Homological systems and bocses

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Abstract

We show that, up to Morita equivalence, any finite-dimensional algebra with a suitable homological system, admits an exact Borel subalgebra. This generalizes a theorem by Koenig, Külshammer, and Ovsienko, which holds for quasi-hereditary algebras. Our proof follows the same general scheme proposed by these authors, in a more general context: we associate a differential graded tensor algebra with relations, using the structure of $A_\infty$-algebra of a suitable Yoneda algebra, and use its category of modules to describe the category of filtered modules associated to the given homological system.

1 Introduction

We denote by $k$ a fixed algebraically closed ground field. For every $k$-algebra or any bimodule over given $k$-algebras we consider, we assume that the field $k$ acts centrally on them.

Given a $k$-algebra $\Lambda$, we denote by $\Lambda$-$\text{Mod}$ the category of left $\Lambda$-modules and by $\Lambda$-$\text{mod}$ its full subcategory of finitely generated $\Lambda$-modules.

We recall that a preordered set $(\mathcal{P}, \leq)$ is a non-empty set $\mathcal{P}$ equipped with a relation $\leq$ such that $i \leq i$, for all $i \in \mathcal{P}$ and such that, whenever we have $i, j, s \in \mathcal{P}$ with $i \leq j$ and $j \leq s$, we also have $i \leq s$. Two elements $i, j \in \mathcal{P}$ are equivalent iff $i \leq j$ and $j \leq i$. In this case, we write $i \sim j$.

Remark 1.1. Let $\overline{\mathcal{P}} = \mathcal{P}/\sim$ be the set of equivalence classes of $\mathcal{P}$ modulo the equivalence relation $\sim$. For any $i \in \mathcal{P}$, denote by $\overline{i}$ its equivalence class. Then, $\overline{\mathcal{P}}$ is a partially ordered set with the relation defined by $\overline{i} \leq \overline{j}$ iff $i \leq j$.

We recall the following terminology from [13], in the particular context we are interested in.

Definition 1.2. Given a finite-dimensional $k$-algebra $\Lambda$, a (finite) homological system $(\mathcal{P}, \leq, \{\Delta_i\}_{i \in \mathcal{P}})$ consists of a finite preordered set $(\mathcal{P}, \leq)$ and a family of
pairwise non-isomorphic indecomposable finite-dimensional $\Lambda$-modules $\{\Delta_i\}_{i \in P}$ satisfying the following two conditions:

1. $\text{Hom}_\Lambda(\Delta_i, \Delta_j) \neq 0$ implies $i \leq j$;
2. $\text{Ext}^1_\Lambda(\Delta_i, \Delta_j) \neq 0$ implies $i \leq j$ and $i \not\sim j$.

Given such an homological system, we write $\Delta := \bigoplus_{i \in P} \Delta_i$ and denote by $F(\Delta)$ the full subcategory of $\Lambda$-mod consisting of the trivial module 0 and all those $M \in \Lambda$-mod which admit a $\Delta$-filtration, that is a filtration of submodules

$$0 = M_t \subseteq M_{t-1} \subseteq \cdots \subseteq M_1 \subseteq M_0 = M$$

such that $M_j/M_{j+1}$ is isomorphic to some module in $\{\Delta_i \mid i \in P\}$, for each $j \in [0, t-1]$.

The following special homological systems will be relevant for us.

**Definition 1.3.** A finite homological system $\mathcal{H} = (P, \leq, \{\Delta_i\}_{i \in P})$ for a finite-dimensional $k$-algebra $\Lambda$ is called *admissible* if $\Lambda \in F(\Delta)$ and the number of isomorphism classes of indecomposable projective $\Lambda$-modules coincides with the cardinality of $P$.

An admissible homological system $\mathcal{H} = (P, \leq, \{\Delta_i\}_{i \in P})$ will be called *strict* if the following condition is satisfied:

- For each $n \geq 1$, we have that $\text{Ext}^n_\Lambda(\Delta_i, \Delta_j) \neq 0$ implies $i \leq j$ and $i \not\sim j$.

**Proposition 1.4.** Let $\mathcal{H} = (P, \leq, \{\Delta_i\}_{i \in P})$ be an admissible homological system for some finite-dimensional algebra $\Lambda$. Denote by $\text{proj}_\Lambda$ a fixed set of representatives of the isoclasses of the indecomposable projective $\Lambda$-modules. Then, we can index $\text{proj}_\Lambda$ with $P$ in such a way that each $P_i \in \text{proj}_\Lambda$ is the projective cover of $\Delta_i$, for all $i \in P$ and, moreover, there is an exact sequence

$$0 \rightarrow Q_i \rightarrow P_i \rightarrow \Delta_i \rightarrow 0,$$

in $\Lambda$-mod such that the module $Q_i$ admits a $\Delta$-filtration with factors of the form $\Delta_j$, with $j > t$.

**Proof.** From [13](3.16), we know that $F(\Delta)$ is closed under direct summands, so every indecomposable projective $\Lambda$-module $P$ belongs to $F(\Delta)$. For each one of these indecomposable projectives $P$ we can choose a $\Delta$-filtration

$$0 = M_t \subseteq M_{t-1} \subseteq \cdots \subseteq M_1 \subseteq M_0 = P$$

with $P/M_1 \cong \Delta_{\sigma(P)}$, for some index $\sigma(P) \in P$. So, there is a short exact sequence

$$0 \rightarrow M_1 \rightarrow P \rightarrow \Delta_{\sigma(P)} \rightarrow 0,$$

so $\text{top}(P) \cong \text{top}(\Delta_{\sigma(P)})$ and $g$ is the projective cover of $\Delta_{\sigma(P)}$. So, given two non-isomorphic indecomposable projectives $P$ and $P'$, their tops $\text{top}(P)$ and $\text{top}(P')$
top($P'$) are not isomorphic, so the modules $\Delta_{\sigma(P)}$ and $\Delta_{\sigma(P')}$ are not isomorphic. Since the cardinality of the set of isomorphism classes of the indecomposable projectives coincides with the cardinality of $\mathcal{P}$, the map $P \mapsto \Delta_{\sigma(P)}$ determines a bijection between proj$_k\Lambda$ and the set $\{\Delta_i \mid i \in \mathcal{P}\}$. Then, we can index with $\mathcal{P}$ the set proj$_\Lambda$ in such a way that we have short exact sequences

$$0 \to M_i \overset{f_i}{\longrightarrow} P_i \overset{g_i}{\longrightarrow} \Delta_i \to 0,$$

with $P_i$ indecomposable projective, for all $i \in \mathcal{P}$.

From (3.12) and (3.4), for each $i \in \mathcal{P}$, there is an exact sequence

$$0 \to U_i \overset{u_i}{\longrightarrow} X_i \overset{v_i}{\longrightarrow} \Delta_i \to 0$$

such that: the module $X_i \in \mathcal{F}(\Delta)$ is indecomposable and satisfies $\text{Ext}^1_\Lambda(X_i, Y) = 0$, for all $Y \in \mathcal{F}(\Delta)$; and the module $U_i \in \mathcal{F}(\Delta)$ is filtered by modules in $\{\Delta_j \mid j \geq i \text{ and } j \neq i\}$.

Since $P_i$ is projective, there is a morphism $s_i : P_i \longrightarrow X_i$ such that $v_is_i = g_i$. Moreover, we have a pull-back diagram

$$
\begin{array}{ccc}
0 & \longrightarrow & M_i \\
\downarrow{id_{M_i}} & & \downarrow{h_i} \\
0 & \longrightarrow & P_i \\
\end{array}
\begin{array}{ccc}
\downarrow{f_i} & & \downarrow{v_i} \\
\longrightarrow & & \longrightarrow \\
\longrightarrow & & \longrightarrow \\
E_i & \longrightarrow & X_i \\
\pi_i & \longrightarrow & \Delta_i \\
\end{array}
$$

Since $\text{Ext}^1_\Lambda(X_i, M_i) = 0$, the exact sequence of the first row in the preceding diagram splits, and there is a morphism $\sigma_i : X_i \longrightarrow E_i$ with $\pi_i\sigma_i = id_{X_i}$. The morphism $t_i := h_i\sigma_i : X_i \longrightarrow P_i$ satisfies $g_it_i = g_ih_i\sigma_i = v_i\pi_i\sigma_i = v_i$. Then, we have that $v_i = g_it_i = v_i\sigma_it_i$ and $g_i = v_i\sigma_i = g_i\sigma_t$, or equivalently $v_i(id_{X_i} - s_it_i) = 0$ and $g_i(id_{P_i} - t_is_i) = 0$, so $id_{X_i} - s_it_i$ and $id_{P_i} - t_is_i$ are not isomorphisms. Since the endomorphism algebras of $P_i$ and $X_i$ are local, we get that $s_it_i$ and $t_is_i$ are isomorphisms. From this, we get that $s_i$ is an isomorphism and $P_i \cong X_i$.

**Definition 1.5.** Assume that $\mathcal{H} = (\mathcal{P}, \leq, \{\Delta_i\}_{i \in \mathcal{P}})$ is an admissible homological system for a finite-dimensional $k$-algebra $\Lambda$ and keep the index assignment imposed on proj$_\Lambda$ in (3.4). Consider a non-zero $M \in \Lambda$-mod and any projective resolution

$$P_M : \cdots \longrightarrow P_M^{-1} \longrightarrow P_M^0 \longrightarrow M \longrightarrow 0$$

of $M$. Then, for $i \in \mathcal{P}$, we will say that the projective resolution $P_M$ is $i$-bounded if it is finite and, whenever $P_j$ is a direct summand of $P_M^t$, we have $j \geq t$ for $t \geq 0$, with $j > t$ if $t > 0$.

**Lemma 1.6.** Assume that $\mathcal{H} = (\mathcal{P}, \leq, \{\Delta_i\}_{i \in \mathcal{P}})$ is an admissible homological system for a finite-dimensional $k$-algebra $\Lambda$. Then, for each $i \in \mathcal{P}$, the $\Lambda$-module $\Delta_i$ admits an $i$-bounded minimal projective resolution $P_{\Delta_i}$ of the form

$$P_{\Delta_i} : \cdots \longrightarrow P_{\Delta_i}^{-1} \longrightarrow P_{\Delta_i}^0 \longrightarrow \Delta_i \longrightarrow 0.$$
Proof. Notice that if \( i \in P \) is such that \( \bar{i} \) is maximal in \( \overline{P} \), in the sequence of (1.4), we get \( Q_i = 0 \), so \( \Delta_i \cong P_i \) is projective and \( \Delta_i \) has a trivial \( i \)-bounded projective resolution \( P_{\Delta_i} \) with \( P^0_{\Delta_i} = P_i \).

Observe that given \( i \in P \) and an exact sequence \( 0 \rightarrow M \rightarrow E \rightarrow N \rightarrow 0 \) in \( \Lambda \)-mod, where \( M \) admits an \( i_1 \)-bounded projective resolution \( P_M \) and \( N \) admits an \( i_2 \)-bounded projective resolution \( P_N \), with \( i_1, i_2 > i \), then the horseshoe lemma yields an \( i \)-bounded projective resolution \( P_E \) of \( E \) such that: \( P_j \) direct summand of \( P^0_E \) implies that \( j > i \).

An easy induction using the last observation, shows that whenever \( E \in \mathcal{F}(\Delta) \) has a \( \Delta \)-filtration with factors \( \{ \Delta_j \}_{u \in [1,s]} \) with \( \overline{j} > \bar{i} \) for all \( u \in [1,s] \), which have \( i_u \)-bounded projective resolutions \( \{ P_{\Delta_{j_u}} \}_{u \in [1,s]} \), with \( P_{\Delta_{j_u}}^{0_{j_u}} = P_{j_u} \), we have an \( i \)-bounded projective resolution \( P_E \) of \( E \) such that: \( P_j \) direct summand of \( P^0_E \) implies that \( j > \overline{i} \).

Finally, fix \( i \in P \) such that \( \bar{i} \) is not maximal in \( \overline{P} \) and assume that \( \Delta_j \) admits an \( i \)-bounded projective resolution \( P_{\Delta_j} \) with \( P^0_{\Delta_j} = P_j \), for all \( j \in \overline{P} \) with \( \overline{j} > \bar{i} \). Consider the exact sequence of (1.4) for \( \Delta_i \). Then the preceding paragraph yields an \( i \)-bounded projective resolution \( P_{Q_i} \) for \( Q_i \), such that: \( P_j \) direct summand of \( P^0_{Q_i} \) implies that \( j > \overline{i} \). Then, splicing this exact sequence with the projective resolution \( P_{Q_i} \), we obtain the wanted \( i \)-bounded projective resolution of \( \Delta_i \)

\[
P_{\Delta_i} : \cdots \rightarrow P^{-1}_{Q_i} \rightarrow P^0_{Q_i} \rightarrow P_i \rightarrow \Delta_i \rightarrow 0.
\]

Finally, any minimal projective resolution of \( \Delta_i \) is a direct summand of \( P_{\Delta_i} \) and, therefore, is also \( i \)-bounded.

Recall from [13] the following.

Definition 1.7. Assume that \( \Lambda \) is a finite dimensional \( k \)-algebra, equipped with a preordered set of indexes \( (P, \leq) \) for the family \( \{ P_i \}_{i \in P} \) of representatives of the non-isomorphic indecomposable projective \( \Lambda \)-modules. For \( i \in P \), denote by \( \Delta_i \) the \( i^{th} \)-standard \( \Lambda \)-module, that is \( \Delta_i := P_i/T_i \), where

\[
T_i := \sum_{f \in \text{Hom}_\Lambda(P_j, P_i) \atop \overline{j} \leq \bar{i}} \text{Im}(f).
\]

Then, the algebra \( \Lambda \) is called a prestandardly stratified algebra if, for each \( i \in P \), there is an exact sequence

\[
0 \rightarrow Q_i \rightarrow P_i \rightarrow \Delta_i \rightarrow 0,
\]

in \( \Lambda \)-mod such that the module \( Q_i \) admits a \( \Delta \)-filtration with factors of the form \( \Delta_j \), with \( \overline{j} > \bar{i} \). A prestandardly stratified algebra \( \Lambda \), with preordered index set \( (P, \leq) \) is called standardly stratified if \( (P, \leq) \) is a partial order. A standardly stratified algebra \( \Lambda \) is quasi-hereditary iff \( (P, \leq) \) is linearly ordered and \( \text{End}_\Lambda(\Delta_i) \cong k \), for each \( i \in P \).
Remark 1.8. If $\Lambda$ is a prestandardly stratified algebra with preorder $(P, \leq)$ and standard modules $\{\Delta_i\}_{i \in I}$, then $\mathcal{H} = (P, \leq, \{\Delta_i\}_{i \in I})$ is an admissible homological system, see [13](2.3). If, furthermore, the algebra $\Lambda$ is quasi-hereditary, then the homological system $\mathcal{H}$ is strict, see [12].

The following statement is interesting because it assumes no requirement on the endomorphism algebras of the standard modules.

Corollary 1.9. Assume that $\Lambda$ is prestandardly stratified with homological system of standard modules $H = (P, \leq, \{\Delta_i\}_{i \in I})$, where $(P, \leq)$ is linearly ordered. Then the homological system $H$ is strict.

Proof. In the exact sequence granted by (1.4) for the standard module $\Delta_i$, we can assume that the epimorphism $P_i \longrightarrow \Delta_i$ is the canonical projection from $P_i$ onto $\Delta_i = P_i/T_i$, where $T_i = \sum_{f \in \text{Hom}_\Lambda(P_j, P_i), j > i} \text{Im}(f)$. From (1.6), we have an $i$-bounded projective resolution

$$P_{\Delta_i} : \cdots \longrightarrow P_{\Delta_i}^2 \longrightarrow P_{\Delta_i}^1 \longrightarrow P_{\Delta_i}^{\nu_i} \longrightarrow \Delta_i \longrightarrow 0,$$

where $\nu_i$ is the canonical projection. Fix $n \in \mathbb{N}$ and assume that $\text{Ext}_\Lambda^n(\Delta_i, \Delta_j) \neq 0$. This implies that $\text{Hom}_\Lambda(P_{\Delta_i}^n, \Delta_j) \neq 0$. So, there is an indecomposable direct summand $P_s$ of $P_{\Delta_i}^n$, which must satisfy $i < s$, with $\text{Hom}_\Lambda(P_s, \Delta_j) \neq 0$. Take any non-zero morphism $f : P_s \longrightarrow \Delta_j$. Since $P_s$ is projective, there is some morphism $h : P_s \longrightarrow P_j$ with $\nu_j f = h$. If $s > j$, we have $\nu_j(\text{Im}(f)) = 0$, and $h = 0$, which is not the case. So we have $i < s \leq j$.

The study of the category $\mathcal{F}(\Delta)$ in the mentioned quasi-hereditary case, and in more general settings, has been of great interest in representation theory of algebras, see [13] and its references.

In this paper we generalize the main results of S. Koenig, J. Külshammer and S. Ovsienko proved in [9] for the quasihereditary case. Here we extend their arguments to arbitrary strict admissible homological systems. In the following lines we give the layout of the contents of this article.

After section 2, which contains technical remarks on graded duals and tensor products, we show in section 3 that the Yoneda algebra $A$ of the $\Lambda$-module $\Delta$, associated to a strict homological system $\mathcal{H}$ for a finite dimensional $k$-algebra $\Lambda$, admits a strict structure of $A_\infty$-algebra, see (3.7). For this, we proceed as in [9], we reelaborate Keller’s argument in [8] supported by Kadeishvili Theorem (see also Merkulov [14]), in our more general context.

In section 4, we construct the interlaced weak ditalgebra $A(\Delta) = (A(\Delta), I)$, see (4.19) and (4.19), associated to the Yoneda $A_\infty$-algebra $A$ of $\Delta$. Throughout the whole paper we use the language of ditalgebras (differential tensor algebras) and their categories of modules, as in [8], enriched with the possibility of considering “ditalgebras with relations” called here interlaced weak ditalgebras and their categories of modules denoted by $(A, I)$-mod, see (8.1) and [1]. We prefer this to the alternative language of bocses, used in [9], because it seems to ease many computations.
In section 5, 6 and 7, we review elementary properties of $A_\infty$-categories and review the $A_\infty$-categories $\text{ad}(A)$, $\text{cv}(A)$, and $\text{tw}(A)$ with some of their relationships. All this supports the work of section 8 which exhibits an equivalence of categories $F(\Delta) \simeq H^0(\text{tw}(A)) \simeq \mathcal{A}(\Delta)$-mod. This generalizes theorem (8.2) of [9] to the case of an arbitrary strict homological system. In their domain of study, the homological system is of standard modules over a quasi-hereditary algebra, and they obtain a directed bocs. This is no longer true in our case of a general strict homological system, but we still obtain a manageable strict interlaced weak ditalgebra $\mathcal{A}(\Delta)$, see (12.1).

We study in section 10 and 11 some special kind of interlaced weak ditalgebras $\mathcal{A}$. We equip their categories of modules $\mathcal{A}$-mod with a natural exact structure and then show that Burt-Butler theory on their right and left algebras still works, see [5] and [6]. All this is applied later to the interlaced weak ditalgebra $\mathcal{A}(\Delta)$. So, its right algebra $\Gamma$ admits a natural strict homological system $\mathcal{H}' = (\mathcal{P}, \leq, \{\Delta'_i\}_{i \in \mathcal{P}})$ and, moreover, the algebra $\Gamma$ is Morita equivalent to $\Lambda$.

In the last section, we show that the Yoneda algebra associated to an admissible homological system $\mathcal{H}$ admits a strict structure of $A_\infty$-algebra, as in (5.3), if and only if the homological system $\mathcal{H}$ is strict, see (12.3).

Moreover, generalizing the argumentation of [9], we prove that any finite dimensional $k$-algebra $\Lambda$ with a strict homological system $\mathcal{H} = (\mathcal{P}, \leq, \{\Delta_i\}_{i \in \mathcal{P}})$ is Morita equivalent to a finite-dimensional $k$-algebra $\Gamma$ which admits a strict homological system $\mathcal{H}' = (\mathcal{P}, \leq, \{\Delta'_i\}_{i \in \mathcal{P}})$ such that $\Gamma$ has an exact Borel subalgebra $B$, which is regular and homological, as in the following definition (1.14). Moreover, there is an equivalence $\Omega: \Gamma\text{-mod} \rightarrow \Lambda\text{-mod}$ such that $\Omega(\Delta'_i) \cong \Delta_i$, for all $i \in \mathcal{P}$. In the next lines we give some terminology required for this definition.

**Remark 1.10.** Let $B$ be a basic finite-dimensional $k$-algebra. Let us recall that the (Gabriel) quiver $Q$ of the algebra $B$ is constructed as follows. Consider a splitting $B = S \oplus J$ of the algebra $B$ over its radical $J$, then $S$ is a finite product of copies of the field $k$ and we can write $S = \bigoplus_{i \in \mathcal{P}} ke_i$, where $1 = \sum_{i \in \mathcal{P}} e_i$ is a decomposition of the unit of $B$ as a finite sum of primitive orthogonal idempotents. So, we get that $\{P_i\}_{i \in \mathcal{P}}$, where $P_i = Be_i$, is a complete family of representatives of the isoclasses of the indecomposable projective $B$-modules, and the family $\{S_i\}_{i \in \mathcal{P}}$, where $S_i = P_i/JP_i$ is a complete family of representatives of the isoclasses of the indecomposable simple $B$-modules. Then, by definition, the set of points of $Q$ is $\mathcal{P}$ and, given $i, j \in \mathcal{P}$, there are exactly dim$k$ $\text{Ext}^1_B(S_i, S_j)$ arrows from $i$ to $j$ in $Q$.

Consider the preorder of precedence $\preceq$ in the set $\mathcal{P}$ determined by $i \prec j$ if there is an arrow $i \rightarrow j$ in $Q$ or, equivalently, if $\text{Ext}^1_B(S_i, S_j) \neq 0$.

Recall that $B$ can be identified with a quotient $k(Q)/I$ of the path algebra $k(Q)$ by some admissible ideal $I$, where the idempotents $e_i$ are identified with the classes of the trivial paths of $Q$. Then, we have for different $i, j \in \mathcal{P}$ that $e_jBe_i \neq 0$ implies $i \prec j$.

If the algebra $B$ is directed, that is if its quiver $Q$ has no oriented cycle, then $\preceq$ is a partial order on $\mathcal{P}$. In this case, we have $e_iBe_i = ke_i$, for all $i \in \mathcal{P}$. 

The following lemma illustrates the preceding notions, but it is convenient to make some precisions first.

**Remark 1.11.** For any finite poset \((X, \leq)\), its height map \(h : X \longrightarrow \mathbb{N} \cup \{0\}\) is defined for minimal elements \(x \in X\), as \(h(x) = 0\), and, inductively, for any non-minimal element \(x \in X\), as \(h(x) := \max\{h(y) \mid y < x\} + 1\).

Clearly, whenever we have \(y < x\) in \(X\), we get \(h(y) < h(x)\).

**Remark 1.12.** A linearization \((X, \leq^L)\) of a finite poset \((X, \leq)\) is defined as follows. Consider the subsets \(X_t := \{i \in X \mid h(i) = t\}\), so \(X = \bigcup_{t=1}^s X_t\), for some finite \(s\). Then, consider for each \(t \in [1, s]\) any linear order \(\leq^t\) in the set \(X_t\). Then, by definition, for any elements \(i, j \in X\), we make \(i \leq^L j\) if \(h(i) < h(j)\) or \((h(i) = h(j)\) and \(i \leq^t j\).

**Lemma 1.13.** Assume that \(B\) is a directed finite-dimensional \(k\)-algebra and adopt the notations of \((1.10)\). Then, the following holds.

1. The triple \(\mathcal{H} = (\mathcal{P}, \leq, \{S_i\}_{i \in \mathcal{P}})\) is a strict homological system for the directed algebra \(B\).

2. For each \(i \in \mathcal{P}\), the \(i\)th-standard \(B\)-module associated to the partially ordered set \((\mathcal{P}, \leq)\) is \(\Delta_i = S_i\). Clearly, we have \(\text{End}_B(S_i) \cong k\).

3. For any linearization \(\leq^L\) of \(\mathcal{P}\) with the partial order of precedence \(\preceq\), we have that \(B\) is a quasi-hereditary algebra with homological system of standard modules \(\mathcal{H}^L = (\mathcal{P}, \leq^L, \{S_i\}_{i \in \mathcal{P}})\).

**Proof.** (1): We already know that \((\mathcal{P}, \preceq)\) is a finite partially ordered set and that \(\{S_i\}_{i \in \mathcal{P}}\) is a family of non-isomorphic indecomposable \(B\)-modules. Clearly, \(\text{Hom}_B(S_i, S_j) = 0\), unless \(i = j\). By definition of the order of precedence \(\preceq\), we have that \(\text{Ext}^1_B(S_i, S_j) \neq 0\) implies that \(i \npreceq j\). Here, we have \(\mathcal{P} = \mathcal{P}\) and \(\Lambda \in \mathcal{F}(\oplus_{i \in \mathcal{P}} S_i)\), so \(\mathcal{H}\) is an admissible homological system for the algebra \(B\).

Let us show that \(\mathcal{H}\) is strict. Assume that \(\text{Ext}^1_B(S_i, S_j) \neq 0\) and consider the \(i\)-bounded projective resolution of \(S_i\)

\[
\cdots \longrightarrow P_{S_i}^{-n-1} \xrightarrow{d^{-n}} P_{S_i}^{-n} \xrightarrow{d^{-n+1}} P_{S_i}^{-n+1} \longrightarrow \cdots \longrightarrow P_{S_i}^{-1} \xrightarrow{d^0} P_{S_i}^0 \xrightarrow{\epsilon} S_i \longrightarrow 0
\]

given by \((1.6)\). Then, for each \(n \geq 1\), we have

\[P_{S_i}^{-n} \cong \bigoplus_{i < t} m_{i, t} P_t, \text{ for some } m_{i, t} \geq 0.\]

In order to compute \(\text{Ext}^1_B(S_i, S_j)\), apply the functor \(\text{Hom}_B(-, S_j)\) to the given projective resolution of \(S_i\) and then look at the \(n\)th-homology space, which is a quotient of the space \(Z_n(P_{S_i}^{-n}, S_j) \subseteq \text{Hom}_B(P_{S_i}^{-n}, S_j)\). From the given decomposition of \(P_{S_i}^{-n}\), we see that this space is zero whenever \(i \neq j\). Thus, \(\text{Ext}^1_B(S_i, S_j) \neq 0\) implies \(i \preceq j\), and \(\mathcal{H}\) is a strict homological system for the algebra \(B\).
Observe that if \( i, j \in P \) are incomparable in the poset \((P, \preceq)\), then we have \( \text{Hom}_B(P_j, P_i) = e_j B e_i = 0 \). Therefore, we get

\[
T_i = \sum_{f \in \text{Hom}_B(P_j, P_i) \atop j \not\preceq i} \text{Im} (f) = \sum_{f \in \text{Hom}_B(P_j, P_i) \atop i \prec j} \text{Im} (f) = \text{rad}(P_i),
\]

and we obtain \( \Delta_i = P_i / T_i = S_i \).

By definition of the linearization \( L \), we have that \( i \prec L j \) implies that \( i \prec j \) or \( i, j \) are \( \preceq \)-incomparable elements of \( P \). Then, we obtain

\[
T_i = \sum_{f \in \text{Hom}_B(P_j, P_i) \atop i \prec j} \text{Im} (f) = \sum_{f \in \text{Hom}_B(P_j, P_i) \atop i \prec L j} \text{Im} (f) = \sum_{f \in \text{Hom}_B(P_j, P_i) \atop j \not\preceq L i} \text{Im} (f).
\]

Then, if \( \Delta'_i \) denotes the standard \( B \)-module associated to the linear order \( \preceq_L \), we get \( \Delta'_i = S_i \), for all \( i \in P \).

Since \( i \prec j \) implies \( i \prec L j \), the fact that \( H \) is a strict homological system for \( B \) implies that \( H^L \) is a strict homological system for \( B \). Hence, \( B \) is a quasi-hereditary algebra.

The following definition adapts slightly the definition of exact Borel subalgebra given in [4] (3.4).

**Definition 1.14.** Let \( \Gamma \) be a finite-dimensional \( k \)-algebra with a strict homological system \( \mathcal{H}' = (P, \preceq, \{\Delta'_i\}_{i \in P}) \). Then, a subalgebra \( B \) of \( \Gamma \) is an exact Borel subalgebra if

1. The finite-dimensional \( k \)-algebra \( B \) is directed and we can index a complete family \( \{S_i\}_{i \in P} \) of representatives of the isoclasses of the simple \( B \)-modules with the same index set \( P \). Thus (1.13) applies to \( B \), in particular \( B \) is quasi-hereditary with simple standard modules.
2. The right \( B \)-module \( \Gamma \) is projective.
3. For each \( i \in P \), we have that \( \Delta'_i \cong \Gamma \otimes_B S_i \).

The subalgebra \( B \) is called an homological exact Borel subalgebra if the morphisms \( \text{Ext}^n_B(M, N) \rightarrow \text{Ext}^n_B(\Gamma \otimes_B M, \Gamma \otimes_B N) \) induced by the tensor product functor \( \Gamma \otimes_B - \) are epimorphisms for \( n \geq 1 \) and isomorphisms for \( n \geq 2 \).

The subalgebra \( B \) is called a regular exact Borel subalgebra if the morphisms \( \text{Ext}^n_B(S_i, S_j) \rightarrow \text{Ext}^n_B(\Gamma \otimes_B S_i, \Gamma \otimes_B S_j) \) induced by the tensor product functor \( \Gamma \otimes_B - \) are isomorphisms for all \( n \in \mathbb{N} \) and \( i, j \in P \).

2 Graded duals and tensor product

Let \( S \) be a fixed basic finite-dimensional semisimple \( k \)-algebra. Recall the following basic isomorphism involving left duals of \( S \)-\( S \)-bimodules.
Lemma 2.1. Given an $S$-$S$-module $V$, we have the left dual $S$-$S$-bimodule $D(V) := \text{Hom}_S(V, S)$. Then, given two $S$-$S$-bimodules $V$ and $W$, where $W$ is finitely generated as a left $S$-module, there is a natural isomorphism of $S$-$S$-bimodules

$$\theta_{V,W} : D(W) \otimes_S D(V) \longrightarrow D(V \otimes_S W)$$

satisfying that $\theta(\beta \otimes \alpha)[v \otimes w] = \alpha(v\beta(w))$, for $\alpha \in D(V)$, $\beta \in D(W)$, $v \in V$, and $w \in W$. Its inverse $\tau : D(V \otimes_S W) \longrightarrow D(W) \otimes_S D(V)$ can be described by $\tau(\gamma) = \sum_{i=1}^{n} w_i^* \otimes \gamma_i$, where $(w_i, w_i^*)_{i \in [1,n]}$ is a dual basis for the projective left $S$-module $W$, $\gamma \in D(V \otimes_S W)$, and $\gamma_i \in D(V)$ is the morphism of left $S$-modules given by $\gamma_i(v) = \gamma(v \otimes w_i)$, for $v \in V$ and $i \in [1,n]$.

Proof. It is easy to see that both, $\theta$ and $\tau$ are well defined maps and that $\theta$ is a morphism of $S$-$S$-bimodules. For each $\gamma \in D(V \otimes_S W)$ we have that $\theta\tau(\gamma)[v \otimes w] = \theta(\sum_i w_i^* \otimes \gamma_i)(v \otimes w) = \sum_i \gamma_i(vw_i^*(w)) = \sum_i \gamma_i(vw_i^*(w) \otimes w_i) = \gamma(v \otimes \sum_i w_i^*(w)w_i) = \gamma(v \otimes w)$, thus $\theta\tau = \text{id}$. Moreover, we have for each $i \in [1,n]$ that $\theta(\beta \otimes \alpha)_i[v \otimes w] = \alpha(v\beta(w_i)) = \alpha(v\beta(w_i)) = \beta(w_i)\alpha[v]$ and so $\theta(\beta \otimes \alpha)_i = \beta(w_i)\alpha$. Therefore, we get $\tau\theta(\beta \otimes \alpha) = \sum_i w_i^* \otimes \beta(w_i)\alpha = \sum_i w_i^* \beta(w_i) \otimes \alpha = \beta \otimes \alpha$. Then, we also have that $\tau\theta = \text{id}$ and $\theta$ is an isomorphism.

For the naturality of the isomorphism, it is clear that, given morphisms of $S$-$S$-bimodules $f : V \longrightarrow V'$ and $g : W \longrightarrow W'$, the following squares commute

$$\begin{array}{cccc}
D(W) \otimes D(V) & \overset{\theta}{\longrightarrow} & D(V' \otimes W) & \overset{\theta}{\longrightarrow} & D(V \otimes W') \\
\downarrow \text{id}_{D(W) \otimes D(f)} & & \downarrow \text{id}_{D(f \otimes D(V))} & & \downarrow \text{id}_{D(g \otimes D(V))} \\
D(W) \otimes D(V) & \overset{\theta}{\longrightarrow} & D(V' \otimes W) & \overset{\theta}{\longrightarrow} & D(V \otimes W').
\end{array}$$

Notation 2.2. By assumption, the $k$-algebra $S$ is a finite product of copies of the field $k$. Consider the decomposition of its unit $1 = \sum_{i \in P} e_i$ as a sum of primitive central orthogonal idempotents.

Given any $S$-$S$-bimodule $B$, an element $z \in B$ is called directed iff $z \in e_jBe_i$, for some $i,j \in P$. In this case, we write $s(z) = i$ for its source point and $t(z) = j$ for its terminal point.

In the following, we consider the algebra $S$ as a graded $S$-$S$-bimodule concentrated at degree zero.

Definition 2.3. Let $B = \bigoplus_{n \in \mathbb{Z}} B_n$ be a graded $S$-$S$-bimodule. Then, for each $n \in \mathbb{Z}$, the graded (left) dual $S$-$S$-bimodule $D(B)$ is defined as

$$D(B) = \bigoplus_{n \in \mathbb{Z}} D(B)_n,$$

where $D(B)_n := \text{Hom}_S(B^{-n}, S) = D(B_{-n})$, for $n \in \mathbb{Z}$. Thus, we have

$$D(B) = \bigoplus_{n \in \mathbb{Z}} \text{Hom}_S(B_{-n}, S) = \bigoplus_{n \in \mathbb{Z}} \text{Hom}^n_{\text{Mod}-S}(B, S) = \text{Hom}_{\text{Mod}-S}(B, S).$$
Given a homogeneous morphism of graded $S$-$S$-bimodules $f : A \longrightarrow B$ of degree $|f| = d$, we have the restriction morphism $f_n : A_{-n-d} \longrightarrow B_{-n}$ which determines a morphism of $S$-$S$-bimodules

$$D(f_n) : \text{Hom}_S(B_{-n}, S) \longrightarrow \text{Hom}_S(A_{-n-d}, S).$$

Hence, we have the morphism $D(f) = \bigoplus_{n \in \mathbb{Z}} D(f_n)$ of graded $S$-$S$-bimodules

$$D(B) = \bigoplus_{n \in \mathbb{Z}} \text{Hom}_S(B_{-n}, S) \longrightarrow \bigoplus_{n \in \mathbb{Z}} \text{Hom}_S(A_{-n-d}, S) = D(A),$$

which is homogeneous with degree $|D(f)| = d = |f|$.

**Remark 2.4.** A graded $S$-$S$-bimodule $B = \bigoplus_{n \in \mathbb{Z}} B_n$ is called **locally finite** iff each $B_n$ is finite-dimensional over $k$. We will say that a graded $S$-$S$-bimodule $B$ is **bounded below** (resp. **bounded above**) if there is some $\ell_B \in \mathbb{Z}$ such that $B_n = 0$, for all $n < \ell_B$ (resp. for all $n > \ell_B$).

Notice that $D(B)$ is a graded locally finite bounded above (resp. bounded below) $S$-$S$-bimodule, whenever $B$ is a graded locally finite bounded below (resp. bounded above) $S$-$S$-bimodule.

Moreover, if $A$ and $B$ are graded locally finite bounded below (resp. bounded above) $S$-$S$-bimodules, then the graded $S$-$S$-bimodule $A \otimes_S B$ is locally finite and bounded below (resp. bounded above). Indeed, if $A$ and $B$ are bounded below by $\ell_A$ and $\ell_B$, respectively then $A \otimes_S B$ is bounded below by $\ell_A + \ell_B$: Each homogeneous component $[A \otimes_S B]_n = \bigoplus_{t+n=r} A_t \otimes_S B_r$ is a finite direct sum, because $A_{n-t} \otimes_S B_t \neq 0$ implies that $\ell_B \leq t \leq n - \ell_A$. If $n < \ell_A + \ell_B$ and $A_{n-t} \otimes_S B_t \neq 0$ we get a contradiction. Thus, we get $[A \otimes_S B]_n = 0$, for $n < \ell_A + \ell_B$, and then $A \otimes_S B$ is bounded below.

We have the graded category $\text{Gb}_S$ (resp. $\text{Gfb}_S$) of graded locally finite bounded below (resp. bounded above) $S$-$S$-bimodules, with hom set

$$\text{Hom}_{\text{Gb}_S}(A, B) = \bigoplus_{n \in \mathbb{Z}} \text{Hom}_{\text{Gfb}_S}(A, B),$$

where $\text{Hom}_{\text{Gfb}_S}(A, B)$ is the set of homogeneous morphisms of $S$-$S$-bimodules of degree $n$. Then, we have the contravariant equivalence of graded categories

$$D : \text{Gb}_S \longrightarrow \text{Gfb}_S.$$

**Lemma 2.5.** Let $A, B \in \text{Gb}_S$ and consider the bounded below locally finite graded $S$-$S$-bimodule $A \otimes_S B$. Then, we have a canonical isomorphism of graded $S$-$S$-bimodules

$$D(B) \otimes_S D(A) \overset{\theta_{A,B}}{\longrightarrow} D(A \otimes_S B)$$

such that $\theta_{A,B}(\beta \otimes \alpha)[a \otimes b] = \alpha(a)\beta(b)$, for any homogeneous elements $\beta \in D(B)$, $\alpha \in D(A)$, $a \in A$, and $b \in B$. Its inverse is again an isomorphism of graded $S$-$S$-bimodules

$$D(A \otimes_S B) \overset{\tau_{A,B}}{\longrightarrow} D(B) \otimes_S D(A).$$
Proof. The morphism $\theta_{A,B}$ maps, for each $n \in \mathbb{Z}$, the finite direct sum

$$[D(B) \otimes_S D(A)]_n = \bigoplus_{r+s=n} D(B)_r \otimes_S D(A)_s = \bigoplus_{r+s=n} \text{Hom}_S(B_{-r}, S) \otimes_S \text{Hom}_S(A_{-s}, S)$$

onto the direct sum

$$[D(A \otimes_S B)]_n = \text{Hom}_S([A \otimes_S B]_{-n}, S) = \bigoplus_{r+s=n} \text{Hom}_S(A_{-s} \otimes_S B_{-r}, S)$$

with the direct sum of isomorphisms $\bigoplus_{r+s=n} \theta_{A_{-s}, B_{-r}}$, described in (2.1). The latter can be applied because $B_{-r}$ is a finitely generated left $S$-module. \qed

**Definition 2.6.** Consider the contravariant equivalence

$$\hat{D} : \text{Gl}_B S \longrightarrow \text{Gl}_S,$$

defined by $\hat{D}(A) = D(A)^{\text{op}}$, the opposite $S$-$S$-bimodule of $D(A)$. Thus, $D(A)^{\text{op}} = D(A)$ as vector spaces and, whenever $s, s' \in S$, the action of $S$ on $\hat{D}(A)$, denoted with a star $*$ is given by $s * \beta * s' = s' \beta s$, for $\beta \in \hat{D}(A)$. Moreover, $\hat{D}(f) : \hat{D}(B) \longrightarrow \hat{D}(A)$ has the same recipe than $D(f)$.

Then, for any $A, B \in \text{Gl}_B S$ we have an isomorphism of graded $S$-$S$-bimodules

$$\hat{D}(A) \otimes_S \hat{D}(B) \xrightarrow{\hat{\theta}_{A,B}} \hat{D}(A \otimes_S B)$$

such that $\hat{\theta}_{A,B}(\alpha \otimes \beta)[a \otimes b] = \alpha(a \beta(b))$, for any homogeneous elements $\beta \in \hat{D}(B)$, $\alpha \in \hat{D}(A)$, $a \in A$, and $b \in B$.

The map $\hat{\theta}_{A,B}$ is obtained as the composition

$$D(A)^{\text{op}} \otimes_S D(B)^{\text{op}} \xrightarrow{\zeta} (D(B) \otimes_S D(A))^{\text{op}} \xrightarrow{\theta_{A,B}} D(A \otimes_S B)^{\text{op}}$$

where $\zeta$ is the flip isomorphism $\alpha \otimes \beta \mapsto \beta \otimes \alpha$.

**Lemma 2.7.** Given two homogeneous morphisms $f : A \longrightarrow A'$ and $g : B \longrightarrow B'$ in $\text{Gl}_B S$, we have the homogeneous morphism $f \otimes g : A \otimes_S B \longrightarrow A' \otimes_S B'$ in $\text{Gl}_S$. Then, the following diagram commutes in $\text{Gl}_S$:

$$
\begin{array}{c}
\hat{D}(A') \otimes_S \hat{D}(B') & \xrightarrow{\hat{\theta}_{A',B'}} & \hat{D}(A' \otimes_S B') \\
\hat{D}(f) \otimes \hat{D}(g) & \downarrow & \downarrow (-1)^{|f||g|} \hat{D}(f \otimes g) \\
\hat{D}(A) \otimes_S \hat{D}(B) & \xrightarrow{\hat{\theta}_{A,B}} & \hat{D}(A \otimes_S B).
\end{array}
$$

Proof. Take a typical homogeneous generator $\alpha \otimes \beta$ of $\hat{D}(A') \otimes_S \hat{D}(B')$, so $\alpha \in \hat{D}(A')$ and $\beta \in \hat{D}(B')$ are homogeneous, and take $a \otimes b \in A \otimes_S B$ a typical homogeneous generator, so $a \in A$ and $b \in B$ are homogeneous. Make $\Delta_1 = (-1)^{|f||g|} \hat{D}(f \otimes g) \hat{\theta}_{A',B'}(\alpha \otimes \beta)(a \otimes b)$, then we have

$$
\Delta_1 = (-1)^{|f||g|} \hat{\theta}_{A',B'}(\alpha \otimes \beta)(f \otimes g)(a \otimes b)
= (-1)^{|f||g|+|a|} \hat{\theta}_{A',B'}(\alpha \otimes \beta)(f(a) \otimes g(b))
= (-1)^{|f||g|+|a|} \alpha[f(a), \beta(g(b))].
$$
Now make $\Delta_2 := [\hat{\theta}_{A,B}(\hat{D}(f) \otimes \hat{D}(g))(\alpha \otimes \beta)][(a \otimes b)$, then we get

$$\Delta_2 = [\hat{\theta}_{A,B}(\hat{D}(f) \otimes \hat{D}(g))(\alpha \otimes \beta)][(a \otimes b)$$

$$= (-1)^{|D(g)||\alpha|}\hat{\theta}_{A,B}(\hat{D}(f)(\alpha) \otimes \hat{D}(g)(\beta))(a \otimes b)$$

$$= (-1)^{|g||\alpha|}\hat{\theta}_{A,B}(\alpha f \otimes \beta g)(a \otimes b)$$

$$= (-1)^{|g||\alpha|}\alpha f(a \beta g(b)) = (-1)^{|g||\alpha|}[f(a \beta g(b))].$$

If $-|\alpha| \neq |f(a)|$, since $\alpha \in \text{Hom}_S^{\alpha}(A', S) = \text{Hom}_S(A'_{-\alpha}, S)$, we obtain that $\alpha[f(a \beta g(b))] = 0$ and, hence, $\Delta_1 = \Delta_2$. Otherwise, we have $-|\alpha| = |f| + |a|$ and hence

$$(-1)^{|f|+|g||\alpha|} = (-1)^{|g||\alpha|}.$$

\[\square\]

**Definition 2.8.** Given $A_1, \ldots, A_n \in \mathcal{G}_{\hat{S}}$, the isomorphism

$$\hat{D}(A_1) \otimes_S \cdots \otimes_S \hat{D}(A_n) \xrightarrow{\hat{\theta}_{A_1, \ldots, A_n}} \hat{D}(A_1 \otimes \cdots \otimes_S A_n)$$

is defined recursively, for $n \geq 3$, as the composition of

$$\hat{D}(A_1) \otimes_S \cdots \otimes_S \hat{D}(A_n) \xrightarrow{id_{\hat{D}(A_2)} \otimes \hat{\theta}_{A_2, \ldots, A_n}} \hat{D}(A_1) \otimes_S \hat{D}(A_2 \otimes \cdots \otimes_S A_n)$$

with

$$\hat{D}(A_1) \otimes_S \hat{D}(A_2 \otimes \cdots \otimes_S A_n) \xrightarrow{\hat{\theta}_{A_1, A_2} \otimes \otimes \hat{\theta}_{A_3, \ldots, A_n}} \hat{D}(A_1 \otimes \cdots \otimes_S A_n).$$

Moreover, for any $A \in \mathcal{G}_{\hat{S}}$, we make $\hat{\theta}_A := id_{\hat{D}(A)}$, by definition.

Then, given $n \geq 1$ and a typical homogeneous generator $\alpha_1 \otimes \cdots \otimes \alpha_n$ of the graded tensor product $\hat{D}(A_1) \otimes_S \cdots \otimes_S \hat{D}(A_n)$ and a typical homogeneous generator $a_1 \otimes \cdots \otimes a_n$ of the graded tensor product $A_1 \otimes \cdots \otimes S A_n$, we have

$$\hat{\theta}_{A_1, \ldots, A_n}((\alpha_1 \otimes \cdots \otimes \alpha_n)(a_1 \otimes \cdots \otimes a_n) = \alpha_1(a_1 \alpha_2(a_2(a_3 \cdots a_{n-1} a_n)) \cdots).$$

The inverse

$$\hat{\tau}_{A_1, \ldots, A_n} : \hat{D}(A_1 \otimes \cdots \otimes_S A_n) \longrightarrow \hat{D}(A_1) \otimes_S \cdots \otimes_S \hat{D}(A_n)$$

of $\hat{\theta}_{A_1, \ldots, A_n}$ will be relevant in the following.

**Lemma 2.9.** Given $A_1, \ldots, A_n \in \mathcal{G}_{\hat{S}}$ and an increasing sequence natural numbers $\vartheta = i_0 < i_1 < \cdots < i_{\ell-1} < i_\ell = n$, $\ell \geq 2$, we have the following associativity property for the isomorphisms $\hat{\tau}$. Make $I_j := (A_{i_j-1+1}, A_{i_j-1+2}, \ldots, A_{i_j})$ and $A^{\otimes I_j} := A_{i_j-1+1} \otimes A_{i_j-1+2} \otimes \cdots \otimes A_{i_j}$, for $j \in [1, \ell]$. Then, we have

$$\hat{\theta}_{A^{\otimes I_1}, \ldots, A^{\otimes I_\ell}}(\hat{\theta}_{I_1} \otimes \hat{\theta}_{I_2} \otimes \cdots \otimes \hat{\theta}_{I_\ell}) = \hat{\theta}_{A_1, \ldots, A_n}.$$ 

Taking inverses the formula becomes

$$(\hat{\tau}_{I_1} \otimes \hat{\tau}_{I_2} \otimes \cdots \otimes \hat{\tau}_{I_\ell})(\hat{\tau}_{A^{\otimes I_1}, \ldots, A^{\otimes I_\ell}} = \hat{\tau}_{A_1, \ldots, A_n}.$$
Proof. For $\ell = 2$, if $\alpha_1 \otimes \cdots \otimes \alpha_n \in \check{D}(A_1) \otimes \cdots \otimes \check{D}(A_n)$ is a typical homogeneous generator, we have

$$\check{T}_{A \otimes i_1, A \otimes i_2} (\check{T}_{i_1} \otimes \check{T}_{i_2})(\alpha_1 \otimes \cdots \otimes \alpha_n) = \check{T}_{A \otimes i_1, A \otimes i_2} (\alpha_1 \otimes \cdots \otimes \alpha_1) \otimes \check{T}_{i_2}(\alpha_{i_1+1} \otimes \cdots \otimes \alpha_n).$$

Evaluating at a typical homogeneous generator $\alpha_1 \otimes \cdots \otimes \alpha_n$ we get

$$\check{T}_{i_1}(\alpha_1 \otimes \cdots \otimes \alpha_1)[\alpha_1 \otimes \cdots \otimes \alpha_{i_1} \otimes \check{T}_{i_2}(\alpha_{i_1+1} \otimes \cdots \otimes \alpha_n)(\alpha_{i_1+1} \otimes \cdots \otimes \alpha_n)]$$

which clearly coincides with $\check{T}_{A_1, \ldots, A_n}(\alpha_1 \otimes \cdots \otimes \alpha_n)(\alpha_1 \otimes \cdots \otimes \alpha_n)$. So we can proceed by induction. Assume that the lemma holds for sequences with length $\ell - 1$ and let us show it for a sequence of length $\ell$.

We have that $\check{T}_{A \otimes i_1, A \otimes i_2, \ldots, A \otimes i_\ell} (\check{T}_{i_1} \otimes \check{T}_{i_2} \otimes \cdots \otimes \check{T}_{i_\ell})(\alpha_1 \otimes \cdots \otimes \alpha_n)$ equals

$$\check{T}_{A \otimes i_1, A \otimes i_2, \ldots, A \otimes i_\ell} [\check{T}_{i_1}(\alpha_1 \otimes \cdots \otimes \alpha_1) \otimes \check{T}_{i_2}(\alpha_{i_1+1} \otimes \cdots \otimes \alpha_2) \otimes \cdots \otimes \check{T}_{i_\ell}(\alpha_{i_{\ell-1}+1} \otimes \cdots \otimes \alpha_n)].$$

Evaluating at a typical homogeneous generator $\alpha_1 \otimes \cdots \otimes \alpha_n$, we obtain

$$\check{T}_{i_1}(\alpha_1 \otimes \cdots \otimes \alpha_1)[\alpha_1 \otimes \cdots \otimes \alpha_{i_1} \otimes \check{T}_{\alpha_{i_1+1} \otimes \cdots \otimes \alpha_n}(\alpha_{i_1+1} \otimes \cdots \otimes \alpha_n)].$$

where $H = \check{T}_{A \otimes i_1, A \otimes i_2, \ldots, A \otimes i_\ell}(\check{T}_{i_1} \otimes \check{T}_{i_2} \otimes \cdots \otimes \check{T}_{i_\ell})(\alpha_{i_1+1} \otimes \cdots \otimes \alpha_n)$. By induction hypothesis, we have

$$H = \check{T}_{A_1, \ldots, A_n}(\alpha_{i_1+1} \otimes \cdots \otimes \alpha_n)(\alpha_{i_1+1} \otimes \cdots \otimes \alpha_n).$$

Then, we have the wanted formula. \qed

Remark 2.10. For any natural number $n \geq 2$, and $d_1, \ldots, d_n \in \mathbb{Z}$, we can define

$$\lambda_n(d_1, \ldots, d_n) = \sum_{1 \leq i < j \leq n} d_i d_j.$$ 

So, we have $\lambda_n(d_1, \ldots, d_n) = \lambda_{n-1}(d_2, \ldots, d_n) + d_1(d_2 + \cdots + d_n)$. Then, from [2,7], we obtain the following lemma by induction on the natural number $n$.

Lemma 2.11. Given homogeneous morphisms $A_1 \stackrel{f_1}{\longrightarrow} B_1, \ldots, A_n \stackrel{f_n}{\longrightarrow} B_n$ in $\mathbf{GbS}$ we have a commutative diagram

$$
\begin{array}{ccc}
\check{D}(B_1) \otimes_S \cdots \otimes_S \check{D}(B_n) & \longrightarrow & \check{D}(B_1) \otimes_S \cdots \otimes_S \check{D}(B_n) \\
\check{D}(f_1) \otimes_S \cdots \otimes_S \check{D}(f_n) & \downarrow & \downarrow (-1)^{\lambda_n(\sum f_i)} \check{D}(f_1) \otimes_S \cdots \otimes_S \check{D}(f_n) \\
\check{D}(A_1) \otimes_S \cdots \otimes_S \check{D}(A_n) & \longrightarrow & \check{D}(A_1) \otimes_S \cdots \otimes_S \check{D}(A_n) \end{array}
$$

Taking inverses, we obtain the formula:

$$(\check{D}(f_1) \otimes \cdots \otimes \check{D}(f_n)) \check{T}_{B_1, \ldots, B_n} = (-1)^{\lambda_n(\sum f_i)} \check{T}_{A_1, \ldots, A_n} \check{D}(f_1 \otimes \cdots \otimes f_n).$$

Definition 2.12. Once we fix some $B \in \mathbf{GbS}$, we can consider the following sequence of canonical isomorphisms:
1. For $n = 1$, we make $\hat{\theta}_1 := id_{\hat{D}(B)} : \hat{D}(B) \rightarrow \hat{D}(B)$; and, for $n \geq 2$, we make $\hat{\theta}_n := \hat{\theta}_{B,...,B} : \hat{D}(B)^{\otimes n} \rightarrow \hat{D}(B)^{\otimes n}$.
and their inverses:

2. For $n = 1$, we make $\hat{\tau}_1 := id_{\hat{D}(B)} : \hat{D}(B) \rightarrow \hat{D}(B)$; and, for $n \geq 2$, we make $\hat{\tau}_n := \hat{\tau}_{B,...,B} : \hat{D}(B)^{\otimes n} \rightarrow \hat{D}(B)^{\otimes n}$.

Lemma 2.13. Given $B \in \mathcal{G}_{\mathcal{B}-S}$, non-negative integers $r,s,t,n$ with $r + s + t = n$ and $s \geq 1$, and any homogeneous morphism of graded $S-S$-bimodules $b_{s} : B^{\otimes s} \rightarrow B$ of degree $|b_{s}| = 1$, we have the following equality of morphisms from $\hat{D}(B)^{\otimes (r+1+t)}$ to $\hat{D}(B)^{\otimes n}$

\[
(id_{\hat{D}(B)}^{\otimes r} \otimes \hat{\tau}_{s} \hat{D}(b_{s}) \otimes id_{\hat{D}(B)}^{\otimes t})\hat{\tau}_{r+1+t} = \hat{\tau}_{n} \hat{D}(id_{B}^{\otimes r} \otimes b_{s} \otimes id_{B}^{\otimes t}).
\]

Proof. Make $A_{i} = B$, for all $i \in [1,n]$; let $\ell := r + 1 + t$ and consider the sequence $0 = i_{0} < i_{1} < \cdots < i_{\ell} = n$ defined by $i_{j} := j$, for $j \in [1,r]$; $i_{r+1} := r + s$; and $i_{j} := j$, for $j \in [r+2,\ell]$. Then, from the associativity formula in (2.11), we have

\[
(id_{\hat{D}(B)}^{\otimes r} \otimes \hat{\tau}_{s} \hat{D}(b_{s}) \otimes id_{\hat{D}(B)}^{\otimes t})\hat{\tau}_{r+1+t} = \hat{\tau}_{n} \hat{D}(id_{B}^{\otimes r} \otimes b_{s} \otimes id_{B}^{\otimes t}),
\]

where, for $m \geq 0$, we denote by $J_{m} := (B, B, \ldots, B)$, a sequence with $m$ terms.

Now, apply the formula in (2.11) to the family $f_{1}, \ldots, f_{r+1+t}$ defined by $f_{i} := id_{B}$, for $i \in [1,r] \cup [r+2, r+1+t]$, and $f_{r+1} := b_{s} : B^{\otimes s} \rightarrow B$. Thus $|f_{r+1}| = 1$ and $|f_{i}| = 0$ for $i \neq r + 1$. Hence $\lambda_{r+1+t}([f_{1}, \ldots, |f_{r+1+t}|) = 0$, and we get the formula

\[
(id_{\hat{D}(B)}^{\otimes r} \otimes \hat{\tau}_{s} \hat{D}(b_{s}) \otimes id_{\hat{D}(B)}^{\otimes t})\hat{\tau}_{r+1+t} = \hat{\tau}_{n} \hat{D}(id_{B}^{\otimes r} \otimes b_{s} \otimes id_{B}^{\otimes t}).
\]

Then, if $\Delta = (id_{\hat{D}(B)}^{\otimes r} \otimes \hat{\tau}_{s} \hat{D}(b_{s}) \otimes id_{\hat{D}(B)}^{\otimes t})\hat{\tau}_{r+1+t}$, we have

\[
\Delta = (id_{\hat{D}(B)}^{\otimes r} \otimes \hat{\tau}_{s} \otimes id_{\hat{D}(B)}^{\otimes t})(id_{\hat{D}(B)}^{\otimes r} \otimes \hat{D}(b_{s}) \otimes id_{\hat{D}(B)}^{\otimes t})\hat{\tau}_{r+1+t}
\]

\[
= (id_{\hat{D}(B)}^{\otimes r} \otimes \hat{\tau}_{s} \otimes id_{\hat{D}(B)}^{\otimes t})\hat{\tau}_{r+1+t} \hat{D}(id_{B}^{\otimes r} \otimes b_{s} \otimes id_{B}^{\otimes t})
\]

\[
= \hat{\tau}_{n} \hat{D}(id_{B}^{\otimes r} \otimes b_{s} \otimes id_{B}^{\otimes t}).
\]

\[\square\]

3 The Yoneda $A_{\infty}$-algebra

Assume that $S$ is a finite product of copies of the field $k$ and $1 = \sum_{i \in \mathcal{P}} e_{i}$ is a decomposition of the unit element of $S$ as a sum of orthogonal idempotents.
We recall the following basic definition.

Definition 3.1. An $A_{\infty}$-algebra $A$ is a graded $S$-$S$-bimodule $A$, equipped with a sequence of homogeneous morphisms of $S$-$S$-bimodules

\[
\{m_{n} : A^{\otimes n} \rightarrow A\}_{n \in \mathbb{N}},
\]
where each \(m_n\) has degree \(|m_n| = 2 - n\), such that, for each \(n \in \mathbb{N}\), the following Stasheff identity holds.

\[
S_n := \sum_{s \geq 1; r, t \geq 0} (-1)^{r+t}s m_{r+1+t}(id^{\otimes r} \otimes m_s \otimes id^{\otimes t}) = 0.
\]

**Lemma 3.2.** Let \(\hat{A}\) be an augmented graded S-algebra, that is the \(S\)-\(S\)-bimodule \(A\) admits a decomposition of graded \(S\)-algebras \(A = S \oplus \hat{A}\) and has \(1_S\) as unit element; thus the product of \(\hat{A}\) restricts to a product on the graded \(S\)-algebra \(A\). Then, whenever \(\hat{A}\) admits a structure of \(A\)-\(S\) structure \(\{m_n : A^{\otimes n} \to A\}_{n \in \mathbb{N}}\) such that \(m_1 = 0\) and \(m_2\) is induced by the product of \(A\), we have that \(\hat{A}\) admits a unique structure of \(A\)-\(S\) structure \(\{\hat{m}_n : \hat{A}^{\otimes n} \to \hat{A}\}_{n \in \mathbb{N}}\) such that \(\hat{m}_1 = 0\) and \(\hat{m}_2\) is induced by the product of \(\hat{A}\) and, moreover: \(\hat{m}_n(z_1 \otimes z_2 \otimes \cdots \otimes z_n) = 0\), for all \(n \geq 3\) and \(z_1, \ldots, z_n \in \hat{A}\) with \(z_u \in \{e_i \mid i \in P\}\), for some \(u \in [1, n]\).

**Proof.** Make \(\hat{m}_1 = 0\) and let \(\hat{m}_2 : \hat{A} \otimes S \hat{A} \to \hat{A}\) be the morphism of \(S\)-\(S\)-bimodules induced by the product of \(\hat{A}\). For \(n \geq 3\), we have an \(S\)-\(S\)-bimodule decomposition \(\hat{A}^{\otimes n} = A^{\otimes n} \oplus E_n\), where \(E_n\) denotes the \(S\)-\(S\)-subbimodule generated by tensors with some factor in \(\{e_i \mid i \in P\}\). Then, consider the extension \(\hat{m}_n : \hat{A}^{\otimes n} \to \hat{A}\) of \(m_n\) which maps \(E_n\) to zero. Thus \(|m_n| = |\hat{m}_n|\) for all \(n\).

We claim that \(\hat{S}_n = 0\), for all \(n \geq 1\), where

\[
\hat{S}_n = \sum_{s \geq 1; r, t \geq 0} (-1)^{r+t}s \hat{m}_{r+1+t}(id^{\otimes r} \otimes \hat{m}_s \otimes id^{\otimes t}).
\]

Since \(\hat{m}_1 = 0\), we clearly have that \(\hat{S}_1 = \hat{m}_1^2 = 0\) and

\[
\hat{S}_2 = -\hat{m}_2(\hat{m}_1 \otimes id) - \hat{m}_2(id \otimes \hat{m}_1) + \hat{m}_1 \hat{m}_2 = 0.
\]

For \(n \geq 3\), we consider a typical non-zero generator \(z = z_1 \otimes \cdots \otimes z_n \in \hat{A}^{\otimes n}\), where we assume that all the elements \(z_1, \ldots, z_n \in \hat{A}\) are directed, see \((2.2)\). Let us examine \(\hat{S}_n(z)\). If all \(z_i \in A\), we have \(\hat{S}_n(z) = S_n(z) = 0\). So, assume that \(z_u \in \{e_i \mid i \in P\}\), for some \(u \in [1, n]\). Consider the index set \(I = \{(r, s, t) \mid r + s + t = n; s \geq 1; r, t \geq 0\}\). Then, \(\hat{S}_n = \sum_{(r, s, t) \in I} \hat{S}_n(r, s, t)\), where

\[
\hat{S}_n(r, s, t) := (-1)^{r+s+t}\hat{m}_{r+1+t}(id^{\otimes r} \otimes \hat{m}_s \otimes id^{\otimes t}).
\]

We prove that \(\hat{S}_n(z) = 0\) in each one of the following three possible cases.

**Case 1:** \(u = 1\).

In each one of the following situations, we have that \(\hat{S}_n(r, s, t)(z) = 0\).

1.1 If we have \(r > 1\);

1.2 If we have \(r = 1\) and \(t \neq 0\);

1.3 If we have \(r = 0\) and \((s = 1\) or \(s \geq 3\));
There are only two other possible situations:

1.4 If we have $r = 0$ and $s = 2$. Thus, $t = n - 2$ and
\[
\hat{S}_n(r, s, t)(z) = (−1)^{r+st}\hat{m}_{r+1+t}(\hat{m}_2(e_i \otimes z_2) \otimes z_3 \otimes \cdots \otimes z_n)
\]
\[
= \hat{m}_{n-1}(z_2 \otimes z_3 \otimes \cdots \otimes z_n).
\]
1.5 If we have $r = 1$ and $t = 0$. Thus, $s = n - 1$ and
\[
\hat{S}_n(r, s, t)(z) = (−1)^{r+st}\hat{m}_{r+1+t}(e_i \otimes \hat{m}_s(z_2 \otimes z_3 \otimes \cdots \otimes z_n))
\]
\[
= −\hat{m}_2(e_i \otimes \hat{m}_s(z_2 \otimes z_3 \otimes \cdots \otimes z_n))
\]
\[
= −\hat{m}_{n-1}(z_2 \otimes z_3 \otimes \cdots \otimes z_n).
\]
Therefore, we get $\hat{S}_n(z) = \hat{S}_n(0, 2, 2)(z) + \hat{S}_n(1, n - 1, 0)(z) = 0$.

Case 2: $u = n$.

This case is similar to the preceding one. Indeed, in each one of the following situations, we have that $\hat{S}_n(r, s, t)(z) = 0$:

2.1 If we have $t > 1$;
2.2 If we have $t = 1$ and $r \neq 0$;
2.3 If we have $t = 0$ and $(s = 1$ or $s \geq 3)$;

There are only two other possible situations:

2.4 If we have $t = 0$ and $s = 2$. Thus, $r = n - 2$ and
\[
\hat{S}_n(r, s, t)(z) = (−1)^{r+st}\hat{m}_{r+1+t}(z_1 \otimes \cdots \otimes z_r \otimes \hat{m}_2(z_{n-1} \otimes e_i))
\]
\[
= (−1)^{n-2}\hat{m}_{n-1}(z_1 \otimes z_2 \otimes \cdots \otimes z_{n-1}).
\]
2.5 If we have $t = 1$ and $r = 0$. Thus, $s = n - 1$ and
\[
\hat{S}_n(r, s, t)(z) = (−1)^{r+st}\hat{m}_{r+1+t}(\hat{m}_s(z_1 \otimes \cdots \otimes z_{n-1}) \otimes e_i)
\]
\[
= (−1)^{n-1}\hat{m}_2(\hat{m}_{n-1}(z_1 \otimes \cdots \otimes z_{n-1}) \otimes e_i)
\]
\[
= (−1)^{n-1}\hat{m}_{n-1}(z_1 \otimes z_2 \otimes \cdots \otimes z_{n-1}).
\]
Therefore, we get $\hat{S}_n(z) = \hat{S}_n(n - 2, 2, 0)(z) + \hat{S}_n(0, n - 1, 1)(z) = 0$.

Case 3: $1 < u < n$.

Again, in each one of the following situations, we have that $\hat{S}_n(r, s, t)(z) = 0$:

3.1 If we have $r \geq u$;
3.2 If we have $t \geq n - u + 1$ (so $u \in [n - t + 1, n - 1]$);
3.3 If we have $r < u$ and $t < n - u + 1$, and $(s = 1$ or $s \geq 3)$;

If we are not in these situations, we have $r < u$, $t < n - u + 1$ and $s = 2$. Thus, we are left with two cases:
3.4 We have $r = u - 1$, and $s = 2$, thus $t = n - u - 1$; or

3.5 We have $r = u - 2$, and $s = 2$, thus $t = n - u$.

In case 3.4, we have that $\tilde{S}_n(r, s, t)(z) = \tilde{S}_n(u - 1, 2, n - u - 1)(z)$ equals

$$(-1)^{u-1} \tilde{m}_{n-1}(z_1 \otimes \cdots \otimes z_{u-1} \otimes \tilde{m}_2(e_i \otimes z_{u+1}) \otimes \cdots \otimes z_n)$$

that is $(-1)^{u-1} \tilde{m}_{n-1}(z_1 \otimes \cdots \otimes z_{u-1} \otimes z_{u+1} \otimes \cdots \otimes z_n)$.

In case 3.5, we have that $\tilde{S}_n(r, s, t)(z) = \tilde{S}_n(u - 2, 2, n - u)(z)$ equals

$$(-1)^{u-2} \tilde{m}_{n-1}(z_1 \otimes \cdots \otimes z_r \otimes \tilde{m}_2(z_{u-1} \otimes e_i) \otimes z_{n-t+1} \otimes \cdots \otimes z_n)$$

that is $(-1)^{u-2} \tilde{m}_{n-1}(z_1 \otimes \cdots \otimes z_{u-1} \otimes z_{u+1} \otimes \cdots \otimes z_n)$. Therefore, we get

$$\tilde{S}_n(z) = \tilde{S}_n(u - 1, 2, n - u - 1)(z) + \tilde{S}_n(u - 2, 2, n - u)(z) = 0.$$

\[\square\]

**Definition 3.3.** Assume that $\{\Delta_i\}_{i \in P}$ is a family of non-isomorphic indecomposable $\Lambda$-modules, where $\Lambda$ is a finite-dimensional $k$-algebra. Make $\Delta = \bigoplus_{i \in P} \Delta_i$. Then, since $k$ is algebraically closed, there is a splitting of the endomorphism algebra $\text{End}_\Lambda(\Delta) = S \bigoplus J$ over its radical $J$. Moreover, the semisimple subalgebra $S$ of $\text{End}_\Lambda(\Delta)$ can be described as $S = \bigoplus_{i \in P} k e_i$, where $e_i = \sigma_i \pi_i \in \text{End}_\Lambda(\Delta)$ is the idempotent defined by the composition of the canonical projection $\pi_i : \Delta \longrightarrow \Delta_i$ and the canonical injection $\sigma_i : \Delta_i \longrightarrow \Delta$.

The *Yoneda algebra of $\Delta$* is the graded $S$-algebra

$$A = \bigoplus_{n=0}^{\infty} A_n,$$

where $A_n = \text{Ext}_n^\Lambda(\Delta, \Delta)$, for $n \geq 0$.

**Definition 3.4.** Assume, as before, that $\{\Delta_i\}_{i \in P}$ is a family of non-isomorphic indecomposable $\Lambda$-modules, where $\Lambda$ is a finite-dimensional $k$-algebra. Make $\Delta = \bigoplus_{i \in P} \Delta_i$. Let $P_{\Delta_i}$ be a fixed projective resolution of $\Delta_i$, for each $i \in P$, and consider them as graded differential $\Lambda$-modules, where $\Lambda$ is considered as a differential graded $k$-algebra concentrated at degree 0. Make $P_{\Delta} := \bigoplus_{i \in P} P_{\Delta_i}$, and

$$\mathcal{E} = \mathcal{E}(P_{\Delta}) := \bigoplus_{n \in \mathbb{Z}} \text{End}_\Lambda^n(P_{\Delta}),$$

where $\text{End}_\Lambda^n(P_{\Delta})$ is the space of homogeneous morphisms $f : P_{\Delta} \longrightarrow P_{\Delta}$ of graded $\Lambda$-modules with degree $|f| = n$; it can be identified with $\text{Hom}_\Lambda^n(P_{\Delta}, P_{\Delta}[n])$.

The $k$-algebra $\mathcal{E}$ admits a differential $d : \mathcal{E} \longrightarrow \mathcal{E}$ given on homogeneous elements $f \in \mathcal{E}$ by the formula $d(f) = d_{P_{\Delta}} f - (-1)^{|f|} f d_{P_{\Delta}}$. So we have a differential graded $k$-algebra $\mathcal{E}$.

For each $i \in P$, denote by $\pi_i^* : P_{\Delta_i} \longrightarrow P_{\Delta}$ and $\sigma_i^* : P_{\Delta_i} \longrightarrow P_{\Delta}$ the projection and the injection, respectively, associated to the direct sum $P_{\Delta} = \bigoplus_{i \in P} P_{\Delta_i}$, and make $e_i^* := \sigma_i^* \pi_i^*$. Then, we obtain a decomposition $id_{P_{\Delta}} = \sum_{i \in P} e_i^*$ of the unit of the algebra $\mathcal{E}$ as a sum of orthogonal idempotents.
Assume that Proposition 3.6.

Remark 3.5. For any family \( \{ \Delta_i \}_{i \in \mathcal{P}} \) as above, a well known result of homological algebra states that there is an isomorphism of graded \( k \)-algebras

\[
\Phi : H(\mathcal{E}(P_{\Delta})) \longrightarrow \bigoplus_{n \in \mathbb{Z}} \text{Ext}^n_{\Lambda}(\Delta, \Delta),
\]

where we take \( \text{Ext}^n_{\Lambda}(\Delta, \Delta) := 0 \), for \( n < 0 \).

Moreover, it is easy to see that \( \Phi(e_i) = e_i \), for all \( i \in \mathcal{P} \), where \( e_i \) denotes the class of the 0-cycle \( e_i^* \) in the homology graded \( S \)-algebra \( H(\mathcal{E}(P_{\Delta})) \).

Proposition 3.6. Assume that \( \mathcal{H} = (\mathcal{P}, \leq, \{ \Delta_i \}_{i \in \mathcal{P}}) \) is a strict homological system for the \( k \)-algebra \( \Lambda \). Consider a fixed \( i \)-bounded projective resolution \( P_{\Delta} \), of \( \Delta_i \), for each \( i \in \mathcal{P} \), as in (1.6), and adopt the notation of the preceding definition (3.4). The following holds.

1. We have a \( d \)-invariant decomposition \( \mathcal{E} = \mathcal{E}' \oplus \mathcal{E}'' \) of graded \( S \)-\( S \)-bimodules, where

\[
\mathcal{E}' = \bigoplus_{i,j \in \mathcal{P}, \tau \leq j} e_i^* \mathcal{E}^*_{\tau j} \quad \text{and} \quad \mathcal{E}'' = \bigoplus_{i,j \in \mathcal{P}, \tau \not\leq j} e_i^* \mathcal{E}^*_{\tau j}.
\]

Moreover, \( \mathcal{E}' \) is a differential graded \( S \)-subalgebra of \( \mathcal{E} \) and \( H(\mathcal{E}'') = 0 \).

2. We have a \( d \)-invariant decomposition \( \mathcal{E}' = \big[ \bigoplus_{i \in \mathcal{P}} k e_i^* \big] \oplus \mathcal{R}' \) of graded \( S \)-\( S \)-bimodules, where \( \mathcal{R}' \) is a differential graded \( S \)-subalgebra of \( \mathcal{E}' \). Therefore,

\[
H(\mathcal{E}) = H(\mathcal{E}') = \bigoplus_{i \in \mathcal{P}} k e_i^* \oplus H(\mathcal{R}').
\]

Proof. (1): We will use the graded isomorphism \( \Phi \) of (3.5). Clearly, there is a decomposition as in (1) of graded vector spaces. Since \( e_i^* \text{Ext}^n_{\Lambda}(\Delta, \Delta)e_i = \text{Ext}^n_{\Lambda}(\Delta_i, \Delta_j) = 0 \), for all \( n \geq 0 \) and all \( \tau \not\leq j \), the restriction of the isomorphism \( \Phi \) gives

\[
H(\mathcal{E}'') \cong \bigoplus_{i,j \in \mathcal{P}, \tau \not\leq j} e_j \text{Ext}^n_{\Lambda}(\Delta, \Delta)e_i = 0.
\]

Moreover, the spaces \( \mathcal{E}' \) and \( \mathcal{E}'' \) are invariant under the differential, because \( d(e_i^*) = 0 \), for all \( i \in \mathcal{P} \).

(2) In order to describe the ideal \( \mathcal{R}' \) of \( \mathcal{E}' \), observe that we have the following isomorphisms for \( i, j \in \mathcal{P} \):

\[
\phi_{i,j}^{i,j} : e_j^* \text{Hom}^n_{\Lambda}(P_{\Delta}, P_{\Delta}) \longrightarrow \text{Hom}^n_{\Lambda}(P_{\Delta}, P_{\Delta}), \quad \text{given by} \; f \mapsto f_{i,j} := \pi_j^* f \sigma_i^*,
\]

and

\[
\psi_{i,j} : \bigoplus_{n \in \mathbb{Z}} \text{Hom}^n_{\Lambda}(P_{\Delta}, P_{\Delta}) \cong \bigoplus_{s, n \in \mathbb{Z}} \text{Hom}_{\Lambda}(P_{\Delta}^s, P_{\Delta}^{s+n}) = \bigoplus_{s, t \in \mathbb{Z}} \text{Hom}_{\Lambda}(P_{\Delta}^s, P_{\Delta}^t)
\]
given, for \( f_{i,j} \in \text{Hom}_n^0(P_{\Delta_i}, P_{\Delta_i}) \), by \( \psi_{i,j}(f_{i,j}) = \sum s f_{i,j,s,s+n} \) where \( f_{i,j,s,s+n} : P_{\Delta_i} \longrightarrow P_{\Delta_i}^{s+s+n} \) is the restriction of \( f_{i,j} \). Then, we obtain the following sequence of isomorphisms

\[
\mathcal{E}' = \bigoplus_{i,j \in P} e_i^j \bigoplus_{n \in \mathbb{Z}} \text{Hom}_n^0(P_{\Delta_i}, P_{\Delta_i}) e_i^j
\]

\[
\cong \bigoplus_{i,j \in P} \bigoplus_{n \in \mathbb{Z}} \text{Hom}_n^0(P_{\Delta_i}, P_{\Delta_i}),
\]

\[
\cong \bigoplus_{i,j \in P} \bigoplus_{s,t \in \mathbb{Z}} \text{Hom}_\Lambda(P_{\Delta_i}^s, P_{\Delta_i}^t) =: \mathcal{E}'.
\]

Denote by \( \phi = \bigoplus_{i,j,n} \phi_{i,j}^n \) the first isomorphism and by \( \psi = \bigoplus_{i,j} \psi_{i,j} \) the second one. Then, we have that \( \phi(e_i^j) = \text{id}_{P_{\Delta_i}} \in \text{Hom}_0^0(P_{\Delta_i}, P_{\Delta_i}) \), thus we have \( \psi(\phi(e_i^j)) = \sum s \text{id}_{P_{\Delta_i}}^s \in \bigoplus_{s \in \mathbb{Z}} \text{Hom}_\Lambda(P_{\Delta_i}^s, P_{\Delta_i}^s) \).

The space \( \mathcal{E}' \) has a natural structure of \( k \)-algebra, given typical generators

\[
P_{\Delta_i} f_{i,u,v,r} P_{\Delta_u} \quad \text{and} \quad P_{\Delta_v} g_{i,u,v,t} P_{\Delta_v},
\]

the product \( g_{i,u,v,t} : f_{i,u,v,r} \circ f_{i,u,v,r} \) if \( v = u \) and \( l = r \), and it is zero otherwise. The morphism \( \psi : \mathcal{E}' \longrightarrow \mathcal{E}' \) is in fact an isomorphism of \( k \)-algebras. Indeed, given \( f \in e_i^j \text{Hom}_\Lambda^0(P_{\Delta_i}, P_{\Delta_i}) e_i^j \) and \( g \in e_i^j \text{Hom}_\Lambda^0(P_{\Delta_i}, P_{\Delta_i}) e_i^j \), we have \( (gf)_{i,j} = \sum l g_{l,j,f_l,v} = g_{i,j,f_{i,t},v} \in \text{Hom}_{\Lambda}^n (P_{\Delta_i}, P_{\Delta_i}) \), so

\[
\psi(\phi(gf)) = \psi((gf)_{i,j}) = \psi(g_{t,j,f_{i,t}}) = \sum s (g_{t,j,f_{i,t}})_{s,s+n+m} = \sum s g_{t,j,s+s+n+m} f_{i,t,s,s+n}.
\]

Therefore,

\[
\psi(\phi(gf)) = \psi((gf)_{i,j}) = \psi(g_{t,j,f_{i,t}}) = \sum s (g_{t,j,f_{i,t}})_{s,s+n+m} = \sum s g_{t,j,s+s+n+m} f_{i,t,s,s+n} = \psi(gf).
\]

Since \( k \) is algebraically closed and \( P_{\Delta_i}^0 \cong P_1 \), its endomorphism algebra splits over its radical, so we obtain

\[
\text{Hom}_\Lambda(P_{\Delta_i}^0, P_{\Delta_i}^0) = \text{kid}_{P_{\Delta_i}^0} \oplus \text{rad}_\Lambda(P_{\Delta_i}^0, P_{\Delta_i}^0),
\]

where \( \text{rad}_\Lambda(P_{\Delta_i}^0, P_{\Delta_i}^0) \) consists of the non-isomorphisms of \( \text{Hom}_\Lambda(P_{\Delta_i}^0, P_{\Delta_i}^0) \).

Now consider the linear subspace \( \mathcal{R}' = (\psi\phi)^{-1} (\mathcal{R}') \) of \( \mathcal{E}' \), where \( \mathcal{R}' \) is the linear subspace of \( \mathcal{E}' \) generated by the morphisms of \( \Lambda \)-modules \( f : P_{\Delta_i}^s \longrightarrow P_{\Delta_i}^t \), such that \( s \neq 0 \) or \( t \neq 0 \), or, if \( s = 0 = t \), then \( f \) is not an isomorphism.

In order to show that \( \mathcal{R}' \) is an ideal of \( \mathcal{E}' \), take any typical linear generator \( g : P_{\Delta_i}^s \longrightarrow P_{\Delta_i}^t \) of \( \mathcal{R}' \) and any typical generator \( f : P_{\Delta_i}^t \longrightarrow P_{\Delta_i}^r \) of \( \mathcal{E}' \) with possible non-zero product \( fg : P_{\Delta_i}^s \longrightarrow P_{\Delta_i}^{r+s} \). If \( s \neq 0 \) or \( r \neq 0 \), we get \( fg \in \mathcal{R}' \).
So assume that \( s = 0 = r \) and that \( fg \) is an isomorphism. Then, \( P_i = P^0_{\Delta_i} \) is a direct summand of \( \sum_{i \in P} \Lambda e_i^* \). Since \( P_{\Delta_j} \) is a \( j \)-bounded minimal projective resolution, if \( t < 0 \), we get that \( t > J \), which is not the case because \( g \in \mathcal{E}' \) implies that \( t \leq J \). Thus \( t = 0 \), so \( P_i \cong P_j \), and we obtain \( i = j \). Then, the composition \( fg \) belongs to the radical \( \text{rad}_A(P_i, P_i) \), because \( g \) does, a contradiction. Hence \( fg \) is not an isomorphism and, so, \( fg \in \mathcal{E}' \). This shows that \( \mathcal{E}' \) is a left ideal of \( \mathcal{E}' \). Similarly, one shows that \( \mathcal{E}' \) is a right ideal of \( \mathcal{E}' \).

Let us show that \( \mathcal{E}' = \left[ \sum_{i \in P} k e_i^* \right] + \mathcal{R}' \). Any \( f = \sum_{i,j,s,t} f_{i,j,s,t} \in \mathcal{E}' \), can be written as follows:

\[
f = \sum_{s \neq 0 \text{ or } t \neq 0} f_{i,j,s,t} + \sum_{i \neq j} f_{i,j,0,0} + \sum_i c_i^i \text{id}_{P^0_{\Delta_i}} + \sum_i \rho_i^i,
\]

where \( f_{i,j,0,0} = c_i^i \text{id}_{P^0_{\Delta_i}} + \rho_i^i \in \text{id}_{P^0_{\Delta_i}} \perp \text{rad} \left( \mathcal{E}' \right) \). Moreover, we have

\[
\psi(e_i^*) = \sum_s \text{id}_{P^0_{\Delta_i}} = \text{id}_{P^0_{\Delta_i}} - \rho_i, \quad \text{where } \rho_i = \sum_{s < 0} \text{id}_{P^0_{\Delta_i}}.
\]

Then, in the preceding expression for \( f \), we can replace

\[
\sum_i c_i^i \text{id}_{P^0_{\Delta_i}} \quad \text{by} \quad \sum_i c_i^i \psi(e_i^*) + \sum_i c_i^i \rho_i,
\]

to obtain that \( f = \sum_i c_i^i \psi(e_i^*) + \sum_i \rho_i, \) for some \( \rho_i \in \mathcal{R}' \). Therefore, we obtain \( \mathcal{E}' = \left[ \sum_{i \in P} e_i^* \right] + \mathcal{R}' \).

If \( \sum_i c_i e_i^* = 0 \), we get \( \psi(\sum_i c_i e_i^*) = \sum_i \sum_s c_i \text{id}_{P^0_{\Delta_i}} \), a combination of linearly independent elements in \( \mathcal{E}' \), thus all the coefficient are zero and \( \sum_{i \in P} k e_i^* = \bigoplus_{i \in P} k e_i^* \). Moreover, it is clear that \( \left( \bigoplus_{i \in P} \text{id}_{P^0_{\Delta_i}} \right) \cap \mathcal{R}' = 0 \), thus if \( \sum_i c_i e_i^* \in \mathcal{R}' \), we obtain that \( \sum_i \sum_s c_i \text{id}_{P^0_{\Delta_i}} - \sum_i c_i \rho_i = \psi(\sum_i c_i e_i^*) \in \mathcal{R}' \), thus \( \sum_i c_i \text{id}_{P^0_{\Delta_i}} \in \mathcal{R}' \) and all the coefficients are zero, so we finally get \( \mathcal{E}' = \left[ \bigoplus_{i \in P} e_i^* \right] + \mathcal{R}' \) as we wanted.

We claim that \( \mathcal{R}' \) is a \( d \)-invariant subalgebra of \( \mathcal{E}' \). Notice first that \( \psi(\text{id}_{P_{\Delta}}) = \sum_i \psi(\text{id}_{P_{\Delta_i}}) = \sum_i \psi(\text{id}_{P_{\Delta_i}}) = \sum_{i,s} \text{id}_{P^0_{\Delta_i}} \). Now, take any typical generator \( f = f_{i,j,s,t} \in \mathcal{E}' \), so \( f' = (\psi \phi)^{-1}(f) \) is a typical generator of \( \mathcal{E}' \), which has degree \( n := s - r \). Then we have

\[
\psi(\phi(f')) = \psi(\phi(\text{id}_{P_{\Delta}} f' - (-1)^n f' \text{id}_{P_{\Delta}})) = \\
\psi(\phi(\text{id}_{P_{\Delta}})) f - (-1)^n f \psi(\phi(\text{id}_{P_{\Delta}})) = \\
\sum_{u,v} d_{P_{\Delta}} f - (-1)^n \sum_{v,b} f d_{P_{\Delta}} = \\
d_{P_{\Delta_j}} (f - (-1)^n f d_{P_{\Delta_i}}).
\]

These maps are disposed as follows

\[
P^{-1}_{\Delta_j} \xrightarrow{d_{P_{\Delta_j}}^{-1}} P^0_{\Delta_i} \xrightarrow{f} P^0_{\Delta_j} \xrightarrow{d_{P_{\Delta_j}}} P^{s+1}_{\Delta_j}.
\]

Let us examine \( d_{P_{\Delta_j}}^s f \). If \( r \neq 0 \) or \( s + 1 \neq 0 \), then \( d_{P_{\Delta_j}}^s f \in \mathcal{R}' \). If \( r = 0 \) and \( s = -1 \), and \( d_{P_{\Delta_j}}^s f \) is an isomorphism, we get that \( P_i \) is a direct summand of
$P_{\Delta}^{-1}$ and, since $P_{\Delta}$ is a $j$-bounded projective resolution, we obtain $\overline{t} > j$, which is not the case. Thus $d_{P_{\Delta}^{-1}}^i f$ is not an isomorphism and belongs to $R'$. Similarly, one shows that $f d_{P_{\Delta}^{-1}}^i \in R'$. Therefore, we get $d(f') \in R'$. So, we finally obtain that $d(R') \subseteq R'$.

Then, we have a $d$-invariant decomposition $E' = \bigoplus_{i \in \mathcal{P}} \ker e_i^* \oplus R'$, where $R'$ is a differential graded $k$-algebra and then $Z(E') = \bigoplus_{i \in \mathcal{P}} \ker e_i^* \oplus Z(R')$ and $B(E') = B(R')$, thus

$$H(E) = H(E') = \bigoplus_{i \in \mathcal{P}} \ker e_i^* \oplus H(R').$$

Notice that if we identify $S$ with $\bigoplus_{i \in \mathcal{P}} \ker e_i^*$, we obtain that $R'$ is differential graded (without unit) $S$-algebra.}

The following theorem generalizes the argument of \cite{8}(3.5), which relies on a theorem by Kadeishvili, see \cite{3.3}, \cite{17} and \cite{14}.

**Theorem 3.7.** Assume that $\mathcal{H} = (\mathcal{P}, \leq, \{\Delta_i\} \subseteq \mathcal{P})$ is a strict homological system for the $k$-algebra $\Lambda$. Then, the Yoneda graded $S$-algebra $A$ of the $\Lambda$-module $\Delta = \bigoplus_{i \in \mathcal{P}} \Delta_i$ admits a structure of $A_\infty$-algebra with $S$-$S$-bimodule higher multiplications $\{m_n : A^{\otimes n} \longrightarrow A\}_{n \in \mathbb{N}}$ such that $m_1 = 0$, $m_2$ is induced by the product of the Yoneda algebra and, if $n \geq 3$, they satisfy that $m_n(z_1 \otimes z_2 \otimes \cdots \otimes z_n) = 0$, whenever $z_i \in \{e_j \mid j \in \mathcal{P}\}$, for some $i \in [1, n]$.

*Proof.* Adopt the notation of the preceding Proposition. Apply Kadeishvili theorem to the differential graded $S$-algebra $R'$, to obtain a structure $\{m_n\}_{n \in \mathbb{N}}$ of $A_\infty$-algebra on the graded homology $S$-algebra $H(R')$, such that $m_1 = 0$ and $m_2$ is induced by the product of $H(R')$.

Then, apply \cite{3.2} to the augmented graded $S$-algebra $H(E') = \bigoplus_{i \in \mathcal{P}} \ker e_i^* \oplus H(R')$, to obtain a structure of $A_\infty$-algebra $\{\tilde{m}_n\}_{n \in \mathbb{N}}$ on $H(E')$ such that $\tilde{m}_1 = 0$, $\tilde{m}_2$ is induced by the product of $H(E')$, and $\tilde{m}_n(z_1 \otimes \cdots \otimes z_n) = 0$, for $n \geq 3$ and $z_1, \ldots, z_n \in H(E')$, whenever $z_i \in \{e_j \mid j \in \mathcal{P}\}$, for some $i \in [1, n]$.

Finally, we can derive the statement of the theorem by transferring the structure of $A_\infty$-algebra of $H(E) = H(E')$ onto the Yoneda algebra $A$ with the help of the graded isomorphism $\Phi : H(E) \longrightarrow A$ mentioned in \cite{3.5}.}

**Remark 3.8.** We can consider the Yoneda algebra $A$ associated to any admissible homological system $\mathcal{H} = (\mathcal{P}, \leq, \{\Delta_i\} \subseteq \mathcal{P})$ for a finite-dimensional $k$-algebra $\Lambda$. Assume that $A$ admits a structure of $A_\infty$-algebra with higher multiplications $\{m_n : A^{\otimes n} \longrightarrow A\}_{n \in \mathbb{N}}$. Notice that, for $n \geq 3$, the following are equivalent:

1. $m_n(z_1 \otimes z_2 \otimes \cdots \otimes z_n) = 0$, whenever $z_i \in \{e_j \mid j \in \mathcal{P}\}$, for some $i \in [1, n]$;
2. $m_n(z_1 \otimes z_2 \otimes \cdots \otimes z_n) = 0$, whenever $z_i = i d_{\Delta}$, for some $i \in [1, n]$.

We will say that the $A_\infty$-algebra $A$ has a strict unit or that it has a strict structure of $A_\infty$-algebra (or even that $A$ is strict) if $m_1 = 0$, $m_2$ is induced by the product of the Yoneda algebra, and the preceding property of the higher multiplications of $A$ is satisfied.
4 Algebras linked to strict Yoneda $A_{\infty}$-algebras

From now on, unless otherwise specified, we will keep the notation of the following remarks.

Remark 4.1. We denote by $\mathcal{H} = (\mathcal{P}, \leq, \{\Delta_i\}_{i \in \mathcal{P}})$ a fixed admissible homological system for a given finite-dimensional $k$-algebra $\Lambda$, as in (1.3). Consider the $\Lambda$-module $\Delta = \bigoplus_{i \in \mathcal{P}} \Delta_i$. Then, we have a decomposition $\text{End}_\Lambda(\Delta) = S \bigoplus J$, where $J$ is the Jacobson radical of $\text{End}_\Lambda(\Delta)$ and $S = \bigoplus_{i \in \mathcal{P}} ke_i$, where $\{e_i\}_{i \in \mathcal{P}}$ are the canonical central idempotents of $S$ considered in (3.3).

For the study of the category $\mathcal{F}(\Delta)$, the Yoneda algebra $A = \bigoplus_{n \geq 0} A_n$, where $A_n = \text{Ext}^n_\Lambda(\Delta, \Delta)$, for $n \geq 0$, of $\Delta$ is an important tool. From now on, we assume that the Yoneda algebra $A$ admits a strict structure $\{m_n : A \otimes n \rightarrow A\}_{n \in \mathbb{N}}$ of $A_{\infty}$-algebra as in (3.8), thus $m_1 = 0$, $m_2$ is induced by the product of $A$, and, for $n \geq 3$, we have $m_n(z_1 \otimes \cdots \otimes z_n) = 0$, whenever $z_u \in \{e_i \mid i \in \mathcal{P}\}$ for some $u \in [1, n]$.

From (3.7), we know that this is the case whenever $\mathcal{H}$ is a strict homological system. We will prove in (12.8) that this is the only possible case.

Definition 4.2. We refer to the preceding $A_{\infty}$-algebra $A$ as the Yoneda $A_{\infty}$-algebra of $\Delta$. Now, we consider the bar construction $B$ of this $A_{\infty}$-algebra. That is we consider the shifted graded $S$-$S$-bimodule $B := A[1]$ endowed with the sequence $\{b_n : B \otimes n \rightarrow B\}_{n \in \mathbb{N}}$ of $S$-$S$-bimodule morphisms given by the following commutative squares

$$
\begin{array}{ccc}
A \otimes n & \xrightarrow{\sigma \otimes n} & B \otimes n \\
\downarrow{m_n} & & \downarrow{b_n} \\
A & \xrightarrow{\sigma} & B,
\end{array}
$$

where $\sigma : A \rightarrow A[1] = B$ is the identity map, considered as a homogeneous morphism of graded $S$-$S$-bimodules of degree $-1$. Thus, each $b_n : B \otimes n \rightarrow B$ is a homogeneous morphism of degree $|b_n| = 1$, and $b_1 = 0$. Then, the family $\{b_n : B \otimes n \rightarrow B\}_{n \in \mathbb{N}}$ satisfies the basic relations

$$\sum_{r + s + t = n, r, t \geq 0; s \geq 1} b_{r+1+t}(id \otimes r \otimes b_s \otimes id \otimes t) = 0,$$

for all $n \in \mathbb{N}$.

Remark 4.3. Adopt the preceding notation.

1. The homogeneous summands of the graded $S$-$S$-bimodule $B \in \text{G}_2 \cdot S$ are

- $B_j = A_{j+1} = 0$, for $j < -1$;
- $B_{-1} = A_0 = \text{End}_\Lambda(\Delta)$;
- $B_0 = A_1 = \text{Ext}_\Lambda^1(\Delta, \Delta)$;
- $B_j = A_{j+1} = \text{Ext}_\Lambda^{j+1}(\Delta, \Delta)$, for $j \geq 1$. 

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2. For the graded dual $S\text{-}S$-bimodule $\hat{D}(B) \in G\text{p}\text{-}S$ we have:

\[
\begin{align*}
\hat{D}(B)_j &= \hat{D}(B_{-j}) = 0, \text{ for } j > 1; \\
\hat{D}(B)_1 &= \hat{D}(B_{-1}) = \hat{D}(\text{End}_A(\Delta)); \\
\hat{D}(B)_0 &= \hat{D}(B_0) = \hat{D}(\text{Ext}^1_A(\Delta, \Delta)); \\
\hat{D}(B)_{-j} &= \hat{D}(B_j) = \hat{D}(\text{Ext}^{j+1}_A(\Delta, \Delta)), \text{ for } j > 0.
\end{align*}
\]

3. For $i, j \in \mathcal{P}$, we have 

\[
e_j \cdot \hat{D}(B)_1 * e_i = e_i \text{Hom}_A(\Delta, \Delta)e_i \cong \text{Hom}_A(\Delta_i, \Delta_j),
\]

hence

\[
e_j \cdot \hat{D}(B)_1 * e_i = e_i \text{D}(B)_1 e_j = e_i \text{D}(B_{-1}) e_j \cong D(e_j B_{-1} e_i) = D\text{Hom}_A(\Delta_i, \Delta_j).
\]

Similarly, we have 

\[
e_j \cdot \hat{D}(B)_0 * e_i = e_i \text{D}(B)_0 e_j = D(e_j B_0 e_i) \cong D(\text{Ext}^1_A(\Delta_i, \Delta_j)).
\]

From the definition of homological system \((1.2)\), we have 

\[
\begin{align*}
0 \neq z \in e_j \cdot \hat{D}(B)_1 * e_i \text{ implies } s(z) &= i \leq j = t(z), \text{ and } \\
0 \neq z \in e_j \cdot \hat{D}(B)_0 * e_i \text{ implies } s(z) &= i \leq j = t(z) \text{ and } s(z) = i \neq j = t(z).
\end{align*}
\]

**Definition 4.4.** Given any $S\text{-}S$-bimodule $B$, the formal tensor series $S$-algebra 

\[
\widehat{T}_S(B)
\]

over $B$ is defined as an $S\text{-}S$-bimodule by 

\[
\widehat{T}_S(B) = \prod_{i=0}^{\infty} B^{\otimes i}.
\]

We write the elements $f \in \widehat{T}_S(B)$ as formal series $f = \sum_{i=0}^{\infty} f_i$, where $f_i \in B^{\otimes i}$. The product of the $S$-algebra $\widehat{T}_S(B)$ is given by the formula:

\[
\left( \sum_{i=0}^{\infty} f_i \right) \left( \sum_{j=0}^{\infty} g_j \right) = \sum_{i=0}^{\infty} \left( \sum_{i+j=t} f_i g_j \right),
\]

where $f_i g_j$ is the usual product in the tensor algebra $T_S(B)$.

**Remark 4.5.** The algebra defined above is indeed a unital associative $S$-algebra. Moreover:

1. If, for $0 \neq f \in \widehat{T}_S(B)$, we denote by $\nu(f)$ the minimal integer $i \geq 0$ such that $f_i \neq 0$, then the map $\nu$ determines a metric 

\[
d : \widehat{T}_S(B) \times \widehat{T}_S(B) \longrightarrow \mathbb{R}
\]

such that $d(f, g) = 2^{-\nu(f-g)}$, if $f \neq g$ and $d(f, f) = 0$. With the metric $d$, the algebra $\widehat{T}_S(B)$ becomes a topological algebra.
2. If we denote by \( \langle B \rangle \) the ideal of \( T_S(B) \) generated by \( B \), then \( \hat{T}_S(B) \) is the \( \langle B \rangle \)-adic completion of the tensor algebra \( T_S(B) \).

When \( B \) is a graded \( S \)-\( S \)-bimodule, although the algebra \( \hat{T}_S(B) \) is not in general, in a natural way, a graded \( S \)-algebra, we can introduce the following terminology.

**Definition 4.6.** Let \( B \) be a graded \( S \)-\( S \)-bimodule. Given \( n \in \mathbb{Z} \), an element \( f = \sum_{i=0}^{\infty} f_i \in \hat{T}_S(B) \) is called **homogeneous of degree** \( n \) iff \( |f_i| = n \), for all \( i \geq 0 \). Then, the degree of such an homogeneous \( f \) is denoted by \( |f| \). Clearly, the product of two homogeneous elements \( f, g \in \hat{T}_S(B) \) is again homogeneous and \( |fg| = |f| + |g| \).

A morphism of \( S \)-\( S \)-bimodules \( d : \hat{T}_S(B) \longrightarrow \hat{T}_S(B) \) will be called a **differential on** \( \hat{T}_S(B) \) if the following holds:

1. Whenever \( f \in \hat{T}_S(B) \) is homogeneous, we have that \( d(f) \) is homogeneous of degree \( |f| + 1 \).
2. Given homogeneous elements \( f, g \in \hat{T}_S(B) \), the Leibniz formula holds:
   \[
   d(fg) = d(f)g + (-1)^{|f|} fd(g).
   \]
3. We have \( d^2 = 0 \).

Let us come back now to the notation used before (4.4). We want to construct a differential on \( \hat{T}_S(D(B)) \) using the family of maps \( \{b_n : B^\otimes n \longrightarrow B\}_{n \in \mathbb{N}} \).

**Proposition 4.7.** With the preceding notation, consider for \( s \geq 1 \), the homogeneous morphism of \( S \)-\( S \)-bimodules of degree 1

\[
\hat{d}(b_s) := (\hat{D}(B) \xrightarrow{\hat{D}(b_s)} \hat{D}(B^{\otimes s}) \xrightarrow{\tau_s} \hat{D}(B)^{\otimes s}).
\]

The family \( \{\hat{d}(b_j)\}_{j \geq 1} \) determines a morphism of \( S \)-\( S \)-bimodules

\[
d_1 : \hat{D}(B) \longrightarrow \hat{T}_S(\hat{D}(B)) \text{ such that } d_1(h) = \sum_{s=1}^{\infty} \hat{d}(b_s)(h), \text{ for } h \in \hat{D}(B).
\]

For each \( j \geq 2 \), using Leibniz formula for homogeneous elements, the morphism \( d_1 \) induces a morphism of \( S \)-\( S \)-bimodules

\[
d_j : \hat{D}(B)^{\otimes j} \longrightarrow \hat{T}_S(\hat{D}(B)).
\]

Thus, \( d_j \) has the following association recipe:

\[
d_j = \sum_{r+t=j, r, t \geq 0} id^{\otimes r} \otimes d_1 \otimes id^{\otimes t} = \sum_{s=1}^{\infty} \sum_{r+t=j, r, t \geq 0} id^{\otimes r} \otimes \hat{d}(b_s) \otimes id^{\otimes t}.
\]

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More explicitly, given \( j \geq 1 \) and a homogeneous generator \( h_1 \otimes \cdots \otimes h_j \in \hat{D}(B)_{i_1} \otimes \cdots \otimes \hat{D}(B)_{i_j} \subseteq \hat{D}(B)^{\otimes j} \) of degree \( m \), with \( i_1 + \cdots + i_j = m \), the recipe for computing \( d_j(h_1 \otimes \cdots \otimes h_j) \) is

\[
\sum_{r=1}^{j} (-1)^{\sum_{s=1}^{r-1} |i_s|} h_1 \otimes \cdots \otimes h_{r-1} \otimes d_1(h_r) \otimes h_{r+1} \otimes \cdots \otimes h_j,
\]

which coincides with

\[
\sum_{s=1}^{\infty} \sum_{r=1}^{j} (-1)^{\sum_{s=1}^{r-1} |i_s|} h_1 \otimes \cdots \otimes h_{r-1} \otimes \hat{d}(b_s)(h_r) \otimes h_{r+1} \otimes \cdots \otimes h_j.
\]

The family \( \{d_j : \hat{D}(B)^{\otimes j} \longrightarrow \hat{T}_S(\hat{D}(B))\}_{j \geq 0} \), where \( d_0 = 0 : S \longrightarrow \hat{T}_S(\hat{D}(B)) \), determines a morphism of \( S \)-\( S \)-bimodules \( d : T_S(\hat{D}(B)) \longrightarrow \hat{T}_S(\hat{D}(B)) \) by the universal property of the direct sums. Then, we can extend \( d \) to a new map \( d : \hat{T}_S(\hat{D}(B)) \longrightarrow \hat{T}_S(\hat{D}(B)) \), which we denote with the same symbol \( d \), given by

\[
d(\sum_{j=0}^{\infty} f_j) = \sum_{j=0}^{\infty} d_j(f_j).
\]

Moreover, we have that \( d(S) = 0 \) and \( d \) is a differential on \( \hat{T}_S(\hat{D}(B)) \). Thus, given \( f = \sum_{j=0}^{\infty} f_j \in \hat{T}_S(\hat{D}(B)) \), the element \( d(f) = \sum_{n=0}^{\infty} d(f)_n \in \hat{D}(B) \) has components \( d(f)_n \in \hat{D}(B)^{\otimes n} \), for each \( n \geq 0 \), given by

\[
d(f)_n = \sum_{j=1}^{n} \sum_{r+1+i=j} (id^{\otimes r} \otimes \hat{d}(b_{n-j+1}) \otimes id^{\otimes i})(f_j).
\]

**Proof.** Since each \( b_i \) is homogeneous with \( |b_i| = 1 \), as remarked in (2.3), we have \( \hat{d}(b_i) = \hat{\tau}_i \hat{D}(b_i) \) is homogeneous of degree \( |\hat{\tau}_i| + |\hat{D}(b_i)| = |b_i| = 1 \). It follows that \( d(f_i) \) is homogeneous of degree \( |f_i| + 1 \), for all \( i \geq 0 \) and \( f_i \in \hat{D}(B)^{\otimes i} \) homogeneous. Hence we obtain that \( d(f) \) is homogeneous with \( |d(f)| = |f| + 1 \), for all homogeneous \( f \in \hat{T}_S(\hat{D}(B)) \).

**Step 1:** The morphism \( d \) satisfies Leibniz formula.

Assume that \( f, g \in \hat{T}_S(\hat{D}(B)) \) are homogeneous. We want to show that

\[
d(fg) = d(f)g + (-1)^{|f|} fd(g).
\]

Fix \( n \geq 0 \), and let us show that their \( n \)-components coincide

\[
d(fg)_n = [d(f)g + (-1)^{|f|} fd(g)]_n.
\]

So, we have to show that

\[
d(fg)_n = \sum_{u+v=n} d(f)_u g_v + (-1)^{|f|} f u d(g)_v.
\]
If $n = 0$, then $u = 0 = v$ and the equality is clear. So, from now on, we assume that $n \geq 1$. By definition, for $j \geq 0$, we have

$$(fg)_j = \sum_{a+c=j} f_ag_c = \sum_{a+c=j} f_a \otimes g_c + f_0g_j + f_jg_0,$$

then $d(fg)_n = S + S_0 + S^0$, where

$$S = \sum_{j=1}^n \sum_{r+1+t=j} \sum_{a+c=j} (id^{\otimes r} \otimes \hat{d}(b_{n-j+1}) \otimes id^{\otimes t})(f_a \otimes g_c)$$

and

$$S_0 = \sum_{j=1}^n \sum_{r+1+t=j} (id^{\otimes r} \otimes \hat{d}(b_{n-j+1}) \otimes id^{\otimes t})(f_0g_j)$$

where $S' = \sum_{u+v=n} d(f)u_g v + (-1)^{|f|} f_0d(g)v = S' + (-1)^{|f|} f_0d(g)_n + d(f)_n g_0$, with $\sum_{u,v \geq 1}$.

Since $id^{\otimes r} \otimes \hat{d}(b_{n-j+1}) \otimes id^{\otimes t}$ is a morphism of $S$-$S$-bimodules, we have $d(f)_n g_0 = S^0$. Similarly, we have that $(-1)^{|f|} f_0d(g)_n = S_0$: indeed, if $f_0 \neq 0$, then $|f| = |f_0| = 0$, thus we can use that $id^{\otimes r} \otimes \hat{d}(b_{n-j+1}) \otimes id^{\otimes t}$ is a morphism of $S$-$S$-bimodules; if $f_0 = 0$, the equality is clear.

So, we are reduced to prove that $S = S'$. We have $S' = S'_g + S'_f$, where

$$S'_g = \sum_{u+v=n} \sum_{j=1}^n \sum_{r+1+t=j} (id^{\otimes r_1} \otimes \hat{d}(b_{n-j_1+1}) \otimes id^{\otimes t_1})(f_{j_1}) \otimes g_v$$

and

$$S'_f = \sum_{u+v=n} \sum_{j=1}^n \sum_{r_2+1+t_2=j_2} (-1)^{|f_u|} f_u \otimes (id^{\otimes r_2} \otimes \hat{d}(b_{n-j_2+1}) \otimes id^{\otimes t_2})(g_{j_2}),$$

because $|f| = |f_u|$, for each $u \geq 1$.

Moreover, we have $S = S_g + S_f$, where

$$S_g = \sum_{j=1}^n \sum_{r+1+t=j} \sum_{a+c=j} (id^{\otimes r} \otimes \hat{d}(b_{n-j+1}) \otimes id^{\otimes t})(f_a \otimes g_c)$$

and

$$S_f = \sum_{j=1}^n \sum_{r+1+t=j} \sum_{a+c=j} (id^{\otimes r} \otimes \hat{d}(b_{n-j+1}) \otimes id^{\otimes t})(f_a \otimes g_c).$$
and
\[ S_f = \sum_{j=1}^{n} \sum_{r=1+t=j}^{n} \sum_{a, c \geq 1; a \leq r} (id^{\otimes r} \otimes \hat{d}(b_{n-j+1}) \otimes id^{\otimes t})(f_a \otimes g_c). \]

Indeed, we have \( a \leq r \) iff \( j - c = a \leq r = j - 1 - t \) iff \( c \geq t + 1 \). So, \( a \leq r \) iff \( c \leq t \). It will be enough to show that \( S_f = S'_f \) and \( S_g = S'_g \). Consider the sets of indices of the sum \( S_g \)
\[ I_g = \left\{(j, r, t, a, c) \in \mathbb{Z}^5 \mid 1 \leq j \leq n; r + 1 + t = j; a + c = j; a, c \geq 1; r, t \geq 0; c \leq t \right\}, \]
and the set of indices of the sum \( S'_g \)
\[ I'_g = \left\{(j_1, r_1, t_1, u, v) \in \mathbb{Z}^5 \mid u + v = n; 1 \leq j_1 \leq u; r_1 + 1 + t_1 = j_1; u, v \geq 1; r_1, t_1 \geq 0 \right\}. \]

Since we have
\[ S_g = \sum_{j=1}^{n} \sum_{r=1+t=j}^{n} \sum_{a, c \geq 1; a \leq r} (id^{\otimes r} \otimes \hat{d}(b_{n-j+1}) \otimes id^{\otimes t})(f_a \otimes g_c), \]
we obtain \( S_g = S'_g \), because we have the bijection \( \phi_g : I_g \longrightarrow I'_g \) of indices given by \( (j, r, t, a, c) \mapsto (j_1, r_1, t_1, u, v) = (j - c, r, t - c, n - c, c) \). Similarly, if we consider the set of indices of the sum \( S_f \)
\[ I_f = \left\{(j, r, t, a, c) \in \mathbb{Z}^5 \mid 1 \leq j \leq n; r + 1 + t = j; a + c = j; a, c \geq 1; r, t \geq 0; a \leq r \right\}, \]
and the set of indices of the sum \( S'_f \)
\[ I'_f = \left\{(j_2, r_2, t_2, u, v) \in \mathbb{Z}^5 \mid u + v = n; 1 \leq j_2 \leq v; r_2 + 1 + t_2 = j_2; u, v \geq 1; r_2, t_2 \geq 0 \right\}. \]

Since we have
\[ S_f = \sum_{j=1}^{n} \sum_{r=1+t=j}^{n} \sum_{a, c \geq 1; a \leq r} (-1)^{|f_a|} f_a \otimes (id^{\otimes (r-a)} \otimes \hat{d}(b_{n-j+1}) \otimes id^{\otimes t})(g_c), \]
we obtain \( S_f = S'_f \), because we have the bijection \( \phi_f : I_f \longrightarrow I'_f \) of indices given by \( (j, r, t, a, c) \mapsto (j_2, r_2, t_2, u, v) = (j - a, r - a, t, a, n - a) \).

**Step 2:** We have \( d^2 = 0 \).

Since \( d^2 : \hat{T}_S(\hat{D}(B)) \longrightarrow \hat{T}_S(\hat{D}(B)) \) is an \( S \)-\( S \)-bimodule morphism satisfying \( d^2(fg) = d^2(f)g + fd^2(g) \), for any homogeneous \( f, g \in \hat{T}_S(\hat{D}(B)) \), it is determined by its values on \( \hat{D}(B) \). So, it will be enough to show that its restriction to \( \hat{D}(B) \) is zero. This restriction is the composition
\[ \hat{D}(B) \xrightarrow{d_1} \hat{T}_S(\hat{D}(B)) \xrightarrow{d} \hat{T}_S(\hat{D}(B)). \]
From (2.13) and (4.2), we have

\[ dd_1 = \sum_{j=1}^{\infty} d_j \delta(b_j) \]
\[ = \sum_{j=1}^{\infty} \sum_{s=1}^{\infty} \sum_{r,t \geq 0} \tau_{r+s+t} = j(id^{\otimes r} \otimes \delta(b_s) \otimes id^{\otimes t}) \delta(b_j) \]
\[ = \sum_{j,s=1}^{\infty} \tau_{r+s+t} = j(id^{\otimes r} \otimes \delta(b_s) \otimes id^{\otimes t}) \tau_{i,j} \delta(b_j) \]
\[ = \sum_{n=1}^{\infty} \tau_{n} \delta(id^{\otimes r} \otimes b_s \otimes id^{\otimes t}) \delta(b_{r+s+t}) \]
\[ = \sum_{n=1}^{\infty} \tau_{n} \delta(id^{\otimes r} \otimes b_s \otimes id^{\otimes t}) ] = 0. \]

Lemma 4.8. Consider the quotient poset \( \mathcal{P} \) of the preordered set \( \mathcal{P} \), as in (1.11), and make \( \ell := \lceil \mathcal{P} \rceil \). Then, with the preceding notation, we have:

1. If \( 0 \neq h \) is a homogeneous element in \( (D(B))_0 \oplus \delta(B)_1 \) for some \( n \geq 0 \), then \( |h| \geq n - \ell \).

2. If \( h = \sum_{n=0}^{\infty} h_n \in \mathcal{T}_n(D(B)_0 \oplus \delta(B)_1) \) is a homogeneous element, then \( h_n = 0 \) for \( n > |h| + \ell \), thus \( h \in T_n(D(B)_0 \oplus \delta(B)_1) \).

Proof. (1): If \( r := |h| \), we can assume that

\[ h = \alpha^r_1 \otimes \cdots \otimes \alpha^r_1 \otimes x_1 \otimes \cdots \otimes x_2 \otimes \alpha^1_1 \otimes \cdots \otimes \alpha^1_1 \otimes x_1 \otimes \alpha^0_0 \otimes \cdots \otimes \alpha^0_0, \]

where \( \alpha^r_i \) are directed elements in \( D(B)_0 \) and \( x_1, \ldots, x_r \) are directed elements in \( D(B)_1 \); \( t_0, t_1, \ldots, t_r \geq 0 \) and \( r + \sum_{j=0}^{r} t_j = n \). Then, from (1.13) (3), we have in the poset \( \mathcal{P} \)

\[ s(\alpha^1_1) < \ell(\alpha^1_1) < s(\alpha^0_0) < \cdots < \ell(\alpha^0_0) = s(x_1) \leq \ell(x_1) = s(\alpha^1_1), \]
\[ s(\alpha^1_1) < \ell(\alpha^1_1) < s(\alpha^1_1) < \cdots < \ell(\alpha^1_1) = s(x_2) \leq \ell(x_2) = s(\alpha^2_2), \]
\[ \cdots \]
\[ s(\alpha^1_1) < \ell(\alpha^1_1) < s(\alpha^1_1) < \cdots < \ell(\alpha^1_1) = s(x_2) \leq \ell(x_2) = s(\alpha^2_2). \]

Thus, \( s(\alpha^1_1) < s(\alpha^0_0) < \cdots < s(\alpha^0_0) < \cdots < s(\alpha^1_1) < s(\alpha^0_0) < \cdots < s(\alpha^1_1) < s(\alpha^0_0) < \cdots < s(\alpha^1_1) < s(\alpha^0_0) < \cdots < s(\alpha^1_1) < s(\alpha^0_0). \) This implies that \( \sum_{j=0}^{r} t_j \leq \ell \). Therefore, we have \( n = r + \sum_{j=0}^{r} t_j \leq r + \ell \), and (1) follows. (2) is clearly follows from (1). \( \square \)

Remark 4.9. It is convenient to fix a special directed basis for the graded \( S \)-\( S \)-bimodules \( B \) and \( D(B) \). They are chosen as follows. For each \( t \geq 0 \) and \( i, j \in \mathcal{P} \), we choose a \( k \)-basis \( B_i(i,j) \) for the space \( e_j B_i e_i \); then, we consider the basis \( B_i = \bigcup_{t \geq 0} B_i(i,j) \) of \( B_i \); for \( t = -1 \), we have \( B_{-1} = \text{End}_A(\Delta) = S \otimes J = \bigoplus_{i \in \mathcal{P}} k e_i \otimes J \), so we choose, for any indices \( i, j \in \mathcal{P} \), a \( k \)-basis \( \mathcal{J}(i,j) \)
for \( e_j, f e_i \); then, make \( J = \bigcup_{i,j \in \mathcal{P}} J(i,j) \); for \( i \in \mathcal{P} \), we consider the \( k \)-basis \( \mathbb{B}_{-1}(i,i) = \mathbb{J}(i,i) \cup \{ e_i \} \) of \( e_i, \mathbb{B}_{-1} e_i \); then take \( \mathbb{B}_{-1} = \{ e_i \mid i \in \mathcal{P} \} \cup \mathbb{J} \). Finally, we can consider the \( k \)-basis \( \mathbb{B} = \bigcup_{j \geq 1} \mathbb{B}_j \) of \( B \).

Then, we can consider for each finite-dimensional \( B_t \), the dual basis of \( (\mathbb{B}_t)^* \) of \( \hat{D}(B_t) = \hat{D}(B)_{-t} \subseteq \hat{D}(B) \), with elements \( x^* \in \hat{D}(B_t) = \hat{D}(B)_{-t} \), and then consider the union \( \mathbb{B}^* := \bigcup_{t \leq 1}(\mathbb{B}_t)^* \). By definition, if \( x \in \mathbb{B}_t(i,j) \), the directed element \( x^* \in (\mathbb{B}^*)_t(i,j) \subseteq e_j \ast \hat{D}(B_t) \ast e_i \subseteq \hat{D}(B)_{-t} \) satisfies, for \( y \in \mathbb{B} \), that \( x^*(y) = 0 \), unless \( y \in \mathbb{B}_t(i,j) \) and, in this case, we have \( x^*(y) = \delta_{x,y} e_j = \delta_{x,y} f e_t(y) \).

For instance, we have \( e_i^* \in (\mathbb{B}^*)_1 \subseteq \hat{D}(B)_1 \subseteq \hat{D}(B) \) such that \( e_i^*(B_t) = 0 \), for \( t \neq -1 \), \( e_i^*(J) = 0 \), and \( e_i^*(e_j) = \delta_{i,j} e_j \).

Notice that each tensor power \( \hat{D}(B)^{\otimes n} \) admits as \( k \)-basis the set of elements of the form \( z = x_{i_1}^* \otimes \cdots \otimes x_{i_n}^* \otimes x_{i_1}^* \), where \( x_{i_1}, \ldots, x_{i_n} \) are directed basic elements in \( \mathbb{B}_{i_1}, \ldots, \mathbb{B}_{i_n} \), respectively, such that \( t(x_{i_1}^*) = s(x_{i_2}^*), \ldots, t(x_{i_n}^*) = s(x_{i_n}^*) \), or equivalently, such that \( t(x_{i_1}) = s(x_{i_2}), \ldots, t(x_{i_n}) = s(x_{i_n}) \). We denote this basis by \( \mathbb{T}_n \). Therefore, for \( \gamma \in \hat{D}(B)^{\otimes n} \), we can write

\[
\gamma = \sum_{z \in \mathbb{T}_n} c_z^* z, \quad \text{for some scalars } c_z^* \in k.
\]

**Lemma 4.10.** Consider the following linear spaces:

1. \( \mathcal{N} \) is the linear subspace of \( \mathcal{S}(\hat{D}(B)) \) generated by the elements \( g = \sum_{n \geq 1} g_n \), such that each \( g_n \in \hat{D}(B)_{i_1} \otimes \cdots \otimes \hat{D}(B)_{i_n} \) with \( i_n < 0 \), for some \( j \in [1, n] \);

2. \( \mathcal{E} \) is the linear subspace of \( \mathcal{S}(\hat{D}(B)_0 \oplus \hat{D}(B)_1) \subseteq \mathcal{S}(\hat{D}(B)) \) generated by the elements \( g = \sum_{n \geq 1} g_n \), such that each \( g_n \) is a sum of elements of the form \( \alpha_{i_n} \otimes \cdots \otimes \alpha_1 \) with \( \alpha_i \in \hat{D}(B)_0 \cup \hat{D}(B)_1 \) and \( \alpha_i = e_i^* \), for some \( i \in [1, n] \) and \( s \in \mathcal{P} \).

Then, we have the vector space direct sum decomposition

\[
\mathcal{S}(\hat{D}(B)) = \mathcal{S}(\hat{D}(B)_0 \oplus \hat{D}(B)) \oplus \mathcal{E} \oplus \mathcal{N}.
\]

**Proof.** We first remark that \( \mathcal{S}(\hat{D}(B)) = \mathcal{S}(\hat{D}(B)_0 \oplus \hat{D}(B)_1) \oplus \mathcal{N} \). Indeed, given \( f = \sum_{n \geq 0} f_n \in \mathcal{S}(\hat{D}(B)) \), so \( f_n \in \hat{D}(B)^{\otimes n} \), for each \( n \). Then, \( f_n = \sum_{h} h_j \) is a finite sum with \( h_j \in [\hat{D}(B)^{\otimes n}]_j = \bigoplus_{i_1 + \cdots + i_n = j} \hat{D}(B)_{i_1} \otimes \cdots \otimes \hat{D}(B)_{i_n} \), homogeneous for each \( j \). Then, \( h_j = h'_j + h''_j \), where

\[
h'_j \in \bigoplus_{i \in [1, n]} \hat{D}(B)_{i_1} \otimes \cdots \otimes \hat{D}(B)_{i_n} \subseteq \mathcal{S}(\hat{D}(B)_0 \oplus \hat{D}(B)_1)
\]

and

\[
h''_j \in \bigoplus_{i \in [1, n]} \hat{D}(B)_{i_1} \otimes \cdots \otimes \hat{D}(B)_{i_n} \subseteq \mathcal{N}.
\]
Thus, we get \( f = f' + f'' \), with \( f'_n = \sum_j h'_j \) and \( f''_n = \sum_j h''_j \), thus \( f' \in T_S(D(B)_0 \oplus \hat{D}(B)_{1}) \) and \( f'' \in \mathcal{N} \).

Now, we remark that \( T_S(D(B)_0 \oplus \hat{D}(B)_{1}) = T_S(D(B)_0 \oplus \hat{D}(J)) \oplus \mathcal{E} \). Indeed, given \( f = \sum_{n \geq 0} f_n \in T_S(D(B)_0 \oplus \hat{D}(B)_{1}) \), so \( f_n \in (D(B)_{0} \oplus \hat{D}(B)_{1})^{\otimes n} \), for each \( n \). Hence, we have a finite sum of homogeneous components \( f_n = \sum_j h_j \), with

\[
\hat{h}_j \in \bigoplus_{i_1 + \cdots + i_n = j, 0 \leq i_1, \ldots, i_n \leq 1} \hat{D}(B)_{i_1} \otimes \cdots \otimes \hat{D}(B)_{i_n}.
\]

But \( \hat{D}(B)_{1} = \bigoplus_{s \in \mathbb{P}} ke_s^{*} \oplus \hat{D}(J) \). Thus, each component decomposes as \( h_j = h'_j + h''_j \), with \( h'_j \in T_S(D(B)_0 \oplus \hat{D}(J)) \) and \( h''_j \in \mathcal{E} \).

In the following lemmas, given a homogeneous morphism \( f : M \rightarrow N \) with degree \( m \) of graded \( S\)-\( S \)-bimodules, we denote by \( f^t : M_t \rightarrow N_{t+m} \) the restriction of the morphism \( f \) to the homogeneous components of \( M \) and \( N \), thus \( f = \bigoplus_{t \in \mathbb{Z}} f^t : M \rightarrow N \).

**Lemma 4.11.** Given \( n \geq 2 \) and \( t \in \mathbb{Z} \), consider the homogeneous component of degree \( t \) of \( B^{\otimes n} \), thus, we have a finite direct sum

\[
(B^{\otimes n})_t = \bigoplus_{i_1 + \cdots + i_n = t} B_{i_1} \otimes B_{i_2} \otimes \cdots \otimes B_{i_n}.
\]

We have canonical projections \( \pi_{i_1, \ldots, i_n} : (B^{\otimes n})_t \rightarrow B_{i_1} \otimes B_{i_2} \otimes \cdots \otimes B_{i_n} \) and the injections \( \sigma_{i_1, \ldots, i_n} : B_{i_1} \otimes B_{i_2} \otimes \cdots \otimes B_{i_n} \rightarrow (B^{\otimes n})_t \). Similarly, we have the finite direct sum

\[
(\hat{D}(B)^{\otimes n})_{-t} = \bigoplus_{-i_1 - \cdots - i_n = -t} \hat{D}(B)_{-i_1} \otimes \hat{D}(B)_{-i_2} \otimes \cdots \otimes \hat{D}(B)_{-i_n}
= \bigoplus_{i_1 + \cdots + i_n = t} \hat{D}(B_{i_1}) \otimes \hat{D}(B_{i_2}) \otimes \cdots \otimes \hat{D}(B_{i_n})
\]

and the injections \( \hat{\sigma}_{i_1, \ldots, i_n} : \hat{D}(B_{i_1}) \otimes \hat{D}(B_{i_2}) \otimes \cdots \otimes \hat{D}(B_{i_n}) \rightarrow (\hat{D}(B)^{\otimes n})_{-t} \) and projections \( \hat{\tau}_{i_1, \ldots, i_n} : (\hat{D}(B)^{\otimes n})_{-t} \rightarrow \hat{D}(B_{i_1}) \otimes \hat{D}(B_{i_2}) \otimes \cdots \otimes \hat{D}(B_{i_n}) \).

Moreover, we have

\[
\hat{D}((B^{\otimes n})_t) = \bigoplus_{j_1 + \cdots + j_n = t} \hat{D}(B_{j_1} \otimes B_{j_2} \otimes \cdots \otimes B_{j_n}).
\]

Consider \( \hat{\tau}^{t}_{i_1, \ldots, i_n} := \hat{\tau}_{B_{i_1}, \ldots, B_{i_n}} \) and the restriction of \( \hat{\tau}^{t}_n : \hat{D}(B^{\otimes n}) \rightarrow \hat{D}(B)^{\otimes n} \) to the components of degree \(-t\)

\[
\hat{\tau}^{-t}_n : \hat{D}(B)^{\otimes n}_{-t} \rightarrow (\hat{D}(B)^{\otimes n})_{-t}.
\]

Then, the following squares commute

\[
\begin{array}{ccc}
\hat{D}(B_{j_1} \otimes \cdots \otimes B_{j_n}) & \xrightarrow{\hat{D}(\pi_{j_1, \ldots, j_n})} & \hat{D}(\bigoplus_{j_1 + \cdots + j_n = t} B_{j_1} \otimes \cdots \otimes B_{j_n}) \\
\hat{\tau}^{-t}_n \downarrow & & \downarrow \hat{\tau}^{-t}_n \\
\hat{D}(B_{i_1} \otimes \cdots \otimes \hat{D}(B_{i_n}) & \xrightarrow{\hat{\tau}_{i_1, \ldots, i_n}} & \bigoplus_{j_1 + \cdots + j_n = t} \hat{D}(B_{j_1}) \otimes \cdots \otimes \hat{D}(B_{j_n})
\end{array}
\]

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\[ \hat{D}(B_{i_1} \otimes \cdots \otimes B_{i_n}) \xleftarrow{\hat{D}(\sigma_{i_1, \ldots, i_n})} \hat{D}(\bigoplus_{j_1 + \cdots + j_n = t} B_{j_1} \otimes \cdots \otimes B_{j_n}) \]

Proof. In order to show the commutativity of the first square, it will be enough to show that the following square involving the inverses commutes

\[ \hat{D}(B_{i_1} \otimes \cdots \otimes \hat{D}(B_{i_n})) \xleftarrow{\hat{D}(\sigma_{i_1, \ldots, i_n})} \bigoplus_{j_1 + \cdots + j_n = t} \hat{D}(B_{j_1}) \otimes \cdots \otimes \hat{D}(B_{j_n}) \]

\[ \hat{D}(B_{i_1} \otimes \cdots \otimes B_{i_n}) \xleftarrow{\hat{D}(\sigma_{i_1, \ldots, i_n})} \bigoplus_{j_1 + \cdots + j_n = t} \hat{D}(B_{j_1}) \otimes \cdots \otimes \hat{D}(B_{j_n}) \]

\[ \bigoplus_{j_1 + \cdots + j_n = t} B_{j_1} \otimes \cdots \otimes B_{j_n} \]

For this, take a typical generator \( \alpha_{i_1} \otimes \cdots \otimes \alpha_{i_n} \in \hat{D}(B_{i_1}) \otimes \cdots \otimes \hat{D}(B_{i_n}) \) and a typical generator

\[ \sum_{j_1 + \cdots + j_n = t} a_{j_1} \otimes \cdots \otimes a_{j_n} \in \bigoplus_{j_1 + \cdots + j_n = t} B_{j_1} \otimes \cdots \otimes B_{j_n}. \]

Then, if we make \( \Delta = \hat{D}(\pi_{i_1, \ldots, i_n}) \hat{\theta}_{i_1, \ldots, i_n} (\alpha_{i_1} \otimes \cdots \otimes \alpha_{i_n}) (\sum a_{j_1} \otimes \cdots \otimes a_{j_n}) \), we get

\[ \Delta = \hat{D}(\pi_{i_1, \ldots, i_n}) \hat{\theta}_{i_1, \ldots, i_n} (\alpha_{i_1} \otimes \cdots \otimes \alpha_{i_n}) (\sum a_{j_1} \otimes \cdots \otimes a_{j_n}) \]

Recall that in the definition of \( \hat{D}(B) \), we identified \( \hat{D}(B_{i_n}) \) with \( \hat{D}(B_{i_n}) \) mapping each \( \alpha_{i_n} : B_{i_n} \longrightarrow S \) onto its extension \( \alpha_{i_n} : B \longrightarrow S \) such that \( \alpha_{i_n}(B_{j_v}) = 0 \) whenever \( i_u \neq j_v \).

The commutativity of the second square of the lemma is equivalent to the commutativity of the square

\[ \hat{D}(B_{i_1} \otimes \cdots \otimes \hat{D}(B_{i_n})) \xleftarrow{\hat{D}(\sigma_{i_1, \ldots, i_n})} \bigoplus_{j_1 + \cdots + j_n = t} \hat{D}(B_{j_1}) \otimes \cdots \otimes \hat{D}(B_{j_n}) \]

\[ \hat{D}(B_{i_1} \otimes \cdots \otimes B_{i_n}) \xleftarrow{\hat{D}(\sigma_{i_1, \ldots, i_n})} \bigoplus_{j_1 + \cdots + j_n = t} B_{j_1} \otimes \cdots \otimes B_{j_n} \]

In order to show that this square commutes, take a typical generator

\[ \sum_{j_1 + \cdots + j_n = t} \alpha_{j_1} \otimes \cdots \otimes \alpha_{j_n} \in \bigoplus_{j_1 + \cdots + j_n = t} \hat{D}(B_{j_1}) \otimes \cdots \otimes \hat{D}(B_{j_n}) \]
and a typical generator $a_{i_1} \otimes \cdots \otimes a_{i_n} \in B_{i_1} \otimes \cdots \otimes B_{i_n}$. Then, if we make
$$
\Delta' = \hat{D}(\sigma_{i_1, \ldots, i_n}) \theta_n^{-1}(\sum a_{j_1} \otimes \cdots \otimes a_{j_n})(a_{i_1} \otimes \cdots \otimes a_{i_n}),
$$
we obtain
$$
\Delta' = \hat{D}(\sigma_{i_1, \ldots, i_n}) \theta_n(\sum a_{j_1} \otimes \cdots \otimes a_{j_n}) \sigma_{i_1, \ldots, i_n}(a_{i_1} \otimes \cdots \otimes a_{i_n}) = \hat{\theta}_n(\sum a_{j_1} \otimes \cdots \otimes a_{j_n})(a_{i_1} \otimes \cdots \otimes a_{i_n}) = \hat{\theta}_{i_1, \ldots, i_n}(\sigma_{i_1, \ldots, i_n})(a_{i_1} \otimes \cdots \otimes a_{i_n}) = \hat{\theta}_{i_1, \ldots, i_n} \pi_{i_1, \ldots, i_n}(\sum a_{j_1} \otimes \cdots \otimes a_{j_n})(a_{i_1} \otimes \cdots \otimes a_{i_n}).
$$

\[ \square \]

**Lemma 4.12.** Given $n \geq 2$ and $t \in \mathbb{Z}$. With the notation of [4.11], consider the compositions

$$
b_{i_1, \ldots, i_n} = (B_{i_1} \otimes B_{i_2} \otimes \cdots \otimes B_{i_n} \xrightarrow{\sigma_{i_1, \ldots, i_n}} (B^\otimes)_{t} \xrightarrow{\hat{b}_n^{t}} B_{t+1}).
$$

Then, the components of the morphism

$$
\hat{d}(b_n) = \bigoplus_{t \in \mathbb{Z}} \hat{d}(b_n)^{t-1} : \bigoplus_{t \in \mathbb{Z}} \hat{D}(B)_{-t-1} \longrightarrow \bigoplus_{t \in \mathbb{Z}} \hat{D}(B^\otimes)^{-t-1}
$$

satisfies the following formula for $t \in \mathbb{Z}$,

$$
\hat{d}(b_n)^{t-1} = \sum_{i_1 + \cdots + i_n = t} \hat{\sigma}_{i_1, \ldots, i_n} \hat{d}(b_{i_1, \ldots, i_n}),
$$

where $\hat{d}(b_{i_1, \ldots, i_n})$ is by definition the composition $\hat{\tau}_{i_1, \ldots, i_n} \hat{D}(b_{i_1, \ldots, i_n})$.

**Proof.** We have the morphisms $b_n : B^\otimes \longrightarrow B$ and $\hat{D}(b_n) : \hat{D}(B) \longrightarrow \hat{D}(B^\otimes)$.

Therefore, we have the restriction morphism $\hat{D}(b_n)^{-t} : \hat{D}(B)_{-t} \longrightarrow \hat{D}(B^\otimes)_{-t-1}$.

We also have $\hat{\tau}_n : \hat{D}(B^\otimes)_{-t} \longrightarrow \hat{D}(B^\otimes)_{-t-1}$, so we have the restriction morphism $\hat{\tau}_n : \hat{D}(B^\otimes)_{-t} \longrightarrow \hat{D}(B^\otimes)_{-t-1}$.

Recall that $\hat{d}(b_n) = \hat{\tau}_n \hat{D}(b_n)$, hence $\hat{d}(b_n)^{-t-1} = \hat{\tau}_n^{-1} \hat{D}(b_n)^{t-1}$. We have the following diagram

\[
\begin{array}{ccc}
\hat{D}(B)_{-t-1} & \xrightarrow{\hat{D}(b_n)^{t-1}} & \hat{D}(B^\otimes)_{-t} \\
\downarrow \hat{d}(b_n)^{t-1} & & \downarrow \hat{\tau}_n^{-1} \\
\hat{D}(B^\otimes)_{-t-1} & \xrightarrow{\hat{D}(\sigma_{i_1, \ldots, i_n})^{-t}} & \hat{D}(B_{i_1} \otimes \cdots \otimes B_{i_n}) \\
\end{array}
\]

Since $id_{(B^\otimes)^t} = \sum_{j_1 + \cdots + j_n = t} \sigma_{j_1, \ldots, j_n} \pi_{j_1, \ldots, j_n}$, we get $id_{(B^\otimes)^{t-1}} = \sum_{j_1, \ldots, j_n = 1} \hat{D}(\pi_{j_1, \ldots, j_n}) \hat{D}(\sigma_{j_1, \ldots, j_n})$. Therefore, from (4.11), we obtain,

$$
\hat{d}(b_n)^{t-1} = \hat{\tau}_n^{-1} \hat{D}(b_n)^{t-1} = \sum_{j_1, \ldots, j_n = 1} \hat{\tau}_n^{-1} \hat{D}(\pi_{j_1, \ldots, j_n}) \hat{D}(\sigma_{j_1, \ldots, j_n}) \hat{D}(b_n)^{t-1} = \sum_{j_1, \ldots, j_n = 1} \hat{\sigma}_{j_1, \ldots, j_n} \hat{\tau}_{j_1, \ldots, j_n} \hat{D}(b_{j_1, \ldots, j_n}),
$$

because $\hat{D}(b_n^{t}) = \hat{D}(b_n)^{t-1}$. \[ \square \]
Proposition 4.13. For any \( n \geq 2 \), consider the restriction
\[
b_{n,0} = (B_0^{\otimes n}) \subseteq (B^{\otimes n})_0 \xrightarrow{b_n} B_1 \]
of the morphism \( b_n : B^{\otimes n} \longrightarrow B \). So, \( b_{n,0} = b_0^{i_1,\ldots,i_n} \), where \( i_1 = \cdots = i_n = 0 \) and, by definition, we have \( \hat{d}(b_{n,0}) = \hat{\tau}_{i_1,\ldots,i_n} \hat{D}(b_{n,0}) : \hat{D}(B_1) \longrightarrow \hat{D}(B_0)^{\otimes n} \).
Then, we have a morphism of \( S-S \)-bimodules
\[
\beta : \hat{D}(B_1) \longrightarrow T_S(\hat{D}(B)_0) \subseteq \hat{T}_S(\hat{D}(B)),
\]
defined by \( \beta(x) = \sum_{n \geq 2} \hat{d}(b_{n,0})(x) \), for \( x \in \hat{D}(B_1) \). Moreover, for \( x \in \hat{D}(B_1) \), we have
\[
d(x) = \beta(x) + h(x), \text{ with } h(x) \in N.
\]
Proof. For each \( x \in \hat{D}(B_1) \), we have that \( \hat{d}(b_{n,0})(x) \in \hat{D}(B_0)^{\otimes n} \) is homogeneous of degree 0. Moreover, from (4.13)(1), we get that \( \hat{d}(b_{n,0})(x) = 0 \), for all \( n > \ell \).
Thus we get \( \beta(x) = \sum_{n=2}^{\ell} \hat{d}(b_{n,0})(x) \in T_S(\hat{D}(B)_0) \).
From (4.12) we have
\[
d(x) = \sum_{n \geq 2} \hat{d}(b_{n,0})(x) = \sum_{n \geq 2} \sum_{i_1 + \cdots + i_n = 0} \hat{\tau}_{i_1,\ldots,i_n} \hat{d}(b_0^{i_1,\ldots,i_n})(x).
\]
Here, for each index \((i_1,\ldots,i_n)\) with \( i_1 + \cdots + i_n = 0 \), we have \( \hat{d}(b_0^{i_1,\ldots,i_n})(x) \in \hat{D}(B_{i_1}) \otimes \cdots \otimes \hat{D}(B_{i_n}) = \hat{D}(B)_{-i_1} \otimes \cdots \otimes \hat{D}(B)_{-i_n} \). If some \( i_k > 0 \), then \( \hat{d}(b_0^{i_1,\ldots,i_n})(x) \in N \). If all \( i_1,\ldots,i_n \leq 0 \), having zero sum implies that they are all zero and \( \hat{d}(b_0^{i_1,\ldots,i_n})(x) = \hat{d}(b_{n,0})(x) \). Thus, we get the desired formula. \( \square \)

Lemma 4.14. From (4.13), for \( \alpha \in \hat{D}(B) \) and \( n \geq 2 \), we have
\[
\hat{d}(b_{n,0})(\alpha) = \sum_{z \in \mathbb{T}_n} c_\alpha^z z \in \hat{D}(B)^{\otimes n}, \text{ for some scalars } c_\alpha^z \in k,
\]
where the elements \( z \in \mathbb{T}_n \) have the form \( z = x_{i_n}^* \otimes \cdots \otimes x_{i_1}^* \), for some directed basic elements \( x_{i_1},\ldots,x_{i_n} \in \mathbb{B} \), with each \( x_{i_j} \in \mathbb{B}_{-i_j} \). Moreover, we have that
\[
et(x_{i_n}) c_\alpha^z x_{i_n}^* \otimes \cdots \otimes x_{i_1}^* = \alpha(b_n(x_{i_n} \otimes \cdots \otimes x_{i_1}))
\]
for each \( x_{i_n}^* \otimes \cdots \otimes x_{i_1}^* \in \mathbb{T}_n \).
Proof. From (4.9), we have the expression described for \( \hat{d}(b_{n,0})(\alpha) \) in the statement of our lemma. Thus, applying the inverse \( \hat{\theta} \) of \( \hat{\tau} \), we obtain
\[
\hat{D}(b_{n,0})(\alpha) = \sum_{z \in \mathbb{T}_n} \hat{\theta}(c_\alpha^z z) \in \hat{D}(B^{\otimes n}).
\]
Evaluating both sides in the preceding equality at a fixed homogeneous basic element \( x_{j_n} \otimes \cdots \otimes x_{j_1} \in B^\otimes n \), we get

\[
\alpha(b_n(x_{j_n} \otimes \cdots \otimes x_{j_1})) = \sum_z c^z_n x^*_{j_n} (x_{j_n} x^*_{j_{n-1}} \cdots x^*_{i_1} (x_{j_1}) \cdots) \\
= c^0_n x^*_{j_n} (x_{j_n} x^*_{j_{n-1}} \cdots x^*_{i_1} (x_{j_1}) \cdots) \\
= c^0_n \delta(x_{j_n}),
\]

where \( z_0 = x^*_{j_n} \otimes \cdots \otimes x^*_{j_1} \in \widehat{T}_n \). Indeed, we have \( x^*_{j_1} (x_{j_1}) = \delta_{i_1,j_1} e_{\ell(x_{j_1})} \); \( x^*_{j_2} (x_{j_2} e_{\ell(x_{j_1})}) = x^*_{j_2} (x_{j_2}) = \delta_{i_2,j_2} e_{\ell(x_{j_2})} \) and, in a finite number of steps, we have the last preceding equality.

**Proposition 4.15.** There are morphisms of \( S\)-\( S \)-bimodules

\[
\delta_0 : \hat{D}(B)_0 \longrightarrow [T_S(\hat{D}(B)_0 \oplus \hat{D}(J))]_1 \\
\delta_1 : \hat{D}(J) \longrightarrow [T_S(\hat{D}(B)_0 \oplus \hat{D}(J))]_2
\]

where \([T_S(\hat{D}(B)_0 \oplus \hat{D}(J))]_1 \) and \([T_S(\hat{D}(B)_0 \oplus \hat{D}(J))]_2 \) denote the homogeneous components of the graded tensor algebra \( T_S(\hat{D}(B)_0 \oplus \hat{D}(J)) \) of degree 1 and 2, respectively, such that the differential \( d \) of the algebra \( T_S(\hat{D}(B)) \) satisfies the following.

1. For any \( i \in \mathcal{P} \), we have
   \[
   d(c^i) = c^i \otimes e^i + h(e^i), \quad \text{with } h(e^i) \in \mathcal{N}.
   \]

2. For any directed basic element \( x \in e_j \mathbb{B}_0 e_i \), hence with \( i \neq j \), we have
   \[
   d(x^*) = e^*_j \otimes x^* - x^* \otimes e^*_i + \delta_0 (x^*) + h(x^*), \quad \text{with } h(x^*) \in \mathcal{N}.
   \]

3. For any directed basic element \( x \in e_j \mathbb{B}_1 e_i \), we have
   \[
   d(x^*) = e^*_j \otimes x^* + x^* \otimes e^*_i + \delta_1 (x^*) + h(x^*), \quad \text{with } h(x^*) \in \mathcal{N}.
   \]

**Proof.** By definition of the differential \( d \), for any homogeneous element \( \alpha \in \hat{D}(B) \), we have \( d(\alpha) = \sum_{n \geq 2} \hat{d}(b_n)(\alpha) \). If \( |\alpha| \in \{0, 1\} \), from \((4.14)\), we have

\[
\hat{d}(b_n)(\alpha) = \sum_{z \in \widehat{T}_n} c^z_n z.
\]

Let \( \widehat{T}_n(\alpha) \) be the subset of \( \widehat{T}_n \) formed by the basic elements \( z = x^*_{i_n} \otimes \cdots \otimes x^*_{i_1} \in \widehat{T}_n \) with \( \sum_{n=1}^n |x^*_{i_n}| = |x^*_{i_n} \otimes \cdots \otimes x^*_{i_1}| = |\alpha| + 1 \) and \( |x^*_{i_1}|, \ldots, |x^*_{i_n}| \in \{0, 1\} \). Then, we have

\[
\hat{d}(b_n)(\alpha) = g_n(\alpha) + h_n(\alpha),
\]

where \( h_n(\alpha) = \sum_{z \in \widehat{T}_n(\alpha)} c^z_n z \in \mathcal{N} \) and \( g_n(\alpha) = \sum_{z \in \widehat{T}_n(\alpha)} c^z_n z \). If we make \( h(\alpha) = \sum_{n \geq 2} h_n(\alpha) \) and \( g(\alpha) = \sum_{n \geq 2} g_n(\alpha) \), we get \( h(\alpha) \in \mathcal{N} \) and we are
reduced to the computation of }g(α) = d(α) - h(α),\text{ that is of }g_n(α),\text{ for }n \geq 2,\text{ in each one of the items of the proposition.}

(1): Assume that }α = e_i^∗.

We have an element with }|e_i^∗| = 1. The subset }T_n(e_i^∗)\text{ of }T_n\text{ consists of the basic elements }z = x_{i_1}^∗ \otimes \cdots \otimes x_{i_n}^∗ \in T_n\text{ with }\sum_{s=1}^{n} |x_{i_s}^∗| = |x_{i_1}^∗ \otimes \cdots \otimes x_{i_n}^∗| = 2 \text{ and }|x_{i_1}^∗|, \ldots, |x_{i_n}^∗| \in \{0, 1\}.

For }n = 2,\text{ we want to compute }g_2(e_1^∗) = \sum_{z \in T_n(e_1^∗)} c_z^∗ z.\text{ If }z = x_{i_2}^∗ \otimes x_{i_1}^∗ \in T_2(e_1^∗),\text{ we have }|x_{i_1}^∗| = 1 = |x_{i_2}^∗|.\text{ So, }x_{i_1}, x_{i_2} \in B_{-1} = A_0 = \text{End}_A(Δ).\text{ So they belong to }\{e_j \mid j \in P\}\text{ or they belong to the radical }J.\text{ Having in mind }[4, 14]\text{ and the definition of }b_2 \text{ in }[4.2],\text{ we have}

\[e_t(x_{i_2})c_z^∗ = e_t^∗(b_2(x_{i_2} \otimes x_{i_1})) = -(-1)^{|x_{i_2}^∗|} x_{i_2}^∗ \sigma(\sigma^{-1}(x_{i_2})\sigma^{-1}(x_{i_1})).\]

The element }e_t^∗σ(\sigma^{-1}(x_{i_2})\sigma^{-1}(x_{i_1}))\text{ is not zero iff }x_{i_2}^∗ = e_t = x_{i_1}.\text{ Therefore, we obtain }g_2(e_1^∗) = \sum_{z \in T_n(e_1^∗)} c_z^∗ z = e_t^∗ \otimes e_i^∗.

For }n \geq 3,\text{ we want to compute }g_n(e_1^∗) = \sum_{z \in T_n(e_1^∗)} c_z^∗ z.\text{ If we consider an element }z = x_{i_1}^∗ \otimes \cdots \otimes x_{i_n}^∗ \in T_n(e_1^∗),\text{ we must have some }|x_{i_n}^∗| = 0.\text{ This implies, by }[4.3],\text{ that }s(x_{i_n}^∗) < t(x_{i_n}^∗).\text{ Then, }s(x_{i_n}^∗) < t(x_{i_n}^∗)\text{ and }s(b_n(x_{i_n} \otimes \cdots \otimes x_{i_1})) < t(b_n(x_{i_n} \otimes \cdots \otimes x_{i_1})).\text{ Hence, }e_t(x_{i_n})c_z^∗ = e_t^∗(b_n(x_{i_n} \otimes \cdots \otimes x_{i_1})) = 0,\text{ and }c_z^∗ = 0.\text{ Thus, }g_n(e_1^∗) = \sum_{z \in T_n(e_1^∗)} c_z^∗ z = 0.

As a consequence, we obtain }d(e_1^∗) - h(e_1^∗) = \sum_{n \geq 2} g_n(e_1^∗) = e_t^∗ \otimes e_i^∗.

(2): Suppose }x\text{ is a directed basic element in }e_j \mathbb{B}_0 e_i.

We know that }i \neq j,\text{ due to }[4.3](3).

Here }α = x^*\text{ has degree }|x^*| = 0\text{ and }T_n(x^*)\text{ consists of the basic elements }z = x_{i_n}^∗ \otimes \cdots \otimes x_{i_1}^∗ \in T_n\text{ with }\sum_{s=1}^{n} |x_{i_s}^∗| = |x_{i_1}^∗ \otimes \cdots \otimes x_{i_n}^∗| = 1 \text{ and }|x_{i_1}^∗|, \ldots, |x_{i_n}^∗| \in \{0, 1\}.

For }n = 2,\text{ we want to compute }g_2(x^*) = \sum_{z \in T_2(x^*)} c_z^∗ z.\text{ Given any element }z = x_{i_2}^∗ \otimes x_{i_1}^∗ \in T_2(x^*),\text{ we have}

\[e_t(x_{i_2})c_z^∗ = x^*(b_2(x_{i_2} \otimes x_{i_1})) = -(-1)^{|x_{i_2}^∗|} x^* \sigma(\sigma^{-1}(x_{i_2})\sigma^{-1}(x_{i_1}))).\]

There are the following two possibilities:

(a) If }|x_{i_2}^∗| = 1 \text{ and }|x_{i_1}^∗| = 0.\text{ Then, we have }x_{i_2} \in B_{-1} = A_0 = \text{End}_A(Δ).

Thus, if moreover, }x_{i_2} \in J\text{ we have }z = x_{i_1}^∗ \otimes x_{i_2}^∗ \in T_s(\hat{D}(B)0 \oplus \hat{D}(J)).\text{ If }x_{i_2} \notin J\text{, then }x_{i_2} = e_s,\text{ for some }s \in P.\text{ Then,}

\[e_t(x_{i_2})c_z^∗ = x^*(\sigma(\sigma^{-1}(e_s)\sigma^{-1}(x_{i_1}))).\]

As before, if this expression is not zero, we must have }x_{i_1} = x \text{ and }e_s = e_j.\text{ Therefore, }z = e_j^∗ \otimes x^* \text{ and }c_z^∗ = 1.
There are the following three possibilities:

(b) If $|x_i^*| = 0$ and $|x_i^*| = 1$. Here we have $x_i^* \in B_{-1} = A_0 = \text{End}_A(\Delta)$.

  Thus, if moreover, $x_i^* \in J$ we have $z = x_i^* \otimes x_i^* \in T_S(\hat{D}(B) \oplus \hat{D}(J))$. If $x_i^* \notin J$, then $x_i^* = e_s$, for some $s \in \mathcal{P}$. Then,

  \[ e_i(x_{i^*}) c_{i^*}^{x^*} = -x^* (\sigma(\sigma^{-1}(x_i^*) \sigma^{-1}(e_s))). \]

  As before, if this element is not zero, we must have $x_{i^*} = x$ and $e_s = e_i$.

  Therefore, $z = x^* \otimes e_i^*$ and $c_{i^*}^{x^*} = -1$.

From the preceding argument, we obtain

\[ g_2(x^*) = e_j^* \otimes x^* - x^* \otimes e_i^* + \delta_0(x^*), \quad \text{with } \delta_0(x^*) \in [T_S(\hat{D}(B) \oplus \hat{D}(J))]_1. \]

For $n \geq 3$, we want to compute $g_n(x^*) = \sum_{z \in \hat{T}_n(x^*)} c_{i^*}^{x^*} z$. Here, the basic elements $z = x_{i_1}^* \otimes \cdots \otimes x_{i_n}^* \in \hat{T}_n(x^*)$ have degree $|z| = 1$, thus only one of $x_{i_1}^*, \ldots, x_{i_n}^*$ has degree 1 and the others have degree 0. If $|x_i^*| = 1$, then $x_i^* = e_r$, for some $r \in \mathcal{P}$, or $x_i^* \in J$. The first possibility implies that we have $b_n(x_{i_1} \otimes \cdots \otimes x_{i_n}) = 0$, see (1.1); therefore, $c_{i^*}^{x^*} = 0$. Thus, if $c_{i^*}^{x^*} \neq 0$, we get $g_n(x^*) \in [T_S(\hat{D}(B) \oplus \hat{D}(J))]_1$. It follows that

\[ d(x^*) - h(x^*) = \sum_{n \geq 2} g_n(x) = e_j^* \otimes x^* - x^* \otimes e_i^* + \delta_0(x^*), \]

where $\delta_0(x^*) \in [T_S(\hat{D}(B) \oplus \hat{D}(J))]_1$.

(3): Take any directed basic element $x \in e_j \mathcal{P} e_i$.

Here $\alpha = x^*$ has degree $|x^*| = 1$ and the set $\hat{T}_n(x^*)$ consists of the basic elements $z = x_{i_1}^* \otimes \cdots \otimes x_{i_n}^* \in \hat{T}_n$ with $\sum_{s=1}^n |x_{i_s}^*| = |x_{i_1}^* \otimes \cdots \otimes x_{i_n}^*| = 2$ and $|x_{i_1}^*|, \ldots, |x_{i_n}^*| \in \{0, 1\}$.

For $n = 2$, we want to compute $g_2(x^*) = \sum_{z \in \hat{T}_2(x^*)} c_{i^*}^{x^*} z$. Given any element $z = x_{i^*_1} \otimes x_{i^*_2} \in \hat{T}_2(x^*)$, we have that $x_{i_1}$ and $x_{i_2}$ both have degree 1 and

\[ e_i(x_{i^*_1}) c_{i^*_2}^{x^*} = x^*(b_2(x_{i_1} \otimes x_{i_2})) = x^*(\sigma^{-1}(x_{i^*_2}) \sigma^{-1}(x_{i^*_1}))). \]

There are the following three possibilities:

(a) We have $x_{i^*_2} = e_s$, for some $s \in \mathcal{P}$. Then,

\[ e_s c_{i^*_2}^{x^*} = x^*(\sigma^{-1}(e_s) \sigma^{-1}(x_{i^*_1}))). \]

  As before, if this expression is not zero, we must have $x_{i^*_1} = x$ and $e_s = e_j$.

  Therefore, $z = e_j^* \otimes x^*$ has coefficient $c_{i^*_2}^{x^*} = 1$.

(b) We have $x_{i^*_i} = e_s$, for some $s \in \mathcal{P}$. Then,

\[ e_i(x_{i^*_2}) c_{i^*_2}^{x^*} = x^*(\sigma^{-1}(x_{i^*_2}) \sigma^{-1}(e_s))), \]

  and, if this element is not zero, we must have $x_{i^*_2} = x$ and $e_s = e_i$.

  Therefore, $z = x^* \otimes e_i^*$ has coefficient $c_{i^*_2}^{x^*} = 1$.
(c) We have \(x_{i_1}, x_{i_2} \in J\). Here, we get \(z = x_{i_1}^* \otimes x_{i_1}^* \in \hat{D}(J)^{\otimes 2}\).

From the preceding argument, we obtain
\[
g_2(x^*) = e_j^* \otimes x^* + x^* \otimes e_i + u_2(x^*), \text{ with } u_2(x^*) \in [T_S(\hat{D}(B)_0 \oplus \hat{D}(J))]_2.
\]

For \(n \geq 3\), we want to compute \(g_n(x^*) = \sum_{x \in T_n(x^*)} e_x^* z\). Here, the basic elements \(z = x_{i_1}^* \otimes \cdots \otimes x_{i_n}^* \in \hat{T}_n(x^*)\) have degree \(|z| = 2\), thus only two of \(x_{i_1}^*\), \ldots, \(x_{i_n}^*\) have degree 1, say \(x_{i_k}^*\) and \(x_{i_l}^*\) have degree 1, and the others have degree 0. From (4.1), we know that if \(e_t(x_{ii})e_x^* = x^*(b_n(x_{ii} \otimes \cdots x_{ii}))\) is not zero, we have \(x_{i_k}^*, x_{i_l}^* \in J\) and, hence, \(z \in [T_S(\hat{D}(B)_0 \oplus \hat{D}(J))]_2\). Therefore, \(g_n(x^*) \in [T_S(\hat{D}(B)_0 \oplus \hat{D}(J))]_2\).

It follows that
\[
d(x^*) - h(x^*) = \sum_{n \in \mathbb{N}} g_n(x^*) = e_j^* \otimes x^* + x^* \otimes e_i^* + \delta_1(x^*),
\]
where \(\delta_1(x^*) \in [T_S(\hat{D}(B)_0 \oplus \hat{D}(J))]_2\).

\(\square\)

**Corollary 4.16.** Consider the differential \(\delta\), with \(\delta(S) = 0\), on the graded tensor algebra \(T_S(\hat{D}(B)_0 \oplus \hat{D}(J))\) induced by the morphisms \(\delta_0\) and \(\delta_1\) of the preceding proposition, see (3(4,4)). The following holds.

1. For any \(i \in P\), we have
   \[
d(e_i^*) = e_i^* \otimes e_i^* + h(e_i^*), \text{ where } h(e_i^*) \in \mathcal{N}.
   \]

2. For any directed element \(\gamma \in e_j \ast T_S(\hat{D}(B)_0) \ast e_i \setminus S\), we have
   \[
d(\gamma) = e_j^* \otimes \gamma - \gamma \otimes e_i^* + \delta(\gamma) + h(\gamma), \text{ where } h(\gamma) \in \mathcal{N}.
   \]

3. For any directed element \(\gamma \in e_j \ast [T_S(\hat{D}(B)_0 \oplus \hat{D}(J))]_1 \ast e_i\), we have
   \[
d(\gamma) = e_j^* \otimes \gamma + \gamma \otimes e_i^* + \delta(\gamma) + h(\gamma), \text{ where } h(\gamma) \in \mathcal{N}.
   \]

**Proof.** (1): This is clear.

(2): If the statement (2) holds for \(\gamma_1 = e_j \ast \gamma_1 \ast e_i\) and \(\gamma_2 = e_i \ast \gamma_2 \ast e_j\) in \(T_S(\hat{D}(B)_0)\), we have
\[
d(\gamma_2 \otimes \gamma_1) = d(\gamma_2) \otimes \gamma_1 + \gamma_2 \otimes d(\gamma_1)
= e_i^* \otimes \gamma_2 \otimes \gamma_1 - \gamma_2 \otimes e_i^* \otimes \gamma_1 + \delta(\gamma_2) \otimes \gamma_1 + h(\gamma_2) \otimes \gamma_1
+ \gamma_2 \otimes e_i^* \otimes \gamma_1 - \gamma_2 \otimes \gamma_1 \otimes e_i^* + \gamma_2 \otimes \delta(\gamma_1) + \gamma_2 \otimes h(\gamma_1)
= e_i^* \otimes (\gamma_2 \otimes \gamma_1) - (\gamma_2 \otimes \gamma_1) \otimes e_i^* + \delta(\gamma_2 \otimes \gamma_1) + h(\gamma_2 \otimes \gamma_1),
\]
where \(h(\gamma_2 \otimes \gamma_1) = \gamma_2 \otimes h(\gamma_1) + h(\gamma_2) \otimes \gamma_1 \in \mathcal{N}\). The elements in the vector space \(e_j \ast T_S(\hat{D}(B)_0) \ast e_i \setminus S\) are linear combinations of tensors of the form \(\gamma = x_{i_1}^* \otimes \cdots \otimes x_{i_n}^*\), where \(n \geq 1\) and each \(x_{i_1}, \ldots, x_{i_n} \in \mathbb{B}_0\) are directed.
elements with distinct source and target points by (4.3). Hence, by (4.15)(2), we have \( d(x^*) = e_i^* \otimes x - x \otimes e_i^* + \delta(x^*) + h(x^*), \) with \( h(x^*) \in \mathcal{N}, \) for each \( x \in \{x_i, \ldots, x_n\}. \) Then, we can use induction and the preceding argument to obtain (2).

(3) If \( \gamma = \gamma_2 \otimes \gamma_1 \) has degree \( |\gamma| = 1, \) with \( \gamma_1 = e_j \ast \gamma_1 \ast e_i \) and \( \gamma_2 = e_i \ast \gamma_2 \ast e_j \) in \( T_S(\hat{D}(B)_0 \oplus \hat{D}(J)), \) one of them has degree 0, so item (2) holds for it and we assume that (3) holds for the other one. We have the following two cases.

Case 1: \( |\gamma_2| = 1 \) and \( |\gamma_1| = 0. \)

We have
\[
\begin{align*}
\delta(\gamma_2 \otimes \gamma_1) &= \delta(\gamma_2) \otimes \gamma_1 - \gamma_2 \otimes \delta(\gamma_1) \\
&= e_i^* \otimes \gamma_2 \otimes \gamma_1 + \gamma_2 \otimes e_i^* \otimes \gamma_1 + \delta(\gamma_2) \otimes \gamma_1 + h(\gamma_2) \otimes \gamma_1 \\
&\quad - \gamma_2 \otimes e_i^* \otimes \gamma_1 + \gamma_2 \otimes \gamma_1 \otimes e_i^* - \gamma_2 \otimes \delta(\gamma_1) - \gamma_2 \otimes h(\gamma_1) \\
&= e_i^* \otimes (\gamma_2 \otimes \gamma_1) + (\gamma_2 \otimes \gamma_1) \otimes e_i^* + \delta(\gamma_2 \otimes \gamma_1) + h(\gamma_2 \otimes \gamma_1),
\end{align*}
\]

where \( h(\gamma_2 \otimes \gamma_1) = -\gamma_2 \otimes h(\gamma_1) + h(\gamma_2) \otimes \gamma_1 \in \mathcal{N}. \)

Case 2: \( |\gamma_2| = 0 \) and \( |\gamma_1| = 1. \)

We have
\[
\begin{align*}
\delta(\gamma_2 \otimes \gamma_1) &= \delta(\gamma_2) \otimes \gamma_1 + \gamma_2 \otimes \delta(\gamma_1) \\
&= e_i^* \otimes \gamma_2 \otimes \gamma_1 - \gamma_2 \otimes e_i^* \otimes \gamma_1 + \delta(\gamma_2) \otimes \gamma_1 + h(\gamma_2) \otimes \gamma_1 \\
&\quad + \gamma_2 \otimes e_i^* \otimes \gamma_1 + \gamma_2 \otimes \gamma_1 \otimes e_i^* + \gamma_2 \otimes \delta(\gamma_1) + \gamma_2 \otimes h(\gamma_1) \\
&= e_i^* \otimes (\gamma_2 \otimes \gamma_1) + (\gamma_2 \otimes \gamma_1) \otimes e_i^* + \delta(\gamma_2 \otimes \gamma_1) + h(\gamma_2 \otimes \gamma_1),
\end{align*}
\]

where \( h(\gamma_2 \otimes \gamma_1) = \gamma_2 \otimes h(\gamma_1) + h(\gamma_2) \otimes \gamma_1 \in \mathcal{N}. \)

As before, the elements in the vector space \( e_j \ast [T_S(\hat{D}(B)_0 \oplus \hat{D}(J))]_1 \ast e_i \) are linear combinations of tensors of the form \( \gamma = x_{i_n}^* \otimes \cdots \otimes x_{i_1}^*; \) \( \gamma \) has degree \( \gamma \) and, with only one exception, each \( x_{i_1}, \ldots, x_{i_n} \in \mathbb{B}_0 \) are directed elements with distinct source and target points by (4.3). If \( x = x_{i_n} \) is the exception, we have \( x_{i_n} \in e_j \ast e_i \) and, hence, by (4.15)(3), we get \( d(x^*) = e_i^* \otimes x^* + x^* \otimes e_i^* + \delta(x^*) + h(x^*), \) with \( h(x^*) \in \mathcal{N}. \) Then, we can use induction and the preceding argument to obtain (3).

\[\square\]

**Proposition 4.17.** Consider the ideal \( I \) of the algebra \( T_S(\hat{D}(B)_0) \) generated by \( \text{Im} \beta, \) where \( \beta \) is the map introduced in (4.13). Consider also the S-Ssubbimodule \( V := [T_S(\hat{D}(B)_0 \oplus \hat{D}(J))]_1 \) of \( T_S(\hat{D}(B)_0 \oplus \hat{D}(J)) \) and the differential \( \delta \) on this algebra introduced in (4.16). Then the ideal \( I \) has the following properties.

1. \( \delta(I) \subseteq IV + VI \)
2. \( \delta^2(T_S(\hat{D}(B)_0)) \subseteq IV^2 + VI^2 + V^2I, \) and
3. \( \delta^2(V) \subseteq IV^3 + VI^3 + V^2IV + V^3I. \)
Proof. Consider the linear projection
\[ \pi : \hat{T}_S(\hat{D}(B)) \longrightarrow \hat{T}_S(\hat{D}(B)_0 + \hat{D}(J)) \]
associated to the vector space decomposition of (4.10). We proceed in four steps.

**Step 1:** For any homogeneous element \( h \in \mathcal{N} \) we have
\[ \pi d(h) \in \sum_{r + s = |h| + 1} V^r IV^s. \]

We have \( h = \sum_{s \geq 1} h_s \), with \( h_s \in \hat{D}(B)^{\otimes s} \) and \( |h_s| = |h| \), for all \( s \). By definition, \( d(h) = \sum_{s \geq 1} d(h_s) \). Each \( h_s \) is a finite \( k \)-linear combination of elements of the form \( x^*_i \otimes \cdots \otimes x^*_r \), with \( x_i, \ldots, x_r \in \mathbb{B} \). Then, \( d(h_s) \) is a finite \( k \)-linear combination of elements of the form
\[ x^*_i \otimes \cdots \otimes x^*_r \otimes d(x^*_r) \otimes x^*_r \otimes \cdots \otimes x^*_s, \]
with \( r \in [1, s] \).

If \( |x_i| < -1 \), then \( |d(x^*_i)| < 0 \) and this element belongs to \( \mathcal{N} \). If \( |x_i| = -1 \), then, by (4.13), we have \( d(x^*_i) = \beta(x^*_i) + h(x^*_i) \), with \( h(x^*_i) \in \mathcal{N} \). This implies that \( \pi d(h) \) is a finite linear combination of elements with degree \( |h| + 1 \) of the form
\[ x^*_i \otimes \cdots \otimes x^*_r \otimes \beta(x^*_r) \otimes x^*_r \otimes \cdots \otimes x^*_s, \]
such that, for \( j \neq r \), we have \( |x_{ij}| \geq 0 \) and if \( |x_{ij}| = 1 \), we have \( x_{ij} \in J \).

**Step 2:** We have \( \delta(I) \subseteq VI + IV \).

By Leibniz rule, it will be enough to show that \( \delta(\text{Im} \beta) \subseteq VI + IV \). Take any \( x^* \in e_j \ast \hat{D}(B_0) \ast e_i \), then from (4.13), we have
\[ d(x^*) = \beta(x^*) + h(x^*), \quad \text{with } h(x^*) \in \mathcal{N}. \]

Then, from (4.16), we have
\[ 0 = d^2(x^*) = \beta(\beta(x^*)) + d(h(x^*)) = e_j^* \otimes \beta(x^*) - \beta(x^*) \otimes e_j^* + d(\beta(x^*)) + h(x^*). \]

Applying \( \pi \) to this equality we obtain \( 0 = \delta(\beta(x^*)) + \pi d(h(x^*)) \). Therefore, we obtain \( \delta(\beta(x^*)) = -\pi d(h(x^*)) \in VI + IV \), by the first step of this proof.

**Step 3:** We have \( \delta^2(T_S(\hat{D}(B)_0)) \subseteq IV^2 + VI^1 + V^2I \).

It will be enough to prove that \( \delta^2(\gamma) \in IV^2 + VI^1 + V^2I \), for any directed element \( \gamma \in e_j \ast \hat{D}(B)_0 \ast e_i \). From (4.10), we have
\[ d(\gamma) = e_j^* \otimes \gamma - \gamma \otimes e_j^* + h(\gamma), \quad \text{where } h(\gamma) \in \mathcal{N}. \]

Then, we get
\[ 0 = d^2(\gamma) = d(e_j^* \otimes \gamma) - d(\gamma \otimes e_j^*) + d(\delta(\gamma)) + d(h(\gamma)). \]

Using Leibniz rule on the first two summands, we obtain
\[ d(e_j^* \otimes \gamma) = d(e_j^* \otimes \gamma - e_j^* \otimes d(\gamma)) = e_j^* \otimes e_j^* \otimes \gamma + h(e_j^*) \otimes \gamma - e_j^* \otimes d(\gamma), \quad \text{and} \]
\[ 39 \]
Given any directed basic element \( x \in \mathbb{B}_0 \cup \mathbb{I} \), the value of the differential \( \delta(x^*) \) is a finite sum of the form

\[
\delta(x^*) = \sum_{n=2}^{\ell+1} \sum_{z \in \mathbb{T}_n} c_z^x z \in T_S(\hat{D}(B)_0 \oplus \hat{D}(J)), \text{ for some scalars } c_z^x \in k,
\]

where the elements \( z \in \mathbb{T}_n \) have the form \( z = x_1^* \otimes \cdots \otimes x_{i_n}^* \), for some directed basic elements \( x_1, \ldots, x_{i_n} \in \mathbb{B}_0 \cup \mathbb{I} \), and \( \ell = |\mathbb{P}| \). Moreover, we have that

\[
\epsilon_{(i(x_n))} c_{x_1^* \otimes \cdots \otimes x_{i_1}^*}^x = x^*(b_0(x_{i_n} \otimes \cdots \otimes x_1))
\]

for each \( x_1^* \otimes \cdots \otimes x_{i_n}^* \in \mathbb{T}_n \).
Proof. We have \( \delta(x^*) = \sum_{n \geq 1} \delta_n(x^*) \), where \( \delta_n(x^*) \) denotes the component of \( \delta(x^*) \) in \( (\hat{D}(B)_0 \oplus \hat{D}(J))^\otimes n \), which is zero for all \( n > |x^*| + \ell \), by (4.8). From (4.14), we also know that \( d(x^*) = \sum_{n \geq 1} \hat{d}(b_n)(x^*) \) with \( \hat{d}(b_1) = 0 \), and
\[
\hat{d}(b_n)(x^*) = \sum_{z \in \hat{T}_n} c_z^* \, z \in \hat{D}(B) \otimes^n, \tag{4.16}
\]
for some scalars \( c_z^* \in k \), where the elements \( z \in \hat{T}_n \) have the form \( z = x_{i_n}^* \otimes \cdots \otimes x_{i_1}^* \), for some directed basic elements \( x_{i_1}, \ldots, x_{i_n} \in B \). Moreover, we have that
\[
\epsilon_t(x_{i_n}^*) = x_{i_n}^*(x_{i_n}^* \otimes \cdots \otimes x_{i_1}^*) = x^*_n(x_{i_n}^* \otimes \cdots \otimes x_{i_1}^*)
\]
for each \( x_{i_n}^* \otimes \cdots \otimes x_{i_1}^* \in \hat{T}_n \).

By (4.16), the projection \( \pi : \hat{S}(\hat{D}(B)) \longrightarrow \hat{S}(\hat{D}(B)_0 \oplus \hat{D}(J)) \) maps \( \hat{d}(b_n)(x^*) \) on \( \delta_n(x^*) \) and it maps to zero every element in \( \hat{T}_n \setminus T_n \). \( \square \)

**Definition 4.19.** Consider the graded tensor algebras \( A(\Delta) := T_S(\hat{D}(B)_0) \) and \( T(\Delta) := T_S(\hat{D}(B)_0 \oplus \hat{D}(J)) \). There is a canonical isomorphism of graded \( S \)-algebras
\[
T(\Delta) \cong T_{A(\Delta)}(V(\Delta)),
\]
where \( V(\Delta) := A(\Delta) \otimes_S \hat{D}(J) \otimes_S A(\Delta) \). Consider the homogeneous ideal \( I(\Delta) \) of \( T(\Delta) \) generated by its subsets \( I \) and \( IV := IV + VI \). From (4.17)(1) and \[\]§3 we know that there is a canonical isomorphism
\[
T(\Delta)/I(\Delta) \cong T_{A(\Delta)/I}(V/I_V).\]
Moreover, from \[\]§3, we know that the differential \( \delta \) of \( T(\Delta) \), see (4.16), induces a differential \( \widetilde{\delta} \) on the quotient tensor algebra \( T(\Delta)/I(\Delta) \) with \( \widetilde{\delta}^2 = 0 \). Hence, we have a ditalgebra \( (T(\Delta)/I(\Delta), \widetilde{\delta}) \) and we can consider its category of modules as in \[\]§3. We use the notation \( A(\Delta) := (T(\Delta), \delta) \) and recall that there is an equivalence of categories
\[
(T(\Delta)/I(\Delta), \widetilde{\delta})\text{-Mod} \longrightarrow (A(\Delta), I)\text{-Mod},
\]
see (8.1). We refer to the properties relating the differential \( \delta \) with the ideal \( I \) described in (4.17), by saying that the ditalgebra \( A(\Delta) \) is **interlaced with the ideal** \( I \).

In the following sections, we review the construction of an equivalence from the category \( \mathcal{F}(\Delta) \) to the preceding module category.

### 5 \( A_\infty \)-categories and \( b \)-categories

We need to recall the notion of an \( A_\infty \)-category and a \( b \)-category, which are (in some sense equivalent) generalizations of the notion of category.
Definition 5.1. An $A_{\infty}$-category $\mathcal{A}$ consists of the following: a class of objects $\text{Ob}(\mathcal{A})$; for each pair of objects $X,Y \in \text{Ob}(\mathcal{A})$, there is a graded $k$-vector space

$$\mathcal{A}(X,Y) = \bigoplus_{s \in \mathbb{Z}} \mathcal{A}(X,Y)_s;$$

and for each finite sequence $X_0, X_1, \ldots, X_n \in \text{Ob}(\mathcal{A})$, with $n \geq 1$, there is a homogeneous morphism $m_n$ of graded $k$-vector spaces with degree $|m_n| = 2 - n$:

$$\mathcal{A}(X_{n-1}, X_n) \otimes_k \cdots \otimes_k \mathcal{A}(X_1, X_2) \otimes_k \mathcal{A}(X_0, X_1) \xrightarrow{m_n} \mathcal{A}(X_0, X_n).$$

It is required that the maps $m_n$, called the higher composition maps, satisfy the following relation for each $n \in \mathbb{N}$ and each finite sequence of objects $X_0, X_1, \ldots, X_n$ of $\mathcal{A}$:

$$S^n_\infty : \sum_{r+s+t = n \atop s \geq 1; r,t \geq 0} (-1)^{r+s} m_{r+1+t}(id \otimes id^{s+t}) = 0.$$

These are called the Stasheff identities.

Definition 5.2. A $b$-category $\mathcal{B}$ consists of the following: a class of objects $\text{Ob}(\mathcal{B})$; for each pair of objects $X,Y \in \text{Ob}(\mathcal{B})$ a graded $k$-vector space

$$\mathcal{B}(X,Y) = \bigoplus_{s \in \mathbb{Z}} \mathcal{B}(X,Y)_s;$$

and for each finite sequence $X_0, X_1, \ldots, X_n \in \text{Ob}(\mathcal{B})$, with $n \geq 1$, there is a homogeneous morphism $b_n$ of graded $k$-vector spaces with degree $|b_n| = 1$:

$$\mathcal{B}(X_{n-1}, X_n) \otimes_k \cdots \otimes_k \mathcal{B}(X_1, X_2) \otimes_k \mathcal{B}(X_0, X_1) \xrightarrow{b_n} \mathcal{B}(X_0, X_n).$$

It is required that the maps $b_n$ satisfy the following relation for each $n \in \mathbb{N}$ and each finite sequence of objects $X_0, X_1, \ldots, X_n$ of $\mathcal{B}$:

$$S^n_b : \sum_{r+s+t = n \atop s \geq 1; r,t \geq 0} b_{r+1+t}(id^{s+t} \otimes id^{s+t}) = 0.$$

These notions are related by the bar construction detailed in the following statement.

Proposition 5.3. Given an $A_{\infty}$-category $\mathcal{A}$, a $b$-category $\mathcal{B}$ associated to $\mathcal{A}$ is constructed as follows. By definition, $\text{Ob}(\mathcal{B}) = \text{Ob}(\mathcal{A})$. For $X,Y \in \text{Ob}(\mathcal{B})$, we consider $\mathcal{B}(X,Y) := \mathcal{A}(X,Y)[1]$, the shifted graded $k$-vector space of $\mathcal{A}(X,Y)$, and the canonical isomorphism $\sigma_{X,Y} : \mathcal{A}(X,Y) \xrightarrow{\cong} \mathcal{B}(X,Y)$ determined by the identity map. Given a finite sequence of objects $X_0, X_1, \ldots, X_n$ of $\mathcal{B}$, the morphism $b_n$ is defined by the commutativity of the following diagram

$$\begin{array}{ccc}
\mathcal{A}(X_{n-1}, X_n) \otimes_k \cdots \otimes_k \mathcal{A}(X_1, X_2) & \otimes_k \mathcal{A}(X_0, X_1) & \xrightarrow{m_n} \mathcal{A}(X_0, X_n) \\
\sigma_{X_{n-1}, X_n} \otimes \cdots \otimes \sigma_{X_0, X_1} & & \sigma_{X_0, X_n} \\
\mathcal{B}(X_{n-1}, X_n) \otimes_k \cdots \otimes_k \mathcal{B}(X_1, X_2) & \otimes_k \mathcal{B}(X_0, X_1) & \xrightarrow{b_n} \mathcal{B}(X_0, X_n).
\end{array}$$

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The preceding construction is reversible, given a $b$-category $\mathcal{B}$, we can construct an $A_\infty$-category with the same objects and hom graded vector spaces with higher composition maps determined by the preceding commutative square.

**Proof.** It is enough to show that for any $n \in \mathbb{N}$ and for any sequence of objects $X_0, X_1, \ldots, X_n$ we have

$$\sigma_{X_0, X_n} S^\infty_n = S_n^b(\sigma_{X_{n-1}, X_n} \otimes \cdots \otimes \sigma_{X_0, X_1}).$$

This holds because for each summand, whenever $n = r + s + t$ with $s \geq 1$ and $s, t \geq 0$, we have

$$(-1)^{r+st} \sigma_{r+1,t}(id^{\otimes r} \otimes m_s \otimes id^{\otimes t}) = b_{r+1,t}(id^{\otimes r} \otimes b_s \otimes id^{\otimes t}) \sigma^{\otimes n},$$

where we avoided subindices for notational simplicity.

The advantage of working with $b$-categories instead of $A_\infty$-categories is that we do not have to deal with the signs appearing in Stasheff identities.

**Proposition 5.4.** Consider the Yoneda algebra $A$ associated to the $\Lambda$-module $\Delta$, with a strict structure of $A_\infty$-algebra, as specified in (4.1). Then an $A_\infty$-category, denoted by $\text{ad}(A)$, is defined by the following. The objects of $\text{ad}(A)$ are the right $S$-modules; the spaces of morphisms are given by

$$\text{ad}(A)(X,Y) := \bigoplus_{i,j \in \mathcal{P}} \text{Hom}_k(Xe_i, Ye_j) \otimes_k e_j Ae_i,$$

with the canonical grading of the tensor product where $\text{Hom}_k(Xe_i, Xe_j)$ is considered as a graded vector space concentrated at degree 0; finally the higher multiplications $m_n^{\text{ad}}$ are defined, for $n \in \mathbb{N}$ and a sequence of right $S$-modules $X_0, X_1, \ldots, X_n$, on typical generators by

$$\text{ad}(A)(X_{n-1}, X_n) \otimes_k \cdots \otimes_k \text{ad}(A)(X_1, X_2) \otimes_k \text{ad}(A)(X_0, X_1) \xrightarrow{m_n^{\text{ad}}} \text{ad}(A)(X_0, X_n)$$

$$(f_n \otimes a_n) \otimes \cdots \otimes (f_2 \otimes a_2) \otimes (f_1 \otimes a_1) \mapsto f_n \cdots f_2 f_1 \otimes m_n(a_n \otimes \cdots \otimes a_1).$$

In the preceding recipe, since for given right $S$-modules $X$ and $Y$, we have

$$\text{Hom}_k(X,Y) = \bigoplus_{i,j \in \mathcal{P}} e_j \text{Hom}_k(X,Y)e_j = \bigoplus_{i,j \in \mathcal{P}} \text{Hom}_k(Xe_i, Ye_j),$$

we identify the elements of $\text{Hom}_k(Xe_i, Ye_j)$ with the corresponding elements in $\text{Hom}_k(X,Y)$, so the composition $f_n \cdots f_2 f_1$ makes sense.

The proof of this proposition follows from the next result, where we consider the bar category of $\text{ad}(A)$.

**Proposition 5.5.** Consider the Yoneda algebra $A$ associated to the $\Lambda$-module $\Delta$, with the structure of $A_\infty$-algebra specified in (4.1). Then, consider its bar construction $B$, as in (4.2). Then a $b$-category, denoted by $\text{ad}(B)$, is defined.
by the following. The objects of $\text{ad}(B)$ are the right $S$-modules; the spaces of morphisms are given by

$$\text{ad}(B)(X, Y) := \bigoplus_{i,j \in P} \text{Hom}_k(Xe_i, Ye_j) \otimes_k e_j Be_i,$$

with the canonical grading of the tensor product where $\text{Hom}_k(Xe_i, Ye_j)$ is considered as a graded vector space concentrated at degree 0; finally the morphisms $b_{n}^{ad}$ are defined, for $n \in \mathbb{N}$ and a sequence of right $S$-modules $X_0, X_1, \ldots, X_n$, on typical generators by

$$\text{ad}(B)(X_{n-1}, X_n) \otimes_k \cdots \otimes_k \text{ad}(B)(X_1, X_2) \otimes_k \text{ad}(B)(X_0, X_1) \xrightarrow{b_{n}^{ad}} \text{ad}(B)(X_0, X_n)$$

$$(f_n \otimes a_n) \otimes \cdots \otimes (f_2 \otimes a_2) \otimes (f_1 \otimes a_1) \mapsto f_n \cdots f_2 f_1 \otimes b_n(a_n \otimes \cdots \otimes a_1).$$

Moreover, the $b$-category $\text{ad}(B)$ is the bar construction of $\text{ad}(A)$.

**Proof.** For $n \in \mathbb{N}$ and a sequence of right $S$-modules $X_0, X_1, \ldots, X_n$, consider a typical homogeneous generator $(f_n \otimes a_n) \otimes \cdots \otimes (f_2 \otimes a_2) \otimes (f_1 \otimes a_1)$ in the space $\text{ad}(B)(X_{n-1}, X_n) \otimes_k \cdots \otimes_k \text{ad}(B)(X_1, X_2) \otimes_k \text{ad}(B)(X_0, X_1)$. We want to show that

$$0 = \sum_{r + s + t = n \geq 1; r, t \geq 0} b_{r+1+t}^{ad}(id \otimes r \otimes b_s \otimes id \otimes t)([f_n \otimes a_n] \otimes \cdots \otimes (f_1 \otimes a_1]).$$

But this equals

$$(f_n \cdots f_2 f_1) \otimes \sum_{r + s + t = n \geq 1; r, t \geq 0} b_{r+1+t}^{ad}(id \otimes r \otimes b_s \otimes id \otimes t)[a_n \otimes \cdots \otimes a_1] = 0.$$

Hence, we get that $\text{ad}(B)$ is a $b$-category.

If we denote by $A$ the $A_\infty$-category $\text{ad}(A)$ and by $B$ its associated $b$-category, we have, for any right $S$-modules $X$ and $Y$, that

$$\text{ad}(B)(X, Y) = \bigoplus_{i,j \in P} \text{Hom}_k(Xe_i, Ye_j) \otimes e_j Be_i = A(X, Y)[1] = B(X, Y).$$

For each $n \geq 1$, we have $b_n^B = \sigma m_n^{ad}(\sigma \otimes n)^{-1}$. Take a typical homogeneous generator $(f_n \otimes a_n) \otimes \cdots \otimes (f_1 \otimes a_1)$ in $\text{ad}(B)(X_{n-1}, X_n) \otimes_k \cdots \otimes_k \text{ad}(B)(X_0, X_1)$ and make $E_n = b_n^B[(f_n \otimes a_n) \otimes \cdots \otimes (f_1 \otimes a_1)]$, then

$$E_n = \sigma m_n^{ad}(\sigma \otimes n)^{-1}[(f_n \otimes a_n) \otimes \cdots \otimes (f_1 \otimes a_1)]$$

$$= (-1)^{sgn} \sigma m_n^{ad}[(f_n \otimes \sigma^{-1}(a_n)) \otimes \cdots \otimes (f_1 \otimes \sigma^{-1}(a_1))],$$

$$= (-1)^{sgn}[f_n \cdots f_2 f_1 \otimes m_n(\sigma^{-1}(a_n) \otimes \cdots \otimes \sigma^{-1}(a_1))],$$

$$= f_n \cdots f_2 f_1 \otimes \sigma m_n(\sigma \otimes n)^{-1}(a_n \otimes \cdots \otimes a_1)$$

$$= f_n \cdots f_2 f_1 \otimes b_n(a_n \otimes \cdots \otimes a_1)$$

$$= b_n^{ad}[(f_n \otimes a_n) \otimes \cdots \otimes (f_1 \otimes a_1)],$$

where the number $sgn$ is determined by the degrees of the homogeneous elements $a_1, \ldots, a_n$. Thus, we get $b_n^B = b_n^{ad}$, for each $n \geq 1$. So $\text{ad}(B)$ is the $b$-category of the $A_\infty$-category $\text{ad}(A)$. 

\[\square\]
Definition 5.6. Given an $A_\infty$-category $\mathcal{A}$, for each pair of objects $X, Y$ of $\mathcal{A}$, the homogeneous map $m_1 : \mathcal{A}(X, Y) \longrightarrow \mathcal{A}(X, Y)$ has degree $1$ and satisfies $m_1^2 = 0$. Then, we have a cochain complex $\mathcal{A}(X, Y)$ with boundary operator $m_1$ and the space of $0$-cocycles $Z_0(\mathcal{A})(X, Y) \subseteq \mathcal{A}(X, Y)_0$ and $0$-coboundaries $I_0(\mathcal{A})(X, Y) \subseteq Z_0(\mathcal{A})(X, Y)$. Then, we can consider the cohomology space

$$H^0(\mathcal{A})(X, Y) = Z_0(\mathcal{A})(X, Y)/I_0(\mathcal{A})(X, Y).$$

The morphism $m_2$ induces composition maps

$$H^0(\mathcal{A})(Y, Z) \times H^0(\mathcal{A})(X, Y) \longrightarrow H^0(\mathcal{A})(X, Z),$$

If $\zeta_1 \in Z_0(\mathcal{A})(X, Y)$ and $\zeta_2 \in Z_0(\mathcal{A})(Y, Z)$, by definition, $\overline{\zeta_2 \zeta_1} = m_2(\zeta_2 \otimes \zeta_1)$. The preceding composition is well defined because $S_2^\infty = 0$, and it is associative because $S_3^\infty = 0$.

The $A_\infty$-category $\mathcal{A}$ is called homologically unitary if, for each object $X$ in $\mathcal{A}$, there is $1_X \in H^0(\mathcal{A})(X, X)$, such that $1_Y \zeta = \zeta$ and $\zeta 1_X = \zeta$, for any $\zeta \in Z_0(\mathcal{A})(X, Y)$. For such homologically unitary $A_\infty$-category $\mathcal{A}$, we can consider the category $H^0(\mathcal{A})$ with the same objects than $\mathcal{A}$ and the hom spaces $H^0(\mathcal{A})(X, Y)$, for any pair of objects $X, Y$.

Similarly, we have the following.

Definition 5.7. Given a $b$-category $\mathcal{B}$, for each pair of objects $X, Y$ of $\mathcal{B}$, the homogeneous map $b_1 : \mathcal{B}(X, Y) \longrightarrow \mathcal{B}(X, Y)$ has degree $1$ and satisfies $b_1^2 = 0$. Then, we have a cochain complex $\mathcal{B}(X, Y)$ with boundary operator $b_1$ and the space of $-1$-cocycles $Z_{-1}(\mathcal{B})(X, Y) \subseteq \mathcal{B}(X, Y)_{-1}$ and $-1$-coboundaries $I_{-1}(\mathcal{B})(X, Y)$. Then, we can consider the cohomology space

$$H^{-1}(\mathcal{B})(X, Y) = Z_{-1}(\mathcal{B})(X, Y)/I_{-1}(\mathcal{B})(X, Y).$$

The morphism $b_2$ induces composition maps

$$H^{-1}(\mathcal{B})(Y, Z) \times H^{-1}(\mathcal{B})(X, Y) \longrightarrow H^{-1}(\mathcal{B})(X, Z),$$

If $\zeta_1 \in Z_{-1}(\mathcal{B})(X, Y)$ and $\zeta_2 \in Z_{-1}(\mathcal{B})(Y, Z)$, by definition, $\overline{\zeta_2 \zeta_1} = b_2(\zeta_2 \otimes \zeta_1)$. The $b$-category $\mathcal{B}$ is called homologically unitary if, for each object $X$ in $\mathcal{B}$, there is $1_X \in H^{-1}(\mathcal{B})(X, X)$, such that $1_Y \zeta = \zeta$ and $\zeta 1_X = \zeta$, for any $\zeta \in Z_{-1}(\mathcal{B})(X, Y)$. For such an homologically unitary $b$-category $\mathcal{B}$, we can consider the category $H^{-1}(\mathcal{B})$ with the same objects than $\mathcal{B}$ and the hom spaces $H^{-1}(\mathcal{B})(X, Y)$, for any pair of objects $X, Y$.

Lemma 5.8. Let $\mathcal{A}$ be an $A_\infty$-category and $\mathcal{B}$ the $b$-category obtained from $\mathcal{A}$ by the bar construction. Then, $\mathcal{A}$ is homologically unitary iff so is $\mathcal{B}$ and we have an isomorphism of categories

$$H^0(\mathcal{A}) \simeq H^{-1}(\mathcal{B}).$$
Proof. The maps $\sigma = \sigma_{X,Y} : \mathcal{A}(X,Y) \longrightarrow \mathcal{B}(X,Y)$ are isomorphisms of complexes of degree $-1$, so they induce linear isomorphisms on the cohomologies

$$\varpi : H^0(\mathcal{A})(X,Y) \longrightarrow H^{-1}(\mathcal{B})(X,Y).$$

We consider the association rule $E : H^0(\mathcal{A}) \longrightarrow H^{-1}(\mathcal{B})$ which maps every object $X$ to the same object $X$ and the action on morphisms is given by these $\varpi$’s.

Then, for $\zeta_1 \in Z_0(\mathcal{A})(X,Y)$ and $\zeta_2 \in Z_0(\mathcal{A})(Y,Z)$, we have

$$\sigma m_2(\zeta_2 \otimes \zeta_1) = b_2 \sigma \otimes^g 2 (\zeta_2 \otimes \zeta_1) = b_2 (\sigma(\zeta_2) \otimes \sigma(\zeta_1)).$$

Taking classes in the homology and applying $\varpi$, we obtain

$$E(\zeta_1 \zeta_2) = \varpi (m_2(\zeta_2 \otimes \zeta_1)) = b_2 (\sigma(\zeta_2) \otimes \sigma(\zeta_1)) = E(\zeta_2)^{E(\zeta_1)}.$$

Thus, $E$ preserves composition.

Since $E$ is surjective on hom spaces and preserves composition, we obtain that $1_X$ is the identity of the object $X$ in $H^0(\mathcal{A})$ iff $E(1_X)$ is the identity of the object $X$ in $H^{-1}(\mathcal{B})$. Hence $E$ is an isomorphism of categories. \qed

The preceding statement allows us to work with the $b$-category $\mathcal{B}$ associated to an $A_{\infty}$-category $\mathcal{A}$, instead of working directly with $\mathcal{A}$, if we are ultimately interested in $H^0(\mathcal{A})$.

6 The $b$-category $\text{tw}(B)$ and $\mathcal{F}(\Delta)$

From now on, we retake our preceding terminology where $A$ is the Yoneda $A_{\infty}$-algebra associated to $\Delta$ and $B$ is its bar construction, as in (4.1) and (4.2).

Moreover, we keep the directed vector space basis $\mathbb{B}$ for $B$ chosen in (4.9).

Proposition 6.1. There is a $b$-category $\text{tw}(B)$ described by the following. The objects of $\text{tw}(B)$ are the pairs $\underline{X} = (X, \delta_X)$ where $X$ is a right $S$-module and $\delta_X \in \text{ad}(B)(X,X)_0$. Moreover we ask the following conditions on $\delta_X$:

1. There is a finite filtration $0 = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_{\ell(X)} = X$ of right $S$-modules such that if we express $\delta_X = \sum_{x \in \mathbb{B}} f_x \otimes x$, where the maps $f_x \in \text{Hom}_k(X,X)$ are uniquely determined, we have $f_x(X_r) \subseteq X_{r-1}$, for all $r \in [1, \ell(X)]$.

2. We have $\sum_{s=1}^{\infty} b_s^{ad} ((\delta_X)^{\otimes s}) = 0$, where we notice that the preceding condition 1 implies that $b_s^{ad} ((\delta_X)^{\otimes s}) = 0$ for $s \geq \ell(X)$, so we are dealing with a finite sum.

Given $\underline{X}, \underline{Y} \in \text{Ob}(\text{tw}(B))$, we have the hom graded $k$-vector space

$$\text{tw}(B)(\underline{X}, \underline{Y}) = \text{ad}(B)(X,Y) = \bigoplus_{i,j \in \mathcal{P}} \text{Hom}_k(X_{e_i}, Y_{e_j}) \otimes_k e_j Be_i.$$
If \( n \geq 1 \) and \( \underline{\mathbf{x}}, \mathbf{x}_1, \ldots, \mathbf{x}_n \in \text{Ob}(\text{tw}(B)) \), we have the following homogeneous linear map of degree 1

\[ \text{tw}(B)(\mathbf{x}_{n-1}, \mathbf{x}_n) \otimes_k \cdots \otimes_k \text{tw}(B)(\mathbf{x}_1, \mathbf{x}_2) \otimes_k \text{tw}(B)(\mathbf{x}_n, \mathbf{x}_1) \overset{b^w_{n}}{\longrightarrow} \text{tw}(B)(\mathbf{x}_0, \mathbf{x}_n) \]

which maps each homogeneous generator \( t_n \otimes \cdots \otimes t_2 \otimes t_1 \) on

\[
\sum_{i_0, \ldots, i_n \geq 0} b^{ad}_{i_0 + \cdots + i_n + n}(\delta^{\otimes r}_{X_n} \otimes t_n \otimes \delta^{\otimes i_{n-1}}_{X_{n-1}} \otimes t_{n-1} \otimes \cdots \otimes \delta^{\otimes i_1}_{X_1} \otimes t_1 \otimes \delta^{\otimes i_0}_{X_0}),
\]

which is a finite sum.

**Proof.** Given any object \((X, \delta_X)\) of \(\text{tw}(B)\), we can write \(\delta_X = \sum_{x_i \in \mathcal{B}} f_{x_i} \otimes x_i\), for any mute variable \(x_i\). Then,

\[
\delta^{\otimes s}_X = \sum_{x_1, \ldots, x_s \in \mathcal{B}} (f_{x_1} \otimes x_1) \otimes \cdots \otimes (f_{x_s} \otimes x_s) \otimes (f_{x_s} \otimes x_1)
\]

and

\[
b^d_s(\delta^{\otimes s}_X) = \sum_{x_1, \ldots, x_s \in \mathcal{B}} f_{x_1} \cdots f_{x_s} f_{x_1} \otimes b_s (x_s \otimes \cdots \otimes x_2 \otimes x_1).
\]

By condition (1), we see that any composition of more than \(\ell(X)\) linear operators \(f_{x_i}\) of \(X\) is zero, thus for \(s \geq \ell(X)\), we get \(f_{x_s} \cdots f_{x_2} f_{x_1} = 0\), hence \(b^d_s(\delta^{\otimes s}_X) = 0\). Hence, the sum in (2) is finite. For the same reason, the sum defining \(b^w_n(t_n \otimes \cdots \otimes t_1)\) is finite.

In order to show that

\[
\sum_{r + s + t = n \atop s \geq 1; r, t \geq 0} b^{tw}_{r+1+t}(id^{\otimes r} \otimes b^w_s \otimes id^{\otimes t})(t_n \otimes \cdots \otimes t_1) = 0,
\]

we compute each summand \(S^{tw}_{n, r, t} := b^{tw}_{r+1+t}(id^{\otimes r} \otimes b^w_s \otimes id^{\otimes t})(t_n \otimes \cdots \otimes t_1)\), where \(n = r + s + t\) with \(s \geq 1\) and \(r, t \geq 0\). We have

\[
S^{tw}_{n, r, t} = (-1)^{|r^*_1|} t^{tw}_{r+1+t}(x^r \otimes b^s_{x^s} (x^t_{2} \otimes x^t_{1})),
\]

where

\[
\tau^s_2 = t_n \otimes \cdots \otimes t_{r+1},
\]

\[
\tau^s_3 = t_n \otimes \cdots \otimes t_{r+1}.
\]

Consider sequences of non-negative integers

\[
\mathbf{j}^1_t = (j_t, \ldots, j_0) \quad \text{and} \quad \mathbf{j}^3_t = (j_t, \ldots, j_{t+s}).
\]

and make

\[
\begin{align*}
\tau^1_{j_1} &= \delta^{\otimes j_1}_{X_t} \otimes x_t \otimes \cdots \otimes x_2 \otimes \delta^{\otimes j_1}_{X_1} \otimes x_1 \otimes \delta^{\otimes j_0}_{X_0}, \\
\tau^3_{j_3} &= \delta^{\otimes j_3}_{X_n} \otimes t_n \otimes \cdots \otimes t_{s+2} \otimes \delta^{\otimes j_3}_{X_{s+1}} \otimes t_{s+1} \otimes \delta^{\otimes j_{s+2}}_{X_{s+2}}.
\end{align*}
\]
Then, we obtain
\[
S_{r,s,t}^{tw} = \sum_{j_1, j_3} (−1)^{|j_2|} b_{j_1}^{ad} b_{|j_1|+|j_3|+r+t+1}^{id} (\tilde{\gamma}_{j_3}^{j_1} \otimes b_{j_2}^{bw} (\tilde{\gamma}_{j_2}^{j_1} \otimes \tilde{\gamma}_{j_1}^{j_1})),
\]
where \(|j_1|^1\) and \(|j_3|^1\) denote the sum of their components respectively. Now, make
\[
\begin{align*}
\hat{j}_1^2 &= \{j_1^{s+1}, j_1^{s+2}, \ldots, j_1^{s+\hat{t}}\}, \\
\hat{j}_2^3 &= \delta_{X^{s+1}} \otimes t_{r+s} \otimes \delta_{X^{s+2}}^{j_1^1} \otimes \cdots \otimes t_{r+2} \otimes \delta_{X^{s+\hat{t}+1}}^{j_1^1} \otimes t_{r+1} \otimes \delta_{X^{\hat{t}+1}}^{j_1^1}.
\end{align*}
\]
Then, we have
\[
S_{r,s,t}^{tw} = \sum_{j_1, j_3} (−1)^{|j_2|} b_{j_1}^{ad} b_{|j_1|+|j_3|+r+t+1}^{id} (\tilde{\gamma}_{j_3}^{j_1} \otimes \tilde{\gamma}_{j_2}^{j_1} \otimes \tilde{\gamma}_{j_1}^{j_1}),(\tilde{\gamma}_{j_3}^{j_1} \otimes \tilde{\gamma}_{j_2}^{j_1} \otimes \tilde{\gamma}_{j_1}^{j_1})),
\]
where \(|j_3|^1\) denotes the sum of its components. Then, we have
\[
S_{r,s,t}^{tw} = \sum_{j_1, j_3} b_{j_1}^{ad} b_{|j_1|+|j_3|+r+t+1}^{id} \otimes b_{j_2}^{ad} \otimes \delta_{X^{s+1}}^{j_3^1} \otimes \delta_{X^{s+2}}^{j_1^1} \otimes \cdots \otimes \delta_{X^{s+\hat{t}+1}}^{j_1^1} \otimes \delta_{X^{\hat{t}+1}}^{j_1^1} \otimes \delta_{X^{\hat{t}+1}}^{j_1^1} \otimes \delta_{X^{0}}^{j_1^1}.
\]
Now, consider the following sequence of non-negative integers
\[
\hat{j}_1^2 \ast j_2^3 \ast j_1^1 = \{j_1^{s_1}, \ldots, j_1^{s_1+t}, j_1^{t_1}, j_1^{s_2+1}, \ldots, j_1^{s_2+t_2}, \ldots, j_1^{t_1}, j_1^{t_1-1}, \ldots, j_1^{t_1}\}.
\]
Then, we have
\[
S_{r,s,t}^{tw} = \sum_{j_1, j_3} b_{j_1}^{ad} b_{|j_1|+|j_3|+r+t+1}^{id} \otimes b_{j_2}^{ad} \otimes \delta_{X^{s+1}}^{j_1^1} \otimes \delta_{X^{s+2}}^{j_1^1} \otimes \cdots \otimes \delta_{X^{s+\hat{t}+1}}^{j_1^1} \otimes \delta_{X^{\hat{t}+1}}^{j_1^1} \otimes \delta_{X^{\hat{t}+1}}^{j_1^1} \otimes \delta_{X^{0}}^{j_1^1}.
\]
where \(\tilde{\gamma}_{j_3}^{j_1} \ast j_2^3 \ast j_1^1\) is defined using the sequence on non-negative integers \(j_1^2 \ast j_2^3 \ast j_1^1\) in a similar way as \(\tilde{\gamma}_{j_1^1} \otimes j_1^1\) is defined using the sequence \(j_1^1\).

For any sequence \(i_n = (i_n, \ldots, i_0)\) of non-negative integers and any integers \(r', s', t'\) such that \(n + |i_n| = r' + s' + t'\) with \(r', t' \geq 0\) and \(s' \geq 1\), make
\[
S_{r',s',t'}^{i_n} = \tilde{\gamma}_{X^{s+1}}^{i_n} \otimes t_{n} \otimes \cdots \otimes \delta_{X}^{i_1} \otimes t_{1} \otimes \delta_{X^{0}}^{i_0}. \quad \text{Hence, we have}
\]
\[
S_{r',s',t'}^{i_n} = \tilde{\gamma}_{X^{s+1}}^{i_n} \otimes t_{n} \otimes \cdots \otimes \delta_{X^{s+1}}^{i_1} \otimes t_{1} \otimes \delta_{X^{0}}^{i_0}.
\]
for unique integers \(r, s, t, \) such that \(r, t \geq 0, s \geq 1, \) and \(r + s + t = n, \) and unique sequences \(j_1^2, j_2^3, j_1^1\) with \(r' = |j_1^2| + r, s' = |j_2^3| + s\) and \(t' = |j_1^1| + t, \) thus \(\tilde{\gamma}_{j_3}^{j_1} \ast j_2^3 \ast j_1^1.\) In fact, we have the following identity
\[
\sum_{r+s+t=n} S_{r,s,t}^{tw} = \sum_{i_n} S_{r',s',t'}^{i_n},
\]
where \(s \geq 1; r, t \geq 0.\)
Now, since \( \text{ad}(B) \) is a \( b \)-category, for fixed \( n \) and \( i_n \), we have
\[
\sum_{r' + s' + t' = n + |i_n|} S^b_{r',s',t'} = \sum_{r' + s' + t' = n + |i_n|} b^d_{r'+1+t'}(id^\otimes r' \otimes b^d_{s'} \otimes id^\otimes t')(\hat{\nu}_n) = 0.
\]
Hence, we finally obtain that \( \sum_{r + s + t = n} S^c_{r,s,t} = 0. \)
\( \square \)

**Remark 6.2.** The \( A_\infty \)-category \( t(A) \) can be defined naturally in such a way that \( t(A) \) is the \( A_\infty \)-category of \( t(B) \). Thus, \( H^0(t(A)) \simeq H^{-1}(t(B)) \).

The following theorem of Keller and Lefèvre-Hasegawa, see [10], §7, plays an essential role in our later argumentation, which follows closely that of [9].

**Theorem 6.3.** If \( A \) is the Yoneda \( A_\infty \)-algebra associated to the \( \Lambda \)-module \( \Delta \), then there is an equivalence of categories
\[
\mathcal{F}(\Delta) \xrightarrow{\sim} H^0(t(A)).
\]

### 7 The \( b \)-category \( \text{cv}(B) \)

In this section \( A \) is the Yoneda \( A_\infty \)-algebra associated to \( \Delta \), the algebra \( B \) denotes its bar construction, and \( \hat{D}(B) \) its graded dual \( S \)-\( S \)-bimodule. The following construction, which was introduced in [9], plays an important role in the passage from \( H^{-1}(t(B)) \) to the module category of a special \( b \)-ocs, see [8].

**Proposition 7.1.** There is a \( b \)-category \( \text{cv}(B) \) described by the following. The objects of \( \text{cv}(B) \) are the left \( S \)-modules, given two left \( S \)-modules \( X \) and \( Y \), we have the graded hom space
\[
\text{cv}(B)(X,Y) = \bigoplus_{n \in \mathbb{Z}} \text{Hom}^0_{\mathcal{G}Mod-S-S}(\hat{D}(B), \text{Hom}_{k}(X,Y))
\]
\[
= \bigoplus_{n \in \mathbb{Z}} \text{Hom}_{\mathcal{G}Mod-S-S}(\hat{D}(B)^{-n}, \text{Hom}_{k}(X,Y))
\]
\[
= \bigoplus_{n \geq -1} \text{Hom}_{\mathcal{G}Mod-S-S}(\hat{D}(B)_n, \text{Hom}_{k}(X,Y)).
\]

If \( n \geq 1 \) and \( X_0, X_1, \ldots, X_n \in \text{Ob}(\text{cv}(B)) \), we have the following homogeneous linear map of degree 1
\[
\text{cv}(B)(X_{n-1}, X_n) \otimes_k \cdots \otimes_k \text{cv}(B)(X_1, X_2) \otimes_k \text{cv}(B)(X_0, X_1) \xrightarrow{d^c_n} \text{cv}(B)(X_0, X_n)
\]
which maps each homogeneous generator \( F_n \otimes \cdots \otimes F_2 \otimes F_1 \) on
\[
b^c_n(F_n \otimes \cdots \otimes F_2 \otimes F_1) = (-1)^{\lambda_n(|F_n|, \ldots, |F_1|)} \nu_n(F_n \otimes \cdots \otimes F_1) \hat{d}(b_n),
\]
here we have each \( F_i : \hat{D}(B) \longrightarrow \text{Hom}_{k}(X_{i-1}, X_i) \), so we have
\[
\hat{D}(B)^\otimes_n F_n \otimes \cdots \otimes F_1 \otimes \text{Hom}_{k}(X_{n-1}, X_n) \otimes_S \cdots \otimes_S \text{Hom}_{k}(X_0, X_1) \longrightarrow \text{Hom}_{k}(X_0, X_n),
\]
where \( \nu_n \) denotes the map induced by composition, \( \hat{d}(b_n) \) is defined in [4, 7], and \( \lambda_n \) is defined in [2, 10].
Proof. In order to show that
\[ \sum_{s \geq 1; r, t \geq 0} b^{cv}_{r+1+t}(id^{\otimes r} \otimes b^{cv}_{s} \otimes id^{\otimes t})(F_n \otimes \cdots \otimes F_1) = 0, \]
we compute each summand \( S^{cv}_{r,s,t} := b^{cv}_{r+1+t}(id^{\otimes r} \otimes b^{cv}_{s} \otimes id^{\otimes t})(F_n \otimes \cdots \otimes F_1) \), where \( n = r + s + t \) with \( s \geq 1 \) and \( t, r \geq 0 \). We proceed as we did in the proof that \( tw(B) \) is a \( b \)-category. Make
\[
\begin{align*}
\phi_1 &= F_t \otimes \cdots \otimes F_1, \\
\phi_2 &= F_{n-r} \otimes \cdots \otimes F_{t+1}, \\
\phi_3 &= F_n \otimes \cdots \otimes F_{n-r+1}.
\end{align*}
\]

Then, we have \( S^{cv}_{r,s,t} = (-1)^{|\phi_2|} b^{cv}_{r+1+t}(\phi_2 \otimes b^{cv}_{s} \otimes \phi_1) \). From the equality \( |b^{cv}_{s}(\phi_2)| = |F_{n-r}| + \cdots + |F_{t+1} - 1| \), we obtain
\[
\begin{align*}
S^{cv}_{r,s,t} &= (-1)^{|\phi_2|} \nu_{r+1+t} \nu_s \nu_t (\phi_2 \otimes b^{cv}_{s} \otimes \phi_1) \hat{d}(b_{r+1+t}),
\end{align*}
\]
where \( u_1 = \lambda_{r+1+t}(|F_n|, \ldots, |F_{n-r}|, |F_{n-r-1}| + \cdots + |F_{t+1} - 1|, |F_t|, \ldots, |F_1|) \).

Now, make \( u_2 = \lambda_s(|F_{n-r}|, \ldots, |F_{t+1}|) \), then we have
\[
\begin{align*}
S^{cv}_{r,s,t} &= (-1)^{|\phi_2| + u_1 + u_2} \nu_{r+1+t} \nu_s (\phi_2 \otimes \nu_t \phi_1) \hat{d}(b_t) \otimes \phi_1 \hat{d}(b_{r+1+t}),
\end{align*}
\]
which coincides with
\[
(-1)^u \nu_{r+1+t} (id^{\otimes r} \otimes \nu_s \otimes id^{\otimes t})(\phi_1 \otimes \nu_s \otimes \phi_1) (id^{\otimes r} \otimes \hat{d}(b_s) \otimes id^{\otimes t}) \hat{d}(b_{r+1+t}),
\]
where \( u = |\phi_2| + u_1 + u_2 + |\phi_1| \).

Notice that the map \( \lambda_n : \mathbb{Z}^n \rightarrow \mathbb{Z} \), given by \( \lambda_n(d_n, \ldots, d_1) = \sum_{n \geq i > j \geq 1} d_id_j \), always satisfies that \( \lambda_n(d_n, \ldots, d_1) \) coincides with
\[
\begin{align*}
&\lambda_{r+1+t}(d_n, \ldots, d_{n-r+1}, d_{n-r} + \cdots + d_{n-r-s+1} - 1, d_{n-r-s}, \ldots, d_1) \\
&\quad + \lambda_s(d_{n-r}, \ldots, d_{n-r-s+1} + d_n + \cdots + d_{n-r+1} + d_{n-r-s} + \cdots + d_1).
\end{align*}
\]
Therefore, we obtain that \( u = \lambda_n(|F_n|, \ldots, |F_1|) \). From \( \textbf{2.13} \), we see that the last description of \( S^{cv}_{r,s,t} \) coincides with
\[
(-1)^u \nu_{r+1+t} (id^{\otimes r} \otimes \nu_s \otimes id^{\otimes t})(\phi_1 \otimes \nu_s \otimes \phi_1) (id^{\otimes r} \otimes \hat{d}(b_s) \otimes id^{\otimes t}) \hat{d}(b_{r+1+t}),
\]
and with
\[
(-1)^u \nu_{r+1+t} (id^{\otimes r} \otimes \nu_s \otimes id^{\otimes t})(\phi_1 \otimes \phi_2 \otimes \phi_1) \hat{d}(id^{\otimes r} \otimes b_s \otimes id^{\otimes t}) \hat{d}(b_{r+1+t}).
\]
Then, we have
\[
S^{cv}_{s,t} = (-1)^u \nu_n (\phi_1 \otimes \phi_2 \otimes \phi_1) \hat{d}(id^{\otimes r} \otimes b_s \otimes id^{\otimes t}) \hat{d}(b_{r+1+t})
\]
and
\[
(-1)^u \nu_n (F_n \otimes \cdots \otimes F_1) \hat{d}(b_{r+1+t}(id^{\otimes r} \otimes b_s \otimes id^{\otimes t})).
\]
Then, adding up, we obtain
\[
\sum_{s \geq 1; r, t \geq 0} (-1)^n \nu_n (F_n \otimes \cdots \otimes F_1) \tau_n \hat{D} (b_{r+1+t} (id^{\otimes r} \otimes b_s \otimes id^{\otimes t})),
\]
where the sum at the right can be rewritten as
\[
(-1)^n \nu_n (F_n \otimes \cdots \otimes F_1) \tau_n \hat{D} \left( \sum_{s \geq 1; r, t \geq 0} b_{r+1+t} (id^{\otimes r} \otimes b_s \otimes id^{\otimes t}) \right) = 0,
\]
so \(cv(B)\) is a b-category. \(\Box\)

Now, we describe an equivalence between \(ad(B)\) and \(cv(B)\), which plays an important role in the next section.

**Proposition 7.2.** For each pair of right \(S\)-modules \(X\) and \(Y\), there is an isomorphism of graded \(k\)-vector spaces
\[
\Psi_{X,Y} : ad(B)(X,Y) \longrightarrow cv(B)(X^{op}, Y^{op}).
\]

The recipe of \(\Psi\) on each homogeneous component of degree \(n \in \mathbb{Z}\) is the following. Given
\[
f = \sum_{x \in B_n} f_x \otimes x \in ad(B)(X,Y)_n = \bigoplus_{i,j \in \mathcal{P}} \text{Hom}_k(Xe_i, Ye_j) \otimes_k e_j B_n e_i
\]
we have \(\Psi(f) \in cv(B)(X^{op}, Y^{op})_n = \text{Hom}_{S,S} (\hat{D}(B_n), \text{Hom}_k(X^{op}, Y^{op}))\), defined by
\[
\Psi(f)(\zeta) = \sum_{x \in B_n} f_x \zeta(x),
\]
where \(\zeta \in \hat{D}(B_n)\). As before, we consider \(\zeta \in \hat{D}(B_n) \subseteq \hat{D}(B)\) and \(f_x \in \text{Hom}_k(Xe_{s(x)}, Ye_{t(x)}) \subseteq \text{Hom}_k(X, Y)\). Moreover, for each sequence of right \(S\)-modules \(X_0, X_1, \ldots, X_n\), we have the following commutative diagram
\[
\text{ad}(B)(X_{n-1}, X_n) \otimes_k \cdots \otimes_k \text{ad}(B)(X_0, X_1) \xrightarrow{\psi_{X_{n-1},X_n} \cdots \otimes \psi_{X_0,X_1}} \text{ad}(B)(X_0, X_n) \\
\text{cv}(B)(X_{n-1}^{op}, X_n^{op}) \otimes_k \cdots \otimes_k \text{cv}(B)(X_0^{op}, X_1^{op}) \xrightarrow{\psi_{X_0,X_n}} \text{cv}(B)(X_0^{op}, X_n^{op}).
\]

**Proof.** Given any elements \(f = \sum_{x \in B_n} f_x \otimes x \in \text{ad}(B)(X,Y)_n\) and \(\zeta \in \hat{D}(B_n)\), the product \(f_x \zeta(x)\) is taken in the \(S\)-\(S\)-bimodule \(\text{Hom}_k(X,Y)\), but belongs to \(\text{Hom}_k(X^{op}, Y^{op}) = \text{Hom}_k(X, Y)^{op}\). In fact, we have
\[
\text{cv}(B)(X^{op}, Y^{op})_n = \text{Hom}_{S,S} (\hat{D}(B_n), \text{Hom}_k(X^{op}, Y^{op})) = \text{Hom}_{S,S} (\hat{D}(B_n), \text{Hom}_k(X, Y)^{op}) = \text{Hom}_{S,S} (\hat{D}(B_n), \text{Hom}_k(X, Y)).
\]
In order to show that $\Psi(f) \in \text{cv}(B)(X^{op}, Y^{op})_n$, we have to verify that

$$\Psi(f) \in \text{Hom}_{S,S}(D(B_n), \text{Hom}_k(X, Y)),$$

so take any $\zeta \in D(B_n) = \text{Hom}_S(sB_n, sS)$ and let us show that, for any $s, s' \in S$, we have $\Psi(f)(s\zeta s') = s[\Psi(f)(\zeta)]s'$. It is enough to verify this on any generator $f = f_x \otimes x$ of $\text{ad}(B)(X, Y)_n$. We have

$$\Psi(f_x \otimes x)(s\zeta s') = f_x[(s\zeta s')(x)] = f_x[\zeta(xs)s']$$

$$= [f_x\zeta(xs)]s' = \Psi(f_x \otimes xs)(\zeta)s'$$

$$= \Psi(sf_x \otimes x)(\zeta)s' = sf_x\zeta(x)s'$$

$$= s[\Psi(f_x \otimes x)(\zeta)]s',$$

because $x \in e_{i(x)}Be_{s(x)}$ and $f_x \in e_{s(x)}\text{Hom}_k(X, Y)e_{i(x)}$. So $\Psi_{X,Y}$ is a well defined linear map, which restricts to linear maps

$$\text{Hom}_k(Xe_i, Ye_j) \otimes_k e_jB_ne_i \xrightarrow{\Psi_{n,j}^{i}} \text{Hom}_k(D(e_jB_ne_i), \text{Hom}_k(Xe_i, Ye_j)).$$

So, in order to verify that $\Psi_{X,Y}$ is an isomorphism, we verify that each $\Psi_{n,j}^{i}$ is an isomorphism. For this notice that $D(e_jB_ne_i) \cong \text{Hom}_k(e_jB_ne_i, k) = D_k(e_jB_ne_i)$ and then we have an isomorphism

$$\text{Hom}_k(D(e_jB_ne_i), \text{Hom}_k(Xe_i, Ye_j)) \xrightarrow{\phi} \text{Hom}_k(D_k(e_jB_ne_i), \text{Hom}_k(Xe_i, Ye_j)).$$

Recall that, if $V, W$ are $k$-vector spaces, there is a natural morphism

$$\psi : W \otimes_k V \longrightarrow \text{Hom}_k(V^*, W),$$

where $V^* = \text{Hom}_k(V, k)$ and $\psi(w \otimes v)(\zeta) = w\zeta(v)$, for $w \in W$, $v \in V$, $\zeta \in V^*$. The map $\psi$ is an isomorphism when $V$ is finite-dimensional. If we take $W = \text{Hom}_k(Xe_i, Ye_j)$ and $V = e_jB_ne_i$, we have that $\phi \Psi_{n,j}^{i} = \psi$ is an isomorphism. So $\Psi_{X,Y}$ is an isomorphism.

Now, take a sequence of right $S$-modules $X_0, X_1, \ldots, X_n$. We want to show that

$$b_n^{cr}(\Psi_{X_{n-1}, X_n} \otimes \cdots \otimes \Psi_{X_0, X_1}) = \Psi_{X_0, X_n}b_n^{nd}.$$  

Consider typical homogeneous generators

$$f_n \otimes x_n \in \text{ad}(B)(X_{n-1}, X_n), \ldots, f_1 \otimes x_1 \in \text{ad}(B)(X_0, X_1),$$

$\zeta \in D(B_n)$, and let us compute

$$F_n := b_n^{cr}(\Psi_{X_{n-1}, X_n}(f_n \otimes x_n) \otimes \cdots \otimes \Psi_{X_0, X_1}(f_1 \otimes x_1))(\zeta).$$

From (4.14), we have $\hat{d}(b_n)(\zeta) = \sum_{z \in \hat{T}_n} c_z z$, where $\hat{T}_n$ consists of products of the form $z = x_n^* \otimes \cdots \otimes x_1^*$, and $c_{e_{i(x_n)}^*} = \zeta(b_n(x_{n-1} \otimes \cdots \otimes x_1))$. Since

$$F_n = (-1)^{\lambda_n([x_n, \ldots, x_1])} \nu_n(\Psi_{X_{n-1}, X_n}(f_n \otimes x_n) \otimes \cdots \otimes \Psi_{X_0, X_1}(f_1 \otimes x_1))\hat{d}(b_n)(\zeta),$$

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Proof. We have
\[ F_n = (-1)^{\lambda_n((x_n, \ldots, x_1))} \sum_z c_z \nu_n(\Psi_{X_{n-1}, X_n}(f_n \otimes x_n) \otimes \cdots \otimes \Psi_{X_0, X_1}(f_1 \otimes x_1))[z]. \]

Hence, we have
\[
F_n = \sum x_i^* \otimes \cdots \otimes x_1^* \sum \nu_s(\Psi(\delta_Y) \otimes \Psi(t) \otimes \Psi(\delta_X)^{\otimes i_0})[\delta(\alpha)].
\]

This finishes the proof. \(\square\)

**Lemma 7.3.** Assume that \(t \in \text{tw}(B)(\Sigma_X, \Sigma_Y)_{-1}\), then \(t\) is a cocycle of the cochain complex \(\text{tw}(B)(\Sigma_X, \Sigma_Y)\) if and only if, for any \(\alpha \in e_j \hat{D}(B_0)e_i\), we have
\[
\Psi(\delta_Y)(\alpha)\Psi(t)(e_i^*) = \Psi(t)(e_i^*)\Psi(\delta_X)(\alpha) + S_{\delta(\alpha)}(t),
\]
where
\[
S_{\delta(\alpha)}(t) = \sum_{s=2}^{\infty} \sum_{i_0, i_1 \geq 0} \nu_s(\Psi(\delta_Y)^{\otimes i_1} \otimes \Psi(t) \otimes \Psi(\delta_X)^{\otimes i_0})[\delta(\alpha)].
\]

**Proof.** We have \(t \in \text{ad}(B)(X, Y)_{-1}\). By definition, the \(\text{tw}(B)\)-morphism \(t\) is a \(-1\)-cocycle iff we have
\[
b_1^{tw}(t) = \sum b_{i_0, i_1}^{\otimes i_1 + 1} \Psi^i_{i_0 + i_1 + 1}(\delta_Y^{\otimes i_1} \otimes t \otimes \delta_X^{\otimes i_0}) = 0.
\]

From (7.2), the preceding formula holds iff the following holds
\[
0 = \sum_{i_0, i_1 \geq 0} \Psi_{i_0 + i_1 + 1} b_{i_0, i_1}^{\otimes i_1} \delta_Y^{\otimes i_1} \otimes t \otimes \delta_X^{\otimes i_0} = \sum_{i_0, i_1 \geq 0} \Psi_{i_0 + i_1 + 1} b_{i_0, i_1}^{\otimes i_1} \delta_Y^{\otimes i_1} \otimes t \otimes \delta_X^{\otimes i_0}.
\]

For any directed element \(\alpha \in e_j \hat{D}(B_0)e_i\), we have
\[
\sum_{i_0, i_1 \geq 0} \nu_{i_0 + i_1 + 1}(\Psi(\delta_Y)^{\otimes i_1} \otimes \Psi(t) \otimes \Psi(\delta_X)^{\otimes i_0})d(b_{i_0 + i_1 + 1})(\alpha) = 0,
\]
or, equivalently,
\[
\sum_{s=2}^{\infty} \sum_{i_0, i_1 \geq 0} \nu_s(\Psi(\delta_Y)^{\otimes i_1} \otimes \Psi(t) \otimes \Psi(\delta_X)^{\otimes i_0})d(\alpha) = 0.
\]
From (4.10), we have $d(\alpha) = e^*_j \otimes \alpha - \alpha \otimes e^*_i + \delta(\alpha) + h(\alpha)$, with $h(\alpha) \in \mathcal{N}$. Evaluating at this expression of $d(\alpha)$ the preceding equation, we obtain

$$
0 = \nu_2(\Psi(t) \otimes \Psi(\delta_X))(e^*_j \otimes \alpha) - \nu_2(\Psi(\delta_Y) \otimes \Psi(t))(\alpha \otimes e^*_i) + S_{\delta(\alpha)}(t) \\
= \nu_2(\Psi(t)(e^*_j) \otimes \Psi(\delta_X)(\alpha)) - \nu_2(\Psi(\delta_Y)(\alpha) \otimes \Psi(t)(e^*_i)) + S_{\delta(\alpha)}(t),
$$

where $S_{\delta(\alpha)}$ is defined in the statement of this lemma. Then, we get

$$
\Psi(\delta_Y)(\alpha)\Psi(t)(e^*_i) = \Psi(t)(e^*_j)\Psi(\delta_X)(\alpha) + S_{\delta(\alpha)}(t),
$$
as we wanted to show.

**Lemma 7.4.** The b-category $\text{tw}(B)$ is cohomologically unitary, as in [5.7], so $H^{-1}(\text{tw}(B)) = Z_{-1}(\text{tw}(B))$ is a category. Moreover, for each object $X$, the identity morphism $1_X$ in $Z_{-1}(\text{tw}(B))$ satisfies $\Psi(1_X)(e^*_i) = \text{id}_{X_{e_i}}$, for $i \in \mathcal{P}$, and $\Psi(1_X)(\xi) = 0$, for $\xi \in \mathcal{D}(J)$.

**Proof.** Take $X = (X, \delta_X) \in \text{Ob}(\text{tw}(B))$ and consider $\tau_X := \sum_{j \in \mathcal{P}} \text{id}_{X_{e_j}} \otimes e_j$, thus $\tau_X \in \text{tw}(B)(X, X_{-1}) = \text{ad}(B)(X, X_{-1})$. From (7.3), in order to prove that $b^{1u}_t(\tau_X) = 0$, we have to show, for $\alpha \in e_j \text{ad}(B_0)e_i$, the following equality

$$
\Psi(\delta_Y)(\alpha)\Psi(\tau_X)(e^*_i) = \Psi(\tau_X)(e^*_j)\Psi(\delta_X)(\alpha) + S_{\delta(\alpha)}(\tau_X).
$$

Recall that $\Psi(\tau_X) \in \text{Hom}_{S,S}(\mathcal{D}(B_{-1}), \text{Hom}_{k}(X_{\text{op}}, X_{\text{op}}))$. For $\xi \in \mathcal{D}(J)$, we get

$$
\Psi(\tau_X)(\xi) = \Psi(\sum_{j \in \mathcal{P}} \text{id}_{X_{e_j}} \otimes e_j)(\xi) = \sum_{j \in \mathcal{P}} \text{id}_{X_{e_j}}(\xi) = 0,
$$

and, for any $i \in \mathcal{P}$, we have $\Psi(\tau_X)(e^*_i) = \Psi(\sum_j \text{id}_{X_{e_j}} \otimes e_j)(e^*_i) = \text{id}_{X_{e_i}}$. These equalities imply the wanted equality, because, from (4.13), it follows that $S_{\delta(\alpha)}(\tau_X) = 0$. Hence, we have $\tau_X \in Z_{-1}(\text{tw}(B))(X, X)$. Now, given $t \in \text{tw}(B)(X, Y_{-1}) = \text{ad}(B)(X, Y_{-1})$, we want to show that $t\tau_X = t$ and $t\tau_X = t$ in $H^{-1}(\text{tw}(B)) = Z_{-1}(\text{tw}(B))$. That is $b^{1u}(t \otimes \tau_X) = t$ and $b^{1u}(t \otimes \tau_X) = t$. In order to prove that $t\tau_X = t$, we have to show that

$$
\sum \quad b^{ad}_{i_0 + i_1 + i_2 + 2}(\delta^{\otimes i_2}_Y \otimes t \otimes \delta^{\otimes i_1}_X \otimes \tau_X \otimes \delta^{\otimes i_0}_X) = t.
$$

Recall that, for $n \geq 3$, we have $b_n(z_n \otimes \cdots \otimes z_2 \otimes z_1) = 0$ when some factor $z_i \in \{e_j \mid j \in \mathcal{P}\}$, see (4.11). Then we are reduced to show that $b^{ad}_2(t \otimes \tau_X) = t$. The last equality holds for the typical generators $f_s \otimes x_s \in \text{ad}(B)(X, Y_{-1})$, where $x_s \in e_v B_{-1} e_u$ and $f_s \in \text{Hom}_k(X e_u, Y e_v)$, for some $u, v \in \mathcal{P}$, because

$$
b^{ad}_2((f_s \otimes x_s) \otimes \tau_X) = \sum b^{ad}_2((f_s \otimes x_s) \otimes (\text{id}_{X_{e_j}} \otimes e_j)) = \sum b^{ad}_2(f_s \text{id}_{X_{e_j}} \otimes b_2(x_s \otimes e_j) = f_s \otimes (-1)^{1+|x_s|} s \text{m}_2(\sigma^{-1}(x_s) \otimes \sigma^{-1}(e_u)) = f_s \otimes x_s e_u = f_s \otimes x_s.
$$

Hence $b^{ad}_2(t \otimes \tau_X) = t$, and $t\tau_X = t$ in $Z_{-1}(\text{tw}(B))$. Similarly, we have that $b^{ad}_2(\tau_Y \otimes t) = t$, and then $\tau_Y t = t$ in $Z_{-1}(\text{tw}(B))$. □
Lemma 8.2. We have the following equivalences.

8. The category of \((\mathcal{A}(\Delta), I)\)-modules

In this section we describe an equivalence of categories from \(H^{-1}(\text{tw}(B))\) onto \((\mathcal{A}(\Delta), I)\)-Mod. We recall from [1]§2, the following.

Definition 8.1. Assume that \(\mathcal{A} = \mathcal{A}(\Delta)\) is the weak ditalgebra associated to \(\Delta\), which is interlaced with the ideal \(I\), as in [4.19], see [9.1] and [9.2]. The category \((\mathcal{A}(\Delta), I)\)-Mod is the following. The class of objects of \((\mathcal{A}, I)\)-Mod is the class of left \(\mathcal{A}(\Delta)\)-modules \(M\) such that \(IM = 0\); given two \(\mathcal{A}(\Delta)\)-modules \(M\) and \(N\) annihilated by \(I\), the set of morphisms \(\text{Hom}_{\mathcal{A}(\Delta), I}(M, N)\) from \(M\) to \(N\) in \((\mathcal{A}, I)\)-Mod is, by definition, the collection of pairs \(f = (f^0, f^1)\), with \(f^0 \in \text{Hom}_k(M, N)\) and \(f^1 \in \text{Hom}_{\mathcal{A}(\Delta)-\mathcal{A}(\Delta)}(V, \text{Hom}_k(M, N))\), where \(V\) denotes the bimodule \(V = \mathcal{A}(\Delta) \otimes S D(J) \otimes S A(\Delta)\), such that, for any \(a \in \mathcal{A}(\Delta)\) and \(m \in M\), the following holds

\[
a f^0[m] = f^0[am] + f^1(\delta(a))[m].
\]

If \(f^1 \in \text{Hom}_{\mathcal{A}(\Delta)-\mathcal{A}(\Delta)}(V, \text{Hom}_k(M, N))\) and \(g^1 \in \text{Hom}_{\mathcal{A}(\Delta)-\mathcal{A}(\Delta)}(V, \text{Hom}_k(N, L))\), we consider the morphism \(g^1 \ast f^1 \in \text{Hom}_{\mathcal{A}(\Delta)-\mathcal{A}(\Delta)}(V \otimes^2, \text{Hom}_k(M, L))\) defined, for any \(\sum_j u_j \otimes v_j \in V \otimes_A V\), by

\[
(g^1 \ast f^1)(\sum_j u_j \otimes v_j) = \sum_j g^1(u_j)f^1(v_j) : M \longrightarrow L.
\]

Given \(f = (f^0, f^1) \in \text{Hom}_{\mathcal{A}(\Delta), I}(M, N)\) and \(g = (g^0, g^1) \in \text{Hom}_{\mathcal{A}(\Delta), I}(N, L)\), by definition \(g f := (g^0 f^0, (gf)^1) \in \text{Hom}_{\mathcal{A}(\Delta), I}(M, L)\), where

\[
(g f)^1(v) := g^0 f^1(v) + g^1(v) f^0 + (g^1 \ast f^1)(\delta(v)), \text{ for } v \in V.
\]

It is convenient to think an \(\mathcal{A}(\Delta)\)-module \(M\) as a pair \((M, \rho_M)\), where \(M\) is a left \(S\)-module and \(\rho_M : \mathcal{A}(\Delta) \longrightarrow \text{End}_k(M)\) is a morphism of \(S\)-algebras. Moreover, such \(\rho_M\) is uniquely determined by a morphism of \(S\)-\(S\)-bimodules, which we denote with the same symbol, \(\rho_M : \hat{D}(B_0) \longrightarrow \text{Hom}_k(S M, S M)\). In the next statement we reformulate the conditions defining \((\mathcal{A}, I)\)-Mod.

Lemma 8.2. We have the following equivalences.

1. An \(\mathcal{A}(\Delta)\)-module \(M\) satisfies \(IM = 0\) iff

\[
\sum_{n=2}^{\infty} \nu_n(\rho_M^\otimes) \hat{a}(b_{n,0})(\zeta) = 0, \text{ for all } \zeta \in \hat{D}(B)_{-1}.
\]

2. Given \(M, N \in (\mathcal{A}, I)\)-Mod, a pair \((f^0, f^1)\), where \(f^0 \in \text{Hom}_k(M, N)\) and \(f^1 \in \text{Hom}_{\mathcal{A}(\Delta)-\mathcal{A}(\Delta)}(V, \text{Hom}_k(M, N))\), is a morphism in \((\mathcal{A}, I)\)-Mod iff for any directed element \(\alpha \in \hat{D}(B_0)\), we have

\[
\rho_N(\alpha)f^0_s(\alpha) = f^0_t(\alpha)\rho_M(\alpha) + \sum_{n=2}^{\infty} \sum_{i_0+i_1+i_2+n} \nu_n(\rho_N^\otimes f^1 \otimes \rho_M^\otimes)(\delta(\alpha)),
\]

where \(f^0_i = e_i f^0 e_i : e_i M \longrightarrow e_i N\) denotes the restriction of \(f^0\), for \(i \in \mathcal{P}\).
3. Given morphisms \( f = (f^0, f^1) : (M, \rho_M) \longrightarrow (N, \rho_N) \) and \( g = (g^0, g^1) : (N, \rho_N) \longrightarrow (L, \rho_L) \) in \((\mathcal{A}, \mathcal{I})\)-Mod, then

\[
(gf)^1 = g^0 f^1 + g^1 f^0 - \sum_{n=2}^{\infty} \sum_{i_0 + i_1 + i_2 = n} \nu_n (\rho_L^{\otimes i_2} \otimes g^1 \otimes \rho_N^{\otimes i_1} \otimes f^1 \otimes \rho_M^{\otimes i_0}) \delta.
\]

Proof. (1): Given an \( A(\Delta) \)-module \( M \), the action of some element \( \alpha \in \hat{D}(B_0) \) on \( m \in M \) is given by \( \alpha m = \rho_M(\alpha)(m) \). For an element \( \gamma_s \in \hat{D}(B_0)^{\otimes s} \) of the form \( \gamma_s = \alpha_s \otimes \cdots \otimes \alpha_1 \), with \( \alpha_1, \ldots, \alpha_s \in \hat{D}(B_0) \), we have

\[
\gamma_s m = [\rho_M(\alpha_s) \cdots \rho_M(\alpha_1)](m) = \nu_s \rho_M^{\otimes s}(\alpha_s \cdots \otimes \alpha_1)(m).
\]

Thus, if \( \gamma = \sum_s \gamma_s \), we have \( \gamma m = \sum_s \nu_s \rho_M^{\otimes s}(\gamma_s)[m] \). Clearly, \( IM = 0 \) iff \( \beta(\zeta)M = 0 \), for all \( \zeta \in \hat{D}(B_{-1}) \). This is equivalent to \( \rho_M(\beta(\zeta)) = 0 \). Recall that \( \beta(\zeta) = \sum_{n=2}^{\infty} \delta(b_{n,0})(\zeta) \), a finite sum with \( \delta(b_{n,0})(\zeta) \in \hat{D}(B_0)^{\otimes n} \). So, \( \rho_M(\beta(\zeta)) = \sum_{n=2}^{\infty} \nu_n \rho_M^{\otimes n} \delta(b_{n,0})(\zeta) \) and (1) holds.

(2): Take \( f^0 \in \text{Hom}_k(M, N) \) and \( f^1 \in \text{Hom}_{A(\Delta)-A(\Delta)}(V, \text{Hom}_k(M, N)) \). The morphism \( f^1 \) is uniquely determined by its restriction \( f^1 : \hat{D}(J) \longrightarrow \text{Hom}_k(M, N) \), by the formula

\[
f^1(\gamma_r \otimes \xi \otimes \gamma_t) = (\nu_r \rho_N^{\otimes r} \gamma_r)f^1(\xi)(\nu_t \rho_M^{\otimes t} \gamma_t) = \nu_{r+1+t}(\rho_N^{\otimes r} \otimes f^1 \otimes \rho_M^{\otimes t})(\gamma_r \otimes \xi \otimes \gamma_t),
\]

for \( \gamma_r \in \hat{D}(B_0)^{\otimes r} \), \( \xi \in \hat{D}(J) \) and \( \gamma_t \in \hat{D}(B_0)^{\otimes t} \). The condition for \( (f^0, f^1) \) to be a morphism from \( M \) to \( N \) in \((\mathcal{A}, \mathcal{I})\)-Mod is equivalent to

\[
\alpha f^0[m] = f^0[\alpha m] + f^1(\delta(\alpha))[m],
\]

for any \( \alpha \in \hat{D}(B_0) \) and \( m \in M \). We can write \( \delta(\alpha) = \sum_n \delta(\alpha)_n \), where

\[
\delta(\alpha)_n = \sum_i \gamma_i^2 \otimes \xi_i \otimes \gamma_i^1,
\]

with \( \gamma_i^2 \in \hat{D}(B_0)^{\otimes r} \), \( \xi_i \in \hat{D}(J) \), \( \gamma_i^1 \in \hat{D}(B_0)^{\otimes t} \) satisfying that \( r + 1 + t = n \). Then, we have

\[
f^1(\delta(\alpha)) = \sum_{n=2}^{\infty} \sum_{r+1+t=n} \nu_n (\rho_N^{\otimes r} \otimes f^1 \otimes \rho_M^{\otimes t})(\delta(\alpha)).
\]

Then, (2) holds.

(3): Consider morphisms \( f = (f^0, f^1) : (M, \rho_M) \longrightarrow (N, \rho_N) \) and \( g = (g^0, g^1) : (N, \rho_N) \longrightarrow (L, \rho_L) \) in \((\mathcal{A}, \mathcal{I})\)-Mod. Given \( v \in V \), we can write \( \delta(v) = \sum_n \delta(v)_n \), where

\[
\delta(v)_n = \sum_i \gamma_i^3 \otimes \xi_i^2 \otimes \gamma_i^2 \otimes \xi_i^1 \otimes \gamma_i^1,
\]

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with $\gamma_1^2 \in \hat{D}(B_0)^{\circ^r}$, $\gamma_1^2 \in \hat{D}(B_0)^{\circ^s}$, $\gamma_1^2 \in \hat{D}(B_0)^{\circ^t}$, and $\xi_1^1, \xi_1^2 \in \hat{D}(J)$, satisfying that $r+s+t+2 = n$. Then, since the morphism $f^1 \in \text{Hom}_{\mathcal{S}_S}(\hat{D}(J), \text{Hom}_k(M, N))$ has degree 1, we obtain

$$ (g^1 \ast f^1)(\delta(v)) = -\sum_{n=2}^{\infty} \sum_{\nu=2}^{\infty} \nu_n(\rho_{\mathcal{L}}^{\circ^r} \otimes g^1 \otimes \rho_{\mathcal{N}}^{\circ^s} \otimes f^1 \otimes \rho_{\mathcal{M}}^{\circ^t})(\delta(v)). $$

Then, we get the formula in (3).

**Theorem 8.3.** Let $A$ be the Yoneda $A_{\infty}$-algebra associated to $\Delta$, $B$ its bar construction, and $\Lambda = (\Lambda(\Delta), \delta)$ the corresponding weak ditalgebra with the ideal $I$ fixed in [1.17]. Then, there is an equivalence of categories

$$ H^{-1}(\text{tw}(B)) \simeq (\Lambda, I)\text{-Mod}. $$

Since $\text{tw}(B)(X, Y)_n = \text{ad}(B)(X, Y)_n = 0$, for $n < -1$, the category $H^{-1}(\text{tw}(B))$ coincides with the category $Z_{-1}(\text{tw}(B))$ which has the same objects than $\text{tw}(B)$, and the morphisms $t : X \rightarrow Y$ of $Z_{-1}(\text{tw}(B))$ are the elements in $\text{ad}(B)(X, Y)_{-1}$ such that

$$ b^t_{i_0i_1i_2}(t) = \sum_{i_0, i_1, i_2 \geq 0} \nu_n(\delta_{\mathcal{X}}^{\circ^t} \otimes t \otimes \delta_{\mathcal{X}}^{\circ^i_0} \otimes t \otimes \delta_{\mathcal{X}}^{\circ^i_1} \otimes t \otimes \delta_{\mathcal{X}}^{\circ^i_2}). $$

The composition of two morphisms $t_1 : X_0 \rightarrow X_1$ and $t_2 : X_1 \rightarrow X_2$ in $Z_{-1}(\text{tw}(B))$ is given by

$$ t_2t_1 = b^{tw}_{i_0i_1i_2}(t_2 \otimes t_1) = \sum_{i_0, i_1, i_2 \geq 0} \nu_n(\delta_{\mathcal{X}}^{\circ^t_2} \otimes t_2 \otimes \delta_{\mathcal{X}}^{\circ^i_1} \otimes t_1 \otimes \delta_{\mathcal{X}}^{\circ^i_0}). $$

The equivalence functor $M : Z_{-1}(\text{tw}(B)) \rightarrow (\Lambda, I)\text{-Mod}$ is defined as follows. Given $X \in Z_{-1}(\text{tw}(B))$, its image is the $\Lambda(\Delta)$-module $M(X) = (X^{\circ^p}, \Psi(\delta_X))$, where $\Psi$ is the isomorphism introduced in [7.2]. Given a morphism $t : X \rightarrow Y$ in $Z_{-1}(\text{tw}(B))$, its image $M(t) = (M(t)^0, M(t)^1) : M(X) \rightarrow M(Y)$ is given by

$$ M(t)^0 = \Psi(t)(\sum_{j \in P} c_j^t) \quad \text{and} \quad M(t)^1(\xi) = \Psi(t)(\xi), \quad \text{for} \ \xi \in \hat{D}(J). $$

**Proof.** Step 1: According to [5.2], in order to show that $IM(X) = 0$, we have to prove that, for any $\xi \in \hat{D}(B)_{-1}$, we have $\sum_{s=2}^{\infty} \nu_s(\beta_M^{\circ^s})d(b_{s,0})(\zeta) = 0$. But, since $X$ is an object of $\text{tw}(B)$, it satisfies $\sum_{s=2}^{\infty} b^{\circ^t}_s(\delta_X^{\circ^s}) = 0$. From the commutativity of the square in [7.2], using [1.12] and the fact that $|\delta_X| = 0$, we obtain

$$ 0 = \sum_{s=2}^{\infty} \Psi b^{ad}(\delta_X^{\circ^s})(\zeta) = \sum_{s=1}^{\infty} b^{ad}_s(\Psi(\delta_X^{\circ^s}))(\zeta) = \sum_{s=2}^{\infty} \nu_s(\beta_M^{\circ^s})d(b_{s,0})(\zeta) = \sum_{s=2}^{\infty} \nu_s(\beta_M^{\circ^s})d(b_{s,0})(\zeta). $$

So, indeed we have $M(X) \in (\Lambda, I)\text{-Mod}$.  

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Step 2: Take a morphism $t : X \to Y$ of $Z_1(tw(B))$. From (7.3), we have

$$\Psi(\delta_Y)(\alpha)\Psi(t)(e_i^*) = \Psi(t)(e_i^*)\Psi(\delta_X)(\alpha) + S_{\delta(\alpha)}(t),$$

for any $\alpha \in e_jB_0c_i$, where

$$S_{\delta(\alpha)}(t) = \sum_{i_0, i_1 \geq 0} \sum_{s=2}^{\infty} v_s(\Psi(\delta_Y)^{\otimes i_1} \otimes \Psi(t) \otimes \Psi(\delta_X)^{\otimes i_0})(\delta(\alpha)).$$

We want to show that the pair $M(t) = (M(t)^0, M(t)^1)$ satisfies the equation of (3.2)(2), where $M(t)^1$ denotes also the extension of the morphism $M(t)^1 : \hat{D}(J) \to \text{Hom}_k(M(X), M(Y))$ to $V$.

We have that $\Psi(t) \in \text{Hom}_{\mathcal{S}_S}(\hat{D}(B_{-1}), \text{Hom}_k(X^{op}, Y^{op}))$. For each $i \in \mathcal{P}$, we have $M(t)^0_i = e_iM(t)^0e_i = e_i\Psi(t)(\sum_j e_j^*)e_i = \Psi(t)(e_i^*)$. Then, we get

$$\Psi(\delta_Y)(\alpha)M(t)^0_i = M(t)^1_i \Psi(\delta_X)(\alpha) + S_{\delta(\alpha)}(t),$$

which means that $M(t) : M(X) \to M(Y)$ is a morphism in $(A, I)$-Mod, according to (8.2)(2).

Step 3: Take morphisms $t_1 : X_0 \to X_1$ and $t_2 : X_1 \to X_2$ in $Z_1(tw(B))$, and let us show that the given association rule $M$ preserves their composition: $M(t_2t_1) = M(t_2)M(t_1)$. Applying $\Psi$ to the formula of the composition $t_2t_1$ in $Z_{-1}(tw(B))$ recalled in the statement of this theorem, we have

$$\Psi(t_2t_1) = \sum_{i_0, i_1, i_2 \geq 0} \Psi(t^{ad}_{i_0+i_1+i_2+2}(\delta^{\otimes i_2} \otimes t_2 \otimes \delta^{\otimes i_1} \otimes t_1 \otimes \delta^{\otimes i_0})).$$

which coincides with

$$\sum_{i_0, i_1, i_2 \geq 0} b^{cv}_{i_0+i_1+i_2+2}(\Psi(\delta_X)^{\otimes i_2} \otimes \Psi(t_2) \otimes \Psi(\delta_X)^{\otimes i_1} \otimes \Psi(t_1) \otimes \Psi(\delta_X)^{\otimes i_0}).$$

For simplicity, we write $M_u$ and $\rho_u$ instead of $M(X_u)$ and $\Psi(\delta_X)$, respectively, for $u \in \{0, 1, 2\}$. Then, for a basic directed element $x \in c_2B_{-1}c_i$, we have that $\Psi(t_2t_1)(x^*)$ coincides with

$$\sum_{s=2}^{\infty} \sum_{i_0, i_1, i_2 \geq 0} (-1)^{u(i_2, i_1, i_0)} \nu_s(\rho_2^{\otimes i_2} \otimes \Psi(t_2) \otimes \rho_1^{\otimes i_1} \otimes \Psi(t_1) \otimes \rho_0^{\otimes i_0})d(x^*),$$

with $u(i_2, i_1, i_0) = \lambda_u(0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0) = 1$, where the first sequence of ceros, from left to right, has length $i_2$, the second has length $i_1$, and the third one has length $i_0$.

If $x = e_i$, then $d(e_i^*) = e_i^* \otimes e_i^* + h(e_i^*)$, with $h(e_i^*) \in N$. It follows that $M(t_2t_1)^0 = \Psi(t_2t_1)(e_i^*) = -\nu_2(\Psi(t_2) \otimes \Psi(t_1))(e_i^* \otimes e_i^*) = \Psi(t_2)(e_i^*)\Psi(t_1)(e_i^*) = M(t_2)^0M(t_1)^0$, for all $i \in \mathcal{P}$, so $M(t_2t_1)^0 = M(t_2)^0M(t_1)^0$.

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If $x \in J$, we have $d(x^*) = e_j^* \otimes x^* + x^* \otimes e_i^* + \delta(x^*) + h(x^*)$, with $h(x^*) \in N$. Then, we have that $M(t_2t_1)(x^*) = \Psi(t_2t_1)(x^*) = S_0 + S_{\delta(x^*)}$, where

$$S_0 = -\nu_2(\Psi(t_2) \otimes \Psi(t_1))(e_j^* \otimes x^* + x^* \otimes e_i^*),$$

and

$$S_{\delta(x^*)} = -\sum_{s=2}^{\infty} \sum_{i_0, i_1, i_2 \geq 0} \nu_s(\rho_2^{\otimes i_2} \otimes \Psi(t_2) \otimes \rho_1^{\otimes i_1} \otimes \Psi(t_1) \otimes \rho_0^{\otimes i_0})\delta(x^*)\delta(y^*) .$$

Moreover, we have

$$S_0 = \Psi(t_2)(e_j^*)\Psi(t_1)(x^*) + \Psi(t_2)(x^*)\Psi(t_1)(e_i^*) = M(t_2)^0M(t_1)^1(x^*) + M(t_2)^1(x^*)M(t_1)^0$$

$$= [M(t_2)^0M(t_1)^1 + M(t_2)^1M(t_1)^0](x^*) .$$

By the definition of the composition in $(\mathcal{A}, I)$-Mod, we have

$$(M(t_2)M(t_1))^1 = M(t_2)^0M(t_1)^1 + M(t_2)^1M(t_1)^0 + (M(t_2)^1 \ast M(t_1)^1)\delta .$$

So, we only have to verify that $(M(t_2)^1 \ast M(t_1)^1)(\delta(x^*)) = S_{\delta(x^*)}$. But, according with (5.2.13), we have this equality. Hence, we get that $M$ preserves composition of morphisms.

According with (7.4), the identity morphism $1_X$ in $Z_{-1}(\text{tw}(B))(\underline{X}, \underline{X})$ of the object $X$ satisfies $\Psi(1_X)(e_i^* \otimes x^*) = \text{id}_{X_{ei}}$ and $\Psi(1_X)(\xi) = 0$, for all $\xi \in \hat{D}(J)$. Then, we have $M(1_X)^0 = \Psi(1_X)(e_i^*) = \text{id}_{X_{ei}}$, and, if $\xi \in \hat{D}(J)$, we have $M(1_X)^1(\xi) = \Psi(1_X)(\xi) = 0$. So $M$ preserves identities and we have a functor $M : Z_{-1}(\text{tw}(B)) \longrightarrow (\mathcal{A}, I)$-Mod.

Step 4: Let us show that the functor $M$ is dense. So, consider an object in $(N, \rho_N) \in (\mathcal{A}, I)$-Mod. Then, we can write it as $N = X^{\text{op}}$, where $X$ is a right $S$-module. We have $\rho_N \in \text{Hom}_{S-S}(\hat{D}(B)_0, \text{Hom}_k(X^{\text{op}}, X^{\text{op}}))$, but we know that $\Psi : \text{ad}(B)(X, X) \longrightarrow \text{ev}(B)(X^{\text{op}}, X^{\text{op}})$ is an isomorphism, so there is some $\delta_X \in \text{ad}(B)(X, X)$ homogeneous with degree 0 such that $\Psi(\delta_X) = \rho_N$.

By (1.3), we have that $\hat{D}(B)_0$ is finite dimensional and $\hat{D}(B)^{\otimes n} = 0$, then the $k$-algebra $\Gamma = A(\Delta) = T_S(\hat{D}(B)_0)$ is finite dimensional. Consider the Jacobson radical $\text{rad}(\Gamma)$ of $\Gamma$, so we have $\text{rad}(\Gamma)^n = 0$ for its nilpotence index $n$. Consider the filtration of left $S$-modules

$$0 = \text{rad}(\Gamma)^0N \subseteq \cdots \subseteq \text{rad}(\Gamma)^nN \subseteq \text{rad}(\Gamma)N \subseteq N$$

or, equivalently, the filtration of right $S$-modules

$$0 = X\text{rad}(\Gamma^{op})^0 \subseteq \cdots \subseteq X\text{rad}(\Gamma^{op})^n \subseteq X\text{rad}(\Gamma^{op}) \subseteq X.$$

If we write $\delta_X = \sum_{x \in B_0} f_x \otimes x$ and take a directed basic element $y \in B_0$, we obtain $\rho_N(y)^* = \Psi(\delta_X)(y)^* = \sum_{x \in B_0} f_x y^* = f_y e_{t(y)} = f_y$. Since $y^* \in \hat{D}(B)_0 \subseteq \text{rad}(\Gamma)$, it annihilates the semisimple right $\Gamma^{op}$-module $X\text{rad}(\Gamma^{op})^s/X\text{rad}(\Gamma^{op})^{s+1}$, for all $s \in [0, n - 1]$. Hence, we have

$$f_y(X\text{rad}(\Gamma^{op})^s) = \rho_N(y)^*X\text{rad}(\Gamma^{op})^s \subseteq X\text{rad}(\Gamma^{op})^{s+1}.$$
and the pair \((X, \delta_X)\) satisfies the first condition of \([6.1]\). For the second one, since \((N, \rho_N) \in \langle A, I \rangle\text{-Mod}, \text{ the } A(\Delta)\text{-module } N \text{ satisfies } IN = 0. \text{ As we saw in } [\S 2], \text{ this means that } \sum_{n=2}^{\infty} \nu_n(\rho_N^\otimes n) d(b_{n,0}) = 0. \text{ Then, we have }

\[
0 = \sum_{n=2}^{\infty} b_n^{\text{ad}}(\delta_X^\otimes n) = \Psi(\sum_{n=2}^{\infty} b_n^{\text{ad}}(\delta_X^\otimes n)).
\]

Since \(\Psi\) is an isomorphism, we obtain that \(\sum_{n=2}^{\infty} b_n^{\text{ad}}(\delta_X^\otimes n) = 0\). Therefore, the pair \((X, \delta_X)\) is an object of \(\text{tw}(B)\).

**Step 5:** Let us show that \(M\) is a faithful and full functor. Assume that \(t : X \longrightarrow Y\) is a morphism in \(Z_{-1}(\text{tw}(B))\) such that \(M(t) = 0\). Thus, we have \(\Psi(t)(e_i^*|_i = M(t)^j|_0 = 0\), for \(i \in \mathcal{P}\), and \(\Psi(t)(\xi) = M(t)^j(\xi) = 0\), for \(\xi \in \hat{D}(J)\).

Here, the domain of \(\Psi(t)\) is \(\hat{D}(B)_1 = \bigoplus_{j \in \mathbb{P}} ke_j^* \oplus \hat{D}(J)\), so we get \(\Psi(t) = 0\). Since \(\Psi\) is an isomorphism we obtain \(t = 0\), and \(M\) is a faithful functor.

Let \(f = (f^0, f^1) : M(X, \delta_X) \longrightarrow M(Y, \delta_Y)\) be a morphism in \(\langle A, I \rangle\text{-Mod}\). Consider the morphism of \(S\text{-}S\) bimodules \(\phi : \hat{D}(B)_1 \longrightarrow \text{Hom}_k(X^{\text{op}}, Y^{\text{op}})\) defined by

\[
\phi(e_i^*) = e_i f^0 e_i \in \text{Hom}_k(Xe_i, Ye_i) \quad \text{and} \quad \phi(\xi) = e_j f^1(\xi) e_i \in \text{Hom}_k(Xe_i, Ye_j),
\]

for \(i \in \mathcal{P}\) and \(\xi \in e_j J e_i \cap \mathbb{B}\). Since \(\Psi\) is surjective, there is some morphism \(t \in \text{ad}(B)(X^{\text{op}}, Y^{\text{op}})_{-1}\) such that \(\Psi(t) = \phi\) since \(f = (f^0, f^1)\) is a morphism in \(\langle A, I \rangle\text{-Mod}\), we have the equation \([\S 2](2)\), for all directed elements \(\alpha \in \hat{D}(B)_0\).

Then, we can reverse the argument given in **Step 2** to obtain that \(t\) satisfies

\[
b_4^{\text{tw}}(t) = \sum_{i_0, i_1 \geq 0} b_4^{\text{ad}}(\delta_Y^{i_1} \otimes t \otimes \delta_X^{i_0}) = 0.
\]

So, \(t : (X, \delta_X) \longrightarrow (Y, \delta_Y)\) is a morphism in \(Z_{-1}(\text{tw}(B))\). Then, we have that \(M\) is a full functor.

\[\square\]

### 9 The weak ditalgebra \(A(\Delta)\) and the ideal \(I\)

We recall some terminology from \([3]\) and \([1]\).

**Definition 9.1.** A weak ditalgebra \(A\) is a pair \(A = (T, \delta)\), where \(T\) is a graded tensor algebra and \(\delta\) is a differential on \(T\), that is \(\delta : T \longrightarrow T\) is a homogeneous linear map of degree 1 and satisfies Leibniz rule \(\delta(ab) = \delta(a)b + (-1)^{|a|}a\delta(b)\) on homogeneous elements \(a, b \in T\). The weak ditalgebra \(A\) is called a ditalgebra if, furthermore, we have \(\delta^2 = 0\).

A weak ditalgebra \(A = (T, \delta)\) has layer \((R, W)\) iff \(R\) is a \(k\)-algebra and \(W\) is an \(R\text{-}R\) bimodules equipped with an \(R\text{-}R\) bimodule decomposition \(W_0 \oplus W_1\) such that \(W_0 \subseteq T_0, W_1 \subseteq T_1\), the algebra \(T\) is freely generated by the pair \((R, W)\), and \(\delta(R) = 0\), see \([3](4.1)\). When there is no danger of confusion, we write \(A := T_0\) and \(V := T_1\). Thus, we have \(A \cong T_R(W_0)\) and \(V \cong A \otimes_R W_1 \otimes_R A\).

A layer \((R, W)\) of a weak ditalgebra \(A = (T, \delta)\) is called triangular if
1. There is a filtration of $\text{R-R}$-submodules $0 = W_0^0 \subseteq W_0^1 \subseteq \cdots \subseteq W_0^r = W_0$ such that $\delta(W_0^{i+1}) \subseteq A_iW_1A_i$, for all $i \in [0, r - 1]$, where $A_i$ denotes the $\text{R}$-subalgebra $A$ generated by $W_0^i$. 

2. There is a filtration of $\text{R-R}$-submodules $0 = W_1^0 \subseteq W_1^1 \subseteq \cdots \subseteq W_1^r = W_1$ such that $\delta(W_1^{i+1}) \subseteq AW_1^iAW_1^iA$, for all $i \in [0, s - 1]$.

**Definition 9.2.** Let $\mathcal{A} = (T, \delta)$ be a weak ditalgebra, make $\mathcal{A} = T_0$ and $V = T_1$. Assume that $I$ is an ideal of $\mathcal{A}$, satisfying that $\delta(I) \subseteq IV + VI$. Then, the weak ditalgebra $\mathcal{A}$ is called *interlaced with the ideal* $I$ iff

1. $\delta^2(A) \subseteq IV^2 + VIV + V^2I$, and
2. $\delta^2(V) \subseteq IV^3 + VIV^2 + V^2IV + V^3I$.

The pair $(\mathcal{A}, I)$ is called then an *interlaced weak ditalgebra*.

**Definition 9.3.** Let $\mathcal{A} = (T, \delta)$ be a weak ditalgebra, make $\mathcal{A} = T_0$ and $V = T_1$, and assume that $I$ is an ideal of $\mathcal{A}$. Then, we say that $I$ is an *$\mathcal{A}$-triangular ideal* of $\mathcal{A}$ iff there is a sequence of $k$-subspaces

\[0 = H_0 \subseteq H_1 \subseteq \cdots \subseteq H_t = I\]

such that $\delta(H_i) \subseteq AH_{i-1}V + VH_{i-1}A$, for all $i \in [1, t]$.

**Definition 9.4.** A pair $(\mathcal{A}, I)$ is called a *triangular interlaced weak ditalgebra* if $\mathcal{A} = (T, \delta)$ is a weak ditalgebra interlaced with the ideal $I$ of $T_0$, where $\mathcal{A}$ has a triangular layer and $I$ is an $\mathcal{A}$-triangular ideal.

The importance of these triangular interlaced weak ditalgebras $(\mathcal{A}, I)$ is that the reduction procedures introduced by the Kiev school of representation theory can be applied to the study of their module categories $(\mathcal{A}, I)$-$\text{Mod}$, see [1]. In order to show that the pair $(\mathcal{A}(\Delta), I)$ constructed in §4, see [1,19], is a triangular interlaced weak ditalgebra, we need some preliminary facts.

**Remark 9.5.** We will use a basis $\mathcal{B}$ of $B$ as constructed in [19], but we refine the choice of the basis of the vector space $J$ as follows. Since $J$ is nilpotent, we have a filtration $0 = J^0 \subseteq \cdots \subseteq J^2 \subseteq J$, and an $S-S$-bimodule decomposition $J = L_1 \oplus L_2 \oplus \cdots \oplus L_{\ell_0-1}$, such that $J^u = J^{(u+1)} \oplus L_u$, for all $u \in [1, \ell_0 - 1]$. Then, we can choose directed $k$-vector space basis $\mathcal{J}_u(i, j) \subseteq e_jL_ue_i$, for $i, j \in \mathcal{P}$ and $u \in [1, \ell_0 - 1]$. Then, we make $\mathcal{J}_u = \bigcup_{i,j} \mathcal{J}_u(i, j)$ and $\mathcal{J} = \bigcup_u \mathcal{J}_u$. For the construction of $\mathcal{B}$ in [19], we use this special choice for the basis of $J$.

As usual, the *depth* $\nu_1(x)$ of an element $x \in J$ is the number such that $x \in J^{\nu_1(x)} \setminus J^{\nu_1(x)+1}$. Here, the set of basic elements in $x \in \mathcal{J}$ with $\nu_1(x) = u$ is precisely $\mathcal{J}_u$. Thus, for $x \in \mathcal{J}$, we have that $x^*(L_u) = 0$, for all $v \neq \nu_1(x)$. If $x \in J$, we make $\nu_1(x^*) = \nu_1(x)$.

We will also refine the choice of the basis $\mathcal{B}_0$ as follows. For each $t \in \mathbb{N}$, consider the $S-S$-subbimodule $U_t := \sum_{r+s=t} J^rB_0J^s$ of $B_0$. Then, we have an $S-S$-bimodule filtration $0 = U_0 \subseteq \cdots \subseteq U_1 \subset U_t \subset U_{t+1} = B_0$, and an $S-S$-bimodule
decomposition $B_0 = G_0 \oplus G_1 \oplus \cdots \oplus G_{\ell_1 - 1}$, with $U_v = U_{v+1} \oplus G_v$, for all $v \in [1, \ell_1 - 1]$. As before, we choose directed $k$-vector space basis $G_v(i, j) \subseteq e_j G_v e_i$, for $i, j \in \mathcal{P}$ and $v \in [1, \ell_1 - 1]$. Then, we make $G_v = \bigcup_{i, j} G_v(i, j)$ and $B_0 = \bigcup_v G_v$. For the construction of $\mathcal{B}$ in (1.13), we use this special choice for the basis of $B_0$.

As before, the depth $\nu_0(x)$ of an element $x \in B_0$ is the number such that $x \in U_{\nu_0(x)} \setminus U_{\nu_0(x)+1}$. Thus, for $x \in B_0$, we have that $x^*(G_v) = 0$ if $v \neq \nu_0(x)$. Also, for $x \in B_0$, we make $\nu_0(x^*) = \nu_0(x)$.

Remark 9.6. Notice that the pair $(S, W)$, where $W_0 = \bar{D}(B)_0$ and $W_1 = \bar{D}(J)$, is a layer for the weak ditalgebra $\mathcal{A}(\Delta) = (T(\Delta), \delta)$, see (4.19).

Let us consider the bigraph $\mathcal{B}$ of the layered ditalgebra $\mathcal{A}(\Delta)$, it has set of points $\mathcal{P}$; given $i, j \in \mathcal{P}$, the set of solid arrows from $i$ to $j$ is $\mathcal{B}_0(i, j)$ and the set of dashed arrows from $i$ to $j$ is $\mathcal{J}^*(i, j)$. The starting point of any path $\gamma$ is denoted by $s(\gamma)$ and its terminal point by $t(\gamma)$. A path $\gamma$ is called a cycle iff $s(\gamma) = t(\gamma)$. A cycle $\gamma$ is called a loop if $\gamma$ has only one arrow.

Notice that the graded tensor algebra $T(\Delta) = T_S(W_0 \oplus W_1)$ can be identified with the path algebra $k(\mathcal{B})$ of the bigraph $\mathcal{B}$, whose underlying vector space has as basis the set of paths (of any kind of arrows) of $\mathcal{B}$ (including one trivial path for each point $i \in \mathcal{P}$) and the product is induced by the concatenation of paths. Each idempotent $e_i$ of $S$ is identified with the trivial path at the point $i$. The homogeneous elements $h \in k(\mathcal{B})_u$ of degree $u$ are the linear combinations of paths containing exactly $u$ dashed arrows.

We visualize the products of basic directed elements of $\mathcal{B}^* \cup \mathcal{J}^*$ in $T(\Delta)$ as paths of the bigraph $\mathcal{B}$. An important observation is the following.

1. If $\alpha : i \rightarrow j$ is solid arrow of $\mathcal{B}$, we have $\bar{i} < \bar{j}$ in the quotient poset $\overline{\mathcal{P}}$.

2. If $\xi : i \rightarrow j$ is a dashed arrow of $\mathcal{B}$, we have $\bar{i} \leq \bar{j}$ in the quotient poset $\overline{\mathcal{P}}$.

Lemma 9.7. A path $\gamma$ from $i$ to $j$ in the bigraph $\mathcal{B}$ is called a precycle iff $i \sim j$. A precycle of length 1 is called a preloop. Then, any precycle $\gamma$ of length $\geq 1$ consists of dashed arrows only.

Proof. Assume that $\gamma = \alpha_n \alpha_{n-1} \cdots \alpha_1$ is a precycle from $i$ to $j$, where $\alpha_1, \ldots, \alpha_n$ are (solid or dashed) arrows of $\mathcal{B}$, with $t(\alpha_u) = i_u$, for $u \in [1, n]$. Then, by the last observation in (4.19), we have $\bar{i} \leq \bar{i_1} \leq \cdots \leq \bar{i_n} = \bar{j}$ with $\bar{i} = \bar{i_1} = \cdots = \bar{i_n}$. Then all the arrows $\alpha_1, \ldots, \alpha_n$ must be dashed arrows.

In the following we use the height maps $h : X \rightarrow \mathbb{N} \cup \{0\}$ associated to various finite posets $(X, \leq)$, as defined in (1111). Recall that every height map $h$ satisfies: $y < x$ implies $h(y) < h(x)$, for all $x, y \in X$.

Lemma 9.8. Consider the quotient poset $\overline{\mathcal{P}}$ associated to $\mathcal{P}$, and its height map $h : \overline{\mathcal{P}} \rightarrow \mathbb{N} \cup \{0\}$. Consider the set of paths $\Gamma$ of $\mathcal{B}$ and the map $d : \Gamma \rightarrow \mathbb{N} \cup \{0\}$ defined, for each path $\gamma$ in $\mathcal{B}$, as $d(\gamma) = h(t(\gamma)) - h(s(\gamma))$. We have the following.

1. If $\alpha$ is a solid arrow, then $d(\alpha) > 0$. 

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2. If \( \xi \) is a dashed arrow, then \( d(\xi) \geq 0 \) and \( d(\xi) = 0 \) if \( \xi \) is a preloop.

3. Whenever \( \gamma = \gamma'_\gamma\gamma'' \) is a factorization of the path \( \gamma \) as a product of non-trivial subpaths, where exactly one of them is a precycle, we have

\[
d(\gamma) = d(\eta), \quad \text{where } \eta \in \{\gamma', \gamma''\} \text{ is not a precycle.}
\]

4. Whenever \( \gamma = \gamma'_\gamma\gamma''\gamma''' \) is a factorization of the path \( \gamma \) as a product of non-trivial subpaths, where at most one of them is a precycle, we have

\[
d(\gamma) > d(\eta), \quad \text{for each } \eta \in \{\gamma', \gamma''\gamma''', \gamma''''\} \text{ not a precycle.}
\]

Proof. (1), (2) and (3) are clear. In (4), we have three cases. Assume, for instance, that \( \gamma''' \) is the only precycle. Then, we have

\[
\overline{s(\gamma)} = \overline{s(\gamma''')} = \overline{t(\gamma''')} = \overline{t(\gamma')} < \overline{t(\gamma')} = \overline{t(\gamma)} = \overline{t(\gamma)},
\]

thus \( d(\gamma') < d(\gamma) \) and \( d(\gamma''') < d(\gamma) \). The other cases are similar.

**Proposition 9.9.** The pair \((A(\Delta), I)\) constructed in §4, see (4.19), is a triangular interlaced weak ditalgebra.

Proof. We adopt the preceding notations.

**Step 1: Description of the triangular filtration of \( W_0 \), see (9.1)(1).**

Consider the partial order on the set \( \mathcal{P}(B_0^*) = \{(d(\alpha), \nu_0(\alpha)) \mid \alpha \in B_0^*\} \) defined by \((i, j) < (i', j')\) iff \( i < i' \) or (if \( i = i' \), then \( j < j' \)), and its corresponding height map \( h_0 : \mathcal{P}(B_0^*) \longrightarrow \mathbb{N} \cup \{0\} \). Write \( h_0(\alpha) := h_0(d(\alpha), \nu_0(\alpha)) \), for any \( \alpha \in B_0^* \). Thus, \( (d(\alpha), \nu_0(\alpha)) < (d(\beta), \nu_0(\beta)) \) implies that \( h_0(\alpha) < h_0(\beta) \).

Consider the filtration \( 0 = W_0^0 \subset W_0^1 \subset \cdots \subset W_0^r = W_0 \), where \( W_0^0 \) is the \( S\)-\( S \)-submodule of \( W_0 \) generated by \( \{\alpha \in B_0^* \mid h_0(\alpha) < u\} \).

Assume that we have a solid arrow \( \alpha \in W_0^w \), thus \( h_0(\alpha) < u \) and \( u \geq 1 \). We want to show that \( \delta(\alpha) \in A_{u-1} D(J) A_{u-1} \). We can write

\[
\delta(\alpha) = \delta(\alpha)_2 + \delta(\alpha)_3 + \cdots + \delta(\alpha)_w,
\]

for some \( w \geq 2 \) and each \( \delta(\alpha)_v \) is a linear combination of paths of length \( v \) and degree 1. Recall that \( \alpha \) is a solid arrow, so it can not be a cycle.

Then, if \( \delta(\alpha)_v \neq 0 \), for \( v \in [3, w] \), there is a path (non-cycle) \( \gamma_2 \xi \gamma_1 \) in its expression as linear combination of paths, with non-zero coefficient, where \( \gamma_2 \) and \( \gamma_1 \) are paths composed by solid arrows and \( \xi \) is a dashed arrow. Here, from (4.18), we know that one of \( \gamma_1 \) or \( \gamma_2 \) must be non-trivial. Then, (9.8) implies that whenever \( \gamma_1 \) is not trivial (resp. \( \gamma_2 \) non-trivial) we have \( d(\gamma_1) < d(\gamma_2 \xi \gamma_1) = d(\alpha) \) (resp. \( d(\gamma_2) < d(\gamma_2 \xi \gamma_1) = d(\alpha) \)), then, in any case, the arrows \( \beta \) composing \( \gamma_1 \) or \( \gamma_2 \) satisfy \( d(\beta) < d(\alpha) \), so \( h_0(\beta) < h_0(\alpha) \), which implies that \( \gamma_2 \xi \gamma_1 \in A_{u-1} D(J) A_{u-1} \). So \( \delta(\alpha)_v \in A_{u-1} D(J) A_{u-1} \).

It remains to consider the case \( v = 2 \). If \( \delta(\alpha)_2 \neq 0 \), there is a path of the form \( \xi \beta \) (or \( \beta \xi \)), where \( \beta \) is a solid arrow and \( \xi \) is a dashed arrow, appearing in the
expression of $\delta(\alpha)_2$ with non-zero coefficient. If $\xi$ is not a preloop, we proceed as before to obtain that $\beta\xi \in A_{u-1}\hat{D}(J)A_{u-1}$ (or $\xi\beta \in A_{u-1}\hat{D}(J)A_{u-1}$). Assume that $\xi$ is a preloop and that the path $\xi\beta$ has appeared in $\delta(\alpha)_2$. Here, we have $d(\beta) = d(\alpha)$. Let us examine its coefficient $c^2_{\xi\otimes\beta}$ in the explicit description of $\delta(\alpha)$ given in (4.18). We have $\alpha = x^*, \beta = y^*$ and $\xi = r^*$, where $x, y \in \mathcal{B}$ and $r \in \mathcal{J}$, then

$$0 \neq e_{t(\alpha)}c^2_{\xi\otimes\beta} = x^*[h_2(r \otimes y)] = x^*[\sigma(\sigma^{-1}(r)\sigma^{-1}(y))] = x^*(ry).$$

We know that $r \in \mathcal{J}$, thus $r \in \mathcal{J}_s \subseteq \mathcal{J}^*$, for some $s \in [1, \ell_0 - 1]$. Thus, we get $ry \in U_{s+v_0(y)} = \bigoplus_{l \geq s+v_0(y)} G_l$. Since $x^*(ry) \neq 0$, we have $x^*(U_{s+v_0(y)}) \neq 0$, thus $x^*(G_l) \neq 0$, for some $l \geq s + v_0(y)$. This implies that $v_0(\beta) = v_0(y) < s + v_0(y) \leq l = v_0(x) = v_0(\alpha)$, so $h_0(\beta) < h_0(\alpha)$ and $\xi\beta \in A_{u-1}\hat{D}(J)A_{u-1}$. The case where $\beta\xi$ appears in the expression of $\delta(\alpha)$ with non-zero coefficient is treated similarly. Then, we get $\delta(\alpha)_2 \in A_{u-1}\hat{D}(J)A_{u-1}$. Adding up, we get that $\delta(\alpha) \in A_{u-1}\hat{D}(J)A_{u-1}$, and we have the triangularity condition for $W_0$.

Step 2: Description of the triangular filtration of $W_1$, see (9.1)(2).

Consider the partial order on the set $\mathcal{P}(\mathcal{J}^*) = \{ (d(\xi), \nu_1(\xi)) \ | \ \xi \in \mathcal{J}^* \}$ defined by $(i, j) < (i', j')$ iff $i < i'$ or (if $i = i'$, then $j < j'$). The poset $\mathcal{P}(\mathcal{J}^*)$ admits a height map $h_1 : \mathcal{P}(\mathcal{J}^*) \rightarrow s\mathbb{N} \cup \{0\}$. Write $h_1(\xi) = h_1(d(\xi), \nu_1(\xi))$, for any $\xi \in \mathcal{J}^*$. Thus, as before, $(d(\xi), \nu_1(\xi)) < (d(\zeta), \nu_1(\zeta))$ implies that $h_1(\xi) < h_1(\zeta)$.

Consider the filtration $0 = W_0^1 \subseteq W_1^1 \subseteq \cdots \subseteq W^1_i = W_1$ where $W_i^1$ is the $S$-submodule of $W_1$ generated by $\{ \xi \in \mathcal{J}^* | h_1(\xi) < u \}$. Assume that we have a dashed arrow $\xi \in W_i^1$, thus $h_1(\xi) < u$ and $u \geq 1$. We want to show that $\delta(\xi) \in AW_{u-1}^1AW_{u-1}^1A$. We can write

$$\delta(\xi) = \delta(\xi)_2 + \delta(\xi)_3 + \cdots + \delta(\xi)_w,$$

for some $w \geq 2$ and each $\delta(\xi)_v$ is a linear combination of paths of length $v$ and degree 2. Now, the dashed arrow $\xi$ may be a loop.

Case 1: $\xi$ is not a preloop (or, equivalently, $d(\xi) > 0$).

If $\delta(\xi)_v \neq 0$, for $v \in [3, w]$, there is a path $\gamma_3\xi_2\gamma_2\xi_1\gamma_1$ in its expression as linear combination of paths, with non-zero coefficient, where $\gamma_1$, $\gamma_2$, and $\gamma_3$ are paths composed by solid arrows and $\xi_1$ and $\xi_2$ are dashed arrows. Moreover, at least one of the paths $\gamma_1$, $\gamma_2$ or $\gamma_3$ is not trivial.

In each possible case, using (9.3), we can show that the dashed arrows $\xi_1$ and $\xi_2$ satisfy $d(\xi_1), d(\xi_2) < d(\xi)$, so $h_1(\xi_1), h_1(\xi_2) < h_1(\xi)$, which implies that $\gamma_3\xi_2\gamma_2\xi_1\gamma_1 \in AW_{u-1}^1AW_{u-1}^1A$. Thus, we get $\delta(\xi)_v \in AW_{u-1}^1AW_{u-1}^1A$.

It remains to consider the case $v = 2$. If $\delta(\xi)_2 \neq 0$, there is a path of the form $\xi_3\xi_2$, where $\xi_1$ and $\xi_2$ are dashed arrows, appearing in the expression of $\delta(\xi)$ with non-zero coefficient. If both $\xi_1$ and $\xi_2$ are not preloops, we proceed as before, applying (9.3) to $\xi_2\xi_1$, to obtain that $\xi_2\xi_1 \in W_{u-1}^1AW_{u-1}^1$. If some of $\xi_1$ or $\xi_2$ is a preloop, so $d(\xi) = d(\xi_1\xi_2)$, we can examine its coefficient $c^2_{\xi_2\otimes\xi_1}$ in the explicit description of $\delta(\xi)$ given in (4.18) and deduce, as before, that $\nu_1(\xi_2), \nu_1(\xi_1) < \nu_1(\xi)$. Indeed, we have $\xi = x^*, \xi_1 = x^1$ and $\xi_2 = x^2$, for $x \in \mathcal{J}$.
Case 2: \( \xi \) is a preloop (or, equivalently, \( d(\xi) = 0 \)).

If \( \xi \) is a preloop, since the only possible precycles in the bigraph \( B \) are composed by dashed arrows, we obtain that \( \delta(\xi) \) is a finite sum of elements of the form \( \xi_\delta^{\xi} \), for some dashed preloops \( \xi_1 \) and \( \xi_2 \), and \( \xi_\delta^{\xi} \in k \). Here again, examining the coefficient \( \xi_\delta^{\xi} \) in case it is not zero, we obtain that \( \delta(\xi) \in AW_1^{u-1}AW_1^{u-1}A \).

Step 3: Description of the triangular filtration of \( I \), as in (4.3).

Consider the quotient poset \( \overline{P} \) associated to the preordered set \( \mathcal{P} \) and define the following relation on \( S := \overline{P} \times \overline{P} \). We make \((\overline{i}, \overline{j}) < (\overline{i}^\prime, \overline{j}^\prime) \) iff \( \overline{i} < \overline{i}^\prime \) and \([\overline{i} = \overline{i}^\prime, \text{then} \overline{j} < \overline{j}^\prime] \). With this relation, the set \( S \) becomes a poset. Consider the corresponding height map \( h : S \longrightarrow \mathbb{N} \cup \{0\} \). Then, given \( i, j \in \mathcal{P} \) and \( \eta \in e_jIe_i \), we write \( h(\eta) := h(i,j) \).

Consider the map \( \phi : I \longrightarrow \mathbb{N} \cup \{0\} \) which associates to an element \( \eta \in I \) the minimal length of the paths which appear with non-zero coefficient in the expression of \( \eta \) as a linear combination of paths formed by solid arrows of the bigraph \( B \). Finally, define the map \( p : I \longrightarrow \mathbb{N} \cup \{0\} \) by the following rule, given \( \eta \in I \) with \( \phi(\eta) = n \),

\[
p(\eta) = \max\{h_0(\alpha_n) + \cdots + h_0(\alpha_1) \mid \alpha_n \cdots \alpha_2\alpha_1 \in \Gamma(\eta)\},
\]

where the elements \( \alpha_n \cdots \alpha_2\alpha_1 \) run in the set of paths \( \Gamma(\eta) \) of length \( n \) which appear in the expression of \( \eta \) as a linear combination of paths composed by solid arrows with non-zero coefficient.

Notice that, since the lengths of the paths of solid arrows in \( B \) are bounded, there are bounds for the numbers \( h(\eta) \), \( \phi(\eta) \) and \( p(\eta) \), for \( \eta \in I \). Consider the finite set \( \mathcal{P}(I) = \{([\eta], \phi(\eta), p(\eta)) \mid \eta \in I \text{ directed}\} \) and the partial order relation on it defined by \((r, s, t) < (r^\prime, s^\prime, t^\prime) \) iff \( r < r^\prime \) or \( [r = r^\prime \text{ and } s > s^\prime \text{ or } s = s^\prime \text{ and } t < t^\prime] \). Then, consider the associated height map \( \mathbf{h} : \mathcal{P}(I) \longrightarrow \mathbb{N} \cup \{0\} \), and make \( \mathbf{h}(\eta) = \mathbf{h}(\phi(\eta), \phi(\eta), p(\eta)) \), for any directed element \( \eta \in I \). For each \( u \geq 0 \), consider the vector subspace \( H_u \) of \( I \) generated by the directed elements \( \eta \in I \) with \( \mathbf{h}(\eta) < u \). We have a finite vector space filtration of \( I \) of the form

\[
0 = H_0 \subseteq H_1 \subseteq \cdots \subseteq H_{\ell_I} = I.
\]

Take \( u \in [1, \ell_I] \), we have to verify that \( \delta(H_u) \subseteq AH_{u-1}V + VH_{u-1}A \). It will be enough to verify that \( \delta(\eta) \in AH_{u-1}V + VH_{u-1}A \), for any directed element \( \eta \in I \) with \( \mathbf{h}(\eta) < u \). Suppose that \( \eta \neq e_jIe_i \). The fact that there are no precycles in \( B \) formed by solid arrows implies that \( i \neq j \). By (4.17), we have
notice that component whenever we examine the component, denoted by \( \xi \), which can also be considered as an element \( \lambda \), determines its dual map \( \xi \rightarrow \lambda \). This implies that \( \hat{h}(\lambda) < \hat{h}(\eta) \). Thus, this component of \( \delta(\eta) \) belongs to \( H_{u-1}V \). Similarly, we can show that the component of \( \delta(\eta) \) in \( e_j V \) is \( e_j V \) and its graded right dual bimodule \( \check{\mathcal{B}}(W) \), which can also be considered as an element \( \xi \in \check{\mathcal{B}}(W) \) with \( \xi^\oplus(0,0) = 0 \). Consider the linear map \( \lambda : k(B) \rightarrow k(B) \) which is defined on the basis of \( k(B) \) formed by the paths of \( B \) by

\[
\lambda_\xi(\alpha_n \alpha_{n-1} \cdots \alpha_1) = \xi^\oplus(\alpha_n) \alpha_{n-1} \cdots \alpha_1.
\]

Thus, we get \( \lambda_\xi(\alpha_n \alpha_{n-1} \cdots \alpha_1) = \xi \alpha_{n-1} \cdots \alpha_1 \), if \( \alpha_n = \xi \), and \( \lambda_\xi(\alpha_n \cdots \alpha_1) = 0 \), whenever \( \alpha_n \neq \xi \). We have

\[
e_j V e_r \xi = \bigoplus_{\xi} e_r \xi,
\]

where \( \xi \) runs in the set of (dashed) preloops from \( r \) to \( j \) of the bigraph \( B \). Let us examine the component, denoted by \( \delta(\eta)_\xi \), of \( \delta(\eta) \) in the direct summand \( \xi e_r \), for a fixed dashed preloop \( \xi \) from \( r \) to \( j \). Assume that \( \delta(\eta)_\xi \neq 0 \) and notice that \( \delta(\eta)_\xi = \xi \lambda_\xi(\delta(\eta)) \), so the element \( \lambda_\xi(\delta(\eta)) \) \( \in e_r \xi \), determines the component \( \delta(\eta)_\xi \).

Make \( m := \phi(\eta) \) and \( n := \phi(\lambda_\xi(\delta(\eta))) \). So \( n \geq m \), because, from [4.18], we have \( n + 1 = \phi(\lambda_\xi(\delta(\eta))) = \phi(\delta(\eta)_\xi) \geq \phi(\delta(\eta)) > \phi(\eta) = m \). We have \( \hat{h}(\lambda_\xi(\delta(\eta))) = \hat{h}(\xi, j) \), so if \( n > m \), we get

\[
(\hat{h}(\lambda_\xi(\delta(\eta))), \phi(\lambda_\xi(\delta(\eta)))) = (\hat{h}(\eta), \phi(\eta), p(\eta)).
\]

This implies that \( H(\lambda_\xi(\delta(\eta))) < H(\eta) \), and \( \delta(\eta)_\xi = \xi \lambda_\xi(\delta(\eta)) \in V H_{u-1} \).

Now, assume that \( n = m \). We can write \( \eta = \eta_m + \eta_{m+1} + \cdots \), where each \( \eta_{m+t} \) is a linear combination of paths of length \( m + t \). Moreover, we have

\[
\lambda_\xi(\delta(\eta)) = \lambda_\xi(\delta(\eta_m)) + \lambda_\xi(\delta(\eta_{m+1})) + \cdots
\]

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Then, we have \( \phi(\lambda_\xi(\delta(\eta_n))) = n \), for some \( s \geq m \). Therefore, we have \( n = \phi(\lambda_\xi(\delta(\eta_n))) \geq s \geq m \), which implies that \( n = s = m \).

Write \( \eta_n = \sum_v c_v \gamma_v \), where each \( \gamma_v \) is a path formed by \( n \) solid arrows and \( c_v \in k \), thus \( \delta(\eta_n) = \sum_v c_v \delta(\gamma_v) \). If \( \gamma_v = \alpha_n^v \cdots \alpha_2^v \alpha_1^v \), with \( \alpha_n^v, \ldots, \alpha_1^v \) solid arrows, then

\[
\delta(\gamma_v) = \phi(\lambda_\xi(\delta(\eta_n))) = \phi(\lambda_\xi(\delta(\gamma_v))) = (\sum_v c_v \delta(\gamma_v)) = \sum_v c_v \delta(\gamma_v).
\]

Notice that \( \lambda_\xi(\delta(\eta_n)) = \lambda_\xi(\delta(\alpha_n^v \cdots \alpha_2^v \alpha_1^v)) = \lambda_\xi(\delta(\alpha_n^v))\alpha_{n-1}^v \cdots \alpha_2^v \alpha_1^v \), because \( \alpha_n^v \) is a solid arrow, so \( \xi^\#(\alpha_n^v) = 0 \). Now, write \( \delta(\alpha_n^v) = \sum_w \xi_w^\# \xi_v^\# \gamma_n^w \), where each \( \gamma_n^w \) and \( \xi_v^\# \) are paths of degree 0, \( \xi_v^\# \) are dashed arrows, and \( c_w^v \in k \). Then, \( \lambda_\xi(\delta(\alpha_n^v)) \neq 0 \) implies that some \( \xi_v^\# \) is trivial, again because, otherwise, \( \xi^\# \) evaluated at the last arrow of \( \xi_v^\# \) would be zero. Then, we have \( \lambda_\xi(\delta(\alpha_n^v)) = \sum_w c_w^v \lambda_\xi(\xi_v^w \gamma_n^w) \), where now \( w \) runs in the set of indices with \( \xi_v^w \) trivial, and therefore

\[
n = \phi(\lambda_\xi(\delta(\eta_n))) = \phi(\sum_v c_v \lambda_\xi(\delta(\gamma_v))).
\]

Then, there is at least one path \( \gamma_n^w \) which is a solid arrow \( \beta_n^w \), and we have \( \lambda_\xi(\delta(\eta_n)) = \sum_w c_w^v \lambda_\xi(\xi_v^w \gamma_n^w) \). Then, we obtain

\[
p(\lambda_\xi(\delta(\eta_n))) = \phi(\sum_w c_w^v \lambda_\xi(\delta(\gamma_v))).
\]

Since \( \phi(\lambda_\xi(\delta(\eta_n))) \geq l \), for all \( l > n \), we have \( (\lambda_\xi(\delta(\eta_n))) = \langle (\lambda_\xi(\delta(\eta_n))) \rangle \). Therefore, we get \( p(\lambda_\xi(\delta(\eta_n))) = p(\lambda_\xi(\delta(\eta_n))) \), and this implies that \( \langle (\lambda_\xi(\delta(\eta_n))) \rangle \geq \langle (\lambda_\xi(\delta(\eta_n))) \rangle \). Then, also in this case, we get \( \delta(\eta_\xi) = \lambda_\xi(\delta(\eta_n)) \in VH_{u-1} \).

We also have that the component of \( \delta(\eta) \) in \( e_j Ie_r V e_i \) lies in \( H_{u-1} V \). This can be shown in a similar way. We consider the decomposition

\[
e_j Ie_r V e_i = \bigoplus \xi \in \xi_j Ie_r \xi,
\]

where \( \xi \) runs in the set of (dashed) preloops from \( i \) to \( r \) of the bigraph \( B \) and then examine each component \( \delta(\eta_\xi) \) in the summand \( e_j Ie_r \xi \). For this analysis, we can consider the left dual \( S\text{-}S\)-bimodule \( \tilde{D}(W_1) \) of the \( S\text{-}S\)-bimodule \( W_1 \) and a dual basis for the fixed basis \( \mathbb{I} \) of \( W_1 \). Thus, for a fixed basic element \( \xi \in \mathbb{J}^\ast \), we have its dual basic element \( \xi^\ast : W_1 \longrightarrow S \) is given by \( \xi^\ast(\zeta) = \delta_{\xi, \zeta} e_i(\zeta) \), and the linear map \( \rho_\xi : k(B) \longrightarrow k(B) \) defined on the basis of \( k(B) \) formed by the paths of \( B \) by

\[
\rho_\xi(\alpha_n \cdots \alpha_1) = \alpha_n \cdots \alpha_1 \xi^\ast(\alpha_1).
\]

Finally, since every component of \( \delta(\eta) \) lies in \( H_{u-1} V + VH_{u-1} \), so does \( \delta(\eta) \). This finishes the proof. \( \square \)
10 The exact category \((\mathcal{A}, I)\)-Mod

The category of modules \(\mathcal{A}\)-Mod for an interlaced weak ditalgebra \(\mathcal{A} = (\mathcal{A}, I)\) is defined in general in a similar way than the category \((\mathcal{A}(\Delta), I)\)-Mod, see (8.1) and [1]§2. In this section, we show that, for a Roiter interlaced weak ditalgebra \(\mathcal{A} = (\mathcal{A}, I)\), its category of modules \(\mathcal{A}\)-Mod admits an exact structure. We only have to adapt [3]§6 to this context. We will apply this section to \((\mathcal{A}(\Delta), I)\), but we give a more general presentation. Let us first recall some facts from [1]§5.

**Definition 10.1.** A triangular interlaced weak ditalgebra \((\mathcal{A}, I)\), with layer \((R, W)\), is called a Roiter interlaced weak ditalgebra iff the following property is satisfied: for any isomorphism \(f^0 \) of \(R\)-modules \(f^0 : M \rightarrow N\) and any \(f^1 \in \text{Hom}_R(W_1, \text{Hom}_R(M, N))\), if one of \(M\) or \(N\) has a structure of left \(A/I\)-module, then the other one admits also a structure of left \(A/I\)-module such that \((f^0, f^1) \in \text{Hom}_{(\mathcal{A}, I)}(M, N)\).

Notice that, whenever \((\mathcal{A}, I)\) is a triangular interlaced weak ditalgebra with layer \((R, W)\) such that \(R\) is semisimple, from [1](5.3), we know that \((\mathcal{A}, I)\) is a Roiter interlaced weak ditalgebra.

**Lemma 10.2.** Assume that \((\mathcal{A}, I)\) is a Roiter interlaced weak ditalgebra with layer \((R, W)\). Suppose that \(f = (f^0, f^1) : M \rightarrow N\) in \((\mathcal{A}, I)\)-Mod. Then,

1. If \(f^0\) is a retraction in \(R\)-Mod, there is a morphism \(h : M' \rightarrow M\) in \((\mathcal{A}, I)\)-Mod with \(h^0\) isomorphism and \((fh)^1 = 0\).

2. If \(f^0\) is a section in \(R\)-Mod, there is a morphism \(g : N \rightarrow N'\) in \((\mathcal{A}, I)\)-Mod with \(g^0\) isomorphism and \((gf)^1 = 0\).

**Proof.** We only prove (1). Adopt the notation of [9,1] and notice that the proof follows by induction from the following.

**Claim:** Given \(i \in [1, s]\), if \(f^1(W_i^{-1}) = 0\) then there exists a morphism \(h = (h^0, h^1) : M' \rightarrow M\) in \((\mathcal{A}, I)\)-Mod with \(h^0\) isomorphism and \((fh)^1(W_i^{-1}) = 0\).

**Proof of the Claim:** Let \(f^0 \in \text{Hom}_R(N, M)\) be a right inverse for \(f^0\) in \(R\)-Mod. Denote by \(M'\) the underlying \(R\)-module of \(M\). Define, for each \(w \in W_1\), \(h^1(w) := -t^0f^1(w) \in \text{Hom}_R(M', M)\). Then, \(h^1 \in \text{Hom}_R(W_1, \text{Hom}_R(M', M))\). Since \((\mathcal{A}, I)\) is a Roiter interlaced weak ditalgebra, we know that \(M'\) admits a left \(A/I\)-module structure such that \(h = (1_M, h^1)\) is a morphism in \((\mathcal{A}, I)\)-Mod. Since \(f^1(W_1^{-1}) = 0\), then \(h^1(W_1^{-1}) = 0\). Now, if \(w \in W_1\), we obtain from the triangularity condition that \(\delta(w) = \sum a_j w_j b_j w'_j c_j\), for some \(a_j, b_j, c_j \in A\) and \(w_j, w'_j \in W_1^{-1}\). Then, \((fh)^1(w) = f^0h^1(w) + f^1(w)1_M + \sum a_j f^1(w_j) b_j h^1(w'_j) c_j\). Therefore \((fh)^1(w) = -f^0t^0 f^1(w) + f^1(w)1_M = 0\), and the claim is proved. \(\Box\)

**Corollary 10.3.** Assume that \((\mathcal{A}, I)\) is a Roiter interlaced weak ditalgebra with layer \((R, W)\) and consider any morphism \(f = (f^0, f^1) : M \rightarrow N\) in \((\mathcal{A}, I)\)-Mod. Then, \(f\) is an isomorphism in \((\mathcal{A}, I)\)-Mod iff \(f^0\) is an isomorphism in \(R\)-Mod.
Proof. It is clear that if \( f \) is an isomorphism in \((\mathcal{A}, I)\)-Mod then \( f^0 \) is an isomorphism in \( R\)-Mod. Conversely, assume that \( f^0 \) is an isomorphism in \( R\)-Mod. By (10.2), there is morphism \( g = (g^0, g^1) : N \rightarrow N' \) with \((gf)^1 = \mathbf{0}\) and \( g^0 \) isomorphism in \( R\)-Mod. This implies that \( g^0 f^0 : M \rightarrow N' \) is an \( A/I\)-module isomorphism, hence \( g = \mathbf{0} \) is an isomorphism in \((\mathcal{A}, I)\)-Mod, and so \( f \) is a section in \((\mathcal{A}, I)\)-Mod. Dually, one shows that \( f \) is a retraction in \((\mathcal{A}, I)\)-Mod.

Proposition 10.4. Let \((\mathcal{A}, I)\) be a Roiter interlaced weak ditalgebra, then idempotents split in \((\mathcal{A}, I)\)-Mod. That is, for any idempotent \( e \in \text{End}_{(\mathcal{A}, I)}(M) \), there is an isomorphism \( h : M_1 \oplus M_2 \rightarrow M \) in \((\mathcal{A}, I)\)-Mod such that

\[
h^{-1}eh = \begin{pmatrix} 1_{M_1} & 0 \\ 0 & 0 \end{pmatrix}.
\]

Lemma 10.5. Let \((\mathcal{A}, I)\) be a Roiter interlaced weak ditalgebra. Suppose that \( M \rightarrow E \rightarrow N \) is a pair of composable morphisms in \((\mathcal{A}, I)\)-Mod with \( gf = \mathbf{0} \) and

\[
0 \rightarrow M \xrightarrow{f^0} E \xrightarrow{g^0} N \rightarrow 0
\]

is a split exact sequence of left \( R\)-modules. Then, there is an isomorphism \( h : E' \rightarrow E \) in \((\mathcal{A}, I)\)-Mod such that \((gh)^1 = \mathbf{0}\) and \((h^{-1}f)^1 = \mathbf{0}\).

Proof. By (10.2), we may assume that \( f^1 = \mathbf{0} \). Once again, we adopt the notation of (9.1). The proof will follow by induction from the following.

Claim: Given \( i \in [1, s] \), if \( f^i = \mathbf{0} \) and \( g^1(W_i^{i-1}) = \mathbf{0} \) then there exists an isomorphism \( h : E' \rightarrow E \) in \((\mathcal{A}, I)\)-Mod with \((h^{-1}f)^i = \mathbf{0}\) and \((gh)^i(W_i^i) = \mathbf{0}\).

Proof of the Claim: First follow the argument in the proof of the claim in (10.2), where using a right inverse \( t^0 \) for \( g^0 \) we constructed an isomorphism \( h = (1_E, h^1) : E' \rightarrow E \) in \((\mathcal{A}, I)\)-Mod, where \( E' \) has the same underlying \( R\)-module that \( E \) has, \( h^1(w) = t^0 g^1(w) \), for \( w \in W_1 \), and \((gh)^1(W_1^i) = \mathbf{0}\). We will show that \((h^{-1}f)^i = \mathbf{0}\) for this, we first notice that \((gf)^i = \mathbf{0}\) implies \(0 = (gf)^i(w) = g^i(w)f^0\), for any \( w \in W_1 \). Thus, \( h^1(w)f^0 = \mathbf{0}\), for any \( w \in W_1 \), too. Since \( f^1 = \mathbf{0}\), we know that \( f^0 \in \text{Hom}_{\mathcal{A}}(M, E) \). Then, if \( a, b, a, w \in W_1 \) and \( m \in M \), we have \( (h^1(abf^0)[m] = (ah^1(w)b) [f^0[m]] = ah^1(w)[bf^0[m] = a(h^1(w)f^0[bm]) = 0 \). Thus, \( h^1(v)f^0 = \mathbf{0}\) holds for any \( v \in AW_1 \). Now, if \( u := h^{-1} \), we get that \( u^0 = 1_E \). Moreover, if \( w \in W_1 \) and we write \( \delta(w) = \sum_j v_jv'_j \), with \( v_j, v'_j \in AW_1 \), then \( 0 = (uf)^i(w) = u^1(w) + u^1(w) + \sum_j u^1(v_j)h^1(v'_j) \). Then, \((uf)^i(w) = u^0f^1(w) + u^1(w)f^0 + \sum_j u^1(v_j)f^1(v'_j) = u^1(w)f^0 = [-h^1(w) - \sum_j u^1(v_j)h^1(v'_j)]f^0 = \mathbf{0}\). Which implies that \((uf)^1 = \mathbf{0}\), as wanted.

Definition 10.6. If \( \mathcal{C} \) is an additive \( k\)-category where idempotents split, a pair \((s, d)\) of composable morphisms \( M \xrightarrow{s} E \xrightarrow{d} N \) in \( \mathcal{C} \) is called exact iff \( s \) is the kernel of \( d \) and \( d \) is the cokernel of \( s \). A morphism of exact pairs
from $M \rightarrow E \rightarrow N$ to $M' \rightarrow E' \rightarrow N'$ is a triple of morphisms $(u, v, w)$ in $\mathcal{C}$ making the following diagram commutative

$$
\begin{array}{ccc}
M & \rightarrow & E \rightarrow N \\
\downarrow u & & \downarrow v \\
M' & \rightarrow & E' \rightarrow N'.
\end{array}
$$

If $u$, $v$ and $w$ are isomorphisms, $(u, v, w)$ is an isomorphism of exact pairs. If there is such an isomorphism with $u = 1_M$ and $w = 1_N$, the exact pairs are called equivalent.

Once we have fixed a class of exact pairs $\mathcal{E}$ closed under isomorphisms, if $(s, d) \in \mathcal{E}$, $s$ is called an inflation and $d$ is called a deflation.

**Lemma 10.7.** Assume that $(\mathcal{A}, I)$ is a Roiter interlaced weak ditalgebra. Then, whenever

$$
0 \rightarrow M \xrightarrow{f^0} E \xrightarrow{g^0} \mathcal{N} \rightarrow 0
$$

is an exact sequence in $\mathcal{A}/I$-Mod, we have that $M \xrightarrow{(f^0, 0)} E \xrightarrow{(g^0, 0)} \mathcal{N}$ is an exact pair in $(\mathcal{A}, I)$-Mod.

**Proof.** Write $f = (f^0, 0)$ and $g = (g^0, 0)$. We only prove that $f = \text{Ker} g$ (the equality $g = \text{Cok} f$ is proved dually). So suppose that $Z \xrightarrow{t} E$ satisfies $gt = 0$ in $(\mathcal{A}, I)$-Mod. Then, $q^0 h^0 = 0$ and $q^0 t^1 (v) = 0$, for all $v \in V$. Then, there exist a unique $h^0 \in \text{Hom}_R(Z, M)$ with $f^0 h^0 = t^0$ and, for each $v \in V$, there exists a unique $h^1 (v) \in \text{Hom}_R(Z, M)$ with $f^0 h^1 (v) = t^1 (v)$. Since $f^0$ is an injective morphism of $A$-modules, we get that $h^1 \in \text{Hom}_{\mathcal{A}-\mathcal{A}}(V, \text{Hom}_R(Z, M))$. Moreover, for $a \in A$ and $z \in Z$,

$$
f^0 (ah^0 (z)) = at^0 (z) = t^0 (az) + t^1 (\delta (a)) (z) = f^0 [h^0 (az) + h^1 (\delta (a)) (z)].
$$

Using again that $f^0$ is injective, we get that $h = (h^0, h^1) \in \text{Hom}_{\mathcal{A}-\mathcal{A}}(Z, M)$. Clearly, $h$ is unique such that $fh = t$ in $(\mathcal{A}, I)$-Mod. \qed

For the sake of completeness, we recall the following definition.

**Definition 10.8.** Assume that $\mathcal{C}$ is an additive $k$-category and $\mathcal{E}$ is a class of exact pairs in $\mathcal{C}$ closed under isomorphisms. Then, the class $\mathcal{E}$ is called an exact structure on $\mathcal{C}$ iff

1. The composition of deflations is a deflation.
2. For each morphism $f : Z' \rightarrow Z$ and each deflation $d : Y \rightarrow Z$ there exists a morphism $f' : Y' \rightarrow Y$ and a deflation $d' : Y' \rightarrow Z'$ such that $df' = fd'$.
3. Identities are deflations. If $gf$ is a deflation, then so is $g$.
4. Identities are inflations. If $gf$ is an inflation, then so is $f$. 

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An exact category is a category $\mathcal{C}$, as above, endowed with an exact structure $\mathcal{E}$. A functor $F : \mathcal{C} \rightarrow \mathcal{C}'$, where the categories $\mathcal{C}$ and $\mathcal{C}'$ are endowed with exact structures $\mathcal{E}$ and $\mathcal{E}'$, respectively, is called exact if it maps $\mathcal{E}$ into $\mathcal{E}'$.

**Definition 10.9.** Assume that $\mathcal{A} = (\mathcal{A}, I)$ is a Roiter interlaced weak ditalgebra with layer $(R, W)$. Consider the class $\mathcal{E} = \mathcal{E}(\mathcal{A})$ of composable pairs of morphisms $M \xrightarrow{f} E \xrightarrow{g} N$ such that there is a commutative diagram in $\mathcal{A}$-Mod

\[
\begin{array}{ccc}
M & \xrightarrow{f} & E \\
\downarrow_{\cong} & & \downarrow_{\cong} \\
M' & \xrightarrow{(\varphi^0, 0)} & E' \\
\end{array}
\]

such that $0 \rightarrow M' \xrightarrow{\varphi^0} E' \xrightarrow{\psi^0} N' \rightarrow 0$ is a split exact sequence in $R$-Mod.

We will establish that $\mathcal{E}$ is an exact structure of $\mathcal{A}$-Mod, for any Roiter interlaced weak ditalgebra $\mathcal{A}$. The following statement gives a simple description of the class $\mathcal{E}$.

**Lemma 10.10.** Let $\mathcal{A}$ be a Roiter interlaced weak ditalgebra. Then, $\mathcal{E}$ is a class of exact pairs in $\mathcal{A}$-Mod closed under isomorphisms. Moreover, if we consider morphisms $f : M \rightarrow E$ and $g : E \rightarrow N$ in $\mathcal{A}$-Mod. Then:

1. $M \xrightarrow{f} E \xrightarrow{g} N$ is a conflation iff $gf = 0$ and $0 \rightarrow M' \xrightarrow{\varphi^0} E' \xrightarrow{\psi^0} N' \rightarrow 0$ is a split exact sequence in $R$-Mod.

2. $g : E \rightarrow N$ is a deflation iff $g^0 : E \rightarrow N$ is a retraction in $R$-Mod.

3. $f : M \rightarrow E$ is an inflation iff $f^0 : M \rightarrow E$ is a section in $R$-Mod.

**Proof.** We will prove (1) and, at the same time, that $\mathcal{E}$ is a class of exact pairs in $\mathcal{A}$-Mod. If $M \xrightarrow{f} E \xrightarrow{g} N \in \mathcal{E}$, there is a commutative diagram in $\mathcal{A}$-Mod

\[
\begin{array}{ccc}
M & \xrightarrow{f} & E \\
\downarrow_{t} & & \downarrow_{s} \\
M' & \xrightarrow{(\varphi^0, 0)} & E' \\
\end{array}
\]

such that $r, s, t$ are isomorphisms and $0 \rightarrow M' \xrightarrow{\varphi^0} E' \xrightarrow{\psi^0} N' \rightarrow 0$ is a split exact sequence in $R$-Mod. In particular, we have the commutative diagram in $R$-Mod:

\[
\begin{array}{ccc}
M & \xrightarrow{f^0} & E \\
\downarrow_{t^0} & & \downarrow_{s^0} \\
M' & \xrightarrow{\varphi^0} & E' \\
\end{array}
\]

where $r^0, s^0, t^0$ are isomorphisms, which implies that $0 \rightarrow M' \xrightarrow{\varphi^0} E' \xrightarrow{\psi^0} N' \rightarrow 0$ is a split exact sequence in $R$-Mod.
Conversely, if \( M \xrightarrow{f} E \xrightarrow{g} N \) is a pair of composable morphisms in \( A\text{-Mod} \) such that \( gf = 0 \) and

\[
0 \longrightarrow M \xrightarrow{f^0} E \xrightarrow{g^0} N \longrightarrow 0
\]

is a split exact sequence in \( R\text{-Mod} \), then, by (10.2)(1), there is a commutative square in \( A\text{-Mod} \)

\[
\begin{array}{ccc}
E & \xrightarrow{g} & N \\
\downarrow s & & \downarrow r \\
E' & \xrightarrow{\psi^0,0} & N'
\end{array}
\]

where \( r, s \) are isomorphisms. It follows that \( \psi^0 \) is a retraction in \( R\text{-Mod} \). Now, consider the kernel \( \phi^0 : M' \longrightarrow E' \) of the morphism \( \psi^0 : E' \longrightarrow N' \) in \( A/I\text{-Mod} \). Since \( gf = 0 \), from (10.7), we know that there is an induced morphism of \( A\)-modules \( t : M \longrightarrow M' \) such that the following diagram commutes:

\[
\begin{array}{ccc}
M & \xrightarrow{f} & E & \xrightarrow{g} & N \\
\downarrow t & & \downarrow s & & \downarrow r \\
M' & \xrightarrow{\phi^0,0} & E' & \xrightarrow{\psi^0,0} & N'
\end{array}
\]

where the lower row is an exact pair in \( A\text{-Mod} \). In particular, we have a commutative diagram

\[
\begin{array}{ccc}
M & \xrightarrow{f^0} & E & \xrightarrow{g^0} & N \\
\downarrow t^0 & & \downarrow s^0 & & \downarrow r^0 \\
M' & \xrightarrow{\psi^0} & E' & \xrightarrow{\psi^0} & N'
\end{array}
\]

which implies that \( t^0 \) is an isomorphism. Since \( A \) is a Roiter interlaced weak ditalgebra, \( t \) is an isomorphism too and \( M \longrightarrow E \xrightarrow{g} N \) is an exact pair of \( A\text{-Mod} \), which belongs to \( E \).

(2): If \( g : E \longrightarrow N \) is a deflation, clearly, we get that \( g^0 \) is a retraction in \( R\text{-Mod} \). On the other hand, if we start from a morphism \( g : E \longrightarrow N \) in \( A\text{-Mod} \) such that \( g^0 : E \longrightarrow N \) is a retraction in \( R\text{-Mod} \), since \( A \) is a Roiter interlaced weak ditalgebra, by (10.2)(1), there is a commutative square in \( A\text{-Mod} \)

\[
\begin{array}{ccc}
E & \xrightarrow{g} & N \\
\downarrow s & & \downarrow r \\
E' & \xrightarrow{\psi^0,0} & N'
\end{array}
\]

where \( r, s \) are isomorphisms. Then, as before, we consider the kernel \( \phi^0 \) of \( \psi^0 \) in \( A/I\text{-Mod} \) and construct a commutative square in \( A\text{-Mod} \)

\[
\begin{array}{ccc}
M & \xrightarrow{f} & E & \xrightarrow{g} & N \\
\downarrow t & & \downarrow s & & \downarrow r \\
M' & \xrightarrow{\phi^0,0} & E' & \xrightarrow{\psi^0,0} & N'
\end{array}
\]

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where \( t \) is an isomorphism, so \( g \) is a deflation. The proof of (3) is dual to the proof of (2).

**Proposition 10.11.** If \( A \) is a Roiter interlaced weak ditalgebra, then \( E \) is an exact structure on \( A Mod \).

**Proof.** From (10.10), items (1), (3) and (3) of Definition (10.6) are clear. Let us prove the remaining one (2). Take a morphism \( f : Z' \to Z \) and any deflation \( d : Y \to Z \). Consider the morphism \((f, d) : Z' \oplus Y \to Z\) in \((A, I) Mod\). Since \( d \) is a deflation, \( d^0 \) is a retraction in \( R Mod \) and, therefore, so is \((f^0, d^0)\). Then, \((f, d) : Z' \oplus Y \to Z\) is a deflation and its kernel has the form \((d', -f')^t : Y' \to Z' \oplus Y\). Therefore, we have \( fd' = df' \) and a pull-back diagram in \( R Mod \):

\[
\begin{array}{ccc}
Y' & \xrightarrow{(d')^0} & Z' \\
\downarrow{(f')^0} & & \downarrow{f^0} \\
Y & \xrightarrow{d^0} & Z.
\end{array}
\]

Since \( d^0 \) is a retraction in \( R Mod \), the same is true for \((d')^0\). Therefore, \( d' \) is a deflation, proving our statement.

The preceding result applies to the module category of the Roiter interlaced weak ditalgebra \( A(\Delta) = (A(\Delta), I) \) constructed before, see (4.19). The following theorem combines the equivalence of (6.3) with the one constructed in (8.3). It generalizes the result given in [9] for the case of quasi-hereditary algebras. It relies on the explicit description of the equivalence of categories \( F(\Delta) \cong H^0(\text{tw}(A)) \) mentioned in (6.3). The detailed proof of this result is beyond the scope of this paper.

**Theorem 10.12.** Consider the equivalence functors \( K : F(\Delta) \to H^0(\text{tw}(A)) \) of (6.3), \( E : H^0(\text{tw}(A)) \to H^{-1}(\text{tw}(B)) \), of (7.3), and the equivalence \( M : H^{-1}(\text{tw}(B)) \to A(\Delta) Mod \) constructed in (8.3). Then the composition

\[
G := MEK : F(\Delta) \to A(\Delta) - \text{mod}
\]

is an equivalence mapping short exact sequences to conflations and such that \( G(\Delta_i) \cong S_i \), for all \( i \in P \), where \( S_i = (ke_i, \rho_{S_i}) \) and \( \rho_{S_i} = 0 \), with the notation of (8.2).

### 11 The right algebra of \((A, I)\)

In this section we recall some results by W.L. Burt and M.C. Butler on bocses and reformulate them in our context for their application in the next section.

**Remark 11.1.** In this section we assume that \( A = (A, I) \) is a special triangular interlaced weak ditalgebra, in the following sense. The weak ditalgebra \( A = (T, \delta) \) admits a triangular layer \((S, W_0 \oplus W_1)\), where \( S \) is a finite product of...
fields. We write $S = \bigoplus_{i \in P} ke_i$, where $1 = \sum_{i \in P} e_i$ is a decomposition of the unit of $S$ as a sum of primitive orthogonal central idempotents. We adopt the notation of (9.1)–(9.4) with $R = S$. Thus, we have $T = T_S(W_0 \oplus W_1)$, $A = T_S(W_0)$, $V = A \otimes_S W_1 \otimes_S A$, and $I$ is an ideal of $A$. We will assume that $W_0$ and $W_1$ are finite dimensional $S$-$S$-bimodules and we will furthermore assume that the quotient algebra $A := A/I$ is finite-dimensional.

Recall, from (4.3), that

$$V := V/(IV + VI) = \frac{A \otimes_S W_1 \otimes_S A}{I \otimes W_1 \otimes A + A \otimes W_1 \otimes I} \cong A \otimes_S W_1 \otimes_S A.$$ 

We denote by $\pi = p \otimes \text{id}_{W_1} : A \otimes W_1 \otimes A \longrightarrow A \otimes W_1 \otimes A$, where $p : A \longrightarrow A$ is the canonical projection.

The pair $A$ is a Roiter interlaced weak ditalgebra, as in (10.1), and §10 applies to its module category $A$-$\text{Mod}$. In particular, if we make $A = A/I$, the embedding functor $L = L_A : A$-$\text{mod} \longrightarrow A$-$\text{mod}$, which maps each $A$-module $M$ onto itself and any morphism $f^0 : M \longrightarrow N$ in $A$-$\text{mod}$ onto $(f^0, 0) : M \longrightarrow N$ in $A$-$\text{mod}$, is an exact functor, when we consider the exact structure $\mathcal{E}$ on $A$-$\text{mod}$ introduced in (10.9), see (10.10).

**Definition 11.2.** Assume that $A$ is a special interlaced weak ditalgebra as above. Then, the right algebra of $A$ is the finite-dimensional $k$-algebra $\Gamma := \text{End}_A(A)^{\text{op}}$. There is an embedding of $A$ in $\Gamma$ given by the composition

$$A \longrightarrow \text{End}_A(A)^{\text{op}} \longrightarrow \text{End}_A(A)^{\text{op}} = \Gamma,$$

so $\Gamma$ becomes naturally an $A$-algebra.

The statements in (11.13) outline important relations between the algebras $A$ and $\Gamma$, they translate results of W.L. Burt and M.C.R. Butler to the terminology we use here, see [5] and [6]. For the sake of completeness, we give a sketch-review of their proofs. In the rest of this section, $A$ is a special interlaced weak ditalgebra, $\Gamma$ is its right algebra, and we adopt the preceding notation.

**Lemma 11.3.** For any $M \in A$-$\text{mod}$, there is an exact sequence of left $A$-modules

$$0 \longrightarrow \text{Hom}_A(A, M) \longrightarrow \text{Hom}_A(A, M) \longrightarrow 0,$$

where the space $\text{Hom}_A(A, M)$ is a left $A$-module through $A \longrightarrow \text{End}_A(A)^{\text{op}}$.

**Proof.** Since $M \in A$-$\text{mod}$, any $f^1 \in \text{Hom}_A(A, V, \text{Hom}_k(A, M))$ determines an $f^1 \in \text{Hom}_A(A, V, \text{Hom}_k(A, M))$ such that $f^1 \pi = f^1$. Then, we can define

$$\xi : \text{Hom}_A(A, M) \longrightarrow \text{Hom}_A(V, M) \quad f \longmapsto \xi(f) : V \longrightarrow M \quad v \longmapsto f^1(v)[1].$$
Thus, $\xi$ is a morphism of left $A$-modules. Suppose that $f \in \text{Hom}_A(A, M)$ satisfies that $\xi(f) = 0$. Thus, $f^1(v)[1] = 0$, for all $v \in V$. Hence, $f^1(v)[b] = f^1(vb)[1] = 0$, for all $b \in A$. Thus, $f^1(v) = 0$, for all $v \in V$, and $f^1 = 0$, thus $f = 0$. This means that $f \in \text{Im} L$. Therefore, $\text{Im} L = \text{Ker} \xi$.

Take $g \in \text{Hom}_A(V, M)$ and consider its image $g^1$ under the canonical isomorphism

$$\text{Hom}_A(V, M) \cong \text{Hom}_A(V \otimes_A A, M) \cong \text{Hom}_{A-A}(V, \text{Hom}_k(A, M)).$$

Thus, $g^1 : V \longrightarrow \text{Hom}_k(A, M)$ is a morphism of $A$-bimodules satisfying $g^1(v)[b] = g(vb)$, for $v \in V$ and $b \in A$. Consider the composition morphism $g^1 := g^1 \pi \in \text{Hom}_{A-A}(V, \text{Hom}_k(A, M))$. Then, fix any $m_0 \in M$ and define, for $a \in A$,

$$g^0(a) := am_0 - g^1(\delta(a))[1].$$

Thus, $g^0 \in \text{Hom}_k(A, M)$. Moreover, $(g^0, g^1) \in \text{Hom}_A(A, M)$. Indeed, given $a \in A$ and $b \in A$, we have

$$bg^0(a) = bam_0 - bg^1(\delta(a))[1] = bam_0 - g^1(b\delta(a))[1] = bam_0 - g^1(\delta(b)a - \delta(ba))[1] = bam_0 - g^1(\delta(b)a)[1] + g^1(\delta(b))[a] = g^0(ba) + g^1(\delta(b))[a].$$

Since $\xi(g^0, g^1)(v) = g^1(v)[1] = g(v)$, for all $v \in V$, $\xi$ is surjective and we are done.

**Remark 11.4.** Using the same argument in the proof of [3](6.20), we obtain that any projective $A$-module is an $\mathcal{E}$-projective $A$-module. Moreover, every $\mathcal{E}$-projective $A$-module is isomorphic to a direct summand in $\mathcal{A}$-mod of some projective $A$-module.

Then, the functor $F = \text{Hom}_A(A, -) : \mathcal{A}$-mod $\longrightarrow \Gamma$-mod is exact. The following lemma implies that the right $A$-module $\Gamma$ is flat. Indeed, the functor $\Gamma \otimes_A -$ is exact because $\Gamma \otimes_A - \cong FL$ and we know that $L$ and $F$ are exact functors, see [10.7].

It is not hard to see that the functor $F$ is in fact full and faithful. The proof of [3](7.12) can be adapted to this situation.

**Lemma 11.5.** For $M \in \mathcal{A}$-mod, the map $\sigma_M : \Gamma \otimes_A M \longrightarrow \text{Hom}_A(A, M)$ given by $\sigma(f \otimes m) = f_m = (f^0_m, f^1_m)$, where $f^0_m(a) = f^0_m(a)m$ and $f^1_m(v)[a] = f^1(v)[a]m$ defines a natural isomorphism

$$\sigma_M : \Gamma \otimes_A M \longrightarrow \text{Hom}_A(A, L(M)).$$

**Proof.** Notice that $\sigma_A : \Gamma \otimes_A A \longrightarrow \Gamma$ is just the isomorphism given by multiplication. From the additivity of the functors $\Gamma \otimes_A -$ and $\text{Hom}_A(A, L(-))$, we obtain that $\sigma_{A^n}$ is an isomorphism too. Finally, if we consider a free resolution $A^m \rightarrow A^n \rightarrow M \rightarrow 0$ of $M$ in $\mathcal{A}$-mod, we can apply the exact functors $\Gamma \otimes_A -$
and \( \text{Hom}_A(\mathcal{A}, L(-)) \) to it, and then \( \sigma_M \) is an isomorphism because the following diagram commutes

\[
\begin{array}{cccccc}
\Gamma \otimes A \mathcal{A}^m & \rightarrow & \Gamma \otimes A \mathcal{A}^n & \rightarrow & \Gamma \otimes A M & \rightarrow & 0 \\
\sigma_{\mathcal{A}^m} & \downarrow & \sigma_{\mathcal{A}^n} & \downarrow & \sigma_M & \\
\text{Hom}_A(\mathcal{A}, \mathcal{A}^m) & \rightarrow & \text{Hom}_A(\mathcal{A}, \mathcal{A}^n) & \rightarrow & \text{Hom}_A(\mathcal{A}, M) & \rightarrow & 0.
\end{array}
\]

\[\square\]

Lemma 11.6. The functor \( T := \Gamma \otimes_A - : \mathcal{A}-\text{mod} \rightarrow \Gamma-\text{mod} \) admits as a right adjoint the restriction functor \( S : \Gamma-\text{mod} \rightarrow \mathcal{A}-\text{mod} \). Consider the adjunction isomorphism

\[\zeta : \text{Hom}_\Gamma(TM, N) \rightarrow \text{Hom}_A(M, SN),\]

and its unit \( \alpha \), that is the natural transformation \( \alpha : 1_{\mathcal{A}-\text{mod}} \rightarrow ST \) defined, for \( M \in \mathcal{A}-\text{mod} \), by \( \alpha_M := \zeta(1_{TM}) : M \rightarrow STM \). Then, for each \( M \in \mathcal{A}-\text{mod} \), we have the commutative square

\[
\begin{array}{ccc}
M & \xrightarrow{\alpha_M} & \Gamma \otimes A M \\
\psi_M & \downarrow & \sigma_M \\
\text{Hom}_A(\mathcal{A}, M) & \xrightarrow{L} & \text{Hom}_A(\mathcal{A}, M),
\end{array}
\]

where \( \psi_M \) is the canonical isomorphism and \( \sigma_M \) is the isomorphism introduced in (11.5). In particular, \( \alpha_M \) is a monomorphism.

Proof. For \( M \in \mathcal{A}-\text{mod} \) and \( N \in \Gamma-\text{mod} \), we have the adjunction isomorphism

\[
\zeta : \text{Hom}_\Gamma(TM, N) \rightarrow \text{Hom}_A(M, SN)
\]

\[h \mapsto \zeta(h) : M \rightarrow N \]

\[m \mapsto h(1 \otimes m),\]

with inverse

\[
\zeta' : \text{Hom}_A(M, SN) \rightarrow \text{Hom}_\Gamma(TM, N)
\]

\[g \mapsto p(1 \Gamma \otimes g),\]

where \( p : \Gamma \otimes_A N \rightarrow N \) is the product morphism. The verification of the naturality of \( \zeta \) is straightforward.

Notice that \( \alpha_M(m) = 1 \otimes m \), for \( m \in M \). Make \( \theta := \sigma_M \alpha_M \). Then, we have \( \theta(m) = \sigma_M \alpha_M(m) = \sigma_M(id \mathcal{A} \otimes m) = (\theta(m)^0, \theta(m)^1) \), where \( \theta(m)^0[a] = am \) and \( \theta(m)^1(v)[a] = id \mathcal{A}(v)[a]m = 0 \). Then, \( \sigma_M \alpha_M(m) = \theta(m) = (\psi_M(m), 0) = L \psi_M(m) \), and we are done. \( \square \)

Corollary 11.7. For each \( M \in \mathcal{A}-\text{mod} \), there is the following exact sequence in \( \mathcal{A}-\text{mod} \)

\[0 \rightarrow M \xrightarrow{\alpha_M} \Gamma \otimes_A M \rightarrow \text{Hom}_A(V, M) \rightarrow 0.\]
Proof. It follows from (11.3) and (11.6). \qed

**Definition 11.8.** An $\mathcal{A}$-bimodule $V$ is called projectivizing iff $V \otimes A X$ is projective for all $X \in \mathcal{A}$-mod and $Y \otimes A V$ is projective for all $Y \in \text{mod-} \mathcal{A}$.

**Lemma 11.9.** With our fixed notation, the $S\mathcal{S}$-bimodule $W_1$ is freely generated by a set $B_1$. Hence, the $\mathcal{A}$-bimodule $V$ is projectivizing.

Proof. Since $S$ is a finite product of fields and $W_1$ is finite-dimensional, there is a directed $k$-basis $B_1$ of $W_1$ and an isomorphism $W_1 \cong \bigoplus_{w \in B_1} A \epsilon_t(w) \otimes_k e_s(w) S$. Then, we have

\[
V \cong \mathcal{A} \otimes S W_1 \otimes_S A \\
\cong \bigoplus_{w \in B_1} \mathcal{A} \otimes S A \epsilon_t(w) \otimes_k e_s(w) S \otimes_S A \\
\cong \bigoplus_{w \in B_1} \mathcal{A} \epsilon_t(w) \otimes_k e_s(w) \mathcal{A}
\]

Now notice that, given a family $\{Q_i\}_{i=1}^n$ of projective left $\mathcal{A}$-modules and a family $\{P_i\}_{i=1}^n$ of projective right $\mathcal{A}$-modules, then the $\mathcal{A}$-bimodule $V := \bigoplus_{i=1}^n P_i \otimes_k Q_i$ is projectivizing. Thus, the $\mathcal{A}$-bimodule $V$ is projectivizing. \qed

**Corollary 11.10.** The right $\mathcal{A}$-module $\Gamma$ is projective.

Proof. Taking $M = A$, from (11.3), we have the exact sequence of vector spaces

\[
0 \rightarrow \text{Hom}_A(A, A) \rightarrow \text{Hom}_A(A,A) \xrightarrow{\xi} \text{Hom}_A(V,A) \rightarrow 0.
\]

Thus, we have the exact sequence of vector spaces

\[
0 \rightarrow A \xrightarrow{s} \Gamma \xrightarrow{\xi} \text{Hom}_A(V,A) \rightarrow 0,
\]

where $s$ is the canonical embedding. This is, in fact, an exact sequence of right $\mathcal{A}$-modules. Since the left $\mathcal{A}$-module $V$ is finitely generated projective, its dual right $\mathcal{A}$-module $\text{Hom}_A(V,A)$ is finitely generated projective, thus the last exact sequence splits, and $\Gamma_{\mathcal{A}}$ is projective. \qed

From [5](3.6) or [6](11.8), we have the following.

**Lemma 11.11.** If $V$ is a projectivizing $\mathcal{A}$-bimodule, then $\text{Hom}_A(V, M)$ is an injective $\mathcal{A}$-module, for all $M \in \text{mod-} \mathcal{A}$.

**Proposition 11.12.** Consider the tensor functor $T := \Gamma \otimes_{\mathcal{A}} - : \text{mod-} \mathcal{A} \rightarrow \text{mod-} \Gamma$. Then, every extension $e : 0 \rightarrow TM \rightarrow E \rightarrow TN \rightarrow 0$ in $\Gamma$-mod is equivalent to an extension $Te'$ obtained by applying $T$ to an extension $e'' : 0 \rightarrow M \rightarrow E'' \rightarrow N \rightarrow 0$ in $\text{mod-} \mathcal{A}$.
Proof. We have the adjoint pair of exact functors

\[ \mathbf{A}\text{-mod} \xleftarrow{S} \mathbf{T_{\mathbf{A}} \text{-mod}} \xrightarrow{T} \Gamma\text{-mod}, \]

where \( S \) is the restriction functor. Moreover, the morphism \( \alpha_M : M \rightarrow STM \) is a monomorphism with injective cokernel. Then, we can follow the argument of [6](10.3) to obtain the wanted extension equivalent to \( e \).

\[ \square \]

Proposition 11.13. Given a special interlaced weak ditalgebra \( \mathbf{A} \), we have:

1. The functor \( F = \text{Hom}_\mathbf{A}(\mathbf{A}, -) : \mathbf{A}\text{-mod} \rightarrow \Gamma\text{-mod} \) is a full and faithful exact functor.

2. The \( \mathbf{A}\text{-algebra} \Gamma \) is a right \( \mathbf{A}\text{-module} \) by restriction, it determines an exact functor \( \Gamma \otimes_\mathbf{A} - : \mathbf{A}\text{-mod} \rightarrow \Gamma\text{-mod} \).

and, we have \( \Gamma \otimes_\mathbf{A} - \cong \text{Hom}_\mathbf{A}(\mathbf{A}, L(-)) \).

3. The functor \( F \) restricts to an equivalence of categories \( F : \mathbf{A}\text{-mod} \rightarrow I \), where \( I \) is the full subcategory of \( \Gamma\text{-mod} \) of modules induced from \( \mathbf{A}\text{-mod} \), that is by the class of \( \Gamma\text{-modules} \) isomorphic to some \( \Gamma \otimes_\mathbf{A} N \), for some \( N \in \mathbf{A}\text{-mod} \).

Moreover, the subcategory \( I \) of \( \Gamma\text{-mod} \) is closed under extensions.

4. The functor \( \Gamma \otimes_\mathbf{A} - : \mathbf{A}\text{-mod} \rightarrow \Gamma\text{-mod} \) induces epimorphisms

\[ \text{Ext}^n_A(M, N) \rightarrow \text{Ext}^n(\Gamma \otimes_\mathbf{A} M, \Gamma \otimes_\mathbf{A} N), \]

for all \( M, N \in \mathbf{A}\text{-mod} \) and \( n \geq 1 \). They are isomorphisms for \( n \geq 2 \).

Proof. (1) and (2) were remarked in [11.14], and (3) is [11.12].

(4): Recall that, we have an exact sequence \( 0 \rightarrow N \xrightarrow{\alpha_N} \Gamma \otimes_\mathbf{A} N \rightarrow I \rightarrow 0 \) in \( \mathbf{A}\text{-mod} \), with \( I \) injective, for any \( N \in \mathbf{A}\text{-mod} \). The corresponding long exact sequence gives epimorphisms

\[ \text{Ext}^n_A(M, N) : \text{Ext}^n_A(M, N) \rightarrow \text{Ext}^n_A(M, \Gamma \otimes_\mathbf{A} N), \]

for \( n \geq 1 \) and isomorphisms for \( n \geq 2 \). We have a commutative diagram

\[ \begin{array}{ccc}
\text{Ext}^n_A(M, N) & \xrightarrow{\gamma^n} & \text{Ext}^n_A(M, \Gamma \otimes_\mathbf{A} N) \\
\text{Ext}^n_M(M, \alpha_N) & & \text{Ext}^n_A(M, \beta \otimes_\mathbf{A} N)
\end{array} \]

where \( \gamma^n \) denotes the morphisms induced by the application of the exact functor \( T = \Gamma \otimes_\mathbf{A} - \), and \( \beta \otimes_\mathbf{A} N : \Gamma \otimes_\mathbf{A} N \rightarrow \Gamma \otimes_\mathbf{A} N \) is the product map. Since the composition \( \text{Ext}^n_A(\Gamma \otimes_\mathbf{A} M, \beta \otimes_\mathbf{A} N) \gamma^n \) is an isomorphism, see [6](20.9), we are done. \[ \square \]
12 Strict interlaced weak ditalgebras

Definition 12.1. Let $P = (P, \preceq)$ be a finite preordered set. Let $A = (A, I)$ be a triangular interlaced weak ditalgebra, thus $A = (T_S(W_0 \oplus W_1), \delta)$ is a weak ditalgebra with triangular layer $(S, W_0 \oplus W_1)$ and $I$ is an $A$-triangular ideal of $A := T_S(W_0)$. We will say that $A$ is $P$-strict interlaced weak ditalgebra iff the following holds:

1. The algebra $S$ is a finite product of $|P|$ copies of the field $k$, thus $S = \bigoplus_{i \in P} k e_i$, where $1 = \sum_{i \in P} e_i$ is a decomposition of the unit of $S$ as a sum of primitive orthogonal central idempotents.
2. We have that $e_j W_1 e_i \neq 0$ implies $i \preceq j$,
3. We have that $e_j W_0 e_i \neq 0$ implies $i < j$.
4. $I \subseteq \text{rad}(A)^2$.

We will denote by $A$ the quotient $k$-algebra $A/I$.

Remark 12.2. If $A$ be a $P$-strict interlaced weak ditalgebra as above, we will identify the graded tensor algebra $A = T_S(W_0)$ with the path algebra $k(B_0)$ of the quiver $B_0$ with set of points $P$; such that for $i, j \in P$, the set of solid arrows from $i$ to $j$ is a fixed basis $B_0(i,j)$ of the vector space $e_j W_0 e_i$.

Notice that the quiver $B_0$ coincides with the Gabriel quiver of $A$ and has not oriented cycles. Then, we have the partial order of precedence $\preceq$ in $P$, as defined in (1.10). So, for $i, j \in P$, we have $i < j$ iff there is a non-trivial path from $i$ to $j$ in the quiver $B_0$. In fact, we always have that $i < j$ implies $i \preceq j$.

Since the quiver $B_0$ has no oriented cycle, the algebra $A$ is finite-dimensional. Let us write $P_i := A e_i$, for all $i \in P$. Then, the family $\{P_i\}_{i \in P}$ is a complete family of representatives of the indecomposable projective $A$-modules (we denote by $\{S_i\}_{i \in P}$ the corresponding family of simple $A$-modules $S_i = P_i/\text{rad}(P_i)$, which is a complete family of representatives of the isoclasses of the simple $A$-modules).

Lemma 12.3. Assume that $A$ is a $P$-strict interlaced weak ditalgebra, then the algebra $A$ is directed. Moreover, the triple $\mathcal{H} = (P, \preceq, \{S_i\}_{i \in P})$ is a strict homological system for the algebra $A$.

Proof. From (1.13), we already know that $\mathcal{H} = (P, \preceq, \{S_i\}_{i \in P})$ is a strict homological system for $A$. Since $i \not\preceq j$ implies $i < j$, we immediately obtain that $\mathcal{H}$ is a strict homological system for $A$. \hfill $\square$

Remark 12.4. Let $A$ be a $P$-strict interlaced weak ditalgebra and adopt the notation of (12.1). Then, we have a special interlaced weak ditalgebra and the results of sections §10 and §11 hold.

Lemma 12.5. Let $A$ be a $P$-strict interlaced ditalgebra. Then, there is a family $\{Q_i\}_{i \in P}$ of non-isomorphic indecomposable $A$-projective $A$-modules, which is a
complete family of representatives of the indecomposable $\mathcal{E}$-projective $\mathbf{A}$-modules and, moreover, for each $i \in \mathcal{P}$, we have an isomorphism in $\mathbf{A}$-mod

$$P_i \cong \mathcal{Q}_i \oplus \bigoplus_{j < i} \mathcal{Q}_{i,j}, \text{ for some } m_{i,j} \geq 0.$$  

Proof. Given $i \in \mathcal{P}$, as we remarked in (11.4), the $\mathbf{A}$-module $P_i$ is $\mathcal{E}$-projective. Consider its decomposition as a direct sum of indecomposables. So we have an isomorphism in $\mathbf{A}$-mod:

$$g = (g^0, g^1) : P_i \longrightarrow \bigoplus_{j=1}^{n_i} \mathcal{Q}_{i,j},$$

a direct sum of indecomposable $\mathcal{E}$-projectives. Then, we have the linear isomorphism $g^0 : e_i \mathbf{A} e_i \longrightarrow \bigoplus_{j=1}^{n_i} e_i \mathcal{Q}_{i,j}$, where the domain is one dimensional. Then, there is a unique $\mathcal{Q}_i : = \mathcal{Q}_{i,j}$ with $\dim e_i \mathcal{Q}_{i,j} = 1$. We shall prove our lemma by induction on the partial order $\prec$ on $\mathcal{P}$.

If $i$ is maximal (that is if $i$ is a sink in $\mathcal{B}_0$), then $P_i \cong \mathcal{Q}_i$ is indecomposable in $\mathbf{A}$-mod and we are done. Assume we have $P_s \cong \mathcal{Q}_s \oplus \bigoplus_{j=1}^{n_i} m_{s,j} \mathcal{Q}_j$, for all $s \in \mathcal{P}$ with $i \prec s$, for a fixed $i \in \mathcal{P}$. Consider the indecomposable decomposition of $P_i \cong \mathcal{Q}_i \oplus \bigoplus_{j=1}^{n_i} \mathcal{Q}_{i,j}$ of $P_i$ in $\mathbf{A}$-mod. Fix any direct summand $Q : = \mathcal{Q}_{i,j}$, not isomorphic to $\mathcal{Q}_i$, then $Q$ is an indecomposable $\mathcal{E}$-projective $\mathbf{A}$-module such that $e_i Q = 0$. Moreover, if $e_j Q \neq 0$, we have $e_i \mathbf{A} e_i \neq 0$, so $i \prec j$. Now, consider the projective cover $q : P \longrightarrow Q$ of $Q$ in $\mathbf{A}$-mod, then we have the commutative diagram

$$\begin{array}{ccc}
P & \xrightarrow{q} & Q \\
\downarrow & & \downarrow \\
Q / \text{rad}(Q) & & \\
\end{array}$$

There, the module $P_i$ is a direct summand of $P$ iff $S_j$ is a direct summand of $Q / \text{rad}(Q)$, which is only possible if $e_j Q \neq 0$, that is if $i \prec j$. This means that $P_i \cong \bigoplus_{s \prec j} s_j P_j$ in $\mathbf{A}$-mod, for some $j \geq 0$. Then, if $\sigma : K \longrightarrow P$ is the kernel of $q$ in $\mathbf{A}$-mod, we have the conflation

$$K \xrightarrow{(\sigma, 0)} P \xrightarrow{(q, 0)} Q$$

in the exact category $(\mathbf{A}$-mod, $\mathcal{E})$, where $Q$ is an $\mathcal{E}$-projective module. Hence, $Q$ is a direct summand of $P \cong \bigoplus_{s \prec j} s_j P_j$ in $\mathbf{A}$-mod. Then, by induction hypothesis, $Q \cong \mathcal{Q}_i$, for some $i \prec j \prec t$, and this finishes the proof. \hfill \Box

Proposition 12.6. Let $\mathbf{A}$ be a $\mathcal{P}$-strict interlaced weak ditalgebra and consider its right algebra $\Gamma$. For $i \in \mathcal{P}$, make $P_i' : = \Gamma \otimes \mathbf{A} \mathcal{Q}_i$ and $\Delta_i' : = \Gamma \otimes \mathbf{A} \mathcal{S}_i$. Then, $(P_i')_{i \in \mathcal{P}}$ is a complete family of representatives of the indecomposable projective $\Gamma$-modules and $\mathcal{H}' : = (\mathcal{P}, \leq, \{\Delta_i'\}_{i \in \mathcal{P}})$ is a strict homological system for $\Gamma$.

Proof. Since $F$ is full and faithful, each $P_i' = \Gamma \otimes \mathbf{A} \mathcal{Q}_i \cong F(\mathcal{L}(Q_i)) = F(Q_i)$ and each $\Delta_i' = \Gamma \otimes \mathbf{A} \mathcal{S}_i \cong F(\mathcal{L}(S_i)) = F(S_i)$ are indecomposable $\Gamma$-modules.

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Moreover, using again that \( F \) is full and faithful, and (12.5), we know that \( \{ P'_i \}_{i \in \mathcal{P}} \) and \( \{ \Delta'_i \}_{i \in \mathcal{P}} \) are families of non-isomorphic \( \Gamma \)-modules.

From (12.5), we know that \( \Gamma \cong \Gamma \otimes_{\mathcal{A}} \mathcal{A} \cong \Gamma \otimes_{\mathcal{A}} (\bigoplus_{i \in \mathcal{P}} \Delta_i) \cong F(\bigoplus_{i \in \mathcal{P}} P_i) \cong \bigoplus_{i \in \mathcal{P}} F(Q_i) \oplus (\bigoplus_{i \neq j} m_{ij} F(Q_j)) \), so every \( P'_i \) is a projective indecomposable \( \Gamma \)-module, and \( \{ P'_i \}_{i \in \mathcal{P}} \) is a complete family of representatives of the indecomposable projective \( \Gamma \)-modules.

Assume that, for \( i, j \in \mathcal{P} \), we have \( \text{Hom}_{\mathcal{A}}(\Delta'_i, \Delta'_j) \neq 0 \). Then, we have that \( \text{Hom}_{\mathcal{A}}(S_i, S_j) \neq 0 \). If we assume that \( i \neq j \), and take \( 0 \neq f = (f^0, f^1) \in \text{Hom}_{\mathcal{A}}(S_i, S_j) \), we have that \( f^0 = 0 \), but \( 0 \neq f^1 \in \text{Hom}_{\mathcal{A}}(S_i, S_j) \) \( \cong e_j W_1 e_i \), therefore, since \( \mathcal{A} \) is strict, we obtain that \( i \leq j \).

Now, suppose that, for \( i, j \in \mathcal{P} \) and \( n \in \mathbb{N} \), we have that \( \text{Ext}^n_{\mathcal{A}}(\Delta'_i, \Delta'_j) \neq 0 \). From (11.13), we obtain \( \text{Ext}^n_{\mathcal{A}}(S_i, S_j) \neq 0 \) and, from (12.3), we get \( \overline{i} < \overline{j} \).

So, once we show that \( \mathcal{H}_{\overline{i}} \) is an admissible homological system for the algebra \( \Gamma \), we get that it is strict.

Fix \( i \in \mathcal{P} \) and consider the exact sequence \( 0 \longrightarrow \text{rad}(P_i) \longrightarrow P_i \longrightarrow S_i \longrightarrow 0 \) in \( \mathcal{A}\text{-mod} \). Applying \( L_{\mathcal{A}} \), we have a conflation \( \text{rad}(P_i) \longrightarrow P_i \longrightarrow S_i \) in \( \mathcal{A}\text{-mod} \). Consider the following composition of morphisms in \( \mathcal{A}\text{-mod} \)

\[
g_i = (g_0^i, g_1^i) = (Q_i \xrightarrow{(\sigma_i, 0)} Q_i \oplus (\bigoplus_{i \neq j} m_{ij} Q_j) \cong P_i \xrightarrow{(\nu^0, 0)} S_i).
\]

Since \( g_0^i \neq 0 \), we get that \( g_0^i \) is a retraction, so \( g_i \) is a deflation and we have a conflation \( K_i \xrightarrow{f_i} Q_i \xrightarrow{g_i} S_i \) in \( \mathcal{E} \). Here, we know that \( e_j K_i \neq 0 \) implies \( e_j Q_i \neq 0 \), hence that \( i < j \). By the definition of \( \mathcal{E} \) in (11.19), we know there is a commutative diagram

\[
\begin{array}{cccc}
K_i & \xrightarrow{f_i} & Q_i & \xrightarrow{g_i} & S_i \\
\cong & & \cong & & \cong \\
K'_i & \xrightarrow{(\nu^0, 0)} & E'_i & \xrightarrow{(\nu^0, 0)} & S'_i,
\end{array}
\]

where \( 0 \longrightarrow K'_i \longrightarrow E'_i \longrightarrow S'_i \longrightarrow 0 \) is exact in \( \mathcal{A}\text{-mod} \). Applying \( F \) to the diagram, from (11.13), we get the commutative diagram in \( \Gamma\text{-mod} \) with exact lower row

\[
\begin{array}{cccc}
F(K_i) & \xrightarrow{F(f_i)} & F(Q_i) & \xrightarrow{F(g_i)} & F(S_i) \\
\cong & & \cong & & \cong \\
F(K'_i) & \xrightarrow{F(\nu^0, 0)} & F(E'_i) & \xrightarrow{F(\nu^0, 0)} & F(S'_i).
\end{array}
\]

Hence, we obtain an exact sequence of \( \Gamma \)-modules

\[
0 \longrightarrow \Gamma \otimes_\mathcal{A} K_i \longrightarrow \Gamma \otimes_\mathcal{A} Q_i \longrightarrow \Gamma \otimes_\mathcal{A} S_i \longrightarrow 0,
\]

where \( P'_i = \Gamma \otimes_\mathcal{A} Q_i \) and \( \Delta'_i = \Gamma \otimes_\mathcal{A} S_i \).

Now, we show that \( \Gamma \otimes_\mathcal{A} K_i \) is filtered by \( \Gamma \)-modules \( \Delta'_i \) with \( \overline{i} < \overline{j} \). Recall that the \( \mathcal{A} \)-module \( K_i \) admits a simple \( \mathcal{A} \)-module \( S_j \) as a composition factor if \( \text{Hom}_{\mathcal{A}}(\Delta_j, K_i) \neq 0 \), that is iff \( e_j K_i \neq 0 \), thus \( i < j \). Consider a composition
series $0 \subseteq E_\ell \subseteq \cdots \subseteq E_1 \subseteq E_0 = K_i$ of the $A$-module $K_i$, then we have exact sequences

$$0 \rightarrow E_\ell \rightarrow E_{\ell-1} \rightarrow S_{j_2} \rightarrow 0$$
$$0 \rightarrow E_{\ell-1} \rightarrow E_{\ell-2} \rightarrow S_{j_3} \rightarrow 0$$
$$\vdots$$
$$0 \rightarrow E_1 \rightarrow E_0 \rightarrow S_{j_t} \rightarrow 0$$

with $i < j_1, \ldots, j_t$ and $E_\ell \cong S_{j_1}$. Applying the exact functor $\Gamma \otimes_A -$ we obtain exact sequences

$$0 \rightarrow \Gamma \otimes_A S_{j_1} \rightarrow \Gamma \otimes_A E_{\ell-1} \rightarrow \Gamma \otimes_A S_{j_2} \rightarrow 0$$
$$0 \rightarrow \Gamma \otimes_A E_{\ell-1} \rightarrow \Gamma \otimes_A E_{\ell-2} \rightarrow \Gamma \otimes_A S_{j_3} \rightarrow 0$$
$$\vdots$$
$$0 \rightarrow \Gamma \otimes_A E_1 \rightarrow \Gamma \otimes_A E_0 \rightarrow \Gamma \otimes_A S_{j_t} \rightarrow 0.$$  

Then, the module $\Gamma \otimes_A K_i$ is filtered by $\Gamma$-modules $\Delta_i'$ with $i < j_1, \ldots, j_t$, thus with $\bar{i} < j_1, \ldots, j_t$. Thus, $P_i' \in \mathcal{F}(\Delta')$, for all $i \in \mathcal{P}$, and $\mathcal{H}'$ is a strict homological system.

**Lemma 12.7.** Adopt the notation of (11.13). Then, we have $\mathcal{I} = \mathcal{F}(\Delta')$. Moreover, the family $\{P_i'\}_{i \in \mathcal{P}}$ is a complete family of representatives of the indecomposable $\mathcal{F}(\Delta')$-projective objects.

**Proof.** Notice first that since $\mathcal{I}$ is closed under extensions and $\Delta_i' = \Gamma \otimes_A S_i \in \mathcal{I}$, an easy induction shows that $\mathcal{F}(\Delta') \subseteq \mathcal{I}$. Conversely, given $\Gamma \otimes_A M \in \mathcal{I}$, a composition series for the $A$-module $M$ is transformed by the application of $\Gamma \otimes_A -$ into a $\Delta$-filtration of $\Gamma \otimes_A M$, so it belongs to $\mathcal{F}(\Delta')$. So, indeed we have $\mathcal{I} = \mathcal{F}(\Delta')$.

From (12.6), we know that each $P_i' \in \mathcal{F}(\Delta')$, and it is an indecomposable projective $\Gamma$-module, so, it is an indecomposable $\mathcal{F}(\Delta')$-projective. Assume that $Q' \in \mathcal{F}(\Delta')$ is an indecomposable $\mathcal{F}(\Delta')$-projective. Since $\mathcal{F}(\Delta')$ consists of induced $\Gamma$-modules, we know that $Q' \cong \Gamma \otimes_A Q$, for some $Q \in A$-mod. Consider any exact sequence in $A$-mod

$$0 \rightarrow K \xrightarrow{f} P \xrightarrow{g} Q \rightarrow 0$$

with $P$ a projective $A$-module. Then, we have the conflation $K \xrightarrow{(f,0)} P \xrightarrow{(g,0)} Q$ in $A$-mod and an exact sequence

$$0 \rightarrow \Gamma \otimes_A K \rightarrow \Gamma \otimes_A P \rightarrow Q' \rightarrow 0,$$

in $\mathcal{I}$, which lies in $\mathcal{I} = \mathcal{F}(\Delta')$. Moreover, we know that the projective $A$-module $P$ is $E$-projective and, so, it has the form $P \cong \bigoplus_{i \in \mathcal{P}} m_i Q_i$ in $A$-mod. Then, $\Gamma \otimes_A P \cong \bigoplus_{i \in \mathcal{P}} m_i \Gamma \otimes_A Q_i = \bigoplus_{i \in \mathcal{P}} m_i P_i'$. Since $Q'$ is an indecomposable $\mathcal{F}(\Delta')$-projective, the preceding exact sequence splits and we obtain that $Q' \cong P_i'$, for some $i \in \mathcal{P}$.

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Theorem 12.8. Assume that $\mathcal{H} = (\mathcal{P}, \leq, \{\Delta_i\}_{i \in \mathcal{P}})$ is an admissible homological system for $\Lambda$. Then, the following are equivalent.

1. The Yoneda algebra $A(\Delta) = \bigoplus_{n \geq 0} \text{Ext}^n_H(\Delta, \Delta)$ admits a strict structure of $A_\infty$-algebra over $S = \bigoplus_{i \in \mathcal{P}} ke_i$.

2. The homological system $\mathcal{H}$ is strict.

Moreover, in this case the algebra $\Lambda$ and the right algebra $\Gamma$ of the interlaced weak ditalgebra $A(\Delta)$ of $\Delta$ are Morita equivalent.

Proof. (2) implies (1): If $\mathcal{H}$ is a strict homological system, from (3.7), we know that the Yoneda algebra of $\Delta$ admits a strict structure of $A_\infty$-algebra over $S$.

(1) implies (2): The construction of the triangular interlaced weak ditalgebra $A(\Delta) = (A(\Delta), I)$ accomplished in (4.19), only required an admissible homological system $\mathcal{H}$ for $\Lambda$ and a strict structure of $A_\infty$-algebra on the Yoneda algebra of $\Delta$, as we assumed at the beginning of §4. It is clear that the resulting interlaced weak ditalgebra $A(\Delta)$ is $\mathcal{P}$-strict. Then, we have the strict homological system $\mathcal{H}' = (\mathcal{P}, \leq, \{\Delta'_i\}_{i \in \mathcal{P}})$ for its right algebra $\Gamma$ described in (12.6).

We have the exact functor $F : A(\Delta)-\text{mod} \longrightarrow \Gamma-\text{mod}$ such that $F(S_i) = \Delta'_i$, for all $i \in \mathcal{P}$. From (11.13) and (12.7), we know that this functor restricts to an equivalence of categories $F' : A(\Delta)-\text{mod} \longrightarrow \mathcal{F}(\Delta')$, which maps conflations onto short exact sequences.

From (10.12), we know that there is an equivalence of categories $G : \mathcal{F}(\Delta) \longrightarrow A(\Delta)-\text{mod}$, with $G(\Delta_i) \cong S_i$, for $i \in \mathcal{P}$, and such that it maps short exact sequences onto conflations. The equivalence $\Theta = F'G : \mathcal{F}(\Delta) \longrightarrow \mathcal{F}(\Delta')$, satisfies $\Theta(\Delta_i) \cong \Delta'_i$, for all $i \in \mathcal{P}$, and it maps short exact sequences with terms in $\mathcal{F}(\Delta)$ onto short exact sequences with terms in $\mathcal{F}(\Delta')$. Its quasi-inverse $\Theta' : \mathcal{F}(\Delta') \longrightarrow \mathcal{F}(\Delta)$ satisfies $\Theta'(\Delta'_i) \cong \Delta_i$, for $i \in \mathcal{P}$.

We clearly have that $\{P_i\}_{i \in \mathcal{P}}$ is a family of non-isomorphic indecomposable $\mathcal{F}(\Delta)$-projectives. Since $\Theta$ is an equivalence, from (12.7), we know that $\{\Theta'(P'_i)\}_{i \in \mathcal{P}}$ is a complete family of non-isomorphic indecomposable $\mathcal{F}(\Delta)$-projectives. Suppose that $P_i \cong \Theta'(P'_j)$, thus $\Theta(P_i) \cong \Theta\Theta'(P'_j) \cong P'_j$. From (11.4), we have the exact sequence in $\mathcal{F}(\Delta)$

$$0 \longrightarrow Q_i \longrightarrow P_i \overset{\nu_i}{\longrightarrow} \Delta_i \longrightarrow 0.$$

Thus, we have an exact sequence in $\mathcal{F}(\Delta')$

$$0 \longrightarrow \Theta(Q_i) \longrightarrow \Theta(P_i) \longrightarrow \Theta(\Delta_i) \longrightarrow 0.$$

This gives a surjective morphism $P'_j \longrightarrow \Delta'_i$ in $\Gamma$-mod, which has to factor through the projective cover $\nu'_i : P'_i \longrightarrow \Delta'_i$, see again (11.4). Then, there is a surjective morphism $P'_i \longrightarrow P'_j$ and, so, we get $P'_i \cong P'_j$, and $i = j$. So we have that $\Theta'(P'_i) \cong P'_i$, for all $i \in \mathcal{P}$. 

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Thus the equivalence $\Theta : \mathcal{F}(\Delta) \longrightarrow \mathcal{F}(\Delta')$ is such that $\Theta(\Delta_i) \cong \Delta'_i$ and $\Theta(P_i) \cong P'_i$, for all $i \in \mathcal{P}$. After adjusting $\Theta$, if necessary, we can assume that $\Theta(\Delta_i) = \Delta'_i$ and $\Theta(P_i) = P'_i$, for all $i \in \mathcal{P}$. Make $P := \bigoplus_{i \in \mathcal{P}} P_i$ and $P' := \bigoplus_{i \in \mathcal{P}} P'_i$, then $\Theta$ determines an isomorphism of $k$-algebras

$$\Lambda' := \text{End}_\Lambda(P')^{\op} = \text{End}_\Lambda\left(\bigoplus_{i \in \mathcal{P}} P_i\right)^{\op} \theta \text{End}_\Gamma\left(\bigoplus_{i \in \mathcal{P}} P'_i\right)^{\op} = \text{End}_\Gamma(P')^{\op} =: \Gamma'.$$

We know that $\Gamma$ is Morita equivalent to $\Gamma'$ and that $\Lambda$ is Morita equivalent to $\Lambda'$. Consider the following series of equivalences

$$\Gamma\text{-mod} \xrightarrow{\Phi} \Gamma'\text{-mod} \xrightarrow{F_\theta} \Lambda'\text{-mod} \xrightarrow{\Psi} \Lambda\text{-mod}$$

where $\Phi = \text{Hom}_\Gamma(P', -)$, $\Psi = \text{Hom}_\Lambda(P, -)$, and $F_\theta$ is the restriction functor determined by the isomorphism $\theta$. Once the equivalence $\Theta$ has been fixed as above, we can choose wisely a quasi-inverse $\Theta' : \mathcal{F}(\Delta') \longrightarrow \mathcal{F}(\Delta)$ of $\Theta$ such that $\Theta'(\Delta'_i) = \Delta_i$, $\Theta'(P'_i) = P_i$, for $i \in \mathcal{P}$ and, furthermore, such that $\Theta'(\lambda') = \lambda'$, for any $\lambda' \in \Lambda'$.

We claim that $F_\theta \Phi(\Delta'_i) \cong \Psi(\Delta_i)$, for all $i \in \mathcal{P}$. Indeed, we have $F_\theta \Phi(\Delta'_i) = \text{Hom}_\Gamma(P'_i, \Delta'_i) = \text{Hom}_{\mathcal{F}(\Delta)}(P'_i, \Delta'_i)$. Similarly, we have $\Psi(\Delta_i) = \text{Hom}_\Lambda(P_i, \Delta_i) = \text{Hom}_{\mathcal{F}(\Delta)}(P, \Delta_i)$. By the preceding considerations, we have an isomorphism of vector spaces

$$F_\theta \Phi(\Delta'_i) = \text{Hom}_{\mathcal{F}(\Delta)}(P'_i, \Delta'_i) \xrightarrow{\Theta'} \Lambda' \text{Hom}_{\mathcal{F}(\Delta)}(P, \Delta_i) \cong \Psi(\Delta_i).$$

So we only have to verify that $\Theta'$ is an isomorphism of $\Lambda'$-modules. Take any morphism $f : P' \longrightarrow \Delta'_i$ in $\Gamma$-mod, $\lambda' \in \Lambda'$ and $p \in P$, then we have

$$\Theta'(\lambda' f)(p) = \Theta'(f \circ \Theta(\lambda'))(p) = (\Theta'(f) \circ \lambda')(p) = \Theta'(f)(p\lambda') = (\lambda' \Theta'(f))(p).$$

Therefore, if $\Psi' : \Lambda'\text{-mod} \longrightarrow \Lambda\text{-mod}$ is any quasi-inverse of $\Psi$, we have that the composition $\Omega := \Psi' F_\theta \Phi : \Gamma\text{-mod} \longrightarrow \Lambda\text{-mod}$ is an equivalence of categories with $\Omega(\Delta'_i) = \Psi' F_\theta \Phi(\Delta'_i) \cong \Delta_i$, for all $i \in \mathcal{P}$. From this we obtain the strictness condition for the homological system $\mathcal{H}$ from the corresponding condition for the homological system $\mathcal{H}'$, see (12.6).

Now, we can derive the following.

**Theorem 12.9.** Let $k$ be an algebraically closed field and let $\Lambda$ be a finite dimensional $k$-algebra with a strict homological system $\mathcal{H} = (\mathcal{P}, \leq, \{\Delta_i\}_{i \in \mathcal{P}})$. Then, the algebra $\Lambda$ is Morita equivalent to a finite-dimensional $k$-algebra $\Gamma$ which admits a strict homological system $\mathcal{H}' = (\mathcal{P}, \leq, \{\Delta'_i\}_{i \in \mathcal{P}})$ such that $\Gamma$ has a regular homological exact Borel subalgebra $\underline{\Lambda}$, as in (1.14). Moreover, there is an equivalence $\Omega : \Gamma\text{-mod} \longrightarrow \Lambda\text{-mod}$ such that $\Omega(\Delta'_i) \cong \Delta_i$, for all $i \in \mathcal{P}$.
Proof. Consider the $\mathcal{P}$-strict interlaced weak ditalgebra $\underline{A}(\Delta)$, adopt the notations of this section. From (12.6) and (11.10), the algebra $\underline{A}$ is an exact Borel subalgebra of $\Gamma$, the right algebra of $\underline{A}(\Delta)$. From (11.13)(4), the Borel subalgebra $\underline{A}$ is homological. In order to show that $\underline{A}$ is regular, consider the epimorphisms

$$\text{Ext}^1_\underline{A}(S_i, S_j) \longrightarrow \text{Ext}^1_\underline{A}(\Gamma \otimes \underline{A} S_i, \Gamma \otimes \underline{A} S_j) = \text{Ext}^1_\underline{A}(\Delta_i', \Delta_j') \cong \text{Ext}^1_\underline{A}(\Delta_i, \Delta_j),$$

for $i, j \in \mathcal{P}$, where the last isomorphism is obtained from the equivalence $\Omega : \Gamma\text{-mod} \longrightarrow \Lambda\text{-mod}$ constructed in the proof of (12.3). Then, since $e_j \dot{D}(B)_0 e_i \cong e_j \text{Ext}^1_\underline{A}(\Delta, \Delta)e_i \cong \text{Ext}^1_\underline{A}(\Delta_i, \Delta_j)$, we get $\dim_k \text{Ext}^1_\underline{A}(S_i, S_j) = \dim_k e_j \dot{D}(B)_0 e_i = \dim_k e_j \dot{D}(B)_0 e_i = \dim_k \text{Ext}^1_\underline{A}(\Delta_i, \Delta_j)$.

Example 12.10. Let $\Lambda$ be a finite-dimensional $k$-algebra and $\mathcal{D} = (T, \delta)$ its Drozd’s ditalgebra, see §319 and (23.25). We recall some of its features. Here, we have a splitting $\Lambda \cong R \oplus J$ of the algebra $\Lambda$ over its radical $J$, so the algebra $R$ is a finite product of fields and we can write $R = \bigoplus_{e_i \in F} ke_i$, where $1_R = \sum_{e_i \in F} ke_i$ is a decomposition of the unit element as a sum of primitive orthogonal idempotents of $R$. Consider the product algebra $S = R \times R$, so that $S$ is again a finite product of fields and we can write $S = \bigoplus_{e_i \in P} ke_i$, with $\mathcal{P} := \mathcal{P}' \cup \mathcal{P}''$, where $\mathcal{P}' = \{i' | i \in F\}$ and $\mathcal{P}'' = \{i'' | i \in F\}$ are disjoint copies of the set $F$, and $1 = \sum_{e_i \in F} e_i$ is a decomposition of the unit as a sum of primitive central orthogonal idempotents of $S$. Let us fix a basis $B(i, j)$ of the space $e_j e_i$, for each $i, j \in F$. Then, the tensor algebra $T$ mentioned above can be identified with the graded path algebra $k(B)$ of the bigraph $B$ defined by the following. The set of points of $B$ is $P$; and, for each $\alpha \in B(i, j)$, there is a solid arrow $\alpha : i' \to j''$ and two dashed arrows $\alpha' : i' \to j'$ and $\alpha'' : i'' \to j''$ in $B$.

The ditalgebra $\mathcal{D}$ has no relations, that is we take $I = 0$. Consider the preorder $\leq$ in $\mathcal{P}$ defined by $i' \leq j'$ and $i'' \leq j''$, for all $i, j \in F$ and $i < j$ iff $B(i, j) \neq \emptyset$. Thus, the poset $\mathcal{P}$ has only two elements. Then, the triangular interlaced weak ditalgebra $\underline{D} = (\mathcal{D}, 0)$ is strict with preordered set $(\mathcal{P}, \leq)$ and this section applies to it. Its right algebra is not quasi-hereditary in general.

Similarly, given any finite partially ordered sets $S$ and $T$, the associated ditalgebra $A(S, T)$, see §(34.1), determines a triangular interlaced weak ditalgebra $\underline{A}(S, T) = (A(S, T), 0)$, with a preordered set $(\mathcal{P}, \leq)$ defined similarly.

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References

[1] Bautista R., Pérez E., Salmerón L., Tame and wild theorem for the category of filtered by standard modules for a quasi-hereditary algebra, Preprint, June, 2017.
[2] Bautista R., Pérez E., Salmerón L., *Differential graded bocses and $A_\infty$-algebras*, Preprint, June, 2019.

[3] Bautista R., Salmerón L., Zuazua R., *Differential Tensor Algebras and their Module Categories*. London Math. Soc. Lecture Note Series 362. Cambridge University Press, 2009.

[4] Brzeziński T., Koenig S., Külshammer J., *From quasi-hereditary algebras with exact Borel subalgebras to directed bocses*. Bull. Lond. Math. Soc. 52 (2020), no. 2, 367–378.

[5] Burt W.L., Butler M.C.R., *Almost split sequences for bocses*. Representations of finite dimensional algebras. Canadian Math. Soc. Conference Proceedings, 11 (1990) 89–121.

[6] Burt W.L., *Almost split sequences and bocses*, Unpublished Monograph (2005).

[7] Kadeishvili T.V., *On the homology theory of fibre spaces*. Russ. Math. Surv. 35 (1980) 231–238.

[8] Keller B., *Introduction to $A_\infty$-algebras and modules*. Homology, Homotopy and Applications, vol. 3, 1 (2001) 1–35.

[9] Koenig S., Külshammer J., Ovsienko S., *Quasi-hereditary algebras, exact Borel subalgebras, $A_\infty$-categories and boxes*. Advances in Mathematics 262 (2014) 546–592.

[10] Lefèvre-Hasegawa K., Sur les $A_\infty$-catégories, Thèse de doctorat, Université Denis Diderot–Paris 7, 2003.

[11] Lu D.M., Palmieri J.H., Wu Q.S., Zhang J.J., *A-infinity structure on Ext-algebras*. Journal of Pure and Applied Algebra 213 (2009) 2017–2037.

[12] Madsen D.O., *Quasi-hereditary algebras and the category of modules with standard filtration*. São Paulo J. Math. Sci. 11 (2017), no. 1, 68–80.

[13] Mendoza O., Sáenz C., Xi Ch., *Homological systems in module categories over preordered sets*. Quarterly J. Math. 60, 1, (2009) 75–103.

[14] Merkulov A., *Strong homotopy algebras of a Kähler manifold*. Internat. Math. Res. Notices, (3) (1999) 153–164.

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