 2, b_m \rangle$, then:

$$f_{\neg G}(N(m)) \geq k(m) + 1,$$

by the definition of $f_{\neg G}$.

• $m \in B$. Since $B$ is PA-provably recursive, this means that the $k(m)$ are also PA-provably recursive over $B'$.

• Therefore, so are the $b_m$ by construction, and as a result $B'$ has a PA-provably recursive characteristic function in BGS. Thus $B'$ is PA-recursive as a subset of BGS.

• Conclusion. Now go back to $B$ and trivially obtain the values of $f_{\neg G}|B$ from those of $f_{\neg G}|B'$.

Proceed as follows: for $\langle m, 2, b_m \rangle \in B'$ and $m \in B$, we have (Remark 2.6) that:

$$f_{\neg G}^*(m)|_B = f_{\neg G}(\langle m, 2, b_m \rangle).$$

**Corollary 2.8** If there is a restriction $f_{\neg G}^*|B$ which isn’t PA-provably total, then $f_{\neg G}$ (over BGS) cannot be PA-provably total. □

Therefore, if we manage to show that at least one such restriction isn’t PA-provably total recursive, we have that the counterexample function over BGS cannot be PA-provably total recursive. Consequence is:

**Corollary 2.9** If there is a restriction $f_{\neg G}^*|B$ which isn’t PA-provably total, then PA cannot prove $P < NP$.

*Proof:* Follows from the fact that $P < NP$ is a $\Pi^0_2$ sentence in PA, and from Kreisel’s theorem ([3], p. 885ff). □

On that last possibility see [2, 3, 5].

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References

[1] T. Baker, J. Gill, R. Solovay, “Relativizations of the \( P =? NP \) question,” *SIAM Journal of Computing* **4**, 431 (1975).

[2] N. C. A. da Costa and F. A. Doria, “On a total function which overtakes all total recursive functions,” preprint 01–RGC–IEA (2001).

[3] N. C. A. da Costa and F. A. Doria, “Why is the \( P =? NP \) question so difficult?” preprint 03–RGC–IEA (2001).

[4] F. A. Doria, “Is there a simple, pedestrian, arithmetic sentence which is independent of \( ZFC \)?” *Synthèse* **125**, # 1/2, 69 (2000).

[5] F. A. Doria, posts to the forum *theory–edge* at Yahoo Groups (November–December 2000 and February–April 2001).

[6] H. Schwichtenberg, “Proof theory: some applications of cut–elimination,” in J. Barwise, ed., *Handbook of Mathematical Logic*, North–Holland (1989).