POISSON LIMIT THEOREMS
FOR THE ROBINSON–SCHENSTED CORRESPONDENCE
AND FOR THE MULTI-LINE HAMMERSLEY PROCESS

MIKOŁAJ MARCINIAK, ŁUKASZ MAŚLANKA, AND PIOTR ŚNIADY

ABSTRACT. We consider the Robinson–Schensted–Knuth algorithm applied to a random input and study the growth of the bottom rows of the corresponding Young diagrams. We prove a multidimensional Poisson limit theorem for the resulting Plancherel growth process. In this way we extend the result of Aldous and Diaconis to more than just one row. This result can be interpreted as convergence of the multi-line Hammersley process to its stationary distribution which is given by a collection of independent Poisson point processes.

1. INTRODUCTION

1.1. Notations. The set of Young diagrams will be denoted by $\mathbb{Y}$; the set of Young diagrams with $n$ boxes will be denoted by $\mathbb{Y}_n$. The set $\mathbb{Y}$ has a structure of an oriented graph, called the Young graph; a pair $\mu \not\rightarrow \lambda$ forms an oriented edge in this graph if the Young diagram $\lambda$ can be created from the Young diagram $\mu$ by addition of a single box.

We will draw Young diagrams and tableaux in the French convention with the Cartesian coordinate system $Oxy$, see Figure 1a. We index the rows and the columns of tableaux by non-negative integers from $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. In particular, if $\square$ is a box of a tableau, we identify it with the Cartesian coordinates of its lower-left corner: $\square = (x, y) \in \mathbb{N}_0 \times \mathbb{N}_0$. For a tableau $\mathcal{T}$ we denote by $\mathcal{T}_{x,y}$ its entry which lies in the intersection of the row $y \in \mathbb{N}_0$ and the column $x \in \mathbb{N}_0$.

Also the rows of any Young diagram $\lambda = (\lambda_0, \lambda_1, \ldots)$ are indexed by the elements of $\mathbb{N}_0$; in particular the length of the bottom row of $\lambda$ is denoted by $\lambda_0$.

Let $X: \Omega \to V$ be a random variable with values in some set $V$. When we want to phrase this statement without mentioning the sample space $\Omega$ explicitly, we will write $X \overset{\text{rv}}{\in} V$.

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Figure 1. (a) The original tableau $T$. (b) We consider the Schensted row insertion of the number 18 to the tableau $T$. The highlighted boxes form the corresponding bumping route. The small numbers on the left (next to the arrows) indicate the inserted/bumped numbers. (c) The output $T \leftarrow 18$ of the Schensted insertion.

If $E$ is a random event, we denote by $1_E$ its indicator, which is the random variable given by

$$1_E(\omega) = \begin{cases} 1 & \text{if } \omega \in E, \\ 0 & \text{otherwise}. \end{cases}$$

1.2. Schensted row insertion. The Schensted row insertion is an algorithm which takes as an input a tableau $T$ and some number $a$. The number $a$ is inserted into the first row (that is, the bottom row, the row with the index $0$) of $T$ to the leftmost box which contains an entry which is strictly bigger than $a$.

In the case when the row contains no entries which are bigger than $a$, the number $a$ is inserted into the leftmost empty box in this row and the algorithm terminates.

If, however, the number $a$ is inserted into a box which was not empty, the previous content $a'$ of the box is bumped into the second row (that is, the row with the index 1). This means that the algorithm is iterated but this time the number $a'$ is inserted into the second row to the leftmost box which contains a number bigger than $a'$. We repeat these steps of row insertion and bumping until some number is inserted into a previously empty box. This process is illustrated on Figures 1b and 1c. The outcome of the Schensted insertion is defined as the result of the aforementioned procedure; it will be denoted by $T \leftarrow a$.

1.3. Robinson–Schensted–Knuth algorithm. For the purposes of this article we consider a simplified version of the Robinson–Schensted–Knuth
algorithm; for this reason we should rather call it the Robinson–Schensted algorithm. Nevertheless, we use the first name because of its well-known acronym RSK. The RSK algorithm associates to a finite sequence \( w = (w_1, \dotsc, w_\ell) \) a pair of tableaux: the insertion tableau \( P(w) \) and the recording tableau \( Q(w) \).

The insertion tableau

\[
P(w) = \left( \left( \emptyset \leftarrow w_1 \right) \leftarrow w_2 \right) \leftarrow \cdots \right) \leftarrow w_\ell
\]

is defined as the result of the iterative Schensted insertion applied to the entries of the sequence \( w \), starting from the empty tableau \( \emptyset \).

The recording tableau \( Q(w) \) is defined as the standard Young tableau of the same shape as \( P(w) \) in which each entry is equal to the number of the iteration of (1) in which the given box of \( P(w) \) stopped being empty; in other words the entries of \( Q(w) \) give the order in which the entries of the insertion tableau were filled.

The tableaux \( P(w) \) and \( Q(w) \) have the same shape; we will denote this common shape by \( \text{RSK} \( w \) \) and call it the RSK shape associated to \( w \).

The RSK algorithm is of great importance in algebraic combinatorics, especially in the context of the representation theory [Ful 97].

1.4. Plancherel measure, Plancherel growth process. Let \( \mathfrak{S}_n \) denote the symmetric group of order \( n \). We will view each permutation \( \pi \in \mathfrak{S}_n \) as a sequence \( \pi = (\pi_1, \dotsc, \pi_n) \) which has no repeated entries, and such that \( \pi_1, \dotsc, \pi_n \in \{1, \dotsc, n\} \). A restriction of RSK to the symmetric group is a bijection which to a given permutation from \( \mathfrak{S}_n \) associates a pair \((P, Q)\) of standard Young tableaux of the same shape, consisting of \( n \) boxes. A fruitful area of study concerns the RSK algorithm applied to a uniformly random permutation from \( \mathfrak{S}_n \), especially asymptotically in the limit \( n \to \infty \), see [Rom15] and the references therein.

The Plancherel measure on \( \mathbb{Y}_n \), denoted \( \text{Plan}_n \), is defined as the probability distribution of the random Young diagram \( \text{RSK}(w) \) for a uniformly random permutation \( w \) selected from \( \mathfrak{S}_n \).

An infinite standard Young tableau [Ker99, Section 2.2] is a filling of the boxes in a subset of the upper-right quarterplane with positive integers, such that each row and each column is increasing, and each positive integer is used exactly once. There is a natural bijection between the set of infinite standard Young tableaux and the set of infinite sequences of Young diagrams

\[
\lambda^{(0)} \nearrow \lambda^{(1)} \nearrow \cdots \quad \text{with} \quad \lambda^{(0)} = \emptyset.
\]

this bijection is given by setting \( \lambda^{(n)} \) to be the set of boxes of a given infinite standard Young tableau which are \( \leq n \).
If \( w = (w_1, w_2, \ldots) \) is an infinite sequence, the recording tableau \( Q(w) \) is defined as the infinite standard Young tableau in which each non-empty entry is equal to the number of the iteration in the infinite sequence of Schensted insertions

\[
( (\emptyset \leftarrow w_1) \leftarrow w_2) \leftarrow \cdots
\]

in which the corresponding box stopped being empty, see [RS15, Section 1.2.4]. Under the aforementioned bijection, the recording tableau \( Q(w) \) corresponds to the sequence (2) with

\[
\lambda^{(n)} = \text{RSK}(w_1, \ldots, w_n).
\]

Let \( \xi = (\xi_1, \xi_2, \ldots) \) be an infinite sequence of independent, identically distributed random variables with the uniform distribution \( U(0, 1) \) on the unit interval \([0, 1]\). The Plancherel measure on the set of infinite standard Young tableaux is defined as the probability distribution of \( Q(\xi) \). Any sequence with the same probability distribution as (2) with \( \lambda^{(n)} \) will be called the Plancherel growth process [Ker99]. For a more systematic introduction to this topic we recommend the monograph [Rom15, Section 1.19].

**1.5. The main result: Poisson limit theorem for the Plancherel growth process.**

**Theorem 1.1.** Let \( \lambda^{(0)} \nearrow \lambda^{(1)} \nearrow \cdots \) be the Plancherel growth process. Let us fix \( k \in \mathbb{N}_0 \). We denote by

\[
\Lambda^{(n)} = \left( \lambda_0^{(n)}, \ldots, \lambda_k^{(n)} \right) \stackrel{\text{rv}}{\in} (\mathbb{N}_0)^{k+1}
\]

the random vector formed by the lengths of the bottom \( k + 1 \) rows of the random diagram \( \lambda^{(n)} \). For each \( n \in \mathbb{N}_0 \) we consider the random function \( \Delta_n : \mathbb{R} \to \mathbb{Z}^{k+1} \) given by

\[
\Delta_n(t) = \Lambda^{(n_t)} - \Lambda^{(n)},
\]

where

\[
n_t = \max \left( n + \lfloor t \sqrt{n} \rfloor, 0 \right).
\]

Then, for \( n \to \infty \), the random function \( \Delta_n \) converges in distribution to a tuple \( (N_0, \ldots, N_k) \) of \( k + 1 \) independent copies of the Poisson process with unit intensity. This convergence is understood as convergence (with respect to the topology given by the total variation distance) of all finite-dimensional marginals \( (\Delta_n(t_1), \ldots, \Delta_n(t_\ell)) \) over all choices of \( t_1, \ldots, t_\ell \in \mathbb{R} \).
The proof is postponed to Section 3.1.

In Sections 1.6 to 1.9 we shall discuss the connections of this theorem with the (multi-line) Hammersley process and the assumption $\alpha$ of Hammersley, and in Section 1.10 the connections with the work of Aldous and Diaconis [AD95, Theorem 5(b)]. In Sections 1.11 and 1.12 we will discuss links with some other areas of mathematics.

**Remark 1.2.** The fact that the intensity of the limiting Poisson process is equal to 1 can be justified by the following heuristic argument. The first order approximation for the length of the given fixed row is given by

$$E\lambda_i^{(n)} \approx 2\sqrt{n},$$

cf. (8) for the special case of the bottom row. By considering the derivative of the right-hand side we can hope that

$$E\lambda_i^{(n+c\sqrt{n})} - E\lambda_i^{(n)} \approx c\sqrt{n} \frac{1}{\sqrt{n}} = c$$

which is consistent with the expected growth of the Poisson process with the unit intensity.

1.6. **Local spacings in the bottom rows of the recording tableau.** Theorem 1.1 can be interpreted as a result about the random sets of points in which the coordinates of the function (4) have jumps; this interpretation gives the following immediate corollary. For an alternative proof see Section 3.2.

**Corollary 1.3.** Let $\xi = (\xi_1, \xi_2, \ldots)$ be an infinite sequence of independent, identically distributed random variables with the uniform distribution $U(0,1)$ on the unit interval $[0,1]$ and let $Q(\xi) = [Q_{x,y}]_{x,y \geq 0}$ be the corresponding random recording tableau with the Plancherel distribution.

Then for any integer $k \in \mathbb{N}_0$ the collection of $k + 1$ random sets

$$(6) \left\{ \left\{ \frac{Q_{x,y} - n}{\sqrt{n}} : x \in \mathbb{N}_0 \right\} : y \in \{0, \ldots, k\} \right\}$$

converges in distribution, as $n \to \infty$, to a family of $k + 1$ independent Poisson point processes on $\mathbb{R}$ with the unit intensity.

1.7. **The Hammersley process.** The information about the sequence $w = (w_1, \ldots, w_\ell)$ can be encoded by a collection of points $(w_1, 1), \ldots, (w_\ell, \ell)$ on the plane (marked as small disks on Figure 2a). The time evolution of the bottom row of the insertion tableau in the process of insertions (1) can be encoded by the time evolution of a collection of particles on the real line (their trajectories are marked on Figure 2a as blue zig-zag lines) which is subject to the following dynamics. When we have reached one of the disks $(x, t)$
The dynamics of the particles in the Hammersley process with some initial configuration of the particles. The time flows from bottom to top. The second line of the multi-line Hammersley process.

(translation: at time \( t \), when a number \( x \) is inserted into the bottom row of the insertion tableau...) one of the following happens: (i) a particle, which is first to the right of \( x \), jumps left to \( x \) (translation: ...the newly inserted number \( x \) bumps from the bottom row the smallest number which is bigger than \( x \)), or (ii) a new particle is created in \( x \) (translation: the number \( x \) is appended at the end of the bottom row), see Figure 2a for an illustration.

If the locations of the disks on the upper halfplane are random, sampled according to the Poisson point process on \( I \times \mathbb{R}_+ \) (for some specified set \( I \subseteq \mathbb{R} \)) we obtain in this way the celebrated Hammersley process on \( I \) [Ham72; AD95].

The information about all bumpings from the bottom row of the insertion tableau can be encoded by the dual corners [FM09] (marked on Figure 2a by red X crosses). These crosses are used as an input for the dynamics of the second row of the insertion tableau in an analogous way as the disks were used for the dynamics of the bottom row, see Figure 2b. In other words, the output of the Hammersley process (which will be the first line of the multi-line Hammersley process which we will construct) is used as the input for the second line of the multi-line Hammersley process.

This procedure can be iterated; in this way the dynamics of all rows of the insertion tableau is fully encoded by the multi-line Hammersley process [FM09]. The name is motivated by the analogy with the tandem queues where the happy customers who exit one waiting line are the input for the second line.
1.8. **Limit distribution of the multi-line Hammersley process.** As we already mentioned, the entries of the bottom row of the insertion tableau can be interpreted as positions of the particles in (the de-Poissonized version of) the Hammersley interacting particle process on the unit interval \([0, 1]\). Therefore the following result (Corollary 1.4 below) is a generalization of the result of Aldous and Diaconis [AD95, Theorem 5(b)] which concerned only the special case \(k = 0\) of the single-line Hammersley process (in the Poissonized setup). For a more detailed discussion of the link between these results see Section 1.10.

The general case \(k \geq 0\) can be interpreted as a statement about the convergence of the multi-line version of the Hammersley process on the unit interval \([0, 1]\) to its stationary distribution on the whole real line \(\mathbb{R}\) which was calculated by Fan and Seppäläinen [FS20, Theorem 5.1].

Note that in his original paper [Ham72, Section 9] Hammersley considered the particle process with a discrete time parameter indexed by non-negative integers. Slightly confusingly, this process with the modern terminology would be referred to as the de-Poissonized version of the Hammersley process (as opposed to the Hammersley process in which the time is continuous and the input is given by the Poisson point process on the quarterplane). It follows that the setup which we consider in Corollary 1.4 coincides with the one from the original paper of Hammersley. The special case \(k = 0\) of Corollary 1.4 was conjectured already by Hammersley [Ham72, “assumption α” on page 371] who did not predict the exact value of the intensity of the Poisson process.

**Corollary 1.4.** Let \(\xi = (\xi_1, \ldots, \xi_n)\) be a sequence of independent, identically distributed random variables with the uniform distribution \(U(0, 1)\) on the unit interval \([0, 1]\) and let

\[
\begin{bmatrix}
P^{(n)}_{x,y} \
\end{bmatrix}_{y \in \mathbb{N}_0, 0 \leq x < \lambda_y^{(n)}} = P(\xi_1, \ldots, \xi_n)
\]

be the corresponding insertion tableau; we denote by \(\lambda^{(n)}\) its shape.

For any integer \(k \in \mathbb{N}_0\) and any real number \(0 < w < 1\) the collection of \(k + 1\) random sets \(P_0^{(n)}, \ldots, P_k^{(n)}\) with

\[
P^{(n)}_y := \left\{ \sqrt{w} \left( P^{(n)}_{x,y} - w \right) : 0 \leq x < \lambda_y^{(n)} \right\}
\]

(7)

converges in distribution, as \(n \to \infty\), to a family of \(k + 1\) independent Poisson point processes on \(\mathbb{R}\) with the intensity \(\frac{1}{\sqrt{w}}\).

The above statement remains true for \(w = 1\) but the limit in this case is a family of \(k + 1\) independent Poisson point processes on the negative halfline \(\mathbb{R}_-\) with the unit intensity.
The key ingredient of the proof is to use some symmetries of the RSK algorithm which allow to interchange the roles of the insertion tableau and the recording tableau, see Section 3.3 for the details.

We were inspired to state Corollaries 1.3 and 1.4 by the work of Azangulov [Aza20] who studied fluctuations of the last entry in the bottom row of $P^{(n)}$ around $w = 1$; more specifically he proved that the (shifted and rescaled) last entry in the bottom row

$$
\sqrt{n} \left(1 - \frac{P^{(n)}_{0,\lambda_0^{(n)} - 1}}{\lambda_0^{(n)}}\right)
$$

converges in law to the exponential distribution $\text{Exp}(1)$.

1.9. The idea behind the proof: the link between the Ulam’s problem and the Hammersley’s assumption $\alpha$. The key idea behind the proof of Theorem 1.1 lies in the intimate interplay between the Ulam’s problem and the Hammersley’s assumption $\alpha$ which was already subject to investigation by several researchers in this field.

1.9.1. Ulam’s problem. Recall that Ulam [Ula61] asked about the value of the limit

$$
c = \lim_{n \to \infty} \frac{\mathbb{E}\lambda_0^{(n)}}{\sqrt{n}}.
$$

The first solution to this problem consisted of two components: proving the lower bound $c \geq 2$ and the upper bound $c \leq 2$; interestingly these two components have quite different proofs.

The lower bound $c \geq 2$ was proved independently by Logan and Shepp [LS77] as well as by Vershik and Kerov [VK77] by finding explicitly the limit shape of typical random Young diagrams distributed according to the Plancherel measure. Both proofs were based on the hook-length formula for the number of standard Young tableaux of prescribed shape and finding the minimizer of the corresponding functional. An alternative approach which avoids the variational calculus is to use the results of Biane [Bia01] in order to show that the (scaled down) transition measure of a Plancherel-distributed random diagram $\lambda^{(n)}$ converges in probability to the semicircle distribution and to deduce that the probability of the event $\lambda_0^{(n)} < (2 - \epsilon)\sqrt{n}$ converges to zero for each $\epsilon > 0$.

This lower bound $c \geq 2$ will play an important role in our paper and we will use it in order to show Proposition 2.8 (more specifically, we use it in Lemma 2.6).
The upper bound $c \leq 2$ is due to Vershik and Kerov \cite{VK85a}. This upper bound plays an even more important role in our paper. We will come back to this topic in Section 1.9.3 below.

1.9.2. Hammersley’s assumption $\alpha$. Hammersley formulated his assumption $\alpha$ \cite[page 371]{Ham72} as a rather vague statement ("It is reasonable to assume that the distribution of the discontinuities $y_i$ is locally homogenous and random") which we interpret as a conjecture that the local behavior of the numbers in the bottom row of the insertion tableau $P^{(n)}$ after appropriate rescaling converges to some Poisson point process with unspecified intensity, cf. Corollary 1.4. Hammersley also gave an informal argument which explained how the assumption $\alpha$ would give solution to the Ulam’s problem and he correctly predicted the value of the constant $c = 2$.

The first proof of a result of the flavor of the assumption $\alpha$ is the aforementioned work of Aldous and Diaconis \cite[Theorem 5(b)]{AD95}. Interestingly, the proof starts with two separate parts: one which happens to give an alternative proof for the upper bound $c \leq 2$ for the constant in the Ulam’s problem, and one which happens to give an alternative proof for the lower bound $c \geq 2$. Finally, the combination of these two results gives the desired proof of the assumption $\alpha$.

The arguments in the aforementioned papers \cite{Ham72,AD95} were based on a probabilistic analysis of the Hammersley process viewed as an interacting particle system (see Section 1.7) and thus were quite different from those mentioned in Section 1.9.1.

1.9.3. The idea of the proof. As we can see from the aforementioned papers \cite{Ham72,AD95}, the Ulam’s problem and the assumption $\alpha$ are intimately related one with another and a solution to one of them gives (at least heuristically) the solution to the other one. From this perspective it is somewhat surprising that the original solution to the Ulam’s problem contained in the papers \cite{LS77,VK77,VK85a} did not result with a corresponding proof of the assumption $\alpha$ in the language of the Plancherel growth process and random Young diagrams. The current paper fills this gap.

Our strategy is to revisit the proof of the upper bound $c \leq 2$ which is due to Vershik and Kerov \cite[Раздел 3, Лемма 6]{VK85a} (see also \cite[Section 3, Lemma 6]{VK85b} for the English translation; be advised that there are two lemmas having number 6 in this paper). With the notations used in our paper (see the proof of Lemma 2.9), this proof can be rephrased as an application of the Cauchy–Schwarz inequality for a clever choice of a pair of vectors $X$ and $Y$ of (approximately) unit length. It is somewhat surprising that such a coarse bound as Cauchy–Schwarz inequality gives the optimal upper bound $c \leq 2$ for the Ulam’s constant. This phenomenon is an
indication that the Cauchy–Schwarz inequality is applied here in a setting in which it becomes (asymptotically) saturated, which implies that the vectors $X$ and $Y$ are (approximately) multiples of one another and therefore $X \approx Y$. It follows in particular that

$$\langle X - Y, Y \rangle \approx 0.$$

Let $\lambda^{(0)} \nearrow \lambda^{(1)} \nearrow \cdots$ be the Plancherel growth process. It turns out that a slight modification of the left-hand side of (9) has a natural probabilistic interpretation as the total variation distance between:

- the probability distribution of the Young diagram $\lambda^{(n)}$, and
- the *conditional* probability distribution of the Young diagram $\lambda^{(n)}$ *under the condition* that the growth between the diagrams $\lambda^{(n-1)}$ and $\lambda^{(n)}$ occurred in a specified row.

In particular, (9) implies that this total variation distance converges to zero as $n \to \infty$, see Lemma 2.9 for a precise statement.

Heuristically, this means that the information about the number of the row $r^{(n)}$ in which the growth occurred between $\lambda^{(n-1)}$ and $\lambda^{(n)}$ does not influence too much the distribution of the resulting random Young diagram $\lambda^{(n)}$. Since the Plancherel growth process is a Markov process, this argument can be iterated to show that the numbers of the rows

$$r^{(n+1)}, \ldots, r^{(n+\ell)}$$

in which the growths occur in the part of the Plancherel growth process $\lambda^{(n)} \nearrow \cdots \nearrow \lambda^{(n+\ell)}$ are approximately independent random variables, see Theorem 2.2 for a precise statement. Various variants of the assumption $\alpha$ are now simple corollaries.

This approach gives some additional information — which does not seem to be available by the hydrodynamic approach [AD95; CG05] — about the asymptotic independence of the rows (10) and the shape of the final Young diagram $\lambda^{(n+\ell)}$, see Theorem 2.2 for more detail. This additional information will be essential in our forthcoming paper devoted to the refined asymptotics of the bumping routes [MMŚ21].

The aforementioned Lemma 2.9 can be seen as an additional step in the reasoning which was overlooked by the authors of [VK85a]. The monograph of Romik [Rom15, Section 1.19] contains a more pedagogical presentation of these ideas of Vershik and Kerov; in the following we will use Romik’s notations with some minor adjustments.

Note that, analogously as the proof of Aldous and Diaconis [AD95, Theorem 5(b)], our proof of the assumption $\alpha$ is based on combining two components: the one which is related to the lower bound $c \geq 2$ in the Ulam’s problem (see Lemma 2.6) and the one related to the upper bound $c \leq 2$ (see
the proof of Lemma 2.9 and only combination of these two components completes the proof.

1.10. The link with the work of Aldous and Diaconis. Following the notations from [AD95] we consider the Hammersley process on \( \mathbb{R}_+ \) (starting from the empty configuration of the particles) and denote by \( N^+(x, t) \) the number of particles at time \( t \) which have their spacial coordinate \( \leq x \). Aldous and Diaconis [AD95, Theorem 5(b)] proved that for each fixed \( w > 0 \) the counting process

\[
(N^+(ws + y, s) - N^+(ws, s), \ y \in \mathbb{R})
\]

converges in distribution, as \( s \to \infty \), to the Poisson counting process with intensity \( w^{-1/2} \).

1.10.1. The link of Corollary 1.4. In the following we sketch very briefly the proof that the special case of Corollary 1.4 which corresponds to \( k = 0 \) is equivalent to the aforementioned result of Aldous and Diaconis.

Due to the space-time scale invariance [AD95, Lemma 4], the stochastic process (11) has the same distribution as

\[
\left( N^+ \left( w + \frac{y}{s}, s^2 \right) - N^+ (w, s^2), \ y \in \mathbb{R} \right)
\]

Let \( \eta_1(s^2) < \eta_2(s^2) < \cdots \) denote the positions of the particles at time \( s^2 \). The result of Aldous and Diaconis can be therefore rephrased as convergence in distribution of the set of jumps of the function (12) which is equal to

\[
\left\{ s \cdot \left[ \eta_i(s^2) - w \right] : i \in \mathbb{N} \right\}
\]

towards the Poisson point process with the intensity \( w^{-1/2} \).

For simplicity we restrict our attention to \( 0 < w < 1 \) (the general case \( w > 0 \) can be obtained by an application of a slightly more involved space-time scale invariance). The above result does not change if we modify our setup and consider the Hammersley process on the unit interval \([0, 1]\). The number of disks (which are the input for the Hammersley process) in the rectangle \([0, 1] \times [0, t]\) is equal to the value \( N(t) \) of the Poisson process at time \( t \). With the notations of Corollary 1.4, the entries \( P^{(n)}_{x, 0} : 0 \leq x < \lambda^{(n)}_0 \) of the bottom row in the insertion tableau \( P^{(n)} \) can be interpreted as the coordinates of the particles in the Hammersley process at the time when \( n \) disks appeared; it follows that the set (13) is equal to

\[
\left\{ s \cdot \left[ P^{(N(s^2))}_{x, 0} - w \right] : 0 \leq x < \lambda^{(n)}_0 \right\}.
\]
In this way we proved that the distribution of the random set $\{14\}$ (which appears in a reformulation of the result of Aldous and Diaconis) is the mixture of the probability distribution of the random set $P_0^{(n)}$ given by $\{7\}$ which corresponds to the bottom row, rescaled by the factor $\frac{1}{\sqrt{n}}$. The mixture is taken over $n := N(s^2)$ which has the Poisson distribution $\text{Pois}(s^2)$. Since the scaling factor $\frac{1}{\sqrt{n}}$ converges in probability to 1 as $s \to \infty$, the scaling is asymptotically irrelevant.

We proved in this way that the result of Aldous and Diaconis [AD95, Theorem 5(b)] is a consequence of the special case of Corollary $\{14\}$ for $k = 0$, obtained by a straightforward Poissonization procedure.

The implication in the opposite direction is more challenging, but general de-Poissonization techniques [JS98] can be used to show that the result of Aldous and Diaconis implies the special case $k = 0$ of our Corollary $\{14\}$.

1.10.2. The link of Theorem $\{11\}$ In the following we sketch the proof that the special case of Theorem $\{11\}$ which corresponds to $k = 0$ is equivalent to the result of Aldous and Diaconis $\{11\}$. For simplicity we will consider only the special case $w = 1$; the general case $w > 0$ would follow from a slightly more complex scaling of the space-time.

Due to the space-time scale invariance [AD95, Lemma 4], the stochastic process $\{11\}$ in the special case $w = 1$ has the same distribution as

\[
\left( N^+(s^2 + ys, 1) - N^+(s^2, 1), y \in \mathbb{R} \right)
\]

and, due to the space-time interchange property [AD95, Lemma 3], the same distribution as

\[
\left( N^+(1, s^2 + ts) - N^+(1, s^2), t \in \mathbb{R} \right).
\]

By [AD95, Theorem 5(b)] the process $\{11\}$ for $w = 1$ or, equivalently, the process $\{15\}$ converges for $s \to \infty$ to the Poisson process with the unit intensity. This result does not change if we modify our setup and consider the Hammersley process on the unit interval $[0, 1]$.

For each $t \geq 0$ the number of disks (which are the input for the Hammersley process) in the rectangle $[0, 1] \times [0, t]$ is equal to the value $N(t)$ of the Poisson process at time $t$. Thus the number of all particles at time $t$, given by $N^+(1, t) = \lambda_0^{(N(t))}$, is equal to the length of the bottom row of the insertion tableau after $N(t)$ disks appeared. In this way we proved that the probability distribution of the process $\{15\}$ coincides with the probability distribution of the process

\[
\left( \lambda_0^{(N(s^2 + ts))} - \lambda_0^{(N(s^2))}, t \in \mathbb{R} \right).
\]
We set $n := N(s^2)$ and

$$\tau(t) = \frac{N(s^2 + ts) - N(s^2)}{\sqrt{N(s^2)}}$$

so that $N(s^2 + ts) = n_{\tau(t)}$ with the notations from (5). In the case when $N(s^2) = 0$ and (17) is not well-defined, we declare that $n_{\tau(t)} = N(s^2 + ts)$ by definition. Since the probability of the event that $N(s^2) = 0$ converges to zero as $s \to \infty$, this will not create problems in the following.

The random function $\tau$ converges in probability, as $s \to \infty$, to the identity map $t \mapsto t$ uniformly over compact subsets. With these notations the probability distribution of the process (15) coincides with the probability distribution of the process

$$\left(\lambda_0^{(n_{\tau(t)})} - \lambda_0^{(n)}, \ t \in \mathbb{R}\right).$$

By taking the average over the random values of $n$ and $\tau$ it follows that the special case of Theorem 1.1 for $k = 0$ implies that (18) indeed converges to the Poisson process. We proved in this way that the result of Aldous and Diaconis [AD95, Theorem 5(b)] (at least in the special case $w = 1$) is a consequence of the special case of Theorem 1.1 for $k = 0$, obtained by a rather straightforward Poissonization procedure.

The implication in the opposite direction is more challenging, but again general de-Poissonization techniques [JS98] can be applied.

1.11. Asymptotics of the bottom rows. The research related to the Ulam’s problem culminated in the works of Baik, Deift and Johansson [BDJ99; BDJ00] as well as its extensions [BOO00; Oko00; Joh01]. Roughly speaking, these results say that the suitably normalized lengths of the bottom rows of a Plancherel distributed random Young diagram converge to an explicit non-Gaussian limit which is related to the eigenvalues of a large GUE random matrix, see the monograph [Rom15] for a more pedagogical introduction.

Such a non-Gaussian limit behavior for the lengths of the bottom rows is at a sharp contrast with our Theorem 1.1 which states that the growths of the bottom rows $\Delta_n(t)$ are given by the Poisson process which with the right scaling converges to the Brownian motion. This discrepancy is an indication that the process $\Delta_n(t)$ considered in that theorem cannot be approximated by the Poisson process in the scaling as $t = t(n) \gg 1$. It would be interesting to find the scaling in which this passage from the regime of independent growths to the non-Gaussian regime related to the random matrices occurs. See also Problem 3.2.
1.12. Possible generalization of Corollary 1.4: the bottom rows in the Schur–Weyl insertion tableau. We suspect that Corollary 1.4 is a special case of a conjectural result which would hold in a much wider generality. In the current section we will present the details.

1.12.1. Schur–Weyl insertion tableau. For given positive integers \( d \) and \( n \) let \( w = (w_1, \ldots, w_n) \) be a sequence of independent, identically distributed random variables with the uniform distribution on the discrete set \( \{1, \ldots, d\} \) with \( d \) elements; we denote by \( P = P(w_1, \ldots, w_n) \) the corresponding insertion tableau. This random tableau appears naturally in the context of the Schur–Weyl duality which we review in the following. The tensor product \( \mathbb{C}^d \otimes \mathbb{C}^n \) has a natural structure of a \( \mathfrak{S}_n \times \text{GL}_d(\mathbb{C}) \)-module; the irreducible components are naturally indexed by Young diagrams. The Schur–Weyl measure is defined as the probability distribution on Young diagrams which corresponds to sampling a random irreducible component of \( \mathbb{C}^d \otimes \mathbb{C}^n \) with the probability proportional to the dimension of the component, see [Mél11] for the details. The probability distribution of the shape of \( P \) coincides with the Schur–Weyl measure; for this reason we will call the random tableau \( P \) itself the Schur–Weyl insertion tableau.

In the scaling when \( d, n \to \infty \) tend to infinity in such a way that \( d \approx c \sqrt{n} \) for some \( c > 0 \), there is a law of large numbers for the global form of the scaled down insertion tableau \( P \), see [MS20, Remark 1.6 and Section 1.7.3].

In the following we will concentrate on another aspect of the Schur–Weyl insertion tableau \( P \), namely in the entries which are located in the bottom row, near its end. For an integer \( i \in \{1, \ldots, d\} \) we denote by \( M_i = M_i(P) \) the number of occurrences of \( i \) in the bottom row of \( P \); our problem is to understand the joint distribution of the family of random variables \( M_i \) indexed by \( i \in I \) in some interval of the form

\[
I = \{d + 1 - \ell, d + 2 - \ell, \ldots, d\}
\]

for some choice of an integer \( \ell \geq 1 \).

There are two interesting limits which one can consider in this setup, namely the one when the size of the alphabet \( d \to \infty \) tends to infinity, and the one when the length of the sequence \( n \to \infty \) tends to infinity. In the following we will review these two limits and discuss what happens when these two limits are iterated.

1.12.2. The limit \( d \to \infty \). When \( d \to \infty \) and the length \( n \) of the sequence is fixed, the probability that \( w_1, \ldots, w_n \) are all distinct converges to \( 1 \). In the following we consider the conditional probability space for which \( w_1, \ldots, w_n \) are all distinct; it is easy to check that such a conditional joint distribution of the ratios \( \frac{w_1}{d}, \ldots, \frac{w_n}{d} \) converges to that of a tuple \( \xi_1, \ldots, \xi_n \)
of independent random variables with the uniform distribution \( U(0, 1) \). It follows that the (conditional, and hence unconditional as well) probability distribution of the tableau \( \frac{1}{d}P \) (which is obtained from \( P \) by dividing each entry by \( d \)) converges to the probability distribution of the insertion tableau \( P(\xi_1, \ldots, \xi_n) \).

We see that the iterated limit in which we first take the limit as the size of the alphabet \( d \to \infty \) tends to infinity and then the limit \( n \to \infty \) as the length of the sequence tends to infinity, is the one in which we recover Corollary \([14]\). Heuristically, we can expect that if \( n \to \infty \) and \( d = d(n) \gg n^2 \) tends to infinity fast enough, each integer from the interval \( I \) given by \([19]\) of moderate length \( \ell = O \left( \frac{d}{\sqrt{n}} \right) \) will not appear more than once (except for asymptotically negligible probability); and that the probability that a given integer \( i \in I \) will appear in the bottom row is of order

\[
q = \frac{n}{d} \cdot \frac{1}{\sqrt{n}} = \frac{\sqrt{n}}{d} \approx \frac{\sqrt{n}}{d + \sqrt{n}} \ll 1
\]

which is approximately the product of the probability that \( i \) belongs to the sequence \( w \) (which is roughly \( \frac{n}{d} \)) and the probability that a given large entry of the sequence \( w \) will be in the bottom row (which is roughly \( \frac{1}{\sqrt{n}} \) by Corollary \([14]\)).

Such a Bernoulli probability distribution on the set \( \{0, 1\} \) with the success probability \( q \) can be approximated by the geometric distribution on the set of non-negative integers with the parameter \( p = 1 - q \). We formalize this informal discussion as the following conjecture.

**Conjecture 1.5.** Let \( d = d(n) \) and \( \ell = \ell(n) \) be as above.

Then the total variation distance between:

- the random vector of the multiplicities

\[
(M_i : i \in I),
\]

and

- the collection of \( \ell \) independent random variables, each with the geometric distribution \( \text{Geo}(p) \) with the parameter

\[
p = 1 - \frac{\sqrt{n}}{d + \sqrt{n}} = \frac{d}{d + \sqrt{n}}
\]

converges to zero, as \( n \to \infty \).

In the following we will discuss whether this conjecture is plausible also in some other choices of the scaling for the parameters \( d \) and \( n \).
1.12.3. The limit $n \to \infty$. For the purposes of the current section by the $d \times d$ GUE random matrix we understand the random matrix with the Gaussian distribution supported on Hermitian $d \times d$ matrices, which is invariant under conjugation by unitary matrices, and normalized in such a way that the variance of each diagonal entry is equal to $d$. The traceless GUE random matrix is obtained from the above random matrix by subtracting a multiple of the identity matrix in such a way that the trace of the outcome is equal to zero.

For $m \in \{0, 1, \ldots, d\}$ we denote by
\[ \lambda^{(im)} = (\lambda_0^{(im)}, \ldots, \lambda_{m-1}^{(im)}) = \text{sh} P_{\leq m} \]
the shape of the tableau $P_{\leq m}$ which is obtained from the Schur–Weyl random insertion tableau $P$ by keeping only the boxes which are at most $m$; in particular $\lambda^{(id)}$ is the shape of $P$. Recall that $P$ and, as a consequence, $\lambda^{(im)}$ as well, depend implicitly on the choice of the positive integers $d$ and $n$. In the following we fix the value of the integer $d \geq 1$ and let $n$ vary. Johansson [Joh01, Theorem 1.6] proved that the distribution of the random vector
\[ \frac{d\lambda^{(id)} - n}{\sqrt{n}} \]
converges, as $n \to \infty$, to the joint distribution of the eigenvalues $\text{spec} X$ of the traceless $d \times d$ GUE random matrix $(X_{ij})_{1 \leq i, j \leq d}$. This result can be further extended using the ideas of Kuperberg [Kup02], one can show that the joint distribution of the random vectors
\[ \frac{d\lambda^{(id)} - n}{\sqrt{n}}, \frac{d\lambda^{(id-1)} - n}{\sqrt{n}}, \ldots, \frac{d\lambda^{(1)} - n}{\sqrt{n}} \]
converges in distribution to the joint distribution of the eigenvalues of $X$, together with the eigenvalues of the minors of $X$ obtained by iterative removal of the last row and the last column:
\[ \text{spec}(X_{ij})_{1 \leq i, j \leq d}, \text{spec}(X_{ij})_{1 \leq i, j \leq d-1}, \ldots, \text{spec}(X_{ij})_{1 \leq i, j \leq 1}. \]

Since the content of the current section is mostly heuristic, we skip the details of the proof; the key point is to use [CŚ09, Corollary 5.2].

With the above notations, for $m \in \{2, \ldots, d\}$ we have that $M_m = \lambda_0^{(im)} - \lambda_0^{(im-1)}$ is the number of entries in the bottom row of the tableau $P$ which are equal to $m$. Also, for $m \in \{1, \ldots, d\}$ let $\mu^{(m)} = \max \left( \text{spec}(X_{ij})_{1 \leq i, j \leq m} \right)$ be the largest eigenvalue of the minor $(X_{ij})_{1 \leq i, j \leq m}$. The aforementioned result implies, in particular, that the probability distribution of the random vector
\[ \frac{d}{\sqrt{n}} \left( M_d, M_{d-1}, \ldots, M_2 \right) \]
which describes the content of the bottom row of $P$ (after disregarding all entries equal to 1) converges, as $n \to \infty$, to the joint distribution of the entries of the random vector
\begin{equation}
(\mu^{(d)} - \mu^{(d-1)}, \mu^{(d-1)} - \mu^{(d-2)}, \ldots, \mu^{(2)} - \mu^{(1)})
\end{equation}

Gorin and Shkolnikov [GS17, Corollary 1.3] studied the asymptotics $d \to \infty$ of the probability distribution of any prefix of a fixed length $\ell$ of the random vector (22). In this scaling the difference between the GUE random matrix and the traceless GUE random matrix turns out to be irrelevant. More specifically, they proved that for each fixed value of an integer $\ell \geq 1$ the joint distribution of the random variables
\begin{equation}
(\mu^{(d)} - \mu^{(d-1)}, \mu^{(d-1)} - \mu^{(d-2)}, \ldots, \mu^{(d+1-\ell)} - \mu^{(d-\ell)})
\end{equation}
converges to the distribution of $\ell$ independent random variables, each with the exponential distribution $\text{Exp}(1)$.

By combining these results it follows that for each fixed integer $\ell \geq 1$ the probability distribution of the random vector
\begin{equation}
\frac{d}{\sqrt{n}} (M_d, M_{d-1}, \ldots, M_{d+1-\ell})
\end{equation}
converges to that of independent exponential random variables in the iterated limit in which we first take the limit $n \to \infty$ as the length of the sequence tends to infinity, and then the limit as the size of the alphabet $d \to \infty$ tends to infinity. Since the exponential distribution arises naturally as a limit of the geometric distribution $\text{Geo}(p)$ in the scaling as $p \to 0$, we suspect that the following stronger result is true.

**Conjecture 1.6.** Conjecture 1.5 remains true if $d = d(n)$ is a sequence of positive integers which tends to infinity in a sufficiently slow way and $\ell \geq 1$ is a fixed integer.

1.12.4. The joint limit $d, n \to \infty$. Heuristically, each of the iterated limits considered at the very end of the Sections 1.12.2 and 1.12.3 can be seen as a limit in which both variables $d, n \to \infty$ tend to infinity in such a way that one of these variables grows much faster than the other one. With these two extreme cases covered one can wonder whether Conjecture 1.5 holds true in general.

**Conjecture 1.7.** Conjecture 1.5 remains true if $d = d(n)$ is an arbitrary sequence of positive integers which tends to infinity and $\ell = \ell(n)$ is a sequence of positive integers which is either constant or tends to infinity in such a way that $\ell = O\left(\frac{d}{\sqrt{n}}\right)$. 
Particularly interesting is the balanced scaling in which \( n \) and \( d \) tend to infinity in such a way that \( \frac{d}{\sqrt{n}} \) converges to some limit so that the law of large numbers applies to the shape of \( P \) [Bia01, Section 3] as well as to the tableau \( P \) itself [MS20, Remark 1.6 and Section 1.7.3]. Conjecture 1.7 concerns the entries in the bottom row; we conjecture that an analogous result for any fixed number of bottom rows remains true.

2. Estimates for the total variation distance

Our main tool for proving the main results of the paper is Theorem 2.2. It gives an insight into the way in which the first rows of a Young diagram develop in the Plancherel growth process (thanks to this part we will have Theorem 1.1 as a straightforward corollary), together with the information about the global shape of the Young diagram. This latter additional information will be key for the developments in our forthcoming paper [MMS21].

2.1. Total variation distance. Suppose that \( \mu \) and \( \nu \) are probability measures on the same discrete set \( S \). Such measures can be identified with real-valued functions on \( S \). We define the total variation distance between the measures \( \mu \) and \( \nu 

(23) \quad \delta(\mu, \nu) := \frac{1}{2} \| \mu - \nu \|_{\ell^1} = \max_{X \in S} \left| \mu(X) - \nu(X) \right|

as half of their \( \ell^1 \) distance as functions. If \( X \) and \( Y \) are two random variables with values in the same discrete set \( S \), we define their total variation distance \( \delta(X, Y) \) as the total variation distance between their probability distributions.

Several times we will use the following simple lemma.

Lemma 2.1.

(a) Let \( X = (X_1, X_2) \) and \( Y = (Y_1, Y_2) \) be random vectors with independent coordinates and such that their first coordinates have equal distribution: \( X_1 \overset{d}{=} Y_1 \). Then the total variation distance between the vectors is equal to the total variation distance between their second coordinates:

\[ \delta(X, Y) = \delta(X_2, Y_2). \]

(b) Let \( X = (X_1, \ldots, X_\ell) \) and \( Y = (Y_1, \ldots, Y_\ell) \) be random vectors with independent coordinates. Then the total variation distance between the random vectors is bounded by the sum of the coordinate-wise total variation distances:

\[ \delta(X, Y) \leq \sum_{1 \leq i \leq \ell} \delta(X_i, Y_i). \]
If \(\mu_1, \ldots, \mu_\ell\) and \(\nu_1, \ldots, \nu_\ell\) are discrete measures on some vector space then the total variation distance between their convolutions is bounded by the sum of the summand-wise total variation distances:

\[
\delta (\mu_1 * \cdots * \mu_\ell, \nu_1 * \cdots * \nu_\ell) \leq \sum_{1 \leq i \leq \ell} \delta (\mu_i, \nu_i).
\]

This result seems to be folklore wisdom [pik15], nevertheless we failed to find a conventional reference and we provide a proof below.

Proof. For part (a) we view the total variation distance \(\delta(X, Y)\) as half of the appropriate \(\ell^1\) norm. This double sum factorizes thanks to independence.

For the part (b) we consider a collection of random vectors given by

\[
Z^i = (Y_1, \ldots, Y_i, X_{i+1}, \ldots, X_\ell) \quad \text{for } i \in \{0, \ldots, \ell\}.
\]

In this way \(Z^0 = X\) and \(Z^\ell = Y\); the neighboring random vectors \(Z^{i-1}\) and \(Z^i\) differ only on the \(i\)-th coordinate. By the triangle inequality

\[
\delta(X, Y) \leq \sum_{1 \leq i \leq \ell} \delta(Z^{i-1}, Z^i) = \sum_{1 \leq i \leq \ell} \delta(X_i, Y_i),
\]

where the last equality is a consequence of part (a).

In order to prove part (c) we shall use (b) with the special choice that \(X_i\) is a random variable with the distribution \(\mu_i\) and \(Y_i\) is a random variable with the distribution \(\nu_i\). An application of the same measurable map to both arguments \(X = (X_1, \ldots, X_\ell)\) and \(Y = (Y_1, \ldots, Y_\ell)\) cannot increase the total variation distance between them, so

\[
\delta (\mu_1 * \cdots * \mu_\ell, \nu_1 * \cdots * \nu_\ell) = \delta (X_1 + \cdots + X_\ell, Y_1 + \cdots + Y_\ell) \leq \delta(X, Y).
\]

The application of part (b) to the right-hand side completes the proof. \(\square\)

2.2. Growth of rows in Plancherel growth process. Let us fix an integer \(k \in \mathbb{N}_0\). We define the finite set \(\mathcal{N} = \{0, 1, \ldots, k, \infty\}\) which can be interpreted as the set of the natural numbers from the perspective of a person who cannot count on numbers bigger than \(k\) (for example, for \(k = 3\) we would have “zero, one, two, three, many”).

Let \(\lambda^{(0)} \nearrow \lambda^{(1)} \nearrow \cdots\) be the Plancherel growth process. For integers \(n \geq 1\) and \(r \in \mathbb{N}_0\) we denote by \(E_r^{(n)}\) the random event which occurs if the unique box of the skew diagram \(\lambda^{(n)}/\lambda^{(n-1)}\) is located in the row with the index \(r\). For \(n \geq 1\) we define the random variable \(R^{(n)}\) which takes values in \(\mathcal{N}\) and which is given by

\[
R^{(n)} = \begin{cases} 
  r & \text{if the event } E_r^{(n)} \text{ occurs for } 0 \leq r \leq k, \\
  \infty & \text{if the event } E_r^{(n)} \text{ occurs for some } r > k,
\end{cases}
\]
and which — from the perspective of the aforementioned person with limited counting skills — gives the number of the row in which the growth occurred.

Let \( \ell = \ell(m) \) be a sequence of non-negative integers such that
\[
\ell = O\left(\sqrt{m}\right).
\]
For a given integer \( m \geq (k + 1)^2 \) we focus on the specific part of the Plancherel growth process
\[
\lambda^{(m)} \to \ldots \to \lambda^{(m+\ell)}
\]
We will encode some partial information about the growths of the rows as well as about the final Young diagram in (24) by the random vector
\[
V^{(m)} = \left( R^{(m+1)}, \ldots, R^{(m+\ell)}, \lambda^{(m+\ell)} \right) \text{ rv } \mathbb{N}^{\ell} \times \mathbb{Y}.
\]
We also consider the random vector
\[
\bar{V}^{(m)} = \left( \bar{R}^{(m+1)}, \ldots, \bar{R}^{(m+\ell)}, \bar{\lambda}^{(m+\ell)} \right) \text{ rv } \mathbb{N}^{\ell} \times \mathbb{Y}
\]
which is defined as a sequence of independent random variables; the random variables \( \bar{R}^{(m+1)}, \ldots, \bar{R}^{(m+\ell)} \) have the same distribution given by
\[
\mathbb{P}\left\{ \bar{R}^{(m+i)} = r \right\} = \frac{1}{\sqrt{m}} \quad \text{for } r \in \{0, \ldots, k\},
\]
\[
\mathbb{P}\left\{ \bar{R}^{(m+i)} = \infty \right\} = 1 - \frac{k + 1}{\sqrt{m}}
\]
and \( \bar{\lambda}^{(m+\ell)} \) is distributed according to Plancherel measure Plan\( _{m+\ell} \); in particular the random variables \( \lambda^{(m+\ell)} \) and \( \bar{\lambda}^{(m+\ell)} \) have the same distribution.

Heuristically, the following result states that when Plancherel growth process is in an advanced stage and we observe a relatively small number of its additional steps, the growths of the bottom rows occur approximately like independent random variables. Additionally, these growths do not affect too much the final shape of the Young diagram.

**Theorem 2.2.** With the above notations, for each fixed \( k \in \mathbb{N}_0 \) the total variation distance between \( V^{(m)} \) and \( \bar{V}^{(m)} \) converges to zero, as \( m \to \infty \); more specifically
\[
\delta\left( V^{(m)}, \bar{V}^{(m)} \right) = o\left( \frac{\ell}{\sqrt{m}} \right).
\]
The proof is postponed to Section 2.6; in the forthcoming Sections 2.3 to 2.5 we will gather the tools which are necessary for this goal. In fact, we conjecture that the following stronger result is true.
Conjecture 2.3. The distance (29) is equal to
\[ O \left( \frac{\ell}{m^\frac{5}{6}} \right). \]

We will discuss this stronger result in Section 2.7.

2.3. Asymptotics of growth of a given row. Our main result in this sub-
section is Proposition 2.8 which gives asymptotics of the probability of
a growth of a given row in the Plancherel growth process. This result is
not new; it was proved by Okounkov [Oko00, Proposition 2]. Neverthe-
less we provide an alternative (hopefully simpler) proof below. As a prepa-
ration, we start with some auxiliary lemmas.

In the following we keep the notations from the beginning of Section 2.2;
in particular \( \lambda^{(0)} \to \lambda^{(1)} \to \cdots \) is the Plancherel growth process. Let
\( K \in \mathbb{N}_0 \) be fixed. For \( n \geq 1 \) we define
\[ s^{(n)}_K = \sum_{0 \leq r \leq K} \mathbb{P} \left( E^{(n)}_r \right), \]
\[ (30) \]
i.e., \( s^{(n)}_K \) is defined as the probability that the unique box of the skew
diagram \( \lambda^{(n)}/\lambda^{(n-1)} \) is located in one of the rows \( 0, 1, \ldots, K \).

Lemma 2.4. For each \( K \in \mathbb{N}_0 \) the sequence \( s^{(1)}_K, s^{(2)}_K, \ldots \) is weakly decrea-
sing.

Proof. For \( n \geq 1 \) let \( d_n \) denote the unique box of the skew diagram
\( \lambda^{(n)}/\lambda^{(n-1)} \). Then \( s^{(n)}_K \) is equal to the probability that the box \( d_n \)
is located in one of the rows \( 0, 1, \ldots, K \).

Romik and the last named author [RS15, Section 3.3] constructed a ran-
dom sequence of boxes \( q_1, q_2, \ldots \) (which is “the jeu de taquin trajectory
in the lazy parametrization”) such that for each \( n \geq 1 \) we have equality of
distributions [RS15, Lemma 3.4]
\[ d_n \overset{d}{=} q_n \]
and, furthermore, each box \( q_{n+1} \) is obtained from the previous one \( q_n \) by
moving one node to the right, or node up, or by staying put. In this way
\( (\text{the number of the row of } q_n)_{n \geq 1} \)
is a weakly increasing sequence of random variables. It follows that the cor-
responding cumulative distribution functions evaluated in point \( K \)
\[ s^{(n)}_K = \mathbb{P}\{ (\text{the number of the row of } d_n) \leq K \} = \mathbb{P}\{ (\text{the number of the row of } q_n) \leq K \} \]
form a weakly decreasing sequence, which completes the proof. \( \square \)
Lemma 2.5. For each $K \in \mathbb{N}_0$ and $n \geq 1$

$$s_K^{(n)} \leq \frac{K + 1}{\sqrt{n}}.$$  

Proof. The monograph of Romik [Rom15, Section 1.19] contains the proof (which is based on the work of Vershik and Kerov [VK85a, VK85b, Section 3, Lemma 6]) of the inequality

$$\mathbb{P}(E_r^{(n)}) \leq \frac{1}{\sqrt{n}}$$

in the special case of the bottom row $r = 0$. After some minor adjustments this proof is applicable to the general case of $r \in \mathbb{N}_0$ (in fact, these adjustments are explicitly explained in the proof of Eq. (36) later on). The summation over $r \in \{0, \ldots, K\}$ concludes the proof. □

Lemma 2.6. For each $K \in \mathbb{N}_0$

$$\lim \inf_{n \to \infty} \frac{s_K^{(1)} + \cdots + s_K^{(n)}}{\sqrt{n}} \geq 2(K + 1).$$

Proof. We write $\lambda^{(n)} = (\lambda_0^{(n)}, \lambda_1^{(n)}, \ldots)$ so that $\lambda_r^{(n)}$ is the length of the appropriate row of the Young diagram $\lambda^{(n)}$. The work of Logan and Shepp [LS77] as well as the work of Vershik and Kerov [VK77] contains the proof that for each $\epsilon > 0$

$$\lim_{n \to \infty} \mathbb{P} \left\{ \frac{\lambda_r^{(n)}}{\sqrt{n}} < 2 - \epsilon \right\} = 0$$

in the special case of the bottom row $r = 0$. We will revisit this proof and explain how to adjust it for the general case $r \in \mathbb{N}_0$.

With the notations of Romik [Rom15, the proof of Theorem 1.23], if (32) were not true for some $\epsilon > 0$ and $r \in \mathbb{N}_0$, then for infinitely many values of $n$ the corresponding function $\psi_n$ (which encodes the Young diagram $\lambda^{(n)}$ in the Russian coordinate system [Rom15, Section 1.17]) would be bounded from above by the shifted absolute value function $|u| + r \sqrt{\frac{2}{n}}$ on the interval $\left[\sqrt{2} - \frac{\epsilon}{n}, \sqrt{2}\right]$. Clearly this would prevent $\psi_n$ from converging uniformly to the limit shape in contradiction to Logan–Shepp–Vershik–Kerov limit theorem [Rom15, Theorem 1.22].

Equation (32) implies that for each $r \in \mathbb{N}_0$

$$\lim \inf_{n \to \infty} \frac{\mathbb{E}\lambda_r^{(n)}}{\sqrt{n}} \geq 2.$$
We revisit the ideas of Vershik and Kerov [VK85a, VK85b, Section 3, Lemma 6], see also [Rom15, Section 1.19]. Since the indicator of the event $E^{(m)}_r$ fulfills
\[ I_{E^{(m)}_r} = \lambda^{(m)}_r - \lambda^{(m-1)}_r \]
it follows that
\[ \mathbb{P} (E^{(m)}_r) = \mathbb{E} I_{E^{(m)}_r} = \mathbb{E} \lambda^{(m)}_r - \mathbb{E} \lambda^{(m-1)}_r. \]
By summing over $1 \leq m \leq n$ and over $0 \leq r \leq K$ it follows that
\[ s^{(1)}_K + \cdots + s^{(n)}_K = \mathbb{E} \lambda^{(n)}_0 + \cdots + \mathbb{E} \lambda^{(n)}_K. \]
Application of (33) completes the proof. □

**Lemma 2.7.** For each $K \in \mathbb{N}_0$
\[ \lim_{n \to \infty} \sqrt{n} s^{(n)}_K = K + 1. \]

**Proof.** We will use a simplified notation and write $s^{(i)} = s^{(i)}_K$.

For the lower bound, suppose \textit{a contrario} that for some $\epsilon > 0$ there exist infinitely many values of an integer $n \geq 1$ for which
\[ \sqrt{n} s^{(n)} < (1 - \epsilon)(K + 1). \]
Let $C > 0$ be a number which will be fixed later in the proof and set $m = n + \lfloor Cn \rfloor$. Lemmas 2.4 and 2.5 imply that
\[ \frac{s^{(1)} + \cdots + s^{(m)}}{(K + 1)\sqrt{n}} \leq \left( \frac{s^{(1)} + \cdots + s^{(n)}}{(K + 1)\sqrt{n}} + \frac{(m - n)s^{(n)}}{(K + 1)\sqrt{n}} \right) < \frac{1}{\sqrt{n} \sqrt{1 + \cdots + \frac{1}{\sqrt{n}}}} + (1 - \epsilon)\frac{m - n}{n}. \]
On the right-hand side we may bound the sum $\frac{1}{\sqrt{1}} + \cdots + \frac{1}{\sqrt{n}}$ by the corresponding integral, thus
\[ \sqrt{\frac{m}{n}} \cdot s^{(1)} + \cdots + s^{(m)} = \left( \frac{s^{(1)} + \cdots + s^{(n)}}{(K + 1)\sqrt{n}} \right) < \frac{1}{\sqrt{n}} \int_0^n \frac{1}{\sqrt{x}} \, dx + (1 - \epsilon)\frac{m - n}{n} = 2 + (1 - \epsilon)\frac{m - n}{n}. \]
Passing to the limit $n \to \infty$ for the values of $n$ for which (35) holds true, Lemma 2.6 and the above inequality imply that
\[ 2\sqrt{1 + C} \leq 2 + (1 - \epsilon)C. \]
However, the above inequality is not fulfilled for any
\[ 0 < C < \frac{4\epsilon}{(1 - \epsilon)^2} \]
which completes the proof \textit{a contrario}. \qed

The following result is due to Okounkov [Ok00, Proposition 2]. We provide an alternative, hopefully simpler proof below.

**Proposition 2.8.** For each \( i \in \mathbb{N}_0 \)
\[ \lim_{n \to \infty} \sqrt{n} \mathbb{P} \left( E_i^{(n)} \right) = 1. \]

**Proof.** From (30) it follows that
\[ \sqrt{n} \mathbb{P} \left( E_i^{(n)} \right) = \sqrt{n} \left( s_i^{(n)} - s_{i-1}^{(n)} \right) = \sqrt{n} s_i^{(n)} - \sqrt{n} s_{i-1}^{(n)}. \]

To each of the two summands on the right-hand side we apply Lemma 2.7 which completes the proof. \qed

### 2.4. What happens after just one step?

We will prove Lemma 2.9 and Lemma 2.10 which show that the (rough) information about the number of the row in which the growth of a Young diagram occurred does not influence too much the probability distribution of the resulting Young diagram.

**Lemma 2.9.** For each \( r \in \mathbb{N}_0 \) the total variation distance between:
- the probability distribution of \( \lambda^{(n)} \) (i.e., the Plancherel measure on the set \( \mathbb{Y}_n \)), and
- the conditional probability distribution of \( \lambda^{(n)} \) under the condition that the event \( E_r^{(n)} \) occurred,
converges to zero, as \( n \to \infty \).

In Section 2.7 we will discuss the conjectural rate of convergence in this result.

**Proof.** For a Young diagram \( \mu = (\mu_0, \mu_1, \ldots) \) and \( r \in \mathbb{N}_0 \) we denote by
\[ \text{del}_r \mu = (\mu_0, \ldots, \mu_{r-1}, \mu_r - 1, \mu_{r+1}, \ldots) \]
the Young diagram obtained from \( \mu \) by removing a single box from the row with the index \( r \). The Young diagram \( \text{del}_r \mu \) is well-defined only if \( \mu_r > \mu_{r+1} \).

We consider the finite-dimensional vector space of real-valued functions on the set \( \mathbb{Y}_n \) of Young diagrams with \( n \) boxes. For any subset \( A \subseteq \mathbb{Y}_n \) we consider the non-negative bilinear form on this space
\[ \langle f, g \rangle_A = \sum_{\mu \in A} f_{\mu} g_{\mu} \]
and the corresponding seminorm
\[ \|f\|_A := \sqrt{\langle f, f \rangle_A}. \]

An important special case is \( A = \mathbb{Y}_n \) with the corresponding norm \( \| \cdot \|_{\mathbb{Y}_n} \).

We consider two special vectors \( X, Y \) in this space:
\[ X_\mu := \begin{cases} \frac{1}{\sqrt{(n-1)!}} d_{\Delta_r \mu}, & \text{if } \Delta_r \mu \text{ is well-defined}, \\ 0, & \text{otherwise}, \end{cases} \]
\[ Y_\mu := \frac{d_\mu}{\sqrt{n!}} \]
where \( d_\mu \) denotes the number of standard Young tableaux of shape \( \mu \). Observe that the vector \( X_\mu \) depends on \( \tau \). An important feature of these vectors is that for any set \( A \subseteq \mathbb{Y}_n \)
\[ \|Y\|^2_A = \mathbb{P} \left\{ \lambda^{(n)} \in A \right\}, \]
\[ \langle X, Y \rangle_A = \sqrt{n} \mathbb{P} \left\{ \lambda^{(n)} \in A \text{ and } E_r^{(n)} \right\}, \]
\[ \|X\|^2_A = \mathbb{P} \left\{ \lambda^{(n-1)} \in \Delta_r A \right\}, \]
see [Rom15, Section 1.19, Proof of Lemma 1.25].

In particular, for the special case \( A = \mathbb{Y}_n \)
\[ c_n := \sqrt{n} \mathbb{P} \left( E_r^{(n)} \right) = \langle X, Y \rangle_{\mathbb{Y}_n} \leq \|X\|_{\mathbb{Y}_n} \cdot \|Y\|_{\mathbb{Y}_n} \leq 1. \]

By Proposition 2.8 the left-hand side converges to 1 as \( n \to \infty \). Since \( \|X\|_{\mathbb{Y}_n} \leq 1 \) and \( \|Y\|_{\mathbb{Y}_n} = 1 \), it follows that
\[ \lim_{n \to \infty} c_n = \lim_{n \to \infty} \langle X, Y \rangle_{\mathbb{Y}_n} = \lim_{n \to \infty} \|X\|_{\mathbb{Y}_n} = 1. \]

As a consequence, a simple calculation using bilinearity of the scalar product shows that
\[ \lim_{n \to \infty} \|c_n^{-1} X - Y\|_{\mathbb{Y}_n} = 0. \]

For any \( A \subseteq \mathbb{Y}_n \) it follows therefore that
\[ \left| \mathbb{P} \left\{ \lambda^{(n)} \in A \big| E_r^{(n)} \right\} - \mathbb{P} \left\{ \lambda^{(n)} \in A \right\} \right| = \left| \frac{\langle X, Y \rangle_A}{\sqrt{n} \mathbb{P} \left( E_r^{(n)} \right)} - \langle Y, Y \rangle_A \right| \leq \left| \langle c_n^{-1} X - Y, Y \rangle_A \right| \leq \|c_n^{-1} X - Y\|_{\mathbb{Y}_n} \cdot \|Y\|_{\mathbb{Y}_n}. \]
The right-hand side does not depend on the choice of $A$ and converges to zero which concludes the proof. □

**Lemma 2.10.** For each $k \in \mathbb{N}_0$ the total variation distance between:

- the probability distribution of $\lambda^{(n)}$ (i.e., the Plancherel measure on the set $\mathbb{Y}_n$), and
- the conditional probability distribution of $\lambda^{(n)}$ under the condition that the event $\left( E_0^{(n)} \cup \cdots \cup E_k^{(n)} \right)^c$ occurred,

is of order $o \left( \frac{1}{\sqrt{n}} \right)$, as $n \to \infty$.

**Proof.** For real numbers $x$ and $c > 0$ we will denote by $x \pm c$ some unspecified real number in the interval $[x - c, x + c]$. In the following we will use the quantity $s_k^{(n)}$ defined in (30). We denote

$$F^{(n)} = \left( E_0^{(n)} \cup \cdots \cup E_k^{(n)} \right)^c.$$

Let $C_n$ be the maximum (over $r \in \{0, \ldots, k\}$) of the total variation distance considered in Lemma 2.9. The law of total probability implies that for any set $A \subseteq \mathbb{Y}_n$

$$\mathbb{P} \left\{ \lambda^{(n)} \in A \right\} = \sum_{0 \leq r \leq k} \mathbb{P} \left\{ \lambda^{(n)} \in A \mid E_r^{(n)} \right\} \mathbb{P} \left( E_r^{(n)} \right) + \mathbb{P} \left\{ \lambda^{(n)} \in A \mid F^{(n)} \right\} \mathbb{P} \left( F^{(n)} \right) = \sum_{0 \leq r \leq k} \left[ \mathbb{P} \left\{ \lambda^{(n)} \in A \right\} + C_n \right] \mathbb{P} \left( E_r^{(n)} \right) + \mathbb{P} \left\{ \lambda^{(n)} \in A \mid F^{(n)} \right\} \mathbb{P} \left( F^{(n)} \right) = \mathbb{P} \left\{ \lambda^{(n)} \in A \right\} s_k^{(n)} + C_n s_k^{(n)} + \mathbb{P} \left\{ \lambda^{(n)} \in A \mid F^{(n)} \right\} \left( 1 - s_k^{(n)} \right).$$

By solving the above equation for the conditional probability we get

$$\mathbb{P} \left\{ \lambda^{(n)} \in A \mid F^{(n)} \right\} = \frac{\mathbb{P} \left\{ \lambda^{(n)} \in A \right\} \left( 1 - s_k^{(n)} \right) + C_n s_k^{(n)}}{1 - s_k^{(n)}}.$$

In this way we proved that the total variation distance considered in the statement of the lemma is bounded from above by

(37) $$\frac{C_n s_k^{(n)}}{1 - s_k^{(n)}}.$$

The asymptotics of the individual factors in (37) is provided by Lemma 2.9 (which gives $C_n = o(1)$) and by Lemma 2.7 or, equivalently, by Okounkov’s result Proposition 2.8 (which gives $s_k^{(n)} = O \left( \frac{1}{\sqrt{n}} \right)$); this completes the proof. □
2.5. Asymptotic independence. As an intermediate step towards the proof of Theorem 2.2 we consider a sequence of independent random variables

\[ \tilde{V}^{(m)} = \left( \tilde{R}^{(m+1)}, \ldots, \tilde{R}^{(m+\ell)}, \tilde{\lambda}^{(m+\ell)} \right) \in \mathcal{N}^\ell \times \mathcal{Y} \]

which is independent with the vectors \( V^{(m)} \) and \( \tilde{V}^{(m)} \) (recall the definitions in (25) and (26)), and such that the marginal distributions of \( V^{(m)} \) and (38) coincide:

\[ \tilde{R}^{(m+i)} \overset{d}{=} R^{(m+i)}, \quad \text{for all } 1 \leq i \leq \ell \]

\[ \tilde{\lambda}^{(m+\ell)} \overset{d}{=} \lambda^{(m+\ell)}. \]

In particular, the probability distribution of \( \tilde{V}^{(m)} \) depends implicitly on \( k \) which is the number of the rows of Young diagrams which we observe less 1.

**Lemma 2.11.** For each \( k \in \mathbb{N}_0 \) there exists a sequence \( b_n = o \left( \frac{1}{\sqrt{n}} \right) \) with the property that for all \( m \geq (k+1)^2 \) and \( \ell \geq 1 \) and \( i \in \{1, \ldots, \ell\} \)

\[ \delta \left( \left( \tilde{R}^{(m+1)}, \ldots, \tilde{R}^{(m+i-1)}, \tilde{R}^{(m+i)}, R^{(m+i+1)}, \ldots, R^{(m+\ell)}, \lambda^{(m+\ell)} \right), \right. \]

\[ \left. \left( \tilde{R}^{(m+1)}, \ldots, \tilde{R}^{(m+i-1)}, \tilde{R}^{(m+i)}, R^{(m+i+1)}, \ldots, R^{(m+\ell)}, \lambda^{(m+\ell)} \right) \right) \leq b_{m+i}. \]

The only difference between the two random vectors considered in (39) lies in the \( i \)-th coordinate: in the first vector this coordinate is equal to \( \tilde{R}^{(m+i)} \) while in the second to \( R^{(m+i)} \).

**Proof.** For an integer \( n \geq 1 \) we define

\[ b_n := \sum_{r \in \mathcal{N}} \mathbb{P} \left( R^{(n)} = r \right) \times \]

\[ \sum_{\lambda \in \mathcal{Y}_n} \left| \mathbb{P} \left( \lambda^{(n)} = \lambda \mid R^{(n)} = r \right) - \mathbb{P} \left( \lambda^{(n)} = \lambda \right) \right|. \]

For the summands corresponding to \( r \in \{0, \ldots, k\} \) we note that the random events \( \{ R^{(n)} = r \} \) and \( E_r^{(n)} \) are equal and we apply Proposition 2.8 and Lemma 2.9. For the summand \( r = \infty \) we apply Lemma 2.10. This gives the desired asymptotics \( b_n = o \left( \frac{1}{\sqrt{n}} \right) \). In the following we will show that this sequence indeed fulfills (39).
An iterative application of Lemma 2.1(a) shows that the left-hand side of (39) is equal to the total variation distance of the suffixes
\[
\delta \left( \left( R^{(m+i)}, R^{(m+i+1)}, \ldots, R^{(m+\ell)}, \lambda^{(m+\ell)} \right), \left( \tilde{R}^{(m+i)}, R^{(m+i+1)}, \ldots, R^{(m+\ell)}, \lambda^{(m+\ell)} \right) \right).
\]

In order to evaluate the latter we consider an arbitrary set \( X \subseteq N_{\ell+1-i} \times \mathbb{Y} \). We can write
\[
X = \bigcup_{r \in N} \{r\} \times X_r
\]
for some family of sets \( X_r \subseteq N_{\ell-i} \times \mathbb{Y} \) indexed by \( r \in N \). Since the Plancherel growth process (24) is a Markov process [Ker99, Sections 2.2 and 2.4],
\[
\mathbb{P} \left( \left( R^{(m+i)}, R^{(m+i+1)}, \ldots, R^{(m+\ell)}, \lambda^{(m+\ell)} \right) \in X \right) = \sum_{r \in N} \sum_{\lambda \in \mathbb{Y}_{m+i}} \mathbb{P} \left( R^{(m+i)} = r \text{ and } \lambda^{(m+i)} = \lambda \right) \times \mathbb{P} \left( \left( R^{(m+i+1)}, \ldots, R^{(m+\ell)}, \lambda^{(m+\ell)} \right) \in X_r \bigg| \lambda^{(m+i)} = \lambda \right) = \sum_{r \in N} \sum_{\lambda \in \mathbb{Y}_{m+i}} \mathbb{P} \left( R^{(m+i)} = r \right) \mathbb{P} \left( \lambda^{(m+i)} = \lambda \bigg| R^{(m+i)} = r \right) \times \mathbb{P} \left( \left( R^{(m+i+1)}, \ldots, R^{(m+\ell)}, \lambda^{(m+\ell)} \right) \in X_r \bigg| \lambda^{(m+i)} = \lambda \right).
\]

An analogous, but simpler calculation shows that
\[
\mathbb{P} \left( \left( \tilde{R}^{(m+i)}, R^{(m+i+1)}, \ldots, R^{(m+\ell)}, \lambda^{(m+\ell)} \right) \in X \right) = \sum_{r \in N} \sum_{\lambda \in \mathbb{Y}_{m+i}} \mathbb{P} \left( R^{(m+i)} = r \right) \mathbb{P} \left( \lambda^{(m+i)} = \lambda \right) \times \mathbb{P} \left( \left( R^{(m+i+1)}, \ldots, R^{(m+\ell)}, \lambda^{(m+\ell)} \right) \in X_r \bigg| \lambda^{(m+i)} = \lambda \right).
\]

The first and the third factor on the right-hand side of (40) coincide with their counterparts on the right-hand side of (41), and the third factor is bounded from above by 1. It follows that the absolute value of the difference between (40) and (41) is bounded from above by \( b_{m+i} \), as required. □
2.6. **Proof of Theorem 2.2**

**Proof.** An iterative application of the triangle inequality combined with Lemma 2.11 implies that

\[
\delta \left( \left( R^{(m+1)}, \ldots, R^{(m+\ell)}, \lambda^{(m+\ell)} \right), \left( \tilde{R}^{(m+1)}, \ldots, \tilde{R}^{(m+\ell)}, \tilde{\lambda}^{(m+\ell)} \right) \right) = o \left( \frac{\ell}{\sqrt{m}} \right).
\]

On the other hand, thanks to the independence of the coordinates, Lemma 2.1(b) gives

\[
\delta \left( \left( \tilde{R}^{(m+1)}, \ldots, \tilde{R}^{(m+\ell)}, \tilde{\lambda}^{(m+\ell)} \right) \right) \leq \sum_{1 \leq i \leq \ell} \delta \left( \tilde{R}^{(m+i)}, R^{(m+i)} \right).
\]

In the remaining part of the proof we will investigate the individual summand which corresponds to \( n := m + \ell \). We have

\[
\delta \left( \tilde{R}^{(n)}, R^{(n)} \right) = \frac{1}{2} \sum_{0 \leq r \leq k} \left| \mathbb{P} \left( \tilde{R}^{(n)} = r \right) - \mathbb{P} \left( R^{(n)} = r \right) \right| + \frac{1}{2} \left| \mathbb{P} \left( \tilde{R}^{(n)} = \infty \right) - \mathbb{P} \left( R^{(n)} = \infty \right) \right|.
\]

The equality

\[
\mathbb{P} \left( \tilde{R}^{(n)} = \infty \right) = 1 - \sum_{0 \leq r \leq k} \mathbb{P} \left( \tilde{R}^{(n)} = r \right)
\]

and the analogous equality for \( R^{(n)} \) imply that the right-hand side of (44) can be bounded as follows:

\[
\delta \left( \tilde{R}^{(n)}, R^{(n)} \right) \leq \sum_{0 \leq r \leq k} \left| \mathbb{P} \left( \tilde{R}^{(n)} = r \right) - \mathbb{P} \left( R^{(n)} = r \right) \right|.
\]

Below we will find the asymptotics of the individual summands on the right-hand side.

For any \( 0 \leq r \leq k \), by the definitions

\[
\mathbb{P} \left( \tilde{R}^{(n)} = r \right) = \mathbb{P} \left( R^{(n)} = r \right) = \mathbb{P} \left( E_r^{(n)} \right)
\]
and by Proposition 2.8 we get the asymptotics of the probability $\mathbb{P}(\overline{R}^{(n)} = r)$. The asymptotics of the probability $\mathbb{P}(\overline{R}^{(n)} = r)$ is given by its definition (27).

It follows that the total variation distance in (43) is of order $o\left(\frac{1}{\sqrt{n}}\right)$.

In this way we proved that

$$
\delta\left(\overline{R}^{(m+1)}, \ldots, \overline{R}^{(m+\ell)}, \overline{\lambda}^{(m+\ell)}\right),
\left(\overline{R}^{(m+1)}, \ldots, \overline{R}^{(m+\ell)}, \overline{\lambda}^{(m+\ell)}\right) \leq o\left(\frac{\ell}{\sqrt{m}}\right).
$$

The triangle inequality combined with (42) and (46) completes the proof. □

The following problem was asked by Maciej Dołęga.

**Question 2.12.** Plancherel growth process may be defined in terms of Schur polynomials and the corresponding Pieri rule. Is it possible to apply the ideas presented in the current section in the context of some other growth processes on $\mathbb{Y}$ (such as Jack–Plancherel growth process [Ker00]) which are related to other classical families of symmetric polynomials (such as Jack polynomials)?

2.7. **What is the rate of convergence?** A more refine asymptotics for the length of the given row

$$
\mathbb{E} \lambda_i^{(n)} = 2\sqrt{n} + c_i n^{\frac{1}{2}} + o\left(n^{\frac{1}{2}}\right)
$$

(see [BDJ99, Theorem 1.2] for the case of the bottom row) suggests that the probability of the event $E_i^{(n)}$ can be approximated by the derivative of the right-hand side of (47), thus

$$
1 - \sqrt{n} \mathbb{P}(E_i^{(n)}) = O\left(n^{-\frac{1}{2}}\right);
$$

note that the same kind of argument appeared already in Remark 1.2. If this stronger version of Proposition 2.8 indeed holds true, then also the total variation distance in Lemma 2.9 is at most $O\left(n^{-\frac{1}{2}}\right)$. Consequently, the total variation distance in Lemma 2.10 as well as the sequence $b_n$ in Lemma 2.11 can be better bounded as $O\left(n^{-\frac{1}{2}}\frac{1}{2}\right) = O\left(n^{-\frac{3}{2}}\right)$. This would imply that Conjecture 2.3 indeed holds true.
3. Proofs of the main results

3.1. Proof of Theorem 1.1. Let \( N(t) = (N_0(t), \ldots, N_k(t)), t \in \mathbb{R} \), be a collection of \( k + 1 \) independent copies of the standard Poisson process. Let us fix some real number \( c > 0 \); in the following we assume that \( n \) is big enough so that \( n - c \sqrt{n} \geq (k + 1)^2 \). It follows in particular that
\[
N_t = n + \lfloor t \sqrt{n} \rfloor \quad \text{for } t \in [-c, c].
\]

We denote
\[
L = L(n) = n_c - n_{-c} = [c \sqrt{n}] - [-c \sqrt{n}].
\]

Lemma 3.1. For each \( c > 0 \) the total variation distance between the random vector
\[
\Lambda^{(n+i)} - \Lambda^{(n)} : [-c \sqrt{n}] \leq i \leq [c \sqrt{n}] \quad \text{rv} \in \mathbb{Z}^{k+1}
\]
and the corresponding random vector
\[
N \left( \frac{i}{\sqrt{n}} \right) : [-c \sqrt{n}] \leq i \leq [c \sqrt{n}] \quad \text{rv} \in \mathbb{Z}^{k+1}
\]
converges to zero, as \( n \to \infty \) tends to infinity.

Proof. We consider the bijection
\[
\mathbb{Z}^L \ni \left( a_i : [-c \sqrt{n}] \leq i \leq [c \sqrt{n}] \quad \text{with} \quad a_0 = 0 \right) \sim
\left( a_i - a_{i-1} : [-c \sqrt{n}] < i \leq [c \sqrt{n}] \right) \in \mathbb{Z}^L.
\]
Since an application of a bijection does not change the total variation distance, the aforementioned total variation distance between \((48)\) and \((49)\) is equal to the total variation distance \( \delta(A, B) \) between the corresponding sequences of the increments, i.e.,
\[
A := \left( \Lambda^{(n+i)} - \Lambda^{(n+i-1)} : [-c \sqrt{n}] < i \leq [c \sqrt{n}] \right)
= \left( \mathbb{1}_{R^{(n+i)} = 0}, \ldots, \mathbb{1}_{R^{(n+i)} = k} : [-c \sqrt{n}] < i \leq [c \sqrt{n}] \right)
\]
and the sequence of independent random vectors
\[
B := \left( N \left( \frac{i}{\sqrt{n}} \right) - N \left( \frac{i-1}{\sqrt{n}} \right) : [-c \sqrt{n}] < i \leq [c \sqrt{n}] \right).
\]
In the following we use the notations from Section 2.2 with $m := \lfloor n - c\sqrt{n} \rfloor$ and $\ell := L$. In particular we consider the collection of random variables $\mathbf{R}^{(m)}$ with the probability distribution given by (27) and (28) for this specific value of $m$. We define

$$\overline{A} := \left( \mathbf{1}_{\mathbf{R}^{(m_1)}} = 0, \ldots, \mathbf{1}_{\mathbf{R}^{(m_k)}} = k \right) : \lfloor -c\sqrt{n} \rfloor < i \leq \lfloor c\sqrt{n} \rfloor;$$

our strategy will be to apply the triangle inequality

$$\delta(A, B) \leq \delta(A, \overline{A}) + \delta(\overline{A}, B).$$

In order to bound the first summand on the right-hand side of (50) we apply Theorem 2.2 for the aforementioned values of $m$ and $\ell$; it follows that

$$\delta(A, \overline{A}) \leq \delta \left( \mathbf{V}^{(m)}, \mathbf{V}^{(m)} \right) = o(1).$$

For the second summand on the right-hand side of (50) we apply Lemma 2.1(b)

$$\delta(\overline{A}, B) = \sum_{\lfloor -c\sqrt{n} \rfloor < i \leq \lfloor c\sqrt{n} \rfloor} \delta \left( \mathbf{1}_{\mathbf{R}^{(m_1)}} = 0, \ldots, \mathbf{1}_{\mathbf{R}^{(m_k)}} = k \right),$$

$$\text{Pois} \left( \frac{1}{\sqrt{n}} \right) \times \cdots \times \text{Pois} \left( \frac{1}{\sqrt{n}} \right) = O \left( \frac{1}{\sqrt{n}} \right),$$

where the last bound follows from a direct calculation of the total variation distance of specific probability distributions on $\mathbb{N}_0^{k+1}$. □

**Proof of Theorem 1.1.** For given $t_1, \ldots, t_\ell \in \mathbb{R}$ we select arbitrary $c > \max(|t_1|, \ldots, |t_\ell|)$. Let $n$ be big enough so that $n - c\sqrt{n} \geq (k + 1)^2$. We apply Lemma 3.1 from the two random vectors which appear in this lemma we select the coordinates which correspond to $i \in \{\lfloor t_1\sqrt{n} \rfloor, \ldots, \lfloor t_\ell\sqrt{n} \rfloor \}$. It follows that the total variation distance between the law of the finite-dimensional marginal

$$\left( \Lambda^{(n_{t_1})}, \ldots, \Lambda^{(n_{t_\ell})} - \Lambda^{(n)} \right)$$

and the law of the appropriate marginal of $\mathbf{N}$, that is

$$\left( \mathbf{N} \left( \frac{\lfloor t_1\sqrt{n} \rfloor}{\sqrt{n}} \right), \ldots, \mathbf{N} \left( \frac{\lfloor t_\ell\sqrt{n} \rfloor}{\sqrt{n}} \right) \right)$$

converges to zero as $n \to \infty$. 

On the other hand, by the maximal coupling lemma [Tho00, Section 8.3, Eq. (8.19)] (the definition of the total variance distance therein differs from ours by the factor 2), the total variation distance between \((N(t_1), \ldots, N(t_\ell))\) is bounded from above by

\[
P \left\{ N \left( \frac{t_i \sqrt{n}}{\sqrt{n}} \right) \neq N(t_i) \text{ for some } i \in \{1, \ldots, \ell\} \right\} = O \left( \frac{1}{\sqrt{n}} \right).
\]

An application of the triangle inequality for the total variation distance completes the proof. □

**Problem 3.2.** Find the precise rate of convergence in Lemma 2.9 and Theorem 2.2. This convergence probably cannot be too fast because this would imply that an analogue of Theorem 1.1 holds true also in the scaling when in (4) we study \(t \gg 1\), and the latter would potentially contradict the non-Gaussianity results for the lengths of the rows of Plancherel-distributed Young diagrams [BDJ99; BDJ00; BOO00; Joh01].

### 3.2. Proof of Corollary 1.3

**Proof.** Let \(c > 0\) be arbitrary. We consider \(n \in \mathbb{N}_0\) which is big enough so that \(n - c\sqrt{n} - 1 \geq (k + 1)^2\). We denote

\[
I_n = \left\{ \left\lfloor n - c\sqrt{n} \right\rfloor, \ldots, \left\lceil n + c\sqrt{n} \right\rceil \right\}.
\]

Our strategy is to apply Theorem 2.2 for \(m := \left\lfloor n - c\sqrt{n} \right\rfloor - 1\) and \(\ell := |I_n|\); in particular in the following we will use the collection of random variables \(R^{(i)}\) over \(i \in I_n\) with the probability distribution given by (27) and (28) for this specific value of \(m\).

We consider the following three collections of \(k + 1\) random subsets of \(I_n\):

- the sequence \(A = (A_0, \ldots, A_k)\) with

  \[
  A_y = \{ Q_{x,y} : x \in \mathbb{N}_0 \} \cap I_n = \left\{ i \in I_n : R^{(i)} = y \right\}
  \]

  obtained by selecting the entries of the bottom \(k + 1\) rows of the recording tableau which belong to the specified interval,

- the sequence \(\overline{A} = (\overline{A}_0, \ldots, \overline{A}_k)\) with

  \[
  \overline{A}_y = \left\{ i \in I_n : \overline{R}^{(i)} = y \right\},
  \]
• the sequence \( B = (B_0, \ldots, B_k) \) obtained by independent sampling, i.e., such that the family of random events \( \{ i \in B_y \} \) indexed by \( i \in I \) and \( y \in \{0, \ldots, k\} \) is a family of independent events, each having equal probability \( \frac{1}{n} \).

Theorem 2.2 implies that the total variation distance \( \delta(A, \overline{A}) \) converges to zero, as \( n \to \infty \).

The information about the sequence \( \overline{A} \) can be alternatively encoded by the sequence of independent random variables \( (v_i : i \in I_n) \) given by

\[
v_i = \left( \mathbb{1}_{i \in \overline{A}_0}, \ldots, \mathbb{1}_{i \in \overline{A}_k} \right) = \left( \mathbb{1}_{\overline{R}^{(i)} = 0}, \ldots, \mathbb{1}_{\overline{R}^{(i)} = k} \right) \quad \text{for } i \in I_n.
\]

Analogously the information about \( B \) can be encoded by the sequence of independent random variables \( (w_i : i \in I_n) \), where

\[
w_i = \left( \mathbb{1}_{i \in B_0}, \ldots, \mathbb{1}_{i \in B_k} \right) \quad \text{for } i \in I_n.
\]

By Lemma 2.1(b) it follows that the total variation distance for the vectors

\[
\delta(A, B) = \delta(v, w) \leq \sum_{i \in I_n} \delta(v_i, w_i)
\]

is bounded by the sum of the coordinatewise total variation distances. The asymptotics of the individual summand

\[
\delta(v_i, w_i) = o\left( \frac{1}{\sqrt{n}} \right)
\]

is a consequence of a direct calculation based on the explicit form of the two probability distributions involved here. In this way we proved that the total variation distance \( \delta(\overline{A}, B) \) converges to 0 as \( n \to \infty \).

We apply a shift and a scaling to the random sets which form the collection \( A \) and the collection \( B \); it follows that the total variation distance between

• the collection of random subsets of \( \mathbb{R} \)

\[
\left( \left\{ \frac{Q_{x,y} - n}{\sqrt{n}} : x \in \mathbb{N}_0 \right\} \cap [-c, c] : y \in \{0, \ldots, k\} \right)
\]

obtained by truncating (6), and

• the collection of random subsets of \( \mathbb{R} \)

\[
\left( \left\{ \frac{j - n}{\sqrt{n}} : j \in B_y \right\} : y \in \{0, \ldots, k\} \right)
\]

(52)

converges to zero as \( n \to \infty \).

Each random set from the collection (52) converges in distribution to the Poisson point process on the interval \( [-c, c] \), see [DVJ08, Proposition 11.3.1], which concludes the proof. \( \square \)
3.3. **Proof of Corollary [1.4]** We start with an auxiliary result.

**Lemma 3.3.** For real numbers \( p, \lambda \geq 0 \) and for integers \( k \geq 0 \) and \( n \) such that \( p(k + 1) \leq 1 \) let \( \xi_1, \ldots, \xi_n \) be independent, identically distributed random variables with the uniform distribution \( \mathbb{U}(0, 1) \) and let \( \overline{R}^{(1)}, \ldots, \overline{R}^{(n)} \) be independent, identically distributed random variables with the distribution

\[
\begin{align*}
\mathbb{P}\left\{ \overline{R}^{(i)} = r \right\} &= p & \text{for } r \in \{0, \ldots, k\}, \\
\mathbb{P}\left\{ \overline{R}^{(i)} = \infty \right\} &= 1 - (k + 1)p,
\end{align*}
\]

(53) cf. (27) and (28) for an analogous distribution in a similar context.

Then for any real numbers \( a, b \) such that \( 0 \leq a \leq b \leq 1 \) the total variation distance between:

(a) the collection of \( k + 1 \) random sets

\[
\left( [a, b] \cap \{ \xi_i : \overline{R}^{(i)} = y \} : y \in \{0, \ldots, k\} \right),
\]

and

(b) the collection of \( k + 1 \) independent Poisson point processes \( N_0, \ldots, N_k \) on the interval \( [a, b] \) with the intensity \( \lambda \),

is bounded from above by

\[
(k + 1)^2 n p^2 l^2 + (k + 1) \cdot |\lambda l - np|,
\]

where \( l = b - a \) is the length of the interval.

**Proof.** For each \( i \in \{0, \ldots, k\} \) the corresponding Poisson point process \( N_i \) can be generated by the following two-step procedure. Firstly, we sample the number of points \( n_i \); it is a random variable with the Poisson distribution with the parameter \( \lambda l \), where \( l := b - a \) is the length of the interval. Secondly, we take \( n_i \) independent random elements of the unit interval \( [a, b] \) with the uniform distribution. The random variables \( n_0, \ldots, n_k \) which correspond to independent Poisson processes are independent.

A similar construction can be performed for the collection (54) of random sets: we first sample the vector \( (m_0, \ldots, m_k) \) of the cardinalities of the sets from (54) and then for each index \( i \in \{0, \ldots, k\} \) we sample \( m_i \) elements of the interval \( [a, b] \). In this case, however, the random variables \( m_0, \ldots, m_k \) are not independent.

From the above discussion it follows that the total variation distance considered in the statement of this lemma between [a] and [b] is equal to the total variation distance between the random vectors \( m = (m_0, \ldots, m_k) \) and \( n = (n_0, \ldots, n_k) \). In the following we will bound the latter distance.
The distribution of the random vector

\[ m = \sum_{1 \leq j \leq n} \left( \mathbf{1}_{e_j = 0}, \ldots, \mathbf{1}_{e_j = k} \right) \]

is the \( n \)-fold additive convolution of the discrete probability measure \( \mathcal{M} \) on \( \mathbb{Z}^{k+1} \) which to each basis vector \( e_i = [0, \ldots, 0, 1, 0, \ldots, 0] \in \mathbb{Z}^{k+1} \) associates the probability \( pl \) and to the zero vector associates the remaining probability \( 1 - (k + 1)pl \).

On the other hand, the distribution of the random vector \( n \) can be alternatively seen as the \( n \)-fold additive convolution of the product measure \( \mathcal{N} = \text{Pois} \left( \frac{\lambda}{n} \right) \times \cdots \times \text{Pois} \left( \frac{\lambda}{n} \right) \) on \( \mathbb{Z}^{k+1} \). We also consider an auxiliary product measure \( \mathcal{N}' = \text{Pois} (pl) \times \cdots \times \text{Pois} (pl) \) on \( \mathbb{Z}^{k+1} \).

By Lemma 2.1(c) and the triangle inequality it follows that

\[ \delta(m, n) \leq n \delta(\mathcal{M}, \mathcal{N}) \leq n \delta(\mathcal{M}, \mathcal{N}') + n \delta(\mathcal{N}', \mathcal{N}). \]

For the first summand on the right-hand side, by a direct calculation of the positive part of the difference of the two measures and its \( \ell^1 \) norm, we have

\[ n \delta(\mathcal{M}, \mathcal{N}') = n(k + 1) \left[ pl - ple^{-pl(k+1)} \right] \leq n(k + 1)^2 p^2 l^2, \]

where the inequality follows from an elementary bound on the exponential function. For the second summand on the right-hand side of (55) we apply Lemma 2.1(c) in order to bound the total variation distance between two Poisson distributions; it follows that

\[ n \delta(\mathcal{N}', \mathcal{N}) \leq (k + 1) \cdot |\lambda - npl| \]

which completes the proof. \( \square \)

Proof of Corollary 1.4: We start with the case when \( 0 < w < 1 \). We will show a stronger result that for each \( A > 0 \) the total variation distance between:

(i) the collection of \( k + 1 \) sets

\[ \left( \mathcal{P}^{(n)}(y) \cap [-A, A] : y \in \{0, \ldots, k\} \right), \]

(cf. (7)), and

(ii) the collection of \( k + 1 \) independent Poisson point processes on the interval \([-A, A]\) with the intensity \( \frac{1}{\sqrt{w}} \)

converges to zero, as \( n \to \infty \).

As the first step, let us fix \( \epsilon > 0 \). Let \( Z_1 \) be the number of the entries of the sequence \( \xi_1, \ldots, \xi_n \) which are weakly smaller than \( w - \frac{A}{\sqrt{w}} \) and let \( Z_2 \) be the number of the entries of this sequence which are strictly smaller.
than \( w + \frac{A}{\sqrt{n}} \); clearly the probability distribution of \( Z_1 \) and \( Z_2 \) is a binomial distribution. By Bienaymé–Chebyshev inequality it follows that the constant

\[
B := A + \frac{1}{\sqrt{\epsilon}}
\]

has the property that for each positive integer \( n \)

\[
P(Z_1 < n_{\min}) \leq \epsilon, \quad P(Z_2 \geq n_{\max}) \leq \epsilon
\]

with

\[
n_{\min} := \left\lfloor nw - B \sqrt{n} \right\rfloor, \quad n_{\max} := \left\lceil nw + B \sqrt{n} \right\rceil.
\]

In the following we assume that \( n \) is big enough so that

\[1 \leq n_{\min} \leq n_{\max} \leq n.\]

Without loss of generality we may assume that the entries of the sequence \( \xi_1, \ldots, \xi_n \) are not repeated. It follows that this sequence can be encoded by two pieces of information:

- the sequence of order statistics \( 0 < \xi^{(1)} < \cdots < \xi^{(n)} < 1 \), and
- the permutation \( \pi = (\pi_1, \ldots, \pi_n) \) which encodes the order of the entries, i.e. \( \pi_i < \pi_j \) if and only if \( \xi_i < \xi_j \);

these two pieces of information are clearly independent and the permutation \( \pi \) is a uniformly random element of the symmetric group.

Since RSK algorithm is sensitive only to the relative order of the entries and not to their exact values, the insertion tableaux \( P(\xi_1, \ldots, \xi_n) \) can be obtained from the insertion tableau \( P(\pi_1, \ldots, \pi_n) \) by replacing each entry by the corresponding order statistic. It follows that the entries of a given row \( y \in \mathbb{N}_0 \) of the insertion tableau can be alternatively described as follows:

\[
\{ P^{(n)}_{x,y} : 0 \leq x < \lambda^{(n)}_y \} = \{ \xi^{(i)} : i \text{ is in the row } y \text{ of the tableau } P(\pi_1, \ldots, \pi_n) \}.
\]

It follows in particular that the intersection \( P^{(n)}_y \cap [-A, A] \) is defined in terms of (a certain subset of) the set of these order statistics \( \xi^{(i)} \) which belong to the interval

\[ I := [a, b] \]

with

\[
a := w - \frac{A}{\sqrt{n}}, \quad b := w + \frac{A}{\sqrt{n}}.
\]

If neither of the two random events appearing in (57) holds true then this set of order statistics fulfills

\[ I \cap \{ \xi^{(1)}, \ldots, \xi^{(n)} \} \subseteq \{ \xi^{(n_{\min})}, \ldots, \xi^{(n_{\max})} \}; \]
under this condition it follows that in order to find the number of the row of 
$P^{(n)}$ which contains a given order statistic $\xi_{(i)} \in I$ it is enough to know the
number of the row of the tableau $P(\pi_1, \ldots, \pi_n)$ which contains a given number $j$, over all choices of $j \in \{n_{\text{min}}, \ldots, n_{\text{max}}\}$. Here and in the following by the number of the row we will understand the element of the fixed finite set $N = \{0, \ldots, k, \infty\}$ with the convention that the element $\infty$ corresponds to all rows above the bottom $k + 1$ rows.

The insertion tableau $P(\pi) = Q(\pi^{-1})$ is equal to the recording tableau of the inverse permutation; as a consequence the probability distribution of 
$P(\pi)$ is given by the Plancherel measure, can be interpreted as (a part of) the Plancherel growth process and thus Theorem 2.2 is applicable to this tableau. It follows that the probability distribution of the vector formed by the so understood numbers of the rows of the boxes $n_{\text{min}}, \ldots, n_{\text{max}}$ can be approximated (up to an error $o(1)$ with respect to the total variation distance) by a sequence of independent random variables with the probability distribution given by $(53)$ for $p = \frac{1}{\sqrt{n_{\text{min}}}}$. Here and in the following we assume that $n$ is big enough so that $(k + 1)p \leq 1$.

The above two paragraphs show that the total variation distance between the collection

$$\left( [a, b] \cap \left\{ P_{x,y}^{(n)} : x \geq 0 \right\} : y \in \{0, \ldots, k\} \right),$$

of truncated entries of the bottom rows and the collection $(54)$ is bounded from above by $2\epsilon + o(1)$. On the other hand, Lemma 3.3 applied to $\lambda = \sqrt{\frac{\pi}{\sqrt{w}}}$ shows that the total variation distance between $(54)$ and the collection of $k + 1$ independent Poisson point processes $(b)$ with the intensity $\lambda$ on the interval $I$ converges to zero. We combine these two bounds by the triangle inequality; as a result the total variation distance between $(58)$ and $(b)$ is bounded from above by $2\epsilon + o(1)$.

We consider the affine transformation $x \mapsto \sqrt{n} (x - w)$ which maps the interval $[a, b]$ to $[-A, A]$. This affine transformation also maps the random collection $(58)$ to $(56)$ from $(i)$; it also maps the collection of Poisson processes $(b)$ from Lemma 3.3 with the intensity $\lambda$ to the collection $(ii)$ of Poisson processes with the intensity $\lambda \sqrt{\frac{n}{\sqrt{w}}} = \sqrt{\frac{\pi}{\sqrt{w}}}$. The application of this affine transformation preserves the total variation distance between the random variables, so the inequality from the previous paragraph completes the proof.

In the case when $w = 1$ the above proof can be easily adjusted by changing the definitions of $n_{\text{max}} := n$ and $b := 1$. □
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INTERDISCIPLINARY DOCTORAL SCHOOL “ACADEMIA COPERNICANA”, FACULTY OF MATHEMATICS AND COMPUTER SCIENCE, NICOLAUS COPERNICUS UNIVERSITY IN TORUŃ, UL. CHOPINA 12/18, 87-100 TORUŃ, POLAND
Email address: marciniak@mat.umk.pl

INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES, UL. ŚNIADECKICH 8, 00-656 WARSAWA, POLAND
Email address: lmaslanka@impan.pl

INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES, UL. ŚNIADECKICH 8, 00-656 WARSAWA, POLAND
Email address: psniady@impan.pl