Distribution of Chern–Simons invariants

Julien Marche

To cite this version:

Julien Marche. Distribution of Chern–Simons invariants. Annales de l’Institut Fourier, 2019, 69 (2), pp.753-762. 10.5802/aif.3256 . hal-02171919

HAL Id: hal-02171919
https://hal.sorbonne-universite.fr/hal-02171919v1
Submitted on 3 Jul 2019

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
DISTRIBUTION OF CHERN-SIMONS INVARIANTS

JULIEN MARCHÉ

Abstract. Let $M$ be a 3-manifold with a finite set $X(M)$ of conjugacy classes of representations $\rho : \pi_1(M) \to SU_2$. We study here the distribution of the values of the Chern-Simons function $CS : X(M) \to \mathbb{R}/2\pi\mathbb{Z}$. We observe in some examples that it resembles the distribution of quadratic residues. In particular for specific sequences of 3-manifolds, the invariants tends to become equidistributed on the circle with white noise fluctuations of order $|X(M)|^{-1/2}$. We prove that for a manifold with toric boundary the Chern-Simons invariants of the Dehn fillings $M_{p/q}$ have the same behaviour when $p$ and $q$ go to infinity and compute fluctuations at first order.

1. Introduction

1.1. Distribution of quadratic residues. Let $p$ be a prime number. We consider the weighted counting measure on the circle $T = \mathbb{R}/2\pi\mathbb{Z}$ defined by quadratic residues modulo $p$, that is:

$$\mu_p = \frac{1}{p} \sum_{k=0}^{p-1} \delta_{\frac{2\pi k^2}{p}}.$$ 

We investigate the limit of $\mu_p$ when $p$ goes to infinity and to that purpose, we consider its $\ell$-th momentum i.e $\muP = \int e^{i\ell \theta} d\mu_p(\theta) = \frac{1}{p} \sum_{k=0}^{p-1} \exp(2\pi i \ell k^2 / p)$. We have $\muP = 1$ if $p \mid \ell$, and else by the Gauss sum formula, $\muP = \left( \frac{\ell}{p} \right) \frac{1}{\sqrt{p}}$ where $\left( \frac{\ell}{p} \right)$ is the Legendre symbol.

This shows that $\mu_p$ converges to the uniform measure $\mu_\infty$ whereas the renormalized measure $\sqrt{p}(\mu_p - \mu_\infty)$ - that we call fluctuation- has $l$-th momentum $\pm 1$ depending on the residue of $l$ modulo $p$ and hence is a kind of “white noise”.

1.2. Distribution of Chern-Simons invariants. On the other hand, such Gauss sums appear naturally in the context of Chern-Simons invariants of 3-manifolds. Consider an oriented and compact 3-manifold $M$ and define its character variety as the set $X(M) = \text{Hom}(\pi_1(M), SU_2)/SU_2$. In what follows, we will confuse between representations and their conjugacy classes. The Chern-Simons invariant may be viewed as a locally constant map $CS : X(M) \to \mathbb{T}$. We refer to [3] for background on Chern-Simons invariants and give here a quick definition for the convenience of the reader.
Let $\nu$ be the Haar measure of $\text{SU}_2$ normalised by $\nu(\text{SU}_2) = 2\pi$ and let $\pi : \hat{M} \to M$ be the universal cover of $M$. There is an equivariant map $F : \hat{M} \to \text{SU}_2$ in the sense that $F(\gamma x) = \rho(\gamma) F(x)$ for all $\gamma \in \pi_1(M)$ and $x \in M$. The form $F^*\nu$ is invariant hence can be written $F^*\nu = \pi^*\nu_F$. We set $\text{CS}(\rho) = \int_M \nu_F$ and claim that it is independent on the choice of equivariant map $F$ modulo $2\pi$.

**Definition 1.1.** Let $M$ be a 3-manifold whose character variety is finite. We define its Chern-Simons measure as $\mu_M = \frac{1}{|X(M)|} \sum_{\rho \in X(M)} \delta_{\text{CS}(\rho)}$.

1.2.1. Lens spaces. For instance, if $M = L(p, q)$ is a lens space, then $\pi_1(M) = \mathbb{Z}/p\mathbb{Z}$ and $X(M) = \{\rho_n, n \in \mathbb{Z}/p\mathbb{Z}\}$ where $\rho_n$ maps the generator of $\mathbb{Z}/p\mathbb{Z}$ to a matrix with eigenvalues $e^{\pm \frac{2\pi i n}{p}}$. We know from [3] that $\text{CS}(\rho_n) = 2\pi \frac{n^2}{p}$ where $qq^* = 1 \mod p$. Hence, the Chern-Simons invariants of $L(p, q)$ behave exactly like quadratic residues when $p$ goes to infinity.

1.2.2. Brieskorn spheres. To give a more complicated but still manageable example, consider the Brieskorn sphere $M = \Sigma(p_1, p_2, p_3)$ where $p_1, p_2, p_3$ are distinct primes. This is a homology sphere whose irreducible representations in $\text{SU}_2$ have the form $\rho_{n_1, n_2, n_3}$ where $0 < n_1 < p_1, 0 < n_2 < p_2, 0 < n_3 < p_3$. From [3] we have

$$\text{CS}(\rho_{n_1, n_2, n_3}) = 2\pi \frac{(n_1p_2p_3 + n_1n_2p_3 + n_1p_2n_3)^2}{4p_1p_2p_3}$$

Setting $n = n_1p_2p_3 + n_1n_2p_3 + n_1p_2n_3$, we observe that -due to Chinese remainder theorem- $n$ describes $(\mathbb{Z}/p_1p_2p_3\mathbb{Z})^\times$ when $n_i$ describes $(\mathbb{Z}/p_i\mathbb{Z})^\times$ for $i = 1, 2, 3$. Hence, we compute that the following $\ell$-th momentum:

$$\mu_{p_1p_2p_3}^\ell = \frac{1}{|X(\Sigma(p_1,p_2,p_3))|} \sum_{\rho \in X(M)} \exp(i\ell \text{CS}(\rho)) \sim \frac{1}{p_1p_2p_3} \sum_{n=0}^{p_1p_2p_3-1} e^{\frac{2\pi i n^2}{p_1p_2p_3}}$$

Assuming $\ell$ is coprime with $p = p_1p_2p_3$ we get from [1] the following estimates where $\epsilon_n = 1$ is $n = 1 \mod 4$ and $\epsilon_n = i$ if $n = 3 \mod 4$:

$$\mu_p^\ell \sim \begin{cases} \sqrt{p} \frac{\ell^2/4}{p} & \text{if } \ell = 0 \mod 4 \\ 0 & \text{if } \ell = 2 \mod 4 \\ \frac{1+i}{\sqrt{2}} \frac{\ell}{\sqrt{p}} & \text{else.} \end{cases}$$

Again we obtain that $\mu_p$ converges to the uniform measure when $p$ goes to infinity. The renormalised measure $\sqrt{p}(\mu_p - \mu_\infty)$ have $\ell$-th momentum with modulus equal to $1, \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}$ depending on $\ell \mod 4$.

1.3. Dehn Fillings. The main question we address in this article is the following: fix a manifold $M$ with boundary $\partial M = \mathbb{T} \times T$. For any $\frac{p}{q} \in \mathbb{P}^1(Q)$, we denote by $\mathbb{T}_{p/q}$ the curve on $\mathbb{T}^2$ parametrised by $(pt, qt)$ for $t$ in $\mathbb{T}$. We define the manifold $M_{p/q}$ by Dehn filling i.e the result of gluing $M$ with a solid torus such that $\mathbb{T}_{p/q}$ bounds a disc.
We recall from [3] that in the case where $M$ has boundary, there is a principal $\mathbb{T}$-bundle with connection $L \to X(\partial M)$ such that the Chern-Simons invariant is a flat section of $\text{Res}^* L$

\[
\begin{array}{c}
\text{CS} \\
X(M) \xrightarrow{\text{Res}} X(\partial M)
\end{array}
\]

where $\text{Res}(\rho) = \rho \circ i_*$ and $i : \partial M \to M$ is the inclusion.

We will denote by $\left\| d\theta \right\|$ the natural density on $X(\mathbb{T}) = \mathbb{T}/(\theta \sim -\theta)$.

We also have $X(\mathbb{T}^2) = \mathbb{T}^2/(x, y) \sim (-x, -y)$ and for any $p, q$ the map $\text{Res}_{p/q} : X(\mathbb{T}^2) \to X(\mathbb{T}_{p/q})$ is given by $(x, y) \mapsto px + qy$.

Moreover, for any $\frac{k}{q}, \ell > 0$ and $0 \leq k \leq \ell$, there are natural flat sections $\text{CS}^{k/\ell}_{p/q}$ of $L^\ell$ over the preimage $\text{Res}^{-1}_{p/q}((\frac{k}{q}))$. These sections are called Bohr-Sommerfeld sections and they coincide for $k = 0$ with $\text{CS}^\ell$. See [3] or [2] for a detailed description.

**Theorem 1.2.** Let $M$ be a 3-manifold with $\partial M = \mathbb{T}^2$ satisfying the hypothesis of Section 2.2. Let $p, q, r, s$ be integers satisfying $ps - qr = 1$ and for any integer $n$, set $p_n = pn - r$ and $q_n = qn - s$. Then setting

\[
\mu_n^\ell = \frac{1}{n} \sum_{\rho \in X(M_{p_n/q_n})} e^{i \ell \text{CS}(\rho)}
\]

we get first

\[
\mu_n^0 = \int_{X(M)} \text{Res}^* |d\theta| + O\left(\frac{1}{n}\right)
\]

and for $\ell > 0$

\[
\mu_n^\ell = \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\ell} \sum_{\rho, k/\text{Res}_{r/s}(\rho) = \pi \frac{k}{4\ell}} \exp(-2i\pi n \frac{k^2}{4\ell} + i\ell \text{CS}(\rho) - i \text{CS}^{k/\ell}_{r/s}(\rho)) + O\left(\frac{1}{n}\right)
\]

Hence, we recover the behaviour that we observed for Lens spaces and Brieskorn spheres. The measure converges to a uniform measure $\mu_\infty$ and the renormalised measure $\sqrt{n}(\mu_n - \mu_\infty)$ has an oscillating behaviour controlled by representations in $X(M)$ with rational angle along $\mathbb{T}_{r/s}$.

1.4. **Intersection of Legendrian subvarieties.** We will prove Theorem 1.2 in the more general situation of curves immersed in a torus. Indeed, the problem makes sense in an even more general setting that we present here.

1.4.1. **Prequantum bundles.**

**Definition 1.3.** Let $(M, \omega)$ be a symplectic manifold. A prequantum bundle is a principal $\mathbb{T}$-bundle with connection whose curvature is $\omega$. 
It is well-known that the set of isomorphism classes of prequantum bundles is homogeneous under $H^1(M,T)$ and non-empty if and only if $\omega$ vanishes in $H^2(M,T)$. Let us give three examples:

**Example 1.4.**  
(i) Take $\mathbb{R}^2 \times \mathbb{T}$ with $\lambda = d\theta + \frac{1}{2\pi}(x\,dy - y\,dx)$. This gives a prequantum bundle on $\mathbb{R}^2$. Dividing by the action of $\mathbb{Z}^2$ given by

$$ (m, n) \cdot (x, y, \theta) = (x + 2\pi m, y + 2\pi n, \theta + my - nx) $$

gives a prequantum bundle $\pi : L \to \mathbb{T}^2$.

(ii) Any complex projective manifold $M \subset \mathbb{P}^n(\mathbb{C})$ has such a structure by restricting the tautological bundle whose curvature is the restriction of the Fubini-Study metric.

(iii) The Chern-Simons bundle over the character variety of a surface.

In all these cases, there is a natural subgroup of the group of symplectomorphisms of $(M, \omega)$ which acts on the prequantum bundle. The group $SL_2(\mathbb{Z})$ acts in the first case and the mapping class group in the third case. In the second case, a group acting linearly on $\mathbb{C}^{n+1}$ and preserving $M$ will give an example.

1.4.2. **Legendrian submanifolds and their pairing.** Consider a prequantum bundle $\pi : L \to M$ where $M$ has dimension $2n$ and denote by $\lambda \in \Omega^1(L)$ the connection 1-form. By Legendrian immersion we will mean an immersion $i : N \to L$ where $N$ is a manifold of dimension $n+1$ such that $i^*\lambda = 0$. This condition implies that $i$ is transverse to the fibres of $\pi$ and hence $\pi \circ i : N \to M$ is a Lagrangian immersion.

**Definition 1.5.**  
(1) Given $i_1 : N_1 \to L$ and $i_2 : N_2 \to L$ two Legendrian immersions, we will say that they are transverse if it is the case of $\pi \circ i_1$ and $\pi \circ i_2$.

(2) Given such transverse Legendrian immersions and an intersection point, i.e. $x_1 \in N_1$ and $x_2 \in N_2$ such that $\pi(i_1(x_1)) = \pi(i_2(x_2))$ we define their phase $\phi(i_1(x_1), i_2(x_2))$ as the element $\theta \in \mathbb{T}$ such that $i_2(x_2) = i_1(x_1) + \theta$.

(3) The phase measure $\phi(i_1, i_2)$ is the measure on the circle defined by

$$ \phi(i_1, i_2) = \sum_{\pi(i_1(x_1)) = \pi(i_2(x_2))} \delta_{\phi(i_1(x_1), i_2(x_2))}. $$

If $M$ is a 3-manifold obtained as $M = M_1 \cup M_2$ then, assuming transversality, the Chern-Simons measure of $M$ is given by $\mu_M = \phi(CS_1, CS_2)$ where $CS_i : X(M_i) \to L$ is the Chern-Simons invariant with values in the Chern-Simons bundle.

2. **The torus case**

2.1. **Immersed curves in the torus.** Consider the prequantum bundle $\pi : L \to \mathbb{T}^2$ given in the first item of Example 1.4. We consider a fixed
Legendrian immersion $i : [a, b] \to L$ and for any coprime integers $p, q$ the Legendrian immersion

$$i_{p/q} : \mathbb{T} \to L, i_{p/q}(t) = (pt, qt, 0).$$

Our aim here is to study the behaviour of $\phi(i, i_{p/q})$ when $(p, q) \to \infty$.

We first lift $i$ to an immersion $I : [a, b] \to \mathbb{R}^2 \times \mathbb{R}$ of the form $I(t) = (x(t), y(t), \theta(t))$. By assumption we have $\dot{\theta} = -\frac{1}{2\pi}(xy' - yx')$. For instance, lifting $i_{p/q}$ we get simply the map $I_{p/q} : t \mapsto (pt, qt, 0)$.

Let $r, s$ be integers such that $A = \begin{pmatrix} p & r \\ q & s \end{pmatrix}$ has determinant 1. Take $F_A : \mathbb{R}^2 \to \mathbb{R}$ the function

$$F_A(x, y) = \frac{1}{2\pi}(sx - ry)(qx - py).$$

A direct computation shows that this function satisfies $(m, n).I_{p/q}(t) = (pt + 2\pi m, qt + 2\pi n, F(pt + 2\pi m, qt + 2\pi n))$. We obtain from it the following formula:

$$\phi(i, i_{p/q}) = \sum_{a \leq t \leq b, qx(t) - py(t) \in 2\pi\mathbb{Z}} \delta_{\theta(t) - F(x(t), y(t))}. \tag{2}$$

If we put $i = i_{0/1}$ this formula becomes $\phi(i_{0/1}, i_{p/q}) = \sum_{k=0}^{p-1} \delta_{(x(t) + k\frac{p}{q})}$. This measure is related to the usual Gauss sum in the sense that denoting by $q^*$ an inverse of $q$ mod $p$ we have:

$$\int e^{i\theta} d\phi(i_{0/1}, i_{p/q})(\theta) = \sum_{k \in \mathbb{Z}/q\mathbb{Z}} \exp(2i\pi q^*k^2/p).$$

Suppose that $p_n = pn - r$ and $q_n = qn - s$. A Bézout matrix is given by $A_n = \begin{pmatrix} pm - r & p \\ qn - s & q \end{pmatrix}$. Up to the action of $\text{SL}_2(\mathbb{Z})$, we can suppose that $p = s = 1$ and $q = r = 0$ in which case $F_{A_n}(x, y) = -\frac{y}{2\pi}(x + ny)$. We get from Equation (2) the following formula for $\mu_n^\ell = \frac{1}{\pi} \int e^{i\theta} d\phi(i, i_{p_n/q})(\theta)$:

$$\mu_n^\ell = \frac{1}{n} \sum_{a \leq t \leq b} \exp \left( i\ell(\theta(t) + y(t)/2\pi) (x(t) + ny(t)) \right). \tag{3}$$

Taking $\ell = 0$, we are simply counting the number of solutions of $x(t) + ny(t) \in 2\pi\mathbb{Z}$ for $t \in [a, b]$. Assuming that $y$ is monotonic, the number of solutions for $t \in [a, b]$ is asymptotic to $|y(b) - y(a)|$. Hence the asymptotic density of intersection points is $i^*|dy|$ and we get

$$\lim_{n \to \infty} \mu_n^0 = \int_a^b i^*|dy|.$$ 

To treat the case $\ell > 0$, we need the following version of the Poisson formula:
Lemma 2.1. If $f, g : [a, b] \to \mathbb{R}$ are respectively $C^1$ and continuous and $f$ is piecewise monotonic, then if further $f(a), f(b) \notin 2\pi \mathbb{Z}$ we have

$$\sum_{a \leq t \leq b, f(t) \in 2\pi \mathbb{Z}} g(t) = \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} \int_a^b e^{-ikf(t)}|f'(t)|g(t)dt$$

Applying it here, we get

$$\mu_n^l = \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} \int_a^b e^{-ik(x+ny)+il(x+ny)}|\dot{x} + \dot{y}|dt$$

We apply a stationary phase expansion in this integral, the phase being $\Phi = -ky + \frac{l}{2}y^2/2\pi$ and its derivative being $\dot{\Phi} = (\frac{l}{2} + \frac{y}{\pi}k)y$. We find two types of critical points: the horizontal tangents $\dot{y} = 0$ and the points of rational height $y = \frac{k}{l}$. We observe that when $\dot{y} = 0$ the amplitude is $O(\frac{1}{n})$ and hence these contributions can be neglected compared with the other ones, where $y = \frac{k}{l}$.

We compute $\dot{\Phi} = \frac{l}{2}\dot{y}^2 + (-k + \frac{l}{\pi}y)\dot{y} = \frac{l}{2}\dot{y}^2$ and $\Phi = -\frac{k^2}{2l}$. As $\dot{\Phi} > 0$, the stationary phase approximation gives

$$\mu_n^l = \frac{1}{\sqrt{2n}} \sum_{y = \frac{k}{l}} e^{-in(k^2/2l - i\frac{k}{l} + il\theta)} + O(\frac{1}{n})$$

In order to give the final result, observe that the map $t \mapsto (t, \pi\frac{k}{l}, \frac{k}{l})$ defines a flat section of $L^l$ that we denote by $i_{\frac{k}{l}}^l$.

We can sum up the discussion by stating the following proposition.

Proposition 2.2. Let $i : \mathbb{T} \to L$ be a Legendrian immersion and suppose that $\pi \circ i$ is transverse to $i_{pn/\pi-1}$ for $n$ large enough and to the circles of equation $y = \pi\xi$ for $\xi \in \mathbb{Q}$.

Then writing $i(t) = (x(t), y(t), \theta(t))$ and $\mu_n^\ell = \frac{1}{n} \int e^{i\ell\theta}d\phi(i, i_{pn/\pi-1})(\theta)$ we have for all $\ell > 0$:

$$\mu_n^\ell = \frac{1}{\sqrt{2n}} \sum_{k \in \mathbb{Z}/2\mathbb{Z}} \sum_{t \in \mathbb{T}, y(t) = \pi\frac{k}{\ell}} e^{-in\pi(\frac{k^2}{2\ell} - i\frac{k}{\ell} + i\phi(i, i_{\pi/0}(x(t))}) + O(\frac{1}{n})$$

2.2. Application to Chern-Simons invariants. Let $M$ be a 3-manifold with $\partial M = \mathbb{T} \times \mathbb{T}$. We assume that $X(M)$ is at most 1-dimensional and that the restriction map $\text{Res} : X(M) \to X(\partial M)$ is an immersion on the smooth part and map the singular points to non-torsion points. Then we know that $\text{Res}(X(M))$ is transverse to $\mathbb{T}_{p/q}$ for all but a finite number of $p/q$, see [4].

Consider the projection map $\pi : \mathbb{T}^2 \to X(\partial M)$ which is a 2-fold ramified covering. We may decompose $X(M)$ as a union of segments $[a_i, b_i]$ whose extremities contain all singular points. The restriction map $\text{Res}$ can be lifted to $\mathbb{T}^2$ and the Chern-Simons invariant may be viewed as a map $CS$:
DISTRIBUTION OF CHERN-SIMONS INVARIANTS

\[ [a_i, b_i] \to L. \] Hence, we may apply it the results of Proposition \[2.2\] and obtain Theorem \[1.2\].

We may comment that the flat sections \[i_{k/0}^{k/l}\] of \[L^\ell\] over the line \[y = \pi k\ell\]
induces through the quotient \((x, y, \theta) \sim (−x, −y, −\theta)\) a flat section of \[L^\ell\]
that we denoted \[CS_{0/1}^{k/l}\] over the subvariety \[Res_{0/1}(\pi k\ell)\].

3. Chern-Simons invariants of coverings

3.1. General setting. Beyond Dehn fillings, we can ask for the limit of the
Chern-Simons measure of any sequence of 3-manifolds. A natural class to
look at is the case of coverings of a same manifold \[M\]. Among that category,
one can restrict to the family of cyclic coverings. One can even specify the
problem to the following case.

**Question:** Let \(p : M \to \mathbb{T}\) be a fibration over the circle and \(M_n\) be the
pull-back of the self-covering of \(\mathbb{T}\) given by \(z \mapsto z^n\). What is the asymptotic
behaviour of \(\mu_{M_n}\)?

This problem can be formulated in the following way. Let \(\Sigma\) be the fiber
of \(M\) and \(f \in \text{Mod}(\Sigma)\) be its monodromy. Any representation \(\rho \in X(M)\)
restricts to a representation \(\text{Res}(\rho) \in X(\Sigma)\) invariant by the action \(f^*_s\) of \(f\)
on \(X(\Sigma)\). Reciprocally, any irreducible representation \(\rho \in X(\Sigma)\) fixed by \(f^*_s\)
correspond to two irreducible representations in \(X(M)\).

The Chern-Simons invariant corresponding to a fixed point may be com-
puted in the following way: pick a path \(\gamma : [0, 1] \to X(\Sigma)\) joining the trivial
representation to \(\rho\) and consider the closed path obtained by composing
\(\gamma\) with \(f(\gamma)\) in the opposite direction. Then its holonomy along \(L\) is the
Chern-Simons invariant of the corresponding representation.

Understanding the asymptotic behaviour of \(\mu_{M_n}\) consists in understand-
ing the fixed points of \(f^*_n\) on \(X(\Sigma)\) and the distribution of Chern-Simons
invariants of these fixed points, a problem which seems to be out of reach
for the moment.

3.2. Torus bundles over the circle. In this elementary case, the compu-
tation can be done. Let \(A \in \text{SL}_2(\mathbb{Z})\) act on \(\mathbb{R}^2/\mathbb{Z}^2\). Its fixed points form
a group \(G_A = \{v \in \mathbb{Q}^2, Av = v \mod \mathbb{Z}^2\}/\mathbb{Z}^2\). If \(\text{Tr}(A) \neq 2\), which we
suppose from now, \(G_A\) is isomorphic to \(\text{Coker}(A - \text{Id})\) and has cardinality
\(|\det(A - \text{Id})|\).

Following the construction explained above, the phase is a map \(f : G_A \to \mathbb{Q}/\mathbb{Z}\) given by \(f([v]) = \det(v, Av) \mod \mathbb{Z}\). Hence, the measure we are trying
to understand is the following:

\[ \mu_A = \frac{1}{|\det(A - \text{Id})|} \sum_{v \in G_A} \delta_{2\pi \det(v, Av)}. \]

Consider the \(\ell\)-th moment \(\mu_A^\ell\) of \(\mu_A\). It is a kind of Gauss sum that can
be computed explicitly. The map \(f\) is a quadratic form on \(G_A\) with values
in \(\mathbb{Q}/\mathbb{Z}\). Its associated bilinear form is \(b(v, w) = \det(v, Aw) + \det(w, Av) = \)

det(\(v, (A - A^{-1})w\)). As \(A + A^{-1} = \text{Tr}(A) \text{Id}\) and \(\det(A - \text{Id}) = 2 - \text{Tr}(A)\) we get \(b(v, w) = 2 \det(v, (A - \text{Id})w) \mod \mathbb{Z}\). Hence, if \(2\ell\) is invertible in \(G_A\), then \(\ell b\) is non-degenerate and standard arguments (see \([5]\) for instance) show that \(|\mu_A^\ell| = |\det(A - \text{Id})|^{-1/2}\). Hence we still get the same kind of asymptotic behaviour for the Chern-Simons measure of the torus bundles over the circle.

References

[1] B. C. Berndt, R. J. Evans and K. S. Williams *Gauss and Jacobi sums*, Canadian Mathematical Society Series of Monographs and Advanced Texts (1998).
[2] L. Charles and J. Marché, *Knot asymptotics II, Witten conjecture and irreducible representations*, Publ. Math. Inst. Hautes Études Sci. 121 (2015), 323–361.
[3] P. Kirk and E. Klassen, *Chern-Simons invariants of 3-manifolds and representation spaces of knot groups*, Math.Ann. 287 (1990), 343–367.
[4] J. Marché and G. Maurin, *Singular intersections of subgroups and character varieties*, arXiv:1406.2862.
[5] V. Turaev, *Reciprocity for Gauss sums on finite abelian groups*, Math. Proc. Camb. Phil. Soc. 124 no. 2 (1998), 205–214.

Institut de Mathématiques de Jussieu - Paris Rive Gauche, Université Pierre et Marie Curie, 75252 Paris cédex 05, France
E-mail address: julien.marche@imj-prg.fr