LAX OPERATOR ALGEBRAS AND HAMILTONIAN INTEGRABLE HIERARCHIES

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Abstract. We consider the theory of Lax equations in complex simple and reductive classical Lie algebras with the spectral parameter on a Riemann surface of finite genus. Our approach is based on the new objects — the Lax operator algebras, and develops the approach of I.Krichever treating the $\mathfrak{gl}(n)$ case. For every Lax operator considered as the mapping sending a point of the cotangent bundle on the space of extended Tyurin data to an element of the corresponding Lax operator algebra we construct the hierarchy of mutually commuting flows given by Lax equations and prove that those are Hamiltonian with respect to the Krichever-Phong symplectic structure. The corresponding Hamiltonians give integrable finite-dimensional Hitchin-type systems. For example we derive elliptic $A_n$, $C_n$, $D_n$ Calogero-Moser systems in frame of our approach.

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1. Introduction

In [4, 5] I.M. Krichever and S.P. Novikov proposed the technique of finding high rank finite-zone solutions to Kadomtsev-Petviashvili and Schrödinger equations. Based on the ideas of those works and on his results on effective classification of high rank pairs of commuting differential operators [6] I.M. Krichever proposed the theory of Lax operators with the spectral parameter on a Riemann surface [2]. In [8] I.M. Krichever and the author found that these Lax operators form an associative algebra, and constructed their orthogonal and symplectic analogs which form Lie algebras. They were called Lax operator algebras. Lax operator algebras form a new class of one-dimensional current algebras.

The applications of current algebras to the theory of Lax equations have a long history. They are initiated in the works of I. Gelfand, L. Dikii, I. Dorfman, A. Reyman, M. Semenov-Tian-Shanskii, V. Drinfeld, V. Sokolov, V. Kac, P. van Moerbeke. Basically these applications are related to Kac-Moody algebras which appear quite naturally in the context of Lax equations with rational spectral parameter. In [2] the theory of conventional Lax and zero curvature representations with rational spectral parameter was generalized to the case of algebraic curves $\Sigma$ of arbitrary genus $g$. Such representations arise in several ways in the theory of integrable systems, c.f. [4] where a zero curvature representation of the Krichever-Novikov equation is introduced, or [2] where a field analog of the Calogero-Moser system on an elliptic curve is presented. Lax operator algebras appear as a corresponding generalization of Kac-Moody algebras.

In [12, 13] we have posed the problem of generalization of the Krichever theory [2] to all Lax operator algebras, and have done the first steps in that direction including the construction of integrable hierarchies of Lax equations. In the present article we undertake the concluding step: prove that those Lax equations are Hamiltonian and construct the corresponding Hamiltonians. We consider certain wellknown examples (the elliptic Calogero-Moser systems) in the context of our approach.

The concept of Lax operators on algebraic curves is closely related to A. Tyurin results on the classification of holomorphic vector bundles on algebraic curves [14]. It uses Tyurin data modelled on Tyurin parameters of such bundles. Tyurin data consist of points $\gamma_s \in \Sigma$ ($s = 1, \ldots, ng$), and associated elements $\alpha_s \in \mathbb{C} P^n$ where $g$ denotes the genus of the Riemann surface $\Sigma$, and $n$ corresponds to the rank of the bundle. Every Lax operator algebra is associated with fixed Tyurin data and a set of marked points on $\Sigma$. For a finite-dimensional simple or reductive Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ we denote this algebra by $\mathfrak{g}$. It is an almost graded Lie algebra. Its elements are parameterized by cotangent vectors to the space of Tyurin data at the point. Those elements are defined in [2, 8] as meromorphic $(n \times n)$ matrix-valued functions on $\Sigma$ having arbitrary poles at the points $P_k$, $k = 1, \ldots, N$ (which are assumed to be fixed), and poles of the order at most two at $\gamma_s$’s. The coefficients of the Laurent
expansion of those matrix-valued functions in the neighborhood of a point $\gamma_s$ have to obey certain constraints parametrized by $\alpha_s$ (relations (3.4) below). In the case of absence of the points $\gamma_s$ (which corresponds to trivial vector bundles) we return to the known class of Krichever-Novikov algebras (see [11] for a review). If, in addition, the genus of $\Sigma$ is equal to 0, $\sharp\{P_i\} = 2$, and these two points are 0 and $\infty$ we obtain the loop algebras. It may be mentioned here that all these three types of current algebras have quite similar theory of central extensions [8, 10].

By Lax operator we mean the mapping sending a point of the cotangent bundle on the space of Tyurin data (extended by marked points) to an element of the corresponding Lax operator algebra. With every positive divisor $D = \sum m_i P_i$ we associate the subspace $\mathcal{L}^D$ of the cotangent bundle such that the Lax operator, as a function on $\Sigma$, has a pole of order at most $m_i$ at $P_i$ for every fixed point in that subspace. We consider a certain dynamical system on $\mathcal{L}^D$ given by the Lax equation. We prove that these dynamical systems are Hamiltonian and integrable with respect to the Krichever-Phong symplectic structure.

In Section 2 following [2, 13] we introduce $M$-operators (partners of Lax operators in Lax pairs) and study their analytic and algebraic properties. Every $M$-operator defines a flow in the cotangent bundle on the space of extended Tyurin data. The corresponding dynamics of the Tyurin parameters (2.7) plays a fundamental role. It appeared in [4] as an ansatz for effective integration of KP equation in the class of high rank solutions, and is hardly used in [2].

In Section 3 we give a brief survey of Lax operator algebras. Following [2, 13] we introduce their elements (L-operators) as $M$-operators yielding a trivial dynamics of Tyurin data. There is also an independent definition of $L$-operators given in [8]. We refer to that article for more detail.

In Section 4 we prove that a Lax equation is equal to the system of equations on the main parts of the Lax operator at the points $P_i$, and of the above mentioned dynamics equations of Tyurin data. We prove a criteria of well-definiteness of the dynamical system given by a Lax equation on a space $\mathcal{L}^D$.

In Section 5 we construct hierarchies of commuting flows on the cotangent bundle on $\mathcal{L}^D$ given by a Lax operator. A key role is played by the Lemma 5.1 which treats the dimension of the space of $M$-operators satisfying the criteria of Section 4. Modulo that lemma we follow the lines of [2]. We prove the final result for all classical Lie algebras in question, in particular for $\mathfrak{sp}(2n)$ which was not done in the earlier authors article [13] on hierarchies. We would like to stress here that in general $L$ and $M$ do not belong to the same Lie algebra and give the list of the corresponding pairs of Lie algebras [12]. Observe that the method of constructing a hierarchy given in this section is not unique. In Section 8 we give another one for elliptic curves enabling us to incorporate certain Calogero-Moser systems in frame of our approach.
In Sections 6, 7 we develop a Hamiltonian theory for the Lax equations in question. In Section 6 we construct the analog of the Krichever-Phong symplectic structure on the cotangent bundle of a certain subspace $P^D \subset \mathcal{L}^D/G$ (where $g = \text{Lie } G$) invariant with respect to the flows of our commuting hierarchy. In Section 7 we prove that the hierarchies of Section 5 are Hamiltonian with respect to that structure and construct the corresponding Hamiltonians. Again, the presentation is similar for all classical Lie algebras in question, and follows the lines of [2] except for the proofs of non-degenerateness of the Krichever-Phong symplectic form (Section 6) and of holomorphy of spectra of operators in Lax pairs (Section 7). The latter is done for every class of Lie algebras individually. We also give the proofs of the statements in [2] in slightly more detail.

In Section 8 we consider three examples of integrable systems in frame of our approach. The first one is the elliptic Calogero-Moser system for $g = \mathfrak{gl}(n)$. It is considered earlier in [2, 3]. Further on, by modifying the technique of Section 5 we obtain the Lax operator and the corresponding Hamiltonian integrable hierarchy for the elliptic Calogero-Moser $\mathfrak{so}(2n)$ and $\mathfrak{sp}(2n)$ systems. Unfortunately we don’t know any explicit form of Lax operators for $\mathfrak{so}(2n + 1)$ systems in frame of our approach. In particular, such Lax operators must be meromorphic in the spectral parameter. Observe that the Lax representations of those systems with the Lax operators of Backer-Akhieser type are known [1].

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2. $M$-OPERATORS AND TIMES

Let $\Sigma$ be a compact Riemann surface of genus $g$ with two marked points $P_+, P_-$, and $g$ be a Lie algebra over $\mathbb{C}$ from the following list: $\mathfrak{gl}(n), \mathfrak{sl}(n), \mathfrak{so}(2n), \mathfrak{so}(2n + 1), \mathfrak{sp}(2n), \mathfrak{s}(n), \mathfrak{tsp}(2n)$ where $\mathfrak{s}(n)$ is the algebra of scalar matrices, and $\mathfrak{tsp}(2n)$ is a Lie subalgebra of $\mathfrak{sp}(2n + 2)$ consisting of the matrices having zero first column and last row. This is the list of the previous work [13] extended with $\mathfrak{tsp}(2n)$ which appears in the construction of integrable hierarchies in the symplectic case.

Let us fix $K$ additional points $\gamma_s \in \Sigma$, and let

$$W := \{\gamma_s \in \Sigma \setminus \{P_+, P_-\} \mid s = 1, \ldots, K\}$$

($K$ will be specified in Section 5). To every point $\gamma_s$ we assign a vector $\alpha_s \in \mathbb{C}^p$ given up to a scalar factor where $p$ is the dimension of the standard (vector) representation of the corresponding $g$ (i.e. the elements of $g$ are $p \times p$-matrices). The system

$$T := \{ (\gamma_s, \alpha_s) \mid s = 1, \ldots, K \}$$
is called Tyurin data below. This data is related to the moduli of holomorphic vector bundles over Σ. In particular, for generic values of \((γ_s, α_s)\) with \(α_s \neq 0\) and \(K = ng\) Tyurin data parameterize the semistable rank \(n\) degree \(ng\) framed holomorphic vector bundles over Σ, see [14].

Let \(M : Σ \to g\) be a meromorphic function. We require that at a point \(γ = γ_s\) \(M\) has the expansion

\[
M = \frac{M_{-2}}{(z - z_γ)^2} + \frac{M_{-1}}{z - z_γ} + M_0 + \ldots, \tag{2.3}
\]

where \(z\) is a fixed local coordinate in the neighborhood of \(γ\), \(z_γ\) is the coordinate of \(γ\) itself, \(M_{-2}, M_{-1}, M_0, M_1, \ldots \in g\) and

\[
M_{-2} = λαα^tσ, \quad M_{-1} = (αμ^t + εμα^t)σ \tag{2.4}
\]

where \(λ ∈ C\), \(μ ∈ C^n\), \(σ\) is a \(n \times n\) matrix, the upper \(t\) denotes the matrix transposition

\[
λ \equiv 0, \quad ε = 0, \quad σ = id \quad \text{for } g = gl(n), \quad sl(n), \tag{2.5}
\]

\[
λ \equiv 0, \quad ε = -1, \quad σ = id \quad \text{for } g = so(n), \quad ε = 1 \quad \text{for } g = sp(2n),
\]

and \(σ\) is a matrix of the symplectic form for \(g = sp(2n)\). In the case \(g = tsp(2n)\) we require that \(α\) and \(μ\) were of the form \(α = (α_0, \tilde{α}^t, 0)^t\) \((α_0 ∈ C, \tilde{α} ∈ C^{2n})\), \(μ = (μ_0, \tilde{μ}^t, 0)^t\) \((μ_0 ∈ C, \tilde{μ} ∈ C^{2n})\), and \(σ = \begin{pmatrix} 0 & 0 & \tilde{α}^t & 1 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0\end{pmatrix}\) where \(\tilde{σ}\) is the matrix of the symplectic form corresponding to \(sp(2n)\). For such \(σ\) the elements of \(tsp(2n)\) are of the form \(X = \begin{pmatrix} 0 & u^t & ε \\ 0 & X & 0 \\ 0 & 0 & 0\end{pmatrix}\) where \(\tilde{X} ∈ sp(2n), \quad u^t = (a^t, b^t), \quad v^t = (b^t, -a^t)\), and \(a, b ∈ C^n\). In other words, \(tsp(2n)\) is the semidirect sum of the \(sp(2n)\) and the Heisenberg algebra. Here and below we omit the subscripts \(s, γ\) indicating the point \(γ\) except for \(z_γ\). In the cases \(g = sp(2n), \quad g = tsp(2n)\) we require that

\[
α^tσM_1α = 0. \tag{2.6}
\]

Under above requirements we call \(M\) a \(g\)-valued \(M\)-operator. In [13] we did not require (2.6) for \(M\)-operators doing that for \(L\)-operators only (see Section 3).

Every \(M\)-operator and a complex number \(κ\) define a dynamical system on the space of Tyurin data:

\[
\dot{z}_γ = -μ^tσα, \quad \dot{α} = -M_0α + κα, \tag{2.7}
\]

where the upper dot means the time derivative, \(κ ∈ C\). We comment on these equations in Lemma 4.1 and subsequent remarks.
Lemma 2.1. For any two $M$-operators $M_a$, $M_b$ and corresponding times the expression

$$M_{ab} = \partial_a M_b - \partial_b M_a + [M_a, M_b]$$

is an $M$-operator too.

For all Lie algebras from our list the lemma is proved in [13], except for $\mathfrak{g} = \mathfrak{sp}(2n), \mathfrak{tsp}(2n)$. We reproduce the proof of that work in order to show the place for additional arguments needed in the case $\mathfrak{g} = \mathfrak{tsp}(2n)$. We prove also that $M_{ab}$ satisfies the relation (2.6) which was not needed in the setting of [13].

Proof. Let us verify that $M_{ab}$ satisfies (2.4).

For an arbitrary $g$ from our list we have

$$M_a = \frac{\lambda_a \alpha \alpha t \sigma}{(z-z_\gamma)^2} + \frac{(\alpha \mu_a + \varepsilon \mu_a \alpha t \sigma)}{z - z_\gamma} + M_{0a} + \ldots$$

and similar expression for $M_b$ where $\lambda_a, \lambda_b, \varepsilon$ and $\sigma = id$ are subjected to (2.5).

Next we have

$$\partial_a M_b = 2(\partial_a z_\gamma) \frac{\lambda_b \alpha \alpha t \sigma}{(z-z_\gamma)^3} + \frac{(\partial_a \lambda_b) \alpha \alpha t \sigma + \lambda_b \partial_a (\alpha \alpha t \sigma) + (\partial_a z_\gamma) M_{-1,b}}{(z-z_\gamma)^2} +$$

$$\frac{((\partial_a \mu_b)^t + \varepsilon \mu_b (\partial_a \alpha t \sigma) + \alpha (\partial_a \mu_b) + \varepsilon (\partial_a \mu_b) \alpha t \sigma)}{z - z_\gamma} + \ldots$$

and similar expression for $\partial_b M_a$.

For the commutator we have

$$[M_a, M_b] = \frac{(1 + \varepsilon^2)(\lambda_b \cdot \mu_a \sigma \alpha - \lambda_a \cdot \mu_b \sigma \alpha) \alpha \alpha t \sigma}{(z-z_\gamma)^3} +$$

$$\frac{(\lambda_a \partial_b - \lambda_b \partial_a) \alpha \alpha t \sigma + \lambda_a \alpha \alpha t \sigma + (\mu_a \sigma \alpha) M_{-1,b} - (\mu_b \sigma \alpha) M_{-1,a}}{(z-z_\gamma)^2} +$$

$$\frac{((\partial_a \mu_b)^t + \varepsilon \mu_a (\partial_a \alpha t \sigma) - ((\partial_a \alpha) \mu_b + \varepsilon \mu_b (\partial_a \alpha t \sigma)) \sigma}{z - z_\gamma} +$$

$$\frac{(\alpha \mu_{ab}^t + \varepsilon \mu_{ab} \alpha t \sigma)}{z - z_\gamma} + \ldots$$

where $\lambda_{ab} = 2\lambda_b \kappa_a - 2\lambda_a \kappa_b + \varepsilon(\mu_a^t \sigma \mu_b - \mu_b^t \sigma \mu_a)$, $\mu_{ab} = \kappa_a \mu_b - \kappa_b \mu_a - \lambda_a M_{1b} \alpha + \lambda_b M_{1a} \alpha - M_{0b} \mu_a + M_{0a} \mu_b$. To obtain this relation we used the equations (2.7) and some additional relations, in particular $\varepsilon \alpha^t \sigma \mu = -\varepsilon^2 \mu^t \sigma \alpha$ and $\lambda \alpha^t \sigma \alpha = 0$ which are fulfilled in all cases. In the computation of $[M_a, M_b]_{-2}$ we also used the relation

$$[M_{-1,a}, M_{-1,b}] = (\mu_b^t \sigma \alpha) M_{-1,b} - (\mu_a^t \sigma \alpha) M_{-1,a} + \varepsilon(\mu_a^t \sigma \mu_b - \mu_b^t \sigma \mu_a) \alpha \alpha t \sigma$$
which can be verified using (2.4). To obtain $[M_a, M_b]_{-1}$ in the form (2.9) it is heavily used that $M_i^{t,a} = -\sigma M_{i,a}^{-1}$ for $\varepsilon \neq 0$ (which follows from (2.5)), and the same for $M_{i,b}$.

In the case $g = \mathfrak{sp}(2n)$ we need to prove that $\mu_{ab}$ has a zero last coordinate. This is obviously the case because the matrices $M_{1a}$, $M_{1b}$, $M_{0a}$, $M_{0b}$ have zero last rows.

Comparing (2.9) and (2.8) (and the corresponding relation for $\partial_b M_a$) and using (2.7) we obtain

$$M_{ab} = \frac{\tilde{\lambda}_{ab} \alpha \lambda \sigma}{(z - z_\gamma)^2} + \frac{(\alpha \tilde{\mu}_{ab} + \varepsilon \tilde{\mu}_{ab} \alpha \lambda \sigma)}{z - z_\gamma} + \ldots$$

where $\tilde{\lambda}_{ab} = \partial_a \lambda_b - \partial_b \lambda_a + \lambda_{ab}$, $\tilde{\mu}_{ab} = \partial_a \mu_b - \partial_b \mu_a + \mu_{ab}$. We observe that $M_{ab}$ has the form (2.3), (2.4). In particular the $-3$ order term vanishes because either $\lambda_a = \lambda_b = 0$ or $\varepsilon^2 = 1$ (which follows from (2.5)).

Let us prove that $M_{ab}$ satisfies (2.6) in the symplectic case. We have

$$(\partial_a M_b)_1 = \partial_a M_{1b} - 2(\partial_a z_\gamma) M_{2b}, \quad (\partial_b M_a)_1 = \partial_b M_{1a} - 2(\partial_b z_\gamma) M_{2a}.$$  

Applying $\partial_a$ to both parts of the relation $\alpha^t \sigma M_{1,b} \alpha = 0$, and $\partial_b$ to the corresponding relation for $M_{1,a}$, and using (2.7) we obtain

$$(\alpha^t \sigma (\partial_a M_{b1}) \alpha = -\alpha^t \sigma [M_{0a}, M_{1b}] \alpha, \quad \alpha^t \sigma (\partial_b M_{a1}) \alpha = -\alpha^t \sigma [M_{0b}, M_{1a}] \alpha.$$  

Further on, we have

$$[M_a, M_b]_{1} = \lambda_a [\alpha \alpha^t \sigma, M_{3b}] + [(\alpha \mu_a^t + \mu_a \alpha^t) \sigma, M_{2b}] + [M_{0a}, M_{1b}]$$

$$+ [M_{1a}, M_{0b}] + [M_{a2}, (\alpha \mu_b^t + \mu_b \alpha^t \sigma)] + \lambda_b [M_{3a}, \alpha \alpha^t \sigma].$$

After taking $\alpha^t \sigma [M_a, M_b]_{1} \alpha$ the first and the last commutators in the last relation vanish by $\alpha^t \sigma \alpha = 0$ (as well as some terms of the second and the fifth commutators). The two commutators in the middle annihilate with the commutators on the right hand sides of (2.11). The remaining terms of the second and the fifth commutators give

$$\alpha^t \sigma (\mu_a \alpha^t \sigma M_{2b} - M_{2b} \alpha \mu_a^t \sigma + M_{2a} \alpha \mu_b \lambda \sigma - \mu_b \lambda \sigma M_{2a}) \alpha =$$

$$= 2 \alpha^t \sigma \mu_a (\alpha^t \sigma M_{2b} \alpha) - 2 \alpha^t \sigma \mu_b (\alpha^t \sigma M_{2a} \alpha)$$

which annihilate with the corresponding terms coming from (2.10) due to (2.7). □

3. L-operators and Lax operator algebras

We define $L$-operators as $M$-operators yielding trivial dynamics by (2.7). Thus, by definition, every $L$-operator $L$ is a meromorphic $g$-valued function on $\Sigma$ holomorphic outside $W \cup \{P_+, P_-\}$ such that at a point $\gamma = \gamma_s$

$$(3.1) \quad L = \frac{L_{-2}}{(z - z_\gamma)^2} + \frac{L_{-1}}{z - z_\gamma} + L_0 + \ldots ,$$
where \( z \) is a fixed local coordinate in the neighborhood of \( \gamma \), \( z_\gamma \) is the coordinate of \( \gamma \) itself, \( L_{-2}, L_{-1}, L_0, L_1, \ldots \in \mathfrak{g} \) and

\[
L_{-2} = \nu \alpha \sigma, \quad L_{-1} = (\alpha \beta + \varepsilon \beta \sigma)\sigma
\]

where \( \nu \in \mathbb{C}, \beta \in \mathbb{C}^n, \sigma \) is a \( n \times n \) matrix,

\[
\nu \equiv 0, \quad \varepsilon = 0, \quad \sigma = \text{id} \quad \text{for} \quad \mathfrak{g} = \mathfrak{gl}(n), \mathfrak{sl}(n),
\]

\[
\nu \equiv 0, \quad \varepsilon = -1, \quad \sigma = \text{id} \quad \text{for} \quad \mathfrak{g} = \mathfrak{so}(n),
\]

\[
\varepsilon = 1 \quad \text{for} \quad \mathfrak{g} = \mathfrak{sp}(2n),
\]

and \( \sigma \) is a matrix of the symplectic form for \( \mathfrak{g} = \mathfrak{sp}(2n) \). The case \( \mathfrak{g} = \mathfrak{tsp}(2n) \) is not considered in this section; nevertheless, such consideration is possible.

Further on, the requirement of triviality of the dynamics (2.7) writes as

\[
\beta^t \sigma \alpha = 0, \quad L_0 \alpha = \kappa \alpha.
\]

In addition we assume that

\[
\alpha^t \alpha = 0 \quad \text{for} \quad \mathfrak{g} = \mathfrak{so}(n)
\]

and

\[
\alpha^t \sigma L_1 \alpha = 0 \quad \text{for} \quad \mathfrak{g} = \mathfrak{sp}(2n).
\]

**Theorem 3.1.** The space \( \overline{\mathfrak{g}} \) of L-operators is a Lie algebra under the point-wise matrix commutator. For \( \mathfrak{g} = \mathfrak{gl}(n) \) it is an associative algebra under the point-wise matrix multiplication as well.

The theorem is proven in [8]. Another proof is given in [13] where the theorem is derived from the Lemma 2.1. It can be easy proven also for \( \mathfrak{g} = \mathfrak{tsp}(2n) \). In the last case, making use of the already proven result in the symplectic case, we obtain that given \( L_1, L_2 \in \mathfrak{sp}(2n + 2) \), the commutator \([L_1, L_2]\) belongs to \( \mathfrak{sp}(2n + 2) \) too. If \( L_1, L_2 \in \mathfrak{tsp}(2n) \) at every point then \([L_1, L_2]\) does as well, and according to the proof of the Lemma 2.1 the corresponding \( \beta \) has a zero last coordinate. Hence \([L_1, L_2] \in \mathfrak{tsp}(2n)\).

The Lie algebra \( \overline{\mathfrak{g}} \) is called a Lax operator algebra. It \( \overline{\mathfrak{g}} \) depends both on the choice of Tyurin parameters and of the points \( P_+ \) and \( P_- \) but we omit any indication on this dependence in our notation.

Consider \( \overline{\mathfrak{g}}(n) \) in more detail.

In this case \( L_{-2} = 0, L_{-1} = \alpha \beta \) where \( \beta^t \alpha = 0 \) and \( L_0 \alpha = \kappa \alpha \). These constraints imply that the elements of the Lax operator algebra \( \overline{\mathfrak{gl}}(n) \) can be considered as sections of the endomorphism bundle \( \text{End}(B) \), where \( B \) is the holomorphic vector bundle corresponding to the Tyurin data \( T \).

The splitting \( \mathfrak{gl}(n) = \mathfrak{s}(n) \oplus \mathfrak{s}(n) \) given by

\[
X \mapsto \left( \frac{\text{tr}(X)}{n} I_n, X - \frac{\text{tr}(X)}{n} I_n \right)
\]
(where \( I_n \) is the \( n \times n \) unit matrix) induces a corresponding splitting for \( \mathfrak{gl}(n) \):

\[
(3.8) \quad \mathfrak{gl}(n) = \mathfrak{s}(n) \oplus \mathfrak{s}(n).
\]

For \( \mathfrak{s}(n) \) all coefficients in \( (3.1) \) are scalar matrices. For this reason, the coefficients \( L_{-1} \) vanish for all \( \gamma \in W \), hence the elements of \( \mathfrak{s}(n) \) are holomorphic at \( W \). Also \( L_{s,0} \), as a scalar matrix, has any \( \alpha_s \) as an eigenvector. This means that, by definition,

\[
(3.9) \quad \mathfrak{s}(n) \cong \mathfrak{s}(n) \otimes A \cong A
\]
as associative algebras.

Any Lax operator algebra \( \mathfrak{g} \) possesses an almost-graded structure (see Theorem 3.2 below for the definition).

Assume, all our marked points (including the points in \( W \)) are in generic position, and \( W \neq \emptyset \). Let us choose local coordinates \( z_\pm \) at \( P_\pm \), and \( z_s \) at \( \gamma_s, s = 1, \ldots, K \). Assume \( \mathfrak{g} \) to be a simple Lie algebra from our list. For an arbitrary \( m \in \mathbb{Z} \) consider the subspace

\[
(3.10) \quad \mathfrak{g}_m := \{ L \in \mathfrak{g} | \exists X_+, X_- \in \mathfrak{g} \text{ such that } L(z_+) = X_+ z^m + O(z^{m+1}), L(z_-) = X_- z^{-m-g} + O(z^{-m-g+1}) \}.
\]

For \( \mathfrak{g} = \mathfrak{gl}(n) \) it is proven above that \( \mathfrak{gl}(n) = \mathfrak{s}(n) \oplus A \cdot \text{id} \) where \( A \) is the Krichever-Novikov function algebra. In this case we set

\[
(3.11) \quad \mathfrak{gl}(n)_m = \mathfrak{s}(n)_m \oplus A_m \cdot \text{id}
\]
where \( A_m \) is the corresponding homogeneous subspace for \( A \) \([7]\). If \( W = \emptyset \), we are in the setup of Krichever-Novikov algebras and use the corresponding prescriptions \([7, 11]\).

We call \( \mathfrak{g}_m \) a (homogeneous) subspace of degree \( m \) in \( \mathfrak{g} \).

**Theorem 3.2** \(([8])\). The subspaces \( \mathfrak{g}_m \) give the structure of an almost-graded Lie algebra on \( \mathfrak{g} \). More precisely,

1. \( \dim \mathfrak{g}_m = \dim \mathfrak{g} \);
2. \( \mathfrak{g} = \bigoplus_{m \in \mathbb{Z}} \mathfrak{g}_m \);
3. \( [\mathfrak{g}_m, \mathfrak{g}_k] \subseteq \bigoplus_{h=m+k}^{m+k+M} \mathfrak{g}_h \),

where \( M = g \) for \( \mathfrak{s}(n), \mathfrak{s}(n), \mathfrak{s}(2n) \), \( M = g + 1 \) for \( \mathfrak{gl}(n) \).

**Corollary 3.3.** Let \( X \) be an element of \( \mathfrak{g} \). For each \( m \) there is a unique element \( X_m \) in \( \mathfrak{g}_m \) such that

\[
(3.12) \quad X_m = X z^m + O(z^{m+1}).
\]

**Proof.** From the first statement of Theorem 3.2 i.e. that \( \dim \mathfrak{g}_m = \dim \mathfrak{g} \) it follows that there is a unique combination of the basis elements such that \( (3.12) \) is true. \( \square \)
4. g-valued Lax equations

In this section, we consider consistency of Lax equations of the form
\begin{equation}
L_t = [L, M], \quad L \in \mathfrak{g}, \quad M \in \mathfrak{g}^\circ
\end{equation}
where $L, M$ are an $L$-operator and an $M$-operator respectively, and the correspondence between $\mathfrak{g}$ and $\mathfrak{g}^\circ$ is given as follows:
\begin{equation}
\mathfrak{g}^\circ = \begin{cases} 
\mathfrak{gl}(n) & \text{if } \mathfrak{g} = \mathfrak{gl}(n), \mathfrak{sl}(n) \\
\mathfrak{so}(2n + 1) & \text{if } \mathfrak{g} = \mathfrak{so}(2n), \mathfrak{so}(2n + 1) \\
\mathfrak{tsp}(2n) & \text{if } \mathfrak{g} = \mathfrak{sp}(2n).
\end{cases}
\end{equation}
In all cases $\mathfrak{g}$ is assumed to be embedded into $\mathfrak{g}^\circ$ in a standard way, thus the commutator $[L, M]$ is well-defined.

Following [2] let us assign every effective divisor $D = \sum_i m_i P_i$ on $\Sigma$ with the space $\mathcal{L}^D = \bigcup_{(\alpha, \gamma)} L^D_{\alpha, \gamma}$ (over all Tyurin parameters $(\alpha, \gamma)$ satisfying (3.5)) where
\begin{equation}
L_{\alpha, \gamma}^D = \{ L \in \mathfrak{g}_{\alpha, \gamma} \mid (L) + D \geq 0 \}
\end{equation}
and $\mathfrak{g}_{\alpha, \gamma}$ is the Lax operator algebra corresponding to $(\alpha, \gamma)$.

Under certain conditions given by the Lemma 4.3 below, the Lax equation (4.1) gives a well-defined dynamical system on $\mathcal{L}^D$.

Let the upper dot mean the time derivative, and the term $\gamma$-points be reserved for the points $\gamma_s$.

**Lemma 4.1.** At $\gamma$-points, the equations on main parts of $L$ and $M$ following from (4.1) are fulfilled under the following (sufficient) conditions:
\begin{equation}
\dot{z}_\gamma = -\mu^t \sigma \alpha, \quad \dot{\alpha} = -M_0 \alpha + \kappa \alpha,
\end{equation}
\begin{equation}
\dot{\beta} = M_0^t \beta - L_0^t \mu + \kappa_L \mu - \kappa \beta \text{ for } \mathfrak{g} = \mathfrak{gl}(n), \mathfrak{sl}(n),
\end{equation}
\begin{equation}
\dot{\beta} = -M_0 \beta + L_0 \mu + \kappa_L \mu - \kappa \beta \text{ for } \mathfrak{g} = \mathfrak{so}(n),
\end{equation}
\begin{equation}
\dot{\beta} = -M_0 \beta + L_0 \mu + \kappa_L \mu - \kappa \beta - \nu M_1 \alpha + \lambda L_1 \alpha \text{ for } \mathfrak{g} = \mathfrak{sp}(2n),
\end{equation}
\begin{equation}
\dot{\nu} = 2(\beta^t \sigma \mu + \lambda \kappa_L - \nu \kappa) \text{ for } \mathfrak{g} = \mathfrak{sp}(2n)
\end{equation}
where $\kappa_L$ is defined by $L_0 \alpha = \kappa_L \alpha$. Moreover conditions $\dot{z}_\gamma = -\mu^t \sigma \alpha$ and (4.4) are necessary.

**Proof.** By a straightforward computation we have
\begin{equation}
\dot{L} = 2\dot{z}_\gamma \frac{\nu \alpha \beta^t \sigma}{(z - z_{\gamma})} + \frac{\dot{\nu} \alpha \beta^t \sigma + \nu \dot{\alpha} \beta^t \sigma + \nu \alpha \dot{\beta}^t \sigma + \dot{z}_\gamma (\alpha \beta^t + \varepsilon \beta \alpha^t) \sigma}{(z - z_{\gamma})^2} + \\
+ \frac{\dot{\alpha} \beta^t \sigma + \alpha \dot{\beta}^t \sigma + \varepsilon \dot{\beta} \alpha^t \sigma + \varepsilon \beta \dot{\alpha}^t \sigma}{(z - z_{\gamma})} + (\dot{L}_0 - \dot{z}_\gamma L_1) + \ldots
\end{equation}
Using (2.9) for $M_a = L$, $M_b = M$ we obtain

$$[L, M] = (1 + \varepsilon)^2 \frac{(-\nu \cdot \mu' \sigma \alpha \alpha' \sigma)}{(z - z \gamma)^3} + \frac{\nu(\dot{\alpha} \alpha' + \alpha' \dot{\alpha}) \sigma + \lambda_{ab} \alpha \alpha' \sigma - (\mu' \sigma \alpha)L_{-1}}{(z - z \gamma)^2} + \frac{(\dot{\alpha} \beta' + \varepsilon \beta \dot{\alpha'}) \sigma + (\alpha \mu'_{ab} + \varepsilon \mu_{ab} \alpha') \sigma}{(z - z \gamma)} + \ldots.$$  

Note that the second of the relations (4.3) is used in deriving the last relation.

If $\nu \neq 0$ (i.e. $g = \text{sp}(2n)$) then the order $-3$ terms are equal if and only if

$$\dot{z} \gamma = -\mu' \sigma \alpha.$$  

If $\nu \equiv 0$ then the order $-3$ terms of (4.6) and (4.7) both are equal to 0. The order $-2$ terms are equal if and only if

$$\dot{\nu} \alpha' \sigma + \dot{z} \gamma (\alpha \beta' + \varepsilon \beta \alpha') \sigma = \lambda_{ab} \alpha \alpha' \sigma - (\mu' \sigma \alpha)L_{-1}$$  

where $\lambda_{ab}$ is defined by (2.9). By the previous relation we have

$$\dot{\nu} \alpha' \sigma = \lambda_{ab} \alpha \alpha' \sigma$$  

which is fulfilled if $\dot{\nu} = \lambda_{ab}$. Note that this relation is trivial except for $g = \text{sp}(2n)$, in which case $\lambda_{ab} = 2(\lambda \kappa_L - \nu \kappa + 2 \beta' \sigma \mu$) and our relation coincides with (4.5).

In a similar way, comparing the $-1$ order terms of the relations (4.6) and (4.7) we observe that they are equal if $\dot{\beta} = \mu_{ab}$ where $\mu_{ab}$ are defined by (2.9). This gives the relations (4.4). Since the relations (2.9) themselves are derived under the assumptions (4.3) we obtain the Lemma. □

**Remark.** The equation (4.4) for $g = \text{so}(n)$ follows from the corresponding equation for $gl(n)$ by relations $M_0' = -M_0$, $L_0' = -L_0$. It follows also from the equation for $g = \text{sp}(2n)$, with the corresponding replacement of the matrix $\sigma$, if $\lambda = \nu = 0$ which is true for $g = \text{so}(n)$.

**Remark.** The second condition in (4.3) and conditions (4.4) are not necessary. The statement remains true if we take $\dot{\alpha} = -M_0 \alpha$ in (4.3) and exclude the term $\kappa \beta$ in (4.4).

**Corollary 4.2.** The following relation holds along solutions of the Lax equation in the symplectic case regardless to the requirements (2.6) and (3.6):

$$\nu \cdot \alpha' \sigma M_1 \alpha = \lambda \cdot \alpha' \sigma L_1 \alpha.$$  

**Proof.** Let us multiply the last of the relations (4.4) by $\alpha' \sigma$ from the left:

$$\alpha' \sigma \dot{\beta} = -\alpha' \sigma M_0 \beta + \alpha' \sigma L_0 \mu + \kappa_L \alpha' \sigma \mu - \kappa \alpha' \sigma \beta - \nu \alpha' \sigma M_1 \alpha + \lambda \alpha' \sigma L_1 \alpha$$
and perform the following replacements: \( \alpha^t \sigma M_0 = \dot{\alpha}^t \sigma, \alpha^t \sigma L_0 = -\kappa_L \alpha^t \sigma, \alpha^t \sigma \beta = 0. \)

We will obtain

\[
\nu \alpha^t \sigma M_1 \alpha \sigma - \lambda \alpha^t \sigma L_1 \alpha = \frac{d}{dt} (\alpha^t \sigma \beta) = 0.
\]

The next Lemma shows that the equations (4.4), (4.5) can be thrown off. The equations (4.3) are most important. These are the equations of motion for Tyurin parameters. They are heavily used in [2]. Originally, the concept of moving Tyurin parameters was invented in [4], where it served for effective solution of Kadomtsev-Petviashvili equations in certain cases.

Let \( T_L T^D \) denote the tangent space to \( L^D \) at a point \( L \).

**Lemma 4.3.** \([L, M] \in T_L T^D \iff ([L, M]) + D \geq 0 \) outside \( \gamma 's \) and the equations (4.3) are fulfilled at every \( \gamma \).

**Proof.** In our proof we follow the lines of [2] where the Lemma was formulated and proved for \( g = gl(n) \).

Let \( z \) be a local coordinate in an open set containing a point \( \gamma \), and \( z_\gamma \) be the corresponding coordinate of \( \gamma \).

Identify \( T_L T^D \) with the space \( T^D \) of all meromorphic \( g \)-valued functions \( T \) such that at every \( \gamma \)

\[
T = 2z_\gamma \frac{\nu \alpha^t \sigma}{(z - z_\gamma)^3} + \frac{\dot{\nu} \alpha^t \sigma + \nu (\dot{\alpha} \alpha^t + a \dot{\alpha}^t) \sigma}{(z - z_\gamma)^2} + \frac{\dot{z}_\gamma (\alpha \beta^t + \epsilon \beta \alpha^t) \sigma}{z - z_\gamma} + T_0 + \ldots,
\]

(4.8)

(4.9)

\[
\dot{\alpha}^t \sigma \beta + \alpha^t \sigma \dot{\beta} = 0
\]

(4.10)

\[
 T_0 \alpha = \kappa \dot{\alpha} + \dot{\kappa} \alpha - L_0 \dot{\alpha} - \dot{z}_\gamma L_1 \alpha
\]

where \( \dot{z}_\gamma, \dot{\kappa} \) are constants, \( \dot{\alpha}, \dot{\beta} \) are constant vectors fulfilling the relations (4.3), and the divisor of \( T \) outside the points \( \gamma \) is greater or equal to \( -D \).

The relation (4.3) is modelled on (4.6) which is obtained by the time derivation of (3.1). In particular

\[
 T_0 = \dot{L}_0 - \dot{z}_\gamma L_1.
\]

Together with the time derivation of (3.4) this gives (4.10). Thus \( T_L T^D \) embeds to \( T^D \). Let us check coincidence of dimensions of those spaces. It can be done in a quite uniform way, so we show it in the most difficult case \( g = sp(2n) \). We have \( (T) + \tilde{D} \geq 0 \) where \( \tilde{D} = D + 3 \sum \gamma \), and \( \deg \tilde{D} = \deg D + 3K \). By Riemann-Roch theorem \( \dim \{ T \mid (T) + \tilde{D} \geq 0 \} = (\dim g)(\deg D + 3K - g + 1) \). The elements of \( T^D \)
are distinguished in the space \( \{ T| (T) + \tilde{D} \geq 0 \} \) by the following relations. First at every point \( \gamma \) we have

\[
\begin{align*}
T_{-3} &= 2 \dot{z}_\gamma \nu \alpha \alpha^t \sigma , \\
T_{-2} &= \dot{\sigma} + (\dot{\alpha} \alpha^t + \alpha \dot{\alpha}^t + \alpha \beta \sigma ) , \\
T_{-1} &= (\dot{\alpha} \beta^t + \dot{\alpha} \beta^t + \dot{\beta} \alpha^t + \beta \dot{\alpha}^t) .
\end{align*}
\]

(4.11)

Since the elements on the left hand side belong to \( g \), (4.11) gives \( 3 \dim g \) relations. Taking account of (4.9) and (4.10) gives \( 2n + 1 \) relations, and (4.3) give another \( 2n + 1 \) relations. Thus we have \( 3 \dim g + 4n + 2 \) relations. These relations contain \( 4n + 2 \) free parameters \( \dot{\gamma} \), \( \dot{\nu} \), \( \dot{\alpha} \), \( \dot{\beta} \). Thus we actually obtained \( 3 \dim g + 4n + 2 \) relations at every point \( \gamma \), and the number of those points is \( K \), hence we have \( 3 \dim (\dim g)(deg D - g + 1) \).

But \( T_D \) has the same dimension. We can count this in a quite similar way or make use of the Theorem 3.2. Assume, we are in the two-point situation, i.e. \( D = -m_+ P_+ + (m_- + g) P_- \) where \( m_- > m_+ \) for simplicity. Then \( T_D = g_{m_+} \oplus \ldots \oplus g_{m_-} \). By Theorem 3.2 \( \dim T_D = (\dim g)(m_- - m_+ + 1) \) which exactly is equal to \( (\dim g)(deg D - g + 1) \). We conclude that \( \dim T_D = \dim L D \), hence these linear spaces coincide.

Next we prove that if \( L, M \) are as above then \([L,M]\) possesses the properties (4.8)–(4.10), i.e. belongs to \( T_D \). The proof is straightforward again. For example, show (4.10). Denote the degree zero term \([L,M]_0\) of the commutator by \( T_0 \). Then in the case \( g = gl(n) \) we find by a computation

\[
T_0 \alpha = \alpha (\beta^t M_1 \alpha - \mu^t L_1 \alpha) + (L_0 - \kappa) M_0 \alpha + L_1 \alpha (\mu^t \alpha) .
\]

(4.12)

If we replace \( \mu^t \alpha \) with \(- \dot{\gamma} \), \( M_0 \alpha \) with \(- \dot{\alpha} \) and denote \( \beta^t M_1 - \mu^t L_1 \alpha \) by \( \kappa \), we obtain (4.10). For other types of \( g \) the expression for \( T_0 \alpha \) is more complicated and we use the relations (3.3)–(3.6) to identify it with (4.10).

In the case \( g = sp(2n) \) we get

\[
T_0 \alpha = \alpha (\nu \alpha^t \sigma M_2 \alpha + \beta^t \sigma M_1 \alpha - \mu \alpha^t \sigma L_2 \alpha + (L_0 - \kappa) M_0 \alpha + L_1 \alpha (\mu^t \sigma \alpha) + \beta \alpha^t \sigma M_1 \alpha - \mu \alpha^t \sigma L_1 \alpha
\]

instead (4.12). Making use of the relations \( \alpha^t \sigma M_1 \alpha = 0, \alpha^t \sigma L_1 \alpha = 0 \) we obtain the same result. \( \square \)

Lemma 4.3 directly implies that whenever \([L,M] + D \geq 0\) outside \( \gamma \)'s, and the equations of moving poles are fulfilled the Lax equation (4.1) is consistent.
5. Hierarchies of commuting flows

In the case of \( \mathfrak{g} = \mathfrak{gl}(n) \) I. Krichever [2] has shown that the Lax operator considered as a function on \( L^D \) yields the hierarchy of commuting flows on \( L^D \). The generalization of that result on the classic Lie algebras appeared to be not unique.

In this article, we present two methods of constructing hierarchies of commuting flows given by Lax operators. They differ by the space of \( M \)-operators and the number of \( \gamma \)-points. By the first method we obtain below (following [2]) the \( A_n \) elliptic Calogero-Moser system. By the second method we obtain the \( C_n \) and \( D_n \) elliptic Calogero-Moser systems.

The first method is as follows. For a divisor \( D = \sum m_i P_i \) define a divisor \( \tilde{D} = D + \delta \sum_{s=1}^{K} \gamma_s \) where

\[
K = \begin{cases} 
ng, & \mathfrak{g} = \mathfrak{gl}(n), \mathfrak{sl}(n), \mathfrak{so}(2n), \mathfrak{so}(2n+1), \\
(n+1)g, & \mathfrak{g} = \mathfrak{sp}(2n)
\end{cases}
\]

and

\[
\delta = \begin{cases} 
1, & \mathfrak{g} = \mathfrak{gl}(n), \mathfrak{sl}(n), \mathfrak{so}(2n), \mathfrak{so}(2n+1), \\
2, & \mathfrak{g} = \mathfrak{sp}(2n),
\end{cases}
\]

(\( K \) and \( \delta \) depend on \( \mathfrak{g}^o \) actually).

Let us define \( \mathcal{N}^D \subset \mathfrak{g}^o \) as a subspace of \( M \)-operators such that \( (M) + \tilde{D} \geq 0 \).

**Lemma 5.1.** \( \dim \mathcal{N}^D = (\dim \mathfrak{g}^o)(\deg D + 1) \).

**Proof.** We compute \( \dim \mathcal{N}^D \) by Riemann-Roch theorem taking account of additional relations at the points \( \gamma \). Those are relations determining \( M_{-2} \), \( M_{-1} \) and \( M_1 \) (the latter only in the case \( \mathfrak{g} = \mathfrak{sp}(2n) \)). The number of those relations at every point \( \gamma \) is equal to \( \delta \dim \mathfrak{g}^o \) if \( \mathfrak{g} \neq \mathfrak{sp}(2n) \), and is equal to \( \delta \dim \mathfrak{g}^o + 1 \) in the last case. We write down that number in the form \( \delta \dim \mathfrak{g}^o + r_{sp} \) where \( r_{sp} = 1 \) for \( \mathfrak{g} = \mathfrak{sp}(2n) \) and \( r_{sp} = 0 \) otherwise. We also have free parameters \( \mu, \lambda \) (\( \lambda \) appears only in the case \( \mathfrak{g} = \mathfrak{sp}(2n) \)). Let \( r \) be the number of those parameters for a fixed \( \gamma \). We can think that at every \( \gamma \) there are \( \delta \dim \mathfrak{g} - r + r_{sp} \) relations.

Let us write \( K \) in the form \( K = lg \) where \( l \) is equal to \( n \) or to \( n + 1 \) depending on \( \mathfrak{g}^o \). We have

\[
\dim N^D = (\dim \mathfrak{g}^o)(\deg D + \delta lg - g + 1) - (\delta \dim \mathfrak{g}^o - r + r_{sp})lg
\]

(5.1)

\[
= (\dim \mathfrak{g}^o)(\deg D + 1) - (\dim \mathfrak{g}^o - (r - r_{sp})l)g.
\]

Next verify that

(5.2) \( \dim \mathfrak{g}^o = (r - r_{sp})l. \)

Indeed, for \( \mathfrak{g}^o = \mathfrak{gl}(n) \) we have \( r = l = n, r_{sp} = 0 \) hence \( (r - r_{sp})l = n^2 \). If \( \mathfrak{g}^o = \mathfrak{so}(2n+1) \) then \( r = 2n + 1, l = n, r_{sp} = 0 \) and \( (r - r_{sp})l = (2n + 1)n. \) At last,
if \( \mathfrak{g}^\circ = \mathfrak{tsp}(2n) \) then \( \dim \mathfrak{g}^\circ = n(2n+1) + (2n+1) = (2n+1)(n+1) \) (the sum of the dimensions of \( \mathfrak{sp}(2n) \) and the Heisenberg algebra, see page 5). We have \( r = 2n + 2 \) in this case (2n + 1 free parameters come from \( \mu \) and 1 corresponds to \( \lambda \)). Hence \( r - r_{\mathfrak{sp}} = 2n + 1 \) and \( (r - r_{\mathfrak{sp}})l = (2n + 1)(n+1) \).

In all cases (5.2) is true. □

Following [2] let us fix a point \( P_0 \in \Sigma \) and local coordinates \( w_0, w_i \) in the neighborhoods of the points \( P_0, P_i \). Our next goal is to define gauge invariant functions \( M_a \) that satisfy the assumptions of Lemma 4.3. Let us define \( a \) as a triple

\[
(5.3) \quad a = (P_i, k, m), \quad k > 0, \quad m > -m_i,
\]

where \( k, m \) are integers, \( k \equiv 1(\text{mod 2}) \) for \( \mathfrak{g} = \mathfrak{so}(n) \) and \( \mathfrak{g} = \mathfrak{sp}(2n) \).

By Lemma 5.1 for generic \( L \in \mathfrak{g} \) there is a unique \( \mathfrak{g}^\circ \)-valued meromorphic function \( M_a \) such that

- (i) \( M_a \) is an \( M \)-operator;
- (ii) outside the points \( \gamma \) it has pole at the point \( P_i \) only, and

\[
M_a(q) = w_i^{-m}L^n(q) + O(1),
\]

i.e. singular parts of \( M_a \) and \( w_i^{-m}L^n \) coincide;
- (iii) \( M_a \) is normalized by the condition \( M_a(P_0) = 0 \).

Theorem 5.2. The equations

\[
(5.4) \quad \partial_a L = [L, M_a], \quad \partial_a = \partial/\partial t_a
\]

define a hierarchy of commuting flows on an open set of \( \mathcal{L}^D \).

For \( \mathfrak{g} = \mathfrak{gl}(n) \) the theorem is formulated and proved in [2]. For \( \mathfrak{g} = \mathfrak{so}(2n), \mathfrak{so}(2n+1) \) it is proved in [13] under slightly different assumptions. Here we formulate and prove the theorem for all classical Lie algebras in question including \( \mathfrak{g} = \mathfrak{sp}(2n) \).

Proof. It follows from (ii) that \( ([L, M_a]) + D \geq 0 \), hence by Lemma 4.3 \([L, M_a] \in T_L \mathcal{L}^D \) and the equation \( \partial_a L = [L, M_a] \) defines a flow on \( \mathcal{L}^D \).

To prove commutativity of such flows it is sufficient to verify that \( M_{ab} = \partial_a M_b - \partial_b M_a + [M_a, M_b] = 0 \) identically. By Lemma 2.1 \( M_{ab} \) is an \( M \)-operator. Below, we prove that this \( M \)-operator is regular at the points of the divisor \( D \). By Lemma 5.1 the space of such operators has the same dimension as \( \mathfrak{g}^\circ \). Due to (iii) we obtain \( M_{ab} = 0 \).

Let us prove that \( M_{ab} \) is regular at the points of the divisor \( D \). We repeat here the corresponding part of the proof of [2] Theorem 2.1. First assume that indices \( a, b \) correspond to the same point \( P_i \), i.e. \( a = (P_i, n, m), \ b = (P_i, n', m') \). Denote \( M_a - w^{-m}L^n \) by \( M_a^- \) and \( M_b - w^{-m'}L^{n'} \) by \( M_b^- \), then by (ii) \( M_a^- \) and \( M_b^- \) are regular.
in the neighborhood of $P_i$. We have
\[
\partial_a M_b = w^{-m'} \partial_a L^{n'} + \partial_a M_b^- = w^{-m'} [L^{n'}, M_a] + \partial_a M_b^-
\]
and
\[
[M_a, M_b] = [M_a^- + w^{-m} L^n, M_b^- + w^{-m'} L^{n'}]
\]
\[= w^{-m} [L^n, M_b^-] - w^{-m'} [L^{n'}, M_a^-] + [M_a^-, M_b^-].
\]
Hence $M_{ab} = \partial_a M_b^- - \partial_b M_a^- + [M_a^-, M_b^-]$ at the point $P_i$, which is a regular expression at that point. By definition $M_{ab}$ is regular also at the other points of $D$.

The proof is similar in the case when $a$ and $b$ correspond to the different points of $D$. □

6. SYMPLECTIC STRUCTURE

Following the lines of [2] we introduce here a symplectic structure on a certain subspace $\frak{P}^D \subset \frak{L}^D/G$ where $G = \exp \frak{g}$. We call it Krichever-Phong symplectic structure.

Let $\Psi$ be the matrix formed by the canonically normalized left eigenvectors of $L$ (we consider a vector $\psi$ to be canonically normalized if $\sum \psi_i = 1$). It is defined modulo permutations of its rows. We consider $L$ and $\Psi$ as matrix-valued functions on $\frak{L}^D$. Let $\delta L$ and $\delta \Psi$ denote their external derivatives which are 1-forms on $\frak{L}^D$.

In the same way we consider the diagonal matrix $K$ defined by
\[
\Psi L = K \Psi,
\]
i.e. formed by the eigenvalues of $L$, and the matrix-valued 1-form $\delta K$. Let $\Omega$ be a 2-form on $\frak{L}^D$ with values in the space of meromorphic functions on $\Sigma$ defined by the relation
\[
\Omega = \text{tr}(\delta \Psi \wedge \delta L \cdot \Psi^{-1} - \delta K \wedge \delta \Psi \cdot \Psi^{-1}).
\]
$\Omega$ does not depend on the order of the eigenvalues, hence it is well-defined on $\frak{L}$.

Fix a holomorphic differential $dz$ on $\Sigma$ and define a scalar-valued 2-form $\omega$ on $\frak{L}^D$ by the relation
\[
\omega = -\frac{1}{2} \left( \sum_{s=1}^{K} \text{res}_{\gamma_s} \Omega dz + \sum_{P_i \in D} \Omega dz \right).
\]
There is another representation for $\Omega$:
\[
\Omega = 2 \delta \text{tr} (\delta \Psi \cdot \Psi^{-1} K)
\]
which implies that $\omega$ is apparently closed. First we want prove that it is nondegenerate when restricted to the space of Tyurin parameters, i.e $\omega$ yields a symplectic form on this space. We will point out a canonical form of that restriction.
Lemma 6.1. The restriction of $\omega$ to the space of Tyurin parameters is of the form

$$\omega_0 = \sum_{s=1}^{K} (a \delta z_s \wedge \delta \kappa_s + \delta \beta_s^t \wedge \delta \alpha_s)$$

where $a = 1$ for $g = \mathfrak{gl}(n)$, $a = 2$ for $g = \mathfrak{so}(n)$ and $g = \mathfrak{sp}(2n)$.

Proof. For $g = \mathfrak{gl}(n)$ the corresponding statement is contained in [2, Lemma 4.3]. It is instructive to reproduce the proof here. Let $g_s$ be a constant nondegenerate matrix such that $g_s^{-1} \alpha = e_1$ where $e_1 = (1,0,\ldots,0)$. Then for $L'_s = g_s^{-1} L_s g_s$ we have $(L'_s)_{-1} = e_1 f^t$ where $f^t = \beta^t g_s$, and $f_1 = 0$ since $f^t e_1 = 0$. Apparently, only the entries of the first row of the matrix $L'_{s,-1}$ are nonzero, but $(L'_{s,-1})_{11} = 0$.

The $e_1$ is an eigenvector for $L_{s0}$ with the eigenvalue $\kappa$. For that reason $(L_{s0})^{11} = \kappa$, $(L_{s0})^{i1} = 0 (i > 1)$.

It follows from those remarks that the conjugation by the matrix $f_s = \text{diag}(z-z_s,1,\ldots,1)$ takes $(z-z_s)^{-1} L_{-1}$ and $L_0$ to holomorphic matrix-valued functions. Hence the same is true for $L'_s$, and the lemma is proven with $\Phi_s = f_s g_s$. Let us note for the future that $f_s$ is diagonal.

Let us watch now for the transformations of $\Omega$ corresponding to the just performed transformations of $L$.

Under the gauge transformation $L' = g^{-1} L g$, $\Psi' = \Psi g$ the form $\Omega$ transforms to $\Omega'$ where $\Omega' = \Omega - 2 \text{tr}(\delta L \wedge \delta g g^{-1} + L \delta g g^{-1} \wedge \delta gg^{-1})$.

After the first of the above transformation (by the matrix $g_s$) we obtain

$$\text{res}_{\gamma_s} \Omega' dz = \text{res}_{\gamma_s} \Omega dz - 2 \text{res}_{\gamma_s} \text{tr}(\delta L \wedge \delta g_s g_s^{-1} + L \delta g_s g_s^{-1} \wedge \delta g_s g_s^{-1})$$

Since $g_s$ is constant we have

$$\text{res}_{\gamma_s} \Omega' dz = \text{res}_{\gamma_s} \Omega dz - 2 \text{res}_{\gamma_s} \text{tr}(\delta \alpha_s \cdot \beta_s^t + \alpha_s \cdot \delta \beta_s^t) \wedge \delta g_s g_s^{-1} + \alpha_s \beta_s^t \delta g_s g_s^{-1} \wedge \delta g_s g_s^{-1})$$

By differentiating the relation $g_s^{-1} \alpha = e_1$ we obtain $\delta \alpha = \delta g_s g_s^{-1} \alpha$. Substituting that to the previous relation we obtain

$$\text{res}_{\gamma_s} \Omega' dz = \text{res}_{\gamma_s} \Omega dz + 2 \text{tr}(\delta \alpha_s \wedge \delta \beta_s^t).$$

The matrix $L'$ becomes holomorphic under the transformation $\tilde{L}_s = f L' f^{-1}$, hence

$$0 = \text{res}_{\gamma_s} \Omega' dz + 2 \text{res}_{\gamma_s} \text{tr}(\delta L \wedge f_s^{-1} \delta f_s + L \delta f_s^{-1} f_s \wedge \delta f_s^{-1} f_s).$$

As $f$ is diagonal the last term vanishes. Making use of the above obtained special form of $L'$ and $f_s$ we conclude that

$$\text{res}_{\gamma_s} \Omega' dz = 2 \delta z_s \wedge \delta \kappa_s.$$ 

Thus the contribution of the point $\gamma_s$ to $\Omega$ is equal to

$$\text{res}_{\gamma_s} \Omega = -2 \text{tr}(\delta \alpha_s \wedge \delta \beta_s^t) - 2 \delta z_s \wedge \delta \kappa_s.$$
and the corresponding contribution to $\omega$ is equal to

$$\omega_s = -\frac{1}{2} \text{res}_{\gamma_s} \Omega = \delta \beta_s^t \wedge \delta \alpha_s + \delta \z_s \wedge \delta \kappa_s.$$  

The proof for $\mathfrak{so}(n)$ and $\mathfrak{sp}(2n)$ is more complicated but similar. In the symplectic case we choose

$$\sigma = \begin{pmatrix} \sigma' & 0 \\ 0 & \sigma'' \end{pmatrix}$$

where $\sigma' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

as a matrix giving the symplectic form.

We can send $\alpha$ to any vector by a nondegenerate matrix $g$ so that $\sigma$ is invariant, i.e. $g$ is symplectic. Let us send $\alpha$ to $e_1$ where $e_1^t = (1, 0, \ldots, 0)^t$. Then

$$\alpha \alpha^t \sigma = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix}, \quad (\alpha \beta^t + \beta \alpha^t)\sigma = \begin{pmatrix} -\beta_2 & 2\beta_1^* & \ldots & \ast \\ 0 & \beta_2 & 0 & \ldots & 0 \\ 0 & \beta_3 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \beta_{2n} & 0 & \ldots & 0 \end{pmatrix}$$

where stars denote constants linearly depending on $\beta$. Observe that by $\beta^t \sigma \alpha = 0$ we have $\beta_2 = 0$.

It is easy to check that conjugation by $f = \begin{pmatrix} Z & 0 \\ 0 & E \end{pmatrix}$, where $Z = \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}$, multiplies the only nonzero entry of $\alpha \alpha^t \sigma$ by $z^2$, and the matrix $(\alpha \beta^t + \beta \alpha^t)\sigma$ by $z$ (except for the entry $2\beta_1$ multiplied by $z^2$). Hence $f \cdot (\frac{\alpha \alpha^t \sigma}{z^2} + \frac{\alpha \beta^t + \beta \alpha^t}{z}) \cdot f^{-1}$ is holomorphic.

The matrix $L_0$ is similar to $(\alpha \beta^t + \beta \alpha^t)\sigma$ by structure. Its first column is equal to $(\kappa, 0, \ldots, 0)^t$ by $L_0 e_1 = \kappa e_1$. Its second row is equal to $(0, -\kappa, \ldots, 0)^t$ by $(e_1^t \sigma)L_0 = -\kappa(e_1^t \sigma)$ (which follows from $L_0^2 \sigma + \sigma L_0 = 0$). Nonzero entries of $L_0$ are multiplied by positive degrees of $z$ under the conjugation by $f$, except for the right lower $(2n-2) \times (2n-2)$ corner block which is invariant. Hence $f L_0 f^{-1}$ is holomorphic.

Thus, a singularity could only come from $f \cdot (z L_1) f^{-1}$ because $(L_1)_{21}$ is multiplied by $z^{-2}$. But $\alpha^t \sigma L_1 \alpha = 0$ implies $(L_1)_{21} = 0$. All other entries of $f \cdot L_1 f^{-1}$ are of order $z^{-1}$ and more, hence multiplied by $z$ become holomorphic.

The matrix $\delta f \cdot f^{-1}$ has null entries except for the left upper $2 \times 2$ corner block which is equal to $\begin{pmatrix} \delta z \cdot z^{-1} & 0 \\ 0 & -\delta z \cdot z^{-1} \end{pmatrix}$. As it follows from what was said of the structure of $L_0$ the corresponding block of $\delta L_0$ is equal to $\begin{pmatrix} \delta \kappa & \ast \\ 0 & -\delta \kappa \end{pmatrix}$. Thus, at a point $\gamma_s$, we have the contribution $\delta \beta_s^t \wedge \delta \alpha_s$ from the gauge transformation corresponding to $g$, and $2 \delta \z_s \wedge \delta \gamma_s$ from the gauge transformation corresponding to $f$. The total contribution of a point $\gamma_s$ to $\omega$ is equal to $2 \delta \z_s \wedge \delta \kappa_s + \delta \beta_s^t \wedge \delta \alpha_s$. 


In the orthogonal case we must additionally satisfy the relation \( e_1^t \sigma e_1 = 0 \). Consider \( g = \mathfrak{so}(2n) \) for example. Let us choose \( \sigma \) as in the symplectic case but take \( \sigma' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) and assume \( \sigma'' \) to be positive definite. Then \( e_1 \) (where \( e_1^t = (1, 0, \ldots, 0)^t \)) satisfies the required relation and we proceed as in the symplectic case, with the almost only difference that the left upper corner \( 2 \times 2 \) block is of the form \( \begin{pmatrix} \beta_2 & * \\ 0 & -\beta_2 \end{pmatrix} \). This difference is technical and does not affect the result.

Let us consider now the contribution of the points \( P_1, \ldots, P_N \). Let \( \omega_m = -\frac{1}{2} \text{res}_{P_m} \Omega dz \). Define \( \mathcal{P}_0^D \) as a subspace in \( \mathcal{L}^D \) where the 1-form \( \delta \kappa dz \) is holomorphic. It is the same as the set of common zeroes of the following functions on \( \mathcal{L}^D \):

\[
T_{ijl} = \text{res}_{P_l} ((z - z(P_i))^j k dz), \quad j = 0, \ldots, (m_i - d_i),
\]

where \( l \) enumerates sheets of the spectral curve (as a branch cover of \( \Sigma \)), \( d_i = \text{ord}_{P_i} dz \). Let us notice that the foliation given by the common level sets of those functions is invariant with respect to the flows of our commuting hierarchy since those preserve the spectrum \( k \) (see also the next Section).

**Lemma 6.2.** On the space \( \mathcal{P}_0^D \)

\[
\omega_m = \text{res}_{P_m} \text{tr}(L \Psi^{-1} \delta \Psi \wedge \Psi^{-1} \delta \Psi) dz.
\]

**Proof.** By differentiating the relation \( \Psi L = K \Psi \) exclude \( \delta L \) in the definition of \( \Omega \). We will obtain

\[
\omega_m = \text{res}_{P_m} \text{tr}(L \Psi^{-1} \delta \Psi \wedge \Psi^{-1} \delta \Psi - \delta K \wedge \delta \Psi \cdot \Psi^{-1} + \frac{1}{2} K \delta \Psi \cdot \Psi^{-1} \wedge \delta \Psi \cdot \Psi^{-1}) dz.
\]

That \( K \) is diagonal implies \( \text{tr}(K \delta \Psi \cdot \Psi^{-1} \wedge \delta \Psi \cdot \Psi^{-1}) = 0 \), thus the third summand in the expression for \( \omega_m \) vanishes. The second summand is holomorphic on the space \( \mathcal{P}_0^D \), because such is \( \delta K dz \), and the form \( \delta \Psi \cdot \Psi^{-1} \) is holomorphic always since \( \delta \Psi \) and \( \Psi \) have the same order at \( P_m \) (the point \( P_m \) is immovable and there is no variation in the local coordinate on \( \Sigma \) while taking \( \delta \Psi \)).

Let \( \xi \) be a tangent vector to \( \mathcal{P}_0^D \) at a point \( L \). It is a \( g \)-valued meromorphic function on \( \Sigma \), and \( \delta \Psi(\xi) \cdot \Psi^{-1} = \xi \). Hence for any pair \( \xi, \eta \) of tangent vectors we have \( \omega_m(\xi, \eta) = \text{res}_{P_m} \text{tr}(L[\xi, \eta]) \). This is a 2-form of Kirillov type. In general it is degenerate and has symplectic leaves. In particular in the case when \( Ldz \) has a simple pole at \( P_m \) with the residue \( L_m \) the \( \omega_m \) descends to the canonical Kirillov form on the orbit \( O_m \) of the element \( L_m \in g \). We omit details and refer to the full analogy with \([2]\) with this respect.
Since $\omega$ is $G$-invariant it is actually defined on $\mathcal{P}^D = \mathcal{P}_0^D / G$. Together with what has been already proven in this section we have the following statement quite similar to the [2, Theorem 4.1].

**Theorem 6.3.** The form

$$
\omega = \sum_s \left( a \delta \kappa_s \wedge \delta z_s + \delta \alpha_s \wedge \delta \beta_s \right) + \sum_m \omega_m.
$$

is nondegenerate on $\mathcal{P}^D$. Thus it gives a symplectic structure on $\mathcal{P}^D$.

7. Hamiltonian theory

Following the lines of [2] we show here that hierarchies in the Theorem 5.2 are Hamiltonian for all classical Lie algebras in question, and compute the corresponding Hamiltonians.

For a vector field $e$ on $\mathcal{L}^D$ let $i_e \omega$ be the 1-form defined by $i_e \omega(X) = \omega(e, X)$ (where $X$ is an arbitrary vector field). By definition, a vector field $\partial_t$ is Hamiltonian if $i_{\partial_t} \omega = \delta H$ where $H$ is some function called the Hamiltonian of $\partial_t$.

**Theorem 7.1.** Let $\partial_a$ be a vector field defined by (5.4). Then

$$i_{\partial_a} \omega = \delta H_a$$

where

$$H_a = -\frac{1}{k+1} \text{res}_\gamma \text{tr}(w^{-m}L^{k+1})dz, \quad a = (P, k, m).$$

Before we start proving the theorem let us consider some properties of $L$, $\partial_a + M_a$ and their spectra. First of all these operators commute due to the Lax equation, hence the rows of $\Psi$ are eigenvectors for both of them. The diagonal forms of those two operators are defined as follows:

$$K = \Psi L \Psi^{-1}, \quad F_a = \Psi(\partial_a + M_a) \Psi^{-1}.$$  

Equivalently

$$\Psi L = K \Psi, \quad \partial_a \Psi = \Psi M_a - F_a \Psi.$$  

These relations imply

$$\partial_a K = 0.$$  

As shown above $L$ is conjugated to a function holomorphic at the $\gamma$-points, hence its spectrum $K$ is also holomorphic there. But $L$ itself is singular, hence for $L$ in a generic position det $\Psi$ has a (simple) zero at every $\gamma$-point. Assume

$$\Psi(z - z_\gamma) = \Psi_0 + \Psi_1(z - z_\gamma) + \ldots, \quad \Psi^{-1}(z - z_\gamma) = \frac{\Psi^{-1}_0}{(z - z_\gamma)} + \tilde{\Psi}_0 + \ldots.$$
Then in a generic position the holomorphy of $\Psi L$ and the relations $\Psi \Psi^{-1} = id$ are equivalent to the following relations

\begin{equation}
(7.4) \quad \Psi_0 \alpha = 0, \quad \epsilon \alpha^t \sigma \Psi_0 = 0, \quad \bar{\Psi}_{-1} = \alpha \bar{\beta}^t.
\end{equation}

Observe that if $\epsilon = 0$ (i.e. $g = g(n)$) then $\nu = \lambda = 0$. Hence $\nu \alpha^t \sigma \bar{\Psi}_0 = \lambda \alpha^t \sigma \bar{\Psi}_0 = 0$ as well. For the same reason $\epsilon \alpha^t \sigma \alpha = \nu \alpha^t \sigma \alpha = \lambda \alpha^t \sigma \alpha = 0$.

**Remark.** In the presence of antiinvolution $\Psi$ is always holomorphic at the $\gamma$-points.

Indeed, if $\Psi$ has a pole at a point $\gamma$, let us operate by the antiinvolution on the above relation. We obtain $\sigma (\Psi^t)^{-1} (\sigma L^t \sigma^{-1}) (\sigma \Psi^t \sigma^{-1}) = \sigma K^t \sigma^{-1}$. By replacing $\sigma L^t \sigma^{-1}$ with $-L$ we obtain $\sigma (\Psi^t)^{-1} L (\sigma \Psi^t) = -K^t$. In the last relation, $-K^t$ still is diagonal, and the matrix to the left of $L$ is holomorphic.

**Remark.** If $L$ has a second order pole at a $\gamma$, the relations $\Psi_0 \alpha = 0$ and $\alpha^t \sigma \bar{\Psi}_0 = 0$ hold always at the $\gamma$. Indeed, by $\Psi L = K \Psi$ we have

\begin{equation}
(\Psi_0 + \Psi_1 (z - z_\gamma) + \ldots) \left( \nu \frac{\alpha \alpha^t \sigma}{(z - z_\gamma)^2} + \ldots \right) = O(1).
\end{equation}

Hence $\Psi_0 \alpha \alpha^t \sigma = 0$. Since $\alpha \neq 0$ and $\sigma$ is a nondegenerate quadratic form, we have $\Psi_0 \alpha = 0$.

Consider now the same relation in the form $L \Psi^{-1} = K \Psi^{-1}$. For the left hand side we obtain by computation $L \Psi^{-1} = \nu \frac{\alpha^t \Psi_0}{(z - z_\gamma)^2} + O((z - z_\gamma)^{-1})$ while the right hand side has the pole of order $-1$, at most. As above, we conclude that $\alpha^t \sigma \bar{\Psi}_0 = 0$.

**Lemma 7.2.** The matrix-valued functions $K$ and $F_a$ are holomorphic at all $\gamma$-points provided (7.4) hold there.

**Proof.**

\begin{align*}
\Psi L \Psi^{-1} &= \frac{\nu \Psi_0 \alpha \alpha^t \sigma \bar{\Psi}_{-1}}{(z - z_\gamma)^3} + \frac{\nu \Psi_0 \alpha \alpha^t \sigma \bar{\Psi}_0 + \Psi_0 (\alpha \beta^t + \epsilon \beta \alpha^t) \sigma \bar{\Psi}_{-1} + \nu \Psi_1 \alpha \alpha^t \sigma \bar{\Psi}_{-1}}{(z - z_\gamma)^2} \\
&\quad + \frac{\nu \Psi_0 \alpha \alpha^t \sigma \bar{\Psi}_1 + \Psi_0 (\alpha \beta^t + \epsilon \beta \alpha^t) \sigma \bar{\Psi}_0 + \Psi_0 L_0 \bar{\Psi}_{-1} + \nu \Psi_1 \alpha \alpha^t \sigma \bar{\Psi}_0}{z - z_\gamma} \\
&\quad + \frac{\Psi_1 (\alpha \beta^t + \epsilon \beta \alpha^t) \sigma \bar{\Psi}_{-1} + \nu \Psi_2 \alpha \alpha^t \sigma \bar{\Psi}_{-1}}{z - z_\gamma} + O(1).
\end{align*}

The singular part of that expression obviously vanishes under the conditions (7.4).

Let us prove now holomorphy of the spectrum of $\partial_a + M_a$. Indeed, this spectrum is expressed by the matrix $F_a = \Psi (\partial_a + M_a) \Psi^{-1}$. Using the expansions for $M_a$ and
\(\Psi^{-1}\) at \(\gamma\) we obtain
\[
(\partial_a + M_a)\Psi^{-1} = \frac{(\lambda \alpha \alpha' \sigma)(\alpha \hat{\beta}')}{(z - z_\gamma)^3} + \frac{(\partial_a z_\gamma)(\alpha \hat{\beta}')}{(z - z_\gamma)^2} + \frac{(\partial_a \alpha)(\alpha \hat{\beta}')}{(z - z_\gamma)} + O(1)
\]
\[
+ \frac{(\alpha \hat{\beta}' + \alpha (\partial_a \hat{\beta}') + \lambda \alpha \alpha' \sigma \Phi_0 + (\alpha \mu_a + \varepsilon \mu_a \alpha') \sigma \alpha \hat{\beta}'+ M_0 \alpha \hat{\beta}'}{z - z_\gamma} + O(1)
\]
After omitting the terms vanishing by the relations (7.4) (see also the relations on \(\alpha' \sigma \alpha\) there) we obtain
\[
(\partial_a + M_a)\Psi^{-1} = \frac{(\partial_a z_\gamma + \mu_a' \sigma \alpha)(\alpha \hat{\beta}')}{(z - z_\gamma)^2} + \frac{(\partial_a \alpha + M_0 \alpha)(\alpha \hat{\beta}')}{(z - z_\gamma)} + O(1)
\]
where \(\hat{\beta}' = \partial_a \hat{\beta}' + \lambda \alpha \sigma \hat{\Psi}_1 + \mu_a' \sigma \Phi_0\). Due to (4.3) the first summand vanishes, and the second summand is equal to \(\kappa \alpha \hat{\beta}'\). Hence \((\partial_a + M_a)\Psi^{-1} = \frac{\alpha \hat{\beta}' + \kappa \alpha \hat{\beta}'}{z - z_\gamma} + O(1),\) and \(\Psi(\partial_a + M_a)\Psi^{-1} = O(1)\) by \(\Psi \sigma \alpha = 0\).

**Proof of the Theorem 7.1.** Modulo Lemma 7.2 the proof of the theorem is the same as in [2]. We give it here for completeness.

By definition
\[
i_{\partial_a} \omega = \omega(\partial_a, \cdot) = -\frac{1}{2} \left( \sum_{s=1}^{K} \text{res}_{\gamma_s} \Lambda + \sum_{i=1}^{N} \text{res}_{\rho_i} \Lambda \right),
\]
where \(\Lambda = \Omega(\partial_a, \cdot)\). By \(\delta \Psi(\partial_a) = \partial_a \Psi\) and \(\delta L(\partial_a) = \partial_a L\) (the evaluation of a differential on a vector field is the derivative along that vector field) we have
\[
\Lambda = \text{tr} \left( \partial_a \Psi \cdot \delta L \cdot \Psi^{-1} - \delta \Psi \cdot \partial_a L \cdot \Psi^{-1} - \partial_a K \cdot \delta \Psi \cdot \Psi^{-1} + \delta K \cdot \partial_a \Psi \cdot \Psi^{-1} \right).
\]

By (7.2) and the Lax equation
\[
\Lambda = \text{tr} \left( (\Psi M_a - F_a \Psi) \delta L \cdot \Psi^{-1} - \delta \Psi [L, M_a] \Psi^{-1} + \delta K (\Psi M_a - F_a \Psi) \Psi^{-1} \right)
\]
\[
= \text{tr} \left( M_a \delta L - F_a \Psi \delta L \cdot \Psi^{-1} - \delta \Psi [L, M_a] \Psi^{-1} + \delta K \Psi M_a \Psi^{-1} - \delta K F_a \right).
\]
Let us transform the middle term. By \(\Psi L = K \Psi\) we have \(\delta \Psi \cdot L = -\Psi \delta L + \delta K \Psi + K \delta \Psi\). Hence
\[
\text{tr} \delta \Psi [L, M_a] \Psi^{-1} = \text{tr} \left( (\delta \Psi \cdot L) M_a \Psi^{-1} - \delta \Psi M_a L \Psi^{-1} \right)
\]
\[
= \text{tr} \left( (\delta \Psi \cdot L + \delta K \Psi + K \delta \Psi) M_a \Psi^{-1} - \delta \Psi M_a L \Psi^{-1} \right)
\]
\[
= \text{tr} \left( \delta K \Psi M_a \Psi^{-1} + \delta K \Psi M_a \Psi^{-1} - \delta \Psi M_a L \Psi^{-1} - \delta \Psi M_a L \Psi^{-1} \right).
\]
The last two terms annihilate because
\[
\text{tr} (\delta \Psi M_a L \Psi^{-1}) = \text{tr} (\delta \Psi M_a \Psi^{-1} (\Psi L \Psi^{-1})) = \text{tr} (\delta \Psi M_a \Psi^{-1} K),
\]
and we obtain
\[ \text{tr} \delta \Psi [L, M_a] \Psi^{-1} = \text{tr} \left( -\delta L M_a + \delta K \Psi M_a \Psi^{-1} \right). \]
Substituting that to the last expression for \( \Lambda \) we obtain
\[ \Lambda = \text{tr} \left( 2M_a \delta L - F_a \Psi \delta L \cdot \Psi^{-1} - \delta K F_a \right). \]
The last two terms are equal (under the symbol of tr) which can be obtained by replacing \( \Psi \delta L \) with \(-\delta \Psi L + \delta K \Psi + K \delta \Psi \). Hence we finally obtain
\[ \Lambda = \text{tr} \left( 2M_a \delta L - 2 \delta K F_a \right). \]
This gives
\[ (7.5) \quad i \partial_a \omega = \sum_{j=1}^{N} \text{res}_{P_j} \text{tr} (\delta K F_a) dz - R_a, \]
where
\[ (7.6) \quad R_a = \sum_{s=1}^{K} \text{res}_{\gamma_s} \text{tr} (\delta L M_a) dz + \sum_{j=1}^{N} \text{res}_{P_j} \text{tr} (\delta L M_a) dz. \]
Observe that the sum over the \( \gamma \)-points in (7.5) vanishes because \( \delta K \) and \( F_a \) are holomorphic at the \( \gamma \)-points (Lemma 7.2). Observe also that the \( \gamma \)-points and the points \( P_j \) are not all singularities of the function \( F_a \). Indeed \( F_a = -\partial_a \Psi \cdot \Psi^{-1} - \Psi M_a \Psi^{-1} \), and \( \Psi^{-1} \) has poles at the branching points of eigenvalues of \( L \).

On the contrary, \( L \) and \( M_a \) are holomorphic everywhere except at the points \( \gamma_s \) and \( P_j \). For this reason all singularities of the the 1-form \( M_a \delta L \) are located at those points. Hence \( R_a = 0 \) as the sum of residues of a meromorphic 1-form over all its poles. Moreover, by the construction of \( M_a \) the matrix \( F_a \) is holomorphic at \( P_j \), \( j \neq i \) for all times \( a \) corresponding to the point \( P_i \). For this reason for such times
\[ i \partial_a \omega = \text{res}_{P_i} \text{tr} (\delta K F_a) dz. \]
But \( F_a = -\partial_a \Psi \cdot \Psi^{-1} - \Psi M_a \Psi^{-1} \), and \( \partial_a \Psi \cdot \Psi^{-1} \) is holomorphic at \( P_i \) because the coordinate of \( P_i \) is independent of any time, hence \( \partial_a \Psi \) and \( \Psi \) have the same order at \( P_i \). Hence \( F_a = -\Psi M_a \Psi^{-1} + O(1) \). By definition \( M_a = w^{-m} L^k + O(1) \) at \( P_i \) for \( a = (P_i, k, m) \). That implies \( \Psi M_a \Psi^{-1} = w^{-m} \Psi L^k \Psi^{-1} + O(1) \) since \( \Psi \) is holomorphically invertible at \( P_i \). But \( \Psi L^k \Psi^{-1} = K^k \), hence \( F_a = w^{-m} K^k + O(1) \) at \( P_i \). Since \( \delta K dz \) is holomorphic at \( P_i \) we obtain
\[ i \partial_a \omega = - \text{res}_{P_i} \text{tr} (w_i^{-m} \delta K K^k) dz = - \frac{1}{k+1} \text{res}_{P_i} \delta \text{tr} (w_i^{-m} K^{k+1}) = \]
\[ = - \frac{1}{k+1} \delta \text{res}_{P_i} \text{tr} (w_i^{-m} L^{k+1}) = \delta H_a. \]
The Hamiltonians $H_a$ are in involution since they depend only on the spectral parameters. We refer to [2] for the details.

8. Examples: Calogero-Moser systems

Let us start with the example considered in [2] — the elliptic Calogero-Moser model for $\mathfrak{g} = \mathfrak{gl}(n)$. Define the Lax operator by

\begin{equation}
L_{ij} = f_{ij} \frac{\sigma(z + q_j - q_i)\sigma(z - q_j)\sigma(q_i)}{\sigma(z)\sigma(z - q_i)\sigma(q_j - q_i)\sigma(q_j)} \quad (i \neq j), \quad L_{jj} = p_j
\end{equation}

where $f_{ij} \in \mathbb{C}$ are constant. This form of $L$ is determined by two requirements: that $L$ is elliptic and that it has poles at the points $z = q_i \ (i = 1, \ldots, n)$ and $z = 0$. The last is the only immovable pole. By reduction of the remaining gauge freedom it is obtained in [2] that $f_{ij}f_{ji} = 1$. For the second order Hamiltonian corresponding to that pole we have according to Theorem 7.1, up to normalization,

$$
H = \text{res}_{z=0} z^{-1} \left( -\frac{1}{2} \sum_{j=1}^{n} p_j^2 - \sum_{i<j} L_{ij}L_{ji} \right).
$$

By the addition theorem for Weierstrass functions

$$
-L_{ij}L_{ji} = \frac{\sigma(z + q_i - q_j)\sigma(z + q_j - q_i)}{\sigma(z)^2\sigma(q_i - q_j)^2} = \wp(q_i - q_j) - \wp(z),
$$

hence

$$
H = -\frac{1}{2} \sum_{j=1}^{n} p_j^2 + \sum_{i<j} \wp(q_i - q_j).
$$

Let us consider now the case of $\mathfrak{so}(2n)$. We present here another method of constructing the hierarchies on elliptic curves. Let $M$-operators take values in the same algebra, i.e. an $M$-operator is of the form

$$
M = \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix}, \quad B^t = -B, \quad C^t = -C.
$$

We assume that $K = 2n$, $(A) + D + \sum_{i=1}^{n} q_i \geq 0$, $(C) + D + \sum_{i=1}^{n} q_i \geq 0$ and $(B) + D + \sum_{i=1}^{n} (-q_i) \geq 0$ where $D = \sum m_i P_i$ as earlier. Hence the submatrices $A, C$ are holomorphic at the points $-q_i$, and the submatrix $B$ is holomorphic at the points $q_i$. Let us denote the space of such $M$-operators by $\mathcal{N}^D$ again.

**Lemma 8.1.** $\dim \mathcal{N}^D = (\dim \mathfrak{g})(\deg D + 1)$. 
Proof. Denote the dimension of the subspace of elements in \( g \) with \( B = 0 \) by \( d' \), and of the subspace of elements in \( g \) with \( A = C = 0 \) by \( d'' \). Thus \( \dim g = d' + d'' \). By Riemann-Roch theorem for \( g = 1 \) we obtain
\[
(8.2) \quad \dim \mathcal{N}^D = d'(\deg D + n) + d''(\deg D + n) - (d' - n)n - (d'' - n)n - n.
\]
The last three summands correspond to the relations. At a point \( q_i \) we have \( d' \) relations \( \text{res}_{q_i} \left( \begin{array}{cc} A & 0 \\ C & -A^t \end{array} \right) = \left( \begin{array}{c} \alpha_i' \\ \alpha_i'' \end{array} \right) \left( \begin{array}{c} \mu_i' \\ \mu_i'' \end{array} \right) \sigma - \left( \begin{array}{c} \mu_i' \\ \mu_i'' \end{array} \right) (\alpha_i'^n, \alpha_i''^n) \sigma \) where \( \sigma = \left( \begin{array}{cc} 0 & E \\ E & 0 \end{array} \right) \), \( E \) is the unit matrix. We also have \( 2n \) parameters \( \mu_i', \mu_i'' \) subjected to \( n \) relations \( \alpha_i'^n - \mu_i'^n = 0 \) which are equivalent to vanishing the \( B \)-block.
At a point \( -q_i \) we have \( d'' \) relations \( \text{res}_{-q_i} B = (\alpha_{-i} \mu_{-i} - \mu_{-i} \alpha_{-i}) \sigma \) (where \( \alpha_{-i} = (\alpha_{-i}', 0), \mu_{-i} = (\mu_{-i}', 0) \), and \( n \) free parameters coming from \( \mu_{-i} \).
The last \( n \) relations arise due to the fact that the motions of \( q_i \) and \( -q_i \) are related. Since \( \dot{q}_i = -\mu_i'^n \alpha_i'^n - \mu_i'' \alpha_i', -\dot{q}_i = -\mu_i'' \alpha_{-i} \), and \( \dot{q}_i + (-\dot{q}_i) = 0 \) we obtain those additional \( n \) relations.
Finally
\[
\dim \mathcal{N}^D = (\dim g) \deg D + (2n^2 - n) = (\dim g)(\deg D + 1).
\]
\[\square\]

We take \( L \) in the same form as \( M \) where \( A \) is given by \( [8.1] \). For \( i < j \) we take
\[
(8.3) \quad B_{ij} = f_{ij}^B \frac{\sigma(z + q_j + q_i)\sigma(z - q_j)}{\sigma(z)\sigma(z + q_i)\sigma(z - q_j)} \quad C_{ji} = f_{ji}^C \frac{\sigma(z - q_j + q_i)\sigma(z + q_i)}{\sigma(z)\sigma(z - q_j)\sigma(q_i + q_j)}
\]
where \( f_{ij}^B, f_{ij}^C \in \mathbb{C} \) are constant. These relations determine the matrices \( B \) and \( C \) due to skew-symmetry. Similar to the case of \( \mathfrak{gl}(n) \) we obtain \( f_{ij}^B f_{ji}^C = -1 \) by reduction of the remaining gauge freedom (taking account of the relation \( \alpha'^t \sigma \alpha = 0 \) which descends to \( \alpha'^t \alpha'' = 0 \) in this case). For the Hamiltonian we have
\[
H = -\text{res}_{z=0} z^{-1} \left( \sum_{i=1}^n p_i^2 + 2 \sum_{i<j} A_{ij} A_{ji} + 2 \sum_{i<j} B_{ij} C_{ji} \right)
\]
\[
= - \sum_{i=1}^n p_i^2 + 2 \sum_{i<j} \varphi(q_i - q_j) + 2 \sum_{i<j} \varphi(q_i + q_j).
\]
Let us consider now the case \( g = \mathfrak{sp}(2n) \). Define \( \mathcal{N}^D \) as the space of \( M \)-operators taking values in \( \mathfrak{sp}(2n) \), with \( n + 1 \) pairs of double poles \( \pm q_i \) \((i = 1, \ldots, n + 1)\).
Thus \( M = \begin{pmatrix} \begin{array}{cccc} 0 & -A^t & B^t & c \\ A & B & C - A^t & b \\ 0 & C & -A^t & a \\ 0 & 0 & 0 & 0 \end{array} \end{pmatrix} \) where \( a, b, c \in \mathbb{C}^n \), \( c \in \mathbb{C} \), \( A, B, C \) are \( n \times n \) matrices,
\( B = B', \ C = C'. \) We assume that \((A) + D + \sum_{i=1}^{n+1} 2q_i \geq 0 \) (and the same for the divisors of \( C, a, b, c \)) and \((B) + D + \sum_{i=1}^{n+1} 2(-q_i) \geq 0.\)

The counterpart of the relation (8.2) writes
\[
\dim \mathcal{N}^D = d'(\deg D + 2n + 2) + d''(\deg D + 2n + 2) \\
- (2d' - n - 1)(n + 1) - (2d'' + 1 - n)(n + 1) - n.
\]

where \(d''\) corresponds to the (matrix) dimension of the \(B\)-block and \(d'\) corresponds to the remainder of the matrix. To explain the second line let us notice that at each pole \(q_1, \ldots, q_n\) we have \(2d'\) relations corresponding to the fact that the form of \(-1\) and \(-2\) order terms is prescribed. We want to have \(\nu = 0\) which leads to one more relation following from (4.5) but this relation is compensated as follows: we actually don’t care of asymptotical behavior of the entry \(c\) in the matrix \(M\) because it corresponds to the center of \(\mathfrak{sp}(2n)\), so we can omit the relation on the second order pole for this entry. We also have \(2n + 1\) parameters, given by \(\mu\), subjected to \(n\) vanishing conditions for the \(\text{res}_q B\). Thus we have \(2d' - (n + 1)\) effective relations at every \(q_i\). Let us notice also that the parameters \(\lambda\) are compensated by the relations 
\[
\alpha^i \sigma M_1 \alpha = 0.
\]
At a point \(-q_i\) we have \(2d'' + 1 - n\) relations since \(\mu\) is \(n\)-dimensional for the block \(B\), and there is nothing to compensate the relation following from (4.5). The last \((-n)\) corresponds to the relations \(\dot{q}_i + (-\dot{q}_i) = 0, \ i = 1, \ldots, n\) (and we do not care of the behavior of the last pair of poles).

Thus we obtain
\[
\dim \mathcal{N}^D = (\dim g^\circ)(\deg D) + 2n^2 + n.
\]

Let us notice that \(2n^2 + n = \dim g\), i.e. this is the dimension of the submatrix \((A' \ C \ -A')\) of \(M\). Thus we can require that this submatrix vanished at \(P_0\) instead of the normalization condition (iii), page 15, in the construction of the flows \(M_a\).

Let us take \(L\) in the same form as \(M\) where the corresponding elements \(A_{ij}, B_{ij}, C_{ij}\) are defined by (8.1) (with \(A\) instead \(L\)),(8.3). The relations (8.3) make sense for \(i = j\) and their contribution to the second order Hamiltonian is equal to
\[
B_{ii}C_{ii} = f_{ii}^B f_{ii}^C (\varphi(2q_i) - \varphi(z))
\]
where we can set \(f_{ii}^B f_{ii}^C\) to a constant \(\varepsilon\). The submatrices \(a\) and \(b\) of the matrix \(L\) do not contribute into the Hamiltonians regardless to any explicit form of them. Hence
\[
H = -\sum_{i=1}^{n} p_i^2 + 2 \sum_{i<j} \varphi(q_i - q_j) + 2 \sum_{i<j} \varphi(q_i + q_j) + \varepsilon \sum_{i=1}^{n} \varphi(2q_i).
\]

which is a conventional form of the second order Hamiltonian of the elliptic Calogero-Moser model in the symplectic case.
References

[1] D’Hocker, E., Phong, D.H. Calogero-Moser Lax pairs with spectral parameter for general Lie algebras. [hep-th/9804124]

[2] Krichever, I.M. Vector bundles and Lax equations on algebraic curves. Comm. Math. Phys. 229, 229–269 (2002).

[3] Krichever, I.M. Elliptic solutions to the Kadomtsev-Petviashvili equation and integrable systems of particles. Funct. Analysis and Appl. 4, 44 (1980), p. 45–54.

[4] Krichever I.M., Novikov S.P. Holomorphic bundles on algebraic curves and nonlinear equations. Uspekhi Math. Nauk (Russ. Math.Surv), 35 (1980), 6, 47–68.

[5] Krichever I.M., Novikov S.P. Holomorphic bundles on Riemann surfaces and Kadomtsev-Petviashvili equation. Funct. Anal. and Appl., 12 (1978), 4, 41–52.

[6] Krichever I.M. Commutative rings of ordinary linear differential operators. Functional Anal. and Appl. (Russ.), 12 (1978), n 3, 20–31.

[7] Krichever, I.M. Novikov, S.P. Algebras of Virasoro type, Riemann surfaces and structures of the theory of solitons. Funktional Anal. i Prilozhen. 21, No.2 (1987), 46-63.

[8] Krichever, I.M., Sheinman, O.K. Lax operator algebras. Funct. Anal. i Prilozhen., 41 (2007), no. 4, p. 46-59. [math.RT/0701648]

[9] Perelomov, A.M. Integrable systems of classical mechanics and Lie algebras, Birkhäuser Verlag, Basel, 1990.

[10] Schlichenmaier, M., Sheinman, O.K. Central extensions of Lax operator algebras. Russ. Math. Surv., 63, no.4, p. 131-172. [arXiv:0711.4688]

[11] Sheinman, O.K. Krichever-Novikov algebras, their representations and applications. In: Geometry, Topology and Mathematical Physics. S.P.Novikov’s Seminar 2002-2003, V.M.Buchstaber, I.M.Krichever, eds., AMS Translations, Ser.2, v. 212 (2004), 297–316, [math.RT/0304020]

[12] Sheinman, O.K. On certain current algebras related to finite-zone integration. In: Geometry, Topology and Mathematical Physics. S.P.Novikov’s Seminar 2004-2008, V.M.Buchstaber, I.M.Krichever, eds., AMS Translations, Ser.2, v.224 (2008).

[13] Sheinman, O.K. Lax operator algebras and integrable hierarchies. In: Proc. of the Steklov Institute of Mathematics, 2008, v.263.

[14] Tyurin, A.N. Classification of vector bundles on an algebraic curve of an arbitrary genus. Soviet Izvestia, ser. Math., 29, 657–688.

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