Quantum to classical crossover in many-body chaos in a glass

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Chaotic quantum systems with Lyapunov exponent $\lambda_L$ obey an upper bound $\lambda_L \leq 2\pi k_B T/h$ at temperature $T$, implying a divergence of the bound in the classical limit $h \to 0$. Following this trend, does a quantum system necessarily become ‘more chaotic’ when quantum fluctuations are reduced? We explore this question by computing $\lambda_L(h, T)$ in the quantum spherical $p$-spin glass model, where $h$ can be continuously varied. We find that quantum fluctuations, in general, make paramagnetic phase less chaotic and the spin glass phase more chaotic. We show that the approach to the classical limit could be non-trivial, with non-monotonic dependence of $\lambda_L$ on $h$ close to the dynamical glass transition temperature $T_d$. Our results in the classical limit ($h \to 0$) naturally describe chaos in super-cooled liquid in structural glasses. We find a crossover from strong to weak chaos substantially above $T_d$, concomitant with the onset of two-step glassy relaxation. We further show that $\lambda_L \sim T^\alpha$, with the exponent $\alpha$ varying between 2 and 1 from quantum to classical limit, at low temperatures in the spin glass phase. Our results reveal intricate interplay between quantum fluctuations, glassy dynamics and chaos.

Understanding of the rates of thermalization or transport and fundamental bounds on these rates in quantum many-body systems have far-reaching implications for a remarkably wide range of topics such as information scrambling in black holes¹,² and quantum circuits³, strange metals and Planckian dissipation⁴,⁵. Surprisingly, only concrete example of a fundamental bound on such a rate pertains to the Lyapunov exponent or the growth rate of chaos, which is predominantly a classical, or at best a semiclassical concept. The bound, $\lambda_L \leq 2\pi k_B T/h$, is conjectured² to hold for any thermal quantum system at a temperature $T$. The systems that saturate the bound have led to interesting links between black holes and certain non-Fermi liquid metals, described by the celebrated Sachdev-Ye-Kitaev (SYK) model.

In quantum systems the Lyapunov exponent is defined from the exponential growth of out-of-time-ordered commutator⁷,⁹. However, for generic quantum systems, the exponential growth is only observed for systems with a suitable semiclassical limit²,⁷,⁸,¹⁰,¹¹. In the strongly interacting SYK model, where $\lambda_L = 2\pi k_B T/h$ for $T \to 0$, the semiclassical limit corresponds to the large-$N$ limit, related to the dual semiclassical Einstein gravity⁸,¹²-¹⁷. However, $h$ cannot be tuned in the fermionic SYK model. Thus, it would be desirable to study chaos in a model where $h$ can be varied explicitly and explore how quantum mechanics intervenes in the evolution of chaos between the classical and quantum limits.

To this end, we study many-body chaos in a quantum version¹⁸-²¹ of a paradigmatic model of glasses, namely the spherical $p$-spin glass model²²,²³ of $N$ spins interacting with random all-to-all $p$-spin interactions. The model, which has many features similar to the SYK model²⁴, is solvable in the large-$N$ limit, like the SYK model, but $h$ can be tuned continuously here. As shown in Fig.1, the model has thermodynamic transition, $T_c(h)$, between paramagnetic (PM) and replica-symmetry broken (RSB) spin glass (SG) phase for $p \geq 2^{²⁰,²¹}. Moreover, there is a dynamical transition at $T_d > T_c$ from slow glassy thermalization to complete lack of it below $T_d$. The classical glassy dynamics in the model has been extensively studied²⁵-²⁸ and is identical to the mode coupling theory (MCT) dynamics of super-cooled liquid in real glasses³⁰-³².

In this work, we find out how quantum fluctuations, glassy dynamics and (replica) symmetry breaking influence chaos. We obtain $\lambda_L(h, T)$ for the quantum spin glass by varying $h$ within the PM and SG phases, and across phase transitions [Fig.1]. In general, quantum fluctuations is found to reduce chaos in the PM, in contrast to the SG phase, where increasing $h$ makes the system more chaotic. However, we show that $\lambda_L$, over certain temperature range close to $T_d(h)$ in the PM phase, is a non-monotonic function of $h$. This indicates non-trivial nature of quantum corrections to $\lambda_L$. Our results for $\lambda_L$ in the $h \to 0$ limit for $T > T_d$ capture the temperature dependence of the Lyapunov exponent of a super-cooled liquid. Surprisingly, we find that the Lyapunov time $\lambda_L^{-1}$ has a broad minimum at $T = T_m > T_d$ [Fig.1], unlike the usual relaxation time $\tau_\alpha$ extracted from the spin-spin correlation function. The relaxation time diverges approaching $T_d$ from above. We show that the temperature scale $T_m$ is correlated with the onset of the $\beta$-regime³⁰-³² or two-step glassy relaxation. For computing $\lambda_L$ in the SG phase, we obtain the out-of-time-ordered correlation (OTOC) function in a replica-symmetry broken marginal SG (mSG) phase³⁰,²¹. We find $\lambda_L \sim T^\alpha$ at low temperature in the mSG phase, with the exponent $\alpha$ varying between $\sim 2 - 1$ from quantum to the classical limit.

We note that earlier works³³,³⁴ have studied chaos in a solvable glass model of quantum rotors³⁵, but only in the PM phase in the quantum limit, and for 2-rotor interaction, which has the same thermodynamic phase diagram as the $p = 2$ spin glass model. The SG phase in these models are replica symmetric³⁶ and somewhat
trivial. Similar SG phase is also realized in a version of SYK model represented in terms of \( SO(N) \) spins. In this model, the Lyapunov exponent has been computed via numerical simulation in the classical large spin limit\textsuperscript{47}.

**Model.**— We study the quantum spherical \( p \)-spin glass model\textsuperscript{18–21}, described by the Hamiltonian,

\[
    \mathcal{H} = \sum_i \frac{\pi_i^2}{2M} + \sum_{p,i_1<...<i_p} J_{i_1...i_p}^p s_{i_1}...s_{i_p},
\]

which contains random all-to-all interactions among \( p = 2, 3, \ldots \) spins on \( i = 1, \ldots, N \) sites with the couplings\textsuperscript{18} \( J_{i_1...i_p}^p \) drawn from Gaussian distribution with variance \( J_{p,p}^2/2N^{p-1} \). The quantum dynamics in the model results from the commutation relation \( [s_i, s_j] = i\hbar \delta_{ij} \). The model is made non-trivial by imposing the spherical constraint \( \sum_j s_j^2 = N \). The Hamiltonian describes a particle moving on the surface of an \( N \)-dimensional hypersphere where \( M \) is the mass of the particle. The model, exactly solvable in large-\( N \) limit, describes a mean field spin glass and shares many common features with other models of quantum spin glass\textsuperscript{38–41}. We mainly study chaos in the model with \( p = 3 \) (\( J_3 = J \)). For \( p = 2 \), the model is effectively non-interacting\textsuperscript{36} and non-chaotic, i.e. \( \lambda_L = 0 \).

![FIG. 1. The dependence of Lyapunov exponent \( \lambda_L \) (in units of \( J \)) on \( T \) and \( h \) in the thermodynamic phase diagram is shown through colormap. The paramagnet (PM) to spin glass (SG) thermodynamic phase transition \( T_m(h) \) line (thin red solid line) is second order up to a tri-critical point (black star) and then first order (thick dashed red line). The mSG to PM transition is demarcated by the dynamical transition line \( T_d(h) \) (thick solid red line). The locus of the broad maximum of \( \lambda_L(T,h) \) is shown as \( T_m(h) \) (solid black line) and compared with the onset temperature \( T_\beta(h) \) of the two-step glassy relaxation regime (blue line).](image)

**Large-\( N \) saddle points and phase diagram.**— The equilibrium and dynamical phase-diagrams of the model [Eq.\textsuperscript{(1)}] have been analyzed in detail in earlier works\textsuperscript{18–21,42}. In the \( N \to \infty \) limit, the equilibrium phases are characterized in terms of disorder averaged time-ordered (\( T \)) correlation function, \( Q_{ab}(\tau) = \langle 1/N \rangle \sum_i \langle T_\tau s_{ia}(\tau) s_{ib}(0) \rangle \), obtained from the following saddle point equations of the imaginary time (\( \tau \)) path integral [see Supplementary Material (SM), Sec. S1],

\[
    Q_{ab}^{-1}(\omega_k) = \left( \frac{\omega_k^2}{T} + z \right) \delta_{ab} - \Sigma_{ab}(\omega_k) \tag{2a}
\]

\[
    \Sigma_{ab}(\tau) = \sum_p \frac{pJ_p^2}{2} [Q_{ab}(\tau)]^{p-1}. \tag{2b}
\]

The replicas \( a = 1, \ldots, n \) are introduced to perform the disorder averaging and \( \omega_k = 2k\pi T \) is bosonic Matsubara frequency with \( k \) an integer \((k_B = 1)\); \( J_p = J_p/J \), and temperature, time and frequency are in units of \( J \), \( h/J \) and \( T/j \), respectively. \( Q_{ab}(\omega_k) = \int_0^\beta d\tau e^{\omega_k \tau} Q_{ab}(\tau) \) \((\beta = 1/T)\) is matrix in replica space and the spherical constraint, \((1/N) \sum_i s_{i\alpha}^2 = Q_{aa}(\tau = 0) = 1\), is imposed via the Lagrange multiplier \( z \). The quantum fluctuations is tuned through the dimensionless parameter \( \Gamma = h^2/MJ \) by changing \( h \) with fixed \( M \).

As in the earlier work\textsuperscript{20,21}, we obtain the phase diagram [Fig.\textsuperscript{1}] by numerically solving the saddle-point equations [Eqs.\textsuperscript{(2)}] self-consistently [see SM, Sec.S1, S13]. The replica structure of the order parameter \( Q_{ab}(\tau) \) in the \( n \to 0 \) limit characterizes PM and SG phases, namely — (a) in the PM phase [Fig.\textsuperscript{1}], \( Q_{ab}(\tau) = Q(\tau) \delta_{ab} \) is replica symmetric and diagonal, and (b) for the SG phase, the order parameter has an exact one-step replica symmetry breaking (1-RSB) structure where \( n \) replicas are broken down into diagonal blocks with \( m \) replicas and \( Q_{ab}(\tau) = (q_d(\tau) - q_{EA}) \delta_{ab} + q_{EA} \delta_{ab}; \epsilon_{ab} = 1 \) if \( a \neq b \) are in diagonal block else \( \epsilon_{ab} = 0 \). The Edward-Anderson (EA) order parameter \( q_{EA} \) is finite in the SG phase and vanishes in the PM phase.

As shown in Fig.\textsuperscript{1}, the PM to SG phase transition \( T_m(h) \) is second order up to a tricritical point and then first order till the PM to SG quantum phase transition at \( T = 0\textsuperscript{20,21} \). There are, in fact, two PM phases, a classical PM (cPM), abidically connected to PM at \( h = 0 \), and a quantum PM (qPM) phase for \( T \lesssim T^* \) and above the first-order line. We mainly compute \( \lambda_L \) in the cPM region since the qPM is strongly gapped\textsuperscript{21}, and hence very weakly chaotic. For studying chaos in the SG phase, we only consider the so-called marginal spin glass phase\textsuperscript{21}, where the block size or the break point \( m \) is obtained by the marginal stability criterion\textsuperscript{21} [see SM, Sec.S1.2]. The mSG phase is demarcated by the dynamical phase transition line \( T_d(h) > T_m(h) \) [Fig.\textsuperscript{1}]. The mSG and \( T_d \) are naturally realized as ergodicity broken aging regime and its onset temperature while cooling the system from the PM phase in the presence of a bath\textsuperscript{21}.

For computing the Lyapunov exponent \( \lambda_L \), as discussed below, we also need dynamical correlation and response functions in real time (\( t \) and/or real frequency (\( \omega \)). These are obtained using the spectral function \( \rho(\omega) = -\text{Im} Q^{R}_{ab}(\omega)/\pi \), where the retarded propagator, \( Q^{R}_{ab}(\omega) = Q_{ab}(\omega k \to \omega + i0^+) = Q^{R}(\omega) \delta_{ab} \), is obtained by solving the saddle-point equations [Eqs.\textsuperscript{(2)}] after analytical continuation [SM, Sec.S1.1] to \( \omega + i0^+ \).
FIG. 2. (a) The four real-time branches (separated by imaginary time $\beta/4$) of the Schwinger-Keldysh contour used for computing the OTOC is shown. (b) The ladder diagram for the self-consistency equation (3) for $Q_{N}^{7,13,44}$, with four branches, as shown in Fig. 2. However, in contrast to the SYK model, where the large-$N$ saddle point is always replica symmetric, here we need to incorporate the non-trivial 1-RSB structure in the OTOC. We achieve this by using a replicated SK path integral as discussed in the SM, Sec.S2. We define the following regularized disorder-averaged OTOC $Q_{N}^{12,43,47}$, $F_{a}(t_{1},t_{2}) = (1/N^{2}) \sum_{i,j} \text{Tr}(y_{i}a_{i}(t_{1})y_{j}a_{j}(0)y_{i}a_{i}(t_{2})y_{j}a_{j}(0))$, where $y_{i} = \exp(-\beta H)/\text{Tr}[\exp(-\beta H)]$. The Lyapunov exponent $\lambda_{L}$ is extracted from the chaotic growth, $F_{a}(t_{1},t_{2}) = \int dt_{3}dt_{4}K_{a}(t_{1},t_{2},t_{3},t_{4})F_{a}(t_{3},t_{4})$. The ladder Kernel $K$, e.g., for $p = 3$, $K_{a}(t_{1},t_{2},t_{3},t_{4}) = 3J^{2}Q_{aa}(t_{13})Q_{aa}(t_{24})Q_{aa}(t_{34})$ (see SM, Sec.S2.1), is obtained by numerically diagonalizing the Kernel $K$ [SM, Sec.S2.2].

The information about the PM and SG phases are encoded in the ladder Kernel and a crucial difference is seen in the Wightman correlator, namely for the SG phase, $Q_{aa}^{W}(\omega) = [e_{aa}2\pi \delta(\omega)q_{EE} - \delta_{ab}(\pi q(\omega)/\sinh(\beta q/2))]$, whereas the first term is absent for $Q_{aa}^{W}$ in the PM phase. Thus, the EA order parameter appears in the Kernel of the SG phase.

**Chaos in the PM phase.**— We first discuss the dependence of $\lambda_{L}$ on $T$ and $h$ in the cPM phase, as shown through the colormap in Fig.1 for $p = 3$. Overall, $\lambda_{L}$ becomes small when temperature or quantum fluctuations are large. $\lambda_{L}$ exhibits a broad maximum at $T_{m}(h)$, substantially above $T_{d}$, albeit tracking $T_{d}(h)$ line and merging with it at the tricritical point. To understand the feature better, $\lambda_{L}$ is plotted in Fig.3(a) as function of $h$ for several temperatures. For high and intermediate temperatures ($T \gtrsim J$), $\lambda_{L}$ is a monotonically decreasing function of $h$, approaching a constant value in the classical limit $h \to 0$. $\lambda_{L}$ decreases rapidly for $h \gtrsim 1$ since the system acquires a large spectral gap $\sim \Gamma$ for strong quantum fluctuations $\Gamma \gg T_{J}$ [SM, Sec.S4.2], making the interaction effects, and thus the chaos, very weak ($\lambda_{L} \sim e^{-\Gamma/\Gamma}$).

Remarkably, when temperature is close to $T_{d}(0)$, $\lambda_{L}$ is a non-monotonic function of $h$. This implies that the approach to classical limit, even without encountering a phase transition, could be non-trivial for chaotic properties. As discussed later, this phenomenon arises from the proximity to the dynamical transition. A non-monotonic dependence is also seen as a function of temperature, as shown in Fig.3(a) (inset) for $h = \sqrt{0.001}$. Start-
ing from $T \gtrsim T_d(h)$, $\lambda_L$ initially increases reaching the maximum at $T_m$ and then decreases with increasing $T$, as $\sim 1/T^2 e^{-\sqrt{T/T'}}$, e.g., at high temperature $T \gg J$ [SM, Sec.S4.1]. In this limit, system has a small gap $\sim \sqrt{T/T'} < T$ for $T > \Gamma$, whereas in the intermediate regimes $T, \Gamma \gtrsim T_m$, the system is soft-gapped and becomes gapless in the classical limit $\Gamma \to 0$ [SM, Sec.S3].

The Lyapunov exponent $\lambda_t$ and corresponding to the bound [see SM, Sec.S4.3].

The temperature dependence $\lambda_L \sim T^2$, for large $h$ [Fig.1] within the mSG phase, is similar to that in a Fermi liquid\textsuperscript{44,48,49}. This $T$-dependence in the mSG phase can be analytically understood based on the observation that the self-consistent equations for the time-dependent part of $Q_{ab}(\tau)$ [Eq.(2)], and the Kernel [Eq.(3)], in the presence of 1-RSB are equivalent to those in the PM phase of an effective model with both $p = 3$ ($J_3 = J$) and $p = 2$ ($J_2 = J \sqrt{4E_A}$) terms in Eq.(1) [SM, Sec.S2.1].

The temperature dependence $\lambda_t$, implying $\lambda_t \sim T^2$ $\sim \sqrt{T/T'} < T$, as shown in Fig.3(c), the correlation function $C(t)$ oscillates around a slowly decaying plateau-like regime before finally decaying to zero over a much longer time scale $\tau_\alpha$. (d) The $\alpha$-relaxation time scale $\tau_\alpha$ extracted from $C(t)$ and scrambling time scale $\lambda_t^{-1}$ as function of time are shown in the classical limit $h = \sqrt{0.001}T$. When $T$ approaches $T_d$, $\tau_\alpha$ diverges but $\lambda_t^{-1}$ remains finite.

Chaos in mSG phase and across the dynamical transition.— In contrast to the PM phases, the mSG phase is gapless\textsuperscript{20,21}. Moreover, unlike that in the cPM phase [Fig.3(a)], $\lambda_L(h)$, in general, monotonically increases with $h$, as shown in Fig.3(b), apart from some weak non-monotonic dependence on $h$ at low temperatures. Thus, somewhat counterintuitively, quantum fluctuations makes the system more chaotic in the mSG phase. Fig.3(c) shows $\lambda_L(T)$ for several $h$s varying from the quantum to classical limit. The Lyapunov exponent follows a power-law temperature dependence, $\lambda_L \sim T^\alpha$, with exponent $\alpha$ varying from 2 to 1 [Fig.3(d)] with decreasing $h$, implying $\lambda_L \sim T^{\alpha}$ in the classical limit. However, the pre-factor of linear $T$ is much smaller than $2\pi/\hbar$ corresponding to the bound [see SM, Sec.S4.3].

The emergence of the two-step relaxation close to $T_d$ is seen in Fig.4 where $C(t)$ oscillates around a plateau-like regime before finally decaying to zero [see also SM, Sec.S5]. Far above $T_d$, $C(t)$ oscillates around zero after the fast decay. In Fig.4(d), we plot $\tau_\alpha$ extracted from the numerical fit to $C(t)$ [Fig.4(c)] in the $\alpha$ regime and compare with Lyapunov time $\lambda_t^{-1}$. In contrast to $\tau_\alpha$, which diverges approaching $T_d$, $\lambda_t^{-1}$ has a minimum at $T_m$, substantially above $T_d$. We find that $T_m$ is correlated with the onset temperature ($T_\beta$) of $\beta$-relaxation. We approximately estimate $T_\beta$ from the temperature where $C(t)$ at the first minimum of oscillations in the $\beta$ plateau turns positive SM, Sec.S5. The correlation between $T_m$ and $T_\beta$ can be rationalized from the fact that the interaction effects, which give rise to chaos, are weak for $T \gtrsim T_\beta$, where the kinetic energy term in Eq.(1) dominates over interaction term, and the system becomes a strongly correlated paramagnet (or liquid) below $T_\beta$, as manifested by the onset of $\beta$ plateau. Thus the peak of $\lambda_t$ at $T_\beta \approx T_m$ marks a crossover from weak...
to strong chaos. In the latter regime the inelastic effects due to interaction increases with temperature giving rise to a $\lambda_T$ that increases with $T$. On the other hand, in the weak chaos regime above $T_m \approx T_\beta$, $\lambda_T$ decreases with $T$ leading to a broad maximum in $\lambda_T$ at $T_m$.

Conclusions.— In this work, we have explored how quantum mechanics influences chaos in a solvable quantum spin glass model, where the quantum fluctuations can be tuned continuously to go from the quantum to the classical limit. We find that, depending on subelliptic interplay of quantum fluctuations, glassy dynamics, dynamical SG transitions and replica symmetry breaking, quantum fluctuations can either reduce or enhance chaos. Our results in the classical limit reveal a crossover from weak to strong chaos, concomitant with the onset of complex two-step glassy relaxation, substantially above the dynamical glass transition. This result is relevant for classical supercooled liquids, given the mathematical equivalence of their dynamics above the MCT temperature with that of the $p$-spin glass model. In future, it would be interesting to get an analytical understanding of the chaos in this regime using the power-law time dependence of correlation functions around the $\beta$-relaxation plateau. The $\lambda_T$ computed in the $p$-spin glass model, with the particular quantum dynamics considered here, turns out to be much smaller than the quantum bound. Hence it would be worthwhile to study the quantum to classical crossover in a model, e.g. the $SO(N)$ spin model of Ref.37, where $\lambda_T$ already starts at the fundamental upper bound $2\pi T/\hbar$ in the quantum limit, which corresponds to the SYK model. This will enable us to get insights into an explicit mechanism by which quantum mechanics brings forth fundamental limitation on the time scales of interacting many-body systems.

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\[1\] Yasuhiro Sekino and L. Susskind, “Fast scramblers,” Journal of High Energy Physics 2008, 065 (2008), arXiv:0808.2096 [hep-th].

\[2\] Juan Maldacena, Stephen H. Shenker, and Douglas Stanford, “A bound on chaos,” Journal of High Energy Physics 2016, 1–17 (2016).

\[3\] Pavan Hosur, Xiao Liang Qi, Daniel A. Roberts, and Beni Yoshida, “Chaos in quantum channels,” Journal of High Energy Physics 2016, 1–49 (2016), arXiv:1511.04021.

\[4\] J. A. N. Bruin, H. Sakai, R. S. Perry, and A. P. Mackenzie, “Similarity of Scattering Rates in Metals Showing T-Linear Resistivity,” Science 339, 804–807 (2013), https://science.sciencemag.org/content/339/6121/804.full.pdf.

\[5\] Sean A. Hartnoll, “Theory of universal incoherent metallic transport,” Nature Physics 11, 54–61 (2014).

\[6\] Subir Sachdev, “Bekenstein-hawking entropy and strange metals,” Physical Review X 5, 1–13 (2015).

\[7\] A. Kitaev, “A simple model of quantum holography,” Talks at KITP, April 7, 2015 and May 27, 2015.

\[8\] Alexei Kitaev and S. Josephine Suh, “Statistical mechanics of a two-dimensional black hole,” (2018), arXiv:1808.07032 [hep-th].

\[9\] A I Larkin and Yu N Ovchinnikov, “Quasiclassical Method in the Theory of Superconductivity,” JETP 28, 1200–1205 (1969).

\[10\] Igor L. Aleiner, Lara Faoro, and Lev B. Ioffe, “Microscopic model of quantum butterfly effect: Out-of-time-order correlators and traveling combustion waves,” Annals of Physics 375, 378 – 406 (2016).

\[11\] Avishkar A. Patel, Debanjan Chowdhury, Subir Sachdev, and Brian Swingle, “Quantum butterfly effect in weakly interacting diffusive metals,” Phys. Rev. X 7, 031047 (2017).

\[12\] Subir Sachdev, “Holographic metals and the fractionalized fermi liquid,” Phys. Rev. Lett. 105, 151602 (2010).

\[13\] Juan Maldacena and Douglas Stanford, “Remarks on the sachdev-ye-kitaev model,” Phys. Rev. D 94, 106002 (2016).

\[14\] Juan Maldacena, Douglas Stanford, and Zhenbin Yang, “Conformal symmetry and its breaking in two dimensional nearly anti-de-sitter space,” (2016), arXiv:1606.01857 [hep-th].

\[15\] Kristan Jensen, “Chaos in ads2 holography,” Phys. Rev. Lett. 117, 111601 (2016).

\[16\] Julius Engelsöy, Thomas G. Mertens, and Herman Verlinde, “An investigation of ads2 backreaction and holography,” Journal of High Energy Physics 2016 (2016), 10.1007/jhep07(2016)139.

\[17\] Gautam Mandal, Pranjal Nayak, and Spenta R. Wadia, “Coadjoint orbit action of virasoro group and two-dimensional quantum gravity dual to syk/tensor models,” Journal of High Energy Physics 2017 (2017), 10.1007/jhep11(2017)046.

\[18\] Leticia F. Cugliandolo and Gustavo Lozano, “Quantum aging in mean-field models,” Phys. Rev. Lett. 80, 4979–4982 (1998).

\[19\] Leticia F. Cugliandolo and Gustavo Lozano, “Real-time nonequilibrium dynamics of quantum glassy systems,” Phys. Rev. B 59, 915–942 (1999).

\[20\] Leticia F. Cugliandolo, Daniel R. Grempel, and Constantino A. da Silva Santos, “From second to first order transitions in a disordered quantum magnet,” Phys. Rev. Lett. 85, 2589–2592 (2000).

\[21\] Leticia F. Cugliandolo, D. R. Grempel, and Constantino A. da Silva Santos, “Imaginary-time replica formalism study of a quantum spherical p-spin-glass model,” Phys. Rev. B 64, 014403 (2001).

\[22\] B. Derrida, “Random-energy model: Limit of a family of disordered models,” Phys. Rev. Lett. 45, 79–82 (1980).

\[23\] Bernard Derrida, “Random-energy model: An exactly solvable model of disordered systems,” Phys. Rev. B 24, 2613–2626 (1981).

\[24\] Davide Facetti, Giulio Biroli, Jorge Kurchan, and David R. Reichman, “Classical glasses, black holes, and strange quantum liquids,” Phys. Rev. B 100, 205108 (2019).

\[25\] T. R. Kirkpatrick and D. Thirumalai, “Dynamics of the structural glass transition and the p-spin—interaction
spin-glass model,” Phys. Rev. Lett. 58, 2091–2094 (1987).
26 T. R. Kirkpatrick and D. Thirumalai, “p-spin-interaction spin-glass models: Connections with the structural glass problem,” Phys. Rev. B 36, 5388–5397 (1987).
27 A. Crisanti, H. Horner, and H. J. Sommers, “The spherical p-spin interaction spin-glass model,” Zeitschrift fur Physik B Condensed Matter 92, 257–271 (1993).
28 L. F. Cugliandrò and J. Kurchan, “Analytical solution of the off-equilibrium dynamics of a long-range spin-glass model,” Phys. Rev. Lett. 71, 173–176 (1993).
29 Jean-Philippe Bouchaud, Leticia Cugliandolo, Jorge Kurchan, and Marc Mzard, “Mode-coupling approximations, glass theory and disordered systems,” Physica A: Statistical Mechanics and its Applications 226, 243–273 (1996).
30 W. Gtz, Complex Dynamics of Glass-Forming Liquids (Oxford University Press, Oxford, 2008).
31 W. Kob, “The Mode-Coupling Theory of the Glass Transition,” arXiv e-prints, cond-mat/9702073 (1997), arXiv:cond-mat/9702073 [cond-mat.stat-mech].
32 David R. Reichman and Patrick Charbonneau, “Mode-coupling theory,” Journal of Statistical Mechanics: Theory and Experiment 2005, 05013 (2005), arXiv:cond-mat/0511407 [cond-mat.soft].
33 Dan Mao, Debajan Chowdhury, and T. Senthil, “Slow scrambling and hidden integrability in a random rotor model,” Phys. Rev. B 102, 094306 (2020).
34 Gong Cheng and Brian Swingle, “Chaos in a quantum rotor model,” (2019), arXiv:1901.10446 [cond-mat.dis-nn].
35 J. Ye, S. Sachdev, and N. Read, “Solvable spin glass of quantum rotors,” Phys. Rev. Lett. 70, 4011–4014 (1993).
36 J. M. Kosterlitz, D. J. Thouless, and Raymund C. Jones, “Spherical model of a spin-glass,” Phys. Rev. Lett. 58, 4011–4014 (1993).
37 J. Ye, S. Sachdev, and N. Read, “Solvable spin glass of quantum rotors,” Phys. Rev. Lett. 70, 4011–4014 (1993).
38 J. M. Kosterlitz, D. J. Thouless, and Raymund C. Jones, “Spherical model of a spin-glass,” Phys. Rev. Lett. 58, 4011–4014 (1993).
39 Thomas Scaffidi and Ehud Altman, “Chaos in a classical limit of the sachdev-ye-kitaev model,” Phys. Rev. B 100, 155128 (2019).
40 Yadin Y. Goldschmidt, “Solvable model of the quantum spin glass in a transverse field,” Phys. Rev. B 41, 4858–4861 (1990).
41 V. Dobrosavljevic and D. Thirumalai, “1/p expansion for a p-spin interaction spin-glass model in a transverse field,” Journal of Physics A Mathematical General 23, L767–L774 (1990).
42 Theo M. Nieuwenhuizen and Felix Ritort, “Quantum phase transition in spin glasses with multi-spin interactions,” Physica A: Statistical Mechanics and its Applications 250, 8–45 (1998).
43 Tomoyuki Obuchi, Hidetoshi Nishimori, and David Sherrington, “Phase diagram of the p-spin-interacting spin glass with ferromagnetic bias and a transverse field in the infinite- p limit,” Journal of the Physical Society of Japan 76, 054002 (2007).
44 S. J. Thomson, P. Urbani, and M. Sbró, “Quantum quenches in isolated quantum glasses out of equilibrium,” Phys. Rev. Lett. 125, 120602 (2020).
45 The classical limit Γ → 0 can be also taken by M → ∞ with fixed ℏ, but this limit trivially corresponds to infinitely massive particle and there is no dynamics in the classical limit. For all our results, we calculate λ noted (in units of ℏJ), as a function of Γ and obtain λ noted in units of J by dividing with √ΓJ for M = 1.
46 Sumilan Banerjee and Ehud Altman, “Solvable model for a dynamical quantum phase transition from fast to slow scrambling,” Phys. Rev. B 95, 134302 (2017).
47 A. Houghton, S. Jain, and A. P. Young, “Role of initial conditions in the mean-field theory of spin-glass dynamics,” Phys. Rev. B 28, 2630–2637 (1983).
48 Leticia F Cugliandolo, Gustavo S Lozano, and Nicolás Nessi, “Role of initial conditions in the dynamics of quantum glassy systems,” Journal of Statistical Mechanics: Theory and Experiment 2019, 023301 (2019).
49 Douglas Stanford, “Many-body chaos at weak coupling,” Journal of High Energy Physics 2016, 9 (2016).
50 Jaewon Kim and Xiangyu Cao, “Comment on “Chaotic Integrable Transition in the Sachdev-Ye-Kitaev Model’’”, arXiv e-prints, arXiv:2004.05313 (2020), arXiv:2004.05313 [cond-mat.stat-mech].
51 Jaewon Kim, Xiangyu Cao, and Ehud Altman, “Scrambling versus relaxation in fermi and non-fermi liquids,” Phys. Rev. B 102, 085134 (2020).
52 G. Pöschl and E. Teller, “Bemerkungen zur Quantenmechanik des anharmonischen Osillators,” Zeitschrift fur Physik 83, 143–151 (1933).
53 Paolo Butera and Giovanni Caravati, “Phase transitions and Lyapunov characteristic exponents,” Phys. Rev. A 36, 962–964 (1987).
54 Lando Caiani, Lapo Casetti, Cecilia Clementi, and Marco Pettini, “Geometry of Dynamics, Lyapunov Exponents, and Phase Transitions,” Phys. Rev. Lett. 79, 4361–4364 (1997).
55 Lando Caiani, Lapo Casetti, Cecilia Clementi, Giulio Pettini, Marco Pettini, and Raoul Gatto, “Geometry of dynamics and phase transitions in classical lattice φ4 theories,” Phys. Rev. E 57, 3886–3899 (1998).
56 Xavier Leoncini, Alberto D. Verga, and Stefano Ruffo, “Hamiltonian dynamics and the phase transition of the XY model,” Physical Review E 57, 6377–6389 (1998).
57 A. S. de Wijn, B. Hess, and B. V. Fine, “Chaotic properties of spin lattices near second-order phase transitions,” Physical Review E 92 (2015), 10.1103/physreve.92.062209.
58 Moupriya Das and Jason R. Green, “Critical fluctuations and slowing down of chaos,” Nature Communications 10 (2019), 10.1038/s41467-019-10040-3.
59 Alexander Schuckert and Michael Knap, “Many-body chaos near a thermal phase transition,” SciPost Phys. 7, 22 (2019).
60 Siaram Ruidas and Sumilan Banerjee, “Many-body chaos and anomalous diffusion across thermal phase transitions in two dimensions,” arXiv e-prints, arXiv:2007.12708 (2020), arXiv:2007.12708 [cond-mat.stat-mech].
61 A. Kamenev, Field Theory of Non-Equilibrium Systems, 1st ed. (Cambridge University Press, 2011).
62 Piers Coleman, Introduction to Many-Body Physics (Cambridge University Press, 2015).
S1: Imaginary-time path integral and saddle point equations

The imaginary time path-integral and saddle point equations for $p$-spin glass model are discussed in Refs. 20 and 21. Here we sketch the basic steps of the calculations. The thermodynamic properties of the model [Eq.(1)] is obtained from the free-energy,

$$F = -k_B T \ln Z,$$

where the overline denotes disorder average over configuration $\{J^{(p)}_{i_1\ldots i_p}\}$ of the random $p$-spin couplings and $T$ is the temperature of the system. We use the replica trick, $\ln \overline{Z} = \lim_{n \to 0} (\overline{Z^n} - 1)/n$, to obtain disorder average free energy. The replicated imaginary-time path integral is obtained as

$$Z^n = \int [\prod_a D s_a(\tau)] \delta \left( \sum_i s_{i,a}(\tau) - N \right) \exp \left[ -\frac{1}{\hbar} \int_0^{\beta \hbar} d\tau \left( \sum_{i,a} \frac{M}{2} \left( \frac{\partial s_{i,a}}{\partial \tau} \right)^2 + \sum_{p,i_1,\ldots,i_p} J^{(p)}_{i_1\ldots i_p} s_{i_1,a}(\tau) \cdots s_{i_p,a}(\tau) \right) \right]$$

(S1.2)

where $a = 1, \ldots, n$ is the replica index and the $\delta$-function in the above imposes the spherical constraint. We write the $\delta$-function as $\int D s_a(\tau) \exp \left[ -\frac{1}{\beta \hbar} z_a(\tau) \left( \sum_i s_{i,a}(\tau) - N \right) \right]$. After averaging over disorder, introducing bi-local order parameter field $Q_{ab}(\tau, \tau') = (1/N) \sum_i s_{i,a}(\tau) s_{i,b}(\tau')$ and integrating out the fields $\{s_{i,a}(\tau)\}$, we obtain an effective action in terms of $Q_{ab}(\tau, \tau')$, i.e.

$$\overline{Z^n} = \int DQ_{ab} \exp \left[ -\frac{S_{\text{eff}}[Q_{ab}]/\hbar}{} \right].$$

(S1.3)

For the equilibrium saddle point $Q_{ab}(\tau, \tau') = Q_{ab}(\tau - \tau')$ and we use Matsubara frequencies, $\omega_k = 2\pi k T$ with integers $k$, to write down the effective action as

$$-\frac{S_{\text{eff}}}{\hbar} = \frac{N}{2} \sum_k \text{Tr} \ln[(\beta \hbar)^{-1} Q(\omega_k)] + \frac{N}{2} \sum_{k,a,b} \left[ \delta_{ab} - \frac{1}{\hbar} (M \omega_k^2 + z) \delta_{ab} Q_{ab}(\omega_k) \right]$$

$$+ \frac{J^2 N \beta}{4 \hbar} \sum_{a,b} \int_0^{\beta \hbar} d\tau \left[ \frac{1}{\beta \hbar} \sum_k \exp \left( -i \omega_k \tau \right) Q_{ab}(\omega_k) \right]^p + \frac{N n \beta}{2} z^p$$

(S1.4)

where $z_a(\tau) = z$ and $Q_{ab}(\omega_k) = \int_0^{\beta \hbar} d\tau \exp(i \omega_k \tau) Q_{ab}(\tau)$. For $N \to \infty$ limit the following saddle-point equation is obtained by minimizing the above effective action as

$$\frac{1}{\hbar} (M \omega_k^2 + z) \delta_{ab} = (Q^{-1})_{ab}(\omega_k) + \sum_p J_p^2 \frac{p}{2 \hbar^2} \int_0^{\beta \hbar} d\tau \exp(i \omega_k \tau) [Q_{ab}(\tau)]^{p-1}$$

(S1.5)

We work with the dimensionless quantities - $\tilde{J}_p = J_p/J$, energy and temperature in units of $J$, and time $\tilde{\tau} = J \tau/\hbar$ and frequency $\tilde{\omega}_k = \omega_k \hbar/J$, to obtain

$$(Q^{-1})_{ab}(\tilde{\omega}_k) = Q_{0}^{-1}(\tilde{\omega}_k) \delta_{ab} - \Sigma_{ab}(\tilde{\omega}_k)$$

(S1.6)

where

$$Q_{0}^{-1}(\tilde{\omega}_k) = \frac{\tilde{\omega}_k^2}{\Gamma} + \tilde{z}, \quad \Gamma = \hbar^2 / M J$$

$$\Sigma_{ab}(\tilde{\omega}_k) = \sum_p J^2 p \frac{p}{2} \int_0^{\beta} d\tilde{\tau} \exp(i \tilde{\omega}_k \tilde{\tau}) [Q_{ab}(\tilde{\tau})]^{p-1}$$

(S1.7)

with $\tilde{z} = z/J$. The quantum fluctuation parameter $\hbar$ enters through the parameter $\Gamma = \hbar^2 / M J$ in the saddle point equation. From here on, we work with these dimensionless variables and, to simplify the notation, we omit the hat from the symbols.
1. Saddle point equation and spectral function calculation in the PM phase

In the PM phase, the order parameter is replica symmetric and diagonal, i.e. $Q_{ab}(\omega_k) = Q(\omega_k)\delta_{ab}$. So, the saddle point Eqs. (S1.6) and (S1.7) in the PM phase, e.g., for $p=3$ spin glass model simplify to

$$Q^{-1}(\omega_k) = \frac{\omega_k^2}{\Gamma} + z - \Sigma(\omega_k)$$  \hspace{1cm} (S1.8a) \\
$$\Sigma(\tau) = \frac{3\tilde{J}_3^2}{2\beta}[Q(\tau)]^2$$  \hspace{1cm} (S1.8b)

To obtain the spectral function, we solve the above equations numerically after analytical continuation from Matsubara to real frequency i.e $\omega_k \rightarrow \omega + i0^+$, i.e. in terms of the retarded functions

$$(Q^R(\omega))^{-1} = -\frac{\omega^2}{\Gamma} + z - \Sigma^R(\omega)$$  \hspace{1cm} (S1.9)

where $\Sigma^R(\omega)$ is obtained from $\Sigma(\omega_k)$, i.e.

$$\Sigma(\omega_k) = \int_0^\beta d\tau e^{i\omega_k\tau}\Sigma(\tau) = \frac{3\tilde{J}_3^2}{2\beta} \sum_k Q(\omega_k)Q(\omega_n - \omega_k)$$  \hspace{1cm} (S1.10)

We perform the Matsubara summation using the spectral representation $Q(\omega_k) = \int_{-\infty}^{\infty} d\omega \rho(\omega)/(i\omega_k - \omega)$, where $\rho(\omega) = -\text{Im}Q^R(\omega)/\pi$ is the spectral function. After analytical continuation, we obtain

$$\Sigma^R(\omega) = \frac{3\tilde{J}_3^2}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\omega_1 d\omega_2 \rho(\omega_1)\rho(\omega_2) \frac{n_B(-\omega_1)n_B(-\omega_2) - n_B(\omega_1)n_B(\omega_2)}{\omega_1 + \omega_2 - \omega - i0^+}.$$  \hspace{1cm} (S1.11)

Here $n_B(\omega) = 1/(e^{\beta\omega} - 1)$ is the Bose function. To numerically evaluate the self-energy $\Sigma^R(\omega)$ efficiently, we use the identity $1/(\omega - \omega_1 - \omega_2 + i0^+) = \frac{1}{\omega} \int_0^{\infty} d\tau e^{i\omega(-\omega_1 - \omega_2 + i0^+)^t}$ to rewrite the above equation as follows

$$\Sigma^R(\omega) = i\tilde{J}_3^2 \int_0^{\infty} dt e^{i\omega t}[n_1(t)^2 - n_2(t)^2],$$  \hspace{1cm} (S1.12)

where $n_1(t) = \int_{-\infty}^{\infty} d\omega e^{-i\omega t}\rho(\omega)n_B(-\omega)$ and $n_2(t) = \int_{-\infty}^{\infty} d\omega e^{-i\omega t}\rho(\omega)n_B(\omega)$.

The Lagrange multiplier $z$ in Eq. (S1.9) can be determined either from the imaginary-time calculation using Eqs. (S1.8) subjected to the spherical constraint $Q_{ab}(\tau = 0) = 1$, or we can determine $z$ from real frequency calculation itself. To determine $z$ from real frequency calculation, we can vary $z$ and the correct $z$ will give physical solution which satisfies the spherical constraint expressed in terms of the spectral function,

$$-\int_{-\infty}^{\infty} d\omega \rho(\omega)n_B(\omega) = 1.$$  \hspace{1cm} (S1.13)

We use the above sum rule condition Eq. (S1.13) to check the accuracy of the spectral function obtained numerically by iterating the saddle-point equations.

We solve the self-consistent saddle point equations (S1.9), (S1.12) by discretizing over frequency ($\omega$) and starting with some initial guesses for Lagrange multiplier $z$ and $Q^R(\omega)$, e.g. the non-interacting retarded function for $J_3 = 0$. We calculate the retarded self-energy $\Sigma^R(\omega)$ and $n_i(t)$, $i = 1, 2$ using fast-Fourier transform (FFT) and iterate for $Q^R(\omega)$ until convergence with a required numerical accuracy. We repeat this process by varying $z$ till the sum rule condition of Eq. (S1.13) is satisfied. We find that the process is much more efficient if we split $z$ as $z = z' + \Sigma^R(\omega = 0)$ and vary $z'$.

2. Saddle point equation in the SG phase

In the quantum spherical $p$-spin glass model, there are two types of SG phases – a thermodynamic SG phase and a marginal SG phase. A detailed discussion can be found in Refs. 20 and 21. Here we briefly discuss these two different SG phases for completeness and then mainly focus on the saddle-point solution for the marginal SG phase.
The replica-symmetric spin-glass phase, unlike that for $p = 2^{\text{36}}$, is unstable for $p \geq 3$ and one needs to consider the replica symmetry breaking. In this model, one step replica symmetry (1-RSB) breaking is exact$^{21}$ and the order parameter in the imaginary time is given by

$$Q_{ab}(\tau) = (q_d(\tau) - q_{EA})\delta_{ab} + q_{EA}\epsilon_{ab}$$

(S1.14)

where $q_{EA}$ is Edward-Anderson order parameter and $\epsilon_{ab} = 1$ for the diagonal blocks and zero otherwise, as described in the main text also. The above 1-RSB order parameter in Matsubara frequency reads as

$$Q_{ab}(\omega_k) = (q_d(\omega_k) - \tilde{q}_{EA})\delta_{ab} + \tilde{q}_{EA}\epsilon_{ab}.$$  

(S1.15)

where $\tilde{q}_{EA} = \beta q_{EA}\delta_{\omega_k,0}$ and $q_d(\omega_k)$ is Matsubara Fourier transformation of $q_d(\tau) = Q_{aa}(\tau)$. Now, it is convenient to write $q_d(\tau) = q_{\text{reg}}(\tau) + q_{EA}$. The inverse matrix $Q^{-1}(\omega_k)$ has the following structure

$$(Q^{-1})_{ab}(\omega_k) = A(\omega_k)\delta_{ab} + B(\omega_k)\epsilon_{ab}$$

(S1.16)

with

$$A(\omega_k) = \frac{1}{q_d(\omega_k) - \tilde{q}_{EA}}, \quad B(\omega_k) = \frac{-\tilde{q}_{EA}}{q_d(\omega_k)^2 - (m-1)\tilde{q}_{EA}^2 + (m-2)q_d(\omega_k)\tilde{q}_{EA}}.$$  

(S1.17)

Here $m$ is the break point$^{21}$. The diagonal element of the inverse matrix $(Q^{-1})$ is given by

$$(Q^{-1})_{aa} = A(\omega_k) + B(\omega_k) = \frac{q_d(\omega_k) + (m-2)\tilde{q}_{EA}}{(q_d(\omega_k)^2 + (m-2)\tilde{q}_{EA}q_d(\omega_k) - (m-1)\tilde{q}_{EA}^2)}.$$  

(S1.18)

The off-diagonal element $(a \neq b)$ is

$$(Q^{-1})_{ab}(\omega_k) = B(\omega_k).$$  

(S1.19)

The saddle-point equation for diagonal component $q_d(\omega_k)$ can be obtained by using equations (S1.6) and (S1.18) as

$$\frac{\omega_k^2}{\Gamma} + z = \frac{q_d(\omega_k) + (m-2)\tilde{q}_{EA}}{(q_d(\omega_k)^2 + (m-2)\tilde{q}_{EA}q_d(\omega_k) - (m-1)\tilde{q}_{EA}^2)} + \Sigma(\omega_k),$$  

(S1.20)

with $\Sigma(\omega_k) = \Sigma_{aa}(\omega_k)$, the Fourier transform of $\Sigma_{aa}(\tau) = \frac{j^2p}{2}[q_d(\tau)]^{p-1}$. Using equations (S1.6) and (S1.19), the saddle point equation for $q_{\text{reg}}$ is obtained from

$$0 = -\frac{q_{\text{reg}}}{q_d(0)^2 - (m-1)\beta^2q_{\text{reg}}^2 + (m-2)\beta q_d(0)q_{\text{reg}}} + \frac{p}{2}q_{\text{reg}}^{p-1}. $$  

(S1.21)

It is convenient to separate out constant and $\tau$–dependent part of $q_d(\tau)$ and self-energy $\Sigma_{aa}(\tau)$. So, we write these as below

$$q_{\text{reg}}(\tau) = q_d(\tau) - q_{\text{EA}}, \quad \Sigma_{\text{reg}}(\tau) = \frac{j^2p}{2}[q_d(\tau)]^{p-1} - \frac{j^2p}{2}q_{\text{EA}}^{p-1}. $$  

(S1.22)

Using saddle point Eq.(S1.20), we can rewrite Eq. (S1.20) as follows

$$\frac{\omega_k^2}{\Gamma} + z = \frac{1}{q_{\text{reg}}(\omega_k)} + \Sigma_{\text{reg}}(\omega_k) + \left[ \frac{-\beta q_{\text{reg}}\delta_{\omega_k,0}}{(q_d(\omega_k)^2 - (m-1)\tilde{q}_{EA}^2 + (m-2)q_d(\omega_k)q_{\text{reg}}\beta\delta_{\omega_k,0})} + \frac{p}{2}q_{\text{reg}}^{p-1} \right].$$  

The term inside the third bracket above is zero due to Eq.(S1.21), and we obtain the simplified saddle point equation

$$\frac{\omega_k^2}{\Gamma} + z = \frac{1}{q_{\text{reg}}(\omega_k)} + \Sigma_{\text{reg}}(\omega_k),$$  

(S1.23)

where

$$\Sigma_{\text{reg}}(\omega_k) = \frac{j^2p}{2} \int_0^\infty d\tau e^{i\omega_k\tau} \times (q_d^{p-1}(\tau) - q_{\text{EA}}^{p-1}).$$  

(S1.24)
Moreover, it is convenient to split the Lagrange multiplier $z$ as $z = z' + \Sigma_{\text{reg}}(\omega_k = 0)$ with
\[
z' = \frac{1}{q_d(0)(1 - y)} = \frac{p}{2} \beta m q_{\text{EA}}^p x_p, \tag{S1.25}
\]
where $y = \beta q_{\text{EA}}/q_d(0)$ and $x_p = my/(1 - y)$. The saddle point equation [Eq.(S1.20)] for the Edward-Anderson parameter $q_{\text{EA}}$ can be rewritten as
\[
\frac{p(\beta m)^2}{2} q_{\text{EA}}^p = \frac{x_p^2}{1 + x_p}. \tag{S1.26}
\]

In the thermodynamic SG phase, the break point $m$ is determined by extremizing the free-energy functional $F[m]$\textsuperscript{21}, leading to
\[
\ln \left[ \frac{1}{1 + x_p} \right] + \frac{x_p}{1 + x_p} + \frac{x_p^2}{p(1 + x_p)} = 0. \tag{S1.27}
\]

On the other hand, in the marginal SG phase, the marginal stability condition\textsuperscript{21} is applied, i.e the replicon eigenvalue or the transverse eigenvalue of gaussian fluctuation matrix around the 1-RSB saddle point is set to zero to determine the break point $m$. The replicon eigenvalue $\Lambda_T$ is given by\textsuperscript{21}
\[
\Lambda_T = \beta^2 \left( \frac{1}{q_d(0) - \beta q_{\text{EA}}} - \frac{p(p - 1)}{2} q_{\text{EA}}^{p-2} \right). \tag{S1.28}
\]
Setting $\Lambda_T = 0$, we obtain $x_p = p - 2$, which determines $m$ in marginal SG phase.

### 3. Numerical solution of the saddle point equation in the marginal SG phase

To solve the saddle point equation for the marginal SG phase, e.g. for $p = 3$, using $q_d(\tau) = q_{\text{reg}}(\tau) + q_{\text{EA}}$, we rewrite Eq.(S1.24) as
\[
\Sigma_{\text{reg}}(\omega_k) = \Sigma_3(\omega_k) + 3 J^2 q_{\text{EA}} q_{\text{reg}}(\omega_k), \quad \Sigma_3(\omega_k) = \frac{3J^2}{2} \int_0^\beta d\tau e^{i\omega_k \tau} [q_{\text{reg}}(\tau)]^2 \tag{S1.29}
\]
Therefore, the self-energy $\Sigma_{\text{reg}}(\omega_k) = \Sigma_3(\omega_k) + \Sigma_2(\omega_k)$, and thus the saddle point equation, formally resemble those in the PM phase of the model [Eq.(1)] with both $p = 2$ and $p = 3$ terms with $J_3 = J$ and $J_2 = \sqrt{3} q_{\text{EA}} J$.

Now, to obtain the spectral function we analytical continue Eq.(S1.23) to real frequency to get
\[
(q_{\text{reg}}^R)^{-1}(\omega) = - \frac{\omega^2}{\Gamma} + z - \Sigma_3^R(\omega) - 3J^2 q_{\text{EA}} q_{\text{reg}}^R(\omega). \tag{S1.30}
\]

The spectral function in this phase is defined as
\[
\rho(\omega) = -\frac{1}{\pi} \text{Im} [q_{\text{reg}}^R(\omega)]. \tag{S1.31}
\]
So, we can express $q_{\text{reg}}(\tau)$ in terms spectral representation as
\[
q_{\text{reg}}(\tau) = - \int_{-\infty}^\infty d\omega \rho(\omega) n_B(\omega) e^{\omega(\beta - \tau)}. \tag{S1.32}
\]
The spherical constraint $Q_{aa}(\tau = 0) = q_d(\tau = 0) = 1$ in terms of spectral function is obtained as
\[
- \int_{-\infty}^\infty d\omega \rho(\omega) n_B(\omega) = 1 - q_{\text{EA}}. \tag{S1.33}
\]
So, this is a sum rule condition for spectral function in the mSG phase. Similar to the PM case, using spectral representation we can rewrite $\Sigma_3^R(\omega)$ in the form of Eq.S1.12. To determine $q_{\text{EA}}$, we use Eq.(S1.26) and marginal stability criterion $x_p = 1$, e.g., for $p = 3$. We follow similar steps, as discussed for the PM phase in Sec.S1 1, to obtain numerically converged solution for which the above sum rule condition is satisfied.
Here we briefly discuss the Schwinger-Keldysh (SK) formulation for the disorder averaged regularized OTOC\(^2\),

\[
F(t_1, t_2) = \frac{1}{N^2} \sum_{ij} \text{Tr}[y_i(t_1)y_j(0)y_i(t_2)y_j(0)],
\]

where \(y^4 = \exp(-\beta H)/\text{Tr}[\exp(-\beta H)]\). The above correlation function can be formulated as a many-body path integral\(^{59}\) on a Schwinger-Keldysh (SK) contour with four real-time branches that are separated by \(\beta/4\) in the imaginary-time\(^{13,44,47}\), namely by writing the path integral representation for the following generating function

\[
Z = \frac{1}{Z} \text{Tr} \left[ e^{-\beta H/4} U(t_0, t_f) e^{-\beta H/4} U(t_f, t_0) e^{-\beta H/4} U(t_0, t_f) e^{-\beta H/4} U(t_f, t_0) \right].
\]

Here \(Z = \text{Tr}[\exp(-\beta H)]\) is the equilibrium partition function and \(U(t, t_0) = e^{-\beta H/4} T \exp[-(i/\hbar) \int_{t_0}^{t} H(t') dt']\) is the time evolution operator with \(T\) denoting time ordering; \(t_0\) is the initial and \(t_f\) some final time. In the Keldysh formalism, typically the disorder averaging can be performed without introducing replicas\(^{59}\). This procedure can be used to describe the real-time dynamics and compute dynamical correlations such as OTOC, e.g., in the PM phase of the quantum p-spin glass model. However, to obtain the dynamical correlations in the SG phase, one needs to consider the actual physical situation, e.g., the initial condition or the initial density matrix more carefully\(^{45,46}\). For instance, to have a clear physical situation in mind, one can weakly couple the system at \(T \gg T_{d}\) to a bath. In this case the system could be equilibrated up to \(T \gtrsim T_{d}\) in the PM phase. Hence, to compute the OTOC in the PM phase above \(T_{d}\), we can directly disorder average the generating function \(Z\) [Eq.(S2.2)] and take \(t_0 \to -\infty\) and \(t_f \to \infty\) limits and vanishing coupling with the bath. In this way, the correlation and retarded functions become time-translational invariant and identical to the equilibrium ones obtained by solving the saddle-point equations in the PM phase in Sec.S1.1.

Below \(T_{d}\), while getting cooled in the presence of the bath, the system enters the so-called aging regime where, e.g., the spin-spin correlation \(C(t_w, t_w + t)\) depends on the waiting time \(t_w\) even for asymptotically large \(t_w\). However, by formally taking \(t_w \to \infty\) and then coupling to the bath to zero in the saddle-point equations on the SK contour\(^{21}\), one arrives at solutions identical to the 1-LSB mSG in Sec.S1.3. Hence, to compute OTOC using the SK contour of Fig.S1, rather than taking the above more complicated route of taking \(t_w \to \infty\) then coupling to the bath to zero, one can simply replicat\(^{45,46}\) the generating function \(Z\) and take \(t_0 \to -\infty, t_f \to \infty\) limits, i.e.

\[
Z^n = \frac{1}{Z^n} \int \left[ \prod_a \mathcal{D}s_a(z) \delta \left( \sum_i s_{i,a}^2(z) - N \right) \right] \exp \left[ \frac{i}{\hbar} \int_C dz \left( \sum_{i,a} \frac{M}{2} \left( \frac{\partial s_{i,a}}{\partial z} \right)^2 + \sum_{i_1 < \ldots < i_p} J_{i_1 \ldots i_p}(s_{i_1,a}(z) \ldots s_{i_p,a}(z)) \right) \right].
\]

Here \(z\) is the variable on the SK contour \(C\) shown in Fig.S1, where the four horizontal real-time branches are denoted as 1,2,3,4 from the top to bottom. We can perform the disorder average over the above replicated generating function. In this case, after taking the \(n \to 0\) limit, the resulting time-translational invariant dynamical correlations and responses are identical to that obtained from the 1-LSB saddle point solutions in the mSG phase (Sec.S1.3). Using the SK contour, the OTOC of Eq.(S2.1) can be written as

\[
F_{aa}(t_1, t_2) = \frac{1}{N^2} \sum_{ij} \langle y_i(t_1) y_j(0) y_i(t_2) y_j(0) \rangle,
\]

where the average \(\langle \ldots \rangle\) is with respect to the generating function and the superscripts in \(s_{i,a}\) refers to the real-time branch.

1. Kernel equation for OTOC in the PM and marginal SG phase

We first discuss the real-time retarded and Wightmann functions, which we need to obtain OTOC using the Schwinger-Keldysh contour in Fig.S1. The retarded function is obtained via analytical continuation as

\[
Q_{ab}^R(\omega) = Q_{ab}(i\omega_k \to \omega + i0^+) = q_{reg}(\omega)\delta_{ab},
\]
We note that the $q_{\text{EA}}\epsilon_{ab}$ term in $Q_{ab}(\omega)$ [Eq.(S1.15)] does not contribute to the retarded function, which is replica diagonal. The spectral representation of $Q_{ab}(\tau)$ is given by the relation Eqs.(S1.32) and (S1.14), namely

$$Q_{ab}(\tau) = q_{\text{reg}}(\tau)\delta_{ab} + q_{\text{EA}}\epsilon_{ab} = q_{\text{EA}}\epsilon_{ab} - \delta_{ab} \int_{-\infty}^{\infty} d\omega \rho(\omega)n_B(\omega)e^{\omega(\beta - \tau)}$$ (S2.6)

The Wightmann function is defined as $Q^W(\tau,t):=Q_{ab}(\tau=it+\beta/2)$. Now, using the above Eq.(S2.6), we obtain the Wightmann function after doing Fourier transformation as follows

$$Q^W_{\text{reg}}(\omega) = \left[\epsilon_{ab}2\pi\delta(\omega)q_{\text{EA}} + \delta_{ab}q^W_{\text{reg}}(\omega)\right], \quad q^W_{\text{reg}}(\omega) = -\frac{\pi\rho(\omega)}{\sinh(\beta\omega/2)}$$ (S2.7)

The replica diagonal component of Wightmann function contains Edward-Anderson parameter $q_{\text{EA}}$ as well as the regular component $q^W_{\text{reg}}$. The replica off-diagonal component of $Q^W$ contains only Edward-Anderson term $\epsilon_{ab}2\pi\delta(\omega)q_{\text{EA}}$.

The exponential growth $\sim e^{\lambda_{t}\tau}$ appears in the OTOC $F_{aabb}(t_1,t_2)$ [Eq.(S2.4)] at $O(1/N)$, namely $F_{aabb}(t_1,t_2) = O(1) + (1/N)F_{aabb}(t_1,t_2)$ with $F_{aabb}(t,t) \sim e^{\lambda_{t}\tau}$. The quantity $F_{aabb}(t_1,t_2)$ can be obtained via the ladder series\cite{7,13,44} shown in Fig.2(b) of the main text, i.e. the kernel equation

$$F_{aabb}(t_1,t_2) = \int dt_3dt_4 \sum_n K_{aacc}(t_1,t_2,t_3,t_4)F_{ccbb}(t_3,t_4)$$ (S2.8)

In the intermediate but long time scale of chaotic growth regime $\lambda_{t}^{-1} \lesssim t \lesssim \lambda_{L}^{-1}\ln(N)$, the propagators along the horizontal lines from $t_1$ to $t_3$ and from $t_2$ to $t_4$ in Fig.2(b) (main text) can be approximated by the retarded propagator\cite{7}. In this case, since the retarded propagator is replica diagonal both in the PM and SG phases, the above equation reduces to

$$F_a(t_1,t_2) = \int dt_3dt_4 K_a(t_1,t_2,t_3,t_4)F_a(t_3,t_4)$$ (S2.9)

with $F_a = F_{aaag}$. Here the kernel for $p=3$ is $K_a(t_1,t_2,t_3,t_4) = 3J^2Q^R_{aa}(t_13)Q^R_{aa}(t_24)Q^W_a(t_{34})$, where $t_{ij} = t_i - t_j$, $i,j = 1,2,3,4$ and $Q^R(t_{ij})$ is retarded propagator and $Q^W(t_{34})$ is Wightmann function in real time. In the chaos regime using exponential growth ansatz, $F_a(t_1,t_2) = f_a(t_1,t_2)e^{\lambda_{t}(t_1+t_2)/2}$, and doing Fourier transformation of Eq.(S2.9), we obtain

$$f_a(\omega) = \int \frac{d\omega'}{2\pi} K_a(\omega,\omega')f_a(\omega')$$ (S2.10)
with the kernel in the frequency domain
\[ K_a(\omega, \omega') = 3J^2 Q_{aa}^R \left( \omega + i\frac{\lambda L}{2} \right) Q_{aa}^R \left( -\omega + i\frac{\lambda L}{2} \right) Q_{aa}^W(\omega - \omega'). \] (S2.11)

We can think of the Eq. (S2.10) as an eigenvalue equation in the frequency space. We discuss in next section the numerical diagonalization of the kernel to obtain Lyapunov exponent.

The kernel in PM phase is different from SG phase as the Wightmann function in PM phase doesn’t contain Edward-Anderson term $\epsilon_a 2\pi \delta(\omega) q_{EA}$ since $q_{EA} = 0$ in PM phase. The information of the PM phase and SG phase is encoded therefore in the Wightmann function.

2. Numerical diagonalization of kernel

We can view Eq. (S2.10) as an eigenvalue equation, $\int \frac{d\omega'}{2\pi} K_a(\omega, \omega') f_a(\omega') = \lambda f_a(\omega)$, with eigenvalue $\lambda = 1$. The kernel Eq. (S2.11) is not symmetric with respect to frequencies $(\omega, \omega')$. We symmetrize the kernel using particle-hole symmetry of the Hamiltonian. Using this symmetry it is easy to show that
\[ Q_{aa}^R \left( \omega + i\frac{\lambda L}{2} \right) Q_{aa}^R \left( -\omega + i\frac{\lambda L}{2} \right) = \left| Q_{aa}^R \left( \omega + i\frac{\lambda L}{2} \right) \right|^2 \] (S2.12)

We can now define a symmetric kernel by defining $f_a(\omega) = |Q_{aa}^R(\omega + i\lambda L/2)| \tilde{f}_a(\omega)$ and obtain the symmetric kernel as
\[ \tilde{K}_a(\omega, \omega') = 3J^2 \left| Q_{aa}^R \left( \omega + i\frac{\lambda L}{2} \right) \right|^2 Q_{aa}^W(\omega - \omega') \left| Q_{aa}^R \left( \omega' + i\frac{\lambda L}{2} \right) \right|^2 \] (S2.13)

Hence the eigenvalue equation becomes
\[ \int_{-\infty}^{+\infty} \frac{d\omega'}{2\pi} \tilde{K}_a(\omega, \omega') \tilde{f}_a(\omega') = \lambda \tilde{f}_a(\omega) \] (S2.14)

To solve the above eigenvalue equation we first discretize frequency, namely $\omega_n$, $n$ being integer, ranging from $-\omega_{\text{max}}$ to $\omega_{\text{max}}$ with spacing $2\omega_{\text{max}}/N_\omega$, where $N_\omega$ is total number of discretized frequency points and $\omega_{\text{max}}$ is a upper frequency cutoff. We diagonalize the matrix $\tilde{K}_a$ and find the value of $\lambda_L$ such that $\tilde{K}_a$ has at least has one eigenvalue $\lambda = 1$.

S3: Spectral function in the cPM, qPM and mSG phases and across the crossover from strong to weak chaos

In this section we discuss the evolution spectral function as a function of $T$ and $\hbar$ over the phase diagram in Fig.1. Thermodynamically the system can be characterized as gapped or gapless only at low temperature ($T \ll J, \Gamma$) based on the temperature dependence of specific heat or susceptibility. As discussed in Ref.21, the qPM phase is strongly gapped with specific heat $C_v \sim e^{-E_g/T}$ and gap $E_g$, whereas the mSG phase is gapless with $C_v \sim T$ at low temperature. Here, however, we are also interested to study chaos at intermediate temperatures. Hence to relate the nature of the excitations in the system with chaos we look into the spectral function $\rho(\omega)$ defined in Sections S1.1 and S1.3 for the PM and SG phases, respectively.

For bosonic systems $\omega \text{Im} Q_0^R(\omega) \geq 0$ and thus the spectral function either approaches zero or has a discontinuity at $\omega = 0$. We operationally characterize, albeit in somewhat ad hoc way, whether the system is gapped (or soft gapped) or gapless as follows. The gapped free retarded propagator in this model for $J_p = 0$ is $Q_0^R(\omega) = -\Gamma/[(\omega + i\eta)^2 - (E_g^0)^2]$ where $E_g^0 = \sqrt{\Gamma^2 z}$ is the gap, where $z$ is Lagrange multiplier that appears in Eq.(2) and $\eta > 0$ is a small broadening. Hence, the free spectral function $\rho^0(\omega) = -(1/\pi)\text{Im} Q_0^R(\omega)$ can be written as
\[ \rho^0(\omega) = \frac{\Gamma}{2\pi E_g^0} \left[ \frac{\eta}{(\omega + E_g^0)^2 + \eta^2} - \frac{\eta}{(\omega - E_g^0)^2 + \eta^2} \right] \] (S3.1)

Therefore, we see that the gapped free spectral function has peaks at $\omega = \pm E_g^0$ with Lorentzian decay around these peaks. We can estimate the spectral weight at $\omega \to 0$ as $\rho_0 = \rho^0(\omega \to 0) = (2\eta\omega\Gamma)/((\pi(E_g^0)^4})$. Numerically we can take $\eta$ as the spacing of discrete frequency points i.e $\Delta \omega$. The slope of spectral function at $\omega \to 0$ is therefore $A_0 = (\partial \rho^0(\omega)/\partial \omega)|_{\omega=0} = 2\eta\Gamma/((\pi(E_g^0)^4})$. 
After numerically calculating the spectral function from the solution of the saddle point equations, we compute the slope of $\rho(\omega)$ at all discrete frequency points between the peak at $\omega = -E_g$, where $\rho(\omega)$ has maximum value, and $\omega = 0$. Due to particle-hole symmetry of the model, $\rho(\omega)$ has odd parity and hence we do not need to check the slopes for $\omega > 0$. We count the number of points where $\left(\frac{\delta \rho(\omega)}{\delta \omega}\right)_{\omega=0} < A_0$. If the number of counts is greater than 10, then we define the spectral function as gapped otherwise it is characterized as gapless. Using this procedure we mark the gapless and gapped regions in the phase diagram in Fig. S5. The spectral function is always gapless in the mSG phase. The system is gapped in the qPM phase, and for finite $\Gamma$, the system becomes gapped at high temperatures in the cPM phase. The spectrum is expected to be gapless strictly at the classical limit $\Gamma = 0^{21}$. We note that the nature of the spectrum does not undergo any qualitative change across the crossover from strong to weak chaos around $T_m(h)$.

The nature of the excitation spectrum can be understood easily at large $\Gamma$ or $T$ where the system effectively becomes non-interacting. For the $\eta \to 0$ limit the free spectral function $\rho^0(\omega)$ in Eq. (S3.1) has two delta function peaks, i.e. $\rho^0(\omega) = \sum_{s=\pm}(s\Gamma/2E_g^0)\delta(\omega + sE_g^0)$. Using the sum rule condition of Eq. (S1.13), we obtain $\tanh(\beta E_g^0/2) = \Gamma/2E_g^0$. At very low temperatures ($\beta \to \infty$), $\tanh(\beta E_g^0/2) \simeq 1$ and thus $E_g^0 = \Gamma/2$. Due to the large gap, the perturbative effect of self-energy is very small for $\Gamma \gg T, J$ i.e. in the qPM phase. On the other hand, in the high temperature limit ($\beta \to 0$) $\tanh(\beta E_g^0/2) \simeq E_g^0/2$ and hence $E_g^0 = \sqrt{\Gamma T} \ll T$ for $T \gg \Gamma$. The self-energy effect will lead to the broadening of the peak, however the gap $E_g \approx E_g^0$ for $T \gg J$. 

**FIG. S2.** The imaginary part of $Q_{aa}^R(\omega)$ [$\text{Im} Q_{aa}^R(\omega) = -\pi \rho(\omega)$] as a function of $\omega/\sqrt{\Gamma}$ is shown for different $T$ at $\hbar = \sqrt{0.001}$ in the cPM phase. The spectral function is gapless for $T \gtrsim 0.7$ and becomes progressively more gapped at higher temperatures.

**FIG. S3.** The imaginary part of $Q_{aa}^R(\omega)$ [$\text{Im} Q_{aa}^R(\omega) = -\pi \rho(\omega)$] as a function of $\omega/\sqrt{\Gamma}$ is shown for different quantum fluctuation $h$ at temperature $T = 0.65$ close to $T_d$ in the cPM phase. The spectral function becomes more gapped with increasing $h$. 
FIG. S4. \[ \text{Im}Q_{\text{R}}(\omega) = -\pi \rho(\omega) \] vs \[ \omega/\sqrt{\Gamma} \] is shown for different quantum fluctuation \( h \) at inverse temperature \( \beta = 5 \) in the mSG phase. The spectral function remains gapless throughout the mSG phase.

FIG. S5. The phase diagram based on nature of spectral function, i.e. gapped or gapless, in the \( p = 3 \) spin glass model. \( T_d \) denotes dynamical transition line between mSG and PM phase. \( T_m \) is temperature where \( \lambda_L \) has a broad minimum. The yellow color represents a non-zero number for the above counts \( \leq 10 \) and the blue color represents zero value for the counts \( > 10 \).

S4: Lyapunov exponent

1. \( \lambda_L(T) \) in the cPM phase at high temperature

As shown in Fig.3(a) (inset) in the main text, \( \lambda_L(T) \) decreases with temperature for \( T > T_m \). As discussed in the preceding section, at high temperature \( (T \gtrsim J) \) in the cPM phase, the system has a spectral gap \( E_g \approx \sqrt{\Gamma T} \), which is not affected much by interaction in this temperature regime. Motivated by this, we fit the temperature dependence of \( \lambda_L \) with \( (A/T^a) \exp(-b/\sqrt{T}) \) for various \( \Gamma \) values ranging from the classical \((\Gamma \rightarrow 0)\) to the quantum limit \( \Gamma > T \), as shown in Fig.S6(a). We find \( a \approx 2 \) implying that \( \lambda_L \) decreases as \( 1/T^2 \) at high temperature.

2. \( \lambda_L(T) \) in the PM phase at low temperature

As discussed earlier, the spectral function becomes strongly gapped for \( \Gamma \gg T, J \) with a very weak self energy effect. Therefore, we expect the Lyapunov exponent to decrease exponentially as \( \sim \exp(-E_g/T) \) with \( E_g \approx E_g^0 = \Gamma/2 \). We fit \( \lambda_L \) with \( A \exp(-b/T) \), where \( A \) and \( b \) are fitting parameters. The results for the fitting and the extracted values of the parameter \( b \) are shown in Fig.S7 for several values of \( \Gamma \gg T \).
3. \(\lambda_L(T)\) in the mSG phase

As discussed in the main text, the Lyapunov exponent follows a power-law temperature dependence \(\lambda_L \sim T^\alpha\) in the mSG phase at low temperature with \(\alpha\) varying between 2 to 1. In particular, in the classical limit \(\hbar \to 0\), \(\lambda_L\) has a linear temperature dependence, analogous to that expected from the chaos bound \(\lambda_L^{(b)} = 2\pi T/\hbar\). For the latter, the linear-\(T\) coefficient diverges as \(1/\hbar\) in the classical limit. On the contrary, the linear-\(T\) coefficient of \(\lambda_L\) in the mSG phase approaches a finite constant in the classical limit, implying \(\lambda_L h/2\pi T \to 0\) for \(\Gamma \to 0\), as shown in Fig.S8. We also plot \(\lambda_L/\lambda_L^{(b)}\) for a few other values of \(\Gamma\) for comparison.

4. Perturbative analysis in the mSG phase

We have shown in Sections S13 and S21 that the saddle point equations and the kernel equation for OTOC in the mSG phase have forms similar to those in the PM phase of the model [Eq.(1)] with \(p = 2\) and \(p = 3\) terms with couplings \(J_2 = \sqrt{3}\eta EA J\) and \(J_3 = J\). Here, we obtain \(\lambda_L(T)\) by treating the \(p = 3\) term as perturbation around the \(p = 2\) saddle point in the low-temperature quantum limit \(T \ll \Gamma\). The analysis is similar to that done in the Fermi liquid (FL) phase\(^{48,49}\) of a large-\(N\) fermionic model related to the SYK model. The perturbative analysis is well controlled, even though the ratio \(J_3/J_2 = \sqrt{3}\eta EA\) is not necessarily small, since the \(p = 3\) spin interaction term is irrelevant at low energies.
The kernel equation for the PM phase of the $p = 2 + p = 3$ model can be written as

$$K = K_1 K_2$$ \hfill (S4.1)

$$K_2 f(\omega) = \left| Q^R \left( \omega + \frac{i \lambda L}{2} \right) \right|^2 f(\omega)$$ \hfill (S4.2)

$$K_1 f(t) = [J^2_2 + 3J^2_3 Q^W(t)] f(t)$$ \hfill (S4.3)

where $Q^R$ and $Q^W$ are retarded and Wightmann functions, respectively. The kernel $K_1$ is diagonal in the time domain and $K_2$ is diagonal in the frequency domain.

To calculate $\lambda_L$ we need to obtain $\left| Q^R \right|^2$ and $Q^W$. The saddle point equation in real frequency is given by

$$(Q^R)^{-1}(\omega) = -\frac{\omega^2}{\Gamma} + z - J^2_2 Q^R(\omega) - \Sigma^R_3(\omega)$$ \hfill (S4.4)

where $\Sigma^R_3(\omega)$ is the self-energy in real frequency for $p = 3$ spin interaction term. The self-energy in imaginary time is given as $\Sigma^R_3(\tau) = (3/2)J^2_3 \Sigma^W(\tau)$. When $J_3 = 0$, the saddle-point equation above can be solved exactly. The spectral function can be made gapless, like in the mSG phase, by choosing $z = 2J_2$. In this case the solution is given by

$$Q^R_0(\omega) = \frac{2}{J_2} \left[ \frac{1}{2} - \tilde{\omega}^2 + i \tilde{\omega} \sqrt{1 - \tilde{\omega}^2} \right]$$ \hfill (S4.5)

where $\tilde{\omega} = \frac{\omega}{2\sqrt{J_2} \Gamma}$. At low energies $\tilde{\omega} << 1$, using $\sqrt{1 - \tilde{\omega}^2} \approx 1$ we get

$$Q^R_0(\omega) = \frac{2}{J_2} \left[ \frac{1}{2} - \tilde{\omega}^2 + i \tilde{\omega} \right] = q^0_1 + iq^0_2,$$ \hfill (S4.6)

where $q^0_1$ and $q^0_2$ are real and imaginary parts of $Q^R_0(\omega)$, respectively, leading to

$$\rho^0(\omega) = -\frac{1}{\pi} \text{Im} Q^R_0(\omega) = -\frac{\omega}{\pi J_2 \sqrt{J_2} \Gamma}$$ \hfill (S4.7)

To calculate the Lyapunov exponent perturbatively, we need to compute $|Q^R|^2$ and $Q^W$ perturbatively around the $J_3 = 0$ gapless solution. For $|Q^R|^2$, the zeroth order $J_3 = 0$ solution is not enough and we have to incorporate the quasi-particle decay contribution, coming from the imaginary part of self-energy term $\Sigma_3(\tau)$. The imaginary part of $\Sigma^R_3(\omega)$ can be computed perturbatively using the zeroth order solution. At the leading order and small frequency we get

$$\text{Im} \Sigma^R_3(\omega) \approx \frac{J^2_3}{2\pi J_2 \Gamma} \left[ \frac{\omega^3}{2} + 2\pi^2 \omega T^2 \right] + O(\omega^5)$$ \hfill (S4.8)

Using the saddle-point equation [Eq.(S4.4)] and $\Sigma^R_3(\omega)$ above, we obtain

$$J^2_2 \left| Q^R \left( \omega + \frac{i \lambda L}{2} \right) \right|^2 = 1 + \frac{1}{J^2_2 q^0_2} \left[ \frac{\omega \lambda L}{\Gamma} - \frac{J^2_3}{2\pi J^2_2 \Gamma \left( \frac{\omega^2}{2} + 2\pi^2 T^2 \omega \right) \left( \omega^2 + 2\pi^2 T^2 \omega \right)} \right]$$ \hfill (S4.9)
For $Q^W$, the $J_3 = 0$ solution is sufficient and the Wightmann function $Q^W(t) = Q(it + \beta/2)$ in real frequency is given by

$$Q^W(\omega) = -\frac{\pi \rho^0(\omega)}{\sinh[\beta\omega/2]} = \frac{\omega}{J_2 \sqrt{J_2^2 + \beta^2}}$$  \hspace{1cm} (S4.10)

By doing Fourier transformation we obtain

$$Q^W(t) = \frac{\pi T^2}{J_2 \sqrt{J_2^2}} \text{sech}^2(T \pi t)$$  \hspace{1cm} (S4.11)

Using equations (S4.9) and (S4.10), the kernel equation (S4.1) is written as

$$K = 1 - \frac{\lambda_L}{\sqrt{J_2^2}} - \frac{J_3^2 \pi T^2}{J_2^2 \sqrt{J_2^2}} + \frac{3J_3^2 \pi T^2}{2J_2^2 \sqrt{J_2^2}} \frac{\partial^2}{\partial s^2} + \frac{3J_3^2 \pi T^2}{2J_2^2 \sqrt{J_2^2}} \text{sech}^2(t \pi T)$$  \hspace{1cm} (S4.12)

Using rescaled variable $s = t \pi T$ and equating the maximum eigenvalue of $K$ to 1, we obtain a simplified expression

$$\left( -\frac{1}{2} \frac{\partial^2}{\partial s^2} - \text{sech}^2 s \right) f(s) = \frac{1}{3} \left( \lambda_L \frac{J_3^2 \pi T^2}{\pi J_3^2 T^2} + 1 \right) f(s)$$  \hspace{1cm} (S4.13)

The bracketed term in the left-hand side of the above equation is the Schrödinger Hamiltonian with Pöschl-Teller potential\textsuperscript{50} whose eigenvalues are well known. The ground-state energy eigenvalue which maximizes $\lambda_L$ is $-1/2$. Using this, we obtain the Lyapunov exponent

$$\lambda_L \simeq \frac{\pi J_3^2 T^2}{2J_2^2}$$  \hspace{1cm} (S4.14)

From Fig:S9 we see that the coefficient $J_2 \propto \sqrt{q_{EA}}$ has weak dependence on temperature for low temperature regime in the quantum limit $\hbar \gtrsim 1$. Therefore, due to quasi-particle decay as in a Fermi liquid\textsuperscript{44,48,49}, we also find the Lyapunov exponent $\lambda_L \propto T^2$ in the quantum limit at low temperature. This analytical result agrees well with the numerically obtained power-law form $\lambda_L \sim T^\alpha$ in Fig.3(d) in the main text. From numerical fitting, we find that the exponent $\alpha$ is between 1.8 to 2 for $\hbar \gtrsim 1$.

On the contrary, we find $\lambda_L \propto T^{-1.1} \sim T$ [Fig.3(d), main text] in the classical limit ($\hbar \ll 1$) for the low temperature regime. The above analytical calculation therefore does not hold in the classical regime. There are a few possible reasons behind this. Our perturbative analysis in the mSG phase is done by drawing analogy to the $p = 2 + p = 3$ PM phase at the level of the saddle point and the OTOC kernel. However, the sum rule condition is $-\int d\omega \rho(\omega)n_B(\omega) = 1$ for the PM phase in contrast to the mSG phase where $-\int d\omega \rho(\omega)n_B(\omega) = 1 - q_{EA}$. The violation of this sum rule is small for $\hbar \gtrsim 1$ since $q_{EA} < 1$, as seen can be seen in Fig:S9, and the analytical result agrees with the numerical result for $\alpha$ here. However, for $\hbar \lesssim 1$, the sum rule violation is large and the temperature dependence of $q_{EA}$ is also stronger. Presumably due to these, the analytical result ($\lambda_L \propto T^2$) in this section is not applicable for the classical limit where $\lambda_L \propto T$.

S5: Correlation function and the onset of two-step relaxation

In this section we discuss the two-step glassy relaxation in the correlation function $C(t)$ in the PM phase above the dynamical transition temperature ($T_d$). In equilibrium, the correlation function can be obtained from retarded function using fluctuation-dissipation theorem. Using the fluctuation-dissipation theorem,

$$C(\omega) = \coth \left( \frac{\beta \omega}{2} \right) \text{Im} Q^R(\omega),$$  \hspace{1cm} (S5.1)

in the frequency domain. We obtain the correlation function $C(t)$ by doing Fourier transform, i.e. $C(t) = \int_{-\infty}^{\infty} (d\omega/2\pi)e^{-i\omega t} C(\omega)$. A two-step relaxation or decay of correlation function is usually seen in classical glasses in the so-called $\beta$-relaxation regime above the glass transition temperature. The correlation functions initially has a faster decay to a plateau-like regime\textsuperscript{30-32}, where correlation decays with slow power laws, before eventually decaying as a stretched exponential in the $\alpha$-relaxation regime. In the quantum $p$-spin glass model, the $C(t)$ also exhibits two-step relaxation as shown in Fig.4(c) (main text) and in Fig.S10(a). However, here the $\beta$-plateaus are mixed with
oscillations, presumably due to quantum fluctuations and the existence of a soft gap (see Sec.S3) in the spectrum. To describe the overall decay profile of \( C(t) \) [Fig.S10(a)], we take the following general two-step relaxation form

\[
f(t) = A \exp\left[-\left(t/\tau_1\right)^{\beta_1}\right] + B \exp\left[-\left(t/\tau_\alpha\right)^{\beta}\right].
\] (S5.2)

We fit numerically the correlation function \( C(t) \) with the above function. The second stretched exponential describes the final \( \alpha \)-decay of the correlation function \( C(t) \). We show the extracted fitting parameters \( A, B, \tau_1, \tau_\alpha, \beta_1, \beta \) in Figs.S10(b),(c) and (d) for a few temperatures for the classical limit \( h = \sqrt{0.001} \). We find that initial decay time \( \tau_1 \) more or less remains constant approaching \( T_d \), whereas the \( \alpha \)-relaxation time \( \tau_\alpha \) tends to diverge for \( T \to T_d \) [Fig.S10(b)]. The contributions \( A \) and \( B \) for the two decays change with temperature with the \( \alpha \)-relaxation starting to dominate close to \( T_d \) [Fig.S10(c)]. We find both the initial and later relaxations to be stretched exponentials as shown in Fig.S10(d). (Note that we denote the stretched exponent \( \beta \) for the \( \alpha \)-relaxation following the standard notation\textsuperscript{32} in the literature. This symbol should not be confused with inverse temperature ‘\( \beta \)’).

As the correlation function has oscillations around the \( \beta \) plateaus, so it is slightly tricky to obtain the onset temperature \( (T_{\beta}) \) of the two-step relaxation. We approximately estimate the \( T_{\beta} \) from the temperature where the value of \( C(t) \) at the first minimum of oscillation in the \( \beta \) plateau turns negative to positive approaching from high temperature. In Fig.S10(a), we see that the first minimum of \( C(t) \) for temperature \( T \sim 0.97 \) turns positive and then for all temperatures \( T < 0.97 \), \( C(t) \) remains positive. We thus estimate \( T_{\beta} \approx 0.97 \) for \( h = \sqrt{0.001} \). Similarly, we find \( T_{\beta} \)'s for other values of \( h \) and obtain \( T_{\beta}(h) \) curve as shown in Fig.1 in the main text. The two-step relaxation cannot be clearly distinguished for \( h \gtrsim 1.4 \).
FIG. S10. (a) The correlation function $C(t)$ (solid line) is shown as a function of time ($t$) for several values of temperature in the classical limit, $\hbar = \sqrt{0.001}$. The fits to the correlation functions (dashed line with the same color) with the two-step relaxation function [Eq.(S5.2)] is also shown for $T = 0.65, 0.80, 0.97$, where $T = 0.65$ is the closest to $T_d$. (b) The behaviour of the relaxation times $\tau_1$ and $\tau_\alpha$ are shown as a function of temperatures $T$. Though $\tau_1$ is almost constant, the $\alpha$-relaxation time ($\tau_\alpha$) diverges $T_d \approx 0.62$ is approached. (c) The dependence of coefficients $A$ and $B$ are shown as a function of $T$. (d) The dependence of stretched exponents $\beta_1$ and $\beta$ are shown as a function of $T$. 