LAGRANGIAN FILLINGS FOR LEGENDRIAN LINKS OF FINITE TYPE

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Abstract. We prove that there are at least seeds many exact embedded Lagrangian fillings for Legendrian links of type ADE. We also provide seeds many Lagrangian fillings with certain symmetries for type BCFG. Our main tools are $N$-graphs and the combinatorics of seed patterns of finite type.

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1. Introduction

1.1. Backgrounds. Legendrian knots are central object in the study of contact 3-dimensional contact manifolds. Classification of Legendrian knots are important as its own right, and also play a prominent role in constructing 4-dimensional Weinstein manifold.

Classical Legendrian knot invariants are Thurston–Bennequin number and rotation number [20] which distinguish the pair of Legendrian knots with the same knot type. There are non-classical
invariants including the Legendrian contact algebra via the method of Floer theory [12, 10], and the space of constructible sheaves using microlocal analysis [21, 29]. These non-classical invariants distinguish the Chekanov pair, a pair of Legendrian knots of type $m\delta_2$ having the same classical invariants.

Recently, the study of exact Lagrangian fillings for Legendrian links has been extremely plentiful. In the context of Legendrian contact algebra, an exact Lagrangian filling gives an augmentation through the functorial view point [11]. There are several level of equivalence between augmentations and the constructible sheaves for Legendrian links from counting to categorical equivalence [24]. By using these idea of augmentations and constructible sheaves, people construct infinitely many fillings for certain Legendrian links [7, 19, 8]. Here is the summarized list of methods of constructing Lagrangian fillings for Legendrian links:

1. Decomposable Lagrangian fillings via pinching sequences [11],
2. Alternating Legendrians and its conjugate Lagrangian fillings [28],
3. Legendrian weaves via $N$-graphs [30, 8],
4. Donaldson–Thomas transformation on augmentation varieties [27, 18, 19].

The cluster structure introduced by [14] plays a crucial role in the above constructions and applications. More precisely, the space of augmentations, or equally the moduli of constructible sheaves adapted to Legendrian links, admits a structure of cluster algebra [28]. Note that a seed of cluster algebra consists of a quiver whose vertices are decorated with cluster variables. An involutory operation at each vertex, called mutation, generates all seeds of the cluster pattern. The main point is to identify the mutation in the cluster pattern and an operation in the space of Lagrangian fillings. This geometric operation is deeply related to the Lagrangian surgery [26] and the wall-crossing phenomenon [2].

Indeed, a Legendrian torus link of type $(2, n)$ admits Catalan number many Lagrangian fillings up to exact Lagrangian isotopy [25, 28, 30]. Interestingly enough, the Catalan number is the number of seeds in a cluster pattern of Dynkin type $A_{n-1}$. There are also Legendrian links corresponding to Dynkin type $D_n, E_6, E_7,$ and $E_8$ [19]. A conjecture in [6, Conjecture 5.1] says that the number of distinct exact embedded Lagrangian fillings (up to exact Lagrangian isotopy) for Legendrian links of type $ADE$ is exactly the same as the number of seeds of the corresponding cluster algebras.

Furthermore, it is also conjectured in [6, Conjecture 5.4] that for Legendrian links of type $A_{2n-1}, D_{n+1}, E_6$ and $D_4$, Lagrangian fillings having certain $\mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/3\mathbb{Z}$-symmetry form the cluster patterns of type $B_n, C_n, F_4$ and $G_2$, which are Dynkin diagrams obtained by folding introduced in [16].

1.2. The results. Our main result is that there are seeds many Lagrangian fillings for Legendrian links of finite type. We deal with $N$-graphs in [8] to construct the Lagrangian fillings. An $N$-graph $\mathcal{S}$ on $\mathbb{D}^2$ gives a Legendrian surface $\Lambda(\mathcal{S})$ in $J^1\mathbb{D}^2$ while the boundary $\partial \mathcal{S}$ on $S^1$ induces a Legendrian link $\lambda(\partial \mathcal{S})$. Then projection of $\Lambda(\mathcal{S})$ along the Reeb direction becomes a Lagrangian filling of $\lambda(\partial \mathcal{S})$.

As mentioned above, we interpret an $N$-graph as a seed in the corresponding cluster pattern. A one-cycle in the Legendrian surface $\Lambda(\mathcal{S})$ corresponds to a vertex of the quiver, and a signed intersection between one-cycles gives an arrow between corresponding vertices. From constructible sheaves adapted to $\Lambda(\mathcal{S})$, one can assign a monodromy to each one-cycle which becomes the cluster variable at each vertex.

There is an operation so called a Legendrian mutation $\mu_\gamma$ on an $N$-graph $\mathcal{S}$ along one-cycle $[\gamma] \in H_1(\Lambda(\mathcal{S}))$ which is the counterpart of the mutation on the cluster pattern, see Proposition 4.19. The delicate and challenging part is that we do not know whether Legendrian mutations are always possible or not. Simply put, this is because the mutation in cluster side is algebraic, whereas the Legendrian mutation is rather geometric.

The main idea of our construction is to consider the following bichromatic (blue and red) graph $\mathcal{S}(a, b, c)$, i.e. $N$-graph with $N = 3$, bounding a Legendrian link $\lambda(a, b, c)$, which is the closure of
the braid $\beta(a,b,c)$ as follows:

$$\lambda(a,b,c) = Cl(\beta(a,b,c)) \subset J^1S^1,$$

$$\beta(a,b,c) = \sigma_2^a\sigma_1^{b+1}\sigma_2\sigma_1^{c+1}.$$ 

Then $\beta(a,b,c)$ corresponds to the rainbow closure of the braid $\beta_0(a,b,c) = \sigma_1\sigma_2^{a+1}\sigma_1\sigma_2^{b+1}\sigma_1\sigma_2^{c+1}.$

![Braid Closure](image)

**Figure 1.** The rainbow closure of the braid $\beta_0(a,b,c)$ in $\mathbb{R}^3$

One-cycles $\mathcal{B}(a,b,c)$ of the Legendrian surface $\Lambda(\mathcal{G}(a,b,c))$ are given by the yellow- and green-shaded edges as depicted in Figure 2. See §4.1 for the detail.

![Tripod N-Graph](image)

**Figure 2.** The tripod $N$-graph $\mathcal{G}(a,b,c)$ and the good tuple $\mathcal{B}(a,b,c)$ of cycles

There are several good properties of $\mathcal{G}(a,b,c)$ as follows:

1. The geometric- and algebraic intersection numbers of the one-cycles in $\mathcal{B}(a,b,c)$ coincide.
2. The corresponding quiver $Q(a,b,c)$ is bipartite, see §4.3 for the details.
3. It covers Legendrian links of type ADE. More precisely, the underlying graphs of $Q(1,b,c)$ for $b+c-1 = n$, $Q(n-2,2,2)$, and $Q(2,3,n-3)$ are the same as Dynkin diagrams of type $A_n$, $D_n$, and $E_n$, respectively.

Let us consider the finite type $N$-graph $\mathcal{G}(a,b,c)$, that is, $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} > 1$. Denote the corresponding rank $n$-root system by $\Phi = \Phi(a,b,c)$, where $n = a+b+c-2$. Let us consider an exchange graph $E(\Phi)$ of the corresponding cluster pattern whose vertices are the seeds and whose edges connect the vertices are given by a single mutation. Note from [9] that the exchange graph $E(\Phi)$ can be realized as vertices and edges of a polytope $P(\Phi) \subset \mathbb{R}^n$ called a generalized associahedron.

The combinatorics of the exchange graph $E(\Phi)$ is the key ingredient in investigating the Legendrian mutability. All facets of the polytope $P(\Phi)$ can be recovered from a sequence of mutations obtained by a Coxeter element together with a subset of facets of $P(\Phi)$ corresponding to $P(\Phi([n] \setminus \{i\}))$, see [16] or Proposition 3.15. We call this specific sequence of mutations a Coxeter mutation. In order to interpret a Coxeter mutation in terms of $N$-graphs, let us consider a partition $\mathcal{B}_+, \mathcal{B}_-$ of the one-cycles $\mathcal{B}$, consisting of yellow- and green-shaded edges, respectively. Then
the $N$-graph realization of the Coxeter mutation is called the *Legendrian Coxeter mutation* and given by the sequence of Legendrian mutations:

$$
\mu_G = \prod_{\gamma \in B^-} \mu_{\gamma} \cdot \prod_{\gamma \in B^+} \mu_{\gamma}.
$$

Then the resulting $N$-graph $\mu_G(\mathcal{G}(a,b,c), \mathcal{B}(a,b,c))$ becomes the $N$-graph shown in Figure 3 up to a sequence of Move (II) in Figure 7.

![Figure 3. Legendrian Coxeter mutation on $(\mathcal{G}(a, b, c), \mathcal{B}(a, b, c))$](image)

Removing the gray-shaded annulus region, the only difference between $(\mathcal{G}(a, b, c), \mathcal{B}(a, b, c))$ and $\mu_G(\mathcal{G}(a, b, c), \mathcal{B}(a, b, c))$ is the reversing of the color. Note that the intersection pattern between one-cycles and the Legendrian mutability are preserved under the action of the Legendrian Coxeter mutation $\mu_G$. Moreover, the operation $\mu_Q$ also acts on the face poset of the generalized associahedron of the root system $\Phi$. By the induction argument on the rank of root system, we conclude that there in no (geometric) obstruction to realize each seed via the $N$-graph, especially for finite type case. This guarantees that there are at least seeds many Lagrangian fillings for $\lambda(a,b,c)$.

For the infinite type, i.e. $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq 1$, the operation $\mu_Q$ is of infinite order and so is $\mu_G$, hence Legendrian weaves $\Lambda(\mu_G(\mathcal{G}(a, b, c), \mathcal{B}(a, b, c)))$

produce infinitely many distinct Lagrangian fillings. Indeed, the quiver $Q(a,b,c)$ is also bipartite and the one can perform the Legendrian Coxeter mutation $\mu_G$ on the $N$-graph $\mathcal{G}(a,b,c)$ by stacking the gray-shaded annulus like as before. Therefore, there is no obstruction to realize seeds obtained by mutations $\mu_Q$ via the $N$-graphs. Since the order of the Legendrian Coxeter mutation is infinite (see Lemma 3.20), we obtain infinitely many $N$-graphs and hence infinitely many exact embedded Lagrangian fillings for the Legendrian link $\lambda(a,b,c)$ with $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq 1$.

**Theorem 1.1** (Theorem 5.12). For each $a, b, c \geq 1$, the Legendrian knot or link $\lambda(a,b,c)$ has distinct infinitely many Lagrangian fillings if

$$
\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq 1,
$$

or equivalently, the tripod $Q(a,b,c)$ is of infinite type.

**Theorem 1.2** (Theorem 5.13). There are at least seeds many distinct exact embedded Lagrangian fillings for Legendrian links of type $ADE$.

There are several way of constructing exact embedded Lagrangian fillings as mentioned above. Especially in $D_4$ case, there are 34 distinct Lagrangian fillings constructed by the method of the alternating Legendrians in [3, 28], while the above $N$-graphs give seeds many 50 Lagrangian fillings.

The remaining finite type Dynkin diagrams, which are non-simply laced, are of type $BCFG$, obtain by the folding procedure from type $ADE$, see §3.3. By keep tracking the folding process,
seeds and mutations in $B_n$, $C_n$, $F_4$, and $G_2$ cluster patterns can be regarded as certain subsets of seeds and sequences of mutations in $A_{2n-1}$, $D_{n-1}$, $E_6$, and $D_4$, respectively. Those specified seeds of type ADE admit $N$-graphs with certain symmetries given by an action of a finite group $G$, and we call such seeds and $N$-graphs $G$-admissible. If a seed (or an $N$-graph) is again $G$-admissible after performing a sequence of mutations indexed by vertices in the same $G$-orbit, then we call it globally foldable with respect to $G$.

The following four $N$-graphs are examples of type BCFG. Indeed, they are $\mathfrak{g}(1, 3, 3)$, $\mathfrak{g}(3, 2, 2)$, $\mathfrak{g}(2, 3, 3)$, and $\mathfrak{g}(2, 2, 2)$, respectively.

The colored regions represent how the group $G$ acts on the $N$-graphs and the induced Lagrangian fillings. The first three $N$-graphs are globally foldable with respect to $\mathbb{Z}/2\mathbb{Z}$ by folding orange- and violet-colored regions in an orientation preserving way. Similarly, the $\mathbb{Z}/3\mathbb{Z}$-rotational symmetry of the last one implies that it is globally foldable with respect to $\mathbb{Z}/3\mathbb{Z}$ by folding three colored regions.

**Theorem 1.3** (Theorem 6.11). The following holds:

1. The Legendrian link $\lambda(A_{2n-1})$ has $\binom{2n}{n} \mathbb{Z}/2\mathbb{Z}$-admissible $N$-graphs which admits the cluster pattern of type $B_n$.
2. The Legendrian link $\lambda(D_{n+1})$ has $\binom{2n}{n} \mathbb{Z}/2\mathbb{Z}$-admissible $N$-graphs which admits the cluster pattern of type $C_n$.
3. The Legendrian link $\lambda(E_6)$ has 105 $\mathbb{Z}/2\mathbb{Z}$-admissible $N$-graphs which admits the cluster pattern of type $F_4$.
4. The Legendrian link $\lambda(D_4)$ has 8 $\mathbb{Z}/3\mathbb{Z}$-admissible $N$-graphs which admits the cluster pattern of type $G_2$.

**Acknowledgement.** B. An was supported by Kyungpook National University Research Fund, 2020. Y. Bae was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (No. 2020R1A2C1A0100320). E. Lee was supported by IBS-R003-D1.

2. Legendrians and $N$-graphs

We recall from [8] the notion of $N$-graphs and their combinatorial moves which encode the Legendrian isotopy data of corresponding Legendrian surfaces. As an application, we review how $N$-graphs can be used to find and to distinguish Lagrangian fillings for Legendrian links.

### 2.1. Geometric setup.

Let us start with the standard contact structure on $\mathbb{R}^3$ whose contact structure is given by $\xi_{st}^3 = \ker(dz - ydx)$. Consider the symplectization $(\mathbb{R}^4, d(e^s(dz - ydx)))$ and its contactization $(\mathbb{R}^5, \alpha_{st}^5 = e^s(dz - ydx) - dt)$.

Now consider a contact 3-dimensional space

$$(\mathbb{R}^5, \alpha_{st}^5) := \{(x, y, z, s, t) \in \mathbb{R}^5 \mid (s, t) = (\ell, 1)\}$$
for each symplectization level $s = \ell$. Take Legendrians $\lambda_1 \subset (\mathbb{R}^3, \xi_{st}^3)$ and $\lambda_2 \subset (\mathbb{R}^2, \xi_{st}^2)$ and consider a Legendrian surface

$$\Lambda \subset (\mathbb{R}^5, \xi_{st}^5) = \ker \alpha_{st}^5 \cap \{1 \leq s \leq 2\}$$

whose boundary is $\lambda_1 \cup \lambda_2$, i.e.,

$$\Lambda \cap (\mathbb{R}^1, \xi_{st}^1) = \lambda_1, \quad \Lambda \cap (\mathbb{R}^2, \xi_{st}^2) = \lambda_2.$$

Let $\pi : \mathbb{R}^5 \to \mathbb{R}^4$ be the projection along the contactization coordinate $t$, then $\pi(\Lambda)$ becomes an exact Lagrangian (possibly immersed) cobordism from $\pi(\lambda_2)$ to $\pi(\lambda_1)$. Note that $\partial_t$ is the Reeb vector field of $(\mathbb{R}^5, \alpha_{st}^5)$ and the above immersed points on $\pi(\Lambda)$ correspond to Reeb chords in $\Lambda$.

Relating the construction of the current article, any Legendrian link in $(\mathbb{R}^3, \xi_{st}^3)$ can be seen as a satellite link of the standard Legendrian unknot $\lambda_{unknot} \subset (\mathbb{R}^3, \xi_{st}^3)$. Note that a neighborhood of $\lambda_{unknot}$ is contactomorphic to $(J^1S^1, \ker(dz - p_1d\theta))$. Denote the corresponding contact embedding by $\iota : J^1S^1 \to \mathbb{R}^5$.

Let us denote the corresponding satellite links of $\lambda_1, \lambda_2$ by

$$\lambda_{\beta_i} = \text{Cl}(\beta_i) \subset (J^1S^1 \times \{i\}, \ker(dz - p_1d\theta)), \quad i = 1, 2.$$

Here $\beta_i$ is a positive braid word for the satellite link of $\lambda_i$, and $\text{Cl}(\beta)$ denotes the closure of a braid word $\beta$. Extending the contact embedding $\iota : J^1S^1 \to \mathbb{R}^5$, by abuse of notation, we have $\iota : J^1S^1 \times \{1, 2\} \hookrightarrow \mathbb{R}^5 \cap \{1 \leq s \leq 2\}$. Denote the corresponding Legendrian surface by

$$\Lambda \subset (J^1S^1 \times \{1, 2\}, \ker(dz - p_1d\theta - p_3d\sigma)),$$

where $\sigma$ is the coordinate for the interval $[1, 2]$ which corresponds to the $e^s$-coordinate. By a strict contactomorphism, we can regard

$$\Lambda \subset (J^1S^1 \times \mathbb{R}_s \times \mathbb{R}_t, \ker(e^s(dz - p_1d\theta) - dt)).$$

Then its Lagrangian projection $\pi \circ \iota(\Lambda)$ gives an exact Lagrangian cobordism from $\lambda_2$ to $\lambda_1$, where $\pi$ is the projection along the $t$-coordinate. Especially when $\lambda_1 = \emptyset$, the boundary $J^1S^1 \times \{1\}$ of $J^1S^1 \times \{1, 2\}$ can be compactified by $J^1D^2$. Under the Lagrangian projection, this corresponds to a exact symplectic filling $(\mathbb{D}^4, \omega_{st})$ of $(S^3, \xi_{st}) = (\mathbb{R}^3, \xi_{st}) \cup \{\infty\}$. We end this section by stating the relation between Legendrian- and Lagrangian fillings.

**Lemma 2.1.** As in the above setup, let $\lambda_{\beta_i} \subset J^1S^1$ be a Legendrian link, and let $\iota(\lambda_{\beta_i})$ be an induced Legendrian link in $(S^3, \xi_{st})$. Let $\Lambda, \Lambda' \subset J^1D^2$ be two Legendrian surfaces, without Reeb chords, bounding $\lambda_{\beta_i}$. If the corresponding exact Lagrangian fillings $\pi \circ \iota(\Lambda), \pi \circ \iota(\Lambda') \subset (\mathbb{D}^4, \omega_{st})$ of $\iota(\lambda_{\beta_i})$ are exact Lagrangian isotopic relative to the boundary, then $\Lambda, \Lambda'$ are Legendrian isotopic relative to the boundary.

### 2.2. $N$-graphs and Legendrian weaves.

**Definition 2.2.** [8, Definition 2.2.2] An $N$-graph $\mathcal{G}$ on a smooth surface $S$ is an $(N - 1)$-tuple of graphs $(\mathcal{G}_1, \ldots, \mathcal{G}_{N-1})$ satisfying the following conditions:

1. Each graph $\mathcal{G}_i$ is embedded, trivalent, possibly empty and non necessarily connected.
2. Any consecutive pair of graphs $(\mathcal{G}_i, \mathcal{G}_{i+1})$, $1 \leq i \leq N - 2$, intersects only at hexagonal points depicted as in Figure 4.
3. Any pair of graphs $(\mathcal{G}_i, \mathcal{G}_j)$ with $1 \leq i, j \leq N - 1$ and $|i - j| > 1$ intersects transversely at edges.

![Figure 4. A hexagonal point](image-url)
Remark 2.3. For the result of the current article, we mainly consider the case $N = 3$ and $S = \mathbb{D}^2$. In other words, we focus bicolored graphs with monochromatic trivalent vertices and bichromatic hexagonal points as in Figure 4.

For any $N$-graph $\mathcal{G}$ on a surface $S$, we associate a Legendrian surface $\Lambda(\mathcal{G}) \subset J^1 \mathcal{G}$. Basically, we construct the Legendrian surface by weaving the wavefronts in $S \times \mathbb{R}$ constructed from a local chart of $S$.

Let $\mathcal{G} \subset S$ be an $N$-graph. A finite cover $\{U_i\}_{i \in I}$ is called $\mathcal{G}$-compatible if

1. each $U_i$ is diffeomorphic to the open disk $\mathbb{D}^2$,
2. $U_i \cap \mathcal{G}$ is connected, and
3. $U_i \cap \mathcal{G}$ contains at most one vertex.

For each $U_i$, we associate a wavefront $\Gamma(U_i) \subset U_i \times \mathbb{R} \subset S \times \mathbb{R}$. Note that there are only four types of local charts for any $N$-graph $\mathcal{G}$ as follows:

Type 1. A chart without any graph component whose corresponding wavefront becomes

$$ \bigcup_{i=1, \ldots, N} \mathbb{D}^2 \times \{i\} \subset \mathbb{D}^2 \times \mathbb{R}.$$ 

Type 2. A chart with single edge. The corresponding wavefront is the union of the $A_2^3$-germ along the two sheets $\mathbb{D}^2 \times \{i\}$ and $\mathbb{D}^2 \times \{i+1\}$, and trivial disks $\mathbb{D}^2 \times \{i\}$, $i \in \{1, \ldots, N\} \backslash \{i, i+1\}$. The local model of $A_2^3$ comes from the origin of the singular surface

$$ \Gamma(A_2^3) = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 - z^2 = 0\}.$$

See Figure 5(a).

Type 3. A chart with a monochromatic trivalent vertex whose wavefront is the union of the $D_4^-$-germ along the two sheets $\mathbb{D}^2 \times \{i\}$, $i \in \{1, \ldots, N\}$ \{i, i+1\}, and trivial disks $\mathbb{D}^2 \times \{i\}$, $i \in \{1, \ldots, N\} \backslash \{i, i+1\}$. The local model for Legendrian singularity of type $D_4^-$ is given by the image at the origin of

$$ \delta^-_4 : \mathbb{R}^2 \to \mathbb{R}^3 : (x, y) \mapsto \left(x^2 - y^2, 2xy, \frac{2}{3}(x^3 - 3xy^2)\right).$$

See Figure 5(c).

Type 4. A chart with a bichromatic hexagonal point. The induced wavefront is the union of the $A_2^3$-germ along the three sheets $\mathbb{D}^2 \times \{\ast\}$, $\ast = i, i+1, i+2$, and trivial disks $\mathbb{D}^2 \times \{i\}$, $i \in \{1, \ldots, N\} \backslash \{i, i+1, i+2\}$. The local model of $A_2^3$ is given by the origin of the singular surface

$$ \{(x, y, z) \in \mathbb{R}^3 \mid (x^2 - z^2)(y - z) = 0\}.$$ 

See Figure 5(b).

![Figure 5. Three-types of wavefronts of Legendrian singularities.](image)

**Definition 2.4.** [8, Definition 2.7] Let $\mathcal{G}$ be an $N$-graph on a surface $S$. The **Legendrian weave** $\Lambda(\mathcal{G}) \subset J^1 \mathcal{G}$ is an embedded Legendrian surface whose wavefront $\Gamma(\mathcal{G}) \subset S \times \mathbb{R}$ is constructed by weaving the wavefronts $\{\Gamma(U_i)\}_{i \in I}$ from a $\mathcal{G}$-compatible cover $\{U_i\}_{i \in I}$ with respect to the gluing data given by $\mathcal{G}$.

**Remark 2.5.** Note that $\Lambda(\mathcal{G})$ is well-defined up to the choice of cover and up to planar isotopies. Let $\{\varphi_t\}_{t \in [0,1]}$ be a compactly supported isotopy of $S$. Then this induces a Legendrian isotopy of Legendrian surface $\Lambda(\varphi_t(\mathcal{G})) \subset J^1 \mathcal{G}$ relative to the boundary.
2.3. **Legendrian isotopies and moves on $N$-graphs.** The idea of $N$-graph is useful in the study of Legendrian surface, because the Legendrian isotopy of the Legendrian weave $\Lambda(\mathcal{G})$ can be encoded in combinatorial moves of $N$-graphs.

**Theorem 2.6.** [8, Theorem 1.1] Let $\mathcal{G}$ be a local $N$-graph. The combinatorial moves in Figure 7 and Figure 8 are Legendrian isotopies for $\Lambda(\mathcal{G})$.

![Figure 7](image)

**Figure 7.** Combinatorial moves for Legendrian isotopies of surface $\Lambda(\mathcal{G})$. Here the pairs (blue, red) and (red, green) are consecutive. Other pairs are not.

2.3.1. **$N$-graphs on $D^2$.** Let $\lambda_\beta \subset J^1 S^1$ be a Legendrian link obtained from a Legendrian line in $\mathbb{R}^3$ by satelliting the Legendrian unknot. Here $\lambda_\beta$ is the closure of a positive $N$-strand braid $\beta$. The braid word $\beta$ consist of alphabets $\sigma_1, \ldots, \sigma_{N-1}$, and these give an $(N-1)$-tuple of sets of points in $S^1$ which can be regarded as a boundary data of $N$-graphs on $D^2$. By the setup in §2.1, $\pi \circ \iota(\Lambda(\mathcal{G}))$ induces an exact (possibly immersed) Lagrangian filling in $(\mathbb{R}^4, \omega_{st})$ of $\iota(\lambda_\beta)$. Let us denote the equivalence class of a $N$-graph $\mathcal{G} \subset D^2$ up to the moves (I), \ldots, (XII) by $[\mathcal{G}]$. 

![Figure 6](image)

**Figure 6.** Four-types of local charts for $N$-graphs.
Figure 8. Combinatorial moves for Legendrian isotopies of surface $\Lambda(\mathcal{G})$: These are moves involving $A_3$-swallowtail singularities, the orange vertex. Here the orange lines are locus of cusp singularities.

Remark 2.7. For $N$-graphs $\mathcal{G}$ on $\mathbb{D}^2$ the stabilization, becomes Move (S) in Figure 9:

Definition 2.8. An $N$-graph $\mathcal{G} \subset \mathbb{D}^2$ is called free if the induced Legendrian weave $\Lambda(\mathcal{G}) \subset J^1\mathbb{D}^2$ can be woven without interior Reeb chord.

Example 2.9. [8, Example 7.3] Let $\mathcal{G} \subset \mathbb{D}^2$ be a 2-graph such that $\mathbb{D}^2 \setminus \mathcal{G}$ is simply connected relative to the boundary $\partial \mathbb{D}^2 \cap (\mathbb{D}^2 \setminus \mathcal{G})$. Then $\mathcal{G}$ is free if and only if $\mathcal{G}$ has no faces contained in $\mathring{\mathbb{D}}^2$. Note that each of such faces admits at least one Reeb chord, see Figure 10.

Figure 10. $N$-graphs with Reeb chords
To investigate the Reeb chords of $\Lambda(\mathcal{G})$ in $J^1 \mathbb{D}^2$, let us consider the wavefront $\Gamma(\mathcal{G})$ in $\mathbb{D}^2 \times \mathbb{R}$. Label the sheets of the wavefront

$$\Gamma(\mathcal{G}) = \bigcup_{i=1}^{N} \Gamma_i$$

by the $z$-coordinate from the bottom to top. Let $f_i : \mathbb{D}^2 \to \mathbb{R}$ be a function whose graph becomes $\Gamma_i$, and let $h_{ij} : \mathbb{D}^2 \to \mathbb{R}$ be a difference function given by $f_j - f_i$ for any $i, j \in [N]$ with $i < j$.

By the construction $h_{i+1}^{-1}(0)$ gives $\mathcal{G}_i \subset \mathcal{G}$. The critical points of $h_{ij}$ on $\partial \mathbb{D}^2 \setminus \mathcal{G}$ are the possible candidates for the Reeb chords. In other words, to guarantee that $\mathcal{G}$ is free, it suffices to show that $h_{ij}$ has no critical point on $\partial \mathbb{D}^2 \setminus \mathcal{G}$.

Now apply this idea to the 3-graph $\mathcal{G}(a, b, c)$ in the introduction. In order to construct $h_{12}$ and $h_{23}$, consider the following graph complements $\mathbb{D}^2 \setminus \mathcal{G}_i$ for $i = 1, 2$. Let us denote the closure of connected components of $\mathbb{D}^2 \setminus \mathcal{G}_i$ by $\{F_{i,k}\}_{k \in K_i}$. Each $F_{i,k}$ is a polygon and exactly one edge comes from the boundary $\partial \mathbb{D}^2$.

By the construction of $h_{i+1}$ and the definition of Reeb chord, there is no Reeb chord connecting $\Gamma_i$ and $\Gamma_{i+1}$ for $i = 1, 2$. Now consider the following gradient flow lines of $h_{12} + h_{23}$ to see the Reeb chords from $\Gamma_1$ to $\Gamma_3$. Without loss of generality, we may assume that $\| \nabla h_{12} \| < \| \nabla h_{23} \|$ except a small neighborhood of $\mathcal{G}_2$. Then by the configuration of $\mathcal{G}(a, b, c) = \mathcal{G}_1 \cup \mathcal{G}_2$ the gradient flow lines of $h_{12} + h_{23}$ never vanish except the hexagonal point. In conclusion, we can construct the wavefront $\Gamma(\mathcal{G}(a, b, c))$ without interior Reeb chords.

**Lemma 2.10.** The 3-graph $\mathcal{G}(a, b, c)$ is free.
2.3.2. $N$-graphs on $\mathbb{A}$. Let $\mathbb{A}$ be the oriented annulus with two boundary components $\partial_+ \mathbb{A}$ and $\partial_- \mathbb{A}$, and let $\mathcal{G}$ be a $N$-graph on $\mathbb{A}$. We say that $\mathcal{G}$ is of type $(\lambda_+, \lambda_-)$ if $\mathcal{G}$ on $\partial_+ \mathbb{A}$ and $\partial_- \mathbb{A}$ are given by Legendrian links $\lambda_+$ and $\lambda_-$, respectively. We may regard the $N$-graph $\mathcal{G}$ of type $(\lambda_+, \lambda_-)$ as a cobordism between $\lambda_+$ and $\lambda_-$. Suppose that two annular $N$-graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ are of type $(\lambda_1, \lambda_2)$ and $(\lambda_2, \lambda_3)$. Then two $N$-graphs can be merged or piled in a natural way to obtain the annular $N$-graph, denoted by $\mathcal{G}_1 \cdot \mathcal{G}_2$ of type $(\lambda_1, \lambda_3)$. Let $\mathcal{G} \subset \mathbb{D}^2$ be an $N$-graph with $\partial \mathcal{G} = \lambda_1$, then the padding operation $\mathcal{G}_1 \mathcal{G}$ is defined by glueing along the boundary $\lambda_1$. Note that if there is a rotational symmetry on $\lambda_1$, then the operation $\mathcal{G}_1 \mathcal{G}$ is well-defined only up to that symmetry.

![Figure 13. Reidemeister moves in $J^1S^1$ or $\mathbb{R}^3$ avoiding cusp singularities.](image_url)

Let us illustrate elementary annulus $N$-graphs coming from the Legendrian isotopies in $J^1S^1$. The following two Legendrian Reidemeister moves (RIII) and (R0) can be interpreted as $N$-graphs $\mathcal{G}_{(RIII)}$ and $\mathcal{G}_{(R0)}$ on the annulus $\mathbb{A}$, respectively, as depicted in Figure 14. The Move (I) and (V) of $N$-graphs in Figure 7 imply that the inverses $\mathcal{G}_{(RIII)}^{-1}$ and $\mathcal{G}_{(R0)}^{-1}$ can be obtained by reversing the role of the inner- and outer boundaries.

Suppose that there are certain rotational symmetry on $N$-graphs. Let us consider a rotational annulus $N$-graph which is trivial as an $N$-graph but rotated respecting the symmetry. A typical example comes from Legendrian torus link $\lambda(n, m)$ of maximal Thurston-Bennequin number. The right one in Figure 14 is a rotational annulus $N$-graph for $\lambda(3, 3)$. This type of annular $N$-graphs play a crucial role in producing a sequence of distinct exact Lagrangian fillings of positive braid Legendrian links, see [23, 7, 19].

3. Cluster algebras

Cluster algebras, introduced by Fomin and Zelevinsky [14], are commutative algebras with specific generators, called cluster variables, defined recursively. In this section, we recall basic notions in the theory of cluster algebras. For more details, we refer the reader to [14, 15]. Throughout this section, we fix $m, n \in \mathbb{Z}_{>0}$ such that $n \leq m$, and we let $\mathbb{F}$ be the rational function field with $m$ independent variables over $\mathbb{C}$.

3.1. Basics on cluster algebras.

**Definition 3.1** (cf. [14, 15]). A seed $\Sigma = (x, B)$ is a pair of

- a tuple $x = (x_1, \ldots, x_m)$ of algebraically independent generators of $\mathbb{F}$, that is, $\mathbb{F} = \mathbb{C}(x_1, \ldots, x_m)$;
- an $n \times m$ integer matrix $B = (b_{i,j})_{i,j}$ such that the principal part $B^{pr} := (b_{i,j})_{1 \leq i, j \leq n}$ is skew-symmetrizable, that is, there exist positive integers $d_1, \ldots, d_n$ such that

$$\text{diag}(d_1, \ldots, d_n) \cdot B^{pr}$$

is a skew-symmetric matrix.

We call elements $x_1, \ldots, x_m$ cluster variables and call $B$ exchange matrix. Moreover, we call $x_1, \ldots, x_n$ unfrozen (or, mutable) variables and $x_{n+1}, \ldots, x_m$ frozen variables.

To define cluster algebras, we introduce mutations on seeds, exchange matrices, and quivers as follows.
Figure 14. Elementary annulus operations on $N$-graphs on $\mathbb{B}^2$, and a rotational annulus $N$-graph.

(1) (Mutation on seeds) For a seed $\Sigma = (x, B)$ and an integer $1 \leq k \leq n$, the mutation $\mu_k(\Sigma) = (x', B')$ is defined as follows:

$$x'_i = \begin{cases} x_i & \text{if } i \neq k, \\ x_k^{-1} \left( \prod_{b_{k,j} > 0} x_j^{b_{k,j}} + \prod_{b_{k,j} < 0} x_j^{-b_{k,j}} \right) & \text{otherwise}. \end{cases}$$

$$b'_{i,j} = \begin{cases} -b_{i,j} & \text{if } i = k \text{ or } j = k, \\ b_{i,j} + \frac{|b_{i,k}b_{k,j} + b_{i,k}b_{k,j}|}{2} & \text{otherwise}. \end{cases}$$

(2) (Mutation on exchange matrices) We define $\mu_k(B) = (b'_{i,j})$, and say that $B' = (b'_{i,j})$ is the mutation of $B$ at $k$.

(3) (Mutation on quivers) We call a finite directed multigraph $Q$ a quiver if it does not have oriented cycles of length at most 2. The adjacency matrix $B(Q)$ of a quiver is always skew-symmetric. Moreover, $\mu_k(B(Q))$ is again the adjacency matrix of a quiver $Q'$. We define $\mu_k(Q)$ to be the quiver satisfying

$$B(\mu_k(Q)) = \mu_k(B(Q)),$$

and say that $\mu_k(Q)$ is the mutation of $Q$ at $k$.

We say a quiver $Q'$ is mutation equivalent to another quiver $Q$ if there exists a sequence of mutations $\mu_{j_1}, \ldots, \mu_{j_l}$ which connects $Q'$ and $Q$, that is,

$$Q' = (\mu_{j_l} \cdots \mu_{j_1})(Q).$$

Also, we say a quiver $Q$ is acyclic if there is no directed cycle.
Example 3.4. Let \( \Sigma \) be the initial seed \( \Sigma \) such that if \( \mu \) is a mutation, then there exists a sequence of mutations from one to other such that intermediate quivers are all acyclic. Indeed, two mutation equivalent acyclic quivers have the same underlying (undirected) graph.

An immediate check shows that \( \mu_k(\Sigma) \) is again a seed, and a mutation is an involution, that is, its square is the identity. Also, note that the mutation on seeds does not change frozen variables \( x_{n+1}, \ldots, x_m \). Let \( T_n \) denote the \( n \)-regular tree whose edges are labeled by \( 1, \ldots, n \). Except for \( n = 1 \), there are infinitely many vertices on the tree \( T_n \). For example, we present regular trees \( T_2 \) and \( T_3 \) in Figure 15. A cluster pattern (or seed pattern) is an assignment

\[
T_n \to \{ \text{seeds in } F \}, \quad t \mapsto \Sigma_t = (x_t, B_t)
\]

such that if \( t \overset{k}{\to} t' \) in \( T_n \), then \( \mu_k(\Sigma_t) = \Sigma_{t'} \). Let \( \{ \Sigma_t = (x_t, B_t) \}_{t \in T_n} \) be a cluster pattern with \( x_t = (x_{1:t}, \ldots, x_{m:t}) \). Since the mutation does not change frozen variables, we may let \( x_{n+1} = x_{n+1:t}, \ldots, x_m = x_{m:t} \).

Definition 3.3 (cf. [15]). Let \( \{ \Sigma_t = (x_t, B_t) \}_{t \in T_n} \) be a cluster pattern with \( x_t = (x_{1:t}, \ldots, x_{m:t}) \). The cluster algebra \( A(\Sigma_t) \) is defined to be the \( \mathbb{C}[x_{n+1}, \ldots, x_m] \)-subalgebra of \( F \) generated by all the cluster variables \( \bigcup_{t \in T_n} \{ x_{1:t}, \ldots, x_{m:t} \} \).

If we fix a vertex \( t_0 \in T_n \), then a cluster pattern \( \{ \Sigma_t \}_{t \in T_n} \) is constructed from the seed \( \Sigma_{t_0} \). In this case, we call \( \Sigma_{t_0} \) an initial seed. Because of this reason, we simply denote by \( A(\Sigma_{t_0}) \) the cluster algebra given by the cluster pattern constructed from the initial seed \( \Sigma_{t_0} \).

Example 3.4. Let \( n = m = 2 \). Suppose that an initial seed is given by

\[
\Sigma_{t_0} = \left( \left( x_1, x_2 \right), \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right).
\]

We present a part of the cluster pattern obtained by the initial seed \( \Sigma_{t_0} \).

\[
\begin{align*}
\Sigma_{t_0} &= \left( \left( x_1, x_2 \right), \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) \quad \overset{\mu_1}{\longrightarrow} \quad \left( \left( \frac{1 + x_1}{x_2}, x_1 \right), \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) \\
&\quad \quad \quad \quad \quad \overset{\mu_2}{\longrightarrow} \quad \left( \left( \frac{1 + x_1}{x_2}, \frac{1 + x_1 + x_2}{x_1 x_2} \right), \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right)
\end{align*}
\]

Remark 3.5. There is another mutation operation called the cluster \( \lambda \)-mutation. Let \( \{ \Sigma_t = (x_t, B_t) \}_{t \in T_n} \) be a cluster pattern with \( x_t = (x_{1:t}, \ldots, x_{m:t}) \). For \( t \in T_n \) and \( i \in [n] \), we set \( y_i = (y_{i:t}, \ldots, y_{n:t}) \) by

\[
y_i = \prod_{j \in [m]} b_j^{(i)} x_{j:t}
\]
where \( B_t = (b_{t}^{i,j}) \). Then the assignment \( t \mapsto (y_t, B_t) \) is called a cluster \( Y \)-pattern and for \( t \rightarrow_k t' \) in \( \mathbb{T}_n \), we have
\[
y_{i,t'} = \begin{cases} y_{i,t} (1 + y_{k,t})^{-b_{k,i}^{i,j}} & \text{if } i \neq k, \\ y_{k,t} & \text{otherwise}; \end{cases}
\]
see [17, Proposition 3.9]. For \( t \rightarrow_k t' \) in \( \mathbb{T}_n \), the operation sends \( (y_t, B_t) \) to \( (y_{t'}, B_{t'}) \) is called the cluster \( X \)-mutation (or, \( X \)-cluster mutation). For exchange matrices and quivers, the cluster \( X \)-mutation is defined the same as before.

### 3.2. Cluster algebras of finite type.

The number of cluster variables in Example 3.4 is finite even though the number of vertices in the graph \( \mathbb{T}_2 \) is infinite. We call such cluster algebras of finite type. More precisely, we recall the following definition.

**Definition 3.6 ([15])**. A cluster algebra is said to be of finite type if it has finitely many cluster variables.

It has been realized that classifying finite type cluster algebras is related to studying exchange matrices. The Cartan counterpart \( C(B_{t_0}^{pr}) = (c_{i,j}) \) of the principal part \( B_{t_0}^{pr} \) of an exchange matrix is defined by
\[
c_{i,j} = \begin{cases} 2 & \text{if } i = j, \\ -|b_{i,j}| & \text{otherwise}. \end{cases}
\]
Since \( B_{t_0}^{pr} \) is skew-symmetrizable, its Cartan counterpart \( C(B_{t_0}^{pr}) \) is symmetrizable. Note that two mutation equivalent acyclic quivers produce the same Cartan counterpart (cf. Remark 3.2). The following theorem presents a classification of cluster algebras of finite type.

**Theorem 3.7 ([15])**. Let \( \{\Sigma_t = (x_t, B_t)\}_{t \in \mathbb{T}_n} \) be a cluster pattern with an initial seed \( \Sigma_{t_0} = (x_{t_0}, B_{t_0}) \). Let \( \mathcal{A}(B_{t_0}) \) be the corresponding cluster algebra. Then we have the following.

1. The cluster algebra \( \mathcal{A}(B_{t_0}) \) is of finite type if and only if \( C(B_{t_0}^{pr}) \) is a Cartan matrix of finite type.

2. If the cluster algebra \( \mathcal{A}(B_{t_0}) \) is of finite type, then there is a bijective correspondence between the set of positive roots for \( C(B_{t_0}^{pr}) \) and the set of noninitial cluster variables. More precisely, for the set \( \Pi = \{\alpha_1, \ldots, \alpha_n\} \) of simple roots, a positive root \( \sum_{i=1}^n d_i \alpha_i \) is associated to a cluster variable of the form
\[
\frac{f(x_{t_0})}{x_{1,t_0}^{d_1} \cdots x_{n,t_0}^{d_n}}, \quad f(x_{t_0}) \in \mathbb{C}[x_{1,t_0}, \ldots, x_{m,t_0}].
\]
Here, \( x_{1,t_0}, \ldots, x_{m,t_0} \) are cluster variables in the initial seed \( x_{t_0} \). Accordingly, there is a bijective correspondence between the set of cluster variables and the set \( \Phi_{\geq -1} \) of almost positive roots \( \Phi_{\geq -1} := \Phi^{+} \cup (-\Pi) \), where \( \Phi \) is the root system whose Cartan matrix is \( C(B_{t_0}^{pr}) \).

We provide a list of finite type root systems and their Dynkin diagram in Table 1. In what follows, we fix an ordering on the simple roots as in Table 1; our conventions agree with that in the standard textbook of Humphreys [22]. In Table 2, we provide enumeration on the number of cluster variables and clusters in each cluster algebra of finite (irreducible) type (cf. [13, Figure 5.17]).

**Definition 3.8.** For a quiver \( Q \), we say that \( Q \) is of type \( X \) if it is mutation equivalent to a quiver \( Q' \) whose underlying unoriented graph on mutable vertices is the Dynkin diagram of type \( X \). Equivalently, \( Q \) is of type \( X \) if it is mutation equivalent to a quiver \( Q' \) whose Cartan counterpart \( C(B^{pr}(Q')) \) of the principal part of the adjacency matrix is the Cartan matrix of type \( X \).

**Example 3.9.** Continuing Example 3.4, the Cartan counterpart of the principal part \( B_{t_0}^{pr} \) is given by
\[
C(B_{t_0}^{pr}) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.
\]
which is the Cartan matrix of Lie type $A_2$. Accordingly, by Theorem 3.7, the cluster algebra $\mathcal{A}(\Sigma_{\alpha_0})$ is of finite type. Indeed, there are five cluster variables and we present the bijective correspondence between them and the set of almost positive roots as described in Theorem 3.7(2).

$$
\begin{align*}
x_1 & \quad x_2 & \quad \frac{1+x_2}{x_1} & \quad \frac{1+x_1}{x_2} & \quad \frac{1+x_1+x_2}{x_1x_2} \\
-\alpha_1 & \quad -\alpha_2 & \quad \alpha_1 & \quad \alpha_2 & \quad \alpha_1 + \alpha_2
\end{align*}
$$

Here, $\alpha_1$ and $\alpha_2$ are simple roots of the Lie algebra of type $A_2$.

### 3.3. Folding

Under certain conditions, one can fold seed patterns to produce new ones. This procedure is used to study cluster algebras of type $BCFG$ from those of simply-laced type $ADE$. As before, we fix $m, n \in \mathbb{Z}_{>0}$ such that $n \leq m$.

| $\Phi$ | Dynkin diagram |
|--------|----------------|
| $A_n \ (n \geq 1)$ | \begin{tikzpicture}
  \node (1) at (0,0) [circle, fill=black] {1};
  \node (2) at (1,0) [circle, fill=black] {2};
  \node (3) at (2,0) [circle, fill=black] {3};
  \node (n-1) at (n-2,0) [circle, fill=black] {n-1};
  \node (n) at (n,0) [circle, fill=black] {n};
  \draw (1) -- (2);
  \draw (2) -- (3);
  \draw (3) -- (n-1);
  \draw (n-1) -- (n);
\end{tikzpicture} |
| $B_n \ (n \geq 2)$ | \begin{tikzpicture}
  \node (1) at (0,0) [circle, fill=black] {1};
  \node (2) at (1,0) [circle, fill=black] {2};
  \node (n-2) at (n-3,0) [circle, fill=black] {n-2};
  \node (n-1) at (n-2,0) [circle, fill=black] {n-1};
  \node (n) at (n,0) [circle, fill=black] {n};
  \draw (1) -- (2);
  \draw (2) -- (n-2);
  \draw (n-2) -- (n-1);
  \draw (n-1) -- (n);
\end{tikzpicture} |
| $C_n \ (n \geq 3)$ | \begin{tikzpicture}
  \node (1) at (0,0) [circle, fill=black] {1};
  \node (2) at (1,0) [circle, fill=black] {2};
  \node (n-2) at (n-3,0) [circle, fill=black] {n-2};
  \node (n-1) at (n-2,0) [circle, fill=black] {n-1};
  \node (n) at (n,0) [circle, fill=black] {n};
  \draw (1) -- (2);
  \draw (2) -- (n-2);
  \draw (n-2) -- (n-1);
  \draw (n-1) -- (n);
\end{tikzpicture} |
| $D_n \ (n \geq 4)$ | \begin{tikzpicture}
  \node (1) at (0,0) [circle, fill=black] {1};
  \node (2) at (1,0) [circle, fill=black] {2};
  \node (n-3) at (n-4,0) [circle, fill=black] {n-3};
  \node (n-2) at (n-3,0) [circle, fill=black] {n-2};
  \node (n-1) at (n-2,0) [circle, fill=black] {n-1};
  \node (n) at (n,0) [circle, fill=black] {n};
  \draw (1) -- (2);
  \draw (2) -- (n-3);
  \draw (n-3) -- (n-2);
  \draw (n-2) -- (n-1);
  \draw (n-1) -- (n);
\end{tikzpicture} |
| $E_6$ | \begin{tikzpicture}
  \node (1) at (0,0) [circle, fill=black] {1};
  \node (2) at (1,0) [circle, fill=black] {2};
  \node (3) at (2,0) [circle, fill=black] {3};
  \node (4) at (3,0) [circle, fill=black] {4};
  \node (5) at (4,0) [circle, fill=black] {5};
  \node (6) at (5,0) [circle, fill=black] {6};
  \draw (1) -- (2);
  \draw (2) -- (3);
  \draw (3) -- (4);
  \draw (4) -- (5);
  \draw (5) -- (6);
\end{tikzpicture} |
| $E_7$ | \begin{tikzpicture}
  \node (1) at (0,0) [circle, fill=black] {1};
  \node (2) at (1,0) [circle, fill=black] {2};
  \node (3) at (2,0) [circle, fill=black] {3};
  \node (4) at (3,0) [circle, fill=black] {4};
  \node (5) at (4,0) [circle, fill=black] {5};
  \node (6) at (5,0) [circle, fill=black] {6};
  \node (7) at (6,0) [circle, fill=black] {7};
  \draw (1) -- (2);
  \draw (2) -- (3);
  \draw (3) -- (4);
  \draw (4) -- (5);
  \draw (5) -- (6);
  \draw (6) -- (7);
\end{tikzpicture} |
| $E_8$ | \begin{tikzpicture}
  \node (1) at (0,0) [circle, fill=black] {1};
  \node (2) at (1,0) [circle, fill=black] {2};
  \node (3) at (2,0) [circle, fill=black] {3};
  \node (4) at (3,0) [circle, fill=black] {4};
  \node (5) at (4,0) [circle, fill=black] {5};
  \node (6) at (5,0) [circle, fill=black] {6};
  \node (7) at (6,0) [circle, fill=black] {7};
  \node (8) at (7,0) [circle, fill=black] {8};
  \draw (1) -- (2);
  \draw (2) -- (3);
  \draw (3) -- (4);
  \draw (4) -- (5);
  \draw (5) -- (6);
  \draw (6) -- (7);
  \draw (7) -- (8);
\end{tikzpicture} |
| $F_4$ | \begin{tikzpicture}
  \node (1) at (0,0) [circle, fill=black] {1};
  \node (2) at (1,0) [circle, fill=black] {2};
  \node (3) at (2,0) [circle, fill=black] {3};
  \node (4) at (3,0) [circle, fill=black] {4};
  \draw (1) -- (2);
  \draw (2) -- (3);
  \draw (3) -- (4);
\end{tikzpicture} |
| $G_2$ | \begin{tikzpicture}
  \node (1) at (0,0) [circle, fill=black] {1};
  \node (2) at (1,0) [circle, fill=black] {2};
  \draw (1) -- (2);
\end{tikzpicture} |

Table 1. Dynkin diagrams of finite type

Here, $\alpha_1$ and $\alpha_2$ are simple roots of the Lie algebra of type $A_2$.

| $\Phi$ | $A_n$ | $B_n$ | $C_n$ | $D_n$ | $E_6$ | $E_7$ | $E_8$ | $F_4$ | $G_2$ |
|--------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| $\#\text{seeds}$ | $\frac{1}{n+2} \binom{2n+2}{n+1}$ | $\binom{2n}{n}$ | $\binom{2n}{n}$ | $\frac{3n-2}{n} \binom{2n-2}{n-1}$ | 833 | 4160 | 25080 | 105 | 8 |
| $\#\text{clvar}$ | $\frac{n(n+3)}{2}$ | $n(n+1)$ | $n(n+1)$ | $n^2$ | 42 | 70 | 128 | 28 | 8 |

Table 2. Enumeration of seeds and cluster variables
Let $Q$ be a labeled quiver having $m$ vertices labeled $1, \ldots, m$. Let $G$ be a finite group acting on the set $[m]$. The notation $i \sim i'$ will mean that $i$ and $i'$ lie in the same $G$-orbit. To study folding of cluster algebras, we prepare some terminologies.

**Definition 3.10** (cf. [13, §4.4]). Let $Q$ be a labeled quiver having $m$ vertices and $G$ a finite group acting on the set $[m]$.

1. The quiver $Q$ (or the corresponding $m \times n$ exchanged matrix $B = B(Q)$) is $G$-admissible if
   a. for any $i \sim i'$, index $i$ is mutable if and only if so is $i'$;
   b. for any indices $i$ and $j$, and any $g \in G$, we have $b_{i,j} = b_{g(i),g(j)}$;
   c. for mutable indices $i \sim i'$, we have $b_{i,i'} = 0$;
   d. for any $i \sim i'$, and any mutable $j$, we have $b_{i,j}b_{i',j} \geq 0$.

2. For a $G$-admissible quiver $Q$, we call a $G$-orbit mutable (respectively, frozen) if it consists of mutable (respectively, frozen) vertices.

For a $G$-admissible quiver $Q$, we define the matrix $B^G = B(Q)^G = (b_{i,j}^G)$ whose rows (respectively, columns) are labeled by the $G$-orbits (respectively, mutable $G$-orbits) by

$$b_{i,j}^G = \sum_{i \in I} b_{i,j},$$

where $j$ is an arbitrary index in $J$. We then say $B^G$ is obtained from $B$ (or from the quiver $Q$) by folding with respect to the given $G$-action.

**Example 3.11.** Let $Q$ be a quiver of type $D_4$ given as follows.

\[
\begin{array}{ccc}
2 & \rightarrow & 3 \\
\downarrow & & \downarrow \\
4 & \rightarrow & 1
\end{array}
\]

\[
B(Q) = \begin{pmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 1 & 1 \\
0 & -1 & 0 & 0 \\
0 & -1 & 0 & 0
\end{pmatrix}
\]

The finite group $G = \mathbb{Z}/3\mathbb{Z}$ acts on $[4]$ by sending $1 \mapsto 3 \mapsto 4 \mapsto 1$ and $2 \mapsto 2$. Here, we decorate vertices of the quiver $Q$ with green and yellow colors for presenting sources and sinks, respectively. One may check that the quiver $Q$ is $G$-admissible. By setting $I_1 = \{2\}$ and $I_2 = \{1, 3, 4\}$, we obtain

$$b_{I_1,I_2}^G = \sum_{i \in I_1} b_{i,1} = b_{2,1} = 1,$$

$$b_{I_2,I_1}^G = \sum_{i \in I_2} b_{i,2} = b_{1,2} + b_{3,2} + b_{4,2} = -3.$$

Accordingly, we obtain the matrix $B^G = \begin{pmatrix} 0 & 1 \\ -3 & 0 \end{pmatrix}$ whose Cartan counterpart is the Cartan matrix of type $G_2$.

For a $G$-admissible quiver $Q$ and a mutable $G$-orbit $I$, we consider a composition of mutations given by

$$\mu_I = \prod_{i \in I} \mu_i,$$

which is well-defined because of the definition of admissible quivers. If $\mu_I(Q)$ is again $G$-admissible, then we have that

$$(\mu_I(B))^G = \mu_I(B^G).$$

We notice that the quiver $\mu_I(Q)$ is not $G$-admissible in general. Therefore, we present the following definition.

**Definition 3.12.** Let $G$ be a group acting on the vertex set of a quiver $Q$. We say that $Q$ is globally foldable with respect to $G$ if $Q$ is $G$-admissible and moreover for any sequence of mutable $G$-orbits $I_1, \ldots, I_\ell$, the quiver $(\mu_{I_\ell} \cdots \mu_{I_1})(Q)$ is $G$-admissible.
For a globally foldable quiver, we can fold all the seeds in the corresponding seed pattern. Let $m^G$ denote the number of orbits of the action of $G$ on $[m]$. Let $\mathbb{F}^G$ be the field of rational functions in $m^G$ independent variables. Let $\psi: \mathbb{F} \to \mathbb{F}^G$ be a surjective homomorphism. A seed $\Sigma = (x, \mathcal{B}(\mathbb{Q}))$ is called $(G, \psi)$-admissible if

- $Q$ is a $G$-admissible quiver;
- for any $i \sim i'$, we have $\psi(x_i) = \psi(x_{i'})$.

In this situation, we define a new "folded" seed $\Sigma^G = (x^G, \mathcal{B}^G)$ in $\mathbb{F}^G$ whose exchange matrix is given as before and cluster variables $x^G = (x_I)$ are indexed by the $G$-orbits and given by $x_I = \psi(x_I)$.

**Proposition 3.13** (cf. [13, Corollary 4.4.11]). Let $Q$ be a quiver which is globally foldable with respect to a group $G$ acting on the set of its vertices. Let $\Sigma = (x, \mathcal{B}(\mathbb{Q}))$ be a seed in the field $\mathbb{F}$ of rational functions freely generated by a cluster $x = (x_1, \ldots, x_m)$. Define $\psi: \mathbb{F} \to \mathbb{F}^G$ so that $\Sigma$ is a $(G, \psi)$-admissible seed. Then, for any mutable $G$-orbits $I_1, \ldots, I_n$, the seed $(\mu_{I_1} \ldots \mu_{I_n})(\Sigma)$ is $(G, \psi)$-admissible, and moreover the folded seeds $((\mu_{I_1} \ldots \mu_{I_n})(\Sigma))^G$ form a seed pattern in $\mathbb{F}^G$ with the initial seed $\Sigma^G = (x^G, (\mathcal{B}(\mathbb{Q}))^G)$.

**Example 3.14.** The quiver in Example 3.11 is globally foldable, and moreover the corresponding seed pattern is of type $G_2$. In fact, seed patterns of type $BCFG$ are obtained by folding quivers of type $ADE$ in general. In Figure 16, we present the corresponding quivers of type $ADE$. We decorate vertices of quivers with yellow and green colors for presenting source and sink, respectively. As one may see, we have to put arrows on the Dynkin diagram alternatingly. For each case, the finite group action that makes each quiver globally foldable is given as follows.

1. $A_{2n-1} \rightsquigarrow B_n$: The finite group $G = \mathbb{Z}/2\mathbb{Z}$ acts on the set $[2n-1]$ of vertices of the quiver of type $A_{2n-1}$ by

   $$i \mapsto 2n - i \quad \text{for } i \in [2n-1].$$

   There are $n$ orbits: $I_i = \{i, 2n - i\}$ for $i \in [n]$.

2. $D_{n+1} \rightsquigarrow C_n$: The finite group $G = \mathbb{Z}/2\mathbb{Z}$ acts on the set $[n+1]$ of vertices of the quiver of type $D_{n+1}$ by

   $$i \mapsto i \quad \text{for } i \in [n-1],$$
   $$n \mapsto n + 1, \quad n + 1 \mapsto n.$$

   There are $n$ orbits: $I_i = \{i\}$ for $i \in [n-1]$, and $I_n = \{n, n+1\}$.

3. $E_6 \rightsquigarrow F_4$: The finite group $G = \mathbb{Z}/2\mathbb{Z}$ acts on the set $[6]$ of vertices of the quiver of type $E_6$ by

   $$i \mapsto i \quad \text{for } i = 2, 4,$$
   $$1 \mapsto 6, \quad 6 \mapsto 1,$$
   $$3 \mapsto 5, \quad 5 \mapsto 3.$$

   There are 4 orbits: $I_1 = \{1, 6\}$, $I_2 = \{3, 5\}$, $I_3 = \{4\}$, and $I_4 = \{2\}$.

4. $D_4 \rightsquigarrow G_2$: The finite group $G = \mathbb{Z}/3\mathbb{Z}$ acts on the set $[4]$ of vertices of the quiver of type $D_4$ by

   $$2 \mapsto 2,$$
   $$1 \mapsto 3, \quad 3 \mapsto 4, \quad 4 \mapsto 1.$$
3.4. Combinatorics of exchange graphs. The exchange graph of a cluster pattern is the \(n\)-regular (finite or infinite) connected graph whose vertices are the seeds of the cluster pattern and whose edges connect the seeds related by a single mutation. For example, the exchange graph in Example 3.4 is a cycle graph with 5 vertices. In this section, we recall the combinatorics of exchange graphs which will be used later. For more details, we refer the reader to [15, 16, 17].

As we already have seen in Theorem 3.7, cluster algebras of finite type are classified by Cartan matrices of finite type. Moreover, for a cluster algebra of finite type, the exchange graph depends only on the exchange matrix (see [15]). Because of this reason, we denote by \(E(\Phi)\) the exchange graph of a cluster pattern corresponding to the root system \(\Phi\).

To study the combinatorics of exchange graphs of cluster algebras, we prepare some terminologies. We call a graph over \(\{m\}\) bipartite if there is a function \(\varepsilon: \{m\} \rightarrow \{+, -\}\), called a coloring, such that for all \(i\) and \(j\) in \(\{m\}\),

\[
b_{i,j} \neq 0 \implies \begin{cases} \varepsilon(i) = +, \\ \varepsilon(j) = -. \end{cases}
\]

Here, \(b_{i,j}\) is the adjacency matrix of the graph. For example, every tree is bipartite, but cycle graphs with an odd number of vertices are not bipartite. Dynkin diagrams of finite type are bipartite since they are trees.

Let \(\Phi\) be a rank \(n\) root system of finite type with the set of simple roots \(\Pi = \{\alpha_i \mid i \in [n]\}\) and the set of positive roots \(\Phi^+\). For every subset \(J \subset [n]\), let \(\Phi(J)\) denote the root subsystem of \(\Phi\) spanned by the set of simple roots \(\{\alpha_i \mid i \in J\}\). Note that \(\Phi(J)\) may not be irreducible even if \(\Phi\) is. Let \(W\) be the Weyl group of \(\Phi\) which is generated by the simple reflections \(s_i = s_{\alpha_i}\). Since the Dynkin diagram of \(\Phi\) is a bipartite graph, let \(I_+\) and \(I_-\) be two parts of the set \([n]\); they are determined uniquely up to renaming. Recall that a Coxeter element is the product of all simple reflections. The order \(h\) of a Coxeter element in \(W\) is called the Coxeter number of \(\Phi\). We present the known formula of Coxeter numbers \(h\) in Table 3 (see [4, Appendix]).

| \(\Phi\) | \(A_n\) | \(B_n\) | \(C_n\) | \(D_n\) | \(E_6\) | \(E_7\) | \(E_8\) | \(F_4\) | \(G_2\) |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| \(h\) | \(n + 1\) | \(2n\) | \(2n\) | \(2n - 2\) | \(12\) | \(18\) | \(30\) | \(12\) | \(6\) |

Table 3. Coxeter numbers
Let $\Delta(\Phi)$ be a simplicial complex whose ground set is $\Phi_{\geq 1}$ and maximal simplices are called clusters. The dual graph of $\Delta(\Phi)$ is known to be the exchange graph $E(\Phi)$. Recall from [9, 16] that there is a polytopal realization of the simplicial complex $\Delta(\Phi)$, that is, there is a simple convex polytope $P(\Phi) \subset \mathbb{R}^n$ such that the dual complex of $P(\Phi)$ agrees with $\Delta(\Phi)$. The polytope $P(\Phi)$ is called the generalized associahedron. We denote by $F_\beta$ the facet of the polytope $P(\Phi)$ corresponding to a root $\beta \in \Phi_{\geq 1}$. Here, a facet of a polytope of dimension $n$ is a face of dimension $n - 1$.

Consider the composition $\mu_Q = \mu_+ \mu_-$ of a sequence of mutations where

$$\mu_e = \prod_{\ell \in I_+} \mu_\ell \quad \text{for } e \in \{+, -\}.$$  

We call $\mu_Q$ a Coxeter mutation. It is known from [16, Proposition 3.2] that both $\mu_+$ and $\mu_-$ act on the face poset of the polytope $P(\Phi)$. Moreover, we have the following properties.

**Proposition 3.15** (cf. [16, Propositions 2.5, 3.2, and 3.7]). The following holds.

1. Both $\mu_+$ and $\mu_-$ act on the face poset of the polytope $P(\Phi)$.
2. Suppose that $h = 2e$ is even. The map $(r, i) \mapsto \mu_Q^r(F_{-\alpha_i})$ induces a bijection

$$\{0, 1, \ldots, e\} \times I \to \mathcal{F}(P(\Phi))$$

where $\mathcal{F}(P(\Phi))$ is the set of facets, which are codimension one faces, of the polytope $P(\Phi)$.
3. The face poset of a facet $\mu_Q^r(F_{-\alpha_i})$ in $P(\Phi)$ is the same as that of the generalized associahedron $P(\Phi([n] \setminus \{i\}))$ of dimension $n - 1$.
4. The facets corresponding to negative simple roots $-\alpha_1, \ldots, -\alpha_n$ intersect at a vertex.

As a direct consequence of Proposition 3.15, we have the following lemma which will be used later.

**Lemma 3.16.** Let $\Sigma$ be a seed in a cluster pattern of finite type with even Coxeter number $h = 2e$. Suppose that $\Sigma \in \mu_Q^r(F_{-\alpha_i})$ for $r \in \{0, 1, \ldots, e\}$ and $\alpha_i \in \Pi$. Then, there exists a sequence $j_1, \ldots, j_\ell \in [n] \setminus \{i\}$ which gives a sequence $\mu_{j_1}, \ldots, \mu_{j_\ell}$ of mutations from $\mu_Q^r(\Sigma_{t_0})$ to $\Sigma$ inside a facet $\mu_Q^r(F_{-\alpha_i})$, that is,

$$\mu_Q^r(\Sigma_{t_0}), \mu_{j_1}(\mu_Q^r(\Sigma_{t_0})), (\mu_{j_2}, \mu_{j_1})(\mu_Q^r(\Sigma_{t_0})), \ldots, (\mu_{j_\ell}, \ldots, \mu_{j_1})(\mu_Q^r(\Sigma_{t_0})) \in \mu_Q^r(F_{-\alpha_i})$$

and

$$\Sigma = (\mu_{j_\ell} \cdots \mu_{j_1})(\mu_Q^r(\Sigma_{t_0})).$$

**Proof.** Since $\Sigma_{t_0} \in F_{-\alpha_i}$, we have $\mu_Q^r(F_{-\alpha_i}) \in F_{-\alpha_i}$. Accordingly, both seeds $\mu_Q^r(\Sigma_{t_0})$ and $\Sigma$ are contained in the same facet $\mu_Q^r(F_{-\alpha_i})$, so there exists a sequence $\mu_{j_1}, \ldots, \mu_{j_\ell}$ of mutations from $\mu_Q^r(\Sigma_{t_0})$ to $\Sigma$ inside $\mu_Q^r(F_{-\alpha_i})$ as desired.

**Example 3.17.** Consider the root system $\Phi$ of type $A_3$. In this case, the Coxeter number is 4, which is even (cf. Table 3). In Table 4, we present how $\mu_Q$ acts on the set of facets. Here, we use the convention that $I_+ = \{1, 3\}$ and $I_- = \{2\}$. The corresponding generalized associahedron is presented in Figure 17. We label each facet the corresponding almost positive root. The backside facets are associated with the set of negative simple roots. As one may see that the face posets of $\mu_Q^r(F_{-\alpha_1})$ are the same as that of the generalized associahedron $P(\Phi([n] \setminus \{i\}))$. Indeed, the facets $\mu_Q^r(F_{-\alpha_1})$ and $\mu_Q^r(F_{-\alpha_3})$ are pentagons, and the facets $\mu_Q^r(F_{-\alpha_2})$ are squares. For $\Sigma = F_{-\alpha_1} \cap F_{-\alpha_2} \cap F_{-\alpha_3}$, we decorate the vertices $\{\mu_Q^k(\Sigma) \mid k = 0, 1, 2\}$ with green.

| $r$ | $\mu_Q^r(F_{-\alpha_1})$ | $\mu_Q^r(F_{-\alpha_2})$ | $\mu_Q^r(F_{-\alpha_3})$ |
|-----|-------------------|-------------------|-------------------|
| 0   | $F_{-\alpha_1}$   | $F_{-\alpha_2}$   | $F_{-\alpha_3}$   |
| 1   | $F_{\alpha_1+\alpha_2}$ | $F_{\alpha_2}$   | $F_{\alpha_2+\alpha_3}$ |
| 2   | $F_{\alpha_3}$   | $F_{\alpha_1+\alpha_2+\alpha_3}$ | $F_{\alpha_1}$   |

**Table 4.** Computation $\mu_Q^r(F_{-\alpha_1})$ for type $A_3$
Example 3.18. We consider the generalized associahedron of type $D_4$ and present four facets corresponding to the negative simple roots in Figure 18. The facet corresponding to $-\alpha_2$ is combinatorially equivalent to $P(\Phi(\{1\})) \times P(\Phi(\{3\})) \times P(\Phi(\{4\}))$, which is a 3-cube presented in the boundary. The intersection of these four facets is a vertex sits in the bottom colored in green. The Coxeter mutation $\mu_Q$ acts on the face poset of the permutohedron, especially, four green vertices are in the same orbit.

Remark 3.19. As we have seen in Example 3.14, bipartite coloring on quivers of type $ADE$ induce that on quivers of type $BCFG$. Accordingly, if a seed pattern of simply-laced type $X$ gives a seed pattern of type $Y$ via the folding procedure, then the Coxeter mutation of type $Y$ is the same as that of type $X$. More precisely, for a globally foldable seed $\Sigma$ with respect to $G$ defining a cluster algebra of type $X$ and its Coxeter mutation $\mu_X^G$, we have

$$\mu_Y^G(\Sigma) = (\mu_X^G(\Sigma))^G.$$ 

Here, $\mu_Y^G$ is the Coxeter mutation on the seed pattern determined by $\Sigma^G$.

Moreover, Coxeter numbers of $X$ and $Y$ are the same. Indeed,

$$h(A_{2n-1}) = h(B_n) = 2n,$$
$$h(D_{n+1}) = h(C_n) = 2n,$$
$$h(E_6) = h(F_4) = 12,$$
$$h(D_4) = h(G_2) = 6.$$

In the remaining part of this section, we recall [17] which considers the combinatorics on mutations in a more general setting. Let $Q$ be a bipartite quiver and $I_+$ and $I_-$ be the bipartite decomposition of the vertex set of $Q$. Consider the composition $\mu_Q = \mu_- \mu_+$ of a sequence of mutations where

$$\mu_\varepsilon = \prod_{i \in I_\varepsilon} \mu_i \quad \text{for } \varepsilon \in \{+, -\}.$$ 

We call $\mu_Q$ a Coxeter mutation as before. We enclose this section by recalling the following result which will be used later.

Lemma 3.20 ([17, Theorem 8.8]). Let $\Sigma_{i_0} = (x_{i_0}, B_{i_0})$ be an initial seed. Suppose that the exchange matrix $B_{i_0}$ is the adjacency matrix of a bipartite quiver $Q$. Then the set $\{\mu_Q^G(\Sigma_{i_0})\}_{r \in \mathbb{Z}_{\geq 0}}$ of seeds is finite if and only if the Cartan counterpart $C(B_{i_0}^{pr})$ is a Cartan matrix of finite type.
Moreover, for a quiver $Q$ of finite type, the order the $\mu_Q$-action is given by $(h + 2)/2$ if $h$ is even, or $h + 2$ otherwise.

4. $N$-graphs and seeds

Let us recall from [8] how to construct a seed from an $N$-graph $G$. Each one-cycle in $\Lambda(G)$ corresponds to a vertex of the quiver, and a monodromy along that cycle gives a coordinate
function at that vertex. The quiver is obtained from the intersection data among one-cycles. Moreover, there is an operation in $N$-graph, called Legendrian mutation, which is a counterpart of the mutation in the cluster structure. The Legendrian mutation is crucial in constructing and distinguishing $N$-graphs. In turn, these will give seeds many Lagrangian fillings of Legendrian links.

4.1. One-cycles in Legendrian weaves. Let $\mathcal{G} \subset \mathbb{D}^2$ be a free $N$-graph and $\Lambda(\mathcal{G})$ be the induced Legendrian weave. We express one-cycles of $\Lambda(\mathcal{G})$ in terms of subgraphs of $\mathcal{G}$.

Definition 4.1. A subgraph $T \subset \mathcal{G}$ is said to be admissible if it satisfies the following conditions:

- every vertex of $T$ is at most trivalent,
- each univalent vertex in $T$ is a trivalent vertex in $\mathcal{G}$,
- each bivalent vertex in $T$ corresponding to a hexagonal point in $\mathcal{G}$ connects two opposite edges in $\mathcal{G}$, and
- each trivalent vertex in $T$ corresponding to a hexagonal point in $\mathcal{G}$ and connects three edges in the same color.

An admissible graph $T \subset \mathcal{G}$ is good if it is a (connected) tree and only univalent vertices of $T$ are trivalent vertices in $\mathcal{G}$.

For each admissible subgraph $T$, we can define an oriented immersed loop $\ell(T) \subset \mathbb{D}^2$ is defined by paths whose local pictures look as depicted in Figure 19. Each arc $\ell_j(T) \subset \ell(T)$ cut by $\mathcal{G}$ is labelled as $s_j \in \{1, \ldots, N\}$, which lifts to the $s_j$-th sheet $\Gamma_{s_j}$ via $\pi \colon \Gamma(\mathcal{G}) \to \mathbb{D}^2$. By concatenating the lifts, we have an oriented embedded loop $\gamma(T)$ in $\Lambda(\mathcal{G})$ and a one-cycle $[\gamma] \in H_1(\Lambda(\mathcal{G}), \mathbb{Z})$ is called a $T$-cycle if $[\gamma] = [\gamma(T)]$.

![Figure 19. Local configurations on cycles and corresponding arcs of $\mathcal{G} \subset \mathbb{D}^2$](image)

**Example 4.2** ((Long) l-cycles). For an edge $e$ of $\mathcal{G}$ connecting two trivalent vertices, let $l(e)$ be the subgraph of $\mathcal{G}$ consisting of a single edge $e$. Then $l(e)$ is a good subgraph of $\mathcal{G}$ and the cycle $[\gamma(l(e))]$ depicted in Figure 20(a) is called an $l$-cycle.

In general, a linear chain of edges $(e_1, e_2, \ldots, e_n)$ satisfying
• $e_i$ connects a trivalent vertex and a hexagonal point for $i = 1, n$;
• $e_i$ and $e_{i+1}$ meet at a hexagonal point in the opposite way, see Figure 20(b), for $i = 2, \ldots, n - 1$
forms a good subgraph $\Gamma(e_1, \ldots, e_n)$, and the cycle $[\gamma(\Gamma(e_1, \ldots, e_n))]$ is called a long $I$-cycle. See Figure 20(b).

**Example 4.3 (Y-cycles).** Let $e_1, e_2, e_3$ be monochromatic edges joining a hexagonal point $h$ and trivalent vertices $v_i$ for $i = 1, 2, 3$. Then the subgraph $Y(e_1, e_2, e_3)$ consisting of three edges $e_1, e_2$ and $e_3$ is a good subgraph of $\Sigma$ and it defines a cycle $[\gamma(Y(e_1, e_2, e_3))]$ called an upper or lower $Y$-cycle according to the relative position of sheets that edges represent. See Figures 20(c) and 20(d).

![Figure 20: (Long) $I$- and $Y$-cycles](image)

One of the benefits of cycles from admissible subgraphs is that one can keep track how cycles are changed under the $N$-graph moves described in Figure 7, especially under Move (I) and Move (II). Note that Move (III) can be decomposed into a sequence of Move (I) and Move (II). Some of such changes are given in Figure 21. Then it is easy to check that any $T$-cycle coming from a good subgraph $T$ can be transformed to an $I$-cycle.

**Remark 4.4.** It is important to note that not every cycle can be represented by a subgraph. For example, the cycle on the left of the following picture can not be expressed by a subtree but it can be after Move (I).

![Diagram](image)
On the other hand, there might be a one-cycle having two different subgraph presentations as follows:

Therefore, there is a bit subtle issue for picking up nice cycles in a consistent way.

**Definition 4.5.** Let $G \subset D^2$ be an $N$-graph, and $\Lambda(\mathcal{G})$ be an induced Legendrian surface in $J^1D^2$. A cycle $[\gamma] \in H_1(\Lambda(\mathcal{G}))$ is **good** if $[\gamma] = [\gamma(T)]$ for some good $T \subset \mathcal{G}$.

A tuple of linearly independent good cycles $B = \{[\gamma_i]\}_{i \in I}$ in $H_1(\Lambda(\mathcal{G}))$ is **good** if for any pair of cycles $[\gamma_i]$ and $[\gamma_j]$, $\mathcal{G}$ can be transformed into $\mathcal{G}'$ via $N$-graph moves so that two cycles $[\gamma_i]$ and $[\gamma_j]$ become $I$-cycles in $H_1(\Lambda(\mathcal{G}'))$.

**Remark 4.6.** Suppose $[\gamma_i]$ and $[\gamma_j]$ be a pair of cycles in a good tuple $B$ of $H_1(\Lambda(\mathcal{G}))$. If $\mathcal{G}$ is free, then by Example 2.9, there is no bigon in $\mathcal{G}$ up to $N$-graph moves. So, under this assumption, two corresponding $I$-cycles in Definition 4.5 intersect at most one trivalent vertex.

**Definition 4.7.** Let $(\mathcal{G}, B)$ and $(\mathcal{G}', B')$ be pairs of an $N$-graph and good tuples of one-cycles. We say that $(\mathcal{G}, B)$ and $(\mathcal{G}', B')$ are **equivalent** if there is a sequence of moves between $\mathcal{G}$ and $\mathcal{G}'$ inducing moves as depicted in Figure 21 between representatives of cycles in $B$ and $B'$. We denote the equivalent class of $(\mathcal{G}, B)$ by $[\mathcal{G}, B]$.

**Remark 4.8.** For two equivalent pairs $(\mathcal{G}, B)$ and $(\mathcal{G}', B')$, all moves between $N$-graphs $\mathcal{G}$ and $\mathcal{G}'$ can be realized by isotopies between Legendrian weaves $\Lambda(\mathcal{G})$ and $\Lambda(\mathcal{G}')$ by Theorem 1.1 in [8], and then the induced isomorphism $H_1(\Lambda(\mathcal{G})) \cong H_1(\Lambda(\mathcal{G}'))$ identifies $B$ with $B'$.

**4.2. $N$-graphs and flag moduli space.** We recall from [8] a central algebraic invariant $\mathcal{M}(\mathcal{G})$ of the Legendrian weave $\Lambda(\mathcal{G})$. The main idea is to consider moduli spaces of constructible sheaves.
associated to $\Lambda(\mathcal{S})$. To introduce a legible model for such constructible sheaves, let us consider a full flag, i.e. a nested sequence of subspaces in $C^N$:

\[ \mathcal{F}^* \in \{(\mathcal{F}^i)_{i=0}^{N} \mid \dim \mathcal{F}^j = i, \mathcal{F}^j \subset \mathcal{F}^{j+1}, 1 \leq j \leq N-1, \mathcal{F}^N = C^N \}. \]

**Definition 4.9.** [8] Let $\mathcal{S} \subset \mathbb{D}^2$ be an N-graph. Let \( \{F_i\}_{i \in I} \) be a set of closures of connected components of $\mathbb{D}^2 \setminus \mathcal{S}$, call each closure a face. The framed flag moduli space $\tilde{\mathcal{M}}(\mathcal{S})$ is a collection of flags $\mathcal{F}_\Lambda(\mathcal{S}) = \{(\mathcal{F}^*(F_i))_{i \in I}\}$ in $C^N$ satisfying the following:

Let $F_1, F_2$ be a pair of faces sharing an edge in $\Gamma_i$. Then the corresponding flags $\mathcal{F}^*(F_1), \mathcal{F}^*(F_2)$ satisfy

\[
\begin{align*}
F^j(F_1) &= F^j(F_2), & 0 \leq j \leq N, & j \neq i; \\
F^i(F_1) &\neq F^i(F_2). 
\end{align*}
\]

(4.1)

Let us consider the general linear group $GL_N$ action on $\mathcal{M}(\mathcal{S})$ by acting on all flags at once. The flag moduli space of the N-graph $\mathcal{S}$ is defined by the quotient space (a stack, in general)

\[ \mathcal{M}(\mathcal{S}) := \tilde{\mathcal{M}}(\mathcal{S})/GL_N. \]

Let $Sh(\mathbb{D}^2 \times \mathbb{R})$ be the category of constructible sheaves on $\mathbb{D}^2 \times \mathbb{R}$. Under the identification $J^1\mathbb{D}^2 \cong T^{\infty,-}(\mathbb{D}^2 \times \mathbb{R})$, an N-graph $\mathcal{S} \subset \mathbb{D}^2$ gives a Legendrian

\[ \Lambda(\mathcal{S}) \subset J^1\mathbb{D}^2 \cong T^{\infty,-}(\mathbb{D}^2 \times \mathbb{R}) \subset T^{\infty}(\mathbb{D}^2 \times \mathbb{R}). \]

This can be used to define a Legendrian isotopy invariant $Sh^1(\Lambda(\mathcal{S}) \mathbb{D}^2 \times \mathbb{R})_0$ of $Sh(\mathbb{D}^2 \times \mathbb{R})$ consisting of constructible sheaves

- whose singular support at infinity lies in $\Lambda(\mathcal{S}) \subset T^{\infty}(\mathbb{D}^2 \times \mathbb{R})$,
- whose microlocal rank is one, and
- which are zero near $\mathbb{D}^2 \times \{ -\infty \}$.

See [21, 29] for the detail.

**Theorem 4.10 ([8, Theorem 5.3]).** The flag moduli space $\mathcal{M}(\mathcal{S})$ is isomorphic to $Sh^1(\Lambda(\mathcal{S}) \mathbb{D}^2 \times \mathbb{R})_0$. Hence $\mathcal{M}(\mathcal{S})$ is a Legendrian isotopy invariant of $\Lambda(\mathcal{S})$.

**Remark 4.11.** Indeed, the actual theorem is about a connected surface, not only for $\mathbb{D}^2$.

Let $\lambda = \lambda_\beta$ be a Legendrian in $J^1S^1$, which gives us an $(N-1)$-tuple of points $X = (X_1, \ldots, X_{N-1})$ in $S^1$ which given by the alphabet $\sigma_1, \ldots, \sigma_{N-1}$ of the braid word $\beta$. Let $\{f_i\}_{i \in I}$ be the set of closures of connected components of $S^1 \setminus X$. The flags $\mathcal{F}_\lambda = \{(\mathcal{F}^*(f_i))_{i \in I}\} \subset C^N$ satisfying exactly the same conditions in (4.1) will be called simply flags on $\lambda$. It is well known that the moduli space $\mathcal{M}(X)$ of such flags $\mathcal{F}_\lambda$ up to $GL_N$ is isomorphic to $Sh^1(S^1 \times \mathbb{R})_0$ which is a Legendrian isotopy invariant, see [29, Theorem 1.1].

**Definition 4.12.** Let $\mathcal{S} \subset \mathbb{D}^2$ be an N-graph, and let $\mathcal{F}_\lambda$ be flags adapted to $\lambda \subset J^1\partial\mathbb{D}^2$ given by $\partial\mathcal{S}$. An N-graph $\mathcal{S}$ is good, if the flags $\mathcal{F}_\lambda$ uniquely determine flags $\mathcal{F}_{\Lambda(\mathcal{S})}$ in Definition 4.9.

Note that $\mathcal{S}(a, b, c)$ in the introduction is good in an obvious way. If an N-graph $\mathcal{S} \subset \mathbb{D}^2$ is good and $[\mathcal{S}] = [\mathcal{S}']$, then $\mathcal{S}'$ is also good.

### 4.3. N-graphs and seeds.

Let $\mathcal{S} \subset \mathbb{D}^2$ be an N-graph, and $\mathcal{B} = \{[\gamma_i]_{i \in [n]} \subset H_1(\Lambda(\mathcal{S}))\}$ be a good tuple of cycles. For two cycles $[\gamma_i]$ and $[\gamma_j]$, let $i([\gamma_i], [\gamma_j])$ be the algebraic intersection number in $H_1(\Lambda(\mathcal{S}))$ which can be computed explicitly as follows: without loss of generality, we may assume that both $[\gamma_i]$ and $[\gamma_j]$ are 1-cycles represented by $\gamma_l(e)$ and $\gamma_l(e')$ for some edges $e$ and $e'$ in $\mathcal{S}$, respectively. Suppose that $e$ and $e'$ intersect at the vertex in $\mathcal{S}$. Then two representatives of $\gamma_i$ and $\gamma_j$ look locally as depicted in Figure 22 and their intersection is defined to be $\pm 1$ by using the clockwise rotation convention.

**Definition 4.13.** For each a pair $(\mathcal{S}, \mathcal{B})$ of an N-graph and a good tuple of cycles, we define a quiver $\mathcal{Q} = Q(\mathcal{S}, \mathcal{B})$ as follows:

1. the set of vertices is $[n]$ where $\mathcal{B} = \{[\gamma_i] \mid i \in [n]\} \subset H_1(\Lambda(\mathcal{S}))$, and
(2) the \((i, j)\)-entry \(b_{i,j}\) for \(\mathcal{B}(Q) = (b_{i,j})\) is the algebraic intersection number between \([\gamma_i]\) and \([\gamma_j]\).

In order to assign a cluster variable to each one-cycle. Let us review the microlocal monodromy functor from \([29]\)

\[
\mu_{\text{mon}} : \text{Sh}^1_{\Lambda} \rightarrow \text{Loc}^*(\Lambda).
\]

In our case, this functor sends microlocal rank-one sheaves \(\mathcal{F}_{\Lambda(\mathfrak{g})} \in \text{Sh}^1_{\Lambda(\mathfrak{g})}(\mathbb{D}^2 \times \mathbb{R})_0\), or equivalently, flags \(\{\mathcal{F}^*(F_i)\}_{i \in I} \in \mathcal{M}(\mathfrak{g})\) to rank-one local systems \(\mu_{\text{mon}}(\mathcal{F}_{\Lambda(\mathfrak{g})})\) on the Legendrian surface \(\Lambda(\mathfrak{g})\). Then (cluster) variables \(x\) for the triple \((\mathfrak{g}, \mathcal{B}, \mathcal{F}_{\Lambda(\mathfrak{g})})\) are defined by

\[
x = (\mu_{\text{mon}}(\mathcal{F}_{\Lambda(\mathfrak{g})})([\gamma_1]), \ldots, \mu_{\text{mon}}(\mathcal{F}_{\Lambda(\mathfrak{g})})([\gamma_n])).
\]

Let us denote the above assignment by

\[
\Psi(\mathfrak{g}, \mathcal{B}, \mathcal{F}_{\Lambda(\mathfrak{g})}) = (x(\Lambda(\mathfrak{g}), \mathcal{B}, \mathcal{F}_{\Lambda(\mathfrak{g})}), Q(\Lambda(\mathfrak{g}), \mathcal{B})).
\]

By the Legendrian isotopy invariance of \(\text{Sh}^1_{\Lambda(\mathfrak{g})}(\mathbb{D}^2 \times \mathbb{R})_0\) in \([21]\), and the functorial property of the microlocal monodromy functor \(\mu_{\text{mon}}\) \([29]\), the assignment \(\Psi\) is well-defined up to isotopy of \(\Lambda(\mathfrak{g})\). That is, if two triples \((\mathfrak{g}, \mathcal{B}, \mathcal{F}_{\Lambda(\mathfrak{g})})\) and \((\mathfrak{g}', \mathcal{B}', \mathcal{F}_{\Lambda(\mathfrak{g}')}\)) are Legendrian isotopic, then they give us the same seed via \(\Psi\).

Especially when an \(N\)-graph \(\mathfrak{g}\) is good, see Definition 4.12, \(\mathcal{F}_{\Lambda(\mathfrak{g})}\) is determined by the flags \(\mathcal{F}_\lambda \in \text{Sh}^1_{\Lambda}(\partial \mathbb{D}^2 \times \mathbb{R})\) at the boundary, where the Legendrian link \(\lambda\) is given by \(\partial \mathfrak{g}\). So we have

**Theorem 4.14.** \([8, \S 7.2.1]\) Let \(\mathfrak{g} \subset \mathbb{D}^2\) be a good \(N\)-graph with a good tuple \(\mathcal{B}\) of cycles in \(H_1(\Lambda(\mathfrak{g}))\), and with flags \(\mathcal{F}_\lambda\) on \(\lambda \subset J^1 \mathbb{S}^1\) at the boundary. Then the assignment \(\Psi\) to a seed in a cluster structure

\[
\Psi(\mathfrak{g}, \mathcal{B}, \mathcal{F}_\lambda) = (x(\Lambda(\mathfrak{g}), \mathcal{B}, \mathcal{F}_\lambda), Q(\Lambda(\mathfrak{g}), \mathcal{B}))
\]

is well-defined up to Legendrian isotopy.

As a corollary, the seed \(\Psi(\mathfrak{g}, \mathcal{B}, \mathcal{F}_\lambda)\) can be used to distinguish a pair of Legendrian surfaces and hence, by Lemma 2.1, a pair of Lagrangian fillings.

**Corollary 4.15.** As in the above setup, if two triples \((\mathfrak{g}, \mathcal{B}, \mathcal{F}_\lambda), (\mathfrak{g}', \mathcal{B}', \mathcal{F}_\lambda)\) with the same boundary condition define different seeds, then two induced Lagrangian fillings \(\pi \circ \iota(\Lambda(\mathfrak{g})), \pi \circ \iota(\Lambda(\mathfrak{g}'))\) bounding \(\iota(\lambda)\) are not exact Lagrangian isotopic to each other.

The monodromy \(\mu_{\text{mon}}(\mathcal{F}_{\Lambda(\mathfrak{g})})\) along a loop \([\gamma] \in H_1(\Lambda(\mathfrak{g}))\) can be obtained by restricting the constructible sheaf \(\mathcal{F}_{\Lambda(\mathfrak{g})}\) to a tubular neighborhood of \(\gamma\). Let us investigate how the monodromy can be computed explicitly in terms of flags \(\{\mathcal{F}^*(F_i)\}_{i \in I}\).

Let us consider an \(I\)-cycle \([\gamma]\) represented by a loop \(\gamma(e)\) for some monochromatic edge \(e\) as in Figure 23(a). Let us denote four flags corresponding to each region by \(F_1, F_2, F_3, F_4\), respectively.
Suppose that \( e \subset \mathcal{G}_i \), then by the construction of flag moduli space \( \mathcal{M}(\mathcal{G}) \), a two-dimensional vector space \( V := \mathcal{F}^{i+1}(F_*)/\mathcal{F}^{i-1}(F_*) \) is independent of \( * = 1, 2, 3, 4 \). Moreover, \( \mathcal{F}^i(F_*)/\mathcal{F}^{i-1}(F_*) \) defines a one-dimensional subspace \( v_* \subset V \) for \( * = 1, 2, 3, 4 \), satisfying
\[
v_1 \neq v_2 \neq v_3 \neq v_4 \neq v_1.
\]

Then \( \mu_{\text{mon}}(\mathcal{F}_{\Lambda(\mathcal{G})})([\gamma]) \) along the one-cycle \([\gamma(e)]\) is defined by the cross ratio
\[
\mu_{\text{mon}}(\mathcal{F}_{\Lambda(\mathcal{G})})([\gamma]) := \langle v_1, v_2, v_3, v_4 \rangle = \frac{v_1 \wedge v_2}{v_2 \wedge v_3} \cdot \frac{v_3 \wedge v_4}{v_4 \wedge v_1}.
\]

Suppose that local flags \( \{ F_j \}_{j \in J} \) near the upper \( \gamma \)-cycle \([\gamma_U]\) look like in Figure 23(b). Let \( \mathcal{G}_i \) and \( \mathcal{G}_{i+1} \) be the \( N \)-subgraphs in red and blue, respectively. Then the 3-dimensional vector space \( V = \mathcal{F}^{i+2}(F_*)/\mathcal{F}^{i-1}(F_*) \) is independent of \( * \in J \). Now regard \( a, b, c \) and \( A, B, C \) are subspaces of \( V \) of dimension one and two, respectively. Then the microlocal monodromy along the \( \gamma \)-cycle \([\gamma_U]\) becomes
\[
\mu_{\text{mon}}(\mathcal{F}_{\Lambda(\mathcal{G})})([\gamma_U]) := \frac{B(a)C(b)A(c)}{B(c)C(a)A(b)}.
\]

Here \( B(a) \) can be seen as a paring between the vector \( a \) and the covector \( B \).

Now consider the lower \( \gamma \)-cycle \([\gamma_L]\) whose local flags given as in Figure 23(c). We already have seen that the orientation convention of the loop in Figure 20 for the upper and lower \( \gamma \)-cycle is different. Then microlocal monodromy along \([\gamma_L]\) follows the opposite orientation and becomes
\[
\mu_{\text{mon}}(\mathcal{F}_{\Lambda(\mathcal{G})})([\gamma_L]) := \frac{C(a)B(c)A(b)}{C(b)B(a)A(c)}.
\]

Here, \( B(a) \) is a pairing between the vector \( B \) and covector \( a \) which is the same as the above.

**Figure 23.** I- and \( \gamma \)-cycles with flags.

### 4.4. Legendrian mutations in \( N \)-graphs

Let us define an operation called (Legendrian) mutation on \( N \)-graphs \( \mathcal{G} \) which corresponds to a geometric operation on the induced Legendrian surface \( \Lambda(\mathcal{G}) \) that producing a smoothly isotopic but not necessarily Legendrian isotopic to \( \Lambda(\mathcal{G}) \), see [8, Definition 4.19]. Note that operation has an intimate relation with the wall-crossing phenomenon [2], Lagrangian surgery [26], and quiver (or cluster) mutations [14].

**Definition 4.16.** [8] Let \( \mathcal{G} \) be a (local) \( N \)-graph and \( e \in \mathcal{G}_i \subset \mathcal{G} \) be an edge between two trivalent vertices corresponding to an I-cycle \([\gamma] = [\gamma(e)]\). The mutation \( \mu_\gamma(\mathcal{G}) \) of \( \mathcal{G} \) along \( \gamma \) is obtained by applying the local change depicted in the left of Figure 24.

For the \( \gamma \)-cycle, the Legendrian mutation becomes as in the right of Figure 24. Note that the mutation at \( \gamma \)-cycle can be decomposed into a sequence of Move (I) and Move (II) together with a mutation at I-cycle.

Let us remind our main purpose of finding exact embedded Lagrangian fillings for a Legendrian links. The following lemma guarantees that Legendrian mutation preserves the embedding property of Lagrangian fillings.
Proposition 4.17. [8, Lemma 7.4] Let $\mathcal{G} \subset \mathbb{D}^2$ be a free $N$-graph. Then mutation $\mu(\mathcal{G})$ at any $I$- or $Y$-cycle is again free $N$-graph.

Proposition 4.18. Let $\mathcal{G} \subset \mathbb{D}^2$ be a good $N$-graph. Then mutation $\mu_\gamma(\mathcal{G})$ at $I$-cycle $\gamma$ is again good $N$-graph.

Proof. The proof is straightforward from the notion of the good $N$-graph in Definition 4.12 and of the Legendrian mutation depicted in Figure 24(a). Note that the Legendrian mutation $\mu_\gamma(\mathcal{G})$ at $Y$-cycle $\gamma$ is also good, since $\mu_\gamma(\mathcal{G})$ is a composition of Moves (I) and (II), and a mutation at $I$-cycle. \hfill \Box

An important observation is the Legendrian mutation on $(\mathcal{G}, B)$ induces a cluster mutation on the induced seed $(x(\Lambda(\mathcal{G}), B, F_{\lambda}), Q(\Lambda(\mathcal{G}), B))$.

Proposition 4.19 ([8, §7.2]). Let $\mathcal{G} \subset \mathbb{D}^2$ be a good $N$-graph and $B$ be a good tuple of cycles in $H_1(\Lambda(\mathcal{G}))$. Let $\mu_{\gamma_i}(\mathcal{G}, B)$ be a Legendrian mutation of $(\mathcal{G}, B)$ along a one-cycle $\gamma_i$ then the following holds: for flags $F_{\lambda}$ on $\lambda$,

$$\Psi(\mu_{\gamma_i}(\mathcal{G}, B), F_{\lambda}) = \mu_i(\Psi(\mathcal{G}, B, F_{\lambda})).$$

Here, $\mu_i$ is the cluster $X$-mutation at the vertex $i$ (cf. Remark 3.5).

5. Lagrangian fillings for of Legendrians type ADE

5.1. Tripods. Let $\lambda \subset J^1\mathbb{S}^1$ be a Legendrian knot or link which bounds a Legendrian surface $\Lambda(\mathcal{G})$ in $J^1\mathbb{D}^2$ for some free $N$-graph $\mathcal{G}$. We fix a good tuple $B$ of cycles in the sense of Definition 4.5, and fix flags $F_{\lambda}$ on $\lambda$. Then by Theorem 4.14, we obtain a seed $\Psi(\mathcal{G}, B, F_{\lambda})$ which is a pair of a set of cluster variables $x(\Lambda(\mathcal{G}), B, F_{\lambda})$ and a quiver $Q(\Lambda(\mathcal{G}), B)$.

We say that the pair $(\mathcal{G}, B)$ is of finite type or of infinite type if so is the cluster algebra defined by $Q(\Lambda(\mathcal{G}), B)$. In particular, it is said to be of type ADE if the quiver $Q(\Lambda(\mathcal{G}), B)$ is of type $A_n, D_n, E_6, E_7$ or $E_8$ (see Definition 3.8). Braid words $\beta$ of Legendrians, $N$-graphs and good tuples of cycles $(\mathcal{G}, B)$ of type ADE are depicted in Table 5.

One can generalize these quivers of type ADE as follows:

Definition 5.1 (Tripod quiver). For $a, b, c \geq 1$, the tripod $Q(a, b, c)$ of type $(a, b, c)$ is a bipartite quiver such that

1. the set of vertices is $|n|$ for $n = a + b + c - 2$,
2. the underlying graph is a boundary wedge sum of three quivers $A_n, A_b$, and $A_c$, and
3. the vertex where $A_n, A_b$, and $A_c$ are glued together is called the central vertex, labelled as $+$.

We define an $N$-graph $\mathcal{G}(a, b, c)$ on $\mathbb{D}^2$ as the concatenation of three $N$-graphs $\mathcal{G}(A_n), \mathcal{G}(A_b)$ and $\mathcal{G}(A_c)$ by making one $Y$-cycle $\gamma_1$ and define a good tuple $B(a, b, c)$ of cycles as the union of cycles in three $N$-graphs. The $N$-graph obtained by switching colors from $\mathcal{G}(a, b, c)$ and the induced set of chosen cycles will be denoted by $\mathcal{G}(a, b, c)$ and $B(a, b, c)$, respectively.

The pictorial definitions of $Q(a, b, c), \mathcal{G}(a, b, c)$ and $B(a, b, c)$ are depicted in Figure 25.
It is obvious that if $a, b$ or $c$ is one, then it is the same as $A_n$ for $n = a + b + c - 2$ up to relabelling. Similarly, the cases that $a, b$ or $c$ is two include quivers of type $D_n$ and $E_n$.

Notice that the boundary of the Legendrian weave $\Lambda(G(a, b, c))$ denoted by $\lambda(a, b, c)$ is expressed as the braid word

$$
\lambda(a, b, c) = \text{Cl}(\beta(a, b, c)), \quad \beta(a, b, c) = \sigma_2 \sigma_1^{a+1} \sigma_2 \sigma_1^{b+1} \sigma_2 \sigma_1^{c+1}.
$$

Then this braid is equivalent to the following:

$$
\beta(a, b, c) = \sigma_2 \sigma_1^{a+1} \sigma_2 \sigma_1^{b+1} \sigma_2 \sigma_1^{c+1} = \sigma_2 \sigma_1 (\sigma_2 \sigma_1) \sigma_2 \sigma_1^{b-1} \sigma_2 \sigma_1 (\sigma_1 \sigma_2) \sigma_1 = \Delta \sigma_2 \sigma_1 \sigma_2^{b-1} \sigma_2 \Delta.
$$
Hence $\lambda(a, b, c) \subset J^1S^1$ corresponds to the rainbow closure of the braid $\beta_0(a, b, c) = \sigma_1 \sigma_2 \sigma_1^{-1} \sigma_2^{-1} \sigma_1$.

**Remark 5.2.** One can easily check the quiver $Q^{\text{brick}}(a, b, c)$ from the brick diagram of $\beta_0(a, b, c)$ described in [19] looks as follows:

Then this quiver $Q^{\text{brick}}(a, b, c)$ is obviously mutation equivalent to the bipartite quiver $Q(a, b, c)$. 

---

Figure 25. The tripod $N$-graph $\mathcal{G}(a, b, c)$ and the chosen set $\mathcal{B}(a, b, c)$ of cycles
It is not hard to check that $\beta(1, b, c)$ is a stabilization of $\beta(A_n) = \sigma_1^{n+3}$ for $n = b + c - 1$ since
\[
\beta(A_n) = \sigma_1^{n+3} = \sigma_1^{b+1}\sigma_1^{c+1} \xrightarrow{(S)} \sigma_2\sigma_1^2\sigma_2\sigma_1^{b+1}\sigma_2\sigma_1^{c+1} = \beta(1, b, c).
\]

**Lemma 5.3.** The N-graph $G(1, b, c)$ is a stabilization of $G(A_n)$ for $n = b + c - 1$.

**Proof.** According to Remark 2.7 and Figure 9, a stabilization of $G(A_n)$ is given as the second picture in Figure 26. Then by adding an annular N-graph corresponding to a sequence of (RIII), we obtain the third, which produces the fourth by applying the following generalized push-through move.

Now we add an annular N-graph consisting of (RIII)’s as above to obtain the fifth N-graph in Figure 26, which is the same as $G(1, b, c)$ as desired up to Move (II) at the center. \qed

**Definition 5.4.** Let $(\mathcal{G}, \mathcal{B})$ and $(\mathcal{G}', \mathcal{B}')$ be pairs of N-graphs and good tuples of cycles. We say that $(\mathcal{G}, \mathcal{B})$ is Legendrian mutation equivalent to $(\mathcal{G}', \mathcal{B}')$ if there exists a sequence $\mu$ of Legendrian mutations which sends $(\mathcal{G}, \mathcal{B})$ to $(\mathcal{G}', \mathcal{B}')$ up to equivalence. That is,
\[
[\mu(\mathcal{G}, \mathcal{B})] = [(\mathcal{G}', \mathcal{B}')].
\]

In particular, $(\mathcal{G}, \mathcal{B})$ is said to be of type X or of type $(a, b, c)$ if it is Legendrian mutation equivalent to $(\mathcal{G}(X), \mathcal{B}(X))$ or $(\mathcal{G}(a, b, c), \mathcal{B}(a, b, c))$, respectively.

5.2. **Coxeter mutation for tripods.** For a bipartite quiver $Q$, we have two sets of vertices $I_+$ and $I_-$ so that all edges are oriented from $I_+$ to $I_-$. By definition, for a tripod $Q(a, b, c)$, the central vertex 1 is lying in $I_+$. Let $\mu_+$ and $\mu_-$ be sequences of mutations defined by compositions of mutations corresponding to each and every vertex in $I_+$ and $I_-$, respectively. A Coxeter mutation $\mu_Q$ is the composition
\[
\mu_Q = \mu_+ \cdot \mu_- = \prod_{i \in I_-} \mu_i \cdot \prod_{i \in I_+} \mu_i.
\]

**Remark 5.5.** For any sequence $\mu$ of mutations, we will use the right-to-left convention. Namely, the rightmost mutation will be applied first on the quiver $Q$.

Similarly, we define the Legendrian Coxeter mutation, which will be denoted by $\mu_\mathcal{G}$, on a bipartite N-graph $\mathcal{G}$ as follows:

**Definition 5.6 (Legendrian Coxeter mutation).** For a bipartite N-graph $\mathcal{G}$ with decomposed sets of cycles $\mathcal{B} = \mathcal{B}_+ \cup \mathcal{B}_-$, we define the Legendrian Coxeter mutation $\mu_\mathcal{G}$ as the composition of Legendrian mutations
\[
\mu_\mathcal{G} = \prod_{\gamma \in \mathcal{B}_-} \mu_\gamma \cdot \prod_{\gamma \in \mathcal{B}_+} \mu_\gamma.
\]

**Lemma 5.7.** The effect of the Legendrian Coxeter mutation on $(G(A_n), \mathcal{B}(A_n))$ is the clockwise $\frac{2\pi}{n+3}$-rotation.

**Proof.** We may assume that the Coxeter element $\mu_\mathcal{G}$ can be represented by the sequence
\[
\mu_\mathcal{G} = \mu_+ \cdot \mu_- = (\mu_{\gamma_2} \mu_{\gamma_4} \mu_{\gamma_6} \cdots) (\mu_{\gamma_4} \mu_{\gamma_6} \mu_{\gamma_8} \cdots).
\]

Then the action of $\mu_\mathcal{G}$ on $G(A_n)$ is as depicted in Figure 27, which is nothing but the clockwise $\frac{2\pi}{n+3}$-rotation of the original N-graph $(G(A_n), \mathcal{B}(A_n))$ as claimed. \qed

**Remark 5.8.** The order of the Coxeter mutation is either $(n + 3)/2$ if $n$ is odd or $n + 3$ otherwise. Since the Coxeter number $h = n + 1$ for $A_n$, this verifies Lemma 3.20.
Let $(\mathcal{G}, \mathcal{B}, \mathcal{F}_\lambda)$ be a triple of a good N-graph, a good tuple of cycles and flags $\mathcal{F}_\lambda$ on $\lambda$. Suppose that the quiver $Q(\mathcal{G}, \mathcal{B})$ is bipartite and $\mu_{\mathcal{G}}(\mathcal{G}, \mathcal{B})$ is well-defined. Then by Proposition 4.19, we
This follows directly from the above observation.

**Proof.**

Consider the Legendrian Coxeter mutation on \( \mathcal{G}(A_n) \), which is denoted by \( \mu \). The Legendrian Coxeter mutation on \( \mathcal{G}(A_n) \) is given as the concatenation of \( \mu \) at \( Y \)-mutation at the central \( (\mathcal{G}(A_n), \mathcal{B}(A_n)) \) as we want.

Figure 27. Legendrian Coxeter mutation \( \mu \) on \( \mathcal{G}(A_n), \mathcal{B}(A_n) \)

have

\[
\Psi(\mu_\mathcal{G}(\mathcal{G}, \mathcal{B}), \mathcal{F}_\lambda) = \mu_\mathcal{Q}(\Psi(\mathcal{G}, \mathcal{B}, \mathcal{F}_\lambda)).
\]

In particular, for quivers of type \( A_n \) or tripods we have the following corollary.

**Corollary 5.9.** For each \( n \geq 1 \) and \( a, b, c \geq 1 \), the Legendrian Coxeter mutation \( \mu_\mathcal{G} \) on \( \mathcal{G}(A_n), \mathcal{B}(A_n) \) or \( \mathcal{G}(a, b, c), \mathcal{B}(a, b, c) \) corresponds to the Coxeter mutation \( \mu_\mathcal{Q} \) on \( Q(A_n) \) or \( Q(a, b, c) \), respectively.

By the mutation convention mentioned above, for each tripod \( \mathcal{G}(a, b, c) \), we always take a mutation at the central \( Y \)-cycle \( \gamma_1 \) first. After the Legendrian mutation on \( \mathcal{G}(a, b, c), \mathcal{B}(a, b, c) \) at \( \gamma_1 \), we have the \( N \)-graph on the left in Figure 28(a). Then there are three shaded regions that we can apply the generalized push-through moves so that we obtain the \( N \)-graph on the right in Figure 28(a). Notice that in each triangular shaded region, the \( N \)-subgraph looks like the \( N \)-graph of type \( A_n-1 \), \( A_{b-1} \) or \( A_{c-1} \). Moreover, the mutations corresponding to the rest sequence is just a composition of Coxeter mutations of type \( A_n-1 \), \( A_{b-1} \) and \( A_{c-1} \), which are essentially the same as the clockwise rotations. Therefore, the result of the Coxeter mutation will be given as depicted in Figure 28(b).

Then one can observe that this is very similar to the original \( N \)-graph \( \mathcal{G}(a, b, c) \). Indeed, the inside is identical to \( \mathcal{G}(a, b, c) \) but the colors are switched, which is \( \bar{\mathcal{G}}(a, b, c) \) by definition. The complement of \( \mathcal{G}(a, b, c) \) in \( \mu_\mathcal{Q}(\mathcal{G}(a, b, c), \mathcal{B}(a, b, c)) \) is an annular \( N \)-graph.

**Definition 5.10** (Coxeter padding). For each triple \( a, b, c \), the annular \( N \)-graph depicted in Figure 29 is denoted by \( \mathcal{E}(a, b, c) \) and called the **Coxeter padding** of type \( (a, b, c) \). We also denote the Coxeter padding with color switched by \( \bar{\mathcal{E}}(a, b, c) \).

Notice that two Coxeter paddings \( \mathcal{E}(a, b, c) \) and \( \bar{\mathcal{E}}(a, b, c) \) can be glued without any ambiguity and so we can also pile up Coxeter paddings \( \mathcal{E}(a, b, c) \) and \( \bar{\mathcal{E}}(a, b, c) \) alternatively as many times as we want.

We also define the concatenation of the Coxeter padding \( \bar{\mathcal{E}}(a, b, c) \) on the pair \( \mathcal{G}(a, b, c), \mathcal{B}(a, b, c) \) as the pair \( (\mathcal{G}', \mathcal{B}') \) such that

1. the \( N \)-graph \( \mathcal{G}' \) is obtained by gluing \( \bar{\mathcal{E}}(a, b, c) \) on \( \mathcal{G}(a, b, c) \), and
2. the tuple \( \mathcal{B}' \) of cycles is the set of \( \mathcal{L} \)- and \( \mathcal{Y} \)-cycles identified with \( \mathcal{B}(a, b, c) \) in a canonical way.

**Proposition 5.11.** The Legendrian Coxeter mutation on \( \mathcal{G}(a, b, c), \mathcal{B}(a, b, c) \) or \( \mathcal{G}(a, b, c), \mathcal{B}(a, b, c) \) is given as the concatenation

\[
\mu_\mathcal{G}(\mathcal{G}(a, b, c), \mathcal{B}(a, b, c)) = \mathcal{E}(a, b, c)(\bar{\mathcal{E}}(a, b, c), \bar{\mathcal{B}}(a, b, c)),
\]

\[
\mu_\mathcal{G}(\bar{\mathcal{G}}(a, b, c), \bar{\mathcal{B}}(a, b, c)) = \bar{\mathcal{E}}(a, b, c)(\bar{\mathcal{G}}(a, b, c), \bar{\mathcal{B}}(a, b, c)).
\]

**Proof.** This follows directly from the above observation. \( \square \)
It is important that this proposition holds only when we take the Legendrian Coxeter mutation on the very standard $N$-graph with the tuple of cycles $(\mathcal{G}(a, b, c), \mathcal{B}(a, b, c))$. Otherwise, the Legendrian Coxeter mutation will not be expressed as simple as above.

**Theorem 5.12.** For $a, b, c \geq 1$ with $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq 1$, the Legendrian knot or link $\lambda(a, b, c)$ in $J^1S^1$ admits infinitely many distinct exact embedded Lagrangian fillings.

**Proof.** By Proposition 5.11, the effect of the Legendrian Coxeter mutation on $(\mathcal{G}(a, b, c), \mathcal{B}(a, b, c))$ is just to attach the Coxeter padding on $(\bar{\mathcal{G}}(a, b, c), \bar{\mathcal{B}}(a, b, c))$. In particular, as mentioned earlier, for each $r \geq 0$, the iterated Legendrian Coxeter mutation

$$\mu^r_{\alpha}(\mathcal{G}(a, b, c), \mathcal{B}(a, b, c))$$

is well-defined. Each of these $N$-graphs define a Legendrian weave $\Lambda(\mu^r_{\alpha}(\mathcal{G}(a, b, c), \mathcal{B}(a, b, c)))$, whose Lagrangian projection is a Lagrangian filling

$$L_\alpha(a, b, c) := (\pi \circ \iota)(\Lambda(\mu^r_{\alpha}(\mathcal{G}(a, b, c), \mathcal{B}(a, b, c))))$$

as desired. Therefore it suffices to prove that Lagrangians $L_\alpha(a, b, c)$ for $r \geq 0$ are pairwise distinct up to exact Lagrangian isotopy, when $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq 1$.

Now suppose that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq 1$, or equivalently, $Q(a, b, c)$ is of infinite type. Then the order of the Coxeter mutation is infinite by Lemma 3.20 and so is the order of the Legendrian Coxeter
mutation by Corollary 5.9. In particular, for fixed flags $\mathcal{F}_\lambda$ on $\lambda$, the set
\[
\{ \Psi(\mu_r^\lambda(\mathcal{G}(a, b, c), \mathcal{B}(a, b, c)), \mathcal{F}_\lambda) \mid r \geq 0 \}
\]
is the set of infinitely many pairwise distinct seeds in the cluster pattern for $Q(a, b, c)$. Hence by Corollary 4.15, we have pairwise distinct Lagrangian fillings $L_r(a, b, c)$. □

5.3. $N$-graphs of type ADE. In this section, we will prove one of the main theorem.

Theorem 5.13. Let $\lambda$ be a Legendrian knot or link which is either $\lambda(A_n)$ or $\lambda(a, b, c)$ of type ADE. Then it admits exact embedded Lagrangian fillings as many as seeds in its seed pattern of the same type.

Indeed, this theorem follows from the generalized questions.

Question 5.14. For given $N$-graph $\mathcal{G}$ with a chosen set $\mathcal{B}$ of cycles, can we take a Legendrian mutation as many times as we want? Or equivalently, after applying a mutation $\mu_k$ on $(\mathcal{G}, \mathcal{B})$, is the tuple $\mu_k(\mathcal{B})$ still good in $\mu_k(\mathcal{G})$?

This question has been raised previously in [8, Remark 7.13]. One of the main reason making the question nontrivial is that the potential difference of geometric and algebraic intersections between two cycles. More concretely, two cycles $\gamma_1$ and $\gamma_2$ as shown in Figure 30, can never be isotoped off to each other but their signed intersections following the rule in Figure 22 vanishes. Hence in the corresponding quiver to the first local $N$-graph, there are no arrows between the corresponding vertices 1 and 2. However, after a sequence of Move (II), we can deform $\gamma_2$ into $\gamma(e)$ for an edge $e$ as depicted in the third picture of Figure 30. The mutation $\mu_{\gamma(e)}$ transforms $\gamma_1$ to $\gamma_1'$, which is not good and so it is not clear how to define a mutation $\mu_{\gamma_1}$.

Instead of attacking this question directly, we will prove the following:

Proposition 5.15. Let $\lambda$ be as above and $\mathcal{F}_\lambda$ be flags on $\lambda$. Suppose that $\Sigma$ is a seed in the seed pattern of the same type with the initial seed
\[
\Sigma_{t_0} = \begin{cases} 
\Psi(\mathcal{G}(A_n), \mathcal{B}(A_n), \mathcal{F}_\lambda) & \lambda = \lambda(A_n); \\
\Psi(\mathcal{G}(a, b, c), \mathcal{B}(a, b, c), \mathcal{F}_\lambda) & \lambda = \lambda(a, b, c).
\end{cases}
\]
Then $\lambda$ admits either an $N$-graph $(\mathcal{G}, \mathcal{B})$ on $\mathbb{D}^2$ such that $\mathcal{G}$ is either a 2-graph if $\lambda = \lambda(A_n)$ or a 3-graph if $\lambda = \lambda(a, b, c)$, and

$\Sigma = \Psi(\mathcal{G}, \mathcal{B}, \mathcal{F}_\lambda)$.

Under the aid of this proposition, one can prove Theorem 5.13.
\( \gamma_1 \) and \( \gamma_2 \) with signed intersection number zero, and a mutation \( \mu_e \).

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure30.png}
\caption{Non-disjoint cycles \( \gamma_1 \) and \( \gamma_2 \) with signed intersection number zero, and a mutation \( \mu_e \).}
\end{figure}

\textbf{Proof of Theorem 5.13.} Let \( \lambda \) be given as above. Then by Proposition 5.15, we have pairs of \( N \)-graphs and good tuples of cycles which have a one-to-one correspondence \( \Psi \) with seeds in the seed pattern of \( \mathcal{Q}(a, b, c) \). Hence any pair of the Lagrangian fillings coming from these \( N \)-graphs is never exact Lagrangian isotopic by Corollary 4.15. This completes the proof. \hfill \Box

We will use the following observations: let \( P(\Phi) \) be the generalized associahedron for the root system \( \Phi \) of type ADE (cf. Theorem 3.7 and \S 3.4).

1. There is one-to-one correspondence between the sets of vertices and seeds.
2. There is one-to-one correspondence between the set of facets, faces of codimension 1, and the set of almost positive roots \( \Phi_{\geq -1} \).
3. For the initial seed \( \Sigma_{t_0} \), we may assume that the facets of codimension one including \( \Sigma_{t_0} \) correspond to negative simple roots. Namely, there are exactly \( n \)-facets
   \[ \mathcal{F} = \{ F_{-\alpha_i} \mid \alpha_i \in \Pi \} \]
4. The orbits of \( \mathcal{F} \) under the action of the Legendrian Coxeter mutation \( \mu_{\mathcal{Q}} \) exhaust all facets.

\textbf{Proof of Proposition 5.15.} For a Legendrian link \( \lambda \) of type ADE, we fix flags \( \mathcal{F}_\lambda \) on \( \lambda \). Let us define the initial 2-graph or 3-graph with the chosen tuple of cycles \(( \mathcal{G}_{t_0}, \mathcal{B}_{t_0} )\) as
\[
( \mathcal{G}_{t_0}, \mathcal{B}_{t_0} ) = \begin{cases} 
( \mathcal{G}(A_n), \mathcal{B}(A_n) ) & \lambda = \lambda(A_n); \\
( \mathcal{G}(a, b, c), \mathcal{B}(a, b, c) ) & \lambda = \lambda(a, b, c),
\end{cases}
\]
which defines the initial seed \( \Sigma_{t_0} \) via \( \Psi \)
\[
\Sigma_{t_0} = \Psi(\mathcal{G}_{t_0}, \mathcal{B}_{t_0}, \mathcal{F}_\lambda) = (x(\lambda(\mathcal{G}_{t_0}), \mathcal{B}_{t_0}, \mathcal{F}_\lambda), \mathcal{Q}(\lambda(\mathcal{G}_{t_0}), \mathcal{B}_{t_0})).
\]

Suppose that \( \Sigma \) is a seed in the cluster pattern. Then we need to to prove that there exists an \( N \)-graph \(( \mathcal{G}, \mathcal{B} )\) such that \( \Psi(\mathcal{G}, \mathcal{B}, \mathcal{F}_\lambda) = \Sigma \).

By Proposition 3.15 and Lemma 3.16, there exist an integer \( r \) and a sequence \( \mu' \) of mutations such that
\[
\Sigma = \mu'(\mu_{\mathcal{Q}}^r(\Sigma_{t_0})),
\]
where \( \mu' \) joins \( \mu_{\mathcal{Q}}^r(\Sigma_{t_0}) \) and \( \Sigma \) inside a facet.

If \( \lambda = \lambda(A_n) \), then \( \mu_{\mathcal{G}} \) acts on \(( \mathcal{G}_{t_0}, \mathcal{B}_{t_0} )\) as the \( \left( \frac{2\pi}{n+2} \right) \)-rotation, which obviously commutes with Legendrian mutation \( \mu' \). Hence it suffices to show the well-definedness of \( \mu'(\mathcal{G}_{t_0}, \mathcal{B}_{t_0}) \).
Otherwise, as seen earlier, the action of the Legendrian Coxeter mutation $\mu_r^\tau$ on $(G_{t_0}, B_{t_0})$ is obtained by the concatenation of sequences of $C = C(a, b, c)$ and $\hat{C} = \hat{C}(a, b, c)$ to either $(G_{t_0}, B_{t_0})$ or $(\hat{G}_{t_0}, \hat{B}_{t_0})$.

$$\mu_r^\tau(G_{t_0}, B_{t_0}) = \begin{cases} C\hat{C} \cdots \hat{C}(G_{t_0}, B_{t_0}) & r \text{ is even,} \\ C\hat{C} \cdots \hat{C}(G_{t_0}, B_{t_0}) & r \text{ is odd.} \end{cases}$$

Let us regard the sequence $\mu'$ of mutations as the sequence of Legendrian mutations. Since the concatenation of $C$ or $\hat{C}$ do not touch any chosen cycle in $B_{t_0}$, two operations—the concatenation of $C$ or $\hat{C}$, and the mutation $\mu'$—commute. Therefore

$$\mu'(\mu_r^\tau(G_{t_0}, B_{t_0})) = \begin{cases} C\hat{C} \cdots \hat{C}(\mu'(G_{t_0}, B_{t_0})) & r \text{ is even,} \\ C\hat{C} \cdots \hat{C}(\mu'(G_{t_0}, B_{t_0})) & r \text{ is odd,} \end{cases}$$

and the proposition follows if $\mu'(G_{t_0}, B_{t_0})$ and $\mu'(\hat{G}_{t_0}, \hat{B}_{t_0})$ are well-defined. Since $G_{t_0}$ and $\hat{G}_{t_0}$ are essentially the same, it suffices to show the well-definedness of $\mu'(G_{t_0}, B_{t_0})$ as before.

Now we will prove the well-definedness of $\mu'(G_{t_0}, B_{t_0})$ in both cases by using induction on $n$. Suppose that $\mu'$ is a sequence of mutations in a facet $F_\beta$ for some $\beta \in \Phi_{\geq 1}$, and $\mu_r^\tau(F_{\alpha_1}) = F_\beta$ for some $\alpha_1 \in \Pi$. Then the facet $F_\beta$ is combinatorially equivalent to the lower dimensional generalized associahedron

$$F_\beta \cong P(\Phi([n] \setminus \{i\})) \cong P(\Phi_1) \times \cdots \times P(\Phi_m).$$

Here, $\Phi([n] \setminus \{i\})$ is not necessarily irreducible and we denote by $\Phi_1, \ldots, \Phi_\ell$, $\ell \leq 3$ the root systems satisfying that $\Phi([n] \setminus \{i\}) = \Phi_1 \times \cdots \times \Phi_\ell$. Moreover, in terms of quivers, if we denote the connected components of $Q \setminus \{i\}$ by $Q^{(1)}, \ldots, Q^{(\ell)}$, then we may say that $\Phi_j$ and $Q^{(j)}$ are of the same type. Therefore the sequence $\mu'$ of mutations can be decomposed into $\mu^{(1)}, \ldots, \mu^{(\ell)}$ on $Q^{(1)}, \ldots, Q^{(\ell)}$, respectively.

Similarly, in $N$-graph $G_{t_0}$, the $i$-th cycle $[\gamma_i]$ separates $(G_{t_0}, B_{t_0})$ into at most three parts 

$$(G_{t_0}, B_{t_0}), (G_{t_0}, B_{t_0}), \ldots, (G_{t_0}, B_{t_0}),$$

as seen in Figure 31. This means that $\mu_j^{(3)}(G_{t_0}, B_{t_0})$ is well-defined for all $1 \leq j \leq \ell \implies \mu'(G_{t_0}, B_{t_0})$ is well-defined.

Indeed, if $\lambda = \lambda(A_n)$, then we have the following two cases:

1. If $\gamma_i$ corresponds to a bivalent vertex, then for some $1 \leq r, s$ with $r + s + 1 = n$, we have two 2-subgraphs

$$\{(G(A_r), B(A_r)), (G(A_s), B(A_s))\};$$

2. If $\gamma_i$ corresponds to a leaf, then we have the 2-subgraph

$$\{(G(A_{n-1}), B(A_{n-1}))\}.$$

Otherwise, if $\lambda = \lambda(a, b, c)$, then we have the following three cases:

1. If $\gamma_i$ corresponds to the central vertex, then we have three 3-subgraphs

$$\{(G(a), B(a)), (G(b), B(b)), (G(c), B(c))\};$$

2. If $\gamma_i$ corresponds to a bivalent vertex, then for some $1 \leq r, s$ with $r + s + 1 = a$, up to permuting indices $a, b, c$, we have two 3-subgraphs

$$\{(G(r), B(r)), (G(b), B(b))\};$$

3. Otherwise, if $\gamma_i$ corresponds to a leaf, then up to permuting indices $a, b, c$, we have the 3-subgraph

$$\{(G(a - 1), B(a - 1), b, c))\}.$$
Since there are no obstructions for mutations on these \( N \)-graphs, we are done for the initial condition for the induction.

\[ \xrightarrow{\text{G}(3)(A_{a-1}), \text{B}(3)(A_{a-1})} \]

\[ \xrightarrow{\text{G}(3)(A_{b-1}), \text{B}(3)(A_{b-1})} \]

\[ \xrightarrow{\text{G}(3)(A_{c-1}), \text{B}(3)(A_{c-1})} \]

\[ \xrightarrow{\text{G}(r,b,c), \text{B}(r,b,c)} \]

**Figure 31.** Separations of \((G(a,b,c), B(a,b,c))\) at \( \gamma_i \)

**Remark 5.16.** In the above proof, it is not claimed that two mutations \( \mu' \) and \( \mu_B \) commute. Indeed, if we first mutate \((G_{t_0}, B_{t_0})\) via \( \mu' \), then the result may not look like either \((G_{t_0}, B_{t_0})\) or \((\bar{G}_{t_0}, \bar{B}_{t_0})\) and hence \( \mu_B \) will not work as expected.

6. Lagrangian fillings admitting cluster structures of type \BCFG

In this section, we will construct cluster structures of type \BCFG on certain \( N \)-graphs by using the folding of \( N \)-graphs. Throughout this section, let us assume that a triple \((X, Y, G)\) is one of

\[ (A_{2n-1}, B_n, Z/2Z), \quad (D_{n+1}, C_n, Z/2Z), \quad (E_6, F_4, Z/2Z), \quad (D_4, G_2, Z/3Z) \]

and that the group \( G \) is generated by \( \tau \).

**Remark 6.1.** For \( Q(D_{n+1}) = Q(n - 1, 2, 2), Q(E_6) = Q(2, 3, 3) \) and \( Q(D_4) = Q(2, 2, 2) \), we will use the labelling convention of tripods depicted in Figure 25(a), which is different from the usual convention of Dynkin diagrams shown in Table 1.
6.1. Rotation actions on \(N\)-graphs of type \(A_{2n-1}\) and \(D_4\). Let \((\mathcal{G}, \mathcal{B})\) be a pair of a 2- or 3-graph and a good tuple of cycles of type \(X = A_{2n-1}\) or \(D_4\), respectively. We define a new pair \((\tau(\mathcal{G}), \tau(\mathcal{B}))\) such that \(\tau(\mathcal{G})\) and \(\tau(\mathcal{B})\) are obtained by the \((2\pi/N)\)-rotation on \((\mathcal{G}, \mathcal{B})\).

We say that \((\mathcal{G}, \mathcal{B})\) is \(G\)-admissible if

1. the \(N\)-graph \(\mathcal{G}\) has the \((2\pi/N)\)-rotation symmetry so that \(\tau(\mathcal{G}) = \mathcal{G}\),
2. the tuples of cycles \(\mathcal{B}\) and \(\tau(\mathcal{B})\) are identical up to relabelling as follows: if \(X = A_{2n-1}\),
   \[\gamma_i \leftrightarrow \gamma_{2n-i},\]
   and if \(X = D_4\),
   \[\gamma_2 \rightarrow \gamma_3, \quad \gamma_3 \rightarrow \gamma_4, \quad \gamma_4 \rightarrow \gamma_2.\]

In particular, \(\tau\) preserves \(\gamma_n\) if \(X = A_{2n-1}\) and \(\gamma_1\) if \(X = D_4\). Figure 33 shows examples and non-examples of \(G\)-admissible \(N\)-graphs.

\begin{figure}[h]
\centering
\includegraphics[width=0.7\textwidth]{fig32.png}
\caption{Rotation actions on \(N\)-graphs of type \(A_{2n-1}\) and \(D_4\)}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.7\textwidth]{fig33.png}
\caption{\(G\)-admissible or non-\(G\)-admissible \(N\)-graphs for Legendrians of type \(A_{2n-1}\) and \(D_4\)}
\end{figure}

Remark 6.2. Notice that the Coxeter padding \(\mathcal{C}\) for \(\mathcal{G}(A_{2n-1})\) is empty and \(\mathcal{C}(2, 2, 2)\) has obviously the \(\mathbb{Z}/3\mathbb{Z}\)-rotational symmetry. For each \(G\)-admissible \((\mathcal{G}, \mathcal{B})\) of type \(A_{2n-1}\) or \(D_4\), so is the
following
\[ \mathcal{C} \cdots \mathcal{C}(\mathcal{G}, \mathcal{B}) \quad \text{or} \quad \bar{\mathcal{C}} \cdots \bar{\mathcal{C}}(\mathcal{G}, \mathcal{B}). \]

**Lemma 6.3.** Let \( Q \) be a quiver of type \( A_{2n-1} \). Suppose that \( Q \) is invariant under the action
\[ \tau(i) = 2n - i \]
for all \( i \in [2n-1] \). Then there is no oriented cycle of the form
\[ j \to i \to \tau(j) \to \tau(i) \to j \]
for any \( i, j \neq n \).

**Proof.** It is well known that any minimal cycle in \( Q \) is of length 3. Therefore, if such an oriented cycle exists, then there must be an edge \( i - \tau(i) \) or \( j - \tau(j) \) in \( Q \). Hence \( b_i,\tau(i) \neq 0 \) or \( b_j,\tau(j) \neq 0 \) for \( B = (b_{i,j}) = B(Q) \).

This is impossible because \( Q \) is \( \mathbb{Z}/2\mathbb{Z} \)-admissible and so
\[ b_i,\tau(i) = b_{\tau(i),\tau(j)} = b_{\tau(i),i} = -b_i,\tau(i) \quad \implies \quad b_i,\tau(i) = 0. \]

Therefore we are done. \( \square \)

**Proposition 6.4.** Let \( (\mathcal{G}, \mathcal{B}) \) be of type \( A_{2n-1} \) as above. If \( (\mathcal{G}, \mathcal{B}) \) is \( \mathbb{Z}/2\mathbb{Z} \)-admissible, then so is the quiver \( Q(\Lambda(\mathcal{G}), \mathcal{B}) \).

**Proof.** For \( Q = Q(\Lambda(\mathcal{G}), \mathcal{B}) \), since the generator \( \tau \in \mathbb{Z}/2\mathbb{Z} \) acts on \( \mathcal{B} \) as \( \tau(\gamma_i) = \gamma_{2n-i} \), we have the \( \mathbb{Z}/2\mathbb{Z} \)-action on the set \([2n-1]\) of vertices of \( Q \) as \( \tau(i) = 2n - i \) and therefore for each \( i \in [2n-1] \),
\[ i = 2n - i. \]

We will check the conditions (a), (b), (c), and (d) for admissibility according to Definition 3.10.

(a) Since all vertices in \( Q \) are mutable, the condition (a) is obviously satisfied.

(b) Let \( B = (b_{i,j}) = B(Q) \). Then for each \( i, j \in [2n-1] \), the entry \( b_{i,j} \) is given by the algebraic intersection number \((\gamma_i, \gamma_j)\), which is the same as \((\gamma_{2n-i}, \gamma_{2n-j})\) since \( \mathcal{G} \) has the \( \pi \)-rotation symmetry. Hence
\[ b_{i,j} = b_{\tau(i),\tau(j)}. \]

(c) On the other hand, for each \( i \in [2n-1] \), we have
\[ b_{i,\tau(i)} = (\gamma_i, \gamma_{\tau(i)}) = (\gamma_{\tau(i)}, \gamma_{\tau(\tau(i))}) = (\gamma_{\tau(j)}, \gamma_i) = -b_i,\tau(i), \]

which implies that
\[ b_i,\tau(i) = 0. \]

(d) Finally, we need to prove that for each \( i, j \),
\[ b_{i,j}b_{\tau(i),j} \geq 0. \]

If \( j = n \), then since \( \tau(n) = n \), we have
\[ b_{i,n}b_{\tau(i),n} = b_{i,n}b_{\tau(i),\tau(n)} = b_{i,n}b_{i,n} \geq 0. \]

Similarly, if \( i = n \), then
\[ b_{n,j}b_{\tau(n),j} = b_{n,j}b_{n,j} \geq 0. \]

Suppose that for some \( i, j \neq n \),
\[ b_{i,j}b_{\tau(i),j} < 0. \]

By changing the roles of \( i \) and \( \tau(i) \) if necessary, we may assume that \( b_{i,j} < 0 \). Then we also have
\[ b_{\tau(i),\tau(j)} < 0 < b_{i,\tau(j)}, \]

which implies that there is an oriented cycle in \( Q \)
\[ j \to i \to \tau(j) \to \tau(i) \to j. \]

However, this contradicts to Lemma 6.3 and therefore \( Q \) satisfies all conditions in Definition 3.10. \( \square \)

Similarly, we have the following proposition as well.
**Definition 6.6** (Ray symmetry). We say that the pair \((\mathcal{G}, \mathcal{B})\) is *ray-symmetric* if the intersections \((\mathcal{G}, \mathcal{B}) \cap R_\theta\) for \(\theta = 0, 2\pi/3, 4\pi/3\) are the same up to rotation and avoid all trivalent vertices and hexagonal points except at the origin.

\[
(R_0, (\mathcal{G}, \mathcal{B}) \cap R_0) \cong (R_{2\pi/3}, (\mathcal{G}, \mathcal{B}) \cap R_{2\pi/3}) \cong (R_{4\pi/3}, (\mathcal{G}, \mathcal{B}) \cap R_{4\pi/3}). \tag{6.1}
\]

\[\begin{array}{c}
\text{Figure 34. Ray-symmetrylicity}
\end{array}\]

**Proposition 6.5.** Let \((\mathcal{G}, \mathcal{B})\) be of type \(D_4\). If \((\mathcal{G}, \mathcal{B})\) is \(\mathbb{Z}/3\mathbb{Z}\)-admissible, then so is the quiver \(Q(\mathcal{G}, \mathcal{B})\).

**Proof.** Let \(Q = Q(\mathcal{G}, \mathcal{B})\). Then by definition of the \(\mathbb{Z}/3\mathbb{Z}\)-action, we have

\[2 \sim 3 \sim 4.\]

(a) and (b) This is obvious as before.

(c) Let \(B = (b_{i,j}) = B(Q)\). Suppose that \(b_{2,3} \neq 0\). Then by (b),

\[b_{2,3} = b_{3,4} = b_{4,2} \neq 0\]

and so \(Q\) has a directed cycle either

\[2 \to 3 \to 4 \to 2 \text{ or } 2 \to 4 \to 3 \to 2.\]

Then according to \(b_{1,2}\), the underlying graph of the quiver \(Q\) is either the complete graph \(K_4\) or a disconnected graph, but both are impossible. Therefore

\[b_{2,3} = b_{3,4} = b_{4,2} = 0.\]

(d) The only entries we need to check are \(b_{1,j}\)'s, which are all equal by (b). Therefore

\[b_{1,j} b_{1,j'} \geq 0. \quad \square\]
Then we define a $\mathbb{Z}/2\mathbb{Z}$-action on a ray-symmetric $(\mathcal{G}, \mathcal{B})$ as follows:

1. cut $D^2$ into three sectors $D^2_1$, $D^2_2$, and $D^2_3$ along the rays $R_\theta$ for $\theta = 0, 2\pi/3$ and $4\pi/3$ so that $(\mathcal{G}, \mathcal{B})$ gives us three 3-subgraphs

\[ \{(G_1, B_1), (G_2, B_2), (G_3, B_3)\}, \quad G_i = \mathcal{G} \cap D^2_i, \]

2. change two subgraphs contained in sectors whose angles are in between $[2\pi/3, 4\pi/3]$ and $[4\pi/3, 2\pi]$ by rotating certain angles.

3. The result will be denoted by $(\tau(G), \tau(B))$.

Notice that each subgraph $G_i$ may not satisfy the condition of $N$-graphs but the final result will be a well-defined 3-graph since $G$ is ray-symmetric. However, if $G$ is not ray-symmetric, then the $\mathbb{Z}/2\mathbb{Z}$-action is never well-defined. We call this action the partial rotation and see Figure 35 for the pictorial definition.

We say that $(\mathcal{G}, \mathcal{B})$ is $\mathbb{Z}/2\mathbb{Z}$-admissible if it is invariant under the partial rotation up to relabeling of cycles as follows:

1. if $X = D_{n+1}$, then

\[ \gamma_n \leftrightarrow \gamma_{n+1}, \]

2. if $X = E_6$, then

\[ \gamma_3 \leftrightarrow \gamma_5, \quad \gamma_4 \leftrightarrow \gamma_6. \]

Remark 6.7. Similar to Remark 6.2, the Coxeter padding $\mathcal{C} = \mathcal{C}(n-1, 2, 2)$ for $\mathcal{G}(D_{n+1})$ or $\mathcal{C}(2, 3, 3)$ for $\mathcal{G}(E_6)$ has also the $\mathbb{Z}/2\mathbb{Z}$-symmetry under the partial rotation. Therefore for any $\mathbb{Z}/2\mathbb{Z}$-admissible $(\mathcal{G}, \mathcal{B})$ under the partial rotation of type $D_{n+1}$ or $E_6$, so is the following

\[ \mathcal{C} \cdots \mathcal{C}(\mathcal{G}, \mathcal{B}) \quad \text{or} \quad \bar{\mathcal{C}} \cdots \bar{\mathcal{C}}(\mathcal{G}, \mathcal{B}). \]

Lemma 6.8. Let $Q$ be a quiver of type $E_6$. Suppose that $Q$ is invariant under the action

\[ \tau(3) = 5, \quad \tau(4) = 6. \]
Then there is no oriented cycle, which is either
\[ 3 \to 4 \to 5 \to 6 \to 3 \quad \text{or} \quad 3 \to 6 \to 5 \to 4 \to 3. \]

**Proof.** We will use the essentially same argument as the proof of Lemma 6.3.

If \( Q = \mu(Q(E_6)) \), then by Proposition 3.15 and Lemma 3.16, there exist an integer \( r \) and a sequence \( \mu' \) of mutations such that
\[ Q = \mu'(\mu_r^r(Q(E_6))) = \mu'(Q(E_6)). \]
Moreover, \( \mu' \) misses at least one mutation \( \mu_i \).

1. If \( i = 1 \), then \( Q(E_6) \setminus \{1\} \) consists of three quivers
   \[ \{2\} \subset Q(A_1), \quad \{3, 4\} \subset Q(A_2), \quad \{5, 6\} \subset Q(A_2). \]
   In particular, there are no direct edges between two sets of vertices \( \{3, 4\} \) and \( \{5, 6\} \).

2. If \( i = 2 \), then \( \mu' \) can be regarded as a sequence of mutations of type \( A_5 \). By Lemma 6.3, we are done.

3. If \( i = 3 \), then we may assume that \( \mu' \) also misses \( \mu_5 \) due to the symmetry. Hence we have separated quivers
   \[ \{1, 2\} \subset Q(A_2), \quad \{4\} \subset Q(A_1), \quad \{6\} \subset Q(A_1). \]
   Then after the mutation, the vertices 4 and 6 can be joined only with 3 and 5, respectively, and so we never have an edge between 3 and 6 or between 4 to 5.

4. If \( i = 4 \), then as above, we may assume that \( \mu' \) misses \( \mu_6 \) as well and we may regard \( \mu' \) as a sequence of mutations on \( Q(D_4) \)
   \[ \{1, 2, 3, 5\} \subset Q(D_4). \]
Moreover, \( \mu' \) consists of mutations corresponding to \( \mathbb{Z}/2\mathbb{Z} \)-orbits, which are \( \mu_1, \mu_2 \) and \( \mu_{3,5} = (\mu_3 \mu_5) \). Here the group \( \mathbb{Z}/2\mathbb{Z} \) folds \( D_4 \) onto \( C_3 \), and up to Coxeter mutations, there are only three facets
\[ F_{C_3}^{C_3} \cong P(\Phi(C_3)), \quad F_{C_3}^{C_3} \cong P(\Phi(A_1)) \times P(\Phi(A_1)), \quad F_{C_3}^{C_3} \cong P(\Phi(A_2)). \]
Hence all possible quivers are obtained by one of the following ways:
\[
\begin{align*}
\mu_1 \mu_3 \mu_5 \mu_1 \cdots (Q(D_4)), & \quad 0 \leq k \leq 11 \\
\mu_2 \mu_3 \mu_2 \cdots (Q(D_4)), & \quad 0 \leq k \leq 3 \\
\mu_1 \mu_2 \mu_1 \cdots (Q(D_4)), & \quad 0 \leq k \leq 9
\end{align*}
\]
Note that two quivers
\[
(\mu_1 \mu_3 \mu_5 \mu_1 \mu_3 \mu_1)(Q(D_4)) \quad \text{and} \quad (\mu_1 \mu_2 \mu_1 \mu_2 \mu_1)(Q(D_4))
\]
are obtained from \( Q(D_4) \) by permuting vertices \( 3 \leftrightarrow 5 \) and \( 1 \leftrightarrow 2 \), respectively. Finally, one can directly check that we have no such cycles by the exhaustive search in this full list. \( \square \)

**Proposition 6.9.** Let \( (\mathcal{S}, \mathcal{B}) \) be of type \( X = D_{n+1} \) or \( E_6 \). If \( (\mathcal{S}, \mathcal{B}) \) is \( \mathbb{Z}/2\mathbb{Z} \)-admissible, then so is the quiver \( Q(\mathcal{S}, \mathcal{B}) \).

**Proof.** (a) and (b): This is obvious as before.
(c) Let \( \mathcal{B} = (b_{i,j}) = \mathcal{B}(Q) \). Then by (b),
\[ b_{i,\tau(i)} = b_{\tau(i),\tau(i)} = b_{\tau(i),i} = -b_{i,\tau(i)} \quad \Rightarrow \quad b_{i,\tau(i)} = 0. \]
(d) If \( X = D_{n+1} \), then we only need to show
\[ b_{i,n} b_{i,n+1} \geq 0 \]
for $i < n$. This is obvious since 

$$b_{i,n+1} = b_{r(i),r(n+1)} = b_{i,n}.$$ 

If $X = E_6$, then all we need to show are inequalities 

$$b_{i,j}b_{i,j+2} \geq 0,$$ 

$$b_{3,4}b_{3,6} \geq 0$$

for $i = 1, 2$ and $j = 3, 4$.

The first inequality is obvious since 

$$b_{i,j+2} = b_{r(i),r(j+2)} = b_{i,j}.$$ 

Suppose that $b_{3,4}b_{3,6} < 0$. Then since $b_{3,4} = b_{5,6}$ and $b_{3,6} = b_{5,4}$, the $Q$ has a loop either 

$$3 \to 4 \to 5 \to 6 \to 3 \quad \text{or} \quad 3 \to 6 \to 5 \to 4 \to 3.$$ 

$\square$

6.2.1. $Q(A_{2n-1})$ as a tripod $Q(1, n, n)$. As observed in Lemma 5.3, one can think $Q(1, n, n)$ and $S(1, n, n)$ for $A_{2n-1}$ instead of $Q(A_{2n-1})$ and $S(A_{2n-1})$.

The major difference is now we have to use 3-graphs and partial rotations instead of 2-graphs and $\pi$-rotations. Then it can be easily checked that the above two notions are identical and so the $Z/2Z$-admissibility for 3-graphs of type $A_{2n-1}$ is also well-defined. Moreover, the 3-graph analogue under the partial rotation of Proposition 6.4 will be true.

6.3. Global foldability of $N$-graphs. Let $(S, B)$ be of type $X$. We say that $(S, B)$ is globally foldable with respect to $G$ if $(S, B)$ is $G$-admissible and for any sequence of mutable $G$-orbits $I_1, \ldots, I_t$, there exists a $G$-admissible $(S', B')$ such that 

$$Q(\Lambda(S'), B') = (\mu_{I_1} \cdots \mu_{I_t})(Q(\Lambda(S), B)).$$

Theorem 6.10. The $N$-graph with a good tuple of cycles $(S(X), B(X))$ is globally foldable with respect to $G$.

Proof. Let us define the initial quiver $Q_{i_0}$ by 

$$Q_{i_0} = Q(\Lambda(S(X)), B(X)).$$

For a sequence of mutable $G$-orbits $I_1, \ldots, I_t$, we have an integer $r$ and $\mu^Y$ by Proposition 3.15 and Lemma 3.16 such that in the cluster pattern of type $Y$, two sequences of mutations 

$$(\mu_{I_{1_t}} \cdots \mu_{I_1})(Q_{i_0})^G = \mu^N((\mu_{Q}^X)^r (Q_{i_0}^G))$$

will produce the same seed.

On the other hand, as seen in Remark 3.19, the Coxeter mutation $\mu_{Q}^X$ will correspond to the Coxeter mutation $\mu_{Q}^X$ via the folding. Moreover, $\mu^N$ comes from a sequence $\mu^X$ of mutations via the folding such that $\mu^X$ is the composition of mutations at $G$-orbits and happens inside some facet $F_\beta \subset P(\Phi(X))$. Hence we have 

$$\mu^N((\mu_{Q}^X)^r (Q_{i_0}^G)) = \mu^N(((\mu_{Q}^X)^r (Q_{i_0}))^G) = (\mu^X((\mu_{Q}^X)^r (Q_{i_0}))^G).$$

By Proposition 5.15, there exists a pair $(S', B')$ satisfying that 

$$Q(\Lambda(S'), B') = Q(\mu^X((\mu_{Q}^X)^r (Q_{i_0}))) = (\mu_{I_{1_t}} \cdots \mu_{I_1})(Q_{i_0}).$$

Finally, we need to show that $(S', B')$ can be assumed to be $G$-admissible. As in the proof of Proposition 5.15, $(S', B')$ is obtained by taking a $\mu^X$ on either $(S(X), B(X))$ or $(S(X), B(X))$ and attaching Coxeter paddings. As observed in Remarks 6.2 and 6.7, the Coxeter paddings themselves are already $G$-admissible and the process attaching them preserve the $G$-admissibility in each case. Therefore we only need to show the $G$-admissibility of $\mu^X(S(X), B(X))$.

We will use the essentially same strategy as the proof of Proposition 5.15. Since $\mu^X$ misses some $\mu_{r_i}$, it misses all $\mu_{r_{i'}}$ for $i < i'$. Then one can split $(S(X), B(X))$ in a $G$-admissible way. That is, the set of $N$-subgraphs 

$$\{(S_1, B_1), \ldots, (S_\ell, B_\ell)\}$$
is closed under the $G$-action. In this case, $G$ may permute $N$-subgraphs as well. Now we split $\mu^X$ into $\{\mu_1', \ldots, \mu_\ell'\}$ such that each $\mu_i'$ is a sequence of mutations of $Q(S_i, B_i)$. Then $\mu^X(S(X), B(X))$ is $G$-admissible if so is $\mu_i'(S_i, B_i)$ for each $1 \leq i \leq \ell$. 

Since each $(S_i, B_i)$ is strictly simpler than $(S(X), B(X))$ in terms of the number of vertices and is again of type ADE, the rest of the proof follows from induction and we omit the detail. \hfill \Box

As a direct consequence, we will prove the following theorem:

**Theorem 6.11.** The following holds:

1. The Legendrian link $\lambda(A_{2n-1})$ has $(2^n) \mathbb{Z}/2\mathbb{Z}$-admissible $N$-graphs which admits the cluster pattern of type $B_n$.
2. The Legendrian link $\lambda(D_{n+1})$ has $(2^n) \mathbb{Z}/2\mathbb{Z}$-admissible $N$-graphs which admits the cluster pattern of type $C_n$.
3. The Legendrian link $\lambda(E_6)$ has $105 \mathbb{Z}/2\mathbb{Z}$-admissible $N$-graphs which admits the cluster pattern of type $F_4$.
4. The Legendrian link $\lambda(D_4)$ has $8 \mathbb{Z}/3\mathbb{Z}$-admissible $N$-graphs which admits the cluster pattern of type $G_2$.

**Proof.** By Theorem 6.10, it is already known that for each $X$, the quiver of type $X$ is globally foldable with respect to $G$. By Propositions 6.4, 6.5 and 6.9, the quiver $Q(S(X), B(X))$ is also globally foldable with respect to $G$.

Let $\Sigma_0 = \Psi(S(X), B(X), F_X) = (x(\Lambda(S(X)), B(X), F_X), Q(\Lambda(S(X)), B(X)))$ be the initial seed. Without loss of generality, we may denote cluster variables in $x$ by

$$x = (x_1, \ldots, x_{rk(X)})$$

and we define a field homomorphism $\psi : F = \mathbb{C}(x_1, \ldots, x_{rk(X)}) \to F^G = \mathbb{C}(x_1, \ldots, x_{rk(Y)})$ by

$$\psi(x_i) = x_I$$

for any $i$ in a $G$-orbit $I$. Then by construction, the initial seed $\Sigma_0$ is $(G, \psi)$-admissible. See §3.3.

Finally, by Proposition 3.13, folded seeds form a seed pattern of type $Y$ as desired, and we are done. \hfill \Box

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