Abstract. We consider divergence form, second-order strongly parabolic systems in a cylindrical domain with a finite number of subdomains under the assumption that the interfacial boundaries are $C^{1}$, $Dini$ and $C^{\gamma}_{0}$ in the spatial variables and the time variable, respectively. Gradient estimates and piecewise $C^{1/2}$-regularity are established when the leading coefficients and data are assumed to be of piecewise Dini mean oscillation or piecewise Hölder continuous. Our results improve the previous results in [26, 19] to a large extent. We also prove a global weak type-$(1, 1)$ estimate with respect to $A_{1}$ Muckenhoupt weights for the parabolic systems with leading coefficients which satisfy a stronger assumption. As a byproduct, we give a proof of optimal regularity of weak solutions to parabolic transmission problems with $C^{1,\mu}$ or $C^{1,Dini}$ interfaces. This gives an extension of a recent result in [5] to parabolic systems.

1. Introduction

We are concerned with second-order parabolic systems in divergence form arising from composite materials. We are interested in obtaining the gradient estimates for such systems when the domain can be decomposed into a finite number of time-dependent subdomains with coefficients and data which are of piecewise Dini mean oscillation. See the specific definitions given in the next section. These estimates will be shown to be independent of the distance between subdomains. Such a problem also appears in the study of the evolution of fronts in fluid dynamics, where the interfacial boundaries are typically time-dependent. See, for instance, [20].

The well-known theory of De Giorgi-Nash-Moser states that weak solutions for divergence form second-order elliptic and parabolic equations are Hölder continuous when the leading coefficients are bounded and measurable. On the other hand, we recall that the examples in [29, 30] reveal that solutions to second-order elliptic and parabolic equations with bounded and measurable coefficients are in general not Lipschitz continuous. A natural question is that what is the minimal regularity assumption of the coefficients for the $C^{1}$ or Lipschitz regularity of weak solutions. See [1, 4] for results in this direction. In [25], Li proved $C^{1}$-regularity of solutions to divergence form elliptic systems

$$
D_{\alpha}(A^{\alpha\beta}D_{\beta}u) = 0,
$$

provided that the modulus of the continuity of coefficients in the $L^{\infty}$ sense satisfies the Dini condition. This result was extended in [16] to non-homogeneous equations

$$
D_{\alpha}(A^{\alpha\beta}D_{\beta}u) = \text{div} g,
$$

(1.1)
where the coefficients and data are only assumed to be of Dini mean oscillation. See also [11] for the corresponding boundary estimates. Recently, Dong, Escauriaza, and Kim [12] considered parabolic equations in divergence form with zero Dirichlet boundary conditions and showed that weak solutions are continuously differentiable in the space variables and $C^{1/2}$ in the time variable up to the boundary when the leading coefficients have Dini mean oscillation with respect to the space variables and the lower-order coefficients satisfy certain conditions.

There are also many works in the literature concerning the case when the domain contains subdomains and the coefficients are piecewise regular. See, for instance, [2, 3, 31]. Chipot, Kinderlehrer, and Vergara-Caffarelli [7] showed that any weak solution $u$ of (1.1) is locally Lipschitz if $A^{\alpha\beta}$ are piecewise constants and $g \in C^k$ for $k \geq \lfloor d/2 \rfloor$, when the domain consists of a finite number of linearly elastic, homogeneous, and parallel laminae. Li and Vogelius [28] considered scalar elliptic equations

$$\nabla_a(a^{\alpha\beta}\nabla u) = \text{div} g + f,$$

where the matrix $(a^{\alpha\beta})$ and data are assumed to be $C^0$ up to the boundary in each subdomain with $C^{1,\mu}, 0 < \mu \leq 1$, boundary, but may have jump discontinuities across the boundaries of the subdomains. The authors derived global Lipschitz and piecewise $C^{1,\delta'}$ estimates of the solution $u$ for any $\delta' \in (1, \min[\delta, \frac{\mu}{2(\mu+1)}])$, with the estimates independent of the distance between subdomains. Li and Nirenberg [27] later extended their results to elliptic systems under the same conditions when $\delta'$ is in a larger range $(0, \min[\delta, \frac{\mu}{2(\mu+1)}])$. Recently, Dong and Xu [18] improved the regularity of $u$ by further extending the range of $\delta'$ from $(0, \min[\delta, \frac{\mu}{2(\mu+1)}])$ to $(0, \min[\delta, \frac{\mu}{\mu+1}])$, which seems to be sharp. The proof is based on a weak type-(1,1) estimate and Campanato’s method, which are different from the $L^2$-estimates used in [27, 28].

Parabolic equations have also been studied in this setting. In [26], Li and Li extended the interior estimates in [27] to parabolic systems with coefficients which are piecewise Hölder continuous in the space variables and smooth in the time variable, when the subdomains are cylindrical. See also [19], where the coefficients are independent of the time variable and the subdomains are also assumed to be cylindrical. We would like to mention that in [9] optimal regularity of weak solutions was obtained when the coefficients and data are Dini continuous in the time variable and all but one spatial variable.

The current paper is a natural extension of [18] from the elliptic case to the parabolic case. We substantially improve the results in the aforementioned papers [26, 19] in the following two aspects. First, we allow the subdomains to be non-cylindrical, and the interfacial boundaries to be $C^{1,\text{Dini}}$ in the spatial variables and $C^{\gamma_0}$ in the time variable, where $\gamma_0 > 1/2$. Second, we relax the regularity assumption on the leading coefficients, particularly in the time variable. We show in Theorem 2.1 that $\mathcal{H}^1_p (1 < p < \infty)$ weak solutions to parabolic systems in divergence form are Lipschitz in all spatial variables and piecewise $C^{1/2,1}$ when the leading coefficients and data are of piecewise Dini mean oscillation and the lower-order coefficients are bounded. Besides, we obtain the local $L^p$-estimate for $\mathcal{H}^1$ weak solutions in Corollary 2.2 by adapting the idea in [1, 4], and thus the results in Theorem 2.1 also hold for these solutions. When the leading coefficients and data have piecewise Hölder regularity, we prove the piecewise $C^{1,\delta'}$-regularity in...
the spatial variables and $C^{(1+\delta')/2}$-regularity in the time variable of weak solutions in Theorem 2.3, where $\delta'$ is in the optimal range $(1, \min\{\delta, \frac{2}{\gamma_0 + 1}, 2\gamma_0 - 1\}]$.

Our arguments in proving Theorems 2.1 and 2.3 are different from those in [19, 26, 27, 28]. The proofs below are based on Campanato’s method, which was also used recently in [18]. The key point is to show the mean oscillation of $Du$ in balls or cylinders vanishes in a certain order as the radii of the balls or cylinders go to zero. However, this method cannot be employed directly because $Du$ is discontinuous across the interfacial boundaries and we only impose the assumption on the $L_1$-mean oscillation of the coefficients and data, so that the usual argument based on $L_p (p > 1)$ estimates does not work here. To overcome these difficulties, we first fix the coordinate system and derive weak type-$(1, 1)$ estimates by using a duality argument. We then establish some interior Hölder regularity of $D_x^\beta u$ and $\bar{U} := \bar{A}^\beta D_\beta u$ for parabolic systems with coefficients depending on one variable, say, $x^d$. The desired results in Theorem 2.1 are proved by adapting Campanato’s approach in the $L_p$ setting for some $p \in (0, 1)$. The proof of the $C^{(1+\delta')/2}$-Hölder continuity in $t$ of weak solutions in Theorem 2.3 is more involved. We prove a weak type-$(1, 1)$ estimate and apply Campanato’s idea to $u$ itself instead of its first derivatives. For this, we introduce a set consisting of functions in $x$ which are linear in $x'$ and prove Lemma 5.3 which plays a key role in estimating the difference between $u$ and its approximations in the $L_q$-mean sense, $q \in (0, 1)$. Compared to [18], this is new and can be considered as the main contribution of the current paper.

As a byproduct, in Theorem 7.1 we prove the existence, uniqueness, and $C^{1,\mu}$ regularity of weak solutions to transmission problems with $C^{1,\mu}$ interfaces in the parabolic setting, which is an extension of a recent result in [5]. We also consider a more general case when the interfaces are $C^{1,\text{Dini}}$. See Theorem 7.3. For the proof, we adapt an idea in [10] by solving certain auxiliary equations in subdomains with conormal boundary data and then reducing the transmission problem to a parabolic equation with piecewise Hölder (or Dini) inhomogeneous terms. In contrast to the elliptic case, here we cannot treat the derivatives of solutions to the auxiliary equations as inhomogeneous terms because their time derivatives are in Sobolev spaces of negative order. In this paper, we modify the argument in [10] by considering the difference of $u$ and these auxiliary solutions. As such, we need to extend these solutions to the whole domain, which is achieved by a partition of unity argument applied to each subdomain together with a flattening-reflection technique.

Throughout this paper, unless otherwise stated, $N$ denotes a constant, whose value may vary from line to line and independent of the distance between subdomains. We call it a universal constant.

The rest of this paper is organized as follows. We formulate the problem and state our main results, Theorems 2.1 and 2.3, in Section 2. In Section 3, we introduce some notation, definitions, and auxiliary lemmas used in this paper. The main result Theorem 2.1 under the assumptions that the leading coefficients and data are of piecewise Dini mean oscillation is proved in Section 4, where we also give the proof of Corollary 2.2. We prove Theorem 2.3 in Section 5. Section 6 is devoted to a global weak type-$(1, 1)$ estimate with respect to $A_1$ Muckenhoupt weights for solutions to parabolic systems. In Section 7, we state and prove Theorems 7.1 and 7.3 by adapting the method in [10]. In the Appendix, we prove a weighted
\( \mathcal{H}_p\)-solvability and estimate for divergence form parabolic systems in nonsmooth domains with partially VMO coefficients.

2. Problem formulation and main results

2.1. Problem formulation. In this paper, we aim to establish gradient estimates for strongly parabolic systems in divergence form

\[
\mathcal{P}u := -u_t + D_{\alpha}(A^{\alpha\beta}D_{\beta}u + B^\alpha u) + \hat{B}^\alpha D_{\alpha}u + Cu = \text{div} \ g + f
\]

(2.1)
in a cylindrical domain \( Q := (-T, 0) \times \mathcal{D} \), where \( T \in (0, \infty) \) and \( \mathcal{D} \) is a bounded domain in \( \mathbb{R}^d \). We assume that \( Q \) contains \( M \) disjoint time-dependent subdomains \( Q_j, j = 1, \ldots, M \), and the interfacial boundaries are \( C^{1, \text{Dini}} \) in the spatial variables and \( C^{\gamma_0} \) in the time variable, where \( \gamma_0 > 1/2 \). See the details in Definition 3.2. We also assume that any point \((t, x) \in Q \) belongs to the boundaries of at most two of the \( Q_j \)'s. Moreover, the Einstein summation convention over repeated indices are assumed throughout this paper. Here

\[
A = (u^1, \ldots, u^n)^\top, \quad \gamma = (\gamma^1, \ldots, \gamma^n)^\top, \quad f = (f^1, \ldots, f^n)^\top
\]

are (column) vector-valued functions, \( A^{\alpha\beta}, B^\alpha, \hat{B}^\alpha \) (often denoted by \( A, B, \hat{B} \) for abbreviation), and \( C \) are \( n \times n \) matrices, which are bounded by a positive constant \( \Lambda \). The leading coefficients matrices \( A^{\alpha\beta} \) satisfy the strong parabolicity condition: there exists a number \( v > 0 \) such that for any \( \xi = (\xi^i) \in \mathbb{R}^{n\times d} \),

\[
|v| |\xi|^2 \leq A^{\alpha\beta}_{ij} \xi^i \xi^j \quad |A^{\alpha\beta}| \leq v^{-1}.
\]

To localize the problem, we slightly abuse the notation by taking \( Q \) to be a unit cylinder \( Q^- := (-1, 0) \times B_1 \), and \( z_0 = (t_0, x_0) \in (-9/16, 0) \times B_{3/4} \). The domain is fixed as follows. By suitable rotation and scaling, we may suppose that a finite number of subdomains lie in \( Q^- \) and that they can be represented by

\[
x^d = h_j(t, x') , \quad \forall \ t \in (-1, 0), \ x' \in B_1', \ j = 1, \ldots, L < M,
\]

where

\[
-1 < h_1(t, x') < \cdots < h_l(t, x') < 1, \quad (2.2)
\]

\[
h_j(t, \cdot) \in C^{1, \text{Dini}}(B_1'), \text{ and } h_j(t, x') \in C^{\gamma_0}(-1, 0), \text{ where } \gamma_0 > 1/2. \text{ Set } h_0(t, x') = -1 \text{ and } h_{l+1}(t, x') = 1. \text{ Then we have } l + 1 \text{ regions:}
\]

\[
Q_j := \{(t, x) \in Q : h_{j-1}(t, x') < x^d < h_j(t, x') \}, \quad 1 \leq j \leq l + 1.
\]

We may suppose that there exists some \( Q_{b_j} \), such that \((t_0, x_0) \in (-9/16, 0) \times B_{3/4} \) \( \cap \) \( Q_{b_j} \) and the closest point on \( \partial_{\nu} Q_{b_j} \cap \{t = t_0\} \) to \((t_0, x_0)\) is \((t_0, x_0', h_{b_j}(t_0, x_0'))\), and \( \nu_{x'} h_{b_j}(t_0, x_0') = 0' \). We introduce the \( l + 1 \) “strips”

\[
\Omega_j := \{(t, x) \in Q : h_{j-1}(t_0, x_0') < x^d < h_j(t_0, x_0') \}, \quad 1 \leq j \leq l + 1.
\]

Denote by \( \mathcal{A} \) the set of piecewise constant functions in each \( \Omega_j \). We then further assume that \( A \) is of piecewise Dini mean oscillation in \( Q \), that is,

\[
\omega_A(r) := \sup_{z \in Q} \inf_{A \in \mathcal{A}} \int_{Q^r(z)} |A(z) - \hat{A}| \ dz
\]

(2.3)
satisfies the Dini condition, where \( Q^r(z_0) := (t_0 - r^2, t) \times B_r(x_0) \subset Q \). The reader can refer to Definition 3.1 about the Dini condition. For \( \epsilon > 0 \) small, we set

\[
\mathcal{D}_\epsilon := \{x \in \mathcal{D} : \text{dist}(x, \partial \mathcal{D}) > \epsilon\}.
\]
2.2. Main results. We state the main results of this paper.

**Theorem 2.1.** Let $Q$ be defined as above. Let $\varepsilon \in (0,1)$, $p \in (1,\infty)$, and $\gamma \in (0,1)$. Assume that $A$, $B$, and $g$ are of piecewise Dini mean oscillation in $Q$, and $f, g \in L_\infty(Q)$. If $u \in \mathcal{H}_1^1(Q)$ is a weak solution to (2.1) in $Q$, then $u \in C^{1/2,1}(Q) \cap ((-T+\varepsilon,0) \times \mathcal{D}_j)$, $j = 1,\ldots,M$. Moreover, for any fixed $z_0 \in (-T+\varepsilon,0) \times \mathcal{D}_j$, there exists a coordinate system associated with $z_0$, such that for all $z \in (-T+\varepsilon,0) \times \mathcal{D}_j$, we have

\[
|\langle D_x u(z_0), U(z_0) \rangle - \langle D_x u(z), U(z) \rangle| 
\leq N \int_0^{z_0-z} \frac{\omega(s)}{s} ds + N|z_0 - z|_{p_0}^{1/\gamma} \left( \|Du\|_{L_1(Q)} + \|g\|_{L_\infty(Q)} + \|f\|_{L_\infty(Q)} + \|u\|_{L_1(Q)} \right) 
+ N \int_0^{z_0-z} \frac{\omega_A(s)}{s} ds \left( \|Du\|_{L_1(Q)} + \|g\|_{L_\infty(Q)} + \|f\|_{L_\infty(Q)} + \|u\|_{L_1(Q)} \right),
\]

where $U = A^{1/2} D_x u + B^{1/2} u - g_d$, $\| \cdot \|_p$ is the parabolic distance defined in (3.1), $N$ depends on $n, d, M, p, \Lambda, \nu, \varepsilon, \omega_0$, and the $C^{1,\mu}$ and $C^{\gamma}$ characteristics of $Q_j$ with respect to $x$ and $t$, respectively, and $\omega_*(t)$ is a Dini function derived from $\omega_*(t)$. See (4.10).

By using a duality argument and Theorem 2.1, we obtain the following result.

**Corollary 2.2.** Under the same conditions as in Theorem 2.1, if $u \in \mathcal{H}_1^1(Q)$ is a weak solution to (2.1) in $Q$, then $u \in \mathcal{H}_1^{1,p_{\text{loc}}}(Q)$ for some $p \in (1,\infty)$ and for any $Q' \subset Q$,

\[
\|u\|_{\mathcal{H}_1^1(Q')} \leq N \left( \|g\|_{L_\infty(Q)} + \|f\|_{L_\infty(Q)} + \|u\|_{\mathcal{H}_1^1(Q)} \right).
\]

Furthermore, the conclusion of Theorem 2.1 still holds true.

The next theorem shows that if we impose piecewise Hölder regularity assumptions on the coefficients and data, then $D_x u$ and $U$ are Hölder continuous.

**Theorem 2.3.** Let $Q$ be defined as above and the boundary condition on each subdomain $\mathcal{D}_j$ be replaced with $C^{1,\mu}$. Let $\varepsilon \in (0,1)$ and $p \in (1,\infty)$. Assume that $A, B, g \in C^{0,1/\mu}(\overline{Q})$ with $\delta \in (0,\mu/(1+\mu)]$, and $f \in L_\infty(Q)$. If $u \in \mathcal{H}_1^1(Q)$ is a weak solution to (2.1) in $Q$, then $u \in C^{1,\mu}(Q) \cap ((-T+\varepsilon,0) \times \mathcal{D}_j)$, $j = 1,\ldots,M$. Moreover, for any fixed $z_0 \in (-T+\varepsilon,0) \times \mathcal{D}_j$, there exists a coordinate system associated with $z_0$ such that for all $z \in (-T+\varepsilon,0) \times \mathcal{D}_j$, we have

\[
|\langle D_x u(z_0) - D_x u(z), U(z_0) - U(z) \rangle| 
\leq N|z_0 - z|_{p_0}^{1/\gamma} \left( \sum_{j=1}^M \|g\|_{L_\infty(Q_j)} + \|f\|_{L_\infty(Q_j)} + \|u\|_{L_1(Q_j)} + \|Du\|_{L_1(Q_j)} \right),
\]

and

\[
\langle u\rangle_{1+\delta',Q_j/2} \leq N \left( \sum_{j=1}^M \|g\|_{L_\infty(Q_j)} + \|f\|_{L_\infty(Q_j)} + \|u\|_{L_1(Q_j)} + \|Du\|_{L_1(Q_j)} \right),
\]

where $\delta' = \min\{\delta, 2\gamma_0 - 1\}$, $N$ depends on $n, d, M, \delta, \mu, \Lambda, \varepsilon, p, \|A\|_{C^{0,1/\mu}(\overline{Q})}$, $\|B\|_{C^{0,1/\mu}(\overline{Q})}$, and the $C^{1,\mu}$ and $C^{\gamma}$ norms of $Q_j$ with respect to $x$ and $t$, respectively.
3. Preliminaries

In this section, we first introduce the notation and definitions. Then we prove some properties of our domain, coefficients, and data. Finally we establish the existence and $L_p$-estimates of solutions to parabolic systems with coefficients satisfying certain regularity assumptions. Besides, we also prove some auxiliary estimates that will be used in the proofs of our main results.

3.1. Notation and definitions. We follow most of the notation in [18] and [9]. We write $z = (t, x)$, $z' = (t, x')$, and $x = (x^1, \ldots, x^d) = (x', x'')$, where $d \geq 2$. We denote

\[ B_r(x) := \{ y \in \mathbb{R}^d : |y - x| < r \}, \quad B'_r(x') := \{ y' \in \mathbb{R}^{d-1} : |y' - x'| < r \} , \]

\[ Q_r^+(t,x) := (t-r^2,t) \times B_r(x), \quad Q_r^- (t,x) := (t-r^2,t) \times B'_r(x') , \]

\[ Q_r^-(t,x) := (t,t+r^2) \times B_r(x) , \quad Q_r(t,x) := (t-r^2,t+r^2) \times B_r(x) , \]

and the parabolic distance between two points $z_1 = (t_1, x_1)$ and $z_2 = (t_2, x_2)$ by

\[ |z_1 - z_2|_p := \max \{|t_1 - t_2|^{1/2}, |x_1 - x_2|\} . \] (3.1)

We use $B_r := B_r(0)$, $B'_r := B'_r(0')$, $Q_r^+ := Q_r^+(0,0)$, $Q_r^- := Q_r^-(0,0)$, $Q_r(t,x) := Q \cap Q_r(t,x)$, and $Q(t,x) := Q \cap Q_r(t,x)$ for abbreviation, respectively. The parabolic boundary of $Q = (a, b) \times \mathcal{D}$ is defined by

\[ \partial_p Q = ((a,b) \times \partial \mathcal{D}) \cup ([a] \times \mathcal{D}) . \]

The following notation will also be used:

\[ D_t u = u_t , \quad D_x u = u_x , \quad DD_x u = u_{xx} . \]

For a function $f$ defined in $\mathbb{R}^{d+1}$, we set

\[ (f)_Q = \frac{1}{|Q|} \int_Q f(t,x) \, dx \, dt = \frac{1}{f} \int_Q f(t,x) \, dx \, dt , \]

where $|Q|$ is the $d+1$-dimensional Lebesgue measure of $Q$. For $\gamma \in (0,1]$, we denote the $C^{\gamma/2}$ semi-norm by

\[ [u]_{\gamma/2,\gamma,Q} := \sup_{(t,x), (s,y) \in Q} \frac{|u(t,x) - u(s,y)|}{|t-s|^{\gamma/2} + |x-y|^\gamma} , \]

and the $C^{\gamma/2,\gamma}$ norm by

\[ |u|_{\gamma/2,\gamma,Q} := [u]_{\gamma/2,\gamma,Q} + |u|_{0,Q} , \quad \text{where } |u|_{0,Q} = \sup_{Q} |u| . \]

We define

\[ [u]_{(1+\gamma)/2,1+\gamma,Q} := [Du]_{\gamma/2,\gamma,Q} + \langle u \rangle_{1+\gamma,Q} \]

and

\[ |u|_{(1+\gamma)/2,1+\gamma,Q} := [u]_{(1+\gamma)/2,1+\gamma,Q} + |u|_{0,Q} , \]

where

\[ \langle u \rangle_{1+\gamma,Q} := \sup_{(t,x),(s,y) \in Q} \frac{|u(t,x) - u(s,x)|}{|t-s|^{(1+\gamma)/2}} . \]

Next we define the semi-norm

\[ [u]_{\gamma',(1+\gamma)/2,1+\gamma,Q} := [D^\gamma u]_{\gamma'/2,\gamma,Q} + \langle u \rangle_{1+\gamma,Q} \]
and the norm
\[ |u|_{C^{(2,1)}(D;\mathbb{R})} := |u|_{C^{(2,1)}(D;\mathbb{R})} + |Du|_{C^{(2,1)}(D;\mathbb{R})} \]
where
\[ [Du]_{C^{(2,1)}(D;\mathbb{R})} := \sup_{\substack{\xi,\eta \in D; \|\xi\| + \|\eta\| \leq 2|u|^{2/3}}} \frac{|Du(t,\xi)| + |Du(t,\eta)|}{|\xi - \eta|^2}. \]

Denote \( C^{(2,1)}(D;\mathbb{R}) \) by the set of all bounded measurable functions \( u \) for which \( Du \) are bounded and continuous in \( D \) and \( |u|_{C^{(2,1)}(D;\mathbb{R})} < \infty \).

We introduce some Lebesgue spaces which will be utilized throughout the paper. For \( p \in (1, \infty) \), we denote
\[ W^{1,2}_p(D;\mathbb{R}) := \{ u : u, Du, D^2u \in L^p(D;\mathbb{R}) \}. \]
We define the solution spaces \( H^1_p(D;\mathbb{R}) \) as follows. Set
\[ H^1_p(D;\mathbb{R}) := \{ f : f = \sum_{|\alpha| \leq 1} D^\alpha f, f \in L^p(D;\mathbb{R}) \}, \]
\[ \|f\|_{H^1_p(D;\mathbb{R})} := \inf \left\{ \sum_{|\alpha| \leq 1} \|D^\alpha f\|_{L^p(D;\mathbb{R})} : f = \sum_{|\alpha| \leq 1} D^\alpha f \right\}, \]
and
\[ \mathcal{H}^1_p(D;\mathbb{R}) := \{ u : u \in H^1_p(D;\mathbb{R}), D^2u \in L^p(D;\mathbb{R}), 0 \leq |\alpha| \leq 1 \}, \]
\[ \|u\|_{\mathcal{H}^1_p(D;\mathbb{R})} := \|u\|_{H^1_p(D;\mathbb{R})} + \sum_{|\alpha| \leq 1} \|D^\alpha u\|_{L^p(D;\mathbb{R})}. \]

Define \( C^0([-1, 0] \times D) \) to be the collection of infinitely differentiable functions \( \phi := \phi(t, x) \) with compact supports in \([-1, 0] \times D\). Finally, we set \( \mathcal{H}^1_p((-1, 0) \times D) \) to be the closure of \( C^0([-1, 0] \times D) \) in \( \mathcal{H}^1_p((-1, 0) \times D) \).

**Definition 3.1.** We say that a continuous increasing function \( \omega : [0, 1) \to \mathbb{R} \) satisfies the Dini condition provided that \( \omega(0) = 0 \) and
\[ \int_0^r \frac{\omega(s)}{s} ds < +\infty, \quad \forall r \in (0, 1). \]

**Definition 3.2.** Let \( D \subset \mathbb{R}^d \) be open and bounded. We say that \( \partial D \) is \( C^{1, \text{Dini}} \) if for each point \( x_0 \in \partial D \), there exists \( R_0 \in (0, 1/8) \) independent of \( x_0 \) and a \( C^{1, \text{Dini}} \) function (i.e., \( C^1 \) function whose first derivatives are Dini continuous) \( \varphi : B_{R_0} \to \mathbb{R} \) such that (upon relabeling and reorienting the coordinates if necessary) in a new coordinate system \((x', x'')\), \( x_0 \) becomes the origin,
\[ D_{R_0}(0) = \{ x \in B_{R_0} : x'' > \varphi(x') \}, \quad \varphi(0') = 0, \quad \nabla_x \varphi(0') = 0, \]
and \( \nabla_x \varphi \) has a modulus of continuity \( \omega_{x_0} \), which is increasing, concave, independent of \( x_0 \), and satisfies the Dini condition.
3.2. Some auxiliary estimates. Under the same setting as in Section 2.1, we have the following result.

Lemma 3.3. There exists a constant $N$, depending on $d, l$, the $C^{1, \text{Dini}}$ characteristics of $h_j(t, \cdot)$ for fixed $t \in (-1, 0)$, and $C^{\gamma}$-norms of $h_j(\cdot, x')$ for fixed $x' \in B'_r(0)$, $1 \leq j \leq l + 1$, such that

$$|\{Q_j \Delta \Omega_j \cap Q_r^-(z_0)\} | \leq N\alpha_1(\gamma) d^{d+2} + N r^{d+2 \gamma}$$

when $0 < r < r_0 := \frac{2}{3} \int_0^{R_0/2} a'(s) s ds$,

where $Q_j \Delta \Omega_j = (Q_j \setminus \Omega_j) \cup (\Omega_j \setminus Q_j)$, $\omega'_0$ denotes the left derivative of $\omega_0$, and $\omega_1 = \alpha_1(\gamma)$ is a Dini function derived from $\omega_0$ in Definition 3.2.

Proof. Let $(t_0, x', h_j(t_0, x')) \in Q_r^-(t_0, x_0)$ for some $x' \in B'_r$. We first prove that $|\nabla_x h_j(t_0, x')|$ in $B'_r(x'_0)$ is bounded by $\alpha_1(\gamma) := 2\alpha_0(2r + R)$, where $R$ is a fixed number only depending on $r$. This is based on the arguments in the proof of [18, Lemma 2.3]. Indeed, we denote the supremum of $|\nabla_x h_j(t_0, x')|$ in $B'_r(x'_0)$ by $S$. Then for fixed $(t_0, y', h_j(t_0, y')) \in Q_r^-(t_0, x_0)$, we obtain from (2.2) and $\nabla_x h_j(t_0, y') = 0$ that

$$|h_j(t_0, y') - h_j(t_0, x'_0)| \leq r, \quad |\nabla_x h_j(t_0, y')| \leq \alpha_0(2r), \quad |\nabla_x h_j(t_0, y')| \geq S - \alpha_0(2r).$$

Then in view of (2.2), we have for any $R \in (0, 1/8)$,

$$\int_0^R (S - 2\alpha_0(2r + s)) ds \leq 3r. \quad (3.2)$$

The maximum of the left-hand side of (3.2) with respect to $R$ is attained when $2\alpha_0(2r + R) = S$. This yields

$$R\alpha_0(2r + R) \geq \int_0^R \alpha_0(2r + s) ds \leq 3r/2.$$

So we have

$$\int_0^R \alpha'_0(2r + s) ds \leq 3r/2. \quad (3.3)$$

In order to obtain an upper bound for $S$, we use (3.3) to fix the number $R = R(r)(> 2r)$ such that

$$\int_0^R \alpha'_0(2r + s) ds = 3r/2.$$

We henceforth get

$$S = 2\alpha_0(2r + R) = : \alpha_1(\gamma),$$

which is a Dini function on $(0, r_0)$. See the proof of [18, Lemma 2.3].

Now, for any $t \in (t_0 - r^2, t_0)$, by using the triangle inequality and $h_j(\cdot, x') \in C^\gamma$, we have

$$|h_j(t, x') - h_j(t_0, x'_0)| \leq |h_j(t, x') - h_j(t_0, x')| + |h_j(t_0, x') - h_j(t_0, x'_0)| \leq N r^{2\gamma} + \alpha_1(\gamma).$$

We thus obtain

$$|\{Q_j \Delta \Omega_j \cap Q_r^-(t_0, x_0)\} | \leq N(r^{2\gamma} + \alpha_1(\gamma) r^{d+1} \leq N r^{d+2} + 2N \alpha_1(\gamma) r^{d+1}.$$ 

The lemma is proved. \qed
Let $\tilde{A}^{(i)} \in \mathcal{A}$ be a constant function in $Q$, which corresponds to the definition of $\omega_A(r)$ in (2.3). Similarly, $\tilde{B}^{(i)}$ and $\tilde{g}^{(i)}$ are defined in $Q$. We define the piecewise constant (matrix-valued) functions

$$
\tilde{A}(t, x) = \tilde{A}^{(i)}, \quad (t, x) \in \Omega.
$$

We remark that $\tilde{A}(t, x)$ only depends on $x^d$. Using $\tilde{B}^{(i)}$ and $\tilde{g}^{(i)}$, we similarly define piecewise constant functions $\tilde{B}$ and $\tilde{g}$. From Lemma 3.3 and the boundedness of $\tilde{A}$, we have

$$
\int_{Q_r(z_0)} |A - \tilde{A}| dz \leq Nr^{-d-2} \sum_{j=1}^{l+1} |(Q_j \cup \Omega) \cap Q^+_r(z_0)|
$$

$$
\leq N\omega_1(r) + Nr^{2\gamma_0-1} =: N\tilde{\omega}_1(r),
$$

(3.4)

where $\tilde{\omega}_1(r) := \omega_1(r) + r^{2\gamma_0-1}$ is a Dini function. This is also true for $\tilde{B}$ and $\tilde{g}$.

We now turn to the $\mathcal{H}_1$-estimate for parabolic equations with various partially small BMO (bounded mean oscillation) coefficients (see [14]): there exists a sufficiently small constant $\gamma_0 = \gamma_0(d, n, p, \nu) \in (0, 1/2)$ and a constant $r_0 \in (0, 1)$ such that for any $r \in (0, r_0)$ and $(t_0, x_0) \in Q_1^-$ with $B_r(x_0) \subset B_1$, in a coordinate system depending on $(t_0, x_0)$ and $r$, one can find a $\tilde{A} = \tilde{A}(x^d)$ satisfying

$$
\int_{Q_r(x_0)} |A(t, x) - \tilde{A}(x^d)| dx dt \leq \gamma_0.
$$

(3.5)

We obtain the following lemma from [14, Theorem 8.2] by a similar localization argument that led to [9, Lemma 4, Corollary 3], the interpolation inequality, and iteration arguments.

**Lemma 3.4.** Let $0 < p < 1 < q < \infty$. Assume $A$ satisfies (3.5) with a sufficiently small constant $\gamma_0 = \gamma_0(d, n, p, \nu, \Lambda) \in (0, 1/2)$ and $u \in \mathcal{H}_1^{\gamma_{loc}}$ satisfies (2.1) in $Q_1^-$, where $f, g \in L_q(Q_1^-)$. Then

$$
||u||_{\mathcal{H}_1^{\gamma}(Q_1^-)} \leq N(||u||_{L_q(Q_1^-)} + ||g||_{L_q(Q_1^-)} + ||f||_{L_q(Q_1^-)}).
$$

In particular, if $q > d + 2$, it holds that

$$
||u||_{L_q^{2\gamma} Q_1^-} \leq N(||u||_{L_q(Q_1^-)} + ||g||_{L_q(Q_1^-)} + ||f||_{L_q(Q_1^-)}),
$$

where $\gamma = 1 - (d + 2)/q$ and $N$ depends on $n, d, \nu, \Lambda, p, q$, and $r_0$.

In the proofs below, we will also use the $\mathcal{H}_1^{\gamma}$-solvability for parabolic systems with leading coefficients which satisfy (3.5) in $Q_1^-$. For this we choose a cut-off function $\eta \in C_0^\infty(B_1)$ with

$$
0 \leq \eta \leq 1, \quad \eta \equiv 1 \text{ in } B_{3/4}, \quad |\nabla \eta| \leq 8.
$$

Let $\tilde{P}$ be the parabolic operator defined by

$$
\tilde{P} u := -u_t + D_\alpha(\tilde{A}^{\alpha\beta}D_\beta u),
$$

where $\tilde{A}^{\alpha\beta} = \eta A^{\alpha\beta}(t, x) + \nu(1 - \eta)\delta_{\alpha\beta}$, $\delta_{\alpha\beta}$ and $\delta_{ij}$ are the Kronecker delta symbols. Then for sufficiently small $\gamma$, the coefficients $\tilde{A}^{\alpha\beta}(t, x)$ and the boundary $\partial B_1$ satisfy the Assumption 8.1 ($\gamma$) in [14]. By [14, Theorem 8.2] (or Lemma 8.2 below), we have

**Lemma 3.5.** For any $p \in (1, \infty)$, $g, f \in L_p(Q_1^-)$, the following hold.
(1) For any \( u \in \mathcal{H}_p(Q_1^-) \) satisfying
\[
\mathcal{P} u = \text{div} \ g + f \quad \text{in} \ Q_1^-,
\]
we have
\[
\|u\|_{\mathcal{H}_p(Q_1^-)} \leq N\left(\|g\|_{L_p(Q_1^-)} + \|f\|_{L_p(Q_1^-)}\right),
\]
where \( N \) depends on \( d, n, p, \nu, \Lambda, \) and \( r_0 \).
(2) For any \( g, f \in L_p(Q_1^-) \), there exists a unique solution \( u \in \mathcal{H}_p(Q_1^-) \) of (3.6) with the initial data \( u(-1, \cdot) = 0 \) in \( B_1 \). Furthermore, \( u \) satisfies (3.7).

In addition to the above estimates, we will also need to consider systems with coefficients depending only on \( x' \). Denote
\[
\mathcal{P}_0 u := -u_t + D_x(\tilde{A}^{dp}(x')D\mu),
\]
and
\[
\tilde{U} := \tilde{A}^{dp}(x')D\mu.
\]

**Lemma 3.6.** Let \( p \in (0, \infty) \). Assume \( u \in C^{0,1}_{\text{loc}} \) satisfies \( \mathcal{P}_0 u = 0 \) in \( Q_1^- \). Then there exists a constant \( N = N(u, d, p, \nu, \Lambda) \) such that
\[
[u]_{C^{1,2}((Q_1^-) \cap Q_{1/2})} \leq N\|u\|_{L_p(Q_1^-)} \quad \text{and} \quad [D_{x'} u]_{C^{1,2}((Q_1^-) \cap Q_{1/2})} \leq N\|D_{x'} u\|_{L_p(Q_1^-)},
\]
and
\[
[\tilde{U}]_{C^{1,2}((Q_1^-) \cap Q_{1/2})} \leq N\|Du\|_{L_p(Q_1^-)}. \tag{3.9}
\]

**Proof.** By using Lemma 3.4, the Sobolev embedding theorem, the interpolation inequality, and iteration arguments, we have
\[
\|u\|_{L_p(Q_{1/2})} \leq N\|u\|_{L_p(Q_1^-)} \quad p > 0. \tag{3.10}
\]
For fixed \( t \in (-1, 0) \), we define the finite difference quotient
\[
\delta_h k f(t,x) := \frac{f(t,x + h\epsilon_k) - f(t,x)}{h},
\]
where \( k = 1, \ldots, d - 1, 0 < |h| < 1/12 \). Since \( \tilde{A}^{dp}(x') \) are independent of \( x' \), we have \( \mathcal{P}_0(\delta_h k u) = 0 \) in \( Q_1^- \). Then in view of Lemma 3.4 and (3.10), we obtain
\[
\|\delta_h k u\|_{\mathcal{H}_p(Q_{1/2})} \leq N\|\delta_h k u\|_{L_2(Q_{1/2})} \leq N\|D_{x'} u\|_{L_2(Q_{1/2})}, \quad \forall \ q > 1.
\]

Letting \( h \to 0 \), we obtain
\[
\|D_{x'} u\|_{\mathcal{H}_p(Q_{1/2})} \leq N\|D_{x'} u\|_{L_2(Q_{1/2})}, \quad \forall \ q > 1. \tag{3.11}
\]

On the other hand, from [15, Lemma 3.3], we have
\[
\|u_t\|_{L_2(Q_{1/2})} \leq N\|Du\|_{L_2(Q_{1/2})}. \tag{3.12}
\]
Observing that \( \mathcal{P}_0(u_t) = 0 \) in \( Q_1^- \), and using Lemma 3.4, (3.10), and (3.12), we get
\[
\|u_t\|_{\mathcal{H}_p(Q_{1/2})} \leq N\|u_t\|_{L_2(Q_{1/2})} \leq N\|Du\|_{L_2(Q_{1/2})}, \quad \forall \ q > 1. \tag{3.13}
\]

Hence, by the Sobolev embedding theorem for \( q > d + 2 \), we have
\[
\|u_t\|_{L_\infty(Q_{1/2})} \leq N\|Du\|_{L_2(Q_{1/2})}. \tag{3.14}
\]
Now notice that in $Q_t^-$,
\[
D_t \bar{U} = u_t - \sum_{\alpha=1}^{d-1} \sum_{\beta=1}^{d} \bar{A}^{\alpha \beta} D_\alpha u, \quad D_x \bar{U} = \sum_{\beta=1}^{d} \bar{A}^{d \beta} D_\beta u, \quad U_t = \sum_{\beta=1}^{d} \bar{A}^{d \beta} D_\beta u. \tag{3.15}
\]

Therefore, it follows from (3.11), (3.13), and (3.15) that
\[
\|D_x u\|_{L^q(Q_{t/2}^+)} + \|\bar{U}\|_{H^1(Q_{t/2}^+)} \leq N\|Du\|_{L^q(Q_{3t/4}^+)}.
\]
Then by the Sobolev embedding theorem for $q > d + 2$, $\bar{A}^{dd}(x^d) \geq \nu$, and the definition of $\bar{U}$, we have
\[
\|Du\|_{L^\infty(Q_{t/2}^+)} \leq N\|Du\|_{L^2(Q_{3t/4}^+)}. \tag{3.16}
\]

By using (3.14), (3.16), Lemma 3.4, and (3.10), we have
\[
\|u_t\|_{L^\infty(Q_{t/2}^+)} + \|Du\|_{L^\infty(Q_{t/2}^+)} \leq N\|Du\|_{L^2(Q_{3t/4}^+)} \leq C\|u\|_{L^2(Q_{3t/4}^+)} \leq C\|u\|_{L^p(Q_t^+)}, \quad \forall \, p > 0. \tag{3.17}
\]

Recalling that the coefficients of $P_0$ are independent of $x'$, we henceforth have
\[
P_0(D_x u) = 0 \quad \text{in} \quad Q_t^-.
\]

Replacing $u$ with $D_x u$ in (3.17), we get
\[
\|D_x u_t\|_{L^\infty(Q_{t/2}^+)} + \|DD_x u\|_{L^\infty(Q_{t/2}^+)} \leq N\|D_x u\|_{L^p(Q_t^-)}, \quad \forall \, p > 0. \tag{3.18}
\]

We thus obtain (3.8) by using (3.17) and (3.18).

Next we prove (3.9). By using (3.16), the interpolation inequality, and iteration arguments, we obtain
\[
\|Du\|_{L^\infty(Q_{t/2}^+)} \leq N\|Du\|_{L^p(Q_t^-)}, \quad \forall \, p > 0. \tag{3.19}
\]

Now by using the fact that the coefficients of $P_0$ are independent of $t$, we have
\[
P_0(u_t) = 0 \quad \text{in} \quad Q_t^-.
\]

Replacing $u$ with $u_t$ in (3.17) with a slightly smaller domain, using (3.14) and (3.19), we have
\[
\|Du_t\|_{L^\infty(Q_{t/2}^+)} \leq N\|Du\|_{L^p(Q_t^-)}, \quad \forall \, p > 0. \tag{3.20}
\]

Therefore, (3.9) is a consequence of (3.8), (3.15), and (3.20). The lemma is proved.

We remark that the same proofs of Lemmas 3.4–3.6 give similar results for the adjoint operator of $P$. We close this section by giving the following two lemmas.

**Lemma 3.7.** [16, Lemma 2.7] Let $\omega$ be a nonnegative bounded function. Suppose there is $c_1, c_2 > 0$ and $0 < \kappa < 1$ such that for $\kappa t \leq s \leq t$ and $0 < t < r$,
\[
c_1\omega(t) \leq \omega(s) \leq c_2\omega(t). \tag{3.21}
\]

Then, we have
\[
\sum_{i=0}^{\infty} \omega(\kappa^i r) \leq N \int_0^r \frac{\omega(t)}{t} \, dt,
\]
where $N = N(\kappa, c_1, c_2)$. 
Lemma 3.8. Let \( Q = (-T, 0) \times D \) be a bounded domain in \( \mathbb{R}^{d+1} \). For fixed \( t \) and all \( x \in \overline{D} \),
\[
|D \cap B_r(x)| \geq A_0 r^d \quad \text{for} \ r \in (0, \text{diam } D),
\]
where \( A_0 > 0 \) is a constant. Let \( p \in (1, \infty) \) and \( S \) be a bounded linear operator on \( L_p(Q) \). Suppose that for any \( z \in Q \) and \( 0 < r < \mu \text{ diam } D \), we have
\[
\int_{Q \cap (D - (t))} |Sb| \leq C_0 \int_{Q \cap (D - (t))} |b|
\]
whenever \( b \in L_p(Q) \) is supported in \( Q \cap (D - (t)) \), \( \int_Q b = 0 \), and \( c > 1 \), \( C_0 > 0 \), \( \mu \in (0, 1) \) are constants. Then for \( g \in L_p(Q) \) and any \( s > 0 \), we have
\[
\|(t, x) \in Q : |Sg(t, x)| > s\| \leq \frac{N}{s} \int_Q |g|,
\]
where \( N = N(d, c, C_0, D, A_0, \mu, ||S||_L^1 \rightarrow L^\infty) \) is a constant.

Lemma 3.8 is similar to [11, Lemma 4.1], where the proof is based on the Calderón-Zygmund decomposition. Here we can modify the proof there by using the “dyadic parabolic cube” decomposition of \( Q \). See also [8, Theorem 11].

4. Proofs of Theorem 2.1 and Corollary 2.2

In this section, we give the proofs of Theorem 2.1 and Corollary 2.2. First of all, by using Campanato’s characterization of Hölder continuous functions, the global \( W^{1,2}_p \) estimate for the heat equation, the Sobolev embedding theorem, and Lemma 3.4, we reduce the proof of Theorem 2.1 to that of Proposition 4.2, which is about the parabolic systems without lower-order terms. Then we prove some auxiliary estimates that play key roles in deriving an a priori estimate of the modulus of continuity of \( (D_{x,t} u, u) \). With the above preparations, we complete the proof of Proposition 4.2 by discussing two cases since our argument and estimates depend on the coordinate system. Finally, we prove Corollary 2.2 using a duality argument and Theorem 2.1.

4.1. Simplified problem. We first reduce the estimate of \( [u]_{1/2,1} \) to the estimate of \( ||Du||_{L^\infty} \) by using the following lemma.

Lemma 4.1. Let \( u \) be a weak solution to (2.1) in \( Q_1^- \). Suppose that \( ||u||_{L^\infty(Q_{1/2})} < \infty \) and \( ||Du||_{L^\infty(Q_{1/2})} < \infty \). Then
\[
[u]_{1/2,1,Q_{1/4}} \leq N \left( ||u||_{L^\infty(Q_{1/2})} + ||Du||_{L^\infty(Q_{1/2})} + ||f||_{L^\infty(Q_{1/2})} + ||g||_{L^\infty(Q_{1/2})} \right).
\]

Proof. The lemma follows from a similar argument that led to [9, Lemma 6] by using Campanato’s characterization of Hölder continuous functions (see [23, Lemma 4.3]) and a variant of the parabolic Poincaré inequality (see [9, Lemma 3]).

Next we show that it suffices to consider the parabolic systems without lower-order terms. Rewrite (2.1) as
\[
-u_t + D_{x,t}(A^0 D_{x,t} u) = \text{div}(g - Bu) + f - \tilde{B}^a D_{x,t} u - Cu \quad \text{in } Q = Q_1^-.
\]

Let \( v \in W^{1,2}_p(Q_1^-) \) satisfy
\[
\begin{cases}
-v + \Delta v = (f - \tilde{B}^a D_{x,t} u - Cu)\chi_{Q_{1/2}} & \text{in } Q_1^-,
\quad
-v = 0 & \text{on } \partial \rho Q_1^-.
\end{cases}
\]
Then by the global $W_p^{1,2}$ estimate for the heat equation, we have

$$
||v||_{W_p^{1,2}(Q_1)} \leq N \left(||u||_{L_p(Q_1)} + ||Du||_{L_p(Q_1)} + ||f||_{L_p(Q_1)}\right). \tag{4.1}
$$

By Lemma 3.4, we have for some $q > d + 2$,

$$
||u||_{H^q(Q_{1/2})} \leq N \left(||u||_{L_p(Q_{1/2})} + ||g||_{L_p(Q_{1/2})} + ||f||_{L_p(Q_{1/2})}\right). \tag{4.2}
$$

By the Sobolev embedding theorem for $q > d + 2$, we obtain $u \in C^{1/2,\delta}(Q_{1/2})$ and

$$
||u||_{C^{1/2,\delta}(Q_{1/2})} \leq N \left(||u||_{L_p(Q_{1/2})} + ||g||_{L_p(Q_{1/2})} + ||f||_{L_p(Q_{1/2})}\right),
$$

where $\delta = 1 - (d + 2)/q$. Now coming back to (4.1), replacing $p$ with $q$, and using (4.2), we get $v \in C^{1+\delta/2,1+\epsilon/2}(Q_{3/2})$ with

$$
||v||_{C^{1+\delta/2,1+\epsilon/2}(Q_{3/2})} \leq N \left(||u||_{L_p(Q_{1/2})} + ||g||_{L_p(Q_{1/2})} + ||f||_{L_p(Q_{1/2})}\right). \tag{4.3}
$$

Denote $g' := g - Bu + (I-A)Dv$ and $w := u - v$, then $w$ satisfies

$$
-w_t + D_\alpha(A^{\alpha\beta}D_\beta w) = \text{div} \ g' \quad \text{in } Q_{1/2},
$$

where

$$
||g'||_{L_p(Q_{1/2})} \leq N \left(||u||_{L_p(Q_{1/2})} + ||g||_{L_p(Q_{1/2})} + ||f||_{L_p(Q_{1/2})}\right).
$$

Moreover, $g'$ is of piecewise Dini mean oscillation satisfying

$$
\omega_{g'}(r) \leq N(\Lambda)(\omega_g(r) + \omega_A(r)||u||_{L_p(Q_{1/2})} + r^\beta[||u||_{L_p} + \omega_A(r)||Dv||_{L_p} + r^\beta[||Dv||_{L_p}]) \leq N\left(\omega_g(r) + \omega_A(r)||u||_{L_p(Q_{1/2})} + \omega_A(r) + r^\beta\cdot(||u||_{L_p(Q_{1/2})} + ||g||_{L_p(Q_{1/2})} + ||f||_{L_p(Q_{1/2})})\right).
$$

Therefore, bearing in mind that $u = w + v$ and $v$ satisfies (4.3), the results for $w$ yield these for $u$.

Finally, we conclude that to finish the proof of Theorem 2.1, we only need to prove the following proposition.

**Proposition 4.2.** Let $\epsilon \in (0,1)$ and $p \in (1,\infty)$. Suppose that $A$ and $g$ are of piecewise Dini mean oscillation in $Q$, and $g \in L_\infty(Q)$. If $u \in \mathcal{H}_p^1(Q)$ is a weak solution to

$$
-u_t + D_\alpha(A^{\alpha\beta}D_\beta u) = \text{div} \ g \quad \text{in } Q,
$$

then $u \in C^{1/2,1}(\overline{Q} \cap ((-T + \epsilon,0) \times D_\epsilon)), j = 1, \ldots, M$, and for any fixed $t \in (-T + \epsilon,0)$, $u(t,\cdot)$ is Lipschitz in $D_\epsilon$.

We will establish an a priori estimate of the modulus of continuity of $(D_\epsilon u, U)$ by assuming that $u \in C^{0,1}(Q_{3/4}, B_{3/4})$, i.e., for each $t \in (-9/16,0)$, $u(t,\cdot) \in C^{0,1}(B_{3/4})$. The proof of Proposition 4.2 is mainly based on Campanato’s approach [6, 21]. The general case follows from an approximation argument and the technique of locally flattening the boundaries [18, p. 2466].

Fix $z_0 = (t_0, x_0) \in \left((-9/16,0) \times B_{3/4}\right) \cap Q_{3/4}$, $0 < r \leq 1/4$, and take a coordinate system associated with $(t_0, x_0)$ as in Subsection 3.2.

Denote

$$
\mathcal{P}_{z_0} u := -u_t + D_\alpha(A^{\alpha\beta}(z_0, x^{\beta})D_\beta u),
$$

where $z'_0 = (t_0, x'_0)$, Next we prove several auxiliary lemmas which play important roles in the proof of Proposition 4.2.
4.2. Auxiliary lemmas. We will begin with a weak type-(1, 1) estimate. Before that, we need to modify the coefficients $\tilde{A}^{\alpha\beta}(z^r, x^d)$ to get the following parabolic operator defined by

$$\tilde{P} u := -u_t + D_\alpha(\tilde{A}^{\alpha\beta} D_\beta u),$$

where $\tilde{A}^{\alpha\beta} = \eta A^{\alpha\beta}(z^r, x^d) + \nu(1-\eta)\delta_{\alpha\beta}\delta_{ij}$ with $\eta \in C^\infty_0(B_r(x_0))$ satisfying

$$0 \leq \eta \leq 1, \quad \eta \equiv 1 \text{ in } B_{2r/3}(x_0), \quad |\nabla \eta| \leq 6/r.$$

Then we can apply Lemma 3.5 with a scaling to the operator $\tilde{P}$.

**Lemma 4.3.** Let $p \in (1, \infty)$. Let $v \in \mathcal{H}^1_p(Q_r^{-}(z_0))$ be a weak solution to the problem

$$\begin{cases}
\tilde{P} v = \text{div}(F \chi_{Q_r^-}(z_0)) & \text{in } Q_r^{-}(z_0), \\
v = 0 & \text{on } \partial_r Q_r^{-}(z_0),
\end{cases}$$

where $F \in L_p(Q_r^{-}(z_0))$. Then for any $s > 0$, we have

$$|\{z \in Q_r^{-}(z_0) : |Dv(z)| > s\}| \leq \frac{N}{s}||F||_{L_1((Q_r^{-}(z_0)))},$$

where $N = N(n, d, p, \nu)$.

**Proof.** The proof is a modification of [18, Lemma 3.2]. We set $z_0 = 0$, $r = 1$, $\tilde{A}^{\alpha\beta}(x^d) := A^{\alpha\beta}(0, x^d)$, and $\tilde{P} := \tilde{P}_0$ for simplicity. Suppose $E = (E^{\alpha\beta}(x^d))$ is a $d \times d$ matrix with

$$E_{\alpha\beta}(x^d) = \delta_{\alpha\beta} \text{ for } \alpha, \beta \in \{1, \ldots, d-1\}; \quad E_{\alpha \beta}(x^d) = A_{\alpha \beta}^{\text{ad}}(x^d) \text{ for } \alpha \in \{1, \ldots, d\};$$

$$E_{\beta \alpha}(x^d) = 0 \text{ for } \beta \in \{1, \ldots, d-1\}.$$

For any $\hat{F} \in L_p(Q_{1/2}^{-})$, let $F = E\hat{F}$ and solve for $v$. It follows from Lemma 3.5 that $S : \hat{F} \rightarrow Dv$ is a bounded linear operator on $L_p(Q_{1/2}^{-})$. So we only need to prove that $S$ satisfies the hypothesis of Lemma 3.8. Set $\epsilon = 24$ and fix $\tilde{z} = (\tilde{t}, \tilde{x}) \in Q_{1/2}^{-}$,

$$0 < r < 1/4.$$ 

Let $\hat{b} \in L_p(Q_{1/2}^{-})$ be supported in $Q_r^{-}(\tilde{z}) \cap Q_{1/2}^{-}$ with mean zero, $b = E\hat{b}$, and $v_1 \in \mathcal{H}^1_p(Q_1^{-})$ be the unique weak solution of

$$\begin{cases}
\tilde{P} v_1 = \text{div } b & \text{in } Q_1^{-}, \\
v_1 = 0 & \text{on } \partial_r Q_1^{-}.
\end{cases}$$

For any $R \geq cr$ such that $Q_{1/2}^{-} \setminus Q_R(\tilde{z}) \neq \emptyset$ and $h \in C^\infty_0((Q_{2R}(\tilde{z}) \setminus Q_R(\tilde{z})) \cap Q_{1/2}^{-})$, let $v_0 \in \mathcal{H}^1_p(Q_1^{-})$ be a weak solution of

$$\begin{cases}
\tilde{P} v_0 = \text{div } h & \text{in } Q_1^{-}, \\
v_0 = 0 & \text{on } (-1, 0) \times \partial B_1 \cup (\{0\} \times \overline{B_1}),
\end{cases}$$

where $1/p + 1/p' = 1$ and $\tilde{P}^*$ is the adjoint operator of $\tilde{P}$ defined by

$$\tilde{P}^* u := u_t + D_\alpha(\tilde{A}^{\alpha\beta} D_\beta u).$$

In view of the definition of weak solutions and the assumption of $\hat{b}$, we have

$$\int_{Q_{1/2}^{-}} D v_1 \cdot h = \int_{Q_{1/2}^{-}} D v_0 \cdot b = \int_{Q_r^{-}(\tilde{z})\cap Q_{1/2}^{-}} (D_x v_0, V_0) \cdot \hat{b} = \int_{Q_r^{-}(\tilde{z})\cap Q_{1/2}^{-}} (D_x v_0 - D_x v_0(\tilde{z}), V_0 - V_0(\tilde{z})) \cdot \hat{b},$$

where $\hat{b}$ is a bounded linear operator on $L_p(Q_{1/2}^{-})$. So we only need to prove that $S$ satisfies the hypothesis of Lemma 3.8. Set $\epsilon = 24$ and fix $\tilde{z} = (\tilde{t}, \tilde{x}) \in Q_{1/2}^{-}$,
where \( V_0 = \tilde{A} d\phi(x^2) D\rho v_0 \). Hence, we have
\[
\left| \int_{(Q_{2r}(t))(Q_{3r}(t)) \cap Q_{r/12}} Dv_1 \cdot h \right| \\
\leq \| \hat{b} \|_{L^1(Q_r(\xi))} \| (Dx^2 \nu_0 - Dx^2 \nu_0(\xi), V_0 - V_0(\xi)) \|_{L^\infty(Q_r(\xi))},
\]
(4.4)
Moreover, we find that \( v_0 \in H^1_p(Q_r^\eta) \) satisfies
\[
\phi^\eta v_0 = 0 \quad \text{in} \quad Q_{r/12}(\xi),
\]
where we recalled that \( \eta \equiv 1 \) in \( B_{2/3} \) and \( B_{2/12}(\tilde{y}) \subset B_{2/3} \). By applying a similar argument that led to (3.8) and (3.9) to the adjoint operator, and using a suitable scaling, \( r \leq R/24 \), and the \( H^1_p \) estimate, we have
\[
\| Dx^2 \nu_0 - Dx^2 \nu_0(\xi) \|_{L^\infty(Q_r(\xi))} + \| V_0 - V_0(\xi) \|_{L^\infty(Q_r(\xi))} \\
\leq Nr \| [Dx^2 \nu_0]_{C^{1,2}(Q_{3r/2}(\xi))} \| + \| V_0 \|_{C^{1,2}(Q_{3r/2}(\xi))} \\
\leq Nr R^{-1-(d+2)/p} \| D\nu_0 \|_{C^{1,2}(Q_{3r/2}(\xi))} \\
\leq Nr R^{-1-(d+2)/p} \| b \|_{L^{p,\gamma}(Q_{2r}(\xi))}.
\]
(4.5)
Substituting the above estimate (4.5) into (4.4) and using the duality and Hölder’s inequality, we have
\[
\| Dv_1 \|_{L^1(Q_r(\xi))} \leq Nr^{-1} \| \hat{b} \|_{L^1(Q_r(\xi))}.
\]
(4.6)
Let \( N_0 \) be the smallest positive integer such that \( Q_{1/2} \subset Q_{2^{N_0-1}r} \). By taking \( R = cr, 2cr, \ldots, 2^{N_0-1}cr \) in (4.6) and summarizing, we obtain
\[
\int_{Q_{1/2}(\xi)} |Dv_1| \, dx \, dt \leq N \sum_{k=1}^{N_0} 2^{-k} \| \hat{b} \|_{L^1(Q_r(\xi))} \leq N \int_{Q_{1/2}(\xi)} |\hat{b}| \, dx \, dt.
\]
Therefore, \( S \) satisfies the hypothesis of Lemma 3.8. The proof of this lemma is finished. \( \square \)

Denote
\[
\phi(z_0, r) := \inf_{q \in \mathbb{R} \cap q < 1} \left( \int_{Q_r(z_0)} \| (Dx^2 u, U) - q \|^q \, dx \, dt \right)^{1/q},
\]
where \( 0 < q < 1 \) is some fixed exponent. We are going to use Lemma 4.3 to prove an iteration formula about the function \( \phi(z_0, r) \), from which we can derive the following

**Lemma 4.4.** For any \( \gamma \in (0, 1) \) and \( 0 < \rho \leq r \leq 1/4 \), we have
\[
\phi(z_0, r) \leq N \left( \frac{d}{r} \right)^{d-2} \| (Dx^2 u, U) \|_{L^1(Q_r(z_0))} + N \tilde{a}_\lambda(p) \| Du \|_{L^\infty(Q_r(z_0))} + N \tilde{a}_\gamma(p),
\]
(4.7)
where \( N = N(n, d, p, \nu, \gamma) \), and \( \tilde{a}_\lambda(t) \) is a Dini function derived from \( \omega_\lambda(t) \).

**Proof.** We apply Lemma 4.3 with
\[
F = (\tilde{A} z_0^2 x^2) - A(t, x) Du + g(t, x) - \bar{g}(z_0', x'),
\]
(3.4), and follow the same argument as in deriving [18, (3.7)] to obtain that
\[
\left( \int_{Q_{1/2}(z_0)} |Dx^2 v|^q \, dx \, dt + \int_{Q_{1/2}(z_0)} |V|^q \, dx \, dt \right)^{1/q} \leq N \left( \tilde{a}_\lambda(r) + \tilde{a}_\gamma(r) \right),
\]
(4.8)
where \( \tilde{\omega}_*(r) = \omega_*(r) + \tilde{\omega}_1(r) \) and \( V = \tilde{A}^{dp}(z_0', x^d)D_{p}v(t, x) \).

We now claim that
\[
\phi(z_0, \kappa^j r) \leq \kappa^{j^0} \phi(z_0, r) + N\|Du\|_{L_\infty(Q_{\tau}(z_0))}\tilde{\omega}_A(\kappa^j r) + N\tilde{\omega}_A(\kappa^j r), \tag{4.9}
\]
where \( \kappa \in (0, 1/2) \) is some fixed constant and
\[
\tilde{\omega}_*(t) = \sum_{i=1}^{\infty} \kappa^{j^0}\left( \tilde{\omega}_*(\kappa^{-i} t)\chi_{x^{-i} \leq 1} + \tilde{\omega}_*(1)\chi_{x^{-i} > 1} \right). \tag{4.10}
\]
Furthermore, \( \tilde{\omega}_*(t) \) is a Dini function (see Lemma 1 in [9]) and satisfies (3.21). Then for any \( \rho \) satisfying \( 0 < \rho \leq r \leq 1/4 \), we take \( j \) to be the integer with \( \kappa^{j+1} < \rho/r \leq \kappa^j \). By using (4.9) and (3.21), we have
\[
\phi(z_0, \rho) \leq N\left( \frac{\rho}{r} \right)^{\nu} \phi(z_0, r) + N\tilde{\omega}_A(\rho)\|Du\|_{L_\infty(Q_{\tau}(z_0))} + N\tilde{\omega}_A(\rho), \tag{4.11}
\]
where, it follows from Hölder’s inequality that
\[
\phi(z_0, r) \leq \left( \int_{Q_{\tau}(z_0)} \| (D_{x'}u, U) \|^q \, dx \, dt \right)^{1/q} \leq Nr^{-d-2}\| (D_{x'}u, U) \|_{L_4(Q_{\tau}(z_0))}. \tag{4.12}
\]
Combining (4.12) and (4.11), we get (4.7).

Finally, we prove the claim (4.9). Let
\[
\bar{u}_1(x^d) = \int_{x_0^d}^{x^d} (\tilde{A}^{dp}(z_0', s))^{-1} \tilde{g}_d(z_0', s) \, ds, \quad \bar{u} = u - u_1, \quad w = \bar{w} - v. \tag{4.13}
\]
Then a direct calculation yields \( \tilde{P}^{\gamma}_{x'}w = 0 \) in \( Q_{\tau/2}(z_0) \). For any \( \kappa \in (0, 1/4) \), by Lemma 3.6 with a suitable scaling, we have
\[
\begin{align*}
\|D_{x'}w - (D_{x'}\bar{w})_{Q_{\tau}(z_0)}\|_{L_\infty(Q_{\tau}(z_0))} &+ \|W - (W)_{Q_{\tau}(z_0)}\|_{L_\infty(Q_{\tau}(z_0))} \\
&\leq N(\kappa r)^{d+2+q} \left( \|D_{x'}w\|_{C^{1/2}(Q_{\tau}(z_0))} + [W]_{C^{1/2}(Q_{\tau/4}(z_0))} \right) \\
&\leq N\kappa^{d+2+q} \int_{Q_{\tau/2}(z_0)} |Dw| \, dx \, dt \\
&\leq N\kappa^{d+2+q} \int_{Q_{\tau/2}(z_0)} |(D_{x'}w, W)| \, dx \, dt, \tag{4.14}
\end{align*}
\]
where \( W = \tilde{A}^{dp}(z_0', x^d)D_{p}w \). Define
\[
h(x^d) := \int_{x_0^d}^{x^d} (\tilde{A}^{dp}(z_0', s))^{-1} \left( q_d - \sum_{j=1}^{d-1} \tilde{A}^{dp}(z_0', s)q_{\beta} \right) \, ds, \quad q = (q', q_d) \in \mathbb{R}^{m+d},
\]
and
\[
\bar{w} := w - \sum_{j=1}^{d-1} x^{\beta}q_{\beta} - h(x^d).
\]
Then
\[
D_{x'}\bar{w} = D_{x'}w - q', \quad \bar{W} := \tilde{A}^{dp}(z_0', x^d)D_{p}\bar{w} = W - q_d. \]
Moreover, \( \mathcal{P}_t \tilde{w} = 0 \) in \( Q_{r/2}(z_0) \). Now replacing \( w \) and \( W \) with \( \tilde{w} \) and \( \tilde{W} \) in (4.14), respectively, we get

\[
\|D_x w - (D_x \tilde{w})_{Q_r(z_0)}\|_{L^q(Q_r(z_0))} + \|W - (W)_{Q_r(z_0)}\|_{L^q(Q_r(z_0))} \\
\leq N_0 \epsilon^{d+2} \int_{Q_{r/2}(z_0)} |(D_x w - q', W - q)|^q \, dx \, dt \\
= N_0 \epsilon^{d+2} \int_{Q_{r/2}(z_0)} |(D_x w, W) - q|^q \, dx \, dt,
\]

which implies

\[
\left( \int_{Q_r(z_0)} |D_x w - (D_x \tilde{w})_{Q_r(z_0)}|^q \, dx \, dt + \int_{Q_r(z_0)} |W - (W)_{Q_r(z_0)}|^q \, dx \, dt \right)^{1/q} \\
\leq N_0 \epsilon \left( \int_{Q_{r/2}(z_0)} |(D_x w, W) - q|^q \, dx \, dt \right)^{1/q}, \tag{4.15}
\]

where \( N_0 = N_0(n, d, p, \nu, \Lambda) \). Recalling that \( \tilde{u} = w + v \), we obtain from (4.15) that

\[
\left( \int_{Q_r(z_0)} |D_x \tilde{u} - (D_x w)_{Q_r(z_0)}|^q + |(U - (W))_{Q_r(z_0)}|^q \, dx \, dt \right)^{1/q} \\
\leq 2^{1/q-1} \left( \int_{Q_r(z_0)} |D_x w - (D_x \tilde{w})_{Q_r(z_0)}|^q + |W - (W)_{Q_r(z_0)}|^q \, dx \, dt \right)^{1/q} \\
+ N \left( \int_{Q_r(z_0)} |D_x v|^q + |V|^q \, dx \, dt \right)^{1/q} \\
\leq N_0 \epsilon \left( \int_{Q_{r/2}(z_0)} |(D_x \tilde{u}, \tilde{U}) - q|^q \, dx \, dt \right)^{1/q} + N \epsilon^{-(d+2)/q} \left( \int_{Q_r(z_0)} |D_x v|^q + |V|^q \, dx \, dt \right)^{1/q}, \tag{4.16}
\]

where \( \tilde{U} = \tilde{A}(z', x^d)D_{\tilde{u}} \). Recalling that

\( D_x \tilde{u} = D_x u, \quad U = A(t, x)D_{\tilde{u}} + g_d(t, x), \) and \( \tilde{U} = \tilde{A}(z', x^d)D_{\tilde{u}} + \tilde{g}_d(z', x^d) \),

we have for \( z \in Q_r(z_0) \),

\[ |U - \tilde{U}| \leq |D u|_{L^\infty(Q_r(z_0))}|A(z) - \tilde{A}(z', x^d)| + |g_d(z) - \tilde{g}_d(z', x^d)|. \]

Thus, substituting (3.4) and (4.8) into (4.16), we have

\[
\left( \int_{Q_r(z_0)} |(D_x u, U) - (D_x w, W)_{Q_r(z_0)}|^q \, dx \, dt \right)^{1/q} \\
\leq N_0 \epsilon \left( \int_{Q_{r/2}(z_0)} |(D_x u, U) - q|^q \, dx \, dt \right)^{1/q} + N \epsilon^{-(d+2)/q} \left( \int_{Q_r(z_0)} |U - \tilde{U}||^q \, dx \, dt \right)^{1/q} \\
+ N \epsilon^{-(d+2)/q} \left( \int_{Q_r(z_0)} |D_x v|^q + |V|^q \, dx \, dt \right)^{1/q} \\
\leq N_0 \epsilon \left( \int_{Q_{r/2}(z_0)} |(D_x u, U) - q|^q \, dx \, dt \right)^{1/q} + N \epsilon^{-(d+2)/q} |Du|_{L^\infty(Q_r(z_0))}. \]
The proof is similar to that in [18], so we only list the main differences. We recall that for each $\Omega \subset \mathbb{R}^n$ we get Lemma 4.4. Once we get Lemma 4.4, we can obtain the local boundedness of $Du$ in Lemma 4.5 below. The proof of it is the same as that of [18, Lemma 3.4] and thus omitted.

Lemma 4.5. We have

$$
\|Du\|_{L^q(Q_{r/4})} \leq N \|(Du, U)\|_{L^1(Q_{r/4})} + N \left( \int_0^r \frac{\omega_A(s)}{s} \, ds + \|g\|_{L^q(Q_{r/4})} \right),
$$

(4.18)

where $N > 0$ is a constant depending only on $n, d, p, v, \gamma, \omega_A$, and $\omega_1$.

4.3. Proof of Proposition 4.2.

Proof. We recall that for each $z_0$, the coordinate system is chosen according to it. The proof is similar to that in [18], so we only list the main differences. We claim that for a.e. $z_0 \in Q_{3/4}$

$$
\|(Du, U(z_0)) - q_{z_0,r}\|
$$

$$
\leq N \left( \phi(z_0, r) + \|Du\|_{L^q(Q_{r/4})} \int_0^r \frac{\omega_A(s)}{s} \, ds + \|g\|_{L^q(Q_{r/4})} \right),
$$

(4.19)

where $q_{z_0,r} \in \mathbb{R}^{n,d}$ satisfying

$$
\phi(z_0, r) = \left( \int_{Q_r} (|Du, U| - q_{z_0,r})^q \, dx \, dt \right)^{1/q}.
$$

Note that (4.19) is similar to [18, (3.16)]. One can prove it by iteration, (4.9), the assumption that $u \in C^{0,1}(\overline{Q_{3/4}})$, and Lemma 3.7. Then for $0 < r < 1/8$,

$$
\sup_{z_0 \in Q_{3/8}} \| (Du, U(z_0)) - q_{z_0,r} \|
$$

$$
\leq N \left( \phi(z_0, r) + \|Du\|_{L^q(Q_{3/8})} \int_0^r \frac{\omega_A(s)}{s} \, ds + \|g\|_{L^q(Q_{3/8})} \right),
$$

(4.20)

where $N$ is a constant depending only on $n, d, p, v, \gamma, \omega_A$, and $\omega_1$. 

Once we get Lemma 4.4, we can obtain the local boundedness of $Du$ in Lemma 4.5 below. The proof of it is the same as that of [18, Lemma 3.4] and thus omitted.
where, it follows from Lemma 4.4 that for any $0 < r < 1/8$,
\[
\sup_{z_0 \in \mathcal{Q}_{1,8}} \phi(z_0, r) \leq N \left( r^\gamma \|\mathcal{D}_x u, U\|_{L^1(\mathcal{Q}_{1,4})} + \alpha_A(r) \|\mathcal{D}u\|_{L^\infty(\mathcal{Q}_{1,4})} + \alpha_b(r) \right). \tag{4.20}
\]

Now suppose that $z_1 = (t_1, x_1) \in \mathcal{Q}_{1,8} \cap \mathcal{Q}_{ji}$ for some $j_1 \in [1, l + 1]$. If $|z_0 - z_1|_p \geq 1/32$, then by
\[
|\mathcal{D}_x u(z_0), U(z_0)) - (\mathcal{D}_x u(z_1), U(z_1))| \leq 2 \|\mathcal{D}u\|_{L^\infty(\mathcal{Q}_{1,4})} + \|g\|_{L^\infty(\mathcal{Q})}
\]
and (4.18), we have
\[
|\mathcal{D}_x u(z_0), U(z_0)) - (\mathcal{D}_x u(z_1), U(z_1))|
\]
\[
\leq N |z_0 - z_1|_p^\gamma \left( \|\mathcal{D}_x u, U\|_{L_1(\mathcal{Q}_{1,4})} + \int_0^1 \frac{\alpha_g(s)}{s} \, ds + \|g\|_{L^\infty(\mathcal{Q})} \right), \tag{4.21}
\]
where $\gamma \in (0, 1)$ is a constant. If $|z_0 - z_1|_p < 1/32$, we set $r = |z_0 - z_1|_p$ and claim that dist$(z_0, \partial_{\mathcal{D}} \mathcal{Q}_{ji} \cap \{ t = t_0 \})$ and dist$(z_0, \partial_{\mathcal{D}} \mathcal{Q}_{ji})$ are comparable. Indeed, on one hand, clearly
\[
\text{dist}(z_0, \partial_{\mathcal{D}} \mathcal{Q}_{ji} \cap \{ t = t_0 \}) \geq \text{dist}(z_0, \partial_{\mathcal{D}} \mathcal{Q}_{ji}).
\]
On the other hand, we may suppose that
\[
\text{dist}(z_0, \partial_{\mathcal{D}} \mathcal{Q}_{ji} \cap \{ t = t_0 \}) = |z_0 - (t_0, x_0, h_{ji}(t_0, x_0))|_p
\]
and
\[
\text{dist}(z_0, \partial_{\mathcal{D}} \mathcal{Q}_{ji}) = |z_0 - (t, x', h_{ji}(t, x'))|_p.
\]
Then by using the triangle inequality and $h_{ji} \in C^\gamma_0$ with $\gamma_0 > 1/2$, we have
\[
|z_0 - (t_0, x_0, h_{ji}(t_0, x_0))|_p
\]
\[
\leq |z_0 - (t, x', h_{ji}(t, x'))|_p + |(t - t_0, x' - x_0, h_{ji}(t, x') - h_{ji}(t_0, x_0))|_p
\]
\[
\leq |z_0 - (t, x', h_{ji}(t, x'))|_p + N|t - t_0|^{1/2} + |x' - x_0| + |t - t_0|^{\gamma_0}
\]
\[
\leq N \text{dist}(z_0, \partial_{\mathcal{D}} \mathcal{Q}_{ji}).
\]
Now we continue the proof by discussing the following two cases.

**Case 1.** If
\[
r > 1/16 \max \{ \text{dist}(z_0, \partial_{\mathcal{D}} \mathcal{Q}_{ji} \cap \{ t = t_0 \}), \text{dist}(z_1, \partial_{\mathcal{D}} \mathcal{Q}_{ji} \cap \{ t = t_1 \}) \},
\]
then without loss of generality, we assume that $z_0$ is above $z_1$. By the triangle inequality, we have for $\forall z \in \mathcal{Q}_r(z_1)$,
\[
|\mathcal{D}_x u(z_0), U(z_0)) - (\mathcal{D}_x u(z_1), U(z_1))|_p
\]
\[
\leq |\mathcal{D}_x u(z_0), U(z_0)) - \mathcal{Q}_{z_0,2r}|_p + |\mathcal{Q}_{z_0,2r} - \mathcal{Q}_{z_1,2r}|_p + |(\mathcal{D}_x u(z_1), U(z_1)) - \mathcal{Q}_{z_1,2r}|_p
\]
\[
+ |(\mathcal{D}_x u(z_1), U(z_1)) - (\mathcal{D}_x u(z_1), U(z_1))|_p
\]
\[
\leq N|z_0 - (2r) + |(\mathcal{D}_x u(z_1), U(z_1)) - \mathcal{Q}_{z_0,2r}|_p + |(\mathcal{D}_x u(z), U(z)) - \mathcal{Q}_{z_0,2r}|_p
\]
\[
+ |(\mathcal{D}_x u(z), U(z)) - (\mathcal{D}_x u(z), U(z))|_p + |(\mathcal{D}_x u(z_1), U(z_1)) - (\mathcal{D}_x u(z_1), U(z_1))|_p.
\]  \tag{4.22}
where $D_y$ denotes the first derivatives with respect to the first $d - 1$ space variables in the coordinate system associated with $z_1$ and in this coordinate system, we use $D_y$ to define $\tilde{U}$. For the last term, one can see that

$$D_y u(z_1) - D_y u(z_1) = (D_y u(z_1), D_y u(z_1))(I - X^{-1})I_0,$$

where $I_0 = (P_{\alpha\beta})$ is a $d \times (d - 1)$ matrix with

$$P_{\alpha\beta} = \delta_{\alpha\beta} \quad \text{for } \alpha, \beta \in \{1, \ldots, d - 1\}; \quad P_{\alpha\beta} = 0 \quad \text{for } \beta \in \{1, \ldots, d - 1\},$$

$X = (X_{\alpha\beta})$ is a $d \times d$ matrix with

$$X_{\alpha\beta} = \frac{\partial y^i}{\partial \bar{x}^\beta} \quad \text{for } \alpha, \beta = 1, \ldots, d,$$

and $I$ is a $d \times d$ identity matrix. We henceforth need to estimate $I - X^{-1}$. To end this, we suppose that for the fixed $t_1$, the closest point on $\partial_\nu Q_{\tau_1} \cap \{t = t_1\}$ to $z_1 = (t_1, x', x^d)$ is $(\bar{z}_1', h_1(\bar{z}_1'))$, and let

$$n_2 = \frac{\left(\bar{\nabla}_x h_1(\bar{z}_1'), 1\right)^\top}{\sqrt{1 + |\bar{\nabla}_x h_1(\bar{z}_1')|^2}}$$

be the unit normal vector at $(\bar{z}_1', h_1(\bar{z}_1'))$ on the surface $\{(t_1, x', x^d) : x^d = h_1(t_1, x')\}$. The corresponding tangential vectors are given by

$$\tau_{2,1} = (1, 0, \ldots, 0, D_{x'} h_1(\bar{z}_1'))^\top, \ldots, \tau_{2,d-1} = (0, 0, \ldots, 1, D_{x'} h_1(\bar{z}_1'))^\top,$$

from which we can use the Gram-Schmidt process to find an orthonormal basis $\{\hat{\tau}_{2,1}, \ldots, \hat{\tau}_{2,d-1}\}$ of the tangent space. Similarly, we denote

$$n_1 = \frac{\left(\bar{\nabla}_x h_0(\bar{z}_0'), 1\right)^\top}{\sqrt{1 + |\bar{\nabla}_x h_0(\bar{z}_0')|^2}} = (0', 1)^\top$$

to be the unit normal vector at $(\bar{z}_0', h_0(\bar{z}_0'))$, and the corresponding tangential vectors are

$$\tau_{1,1} = (1, 0, \ldots, 0)^\top, \ldots, \tau_{1,d-1} = (0, 0, \ldots, 1, 0)^\top.$$

It follows from the proof of Lemma 3.3 that $|\bar{\nabla}_x h_1(\bar{z}')|$ is bounded from above by $N\omega_1(r)$. Then we have

$$|n_1 - n_2| = \left|\left(0', 1\right)^\top - \left(-\bar{\nabla}_x h_1(\bar{z}_1'), 1\right)^\top\right|\frac{1}{\sqrt{1 + |\bar{\nabla}_x h_1(\bar{z}_1')|^2}} \leq N\omega_1(N_0|z_0 - z_1|_p) \leq N\omega_1(|z_0 - z_1|_p) \leq N\tilde{\omega}_1(|z_0 - z_1|_p),$$

where we used $\omega_1(N_0r) \leq N_0\omega_1(r)$ in the second inequality, which can be derived from the fact that $\omega_0$ is an increasing and concave function, $R$ is a monotonically increasing function with respect to $r$, and the definition of $\omega_1(r) = 2\omega_0(2r + R)$ in the proof of Lemma 3.3. This is also true for $|\tau_{1,i} - \bar{\tau}_{2,i}|, i = 1, \ldots, d - 1$. We thus obtain

$$|D_y u(z_1) - D_y u(z_1)| \leq N\|Du\|_{\nu(\partial_\nu Q_{\tau_0})}\tilde{\omega}_1(|z_0 - z_1|_p).$$
We similarly can estimate the difference of $U$ in different coordinate systems. Hence, we obtain
\[
\| (D_{x'} u(z_1), U(z_1)) - (D_{x''} u(z_1), \bar{U}(z_1)) \| \leq N \| Du \|_{L_{\infty}(Q_{j_1})} \tilde{a}_1(|z_0 - z_1|_p). \tag{4.23}
\]
Also, (4.23) is satisfied by the penultimate term of (4.22). Coming back to (4.22), we take the average over $z \in Q_j(z_1)$ and take the $q$-th root to get
\[
\| (D_{x'} u(z_0), U(z_0)) - (D_{x''} u(z_1), U(z_1)) \|
\leq N \left( \psi(2r) + \phi(z_0, 2r) + \phi(z_1, 2r) + \| Du \|_{L_{\infty}(Q_{j_1})} \tilde{a}_1(|z_0 - z_1|_p) \right).
\]
Therefore, we obtain from (4.18), (4.20), and (3.21) that
\[
\| (D_{x'} u(z_0), U(z_0)) - (D_{x''} u(z_1), U(z_1)) \|
\leq N \| z_0 - z_1 \|_p \sum \| (D_{x'} u, U) \|_{L_{1}(Q_{j_1})} + N \int_0^{|z_0 - z_1|_p} \frac{\tilde{a}_J(s)}{s} \, ds
\]
\[+ N \int_0^{|z_0 - z_1|_p} \frac{\tilde{a}_A(s)}{s} \, ds \cdot \left( \| (D_{x'} u, U) \|_{L_{1}(Q_{j_1})} + \int_0^{1} \frac{\tilde{a}_J(s)}{s} \, ds + \| g \|_{L_{\infty}(Q)} \right). \tag{4.24}
\]

Case 2. If $r \leq 1/16 \max \{ \text{dist}(z_0, \partial_p Q_{j_1} \cap \{ t = t_0 \}), \text{dist}(z_1, \partial_p Q_{j_1} \cap \{ t = t_1 \}) \}$, then $j_0 = j_1$. Then we follow the same arguments as in [18, Case 1.] to obtain
\[
\| (D_{x'} u(z_0), U(z_0)) - (D_{x''} u(z_1), U(z_1)) \|
\leq N \| z_0 - z_1 \|_p \sum \| (D_{x'} u, U) \|_{L_{1}(Q_{j_1})} + N \int_0^{|z_0 - z_1|_p} \frac{\tilde{a}_J(s)}{s} \, ds
\]
\[+ N \int_0^{|z_0 - z_1|_p} \frac{\tilde{a}_A(s)}{s} \, ds \cdot \left( \| (D_{x'} u, U) \|_{L_{1}(Q_{j_1})} + \int_0^{1} \frac{\tilde{a}_J(s)}{s} \, ds + \| g \|_{L_{\infty}(Q)} \right). \tag{4.25}
\]
Thus, Proposition 4.2 is proved. \hfill \Box

4.4. Proof of Corollary 2.2. The proof is a modification of [18, Corollary 1.6], which in turn is based on the approach in [1, 4]. By the Sobolev embedding theorem in the parabolic setting (see, for instance, [22, Lemma 8.1]), we have $u \in L_{\frac{d}{d-2}}(Q)$. Fix some $p \in (1, \frac{2d}{d+2})$ such that $d + 2 < p' < \infty$, where $p' = p/(p - 1)$, we next prove that $Du \in L_{p, \text{loc}}(Q)$. Let $h \in C_c^\infty(Q)$ and $v \in \mathcal{H}_2^1(Q)$ be the solution of
\[
\begin{cases}
P^* v = \text{div} h & \text{in } Q \\
v = 0 & \text{on } ((-T, 0] \times \partial D) \cup ([0] \times \overline{D}),
\end{cases}
\tag{4.26}
\]
where $P^*$ is the adjoint operator of $P$ defined by
\[P^* v := v_t + D_{\beta}((A^{\alpha \beta})^T D_{\alpha} v) - D_{\alpha}((B^\alpha)^T v) - (B^\alpha)^T D_{\alpha} v + C^T v.
\]
Then by Theorem 2.1, we obtain $Dv \in L_{\infty}((-T + \varepsilon, 0) \times D_{\alpha})$. By the $\mathcal{H}_2^1$-estimate and $p' > 2$, we have
\[
\| v \|_{\mathcal{H}_2^1(Q)} \leq N \| h \|_{L_{\infty}(Q)} \leq N \| h \|_{L^{p'}(Q)}. \tag{4.27}
\]
By Lemma 3.4 and (4.27), we have
\[ \|v\|_{L^p((T+\varepsilon,0)\times D)} \leq N \left( \|h\|_{L^p(Q)} + \|v\|_{L^p(Q)} \right) \leq N \|h\|_{L^p(Q)}. \]
This together with Sobolev-Morrey theorem and \( p' > d + 2 \) implies that
\[ \|v\|_{L^p((-T+\varepsilon,0)\times D)} \leq N \|h\|_{L^p(Q)}. \]
Fix \( \zeta \in C_c^\infty((-T+\varepsilon,0)\times D) \) with \( \zeta \equiv 1 \) on \( Q' \subset \subset (-T+\varepsilon,0)\times D_u. \) Then we use \( \zeta u \) as a test function to (4.26) and obtain
\[
\int_Q -v_i u\zeta + (A^{i\beta})^T D_\alpha v \left( \zeta D_\beta u + u D_\beta \zeta \right) + (B^a)^T D_a v u \zeta \\
- (B^a)^T v \left( \zeta D_a u + u D_a \zeta \right) - C^T v u \zeta = \int_Q h_a D_a (u \zeta). \tag{4.28}
\]
On the other hand, recalling that \( u \in \mathcal{H}_1^l(Q) \) is a weak solution of (2.1), we choose \( \zeta v \) as a test function and get
\[
\int_Q u_i \zeta v + A^{i\beta} D_\beta u \left( \zeta D_\alpha v + v D_\alpha \zeta \right) + B^a u \left( \zeta D_a v + v D_a \zeta \right) - \hat{B}^a D_a u \zeta - C v \zeta \\
= \int_Q g_a \left( \zeta D_a v + v D_a \zeta \right) - f \zeta v. \tag{4.29}
\]
Combining (4.28) and (4.29), we obtain
\[
\int_Q h_a D_a (u \zeta) = \int_Q u v \zeta - \int_Q A^{i\beta} v D_\beta u D_\alpha \zeta + \int_Q (A^{i\beta})^T u D_\alpha v D_\beta \zeta - u v B^a D_a \zeta \\
- (\hat{B}^a)^T u v D_a \zeta + \int_Q g_a \left( \zeta D_a v + v D_a \zeta \right) - f \zeta v,
\]
which is similar to [18, (4.8)]. Then by replicating the argument in the proof of [18, Corollary 1.6], we have
\[
\left| \int_Q h_a D_a (u \zeta) \right| \leq N \left( \|g\|_{L^p(Q)} + \|f\|_{L^p(Q)} + \|u\|_{\mathcal{H}_1^l(Q)} \right) \|h\|_{L^p(Q)}
\]
for all \( h \in C_c^\infty(Q) \). Hence, \( u \in \mathcal{H}_1^l(Q') \) and
\[
\|u\|_{\mathcal{H}_1^l(Q')} \leq N \left( \|g\|_{L^p(Q)} + \|f\|_{L^p(Q)} + \|u\|_{\mathcal{H}_1^l(Q)} \right).
\]
The corollary is proved.

5. Proof of Theorem 2.3

5.1. The continuity of \( D_x u \) and \( U \). We first prove (2.4). Similar to the proof of Theorem 2.1, we take \( \Omega = Q_{3/4} \cap \Omega_0 \). Let \( A^{i(j)} \in \mathcal{C}^{(1/2,0)}(\overline{Q}_0) \), \( 1 \leq j \leq l + 1 \), be matrix-valued functions, and \( B^{(l)} \) be in \( \mathcal{C}^{(1/2,0)}(\overline{Q}_0) \). Define the piecewise constant (matrix-valued) functions
\[
\bar{A}(z) = A^{i(j)}(z_0), \quad z \in \Omega_0, \quad \bar{A}(z) = A^{i(j)}(z_0,h_j(z_0)), \quad z \in \Omega_j, \quad j \neq 0.
\]
From \( B^{(l)} \) and \( g^{(l)} \), we similarly define piecewise constant functions \( \bar{B} \) and \( \bar{g} \). Notice that these functions only depend on the center \( z_0 \) but are independent of the radius of the cylinder \( r \). Using Lemma 3.3, we immediately get the following result.
Lemma 5.1. Let $A, \tilde{A}, B, \tilde{B}, g$, and $\tilde{g}$ be defined as above, there exists a positive constant $N$, depending only on $d, l, \mu, \delta, \nu, \Lambda, \max_{1 \leq j \leq l+1} \|A\|_{C^{2}\Theta(Q)}\gamma \max_{1 \leq j \leq l+1} \|B\|_{C^{2}\Theta(Q)}\gamma \max_{1 \leq j \leq l+1} \|g\|_{C^{2}\Theta(Q)}$ and $\max_{1 \leq j \leq l+1} \|h_{j}\|_{\mathcal{C}(\mathbb{R}^{n})y}$ such that for $0 < r \leq 1$,
\[
\int_{Q_{r}(z_{0})} |A - \tilde{A}| \, dx \, dt + \int_{Q_{r}(z_{0})} |B - \tilde{B}| \, dx \, dt + \int_{Q_{r}(z_{0})} |g - \tilde{g}| \, dx \, dt \leq Nr^{\beta'},
\]
where $\beta' = \min\{\delta, 2\gamma_{0} - 1\}$. Thus, (2.4) directly follows from (4.25), (4.24), and (4.21) by taking $\gamma \in (\delta', 1)$.

Next, we observe from (2.4) that for each $j = 1, \ldots, M$,
\[
D_{d}u, \ U \in C^{\beta'/2, \beta'}(\overline{Q}_{r} \cap ((-T + \varepsilon, 0) \times \Omega_{j})).
\]

On the other hand, since
\[
D_{d}u = (A^{dd})^{-1} \left( U + g_{d} - B^{d}u - \sum_{\beta=1}^{d-1} A^{d\beta}D_{\beta}u \right),
\]
we conclude that $D_{d}u \in C^{\beta'/2, \beta'}(\overline{Q}_{r} \cap ((-T + \varepsilon, 0) \times \Omega_{j})).$

5.2. The estimate of $\langle u \rangle_{1+\beta}$. The proof is again based on the Campanato’s method, but we work on $u$ itself instead of its first derivatives. The key point is to prove that the mean oscillation of $u$ in cylinders vanishes in the order $r^{1+\beta'}$ as the radii $r$ of cylinders go to zero. In order to derive this, as shown in Subsection 4.1, we only need to treat the case without lower-order terms and the data $f$. Then we prove a weak type-(1,1) estimate for solutions to parabolic systems with coefficients are piecewise Dini mean oscillation. Finally, we introduce a set consisting of polynomials with respect to $x$ and use it to prove an estimate of the difference between $u$ and some polynomial in the $L_{q}$-mean sense, $q \in (0, 1)$.

Fix $z_{0} \in \left((-9/16, 0) \times B_{3/4}\right) \cap Q_{\delta}$, and take $0 < r < R \leq 1/4$, we take the coordinate system associated with $z_{0}$ and follow the proof of Theorem 2.1. As in Section 4, we denote
\[
\mathcal{P}u := -u_{t} + D_{a}(\tilde{A}^{a}(z_{0}^{0}, x^{0})D_{\beta}u).
\]
Then
\[
\mathcal{P}u = \text{div}(g + (\tilde{A}(z_{0}^{0}, x^{0}) - A(z))Du).
\]
Let $\mathcal{P}$ be the modified operator corresponding to $\mathcal{P}$ as in Section 4. Let $v \in \mathcal{H}_{\rho}(Q_{r}(z_{0}))$ be a weak solution to
\[
\left\{
\begin{array}{ll}
\mathcal{P} v = \text{div}(g - \tilde{g} + (\tilde{A}(z_{0}^{0}, x^{0}) - A(z))Du) & \text{in } Q_{r}(z_{0}), \\
v = 0 & \text{on } \partial_{p}Q_{r}(z_{0}),
\end{array}
\right.
\]
where $\tilde{g} := \tilde{g}(z_{0}^{0}, x^{0})$ is the piecewise constant function corresponding to $g$ defined in Subsection 3.2. Next we give two lemmas which are the key ingredients of the proof of the estimate of $\langle u \rangle_{1+\beta'}$. 


Lemma 5.2 (Weak type-(1, 1) estimate). Let $R \in (0, 1/4)$ and $p \in (1, \infty)$. Let $v \in H^1_p(Q^+_R(0))$ be a weak solution to the problem

$$
\begin{cases}
\mathcal{P} v = \text{div}(F_X Q^+_R(0)) & \text{in } Q^+_R(0), \\
v = 0 & \text{on } \partial Q^+_R(0),
\end{cases}
$$

where $F \in L_p(Q^+_R(0))$. Then for any $s > 0$, we have

$$
||z \in Q^+_R(0) : |v(z)| > s|| \leq \frac{NR}{s} ||F||_{L_1(Q^+_R(0))},
$$

where $N = N(n, d, p, v)$.

Proof. As in the proof of Lemma 4.3, we set $z_0 = 0$, $R = 1$, $A^\alpha(\cdot) := A^\alpha(0', \cdot')$, $\mathcal{P} := \mathcal{P}_\alpha$ for simplicity, and follow the same notation there. We are going to prove that the hypothesis of Lemma 3.8 is satisfied. Set $c = 24$ and fix $\hat{z} = (\hat{t}, \hat{y}) \in Q^+_R(0)$, $0 < r < 1/4$. Let $\hat{b} \in L^p(Q^+_1)$ be supported in $Q^+_1(\hat{z}) \cap Q^+_1(0)$ with mean zero, $b = E\hat{b}$, and $v_1 \in H^1_p(Q^+_1)$ be the unique weak solution of

$$
\begin{cases}
\mathcal{P} v_1 = \text{div } b & \text{in } Q^+_1, \\
v_1 = 0 & \text{on } \partial Q^+_1.
\end{cases}
$$

For any $R \geq cr$ such that $Q^+_1 \setminus Q^+_R(\hat{z}) \neq \emptyset$ and $h \in C^\infty_0((Q^+_2(\hat{z}) \setminus Q^+_R(\hat{z})) \cap Q^+_1(\hat{z}))$, let $v_0 \in H^1_p(Q^+_1)$ be a weak solution of

$$
\begin{cases}
\mathcal{P}^* v_0 = h & \text{in } Q^+_1, \\
v_0 = 0 & \text{on } ((-1, 0] \times \partial B_1) \cup (\{0\} \times B_1),
\end{cases}
$$

where $1/p + 1/p' = 1$ and $\mathcal{P}^*$ is the adjoint operator of $\mathcal{P}$ defined by $\mathcal{P}^* u := u_t + D_\beta(\tilde{A}^{\alpha \beta} D_\alpha u)$. In view of the definition of weak solutions and the assumption of $\hat{b}$, we have

$$
\int_{Q^+_1(\hat{z})} v_1 h = \int_{Q^+_1(\hat{z})} Dv_0 \cdot b = \int_{Q^+_1(\hat{z}) \cap Q^+_1(\hat{z})} (D_x v_0, V_0) \cdot \hat{b} = \int_{Q^+_1(\hat{z}) \cap Q^+_1(\hat{z})} (D_x v_0 - D_x V_0(\hat{z}), V_0 - V_0(\hat{z})) \cdot \hat{b},
$$

where $V_0 = \tilde{A}^{\alpha \beta}(x') D_\beta v_0$. Hence, as before we have

$$
||v_1||_{L_1((Q^+_2(\hat{z}) \setminus Q^+_R(\hat{z})) \cap Q^+_1(\hat{z}))} \leq NrR^{-1} ||\hat{b}||_{L_1(Q^+_1(\hat{z}) \cap Q^+_1(\hat{z}))},
$$

(5.3)

where we used (4.5), the duality, and Hölder’s inequality. Let $N_0$ be the smallest positive integer such that $Q^+_1 \subset Q^+_2 \subset \ldots \subset Q^+_n \subset (\hat{z})$. By taking $R = cr, 2cr, \ldots, 2^{N_0-1} cr$ in (5.3) and summarizing, we obtain

$$
\int_{Q^+_1(\hat{z}) \cap Q^+_1(\hat{z})} |v_1| \, dx \, dt \leq N \sum_{k=1}^{N_0} 2^{-k} ||\hat{b}||_{L_1(Q^+_1(\hat{z}) \cap Q^+_1(\hat{z}))},
$$

Therefore, the hypothesis of Lemma 3.8 is satisfied. The proof of this lemma is finished. \hfill \square
Denote
\[ P_1 = \{ p : p(x) = \sum_{\beta=1}^{d-1} \ell^\beta p x^\beta + \delta(x^d) \}, \]
where \( \ell^\beta \)'s are constants and \( \delta(\cdot) \) is a measurable function. For any \( z_0 \in (-9/16, 0) \times B_{3/4} \), we also denote
\[ P_{10} = \{ p : p(x) = \ell_0 + \sum_{\beta=1}^{d-1} \ell^\beta p (x^\beta - x^\beta_0) + \int_{x_0}^{x^d} (\bar{A}^{dd}(z_0', s))^{-1}(\ell_d - \sum_{\beta=1}^{d-1} \bar{A}^{d\beta}(z_0', s)\ell^\beta) \, ds \}, \]
where \( \ell^\beta \)'s are constants. Clearly, \( P_{10} \subset P_1 \).

**Lemma 5.3.** Let \( r > 0 \) and \( p \in P_{10} \). Suppose that
\[ \int_{B_{r}(x_0)} |p(x)|^q \, dx \leq C_0 r^{p(1+\beta')}, \tag{5.4} \]
where \( C_0 \geq 0 \) is a constant. Then we have
\[ |\ell_0| \leq NC_{0} r_0^{\beta'}, |\ell^\beta| \leq NC_0 r_0^{\beta'}, \beta = 1, \ldots, d, \]
where \( N > 0 \) depends only on \( d, n, \nu, q, \) and \( \delta' \).

**Proof.** Without loss of generality, we may assume that \( x_0 = 0 \). Since \( p(x) - p(-x^1, x^2, \ldots, x^d) = 2\ell_1 x^1 \), we have
\[ \int_{B_r} |2\ell_1 x^1|^q \, dx \leq 2 \int_{B_r} |p(x)|^q + |p(-x^1, x^2, \ldots, x^d)|^q \, dx \leq C_0 r^{p(1+\beta')}, \]
which implies that \( |\ell_1| \leq NC_0 r_{0}^{\beta'} \). Similarly, we get
\[ |\ell^\beta| \leq NC_0 r_{0}^{\beta'} \quad \text{for} \quad \beta = 2, \ldots, d - 1. \tag{5.5} \]
By using the parabolicity condition of \( \bar{A}^{dd} \), we have
\[ |p(x) - p(x', x^d/2)| \geq (N^{-1} \nu |\ell_d| - N \nu^{-1} |\ell'|) |x^d|, \]
where \( \ell' = (\ell_1, \ldots, \ell_{d-1}) \). This together with (5.5) gives
\[ \int_{B_r} |\ell_d|^q |x^d|^q \, dx \leq C_0 r^{p(1+\beta')} + \int_{B_r} |p(x) - p(x', x^d/2)|^q \, dx \leq NC_0 r^{p(1+\beta')}, \]
which implies
\[ |\ell_d| \leq NC_0 r_0^{\beta'}. \]
Finally, the bound of \( \ell_0 \) follows from (5.4) and the bounds of \( \ell^\beta \), where \( \beta = 1, \ldots, d \). The lemma is proved. \( \square \)

**Proof of (2.5).** We divide the proof into three steps.

**Step 1. Claim:** for any \( z_0 \in (-9/16, 0) \times B_{3/4} \) and \( r \in (1, 1/4) \), in the coordinate system associated with \( z_0 \) we can find \( p^{z_{00}} = p^{z_{00}}(x) \) in the form
\[ \ell^0_{z_{00}} + \sum_{\beta=1}^{d-1} \ell^\beta_{z_{00}} (x^\beta - x^\beta_0) + \int_{x_0}^{x^d} (\bar{A}^{dd}(z_0', s))^{-1}(\ell_d - \sum_{\beta=1}^{d-1} \bar{A}^{d\beta}(z_0', s)\ell^\beta) \, ds \]
\[ \in u_1 + P_{10}, \]

```
where \( t^{z_0}_R \) are constants and \( u_1 \) is defined in (4.13), such that

\[
\int_{Q^c(z_0)} |u - p^{x, z_0}|^q \leq NC_0^q q^{(1+\beta)},
\]

where

\[
C_0 = \sum_{j=1}^{M} |g|_{\delta/2, \delta Q_j} + \|u\|_{L^q(Q)} + \|Du\|_{L^1(Q)}.
\]

For simplicity, assume that \( x_0 = 0 \). Applying Lemma 5.2 to (5.2) and using the same argument that led to (4.8), we obtain

\[
\left( \int_{Q^c(z_0)} |v|^q \, dx \, dt \right)^{1/q} \leq Nr^{1+\beta'} \left( \|Du\|_{L^q(Q^c(z_0))} + \sum_{j=1}^{M} |g|_{\delta/2, \delta Q_j} \right),
\]

where \( q \in (0, 1) \). Recall that \( w = u - u_1 - v \) satisfies \( \mathcal{T}w = 0 \) in \( Q^c_{1/2}(z_0) \). Define

\[
p_1(x) = \mathcal{T}w = w(z_0) + x' \cdot D_x w(z_0) + \int_0^{x'} \left( \bar{A}_{dd}(z'_0, s) \right)^{-1} \left( W(z_0) - \sum_{\beta=1}^{d-1} \bar{A}_{d\beta}(z'_0, s) D\beta w(z_0) \right) \, ds \in \mathbb{P}^2_1,
\]

where \( W = \sum_{\beta=1}^{d} \bar{A}_{d\beta}(z'_0, x^d) D\beta w \). Then we have

\[
\|D_x(w - p_1)\|_{L^q(Q^c_0(z_0))} = \|D_x w - D_x w(z_0)\|_{L^q(Q^c_0(z_0))} \leq Nkr[D_x w]_{C^{1/2}(Q^c_0(z_0))}
\]

and

\[
\|\bar{A}_{d\beta}(z'_0, x^d) D\beta (w - p_1)\|_{L^q(Q^c_0(z_0))} = \|W - W(z_0)\|_{L^q(Q^c_0(z_0))} \leq Nkr[W]_{C^{1/2}(Q^c_0(z_0))},
\]

which together with Lemma 3.6 with a suitable scaling imply

\[
\|D(w - p_1)\|_{L^q(Q^c_0(z_0))} \leq Nkr^{-1-(d+2)/q} \|w\|_{L^q(Q^c_{1/2}(z_0))}.
\]

Since \( w - p_1(z_0) = 0 \), by Lemma 3.6 with a suitable scaling, we have

\[
\int_{Q^c_0(z_0)} |w - p_1|^q \, dx \, dt \leq N \int_{Q^c_0(z_0)} (|w - w(t_0, x)| + |w(t_0, x) - p_1|^q) \, dx \, dt
\]

\[
\leq N(xr)^2 \|w\|_{L^q(Q^c_0(z_0))} + N(xr)^q \|D(w - p_1)\|_{L^q(Q^c_0(z_0))}^q
\]

\[
\leq Nkr^2 \int_{Q^c_0(z_0)} |w|^q \, dx \, dt.
\]

Noting that \( w - p \) satisfies the same equation as \( w \) for any \( p \in \mathbb{P}^2_1 \) and \( \mathcal{T}^1p = p \), we then infer from (5.9) that for any \( p \in \mathbb{P}^2_1 \),

\[
\int_{Q^c_0(z_0)} |w - p_1|^q \, dx \, dt \leq Nkr^2 \int_{Q^c_0(z_0)} |w - p|^q \, dx \, dt.
\]

By using (5.8), (5.10), \( w = u - u_1 - v \), the triangle inequality, (4.18), and the proof of Lemma 4.4 (cf. (4.16) and (4.17)), we have

\[
\int_{Q^c_0(z_0)} |u - u_1 - p_1|^q \, dx \, dt
\]

\[
\leq \int_{Q^c_0(z_0)} |w - p_1|^q \, dx \, dt + \int_{Q^c_0(z_0)} |v|^q \, dx \, dt
\]
where $C_0$ is defined in (5.7).

Denote

$$F(r) := \inf_{p \in \mathbb{P}_1} \iint_{Q_r(z_0)} |u - u_1 - p|^q \, dx \, dt.$$ 

Then from (5.11) we have

$$F(kr) \leq \kappa^{2q} F(r) + NC_0^q \kappa^{-d} r^q(1 + b'),$$

for any $r \in (0, 1/4)$ and $k \in (0, 1/2)$. Then by using a well-known iteration argument (see, for instance, [21, Lemma 2.1, p. 86]), we have

$$F(r) \leq r^q(1+b') F(1/4) + NC_0^q r^q(1+b') \leq NC_0^q r^q(1+b').$$

Thus, we conclude (5.6).

**Step 2. Convergence of $\ell_{\beta}^{d,2}$, $\beta = 1, \ldots, d$.**

By using the triangle inequality and (5.6), we have

$$\iint_{Q_r(z_0)} |p_{\beta}^{d,2} - p_{\beta}^{d,2}| \leq \iint_{Q_r(z_0)} |u - p_{\beta}^{d,2}| \leq NC_0^q r^q(1+b').$$

Since

$$p_{\beta}^{d,2} - p_{\beta}^{d,2} = \ell_{\beta}^{d,2} - \ell_{\beta}^{d,2} + \sum_{\beta=1}^{d-1} \left( \ell_{\beta}^{d,2} - \ell_{\beta}^{d,2} \right) \chi_{\beta}^{d,2} + \sum_{\beta=1}^{d-1} \left( \ell_{\beta}^{d,2} - \ell_{\beta}^{d,2} \right) \right) ds \in \mathbb{P}_1,$$ 

we can apply Lemma 5.3 to get

$$|\ell_{\beta}^{d,2} - \ell_{\beta}^{d,2}| \leq NC_0^q r^{1+b'}, \quad |\ell_{\beta}^{d,2} - \ell_{\beta}^{d,2}| \leq NC_0^q r^{1+b'}, \beta = 1, \ldots, d.$$ (5.12)

Therefore, the limits of $\ell_{\beta}^{d,2}$ exist and are denoted by $\ell_{\beta}^{d,2}$ for $\beta = 0, 1, \ldots, d$. Since $u$ is continuous, it is easily seen that for any $r \in (0, 1/4)$,

$$u(z_0) = \ell_{0,0}^{d,2}, \quad |u(z_0) - \ell_{0,0}^{d,2}| \leq NC_0^q r^{1+b'}.$$ (5.13)

Next, we claim that for $\beta = 1, \ldots, d - 1,$

$$\iint_{Q_{r/2}(z_0)} |D_\beta u - \ell_{\beta}^{d,2}|^q + \iint_{Q_{r/2}(z_0)} |U - \ell_{d}^{d,2}|^q \leq NC_0^q r^{1+b'}.$$ (5.14)

In fact, recalling the definition of the operator $\hat{\mathcal{P}}$ in (5.1), we have

$$\hat{\mathcal{P}}(u - p_{\beta}^{d,2}) = \text{div}(g - \tilde{g} + (\tilde{A}(z_0', x^d) - A(z)) Du).$$

Then by using Lemma 3.4 with $0 < q < 1 < p < \infty$, the interpolation inequality, Lemma 4.5, (5.6), and Lemma 5.1, we have

$$\|D(u - p_{\beta}^{d,2})\|_{L_q(Q_{r/2}(z_0))} \leq N r^{1+b' - \frac{d}{q} + \frac{d}{p}} \|u - p_{\beta}^{d,2}\|_{L_q(Q_{r/2}(z_0))} + N \|g - \tilde{g} + (\tilde{A}(z_0', x^d) - A) Du\|_{L_p(Q_{r/2}(z_0))},$$
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\[
\leq N r^{\frac{1}{r}} \left( \int_{Q_{r/2}(z_0)} |u - p^{z_0}|^q \right) + NC_0 r^{\frac{d-1}{r}} (d+2) \|g - \bar{g} + (\bar{A}(z_0^\beta, x^d) - A) Du\|_{L^q(Q_{r/2}(z_0))} \\
\leq NC_0^q r^{\frac{1+4}{r}} + \delta^q.
\]

By using Hölder’s inequality, we obtain

\[
\int_{Q_{r/2}(z_0)} |D(u - p^{z_0})|^q \leq \left( \int_{Q_{r/2}(z_0)} |D(u - p^{z_0})|^p \right)^{\frac{q}{p}} \\
\leq NC_0^q r^{1+q} = NC_0^q r^{\frac{1+4}{r}}.
\]

We thus obtain

\[
\int_{Q_{r/2}(z_0)} |D\bar{u} - p^{z_0}|^q + \int_{Q_{r/2}(z_0)} |\bar{A}^{d\beta} D_\beta (u - p^{z_0})|^q \leq NC_0^q r^{\frac{1+4}{r}}.
\] (5.15)

A direct calculation yields

\[
D_\beta p^{z_0} = \ell^{z_0}_\beta, \quad \beta = 1, \ldots, d - 1, \quad \sum_{\beta=1}^d |\bar{A}^{d\beta} D_\beta p^{z_0}| = \bar{g}_d + \ell^{z_0}_d.
\]

Combining \( U = \sum_{\beta=1}^d \bar{A}^{d\beta} D_\beta u - \bar{g}_d \), (5.15), the triangle inequality, and Lemma 5.1, we have

\[
\int_{Q_{r/2}(z_0)} |D_\beta u - \ell^{z_0}_\beta|^q + \int_{Q_{r/2}(z_0)} |U - \ell^z_d|^q \\
\leq NC_0^q r^{\frac{1+4}{r}} + \int_{Q_{r/2}(z_0)} |\bar{A}^{d\beta} D_\beta u - \bar{A}^{d\beta}(x^d) D_\beta u + \bar{g}_d - \bar{g}_d|^q \\
\leq NC_0^q r^{\frac{1+4}{r}}.
\]

Therefore, we prove the claim (5.14). It follows from (5.14), (5.12) and the continuity of \( D_\beta u \) and \( U \) in (2.4) that

\[
D_\beta u(z_0) = \ell^{z_0}_\beta, \quad |D_\beta u(z_0) - \ell^{z_0}_\beta| \leq NC_0 r^{\frac{1+4}{r}}, \quad \beta = 1, \ldots, d - 1,
\] (5.16)

and

\[
U(z_0) = \ell^{z_0}_d, \quad |U(z_0) - \ell^z_d| \leq NC_0 r^{\frac{1+4}{r}}.
\] (5.17)

Let \( z_1 = (t_0 - r^2, 0) \). Similar to (5.13), (5.16), and (5.17), we have for any \( r \in (0, 1/4) \), under the coordinate system associated with \( z_1 \),

\[
u(z_1) = \ell^{z_1}_\beta, \quad |u(z_1) - \ell^{z_1}_\beta| \leq NC_0 r^{\frac{1+4}{r}},
\]

\[
D_\gamma u(z_1) = \ell^{z_1}_\beta, \quad |D_\gamma u(z_1) - \ell^{z_1}_\beta| \leq NC_0 r^{\frac{1+4}{r}}, \quad \beta = 1, \ldots, d - 1,
\] (5.18)

and

\[
\bar{U}(z_1) = \ell^{z_1}_d, \quad |\bar{U}(z_1) - \ell^z_d| \leq NC_0 r^{\frac{1+4}{r}},
\] (5.19)

where \( D_\gamma \) denotes the derivatives with respect to the space variables in the coordinate system associated with \( z_1 \) and \( U = A^{d\beta} D_\gamma u - \bar{g}_d \).

**Step 3. Estimate of \( (u)_1^{1+4} \).** Finally, we are going to estimate \( (u)_1^{1+4} \). If \( |t_0 - t_1|^{1/2} = r > 1/32 \), then (2.5) is obvious. If \( r \leq 1/32 \), then by the triangle inequality, we have

\[
|u(z_0) - u(z_1)|^q \leq |u(z_0) - \ell^{z_0}_0|^q + |\ell^{z_0}_0 - \ell^{z_1}_0|^q + |u(z_1) - \ell^{z_1}_0|^q.
\] (5.20)
Next we estimate \( |\ell^2_{0,0} - \ell^2_{0,1}| \). Noting that

\[
|\ell^2_{0,0} - \ell^2_{0,1}|^q \leq |p^2_{0,0} - p^2_{0,1}|^q + \left| \sum_{\beta=1}^{d-1} \ell^2_{0,0}\tilde{x}^\beta - \sum_{\beta=1}^{d-1} \ell^2_{0,1}\tilde{y}^\beta \right|^q
\]

\[
+ \left| \int_0^\nu \left( \tilde{A}_{dd}(z'_0, s) \right)^{-1} \left( \tilde{g}_d(z'_0, s) + \ell^2_{d,0} - \sum_{\beta=1}^{d-1} \tilde{A}_{d\beta}(z'_0, s) \ell^2_{0,0} \right) ds \right|^q
\]

\[
- \left| \int_0^\nu \left( \tilde{A}_{dd}(z'_1, s) \right)^{-1} \left( \tilde{g}_d(z'_1, s) + \ell^2_{d,1} - \sum_{\beta=1}^{d-1} \tilde{A}_{d\beta}(z'_1, s) \ell^2_{0,1} \right) ds \right|^q.
\]

For the first term, by using the triangle inequality and (5.6), we have

\[
\int_{\Omega_0(z_1)} |p^2_{0,0} - p^2_{0,1}| \leq \int_{\Omega_0(z_1)} |u(z) - p^2_{0,0}| + \int_{\Omega_0(z_1)} |u(z) - p^2_{0,1}|^q
\]

\[
\leq \int_{\Omega_0(z_0)} |u(z) - p^2_{0,0}| + NC_0 q \rho^{(1+\delta')} \leq NC_0 q \rho^{(1+\delta')} \quad (5.21).
\]

In order to estimate the last two terms, we first obtain from the estimate \( I - X^{-1} \) in the case 1 of the proof of Proposition 4.2 that

\[
|x - y| \leq N \rho^{1+\delta'}. \quad (5.22)
\]

Then for the second term, under the coordinate system associated with \( z_0 \), it follows from the triangle inequality, (5.22), (5.13), (5.18), (2.4), and (4.23) that for \( \beta = 1, \ldots, d - 1 \),

\[
|\ell^2_{0,0}\tilde{x}^\beta - \ell^2_{0,1}\tilde{y}^\beta| \leq \left| \left| (\ell^2_{0,0} - \ell^2_{0,1})\tilde{x}^\beta + \ell^2_{0,1}(\tilde{x}^\beta - \tilde{y}^\beta) \right| \right|
\]

\[
\leq \rho \left| |\ell^2_{0,0} - D_\beta u(z_0)| + |D_\beta u(z_0) - D_\beta u(z_1)| + |D_\beta u(z_1)| \right|
\]

\[
+ |D_\beta \rho u(z_1) - \ell^2_{0,1}| + N \rho^{1+\delta'}
\]

\[
\leq NC_0 \rho^{1+\delta'}. \quad (5.23)
\]

For the third term, by using the triangle inequality, \( \tilde{A}_{dd} \geq \nu \), Lemma 5.1, and (5.22), we have

\[
\left| \int_0^\nu \left( \tilde{A}_{dd}(z'_0, s) \right)^{-1} \tilde{g}_d(z'_0, s) ds - \int_0^\nu \left( \tilde{A}_{dd}(z'_1, s) \right)^{-1} \tilde{g}_d(z'_1, s) ds \right|^q
\]

\[
\leq \left| \int_0^\nu \left( \tilde{A}_{dd}(z'_0, s) \right)^{-1} \left( \tilde{g}_d(z'_0, s) - \tilde{g}_d(z'_1, s) \right) ds \right|^q
\]

\[
+ \left| \int_0^\nu \left( \tilde{A}_{dd}(z'_0, s) \right)^{-1} - \left( \tilde{A}_{dd}(z'_1, s) \right)^{-1} \tilde{g}_d(z'_1, s) ds \right|^q
\]

\[
+ \left| \int_\nu^\nu \left( \tilde{A}_{dd}(z'_1, s) \right)^{-1} \tilde{g}_d(z'_1, s) ds \right|^q \leq NC_0 q \rho^{(1+\delta')} \quad (5.24).
\]
Similarly, by using the triangle inequality, \( \bar{A}^{dd} \geq \nu \), Lemma 5.1, (5.22), (5.17), (2.4), (4.23), and (5.19), we have

\[
\left| \int_0^{y}\left( \bar{A}^{dd}(z_0', s) \right)^{-1} t_d^{2r_{2z_0}} ds - \int_0^{y}\left( \bar{A}^{dd}(z'_1, s) \right)^{-1} t_d^{2r_{2z_1}} ds \right|^q \\
\leq \int_0^{y}\left( \bar{A}^{dd}(z_0', s) \right)^{-1} \left( t_d^{2r_{2z_0}} - t_d^{2r_{2z_1}} \right) ds \\
+ \int_0^{y}\left( \bar{A}^{dd}(z'_1, s) \right)^{-1} \left( t_d^{2r_{2z_1}} - \bar{A}^{dd}(z'_1, s) \right)^{-1} t_d^{2r_{2z_1}} ds \\
\leq N|x|^q |t_d^{2r_{2z_0}} - U(z_0)|^q + N|x|^q |U(z_0) - U(z_1)|^q + N|x|^q |U(z_1) - t_d^{2r_{2z_1}}|^q + N\rho^{(1+q')}
\leq NC_0^q \rho^{(1+q')} + N|x|^q |U(z_1) - \bar{U}(z_1)|^q + N|x|^q |\bar{U}(z_1) - t_d^{2r_{2z_1}}|^q
\leq NC_0^q \rho^{(1+q')}.
\]

(5.25)

Also, we have

\[
\left| \int_0^{y}\left( \bar{A}^{dd}(z_0', s) \right)^{-1} \sum_{\beta=1}^{d-1} \bar{A}^{\beta\beta}(z_0', s) t_d^{2r_{2z_0}} ds \\
+ \int_0^{y}\left( \bar{A}^{dd}(z'_1, s) \right)^{-1} \sum_{\beta=1}^{d-1} \bar{A}^{\beta\beta}(z'_1, s) t_d^{2r_{2z_1}} ds \right|^q \leq NC_0^q \rho^{(1+q')}.
\]

(5.26)

Now, coming back to (5.20), taking the average over \( z \in Q_r(z_1) \) and taking the \( q \)th root, using (5.21), and (5.23)–(5.26), we have

\[ |u(z_0) - u(z_1)| \leq NC_0^{1+q'}. \]

Therefore, we finish the proof of (2.5).

\( \square \)

6. Weighted weak type-(1, 1) estimates

This section is devoted to the proof of a global weak type-(1, 1) estimate with respect to \( A_1 \) Muckenhoupt weights for solutions to

\[ \mathcal{P}u = \text{div} \ f, \]

with the coefficients \( A = (A^{\alpha\beta}) \) satisfying the following condition.

**Assumption 6.1.**

1. \( A \) is of piecewise Dini mean oscillation in \( Q \), and there exists some constant \( c_0 > 0 \) such that for any \( r \in (0, 1/2) \), \( \omega_A(r) \leq c_0 (\ln r)^{-2} \).

2. For some constant \( c_1, c_2 > 0 \), \( \omega_A(R_0) \geq c_1 \) and for any \( R \in (0, R_0/2) \), \( \omega_R(R) \leq c_2 (\ln R)^{-2} \).

We say \( w : \mathbb{R}^{d+1} \to [0, \infty) \) belongs to \( A_1 \) if there exists some constant \( C \) such that for all parabolic cylinders \( Q \) in \( \mathbb{R}^{d+1} \),

\[ \int_Q w(s, y) \, dy \leq C \inf_{z \in Q} w(z). \]

The \( A_1 \) constant \( [w]_{A_1} \) of \( w \) is defined as the infimum of all such \( C \)'s. In order to state our result, we first denote

\[ w(Q) := \int_Q w(z) \, dz, \quad \|f\|_{L^p_w(Q)} := \left( \int_Q |f|^p w \, dz \right)^{1/p}, p \in [1, \infty), \]

\[ \int_Q w^{1/p} \, dz \]
and then introduce the weighted Sobolev space:

\[ \mathcal{H}^1_{p,w}(Q) := \{ u : u_t, u \in \mathcal{H}^{-1}_{p,w}(Q), u, Du \in L_{p,w}(Q) \}, \]

where

\[ \mathcal{H}^{-1}_{p,w}(Q) := \left\{ f : f = \sum_{|a| \leq 1} D^a f_a, f_a \in L_{p,w}(Q) \right\}, \]

\[ \|f\|_{\mathcal{H}^{-1}_{p,w}(Q)} := \inf \left\{ \sum_{|a| \leq 1} \|f_a\|_{L_{p,w}(Q)} : f = \sum_{|a| \leq 1} D^a f_a \right\}, \]

and

\[ \|u\|_{\mathcal{H}^1_{p,w}(Q)} := \|u_t\|_{\mathcal{H}^{-1}_{p,w}(Q)} + \sum_{|a| \leq 1} \|D^a u\|_{L_{p,w}(Q)}. \]

Denote \( \delta_0 := \min_{1 \leq i \leq M-1} \text{dist}(\partial \Omega, (-T, 0) \times \partial \mathcal{D}). \)

Recall that in the proof of Lemma 4.3, Lemmas 3.8 and 3.5 are the key points. We will use generalizations of the two lemmas since our estimate and argument depend on the coordinate system, one of them is stated below. Let \( \{Q^k_a\} \) be a collection of dyadic “parabolic cubes” in \( Q \). See [8, Theorem 11] and also the proof of [11, Lemma 4.1]. Let \( p, c \in (1, \infty). \)

**Assumption 6.2.** i) \( S \) is a bounded linear operator on \( L_{p,w}(Q) \).

ii) If for some \( f \in L_{p,w}(Q), s > 0, \) and some cube \( Q^k_a \) we have

\[ s < \frac{1}{w(Q^k_a)} \int_{Q^k_a} |f| w \, dz \leq C_0 s, \]

then \( f \) admits a decomposition \( f = g + b \) in \( Q^k_a \), where \( g \) and \( b \) satisfy

\[ \int_{Q^k_a} |g|^p w \, dz \leq C_1 s^p w(Q^k_a), \]

\[ \int_{Q^k_a} |S(b_{Q^k_a})| w \, dz \leq C_1 s w(Q^k_a) \]

with \( z_0 \in Q^k_a \) and \( r = \text{diam} Q^k_a \).

Then the same proof as in [18, Lemma 6.3] gives the following result.

**Lemma 6.3.** Under Assumption 6.2, for any \( f \in L_{p,w}(Q) \) and \( s > 0, \) we have

\[ w(\{ z \in Q : |Sf(z)| > s \}) \leq \frac{N}{s} \int_Q |f| w \, dz, \]

where \( N = N(d, c, Q, C_1, ||S||_{L_{p,w}(Q)}) \) is a constant. Moreover, \( S \) can be extended to a bounded operator from \( L_{1,w}(Q) \) to weak-\( L_{1,w}(Q) \).

The generalization of Lemma 3.5 and its proof will be given in the Appendix. Now we state our global weak type-(1, 1) estimate with \( A_1 \) weights.

**Theorem 6.4.** Let \( p \in (1, \infty), Q := (-T, 0) \times \mathcal{D}, \mathcal{D} \) have a \( C^1, \text{Dini boundary}, Q_1, \ldots, Q_{M-1} \) be away from \((T, 0) \times \partial \mathcal{D} \) and satisfy the conditions in Theorem 2.1. Let \( w \) be an \( A_1 \) Muckenhoupt weight and Assumption 6.1 be satisfied. For \( f \in L_{p,w}(\mathcal{D}) \) with \( T \in (0, \infty), \) let \( u \in \mathcal{H}^1_{p,w}(Q) \) be a weak solution to

\[ \begin{cases} \mathcal{P}u = \text{div} f & \text{in } Q, \\ u = 0 & \text{on } \partial_p Q. \end{cases} \]
Then for any $s > 0$, we have

$$
\left| \{(t, x) \in Q : |Du(t, x)| > s\} \right| \leq \frac{N}{s} \| f \|_{L^1_w(Q)},
$$

where $N$ depends on $n, d, M, \omega, \Lambda, p_i, 0, [w] \Lambda, T$, the $C^{1,0}$ characteristics of $D$, $Q_j$, and $C^{1,0}$ norms of $Q_j$ with respect to $x$ and $t$, respectively. Moreover, the linear operator $S : f \mapsto Du$ can be extended to a bounded operator from $L_{1,w}(Q)$ to weak-$L_{1,w}(Q)$.

**Proof.** We follow a similar argument in the proof of [18, Theorem 5.2], in which the weighted $W^{1,p}_w$-solvability and estimates for divergence form elliptic systems are the important ingredients. Here we use the $H^1_{p,w}$-solvability and estimates with $A_p$ weights, Lemma 8.2, for divergence form parabolic systems, to conclude that the map $S : f \mapsto Du$ is a bounded linear map on $L_{p,w}(Q)$. We need to show that $S$ verifies the conditions of Lemma 6.3.

For a fixed $z_k = (t_k, x_k) \in Q^{h}_a$, we associate $Q^k_j$ with a parabolic cylinder $Q^k = Q^k(x_k)$ such that $z_k \in Q^k_j \subset Q^k$, where $r_k = \text{diam} Q^k \leq \delta_0/2$. Suppose for some $Q^k_j$ and $s > 0$,

$$
s < \frac{1}{w(Q^k_j)} \int_{Q^k_j} |f| w dz \leq C_0 s. \quad (6.1)
$$

We need to check that $f$ enjoys the Assumption 6.2, i.e., $f$ admits a decomposition in $Q^k_j$.

(i) If $\text{dist}(x_k, \partial D) \leq \delta_0/2$, then $Q^k$ does not intersect with subdomains $Q_j, j = 1, \ldots, M - 1$. In this case, we choose the coordinate system according to $\tilde{z}_k \in (-T, 0) \times \partial D$, which satisfies $|z_k - \tilde{z}_k|_p = \text{dist}(x_k, \partial D)$. Let

$$
g := \int_{Q^k_a} f dz, \quad b = f - g \quad \text{in } Q^k_a.
$$

Then

$$
\int_{Q^k_j} b dz = 0
$$

and

$$
|g| \leq \int_{Q^k_j} f dz \leq \frac{1}{|Q^k_j| \inf_{Q^k_j} w} \int_{Q^k_j} |f| w dz \leq \frac{N}{w(Q^k_j)} \int_{Q^k_j} |f| w dz \leq NC_0 s,
$$

where we used the definition of $w$ and (6.1). Hence,

$$
\int_{Q^k_j} |g|^p w dz \leq NC_0^{p'} s^{p'} w(Q^k_j).
$$

Let $u_1 \in H^1_{p,w}(Q)$ be the unique weak solution of

$$
\begin{cases}
P u_1 = \text{div } b & \text{in } Q, \\
u_1 = 0 & \text{on } \partial_D Q.
\end{cases}
$$

Let $p' = p/(p-1)$ and $P'$ be the adjoint operator of $P$. Set $c = \frac{4R_0}{27}$ with $R_0 = \text{diam } Q$. Then for any $R \geq c r_k$ such that $Q \setminus Q_R(z_k) \neq \emptyset$ and $h \in C_{\infty}^0(Q_{2R}(z_k) \setminus Q_R(z_k))$, let $u_2 \in H^1_{p',w}(Q)$ be a weak solution of

$$
\begin{cases}
P' u_2 = \text{div } h & \text{in } Q, \\
u_2 = 0 & \text{on } ((-T, 0) \times \partial D) \cup (\{t = 0\} \times \overline{D}).
\end{cases}
$$
which satisfies
\[
\left( \int_\Omega |Du_2|^{\alpha(p)} w^{-\frac{p}{\alpha(p)-1}} \, dz \right)^{\frac{1}{\alpha(p)}} \leq N \left( \int_\Omega \|w\|^{\alpha(p)} v^{-\frac{p}{\alpha(p)-1}} \, dz \right)^{\frac{1}{\alpha(p)}} = N \left( \int_{Q_{2R}(z_k)} |hw(w)^{\frac{1}{\alpha(p)}} v^{-\frac{p}{\alpha(p)-1}} \, dz \right)^{\frac{1}{\alpha(p)}}.
\]

See Lemma 8.2. Then by using the definition of adjoint solutions, the fact that \(b\) is supported in \(Q_n^k\) with mean zero, and \(h \in C_0(\Omega_{2R}(z_k) \setminus \Omega_R(z_k))\), we obtain
\[
\int_{Q_{2R}(z_k) \setminus Q_R(z_k)} Du_1 \cdot h = \int_{Q_n^k} Du_2 \cdot b = \int_{Q_n^k} (Du_2 - Du_2(z_k)) \cdot b.
\]

(6.2)

Since \(R \leq R_0\), \(Q_{\alpha^2}(z_k)\) does not intersect with subdomains \(Q_j\), \(j = 1, \ldots, M - 1\). Because \(P^*u_2 = 0\) in \(Q_R(z_k)\), by flattening the boundary and using a similar argument that led to an a priori estimate of the modulus of continuity of \(Du_2\) in the proof of [12, (4.22)], we have
\[
|Du_2(z) - Du_2(z_k)| \leq N \left( \frac{|z - z_k|}{R} \right)^{\gamma} + \int_0^{2|z - z_k|} \omega(s) |Du_2|_{L_1(|Q^{k}_{\alpha^{2}}(z_k))} \, ds \leq N \omega(Q_{\alpha^{2}}(z_k))
\]

for any \(z \in Q^k_{\alpha^{2}} \subset Q_{\alpha^2}(z_k)\), where \(\gamma \in (0, 1)\) is a constant and \(\omega_A(s)\) is defined as in [12, (4.15)], which is derived from \(\alpha_A(s)\). Then, coming back to (6.2), using a similar argument in the proof of [18, Theorem 5.2] and Lemma 4.3, we obtain
\[
\int_{Q^{k}_{\alpha^{2}}(z_k)} |Du_1|w \, dz \leq Nsw(Q_{\alpha^{2}}^k).
\]

(6.3)

(ii) If dist\((x_k, \partial \Omega) \geq \delta_0/2\), then \(Q_k\) does not intersect with \((-T, 0) \times \partial \Omega\). In this case, we choose the coordinate system according to \(z_k\). The rest proof is the same as that in [18, Theorem 5.2] and we also obtain (6.3). Therefore, \(S\) satisfies the hypothesis of Lemma 6.3, and thus for any \(s > 0\),
\[
\omega(\{z \in \Omega : |Du(z)| > s\}) \leq \frac{N}{s} ||f||_{L_1(\Omega)}.
\]

The theorem is proved. \(\square\)

7. Application: Regularity for parabolic transmission problems

Caffarelli, Soria-Carro, and Stinga [5] recently proved existence, uniqueness, and optimal regularity of solutions to transmission problems for harmonic functions with \(C^{1,\alpha}\) interfaces. Their argument is mainly based on the mean value property and the maximum principle for harmonic functions and an approximation argument. In [10], an alternative proof of the result in [5] is given, which works for more general elliptic systems with multiple subdomains and \(C^{1,\text{Dini}}\) interfaces. The main idea of the proof in [10] is to reduce the transmission problems to elliptic systems with piecewise Hölder or Dini continuous non-homogeneous terms by solving a Laplace equation with conormal boundary data. Then the results follows by these in [9, 18]. In this section, we extend the results in [5, 10] to parabolic systems by using Theorems 2.1 and 2.3.
7.1. Main results for the transmission problem. We first introduce some notation. For $Q := (-T,0) \times D$, we denote
\[
BQ := \{ t = -T \} \times D, \quad SQ := (-T,0) \times \partial D, \quad \partial_p Q := BQ \cup SQ.
\]
In the following, we assume that $Q_j = (-T,0) \times D_j$ are cylindrical, $j = 1, \ldots, M$, and similarly define $BQ_j$ and $SQ_j$. Without loss of generality, we assume that $D_j \subset D$ for $j = 1, \ldots, M - 1$ and $\partial D \subset \partial D_M$. For $\gamma \in (0,1]$, we denote $C^{(1+\gamma)/2,1+\gamma}(Q)$ to be the space of functions with finite norm $|u|_{(1+\gamma)/2,1+\gamma,Q}$. The transmission problem is given by
\[
\begin{aligned}
\mathcal{P}_0 u &= -u_1 + D_\alpha(A^{\alpha\beta}D_\beta u) = \text{div} F + f \quad \text{in } \cup_{j=1}^M Q_j, \\
u^+ &= -u^+ \quad \text{on } \partial_p Q_j,
\end{aligned}
\]
where $\nu$, $\nu^+$ is the unit normal vector on $SQ_j$ pointing inside $SQ_j$, $u^+$ and $u^-$ are the left and right limits of $u$ (its conormal derivatives) on $SQ_j$, respectively, $j = 1, \ldots, M - 1$.

The first result of this section is about the case when the interfaces are $C^{1,\mu}$ in the spatial variables and the coefficients and data are piecewise Hölder continuous.

**Theorem 7.1.** Assume that $\partial D_j$ are $C^{1,\mu}$, $A^{\alpha\beta}$ and $F$ are piecewise $C^{0,\mu}$ with $\mu \in \big(0, \mu/(1 + \mu)\big)$, and $g_j \in C^{0,2,\delta}(SQ_j)$, $j = 1, \ldots, M - 1$. Then there exists a unique weak solution $u \in H^1_2(Q)$ to (7.1), which is piecewise $C^{(1+\delta)/2,1+\delta}$ up to $SQ_j$, $j = 1, \ldots, M$, and satisfies
\[
\sum_{j=1}^M |u|_{(1+\delta)/2,1+\delta,Q_j} \leq N\left( \sum_{j=1}^{M-1} |g_j|_{0,2,\delta,Q_j} + \sum_{j=1}^M |F_j|_{0,2,\delta,Q_j} + |f|_{L^\infty(Q)} \right),
\]
where $N$ depends on $n, d, M, \delta, \mu, \nu, \Lambda, Q_j$, and $\|A\|_{C^{0,2,\delta}(Q)}$.

**Remark 7.2.** In the special case when $M = 2$ or when $A^{\alpha\beta}$ and $F$ are Hölder continuous in the whole domain, by using [9] and the linearity the result of Theorem 7.1 still holds with $\delta = \mu$.

Our second result is concerned with the case when the interfaces are $C^{1,Dini}$ in the spatial variables and the coefficients and data are of piecewise Dini mean oscillation.

**Theorem 7.3.** Assume that $\partial D_j$ are $C^{1,Dini}$, $A^{\alpha\beta}$ and $F$ are of piecewise Dini mean oscillation in $Q$, $F, f \in L^\infty(Q)$, and $g_j$ are Dini continuous on $SQ_j$, $j = 1, \ldots, M - 1$. Then there exists a unique weak solution $u \in H^1_2(Q)$ to (7.1), which is piecewise $C^{1/2}$ up to $SQ_j$, $j = 1, \ldots, M$.

7.2. Proofs of Theorems 7.1 and 7.3. The proof is a modification of [10].

**Proof of Theorem 7.1.** For a fixed $j = 1, \ldots, M - 1$ and a point $x_{jk} \in \partial D_j$, there is a neighbourhood $V_{jk}$ of $x_{jk}$ and a $C^{1,\mu}$ diffeomorphism $\Phi_{jk}$ from $V_{jk}$ onto a unit ball $B := B_1(0) \subset \mathbb{R}^d$ such that
\[
\Phi_{jk}(V_{jk} \cap \partial D_j) \subset \partial B_1, \quad \det D\Phi_{jk} = 1.
\]
Let
\[ y = \Phi_j^k(x) = (\Phi_j^1(x), \ldots, \Phi_j^l(x)), \quad x = (\Phi_j^k)^{-1}(y) =: \Psi_j^k(y) \quad (\det D\Psi_j^k = 1), \]
and
\[ g := g(t, y') = g_j(t, x'). \]
Since \( \partial D \) is \( C^{1,\gamma} \), there exist finitely many points \( x_j \in \partial D \), and \( V_j \subset D_j, j = 1, \ldots, m \), such that \( \partial D_j \subset \bigcup_{j=1}^m V_j \). Let \( [\zeta_j]_{k=1}^m \) be a smooth partition of unity subordinate to \( V_j \). Denote \( B^+ := B \cap \{ y^d > 0 \} \), \( Q^+ := (-T, 0) \times B^+ \), \( \Gamma := B \cap \{ y^d = 0 \} \), and \( S Q^+ := (-T, 0) \times \Gamma \). Let \( v_j \in H^1_2(Q^+) \) be the weak solution to
\[
\begin{align*}
-\partial_t v_j + \Delta v_j &= 0 & \text{in} \ Q^+, \\
\partial_v v_j |_{SQ^+} &= \frac{1}{2} \zeta_j \circ \Psi_j^k g, \\
v_j &= 0 & \text{on} \ \partial Q^+ \setminus SQ^+.
\end{align*}
\]
The existence and uniqueness follows from [23, Theorem 6.46]. We take the even extension \( \tilde{v}_j \) of \( v_j \) with respect to \( \{ y^d = 0 \} \) defined by
\[
\tilde{v}_j(t, y) = \begin{cases} v_j(t, y) & \text{in} \ (-T, 0) \times B^+, \\
v_j(t, y', -y^d) & \text{in} \ (-T, 0) \times B^-.
\end{cases}
\]
where \( B^- := B \cap \{ y^d < 0 \} \). Then \( \tilde{v}_j \) satisfies
\[
\begin{align*}
-\partial_t \tilde{v}_j + \Delta \tilde{v}_j &= 0 & \text{in} \ (-T, 0) \times B, \\
\partial_v \tilde{v}_j |_{SQ^{-}} - \partial_v \tilde{v}_j |_{SQ^+} &= \zeta_j \circ \Psi_j^k g, \\
\tilde{v}_j |_{SQ^-} &= \tilde{v}_j |_{SQ^+}, \\
\tilde{v}_j &= 0 & \text{on} \ \partial B((-T, 0) \times B).
\end{align*}
\]
Next we transform back to the \( x \)-variables. Let \( \eta_j \) be a cut-off function \( \eta_j \in C^\infty_0(V_j) \) satisfying \( 0 \leq \eta_j \leq 1 \) and \( \eta_j \equiv 1 \) on the support of \( \zeta_j \). From (7.2), we obtain
\[
\begin{align*}
-\partial_t \tilde{v}_j + D_{x}(a^{q2}(\partial_v \tilde{v}_j)) &= 0 & \text{in} \ (-T, 0) \times V_j, \\
a^{q2} \partial \tilde{v}_j |_{\{ -T, 0 \} \times (V_j \cap \partial D_j)} - a^{q2} \partial \tilde{v}_j |_{\{ -T, 0 \} \times (V_j \cap \partial D_j)} &= \zeta_j g_j, \\
\tilde{v}_j |_{\{ -T, 0 \} \times (V_j \cap \partial D_j)} &= \tilde{v}_j |_{\{ -T, 0 \} \times (V_j \cap \partial D_j)}, \\
\tilde{v}_j &= 0 & \text{on} \ \partial B((-T, 0) \times V_j),
\end{align*}
\]
which together with a direct calculation yields
\[
\begin{align*}
-\partial_t (\tilde{v}_j \eta_j) + D_a a^{q2} (D_{\tilde{v}_j} (\partial_v \eta_j)) &= D_a (a^{q2} D_{\tilde{v}_j} \eta_j) + a^{r1} D_{\tilde{v}_j} \eta_j, & \text{in} \ Q, \\
a^{q2} \partial (\tilde{v}_j \eta_j) |_{\{ -T, 0 \} \times (V_j \cap \partial D_j)} - a^{q2} \partial \tilde{v}_j |_{\{ -T, 0 \} \times (V_j \cap \partial D_j)} &= \zeta_j g_j, \\
\tilde{v}_j \eta_j |_{\{ -T, 0 \} \times (V_j \cap \partial D_j)} &= \tilde{v}_j \eta_j |_{\{ -T, 0 \} \times (V_j \cap \partial D_j)}, \\
\tilde{v}_j \eta_j &= 0 & \text{on} \ \partial B Q.
\end{align*}
\]
Applying the trace lemma, \( H^1_2 \)-estimate, and [24, Theorem 1.1] to (7.3), we have
\[
\| \tilde{v}_j \eta_j \|_{H^1_2((-T, 0) \times V_j)} \leq N \| g \|_{L^2(SQ)}.
\]
and
\[ |\bar{\theta}_k\eta_k|(1+b)/(2,1+\epsilon_0)\times v_\mu \leq N|g|_{b/2,b;\mathcal{S}Q_1}. \tag{7.6} \]

Denote
\[ v_j := \sum_{k=1}^m \bar{\theta}_{jk} \eta_{jk} \quad \text{in } Q. \]

Then we have from (7.4) and the identity \( \sum_{k=1}^m \zeta_{jk} = 1 \) on \( \partial D_j \) that
\[
\begin{cases}
-\partial v_j + D_a(a a^g D_j v_j) = \sum_{k=1}^m D_a(\bar{\theta}_{jk} a a^g D_j \eta_{jk}) + \sum_{k=1}^m a^{ag}_\alpha D_a \eta_{jk} D_j \bar{\sigma}_{jk} & \text{in } Q, \\
\alpha \beta D_j v_j \nu_{\alpha} \big|_{\mathcal{S}Q_j} - a a^g D_j v_j \nu_{\alpha} \big|_{\mathcal{S}Q_j} = g_j, \\
v_j \big|_{\mathcal{S}Q_j} = v_j \big|_{\mathcal{S}Q_j}, \\
v_j = 0 \\
\end{cases}
\tag{7.7}
\]

Furthermore, we have from (7.5) and (7.6) that
\[ \|v_j\|_{\mathcal{H}_2^1(Q)} \leq N\|g_j\|_{L^2(\mathcal{S}Q_j)}, \tag{7.8} \]

and
\[ \|v_j\|_{(1+b)/(2,1+\epsilon_0)\times Q} \leq N|g_j|_{b/2,b;\mathcal{S}Q_1}. \tag{7.9} \]

Denote
\[ w := u - \sum_{j=1}^{M-1} v_j \tag{7.10} \]

then it follows from the definition of weak solutions and (7.7) that \( w \) satisfies
\[
\begin{cases}
\mathcal{P}_0 w = \text{div } \bar{F} + \bar{f} \quad \text{in } Q, \\
w = 0 \quad \text{on } \partial D_j Q, \\
\end{cases}
\tag{7.11}
\]

where
\[ \bar{F} = 1_{\bigcup_{j=1}^M Q_j} F - \sum_{j=1}^{M-1} (A - a) D v_j - \sum_{j=1}^{M-1} \sum_{k=1}^m \bar{\theta}_{jk} D_a \eta_{jk}, \quad a = (a a^g), \]

and
\[ \bar{f} = f - \sum_{j=1}^{M-1} \sum_{k=1}^m a^{ag}_\alpha D_a \eta_{jk} D_j \bar{\sigma}_{jk}. \]

By using the Galerkin method, we find that there is a unique solution \( w \in \mathcal{H}_2^1(Q) \) to (7.11) and thus the existence and uniqueness of \( u \) is proved. Moreover,
\[
\|w\|_{\mathcal{H}_2^1(Q)} \leq N\left(\|\bar{F}\|_{L^2(Q)} + \|\bar{f}\|_{L^2(Q)}\right)
\leq N\left(\|F\|_{L^2(Q)} + \sum_{j=1}^{M-1} \| (A - a) D v_j \|_{L^2(Q)} + \sum_{j=1}^{M-1} \sum_{k=1}^m \| \bar{\theta}_{jk} a a^g D_j \eta_{jk} \|_{L^2(Q)} \right)
\]
\[ + \|f\|_{L^2(Q)} + \sum_{j=1}^{M-1} \sum_{k=1}^m \| a^{ag}_\alpha D_a \eta_{jk} D_j \bar{\sigma}_{jk} \|_{L^2(Q)} \]
\[
\leq N\left(\|F\|_{L^2(Q)} + \sum_{j=1}^{M-1} \|g_j\|_{L^2(\mathcal{S}Q_j)} + \|f\|_{L^2(Q)}\right), \tag{7.12}
\]
where we used (7.8), the boundedness of $A^{a\xi}$ and $a^{a\xi}$, and (7.5) in the third inequality. Recalling that $\partial D_j$ are $C^{1,\mu}$, $A^{a\xi}$ and $F$ are piecewise $C^{0/2,\delta}$, by using Theorem 2.3, (7.6), (7.9), and (7.12), we obtain

$$\sum_{j=1}^{M} |u|_{(1+\delta)/2,1+\delta,\overline{Q_j}} \leq N\left( \sum_{j=1}^{M-1} |(A^{a\xi} - a^{a\xi})D_{\beta}v|_{b/2,\delta,\overline{Q_j}} + \sum_{j=1}^{M-1} \sum_{k=1}^{m} ||\tilde{D}_{\beta}a^{a\xi}D_{\rho}f||_{b/2,\delta,\overline{Q_j}} + \sum_{j=1}^{M} |F|_{b/2,\delta,\overline{Q_j}} + ||f||_{L^\infty(Q_j)} \right) + ||f||_{L^\infty(Q_j)} + \sum_{j=1}^{M} |g|_{b/2,\delta,\overline{Q_j}} + \sum_{j=1}^{M} |f|_{b/2,\delta,\overline{Q_j}}$$

(7.13)

Combining (7.10), the triangle inequality, (7.9), and (7.13), we have

$$\sum_{j=1}^{M} |u|_{(1+\delta)/2,1+\delta,\overline{Q_j}} \leq N\left( \sum_{j=1}^{M-1} |g|_{b/2,\delta,\overline{Q_j}} + \sum_{j=1}^{M} |f|_{b/2,\delta,\overline{Q_j}} + ||f||_{L^\infty(Q_j)} \right).$$

The theorem is proved.

We say that a function $f$ is of $L^2$-Dini mean oscillation in $Q$ if

$$\tilde{\omega}_f(r) := \sup_{z_0 \in Q} \left( \int_{Q \cap (Q(r) \cap Q)'} |f(t,x) - (f)(Q \cap (Q(r) \cap Q))|^2 \, dz \right)^{1/2}$$

satisfies the Dini condition.

**Proof of Theorem 7.3.** If we can prove the claim: $\nabla u$, satisfies the $L^2$-Dini mean oscillation condition in $Q$, then the rest of the proof follows from that of Theorem 7.1 by using Theorem 2.1. Hence, it suffices to prove the claim. The proof is a modification of [17, Theorem 1.7] and [10, Theorem 1.4], and we only need to prove the boundary estimate since the interior estimate is simpler.

For any $z_0 \in \partial \mathbb{R}^d = \{x^d = 0\}$,

$$Q_{r^+}(z_0) := Q_{r^+}^-(z_0) \cap \{x^d > 0\} \quad \text{and} \quad \Gamma_r(z_0) := \Gamma_r(\Gamma_r(z_0) \cap \{x^d = 0\}).$$

Recalling that $SQ_j$ is $C^{1,\text{Dini}}$, and as in the proof of Theorem 7.1, we only need to prove that for any weak $u$ solution to

$$\begin{cases}
-u_t + \Delta u = D_d g^d & \text{in } Q_{r^+}^-(0), \\
D_d u = g^d & \text{on } \Gamma_4(0),
\end{cases}$$

(7.14)

$\nabla u$ satisfies the $L^2$-Dini mean oscillation condition in $Q_{r^+}^-(0)$, provided that $g^d = g^d(t,x) = g^d(t,x')$ is Dini continuous satisfying $D_d g^d = 0$. 
For \( z \in Q_{r,+}^-(0) \) and \( r > 0 \), we define

\[
\phi(z, r) := \left( \int_{Q_r^-(z) \cap Q_{r,0}^+(0)} |Du - (Du)_{Q_r^-(z) \cap Q_{r,0}^+(0)}|^2 \right)^{1/2}.
\]

As in [12], we denote \( B_+^+(x_0) = B_+(x_0) \cap \{ x^d > 0 \} \) and fix a smooth set \( D \subset \mathbb{R}^d \) satisfying \( B_4^+(0) \subset D \subset B_2^+(0) \) and for \( z_0 = (t_0, x_0) \in \partial \mathbb{R}^{d+1} \), we denote

\[
\mathcal{D}_r(x_0) := rD + x_0.
\]

Now we decompose \( u = w + v \), where \( w \in H_2^1 \) is the weak solution of

\[
\begin{align*}
-w_t + \Delta w &= D_d(g^d - \bar{g}^d) \quad \text{in } (t_0 - 4r^2, t_0) \times \mathcal{D}_2/(x_0), \\
D_d w_{x_a} &= (g^d - \bar{g}^d)v_d \quad \text{on } (t_0 - 4r^2, t_0) \times \partial \mathcal{D}_2/(x_0), \\
w &= 0 \quad \text{on } \{ t = t_0 - 4r^2 \} \times \mathcal{D}_2/(x_0),
\end{align*}
\]

where

\[
\bar{g}^d = \int_{Q_{4r}^-(z_0)} g^d(t, x) \, dx \, dt.
\]

By using the \( H_2^1 \)-estimate and \( Q_{r,0}^-(z_0) \subset (t_0 - 4r^2, t_0) \times \mathcal{D}_2/(x_0) \subset Q_{2r,0}^-(z_0) \), we have

\[
\left( \int_{Q_{r}^-(z_0)} |Dw|^2 \right)^{1/2} \leq Na_{2r}(g). \tag{7.15}
\]

Here \( a_r(g) \) denotes the modulus of continuity of \( g \) in the \( L_{\infty} \) sense. Notice that \( v := u - w \) satisfies

\[
\begin{align*}
-v_t + \Delta v &= D_d \bar{g}^d \quad \text{in } Q_{r,0}^-(z_0), \\
D_d v &= \bar{g}^d \quad \text{on } \Gamma_r(z_0).
\end{align*}
\]

For any \( q = (q_1, \ldots, q_d) \in \mathbb{R}^d \), \( \bar{q} := D_x v - q \) satisfies

\[
\begin{align*}
-\bar{q}_t + \Delta \bar{q} &= 0 \quad \text{in } Q_{r,0}^-(z_0), \\
D_d \bar{q} &= 0 \quad \text{on } \Gamma_r(z_0).
\end{align*}
\]

Then by the standard parabolic estimates for equations with constant coefficients and zero conormal boundary data, we have

\[
[D_d v]_{C^{1/2}((Q_{r,0}^-(z_0))} \leq Nr^{-1} \left( \int_{Q_{r}^-(z_0)} |D_x v - q|^2 \right)^{1/2}. \tag{7.16}
\]

Now observe that \( D_d v \) satisfies a heat equation in \( Q_{r,0}^-(z_0) \) with a constant Dirichlet boundary condition on \( \Gamma_r(z_0) \). Applying [12, Lemma 4.15], we get

\[
[D_d v]_{C^{1/2}((Q_{r,0}^-(z_0))} \leq Nr^{-1} \left( \int_{Q_{r}^-(z_0)} |D_d v - q_d|^2 \right)^{1/2}. \tag{7.17}
\]

Let \( \kappa \in (0, 1/2) \) be a small constant, then combining (7.16) and (7.17), we have

\[
\left( \int_{Q_{r}^-(z_0)} |Dv - (Dv)_{Q_{r,0}^-(z_0)}|^2 \right)^{1/2} \leq 2 \kappa r |Dv|_{C^{1/2}((Q_{r,0}^-(z_0))}\].
\]
where $N_0 = N_0(d, v, \Lambda)$. By using $u = v + w$, the triangle inequality, (7.15), and (7.18), we obtain

\[
\left( \int_{Q_{r_0}(z_0)} |Du - (Dv)_{Q_{r_0}(z_0)}|^2 \right)^{1/2} \leq N_0 \kappa \left( \int_{Q_{r_0}(z_0)} |Dv - (Dw)_{Q_{r_0}(z_0)}|^2 \right)^{1/2} + \left( \int_{Q_{r_0}(z_0)} |Dw|^2 \right)^{1/2}.
\]

Choosing $q = (Du)_{Q_{r_0}(z_0)}$, we reach

\[
\phi(x_0, kr) \leq N_0 \kappa \phi(x_0, r) + N \kappa^{-(d+2)/2} \omega_2(g).
\]

The rest of the proof follows from an iteration argument. See, for example, the proof of [12, Proposition 4.2]. We omit the details. Therefore, we show that $Du$ is of $L^2$-Dini mean oscillation.

8. Appendix

In the appendix, we prove a generalization of Lemma 3.5. We say that $w : \mathbb{R}^{d+1} \to [0, \infty)$ belongs to $A_p$ for $p \in (1, \infty)$ if

\[
\sup_Q \frac{w(Q)}{|Q|} \left( \frac{w^{p-1}(Q)}{|Q|} \right)^{p-1} < \infty,
\]

where the supremum is taken over all parabolic cylinders $Q$ in $\mathbb{R}^{d+1}$. The value of the supremum is the $A_p$ constant of $w$, and will be denoted by $[w]_{A_p}$.

We consider the parabolic systems on $Q := (-T, 0) \times D$, where $D$ is a Reifenberg flat domain and the boundary $\partial D$ satisfies the following assumption with a parameter $\gamma_0 \in (0, 1/4)$ to be specified later.

**Assumption 8.1 ($\gamma_0$).** There exists a constant $r_0 \in (0, 1]$ such that the following conditions hold.

1. In the interior of $D$, $A_{\alpha \beta}^i$ satisfy (3.5) in some coordinate system depending on $(t_0, x_0)$ and $r$.
2. For any $x_0 \in \partial D$, $t \in \mathbb{R}$, and $r \in (0, r_0]$, there is a coordinate system depending on $(t_0, x_0)$ and $r$ such that in this new coordinate system, we have

\[
\{(y', y) : x_0^d + \gamma_0 r < y^d \} \cap B_R(x_0) \subset D \cap B_R(x_0) \subset \{(y', y) : x_0^d - \gamma_0 r < y^d \} \cap B_R(x_0)
\]

and

\[
\int_{Q_{r_0}(z_0)} |\langle A(t, x) - (A)_{Q_{r_0}(z_0)} \rangle | dx \, dt \leq \gamma_0,
\]

where $(A)_{Q_{r_0}(z_0)} = \int_{Q_{r_0}(z_0)} A(z', x^d) \, dz'$.
Lemma 8.2. Let \( p \in (1, \infty) \) and \( w \) be an \( A_p \) weight. There exists a constant \( \gamma_0 \in (0, 1/4) \) depending on \( d, p, \nu, \Lambda, \) and \([w]_{A_p}\) such that, under Assumption 8.1, for any \( u \in \mathcal{H}^1_{p,w}((-T, 0) \times \mathcal{D}) \) satisfying

\[
\begin{aligned}
Pu - \lambda u &= \operatorname{div} f \quad \text{in } Q, \\
u &= 0 \quad \text{on } \partial_p Q,
\end{aligned}
\]

(8.1)

where \( \lambda \geq 0 \) and \( f \in L_{p,w}(Q) \), we have

\[
\|u\|_{\mathcal{H}^1_{p,w}(Q)} \leq N\|f\|_{L_{p,w}(Q)},
\]

(8.2)

where \( N = N(n, d, p, \nu, \Lambda, [w]_{A_p}, r_0, T) \). Moreover, for any \( f \in L_{p,w}(Q) \), (8.1) admits a unique solution \( u \in \mathcal{H}^1_{p,w}(Q) \).

Proof. The case when \( \lambda > \lambda_0 \) is proved in [13, Section 8] and [13, Theorem 7.2], where \( \lambda_0 > 0 \) is a sufficiently large constant depending on \( n, d, p, \nu, \Lambda, [w]_{A_p} \), and \( r_0 \). For \( 0 \leq \lambda \leq \lambda_0 \), we set

\[
\nu := ue^{-\lambda t}.
\]

Then we have

\[
\begin{aligned}
P\nu - (\lambda + \lambda_0)\nu &= e^{-\lambda t} \operatorname{div} f \quad \text{in } Q, \\
u &= 0 \quad \text{on } \partial_p Q.
\end{aligned}
\]

By using [13, Theorem 7.2], we have

\[
\|\nu\|_{\mathcal{H}^1_{p,w}(Q)} \leq N\|e^{-\lambda t} f\|_{L_{p,w}(Q)} \leq N\|f\|_{L_{p,w}(Q)},
\]

where \( N = N(n, d, p, \nu, \Lambda, [w]_{A_p}, r_0, T) \). Hence, we obtain

\[
\|u\|_{\mathcal{H}^1_{p,w}(Q)} = \|ue^{\lambda t}\|_{\mathcal{H}^1_{p,w}(Q)} \leq N\|f\|_{L_{p,w}(Q)}.
\]

The theorem is proved.

\( \square \)

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