Two Fractal Overlap Time Series: Earthquakes and Market Crashes

Bikas K. Chakrabarti,1 Arnab Chatterjee,1,∗ and Pratip Bhattacharyya1,2
1Theoretical Condensed Matter Physics Division and Centre for Applied Mathematics and Computational Science, Saha Institute of Nuclear Physics, 1/AF Bidhannagar, Kolkata 700 064, India.
2Physics Department, Gurudas College, Narkeldanga, Kolkata 700 054, India.

We find prominent similarities in the features of the time series for the (model earthquakes or) overlap of two Cantor sets when one set moves with uniform relative velocity over the other and time series of stock prices. An anticipation method for some of the crashes have been proposed here, based on these observations.

I. INTRODUCTION

Capturing dynamical patterns of stock prices are major challenges both for epistemologists as well as for financial analysts.[1] The statistical properties of their (time) variations or fluctuations[2] are now well studied and characterized (with established fractal properties), but are not very useful for studying and anticipating their dynamics in the market. Noting that a single fractal gives essentially a time averaged picture, a minimal two-fractal overlap time series model was introduced [2, 3, 4] to capture the time series of earthquake magnitudes. We find that the same model can be used to mimic and study the essential features of the time series of stock prices.

II. THE TWO FRACTAL-OVERLAP MODEL OF EARTHQUAKE

Let us consider first a geometric model[2, 3, 4] of the fault dynamics occurring in overlapping tectonic plates that form the earth’s lithosphere. A geological fault is created by a fracture in the earth’s rock layers followed by a displacement of one part relative to the other. The two surfaces of the fault are known to be self-similar fractals. In the model considered here[2, 3, 4, 5] a fault is represented by a pair of overlapping identical fractals and the fault dynamics arising out of the relative motion of the associated tectonic plates is represented by sliding one of the fractals over the other; the overlap O between the two fractals represents the energy released in an earthquake whereas log O represents the magnitude of the earthquake. In the simplest form of the model each of the two identical fractals is represented by a regular Cantor set of fractal dimension log 2/ log 3 (see Fig. 1). This is the only exactly solvable model for earthquakes known so far. The exact analysis of this model[2] for a finite generation n of the Cantor sets with periodic boundary conditions showed that the probability of the overlap O, which assumes the values O = 2n−k(k = 0, . . . , n), fol-

FIG. 1: The overlap of two identical Cantor sets of dimension log 2/ log 3 at generation n = 2 as one moves over the other with uniform velocity. The total measure O of the overlap (total shaded region) varies with time and are shown for two different time instances.

low the binomial distribution F of log2 O = n − k [6]:

\[ \Pr (O = 2^{n-k}) = \Pr (\log O = n-k) = \binom{n}{n-k} \left( \frac{1}{3} \right)^{n-k} \left( \frac{2}{3} \right)^k = F(n-k). \]  (1)

Since the index of the central term (i.e., the term for the most probable event) of the above distribution is n/3 + δ, −2/3 < δ < 1/3, for large values of n Eq. (1) may be written as

\[ F \left( \frac{n}{3} \pm r \right) \approx \binom{n}{n \pm r} \left( \frac{1}{3} \right)^{r\pm r} \left( \frac{2}{3} \right)^{2r} \]  (2)

by replacing n − k with n/3 ± r. For r ≪ n, we can write the normal approximation to the above binomial distribution as

\[ F \left( \frac{n}{3} \pm r \right) \approx \frac{3}{\sqrt{2\pi n}} \exp \left( -\frac{9r^2}{2n} \right) \]  (3)

Since log2 O = n − k = \frac{n}{3} ± r, we have

\[ F (\log_2 O) \approx \frac{1}{\sqrt{n}} \exp \left[ -\left( \frac{\log_2 O}{n} \right)^2 \right], \]  (4)

not mentioning the factors that do not depend on O. Now

\[ F (\log_2 O) d (\log_2 O) \equiv G(O)dO \]  (5)

where

\[ G(O) \equiv \frac{1}{O} \exp \left[ -\left( \frac{\log_2 O}{n} \right)^2 \right] \]  (6)
is the log-normal distribution of $O$. As the generation index $n \to \infty$, the normal factor spreads indefinitely (since its width is proportional to $\sqrt{n}$) and becomes a very weak function of $O$ so that it may be considered to be almost constant; thus $G(O)$ asymptotically assumes the form of a simple power law with an exponent that is independent of the fractal dimension of the overlapping Cantor sets $[4]$:

$$G(O) \sim \frac{1}{O} \quad \text{for} \quad n \to \infty. \quad (7)$$

### III. THE CANTOR SET OVERLAP TIME SERIES

We now consider the time series $O(t)$ of the overlap set (of two identical fractals [4, 5]), as one slides over the other with uniform velocity. Let us again consider two regular cantor sets at finite generation $n$. As one set slides over the other, the overlap set changes. The total overlap $O(t)$ at any instant $t$ changes with time (see Fig. 2(a)). In Fig. 2(b) we show the behavior of the cumulative overlap $[4] Q^o(t) = \int_{t_{i-1}}^{t_i} O(\tilde{t})d\tilde{t}$, $i \leq t_i$, has got the peak value ‘quantization’ as shown in Fig. 2(c). The reason is obvious. This justifies the simple thumb rule: one can simply count the cumulative $Q^o(t)$ of the overlaps since the last ‘crash’ or ‘shock’ at $t_{i-1}$ and if the value exceeds the minimum value ($q_o$), one can safely extrapolate linearly and expect growth upto $\alpha q_o$ here and face a ‘crash’ or overlap greater than $\Delta = 150$ in Fig. 2. If nothing happens there, one can again wait up to a time until which the cumulative grows upto $\alpha^2 q_o$ and feel a ‘crash’ and so on ($\alpha = 5$ in the set considered in Fig. 2).

### IV. THE STOCK PRICE TIME SERIES

We now consider some typical stock price time-series data, available in the internet. The data analyzed here

![FIG. 2: (a) The time series data of overlap size $O(t)$ for a regular Cantor set of dimension $\ln 4/\ln 5$ at generation $n = 4$. (b) Cumulative overlap $Q^o(t)$ and (c) the variation of the cumulative overlap $Q^o(t)$ for the same series, where $Q$ is reset to zero after any big event of size greater than $\Delta = 150$.](image)

![FIG. 3: Data from New York Stock Exchange from January 1966 to December 1979: industrial index [5]. (a) Daily closing index $S(t)$ (b) integrated $Q^o(t)$, (c) daily changes $\delta S(t)$ of the index $S(t)$ defined as $\delta S(t) = S(t+1) - S(t)$, and (d) behavior of $Q^o(t)$ where $\delta S(t_i) > \Delta$. Here, $\Delta = -1.0$ as shown in (c) by the dotted line (from [5]).](image)
A simple ‘anticipation strategy’ for some of the crashes may be as follows: If the cumulative $Q_i(t)$ since the last crash has grown beyond $q_0 \simeq 8000$ here, wait until it grows (linearly with time) until about 17,500 ($\simeq 2.5q_0$) and expect a crash there. If nothing happens, then wait until $Q_i(t)$ grows (again linearly with time) to a value of the order of 39,000 ($\simeq (2.2)^2q_0$) and expect a crash, and so on.

The same kind of analysis for the NYSE ‘utility index’, for the same period, is shown in Figs. 4b.

V. EARTHQUAKE MAGNITUDE TIME SERIES

Unlike in the case of stock price time series where accurate data are easily available, the time series for earthquake magnitudes $M(t)$ at any fault involves considerably coordinated measurements and comparable accuracies are not easily achievable. Still from the available data, as in the case of stock market (where the integrated stock price $Q^s(t)$ shows clear linear variations with time and this fits well with that for the cumulative overlap $Q^m(t)$ for the fractal overlap model; see also [9]), the integrated earthquake magnitude $Q^m(t) = \int_0^t M(t)dt$ of the aftershocks does also show such prominent linear variations (see Fig. 5). We believe, the slopes of these linear $Q^m(t)$ vs. $t$ curves for different faults would give us the signature of the corresponding fractal structure of the underlying fault. It may be noted in this context, in our model, the slope becomes $[(a-1)^2/u]^n$ for an $n$th generation Cantor set, formed out of the remaining $a-1$ blocks having the central block removed.

VI. SUMMARY

Based on the formal similarity between the two-fractal overlap model of earthquake time series and of the stock market, we considered here a detailed comparison. We find, the features of the time series for the overlap of two Cantor sets when one set moves with uniform relative
velocity over the other looks somewhat similar to the time series of stock prices. We analyze both and explore the possibilities of anticipating a large (change in Cantor set) overlap or a large change in stock price. An anticipation method for some of the crashes has been proposed here, based on these observations.

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