ON $\mathcal{H}$-COMPLETE TOPOLOGICAL SEMILATTICES

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Abstract. In the paper we describe the structure of $\mathcal{A}\mathcal{H}$-completions and $\mathcal{H}$-completions of the discrete semilattices $(\mathbb{N}, \min)$ and $(\mathbb{N}, \max)$. We give an example of an $\mathcal{H}$-complete topological semilattice which is not $\mathcal{A}\mathcal{H}$-complete. Also we construct an $\mathcal{H}$-complete topological semilattice of cardinality $\lambda$ which has $2^\lambda$ many open-and-closed continuous homomorphic images which are not $\mathcal{H}$-complete topological semilattices. The constructed examples give a negative answer to Question 17 from [12].

In this paper all topological spaces will be assumed to be Hausdorff. We shall follow the terminology of [1, 3], and [4]. For a subset $A$ of a topological space $X$ by $\text{cl}_X(A)$ we denote the closure of $A$ in $X$. A filter $\mathcal{F}$ on a set $S$ is called free if $\bigcap \mathcal{F} = \emptyset$.

A semilattice is a set endowed with a commutative idempotent associative operation. If $E$ is a semilattice, then the semilattice operation on $E$ determines the partial order $\leq$ on $E$: $e \leq f \quad \text{if and only if} \quad ef = fe = e.$

This order is called natural. An element $e$ of a semilattice $E$ is called minimal (maximal) if $f \leq e$ ($e \geq f$) implies $f = e$ for $f \in E$. A semilattice $E$ is said to be linearly ordered or a chain if the natural order on $E$ is linear.

If $S$ is a topological space equipped with a commutative idempotent associative operation then $S$ is called a topological semigroup. A topological semilattice is a topological semigroup which is algebraically a semilattice.

Let $\mathcal{T}\mathcal{S}$ be a category whose objects are topological semigroups and morphisms are homomorphisms between topological semigroups. A topological semigroup $X \in \text{Ob} \mathcal{T}\mathcal{S}$ is called $\mathcal{T}\mathcal{S}$-complete if for each object $Y \in \text{Ob} \mathcal{T}\mathcal{S}$ and a morphism $f: X \to Y$ of the category $\mathcal{T}\mathcal{S}$ the image $f(X)$ is closed in $Y$.

By a $\mathcal{T}\mathcal{S}$-completion of a topological semigroup $X$ we understand any $\mathcal{T}\mathcal{S}$-complete topological semigroup $\tilde{X} \in \text{Ob} \mathcal{T}\mathcal{S}$ containing $X$ as a dense subsemigroup. A $\mathcal{T}\mathcal{S}$-completion $\tilde{X}$ of $X$ is called universal if each continuous homomorphism $h: X \to Y$ to a $\mathcal{T}\mathcal{S}$-complete topological semigroup $Y \in \text{Ob} \mathcal{T}\mathcal{S}$ extends to a continuous homomorphism $\tilde{h}: \tilde{X} \to Y$.

It is well-known that for the category $\mathcal{T}\mathcal{G}$ of topological groups and their continuous homomorphisms, each object $G \in \text{Ob} \mathcal{T}\mathcal{G}$ has a $\mathcal{T}\mathcal{G}$-completion and each $\mathcal{T}\mathcal{G}$-completion of $G$ is universal [10].

In the category of topological semigroups the situation is totally different. We show this on the example of the discrete topological semigroups $(\mathbb{N}, \min)$ and $(\mathbb{N}, \max)$. We shall study $\mathcal{H}$-completions and $\mathcal{A}\mathcal{H}$-completions of discrete topological semigroup $(\mathbb{N}, \min)$ and $(\mathbb{N}, \max)$ in the category $\mathcal{A}\mathcal{H}$ (resp. $\mathcal{H}$) whose objects are Hausdorff topological semigroups and morphisms are continuous homomorphisms (resp. isomorphic topological embeddings) between topological semigroups.

The notion of $\mathcal{H}$-completion was introduced by Stepp in [11], where he showed that for each locally compact topological semigroup $S$ there exists an $\mathcal{H}$-complete topological semigroup $T$ which contains $S$ as a dense subsemigroup.

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Stepp [12] proved that a discrete semilattice $E$ is $\mathcal{H}$-complete if and only if any maximal chain in $E$ is finite. In [7] Gutik and Pavlyk remarked that a topological semilattice is $\mathcal{H}$-complete ($\mathcal{A}\mathcal{H}$-complete) if and only if it is ($\mathcal{A}\mathcal{H}$-complete) as a topological semigroup. In [3] Gutik and Repovš studied properties of linearly ordered $\mathcal{H}$-complete topological semilattices and proved the following characterization theorem:

**Theorem 1** ([9, Theorem 2]). A linearly ordered topological semilattice $E$ is $\mathcal{H}$-complete if and only if the following conditions hold:

(i) $E$ is complete;
(ii) $x = \sup A$ for $A = \downarrow A$ implies $x \in \text{cl}_E A$; and
(iii) $x = \inf B$ for $B = \uparrow B$ implies $x \in \text{cl}_E B$.

Also, in [2] Gutik and Repovš proved that each linearly ordered $\mathcal{H}$-complete topological semilattice is $\mathcal{A}\mathcal{H}$-complete and showed that every linearly ordered semilattice is a dense subsemilattice of an $\mathcal{H}$-complete topological semilattice. In [2] Chuchman and Gutik proved that any $\mathcal{H}$-complete topologically weakly U-semilattice contain minimal idempotents.

In [12, Question 17] Stepp asked the following question: Is each $\mathcal{H}$-complete topological semilattice $\mathcal{A}\mathcal{H}$-complete? In the present paper we answer this Stepp’s question in the negative by constructing an example of an $\mathcal{H}$-complete topological semilattice which is not $\mathcal{A}\mathcal{H}$-complete. Also we construct an $\mathcal{H}$-complete topological semilattice of arbitrary infinite cardinality $\lambda$ which has $2^\lambda$ many open-and-closed continuous homomorphic images which are not $\mathcal{H}$-complete topological semilattices.

Let $\mathbb{N}$ denote the set of positive integers. For each free filter $\mathcal{F}$ on $\mathbb{N}$ consider the topological space $\mathbb{N}_\mathcal{F} = \mathbb{N} \cup \{\mathcal{F}\}$ in which all points $x \in \mathbb{N}$ are isolated while the sets $F \cup \{\mathcal{F}\}$, $F \in \mathcal{F}$, form a neighbourhood base at the unique non-isolated point $\mathcal{F}$.

The semilattice operation $\min$ (resp., $\max$) of $\mathbb{N}$ extends to a continuous semilattice operation $\max$ (resp., $\min$) on $\mathbb{N}_\mathcal{F}$ such that $\min\{n, \mathcal{F}\} = \min\{\mathcal{F}, n\} = n$ and $\min\{\mathcal{F}, \mathcal{F}\} = \mathcal{F}$ (resp., $\max\{n, \mathcal{F}\} = \max\{\mathcal{F}, n\} = \mathcal{F} = \max\{\mathcal{F}, \mathcal{F}\}$) for all $n \in \mathbb{N}$. By $\mathbb{N}_{\mathcal{F}, \min}$ (resp., $\mathbb{N}_{\mathcal{F}, \max}$) we shall denote the topological space $\mathbb{N}_\mathcal{F}$ with the semilattice operation $\min$ (resp., $\max$). Simple verifications show that $\mathbb{N}_{\mathcal{F}, \min}$ and $\mathbb{N}_{\mathcal{F}, \max}$ are topological semilattices.

**Theorem 2.** (i) For each free filter $\mathcal{F}$ on $\mathbb{N}$ the topological semilattices $\mathbb{N}_{\mathcal{F}, \min}$ and $\mathbb{N}_{\mathcal{F}, \max}$ are $\mathcal{A}\mathcal{H}$-complete.

(ii) Each $\mathcal{H}$-completion of the discrete semilattice $(\mathbb{N}, \min)$ (resp., $(\mathbb{N}, \max)$) is topologically isomorphic to the topological semilattice $\mathbb{N}_{\mathcal{F}, \min}$ (resp., $\mathbb{N}_{\mathcal{F}, \max}$) for some free filter $\mathcal{F}$ on $\mathbb{N}$.

(iii) The topological semilattice $(\mathbb{N}, \min)$ (resp., $(\mathbb{N}, \max)$) has no universal $\mathcal{A}\mathcal{H}$-completion.

**Proof.** (i) By Theorem 1 we have that the topological semilattices $\mathbb{N}_{\mathcal{F}, \min}$ and $\mathbb{N}_{\mathcal{F}, \max}$ are $\mathcal{H}$-complete. Since $\mathbb{N}_{\mathcal{F}, \min}$ and $\mathbb{N}_{\mathcal{F}, \max}$ are linearly ordered semilattices, Theorem 3 of [9] implies that the topological semilattices $\mathbb{N}_{\mathcal{F}, \min}$ and $\mathbb{N}_{\mathcal{F}, \max}$ are $\mathcal{A}\mathcal{H}$-complete.

(ii) We shall prove the statement for the semilattice $(\mathbb{N}, \min)$. In the case of $(\mathbb{N}, \max)$ the proof is similar.

Let $S$ be an $\mathcal{H}$-complete topological semilattice containing $(\mathbb{N}, \min)$ as a dense subsemilattice. Since the closure of a linearly ordered subsemilattice in a Hausdorff topological semigroup is a linearly ordered topological semilattice (see [7, Corollary 19] and [9, Lemma 1]), we conclude that $S$ is linearly ordered and $S \setminus \mathbb{N}$ is a singleton $\{a\}$. Then since $(\mathbb{N}, \min)$ is a dense subsemilattice of $S$, the continuity of the semilattice operation in $S$ implies that $a \cdot a = a$ and $a \cdot n = n \cdot a = n$ for any $n \in \mathbb{N}$. Let $\mathcal{B}(a)$ be the filter of neighborhoods of the point $a$ in $S$. This filter induces the free filter $\mathcal{F} = \{F \subseteq \mathbb{N} : F \cup \{a\} \in \mathcal{B}(a)\}$. Then we can identify the topological semilattice $S$ with $\mathbb{N}_{\mathcal{F}, \min}$ by the topological isomorphism $f : S \to \mathbb{N}_{\mathcal{F}, \min}$ such that $f(a) = \mathcal{F}$ and $f(n) = n$ for every $n \in \mathbb{N}$. 
Theorem 3. Let $F$ be a free filter on $\mathbb{N}$ and $F \in \mathcal{F}$ be a set with infinite complement $\mathbb{N} \setminus F$. Then the following statements hold:

(i) the closed subsemilattice $E = (\mathbb{N}_{\mathcal{F},\min} \times \{0\}) \cup ((\mathbb{N} \setminus F) \times \{1\})$ of the direct product $\mathbb{N}_{\mathcal{F},\min} \times E_2$ is $\mathcal{H}$-complete;

(ii) the subset $I = \mathbb{N}_{\mathcal{F},\min} \times \{0\}$ is an open-and-closed ideal in $E$, and the quotient semilattice $E/I$ with the quotient topology is discrete and not $\mathcal{H}$-complete;

(iii) the semilattice $E$ is not $\mathcal{A}\mathcal{H}$-complete.

Proof. (i) The definition of the topological semilattice $\mathbb{N}_{\mathcal{F},\min} \times E_2$ implies that $E$ is a closed subsemilattice of $\mathbb{N}_{\mathcal{F},\min} \times E_2$.

Suppose the contrary: the topological semilattice $E$ is not $\mathcal{H}$-complete. Since the closure of a subsemilattice in a topological semigroup is a semilattice (see Corollary 19 of [7]), we conclude that there exists a topological semilattice $S$ which contains $E$ as a dense subsemilattice and $S \setminus E \neq \emptyset$. We fix an arbitrary $a \in S \setminus E$. Then for every open neighbourhood $U(a)$ of the point $a$ in $S$ we have that the set $U(a) \cap E$ is infinite. By Theorem 2, the subspace $\mathbb{N}_{\mathcal{F},\min} \times \{0\}$ of $E$ with the induced semilattice operation from $E$ is an $\mathcal{H}$-complete topological semilattice. Therefore, there exists an open neighbourhood $U(a)$ of the point $a$ in $S$ such that $U(a) \cap E \subseteq (\mathbb{N} \setminus F) \times \{1\}$ and hence the set $U(a) \cap ((\mathbb{N} \setminus F) \times \{1\})$ is infinite.

Next we shall show that $a \cdot x = x$ for any $x \in E \setminus \{ (\mathcal{F},0) \}$. Since the set $U(x) \cap ((\mathbb{N} \setminus F) \times \{1\})$ is infinite, the continuity of the semilattice operation in $E$ implies that $a \cdot x = x$ for any $x \in (\mathbb{N} \setminus F) \times \{1\}$. Now fix any point $y \in \mathbb{N} \times \{0\} \subset E$. By the definition of the semilattice operation on $E$, we can find a point $x_y \in (\mathbb{N} \setminus F) \times \{1\}$ with $x_y \cdot y = y$ and conclude that

$$a \cdot y = a \cdot (x_y \cdot y) = (a \cdot x_y) \cdot y = x_y \cdot y = y.$$ 

Since $(\mathcal{F},0)$ is a cluster point of the set $\mathbb{N} \times \{0\}$, the continuity of the semilattice operation implies that $a \cdot (\mathcal{F},0) = (\mathcal{F},0)$.

Since $W(\mathcal{F},0) = (F \cup \{\mathcal{F}\}) \times \{0\}$ is a neighborhood of the point $(\mathcal{F},0) = a \cdot (\mathcal{F},0)$, the continuity of the semilattice operation yields the existence of neighborhoods $U(a)$ and $V(\mathcal{F},0)$ of the points $a$ and $(\mathcal{F},0)$ in $S$ such that $U(a) \cdot V(\mathcal{F},0) \subset W(\mathcal{F},0)$. Now choose any point $(n,1) \in U(a) \cap ((\mathbb{N} \setminus F) \times \{1\})$ and find a point $(m,0) \in V(\mathcal{F},0)$ such that $m \geq n$. Then

$$(n,0) = (n,1) \cdot (m,1) \in U(a) \cdot U(\mathcal{F},0) \subset W(\mathcal{F},0) = (F \cup \{\mathcal{F}\}) \times \{0\},$$

which contradicts the choice of $n \in \mathbb{N} \setminus F$.

(ii) The definition of the semilattice $E$ implies that $I = \mathbb{N}_{\mathcal{F},\min} \times \{0\}$ is an open-and-closed ideal in $E$. Then the quotient semilattice $E/I$ (endowed with the quotient topology) is a discrete topological semilattice, topologically isomorphic to the discrete semilattice $(\mathbb{N}, \min)$. By Theorem 11, the semilattice $E/I$ is not $\mathcal{H}$-complete.

Statement (iii) follows from statement (ii). □

Corollary 4. For a free filter $\mathcal{F}$ on $\mathbb{N}$, each closed subsemilattice of the semilattice $\mathbb{N}_{\mathcal{F},\min} \times E_2$ is $\mathcal{A}\mathcal{H}$-complete if and only if $\mathcal{F}$ is the filter of cofinite sets on $\mathbb{N}$.
Proof. (⇒) If $\mathcal{F}$ is the filter of cofinite sets on $\mathbb{N}$, then the space $\mathbb{N}_{\mathcal{F}_{\min}} \times E_2$ is compact. Then each closed subset of $\mathbb{N}_{\mathcal{F}_{\min}} \times E_2$ is compact and hence each closed subsemilattice of the semilattice $\mathbb{N}_{\mathcal{F}_{\min}} \times E_2$ is $\mathcal{H}$-complete.

(⇒) If $\mathcal{F}$ is a free filter on $\mathbb{N}$ containing a set $F \subseteq \mathbb{N}$ with the infinite complement $\mathbb{N} \setminus F$, then $E = (\mathbb{N}_{\mathcal{F}_{\min}} \times \{0\}) \cup ((\mathbb{N} \setminus F) \times \{1\})$ is a closed subsemilattice of the topological semilattice $\mathbb{N}_{\mathcal{F}_{\min}} \times E_2$ and Theorem 3 implies that $E$ is not $\mathcal{H}$-complete. \hfill $\square$

The proof of the following theorem is similar to the proof of Theorem 3 with some simple modifications.

**Theorem 5.** Let $\mathcal{F}$ be a free filter on $\mathbb{N}$ and $F \in \mathcal{F}$ be a set with infinite complement $\mathbb{N} \setminus F$. Then the following assertions hold:

(i) the closed subsemilattice $E = (\mathbb{N}_{\mathcal{F}_{\max}} \times \{0\}) \cup ((\mathbb{N} \setminus F) \times \{1\})$ of the direct product $\mathbb{N}_{\mathcal{F}_{\max}} \times E_2$ is $\mathcal{H}$-complete;

(ii) the subset $I = \mathbb{N}_{\mathcal{F}_{\max}} \times \{0\}$ is an open-and-closed ideal in $E$, and the quotient semilattice $E/I$ with the quotient topology is discrete and not $\mathcal{H}$-complete;

(iii) the semilattice $E$ is not $\mathcal{AH}$-complete.

The proof of the following corollary is similar to the proof of Corollary 4 and follows from Theorem 3.

**Corollary 6.** For a free filter $\mathcal{F}$ on $\mathbb{N}$, each closed subsemilattice of the semilattice $\mathbb{N}_{\mathcal{F}_{\max}} \times E_2$ is $\mathcal{AH}$-complete if and only if $\mathcal{F}$ is the filter of cofinite sets on $\mathbb{N}$.

We remark that Theorems 3 and 4 give negative answers on Question 17 from [12].

Also, Theorems 3 and 5 imply the following corollary:

**Corollary 7.** There exists a countable locally compact $\mathcal{H}$-complete topological semilattice $E$ with an open-and-closed ideal $I$ such that $I$ is a $\mathcal{AH}$-complete semilattice and the Rees quotient semigroup $E/I$ with the quotient topology is not $\mathcal{H}$-complete.

**Remark 8.** A Hausdorff partially ordered space $X$ is called $\mathcal{H}$-complete if $X$ is a closed subspace of every Hausdorff partially ordered space in which it is contained [5]. A linearly ordered topological semilattice $E$ is $\mathcal{H}$-complete if and only if $E$ is an $\mathcal{H}$-complete partially ordered space [9]. In [13] Yokoyama showed that a partially ordered space $X$ without an infinite antichain is an $\mathcal{H}$-complete partially ordered space if and only if $X$ is a directed complete and down-complete poset such that sup $L$ and inf $L$ are contained in the closure of $L$ for any nonempty chain $L$ in $X$. Theorems 3 and 5 imply that there exists an $\mathcal{H}$-complete topological semilattice without an infinite antichain which is not an $\mathcal{H}$-complete partially ordered space. Also Theorems 3 implies that there exists a countable $\mathcal{H}$-complete locally compact topological semilattice $E$ without an infinite antichain which contains a maximal chain $L$ which is not directed complete, and $L$ does not have a maximal element.

Let $\lambda$ be any infinite cardinal and let $0 \notin \lambda$. On the set $E_\lambda = \{0\} \cup \lambda$ endowed with the discrete topology we define the semilattice operation by the formula:

$$x \cdot y = \begin{cases} x, & \text{if } x = y; \\ 0, & \text{if } x \neq y. \end{cases}$$

**Theorem 9.** Let $\mathcal{F}$ be a free filter on $\mathbb{N}$ and $F \in \mathcal{F}$ be a set with infinite complement $\mathbb{N} \setminus F$. Then for each infinite cardinal $\lambda$ the following statements hold:

(i) the closed subsemilattice $E = (\mathbb{N}_{\mathcal{F}_{\max}} \times \{0\}) \cup ((\mathbb{N} \setminus F) \times \lambda)$ of the direct product $\mathbb{N}_{\mathcal{F}_{\max}} \times E_\lambda$ is $\mathcal{H}$-complete;

(ii) for each subset $\kappa \subset \lambda$ the subset $I_\kappa = (\mathbb{N}_{\mathcal{F}_{\max}} \times \{0\}) \cup ((\mathbb{N} \setminus F) \times \kappa)$ is an open-and-closed ideal in $E$, and the quotient semilattice $E/I_\kappa$ with the quotient topology is discrete and not $\mathcal{H}$-complete.
(iii) the semilattice $E$ is not $\mathcal{H}$-complete.

Proof. (i) Assuming that the topological semilattice $E$ is not $\mathcal{H}$-complete, find a topological semilattice $T$ containing $E$ as a dense subsemilattice with non-empty complement $T \setminus E$. Fix any element $e \in T \setminus E$. By Theorem 9, the topological semilattice $E^0 = \mathbb{N}_{\mathcal{F}_{\max}} \times \{0\}$ is $\mathcal{H}$-complete and hence is closed in $T$. Then there exists an open neighbourhood $U(e)$ of the point $e$ in $T$ such that $U(e) \cap E^0 = \emptyset$. By the continuity of the semilattice operation in $T$, there exists an open neighbourhood $V(e) \subseteq U(e)$ of the point $e$ in $T$ such that $V(e) \cdot V(e) \subseteq U(e)$. By Theorem 5, for each $a \in \lambda$, the subsemilattice $E_a = (\mathbb{N}_{\mathcal{F}_{\max}} \times \{0\}) \cup ((\mathbb{N} \setminus F) \times \{a\})$ of the direct product $\mathbb{N}_{\mathcal{F}_{\max}} \times E_\lambda$ is $\mathcal{H}$-complete and hence is closed in $T$. This implies that $V(e) \cap E_a \neq \emptyset$ for infinitely many points $a \in \lambda$, and hence $(V(e) \cdot V(e)) \cap E^0 \neq \emptyset$. This contradicts the choice of the neighbourhood $U(e)$. The obtained contradiction implies that the topological semilattice $E$ is $\mathcal{H}$-complete.

(ii) The definition of the semilattice $E$ implies that $I_\kappa = (\mathbb{N}_{\mathcal{F}_{\max}} \times \{0\}) \cup ((\mathbb{N} \setminus F) \times \kappa)$ is an open-and-closed ideal in $E$. Then we have that the quotient semilattice $E/I_\kappa$ with the quotient topology is a discrete topological semilattice. Also, $E/I_\kappa$ is topologically isomorphic to the orthogonal sum of $\lambda$ infinitely many of $(\mathbb{N}, \max)$ with isolated zero. This implies that the semilattice $E/I_\kappa$ is not $\mathcal{H}$-complete.

Statement (iii) follows from statement (ii).

Remark 10. The topological semilattices $E$ and $I_\kappa$ from Theorem 9 are metrizable locally compact spaces for each free countably generated filter $\mathcal{F}$ on $\mathbb{N}$ and any $\kappa \subseteq \lambda$.

Remark 11. It can be shown that continuous homomorphisms into the discrete semilattice $\{0, 1\}$ separate points of the topological semilattices $E$ considered in Theorems 5 and 9.

Since for each subset $\kappa \subseteq \lambda$ the natural homomorphism $\pi: E \to E/I_\kappa$ is an open-and-closed map, Theorem 9 implies the following corollary:

Corollary 12. Let $\mathcal{F}$ be a free filter on $\mathbb{N}$ containing a set $F \in \mathcal{F}$ with infinite complement $\mathbb{N} \setminus F$. Then for each infinite cardinal $\lambda$ there exist $2^\lambda$ many continuous open-and-closed surjective homomorphic images of the topological semilattice $E = (\mathbb{N}_{\mathcal{F}_{\max}} \times \{0\}) \cup ((\mathbb{N} \setminus F) \times \lambda) \subset \mathbb{N}_{\mathcal{F}_{\max}} \times E_\lambda$, which are not $\mathcal{H}$-complete.

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