On the Complexity of Robust Bilevel Optimization With Uncertain Follower’s Objective*

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We investigate the complexity of bilevel combinatorial optimization with uncertainty in the follower’s objective, in a robust optimization approach. We show that the robust counterpart of the bilevel problem under interval uncertainty can be \( \Sigma_2^P \)-hard, even when the certain bilevel problem is NP-equivalent and the follower’s problem is tractable. On the contrary, in the discrete uncertainty case, the robust bilevel problem is at most one level harder than the follower’s problem.

Keywords: bilevel optimization, robust optimization, complexity

1 Introduction

When addressing combinatorial optimization problems, it is usually assumed that there is only one decision maker that has control over all variables, and that all parameters of the problem are known precisely. However, these two assumptions are not always satisfied in real-world applications. The latter issue of data uncertainty has led to the growing research fields of stochastic and robust optimization. In this paper, we focus on the robust optimization approach to combinatorial optimization and assume that only objective functions are uncertain. The complexity of the resulting robust counterparts highly depends on the chosen uncertainty sets, which are supposed to contain all possible scenarios, while the objective function value in the worst-case scenario is optimized. Typical classes of uncertainty sets are finite sets (discrete uncertainty) or hyper boxes (interval uncertainty). In the standard setting, interval uncertainty preserves the complexity of the problem, while discrete uncertainty often leads to NP-hard robust counterparts even when the underlying certain problem is tractable. For these

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and other complexity results, we refer to [13] or to the recent survey [7]. For a general introduction to robust optimization, see, e.g., [2].

A common approach to address optimization problems involving more than one decision maker is multilevel optimization (or bilevel optimization in case of two decision makers). A multilevel optimization problem models the interplay between several decision makers, each of them having their own decision variables, objective function and constraints. The decisions may depend on each other and are made in a hierarchical order: first, the highest-level decision maker decides. Based on this, the decision maker on the next level takes his choice, followed by the third-level decision maker, and so on. Usually, the problem is viewed from the perspective of the highest-level decision maker, who has perfect knowledge about all the lower-level problems. In particular, she has to anticipate the entire sequence of optimal lower-level responses to her decision. One has to distinguish between two kinds of behavior in case some lower-level decision maker has more than one optimal solution: in the optimistic setting, he acts in favor of the higher-level decision maker (or another upper-level decision maker), and in the pessimistic setting, he chooses a worst-case solution for her.

Jeroslow [12] showed that an \( \ell \)-level program with only linear constraints and objectives is already \( \Sigma^P_\ell-1 \)-hard in general, and thus NP-hard in the bilevel case, where he assumes the optimistic setting for every pair of consecutive players. Furthermore, if the variables of all decision makers are restricted to be binary, then the problem turns out to be \( \Sigma^P_\ell \)-hard in general. Many bilevel variants of classical combinatorial optimization problems have also shown to be NP-hard, e.g., a bilevel assignment problem [11] or a bilevel minimum spanning tree problem [6]. For a thorough overview of bilevel and multilevel optimization, we refer to the surveys [17, 8, 9, 14] or to the book [10].

In this paper, our aim is to bring together robust and bilevel optimization and to investigate the result in terms of complexity. More precisely, we address bilevel combinatorial optimization problems where the lower-level objective function is unknown to the upper-level decision maker. However, the latter knows an uncertainty set to which this objective function will belong, and she aims at optimizing her worst-case objective. While the two decision makers are often referred to as leader and follower, motivated by the origins of bilevel optimization in Stackelberg games [18], our situation can be illustrated by considering three decision makers: the leader first chooses a solution, then an adversary chooses an objective function for the follower, and the follower finally computes his optimal solution according to this objective function and depending on the leader’s choice. The aim of the leader is to optimize her own objective value, which depends on the follower’s choice, while the adversary has the opposite objective.

We start our investigation with a certain combinatorial bilevel problem of the form

\[
\begin{align*}
\max_x \quad & d^\top y_c \\
\text{s.t.} \quad & x \in X \\
y_c \in & \arg \max_y \quad c^\top y \\
\text{s.t.} \quad & y \in Y(x)
\end{align*}
\]

(P)
with $X \subseteq \{0,1\}^p$, $Y(x) \subseteq \mathbb{R}^n$ for all $x \in X$, and $c,d \in \mathbb{R}^n$. We note that the notation of (P) (and all other bilevel programs throughout this paper) is convenient, but not precise in case the follower does not have a unique optimum solution. Formally, the optimistic or pessimistic case could be modeled by adding a maximization or minimization over $y_c$, respectively, to the leader’s objective function.

We concentrate on the most basic setting where the follower’s problem is a linear program with only the right hand side depending (affine-linearly) on the leader’s decision $x$. More formally, we assume that there are $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{m \times p}$, and $b \in \mathbb{R}^m$ such that $Y(x) = \{y \in \mathbb{R}^n \mid Ay \leq Bx + b\}$ for all $x \in X$. For simplicity, we assume throughout that $Y(x)$ is non-empty and bounded for all $x \in X$, so that an optimal follower’s response always exists.

In particular, for given $x$ and $c$, the follower’s problem is a linear program and thus always tractable. This holds not only in terms of finding any optimal solution, but also if the task includes finding an optimal solution that is best or worst possible for the leader. This amounts to optimizing two linear objective functions lexicographically, which can be done efficiently [15]. It follows that the leader’s optimization problem (P) is NP-easy, since its decision variant belongs to NP, the solution $x \in X$ being the certificate.

Turning our attention to the robust optimization setting, we now assume that the follower’s objective $c$ is uncertain. However, we are given an uncertainty set $U \subseteq \mathbb{R}^n$ containing all relevant realizations of $c$. The robust counterpart of (P) then reads

$$\max_x \min_{c \in U} d^\top y_c$$

subject to:

$$x \in X$$

$$y_c \in \arg \max_y c^\top y$$

subject to:

$$y \in Y(x).$$

Our aim is to investigate the complexity of (R) relative to the complexity of (P). In case the follower’s solution $y_c$ for a given objective $c$ is not unique, we will consider both the optimistic and the pessimistic approach.

Besides the robust counterpart (R), we will also consider the adversary’s subproblem, i.e., the problem of evaluating the leader’s objective function in (R). For a fixed leader’s choice $x \in X$, this problem thus reads

$$\min_c d^\top y_c$$

subject to:

$$c \in U$$

$$y_c \in \arg \max_y c^\top y$$

subject to:

$$y \in Y(x).$$

which turns out to be a bilevel optimization problem again, however with the leader’s constraints having a very specific structure now, depending only on the uncertainty
Finally, for fixed $x$ and $c$, we will consider the certain follower’s problem

\[ \max_y c^\top y \]

\[ \text{s.t. } y \in Y(x), \] (F)

which, as already mentioned, is a linear program in our setting.

It is easily verified that the robust problem (R) can be at most one level harder, in the polynomial hierarchy, than the adversary’s problem (A), i.e., the evaluation of the objective function of (R). The complexity of (R) and (A) strongly depends on the uncertainty set $U$. We will see that (R) can be $\Sigma^P_2$-hard and (A) NP-hard in case of interval uncertainty, while in case of discrete uncertainty, (A) is tractable and thus (R) is NP-easy. This is a remarkable difference to classical (single-level) robust optimization, where, as mentioned above, interval uncertainty in the objective function does not make a problem harder, while many tractable combinatorial optimization problems have an NP-hard robust counterpart when the set $U$ is finite.

Addressing similar complexity questions, Buchheim and Henke [5] investigate a bilevel continuous knapsack problem where the leader controls the continuous capacity of the knapsack and, as in (R), the follower’s profits are uncertain. They show that (the analoga of) both (F) and (P) can be solved in polynomial time in this case, while (A) and (R) turn out to be NP-hard for, e.g., budgeted and ellipsoidal uncertainty. On the other hand, both (A) and (R) with respect to discrete and interval uncertainty remain tractable. The latter result rises the question whether or not interval or discrete uncertainty, in general, can increase the complexity when going from (P) to (R).

The complexity of bilevel optimization problems under uncertainty, in a robust optimization framework, has also been addressed in a few recent articles. However, the assumptions about what is uncertain are different to our setting. In [4], multilevel optimization problems are investigated in which some follower’s decision does not have to be optimal according to his objective, but can deviate from the optimum value by a small amount; see also [3] and the references therein. In [1], bilinear bilevel programming problems are considered and it is assumed that the follower cannot observe the leader’s decision precisely. In both settings, it turns out that the complexity does not increase significantly with respect to the corresponding model without uncertainty.

## 2 Interval uncertainty

We start by considering the case of interval uncertainty, i.e., uncertainty sets of the form $U = [c^-_1, c^+_1] \times \cdots \times [c^-_n, c^+_n]$ for given $c^-, c^+ \in \mathbb{R}^n$ with $c^- \leq c^+$. We will prove that, in this case, the robust counterpart (R) is significantly harder than the underlying problem (P) in general, while the adversary’s problem (A) is significantly harder than the follower’s problem (F). For the reductions, we will use the well-known satisfiability problem (SAT) as well as the less-known quantified satisfiability problem (QSAT), defined as follows:

(SAT) Given a Boolean formula $f: \{0, 1\}^n \to \{0, 1\}$, decide whether there exists a satisfying assignment, i.e., a $y \in \{0, 1\}^n$ such that $f(y) = 1$. 
(QSAT) Given a Boolean formula $f : \{0, 1\}^p \times \{0, 1\}^n \to \{0, 1\}$, decide whether there exists some $x \in \{0, 1\}^p$ such that $f(x, y) = 1$ for all $y \in \{0, 1\}^n$.

For both problems, we may assume that the function $f$ is given in the input by a recursive definition using negations and bivariate conjunctions and disjunctions. The problem SAT is well-known to be NP-complete, while QSAT is an important example of a $\Sigma^P_2$-complete problem [16]. The complexity class $\Sigma^P_2$ contains all decision problems which can be solved by a nondeterministic Turing machine equipped with an oracle for some NP-complete decision problem; see [16] for an introduction to the polynomial-time hierarchy and a formal definition of the complexity classes $\Sigma^P_\ell$. For later use, we also recall the following definitions: a decision or optimization problem is $\text{NP-easy}$ if it can be polynomially reduced to an NP-complete decision problem, and $\text{NP-equivalent}$ if it is both NP-hard and NP-easy.

**Theorem 1.** In case of interval uncertainty, the robust counterpart (R) can be $\Sigma^P_2$-hard and the adversary’s problem (A) NP-hard, even if $X = \{0, 1\}^p$.

**Proof.** We prove the $\Sigma^P_2$-hardness of (R) by a polynomial reduction from the $\Sigma^P_2$-complete problem QSAT. Since (R) can be at most one level harder than (A), as mentioned in the introduction, the NP-hardness of (A) directly follows. Furthermore, a reduction from the complement of SAT to (A) is implicitly contained in our reduction from QSAT to (R).

For a given Boolean formula $f : \{0, 1\}^p \times \{0, 1\}^n \to \{0, 1\}$, consider the bilevel problem

$$\max_x \min_{c \in [-1, 1]^n} f(x, y_c)$$

s.t.

\[
\begin{align*}
x &\in \{0, 1\}^p \\
y_c &\in \arg \max_y c^\top y \\
s.t. & \quad y \in [0, 1]^n.
\end{align*}
\]

(Q)

This problem can be linearized in the standard way: introduce a new variable with domain $[0, 1]$ for each intermediate term in the recursive definition of $f$, including $f$ itself, and model the corresponding nonlinear relations between all variables by appropriate linear inequalities (for conjunctions and disjunctions) or linear equations (for negations); see, e.g., [12]. In order to obtain a reformulation of (Q) in the form of (R) with $X = \{0, 1\}^p$, we add all new variables and linear constraints to the follower’s problem. For all newly introduced variables, we set the corresponding entries of $c^-$ and $c^+$ to 0.

We first consider the optimistic setting. Assume that $f$ is a no-instance of QSAT. Then for all leader’s choices $x \in \{0, 1\}^p$, there exists some $y \in \{0, 1\}^n$ such that $f(x, y) = 0$. By construction of (Q), the adversary can enforce any follower’s response $y \in \{0, 1\}^n$ by appropriately choosing $c \in \{-1, 1\}^n$. In that case, since $x$ and $y$ are binary, the linearization forces all additional variables to the corresponding binary values as well, including the variable representing the objective function. This shows that the optimal value of (Q) is 0 in this case.

If $f$ is a yes-instance, there exists $x \in \{0, 1\}^p$ such that $f(x, y) = 1$ for all $y \in \{0, 1\}^n$. Then we claim that this leader’s choice $x$ yields an objective value of 1 in the
Figure 1: Illustration of the follower’s objective function in the pessimistic setting in the proof of Theorem 1. For a fixed dimension $i \in \{1, \ldots, n\}$, the value $c_i y_i + \bar{y}_i$ is displayed in dependence of the value $y_i \in [0, 1]$ the follower chooses, for different values of $c_i$ chosen by the adversary.

linearization of (Q). Indeed, for every adversary’s choice $c_i$, there is an optimal follower’s solution such that $y_i$ is binary. By construction, each such solution leads to $f(x, y_c) = 1$. Since we adopt the optimistic approach, the follower will thus choose some solution $y_c$ with $f(x, y_c) = 1$.

Finally, assume that we adopt the pessimistic approach. In this case, we need to modify the construction of (Q) as follows. For every follower’s variable $y_i$, we introduce a new follower’s variable $\bar{y}_i$, along with linear constraints

$$\bar{y}_i \geq 0, \quad \bar{y}_i \leq y_i, \quad \bar{y}_i \leq 1 - y_i.$$ 

Each new variable $\bar{y}_i$ has coefficient $M$ in the leader’s objective, where $M \geq 3$ is at least the number of atomic terms in $f$, and a certain coefficient of 1 in the follower’s objective. This ensures that $\bar{y}_i = \min\{y_i, 1 - y_i\}$ holds in every optimal follower’s solution, i.e., the variable $\bar{y}_i$ models the deviation of $y_i$ from the closest binary value.

The resulting follower’s objective $c_i y_i + \bar{y}_i$, in every dimension $i \in \{1, \ldots, n\}$, depends on the adversary’s choice $c_i$ and always consists of two linear pieces, as a function in $y_i$; see Figure 1. If the adversary chooses $c_i \in \{-1, 1\}^n$, the follower’s optimum solutions satisfy $y_i = \bar{y}_i \in [0, 1/2]$ or $y_i = 1 - \bar{y}_i \in [1/2, 1]$, respectively. By setting all $y_i$ to binary values, the follower can achieve a leader’s objective value of at most 1 then, due to the pessimistic assumption. If the adversary chooses some $c_i \in (-1, 1)$, then by construction, the value of $(y_i, \bar{y}_i)$ in any optimal follower’s solution is $(1/2, 1/2)$, so that the leader’s objective is at least $M/2 > 1$, which is never optimal for the adversary.

Hence, the adversary must always choose $c_i \in \{-1, 1\}^n$.

For a no-instance $f$ and any $x \in \{0, 1\}^p$, the adversary can now choose $c_i \in \{-1, 1\}^n$ such that the follower has a binary vector $y \in \{0, 1\}^n$ in his set of optimum solutions for which $f(x, y) = 0$ holds. By the pessimistic assumption, the follower actually chooses $y$ such that the leader’s objective value is 0.

Assume, on the contrary, that $f$ is a yes-instance, and consider some leader’s solution $x \in \{0, 1\}^p$ such that $f(x, y) = 1$ for all $y \in \{0, 1\}^n$. Any adversary’s choice
$c \in \{-1, 1\}^n$, together with any binary follower’s choice $y$ then results in an objective value of 1 for the leader. Since we adopt the pessimistic approach, it remains to show that the follower cannot achieve a smaller objective value for the leader by choosing $y_i$ nonbinary and exploiting the resulting flexibility when setting the linearization variables. However, one can verify that this is not possible, because the gain in the variable representing $f(x, y)$ would be exceeded by the punishment due to the terms $M\bar{y}_i$, by definition of $M$; see, e.g., Lemma 4.1 in [12]. Thus, the follower chooses $y \in \{0, 1\}^n$ and the leader’s optimal value is indeed 1.

Note that the underlying certain problem of (Q) is NP-equivalent. Indeed, it is NP-hard even when $n = 0$, i.e., when the follower only controls the additional variables introduced for linearizing $f$, which are uniquely determined by the leader’s choice $x \in \{0, 1\}^p$. The problem is then equivalent to SAT. On the other hand, the problem is NP-easy, for any $n$, as argued in the introduction. We emphasize however that the optimal follower’s choice as well as the leader’s optimal value in the underlying certain problem might not be binary if $n \neq 0$, for some values of $c$.

In order to strengthen the statement of Theorem 1, the integrality constraints in the leader’s problem can be relaxed without changing the result, using a similar construction as in the pessimistic case (and in [12]): for each leader’s variable $x_i$, we can introduce a new follower’s variable $\bar{x}_i \in [0, 1]$ with coefficient $-M$ in the leader’s objective, where $M \geq 3$ is again at least the number of atomic terms in $f$, with certain coefficient 1 in the follower’s objective, and with additional follower’s constraints $\bar{x}_i \geq 0$, $\bar{x}_i \leq x_i$, $\bar{x}_i \leq 1 - x_i$. This ensures $\bar{x}_i = \min\{x_i, 1 - x_i\}$ in any optimal follower’s solution and, by the choice of $M$, that $x \in \{0, 1\}^p$ holds in any optimal leader’s solution. We obtain a problem of type (R) again, but without integrality constraints. Note that this construction does not interfere with the one in the proof of Theorem 1 in the pessimistic case because the latter is relevant only in case of a yes-instance, whereas the former is relevant only in case of a no-instance.

Remark 2. If binarity constraints were allowed also in the follower’s problem, a similar construction as in Theorem 1 would actually yield a $\Sigma^P_3$-hard problem, in the optimistic setting. For this, consider the $\Sigma^P_3$-complete problem

\[(\text{QSAT}_3)\] Given a Boolean formula $f : \{0, 1\}^p \times \{0, 1\}^n \times \{0, 1\}^m \rightarrow \{0, 1\}$, decide whether there exists some $x \in \{0, 1\}^p$ such that for all $y \in \{0, 1\}^n$ there exists some $z \in \{0, 1\}^m$ with $f(x, y, z) = 1$.

One could reduce QSAT$_3$ to (R) by considering two separate sets of variables in the follower’s problem, the first one corresponding to $y$ and being “controlled” by the adversary (i.e., with $c_i^- = -1$, $c_i^+ = 1$) and the second one corresponding to $z$ and being “controlled” by the follower (i.e., with $c_i^- = c_i^+ = 0$). The adversary’s problem is then equivalent to the complement of QSAT and thus $\Sigma^P_3$-hard. The underlying certain problem is equivalent to SAT, similarly as in Theorem 1 and thus NP-equivalent. However, in contrast to the setting of Theorem 1, the same holds for the follower’s problem here. Hence, the increase in complexity of the robust problem relative to the underlying certain problem is NP-hard even when $n = 0$, i.e., when the follower only controls the additional variables introduced for linearizing $f$, which are uniquely determined by the leader’s choice $x \in \{0, 1\}^p$. The problem is then equivalent to SAT. On the other hand, the problem is NP-easy, for any $n$, as argued in the introduction. We emphasize however that the optimal follower’s choice as well as the leader’s optimal value in the underlying certain problem might not be binary if $n \neq 0$, for some values of $c$.
problem is strictly greater here than in Theorem \[\text{I}\] while it remains the same relative to the follower’s problem.

**Remark 3.** The result of Theorem \[\text{I}\] does not only hold for interval uncertainty sets, but also for many other uncertainty sets \(U\) where each entry of \(c\) can be chosen to be positive or negative independently of each other. In particular, the proof of Theorem \[\text{I}\] can be easily adapted for the case of uncorrelated discrete uncertainty, as considered in \[\text{B}].

### 3 Discrete Uncertainty

We next address the case of a finite uncertainty set \(U\). For this case, we first show that robust counterparts of bilevel problems are at least as hard as robust counterparts of single-level problems, for each underlying feasible set of the leader.

**Lemma 4.** For any class of underlying sets \(X \subseteq \{0, 1\}^P\), the robust counterpart of linear optimization over \(X\) subject to discrete uncertainty can be polynomially reduced to a problem of type \(\text{(R)}\), with the same number of scenarios.

**Proof.** Starting from the robust single-level counterpart

\[
\begin{align*}
\max_x \min_{c \in U} \ c^\top x \\
\text{s.t. } x \in X
\end{align*}
\]

of linear optimization over some set \(X \subseteq \{0, 1\}^P\), with \(U = \{c_1, \ldots, c_m\} \subseteq \mathbb{R}^P\), we construct the robust bilevel problem

\[
\begin{align*}
\max_x \min_{\tilde{c} \in \tilde{U}} \ y_{\tilde{c}} \\
\text{s.t. } x \in X
\end{align*}
\]

\[
(y_{\tilde{c}}, z_{\tilde{c}}) \in \arg \max_{y,z} \tilde{c}^\top z
\]

\[
\text{s.t. } y = \sum_{j=1}^m \sum_{i=1}^P c_{ji} x_i z_j \\
y \in \mathbb{R}, \quad z \in \mathbb{R}^m_{\geq 0}, \quad \sum_{j=1}^m z_j = 1
\]

with uncertainty set \(\tilde{U} = \{e_1, \ldots, e_m\} \subseteq \mathbb{R}^m\), where \(e_i\) denotes the \(i\)-th unit vector. When the adversary chooses \(\tilde{c} = e_j\), the follower’s unique optimum solution consists of \(z = e_j\) and \(y = c_j^\top y\). Thus \(\text{(S)}\) and \(\text{(R(S))}\) are equivalent.

It remains to show that we can linearize \(\text{(R(S))}\) and thus turn it into the form required in \(\text{(R)}\). For this, we introduce follower’s variables \(u_{ji}\) and constraints

\[
u_{ji} \geq 0, \quad u_{ji} \geq x_i + z_j - 1, \quad u_{ji} \leq x_i, \quad u_{ji} \leq z_j
\]
for all $j \in \{1, \ldots, m\}$ and $i \in \{1, \ldots, p\}$, and replace the first constraint in the follower’s problem by

$$y = \sum_{j=1}^{m} \sum_{i=1}^{p} c_{ji} u_{ji}.$$  

For every $\tilde{c} \in \tilde{U}$, the unique optimum follower’s solution satisfies $z \in \{0, 1\}^m$. Hence this linearization ensures $u_{ji} = x_i z_j$ and thus yields a problem equivalent to $\langle R(S) \rangle$, having the form of $\langle R \rangle$. Formally, all scenarios $\tilde{c} \in \tilde{U}$ must be extended by zeros in the entries corresponding to the variables $y$ and $u_{ji}$. 

Note that all optimal follower’s solutions in the proof of Lemma 4 are uniquely determined, so that the result holds for both the optimistic and the pessimistic setting.

The robust counterpart of a single-level combinatorial optimization problem under discrete uncertainty often turns out to be NP-hard even for two scenarios, and strongly NP-hard when the number of scenarios is not fixed; this holds true even for unrestricted binary optimization [7]. Together with Lemma 4, this immediately implies

**Corollary 5.** Problem $\langle R \rangle$ with $X = \{0, 1\}^p$ can be NP-hard for $|U| = 2$ and strongly NP-hard for finite $U$, even when $\langle P \rangle$ is solvable in polynomial time.

On the other hand, the complexity of $\langle R \rangle$ can be bounded from above as follows: as mentioned in the introduction, $\langle R \rangle$ can be at most one level harder than $\langle A \rangle$. Moreover, $\langle A \rangle$ can be polynomially reduced to $\langle P \rangle$ in case of discrete uncertainty by enumerating all scenarios and solving $\langle P \rangle$ for each of them. For our setting, in which $\langle P \rangle$ is solvable in polynomial time, this immediately implies the following result:

**Theorem 6.** Let $U$ be a finite set, given explicitly as part of the input. Then the adversary’s problem $\langle A \rangle$ is solvable in polynomial time and the robust problem $\langle R \rangle$ is NP-easy.

When replacing the uncertainty set $U$ in Theorem 6 by its convex hull, we obtain a significantly harder problem:

**Theorem 7.** Let $U$ be defined as the convex hull of a finite set of vectors, which are given explicitly as input. Then the robust counterpart $\langle R \rangle$ can be $\Sigma_2^P$-hard and the adversary’s problem $\langle A \rangle$ NP-hard, even if $X = \{0, 1\}^p$.

**Proof.** In the proof of Theorem 6, we have seen that the linearized version of $\langle Q \rangle$ is $\Sigma_2^P$-hard for uncertainty sets of the form $[-1, 1]^n \times \{\tilde{c}\}$, where $\tilde{c}$ is a vector collecting all certain entries of $c$, corresponding to linearization variables and to artificial variables $\tilde{y}_i$ in the pessimistic case. It is easy to verify that the problem does not change when replacing $[-1, 1]^n$ by a superset, e.g., by the simplex convex $\{ -e, -e + 2ne_1, \ldots, -e + 2ne_n \}$, where $e$ is the all-ones vector and $e_i$ is the $i$-th unit vector. Since the latter is defined by polynomially many vertices, which can all be extended by $\tilde{c}$, we obtain the desired result for $\langle R \rangle$. The NP-hardness of $\langle A \rangle$ again follows directly from the $\Sigma_2^P$-hardness of $\langle R \rangle$. 

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4 Conclusion

We have shown that bilevel optimization problems become significantly harder when considering uncertain follower’s objective functions in a robust optimization approach. However, our results highlight a fundamental difference between the interval uncertainty case and the case of discrete uncertainty. Indeed, the construction in Theorem 1 yields a problem where $(R)$ is $\Sigma^P_2$-hard and $(A)$ is NP-hard, while both problems are at least one level easier in the setting of Theorem 6. In this sense, interval uncertainty renders bilevel optimization significantly harder than discrete uncertainty, in contrast to classical single-level robust optimization. However, Theorem 1 and Corollary 5 show that $(R)$ can be one level harder than $(P)$ for both interval and discrete uncertainty.

Moreover, Theorem 6 and Theorem 7 together imply that the complexity of both the robust counterpart $(R)$ and the adversary’s problem $(A)$ may increase when replacing $\mathcal{U}$ by its convex hull. This again is in contrast to the single-level case, where it is well-known that replacing $\mathcal{U}$ by its convex hull essentially does not change the robust counterpart.

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