ON THE COMPLEX STRUCTURE OF KÄHLER MANIFOLDS WITH NONNEGATIVE CURVATURE

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Abstract. We study the asymptotic behavior of the Kähler-Ricci flow on Kähler manifolds of nonnegative holomorphic bisectional curvature. Using these results we prove that a complete noncompact Kähler manifold with nonnegative bounded holomorphic bisectional curvature and maximal volume growth is biholomorphic to complex Euclidean space $\mathbb{C}^n$. We also show that the volume growth condition can be removed if we assume $(M, g)$ has average quadratic scalar curvature decay (see Theorem 2.1) and positive curvature operator.

1. Introduction

The classical uniformization theorem says that a simply connected Riemann surface is either the Riemann sphere, the open unit disk or the complex plane. On the other hand, there is a close relation between the complex structure and the geometry of a Riemann surface. An important case of this is that a complete noncompact Riemannian surface with positive Gaussian curvature is necessarily conformally equivalent to the complex plane. In higher dimensions, there is a long standing conjecture predicting similar results. In its most general form, the conjecture is due to Yau [41], and it states: A complete noncompact Kähler manifold with positive holomorphic bisectional curvature is biholomorphic to $\mathbb{C}^n$. In fact, the conjecture is part of a program proposed by Yau [41] in 1974 to study complex manifolds of parabolic type.

The first result supporting this conjecture was by Mok-Siu-Yau [26]. There, the authors proved that if $M^n$ is a complete noncompact Kähler manifold with nonnegative bisectional curvature, maximal volume growth and faster than quadratic scalar curvature decay, then $M$ is isometrically biholomorphic to $\mathbb{C}^n$. Later, Mok [24] proved that if $M$ has positive bisectional curvature, maximal volume growth and quadratic scalar curvature decay, then $M$ is an affine algebraic variety. As a consequence, if $n = 2$ and the sectional curvature is positive, then $M$ is biholomorphic to $\mathbb{C}^2$ by a result of Ramanujan [33]. In this case, dimension 2, it is known that the condition on the sectional curvature can be relaxed and the decay of the scalar curvature can also be removed, see [7, 9, 28]. In higher dimensions...
and in general, the conjecture is still very open, and until now, this has been so even if $M$ is assumed to have bounded curvature and maximal volume growth. In this paper (Corollary 1.1) we show that the conjecture is true in all dimensions provided $M$ has bounded curvature and maximal volume growth.

In his thesis [36], Shi used the following Ricci flow of Hamilton [19] to better understand the uniformization conjecture in the case of $(M, g)$ as in Mok’s paper [24]:

$$
\frac{\partial}{\partial t} \tilde{g}_{ij}(x, t) = -\tilde{R}_{ij}(x, t)
$$

$$
\tilde{g}_{ij}(x, 0) = \tilde{g}_{ij}(x).
$$

(1.1)

On a Kähler manifold, (1.1) is referred to as the Kähler-Ricci flow. In [36, 35], Shi obtained several important results for this flow including short time existence for general solutions, and long time existence together with many useful estimates in the above case; see Theorem 2.1 for more details. Although the results in [36] did not actually prove uniformization in this case\(^1\), their importance has remained fundamental to the field; in particular, in the above mentioned works [7], [9], [28] as well as the present paper.

In this paper, by studying the asymptotic behavior of the Kähler-Ricci flow (1.1) in more detail, we will prove the following uniformization theorem:

**Theorem 1.1.** Let $(M^n, \tilde{g})$ be a complete noncompact Kähler manifold with nonnegative and bounded holomorphic bisectional curvature. Suppose

(i) $\text{Vol}(B(p, r)) \geq C_1 r^{2n}; \quad \forall r \in [0, \infty)$ for some $p \in M$,

(ii) $\frac{1}{V_{2n}(r)} \int_{B_r(x)} R \leq C_2$ for all $x \in M$ and for all $r > 0$,

for some positive constants $C_1, C_2$. Then $M$ is biholomorphic to $\mathbb{C}^n$. Moreover, condition (i) can be removed if $M$ has positive curvature operator.

The assumption is a little bit weaker than in [36]. Here we only assume the holomorphic bisectional curvature is nonnegative.

In [11], Yau conjectured that (i) actually implies (ii). This has been confirmed by Chen-Tang-Zhu [7] for the case of dimension 2, and Chen-Zhu [9] for higher dimensions under the additional condition that the curvature operator is nonnegative and recently by Ni [28] for all dimensions. Hence we have:

**Corollary 1.1.** Let $(M^n, \tilde{g})$ be a complete noncompact Kähler manifold with nonnegative and bounded holomorphic bisectional curvature and maximal volume growth. Then $M$ is biholomorphic to $\mathbb{C}^n$.

Also, if we only assume that the curvature operator is nonnegative and if the scalar curvature is still assumed to satisfy (ii), then one can prove that the universal cover of $M$ is biholomorphic to $\mathbb{C}^n$.

\(^1\)This was observed later on in [6]. Also see [10].
In order to prove Theorem 1.1, we will first obtain some results on the long time behavior of the Kähler-Ricci flow (1.1). We believe these results may be of independent interest. It will be more convenient to consider the normalized Kähler-Ricci flow

\[
\frac{\partial}{\partial t} g(t) = -\text{Rc}(t) - g(t)
\]

where \( g(t) = e^{-t} \tilde{g}(e^t) \) and \( \text{Rc}(t) \) is the Ricci curvature of \( g(t) \). Under the assumptions of Theorem 1.1, we have:

**Theorem 1.2.** Let \((M^n, \tilde{g})\) be as in Theorem 1.1 with maximal volume growth or with positive curvature operator and let \( g(x, t) \) be as in (1.2). Let \( p \in M \) be any point. Then the eigenvalues of \( \text{Rc}(p, t) \) with respect to \( g(p, t) \) will converge as \( t \to \infty \). Moreover, if \( \mu_1 > \mu_2 > \cdots > \mu_l \) are the distinct limits of the eigenvalues, then \( V = T_p^{(1,0)}(M) \) can be decomposed orthogonally with respect to \( g(0) \) as \( V_1 \oplus \cdots \oplus V_l \) so that the following are true:

(i) If \( v \) is a nonzero vector in \( V_i \) for some \( 1 \leq i \leq l \), then

\[
\lim_{t \to \infty} \text{Rc}(v(t), \bar{v}(t)) = \mu_i
\]

and thus

\[
\lim_{t \to \infty} \frac{1}{t} \log \frac{|v(t)|^2}{|v(0)|^2} = -\mu_i - 1.
\]

Moreover, both convergences are uniform over all \( v \in V_i \setminus \{0\} \). Here \( v(t) = v/|v|_{g(t)} \).

(ii) For \( 1 \leq i, j \leq l \) and for nonzero vectors \( v \in V_i \) and \( w \in V_j \) where \( i \neq j \), \( \lim_{t \to \infty} \langle v(t), w(t) \rangle_t = 0 \) and the convergence is uniform over all such nonzero vectors \( v, w \).

(iii) \( \dim \mathbb{C}(V_i) = n_i - n_{i-1} \) for each \( i \).

(iv)

\[
\sum_{i=1}^l (-\mu_i - 1) \dim \mathbb{C}(V_i) = \lim_{t \to \infty} \frac{1}{t} \log \frac{\det(g_{ij}(t))}{\det(g_{ij}(0))}.
\]

The theorem says that at \( p \), \((M, g(t))\) behaves like a gradient Kähler-Ricci soliton of expanding type, see Proposition 3.2 for more details. Moreover, conclusions (i) and (ii) mean in some sense that \( \text{Rc}(p, t) \) can be ‘simultaneously digonalized’ near infinity. Conditions (i), (iii) and (iv) basically say that \( g(t) \) is Lyapunov regular in the sense of dynamical systems, see [2].

On the other hand, by Theorem 2.1, we can construct biholomorphisms from a sequence of open sets which exhaust \( M \) onto a fixed ball in \( \mathbb{C}^n \). By identifying these open sets, the results in Theorem 1.2 can be interpreted in terms of the dynamics of a randomly iterated sequence of biholomorphisms as in [22]. Using the results of Theorem 1.2 in this setting, and using techniques developed by Rosay-Rudin [34] and Jonsson-Varolin [22], we then
proceed to assemble these biholomorphisms into a global biholomorphism from \( M \) to \( \mathbb{C}^n \).

The paper is organized as follows. In §2 we review the main results on the Kähler-Ricci flow (2.1) which we use later. In §3 and §4 we study the asymptotic behavior of the Kähler-Ricci flow on \( M \) as \( t \to \infty \). The focus of §3 will primarily be on the global asymptotics of the Kähler-Ricci flow on \( M \) while that of §4 will be purely local. We believe that these asymptotics should be of independent interest to the study of the Kähler-Ricci flow.

Finally, in §5 we will prove Theorem 1.1 and its corollaries.

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2. the Kähler Ricci flow

In this section, we will collect some known results on Kähler-Ricci flow which will be used in this work. Recall that on a complete noncompact Kähler manifold \((M^n, \tilde{g}_{i\bar{j}}(x))\), the Kähler-Ricci flow equation is:

\[
\frac{\partial}{\partial t} \tilde{g}_{i\bar{j}}(x, t) = -\tilde{R}_{i\bar{j}}(x, t),
\]

\[
\tilde{g}_{i\bar{j}}(x, 0) = \tilde{g}_{i\bar{j}}(x).
\]

**Theorem 2.1.** Let \((M^n, \tilde{g})\) be a complete noncompact Kähler manifold with bounded nonnegative holomorphic bisectional curvature. Suppose there is a constant \( C > 0 \) such that its scalar curvature satisfies

\[
\frac{1}{V_x(r)} \int_{B_x(r)} \tilde{R} \, dV_{\tilde{g}} \leq \frac{C}{1 + r^2}
\]

for all \( x \in M \) and for all \( r > 0 \) where \( \tilde{R} \) is the scalar curvature. Then the Kähler-Ricci flow (2.1) has a long time solution \( \tilde{g}_{\alpha\bar{\beta}}(x, t) \) on \( M \times [0, \infty) \). Moreover, the following are true:

(i) For any \( t \geq 0 \), \( \tilde{g}(x, t) \) is Kähler with nonnegative holomorphic bisectional curvature.

(ii) For any integer \( m \geq 0 \), there is a constant \( C_1 \) depending only on \( m \) and the initial metric such that

\[
\|\nabla^m \tilde{R}m\|^2(x, t) \leq \frac{C_1}{t^{2+m}},
\]

for all \( x \in M \) and for all \( t \geq 0 \), where \( \nabla \) is the covariant derivative with respect to \( \tilde{g} \) and the norm is also taken in \( \tilde{g} \).

(iii) If in addition \((M, \tilde{g}(0))\) has either maximum volume growth or positive curvature operator, then there exists a constant \( C_2 > 0 \) depending only on the initial metric such that the injectivity radius of \( \tilde{g}(t) \) is bounded below by \( C_2 t^{1/2} \) for all \( t \geq 1 \).

**Proof.** (i) and (ii) are mainly obtained by Shi [35, 36, 38] (also see [31]). To prove (iii), suppose \( \tilde{g}(0) \) has positive curvature operator. Then by [21] we know that positive curvature operator is preserved under (2.1), and thus \( g(t) \)
has positive sectional curvature at every time $t$. From this and the estimates in (ii), we can conclude by the results in [18] that (iii) is true in the case of positive curvature operator (See [12] p. 14 for a description of how to prove this). In the case of maximal volume growth, (iii) has been observed in [6].

In fact, if $\text{Vol}_0(B_x(r)) \geq Cr^2n$ for some $C > 0$ for the initial metric, then we also have $\text{Vol}_t(B_x(r)) \geq Cr^2n$ for the metric $g(t)$ for all $t \geq 0$ with the same constant $C$, see [6] for example. Combining this with the curvature estimates (ii) and the injectivity radius estimates in [13], (iii) follows in this case.

□

We now consider the following normalization of (2.1):

$$\frac{\partial}{\partial t} g_i^{\bar{j}}(x, t) = -R_i^{\bar{j}}(x, t) - g_i^{\bar{j}}(x, t).$$

(2.3)

It is easy to verify that if $\tilde{g}(x, t)$ solves (2.1), then

$$g(x, t) = e^{-t} \tilde{g}(x, e^t)$$

is a solution to (2.3). Thus for $\tilde{g}(x, t)$ as in Theorem 2.1, $g(x, t)$ in (2.4) is defined for $-\infty < t < \infty$. Note that $\lim_{t \to -\infty} g(x, t) = \tilde{g}(x)$ which is the initial data of (2.1). The results in Theorem 2.1 can be translated to the following results for a solution to (2.3):

**Corollary 2.1.** Let $\tilde{g}(x, t)$ be as in Theorem 2.1 and let $g(x, t)$ be given by (2.4). Then the following are true:

(i) For any $-\infty < t < \infty$, $g(x, t)$ is Kähler with nonnegative holomorphic bisectional curvature.

(ii) For any integer $m \geq 0$, there is a constant $C_1$ depending only on $m$ and the initial metric such that

$$||\nabla^m Rm||^2(x, t) \leq C_1,$$

for all $x \in M$ and for all $t \geq 0$, where $\nabla$ is the covariant derivative with respect to $g(t)$ and the norm is also taken in $g(t)$.

(iii) If in addition $(M, \tilde{g}(0))$ has either maximum volume growth or positive curvature operator, then there exists a constant $C_2 > 0$ depending only on the initial metric such that the injectivity radius of $g(t)$ is bounded below by $C_2$ for all $t \geq 0$.

We shall need the following.

**Proposition 2.1.** Let $(M^n, g)$ be a complete Kähler manifold with nonnegative holomorphic bisectional curvature such that $|Rm| + |\nabla Rm| \leq C_1$ and the injectivity of $M$ is larger than $r_0$. Then there exist positive constants $r_1, r_2$ and $C_2$ depending only on $C_1, r_0$ and $n$ such that for each $p \in M$, there is a holomorphic map $\Phi$ from the Euclidean ball $\mathbb{B}_0(r_1)$ at the origin of $\mathbb{C}^n$ to $M$ satisfying the following:

(i) $\Phi$ is a biholomorphism from $\mathbb{B}_0(r_1)$ onto its image;

(ii) $\Phi(0) = p$;

(iii) $\Phi^*(g)(0) = g_c$. 


(iv) \( \frac{1}{2} \rho \leq \Phi'(\rho) \leq r_2 \rho \) in \( \hat{B}(0, r_1) \).

where \( \rho \) is the standard metric on \( \mathbb{C}^n \).

Proof. This is in fact a special case of Proposition 1.2 in [40], see also [38, 6]. For the sake of completeness, we sketch the proof as follows.

By the assumption on the injectivity radius, let \( x_1, \ldots, x_{2n} \) be normal coordinates on \( B_p(r_0) \) so that if \( z_i = x_i + \sqrt{-1}x_{n+i} \) are standard complex coordinates of \( \mathbb{C}^n \), then \( \frac{\partial}{\partial z_i} \) form a basis for \( T_p(M) \) at \( p \). Hence there is a diffeomorphism \( F \) from \( B_p(r_0) \) onto \( \hat{B}_0(r_0) \) such that \( F(p) = 0 \) and \( dF \circ J = J \circ dF \) at 0 where \( J \) is the standard complex structure on \( \mathbb{C}^n \) and \( J \) is the complex structure of \( M \). By [20], the components of the metric \( g \) with respect to coordinates \( x_i \) satisfies

\[
|\delta_{ij} - g_{ij}| \leq C_2|x|^2, \quad \frac{\partial^2}{\partial x_k \partial x_l} g_{ij} \leq C_2
\]

and

\[
|\frac{\partial}{\partial x_k} g_{ij}(x)| \leq C_2|x|
\]

in \( B_p(r_1) \) for some positive constants \( r_1, C_2 \) depending only on \( C_1, r_0 \) and \( n \). Here \( |x|^2 = \sum_{i} (x_i)^2 \). In the following \( C_i \)'s and \( r_i \)'s always denote positive constants depending only on \( C_1, r_0 \) and \( n \). Hence if \( r_1 \) small enough, \( \sqrt{-1} \partial \bar{\partial} \log \rho^2 \geq -C_3 \omega \) and the eigenvalues of the Hessian of \( \rho^2 \) are bounded below by \( C_4 \). Here \( \rho \) is the distance from \( p \) and \( \omega \) is the Kähler form. One can prove that \( |J - \hat{J}| \leq C_5 \rho^2 \), where we also denote the pull back of \( \hat{J} \) under \( F \) with \( \hat{J} \), see [3]. The \( i \)-th component \( z_i = x_i + \sqrt{-1}x_{n+i} \) of the map \( F \) when considered as a map from \( B_p(r_0) \) to \( \mathbb{C}^n \) satisfies

\[
|\bar{\partial} z_i| \leq C_6 \rho^2.
\]

As in [27], by Corollary 5.3 in [14], using the weight function \( \varphi = (n + 2) \log \rho^2 + C_7 \rho^2 \) for some \( C_7 \) so that \( \sqrt{-1} \partial \bar{\partial} \varphi \geq C_8 \omega \), one can solve \( \bar{\partial} u_i = \bar{\partial} z_i \) in \( B_p(r_1) \) with

\[
\int_{B_p(r_1)} |u_i|^2 e^{-\varphi} \leq \frac{1}{C_8} \int_{B_p(r_1)} |\bar{\partial} z_i|^2 e^{-\varphi} \leq C_9
\]

for some \( C_9 \). Here we have used the fact that \( R\text{ic} \geq 0 \) and (2.5). From this, it is easy to see that \( u_i(p) = 0 \) and \( du_i(p) = 0 \). Moreover, from the fact that \( z_i - u_i \) is holomorphic one can prove that on \( B_p(r_1/2) \),

\[
|u_i| + |\nabla u_i| + |\nabla^2 u_i| \leq C_{10}
\]

by [26], mean value inequality, gradient estimates and Schauder estimates. Hence we have \( |\nabla u_i| \leq C_{11} \rho \) and \( |u_i| \leq C_{11} \rho^2 \). So the map \( \Phi \) given by \( \Phi^{-1} = (z_1 - u_1, \ldots, z_n - u_n) \) will satisfy the conditions in the proposition if \( r_1 \) is small enough and \( r_2 \) is large enough. \( \Box \)
Using this and Corollary 2.1 we have the following (also see [31, 38]).

**Corollary 2.2.** Let \((M^n, \bar{g}(0))\) and \(g(x,t)\) be as in Corollary 2.1 such that \((M, \bar{g}(0))\) has either maximum volume growth or positive curvature operator. Let \(p \in M\) be a fixed point. Then there are constants \(r_1\) and \(r_2\) depending only on the initial metric such that for every \(t > 0\) there exists a holomorphic map \(\Phi_t : \tilde{B}_0(r_1) \subset \mathbb{C}^n \to M\) satisfying:

1. \(\Phi_t\) is a biholomorphism from \(\tilde{B}_0(r_1)\) onto its image;
2. \(\Phi_t(0) = p;\)
3. \(\Phi_t(t)(0) = g_c;\)
4. \(\frac{1}{r_2}g_c \leq \Phi_t(t) \leq r_2g_c\) in \(\tilde{B}_0(r_1);\)

where \(g_c\) is the standard metric on \(\mathbb{C}^n\), and \(\tilde{B}_0(r_1)\) is the Euclidean ball of radius \(r_1\) with center at the origin in \(\mathbb{C}^n\). Moreover, the following are true:

1. For any \(t \to \infty\) and for any \(0 < r < r_1\), the family \(\{\Phi_t(\tilde{B}_0(r))\}_{t \geq 1}\) exhausts \(M\) and hence \(M\) is simply connected.
2. If \(T\) is large enough, then \(F_{i+1} = \Phi^{-1}_{(i+1)T} \circ \Phi_{iT}\) maps \(\tilde{B}_0(r_1)\) into \(\tilde{B}_0(r_1)\) for each \(i\), and there is \(0 < \delta < 1\), \(0 < a < b < 1\) such that
   \[ |F_{i+1}(z)| \leq \delta|z| \]
   for all \(z \in \tilde{B}_0(r_1)\), and
   \[ a|v| \leq |F_{i+1}'(0)(v)| \leq b|v| \]
   for all \(v\) for all \(i\).

**Proof.** (i)–(iv) follows immediately from Proposition 2.1 and Corollary 2.1.

To prove (v), observe that \(B_1^i(r/r_2) \subset \Phi_t(\tilde{B}_0(r))\) by (i) and (iv), where \(B_1^i(R)\) is the geodesic ball of radius \(R\) with respect to \(g(t)\) with center at \(p\). On the other hand, by (2.3), \(g(t) \leq e^{-t/2}g(0)\) and so \(B_1^0(R) \subset B_1^0(e^{-t}R)\). From this it is easy to see that (v) is true.

To prove (vi), let \(v\) be a \((1,0)\) vector on \(M\) and denote \(|v|_t\) to be the length of \(v\) with respect to \(g(t)\). By (2.3) and Corollary 2.1

\[
-|v|_t^2 \geq \frac{d}{dt}|v|_t^2 \\
= -Rc_\bar{g}(v,v) - \bar{g}(v,v) \\
\geq -C_1 \bar{g}(v,v) - \bar{g}(v,v) \\
\geq -(C_1 + 1)|v|_t^2
\]

for some constant \(C_1 > 0\) which is independent of \(v\) and \(t\). Hence for any \(T > 0\) and \(i \geq 1,

\[
e^{-T} \geq \frac{|v|_{(i+1)T}^2}{|v|_T^2} \geq e^{-(C_1+1)T}.
\]

Since

\[
\Phi_{iT}(\tilde{B}_0(r_1)) \subset B_{iT}^i(r_2r_1) \subset B_{iT}^{(i+1)}(e^{-T/2}r_2r_1),
\]
and \( \Phi_{i+1}(\tilde{B}_0(r_1)) \supset B_p^{(i+1)}(r_1/r_2) \). Hence \( F_{i+1} \) is defined on \( \tilde{B}_0(r_1) \) and \( F_{i+1}(\tilde{B}_0(r_1)) \subset \tilde{B}_0(r_1) \) if \( T \) is large enough. From (iv) and (2.8), it is easy to see and there is \( 0 < \delta < 1 \), such that
\[
|F_{i+1}(z)| \leq \delta |z|
\]
for all \( z \in \tilde{B}_0(r_1) \) for all \( i \) if \( T \) is large. From (ii), (iii) and (2.8), we can also find \( 0 < a < b < 1 \) such that
\[
a|v| \leq |F'_{i+1}(0)(v)| \leq b|v|
\]
for all \( v \) and for all \( i \). This completes the proof of the corollary. \( \square \)

In §5, we will use the maps \( \Phi_t \) to construct a biholomorphism from \( M \) to \( \mathbb{C}^n \).

3. Asymptotic behavior of Kähler Ricci flow (I)

Let \((M^n, \bar{g}_{ij}(x))\) be as in Theorem 2.1 satisfying (2.2). Let \( \bar{g}(x, t) \) and \( g(x, t) \) be the corresponding solutions to (2.1) and (2.3) respectively. We will show that the eigenvalues of \( Rc(p, t) \) relative to \( g(p, t) \) actually converge to a fixed set of numbers as \( t \to \infty \). Here \( Rc(p, t) \) is the Ricci tensor of \( g(t) \) at \( p \). If in addition that \((M, \bar{g})\) has maximal volume growth with positive Ricci curvature or has positive curvature operator, then we will show that for any \( p \in M \), \((M, g(x, t), p)\) approaches an expanding gradient Kähler-Ricci soliton as \( t \to \infty \) in the sense of limiting solutions to the Kähler-Ricci flow (2.0).

**Proposition 3.1.** Let \((M^n, g_{ij}(x)), \bar{g}(x, t), g(x, t)\) be as in Theorem 2.1 satisfying (2.2). Let \( \bar{g}(x, t) \) and \( g(x, t) \) be a fixed point in \( M \) and let \( \lambda_1(t) \geq \cdots \lambda_n(t) \geq 0 \) be the eigenvalues of \( R_{ij}(p, t) \) relative to \( g_{ij}(p, t) \).

(i) For any \( \tau > 0 \),
\[
\frac{\det(R_{ij}(p, t) + \tau \delta_{ij})}{\det(g_{ij}(p, t))}
\]
is nondecreasing in \( t \).

(ii) Assume in addition that \( \bar{g}_{ij}(x) \) has positive Ricci curvature. Then there is a constant \( C > 0 \) such that \( \lambda_n(t) \geq C \) for all \( t \).

(iii) For \( 1 \leq i \leq n \) the limit \( \lim_{t \to \infty} \lambda_i(t) \) exists.

(iv) Let \( \mu_1 > \cdots > \mu_l \geq 0 \) be the distinct limits in (iii) and let \( \rho > 0 \) be such that \( |\mu_k - \rho, \mu_k + \rho|, 1 \leq k \leq l \) are disjoint. For any \( t \), let \( E_k(t) \) be the sum of the eigenspaces corresponding to the eigenvalues \( \lambda_i(t) \) such that \( \lambda_i(t) \in (\mu_k - \rho, \mu_k + \rho) \). Let \( P_k(t) \) be the orthogonal projection (with respect to \( g(t) \)) onto \( E_k(t) \). Then there exists \( T > 0 \) such that if \( t > T \) and if \( \omega \in T_p^{(1)}(M) \), \( |P_k(t)(\omega)|_t \) is continuous in \( t \), where \( |\cdot|_t \) is the length measured with respect to the metric \( g(p, t) \).
Proof. (i): By the Li-Yau-Hamilton (LYH) inequality in [3] and in [4], Theorem 2.1, if

\[ Z_{ij} = \frac{\partial R_{ij}}{\partial t} + g^{kl} R_{ik} R_{lj} + R_{ij} \tag{3.1} \]

then

\[ Z_{ij} w^i w^j \geq 0 \tag{3.2} \]

for any \( w \in T^{(1,0)}(M) \). For any \( \tau > 0 \), denote

\[ \phi(t) = \frac{\det(R_{ij} + \tau g_{ij})}{\det(g_{ij})} \]

at \((p, t)\). Denote \( p_{ij} = R_{ij} + \tau g_{ij} \) as in [3] and note that \( (p_{ij}) \) is invertible and its inverse is denoted by \((p^{ij})\). We have

\[
\frac{\partial}{\partial t} \log \phi = p^{ij} \frac{\partial}{\partial t} p_{ij} - g^{ij} \frac{\partial}{\partial t} g_{ij}
\]

\[
= p^{ij} \left( \frac{\partial}{\partial t} R_{ij} - \tau (R_{ij} + g_{ij}) \right) + g^{ij} (R_{ij} + g_{ij})
\]

\[
\geq p^{ij} \left( -g^{kl} R_{ik} R_{kj} - R_{ij} - \tau (R_{ij} + g_{ij}) \right) + g^{ij} (R_{ij} + g_{ij})
\]

\[
= p^{ij} \left( -g^{kl} R_{ik} R_{kj} - (\tau + 1)p_{ij} \right) + \tau^2 p^{ij} g_{ij} + g^{ij} (R_{ij} + g_{ij})
\]

where we have used (3.1) and (3.2). Now at the point \((p, t)\), we choose a unitary basis such that \( g_{ij} = \delta_{ij} \) and \( R_{ij} = \lambda_i \delta_{ij} \). Then \( p_{ij} = (\lambda_i + \tau) \delta_{ij} \) and \( p^{ij} = (\lambda_i + \tau)^{-1} \delta_{ij} \). Hence we have

\[
\frac{\partial}{\partial t} \log \phi \geq - \sum_{i=1}^{n} \frac{\lambda_i^2}{\lambda_i + \tau} - (\tau + 1)n + \sum_{i=1}^{n} \frac{\tau^2}{\lambda_i + \tau} + \sum_{i=1}^{n} \lambda_i + n
\]

\[
= \sum_{i=1}^{n} \left( -\frac{\lambda_i^2}{\lambda_i + \tau} - \tau + \frac{\tau^2}{\lambda_i + \tau} + \lambda_i \right)
\]

\[
= 0.
\]

From this (i) follows.

(ii): By (i), we conclude that \( \frac{\det(R_{ij}(p, t))}{\det(g_{ij}(p, t))} \) is nondecreasing. (This fact has been proved in [3].) Moreover,

\[
\lim_{t \to -\infty} \frac{\det(R_{ij}(p, t))}{\det(g_{ij}(p, t))} = \frac{\det(R_{ij}(p))}{\det(g_{ij}(p))}
\]

where the right side is in terms of the initial metric \( g \) for [2, 1]. Since the Ricci curvature is assumed to be positive, \( \frac{\det(R_{ij}(p, t))}{\det(g_{ij}(p, t))} \geq C_1 \) for some positive constant \( C_1 \) for all \( t \). On the other hand, by Corollary 2.1 there is a constant \( C_2 \) independent of \( t \) such that \( \lambda_1(t) \leq C_2 \). From these two facts, part (ii) of the proposition follows.
(iii): Choose a unitary basis \(v_1, \ldots, v_n\) for \(T_p^{(1,0)}(M)\) with respect to the metric \(g(p, 0)\). Using the Gram-Schmidt process, we can obtain a unitary basis \(v_1(t), \ldots, v_n(t)\) for \(g(p, t)\). Since \(g(t)\) is smooth in \(t\), we conclude that the \(v_i(t)\)'s are smooth in \(t\). That is to say, \(v_i(t)\) is a linear combination of a fixed basis of \(T_p^{(1,0)}(M)\) with smooth coefficients. Denote by \(R_{ij}(t) = Rc(v_i(t), \bar{v}_j(t))\) the components of \(Rc(p, t)\) with respect to this basis. Then \(R_{ij}(t)\) is also smooth in \(t\). By (i) and Corollary 2.1, for any \(\tau > 0\),

\[
\lim_{t \to \infty} \det(R_{ij}(t) + \tau \delta_{ij}) = c(\tau)
\]

exists.

Now \(\lambda_i(t)\) are uniformly bounded functions in \(t\). To prove (iii), it is sufficient to prove that if \(t_k \to \infty, t_k' \to \infty\) and

\[
\lim_{k \to \infty} \lambda_i(t_k) = \tau_i, \quad \lim_{k \to \infty} \lambda_i(t_k') = \tau_i'
\]

for all \(i\), then \(\tau_i = \tau_i'\).

By (3.5), we have

\[
\prod_{i=1}^{n}(\tau_i + \tau) = \prod_{i=1}^{n}(\tau_i' + \tau)
\]

for all \(\tau > 0\). Since \(\tau_1 \geq \cdots \geq \tau_n\) and \(\tau_1' \geq \cdots \geq \tau_n'\), we must have \(\tau_i = \tau_i'\).

This completes the proof of (iii).

(iv): By (iii), if \(T\) is large enough, for each \(i\) we have \(\lambda_i(t) \in (\mu_k - \rho, \mu_k + \rho)\) for some \(k\) for all \(t \geq T\). Hence \(\dim E_k(t)\) is constant in \(t\) for \(t \geq T\). Let \(P_k(t)\) be the orthogonal projection (with respect to \(g(t)\)) onto \(E_k(t)\). We also denote the matrix of this projection, with respect to the basis \(v_1(t), \ldots, v_n(t)\) in (iii), by \(P_k(t)\). Then

\[
P_k(t) = -\frac{1}{2\pi i} \int_{C} (R_{ij} - z\delta_{ij})^{-1} dz.
\]

where \(C\) is a circle on the complex plane with center at \(\mu_k\) and radius \(\rho\), see [23] p. 40 for example. It is easy to see that the matrix valued function \(P_k(t)\) is continuous in \(t\). Hence (iv) is true.

\[\square\]

Remark 1. The facts that the scalar curvature \(R(t)\) and \(\det(R_{ij}(t))/\det(g_{ij}(t))\) are nondecreasing have been proved in [3].

Next, we will study the asymptotic behavior of the manifolds \((M^n, g(t))\) as \(t \to \infty\). We will need the following lemma from [15]:

**Lemma 3.1.** Let \((M^n, g_{ij})\) be a complete noncompact Kähler manifold with bounded curvature. Suppose there is a smooth function \(f\) such that \(\sqrt{-1} \partial \bar{\partial} f = Rc\). Let \(g_{ij}(t)\) and \(\tilde{g}_{ij}(t)\) be two solutions of \((2.7)\) on \(M \times [0, T]\), \(T > 0\) with the same initial data \(g_{ij}\) such that

\[
c^{-1} g_{ij}(x) \leq g_{ij}(x, t), \tilde{g}_{ij}(x, t) \leq cg_{ij}(x)
\]

for some constant \(c > 0\) for all \((x, t) \in M \times [0, T]\). Then \(g_{ij}(x, t) = \tilde{g}_{ij}(x, t)\) on \(M \times [0, T]\).
In [4] it was proved by Cao that for any \( t_k \to \infty \), if \( |R(p_k, t_k)| \) is the maximum of the scalar curvature on \( M \) at \( t_k \), then the blow down limit of \( g(t) \) along \( (p_k, t_k) \) is an expanding gradient Kähler-Ricci soliton. Recently, it is shown by Ni in [29] that the result is still true for an arbitrary sequence \( p_k \in M, t_k \to \infty \). In the special case that the sequence \( p_k = p \) is fixed at an arbitrary \( p \in M \), the result follows from a rather simple observation and the argument in [4], which we present below.

**Proposition 3.2.** Assume the conditions and notation of Proposition 3.1. In addition, assume the initial metric \( \bar{g}(x, 0) = \bar{g}_{ij}(x) \) of (2.1) has either maximal volume growth with positive Ricci curvature or has positive curvature operator. Let \( p \in M \) be a fixed point. The given any \( t_k \to \infty \), we can find a subsequence also denoted by \( t_k \), a complete noncompact complex manifold \( N^n \), and a family of Kähler metrics \( h(t) \) on \( N \) satisfying (2.8) for all \( t \in \mathbb{R} \) such that \( (M^n, g_k(t)) \), where \( g_k(t) = g(t_k + t) \) for all \( t \in \mathbb{R} \), converges to \((N, h(t))\) in the following sense: There exists a family of diffeomorphisms \( F_k : U_k \subset N \to M \) with the following properties.

(i) Each \( U_k \) contains the fixed point \( o \) of \( N \) is a fixed point and \( F_k(o) = p \).
(ii) \( U_k \) is open and the \( U_k \)'s exhaust \( N \).
(iii) \( (U_k, F_k^*(g_k(t))) \) converges in \( C^\infty \) norm uniformly to \( h(t) \) in \( N \times \mathbb{R} \).

Moreover \((N, h(t))\) is a gradient Kähler-Ricci soliton. More precisely, there is a family of biholomorphisms \( \phi_t \) of \( N \) determined by the gradient of some real valued function such that \( o \) is a fixed point of each \( \phi_t \) and \( \phi_t^*(h(0)) = h(t) \) for all \( t \geq 0 \).

**Proof.** The existence of \( t_k, N, h(t) \) and \( F_k \) satisfying (i)–(iii) is a consequence of Theorem 2.1 and the compactness theorem of Hamilton [20].

We now prove the last assertion in the Proposition. Begin by noting that \( \lim_{t \to \infty} R(t) \) exists by Proposition 3.1, where \( R(t) \) is the scalar curvature of \( g(t) \) at \( p \). Let \( R^h(t) \) be the scalar curvature of \( h(t) \) at \( o \). Then for any \( t, t' \)

\[
(3.7) \quad R^h(t) = \lim_{k \to \infty} R(t_k + t) = \lim_{k \to \infty} R(t_k + t') = R^h(t').
\]

Now consider the metric \( \tilde{h}(t) = th(\log t) \) for \( t \geq 1 \). Then \( \tilde{h} \) is a solution to (2.1) on \( N \times [0, \infty) \). Also, since \( g(t) \) has uniformly bounded curvature in spacetime by Corollary 2.1, \( h(t) \) also has uniformly bounded curvature in spacetime. By Proposition 3.1 (ii), the Ricci curvature of \( h(t) \) at \( p \) is positive. Moreover, by Theorem 2.1, the facts that \( M \) is simply connected and that the metrics \( g(t) \) are decreasing in \( t \), we can conclude that \( N \) is simply connected. By [5], it is easy to see that \( h(t) \) and hence \( \tilde{h}(t) \) have positive Ricci curvature. Now (3.7) implies that \( t \tilde{R}(t) \) is constant where \( \tilde{R}(t) \) is the scalar curvature of \( \tilde{h}(t) \) at \( p \). Hence \( \frac{d}{dt}(t \tilde{R}) = 0 \) for all \( t \), and by the proof of Theorem 4.2 in [4], there is a real valued function \( f \) such that \( f_{ij}(x) = \tilde{R}_{ij}(x, 1) + \tilde{h}_{ij}(x, 1) \) on \( N \) with \( f_{ij} \equiv 0 \) and \( \nabla f(o) = 0 \).
Let $\phi_t(x)$ be the integral curve of $-\frac{1}{2} \nabla f$ on $N$ with initial point $x$. We claim that $\phi_t(x)$ is defined for all $x$ and $t$. Let $\tilde{h}_{AB}$ and $\tilde{R}_{AB}$ be the Riemannian metric $2 \text{Re}(\tilde{h}_{ij})$ and Ricci curvature of $\tilde{h}_{AB}$. Then $f_{AB} = \tilde{R}_{AB} + \tilde{h}_{AB}$. Observe that as in ([21] §20), we have

$$|\nabla f|^2 + \tilde{R} = 2f + 2C_1$$

where $\tilde{R}$ is the scalar curvature of $\tilde{h}(1)$ and $C_1$ is a constant.

Now, as long as $\phi_t(x)$ is defined in on $[-T, 0]$ for $T > 0$, then for $0 \leq t \leq T$

$$f(\phi_{-t}(x)) - f(x) = \int_0^{-t} ds \frac{d}{ds} f(\phi_s(x))ds$$

$$= \int_0^{-t} \langle \nabla f(\phi_s(x)), \frac{d}{ds} \phi_s(x) \rangle ds$$

$$= \frac{1}{2} \int_0^{t} |\nabla f(\phi_{-s}(x))|^2 ds$$

$$\leq \int_0^{t} f(\phi_{-s}(x))ds + C_1 t$$

by (3.8). Hence we have $f(\phi_{-t}(x)) \leq C_2$ for some constant depending only on $T$, $C_1$ and $f(x)$. One can also prove that $f(\phi_t(x)) \leq f(x)$ for $t > 0$ as long as $\phi_t(x)$ is defined up to $t$. Since $f$ is an exhaustion function by ([10], Lemma 3.1), we conclude that $f(\phi_t(x))$ remains in a fixed compact set on any bounded interval of $\mathbb{R}$ as long as $\phi_t$ is defined on that interval. From this it is easy to see that $\phi_t(x)$ is defined for all $t$. Since $\nabla f$ is a holomorphic vector field, $\phi_t$ is in fact a biholomorphism on $N$ for all $t$.

Let $h_1(t) = \phi_t^*(\tilde{h}(1)) = \phi_{t}^*(h(0))$ and let $\tilde{h}_1(t) = th_1(\log t)$ for $t \geq 1$. We will show that $h_1(1) = h(1)$. Since $h(t)$ has nonnegative holomorphic bisectional curvature such that its scalar curvature is uniformly bounded in spacetime, $\tilde{h}(t)$ also has nonnegative holomorphic bisectional curvature with $t\tilde{R}(t)$ being uniformly bounded in spacetime where $\tilde{R}(t)$ is the scalar curvature of $\tilde{h}(t)$. By [31] Theorem 2.1 and [30] Theorem 5.1, we can find a potential function for the Ricci tensor of $h(1)$. Since the curvature of $\tilde{h}$ and $\tilde{h}_1$ are uniformly bounded on $M \times [0, T]$ for fixed $T > 0$, it is easy to see that they satisfy [36]. By Lemma 3.4 we conclude that $\tilde{h}_1(t) = h(1)$ for $t \geq 1$. Hence $h_1(t) = h(t)$ for all $t \geq 0$. This completes the proof of the proposition. \qed

Let $t_k \to \infty$ such that $(M, g_k(t))$ converges to $(N, h(t))$ as in Proposition 3.2. We will describe this convergence in terms of the convergence of certain specific quantities. For simplicity, we identify $(M, g_k(t))$ near $p$ with $(U, F_k^*(g_k(t)))$ for some open set $U \subset N$ containing $o$. Let $J$ be the complex structure on $U$ given by the pullback of the complex structure of $M$ under $F_k$ and let $J$ be the complex structure of $N$. By taking a subsequence we
may also assume that $J_k \to J$. Let $w_k \in T^{(1,0)}_p(M)$ with $|w_k|_{g_k(0)} = 1$ and let $w_k(t) = w_k/|w_k|_{g_k(t)}$ for $t \geq 0$. Denote $w_k = x_k - \sqrt{-1}J_k(x_k)$ where $x_k$ is in the real tangent space of $M$ at $p$ which is identified with the real tangent space of $N$ at $o$. Assume that $x_k \to x$. Then $J_k(x_k) \to J(x)$. Let $u = x - \sqrt{-1}J(x)$ and let $u(t) = u/|u|_{h(t)}$ for $t \geq 0$. Note that $|u|_{h(0)} = 1$.

Assume the conditions and notation of Proposition 3.2 and Proposition 3.1. Then we can see that by the propositions, the eigenvalues of the Ricci curvature of $h(t)$ with respect to $h(t)$ at $o$ are $\mu_1 > \cdots > \mu_l > 0$ such that the multiplicity of $\mu_i$ is $\dim E_i(t)$ for $t$ large enough.

Let $E^k_i(t)$ be the eigenspace of the Ricci tensor of $h(t)$ corresponding to the eigenvalue $\mu_i$.

We want to prove the following:

**Lemma 3.2.** With the assumptions as in Proposition 3.2 and with the above notations. Suppose $w_k(t) = \sum_{i=1}^l w_{k,i}(t)$ where $w_{k,i}(t)$ is the orthogonal projection of $w_k(t)$ onto $E_i(t + t_k)$ with respect to $g_k(t) = g(t_k + t)$ and suppose $u(t) = \sum_{i=1}^l u_i(t)$ where $u_i(t)$ is the orthogonal projection of $u(t)$ onto $E^k_i(t)$ with respect to $h(t)$. Then for any $T > 0$, the following are true:

(i) $w_k(t)$ converges uniformly to $u(t)$ on $t \in [0, T]$ in the sense that the real parts and the imaginary parts of $w_k(t)$ converge uniformly to the real part and imaginary part of $u(t)$ respectively.

(ii) $Rc^k_i(w_k(t), \bar{w}_k(t))$ converges uniformly to $Rc^k_i(u(t), \bar{u}(t))$ on $t \in [0, T]$ where $Rc^k_i$ is the Ricci tensor of $g_k(t)$ at $p$ and $Rc^k_i$ is the Ricci tensor of $h(t)$ at $o$.

(iii) By passing to a subsequence if necessary, for $1 \leq i \leq l$, $|w_{k,i}(t)|_{g_k(t)}$ converge uniformly to $|u_i(t)|_{h(t)}$ on $t \in [0, T]$.

**Proof.** (i): Since $g_k(t)$ converges uniformly to $h(t)$ on $[0, T]$ at $o$ and since $w_k \to u$, $|w_k|_{g_k(t)}$ converge to $|u|_{h(t)}$ uniformly on $[0, T]$. From this it is easy to see that (i) is true.

(ii): Since $g_k(t)$ converges uniformly on $U \times [0, T]$ in $C^\infty$ norm, by (i) it is easy to see that (ii) is true.

(iii): Let $v_k^{(1)}, \ldots, v_k^{(n)}$ be a unitary basis for $T^{(1,0)}_p(M)$ with respect to $g_k(0)$. Passing to a subsequence if necessary, we may assume that they converge to a unitary basis $u^{(1)}, \ldots, u^{(n)}$ of $T^{(1,0)}_o(N)$ with respect to $h(0)$. Using the Gram-Schmidt process, we can obtain $v_k^{(1)}(t), \ldots, v_k^{(n)}(t)$ to be a unitary basis for $T^{(1,0)}_p(M)$ with respect to $g_k(0)$ and a unitary basis $u^{(1)}(t), \ldots, u^{(n)}(t)$ of $T^{(1,0)}_o(N)$ with respect to $h(t)$. We claim that $v_k^{(i)}(t)$ converges to $u^{(i)}(t)$ uniformly on $[0, T]$. In fact, since $g_k(t)$ converge to $h(t)$ uniformly on $[0, T]$ and $v_k^{(i)} \to u^{(i)}$, $|v_k^{(i)}|_{g_k(t)}$ to $|u^{(i)}|_{h(t)}$ uniformly on $[0, T]$. Define $v_k^{(1)}(t) = v_k^{(1)}/|v_k^{(1)}|_{g_k(t)}$ and $u^{(1)}(t) = u^{(1)}/|u^{(1)}|_{h(t)}$. Then $v_k^{(1)}(t)$ converge to $u^{(1)}(t)$ uniformly on $[0, T]$. Suppose we have found $v_k^{(i)}(t), 1 \leq i \leq m$ and $u^{(i)}(t), 1 \leq i \leq m$ such that (a) $v_k^{(i)}(t), 1 \leq i \leq m$ are unitary with
respect to $g_k(t)$ and are linear combinations of $v_k^{(i)}$, $1 \leq i \leq m$; (b) $u^{(i)}(t)$, $1 \leq i \leq m$ are unitary with respect to $h(t)$ and are linear combinations of $u^{(i)}$, $1 \leq i \leq m$; and (c) $v_k^{(i)}(t)$ converge to $u^{(i)}(t)$ uniformly on $[0, T]$ for $1 \leq i \leq m$. Define

$$v_k^{(m+1)}(t) = \frac{v_k^{(m+1)} - \sum_{i=1}^{m} (v_k^{(i)}(t)) g_k(t) v_k^{(i)}(t)}{|v_k^{(m+1)} - \sum_{i=1}^{m} (v_k^{(i)}(t)) g_k(t) v_k^{(i)}(t)|}$$

and define

$$u^{(m+1)}(t) = \frac{u^{(m+1)} - \sum_{i=1}^{m} (u^{(i)}(t)) h(t) u^{(i)}(t)}{|u^{(m+1)} - \sum_{i=1}^{m} (u^{(i)}(t)) h(t) u^{(i)}(t)|}$$

Then it is easy (a), (b) and (c) are still true with $m$ replaced by $m + 1$. Hence by induction, we can construct $v_k^{(i)}(t)$ and $u^{(i)}(t)$ as claimed.

Let $R^k_{ij}(t) = R^k_{ij}(v_k^{(i)}(t), z_k^{(i)}(t))$ and let $R^h_{ij}(t) = R^h_{ij}(u^{(i)}(t), z^{(i)}(t))$. Then as in (ii), we can prove that $R^k_{ij}(t)$ converge to $R^h_{ij}(t)$ uniformly on $[0, T]$. Denote $P^k_{ij}(t)$ to be the matrix with respect to the basis $v_k^{(1)}(t), \ldots, v_k^{(m)}(t)$ of the orthogonal projection onto $E_i(t + t_k)$ with respect to $g_k(t)$. Denote $P_i(t)$ to be the matrix with respect to the basis $u^{(1)}(t), \ldots, u^{(n)}(t)$ of the orthogonal projection onto $E_i(t)$ with respect to $h(t)$. As in the proof of Proposition 3.1(iv),

\begin{equation}
P^k_{ij}(t) = -\frac{1}{2\pi \sqrt{-1}} \int_C (R^k_{ij}(t) - z\delta_{ij})^{-1}dz
\end{equation}

and

\begin{equation}
P_i(t) = -\frac{1}{2\pi \sqrt{-1}} \int_C (R^h_{ij}(t) - z\delta_{ij})^{-1}dz
\end{equation}

where $C$ is a circle on the complex plane with center at $\mu_s$ and radius $\rho$. Since $R^k_{ij}(t)$ converge to $R^h_{ij}(t)$ uniformly on $[0, T]$, (iii) follows from (3.10), (3.11) and (i).

\[\square\]

4. Asymptotic behavior of Kähler Ricci flow (II)

Let $(M^n, \bar{g})$ be as in Theorem 2.1 with maximal volume growth or with positive curvature operator and let $g(x, t)$ be the corresponding solution to (2.3). As before, we denote the eigenvalues of $Rc(p, t)$ by $\lambda_i(t)$ for $i = 1, \ldots, n$ and we let $\mu_k$, $E_k(t)$ and $P_k(t)$ for $k = 1, \ldots, l$ be as in Proposition 3.1. We let $n_m$ for $m = 0, \ldots, l - 1$ be such that $\lambda_k(t) \in (\mu_m - \rho, \mu_m + \rho)$ for all $n_m \leq k \leq n_{m+1}$ and $t$ sufficiently large such that the intervals $[\mu_m - \rho, \mu_m + \rho]$ are disjoint as in Proposition 3.1 part (iv). For any nonzero vector $v \in T_p^{1,0}(M)$, let $v(t) = v/|v|_t$ where $|v|_t$ is the length of $v$ with respect to $g(t)$ and $v(t) = P_i(t)v(t)$.
The goal of this section will be to prove that $Rc(p, t)$ can be ‘diagonalized’ simultaneously near infinity in a certain sense and that $g(t)$ is ‘Lyapunov regular’, to borrow a notion from dynamical systems (see [2]).

In the following lemmas we assume that the initial metric $\bar{g}(0)$ is \(g(0)\) in \(\mathbb{R}^m\), and thus by Proposition 3.1, \(g(x, t)\) for all \((x, t)\), has positive Ricci curvature.

Now let \((N, h(t))\) be a gradient Kähler-Ricci soliton as in Proposition 3.2 and let \(o \in N\), \(\phi_0\) and \(E^h(t)\) also be as in the Proposition. For any nonzero vector \(w \in T^o(N)\) let \(w(t) = w/|w|_{h(t)}\) and \(w_i(t)\) be the projection of \(w(t)\) onto \(E^h_{\bar{\xi}}(t)\). We begin by making the following observation.

Let \(\phi_t\) be given by \(-\frac{1}{2} \nabla f\) such that \(f_{ij}(x) = R^h_{ij}(x, 0) + h_{ij}(x, 0)\) and \(f_{ij} = 0\). Near \(o\), we may choose local coordinates \(z_i\) such that \(\partial_i = \frac{\partial}{\partial z_i}\) are unitary at \(o\), which diagonalize \(f_{ij}\) at \(o\). We also assume that the origin corresponds to \(o\). Then \(\mu_1 > \mu_2 > \cdots > \mu_l > 0\) are distinct eigenvalues of $Ric^h$ at \(t = 0\) with respect to $h(0)$. Since \(\partial_i\) are eigenvectors of \(f_{ij}\), for each \(i\) we have

\[
(\phi_t)_*(\partial_i) = e^{-\frac{1}{2}(\mu_j+1)t} \partial_i
\]

for some \(j\) at \(o\). Because of \(\phi_t^*\) and the fact that \(\partial_i\) are also eigenvectors of \(R_{ij}\) at \(o\) and \(t = 0\), \(E^h_{\bar{\xi}}(0) = E^h_{\bar{\xi}}(t)\) and \(w_i(t) = w_i(0)/|w|_{h(t)}\).

**Lemma 4.1.** Let \((N, h(t))\) be a gradient Kähler-Ricci soliton and \(w \in T^{(1,0)}(N)\) with \(|w|_{h(0)} = 1\) as above. Let \(1 < m < l\), and suppose \(a < \sum_{j=m+1}^l |w_j(0)|^2_{h(0)} < 1 - a\) for some \(0 < a < 1\). Then for \(t \geq 0\),

\[
\frac{\sum_{j=m+1}^l |w_j(t)|^2_{h(t)}}{\sum_{j=1}^m |w_j(t)|^2_{h(t)}} \geq \frac{\sum_{j=m+1}^l |w_j(0)|^2_{h(0)}}{\sum_{j=1}^m |w_j(0)|^2_{h(0)}} \cdot e^{(\mu_m - \mu_{m+1})t}.
\]

In particular,

\[
\sum_{j=m+1}^l |w_j(t)|^2_{h(t)} \geq \sum_{j=m+1}^l |w_j(0)|^2_{h(0)}
\]

for \(t \geq 0\). Moreover, for any \(\delta > 0\), there is a \(t_0\) depending only on the \(a\), \(\mu_m\), \(\mu_{m+1}\) and \(\delta\) such that for all \(t \geq t_0\),

\[
\sum_{j=m+1}^l |w_j(t)|^2_{h(t)} \geq 1 - \delta.
\]

**Proof.** For simplicity, let us denote \(| \cdot |_{h(t)}\) simply by \(| \cdot |_t\).

\[
|w_j(t)|^2_t = \frac{|(\phi_t)_*(w_j(0))|^2_0}{|(\phi_t)_*(w)|^2_0} = \frac{e^{(-\mu_j-1)t}|w_j(0)|^2_0}{|w|^2_t}.
\]

\[\text{(4.2)}\]
Hence for \( t \geq 0 \)

\[
\sum_{j=1}^{m} |w_j(t)|_t^2 = \sum_{j=1}^{m} e^{(\mu_j - 1)t} |w_j(0)|_0^2 \leq \frac{e^{(\mu_m - 1)t} \sum_{j=1}^{m} |w_j(0)|_0^2}{|w|_t^2}
\]

because \( \mu_1 > \cdots > \mu_l \). Similarly,

\[
\sum_{j=m+1}^{l} |w_j(t)|_t^2 = \sum_{j=m+1}^{l} e^{(\mu_j - 1)t} |w_j(0)|_0^2 \geq \frac{e^{(\mu_{m+1} - 1)t} \sum_{j=m+1}^{l} |w_j(0)|_0^2}{|w|_t^2}.
\]

The lemma then follows from (4.3) and (4.4).

Because of Proposition 3.2 and Lemma 3.2, we expect to have similar behavior for \( g(t) \) for \( t \) large. More precisely, we have the following:

**Lemma 4.2.** Let \( v_k \in T^{(1,0)}_p(M) \) be a sequence such that \( |v_k|_0 = 1 \) for each \( k \). Let \( t_k \to \infty \) be a sequence in time. Define \( f_{ik}(t) := |P_i(t)v_k(t)|_t^2 \).

(i) Suppose there exists \( a > 0 \) and \( 1 \leq m \leq l \) for which

\[
\sum_{i \geq m} f_{ik}(t_k) = a
\]

for all \( k \). Then for any sequence \( s_k > t_k \) we have

\[
\liminf_{k \to \infty} \sum_{i \geq m} f_{ik}(s_k) \geq a.
\]

(ii) Suppose there exists \( 1 > a > 0 \) and \( 1 \leq m \leq l \) for which

\[
a \leq \sum_{i \geq m} f_{ik}(t_k) \leq 1 - a.
\]

for all \( k \). Then for any \( 1 > \delta > 0 \) there exists \( T > 0 \) such that

\[
\liminf_{k \to \infty} \sum_{i \geq m} f_{ik}(t_k + T) \geq 1 - \delta.
\]

**Proof.** Suppose (i) is false. Then \( m > 1 \) and there exists a subsequence of \( t_k \) which we will also denote by \( t_k \), a sequence \( s_k > t_k \), and some \( \epsilon > 0 \) for which

\[
\sum_{i \geq m} f_{ik}(s_k) \leq a - \epsilon
\]

for all \( k \). Thus by the continuity of \( f_{ik}(t) \) in \( t \) for each \( i \) (see Proposition 3.1(iv)), there is a sequence \( t_k < T_k < s_k \) such that

\[
\sum_{i \geq m} f_{ik}(T_k) = a - \frac{\epsilon}{2}
\]

and

\[
\sum_{i \geq m} f_{ik}(t) \leq a - \frac{\epsilon}{2}
\]
for all \( t \in [T_k, s_k] \).

Now define \( g_k(t) = g(T_k + t) \). Then we may assume that \((M, g_k(t))\) converges to a soliton \((N, h(t))\) as in Proposition 3.2 such that \( p \) corresponds to the stationary point \( o \). We may also assume that \( v_k(T_k) \) converges to a vector \( w \) in \( T_o(N) \) where \( w \) has length 1 with respect to \( h(0) \). Then by Lemma 3.2(iii), for any \( T > 0 \), we have

\[
\lim_{k \to \infty} \sum_{i \geq m} f_{ik}(T_k + t) = \sum_{i \geq m} |w_i(t)|_{h(t)}
\]

uniformly for all \( t \in [0, T] \), where \( w(t) = w/|w|_{h(t)} \) and \( w_i(t) \) is the orthogonal projection of \( w(t) \) onto the eigenspace of \( \text{Ric}^h(t) \) at \( o \) of the eigenvalue \( \mu_i \) with respect to \( h(t) \).

We claim that \( s_k - T_k > \tau \) for some \( \tau > 0 \). Otherwise, we may assume that \( s_k - T_k \to 0 \), and thus from (4.9), (4.10) and (4.12) we may draw the contradiction that

\[
a - \frac{\epsilon}{2} = \sum_{i \geq m} |w_i(0)|_{h(0)} \leq a - \epsilon.
\]

This proves the claim. Thus from (4.10), (4.11) and (4.12) we may conclude that

\[
\sum_{i \geq m} w_i(0) = a - \frac{\epsilon}{2}
\]

and

\[
\sum_{i \geq m} w_i(t) \leq a - \frac{\epsilon}{2}
\]

for all \( t \in [0, \tau] \). But (4.13) and (4.14) contradict Lemma 4.1. This completes the proof of (i) by contradiction.

We now suppose (ii) is false. Note that \( m > 1 \) because \( 0 < a < 1 \). Then there exists a \( \delta > 0 \) with the property that: given any \( T > 0 \), there exists a subsequence of \( t_k \), which we also denote by \( t_k \), for which

\[
\sum_{i \geq m} f_{ik}(t_k + T) \leq 1 - \delta.
\]

for all \( k \).

Now we define \( g_k(t) = g(t_k + t) \) and assume \((M, g_k(t))\) converges to a soliton \((N, h(t))\) as in the proof of (i). We also assume that \( v_k(T_k) \) converges to a vector \( w \) in \( T_o(N) \) where \( w \) has length 1 with respect to \( h(0) \). Then by taking a limit as in the proof of (i), using Lemma 3.2(iii), (4.7) and (4.15), we have

\[
a \leq \sum_{i \geq m} w_i(0) \leq 1 - a
\]
and
\[
\sum_{i \geq m} w_i(T) \leq 1 - \delta.
\]

But for \( T \) sufficiently large depending only on \( a, \mu_{m-1}, \mu_m \) and \( \delta \), (4.16) and (4.17) contradict Lemma 4.1. This complete our proof of (ii) by contradiction. \( \square \)

We are ready to prove the main theorem in this section.

**Theorem 4.1.** Let \((M^n, \tilde{g})\) be as in Theorem 2.1 with maximal volume growth or with positive curvature operator and let \(g(x, t)\) be the corresponding solution to (2.3). With the same notation as above, \( V = T_p^{(1,0)}(M) \) can be decomposed orthogonally with respect to \( g(0) \) as \( V_1 \oplus \cdots \oplus V_l \) so that the following are true:

(i) If \( v \) is a nonzero vector in \( V_i \) for some \( 1 \leq i \leq l \), then \( \lim_{t \to \infty} |v_i(t)| = 1 \) and thus \( \lim_{t \to \infty} Rc(v(t), \bar{v}(t)) = \mu_i \) and
\[
\lim_{t \to \infty} \frac{1}{t} \log \frac{|v|^2}{|v_0|^2} = -\mu_i - 1.
\]

Moreover, the convergences are uniform over all \( v \in V_i \setminus \{0\} \).

(ii) For \( 1 \leq i, j \leq l \) and for nonzero vectors \( v \in V_i \) and \( w \in V_j \) where \( i \neq j \), \( \lim_{t \to \infty} \langle v(t), \bar{v}(t) \rangle_t = 0 \) and the convergence is uniform over all such nonzero vectors \( v, w \).

(iii) \( \dim \mathbb{C}(V_i) = n_i - n_{i-1} \) for each \( i \).

(iv)
\[
\sum_{i=1}^{l} (-\mu_i - 1) \dim \mathbb{C}(V_i) = \lim_{t \to \infty} \frac{1}{t} \log \frac{\det(g_{ij}(t))}{\det(g_{ij}(0))}.
\]

**Proof.** We first assume that the initial metric \( \tilde{g}(0) \) in (2.1) and thus \( g(x, t) \) for all \((x, t)\), has positive Ricci curvature by Proposition 3.1.

To prove (i), let \( v \in T_p(M) \) be a fixed nonzero vector and let \( f_j(t) = |v_j(t)|^2 \). We claim that \( \lim_{t \to \infty} f_m(t) = 1 \) for some \( m \), and thus \( \lim_{t \to \infty} f_k(t) = 0 \) for all \( k \neq m \). To prove our claim it will be sufficient to prove the following for every \( m \): Suppose \( \lim_{t \to \infty} f_j(t) = 0 \) for all \( j < m \). Then either
\[
\lim_{t \to \infty} f_m(t) = 1
\]
or
\[
\lim_{t \to \infty} f_m(t) = 0.
\]

If \( m = l \), then we must have \( \lim_{t \to \infty} f_m(t) = 1 \). Suppose \( 1 \leq m < l \) and \( \lim_{t \to \infty} f_j(t) = 0 \) for all \( j < m \) and that neither (4.18) nor (4.19) holds. By the continuity of \( f_i(t) \), we can find \( t_k \to \infty \) such that
\[
a \leq \sum_{i \geq m+1} f_i(t_k) \leq 1 - a
\]
for some $0 < a < 1$. By letting $v_k = v$ for all $k$, it follows from Lemma 4.2(ii), we can find $T > 0$, such that passing to a subsequence if necessary we have

$$\sum_{i \geq m+1} f_i(t_k + T) \geq 1 - \frac{a}{2}. \tag{4.21}$$

For each $j$, we can find $k_j$ such that $t_{k_j} > t_j + T$. Since

$$\sum_{i \geq m+1} f_i(t_j + T) \geq 1 - \frac{a}{2}$$

and

$$\sum_{i \geq m+1} f_i(t_{k_j}) \leq 1 - a$$

for all $j$, we may derive a contradiction from part (i) of Lemma 4.1. Thus our initial assumption was false, and for any $v \in T_p(M)$ and $m$, either (4.18) or (4.19) holds. Thus for any nonzero $v \in T_p(M)$ we have $\lim_{t \to \infty} f_m(t) = 1$ for some $m$.

Now suppose $\lim_{t \to \infty} f_m(t) = 1$. Using (2.3), Proposition 3.1, the definition of $\mu_i$ and the definition of $f_i(t)$, a straightforward calculation gives

$$\lim_{t \to \infty} \frac{1}{t} \log |v|^2_t = -\mu_m - 1.$$

Note that if

$$\lim_{t \to \infty} \frac{1}{t} \log |v|^2_t = -\mu_i - 1$$

and

$$\lim_{t \to \infty} \frac{1}{t} \log |w|^2_t = -\mu_j - 1$$

and $i \leq j$ (so that $-\mu_j \geq -\mu_i$), then

$$\lim_{t \to \infty} \frac{1}{t} \log |av + bw|^2_t \leq -\mu_j - 1. \tag{4.22}$$

provided $av + bw \neq 0$.

Let $V_1$ be the subspace of $V = T_p^{(1,0)}(M)$ defined by

$$V_1 = \{ v \in V \setminus \{0\} | \lim_{t \to \infty} \frac{1}{t} \log |v|^2_t = -\mu_1 - 1 \} \cup \{0\}. $$

It is easy to see that $V_1$ is a subspace by (4.22). Let $V_1^\perp$ be the orthogonal complement of $V_1$ with respect to $g(0)$. Then by the definition of $V_1$, for any nonzero $v \in V_1^\perp$, we have

$$\lim_{t \to \infty} \frac{1}{t} \log |v|^2_t = -\mu_j - 1$$

for some $j > 1$. Define

$$V_2 = \{ v \in V_1^\perp \setminus \{0\} | \lim_{t \to \infty} \frac{1}{t} \log |v|^2_t = -\mu_2 - 1 \} \cup \{0\}. $$
Continuing in this way, we can decompose $V$ as $V = V_1 \oplus \cdots \oplus V_l$ orthogonally with respect to $g(0)$, such that if $v \in V_m$, then
\begin{equation}
\lim_{t \to \infty} f_m(t) = 1,
\end{equation}
and
\begin{equation}
\lim_{t \to \infty} \frac{1}{t} \log \frac{|v|^2_t}{|v_0|^2} = -\mu_m - 1.
\end{equation}

It remains to prove that both convergences are uniform on $V_m \setminus \{0\}$. It is sufficient to prove the convergence in (4.23) is uniform. Suppose the convergence is not uniform over $V_m \setminus \{0\}$. Then there exist $v_k \in V_m$, $t_k \to \infty$, $\epsilon > 0$ such that
\begin{equation}
|v_k_0| = 1, \quad v_k \to v \in V_m\text{ and }\quad (4.25)\quad f_{mk}(t) = |P_m(t)v_k(t)|^2_{t_k} \leq 1 - 5\epsilon.
\end{equation}

Since $f_{mk}(t) = |P_m(t)v_k(t)|^2_t \to 1$ as $t \to \infty$ for all $k$, we can find $r_k > t_k$ such that
\begin{equation}
(4.26)\quad f_{mk}(r_k) \geq 1 - \epsilon.
\end{equation}

On the other hand, for each fixed $s$, $\lim_{k \to \infty} f_{mk}(s) = |P_m(s)v(s)|^2_s$. Moreover, $\lim_{s \to \infty} |P_m(s)v(s)|^2_s = 1$ because $v \in V_m$ and $|v_0| = 1$. Hence passing to a subsequence if necessary, we can find $s_k \to \infty$ such that $s_k < t_k$ and
\begin{equation}
(4.27)\quad f_{mk}(s_k) \geq 1 - \epsilon.
\end{equation}

Now we claim that there exists $k_0$ such that if $k \geq k_0$ then
\begin{equation}
(4.28)\quad \sum_{i \geq m} f_{ik}(t) \geq 1 - 2\epsilon
\end{equation}
for all $t > s_k$. Otherwise, we can find $s'_k > s_k$ for infinitely many $k$ such that
\begin{equation}
(4.29)\quad \sum_{i \geq m} f_{ik}(s'_k) \leq 1 - 2\epsilon.
\end{equation}

But (4.27), (4.29) and the fact that $s'_k > s_k$ contradicts Lemma 4.2(i).

If $m = l$, then for $k \geq k_0$ (4.28) contradicts (4.26) because $t_k > s_k$. Suppose $1 \leq m < l$, then by (4.25) and (4.28), for $k \geq k_0$, we have
\begin{equation}
(4.30)\quad \sum_{i \geq m+1} f_{mk}(t_k) \geq 3\epsilon.
\end{equation}

and from (4.26)
\begin{equation}
(4.31)\quad \sum_{i \geq m+1} f_{mk}(r_k) \leq \epsilon.
\end{equation}

for $k$ large enough. Since $r_k > t_k$, (4.30) and (4.31) contradicts Lemma 4.2(i) again. This completes the proof of part (i).

Part (ii) of the theorem follows directly from the definition of $v(t)$ and $w(t)$, the orthogonality of the spaces $E_i(t)$ with respect to $g(t)$ and part (i).
To prove (iii), we begin by showing the following: Fix $1 \leq m \leq l$. Let $v_k \in E_1(s_k) + \cdots + E_m(s_k)$ with $s_k \to \infty$ such that $|v_k|_0 = 1$ and $v_k$ converge to a vector $u \in T_p^{1,0}(M)$ of unit length with respect to $g(0)$. Then

$$
(4.32) \quad \lim_{t \to \infty} |u_j(t)| = 0
$$

for all $j > m$, where $u_j(t) = P_j(t)u(t)$ and $u(t) = u/|u|_t$ as before.

Suppose this is false. Then by (i), we have

$$
(4.33) \quad \lim_{t \to \infty} \sum_{j \geq m+1} |u_j(t)|^2 = 1.
$$

Let $f_{jk}(t) = |P_j(t)v_k(t)|^2$. Since for fixed $t$,

$$
\lim_{k \to \infty} f_{jk}(t) = |u_j(t)|^2,
$$

as before, given any $\frac{1}{2} > \epsilon > 0$ we may choose a subsequence of $s_k$ also denoted by $s_k$, and a sequence $t_k < s_k$ for which $t_k \to \infty$ and

$$
(4.34) \quad \sum_{j \geq m+1} f_{jk}(t_k) \geq 1 - \epsilon.
$$

for all $k$. But $\sum_{j \geq m+1} f_{jk}(s_k) = 0$ for all $k$ by definition. This is impossible by Lemma 4.2(i). Thus (4.32) is true for all $j > m$.

We now show that for all $1 \leq m \leq l$, $	ext{dim}_C V_m = n_m - n_{m-1}$ which is equal to $	ext{dim}_C E_m(t)$ for $t$ large enough. Let $d_i = \text{dim} V_i$. We claim that for any $1 \leq m \leq l$,

$$
(4.35) \quad d_1 + \cdots + d_m \geq n_m.
$$

Fix $1 \leq m \leq l$. Choose $t_k \to \infty$. We may assume that $\text{dim} E_j(t_k) = n_j - n_{j-1}$ for all $j$ and $k$. Hence we can choose a basis $v_1(t_k), \ldots, v_{n_m}(t_k)$ of $\sum_{j=1}^m E_j(t_k)$. Using Gram-Schmidt process, we may assume that

$$
v_1(t_k)/|v_1(t_k)|_{g(0)}, \ldots, v_{n_m}(t_k)/|v_{n_m}(t_k)|_{g(0)}
$$

are unitary with respect to $g(0)$. Moreover, we may assume that for $k \to \infty$, $v_j(t_k)/|v_j(t_k)|_0$ converge to some $w_j$ for all $1 \leq j \leq n_m$. Hence we have $n_m$ vectors $w_1, \ldots, w_{n_m}$. They satisfy the following:

(a) They are unitary with respect to $g(0)$ by construction.

(b) For each $1 \leq j \leq n_m$

$$
\lim_{t \to \infty} \frac{1}{t} \log |w_j(t)|_t^2 \leq -\mu_m - 1
$$

by (4.32).

For each $w_j$ ($1 \leq j \leq n_m$), $w_j = \sum_{k=1}^l w_{j,k}$ where $w_{j,k} \in V_k$. If there is a $k > m$ such that $w_{j,k} \neq 0$, then by (i) and the fact that $-\mu_k > -u_m$ and the definition of $V_k$, we have

$$
\lim_{t \to \infty} \frac{1}{t} \log |w_j(t)|_t^2 \geq -\mu_k - 1 > -\mu_m - 1,
$$

by (4.35).
contradicting (a). Thus \( w_j \in V_1 \oplus \cdots \oplus V_m \) for \( 1 \leq j \leq n_m \). From this follows because the \( w_j \) are linearly independent by (a).

Choose a unitary basis \( v_{j,1}, \ldots, v_{j,d_j} \) of \( V_j \) with respect to \( g(0) \) for all \( 1 \leq j \leq l \). This gives a unitary basis with respect to \( g(0) \) for \( T_p^{(1,0)}(M) \). Let \( g_{ij}(t) \) be components of \( g(t) \) with respect to this basis. Then

\[
\det(g_{ij}(t)) \leq \prod_{j=1}^{l} \prod_{k=1}^{d_j} |v_{j,k}|^2_{g(t)}.
\]

Since \( \lim_{t \to \infty} R(t) = \sum_{j=1}^{l} (n_j - n_{j-1})\mu_j \) by Proposition 3.1 where \( R(t) \) is the scalar curvature of \( g(t) \), by (2.3) and the above inequality we have

\[
\sum_{j=1}^{l} (n_j - n_{j-1})(-\mu_j - 1) = -\lim_{t \to \infty} R(t) - n
\]

\[
= \lim_{t \to \infty} \frac{1}{t} \log \frac{\det(g_{ij}(t))}{\det(g_{ij}(0))}
\]

\[
\leq \sum_{j=1}^{l} \sum_{k=1}^{d_j} \lim_{t \to \infty} \frac{1}{t} \log |v_{j,k}|^2_{g(t)}
\]

\[
= \sum_{j=1}^{l} d_j (-\mu_j - 1).
\]

Let us denote \( n_j - n_{j-1} \) by \( k_j \), then we have

\[
\sum_{j=1}^{l} k_j (-\mu_j) \leq \sum_{j=1}^{l} d_j (-\mu_j)
\]

and \( \sum_{j=1}^{l} d_j \geq \sum_{j=1}^{l} k_j \) for all \( 1 \leq m \leq l \) by (1.35). Also \( \sum_{j=1}^{l} d_j = \sum_{j=1}^{l} k_j = n \). Since \( -\mu_1 < -\mu_2 < \cdots < -\mu_l \), we must have \( d_j = k_j \) for all \( j \).

In fact, if this is not the case, since \( d_1 \geq k_1 \), and \( \sum_{j=1}^{m} d_j \geq \sum_{j=1}^{m} k_j \) for all \( 1 \leq m \leq l \), then we can find \( m \) to be the first \( m \) such that \( d_m > k_m \) and \( d_j = k_j \) for \( j < m \). We have

\[
\sum_{j=1}^{l} d_j (-\mu_j) = \sum_{j<m} k_j (-\mu_j) + k_m (-\mu_m) + (d_m - k_m) (-\mu_m) + \sum_{j>m} d_j (-\mu_j)
\]

\[
< \sum_{j \leq m} k_j (-\mu_j) + (d_m - k_m + d_{m+1}) (-\mu_{m+1}) + \sum_{j>m+1} d_j (-\mu_j)
\]

(4.37)
because \(-\mu_m < -\mu_{m+1}\) and \(d_m - k_m > 0\). If we let \(d'_j = k_j\) for \(1 \leq j \leq m\), \(d'_j = d_j\) for \(j > m + 1\), and \(d'_{m+1} = d_m - k_m + d_{m+1}\) then we have

\[
\sum_{j=1}^{l} k_j(-\mu_j) < \sum_{j=1}^{l} d'_j(-\mu_j)
\]

and \(\sum_{j=1}^{p} d'_j \geq \sum_{j=1}^{p} k_j\) for all \(1 \leq p \leq l\) by (4.36). Also \(\sum_{j=1}^{l} d'_j = \sum_{j=1}^{l} k_j = n\). Moreover, \(d'_j = k_j\) for all \(1 \leq j \leq m\). By induction, we will end up with

\[
\sum_{j=1}^{l} k_j(-\mu_j) < \sum_{j=1}^{l} k_j(-\mu_j)
\]

which is impossible. This completes the proof of part (iii).

Part (iv) follows directly from part (iii) and the first two equalities in (4.36).

We have thus proved that Theorem in the case that \((M, g)\) satisfied the additional assumption of positive Ricci curvature. Now if the Ricci curvature is not strictly positive on \(M\), we can use the results in [5] to reduce back to the case of positive Ricci curvature. This completes the proof of the theorem.

\[\square\]

5. Uniformization

Let \((M^n, \bar{g})\) be as in Theorem 2.1 and assume \((M, \bar{g})\) has either maximum volume growth or positive curvature operator. Let \(\bar{g}(t)\) be the solution of the Kähler-Ricci flow (2.1) and let \(g(t)\) be the corresponding solution of the normalized flow (2.3). Fix a point \(p \in M\). Then by Corollary 2.2 there exist \(1 > r_1, r_2 > 0\) such that for all \(t > 0\), there is a holomorphic map \(\Phi_t : D(r_1) \to M\) (where \(D(r_1) = \{z \in \mathbb{C}^n \mid |z| < r_1\}\)), satisfying the following:

\[
(5.1) \begin{cases} 
\Phi_t \text{ is biholomorphism from } D(r_1) \text{ onto its image.} \\
\Phi_t(0) = p. \\
\Phi_t^*(g(t))(0) = g_e, \text{ where } g_e \text{ is the standard Euclidean metric of } \mathbb{C}^n. \\
\frac{1}{r_2} g_e \leq \Phi_t^*(g(t)) \leq r_2 g_e \text{ in } D(r_1).
\end{cases}
\]

Let \(T > 0\) and let \(F_{i+1} = \Phi_{(i+1)T}^{-1} \circ \Phi_{iT}\). Then for each \(i\), \(F_i\) is a holomorphic map from \(D(r_1)\) into \(\mathbb{C}^n\) and is a biholomorphism onto its image. Let \(A_i = F_i'(0)\) be the Jacobian matrix of \(F_i\) at 0. By Corollary 2.2 we will choose \(T > 0\) large enough such that

\[
(5.2) \quad F_i(D(r_1)) \subset D(r_1), \quad |F_i(z)| \leq \delta|z| \quad \text{for some } 0 < \delta < 1.
\]

Since \(R_{ij} \geq 0\) for all \(t\), we have

\[
(5.3) \quad a|v| \leq |A_i(v)| \leq b|v|
\]
for some $0 < a < b < 1$ for all $i$. Here $a, b, \delta$ are independent of $i$. We will now modify (decompose) the maps $F_i$ as in \cite{19} and \cite{22}, then assemble them to obtain a global biholomorphism from $M$ to $\mathbb{C}^n$.

We begin by fixing some notation. By Proposition 3.1, let $0 \leq \lambda_1(t) \leq \lambda_2(t) \leq \ldots \leq \lambda_n(t)$ be the eigenvalues of $R_{ij}(t)$ with respect to $g(t)$ and let $0 \leq \mu_1 < \mu_2 \cdots < \mu_l$ be their limits. Let $\rho > 0$ and $E_k(t), 1 \leq k \leq l$ be as in Proposition 3.1 and let $P_k(t)$ be the orthogonal projection onto $E_k(t)$ with respect to $g(t)$. Let $\tau_k = e^{-\mu_k}T, 1 \leq k \leq l$. Note that for convenience, we have reversed the order of $\lambda_i$ and hence the order of $\mu_k$.

By Theorem \ref{thm:3.1}, $T_p^{(1,0)}(M)$ can be decomposed orthogonally with respect to the initial metric as $E_1 \oplus \cdots \oplus E_l$ such that if $v \in E_k$ and $w \in E_j$ are nonzero vectors and if $v(t) = v/|v|_t, w(t) = w/|w|_t$ where $|v|_t$ is the norm taken with respect to $g(t)$, then

$$(5.4) \quad \lim_{t \to \infty} |P_k(t)v(t)|_t = 1, \text{ for } 1 \leq k \leq l \text{ and } \lim_{t \to \infty} \langle v(t), w(t) \rangle_t = 0 \text{ for all } j \neq k.$$  

where $\langle \cdot, \cdot \rangle_t$ is the inner product with respect to $g(t)$. Moreover, the convergence is uniform.

For any $i$, let $E_{i,k} = d\Phi_{iT}^{-1}(E_k), 1 \leq k \leq l$. Denote $A(i) = A_i \cdots A_1$ and $A(i+j, i) = A_{i+j} \cdots A_{i+1}$. Then $E_{i,k} = A(i)(E_{1,k})$ and $A_{i+1}(E_{i,k}) = E_{i+1,k}$.

**Lemma 5.1.** Given $\epsilon > 0$, there exists $i_0$ such that if $i \geq i_0$, then the following are true:

(i) $(1 - \epsilon)\tau_k |v|^2 \leq |A_i(v)|^2 \leq (1 + \epsilon)\tau_k |v|^2$ for all $v \in E_{i,k}$ and $1 \leq k \leq l$, where $\tau_k = e^{-(\mu_k+1)T}$.

(ii) For any nonzero vector $v \in \mathbb{C}^n$

$$\left(1 - \epsilon\right) \leq \frac{|v|^2}{\sum_{k=1}^l |v_k|^2} \leq \left(1 + \epsilon\right)$$

where $v = \sum_{k=1}^l v_k$ is the decomposition of $v$ in $E_{i,1} \oplus \cdots \oplus E_{i,l}$.

**Proof.** (i) Let $1 \leq k \leq l$. By (5.4), given $\epsilon > 0$, there exists $t_0 > 0$ such that

$$|P_k(t)(w)|_t \geq 1 - \epsilon$$

for all $w \in E_k \setminus \{0\}$ and for all $t \geq t_0$. By the definition of $E_k$ and Proposition \ref{prop:3.1}, we have that $|\text{Ric}(w(t), \tilde{w}(t)) - \mu_k| \leq \epsilon$ for all $w \in E_k \setminus \{0\}$, provided $t_0$ is large enough. Suppose $i_0 > t_0/T$. Then for $i \geq i_0$ and $v \in E_{i,k} \setminus \{0\}$, there is $w \in E_k \setminus \{0\}$ with $d\Phi_{iT}^{-1}(w) = v$. Hence $A_i(v) = d\Phi_{(i+1)T}^{-1}(w)$. By (5.1), $|v| = |w|_{(i+1)T}$ and $|A_i(v)| = |w|_{(i+1)T}$. By the Kähler-Ricci flow equation we have

$$\log \left[ \frac{|A_i(v)|^2}{|v|^2} \right] + (\mu_k + 1)T = (\mu_k + 1)T$$

$$= \int_{iT}^{(i+1)T} (\mu_k - \text{Ric}(w(t), \tilde{w}(t))) \, dt.$$
Hence if $v$ where

Proof. Let $v = \sum_{k=1}^{l} v_k$ be the decomposition of $v$ in $E_{i,1} \oplus \cdots \oplus E_{i,l}$. Let $w \in T_{0}^{(1,0)}(M)$ be such that $d\Phi_{iT}^{-1}(w) = v$ and similarly decompose $w = \sum_{k=1}^{l} w_k$ with respect to $E_1 \oplus \cdots \oplus E_l$. Then $v_k = d\Phi_{iT}^{-1}(w)$. Since $\langle v_j, v_k \rangle = \langle w_j, w_k \rangle_{g(iT)}$ and $|v|^2 = |w|^2_{g(iT)}$ by (5.1), (ii) follows from (5.4).

Let us fix more notation. Let $\Phi$ be a polynomial maps from $\mathbb{C}^n$ into $\mathbb{C}^n$, which means that each component of $\Phi$ is a polynomial. Suppose $\Phi$ is of homogeneous of degree $m$. That is to say, each component of $\Phi$ is a homogeneous polynomial of degree $m \geq 1$. We define

$$||\Phi|| = \sup_{v \in \mathbb{C}^n, v \neq 0} \frac{|\Phi(v)|}{|v|^m}.$$

In general, if $\Phi$ is a polynomial map with $\Phi(0) = 0$, let $\Phi = \sum_{m=1}^{q} \Phi_m$ be the decomposition of $\Phi$ such that $\Phi_m$ is homogeneous of degree $m$, then $||\Phi||$ is defined as

$$||\Phi|| = \sum_{m=1}^{q} ||\Phi_m||.$$

If we decompose $\mathbb{C}^n$ as $E_{i,1} \oplus \cdots \oplus E_{i,l}$, we will denote $\mathbb{C}^n$ by $\mathbb{C}^n_i$. Let $\Phi : \mathbb{C}^n_i \to \mathbb{C}^n_{i+1}$ be a map. Then we decompose $\Phi$ as $\Phi(v) = \sum_{k=1}^{l} \Phi_k(v) = \Phi_1 \oplus \cdots \oplus \Phi_l$ where $\Phi_k(v) \in E_{i+1,k}$. Let $\alpha = (\alpha_1, \ldots, \alpha_l)$ be a multi-index such that $|\alpha| = \sum_{k=1}^{l} \alpha_k = m \geq 1$. Then a polynomial map $\Phi$ is said to be homogeneous of degree $\alpha$ if

$$\Phi(c_1 v_1 \oplus \cdots \oplus c_l v_l) = c^\alpha \Phi(v_1 \oplus \cdots \oplus v_l),$$

where $v_k \in E_{i,k}$. Note that if $\Phi$ homogeneous of degree $\alpha$, then $\Phi$ is homogeneous of degree $|\alpha|$ in the usual sense. $\Phi$ is said to be lower triangular, if $\Phi_k(v_1 \oplus \cdots \oplus v_l) = c_k v_k + \Psi_k(v_1 \oplus \cdots \oplus v_{k-1})$.

**Lemma 5.2.** Let $\Phi : \mathbb{C}^n_i \to \mathbb{C}^n_{i+1}$ be homogeneous of degree $\alpha = (\alpha_1, \ldots, \alpha_l)$ with $|\alpha| = m$. Then

$$|\Phi(v_1 \oplus \cdots \oplus v_l)| \leq l^m ||\Phi|| |v_1|^{\alpha_1} \cdots |v_l|^{\alpha_l}.$$

Here by convention if $\alpha_i = 0$, then $|v_i|^{\alpha_i} = 1$ for all $v_i$.

**Proof.** Let $v = v_1 \oplus \cdots \oplus v_l$ such that $|v_k| = 1$ for all $1 \leq k \leq l$, then

$$|\Phi(v)| \leq ||\Phi|| |v|^m \leq l^m ||\Phi||.$$

Hence if $v_k \neq 0$ for all $k$, then

$$|\Phi(v)| = |\Phi(v_k)| \leq l^m ||\Phi|| |v_1|^{\alpha_1} \cdots |v_l|^{\alpha_l}.$$

From this the lemma follows. \(\square\)
Note that $\tau_1 > \cdots > \tau_l$. Choose $1 > \epsilon > 0$ small enough such that $b^2(1 - \epsilon)^{-1}(1 + \epsilon) < 1$ where $b < 1$ is the constant in (5.3). Since we are interested in the maps $F_i$ for large $i$, without loss of generality, we assume the conclusions of Lemma 5.1 are true for all $i$ with this $\epsilon$. Let $m_0 \geq 2$ be a positive integer such that $a^{-1}b^m_0 < \frac{1}{2}$, where $0 < a < b < 1$ are the constants in (5.3).

We now begin to assemble the maps $F_i$ to produce a global biholomorphism from $M$ to $C^n$. The constructions follow those in [31] and [22]; in particular those in [22] where the authors study the dynamics of a randomly iterated sequence of biholomorphisms.

**Lemma 5.3.** Let $\Phi_{i+1} : C^n_i \to C^n_{i+1}$, $1 \leq i < \infty$, be a family homogeneous polynomial maps of degree $m \geq 2$ such that $\sup_i ||\Phi_i|| < \infty$. Then there exist homogeneous polynomial maps $H_{i+1}$ and $Q_{i+1}$ from $C^n_i$ to $C^n_{i+1}$ such that $\Phi_{i+1} = Q_{i+1} + H_{i+1} - A_{i+1}^{-1}H_{i+2}A_{i+1}$. Moreover, $H_{i+1}$ and $Q_{i+1}$ satisfy the following:

(i) $\sup_i ||H_i|| < \infty$ and $\sup_i ||Q_i|| < \infty$.

(ii) $Q_{i+1} = 0$ if $m \geq m_0$.

(iii) $Q_{i+1}$ is lower triangular:

$$Q_{i+1}(v_1 \oplus \cdots \oplus v_l) = 0 \oplus Q_{i+1,2}(v_1) \oplus Q_{i+1,3}(v_1 \oplus v_2) \oplus \cdots \oplus Q_{i+1,l}(v_1 \oplus \cdots \oplus v_{l-1})$$

where $v_k \in E_{i,k}$ and $Q_{i+1,k} : C^n_i \to E_{i+1,k}$.

**Proof.** For each $i$, let $\beta_k$ be a unitary basis for $E_{i,k}$ with respect to the standard metric of $C^n_i$. Let $v \in C^n_i$ and if $v = \sum_{k=1}^l \sum_{w \in \beta_k} a_w w$, then

$$C_1^{-1}|v|^2 \leq \sum_{k=1}^l \sum_{w \in \beta_k} |a_w|^2 \leq C_1|v|^2$$

for some constant $C_1$ independent of $i$ by Lemma 5.1(ii). Hence if we decompose $\Phi_{i+1}$ into $\alpha$-homogeneous parts $\Phi_{i+1,\alpha}$, $|\alpha| = m$, then $||\Phi_{i+1,\alpha}|| \leq C_2||\Phi_{i+1}||$ for some constant $C_2$ independent of $\Phi_{i+1}$ and $i$. Moreover, if we decompose $\Phi_{i+1,\alpha} = \Phi_{i+1,\alpha,1} \oplus \cdots \oplus \Phi_{i+1,\alpha,l}$ with $\Phi_{i+1,\alpha,k}(v) \in E_{i+1,\alpha}$, then by Lemma 5.1(ii) again, $||\Phi_{i+1,\alpha,k}|| \leq C_3||\Phi_{i+1,\alpha}||$ for some constant $C_3$ independent of $i$. Hence in order to prove the lemma, we may assume that $\Phi_{i+1}$ is homogeneous of degree $\alpha = (\alpha_1, \ldots, \alpha_l)$ with $|\alpha| = m$ and $\Phi_{i+1}(v) \in E_{i+1,k}$ for all $i$ for some $1 \leq k \leq l$.

Suppose $m \geq m_0$. Then we define $Q_{i+1} = 0$ and let

$$H_{i+1} = \Phi_{i+1} + \sum_{s=0}^{\infty} A_{i+2}^{-1} \cdots A_{i+s+2}^{-1} \Phi_{i+s+2}A_{i+s+1} \cdots A_{i+1}.$$  

To see $H_{i+1}$ is well-defined, by (5.3) we have that for any $v \in C^n_i$,

$$|\Phi_{i+s+2}A_{i+s+1} \cdots A_{i+1}(v)| \leq ||\Phi_{i+s+2}||(b^{s+1}|v|)^m$$

This completes the proof of Lemma 5.3.
and
\[ |A_{i+2}^{-1} \cdots A_{i+s+2}^{-1} \Phi_{i+s+2} A_{i+s+1} \cdots A_{i+1}(v)| \leq ||\Phi_{i+s+2}|| (a^{-1}b^m)^s + 1|v|^m \]
\[ \leq 2^{-s-1} ||\Phi_{i+s+2}|| |v|^m. \]

Hence $H_{i+1}$ is well-defined, homogeneous of degree $m$ and $||H_{i+1}|| \leq C_4$ for some constant $C_4$ independent of $i$. It is easy to see that $H_{i+1}$ and $Q_{i+1}$ satisfy the required conditions.

Now consider the case that $2 \leq m < m_0$. Decompose $\Phi_{i+1}$ as $\Phi_{i+1}^{(1)} + \Phi_{i+1}^{(2)}$ where $\Phi_{i+1}^{(1)}(v_1 \oplus \cdots \oplus v_l) = \Phi_{i+1}(v_1 \oplus \cdots \oplus v_k \oplus 0 \oplus 0)$ consisting all terms that depending only on $v_1, \ldots, v_k$ and $\Phi_{i+1}^{(2)} = \Phi_{i+1} - \Phi_{i+1}^{(1)}$. Let $Q_{i+1} = \Phi_{i+1}^{(1)}$. Since $\Phi_{i+1}(v) \in E_{i+1,k}$, it is easy to see that $Q_{i+1}$ satisfies condition (iii) in the lemma. It is also easy to see that $||Q_{i+1}|| \leq ||\Phi_{i+1}||$.

Suppose $\alpha_j = 0$ for all $j \geq k$, then $\Phi_{i+1}^{(2)} = 0$. In this case, let $H_{i+1} = 0$. Then $Q_{i+1}$ and $H_{i+1}$ satisfy the required conditions.

Suppose there is $j \geq k$ with $\alpha_j \geq 1$. Then define
\[ (5.5) \quad H_{i+1} = \Phi_{i+1}^{(2)} + \sum_{s=0}^{\infty} A_{i+2}^{(s)} \cdots A_{i+s+2}^{(s)} \Phi_{i+s+2}^{(2)} A_{i+s+1} \cdots A_{i+1}. \]

To prove $H_{i+1}$ is well-defined and $||H_{i+1}||$ is uniformly bounded, we observe that
\[ (5.6) \quad ||\Phi_{i+s+2}^{(2)}|| \leq ||\Phi_{i+s+2}^{(1)}|| + ||\Phi_{i+s+2}|| \leq 2||\Phi_{i+s+2}||. \]

Let $v \in \mathbb{C}^n$ and let $w = w_1 \oplus \cdots \oplus w_l = A(i+s+1,i)(v)$ and let $u = A(i+s+2,i+1)^{-1}(\Phi_{i+s+2}^{(2)}(w))$. Note that if $v = v_1 \oplus \cdots \oplus v_l$ with $v_i \in E_{i,q}$, then $A_{i+r}(v_q) \in E_{i+r,q}$. Hence by Lemma 5.2, Lemma 5.1(i) and (5.3) \[ \Phi_{i+s+2}^{(2)}(w) \leq l^n ||\Phi_{i+s+2}^{(2)}|| ||w_1|^{n_1} \cdots ||w_l|^{n_l} \]
\[ \leq 2l^n ||\Phi_{i+s+2}|| ||w|^{m-1} ||w_j|| \]
\[ \leq 2l^n ||\Phi_{i+s+2}|| b^{i(m+1)} [(1+\epsilon)\tau_j \frac{b^{i+1}}{2} |v|^m. \]

Since $\Phi_{i+s+2}^{(2)} w \in E_{i+s+2,k}$, by Lemma 5.1(i) and the fact that $A_{i+1}^{-1}(E_{r+1,k}) = E_{r,k}$ for all $r$, we have
\[ (5.7) \quad |u| = |A(i+s+2,i+1)^{-1}(\Phi_{i+s+2}^{(2)} w)| \]
\[ \leq [(1-\epsilon)\tau_k]^{-\frac{i+1}{2}} ||\Phi_{i+s+2}^{(2)}|| \]
\[ \leq 2l^n ||\Phi_{i+s+2}|| b^{i(m+1)} [(1-\epsilon)\tau_k]^{-\frac{i+1}{2}} ||w|| \]
\[ \leq 2l^n ||\Phi_{i+s+2}|| b^{i(1-\epsilon)^{-1}(1+\epsilon)^{i+1}} |v|^m \]

since $\tau_k \geq \tau_j$ for $j \geq k$, $m \geq 2$ and $b < 1$. Since we have chosen $\epsilon$ such that $b^{i(1-\epsilon)^{-1}(1+\epsilon)} < 1$, from (5.5)–(5.7), we conclude that $H_{i+1}$ is well-defined and $||H_{i+1}||$ are uniformly bounded. Note that $H_{i+1}$ is homogeneous of degree $m$. Then $Q_{i+1}$ and $H_{i+1}$ satisfy the required conditions. \qed
Lemma 5.4. Given any \( m \geq 2 \), we can find constants \( C(m) \) > 0 and \( r_1 \geq r_m > 0 \) and families of holomorphic maps \( T_{i,m} \) from \( D(r_m) \subset \mathbb{C}^n_i \) to \( D(r_m) \subset \mathbb{C}^n_i \) and \( G_{i+1,m} \) from \( \mathbb{C}^n_i \) to \( \mathbb{C}^n_i \) with the following properties:

(i) For each \( i \), \( T_{i+1,m} \) is a polynomial map of degree \( m-1 \) which is biholomorphic to its image, \( T_{i+1,m}(0) = 0 \), \( T_{i+1,m}'(0) = \text{Id} \) and \( |T_{i+1,m}| \leq C(m) \).

(ii) \( G_{i+1,m} = A_{i+1} + G_{i+1,m} \) where \( G_{i+1,m} \) is a polynomial map of degree \( m-1 \).

\[ \tilde{G}_{i+1,m}(v_1 \oplus \cdots \oplus v_l) = 0 \oplus \tilde{G}_{i+1,m,2}(v_1) \oplus \cdots \oplus \tilde{G}_{i+1,m,2}(v_1 \oplus \cdots \oplus v_{l-1}) \]

is lower triangular, and \( ||G_{i+1,m}|| \leq C(m) \), \( \tilde{G}_{i+1,m}(0) = 0 \) and \( G_{i+1,m}'(0) = 0 \). Moreover, \( G_{i+1,m} = G_{i+1,m_0} \) for all \( m \geq m_0 \), where \( m_0 \) is the integer in Lemma 5.3.

(iii) \( F_{i+1}(D(r_m)) \subset D(r_m) \) and

\[ |T_{i+1,m}F_{i+1}(v) - G_{i+1,m}T_{i,m}(v)| \leq C(m)|v|^m. \]

Here \( T_{i+1,m}F_{i+1} - G_{i+1,m}T_{i,m} \) means \( T_{i+1,m} \circ F_{i+1} - G_{i+1,m} \circ T_{i,m} \).

Proof. Note that since \( A_{i+1} \) is nonsingular, \( G_{i+1,m} \) will be a biholomorphism. We will construct the maps by induction. For \( m = 2 \), let \( T_{i+1,m} = \text{Id} \), \( G_{i+1,m} = A_{i+1} \). Since \( F_{i+1}(D(r_1)) \subset D(r_1) \) and is holomorphic, by (5.2) we can take \( r_2 = \frac{1}{2}r_1 \), then it is easy to see that one can find \( C_2 \) satisfies the required conditions. Suppose we have found \( T_{i+1,m}, G_{i+1,m}, C(m) \) and \( r_m \) which have the required properties. Since

\[ |T_{i+1,m}F_{i+1}(v) - G_{i+1,m}T_{i,m}(v)| \leq C(m)|v|^m \]

we have \( ||\Phi_{i+1}|| \leq C_1 \) for some \( C_1 \) which is independent of \( i \), where \( \Phi_{i+1} \) is the homogeneous polynomial of degree \( m \) which is the \( m \)-th power terms of the Taylor series of \( T_{i+2,m}F_{i+1} - G_{i+1,m+1}T_{i+1,m} \). By Lemma 5.3, we can find \( H_{i+1} \) and \( Q_{i+1} \) such that both are homogeneous of degree \( m \), \( H_{i+1} \) and \( Q_{i+1} \) satisfies conditions (i)–(iii) in Lemma 5.3 and

\[ \Phi_{i+1} = Q_{i+1} + H_{i+1} - A_{i+1}^{-1}H_{i+2}A_{i+1}. \]

Now define \( T_{i,m+1} = T_{i,m} + A_{i+1}^{-1}H_{i+1} \) and \( G_{i+1,m+1} = G_{i+1,m} + Q_{i+1} \). Note that if \( m \geq m_0 \), then \( Q_{i+1} = 0 \). By the induction hypothesis, Lemma 5.3 and (5.3), it is easy to see that \( T_{i+1,m+1} \) and \( G_{i+1,m+1} \) satisfy (i) and (ii) of the lemma for some constants \( C(m+1) \) and \( r_{m+1} \leq \frac{1}{2}r_m \). It remains to check condition (iii). We proceed as in (31).
In the following, \( O(m + 1) \) will denote some function \( h \) such that \( |h(v)| \leq C|v|^{m+1} \) for \( |v| \leq \frac{1}{2}r_m \), where \( C \) is a constant independent of \( i \).

(5.8)
\[
T_{i+1,m+1}F_{i+1} - G_{i+1,m+1}T_{i,m+1} = (T_{i+1,m} + A_{i+1}^{-1}H_{i+2})F_{i+1} - (G_{i+1,m} + Q_{i+1})(T_{i,m} + A_{i+1}^{-1}H_{i+1})
\]
\[
= [T_{i+1,m}F_{i+1} - G_{i+1,m}T_{i,m} + G_{i+1,m}T_{i,m} - G_{i+1,m}(T_{i,m} + A_{i+1}^{-1}H_{i+1})]
- Q_{i+1}(T_{i,m} + A_{i+1}^{-1}H_{i+1}) + A_{i+1}^{-1}H_{i+2}F_{i+1}
\]

Since \( F_i(D(r_m)) \subset D(r_m) \), and \( ||T_{i,m}|| \) and \( ||G_{i,m}|| \) are uniformly bounded,

\[
T_{i+1,m}F_{i+1} - G_{i+1,m}T_{i,m} = \Phi_{i+1} + O(m + 1)
\]
\[
= Q_{i+1} + H_{i+1} - A_{i+2}^{-1}H_{i+2}A_{i+1} + O(m + 1).
\]

Combining this with (5.8), we have

(5.9)
\[
T_{i+1,m+1}F_{i+1} - G_{i+1,m+1}T_{i,m+1} = Q_{i+1} + H_{i+1} - A_{i+2}^{-1}H_{i+2}A_{i+1} + O(m + 1)
\]
\[
= [Q_{i+1}(T_{i,m} + A_{i+1}^{-1}H_{i+1}) + A_{i+1}^{-1}H_{i+2}F_{i+1} + O(m + 1)]
+ [Q_{i+1}(T_{i,m} + A_{i+1}^{-1}H_{i+1}) + A_{i+1}^{-1}H_{i+2}F_{i+1} + O(m + 1)]
\]

Denote the differential of a map \( h \) by \( h' \). Then

\[
H_{i+2} \circ F_{i+1} - H_{i+2}A_{i+1} = \int_0^1 \frac{d}{ds}(H_{i+1} \circ (sF_{i+1} - (1 - s)A_{i+1}))ds
\]
\[
= \int_0^1 [H_{i+1}'(sF_{i+1} - (1 - s)A_{i+1})] (F_{i+1} - A_{i+1})ds
\]

where the multiplication of the terms under integral sign is matrix multiplication. By (5.2), (5.3), the definition of \( A_{i+1} \) and the fact that \( ||H_{i+1}|| \) are uniformly bounded and homogeneous of degree \( m \geq 2 \), we have

(5.10)
\[
H_{i+2} \circ F_{i+1} - H_{i+2}A_{i+1} = O(m + 1).
\]

Using (5.3) and the facts that \( ||Q_{i+1}||, ||T_{i,m}|| \) and \( ||H_{i+1}|| \) are uniformly bounded, \( Q_{i+1} \) is homogeneous of degree \( m \geq 2 \) and that \( T_{i,m}(0) = Id \), we can prove similarly that

(5.11)
\[
Q_{i+1} - Q_{i+1} \circ (T_{i,m} + A_{i+1}^{-1}H_{i+1}) = O(m + 1).
\]
Finally,
\[ G_{i+1,m} \circ T_{i,m} - G_{i+1,m} \circ (T_{i,m} + A_{i+1}^{-1}H_{i+1}) + H_{i+1} \]
\[ = - \int_0^1 \frac{d}{ds} \left(G_{i+1,m} \circ (T_{i,m} + sA_{i+1}^{-1}H_{i+1})\right) ds + H_{i+1} \]
\[ = - \int_0^1 \left([G'_{i+1,m}(T_{i,m} + sA_{i+1}^{-1}H_{i+1})] (A_{i+1}^{-1}H_{i+1}) - A_{i+1}A_{i+1}^{-1}H_{i+1}\right) ds \]
\[ = - \int_0^1 \left([G'_{i+1,m}(T_{i,m} + sA_{i+1}^{-1}H_{i+1}) - A_{i+1}] (A_{i+1}^{-1}H_{i+1})\right) ds \]

Using (5.3), and the facts that \( G'_{i+1,m}(0) = A_{i+1} \), that \( ||G_{i+1,m}||, ||H_{i+1}|| \) are uniformly bounded, and that \( H_{i+1} \) is homogeneous of degree \( m \) we conclude that

(vi) \( G_{i+1,m} \circ T_{i,m} - G_{i+1,m} \circ (T_{i,m} + A_{i+1}^{-1}H_{i+1}) + H_{i+1} = O(m+1). \)

From (5.9)–(5.12), we conclude that

\[ |T_{i+1,m+1}F_{i+1}(v) - G_{i+1,m+1}T_{i,m+1}(v)| \leq C(m+1)|v|^{m+1}. \]

This completes the proof of the lemma. \( \square \)

Let \( m \geq m_0 \) and denote \( G_{i+1,m} \) simply by \( G_{i+1} \) and denote \( \tilde{G}_{i+1,m} \) by \( \tilde{G}_{i+1} \) etc. Note that \( G_{i+1} \) is independent of \( m \) and is a biholomorphism on \( \mathbb{C}^n \). Degree of each \( G_{i+1} \) is \( m-1 \). For any positive integers \( i, j \), let \( G(i+j,i) = G_{i+j} \cdots G_{i+1} \).

**Lemma 5.5.** Let \( G_{i+1} \) as above, then its inverse is a polynomial map of degree \( (m-1)^{1-1} \) and satisfies:

\[ G_{i+1}^{-1} = A_{i+1}^{-1} + S_{i+1} \]

where \( S_{i+1} : \mathbb{C}^n_{i+1} \to \mathbb{C}^n_i \) with

\[ S_{i+1}(w_1 \oplus \cdots \oplus w_l) = 0 \oplus S_{i+1,2}(w_1) \oplus \cdots \oplus S_{i+1,i}(w_1 \oplus \cdots \oplus w_{l-1}). \]

Moreover, \( ||G_{i+1}^{-1}|| \) is bounded by a constant independent of \( i \).

**Proof.** Let \( w_1 \oplus \cdots \oplus w_l = E_{i+1,1} \oplus \cdots \oplus E_{i+1,l} \in \mathbb{C}^n_{i+1} \). Let \( v_1 = A_{i+1}^{-1}w_1 \), \( v_2 = A_{i+1}^{-1}(w_2 - \tilde{G}_{i+1,2}(v_1)), \ldots, v_l = A_{i+1}^{-1}(w_l - \tilde{G}_{i+1,l}(v_1 \oplus \cdots \oplus v_{l-1})) \). Let \( S_{i+1,k}(w_1 \oplus \cdots \oplus w_{k-1}) = -A_{i+1}^{-1}G_{i+1,k}(v_1 \oplus \cdots \oplus v_{k-1}), 2 \leq k \leq l \). It is easy to see that \( S_{i+1,k} \) is well-defined and \( S_{i+1,k}(w_1 \oplus \cdots \oplus w_{k-1}) \in E_{i,k} \) because \( A_{i+1}(E_{i,k}) = E_{i+1,k} \). Moreover, the degree of each \( S_{i+1,k} \) is at most \( (m-1)^{k-1} \). It is also easy to see that

\[ G_{i+1}^{-1} = A_{i+1}^{-1} + S_{i+1} \]

where \( S_{i+1} = 0 \oplus S_{i+1,2} \oplus \cdots \oplus S_{i+1,l} \).

Let \( w_1 \oplus \cdots \oplus w_l \in \mathbb{C}^n_{i+1} \) with \( |w_k| \leq 1 \) and \( v_1 \oplus \cdots \oplus v_l = G_{i+1}^{-1}(w_1 \oplus \cdots \oplus w_l) \).

We claim that \( |v_k| \) is bounded by a constant independent of \( i \) for each \( k \).

If this is true, then by Lemma 5.1 and (5.3) again, we can conclude that
\[ \|G_{i+1}^{-1}\| \] is bounded by a constant independent of \( i \). To prove the claim, by (5.3) have \( |v_1| = |A_{i+1}^{-1}(w_1)| \) is uniformly bounded for \( |w_1| \leq 1 \). Since \( \|G_{i+1}^{-1}\| \) is uniformly bounded by a constant independent of \( i \), \( \|G_{i+1,k}\| \) is also uniformly bounded by a constant independent of \( i \) by Lemma 5.1(ii) and (5.3). Suppose we have proved that \( |v_1|, \ldots, |v_{k-1}| \) are bounded by a constant independent of \( i \). Then

\[
|S_{i+1,k}(w_1 + \cdots + w_{k-1})| = |A_{i+1}^{-1}\tilde{G}_{i+1,k}(v_1 + \cdots + w_{k-1})|
\]

is also bounded by a constant independent of \( i \). Hence \( |v_k| \) is bounded by a constant independent of \( i \). This completes the proof of the lemma.

**Lemma 5.6.** Let \( D(R) \) be a ball in \( \mathbb{C}^n \) with radius \( R \) with center at the origin. Then the following are true:

(i) There exist \( \beta > 0 \) such that for all \( z, z' \in D(R) \) and for any positive integers \( i \) and \( j \),

\[
|G(i+j,i)^{-1}(z) - G(i+j,i)^{-1}(z')| \leq \beta |z - z'|.
\]

(ii) For any positive integer \( i \) and for any open set \( U \) containing the origin,

\[
\bigcup_{j=1}^{\infty} G(i+j,i)^{-1}(U) = \mathbb{C}^n.
\]

**Proof.** (i) For simplicity, let us assume that \( R = 1 \) and first assume that \( i = 0 \). Let us write

(5.13) \[ G(j,0)^{-1} = G_1^{-1} \cdots G_j^{-1} = H_{j,1} \oplus \cdots \oplus H_{j,l} \]

with \( H_{j,k}(v) \in E_{l,k} \). By Lemma 5.2 and the Schwartz lemma, it is sufficient to prove that

\[
|H_{j,k}(v)| \leq \beta^j
\]

for some constant \( \beta \) for all \( k \) and \( j \) provided \( |v| \leq 1 \). By Lemma 5.5, \( G_i^{-1} = A_i + S_i \) where \( S_i \) satisfies the conclusions in the lemma. Let \( v = v_1 \oplus \cdots \oplus v_l \in \mathbb{C}^j \). Then \( H_{j,1}(v) = A_j^{-1} \cdots A_1^{-1}(v_1) \leq a^j |v_1| \leq 2a^j \), where we have used Lemma 5.2. Hence (5.13) is true for \( k = 1 \). Suppose (5.13) is true for \( 1, \ldots, k-1 \). We may assume that \( \beta > a^{-1} \). By Lemma 5.2 and (5.3) we know that \( |S_j| \) is uniformly bounded. Let \( C_j = \max_k \{ \max_{|v| \leq 1} |H_{j,k}(v)|, 1 \} \). Since \( G_{j,k}(w) = A_j^{-1}(w_k) + S_j(w_1 + \cdots + w_{k-1}) \), we have

\[
C_j \leq a^{-1}C_{j-1} + C \beta^{(j-1)} N
\]

\[
\leq 2C_{j-1} \beta_1^{j-1}
\]

where \( N = (m - 1)^{l-1} \) which is the degree of \( S_i \), \( C \geq 1 \) is a constant dependent only on \( |S_i| \) and \( N \), and \( \beta_1 = C \beta^N \geq a^{-1} \), where we have used the fact that \( C_{j-1} \geq 1 \). Hence \( C_j \leq (2\beta_1)^{j-1} C_1 \). From this the lemma follows for \( i = 0 \). For general \( i \), the proof is similar. Note that the constants in the proof do not depend on \( i \).
(ii) The proof is similar to the proof of (i). Let us write $G_j \cdots G_1 = K_{j,1} \oplus \cdots \oplus K_{j,l}$. Then $K_{j,1}(v_1 \oplus \cdots \oplus v_l) = A_j \cdots A_1(v_1)$. Hence $K_{j,1}(v)$ converge to zero uniformly on compact sets. Suppose $K_{j,1}, \ldots, K_{j,k-1}$ converge uniformly to $0$ on compact sets. Let $\Omega$ be a compact set and let $s_{j} = \sup_{v \in \Omega} |K_{j,k}|$. Then as before,

$$s_{j} \leq bs_{j-1} + \sup_{v \in \Omega} |\tilde{G}_{j,k}(K_{j-1,1}(v), \ldots, K_{j-1,k-1}(v))|.$$ 

Hence

$$\limsup_{j \to \infty} s_{j} \leq b \limsup_{j \to \infty} s_{j-1}$$

because $||\tilde{G}_{j,k}||$ are uniformly bounded with uniformly bounded degrees and $K_{j-1,p}(v) \to 0$ uniformly on $\Omega$ for $1 \leq p \leq k - 1$. From this it is easy to see that $s_{j} \to 0$ as $j \to \infty$. Hence $G_j \cdots G_1 \to 0$ uniformly on compact sets. From this (ii) follows.

\[ \square \]

Let $\beta$ be the constant in Lemma 5.6. Note that $\beta$ does not depend on $i$ and $m$ provided $m \geq m_0$, where $m_0 \geq 2$ is the integer in Lemma 5.3. Fix $m \geq m_0$ such that

$$\delta^m \leq \frac{1}{2}\beta. \tag{5.14}$$

where $1 > \delta > 0$ be the constant in (5.2). Let $G_{i,m}$, $T_{i,m}$ be the maps given in Lemma 5.4 which are defined on $D(r_m)$, $0 < r_m < r_1 < 1$. Let us denote $G_{i,m}$ by $G_i$, $T_{i,m}$ by $T_i$ and $r_m$ be $r$.

In the following, a holomorphic map $\Phi$ from a complex manifold to another is said to be nondegenerate if it is injective and so that it is a biholomorphism onto its image. We apply the method in [34] to obtain the following.

Lemma 5.7. Let $k \geq 0$ be an integer. Then

$$\Psi_k = \lim_{l \to \infty} G_{k+1}^{-1} \circ G_{k+2}^{-1} \circ \cdots \circ G_{k+l}^{-1} \circ T_{k+l} \circ F_{k+l} \circ \cdots \circ F_{k+2} \circ F_{k+1}$$

exists and is a nondegenerate holomorphic map from $D(r)$ to $\mathbb{C}^n$. Moreover, there is a constant $\gamma > 0$ which is independent of $k$ such that

$$\gamma^{-1}D(r) \subset \Psi(D(r)) \subset \gamma D(r). \tag{5.15}$$

Proof. Let $\Phi_l = G_{k+1}^{-1} \circ G_{k+2}^{-1} \circ \cdots \circ G_{k+l}^{-1} \circ T_{k+l} \circ F_{k+l} \circ \cdots \circ F_{k+2} \circ F_{k+1}$. By the construction in Lemma 5.4, $\Phi_l$ is a nondegenerate holomorphic map on $D(r)$ and $\Phi_l(0) = 0$. For any $z \in D(r)$, let $w = F_{k+l} \circ \cdots \circ F_{k+1}(z)$. Then $|w| \leq \delta^l r$ by (5.2). Hence $T_{k+l}(w) \in D(R)$, $T_{k+l+1} \circ F_{k+l+1}(w) \in D(R)$, $G_{k+l+1}^{-1} \circ T_{k+l+1} \circ F_{k+l+1}(w) \in D(R)$, and $G_{k+l+1} \circ T_{k+l+1}(w) \in D(R)$ for some $R$ independent of $k$ and $l$ by Lemmas 5.4 and 5.5. By Lemmas 5.4 (iii) and
we have
\[
\left| G_{k+1}^{-1} \circ \cdots \circ G_{k+l}^{-1} \circ T_{k+l} \circ F_{k+l+1} \circ T_{k+l+1} \circ F_{k+l+1} \cdots \circ F_{k+1}(z) - G_{k+1}^{-1} \circ \cdots \circ G_{k+l}^{-1} \circ T_{k+l} \circ F_{k+l} \circ \cdots \circ F_{k+1}(z) \right|
\]
\[
= \left| G_{k+1}^{-1} \circ \cdots \circ G_{k+l}^{-1} \circ T_{k+l+1} \circ F_{k+l+1} \circ F_{k+l+1} \cdots \circ F_{k+1}(w) - G_{k+1}^{-1} \circ \cdots \circ G_{k+l}^{-1} \circ T_{k+l}(w) \right|
\]
\[
\leq \beta^{l+1} |T_{k+l+1} \circ F_{k+l+1}(w) - G_{k+l+1} \circ T_{k+l}(w)|
\]
\[
\leq C_1 \beta^{l+1} |w|^m
\]
\[
\leq C_1 \beta^{l+1} \delta^{l+m}
\]
\[
\leq C_1 \beta \left( \frac{1}{2} \right)^l
\]
by \((5.14)\). From this it is easy to see that \(\Psi_k = \lim_{l \to \infty} \Phi_l\) exists and is holomorphic on \(D(r)\). Moreover
\[
|\Psi_k(z)| \leq |\Phi_1(z)| + C_1 \beta.
\]
Using \((5.3)\) and the fact that \(||G_i||\) and \(||T_i||\) are uniformly bounded, we can find \(\gamma > 1\) independent of \(k\) and \(l\) such that \(\Psi_k(D(r)) \subset \gamma D(r)\). Since \(\Phi_i(0) = Id, \Psi_i(0) = Id\). By the gradient estimates of holomorphic functions, \(|\Phi_i(z) - Id| \leq C_2|z|\) on \(D(r)\) for some constant \(C_2\) independent of \(k\). Hence there exists \(r > r' > 0\) independent of \(k\) such that \(\Phi_k\) is nondegenerate in \(D(r')\) and \(\Psi_k(D(r)) \supset \gamma^{-1} D(r)\) proved \(\gamma\) is large enough independent of \(k\). To prove that \(\Psi_k\) is nondegenerate on \(D(r)\), let \(l_0\) be such that \(F_{k+l_0} \cdots \circ F_{k+1}(D(r)) \subset D(r')\). Then
\[
\Psi_k = G_{k+1}^{-1} \circ \cdots \circ G_{k+l_0}^{-1} \circ \Psi_{k+l_0} \circ F_{k+l_0} \circ \cdots \circ F_{k+1}.
\]
Since \(F_{k+l_0} \cdots \circ F_{k+1}\) is nondegenerate on \(D(r)\), \(\Psi_{k+l_0}\) is nondegenerate on \(D(r')\), and \(G_{k+1}^{-1} \circ \cdots \circ G_{k+l_0}^{-1}\) is a biholomorphism of \(\mathbb{C}^n\), we conclude that \(\Psi_k\) is nondegenerate on \(D(r)\).

Now we are ready to prove the following uniformization theorem.

**Theorem 5.1.** Let \((M^n, \tilde{g})\) be a complete noncompact Kähler manifold with nonnegative and bounded holomorphic bisectional curvature. Suppose the scalar curvature of \(M\) satisfies
\[
(5.16) \quad \frac{1}{V_x(r)} \int_{B_x(r)} R \leq \frac{C}{1 + r^2}
\]
for some constant $C$ for all $x \in M$ for all $r$. Suppose $(M, g)$ has maximal volume growth. Then $M$ is biholomorphic to $\mathbb{C}^n$. Moreover, the assumption of maximal volume growth can be removed if $M$ has positive curvature operator.

**Proof.** If $\tilde{g}$ satisfies the given conditions, then one can solve the Kähler-Ricci flow (2.3) and construct $\Phi_t$ and $F_i$ as in the beginning of this section. We can also construct $G_i$, $T_i$ as in Lemma 5.3 so that Lemmas 5.6 and 5.7 are true. Let $\Omega_i = \Phi^{-1}_i(D(r))$ where $r > 0$ is the constant in Lemma 5.7. By (5.1) and the fact that the solution $g(t)$ of (2.3) decays exponentially, $\{\Omega_i\}_{i \geq 1}$ exhausts $M$. Consider the following holomorphic maps from $\Omega_i$ to $\mathbb{C}^n$:

$$S_i = G_1^{-1} \circ \cdots \circ G_i^{-1} \circ T_i \circ \Phi^{-1}_i.$$

For each fixed $k$, and $l \geq 1$

$$S_{k+l} = G_1^{-1} \circ \cdots \circ G_k^{-1} \circ G_{k+l}^{-1} \circ T_{k+l} \circ \Phi^{-1}_{k+l}.$$

By Lemma 5.7, we conclude that $S = \lim_{i \to \infty} S_i$ exists and is a nondegenerate holomorphic map from $M$ into $\mathbb{C}^n$. Moreover, $S = G_1^{-1} \circ \cdots \circ G_k^{-1} \circ \Psi_k \circ \Phi^{-1}_k$ on $\Omega_k$ where $\Psi_k$ is the nondegenerate holomorphic map in Lemma 5.7. Hence

$$S(\Omega_k) = G_1^{-1} \circ \cdots \circ G_k^{-1} \circ \Psi_k(D(r)) \supset G_1^{-1} \circ \cdots \circ G_k^{-1} (\gamma^{-1} D(r))$$

by Lemma 5.7 for some $\gamma$ independent of $k$. Therefore $S(M) = \mathbb{C}^n$ by Lemma 5.6 (ii). This completes proof of the theorem. 

By a recent result of Ni [28], if $M$ has maximal volume growth, then (5.16) is satisfied automatically. Hence we have:

**Corollary 5.1.** Let $(M^n, \tilde{g})$ be a complete noncompact Kähler manifold with nonnegative and bounded holomorphic bisectional curvature. Suppose $M$ has maximal volume growth, then $M$ is biholomorphic to $\mathbb{C}^n$.

We also have the following uniformization theorem.

**Theorem 5.2.** Let $(M^n, \tilde{g})$ be a complete noncompact Kähler manifold with nonnegative curvature operator such that the scalar curvature $R$ of $M$ satisfies (5.16). Then the universal cover of $M$ is biholomorphic to $\mathbb{C}^n$.

**Proof.** Let $\tilde{g}(t)$ be the corresponding solution to the Kähler-Ricci flow (2.3). Let $\tilde{M}$ be the universal cover of $M$. We then lift the flow $\tilde{g}(t)$ to $\tilde{M}$ and denote the lifted flow by $\tilde{h}(t)$.

By the result in [5] and the De Rham decomposition theorem, one may assume that $\tilde{M} = \mathbb{C}^k \times N_1 \times \cdots \times N_l$ isometrically and holomorphically so that each $N_j$ is irreducible and has nonnegative curvature operator and positive Ricci curvature. Note that the flow $\tilde{h}(t)$ still satisfies the Kähler-Ricci flow equation when restricted on each $N_j$. Now suppose there is a
positive constant $C$ such that for $t$ large enough, the injectivity radius of $\tilde{h}(t)$ is bounded below by $Ct^{1/2}$. Then by the proof of Theorem 5.1, it is not hard to show that in this case we can still have the results of sections $\S 3$, $\S 4$ and $\S 5$ for the restriction of $\tilde{h}(t)$ to any $N_j$, thus proving Theorem 5.2. We now proceed to show the above injectivity radius bound.

We claim that each $N_j$ is noncompact. In fact, by the curvature assumption on $M$, there exists $u$ such that $\sqrt{-1} \partial \bar{\partial} u = \text{Ric}_M$; see (30). Let $\tilde{u}$ be the lift of $u$ to $\tilde{M}$. Then $\sqrt{-1} \partial \bar{\partial} \tilde{u} = \text{Ric}_{\tilde{M}}$. In particular, $\tilde{u}$ is strictly plurisubharmonic on each $N_j$. Hence $N_j$ is noncompact.

By the proof in [9], p. 25-26, one can conclude that for any $t_0 > 0$, there is a $\delta > 0$ such that $h(t)$ has positive sectional curvature for $t_0 < t \leq t_0 + \delta$ when restricted to $N_j$. Using the result of Gromoll-Meyer as before and using the fact that the curvature of $N_j$ is bounded above by $C_1 t^{-1}$ by Theorem 2.1, one can conclude that the injectivity radius of $\tilde{h}(t)$ on $N_j$ is bounded below by $C_1 t^{1/2}$ for some constant $C_1 > 0$ independent of $t$, $t_0$ and $j$. From this we can conclude that the injectivity radius of $h(t_0)$ on $N_j$ is bounded below by $C_1 t^{1/2}$. Hence the injectivity radius of $h(t)$ on $\tilde{M}$ is bounded below by $Ct^{1/2}$ for some constant $C > 0$ independent of $t$. This completes the proof of the Theorem.

□

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