DIMENSION OF THE SPACE OF CONICS ON FANO HYPERSURFACES

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Abstract. R. Beheshti showed that, for a smooth Fano hypersurface $X$ of degree $\leq 8$ over the complex number field $\mathbb{C}$, the dimension of the space of lines lying in $X$ is equal to the expected dimension.

We study the space of conics on $X$. In this case, if $X$ contains some linear subvariety, then the dimension of the space can be larger than the expected dimension.

In this paper, we show that, for a smooth Fano hypersurface $X$ of degree $\leq 6$ over $\mathbb{C}$, and for an irreducible component $R$ of the space of conics lying in $X$, if the 2-plane spanned by a general conic of $R$ is not contained in $X$, then the dimension of $R$ is equal to the expected dimension.

1. Introduction

Let $X \subset \mathbb{P}^n$ be a hypersurface of degree $d$ over the complex number field $\mathbb{C}$. We define $R_e(X)$ to be the space of smooth rational curves of degree $e$ in $\mathbb{P}^n$ lying in $X$, which is an open subscheme of the Hilbert scheme $\text{Hilb}^{e+1}(X)$. The number

$$(n + 1 - d)e + n - 4$$

is called the expected dimension of $R_e(X)$, where the dimension of $R_e(X)$ at $C$ is greater than or equal to this number if there exists $C \in R_e(X)$ such that $X$ is smooth along $C$.

The space of rational curves on a general Fano hypersurface have been studied by many authors ([5], [8], [9], [14], [4], [12] in characteristic zero; [10, V, §4], [7] in any characteristic). At least for $e = 1, 2$, it is well known that $R_e(X)$ has the expected dimension if $X$ is a general Fano hypersurface.

On the other hand, it is difficult to know about $R_e(X)$ for any smooth $X \subset \mathbb{P}^n$. For $n$ exponentially large in $d$, T. D. Browning and P. Vishe [5] showed that the space of rational curves of any degree $e$ on smooth $X$ has the expected dimension.

For any $n$ in the case of $e = 1$, as an answer of the question which was asked by O. Debarre and J. de Jong independently, R. Beheshti [2] showed that $R_1(X)$ has the expected dimension if $X \subset \mathbb{P}^n$ is a smooth Fano hypersurface of degree $d \leq 6$.
J. M. Landsberg and C. Robles [11] gave another proof for the same degree \( d \leq 6 \). Beheshti [3] later showed the same statement for \( d \leq 8 \).

In the case of \( e = 2 \), A. Collino, J. P. Murre, G. E. Welters [6] studied \( R_2(X) \) for a smooth quadric 3-fold \( X \subset P^4 \); in this case, \( R_2(X) \) has the expected dimension. Note that \( \text{Hilb}^{2d+1}(X) \) is connected for any smooth hypersurface \( X \subset P^n \) if the expected dimension is positive [7, Proposition 5.6].

In this paper, we study the dimension of \( R_2(X) \) for a smooth Fano hypersurface of degree \( d \leq 6 \). Our main result is the following.

**Theorem 1.1.** Let \( X \subset P^n \) be a smooth Fano hypersurface of degree \( d \leq 6 \) over \( \mathbb{C} \). Let \( R \neq \emptyset \) be an irreducible component of \( R_2(X) \) such that

\[ \langle C \rangle \not\subset X \text{ for general } C \in R, \]

where \( \langle C \rangle = P^2 \subset P^n \) is the 2-plane spanned by \( C \). Then \( \dim(R) \) is equal to the expected dimension \( 3n - 2d - 2 \).

The dimension of \( R_2(X) \) can be greater than the expected one when \( X \) contains certain linear varieties (see Example 3.17); this is the reason why we assume the condition (1). The statement of Theorem 1.1 does not hold for \( d \geq 10 \) (see Example 3.18); thus it may need some conditions stronger than (1) for larger \( d \).

The paper is organized as follows. We assume that \( \dim R \) is greater than the expected dimension, and take \( Y := \text{Loc}(R) \subset X \) to be the locus swept out by conics of \( R \). The codimension of \( Y \) is \( \geq 2 \) in \( X \) due to a result of Beheshti [3] (see Remark 2.2). Then it is sufficient to investigate the case when \( (d, \dim Y) = (6, n - 3) \). In [2] we consider the linear subvariety \( \langle Y \rangle \subset P^n \) spanned by \( Y \), and show that the codimension of \( \langle Y \rangle \) is \( \leq 1 \) in \( P^n \) by using projective techniques (Proposition 2.6). Let \( T_xX \subset P^n \) be the embedded tangent space to \( X \) at \( x \). In [33], considering the subset \( R_2^x \subset R \) consisting of conics \( C \) such that \( x \in C \subset T_xX \), we show that \( \langle \text{Loc}(R_2^x) \rangle \) is an \((n - 3)\)-plane (Proposition 3.6), and show that \( \text{Loc}(R_2^x) \) is a quadric hypersurface in \( \langle \text{Loc}(R_2^x) \rangle = P^{n-3} \) (Corollary 3.9). In particular, our problem is reduced to the case \( n = d = 6 \). Using such quadrics, we give the proof of Theorem 1.1 step by step.

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## 2. The Locus Swept Out by Conics

We use the following notations. For a Fano hypersurface \( X \subset P^n \) of degree \( d \leq 6 \), we take an irreducible component \( R \neq \emptyset \) of \( R_2(X) \) such that \( \langle C \rangle \not\subset X \) for
general $C \in R_2(X)$. We denote by $\tilde{R}$ the closure in $\text{Hilb}^{2+1}(X)$. Let

$$\mathcal{U} := \{ (C, x) \in R \times X \mid x \in C \}$$

be the universal family of $R$, and let $\pi : \mathcal{U} \to R$ and $\text{ev} : \mathcal{U} \to X$ be the first and second projections. For a subset $A \subset R$, we write $\mathcal{U}_A := \pi^{-1}(A)$ and $\text{Loc}(A) := \text{ev}(\mathcal{U}_A) = \bigcup_{C \in A} C \subset X$.

We write $R_x \subset R$ to be the set of $C \in R$ passing through $x \in X$, and write $R_{xy} = R_x \cap R_y \subset R$, the set of $C \in R$ passing through $x, y \in X$.

We set $\langle S_1 \cdots S_m \rangle \subset \mathbb{P}^n$ to be the linear variety spanned by subsets $S_1, \ldots, S_m \subset \mathbb{P}^n$. For example, $\langle xy \rangle \subset \mathbb{P}^n$ is the line passing through $x, y \in \mathbb{P}^n$, and $\langle C \rangle \subset \mathbb{P}^n$ is the 2-plane spanned by $C$ for a conic $C \subset \mathbb{P}^n$.

The condition (1) in Theorem 1.1 gives the following basic property for $R_{xy}$.

**Lemma 2.1.** Let $R'_{xy}$ be an irreducible component of $R_{xy}$ such that $\langle C \rangle \not\subset X$ for general $C \in R'_{xy}$. Then

$$\mathcal{U}_{R'_{xy}} \to \text{Loc}(R'_{xy}) \subset X$$

is generically finite; moreover a fiber at $z \in \text{Loc}(R'_{xy})$ is of positive dimension only if $\langle xyz \rangle \subset X$. In particular, $\dim \text{Loc}(R'_{xy}) = \dim(R'_{xy}) + 1$.

**Proof.** If $\dim R'_{xy} = 0$, then the assertion follows immediately. Assume $\dim R'_{xy} \geq 1$. Then $\dim \text{Loc}(R'_{xy}) \geq 2$. Take $C \in R'_{xy}$ and $z \in C$ such that $M := \langle xyz \rangle = \langle C \rangle \not\subset X$. Then $M \cap X$ is a union of finitely many curves. Since any conic $\tilde{C} \in \pi(\text{ev}^{-1}(z) \cap \mathcal{U}_{R'_{xy}})$ satisfies $\tilde{C} \subset M$, it coincides with a component of $M \cap X$. Hence the fiber $\text{ev}^{-1}(z) \cap \mathcal{U}_{R'_{xy}}$ must be a finite set. \qed

We set $Y := \text{Loc}(R) \subset X$, the locus swept out by conics $C \subset R$, which is non-linear since $\langle C \rangle \not\subset X$ for general $C$. Let us consider the projection

$$\text{ev}^{(2)} : \mathcal{U} \times R \mathcal{U} \simeq \{ (C, x, y) \in R \times Y \times Y \mid x, y \in C \} \to Y \times Y$$

whose fiber at $(x, y) \in Y \times Y$ is isomorphic to $R_{xy}$. Considering $\mathcal{U} \times_R \mathcal{U} \to R$, we have $\dim \mathcal{U} \times_R \mathcal{U} = r + 2$. Note that $\text{ev}^{(2)}$ is dominant if and only if $\text{Loc}(R_x) = Y$ holds for general $x \in Y$.

**Remark 2.2.** Assume that $\dim R$ is greater than the expected dimension. Then the locus $Y$ is much smaller than $X$. More precisely, by a result of R. Beheshti [3, Theorem 3.2(b)], it holds that $\dim Y \leq n - 3$.

We immediately have $\dim Y \leq n - 2$; this is because if $X = Y = \text{Loc}(R)$ (i.e., $\dim Y = n - 1$) in characteristic zero, then $R$ has a free curve $C$ and then $R$ must have expected dimension. Beheshti’s result gives the inequity which is sharper than this.
Lemma 2.3. If \( d \leq 6 \) and \( r = \dim R \) is greater than the expected dimension, then \((d, \dim Y) = (6, n - 3)\).

Proof. We have \( \dim Y \leq n - 3 \) due to Beheshti’s result as we saw in Remark 2.2. Let \((C, x, y) \in \mathcal{U} \times_R \mathcal{U}\) be general. Since \( \text{Loc}(R_{xy}) \subset Y \), it follows from the morphism \([2]\) and Lemma 2.1 that \((r + 2 - 2 \dim(Y)) + 1 \leq \dim(Y)\). Hence \( r + 3 - 3 \dim(Y) \leq 0 \). By assumption, \( r \geq 3n - 13 \). Thus

\[
3n - 2d + 2 - 3 \dim(Y) \leq 0.
\]

Since \( \dim Y \leq n - 3 \), we have \( 11 - 2d \leq 0 \); hence \( d = 6 \). Therefore \( n - 10/3 \leq \dim(Y) \); hence \( \dim(Y) = n - 3 \). □

By the above lemma, let us study the case \((d, \dim Y) = (6, n - 3)\), and assume that \( r := \dim R \) is greater than the expected dimension, that is to say,

\[
(3) \quad r \geq 3n - 13.
\]

Lemma 2.4. \( \text{ev}^{(2)} \) is dominant. Therefore \( \text{Loc}(R_x) = Y \) for general \( x \in Y \).

Proof. For general \((x, y) \in \text{im}(\text{ev}^{(2)})\), we have \( \dim R_{xy} = (r + 2) - \dim(\text{im}(\text{ev}^{(2)})) \), which implies \( \dim R_{xy} + \dim(\text{im}(\text{ev}^{(2)})) = (r + 2) \geq 3n - 11 \).

Suppose that \( \text{ev}^{(2)} \) is not dominant, that is, \( \dim(\text{im}(\text{ev}^{(2)})) < 2(n - 3) \). Then \( \text{Loc}(R_{xy}) \neq Y \), which implies \( \dim R_{xy} + 1 < \dim Y = n - 3 \) because of Lemma 2.1. Then \( \dim R_{xy} + \dim(\text{im}(\text{ev}^{(2)})) \leq (n - 5) + (2(n - 3) - 1) = 3n - 12 \), a contradiction. □

Note that for any \( x, y \in Y \)

\[
(4) \quad \dim R_{xy} \geq r + 2 - 2 \dim(Y) \geq n - 5.
\]

Hereafter we will use several projective techniques in order to study \( Y \).

Remark 2.5. We sometimes consider the Gauss map of a variety \( Z \subset \mathbb{P}^n \), which is a rational map

\[
\gamma_Z : Z \dasharrow \mathbb{G}(\dim Z, \mathbb{P}^n),
\]

sending a smooth point \( x \in Z \) to the embedded tangent space \( T_x Z \subset \mathbb{P}^n \) at \( x \). A general fiber of \( \gamma_Z \) is a linear variety of \( \mathbb{P}^n \) in characteristic zero (in particular, irreducible). The map \( \gamma_Z \) is a finite morphism if \( Z \) is smooth. See [15, I, §2].

We write \( (\mathbb{P}^n)^\vee = \mathbb{G}(n - 1, \mathbb{P}^n) \), the space of hyperplanes of \( \mathbb{P}^n \). For a linear subvariety \( A \subset \mathbb{P}^n \), we denote by \( A^* \subset (\mathbb{P}^n)^\vee \) the set of hyperplanes containing \( A \). In addition, for a subset \( B \subset \mathbb{P}^n \), we set \( \text{Cone}_A(B) := \bigcup_{x \in B} \langle A, x \rangle \subset \mathbb{P}^n \), the cone of \( B \) with vertex \( A \).

Let us consider the linear variety \( \langle Y \rangle \subset \mathbb{P}^n \) spanned by the locus \( Y \subset X \). The following proposition states \( \langle Y \rangle \) cannot be so small in \( \mathbb{P}^n \).
Proposition 2.6. Assume \((d, \dim Y) = (6, n - 3)\) and the formula \([3]\). Then \(\langle Y \rangle\) is of dimension \(\geq n - 1\).

Proof. Since \(Y\) is non-linear, we immediately have \(\dim(\langle Y \rangle) > n - 3\). Now assume that \(\langle Y \rangle\) is an \((n - 2)\)-plane. We need to show two claims.

Claim 2.7. It holds that \(Y = \langle Y \rangle \cap X\) and \(\dim(\text{Sing}(Y)) \leq 1\). In particular, 
\(Y \subset \langle Y \rangle = \mathbb{P}^{n-2}\) is a hypersurface whose degree is equal to \(\deg X = d = 6\).

Proof. For \(x \in \langle Y \rangle \cap X\), it holds that \(x \in \text{Sing}(\langle Y \rangle \cap X)\) if and only if \(\langle Y \rangle \subset T_x X\). It means that \(\gamma_X(\text{Sing}(\langle Y \rangle \cap X)) \subset \langle Y \rangle^*\) for the Gauss map \(\gamma_X : X \to (\mathbb{P}^n)^v\), where \(\langle Y \rangle^* \subset (\mathbb{P}^n)^v\) is the set of hyperplanes containing \(\langle Y \rangle\). Since \(X\) is smooth, \(\gamma_X\) is a finite morphism. Since \(\dim(\langle Y \rangle^*) = 1\), we have \(\dim(\text{Sing}(\langle Y \rangle \cap X)) \leq 1\). If there exists an irreducible component \(Y' \subset \langle Y \rangle \cap X \subset \langle Y \rangle = \mathbb{P}^{n-2}\) such that \(Y' \neq Y\), then we have \(\dim(Y' \cap Y) \geq n - 4 \geq 2\), which is a contradiction since \(Y' \cap Y \subset \text{Sing}(\langle Y \rangle \cap X)\). Thus \(\langle Y \rangle \cap X = Y\). \(\square\)

Claim 2.8. A general \(C \in R\) satisfies \(C \cap \text{Sing}Y \neq \emptyset\).

Proof. Suppose that a general conic \(C \in R\) satisfies \(C \cap \text{Sing}Y = \emptyset\). Since \(Y = \text{Loc}(R)\) and the characteristic is zero, \(C\) is free in \(Y\). Then \(R_2(Y)\) is smooth at \(C\) and has the expected dimension \(3(n - 2) - 2\deg Y - 2\). This contradicts that \(R \subset R_2(Y)\) is of dimension \(> 3n - 2d - 2\), where \(\deg Y = \deg X = d\) because of Claim 2.7. \(\square\)

From Claim 2.8, we may indeed assume
\[(5) \quad C \cap S \neq \emptyset\]
for an irreducible component \(S \subset \text{Sing}Y\). Note that \(\dim S \leq 1\).

First we consider the case \(n > 6\). Let \(C_0 \in R\) be a general conic such that \(\langle C_0 \rangle \not\subset X\), and take \(x, y \in C_0 \setminus S\) be general. From \([4]\) we have \(\dim R_{xy} \geq n - 5 \geq 2\). Let \(R_{xy}' \subset R_{xy}\) be an irreducible component containing \(C_0\), and take \((R_{xy}')^0 \subset R_{xy}' \subset X\) to be the set of \(C\) satisfying \(\langle C \rangle \not\subset X\). Then, for any \(C \in (R_{xy}')^0\) and \(s \in C \cap S\), we have \(\langle xys \rangle = \langle C \rangle \not\subset X\). From Lemma 2.1 for the morphism \(\tilde{ev} : \text{ev} \big|_{U(R_{xy}')^0} : U(R_{xy}')^0 \to \text{Loc}(R_{xy}')\), the preimage \(\tilde{ev}^{-1}(S)\) is of dimension \(\leq 1\). Since \(\dim R_{xy}' \geq 2\), we have \(\pi(\tilde{ev}^{-1}(S)) \neq R_{xy}'\), which means that \(C \cap S = \emptyset\) for general \(C \in R_{xy}'\), a contradiction.

Next we consider the case \(n = 6\), and complete the proof in the following four steps. Note that \(\langle Y \rangle = \mathbb{P}^4 \subset \mathbb{P}^6\).

Step 1. We show that \(S \not\subset \langle C_0 \rangle\) for general \(C_0 \in R\), and also show that \(S \not\subset M\) for a general 3-plane \(M \subset \langle Y \rangle\) containing \(C_0\).
Suppose \( S \subset \langle C_0 \rangle \) for general \( C_0 \in R \), and take \( x, y \in Y \) be general points. We can assume \( y \notin \text{Cone}_x(S) \). Since \( \dim R_{xy} \geq n - 5 \geq 1 \), taking general \( C, C' \in R_{xy} \) with \( C \neq C' \), we have \( S \subset \langle C \rangle \cap \langle C' \rangle = \langle xy \rangle \), a contradiction.

If \( S \subset M \) for a general 3-plane \( M \) containing \( C_0 \), then we can also take another general \( \tilde{M} \neq M \), and then \( \langle C_0 \rangle = M \cap \tilde{M} \supset S \), a contradiction.

**Step 2.** We consider \( \langle Y \rangle^\vee := \mathbb{G}(3, \langle Y \rangle) \), the set of 3-planes in \( \langle Y \rangle = \mathbb{P}^4 \). Let

\[
W = \{ (C, M) \in R \times \langle Y \rangle^\vee \mid C \subset M \},
\]

which is a \( \mathbb{P}^1 \)-bundle over \( R \); in particular, \( \dim W \geq 6 \). Let \( \text{pr}_2 : W \to \langle Y \rangle^\vee \) be the projection to the second factor.

For general \((C_0, M) \in W\), take \( R^M \) to be an irreducible component of \( R \cap R_2(M) \) containing \( C_0 \). We may assume that a general conic \( C \in R^M \) satisfies \( \langle C \rangle \nsubseteq X \). Since \( R \cap R_2(M) \simeq \text{pr}_2^{-1}(M) \), we can assume \( \dim R^M \geq 6 - \dim(\text{pr}_2(W)) \). We set the surface

\[
Y^M := \text{Loc}(R^M) \subset Y \cap M.
\]

**Step 3.** Assume \( \dim \text{pr}_2(W) \leq 3 \). Then we have \( \dim R^M \geq 3 \), which implies that \( R^M \to \mathbb{G}(2, M) = (\mathbb{P}^3)^\vee : C \mapsto \langle C \rangle \) is dominant. From Step 1, \( S \cap M \) is a set of finite points. Thus \( L \cap S = \emptyset \) for a general 2-plane \( L \subset M \). Taking a general \( C \in R^M \) such that \( \langle C \rangle = L \), we find that \( C \cap S = \emptyset \), which contradicts the condition \([5]\).

**Step 4.** Assume \( \dim \text{pr}_2(W) = 4 \), that is, \( \text{pr}_2(W) = \langle Y \rangle^\vee \). For general \((C_0, M) \in W\), since \( M \) is general in \( \langle Y \rangle^\vee \), \( Y \cap M \) is irreducible. Then \( Y^M = Y \cap M \subset M = \mathbb{P}^3 \), which is a surface whose degree is equal to \( \deg Y = 6 \).

Since \( S \cap M \) is a finite set, we may assume that there exists \( s \in S \cap M \) such that \( s \in C \) for general \( C \in R^M \). This implies that \( R^M \subset R_s \). Considering \( U_{R^M} \to Y^M \), we find that \( \dim(R^M \cap R_s) \geq 1 \) for general \( x \in Y^M \). Since \( Y^M \) is surface, \( Y^M = \text{Loc}(R^M \cap R_s) = \text{Loc}(R^M \cap R_{xs}) \).

For general \( C \in R^M \), we show that \( \langle C \rangle \cap Y^M \) is scheme-theoretically equal to \( C \), as follows. Write \( (\langle C \rangle \cap Y^M)_{\text{red}} = C \cup \bigcup E_i \) with the irreducible components \( E_i \)'s. Take a general \( x \in C \setminus \bigcup E_i \) and take a general \( y \in E_1 \setminus (C \cup \langle xs \rangle) \). Taking the closure

\[
\overline{R^M \cap R_{xs}} \subset \text{Hilb}^{2t+1}(X)
\]

and consider the surjective map \( U_{R^M \cap R_{xs}} \to Y^M \), we find \( \tilde{C} \in \overline{R^M \cap R_{xs}} \) such that \( x, y, s \in \tilde{C} \). Then \( \langle \tilde{C} \rangle = \langle yxs \rangle = \langle C \rangle \), which implies that \( \tilde{C} \subset Y^M \cap \langle C \rangle \).

By the choice of \( x \), it follows \( C \subset \tilde{C} \). Then \( C = \tilde{C} \), which implies \( y \in C \), a contradiction.
Thus \((\langle C \rangle \cap Y^M)_{\text{red}} = C\). Suppose that \(\langle C \rangle \cap Y^M\) is non-reduced, which means that \(C\) is a contact locus on \(Y^M\) of \(\langle C \rangle\), i.e., \(\gamma(C) = \langle C \rangle \in \mathbb{G}(2, M)\) for the Gauss map

\[
\gamma = \gamma_{Y^M} : Y^M \rightarrow \mathbb{G}(2, M)
\]

sending \(x \mapsto T_x Y^M\). Then \(\dim \gamma(Y^M) = 1\). Since \(Y^M = \text{Loc}(R^M)\), \(\langle C \rangle \in \gamma(Y^M)\) is a general point if so is \(C \in R^M\). As in Remark 2.5, the general fiber \(\gamma^{-1}(\langle C \rangle)\) is a linear variety, which contradicts \(\langle C \rangle \subset \gamma^{-1}(\langle C \rangle)\).

Thus \(\langle C \rangle \cap Y^M = C\) scheme-theoretically. This contradicts \(\deg Y^M = 6\). \(\square\)

3. Special point of a conic: the embedded tangent space at the point containing the conic

We use the notations of §2. From Lemma 2.3, we may assume \((d, \dim Y) = (6, n - 3)\) and the formula (3).

**Lemma 3.1.** Let \((C, x) \in \mathcal{U}\). Then the following holds.

- (a) \(C \not\subset T_x X\) if and only if \(T_x X \cap C = \{x\}\)
- (b) \(C \subset T_x X\) if and only if one of the following conditions holds: (i) \(\langle C \rangle \subset X\); (ii) \(\langle C \rangle \cap X\) is non-reduced along \(C\); (iii) \(x \in C \cap E\) for some irreducible component \(E \neq C\) of \(\langle C \rangle \cap X\).

**Proof.** (a) Since \(X \subset \mathbb{P}^n\) is a hypersurface and \(C\) is a smooth conic, if \(C \not\subset T_x X\), then we have \(T_x X \cap C \subset T_x X \cap (\langle C \rangle \cap C) = T_x C \cap C = \{x\}\).

(b) It is sufficient to consider the case when \(\langle C \rangle \not\subset X\). If \(C \subset T_x X\), then \(\langle C \rangle \subset T_x X\), and then \(\langle C \rangle \cap X\) is singular at \(x\). This means that (ii) or (iii) holds. \(\square\)

For a smooth conic \(C \subset X\), we always have a point \(x \in C\) satisfying \(C \subset T_x X\), as follows. Since \(\deg(X) = 6\), if (i) and (ii) does not hold, then we have a curve \(E \neq C\) in \(\langle C \rangle \cap X\), and have a point \(x \in C \cap E \subset \langle C \rangle = \mathbb{P}^2\) as in (iii).

**Definition 3.2.** We set \(\mathcal{U}^* \subset \mathcal{U}\) to be an irreducible component of \(\{ (C, x) \in \mathcal{U} \mid C \subset T_x X \}\) such that \(\mathcal{U}^* \rightarrow R\) is dominant, and set \(R^*_x := \pi(\text{ev}^{-1}(x) \cap \mathcal{U}^*)\), which consists of conics \(C \subset X\) such that \(x \in C \subset T_x X\). Note that \(\text{Loc}(R^*_x) \subset T_x X\).

**Lemma 3.3.** \(\mathcal{U}^* \neq \mathcal{U}\).

**Proof.** We consider the Gauss map \(\gamma : X \rightarrow (\mathbb{P}^n)^\vee\). If \(\mathcal{U}^* = \mathcal{U}\), then it follows from Lemma 2.4 that \(Y = \text{Loc}(R_x) = \text{Loc}(R^*_x) \subset T_x X\) for general \(x \in Y\), and then \(T_x X = \langle Y \rangle \subset \mathbb{P}^n\) because of Proposition 2.6. Then \(\gamma(Y) = \langle Y \rangle \in (\mathbb{P}^n)^\vee\), which contradicts that \(\gamma\) is a finite morphism as we mentioned in Remark 2.5. \(\square\)

**Lemma 3.4.** Let \(x \neq y \in Y\) satisfy \(y \in T_x X\). Then \(C \subset T_x X\) for any \(C \in R_{xy}\).
Proof. Take \( C \in R_{xy} \) and suppose \( C \not\subset T_x X \). Since \( \langle C \rangle \not\subset T_x X, \langle C \rangle \cap T_x X \) is the line passing through \( x \) and \( y \). On the other hand, we have \( \langle C \rangle \cap T_x X = T_x C \), which does not intersect with any point of \( C \) except \( x \) since \( C \) is a smooth conic, a contradiction. Thus we have \( C \subset T_x X \). \( \square \)

By Lemma 3.3, we may assume that \( U^* \neq U \). Then \( \dim U^* = \dim R = r \). We take the projection

\[
\mu : U^* \times_R U \simeq \{ (C, x, y) \in R \times Y \times Y \mid (C, x) \in U^*, (C, y) \in U \} \to \text{ev}(U^*) \times Y,
\]

such that \( \mu(C, x, y) = (x, y) \), where \( \text{ev} : U \to Y \) is the second projection.

Let \((C, x, y) \in U^* \times_R U \) be general. (Then \( x, y \in C \subset T_x X \).) We can take the unique irreducible component \( R_{xy}' \subset R_x^* \) containing \( C \), and take the unique irreducible component \( R_{xy}^* \subset R_{xy} \) containing \( C \). (The uniqueness comes from the general choice of \( C \).)

Lemma 3.5. In the above setting, we have \( R_{xy}^* \subset R_x^* \).

Proof. It follows from Lemma 3.4. \( \square \)

Moreover, we have the following key proposition, where for the projection, \( \text{im}(\mu) \to \text{ev}(U^*) : (x, y) \mapsto x \), we also consider the following fiber product

\[
\text{im}(\mu) \times_{\text{ev}(U^*)} \text{im}(\mu) \simeq \{ (x, y, z) \in \text{ev}(U^*) \times Y \times Y \mid (x, y), (x, z) \in \text{im}(\mu) \}.
\]

Note that, for an element \((x, y, z)\) of the above set, there exists conics \( C_1, C_2 \subset T_x X \) such that \( x, y \in C_1 \) and \( x, z \in C_2 \). The projection \( \text{im}(\mu) \times_{\text{ev}(U^*)} \text{im}(\mu) \to Y \times Y \) is dominant.

Proposition 3.6. Assume \((d, \dim Y) = (6, n - 3)\) and the formula \([3]\). Then the following holds.

(a) \( \text{Loc}(R_x^*) = \text{Loc}(R_{xy}) \) and the dimension is \( n - 4 \) for general \((C, x, y) \in U^* \times_R U \).

(b) \( \text{Loc}(R_x^*) \) is an \((n - 3)\)-plane of \( \mathbb{P}^n \) for general \((C, x) \in U^* \).

(c) The projection \( U^* \times_R U \times_R U \to \text{im}(\mu) \times_{\text{ev}(U^*)} \text{im}(\mu) \) defined by \((C, x, y, z) \mapsto (x, y, z) \) is dominant.

(d) Let \((C, x, y, z) \in U^* \times_R U \times_R U \) be general (here, \( x, y, z \in C \subset T_x X \)). Then \( \text{Loc}(R_{xy}^*) = \text{Loc}(R_{yz}'), \) where \( R_{yz}' \subset R_{yz} \) is the unique irreducible component containing \( C \).

From (a) and (b), we have that \( \text{Loc}(R_x^*) \) is a hypersurface of \( \text{Loc}(R_x^*) = \mathbb{P}^{n-3} \). Moreover, as a corollary, later we will show that \( \text{Loc}(R_x^*) \) is a quadric hypersurface, and also will see our problem is reduced to the case of \( n = 6 \) (Corollary 3.9).

Remark 3.11.

In order to prove this proposition, we show the following two lemmas.
Lemma 3.7. \( \text{ev}(U^*) \subset Y \) is of dimension \( \geq n - 5 \). Hence \( \text{Loc}(R_x^*) \neq Y \) for general \( x \in \text{ev}(U^*) \).

Proof. Let \((C, x, y) \in U^* \times_R U \) be general such that \( \langle C \rangle \not\subset X \). Considering the morphism \( \mu, R_x^* \cap R_y \subset R_{xy} \) is of dimension \( \geq r + 1 - \dim \text{ev}(U^*) - \dim Y \). From Lemma 2.1, \( r + 2 - \dim \text{ev}(U^*) - \dim Y \leq \dim Y \). Hence \( \dim \text{ev}(U^*) \geq r + 2 - 2 \dim(Y) \), where the right hand side is \( \geq n - 5 \).

Since \( n \geq 6 \), \( \text{ev}(U^*) \) has positive dimension. On the other hand, we have \( \# \{ x \in \text{ev}(U^*) \mid \text{Loc}(R_x^*) = Y \} \leq \infty \) as follows. If \( \text{Loc}(R_x^*) = Y \), then \( Y \subset T_x X \) and then \( \langle Y \rangle = T_x X \) because of Proposition 2.6. Hence, by the finiteness of the Gauss map of smooth \( X \), we have the assertion. \( \square \)

Lemma 3.8. Let \( S \subset \mathbb{P}^n \) be a non-linear projective variety. Assume \( \text{Cone}_x S = \text{Cone}_y S \) for general \( x, y \in S \). Then \( \langle S \rangle \subset \mathbb{P}^n \) is a \( (\dim(S) + 1) \)-plane.

Proof. Take a general point \( x \in S \) and consider the linear projection \( \pi_x : \mathbb{P}^n \rightarrow \mathbb{P}^{n-1} \) from \( x \). Then we have

\[
\pi_x(S) = \pi_x(\text{Cone}_x S) = \pi_x(\text{Cone}_y S) = \text{Cone}_{\pi_x(y)}(\pi_x(S))
\]

for general \( y \in S \). Hence \( \pi_x(S) = \text{Cone}_{y'}(\pi_x(S)) \) for general \( y' \in \pi_x(S) \). This means that \( \pi_x(S) \) is a \( (\dim(S)) \)-plane. Hence \( \text{Cone}_x S \) is a \( (\dim(S) + 1) \)-plane, which implies the assertion. \( \square \)

Proof of Proposition 3.6. (a) We may assume \( \langle C \rangle \not\subset X \). From Lemma 2.1 and the formula \( (4) \), we have \( \dim \text{Loc}(R_{xy}^*) \geq n - 4 \). On the other hand, Lemma 3.7 implies \( \dim \text{Loc}(R_{xy}^*) \leq n - 4 \). Hence Lemma 3.5 implies \( \text{Loc}(R_{xy}^*) = \text{Loc}(R_x^*) \) and the dimension is \( n - 4 \).

(b) From (a), for general \( y \in \text{Loc}(R_x^*) \), it follows \( \text{Loc}(R_x^*) = \text{Loc}(R_{xy}^*) \), where the right hand side is the closure of \( \bigcup_{C \in R_{xy}} \langle C \rangle \subset \mathbb{P}^n \). Since \( \text{Cone}_x C = \langle C \rangle \) is irreducible, it is sufficient to show that, for general \( x \in \text{ev}(U^*) \) and for general \( y, z \in \text{Loc}(R_x^*) \), there exists \( C \) such that \( \langle C, x, y, z \rangle \in U^* \times_R U \times_R U \).

(c) Since \( \text{im}(\mu) \times_{\text{ev}(U^*)} \text{im}(\mu) \) is irreducible, it is sufficient to show that, for general \( x \in \text{ev}(U^*) \) and for general \( y, z \in \text{Loc}(R_x^*) \), there exists \( C \) such that \( \langle C, x, y, z \rangle \in U^* \times_R U \times_R U \).

First we can take a general \( \langle C_0, x, y \rangle \in U^* \times_R U \) with some \( C_0 \in R_x^* \). From (a), we have \( \text{Loc}(R_{xy}^*) = \text{Loc}(R_{xy}^*) \). Since \( z \in \text{Loc}(R_x^*) \) is general, we have a general \( C \in R_{xy}^* \) such that \( z \in C \).

(d) Consider the projection

\[
\text{pr}_{34} : U^* \times_R U \times_R U \rightarrow Y \times Y
\]
sending \((C, x, y, z) \mapsto (y, z)\). Let \(F_{yz}\) be an irreducible component of the fiber of \(\text{pr}_{34}\) at a general \((y, z) \in Y \times Y\). We identify \(F_{yz}\) and its image in \(U^*\) under the projection \(\text{pr}_{12} : (C, x, y, z) \mapsto (C, x)\).

Let us consider
\[
\bigcup_{(C, x) \in F_{yz} : \text{general}} \text{Loc}(R_x^{*'}) \subset \bigcup_{(C, x) \in F_{yz} : \text{general}} \langle \text{Loc}(R_x^{*'}) \rangle.
\]

Suppose that the closure of the left hand side is equal to \(Y\). Then (b) implies
\[
\text{Cone}_y Y = \text{Cone}_y \bigcup_{(C, x) \in F_{yz} : \text{general}} \text{Loc}(R_x^{*'}) = \bigcup_{(C, x) \in F_{yz} : \text{general}} \langle \text{Loc}(R_x^{*'}) \rangle.
\]

In the same way, \(\text{Cone}_z Y = \bigcup_{(C, x) \in F_{yz} : \text{general}} \langle \text{Loc}(R_x^{*'}) \rangle\). Hence \(\text{Cone}_y Y = \text{Cone}_z Y\). Since \(y, z \in Y\) are general, Lemma 3.8 implies that \(\langle Y \rangle \subset \mathbb{P}^n\) is an \((n-2)\)-plane, which contradicts Proposition 2.6.

Hence the closure of \(\bigcup_{(C, x) \in F_{yz} : \text{general}} \text{Loc}(R_x^{*'})\) is not equal to \(Y\). Since \(\text{Loc}(R_x^{*'})\) is of codimension 1 in \(Y\), it means that \(L := \text{Loc}(R_x^{*'})\) is constant for general \((C, x) \in F_{yz}\). Then \(\text{Loc}(R_{yz}^{*'}) = \bigcup_{(C, x) \in F_{yz} : \text{general}} C \subset L\). Since \(\dim \text{Loc}(R_{yz}^{*'}) \geq n-4\), the assertion follows.

**Corollary 3.9.** Assume \((d, \dim Y) = (6, n-3)\) and the formula \([3]\). Then the following holds.

(a) For general \((C, x) \in U^*\), \(\text{Loc}(R_x^{*'}) \subset \langle \text{Loc}(R_x^{*'}) \rangle = \mathbb{P}^{n-3}\) is a quadric hypersurface.

(b) The projection to the second factor
\[
q : \{ (C, H) \in R \times (\mathbb{P}^n)^\vee \mid C \subset H \} \rightarrow (\mathbb{P}^n)^\vee
\]
is dominant. (In particular, a general fiber of \(q\) is of dimension \(r-3\).)

**Proof.** (a) Let \(y, z \in \text{Loc}(R_x^{*'})\) be general. Let \(M \subset \langle \text{Loc}(R_x^{*'}) \rangle\) be a general 2-plane such that \(y, z \in M\).

From Proposition 3.6(c), \((C_0, x, y, z)\) is general in \(U^* \times_R U \times_R U\) with some \(C_0\). From Proposition 3.6(d), it holds that \(\text{Loc}(R_{yz}^{*'}) = \text{Loc}(R_x^{*'})\). We consider
\[
R_{yz}^{*'} \rightarrow \langle yz \rangle^* : C \mapsto \langle C \rangle,
\]
which is generically finite, where \(\langle yz \rangle^* \subset G(2, \langle \text{Loc}(R_x^{*'}) \rangle)\) is the set of 2-planes containing the line \(\langle yz \rangle\). Since \(\dim R_{yz}^{*'} \geq n-5 = \dim \langle yz \rangle^*\), this morphism is dominant.
Thus we can take a general $\tilde{M} \in \langle yz \rangle^* \subset \mathbb{C}^2, \langle \text{Loc}(R^*_y) \rangle$ near $M$ such that $\tilde{M} = \langle \tilde{C} \rangle$ for some $\tilde{C} \in R^*_y$. By generality, $\tilde{M} \cap \text{Loc}(R^*_x)$ is irreducible. Hence $\tilde{C} = \tilde{M} \cap \text{Loc}(R^*_x)$, which means that $\deg(\text{Loc}(R^*_x)) = 2$.

(b) For $M$ in (a) above, we can take a hyperplane $H \subset \mathbb{P}^n$ containing $\tilde{M}$ as a general element of $(\mathbb{P}^n)^\vee$, and then $(\tilde{C}, H) \in R \times (\mathbb{P}^n)^\vee$, which means that $q$ is dominant.

Considering the projection to the first factor, we find that the dimension of the left hand side of $q$ is $r + n - 3$. Thus a general fiber of $q$ is of dimension $r - 3$. □

**Remark 3.10.** It is known that if a smooth hypersurface in $\mathbb{P}^n$ of degree $> 2$ contains an $m$-dimensional quadric hypersurface, then $m \leq (n - 1)/2$. Thus Corollary 3.9(a) implies $n \leq 7$.

**Remark 3.11.** By induction on $n$, it is sufficient to show Theorem 1.1 in the case of $n = 6$. We see the details in the following. Let $n > 6$ and let $R \subset R_2(X)$ be an irreducible component.

Assume $(d, \dim Y) = (6, n - 3)$ and the formula (3) i.e., $r := \dim R$ is greater than the expected one. Since $q$ is dominant as in Corollary 3.9(b), a fiber of $q$ at general $H \in (\mathbb{P}^n)^\vee$, which identified with $R \cap R_2(X \cap H)$, is of dimension $r - 3$. Again since $q$ is dominant, we may take a conic $C \subset H$ as a general member of $R$. Then we may take an irreducible component $R'$ of $R_2(X \cap H)$ containing $C$ such that $\dim R' \geq r - 3 \geq 3n - 16$. Since $C$ satisfies $\langle C \rangle \not\subset T_x X$, we have $\langle C \rangle \not\subset T_x X \cap H = T_x (X \cap H)$. Then a general $\tilde{C} \in R'$ satisfies $\langle \tilde{C} \rangle \not\subset T_x (X \cap H)$.

Since $H$ is general, $X \cap H \subset H = \mathbb{P}^{n-1}$ is smooth. Once Theorem 1.1 is proved for $n - 1$, we have a contradiction since $\dim R'$ must be equal to the expected dimension $3(n - 1) - 14$.

Using the above results and notations, we now prove the main theorem.

**Proof of Theorem 1.1.** Let $R \not= \emptyset$ be an irreducible component of $R_2(X)$ satisfying the condition (1) and assume that $r := \dim R$ is greater than the expected dimension. From Lemma 2.3, Corollary 3.9(b), and Remark 3.11 we may assume $n = d = 6$ and $\dim Y = 3$. Then $r \geq 5$.

**Claim 3.12.** $\deg(Y) = 3$ or $4$ and $\dim \langle Y \rangle \geq 5$.

**Proof.** From Proposition 2.6 we have $\dim \langle Y \rangle \geq 5$. In particular, $\deg(Y) \geq 3$. Let $H \subset \mathbb{P}^6$ be a general hyperplane, and set $Y' = Y \cap H$. It follows from Corollary 3.9(b) again that $R \cap R_2(Y')$ is of dimension $\geq 2$. It is classically known that, if a surface $Y'$ has a 2-dimensional family of conics, then $Y'$ is projectively equivalent to either the Veronese surface $\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$ or its image under linear projections (see [13, p. 130, p. 157]). Hence $\deg(Y) = \deg(Y') \leq 4$. □

Our first goal is to show that $\langle Y \rangle \subset \mathbb{P}^6$ is of dimension 5. Set $Q_{yz} \subset Y$ to be the surface $\text{Loc}(R^*_y)$ for general $(C, y, z) \in U \times_R U$. 

Claim 3.13. $Q_{yz}$ is a quadric surface, and $Q_{yz} = Q_{\tilde{y}}$ holds for general $(\tilde{C}, \tilde{y}) \in U_{R_{yz}}$.

Proof. Since $(\tilde{C}, y, z, \tilde{y})$ is general in $U \times_R U \times_R U$, $R'_{yz}$ (resp. $R_{yz}$) is the unique irreducible component of $R_{yz}$ (resp. $R_{yz}$) containing $\tilde{C}$. We can take $x \in \tilde{C}$ such that $(\tilde{C}, x)$ is general in $U^\ast$. Then [Proposition 3.6(d)] implies $Q_{yz} = \text{Loc}(R^\ast_{\tilde{y}}) = Q_{\tilde{y}}$, which is a quadric as in Corollary 3.9.

For general $y, z_1, z_2 \in Y$ (with some $C_i$ such that $(C_i, y, z_i)$ is general in $U \times_R U$), we write

$$K_{yz}^{z_1z_2} := T_yQ_{yz_1} \cap T_yQ_{yz_2} \subset T_yY = \mathbb{P}^3,$$

whose dimension is $\geq 1$. Then $y \in K_{yz}^{z_1z_2} \subset \langle Q_{yz_1} \rangle \cap \langle Q_{yz_2} \rangle$. (Here we do not know “$z_i \in K_{yz}^{z_1z_2}$”.)

Claim 3.14. $\dim(\langle Q_{yz_1} \rangle \cap \langle Q_{yz_2} \rangle) = 1$. Hence $K_{yz}^{z_1z_2} = \langle Q_{yz_1} \rangle \cap \langle Q_{yz_2} \rangle$.

Proof. Suppose $\dim(\langle Q_{yz_1} \rangle \cap \langle Q_{yz_2} \rangle) \geq 2$ for general $y, z_1, z_2 \in Y$. First we take general points $y_0, z_1, z_2 \in Y$. By generality, $z_1 \notin \langle Q_{yz_2} \rangle$. For general $y \in Q_{yz_2}$, we have $\langle Q_{yz_2} \rangle = \langle Q_{y^0z_2} \rangle$ because of Claim 3.13. Consider an open subset $Y^0 \subset Y$ containing $y_0$ such that $\dim(\langle Q_{yz_1} \rangle \cap \langle Q_{yz_2} \rangle) \geq 2$ for $y \in Y^0$.

Since $Y = \bigcup_{y \in Q_{yz_2} \cap Y^0} Q_{yz_2}$, we have

$$\text{Cone}_{z_1} Y = \bigcup_{y \in Q_{yz_2} \cap Y^0} \langle Q_{yz_1} \rangle,$$

where the right hand side contains $Q_{yz_2}$; in fact, it contains the 3-plane $\langle Q_{yz_2} \rangle$ since each $\langle Q_{yz_1} \rangle$ satisfies $\dim(\langle Q_{yz_1} \rangle \cap \langle Q_{yz_2} \rangle) \geq 2$. Hence $\text{Cone}_{z_1} Y = \text{Cone}_{z_1} \langle Q_{yz_2} \rangle$, which is a 4-plane, a contradiction to $\dim(Y) \geq 5$.

Claim 3.15. $K_{yz}^{z_1z_2} \subset Q_{yz_1}$. Hence there exists an irreducible component $K_{yz}^{z_1}$ of $Q_{yz_1} \cap T_yQ_{yz_2} \subset \langle Q_{yz_1} \rangle$ such that $K_{yz}^{z_1} = K_{yz}^{z_1z_2}$ for general $z_2 \in Y$.

Proof. Suppose $K_{yz}^{z_1z_2} \subset Q_{yz_1}$. Let $z, w \in Y$ be general such that $w \notin Q_{yz}$. For general $\tilde{y} \in Q_{yz}$, we have $Q_{yz} = Q_{\tilde{y}}$.

Since $Y$ is the closure of $\bigcup_{\tilde{y} \in Q_{yz} : \text{general}} Q_{\tilde{y}w}$, we have that $\text{Cone}_w Y$ is the closure of $\bigcup_{\tilde{y} \in Q_{yz} : \text{general}} \langle Q_{\tilde{y}w} \rangle$. Since $\tilde{y} \in K_{\tilde{y}w}^{z_1}$, we have $Q_{yz} \subset \bigcup_{\tilde{y} \in Q_{yz} : \text{general}} K_{\tilde{y}w}^{z_1}$. Moreover, since $Q_{yz}$ is codimension one in $\langle Q_{yz} \rangle$, and since $K_{\tilde{y}w}^{z_1} \subset \langle Q_{yz} \rangle$, we have

$$\langle Q_{yz} \rangle \subset \bigcup_{\tilde{y} \in Q_{yz} : \text{general}} K_{\tilde{y}w}^{z_1}.$$
and the right hand side is of dimension 4. It holds that Cone, Y is a 4-plane, a contradiction.

Note that, since \( K_y^{z_1 + z_2} \) is contained in \( Q_{yz_1} \cap \mathbb{T}_y Q_{yz_1} \subset \langle Q_{yz_1} \rangle \), there exists an irreducible component of \( Q_{yz_1} \cap \mathbb{T}_y Q_{yz_1} \) which is equal to \( K_y^{z_1 + z_2} \) for general \( z_2 \in Y \). Hence the latter statement holds. \( \square \)

**Claim 3.16.** \( \langle Y \rangle \subset \mathbb{P}^6 \) is of dimension 5.

**Proof.** Let \( y, z_1 \in Y \) be general. Since \( Y \) is the closure of \( \bigcup_{z_2 \in Y; \text{ general}} Q_{yz_2} \), it follows that Cone, \( Y \) is the closure of \( \bigcup_{z_2 \in Y; \text{ general}} \langle Q_{yz_2} \rangle \). Since \( K_y^{z_1} = K_y^{z_1 + z_2} \subset \langle Q_{yz_2} \rangle \) for general \( z_2 \in Y \), we have that Cone, \( Y \) is a cone with vertex \( K_y^{z_1} = \mathbb{P}^1 \).

Let \( \tilde{y}, \tilde{z}_1 \in Y \) be general. We may assume \( \tilde{y} \notin K_y^{z_1} \) and \( y \notin K_y^{z_1} \). We show the statement in the following two steps.

**Step 1.** Suppose that two lines \( K_y^{z_1} \) and \( K_y^{z_2} \) intersect at a point \( v \). Then, for general \( s, t \in Y \), the line \( K_s^t \) also intersects with each of \( K_y^{z_1} \) and \( K_y^{z_2} \). If \( v \notin K_s^t \), then \( s \in K_s^t \subset \langle K_y^{z_1}, K_y^{z_2} \rangle \); hence we have \( Y \subset \langle K_y^{z_1}, K_y^{z_2} \rangle = \mathbb{P}^2 \), a contradiction.

If \( v \in K_s^t \), then since \( s \in Y \) is general and \( \overline{sv} = K_s^t \subset Y \), it follows that \( Y \) is a cone with vertex \( v \); hence \( Y \subset \mathbb{T}_v X \), which implies \( \langle Y \rangle = \mathbb{T}_v X = \mathbb{P}^5 \).

**Step 2.** Suppose \( K_y^{z_1} \cap K_y^{z_2} = \emptyset \). We have \( K_y^{z_1} = K_y^{z_1 + \tilde{y}} \subset Q_{\tilde{y}y} \subset \langle Q_{\tilde{y}y} \rangle \). In the same way, \( K_y^{z_2} \subset Q_{\tilde{y}y} \). Since \( K_y^{z_1} = \langle Q_{\tilde{y}z_1} \rangle \cap \langle Q_{\tilde{y}y} \rangle \), we have

\[ K_y^{z_1} \cap \langle Q_{\tilde{y}z_1} \rangle = K_y^{z_1} \cap K_y^{z_2} = \emptyset. \]

For the linear projection \( \pi_y : \mathbb{P}^6 \dashrightarrow \mathbb{P}^5 \), we consider \( \pi_y(\langle Y \rangle) = \pi_y(\text{Cone}, Y) \), a cone with vertex \( w := \pi_y(K_y^{z_1}) \). Since \( w \notin \pi_y(\langle Q_{\tilde{y}z_1} \rangle) \) and since \( \pi_y(\langle Q_{\tilde{y}z_1} \rangle) \) is of codimension 1 in \( \pi_y(\langle Y \rangle) \), it follows that \( \pi_y(\langle Y \rangle) \) is a cone of the quadric \( \pi_y(\langle Q_{\tilde{y}z_1} \rangle) \) with vertex \( w \). Then \( \langle \pi_y(Y) \rangle = \mathbb{P}^4 \). Since \( y \in Y \) is general, it follows that \( Y \) is a 3-fold of degree 3 in \( \langle Y \rangle = \mathbb{P}^5 \). \( \square \)

Let us complete the proof. By the above claim, \( \langle Y \rangle = \mathbb{P}^5 \). Take \( X' := X \cap \langle Y \rangle \). Since the Gauss map \( \gamma = \gamma_X : X \rightarrow (\mathbb{P}^6)^{\vee} \) is finite, \( X' \) is singular at most finitely many points \( (X')' \) is singular at \( x \) if and only if \( \gamma(x) \in \langle Y \rangle \subset (\mathbb{P}^6)^{\vee} \). In particular, \( X' \) is irreducible. (This is because, if \( (X')' = X'_1 \cup X'_2 \subset \langle Y \rangle = \mathbb{P}^5 \), then \( X'_1 \cap X'_2 \subset \text{Sing} X' \).) Here \( Y \subset X' \subset \langle Y \rangle \).

Take a general hyperplane \( M \subset \langle Y \rangle = \mathbb{P}^5 \) such that \( X'' := X' \cap M \) is smooth. Then we have

\[ Y \cap M \subset X'' \subset M = \mathbb{P}^4, \]

where \( Y \cap M \) is a surface of degree \( \leq 4 \) as in Claim 3.12 and \( X'' \) is a smooth 3-fold of degree 6 in \( \mathbb{P}^4 \). This is a contradiction since \( \text{Pic}(\mathbb{P}^4) \simeq \mathbb{Z} \rightarrow \text{Pic}(X'') : \mathcal{O}_\mathbb{P}(1) \rightarrow \mathcal{O}_\mathbb{P}(1)|_{X''} \) is isomorphic due to the Lefschetz theorem. \( \square \)
Example 3.17. Let $X \subset \mathbb{P}^7$ be the Fermat hypersurface of degree 6. Then $\mathbb{P}^3 \subset X$. Thus $R_2(\mathbb{P}^3) \subset R_2(X)$ is of dimension 8, and the expected dimension of $R_2(X)$ is $3n - 2d - 2 = 7$. For a general hyperplane $\mathbb{P}^6 \subset \mathbb{P}^7$, we have $\mathbb{P}^2 \subset X_1 := X \cap \mathbb{P}^6$. Then $R_2(\mathbb{P}^2) \subset R_2(X_1)$ is of dimension 5 and the expected dimension of $R_2(X_1)$ is 4.

In these examples, each $C \in R_2(\mathbb{P}^3)$ (resp. $C \in R_2(\mathbb{P}^2)$) satisfies $\langle C \rangle \subset \mathbb{P}^3 \subset X$ (resp. $\langle C \rangle = \mathbb{P}^2 \subset X_1$).

Example 3.18. Let $X \subset \mathbb{P}^{10}$ be the smooth hypersurface of degree 10 defined by the following polynomial,

$$f := x_0^8(x_0^2 + x_1^2 + x_2^2) + \sum_{i=1}^{5} x_i^{10} - \sum_{i=1}^{5} x_{i+5}^{10}.$$ 

Then the expected dimension of $R_2(X)$ is 8. We consider the 5-plane

$$M := \bigcap_{i=1}^{5} (x_i - x_{i+5} = 0) = \mathbb{P}^5 \subset \mathbb{P}^{10},$$

and take $Y \subset X \cap M$, to be the zero set of $x_0^2 + x_1^2 + x_2^2$ in $M$. Since $Y$ is a cone of the conic $(x_0^2 + x_1^2 + x_2^2 = 0) \subset \mathbb{P}^2$, we have a birational map $R_2(Y) \to G(2, \mathbb{P}^5) : C \mapsto \langle C \rangle$. In particular, $\dim R_2(Y) = 9$. Thus, for an irreducible component $R \subset R_2(X)$ containing $R_2(Y)$, the dimension of $R$ is greater than the expected dimension.

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