General Hamiltonian Representation of ML Detection Relying on the Quantum Approximate Optimization Algorithm

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Abstract

The quantum approximate optimization algorithm (QAOA) conceived for solving combinatorial optimization problems has attracted significant interest since it can be run on the existing noisy intermediate-scale quantum (NISQ) devices. A primary step of using the QAOA is the efficient Hamiltonian construction based on different problem instances. Hence, we solve the maximum likelihood (ML) detection problem for general constellations by appropriately adapting the QAOA, which gives rise to a new paradigm in communication systems. We first transform the ML detection problem into a weighted minimum $N$-satisfiability (WMIN-$N$-SAT) problem, where we formulate the objective function of the WMIN-$N$-SAT as a pseudo Boolean function. Furthermore, we formalize the connection between the degree of the objective function and the Gray-labelled modulation constellations. Explicitly, we show a series of results exploring the connection between the coefficients of the monomials and the patterns of the associated constellation points, which substantially simplifies the objective function with respect to the problem Hamiltonian of the QAOA. In particular, for an M-ary Gray-mapped quadrature amplitude modulation (MQAM) constellation, we show that the specific qubits encoding the in-phase components and those encoding the quadrature components are independent in the quantum system of interest, which allows the in-phase and quadrature components to be detected separately using the QAOA. Furthermore, we characterize the degree of the objective function in the WMIN-$N$-SAT problem corresponding to the ML detection of multiple-input and multiple-output (MIMO) channels. Finally, we evaluate the approximation ratio of the QAOA for the ML detection problem of quadrature phase shift keying (QPSK) relying on QAOA circuits of different depths.

Index Terms

ML detection, QAOA, WMIN-$N$-SAT.

I. INTRODUCTION

Signal detection determines the transmitted signals using mathematical tools, which is an essential component of various communication systems. Maximum likelihood (ML) detection is capable of finding
the optimal solution by a brute-force search over all possible transmitted signal vectors. Hence, its complexity increases according to $M^N$ [1], where $M$ is the modulation order and $N$ is the number of the transmitted antennas/users. Hence, the ML detector has to solve a combinatorial optimization problem, which is proved to be NP-hard [2], [3]. The promise of powerful parallel quantum computing has attracted substantial attention in diverse fields. In the current era of noisy intermediate-scale quantum (NISQ) computers, it is of salient significance to deal with their limited coherence time [4]. To circumvent these practical impediments of NISQ devices, hybrid quantum-classical algorithms that can be readily implemented and tested in NISQ devices have become prominent for demonstrating the much sought-after quantum advantage. Inspired by this ambitious goal, we solve the ML detection problem with general constellations by the quantum approximate optimization algorithm (QAOA).

The QAOA is a variational method designed by Farhi et al. [5] in 2014 to find an approximation of the ground state of the problem Hamiltonian associated with combinatorial optimization problems. More explicitly, the QAOA creates a parameterized quantum circuit by alternately applying the problem Hamiltonian representing the original optimization problem as well as the initial Hamiltonian and it then optimizes the parameters by a classical computer, while conducting the associated function evaluations on a quantum computer [6]–[8]. The QAOA has attracted significant attention both from industry and academia in the context of different hardware platforms based on such as superconducting qubits [9]–[12], trapped-ion qubits [13] and photonics [14], where the problem size was reported to be up to 23 qubits [9]. However, the performance of the QAOA critically depends both on the problem instances [11], [15] as well as on the circuit depth [5], [9], [16]. The lower bounds of the approximation ratio achieved by the QAOA are investigated in [5], [6], [17], [18], revealing that the approximation ratio achieved by the QAOA for a circuit of depth 3 is at least 0.7924 for all the three-regular graphs investigated [18]. Moreover, some analytical and numerical results on the limitations of the QAOA in some problem instances are investigated in [19]–[22]. To overcome the limitations both in terms of the associated state preparation and parameter optimization, the quantum symmetry properties are exploited in [20], [23], [24] for improving the QAOA. As a further advance, the authors of [16] evaluated the QAOA relying on high-depth circuits with the objective of outperforming some classical algorithms in finding the Max-Cut both on a large-girth graph and on the Sherrington-Kirkpatrick Model of [25]. This shed further light on the promising potential of the QAOA.

Against this rich background, we demonstrate the benefits of the QAOA in solving the ML detection problem of high-order modulation schemes, which is a generalization of our previous work on configuring the QAOA for solving the ML detection of BPSK [26]. In contrast to the binary symbols which can be mapped to the qubits directly, the constellation of higher order quadrature phase shift keying (QPSK) and
M-ary quadrature amplitude modulation (MQAM) has multiple constellation points with each representing a complex symbol that might be transmitted by the transmitter. Recall that the basic requirement of using the QAOA is to map the classical optimization problem to the quantum Hamiltonian operator acting on qubits, whose ground state encodes the optimal solution of the classical optimization problem. Explicit Hamiltonian constructions for a bunch of problem instances on a graph are provided in [27]. Then the author of [28] conceived the general representation of the Hamiltonian for handling both pseudo Boolean functions as well as constrained optimization problems. As the optimization of the ML detection problem depends on a set of complex variables, a suitable Hamiltonian construction method was introduced in [29] by treating the real and imaginary components of the complex symbols separately, which requires the design of a bespoke transformation. Given this Hamiltonian representation method, the performance of the uplink ML detection problem in massive multiple-input and multiple-output (MIMO) systems was investigated by harnessing both quantum annealing [29], [30] and the coherent Ising machine of [31].

In contrast to the Hamiltonian representation used in [29]–[31], in this paper we conceive the technique of constructing the Hamiltonian representation of the ML detection problem from the perspective of the Boolean satisfiability (SAT) problem. More specifically, the SAT problem deals with an assignment satisfying a particular ensemble of Boolean expressions, which is NP-complete [32]. The SAT problem can be modelled by pseudo Boolean functions in polynomial representation [33], which helps us to construct the Hamiltonian acting on qubits. Explicitly, it has been widely exploited [27], [28] that the quadratic unconstrained binary optimization (QUBO) relying on a quadratic formalism can be readily transformed to the task of finding the ground state of a quadratic Hamiltonian with respect to a two-body quantum system, which can then be solved by the QAOA [23], [34]. Furthermore, many quantum algorithms have been designed based on the quadratic Hamiltonian such as the variational quantum eigensolver (VQE) [35] and the quantum annealing [36], both of which also compute the eigenstates of the Hamiltonian. Therefore, the efficient representation of the problem Hamiltonian plays a key role in the implementation of these quantum algorithms designed for computing the ground state. However, the Hamiltonian representation of a cubic or even higher-degree pseudo Boolean function contains either three-body or even many-body interactions in the quantum system, which has to be reduced to a quadratic form for using the QAOA.

To expound a little further, the high-degree pseudo Boolean function constitutes a high-degree multilinear polynomial. The corresponding process of reducing the high-degree function into a quadratic form is referred to in parlance as ‘quadratization’ [37], [38]. There are several techniques of reducing the corresponding high-degree polynomials into quadratic forms are summarized in [38]. More explicitly, the techniques of reducing the degree of the corresponding function may rely on auxiliary variables [37], [39]–
but it may also be arranged without introducing auxiliary variables [43]–[46]. A typical technique requiring auxiliary variables is the method of reduction by substitution proposed in [47], which was further improved by numerous researchers [37], [39], [42], [46], [48]. By contrast, for reduction without introducing auxiliary variables, the authors of [44] proposed a technique of iteratively decomposing the high-degree function into a set of quadratic subfunctions corresponding to the leaves of a binary tree. However, this method requires post-processing after implementing the Hamiltonians of the subfunctions for attaining the ground state of the Hamiltonian of the original function. Since there are several efficient quadratization methods for both cubic and quaternary functions [39], [41], [42], a hybrid quadratization framework combining the split-reduction method of [43] and the techniques associated with auxiliary variables was also introduced in [44], which seeks to deal with the limited number of qubits available in NISQ devices. The rich variety of methods introduced for reducing the degree of functions provides an abundance of options for quadratization. Hence, for decomposing the high-degree function representing the ML-detection problem, we can opt for one of these strategies to perform the reduction.

Based on the above discussion, we focus our attention on characterizing the Hamiltonian concerning the constellation diagram as well as the implementation of the QAOA for solving the ML detection of general modulation constellations. Our main contributions are summarized as follows:

1) We provide a new procedure for the Hamiltonian construction of the ML detection problem of general constellations by transforming it into a weighted minimum $N$-satisfiability ($\text{WMIN-}N\text{-SAT}$) problem. Specifically, the objective of our WMIN-$N$-SAT problem can be represented as a pseudo Boolean function, which results in the unconstrained optimization problem of minimizing the pseudo Boolean function of interest.

2) We formalize the connection between the problem Hamiltonian of the ML detection and the Gray-labelled modulation constellations. Explicitly, due to having only a single bit-change between adjacent constellation points in the Gray labelling, we exploit for QPSK that the sum of the squared Euclidean distances from the received signal to the two pairs of diagonal points are the same. By harnessing this property, we demonstrate that the degree of the objective function can be beneficially reduced, hence simplifying the implementation of the Hamiltonian.

3) We also exploit that there are no interactions between the qubits encoding the in-phase and the quadrature bits in a Gray-labelled MQAM constellation, which indicates that the degree of the

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1 A WMIN-$N$-SAT problem can be treated as an extension of the minimum satisfiability (MIN-SAT) problem [49] by attaching a non-negative weight to each clause, where the goal is to minimize the sum of weights of the satisfied clauses by checking all possible assignments [50], [51].
Hamiltonian associated with the MQAM is equal to the number of in-phase bits. We further extend the results provided for a single-input constellation to MIMO channels associated with a joint constellation.

4) We compare the architecture of communication systems relying on the QAOA assisted ML detector (QML) to that of the classical ML detector (CML). Explicitly, the communication system associated with the QML does not need the conventional signal demapping procedure. Finally, we provide the simulation results for quantifying the approximation ratio of the QML for QPSK.

The rest of the paper is organized as follows. In Section II we commence with a brief recap of the ML detection problem. In Section III we focus on the Hamiltonian construction of single-input and single-output (SISO) systems. More explicitly, we transform the ML detection problem of a general constellation to the WMIN-\(N\)-SAT problem in Section III-A, followed by characterizing the Hamiltonian construction in Section III-B, where the explicit Hamiltonians of different constellations and our conclusions on the Gray-labelled MQAM constellations are given. In Section IV we introduce the Hamiltonian construction of the MIMO system. Section V illustrates the QAOA assisted ML detection, followed by the simulation results of the QAOA implementation of QPSK in Section VI. Section VII concludes the paper.

II. System Model

We consider a \(N_r \times N_t\) MIMO communication system having \(N_t\) transmit antennas (TAs) and \(N_r\) receive antennas (RAs). The received vector \(y\) is given by

\[
y = Hs + \eta,
\]

where we have \(y \in \mathbb{C}^{N_r}\), \(H \in \mathbb{C}^{N_r \times N_t}\) is the channel matrix, \(s \in \mathbb{C}^{N_t}\) is the vector of transmit signals, and \(\eta\) is the complex-valued zero-mean noise vector with \(\eta \sim \mathcal{CN}(0, I_{N_r})\). Here, the receiver has perfect knowledge of the random channel \(H\). We consider a general constellation, where \(s = [s_0, \ldots, s_{N_t-1}]\) is independently selected from a finite set of complex numbers \(\mathcal{A}\) according to the data to be transmitted. Hence, \(s \in \mathcal{A}^{N_t}\). Finally, we define \(M = |\mathcal{A}|\) as the number of points in \(\mathcal{A}\), where \(|(\cdot)|\) denotes the cardinality of a set \((\cdot)\). The ML detector can thus be interpreted as finding the closest lattice point in an \(N_t\)-dimensional complex space, which can be expressed as

\[
\hat{s} = \min_{s \in \mathcal{A}^{N_t}} \|y - Hs\|^2.
\]

III. SISO Channel

For a SISO channel, i.e. \(N_t = N_r = 1\), the ML solution for the constellation \(\mathcal{A}\) is given by

\[
\hat{s} = \arg\min_{s \in \mathcal{A}} |y - hs|^2.
\]
Due to the quantum parallelism, quantum computers are capable of evaluating all possible solutions at the same time. Furthermore, quantum computers operate in Hilbert space relying on binary variables. As a result, for solving the ML detection problem on the quantum computers, the solutions to the classical ML detection problem have to be encoded into bit strings. Given a constellation $A$ of $M$ points ($M$ possible solutions associated with the ML detection problem), $N = \log_2(M)$ bits are required for representing each point as an unique bit string. Let $z_0, \ldots, z_{N-1} \in \{0, 1\}^N$ represent the binary variables associated with each possible solution of the ML detection problem. We see that there are $2^N = M$ possible solutions in total with respect to the binary variables $z_0, \ldots, z_{N-1}$.

### A. Problem reformulation

Next, we introduce the mapping rule transforming the constellation points to the binary variables, which allows us to transform the ML detection problem into a WMIN-$N$-SAT problem defined in [50], [51], where a problem instance of WMIN-$N$-SAT is a collection of clauses, with each clause being a Boolean function of $N$ binary variables $z_0, \ldots, z_{N-1}$.

Let $a_i$ denote the $i$-th point ($i$ is a decimal number) in the constellation of $A$ with $i \in \mathcal{M} = \{0, \ldots, M-1\}$, where each $i$ has a $N$-bit binary representation of $[i]_2 = b_0 \cdots b_{N-1}$. The square of the Euclidean distance between the received signal $y$ and $a_i$ is given by

$$d_i = |y - ha_i|^2,$$

where $d_i \geq 0$. Correspondingly, the goal of the ML detection problem is to find the point $a_i$ having the minimum $d_i$, for $i \in \mathcal{M}$.

Here, we define a weighted clause as $w(C_i)$ associated with the non-negative weight $d_i$ obtained from (4), where $C_i$ is a clause associated with the $N$ binary variables $z_0, \ldots, z_{N-1}$. Explicitly, the clause $C_i$ is defined as follows:

$$C_i = B(z_0) \land B(z_1) \land \cdots \land B(z_{N-1}),$$

with $B(z_n)$ being

$$B(z_n) = \begin{cases} 
\neg z_n, & \text{if } b_n = 0, \\
 z_n, & \text{if } b_n = 1,
\end{cases}$$

where $b_n$ is the $n$-th bit of the binary number $b_0 \cdots b_{N-1}$ associated with the decimal number $i$. Here $\land$ and $\neg$ denote the operators ‘And’ and ‘Not’, respectively. We portray the SAT mapping of both QPSK and 8QAM in Fig. 1 for illustrating the connection between the clauses and the constellation points. Fig. 1a shows an example of mapping the QPSK symbols into clauses of a WMIN-2-SAT problem.
The symbols on the QPSK constellation diagram are depicted at the left-hand side of Fig. 1a, where the symbol labels follow a Gray mapping order. The square of the Euclidean distance \( d_i, i = 0, \cdots, 3 \), can be computed by (4). Furthermore, the SAT mapping process from the ordering of the points to the clause is demonstrated at the right-hand side of Fig. 1a. According to (5), we construct four clauses as shown in Fig. 1a, where each clause relies on both of the two binary variables. Furthermore, we also demonstrate the mapping process for a Gray-labelled 8QAM constellation in Fig. 1b, which encodes the decimal index of the constellation points into clauses having three variables. Therefore, the ML detection of the 8QAM constellation can be encoded into a WMIN-3-SAT problem containing 8 clauses in total.

As a result, given an assignment of the binary variables \( z_0, \cdots, z_{N-1} \), the weighted clause \( w(C_i) \) representing a clause \( C_i \) with a weight \( d_i, i \in \mathcal{M} \), can be expressed as

\[
w(C_i) = d_i C_i = d_i [B(z_0) \land B(z_1) \land \cdots \land B(z_{N-1})].
\]

Additionally, the Boolean formula \( C_i(z_0, \cdots, z_{N-1}) \) can be rewritten as a function of binary variables, since \( B(z_i) \land B(z_j) \) and \( \neg z_i \) can be equivalently expressed as \( B(z_i) \cdot B(z_j) \) and \( (1 - z_i) \), respectively. The objective function of the WMIN-N-SAT at hand can thus be formulated as a Boolean pseudo function given by

\[
f(z_0, \cdots, z_{N-1}) = \sum_{i=0}^{M-1} w(C_i) = \sum_{\mathcal{S} \subseteq \mathcal{N}} \tilde{d}_\mathcal{S} \prod_{n \in \mathcal{S}} z_n,
\]

where \( \mathcal{N} = \{0, \cdots, N-1\} \) is the set of all variable indices and \( \mathcal{S} \) is a subset of \( \mathcal{N} \) such that \( \mathcal{S} \subseteq \mathcal{N} \). Furthermore, \( \tilde{d}_\mathcal{S} \) represents the coefficient of the term \( \prod_{n \in \mathcal{S}} z_n \) in the expansion of \( f(z_0, \cdots, z_{N-1}) \). Note that the objective function \( f(z_0, \cdots, z_{N-1}) \) is a multilinear polynomial of degree \( N \), where the degree of \( f(z_0, \cdots, z_{N-1}) \) is the maximum number of distinct variables occurring in any of its monomials [52]. Correspondingly, the degree of a monomial associated with \( \mathcal{S} \) can be denoted as \( |\mathcal{S}| \) representing the number of elements in \( \mathcal{S} \). Note that we omit the constant term in (8), which does not affect the optimal solution of the ML detection. For notational simplicity, the constant terms in the function \( f(z_0, \cdots, z_{N-1}) \) and in the associated Hamiltonian function will be omitted throughout the paper. Therefore, the ML detection problem of (3) can now be equivalently transformed into the WMIN-N-SAT problem as follows:

\[
\min_{z_0, \cdots, z_{N-1} \in \{0,1\}^N} f(z_0, \cdots, z_{N-1})
\]

Following the rules of the polynomial expansion applied to (8), we formulate a pair of remarks from the expansion of \( f(z_0, \cdots, z_{N-1}) \), which will help us to discover some interesting properties of \( f(z_0, \cdots, z_{N-1}) \) concerning different constellations.

**Remark 1.** The expansion of \( C_i \) contains the monomial of the form \( \prod_{n \in \mathcal{S}} z_n, \mathcal{S} \subseteq \mathcal{N} \), if and only if the bit string \( b_0 \cdots b_{N-1} \) such that \( b_l = 0 \), for all \( l \in \mathcal{N} \setminus \mathcal{S} = \{l : l \in \mathcal{N} \text{ and } l \notin \mathcal{S}\} \).
(a) QPSK: A weighted 2-SAT

(b) 8QAM: A weighted 3-SAT

Figure 1: Illustrations of mapping the ML detection problem into SAT for QPSK and 8QAM.

Note that Remark 1 can be equivalently interpreted in a more neat manner as follows. The expansion of $C_i$ contains the monomial of the form $\prod_{n \in S} z_n$, $S \subseteq N$, if and only if $i \in M'$, where

$$M' = \{i : [i]_2 = b_0 \cdots b_{N-1}, \text{ s.t. } b_l = 0, \forall l \in N \setminus S\}. \quad (10)$$

**Remark 2.** There are $2^{|S|}$ clauses containing the monomial of the form $\prod_{n \in S} z_n$, $S \subseteq N$.

**Remark 3.** The coefficient $\tilde{d}_S$ can be expressed as follows:

$$\tilde{d}_S = \sum_{i \in S'} d_i \prod_{n \in S} (-1)^{1-b_n}, \quad (11)$$

with $S' = \{i : [i]_2 = b_0 \cdots b_{N-1}, \text{ s.t. } b_n = 0, \forall n \in N \setminus S\}$ and $|S'| = 2^{|S|}$.

**B. Constructing Hamiltonians**

In the QAOA, the objective function is encoded into a problem Hamiltonian containing a sequence of Pauli Z operators [5], [28], diagonally acting on qubits corresponding to the computational basis vectors. From (7), we can construct the Hamiltonian operator associated with clause $C_i$ as follows:

$$H_{f,C_i}|z_0 \cdots z_{N-1}\rangle = w(C_i)|z_0 \cdots z_{N-1}\rangle, \quad (12)$$

where we see that $w(C_i)$ corresponds to the eigenvalues of $H_{f,C_i}$, associated with the eigenstates $|z_0 \cdots z_{N-1}\rangle$.

Given the expression of binary variables to the clause $C_i(z_0, \cdots, z_{N-1})$, the Hamiltonian $H_{f,C_i}$ can be readily obtained by replacing $z_i$ with $\frac{1-\sigma^{(i)}_2}{2}$. Therefore, the general form for the problem Hamiltonian of $f(z_0, \cdots, z_{N-1})$ in (8) can be expressed as

$$H_f = \sum_{S \in N} g_S \prod_{n \in S} \sigma_z^{(n)}, \quad (13)$$

where $g_S$ is the coefficient of the term $\prod_{n \in S} \sigma_z^{(n)}$, which describes the interactions between the qubits in the quantum system. Given an arbitrary system state $|\psi\rangle$, the problem Hamiltonian $H_f$ obeys $\langle \psi | H_f | \psi \rangle \geq 0$, $H_f | \psi \rangle = f_{\min} | \psi \rangle$ if and only if $|\psi\rangle$ is the ground state of $H_f$ [53], which corresponds to the assignment
that only satisfies the associated constellation point of having the minimum Euclidean distance. The goal of the QAOA is thereby to seek the ground state of $H_f$.

Furthermore, a common choice of the initial Hamiltonian $H_B$ and the initial state $\psi(0)$ used in the QAOA is formulated as follows:

$$H_B = \sum_{i=0}^{N-1} \sigma_x^{(i)},$$

and

$$|\psi(0)\rangle = \frac{1}{2^N} \sum_{z_0} \cdots \sum_{z_{N-1}} |z_0\rangle \cdots |z_{N-1}\rangle,$$

where $\sigma_x^{(i)}$ is the Pauli X operator acting on the $i$-th qubit and $|\psi(0)\rangle$ is a uniform superposition over the computational basis.

1) Problem Hamiltonian for QPSK: We see from Fig. 1a that the QAOA harnessed the ML detection of QPSK requires 2 bits for encoding the 4 symbols of the QPSK constellation, where the Boolean formula for each clause is also given. Based on (5) and (7), we rewrite the weighted clause $w(C_i)$ as a function of binary variables as follows:

$$w(C_0) : d_0(1 - z_0)(1 - z_1), \ w(C_1) : d_1(1 - z_0)z_1, \ w(C_2) : d_2z_0(1 - z_1), \ w(C_3) : d_3z_0z_1. $$

Here each $w(C_i)$ is a quadratic function, which involves a product term $z_0z_1$. The objective function $f_{QPSK}(z_0, z_1)$ is thus given by

$$f_{QPSK}(z_0, z_1) = \bar{d}_0z_0z_1 + \bar{d}_1z_1,$$

where $\bar{d}_{0,1} = d_0 - d_1 - d_2 + d_3$, $\bar{d}_0 = d_2 - d_0$ and $\bar{d}_1 = d_1 - d_0$.

**Proposition 1.** Following the Gray mapping order, the coefficient $\bar{d}_{0,1}$ of the quadratic term is 0, and thus $f(z_0, z_1)$ can be simplified as

$$f_{QPSK}(z_0, z_1) = \bar{d}_0z_0 + \bar{d}_1z_1.$$ 

**Proof.** See Appendix A.

Correspondingly, the problem Hamiltonian $H_{f_{QPSK}}$ can be constructed immediately upon replacing $z_i$ by $\frac{1}{2}(1 - \sigma_z^{(i)})$, $i = 0, 1$, which can be expressed as

$$H_{f_{QPSK}} = -\bar{d}_0\sigma_z^{(0)} - \bar{d}_1\sigma_z^{(1)}.$$

Here we have omitted the common constant coefficient $\frac{1}{2}$ and the constant terms, since they do not affect the ground state of $H_{f_{QPSK}}$. Therefore, we will also omit the common constant coefficient in the Hamiltonian function throughout the paper. We find that there are no quadratic terms containing $\sigma_z^{(0)}\sigma_z^{(1)}$.
in $H_{QPSK}$, which indicates that there are no interactions between the two qubits, as shown in Fig. 2a. Therefore, for implementing the ML detection of QPSK we can use a single-qubit quantum device twice or a pair of single-qubit quantum devices in a parallel manner. As a further advance, we provide another proposition for further extending the property to a rectangle.

**Proposition 2.** If four points of the constellation diagram form a rectangle, then the sum of the squared distance of the received signal from the two pairs of diagonal points are the same.

The proof of **Proposition 1** was formulated for a square, which is shown in **Remark A.1**. We can immediately extend this property to a rectangle in **Proposition 2** following the methods used for proving **Proposition 1**.

**Remark 4.** If using the binary mapping order, the points 0 and 3 are neighbours instead of being diagonal points. Therefore, the quadratic term $z_0 z_1$ in $f_{QPSK}(z_0, z_1)$ cannot be cancelled.

**Theorem 1.** For a Gray-labelled constellation such as a rectangular QAM or a $M$-ary phase-shift keying (MPSK), with $M = 2^N$ and $N \in \mathbb{Z}^+$, the coefficient of the monomial having the highest degree $N$ i.e. the monomial of the form $\prod_{n=0}^{N-1} z_n$, in $f(z_0, \cdots, z_{N-1})$ is always 0.

**Proof.** See Appendix B.

We can immediately infer from **Theorem 1** that the coefficient of the term $z_0 z_1$ in $f_{QPSK}$ of (17) is zero. Fig. 3a illustrates the constellation points associated with the monomials of the form $z_0$, $z_1$ and $z_0 z_1$, respectively. It can be readily seen that the constellation points associated with the monomial of the form $z_0 z_1$ create a rectangle and the pair of the diagonal points (0, 3) has even numbers of 0 in their binary representations, while the binary orders for the other pair of the diagonal points (1, 2) have odd numbers of 0. In communication systems, QAM is widely adopted for data modulation [54], where we will investigate the connection between the degree of the objective function formulated and the associated constellation diagram. Fig. 4a portrays various MQAM constellation diagrams for $M = 16, 32, 64, 128$ and 256, which can be labelled using Gray mapping order. Furthermore, Fig. 4b shows the cross constellations for 32QAM and 128QAM, which cannot be labelled following the Gray mapping rule [55]. The methods of Gray mapping for rectangular QAM constellations and quasi-Gray mapping for cross QAM constellations can be found in [56], [57].

**Corollary 1.** If a constellation is not based on Gray mapping, the monomial of the form $\prod_{n \in N} z_n$ in $f(z_0, \cdots, z_{N-1})$ has a coefficient of zero, if there exists a bunch of rectangles with the constellation points as their vertices, such that the following properties:

1) The vertices of these rectangles cover all points in the constellation;
2) There are no shared vertices between any two rectangles; and
Figure 2: An exemplary illustration of the interactions between the qubits for different modulation schemes, where ↑ and ↓ represent the two possibilities of a bit value in classical systems. In contrast, a single qubit can be represented as a superposition of classical states: $|z_k\rangle = \alpha_k|\uparrow\rangle + \beta_k|\downarrow\rangle$, $\alpha_k, \beta_k \in \mathbb{C}$.

Figure 3: Patterns for the possible monomials in $f(z_0, \cdots, z_{N-1})$ in terms of QPSK, 8QAM and 16QAM using Gray mapping, where the constellation points associated with a monomial of same variables are grouped together. For clarity, Fig. 3c shows the possible monomials of degree 1 and 2, while Fig. 3d shows the possible monomials of degree 3 and 4. Note that the coefficient of a monomial is zero if the set of its associated constellation points is surrounded by dashed lines, and the set of the constellation points associated with the monomials having non-zero coefficients are surrounded by solid lines.

Figure 4: Various constellation diagrams for MQAM.

3) For each rectangle the orders associated with the two pairs of the diagonal points contain odd(even) and even(odd) numbers of 0s, respectively.
**Corollary 2.** Given a constellation with any reasonable mapping order, the monomial of the form $\prod_{n\in S} z_n$ in $f(z_0, \cdots, z_{N-1})$ has a coefficient of zero, if the constellation points of order $i$, for all $i \in M'$, form a bunch of rectangles such that the properties in 2) and 3) of **Corollary 1.**

The proofs of **Corollary 1** and **Corollary 2** are provided in Appendix C. Note that **Corollary 2** shows the zero-coefficient conditions for a monomial of a constellation with general labelling orders.

2) **Problem Hamiltonian for 8QAM Constellation:** As shown in Fig. 1b, there are eight clauses in terms of the eight constellation points of the 8QAM constellation diagram. By combining these clauses into a function of binary variables $z_0, z_1$ and $z_2$, we have the objective function of

$$f_{8QAM}(z_0, z_1, z_2) = d_0(1 - z_0)(1 - z_1)(1 - z_2) + d_1(1 - z_0)(1 - z_1)z_2 + d_2(1 - z_0)z_1(1 - z_2) + d_3(1 - z_0)z_1z_2 + d_4z_0(1 - z_1)(1 - z_2) + d_5z_0(1 - z_1)z_2 + d_6z_0z_1(1 - z_2) + d_7z_0z_1z_2$$

(20)

where $d_{0,1,2} = -d_0 + d_1 - d_2 - d_3 + d_4 - d_5 - d_6 + d_7$, $d_{0,1} = d_0 - d_2 - d_4 + d_6$, $d_{0,2} = d_0 - d_1 - d_4 + d_5$, $d_{1,2} = d_0 - d_1 - d_2 + d_3$, $d_2 = -d_0 + d_1$, $d_3 = -d_0 + d_2$ and $d_4 = -d_0 + d_4$. In Fig. 3b, we demonstrate the sets of constellation points associated with all possible monomials, i.e. for all $S \subseteq N$. Following the property in **Theorem 1** and **Corollary 2**, we have $d_{0,1,2} = 0$, $d_{0,2} = 0$, and $d_{1,2} = 0$. Hence $f_{8QAM}(z_0, z_1, z_2)$ is simplified as:

$$f_{8QAM}(z_0, z_1, z_2) = \bar{d}_{0,1}z_0z_1 + \bar{d}_0z_0 + \bar{d}_1z_1 + \bar{d}_2z_2,$$

(21)

which is a quadratic function involving a product item $z_0z_1$.

Correspondingly, the quantum Hamiltonian $H_f$ can be expressed as

$$H_{f_{8QAM}} = \bar{d}_{0,1}\sigma_z^{(0)}\sigma_z^{(1)} - (2\bar{d}_0 + \bar{d}_{0,1})\sigma_z^{(0)} - (2\bar{d}_1 + \bar{d}_{0,1})\sigma_z^{(1)} - 2\bar{d}_2\sigma_z^{(2)},$$

(22)

which indicates that implementing the ML detection of 8QAM requires 3 qubits, but there is only a single interaction between qubit 0 and qubit 1, as illustrated in Fig. 2(b). As a result, at least two-qubit quantum devices are needed for performing the ML detection of 8QAM.

3) **Problem Hamiltonian for 16QAM:** For the 16QAM constellation having 16 points, we need $N = \log_2(16) = 4$ qubits for encoding the solutions of the ML detection problem. By transforming the clause into a function of binary variables $z_0, z_1, z_2$ and $z_3$, we have the objective function

$$f_{16QAM}(z_0, z_1, z_2, z_3) = d_0(1 - z_0)(1 - z_1)(1 - z_2)(1 - z_3) + d_1(1 - z_0)(1 - z_1)(1 - z_2)z_3$$

$$+ \cdots + d_{15}z_0z_1z_2z_3$$

$$= \bar{d}_{0,1}z_0z_1 + \bar{d}_0z_0 + \bar{d}_1z_1 + \bar{d}_2z_2 + \bar{d}_3z_3,$$

(23)

$$f(z_0, z_1) \quad f(z_1, z_2)$$
where \( \tilde{d}_{0,1} = d_0 - d_4 - d_8 + d_{12} \), \( \tilde{d}_{2,3} = d_0 - d_1 - d_2 + d_3 \), \( \tilde{d}_0 = d_8 - d_0 \), \( \tilde{d}_1 = d_4 - d_0 \), \( \tilde{d}_2 = d_2 - d_0 \) and \( \tilde{d}_3 = d_1 - d_0 \). Given \( f_{16QAM}(z_0, z_1, z_2, z_3) \), we can now immediately write out \( H_{f_{16QAM}} \) explicitly. Here we focus on studying the connection between the patterns in the constellation diagram and the monomials in \( f_{16QAM}(z_0, z_1, z_2, z_3) \). The patterns of the constellation points associated with the monomials from degree 1 to degree 4 in \( f_{16QAM}(z_0, z_1, z_2, z_3) \) are illustrated in Fig. 3c and Fig. 3d. Observe from Fig. 3c that all of the monomials of degree 1 exist in the expansion of \( f_{16QAM}(z_0, z_1, z_2, z_3) \), while there are only two monomials of degree 2. Explicitly, in Fig. 3c the two monomials of the forms \( z_0 z_1 \) and \( z_2 z_3 \) correspond to the constellation points of the bottom row and the far left column in the constellation diagram, respectively. Furthermore, we see from Fig. 3d that all of the coefficients of the monomials with degree 3 and degree 4 are zero, since the set of their associated constellation points form rectangles. Correspondingly, the simplified expansion of \( f_{16QAM}(z_0, z_1, z_2, z_3) \) is obtained in (23). Moreover, upon comparing (23) to (20), we find that the expansion of \( f_{16QAM}(z_0, \cdots, z_3) \) can be viewed as the sum of \( f_{8QAM}(z_0, z_1) \) and \( f_{8QAM}(z_2, z_3) \). This indicates that in the four-qubit system associated with the ML detection problem of 16QAM, there are two pairs of spins in parallel, as illustrated in Fig. 2c, where interactions only happen within each pair. Therefore, it is possible to implement the ML detection of 16QAM in a two-qubit quantum device.

As described in [55], [57], for a rectangular QAM constellation, the bits associated with each point can be split into two groups, denoted as \( b_0 \cdots b_{\lceil \frac{N}{2} \rceil - 1} \) referred to as the in-phase bits and \( b_{\lceil \frac{N}{2} \rceil} \cdots b_{N-1} \) referred to as the quadrature phase bits, respectively. For the simplicity of notations, we define a pair of sets \( \mathcal{N}_I = \{0, \cdots, \lceil \frac{N}{2} \rceil - 1 \} \) and \( \mathcal{N}_Q = \{\lceil \frac{N}{2} \rceil, \cdots, N - 1 \} \) corresponding to the indices of the in-phase and quadrature phase bits, respectively. Hence, we have \( |\mathcal{N}_I| = |\mathcal{N}_Q| = |\mathcal{N}| = N - \lceil \frac{N}{2} \rceil \). The two groups of bits can then be arranged following Gray mapping along each axis and the final order is thus obtained by the combination of the in-phase and quadrature phase bits.

**Theorem 2.** Consider a rectangular MQAM constellation \( A \) following a Gray mapping order, where \( N = \log_2 M \) denotes the number of bits per symbol,

1) The monomials of the form \( \prod_{n \in S} z_n \) in \( f(z_0, \cdots, z_N) \) are zero if there exists at least one pair of bits denoted by \( b_l = 0 \) and \( b_m = 0 \) with \( l, m \in S \), such that \( b_l \) and \( b_m \) belong to different groups (including the group of in-phase bits \( \mathcal{N}_I \) and the group of quadrature bits \( \mathcal{N}_Q \)).

2) The degree of \( f(z_0, \cdots, z_N) \) is \( |\mathcal{N}_I| = \lceil \frac{N}{2} \rceil \).

**Proof.** See Appendix D.

**Remark 5.** Theorem 2 shows that the monomials of the form \( \prod_{n \in S} z_n \) in the expanded \( f(z_0, \cdots, z_{N-1}) \) will be cancelled if \( \exists (l, m) \in S \) such that \( b_l = b_m = 0 \) with \( l \in \mathcal{N}_I \) and \( m \in \mathcal{N}_Q \).
**Remark 6.** Theorem 2 indicates that no interactions happen between the qubits encoding the in-phase and quadrature bits in the quantum system associated with the ML detection problem of interest.

As shown in Fig. 2a-c, we can use a graph to illustrate the interactions among qubits based on the problem Hamiltonian constructed. Note that the graph is disconnected for a Gray-labelled MQAM constellation, since the qubits corresponding to $N_I$ and $N_Q$ are independent. In particular, the graph will contain isolated nodes if $N_I = 1$ or $N_Q = 1$, seen in the graphs generated from QPSK and 8QAM in Fig. 2.

**Remark 7.** Observe from Theorem 2 that the degrees of 64QAM, 256QAM and 1024QAM are 3, 4 and 5, respectively.

Let us now turn to the cross constellation of odd-bit QAM following pseudo-Gray mapping. Corollary 1 demonstrates that the degree of $f(z_0, \cdots, z_N)$ is no higher than $N - 1$, if the cross constellation obeys the properties in 1)-3) of Corollary 1. Furthermore, the pseudo-Gray mapping breaks the pattern of four points forming a rectangle, where a pair of points in each edge share the same in-phase bits or the quadrature phase bits. Therefore, the constellation points associated with a monomial do not always form rectangles. We have to check the coefficient of each monomial individually. For instance, for the cross 32QAM constellation following the quasi-Gray mapping orders of [56], [57], there are always some monomials of degree 4 which cannot be cancelled, this results in the objective function $f_{32QAM}(z_0, \cdots, z_4)$ of degree 4.

4) **Problem Hamiltonian of 64QAM:** We finally consider the objective function $f(z_0, \cdots, z_5)$ for 64QAM, the degree of which is 3 based on Theorem 2. Following Remark 6, we simplify $f(z_0, \cdots, z_5)$ as follows:

$$f_{64QAM}(z_0, \cdots, z_5) = f(z_0, z_1, z_2) + f(z_3, z_4, z_5),$$  \hspace{1cm} (24)

where

$$f(z_l, z_m, z_n) = \vec{d}_{l,m,n} z_l z_m z_n + \bar{d}_{l,m} z_l z_m + \bar{d}_{l,n} z_l z_n + \bar{d}_{m,n} z_m z_n + d_l z_l + \bar{d}_m z_m \bar{d}_n z_n,$$  \hspace{1cm} (25)

with $(l, m, n) = (0, 1, 2)$ and $(3, 4, 5)$, respectively. We can obtain the coefficients $\bar{d}_S$ from Remark 3, which are given by

$$\bar{d}_{0,1,2} = -d_0 + d_8 + d_{16} - d_{24} + d_{32} - d_{40} - d_{48} + d_{56}, \quad \bar{d}_{0,1} = d_0 - d_{16} - d_{32} + d_{48},$$
$$\bar{d}_{0,2} = d_0 - d_8 - d_{32} + d_{40}, \quad \bar{d}_{1,2} = d_0 - d_8 - d_{16} + d_{24}, \quad \bar{d}_0 = -d_0 + d_{32}, \quad \bar{d}_1 = -d_0 + d_{16},$$
$$\bar{d}_2 = -d_0 + d_8, \quad \bar{d}_{3,4,5} = -d_0 + d_1 + d_2 - d_3 + d_4 - d_5 - d_6 + d_7, \quad \bar{d}_{3,4} = d_0 - d_2 - d_4 + d_6,$$
$$\bar{d}_{3,5} = d_0 - d_1 - d_4 + d_5, \quad \bar{d}_{4,5} = d_0 - d_1 - d_2 + d_3, \quad \bar{d}_3 = -d_0 + d_4, \quad \bar{d}_4 = -d_0 + d_2, \quad \bar{d}_5 = -d_0 + d_1.$$  \hspace{1cm} (26)
Then the problem Hamiltonian $H_{f_{4qad}}$ can be obtained upon replacing $z_i$ by $\frac{1}{2}(1 - \sigma_z^{(i)})$ for all $i$. Correspondingly, the Hamiltonians of $f(z_0, z_1, z_2)$ and $f(z_3, z_4, z_5)$ are independent, which can thus be implemented using two independent quantum systems. Note that the degrees of $f(z_0, z_1, z_2)$ and $f(z_3, z_4, z_5)$ are three, which are associated with a 3-local Hamiltonian, respectively. We have to reduce the 3-local Hamiltonian into a two-local Hamiltonian, since the QAOA is typically limited to allowing no more than two-local interactions. This task can be carried out by following the reduction method introduced in Section I, such as the method of reduction by substitution [42] requiring auxiliary variables or the split-reduction method of [44] operating without adding auxiliary variables.

**IV. PROBLEM HAMILTONIAN CONSTRUCTIONS FOR MIMO CHANNELS**

Following the investigations in Section III, in this section we consider the problem Hamiltonian of the $N_r \times N_t$ MIMO channel. Firstly, we rewrite (1) as follows:

$$
\begin{bmatrix}
  y_0 \\
  \vdots \\
  y_{N_r-1}
\end{bmatrix}
= 
\begin{bmatrix}
  h_{0,0} & \cdots & h_{0,N_t-1} \\
  \vdots & \ddots & \vdots \\
  h_{N_r-1,0} & \cdots & h_{N_r-1,N_t-1}
\end{bmatrix}
\begin{bmatrix}
  s_0 \\
  \vdots \\
  s_{N_t-1}
\end{bmatrix}
+ 
\begin{bmatrix}
  \eta_0 \\
  \vdots \\
  \eta_{N_t-1}
\end{bmatrix}
= 
\begin{bmatrix}
  \sum_{k=0}^{N_t-1} h_{1,k}s_k + \eta_0 \\
  \vdots \\
  \sum_{k=0}^{N_t-1} h_{N_r-1,k}s_k + \eta_{N_r-1}
\end{bmatrix},
$$

(27)

which can be viewed as $N_r$ parallel SISO channels. The objective function of the ML detection problem in (2) can be rewritten as

$$
f_{ML}(s_0, \cdots, s_{N_t-1}) = \left\| \begin{bmatrix}
  y_0 \\
  \vdots \\
  y_{N_r-1}
\end{bmatrix} - \begin{bmatrix}
  \sum_{k=0}^{N_t-1} h_{1,k}s_k \\
  \vdots \\
  \sum_{k=0}^{N_t-1} h_{N_r-1,k}s_k
\end{bmatrix} \right\|_2^2 = \left\| \begin{bmatrix}
  y_1 - \sum_{k=0}^{N_t-1} h_{1,k}s_k \\
  \vdots \\
  y_{N_r-1} - \sum_{k=0}^{N_t-1} h_{N_r-1,k}s_k
\end{bmatrix} \right\|_2^2
$$

(28)

Let $\mathcal{A} = \mathcal{A}_0 \times \cdots \times \mathcal{A}_{N_t-1}$ denote the joint constellation, with ‘×’ being the Cartesian product, where a constellation $\mathcal{A}_k$, $k \in N_t = \{0, \cdots, N_t - 1\}$, contains $\mathcal{M}_k$ constellation points. Hence, $\mathcal{A}$ contains $M = \sum_{k \in N_t} \mathcal{M}_k$ constellation points in total, with the indices set $\mathcal{M} = \{0, \cdots, M - 1\}$. Therefore, the ML detection problem is to find a point from the joint constellation $\mathcal{A}$ which has the minimum sum of the distance from the received signal over each antenna. Numbering the points in $\mathcal{A}$ requires $N = \sum_{k \in N_t} N_k$ bits, i.e. the qubits required for representing the solutions, where $N_k = \log_2(\mathcal{M}_k)$ is the number of bits required for $\mathcal{A}_k$, $k \in N_t$. Furthermore, the orders of a point in $\mathcal{A}$ can be constructed by the Cartesian product of $N_t$ sets, in which the element is a $N_t$-tuple denoted as $(i_0, \cdots, i_{N_t-1})$ with $i_k$ being the decimal order for $\mathcal{A}_k$ and $i_k \in \mathcal{M}_k = \{0, \cdots, M_k - 1\}$. As a result, an element $i \in \mathcal{M}$ is defined as

$$
[i]_2 = [i_0]_2 + \cdots + [i_{N_t-1}]_2,
$$

(29)
where ‘+’ represents the concatenation operator which glues the binary orders together. Explicitly, the mapping rule between the $i$-th point in $\mathcal{A}$ and the binary variables $z_0, \cdots, z_{N-1}$ is given by

$$[i]_2 = b_0 \cdots b_{N_0-1} \cdots b_{N-1} \leftrightarrow [b_0, \cdots, b_{N_0-1}], \cdots, z_{N-1}. \quad (30)$$

In addition, for a constellation $\mathcal{A}_k, k \in \mathcal{N}_t$, we define $\mathcal{N}_{k,I} = \{\sum_{l=0}^{k-1} N_l, \cdots, \sum_{l=0}^{k-1} N_l + \lceil N_2 \rceil - 1\}$ and $\mathcal{N}_{k,Q} = \{\sum_{l=0}^{k-1} N_l + \lceil N_2 \rceil, \cdots, \sum_{l=0}^{k} N_l - 1\}$ to be the sets of the in-phase and quadrature phase bit indices, respectively. Hence we have $N_k = N_{k,I} + N_{k,Q}$ with $N_{k,I} = |\mathcal{N}_{k,I}|$ and $N_{k,Q} = |\mathcal{N}_{k,Q}|$.

For the $l$-th antenna, the received signal $y_l = \sum_{k \in \mathcal{N}_t} h_{l,k} s_k + \eta_l$ can be viewed as the output of a multiple access channel (MAC) having $N_t$ users. Similar to (4), the squared distance between $y_l$ and the $i$-th constellation point in $\mathcal{A}$, $i \in \mathcal{M}$, can be expressed as

$$d_{l,i} = |y_l - \sum_{k \in \mathcal{N}_t} h_{l,k} s_k|^2, \quad (31)$$

where the connection between $i$ and $k, k \in \mathcal{N}_t$ satisfies (29). Following the transformations of (5)-(6), the objective function of the ML detection of MIMO systems can be expressed as

$$f(z_0, \cdots, z_{N-1}) = \sum_{l=0}^{N_t-1} \sum_{i \in \mathcal{M}} w_l(C_i) = \sum_{l=0}^{N_t-1} \sum_{s \in \mathcal{N}} ds \prod_{n \in S} z_n = \sum_{l=0}^{N_t-1} f_l(z_0, \cdots, z_{N-1}), \quad (32)$$

where $w_l(C_i)$ and $f_l(z_0, \cdots, z_{N-1})$ represent the weighted function and the corresponding component of the objective function for the $l$-th received antenna, respectively. Similar to (7), $w_l(C_i)$ can be expressed as

$$w_l(C_i) = d_{l,i} C_i = d_{l,i}[B(z_0) \land \cdots \land B(z_{N-1})]. \quad (33)$$

Next we investigate the connection of the zero-coefficient characteristics of a monomial associated with a single constellation studied in Section III and the joint constellation of the MIMO system, which would allow us to simplify the problem Hamiltonian representation. Since it is straightforward to express the problem Hamiltonian from $f(z_0, \cdots, z_{N-1})$, we focus on simplifying $f(z_0, \cdots, z_{N-1})$.

**Proposition 3.** The monomial of the form $\prod_{n \in S} z_n$ in $f(z_0, \cdots, z_{N-1})$ of (32) has a coefficient of zero, if the constellation points in $\mathcal{A}_k$ contributing the component of the variables in $S \cap \mathcal{N}_k$ form a bunch of rectangles obeying the conditions in 2) and 3) of Corollary 2.

**Proof.** Following (32) stating that $f(z_0, \cdots, z_{N-1})$ is the sum of $f_l(z_0, \cdots, z_{N-1})$, we can infer that the zero-coefficient monomial in $f_l(z_0, \cdots, z_{N-1})$ has a coefficient of zero in $f(z_0, \cdots, z_{N-1})$. Therefore, we focus on the monomials in $f_l(z_0, \cdots, z_{N-1})$. The variables $z_n, n \in \mathcal{S}_k = S \cap \mathcal{N}_k$ come from the bits representing $\mathcal{A}_k$ and there are $2^{|S|} = \prod_{k \in \mathcal{N}_t} 2^{|\mathcal{S}_k|}$ constellation points contributing to the monomial of $S$. Fixing the orders of the constellation point in $\mathcal{A}_{k'}, k' \neq k$, there are $2^{|\mathcal{S}_k|}$ constellation points in $\mathcal{A}_k$.
containing the monomial of $S$. We then have that $d_{l,i}$ associated with the clause $C_{l,i}$ for the $l$-th receive antenna is $d_{l,i} = |\tilde{y}_i - h_{l,k}s_k|^2$, where $\tilde{y}_i = y_i - \sum_{k' \neq k} h_{l,k'}s_{k'}$. This indicates that if the constellation points in $A_k$ satisfy the zero-coefficient conditions, the sum of the coefficients for $i \in \mathcal{M}'$ with fixing $i_{k'}, k' \neq k$, is zero, where $\mathcal{M}'$ is given in (10). Furthermore, by traversing $2^{|S|-|S_k|}$ constellation points in $A_{k'}, k' \neq k$, the final coefficient of $\prod_{n \in S} z_n$ can be obtained, which is the sum of $2^{|S|-|S_k|}$ 0s. Hence, the zero-coefficient characteristics of the monomials on a single constellation are suitable for the monomials of a MIMO channel.

Remark 8. For a MIMO system of Gray-labelled constellations, the coefficients of the monomials containing the product of the variables both from the in-phase and from the quadrature phase bits are zero. Therefore, there are still no interactions between the qubits encoding the in-phase and quadrature bits.

Corollary 3. For a $N_r \times N_t$ MIMO system with Gray-labelled QPSK, the degree of the objective function $f(z_0, \cdots, z_N)$ is $N_t$.

Corollary 4. For a $N_r \times N_t$ MIMO system using Gray-labelled MQAM, the degree of the objective function $f(z_0, \cdots, z_N)$ is $N_1N_t$.

The proofs of Corollary 3 and Corollary 4 are given in Appendix E. Based on these two Corollaries, we provide the following theorem for characterizing the degree of the objective function concerning a MIMO system.

Theorem 3. For a $N_r \times N_t$ MIMO channel associated with QAM, the joint constellation is given by $A$, where each constellation follows Gray mapping. The degree of the objective function $f(z_0, \cdots, z_N)$ is $\sum_{k=0}^{N_r-1} N_k1$.

Proof: See Appendix F

Finally, we present an example of an $1 \times 2$ MIMO channel using QPSK for illustrating its problem Hamiltonian, since the number of receiver antennas does not affect the degree of the objective function. Hence, the ML detection problem is defined over the joint constellation $A_{\text{QPSK}}^2$, denoted as $2\times\text{QPSK}$, which requires 4 bits for representing each constellation point. From (32) the objective function of our ML detection problem for the $1 \times 2$ MIMO scenario can be expressed as

$$f_{2\times\text{QPSK}}(z_0, \cdots, z_3) = d_0(1 - z_0)(1 - z_1)(1 - z_2)(1 - z_3) + d_1(1 - z_0)(1 - z_1)(1 - z_2)z_3 + \cdots + d_{15}z_0z_1z_2z_3 = \sum_{(m,n) \in \mathcal{E}} \bar{d}_{m,n}z_mz_n + \sum_{n \in \mathcal{N}} \bar{d}_nz_n,$$

where $\mathcal{E} = \{(0,2), (0,3), (1,2), (1,3)\}$ and $\mathcal{N} = \{0,1,2,3\}$. Following Remark 3, the coefficients can be expressed as $\bar{d}_{0,2} = d_0 - d_2 - d_8 + d_10, \bar{d}_{0,3} = d_0 - d_1 - d_8 + d_0, \bar{d}_{1,2} = d_0 - d_2 - d_4 + d_6, \bar{d}_{1,3} = \frac{d_3}{2}$.
respectively. Furthermore, \( R_q \) the channel realizations. We use 6a and Fig. 6b, respectively. Here, \( \gamma \) is a parameter controlling the depth of the circuits. Moreover, \( q_I \) and \( q_Q \) represent the qubits encoding the in-phase and quadrature phase bits, respectively. For each level \( k \), \( k = 1, \cdots, p \), the parameterized unitary operators \( U(H_f, \gamma_k) \) and \( U(H_B, \beta_k) \) in terms of \( H_f \) and \( H_B \) can be represented as follows:

\[
U(H_f, \gamma_k) = \frac{e^{-i\gamma_k H_f, I} + e^{-i\gamma_k H_f, Q}}{U(H_f, \gamma_k)}
\]

\[
U(H_B, \beta_k) = \frac{e^{-i\beta_k H_B, I} + e^{-i\beta_k H_B, Q}}{U(H_B, \beta_k)}
\]

where \( H_{.I} \) and \( H_{.Q} \) represent the Hamiltonian operators associated with \( q_I \) and \( q_Q \), respectively. Here we discuss the implementation of the QAOA circuits of \( q_I \) as an example, since the QAOA circuits in terms of \( q_I \) and \( q_Q \) have the same structure. For each level \( k \), the unitary evolution of \( U(H_f, l, \gamma_k) \) involves a sequence of two-qubit unitary operators \( R_{zz}(2\gamma_k g_l, m) = e^{-i\gamma_k g_l, m \sigma_z^{(l)} \sigma_z^{(m)}} \) and a set of single-qubit unitary operators \( R_z(2\gamma_k g_l) = e^{-ig_k \gamma_k e_z^{(l)}} \), \( l, m \in \mathcal{N}_I, l \neq m \), which can be realized by the circuits shown in Fig. 6a and Fig. 6b, respectively. Here, \( g_l, m \) and \( g_l \) are the coefficients obtained in (13), which are related to the channel realizations. We use \( q_I \) and \( q_m \) to represent the quantum states of the \( l \)-th and \( m \)-th qubit, respectively. Furthermore, \( R_z(\theta) \) refers to the rotation-Z operator having parameter \( \theta \) with \( R_z(\theta) = e^{-i\theta \sigma_z} \).

More specifically, each unitary operator \( R_{zz} \) as seen in Fig. 6a, consists of two CNOT gates and a \( R_z \) gate. We then have the matrix representation of \( R_{zz}(2\gamma_k d_l, m) \), which is given by

\[
R_{zz}(2\gamma_k d_l, m) = CX_{l,m} R_z(2\gamma_k d_l, m) CX_{l,m} = e^{-i\gamma_k d_l, m \sigma_z^{(l)} \sigma_z^{(m)}} CX_{l,m}
\]

\[
= e^{-i\gamma_k d_l, m \sigma_z^{(l)} \sigma_z^{(m)}} = \text{diag}[e^{-i\gamma_k d_l, m}, e^{-i\gamma_k d_l, m}, e^{-i\gamma_k d_l, m}, e^{-i\gamma_k d_l, m}]
\]

where \( CX_{l,m} \) represents the CNOT gate, with \( q_I \) and \( q_m \) being the control and the target qubits, respectively. In addition, the unitary evolution of \( U(H_B, l, \beta_k) \) for level \( k \) is realized by a series of

\[
\begin{align}
q_0 - d_1 - d_4 + d_5, d_0 &= -d_0 + d_8, \bar{d}_1 &= -d_0 + d_4, \bar{d}_2 &= -d_0 + d_2, \bar{d}_3 &= -d_0 + d_1.
\end{align}
\]
Figure 5: Schematic diagram of the QAOA for Gray-labelled constellations of a SISO channel.

rotation-X operators with parameter \( \beta_k \), where the associated circuit unit is illustrated in Fig. 6c and \( R_x(\theta) = e^{i\theta} \).

Following the unitary evolution associated with the problem Hamiltonian \( H_f \) and the initial Hamiltonian \( H_B \), the parameterized quantum state \( |\psi(\gamma, \beta)\rangle \) in Fig. 5 can be expressed as

\[
|\psi(\gamma, \beta)\rangle = U(H_B, \beta_p)U(H_f, \gamma_p)\cdots U(H_B, \beta_1)U(H_f, \gamma_1)|\psi(0)\rangle,
\]

\[
= U(H_B, \beta_p)U(H_f, \gamma_p)\cdots U(H_B, \beta_1)U(H_f, \gamma_1)|\psi(0)\rangle + U(H_B, \beta_p)U(H_f, \gamma_p)\cdots U(H_B, \beta_1)U(H_f, \gamma_1)|\psi(0)\rangle.
\]

By measuring the state \( |\psi(\gamma, \beta)\rangle \) in the computational basis, we glean an output \( \langle z_0 \cdots z_{N-1} | \psi(\gamma, \beta) \rangle \) associated with a single candidate solution \( |z_0 \cdots z_{N-1}\rangle \). As illustrated in Fig. 5, repeating the preparation of the parameterized state \( |\psi(\gamma, \beta)\rangle \) and the measurements, the expectation value of the objective function to our ML detection can be expressed as

\[
F_p(\gamma, \beta) = \langle \psi(\gamma, \beta) | H_f | \psi(\gamma, \beta) \rangle = \bar{F}_0 + \bar{F}_1,
\]

where

\[
\bar{F}_0 = \bar{d}_0 \sin(2\beta_1) \sin(2\bar{d}_0 \gamma_1) \cos(2\bar{d}_1 \gamma_1) \text{ and } \bar{F}_1 = \bar{d}_1 \sin(2\beta_1) \sin(2\bar{d}_1 \gamma_1) \cos(2\bar{d}_1 \gamma_1)
\]

correspond to the terms of \( \sigma_z^{(0)} \) and \( \sigma_z^{(1)} \) in the \( H_f \), respectively.
B. Quantum Maximum Likelihood (QML) detection

Recall that in the QAOA, the solutions of the ML detection problem are encoded into the eigenstates and the optimal solution corresponds to the ground state of the problem Hamiltonian of the ML detection problem. For the problem Hamiltonian constructed in Section III and Section IV, we encode the candidate solutions (complex symbols represented in the constellation diagram) of the original ML detection problem into binary variables, in which the optimal binary solution to the ground state of the problem Hamiltonian is the specific transmit bit string corresponding to the optimal constellation point. As a result, the output of the QAOA is the estimated binary bit string sent by the transmitter, which is different from the output of the CML detector. Fig. 7 compares the CML and QML receivers. Explicitly, Fig. 7a shows the CML detector assisted uncoded communication system, where the estimated bit string is attained by performing signal demapping after CML detection. We see from Fig. 7b that the estimated bit string is attained directly from the QML detector. Furthermore, quantum mapping in Fig. 7b corresponds to the objective function constructed in (8) for SISO systems or in (32) for MIMO systems.

Finally, we introduce the approximation ratio metric [5], [8] for evaluating the quality of the solution provided by the QAOA. Since the ML detection problem is a minimization problem, we define the approximation ratio as follows:

$$\rho = \frac{f_{CML}}{f_p(\gamma^*, \beta^*)},$$

(41)

where $f_{CML}$ is the objective function value obtained by the classical ML detection method.

VI. SIMULATION RESULTS

In this section, we first visualize the expectation values $F_1$ for QPSK concerning a set of concrete noise and channel coefficients. Then, we quantify the performance of the QAOA assisted QPSK ML detection, where the QAOA is implemented using Qiskit Aer [58].
For the transmit symbols.

Landscapes of expression derived in (40), the landscape of instances are illustrated in terms of the AWGN channel and the Rayleigh channel. Following the analytical given the noise detection problem formulated for QPSK, where each \( \rho \) when solving the ML detection problem.

the channel coefficients as well as the transmit symbols, which complicates the evaluation of the QAOA we see that the factors affecting the positions of the global minima includes both the SNR, the noise and we plot the landscape with different transmit symbols. By comparing the landscapes plotted in Fig. 8, the landscape under an uncorrelated Rayleigh channel with different SNRs. Moreover, in Fig. 8e-Fig. 8g the landscape of \( F \) the minima of \( \gamma \) of the noise and the channel coefficients are generated using a seed value 100 for reproducing the results. In the landscape plots, the smaller (darker) is better. Specifically, the darkest spots indicate the global minima of \( F(\gamma_1, \beta_1) \) that we are seeking to find using the classical optimizer. Fig. 8a and Fig. 8b plots the landscape of \( F(\gamma_1, \beta_1) \) for different SNRs under an AWGN channel, while Fig. 8c and Fig. 8d plot the landscape under an uncorrelated Rayleigh channel with different SNRs. Moreover, in Fig. 8e-Fig. 8g we plot the landscape with different transmit symbols. By comparing the landscapes plotted in Fig. 8, we see that the factors affecting the positions of the global minima includes both the SNR, the noise and the channel coefficients as well as the transmit symbols, which complicates the evaluation of the QAOA when solving the ML detection problem.

Fig. 8 illustrates the landscape of the expectation values \( F(\gamma_1, \beta_1) \) for QPSK, where multiple problem instances are illustrated in terms of the AWGN channel and the Rayleigh channel. Following the analytical expression derived in (40), the landscape of \( F(\gamma_1, \beta_1) \) can be plotted using Mathematica, where the values of the noise and the channel coefficients are generated using a seed value 100 for reproducing the results. In the landscape plots, the smaller (darker) is better. Specifically, the darkest spots indicate the global minima of \( F(\gamma_1, \beta_1) \) that we are seeking to find using the classical optimizer. Fig. 8a and Fig. 8b plots the landscape of \( F(\gamma_1, \beta_1) \) for different SNRs under an AWGN channel, while Fig. 8c and Fig. 8d plot the landscape under an uncorrelated Rayleigh channel with different SNRs. Moreover, in Fig. 8e-Fig. 8g we plot the landscape with different transmit symbols. By comparing the landscapes plotted in Fig. 8, we see that the factors affecting the positions of the global minima includes both the SNR, the noise and the channel coefficients as well as the transmit symbols, which complicates the evaluation of the QAOA when solving the ML detection problem.

Fig. 9 and Fig. 10 show the approximation ratio \( \rho \) achieved by the QAOA when solving the ML detection problem formulated for QPSK, where each \( \rho \) is attained over 100 different channel realizations, given the noise \( \eta \in CN(0,1) \) and the channel \( h \in CN(0,1) \). In the simulations, the expectation values of \( F_p(\gamma, \beta) \) are attained by measuring the output of the QAOA circuits generated by Qiskit Aer [58] and the parameters \( \gamma, \beta \) are updated using the classical optimizer–COYBLA [59] which is a derivative-free
optimization algorithm starting from a given initial point. For obtaining a high-quality solution of the QAOA, we set a complexity budget of 2000 runs for each problem instance with respect to a single channel realization in the simulations, where the COYBLA starts from different random initial points in each run. In Fig. 9, we plots the approximation ratio $\rho$ for different $p$ versus the number of runs at SNR=15dB under the AWGN channel and the Rayleigh channel, respectively. We observe that $\rho$ becomes better and converges to a stable value as the number of runs increases. Moreover, the maximum of $\rho$ achieved tends to one as $p$ grows. Observe in Fig. 9 that $\rho$ achieves 0.7892 for $p = 1$, which increases to 0.9973 for $p = 4$ in the AWGN channel, while $\rho$ reaches 0.7316 for $p = 1$ and tends to 0.9923 for $p = 4$ in the Rayleigh channel. Here high approximation ratio is achieved because the Hamiltonian formulated for QPSK in (19) only includes two spins without interactions, as shown in Fig. 2(a). Additionally, Fig.10 shows the variation of $\rho$ with different $p$ versus SNR in the AWGN channel and the Rayleigh channel, respectively. Specifically, each value of $\rho$ is also attained over 100 channel realizations with respect to 100 different problem instances and 2000 runs are performed. We observe that the impact of the SNR on $\rho$ is more obvious for $p = 1$ than for $p \geq 2$, which indicates that $\rho$ is sensitive to the specific problem instances of different SNRs for $p = 1$, while it becomes less sensitive when $p \geq 2$. 

Figure 9: Comparison of the approximation ratio versus the number of runs parameterized by $p$. 

Figure 10: Comparison of the approximation ratio versus SNR parameterized by $p$. 

VII. Conclusions

A new Hamiltonian construction procedure was conceived for the ML detection problem by using the SAT formulation, which allows us to simplify the problem Hamiltonian’s representation. In our formulation, encoding the ML detection problem of an $M$-dimensional constellation, each constellation point corresponding to a $N = \log_2(M)$ bit string needs at most $N$ qubits. In particular, for an MQAM constellation following a Gray mapping order, the number of qubits required for encoding the problem can be reduced to $\lceil N/2 \rceil$. Furthermore, we demonstrated the connection between the degree of the objective function with respect to the problem Hamiltonian and the Gray-labelled constellation diagram. In contrast to the classical ML detector designed for outputting the estimated complex symbol, the QAOA assisted ML detector directly delivers the estimated input bit string, without any signal demapping procedure required in the classical communication system. Our future work will extensively evaluate the results derived on the available QAOA platforms in NISQ devices and further quantify the performance of the QML detector in comparison to classical detectors.

Appendix A: Proof of Proposition 1

Here, we provide two methods of proving that $d_{0,1} = 0$. We first prove it using geometric properties. Observe from Fig. 1a that the constellation points of QPSK are arranged on a squared grid. For a received signal, its position will always fall inside, outside or on the square formed by the constellation points of Fig. A.1, where we denote the position of the received signal by $s$. If $s$ is on the square, it is straightforward to see that $d_0 + d_3 = d_1 + d_2$, since they are equal to the squared length of the diagonal line. Now we consider more general cases, where $s$ is either inside or outside the square. We add a pair of straight lines passing through $s$, while being parallel to the two edges of a right angle, respectively, represented by the dotted lines in Fig. A.1. The length of the line segments is denoted by $a, b, c$ and $d$, respectively, as shown in Fig. A.1. For $s$ inside the square in Fig. A.1a, we can see that $d_0 = a^2 + d^2$, $d_1 = a^2 + c^2$, $d_2 = b^2 + d^2$ and $d_3 = b^2 + c^2$ following the geometric properties. We see that $d_0 + d_3 = d_1 + d_2 = a^2 + b^2 + c^2 + d^2$. Thus, $\tilde{d}_{0,1} = 0$ is proved. For $s$ outside the square in Fig. A.1b, we observe that $d_0 = a^2 + d^2$, $d_1 = a^2 + c^2$, $d_2 = (a+b)^2 + d^2$ and $d_3 = (a+b)^2 + c^2$. Here we have $d_0 + d_3 = d_1 + d_2 = a^2 + (a+b)^2 + c^2 + d^2$. Hence $\tilde{d}_{0,1}$ is 0. Therefore, we conclude the above proved property of $d_0 + d_3 = d_1 + d_2$ in the following remark.

Remark A.1. Given a point and a square, the sum of the squared distances of the received signal from the two pairs of diagonal points are the same.

In the second method, we prove that $\tilde{d}_{0,1} = 0$ from the perspective of the communication system. We assume that the complex channel coefficient $h$ and the complex received signal $y$ are represented as
\(h = a + bj\) and \(y = c + dj\), respectively, where \(a, b, c\) and \(d\) are real numbers. For the constellation of Figure 1a, we have \(A = \{1 + j, -1 + j, 1 - j, -1 - j\}\). We thus have

\[
d_0 + d_3 = |y - h(1 + j)|^2 + |y - h(-1 - j)|^2
\]

\[= |c + dj - (a + bj)(1 + j)|^2 + |c + dj - (a + bj)(-1 - j)|^2 \tag{A.1}\]

\[= 2(2a^2 + 2b^2 + c^2 + d^2),\]

and

\[
d_1 + d_2 = |y - h(-1 + j)|^2 + |y - h(1 - j)|^2
\]

\[= |c + dj - (a + bj)(-1 + j)|^2 + |c + dj - (a + bj)(1 - j)|^2 \tag{A.2}\]

\[= 2(2a^2 + 2b^2 + c^2 + d^2) = d_0 + d_3.\]

Hence \(d_{0,1} = 0\) is thus proved.

Therefore, we have \(f(z_0, z_1) = d_{0,1}z_0z_1 + d_0z_0 + d_1z_1 + d_0\).

**Appendix B: Proof of Theorem 1**

In our SAT mapping process, given a point \(a_i\) of order \(i\) with a binary representation \(b_0 \cdots b_{N-1}\), there is a clause \(C_i\) associated with point \(i\). From Remark 1 and Remark 2, we observe that there are only two clauses, whose expansions contain the monomial of the form \(z_l, l \in \mathcal{N}\), i.e. \(C_0\) associated with the bit string \(0 \cdots 0\) and \(C_{2^{N-1}-i}\) with the bit string \(b_0 = 0, \cdots, b_{l-1} = 0, b_l = 1, b_{l+1} = 0, \cdots, b_{N-1} = 0\).

Following Remark 3, we can obtain the sign of the coefficient for the monomial of the form \(z_l\) in the expansion of \(w(C_i)\). Explicitly, when expanding a weighted clause \(w(C_i), i \in \mathcal{N}\), a bit \(b_{l'}, l' \in \mathcal{N}\) will contribute \(+1\) to the final coefficient of the monomial of \(z_l\) if \(b_{l'} = 1\) and will contribute \(-1\) if \(b_{l'} = 0\). For the monomial of the form \(z_l\) the sign of the coefficient \(d_i\), which is the squared distance spanning from the received signal points to the \(i\)-th constellation points, can then be known by multiplying these returned \(-1\)s or \(+1\)s together. Applying the above process to the pair clauses \(w(C_0)\) and \(w(C_{2^{N-1}-i})\) that contain the
monomial of the form $z_l$, the signs in front of $d_i, i \in \{0, 2^{N-1}-1\}$, can be determined. Correspondingly, we can obtain the coefficient associated with term $z_l$ in the expansion of $f(z_0, \cdots, z_{N-1})$ by summing the signed squared distances $d_i, i \in \{0, 2^{N-1}-1\}$.

For the expansions of the clauses containing the monomial of the form $z_l z_m$, there are $2^2 = 4$ clauses associated with $S = \{l, m\}$. For the coefficient of the monomial of the form $z_l z_m$, $l, m \in \mathcal{N}, l \neq m$, in the expansion of the clause $w(C_i), i \in S$, the pair of bits $b_l b_m$ will contribute $+1$ to the sign of the coefficient to the monomial of the form $z_l z_m$ if $b_l \times b_m = 1$ and will contribute $-1$ if $b_l \times b_m = 0$. Here, we use $\times$ to denote the product of $b_l$ and $b_m$ for avoiding any confusion. Repeating the above process for all $i \in S$, the signs in front of the squared distances $d_i$, are thus obtained in terms of the monomial of the form $z_l z_m$ in the expansion of $w(C_i)$ for any $i \in S$. We therefore obtain the coefficient associated with the term $z_l z_m$ in the expanded $f(z_0, \cdots, z_{N-1})$ by summing the coefficients for the monomial of the form $z_l z_m$.

Following the above procedures, we can obtain the sign of the coefficient of the monomial having any form of $\prod_{n \in S} z_n$, $S \in \mathcal{N}$, with respect to each clause. As a result, the coefficient of the monomial having the form $\prod_{n \in \mathcal{N}} z_n$ i.e. $S = \mathcal{N}$ in the expanded $f(z_0, \cdots, z_{N-1})$ contains all of the squared distances $d_i, i \in \mathcal{M}$. For a rectangular QAM constellation obeying Gray mapping, the coefficients associated with the diagonal points on the smallest lattice grid constructed by four constellation points have the same sign, while the signs of the coefficients associated with the two pairs of the diagonal points are opposite. This comes from the property of Gray mapping where the orders in binary bits of a pair of neighbouring points differ in only one bit, and the order in bits of the diagonal points thereby differ in two bits. As a result, the coefficient of $\prod_{n \in \mathcal{N}} z_n$ in $f(z_0, \cdots, z_{N-1})$ becomes 0. In terms of a PSK constellation following a Gray mapping order, the PSK constellation would contain $M = 2^N, N \in \mathbb{Z}^+$, constellation points [60], which are positioned at a uniform angular spacing around a circular. Therefore, the smallest grid formed for the PSK constellation can be constructed by choosing a pair of points separated by $\pi$ along with their neighbours having angular spacing $\pi$. Analogous to the smallest grid for the QAM constellation, the signs of the coefficients associated with the two pairs of the diagonal points are opposite corresponding to this rectangle. Therefore, the coefficient of the monomial of the form $\prod_{n \in \mathcal{N}} z_n$ is zero.

**APPENDIX C: PROOFS OF Corollary 1 AND Corollary 2**

**Proof of Corollary 1**: Following the proof of Theorem 1, the coefficient of the monomial having the form $\prod_{n \in \mathcal{N}} z_n$ in the expanded $f(z_0, \cdots, z_{N-1})$ contains all of the squared distances $d_i, i \in \mathcal{M}$. In Corollary 1, the property in 3) indicates that the coefficients associated with a pair of diagonal points of a rectangle share the same sign, while the coefficients associated with the other pair of diagonal points...
have the opposite sign. This makes the sum of the coefficients of $\prod_{n \in \mathcal{N}} z_n$ over a rectangle is zero. Furthermore, the properties in 1) and 2) indicate that all of the points in the constellation can form a set of rectangles without shared vertices between any two of them. Therefore, the final coefficient of $\prod_{n \in \mathcal{N}} z_n$ in the expanded $f(z_0, \cdots, z_{N-1})$ is zero, since the coefficient of $\prod_{n \in \mathcal{N}} z_n$ over a rectangle is zero, which accrues from the property in 3).

**Proof of Corollary 2:** Following the proof of Corollary 1, the property in 3) of Corollary 1 can be extended to demonstrate the coefficient of any monomial in $f(z_0, \cdots, z_{N-1})$. 3) of Corollary 1 indicates that for each rectangle constructed by the constellation points in $\mathcal{S}$, the resultant coefficient for the monomial of the form $\prod_{n \in \mathcal{S}} z_n$ is zero. For the monomial of the form $\prod_{n \in \mathcal{S}} z_n$, $\mathcal{S} \subseteq \mathcal{N}$, we know that it is associated with $2^{\vert \mathcal{S} \vert}$ constellation points, which can form $2^{\vert \mathcal{S} \vert - 1}$ rectangles without sharing vertices. Recall that 2) of Corollary 1 indicates that the rectangles formed by the constellation points which do not share vertices. Hence, the coefficient of the monomial of the form $\prod_{n \in \mathcal{S}} z_n$, $\mathcal{S} \subseteq \mathcal{N}$, is zero.

**APPENDIX D: PROOF OF Theorem 2**

Recall that for a rectangular QAM constellation, the bits associated with each point can be represented as $b_{l}, l \in \mathcal{N}_I$ for the in-phase bits and as $b_m, m \in \mathcal{N}_Q$ for the quadrature phase bits, which are arranged in Gray mapping order along each axis. Therefore, for a Gray-labelled QAM constellation, the points on one line parallel to the in-phase axis or the quadrature axis contains odd and even numbers of ‘0’s, alternatively. Correspondingly, we have the following properties for a rectangular QAM constellation following a Gray mapping:

**Remark D.1.** Given two points in the same line, one of the points has an odd numbers of ‘0’s , while the other point has an odd numbers of ‘0’s if there are even numbers of points between them, and has an even numbers of ‘0’s if they are separated by odd numbers of points.

Now we turn to the proof of Theorem 2. Given a set $\mathcal{S} \subseteq \mathcal{N}$ with $\mathcal{S} \cap \mathcal{N}_I \neq \emptyset$ and $\mathcal{S} \cap \mathcal{N}_Q \neq \emptyset$, the variables in $\prod_{n \in \mathcal{S}} z_n$ come partially from the in-phase bits and partially from the quadrature phase bits. Let us now define $L_I$ and $L_Q$ as the numbers of bits choosing from the in-phase bits and the quadrature phase bits, respectively, where we have $l$ and $m$ such that $1 \leq L_I \leq N_I$, $1 \leq L_Q \leq N_Q$. Following Remark 1, we see that the constellation points generating the term $\prod_{n \in \mathcal{S}} z_n$ are the points lying at the crossings of the $2^{L_I}$ lines parallel to the quadrature axis and $2^{L_Q}$ lines parallel to the in-phase axis. Correspondingly, there are $2^{L_I} \times 2^{L_Q}$ constellation points associated with the term $\prod_{n \in \mathcal{S}} z_n$. Here $2^{L_I}$ and $2^{L_Q}$ are the numbers of points in a line along the in-phase axis sharing the same quadrature phase bits and the quadrature phase axis sharing the same in-phase bits, respectively.
For proving the statement in 1) of Theorem 2, we start by separating the points in a line parallel to the in-phase axis into two groups according to the number of '0' in each binary orders. For the k-th line parallel to the in-phase axis, \( k \in \mathcal{N}_Q \), the group in which the binary orders contain odd numbers of '0's is denoted as \( G^k_{odd} \), while the group in which the binary orders have even numbers of '0's is denoted as \( G^k_{even} \). Thus, for each line there are \( 2^{L_l-1} \) points in each group. Given a \( k \), we randomly choose two constellation points \( a_0 \) and \( a_1 \) such that \( a_0 \in G^k_{odd} \) and \( a_1 \in G^k_{even} \). Because of the symmetric pattern of the Gray mapping, along the quadrature phase axis, there exists a pair of points \( a_2 \in G^k'_{even} \) and \( a_3 \in G^k'_{odd} \) in a line \( k', k' \neq k \), parallel to the in-phase axis as well, such that the binary orders of \( a_2 \) and \( a_3 \) contain the same quadrature phase bits with \( a_0 \) and \( a_3 \), respectively. Since the constellation follows the Gray mapping order, the four points \( a_0, \ldots, a_3 \) form a rectangle such that the binary orders corresponding to one pair of the diagonal points contains odd numbers of '0's and those labelling the other pair of diagonal points contain even numbers of '0's. In the proof of Theorem 1 we discussed the correlation of the sign in front of the squared distance, where in the expansion of a clause \( w(C_i), i \in \{0, 2^{N-1-1}\} \), for the coefficient of a monomial containing the variable \( z_i \), \( l' \in \mathcal{N} \), the bit \( b_{l'} \) will contribute +1 to the final coefficient of the monomial in the expanded \( w(C_i) \) if \( b_{l'} = 1 \) and it will contribute −1 if \( b_{l'} = 0 \). This leads to the fact that the signs in front of the monomial at hand associated with a pair of diagonal points are the same, while the signs of the monomial corresponding to the other pair of diagonal points are the opposite. Accordingly, the sum of the coefficients of the monomial associated with the four points is zero, where Proposition 2 is used. Based on the pattern given in Remark D.1, we can repeat the above process to construct \( 2^{L_l+L_Q-2} \) rectangles such that the sum of the coefficient of the monomials associated with each rectangle is zero. As a result, the coefficient of the monomial of the form \( \prod_{n \in \mathcal{S}} z_n \) in \( f(z_0, \ldots, z_{N-1}) \) is zero, if there exists at least one pair of bits denoted by \( b_l = 0 \) and \( b_m = 0 \) with \( l, m \in \mathcal{S} \), such that \( b_l \) and \( b_m \) belong to different groups \( \mathcal{N}_I \) and \( \mathcal{N}_Q \).

Next we prove the statement in 2). The statement in 1) indicates that the coefficient of the monomial of the form \( \prod_{n \in \mathcal{S}} z_n \) in \( f(z_0, \ldots, z_{N-1}) \), for any \( |\mathcal{S}| > N_I \) is zero. Now we turn to the calculation of the coefficient for the monomial of the form \( \prod_{n \in \mathcal{S}} z_n \), for any \( |\mathcal{S}| = N_I \). Consider the monomial of the form \( \prod_{n \in \mathcal{N}_I} z_n \) associated with the in-phase bits \( b_0 \cdots b_{N_I-1} \). From Remark 1, we know that a constellation point contributes a coefficient of the term \( \prod_{n \in \mathcal{N}_I} z_n \) if and only if the quadrature bits are zeros i.e., \( b_{N_I} = 0, \ldots, b_{N-1} = 0 \). Note that the constellation points corresponding to \( b_0 \cdots b_{N-1} \) such that \( b_0 \cdots b_{N_I-1} \in \{0, 1\}^{N_I} \) and \( b_{N_I} = 0, \ldots, b_{N-1} = 0 \), are lying on the line parallel to the in-phase axis. The coefficient of the monomial having the form \( \prod_{n \in \mathcal{N}_I} z_n \) thus cannot be cancelled. Therefore, there exits at least one monomial of degree \( N_I \) in \( f(z_0, \ldots, z_{N-1}) \), which indicates that the monomial of degree \( N_I \) exits in the expanded \( f(z_0, \ldots, z_{N-1}) \). Accordingly, the degree of \( f(z_0, \ldots, z_{N-1}) \) is \( N_I \).
APPENDIX E: PROOFS OF Corollary 3 and Corollary 4

Proof of Corollary 3: For a $N_r \times N_t$ MIMO channel associated with QPSK, we have $N = 2N_t$ and $N_{0,I} = \cdots = N_{N_t-1,I} = N_I$. We further denote the objective function as $f_{N_r \times QPSK}(z_0, \cdots, z_{N-1})$. Recall that the coefficient of the quadratic term in the objective function $f_{QPSK}(z_0, z_1)$ is zero and the degree of $f_{QPSK}(z_0, z_1)$ of (18) is 1. Following Proposition 3 and Remark 5, we can see that the coefficients of the monomials with a degree higher than $N_t$ in the expanded $f_{N_r \times QPSK}(z_0, \cdots, z_{N-1})$ are zero, since they at least contain a pair of bits $b_l = 0$ and $b_m = 0$ such that $l \in N_I$, $m \in N_Q$ and $N_I, N_Q$ associated with a single QPSK constellation. In the expanded $f_{N_r \times QPSK}(z_0, \cdots, z_{N-1})$, there is a class of the monomials of degree $N_t$, in which the variables come from the $N_t$ QPSK constellations. In this case of monomials of degree $N_t$, the coefficient cannot be cancelled and thus the degree of $f_{N_r \times QPSK}(z_0, \cdots, z_{N-1})$ is $N_t$.

Proof of Corollary 4: For a $N_r \times N_t$ MIMO channel with Gray-labelled MQAM, $M \geq 4$, we have $N_{0,I} = \cdots = N_{N_t-1,I} = N_I$. Recall from Theorem 2 that the degree of the objective function over a single constellation is $N_I$. Following the proof of Corollary 3, the coefficients of the monomials with degree higher than $N_I N_t$ in the expanded $f_{z_0, \cdots, z_{N-1}}$ is zero due to Remark 5. Furthermore, the coefficients of the monomials containing the variables associated with the $N_I$ in-phase bits of each MQAM constellation, cannot be cancelled from the proof of Theorem 2.

APPENDIX F: PROOF OF Theorem 3

It is plausible that the degree of $f(z_0, \cdots, z_N)$ is $N_{k,I} = \lceil \frac{N}{2} \rceil = N_I$ if $N_t = 1$, since this results in the ML detection problem on a single constellation. If $N_t \geq 2$, Corollary 3 and Corollary 4 show that for a $N_r \times N_t$ MIMO channel associated either with QPSK or with MQAM, the degree of the objective function $f(z_0, \cdots, z_N)$ is $N_I N_t$, where $N_I = 1$ for QPSK.

Now we consider the $N_r \times N_t$ MIMO channel of hybrid MQAM, where the number of constellation points can be different from each other. There are $N_I$ QAM constellations with each following a Gray mapping order. Following Remark 8, the in-phase bits and the quadrature phase bits associated with a single constellation are still independent. The highest degree of the monomials produced from a single constellation is $N_{k,I}$, $k \in N_t$, the product of which constitutes the highest degree of the monomials in the expanded $f(z_0, \cdots, z_{N-1})$. Hence, the degree of $f(z_0, \cdots, z_{N-1})$ is $\sum_{k \in N_t} N_{k,I}$.

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