Pattern polynomial graphs

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Abstract

A graph \( X \) is said to be a pattern polynomial graph if its adjacency algebra is a coherent algebra. In this study we will find a necessary and sufficient condition for a graph to be a pattern polynomial graph. Some of the properties of the graphs which are polynomials in the pattern polynomial graph have been studied. We also identify known graph classes which are pattern polynomial graphs.

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1 Introduction and preliminaries

Let \( M_n(\mathbb{C}) \) denote the set of all \( n \times n \) matrices over the field of complex numbers \( \mathbb{C} \). Let \( \mathbb{C}[A] \) denote the set of all matrices which are polynomials in \( A \) with coefficients from \( \mathbb{C} \). Clearly \( \mathbb{C}[A] \) is an algebra over \( \mathbb{C} \) for any \( A \in M_n(\mathbb{C}) \). The dimension of \( \mathbb{C}[A] \) over \( \mathbb{C} \) as a vector space, is the degree of the minimal polynomial of \( A \). If \( A \) is diagonalizable, then from the following lemma, it’s dimension is equal to the number of distinct eigenvalues of \( A \).

**Lemma 1.1** (Hoffman & Kunze [10]). *A matrix is diagonalizable if and only if its minimal polynomial has all distinct linear factors over \( \mathbb{C} \).*

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Definition 1.1. Hadamard product of two \( n \times n \) matrices \( A \) and \( B \) is denoted by \( A \odot B \) and is defined as \( (A \odot B)_{xy} = A_{xy}B_{xy} \).

Two \( n \times n \) matrices \( A \) and \( B \) are said to be disjoint if their Hadamard product is the zero matrix.

Definition 1.2. A subalgebra of \( M_n(\mathbb{C}) \) is called coherent if it contains the matrices \( I \) and \( J \) and if it is closed under conjugate-transposition and Hadamard multiplication. Here \( J \) denotes the matrix with every entry being 1.

Theorem 1.2. \([4]\) Every coherent algebra contains unique basis of mutually disjoint 0, 1- matrices (matrices with entries either 0 or 1).

We call this unique basis of mutually disjoint 0-1 matrices as a standard basis.

Corollary 1.3. Every 0, 1-matrix in a coherent algebra is the sum of one or more matrices in its standard basis.

Proof. Let \( \mathcal{M} \) be a coherent algebra over \( \mathbb{C} \) with its standard basis \( \{M_1, \ldots, M_t\} \). Let \( B \in \mathcal{M} \) be a 0, 1-matrix, then \( B = \sum_{i=1}^{t} a_i M_i \) where \( a_i \in \mathbb{C} \). \( B = B \odot B = \sum_{i=1}^{t} a_i^2 M_i \Rightarrow a_i^2 = a_i \). Hence the result follows.

Corollary 1.4. If \( \mathcal{M} \) is a commutative coherent algebra over \( \mathbb{C} \), then \( \dim(\mathcal{M}) \leq n \).

Proof. Since \( J \) commutes with every element in \( \mathcal{M} \). Hence all the row(column) sums of every matrix in \( \mathcal{M} \) are equal. Consequently the number of elements in the standard basis of \( \mathcal{M} \) is atmost \( n \), as the matrices in the standard basis of \( \mathcal{M} \) are disjoint and whose sum is \( J \).

Observation 1.5. The intersection of coherent algebras is again a coherent algebra.

Definition 1.3. Let \( A \in M_n(\mathbb{C}) \), then coherent closure of \( A \), denoted by \( \langle\langle A\rangle\rangle \) or \( \mathcal{C}(A) \), is the smallest coherent algebra containing \( A \).

In Section 2 we will see a necessary and sufficient condition for any matrix \( A \in M_n(\mathbb{C}) \) such that \( \mathcal{C}[A] = \mathcal{C}(A) \). In the remaining sections we consider \( A \) to be the adjacency matrix of a graph \( X \). If \( A \) is the adjacency matrix of a graph \( X \) and \( \mathcal{C}[A] = \mathcal{C}(A) \), then \( X \) will be called a pattern polynomial.
Some of the properties of pattern polynomial graphs are given in Section 3. In Section 4 we will see a few graph classes which are pattern polynomial graphs. Few partially balanced incomplete block designs from pattern polynomial graphs are constructed in Section 5. Properties of graphs which are polynomials in the pattern polynomial graph are provided in the Section 6.

2 \( \mathbb{C}[A] = \mathbb{C}C(A) \)

In this section we will see a necessary and sufficient condition for a matrix \( A \in M_n(\mathbb{C}) \) such that \( \mathbb{C}[A] = \mathbb{C}C(A) \). For that we construct a vector space which lies in between \( \mathbb{C}[A] \) and \( \mathbb{C}C(A) \) as follows.

Let \( \ell \) be the degree of the minimal polynomial of \( A \). Then \( \{I, A, \ldots, A^{\ell-1}\} \) is a basis for \( \mathbb{C}[A] \) over \( \mathbb{C} \). Let \( y = (y_0, y_1, \ldots, y_{\ell-1}) \in \mathbb{C}^\ell \) be a variable and

\[
B(y) = y_0 I + y_1 A + \ldots + y_{\ell-1} A^{\ell-1} = \begin{bmatrix} p_{11}(y) & p_{12}(y) & \cdots & p_{1n}(y) \\ p_{21}(y) & p_{22}(y) & \cdots & p_{2n}(y) \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1}(y) & p_{n2}(y) & \cdots & p_{nn}(y) \end{bmatrix},
\]

Where \( p_{ij}(y) = p_{ij}(y_0, y_1, \ldots, y_{\ell-1}) \) is a polynomial in the variables \( y_0, y_1, \ldots, y_{\ell-1} \).

Let us assume \( \{q_1(y), q_2(y), \ldots, q_r(y)\} \) be the set of distinct polynomials in the matrix \( B(y) \). We define the matrices called pattern matrices of \( A \), as

\[
(P_j)_{s,t} = \begin{cases} 
1, & \text{if } B(y)_{s,t} = q_j(y), \\
0, & \text{otherwise}.
\end{cases}
\]

Let \( \mathcal{L}(A) = L\{P_1, P_2, \ldots, P_r\} \) denote the linear span of matrices \( P_1, \ldots, P_r \). Then \( \mathcal{L}(A) \) is a subspace of \( M_n(\mathbb{C}) \). From the definition of \( \mathcal{L}(A) \) we have the following observation.

**Observation 2.1.**

1. \( P_i \odot P_j = 0 \) \( \forall i, j \ 1 \leq i, j \leq r, \sum_{i=1}^{r} P_i = J \in \mathcal{L}(X), I \in \mathcal{L}(A) \).

2. \( \mathcal{L}(A) \) is closed under Hadamard product.

**Proof.** Let \( M, N \in \mathcal{L}(A) \), so \( M = \sum_{i=1}^{r} a_i P_i \), \( N = \sum_{i=1}^{r} b_i P_i \) where \( a_i, b_i \in \mathbb{C} \). Then \( M \odot N = \sum_{i=1}^{r} a_i b_i P_i \in \mathcal{L}(A) \). \( \square \)
3. $\mathcal{L}(A)$ is the smallest subspace of $M_n(\mathbb{C})$ closed under Hadamard product and contains all powers of $A$. Consequently $\mathbb{C}[A] \subseteq \mathcal{L}(A) \subseteq \mathcal{C}(A)$ and $\ell \leq r$.

4. If $P_i^T \in \{P_1, P_2, \ldots, P_r\}$ for all $1 \leq i \leq r$, then $\mathcal{L}(A)$ is also closed under conjugate transposition. In particular, if $A$ is symmetric, then all pattern matrices are symmetric hence $\mathcal{L}(A)$ is closed under conjugate transposition.

With the above observation, we are providing the main result of this section.

**Theorem 2.2.** Let $A \in M_n(\mathbb{C})$ be a symmetric matrix. Then $\mathbb{C}[A] = \mathcal{C}(A)$ if and only if $\ell = r$.

**Proof.** If $\mathbb{C}[A] = \mathcal{C}(A)$, then $\mathbb{C}[A] = \mathcal{L}(A)$ hence $\ell = r$. Conversely suppose $\ell = r$ then $\mathbb{C}[A] = \mathcal{L}(A)$. In particular $\mathcal{L}(A)$ is closed under ordinary multiplication. Hence from the above observation $\mathcal{L}(A)$ is a coherent algebra but $\mathcal{C}(A)$ is the smallest coherent algebra containing $\mathbb{C}[A]$. Hence the result follows.

### 3 Pattern polynomial graphs

From now onwards we suppose that $A$ (or $A(X)$) is the adjacency matrix of a graph $X$, hence $A$ is a symmetric 0, 1 matrix. We denote $\mathbb{C}[A]$ by $\mathcal{A}(X)$, $\mathcal{C}(A)$ by $\mathcal{C}(X)$ and $\mathcal{L}(A)$ by $\mathcal{L}(X)$. We call $\mathcal{A}(X)$ as the adjacency algebra of the graph $X$, $\mathcal{C}(X)$ as coherent closure of $X$. First we will provide a few results on adjacency algebra of a graph. Then we will see some additional properties of the vector space $\mathcal{L}(X)$. For two vertices $u$ and $v$ of a connected graph $X$, let $d(u, v)$ denote the length of the shortest path from $u$ to $v$. Then the diameter of a connected graph $X = (V, E)$ is $\max\{d(u, v) : u, v \in V\}$. It is shown in Biggs [3] that if $X$ is a connected graph with diameter $d$, then

$$d + 1 \leq \dim(\mathcal{A}(X)) \leq n \quad (1)$$

where $\dim(\mathcal{A}(X))$ is the dimension of $\mathcal{A}(X)$ as a vector space over $\mathbb{C}$.

A graph $X_1 = (V(X_1), E(X_1))$ is said to be isomorphic to the graph $X_2 = (V(X_2), E(X_2))$, written $X_1 \cong X_2$, if there is a one-to-one correspondence $\rho : V(X_1) \rightarrow V(X_2)$ such that $\{v_1, v_2\} \in E(X_1)$ if and only if
\{\rho(v_1), \rho(v_2)\} \in E(X_2). In such a case, \(\rho\) is called an isomorphism of \(X_1\) and \(X_2\). An isomorphism of a graph \(X\) onto itself is called an automorphism. The collection of all automorphisms of a graph \(X\) is denoted by \(\text{Aut}(X)\). It is well known that \(\text{Aut}(X)\) is a group under composition of two maps. It is easy to see that \(\text{Aut}(X) = \text{Aut}(X^c)\), where \(X^c\) is the complement of the graph \(X\). If \(X\) is a graph with \(n\) vertices we can think of \(\text{Aut}(X)\) as a subgroup of \(S_n\). Under this correspondence, if a graph \(X\) has \(n\) vertices then \(\text{Aut}(X)\) consists of \(n \times n\) permutation matrices and for each \(g \in \text{Aut}(X)\), \(P(g)\) will denote the corresponding permutation matrix. The next result gives a method to check whether a given permutation matrix is an element of \(\text{Aut}(X)\) or not.

**Lemma 3.1** (Biggs [3]). Let \(A\) be the adjacency matrix of a graph \(X\). Then \(g \in \text{Aut}(X)\) is an automorphism of \(X\) if and only if \(P(g)A = AP(g)\).

The following result is very useful in this work, which provides the necessary and sufficient condition for a graph to be connected and regular.

**Lemma 3.2** (Biggs [3]). A graph \(X\) is connected regular if and only if \(J \in \mathcal{A}(X)\).

**Corollary 3.3.** If \(X\) is a regular graph then \(J\) is polynomial in either \(A\) or \(A^c\).

**Proof.** For every graph \(X\), either \(X\) or \(X^c\) is connected. Hence the result follow from the above Lemma. \(\square\)

Let \(X\) be a connected regular graph. Then from the above lemma we have \(\mathcal{A}(X^c) \in \mathcal{A}(X)\). Hence we have the following corollary.

**Corollary 3.4.** Let \(X\) be a connected regular graph. Then \(X^c\) is connected if and only if \(\mathcal{A}(X) = \mathcal{A}(X^c)\).

**Definition 3.1.** A graph \(X\) is said to be a pattern polynomial graph if its pattern matrices are polynomials in the adjacency matrix of \(X\).

**Lemma 3.5.** A graph \(X\) is a pattern polynomial graph if and only if \(\mathcal{A}(X) = \mathcal{CC}(X)\).

Consequently if \(X\) is a pattern polynomial graph, then \(\mathcal{A}(X) = \mathcal{L}(X) = \mathcal{CC}(X)\). For any graph \(X\), we have \(\mathcal{L}(X) = \mathcal{L}(X^c)\) and \(\mathcal{CC}(X) = \mathcal{CC}(X^c)\). Hence we have the following result.

**Corollary 3.6.** Let \(X\) be a pattern polynomial graph and \(X^c\) is also connected. Then \(\mathcal{A}(X) = \mathcal{L}(X) = \mathcal{CC}(X) = \mathcal{A}(X^c) = \mathcal{L}(X^c) = \mathcal{CC}(X^c)\).
3.1 Properties of pattern polynomial graphs

In this subsection we prove that if $X$ is a pattern polynomial graph, then $X$ is necessarily a
a) connected regular graph b) distance polynomial graph c) walk regular graph d) strongly distance-balanced graph. e) edge regular graph whenever $A$ is also a pattern matrix. We also show that every pattern polynomial graph except $K_2$ (complete graph with 2 vertices) has at least one multiple eigenvalue. In particular, if $X$ is a pattern polynomial graph with odd number of vertices, then we show that $\dim(A(X)) \leq \frac{n-1}{2}$. Throughout this section we suppose that $X$ is a pattern polynomial graph that is $A(X) = \mathcal{L}(X) = \mathcal{CC}(X)$ or $\ell = r$. From Lemma 3.2 we have the following result.

**Lemma 3.7.** Every pattern polynomial graph is a connected regular graph.

It is easy to see that the converse of the above lemma is not true. The above lemma provides a necessary condition to have $A(X) = \mathcal{L}(X)$. That is if the graph $X$ is either not connected or not regular, then $A(X) \subsetneq \mathcal{L}(X)$. Now we will state another necessary condition stronger than this.

**Distance polynomial graph**

Let $X = (V, E)$ be a connected graph with diameter $d$. The $k$-th distance matrix $A_k(0 \leq k \leq d)$ of $X$, is defined as $(A_k)_{rs} = \begin{cases} 1, & \text{if } d(v_r, v_s) = k \\ 0, & \text{otherwise.} \end{cases}$

It follows that $A_0 = I$ (Identity matrix), $A_1 = A$, $A_0 + A_1 + \cdots + A_d = J$.

A connected graph $X$ of diameter $d$ is said to be a distance polynomial graph if $A_k \in A(X)$ for $0 \leq k \leq d$. From the Lemma 3.2, the following observation is evident.

**Observation 3.8.** Every distance polynomial graph is a regular connected graph. The converse is generally not true but every regular connected graph of diameter 2 is distance polynomial.

**Lemma 3.9.** If $X$ be a connected graph of diameter $d$, then $A_k \in \mathcal{L}(X)$ ($0 \leq k \leq d$), where $A_k$ is the $k$-th distance matrix of $X$. 

Proof. We prove the result by induction on $d$. $A_0 (= I), A_1 (= A) \in \mathcal{A}(X) \subseteq \mathcal{L}(X)$. So the result is true for $d = 1$. Suppose that $A_0, A_1, \ldots, A_{s-1} \in \mathcal{L}(X)$. Then $J - I - A_0 - A_1 - \cdots - A_{s-1} \in \mathcal{L}(X)$. Let $M = A^s \circ (J - I - A_1 - \cdots - A_{s-1}) \in \mathcal{L}(X)$. Observe that $M_{ij} \neq 0$ if and only if $(A_s)_{ij} = 1$. As $M \in \mathcal{L}(X)$ we have

$$M = \sum_{i=1}^{r} a_i P_i$$

where $a_i \in \mathbb{C}$. \hfill (2)

Hence $A_s = \sum_{i:a_i \neq 0} P_i \in \mathcal{L}(X)$. \hfill \[QED\]

Now from Corollary 1.3 every distance matrix is the sum of one or more pattern matrices.

**Corollary 3.10.** Every pattern polynomial graph is a distance polynomial graph.

From Observation 3.8, every connected regular graph of diameter 2 is distance polynomial. But all connected regular graphs of diameter 2 are not pattern polynomial graphs.

**Definition 3.2** (Paul M. Weichsel [15]). Let $v$ be a vertex in the graph $X$ of diameter $d$. The generalized degree of $v$ is the $d$-tuple $(k_1, k_2, \ldots, k_d)$, where $k_i$ is the number of vertices whose distance from $v$ is $i$. The graph $G$ is called super-regular if each vertex has the same generalized degree.

**Theorem 3.11** (Paul M. Weichsel [15]). Let $X$ be a connected graph of diameter $d$. $X$ is super-regular graph if and only if $(A_i A_j)_{rs} = (A_j A_i)_{rs}$ for all $r, s$.

From the Corollary 3.10 every pattern polynomial graph is a super regular graph. In the present literature super-regular graphs are also called distance-degree regular or strongly distance-balanced graphs for details refer [14].

**Walk-regular graph**

A graph $X$ is said to be walk-regular if for each $s$, the number of closed walks of length $s$ starting at a vertex $v$ is independent of the choice of $v$.

**Theorem 3.12.** [9] Let $A$ be the adjacency matrix of a graph $X$. Then $X$ is walk-regular if and only if the diagonal entries of $A^s \forall s$ are all equal.

The following lemma and corollary can be obtained from the Corollary 1.4 and above theorem.
Lemma 3.13. Every pattern polynomial graph is a walk-regular graph.

Corollary 3.14. If $X$ is pattern polynomial graph then every pattern matrix other than the identity matrix is the adjacency matrix of a regular graph.

Remark 3.15. Let $X$ be a pattern polynomial graph and $P \in \mathcal{A}(X)$ be a permutation matrix. Then from Corollary 1.3 and from the above result $P$ is an element in the standard basis of $\mathcal{A}(X)$. Further it is easy to see that the set of all permutation matrices in $\mathcal{A}(X)$ forms an elementary abelian 2-group since matrices in $\mathcal{A}(X)$ are symmetric.

Let $X$ be a pattern polynomial graph and $\{I = P_1, P_2, \ldots, P_r\}$ be the set of its pattern matrices. Let us call the graph $X_{P_i}$ $2 \leq i \leq r$ as pattern graph of $X$ with adjacency matrix $P_i$. Then form Lemma 3.1, we have $\text{Aut}(X) \subseteq \text{Aut}(X_{P_i})$. In fact in the next section we will show that $\text{Aut}(X) \subseteq \text{Aut}(X_{P_i})$ is true even if $X$ is not a pattern polynomial graph. Now we will show that every pattern polynomial graph except $K_2$ has at least one multiple eigenvalue. In order to prove this, we need the following definition and the result.

Definition 3.3. A graph is said to be vertex transitive if its automorphism group acts transitively on $V$. That is for any two vertices $x, y \in V, \exists g \in G \text{ such that } g(x) = y$.

Lemma 3.16. [Biggs [3]] Let $X$ be a $k$-regular vertex transitive graph, and $\lambda$ be a simple eigenvalue of $X$. Then

$$\lambda = \begin{cases} 
  k, & \text{if } |V| \text{ is odd}, \\
  \text{one of the integers } 2\alpha - k \ (0 \leq \alpha \leq k), & \text{if } |V| \text{ is even}.
\end{cases}$$

Corollary 3.17. If $X$ is a vertex transitive graph and $X \neq K_2$, then $X$ has at least one multiple eigenvalue.

Corollary 3.18. If $X$ is a pattern polynomial graph and $X \neq K_2$, then $X$ has a multiple eigenvalue.

Proof. If all eigenvalues of $X$ are simple, then $\dim(\mathcal{A}(X)) = n$ so every pattern matrix is a symmetric permutation matrix whose sum is $J$. Consequently $X$ is a vertex transitive graph.

We can extend the above result to arbitrary graphs in the following manner.
Corollary 3.19. Let $X$ be a graph with $n > 2$ vertices and $CC(X)$ is a commutative algebra. Then $\dim(\mathcal{A}(X)) \leq n - 1$ and $\dim(\mathcal{CC}(X)) \leq n$.

Proof. First observe that $X$ has at least one multiple eigenvalue if and only if $\dim(\mathcal{A}(X)) \leq n - 1$. Now from the Corollary 1.4, we have $\dim(\mathcal{CC}(X)) \leq n$. If $\dim(\mathcal{A}(X)) = n$, then we will get a contradiction from above corollary.

If $X$ is a pattern polynomial graph with odd number of vertices, then from Corollary 3.14 we have stronger result than above.

Lemma 3.20. If $X$ is a pattern polynomial graph with odd number of vertices, then $\dim(\mathcal{A}(X)) \leq \frac{n+1}{2}$.

Proof. First observe that $\dim(\mathcal{A}(X))$ is the number of pattern matrices. From Corollary 3.14, all pattern graphs of $X$ are regular with odd number of vertices. Consequently each is an even regular graph with regularity $\geq 2$. Hence $X$ has atmost $\frac{n+1}{2}$ pattern graphs.

A graph is said to be an edge-regular graph if any two of its adjacent vertices have the same number of common neighbours. The following result is easy to see.

Lemma 3.21. If $X$ is a pattern polynomial graph and its adjacency matrix itself is a pattern matrix then $X$ is an edge-regular graph.

Proof. Let $X$ be a pattern polynomial graph with adjacency matrix $A$. Let $\{P_0 = I, P_1 = A, P_2, \ldots, P_r\}$ be the standard basis of $\mathcal{A}(X)$. Hence $A^2 = a_0I + a_1A + a_2P_2 + \cdots + a_rP_r$ where $a_i \in \mathbb{C}$. Consequently any two adjacent vertices have exactly $a_1$ common neighbours.

In the following section, we see few classes of graphs which satisfy the condition $\ell = r$. Consequently they are pattern polynomial graphs

4 Some graph classes which are pattern polynomial

In this section we will prove that the following classes of graphs are pattern polynomial graphs a) orbit polynomial graphs b) distance regular graphs hence distance transitive graphs c) connected compact regular graphs.
Definition 4.1. Let $G$ be a subset of $n \times n$ permutation matrices forming a group. Then $V_C(G) = \{ A \in M_n(\mathbb{C}) : PA = AP \ \forall P \in G \}$ forms an algebra over $\mathbb{C}$ called the centralizer algebra of the group $G$.

Definition 4.2. If $G$ is a group acting on a set $V$, then $G$ also acts on $V \times V$ by $g(x, y) = (g(x), g(y))$. The orbits of $G$ on $V \times V$ are called orbitals. In the context of graphs, the orbitals of graph $X$ are orbitals of its automorphism group $\text{Aut}(X)$ acting on the vertex set of $X$. That is, the orbitals are the orbits of the arcs/non-arcs of the graph $X = (V, E)$. The number of orbitals is called the rank of $X$.

An orbital can be represented by a 0,1-matrix $M$ where $M_{ij}$ is 1 if $(i, j)$ belongs to the orbital. We can associate directed graphs to these matrices. If the matrices are symmetric, then these can be treated as undirected graphs.

Observation 4.1. • The ‘1’ entries of any orbital matrix are either all on the diagonal or all are off diagonal.

• The orbitals containing 1’s on the diagonal will be called diagonal orbitals.

Definition 4.3. The centralizer algebra of a graph $X$ denoted by $V(X)$ is the centralizer algebra of its automorphism group acting on the vertex set of $X$.

Theorem 4.2. $V(X)$ is a coherent algebra and orbitals of $\text{Aut}(X)$ acting on the vertex set of $X$ form its unique 0-1 matrix basis.

Since $\text{Aut}(X) = \text{Aut}(X^c)$, we have $V(X) = V(X^c)$. Also $\mathcal{C}(X)$ is the smallest coherent algebra containing $A(X)$ and $V(X)$ is a coherent algebra of $X$ containing $A(X)$ so $\mathcal{C}(X) \subseteq V(X)$.

Orbit polynomial graphs

Definition 4.4. A graph $X = (V, E)$ is orbit polynomial graph if each orbital matrix is a member of $A(X)$. That is, each orbital matrix is a polynomial in $A$.

Lemma 4.3. If $X$ is orbit polynomial graph if and only if $A(X) = V(X)$
If $X$ is an orbit polynomial graph, then $A(X) = CC(X) = V(X)$. Hence every orbit polynomial graph is a pattern polynomial graph.

If $X$ is an orbit polynomial graph and $X^c$ is connected, then from above lemma we have $A(X) = A(X^c) = CC(X) = CC(X^c) = V(X) = V(X^c)$. So we have the following result.

**Corollary 4.4.** If $X$ is an orbit polynomial graph and $X^c$ is connected then $X^c$ is also an orbit polynomial graph.

For any graph $X$, we have $A(X) \subseteq L(X) \subseteq CC(X) \subseteq V(X)$. Consequently from Corollary 1.3 every pattern matrix is the sum of one or more orbital matrices. Further by definition, orbital matrices commute with all automorphisms of $X$ hence we have the following result.

**Lemma 4.5.** Let $X$ be any graph and $\{P_1, P_2, \ldots, P_r\}$ be the set of all pattern matrices of adjacency matrix of $X$. Then $\text{Aut}(X) \subseteq \text{Aut}(X_{P_i}) \ 1 \leq i \leq r$.

Every connected vertex transitive graph of prime order is an orbit polynomial graph, see [Beezer 1]. Following lemma gives a stronger result.

**Lemma 4.6.** If $X$ is a connected graph of prime order, then $X$ is orbit polynomial graph if and only if $V(X)$ is commutative.

**Proof.** If $X$ is an orbit polynomial graph then clearly $V(X)$ is commutative. Conversely suppose that $V(X)$ is commutative, then the identity matrix is in the standard basis of $V(X)$. Consequently $X$ is a vertex transitive graph, hence the result follows. \hfill \Box

An easy consequence of this lemma is that if $X$ is a connected graph of prime order and $V(X)$ is commutative, then $X$ is a pattern polynomial graph.

**Distance transitive graphs**

**Definition 4.5.** A graph $X$ is distance transitive if for all vertices $u, v, x, y$ of $X$ such that $d(u, v) = d(x, y)$ there is a $g$ in $\text{Aut}(X)$ satisfying $g(u) = x$ and $g(v) = y$.

**Remark 4.7.** From the definition of distance transitivity following facts are immediate: a) For a distance transitive graph with diameter $d$, distance matrices and orbital matrices coincide, consequently its rank is $d + 1$. b) From
Equation 1 and the fact that $A(X) \subseteq V(X)$, if $X$ is a distance transitive graph with diameter $d$, then dimension of $A(X)$ is $d + 1$. Further orbital matrices form a basis for $A(X)$. This also implies the following lemma.

Now the following result is immediate from the Theorem 4.11 and the above Remark.

**Lemma 4.8.** Every distance transitive graph is an orbit polynomial graph.

Converse of the above lemma is not true, as every vertex transitive graph of prime order is not a distance transitive graph. so we have

Distance transitive graph $\Rightarrow$ Orbit polynomial graph $\Rightarrow$ Pattern polynomial graph $\Rightarrow$ Distance polynomial graph.

**Compact graphs**

**Definition 4.6.** A graph $X$ is said to be compact if every doubly stochastic matrix which commutes with $A(X)$ is a convex combination of matrices from $Aut(X)$.

A permutation group on a set $X$ is generously transitive if, given any two points, there is a permutation which interchanges them.

**Theorem 4.9.** [8] Let $X$ be a connected regular graph with $r$ distinct eigenvalues. If $X$ is compact, then $Aut(X)$ is a generously transitive permutation group with rank $r$.

In a compact connected regular graph $X$ the number of distinct eigenvalues of $A(X)$ is same as the number of orbitals. It is also the dimension of $A(X)$. Hence we have the following corollary from the fact that $A(X) \subseteq V(X)$.

**Corollary 4.10.** Every compact connected regular graph is orbit polynomial graph.

Godsil [8] showed that if $n \geq 7$, then the line graph of the complete graph $K_n$, is a distance transitive graph but not compact graph. It is also easy to check that if $X$ is compact, then so is its compliment $X^c$. Hence if $X$ is connected compact regular graph and $X^c$ is also connected, then $X^c$ is compact connected regular graph but it need not be a distance transitive graph $X = C_0$(the cycle graph) is such an example.

Now we will see class of graphs which are pattern polynomial graphs but need not be orbit polynomials graphs.
Distance regular Graphs

**Definition 4.7.** A connected graph is distance regular if for any two vertices $u$ and $v$, the number of vertices at distance $i$ from $u$ and $j$ from $v$ depends only on $i$, $j$, and the distance between $u$ and $v$. These graphs are necessarily regular, since $u$ may be equal to $v$.

It is easy to see that every distance transitive graph is distance regular. In fact, there are many distance regular graphs whose automorphism group is trivial [13]. The following theorem establishes that every distance regular graph is a pattern polynomial graph.

**Theorem 4.11.** [Damerell 5] Let $X$ be a distance regular graph with diameter $d$. Then $\{A_0, A_1, \ldots, A_d\}$ is a basis for the adjacency algebra $A(X)$, and consequently the dimension of $A(X)$ is $d + 1$.

**Corollary 4.12.** Every distance regular graph is a pattern polynomial graph.

Observe that if $X$ is a distance regular graph then $A$ itself is a pattern matrix. Consequently if $X$ is a distance regular graph with diameter $\geq 3$, then $X^c$ is not distance regular. Now if $X$ is a distance regular graph and $X^c$ is connected then from Corollary 3.10 $A(X) = A(X^c) = CC(X) = CC(X^c)$. Hence $X^c$ is also pattern polynomial graph but it need not be a distance regular graph for example $C_6^c$. Further there are distance regular graphs whose automorphism group is trivial [13]. So they can’t be orbit polynomial graphs. Finally, if $X$ is a distance regular graph with diameter $\geq 3$ with trivial automorphism group and $X^c$ is connected, then $X^c$ is neither distance regular nor orbit polynomial graph, but it is a pattern polynomial graph.

The following diagram gives the relationship among some of the graph classes which we studied in this work:

[Diagram showing the relationship among different types of graphs]

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5 PBIBD(t) s from pattern polynomial graphs

A design is an ordered pair \((V, B)\) with point set \(V\) and set of blocks \(B\) such that \(B\) is a collection of subsets of \(V\).

A design \((V, B)\) is called \(t-(v, k, \lambda)\) design (some times only \(t\)-design) if \(|V| = v, |B| = k \forall B \in B\) and each subset of \(V\) of cardinality \(t\) is contained in \(\lambda\) blocks. One can show by counting that a \(t\)-design is an \(i\)-design for each \(0 \leq i \leq t\). In fact a \(t-(v, k, \lambda)\) design is an \(i-(v, k, \lambda_i)\) design with \(\lambda_i = \lambda \frac{v-i}{k-i}\) for each \(0 \leq i \leq t\).

A balanced incomplete block design (BIBD) is a 2-design. The parameters \(\lambda_1\) and \(\lambda_0\) are usually denoted by \(r_1\) (replication number) and \(b\) (number of blocks).

**Definition 5.1.** Given \(v\) symbols \(1, 2, \ldots, v\), a relation satisfying the following condition is said to be an symmetric association scheme with \(m\) association classes:

1. Any two symbols \(\alpha\) and \(\beta\) are either first, second, \ldots or \(m\)th associates and this relationship is symmetrical. We denote \((\alpha, \beta) = i\), when \(\alpha\) and \(\beta\) \(i\)th associates.

2. Every symbol \(\alpha\) has \(n_i\) \(i\)th associates, the number \(n_i\) being independent of \(\alpha\).

3. If \((\alpha, \beta) = i\) the number of symbols \(\gamma\) that satisfy simultaneously \((\alpha, \gamma) = j\) and \((\beta, \gamma) = j'\) is \(p_{jj'}^{i}\), and this number is independent of \(\alpha\) and \(\beta\). Further \(p_{jj'}^{i} = p_{j'j}^{i}\)

The numbers \(v, n_i, p_{jj'}^{i}\) are called the parameters of the association scheme. If the relations are not symmetric, then it is called an association scheme.

Let \(R_i = \{(\alpha, \beta)| (\alpha, \beta) = i\}\) be the set of all \(i\)-th associates. Then the relation \(R_i\) of an association scheme can be described by a 0,1-matrices \(A_i\). Hence above definition can be described in terms of matrices as follows.

An association scheme with \(d\) associate classes is a set \(\mathfrak{A} = \{A_0, \ldots, A_d\}\) of 0,1-matrices such that

1. \(A_0 = I\).

2. \(A_0 + A_1 + \cdots + A_d = J\).

3. \(A_i^T \in \mathfrak{A}\).
4. \( A_i A_j = A_j A_i \in \text{span}(\mathfrak{A}) \).

If \( A_i^T = A_i \) \((1 \leq i \leq d)\), then \( \mathfrak{A} \) is a symmetric association scheme also called Bose-Mesner algebra. For example, if \( X \) is a pattern polynomial graph, then \( \mathcal{A}(X) \) is a Bose-Mesner algebra. But every Bose-Mesner algebra can not be obtained in this way. Now we will give an example of a Bose-Mesner algebra \( \mathfrak{A} \) which is not equal to adjacency algebra of any graph.

Let \( G \) be a finite abelian group of order \( n > 2 \). Each element of \( G \) gives rise to a permutation of \( G \), the permutation corresponding to \( 'a \) maps \( g \) in \( G \) to \( ga \)'. Hence for each element \( g \) in \( G \) we have a permutation matrix \( P_g \); the map \( g \rightarrow P_g \) is a group homomorphism. Therefore \( P_g P_h = P_{gh} \), \( P(g^{-1}) = P_g^T \). We have \( P(1) = I \) and \( \sum_{g \in G} P_g = J \). Hence the matrices \( P_g \) forms an association scheme with \( v - 1 \) classes. Note the association scheme obtained in this way is same as centralizer algebra \( \mathcal{V}_c(G) \) which is further equal to group algebra \( C[G] \). The restricted centralizer algebra \( \mathcal{V}_r(G) \), consisting of all real, symmetric matrices in \( \mathcal{V}_c(G) \) is a real subalgebra of \( \mathcal{V}_c(G) \), which is closed under Hadamard product and spanned by the matrices \( P_g + P_g^T \forall g \in G \). That is \( \mathcal{V}_r(G) \) is a symmetric association scheme. Further if we assume that \( G \) is an elementary abelian 2-group, then \( C[G] = \mathcal{V}_r(G) = \mathcal{V}_c(G) \). Consequently \( C[G] \) is a Bose-Mesner algebra and \( \text{dim}(C[G]) = n \). Now from Corollary 3.19 there exists no graph \( X \) with \( n > 2 \) vertices such that \( \mathcal{A}(X) = C[G] \).

**Definition 5.2.** Given an \( m \)-association scheme on \( v \)-symbols a PBIBD\((m)\) with \( m \) associate classes is defined as follows. A PBIBD\((m)\) with \( m \) associate classes is an arrangement of \( v \) symbols in \( b \) sets of size \( k \) \((v)\) such that

1. Every symbol occurs at most once in a set.
2. Every symbol occurs in \( r \) sets.
3. Two symbols \( \alpha \) and \( \beta \) occur in \( \lambda_i \) sets, if \( (\alpha, \beta) = i \) and \( \lambda_i \) is independent of symbols \( \alpha \) and \( \beta \).

The numbers \( v, b, r, k, \lambda_i \) are the parameters of the PBIBD. The PBIBD is usually identified by the association scheme of the symbols. For more information on design theory the reader is referred to Raghavaro [12]. For any PBIBD\((t)\) we will write the parameters as \((v, b, r_1, k_1, \lambda_1, \ldots, \lambda_t)\) where \( v \) is the number of points, \( b \) is the number of blocks, \( r_1 \) is called the replication number, \( k_1 \) is the number of elements in any block of the design.
If $N$ is the incidence matrix of a design $D$, then we say that the design $D$ is obtained from the graph $X$ if $NN^T \in \mathcal{A}(X)$. For example, if $X$ is a pattern polynomial graph with $n$ vertices, then

1. Every BIBD with $n$ points is obtained from $X$. In fact every $t$-design with $n$ points and $t \geq 2$ is obtained from $X$.

2. Let a graph $Y$ be a polynomial in $X$ and $D_1 = (V(Y), E(Y))$ be a design with points as vertices of graph $Y$ and blocks as edges of $Y$. Then $NN^T = D + A(Y) \in \mathcal{A}(X)$ where $N$ is the incidence matrix of design $D_1$, which is also a 0,1-incidence matrix of the graph $Y$ and $D$ is the diagonal matrix with diagonal entries are degree of vertices of $Y$. Note that $D_1$ is a PBIBD($r$), where $r$ is the degree of the minimal polynomial of $X$.

Let $v$ be any vertex in a graph $Z$, $N(v)$ be the set of vertices which are adjacent to $v$ in $Z$, $\mathcal{B}_2 = \{N(v)|v \in V(Z)\}$ and $\mathcal{B}_3 = \{N(v) \cup \{v\}|v \in V(Z)\}$. If $\mathcal{D}_2 = (V(Z), \mathcal{B}_2)$ and $\mathcal{D}_3 = (V(Z), \mathcal{B}_3)$, then $\mathcal{D}_2$ and $\mathcal{D}_3$ are designs on $V(Z)$ with $n = |V(Z)| = |\mathcal{B}_2| = |\mathcal{B}_3| = b$. Hence $A(Z)$ is the incidence matrix of the designs $\mathcal{D}_2$ and $I + A(Z)$ is the incidence matrix of $\mathcal{D}_3$. If $Z$ is a $k$-regular graph, then all blocks in the design $\mathcal{D}_2(\mathcal{D}_3)$ are $k$-subsets ($k + 1$ subsets) of $V(Z)$. And each vertex belongs exactly $k$ ($k + 1$) blocks. In other words $\mathcal{D}_2(\mathcal{D}_3)$ is a 1-design with $r_1 = k_1$. Further if we assume $Z$ is a polynomial in a pattern polynomial graph $X$, then the designs $\mathcal{D}_2$ and $\mathcal{D}_3$ are PBIBD($r$)s where $r$ is the degree of minimal polynomial of $X$. Hence we have the following result.

**Lemma 5.1.** Let a $k$-regular graph $Y$ be a polynomial in a pattern polynomial graph $X$ with $n$ vertices and $\mathcal{D}_i, i = 2, 3$ are designs on $V(Y)$ defined as above. Then $\mathcal{D}_2(\mathcal{D}_3)$ is a PBIBD($r$) obtained from $X$ with parameters \((n, n, k, k, \lambda_1, \ldots, \lambda_r)((n, n, k + 1, k + 1, \lambda'_1, \ldots, \lambda'_r))\) for some $\lambda_i$ and $\lambda'_i$.

6 **On the polynomial of a pattern polynomial graph**

A graph $Y$ is said to be a polynomial in a graph $X$ if $A(Y) \in \mathcal{A}(X)$. For an arbitrary graph $X$ it seems difficult to find whether a given graph is polynomial in $X$ or not. This question is answered for orbit polynomial graph and
distance regular graphs by [Robert A. Beezer 2] and [Paul M. Weichsel 15] respectively. In the following lemma we generalize those results to all pattern polynomial graphs.

**Lemma 6.1.** If $X$ is a pattern polynomial graph with standard basis \{$P_1, P_2, \ldots, P_r$\} where $P_1 = I$, then a graph $Y$ is a polynomial in $X$ if and only if $A(Y) = \sum_{i=2}^{r-1} a_i P_i$ where $a_i \in \{0, 1\}$.

**Proof.** Direct consequence of Corollary 1.3

**Corollary 6.2.** There are $2^{r-1}$ graphs in the adjacency algebra of a pattern polynomial graph $X$, where $r$ is the degree of the minimal polynomial of $A(X)$.

Another trivial fact is as follows.

**Lemma 6.3.** Let a graph $Y$ be a polynomial in a pattern polynomial graph $X$, then $CC(Y) \subseteq CC(X)$.

If a graph $Y$ is a polynomial in a pattern polynomial graph $X$, then $CC(Y)$ is a symmetric (every matrix in $CC(Y)$ is symmetric) commutative algebra. Hence

1. $Y$ is a walk regular graph,
2. $Y$ is a strongly distance-balanced graph, from Lemma 3.9
3. $Y$ has a multiple eigenvalue, whenever $Y \neq K_2$, from Corollary 3.19
4. $\dim(CC(Y)) \leq n - 1$, from Corollary 3.19. Further if the number of vertices in $Y$ is odd, then $\dim(CC(Y)) \leq \frac{n+1}{2}$.

Now it is interesting to answer the following question: If $Y$ is a graph such that $CC(Y)$ is symmetric commutative algebra, then “does there exist a pattern polynomial graph $X$ such that $Y$ is a polynomial in $X$?” For example, if $Y$ is a circulant graph (Cayley graph on cyclic group) with $n$ vertices, then clearly $CC(Y)$ is symmetric commutative algebra and it is also known that $Y$ is a polynomial in cycle graph $C_n$, which is a pattern polynomial graph.

If a graph $Y$ is a polynomial in a graph $X$, then there exists a unique polynomial $p_Y(x) \in \mathbb{C}[x]$, with degree less than the degree of the minimal of $X$, such that $A(Y) = p_Y(A(X))$. It is called representor polynomial of $Y$. If
$X$ is a pattern polynomial graph and $A(Y) = \sum_i a_i P_i$, then $p_Y = \sum_i a_i p_X p_i$. If a graph $Y$ is a polynomial in $X$ with representor polynomial $p_Y(x)$, then the eigenvalues of $A(Y)$ are $p_Y(\lambda_i)$, where $\lambda_i (0 \leq i \leq n - 1)$ are eigenvalues of $A(X)$.

The following result gives whether a graph $Y$ which is a polynomial in a graph $X$ is singular or not. Recall a graph is said to be singular if its adjacency matrix is singular.

**Lemma 6.4.** Let a graph $Y$ be a polynomial in a graph $X$. Then $Y$ is singular if and only if $\deg(\gcd(p(x), p_Y(x))) \geq 1$, where $p(x)$ is the minimal polynomial of $A(X)$ and $p_Y(x)$ is the representor polynomial of $Y$ with respect to $X$.

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