Special Relativistic Magnetohydrodynamics with Gravitation

Hyerim Noh1,2, Jai-chan Hwang3, and Martin Bucher1,4
1 Center for Large Telescope, Korea Astronomy and Space Science Institute, Daejeon, Republic of Korea
2 Department of Astronomy and Atmospheric Sciences, Kyungpook National University, Daegu, Republic of Korea
3 APC, AstroParticule et Cosmologie, Université Paris Diderot, CNRS/IN2P3, CEA/Irfu, Observatoire de Paris, Sorbonne Paris Cité 10, rue Alice Domon et Léonie Duquet, F-75205 Paris Cedex 13, France
4 Astrophysics and Cosmology Research Unit, University of KwaZulu-Natal, Durban, 4041, South Africa

Received 2018 October 9; revised 2019 March 19; accepted 2019 April 8; published 2019 June 3

Abstract

We present a fully nonlinear and exact perturbation formulation of Einstein’s gravity with a general fluid and ideal magnetohydrodynamics (MHD) without imposing the slicing (temporal gauge) condition. Using this formulation, we derive equations of special relativistic (SR) MHD in the presence of weak gravitation. The equations are consistently derived in the limits of weak gravity and action-at-a-distance in the maximal slicing. We show that in this approximation the relativistic nature of gravity does not affect the SR MHD dynamics, but SR effects manifest themselves in the metric, and thus in gravitational lensing. Our formulation can account for strong SR effects, which might dominate over the Newtonian lensing potentials. Neglecting these SR effects may lead to an overestimation of lensing masses.

Key words: hydrodynamics – magnetic fields – magnetohydrodynamics (MHD) – relativistic processes

1. Introduction

Magnetic fields are ubiquitous in celestial objects and in the universe as a whole. Magnetohydrodynamics (MHD) is often a useful approximation for treating fluid and gas coupled to the electromagnetic (EM) field. Relativistic processes play a crucial role in many astrophysical phenomena. Relativistic MHD is required to understand the physical processes in accretion disks, magnetospheres, the plasma winds and astrophysical jets near compact objects (e.g., neutron stars and black holes), and active galactic nuclei where the relativistic effects (of gravity, the gas velocity, the field strength, etc.) are significant. In none of these situations can the astrophysical processes be treated ignoring gravity.

Special relativistic (SR) MHD with non-relativistic gravity has been studied in the literature, as well as fully generally relativistic (GR) MHD, which is implemented in numerical relativity simulations. (For textbook treatments, see Bona et al. 2009; Baumgarte & Shapiro 2010; Gourgoulhon 2012; Shibata 2016.) Full blown simulations using numerical relativity are ultimately the most reliable technique, but they are computationally complicated, time-consuming, and expensive. Moreover, the results depend on the gauge choice (coordinate conditions), often making it difficult to extract the proper physical interpretation.

Here we present two new formulations of GR MHD. The first is the exact and fully nonlinear GR-MHD perturbation equations without imposing a particular slicing (temporal gauge) condition. This is an extension of the fully nonlinear and exact perturbation formulation in the cosmology context (Hwang & Noh 2013a; Noh 2014; Hwang et al. 2016), now including ideal MHD in a Minkowski background.

The complete set of equations of the first formulation is presented in the Appendix. The character of the new formulation is as follows. The equations are a rearrangement of the Arnowitt–Deser–Misner (ADM) equations using exact perturbation variables added to a Minkowski background; see the exact metric and its inverse in Equations (1) and (51). All perturbation variables of the fluid, field, and metric are arbitrary in amplitude. The indices of all (metric, fluid and field) perturbation variables are raised and lowered by the flat space metric $\delta_{ij}$ and its inverse. We have imposed spatial gauge conditions and ignored the transverse-tracefree (TT) perturbations for simplicity, but these conditions can be easily relaxed; see the text after Equation (1). But we have not imposed the temporal gauge condition, which can be chosen at our convenience depending on the problem; all perturbation variables are free from the remnant gauge mode after fixing the slicing condition mentioned after Equation (91). Thus, the variables are equivalent gauge-invariant to fully nonlinear order.

As applications of the formulation we have shown the Newtonian limit (Hwang & Noh 2013b), the post-Newtonian (PN) limit (Noh & Hwang 2013), and the SR hydrodynamics with weak gravity (Hwang & Noh 2016). Whether our new formulation of the GR-MHD equations in the Appendix is to be useful in numerical implementation is open to future investigation.

The second new formulation is the SR MHD with weak gravitation as a consistent limit of the fully nonlinear and exact GR-MHD formulation in the maximal slicing. The asymmetric combination of full SR with weak Newtonian gravity is not a priori guaranteed. Here we prove that such a formulation is possible in a consistent manner in a GR context. The hydrodynamic case of an ideal fluid was previously shown in the maximal slicing (Hwang & Noh 2016), and here we extend the case to a general fluid with SR MHD in the same slicing. We have three new points. First, the equations are derived as a consistent limit from GR; see Section 5. Second, for the dynamics, the SR MHD is valid with a simple combination of Newtonian gravity; see Section 2.2. Third, the SR MHD affects the gravity and thus the metric and the gravitational lensing; see Section 2.3.

Compared with the PN approximation our SR MHD with weak gravity can be regarded as a combination of the infinite-order PN ($\infty$ PN) approximation in the SR fluid and a field with
the zeroth-order PN (OPN) approximation in gravity. Therefore, our new gravitational lensing potential in Equation (22) reveals the
full SR effect on gravitational lensing in which the SR terms can be comparable or even dominate over the rest-mass density in weakly curving the spacetime metric; see Equations (10)–(14).

The equations of SR MHD with weak gravity and the fully nonlinear and exact perturbation of GR MHD are summarized
in Section 2.1 and in the Appendix, respectively. The equations of each formulation are derived in Sections 5 and 4,
respectively. In our approximation the relativistic nature of GR is de

Our spatial gauge conditions, together with neglecting the
χ condition, we end up with an inconsistent result by omitting a
χ term in Equation (2) (Noh et al. 2019 June 1 Noh, Hwang, & Bucher).

We assume the following weak gravity and action-at-a-
distance conditions:

\[ \alpha \equiv \frac{\Phi}{c^2} \ll 1, \quad \varphi \equiv \frac{\Psi}{c^2} \ll 1, \quad \gamma \frac{t_f^2}{t_g^2} \ll 1, \quad (2) \]

where \( t_g \sim 1/\sqrt{G\varphi} \) and \( t_f \sim \ell/c \sim 2\pi/(kc) \) are the gravitational
timescale and the light propagating timescale of a characteristic
length scale \( \ell \), respectively, and \( k \) is the wavenumber with \( \Delta = -k^2 \).

5 We reverse the signs of \( \Phi \) and \( \Psi \) relative to the convention in Hwang &
Noh (2016).

2.1. Results

The equations of motion of SR MHD in the presence of weak gravity are

\[ \frac{\partial}{\partial t} \begin{pmatrix} D \\ E \\ m^i \\ B^i \end{pmatrix} + \nabla \begin{pmatrix} D \nu^j \\ m^i c^2 \\ m^j \\ \nu^i B^j - \nu^j B^i \end{pmatrix} = \begin{pmatrix} 0 \\ -\varpi (2\Phi - \Psi) \delta^{ij} \\ -\varpi \Phi \delta^{ij} \\ 0 \end{pmatrix}, \quad (3) \]

and

\[ B^i = 0. \quad (4) \]

These are the mass, energy, and momentum conservation equations, and the two Maxwell equations, respectively.

The notation is as follows:

\[ D \equiv \varpi g, \quad \varrho \equiv \varpi \left(1 + \frac{\Pi}{c^2}\right). \]

\[ E/c^2 \equiv \left(\varrho + \frac{\rho}{c^2}\right) \gamma^2 - \frac{\rho}{c^2} + \frac{1}{8\pi c^2} \Pi_{ij} \nu^i \nu^j 
+ \frac{1}{2} \frac{1}{4\pi} \left[ B^2 \left(1 + \nu^2/c^2\right) - \frac{1}{c^2} (B^i \nu^j)^2 \right]. \]

\[ m^i \equiv \left(\varrho + \frac{\rho}{c^2}\right) \gamma^2 \nu^i 
+ \frac{1}{2} \frac{1}{4\pi} \left[ \Pi_{ij} \nu^j + \frac{1}{4\pi} \left( B^2 \nu^i - B^i B^j \nu^j \right) \right]. \]

\[ m^j \equiv \left(\varrho + \frac{\rho}{c^2}\right) \gamma^2 \nu^i \nu^j + \rho \delta^{ij} + \Pi^{ij} 
+ \frac{1}{4\pi} \left[ \frac{1}{\gamma} \left( \frac{1}{2} B^2 \delta^{ij} - B^i B^j \right) + \frac{1}{c^2} \left( B^2 \nu^i \nu^j \right) 
+ \frac{1}{2} \left( B^k \nu_k \right)^2 \delta^{ij} - \left( B^i \nu^j + B^j \nu^i \right) B^k \nu_k \right] \right], \quad (5) \]

where \( \rho \) denotes the density, \( \varpi \) is the mass density, \( \varpi \Pi \) is the internal energy, \( \rho \) is the pressure, \( \Pi_{ij} \) is the anisotropic stress, \( \nu^i \) is the velocity, \( \gamma \) is the Lorentz factor

\[ \gamma = \frac{1}{\sqrt{1 - \nu^2/c^2}}, \quad (6) \]

and \( B_i \) is the magnetic field. Notations will be more properly introduced in Section 4. We have

\[ \Pi_{ij} = \Pi_{ij} \nu^i \nu^j. \quad (7) \]

Contributions from gravity appear on the right side of Equation (3). All spatial indices in this section are raised and
dowed using \( \delta_{ij} \) as the metric.

Following the notation typically used in the ADM formulation
of GR, we define \( E \) as the ADM energy density, \( J_i = cm_i \) the ADM flux vector, \( S_i = m_i \) the ADM stress where \( S_i = m_i \) is its isotropic part. Indices of the ADM fluid variables are
raised and lowered by the ADM metric \( h_{ij} \) and this is the same as \( \delta_{ij} \) in our approximation in this section.

Using

\[ E = -\frac{1}{c} \nu \times B \quad (8) \]
in the ideal MHD approximation, the EM parts of above quantities can be written as

\[ E_{\text{MHD}} = S_{\text{MHD}} = \frac{1}{8\pi}(E^2 + B^2), \]

\[ m_{\text{MHD}}^i = \frac{1}{4\pi c}(\mathbf{E} \times \mathbf{B})^i, \]

\[ m_{\text{MHD}}^0 = \frac{1}{4\pi} \left[ -E^i E^j - B^i B^j + \frac{1}{2}(E^2 + B^2)\delta^{ij} \right]. \]  

(9)

The left side of Equation (3) is exactly the same as the equation for SR MHD (Mignone et al. 2007) with an anisotropic stress, and the right side is the source term due to gravity. The two gravitational potentials in the metric satisfy two Poisson-like equations

\[ \Delta \Phi = 4\pi G \left( \varrho + \frac{3p}{c^2} + \frac{2}{c^2} S \right) = \frac{4\pi G E + S}{c^2}, \]  

(10)

\[ \Delta \Psi = 4\pi G \left( \varrho + \frac{1}{c^2} S \right) = \frac{4\pi G E}{c^2}, \]  

(11)

with

\[ S = \left( \varrho + \frac{p}{c^2} \right) \gamma^2 \gamma^2 + \Pi^i \frac{\gamma^2}{c^2} \frac{\gamma^2}{c^2} \frac{\gamma^2}{c^2} \frac{\gamma^2}{c^2} + \frac{1}{8\pi} \left[ B^2 + \frac{1}{c^2} (\varpi \times \mathbf{B}) \right]^2, \]  

(12)

\[ E = \varrho c^2 + S, \quad S = 3p + S. \]  

(13)

The metric component \( \chi_i \) is determined by

\[ \chi_i = -\frac{4\pi G}{c^3} \Delta \delta^i - \Delta (\varrho \nabla^i) m^i. \]  

(14)

Equations (10) and (11) show that the weak gravity conditions imply the action-at-a-distance condition in Equation (2).

Equations (3)–(14) constitute a complete set. The pressure and anisotropic stress should be specified by equations of state, and we do not consider the additional presence of heat flux. All the above equations are consistently derived in Section 5 from a fully GR-MHD formulation derived and presented in Section 4 and in the Appendix.

The presence of a pressure term in the relativistic Poisson’s Equation in (10) has often been noted in the literature (Tolman 1930; Whittaker 1935; McCrea 1951; Harrison 1965). The pressure term, however, does not appear in the zero-shear gauge, and this contradicts the well known Tolman–Oppenheimer–Volkoff equation for a spherically symmetric static solution (Oppenheimer & Volkoff 1939; Tolman 1939). In Hwang & Noh (2016) we argued that when relativistic pressure is present, the zero-shear gauge is not a suitable gauge choice because it leads to such an inconsistent result.

### 2.2. Role of Relativistic Gravity on Dynamics

In the derivation of the first three conservation equations in Equation (3), we have strictly imposed the conditions in Equation (2). All terms in Equations (5), (10), (11), and (14) are of the same order as in the fully SR situation with

\[ 1 - \frac{\varrho}{\varpi} \sim \frac{\Pi}{\varpi c^2} \sim \frac{\gamma^2}{\varpi c^2} \sim \frac{\Pi^i}{\varpi c^2} \sim \frac{B^2}{\varpi c^2} \]  

(15)

Applying the weak gravity condition, we find that Poisson’s equation simply becomes

\[ \Delta \Phi = 4\pi G \varpi, \]  

(16)

with \( \Psi = \Phi \). Thus, the gravity part in Equation (3) becomes

\[ \begin{pmatrix} 0 \\ -\nabla \Phi \cdot \nu^i \\ -\nabla \Phi \\ 0 \end{pmatrix}. \]  

(17)

Therefore, in the framework of our approximation, the relativistic nature of gravity does not alter the dynamics of fluid and fields. We note that effectively \( \Psi = \Phi \) only in the dynamics of the MHD equations, but these two metric potentials in general differ as in Equations (10) and (11).

For a static equilibrium situation, we have \( \nu^i = 0 \), and the momentum conservation equation gives

\[ \nabla \left[ \left( \varrho + \frac{1}{8\pi} B^2 \right) \delta^i + \Pi^i - \frac{1}{4\pi} B^i B^j \right] = -\nabla \Phi. \]  

(18)

For the gravitational potential in the above equation, we have \( \Delta \Phi = 4\pi G \varpi \). However, for the metric we have

\[ \Delta \Phi = 4\pi G \left( \varrho + \frac{3p}{c^2} + \frac{1}{4\pi} B^2 \right), \]  

(19)

\[ \Delta \Psi = 4\pi G \left( \varrho + \frac{1}{8\pi} B^2 \right), \]  

(20)

and \( \chi_i = 0 \). The metric is curved by the internal energy, pressure, and magnetic field contributions, as well as the mass density, and these extra contributions alter the gravitational lensing predictions.

### 2.3. Impact of Special Relativity on Gravitational Lensing

Equations (10)–(14) determine the spacetime metric by assuming weak gravity but taking into account SR effects. Although \( \chi_i \) is non-vanishing in the maximal slicing, we can show that in the weak gravity approximation, the null geodesic equation simply becomes

\[ \frac{d^2 x^i}{dt^2} = -(\Phi + \Psi)^i, \]  

(21)

and thus is the same as it is in the zero-shear gauge, taking \( \chi \equiv 0 \) as the slicing condition. The null geodesic equation to first-order PN (1PN) order can be found in Section 5 of Hwang et al. (2008).

We note that the SR effects of velocity, internal energy, pressure, anisotropic stress, and the magnetic field cause the two potentials \( \Phi \) and \( \Psi \) to differ from each other. This might cause the gravitational lensing to differ from the conventional result, which assumes \( \Psi = \Phi \). In addition to this asymmetric effect (often known as a gravitational slip of the potentials), in the presence of this SR effect, instead of \( \Delta(\Phi + \Psi) = 8\pi G \varpi \),
we have

$$\Delta(\Phi + \Psi) = 4\pi G \left\{ 2\Psi \left( 1 + \frac{\Pi}{c^2} \right) + \frac{3}{c^2} \left[ p + \left( \rho + \frac{p}{c^2} \right) \gamma^2 \right] + 3 + \frac{1}{8\pi} \left( B^2 + \frac{1}{c^2} (v \times B)^2 \right) \right\}. \quad (22)$$

The gravitational potential $2\Phi$ in the commonly used gravitational lensing formulae in Einstein’s gravity should be replaced with $\Phi + \Psi$. For positive pressure and anisotropic stress, all the SR terms lead to an overestimation of the mass.

We would like to emphasize that we have not assumed smallness of the SR terms on the right side. This is in contrast with the PN approximation, where all the SR terms are regarded as the PN corrections and thus are small. In our approximation the SR terms can easily dominate over the Newtonian gravity source. Whether we have such a strongly SR astrophysical situation in nature is a different matter. The strength of the magnetic field in known astrophysical objects reaches up to $B \approx 10^{15}$ Gauss (Kaspi & Beloborodov 2017), which corresponds to the magnetic energy density $B^2 \sim 10^{8} \text{c}^2 \text{g cm}^{-3} \sim 10^{30} \text{erg cm}^{-3}$.

In a homogeneous background medium (as in Friedmann cosmology) with linear perturbations, only the density and pressure terms may contribute to the lensing. The other terms are nonlinear perturbations. For a negative pressure with an equation of state approximating dark energy with $p_{\text{DE}} \simeq -\rho_{\text{DE}}c^2$, we have $\Delta(\Phi + \Psi) \simeq -4\pi G(2\rho_{\text{matter}} - \rho_{\text{DE}})$. If this component is clustered, de-lensing by dispersing the light may result. In the presence of ordinary matter and a (non-cosmological constant) dark energy component, we have $\Delta(\Phi + \Psi) \simeq 4\pi G(2\rho_{\text{matter}} - \rho_{\text{DE}})$, where $\rho_{\text{DE}} \simeq \Lambda c^2/(8\pi G)$ and $\Lambda$ is the cosmological constant.

### 2.4. Non-relativistic MHD Limit

As the non-relativistic limit, we take $c \to \infty$. To get the energy conservation equation properly, we need to consider the next order in $c^{-2}$; this is because our $E$ contains the rest-mass energy density that satisfies the continuity equation separately. In other words, in the $c \to \infty$ limit, $(E - Dc^2) + \nabla_j (m^j - Dv^j)c^2 = -\Psi \nabla \cdot v$ gives

$$\left( \frac{1}{2} \nabla v^2 + \nabla \Pi + \frac{1}{8\pi} B^2 \right) + \nabla_j \left( \frac{1}{2} \nabla v^2 + \nabla \Pi + P \right) = -\Psi \nabla \cdot \nabla \Phi. \quad (23)$$

The complete set of equations is

$$\psi + \nabla \cdot (\nabla v) = 0, \quad (24)$$

$$\Pi + v \cdot \nabla \Pi + \frac{1}{8\pi} (p \nabla \cdot v + v_i \nabla v^i) = 0, \quad (25)$$

$$\nabla(\psi + v \cdot \nabla v) = -\Psi \nabla \Phi - \nabla p - \nabla_j \Pi^j, \quad \quad (26)$$

$$\left. \begin{array}{c}
\nabla \cdot B = 0, \\
\Delta \Phi = 4\pi G \Psi,
\end{array} \right\} \quad (27)$$

and we have

$$E = -\frac{1}{c} v \times B. \quad (29)$$

Combining Equations (24) and (26), we have

$$\left( \nabla v^i \right) + \nabla_j \left[ \nabla v^i \nabla^j + \left( p + \frac{1}{8\pi} B^2 \right) \delta^i_j + \Pi^i_j - \frac{1}{4\pi} B^i B^j \right] = -\frac{1}{\pi} \Phi \psi. \quad (30)$$

where the contributions of magnetic field are interpreted as magnetic pressure and magnetic tension force density, respectively. These differ from the contribution to the pressure and anisotropic stress appearing in the energy-momentum tensor; in the non-relativistic limit, Equation (60) implies that

$$\mu_{\text{MHD}} = \frac{3}{4\pi} \psi M^2, \quad \Pi^i_j = -\frac{1}{B \nabla B_i - \frac{1}{3} \delta_i^j B^2}. \quad (31)$$

By replacing $\mu \to \mu + \mu_{\text{MHD}}$, etc., in the hydrodynamic equations, we can derive the MHD equations.

The above equations can be combined to give

$$\frac{\partial \psi}{\partial t} + \nabla(\psi v^i) - \nabla_j \left( \frac{\psi}{m^i} c^2 + \frac{\psi}{m^i} \frac{v^j}{v^i} B^i \right) = 0, \quad (32)$$

with

$$E = E - Dc^2 = \frac{1}{2} \nabla v^2 + \nabla \Pi + \frac{1}{8\pi} B^2; \quad \mu = m^i v^i c^2 - \frac{1}{4\pi} (\psi v \times B)^2, \quad (33)$$

In the spacetime metric, we have $\Psi = \Phi$, and Equation (14) gives $\chi_i = 0$. Although $\Psi$ does not affect the non-relativistic hydrodynamic or MHD equations directly, the Newtonian gravity $\Phi$ naturally excites the PN potential $\Psi$ (Chandrasekhar 1965). And this curved spacetime metric affects the gravitational lensing.

### 3. General Relativistic Electromagnetism

The complete set of fully nonlinear and exact perturbation equations with a general fluid component is presented in the Appendix of Hwang & Noh (2010). The presence of an EM field can be accommodated in the formulation by interpreting the contribution of the EM as fluid quantities with the Maxwell’s equations appended. Here we present the derivation.

The energy-momentum tensor of EM field is

$$T_{\text{EM}}^{ab} = \frac{1}{4\pi} \left( F_{ac} F_{b}^c - \frac{1}{4} \eta_{ab} F^{cd} F_{cd} \right). \quad (34)$$
The tildes indicate covariant quantities. The EM tensor can be decomposed as
\[ \tilde{F}_{ab} = \tilde{U}_a \tilde{E}_b - \tilde{U}_b \tilde{E}_a - \tilde{\eta}_{abcd} \tilde{U}^c \tilde{U}^d, \]
with \( \tilde{U}_a \tilde{U}^a \equiv 0 \) \( \tilde{U}_a \) is a generic timelike four-vector normalized so that \( \tilde{U}^a \tilde{U}_a = -1 \). With
\[ *\tilde{F}^{ab} \equiv \frac{1}{2} \tilde{\eta}^{abcd} \tilde{F}_{cd} = \tilde{U}^a \tilde{B}^b - \tilde{U}^b \tilde{B}^a + \tilde{\eta}^{abcd} \tilde{U}^c \tilde{E}_d, \]
we have
\[ \tilde{E}_a = \tilde{F}_{ab} \tilde{F}^b, \quad \tilde{B}_a = *\tilde{F}_{ab} \tilde{F}^b. \]
Equation (34) can be written as
\[ \tilde{T}_{ab}^{EM} = \frac{1}{4\pi} \left[ (\tilde{E}^2 + \tilde{B}^2) (\tilde{U}_a \tilde{U}_b + \frac{1}{3} \tilde{\eta}_{abcd} \tilde{U}^c \tilde{U}^d) - \tilde{F}_{a} \tilde{F}_b \right. \]
and the fluid quantities in the \( \tilde{U}^a \)-frame become
\[ \tilde{\rho}_{EM} = 3\tilde{F}_{EM} = \frac{1}{8\pi} (\tilde{E}^2 + \tilde{B}^2), \]
\[ \tilde{q}^{ab}_{EM} = \frac{1}{4\pi} \tilde{\eta}_{abcd} \tilde{E}^b \tilde{B}^c \tilde{U}^d, \]
\[ \tilde{\pi}^{ab}_{EM} = -\frac{1}{4\pi} \left[ \tilde{E}_a \tilde{E}_b + \tilde{B}_a \tilde{B}_b - \frac{1}{3} \tilde{\eta}_{abcd} (\tilde{E}^2 + \tilde{B}^2) \right]. \]
with the fluid quantities in the \( \tilde{U}^a \)-frame introduced as
\[ \tilde{T}_{ab} = \tilde{\rho} \tilde{U}_a \tilde{U}_b + \tilde{p} (\tilde{g}_{ab} + \tilde{U}_a \tilde{U}_b) + \tilde{\eta}_{ab} \tilde{U}_b + \tilde{\bar{\eta}}_{ab} + \tilde{\pi}_{ab}. \]
As we have non-vanishing \( \tilde{q}_{ab} \) for the EM field, in order to have the nonlinear and exact perturbation formulation, we need to consider the \( \tilde{q}_{ab} \) term, which is missing in our previous formulation. In the ideal MHD considered in this work, however, the flux term vanishes for MHD; see Equation (50).

From
\[ *\tilde{F}_{ab} = 0, \quad \tilde{F}_{ab} \tilde{F}^{ab} = 4\pi \tilde{J}_{em} \]
we have the four Maxwell’s equations (Ellis 1973)
\[ \tilde{B}^{ab} \tilde{h}_{ab} = 2\tilde{\omega}^{ab} \tilde{E}^c, \]
\[ \tilde{h}^{ab} \tilde{B}_{bc} \tilde{E}^c = \left( \tilde{\eta}^{abcd} \tilde{U}^d \tilde{U}^e + \tilde{\sigma}^{ab} - \frac{2}{3} \tilde{\xi}^{ab} \tilde{\theta} \right) \tilde{B}^b + \tilde{\xi}^{abcd} \tilde{U}^d (-\tilde{\eta}_{bc} \tilde{E}^c + \tilde{E}_{bc}^e), \]
\[ \tilde{E}^{ab} \tilde{h}_{ab} = 4\pi \tilde{J}_{em} - 2\tilde{\omega}^{ab} \tilde{B}_{ab}, \]
\[ \tilde{h}_{ab} \tilde{E}^{ab} \tilde{U}^c = \left( \tilde{\eta}^{abcd} \tilde{U}^d \tilde{U}^e + \tilde{\sigma}^{ac} - \frac{2}{3} \tilde{\xi}^{ac} \tilde{\theta} \right) \tilde{E}^b + \tilde{\xi}^{abcd} \tilde{U}^d (\tilde{\eta}_{bc} \tilde{B}^e - \tilde{B}_{bc}), \]
with \( \tilde{h}_{ab} \equiv \tilde{g}_{ab} + \tilde{U}_a \tilde{U}_b \) as the projection tensor. We have decomposed the four-current as
\[ \tilde{J}_{em}^{a} = \tilde{\xi}^{em} \tilde{U}^a + \tilde{J}^a, \quad \tilde{J}^a \tilde{U}^a = 0, \]
where the first and the second terms on the right side are the convection and conduction currents, respectively. The covariant kinematic quantities \( \tilde{\omega}_{ab}, \tilde{\sigma}_{ab}, \tilde{\theta}, \) and \( \tilde{a}_{ab} \) are the vorticity vector, shear tensor, expansion scalar, and acceleration vector, respectively, based on the generic four-vector \( \tilde{U}_a \) (Ellis 1971, 1973).

For the fluid (comoving, rest) four-vector we have \( \tilde{U}_a = \tilde{u}_a \). For the laboratory (normal) frame, we have \( \tilde{U}_a = \tilde{n}_a \).
In the following, we set \( \tilde{B}_a \equiv \tilde{B}_a^{(a)} \), \( \tilde{d}_a \equiv \tilde{d}_a^{(a)} \), and do the same for the electric field and the conduction current.

### 4. General Relativistic Ideal MHD: Derivation

The Ohm’s law relates the conduction current in Equation (46) to the electric field in the comoving frame as (Jackson 1975)
\[ \tilde{J}_{a}^{(a)} = \sigma \tilde{E}_{a}^{(a)}, \]
with \( \sigma \) being the electric conductivity. Ideal MHD results from taking a perfectly conducting limit (with \( \sigma \to \infty \)), so that \( \tilde{E}_{a}^{(a)} = 0 \), with non-vanishing \( \tilde{J}_{a}^{(a)} \). In the following we consider ideal MHD.

The ideal MHD equations may also be derived in the following invariant (or coordinate-free) form:
\[ \tilde{L}_{U} \tilde{F} = 0, \]
which physically expresses the fact that the magnetic field lines are frozen into the fluid and thus go with the flow. Here \( \tilde{F} \) is the Maxwell EM tensor regarded as a 2-form, \( \tilde{U} \) is the fluid 4-vector considered as a vector field (and not as a covector field) as is sometimes denoted by \( \tilde{U} \), and \( \tilde{L} \) denotes the Lie derivative. Interestingly, the evolution equation (Equation (48)) thus expressed does not involve the Riemannian (metric) structure of the manifold. We derive Equation (48) as follows. The ideal MHD assumption of infinite conductivity implies that in the fluid rest frame \( \tilde{E} \) vanishes, or equivalently \( \tilde{F} \) contracted with \( \tilde{U} \) vanishes, which can be written as \( i \tilde{q} \tilde{F} = 0 \). Here \( i \) denotes the interior product. The vanishing of the divergence of \( \tilde{B} \) and the Faraday induction equation are expressed as \( \tilde{d} \tilde{F} = 0 \), where \( \tilde{d} \) is the exterior derivative. Equation (48) follows by applying Cartan’s magic formula (Abraham et al. 1988) expressing the Lie derivative acting on a differential form \( \tilde{\alpha} \) in the following way:
\[ \tilde{L}_{\tilde{X}} \tilde{\alpha} = (i \tilde{X} \circ \tilde{d}) \tilde{\alpha} + (\tilde{d} \circ i \tilde{X}) \tilde{\alpha}. \]
The effect of the frozen in flux is to make the fluid behave much like an anisotropic solid given that the flux lines cannot move relative to the fluid. One important consequence is the possibility of anisotropic stresses, which cannot occur for an unmagnetized perfect fluid.

For ideal MHD, with \( \tilde{\varepsilon}_{a} = 0 \), Equations (38) and (39) become
\[ \tilde{T}_{ab}^{\text{MHD}} = \frac{1}{4\pi} \left[ \tilde{B}^{2}(\tilde{a}_{u} \tilde{u}_{b} + \frac{1}{2} \tilde{\eta}_{ab}) - \tilde{B}_{a} \tilde{B}_{b} \right], \]
\[ \tilde{T}_{ab}^{\text{MHD}} = 3\tilde{p}_{\text{MHD}} = \frac{1}{8\pi} \tilde{B}^{2}, \quad \tilde{\pi}_{ab}^{\text{MHD}} = 0, \]
\[ \tilde{\pi}_{ab}^{\text{MHD}} = -\frac{1}{4\pi} \left[ \tilde{h}_{ab} \tilde{B}_{a} - \frac{1}{3} \tilde{B}^{2}(\tilde{g}_{ab} + \tilde{u}_{a} \tilde{u}_{b}) \right]. \]
Now we should express $\tilde{B}_a$ in terms of the magnetic field in the laboratory frame.

In order to express the fluid quantities in terms of the metric notation in Equation (1), the following quantities are useful. The exact inverse metric is (Hwang & Noh 2013a)

$$
\bar{g}^{00} = -\frac{1}{N^2}, \quad \bar{g}^{ab} = -\frac{\chi^a \chi^b}{N^2(1 + 2\varphi)}.
$$

with the index 0 indicating $ct$, $N$ is the lapse function

$$
N = \sqrt{1 + 2\alpha + \frac{\chi^k \chi_k}{1 + 2\varphi}}.
$$

The fluid four-vector becomes

$$
\bar{u}^i = \frac{\gamma v^i}{c}, \quad \bar{u}^0 = -\gamma \left( N + \frac{\chi^i}{1 + 2\varphi} \frac{v^i}{c} \right),
$$

with the Lorentz factor

$$
\gamma = \frac{1}{\sqrt{1 - \frac{1}{1 + 2\varphi} v^2}}.
$$

The normal four-vector is

$$
\bar{n}^i = 0, \quad \bar{n}^0 = -N, \quad \bar{n}^i = \frac{\chi^i}{N(1 + 2\varphi)}, \quad \bar{n}^0 = \frac{1}{N}.
$$

For the field in the laboratory frame, using $\bar{B}_a \bar{n}^a = 0$, we have

$$
\bar{B}_i = B_i, \quad \bar{B}_0 = -\frac{\chi^i B^i}{1 + 2\varphi}, \quad \bar{B}^i = \frac{B^i}{1 + 2\varphi}, \quad \bar{B}^0 = 0,
$$

and similarly for the electric field $\tilde{E}_a$ and the current density $\tilde{J}_a$.

The index of $\bar{B}_i$ is raised and lowered using $\delta_{ij}$ as the metric.

Using $\tilde{B}_a = \bar{F}^{ab}_0 u^b$ and expressing $\bar{F}^{ab}$ in Equation (36) in the laboratory frame, we have

$$
\tilde{B}_i = \frac{1}{\gamma} B_i + \frac{\gamma \chi^j v^j B^i}{c},
$$

$$
\tilde{B}_0 = -\frac{1}{1 + 2\varphi} \left( N\gamma B^i \frac{v^i}{c} + \frac{\chi^i}{1 + 2\varphi} \frac{v^i}{c} \right),
$$

$$
\tilde{B}^i = \frac{1}{1 + 2\varphi} \left[ \frac{B^i}{\gamma} + \frac{\gamma \chi^j v^j B^i}{c} \right],
$$

$$
\tilde{B}^0 = \frac{\gamma}{N(1 + 2\varphi)} B^0.
$$

Using this, Equation (50) gives

$$
\mu_{\text{MHD}} = 3\rho_{\text{MHD}}
$$

$$
= \frac{1}{8\pi} \frac{1}{1 + 2\varphi} \left[ \frac{1}{\gamma^2} B^2 + \frac{1}{1 + 2\varphi} \left( B^i v^i \right)^2 \right],
$$

$$
\Pi_{ij}^{\text{MHD}} = \frac{1}{4\pi} \left\{ \frac{1}{\gamma^2} B_i B_j 
$$

$$
+ \frac{1}{3} \delta_{ij} \left[ \frac{1}{\gamma^2} B^2 + \frac{1}{1 + 2\varphi} \left( B^i v^i \right)^2 \right]
$$

$$
- \frac{1}{1 + 2\varphi} \left( B_i v^i B_j + B_j v^j B_i \right) \frac{B^0}{c},
$$

$$
+ \frac{1}{3} \frac{1}{1 + 2\varphi} \left( B^2 - 2\gamma^2 \frac{B^i v^i}{c} \right) \frac{v_i v_j}{c^2}, \right\}, \quad (60)
$$

where we set $\bar{\mu} = \mu = \rho c^2$, $\bar{p} = \bar{\rho}$ and $\bar{\Pi}_{ij} = \Pi_{ij}$ all in the fluid frame. The indices of $\Pi_{ij}$ are raised and lowered using $\delta_{ij}$ as the metric. Using Equation (57) we have

$$
\mu_{\text{MHD}} = 3\rho_{\text{MHD}} = \frac{1}{8\pi} \frac{1}{1 + 2\varphi} (B^2 - E^2),
$$

$$
\Pi_{ij}^{\text{MHD}} = \frac{1}{4\pi} \left\{ -E_i E_j - B_i B_j + \frac{1}{3} \delta_{ij} (2E^2 + B^2)
$$

$$
+ \frac{2}{3} \frac{1}{1 + 2\varphi} (E^2 - B^2) \gamma^2 \frac{v_i v_j}{c^2}, \right\}, \quad (61)
$$

These relations expressing the fluid quantities in the comoving (energy) frame in terms of the fields in the laboratory (normal) frame, appear asymmetric in the fields. This is because although the energy-momentum tensor in Equation (40) is frame-invariant, the fluid quantities in Equation (39) are not.

Using the fluid quantities in Equation (60), by replacing $\mu \rightarrow \mu + \mu_{\text{MHD}}$ etc., the fully nonlinear and exact perturbation equations in Hwang & Noh (2016) are now complete in the presence of MHD. A complete set of equations is presented in the Appendix.

In the presence of an EM field, we also need to include the Maxwell equations. Taking the laboratory frame, Equations (42–45) give Equations (87–90).

Equations in this section and in the Appendix are fully general in Einstein’s gravity, under the conditions stated after Equation (1), with MHD. We have not yet imposed the temporal gauge condition.
5. Weak Gravity Limit: Derivation

Now, using the fully nonlinear and exact formulation of GR MHD presented in the Appendix, we prove equations in Section 2.1 by taking the weak gravity and action-at-a-distance limit in Equation (2).

The ADM momentum constraint equation in Equation (81) becomes

$$\frac{2}{3}\kappa_{,i} + \frac{c}{N^2}\left(\frac{2}{3}\Delta \chi_{,i} + \frac{1}{2}\Delta \chi_{(v)}_{,i}\right) = -\frac{8\pi G}{c^4}[(\mu + p)\gamma^i\eta]_{,i} + \Pi_{ij}^{(v)} + \frac{1}{4\pi}(B^{2v}_{i} - B_{i}B^{v}_{j}) = -\frac{8\pi G}{c^2}m_{i},$$

(62)

where we decomposed $\chi_i \equiv \chi_{i,} + \chi_{i,v}$. Now we take the maximal slicing as the temporal gauge condition

$$\kappa \equiv 0.$$  

(63)

Thus

$$\chi = -\frac{12\pi G}{c^3}\Delta^{-2}\gamma^i m_{i},$$

(64)

$$\chi_{i,v} = \frac{16\pi G}{c^3}\Delta^{-1}(m_{i} - \Delta^{-1}\gamma^i\gamma^{j}m_{j}).$$

(65)

These give Equation (14).

Considering Equations (15), (64) and (65) give

$$\chi_{i,v} \sim \frac{\gamma^i_{,j} \gamma^j_{,i} \gamma^l_{,l}}{c},$$

(66)

thus we have

$$\chi_{i} \sim \chi_{i,v} \ll \frac{\gamma^i}{c},$$

(67)

and $N = 1$.

The energy and momentum conservation equations in Equations (84) and (85), respectively, give

$$\dot{E} + \nabla m^2 c^2 = -\mathcal{G}(2\Phi - \Psi),\nabla^j v^i,$$

(68)

$$\dot{m} + \nabla m^2 \psi = -\mathcal{G}\Psi.$$

(69)

The continuity equation in Equation (86) gives

$$\dot{D} + \nabla(D v^i) = 0.$$  

(70)

These are three equations in Equation (3). The derivation of Equation (68) deserves a special comment. It is important to carefully keep the gravity term on the right side as explained above Equation (23). As we examine Equation (84) we notice that the first term in the equation leads to $3\mathcal{G}\Psi$ to the gravity part, which is of the same order as we consider $\Psi \sim \Psi_{ij} v^j$. This term, however, exactly cancels the $\chi^i$-term in the second line because of Equation (71).

The trace of ADM propagation and energy constraint equations in Equations (82) and (80), respectively, give Equations (10) and (11). We can show that the traceless part of ADM propagation in Equation (83) simply gives a combination of Equations (10) and (11).

Finally, Equation (79) gives

$$c\Delta \chi = \frac{3\Psi}{c^2},$$

(71)

and using Equations (11), (64), and (68), we can show that this is naturally valid. This calls for comment as the validity of Equation (71) in our approximation misses the gravity term in Equation (68) in the derivation. This is because Equation (71) is already a 1PN order, whereas our approximation is zeroth-order PN(0PN) in gravity while exact in matter part. Equation (79) is the definition of the trace of extrinsic curvature $K_i^i$ and its PN nature is presented in Equation (55) of Hwang et al. (2008).

Thus, using the complete set of Einstein’s equations, we have shown the consistency of our SR MHD equations with weak self-gravity presented in Section 2.1.

In the weak gravity limit, the effect of gravity does not appear in the Maxwell equations. Equations (87)–(90) become

$$B'_{ij} = 0,$$

(72)

$$B' = (v' B)_{j} = [\nabla \times (\nu \times B)]_{j},$$

(73)

$$\nabla \cdot E = 4\pi \phi_{,j},$$

(74)

$$\frac{1}{c} E = \nabla \times B - \frac{4\pi}{c} j,$$

(75)

with

$$E = -\frac{1}{c} \nu \times B,$$

(76)

in the ideal MHD. Equations (72) and (73) can be written as

$$\nabla \cdot B = 0, \quad \frac{1}{c} B = -\nabla \times E.$$  

(77)

These are the well known form of Maxwell’s equations valid for SR MHD with Equation (76).

Thus we have derived the equations in Section 2.1.

6. Discussion

The two formulations of relativistic MHD are the new results in this work. These are (i) the GR MHD in the fully nonlinear and exact perturbation formulation of Einstein’s gravity, and (ii) the SR MHD with weak gravity.

The fully nonlinear and exact perturbation formulation of ideal MHD in Einstein’s gravity is derived in Section 4 and the equations are presented in the Appendix. These are exact equations using perturbation variables added on the Minkowski metric; see Equation (1). We have ignored the transverse-traceless perturbation, but including this as well as not imposing the spatial gauge condition can be trivially achieved; the equations may look complicated though. For a general hydrodynamic fluid in such a general case, see Gong et al. (2017).

By taking the weak gravity and action-at-a-distance limits, we derived a consistent formulation of fully SR MHD with weak gravity; see Sections 2.1 and 5. We show that the role of gravity on the dynamics is effectively the same as in the Newtonian limit. However, the SR effects of the fluid and magnetic field affects the metric, thus gravitational potentials, and these could affect the gravitational lensing, see Section 2.3. The SR effects, if important, might cause overestimation of the lensing mass, see Equation (22).

The weak gravity formulation is derived in the maximal slicing ($\kappa \equiv 0$), which is the unique gauge choice with a consistent weak gravity limit. A similar choice of the zero-shear gauge (often termed as longitudinal or conformal
Newtonian gauge), taking $\chi \equiv 0$ as the slicing condition, leads to an inconsistent result by omitting the pressure term (see Section 2.3 in Hwang & Noh 2016).

Our weak gravity approximation is complementary to the PN approximation. The PN approximation perturbatively expands both gravity and matter consistently. To 1PN order we keep leading order in

$$\frac{\Phi}{c^2} \sim \frac{\Psi}{c^2} \sim \frac{v^2}{c^2} \sim \frac{\Pi}{\nabla c^2} \sim \frac{\Pi_{ij}}{\nabla c^2} \sim \frac{B^2}{\nabla c^2} \ll 1$$

(Chandrasekhar 1965; Greenberg 1971; Hwang et al. 2008; Nazari & Roshan 2018). In this sense our weak gravity approximation with full SR is a 0PN approximation in gravity ($\Phi$, $\Psi$ and $\chi_{ij}$) but exact in the matter part (velocity, internal energy, pressure, stress, magnetic field, etc.), and thus handles the matter part to $\infty$ PN order. It is not a priori clear that such an asymmetric formulation is possible, and here we have shown that it is indeed possible. Extending the program to include gravity to 1PN order might be feasible.

As the derivations presented in Sections 4 and 5 show, our two formulations are robust (consistent in the level of complete equations of GR), but final justification of the usefulness may demand numerical implementation that is beyond our reach at the moment. The complete set of equations in the Appendix is presented without taking the temporal gauge condition and the equations are redundantly complete as we have seen in the derivation of weak gravity limit in Section 5. Including the TT mode is straightforward, as done in the hydrodynamic case in Gong et al. (2017). In the astrophysical region with weak gravity our formulation of SR MHD with weak gravity might be useful and much easier in the numerical implementation, as the equations are similar to the Newtonian MHD. The validity and usefulness of the approximation can be checked by comparing with a full numerical relativity simulation in the same gauge (the maximal slicing together with our spatial gauge condition taken).

Cosmological extension of the present formulations can be made as in the hydrodynamic case presented in Noh et al. (2018). The fully nonlinear and exact perturbation formulation was originally made in the context of cosmology (Hwang & Noh 2013a). The cosmological formulation includes the one in a Minkowski background, but not vice versa. Cosmological MHD might have rich applications explaining the origin and distribution of cosmic magnetic field (for reviews, see Grasso & Rubinstein 2001; Widrow 2002; Giovannini 2004, 2018; Barrow et al. 2007; Subramanian 2016). It is not entirely clear whether we need the heavy machinery of fully nonlinear and exact perturbation theory in any part of the currently popular cosmological scenario. Still nonlinear perturbations are essential to get the secondary (i.e., non-primordial) generation of the magnetic field in the Biermann mechanism or the Harrison mechanism (Biermann 1950; Harrison 1970). The magnetic field also generates density perturbation and affects the cosmic microwave background temperature and polarization anisotropies at nonlinear order (Wasserman 1978; Ade et al. 2016).

We wish to thank Professor Dongsu Ryu for encouraging us to pursue this subject. We thank Professor M. James Lee for a useful discussion on gravitational lensing. We would like to thank the anonymous referee for thoughtful suggestions that helped improving the manuscript. H.N. was supported by the National Research Foundation of Korea, funded by the Korean Government (No. 2018R1A2B6002466). J.H. was supported by Basic Science Research Program through the National Research Foundation (NRF) of Korea funded by the Ministry of Science, ICT and future Planning (No. 2016R1A2B4007964, NRF-2019R1A2C1003031, and No. 2018R1A6A1A06024970). M.B. thanks SKA South Africa as well as an NRF KIC grant for partial support.

### Appendix

#### General Relativistic Ideal MHD Equations

Here we present the complete set of Einstein’s equation for a general fluid with ideal MHD. These equations without MHD are presented in Appendix B of Hwang & Noh (2016). The MHD parts can be included by replacing the fluid quantities as $\mu \to \mu + \mu^\text{MHD}$, $p \to p + p^\text{MHD}$, $\Pi_{ij} \to \Pi_{ij} + \Pi^\text{MHD}_{ij}$ using Equation (60). We have not taken the temporal gauge (slicing, hypersurface) condition. All spatial indices in the Appendix are raised and lowered using $\delta_{ij}$ as the metric.

The definition of $\kappa$ (the trace of extrinsic curvature $K_{ij} \equiv \kappa / c$):

$$\kappa \equiv -\frac{1}{N(1+2\varphi)} \left[ 3\varphi + c \left( \chi_{,k} + \chi_{,k} \varphi_{,k} \right) 1/2 \right].$$

ADM energy constraint:

$$\frac{4\pi G}{c^2} \mu + \frac{c^2 \Delta \varphi}{(1+2\varphi)^2} = -\frac{1}{6} \kappa^2 + \frac{3}{2(1+2\varphi)^3} \frac{c^2 \varphi_{,i} \varphi_{,i}}{4} - \frac{c^2}{4} K_{ij} K_{ij}^\prime$$

$$- \frac{4\pi G}{c^2} \left\{ (\mu + p)(\gamma^2 - 1) + \frac{1}{1+2\varphi} \Pi_{ij} \right\}$$

$$+ \frac{1}{8\pi} \frac{1}{1+2\varphi} \left\{ 2 - \frac{1}{\gamma^2} B_t^2 - \frac{1}{1+2\varphi} \left( B_t^2 \varphi \right)^2 \right\}.$$  

ADM momentum constraint:

$$\frac{2}{3} \kappa_{ij} + \frac{c}{N(1+2\varphi)} \left[ \left( \frac{1}{2} \Delta \chi_i + \frac{1}{6} \chi_{,k} \chi_{,kj} \right) \right] = \frac{c}{N(1+2\varphi)}$$

$$\times \left\{ \left( \frac{N_{,j}}{N} - \frac{\partial_j}{1+2\varphi} \right) \left( \frac{1}{2} \chi_{,j} + \chi_{,j} \varphi_{,j} \right) - \frac{\delta^i_j}{3} \chi_{,k} \chi_{,kj} \right\}$$

$$- \frac{\partial^i_j}{(1+2\varphi)^2} \left( \chi_{,i} \varphi_{,j} + \chi_{,i} \varphi_{,j} - \frac{2}{2} \delta^i_j \chi_{,k} \varphi_{,k} \right)$$

$$\times \left\{ \frac{1}{N} \left( \chi_{,i} \varphi_{,j} + \chi_{,i} \varphi_{,j} - \frac{2}{2} \delta^i_j \chi_{,k} \varphi_{,k} \right) \right\}$$

$$- \frac{8\pi G}{c^4} \left\{ (\mu + p) \gamma^2 \nu_{ij} + \frac{1}{1+2\varphi} \Pi_{ij} \nu_{ij} \right\}$$

$$+ \frac{1}{4\pi} \frac{1}{1+2\varphi} \left( B^2 \nu_{ij} - B_i B_j \nu_{ij} \right).$$
Trace of ADM propagation:

\[
-\frac{4\pi G}{c^2}(\mu + 3\rho) + \frac{1}{N}\kappa + \frac{c^2\Delta N}{N(1 + 2\varphi)} = \frac{1}{3}
\]

\[
- \frac{c}{N(1 + 2\varphi)}(\chi^i\kappa_j + c\varphi_p^i N_j + \frac{c^2\mathcal{K}_j^i}{1 + 2\varphi}) + c^2 \mathcal{K}_j^i
\]

\[
+ \frac{8\pi G}{c^2}\left\{(\mu + p)(\gamma^2 - 1) + \frac{1}{1 + 2\varphi} \Pi^i_j + \frac{1}{8\pi}
\right\} \frac{1}{1 + 2\varphi}\left[\left(2 - \frac{1}{\gamma^2}\right) B^2 - \frac{1}{1 + 2\varphi}\left(\frac{B\psi_j}{c}\right)^2\right]\right\}
\]

(82)

Tracefree ADM propagation:

\[
\left(\frac{1}{N} \frac{\partial}{\partial t} - \kappa + \frac{c\chi^k}{N(1 + 2\varphi) \nabla_k}\right)\left\{\frac{c}{N(1 + 2\varphi)}\left[\frac{1}{2}(\chi^j \psi_j + \chi^j \psi^j)
\right] - \frac{3}{2}\delta^j_k \chi^k - \frac{1}{1 + 2\varphi}\left(\chi^i \psi_j + \chi^i \psi^j - \frac{2}{3}\delta^j_k \chi^i \psi^k\right) - \frac{1}{3} \delta^j_k \Delta N\right\}
\]

\[
\frac{c^2}{(1 + 2\varphi)^2}\left[\frac{1}{1 + 2\varphi}\left(\nabla^i \nabla_j - \frac{1}{3}\delta^i_j \nabla\right) \Delta \psi + \frac{c^2}{N^2(1 + 2\varphi)^2}\left[\frac{1}{2}(\chi^i \psi_j + \chi^i \psi^j - \frac{2}{3}\delta^j_k \chi^i \psi^k) - \frac{2}{(1 + 2\varphi)^2}(\chi^i \psi_j + \chi^i \psi^j - \frac{2}{3}\delta^j_k \chi^i \psi^k)\right]
\]

(84)

ADM energy conservation:

\[
\frac{1}{c}\left\{(1 + 2\varphi)^{3/2}\left[\mu + (\mu + p)(\gamma^2 - 1) + \frac{1}{1 + 2\varphi} \Pi^i_j\right)
\right\}
\]

\[
+ \frac{1}{1 + 2\varphi}\left\{\frac{1}{8\pi}\left[\left(2 - \frac{1}{\gamma^2}\right) B^2 - \frac{1}{1 + 2\varphi}\left(\frac{B\psi_j}{c}\right)^2\right]\right\}
\]

\[
\times \left\{(1 + 2\varphi)^{1/2}N\right\} \left\{\frac{1}{1 + 2\varphi}\left[\frac{1}{8\pi}\left[\left(2 - \frac{1}{\gamma^2}\right) B^2 - \frac{1}{1 + 2\varphi}\left(\frac{B\psi_j}{c}\right)^2\right]\right]\right\}
\]

\[
- \frac{1}{1 + 2\varphi}\left(\frac{(1 + 2\varphi)^{3/2}B^2}{c^2} + \frac{1}{1 + 2\varphi} \left(\frac{B\psi_j}{c}\right)^2\right)\right\}
\]

(83)

ADM momentum conservation:

\[
\frac{1}{c}\left\{(1 + 2\varphi)^{3/2}\left[\mu + (\mu + p)(\gamma^2 - 1) + \frac{1}{1 + 2\varphi} \Pi^i_j\right)
\right\}
\]

\[
+ \frac{1}{1 + 2\varphi}\left\{\frac{1}{8\pi}\left[\left(2 - \frac{1}{\gamma^2}\right) B^2 - \frac{1}{1 + 2\varphi}\left(\frac{B\psi_j}{c}\right)^2\right]\right\}
\]

(84)
Continuity equation, \((\overline{\gamma} u^\gamma)_{,\gamma} = 0:\)

\[
\frac{\partial}{\partial t} + \frac{1}{1 + 2 \varphi} (\overline{\gamma} u^i + c \chi^i) \nabla_i - \overline{\gamma} \kappa + \left( \frac{\overline{\gamma} u^i}{1 + 2 \varphi} + \frac{\overline{\gamma} \varphi u^i}{(1 + 2 \varphi)^2} \right)^2 \overline{\gamma} = 0. \tag{86}
\]

Maxwell’s equations:

\[
\nabla \cdot (\sqrt{1 + 2 \varphi} \mathbf{B}) = 0, \tag{87}
\]

\[
(\sqrt{1 + 2 \varphi} \mathbf{B}) = \nabla \times \left[ \frac{1}{\sqrt{1 + 2 \varphi}} (\overline{\gamma} \mathbf{v} + c \chi) \times \mathbf{B} \right]. \tag{88}
\]

\[
\frac{1}{1 + 2 \varphi} \frac{1}{c} \nabla \cdot (v \times \mathbf{B}) = 4 \pi \varrho_{eb}. \tag{89}
\]

\[
-\frac{1}{c^2} (v \times \mathbf{B}) = \nabla \times \left[ \overline{\gamma} \mathbf{B} - \frac{1}{c} \frac{1}{1 + 2 \varphi} (v \mathbf{B} \cdot \chi - v \cdot \chi \mathbf{B}) \right] - 4 \pi \sqrt{1 + 2 \varphi} (\varrho_{eb} \chi + \frac{1}{c} \overline{\gamma} \varphi). \tag{90}
\]

We have

\[
\overline{\kappa}_i \overline{\kappa}_j = \frac{1}{N^2 (1 + 2 \varphi)^2} \left\{ \frac{1}{2} \frac{\chi_i \chi_j}{(1 + 2 \varphi)} - \frac{1}{3} \chi^i \chi^j \right\} + \frac{4}{1 + 2 \varphi} \left\{ \frac{1}{2} \chi^i \varphi \chi^j + \frac{1}{3} \chi^i \chi^j \varphi \right\}, \]

\[
\Pi_i = \frac{1}{1 + 2 \varphi} \Pi_j \frac{\chi^{ij}}{c^2}.
\]

One of the following conditions can be imposed as the temporal gauge condition: (1) maximal slicing \((\kappa \equiv 0);\)

(2) zero-shear gauge (setting the longitudinal part of \(\chi\) to zero, so that \(\gamma \equiv 0);\)

(3) comoving gauge (setting the longitudinal part of \(\gamma\) to zero). These three gauge conditions, as well as various linear combinations thereof, completely remove both the spatial and temporal gauge modes. As such, all perturbation variables in these gauge conditions can be equivalently regarded as gauge-invariant variables to all perturbation orders. Another possible gauge condition is synchronous gauge, setting \(\alpha = 0\), but this gauge choice fails to fix the gauge modes completely. For a discussion of gauge transformation at linear order, see Equation (79) in Hwang & Noh (2016), and for higher (nonlinear) orders, see Noh & Hwang (2004).

**ORCID iDs**

Hyerim Noh \(\text{ORCID:} \) https://orcid.org/0000-0002-1855-9094

**References**

Abraham, R., Marsden, J. E., & Ratiu, T. 1988, Manifolds, Tensor Analysis, and Applications (2nd ed.; New York: Springer)

Ade, P. A. R., Aghanim, N., Arnaud, M., et al. 2016, A&A, 594, A19

Barrow, J. D., Maartens, R., & Tsagas, C. G. 2007, PhR, 449, 131

Baumgartner, T. W., & Shapiro, S. L. 2010, Numerical Relativity: Solving Einstein’s Equations on the Computer (New York: Cambridge Univ. Press)

Biermann, L. 1950, ZNAnna Phys. 5a, 65

Bona, C., Palenzuela-Luque, C., & Bona-Casas, C. 2009, Elements of Numerical Relativity and Relativistic Hydrodynamics: From Einstein’s Equations to Astrophysical Simulations (2nd ed.; New York: Springer)

Chandrasekhar, S. 1965, ApJ, 142, 148

Ellis, G. F. R. 1971, in Proc. Int. Summer School of Physics Enrico Fermi Course 47, General Relativity and Cosmology, ed. R. K. Sachs (New York: Academic), 104

Ellis, G. F. R. 1973, Cargèse Lectures Phys., 6, 1

Giovannini, M. 2004, JMPD, 13, 391

Giovannini, M. 2018, CQGra, 35, 084003

Gong, J., Hwang, J., Noh, H., Wu, D., & Yoo, J. 2017, ICAP, 10, 027

Gourgoulion, E. 2012, 3 + 1 Formalism in General Relativity: Bases of Numerical Relativity (New York: Springer)

Grasso, D., & Rubinstein, H. R. 2001, PhR, 348, 163

Greenberg, P. J. 1971, ApJ, 164, 589

Harrison, E. R. 1965, AnPhy, 35, 437

Harrison, E. R. 1970, MNRAS, 147, 279

Hwang, J., & Noh, H. 2013a, MNRAS, 433, 3472

Hwang, J., & Noh, H. 2013b, JCAP, 04, 035

Hwang, J., & Noh, H. 2016, ApJ, 833, 180

Hwang, J., Noh, H., & Park, C. G. 2016, MNRAS, 461, 3239

Hwang, J., Noh, H., & Puetzfeld, D. 2008, ICAP, 03, 010

Jackson, J. D. 1975, Classical Electrodynamics (2nd ed.; New York: Wiley)

Kaspi, V. M., & Beloborodov, A. M. 2017, ARA&A, 55, 261

McCrea, W. H. 1951, RSPSA, 206, 562
Mignone, A., Bodo, G., Massaglia, S., et al. 2007, ApJS, 170, 228
Nazari, E., & Roshan, M. 2018, ApJ, 868, 98
Noh, H. 2014, JCAP, 07, 037
Noh, H., & Hwang, J. 2004, PhRvD, 69, 104011
Noh, H., & Hwang, J. 2013, JCAP, 08, 040
Noh, H., Hwang, J., & Park, C.-G. 2018, JCAP, 11, 002
Oppenheimer, J. R., & Volkoff, G. M. 1939, PhRv, 55, 374

Subramanian, K. 2016, RPPh, 79, 076901
Tolman, R. C. 1930, PhRv, 35, 875
Tolman, R. C. 1939, PhRv, 55, 364
Wasserman, L. 1978, ApJ, 224, 337
Whittaker, E. T. 1935, RSPSA, 149, 384
Widrow, L. M. 2002, RvMP, 74, 775