Critical dynamics of self-gravitating Langevin particles and bacterial populations

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We study the critical dynamics of the generalized Smoluchowski-Poisson system (for self-gravitating Langevin particles) or generalized Keller-Segel model (for the chemotaxis of bacterial populations). These models [Chavanis & Sire, PRE, 69, 016116 (2004)] are based on generalized stochastic processes leading to the Tsallis statistics. The equilibrium states correspond to polytropic configurations with index $n$ similar to polytropic stars in astrophysics. At the critical index $n_{3} = d/(d-2)$ (where $d \geq 2$ is the dimension of space), there exists a critical temperature $\Theta_{c}$ (for a given mass) or a critical mass $M_{c}$ (for a given temperature). For $\Theta > \Theta_{c}$ or $M < M_{c}$ the system tends to an incomplete polytrope confined by the box (in a bounded domain) or evaporates (in an unbounded domain). For $\Theta < \Theta_{c}$ or $M > M_{c}$ the system collapses and forms, in a finite time, a Dirac peak containing a finite fraction $M_{c}$ of the total mass surrounded by a halo. We study these regimes numerically and, when possible, analytically by looking for self-similar or pseudo self-similar solutions. This study extends the critical dynamics of the ordinary Smoluchowski-Poisson system and Keller-Segel model in $d = 2$ corresponding to isothermal configurations with $n_{3} \to +\infty$. We also stress the analogy between the limiting mass of white dwarf stars (Chandrasekhar's limit) and the critical mass of bacterial populations in the generalized Keller-Segel model of chemotaxis.

I. INTRODUCTION

For a long time, statistical mechanics was restricted to systems interacting via short-range forces. For example, the case of self-gravitating systems is almost never considered in standard textbooks of statistical mechanics and these systems have been studied exclusively in the context of astrophysics. In the sixties, Antonov [1], Lynden-Bell [2] and Thirring [3] realized that self-gravitating systems have a very special thermodynamics marked by the non-equivalence of statistical ensembles (microcanonical, canonical, grand canonical,...). This is related to the non-additivity of the energy and to the presence of negative specific heats in the microcanonical ensemble. Furthermore, these systems experience a rich diversity of phase transitions (microcanonical and canonical first order phase transitions, zeroth order phase transitions,...) associated with their natural tendency to undergo gravitational collapse [4, 5]. Recently, several researchers have started to consider the dynamics and thermodynamics of systems with long-range interactions at a more general level and to develop the analogies between different systems that may seem a priori of a very different nature.

In a series of papers [1, 2, 3, 4, 5, 11, 12, 13, 14, 15, 16, 17], we have investigated the dynamics and thermodynamics of a system of self-gravitating random walkers. The basic idea is to couple the usual Brownian motion (as introduced by Einstein and Smoluchowski) to the gravitational interaction. In our general model [17], the microscopic dynamics of the particles is described by $N$ coupled stochastic equations including a friction force and a stochastic force in addition to the gravitational interaction. The friction force and the stochastic force model the interaction of the system with a thermal bath of non-gravitational origin. Then, the proper statistical description of this dissipative system is the canonical ensemble. In order to simplify the problem, we have considered a strong friction limit in which the motion of the particles is overdamped. We have also considered a mean field approximation which becomes exact in a proper thermodynamic limit $N \to +\infty$ in such a way that the volume $V \sim 1$ is of order unity and the coupling constant $G \sim 1/N$ goes to zero (alternatively, we can consider that the mass of the individual particles scales like $m \sim 1/N$ so that the total mass $M \sim Nm$ and the gravity constant $G$ remain of order unity). These approximations lead to the Smoluchowski-Poisson (SP) system. The steady states correspond to isothermal distributions associated with the Boltzmann statistics. When coupled to the Poisson equation, we obtain density profiles similar to isothermal stars in astrophysics [8, 9]. In the course of our study, we realized that the SP system is isomorphic to the standard Keller-Segel (KS) model [20, 21] introduced...
in mathematical biology to describe the chemotaxis of bacterial populations. The SP system and the KS model have been extensively studied by physicists and applied mathematicians, with different methods and motivations.

We have also studied a generalized Smoluchowski-Poisson (GSP) system [see Eqs. (13–14)] of this paper including an arbitrary barotropic equation of state \( P(\rho) \). This model has been introduced by Chavanis et al. with a density dependent diffusion coefficient \( D(\rho) \). For an isothermal equation of state \( P = kT \), we recover the standard SP system and KS model (with appropriate notations). Apart from the isothermal equation of state, the GSP system and GKS model have been studied for: (i) a polytropic equation of state \( P = K \rho^n \) \([51]\) (ii) a logotropic equation of state \( P = A \ln \rho - \frac{B}{\rho} \) \([72]\) (iii) a Fermi-Dirac equation of state \( P = P_{E.D.}(\rho) \) \([53, 54]\) (iv) an equation of state \( P = -T \rho_{\text{max}} \ln(1 - \rho/\rho_{\text{max}}) \) taking into account excluded volume effects \([55]\). These are standard equations of state introduced in astrophysics and statistical mechanics so that it is natural to consider these equations of state in connexion to the GSP system and GKS model.

Specializing on the polytropic equation of state \( P = K \rho^n \) with \( \gamma = 1 + 1/n \) \([72]\), the steady states of the GSP system correspond to polytropic distributions associated with the Tsallis statistics \([56]\). When coupled to the Poisson equation, we obtain density profiles similar to polytropic stars in astrophysics. For \( d \geq 2 \), there exists a critical index \( \gamma_{d/3} = 2(d-1)/d \), i.e. \( n_3 = d/(d-2) \) \([51]\). For \( 0 < n < n_3 \), there is no stable equilibrium in an unbounded domain. For \( n > n_3 \), there is no stable equilibrium in an unbounded domain.

### Table I: Summary of the different regimes of the GSP system in \( d > 2 \) with references to the physical literature ([P]: present paper; [N]: not done). The case of negative indices is considered in \([52]\). The links to the mathematical literature are indicated in the main text. Note: for \( n = \infty, T > T_c \) and for \( (n_3 < n < \infty, \Theta > \Theta_c) \) in a bounded domain, the system can either reach a metastable equilibrium state or collapse depending on a notion of basin of attraction (see \([1]\) for more details).

| Index | Temperature | Bounded domain | Unbounded domain |
|-------|-------------|----------------|-----------------|
| \( n = \infty \) | \( T > T_c \) | Metastable equilibrium state (local minimum of free energy): box-confined isothermal sphere \([11, 12, 53]\) | • Evaporation \([14]\): asymptotically free diffusion (gravity negligible) |
| | \( T < T_c \) | Self-similar collapse with \( \alpha = 2 \) \([11, 14, 53]\) followed by a self-similar post-collapse leading to the formation of a Dirac peak of mass \( M \) \([13]\) | • Collapse: pre-collapse and post-collapse as in a bounded domain \([11, 12, 53]\) |
| \( 0 < n < n_3 \) | \( \Theta > \Theta_c \) | Equilibrium state: box-confined (incomplete) polytrope \([51]\) | • Evaporation \([P]\): asymptotically free anomalous diffusion (gravity negligible) |
| | \( \Theta < \Theta_c \) | Equilibrium state: complete polytrope (compact support) \([51]\) |
| \( n_3 < n < \infty \) | \( \Theta > \Theta_c \) | Metastable equilibrium state (local minimum of free energy): box-confined polytropic sphere \([51]\) | • Collapse: pre-collapse and post-collapse as in a bounded domain \([51]\) |
| | \( \Theta < \Theta_c \) | Self-similar collapse with \( \alpha = 2n/(n - 1) \) \([51]\) followed by a post-collapse leading to the formation of a Dirac peak of mass \( M \) \([N]\) |
| \( n = n_3 \) | \( \Theta > \Theta_c \) | Equilibrium state: box-confined (incomplete) polytrope \([51]\) | Self-similar evaporation modified by self-gravity \([P]\) |
| | \( \Theta < \Theta_c \) | Pseudo self-similar collapse leading to a Dirac peak of mass \( \Theta/\Theta_c \) and halo \([P]\). This is followed by a post-collapse leading to a Dirac peak of mass \( M \) \([N]\) |
| | \( \Theta = \Theta_c \) | Infinite family of steady states \([P]\) | Infinite family of steady states \([P]\) |
| Index  | Temperature | Bounded domain | Unbounded domain |
|--------|-------------|----------------|------------------|
| $n = \infty$ | $T > T_c$ | Equilibrium state: analytical solution | Self-similar evaporation modified by self-gravity |
|        | $T < T_c$ | Pseudo self-similar collapse leading to a Dirac peak of mass $(T/T_c)M + \text{halo}$ | Collapse [N] |
| $T = T_c$ | Self-similar collapse leading to a Dirac peak of mass $M$ with exponential growth of $\rho(0,t)$ | Self-similar collapse leading to a Dirac peak of mass $M$ with logarithmic growth of $\rho(0,t)$ |
| $0 < n < \infty$ | $T > T_c$ | Equilibrium state: box-confined (incomplete) polytrope | Equilibrium state: complete polytrope (compact support) |
|        | $T \leq T_c$ | Equilibrium state: complete polytrope (compact support) | |

TABLE II: Summary of the different regimes of the GSP system in $d = 2$. In $d = 1$, the GSP system always relaxes towards a statistical equilibrium state so that there is no evaporation or collapse.

The paper is organized as follows. In Sec. [1], we briefly recall the connexion between white dwarf stars and gaseous polytropes. In Sec. [11], we recall the basic properties of the SP and GSP systems and describe the behavior of the solutions depending on the index $n$ and the dimension of space $d$. As the problem is very rich, involving many different cases ($\sim 30$), a summary of previously obtained results, completed by new results and new discussion, is required to understand the place of the present study in the general problem (see also Tables [1] and [11] for an overview). Then, we consider more specifically the particular index $n = n_3$ which presents a critical dynamics that was mentioned, but not studied, in our previous paper [51]. In Sec. [V], we show that this critical value can be understood from a simple dimensional analysis. In Sec. [V], we study the critical collapse dynamics and extend the results obtained in $d = 2$ for isothermal $(n = +\infty)$ systems to the case of critical polytropes $(n = n_3)$ in $d > 2$. In Sec. [VI], we study the evaporation dynamics in unbounded space. We show that for $n > n_3$, self-gravity becomes negligible for large times so that the evaporation is eventually controlled by the pure (anomalous) diffusion. For $n = n_3$, gravity remains relevant at any time so that there exists a self-similar solution for which all the terms of the GSP system scale the same way. Finally, in Sec. [VII], we transpose our main results to the context of chemotaxis using notations and parameters adapted to this problem (this is to facilitate the comparison with the results obtained in mathematical biology).

Our numerical and analytical study was conducted in parallel to a mathematical work by Blanchet et al. [25] who obtained rigorous results for the critical dynamics of the GSP system and GKS model introduced in our paper [31]. These two independent studies have different motivations and use very different methods so they are complementary to each other.
II. WHITE DWARF STARS AND POLYTROPES

In this section, we briefly recall the connexion between the maximum mass of white dwarf stars (Chandrasekhar’s mass \(M_{\text{Chandra}}\)) and the theory of self-gravitating polytropic spheres \([18, 19]\).

In simplest models of stellar structure, a white dwarf star can be viewed as a degenerate gas sphere in hydrostatic equilibrium. The pressure is entirely due to the equation of state of a degenerate gas of relativistic fermions while the density of the star is dominated by the mass of the protons. The condition of hydrostatic equilibrium coupled to the Poisson equation reads

\[\nabla P = -\rho \nabla \Phi, \quad \Delta \Phi = 4\pi G \rho, \tag{1}\]

and the equation of state of a degenerate gas of relativistic fermions at \(T = 0\) can be written parametrically as follows \([23]\):

\[P = A_2 f(x), \quad \rho = B x^3, \tag{2}\]

where

\[A_2 = \frac{\pi m^4 c^5}{3h^3}, \quad B = \frac{8\pi m^3 c^3 \mu H}{3h^3}, \tag{3}\]

\[f(x) = x(2x^2 - 3)(1 + x^2)^{1/2} + 3 \sinh^{-1}x, \tag{4}\]

where \(m\) is the mass of the electrons, \(H\) is the mass of the protons and \(\mu\) is the molecular weight. The function \(f(x)\) has the asymptotic behaviors \(f(x) \approx (8/5)x^5\) for \(x \ll 1\) and \(f(x) \approx 2x^4\) for \(x \gg 1\). The classical limit corresponds to \(x \ll 1\) and the ultra-relativistic limit to \(x \gg 1\). In these limits, the white dwarf star is equivalent to a polytropic gas sphere with an equation of state \(P = \omega_n \rho^n\). The index \(n\) of the polytrope is defined by \(\gamma = 1 + 1/n\). In \(d = 3\) dimensions, polytropes are self-confined for \(n < 5\) and they are stable (with respect to the Euler-Poisson system) for \(n \leq 3\) (for \(n = 3\) they are marginally stable). The mass-radius relation is given by \([13]\):

\[M^{(n-1)/n} R^{(3-n)/n} = \frac{K (1 + n)}{G (4\pi)^{1/n} \omega_n^{(n-1)/n}}, \tag{5}\]

where \(\omega_n\) is a constant (depending only on the index \(n\) of the polytrope) that can be expressed in terms of the solution of the Lane-Emden equation \([13]\).

In the classical case \(x \ll 1\), the equation of state takes the form

\[P = K_1 \rho^{5/3}, \tag{6}\]

with

\[K_1 = \frac{1}{5} \left(\frac{3}{8\pi}\right)^{2/3} \frac{h^2}{m(\mu H)^{5/3}}, \tag{7}\]

Therefore a classical white dwarf star is equivalent to a polytrope of index \(n = 3/2\). The mass-radius relation is given by

\[M^{1/3} R = \frac{1}{2} \left(\frac{3}{32\pi^2}\right)^{2/3} \frac{h^2}{m(\mu H)^{5/3}} \omega_{3/2}^{1/3}, \tag{8}\]

with \(\omega_{3/2} = 132.3843\ldots\). It exhibits the familiar \(M R^3 \sim 1\) scaling.

In the ultra-relativistic limit \(x \gg 1\), the equation of state takes the form

\[P = K_2 \rho^{4/3}, \tag{9}\]
with
\[ K_2 = \frac{1}{4} \left( \frac{3}{8\pi} \right)^{1/3} \frac{hc}{(\mu H)^{4/3}}. \tag{10} \]

Therefore, an ultra-relativistic white dwarf star is equivalent to a polytrope of index \( n = 3 \). For this index, the relation \([13]\) leads to a unique value of the mass
\[ M_c = \left( \frac{3}{32\pi^2} \right)^{1/2} \frac{\omega_3}{G} \left( \frac{hc}{G} \right)^{3/2} \frac{1}{(\mu H)^2}, \tag{11} \]
with \( \omega_3 = 2.01824 \ldots \). This is the Chandrasekhar mass
\[ M_c = 0.196701 \ldots \left( \frac{hc}{G} \right)^{3/2} \frac{1}{(\mu H)^2} \simeq 5.76M_\odot/\mu^2. \tag{12} \]

Considering the general mass-radius relation of partially relativistic white dwarf stars (see Fig. [1]), we note that, for this limiting value, the radius \( R_\text{of the configuration} \) tends to zero. This leads to a Dirac peak with mass \( M_c \). Thus, the Chandrasekhar mass represents the maximum mass of white dwarf stars (see Fig. [2]). There is no hydrostatic equilibrium configuration for \( M > M_c \).

If we extend Chandrasekhar’s theory to a d-dimensional universe \([13]\), we find that white dwarf stars become unstable in a universe with \( d \geq 4 \) dimensions (in \( d = 4 \), classical white dwarf stars exist for a unique value of the mass \( M = M_c = 0.0143958 \ldots h^4/(m^3G^2\mu^4H^2) \) and they are marginally stable). Therefore, the dimension \( d = 3 \) of our universe is very special regarding the laws of gravity. This is the largest dimension of space at which all the series of equilibrium of white dwarf stars (from classical to ultra-relativistic) is stable. This may have implications regarding the Anthropic Principle.

III. SELF-GRAVITATING LANGEVIN PARTICLES

A. The generalized Smoluchowski-Poisson system

In this paper, we shall study a dynamical model of self-gravitating systems whose steady states reproduce the condition of hydrostatic equilibrium Eq. (1). Specifically, we consider the generalized Smoluchowski-Poisson system \([18]\):
\[ \frac{\partial \rho}{\partial t} = \nabla \cdot \left[ \frac{1}{\xi} (\nabla P + \rho \nabla \Phi) \right], \tag{13} \]
\[ \Delta \Phi = S_d G \rho, \tag{14} \]
where \( P(\rho) \) is a barotropic equation of state, i.e., the pressure \( P(\rho, t) \) depends only on the density of particles \( \rho(\rho, t) \). This model describes a dissipative gas of self-gravitating Langevin particles in an overdamped limit \( \xi \to +\infty \) (where inertial effects are neglected) and in the thermodynamic limit \( N \to +\infty \) (where the mean field approximation becomes exact) \([13, 14]\). The GSP system is a particular example of generalized mean field Fokker-Planck equation \([15]\). It is associated to a stochastic process of the form
\[ \frac{d\mathbf{r}}{dt} = -\frac{1}{\xi} \nabla \Phi + \sqrt{\frac{2P(\rho)}{\rho \xi}} \mathbf{R}(t), \tag{15} \]
where \( \mathbf{R}(t) \) is a white noise with \( \langle \mathbf{R}(t) \rangle = 0 \) and \( \langle \mathbf{R}_i(t)\mathbf{R}_j(t') \rangle = \delta_{ij}\delta(t-t') \). This stochastic process describes the evolution of each of the N Langevin particles interacting through the mean field potential \( \Phi(\mathbf{r}, t) \). For sake of generality, we have allowed the strength of the noise term in Eq. (13) to depend on the local distribution of particles. This gives rise to anomalous diffusion and generalized pressure laws as discussed in \([16, 17]\).

The Lyapunov functional (or generalized free energy) associated with the GSP system is
\[ F = \int \rho \int P(\rho') \frac{d\rho'}{\rho^2} d\rho' d\mathbf{r} + \frac{1}{2} \int \rho \Phi d\mathbf{r}. \tag{16} \]
Easy calculations lead to
\[ \dot{F} = -\int \frac{\xi}{\rho} (\nabla P + \rho \nabla \Phi)^2 d\mathbf{r} \leq 0. \tag{17} \]
The GSP system has the following properties: (i) the total mass is conserved. (ii) \( \dot{F} \leq 0 \). (iii) \( \dot{F} = 0 \Leftrightarrow \nabla P + \rho \nabla \Phi = 0 \) (hydrostatic equilibrium) \( \Rightarrow \partial_t \rho = 0 \). (iv) \( \rho_{\text{eq}}(\mathbf{r}) \) is a steady state of the GSP system iff it is a critical point of \( F(\rho) \) at fixed mass. (v) A steady state of the GSP system is linearly dynamically stable iff it is a (local) minimum of \( F(\rho) \) at fixed mass \([17]\). By Lyapunov’s direct method \([16]\), we know that if \( F(\rho) \) is bounded from below, the GSP system will relax towards a (local) minimum of \( F(\rho) \) at fixed mass \([17]\). By Lyapunov’s direct method \([16]\), we know that if \( F(\rho) \) is bounded from below, the GSP system will relax towards a (local) minimum of \( F(\rho) \) at fixed mass \([17]\). By Lyapunov’s direct method \([16]\), we know that if \( F(\rho) \) is bounded from below, the GSP system will relax towards a (local) minimum of \( F(\rho) \) at fixed mass \([17]\). By Lyapunov’s direct method \([16]\), we know that if \( F(\rho) \) is bounded from below, the GSP system will relax towards a (local) minimum of \( F(\rho) \) at fixed mass \([17]\). By Lyapunov’s direct method \([16]\), we know that if \( F(\rho) \) is bounded from below, the GSP system will relax towards a (local) minimum of \( F(\rho) \) at fixed mass \([17]\). By Lyapunov’s direct method \([16]\), we know that if \( F(\rho) \) is bounded from below, the GSP system will relax towards a (local) minimum of \( F(\rho) \) at fixed mass \([17]\). By Lyapunov’s direct method \([16]\), we know that if \( F(\rho) \) is bounded from below, the GSP system will relax towards a (local) minimum of \( F(\rho) \) at fixed mass \([17]\).
B. Isothermal spheres

For an isothermal equation of state \( P = \rho k_B T/m \), we recover the standard Smoluchowski-Poisson system [11]:

\[
\frac{\partial \rho}{\partial t} = \nabla \cdot \left( \frac{1}{\xi} \left( \frac{k_B T}{m} \nabla \rho + \rho \nabla \phi \right) \right),
\]

Equation (18) is an ordinary mean-field Fokker-Planck equation associated with a Langevin dynamics of the form

\[
\Delta \phi = S_d \varepsilon \rho.
\]

Equation (18) is an ordinary mean-field Fokker-Planck equation associated with a Langevin dynamics of the form

\[
\frac{d\mathbf{r}}{dt} = -\frac{1}{\xi} \nabla \phi + \frac{2k_B T}{\xi m} \mathbf{R}(t),
\]

where the strength of the noise is constant. The Lyapunov functional of the SP system can be written

\[
F = k_B T \int \frac{\rho}{m} \log \frac{\rho}{m} d\mathbf{r} + \frac{1}{2} \int \rho \phi \, d\mathbf{r}.
\]

This is the Boltzmann free energy \( F_B = E - TS_B \) where \( E = (1/2) \int \rho \phi \, d\mathbf{r} \) is the energy and \( S_B = -k_B \int (\rho/m) \log(\rho/m) \, d\mathbf{r} \) is the Boltzmann entropy. The stationary solutions of the SP system are given by the Boltzmann distribution

\[
\rho = A e^{-\beta m \phi},
\]

where \( A \) is a constant determined by the mass \( M \). These steady states can also be obtained by extremizing \( F \) at fixed mass, writing \( \delta F - \alpha \delta M = 0 \), where \( \alpha \) is a Lagrange multiplier. The equilibrium distribution is obtained by substituting Eq. (22) into Eq. (19) leading to the Boltzmann-Poisson equation. Specializing on spherically symmetric distributions and defining

\[
\rho = \rho_0 e^{-\psi(\xi)}, \quad \xi = r/r_0 = (S_d \beta G m \rho_0)^{1/2},
\]

where \( \rho_0 \) is the central density, we find after simple algebra that \( \psi \) is solution of the Emden equation

\[
\frac{1}{\xi^{d-1}} \frac{d}{d\xi} \left( \xi^{d-1} \frac{d\psi}{d\xi} \right) = e^{-\psi},
\]

with \( \psi = 0 \) and \( \psi' = 0 \) at \( \xi = 0 \). The Emden equation can also be obtained from the fundamental equation of hydrostatic equilibrium for an isothermal equation of state [12, 13, 19]. Note that the isothermal spheres have a self-similar structure \( \rho(r)/\rho_0 = e^{-\psi(r/r_0)} \); if we rescale the central density and the radius appropriately, they have the same profile \( e^{-\psi(\xi)} \). This property is called homology [19].

For \( d = 1 \), the SP system is equivalent to the Burgers equation [22, 23] and it relaxes towards the Camm distribution [24] which is a global minimum of free energy for any temperature. For \( d > 2 \), there is no steady state with finite mass in an unbounded domain because the density of an isothermal self-gravitating system decreases as \( \rho \sim r^{-2} \) for \( r \to +\infty \) [12]. We shall thus enclose the system within a box of radius \( R \) [26]. For box-confined systems, we must integrate the Emden equation (24) until the normalized box radius \( \xi = \alpha \) with

\[
\alpha = (S_d \beta G m \rho_0)^{1/2} R.
\]

It is useful to define a dimensionless control parameter

\[
\eta = \frac{\beta G m}{R^{d-2}}.
\]

Using the conservation of mass or the Gauss theorem, we get [12]:

\[
\eta = \alpha \psi'(\alpha).
\]

This equation relates the central density to the mass and the temperature. More precisely, the relation \( \eta(\alpha) \) gives the mass \( M \) as a function of the central density (for a fixed temperature \( T \)) or the temperature \( T \) as a function of the density contrast \( R = \rho(0)/\rho(R) = e^{\psi(\alpha)} \) (for a fixed mass \( M \)). The curve \( \eta(\alpha) \) is plotted in Fig. 3 of [12]. For \( 2 < d < 10 \), the series of equilibrium \( \eta(\alpha) \) oscillates and presents a first turning point at \( \eta_c = \eta(\alpha_1) \) (for \( d \geq 10 \), the series of equilibria does not display any oscillation). According to Poincaré’s turning point argument [16, 17], configurations with \( \alpha > \alpha_1 \) are unstable (saddle points of free energy at fixed mass). This concerns in particular the singular isothermal sphere corresponding to \( \alpha \to +\infty \). Configurations with \( \alpha < \alpha_1 \) are metastable (local minima of free energy at fixed mass) and they exist only for \( \eta \leq \eta_c \). There is no global minimum of free energy for self-gravitating isothermal spheres. For \( \eta < \eta_c \), depending on the form of the initial density profile, the SP system can either relax towards a box-confined isothermal sphere (metastable) or collapse. This behavior has been illustrated numerically in Fig. 16 of [11]. For \( \eta > \eta_c \) the SP system undergoes gravitational collapse. This self-similar collapse, followed by the formation of a Dirac peak, has been studied in detail in [12, 13]. If we remove the box, the SP system can either collapse or evaporate depending on the initial condition (this behavior will be illustrated numerically in Sec. 11A).

The dimension \( d = 2 \) is critical and has been studied in detail in [12, 13]. The solution of the Emden equation is known analytically [19]:

\[
e^{-\psi} = \left( \frac{1}{(1 + \frac{1}{8}\xi^2)} \right)^2.
\]

In an unbounded domain, the density profile extends to infinity but the total mass is finite because the density decreases as \( r^{-4} \) for \( r \to +\infty \). The total mass \( M = \int_0^{+\infty} \rho 2\pi r dr \) is given by

\[
M = \frac{1}{\beta G m} \int_0^{+\infty} e^{-\psi} \xi d\xi = \frac{1}{\beta G m} \lim_{\xi \to +\infty} \xi \psi'(\xi),
\]
where we have used the Emden equation (24) to get the last equality. Using Eq. (28), we find that \( \xi \psi' \to 4 \) for \( \xi \to +\infty \). This yields a unique value of the mass (for a fixed temperature), or equivalently a unique value of the temperature (for a fixed mass) given by

\[
M_c = \frac{4k_BT}{Gm}, \quad k_BT_c = \frac{GMm}{4}.
\]

For \( T = T_c \) or \( M = M_c \), we have an infinite family of steady states

\[
\rho(r) = \frac{\rho_0}{(1 + \frac{C}{8}(r/r_0)^2)^2}, \quad \rho_0 r_0^2 = \frac{k_BT}{2\pi G m^2}
\]

parameterized by the central density \( \rho_0 \). For \( \rho_0 \to +\infty \), we obtain a Dirac peak with mass \( M_c \). The steady states have the same value of the free energy, independently on the central density \( \rho_0 \) (see Appendix A) and they are marginally stable (\( \delta^2 F = 0 \)). For \( T \neq T_c \) or \( M \neq M_c \), there is no steady state in an infinite domain. For \( T > T_c \) or \( M < M_c \), the solution of the SP system evaporates and for \( T < T_c \) or \( M > M_c \), the solution of the SP system collapses. These different regimes have been discussed in detail in [12, 16].

If we consider box confined configurations in \( d = 2 \), we observe that the control parameter \( \eta \) is independent on the box radius and can be written

\[
\eta = \beta G M m = 4M_c = 4 \frac{T_c}{\bar{T}}
\]

Using Eqs. (27) and (28), we obtain the relation \( \eta(\alpha) = (\alpha^2/2)/(1 + \alpha^2/8) \) between the central density, the mass and the temperature. The density profiles are given by Eq. (11) with \( 8(r_0/R)^2 = (T/T_c - 1) = (M_c/M - 1) \) so the central density is now determined by the mass \( M \) or the temperature \( T \). Equilibrium states exist only for \( \eta \leq \eta_c = 4 \), i.e. \( M \leq M_c \) or \( T \geq T_c \) and, since the series of equilibria is monotonic, they are fully stable (global minima of free energy at fixed mass). In that case, the SP system tends to a box-confined isothermal sphere. For \( \eta = \eta_c = 4 \), i.e. \( M = M_c \) or \( T = T_c \), the steady state is a Dirac peak containing all the mass. For \( \eta > \eta_c = 4 \) the SP system undergoes gravitational collapse (see Sec. 5).

The mass-central density (for a fixed temperature) of two-dimensional isothermal spheres is plotted in Fig. 3. We note the striking analogy with the mass-central density of white dwarf stars in Fig. 2. Therefore, the critical mass of isothermal spheres in two dimensions shares some resemblance with the Chandrasekhar mass. We shall show in the next section that this analogy (which is not obvious a priori) bears more significance than is apparent at first sight.

### C. Complete polytropes

If we consider a polytropic equation of state \( P = K \rho^\gamma \) with \( \gamma = 1 + 1/n \), we get the polytropic Smoluchowski-Poisson system

\[
\frac{\partial \rho}{\partial t} = \nabla \cdot \left[ \frac{1}{\xi} (\bar{K} \nabla \rho^\gamma + \rho \nabla \Phi) \right],
\]

\[
\Delta \Phi = S_d G \rho.
\]

Equation (33) is a generalized mean field Fokker-Planck equation associated with the stochastic process

\[
\frac{d\rho}{dt} = -\frac{1}{\xi} \nabla \Phi + \sqrt{\frac{2K}{\xi} \rho^{(\gamma - 1)/2}} R(t),
\]

where the strength of the noise depends on the local density in a power law [30]. The Lyapunov functional of the polytropic SP system can be written

\[
F = \frac{K}{\gamma - 1} \int (\rho^\gamma - \rho) \, d\rho + \frac{1}{2} \int \rho \Phi \, d\rho.
\]

It can be interpreted as a generalized free energy of the form \( F = E - T_{eff} S \) where \( E = (1/2) \int \rho \Phi \, d\rho \) is the energy, \( T_{eff} = K \) is an effective temperature (polytropic temperature) and \( S = -1/(\gamma - 1) \int (\rho^\gamma - \rho) \, d\rho \) is the Tsallis entropy (the polytropic index \( \gamma \) plays the role of the Tsallis q parameter). For \( \gamma = 1 \), i.e. \( n \to +\infty \), the polytropic equation of state \( P = K \rho^\gamma \) reduces to \( P = K \rho \). It coincides with an isothermal equation of state \( P = p_k T/m \) with temperature \( K = k_BT/m \) leading to the standard Smoluchowski-Poisson system (8)-(9).
The stationary solutions of the GSP system are given by the Tsallis distributions

\[ \rho = \rho_0 \theta^n(\xi), \quad \xi = r/r_0, \quad r_0 = \left[ \frac{K(1+n)}{S_dG\rho_0^{1-1/n}} \right]^{1/2}, \]  

where \( \rho_0 \) is the central density, we find after simple algebra that \( \theta \) is solution of the Lane-Emden equation

\[ -\frac{1}{\xi^{d-1}} \frac{d}{d\xi} \left( \xi^{d-1} \frac{d\theta}{d\xi} \right) = -\theta^n, \]

with \( \theta = 1 \) and \( \theta' = 0 \) at \( \xi = 0 \). The Lane-Emden equation can equivalently be derived from the fundamental equation of hydrostatic equilibrium with a polytropic equation. Note that the polytropic spheres have a self-similar structure \( \rho(r)/\rho_0 = \theta^n(r/r_0) \): if we rescale the central density and the radius appropriately, they have the same profile \( \theta^n(\xi) \). This property is called homology.

In this paper, we restrict ourselves to \( n > 0 \). Let us first discuss the case \( d > 2 \). For \( n > n_3 = (d+2)/(d-2) \), unbounded self-gravitating polytropes have infinite mass because their density profile decreases like \( r^{-\alpha} \) for \( r \to +\infty \), with \( \alpha = 2n/(n-1) \). For \( n < n_3 < (d+2)/(d-2) \), they are self-confined. In that case, the function \( \theta \) vanishes at \( \xi = \xi_1 \) and the density vanishes at \( R_s = r_0 \xi_1 \) which defines the radius of the polytrope. The relation between the radius and the central density is

\[ R_s = \left[ \frac{K(1+n)}{S_dG\rho_0^{1-1/n}} \right]^{1/2} \xi_1. \]  

The total mass \( M = \int_0^{R_s} \rho S_d r^{d-1} dr \) can be written as

\[ M = S_d\rho_0 r_0^d \int_0^{\xi_1} \theta^n \xi^{d-1} d\xi = -S_d\rho_0 r_0^d \xi_1^{d-1} \theta'_1, \]

where we have used the Lane-Emden equation to get the last equality. Therefore, the relation between the mass and the central density is

\[ M = -S_d\rho_0 \left[ \frac{K(1+n)}{S_dG\rho_0^{1-1/n}} \right]^{d/2} \xi_1^{d-1} \theta'_1. \]

Eliminating the central density between Eqs. \( (40) \) and \( (42) \) and introducing the index

\[ n_3 = \frac{d}{d-2}, \]  

we get the mass-radius relation

\[ M(n-1/n)R_s^{(d-2)(n_3-n)/n} = \frac{K(1+n)}{G S_d^{1/n}} \omega_n^{(n-1)/n}, \]

where

\[ \omega_n = -\xi_1^{(n+1)/(n-1)} \theta'_1. \]

Let us introduce the polytropic temperature

\[ \Theta = \frac{K(1+n)}{n S_d^{1/n}}. \]

For \( 0 < n < n_3 \) there is one, and only one, steady state for each mass \( M \) and temperature \( \Theta \) and it is fully stable (global minimum of \( F \) at fixed mass). The GSP system will relax towards this complete polytrope (note that for \( n = 1 \) the radius \( R_s \) of the polytrope is independent on the mass). For \( n_3 < n < n_3 \) there is one, and only one, steady state for each mass \( M \) and temperature \( \Theta \) but it is unstable (saddle point of \( F \) at fixed mass). In that case, the system will either collapse or evaporate. The index \( n_3 \) is critical. For \( n = n_3 \), there exists steady solutions for a unique value of the mass (at fixed temperature \( \Theta \)):

\[ M_c = \left( \frac{n \Theta}{G} \right)^{n_3/(n_3-1)} \omega_{n_3}, \]

or for a unique temperature (at fixed mass \( M \)): \[ \Theta_c = \left( \frac{G}{n_3 \omega_{n_3}} \right)^{(n_3-1)/n_3}. \]

For \( d = 3 \), we have \( M_c = (3\Theta/G)^{3/2} \omega_3 = 10.487... \) and \( \Theta_c = (G/3)(M/\omega_3)^{2/3} = 0.20872...(G/M)^{2/3} \). As we have seen in Sec. 4, the Chandrasekhar limiting mass of relativistic white dwarf stars is connected to the limiting mass of critical polytropes. For a polytropic equation of state with critical index \( n = n_3 \), and for \( M = M_c \), we get an infinite family of steady solutions

\[ \rho(r) = \rho_0 \theta^n(\rho(r/r_0)), \quad \rho_0 r_0^d = \left( \frac{\Theta n_3}{G} \right)^{d/2}, \]

parameterized by the central density \( \rho_0 \). For \( \rho_0 \to +\infty \), the density profile tends to a Dirac peak with mass \( M_c \). These solutions have the same equilibrium free energy \( F[\rho_{eq}] = -dKM/(d-2) \) independently on the central density \( \rho_0 \) (see Appendix [8]) and they are marginally stable (\( \delta^2 F = 0 \)). For \( M < M_c \) (at fixed temperature) or \( \Theta > \Theta_c \) (at fixed mass), the solutions of the GSP system
evaporate and for $M > M_c$ (at fixed temperature) or \( \Theta < \Theta_c \) (at fixed mass), they collapse. These different regimes will be studied in detail in Secs. 3 and 4.

For \( d = 2 \), we find that \( n_3 \to +\infty \), so we realize that isothermal systems (\( n = +\infty \)) in two dimensions are similar to critical polytropes (\( n = n_3 \)) in higher dimensions \( d > 2 \). This is why the critical mass of isothermal spheres is valid only for incomplete polytropes whose density profile is arrested by the box (i.e. \( \rho(R) > 0 \)). For \( n \geq n_s \), this is always the case. For \( 0 < n < n_s \), the identity

\[
\frac{\alpha}{\xi_1} = \frac{R}{R_s},
\]

the polytrope is confined by the box if \( R_s \geq R \), i.e. \( \alpha \leq \xi_1 \). For \( R_s < R \), i.e. \( \alpha > \xi_1 \), we have complete polytropes whose density profile vanishes before the wall. In that case, we need to integrate the Lane-Emden equation until the natural polytropic radius \( \xi = \xi_1 \). For \( \alpha > \xi_1 \), the relation (54) is replaced by

\[
\eta = n^{n/(n-1)} \omega_n \left( \frac{R_s}{R} \right)^{(d-2)(n-n_s)/(n-1)} (\alpha > \xi_1),
\]

which is equivalent to the mass-radius relation (14). Using Eq. (55), it can be expressed in terms of \( \alpha \), giving the relation between the mass and the central density (at fixed temperature) for complete polytropes. Finally, the intermediate case is \( R_s = R \), i.e. \( \alpha = \xi_1 \), at which the density profile vanishes precisely at the box radius. In that case, we have

\[
\eta = n^{n/(n-1)} \omega_n \quad (\alpha = \xi_1).
\]

**D. Box confined polytropes**

For systems confined within a box of radius \( R \), we need to integrate the Lane-Emden equation (33) until the normalized box radius \( \xi = \alpha \) with

\[
\alpha = R/r_0 = \left[ \frac{S_d G \rho_0^{1-1/n}}{K(n+1)} \right]^{1/2} R.
\]

(51)

It is useful to define a dimensionless control parameter (the definition of this parameter has been slightly changed with respect to our previous paper [51]):

\[
\eta = M \left[ \frac{nS_d^{1/n} G}{K(1+n)} \right]^{n/(n-1)} \frac{1}{R^{(d-2)(n-n_s)/(n-1)}}.
\]

(52)

In terms of the polytropic temperature (10), it can be rewritten

\[
\eta = \frac{G^{n/(n-1)} M}{\Theta^{n/(n-1)} R^{(d-2)(n-n_s)/(n-1)}}.
\]

(53)

Note that for \( n \to +\infty \), we have \( \Theta = K = k_B T/n \) and the definitions (21) and (23) coincide. Using the conservation of mass or the Gauss theorem, we get [51]:

\[
\eta = -n^{n/(n-1)} \alpha^{(n+1)/(n-1)} \theta'(\alpha), \quad (\alpha < \xi_1).
\]

(54)

This equation relates the central density to the mass (at fixed temperature and box radius). In fact, this relation is valid only for incomplete polytropes whose density profile is arrested by the box (i.e. \( \rho(R) > 0 \)). For \( n \geq n_s \), this is always the case. For \( 0 < n < n_s \), the identity

\[
\frac{\alpha}{\xi_1} = \frac{R}{R_s},
\]

the polytrope is confined by the box if \( R_s \geq R \), i.e. \( \alpha \leq \xi_1 \). For \( R_s < R \), i.e. \( \alpha > \xi_1 \), we have complete polytropes whose density profile vanishes before the wall. In that case, we need to integrate the Lane-Emden equation until the natural polytropic radius \( \xi = \xi_1 \). For \( \alpha > \xi_1 \), the relation (54) is replaced by

\[
\eta = n^{n/(n-1)} \omega_n \quad (\alpha > \xi_1).
\]

(57)

**FIG. 4:** Series of equilibria for box-confined polytropes with different index (the figure is done for \( d = 3 \)). The full lines \((\alpha < \xi_1)\) correspond to incomplete polytropes whose profile is arrested by the box and the dashed lines \((\alpha > \xi_1)\) correspond to complete polytropes that are self-confined.

The relation \( \eta(\alpha) \) defines the series of equilibria containing incomplete (for \( \alpha < \xi_1 \)) and complete (for \( \alpha > \xi_1 \)) polytropes. It gives the mass \( M \) as a function of the central density (for a fixed temperature \( \Theta \) and box radius \( R \)) or the temperature \( \Theta \) as a function of the density contrast (for a fixed mass \( M \) and radius \( R \)). Different examples of curves \( \eta(\alpha) \) are represented in Fig. 4 for various indices in \( d = 3 \):

- For \( n < n_3 \), the series of equilibria \( \eta(\alpha) \) is monotonic. Since polytropic spheres are stable in absence of gravity.
(corresponding to \( \alpha \to 0 \)) and since there is no turning point, the Poincaré argument implies that all the polytropes are stable. It can be shown furthermore that they are fully stable (global minima of free energy at fixed mass) so that the GSP system will tend to a steady state for \( t \to +\infty \). For \( \eta < \eta_1 = \eta(\xi_1) = n^{n/(n-1)} \omega_n \), the GSP system tends to an incomplete polytrope confined by the box. For \( \eta > \eta_1 \), the GSP tends to a complete polytrope with radius \( R_c < R \). This has been illustrated numerically in Fig. 21 for \( n = 3/2 \) in \( d = 3 \). This index corresponds to a classical white dwarf star in astrophysics. If we remove the box, the GSP system always tends to the complete polytrope.

- For \( n > n_3 \), the series of equilibria \( \eta(\alpha) \) presents a turning point at \( \eta_c = \eta(\alpha_1) \). According to the Poincaré turning point argument, configurations with \( \alpha > \alpha_1 \) are unstable (saddle points of free energy at fixed mass).

This concerns in particular the case of complete polytropes for \( n_3 < n < 5 \) (corresponding to \( \alpha = \xi_1 \)), the Schuster polytrope \( n = n_3 \) and the singular polytropic spheres for \( n \geq n_5 \) (corresponding to \( \alpha = +\infty \)). Configurations with \( \alpha < \alpha_1 \) are metastable (local minima of free energy at fixed mass) and they exist only for \( \eta \leq \eta_c \). There is no global minimum of free energy for \( n > n_3 \).

For \( \eta \leq \eta_c \), depending on the form of the initial density profile, the GSP system can either relax towards an incomplete polytrope confined by the box (metastable) or collapse. For \( \eta > \eta_c \), the GSP system undergoes gravitational collapse. This self-similar collapse has been studied in detail in [5]. It is very similar to the self-similar collapse of isothermal systems in \( d > 2 \) corresponding to \( n \to +\infty \). If we remove the box, the GSP system can either collapse or evaporate depending on the initial condition (this will be illustrated numerically in Sec. [4]).

- The case \( n = n_3 \) is critical and will be studied in detail in this paper. For the critical index \( n = n_3 \), the control parameter is independent on the box radius and can be written

\[
\eta = M \left( \frac{G}{\Theta} \right)^{n_3/(n_3 - 1)}. \tag{58}
\]

In terms of the critical mass [47] or critical temperature [48], we have

\[
\eta = n_3^{n_3/(n_3 - 1)} \omega_n \frac{M}{M_c} = n_3^{n_3/(n_3 - 1)} \omega_n \left( \frac{\Theta_c}{\Theta} \right)^{n_3/(n_3 - 1)}. \tag{59}
\]

For incomplete polytropes with \( \alpha < \xi_1 \), the relation \( \eta(\alpha) \) between the central density, the mass and the temperature is given by Eq. [64]. Their density profile is given by Eq. [64] where \( r_0 \) is determined by \( (\Theta_c/\Theta)^{d/2} = M/M_c = (1/\omega_n)(R/r_0)^{d-1} \omega_n (R/r_0) \), equivalent to relation [64], so the central density is now determined by the mass \( M \) or the temperature \( \Theta \). Complete polytropes with \( \alpha \geq \xi_1 \) exist for a unique value of the control parameter

\[
\eta_c = n_3^{n_3/(n_3 - 1)} \omega_n \quad \text{as a function of the density contrast} \quad R(\alpha) \quad \text{for a fixed mass.}
\]

This corresponds to the critical mass \( M = M_c \) or critical temperature \( \Theta = \Theta_c \). Equilibrium states exist only for \( \eta \leq \eta_c \), i.e. \( M \leq M_c \) or \( \Theta \geq \Theta_c \). For \( \eta < \eta_c \), they are fully stable (global minima of free energy at fixed mass). In that case, the GSP system relaxes towards an incomplete polytrope confined by the box. For \( \eta = \eta_c \), i.e. \( M = M_c \) or \( \Theta = \Theta_c \), we have an infinite family of steady states parameterized by their central density \( \alpha \geq \xi_1 \) or equivalently by their radius \( R_c < R \). They are marginally stable (\( \delta^2 F = 0 \)). For \( \eta > \eta_c \), i.e. \( M > M_c \) or \( \Theta < \Theta_c \), the GSP system undergoes gravitational collapse. The collapse dynamics is expected to be similar to the critical collapse of isothermal systems with \( n \to +\infty \) in \( d = 2 \) (see below). If we remove the box, the solution of the GSP system evaporates for \( \eta < \eta_c \), i.e. \( M < M_c \) or \( \Theta > \Theta_c \) and collapses for \( \eta > \eta_c \), i.e. \( M > M_c \) or \( \Theta < \Theta_c \). These different regimes will be studied in detail in Secs. [3] and [4].

\[\text{FIG. 5: Mass as a function of the central density for box-confined self-gravitating polytropic spheres with critical index} \quad n = n_3 \quad \text{in} \quad d = 3. \quad \text{Incomplete polytropes with} \quad \rho_0 \quad \text{are represented by a solid line and complete polytropes with} \quad R_c \leq R \quad \text{are represented by a dashed line.} \]

\[\text{The mass-central density relation (for a fixed temperature) of box-confined self-gravitating polytropic spheres with critical index} \quad n = n_3 = 3 \quad \text{in} \quad d = 3. \quad \text{Incomplete polytropes with} \quad \rho_0 \quad \text{are represented by a solid line and complete polytropes with} \quad R_c \leq R \quad \text{are represented by a dashed line.}\]

We note the striking analogy with the mass-central density relation of white dwarf stars in Fig. [3]. Indeed, ultra-relativistic white dwarf stars are equivalent to polytropes with critical index \( n = n_3 = 3 \) in \( d = 3 \). In this context, the critical mass \( M_c \) corresponds to the Chandrasekhar limit. We emphasize, however, that we are considering here pure critical polytropes enclosed within
a box while in Sec. I we considered self-confined partially relativistic white dwarf stars for which a box is not needed. It is only when $M \rightarrow M_{\text{Chandra}}$ (ultrarelativistic limit) that they become equivalent to pure polytropes. Furthermore, at $M = M_{\text{Chandra}}$ for white dwarf stars, the only steady state is a Dirac peak while at $M = M_c$ for pure critical polytropes, we have an infinite family of steady states with different central densities (the same difference holds between critical polytropes $n = n_3$ in $d = 2$ and isothermal spheres $n = n_3 = +\infty$ in $d = 2$; compare Figs. 3 and 5). Finally, in Fig. 6, we plot the mass as a function of the central density for different dimensions of space. This figure illustrates in particular the connexion between the critical mass in $d = 3$ reached for a finite value of the central density and the critical mass in $d = 2$ reached for an infinite value of the central density.

IV. THE CRITICAL INDEX FROM DIMENSIONAL ANALYSIS

It is instructive to understand the origin of the critical index $\gamma_{4/3} = 2(d-1)/2$ or $n_3 = d/(d-2)$ from simple dimensional analysis. Here we consider unconfined systems in $d$ dimensions with arbitrary value of $\gamma$. The polytropic Smoluchowski-Poisson system can be written

$$\frac{\partial \rho}{\partial t} = \nabla \cdot \left( \frac{1}{\xi} \left( K \gamma \rho^{\gamma - 1} \nabla \rho + \rho \nabla \Phi \right) \right) \equiv -\nabla \cdot \mathbf{J}, \quad (61)$$

$$\Delta \Phi = S_d G \rho. \quad (62)$$

The current $\mathbf{J} = \mathbf{J}_d + \mathbf{J}_g$ appearing in the Smoluchowski equation is the sum of two terms: a diffusion current $\mathbf{J}_d = -K \gamma \rho^{\gamma - 1} \nabla \rho$ and a gravitational drift $\mathbf{J}_g = -\rho \nabla \Phi$.

Based on dimensional analysis, the diffusion current can be estimated by

$$J_d \sim +K \gamma (M/L^d)^{\gamma - 1} (\rho/L) \sim +(1/L)^{d(\gamma - 1)+1}, \quad (63)$$

and the drift term by

$$J_g = -\rho GM/L^{d-1} \sim -(1/L)^{d-1}, \quad (64)$$

where $M$ is the mass of the system and $L$ is the characteristic size of the system.

The system will collapse to a point if gravity overcomes (anomalous) diffusion, i.e. $|J_g| \gg |J_d|$, when $L \rightarrow 0$. This will be the case if $d-1 > d(\gamma - 1) + 1$, i.e. $\gamma < \gamma_{4/3}$. Conversely, if $\gamma > \gamma_{4/3}$, the diffusion term can stabilize the system against gravitational collapse so that the system can be in stable equilibrium. The system will evaporate to infinity if (anomalous) diffusion overcomes gravity, i.e. $|J_d| \gg |J_g|$, when $L \rightarrow +\infty$. This will be the case if $d(\gamma - 1) + 1 < d - 1$, i.e. if $\gamma < \gamma_{4/3}$. Conversely, if $\gamma > \gamma_{4/3}$, the gravitational attraction can prevent evaporation so that the system can be in stable equilibrium. In conclusion, we find that the system can be in a stable equilibrium state iff $\gamma > \gamma_{4/3}$, i.e. $1/n > 1/n_3$. In the opposite case, the system can either collapse to a point or evaporate to infinity. By this very simple argument, we recover the stability criterion of self-gravitating polytropic spheres obtained by other methods (see Appendix B of [11]).

The critical case is obtained when $J_d \sim J_g$ implying $d(\gamma - 1) + 1 = d - 1$, i.e. $\gamma = \gamma_{4/3}$ or, equivalently, $n = n_3$. In that case, the stability of the system will depend on its mass. The system will collapse to a point if gravity overcomes diffusion, i.e. $|J_g| \gg |J_d|$, when $L \rightarrow 0$. This will be the case if $M > M_c$, where $M_c \sim (K/G)^{d/2}$ is a
critical mass. The system will evaporate to infinity (in an unbounded domain) if (anomalous) diffusion overcomes gravity, i.e. \(|J_d| \gg |J_g|\), when \(L \to +\infty\). This will be the case if \(M < M_c\). Therefore, at the critical index \(\gamma = \gamma_4/3\) i.e. \(n = n_3\), the system collapses if \(M > M_c\) and evaporates if \(M < M_c\). Again, this is fully consistent with the results obtained in Appendix B of [1].

V. COLLAPSE DYNAMICS

For \(0 < n < n_3\) in a space with \(d \geq 2\) dimensions, the GSP system tends to an equilibrium state. For \(n \geq n_3\), it can undergo gravitational collapse. For \(n > n_3\) with \(d > 2\), the collapse is self-similar as studied in [32] (the case of negative indices \(n < 0\) is studied in [52]). In the present section, we consider the collapse dynamics of self-gravitating Langevin particles associated with the critical index \(n_3 = d/(d-2)\) in \(d \geq 2\) dimensions which presents non trivial features.

A. Generalities: self-similar analysis

From now on, we adopt normalized variables such that \(G = M = R = \xi = 1\). The unique control parameter is the temperature \(\Theta\). For spherically symmetric solutions, using the Gauss theorem, the GSP system can be written in the form of an integrodifferential equation

\[
\frac{\partial \rho}{\partial t} = \frac{1}{r^{d-1}} \frac{\partial}{\partial r} \left( r^{d-1} \left[ (S_d \rho)^{1/n} \Theta \frac{\partial \rho}{\partial r} + \frac{\rho}{r^{d-1}} \int_0^r \rho(r') S_d r'^{d-1} dr' \right] \right).
\] (65)

Introducing the mass within a sphere of radius \(r\)

\[
M(r,t) = \int_0^r \rho(r') S_d r'^{d-1} dr',
\] (66)

the GSP system can be formulated through a unique non-linear dynamical equation for \(M(r,t)\):

\[
\frac{\partial M}{\partial t} = \Theta \left( \frac{1}{r^{d-1}} \frac{\partial M}{\partial r} \right)^{1/n} \left[ \frac{\partial^2 M}{\partial r^2} - \frac{d-1}{r} \frac{\partial M}{\partial r} \right] + \frac{M}{r^{d-1}} \frac{\partial M}{\partial r}.
\] (67)

If the system of total mass \(M = 1\) is confined within a box of radius \(R = 1\), the appropriate boundary conditions are

\[\tag{68}
M(0,t) = 0, \quad M(1,t) = 1.
\]

If the system is not confined, the second condition should be replaced by

\[\tag{69}
M(\infty,t) = 1.
\]

It is also convenient to introduce the function \(s(r,t) = M(r,t)/r^d\) which has the same dimension as the density and which satisfies

\[
\frac{\partial s}{\partial t} = \Theta \left( r \frac{\partial s}{\partial r} + ds \right)^{1/n} \left( \frac{\partial^2 s}{\partial r^2} + \frac{d+1}{r} \frac{\partial s}{\partial r} \right) + \left( r \frac{\partial s}{\partial r} + ds \right) s.
\] (70)

For \(n \to +\infty\), these equations reduce to those studied in Refs. [11, 12] in the isothermal case.

When the system collapses, it is natural to look for self-similar solutions of the form

\[
\rho(r,t) = \rho_0(t) \left( \frac{r}{r_0(t)} \right)^{1/2}, \quad r_0 = \left( \frac{\Theta}{\rho_0^{1/n}} \right)^{1/2}.
\] (71)

The relation between the core radius \(r_0\) and \(\rho_0\) (proportional to the central density \(\Theta\)) is obtained by requiring that the diffusive term and the drift term in Eq. (65) scale in the same way. This relation can be rewritten \(\rho_0 r_0^\alpha \sim 1\) with

\[\tag{72}
\alpha = \frac{2n}{n-1}.
\]

In terms of the mass profile, we have

\[
M(r,t) = M_0(t) g \left( \frac{r}{r_0(t)} \right), \quad \text{with} \quad M_0(t) = \rho_0^d r_0^d,
\] (73)

and

\[
g(x) = \int_0^x f(x') S_d x'^{d-1} dx'.
\] (74)

In terms of the function \(s\), we have

\[
s(r,t) = \rho_0(t) S \left( \frac{r}{r_0(t)} \right), \quad \text{with} \quad S(x) = \frac{g(x)}{x^d}.
\] (75)

Inserting the ansatz (72) in Eq. (70) and using Eq. (71), we obtain

\[
\frac{1}{\rho_0^\alpha} \frac{d\rho_0}{dt} = \alpha, \quad \text{with} \quad \alpha = \frac{2n}{n-1}.
\] (76)

and

\[
\alpha S + x S' = \left( x S' + dS \right)^{1/n} \left( S'' + \frac{d+1}{x} S' \right) + (x S' + dS) S.
\] (77)

Assuming that Eq. (77) has a solution so that the self-similar solution exists, Eq. (74) is readily integrated in

\[
\rho_0(t) = \frac{1}{\alpha} (t_{\text{coll}} - t)^{-1},
\] (78)

implying a finite time singularity. On the other hand, the invariant profile has the asymptotic behavior \(f(x) \sim x^{-n}\) for \(x \to +\infty\).
B. The two-dimensional isothermal case

In $d = 2$ dimensions, the critical index is $n_3 = +\infty$ corresponding to the isothermal case studied in [12] (in that case $\Theta = T$). Since the study of the critical dynamics is rather complicated, it can be useful to summarize our results, with some complements and amplifications, before treating the case $d > 2$.

In $d = 2$, there exists a critical temperature $T_c = 1/4$. If the system is enclosed within a box and $T > T_c$, it relaxes to an equilibrium distribution confined by the box. If the system is not confined and $T > T_c$, an evaporation process develops which has been studied in [14]. For $T = T_c$, the system undergoes gravitational collapse. The evolution is self-similar and leads to a Dirac peak containing the whole mass $M = 1$ for $t \to +\infty$. In a bounded domain, the central density grows exponentially [12] rapidly with time and in an unbounded domain, the central density increases logarithmically [16] with time (and a tiny fraction of mass is ejected at large distances). In fact, this could have been anticipated from the fact that the scaling functions $s(x)$ and $f(x)$ should decay as $x^{-2} = x^{-d}$ for large $x$. Then, the total mass in the profile is of order $\rho_0 r_0^2 \int_0^{r_0} x^{-2} x \, dx \sim \ln(1/r_0)$, which unphysically diverges when $r_0$ goes to zero. Said differently, the scaling profile at $t = t_{coll}$ is $\rho \propto r^{-2}$ so that the mass $M = \int \rho(r) 2\pi r \, dr$ diverges logarithmically for $r \to 0$. This logarithmic divergence is symptomatic of the formation of a Dirac peak resulting from a pseudo self-similar collapse. In the case $d = 2$, this situation can be analyzed analytically in great detail.

To that purpose, we note that the profile which cancels out the r.h.s. of the SP system is exactly given by

$$M_1(r, t) = 4 T_0 \frac{(r/r_0(t))^2}{1 + (r/r_0(t))^2}, \quad \rho_1(r, t) = \frac{4 \rho_0(t)}{\pi} \frac{1}{(1 + (r/r_0(t))^2)^2},$$

with

$$\rho_0(t) r_0(t)^2 = T.$$  

If we consider time independent solutions ($\partial \rho / \partial t = 0$) and impose the conservation of mass, we recover the steady solutions which exist for $T \geq T_c$ in a bounded domain (in that case $r_0 = (T/T_c - 1)^{1/2}$) and for $T = T_c$ only in an infinite domain (in that case we get a family of distributions parameterized by $r_0$). However, in the present case, we consider the case $T < T_c$ and seek the temporal evolution of $\rho_0(t)$ and $r_0(t)$. We argue that the solution $S_0(r, t)$ gives the leading contribution of the density profile in the core. This profile contains a mass $T/T_c$. We expect that the collapse will lead to $\rho_0(t) \to +\infty$ and $r_0(t) \to 0$ for $t \to t_{coll}$ (finite time singularity). Then, we see that the profile $\rho_0(r, t)$ leads to a Dirac peak with mass $T/T_c$, i.e.,

$$\rho_1(r, t) \to \frac{T}{T_c} \delta(r).$$

The excess of mass will be contained in the profile extending in the halo. Therefore, we look for solutions of the form

$$\rho(r, t) = \rho_1(r, t) + \rho_2(r, t),$$

$$= \rho_0(t) f_1(r/r_0(t)) + \rho_0(t)^{\alpha(t)/2} f_2(r/r_0(t)).$$

The first component has a scaling behavior and dominates in the center of the collapse region. It leads to a Dirac peak containing a fraction $M_0 = T/T_c$ of the total mass $M = 1$ at $t = t_{coll}$. The second component obeys a pseudo-scaling and $f_2(x) \sim x^{-\alpha(t)}$ for large $x$, with an effective scaling exponent $\alpha(t)$ which very slowly approaches the value 2 (expected from the naive self-similar analysis) when $t \to t_{coll}$. Thus, at $t = t_{coll}$, we get

$$\rho(r, t) \to M_0 \delta(r) + \chi(r, t),$$

where $\chi(r)$ is singular at $r = 0$ behaving roughly as $r^{-2}$. In Fig. [2] we illustrate this decomposition of the density profile into two components. It is shown in [2] that the central density satisfies an equation of the form

$$\frac{1}{\rho_0(t)} \frac{d \rho_0(t)}{d t} \propto \rho_0(t)^{\alpha(t)/2},$$

instead of Eq. (43), and that the effective scaling exponent $\alpha(t)$ depends on the central density as

$$\epsilon(t) \equiv 1 - \frac{\alpha(t)}{2} \sim \sqrt{\frac{\ln \ln \rho_0(t)}{2 \ln \rho_0(t)}}.$$

This yields $\rho_0 \sim (t_{coll} - t)^{-1 + \epsilon(t)}$ or equivalently

$$\ln(r_0(t)) \sim -2 \ln(r_0/\sqrt{\tau}) \sim \sqrt{\ln \tau \ln \ln \tau},$$

where we have noted $\tau = t_{coll} - t$.

Prior to our work [2], and unknown to us at that time, Herrero & Velazquez [24] had investigated the same problem in the context of chemotaxis using a different method based on match asymptotics. For $T < T_c$ (as
far as we know, they did not consider the case $T = T_c$ treated in \cite{12}, they showed that the system forms a Dirac peak of mass $M_c = T/T_c$, and the halo obeys pseudo-scaling (dashed line) with an exponent $\alpha(t)$ tending slowly to $d = 2$ as $\rho_0 \to +\infty$.

$\ln(\rho_0 \tau) \sim -2 \ln(r_0/\sqrt{\tau}) \sim \sqrt{2|\ln \tau|}$

$+ \frac{1}{2} \left( 1 - \frac{1}{|\ln \tau|} \right) \ln |\ln \tau|$. \quad (89)

obtained by Herrero & Velazquez (HV) are slightly different from ours (SC). They lead to an effective exponent given by

$1 - \frac{\alpha(t)}{2} \sim \sqrt{\frac{2}{\ln \rho_0}} + \frac{1}{2} \left( 1 - \frac{1}{\sqrt{\ln \rho_0}} \right) \ln \frac{\ln \rho_0}{\ln \rho_0}$. \quad (90)

instead of Eq. (87). For the densities accessible numerically, one gets $\alpha_{SC}(\rho_0 = 10^3) = 1.252...$ while $\alpha_{HV}(\rho_0 = 10^3) = 0.751...$ and $\alpha_{SC}(\rho_0 = 10^5) = 1.348...$ while $\alpha_{HV}(\rho_0 = 10^5) = 1.017...$. Numerical simulations performed in \cite{12} show a good agreement with the predicted values of $\alpha_{SC}$ for the densities accessible. However, in view of the complexity of the problem, and of the logarithmic (and sub-logarithmic!) corrections, it is difficult to understand the origin of the (slight) discrepancy between the two approaches. In any case, they both show that the collapse is not exactly self-similar but that the apparent scaling exponent $\alpha(t)$ is a very slowly varying function of the central density.

C. The critical polytropic case with $d > 2$

We now consider the critical index $n = n_3 = d/(d-2)$ with $d > 2$. There exists a critical temperature $\Theta_c = 1/[n_3 \omega_{n_3}^{(n_3-1)/n_3}]$ (in $d = 3$, we have $\Theta_c = 0.20872...$). If the system is confined within a box and $\Theta > \Theta_c$, it relaxes to an incomplete polytrope. This is illustrated in Fig. 9. If the system is not confined and $\Theta > \Theta_c$, an evaporation process develops which will be studied in the next section. In the confined case, when the generalized temperature $\Theta$ reaches the value $\Theta_c$, the equilibrium density profile vanishes exactly at $R = 1$. For $\Theta < \Theta_c$, and irrespectively of the presence of a confining box, the system collapses.

$S'' + \frac{d+1}{x} S' + (xS' + dS)^{2/d}(S-1) = 0$. \quad (91)

It happens that as in the case ($d = 2, n_3 = \infty$), this equation does not have any physical solution for large $x$. Again, this could have been anticipated from the fact that the scaling functions $s(x)$ and $f(x)$ should decay as $x^{-2n_3/(n_3-1)} = x^{-d}$, for large $x$. Then, the total mass in the profile is of order

$\rho_0^{d}\int_0^{1/r_0} x^{-d} \times x^{d-1} dx \sim \ln(1/r_0) \quad (92)$

which unphysically diverges when $r_0$ goes to zero. Said differently, the scaling profile at $t = t_{coll}$ is $\rho \propto r^{-d}$ so that the mass $M = \int \rho(r) S_0^{d-1} dr$ diverges logarithmically for $r \to 0$ \cite{252}.
tending slowly to 0 by which cancels out the r.h.s. of the GSP system is given in [12]). Using the results of Sec. III C, the profile which cancels out the r.h.s. of the GSP system is given by

$$\rho_1(r, t) = \frac{n_3}{S_d}(\Theta/\Theta_c)^{n_3}(r/r_0(t)), \quad (93)$$

with

$$\rho_0(t)r_0(t)^d = \Theta^{d/2}. \quad (94)$$

If we consider time independent solutions ($\partial\rho/\partial t = 0$) and impose the conservation of mass, we recover the steady solutions that exist for $\Theta \geq \Theta_c$ in a bounded domain (in that case, we have $(\Theta_c/\Theta)^{d/2} = -(1/\omega_n)(R/r_0)^{d-1}\theta'(R/r_0)$ and for $\Theta = \Theta_c$ only in an infinite domain (in that case we get a family of distributions parameterized by $r_0$). However, in the present case, we consider the case $\Theta < \Theta_c$ and seek the temporal evolution of $\rho_0(t)$ and $r_0(t)$. We argue that the solution $(93)$ gives the leading contribution of the density profile in the core. This profile vanishes at $R_*(r) = \xi_1r_0(t)$, has a central density $(n_3^{d/2}/S_d)\rho_0(t)$ and contains a mass (see Sec. III C):

$$M_c = \left(\frac{\Theta}{\Theta_c}\right)^{d/2}. \quad (95)$$

We expect that collapse will lead to $\rho_0(t) \to +\infty$ and $r_0(t) \to 0$. Then, we see that the profile $(93)$ tends to a Dirac peak with mass $M_c$, i.e.

$$\rho_1(r, t) \to \left(\frac{\Theta}{\Theta_c}\right)^{d/2} \delta(r). \quad (96)$$

The excess of mass will be contained in the profile extending in the halo. Therefore, we look for solutions of the form

$$\rho(r, t) = \rho_1(r, t) + \rho_2(r, t),$$

$$= \rho_0(t)f_1(r/r_0(t)) + \rho_0(t)^{\alpha(t)/d}f_2(r/r_0(t)). \quad (97)$$

The first component $(93)$ has a scaling behavior and dominates in the center of the collapse region. It leads to a Dirac peak containing a fraction $M_c = (\Theta/\Theta_c)^{d/2}$ of the total mass at $t = t_{coll}$. The second component obeys a pseudo-scaling and $f_2(x) \sim x^{-\alpha(t)}$ for large $x$, with an effective scaling exponent $\alpha(t)$ which very slowly approaches the value $d$ (expected from the naive self-similar analysis) when $t \to t_{coll}$. At $t = t_{coll}$, the first component $\rho_1(r, t)$ tends to a Dirac peak at the origin containing the mass $M_c$, whereas the second component develops a singularity at $r = 0$. Thus, we have

$$\rho(r, t) \to M_c\delta(r) + \chi(r, t), \quad (98)$$

with $\chi(r)$ behaving roughly as $r^{-d}$. In Fig. 10, we illustrate this decomposition of the density profile into two components.

![Graph](image)

**FIG. 10:** For $d = 3$, $n = n_3 = 3$, and deep into the collapse regime for $\Theta = 0.75\Theta_c$, we plot the density profile (full line), emphasizing its two components: the core is dominated by the bounded invariant scaling profile (complete polytrope of index $n_3$) containing a mass $M_c = (\Theta/\Theta_c)^{3/2}$ (dotted line), and the halo obeys pseudo-scaling (dashed line) with an exponent $\alpha(t)$ tending slowly to $d = 3$ as $\rho_0 \to +\infty$.

![Graph](image)

**FIG. 11:** For $\Theta = 0.75\Theta_c$ (here $d = 3$, $n = n_3 = 3$), we plot $\rho_0^{-1}(t)\frac{d\rho_0}{dt}$ (top full line) and $\rho_0(t)$ (bottom full line) as a function of $\rho_0(t)$. Both grow with an effective exponent $\alpha/3 \approx 0.93$ (dotted lines), which slowly increases and should saturate to unity (the dashed line has slope unity).

In Fig. 10, we show that perfect scaling which would imply $\rho_0^{-1}(t)\frac{d\rho_0}{dt} \sim \rho_0$ is not obeyed. Instead, in the accessible density range, $\rho_0^{-1}(t)\frac{d\rho_0}{dt}$ decays with an apparent power-law of $\rho_0$ which increases very slowly with
time, but remains less than unity. We expect to have a relation of the form
\[ \frac{1}{\rho_0} \frac{d\rho_0}{dt} \propto \rho_0^{\alpha(t)/d}, \tag{99} \]
which is indeed confirmed by the numerics. In Fig. 11, we also plot the central density in the pseudo-scaling component
\[ \hat{\rho}_0(t) = \rho_0^{\alpha(t)/d}(t), \tag{100} \]
which shows that the effective exponent \( \alpha(t) \) slowly converges to \( \alpha = d \).

Finally in Fig. 12, we display the apparent scaling behavior of \( \rho_2(r, t) = \rho_0(t)^{\alpha(t)/d}f_2(r/r_0(t)) \), associated to a value of \( \alpha \approx 2.8 \), fully compatible with the value obtained in Fig. 11 (in \( d = 3 \)).

**VI. EVAPORATION DYNAMICS IN UNBOUNDED SPACE**

**A. The case \( n > n_3 \)**

When the system is not confined to a finite box, the nature of the dynamics crucially depends on the value of the polytropic index \( n \) with respect to \( n_3 \). As before, we consider \( d \geq 2 \) and \( n > 0 \). If \( n < n_3 \), there exists equilibrium solutions (fully stable complete polytropes) which are reached for any initial density profile. If \( n \geq n_3 \), depending on the initial density profile and on the temperature, the system can collapse or evaporate. If \( R_0 \) is the typical extension of the initial density profile containing a mass \( M \), one can form a quantity with the dimension of \( \Theta \):
\[ \Theta_* = \frac{GM^{(n-1)/n}}{R_0^{(d-2)(n-n_3)/n}}, \tag{101} \]
which plays the role of an effective critical temperature.

If \( \Theta \ll \Theta_* \), the system should collapse as it would do if confined in a box of typical radius \( R_0 \). If \( \Theta \gg \Theta_* \), the system should evaporate in the absence of an actual confining box. Hence, for a given initial profile, there exists a non universal \( \Theta_\ast \), separating these two regimes. We present numerical simulations for the case \( n > n_3 \). In Fig. 13, and for a particular initial process, we illustrate the fact that depending on the value of \( \Theta \) with respect to a non universal \( \Theta_\ast \), the system can collapse or evaporate. In the evaporation regime and for \( n > n_3 \), a scaling analysis shows that gravity becomes gradually irrelevant and that this process becomes exclusively controlled by free (anomalous) diffusion. This fact is illustrated in Fig. 14. Indeed, when the evaporation length \( r_0(t) \to +\infty \), we see from Eq. (67) that the gravitational term becomes negligible in front of the diffusion term:
\[ \frac{M \cdot \partial M}{r^{d-1} \partial r} \ll \Theta \left( \frac{1}{r^{d-1} \partial r} \right)^{1/n} \frac{\partial^2 M}{\partial r^2}, \tag{102} \]
if \( d > d/n + 2 \), i.e. \( n > n_3 \). Therefore, for \( t \gg 1 \), the GSP system reduces to the pure anomalous diffusion equation
\[ \frac{\partial \rho}{\partial t} \approx \frac{K}{r^{d-1}} \frac{\partial}{\partial r} \left( r^{d-1} \frac{\partial \rho}{\partial r} \right), \tag{103} \]
with \( K = S_d^{\gamma-1} \Theta/\gamma \). This equation has self-similar solutions that were first discovered by Barenblatt [70] in the context of porous media. These solutions are closely related to the form of generalized thermodynamics introduced by Tsallis [50].

Using the original idea of Plastino & Plastino [71], we look for solution of Eq. (103) in the form of a Tsallis distribution with index \( \gamma \) and time dependent coefficients
\[ \rho(r, t) = \frac{1}{Z} \rho_0(t) \left[ 1 - \frac{(r/r_0(t))^{2(1/(\gamma-1))}}{1+(r/r_0(t))^{2(1/(\gamma-1))}} \right], \tag{104} \]
For \( \gamma > 1 \), i.e. \( n > 0 \), we have a profile with compact support where the density vanishes at \( r_{\text{max}}(t) = r_0(t)/\sqrt{\gamma-1} \). For \( \gamma < 1 \), i.e. \( n < 0 \), the density decreases like \( \rho \sim r^{-2/(1-\gamma)} \) and the total mass is finite provided that \( \gamma > \gamma_1/3 = (d-2)/d \), i.e. \( n < -d/2 \). Requiring that the profile (104) contains all the mass \( M = 1 \), and imposing
\[ \rho_0(t)r_0(t)^d = 1, \tag{105} \]
with find the normalization factor
\[ Z = \int_0^{+\infty} \left[ 1 - \frac{x^{2(1/(\gamma-1))}}{1+(x^{2(1/(\gamma-1))})} \right] S_d x^{d-1} dx. \tag{106} \]
Then, substituting the ansatz (104) with Eq. (105) in Eq. (103), we obtain
\[ \dot{\rho}_0 = -2dS_d^{\gamma-1}\Theta Z^{\gamma-1}r_0^{\gamma+2/d}. \tag{107} \]
We require that all the mass is in the profile (110) and impose the relation (107), implying that
\[ \int_0^{+\infty} f(x) S_d x^{d-1} dx = 1. \]  
(111)

Substituting the ansatz (110) with Eq. (105) in Eq. (103), and imposing the condition (107) where \( Z \) is for the moment an arbitrary constant, we obtain the differential equation
\[ \frac{1}{x^{d-1}} \frac{d}{dx} \left( x^{d-1} f^\gamma \frac{df}{dx} \right) = -2Z^{1-\gamma} (xf' + df). \]  
(112)

Noting the identity \( x^{d-1} (xf' + df)' = (x^d f)' \), this equation can be integrated into
\[ f^{\gamma-2} \frac{df}{dx} + 2Z^{1-\gamma} x = 0. \]  
(113)

This first order differential equation can again be readily integrated. We can choose the constant of integration so as to obtain a solution of the form
\[ f(x) = \frac{1}{Z} [1 - (\gamma - 1)x^2]^{1/(\gamma-1)}. \]  
(114)

Finally, the normalization condition (111) implies that \( Z \) is given by Eq. (109). It is interesting to realize that the \( q \)-exponential function \( e_\gamma(x) = [1 + (q - 1)x]^1_{+}^{1/(q-1)} \) introduced in the context of Tsallis generalized thermodynamics stems from the simple differential equation (113) related to the anomalous diffusion equation (103). Indeed, the scaling solution of this equation can be written
\[ f(x) = \frac{1}{Z} e_\gamma(-x^2), \]  
(115)

which generalizes the gaussian distribution obtained for the ordinary diffusion equation recovered for \( \gamma = 1 \). The moments \( \langle r^k \rangle \) of the distribution (110) are given by
\[ \langle r^k \rangle(t) = r_0(t)^k \int_0^{+\infty} f(x) x^{d+k-1} S_d dx. \]  
(116)

They exist provided that \( k > -d \) for \( \gamma \geq 1 \) and provided that \(-d < k < 2/(1 - \gamma) - d \) for \( \gamma < 1 \). They scale like \( \langle r^k \rangle \propto r_0^k \propto t^{k(d+2n)} \).

The Tsallis entropy is finite for \( \gamma > 3/5 \), and it scales like
\[ S(t) - nM = -n \rho_0^{1/n} \int_0^{+\infty} f(x) \gamma S_d x^{d-1} dx \propto t^{-d/(d+2n)}. \]  
(117)

On the other hand, for \( d > 2 \), the potential energy \( W = -1/(2S_d) \int (\nabla \Phi)^2 dr \) scales like
\[ W \propto \int_0^{+\infty} \left[ \frac{M(r)}{r^{d-1}} \right]^2 r^{d-1} dr \propto \frac{1}{r_0^{d-2}} \propto t^{-n(d-2)/(d+2n)}. \]  
(118)
Comparing Eqs. (117) and (118), we see that the potential energy is always negligible with respect to the entropy for $n > n_3$. Therefore, the Tsallis free energy behaves like

$$ F(t) + nK M \propto t^{-d/(d+2n)}, \quad (119) $$

for $t \to +\infty$. Note that for $n_3 < n < +\infty$, the free energy tends to a finite value $-nK M$ as the system spreads to infinity. Alternatively, for the isothermal case $n = +\infty$, the free energy is given by Eq. (95) of [16] and it tends to $-\infty$.

We can use the identity (42) to derive the first correction in the evolution of the moments $\langle r^k \rangle$ due to self-gravity. To that purpose, we introduce the zeroth order solution $\langle r^k \rangle_0$ in the equation

$$ \frac{d\langle r^k \rangle}{dt} = k(k + d - 2) \int P_r^{k-2} \, dr \quad -k \int_0^{+\infty} r^{k-d} M(r) \frac{\partial M}{\partial r} \, dr. \quad (120) $$

The first term gives, after integration, the pure anomalous scaling

$$ \langle r^k \rangle_0 \propto r^{k/(d+2n)}. \quad (121) $$

The second term gives, after integration, the first correction due to gravity. If we write $\Delta \langle r^k \rangle = \langle r^k \rangle - \langle r^k \rangle_0$, we get

$$ \Delta \langle r^k \rangle \propto t^{\frac{n(k-d)}{d+2n}+1}. \quad (122) $$

Let us consider some particular cases: (i) for $n \to +\infty$, we obtain $\Delta \langle r^k \rangle \propto t^{(k-d)/2d+1}$. If we furthermore consider the second moment $k = 2$ (moment of inertia), we recover the scaling $\Delta \langle r^2 \rangle \propto t^{2-d/2}$ of [16]. (ii) for $k = d$, we find that $\Delta \langle r^d \rangle = -(d/2)t \propto t$ whatever the index $n$ and the dimension of space $d$. (iii) For $n = n_3$, gravitational effects are of the same order as diffusive effects and $\langle r^k \rangle_0 \propto \Delta \langle r^k \rangle \propto r^{k/d}$. This case will be studied in detail in the next section. (iv) Finally, let us introduce the number $k_0 \equiv d - d/n - 2$. For $k < k_0$, $\Delta \langle r^k \rangle \to 0$; for $k = k_0$, $\Delta \langle r^k \rangle \propto 1/t$; for $k > k_0$, $\Delta \langle r^k \rangle \to +\infty$.

**B. The critical case $n = n_3$**

Finally, for $n = n_3$, and since a critical $\Theta_c$ exists irrespectively of the presence of a confining box, the system collapses for $\Theta < \Theta_c$ and evaporates for $\Theta > \Theta_c$. In the latter regime, gravity remains relevant and evaporation is controlled by both gravity and the diffusion process (see Fig. [3]). Mathematically, this arises from the fact that there is an evaporation scaling solution for which all the terms of Eq. (53) scale in the same way. More specifically, we expect an evaporation density profile of the form

$$ \rho(r,t) = \rho_0(t)f \left( \frac{r}{r_0(t)} \right), \quad \rho_0(t)r_0(t)^d = 1. \quad (123) $$

The relation between the evaporation length $r_0$ and $\rho_0$ (proportional to the central density) is obtained by requiring that the diffusive term and the drift term in Eq. (53) scale the same. In terms of the mass profile, we have

$$ M(r,t) = g \left( \frac{r}{r_0(t)} \right) \quad \text{with} \quad g(x) = \int_0^x f(x') S_d x'^{d-1} \, dx', \quad (124) $$

and in terms of the function $s$, we have

$$ s(r,t) = \rho_0(t)S \left( \frac{r}{r_0(t)} \right), \quad \text{with} \quad S(x) = \frac{g(x)}{x^d}. \quad (125) $$

We require that all the mass is contained in the self-similar profile $\rho_0$, which implies that

$$ g(+\infty) = \int_0^{+\infty} f(x) S_d x^{d-1} \, dx = 1. \quad (126) $$

Inserting the ansatz (125) in Eq. (70), using Eq. (123), and imposing

$$ \frac{1}{\rho_0^d} \frac{d\rho_0}{dt} = -d\Theta, \quad \text{i.e.} \quad r_0^{d-1} \frac{dr_0}{dt} = \Theta, \quad (127) $$

we obtain the scaling equation (note the change of sign compared to Eq. (77)) [11]:

$$ S'' + \frac{d+1}{x} S' + (xS' + dS)^{2/d} \left( \frac{1}{S} \right) S + 1 = 0. \quad (128) $$

The evaporation radius is given by

$$ r_0(t) = (d\Theta t)^{1/d}. \quad (129) $$

The moments scale like $\langle r^k \rangle \propto r_0^k \propto (d\Theta t)^{k/d}$ and the free energy scales like $F(t) + n_3 K M \propto t^{-(d-2)/d}$.

If we consider the large temperature limit $\Theta \gg 1$ where the diffusion term dominates on the gravitational drift, the foregoing differential equation reduces to

$$ S'' + \frac{d+1}{x} S' + (xS' + dS)^{2/d} = 0. \quad (130) $$

In terms of the function $f$ it can be written

$$ f^{-2/d} f' + \frac{x}{S_d^{(d-2)/d}} = 0, \quad (131) $$

which is consistent with Eq. (113) up to the changes of notations in Eqs. (107) and (127). We can either solve this equation and impose the normalization condition (226) or make simple transformations in order to directly use the results of Sec. [M1A]. Indeed, let us set $\rho_0 = \sigma \rho_*$ and $r_0 = \mu r_*$. We impose $\rho_* r_*^d = 1$ leading to $\sigma \mu^d = 1$. On the other hand, we choose $\sigma = 2(S_d/Z)^{(d-2)/d}$ where $Z$ is defined by Eq. (108) so that $\dot{\rho}_* = -2d(S_d/Z)^{(d-2)/d} \Theta \rho_*^2$. Then, $\rho = \rho_* f_s(r/r_*)$ with $f_s(x) = \sigma f(x/\mu)$. Now, $\rho_*$, $r_*$ and $f_s$ have been defined so as to coincide with the functions $\rho_0$, $r_0$ and $f$ of Sec. [M1A]. Thus, we get $f(x) = (1/\sigma) f_s(\mu x)$ where $f_s$
is the function \( \langle f \rangle \). Therefore, the normalized solution of Eq. (131) with the present notations can be written

\[
f(x) = \frac{1}{\sigma Z} \left[ 1 - \frac{d - 2}{d} \mu^2 x^2 \right]^{d/(d-2)},
\]

with

\[
\sigma \mu^d = 1, \quad \sigma = 2 \left( \frac{S_d}{Z} \right)^{(d-2)/d}, \quad (133)
\]

and where \( Z \) is given by Eq. (106). Proceeding along the lines of [10], we could expand the solutions of Eq. (128) (or of the equivalent equation for \( f \)) in powers of \( \Theta^{-1} \) in the limit \( \Theta \to +\infty \).

In Fig. 15, we show the form of the evaporation density profile \( f \) as a function of \( \Theta > \Theta_c \). As \( \Theta \) approaches \( \Theta_c \), the central density diverges, whereas the profile tends to the one corresponding to free diffusion for large \( \Theta \). In addition, we present numerical simulations for an intermediate \( \Theta \), showing that dynamical scaling is perfectly obeyed. Moreover, when \( \Theta \to \Theta_c \), we find that the scaling function obeys itself a scaling relation (see insert of Fig. 15). Defining \( \varepsilon = (\Theta - \Theta_c)/\Theta_c \), we find

\[
f(\Theta, x) = \varepsilon^{-1} F(x/\varepsilon^{1/d}),
\]

where \( F \) takes the form of a steady polytropic profile of index \( n_3 \). This scaling relation implies that close to \( \Theta_c \), the \( d \)-th moment of \( r \) scales as

\[
\langle r^d(t) \rangle \sim (\Theta - \Theta_c)t,
\]

which is a generalization of our exact result for \( d = 2 \) \( (n_3 = +\infty, T_c = 1/4) \) [10],

\[
\langle r^2(t) \rangle = 4(T - T_c)t + \langle r^2(0) \rangle.
\]

FIG. 15: In \( d = 3 \) and for \( n = n_3 = 3 \), we compare the evaporation profiles at different times for \( \Theta = 1 > \Theta_c \), for self-gravitating particles (full lines), to the faster evaporation dynamics when gravity is switched off (dashed lines).

FIG. 16: In \( d = 3 \) and for \( n = n_3 = 3 \), we compare the scaling profiles for \( \Theta = 0.21 \) near \( \Theta_c \approx 0.20872 \), \( \Theta = 1 \), and \( \Theta = 100 \) (top to bottom full lines; for clarity, the \( \Theta = 0 \) profile has been scaled down by a factor 150). For \( \Theta \gg 1 \), the invariant profile corresponds to the Barenblatt solution (pure anomalous diffusion) which is a Tsallis distribution with index \( \gamma_{4/3} = 1 + 1/n_3 \). For \( \Theta \to \Theta_c \), the invariant profile tends to the profile of a steady polytrope with index \( n_3 \). For an intermediate temperature \( \Theta = 1 \), we illustrate the perfect observed data collapse by plotting \( r_0(t)\rho(r,t) \) as a function of \( r/r_0(t) \), for \( t = 1.5^n \ (n = 0, ..., 13) \). These 14 curves are indistinguishable from the theoretical scaling profile. In the insert, we illustrate the scaling relation of Eq. (134) obtained for different values of \( \varepsilon = (\Theta - \Theta_c)/\Theta_c \to 0 \).

VII. ANALOGY BETWEEN THE LIMITING MASS OF WHITE DWARF STARS AND THE CRITICAL MASS OF BACTERIAL POPULATIONS

The generalized Smoluchowski-Poisson (GSP) system describing the dynamics of self-gravitating Langevin particles shares many analogies with the generalized Keller-Segel (GKS) model describing the chemotaxis of bacterial populations. Below, we briefly review the basic equations of chemotaxis and show the close link with the present work.

A. The generalized Keller-Segel model

The original Keller-Segel model has the form [20]:

\[
\frac{\partial \rho}{\partial t} = \nabla \cdot (D_2(\rho, c) \nabla \rho) - \nabla \cdot (D_1(\rho, c) \nabla c), \quad (137)
\]

\[
\frac{\partial c}{\partial t} = -k(c)c + h(c)\rho + D_c \Delta c. \quad (138)
\]

The drift-diffusion equation (137) governs the evolution of the density of bacteria \( \rho(r,t) \) and the reaction-diffusion equation (138) governs the evolution of the secreted chemical \( c(r,t) \). The bacteria diffuse with a diffusion
coefficient $D_2$ and they also move in a direction of a positive gradient of the chemical (chemotactic drift). The coefficient $D_1$ is a measure of the strength of the influence of the chemical gradient on the flow of bacteria. On the other hand, the chemical is produced by the bacteria with a rate $h(c)$ and is degraded with a rate $k(c)$. It also diffuses with a diffusion coefficient $D_c$. In the primitive Keller-Segel model, $D_1 = D_1(\rho, c)$ and $D_2 = D_2(\rho, c)$ can both depend on the concentration of the bacteria and of the chemical. This can take into account microscopic constraints, like close-packing effects \cite{51,32,33} or anomalous diffusion \cite{51}.

If we assume a constant diffusion coefficient $D_2 = D$ and a constant mobility $D_1/\rho = \chi$ (we also consider a constant production rate $\lambda$ and a constant degradation rate $k^2$ of the chemical), we obtain the standard Keller-Segel (KS) model

$$\frac{\partial \rho}{\partial t} = \nabla \cdot (D \nabla \rho - \chi \rho \nabla c), \quad (139)$$

$$\epsilon \frac{\partial c}{\partial t} = \Delta c - k^2 c + \lambda \rho. \quad (140)$$

If we now assume that the diffusion coefficient and the mobility depend on the concentration of the bacteria, and if we set $D_2 = Dh(\rho)$ and $D_1 = \chi g(\rho)$, where $h$ and $g$ are positive functions, we obtain the generalized Keller-Segel (GKS) model \cite{51,32,33}:

$$\frac{\partial \rho}{\partial t} = \nabla \cdot (Dh(\rho) \nabla \rho - \chi g(\rho) \nabla c), \quad (141)$$

$$\epsilon \frac{\partial c}{\partial t} = \Delta c - k^2 c + \lambda \rho. \quad (142)$$

Equation \cite{144} can be viewed as a nonlinear mean field Fokker-Planck (NFP) equation \cite{52} associated with a stochastic process of the form

$$\frac{dr}{dt} = \chi(\rho) \nabla c + \sqrt{2D(\rho)R(t)}, \quad (143)$$

with a diffusion coefficient $D(\rho) = (D/\rho) \int_0^\rho h(\rho')d\rho'$ and a mobility $\chi(\rho) = \chi g(\rho)/\rho$. These equations are associated with a notion of effective generalized thermodynamics \cite{45,50}. The Lyapunov functional of the NFP equation \cite{144}-\cite{142} can be written in the form of a generalized free energy $F = E - T_{eff}S$ where

$$E = \frac{1}{2\lambda} \int [(\nabla c)^2 + k^2 c^2] \, d\rho - \int \rho c \, d\rho, \quad (144)$$

is the energy, $T_{eff} = D/\chi$ is an effective temperature given by an Einstein-like relation and

$$S = -\int C(\rho) \, d\rho, \quad C''(\rho) = \frac{h(\rho)}{g(\rho)}, \quad (145)$$

is a generalized entropy. A straightforward calculation shows that

$$\dot{F} = -\frac{1}{\lambda c} \int (-\Delta c + k^2 c - \lambda \rho)^2 \, d\rho - \int \frac{1}{\chi g(\rho)}(Dh(\rho) \nabla \rho - \chi g(\rho) \nabla c)^2 \, d\rho \leq 0, \quad (146)$$

which is the expression of the H-theorem in the canonical ensemble adapted to dissipative systems. If we consider the particular case of a constant mobility $g(\rho) = \rho$ and a power law diffusion $h(\rho) = \gamma \rho^{\gamma - 1}$, with $\gamma = 1 + 1/n$, we obtain the polytropic Keller-Segel model \cite{51}:

$$\frac{\partial \rho}{\partial t} = \nabla \cdot (D \nabla \rho^\gamma - c \rho \nabla c), \quad (147)$$

$$\epsilon \frac{\partial c}{\partial t} = \Delta c - k^2 c + \lambda \rho. \quad (148)$$

The standard Keller-Segel model is recovered for $\gamma = 1$. Finally, if we neglect the degradation of the chemical ($k = 0$) and consider a limit of large diffusivity of the chemical (implying $\epsilon = 0$), we obtain for sufficiently large concentrations (see Appendix C of \cite{73})

$$\frac{\partial \rho}{\partial t} = \nabla \cdot (D \nabla \rho^\gamma - \rho \nabla c), \quad (149)$$

$$\Delta c = -\lambda \rho. \quad (150)$$

These equations are isomorphic to the generalized Smoluchowski-Poisson (GSP) system \cite{53-54} provided that we set

$$D = K/\xi, \quad \chi = 1/\xi, \quad c = -\Phi, \quad \lambda = S_dG. \quad (151)$$

Therefore, the results of the present paper apply to the chemotactic problem provided that the parameters are properly re-interpreted.

\section*{B. Formulation of the results with the biological variables}

In the gravitational context, we usually fix the coefficients $\xi$, $G$ and $M$ and use the temperature $\Theta$ as a control parameter. In the biological context, the coefficients $D$, $\chi$ and $\lambda$ are assumed given and the control parameter is the mass $M$. Therefore, it may be useful to briefly reformulate the previous results in terms of the mass, using notations adapted to the chemotactic problem.

For the critical index $n = n_3 = d/(d - 2)$ in $d \geq 2$, the steady states (polytropes) of the GKS model \cite{143}-\cite{150} exist, in an unbounded domain, for a unique value of the mass given by \cite{59}:

$$M_c = S_d \left( \frac{D(1 + n_3)}{\chi \lambda} \right)^{n_3/(n_3 - 1)} \omega_{n_3}. \quad (152)$$
For \( d = 3 \), we have

\[
M_c = 32\pi\omega_3 \left( \frac{D}{\chi\lambda} \right)^{3/2} \approx 202.8956 \cdots \left( \frac{D}{\chi\lambda} \right)^{3/2}.
\] (153)

For \( d = 2 \), using the identity \([50]\), we recover the critical mass

\[
M_c = \frac{8\pi D}{\chi\lambda},
\] (154)

associated with the two-dimensional standard Keller-Segel (KS) model (see \([50]\) and references therein). It is convenient to introduce rescaled variables so that \( D = \lambda = 1 \). With this system of units the critical mass is \( M_c(d) = S_d(1 + n_3)^{n_3/(n_3-1)}\omega_{n_3} = S_d[2(d-1)/(d-2)]^{d/2}\omega_d/(d-2) \). For example, \( M_c(d = 2) = 8\pi \) and \( M_c(d = 3) = 32\pi\omega_3 = 202.8956 \cdots \). Using the approximate expression of \( \omega_n \) obtained in Eq. (B72) of \([61]\), we can derive an approximate expression of the critical mass in the form

\[
M_c^{\text{approx}}(d) = \frac{S_d}{d}[d(d+2)]^{d/2}.
\] (155)

For \( d = 2 \), it returns the exact result \( M_c^{\text{approx}}(2) = M_c = 8\pi \). On the other hand, \( M_c^{\text{approx}}(d = 3) = 243 \) and \( M_c^{\text{approx}}(d = 4) = 2842 \). Using \( S_d = 2\pi^{d/2}/\Gamma(d/2) \) we find that \( M_c^{\text{approx}}(d) \sim 2\pi^{d/2}d^{d/2}/\Gamma(d/2) \) for \( d \to +\infty \).

Let us briefly discuss the critical dynamics of the GKS system with index \( n = n_3 = d/(d-2) \) for \( d \geq 2 \), depending on the total mass of the bacteria. For \( M < M_c \), a box-confined system tends to an incomplete polytrope confined by the walls of the box. In an unbounded domain, the system evaporates in a self-similar way as discussed in Sec. VI.c. For \( M > M_c \), the system undergoes finite time collapse as discussed in Sec. VII.b. In a finite time \( t = t_{\text{coll}} \), it forms a Dirac peak containing a mass \( M_c \) surrounded by a collapsing halo evolving quasi self-similarly with a time-dependent exponent \( \alpha(t) \) tending extremely slowly to \( \alpha = d/2 \) as \( t \to t_{\text{coll}} \). Thus,

\[
\rho(\mathbf{r}, t) \to M_c\delta(\mathbf{r}) + \chi(\mathbf{r}, t),
\] (156)

where \( \chi(\mathbf{r}) \) behaves roughly as \( r^{-d} \) for \( r \to 0 \). For \( M = M_c \), the situation is delicate and depends on the dimension of space. For \( d = 2 \), in a bounded domain, the steady state of the KS model is a Dirac peak \((\rho_0 = +\infty)\). We have constructed in \([12]\) a self-similar solution tending to this Dirac peak in infinite time. The central density increases exponentially rapidly. In an infinite domain, the KS model admits an infinite family of steady state solutions parameterized by their central density but the Dirac peak \((\rho_0 = +\infty)\) is selected dynamically (the other solutions have an infinite moment of inertia and, since the moment of inertia is conserved when \( M = M_c \), they cannot be reached from an initial condition with a finite moment of inertia). We have constructed in \([14]\) a self-similar solution tending to this Dirac peak in infinite time (and ejecting a small amount of mass at large distances so as to satisfy the moment of inertia constraint).

The central density increases logarithmically rapidly. For \( d > 2 \) and \( M = M_c \), in a bounded domain, the GKS model admits an infinite family of steady state solutions parameterized by their central density or, equivalently, by their natural radius \( R_* \). We have found numerically that the system tends to the polytrope where the density reaches zero at the box radius \((R_0 = R)\).

Due to the analogy between gravity and chemotaxis \([53]\), we find that the critical mass of bacterial populations in the standard Keller-Segel model in \( d = 2 \) and in the generalized Keller-Segel model in \( d > 2 \) for the critical index \( n = n_3 \) shares some resemblance with the Chandrasekhar mass of white dwarf stars. For example, the curves of Figs. \([3\,\text{and 4}\) also represent the mass of the bacterial aggregate as a function of the central density. As we have seen, they are strikingly similar to the mass-central density relation of white dwarf stars in Fig. \([\text{Fig. 4}\). Therefore, bacteria and white dwarf stars share deep analogies despite their very different physical nature \([59]\).

VIII. CONCLUSIONS AND PERSPECTIVES:
THE GSP SYSTEM WITH A RELATIVISTIC EQUATION OF STATE

In this paper, we have studied the critical dynamics, at the index \( n = n_3 \), of the GSP system and GKS model describing self-gravitating Langevin particles and bacterial populations. This study completes our previous investigation \([51]\) that was restricted to the cases \( n < n_3 \) and \( n > n_3 \). We have seen that, at the index \( n = n_3 \), there exists a critical mass \( M_c \) (independent of the size of the system) that is connected to the Chandrasekhar limiting mass of white dwarf stars \([8]\). In order to strengthen this analogy, it would be interesting to study the GSP system \([8\,\text{and 14}\) with the equation of state \([8\) corresponding to relativistic white dwarf stars. In fact, we can already describe qualitatively the behavior of the solutions by using the results obtained here for polytropes (see also the stability results obtained in \([7]\) for relativistic white dwarf stars).

For \( d = 1 \) and \( d = 2 \), there exists an equilibrium state (global minimum of free energy) for all values of the mass \( M \). Therefore, the GSP system relaxes towards that steady state.

For \( d = 3 \), there exists a critical mass \( M_{\text{Chandra}} = 0.196701 \cdots (\hbar c/G)^{3/2}/(\mu H)^2 \). For \( M < M_{\text{Chandra}} \), the GSP system tends to a partially relativistic white dwarf star (global minimum of free energy). For \( M \lesssim M_{\text{Chandra}} \), the density is small so that the equation of state reduces to that of a polytrope of index \( n = 3/2 \) (classical limit). Therefore, the GSP system relaxes towards a classical white dwarf star as described in Fig. 21 of \([7]\). For \( M = M_{\text{Chandra}} \) the density becomes large so that the equation of state reduces to that of a critical polytrope of index \( n = 3 \) (ultra-relativistic limit). We expect that the GSP system forms a Dirac peak of mass \( M_{\text{Chandra}} \) in infinite time. For \( M > M_{\text{Chandra}} \), there is
no equilibrium state and the system collapses. When the density reaches high values, the system becomes equivalent to a polytrope of index \( n = 3 \). Therefore, according to the present study, it forms in a finite time a Dirac peak of mass \( M_{\text{Chand}} \) surrounded by a halo evolving quasi-self-similarly with an exponent \( \alpha(t) \) converging very slowly to \( n = 3 \).

For \( d = 4 \), there exists a critical mass \( m_c = 0.0143968...h^4/(m^2G^2\mu^6H) \) discovered in \cite{31}. For \( M < M_c \), the steady states are unstable and the system can either collapse or evaporate (depending on the form of the initial density profile and on the basin of attraction of the solution). In case of evaporation, when the density reaches high values, the system becomes equivalent to a polytrope of critical index \( n_{3/2} = n = 2 \) (classical limit). In that case, it undergoes a self-similar evaporation similar to that described in Sec. \( \text{VI}B \) where gravity becomes asymptotically negligible. In case of collapse, when the density reaches low values, the system becomes equivalent to a polytrope of index \( n'_{3} = 4 > n_{3} = 2 \) (ultra-relativistic limit). In that case, it undergoes a self-similar collapse similar to that described in \cite{31}. For \( M > M_c \), there is no steady state and the system collapses in the way discussed previously (energy considerations developed in \cite{31} show that there is no evaporation in that case).

For \( d \geq 5 \), there is no steady state and the system can either collapse or evaporate. In case of evaporation, when the density reaches low values, the system becomes equivalent to a polytrope of index \( n_{3/2} > n_{3} \). In that case, it undergoes a self-similar evaporation similar to that described in Sec. \( \text{VI}A \) where gravity becomes asymptotically negligible. In case of collapse, when the density reaches high values, the system becomes equivalent to a polytrope of index \( n'_{3} > n_{3} \). In that case, it undergoes a self-similar collapse similar to that described in \cite{31}.

As we have already mentioned, the real dynamics of white dwarf stars is not described by the GSP system, but is much more complicated. However, we think that the study of this simple dynamical model is an interesting first step before considering more complicated models. At least, it reveals the great richness of the problem. A next step would be to take into account inertial effects and study the (generalized) Kramers-Poisson system and the corresponding hydrodynamic equations \cite{17}.

\section*{APPENDIX A: VIRIAL THEOREM AND FREE ENERGY OF CRITICAL POLYTROPES}

The scalar Virial theorem for the GSP system reads \cite{16}:

\begin{equation}
\frac{1}{2} \xi \frac{dI}{dt} = 2E_{\text{kin}} + W_{ii}, \quad (A1)
\end{equation}

where \( I = \int \rho r^2 d\mathbf{r} \) is the moment of inertia, \( E_{\text{kin}} = (d/2) \int P d\mathbf{r} \) is the kinetic energy of the microscopic motion and \( W_{ii} = -\int \rho r \cdot \nabla \Phi d\mathbf{r} \) is the Virial. For \( d = 2 \), \( W_{ii} = -GM^2/2 \) and for \( d \neq 2 \), \( W_{ii} = (d-2)W \) where \( W = (1/2) \int \rho \Phi d\mathbf{r} \) is the potential energy. If the system is enclosed within a box, we must add a term \(-dP_bV\) on the right hand side, where \( P_b \) is the pressure against the box. In the following, we assume that the system is unbounded so that \( P_b = 0 \).

For a polytropic equation of state \( P = K\rho^n \), with \( \gamma = 1 + 1/n \), the free energy \( (B3) \) can be written

\begin{equation}
F = \frac{2n}{d}E_{\text{kin}} + W - nKM. \quad (A2)
\end{equation}

Therefore, the Virial theorem can be expressed in the form

\begin{equation}
\frac{1}{2} \xi \frac{dI}{dt} = \frac{dF}{n} + W_{ii} - \frac{dW}{n} + dKM. \quad (A3)
\end{equation}

For the critical index \( n = n_3 = d/(d-2) \), we get

\begin{equation}
\frac{1}{2} \xi \frac{dI}{dt} = (d-2)F + W_{ii} - (d-2)W + dKM. \quad (A4)
\end{equation}

For \( d \neq 2 \), it reduces to

\begin{equation}
\frac{1}{2} \xi \frac{dI}{dt} = (d-2)F + dKM. \quad (A5)
\end{equation}

For a steady state (\( \dot{I} = 0 \)), the Virial theorem implies

\begin{equation}
F_{eq} = -\frac{d}{d-2}KM. \quad (A6)
\end{equation}

We have seen in Sec. \( \text{IV}C \) that spherically symmetric steady states of the GSP system with \( n = n_3 \) exist for a unique value of the mass \( M = M_c \) (for fixed \( K \)) or a unique value of the temperature \( \Theta = \Theta_c \) (for fixed \( M \)) and form an infinite family of solutions parameterized by their central density \( \rho_0 \). According to Eq. \( (A7) \), they all have the same free energy, independent on the central density \( \rho_0 \). Therefore, thermodynamical arguments do not allow to select a particular solution among the whole family.

For \( d = 2 \), the critical index \( n_3 \to +\infty \) and the equation of state is isothermal with \( K = k_BT/m \). Then, the Virial theorem \( (A4) \) becomes \( (A7) \):

\begin{equation}
\frac{1}{2} \xi \frac{dI}{dt} = 2Nk_B(T - T_c), \quad (A7)
\end{equation}

with \( k_BT_c = GMm/4 \). For a steady state (\( \dot{I} = 0 \)), the Virial theorem implies \( T = T_c \) or \( M = M_c \). It directly yields the result that unbounded two-dimensional isothermal spheres exist for a unique value of the mass or temperature. The spherically symmetric solution is given by Eq. \( (A7) \) reading

\begin{equation}
\rho(r) = \frac{\rho_0}{(1 + (\pi\rho_0/M)r^2)^2}. \quad (A8)
\end{equation}

This family of steady solutions is parameterized by the central density \( \rho_0 \). The corresponding mass profile is
given by $M(r) = \int_0^r \rho(r') 2\pi r' dr$ and the gravitational potential can be obtained from the Gauss theorem $d\Phi/dr = GM(r)/r$ with the gauge condition $\Phi(r) \sim GM \ln r$ for $r \to +\infty$. This yields

$$M(r) = \frac{\pi \rho_0 r^2}{1 + (\pi \rho_0/M)r^2}, \quad (A9)$$

$$\Phi(r) = \frac{GM}{2} \ln \left( \frac{M}{\pi \rho_0} + r^2 \right). \quad (A10)$$

From these expressions, we find that the potential energy is

$$W = \frac{GM^2}{4} \left[ 1 + \ln \left( \frac{M}{\pi \rho_0} \right) \right]. \quad (A11)$$

On the other hand, the Boltzmann entropy $S_B = -k_B \int (\rho/m) \ln(\rho/m) dr$ can be written

$$S_B = 2Nk_B \left[ 1 - \frac{1}{2} \ln \left( \frac{\rho_0}{m} \right) \right]. \quad (A12)$$

Therefore, the Boltzmann free energy $F_B = W - TS_B$ is given by

$$F_B = -\frac{GM^2}{4} \left[ 1 + \ln \left( \frac{\pi}{N} \right) \right]. \quad (A13)$$

We conclude that the free energy of unbounded isothermal spheres in two dimensions is independent on the central density $\rho_0$.

**APPENDIX B: AN EQUATION FOR THE MOMENTS ($r^k$)**

Let us introduce the moments of order $k$:

$$I_k(t) = \int \rho r^k dr. \quad (B1)$$

For $k = 2$, we recover the moment of inertia. Taking the time derivative of Eq. (B1), using the generalized Smoluchowski equation (B3) and integrating by parts, we obtain

$$\frac{1}{k} \frac{dI_k}{dt} = - \int r^{k-2} \cdot \nabla P \, dr - \int r^{k-2} \rho \cdot \nabla \Phi \, dr. \quad (B2)$$

Integrating by parts the first term, we get

$$\frac{1}{k} \frac{dI_k}{dt} = (k+d-2) \int P r^{k-2} \, dr - \int r^{k-2} \rho r \cdot \nabla \Phi \, dr. \quad (B3)$$

If we take into account boundary effects, we need to introduce a pressure term $-\frac{1}{2} P r^{k-2} \cdot \partial \mathbf{S}$ on the r.h.s. For $k = 2$, we recover the Virial theorem (A1). On the other hand, for a spherically symmetric system, using the Gauss theorem, the second integral can be simplified and we obtain

$$\frac{1}{k} \frac{dI_k}{dt} = (k+d-2) \int P r^{k-2} \, dr - \int r^{k-d} M(r) \frac{\partial M}{\partial r} \, dr. \quad (B4)$$

For $k = d$, the second integral can be calculated explicitly and we get

$$\frac{1}{d} \frac{dI_d}{dt} = 2(d-1) \int P r^{d-2} \, dr - \frac{GM^2}{2}. \quad (B5)$$

For $d = 1$, the first term on the r.h.s. must be replaced by $2P(0, t)$.
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[74] Since the free energy $F[\rho]$ coincides with the energy functional $W[\rho]$ of a barotropic gas (up to a positive macroscopic kinetic term) [52], we conclude that $\rho_0(r)$ is linearly dynamically stable with respect to the GSP system if it is formally nonlinearly dynamically stable with respect to the barotropic Euler-Poisson system [35].
[75] We shall study the problem in $d$ dimensions because: (i) We have found that the structure of the mathematical problem with the dimension of space is very rich [2, 5], exhibiting several characteristic dimensions. (ii) In gravity, the usual dimension is $d = 3$ but, in biology (see Sec. [71]), the bacteria (or cells) are compelled to lie on a plane so that $d = 2$.
[76] It may appear artificial to put the system in a “box”. In gravity, the box delimits the region of space where the system can be assumed isolated from the surrounding and where statistical mechanics applies. In biology (see Sec. [71]), the box has a physical meaning since it represents the container in which the bacteria (or cells) are confined.
[77] The reader should be aware that, in the sections dealing with the dynamics, $\rho_0$ and $r_0$ do not exactly coincide with the quantities of the same name introduced in the sections dealing with the statics (they usually differ by a factor of proportionality).
[78] More generally, for a polytrope of index $n$ we have $P \propto r^{-\alpha}$ at $t = t_{coll}$, with $\alpha = 2n/(n - 1)$ so that the self-similar solution exists provided that $\alpha - d + 1 < 1$ leading to $1/n < 1/n_3$ (i.e. $n > n_3$ for $d > 2$). This is precisely the range of indices for which the complete polytropes are dynamically unstable [3].
[79] Defining $\rho_0(x)$ as the equilibrium density profile at $\Theta = \Theta_c$, then the first component can be written $\rho_1(r, t) = \frac{1}{\Theta_c} \rho_0(r/r_0)$.
[80] Looking for a self-similar solution of the form [2] for any index $n$ and requiring that the diffusion and gravity scale the same, we find that $\rho_0 r_0^\alpha \sim 1$ where $\alpha$ is given by Eq. (4). The profile will contain all the mass provided that $\rho_0 r_0^\alpha \sim 1$ which implies $\alpha = d$ leading to $n = n_3$. Thus, it is only for the critical index that we can have a self-similar solution where the diffusion and gravity scale the same and which contains all the mass.
[81] The scaling equations (1) and (2) have a very different mathematical structure. The scaling equation for collapse (1), valid for $n > n_3$, leads to an eigenvalue problem for $S(x)$ [12, 14]. Indeed, it admits a physical solution for a unique value of $S(0)$ equal to $S_*$ (say). For $S(0) < S_*$, the solution becomes negative at some point, and for $S(0) > S_*$, it diverges at a finite $x_0$. By contrast, the scaling equation for evaporation (2) [2], valid for $n = n_3$, admits a one parameter family of solutions parameterized by $S(0)$. Then, the suitable value $S_*$ is selected by the normalization condition (2).