Augmentation based Approximation Algorithms for Flexible Network Design

Chandra Chekuri∗ Rhea Jain†

September 27, 2022

Abstract

Adjiashvili [1] introduced network design in a non-uniform fault model: the edge set of a given graph is partitioned into safe and unsafe edges. A vertex pair \((s, t)\) is \((p, q)\)-flex-connected if \(s\) and \(t\) have \(p\) edge-connectivity even after the removal/failure of any \(q\) unsafe edges. The goal is to choose a min-cost subgraph \(H\) of a given graph \(G\) such that \(H\) has desired flex-connectivity for a given set of vertex pairs. This model generalizes the well-studied edge-connectivity based network design, however, even special cases are provably much harder to approximate [3, 4].

The approximability of network design in this model has been mainly studied for two settings of interest: (i) single pair setting under the names FTP and FTF (fault tolerant path and fault tolerant flow) [4] (ii) spanning setting under the name FGC (flexible graph connectivity) [2, 3, 5]. There have been several positive results in these papers. However, despite similarity to the well-known network design problems, this new model has been challenging to design approximation algorithms for, especially when \(p, q \geq 2\). We obtain two results that advance our understanding of algorithm design in this model.

• We obtain a 5-approximation for the \((2, 2)\)-flex-connectivity for a single pair \((s, t)\). Previously no non-trivial approximation was known for this setting.

• We obtain \(O(p)\) approximation for \((p, 2)\) and \((p, 3)\)-FGC for any \(p \geq 1\), and for \((p, 4)\)-FGC for any even \(p\). We obtain an \(O(q)\)-approximation for \((2, q)\)-FGC for any \(q \geq 1\). Previously only a \(O(q \log n)\)-approximation was known for these settings [5].

Our results are obtained via the augmentation framework where we identify a structured way to use the well-known 2-approximation for covering uncrossable families of cuts. Our analysis also proves corresponding integrality gap bounds on an LP relaxation that we formulate.

∗Dept. of Computer Science, Univ. of Illinois, Urbana-Champaign, Urbana, IL 61801. chekuri@illinois.edu. Supported in part by NSF grants CCF-1910149 and CCF-1907937.

†Dept. of Computer Science, Univ. of Illinois, Urbana-Champaign, Urbana, IL 61801. rheaj3@illinois.edu. Supported in part by NSF grant CCF-1910149.
1 Introduction

Network design is an important area of research in discrete and combinatorial optimization that is motivated by both practical and theoretical considerations. A broad subclass involves finding a minimum cost subgraph $H$ of a given graph $G$ that satisfies some desired connectivity requirements\(^1\). Special cases include well-studied classical problems such as the shortest $s$-$t$ path problem, minimum spanning tree, Steiner tree, and Steiner forest. Generalizations to higher connectivity requirements such as $k$-edge-connected spanning subgraph ($k$-ECSS) and the survivable network design (EC-SNDP) are important problems in applications to fault-tolerant network design. Many of these problems, including Steiner tree, are NP-Hard and APX-hard to approximate. EC-SNDP (also referred to as the Steiner network problem) is the following: given connectivity requirements specified by $r : V \times V \to \mathbb{Z}_+$ for each pair of vertices $(u,v)$, find a min-cost subgraph $H$ of $G$ such that each pair $(u,v)$ is $r(u,v)$-edge-connected in $H$. EC-SNDP, which captures several connectivity problems as special cases, admits a 2-approximation\(^2\). Network design problems have been instrumental in the development of several fundamental and advanced techniques in the design of approximation algorithms — see \cite{31, 33, 28, 26, 20}.

Over the years, several models have been proposed to capture robustness of a network to faults. In this paper we are interested in a specific model that was first proposed by Adjiashvili\(^3\). Recently there has been important algorithmic progress \cite{2, 3, 4, 5} and we elaborate on the results in these papers after describing the model and relevant technical background. In this model the input is an edge-weighted undirected graph $G = (V,E)$ where the edge set $E$ is partitioned to safe edges $S$ and unsafe edges $U$. The assumption, as the names suggest, is that unsafe edges can fail while safe edges cannot fail. We say that a vertex-pair $(s,t)$ is $(p,q)$-flex-connected\(^2\) in a subgraph $H$ of $G$ if $s$ and $t$ are $p$-edge-connected after deleting from $H$ any subset of at most $q$ unsafe edges. Equivalently, we require that any cut $\delta(S)$ that separates $s$ from $t$ contains $p$ safe edges or $(p+q)$ edges in total. Network design in this fault model takes the following form: given $G = (V,E)$ with $E = S \cup U$ and edge costs $c : E \to \mathbb{R}_+$, find a min-cost subgraph $H$ such that $H$ satisfies flex-connectivity given by specification for some vertex pairs. We observe that this model generalizes standard edge-connectivity problems. If all edges are safe, that is $E = S$, then asking for $(p,0)$-flex-connectivity for vertex pair $(s,t)$ is same as asking for $p$-edge-connectivity between $s$ and $t$. One can model edge-connectivity in an alternate manner; if all edges are unsafe, that is $E = U$, then asking for $(1,q-1)$-flex-connectivity for $(s,t)$ is same as asking for $q$-edge-connectivity. However, $(p,q)$-flex-connectivity is more general and complex.

In analogy with EC-SNDP, we define the Flex-SNDP problem: the input is the graph as above with safe/unsafe partition of the edge set, and a $(p_{u,v}, q_{u,v})$-flex-connectivity requirement for each pair $(u,v)$ of vertices in $G$. The goal is to find a min-cost subgraph $H$ of $G$ such that each $(u,v)$ is $(p_{u,v}, q_{u,v})$-flex-connected in $H$. We let $(p,q)$-Flex-SNDP denote the special case when the requirement for each $(u,v)$ is either $(p,q)$ or $(0,0)$. Two special cases will be of main concern in this paper. The first is the spanning case which requires $(p,q)$-flex-connectivity for all pairs of vertices. We refer to this problem as $(p,q)$-FGC to be consistent with the terminology introduced by Boyd et al. in \cite{5}. They generalized the FGC problem (corresponding to $p = 1, q = 1$) introduced in \cite{2, 3}. The other special case is when the requirement is for a single pair $(s,t)$ — this was the original motivation for the model in \cite{1}. We use the term $(p,q)$-Flex-ST to denote the single pair problem. Previously, $(1,k)$-Flex-ST was referred to as FTP (Fault-Tolerant Path) in \cite{1, 4} and the $(k,1)$-Flex-ST problem was referred to as FTF (Fault-Tolerant Flow) in \cite{4}; these problems were studied in directed graphs but it was noted in \cite{4} that the undirected FTP reduces to directed FTP.

\(^1\)We are mainly interested in edge-induced subgraphs. Thus, if $G = (V,E)$, $H = (V,F)$ for some $F \subseteq E$.

\(^2\)We borrow the terminology from the term flexible graph connectivity coined in \cite{3} and used also in \cite{5}.
It is also not hard to see that undirected FTF can be reduced to directed FTF at a slight loss in the approximation. We use the term \((p, q)\)-Flex-Steiner to refer to an instance where there is a \((p, q)\) requirement for every pair of vertices from some given set \(T \subseteq V\) of terminal vertices. Note that when \(T = V\) we have \((p, q)\)-FGC and when \(T = \{s, t\}\) we have the \((p, q)\)-Flex-ST problem.

**Motivation and recent work:** The model of Adjiashvili is motivated by the desire to incorporate non-uniformity in robust network design. This aims to bridge the gap between the standard connectivity problems that are clean and tractable and the models which are more general but tend to be less tractable — we refer the reader to [1, 4, 3] for further discussion.

Apart from the practical motivation, we find the theoretical aspects of the model compelling for several reasons that we outline below. Adjiashvili, Hommelshelm and Mühlenthaler [2] introduced FGC, pointed out that it generalizes the well-known MST and 2-ECSS problems, and derived a constant factor approximation. Boyd et al. [5] obtained several results on \((p, q)\)-FGC. They improved the approximation ratio for FGC to 2. They showed a \(\min\{(q + 1), O(\log n)\}\)-approximation for \((1, q)\)-FGC and a 4-approximation for \((p, 1)\)-FGC. They obtained an \(O(q \log n)\)-approximation for \((p, q)\)-FGC for any \(q\). Importantly, they show several strong connections between flexible graph connectivity and *capacitated* network design which has been studied in several works [18, 6, 7, 8].

Capacitated network design generalizes the standard edge connectivity setting by allowing each edge \(e\) to have an integer capacity \(u_e \geq 1\). One can reduce capacitated network design to standard edge-connectivity network design by replacing each edge \(e\) by \(u_e\) parallel edges with capacity 1 each. This blows up the approximation factor by \(\max_e u_e\), which is acceptable when this quantity is small. When \(\max_e u_e\) can be large, the complexity of capacitated network design varies. While the single pair problem becomes hard to approximate to almost polynomial factors [8], the spanning case (when one seeks to find a min-cost subgraph that has connectivity at least a given quantity \(R\)) admits an \(O(\log n)\)-approximation [7]. Boyd et al. show that \((1, k)\)-Flex-SNDP and \((k, 1)\)-Flex-SNDP can be reduced to Cap-SNDP such that the maximum capacity is \(k\); this implies that \((1, k)\)-Flex-SNDP and \((k, 1)\)-Flex-SNDP admit an \(O(k)\) approximation. Although these ratios are not necessarily tight in all cases, they provide a first-order and easy approach to solve the \((1, k)\) and \((k, 1)\) cases. Boyd et al. also show that an important technique in network design, namely the augmentation approach based on covering uncrossable families, can be applied in some cases; they obtain a 4-approximation for \((k, 1)\)-FGC via this approach.

Adjiashvili et al. [4] considered the single pair setting, mainly in directed graphs: specifically \((1, k)\)-Flex-ST and \((k, 1)\)-Flex-ST problems (referred to as FTP and FTF in [4]). They obtained a \(k\) approximation for \((1, k)\)-Flex-ST, slightly improving the \((k+1)\)-approximation that can obtained via an LP relaxation; they prove that the undirected graph case reduces to the directed case. They prove that \((1, k)\)-Flex-ST in directed graphs is at least as hard as directed Steiner tree which implies poly-logarithmic factor inapproximability [21]. For \((k, 1)\)-Flex-ST they obtain a \((k+1)\)-approximation; when \(k\) is a fixed constant, they obtain a 2-approximation via an involved dynamic programming based approach. They prove that \((k, 1)\)-Flex-ST in directed and undirected graphs is at least as hard to approximate as directed Steiner forest (which has almost polynomial factor hardness [15]). The hardness results are when \(k\) is part of the input and large, and show that approximability of network design in this model is substantially different from the edge-connectivity model.

The results discussed so far, the practical motivation, and the fact that a natural LP relaxation requires \(n^{O(q)}\)-time to solve, suggest that it is fruitful to focus on the approximability of \((p, q)\)-flex-connectivity network design when \(p, q\) are small constants. It is natural to conjecture that \((p, q)\)-Flex-SNDP admits an \(f(p, q)\)-approximation for a non-negative integer valued function \(f : \mathbb{Z}_+ \times \mathbb{Z}_+ \rightarrow \mathbb{Z}_+\). Note that for \((1, k)\) and \((k, 1)\) cases this is true. A weaker version of the conjecture is that \((p, q)\)-Flex-SNDP admits an \(f(p, q)\text{polylog}(n)\)-approximation. For \((p, q)\)-FGC we already had an \(O(q \log n)\)-
approximation and it is natural to conjecture an $O(q)$-approximation, or even a constant factor approximation. As far as we know there are no hardness results or lower bounds on the LP integrality gap that rule out these conjectures. From a theoretical point of view the model presents novel and interesting challenges for algorithm design.

1.1 Our contribution

Despite the possibility of an $f(p,q)$-approximation for $(p,q)$-Flex-SNDP we did not have any non-trivial approximation for $(2,2)$-Flex-ST problem! We note that $(2,2)$-FGC admits an $O(\log n)$-approximation. Thus $(2,2)$-Flex-ST is the first interesting case where both $p, q > 1$ and is not a spanning problem. Our first result is the following.

**Theorem 1.1.** There is a 5-approximation for $(2,2)$-Flex-ST problem.

A natural question is whether one can obtain a non-trivial approximation for $(2,2)$-Flex-Steiner problem. We do not yet have a direct approach for it. The preceding result easily implies an $O(h)$-approximation where $h$ is the number of terminals. Based on past work on vertex connectivity network design [9, 14, 11, 27], we propose a simple and natural algorithm that we conjecture yields an $O(\log h)$-approximation for $(2,2)$-Flex-Steiner problem. We discuss this in Section 5.3.

Our second set of results are for $(p,q)$-FGC. Recall that Boyd et al. [5] obtained a 4-approximation for $(k,1)$-FGC, a min\{(k+1), O(\log n)\}-approximation for $(1,k)$-FGC, and a $O(q \log n)$-approximation for $(p,q)$-FGC. Thus we did not have a constant factor approximation for $(2,2)$-FGC. We prove several results that, as a corollary, yield constant factor approximation for small values of $p,q$.

**Theorem 1.2.** For any $q \geq 0$ there is a $(2q+2)$-approximation for $(2,q)$-FGC. For any $p \geq 1$ there is a $(2p+4)$-approximation for $(p,2)$-FGC, and a $(4p+4)$-approximation for $(p,3)$-FGC. Moreover, for all even $p \geq 2$ there is an $(6p+4)$-approximation for $(p,4)$-FGC.

We explicitly formulate an LP relaxation for $(p,q)$-Flex-Steiner combining ideas implicit in [3, 4, 5] and capacitated network design where knapsack cover inequalities play an important role [6, 7]. Although we do not use the LP relaxation directly, our approximation bounds can be shown with respect to this relaxation. Proving lower bounds on the integrality gap of this relaxation for various special cases of $(p,q)$-Flex-SNDP could yield insights into the hardness of the problem. For instance we show that the LP gap is $\Omega(k)$ for $(1,k)$-Flex-ST (see Section 5.2). We provide examples showing the need for new ideas to extend our current results for $(p,q)$-FGC.

**Barriers and techniques:** The non-uniform nature of flexible connectivity model makes it technically challenging. The reduction to capacitated network design does not extend when $p,q \geq 2$. Although we can formulate an LP relaxation, it does not have clean structural properties that edge-connectivity network design enjoys. Thus, known techniques such as primal-dual and iterated rounding cannot be applied directly. Another technique in network design is the augmentation approach which has proven to be quite successful — in fact, this is the approach that was used for EC-SNDP prior to the iterated rounding approach. Moreover, it is one of the main techniques that we have in more complex settings such as vertex connectivity network design [25, 30] and node-weighted network design [29, 10]. In the augmentation approach we start with a initial set of edges $F_0$ that partially satisfy the connectivity constraints. We then augment $F_0$ with a set $F$ in the graph $G - F_0$; the augmentation is typically done to increase the connectivity by one unit for pairs that are not yet satisfied. We repeat this process in several stages until all connectivity requirements are met. A crucial advantage of the augmentation approach is that it allows one to reduce a higher-connectivity problem to a series of problems that solve a potentially simpler
\{0,1\}-connectivity problem. An important tool in this area is a 2-approximation for covering an
uncrossable function (a formal definition is given in Section 2) [32].

Boyd et al. [5] used the augmentation approach to obtain a 4-approximation for \((p,1)\)-FGC. They first obtain a 2-approximate solution to the \((p,0)\) problem, which is the same as the \(p\)-ECSS problem. In the second step, they augment the solution to obtain a feasible solution to \((p,1)\)-FGC. An advantage of augmenting from \((p,q)\)-FGC to \((p,q+1)\)-FGC is that one can ignore the distinction between safe and unsafe edges in the augmentation step. [5] shows that the augmentation problem from \((p,0)\) to \((p,1)\) results in a nice problem: the cuts to be covered are uncrossable. One can try this approach for \((p,q)\)-Flex-SNDP wherein we incrementally augment from \((p,q-1)\) to \((p,q)\) by defining an appropriate set of cuts to cover. It is not too hard to show that the augmentation problem from \((2,q-1)\)-FGC to \((2,q)\)-FGC is uncrossable for any \(q\). However, the main difficulty is that for most values of \(p, q\), the augmentation problem does not lead to an uncrossable family of cuts. We overcome this difficulty by decomposing the family of cuts to be covered in the augmentation problem into a sequence of cleverly chosen subfamilies such that each subfamily is uncrossable. The process is very problem specific: our approach for \((2,2)\)-Flex-ST is very different from that for \((p,q)\)-FGC although they share the high-level approach. There is antecedent for such complexity in vertex-connectivity network design; \((0,1,2)\)-VC-SNDP admits a 2-approximation [16, 12] while higher connectivity requires different techniques [9, 14, 13, 30], and the approximability and integrality gaps are not fully resolved yet. We hope that our results will spur additional insights and future work in flexible network design.

**Organization:** The rest of the paper is organized as follows. Section 2 discussed the LP relaxation and some of the technical tools needed for the augmentation framework. Section 3 describes the 5-approximation for \((2,2)\)-Flex-ST. Section 4 contains our results for \((p,q)\)-FGC.

## 2 Preliminaries

Throughout the paper we will assume that we are given an undirected graph \(G = (V, E)\). Unless stated otherwise, we will assume that \(E\) is partitioned into safe \(S\) and unsafe \(U\) edges. We are interested in edge-induced subgraphs of a graph. Thus, when we say that \(H\) is a subgraph of \(G = (V, E)\) we implicitly assume that \(H = (V, F)\) for some \(F \subseteq E\). For any subset of edges \(F \subseteq E\) and any set \(S \subseteq V\) we use the notation \(\delta_F(S)\) to denote the set of edges in \(F\) that have exactly one end point in \(S\). We may drop \(F\) if it is clear from the context.

**LP Relaxation:** We describe an LP relaxation for \((p,q)\)-Flex-Steiner problem. It is straightforward to generalize it for Flex-SNDP but we do not explicitly describe it here. Recall that we are given set of terminals \(T \subseteq V\) and the goal is to choose a min-cost subset of the edges \(F\) such that in the subgraph \(H = (V, F)\), \(u\) and \(v\) are \((p,q)\)-flex-connected for any \(u, v \in T\). Let \(C = \{S \subseteq V \mid S \cap T \neq T\} \) be the set of all vertex sets that separate some terminal pair. For a set of edges \(F\) to be feasible for the given \((p,q)\)-Flex-Steiner instance, we require that for all \(S \in C\), \(|\delta_F(S) \setminus B| \geq p\) for any \(B \subseteq U\) with \(|B| \leq q\). We can write cut covering constraints expressing this condition, but these constraints are not adequate by themselves. To improve this LP, we consider the connection to capacitated network design: we give each safe edge a capacity of \(p+q\), each unsafe edge a capacity of \(p\), and require \(p(p+q)\) connectivity for the terminal pairs; it is not difficult to verify that this is a valid constraint. These two sets of constraints yield the following LP relaxation with variables \(x_e \in [0,1]\), \(e \in E\).
\[
\min \sum_{e \in E} c_e x_e \\
\text{subject to } \sum_{e \in \delta(S) \setminus B} x_e \geq p \quad S \in \mathcal{C}, B \subseteq \mathcal{U}, |B| \leq q \\
(p + q) \sum_{e \in S \cap S} x_e + p \sum_{e \in S \cup \mathcal{U}} x_e \geq p(p + q) \\
x_e \geq [0, 1] \\

\]

It is not hard to see that the preceding LP admits a separation oracle that runs in \(n^{O(q)}\) time; for each \(B \subseteq \mathcal{U}, |B| \leq q\) we remove \(B\) and check that for each \(s, t \in T\), the \(s\)-\(t\) min-cut value in the graph \(G - B\) with edge-capacities given by \(x\) is at least \(p\). For the capacitated constraints, we can give every safe edge \(e\) a weight of \((p + q)x_e\), every unsafe edge \(e\) a weight of \(p(x_e)\), and check that the \(s\)-\(t\) min-cut value is at least \(p(p + q)\) for each \(s, t \in T\). Hence the LP can be solved in \(n^{O(q)}\) time. Our algorithms do not directly use the preceding LP relaxation, however, all of our approximation bounds can be shown with respect to the lower bound provided by it.

For the special case of \((p, q)\)-FGC we can show that the LP can be solved in polynomial time without any dependence on \(q\). We borrow ideas from \([5, 7]\). We defer the proof to the appendix (Section 5).

**Lemma 2.1.** The LP relaxation can be solved in polynomial time for \((p, q)\)-FGC.

**Remark 2.2.** Boyd et al. \([5]\) obtained several results for \((p, q)\)-FGC. Several of them can be shown with respect to the lower bound provided by the LP relaxation above. We defer details to a future version. One can also write a similar LP relaxation for the directed version of problems. In some settings it may be advantageous to use the directed formulation even for undirected problems.

**Augmentation:** The results of this paper rely on the augmentation framework. Suppose \(G = (V, E), T \subseteq V\) is an instance of \((p, q)\)-Flex-Steiner. We observe that \((p, 0)\)-Flex-Steiner instance can be solved via 2-approximation to EC-SNDP hence we are interested in \(q \geq 1\). Let \(F_1\) be a feasible solution for the \((p, q - 1)\)-Flex-Steiner instance on \(G, T\). This implies that for any cut \(S\) that separates two terminals we have \(|\delta_{F_1}(S)| \geq p\) or \(|\delta_{F_1}(S)| \geq p + q - 1\). We would like to augment \(F_1\) to obtain a feasible solution to satisfy the \((p, q)\) requirement. Define a function \(f : 2^{|V|} \to \{0, 1\}\) where \(f(S) = 1\) iff (i) \(S\) separates terminals and (ii) \(|\delta_{F_1}(S)| < p\) and \(|\delta_{F_1}(S)| = p + q - 1\). We call \(S\) a violated cut with respect to \(F_1\). Since \(F_1\) satisfies \((p, q - 1)\) requirement, if \(|\delta_{F_1} \cap S| < p\) it must be the case that \(|\delta_{F_1}(S)| \geq p + q - 1\). The following lemma is simple.

**Lemma 2.3.** Suppose \(F_2 \subseteq E \setminus F_1\) is a feasible cover for \(f\), that is, \(|\delta_{F_2}(S)| \geq f(S)\) for all \(S\). Then \(F_1 \cup F_2\) is a feasible solution to \((p, q)\)-Flex-Steiner for the terminal set \(T\).

The augmentation problem is then to find a min-cost subset of edges to cover \(f\) in \(G - F_1\). The key observation is that the augmentation problem does not distinguish between safe and unsafe edges and hence we can rely on traditional connectivity augmentation ideas. Note that \(f\) is a \(\{0, 1\}\) function. We prove a stronger lemma, showing that the LP relaxation for the original instance provides a valid cut-covering relaxation for the augmentation problem. The proof is deferred to the appendix (Section 5).

**Lemma 2.4.** Let \(x \in [0, 1]^{|E|}\) be a feasible LP solution for a given instance of \((p, q)\)-Flex-Steiner. Let \(F_1\) be a feasible solution that satisfies \((p, q - 1)\) requirements for the terminal. Then, for any violated cut \(S \subseteq V\) in \((V, F_1)\), we have \(\sum_{e \in \delta(S) \setminus F_1} x_e \geq 1\).
One can try to augment from \((p, q)\) to \((p + 1, q)\)-flex-connectivity. However, the resulting augmentation problem is less well-behaved; we still need to distinguish between safe and unsafe edges.

**Uncrossable functions and families:** Uncrossable functions are a general class of requirement functions that are an important ingredient in network design [32, 19, 20, 26].

**Definition 2.5.** A function \(f : 2^{|V|} \rightarrow \{0, 1\}\) is **uncrossable** if for every \(A, B \subseteq V\) such that \(f(A) = f(B) = 1\), one of the following is true: (i) \(f(A \cup B) = f(A \cap B) = 1\), (ii) \(f(A - B) = f(B - A) = 1\). A family of cuts \(C \subseteq 2^V\) is an uncrossable family if the indicator function \(f_C : 2^V \rightarrow \{0, 1\}\) with \(f(S) = 1\) iff \(S \in C\), is uncrossable.

**Definition 2.6.** Let \(G = (V, E)\) be a graph and let \(f : 2^{|V|} \rightarrow \{0, 1\}\) be a requirement function. Let \(A \subseteq E\). A set \(S\) is violated with respect to \(A, f\) if \(f(S) = 1\) and \(\delta_A(S) = \emptyset\). The residual requirement function of \(f\) with respect to \(A\), denoted by \(f_A\) is indicator function of the violated sets of \(f\) with respect to \(A\).

An important result in network design is a 2-approximation for the problem of min-cost covering of an uncrossable function \(f\) by the edge set of a graph [32]; this was later generalized to covering all skew-supermodular requirement functions [22]. This requires a computational assumption on \(f\) and we encapsulate it below.

**Theorem 2.7** ([32]). Let \(G = (V, E)\) be an edge-weighted graph and let \(f : 2^{|V|} \rightarrow \{0, 1\}\) be an uncrossable function. Suppose there is an efficient oracle that for any \(A \subseteq E\) outputs all the minimal violated sets of \(f\) with respect to \(A\). Then there is an efficient 2-approximation for the problem of finding a minimum cost subset of edges that covers \(f\).

A special case of uncrossable family of sets is a **ring family.** We say that an uncrossable family of sets \(C \subseteq 2^V\) is a ring family if the following conditions hold: (i) if \(A, B \in C\) and \(A, B\) properly intersect\(^3\) then \(A \cap B\) and \(A \cup B\) are in \(C\) and (ii) there is a unique minimal set in \(C\). We observe that if we have an uncrossable family \(C\) such that there is a vertex \(s\) contained in every \(A \in C\) then \(C\) is automatically a ring family. Theorem 2.7 can be strengthened for this case. There is an optimum algorithm to find a min-cost cover of a ring family — see [29, 30, 17]

**Enumerating small cuts in a graph:** In order to use Theorem 2.7 in the augmentation framework, we need to be able to first prove that a family of cuts that we intend to cover is uncrossable. Second, we need to be able efficiently find all the minimal violated sets of the family with respect to a given set of edges \(A\). Consider the augmentation problem from \((p, q)\) to \((p, q + 1)\), and the requirement function \(f\) that is induced by it. Recall that \(f(S) = 1\) iff \(\delta_{F_1}(S) = p + q\) and \(\delta_{F_1 \cap S} < p\). Thus, for any fixed \(p, q\) we can enumerate all violated sets in \(n^{O(p+q)}\) time by trying all possible cuts in \(F_1\) with \(p + q\) edges. Consider the graph \(G' = G - F_1\) and any set of edges \(A \subseteq E \setminus F_1\). It follows that given \(A\) and \(F_1\) that is feasible for \((p, q)\), we can find the set of all violated cuts in \(G - F_1\) with respect to \(A, f\) in \(n^{O(p+q)}\) time. In the context of \((p, q)\)-FGC we need a more sophisticated process since we do not limit ourselves to fixed \(p, q\). The following lemma from [5] is useful for this.

**Lemma 2.8** ([5]). Let \(G = (V, E)\) be an instance of \((p, q)\)-FGC where \(p, q \geq 1\). Let \(F_1 \subseteq E\) be a feasible solution to \((p, q - 1)\)-FGC on \(G\). Let \(C\) be the set of all cuts that are violated with respect to the augmentation function \(f\). Then, there are \(O(n^4)\) such cuts and they can be enumerated in polynomial time.

\(^3\)\(A, B\) properly intersect if \(A \cap B \neq \emptyset\) and \(A - B, B - A \neq \emptyset\).
Submodularity and posimodularity of the cut function: It is well-known that the cut function of an undirected graph is symmetric and submodular. Submodularity implies that for all $A, B \subseteq V$, $|\delta(A)| + |\delta(B)| \geq |\delta(A \cap B)| + |\delta(A \cup B)|$. Symmetry and submodularity also implies posimodularity: for all $A, B \subseteq V$, $|\delta(A)| + |\delta(B)| \geq |\delta(A - B)| + |\delta(B - A)|$.

3 A 5-approximation for $(2,2)$-Flex-ST

In this section, we provide a constant factor approximation for $(2,2)$-Flex-ST. We are given a graph $G = (V, E)$ with a cost function on the edges $c : E \to \mathbb{R}_{\geq 0}$, a partition of the edge set $E = S \cup U$ into safe and unsafe edges, and $s, t \in V$. Our goal is to find the cheapest set of edges such that every $s$-$t$ cut has either at least two safe edges or at least four total edges. Let $F^* \subseteq E$ denote the optimal solution. Note that $F^*$ is a feasible solution to the $(2,1)$-Flex-ST instance for the same graph $G$ and the same pair $s, t$. The reduction to the Cap-SNDP problem in [5] implies a 3-approximation for $(2,1)$-Flex-ST. It is also worth noting that Adjiashvili, Hommelheim, Mühlenthaler, and Schaudt [4] describe a 2-approximation to $(k,1)$-Flex-ST in directed graphs for any fixed $k$, which can be modified to give a 3-approximation in undirected graphs.

We follow the augmentation approach outlined earlier. However, instead of finding a feasible solution to $(2,1)$-Flex-ST, we start with something slightly stronger via the reduction to capacitated network design.

**Lemma 3.1.** There exists a set of edges $F \subseteq E$ such that for any $s$-$t$ separating set $A \subseteq V$, exactly one of the following is true: (1) $|\delta_F \cap S(A)| \geq 2$, (2) $|\delta_F(S)| \geq 4$, (3) $\delta_F(A)$ has exactly two unsafe and one safe edge. Furthermore, we can efficiently find such a set $F$ such that cost($F$) $\leq 2 \cdot$ cost($F^*$).

**Proof.** Consider giving every safe edge a capacity of 2, every unsafe edge a capacity of 1, and approximating the solution to the corresponding Cap-SNDP problem with an $s$-$t$ connectivity requirement of 4 (all other pairs have 0 requirement). Suppose $F \subseteq E$ is a feasible solution to this Cap-SNDP instance. Then, one can easily verify that $F$ follows the above requirements by casing on the number of safe edges in $\delta(A)$:

1. if $|\delta_F \cap S(A)| = 0$, since capacity of $\delta_F(A)$ is 4, it contains at least four unsafe edges.
2. else if $|\delta_F \cap S(A)| = 1$, then the safe edge provides capacity of 2 and to reach capacity 4, $\delta_F(A)$ contains at least two unsafe edges
3. else $|\delta_F \cap S(A)| \geq 2$.

We can approximate the Cap-SNDP instance to within a factor of 2 by replacing each edge of capacity 2 by two edges of capacity 1 and with the same cost. This reduces the problem to solving an $s$-$t$ minimum-cost flow problem (flow of value 4 with all edges having capacity 1) which can be solved optimally. Thus we obtain the desired polynomial time 2-approximation.

Finally, we show that any solution $F' \subseteq E$ to $(2,2)$-Flex-ST is also a solution to this Cap-SNDP instance. Let $A \subseteq V$ be an $s$-$t$ cut. Then, $\delta_{F'}(A)$ has two safe edges or four total. In both cases, $\delta_{F'}(A)$ has a total capacity of at least four. Thus the 2-approximation to the Cap-SNDP instance costs at most twice the optimum solution to $(2,2)$-Flex-ST.

Let $F \subseteq E$ be obtained via the algorithm in the preceding lemma, so for every $s$-$t$ cut $A$, $\delta_F(A)$ has at least two safe edges or at least four total edges or exactly two unsafe and one safe edge. If

---

This is not explicitly stated in [4] but follows from an easy observation by bidirecting edges. Unlike the case of $(1,k)$-Flex-ST [4], the directed and undirected graph cases do not seem to equivalent for $(k,1)$-Flex-ST.
any cut has at least two safe or four total edges, it satisfies the requirement on (2, 2)-Flex-ST, so we focus our attention on those cuts with exactly two unsafe and one safe edge. Let \( C \) denote the set of all violated cuts containing \( s \), i.e. \( C = \{ A \subset V : s \in A, t \notin A, |\delta_{F \cap S}(A)| = 1, |\delta_{F \cup U}(A)| = 2 \} \). By symmetry, it suffices to only consider cuts containing \( s \), since covering a set also covers its complement. Thus this is exactly the family of cuts that we need to cover in the augmentation phase. Since all violated cuts have exactly three edges, we can assume without loss of generality that all remaining edges in \( E - F \) are unsafe. Therefore, it suffices to solve the augmentation problem: we want to find the minimum cost subset \( F' \subseteq E \setminus F \) s.t. \( F'(A) \geq 1 \) for all \( A \in C \). Note that for any \( A \in C, t \in V \setminus A \) and hence for any \( A, B \in C, A \cup B \neq V \). This family of violated cuts is unfortunately not uncrossable, as shown in Figure 1. Instead, we will show that we can find three sub-families of \( C \) whose union is \( C \) and each sub-family is a ring family.

We begin with a lemma that characterizes when two violated sets \( A, B \) do not uncross.

**Lemma 3.2.** Suppose \( A, B \in C \) and the sets properly intersect. Then, either \( A \cup B, A \cap B \in C \), or one of \( \delta_F(A \cup B) \) and \( \delta_F(A \cap B) \) has exactly two safe edges (and no unsafe) and the other has exactly four unsafe edges (and no safe). These structures are demonstrated in Figure 2.

**Proof.** Suppose \( A, B \in C \) are two sets that properly intersect. Note that they must each have three edges crossing them, out of which one is safe and two are unsafe. First, suppose neither \( \delta_F(A \cup B) \) nor \( \delta_F(A \cap B) \) contain two safe edges. Then, they must have at least three total edges since \( F \) is feasible for (2, 1) s-t-connectivity. By submodularity,

\[
|\delta_F(A)| + |\delta_F(B)| = 6 \geq |\delta_F(A \cup B)| + |\delta_F(A \cap B)|.
\]

Therefore, \( \delta_F(A \cup B) \) and \( \delta_F(A \cap B) \) must both have exactly three total edges. Since neither of the two have two safe edges, they are both violated, i.e. \( A \cup B, A \cap B \in C \).

Instead, suppose one of \( \delta_F(A \cup B) \) and \( \delta_F(A \cap B) \) has at least two safe edges; without loss of generality, we assume \( \delta_{F \cap S}(A \cap B) \geq 2 \). Then, by submodularity of the cut function,

\[
|\delta_{F \cap S}(A)| + |\delta_{F \cap S}(B)| = 2 \geq |\delta_{F \cap S}(A \cap B)| + |\delta_{F \cap S}(A \cup B)|.
\]

This tells us that \( |\delta_{F \cap S}(A \cap B)| \leq 2 \), so \( A \cap B \) must be crossed by exactly 2 safe edges. Furthermore, if \( |\delta_{F \cap S}(A \cap B)| = 2 \), then \( |\delta_{F \cap S}(A \cup B)| = 0 \). Recall that all cuts in \( F \) are crossed by either at least two safe edges, at least four total edges, or exactly one safe and two unsafe edges. Therefore,
if \( \delta_F(A \cup B) \) has no safe edges, it must be the case that \( |\delta_F(A \cup B)| \geq 4 \). However, we know from above that \( |\delta_F(A \cup B)| + |\delta_F(A \cap B)| \leq 6 \). Therefore, \( |\delta_F(A \cup B)| = 4 \) and \( |\delta_F(A \cap B)| = 2 \). Since \( \delta_F(A \cup B) \) has no safe edges, it must be the case that \( \delta_F(A \cup B) \) has exactly four unsafe edges (and no safe). Similarly, since \( |\delta_{F \cap S}(A \cap B)| = |\delta_F(A \cap B)| = 2 \), \( \delta_F(A \cap B) \) has exactly two edges, both of which are safe. The argument is analogous in the case that \( \delta_F(A \cup B) \) has two safe edges.

**Corollary 3.3.** For each safe edge \( e \in S \), the set of all violated cuts \( A \subseteq V \) with \( e \in \delta_F(A) \) is a ring family.

**Proof.** Let \( A, B \in C \). Recall that this means they must each be crossed by exactly one safe edge. Suppose \( \delta_{F \cap S}(A) = \delta_{F \cap S}(B) = \{e\} \). Any safe edge crossing \( A \cap B \) or \( A \cup B \) also crosses at least one of \( A \) and \( B \), since \( \delta_{F \cap S}(A \cap B) \subseteq \delta_{F \cap S}(A) \cup \delta_{F \cap S}(B) \) and \( \delta_{F \cap S}(A \cup B) \subseteq \delta_{F \cap S}(A) \cup \delta_{F \cap S}(B) \). Therefore, both \( A \cap B \) and \( A \cup B \) are crossed by at most 1 safe edge. By Lemma 3.2, \( A \cup B, A \cap B \notin C \). Because of this, they must both be crossed by exactly one safe edge, so \( e \in \delta_F(A \cap B) \) and \( e \in \delta_F(A \cup B) \).

**Lemma 3.4.** Suppose \( A, B \in C \) and the sets properly intersect, and let \( e_1, e_2 \) denote the safe edges in \( \delta_F(A) \) and \( \delta_F(B) \) respectively. If there is an \( s \)-\( t \) path in the graph \((V, F)\) containing both \( e_1 \) and \( e_2 \), then \( A \cup B, A \cap B \in C \). Furthermore, \( \delta_{F \cap S}(A \cap B) \cap \delta_{F \cap S}(A \cup B) \subseteq \{e_1, e_2\} \).

**Proof.** Let \( A, B \in C \) such that the sets properly intersect, \( \{e_1\} = \delta_{F \cap S}(A) \) and \( \{e_2\} = \delta_{F \cap S}(B) \), and \( P \) be an \( s \)-\( t \) path in the graph \((V, F)\) containing both \( e_1 \) and \( e_2 \). If \( e_1 = e_2 \), then by Corollary 3.3, we are done. Else, without loss of generality, suppose \( e_1 \) comes before \( e_2 \) on this path \( P \). Suppose for the sake of contradiction, \( A \cap B \notin C \) or \( A \cup B \notin C \). By Lemma 3.2, either \( |\delta_{F \cap S}(A \cap B)| = 2 \) or \( |\delta_{F \cap S}(A \cup B)| = 2 \). Without loss of generality, suppose \( |\delta_{F \cap S}(A \cap B)| = 2 \). Since \( \delta_{F \cap S}(A \cap B) \subseteq \delta_{F \cap S}(A) \cup \delta_{F \cap S}(B) \), \( \delta_{F \cap S}(A \cup B) = \{e_1, e_2\} \).

Notice that Lemma 3.2 also tells us that there are no other edges in \( \delta_F(A \cap B) \). Let \( e_1 = \{x_1, y_1\}, e_2 = \{x_2, y_2\} \) where \( x_i \in A \cap B, y_i \notin A \cap B \) for \( i = 1, 2 \). Then, since \( s \in A \cap B \) and \( e_1 \) comes before \( e_2 \) in \( P \), \( x_1 \) must be visited before \( y_1 \) in \( P \). All nodes in \( P \) in between \( y_1 \) and \( e_2 \) must be outside \( A \cap B \), since the only way to re-enter \( A \cap B \) is through \( e_2 \). Therefore, \( y_2 \) must be visited before \( x_2 \) in \( P \). This implies that \( x_2 \) is visited after \( x_1, y_1 \), and \( y_2 \), and since \( P \) is a path, none of them can be revisited. Therefore, all remaining nodes in \( P \) must be in \( A \cap B \), since there are no remaining edges in \( \delta_F(A \cap B) \). This is a contradiction, since \( t \notin A \cap B \).

The case where \( e_1, e_2 \in \delta_F(A \cup B) \) is analogous: \( P \) must traverse \( e_1 \) to leave \( A \cup B \), and traverse \( e_2 \) to re-enter \( A \cup B \), but \( t \notin A \cup B \).

Therefore, \( A \cup B, A \cap B \in C \). Note that any edge crossing \( A \cap B \) or \( A \cup B \) must also cross \( A \) or \( B \). Since \( \delta_{F \cap S}(A) \cup \delta_{F \cap S}(B) \subseteq \{e_1, e_2\} \), \( \delta_{F \cap S}(A \cap B) \cup \delta_{F \cap S}(A \cup B) \subseteq \{e_1, e_2\} \) as well.

Let \( S' \subseteq F \cap S \) be the subset of safe edges cross at least one violated cut. Notice that Corollary 3.3 proves that all violated cuts crossed by the same safe edge form a ring family, and the above
Lemma 3.4 proves that all violated cuts crossed by safe edges on the same s-t path form a ring family. Therefore, it suffices to show that there are three s-t paths whose union covers all edges in $S'$.

**Lemma 3.5.** There exist three s-t paths $P_1, P_2, P_3$ in $(V,F)$ s.t. $S' \subseteq \bigcup_{i=1}^{3} P_i$. Furthermore, we can find these paths in polynomial time.

**Proof.** We can assume that $C$ is not empty and hence there is at least one violated cut. Consider a flow network on the graph $(V,F)$, where each safe edge is given a capacity of 2, and each unsafe edge is given a capacity of 1. Note that all violated cuts have a total capacity of exactly 4, and since violated cuts separate $s$ from $t$, the maximum s-t flow is at most 4. If the maximum flow was less than 4, then by the max-flow min-cut theorem, there would be an s-t cut with total capacity strictly less than 4. However, we constructed $F$ ensuring that all s-t cuts have capacity at least 4. Therefore, the maximum flow must be exactly 4.

All capacities are integral, hence there exists some integral max-flow; let $g$ be such a flow. Since the violated cuts have total capacity exactly 4, all edges crossing them must be fully saturated. In particular, all safe edges $e \in S'$ must have $g(e) = 2$. By flow decomposition, there are four s-t paths $P_1, \ldots, P_4$ each carrying a flow of 1, where all $e \in E$ with $g(e) > 0$ are in at least one path. Since each safe edge $e \in S'$ has $g(e) = 2$, $e$ belongs to at least two of the paths $P_1, \ldots, P_4$. Thus, choosing any three of them would cover all edges in $e \in S'$, hence $S' \subseteq \bigcup_{i=1}^{3} P_i$. □

**Claim 3.6.** There is a 5-approximation for $(2,2)$-Flex-ST.

**Proof.** Let $P_i$ be the paths defined by Lemma 3.5, and let $C_i$ be the set of all violated cuts whose corresponding safe edge is on the path $P_i$, i.e. $C_i = \{ A \in C : \delta_{F \cap S}(A) \subseteq P_i \}$. By definition of $S'$, for all $A \in C$, $\delta_{F \cap S}(A) \subseteq S' \subseteq \bigcup_{i=1}^{3} P_i$. Therefore,

$$C = \{ A \in C : \delta_{F \cap S}(A) \subseteq \bigcup_{i=1}^{3} P_i \} = \bigcup_{i=1}^{3} C_i.$$

We can solve the augmentation problems of finding the minimum cost subset $F_i \subseteq E \setminus F$ s.t. $\delta_{F_i}(A) \geq 1$ for all $A \in C_i$ for each $i = 1, 2, 3$. Since $C \subseteq \bigcup_{i=1}^{3} C_i$, $F' = \bigcup_{i=1}^{3} F_i$ is a feasible solution to the augmentation problem of finding the minimum cost subset of $E \setminus F$ s.t. $\delta_{F'}(A) \geq 1$ for all $A \in C$. Therefore, $F \cup F'$ is a feasible solution to (2,2)-Flex-ST.

As an immediate corollary of Claim 3.4, each $C_i$ is a ring family. From the discussion in Section 2, we see that the three corresponding augmentation problems can be solved exactly, so $\text{cost}(F_i) \leq \text{cost}(F^*)$. Thus, $\text{cost}(F') \leq 3 \cdot \text{cost}(F^*)$. By Lemma 3.1, $\text{cost}(F) \leq 2 \cdot \text{cost}(F^*)$. Overall, $\text{cost}(F \cup F') \leq 5 \cdot \text{cost}(F^*)$.

Finally, note that we can find $F$ in polynomial time by Lemma 3.1. For each $F_i$, we can find the minimal violated set by finding all cuts in in $F$ with $p+q$ (in this case, 4) edges, as explained in Section 2. □

**An approach to approximate (p,q)-Flex-Steiner:** In the appendix (Section 5.3) we outline a candidate approximation algorithm for $(p,q)$-Flex-Steiner that leverages an approximation algorithm for $(p,q)$-Flex-ST. We state a conjecture that can lead to a provable approximation guarantee.
4 Approximation Algorithms for FGC

In this section we prove Theorem 1.2. The theorem encapsulates two results which share the high-level approach but differ in some of the details. We first prove an $O(q)$ approximation for $(2,q)$-FGC and then prove our results for $(p,q)$-FGC for $q \leq 4$. As discussed in Section 2, most of the proofs in this section rely on proving that certain families of violated cuts are uncrossable. Suppose $A, B \in C$ and the sets properly intersect. We say that $A, B$ uncross if $A - B, B - A \in C$ or $A \cap B, A \cup B \in C$. Otherwise we say that $A, B$ do not uncross. Note that if $A \cup B = V$ and $A$ and $B$ are violated, then by symmetry, $V - A$ and $V - B$ are violated as well. In this case, however, $V - A = B - A$ and $V - B = A - B$, so once again, $A$ and $B$ uncross. Therefore, when proving uncrossability of two properly intersecting sets $A$ and $B$, we assume without loss of generality that $(A \cup B) \neq V$.

4.1 An $O(q)$-approximation for $(2,q)$-FGC

We prove that the augmentation problem for increasing flex-connectivity from $(2,q)$-FGC is $O(q)$-approximation for $(2,q)$-FGC as follows. We start with a feasible solution to $(2,q)$-FGC. We can use the known $2$-approximation algorithm in each augmentation stage since
the family is uncrossable. Recall from Section 2 that the the violated cuts can be enumerated in polynomial time, and hence the primal-dual 2-approximation for covering an uncrossable function can be implemented in polynomial-time. This leads to the claimed approximation and running time.

4.2 Approximating \((p, q)\)-FGC for \(q \leq 4\)

We have seen that the augmenting problem from \((2, q)\)-FGC to \((2, q+1)\)-FGC leads to covering an uncrossable function. Boyd et al. [5] showed that augmenting from \((p,0)\)-FGC to \((p,1)\)-FGC also leads to an uncrossable function for any \(p \geq 1\). However this approach fails in general for most cases of augmenting from \((p, q)\)-FGC to \((p, q+1)\)-FGC. We give an example for \((3, 1)\) to \((3, 2)\) in Figure 3 in Section 4.3. However, in certain cases, we can argue that we can take a more sophisticated approach where we solve the augmentation problem by considering the violated cuts in a small number of stages: in each stage we choose a subfamily of the violated cuts that is uncrossable. In such cases, we can get obtain a 2\(k\)-approximation for the augmentation problem, where \(k\) is the upper bound on the number of stages. Here we show that this approach works to augment from \((p, q)\) to \((p, q+1)\) whenever \(q \leq 2\) and also for \(q = 3\) when \(p\) is even. We show in Section 4.3 limitations of our approach for \(q \geq 4\) and for \(q = 3\) when \(p\) is odd.

Suppose we want to augment from \((p, q)\)-FGC to \((p, q+1)\)-FGC. Let \(G = (V, E)\) be the original input graph, and let \(F\) be the set of edges we have already included. Note that in \((V, F)\), all cuts fall into one of the following categories:

- \(\geq p\) safe edges
- \(\geq p + q + 1\) total edges
- Exactly \(p + q\) total edges, with at most \(p - 1\) safe edges

Note that the first two satisfy the constraints of \((p, q+1)\)-FGC, so the last category is exactly the set of all violated cuts. Instead of attempting to cover all violated sets at once, we do it in stages where in each stage we consider the violated cuts based on the number of safe edges. We begin by covering all violated sets with no safe edges, then with one safe edge, and iterate until all violated sets are covered. This is explained in Algorithm 1 below.

**Algorithm 1** Augmenting from \((p, q)\) to \((p, q+1)\) in stages

1. \(H \leftarrow F\)
2. for \(i = 0, \ldots, p - 1\) do
3. \(C_i \leftarrow \{S : \emptyset \neq S \subseteq V, |\delta_H(S)| = p + q, |\delta_{H \cap S}| = i\}\)
4. \(F_i \leftarrow \text{approximation algorithm to cover cuts in } C_i\)
5. \(H \leftarrow H \cup F_i\)
6. end for
7. return \(H\)

The only unspecified part of the algorithm is to cover cuts in \(C_i\) in the \(i\)'th stage. If we can prove that \(C_i\) forms an uncrossable family then we can obtain a 2-approximation in each stage. One could also come up with a more elaborate algorithm to cover \(C_i\). First, we prove a generic and useful lemma regarding cuts in \(C_i\).

For the remaining lemmas, we let \(H_i \subseteq E\) denote the set of edges \(H\) at the start of iteration \(i\). In other words, \(H_i\) is a set of edges such that for all \(\emptyset \neq S \subseteq V\), if \(|\delta_{H_i}(S)| = p + q\), then \(|\delta_{H \cap S}(S)| \geq i\).
Lemma 4.2. Fix an iteration $i \in \{0, \ldots, p - 1\}$. Let $C_i$ be as defined in Algorithm 1. Then, if $A, B \in C_i$ and

1. $|\delta_{H_i}(A \cap B)| = |\delta_{H_i}(A \cup B)| = p + q$, or
2. $|\delta_{H_i}(A - B)| = |\delta_{H_i}(B - A)| = p + q$

then $A$ and $B$ uncross, i.e. $A \cap B, A \cup B \in C_i$ or $A - B, B - A \in C_i$.

Proof. By submodularity and posimodularity of the cut function, $|\delta_{H_i}(A \cap B)| + |\delta_{H_i}(A \cup B)|$ and $|\delta_{H_i}(A - B)| + |\delta_{H_i}(B - A)|$ are both at most $2(p + q)$. First, suppose $|\delta_{H_i}(A \cap B)| = |\delta_{H_i}(A \cup B)| = p + q$. By submodularity of the cut function applied to the safe edges $H_i \cap S$,

$$2i = |\delta_{H_i \cap S}(A)| + |\delta_{H_i \cap S}(B)| \geq |\delta_{H_i \cap S}(A \cup B)| + |\delta_{H_i \cap S}(A \cap B)|.$$ 

Since we are in iteration $i$, neither $\delta_{H_i}(A \cup B)$ nor $\delta_{H_i}(A \cap B)$ can have less than $i$ safe edges. Therefore, $|\delta_{H_i \cap S}(A \cup B)| = |\delta_{H_i \cap S}(A \cap B)| = i$. Then, $A \cap B, A \cup B \in C_i$, so $A$ and $B$ uncross. In the other case, where $|\delta_{H_i}(A - B)| = |\delta_{H_i}(B - A)| = p + q$, we can use posimodularity of the cut function to once again show that $|\delta_{H_i \cap S}(A - B)| = |\delta_{H_i \cap S}(B - A)| = i$, so they are both in $C_i$. 

Note that the preceding lemma holds for the high-level approach. Now we focus on cases where we can prove that $C_i$ is uncrossable.

Lemma 4.3. Fix an iteration $i \in \{0, \ldots, p - 1\}$. Let $C_i$ be as defined in Algorithm 1. Then, for $q \leq 2$, $C_i$ is uncrossable.

Proof. Suppose $A, B \subseteq V$ such that $\delta_{H_i}(A)$ and $\delta_{H_i}(B)$ both have exactly $i$ safe and $p + q - i$ unsafe edges. Suppose for the sake of contradiction that they do not uncross. By Lemma 4.2, one of $\delta_{H_i}(A \cap B)$ and $\delta_{H_i}(A \cup B)$ must have at most $p + q - 1$ edges, and the same holds for $\delta_{H_i}(A - B)$ and $\delta_{H_i}(B - A)$. Without loss of generality, suppose $\delta_{H_i}(A \cap B)$ and $\delta_{H_i}(A - B)$ each have at most $p + q - 1$ edges. By the assumptions on $H_i$, they must both have at least $p$ safe edges, hence they each have at most $q - 1$ unsafe edges. Note that $\delta_{H_i}(A) \subseteq \delta_{H_i}(A - B) \cup \delta_{H_i}(A \cap B)$, hence $\delta_{H_i}(A)$ can have at most $2(q - 1)$ unsafe edges. When $q \leq 2$, $2(q - 1) < q + 1$, which implies that $\delta_{H_i}(A)$ has strictly more than $p - 1$ safe edges, a contradiction. Notice that $\delta_{H_i}(A) \subseteq \delta_{H_i}(B - A) \cup \delta_{H_i}(A \cup B)$, $\delta_{H_i}(B) \subseteq \delta_{H_i}(A - B) \cup \delta_{H_i}(A \cap B)$, and $\delta_{H_i}(B) \subseteq \delta_{H_i}(B - A) \cup \delta_{H_i}(A \cap B)$; therefore the same argument follows regardless of which pair of sets each have strictly less than $p + q$ edges.

Corollary 4.4. For any $p \geq 2$ there is a $(2p + 4)$-approximation for $(p, 2)$-FGC and a $(4p + 4)$-approximation for $(p, 3)$-FGC.

Proof. Via [5] we have a 4-approximation for $(p, 1)$-FGC. We can start with a feasible solution $F$ for $(p, 1)$-FGC and use Algorithm 1 to augment from $(p, 1)$ to $(p, 2)$; each stage can be approximated to within a factor of 2 since $C_i$ is an uncrossable family. Since there are $p$ stages and the cost of each stage can be upper bounded by the cost of $F^*$, an optimum solution to the $(p, 2)$ problem, the total cost of augmentation is $2pOPT$. This leads to the desired $(2p + 4)$-approximation for $(p, 2)$-FGC. For $(p, 3)$-FGC we can augment from $(p, 2)$ to $(p, 3)$ paying an additional cost of $2pOPT$. This leads to the claimed $(4p + 4)$-approximation. Following the discussion in Section 2, covering the uncrossable families that arise in Algorithm 1 can be done in polynomial time.

Can we extend the preceding lemma for $q = 3$? It turns out that it does work when $p$ is even but fails for odd $p \geq 3$. 

14
Lemma 4.5. Fix an iteration $i \in \{0, \ldots, p - 1\}$. Let $C_i$ be as defined in Algorithm 1. Then, for $q = 3$ and any even integer $p$, $C_i$ is uncrossable.

Proof. Suppose $A, B \subseteq V$ such that $\delta_H(A)$ and $\delta_H(B)$ both have exactly $i$ safe and $p + 3 - i$ unsafe edges. Suppose for the sake of contradiction that they do not uncross. By Lemma 4.2, one of $\delta_H(A \cap B)$ and $\delta_H(A \cup B)$ must have at most $p + 2$ edges, and the same holds for $\delta_H(A - B)$ and $\delta_H(B - A)$. Without loss of generality, suppose $|\delta_H(A \cap B)|$ and $|\delta_H(A - B)|$ each have at most $p + 2$ edges. By the assumptions on $H_i$, they must both have at least $p$ safe edges. By submodularity of the cut function,

$$|\delta_{H_i \cap S}(A)| + |\delta_{H_i \cap S}(B)| = 2i \geq |\delta_{H_i \cap S}(A \cap B)| + |\delta_{H_i \cap S}(A \cup B)| \geq p + |\delta_{H_i \cap S}(A \cup B)|.$$  

Similarly, by posimodularity of the cut function,

$$|\delta_{H_i \cap S}(A)| + |\delta_{H_i \cap S}(B)| = 2i \geq |\delta_{H_i \cap S}(A - B)| + |\delta_{H_i \cap S}(B - A)| \geq p + |\delta_{H_i \cap S}(B - A)|.$$  

Therefore, $\delta_{H_i}(A \cup B)$ and $\delta_{H_i}(B - A)$ can each have at most $2i - p < i$ safe edges, so by the requirement on $H_i$, they must each have at least $p + 4$ total edges. Once again applying submodularity of the cut function,

$$|\delta_{H_i}(A)| + |\delta_{H_i}(B)| = 2p + 6 \geq |\delta_{H_i}(A \cap B)| + |\delta_{H_i}(A \cup B)| \geq p + 4 + |\delta_{H_i}(A \cap B)|.$$  

Similarly, by posimodularity of the cut function,

$$|\delta_{H_i}(A)| + |\delta_{H_i}(B)| = 2p + 6 \geq |\delta_{H_i}(A - B)| + |\delta_{H_i}(B - A)| \geq p + 4 + |\delta_{H_i \cap S}(A - B)|.$$  

Thus $\delta_{H_i}(A \cap B)$ and $\delta_{H_i}(A - B)$ can each have a total of at most $p + 2$ edges.

Notice that each safe edge on $\delta_{H_i}(A \cap B)$ or $\delta_{H_i}(A - B)$ must be in either $\delta_{H_i}(A)$ or $\delta_{H_i}(A \cap B) \cap \delta_{H_i}(A - B)$. Let $\ell = |\delta_{H_i \cap S}(A \cap B) \cap \delta_{H_i \cap S}(A - B)|$, i.e. the number of safe edges crossing both $A \cap B$ and $A - B$. Since $|\delta_{H_i \cap S}(A)| = i$,

$$i = |\delta_{H_i \cap S}(A)| \geq |\delta_{H_i \cap S}(A \cap B)| + |\delta_{H_i \cap S}(A - B)| - 2\ell \geq 2p - 2\ell,$$

implying that $\ell \geq \frac{2p - i}{2}$. Note that $\delta_{H_i}(A) \subseteq \delta_{H_i}(A \cap B) \cup \delta_{H_i}(A - B)$. Furthermore, if an edge crosses both $A \cap B$ and $A - B$, then it must have one endpoint in each (since they are disjoint sets), and therefore does not cross $A$. Therefore,

$$|\delta_{H_i}(A)| \leq |\delta_{H_i}(A \cap B)| + |\delta_{H_i}(A - B)| - 2\ell \leq 2(p + 2) - 2 \cdot \left\lceil \frac{2p - i}{2} \right\rceil \leq 4 + i.$$  

When $i < p - 1$, this is at most $p + 2$. When $i = p - 1$, since $p$ is even, $2 \cdot \left\lceil \frac{2p - i}{2} \right\rceil = 2 \cdot \left\lceil \frac{p + 1}{2} \right\rceil = p + 2$, and $2(p + 2) - p + 2 = p + 2$. In either case, we get $|\delta_{H_i}(A)| = p + 2$, a contradiction to the assumption on $A$.

The preceding lemma leads to an $(6p + 4)$-approximation for $(p, 4)$-FGC when $p$ is even by augmenting from a feasible solution to $(p, 3)$, since we pay an additional cost of $2pOPT$. The reasoning in the preceding lemma also shows that $C_i$ is uncrossable when $p$ is odd as well for all $i < p - 1$. Therefore, the bottleneck for odd $p$ is in covering $C_{p-1}$. We show, via an example that the family is indeed not uncrossable (see Section 4.3). However, it may be possible to show that $C_{p-1}$ separates into a constant number of uncrossable families leading to an $O(p)$-approximation for $(p, 4)$-FGC for all $p$. The first non-trivial case is when $p = 3$.  

15
Figure 3: Example graph where violated cuts in augmentation problem from (3, 1)-FGC to (3, 2)-FGC are not uncrossable. Red dashed edges are unsafe, while green solid edges are safe.

Figure 4: Example graph where violated cuts in augmentation problem from (p, 3)-FGC to (p, 4)-FGC are not uncrossable when p is odd. The numbers on edges denote the number of parallel edges. Red dashed edges are unsafe, while green solid edges are safe.

4.3 Examples

We describe several examples that demonstrate that violated cuts that arise in augmentation do not form an uncrossable family.

Augmenting from (3, 1)-FGC to (3, 2)-FGC: Consider the simple example in Figure 3. Let \( A = \{x_1, x_2\}, B = \{x_2, x_3\} \). Notice that \( A \) and \( B \) are violated, since they each have two safe and two unsafe edges, but \( A \cup B = \{x_1, x_2, x_3\} \) and \( B - A = \{x_3\} \) are not, since they each have three safe edges.

Augmenting from (p, 3)-FGC to (p, 4)-FGC when p is odd: We show that the family of cuts with exactly \( p - 1 \) safe edges when augmenting from (p, 3)-FGC to (p, 4)-FGC is not uncrossable when \( p \) is odd. See Figure 4. Let \( A = \{x_1, x_2\}, B = \{x_2, x_3\} \). Notice that both \( A \) and \( B \) have exactly \( p - 1 \) safe edges and 4 unsafe edges. However, \( A \cup B \) and \( B - A \) each have \( p \) safe edges, and \( A \cap B \) and \( A - B \) each have \( p + 4 \) total edges. Note that unlike in the above case, none of the cuts \( A \cup B, A \cap B, A - B, B - A \) are violated.

Augmenting from (4, 4)-FGC to (4, 5)-FGC: Finally, we demonstrate an example where two violated cuts are not uncrossable when augmenting from (4, 4)-FGC to (4, 5)-FGC, in Figure 5. As above, let \( A = \{x_1, x_2\}, B = \{x_2, x_3\} \). Note that \( A \) and \( B \) each only have 3 safe edges crossing them, so the argument that \( C_i \) is uncrossable for \( i < p - 1 \) fails. \( A \cup B \) and \( B - A \) each have at least 4 safe edges, and \( A \cap B \) and \( A - B \) each have 9 total edges, so none are violated.
Figure 5: Example graph where violated cuts in augmentation problem from $(4, 4)$-FGC to $(4, 5)$-FGC are not uncrossable. The numbers on edges denote the number of parallel edges. Red dashed edges are unsafe, while green solid edges are safe.

References

[1] David Adjiashvili. Fault-tolerant shortest paths - beyond the uniform failure model, 2013. URL: https://arxiv.org/abs/1301.6299, doi:10.48550/ARXIV.1301.6299. (document), 1

[2] David Adjiashvili, Felix Hommelsheim, and Moritz Mühlenthaler. Flexible graph connectivity. In *International Conference on Integer Programming and Combinatorial Optimization*, pages 13–26. Springer, 2020. (document), 1

[3] David Adjiashvili, Felix Hommelsheim, and Moritz Mühlenthaler. Flexible graph connectivity. *Mathematical Programming*, 192(1):409–441, 2022. (document), 1, 2, 1.1

[4] David Adjiashvili, Felix Hommelsheim, Moritz Mühlenthaler, and Oliver Schaudt. Fault-tolerant edge-disjoint paths – beyond uniform faults, 2020. URL: https://arxiv.org/abs/2009.05382, doi:10.48550/ARXIV.2009.05382. (document), 1, 1.1, 3, 4, 5.2

[5] Sylvia Boyd, Joseph Cheriyan, Arash Haddadan, and Sharat Ibrahimpur. Approximation algorithms for flexible graph connectivity, 2022. A preliminary version of the paper appeared in Proc. of FSTTCS 2021. URL: https://arxiv.org/abs/2202.13298, doi:10.48550/ARXIV.2202.13298. (document), 1, 2, 1.1, 1.1, 2, 2.2, 2, 2.8, 3, 4.2, 4.2

[6] R. D. Carr, L. K. Fleischer, V. J. Leung, and C. A. Phillips. Strengthening integrality gaps for capacitated network design and covering problems. In *Proceedings of the eleventh annual ACM-SIAM symposium on Discrete algorithms*, pages 106–115. Society for Industrial and Applied Mathematics, 2000. 1, 1.1

[7] Deeparnab Chakrabarty, Chandra Chekuri, Sanjeev Khanna, and Nitish Korula. Approximability of capacitated network design. *Algorithmica*, 72(2):493–514, 2015. 1, 1.1, 2, 5.2

[8] Deeparnab Chakrabarty, Ravishankar Krishnaswamy, Shi Li, and Srivatsan Narayanan. Capacitated network design on undirected graphs. In *Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques*, pages 71–80. Springer, 2013. 1
[9] Tanmoy Chakraborty, Julia Chuzhoy, and Sanjeev Khanna. Network design for vertex connectivity. In *Proceedings of the fortieth annual ACM symposium on Theory of computing*, pages 167–176, 2008. 1.1, 1.1, 5.3, 5.3

[10] C. Chekuri, A. Ene, and A. Vakilian. Node-weighted network design in planar and minor-closed families of graphs. In *Automata, Languages, and Programming*, pages 206–217. Springer, 2012. 1.1

[11] Chandra Chekuri and Nitish Korula. Single-sink network design with vertex connectivity requirements. In *IARCS Annual Conference on Foundations of Software Technology and Theoretical Computer Science*. Schloss Dagstuhl-Leibniz-Zentrum für Informatik, 2008. 1.1, 5.3, 5.3

[12] J. Cheriyan, S. Vempala, and A. Vetta. Network design via iterative rounding of setpair relaxations. *Combinatorica*, 26(3):255–275, 2006. 1.1

[13] J. Chuzhoy and S. Khanna. An \(O(k^3 \log n)\)-approximation algorithm for vertex-connectivity survivable network design. *Theory of Computing*, 8:401–413, 2012. 1.1

[14] Julia Chuzhoy and Sanjeev Khanna. Algorithms for single-source vertex connectivity. In *2008 49th Annual IEEE Symposium on Foundations of Computer Science*, pages 105–114. IEEE, 2008. 1.1, 1.1, 5.3, 5.3

[15] Yevgeniy Dodis and Sanjeev Khanna. Design networks with bounded pairwise distance. In *Proceedings of the thirty-first annual ACM symposium on Theory of computing*, pages 750–759, 1999. 1

[16] L. Fleischer, K. Jain, and D. P. Williamson. Iterative rounding 2-approximation algorithms for minimum-cost vertex connectivity problems. *Journal of Computer and System Sciences*, 72(5):838–867, 2006. 1.1

[17] András Frank. Kernel systems of directed graphs. *Acta Sci. Math.(Szeged)*, 41(1-2):63–76, 1979. 2

[18] M. X. Goemans, A. V. Goldberg, S. Plotkin, D. B. Shmoys, E. Tardos, and D. P. Williamson. Improved approximation algorithms for network design problems. In *Proceedings of the fifth annual ACM-SIAM symposium on Discrete algorithms*, pages 223–232, 1994. 1

[19] M. X. Goemans and D. P. Williamson. The primal-dual method for approximation algorithms and its application to network design problems. *Approximation algorithms for NP-hard problems*, pages 144–191, 1997. 2

[20] A. Gupta and J. Könemann. Approximation algorithms for network design: A survey. *Surveys in Operations Research and Management Science*, 16(1):3–20, 2011. 1, 2

[21] Eran Halperin and Robert Krauthgamer. Polylogarithmic inapproximability. In *Proceedings of the thirty-fifth annual ACM symposium on Theory of computing*, pages 585–594, 2003. 1

[22] K. Jain. A factor 2 approximation algorithm for the generalized Steiner network problem. *Combinatorica*, 21(1):39–60, 2001. 1, 2

[23] David R Karger. Global min-cuts in rnc, and other ramifications of a simple min-cut algorithm. In *Soda*, volume 93, pages 21–30, 1993. 5.1
5 Appendix

5.1 Proofs Omitted from Section 2

Proof of Lemma 2.1. We show a polynomial time separation oracle for the given LP. Suppose we are given some vector \( x \in [0, 1]^{|E|} \). We first check if the capacitated min-cut constraints are satisfied in polynomial-time. This can be done in polynomial time by giving every safe edge a weight of \( p+q \) and every unsafe edge a weight of \( p \), and checking that the min-cut value is at least \( p(p+q) \). If it is not, we can find the minimum cut and output the corresponding violated constraint. Suppose all capacitated constraints are satisfied. Now consider the first set of constraints. If one of them is not satisfied there must be some \( S \subset V \) and some \( B \subset U \), \( |B| \leq q \), such that \( \sum_{e \in \delta(S) - B} x_e < p \). In particular, \( \sum_{e \in S \cap \delta(S) - B} x_e < p \) and \( \sum_{e \in U \cap \delta(S) - B} x_e < p \). We claim that the total weight (according to weights \( p+q \) for safe edges and \( p \) for unsafe edges) going across \( \delta(S) \) is at most \( 2p(p+q) \): at most \( (p+q)p \) from \( S \cap (\delta(S) - B) \), at most \( p^2 \) from \( U \cap (\delta(S) - B) \), and at most \( pq \) from \( B \). Recall that the min-cut of the graph according the weights has already been verified to be at least \( p(p+q) \). Hence, any violated cut from the first set of constraints corresponds to 2-approximate min-cut. It is known via Karger’s theorem that there are at most \( O(n^4) \) 2-approximate min-cuts in a graph, and moreover they can also be enumerated in polynomial time [23, 24]. We can enumerate all 2-approximate min-cuts and check each of them to see if they are violated. To verify whether a candidate cut \( S \) is violated we consider the unsafe edges in \( \delta(S) \cap U \) and sort them in decreasing order of \( x_e \) value. Let \( B' \) be a prefix of this sorted order of size \( \min\{q, |\delta(S) \cap U|\} \). It is easy to
of potential hardness, we show an efficient solution.

Claim 5.1. Any optimal integral solution needs at least \( k+1 \) safe edges.

**Proof.** Consider any integral solution \( F \subseteq E \) where \( |F \cap S| < \frac{k+1}{2} \). Let \( S = \{s\} \cup \{v_i : \{v_i, t\} \notin F\} \). Then, \( \delta_F(S) \) is exactly the set of all unsafe edges from \( s \) to \( v_i \) such that \( \{v_i, t\} \in F \). Thus, \( |\delta_F(S)| \leq 2 \cdot |F| < k+1 \). Since \( \delta_F(S) \) has no safe edges, \( S \) is a violated cut, so \( F \) is not a feasible solution.

**Claim 5.2.** Consider the following fractional solution \( x \in [0,1]|E| : x_e = \begin{cases} \frac{1}{2} & e \in U \\ \frac{2}{k+1} & e \in S \end{cases} \). Then, \( x \) is a feasible LP solution.

**Proof.** Let \( S \subseteq V \) be any arbitrary cut such that \( s \in S, t \notin S \). The capacitated constraint for \((1,k)\)-flex connectivity is:

\[
(k+1) \sum_{e \in \delta_S(S)} x_e + \sum_{e \in \delta_U(S)} x_e \geq k+1.
\]
Substituting the values of \( x_e \), we get

\[
(k + 1) \sum_{e \in \delta_S(S)} x_e + \sum_{e \in \delta_U(S)} \left( \frac{2}{k + 1} \right) |\delta_S(S)| + |\delta_U(S)| = 2|\delta_S(S)| + |\delta_U(S)|.
\]

Notice that \( \delta(S) \) has one safe edge for each \( v_i \in S \), since all safe edges are from \( v_i \) to \( t \) and \( t \notin S \), so \(|\delta_S(S)| = |S| - 1\). Similarly, \( \delta(S) \) has two unsafe edges for every \( v_i \notin S \), since \( s \in S \). Thus \(|\delta_U(S)| = 2(k + 1 - (|S| - 1)) = 2(k - |S| + 2)\) Thus \( 2|\delta_S(S)| + |\delta_U(S)| = 2(|S| - 1) + 2(k - |S| + 2) = 2(k + 1) \geq k + 1 \).

For the remaining set of constraints, let \( B \subseteq U \), \(|B| = k \). We want to show that \( \sum_{e \in \delta(S) - B} x_e \geq 1 \). We case on \(|S| \). If \(|S| \geq \frac{k + 3}{2} \), then by the analysis above, \(|\delta_S(S)| = |S| - 1 \geq \frac{k + 1}{2} \), so \( \sum_{e \in \delta_S(S)} x_e \geq \frac{k + 1}{2} \cdot \frac{k + 1}{2} = 1 \). Since \( B \subseteq U \), \( \sum_{e \in \delta(S) - B} x_e \geq 1 \). Suppose instead that \(|S| < \frac{k + 3}{2} \). Then, since \( x_e = 1 \) for all unsafe edges,

\[
\sum_{\delta_U(S) - B} x_e \geq |\delta_U(S)| - |B| = 2(k - |S| + 2) - k = k + 4 - 2|S| > k + 4 - (k + 3) = 1
\]

In either case, we get our desired result.

Notice that the total cost of any optimal integral solution is at least \((k + 1) \cdot \frac{k + 1}{2} = \frac{(k + 1)^2}{2}\). However, the optimal fractional solution has cost \( k + 1 \) from the unsafe edges and \( 2(k + 1) \) from the safe edges, for a total cost of \( 3(k + 1) \). Thus, we get an \( \Omega(k) \) integrality gap.

### 5.3 An approach to approximate \((p, q)\)-Flex-Steiner via \((p, q)\)-Flex-ST

We consider the \((p, q)\)-Flex-Steiner problem. The input is an edge-weighted graph \( G = (V, E) \) and a set \( T \subseteq V \) of terminals and our goal is to find a min-cost subset of edges \( F \subseteq E \) such that in \((V, F)\) the terminals are \((p, q)\)-flex-connected. We first observed that \((p, q)\)-flex-connectivity is symmetric and transitive. Symmetry is easy to see since we are working in undirected graphs. For transitivity, if \( u, v \) are \((p, q)\)-flex-connected and \( v, w \) are \((p, q)\)-flex-connected then we claim that \( u, w \) are \((p, q)\)-flex-connected. This too easily follows from the cut condition for \((p, q)\)-flex-connectivity. Thus, \((p, q)\)-Flex-Steiner problem is equivalent to the rooted problem. In the rooted problem we are given \( G \), a root vertex \( r \in V \) and set \( T \subseteq V \) of terminals and the goal is to find a min-cost subgraph \( H = (V, F) \) such that each \( t \in T \) is \((p, q)\)-flex-connected to \( r \).

One can of course try to solve \((p, q)\)-Flex-Steiner problem directly. However, other than for \((1, q)\) and \((p, 1)\) cases we do not have any non-trivial results. It does not seem easy to generalize our result for \((2, 2)\)-Flex-ST to \((2, 2)\)-Flex-Steiner problem. Nevertheless, we can ask whether the single-pair algorithm can somehow be used to develop a candidate algorithm for the Steiner problem. Suppose we have an \( \alpha \)-approximation for \((p, q)\)-Flex-ST. Given a rooted \((p, q)\)-Flex-Steiner instance we can easily obtain an \(|T|\alpha\)-approximation: for each \( t \in T \) use the approximation algorithm for \((p, q)\)-Flex-ST to connect \( t \) to \( r \) and take the union of the solutions. Can we do better than this naive approach?

Inspired by the success of a simple random permutation based greedy algorithm for single-source vertex-connectivity problem \([9, 14, 11, 27]\) we propose the following algorithm which randomly permutes the terminals and greedily connects the terminals to the root or previous terminals.
Algorithm 2 Rooted \((p,q)\)-Flex-Steiner via \((p,q)\)-Flex-ST

1: \( F \leftarrow \emptyset \)
2: Let \( t_{i_1}, t_{i_2}, \ldots, t_{i_h} \) be a random permutation of the terminal set \( T \subset V \) where \( h = |T| \)
3: for \( j = 1, \ldots, h \) do
   4: \( G_j \) is graph obtained by contracting \( t_{i_1}, \ldots, t_{i_{j-1}} \) into root \( r \)
   5: Use \((p,q)\)-Flex-ST approx algorithm to connect \( t_{i_j} \) to root \( r \) in \( G_j \). Let \( F_j \) be the set of chosen edges.
   6: \( F \leftarrow F \cup F_j \)
4: end for
7: return \( F \)

Lemma 5.3. Algorithm 2 outputs a feasible solution.

Proof. Suppose for the sake of contradiction that some terminal is not \((p,q)\)-flex-connected to the root \( r \) in \((V,F)\), and let \( j \) be the smallest index such that \( t_{i_j} \) is not connected to the root \( r \). Then, there must be some cut \( S \) with \( t_{i_j} \in S \), \( r \notin S \), where \( \delta_F(S) \) has \(< p \) safe edges and \(< p + q \) total edges.

First, suppose \( t_{i_k} \in S \) for some \( k < j \). Then, \( S \) is an unsatisfied cut separating \( t_{i_k} \) from \( r \), implying that \( t_{i_k} \) is not \((p,q)\)-flex-connected to \( r \). This contradicts the minimality of \( j \).

Therefore, \( t_{i_k} \notin S \) for all \( k < j \). Then, in the \( j \)th iteration of Algorithm 2, \( G_j \) would not have contracted any edges in \( \delta_F(S) \), and \( S \) would have separated \( t_{i_j} \) from the contracted root \( r \), a contradiction.

The analysis for the preceding random greedy algorithm for vertex-connectivity is based on a cost sharing argument. We set up the relevant notion and state a conjecture and its implication. Given an instance of rooted \((p,q)\)-Flex-Steiner instance on \( G = (V,E) \) with root \( r \) and terminal set \( T \) we define the quantity \( \beta_t \) for each \( t \in T \) as follows: \( \beta_t \) is the minimum-cost to \((p,q)\)-flex-connect \( t \) to \((T-t) + r \): in other words this is the cost of \((p,q)\)-flex-connecting \( t \) to the root in the graph \( G_{t} \) where we contract all the terminal in \( T-t \) into \( r \).

Conjecture 5.4. There exists an integer valued function \( f(p,q) \) such that \( \sum_{t \in T} \beta_t \leq f(p,q)OPT \) where \( OPT \) is the optimum cost for the given rooted \((p,q)\)-Flex-Steiner instance.

Using a simple analysis that has been used previously [9, 14, 11, 27], we obtain the following.

Lemma 5.5. If Conjecture 5.4 is true then Algorithm 2 yields a randomized \( O(f(p,q)\alpha(p,q) \ln |T|) \) approximation for rooted \((p,q)\)-Flex-Steiner problem where \( \alpha(p,q) \) is the approximation available for \((p,q)\)-Flex-ST. Moreover there is also a deterministic variant of Algorithm 2 that achieves the same approximation bound.

We omit the proof of the preceding lemma since it is an easy consequence of the ideas in the previous work that we mentioned. We believe that Conjecture 5.4 may be useful in understanding the structure of \((p,q)\)-Flex-Steiner problem.