Stability of black holes in f(R) gravity

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Abstract

We investigate the stability of black holes in the viable model of \( f(R) = R + R^2 \) gravity which was known to be the best fit for inflation. These include Schwarzschild and Kerr black holes. Instead of studying the fourth-order linearized equation around the black hole background, we use the corresponding tensor-scalar theory of the Starobinsky model to perform their stability. The Schwarzschild black hole is stable, while the Kerr black hole is unstable because of superradiant instability.

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1 Introduction

The $f(R)$ gravity [1, 2, 3, 4] has much attentions as a strong candidate for explaining the current and future accelerating phases [5, 6]. On the other hand, $f(R)$ black holes have included the Schwarzschild-de Sitter black hole [7] and Schwarzschild-anti de Sitter black hole [8]. The trace of energy-momentum tensor must be zero to obtain a constant curvature black hole when $f(R)$ gravity couples with other matters of the Maxwell field [8], the Yang-Mills field [9], and a nonlinear Maxwell field [10].

Interestingly, $f(R) = R + R^2/(6M^2)$ gravity [11, 12, 13] has shown a strong evidence for inflation to support recent Planck data [14]. An important feature of this model indicates that the inflationary dynamics were driven by the purely gravitational sector $R^2$ and the scale of inflation is linked to the mass parameter $M^2$. However, it cannot work as a successful model for explaining late-time acceleration.

Most of astrophysical black holes are considered to be a rotating black hole [15]. The stability analysis of the rotating Kerr black hole is not a routine work as one has performed the stability analysis of a spherically symmetric Schwarzschild black hole [18, 19, 20] because it is an axisymmetric spinning black hole. The Kerr black hole has been proven to be stable against a massless graviton [21, 22, 23] and a massless scalar [24]. However, there exist another instability of the superradiance known as the black-hole bomb when one introduces a massive boson like scalar [25, 26, 27, 28, 29, 30] and vector [31].

It was first noted that the Kerr black hole obtained from $f(R) = R + hR^2$ is unstable since it could be transformed into the Brans-Dicke theory [32]. The Kerr solution could be obtained from a limited form of $f(R) = a_1 R + a_2 R^2 + a_3 R^3 + \cdots$ gravity [33]. Importantly, a perturbed Kerr black hole could distinguish Einstein gravity with two degrees of freedom (DOF) from $f(R)$ gravity with three DOF [34]. However, it is worth noting that the stability analysis of $f(R)$-rotating black hole was not completely performed because the linearized equation for $f(R)$ gravity contains fourth-order derivative terms. One way to avoid this difficulty is to transform the limited form of $f(R)$ gravity into a scalar-tensor theory with two auxiliary fields, leading to that the $f(R)$-rotating black hole is unstable against a massive scalar perturbation in the Jordan frame [35]. Further, the linearized Ricci scalar equation obtained from the limited $f(R)$ gravity has shown a superradiant instability if the linearized Ricci scalar is considered as a massive spin-0 graviton propagating on the Kerr spacetime [36].

In this work, we wish to focus on performing the stability of Schwarzschild and Kerr black holes in the specific model of $f(R) = R + R^2/(6M^2)$ gravity.
because its scalar-tensor theory was clearly shown to be the Starobinsky model in the Einstein frame which was extensively investigated as a promising single field inflation model. Even though the Starobinsky potential takes the same form, its role in the black hole physics differs from the inflation.

2 \( f(R) \) black holes

We start with a specific \( f(R) \) gravity

\[
S_f = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} f(R), \quad f(R) = R + \frac{R^2}{6M^2}
\]

with \( \kappa^2 = 8\pi G = 1/M_P^2 \). Here the mass parameter \( M^2 \) is chosen to be a positive value, which is consistent with the stability condition of \( f''(R) > 0 \) \([2]\). The Einstein equation takes the form

\[
R_{\mu\nu} f'(R) - \frac{1}{2} g_{\mu\nu} f(R) + \left( g_{\mu\nu} \nabla^2 - \nabla_\mu \nabla_\nu \right) f'(R) = 0,
\]

where the prime (') denotes the differentiation with respect to its argument. It is well-known that Eq.(2) provides the Kerr black hole solution with \( \bar{R} = \bar{R}_{\mu\nu} = 0 \). Hereafter we denote the background quantities with the overbar.

In this work, we use the Boyer-Lindquist coordinates to represent an axisymmetric Kerr black hole with mass \( \tilde{M} \) and angular momentum \( J \) \([16]\)

\[
ds_{Kerr}^2 = \bar{g}_{\mu\nu} dx^\mu dx^\nu = -\left(1 - \frac{2\tilde{M} r}{\rho^2}\right) dt^2 - \frac{2\tilde{M} r a \sin^2 \theta}{\rho^2} 2 dt d\phi
\]

\[
+ \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \left(r^2 + a^2 + \frac{2\tilde{M} r a^2 \sin^2 \theta}{\rho^2}\right) \sin^2 \theta d\phi^2
\]

with

\[
\Delta = r^2 + a^2 - 2\tilde{M} r, \quad \rho^2 = r^2 + a^2 \cos^2 \theta, \quad a = \frac{J}{\tilde{M}}.
\]

In the non-rotating limit of \( a \rightarrow 0 \), \( 3 \) recovers a spherically symmetric Schwarzschild solution

\[
ds_{Sch}^2 = -\left(1 - \frac{2\tilde{M}}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2\tilde{M}}{r}} + r^2 d\Omega_2^2,
\]
while the limit of \( a \to 1 \) corresponds to the extremal Kerr black hole. From the condition of \( \Delta = 0 (g^{rr} = 0) \), we determine two horizons which are located at
\[
{r_\pm} = \tilde{M} \pm \sqrt{\tilde{M}^2 - a^2}.
\] (6)
In the non-rotating limit \([5]\), the event horizon is given by
\[
r_{EH} = 2\tilde{M}.
\] (7)
The angular velocity at the outer (event) horizon takes the form
\[
\Omega = \frac{a}{2M r_+} = \frac{a}{r_+^2 + a^2}.
\] (8)

3 Black holes in the Starobinsky model

Since the \( f(R) \) gravity provides three DOF, one could represent it as a scalar-tensor theory by introducing one auxiliary field \( \psi \) [12]
\[
S_A = \int d^4x \sqrt{-g} \left( \frac{M_P^2}{2} R + \frac{M_P}{M} R \psi - 3\psi^2 \right),
\] (9)
where the superscript \( J \) means the Jordan frame. Integrating out the field \( \psi \) leads to the original \( f(R) \) gravity (1). Employing the conformal transformation and redefining the scalar field (\( \psi \to \phi \)) to arrive at the Einstein frame
\[
g^J_{\mu\nu} \to e^{-\sqrt{\frac{2}{3}} \psi \frac{M_P}{M}} g^E_{\mu\nu} = \frac{1}{1 + \frac{2\phi}{\sqrt{3} M_P}} g^E_{\mu\nu},
\] (10)
we obtain the Starobinsky model [13]
\[
S_S = \int d^4x \sqrt{-g^E} \left[ \frac{M_P^2}{2} R - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right]
\] (11)
with the Starobinsky potential (see Figure 1 for its graphical form)
\[
V(\phi) = \frac{3M_P^2 M^2}{4} \left( 1 - e^{-\sqrt{\frac{2}{3}} \frac{\phi \frac{M_P}{M}}} \right)^2.
\] (12)
The Einstein and scalar equations are given by
\[
G_{\mu\nu} = \frac{1}{M_P^2} T^\phi_{\mu\nu}, \quad T^\phi_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} \left( (\partial \phi)^2 + V \right)
\] (13)
\[
\nabla^2 \phi - V' = 0, \quad V' = \sqrt{\frac{3}{2}} M_P M^2 e^{-\sqrt{\frac{2}{3}} \frac{\phi \frac{M_P}{M}}} \left( 1 - e^{-\sqrt{\frac{2}{3}} \frac{\phi \frac{M_P}{M}}} \right).
\] (14)
In the case of \( \phi = 0 (V = 0) \), we obtain the Kerr solution \([5]\) and Schwarzschild solution \([5]\) to \([13]\) and \([14]\). We note here that a plateau of \( V \approx \frac{2}{3} M_P^2 M^2 \) appears for \( \phi \gg 1 \), which was used to define a slow-roll inflation.
Figure 1: Starobinsky potential with $3M^2_P/4 = 1$. In the case of $\phi = 0 (V = 0)$, we obtain the Kerr and Schwarzschild black hole solutions, while for $\phi \gg 1$ it is sufficiently flat ($V \approx 1$) to ensure slow-roll conditions for inflation in agreement with the Planck data [14].

4 Linearized equations

We start with the metric perturbation around the Kerr black hole to study the linear stability of the black hole

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}. \tag{15}$$

The Taylor expansions around $\bar{R} = 0$ are employed to define the linearized Ricci scalar [37] as

$$f(R) = f(0) + f'(0)\delta R(h) + \cdots, \tag{16}$$

$$f'(R) = f'(0) + f''(0)\delta R(h) + \cdots \tag{17}$$

with $f(0) = 0$, $f'(0) = 1$, and $f''(0) = 1/3M^2$. The linearized equation to (2) is given by

$$\delta R_{\mu\nu}(h) + \frac{1}{3M^2} \left[ \bar{g}_{\mu\nu} \left( -\frac{3M^2}{2} + \nabla^2 \right) - \bar{\nabla}_\mu \bar{\nabla}_\nu \right] \delta R(h) = 0, \tag{18}$$

where the linearized Ricci tensor and scalar are given by

$$\delta R_{\mu\nu}(h) = \frac{1}{2} \left[ \bar{\nabla}^\rho \bar{\nabla}_\mu h_{\nu\rho} + \bar{\nabla}^\rho \bar{\nabla}_\nu h_{\mu\rho} - \bar{\nabla}^2 h_{\mu\nu} - \bar{\nabla}_\mu \bar{\nabla}_\nu h \right], \tag{19}$$

$$\delta R(h) = \bar{\nabla}^\rho \bar{\nabla}_\rho h - \bar{\nabla}^2 h. \tag{20}$$

When using (19) and (20), the linearized equation (18) becomes a fourth-order differential equation with respect to the metric perturbation $h_{\mu\nu}$. Obviously, it is not a solvable equation. Another expression for (18) takes the form

$$\delta G_{\mu\nu} = \frac{1}{3M^2} \left( \bar{\nabla}_\mu \bar{\nabla}_\nu - \bar{g}_{\mu\nu} \nabla^2 \right) \delta R \tag{21}$$
whose trace equation leads to the linearized Ricci scalar equation \[36\]

\[(\nabla^2 - M^2)\delta R = 0. \quad (22)\]

Choosing the Lorentz gauge of \(\nabla_\nu h^{\mu\nu} = \bar{\nabla}^\mu \bar{h}/2\) and using the trace-reversed perturbation of \(\bar{h}_{\mu\nu} = h_{\mu\nu} - h\bar{g}_{\mu\nu}/2\), Eq. (18) takes a simple form \[34\]

\[
\nabla^2 \bar{h}_{\mu\nu} + 2\bar{R}_{\mu\rho\sigma\tau} \bar{h}^{\rho\sigma} - \frac{1}{3M^2} \left(\bar{g}_{\mu\nu} \nabla^2 - \bar{\nabla}_\mu \bar{\nabla}_\nu\right) \nabla^2 \bar{h} = 0. \quad (23)
\]

Also, one could not solve (23) directly for \(M^2 \neq \infty\) because it is still a fourth-order coupled equation for \(h_{\mu\nu}\) and \(\bar{h}\). However, its trace equation can be simplified into a factorized form for \(\bar{h}\)

\[
\nabla^2 \left(\nabla^2 - M^2\right) \bar{h} = 0 \quad (24)
\]

which implies two second-order equations

\[
\begin{align*}
\nabla^2 \bar{h} &= 0, \\
(\nabla^2 - M^2) \bar{h} &= 0.
\end{align*} \quad (25) \quad (26)
\]

On the other hand, two linearized equations from (13) and (14) take the simple forms with \(\delta T^\phi_{\mu\nu} = 0\) and \(\delta R = 0\)

\[
\begin{align*}
\delta R_{\mu\nu}(h) &= 0, \\
(\nabla^2 - M^2) \varphi &= 0.
\end{align*} \quad (27) \quad (28)
\]

We note that two Eqs. (26) and (28) are the same but tensor equation (23) is quite different from the linearized Einstein equation (27). This implies that the complexity of a fourth-order coupled equation (18) can be reduced to two decoupled second-order equations (27) and (28) if one employs the conformal transformation and redefinition of scalar field after introducing the auxiliary formalism, arriving at a canonical scalar \(\phi\) with the Starobinsky potential in the Einstein frame. This describes a process of \([R^2 \rightarrow R\psi - 3\psi^2 \rightarrow -(\partial\phi)^2 - V]\).

5 Stability of Schwarzschild black hole

It is a formidable task to perform stability of the non-rotating Schwarzschild black hole \([5]\) when one uses the fourth-order coupled equation (23). Actually, the fourth-order derivatives appear because the tensor perturbation \(\bar{h}_{\mu\nu}\) is coupled to its trace \(\bar{h}\). In addition, the Lorentz gauge condition makes the
stability analysis difficult because one needs to choose the Regge-Wheeler (RW) gauge. This amounts to double gauge-fixings. Hence, we must use the linearized equation (18) to analyze the black hole stability with the RW gauge. As was mentioned previously, this task is not available to be performed because Eq.(18) is a fourth-order differential equation with respect to \( h_{\mu\nu} \). One way to perform the stability of the Schwarzschild black hole is to use two Starobinsky’s linearized equations (27) and (28). In this case, it is important to note that the tensor perturbation \( h_{\mu\nu} \) is completely decoupled from the scalar \( \phi \). The stability analysis based on (27) corresponds to that of the Schwarzschild black hole in Einstein gravity [18, 19, 20]. It turned out that the Schwarzschild black hole is stable against the tensor perturbation. Furthermore, the scalar perturbation based on (28) is stable for the mass-squared \( M^2 \geq 0 \) [37]. Consequently, it means that the Schwarzschild black hole is stable against all perturbations in the specific model of \( f(R) = R + R^2/(6M^2) \).

6 Instability of Kerr black hole

The rotating (Kerr) black hole has been proven to be stable against a massless spin-2 graviton [21, 22, 23] and a massless spin-0 scalar [24]. The stability implies that normal mode solutions were allowed for tensor and scalar propagating on the Kerr black hole background. However, there exists another instability of the superradiance when one considers a massive boson like scalar [25, 26, 27, 28, 29, 30] and vector [31] around the Kerr black hole background. We remind the reader that either (26) or (28) is a massive scalar equation around the Kerr background. Here we wish to focus on the latter equation. Reminding the axis-symmetric background [33], it is convenient to separate the scalar into mode [38]

\[
\varphi(t, r, \theta, \phi) = e^{-i\omega t + im\phi} S_{\ell m}(\theta) R_{\ell m}(r),
\]

where \( S_{\ell m}(\theta) \) are spheroidal harmonics with \(-m \leq \ell \leq m\) and \( R_{\ell m}(r) \) satisfies a radial Teukolsky equation. Temporary, we may choose a positive frequency \( \omega \) of the mode. Plugging (29) into (28), one has the angular and radial equations for \( S_{\ell m}(\theta) \) and \( R_{\ell m}(r) \) as

\[
\frac{1}{\sin \theta} \partial_\theta (\sin \theta \partial_\theta S_{\ell m}) + \left[ a^2(\omega^2 - M^2) \cos^2 \theta - \frac{m^2}{\sin^2 \theta} + A_{\ell m} \right] S_{\ell m} = 0,
\]

\[
\Delta \partial_r (\Delta \partial_r R_{\ell m}(r)) = [\Delta U - K^2] R_{\ell m}(r)
\]
with \( U = M^2(r^2 + a^2) - 2am\omega + A_{lm} \) and \( K = \omega(r^2 + a^2) - am \). Here \( A_{lm} \) is the separation constant whose form is given by \[ A_{lm} = l(l + 1) + \sum_{k=1}^{\infty} c_k a^{2k}(M^2 - \omega^2)^k \] for \( \omega \approx M \) only. The Teukolsky equation takes the Schrödinger form \[ -\frac{d^2\psi}{dr_*^2} + V(r, \omega)\psi = E\psi, \quad \psi(r) = \sqrt{\Delta} R(r) \] when the tortoise coordinate \( r_* \) is implemented by \( dr_* = \frac{r^2 + a^2}{\Delta} dr \) and \( E = \omega^2 \). Here, the \( \omega \)-dependent potential \( V_\omega(r) \) is given by \[ V_\omega(r) = \omega^2 + \frac{\Delta U - K^2 - \dot{M}^2 + a^2}{\Delta^2}. \] The approximate form of \( V_\omega - E \) is given when keeping \( 1/r \)-order \[ -E + V_{\omega}^{\text{app}} \rightarrow -\omega^2 + M^2 - \frac{2\dot{M}(2\omega^2 - M^2)}{r}, \quad r_* \rightarrow \infty \quad (r \rightarrow \infty), \] while its form near the event horizon is \[ -E + V_\omega \rightarrow (\omega - m\Omega)^2, \quad r_* \rightarrow -\infty \quad (r \rightarrow r_+). \] At this stage, considering the qualitative shape of potential \( V_\omega(r_*) \) in Fig. 2 (see Fig.15 of Ref.\[17\] and Fig.7 of Ref.\[41\]), we could define quasibound states. We note that the shape of \( V_\omega(r_*) \) is slightly different from \( V_\omega(r) \) because \( r_* \) goes from \(-\infty \) to \( \infty \) while \( r \in \{r_+, \infty\} \). For this purpose, we impose the two boundary conditions of purely ingoing waves near the horizon and a decaying (bounded) solution at spatial infinity \[ \psi_{\{\infty\}} = e^{-\frac{1}{\sqrt{M^2 - \omega^2}}r_*, \quad r_* \rightarrow \infty. \] Since the boundary condition at the event horizon is a purely ingoing wave, the Kerr black holes do not admit bound states with real frequency \( \omega \). But, they do admit quasibound states which have complex \( \omega = \omega_R + i\omega_I \) with a negative imaginary part \( (\omega_I < 0) \), implying that the decaying field of \( e^{-i\omega_I t}e^{\omega_R t} \) is infalling into the black holes. For the Kerr black hole, however,
Figure 2: Qualitative shape of the Starobinsky potential $V_\omega(r_*)$. In the limit of $r_* \to \infty$, one finds $V(r_*) \to M^2$, a trapping potential well, a potential barrier, and a potential well in the ergoregion. Here, quasibound states appear for $\omega^2 < M^2$ because the depth of potential well in the ergoregion is deeper than the depth of trapping potential well.

there exists a critical frequency from (37): $\omega_R = \omega_c (= m\Omega)$ with $\omega_I = 0$, showing that there is no scalar flux into the black hole. For $\omega_R < \omega_c$, $\omega_I$ becomes positive, implying that the growing field is falling into black hole. This is the superradiant regime. It is a feature of the rotating black hole, but all quasibound states on the non-rotating (Schwarzschild) black hole are found to be decaying. Importantly, the existence of superradiant modes can be converted into an instability of the black hole if a mechanism to trap these modes in a vicinity of the black hole is provided. There are two mechanisms to achieve it. If one surrounds the black hole by putting a reflecting mirror, the wave will bounce back and forth between black hole and mirror, amplifying itself each time and eventually producing a non-negligible backreaction on the black hole background. It is not considered as a perturbation, but it shows a signal for instability of the black hole. Secondly, the nature may provide its own mirror when one introduces a massive scalar. For $\omega < M$, the mass term works as a mirror effectively.

We remember that any instability must set in via a real frequency mode and thus, we consider modes with $|\omega_I| \ll \omega_R$ which implies $\omega^2 \approx \omega_R^2$. From (38), a bound state of exponentially decaying mode at spatial infinity is characterized by the condition

$$\omega^2 < M^2. \quad (39)$$

The three boundary conditions (37)-(39) imply a discrete set of resonances $\{\omega_n\}$ which corresponds to bound states of the linearized scalar.
More precisely, according to the Hod’s argument [42], two conditions are necessary to trigger the instability of the Kerr black hole when one considers a massive scalar perturbation: i) The existence of an ergoregion where superradiant amplification of the waves takes place. ii) A trapping potential well for quasibound states should exist between the potential barrier from ergoregion and potential barrier from the mass (see Fig. 2). The first condition is implemented by the superradiance condition of $\omega < m \Omega$. The second one is supplied by the condition of the quasibound states for modes in the regime. This condition can be achieved by considering both (39) which states that $\omega^2$ is less than the potential height $V_{\omega}^\infty = M^2$ at $r = \infty$ and that its approximate derivative must be zero ($dV_{\omega}^{\text{app}}/dr \to 0^+$) as $r \to \infty$. Thus, one has the condition of

$$\frac{M^2}{2} < \omega^2 < V_{\omega}^\infty \to \frac{M^2}{2} < \omega^2 < M^2. \quad (40)$$

Combining the superradiance condition with (40), one finds a restricted range for the mass

$$M < \sqrt{2} \omega < \sqrt{2} m \Omega \quad (41)$$

which implies an inequality between mass $M$ of the scalar and angular velocity $\Omega$ of the Kerr black hole

$$M < \sqrt{2} m \Omega \quad (42)$$

for the instability condition of the rotating black hole. On the other hand, the stability condition is given by

$$M \geq \sqrt{2} m \Omega. \quad (43)$$

7 Discussions

We have started with a specific model of $f(R) = R + R^2/(6M^2)$ which is a fourth-order gravity theory. Considering a process of $[R^2 \to R\psi - 3\psi^2 \to -(\partial \phi)^2 - V]$ have led to the Starobinsky model which is a second-order scalar-tensor theory in the Einstein frame. This is a famous inflation model. Even though the stability of Schwarzschild black hole was not carried out completely within the perturbed $f(R)$ gravity, its stability was confirmed within the perturbed Starobinsky model.

In the same spirit, the stability analysis of the Kerr black hole was performed in the Starobinsky model because the stability analysis is a formidable task in the framework of the perturbed $f(R)$ gravity. The superradiant instability of the Kerr spacetime is a consequence of massive modes that are
trapped inside the potential well which exists between the potential barrier from ergoregion and potential barrier from the mass at infinity. In this case, quasibound states appear because the depth of the potential well (near horizon) outside the ergoregion potential barrier is deeper than the depth of trapping potential well.

Finally, we wish to mention other models of \( f_p(R) = R + \lambda R^p \) [43]. The corresponding Starobinsky potential \( V_p \) for \( 1 < p < 2 \) is steeper than \( p = 2 \) for large \( \phi \) (see Fig.1), while \( V_p \) for \( p > 2 \) decreases for large \( \phi \) and it approaches zero [43]. Although the corresponding potentials have different shapes for large \( \phi \), they have the same behavior around \( \phi = 0 \) with \( V_p(0) = 0 \) which means that the two black holes come out as the solution. However, their stability analysis seems to be unclear because \( \delta V'_p \sim \varphi^{p-1} + \varphi^{p-1} \) provides non-integer power mass terms. The \( p = 2 \) case of our work leads to \( \delta V' \sim \varphi \). In other word, we could not obtain a regular Klein-Gordon equation for \( f_p(R) \) gravity. This is closely related to the fact that \( R^p \) for \( 1 < p < 2 \) could not lead to \( \phi \).

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