Photovoltaic Inverter Controllers Seeking AC Optimal Power Flow Solutions

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Abstract—This paper considers future distribution networks featuring inverter-interfaced photovoltaic (PV) systems, and addresses the synthesis of feedback controllers that seek real- and reactive-power inverter setpoints corresponding to AC optimal power flow (OPF) solutions. The objective is to bridge the temporal gap between long-term system optimization and real-time inverter control, and enable seamless PV-owner participation without compromising system efficiency and stability. The design of the controllers is grounded on a dual \( \varepsilon \)-subgradient method, and semidefinite programming relaxations are advocated to bypass the non-convexity of AC OPF formulations. Global convergence of inverter output powers is analytically established for diminishing stepsize rules and strictly convex OPF costs for cases where: i) computational limits dictate asynchronous updates of the controller signals, and ii) inverter reference inputs may be updated at a faster rate than the power-output settling time.

I. INTRODUCTION

Present-generation residential photovoltaic (PV) inverters typically operate in a distributed and uncoordinated fashion, with the primary objective of maximizing the power extracted from PV arrays. With the increased deployment of behind-the-meter PV systems, an upgrade of medium- and low-voltage distribution-system operations and controls is required to address emerging efficiency, reliability, and power-quality concerns. To this end, several architectural frameworks have been proposed for PV-dominant distribution systems to broaden the objectives of inverter real-time control, and enable inverters to partake in distribution-network optimization tasks [1]–[5].

Past works have addressed the design of decentralized real-time inverter-control strategies to regulate the delivery of real and reactive power based on local measurements, so that terminal voltages are within acceptable levels [1], [2]. On a different time scale, optimal power flow (OPF) formulations have been proposed to compute optimal steady-state inverter setpoints, so that power losses and voltage deviations are minimized and economic benefits to end-users providing ancillary services are maximized [3]–[5].

In an effort to bridge the temporal gap between real-time control and network-wide steady-state optimization, this paper addresses the synthesis of feedback controllers that seek optimal PV-inverter power setpoints corresponding to AC OPF solutions. The guiding motivation is to ensure that PV-system operation and control strategies are adaptable to changing ambient conditions and loads, and enable seamless end-user participation without compromising system efficiency.

Prior efforts in this direction include continuous-time feedback controllers that seek Karush-Kuhn-Tucker conditions for optimality developed in [6], and applied to solve an economic dispatch problem for bulk power systems in [7]. Recently, modified automatic generation and frequency control methods that incorporate optimization objectives corresponding to DC OPF problems have been proposed for lossless bulk power systems in e.g., [8]–[10]. A heuristic based on saddle-point-flow methods is utilized in [11] to synthesize controllers seeking AC OPF solutions. Strategies that integrate economic optimization within droop control for islanded lossless microgrids are developed in [12]. In a nutshell, these approaches are close in spirit to the seminal work [13], where dynamical systems that serve as proxies for optimization variables and multipliers are synthesized to evolve in a continuous-time gradient-like fashion to the saddle points of the Lagrangian function associated with a convex optimization problem. For DC OPF, a heuristic comprising continuous-time dual ascent and discrete-time reference-signal updates is proposed in [14]; where, local stability of the resultant closed-loop system is also established.

Distinct from past efforts [7]–[12], [14], this work leverages dual \( \varepsilon \)-subgradient methods [15], [16], to develop a feedback controller that steers the inverter output powers towards the solution of an AC OPF problem. A semidefinite programming (SDP) relaxation is advocated to bypass the non-convexity of the formulated AC OPF problem [5], [17], [18]. The proposed scheme involves the update of dual and primal variables in a discrete-time fashion, with the latter constituting the reference-input signals for the PV inverters. Convergence of PV-inverter-output powers to the solution of the formulated OPF problem is analytically established for settings where: i) in an effort to bridge the time-scale separation between optimization and control, the reference inputs may be updated at a faster rate than the power-output settling time; and, ii) due to inherent computational limits related to existing SDP solvers, the controller signals are updated asynchronously.

Overall, the proposed framework considerably broadens the approaches of [7]–[12], [14] by: i) considering AC OPF setups; ii) incorporating PV-inverter operational constraints; iii) accounting for communication constraints which naturally lead to discrete-time controller updates; and, iv) accounting for computational limits which involves an asynchronous update.
of the controller signals. It is also shown that the controller affords a decentralized implementation, and requires limited message passing between the PV systems and the utility.

The remainder of this paper is organized as follows. Section II outlines the problem formulation, while the PV controller is developed in Section III. Numerical tests are reported in Section IV and conclusions are provided in Section V.

II. PROBLEM FORMULATION

Dynamical models and relevant formulations for optimizing inverter setpoints are outlined for a general networked dynamical system in Section II-A, and tailored to real-time PV-inverter control in Section II-B.

A. General problem setup

Consider $N_D$ dynamical systems described by

$$\begin{align*}
\dot{x}_i(t) &= f_i(x_i(t), d_i(t), u_i(t)) \quad (1a) \\
y_i(t) &= r_i(x_i(t), d_i(t)) \quad (1b)
\end{align*}$$

where: $x_i(t) \in \mathbb{R}^{n_{x,i}}$ is the state of the $i$-th dynamical system at time $t$; $y_i(t) \in \mathcal{Y}_i \subset \mathbb{R}^{n_{y,i}}$ is the measurement of state $x_i(t)$ at time $t$; $u_i(t) \in \mathcal{U}_i$ is the reference input; and $d_i(t) \in D_i \subset \mathbb{R}^{n_{d,i}}$ is the exogenous input. Finally, $f_i : \mathbb{R}^{n_{x,i}} \times \mathbb{R}^{n_{x,i}} \times \mathbb{R}^{n_{d,i}} \rightarrow \mathbb{R}^{n_{x,i}}$, and $r_i : \mathbb{R}^{n_{x,i}} \times \mathbb{R}^{n_{d,i}} \rightarrow \mathbb{R}^{n_{y,i}}$ are arbitrary (non)linear functions. The dynamical system behavior for given finite exogenous inputs and reference signals is assumed.

**Assumption 1:** For given constant exogenous inputs $\{d_i \in D_i\}_{i \in N_D}$ and reference signals $\{u_i \in \mathcal{U}_i\}_{i \in N_D}$, there exist equilibrium points $\{x_i\}_{i \in N_D}$ for (1) that satisfy:

$$\begin{align*}
0 &= f_i(x_i, d_i, u_i) \quad (2a) \\
u_i &= r_i(x_i, d_i), \quad i \in N_D. \quad (2b)
\end{align*}$$

Notice that in (2), the equilibrium output coincides with the commanded input $u_i$; that is, $y_i = u_i$. These equilibrium points are locally asymptotically stable [19].

For given exogenous inputs $\{d_i \in D_i\}_{i \in N_D}$, consider the following optimization problem:

$$\begin{align*}
\text{(P1)} \quad \min_{\mathbf{v} \in \mathcal{V}, \{u_i \in \mathcal{U}_i\}} & \quad H(\mathbf{V}) + \sum_{i \in N_D} \left( \frac{1}{2} u_i^T \mathbf{A}_i u_i + b_i^T u_i \right) \quad (3a) \\
\text{subject to} & \quad h_i(\mathbf{V}) + g_i(u_i, d_i) = 0, \quad \forall i \in N_D \quad (3b)
\end{align*}$$

Notation. Upper-case (lower-case) boldface letters will be used for matrices (column vectors); $\mathbf{1}$ for transposition; $\mathbf{1}^H$ complex-conjugate; and, $\|\cdot\|$ the Euclidean norm of a number or the cardinality of a set; vec$(\mathbf{x})$ returns a vector stacking the columns of matrix $\mathbf{x}$, and $\text{blkdiag}\{\mathbf{x}_i\}$ forms a block-diagonal matrix. $\mathbb{R}^n$ and $\mathbb{C}^n$ denote the spaces of $n \times 1$ real-valued and complex-valued vectors, respectively; $n$ the set of natural numbers; and, $\mathbb{R}^{n \times N}$ denotes the space of $N \times n$ positive semidefinite Hermitian matrices. Given vector $\mathbf{x}$ and square matrix $\mathbf{X}$, $||\mathbf{x}||_2$ denotes the Euclidean norm of $\mathbf{x}$, and $||\mathbf{X}||_2$ the (induced) spectral norm of matrix $\mathbf{X}$. $[\mathbf{x}]_i$, $((\mathbf{x})_i)$ points to the $i$-th element of a vector $\mathbf{x}$ (vector-valued function $f(\mathbf{x})$, $\mathbf{x}(t)$ is the time derivative of $\mathbf{x}(t)$. Given a scalar function $f(x) : \mathbb{R}^n \rightarrow \mathbb{R}$, $\nabla f(x)$ returns the gradient $\mathbf{1}_n \otimes \nabla f(x)$, and $\Delta$. For a continuous function $f(t)$, $f(t)$ denotes its value sampled at $t_k$. Finally, $\mathbf{1}_N$ denotes the $N \times 1$ identity matrix; and, $\mathbf{O}_{M \times N}, \mathbf{1}_{M \times N}$ the $M \times N$ matrices with all zeroes and, respectively.

where $\mathcal{V} \subset \mathcal{H}^{n_{V} \times n_{V}}$ is a convex, closed, and bounded subset of the cone of positive semidefinite (Hermitian) matrices; function $H(\mathbf{V}) : \mathcal{H}^{n_{V} \times n_{V}} \rightarrow \mathbb{R}$ is known, strictly convex and finite over $\mathcal{V}$; $\mathbf{A}_i \succeq 0$ and $\mathbf{b}_i \in \mathbb{R}^{n_{y,i}}, \forall i \in N_D$; the vector-valued function $h_i(\mathbf{V}) : \mathcal{H}^{n_{V} \times n_{V}} \rightarrow \mathbb{R}^{n_{y,i}}$ is affine; and, $g_i(u_i, d_i) : \mathbb{R}^{n_{x,i}} \times n_{d,i} \rightarrow \mathbb{R}^{n_{y,i}}$ takes the form $g_i(u_i, d_i) = \mathbf{C}_i u_i + \mathbf{D}_i d_i$, with $\mathbf{C}_i \in \mathbb{R}^{n_{x,i} \times n_{y,i}}$ and $\mathbf{D}_i \in \mathbb{R}^{n_{x,i} \times n_{d,i}}$ known. Finally, sets $\{\mathcal{Y}_i\}_{i \in N_D}$, which define the space of possible reference inputs for the dynamical systems, are assumed to comply to the following requirement.

**Assumption 2:** Sets $\{\mathcal{Y}_i\}_{i \in N_D}$ are convex and compact. Further, (P1) has a non-empty feasible set and a finite optimal cost.

With these assumptions, problem (P1) is a convex program; moreover, it can be reformulated into a standard SDP form by resorting to the epigraph form of the cost function.

It is evident from [25] that (P1) defines the optimal operating setpoints of the dynamical systems (1) in terms of steady-state outputs [6], [14]. In fact, by utilizing the optimal solution $\{u_i^\text{opt}\}_{i \in N_D}$ of (P1) as reference inputs, it follows from [25] that each system output will eventually be driven to the point $y_i = u_i^\text{opt}$. Function (3a) captures costs incurred by the steady-state outputs, as well as costs associated with matrix variable $\mathbf{V}$, which couples the steady-state system outputs $y_i = u_i^\text{opt}$ through the linear equality constraints (3b).

In principle, (P1) could be solved centrally by a system-level control unit, which subsequently dispatches the reference signals $\{u_i^\text{opt}\}_{i \in N_D}$, for the dynamical systems. In lieu of a centralized solution of (P1), the objective here is to design a decentralized feedback controller for the dynamical systems (1), so that the resultant closed-loop system is globally convergent to an equilibrium point $\{x_i\}_{i \in N_D}, \{y_i = r_i(x_i, d_i)\}_{i \in N_D}$, where the values $\{y_i\}_{i \in N_D}$ of the steady-state outputs coincide with the optimal solution $\{u_i^\text{opt}\}_{i \in N_D}$ of (P1).

B. PV-inverter output regulation to OPF solutions

The task of regulating the power output of PV inverters is outlined in this section, and cast within the framework of [1]-[5]. In this regard, (1)-(2) will model the inverter dynamics [20, Ch. 8], [21]; while OPF will be formulated in the form [3] by leveraging SDP relaxation techniques [17], [18].

Network. Consider a distribution system comprising $N + 1$ nodes collected in the set $\mathcal{N}$, and lines represented by the set of undirected edges $\mathcal{E} := \{(m, n) : m, n \in \mathcal{N}\}$. The set $\mathcal{N} := \{0, 1, \ldots, N\}$ is partitioned as $\mathcal{N} = \{0\} \cup \mathcal{N}_D \cup \mathcal{N}_O$, where node 0 denotes the secondary of the step-down transformer; inverter-interfaced PV systems are located at nodes $\mathcal{N}_D = \{1, \ldots, N\}$ [cf. (1)]; and, $\mathcal{N}_O := \{N + 1, \ldots, N\}$ collects nodes with no power generation. For simplicity of exposition, and similar to e.g., [4], [8], [9], [22], assume that the system is balanced; however, the proposed framework can be readily extended to unbalanced multi-phase systems [5].

Dynamics of PV inverters. Equation (1a) is utilized to model the dynamics of PV inverters, regulating real- and reactive output powers to prescribed setpoints. For example, relevant dynamical models for inverters operating in a grid-connected
mode are discussed in e.g., [20] Ch. 8] and [21]. These models can be conveniently cast within [1]–[2] as shown next.

Let \( p_i(t) := E_i(t) \cos(\omega t + \phi_i(t))i_i(t) \) and \( q_i(t) := E_i(t) \cos(\omega t + \phi_i(t) - \pi/2)i_i(t) \) denote the instantaneous output real and reactive powers of inverter \( i \in N_D \), respectively, where \( \omega \) is the grid frequency, \( v_i(t) := E_i(t) \cos(\omega t + \phi_i(t)) \) the voltage waveform, and \( i_i(t) \) is the current injected. Further, let \( P_i(t) \) and \( Q_i(t) \) denote averages of the instantaneous output real and reactive powers over an AC cycle; that is,

\[
P_i(t) := \frac{\omega}{2\pi} \int_{t-\pi/\omega}^{t} p_i(\tau)d\tau, \quad Q_i(t) := \frac{\omega}{2\pi} \int_{t-\pi/\omega}^{t} q_i(\tau)d\tau.
\]  

(4)

Then, the state of system [1] is \( x_i(t) := [P_i(t), Q_i(t)]^T \).

Vector \( u_i(t) = u_i \) collects the constant commanded real and reactive powers for inverter \( i \). By [2], inverters regulate the output powers to the commanded setpoints \( u_i \); see e.g., [20] Ch. 8], [21].

Let \( P_{i\ell}(t) \) and \( Q_{i\ell}(t) \) denote the demanded real and reactive loads at node \( i \in N \). Then, vector \( d_i(t) \) is set to be \( d_i(t) := [P_{i\ell}(t), Q_{i\ell}(t)]^T \) for all \( i \in N \setminus \{0\} \).

By setting \( r_i(x_i(t), d_i(t)) = x_i(t) \) (5) [1b] equates the state with the measurement of the inverter output powers.

**Prototypical steady-state OPF problem.** Let \( V_i := (E_i/\sqrt{2})e^{j\omega t} \in \mathbb{C} \) be the phasor representation of the steady-state voltage at node \( i \in N \). Similarly, let \( I_i \in \mathbb{C} \) denote the phasor for the current injected at node \( i \in N \), and define \( i := [I_0, \ldots, I_N]^T \in \mathbb{C}^{N+1} \) and \( v := [V_0, \ldots, V_N]^T \in \mathbb{C}^{N+1} \). Then, using Ohm’s and Kirchhoff’s circuit laws, the linear relationship \( i = Yv \) can be established, where \( Y \in \mathbb{C}^{N+1 \times N+1} \) is the admittance matrix formed based on the distribution-network topology and the \( \pi \)-equivalent circuits of lines \( E \).

To formulate an SDP relaxation of a pertinent steady-state OPF problem, consider expressing steady-state powers and voltage magnitudes as linear functions of the outer-product matrix \( V := vv^H \) [17], and define matrix \( Y_i := e_i e_i^H Y \) per node \( i \), where \( \{e_i\}_{i \in N} \) denotes the canonical basis of \( \mathbb{R}^{|N|} \). Using \( Y_i \), form the Hermitian matrices \( \Phi_i := \frac{1}{2}(Y_i + Y_i^H) \), \( \Psi_i := \frac{1}{2}(Y_i - Y_i^H) \), and \( Y_i := e_i e_i^H Y \). Then, the balance equations for real and reactive powers at node \( i \in N_D \) can be expressed as \( \text{Tr}(\Phi_i V) = P_i - P_{i\ell} \) and \( \text{Tr}(\Psi_i V) = Q_i - Q_{i\ell} \), respectively. It follows that constraint [21] represents the steady-state balance equation by setting \( u_i = [P_i, Q_i]^T \) and:

\[
C_i = -I_{2 \times 2}, \quad D_i = I_{2 \times 2} \\
h_i(V) = [\text{Tr}(\Phi_i V), \text{Tr}(\Psi_i V)]^T.
\]  

(6a) \hspace{1cm} (6b)

To complete the OPF formulation, sets \( \mathcal{V} \) and \( \{\mathcal{Y}_i\}_{i \in N_D} \) are specified next. For prevailing ambient conditions, let \( P_{\text{av}} \geq 0 \) denote the available real power for the inverter \( i \in N_D \). The available power is a function of the incident irradiance, and represents the power that would be delivered in the business-as-usual case when the inverters track the maximum power point of the PV array and operate at unity power factor [4]. Then, for PV inverters providing ancillary services, the set of operating points is given by [20].

\[
\mathcal{Y}_i = \{(P_i, Q_i) : P_{i\min} \leq P_i \leq P_{i\max}, Q_i^2 \leq S_i^2 - P_i^2, \quad \text{and} \quad |Q_i| \leq (\tan \theta)P_i \} \quad (7)
\]

where \( S_i \) is the rated apparent power, and the last inequality is utilized to enforce a minimum power factor of \( \cos \theta \). Parameters \( \theta \) and \( P_{i\min} \) can be conveniently tuned to account for the following strategies (see e.g., [1]–[2]):

\begin{itemize}
  \item (c1) Reactive power compensation: \( P_{i\min} = P_{i\max}, \theta \in (0, \pi/2] \);
  \item (c2) Active power curtailment: \( P_{i\min} \in [0, P_{i\max}], \theta = 0 \); and,
  \item (c3) Active and reactive control: \( P_{i\min} \in [0, P_{i\max}], \theta \in [0, \pi/2] \).
\end{itemize}

It is evident that sets \( \{\mathcal{Y}_i\} \) adhere to Assumption [2]

Denote lower and upper limits for \( \{\{V_{ni}\}_{n \in N}\} \) by \( V_{\text{min}} \) and \( V_{\text{max}} \). Then, matrix \( V \) is confined to lie in the set:

\[
\mathcal{V} = \{V : \text{rank}(V) = 1, \quad V_{\text{min}}^T \leq \text{Tr}(Y_i V) \leq V_{\text{max}}^T, \quad \forall i \text{ in } N \} \quad (8)
\]

and \( \text{Tr}(F_i V) = -P_{i\ell}, \text{Tr}(P_i V) = Q_{i\ell}, \forall i \in N_D \), where the last two equalities capture the power-balance equations for nodes without inverters, and the constraint \( |V_0| = 1 \) is left implicit [4]. For nodes without loads (e.g., utility poles), one clearly has that \( P_{i\ell} = Q_{i\ell} = 0 \).

For given load and ambient conditions, a prototypical OPF formulation is expressed as minimizing the steady-state operation of the distribution system can be obtained by constraining variables \( P_i, Q_i \) and \( V \) to the set defined in (7) and (8), respectively, and, by setting \( H(V) = \frac{1}{2} \text{Tr}(F_i V)^2 + b \text{Tr}(F_i V) \) in (3a), with \( a > 0, b \geq 0 \) denoting coefficients related to the price of power supplied by the utility. For \( \Phi_H = F_i, H(V) \) captures the cost of power drawn from (or supplied to) the substation; alternatively, \( H(V) \) can quantify the losses in the network by setting \( \Phi_H = \sum_{(m,n) \in E} \mathbb{R}(y_{mn})(e_m - e_n)(e_m - e_n)^T, \) with \( y_{mn} \) denoting the admittance of line \( (m, n) \) in \( E \). Finally, terms \( \frac{1}{2}u_i^T A_i u_i + b_i^T u_i \) in (3a) model PV-owners costs/rewards for ancillary service provisioning; see also [3], [4], [12].

As with various AC OPF renditions, the resultant problem is nonconvex because of the constraint \( \text{rank}(V) = 1 \); however in the spirit of the SDP relaxation, the set in (8) can be replaced by the convex set

\[
\mathcal{V} := \{V : V \succeq 0, V_{\text{min}}^T \leq \text{Tr}(Y_i V) \leq V_{\text{max}}^T, \quad \forall i \in N \}
\]

and \( \text{Tr}(F_i V) = -P_{i\ell}, \text{Tr}(P_i V) = Q_{i\ell}, \forall i \in N_D \}. \quad (9)

Relaxing (8) with \( \mathcal{V} \), problem (P1) turns out to be a convex relaxation of the AC OPF problem. If the optimal matrix \( V^{\text{opt}} \) has \( \text{rank}(V^{\text{opt}}) = 1 \), then the resultant power flows are globally optimal [17], [13]. Sufficient conditions for this relaxation to be exact for radial and balanced systems are provided in [22], while its applicability to unbalanced multiphase systems is investigated in [5].

In this setup, the objective of the feedback controller that will be designed in the following section, is to drive the inverter outputs \( \{y_i(t) = [P_i(t), Q_i(t)]^T \}_{i \in N_D} \) to the optimal solution \{\{u_{i\text{opt}}\}_{i \in N_D}\} of the OPF problem (P1).

### III. Feedback Controller

Dual \( \epsilon \)-subgradient methods are leveraged in Section III-A to synthesize controllers for systems (1) whose outputs track...
recursive solvers of (P1). Applications to the real-time PV-inverter control problem are discussed in Section III-B.

To streamline proofs of relevant analytical results, it will be convenient to express the linear equality constraints (3b) in a compact form. To this end, define \( u \) := \( \begin{bmatrix} u_1^T, \ldots, u_{ND}^T \end{bmatrix} \), \( d \) := \( \begin{bmatrix} d_1^T, \ldots, d_{ND}^T \end{bmatrix}^T \), \( h(V) := [h_1^T(V), \ldots, h_{ND}^T(V)]^T \), and
\[
g(u, d) := Cu + Dd
\]
with \( C := \text{bdia}([C_i]_{i \in ND}) \) and \( D \) an appropriate matrix formed using \( \{D_i\}_{i \in ND} \). Then, (3b) can be compactly expressed as \( h(V) + g(u, d) = 0 \).

A. Controller synthesis

Consider the Lagrangian corresponding to (3), namely:
\[
L(V, \{u_i\}, \{\lambda_i\}) := H(V) + \sum_{i \in ND} \left( \frac{1}{2} u_i^T A_i u_i + b_i^T u_i \right) + \sum_{i \in ND} \lambda_i^T (h_i(V) + g_i(u_i, d_i))
\]
where \( \lambda_i \in \mathbb{R}^{n_v,i} \) denotes the Lagrange multiplier associated with (3b). Based on (11), the dual function and the dual problem are defined as follows (see, e.g., [23]):
\[
q(\{\lambda_i\}) := \min_{V \in C, \{u_i \in U_i\}_{i \in ND}} L(V, \{u_i\}, \{\lambda_i\})
\]
\[
q^\text{opt} := \max_{\{\lambda_i\}_{i \in ND}} q(\{\lambda_i\}).
\]

Regarding the optimal Lagrange multipliers, the following technical requirement is presumed in order to guarantee their existence and uniqueness; see e.g., [24].

Assumption 3: Vectors
\[
\nabla_{\text{vec}(V), u} [h(V) + g(u, d)]_i, \quad i = 1, \ldots, \sum_{i \in ND} n_{g,i}
\]
are linearly independent.

Section III-B will elaborate on how condition [24] can be checked in the OPF context.

Under current modeling assumptions, it follows that the duality gap is zero, and the dual function \( q(\{\lambda_i\}) \) is concave, differentiable, and it has a continuous first derivative [25]. Under [24], a gradient-type method [23], [25] is utilized next to solve the dual problem (13).

Consider updates performed at discrete time instants \( t \in \{t_k, k \in \mathbb{N}\} \), with \( V[t_k], \{u_i[t_k]\}_{i \in ND} \), and let \( \{\lambda_i[t_k]\}_{i \in ND} \) denote the values of primal and dual variables, respectively, at time \( t_k \). Further, let \( \{\alpha_k\}_{k \in \mathbb{N}} \), with \( \alpha_k \in \mathbb{R}_+ \forall k \in \mathbb{N} \), be a non-summable but square-summable sequence of stepsizes; i.e., there exist sequences \( \{\gamma_k\}_{k \in \mathbb{N}} \) and \( \{\eta_k\}_{k \in \mathbb{N}} \) such that:
\( (s1) \quad \gamma_k \rightarrow 0 \) as \( k \rightarrow +\infty \), and \( \sum_{k=0}^{+\infty} \gamma_k = +\infty \);
\( (s2) \quad \gamma_k \leq \alpha_k \leq \eta_k \) for all \( k \in \mathbb{N} \); and,
\( (s3) \quad \eta_k \downarrow 0 \) as \( k \rightarrow +\infty \), and \( \sum_{k=0}^{+\infty} \eta_k^2 < +\infty \).

The following method accounts for the system dynamics in (1) while solving (P1) with dual-gradient-based approaches:

At time \( t_k \), the system outputs are sampled as:
\[
y_i[t_k] = r_i(x_i(t_k), d_i), \quad \forall i \in ND
\]
and, they are utilized to update the dual variables as follows:
\[
\lambda_i[t_{k+1}] = \lambda_i[t_k] + \alpha_{k+1} \left( h_i(V[t_k]) + C_i y_i[t_k] + D_i d_i \right), \quad \forall i \in ND.
\]
Given \( \lambda_i[t_{k+1}] \), the primal variables \( V[t_{k+1}] \) and \( \{u_i[t_{k+1}]\}_{i \in ND} \) are then updated as:
\[
V[t_{k+1}] = \arg \min_{V \in C} H(V) + \sum_{i \in ND} \lambda_i^T[t_{k+1}] h_i(V)
\]
\[
u_i[t_{k+1}] = \text{proj}_{U_i} \left( -A_i^{-1} C_i^T [\lambda_i[t_{k+1}] - A_i^{-1} b_i] \right).
\]

Once (15c) is solved, the vector-valued reference signal \( u_i[t_{k+1}] \) is applied to the dynamical system (1a) over the interval \( (t_k, t_{k+1}) \); that is, \( u_i(t) = u_i[t_{k+1}], t \in (t_k, t_{k+1}) \). At time \( t_{k+1} \), the system outputs \( \{y_i[t_{k+1}]\}_{i \in ND} \) are sampled again, and (15b)–(15d) are repeated. Notice that, differently from
standard dual gradient methods, variable $u_i[t_k]$ is replaced by the sampled system output $y_i[t_k]$ in the ascent step (15b).

Steps (15b)–(15d) in effect constitute the controller for the dynamical systems (1). Specifically, the (continuous-time) reference signals $\{u_i(t)\}_{t \in \mathbb{R}^D}$ produced by the controller have step changes at instants $\{k, k \in \mathbb{N}\}$, are left-continuous functions, and take the constant values $\{u_i[t_{k+1}]\}_{t \in \mathbb{R}^D}$ over the time interval $[t_k, t_{k+1})$. It is evident that if $u_i[t_k]$ converges to $u_i^{opt}$ as $k \to \infty$ (and thus $u_i(t) \to u_i^{opt}$ as $t \to \infty$), then $y_i(t) \to u_i^{opt}$ as $t \to \infty$ by virtue of (2).

Suppose now that the interval $[t_{k-1}, t_k]$ is large enough to allow the outputs $\{y_i(t)\}_{t \in \mathbb{N}}$ to converge to the commanded input $\{u_i[t_k]\}_{t \in \mathbb{N}}$ [cf. (2)]. Under this ideal setup with a pronounced and tangible time-scale separation between controller and system dynamics, one has that $\lim_{t \to t_k^-} \|y_i(t) - u_i[t_k]\| = 0$, for all $k$ [cf. (2)], and step (15b) is replaced by $\lambda_i[t_{k+1}] = \lambda_i[t_k] + \alpha_{k+1} (V[y_i[t_k]] + C_i y_i[t_k] + D_i d_i)$. Thus, (15c) coincides with standard dual gradient methods [25], and the convergence results in [23, Prop. 8.2.6], [25] carry over to this ideal setup. In this work, convergence of the system outputs $\{y_i(t)\}_{t \in \mathbb{N}}$ to the solution of (P1) is assessed in the more general case where update of reference signals may be performed faster than the systems’ settling times and asynchronously, in order to achieve the following operational goals:

(O1) **Bridging the time-scale separation:** instead of waiting for the underlying systems to converge to intermediate reference levels $\{u_i[t_k]\}_{t \in \mathbb{N}}$, steps (15b)–(15c) are performed continuously (within the limits of affordable computational burden); i.e., at each instant $t_k$, one may have that $\lim_{t \to t_k^-} \|y_i(t) - u_i[t_k]\| \neq 0$ for at least one dynamical system.

(O2) **Accounting for computational limits:** the computational time required to solve the SDP problem (15c) is typically higher than that required by the projection operation (15d), especially when (15d) affords a closed-form solution (see e.g., [26], and pertinent references therein). Thus, convergence of the system outputs is investigated for the case where the update of the input reference levels $\{u_i[t_k]\}_{t \in \mathbb{N}}$ and the dual variables $\{\lambda_i[t_k]\}_{t \in \mathbb{N}}$ is performed at a faster rate than (15c).

To this end, suppose that the computational required to update matrix $V$ spans $M < +\infty$ time intervals; that is, if the computation of (15c) starts at time $t_k$ based on the most up-to-date dual variables $\{\lambda_i[t_k]\}_{t \in \mathbb{N}}$, its solution becomes available only at time $t_{k+M}$. In contrast, the controller affords the computation of steps (15d) and (15b) at each time $\{t_k\}_{t \in \mathbb{N}}$. To capture this asynchronous operation, consider the mapping $c(k) := M \left\lfloor \frac{k}{M} \right\rfloor \quad k \in \mathbb{N}$. (16)

Using (16), steps (15) for all $i \in \mathbb{N}_D$ are modified as:

\begin{align}
y_i[t_k] &= r_i \left( x_i[t_k], d_i \right) \quad (17a) \\
\lambda_i[t_{k+1}] &= \lambda_i[t_k] + \alpha_{k+1} \left( h_i(V[y_i[t_k]]) + C_i y_i[t_k] + D_i d_i \right) \quad (17b) \\
u_i[t_{k+1}] &= \text{proj}_{Y_i} \left\{ -A_i^{-1} C_i^T \lambda_i[t_{k+1}] - A_i^{-1} b_i \right\} \quad (17c)
\end{align}

for all $t_k, k \in \mathbb{N}$. Further, matrix $V[t_{c(k)}]$ is updated (at the possibly slower rate) as:

\[ V[t_{c(k)}] = \arg \min_{V \in \mathbb{V}} H(V) + \sum_{i \in \mathbb{N}_D} \lambda_i^T[t_{c(k)}] h_i(V). \] (17d)

Since $c(k) = k$ over the interval $\{t_k, \ldots, t_{k+M-1}\}$, (17d) indicates that $V$ is being updated every $M$ time slots. The block diagram for (17) can be readily obtained by replacing step (15c) with (17d), as well as (15b) and (15d) with (17b) and (17c), respectively, in Figure 1.

In the following, convergence of the system outputs to the solution of the steady-state optimization problem (P1) is established when the reference signals are produced by (17). Of course, by setting $M = 1$, steps (17) coincide with (15), and therefore the convergence claims for this more general setting naturally carry over to the synchronous setup in (15).

For brevity, collect the system outputs in the vector $y := [y_1, \ldots, y_N]^T$, and the dual variables in $\lambda := [\lambda_1, \ldots, \lambda_N]^T$. Notice that given the strict convexity of $L(V, u, \lambda[k])$ with respect to $V$ and $u$, the pair $(V[t_{c(k)}], y[t_k])$ represents sub-optimal solutions for the primal updates (17c)–(17d) whenever $\lim_{t \to t_k^-} \|y(t) - u[t_k]\| \neq 0$ and/or $M > 1$: i.e., there exists an $\epsilon[t_k] \geq 0$ such that $L(V[t_k], u[t_k], \lambda[k]) \leq L(V[t_{c(k)}], y[t_k], \lambda[k])$ and $L(V[t_{c(k)}], y[t_k], \lambda[k]) \leq L(V[t_k], u[t_k], \lambda[k]) + \epsilon[t_k]$. Thus, (15b) and (17b) are in fact $\epsilon$-subgradient steps [15, Proposition 2].

Before elaborating further on the error $\epsilon[t_k]$, notice that from the compactness of sets $\mathcal{V}$ and $\{Y_i\}_{i \in \mathbb{N}_D}$, it follows that there exists a constant $0 \leq G \leq +\infty$ such that the following holds:

\[ \|h(V) + g(y, d)\| \leq G, \quad \forall V \in \mathbb{V}, \forall y \in \mathcal{Y} \] (18)

with $\mathcal{Y} := \mathcal{Y}_1 \times \mathcal{Y}_2 \times \cdots \times \mathcal{Y}_N$. Furthermore, given the Lipschitz-continuity of the contraction mapping in (15d), there exists $\hat{\lambda}[t_k]$ satisfying [cf. (15d)]:

\[ y_i[t_k] = \text{proj}_{Y_i} \left\{ -A_i^{-1} C_i^T \hat{\lambda}[t_k] - A_i^{-1} b_i \right\}, \quad \forall i \in \mathbb{N}_D \] (19)

that is, $y_i[t_k]$ would be obtained by minimizing the Lagrangian $L(V, u, \lambda[t_k])$ when $\lambda[t_k] := [\lambda_1^T, \ldots, \lambda_N^T]^T$ replaces $\lambda[t_k]$. The following will be assumed for $\hat{\lambda}[t_k]$.

**Assumption 4:** There exists a scalar $\tilde{G}, 0 \leq \tilde{G} < +\infty$, such that the following bound holds for all $t_k, k \geq 0$

\[ \|\lambda[t_k] - \hat{\lambda}[t_k]\|_2 \leq \tilde{G} \|\lambda[t_k] - \lambda[t_{k-1}]\|_2. \] (20)

Condition (20) implicitly bounds the reference signal tracking error $\|y[t_k] - u[t_k]\|_2$, as specified in the following lemma.

**Lemma 1:** Under Assumption 4, it follows that the tracking error $\|y[t_k] - u[t_k]\|_2, k \in \mathbb{N}$, can be bounded as

\[ \|y[t_k] - u[t_k]\|_2 \leq \| - A_i^{-1} C_i^T \|_2 \| GG \| \|y[t_k] - u[t_k]\|_2. \] (21)

Condition (20) can be re-stated in terms of the output signals $y[t_k]$. Specifically, letting $\xi_i[t_k] := -A_i^{-1} C_i^T \lambda_i[t_k] - A_i^{-1} b_i$ be the unprojected reference signal, and assuming that matrix $A_i^{-1} C_i^T$ is invertible, one has that (20) is implied by the bound $\|\xi_i[t_k] - y[t_k]\|_2 \leq G \|\xi_i[t_k] - \xi_i[t_{k-1}]\|_2$. upon setting $G = G \| - A_i^{-1} C_i^T \|_2 \| (-A_i^{-1} C_i^T)^{-1}\|_2$. 
Proof. From the non-expansive property of the projection operator, the left-hand side of (24) can be bounded as
\[
\|y[t_k] - u[t_k]\|_2 \leq \| - A^{-1}C^T (\hat{\lambda}[t_k] - \lambda[t_k])\|_2 \\
\leq \| - A^{-1}C^T\|_2 \|\hat{\lambda}[t_k] - \lambda[t_k]\|_2 \\
\leq \| - A^{-1}C^T\|_2 \tilde{G}G\alpha_k \\
\tag{22a}
\]
where (22a) is obtained by using the following bound (which originates from Assumption 4):
\[
\|\lambda[t_k] - \hat{\lambda}[t_k]\|_2 \leq \tilde{G}^i\|\lambda[t_k] - \lambda[t_k-1]\|_2 \\
\leq \tilde{G}^i\alpha_k (h(V[y[t_{c(h-1)}]]) + g(y[t_{c(h-1)}], d))_2 \\
\leq \tilde{G}G\alpha_k. \\
\tag{22b}
\]
Note that (22c) follows from the dual update in (17b), and (22d) follows from (18).

It can be noticed from (21) that the tracking error is allowed to be arbitrarily large, but the outputs \(y[t_k]\) should eventually follow the reference signal \(u[t_k]\). In fact, since the sequence \(\{\alpha_k\}\) is majorized by \(\{\eta_k\}\), and \(\eta_k \downarrow 0\), it follows that \(\|y[t_k] - u[t_k]\|_2 \to 0\) as \(k \to \infty\). Based on this assumption, two results that establish convergence of the overall system are in order: Lemma 2 provides an analytical characterization of the \(\epsilon\)-subgradient step, while Theorem 7 establishes asymptotic convergence of the output powers to the optimal solution of (P1).

Lemma 2: Suppose that at least one of the following statements is true: i) \(M > 1\); ii) at time \(t_k\), \(y[t_k] \neq u[t_k]\) for at least one dynamical system. Then, \(h(V[y[t_{c(h)}]]) + g(y[t_{c(h)}], d)\) is an \(\epsilon\)-subgradient of the dual function at \(\lambda[t_k]\). In particular, under Assumption 4 and with \(M < +\infty\), it holds that
\[
(h(V[y[t_{c(h)}]]) + g(y[t_{c(h)}], d))^T (\lambda - \lambda[t_k]) \\
\geq q(\lambda) - q(\lambda[t_k]) - \epsilon[t_k] \quad \forall \lambda \\
\tag{23a}
\]
where the error \(\epsilon[t_k] \geq 0\) can be bounded as
\[
\epsilon[t_k] \leq 2\alpha_k \tilde{G}G^2 + 2G^2 \sum_{h=1}^{k-c(k)} \alpha_{k-h+1}. \\
\tag{23b}
\]

Proof. Recall that \(h(V[y[t_k]]) + g(u[t_k], d)\) is the gradient of the dual function (12) evaluated at \(\lambda[t_k]\). Consider decomposing (12) as \(q(\lambda) = q_V(\lambda) + q_u(\lambda)\), with
\[
q_V(\lambda) := \min_{V \in V}\ H(V) + \Lambda^T h(V), \\
\tag{24a}
q_u(\lambda) := \min_{u \in \mathbb{U}} \frac{1}{2} u^T A u + b^T u + \lambda^T g(u, d). \\
\tag{24b}
\]
Then, it will be shown that
\[
g^T(y[t_k], d)(\lambda - \lambda[t_k]) \geq q_u(\lambda) - q_u(\lambda[t_k]) - \epsilon[t_k] \\
\tag{24c}
\]
\[
h^T(V[y[t_{c(h)}]]) - q_V(\lambda[t_k]) \geq q_V(\lambda[t_k]) - \epsilon_V[t_k] \\
\tag{24d}
\]
with \(\epsilon[t_k] \leq 2\alpha_k \tilde{G}G^2\) and \(\epsilon_V[t_k] \leq 2G^2 \sum_{h=1}^{k-c(k)} \alpha_{k-h+1}\). To show (24c), consider the gradient of \(q_u(\lambda)\) evaluated at \(\lambda[t_k]\), which by definition leads to the inequality
\[
g^T(y[t_k], d)(\lambda - \lambda[t_k]) \geq q_u(\lambda) - q_u(\lambda[t_k]) \quad \forall \lambda; \text{ then, add } \\
g^T(y[t_k], d)(\lambda[t_k] - \lambda[t_k]) \quad \text{on both sides to obtain}
\tag{24c}
\]
and add and subtract \(q_u(\lambda[t_k])\) to the right-hand-side
\[
g^T(y[t_k], d)(\lambda - \lambda[t_k]) \geq q_u(\lambda) - q_u(\lambda[t_k]) \\
+ q_u(\lambda[t_k]) - q_u(\lambda[t_k]) + g^T(y[t_k], d)(\lambda[t_k] - \lambda[t_k]). \\
\tag{24f}
\]
In (24f), define \(\epsilon_u[t_k] := q_u(\lambda[t_k]) - q_u(\lambda[t_k]) + g^T(y[t_k], d)(\lambda[t_k] - \lambda[t_k])\). By using the definition of the gradient of the function \(q_u(\lambda)\) at \(\lambda[t_k]\), and applying the Cauchy-Schwartz inequality, one has that
\[
\epsilon_u[t_k] \leq g^T(u[t_k], d)(\lambda[t_k] - \lambda[t_k]) \\
+ g^T(y[t_k], d)(\lambda[t_k] - \lambda[t_k]) \\
\leq 2G^2 \|\lambda[t_k] - \lambda[t_k]\|_2 \leq 2\alpha_k G^2 \\
\tag{24g}
\]
where (17b), (18), and (20) were used to obtain (24h) from (24g). Next, to show (24d), begin with the inequality
\[
h^T(V[y[t_{c(h)}]])(\lambda - \lambda[t_k]) \geq q_V(\lambda) - q_V(\lambda[t_k]). \\
\]
Adding \(h^T(V[y[t_{c(h)}]])(\lambda[t_k] - \lambda[t_k])\), and subtracting the sequences \(\{q_V(\lambda[t_k])\}_{h=1}^{k-c(k)}\) and \(\{h^T(V[y[t_{c(h)}]])(\lambda[t_k] - \lambda[t_k])\}_{h=1}^{k-c(k)}\) to both sides of the inequality,
\[
h^T(V[y[t_{c(h)}]])(\lambda - \lambda[t_k]) \geq q_V(\lambda) - q_V(\lambda[t_k]) - \epsilon_V[t_k]. \\
\tag{24i}
\]
Using the definition of the gradient, the Cauchy-Schwartz inequality, and (18), (24k) can be bounded as
\[
\epsilon_V[t_k] \leq 2G^2 \sum_{h=1}^{k-c(k)} \|\lambda[t_k] - \lambda[t_k]\|_2. \\
\tag{24l}
\]
Finally, upon using (17b) and (18), (24l) can be further bounded as \(2G^2 \sum_{h=1}^{k-c(k)} \|\lambda[t_k] - \lambda[t_k]\|_2 \leq 2G^2 \sum_{h=1}^{k-c(k)} \alpha_{k-h+1}\).

Theorem 1: Under Assumptions 7-12 and for any \(1 \leq M < +\infty\), the following holds for the closed-loop system (17) when a stepsize sequence \(\{\alpha_k\}_{k \in \mathbb{N}}\) satisfying conditions (s1)-(s3) is utilized:

(i) \(\lambda[t_k] \to \lambda_1^{opt}\) as \(k \to \infty\), \(\forall i \in \mathbb{N}_D\); 
(ii) \(V[y[t_{c(h)}]] \to V^{opt}\) and \(\{u_{[i]}[t_k] \to u_i^{opt}\}_{i \in \mathbb{N}_D}\) as \(k \to \infty\); 
(iii) \(y_i(t) \to u_i^{opt}\) as \(t \to \infty\), \(\forall i \in \mathbb{N}_D\).

Statements (i)-(iii) hold for any initial conditions \(V[0], \{u_i[0]\}_{i \in \mathbb{N}_D}, \{y_i(0)\}_{i \in \mathbb{N}_D}, \{\lambda_i[0]\}_{i \in \mathbb{N}_D}\) and any duration of the intervals \(0 < t_k - t_{k-1} < \infty, k \in \mathbb{N}\).

Proof. (i)-(ii) Boundedness and convergence of the dual iterates can be proved by leveraging the results in (16, Theorem 3.4). In particular, it suffices to show that the following
technical requirement is satisfied in the present setup:
\[
\sum_{k=0}^{+\infty} \alpha_k \epsilon_k(t_k) = \sum_{k=0}^{+\infty} \alpha_k (\epsilon_V(t_k) + \epsilon_u(t_k)) < +\infty. \tag{25a}
\]
From Lemma 2, it can be shown that
\[
\sum_{k=0}^{+\infty} \alpha_k \epsilon_u(t_k) \leq 2G^2 \sum_{k=0}^{+\infty} \alpha_k \sum_{h=0}^{k-c(k)} \alpha_{k-h+1} \tag{25b}
\]
where the second inequality in (25b) follows from the fact that \(\alpha_k \leq \eta_k\) for all \(k\). Since \(\sum_{k=0}^{+\infty} \eta_k^2 < +\infty\), the series \(\sum_{k=0}^{+\infty} \alpha_k \epsilon_u(t_k)\) is finite. As for the error \(\epsilon_V(t_k)\), one has that
\[
\sum_{k=0}^{+\infty} \alpha_k \epsilon_V(t_k) \leq 2G^2 \sum_{k=0}^{+\infty} \alpha_k \sum_{h=1}^{k-c(k)} \alpha_{k-h+1} \tag{25c}
\]
\[
\leq 2G^2 \sum_{k=0}^{+\infty} \alpha_k \sum_{h=1}^{M-1} \alpha_{k-h+1} \leq 2G^2 \sum_{k=0}^{+\infty} \sum_{h=1}^{M-1} \eta_{k-h+1} \tag{25d}
\]
where the fact that \(\max\{k-c(k)\} = M-1\) is utilized in (25d), and (25e) follows from (25a) since the sequence \(\{\eta_k\}_{k \in \mathbb{N}}\) majorizes \(\{\epsilon_k\}_{k \in \mathbb{N}}\), and is monotonic decreasing. Since the series \(\eta_k^2\) is square-summable, \(\sum_{k=0}^{+\infty} \alpha_k \epsilon_V(t_k)\) is finite. (iii) From the strict convexity of the Lagrangian in the primal variables, it follows that optimal primal variables can be uniquely recovered as \(\{\nu^\text{opt}, \mu^\text{opt}\} = \arg\min_{\nu \in \mathbb{E}, \mu \in \mathbb{L}} L(V, \mu, \lambda^\text{opt})\).

(iv) At convergence, the reference signal is constant, with value \(\nu_i^\text{opt}\). Then, \(y_i(t) \rightarrow \nu_i^\text{opt}\) as \(t \rightarrow \infty\) by (2).

\section{PV-inverter controller implementation}

When applied to the PV-inverter regulation problem outlined in Section II-B, the controller (17b)–(17d) endows each PV-inverter \(i \in \mathbb{N}_D\) with the capability of steering its power output \(y_i(t) = [P_i(t), Q_i(t)]^T\) towards the solution \(u_i^\text{opt} = [P_i^\text{opt}, Q_i^\text{opt}]^T\) of the formulated AC OPF problem. Claims (i)–(iv) of Theorem 2 hold for any duration \(0 < t_k - t_{k-1} < \infty\), \(k \in \mathbb{N}\), for any size of the distribution network.

The feedback controller (17b) affords a decentralized implementation, where optimization tasks are distributed between the utility and individual PV systems. In particular: i) at the utility, updates (17d) are performed with the goal of pursuing system-wide optimization objectives such as minimization of power losses and voltage regulation (this step is performed every \(M\) time slots); and, ii) updates (17b)–(17c) are performed at each individual PV system, and \(u_k\) and \(\lambda_k\) are stored locally at the same inverter (these steps are performed continuously, within affordable computational and hardware limits).

To exchange relevant control signals, a bidirectional message passing between the utility and individual PV systems is necessary. This entails the following message exchanges every \(M\) time slots: \(h_i(V[t_k])\) is sent from the utility to inverter \(i\); subsequently, the up-to-date dual variable \(\lambda_i(t_k)\) is sent from inverter \(i\) to the utility company. Notice that customer \(i \in \mathbb{N}_D\) does not share load demand and PV-related information with the utility company; in fact, information about the loads is not necessary when computing the update (17d) at the utility. Exchanging just Lagrange multipliers rather than power iterates ensures a privacy-preserving operation. See also Figure 1.

Finally, since functions \(\{h_i(V)\}_{i \in \mathbb{N}_D}\) are linear in \(V\), the prerequisite (14) solely depends on the topology of the distribution network; thus, (14) can be checked at the utility side once matrix \(Y\) is available.

\section{IV. TEST CASES}

The proposed PV-inverter control scheme is tested using a modified version of the IEEE 37-bus test feeder shown in Fig. 2. The modified network is obtained by considering a single-phase equivalent, and by assuming that 6 PV systems are present in the network. Line impedances, shunt admittances, as well as active and reactive loads are adopted from the dataset available at ewh.ieee.org/soc/pes/dsacom/testfeeders. The package CVX (available at: http://cvxr.com/cvx/) is employed to solve relevant optimization problems in MATLAB. The objective of the test cases is to numerically corroborate the claims (i)–(iii) of Theorem 2.

In the OPF problem (P1), the voltage limits \(V_{\text{min}}\) and \(V_{\text{max}}\) in (9) are set to 0.95pu and 1.05pu, respectively; the voltage magnitude at the point of common coupling is fixed to \(|V_0| = 1\) pu. In this test case, \(H(V)\) models the cost of power drawn from the substation as \(H(V) = (\text{Tr}(\Phi_0 V))^2 + 10 \times \text{Tr}(\Phi_0 V)\). In the quadratic function capturing the cost of ancillary services provisioned by inverters, the parameters are set to \(A_i = [1, 0; 0, .01], b_i = [10, 0.01]^T\), for \(i = 1, \ldots, 4\) and \(A_i = [1, 0; 0, .03], b_i = [10, 0.03]^T\), for \(i = 5, 6\).

Following the technical approach of [20] Ch. 8) and [21], a first-order system is utilized to model the real and reactive power dynamics of each inverter. Further, inverters implement strategy (c3), and their regions of possible operating points is formed based on the inverter power ratings \(\{S_i\}_{i \in \mathbb{N}_D}\) and the available active powers \(\{P_i^\text{av}\}_{i \in \mathbb{N}_D}\). Specifically, the power ratings are assumed to be 50, 120, 50, 100, 120, and 80 kVA, whereas the following values for the available powers \(P_i^\text{av} := [P_i^\text{av}, \ldots, P_{N_D}^\text{av}]^T\) are considered in order to test the adaptability of the feedback controller to changing prevailing conditions (with time intervals normalized with respect to the time constant \(\tau\)):

(11) \(P_i^\text{av}(t) = [22, 67, 21, 50, 68, 40]^T\) kW, \(t/\tau \in [1, 200]\);

(12) \(P_i^\text{av}(t) = [25, 80, 24, 55, 85, 45]^T\) kW, \(t/\tau \in [201, 400]\);
at \( t = 200\tau \), the active and reactive powers converged to the OPF solution 21.8, 66.9, 20.9, 67.9, 39.9 kW and 39.2, 85.6, 40.7, 77.1, 31.4, 39.8 kVar. This corroborates the claims of Theorem 1. Figure 3(b) also provides a snapshot of the evolution of the output reactive power for inverter 2; it can be seen that a new reference level is applied after \( \tau \) seconds, before \( Q_2(t) \) settles around the intermediate setpoint.

Notice that, as expected, the behavior of the overall closed-loop system (17) can be approximated as a first-order system with proven asymptotic converge to the solution of (P1). Similar trajectories would have been obtained when the loads are also time varying. Future efforts will explore variations of load and solar irradiance that may have the same temporal scale of the dynamics of (17). It is also interesting to note that, in the considered setup, the steady-state reactive powers coincide with the available powers \( p^{av}(t) \), and reactive compensation turns out to be the optimal ancillary service strategy. This is because the cost of the active power provided by the inverters is lower than the one of the power drawn from the substation.

**V. Concluding Remarks and Future Work**

This paper considered a distribution network featuring PV systems, and addressed the synthesis of feedback controllers that seek inverter setpoints corresponding to AC OPF solutions. To this end, dual \( \epsilon \)-subgradient methods and SDP relaxations were leveraged. Global convergence of PV-inverter output powers was analytically established and numerically corroborated. Although the focus was on PV systems, the framework naturally accommodates different types of inverter-interfaced energy resources. Future efforts will analyze the application of the proposed framework to islanded systems.

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