Baryon Masses in the 1/$N_c$ Expansion

E. Jenkins

$^{a}$Department of Physics, University of California, San Diego, 9500 Gilman Drive, La Jolla, California 92093-0319 USA

The masses of baryons and heavy quark baryons are studied analytically in an expansion in 1/$N_c$, SU(3) flavor symmetry breaking, and heavy-quark symmetry breaking. The measured baryon masses are in striking agreement with the 1/$N_c$ hierarchy.

1. INTRODUCTION

Large-$N_c$ spin-flavor symmetry has proven to be a useful symmetry for studying the properties of baryons in QCD [1]. Baryon observables can be analyzed in an expansion involving a complete operator basis with each operator transforming according to a definite representation under the unbroken spin $\otimes$ flavor subgroup and suppressed by a definite power of 1/$N_c$. From the 1/$N_c$ expansion, it is possible to determine the symmetry structure of 1/$N_c$ corrections which account for deviations from the large-$N_c$ spin-flavor limit. By including additional parameters to account for explicit flavor symmetry breaking, one obtains a combined operator expansion in 1/$N_c$ and flavor symmetry breaking. The symmetry relations which occur when one neglects individual operators in the combined 1/$N_c$ and flavor-symmetry breaking expansion are predicted to hold at a definite order in 1/$N_c$ and SU(3) flavor symmetry breaking. The hierarchy of symmetry relations predicted by this procedure results in a nontrivial pattern for baryons in QCD since 1/$N_c$ = 1/3 and the SU(3) flavor symmetry breaking $\epsilon \equiv m_s/\Lambda_{\chi}$ are comparable in magnitude, so the pattern of relations is not dominated by either 1/$N_c$ or $\epsilon$, and neither parameter can be neglected relative to the other.

The hierarchy of baryon mass relations predicted by the combined expansion in 1/$N_c$ and flavor symmetry breaking has been obtained for the lowest-lying baryon spin-flavor multiplet, which is the 56 of SU(6) for $N_c = 3$ colors and $N_f = 3$ light quark flavors. This spin-flavor multiplet contains the spin-1/2 octet and spin-3/2 decuplet baryon spin $\otimes$ flavor representations. The mass hierarchy predicted for these ground-state baryons is particularly predictive since the expansion extends to relative order 1/$N_c^3$ in the 1/$N_c$ expansion and to third order in SU(3) flavor symmetry breaking. The mass spectrum of the spin-1/2 octet and the spin-3/2 decuplet baryons is in striking accord with the 1/$N_c$ hierarchy. Suppression factors of 1/$N_c = 1/3$ are clearly evident in the data, which provides the strongest single piece of evidence for the relevance of the 1/$N_c$ expansion for QCD baryons where 1/$N_c = 1/3$. Even more importantly, the 1/$N_c$ expansion resolves the long-standing puzzle of why certain famous mass relations of the spin-1/2 octet and spin-3/2 decuplet baryons, such as the Gell-Mann–Okubo formula, Gell-Mann’s Equal Spacing Rule, the Okubo formula and the Coleman-Glashow relation, work so extraordinarily well, which is to say, better than one would have expected based on flavor symmetry breaking suppression factors alone. The point is that the operators in the baryon mass expansion which are highly suppressed in SU(3) flavor symmetry breaking parameters also are suppressed by powers of 1/$N_c$, so that, generically, mass relations which are highly suppressed in flavor are highly suppressed in 1/$N_c$ as well.

This proceedings provides a brief description of progress that has been made in understanding...
the mass spectrum of QCD baryons in an analytic expansion in $1/N_c$ and flavor symmetry breaking. A brief description of the $1/N_c$ operator expansion for baryons is provided, with the example of the operator expansion for the $I = 0$ masses of the ground state baryons presented in detail. Evidence of the $1/N_c$ mass hierarchy of the baryon octet and decuplet is obvious for the $I = 0$ baryon mass combinations, and recent experimental evidence supporting the predicted $1/N_c$ hierarchy in the $I = 1$ sector is noted for the isospin-breaking Coleman-Glashow mass splitting. Finally, the extension of the $1/N_c$ expansion to baryons containing a single heavy quark in Heavy Quark Effective Theory (HQET) is summarized briefly, and then applied to the charm and bottom baryon mass spectra. Using the $1/N_c$ expansion, a prediction was made for the $\Xi_c$ mass valid to a precision of a couple MeV, which has recently been confirmed experimentally.

2. $1/N_c$ Expansion

A contracted $SU(2N_f)$ spin-flavor algebra for baryons arises in the large-$N_c$ limit. For the simplest case of $N_f = 2$ light quark flavors, the spin-flavor generators are given by baryon spin $J^i$, isospin $I^a$ and commuting generators $X^{ia}$. The contracted large-$N_c$ spin-flavor algebra is given by

$$[X^{ia}, X^{jb}] = 0,$$

$$[J^i, X^{ja}] = i\epsilon^{ijk} J_k X^{ja}, \quad [I^a, X^{ib}] = i\epsilon^{abc} X^{ib}, \quad [J^i, J^j] = i\epsilon^{ijk} J_k, \quad [I^a, I^b] = i\epsilon^{abc} I^c,$$  \hspace{1cm} (1)

where the commuting generators $X^{ia}$ are obtained by rescaling the axial vector baryon currents $A^{ia}$, which grow as $O(N_c)$, by $1/N_c$ and taking the large-$N_c$ limit, which yields a finite well-defined result:

$$X^{ia} \equiv \lim_{N_c \rightarrow \infty} \frac{A^{ia}}{N_c}.$$  \hspace{1cm} (2)

When working at finite $N_c$, it is useful to work with the operator

$$G^{ia} = \left( \bar{q} \gamma^i \gamma^a \frac{\sigma \cdot T}{4} q \right)$$  \hspace{1cm} (3)

of the large-$N_c$ quark model, and to use the uncontracted $SU(2N_f)$ algebra of the quark model. In this case, $G^{ia}/N_c$ has finite baryon matrix elements equal to $X^{ia}$ in the large-$N_c$ limit.

The $1/N_c$ expansion of an arbitrary QCD 1-body operator is given by

$$\bar{q} \Gamma q = N_c \sum_{n=0}^{N_c} \frac{1}{N_c^n} c_n \left( \frac{1}{N_c} \right) O_n,$$  \hspace{1cm} (4)

where $O_n$ is an $n$-body quark operator which equals an $n^{th}$ order polynomial in the spin-flavor generators $J^i$, $T^a$, $G^{ia}$, and the operator coefficients $c_n(1/N_c) = O(1) + \cdots$ are unknown, but have a $1/N_c$ expansion beginning at order unity. The baryon matrix elements of the operators $O_n$ are known, since the baryon matrix elements of the spin and flavor generators are known, as are the matrix elements of $G^{ia}$. (Only the leading $O(1)$ matrix elements of the spin-flavor generators $X^{ia}$ are determined; the matrix elements of the spin-flavor generators at subleading orders are not, but any reasonable choice will do. The large-$N_c$ quark model generator $G^{ia}$ or the Skyrme model collective coordinate $X^{ia}_0$ are two possible choices.) The complete set of independent $n$-body operator products are classified. There is a single 0-body operator 1. The independent 1-body operators are the $SU(6)$ spin-flavor generators $J^i$, $T^a$ and $G^{ia}$, and $n$-body operators are products of these 1-body operators. Since not all $n$-body operator products are independent, operator identities are needed to reduce the operator products to an independent set. The complete set of operator identities is known.

3. Baryon Mass Hierarchy

The baryon mass operator transforms as a spin singlet. $SU(3)$ flavor symmetry breaking proportional to the parameter $\epsilon$ transforms as the eighth component of an $SU(3)$ octet, so the baryon mass operator expansion also contains $SU(3)$ flavor-symmetry breaking operators which extend to third order in $SU(3)$ breaking and transform as the $8$, $27$ and $64$ representations of $SU(3)$. The baryon $I = 0$ masses can be organized according to their $SU(3)$ representation,

$$M = M^1 + M^8 + M^{27} + M^{64},$$  \hspace{1cm} (5)
with $1/N_c$ operator expansions

\[ M^1 = N_c \mathbf{1} + \frac{1}{N_c}J^2, \]
\[ M^8 = T^8 + \frac{1}{N_c} \{ J^p, G^{8s} \} + \frac{1}{N_c^2} \{ J^2, T^8 \}, \]
\[ M^{27} = \frac{1}{N_c} \{ T^8, T^8 \} + \frac{1}{N_c^2} \{ T^8, \{ J^p, G^{8s} \} \}, \]
\[ M^{64} = \frac{1}{N_c^2} \{ T^8, \{ T^8, T^8 \} \}, \]

where each operator in the $1/N_c$ expansion is understood to be accompanied by a coefficient, which has been suppressed in Eq. (6) for simplicity. The above operator expansion consists of eight independent operators corresponding to the eight $I = 0$ baryon masses $N, \Lambda, \Sigma, \Xi, \Delta, \Sigma^*, \Xi^*$ and $\Omega$. Each mass operator occurs at a definite order in $1/N_c$ and $\epsilon$ and corresponds to one specific mass splitting, so the combined expansion in $1/N_c$ and $\epsilon$ predicts a hierarchy of $I = 0$ masses. A similar analysis can be performed for the $I = 1$ and $I = 2$ mass splittings of the octet and decuplet baryons. The analysis introduces two new parameters $\epsilon'$ and $\epsilon''$ which are $I = 1$ and $I = 2$ flavor symmetry breaking parameters, respectively. The two parameters are expected to be similar in magnitude.

Figure 1 plots sixteen independent mass splittings of the spin-1/2 octet and the spin-3/2 decuplet. The plot shows, relative to the leading $O(N_c)$ baryon mass, $I = 0$ mass splittings of order $\frac{\epsilon}{N_c}, \frac{\epsilon^p}{N_c}, \frac{\epsilon}{N_c} T^8, \frac{\epsilon^2}{N_c}, \frac{\epsilon'}{N_c}, \frac{\epsilon'}{N_c} T^8,$ and $\frac{\epsilon''}{N_c}$; $I = 1$ mass splittings of order $\frac{\epsilon'}{N_c}, \frac{\epsilon'}{N_c}, \frac{\epsilon'}{N_c}, \frac{\epsilon'}{N_c}, \frac{\epsilon'}{N_c}, \frac{\epsilon'}{N_c},$ and $\frac{\epsilon'}{N_c};$ and $I = 2$ mass splittings of order $\frac{\epsilon''}{N_c}, \frac{\epsilon''}{N_c}, \frac{\epsilon''}{N_c},$ and $\frac{\epsilon''}{N_c},$ respectively.

The first seven points in Figure 1, the $I = 0$ mass splittings, are in striking agreement with the hierarchy predicted by the $1/N_c$ and $\epsilon$ expansions. The singlet mass splitting of relative order $1/N_c^2$ is comparable in magnitude to the octet mass splitting of relative order $\epsilon/N_c$. The three flavor-octet mass splittings proportional to one power of $\epsilon$ are in accord with the predicted $1/N_c$ hierarchy of $1/N_c, 1/N_c^2,$ and $1/N_c^3$. If one did not know about the $1/N_c$ expansion and arbitrarily looked at three different flavor-octet mass splittings, they would generically all contain some portion of the $1/N_c$ mass combination and therefore be of this relative magnitude. Only when examining the linear combinations picked out by the $1/N_c$ expansion does the $1/N_c$ hierarchy become apparent. The two flavor-27 mass splittings are proportional to two powers of $\epsilon$, and occur at relative orders $1/N_c^2$ and $1/N_c^3$ in the $1/N_c$ expansion. These two flavor-27 mass splittings of the $1/N_c$ expansion are linear combinations of the Gell-Mann–Okubo flavor-27 mass splitting of the spin-1/2 baryon octet

\[ \frac{1}{4} (2N - \Sigma - 3\Lambda + 2\Xi), \]

and the flavor-27 Equal Spacing Rule mass splitting of the spin-3/2 baryon decuplet

\[ \frac{1}{7} (4\Delta - 5\Sigma^* - 2\Xi^* + 3\Omega). \]
Since these two famous mass splittings are linear combinations of the two $1/N_c$ mass splittings, they are both suppressed by a factor of $1/N_c^2$ in addition to the flavor suppression factor $\epsilon^2$. Thus, the $1/N_c$ expansion predicts that these flavor-27 mass splittings are about an order of magnitude smaller than expected from an analysis of flavor-symmetry breaking alone. The single flavor-64 mass splitting
\[
\frac{1}{4} (\Delta - 3\Sigma^* + 3\Xi^* - \Omega)
\] is third order in $SU(3)$ flavor symmetry breaking $\epsilon^3$ and relative order $1/N_c^3$ in the $1/N_c$ expansion. The experimental data clearly show that the mass splitting is suppressed by a greater factor than expected from flavor-symmetry breaking alone, and the observed suppression is consistent with the $1/N_c$ hierarchy. A better measurement of this splitting would reduce the experimental error bar and test the $1/N_c^3$ prediction of the $1/N_c$ expansion.

There also is clear evidence for the $1/N_c$ hierarchy in the $I = 1$ mass splittings $\Xi^0$. A new, very precise, measurement of $\Xi^0 = 1314.82 \pm 0.06 \pm 0.2$ MeV significantly reduces the errors on the $I = 1$ mass combinations plotted in Fig. 1. For the first time, the Coleman-Glashow mass splitting
\[
[p - n] - (\Sigma^+ - \Sigma^-) + (\Xi^0 - \Xi^-)
\] is measured to be non-zero, though only at the $1\sigma$ level. The Coleman-Glashow mass combination, the eleventh point in Fig. 1, is predicted to be of relative order $\frac{\epsilon^3}{N_c^2}$, or suppressed by a factor of $\frac{1}{N_c}$ relative to the leading $I = 1$ mass splittings, the eighth and ninth points, which are of relative order $\frac{\epsilon^2}{N_c}$. A similar suppression factor is predicted and observed for the $I = 1$ mass splitting that is plotted as the tenth point in Fig. 1. The leading $I = 2$ mass splitting, point fourteen, is also in line with the prediction of the $1/N_c$ expansion. Remaining points have large error bars. Improved measurements of a number of isospin mass splittings, particularly of the $\Delta$ baryons, would reduce these errors and provide a further test of the $1/N_c$ hierarchy.

4. HEAVY QUARK BARYONS

Large-$N_c$ baryons containing a single heavy quark respect a large-$N_c$ SU(6)$_f$ × SU(4)$_h$ spin-flavor symmetry $\bar{1}$ with light-quark generators $J^i$, $T^a$ and $G^{ia}$ and heavy-quark generators
\[
J^i_h = Q^n_1 \frac{\sigma^i}{2} Q = J^i_c + J^i_b,
\]
\[
I^a_h = Q^n_1 \frac{\tau^a}{2} Q,
\]
\[
G^{ia}_h = Q^n_1 \frac{\sigma^i\tau^a}{4} Q.
\]
Light-quark spin-flavor symmetry is valid in the $N_c \to \infty$ limit, whereas the heavy-quark spin-flavor symmetry is valid in the large-$N_c$ limit and the heavy-quark limit of HQET.

The $1/N_c$ operator expansion for a baryon with $Qqq$ flavor quantum numbers involves operator products which are at most 1-body in the heavy-quark generators and at most 2-body in the light-quark generators. Light-quark generators are accompanied by a factor $1/N_c$, and heavy-quark generators are accompanied by a factor of $1/N_c$ and $\Lambda/m_Q$ since the heavy-quark spin-flavor generators separately violate large-$N_c$ heavy-quark spin-flavor symmetry and heavy-quark spin-flavor symmetry of HQET. Heavy-quark flavor symmetry breaking for the $Q = c$ and $Q = b$ baryons enters through the generators
\[
I^3_h = \frac{1}{2} (N_{\text{charm}} - N_{\text{bottom}}),
\]
\[
G^{3a}_h = \frac{1}{2} (J^a_b - J^a_b),
\]
whereas light-quark SU(3) flavor symmetry breaking enters through light-quark generators which transform as the eighth or third component of an SU(3) octet. The $1/N_c$ mass hierarchy can be analyzed separately for charm- and bottom-quark baryons using only heavy-quark spin symmetry, or together using heavy-quark spin-flavor symmetry. The joint expansion leads to relations between $Q = b$ and $Q = c$ mass splittings such as
\[
\frac{1}{3} (\Sigma_b + 2\Sigma_c) - \Lambda_b = \frac{1}{3} (\Sigma_c + 2\Sigma_c') - \Lambda_c.
\]
pansion which involves the heavy-quark number operator

\[ N_h = Q^\dagger Q = N_{\text{charm}} + N_{\text{bottom}}. \]  (13)

This analysis leads to relations between heavy-quark baryon mass splittings and mass splittings of the spin-1/2 octet and the spin-3/2 decuplet, such as

\[ \frac{1}{3} (\Sigma_Q + 2\Sigma_Q^*) - \Lambda_Q = \frac{2}{3} (\Delta - N). \]  (14)

The 1/$N_c$ expansion for heavy-quark baryon masses successfully predicted the Ξ$_c^\prime$ mass: the predicted value of Ξ$_c^\prime = 2580.8 \pm 2.1$ MeV is in excellent agreement with the subsequent measured value of Ξ$_c^\prime = 2576.5 \pm 2.3$ MeV. With this measurement, all of the lowest-lying charm baryon masses are measured except for Ω$_c^*$. The Ω$_c^*$ mass can be determined in terms of the other measured charm baryon masses using the mass relation

\[ \frac{1}{4} [(\Sigma_c^* - \Sigma_c) - 2(\Xi_c^* - \Xi_c') + (\Omega_c^* - \Omega_c)] , \]  (15)

which is predicted to hold at the sub-MeV level using the 1/$N_c$ expansion, and therefore is extremely accurate and can be taken to be exact for most purposes. Presently, the extraction of the Ω$_c^*$ mass using this mass relation is dominated by experimental uncertainties in the Ω$_c$ mass, whose PDG value of Ω$_c = 2704 \pm 4$ MeV is 2.5σ away from the new CLEO measurement of Ω$_c = 2694.6 \pm 3.5$ MeV. These two experimental values yield central values for the Ω$_c^*$ of 2768 and 2777 MeV, respectively. Finally, all of the $Q = b$ baryons (given the measured Λ$_b$ mass) can be predicted from the charm baryon masses to a precision of about 10 MeV, which is set by the accuracy of the worst $Q = c$ and $Q = b$ mass relation. Observation of some of the $Q = b$ baryons will allow a determination of the remaining unobserved $Q = b$ baryon masses to greater precision.

5. CONCLUSIONS

The 1/$N_c$ expansion of QCD has provided an understanding of the mass spectrum of the spin-1/2 octet and spin-3/2 decuplet baryons. The mass hierarchy of baryons is well-described by an expansion in 1/$N_c$ about the large-$N_c$ spin-flavor limit and an expansion in $SU(3)$ flavor symmetry breaking about the $SU(3)$ flavor symmetry limit. 1/$N_c$ suppression factors are necessary to understand the relative magnitudes of the various baryon mass splittings of the baryon octet and decuplet. In the case of baryons containing a single heavy quark in HQET, there is an additional expansion parameter $\Lambda/m_Q$ which describes corrections to the heavy-quark spin-flavor symmetry limit. The hierarchy of heavy-quark baryon masses predicted from the violation of large-$N_c$ light- and heavy-quark spin-flavor symmetries is sufficiently restrictive that it can be used to predict heavy-quark baryon masses at a level which is interesting experimentally. The combined 1/$N_c$ and flavor-symmetry breaking expansions successfully predicted the Ξ$_c^\prime$ mass to an accuracy of several MeV. Additional predictions for charm and bottom baryon masses can and will be tested in the future.

REFERENCES

1. E. Jenkins, Ann. Rev. Nucl. Part. Sci. 48 (1998) 81.
2. E. Jenkins and R.F. Lebed, Phys. Rev. D52 (1995) 282.
3. E. Jenkins and R.F. Lebed, Phys. Rev. D62:077901 (2000).
4. NA48 Collaboration, Eur. Phys. J. C12 (2000) 69.
5. E. Jenkins, Phys. Rev. D54 (1996) 4515.
6. E. Jenkins, Phys. Rev. D55 (1997) 10.