ON THE ALGEBRAIC APPROACH TO CUBIC LATTICE POTTS MODELS

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Abstract

We consider Diagram algebras, $D_G(Q)$ (generalized Temperley-Lieb algebras) defined for a large class of graphs $G$, including those of relevance for cubic lattice Potts models, and study their structure for generic $Q$. We find that these algebras are too large to play the precisely analogous role in three dimensions to that played by the Temperley-Lieb algebras for generic $Q$ in the planar case. We outline measures to extract the quotient algebra that would illuminate the physics of three dimensional Potts models.

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1 Introduction

With the benefit of hindsight it is striking how easy it might have been, 15-20 years ago, to identify roots of unity as the values of $q$ that were special for the description of the physics of $Q = (q + q^{-1})^2$ state Potts models in two dimensions, and related spin chains in one dimension. It is the work of a few lines to derive these as the exceptional cases using the Temperley-Lieb algebra introduced by Temperley and Lieb 1971 [1] (see [9]). This could have been done before many of the models were solved. Only the interpretation of this result might have puzzled the early ‘algebraic physicist’. Of course, this is not the way things happened. The location of the special points is revealed in the details of the solution of the models [2][3][7], and it was only after the solution of the models that the significance of the special points and their relation to the cataloguing of models into universality classes was appreciated.

In a sense, we find ourselves heading down the same path now for three and higher dimensional models. There has been some very impressive work done on models whose Boltzmann weights satisfy the tetrahedron equations [8], but that is not the route we follow here. In [20] it was suggested that the Diagram algebras $D_G(Q)$ (defined below) for some sequence of graphs $G^{(j)} = \{G^{(1)}, G^{(2)}, \ldots\}$ would play the role of the Temperley-Lieb algebra for higher dimensions (the Temperley-Lieb algebra is the sequence of Diagram algebras with $G^{(j)} = A_j$, where $A_j$ is the $j$ node chain graph). In this paper we determine the structure of $D_G(Q)$ for enough graphs $G$ to show that a direct analogy with two dimensions is too simplistic in general, and suggest a resolution.

The paper is structured in the following way. We introduce the $Q$-state Potts model on any lattice, and point out the relation between the transfer matrix of the 2-dimensional model and the Temperley-Lieb (TL) algebra. Since we take the algebraic route in this paper, we then state the specific link between representation

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theory (the index set for distinct irreducible representations) and physics (primary fields in the 2-dimensional conformal field theory (CFT)) that we would like to examine in the higher dimensional context. Namely, when the index set is finite, the corresponding CFT is minimal. In 2-dimensions, the index set is finite at the special values of $Q$ called Beraha numbers, which are also the values at which the TL algebras defined for a sequence of chain graphs of increasing length becomes non-semisimple beyond some length. One of our objectives in this paper is to locate the corresponding $Q$-values at which our candidate algebra $D_G(Q)$ becomes non-semisimple in an analogous way.

We define the Diagram algebra as a subalgebra of the Partition algebra in the last part of this section. The basis of the defining representation of the Diagram algebra is taken from the set of partitions of the nodes of two copies of a graph $G$, called ‘top’ and ‘bottom’. Multiplication in the algebra involves stacking one such top and bottom over another, and keeping track of the resulting partitions by transitivity (see figure 1). In section 2, we tackle the problem of classifying the irreducible representations of $D_G(Q)$ for generic $Q$. This is carried out in two steps – first by noting the number of parts with both top and bottom nodes as above (called the number of ‘propagating lines’) and then by the permutations of these lines allowed on a given graph $G$. We do this for a large class of graphs and in particular, for a class of graphs which we call unsplitting (see proposition 3 and the remark following it). We also give necessary and sufficient conditions for a set of partitions to be a basis for these irreducibles in proposition 6. Using this key result, we prove in proposition 7 that the algebras defined for a sequence of unsplitting graphs ceases to be semi-simple for at least all integer values of $Q$. In section 4, we apply the above results for the particular example of an unsplitting graph that is relevant for building the transfer matrix of the 3-dimensional Potts model. We discuss the implication of these results next. The appendix lays out the preliminary steps towards the description of the Bratteli diagram (or the inclusion matrix) for the restriction of modules for the generically semi-simple algebras $D_H \subset D_G$ for graphs $G, H$ and $H \subset G$.

### 1.1 Basic definitions

For any simple, unoriented graph $L$, and natural number $Q$, the partition function of the $Q$-state Potts model on the graph $L$ is

$$Z(L) = \sum_{\sigma_i \in \{1, 2, \ldots, Q\}, \forall i \in \Lambda_0^L} \exp \left( \beta \sum_{(i,j) \in \Lambda_1^L} \delta_{\sigma_i, \sigma_j} \right),$$

(1)

where $\Lambda_0^L$ denotes the set of nodes of $L$, and $\Lambda_1^L$, the set of its edges.

Recall that for graphs $G$ and $H$ then $G \times H$ is a graph such that

$$\Lambda_0^{G \times H} = \Lambda_0^G \times \Lambda_0^H,$$

(2)
and

\[(i,j),(k,l) \in \Lambda_{G \times H}^1 \text{ if } \begin{cases} (i,k) \in \Lambda_G^1 \text{ and } j = l, \\
(j,l) \in \Lambda_H^1 \text{ and } i = k. \end{cases} \quad (3)\]

Let \( \hat{A}_t \) be the \( t \)-node closed chain graph. Then for example \( A_1 \times A_m \times \hat{A}_t \) would be the cubic lattice with periodicity in one direction. For any \( G \) the partition function

\[Z(G \times \hat{A}_t) = \text{Tr}( (T_G)^t), \quad (4)\]

where \( T_G \) is the \((G \text{ shaped layer})\) transfer matrix defined as

\[T_G = \prod_{i \in \Lambda_G^0} \left( (e^{\beta} - 1)I + \sqrt{Q} U_i \right) \prod_{(i,j) \in \Lambda_G^1} \left( I + \frac{(e^{\beta} - 1)}{\sqrt{Q}} U_{i,j} \right). \quad (5)\]

Here

\[I = I_Q \otimes I_Q \otimes \ldots \otimes I_Q \quad (6)\]

(one factor for each node of \( G \), each factor a \( Q \times Q \) unit matrix)

\[U_i = \frac{1}{\sqrt{Q}} (I_Q \otimes I_Q \otimes \ldots \otimes M_{i^{th}} \otimes \ldots I_Q) \quad (i \in \Lambda_G^0), \quad (7)\]

where \( M \) is the \( Q \times Q \) matrix with all entries 1, in the \( i^{th} \) position (note that writing the factors in a row implies a total order on \( \Lambda_G^0 \) - this is physically misleading for general \( G \) and can be chosen arbitrarily, c.f. the two dimensional case \[2\]) and

\[U_{i,j} = \sqrt{Q} (I_Q \otimes I_Q \otimes \ldots \otimes N_{i^{th} \otimes j^{th}} \otimes \ldots I_Q), \quad ((i,j) \in \Lambda_G^1) \quad (8)\]

where \( N \) is the \( Q^2 \times Q^2 \) diagonal matrix acting on the \( i^{th} \) and \( j^{th} \) subspaces (and note that \( j \) is not necessarily adjacent to \( i \) in a given ordering) with index set \( \{1,2,\ldots,Q\} \times \{1,2,\ldots,Q\} \), and

\[N_{(i,j),(i,j)} = \begin{cases} 1 & \text{if } i = j, \\
0 & \text{otherwise.} \end{cases} \quad (9)\]

(see \[1\], \[2\], \[23\]).

Note that these matrices obey

\[U_i^2 = \sqrt{Q} U_i, \quad (10)\]

\[U_{i,j}^2 = \sqrt{Q} U_{i,j} \]

\[U_i U_{i,j} U_i = U_i \]

\[U_{i,j} U_{i,j} = U_{i,j} \]

\([U_i, U_j] = [U_i, U_{i,k}] = [U_{i,j}, U_{k,l}] = 0, \quad i \neq j, k. \quad (11)\]

Recall that for \( G = A_n \) the graph \( L = G \times \hat{A}_t \) is the square lattice on a cylinder, and these matrices give a representation of the Temperley-Lieb algebra \[1\]. It is
known that this representation is faithful except at the Beraha type numbers \([16]\) \(Q = 4 \cos^2 \frac{\pi p}{b}\) (\(p, b\) integers), where it is faithful only on the unitarizable quotient \([15]\). Also, for other \(Q\) values the number of distinct irreducible representations in this Potts representation grows unboundedly with \(n\), whereas for \(p, b\) integer it is finite and fixed by \(b\) (\(a la\) primary fields in rational conformal field theories \([11]\)). The models corresponding to these Beraha-type numbers have as massless Euclidean field theory limits the minimal models of conformal field theory. For \(p = 1\), these lattice models are in the same universality class as the ABF models \([12][5][14][6]\) whose corresponding conformal field theories belong to the unitary series of ref. \([13]\) with \(c = 1 - \frac{6}{b(b-1)}\).

In this paper we address the question of what is the appropriate abstract algebra, in the same sense as above, for arbitrary sequence \(G^{(-)}\). In \([20]\), it has been noted that the algebra with generators and relations simply as in equation (11) (the Full Temperley-Lieb algebra) is too big, as the Potts representation is then never faithful for non-chain graphs. Instead, we shall focus on the following finite dimensional quotients. In order to define these quotients, it is useful to recall the definition of the Partition algebra \(P_n = P_n(Q)\) \([19][20]\).

Let \(S_{2n}\) be the set of partitions of the set \(\{1, 2, \ldots, n, 1', 2', \ldots, n'\}\). The \(C\)-linear extension of the product defined in figure 1 on the vector space with basis \(S_{2n}\) gives the Partition algebra, \(P_n(Q)\).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{The top diagram is \(a\), the one in the middle \(b\), and the one at the bottom is the product \(a \circ b\). Trace the connectivities from bottom to top, and for each discarded part from the middle, pick up a factor of \(Q\) to obtain \(a \circ b\).}
\end{figure}

The Diagram algebra, \(D_G(Q)\), for a graph \(G\) is defined as the subalgebra of the Partition algebra with generators:

\[
1 = (11')(22')\ldots(nn'),
A^i = (11')(22')\ldots(i)i'\ldots(nn'), \quad \forall i \in \Lambda_G^0,
A^{ij} = (11')(22')\ldots(i\ j\ i'\ j')\ldots(nn'), \quad \forall (i,j) \in \Lambda_G^1.
\]

Note that \(1^{ij} = ((11')(22')\ldots(ji')\ldots(ji)\ldots(nn')) \in P_n(Q)\), is \textit{not} in \(D_G(Q)\).
The Diagram algebra may be also be thought of (visualized) on \(G \times A_k\) \((k \text{ large})\) as the restriction of \(P_n(Q)\) to partitions achievable as connectivities (i.e. a set of mutually non-intersecting trees c.f. \([10]\)) between the nodes of the bottom layer (the nodes \((i, 1)\) to be called \(i \forall i \in \Lambda^0_G\)), and those of the top layer (the nodes \((i, k)\) are to be called \(i' \forall i \in \Lambda^0_G\)).

Note that, with \(V = \mathcal{A}^Q\),

\[
\rho_G : D_G(Q) \rightarrow \text{End}(V^{\otimes |\Lambda^0_G|}),
\]

(13)

is a representation of the Diagram algebra called the Potts representation (eqs. 7-8).

The Potts representation is generically faithful for \(G = A_n\), and for this reason, we here try \(D_G(Q)\) as a candidate for the appropriate generalization of the Temperley-Lieb algebra for arbitrary graph \(G\). Note in particular that \(D_{A_n}(Q)\) is isomorphic to the Temperley-Lieb algebra for any \(Q\), including non-integer values.

The partition function \(Z(L)\) may be computed working in \(D_G(Q)\) instead of in the defining Potts representation \([2]\), as in the 2-dimensional case, where the \(D_{A_n}(Q)\) calculation is that of the square lattice dichromatic polynomial \([3][10]\).

In the two dimensional case the exceptional models may be identified directly at the level of algebra by finding the \(Q\) values for which the structure of the Temperley-Lieb algebra departs from the generic semi-simple structure. Our idea is that the departures from generic behaviour would be important for arbitrary \(G\). The structure of \(D_G(Q)\) is important "physically," since it may be used to characterize the spectrum of the transfer matrix, \(T_G\). Thus we proceed to analyse the structure of \(D_G(Q)\). This is already known for some \(G\); in particular, for \(G = A_n\) and for \(G = K_n\), the complete graph on \(n\) nodes \([19]\). In this paper we consider graphs appropriate for higher dimensional Potts models and dichromatic polynomials, including sequences appropriate for the physically crucial cubic lattice Potts models, and the bi-plane lattices to which recent ideas in high \(T_c\) superconductivity have drawn attention \([17]\).

## 2 Generic structure of \(D_G(Q)\)

Mathematically, the first step in determining the structure (representation theory) of an algebra is generally to label the irreducible representations. In what follows we take \(n = |\Lambda^0_G|\). The irreducible representations of \(P_n(Q)\) are labelled by

\[
\mathcal{L}_n = \{\lambda : i = 0, 1, 2, ..., n\}
\]

and since \(D_G(Q) \subset P_n(Q)\) all the irreducibles must be somehow contained in the irreducibles of \(P_n(Q)\).
Consider $P_n(Q)$ as a $D_G(Q)$ module. Clearly any $P_n(Q)$ module is also a $D_G(Q)$ module. Now $P_n(Q)$ has been filtered into invariant subspaces with bases

$$B_i = \{ x \in S_{2n} \mid \#^p(x) \leq i \}$$

where $\#^p(x)$ is the number of parts of $x$ containing both primed and unprimed nodes, called the “propagating number” of $x$.

If we define

$$E_i^{(n)} = \prod_{j=i+1}^{n} \left( \frac{A_j}{Q} \right),$$

then $\mathcal{C}\text{-span}(B_i) = P_n E_i P_n$. For a given $n$, we drop the superscript $(n)$ and write $E_i$. Note that $\#^p(E_i) = i$ and for $a, b \in S_{2n}$,

$$\#^p(ab) \leq \min(\#^p(a), \#^p(b)),$$

and we ignore elements of $\mathcal{C}$ in evaluating $\#^p(z) \forall z \in P_n(Q)$. Thus

$$P_n[i] = P_n E_i P_n / P_n E_{i-1} P_n$$

is a $P_n(Q)$ module with basis $B_i \setminus B_{i-1}$. Note that in the diagrammatic realization of the left action of $D_G(Q)$ on $P_n[i]$ the “bottom” of each $x \in B_i \setminus B_{i-1}$ (i.e. the connectivities of the unprimed nodes of any $x \in S_{2n}$) remains unchanged. That is, all elements with the same bottom form a submodule.

For example $\Delta_i := P_n E_i$ (mod. $P_n E_{i-1} P_n$) is one of the left $P_n$ submodules of $P_n[i]$, and $P_n[i]$ may be decomposed into submodules all of which are isomorphic. Note that $\Delta_i$ has a basis the set of partitions which have each unprimed node in a different part, the last $n - i$ nodes singletons (i.e. in parts on their own), the others connected to primed nodes.

In fact as a left $D_G(Q)$ module $\Delta_i$ breaks as $D_G(Q)E_i \oplus R_i$ where $R_i$ is either empty or a direct sum of one dimensional modules (see Appendix), so we need only focus on $D_G E_i \mod D_G E_{i-1} D_G$.

The final piece of the jigsaw for $P_n(Q)$ is to note that $P_n E_i$ is a projective right $S(i)$ module (i.e. a direct summand of a direct sum of copies of the regular representation of the symmetric group $[S(i)]$) where the action is to permute the first $i$ (unprimed) nodes. For example, see figure 2.
Thus $P_{\lambda} E_i \pmod{D_G E_{i-1} D_G}$ breaks up into simple modules indexed by $\lambda \vdash i$ (from $S(i)$ representation theory [21]).

For $D_G(Q)$, however, the picture is more complicated, since $D_G E_i$ is not always closed under the right action of $S(i)$. For example, whereas $P_n(Q) E_n \equiv S(n) \mod P_n(Q)$, we have $D_G E_n = C E_n = C \cdot 1 \mod D_G E_{n-1} D_G$ for any $G$. To see this note that with $n$ propagating lines from bottom to top of $G \times A_k$ ($k$ large) there is only one possibility, as depicted in figure 3.

Our problem is thus reduced to determining the maximum subgroup $H^i_G \subset S(i)$ for which $D_G E_i \pmod{D_G E_{i-1} D_G}$ is a right module. In general, for $i < n$, the situation depends on $G$.

Before actually determining $H^i_G$, let us first explicitly construct words in the algebra that would implement the group action. As is clear from figure 3, one or more nodes of $G$ need to be disconnected to allow for walks on $G \times A_k$ to realize any permutations of the nodes (except for the identity permutation as in figure 3). Also, since there is no unique, or natural ordering of the nodes of $G$, we need to determine whether $H^i_G$ depends on the choice of the nodes disconnected by $E_i$.

For any subset, $s$ of $\{1, 2, \ldots, n\}$, let $p_s$ be the set difference $\{1, 2, \ldots, n\} \setminus s$, and $E_{\{p_s\}} = \prod_{j \in s} (A^j/Q)$. For example, for $i = \{i + 1, i + 2, \ldots, n\}$,

$$p_2 = \{1, 2, \ldots, i\} \quad \text{and} \quad E_{\{p_2\}} = \prod_{j=i+1}^n \left( \frac{A^j}{Q} \right) = E_i. \quad (15)$$

**Definition 1** The partition basis of $D_G(Q)$, denoted by $S_{2n}^G$ is $S_{2n} \cap D_G$. Also, set $B^G_i := B_i \cap D_G$. 

**Figure 3** There is no space for lateral motion if all of the nodes of $G$ (the hexagon) are propagating. The propagating lines are drawn with double lines.
Definition 2 Let \( s, t \subset \{1, 2, \ldots, n\} \), s.t. \(|s| = |t|\). Then

\[
\Phi^t_s := \{ \varphi^t_s \mid \varphi^t_s = E_{(p_t)}X E_{(p_s)} \}, \quad \forall X \in S_{2n}^G \text{ s.t. } \#(E_{(p_t)}X E_{(p_s)}) = |p_s|. \tag{16}
\]

These elements of (16) may be interpreted as bijections, \( \varphi^t_s : p_s \rightarrow p_t \). Note, in particular, that \( \Phi^t_s \subseteq S(p_s)E_{(p_s)} \).

Let \( \delta = s \triangle t = (s \setminus t) \cup (t \setminus s) \), the symmetric difference of sets \( s, t \), with \(|\delta| = 2d\), i.e., \( d \) elements of the \( n - i \) elements of \( s \) are distinct from those of \( t \). Then there exist partitions of \( \delta \) of shape \( 2^d \), i.e., of the form

\[
((\delta_1 \delta_1^*)(\delta_2 \delta_2^*) \cdots (\delta_d \delta_d^*)),
\]

where the unstarred nodes of \( \delta \) are in \( s \) and the starred ones in \( t \). Consider chain subgraphs, \( A^{(i)}_{\mu_i}, i = 1, \ldots, d \) of the connected graph \( G \), with nodes labelled \( x_j^{(\mu_i)} \), \( j = 1, 2, \ldots, \mu_i \) such that the first node of \( A^{(i)}_{\mu_i} \) is \( x_1^{(\mu_i)} = \delta_i \) and the \( \mu_i \)th, \( x_{\mu_i}^{(\mu_i)} = \delta_i^* \). Let us construct words \( \omega_{A^{(i)}_{\mu_i}} \) of the form

\[
\omega_{A^{(i)}_{\mu_i}} = \prod_{j=0}^{\mu_i-2} (A^{x_j^{(\mu_i)}x_{j+1}^{(\mu_i)}}_{x_{j-1}^{(\mu_i)}x_{j+1}^{(\mu_i)}})A_{x_{\mu_i}^{(\mu_i)}}^{(\mu_i)}.
\]

s.t. \( \omega_{A^{(i)}_{\mu_i}} \) achieves the connectivity which differs from the unit in

\[
(\cdots(x_1^{(\mu_i)}) (x_2^{(\mu_i)} x_2^{(\mu_i)})' (x_3^{(\mu_i)} x_3^{(\mu_i)})' \cdots) \in D_G(Q)
\]

on the sublattice \( A^{(i)}_{\mu_i} \times A_k \), \( k > \mu_i \), where (as before) \( (x_j^{(\mu_i)}, 1) = x_j^{(\mu_i)} \) and \( (x_j^{(\mu_i)}, k) = x_j^{(\mu_i)}' \). (See figure 4.)

\[\text{Figure 4} \text{ The element of the algebra shifting the "hole" from } \delta_i \text{ to } \delta_i^*. \text{ Note the minimum height required to achieve this connectivity is of the order of the distance } |\delta_i - \delta_i^*|.\]

It is useful to view the element of the algebra as one that pushes a “hole” from its location in \( s \) to one in \( t \). The equivalence relation that defines the algebra \( D_G(Q) \subset P_n(Q) \) implies that

\[
\omega_{\{\mu_i\}} = \prod_{i=1}^d \omega_{A_{\mu_i}} \in \Phi^t_s,
\]

independent of the choice of graphs \( A^{(i)}_{\mu_i} \) connecting the nodes of \( \delta \). Thus,

Proposition 1

\[
D_G E_{(p_s)} D_G = D_G E_{t} D_G, \quad \forall s \subset \{1, 2, \ldots, n\}, \quad \text{s.t. } |s| = n - i. \tag{21}
\]
The different choices of pairing the starred and unstarred nodes of \( \delta \) give different bijections \( \phi^i_s \in \Phi^i_s \). Let \( H^p_G := \Phi^p_s \). We then have

**Proposition 2** For any fixed element \( \phi^i_s \in \Phi^i_s \), \( \phi^i_s \Phi^j_s = H^p_G \) and \( H^p_G = H^p_G \) if \( |s| = |t| \).

**Proof**: For sets \( s, t, r \) of the same cardinality, these 'bijections' obey

\[
\rho^i_t \circ \phi^i_s \in \Phi^i_r \quad \forall \rho^i_t \in \Phi^i_r \quad \text{and} \quad \phi^i_s \in \Phi^i_t.
\]

Therefore,

\[
\Phi^i_t \Phi^j_s \subseteq \Phi^j_s \Rightarrow |\Phi^i_t| \leq |\Phi^j_s| \quad \text{and} \quad |\Phi^i_t| \leq |\Phi^j_s| \Rightarrow |\Phi^i_t| = |\Phi^j_s|
\]

for any \( r, s, t, u \in \{1, 2, \ldots, n\} \) with \( |r| = |s| = |t| = |u| \).

In particular, \( |\Phi^i_t| = |\Phi^j_s| \) and \( \phi^i_t \Phi^j_s \subseteq \Phi^j_s \Rightarrow \phi^i_t \Phi^j_s = \Phi^j_s \), for a fixed \( \phi^i_t \Phi^j_s \). Also, \( \rho^i_s \Phi^j_s \phi^i_t = \Phi^j_t \). This implies that \( \Phi^i_s \cong \Phi^j_t \) and depends only on the cardinality, \( |p_s| \).

\[ \square \]

**Corollary 2.1**

\[
D_G E_{\{p_s\}} \cong D_G E_i \quad \text{and} \quad E_i D_G E_i \cong E_i \otimes H^i_G \mod. D_G E_{i-1} D_G
\]

(24)

where \( H^{i-1}_G \) is no smaller than the maximal subgroup of \( S(i-1) \) contained in \( H^i_G \); (so that in particular if any \( H^i_G = S(i) \) then \( H^i_G = S(j) \forall j < i \)).

Hence,

**Theorem 1** Let \( \Gamma^i_G \), \( \Gamma_G \) be index sets for irreducible representations of \( H^i_G \) and \( D_G(Q) \) respectively. Then

\[ \Gamma_G = \cup_i \Gamma^i_G. \]

2.1 **How to compute** \( H^i_G \).

For an arbitrary graph \( G \), \( |A^0_G| = n \), and \( \alpha \in A^0_G \), consider closed chain subgraphs, \( \hat{A}_{p+1} \subseteq G \) with \( \alpha \in A^0_{\hat{A}_{p+1}} \). Setting \( d = 1 \), \( x^{(\mu_i)} = x^{(\mu_i)} \) and \( \mu_i = p + 2 \) in (20), we note \( \Phi^i_{\{\alpha\}} \) contains \( Z_p \). By pushing the hole around, by (15), so as to lie on other closed chain subgraphs, the set of these \( p \)-cycles generate \( H^{n-1}_G \).
Figure 5 An example of a 3-cycle (BAC), with $G = A_2 \times A_2$. If we label the nodes such that at the bottom layer, A,B and C are drawn through 1,2 and 3 respectively, with the “hole” at 4, $p_{\{4\}} = \{1,2,3\}$ and the connectivity drawn is $((1'2')(2'3')(3'1')) \in \Phi^{\{4\}}$.

For any graph $G$, let $G^x$ denote the graph obtained by removing the node $x \in \Lambda^G$ and the bonds connected to it, i.e., $\Lambda^G = \{1,2,\ldots,n\} \setminus \{x\}$ and $\Lambda^G = \Lambda_G \setminus \{(i,x) | (i,x) \in \Lambda^G\}$. Let $G^x_{(x,y)}$ denote the graph obtained by removing the bond $(x,y)$, i.e., $\Lambda^G = \Lambda_G$ and $\Lambda^G \setminus \Lambda^G_{(x,y)} = \{(x,y)\}$. Recall, $E_{n-1}D_G E_{n-1} \cong E_{n-1} \otimes H_G^{n-1}$. Then, we have

**Proposition 3** Let $|\Lambda^G_i| = n_i \forall i$.

i) If $G^x = G_1 \cup G_2$, then $H_G^{n-1} = H_{G_1}^{n-1} \times H_{G_2}^{n-1}$ where the 2 factors act on the $n_1 - 1$ and $n_2 - 1$ nodes in $G_1$ and $G_2$ respectively.

ii) If $G^x = G_1 \cup G_2 \cup \ldots$, then $\exists G_i \supset G_i^x \forall i$ s.t. $\cap_i \Lambda^G_i = \{x\}$ and $\cap_i \Lambda^G_i = \{\emptyset\}$, s.t. $H_G^{n-1} = H_{G_1}^{n-1} \times H_{G_2}^{n-1} \times \ldots$, where $H_{G_i}^{n-1}$ acts on $G_i$ only.

**Remark.** Let us call graphs $G$ that do not decompose in the sense of the previous proposition *unsplittting*. A simple example of an unsplittting graph is the closed chain graph $A_n$. We shall consider other examples below.

**Definition 3** The graph $\Theta^n_{p,q}$ ($q < p \leq n - 1 < 2p + q$) has $n$ nodes labelled $\{1,2,\ldots,n\}$, and bonds,

$$\Lambda^1_{p,q} := \{(1,n), (p+q,n), (p,p+q+1), (i,i+1); i = \{1,2,\ldots,n-1\} \setminus \{p+q\}\}.$$  

(See figure 6.) Note that $\Theta^n_{n-1,0} = A_n$.

**Proposition 4** All unsplittting graphs $G$ that are not closed chain graphs contain $\Theta^n_{p,q}$ as a subgraph for some positive integers $p,q$. 

Corollary 5.1 For $G = \Theta_{p,q}^n$, let $\beta_1 := \{p + q + 1, p + q + 2, \ldots, n - 1\}$, $\beta_2 := \{1, 2, \ldots, p - 1\}$ and $\beta_3 := \{p + 1, p + 2, \ldots, p + q\}$, and let $\alpha_i := \{1, 2, \ldots, n\} \setminus \beta_i$, $i = 1, 2, 3$. Then, for $\alpha_i \in \alpha_i$, $i = 1, 2, 3$, and defining $\Phi_{\alpha_i}^{\{a_i\}} \subseteq \Phi_{\alpha_i}^{\{a_i\}}$ to be the words of the form $\omega_{\alpha_i}$ as in eq. $18$, we have $\Phi_{\alpha_1}^{\{a_1\}} \cong \mathbb{Z}_{p+q}$, $\Phi_{\alpha_2}^{\{a_2\}} \cong \mathbb{Z}_{n-p}$ and $\Phi_{\alpha_3}^{\{a_3\}} \cong \mathbb{Z}_{n-q}$. Also $\varphi_{\{a_i\}}^\alpha(x) = x$, $\forall \varphi_{\{a_i\}}^\alpha \in \Phi_{\alpha_1}^{\{a_i\}}$ and $x \in \beta_i$ for $i = 1, 2, 3$.

Proof: This is achieved by a sequence of words as in the proposition that “moves” one of $x$, $y$, while keeping the other fixed. Such a move is easiest if $x \in \alpha_i$ and $y \in \alpha_j$, $i \neq j$. If $x, y \in \alpha_i$, we first move both until only one of $x, y$ is in $\beta_j$ for some $j$. $\blacksquare$

Corollary 5.2 For $\Theta_{p,q}^n$ as above, and $n \geq 4$,

\[
\begin{align*}
H_{\Theta_{n-2,0}^n}^{-1} &= S(n-1),
H_{\Theta_{n-3,0}^n}^{-1} &= \begin{cases} 
S(n-1) & \text{for } n \text{ odd} \\
A(n-1) & \text{for } n \text{ even.}
\end{cases}
H_{\Theta_{n-3,1}^n}^{-1} &= \begin{cases} 
A(n-1) & \text{for } n \text{ odd} \\
S(n-1) & \text{for } n \text{ even.}
\end{cases}
\end{align*}
\]

(25)

Proof: By the same procedure as in the corollary above, it is possible to construct $\varphi_{\{n\}}^\alpha \in \Phi_{\{n\}}^\alpha$, such that for $G = \Theta_{n-2,0}^n$, $\varphi_{\{n\}}^\alpha(x) = n - 2$ and $\varphi_{\{n\}}^\alpha(y) = n - 1$ for $x, y \in \Lambda_0^G \setminus \{n\}$ and $G = \Theta_{n-3,0}^n, \Theta_{n-3,1}^n$, $\varphi_{\{n\}}^\alpha(x_i) = n - i, i = 1, 2, 3$ and $x_i \in \Lambda_0^G \setminus \{n\}$.

Thus, for $\Theta_{n-2,0}^n$ we can achieve arbitrary transpositions, which generate $S(n-1)$, and for $\Theta_{n-3,0}^n, \Theta_{n-3,1}^n$, we can achieve all 3-cycles, generating $A(n-1)$. However, by the proposition above, we can also realize $\Phi_{\alpha_i}^{\{a_i\}} \cong \mathbb{Z}_{p+q}$, which for $G = \Theta_{n-3,0}^n, \Theta_{n-3,1}^n$, $\alpha_i \subseteq \mathbb{Z}_{n-3}$ and $\mathbb{Z}_{n-2}$ respectively. For $n$ even (odd) these would give even (odd) permutations for $\Theta_{n-3,0}^n, \Theta_{n-3,1}^n$ respectively. Hence the result. $\blacksquare$
2.2 On constructing a partition basis for $D_G$.

In any partition in $S_{2n}$, perform the operations of ignoring either the elements $i'$ or the elements $i$ for $i \in \{1, 2, \ldots, n\}$. These may be viewed as sub-partitions of the nodes $i$ and $i'$, which we call “bottoms” and “tops” respectively. The elements of $S_{2n}^G$, the partition basis of $D_G(Q)$ can be constructed diagrammatically, by figuring out the possible “top” and “bottom” configurations that can be achieved by drawing connectivities on $G \times A_k$ for $k$ large, and the possible ways of gluing the top and bottom by the $\#^p(z) = i$ “propagating lines.” The ways of joining bottom to top are dictated by $H_G^i$, so the next step in this program is to determine the set of allowed tops, the bottoms being isomorphic under up-down transposition.

Proposition 6 For $G = \Theta_{p,q}^n$ ($q \leq p < n - 1 < 2p + q$), a partition basis element $z \in \Delta_i$ ($i < n$) is also in the partition basis $B_i^G$ of a $D_G(Q)E_i$-module iff

i) $\exists$ at least one part of $z$ of the form $(a')$ (a singleton node, $a \in \Lambda_G^0$), or

ii) in the sub-partition of $z$ consisting of nodes $i'$, $i \in \Lambda_G^0$, one of the parts is of the form $(\cdots \cdot a'b' \cdot \cdots)$, where $(a, b) \in \Lambda_G^1$.

and the rest may be partitioned in any arbitrary way.

For all elements of the form i), the configurations of “tops” in $B_i^G$ depend only on $n$ and not on $p, q$.

Proof:

(Only if.) Every word except 1 in $D_G$ must begin with either $A_{j'}$, $j \in \Lambda_G^0$ or $A^i_{j'}$, $(i, j) \in \Lambda_G^1$.

(By construction of such a $z$.) Without loss of generality, let the partition $z$ have the primed nodes in a sub-partition of the shape $(l_1, l_2, \ldots, l_r, 1)$ where the last part is the singleton $(a')$ as in case i). For case ii), all words may be written as $A^{a'b}z$, with $z$ constructed as in case i). For $k = 1, 2, \ldots, i$, the parts are $(a_1^{(k)} a_2^{(k)} \cdots a_{l_k}^{(k)} \pi(k))$, where $a_j^{(k)}$, $j = 1, \ldots, l_k$ are the primed nodes and $\pi(k)$ is the image (under $\pi \in H_G^i$) of the bottom node $k$. For parts numbered $k = i + 1, i + 2, \ldots, r$, the parts are of the form $(a_1^{(k)} a_2^{(k)} \cdots a_{l_k}^{(k)})$, the $(r + 1)^{th}$ part is the singleton $(a')$, and the remaining parts consist of singleton bottom nodes.

For all $\alpha, \beta \in \Lambda_G^0$ such that $|\alpha| = j = |\beta|$, let $\varphi_{j} := \cup_{\alpha, \beta} \psi_{\alpha}^{j}$. For a given $z$, define the words

$$\varphi_{j}^{(k)} \in \Phi_{s(z, j, k)}$$

where $s(z, j, k) = \sum_{u=1}^{k-1} l_u + j - \min(i, k-1)$, and

and $\psi_{j} \in \Phi_{l_j}$, by

$$\begin{align*}
\varphi_{j}^{(k)}(1) &= n, & 1 \leq k \leq r \\
\varphi_{j}^{(k)}(a_{j+1}^{(k)}) &= 1, & \forall j = 1, 2, \ldots l_k - 1, & 1 \leq k \leq r \\
\varphi_{j}^{(k)}(\pi(j)) &= \pi(j), & k > j \\
\psi_{k}(1) &= \pi(k), & k = 1, 2, \ldots, i
\end{align*}$$

(26)
where the domain and ranges of the elements of $\Phi_{s(z,i,k)}$, $\Phi_j$ are nodes of $G$. Note that, $\pi(k)$ indicates the positions of the bottom nodes in $z$ as required. Such a construction is possible by corollary 4.1. The word

$$A^n_a \cdot \left( \prod_{k=1}^i \left( \prod_{j=1}^{l_k-1} \varphi_j^{(k)} A^{1,n} A^n \cdot \right) \psi_k \right) \prod_{k=i+1}^r \left( \prod_{j=1}^{l_k-1} \varphi_j^{(k)} A^{1,n} A^n \cdot A^1 \right) \right) E_i$$

in $D_G$ constructs the required $z$. (See figure 7).

Figure 7 The figure depicts how the word $A^4_a \cdot \left( \prod_{j=1}^3 \varphi_j^{(1)} A^{12,1} A^{12} \cdot \right) \psi_1$ builds the partition $((8' 9' 10' 11' 3' 12' 6')(7' 1)(6' 2')(5' 5')(3' 12')(2' 4')(1')(4')(8')(9')(10')(11))$ on the lattice $G \times A_k$, where $G$ is of the type $\Theta^{n=12}_{p,q}$. The dashed boxes denote specific letters in $D_G$ while those drawn with continuous lines are words in $D_G$. Note that the number of lines in the interior of each of the solid boxes above decreases from top to bottom.

This proposition will then enable us to estimate the cardinality of the partition basis of $D_G(Q)$, which we need for the next sub-section.
2.3 On locating the exceptional values of $Q$:

The next stage in giving the generic structure is to give the dimensions of the irreducibles and an explicit construction of a basis for each. Let $\epsilon_\gamma$ be a primitive idempotent of $H^i_G$, i.e. such that

$$R_\gamma = H^i_G \epsilon_\gamma, \quad \gamma \in \Gamma^i_G$$

is an irreducible representation of $H^i_G$ (all of these are known, from [22] for example). Then

$$W_\gamma = D_G(E_i \otimes \epsilon_\gamma)$$

builds generic irreducible $W_\gamma$.

In fact our present concern is not to determine the generic irreducible dimensions for a given $G$, but to locate the exceptional $Q$ values (by analogy with the Beraha numbers for $G = A_n$ which is relevant for the two dimensional case). To this end it is highly indicative to proceed as follows.

We first decide on a sequence approaching the large graph limit (not the same as the thermodynamic limit, see below). Thus, we take a sequence of graphs $G^{(-)} = \{G^j : j = 1, 2, \ldots\}$ (with, say, $A_l \times A_m$ ($l, m$ large) at the ‘end’ if we were to consider graphs appropriate for cubic lattice Potts models as in the next section) and then determine

$$k_{G^{(-)}}^{i_\gamma} = \lim_{j \rightarrow \infty} \frac{\dim(D_{G(j)}(E_i^{(n_j)} \epsilon_\gamma))}{\dim(D_{G(j-1)}(E_i^{(n_{j-1})} \epsilon_\gamma))}, \quad n_j = |\Lambda^0_{G(j)}|,$$

if it exists.

Since for the $Q$-state Potts model representation,

$$\frac{\dim(\rho_{G(j)})}{\dim(\rho_{G(j-1)})} = Q^m$$

for any sequence such that $|\Lambda^0_{G(j)}| = |\Lambda^0_{G(j-1)}| + m$, $m$ a positive integer, if $k_{G^{(-)}}^{i_\gamma} > Q^m,$ then $Q$ is exceptional by the following argument [9] Since $E_0$ is a primitive idempotent (i.e., $E_0D_GE_0 = \mathcal{C}E_0$), $D_GE_0$ is indecomposable. If $D_G(Q)$ were semisimple, $D_GE_0$ would be contained in $\rho_G$ with multiplicity 1 for all $G$. Thus, the evaluation of the case $\kappa_G := k_{G^{(-)}}^{0(0)}$ is sufficient for any sequence $G^{(-)}$. So, if $\dim(D_GE_0) > Q^n$ (for $Q$ an integer) for $n = |\Lambda^0_{G(j)}|, Q$ is exceptional. For example $k_{A^{(-)}}^{i_\gamma} = 4$ where $A^{(-)} = \{G^j = A_j\}_{j \geq 1},$ and this signals the special nature of the $Q = 1, 2, 3$ state Potts models in two dimensions.

We can now consider the asymptotic growth rate of dimensions of the irreducible representations. In particular, we estimate a lower bound on the dimension of the module $D_GE_0$.

Proposition 7 For $G = \Theta^n_{p,q}, (q \leq p \leq n-2)$, and any $k \in \mathbb{R}$, there exists a natural number $M$, such that $\dim(D_GE_0) > k^n$ for $n > M$. 
Corollary 9.1

For the irreducible representations of the diagram algebras for such graphs with \( \Theta \) non-isomorphic outer automorphism-conjugate representations labeled is the index set for irreducible representations of \( D \).

Proof:

Let \( b_m \) be the number of ways of pairing \( m \) nodes (for some even number \( m \)). For any node (say 1), its partner (in the partition of shape \( (2^{m/2}) \)) can be chosen in \( m - 1 \) ways, while the rest of the pairs can be chosen in \( b_{m-2} \) ways. This determines \( b_m = (m - 1)b_{m-2} \) for all even \( m \), with \( b_1 = 1 \). Thus for any \( k \in \mathbb{R}, \exists M \), a natural number, such that \( b_m > k^m \) for \( m > M \). From proposition 5, for any basis element of \( D \) for \( G = \Theta^n_{p,q} \), \( n - 1 \) primed nodes may be partitioned in any arbitrary way. The number of such possibilities is clearly larger that \( b_{n-1} \) (where we have chosen \( n \) to be odd, without loss of generality). Therefore,

\[
\dim (D_G E_0) > k^n.
\]

Thus, for any integer \( Q \) and \( G \) of the type \( \Theta^n_{p,q} \), the dimension of \( D_G E_0 \) is larger than that of the Potts representation, \( \rho_G \) (> \( Q^n \) for \( Q \) a positive integer).

We thus have

Proposition 8 Consider a sequence \( G^{(-)} := \{G^{(j)}\}_{j \geq 1} \) of unsplitting graphs (except \( G^{(j)} = \hat{A}_j \)). For any positive integer \( Q \), \( \exists \) an integer \( n_1 \) s.t. \( \forall n_2 > n_1, D_{G^{(n_2)}} \) is not semi-simple.

3 Cubic lattice Potts models: \( G = A_l \times A_m \)

This is the case we are most interested in, to which we shall apply the results obtained in section 2.

Proposition 9 For \( G = A_l \times A_m \), \( (l, m \geq 2) \), and \( n = lm \),

i) \( H_G^{l-1} = A(n-1) \)

ii) \( H_G^{l} = \hat{S}(i), i < n - 1. \)

Proof: i) For \( n \) even, \( G \) has a subgraph \( \Theta^n_{n-3,0} \); and for \( n \) odd, it has a subgraph \( \Theta^n_{n-1,1} \). Recall corollary 4.2. ii) It is easy to see that \( H_G^{l-2} \) and \( H_G^{l-2} \) for \( n = 4 \) are isomorphic to each other and to \( H_G^{l-3} \) for \( n = 3 \). For \( i \leq n - 3 \), recall [24].

Recall that the representations of \( A(j) \) are indexed by unordered pairs of partitions, \( \lambda \vdash j \) and its conjugate \( \lambda' \), for \( \lambda \neq \lambda' \). For \( \lambda = \lambda' = \bar{\lambda} \), there are two non-isomorphic outer automorphism-conjugate representations labeled \( \bar{\lambda} \) and \( \bar{\lambda}' \).

Corollary 9.1 For \( G = A_l \times A_m \),

\[ \Gamma_G = \bigcup_{i=0}^{l-2} \{ \lambda \vdash i \} \cup \{ (\lambda, \lambda'), \bar{\lambda}, \bar{\lambda}' \mid \lambda, \bar{\lambda} \vdash n - 1 \} \cup \{ \lambda \vdash n = (n) \}. \]

is the index set for irreducible representations of \( D_G(Q) \).

Note that by filling in one of the diagonals of an elementary plaquette of \( G = A_l \times A_m \), we obtain a graph which has \( \Theta^n_{n-2,0} \) as a subgraph. The index set for the irreducible representations of the diagram algebras for such graphs with \( \Theta^n_{n-2,0} \) subgraphs is the same as that of the complete graph on \( n \) nodes. (See [20].)

Since \( A_l \times A_m \supset \Theta^n_{p,q} \) for some \( p, q \), we have (from proposition 5),
Corollary 9.2 For $G = A_l \times A_m$, $(n = lm)$, a partition basis element $z \in \Delta_i$ $(i < n - 1)$ is also in the partition basis $B^G_i$ of a $D_G(Q)E_i$-module iff

i) $\exists$ at least one part of $z$ of the form $(a')$ (a singleton node, $a \in \Lambda^0_G$), or

ii) in the sub-partition of $z$ consisting of nodes $i'$, $i \in \Lambda^0_G$, one of the parts is of the form $(\cdots a'b' \cdots)$, where $(a, b) \in \Lambda^1_G$, and the rest may be partitioned in any arbitrary way.

Also, since $G = A_l \times A_m$ is of the unsplitting type, we infer

Proposition 10 Consider a sequence $G^{(-)} := \{G^{(j)}\}_{j \in \mathbb{Z}^+}$, where $G^{(j)} = A_l \times A_m$, $l, m > 1, lm = j$. For any positive integer $Q$, $\exists$ an integer $n_1$ s.t. $\forall n_2 > n_1$, $D_G^{(n_2)}$ is not semi-simple.

4 Discussion

For $G = A_l \times A_2$ for instance, it might have naively been expected that for large $l$, the results of the familiar $A_l$ case might be approached. However, instead of the known growth rate of dimensions, i.e. 4, we get an unboundedly large number. This discrepancy might be attributed to the length, $k$ of the graph $A_k$ in the transfer ("time-like") direction of the lattice $L = G \times A_k$, on which the partition function is evaluated. The connectivities of nodes are achieved involve "permuting the nodes," i.e., the action of $[\mathfrak{20}]$, where each shift $[\mathfrak{18}]$ can only be realized for $k > \mu_i$. A restriction of the maximum $k$ allowed will obviously reduce the dimensions and their growth rate $\kappa_G$. This is clearly necessary to define the true thermodynamic limit, where the volume has to increase in a specified fashion, keeping the ratios of lengths in all the directions of the lattice, $L$ fixed to some finite value, unlike in the definition of $\kappa_G$, where the size of the $G$ was increased independent of $k$. In the two-dimensional case, the connectivities can all be achieved on $L = A_n \times A_k$ for $k \sim n$, and the problem does not arise. Thus, it might be useful to define a certain "cutoff” height $k$ of the representations of $D_G(Q)$ to narrow in on the physically relevant sectors of the representation theory.

We have indicated that for the smallest deviations away from chain graphs, e.g., $\Theta_{n-2,0}^n$, the diagram algebra is too large to carry directly useful physical information. Suitable quotients have to be implemented to reduce the size of the representations and an appropriately quotiented algebra would then be the analogous "generic" algebra for the cubic lattice models. The special values of $Q$ for which the algebra ceases to be semi-simple is the obvious place to look for the quotient relations that are relevant for the Potts representation which is defined for integer values of $Q$. These integers are certainly a subset of the special points where the algebra ceases to be semi-simple, as we have shown. We expect that the techniques outlined in the appendix can be extended to obtain the degeneracies of the cubic lattice Potts spectrum, which we would like to report in the future. We have also undertaken preliminary calculations on the location of other $Q$-values for which $D_G(Q)$ becomes non-semi-simple for $G = A_l \times A_m$, and so far found only rationals. Further studies are in progress.
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5 Appendix

Proposition 11 As left $D_G(Q)$ modules, $P_n \mathbf{E}_i \cong D_G \mathbf{E}_i \oplus R^G_i$ modulo $D_G \mathbf{E}_{i-1} D_G$ for any $G$, $|G| = n$, where $R^G_i$ is either empty or a direct sum of $\theta_i$ copies of the trivial $H^i_G$-modules, $\chi(i)$, $(i) \vdash i$, $R^G_i \cong \theta_i \chi(i)$. Further, for $\lambda \vdash i \in \{0, 1, 2, \ldots, n\}$, $n = |\Lambda^0_G|$, 

$$
\text{Res}_{D_G}^{P_G} P_G \mathbf{V}_\lambda = \bigoplus \left\{ \begin{array}{ll}
D_G W_\lambda \oplus \theta_i \chi(i), & \forall i \in \{1, 2, \ldots, n-2\}
\lambda, \lambda' \vdash i = n-1,
D_G W_\lambda \oplus D_G W_{\lambda'} \oplus \theta_i \chi(i) & \text{for } \lambda = \lambda', i = n-1.
\end{array} \right.
$$

Proof: $A^x P_n \mathbf{E}_i = A^{k,k} P_n \mathbf{E}_i = 0 \mod D_G \mathbf{E}_i$, therefore, $D_G x = 1x \mod D_G \mathbf{E}_i \forall x \in P_n \mathbf{E}_i$. $H^i_G$ acts trivially. For $\lambda \vdash n-1$, the labels of the representations of $P_n(Q)$ and $D_G(Q)$ are those of $\mathbb{S}(n-1)$ and $\mathbb{A}(n-1)$ respectively, and $\text{Res}_{\mathbb{A}(n-1)}^{\mathbb{S}(n-1)}$ must be invoked.

To characterize the generic structure of the algebra completely, it is necessary to determine the dimensions of its irreducible representations. Also, it is useful to characterize the inclusion of algebras, while approaching the large graph limit described above, in order to identify the subspaces that carry the information relevant for a physical interpretation. A preliminary step would be to determine how, for $H \subset G$, $D_G(Q)$-modules split up as $D_H(Q)$-submodules. Henceforth, we shall denote a left $R$-module $M$ as $rM$.

Proposition 12 For $G = A_i \times A_m$ and $G \supset H = A_{i-1} \times A_m$,

$$
D_G \mathbf{E}_i \cong \bigoplus_{j=-m}^m (D_H \mathbf{E}_{i+j} \oplus R_{i+j}^{G,H}),
$$

as left $D_H(Q)$ modules, where $R_{i+j}^{G,H}$ is either empty or $\theta_{i+j} \chi(i+j)$, where $\theta_k$ is the multiplicity of the trivial $H^k_H$-module, $\chi(k), (k) \vdash k$.

Proof: Let $\underline{m} := \{n-m+1, n-m+2, \ldots, n\}$ and $p_m := \{1, 2, \ldots, n-m\}$. If $w \in D_G \mathbf{E}_i$ s.t. none of the parts of $w$ is of the form $(\ldots k', \ldots l')$ for $k' \in p_m$ and $l' \in \underline{m}$, $D_H w \cong \bigoplus_{j=0}^m D_H \mathbf{E}_{i-j}$. Each summand indexed by $j$ denotes the number of parts of $w$ which contain only primed nodes.

Similarly, if the nodes of $\underline{m}$ are in some $j \leq m$ parts with nodes of $p_m$, we get $D_H w \cong \bigoplus_{j=0}^m D_H \mathbf{E}_{i+m-j}$, where once again, $j$ counts the number of parts of $w$ containing only primed nodes.
As before, for $x \in D_G \mathbf{E}_i$, $D_H x = 1x \mod D_G \mathbf{E}_i$, and $H_G$ thus acts trivially. □

Let $D_G W_\gamma$ denote an irreducible left $D_G(Q)$ module, $\gamma \in \Gamma_G$. We are interested in the restriction $\text{Res}_{D_H}^{D_G} D_G W_\gamma$. Note the following inclusion of algebras:

\[
D_G(Q) \subset P_{|\Lambda_G^0|} \\
\cup \\
D_H(Q) \subset P_{|\Lambda_H^0|}
\]

and consider the corresponding restrictions of modules:

\[
\begin{align*}
\text{Res}_{P_H}^{P_G} P_G V_\lambda &= \bigoplus_{\mu} \xi_{\lambda \mu} P_H V_\mu, \quad \lambda \in \mathcal{L}_{|\Lambda_G^0|}, \mu \in \mathcal{L}_{|\Lambda_H^0|}, \\
\text{Res}_{P_G}^{P_G} P_G V_\lambda &= \bigoplus_{\gamma} g_{\lambda \gamma} D_G W_\gamma, \quad \lambda \in \mathcal{L}_{|\Lambda_G^0|}, \gamma \in \Gamma_G, \\
\text{Res}_{P_H}^{P_G} P_H V_\mu &= \bigoplus_{\eta} h_{\mu \eta} D_H W_\eta, \quad \lambda \in \mathcal{L}_{|\Lambda_H^0|}, \mu \in \Gamma_H, \\
\text{Res}_{D_H}^{D_G} D_G W_\gamma &= \bigoplus_{\mu} m_{\gamma \eta} D_H W_\eta, \quad \gamma \in \Gamma_G, \eta \in \Gamma_H,
\end{align*}
\]

(28)

where $P_G := P_{|\Lambda_G^0|}$ and (recall) the index sets labelling the irreducible representations of $P_n(Q)$ and $D_G(Q)$ are $\mathcal{L}_n$ and $\Gamma_G$ respectively.

Let $\dim(D_G W_\gamma)$ be denoted $d_\gamma$ and $\dim(D_H W_\eta) := d_\eta$. Since the representations have already been assigned an index set, it is sufficient to determine the inclusion matrix $\mathcal{M}$, whose entries are the multiplicities $m_{\gamma \eta}$ in

\[
d_\gamma = \sum_{\eta} m_{\gamma \eta} d_\eta
\]

in order to complete the study of the generic irreducibles.

To obtain this, recall that $\text{Res}_{P_H}^{P_G}$ is known, i.e., the coefficients $\xi_{\lambda \mu} \in \Xi^m$, where the inclusion matrix $\Xi$ encodes the restriction information $P_{n-1}(Q) \subset P_n(Q)$ has been given in [19] and $m = |\Lambda_G^0| - |\Lambda_H^0|$. For a (left) $P_n(Q)$-module, $P_n V_\lambda, \lambda \vdash i$,

\[
\text{Res}_{P_{n-1}}^{P_n} P_n V_\lambda = \bigoplus \begin{cases} 
P_{n-1} V_\lambda', i - 1 \vdash \lambda' \triangleright \lambda \\
P_{n-1} V_\lambda \oplus P_{n-1} V_\lambda', i - 1 \vdash \lambda' \triangleright \lambda \\
P_{n-1} V_\lambda', i + 1 \vdash \lambda' \triangleright \lambda,
\end{cases}
\]

where $\lambda \triangleright \mu$ denotes the “removal of a box” from $\lambda$ to produce $\mu$, $\lambda \vartriangleleft \mu$, denotes the “addition of a box” to $\lambda$ to produce $\mu$, and $\lambda \triangleright \vartriangleleft \mu$ means that we first remove a box from $\mu$ to obtain some $\nu$ (say), and then add a box to $\nu$ to obtain $\lambda$. Addition and removal of boxes correspond to the induction and restriction rules for symmetric group representations, called the Pieri (or Littlewood-Richardson) rules. Also note that in the above, $P_{n-1} V_\lambda \cong P_n V_\lambda$.

This is the key piece of information which, together with proposition (27), will indicate the way to obtain $m_{\gamma \eta}$. Let us evaluate $\text{Res}_{D_H}^{D_G}$ in two ways, corresponding to the paths in the diagram indicating the inclusion of algebras above (restrictions are transitive).

\[
\text{Res}_{D_H}^{D_G} = \text{Res}_{D_H}^{P_H} \text{Res}_{P_H}^{P_G} = \text{Res}_{D_H}^{P_H} \text{Res}_{P_H}^{P_G}.
\]

Thus, from one path we get,

\[
\text{Res}_{D_H}^{P_G} P_G V_\lambda = \text{Res}_{D_H}^{P_H} \left( \text{Res}_{P_H}^{P_G} P_G V_\lambda \right) = \bigoplus_{\eta} \xi_{\lambda \eta} h_{\mu \eta} D_H W_\eta,
\]

(29)
while from the other,

\[ \text{Res}^{P_G}_{D_H} P_G V_\lambda = \text{Res}^{P_G}_{D_H} \left( \text{Res}^{P_G}_{D_G} P_G V_\lambda \right) = \bigoplus \gamma \oplus \eta \, g_{\lambda \gamma} m_{\gamma \eta} \text{ } D_H W_\eta. \]  \hspace{1cm} (30)

Let the inclusion matrices \( \Sigma_i \) and \( \Upsilon^j_i \) encode the restriction information \( \text{Res}^{A(i)}_{S(i)} \) and \( \text{Res}^{A(j)}_{A(i)} \), with matrix elements \( (\Sigma_i)_{a,b} = \varsigma^{(i)}_{a,b} \) and \( (\Upsilon^j_i)_{a,b} = \upsilon^{(i,j)}_{a,b} \) respectively. Then, the restriction information between representations that are not among the list of one-dimensional representations \( (\chi^{(i)}_i) \), is extracted from the above.

\[ \tilde{m}_{\gamma \eta} = \begin{cases} 
\xi_{\gamma \eta}, \gamma \vdash k < n - 1, \eta \vdash l < n - m - 1, \\
\sum_\mu \xi_{\gamma \mu} \varsigma^{(n-m-1)}_{\mu \eta}, \gamma \vdash k < n - 1, \mu, \eta \vdash l = n - m - 1 \\
\upsilon^{(n-1,n-m-1)}_{\gamma \eta}, \gamma \vdash n - 1, \eta \vdash n - m - 1.
\end{cases} \]

In the above, we have used \( \tilde{m}_{\gamma \eta} \) instead of \( m_{\gamma \eta} \) to indicate that \( \tilde{m}_{\gamma \eta} \) does not give the multiplicities \( \theta_i \) of the one-dimensional representations, \( \chi^{(i)}_i \). The number of such \( \chi^{(i)}_i \) is not known in general. Diagrammatically their determination is a combinatorial problem of enumerating the number of “top” configurations that are characterized by corollary 9.2.

We have constructed an algorithm for their enumeration by using recurrence relations for \( G = A_l \times A_2 \), but we have not been able to solve it in closed form. For arbitrary rectangular graphs, the combinatorics is much more complicated.