GLUING RESTRICTED NERVES OF \(\infty\)-CATEGORIES

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Abstract. In this article, we develop a general technique for gluing subcategories of \(\infty\)-categories. We obtain categorical equivalences between simplicial sets associated to certain multisimplicial sets. Such equivalences can be used to construct functors in different contexts. One of our results generalizes Deligne’s gluing theory developed in the construction of the extraordinary pushforward operation in étale cohomology of schemes. Our results are applied in subsequent articles \([15, 16]\) to construct Grothendieck’s six operations in étale cohomology of Artin stacks.

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Introduction

The extraordinary pushforward, one of Grothendieck’s six operations, in étale cohomology of schemes was constructed in SGA 4 XVII \([6]\). Let \(\mathcal{S}ch\) be the category of quasi-compact and quasi-separated schemes, with morphisms being separated of finite type, and let \(\Lambda\) be a fixed torsion ring. For a morphism \(f: Y \to X\) in \(\mathcal{S}ch\), the extraordinary pushforward by \(f\) is a functor

\[ f!: D(Y, \Lambda) \to D(X, \Lambda), \]

between unbounded derived category of \(\Lambda\)-modules in the étale topoi. The functoriality of this operation is encoded by a pseudofunctor

\[ F: \mathcal{S}ch \to \mathcal{C}at_1 \]

sending a scheme \(X\) in \(\mathcal{S}ch\) to \(D(X, \Lambda)\) and a morphism \(f: Y \to X\) in \(\mathcal{S}ch\) to the functor \(f_!\). Here \(\mathcal{C}at_1\) denotes the \((2, 1)\)-category of categories\(^1\). There are obvious candidates for the restrictions \(F_P\) and \(F_J\) of \(F\) to the subcategories \(\mathcal{S}ch_P\) and \(\mathcal{S}ch_J\) of \(\mathcal{S}ch\) spanned respectively by proper morphisms and open immersions. The construction of \(F\) thus amounts to gluing the two pseudofunctors. For this, Deligne developed a general theory for gluing two pseudofunctors of target \(\mathcal{C}at_1\) \([6, \text{Section 3}]\). Deligne’s gluing theory, together with its variants (\([1, \text{Section 1.3}], [20]\)), have found several other applications (\([1], [4], [19]\)).

In the present article, we study the problem of gluing in a higher categorical context. The technique developed here is used in a subsequent series of articles \([15, 16]\) to construct Grothendieck’s six operations in étale cohomology of (higher) Artin stacks, and, in particular, to prove the Base Change theorem in derived categories. This theorem was previously only established on the level of sheaves (and subject to other restrictions) \([13, 14]\). Our construction of the six operations makes essential use of higher categorical descent, so that even if one is only interested in the six operations for ordinary categories, the higher categorical version is still an indispensable step of the construction. We refer the reader to the Introduction of \([15]\) for a

\(^1\)A \((2, 1)\)-category is a \(2\)-category in which all \(2\)-cells are invertible.
detailed explanation of our approach. As a starting point for the descent procedure, we need an enhancement of the pseudofunctor $F$ above, namely, a functor

$$F^\infty : N(Sch') \to \mathcal{C}at_{\infty}$$

between $\infty$-categories in the sense of [17], where $N(Sch')$ is the nerve of Sch' and $\mathcal{C}at_{\infty}$ denotes the $\infty$-category of $\infty$-categories. For every scheme $X$ in Sch', $F^\infty(X)$ is an $\infty$-category $\mathcal{D}(X, \Lambda)$, whose homotopy category is equivalent to $\mathcal{D}(X, \Lambda)$. For every morphism $f : Y \to X$ in Sch', $F^\infty(f)$ is a functor

$$f^\infty : \mathcal{D}(Y, \Lambda) \to \mathcal{D}(X, \Lambda)$$

such that the induced functor $hf_1^\infty$ between homotopy categories is equivalent to the classical $f_1$.

One major difficulty of the construction of $F^\infty$ is the need to keep track of coherence of all levels. By Nagata compactification [3], every morphism $f$ in Sch' can be factorized as $p \circ j$, where $j$ is an open immersion and $p$ is proper. One can then define $F(f)$ as $F(p(p) \circ F(j))$. The issue is that such a factorization is not canonical, so that one needs to include coherence with composition as part of the data. Since the target of $F$ is a $(2, 1)$-category, in Deligne’s theory coherence up to the level of 2-cells suffices. The target of $F^\infty$ being an $\infty$-category, we need to consider coherence of all levels.

Another complication is the need to deal with more than two subcategories. This need is already apparent in [19]. We will give another illustration in the proof of Corollary 0.4 below.

To handle these complications, we propose the following general framework. Let $\mathcal{C}$ be an (ordinary) category and let $k \geq 2$ be an integer. Let $\mathcal{E}_1, \ldots, \mathcal{E}_k \subseteq \text{Ar}(\mathcal{C})$ be $k$ sets of arrows of $\mathcal{C}$, each containing every identity morphism in $\mathcal{C}$. In addition to the nerve $N(\mathcal{C})$ of $\mathcal{C}$, we define another simplicial set, which we denote by $\delta_2^\infty N(\mathcal{C})_{\mathcal{E}_1, \ldots, \mathcal{E}_k}$. Its $n$-simplices are functors $[n]^k \to \mathcal{C}$ such that the image of a morphism in the $i$-th direction is in $\mathcal{E}_i$ for $1 \leq i \leq k$, and the image of every square in direction $(i, j)$ is a Cartesian square for $1 \leq i < j \leq k$. For example, when $k = 2$, the $n$-simplices of $\delta_2^\infty N(\mathcal{C})_{\mathcal{E}_1, \mathcal{E}_2}$ correspond to diagrams

\[
\begin{array}{cccc}
& c_{00} & \cdots & c_{0n} \\
&  &  & \\
c_{10} & c_{11} & \cdots & c_{1n} \\
&  &  & \\
\vdots & \vdots & \ddots & \vdots \\
&  &  & \\
c_{n0} & c_{n1} & \cdots & c_{nn}
\end{array}
\]

(0.1)

where vertical (resp. horizontal) arrows are in $\mathcal{E}_1$ (resp. $\mathcal{E}_2$) and all squares are Cartesian. The face and degeneration maps are defined in the obvious way. Note that $\delta_2^\infty N(\mathcal{C})_{\mathcal{E}_1, \ldots, \mathcal{E}_k}$ is seldom an $\infty$-category. It is the simplicial set associated to a $k$-simplicial set $N(\mathcal{C})_{\mathcal{E}_1, \ldots, \mathcal{E}_k}$. The latter is a special case of what we call the \textit{restricted multisimplicial nerve} of an ($\infty$)-category with extra data (Definition 3.6).

Let $\mathcal{E}_0 \subseteq \text{Ar}(\mathcal{C})$ be a set of arrows stable under composition and containing $\mathcal{E}_1$ and $\mathcal{E}_2$. Then there is a natural map

\[
g : \delta_2^\infty N(\mathcal{C})_{\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \ldots, \mathcal{E}_k} \to \delta_2^\infty N(\mathcal{C})_{\mathcal{E}_0, \mathcal{E}_3, \ldots, \mathcal{E}_k}
\]

of simplicial sets, sending an $n$-simplex of the source corresponding to a functor $[n]^k \to \mathcal{C}$, to its partial diagonal

$$[n]^{k-1} = [n] \times [n] \xrightarrow{\text{diag} \times \text{id}_{[n]^{k-2}}} [n]^k = [n]^2 \times [n]^{k-2} \to \mathcal{C},$$

which is an $n$-simplex of the target.

We say that a subset $\mathcal{E} \subseteq \text{Ar}(\mathcal{C})$ is \textit{admissible} (Definition 3.9) if $\mathcal{E}$ contains every identity morphism, $\mathcal{E}$ is stable under pullback, and for every pair of composable maps $p \in \mathcal{E}$ and $q$ in $\mathcal{C}$, $p \circ q$ is in $\mathcal{E}$ if and only if $q \in \mathcal{E}$. One main result of the article is the following.

**Theorem 0.1** (Special case of Corollary 5.3 and Remark 5.4). Let $\mathcal{C}$ be a category admitting pullbacks and let $\mathcal{E}_0, \mathcal{E}_1, \ldots, \mathcal{E}_k \subseteq \text{Ar}(\mathcal{C})$, $k \geq 2$, be sets of morphisms containing every identity morphism and stable under pullback (in particular, $\mathcal{E}_i$ contains every isomorphism for $0 \leq i \leq k$) and satisfying the following conditions:
(1) $E_1, E_2 \subseteq E_0$ and $E_1, E_2$ are admissible.
(2) For every morphism $f$ in $E_0$, there exist $p \in E_1$ and $q \in E_2$ such that $f = p \circ q$.

Then the natural map (0.2)

$$g: \delta_k^* N(E)_{E_1, E_2, \ldots, E_k} \to \delta_{k-1}^* N(E)_{E_0, \ldots, E_k}$$

is a categorical equivalence (Definition 1.3).

Taking $k = 2$ and $E_0 = \text{Ar}(E)$ we obtain the following.

**Corollary 0.2.** Let $E$ be a category admitting pullbacks. Let $E_1, E_2 \subseteq \text{Ar}(E)$ be admissible subsets. Assume that for every morphism $f$ of $E$, there exist $p \in E_1$ and $q \in E_2$ such that $f = p \circ q$. Then the natural map

$$g: \delta_2^* N(E)_{E_1, E_2} \to N(E)$$

is a categorical equivalence.

In the situation of Corollary 0.2, for every $\infty$-category $D$, the functor

$$\text{Fun}(N(E), D) \to \text{Fun}(\delta_2^* N(E)_{E_1, E_2}, D)$$

is an equivalence of $\infty$-categories. We remark that such equivalences can be used to construct functors in many different contexts. For instance, we can take $D$ to be $N(\text{Cat}_1)^2, \text{Cat}_\infty$, or the $\infty$-category of differential grades categories.

In the above discussion, we may replace $N(E)$ by an $\infty$-category $E$ (not necessarily the nerve of an ordinary category), and define $\delta_k^* E_{E_1, \ldots, E_k}$. Moreover, in [15], we need to encode information such as the Base Change isomorphism, which involves both pullback and (extraordinary) pushforward. To this end, we will define in Section 3, for every subset $L \subseteq \{1, \ldots, k\}$, a variant $\delta_k^* E_{E_1, \ldots, E_k}$ of $\delta_k^* E_{E_1, \ldots, E_k}$ by “taking the opposite” in the directions in $L$. For $L \subseteq \{3, \ldots, k\}$, the theorem remains valid modulo slight modifications. We refer the reader to Corollary 5.3 for a precise statement.

Next we turn to applications to categories of schemes.

**Corollary 0.3.** Let $P \subseteq \text{Ar}(\text{Sch}'')$ be the subset of proper morphisms and let $I \subseteq \text{Ar}(\text{Sch}')$ be the subset of open immersions. Then the natural map

$$\delta_2^* N(\text{Sch}'')_{P, I} \to N(\text{Sch}')$$

is a categorical equivalence.

*Proof.* This follows immediately from Corollary 0.2 applied to $E = \text{Sch}'$, $E_1 = P$, $E_2 = I$. $\square$

As many important moduli stacks are not quasi-compact, in [15] we work with Artin stacks that are not necessarily quasi-compact. Accordingly, we need the following variant of Corollary 0.3.

**Corollary 0.4.** Let $\text{Sch}''$ be the category of disjoint unions of quasi-compact and quasi-separated schemes, with morphisms being separated and locally of finite type. Let $F = \text{Ar}(\text{Sch}'')$, let $P \subseteq F$ be the subset of proper morphisms, and let $I \subseteq F$ be the subset of local isomorphisms [10, Définition 4.4.2]. Then the natural map

$$\delta_2^* N(\text{Sch}'')_{P, I} \to N(\text{Sch}'')$$

is a categorical equivalence.

Corollary 0.4 still holds if one replaces $I$ by the subset $E \subseteq F$ of étale morphisms.

One might be tempted to apply Corollary 0.2 by taking $E_1 = P$, $E_2 = I$. However, the assumption of Corollary 0.2 does not hold in this case.

*Proof of Corollary 0.4.* Let $E = \text{Sch}'$. We introduce the following auxiliary sets of morphisms. Let $F_R \subseteq F$ be the set of separated morphisms of finite type, and let $I_R = I \cap F_R$. Consider the following commutative diagram

$$\begin{array}{c}
\delta_3^* N(E)_{P, I_R, I} \quad \delta_3^* N(E)_{P, I_R, I} \\
\downarrow \quad \downarrow \\
\delta_2^* N(E)_{P, I} \quad N(E),
\end{array}$$

Here $N(\text{Cat}_1)$ denotes the simplicial nerve [17, Définition 1.1.5.5] of $\text{Cat}_1$, the latter regarded as a simplicial category.
where the upper arrow is induced by “composing morphisms in $P$ and $I_R$”, while the left arrow is induced by “composing morphisms in $I_R$ and $I$”. We will apply Theorem 0.1 three times to show that all arrows in the diagram except the lower arrow are categorical equivalences. It then follows that the lower arrow is also a categorical equivalence.

To show that the upper arrow is a categorical equivalence, we take $k = 3$, $\mathcal{E}_0 = F_R$, $\mathcal{E}_1 = P$, $\mathcal{E}_2 = I_R$ and $\mathcal{E}_3 = I$ in the theorem. Condition (1) is obviously satisfied, and (2) follows from Nagata compactification. Similarly, to show that the left (resp. right) arrow is a categorical equivalence, we take $k = 3$, $\mathcal{E}_0 = \mathcal{E}_1 = I$, $\mathcal{E}_2 = I_R$, $\mathcal{E}_3 = P$ (resp. $k = 2$, $\mathcal{E}_0 = F$, $\mathcal{E}_1 = I$, $\mathcal{E}_2 = F_R$) in the theorem. \square

The proof of Theorem 0.1 consists of two steps. Let us illustrate them in the case of Corollary 0.2. The map $g$ can be decomposed as

$$\delta_2^s N(\mathcal{E})_{\xi_1,\xi_2} \xrightarrow{g'} \delta_2^s N(\mathcal{E})_{\xi_1,\xi_2} \xrightarrow{g''} N(\mathcal{E}),$$

where $\delta_2^s N(\mathcal{E})_{\xi_1,\xi_2}$ is the simplicial set whose $n$-simplices are diagrams (0.1) without the requirement that every square is Cartesian, $g'$ is the natural inclusion and $g''$ is the morphism remembering the diagonal. We prove that both $g'$ and $g''$ are categorical equivalences. The fact that $g''$ is a categorical equivalence is an $\infty$-categorical generalization of Deligne’s result [6, Proposition 3.3.2].

The article is organized as follows. In Section 1, we collect some basic definitions and facts in the theory of $\infty$-categories [17] for the reader’s convenience. In Section 2, we develop a general technique for constructing functors to $\infty$-categories. The method will be used several times in this article and its sequels [15, 16]. In Section 3, we introduce several notions related to multisimplicial sets used in the statements of our main results. In particular, we define the restricted multisimplicial nerve of an $\infty$-category with extra data. In Section 4, we prove a multisimplicial descent theorem, which implies that the map $g''$ is a categorical equivalence. In Section 5, we prove a Cartesian gluing theorem, which implies that the inclusion $g'$ is a categorical equivalence. A Cartesian gluing formalism for pseudofunctors between 2-categories was developed in [20]. Our treatment here is quite different and more adapted to the higher categorical context. In Section 6, we prove some facts about inclusions of simplicial sets used in the previous sections.

**Conventions.** Unless otherwise specified, a category is to be understood as an ordinary category. We will not distinguish between sets and categories in which the only morphisms are identity morphisms. Let $\mathcal{C}, \mathcal{D}$ be two categories. We denote by $\text{Fun}(\mathcal{C}, \mathcal{D})$ the category of functors from $\mathcal{C}$ to $\mathcal{D}$, whose objects are functors and whose morphisms are natural transformations. We let $\text{id}$ denote various identity maps.

Throughout this article, an effort has been made to keep our notation consistent with those in [17].

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1. Simplicial sets and $\infty$-categories

In this section, we collect some basic definitions and facts in the theory of $\infty$-categories developed by Joyal [11, 12] (who calls them “quasi-categories”) and Lurie [17]. For a more systematic introduction to Lurie’s theory, we recommend [9].

For $n \geq 0$, we let $[n]$ denote the totally ordered set $\{0, \ldots, n\}$ and we put $[-1] = \emptyset$. We let $\Delta$ denote the category of combinatorial simplices, whose objects are the totally ordered sets $[n]$ for $n \geq 0$ and whose morphisms are given by (non-strictly) order-preserving maps. For $n \geq 0$ and $0 \leq k \leq n$, the face map $d_k^n: [n-1] \to [n]$ is the unique injective order-preserving map such that $k$ is not in the image; and the degeneracy map $s_k^n: [n+1] \to [n]$ is the unique surjective order-preserving map such that $(s_k^n)^{-1}(k)$ has two elements.

**Definition 1.1** (Simplicial set and $\infty$-category). We let $\textbf{Set}$ denote the category of sets$^3$.

$^3$More rigorously, $\textbf{Set}$ is the category of sets in a universe in which we fix once and for all.
There are several equivalent definitions of categorical equivalence. The one given below (equivalent to
respect to all inner fibrations (resp. Kan fibrations).

An \(\infty\)-category (resp. Kan complex) is a simplicial set \(\mathcal{C}\) such that \(\mathcal{C} \to \ast\) has the right lifting property with respect to all inclusions \(\Lambda^n_k \subseteq \Delta^n\) with \(0 < k < n\) (resp. \(0 \leq k \leq n\)). In other words, a simplicial set \(\mathcal{C}\) is an \(\infty\)-category (resp. Kan complex) if and only if every map \(\Lambda^n_k \to \mathcal{C}\) with \(0 < k < n\) (resp. \(0 \leq k \leq n\)) can be extended to a map \(\Delta^n \to \mathcal{C}\).

Note that a Kan complex is a simplicial set. The lifting property in the definition of \(\infty\)-category was first introduced (under the name of “restricted Kan condition”) by Boardman and Vogt [2, Definition IV.4.8].

The category \(\mathcal{S}\) is Cartesian-closed. For simplicial sets, we let \(\text{Map}(S, T)\) denote the internal mapping object defined by \(\text{Hom}_{\mathcal{S}}(K, \text{Map}(S, T)) \simeq \text{Hom}_{\mathcal{S}}(K \times S, T)\). If \(\mathcal{C}\) is an \(\infty\)-category, we write \(\text{Fun}(S, \mathcal{C})\) instead of \(\text{Map}(S, \mathcal{C})\). It is not difficult to see that \(\text{Fun}(S, \mathcal{C})\) is an \(\infty\)-category [17, Proposition 1.2.7.3 (1)].

Example 1.2. Let \(\mathcal{C}\) be an ordinary category. The nerve \(\mathcal{N}(\mathcal{C})\) of \(\mathcal{C}\) is the simplicial set given by \(\mathcal{N}(\mathcal{C})_n = \text{Fun}(n, \mathcal{C})\). It is easy to see that \(\mathcal{N}(\mathcal{C})\) is an \(\infty\)-category and we can identify \(\mathcal{N}(\mathcal{C})_0\) and \(\mathcal{N}(\mathcal{C})_1\) with the set of objects \(\text{Ob}(\mathcal{C})\) and the set of arrows \(\text{Ar}(\mathcal{C})\), respectively. Conversely, from an \(\infty\)-category \(\mathcal{C}\), one constructs an ordinary category \(\text{h\cal C}\), the homotopy category of \(\mathcal{C}\) [17, Definition 1.2.2.1], ignoring the enrichment), such that \(\text{Ob}(\text{h\cal C}) = \mathcal{C}_0\) and \(\text{Hom}_{\text{h\cal C}}(X, Y)\) consists of homotopy classes in \(\mathcal{C}_1\). By [17, Proposition 1.2.3.1], \(\text{h\cal C}\) is a left adjoint to the nerve functor \(\mathcal{N}\).

The lifting property defining \(\infty\)-category (resp. Kan complex) can be adapted to the relative case. More precisely, a map \(f : T \to S\) of simplicial sets is called an inner fibration (resp. Kan fibration) if it has the right lifting property with respect to all inclusions \(\Lambda^n_k \subseteq \Delta^n\) with \(0 < k < n\) (resp. \(0 \leq k \leq n\)). A map \(i : A \to B\) of simplicial sets is said to be inner anodyne (resp. anodyne) if it has the left lifting property with respect to all inner fibrations (resp. Kan fibrations).

We now recall the notion of categorical equivalence of simplicial sets, which is essential to our article. There are several equivalent definitions of categorical equivalence. The one given below (equivalent to [17, Definition 1.1.5.14] in view of [17, Proposition 2.2.5.8]), due to Joyal [12], will be used in the proofs of our theorems.

Definition 1.3 (Categorical equivalence). A map \(f : T \to S\) of simplicial sets is a \(\text{categorical equivalence}\) if for every \(\infty\)-category \(\mathcal{C}\), the induced functor

\[
\text{hFun}(S, \mathcal{C}) \to \text{hFun}(T, \mathcal{C})
\]

is an equivalence of ordinary categories.

In Section 3, we will introduce the notion of multi-marked simplicial sets, which generalizes the notion of marked simplicial sets in [17, Definition 3.1.0.1]. Since marked simplicial sets play an important role in many arguments for \(\infty\)-categories, we briefly recall its definition.

A marked simplicial set is a pair \((X, \mathcal{E})\) where \(X\) is a simplicial set and \(\mathcal{E} \subseteq X_1\) that contains all degenerate edges. A morphism \(f : (X, \mathcal{E}) \to (X', \mathcal{E}')\) of marked simplicial sets is a map \(f : X \to X'\) of simplicial sets such that \(f(\mathcal{E}) \subseteq \mathcal{E}'\). The category of marked simplicial set will be denoted by \(\text{Set}_\Delta^+\). The underlying functor \(\text{Set}_\Delta^+ \to \text{Set}_\Delta\) carrying \((X, \mathcal{E})\) to \(X\) admits a right adjoint carrying a simplicial set \(S\) to \(S^\sharp = (S, S_1)\) and a left adjoint carrying \(S\) to \(S^\natural = (S, \mathcal{E})\), where \(\mathcal{E}\) is the set of all degenerate edges. For an \(\infty\)-category \(\mathcal{C}\), we let \(\mathcal{C}^\natural\) denote the marked simplicial set \((\mathcal{C}, \mathcal{E})\), where \(\mathcal{E}\) is the set of all edges of \(\mathcal{C}\) that are equivalences. In the following sections, we will use the Cartesian model structure on the category \(\text{Set}_\Delta^+/S\) of marked simplicial sets over \(S^\natural\) constructed in [17, Section 3.1.3].

We conclude this section with the following simple criterion of equivalence.

Lemma 1.4. Let \(K\) be a simplicial set, let \(\mathcal{C}\) be an \(\infty\)-category, let \(f, g : K \to \mathcal{C}\) be two functors and let \(\phi : f \to g\) be a natural transformation, that is, a morphism of \(\text{Fun}(K, \mathcal{C})\). Then \(\phi\) is a natural equivalence [17, Notation 1.2.7.2] if and only if for every vertex \(k\) of \(K\), the map \(\phi(k) : f(k) \to g(k)\) is an equivalence [17, Section 1.2.4].
Proof. The necessity is trivial. We prove the sufficiency. By assumption, $\phi$ is an edge of $\text{Map}_\mathcal{C}^\ast(K^\ast, \mathcal{C}) \subseteq \text{Fun}(K, \mathcal{C})$ [17, Section 3.1.3]. By [17, Lemma 3.1.3.6], $\text{Map}_\mathcal{C}^\ast(K^\ast, \mathcal{C})$ is a Kan complex. It follows that $\phi$ is an equivalence in $\text{Fun}(K, \mathcal{C})$. \qed

2. Constructing functors via the category of simplices

In this section, we develop a general technique for constructing functors to $\infty$-categories, which is the key to several constructions in this article and its sequels. For a functor $F: K \to \mathcal{C}$ from a simplicial set $K$ to an $\infty$-category $\mathcal{C}$, the image $F(\sigma)$ of a simplex $\sigma$ of $K$ is a simplex of $\mathcal{C}$, functorial in $\sigma$. Here we address the problem of constructing $F$ when, instead of having a canonical choice for $F(\sigma)$, one has a weakly contractible simplicial set $N(\sigma)$ of candidates for $F(\sigma)$.

We start with some generalities on diagrams of simplicial sets. Let $I$ be an ordinary category. We say that a morphism $R \to R'$ in the functor category $(\text{Set}_\Delta)^I := \text{Fun}(I, \text{Set}_\Delta)$ is an injective fibration if it has the right lifting property with respect to every morphism $N \to M$ such that $N(\sigma) \to M(\sigma)$ is anodyne for all objects $\sigma$ of $I$. We say that an object $R$ of $(\text{Set}_\Delta)^I$ is injectively fibrant if the morphism from $R$ to the final object $\Delta^0_I$ is an injective fibration. The right adjoint of the diagonal functor $\text{Set}_\Delta \to (\text{Set}_\Delta)^I$ is the global section functor

$$\Gamma: (\text{Set}_\Delta)^I \to \text{Set}_\Delta, \quad \Gamma(N)_q = \text{Hom}_{(\text{Set}_\Delta)^I}(\Delta^q_I, N),$$

where $\Delta^q_I: I \to \text{Set}_\Delta$ is the constant functor of value $\Delta^q$.

Notation 2.1. Let $\Phi: N \to \mathcal{R}$ be a morphism of $(\text{Set}_\Delta)^I$. We let $\Gamma_\Phi(\mathcal{R}) \subseteq \Gamma(\mathcal{R})$ denote the simplicial subset, union of the images of $\Gamma(\Psi): \Gamma(M) \to \Gamma(\mathcal{R})$ for all decompositions

$$N \hookrightarrow M \xrightarrow{\Phi} \mathcal{R}$$

of $\Phi$ such that $N(\sigma) \to M(\sigma)$ is anodyne for all objects $\sigma$ of $I$.

The map $\Gamma(\Phi): \Gamma(N) \to \Gamma(\mathcal{R})$ factors through $\Gamma_\Phi(\mathcal{R})$. For a commutative diagram

$$\begin{array}{ccc}
N & \xrightarrow{\Phi} & \mathcal{R} \\
\downarrow & & \downarrow \\
N' & \xrightarrow{\Phi'} & \mathcal{R}'
\end{array}$$

in $(\text{Set}_\Delta)^I$, $\Gamma(F): \Gamma(\mathcal{R}) \to \Gamma(\mathcal{R}')$ carries $\Gamma_\Phi(\mathcal{R})$ into $\Gamma_{\Phi'}(\mathcal{R}')$.

For a functor $g: I' \to I$, composition with $g$ induces a functor $g^*: (\text{Set}_\Delta)^I \to (\text{Set}_\Delta)^{I'}$. The map $g^*: \Gamma(\mathcal{R}) \to \Gamma(g^*\mathcal{R})$ carries $\Gamma_\Phi(\mathcal{R})$ into $\Gamma_{g^*\Phi}(g^*\mathcal{R})$.

Lemma 2.2. Let $I$ be a category. Let $N, \mathcal{R}$ be objects of $(\text{Set}_\Delta)^I$ such that $N(\sigma)$ is weakly contractible for all objects $\sigma$ of $I$ and $\mathcal{R}$ is injectively fibrant.

\begin{enumerate}
\item For every morphism $\Phi: N \to \mathcal{R}$, $\Gamma_\Phi(\mathcal{R})$ is a weakly contractible Kan complex.
\item For homotopic morphisms $\Phi, \Phi': N \to \mathcal{R}$, $\Gamma_\Phi(\mathcal{R})$ and $\Gamma_{\Phi'}(\mathcal{R})$ lie in the same connected component of $\Gamma(\mathcal{R})$.
\end{enumerate}

The condition in (2) means that there exits a morphism $H: \Delta^1_I \times N \to \mathcal{R}$ such that $H|\Delta^0_I \times N = \Phi$ and $H|\Delta^1_I \times N = \Phi'$. Note that $\Gamma(\mathcal{R})$ is a Kan complex.

Proof. (1) By definition, $\Gamma_\Phi(\mathcal{R})$ is the colimit of the image of $\Gamma(\Psi)$, indexed by the filtered category $\mathcal{D}$ of triples $(M, i, \Psi)$ fitting into commutative diagrams in $(\text{Set}_\Delta)^I$ of the form

$$\begin{array}{ccc}
N & \xrightarrow{\Phi} & \mathcal{R} \\
\downarrow \downarrow & & \downarrow \\
M & \xrightarrow{i \circ \Psi} & \mathcal{R}
\end{array}$$

where $\Phi, \Psi$ are maps of $\text{Set}_\Delta$. The image of $\Gamma(\Psi)|\Delta^0_I \times N$ is a weakly contractible Kan complex, so it is a Kan complex.

Therefore, $\Gamma_\Phi(\mathcal{R})$ is a weakly contractible Kan complex.

(2) For homotopic morphisms $\Phi, \Phi': N \to \mathcal{R}$, the components of $\Gamma(\Phi)$ and $\Gamma(\Phi')$ lie in the same connected component, so they are equal.
such that $\mathcal{M}(\sigma)$ is weakly contractible for all objects $\sigma$ in $\mathcal{J}$. A morphism $(\mathcal{M}, i, \Psi) \to (\mathcal{M}', i', \Psi')$ in $\mathcal{D}$ is a monomorphism $j : \mathcal{M} \to \mathcal{M}'$ such that $j \circ i = i'$ and $\Psi = \Psi' \circ j$. Consider the following lifting problem

$$
\begin{array}{ccc}
\partial \Delta^n & \rightarrow & \Gamma_{\Phi}(\mathcal{R}) \\
\downarrow & & \downarrow \\
\Delta^n & \rightarrow & \\
\end{array}
$$

The upper horizontal arrow corresponds to a morphism $(\partial \Delta^n)_j \to \mathcal{M}$, where $(\mathcal{M}, i, \Psi)$ is an object of $\mathcal{J}$. We define an object $\mathcal{M}' = (\mathcal{M} \coprod_{\partial \Delta^n} \Delta^n)^{\mathcal{J}}$ of $(\mathcal{D}, \mathcal{J})^\mathcal{D}_{\mathcal{J}}$ by $\mathcal{M}'(\sigma) = (\mathcal{M}(\sigma) \coprod_{\partial \Delta^n} \Delta^n)^{\mathcal{J}}$ for all morphisms $d$ of $\mathcal{J}$. Consider the inclusion $\mathcal{M} \to \mathcal{M}'$. Since $\mathcal{M}(\sigma) \to \mathcal{M}'(\sigma)$ is anodyne for all $\sigma$ and $\mathcal{R}$ is injectively fibrant, we obtain a morphism $(\mathcal{M}, i, \Psi) \to (\mathcal{M}', i', \Psi')$ in $\mathcal{D}$. Then the morphism $\Delta^n \to \mathcal{M}'$ provides the dotted arrow.

(2) Since $\mathcal{R}$ is injectively fibrant, we can complete the diagram

$$
\begin{array}{ccc}
\Delta^1 \times \mathcal{N} & \xrightarrow{H} & \mathcal{R} \\
\downarrow & & \downarrow \\
\Delta^1 \times \mathcal{N}^\circ & \xrightarrow{H'} & \\
\end{array}
$$

We denote by $h : \Delta^1 \to \mathcal{R}$ the restriction of $H'$ to the cone point of $\mathcal{N}^\circ$. Then $h(0)$ belongs to $\Gamma_{\Phi}(\mathcal{R})$ and $h(1)$ belongs to $\Gamma_{\Phi'}(\mathcal{R})$. □

Next we relate Lemma 2.2 to the problem of constructing functors. Let $K$ be a simplicial set. Recall that the category of simplices of $K$ [17, Notation 6.1.2.5], denoted by $\Delta_{/K}$, is the strict fiber product $\Delta \times_{\mathcal{K}} (\mathcal{K}_{/K})$. An object of $\Delta_{/K}$ is a pair $(J, \sigma)$, where $J$ is an object of $\Delta$ and $\sigma \in \text{Hom}_{\mathcal{K}}(\Delta J, K)$. A morphism $(J, \sigma) \to (J', \sigma')$ is a map $d : \Delta J \to \Delta J'$ such that $\sigma = \sigma' \circ d$. Note that $d$ is a monomorphism (resp. epimorphism) if and only if the underlying map $J \to J'$ is injective (resp. surjective).

**Notation 2.3.** We define a functor $\text{Map}[K, -] : \mathcal{K} \to (\mathcal{K}_{/K})^\mathcal{K}$ as follows. For a marked simplicial set $M$, we define $\text{Map}[K, M]$ by

$$\text{Map}[K, M](J, \sigma) = (\Delta J)^{\mathcal{K}}(\mathcal{K})^\mathcal{K},$$

for every object $(J, \sigma)$ of $\Delta_{/K}$. A morphism $d : (J, \sigma) \to (J', \sigma')$ in $\Delta_{/K}$ goes to the natural restriction map $\text{Res}^d : \text{Map}(\Delta J, M) \to \text{Map}(\Delta J', M)$. For an $\infty$-category $\mathcal{C}$, we let $\text{Map}[K, \mathcal{C}] = \text{Map}[K, \mathcal{C}]$.

We have

$$\Gamma(\text{Map}[K, M]) \simeq \text{Map}(\Delta^v, M).$$

We note that $\text{Map}(\Delta^v, \mathcal{C})$ is the largest Kan complex [17, Proposition 1.2.5.3] contained in $\text{Fun}(K, \mathcal{C})$. If $g : K' \to K$ is a map, composition with the functor $\Delta_{/K'} \to \Delta_{/K}$ induced by $g$ defines a functor $g^* : (\mathcal{K}_{/K'})^\mathcal{K} \to (\mathcal{K}_{/K})^\mathcal{K}$. We have $g^* \text{Map}[K, M] \simeq \text{Map}[K', M]$.

The functor $\text{Map}[K, -]$ admits a left adjoint carrying $\mathcal{R}$ to the colimit of the diagram $(\Delta_{/K})^{\mathcal{K}} \to \mathcal{K}$ carrying $(J, \sigma)$ to $\mathcal{R}(J, \sigma)^{\mathcal{K}}$. Since the left adjoint preserves monomorphisms, the following proposition shows that the pair is a Quillen adjunction between $\mathcal{K}$ endowed with the Cartesian model structure and $(\mathcal{K}_{/K})^{\mathcal{K}}$ endowed with the injective model structure.

**Proposition 2.4.** Let $f : Z \to T$ be a fibration in $\mathcal{K}^\mathcal{K}$ with respect to the Cartesian model structure, and let $K$ be a simplicial set. Then the morphism $\text{Map}[K, f] : \text{Map}[K, Z] \to \text{Map}[K, T]$ is an injective fibration in $(\mathcal{K}_{/K})^{\mathcal{K}}$. In other words, for every commutative square in $(\mathcal{K}_{/K})^{\mathcal{K}}$ of the form

$$
\begin{array}{ccc}
\mathcal{N} & \xrightarrow{\Phi} & \text{Map}[K, Z] \\
\downarrow & & \downarrow \\
\mathcal{M} & \xrightarrow{\Psi} & \text{Map}[K, T] \\
\end{array}
$$

such that $\mathcal{N}(\sigma) \to \mathcal{M}(\sigma)$ is anodyne for all $\sigma \in \Delta_{/K}$, there exists a dotted arrow as indicated, rendering the diagram commutative.
In particular, the fibers of \( \text{Map}[K,f] \) are injectively fibrant objects of \((\text{Set}_\Delta)^{(\Delta/K)^{op}}\).

**Proof.** For \( n \geq 0 \), we let \( T_n \) denote the full subcategory of \( \Delta/K \) spanned by \([m],\sigma\) for \( m \leq n \). We construct \( \Omega|_{T_n} \) by induction on \( n \). It suffices to construct, for every \( \sigma: \Delta^n \to K \), a map \( \Omega(\sigma) \) as the dotted arrow rendering the following diagram commutative

\[
\begin{array}{ccc}
\mathcal{N}(\sigma) & \xrightarrow{\Phi(\sigma)} & \text{Map}^\sharp((\Delta^n)^\flat,Z) \\
\downarrow{\Omega(\sigma)} & & \downarrow{\text{Map}^\sharp((\Delta^n)^\flat,f)} \\
\mathcal{M}(\sigma) & \xrightarrow{\Psi(\sigma)} & \text{Map}^\sharp((\Delta^n)^\flat,T),
\end{array}
\]

such that for every monomorphism \( d: ([n-1],\rho) \to ([n],\sigma) \) and every epimorphism \( s: ([n],\sigma) \to ([n-1],\tau) \), the following diagrams commute

\[
\begin{array}{ccc}
\mathcal{M}(\sigma) & \xrightarrow{\Omega(\sigma)} & \text{Map}^\sharp((\Delta^n)^\flat,Z) \\
\downarrow{M(d)} & & \downarrow{\text{Res}^d} \\
\mathcal{M}(\rho) & \xrightarrow{\Omega(\rho)} & \text{Map}^\sharp((\Delta^{n-1})^\flat,Z) \\
\downarrow{M(s)} & & \downarrow{\text{Res}^s} \\
\mathcal{M}(\sigma) & \xrightarrow{\Omega(\sigma)} & \text{Map}^\sharp((\Delta^n)^\flat,Z).
\end{array}
\]

By the induction hypothesis, the \( \Omega(\rho) \)'s amalgamate into a map \( \mathcal{M}(\sigma) \to \text{Map}^\sharp((\partial\Delta^n)^\flat,Z) \), and the \( \Omega(\tau) \)'s amalgamate into a map \( \mathcal{M}(\sigma)^{\text{deg}} \to \text{Map}^\sharp((\Delta^n)^\flat,Z) \), where \( \mathcal{M}(\sigma)^{\text{deg}} \subseteq \mathcal{M}(\sigma) \) is the union of the images of \( \mathcal{M}(s): \mathcal{M}(\tau) \to \mathcal{M}(\sigma) \). These maps amalgamate with \( \Phi(\sigma): \mathcal{N}(\sigma) \to \text{Map}^\sharp((\Delta^n)^\flat,Z) \) into a map \( \Omega': X \to Z \), where

\[
X = (\mathcal{N}(\sigma) \cup \mathcal{M}(\sigma)^{\text{deg}})^\flat \times (\Delta^n)^\flat \coprod_{(\mathcal{N}(\sigma) \cup \mathcal{M}(\sigma)^{\text{deg}})^\flat \times (\partial\Delta^n)^\flat} \mathcal{M}(\sigma)^\flat \times (\partial\Delta^n)^\flat,
\]

fitting into the commutative square

\[
\begin{array}{ccc}
X & \xrightarrow{\Omega'} & Z \\
\downarrow{i} & & \downarrow{f} \\
\mathcal{M}(\sigma)^\flat \times (\Delta^n)^\flat & \xrightarrow{\Psi(\sigma)} & T.
\end{array}
\]

It suffices to show that \( i \) is a trivial cofibration in \( \text{Set}_\Delta^+ \) with respect to the Cartesian model structure, so that there exists a dotted arrow rendering the diagram commutative.

Let us first prove that the map \( \mathcal{N}(\sigma)^{\text{deg}} \to \mathcal{M}(\sigma)^{\text{deg}} \) is anodyne, where \( \mathcal{N}(\sigma)^{\text{deg}} \subseteq \mathcal{N}(\sigma) \) is the union of the images of \( \mathcal{N}(s) \). More generally, we claim that, for pairwise distinct epimorphisms \( s_1,\ldots,s_m \), where \( s_j: ([n],\sigma) \to ([n-1],\tau_j) \), \( N(\tau_1) \cup \cdots \cup N(\tau_m) \to M(\tau_1) \cup \cdots \cup M(\tau_m) \) is anodyne. Here by abuse of notation, \( N(\tau_j) \subseteq \mathcal{N}(\sigma) \) denotes the image of \( N(s_j) \) and similarly for \( M(\tau_j) \). We proceed by induction on \( m \) (simultaneously for all \( n \)). By assumption, \( N(\tau_m) \to M(\tau_m) \) is anodyne. Moreover, \( N(\tau_1) \cup \cdots \cup N(\tau_{m-1}) \cap N(\tau_m) = N(\tau_j) \cup \cdots \cup N(\tau_m) \) and similarly for \( M \), where \( ([n-2],\tau_j) \) is the pushout of \( s_j \) and \( s_{m-1} \). Thus the claim follows from the induction hypothesis and Lemma 2.5 below.

It follows that the inclusion \( \mathcal{N}(\sigma) \cup \mathcal{M}(\sigma)^{\text{deg}} \subseteq \mathcal{M}(\sigma) \) is anodyne, because it is a pushout of \( \mathcal{N}(\sigma)^{\text{deg}} \subseteq \mathcal{M}(\sigma)^{\text{deg}} \). Thus the inclusion \( (\mathcal{N}(\sigma) \cup \mathcal{M}(\sigma)^{\text{deg}})^\flat \subseteq \mathcal{M}(\sigma)^\flat \) is a trivial cofibration in \( \text{Set}_\Delta^+ \). The lemma then follows from the fact the trivial cofibrations in \( \text{Set}_\Delta^+ \) are stable under smash products by cofibrations [17, Corollary 3.1.4.3].

**Lemma 2.5.** Consider a pushout square

\[
\begin{array}{ccc}
f_0 & \xrightarrow{u} & f_1 \\
v & & v' \\
f_2 & \xrightarrow{u'} & f_3
\end{array}
\]

in \((\text{Set}_\Delta)^{[1]}\), where \( f_1: Y_i \to X_i \). Assume that \( f_0, f_1, f_2 \) are anodyne and the map \( X_0 \coprod_{Y_0} Y_1 \to X_1 \) is a monomorphism. Then \( f_3 \) is anodyne.
Proof. If \( u \) or \( v \) is a pushout square in \( \mathsf{Set}_{\Delta} \), then so is \( u' \) or \( v' \) and the assertion is trivial. Thus we may assume that \( Y_0 \to Y_1, Y_0 \to Y_2 \) are identities. In this case, since \( f_0, f_1 \) are anodyne and \( X_0 \to X_1 \) is a monomorphism, \( X_0 \to X_1 \) is anodyne, so that its pushout \( X_2 \to X_3 \) is anodyne. It then suffices to note that \( f_3 \) can be identified with the composite \( Y_0 \to X_2 \to X_3 \).

Let \( T \) be a simplicial set. The category \( (\mathsf{Set}_{\Delta}^+)_{/T} \) is Cartesian-closed, with internal mapping object given by

\[
\operatorname{Map}^+_T(X,Y) := (\operatorname{Map}^+_T(X,Y), \operatorname{Map}^+_T(X,Y)_1).
\]

Indeed, for a map of simplicial sets \( T \to S \) and an object \( A \) of \( (\mathsf{Set}_{\Delta}^+)_{/S} \), we have an isomorphism

\[
\operatorname{Map}^+_S(A, \operatorname{Map}^+_T(X,Y)) \simeq \operatorname{Map}^+_T(A \times_S X, Y)
\]

in \( (\mathsf{Set}_{\Delta}^+)_{/S} \). If \( Z \to T \) is a Cartesian fibration [17, Definition 2.4.2.1], we have \( \operatorname{Map}^+_T(X,Z^\sharp) = \operatorname{Map}^+_T(X,Z^\sharp)^Z \), where \( Z^\sharp \) denotes \( Z \) with all Cartesian edges marked [17, Definition 3.1.1.9], and the last \( ^\sharp \) denotes an \( \infty \)-category with all equivalences marked as in Section 1.

**Lemma 2.6.** Let \( T \) be a simplicial set, let \( X \to Y \) be a fibration in \( (\mathsf{Set}_{\Delta}^+)_{/T} \) with respect to the Cartesian model structure, and let \( i: A \to B \) be a cofibration in \( (\mathsf{Set}_{\Delta}^+)_{/T} \) with respect to the Cartesian model structure. Then the induced map

\[
\operatorname{Map}^+_T(B,X) \to \operatorname{Map}^+_T(A,X) \times_{\operatorname{Map}^+_T(A,Y)} \operatorname{Map}^+_T(B,Y)
\]

is a fibration in \( \mathsf{Set}_{\Delta}^+ \) with respect to the Cartesian model structure. In particular, the map

\[
\operatorname{Map}^+_T(B,X) \to \operatorname{Map}^+_T(A,X) \times_{\operatorname{Map}^+_T(A,Y)} \operatorname{Map}^+_T(B,Y)
\]

is a Kan fibration.

Proof. The first assertion follows from the fact that smash product by \( i \) sends trivial cofibrations in \( \mathsf{Set}_{\Delta}^+ \) to trivial cofibrations in \( (\mathsf{Set}_{\Delta}^+)_{/T} \) [17, Corollary 3.1.4.3]. The functor \( \delta^+_1: \mathsf{Set}_{\Delta} \to \mathsf{Set}_{\Delta}^+ \) sending \( Z \) to \( Z^\sharp \) and its right adjoint functor \( \delta^+_1 = \operatorname{Map}^+_((\Delta^0)^\sharp, -) \) (cf. Definition 3.3) define a Quillen adjunction for the Kan model structure on \( \mathsf{Set}_{\Delta} \) and the Cartesian model structure on \( \mathsf{Set}_{\Delta}^+ \). In fact, \( \delta^+_1 \) preserves cofibrations and trivial cofibrations. Applying \( \delta^+_1 \) to the first assertion, we obtain the second assertion.

We can now give the form of the construction technique as used in Sections 4 and 5.

**Corollary 2.7.** Let \( K \) be a simplicial set, let \( \mathcal{C} \) be an \( \infty \)-category, and let \( i: A \to B \) be a monomorphism of simplicial sets. Let \( f: \operatorname{Fun}(B,\mathcal{C}) \to \operatorname{Fun}(A,\mathcal{C}) \) be the morphism induced by \( i \). Let \( N \) be an object of \( (\mathsf{Set}_{\Delta})^{\Delta(K)^\op} \) such that \( N(\sigma) \) is weakly contractible for all \( \sigma \), and let \( \Phi: N \to \operatorname{Map}[K,\operatorname{Fun}(B,\mathcal{C})] \) be a morphism such that \( \operatorname{Map}[K,f] \circ \Phi: N \to \operatorname{Map}[K,\operatorname{Fun}(A,\mathcal{C})] \) factors through \( \Delta^0_{(\Delta(K))^{\op}} \) to give a functor \( a: K \to \operatorname{Fun}(A,\mathcal{C}) \). Then there exists \( b: K \to \operatorname{Fun}(B,\mathcal{C}) \) lifting \( a \), such that for every map \( g: K' \to K \) and every global section \( \nu \) of \( g^*N, b \circ g \) and \( g^*\Phi \circ \nu: K' \to \operatorname{Fun}(B,\mathcal{C}) \) are homotopic over \( \operatorname{Fun}(A,\mathcal{C}) \).

Proof. Since \( \operatorname{Fun}(-,\mathcal{C})^\sharp = \operatorname{Map}^+_((-,\mathcal{C})^\sharp), f^\sharp: \operatorname{Fun}(B,\mathcal{C})^\sharp \to \operatorname{Fun}(A,\mathcal{C})^\sharp \) is a fibration in \( \mathsf{Set}_{\Delta}^+ \) for the Cartesian model structure by Lemma 2.6. Thus, by Proposition 2.4, \( \operatorname{Map}[K,f^\sharp] \) is an injective fibration. By Lemma 2.2 (1), \( \Gamma_\Phi(\operatorname{Map}[K,f^\sharp]_a) \) is a weakly contractible Kan complex, where \( \operatorname{Map}[K,f^\sharp]_a \) is the fiber of \( \operatorname{Map}[K,f^\sharp] \) at \( a \). Any element of \( \Gamma_\Phi(\operatorname{Map}[K,f^\sharp]_a) \) then provides the desired \( b \). Indeed, both \( b \circ g \) and \( g^*\Phi \circ \nu \) are given by elements of the weakly contractible Kan complex \( \Gamma_{g^*\Phi}(\operatorname{Map}[K',f^\sharp]_{g^*\nu}a) \), which are necessarily equivalent.

### 3. Restricted multisimplicial nerves

In this section, we introduce several notions related to multisimplicial sets. The restricted multisimplicial nerve (Definition 3.6) of a multi-tiled simplicial set (Definition 3.5) will play an essential role in the statements of our theorems.

**Definition 3.1 (Multisimplicial set).** Let \( I \) be a set. We define the category of \( I \)-simplicial sets to be \( \mathsf{Set}_{I\Delta} := \operatorname{Fun}(\Delta^I)^{\op}, \mathsf{Set} \), where \( \Delta^I := \operatorname{Fun}(I,\Delta) \). For an integer \( k \geq 0 \), we define the category of \( k \)-simplicial sets to be \( \mathsf{Set}_k := \mathsf{Set}_{I\Delta}, \) where \( I = \{1,\ldots,k\} \).
We denote by \(\Delta^{n_{i}|_{i\in I}}\) the I-simplicial set represented by the object \((\eta_{i})_{i\in I}\) of \(\Delta^{I}\). For an I-simplicial set \(S\), we denote by \(S_{n_{i}|_{i\in I}}\) the value of \(S\) at the object \((\eta_{i})_{i\in I}\) of \(\Delta^{I}\). An \((n_{i})_{i\in I}\)-simplex of an I-simplicial set \(S\) is an element of \(S_{n_{i}|_{i\in I}}\). By Yoneda’s lemma, there is a canonical bijection of the set \(S_{n_{i}|_{i\in I}}\) and the set of maps from \(\Delta^{n_{i}|_{i\in I}}\) to \(S\).

Let \(J \subseteq I\). Composition with the partial opposite functor \(\Delta^{J} \rightarrow \Delta^{I}\) sending \((\ldots, S_{j}, \ldots, S_{j}, \ldots)\) to \((\ldots, S_{j}, \ldots, S_{j}, \ldots)\) (taking \(op\) for \(S_{j}\) when \(j \in J\)) defines a functor \(op_{J}^{*}: \mathcal{S}_{\Delta} \rightarrow \mathcal{S}_{\Delta}\). We define \(\Delta^{n_{i}|_{i\in I}}_{J} = op_{J}^{*}\Delta^{n_{i}|_{i\in I}}\). Although \(\Delta^{n_{i}|_{i\in I}}_{J}\) is isomorphic to \(\Delta^{n_{i}|_{i\in I}}\), it will be useful in specifying the variance of many constructions. When \(I = \{1, \ldots, k\}\), we use the notation \(\eta_{i}^{J}\) and \(\Delta^{n_{1}, \ldots, n_{k}}\).

**Notation 3.2.** Let \(f: J \rightarrow I\) be a map of sets. Composition with \(f\) defines a functor \(\Delta^{f}: \Delta^{I} \rightarrow \Delta^{J}\). Composition with \(\Delta^{f}\) induces a functor \((\Delta^{f})^{*}: \mathcal{S}_{\Delta} \rightarrow \mathcal{S}_{\Delta}^{f}\), which has a right adjoint \((\Delta^{f})_{*}: \mathcal{S}_{\Delta} \rightarrow \mathcal{S}_{\Delta^{f}}\). We will now look at two special cases.

Let \(f: J \rightarrow I\) be an injective map of sets. Then \(\Delta^{f}\) has a right adjoint \(c_{f}: \Delta^{I} \rightarrow \Delta^{J}\) given by \(c_{f}(F) = F_{j}\) if \(f(j) = i\) and \(c_{f}(F) = [0]\) if \(i\) is not in the image of \(f\). We have \(\Delta^{I} \circ c_{f} = id_{\Delta^{J}}\). In this case, we write \((\Delta^{f})^{*}\) can be identified with the functor \(\epsilon_{f}\) induced by composition with \(c_{f}\). We have \(\epsilon_{f} \circ (\Delta^{f})^{*} = id_{\Delta^{f}}\) so that the adjunction map \((\Delta^{f})^{*} \circ \epsilon_{f} \rightarrow id_{\Delta^{f}}\) is a split monomorphism. If \(J = \{1, \ldots, k^{f}\}\), we write \(\epsilon_{f}(\cdot)^{J} = \epsilon_{f}^{J}\).

Let \(f: I \rightarrow \{1\}\). Then \(\delta_{f} := \Delta^{f}: \Delta \rightarrow \Delta\) is the diagonal map. Composition with \(\delta_{f}\) induces a functor \(\delta_{f}^{*} = (\Delta^{f})^{*}: \mathcal{S}_{\Delta} \rightarrow \mathcal{S}_{\Delta}\). For \(J \subseteq I\), we define

\[
\Delta^{n_{i}|_{i\in I}}_{J} := \delta_{J}^{*}\Delta^{n_{i}|_{i\in I}} = \left(\prod_{i \in I - J} \Delta^{n_{i}}\right) \times \left(\prod_{j \in J} \Delta^{n_{j}}\right)^{op}.
\]

When \(J = \emptyset\), we simply write \(\Delta^{n_{i}|_{i\in I}}_{\emptyset}\) for \(\Delta^{n_{i}|_{i\in I}}_{\emptyset} = \prod_{i \in I} \Delta^{n_{i}}\). We define the **multisimplicial nerve** functor to be the right adjoint \(\delta_{J}^{*}: \mathcal{S}_{\Delta} \rightarrow \mathcal{S}_{\Delta}^{J}\) of \(\delta_{J}^{*}\). An \((n_{i})_{i\in I}\)-simplex of \(\delta_{J}^{*}\) is given by a map \(\Delta^{n_{i}|_{i\in I}} \rightarrow X\).

For \(J \subseteq I\), we define the twisted diagonal functor \(\delta_{J}^{*} = \delta_{J}^{*}: \mathcal{S}_{\Delta} \rightarrow \mathcal{S}_{\Delta}^{J}\). When \(J = \emptyset\), \(op_{J}^{f}\) is the identity functor so that \(\delta_{J}^{*} = \delta_{J}^{*}\). When \(I = \{1, \ldots, k\}\), we write \(k\) instead of \(I\) in the previous notation. In particular, we have \(\delta_{k}^{*}: \mathcal{S}_{\Delta} \rightarrow \mathcal{S}_{\Delta}^{k}\) so that \((\delta_{k}^{*}X)_{n} = X_{n_{1}, \ldots, n_{k}}\). Moreover, \((\epsilon_{k}^{J}K)_{n} = K_{n_{1}, \ldots, n_{k}, 0}\), where \(n\) is at the \(j\)-th position and all other indices are 0.

We define a bifunctor

\[\boxtimes: \mathcal{S}_{\Delta} \times \mathcal{S}_{\Delta} \rightarrow \mathcal{S}_{\Delta^{I}}\]

by the formula \(S \boxtimes S' = (\Delta^{*}S) \times (\Delta^{*}S')\), where \(\imath_{J}: I \hookrightarrow I, \imath_{J}: J \hookrightarrow I\) are the inclusions. In particular, when \(I = \{1, \ldots, k\}, J = \{1, \ldots, k'\}\), we have

\[\boxtimes: \mathcal{S}_{\Delta} \times \mathcal{S}_{\Delta} \rightarrow \mathcal{S}_{\Delta^{I}} \times \mathcal{S}_{\Delta^{I}} = \Delta^{*}S \times \Delta^{*}S',\]

where \(\imath: \{1, \ldots, k\} \hookrightarrow \{1, \ldots, k + k'\}\) is the identity and \(\imath': \{1, \ldots, k'\} \hookrightarrow \{1, \ldots, k + k'\}\) sends \(j\) to \(j + k\).

In other words, \((S \boxtimes S')_{n_{1}, \ldots, n_{k+k'}} = S_{n_{1}, \ldots, n_{k}} \times S'_{n_{k+1}, \ldots, n_{k+k'}}\). We have \(\Delta^{n_{1}} \boxtimes \cdots \boxtimes \Delta^{n_{k}} = \Delta^{n_{1}, \ldots, n_{k}}\). For a map \(f: J \rightarrow I\), an \((n_{i})_{i\in I}\)-simplex of \((\Delta^{J})_{*}X\) is given by \(\boxtimes_{i \in I} \Delta^{n_{i}|_{i\in I}} \rightarrow X\).

We next turn to restricted variants of the multisimplicial nerve functor \(\delta_{J}^{*}\). We start with restrictions on edges.

**Definition 3.3 (Multi-marked simplicial set).** An I-marked simplicial set (resp. I-marked \(\infty\)-category) is the data \((X, \mathcal{E}_{i})_{i\in I}\), where \(X\) is a simplicial set (resp. an \(\infty\)-category) and, for all \(i \in I, \mathcal{E}_{i}\) is a set of edges of \(X\) which contains every degenerate edge. A morphism \(f: (X, \mathcal{E}_{i})_{i\in I} \rightarrow (X', \mathcal{E}'_{i})_{i\in I}\) of I-marked simplicial sets is a map \(f: X \rightarrow X'\) having the property that \(f(\mathcal{E}_{i}) \subseteq \mathcal{E}'_{i}\) for all \(i \in I\). We denote the category of I-marked simplicial sets by \(\mathcal{S}_{\Delta}^{I}\). It is the strict fiber product of \(I\) copies of \(\mathcal{S}_{\Delta}\) above \(\mathcal{S}_{\Delta}\).

For a simplicial set \(X\) and an inclusion \(J \subseteq I\), we define an I-marked simplicial set \(X^{J}_{\mathcal{E}_{i}} = (X, E_{i})\) by \((X, E_{j}) = X^{J}_{i}\) for \(j \in J\) and \((X, E_{i}) = X^{J}_{i}\) for \(i \in I - J\). We write \(X^{J}_{\mathcal{E}_{i}} = X^{J}_{i}\). For two I-marked simplicial sets \((X, \mathcal{E})\) and \((X', \mathcal{E}')\), we define a simplicial set Map\(^{\mathcal{E}}_{\mathcal{E}'}((X, \mathcal{E}), (X', \mathcal{E}'))\) by

\[
\operatorname{Map}^{\mathcal{E}}_{\mathcal{E}'}((X, \mathcal{E}), (X', \mathcal{E}'))_{n} = \operatorname{Hom}_{\mathcal{S}_{\Delta}^{I}}(\Delta^{n}_{J} \times (X, \mathcal{E}), (X', \mathcal{E}'))
\]
Consider the functor $\delta^+_\ast : \text{Set}_\Delta \to \text{Set}_\Delta^+$ sending $S$ to $(\delta^+_\ast S, \{\mathcal{E}_i\}_{i \in I})$, where $\mathcal{E}_i$ is the set of edges of $\epsilon_i^+ S \subseteq \delta^+_\ast S$. This functor admits a right adjoint $\delta^+_\ast : \text{Set}_\Delta^+ \to \text{Set}_\Delta$ sending $(X, \{\mathcal{E}_i\}_{i \in I})$ to the $I$-simplicial subset of $\delta^+_\ast X$ whose $(n_i)_{i \in I}$-simplices are maps $\Delta^{[n_i]} \to X$ such that for every $j \in I$ and every map $\Delta^1 \to \epsilon_j^+ \Delta^{[n_i]} \to \Delta^{[n_i]} \to X$

is in $\mathcal{E}_j$. When $I = \{1, \ldots, k\}$, we use the notation $\text{Set}^+_\Delta \overset{4}{\cong} \delta^+_\ast$ and $\delta^{k+}$.

**Definition 3.4 (Restricted multisimplicial nerve).** We define the *restricted $I$-simplicial nerve* of an $I$-marked simplicial set $(X, \mathcal{E} = \{\mathcal{E}_i\}_{i \in I})$ to be the $I$-simplicial set

$$X_\mathcal{E} = X_{\{\mathcal{E}_i\}_{i \in I}} := \delta^+_\ast (X, \{\mathcal{E}_i\}_{i \in I}).$$

In particular, for any marked simplicial set $(X, \mathcal{E})$, $X_\mathcal{E} \subseteq X$ denotes the simplicial subspace spanned by the edges in $\mathcal{E}$.

Next we consider restrictions on squares. By a *square* of a simplicial set $X$, we mean a map $\Delta^1 \times \Delta^1 \to X$. The transpose of a square is obtained by swapping the two $\Delta^1$'s. Composition with the maps $\text{id} \times d^n_0$, $\text{id} \times d^n_1 : \Delta^1 \simeq \Delta^1 \times \Delta^0 \to \Delta^1 \times \Delta^1$ induce maps $\text{Hom}(\Delta^1 \times \Delta^1, X) \to X_1$ and composition with the map $\text{id} \times s^n_0 : \Delta^1 \times \Delta^1 \to \Delta^1 \times \Delta^2 \simeq \Delta^1$ induces a map $X_1 \to \text{Hom}(\Delta^1 \times \Delta^1, X)$.

**Definition 3.5 (Multi-tiled simplicial set).** An *$I$-tilled simplicial set* (resp. $I$-tilled $\infty$-category) is the data $(X, \mathcal{E} = \{\mathcal{E}_i\}_{i \in I})$, where $(X, \mathcal{E})$ is an $I$-marked simplicial set (resp. $\infty$-category) and, for all $i, j \in I$, $i \neq j$, $Q_{ij}$ is a set of squares of $X$ such that $Q_{ij}$ and $Q_{ji}$ are obtained from each other by transposition of squares, and $i \times d^n_0$, $i \times d^n_1$ induce maps $Q_{ij} \to \mathcal{E}_i$, and $i \times s^n_0$ induces $Q_{ij} \to Q_{ji}$. A morphism $f : (X, \mathcal{E}, \Omega) \to (X', \mathcal{E}', \Omega')$ of $I$-tilled simplicial sets is a map $f : X \to X'$ having the property that $f(\mathcal{E}_i) \subseteq f(\mathcal{E}_j) \subseteq (\mathcal{E}_i)^\prime$ and the category of $I$-tilled simplicial sets by $\text{Set}^+_\Delta$. For brevity, we will sometimes write $T = (\mathcal{E}, \Omega)$, $T_i = \mathcal{E}_i$, $T_{ij} = Q_{ij}$.

Consider the functor $\delta^+_\square : \text{Set}_\Delta \to \text{Set}_\Delta^+$ carrying $S$ to $(\delta^+_\ast S, \Omega)$, where $Q_{ij} = (\epsilon_{ij}^+ S)^{11}$. This functor admits a right adjoint $\delta^+_\ast : \text{Set}_\Delta^+ \to \text{Set}_\Delta$ carrying $(X, \mathcal{E})$ to the $I$-simplicial subset of $\delta^+_\ast(X, \mathcal{E}) \subseteq \delta^+_\ast X$ whose $(n_i)_{i \in I}$-simplices are maps $\Delta^{[n_i]} \to X$ satisfying the additional condition that for every pair of elements $j, k \in I$, $j \neq k$, and every map $\Delta^1 \times \Delta^1 \to \epsilon_{jk}^+ \Delta^{[n_i]} \to \Delta^{[n_i]} \to X$.

is in $Q_{jk}$. When $I = \{1, \ldots, k\}$, we use the notation $\text{Set}^+_\Delta \overset{4}{\cong} \delta^+_\ast, \delta^+_\square, \delta^{k+}$.

**Definition 3.6 (Restricted multisimplicial nerve).** We define the *restricted $I$-simplicial nerve* of an $I$-tilled simplicial set $(X, \mathcal{T})$ to be the $I$-simplicial set $\delta^+_\square(X, \mathcal{T})$.

The underlying functor $U : \text{Set}_\Delta^+ \to \text{Set}_\Delta^+$ carrying $(X, \mathcal{E}, \Omega)$ to $(X, \mathcal{E})$ admits a left adjoint $V : \text{Set}_\Delta \to \text{Set}_\Delta^+$ and a right adjoint $W : \text{Set}_\Delta^+ \to \text{Set}_\Delta^+$, which can be described as follows. We have $V(X, \mathcal{E}) = (X, \mathcal{E}, \Omega)$, where $Q_{ij}$ is the union of the image of $\mathcal{E}_i$ under $-\circ (i \times d^n_0)$ and the image of $\mathcal{E}_j$ under $-\circ (s^n_0 \times \text{id})$. For sets of edges $\mathcal{E}_1$ and $\mathcal{E}_2$ of $X$, we denote by $\mathcal{E}_1 \ast_X \mathcal{E}_2$ the set of squares $f : \Delta^1 \times \Delta^1 \to X$ such that $f \circ (i \times d^n_0)$ belongs to $\mathcal{E}_1$ and $f \circ (s^n_0 \times \text{id})$ belongs to $\mathcal{E}_2$, for both $\alpha = 0, 1$. We have $W(X, \mathcal{E}) = (X, \mathcal{E}, \Omega)$, where $Q_{ij} = \mathcal{E}_i \ast_X \mathcal{E}_j$. We have $\delta^+_\square \simeq U \circ \delta^+_\ast, \delta^+_\ast \simeq \delta^+_\square \circ W$.

If $\mathcal{E}$ is an $\infty$-category and $\mathcal{E}_1, \mathcal{E}_2$ are sets of edges of $\mathcal{E}$, we denote by $\mathcal{E}_1 \ast_{\mathcal{E}} \mathcal{E}_2$ the subset of $\mathcal{E}_1 \ast \mathcal{E}_2$ consisting of Cartesian squares. For an $I$-marked $\infty$-category $(\mathcal{E}, \mathcal{E})$, we denote by $(\mathcal{E}, \mathcal{E}_{\text{cart}})$ the $I$-tilted $\infty$-category such that $\mathcal{E}_i^\ast_{\mathcal{E}} = \mathcal{E}_i$ for $i \in I$ and $\mathcal{E}_i^\ast_{\text{cart}} = \mathcal{E}_i \ast_{\mathcal{E}} \mathcal{E}_j$ for $i, j \in I$ and $i \neq j$.

**Definition 3.7.** We define the *Cartesian $I$-simplicial nerve* of an $I$-marked $\infty$-category $(\mathcal{E}, \mathcal{E})$ to be

$$\mathcal{E}_{\text{cart}}^\ast := \delta^+_\square(\mathcal{E}, \mathcal{E}_{\text{cart}}).$$

For future reference, we define a few properties of sets of edges.

**Definition 3.8.** Let $X$ be a simplicial set and let $\mathcal{E}$ be a set of edges of $X$. We say that $\mathcal{E}$ is *stable under composition* for every 2-simplex $\sigma$ of $X$ such that $\sigma \circ d^n_0, \sigma \circ d^n_2 \in \mathcal{E}$, we have $\sigma \circ d^n_1 \in \mathcal{E}$.

---

4In particular, $\text{Set}^+_\Delta$ in our notation is $\text{Set}^+_\Delta$ in [18, Definition 6.2.3.2].
If $E$ contains every degenerate edge, then $E$ is stable under composition if and only if $(X, E)$ has the right lifting property with respect to
\[(\Lambda^n_1)^p \coprod_{(\Lambda^1_1)^p} (\Delta^2)^p \subseteq (\Delta^2)^p.\]

**Definition 3.9.** Let $C$ be an $\infty$-category and let $E$, $F$ be sets of edges of $C$. We say that $E$ is **stable under pullback by $F$** if for every Cartesian square in $C$ of the form

\[
\begin{array}{ccc}
  y' & \rightarrow & y \\
  e' \downarrow & & \downarrow e \\
  x' & \rightarrow & x \\
\end{array}
\]

with $e$ in $E$ and $f$ in $F$, $e'$ is in $E$. We say that $E$ is stable under pullback (cf. [17, Notation 6.1.3.4]) if it is stable under pullback by $E_1$. We say that $E$ is **admissible** if $E$ contains every degenerate edge of $C$, $E$ is stable under pullback, and for every 2-simplex of $E$ of the form

\[
\begin{array}{ccc}
  q & \rightarrow & y \\
  r \rightarrow & & \rightarrow p \\
  z & \rightarrow & x \\
\end{array}
\]

with $p \in E$, we have $q \in E$ if and only if $r \in E$.

If $E$ is admits pullbacks, then $E$ is admissible if and only if it contains every degenerate edge of $E$ and is stable under composition, pullback, and taking diagonal in $E$.

For a simplicial set $X$, consider the isomorphism $\text{Hom}(\Delta^1 \times \Delta^1, X) \simeq \text{Hom}(\Delta^1, \text{Map}(\Delta^1, X))$ carrying $f$ to $a \mapsto (b \mapsto f(a, b))$. We say that a set of squares $Q$ of $X$ is **stable under composition in the first direction** if the resulting set of edges of $\text{Map}(\Delta^1, X)$ is stable under composition. For two sets of squares $Q$, $Q'$ of an $\infty$-category $C$, we say that $Q$ is **stable under pullback by $Q'$ in $C$ in the first direction**, if $Q$ is stable under pullback by $Q'$ in $\text{Map}(\Delta^1, C)$, where $Q$ and $Q'$ are viewed as sets of edges via the above isomorphism.

**Remark 3.10.** Let $E$ be an ordinary category and let $E_1, \ldots, E_k$ be sets of morphisms of $E$ stable under composition and containing identity morphisms. Then $N(E_{E_1, \ldots, E_k})$ and $N(E_{\cup E_{E_1, \ldots, E_k}})$ can be interpreted as the $k$-fold nerves in the sense of Fiore and Paoli [7, Definition 2.14] of suitable $k$-fold categories. More generally, if $(N(E), E, Q)$ is a $k$-tiled $\infty$-category and $Q_{ij}$ is stable under composition in both directions for all $i, j$, then $\delta_k^Q(N(E), E, Q)$ is the $k$-fold nerve of a suitable $k$-fold category.

### 4. Multisimplicial descent

In this section, we study the map of simplicial sets obtained by composing two directions in a multisimplicial nerve. Unlike in Theorem 0.1, the two directions are not subject to the Cartesian restriction. The main result is Theorem 4.5, which can be regarded as a generalization of Deligne’s result [6, Proposition 3.3.2] (see Remark 4.12).

In Deligne’s theory, a fundamental role is played by the category of compactifications of a morphism $f$, whose objects are factorizations of $f$ as $p \circ q$, where $p$, $q$ belong respectively to the two classes of morphisms in question. To properly formulate compactifications of simplices of higher dimensions, we introduce a bit of notation.

We identify **partially ordered sets** with ordinary categories in which there is at most one arrow between each pair of objects, by the convention $p \leq q$ if and only if there exists an arrow $p \to q$. For every element $p \in P$, we identify the overcategory $P_{/p}$ (resp. undercategory $P_{p/}$) with the full partially ordered subset of $P$ consisting of elements $\leq p$ (resp. $\geq p$). For $p, p' \in P$, we identify $P_{p/ \cdot p'}$ with the full partially ordered subset of $P$ consisting of elements both $\geq p$ and $\leq p'$, which is empty unless $p \leq p'$. For a subset $Q$ of $P$, we write $Q_{/p} = Q \cap P_{/p}$, etc.

**Notation 4.1.** Let $n \geq 0$. We consider the bisimplicial set $\Delta^{n \cdot n}$ and the partially ordered set $[n] \times [n]$, related by the natural isomorphisms of simplicial sets $S^2_\Delta \Delta^{n \cdot n} \simeq \Delta^n \times \Delta^n \simeq N([n] \times [n])$. We enumerate their vertices by coordinates $(i, j)$ for $0 \leq i, j \leq n$. We define $\text{Cpt}^n \subseteq \Delta^{n \cdot n}$ to be the bisimplicial subset obtained by
restricting to the vertices \((i, j)\) with \(0 \leq i \leq j \leq n\). We define \(Cpt^n \subseteq [n] \times [n]\) to be the full partially ordered subset spanned by \((i, j)\) with \(0 \leq i \leq j \leq n\). We have

\[
\delta_{2}^{*}Cpt^n \simeq \Box^n \subseteq \mathcal{E}pt^n := N(Cpt^n),
\]

where \(\Box^n = \bigcup_{k=0}^{n} \square^n_k\) and \(\square^n_k = N(Cpt^n_{(0,k)}/(k,n))\) is the nerve of the full partially ordered subset of \([n] \times [n]\) spanned by \((i, j)\) with \(0 \leq i \leq k \leq j \leq n\).

Below is the Hasse diagram of \(Cpt^3\), rotated so that the initial object is shown in the upper-left corner. The dashed box represents \(\square^1\), while bullets represent elements in the image of the diagonal embedding \([3] \to Cpt^3\).

The following lemma is crucial for our argument. The proof will be given at the end of Section 6.

**Lemma 4.2.** The inclusion \(\Box^n \subseteq \mathcal{E}pt^n\) is inner anodyne.

**Definition 4.3.** Let \(K\) be a set and let \((X, \mathcal{T})\) be a \((\{1, 2\} \coprod K)\)-tilled simplicial set. For \(L \subseteq K\), \(n, n_k \geq 0\) \((k \in K)\), a map \(\sigma: \Delta^{n, n_k}_{L, k \in K} \to \delta_{\{1, 2\} \coprod K}^{*} X\), and \(\alpha \in \{1, 2\} \coprod K\), we define \(\mathcal{Kpt}^{\alpha}(\sigma) = \mathcal{Kpt}^{\alpha}_{(X, \mathcal{T})}(\sigma)\), the \(\alpha\)-th simplicial set of compactifications of \(\sigma\), to be the limit of the diagram

\[
\begin{array}{ccc}
\text{Map}(\mathcal{E}pt^n \times \Delta^{n, n_k}_{L, k \in K}, X) & \xrightarrow{\text{res}_1} & \text{Map}(\Box^n \times \Delta^{n, n_k}_{L, k \in K}, X) \\
\downarrow & & \downarrow \\
\{\sigma\} \xrightarrow{\text{res}_2} & \text{Map}(\Delta^{n, n_k}_{L, k \in K}, X) &
\end{array}
\]

in the category \(\mathcal{S}et_\Delta\) of simplicial sets, where \(\text{res}_1\) is induced by the inclusion \(\Box^n \subseteq \mathcal{E}pt^n\), \(\text{res}_2\) is induced by the diagonal map \(\Delta^n \to \mathcal{E}pt^n\), \(g\) is the composition of the commutative diagram

\[
\begin{array}{ccc}
\text{Map}(\mathcal{E}pt^n \times \Delta^{n, n_k}_{L, k \in K}, X) & \xrightarrow{\epsilon^{\{1, 2\} \coprod K}_\alpha} & \text{Map}(\mathcal{E}pt^n \times \Delta^{n, n_k}_{L, k \in K}, \delta_{\{1, 2\} \coprod K}^{*} X) \\
\downarrow & & \downarrow \\
\text{Map}(\Box^n \times \Delta^{n, n_k}_{L, k \in K}, \delta_{\{1, 2\} \coprod K}^{*} X) & \xrightarrow{\epsilon^{\{1, 2\} \coprod K}_\alpha} & \text{Map}(\Box^n \times \Delta^{n, n_k}_{L, k \in K}, X).
\end{array}
\]

Here we used the isomorphism \(\epsilon^{\{1, 2\} \coprod K}_\alpha \circ \delta_{\{1, 2\} \coprod K}^{*} \simeq \text{id}_{\mathcal{S}et_\Delta}\). By Lemma 4.2 and [17, Corollaries 2.3.2.4, 2.3.2.5], \(\text{res}_2\) is a trivial fibration if \(X\) is an \(\infty\)-category and \(\text{res}_1\) is an isomorphism if \(X\) is the nerve of an ordinary category.

We let \(\mathcal{Kpt}^{\alpha}(\sigma)_L = \mathcal{Kpt}^{\alpha}(\sigma)\) if \(\alpha \not\in L\) and \(\mathcal{Kpt}^{\alpha}(\sigma)_L = \mathcal{Kpt}^{\alpha}(\sigma)^{op}\) if \(\alpha \in L\). We have a canonical map

\[
\phi(\sigma): \mathcal{Kpt}^{\alpha}(\sigma)_L \to \text{Map}(\Box^n \times \Delta^{n, n_k}_{L, k \in K}, \delta_{\{1, 2\} \coprod K}^{*}) \subseteq \mathcal{T}_\alpha = \text{Hom}(\delta^{\{1, 2\} \coprod K}_{\{\alpha\} \coprod K}, \Delta^{n, n_k}_{L, k \in K}, X, \mathcal{E}_k \subseteq K),
\]

Let \((X, \mathcal{E})\) be a \((\{1, 2\} \coprod K)\)-marked simplicial set. We put \(\mathcal{Kpt}_b^{\alpha}(\mathcal{E}) = \mathcal{Kpt}^{\alpha}_{b(X, \mathcal{E})}(\sigma)\). Let \(\sigma \in \delta_{\{1, 2\} \coprod K}^{*} X\). Let \(Y = \text{Map}^{\mathcal{B}}(\delta_{k}^{*} \Delta^{n, n_k}_{L, k \in K}, (X, \{\mathcal{E}_k \subseteq K\})), \mathcal{T}_\alpha \to \text{Hom}(\delta_{\{\alpha\} \coprod K}^{*} \Delta^{n, n_k}_{L, k \in K}, (X, \{\mathcal{E}_k \subseteq K\}) \subseteq \mathcal{T}_\alpha, \sigma = 1, 2, n, n_k \in K\).

Then every \(\sigma: \Delta^{n, n_k}_{L, k \in K} \to \delta_{\{1, 2\} \coprod K}^{*} X, X, \{\mathcal{E}_k \subseteq K\}\) induces an \(n\)-simplex \(\tau\) of \(Y\) and \(\mathcal{Kpt}^{\alpha}_{b(X, \mathcal{E})}(\sigma) \simeq \mathcal{Kpt}^{\alpha}_{b(X, \mathcal{E})}(\sigma)\).

**Assumption 4.4.** Let \((X, \mathcal{T})\) be a \((\{1, 2\} \coprod K)\)-tilled simplicial set and let \((X, \mathcal{T}')\) be a \((\{0\} \coprod K)\)-tilled simplicial set. Consider the following assumptions:

1. For \(\sigma: \Delta^2 \to X\) with \(\sigma \circ d^2_0 \in \mathcal{T}_1, \sigma \circ d^2_2 \in \mathcal{T}_2\), we have \(\sigma \circ d^2_1 \in \mathcal{T}_0\)
(2) For \( k \in K \) and \( \sigma : \Delta^2 \times \Delta^1 \to X \) satisfying \( \sigma \circ (d_0^2 \times \text{id}) \in T_{1k}, \sigma \circ (d_2^2 \times \text{id}) \in T_{2k}, \) we have 
\[ \sigma \circ (d_1^2 \times \text{id}) \in T_{0k}. \]
(3) For \( k \in K, T_k \subseteq T'_k. \) For \( k, k' \in K, k \neq k', T_{kk'} \subseteq T'_{kk'}. \)

Assumption (1) is satisfied if \( T_1, T_2 \subseteq T'_0 \) and \( T'_0 \) is stable under composition. Assumption (2) is satisfied if \( T_{1k}, T_{2k} \subseteq T'_0 \) and \( T'_0 \) is stable under composition in the first direction.

Let \( \mu = \mu_0 \prod K \to \{0\} \prod K, \) where \( \mu_0 : \{0\} \to \{0\}. \) Let \( (X, T) \) be a \( \{(1), 2\} \prod K \)-tiled simplicial set and let \( (X, T') \) be a \( \{(0) \prod K \)-tiled simplicial set. Let \( L \subseteq K, \alpha \in \{1, 2\} \prod K. \) We have a commutative diagram

\[
\begin{array}{ccc}
\epsilon^{[0]}_{\mu(\alpha)} \text{Map}(\mathbb{C}^{\longleftarrow} \mathbb{C} \times \Delta^{n_k \{k \in K\}, \mathbb{C}}^{\longleftarrow} (X, T', X)) & \xrightarrow{\epsilon^{[0]}_{\mu(\alpha)}} & \text{Map}(\mathbb{C}^{\longleftarrow} \mathbb{C} \times \Delta^{n_k \{k \in K\}}^{\longleftarrow} (X, T', X)) \\
\delta^{[0]}_{\mu(\alpha)K} & \cong & \delta^{[0]}_{\mu(\alpha)K} \\
\text{Map}(\mathbb{C}^{\longleftarrow} \mathbb{C} \times \Delta^{n_k \{k \in K\}}^{\longleftarrow} (X, T', X)) & \xrightarrow{\delta^{[1]}_{\mu(\alpha)K}} & \text{Map}(\mathbb{C}^{\longleftarrow} \mathbb{C} \times \Delta^{n_k \{k \in K\}, X).}
\end{array}
\]

Under Assumption 4.4, for every map \( \sigma : \Delta^{n_k \{k \in K\}} \to \Delta^{[0]}_{\mu(\alpha)K} X, \) the inclusion \( \mathbb{C}^{\longleftarrow} \mathbb{C} \to \Delta^{[0]}_{\mu(\alpha)K} X \) factorizes through the upper-left corner of the above diagram. We have a canonical map

\( \psi(\sigma) : \mathbb{C}^{\longleftarrow} \mathbb{C} \to \Delta^{[0]}_{\mu(\alpha)K} X \).

Moreover, Assumption 4.4 implies \( \delta^{[1]}_{\mu(\alpha)K} (X, T') \subseteq \Delta^{[0]}_{\mu(\alpha)K} (X, T'), \) so that the adjunction for \( \Delta^{[0]}_{\mu(\alpha)K} \) induces a map

\[
\begin{aligned}
\delta^{[1]}_{\mu(\alpha)K} & \delta^{[0]}_{\mu(\alpha)K} (X, T', X) \\
\delta^{[1]}_{\mu(\alpha)K} & \cong \delta^{[0]}_{\mu(\alpha)K} (X, T') \to \delta^{[0]}_{\mu(\alpha)K} (X, T').
\end{aligned}
\]

**Theorem 4.5** (Multisimplicial descent). Let \( K \) be a set, let \( (X, T) \) be a \( \{1, 2\} \prod K \)-tiled simplicial set, and let \( (X, T') \) be a \( \{0\} \prod K \)-tiled simplicial set satisfying Assumption 4.4. Let \( \alpha \in \{1, 2\} \prod K, L \subseteq K. \) Assume that \( \mathbb{C}^{\longleftarrow} \mathbb{C} (\tau) \) is weakly contractible for every \( n \geq 0 \) and every \( (n, k) \in K \)-cell \( \tau \) of \( \delta^{[0]}_{\mu(\alpha)K} (X, T') \) with \( n_k = n. \) Then the map

\[
f : \delta^{[1]}_{\mu(\alpha)K} (X, T') \to \delta^{[0]}_{\mu(\alpha)K} (X, T'),
\]

composition of (4.1), is a categorical equivalence.

Note that the assumption that the \( \mathbb{C}^{\longleftarrow} \mathbb{C} (\tau) \)'s are nonempty implies \( T_k = T'_k \) for all \( k \in K \) and \( T_{kk'} = T'_{kk'} \) for all \( k, k' \in K, k \neq k'. \)

For the proof of the theorem, we need the following criterion of equivalence.

**Lemma 4.6.** A map of simplicial sets \( f : Y \to Z \) is a categorical equivalence if and only if the following conditions are satisfied for every \( \infty \)-category \( D, \)

(1) For every \( l = 0, 1 \) and every commutative diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{v} & \text{Fun}(\Delta^l, D) \\
\downarrow f & & \downarrow p \\
Z & \xrightarrow{w} & \text{Fun}(\partial\Delta^l, D)
\end{array}
\]

where \( p \) is induced by the inclusion \( \partial\Delta^l \subseteq \Delta^l, \) there exists a map \( u : Z \to \text{Fun}(\Delta^l, D) \) satisfying \( p \circ u = w \) such that \( u \circ f \) and \( v \) are homotopic over \( \text{Fun}(\partial\Delta^l, D). \)

(2) For \( l = 2 \) and every commutative diagram as above, there exists a map \( u : Z \to \text{Fun}(\Delta^l, D) \) satisfying \( p \circ u = w. \)

**Proof.** By definition, that \( f \) is a categorical equivalence means that for every \( \infty \)-category \( D, \) the functor

\[
h\text{Fun}(Z, D) \to h\text{Fun}(Y, D)
\]

induced by \( f \) is an equivalence of categories. On the other hand, the conditions for \( l = 0, 1, 2 \) mean that (4.2) is essentially surjective, full, and faithful, respectively. \( \square \)
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Proof of Theorem 4.5. Let Y and Z be the source and target of f, respectively. Consider a commutative diagram as in Lemma 4.6. For every n-simplex σ of Z, corresponding to a map τ: Δ^n × [n]_k∈K → δ^∗_{(0)UK}(X, T'), where n_k = n, consider the commutative diagram

$$
\begin{array}{ccc}
\mathcal{N}(\sigma) \rightarrow & \text{Fun}(\Delta^l \times \mathcal{C} pt^n \times \Delta^{[n]}_k \in K, \mathcal{D}) \xrightarrow{\text{res}_2} \text{Fun}(\Delta^l \times \Delta^n, \mathcal{D}) \\
\downarrow & \downarrow & \downarrow \\
\mathcal{Kpt}^n(\tau)_L \xrightarrow{h} \text{Fun}(H \times \Delta^{[n]}_k \in K, \mathcal{D}) \xrightarrow{\text{res}_2} \text{Fun}(\partial \Delta^l \times \Delta^n, \mathcal{D})
\end{array}
$$

where res_1 is induce by

$$
j: H = \Delta^l \times \square^n \coprod_{\partial \Delta^l \times \square^n} \Delta^l \times \mathcal{C} pt^n \hookrightarrow \Delta^l \times \mathcal{C} pt^n,
$$

h is the amalgamation of v_*φ(τ) and w_*ψ(τ). N(σ) is defined so that the square on the left is a pullback square, and the maps res_2 are induced by the diagonal embedding Δ^n ⊆ Cpt^n × Δ^{[n]}_k ∈ K. By Lemma 4.2 and [17, Corollaries 2.3.2.4, 2.3.2.5], j × id is inner anodyne and consequently res_1 is a trivial fibration. Thus N(σ) is weakly contractible. The composition of the lower horizontal arrows is constant of value w(σ). Let us denote by Φ(σ) the composition of the upper horizontal arrows. Since f induces a bijection on vertices, the image of Φ(σ), when restricted to (Δ^n)_0 × Δ^l, is constant of value v(σ_0). In particular, the image of Φ(σ) is contained in Map^((Δ^n)_0, Fun(Δ^l, \mathcal{D})) \circ Map((Δ^n)_0, Fun(Δ^l, \mathcal{D})) is weakly contractible. We have the following pullback square

$$
\begin{array}{ccc}
\mathcal{Kpt}^n(\tau) \rightarrow & \text{Fun}(\Delta^l \times \mathcal{C} pt^n \times \Delta^{[n]}_k \in K, \mathcal{D}) \xrightarrow{\text{res}_2} \text{Fun}(\Delta^l \times \Delta^n, \mathcal{D}) \\
\downarrow & \downarrow & \downarrow \\
\mathcal{Kpt}^n(\tau) \xrightarrow{h} \text{Fun}(H \times \Delta^{[n]}_k \in K, \mathcal{D}) \xrightarrow{\text{res}_2} \text{Fun}(\partial \Delta^l \times \Delta^n, \mathcal{D})
\end{array}
$$

Corollary 4.7. Let C be an ∞-category admitting pullbacks, let K be a finite set, and let (C, E_0, E_1, E_2, (E_k)_{k∈K}) be a ((0,1,2) \coprod K)-marked ∞-category such that E_1, E_2 ⊆ E_0, E_0 is stable under composition, and for all k ∈ K, E_1, E_2 are stable under pullback by E_k, and E_k is stable under pullback by E_1. Let α ∈ {1, 2}. Assume that for every simplex τ of C_{E_0} ≤ C, Kpt^α_{E_0}(τ) is weakly contractible. Then, for any L ⊆ K, the map

$$
f: (\mathcal{C} pt^n \times \Delta^{[n]}_k \in K, \mathcal{D}) \xrightarrow{\delta^*_{(1,2)UK,L}} (\mathcal{C} pt^n \times \Delta^{[n]}_k \in K, \mathcal{D}) \xrightarrow{\delta^*_{(0)UK,L}} (\mathcal{C} pt^n \times \Delta^{[n]}_k \in K, \mathcal{D})
$$

is a categorical equivalence, where T = (E_1, E_2, (E_k)_{k∈K}, ϕ), and ϕ is determined by Φ_{ij} = (E_1, E_2, (E_k)_{k∈K})_{ij}^α for i, j ≠ (1,2), (2,1), and Φ_{12} = E_1 * E_2.

Proof. By Theorem 4.5, it suffices to show that for every (n, n_k)_{k∈K}-simplex (n_k = n) σ of the (\{0\} \coprod K)-simplicial set C^α_{E_0}(σ), Kpt^α_{E_0}(σ) is weakly contractible. We have the following pullback square

$$
\begin{array}{ccc}
\mathcal{Kpt}^n_{E_0, τ}(σ) \rightarrow & \text{Fun}(\mathcal{C} pt^n × \Delta^{[n]}_k \in K, \mathcal{D}) \xrightarrow{\text{res}_2} \text{Fun}(\Delta^n, \mathcal{C} pt^n) \\
\downarrow & \downarrow & \downarrow \\
\mathcal{Kpt}^n_{E_1, E_2}(τ) \xrightarrow{\text{res}_2} \text{Fun}(\Delta^n, \mathcal{C} pt^n) \xrightarrow{\text{res}_2} \text{Fun}(\Delta^n, \mathcal{C} pt^n)
\end{array}
$$

where

- τ is an n-simplex of C, restriction of σ to Δ^n × {∞}, where ∞ = (n_k)_{k∈K} denotes the final object of Δ^{[n]}_k \in K;
- Q = [n] × [n]_K ⊆ Cpt^n × [n]_K is induced by the diagonal inclusion [n] ⊆ Cpt^n;
- R = Cpt^n × {∞} ⊆ Cpt^n × [n]_K;
- Fun(\mathcal{C} pt^n × Δ^{[n]}_k \in K, \mathcal{D}) \subseteq \text{Fun}(\mathcal{C} pt^n × Δ^{[n]}_k \in K, \mathcal{D}) is the full subcategory spanned by functors F: Cpt^n × Δ^{[n]}_k \in K → C which are right Kan extensions [17, Definition 4.3.2.2] of F | \mathcal{C} pt^n × \Delta^n K;
- The horizontal arrows are restrictions. The right vertical arrow is the amalgamation of the inclusion \mathcal{Kpt}^n_{E_1, E_2}(τ) ≤ \mathcal{Fun}(\mathcal{C} pt^n × \Delta^n, \mathcal{C} pt^n) with σ ∈ \mathcal{Fun}(\mathcal{C} pt^n × \Delta^n, \mathcal{C} pt^n). See below for the left vertical arrow.

Let (i, j, p) be an object of Cpt^n × [n]_K − Q \cup R. Let

$$
B = B(i, j, p) = \{(i,j), (j,j)\} × \{p, ∞\} \subseteq Cpt^n × [n]_K
$$

Let
be the fully partially ordered subset, and let $A = A(i, j, p) = B \cap (Q \cup R)$. For any object $s$ of $(Q \cup R)_{(i,j,p)/}$, $N(A/s)$ is weakly contractable. It follows that $N(A|^{op}) \subseteq (N(Q \cup R)_{(i,j,p)/})^{op}$ is cofinal by [17, Theorem 4.1.3.1]. Thus a functor $F: \mathbf{Cpt}^{n} \times \Delta^{[n]} \in \mathcal{K} \rightarrow \mathcal{E}$ is a right Kan extension of $F| N(Q \cup R)$ if and only if its restriction to $N(B(i, j, p))$ is a pullback square, for all $(i, j, p)$. This implies that the image of $\mathfrak{Kpt}_{\mathcal{E}, \mathcal{F}}(\sigma) \subseteq \text{Fun}(\mathbf{Cpt}^{n} \times \Delta^{[n]} \in \mathcal{K}, \mathcal{E})$ is contained in $\text{Fun}(\mathbf{Cpt}^{n} \times \Delta^{[n]} \in \mathcal{K}, \mathcal{E})_{\mathcal{K}, \mathcal{E}}$. The left vertical arrow of (4.3) is the induced inclusion. For a functor $F$ in $\text{Fun}(\mathbf{Cpt}^{n} \times \Delta^{[n]} \in \mathcal{K}, \mathcal{E})_{\mathcal{K}, \mathcal{E}}$, for $a \leq b$ in $\mathbf{Cpt}^{n}$, $c \leq d$ in $[n]^{K}$, the restriction of $F$ to the full subcategory $\{a, b\} \times \{c, d\}$ is a pullback square. The assumptions then imply that (4.3) is indeed a pullback square.

By [17, Proposition 4.3.2.15], $\text{res}_{3}$ is a trivial fibration. By Lemma 6.3, the inclusion $N(Q) \cup N(R) \subseteq N(Q \cup R)$ is inner anodyne, so that $\text{res}_{2}$ is a trivial fibration. It follows that the upper horizontal arrow is a trivial fibration. The assumption that $\mathfrak{Kpt}_{\mathcal{E}, \mathcal{F}, \mathcal{P}}(\tau)$ is weakly contractible then implies that $\mathfrak{Kpt}_{\mathcal{E}, \mathcal{F}}(\sigma)$ is weakly contractible.

Before giving a criterion for the weak contractibility, we give a criterion for $\mathfrak{Kpt}^{\alpha}(\sigma)$ to be an $\infty$-category.

**Proposition 4.8.** Let $K$ be a set and let $C, K$ be a $\{(1, 2) \coprod K\}$-marked $\infty$-category. Let $\alpha \in \{1, 2\} \coprod K$. Assume that $C_{\alpha}$ is stable under composition. Then, for every $(n, n_{k})_{k \in K}$-simplex $\sigma$ of $\delta_{\alpha}^{10} \cup K C, \mathfrak{Kpt}_{\mathcal{C}, \mathcal{F}}(\sigma)$ is an $\infty$-category.

If $\mathcal{C}$ is the nerve of an ordinary category, then $\mathfrak{Kpt}^{\alpha}(\sigma)$ is also the nerve of an ordinary category. To show the proposition in general, we need a lemma.

**Lemma 4.9.** Let $I$ be a set and let $j \in I$. The weakly saturated class $[17, \text{Definition A.1.2.2}]$ $S$ of morphisms in $\text{Set}_{\Delta}^{\kappa}$ generated by the inclusions

$$
\begin{align*}
&\text{(1) } (\Lambda_{\eta}^{m})^{i} \subseteq (\Delta^{m})^{i}, \text{ } 0 < i < m, \\
&\text{(2) } (\Lambda_{\eta}^{m})^{i} \coprod (\Lambda_{\eta}^{m})^{j} \subseteq (\Delta^{m})^{i+j},
\end{align*}
$$

is stable under smash products with arbitrary monomorphisms.

**Proof.** The proof is similar to [17, Proposition 3.1.2.3]. The class of monomorphisms is generated by (a) $\partial^{1} \subseteq (\Delta^{k})^{1}$, and (b) $(\Delta^{1})^{i} \subseteq (\Delta^{1})^{i+j}$, $j \in I$. It suffices to show that if $f$ is (1) or (2) and $g$ is (a) or (b), the smash of $f$ and $g$ is in $S$. There are four cases to consider:

(1a) The smash product is in the weakly saturated class generated by (1).

(1b) The smash product is an isomorphism since $\Lambda_{\eta}^{m}$ contains all vertices of $\Delta^{m}$.

(2a) The smash product is an isomorphism for $k > 0$ and is isomorphic to $f$ for $k = 0$.

(2b) The smash product is an isomorphism for $j \neq j$ and is a pushout of (2) for $j = j$.

**Proof of Proposition 4.8.** Let $I = \{1, 2\} \coprod K$. We have commutative diagrams with Cartesian squares:

$$
\begin{array}{c}
\mathfrak{Kpt}^{\alpha}(\sigma) \ar[r] \ar[d] & P \\
\text{Fun}(\mathbf{Cpt}^{n} \times \Delta^{[n]}, \mathcal{E}) \ar[r]^{\text{res}_{1}} & \text{Fun}(\square^{n} \coprod (\Delta^{n})_{0} \Delta^{n} \times \Delta^{[n]}, \mathcal{E})
\end{array}
$$

$$
\begin{array}{c}
P \ar[r] \ar[d] \ar[dr] & \bullet \ar[r] \ar[d] & \text{Map}(\mathbf{Cpt}^{n} \otimes \Delta^{[n]} \in \mathcal{K}, \delta_{\alpha}^{10} \cup (\mathcal{C}, \mathcal{E})) \\
\text{Fun}(\square^{n} \coprod (\Delta^{n})_{0} \Delta^{n} \times \Delta^{[n]}, \mathcal{E}) \ar[r]^{\text{res}_{2}} & \text{Fun}(\square^{n} \times \Delta^{[n]}, \mathcal{E}) \ar[d]^{g} \\
\{\sigma\} \ar[r] & \text{Fun}(\Delta^{n} \times \Delta^{[n]}, \mathcal{E}) \ar[d]^{\text{res}_{2}} \ar[r] & \text{Fun}(\Delta^{n} \times \Delta^{[n]}, \mathcal{E})
\end{array}
$$

where $\text{res}_{2}$ is induced by the diagonal embedding $(\Delta^{n})_{0} \subseteq \square^{n}$, and $\text{res}_{1}$ by the inclusion $\square^{n} \coprod (\Delta^{n})_{0} \Delta^{n} \subseteq \mathbf{Cpt}^{n}$. We claim that $\text{res}_{2} \circ g$ is an inner fibration, which implies that $P$ is an $\infty$-category. This implies the lemma because $\text{res}_{1}$ is an inner fibration by [17, Corollary 2.3.2.5].
To show the claim, note that for any simplicial set $A$, we have

$$\Hom_{\Set}(A, t_{\alpha} \Map(C(pt \times \Delta^{[n]}, \delta_+^{[\alpha]}(E, F))) \simeq \Hom_{\Set}(A^{[n]} \times \Delta^{[n]}, \delta_+^{[\alpha]}(E, F)))$$

$$\simeq \Hom_{\Set}(A^{[m]} \times S, (E, F)),$$

where $t_{\alpha} : \{\alpha\} \to I$ is the inclusion, $T = ((\Delta^n)^{[0]} \cdot \Delta^{[n]}_0)_+ \subseteq \delta_+^{[\alpha]}(C(pt \times \Delta^{[n]}_0 \times \Delta^{[n]}_0)) = S$, compatible with $\text{res}_2 \circ g$. For any map $j : A \to B$ of simplicial sets, $\text{res}_2 \circ g$ has the right lifting property with respect to $j$ if and only if $p : (E, F) \to (\Delta^n)^{[\alpha]}$ has the right lifting property with respect to the lower row of the commutative diagram with pushout square

$$A^{[\alpha]} \times S \coprod_{A^{[\alpha]} \times T} B^{[\alpha]} \times T \to B^{[\alpha]} \times S$$

$$A^{[\alpha]} \times S \coprod_{A^{[\alpha]} \times T} B^{[\alpha]} \times T \to (A^{[\alpha]} \coprod B^{[\alpha]}) \times S \to B^{[\alpha]} \times S,$$

which is obtained from $a_j : A^{[\alpha]} \to B^{[\beta]}$ and $b_j : A^{[\alpha]} \coprod B^{[\beta]} \to B^{[\beta]}$ by composition, pushout, and smash products with monomorphisms. For $m > 2$, $b_{\Lambda \subseteq \Delta^m}$ is an isomorphism. By Lemma 4.9, it suffices to check that $p$ satisfies the right lifting property with respect to $a_{\Lambda \subseteq \Delta^m}$, $0 < k < m$ and $b_{\Lambda \subseteq \Delta^2}$, which follows from the assumptions.

\[\square\]

Remark 4.10. Similarly one can show the following analogue of Proposition 4.8 for multi-tiled $\infty$-categories. Let $K$ be a set, let $(E, F)$ be a $(\{1, 2\} \subseteq K)$-tiled $\infty$-category and $\alpha \in \{1, 2\} \subseteq K$. Assume that $T_{\alpha}$ is stable under composition and $T_{\alpha, \beta}$ is stable under composition in the first direction for all $\beta \in \{1, 2\} \subseteq K$, $\beta \neq \alpha$. Then, for every $(n, n_k) \in K$-simplex $\sigma$ of $\delta_+^{[\alpha]}(E, F)$, $\K(pt^{[\alpha]}_{E, F})$ is an $\infty$-category. In fact, for any set $I$ and any $j \in I$, the weakly saturated class of morphisms in $\Set(I, E)$ generated by the inclusions

1. $V((\Delta^n)^{[\alpha]})) \subseteq V((\Delta^n)^{[\alpha]}), 0 < j < m$,
2. $V((\Delta^n)^{[\alpha]} \coprod (\Delta^n)^{[\alpha]}) \subseteq V((\Delta^n)^{[\alpha]}))$,
3. Smash products of (2) with $V((\Delta^n)^{[\alpha]} \subseteq V((\Delta^n)^{[\alpha]}), j' \in I, j' \neq j$,

is stable under smash products with arbitrary monomorphisms.

Proposition 4.11. Let $(E, F, G)$ be a $2$-marked $\infty$-category. Suppose that the following conditions are satisfied:

1. $E_1$ and $E_2$ are stable under composition.
2. For every morphism $f$ of $E$, there exists a $2$-simplex of $E$ of the form

\[
\begin{array}{ccc}
q & y & p \\
z & f & x
\end{array}
\]

with $p \in E_1$ and $q \in E_2$.

3. The $\infty$-category $E_{E_1}$ admits pullbacks and pullbacks are preserved by the inclusion $E_{E_1} \subseteq E$.

Then, for every simplex $\sigma$ of $E, \K(pt^{[\alpha]}(\sigma)^{op}$ is a filtered $\infty$-category and is weakly contractible. Moreover, the natural map

$$\delta_2^E E_{E_1, E_2} \to E$$

is a categorical equivalence.

Assumption (3) is satisfied, for example, when $E$ admits pullbacks and $E_1$ is admissible (Definition 3.9). We refer to [17, Definition 5.3.1.7] for the definition of filtered $\infty$-category. Recall that an ordinary is filtered if and only if its nerve is a filtered $\infty$-category [17, Proposition 5.3.1.13]. Thus, in the case where $E$ is the nerve of an ordinary category, the first assertion of Proposition 4.11 generalizes [6, Proposition 3.2.6].
Proof. By Proposition 4.8, $\mathcal{Kpt}^1(\sigma)$ is an $\infty$-category. It suffices to show that $\mathcal{Kpt}^1(\sigma)^{op}$ is filtered. In fact, every filtered $\infty$-category is weakly contractible [17, Lemma 5.3.1.18]. The last assertion of the proposition then follows from Theorem 4.5.

By [17, Lemma 5.3.1.12], $\mathcal{Kpt}^1(\sigma)^{op}$ is filtered if and only if for every $n \geq 0$, every map $f : A = \partial \Delta^n \to \mathcal{Kpt}^1(\sigma)$ extends to a map $A^e \to \mathcal{Kpt}^1(\sigma)$. We regard $f$ as a map $A \times \mathcal{Kpt}^n \to \mathcal{C}$ and we are going to extend it to $A^e \times \mathcal{Kpt}^n \to \mathcal{C}$. We proceed by induction on $n$. The case $n = 0$ is trivial. For $n \geq 1$, by the induction hypothesis, $A \xrightarrow{\delta_k} \mathcal{Kpt}^1(\sigma)$ (resp. $D \xrightarrow{\delta_k} \mathcal{Kpt}^1(\sigma)$) extends to $A^e \xrightarrow{\delta_k} \mathcal{Kpt}^1(\sigma)$ (resp. $D^e \xrightarrow{\delta_k} \mathcal{Kpt}^1(\sigma)$), corresponding to a map $A^e \times \mathcal{Kpt}^{n-1} \to \mathcal{C}$. We regard $\mathcal{Kpt}^{n-1}$ as the full subcategory of $\mathcal{Kpt}^n$ spanned by the objects $(i,j)$, $1 \leq i \leq j \leq n-1$. Consider the full subcategory $\mathcal{Kpt}^n_0$ of $\mathcal{Kpt}^n$ spanned by the objects $(i,j)$, $0 \leq i \leq j \leq n-1$ and $(i',n)$, $k \leq i' < n$. We have $\mathcal{Kpt}^{n-1}_0 \subseteq \mathcal{Kpt}^n_0 \subseteq \cdots \subseteq \mathcal{Kpt}^0_0 = \mathcal{Kpt}^0$. Similarly we define $\mathcal{Kpt}_k^n \subseteq \mathcal{Kpt}^n$.

We show by descending induction on $k$ that there exists a map $A^e \to \mathcal{Kpt}^1(\sigma)$ compatible with $f$ and $g$, where $\mathcal{Kpt}^1_0(\sigma)$ is defined similarly to $\mathcal{Kpt}^1(\sigma)$ but with $\mathcal{Kpt}^n_0$, $\mathcal{Kpt}^n$, $\square^n = \delta_k^* \mathcal{Kpt}^n$ replaced by $\mathcal{Kpt}^n_k$, $\mathcal{Kpt}^n_k$, $\delta_k^* \mathcal{Kpt}^n_k$, respectively. The case $k = 0$ will allow us to conclude the proof of the proposition. For $k = n$, it suffices to note that the map $\iota : \mathcal{Kpt}^{n-1} \amalg_{\Delta^{n-1}} \Delta^n \to \mathcal{Kpt}^n_0$ induced by the diagonal inclusion $\Delta^n \subseteq \mathcal{Kpt}^n_0$ is inner anodyne by Lemma 6.3, and so is the smash product of $\iota$ with the inclusion $A \subseteq A^e$. For $0 \leq k \leq n-1$, consider the full subcategory $\Delta^2 \subseteq \mathcal{Kpt}^n_k$ spanned by $(k,n-1), (k,n), (k+1,n)$. The inclusion $\mathcal{Kpt}^n_k \amalg_{\Delta^{n-2}} \Delta^2 \subseteq \mathcal{Kpt}^n_k$ is inner anodyne by Lemma 6.2, and so is its smash product with $A \subseteq A^e$. Thus it suffices to construct $A^e \times \Delta^2 \to \mathcal{C}$. We will construct $A^e \times \Delta^2 \to \mathcal{C}_{/\sigma(n)}$.

We are therefore reduced to showing that for every $m \geq 0$ and every object $X$ of $\mathcal{C}$, every map

$$a : A^{e(1)}_n \times T \bigcup_{A^{e(1)}_n \times D} (A^e)^{n-1} \times D \to (\mathcal{C}_{/X}, \mathcal{E}^1_1, \mathcal{E}^2_1)$$

extends to a map $(A^e)^{e(1)}_n \times T \to (\mathcal{C}_{/X}, \mathcal{E}^1_1, \mathcal{E}^2_1)$, where $A = \partial \Delta^m$, $D = (\Delta^{(0,2)})^e \subseteq (\Delta^2, \mathcal{F}_1, \mathcal{F}_2) = T$, with $\mathcal{F}_1$ (resp. $\mathcal{F}_2$) consisting of the degenerate edges and the edge $1 \to 2$ (resp. $0 \to 1$), and $\mathcal{E}^n_0$ denotes the inverse image of $\mathcal{E}^n$ via the map $\mathcal{C}_{/X} \to \mathcal{C}$, $n = 1, 2$. We let $B \subseteq A^e \times \Delta^2$ denote the full subcategory spanning by all objects except $(c,1)$, where $c$ denotes the cone point of $A^e$. The inclusion $A \times \Delta^2 \amalg_{\Delta^{(0,2)}} A^e \times \Delta^{(0,2)} \subseteq B$ is a pushout of the inclusion $A \times \Delta^2 \amalg_{\Delta^0} (A \times \Delta^0)^e \subseteq (A \times \Delta^2)^e$, which is inner anodyne by [17, Lemma 2.1.2.3], since the inclusion $A \times \Delta^0 \subseteq A \times \Delta^2$ is left anodyne by [17, Corollary 2.1.2.7]. Thus $a$ extends to a map $a' : B \to \mathcal{E}^1_1$. Note that $(\mathcal{C}_{/X})_{\mathcal{E}^1_1}$ admits finite limits and such limits are preserved by $(\mathcal{C}_{/X})_{\mathcal{E}^1_1} \subseteq (\mathcal{C}_{/X})$, by assumption (3) and [17, Corollary 4.4.2.4]. Thus, by [17, Proposition 4.3.2.15], there exists a right Kan extension $b : A^e \times \Delta^2 \to (\mathcal{C}_{/X})_{\mathcal{E}^1_1}$ of $a'$. Moreover, we may assume that the restriction of $b$ to $A^e \times \Delta^{(1,2)}$ factors through $(\mathcal{C}_{/X})_{\mathcal{E}^1_1}$. Note that $b$ does not necessarily satisfy the condition that $b(c,0) \to (c,1) \in \mathcal{E}^2_1$. Consider the totally ordered set $I = \{0 \leq 1 \leq 1 \leq 2\}$. Applying assumption (2) to $b((c,0) \to (c,1))$, we obtain a map $h : K = A^e \times \Delta^2 \amalg_{\Delta^{(1,2)}} A \times \Delta^2 \subseteq (\mathcal{C}_{/X})_{\mathcal{E}^1_1}$ such that $h((c,0) \to (c,1)) \in \mathcal{E}^2_1, h((c,1) \to (c,1)) \in \mathcal{E}^2_1$. Consider the pushout $L = A^e \times \Delta^2 \amalg_{\Delta^{(1,2)}} A \times \Delta^2$ given by the degeneracy map $I \to [2]$ identifying $1$ and $1$. The inclusion $K \subseteq L$ induced by the inclusion $A^e \times \Delta^2$ is a pushout of the inclusion

$$((c) \times \Delta^0) \star (A^e \times \Delta^{(1,2)}), \bigcup_{(c) \times \Delta^0 \times (c) \times \Delta^{(1,2)}} ((c) \times \Delta^{(0,1,2)}) \star ((c) \times \Delta^{(1,2)}) \to ((c) \times \Delta^{(0,1,2)}) \star (A^e \times \Delta^{(1,2)}),$$

which is inner anodyne by [17, Lemma 2.1.2.3]. Thus $h$ extends to a map $h' : L \to (\mathcal{C}_{/X})_{\mathcal{E}^1_1}$. The restriction of $h'$ to $A^e \times \Delta^{(0,1,2)}$ provides the desired extension. □

Remark 4.12. In the situation of Proposition 4.11, for every $\infty$-category $D$, the functor

$$(4.4) \quad \text{Fun}(\mathcal{C}, D) \to \text{Fun}(\mathcal{Kpt}^1(\sigma), D)$$

is an equivalence of $\infty$-categories. This generalizes Deligne’s gluing result [6, Proposition 3.3.2], which can be interpreted as saying that (4.4) induces a bijection between the sets of equivalence classes of objects when $\mathcal{C}$ is the nerve of an ordinary category and $D = \text{N(Cat}_1)$. 

Remark 4.13. Given a 2-marked $\infty$-category $(\mathcal{E}, \mathcal{E}_1, \mathcal{E}_2)$ satisfying certain conditions, Gaitsgory defined an $\infty$-category of correspondences $\mathcal{C}_{\text{corr}}_{\mathcal{E}_1, \mathcal{E}_2}$ [8, 5.1.2] $(\mathcal{E}_1 = \text{vert}, \mathcal{E}_2 = \text{horiz}$ in his notation). More generally, without any assumption, one can define the simplicial set of correspondences $\mathcal{C}_{\text{corr}}_{\mathcal{E}_1, \mathcal{E}_2}$ by $(\mathcal{C}_{\text{corr}}_{\mathcal{E}_1, \mathcal{E}_2})_n = \mathcal{C}_{\text{corr}}_{\mathcal{E}_1, \mathcal{E}_2}$.
Hom_{\text{Set}_{\infty}}(\mathbf{Cpt}^n, \text{op}^2_{\{2\}} \mathcal{C}^\text{cart}_{\tilde{\mathcal{E}}_1, \tilde{\mathcal{E}}_2}). Using arguments similar to part of the proof of Theorem 4.5, one shows that the natural map

$$\delta^*_{\{2\}} \mathcal{C}^\text{cart}_{\tilde{\mathcal{E}}_1, \tilde{\mathcal{E}}_2} \rightarrow \mathcal{C}^\text{corr}_{\tilde{\mathcal{E}}_1, \tilde{\mathcal{E}}_2}$$

given by “forgetting the lower-right corner of the simplices”, is a categorical equivalence.

5. Cartesian Gluing

In this section, we compare Cartesian multisimplicial nerves with bigger multisimplicial nerves. The basic idea is to decompose a square into

$$w \xrightarrow{w'} z \quad y \xrightarrow{\quad} x,$$

where the inner square is Cartesian. To make this idea precise, we introduce the following notation. For sets of edges $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}$ of an $\infty$-category $\mathcal{C}$, we let $\mathcal{E}_1 \star_{\mathcal{E}} \mathcal{E}_2 \subseteq \mathcal{E}_1 \star \mathcal{E}_2$ denote the set of squares that admits a decomposition as above with $w \rightarrow w'$ in $\mathcal{E}$.

**Theorem 5.1** (Cartesian gluing). Let $K$ be a finite set and let $(\mathcal{C}, \mathcal{T}) \subseteq (\mathcal{E}, \mathcal{F})$ be two $(\{1, 2\} \coprod K)$-tiled $\infty$-categories such that $\mathcal{T}_j = \mathcal{T}'_j$ for all $j \in \{1, 2\} \coprod K$, $\mathcal{T}_{jj'} = \mathcal{T}'_{jj'}$, for all $j, j' \in \{1, 2\} \coprod K, j \neq j'$, except when $(j, j') = (1, 2)$ or $(2, 1)$, $\mathcal{T}_{12} = \mathcal{E}_1 \star_{\mathcal{E}} \mathcal{E}_2$ and $\mathcal{T}'_{12} = \mathcal{E}_2$, where $\mathcal{E} \subseteq \mathcal{T}_1 \cap \mathcal{T}_2$ is a set of edges of $\mathcal{E}$. Suppose that the following conditions are satisfied:

1. $\mathcal{C}$ admits pullbacks, $\mathcal{T}_1$ is stable under composition and base change by $\mathcal{T}_2$, $\mathcal{T}_2$ is stable under composition and base change by $\mathcal{T}_1$, and $\mathcal{E}$ is stable under composition and base change by $\mathcal{T}_1 \cup \mathcal{T}_2$.

2. There exists a finite sequence of sets of edges of $\mathcal{C}$,

$$\mathcal{E}_0 \subseteq \mathcal{E}_1 \subseteq \cdots \subseteq \mathcal{E}_m = \mathcal{E},$$

each stable under composition and base change by $\mathcal{T}_1 \cup \mathcal{T}_2$, such that $\mathcal{E}_0$ is the set of all equivalences, and for every edge $y \rightarrow x$ in $\mathcal{E}_i$, $1 \leq i \leq m$, its diagonal $y \times_x y$ is in $\mathcal{E}_{i-1}$.

3. For every $k \in K$, $\mathcal{T}_{1k}$ is stable under composition and base change by $\mathcal{T}_{2k}$ in the first direction, $\mathcal{T}_{2k}$ is stable under composition and base change by $\mathcal{T}_{1k}$ in the first direction, and $\mathcal{T}_{1k} \star_{\text{Fun}(\Delta^1, \mathcal{C})} \mathcal{T}_{2k} = \mathcal{T}_{1k} \star_{\text{Fun}(\Delta^1, \mathcal{C})} \mathcal{T}_{2k}$.

4. For every pair $k, k' \in K, k \neq k'$, and every Cartesian square

$$w \xrightarrow{\quad} y \quad z \xrightarrow{\quad} x$$

of $\text{Fun}(\Delta^1 \times \Delta^1, \mathcal{C})$, with $y \rightarrow x$ given by a $(1, 1, 1)$-simplex of $\delta^1_{\{1, k, k\}} \square (\mathcal{E}, \mathcal{T})$ and $z \rightarrow x$ given by a $(1, 1, 1)$-simplex of $\delta^1_{\{2, k, k\}} \square (\mathcal{E}, \mathcal{T})$ (where the obvious restrictions of $\mathcal{T}$ are still denoted by $\mathcal{T}$), we have $w \in \mathcal{T}_{kk}$.

Then, for any $L \subseteq K$, the inclusion

$$g: \delta^1_{\{1, 2\} \coprod L} \delta^1_{\{1, 2\} \coprod K} \square (\mathcal{E}, \mathcal{T}) \rightarrow \delta^1_{\{1, 2\} \coprod K, L} \delta^1_{\{1, 2\} \coprod K} \square (\mathcal{E}, \mathcal{T})$$

is a categorical equivalence.

**Remark 5.2.** In the case where $\mathcal{T}_{jj'} = \mathcal{T}_j \star_{\mathcal{E}} \mathcal{T}_{j'}$ for all $j, j' \in \{1, 2\} \coprod K, j \neq j'$, if for each $k \in K$, $\mathcal{T}_k$ is stable under base change by either $\mathcal{T}_1$ or $\mathcal{T}_2$, then Conditions (3) and (4) above follow from Condition (1).

Combining the theorem with Corollary 4.7 and Proposition 4.11, we obtain the following.
Corollary 5.3. Let $\mathcal{C}$ be an $\infty$-category admitting pullbacks, let $K$ be a finite set, and let $(\mathcal{C}, \mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2, \{\mathcal{E}_k\}_{k \in K})$ be a $\{(0,1), 2\} \sqcup K$-marked $\infty$-category such that $\mathcal{E}_1, \mathcal{E}_2 \subseteq \mathcal{E}_0$, $\mathcal{E}_0$ is stable under composition, $\mathcal{E}_1, \mathcal{E}_2$ are admissible, and $\mathcal{E}_k$, $k \in K$ are stable under pullback by $\mathcal{E}_1$. Suppose that the following conditions are satisfied:

1. For every morphism $f$ of $\mathcal{C}$, there exists a 2-simplex $\sigma$ of $\mathcal{C}$ such that $f = \sigma \circ d_2^1$ and $\sigma \circ d_0^1 \in \mathcal{E}_1$, $\sigma \circ d_2^0 \in \mathcal{E}_2$.
2. There exists a finite sequence
   \[ \mathcal{E}_0' \subseteq \mathcal{E}_1' \subseteq \cdots \subseteq \mathcal{E}_m' = \mathcal{E}_1 \cap \mathcal{E}_2 \]
   of sets of edges of $\mathcal{C}$ stable under composition and pullback by $\mathcal{E}_0$, where $\mathcal{E}_0'$ is the set of all equivalences, such that for every $1 \leq i \leq m$ and every edge $y \rightarrow x$ in $\mathcal{E}_i'$, its diagonal $y \rightarrow y \times_x y$ is in $\mathcal{E}_{i-1}'$.

Then, for any $L \subseteq K$, the natural map
\[ g : \delta^*_1(1,2) \sqcup L \subset \mathcal{C}_{\mathcal{E}_1, \mathcal{E}_2, \{\mathcal{E}_k\}_{k \in K}} \to \delta^*_0(1,2) \sqcup L \subset \mathcal{C}_{\mathcal{E}_0, \{\mathcal{E}_k\}_{k \in K}} \]
is a categorical equivalence.

Note that Condition (1) above implies that $\mathcal{E}_0$ is admissible.

Remark 5.4. Condition (2) of Corollary 5.3 is satisfied if all morphisms in $\mathcal{E}_1 \cap \mathcal{E}_2$ are $n$-truncated ([17, Definition 5.5.6.8]) for some finite number $n$ (which holds automatically if $\mathcal{C}$ is equivalent to an $(n + 1)$-category [17, Proposition 2.3.4.1] by [17, Proposition 2.3.4.18]). In fact, in this case, we can take $m = n + 2$, and $\mathcal{E}_i' \subseteq \mathcal{E}_1 \cap \mathcal{E}_2$ to be the subset of $(i - 2)$-truncated morphisms for $0 \leq i \leq m = n + 2$. The set $\mathcal{E}_i'$ is admissible since the set of all $(i - 2)$-truncated morphisms is admissible (see [17, Remark 5.5.6.12] for the stability under pullback). Similarly, Condition (2) of Theorem 5.1 is satisfied if all morphisms in $\mathcal{E}$ are $n$-truncated.

The rest of this section is devoted to the proof of Theorem 5.1. A key ingredient in the proof is an analogue of the diagram (5.1) for decompositions of simplices of higher dimensions. Such decompositions are naturally encoded by certain lattices. Let us review some basic terminology.

By a lattice we mean a nonempty partially ordered set admitting products (namely, infima) and coproducts (namely, suprema) of pairs of elements, or equivalently, admitting finite nonempty products and coproducts. We denote products by $\wedge$ and coproducts by $\vee$. A finite lattice admits arbitrary products and coproducts.

A lattice $P$ is said to be distributive if $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$ for all $p, q, r \in P$, or equivalently, $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$ [5, Lemma 4.3].

A map between lattices preserving finite nonempty products and coproducts is called a morphism of lattices. A morphism of lattices necessarily preserves order. A nonempty subset of a lattice is called a sublattice if it is stable under finite nonempty products and coproducts. We endow the subset with the induced lattice structure. A nonempty interval of a lattice $P$, namely a subset of the form $P_{p/q}$ for $p, q \in P$, $p \leq q$, is necessarily a sublattice.

Example 5.5. A subset $Q$ of a partially ordered set $P$ is called an up-set if $q \in Q$ and $p \geq q \in Q$ implies $p \in Q$. The set $\mathcal{U}(P)$ of up-sets of $P$, ordered by containment, is a distributive lattice admitting arbitrary products and coproducts. We have $Q \sqcup Q' = Q \cap Q'$, $Q \wedge Q' = Q \cup Q'$. Let $P'$ be a partially ordered set and let $f : P \rightarrow P'$ be an order-preserving map. The map $\mathcal{U}' : \mathcal{U}(P') \rightarrow \mathcal{U}(P)$ carrying $Q$ to $f^{-1}(Q)$ is a morphism of lattices preserving products and coproducts. The functor $\mathcal{U}'$ admits a right adjoint $\mathcal{F} : \mathcal{U}(P) \rightarrow \mathcal{U}(P')$ carrying an up-set $Q$ of $P$ to the up-set of $P'$ generated by $f(Q)$. In other words, $\mathcal{F}(Q) = \bigcup_{q \in Q} P_{f(q)}$. The functor $\mathcal{F}$ preserves coproducts. If $P$ admits coproducts of pairs of elements and $f$ preserves such coproducts, then $\mathcal{F}$ is a morphism of lattices. In fact, in this case, for up-sets $Q_1, Q_2$ of $P$ and $y \in \mathcal{F}(Q_1) \cap \mathcal{F}(Q_2)$, there exist $x_1 \in Q_1$ and $x_2 \in Q_2$ such that $f(x_1) \leq y$ and $f(x_2) \leq y$, so that $f(x_1 \vee x_2) = f(x_1) \vee f(x_2) \leq y$. The latter implies $y \in \mathcal{F}(Q_1 \cap Q_2)$, since $x_1 \vee x_2 \in Q_1 \cap Q_2$.

We have $\mathcal{U}(P) \simeq \mathcal{U}(P)^\circ$, so that $\mathcal{U}(P)$ can be identified with the sublattice of $\mathcal{U}(P^\circ)$ spanned by nonempty up-sets of $P^\circ$. 
The map \( \varsigma^P : P \to \mathcal{U}(P) \) carrying \( p \) to \( P_p \) is a fully faithful functor. It preserves coproducts whenever they exist in \( P \). For every order-preserving map \( f : P \to P' \), the following diagram commutes:

\[
\begin{array}{ccc}
P & \xrightarrow{\varsigma^P} & \mathcal{U}(P) \\
f \downarrow & & \downarrow \mathcal{U}_f \\
P' & \xrightarrow{\varsigma'^P} & \mathcal{U}(P').
\end{array}
\]

For \( x \in \mathcal{U}(P) \), \( \varsigma^P(P)_{x_f} = \{x\} \star \varsigma^P(x) \simeq x^\triangleleft \). Thus a diagram \( F : N(\mathcal{U}(P)) \to \mathcal{C} \) in an \( \infty \)-category \( \mathcal{C} \) is a right Kan extension along \( N(\varsigma^P) \) (hereafter denoted by \( \varsigma^P \)) if and only if for every \( x \in \mathcal{U}(P) \), the restriction of \( F \) to \( \{x\} \star N(\varsigma^P(x)) \) exhibits \( F(x) \) as the limit of \( F \rvert N(\varsigma^P(x)) \).

By an exact square in a lattice, we mean a square that is both a pushout square and a pullback square, or, equivalently, a square of the form

\[
\begin{array}{ccc}
x \land y & \longrightarrow & x \\
\downarrow & & \downarrow \\
y & \longrightarrow & x \lor y.
\end{array}
\]

The left vertical arrow is called an exact pullback of the right vertical arrow. An exact square in \( \mathcal{U}(P) \) corresponds to a pushout square of sets. By [17, Proposition 4.2.3.8], a right Kan extension \( F : N(\mathcal{U}(P)) \to \mathcal{C} \) along \( \varsigma^P \) carries exact squares to pullback squares. If \( P \) is finite, every morphism \( Q \to Q' \) in \( \mathcal{U}(P) \) is the composition of a sequence of exact pullbacks of the morphisms \( \omega^P(x) : \varsigma^P(x) \to \varsigma^P(x) - \{x\} \) for \( x \in P \). In fact, by induction on the cardinality of \( Q - Q' \) and by choosing a minimal element, we are reduced to the case where \( Q' = Q - \{x\} \). In this case \( Q \to Q' \) is the exact pullback of \( \omega^P(x) \) by \( Q' \to \varsigma^P(x) \lor Q' \).

Let \( f : P \to P' \) be a map preserving nonempty coproducts. Then for \( Q \in \mathcal{U}(P) \), the map \( N(Q) \to N(\mathcal{U}_f(Q)) \) induced by \( f \) is cofinal, by [17, Theorem 4.1.3.1]. In fact, for \( y \in \mathcal{U}_f(Q), Q_{/y} \) admits a final object. Thus if \( F : N(\mathcal{U}(P')) \to \mathcal{C} \) is a right Kan extension along \( \varsigma^{P'} \), then \( F \circ N(\mathcal{U}_f) : N(\mathcal{U}(P)) \to \mathcal{C} \) is a right Kan extension along \( \varsigma^P \).

**Remark 5.6.** Although we do not need it in the sequel, let us recall the correspondence between finite partially ordered sets and finite distributive lattices [5, Chapter 5]. An element \( p \) of a lattice \( L \) is said to be product-irreducible if \( p \) is not a final object and \( p = a \land b \) implies \( p = a \) or \( p = b \) for all \( a, b \in L \). We let \( \mathcal{J}(L) \subseteq L \) denote the subset of product-irreducible elements of \( L \). The map \( \varsigma^P \) factorizes to give a map \( P \to \mathcal{J}(\mathcal{U}(P)) \), which is an isomorphism if \( P \) is finite. The map \( \eta_L : L \to \mathcal{U}(\mathcal{J}(L)) \) carrying \( x \) to \( \mathcal{J}(L)_{x_f} \) is a morphism of lattices preserving initial and final objects. Birkhoff's representation theorem states that \( \eta_L \) is an isomorphism for any finite distributive lattice \( L \).

**Notation 5.7.** For \( n \geq 0 \), we let \( \text{Cart}^n \) denote the sublattice of \( \mathcal{U}([n] \times [n]) \) spanned by nonempty up-sets and we let \( \varsigma^n : [n] \times [n] \to \text{Cart}^n \) denote the map induced by \( \varsigma^{[n] \times [n]} \) carrying \( (p, q) \) to \( ([n] \times [n])_{(p,q)} \). For an order-preserving map \( d : [m] \to [n] \), we let \( \text{Cart}(d) : \text{Cart}^m \to \text{Cart}^n \) denote the map induced by \( \mathcal{U}_{d \times d} \). We write \( \mathcal{C} \text{art}^n = \mathcal{N}(\text{Cart}^n) \), \( \mathcal{C} \text{art}(d) = \mathcal{N}(\text{Cart}(d)) \). We still write \( \varsigma^n \) for \( \mathcal{N}(\varsigma^n) \).

Note that the cardinality of \( \text{Cart}^n \) is \( \binom{2n+2}{n+1} - 1 \) and \( \text{Cart}^n \simeq \mathcal{U}([n] \times [n] - \{(n,n)\}) \). The definition given above has the advantage of being functorial with respect to \( n \). Below are the Hasse diagrams of \( \text{Cart}^1 \) and \( \text{Cart}^2 \), rotated so that the initial objects are shown in the upper-left corners. Bullets represent elements in the images of \( \varsigma^1 \) and \( \varsigma^2 \). The dashed boxes represent \( \text{Cart}^1_{0,1} \) and \( \text{Cart}^2_{1,2} \) (see Notation 5.9 below).

The map \( \text{Cart}(d) \) is a morphism of lattices, while \( \varsigma^n \) preserves coproducts and final objects. In particular, \( \varsigma^n(p,q) = \varsigma^n(p,0) \lor \varsigma^n(0,q) \). The maps \( \varsigma^n \) are compatible with \( d \) in the sense that \( \text{Cart}(d)(\varsigma^n(p,q)) = \varsigma^n(d(p),d(q)) \).
A diagram $F: \mathcal{C} \to \mathcal{C}$ in an infinite category $\mathcal{C}$ is a right Kan extension along $\varsigma^n$ if and only if for every $x \in \mathcal{C}$, the restriction of $F$ to $\{x\} \times N(\varsigma^n(x))$ exhibits $F(x)$ as the limit of $F \mid N(\varsigma^n(x))$. If $F: \mathcal{C} \to \mathcal{C}$ is a right Kan extension along $\varsigma^n$, then $F \circ \mathcal{C}(d): \mathcal{C}(d) \to \mathcal{C}$ is a right Kan extension along $\varsigma^m$.

**Definition 5.8.** Let $\mathcal{C}, \mathcal{D}$ be infinite categories and let $\sigma: \Delta^n \times \Delta^n \to \mathcal{C}$ be a functor. We define $\mathcal{K}(\sigma)$, the simplicial set of Cartesianizations of $\sigma$, to be the fiber of the restriction map

$$\text{Fun}(\mathcal{C} \times \mathcal{D}, \mathcal{C})_{\mathcal{RKE}} \xrightarrow{\text{res}} \text{Fun}(\Delta^n \times \Delta^n \times \mathcal{D}, \mathcal{C})$$

at $\sigma$. Here $\text{Fun}(\mathcal{C} \times \mathcal{D}, \mathcal{C})_{\mathcal{RKE}} \subseteq \text{Fun}(\mathcal{C} \times \mathcal{D}, \mathcal{C})$ is the full subcategory spanned by functors $F: \mathcal{C} \times \mathcal{D} \to \mathcal{C}$ that are right Kan extensions of $F \mid \Delta^n \times \mathcal{D}$ along $\varsigma^n \times \text{id}_\mathcal{D}$.

If $\mathcal{C}$ admits pullbacks, then res is a trivial fibration by [17, Proposition 4.3.2.15], so that $\mathcal{K}(\sigma)$ is a contractible Kan complex.

Similarly to the situation of Section 4, we now introduce a simplicial subset of $\mathcal{C}_n$, to which the gluing data apply.

**Notation 5.9.** We define an order-preserving map $\xi^n: [n] \times [n] \to \mathcal{C}_n$ by $\xi^n(p, q) = \varsigma^n(p, 0) \wedge \varsigma^n(0, q)$. For $0 \leq p, q \leq n$, we define $\mathcal{C}_{n, p, q} = \mathcal{C}_{n}^{\xi^n(p, q)/\varsigma^n(p, q)}$. We define $\Xi_n = \bigcup_{0 \leq p, q \leq n} \Xi_{n, p, q} \subseteq \mathcal{C}_n$, where $\Xi_{n, p, q} = N(\mathcal{C}_{n, p, q})$.

For an order-preserving map $d: [m] \to [n]$, we have

$$\mathcal{C}_n(d)(\xi^n(p, q)) = \varsigma^n(d(p), d(0)) \wedge \varsigma^n(d(0), d(q)) \geq \xi^n(d(p), d(q)).$$

Thus $\mathcal{C}(d)$ induces morphisms of lattices $\mathcal{C}^{m}_{n, p, q} \to \mathcal{C}^{m}_{n, d(p), d(q)}$ and $\mathcal{C}(d)$ induces a map $\Xi^m \to \Xi^n$.

The following lemma is crucial for our argument. The proof will be given in Lemma 6.7.

**Lemma 5.10.** The inclusion $\Xi^n \subseteq \mathcal{C}^n$ is a categorical equivalence.

We define a morphism of lattices $\pi^n = (\pi^n_1, \pi^n_2): \mathcal{C}^n \to [n] \times [n]$ to be the composite of the morphism of lattices $\vee \xi^n(n, n): \mathcal{C}^n \to \mathcal{C}^n$ and the isomorphism $\mathcal{C}^n \simeq [n] \times [n]$ carrying $\varsigma^n(p, n) \wedge \varsigma^n(n, q)$ to $(p, q)$. In other words, $\varsigma^n(x, n) \wedge \varsigma^n(n, x) = x \vee \xi^n(n, n)$. Note that $\pi^n(x) = (\min_{p, q \in x} p, \min_{p, q \in x} q)$, so that $\pi^n \circ \mathcal{C}(d) = (d \times d) \circ \pi^m$. Note also that $\varsigma^n$ is a section of $\pi^n$. We still write $\pi^n$ for $N(\xi^n)$.

**Lemma 5.11.** Let $\mathcal{C}$ be an infinite category. A diagram $F: \Delta^n \times \Delta^n \to \mathcal{C}$ is a right Kan extension of $F \mid N(\xi^n(n, n))$ if and only if $F \circ \pi^n: \mathcal{C}^n \to \mathcal{C}$ is a right Kan extension along $\varsigma^n$.

**Proof.** If $F \circ \pi^n: \mathcal{C}^n \to \mathcal{C}$ is a right Kan extension along $\varsigma^n$, then $F(p, q) = F(\pi^n(\varsigma^n(p, n) \wedge \varsigma^n(n, q)))$ is a limit of $F \mid N(\varsigma^n(p, n) \wedge \varsigma^n(n, q))$. Conversely, if $F$ is a right Kan extension of $F \mid N(\xi^n(n, n))$, then for every $x \in \mathcal{C}^n$, $F \mid N(x)$ is a right Kan extension of $F \mid N(\vee \xi^n(n, n))$, so that $F \circ \pi^n \mid \{x\} \ast N(\varsigma^n(x))$ is a limit diagram.

Let $F: \mathcal{C}^n \to \mathcal{C}$ be a right Kan extension of $G = F \circ \varsigma^n$. Since $F$ carries exact squares to pullback squares, any edge in the image of $F$ is a composition of pullbacks of $F(\omega^n(p, q) \mid (p, q) \in [n] \times [n] - \{(n, n)\}$, where $\omega^n(p, q): \varsigma^n(p, q) \to \varsigma^n(p, q) \setminus \{(p, q)\}$. Moreover, for $(p, q) \in [n-1] \times [n-1]$, $F(\omega^n(p, q)) : G(p, q) \to G(p+1, q) \times G(p, q+1) G(p+1, q)$. To state a consequence of this under the assumptions of Theorem 5.1, we introduce a bit more notation.

We define a 2-marking $\mathcal{T} = (\mathcal{T}_1, \mathcal{T}_2)$ on $\mathcal{C}^n$ as follows. We put $\delta^2_+ \Delta^{n+1} = (\Delta^{n+1} \times \Delta^{n+1}, \mathcal{T}_1, \mathcal{T}_2)$ and we define $\mathcal{T}_i = (\pi^n)^{-1}(\mathcal{T}_i)$, $i = 1, 2$. Every square in $\mathcal{T}_1 \ast \mathcal{C}^n \mathcal{T}_2$ has a canonical decomposition

$$z \xrightarrow{x \wedge y} x \xrightarrow{x} w, \quad y \xrightarrow{x \vee y} x.$$
where the oblique arrows are in fibers of $\pi^n$. Note that $\mathcal{T}_{12}^\Box = \mathcal{T}_1 \ast_{\mathcal{C}art, K} \mathcal{T}_2$ is the set of squares such that $z = x \wedge y$. The map $\text{Cart}(d)$ induces $(\text{Cart}^m, \mathcal{F}) \to (\text{Cart}^n, \mathcal{F})$ and $(\text{Cart}^m, \mathcal{F}_{\text{cart}}) \to (\text{Cart}^n, \mathcal{F}_{\text{cart}})$.

**Remark 5.12.** For a $\bigl(\{1,2\} \coprod K\bigr)$-tilled $\infty$-category $(\mathcal{E}, \mathcal{T})$, consider the diagram

\[
\begin{array}{ccc}
\text{Fun}(\text{Cart}^n \times \Delta^{[n_k]_{k \in K}}, \mathcal{E}) & \xrightarrow{\epsilon_1^{\{1,2\} \coprod K}} & \text{Map}(\delta_2^{2+}(\text{Cart}^n, \mathcal{F}) \otimes \Delta^{[n_k]_{k \in K}}, \delta_2^{\{1,2\} \coprod K}(\mathcal{E}) \\
X := \epsilon_1^{\{1,2\} \coprod K}\text{Map}(\delta_2^{2+}(\text{Cart}^n, \mathcal{F}) \otimes \Delta^{[n_k]_{k \in K}}, \delta_2^{\{1,2\} \coprod K}(\mathcal{E}, \mathcal{T})) & \downarrow & \text{Map}(\delta_2^{2+}(\text{Cart}^n, \mathcal{F}) \times \Delta^{[n_k]_{k \in K}}, \delta_2^{\{1,2\} \coprod K}(\mathcal{E}, \mathcal{T})).
\end{array}
\]

Under the assumptions of Theorem 5.1, let $\mathcal{T}'$ be the tiling between $\mathcal{T}$ and $\mathcal{T}'$ defined by $\mathcal{T}_{12} = \mathcal{T}_1 \ast_{\mathcal{C}art, K} \mathcal{T}_2$. Then for a map $\sigma: \Delta_{n,n,n}^{x,y} \rightarrow \delta_2^{\{1,2\} \coprod K}(\mathcal{E}, \mathcal{T})$, the inclusion $\mathcal{K}art(\sigma) \subseteq \text{Fun}(\text{Cart}^n \times \Delta^{[n_k]_{k \in K}}, \mathcal{E})$ factorizes through $X$ with $\mathcal{T} = \mathcal{T}'$. The same holds with $\delta_2^{2+}(\mathcal{E}, \mathcal{T})$ replaced by $\delta_2^{2\Box}(\mathcal{E}, \mathcal{T}_{\text{cart}})$ and $\mathcal{T} = \mathcal{T}'^{-1}$.

We define a section

\[
\alpha: \text{Cart}^n \rightarrow \delta_2^{2+}(\mathcal{E}, \mathcal{T})
\]

of the adjunction map $\delta_2^{2+}(\mathcal{E}, \mathcal{T}) \rightarrow \text{Cart}^n$ as follows. For $x \leq y$ in $\text{Cart}^n$, we define two elements of $\text{Cart}^n_{\text{ CART}}: \lambda^n(x, y) = \zeta^n(\pi_1^n(y), 0) \vee x, \mu^n(x, y) = \zeta^n(0, \pi_2^n(y)) \vee x$. Note that $\pi^n(\lambda^n(x, y)) = (\pi_1^n(y), \pi_2^n(x)), \pi^n(\mu^n(x, y)) = (\pi_1^n(x), \pi_2^n(y)), \lambda^n(x, y) \wedge \mu^n(x, y) = \xi^n(\pi^n(y)) \vee x$. For an $m$-simplex $\tau: \Delta_m \rightarrow \text{Cart}^n$, we define $\alpha(\tau)$ to be the map $\Delta_m \times \Delta_m \rightarrow \text{Cart}^n$ carrying $(a, b)$ to $\lambda^n(\tau(b), \tau(a))$ for $a \geq b$, and to $\mu^n(\tau(a), \tau(b))$ for $a \leq b$. For $x, y \in \text{Cart}^n_{p,q}$, $\lambda^n(x, y) \wedge \mu^n(x, y) = x$. Thus $\alpha$ induces

\[
\beta: \boxtimes^n \rightarrow \delta_2^{2\Box}(\mathcal{E}, \mathcal{T}_{\text{cart}}).
\]

We have $\text{Cart}(d)(\lambda^n(x, y)) = \lambda^n(\text{Cart}(d)(x), \text{Cart}(d)(y)), \text{Cart}(d)(\mu^n(x, y)) = \mu^n(\text{Cart}(d)(x), \text{Cart}(d)(y))$. Thus $\alpha$ and $\beta$ are compatible with $\text{Cart}(d)$.

Composing the maps in Remark 5.12 with $\alpha$ and $\beta$, we obtain maps

\[
\psi(\sigma): \mathcal{K}art(\sigma) \rightarrow \text{Map}(\text{Cart}^n \times \Delta^{[n_k]_{k \in K}}, Y_{i-1}), \quad \phi(\sigma): \mathcal{K}art(\sigma) \rightarrow \text{Map}(\boxtimes^n \times \Delta^{[n_k]_{k \in K}}, Y_{i-1}),
\]

where $Y_i = \delta_1^{\{1,2\} \coprod K}(\mathcal{E}, \mathcal{T})$.

**Proof of Theorem 5.1.** We have $g: Y_0 \rightarrow Y_1 \rightarrow \cdots \rightarrow Y_m$. By Lemma 4.6 and induction, it suffices to show that for every $1 \leq i \leq m$ and every commutative diagram

\[
\begin{array}{ccc}
Y_0 & \xrightarrow{f'} & Y_{i-1} \\
\downarrow f & & \downarrow p \\
Y_i & \xrightarrow{w} & \text{Fun}(\partial\Delta^l, \mathcal{D})
\end{array}
\]

where $f$ and $f'$ are inclusions and $p$ is induced by the inclusion $\partial\Delta^l \subseteq \Delta^l$, there exists a map $u: Y_i \rightarrow \text{Fun}(\Delta^l, \mathcal{D})$ satisfying $p \circ u = w$ such that $u \circ f \circ f'$ and $u \circ f'$ are homotopic over $\text{Fun}(\partial\Delta^l, \mathcal{D})$. The proof is parallel to the proof of Theorem 4.5. For every $n$-simplex $\sigma$ of $Y_i$, corresponding to a map $\tau: \Delta_{n,n,n}^{x,y} \rightarrow \delta_2^{\{1,2\} \coprod K}(X, \mathcal{T})$, where $n_k = n$, consider the commutative diagram

\[
\begin{array}{ccc}
\text{Fun}(\Delta^l \times \text{Cart}^n \times \Delta^{[n_k]_{k \in K}}, \mathcal{D}) & \xrightarrow{\text{res}_2} & \text{Fun}(\Delta^l \times \Delta^n, \mathcal{D}) \\
\downarrow \text{res}_1 & & \downarrow \\
\text{Kart}(\tau) & \xrightarrow{h} & \text{Fun}(H \times \Delta^{[n_k]_{k \in K}}, \mathcal{D}) \\
\downarrow & & \downarrow \\
\text{Fun}(\partial\Delta^l \times \Delta^n, \mathcal{D})
\end{array}
\]
where res$_1$ is induce by
\[ j : H = \Delta^i \times \mathbb{N} \coprod_{\partial \Delta^i \times \mathbb{N}} \partial \Delta^i \times \text{Cart}^n \hookrightarrow \Delta^i \times \text{Cart}^n, \]
h is the amalgamation of $v, \phi(\tau)$ and $w, \psi(\tau)$, $\mathcal{N}(\sigma)$ is defined so that the square on the left is a pullback square, and the maps res$_2$ are induced by the composite embedding $\Delta^n \xrightarrow{\text{diag}} \Delta^n \times \Delta^n \times \Delta^{[n_k] \in K} \xrightarrow{\text{id} \times x} \text{Cart}^n \times \Delta^{[n_k] \in K}$. By Lemma 5.10 and [17, Corollary 2.2.5.4], $j \times \text{id}$ is a categorical equivalence and consequently res$_1$ is a trivial fibration. Thus $\mathcal{N}(\sigma)$ is a contractible Kan complex. The composition of the lower horizontal arrows is constant of value $w(\sigma)$. Let us denote by $\Phi(\sigma)$ the composition of the upper horizontal arrows. The image of $\Phi(\sigma)$ is contained in $\text{Map}^2(\Delta^n, \text{Fun}(\Delta^i, \mathcal{D}))$. This construction is functorial in $\sigma$, giving rise to a monomorphism $\Phi : \mathcal{N} \to \text{Map}[Z, \text{Fun}(\Delta^i, \mathcal{D})]$ in $(\text{Set}_\Delta)^{(\Delta \times \mathbb{N})^\mathbb{R}}$ such that $\text{Map}[Z, p] \circ \Phi$ factorizes through $w$. Moreover, for every $n$-simplex $\sigma'$ of $Y_0$ corresponding to a map $\Delta^n_{n,n,n|_{[K]} \in K} \to \delta^L_{(1,2)(1,2)}$, by composing with $\pi^n$, we obtain a canonical vertex $\nu(\sigma')$ of $\mathcal{N}(f(\sigma'))$, functorial in $\sigma'$, whose image under $\Phi(f(\sigma'))$ is $v(f'(\sigma'))$. Applying Corollary 2.7 to $\Phi$ and $\nu$, we obtain $u : Z \to \text{Fun}(\Delta^i, \mathcal{D})$ satisfying $p \circ u = w$ such that $u \circ f \circ f'$ and $v \circ f'$ are homotopic over $\text{Fun}(\partial \Delta^i, \mathcal{D})$, as desired. 

\section{6. Some trivial cofibrations}

In this section, we prove that certain inclusions of simplicial sets are inner anodyne or categorical equivalences. In particular, they are trivial cofibrations in $\text{Set}_\Delta$ for the Joyal model structure. Results of this section are used in Sections 4 and 5.

We let $\ast$ denote joins of categories and simplicial sets [17, Section 1.2.8].

\begin{lemma}
Let $A_0 \subseteq A$, $B_0 \subseteq B$, $C_0 \subseteq C$ be inclusions of simplicial sets. If $A_0 \subseteq A$ is right anodyne and $C_0 \subseteq C$ is left anodyne [17, Definition 2.0.0.3], then the induced inclusion
\[ A \ast B_0 \ast C \bigcup_{A_0 \ast B_0 \ast C_0} A_0 \ast B \ast C_0 \subseteq A \ast B \ast C \]
is inner anodyne.
\end{lemma}

\begin{proof}
Consider the commutative diagram of inclusions with pushout square
\[
\begin{array}{ccc}
A \ast B_0 \ast C \bigcup_{A_0 \ast B_0 \ast C_0} A_0 \ast B \ast C_0 & \xrightarrow{f} & A \ast B \ast C_0 \\
\downarrow & & \downarrow \\
A \ast B_0 \ast C \bigcup_{A_0 \ast B_0 \ast C_0} A_0 \ast B \ast C_0 & \xrightarrow{f'} & A \ast B_0 \ast C \bigcup_{A_0 \ast B_0 \ast C_0} A \ast B \ast C_0 \xrightarrow{g} A \ast B \ast C.
\end{array}
\]
By [17, Lemma 2.1.2.3], $f$ and $g$ are inner anodyne. It follows that $g \circ f'$ is inner anodyne. \hfill \Box
\end{proof}

\begin{lemma}
Let $S$ be a partially ordered set and let $Q = [2] \subseteq S$, $R = S - \{1\} \subseteq S$ be full inclusions. Assume that 0 is a final object of $R_1 = 2$ is an initial object of $R_{1/}$. Then the inclusion $\mathcal{N}(Q) \cup \mathcal{N}(R) \subseteq \mathcal{N}(S)$ is inner anodyne.
\end{lemma}

\begin{proof}
By [17, Lemma 4.2.3.6], $\mathcal{N}(\{0\}) \subseteq \mathcal{N}(R_{1/})$ is right anodyne and $\mathcal{N}(\{2\}) \subseteq \mathcal{N}(R_{1/})$ is left anodyne. The inclusion $\mathcal{N}(Q) \cup \mathcal{N}(R) \subseteq \mathcal{N}(S)$ is a pushout of the inclusion $\mathcal{N}(R_{1/} \ast R_{1/}) \bigcup_{\mathcal{N}(\{0\} \ast \{2\})} \mathcal{N}(\{2\}) \subseteq \mathcal{N}(R_{1/} \ast \{1\} \ast R_{1/})$, which is inner anodyne by Lemma 6.1. \hfill \Box
\end{proof}

Let $P \subseteq Q$ and $P \subseteq R$ be full inclusions of partially order sets. We endow $S = Q \coprod_P R$ with the partial order such that $Q \subseteq S$ and $R \subseteq S$ are full inclusions and for $q \in Q$, $r \in R$, $q \leq r$ (resp. $q \geq r$) if and only if there exists $p \in P$ satisfying $q \leq p \leq r$ (resp. $q \geq p \geq r$).

\begin{lemma}
Let $P \subseteq Q$ and $P \subseteq R$ be full inclusions of partially order sets and let $S = Q \coprod_P R$. Suppose that the following conditions are satisfied:
\begin{enumerate}
\item $Q$ admits pushouts and pushouts are preserved by the inclusion $Q \subseteq S$.
\item $P = \{p\}$ is finite.
\item $P$ is an up-set of $Q$ (Example 5.5).
\end{enumerate}
Then the inclusion $\mathcal{N}(Q) \cup \mathcal{N}(R) \subseteq \mathcal{N}(S)$ is inner anodyne.
\end{lemma}


Proof. We proceed by induction on \( n = \#(Q - P) \). The case \( n = 0 \) is trivial. For \( n = 1 \), let \( Q - P = \{ q \} \). Assumption (3) implies that the inclusion \( N(Q) \cup N(R) \subseteq N(S) \) is a pushout of the inclusion \( N(Q_q) \cup N(R_q) \subseteq N(S_q) \), which is isomorphic to the inclusion
\[
N(P_q) \sqcup \prod_{N(P_q)} N(R_q) \subseteq N(R_q).
\]

By assumption (1), for every \( r \in R_q \), \( P_{q/r} \) is filtered, so that \( N(P_{q/r}) \) is weakly contractible by [17, Theorem 5.3.1.13, Lemma 5.3.1.18]. It follows that \( N(P_q)^{op} \subseteq N(R_q)^{op} \) is cofinal by [17, Theorem 4.1.3.1], thus right anodyne by [17, Proposition 4.1.1.1 (4)]. Therefore (6.1) is inner anodyne by [17, Lemma 2.1.2.3].

For \( n \geq 2 \), we choose a minimal element \( q \) of \( Q - P \). Let \( Q' = Q - \{ q \} \), \( S' = S - \{ q \} \). Consider the commutative diagram of inclusions with pushout square
\[
\begin{array}{ccc}
N(Q') \cup N(R) & \xrightarrow{f} & N(S') \\
\downarrow & & \downarrow \ \\
N(Q) \cup N(R) & \xrightarrow{g} & N(S).
\end{array}
\]

By the induction hypothesis, \( f \) and \( g \) are both inner anodyne. \( \square \)

Lemma 6.4. Let \( P \) be a finite partially ordered set admitting pushouts and let \( p_0 \leq \cdots \leq p_s; q_0 \leq \cdots \leq q_s \) be elements of \( P \) such that \( p_i \leq q_{i-1} \) for \( 1 \leq i \leq s \). Then the inclusion
\[
\bigcup_{i=0}^{s} N(P_{p_i/q_i}) \subseteq N\left( \bigcup_{i=0}^{s} P_{p_i/q_i} \right)
\]
is inner anodyne.

Proof. Let \( P_i = P_{p_i/q_i} \). The inclusion can be decomposed as \( Q_0 \subseteq \cdots \subseteq Q_n \), where
\[
Q_j = N\left( \bigcup_{i=0}^{j} P_i \right) \cup \bigcup_{i=j+1}^{n} N(P_i).
\]

For \( 1 \leq j \leq n \), \( Q_{j-1} \subseteq Q_j \) is a pushout of
\[
N\left( \bigcup_{i=0}^{j-1} P_i \right) \cup N(P_j) \subseteq N\left( \bigcup_{i=0}^{j} P_i \right).
\]

It suffices to check that the latter satisfies the assumptions of Lemma 6.3. We denote coproducts in \( P \) by \( \vee \), whenever they exist. Let \( x \in A = \bigcup_{i=0}^{j-1} P_i, y \in P_j \). If \( x \geq y \), then \( x, y \in P_{j-1} \cap P_j \). If \( x \leq y \), then \( x \leq x \vee p_j \leq y \), where \( x \vee p_j \in P_{j-1} \cap P_j \). Thus the bijection \( \bigcup_{i=0}^{j} P_i \simeq A \prod_{A \setminus P} P_j \) is an isomorphism of partially ordered sets. Condition (1) of Lemma 6.3 follows from the fact that for \( x \in P_i, y \in P_j \), \( x \vee y \in P_{\text{max}\{i,j\}} \). Conditions (2) and (3) are clear. \( \square \)

Lemma 6.5. Let \( P \) be a finite lattice and let \( p_0 \leq \cdots \leq p_s; q_0 \leq \cdots \leq q_s \) be elements of \( P \) satisfying \( \bigcup_{i=0}^{s} P_{p_i/q_i} = P \). Let \( Q_1, \ldots, Q_t \) be interval sublattices of \( P \). Then the inclusion
\[
\bigcup_{i=0}^{s} N(P_{p_i/q_i}) \cup \bigcup_{j=1}^{t} N(Q_j) \subseteq N(P)
\]
is a categorical equivalence.

Proof. Let \( P_t = P_{p_t/q_t}, R_j = \bigcup_{i=0}^{s} N(P_i) \cup \bigcup_{k=1}^{j} N(Q_k) \). We proceed by induction on \( t \). By Lemma 6.4, the inclusion \( R_0 \subseteq N(P) \) is inner anodyne, thus a categorical equivalence [17, Lemma 2.2.5.2]. Thus for \( t = 0 \) we are done. For \( t \geq 1 \), it suffices to show that the inclusions \( R_0 \subseteq \cdots \subseteq R_t \) are categorical equivalences. For \( 1 \leq j \leq t \), \( R_{j-1} \subseteq R_j \) is a pushout of
\[
\bigcup_{i=0}^{s} N(P_i \cap Q_j) \cup \bigcup_{k=1}^{j-1} N(Q_k \cap Q_j) \subseteq N(Q_j)
\]
by an inclusion. By [17, Lemma A.2.4.3], it suffices to show that (6.2) is a categorical equivalence, which follows from the induction hypothesis. In fact, if we write $Q_j = P_{j/p}/q_j$, then $P_i \cap Q_j = P_{i\cap j/(p\cap q)}$ and for $0 \leq i, i' \leq s$ such that $P_i \cap Q_j \neq \emptyset$, $P_{i'} \cap Q_j \neq \emptyset$, we have $p_i \vee p \leq q_{i'} \wedge q$.

Now we prove Lemmas 4.2 and 5.10.

**Lemma 6.6.** The inclusion $\square^n \subseteq \mathcal{E}pt^n$ is inner anodyne.

**Proof.** Applying Lemma 6.4 to $P = \mathcal{E}pt^n$, $s = n$, $p_i = (0, i)$ and $q_i = (i, n)$, we get that

$$\square^n = \bigcup_{i=0}^n \mathcal{N} \left( \mathcal{E}pt^n_{(0,i)}/\!(i,n) \right) \subseteq \mathcal{N} \left( \bigcup_{i=0}^n \mathcal{E}pt^n_{(0,i)}/\!(i,n) \right) = \mathcal{N}(\mathcal{E}pt^n) = \mathcal{E}pt^n$$

is inner anodyne.

**Lemma 6.7.** The inclusion $\bigcup_{0 \leq p \leq n} \triangleleft_p^{n} \subseteq \mathcal{C}art^n$ is inner anodyne and the inclusion $\boxplus^n \subseteq \mathcal{C}art^n$ is a categorical equivalence.

**Proof.** Note that $\xi^n(0, n) \leq \cdots \leq \xi^n(n, n) \leq \xi^n(0, n) \leq \cdots \leq \xi^n(n, n)$. By Lemmas 6.4 and 6.5 applied to $P = \mathcal{C}art^n$, $s = n$, $p_i = \xi(i, n)$, $q_i = \xi(i, n)$, it suffices to show that $\mathcal{C}art^n = \bigcup_{p=0}^n \mathcal{C}art^n_{p,n}$. Let $x \in \mathcal{C}art^n$ and let $p = \pi^n_1(x)$. In other words, $w \in \mathcal{C}art^n(0, n) = \mathcal{C}art^n(p, n)$. Since $x$ is an up-set of $[n] \times [n]$, $x \geq \mathcal{C}art^n(p, 0) \geq \mathcal{C}art^n(p, n)$, so that $x \in \mathcal{C}art^n_{p,n}$.

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