On the birational geometry of Fano 4-folds

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September 23, 2011

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1 Introduction

After Mori and Mukai’s classification of Fano 3-folds with Picard number $\rho \geq 2$ in the early 80’s, it has become a classical subject to study Fano manifolds via their contractions using Mori theory. Indeed the Fano condition makes the situation quite special, because the Cone and the Contraction Theorems hold for the whole cone of effective curves.

It has been conjectured by Hu and Keel [HK00], and recently proved by Birkar, Cascini, Hacon, and McKernan [BCHM10], that the special behaviour of Fano manifolds with respect to Mori theory is even stronger: in fact, Fano manifolds are Mori dream spaces.

In particular, this implies that the classical point of view can be extended from regular contractions to rational contractions. If $X$ is a Mori dream space, a rational contraction of $X$ is a rational map $f: X \dashrightarrow Y$ which factors as a finite sequence of flips, followed

\footnote{A contraction is a morphism with connected fibers onto a normal projective variety.}
by a regular contraction. Equivalently, $f$ can be seen as a regular contraction of a small $\mathbb{Q}$-factorial modification of $X$, that is, a variety related to $X$ by a sequence of flips.

In this paper we use properties of Mori dream spaces to study rational contractions of a smooth Fano 4-fold $X$. In particular, we are interested in bounding the Picard number $\rho_X$ of $X$.

We recall that $\rho_X = b_2(X)$ is a topological invariant of Fano 4-folds, whose maximal value is not known. By taking products of Del Pezzo surfaces one gets examples with $\rho \in \{2, \ldots, 18\}$, while all known examples of Fano 4-folds which are not products have $\rho \leq 6$.

Our main result is a bound on $\rho_X$ when $X$ has an elementary rational contraction of fiber type, or more generally, a quasi-elementary rational contraction of fiber type. Let us explain the terminology: as in the regular case, a rational contraction $f : X \to Y$ is of fiber type if $\dim Y < \dim X$, and it is elementary if $\rho_X - \rho_Y = 1$.

Quasi-elementary rational contractions are a special class of rational contractions of fiber type, which includes the elementary ones. They share many useful properties of the elementary case, for instance the target is again a Mori dream space. If $f : X \to Y$ is a contraction of fiber type, then $f$ is quasi-elementary if every curve contracted by $f$ is numerically equivalent to a one-cycle contained in a general fiber of $f$. In the case of rational contractions, we give some equivalent characterizations of being quasi-elementary, see section 2.2 for more details.

Quasi-elementary (regular) contractions of Fano manifolds have been studied in [Cas08]; let us recall what is known in the 4-dimensional case.

**Theorem (Cas08, Cor. 1.2).** Let $X$ be a smooth Fano 4-fold.

If $X$ has an elementary contraction of fiber type, then $\rho_X \leq 11$, with equality only if $X \cong \mathbb{P}^1 \times \mathbb{P}^1 \times S$ or $X \cong F_1 \times S$, where $S$ is a surface.

If $X$ has a quasi-elementary contraction of fiber type, then $\rho_X \leq 18$, with equality only if $X$ is a product of surfaces.

Here is the result in the case of a rational contraction.

**Theorem 1.1.** Let $X$ be a smooth Fano 4-fold.

If $X$ has an elementary rational contraction of fiber type, then $\rho_X \leq 11$.

If $X$ has a quasi-elementary rational contraction of fiber type, which is not regular, then $\rho_X \leq 17$.

The strategy for the proof of Th. 1.1 is similar to the one used in [Cas08], via the study of elementary contractions of the target of the rational contraction of fiber type. We systematically use properties of Mori dream spaces, and a key ingredient is a description of non-movable prime divisors in $X$ when $\rho_X \geq 6$. More precisely, we show the following.

**Theorem 1.2.** Let $X$ be a smooth Fano 4-fold with $\rho_X \geq 6$, and $D \subset X$ a non-movable prime divisor. Then either $D$ is the locus of an extremal ray of type $(3, 2)$ or there exists

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2 See on p. 5 for the terminology.
a diagram:

\[
X \to \tilde{X} \to Y
\]

where \(X \to \tilde{X}\) is a sequence of at least \(\rho_X - 4\) flips, \(f\) is an elementary divisorial contraction with exceptional locus the transform \(\tilde{D}\) of \(D\), and one of the following holds:

- \(Y\) is smooth and Fano, \(f\) is the blow-up of a smooth curve, and \(\tilde{D}\) is a \(\mathbb{P}^2\)-bundle over a smooth curve;
- \(Y\) is smooth and Fano, \(f\) is the blow-up of a point, and \(\tilde{D} \cong \mathbb{P}^3\);  
- \(\tilde{D}\) is isomorphic to a quadric, \(f(\tilde{D})\) is a factorial and terminal singular point, and \(Y\) is Fano.

We finally apply these results to Fano 4-folds \(X\) with \(c_X = 1\) or \(c_X = 2\). Let us recall from [Cas11] that \(c_X\) is an invariant of a Fano manifold \(X\), defined as follows. For any prime divisor \(D \subset X\), we consider the restriction map \(H^2(X, \mathbb{R}) \to H^2(D, \mathbb{R})\), and we set:

\[
c_X := \max \{ \dim \ker (H^2(X, \mathbb{R}) \to H^2(D, \mathbb{R})) \mid D \text{ is a prime divisor in } X \} \in \{0, \ldots, \rho_X - 1\}.
\]

By [Cas11, Th. 3.3] we have \(c_X \leq 8\) for any smooth Fano manifold \(X\), and if \(c_X \geq 4\), then \(X\) is a product of a Del Pezzo surface with another Fano manifold.

In particular, in dimension 4, we have \(\rho_X \leq 18\) as soon as \(c_X \geq 4\). Moreover when \(c_X = 3\) we know after [Cas11] that \(\rho_X \leq 8\) (see Th. 3.11). Therefore in order to study Fano 4-folds with large Picard number, we can reduce to the case \(c_X \leq 2\); this is used throughout the paper. In the last section we show the following.

**Theorem 1.3.** Let \(X\) be a smooth Fano 4-fold with \(c_X \in \{1, 2\}\). Then either \(\rho_X \leq 12\), or \(X\) is the blow-up of another Fano 4-fold along a smooth surface.

**Outline of the paper.** Section 2 concerns Mori dream spaces. In section 2.1 we recall from [HK00] the main geometrical properties of Mori dream spaces; then in section 2.2 we define quasi-elementary rational contractions and explain some of their properties.

In section 3 we move to Fano 4-folds. We first give in section 3.1 some elementary properties of small \(\mathbb{Q}\)-factorial modifications and rational contractions of Fano 4-folds. Then in section 3.2 we recall some results needed from [Cas11], and study the implications on prime divisors in a small \(\mathbb{Q}\)-factorial modification of a Fano 4-fold. Finally in section 3.3 we show Th. 1.2 on non-movable prime divisors.

In section 4 we show Th. 1.1. We study first the case where the target is a surface in section 4.1 and then the case where the target has dimension 3 in section 4.2 (the case where the target is a curve is easier and is treated in section 3.1).

Finally in section 5 we show Th. 1.3.
Acknowledgements. Part of this paper has been written during a stay at the Ludwig-Maximilians University in Munich, in spring 2010. I am grateful to Professor Andreas Rosenschon and to the Mathematisches Institut for the kind hospitality. I also thank the referee for some useful remarks.

Notation and terminology
We work over the field of complex numbers.
A manifold is a smooth algebraic variety.
A divisor is a Weil divisor.
If \( f : X \to Y \) is a rational map, \( \text{dom}(f) \) is the largest open subset of \( X \) where \( f \) is regular.

Let \( X \) be a normal projective variety.
A contraction of \( X \) is a morphism with connected fibers \( f : X \to Y \) onto a normal projective variety. We will sometimes consider the case where \( X \) and \( Y \) are quasi-projective and \( f \) is a projective morphism; in this case we call \( f \) a local contraction.

\( \mathcal{N}_1(X) \) (respectively \( \mathcal{N}_1(X) \)) is the \( \mathbb{R} \)-vector space of Cartier divisors (respectively one-cycles) with real coefficients, modulo numerical equivalence.

\( \text{Nef}(X) \subset \mathcal{N}_1(X) \) is the cone of nef classes.
\( \text{Eff}(X) \subset \mathcal{N}_1(X) \) is the convex cone generated by classes of effective divisors, and \( \overline{\text{Eff}}(X) \) is its closure.

Let \( X \) be a normal and \( \mathbb{Q} \)-factorial projective variety.
The anticanonical degree of a curve \( C \subset X \) is \( -K_X \cdot C \).
For any closed subset \( Z \) of \( X \), \( \mathcal{N}_1(Z,X) := i_*(\mathcal{N}_1(Z)) \subseteq \mathcal{N}_1(X) \), where \( i : Z \hookrightarrow X \) is the inclusion.

\( [D] \) is the numerical equivalence class in \( \mathcal{N}_1(X) \) of a divisor \( D \) in \( X \), and similarly \( [C] \in \mathcal{N}_1(X) \) for a curve \( C \subset X \).
\( \equiv \) stands for numerical equivalence.

For any subset \( H \subseteq \mathcal{N}_1(X) \), \( H^\perp := \{ \gamma \in \mathcal{N}_1(X) \mid h \cdot \gamma = 0 \text{ for every } h \in H \} \), and similarly if \( H \subseteq \mathcal{N}_1(X) \). For any divisor \( D \) in \( X \), \( D^\perp := [D]^\perp \subseteq \mathcal{N}_1(X) \).

A divisor \( D \) in \( X \) is movable if its stable base locus has codimension at least 2. \( \text{Mov}(X) \subset \mathcal{N}_1(X) \) is the convex cone generated by classes of movable divisors.

\( \text{NE}(X) \subset \mathcal{N}_1(X) \) is the convex cone generated by classes of effective curves, and \( \overline{\text{NE}}(X) \) is its closure.

\( \text{ME}(X) \subset \mathcal{N}_1(X) \) is the cone dual to \( \overline{\text{Eff}}(X) \subset \mathcal{N}_1(X) \).

Let \( f : X \to Y \) be a contraction. The exceptional locus \( \text{Exc}(f) \) is the set of points of \( X \) where \( f \) is not an isomorphism. If \( D \) is a divisor in \( X \), we say that \( f \) is \( D \)-positive (respectively \( D \)-negative) if \( D \cdot C > 0 \) (respectively \( D \cdot C < 0 \)) for every curve \( C \subset X \) such that \( f(C) = \{ pt \} \). When \( D = K_X \), we just say \( K \)-positive (or \( K \)-negative).

We consider the push-forward of one-cycles \( f_* : \mathcal{N}_1(X) \to \mathcal{N}_1(Y) \), and set \( \text{NE}(f) := \overline{\text{NE}}(X) \cap \ker f_* \). We also say that \( f \) is the contraction of \( \text{NE}(f) \).

The contraction \( f \) is elementary if \( \rho_X - \rho_Y = 1 \). In this case we say that \( f \) is of type \((a,b)\) if \( \dim \text{Exc}(f) = a \) and \( \dim f(\text{Exc}(f)) = b \).

We will use greek letters \( \sigma, \tau, \eta, \) etc. to denote convex polyhedral cones and their faces in \( \mathcal{N}_1(X) \) or \( \mathcal{N}_1(X) \).
If $\sigma \subseteq N_1(X)$ is a convex polyhedral cone and $\sigma^\vee \subseteq N_1(X)$ its dual cone, there is a natural bijection between the faces of $\sigma$ and those of $\sigma^\vee$, given by $\tau \mapsto \tau^* := \sigma^\vee \cap \tau^\perp$ for every face $\tau$ of $\sigma$. 

An extremal ray of $X$ is a one-dimensional face of $\overline{\text{NE}}(X)$. 

Consider an elementary contraction $f: X \to Y$ and the extremal ray $\sigma := \text{NE}(f)$. We say that $\sigma$ is birational, divisorial, small, or of type $(a, b)$, if $f$ is. We set $\text{Locus}(\sigma) := \text{Exc}(f)$, namely $\text{Locus}(\sigma)$ is the union of the curves whose class belongs to $\sigma$. If $D$ is a divisor in $X$, we say that $D \cdot \sigma > 0$ if $D \cdot C > 0$ for a curve $C$ with $[C] \in \sigma$, equivalently if $f$ is $D$-positive; similarly for $D \cdot \sigma = 0$ and $D \cdot \sigma < 0$. 

Suppose that $f: X \to Y$ is a small elementary contraction, and let $D$ be a divisor in $X$ such that $f$ is $D$-negative. The flip of $f$ is a birational map $g: X \dasharrow \tilde{X}$ which fits into a commutative diagram:

$$
\begin{array}{c}
X \xrightarrow{g} \tilde{X} \\
\downarrow f \\
Y \xleftarrow{\tilde{f}}
\end{array}
$$

where $\tilde{X}$ is a normal and $\mathbb{Q}$-factorial projective variety, $g$ is an isomorphism in codimension one, and $\tilde{f}$ is a $\mathbb{D}$-positive, small elementary contraction ($\mathbb{D}$ the transform of $D$ in $\tilde{X}$). If the flip exists, it is unique and does not depend on $D$, see [KM98, Cor. 6.4 and Def. 6.5]. We also say that $g$ is the flip of the small extremal ray $\text{NE}(f)$, and that $g$ is a $D$-negative flip. Similarly, if $B$ is a divisor on $X$ such that $f$ is $B$-positive, we say that $g$ is a $B$-positive flip. Finally, when $D = K_X$, we just say $K$-positive or $K$-negative.

Suppose that $X$ is a projective 4-fold and that $f: X \to Y$ is an elementary contraction. We say that $f$ is of type $(3, 2)^\text{sm}$ if it is birational and every fiber has dimension at most 1, equivalently if $Y$ is smooth and $f$ is the blow-up of a smooth surface (see Th. 3.1).

2 Mori dream spaces

2.1 A brief survey

In this section we recall from [HK00] the definition and the main geometrical properties of Mori dream spaces. It is meant as a quick introduction, and contains no new results; we provide proofs of some elementary properties for which we could not find an easy reference.

**Definition 2.1.** Let $X$ be a normal and $\mathbb{Q}$-factorial projective variety. A small $\mathbb{Q}$-factorial modification (SQM) of $X$ is a normal and $\mathbb{Q}$-factorial projective variety $\tilde{X}$, together with a birational map $f: X \dasharrow \tilde{X}$ which is an isomorphism in codimension 1.

Flips are examples of SQMs.

**Definition 2.2** ([HK00], Def. 1.10). Let $X$ be a normal and $\mathbb{Q}$-factorial projective variety, with finitely generated Picard group. We say that $X$ is a Mori dream space if there are a finite number of SQMs $f_j: X \dasharrow X_j$ such that:
(i) for every \( j \), \( \text{Nef}(X_j) \) is a polyhedral cone, generated by the classes of finitely many semiample divisors;

(ii) \( \text{Mov}(X) = \bigcup_j f_j^*(\text{Nef}(X_j)) \).

Notice that if \( X \) is a normal and \( \mathbb{Q} \)-factorial projective variety having a SQM \( \tilde{X} \) which is a Mori dream space, then \( X \) itself is a Mori dream space.

Let \( X \) be a Mori dream space. We consider the following cones in \( \mathcal{N}_X \):

\[
\text{Nef}(X) \subseteq \text{Mov}(X) \subseteq \text{Eff}(X).
\]

All three are closed and polyhedral (see \([HK00, \text{Prop. 1.11(2)}]\)), and have dimension \( \rho_X \).

By condition (ii), one of the SQMs \( f_j \) must be the identity of \( X \), and by (i) \( \text{Nef}(X) \) is generated by the classes of finitely many semiample divisors. In particular this implies that the association

\[
(f : X \to Y) \mapsto f^*(\text{Nef}(Y))
\]

yields a bijection between the set of contractions of \( X \) and the set of faces of \( \text{Nef}(X) \).

**Definition 2.3.** Let \( X \) be a Mori dream space. A rational contraction of \( X \) is a rational map \( f : X \dashrightarrow Y \) which factors as \( X \dashrightarrow \tilde{X} \to Y \), where \( X \dashrightarrow \tilde{X} \) is a SQM, and \( \tilde{X} \to Y \) a (regular) contraction.

(In \([HK00]\) the terminology “contracting rational map” is also used.) Let us notice that the definition \([HK00, \text{Def. 1.1}]\) is more general, because \( X \) is just assumed to be a normal projective variety; when \( X \) is a Mori dream space, the two notions coincide, by \([HK00, \text{Prop. 1.11}]\).

Every SQM of \( X \) factors as a finite sequence of flips (see \([HK00, \text{Prop. 1.11}]\)), therefore a rational contraction can equivalently be described as a rational map which factors as a finite sequence of flips followed by a contraction.

**Remark 2.4.** Let \( X \) be a Mori dream space, \( Y \) a normal projective variety, and \( f : X \dashrightarrow Y \) a dominant rational map with connected fibers.\(^4\) If there exist open subsets \( U \subseteq X \) and \( V \subseteq Y \) such that \( \text{codim}(Y \setminus V) \geq 2 \) and \( f_U : U \to V \) is a regular contraction, then \( f \) is a rational contraction. When \( f \) is birational, also the converse holds.

Indeed consider a resolution of \( f \):

\[
\begin{array}{ccc}
\hat{X} & \xrightarrow{g} & X \\
\downarrow \tilde{f} & & \downarrow f \\
Y & & \hat{Y}
\end{array}
\]

where \( \hat{X} \) is normal and projective, and \( g \) is birational and an isomorphism over \( \text{dom}(f) \). Then \( Y \setminus V \supseteq \tilde{f}(\text{Exc}(g)) \), so that \( \text{codim} \tilde{f}(\text{Exc}(g)) \geq 2 \). Hence if \( D \) is an effective, \( g \)-exceptional Cartier divisor in \( \hat{X} \), then \( (\tilde{f})_*\mathcal{O}_{\hat{X}}(D) = \mathcal{O}_Y \) (i.e. \( D \) is \( \tilde{f} \)-fixed, in the terminology of \([HK00]\)). Thus \( f \) is a rational contraction by \([HK00, \text{Def. 1.1 and Prop. 1.11}]\).

\(^3\)The dimension of a cone in \( \mathbb{R}^m \) is the dimension of its linear span.

\(^4\)Namely, a resolution of \( f \) has connected fibers; this does not depend on the resolution, see \([HK00, \text{Def. 1.0}]\).
If \( f : X \rightarrow Y \) is a rational contraction, there is a well-defined injective linear map \( f^* : N^1(Y) \rightarrow N^1(X) \), such that \( f^*(\text{Nef}(Y)) \subseteq \text{Mov}(X) \). The bijection between the contractions of \( X \) and the faces of \( \text{Nef}(X) \) generalizes to rational contractions in the following way. Define \[
\mathcal{M}_X := \{ f^*(\text{Nef}(Y)) \mid f : X \rightarrow Y \text{ is a rational contraction of } X \}.
\]

Then we have the following.

**Proposition 2.5** ([HK00], Prop. 1.1(3)). The set \( \mathcal{M}_X \) is a fan in \( N^1(X) \). The union of the cones in \( \mathcal{M}_X \) is \( \text{Mov}(X) \), and every face of \( \text{Mov}(X) \) is a union of cones in \( \mathcal{M}_X \). Moreover, the association

\[
(f : X \rightarrow Y) \mapsto f^*(\text{Nef}(Y))
\]

gives a bijection between the set of rational contractions of \( X \) and \( \mathcal{M}_X \).

Here are some properties of this bijection:

- if \( \sigma \in \mathcal{M}_X \) and \( f : X \rightarrow Y \) is the corresponding contraction, then \( \dim \sigma = \rho_Y \);
- \( f \) is regular if and only if \( \sigma \subseteq \text{Nef}(X) \); in particular \( \text{Nef}(X) \in \mathcal{M}_X \) corresponds to the identity of \( X \);
- \( f \) is of fiber type (i.e. \( \dim Y < \dim X \)) if and only if \( \sigma \) is contained in the boundary of \( \text{Eff}(X) \);
- \( f \) is a SQM if and only if \( \dim \sigma = \rho_X \);
- given two cones \( \sigma_1, \sigma_2 \in \mathcal{M}_X \) with corresponding rational contractions \( f_i : X \rightarrow Y_i \), then \( \sigma_1 \subseteq \sigma_2 \) if and only if there is a regular contraction \( g : Y_2 \rightarrow Y_1 \) such that the following diagram commutes:

\[
\begin{array}{c}
X \\
\downarrow f_1 \quad \downarrow f_2 \\
Y_1 & \overset{g}{\leftarrow} & Y_2
\end{array}
\]

In particular, given \( f_1 : X \rightarrow Y_1 \), the factorizations \( X \rightarrow \tilde{X} \rightarrow Y_1 \) of \( f_1 \) with \( X \rightarrow \tilde{X} \) a SQM correspond to \( \rho_X \)-dimensional cones in \( \mathcal{M}_X \) containing \( \sigma_1 \).

**Example 2.6** (Elementary rational contractions). Let \( f : X \rightarrow Y \) be a rational contraction. We say that \( f \) is elementary if \( \rho_X - \rho_Y = 1 \), equivalently if \( \dim \sigma = \rho_X - 1 \), where \( \sigma \in \mathcal{M}_X \) is the cone corresponding to \( f \). As in the regular case, we have three possibilities:

- (i) if \( \sigma \) is in the interior of \( \text{Mov}(X) \), then \( f \) is an elementary small contraction of a SQM of \( X \);


\footnote{We recall that a fan \( \Sigma \) in \( \mathbb{R}^m \) is a finite set of convex polyhedral cones in \( \mathbb{R}^m \), with the following properties: 1) for every \( \sigma \in \Sigma \), every face of \( \sigma \) is in \( \Sigma \); 2) for every \( \sigma, \tau \in \Sigma \), \( \sigma \cap \tau \) is a face of both \( \sigma \) and \( \tau \).}
(ii) if $\sigma$ lies on the boundary of $\text{Mov}(X)$ but in the interior of $\text{Eff}(X)$, then $f$ is an elementary divisorial contraction of a SQM of $X$;

(iii) if $\sigma$ lies on the boundary of $\text{Eff}(X)$, then $f$ is an elementary fiber type contraction of a SQM of $X$.

As in the regular case, we will say that $f$ is small in case (i), divisorial in case (ii).

**Example 2.7 (Flips).** Let $f: X \to Y$ be a small elementary contraction, and consider $\sigma := f^*(\text{Nef}(Y)) \in M_X$. The cone $\sigma$ is a facet of $\text{Nef}(X)$ and lies in the interior of $\text{Mov}(X)$, therefore there exists a unique $\rho_X$-dimensional cone $\tau \in M_X$ such that $\sigma = \text{Nef}(X) \cap \tau$.

Let $g: X \dasharrow \tilde{X}$ be the SQM corresponding to $\tau$; then $g$ is the flip of $f$.

**Remark 2.8.** Let $X$ be a Mori dream space and $f: X \to Y$ a rational contraction. Suppose that $Y$ is $\mathbb{Q}$-factorial. Then $Y$ is a Mori dream space, and for every rational contraction $g: Y \to Z$, the composition $g \circ f: X \to Z$ is again a rational contraction.

**Proof.** The statement is clear from the definitions if $f$ is a SQM. In general, we factor $f$ as $X \dasharrow \tilde{X} \xrightarrow{\tilde{f}} Y$, where $\tilde{X}$ is a SQM of $X$, and $\tilde{f}$ is a regular contraction. Since $\tilde{X}$ is a Mori dream space, and $g \circ f: X \to Z$ is a rational contraction if and only if $g \circ \widetilde{f}: \tilde{X} \to Z$ is, we can assume that $f$ is regular.

Now $f^*: \text{Pic}(Y) \to \text{Pic}(X)$ is injective, hence $Y$ has finitely generated Picard group. Then we can define the Cox rings $\text{Cox}(Y)$ and $\text{Cox}(X)$ of $Y$ and $X$, see [HK00, Def. 2.6]. By [HK00, Prop. 2.9] $Y$ is a Mori dream space if and only if $\text{Cox}(Y)$ is a finitely generated $\mathbb{C}$-algebra, and for the same reason $\text{Cox}(X)$ is a finitely generated $\mathbb{C}$-algebra.

We have $f^*(\text{Eff}(Y)) = \text{Eff}(X) \cap f^*(\mathcal{N}^1(Y))$, so that $f^*(\text{Eff}(Y))$ is closed and is a convex polyhedral cone. Moreover, via $f^*$, we can see $\text{Cox}(Y)$ as a subalgebra of $\text{Cox}(X)$, graded by the subsemigroup of integral points of $f^*(\text{Eff}(Y))$. This kind of subalgebra is called a Veronese subalgebra; since $\text{Cox}(X)$ is finitely generated, the same holds for $\text{Cox}(Y)$, see [ADHL10, Prop. 1.2.2]. Thus $Y$ is a Mori dream space.

Let us show that $g \circ f$ is a rational contraction. We factor $g$ as $Y \xrightarrow{h} \tilde{Y} \xrightarrow{\tilde{g}} Z$, where $h$ is a SQM and $\tilde{g}$ a regular contraction, and first consider $h \circ f: X \dasharrow \tilde{Y}$. We have $\text{codim}(\tilde{Y} \setminus \text{dom}(h^{-1})) \geq 2$, and $(h \circ f)\cdot h^{-1}(\text{dom}(h)): f^{-1}(\text{dom}(h)) \to \text{dom}(h^{-1})$ is a regular contraction, so $h \circ f$ is a rational contraction by Rem. 2.4.

It is clear from Def. 2.3 that the composition of a rational contraction with a regular contraction is again a rational contraction; since $g \circ f = \tilde{g} \circ (h \circ f)$, we are done. \hfill $\blacksquare$

**Remark 2.9.** If $X$ is a Mori dream space and $f: X \to Y$ is a contraction, then $(\ker f_*)^\perp = f^*(\mathcal{N}^1(Y))$. In other words, for any divisor $D$ in $X$, one has $D^\perp \supseteq \ker f_*$ if and only if $[D] \in f^*(\mathcal{N}^1(Y))$. Indeed it is easy to see that $(\ker f_*)^\perp \supseteq f^*(\mathcal{N}^1(Y))$, and since both subspaces have dimension $\rho_Y$, they must coincide.

**2.10. Mori programs.** Let $X$ be a Mori dream space, and $D$ a divisor in $X$. A Mori program for $D$ is a sequence of varieties and birational maps

\[(2.11) X = X_0 \xrightarrow{f_0} X_1 \dasharrow \cdots \dasharrow X_{k-1} \xrightarrow{f_{k-1}} X_k \]

such that:
(2.12) every $X_i$ is a normal and $\mathbb{Q}$-factorial projective variety;

(2.13) for every $i = 0, \ldots, k-1$ there is a birational, $D_i$-negative extremal ray $\sigma_i$ of $\text{NE}(X_i)$, such that $f_i$ is either the contraction of $\sigma_i$ (if divisorial), or its flip (if small). The divisor $D_{i+1}$ is defined as $(f_i)_*(D_i)$ if $f_i$ is a divisorial contraction, as the transform of $D_i$ if $f_i$ is a flip;

(2.14) either $D_k$ is semiample, or there exists a $D_k$-negative elementary contraction of fiber type $f_k$:

An important property of Mori dream spaces is that one can run a Mori program for any divisor $D$, see [HK00, Prop. 1.1(1)]; moreover, the choice of the extremal rays $\sigma_i$ is arbitrary among those having negative intersection with $D_i$.

**Remark 2.15.** A Mori program for $D$ ends with a fiber type contraction if and only if $[D] \not\in \text{Eff}(X)$.

**2.16. Cones of curves.** In $\mathcal{N}_1(X)$ we have dual cones:

$$\text{ME}(X) := \text{Eff}(X)^{\vee} \subseteq \text{Nef}(X)^{\vee} = \text{NE}(X).$$

Recall that by [BDPP04], for any projective variety $X$, the dual $\text{ME}(X)$ of the cone $\text{Eff}(X)$ is the closure of the convex cone generated by classes of irreducible curves belonging to a covering family of curves.

When $X$ is a Mori dream space, the cone $\text{ME}(X)$ is polyhedral, because $\text{Eff}(X)$ is. Using the same techniques as in [Ara10] (in a much simpler situation), one can see that every one-dimensional face of $\text{ME}(X)$ contains the class of a curve moving in a covering family. The proof of the following Lemma is adapted from [Ara10, Lemma 5.1 and Th. 5.2]; we write it explicitly for the reader’s convenience.

**Lemma 2.17.** Let $X$ be a Mori dream space and $\sigma$ a one-dimensional face of $\text{ME}(X)$. Then there exists a Mori program on $X$ ending with a fiber type contraction:

$$X \dashrightarrow X' \xrightarrow{f} Y$$

such that if $C \subset X$ is the transform of a general curve in a general fiber of $f$, then $[C] \in \sigma$.

**Proof.** Let $B$ be an effective divisor such that $B^\perp \cap \text{ME}(X) = \sigma$, let $H$ be an ample divisor, and set $D := B - H$. Since $[B]$ lies on the boundary of $\text{Eff}(X)$ and $[H]$ in its interior, we have $[D] \not\in \text{Eff}(X)$. By Rem. 2.15, every Mori program for $D$ ends with a fiber type contraction.

We run a Mori program for $D$ with scaling of $H$, see [BCHM10] § 3.10 and [Ara10] § 3.8. Concretely, this means a sequence as (2.11), where at each step the extremal ray $\sigma_i$ is chosen in a prescribed way. At the first step, we choose a facet of $\text{Nef}(X)$ met by moving from $[D]$ to $[H]$ along the segment $s$ joining them in $\mathcal{N}_1(X)$. This facet corresponds to a $D$-negative extremal ray of $\text{NE}(X)$; this will be $\sigma_0$. This process can be repeated at each step, using $H_i$ in $X_i$, where $H_i := (f_{i-1} \circ \cdots \circ f_0)_*(H)$. 

The segment $s$ meets the boundary of $\text{Eff}(X)$ at the point $[B]/2 = ([D] + [H])/2$. The key remark, made in [Ara10, Lemma 5.1], is that for every $i \in \{1, \ldots, k\}$ the segment from $[D_i]$ to $[H_i]$ in $\mathcal{N}^1(X_i)$ meets the boundary of $\text{Eff}(X_i)$ at the point $([D_i] + [H_i])/2$. Indeed, suppose that this is true for $X_{i-1}$, and consider $f_{i-1} : X_{i-1} \to X_i$. The statement is clear if $f_{i-1}$ is a flip, thus let’s assume that it is a divisorial contraction.

We know that $(1 - t)[D_{i-1}] + t[H_{i-1}] \in \text{Eff}(X_{i-1})$ for $t \in [1/2, 1]$, and $(1 - t)[D_{i-1}] + t[H_{i-1}] \notin \text{Eff}(X_{i-1})$ for $t \in [0, 1/2)$. Moreover $(1 - t)D_i + tH_i = (f_{i-1})_*((1 - t)D_{i-1} + tH_{i-1})$, so that again $(1 - t)[D_i] + t[H_i] \in \text{Eff}(X_i)$ if $t \in [1/2, 1]$.

We have $D_{i-1} \cdot \text{NE}(f_{i-1}) < 0$; moreover, by the choice of $\text{NE}(f_{i-1})$, there exists $t_0 \in [1/2, 1]$ such that $((1 - t_0)D_{i-1} + t_0H_{i-1}) \cdot \text{NE}(f_{i-1}) = 0$. Hence

$$(1 - t)D_i - tH_i \cdot \text{NE}(f_{i-1}) < 0 \quad \text{for every } t < \frac{1}{2}.$$  

Therefore if $(1 - t)[D_i] + t[H_i] \notin \text{Eff}(X_i)$ for some $t \in [0, 1/2)$, we can proceed as in the proof of Rem. 2.15 and get a contradiction.

In the end we get an elementary contraction of fiber type $f_k : X_k \to Y$ such that $((1 - t_k)D_k + t_kH_k) \cdot \text{NE}(f_k) = 0$ for some $t_k \in (0, 1]$. Then $(1 - t_k)D_k + t_kH_k$ lies on the boundary of $\text{Eff}(X_k)$, and by what we proved above, $t_k = 1/2$. This means that if $C \subset X$ is the transform of a general curve in a general fiber of $f_k$, then $B \cdot C = 0$, therefore $[C] \in \sigma$.

2.18. Non-movable prime divisors. We conclude this section by showing that non-movable prime divisors in $X$ are exactly the divisors which become exceptional on some SQM of $X$. Notice that if $D$ is a divisor in $X$, then $D$ is movable (i.e. the stable locus of $D$ has codimension at least 2) if and only if $[D] \in \text{Mov}(X)$.

Remark 2.19. Let $X$ be a Mori dream space, and $D \subset X$ a prime divisor. The following conditions are equivalent:

(i) $D$ is not movable;

(ii) there exists a SQM $X \dashrightarrow \tilde{X}$ such that the transform $\tilde{D} \subset \tilde{X}$ of $D$ is the exceptional divisor of an elementary divisorial contraction $\tilde{X} \to Y$.

Moreover, the association $D \mapsto \mathbb{R}_{\geq 0}[D]$ gives a bijection between:

- the set of non-movable prime divisors in $X$, and
- the set of one-dimensional faces of $\text{Eff}(X)$ not contained in $\text{Mov}(X)$.

Let us point out that after the proof, $X \dashrightarrow \tilde{X} \to Y$ (notation as in (ii)) is a Mori program for $D$ (ending with zero), so that $X \dashrightarrow \tilde{X}$ factors as a sequence of $D$-negative flips. In fact, every Mori program for $D$ takes this form.

Proof. Suppose that $D$ is not movable, and consider a Mori program for $D$. Since $D$ is effective, by Rem. 2.15 the program must end with $D$ becoming nef. On the other hand,
there is no SQM of $X$ where $D$ is nef, because $D$ is not movable. Therefore in the Mori program some divisorial contraction must occur. Let $f: \tilde{X} \to Y$ be the first divisorial contraction: then the previous steps are flips, hence $X \to \tilde{X}$ is a SQM (possibly $\tilde{X} = X$). Moreover since $\tilde{D} \cdot \text{NE}(f) < 0$ and $\tilde{D}$ is a prime divisor, we have $\tilde{D} = \text{Exc}(f)$. Since $f_*(\tilde{D}) = 0$, the divisorial contraction $f: \tilde{X} \to Y$ is the last step of the Mori program.

Conversely, if $(ii)$ holds, then $\tilde{D}$ is not movable, hence neither is $D$. Finally, suppose that $(i)$ and $(ii)$ hold, and let $D_1, D_2 \subset \tilde{X}$ be prime divisors such that $a_1D_1 + a_2D_2 \equiv \tilde{D}$, $a_i \in \mathbb{R}_{>0}$. Since $\tilde{D} \cdot \text{NE}(f) < 0$, there exists $i \in \{1, 2\}$ such that $D_i \cdot \text{NE}(f) < 0$, hence $D_i = \tilde{D}$. This implies that $D_1 = D_2 = \tilde{D}$, therefore $\mathbb{R}_{\geq 0}[\tilde{D}]$ is a one-dimensional face of $\text{Eff}(\tilde{X})$. Similarly, one shows that $\tilde{D}$ is the unique prime divisor whose class belongs to this face. ■

We will also need the following.

Remark 2.20. Let $X$ be a Mori dream space, $g: X \to Z$ a contraction, and $D \subset X$ a non-movable prime divisor such that $g(D) = \{\text{pt}\}$. Then there exists a commutative diagram:

$$
\begin{array}{ccc}
X & \rightarrow & \tilde{X} \\
\downarrow^g & & \downarrow^f \\
Z & \leftarrow & Y
\end{array}
$$

where $X \to \tilde{X}$ is a SQM which factors as a sequence of $D$-negative flips, $f$ is an elementary divisorial contraction with exceptional divisor (the transform of) $D$, and $h$ is a contraction.

Proof. By Rem. 2.19 there are a birational map $X \to \tilde{X}$ which factors as a sequence of $D$-

negative flips, and an elementary divisorial contraction $f: \tilde{X} \to Y$ with exceptional divisor the transform of $D$. If $\sigma$ is a $D$-negative extremal ray of $\text{NE}(X)$, then $\text{Locus}(\sigma) \subseteq D$, so that $g(\text{Locus}(\sigma)) = \{\text{pt}\}$ and $\sigma \subseteq \text{NE}(g)$. Iterating this reasoning, we see that the rational map $h: Y \to Z$ is indeed regular. ■

2.2 Quasi-elementary rational contractions

In this section we introduce a special class of rational contractions of fiber type of Mori
dream spaces, called quasi-elementary contractions, which share many good properties of
elementary rational contractions of fiber type. The notion of quasi-elementary contraction
was first introduced in [Cas08], but in a different context: there the objects were regular,
$K$-negative contractions of a smooth projective variety. Here, since we are considering Mori
dream spaces, we do not need to assume $K$-negativity.

Let $X$ be a Mori dream space and $f: X \to Y$ a contraction. Recall that

$$\text{NE}(f) := \ker f_* \cap \text{NE}(X)$$

is a face of $\text{NE}(X)$, corresponding to the face $f^*(\text{Nef}(Y))$ of $\text{Nef}(X)$. In the same way we can associate to $f$ a face of $\text{ME}(X)$, setting

$$\text{ME}(f) := \ker f_* \cap \text{ME}(X) = \text{NE}(f) \cap \text{ME}(X).$$
Proposition 2.22. Let \( F \) follow a contraction, and let \( \sigma \in \mathcal{M} \) be a one-dimensional face of \( \text{Eff}(X) \) containing \( f^*(\text{Eff}(Y)) \).

**Proof.** We clearly have \( i_*(\text{ME}(F)) \subset \ker f_* \). Let \( D_1, \ldots, D_r \subset X \) be prime divisors whose classes generate \( \text{Eff}(X) \). Then for every \( j \in \{1, \ldots, r\} \) \( D_j \) does not contain \( F \), and if \( \gamma \in \text{ME}(F) \) we have

\[
i_*(\gamma) \cdot D_j = \gamma \cdot (D_j)|_F \geq 0,
\]

so that \( i_*(\gamma) \in \text{Eff}(X)^Y = \text{ME}(X) \). This shows that \( i_*(\text{ME}(F)) \subseteq \ker f_* \cap \text{ME}(X) = \text{ME}(f) \).

Conversely, let \( \sigma \) be a one-dimensional face of \( \text{ME}(f) \). By Rem. 2.21, there is a covering family of curves \( \{C_t\} \) in \( X \) whose numerical class belongs to \( \sigma \). On the other hand, since \( \sigma \subset \ker f_* \), all these curves are contracted to a point by \( f \). This means that a subfamily \( \{C_{t'}\} \) gives a covering family of curves in \( F \), hence \([C_{t'}] \in \text{ME}(F)\) and \( \sigma \subseteq i_*(\text{ME}(F)) \).

Therefore \( \text{ME}(f) = i_*(\text{ME}(F)) \).

Now since \( \text{ME}(F) \) generates \( \mathcal{N}_1(F) \), we get that \( \mathcal{N}_1(F, X) = i_*(\mathcal{N}_1(F)) \) is the linear span of \( \text{ME}(f) \) in \( \mathcal{N}_1(X) \), and \( \dim \text{ME}(f) = \dim \mathcal{N}_1(F, X) \).

For the last statement, let \( \tau \) be a face of \( \text{ME}(X) \) and \( \tau^* \) the corresponding face of \( \text{Eff}(X) \). By the definition of \( \tau^* \), if \( H \subseteq \mathcal{N}_1(X) \) is a linear subspace, then \( \tau \subset H \) if and only if \( \tau^* \supseteq \text{Eff}(X) \cap H^\perp \). Now take \( H = \ker f_* \). Since \( H^\perp = f^*(\mathcal{N}^!(Y)) \) (see Rem. 2.9), we get:

\[
\tau \subseteq \text{ME}(f) \iff \tau^* \supseteq \text{Eff}(X) \cap f^*(\mathcal{N}^!(Y)) = f^*(\text{Eff}(Y)).
\]

\[\blacksquare\]

**Proposition 2.22.** Let \( X \) be a Mori dream space, \( f : X \dashrightarrow Y \) a rational contraction, and \( \sigma \in \mathcal{M}_X \) the corresponding cone. Let \( X \dashrightarrow \tilde{X} \overset{f}{\rightarrow} Y \) be a factorization of \( f \) as a SQM followed by a contraction, and let \( F \subset \tilde{X} \) be a general fiber of \( \tilde{f} \).

The following properties are equivalent:

1. \( \mathcal{N}_1(F, \tilde{X}) = \ker \tilde{f}_* \);
2. \( \dim \mathcal{N}_1(F, \tilde{X}) = \rho_X - \rho_Y \);
3. \( \dim \text{ME}(\tilde{f}) = \rho_X - \rho_Y \);
4. \( \sigma \) is contained in a face of \( \text{Eff}(X) \) of the same dimension as \( \sigma \) (that is, \( \rho_Y \));
5. \( f^*(\text{Eff}(Y)) \) is a face of \( \text{Eff}(X) \).

**Definition 2.23.** We say that \( f \) is quasi-elementary if the equivalent conditions above are satisfied and \( f \) is non-trivial (i.e. \( f \) is not an isomorphism nor constant). In particular, \( f \) must be of fiber type.
Notice that $f$ is quasi-elementary if and only if $\tilde{f}$ is (notation as in Prop. 2.22).

Proof of Prop. 2.22. Up to replacing $X$ by $\tilde{X}$, we can assume that $f: X \to Y$ is regular.

(i) $\Rightarrow$ (iii) This follows from Lemma 2.21.

(iii) $\Rightarrow$ (v) Since $\dim \text{ME}(f) = \dim \ker f_*$, $\ker f_*$ is the linear span of $\text{ME}(f)$. Therefore

$$\text{ME}(f)^* = \text{Eff}(X) \cap (\ker f_*)^\perp = \text{Eff}(X) \cap f^*(\mathcal{N}^1(Y)) = f^*(\text{Eff}(Y)).$$

(v) $\Rightarrow$ (iv) This is because $\sigma = f^*(\text{Nef}(Y)) \subseteq f^*(\text{Eff}(Y))$.

(iv) $\Rightarrow$ (ii) Let $\eta$ be the face of $\text{Eff}(X)$ containing $\sigma$ and such that $\dim \eta = \dim \sigma = \rho_Y$. Then the linear span of $\eta$ is the same as that of $\sigma$, namely $f^*(\mathcal{N}^1(Y))$. This gives

$$\eta^* = \text{ME}(X) \cap (f^*(\mathcal{N}^1(Y)))^\perp = \text{ME}(X) \cap \ker f_* = \text{ME}(f),$$

and by Lemma 2.21 we get $\dim \mathcal{N}_1(F,X) = \dim \eta^* = \rho_X - \rho_Y$.

(ii) $\Rightarrow$ (i) This follows from $\mathcal{N}_1(F,X) \subseteq \ker f_*$ and $\dim \ker f_* = \rho_X - \rho_Y$. ■

Remark 2.24. Let $X$ be a Mori dream space and $f: X \dasharrow Y$ a rational contraction of fiber type with $\dim Y > 0$.

- If $f$ is elementary, then it is also quasi-elementary.
- If $\dim Y = \dim X - 1$, then $f$ is elementary if and only if it is quasi-elementary.
- If $f$ is quasi-elementary and regular, and $F \subset X$ is a general fiber, then $\rho_X - \rho_Y \leq \rho_F$.

If $X$ is a Mori dream space, then $X$ has a (non-trivial) rational contraction of fiber type if and only if the boundaries of $\text{Mov}(X)$ and $\text{Eff}(X)$ meet outside zero. For the quasi-elementary case we have the following criterion.

Corollary 2.25. Let $X$ be a Mori dream space and $r \in \{1, \ldots, \rho_X - 1\}$. Then $X$ has a quasi-elementary rational contraction $f: X \dasharrow Y$ with $\rho_Y = r$ if and only if there exists an $r$-dimensional face of $\text{Mov}(X)$ contained in an $r$-dimensional face of $\text{Eff}(X)$.

Proof. Let $f: X \dasharrow Y$ be a quasi-elementary rational contraction with $\rho_Y = r$, and let $\sigma \in \mathcal{M}_X$ be the corresponding cone. Then $\dim \sigma = r$, and by Prop. 2.22(iv), $\sigma$ is contained in a face $\tau$ of $\text{Eff}(X)$ with $\dim \tau = r$. There exists a face $\eta$ of $\text{Mov}(X)$ with $\sigma \subseteq \eta \subseteq \tau$, and we get $\dim \eta = r$.

Conversely, let $\eta$ be a face of $\text{Mov}(X)$ contained in a face $\tau$ of $\text{Eff}(X)$ with $\dim \eta = \dim \tau = r$. Since $\eta$ is a union of cones in $\mathcal{M}_X$, we can choose $\sigma \in \mathcal{M}_X$ such that $\sigma \subseteq \eta$ and $\dim \sigma = r$. Then the rational contraction corresponding to $\sigma$ is quasi-elementary again by Prop. 2.22(iv), and the target has Picard number $r$. ■

Remark 2.26. Let $X$ be a Mori dream space and $f: X \dasharrow Y$ a quasi-elementary rational contraction. Then $Y$ is a Mori dream space, and if $g: Y \dasharrow Z$ is a quasi-elementary rational contraction, then $g \circ f: X \dasharrow Z$ is again quasi-elementary.
Proof. We first show that $Y$ is $\mathbb{Q}$-factorial. Up to replacing $X$ with a SQM, we can assume that $f$ is regular. Let $D \subset Y$ be a prime divisor in $Y$, and let $D' \subset X$ be a prime divisor such that $f(D') \subseteq D$.

If $F \subset X$ is a general fiber of $f$, then $F \cap D' = \emptyset$, so that for every curve $C \subseteq F$ we have $D' \cdot C = 0$. This gives $(D')^{-1} \supseteq N_1(F, X)$; on the other hand $N_1(F, X) = \ker f_*$ because $f$ is quasi-elementary, so that $[D'] \in (\ker f_*)^{-1} = f^*(N^1(Y))$ (see Rem. 2.9). Hence there exist a Cartier divisor $B$ in $Y$ and an integer $m \in \mathbb{N}$ such that $mD' = f^*(B)$. This shows that $B$ is effective and has support contained in $f(D') \subseteq D$, therefore $B = rD$ for some $r \in \mathbb{N}$. Thus $D$ is $\mathbb{Q}$-Cartier, and $Y$ is $\mathbb{Q}$-factorial.

Now applying Rem. 2.8 we see that $Y$ is a Mori dream space and $g \circ f : X \to Z$ is a rational contraction. Since both $f$ and $g$ are quasi-elementary, Prop. 2.22 (v) says that $f^*(\operatorname{Eff}(Y))$ is a face of $\operatorname{Eff}(X)$, and $g^*(\operatorname{Eff}(Z))$ is a face of $\operatorname{Eff}(Y)$. Then $(g \circ f)^*(\operatorname{Eff}(Z))$ is a face of $\operatorname{Eff}(X)$, thus $g \circ f$ is quasi-elementary again by Prop. 2.22 (v). \end{proof}

3 Non-movable prime divisors in a Fano 4-fold

3.1 Fano 4-folds as Mori dream spaces

Let us recollect some well-known results which will be used in the sequel. We recall that by a 4-fold we always mean a smooth 4-dimensional algebraic variety.

Theorem 3.1 (see [AW97], Th. 4.1 and references therein). Let $X$ be a quasi-projective 4-fold and $f : X \to Y$ a local contraction such that $-K_X$ is $f$-ample. Assume that every fiber of $f$ has dimension at most 1. Then $Y$ is smooth and $f$ is either the blow-up of a smooth surface in $Y$, or a conic bundle.

Theorem 3.2. Let $X$ be a projective variety with canonical singularities and $K_X$ Cartier, $f : X \to Y$ a $K$-negative contraction with $\dim X - \dim Y \leq 1$, and $F \subset X$ an isolated 2-dimensional fiber of $f$.

Let $T$ be a 2-dimensional irreducible component of $F_{\text{red}}$. Then the possibilities for $(T, -K_{X|T})$ are the following:

(i) $(\mathbb{P}^2, O_{\mathbb{P}^2}(e))$ with $e = 1, 2$;

(ii) $(S_r, O_{S_r}(1))$ with $r \geq 2$;

(iii) $(\mathbb{F}_r, C_0 + mB)$ with $r \geq 0$, $m \geq r + 1$.

Here $S_r$ is the cone over a rational normal curve of degree $r$, $B \subset \mathbb{F}_r$ is a fiber of the $\mathbb{P}^1$-bundle, and $C_0 \subset \mathbb{F}_r$ is a section of the $\mathbb{P}^1$-bundle with $C_0^2 = -r$.

If moreover $X$ is smooth, then every irreducible component of $F$ has dimension 2.

Proof. If $f$ is birational, this is [AW97] Th. 1.19 and Prop. 4.3.1. When $f$ is of fiber type, by [Mel99] Th. 2.6 there exists a non-empty open neighbourhood $Y_0$ of $f(F)$ such that $-K_X$ is $f$-spanned on $f^{-1}(Y_0)$. Then [AW97] Th. 1.19 and Prop. 4.3.1 still apply. \end{proof}
Let $X$ be a projective 4-fold. An **exceptional plane** in $X$ is a closed subset $L \subset X$ such that $L \cong \mathbb{P}^2$ and $N_{L/X} \cong \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}$. Notice that if $C_L \subset L$ is a line, we have $-K_X \cdot C_L = 1$. An **exceptional line** in $X$ is a curve $l \cong \mathbb{P}^1$ with $N_{l/X} \cong \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 3}$; notice that $K_X \cdot l = 1$.

**Theorem 3.3** ([Kaw89]). Let $X$ be a projective 4-fold and $f : X \dashrightarrow \tilde{X}$ a $K$-negative flip. Then $\tilde{X}$ is smooth, $X \setminus \text{dom}(f)$ is the disjoint union of $r \geq 1$ exceptional planes, and $\tilde{X} \setminus \text{dom}(f^{-1})$ is the disjoint union of $r$ exceptional lines.

Moreover $f$ factors as $h \circ g^{-1}$, where $g : \tilde{X} \rightarrow X$ is the blow-up of $X \setminus \text{dom}(f)$, and $h : \tilde{X} \rightarrow X$ is the blow-up of $\tilde{X} \setminus \text{dom}(f^{-1})$.

**Remark 3.4.** Let $f : X \dashrightarrow \tilde{X}$ be as in Th. 3.3. Let $C \subset X \setminus \text{dom}(f)$ a line in an exceptional plane, and $l \subset \tilde{X} \setminus \text{dom}(f^{-1})$ an exceptional line. Let $D$ be a divisor in $X$ and $\tilde{D}$ its transform in $\tilde{X}$. Then $D \cdot C = -D : l$. This follows easily from the factorization of $f$ as $h \circ g^{-1}$, by comparing $g^*(D)$ and $h^*(\tilde{D})$ in $\tilde{X}$.

In this paper, our interest in Mori dream spaces is motivated by the following result.

**Theorem 3.5** ([BCHM10], Cor. 1.3.2). Let $X$ be a Fano manifold. Then $X$ is a Mori dream space.

We are now going to explain some elementary properties of SQMs and of rational contractions of Fano 4-folds.

**Remark 3.6.** Let $X$ be a Fano 4-fold and $f : X \dashrightarrow \tilde{X}$ a SQM. Then $\tilde{X}$ is smooth, $X \setminus \text{dom}(f)$ is the disjoint union of $r$ exceptional planes, and $\tilde{X} \setminus \text{dom}(f^{-1})$ is the disjoint union of $r$ exceptional lines.

Moreover if $C \subset \tilde{X}$ is an irreducible curve such that $C \cap \text{dom}(f^{-1}) \neq \emptyset$, and $C_X \subset X$ is its transform, we have

$$-K_{\tilde{X}} \cdot C \geq -K_X \cdot C_X + s \geq 1 + s \geq 1,$$

where $s$ is the number of points of $C$ which belong to an exceptional line. In particular:

1. if $-K_{\tilde{X}} \cdot C = 1$, then $C$ does not intersect any exceptional line; in general we have:

$$s \leq -K_{\tilde{X}} \cdot C - 1;$$
2. for every irreducible curve $C \subset \tilde{X}$, either $-K_{\tilde{X}} \cdot C > 0$ (if $C \cap \text{dom}(f^{-1}) \neq \emptyset$), or $C$ is an exceptional line (if $C \cap \text{dom}(f^{-1}) = \emptyset$);
3. if $L \subset \tilde{X}$ is an exceptional plane and $l \subset \tilde{X}$ is an exceptional line, then $L \cap l = \emptyset$.

**Proof.** The statement is trivial if $f$ is an isomorphism. Otherwise, let $\tilde{D}$ be an ample divisor in $\tilde{X}$, and $D := f^*(\tilde{D})$. Then $D$ is a movable divisor in $X$, and any Mori program for $D$ yields a factorization of $f$ as a sequence of flips. Applying [Cas11] Prop. 2.4], we can factor $f$ as a sequence of $m \geq 1$ $K$-negative flips. In this way we get a factorization:

$$X \dashrightarrow X' \dashrightarrow \cdots \dashrightarrow \tilde{X},$$

15
where $f'$ is the composition of the first $m-1$ flips, and $f_m$ is the last one. By induction, we can assume that the statement holds for $f': X \to X'$.

Since $X'$ is smooth and $f_m$ is a $K$-negative flip, we can apply Th. 3.3 in particular, $\tilde{X}$ is smooth. Moreover $X' \setminus \text{dom}(f_m)$ is the disjoint union of $t$ exceptional planes, and $\tilde{X} \setminus \text{dom}(f_m^{-1})$ is the disjoint union of $t$ exceptional lines. By the induction hypothesis, an exceptional plane and an exceptional line in $X'$ cannot meet, therefore the indeterminacy locus of $f_m$ is disjoint from the indeterminacy locus of $(f')^{-1}$.

We have a factorization

$$
\begin{array}{c}
\tilde{X} \\
\downarrow h \\
X' \xrightarrow{f_m} \tilde{X}
\end{array}
$$

where $\tilde{X}$ is smooth, $g$ is the blow-up of $X' \setminus \text{dom}(f_m)$, and $h$ is the blow-up of $\tilde{X} \setminus \text{dom}(f_m^{-1})$. If $E_1, \ldots, E_t \subset \tilde{X}$ are the exceptional divisors, we have

$$
h^*(-K_{\tilde{X}}) = g^*(-K_{X'}) + \sum_{i=1}^{t} E_i.
$$

Consider now an irreducible curve $C \subset \tilde{X}$ such that $C \cap \text{dom}(f_m^{-1}) \neq \emptyset$, and let $C_X \subset X$, $C' \subset X'$, and $\tilde{C} \subset \tilde{X}$ be its transforms. Suppose that $C'$ has $s'$ points belonging to an exceptional line. Then $-K_{X'} \cdot C' \geq -K_X \cdot C_X + s'$ by induction, and $C$ meets the indeterminacy locus of $f_m^{-1}$ in $s - s'$ points, so we get

$$
-K_{\tilde{X}} \cdot C = -K_{X'} \cdot C' + \sum_{i=1}^{t} E_i \cdot \tilde{C} \geq -K_X \cdot C_X + s' + (s - s'),
$$

which gives the statement. ■

**Remark 3.7.** Let $X$ be a Fano 4-fold and $f: X \to Y$ a rational contraction. Then there exists a factorization of $f$ as

$$
\begin{array}{c}
X \\
\downarrow f \\
\tilde{X} \xrightarrow{\tilde{f}} \tilde{Y}
\end{array}
$$

where $X \to \tilde{X}$ is a SQM, $\tilde{X}$ is smooth, and $\tilde{f}$ is a $K$-negative contraction; in particular, $Y$ has rational singularities.

**Proof.** Consider a factorization $f = g_1 \circ h_1$ where $h_1: X \to X_1$ is a SQM and $g_1: X_1 \to Y$ a contraction. If $g_1$ is not $K$-negative, there exists an extremal ray $\sigma$ of $\text{NE}(X_1)$ such that $K_{X_1} \cdot \sigma \geq 0$ and $\sigma \subseteq \text{NE}(g_1)$. By Rem. 3.3 (2), $\text{Locus}(\sigma)$ is the union of finitely many
exceptional lines; let $h_2$ be the composition of $h_1$ with the flip of $\sigma$, and $g_2 := f \circ (h_2)^{-1}$.

Then $g_2$ is a morphism and $f = g_2 \circ h_2$. Moreover the number of connected components of $X \setminus \text{dom}(h_2)$ is strictly smaller than the number of connected components of $X \setminus \text{dom}(h_1)$. Proceeding in this way, after finitely many steps we get a factorization $f = g_m \circ h_m$ where $g_m$ is $K$-negative. Finally, $Y$ has rational singularities by [Kol86, Cor. 7.4].

**Corollary 3.8.** Let $X$ be a Fano 4-fold and $f : X \dasharrow Y$ a quasi-elementary rational contraction. Then $Y$ is a Mori dream space and moreover:

- $Y$ is smooth if $\dim Y = 2$;
- $Y$ has at most isolated canonical and factorial singularities if $\dim Y = 3$.

**Proof.** The target $Y$ is a Mori dream space by Rem. 2.26. After Rem. 3.7 we can factor $f$ as $X \dasharrow \tilde{X} \dasharrow Y$, where $\tilde{X}$ is a smooth 4-fold, and $\tilde{f}$ is a $K$-negative quasi-elementary contraction. Then the statement follows from [Cast08, Lemma 3.10].

**Corollary 3.9.** Let $X$ be a Fano 4-fold and $f : X \dasharrow Y$ a quasi-elementary rational contraction. Assume that $f$ is not regular.

If $\dim Y = 1$, then $\rho_X \leq 10$. If $\dim Y = 2$, then $\rho_X \leq \rho_Y + 8$.

**Proof.** By Rem. 3.7 we can factor $f$ as $X \dasharrow \tilde{X} \dasharrow Y$ where $X \dasharrow \tilde{X}$ is a SQM, $\tilde{X}$ is smooth, and $\tilde{f}$ is a $K$-negative quasi-elementary contraction. The general fiber $F$ of $\tilde{f}$ is a smooth Fano variety, and $\rho_X = \rho_{\tilde{X}} \leq \rho_Y + \rho_F$ (see Rem. 2.24).

Since $f$ is not a morphism, $\tilde{X}$ contains some exceptional line $l$, which cannot intersect curves of anticanonical degree 1 by Rem. 3.6 (1).

We show that $F$ cannot be covered by curves of anticanonical degree 1. Indeed if it were, since $F$ is a general fiber of $\tilde{f}$, we could find a (proper and irreducible) family of curves in $\tilde{X}$, covering $\tilde{X}$, whose general member is an irreducible curve, of anticanonical degree 1, and contracted by $\tilde{f}$. As $-K_{\tilde{X}}$ is $\tilde{f}$-ample, we deduce that every curve of the family has anticanonical degree 1 and is contracted by $\tilde{f}$. On the other hand the exceptional line $l$ is not contracted by $\tilde{f}$, hence $l$ cannot be contained in a member of the family. Thus $l$ must intersect some curve of the family, and we get a contradiction.

If $\dim Y = 2$, then $F$ is a Del Pezzo surface, thus $\rho_F \leq 9$. Moreover if $\rho_F = 9$, then $F$ is covered by the pencil $| -K_F|$ which contains curves of anticanonical degree 1, a contradiction. Therefore $\rho_F \leq 8$ and $\rho_X \leq \rho_Y + 8$. 


If $Y$ is a curve, then $F$ is a Fano 3-fold, so that $\rho_F \leq 10$. Again, if $\rho_F = 10$, then $F \cong \mathbb{P}^1 \times S$ where $S$ is a Del Pezzo surface with $\rho_S = 9$ (see [IP99, p. 141]), and $F$ is covered by curves of anticanonical degree 1; therefore $\rho_F \leq 9$ and we get the statement. 

**Remark 3.10.** Let $X$ be a Fano 4-fold, $f: X \to Y$ a quasi-elementary rational contraction, and $X \dashrightarrow \tilde{X} \to Y$ a factorization of $f$ as in Rem. [3.7]. If $D \subset Y$ is a non-movable prime divisor, then $(\tilde{f})^*(D)$ is a non-movable prime divisor in $\tilde{X}$.

**Proof.** Let $D' \subset \tilde{X}$ be a prime divisor contained in the support of $(\tilde{f})^*(D)$. By [Cas08, Lemma 3.9] we have $\tilde{f}(D') = D$ and $D' = (\tilde{f})^*(D)$, so that $(\tilde{f})^*(D)$ is a prime divisor. Finally it is not difficult to check that $(\tilde{f})^*(D)$ is not movable. 

### 3.2 Picard number of divisors in Fano 4-folds

Let $X$ be a Fano manifold and $D \subset X$ a prime divisor. If $i: D \to X$ is the inclusion, let us consider $N_i(D, X) = i_*(N_1(D)) \subseteq N_1(X)$. We have $\text{codim} N_1(D, X) = \dim \ker (H^2(X, \mathbb{R}) \to H^2(D, \mathbb{R}))$, therefore the invariant $c_X$ defined in the Introduction can also be described as:

$$c_X = \max \{ \text{codim} N_1(D, X) \mid D \text{ is a prime divisor in } X \}.$$ 

We will need the following result.

**Theorem 3.11** ([Cas11]). Let $X$ be a Fano 4-fold with $c_X \geq 3$. Then one of the following holds:

(i) $X \cong S_1 \times S_2$ where $S_i$ are Del Pezzo surfaces with $\rho_{S_i} = c_X + 1 \geq \rho_{S_1}$;

(ii) $c_X = 3$, $\rho_X = 5$, and $X$ has a quasi-elementary contraction onto $\mathbb{P}^2$;

(iii) $c_X = 3$, $\rho_X = 6$, and $X$ has a quasi-elementary contraction onto $\mathbb{P}_1$ or $\mathbb{P}^1 \times \mathbb{P}^1$.

Moreover every elementary contraction of $X$ is either of type $(3, 2)^{sm}$, or a conic bundle.

**Proof.** If $X \cong S_1 \times S_2$ with $S_i$ Del Pezzo surfaces, we have $c_X = \max \{ \rho_{S_1} - 1, \rho_{S_2} - 1 \}$ (see [Cas11, Ex. 3.1]), so up to exchanging $S_1$ and $S_2$ we get $\rho_{S_1} = c_X + 1 \geq \rho_{S_2}$.

If $X$ is not a product of surfaces, then by [Cas11, Cor. 1.3 and Th. 3.3] we have $c_X = 3$, $\rho_X \leq 6$, and $X$ has a quasi-elementary contraction $f: X \to S$ where $S$ is a smooth Del Pezzo surface with $\rho_X - \rho_S = 4$. Thus $\rho_S \in \{ 1, 2 \}$, and if $\rho_S = 1$ we get (ii).

Suppose that $\rho_S = 2$, and let $g$ be an elementary contraction of $X$. If $\text{NE}(g) \not\subset \text{NE}(f)$, then $f$ is finite on every non-trivial fiber $F$ of $g$. Since $\dim N_1(F, X) = 1$, we cannot have $f(F) = S$, therefore $\dim F \leq 1$. Hence $g$ is either of type $(3, 2)^{sm}$, or a conic bundle, by Th. 3.1.

Suppose that $\text{NE}(g) \subset \text{NE}(f)$. After [Cas11, proof of Prop. 3.3.1, in particular §3.3.15], $f$ factors as $h_2 \circ h_1$ where $h_1: X \to Y$ and $h_2: Y \to S$ are conic bundles, $Y$ is smooth with $\dim Y = 3$ and $\rho_Y = 3$, and $\text{NE}(h_1)$ contains 4 extremal rays, all of type $(3, 2)^{sm}$. Therefore either $\text{NE}(g) \subset \text{NE}(h_1)$ and we are done, or $(h_1)_*(\text{NE}(g)) = \text{NE}(h_2)$. In this last case,
$F$ is a non-trivial fiber of $g$, then $h_1$ is finite on $F$ and $h_1(F)$ is contained in a fiber of $h_2$, therefore $\dim F = 1$ and we get the statement.

Therefore, if we are interested in studying Fano 4-folds which are not products and have large Picard number, we can assume that $c_X \leq 2$, so that for every prime divisor $D \subset X$ we have $\dim N_1(D, X) \geq \rho_X - 2$. Let us also state the following application.

**Corollary 3.12.** Let $X$ be a Fano 4-fold. If $X$ has a small elementary contraction then either $\rho_X = 5$ and $c_X = 3$, or $c_X \leq 2$.

It is natural to ask whether we can deduce similar properties for a SQM of $X$. The following two statements describe how $\dim N_1(D, X)$ varies under a flip or a SQM.

**Remark 3.13.** Let $X$ be a smooth 4-fold, $\sigma$ a $K$-negative small extremal ray of $\text{NE}(X)$, $X \rightarrow \tilde{X}$ the flip of $\sigma$, and $\tilde{\sigma}$ the corresponding small extremal ray of $\text{NE} (\tilde{X})$.

1. Let $Z \subset X$ be a closed subset disjoint from $\text{Locus}(\sigma)$, and $\tilde{Z} \subset \tilde{X}$ its transform. Then $\dim N_1(\tilde{Z}, \tilde{X}) = \dim N_1(Z, X)$.

2. Let $D \subset X$ be a prime divisor, and $\tilde{D} \subset \tilde{X}$ its transform. Then either $\dim N_1(\tilde{D}, \tilde{X}) = \dim N_1(D, X)$, or $\dim N_1(\tilde{D}, \tilde{X}) = \dim N_1(D, X) - 1$. If the last case occurs, then $D \cdot \sigma < 0$, $D \cdot \tilde{\sigma} > 0$, and $\tilde{\sigma} \not\subset N_1(\tilde{D}, \tilde{X})$.

**Proof.** We have the standard flip diagram:

```
X --------+--------+--------
          |          |
          g        \tilde{g}
Y --------+--------+--------
          |          |
          \tilde{Y}   Y
```

where $g$ and $\tilde{g}$ are the contractions of $\sigma$ and $\tilde{\sigma}$ respectively.

1. We have $g(Z) = \tilde{g}(\tilde{Z})$ and $g_*(N_1(Z, X)) = N_1(g(Z), Y) = \tilde{g}_*(N_1(\tilde{Z}, \tilde{X}))$.

   We show that $\sigma \subset N_1(Z, X)$ if and only if $\tilde{\sigma} \subset N_1(\tilde{Z}, \tilde{X})$. Indeed let $B \subset \text{Locus}(\sigma)$ be a line in an exceptional plane, and $l \subset \text{Locus}(\tilde{\sigma})$ an exceptional line. If $\sigma \subset N_1(Z, X)$, then $B \equiv \sum_i \lambda_i C_i$, with $\lambda_i \in \mathbb{Q}$ and $C_i \subset Z$ irreducible curves. Let $\tilde{C}_i \subset \tilde{Z}$ be the transform of $C_i$. Then there exists $\mu \in \mathbb{Q}$ such that $\mu l \equiv \sum_i \lambda_i \tilde{C}_i$. On the other hand, by taking anticanonical degrees, we get

$$1 = -K_X \cdot B = \sum_i \lambda_i (-K_X) \cdot C_i = \sum_i \lambda_i (-K_{\tilde{X}}) \cdot \tilde{C}_i = -\mu,$$

therefore $\mu \neq 0$ and $[l] \in N_1(\tilde{Z}, \tilde{X})$. The other implication is shown in the same way.

Therefore $\ker g_* \subset N_1(Z, X)$ if and only if $\ker \tilde{g}_* \subset N_1(\tilde{Z}, \tilde{X})$, which yields $\dim N_1(Z, X) = \dim N_1(\tilde{Z}, \tilde{X})$.

2. If $\text{Locus}(\sigma) \cap D = \emptyset$, then $\dim N_1(D, X) = \dim N_1(\tilde{D}, \tilde{X})$ by (1). Suppose that $\text{Locus}(\sigma) \cap D \neq \emptyset$. By Th. 3.3 $\text{Locus}(\sigma)$ is a union of exceptional planes, in particular

---

6This is equivalent to $\text{Mov}(X) \supset \text{Nef}(X)$.
there is a curve $C \subseteq \text{Locus}(\sigma) \cap D$. Then $[C] \in \sigma \cap \mathcal{N}_1(D, X)$, so that $\sigma \subset \mathcal{N}_1(D, X)$, and we get:
\[
\dim \mathcal{N}_1(D, X) = \begin{cases}
\dim \mathcal{N}_1(\tilde{D}, \tilde{X}) & \text{if } \tilde{\sigma} \subset \mathcal{N}_1(\tilde{D}, \tilde{X}); \\
\dim \mathcal{N}_1(\tilde{D}, \tilde{X}) + 1 & \text{if } \tilde{\sigma} \not\subset \mathcal{N}_1(\tilde{D}, \tilde{X}).
\end{cases}
\]
In the last case, $\text{Locus}(\tilde{\sigma}) \cap \tilde{D}$ must be a (non-empty) finite set, therefore we have $\tilde{D} \cdot \tilde{\sigma} > 0$ and $D \cdot \sigma < 0$.

**Corollary 3.14.** Let $X$ be a Fano 4-fold, $f : X \dasharrow \tilde{X}$ a SQM, $D \subset X$ a prime divisor, and $\tilde{D} \subset \tilde{X}$ its transform.

- If $f$ factors as a sequence of $m$ $K$-negative flips, then $\dim \mathcal{N}_1(D, X) \leq \dim \mathcal{N}_1(\tilde{D}, \tilde{X}) + m$.
- If $D$ does not contain exceptional planes, then $\dim \mathcal{N}_1(D, X) = \dim \mathcal{N}_1(\tilde{D}, \tilde{X})$.

### 3.3 Characterization of non-movable prime divisors

In this section we give a geometric description of non-movable prime divisors in a Fano 4-fold $X$ with $\rho_X \geq 6$ (Th. 3.15). As noticed in Rem. 2.19, the classes of these divisors generate the one-dimensional faces of $\text{Eff}(X)$ which do not lie in $\text{Mov}(X)$. On the other hand, we show that if a one-dimensional face of $\text{Eff}(X)$ is contained in $\text{Mov}(X)$, then $\rho_X \leq 11$ (Prop. 3.20). Th. 3.15 also allows to describe elementary divisorial rational contractions of $X$, see Cor. 3.19.

We refer the reader to [Bar10] for a study of $\text{Eff}(X)$ and $\text{ME}(X)$ for a Fano 4-fold $X$.

**Theorem 3.15.** Let $X$ be a Fano 4-fold with $\rho_X \geq 6$, and $D \subset X$ a non-movable prime divisor. Then there exists a diagram:

\[
\begin{array}{ccc}
X & \dashrightarrow & \tilde{X} \\
\downarrow & & \downarrow f \\
Y & & 
\end{array}
\]

where $X \dashrightarrow \tilde{X}$ is a SQM whose indeterminacy locus is the union of exceptional planes $L$ such that $D \cdot C_L < 0$ for a line $C_L \subset L$, and $f$ is an elementary divisorial contraction with $\text{Exc}(f) = \tilde{D}$ (the transform of $D$). Moreover one of the following holds:

1. $X = \tilde{X}$, $D$ is the locus of an extremal ray of type $(3, 2)$, and $D$ does not contain any exceptional plane;
2. $Y$ is smooth and Fano, $f$ is the blow-up of a smooth curve, and $\tilde{D}$ is a $\mathbb{P}^2$-bundle over a smooth curve;
3. $Y$ is smooth and Fano, $f$ is the blow-up of a point, and $\tilde{D} \cong \mathbb{P}^3$;
4. $\tilde{D}$ is isomorphic to a quadric, $f(\tilde{D})$ is a factorial and terminal singular point, and $Y$ is Fano.
We will say that $D$ is of type $(3, 2)$, of type $(3, 1)$, of type $(3, 0)^{23}$, or of type $(3, 0)^{Q}$, when we are respectively in case (i), (ii), (iii), or (iv) above.

**Remark 3.16.** In cases (ii) – (iv) we also will show that $c_X \leq 2$, and that the birational map $X \dasharrow \tilde{X}$ factors as a sequence of at least $\rho_X - 4$ $D$-negative and $K$-negative flips (this follows from (3.18)). In particular, Th. 3.15 implies Th. 1.2.

**Example 3.17** (A non-movable prime divisor of type $(3, 0)^3$). Let $Y := (\mathbb{P}^1)^4$ and let $f: \tilde{X} \rightarrow Y$ be the blow-up of a point $p \in Y$. Then $\tilde{X}$ is a toric 4-fold with $\rho_{\tilde{X}} = 5$; in particular $\tilde{X}$ is a Mori dream space.

Let $C_1, C_2, C_3, C_4 \subset Y$ be the irreducible curves of type $\mathbb{P}^1 \times \text{pts}$ through $p$, and $l_i \subset \tilde{X}$ the transform of $C_i$. We have $-K_Y \cdot C_i = 2$, $-K_{\tilde{X}} \cdot l_i = -1$, and $\text{Exc}(f) \cdot l_i = 1$; in particular, $\tilde{X}$ is not Fano.

On the other hand $l_i$ is an exceptional line, $\mathbb{R}_{\geq 0}[l_i]$ is a small extremal ray of $\text{NE}(\tilde{X})$, and it is possible to flip these exceptional lines with a sequence of 4 flips $\tilde{X} \dasharrow X$.

Then $X$ is Fano and the transform $D \subset X$ of $\text{Exc}(f)$ is a smooth divisor, isomorphic to the blow-up of $\mathbb{P}^3$ in 4 points. There are 4 exceptional planes $L_1, \ldots, L_4 \subset D$, and $D \cdot C_{L_i} = -1$ where $C_{L_i} \subset L_i$ is a line.

**Proof of Th. 3.15.** After [Cas11, Prop. 2.4], there exists a Mori program for $D$ such that every extremal ray of the program is $K$-negative. By Rem. 2.19, this gives a SQM $g: X \dasharrow \tilde{X}$, which factors as a sequence of $D$-negative and $K$-negative flips, and an elementary, $K$-negative, divisorial contraction $f: \tilde{X} \rightarrow Y$ with exceptional divisor $D$, the transform of $D$. Notice that $D$ has positive intersection with all exceptional lines in $\tilde{X}$.

If $f$ is of type $(3, 2)$, then $\tilde{D}$ is covered by surfaces of anticanonical degree 1, thus by Rem. 3.10 (1) $\tilde{D}$ cannot intersect any exceptional line. This means that $X = \tilde{X}$ and $D$ is the locus of the extremal ray $\text{NE}(f)$, of type $(3, 2)$.

Suppose that $D$ contains an exceptional plane $L$. After the classification of possible isolated 2-dimensional fibers of $f$ in [AW97, Th. 4.7], we know that an exceptional plane cannot be a component of a fiber of $f$, therefore $f$ is finite on $L$. Thus $f(L) = f(D)$, which implies that $\text{dim} \mathcal{N}_1(f(D), Y) = 1$. On the other hand $\mathcal{N}_1(D, X) = (f_*)^{-1} \mathcal{N}_1(f(D), Y)$, so that $\text{dim} \mathcal{N}_1(D, X) = 2$, and $c_X \geq \rho_X - 2 \geq 4$. Then $X$ should be a product of surfaces by Th. 3.11 thus $X$ cannot contain any exceptional plane, and we get a contradiction. Therefore we have (i).

Suppose that $f$ is not of type $(3, 2)$. Since $\rho_X \geq 6$, by [Cas09, Cor. 1.3] $X$ cannot have elementary divisorial contractions of type $(3, 0)$ or $(3, 1)$, therefore $g$ is not an isomorphism and $X$ has a small elementary contraction. Hence $c_X \leq 2$ by Cor. 3.12 and $\text{dim} \mathcal{N}_1(D, X) \geq 4$.

Let $l_1, \ldots, l_r \subset \tilde{X}$ be the exceptional lines, and suppose that $g$ factors as a sequence of $m \geq 1$ $K$-negative and $D$-negative flips. Then Rem. 3.14 yields $m \geq \text{dim} \mathcal{N}_1(D, X) - \text{dim} \mathcal{N}_1(D, \tilde{X})$, therefore:

$$r \geq m \geq \rho_X - c_X - \text{dim} \mathcal{N}_1(D, \tilde{X}) \geq 4 - \text{dim} \mathcal{N}_1(D, \tilde{X}).$$

---

*This toric Fano 4-fold is described in [Bat99 Prop. 3.5.8(iii)].*
Moreover $\tilde{D} \cdot l_i > 0$ for every $i = 1, \ldots, r$, and $l_i$ can not be contained in a fiber of $f$.

Suppose that $f$ is of type $(3,1)$, so that $\dim N_i(\tilde{D}, \tilde{X}) = 2$ and $r \geq 2$ by (3.18). Let $F \subset \tilde{D}$ be a fiber of $f$ intersecting $l_1 \cup \cdots \cup l_r$. Then $F$ cannot be covered by curves of anticanonical degree 1 by Rem. 3.6 (1). By the classification of elementary $K$-negative contractions of type $(3,1)$ in [Tak99], this is possible only if $Y$ is smooth and $f$ is the blow-up a smooth curve $C \subset Y$, so that $\tilde{D}$ is a $\mathbb{P}^2$-bundle over $C$. Moreover the lines in the fibers of $\tilde{f} | _{\tilde{D}}$ have anticanonical degree 2 in $\tilde{X}$.

If $F$ intersects $l_1 \cup \cdots \cup l_r$ in at least two points, by taking the line through these two points we get a contradiction with Rem. 3.6 (1). Therefore every fiber of $\tilde{f} | _{\tilde{D}}$ intersects $l_1 \cup \cdots \cup l_r$ in at most one point, and the exceptional lines $l_1, \ldots, l_r$ intersect different fibers of $\tilde{f} | _{\tilde{D}}$. Since $r \geq 2$, this implies that no exceptional line is contained in $\tilde{D}$.

Let’s show that $Y$ is Fano. We have $f^*(-K_Y) = -K_X + 2\tilde{D}$ and $-K_Y \cdot f(l_i) = (-K_X + 2\tilde{D}) \cdot l_i = 2\tilde{D} \cdot l_i - 1 > 0$ for every $i = 1, \ldots, r$. If $\sigma$ is an extremal ray of $\text{NE}(\tilde{X})$ with $-K_X \cdot \sigma > 0$ and $\tilde{D} \cdot \sigma \geq 0$, then $-K_X + 2\tilde{D} \cdot \sigma > 0$.

Suppose that $\tilde{X}$ has a $\tilde{D}$-negative extremal ray $\sigma \neq \text{NE}(f)$. Then $\text{Locus}(\sigma) \subseteq \tilde{D}$, so that $-K_X \cdot \sigma > 0$. If $G \subset \tilde{D}$ is a non-trivial fiber of the contraction of $\sigma$, then $f$ must be finite on $G$, hence $\dim G = 1$. Therefore $\sigma$ is of type $(3,2)$ (see Th. 3.1), and $\tilde{D}$ is covered by curves of anticanonical degree 1, a contradiction by Rem. 3.6 (1). We deduce that $-K_X + 2\tilde{D}$ is nef and $(-K_X + 2\tilde{D}) \cap \text{NE}(\tilde{X}) = \text{NE}(f)$, hence $-K_Y$ is ample and we have (ii).

Assume now that $f$ is of type $(3,0)$, so that $\dim N_i(\tilde{D}, \tilde{X}) = 1$ and $r \geq 3$ by (3.18). Suppose that $\tilde{D} \cong \mathbb{P}^3$. Since $-K_X \cdot \text{NE}(f) > 0$, we have $N_{\tilde{D}/\tilde{X}} \cong \mathcal{O}_{\mathbb{P}^3}(-a)$ with $a \in \{1, 2, 3\}$. If $a = 3$, then $\tilde{D}$ is covered by curves of anticanonical degree 1, which is impossible by Rem. 3.6 (1), because $\tilde{D}$ intersects $l_1$. If $a = 2$, the lines in $\tilde{D}$ have anticanonical degree 2 in $\tilde{X}$, and by taking a line which intersects both $l_1$ and $l_2$, we get again a contradiction by Rem. 3.6 (1). Therefore $a = 1$, $Y$ is smooth, and $f$ is the blow-up of a point $p \in Y$.

We have $f^*(-K_Y) = -K_X + 3\tilde{D}$ and $-K_Y \cdot f(l_i) = (-K_X + 3\tilde{D}) \cdot l_i = 3\tilde{D} \cdot l_i - 1 > 0$ for every $i = 1, \ldots, r$, and similarly as before we conclude that $Y$ is Fano, so we get (iii).

Suppose that $\tilde{D} \cong Q$, where $Q \subset \mathbb{P}^4$ is a quadric. Again we have $N_{\tilde{D}/\tilde{X}} \cong \mathcal{O}_Q(-a)$ with $a \in \{1, 2\}$. If $a = 2$, then $\tilde{D}$ is covered by curves of anticanonical degree 1, which is impossible. Thus $a = 1$, and if $C \subset \tilde{D}$ corresponds to a line in $Q$, we have $-K_X \cdot C = 2$ and $\tilde{D} \cdot C = -1$. The point $p = f(\tilde{D}) \in Y$ is a factorial terminal singularity in $Y$, and $f^*(-K_Y) = -K_X + 2\tilde{D}$. As before we see that $-K_Y \cdot f(l_i) = 2\tilde{D} \cdot l_i - 1 > 0$ for every $i = 1, \ldots, r$, and $Y$ is Fano, so we get (iv).

We assume now that $\tilde{D}$ is not isomorphic to $\mathbb{P}^3$ or a quadric, and show that this gives a contradiction. This type of exceptional divisor has been studied by Beltrameetti [Bel87, Bel86] and by Fujita as an application of his theory of Del Pezzo varieties – we refer the reader to [IP99, §3.2] for an overview.
Notice that $\tilde{D}$ is reduced and irreducible. Being a divisor in a smooth variety, it is Cohen-Macaulay and has a locally free dualising sheaf $\omega_{\tilde{D}}$ given by

$$\omega_{\tilde{D}} = \mathcal{O}_X(K_X + \tilde{D})|_{\tilde{D}}.$$ 

Therefore $\tilde{D}$ is Gorenstein, and by Serre’s criterion, it is normal if and only if $\dim \text{Sing} \tilde{D} \leq 1$.

By [Fu90, §3, in particular (3.2)], there exists an ample line bundle $L_\tilde{X} \in \text{Pic}(\tilde{X})$ such that, if $L := (L_\tilde{X})|_{\tilde{D}}$, we have

$$\mathcal{O}_\tilde{X}(-K_\tilde{X})|_{\tilde{D}} = \mathcal{O}_\tilde{X}(-\tilde{D})|_{\tilde{D}} = L,$$

hence $\omega_{\tilde{D}} = L^{\otimes (-2)}$. Moreover the pair $\tilde{(D, L)}$ is a Del Pezzo variety, see [Fu90] (3.3) and [IP99] §3.2 for the definition.

Notice that $\tilde{D}$ cannot be covered by curves having intersection 1 with $L$, because these would have anticanonical degree 1 in $\tilde{X}$, contradicting Rem. 3.6 (1).

Set $d := L^3$. If $d = 1$, then by [IP99] Th. 3.2.5 (i) $\tilde{D}$ is isomorphic to a hypersurface of degree 6 in the weighted projective space $\mathbb{P}(3,2,1,1,1)$.

Since $\mathbb{P}(3,2,1,1,1)$ has two singular points, the generic hypersurface is smooth; if in the smooth case $\tilde{D}$ is covered by curves having intersection 1 with $L$, the same must hold also in the singular case.

Hence suppose that $\tilde{D}$ is smooth. By [IP99] Prop. 3.2.4 (i) the general element $S \in |L|$ is a smooth surface with $-K_S = L|_S$ ample and $(-K_S)^2 = d = 1$. Therefore $S$ is covered by curves of anticanonical degree 1 (the pencil $| - K_S|$) and $\tilde{D}$ is covered by curves having intersection 1 with $L$, which gives a contradiction.

If $d = 2$, then by [IP99] Prop. 3.2.4 (ii)] the linear system $|L|$ determines a double covering $\pi: \tilde{D} \to \mathbb{P}^3$ such that $L = \pi^*\mathcal{O}_{\mathbb{P}^3}(1)$. For $i = 1, 2$ choose $p_i \in \tilde{D} \cap l_i$, and let $C \subset \mathbb{P}^3$ be a line through $\pi(p_1)$ and $\pi(p_2)$. Set $C' := \pi^{-1}(C) \subset \tilde{D}$. Then $p_1, p_2 \in C'$, $\pi_*(C') = 2C$ and $-K_\tilde{X} \cdot C' = (L \cdot C')_{\tilde{D}} = 2$ (where $(\cdot)_{\tilde{D}}$ is intersection in $\tilde{D}$). The curve $C'$ can not be irreducible by Rem. 3.6 (1), but if it is reducible we get a curve of anticanonical degree 1 in $\tilde{X}$ containing one of the points $p_i$, which is again impossible.

Suppose now that $d \geq 3$. Then $L$ is very ample and gives an isomorphism of $\tilde{D}$ with $V \subset \mathbb{P}^{d+1}$ of degree $d$, see [IP99] Prop. 3.2.4 (ii)].

If $d = 3$ then $V$ is a cubic in $\mathbb{P}^4$, thus it is covered by lines, and $\tilde{D}$ is covered by curves having intersection 1 with $L$. Similarly, if $d = 4$, then by [IP99] Th. 3.2.5 (iv)] $V$ is the complete intersection of two quadrics in $\mathbb{P}^5$, and again it is covered by lines.

Assume that $d \geq 5$. Then by [Fu90] (2.6)] $V \subset \mathbb{P}^{d+1}$ is not a cone over another variety. If $\tilde{D}$ is smooth, then it is a Fano 3-fold of index 2, and by [IP99] Th. 3.3.1] the possibilities for $\tilde{D}$ are: the blow-up of $\mathbb{P}^3$ in a point, $(\mathbb{P}^1)^3$, $\mathbb{P}_{\mathbb{P}^2}(T_{\mathbb{P}^2})$, or a linear section of $G(1, 4) \subset \mathbb{P}^9$. In all these cases it is easy to see that $\tilde{D}$ is covered by curves having intersection 1 with $L$.

\*This can be seen for instance using toric geometry, see [Con02] Th. 3.6].
Suppose now that $\dim \Sing(\tilde{D}) = 0$. Then $\tilde{D}$ is normal, and by [Fuj86, Th. (2.1) and (2.9)] we see that the singularities of $\tilde{D}$ are ordinary double points; in particular $\tilde{D}$ has terminal singularities. Therefore by [Nam97, Th. 11] $\tilde{D}$ has a smoothing $T$, where $T$ is a smooth Fano 3-fold with index 2 and anticanonical degree $8d$. By the previous case, we know that $T$ is covered by curves of anticanonical degree 2, hence the same holds for $\tilde{D}$.

If $\dim \Sing(\tilde{D}) = 2$ then $\tilde{D}$ is not normal, and by [Fuj86, Th. (2.1)] $V$ is the projection of a smooth variety of minimal degree in $\mathbb{P}^{d+2}$. In particular $V$ is covered by lines, and we are done.

If instead $\dim \Sing(\tilde{D}) = 1$ (so that $\tilde{D}$ is normal), we follow the construction in [Fuj86, (6), p. 150]. Let $p_0 \in \Sing(V)$, and set

$$W := \bigcup_{q \in V \setminus p_0} \mathbb{P}q \subset \mathbb{P}^{d+1},$$

where $\mathbb{P}q$ denotes the line through $p$ and $q$ in $\mathbb{P}^{d+1}$. Notice that $\dim W = 4$ and $W$ has degree $d - 2$. Set moreover

$$R := \{p \in W | \mathbb{P}q \subset W \text{ for every } q \in W \setminus p\}$$

(so that $p_0 \in R$). By [Fuj86, Lemma (2)], $R \subset \mathbb{P}^{d+1}$ is a linear subspace, and if $M \subset W$ is a section of $W$ with a generic linear subspace of dimension $d - \dim R$, then $W$ is the cone over $M$ with vertex $R$. By [Fuj86, (6)] $M$ is smooth and $R \subset \Sing(V)$, therefore $\dim R \in \{0, 1\}$.

All the possibilities for $V$ are listed in [Fuj86, Th. (2.9)]. Since $\dim V = 3$ and $\dim \Sing(V) = 1$, we see that the possibilities are: (vi), (si2ii), (si3ii), (si2i1-a), (si11o-d), and (si2i1-b). In the cases (vi), (si2ii), (si3ii), (si2i1-a), and (si2i1-b) we have $\dim R = 1$ (see [Fuj86], pages 155, 169, 170, and 163 respectively).

In case (si11o-d) we have $\dim R = 0$, however this variety $V$ is the same as (si2i1-a), see [Fuj86, Remark on p. 167]. By choosing the point $p_0$ in a one-dimensional component of $\Sing(V)$, we can reduce to the case where $\dim R = 1$. The case (si21i) is analogous, see [Fuj86, Remark on p. 171].

Therefore $R$ is a line and $\dim M = 2$. We still follow the construction in [Fuj86, (7)]. Let $\tilde{P} \rightarrow \mathbb{P}^{d+1}$ be the blow-up along $R$, let $V \subset \tilde{P}$ and $W \subset \tilde{P}$ be transforms of $V$ and $W$ respectively, and $\varphi : W \rightarrow W$ the induced morphism.

Then $W$ is smooth and there is a $\mathbb{P}^2$-bundle structure $\tilde{W} \rightarrow M$ such that if $F \subset \tilde{W}$ is a fiber we have $\varphi^*(\mathcal{O}_W(1))|_F = \mathcal{O}_{\mathbb{P}^2}(1)$. On the other hand by [Fuj86, (8)] we also have $\varphi^*(\mathcal{O}_W(1))|_F = \mathcal{O}_{\tilde{W}}(\tilde{V})|_F$, therefore for a generic $F$ the intersection $V \cap F$ is a line in $F$, and $\varphi(\tilde{V} \cap F)$ is a line in $V \subset \mathbb{P}^{d+1}$. This shows that $V$ is covered by lines, and concludes the proof.

\begin{corollary}[Elementary divisorial rational contractions] \label{corollary:elementary_divisorial_rational_contractions}
Let $X$ be a Fano 4-fold with $\rho_X \geq 6$, and consider an elementary divisorial contraction $f : \tilde{X} \rightarrow Y$, where $\tilde{X}$ is a SQM of $X$. Then $Y$ has at most isolated terminal and factorial singularities. Moreover one of the following holds:

(i) $f$ is of type $(3, 2)$, $X \rightarrow \tilde{X}$ is an isomorphism over $\Exc f$, and $\Exc(f)$ does not contain any exceptional plane;
\end{corollary}
Proof. Let $D \subset X$ be the transform of $\text{Exc}(f)$. Then $D$ is a non-movable prime divisor, and by Th. 3.15 there is a diagram

$$
\begin{array}{ccc}
X & \to \tilde{X}_1 & \to \tilde{X} \\
\downarrow f_1 & & \downarrow f \\
Y_1 & \to Y \\
\end{array}
$$

where $X \to \tilde{X}_1$ is a SQM and $f_1 : \tilde{X}_1 \to Y_1$ is an elementary divisorial contraction with exceptional divisor the transform of $\text{Exc}(f)$, and satisfying one of the conditions of Th. 3.15.

The birational map $Y_1 \to Y$ is an isomorphism in codimension 1, i.e. it is a SQM.

The cases $(i) - (iv)$ of the statement correspond to the same cases of Th. 3.15; we will consider $(i)$ and $(ii)$, the other ones being completely analogous.

If $D$ is of type $(3, 2)$, then $X = \tilde{X}_1$, $f_1$ is an elementary contraction of type $(3, 2)$, and $D$ does not contain any exceptional plane; in particular $\dim N_1(D, X) = \dim N_1(\text{Exc}(f), \tilde{X})$ by Rem. 3.14.

If the map $X \to \tilde{X}$ is not an isomorphism, then Cor. 3.12 yields that $c_X \leq 2$. Hence

$$\dim N_1(\text{Exc}(f), \tilde{X}) = \dim N_1(D, X) \geq \rho_X - 2 \geq 4,$$

and $f$ cannot be of type $(3, 0)$ nor $(3, 1)$. Therefore $f$ is of type $(3, 2)$ and $\text{Exc}(f)$ is covered by curves of anticanonical degree 1. By Rem. 3.6 (1) the map $X \to \tilde{X}$ is an isomorphism over $\text{Exc}(f)$, and we get $(i)$.

Suppose now that $D$ is of type $(3, 1)$. Then $Y_1$ is smooth and Fano, so that the birational map $Y_1 \to Y$ is an isomorphism outside a disjoint union of exceptional planes in $Y_1$, see Rem. 3.6. Moreover $f_1$ is the blow-up of a smooth curve $C_1 \subset Y_1$, and $(f_1)^*(-K_{Y_1}) = -K_{\tilde{X}_1} + 2\text{Exc}(f_1)$.

Let $l_1, \ldots, l_r \subset \tilde{X}_1$ be the exceptional lines. Then $f_1(l_i)$ intersects $C_1$ and $-K_{Y_1} \cdot f(l_i) = 2\text{Exc}(f_1) \cdot l_i - 1$, hence $-K_{Y_1} \cdot f(l_i) \geq 3$ unless $-K_{Y_1} \cdot f(l_i) = \text{Exc}(f_1) \cdot l_i = 1$. On the other hand let $C_2 \subset Y_1$ be an irreducible curve different from $C_1$, $f_1(l_1), \ldots, f_1(l_r)$. If $C_1 \cap C_2 \neq \emptyset$, then $-K_{Y_1} \cdot C_2 \geq 3$. 

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Let now \( L \subset Y_1 \) be an exceptional plane. Since \( C_1 \) can intersect at most finitely many curves of anticanonical degree 1, we have \( C_1 \cap L = \emptyset \), and \( f_{1}^{-1}(L) \subset \tilde{X}_1 \) is still an exceptional plane. Then \((l_1 \cup \cdots \cup l_r) \cap f_{1}^{-1}(L) = \emptyset \) by Rem. 3.6(3), thus \((f_1(l_1) \cup \cdots \cup f_1(l_r)) \cap L = \emptyset \).

We conclude that \( C_1 \cup f_1(l_1) \cup \cdots \cup f_1(l_r) \) is contained in the open subset where \( Y_1 \to Y \) is an isomorphism, and \( \text{Exc}(f_1) \cup l_1 \cup \cdots \cup l_r \) is contained in the open subset where \( \tilde{X}_1 \to \tilde{X} \) is an isomorphism.

Therefore \( f \) is the blow-up of a smooth curve \( C \subset Y \), and \( C \) does not meet any exceptional line in \( Y \). Let \( C_0 \subset Y \) be an irreducible curve which meets \( C \), \( C_0 \neq C \), and let \( C'_0 \subset Y_1 \) be its transform. We have \(-K_Y \cdot C_0 \geq -K_{Y_1} \cdot C'_0 \) by Rem. 3.6 so that either \(-K_Y \cdot C_0 \geq 3 \), or \( C'_0 = f_i(l_i) \) for some \( i \in \{1, \ldots, r\} \) and \(-K_{Y_1} \cdot C'_0 = \text{Exc}(f_i) \cdot l_i = 1 \); this gives the statement.

We conclude this section showing that when the cones \( \text{Mov}(X) \) and \( \text{Eff}(X) \) share a one-dimensional face, we can easily bound the Picard number of \( X \). As a consequence, when \( \rho_X \) is large, \( X \) contains plenty of non-movable prime divisors.

**Proposition 3.20.** Let \( X \) be a Fano 4-fold, and suppose that there exists a movable prime divisor \( D \) whose class belongs to a one-dimensional face of \( \text{Eff}(X) \).

Then \( \rho_X \leq 11 \). Moreover if \( \rho_X = 11 \) then \( D \) is a fiber of a quasi-elementary contraction \( X \to \mathbb{P}^1 \), with general fiber \( \mathbb{P}^1 \times S \), where \( S \) is a Del Pezzo surface with \( \rho_S = 9 \).

**Proof.** The cone \( \mathbb{R}_{\geq 0}[D] \) is a common one-dimensional face of \( \text{Mov}(X) \) and \( \text{Eff}(X) \). By Cor. 2.25 this implies the existence of a quasi-elementary rational contraction \( f: X \to Y \) with \( \rho_Y = 1 \).

If \( \dim Y = 3 \), then \( f \) is elementary and \( \rho_X = 2 \) (see Rem. 2.24).

Assume that \( \dim Y \leq 2 \). If \( f \) is not regular, then \( \rho_X \leq 10 \) by Cor. 3.9. If \( f \) is regular, the statement follows as in the proof of Cor. 3.9.

**Corollary 3.21.** Let \( X \) be a Fano 4-fold with \( \rho_X \geq 12 \). Then \( \text{Eff}(X) \) is generated by classes of non-movable prime divisors; in particular \( X \) contains at least \( \rho_X \) such divisors.

4 Rational contractions of fiber type of Fano 4-folds

4.1 Quasi-elementary rational contractions onto surfaces

In this section we study Fano 4-folds having a quasi-elementary rational contraction onto a surface. First of all let us recall what happens in the case of a regular contraction.

**Proposition** ([Cas08], Th. 1.1 (i)). Let \( X \) be a Fano 4-fold and \( f: X \to S \) a quasi-elementary contraction onto a surface. Then \( \rho_S \leq 9 \), \( \rho_X \leq 18 \), and \( \rho_X = 18 \) only if \( X \) is a product of surfaces.

If \( f \) is elementary, then \( \rho_X \leq 10 \), with equality only if \( X \cong \mathbb{P}^2 \times S \).

Here is the result in the rational case.
**Proposition 4.1.** Let $X$ be a Fano 4-fold and $f: X \rightarrow S$ a quasi-elementary rational contraction onto a surface. Assume that $f$ is not a morphism.

If $f$ is not elementary, then $\rho_S \leq 9$ and $\rho_X \leq 17$.

If $f$ is elementary, then $\rho_X \leq 11$.

**Proof.** When $f$ is elementary $\rho_X = \rho_S + 1$, while in general $\rho_X \leq \rho_S + 8$ by Cor. 3.19. Therefore we have to show that $\rho_S \leq 10$ if $f$ is elementary, and $\rho_S \leq 9$ otherwise.

The surface $S$ is smooth and is a Mori dream space by Cor. 3.8, moreover $S$ is rational because $X$ is rationally connected.

We assume that $\rho_X \geq 6$ and $\rho_S \geq 4$. Under these conditions, we are going to show that $-K_S$ is nef, and ample when $f$ is not elementary; since $S$ is a smooth rational surface, this implies the statement. Notice that in order to show that $-K_S$ is nef (respectively, ample), it is enough to show that $-K_S \cdot \sigma \geq 0$ (respectively, $> 0$) for every extremal ray $\sigma$ of $\text{NE}(S)$; moreover, every such extremal ray corresponds to an elementary contraction of $S$.

Thus let $g: S \rightarrow S_1$ be an elementary contraction. The surface $S_1$ has rational singularities by Rem. 3.71 and since $\rho_S \geq 4$, $g$ is birational.

Consider a factorization $X \rightarrow \tilde{X} \rightarrow S$ of $f$ as in Rem. 3.71 and let $C \subset S$ be the irreducible curve contracted by $g$. Since $C$ is a non-movable prime divisor in $S$, by Rem. 3.10 $D := (\tilde{f})^*(C) = (\tilde{f})^{-1}(C)$ is a non-movable prime divisor in $\tilde{X}$. We have $(\tilde{f})_* (\mathcal{N}_1(D, \tilde{X})) = \mathbb{R}[C] \subset \mathcal{N}_1(S)$, hence

\[ \dim \mathcal{N}_1(D, \tilde{X}) \leq 1 + \dim \ker(\tilde{f})_* = 1 + \rho_X - \rho_S \leq \rho_X - 3. \]

Let $D_X \subset X$ be the transform of $D$. Since $f$ is not regular, $X$ has a small elementary contraction, and Cor. 3.12 gives $c_X \leq 2$, hence $\dim \mathcal{N}_1(D_X, X) \geq \rho_X - 2$. We apply Th. 3.15 to $D_X$, and consider the possible types.

We notice at once that $\dim \mathcal{N}_1(D_X, X) > \dim \mathcal{N}_1(D, \tilde{X})$, therefore by Rem. 3.14 $D_X$ must contain some exceptional plane. This implies that $D_X$ cannot be of type $(3, 2)$ (see Th. 3.15 (i)).

We apply Rem. 3.20 to $g \circ \tilde{f}: \tilde{X} \rightarrow S_1$ and $D$, and get a diagram:

\[
\begin{array}{ccc}
\tilde{X} - h & \rightarrow & \tilde{X} \\
\downarrow & & \downarrow \scriptstyle{k} \\
S - g & \rightarrow & S_1 \\
\downarrow & \scriptstyle{\tilde{f}} & \\
\end{array}
\]

where $k$ is an elementary divisorial contraction with exceptional divisor the transform of $D$, $\tilde{f}_1$ is a contraction, and $h$ is a birational map which factors as a sequence of $D$-negative flips. Notice that $\tilde{X}_1$ is factorial by Cor. 3.19 in particular it is again a Mori dream space (see Rem. 2.8).

We show that $\tilde{f}_1: \tilde{X}_1 \rightarrow S_1$ is quasi-elementary. Let $F \subset \tilde{X}$ be a general fiber of $\tilde{f}$, and consider its transforms $\tilde{F} \subset \tilde{X}$ and $F_1 \subset \tilde{X}_1$. Since the indeterminacy locus of $h$ is contained in $D$, it is disjoint from $F$; therefore $F$, $\tilde{F}$, and $F_1$ are isomorphic, and $F_1$ is a general fiber of $\tilde{f}_1$. By Rem. 3.13 (1) we get

\[ \dim \mathcal{N}_1(\tilde{F}, \tilde{X}) = \dim \mathcal{N}_1(F, \tilde{X}) = \rho_X - \rho_S = \rho_{\tilde{X}_1} - \rho_{S_1} = \dim \ker(\tilde{f}_1)_*. \]
(we are using that $\tilde{f}$ is quasi-elementary, see Prop. [2.22 (ii)].

On the other hand $\tilde{F} \cap \text{Exc}(k) = \emptyset$, therefore $\mathcal{N}_1(\tilde{F}, \hat{X}) \subseteq \text{Exc}(k)^{\perp}$ and $\text{NE}(k) \not\subseteq \mathcal{N}_1(\tilde{F}, \hat{X})$. We conclude that $k_\ast: \mathcal{N}_1(\tilde{X}) \to \mathcal{N}_1(\tilde{X}_1)$ is injective on $\mathcal{N}_1(\tilde{F}, \hat{X})$, and since $\mathcal{N}_1(F_1, \tilde{X}_1) = k_\ast(\mathcal{N}_1(\tilde{F}, \hat{X}))$, we get $\dim \mathcal{N}_1(F_1, \tilde{X}_1) = \dim \mathcal{N}_1(\tilde{F}, \hat{X}) = \dim \ker(\tilde{f}_1)_\ast$, and $\tilde{f}_1$ is quasi-elementary by Prop. [2.22 (ii)].

If $D_X$ is of type $(3, 1)$ or $(3, 0)^{2,3}$, then $\tilde{X}_1$ is smooth and it is a SQM of a Fano 4-fold $X_1$ by Cor. [3.19] Since $X_1 \dashrightarrow S_1$ is a quasi-elementary rational contraction, Cor. [3.8] implies that $S_1$ is smooth. Hence $g$ is the blow-up of a smooth point, and $-K_S \cdot C = 1$.

Suppose now that $D_X$ is of type $(3, 0)^Q$. We show that $\tilde{f}_1$ is $K$-negative.

By contradiction, suppose that there exists an irreducible curve $C_0 \subset \tilde{X}_1$ such that $\tilde{f}_1(C_0) = \{pt\}$ and $-K_{\tilde{X}_1} \cdot C_0 \leq 0$. By Cor. [3.19] $C_0$ cannot contain the singular point $p := k(\text{Exc}(k))$, therefore $C_0 = k(l)$ where $l \subset \hat{X}$ is an irreducible curve, disjoint from $\text{Exc}(k)$, with $-K_{\hat{X}} \cdot l \leq 0$. By Rem. [3.6] (2), $l$ is an exceptional line. We need the following.

**Remark 4.2.** Let $X$ be a Fano 4-fold and consider a diagram:

\[ (4.3) \]

![Diagram](image)

where $\varphi$ and $\psi$ are SQMs and $h := \psi \circ \varphi^{-1}$. Let $l \subset \hat{X}$ be an exceptional line.

1. Either $l \subset \text{dom}(h^{-1})$, or $l \cap \text{dom}(h^{-1}) = \emptyset$.

2. Let $D$ be a divisor in $\tilde{X}$, $\hat{D}$ its transform in $\hat{X}$, and suppose that $h$ factors as a sequence of $D$-negative flips. If $l \cap \text{dom}(h^{-1}) = \emptyset$, then $\hat{D} \cdot l > 0$.

**Proof.** By Rem. [3.6] (2) we have $l \cap \text{dom}(\psi^{-1}) = \emptyset$. Therefore if $l$ is not contained in the indeterminacy locus of $h^{-1}$, then its transform $\hat{l} \subset \hat{X}$ must be contained in the indeterminacy locus of $\varphi^{-1}$. Then again by Rem. [3.6] $\hat{l}$ is an exceptional line, and $h^{-1} = \varphi \circ \psi^{-1}$ is an isomorphism on $l$.

For the second statement, we can factor $h$ as $\tilde{X} \overset{h_1}{\dashrightarrow} \tilde{X}_1 \overset{h_2}{\dashrightarrow} \hat{X}$, where $h_2$ is a $D_1$-negative flip ($D_1$ the transform of $D$ in $\tilde{X}_1$). By induction, we can assume that the statement holds for $h_1$. Now if $l \cap \text{dom}(h_2^{-1}) = \emptyset$, we have $\hat{D} \cdot l > 0$, because $h_2^{-1}$ is a $D$-positive flip. Otherwise $l$ is contained in the open subset where $h_2^{-1}$ is an isomorphism, so that $l = h_2(l_1)$, $l_1$ an exceptional line in $\tilde{X}_1$. Moreover $l_1 \cap \text{dom}(h_1^{-1}) = \emptyset$, therefore by induction $D_1 \cdot l_1 > 0$ and $\hat{D} \cdot l > 0$.

We carry on with the proof of Prop. [4.1] and apply Rem. [4.2] to $h$ and $l \subset \hat{X}$. Since $\text{Exc}(k) \cdot l = 0$, we deduce that $h$ is an isomorphism over $l$, so that $\hat{l} = h^{-1}(l) \subset \hat{X}$ is an exceptional line disjoint from $D$ and contracted by $g \circ \tilde{f}$. On the other hand $\dim(\tilde{f}(\hat{l})) = 1$ (because $\tilde{f}$ is $K$-negative), and $\tilde{f}(\hat{l}) \neq C$ (because $\hat{l} \not\in \tilde{f}^{-1}(C) = D$), thus $\dim(g \circ \tilde{f})(\hat{l}) = 1$, a contradiction.
Hence \( \tilde{f}_1 : \tilde{X}_1 \to S_1 \) is a \( K \)-negative quasi-elementary contraction. Since \( \tilde{X}_1 \) is factorial, as in [Cas08, Lemmas 3.9 and 3.10] one shows that \( S_1 \) is factorial. Thus \( S_1 \) is a normal, Gorenstein surface with rational singularities, that is, \( S_1 \) has at most Du Val singularities. Therefore either \( g : S \to S_1 \) is the blow-up of a smooth point and \(-K_S \cdot C = 1\), or \( C \) is a \((-2)\)-curve in \( S \) and \(-K_S \cdot C = 0\).

Summing up, we have shown that \(-K_S \cdot \text{NE}(g) \geq 0\) for every elementary contraction \( g \) of \( S \), therefore \(-K_S \) is nef.

Suppose now that \( f \) is not elementary. To show that \(-K_S \) is ample, we need to show that when \( D_X \) is of type \((3,0)\), \( C \) cannot be a \((-2)\)-curve. For this, we show the existence of an irreducible curve \( C' \subset X \) with \( D \cdot C' = -1 \). This gives:

\[
-1 = (\tilde{f})^*(C) \cdot C' = C \cdot (\tilde{f})_*(C'),
\]

hence \((\tilde{f})_*(C') = C \) and \( C^2 = -1 \).

We know that \( \tilde{f}_1 \) is a non-elementary \( K \)-negative quasi-elementary contraction, so that the general fiber is a smooth Del Pezzo surface with Picard number \( > 1 \). In particular, every fiber of \( \tilde{f}_1 \) is covered by curves of anticanonical degree 2, either irreducible, or a union of two irreducible curves of anticanonical degree 1.

Let \( F_0 \subset X_1 \) be the fiber containing the singular point \( p \). By Cor. 3.19, \( p \) cannot be contained in any irreducible curve of anticanonical degree 2, hence we find an irreducible curve \( C_1 \subset F_0 \) such that \( p \in C_1 \) and \(-K_{\tilde{X}_1} \cdot C_1 = 1 \). Again by Cor. 3.19, \( C_1 = k(l_1) \), where \( l_1 \subset \tilde{X} \) is an exceptional line with \( \text{Exc}(k) \cdot l_1 = 1 \); clearly \( l_1 \not\subset \text{Exc}(k) \).

Notice that \( h \) cannot be an isomorphism over \( l_1 \), otherwise we would get an exceptional line in \( \tilde{X} \), not contained in \( D \), but contracted by \( g \circ \tilde{f} \), which is impossible. Therefore by Rem. 4.2 we have \( l_1 \cap \text{dom}(h^{-1}) = \emptyset \).

Consider now the factorization \( h = \psi \circ \varphi^{-1} \) as in [4.3], where \( \varphi \) and \( \psi \) are SQMs. By Rem. 3.6 (2), \( l_1 \) is contained in the indeterminacy locus of \( \psi^{-1} \); let \( L \subset X \) be the corresponding exceptional plane, and \( C_L \subset L \) a line. Since \( \text{Exc}(k) \cdot l_1 = 1 \) in \( \tilde{X}, \) using Rem. 3.3 we see that \( D_X \cdot C_L = -1 \). Now we cannot have \( L \cap \text{dom}(\varphi) = \emptyset \) (otherwise \( h \) would be an isomorphism over \( l_1 \)), therefore \( L \) intersects the indeterminacy locus of \( \varphi \) in finitely many points and we can choose \( C_L \) disjoint from it. In the end \( C' := \varphi(C_L) \subset \tilde{X} \) is an irreducible curve with \( D \cdot C' = -1 \), and this concludes the proof.

### 4.2 Elementary rational contractions onto 3-folds

In this section we study Fano 4-folds having an elementary rational contraction onto a 3-dimensional variety. Also in this case, we first recall the result about the regular case.

**Theorem** ([Cas08, Cor. 1.2 (iii)]). Let \( X \) be a Fano 4-fold and \( f : X \to Y \) an elementary contraction with \( \dim Y = 3 \). Then \( \rho_X \leq 11 \), with equality only if \( X \cong \mathbb{P}^1 \times \mathbb{P}^1 \times S \) or \( X \cong \mathbb{P}_1 \times S \), where \( S \) is a surface.

Here we show the following.

**Theorem 4.4.** Let \( X \) be a Fano 4-fold and \( f : X \dashrightarrow Y \) an elementary rational contraction with \( \dim Y = 3 \). Then \( \rho_X \leq 11 \).
Before proving the theorem, we need some preliminary lemmas.

Lemma 4.5. Let $X$ be a Fano 4-fold and $X \rightarrow Y$ an elementary rational contraction with $\dim Y = 3$. Suppose that $g: Y \rightarrow Y_0$ is a small elementary contraction.

Then $\text{Exc}(g)$ is the disjoint union of smooth rational curves, lying in the smooth locus of $Y$, with normal bundle $\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$; in particular $K_Y \cdot \text{NE}(g) = 0$.

Proof. By Rem. 3.7 we can factor the map $X \rightarrow Y$ as $X \rightarrow \tilde{X} \rightarrow Y$, where $\tilde{X}$ is a SQM of $X$ and $f$ is a $K$-negative elementary contraction.

By standard properties of $K$-negative elementary contractions, $f$ is equidimensional except possibly at finitely many points of $Y$, where $f$ can have isolated 2-dimensional fibers. Moreover $Y$ can have at most canonical and factorial singularities at these points, and is smooth elsewhere (see Th. 3.1 and Cor. 3.3).

We have $\dim \text{Exc}(g) = 1$ and $g(\text{Exc}(g)) = \{p_1, \ldots, p_r\}$ is a finite set of points. Fix $i \in \{1, \ldots, r\}$; we show that there exists an exceptional line $l_i$ contained in $(g \circ f)^{-1}(p_i)$.

Suppose that this is not the case: then there is an open subset $U$ of $Y_0$, containing $p_i$, such that $U := (g \circ f)^{-1}(U)$ does not contain exceptional lines. In particular $(g \circ f)|_{\tilde{U}}: \tilde{U} \rightarrow U$ is a local contraction and $-K_{\tilde{U}}$ is $(g \circ f)$-ample. Moreover $(g \circ f)|_{\tilde{U}}$ factors as $q_{\tilde{U}/U} \circ f_{\tilde{U}}$, where $U_Y := g^{-1}(U)$, so that $\dim N_1(\tilde{U}/U) = 2$ (we refer the reader to [KMM87] for the notation in the relative setting).

Let $\tau$ be the extremal ray of $\text{NE}(\tilde{U}/U)$ different from $\text{NE}(f_{\tilde{U}})$. We have $f(\text{Locus}(\tau)) \subseteq \text{Exc}(g)$, so that $\dim \text{Locus}(\tau) \leq 2$, and $\tau$ is a small extremal ray. On the other hand $f$ is finite on the fibers of the contraction of $\tau$, which then have dimension at most 1. Anyway this is impossible by Th. 3.1 because $-K_{\tilde{U}} \cdot \tau > 0$.

Therefore we have an exceptional line $l_i \subset (g \circ f)^{-1}(p_i)$, and $g \circ f$ is not $K$-negative.

By flipping the $K$-positive extremal rays contracted by $g \circ f$ as in the proof of Rem. 3.7 we get a diagram:

$$
\begin{array}{ccc}
\tilde{X} & \xrightarrow{h} & \tilde{X} \\
\downarrow & & \downarrow \\
Y & \xrightarrow{g} & Y_0
\end{array}
$$

where $h$ is a composition of $K$-positive flips, and $\varphi$ is a $K$-negative contraction. In particular, as in Rem. 3.6 we see that $\tilde{X} \setminus \text{dom}(h^{-1})$ is a disjoint union of exceptional planes, and $\tilde{X} \setminus \text{dom}(h)$ a disjoint union of exceptional lines.

Since $f$ cannot contract any exceptional line, $h$ is an isomorphism on $(g \circ f)^{-1}(Y_0 \setminus \{p_1, \ldots, p_r\})$, so that $\varphi$ is equidimensional outside a finite subset of $Y_0$.

Fix $i \in \{1, \ldots, r\}$, set $S_i := (g \circ f)^{-1}(p_i)$, and let $\tilde{S}_i \subset \tilde{X}$ be its transform, so that $\tilde{S}_i \subseteq \varphi^{-1}(p_i)$. The fiber $\varphi^{-1}(p_i)$ cannot have dimension 3, because $h$ is an isomorphism in codimension 1 and $g \circ f$ has fibers of dimension at most 2. Since $S_i$ has dimension 2, $\varphi^{-1}(p_i)$ is an isolated 2-dimensional fiber of $\varphi$.

On the other hand $S_i$ contains the exceptional line $l_i$, which lies in the indeterminacy locus of $h$. We conclude that there is an exceptional plane $L_i$, lying in the indeterminacy locus of $h^{-1}$, and contained in $\varphi^{-1}(p_i)$, so that $\varphi^{-1}(p_i) \supseteq L_i \cup \tilde{S}_i$. 

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We use the classification of possible isolated 2-dimensional fibers of \( \varphi \) given in [AW97, Prop. 4.3.1] (notice that we can apply this result to \( \varphi \) using [Me99, Th. 2.6]), as in the proof of Th. [4.2]. In particular, if \( T \) is an irreducible component of \( \varphi^{-1}(p_i) \) which intersects \( L_i \) in a curve, we see that \( T \) is either \( \mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1 \), the Hirzebruch surface \( F_1 \), or the quadric cone. On the other hand \( T \cap L_i \) must be a negative curve in \( T \), therefore the only possibility is \( T \cong F_1 \).

We conclude from [AW97, Prop. 4.3.1] that \( \varphi^{-1}(p_i) = L_i \cup \hat{S}_i \), and either \( \hat{S}_i \cong \mathbb{P}^2 \) intersects \( L_i \) in one point, or \( \hat{S}_i \cong F_1 \) intersects \( L_i \) in a curve which is a line in \( L_i \), and the \((-1)\)-curve in \( \hat{S}_i \).

In particular \( \hat{S}_i \) is irreducible, therefore \( S_i \) is irreducible and \( C_i := g^{-1}(p_i) \) is an irreducible curve, because \( C_i = f(S_i) \). Moreover \( f \) cannot have 2-dimensional fibers over \( C_i \), because \( S_i = f^{-1}(C_i) \), so that \( f \) is a conic bundle over \( C_i \) and \( C_i \subset Y_{reg} \) (see Th. [3.1]). On the other hand \( f(l_i) = C_i \) and \( l_i \) cannot intersect curves of anticanonical degree 1 by Rem. [3.6] (1), therefore \( f \) is smooth over \( C_i \).

The birational map \( h^{-1} \) gives an isomorphism \( S_i \setminus l_i \cong \hat{S}_i \setminus L_i \cong \mathbb{P}^2 \setminus \{pt\} \), and under this isomorphism \( f_{|S_i \setminus l_i} \) is the projection. We conclude that \( C_i \cong \mathbb{P}^1 \), \( S_i \cong F_1 \), and \( l_i \) is the \((-1)\)-curve in \( F_1 \).

We have \( \mathcal{N}_i|_{S_i} \cong \mathcal{O}_{\mathbb{P}^1}(-1) \) and \( \mathcal{N}_i|_{X} \cong \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2} \), which imply that \( (\mathcal{N}_i|_{\hat{X}})|_{l_i} \cong \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2} \). On the other hand \( \mathcal{N}_i|_{\hat{X}} \cong (f_{|S_i})^*\mathcal{N}_{C_i/Y} \), therefore

\[
\mathcal{N}_{C_i/Y} \cong (\mathcal{N}_i|_{\hat{X}})|_{l_i} \cong \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2},
\]

and this concludes the proof.

**Lemma 4.6.** Let \( X \) be a Fano 4-fold with \( \rho_X \geq 6 \) and \( X \dasharrow Y \) an elementary rational contraction, which is not regular, with \( \dim Y = 3 \).

**Suppose that** \( g: Y \to Y_0 \) is a divisorial contraction. **Then** \( g \) is the blow-up of a smooth point of \( Y_0 \); in particular \( -K_Y \cdot NE(g) > 0 \).

**Proof.** As usual, using Rem. [3.7] we factor the map \( X \dasharrow Y \) as \( X \dasharrow \hat{X} \xrightarrow{f} Y \), where \( \hat{X} \) is a SQM of \( X \) and \( f \) is a \( K \)-negative elementary contraction. Moreover the map \( X \dasharrow \hat{X} \) is not an isomorphism. Since \( X \) has a small elementary contraction and \( \rho_X \geq 6 \), we have \( c_X \leq 2 \) by Cor. [3.12].

By Rem. [3.10] the divisor \( D := f^{-1}(\text{Exc}(g)) \) is a non-movable prime divisor in \( \hat{X} \). Moreover \( \mathcal{N}_1(D, X) = (g_*)^{-1}(\mathcal{N}_1(\text{Exc}(g), Y)) \), so that

\[
(4.7) \quad \dim \mathcal{N}_1(D, \hat{X}) = 1 + \dim \mathcal{N}_1(\text{Exc}(g), Y) \leq 3.
\]

Let \( D_X \subset X \) be the transform of \( D \); then \( \dim \mathcal{N}_1(D_X, X) \geq \rho_X - 2 \geq 4 > \dim \mathcal{N}_1(D, \hat{X}) \).

As in the proof of Prop. [1.1] this shows that \( D_X \) cannot be of type \((3, 2)\).

**Step 1: we show that** \( g \) **is of type** \((2, 0)\). **By contradiction, suppose that** \( g \) **is of type** \((2, 1)\); we show that then \( D_X \) **must be of type** \((3, 2)\), **which we have already excluded.**

Consider the (possibly empty) set of exceptional lines \( l_1, \ldots, l_r \subset \hat{X} \) such that \( (g \circ \hat{f})(l_i) = \{pt\} \). Set \( U := Y_0 \setminus (g \circ f)(l_1 \cup \cdots \cup l_r) \), \( U_Y := g^{-1}(U) \), and \( \hat{U} := f^{-1}(U_Y) \). Since

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Y₀ ∖ U is a finite set and g is of type (2, 1), we have Exc(g) ∩ U_Y ≠ ∅, and g_{|U_Y}: U_Y → U is a non-trivial local contraction.

Consider now the local contraction (g ∘ f)|Ŷ: Ŷ → U. As in the proof of Lemma 4.5 we see that there is a birational extremal ray τ of NE(Ŷ/U) such that −KŶ · τ > 0, τ ≠ NE(f|Ŷ), and the associated contraction has fibers of dimension at most 1. Then τ is of type (3, 2) by Th. 3.1 in particular Locus(τ) is a prime divisor in Ŷ. On the other hand f(Locus(τ)) ⊆ Exc(g), therefore Locus(τ) = D ∖ U.

We run a Mori program on Ÿ for D over Y₀. This means that we obtain a commutative diagram:

\[ \tilde{X} = X_0 \xrightarrow{f_0} X_1 \xrightarrow{φ_2} \cdots \xrightarrow{φ_{k-1}} X_{k-1} \xrightarrow{f_{k-1}} X_k \]

satisfying (2.12) and (2.13), where moreover for every i = 0, ..., k there is a contraction φ_i: X_i → Y₀ (with φ₀ = g ∘ f) such that σ_i ⊆ NE(φ_i) for i < k. Instead of (2.14), in the end we get that either D_k is φ_k-nef, or there exists a D_k-negative extremal ray of fiber type σ_k ⊆ NE(φ_k).

In our situation, D_k is effective, therefore in X_k there cannot be a D_k-negative extremal ray of fiber type, and D_k is φ_k-nef.

Let i ∈ {0, ..., k} be such that f_j is a flip for every j ∈ {0, ..., i − 1}, and either i < k and f_i is divisorial, or i = k; in particular X → X_i is a SQM. Then f_j is an isomorphism on φ_j⁻¹(U) for every j ∈ {0, ..., i − 1}. Indeed suppose that i > 0; then σ_0 ⊆ NE(g ∘ f) is a small extremal ray, and NE(Ŷ/U) = τ + NE(f|Ŷ), hence Locus(τ) ∖ U = ∅. Iterating this reasoning, in the end we see that U_i := φ_i⁻¹(U) is isomorphic to Ŷ, and D_i ∖ U_i is the locus of an extremal ray of type (3, 2) in NE(U_i/U).

In particular D_i is not φ_i-nef, so that i < k, and f_i: X_i → X_{i+1} is an elementary divisorial contraction with exceptional divisor D_i. We deduce that that f_i is of type (3, 2), and hence that D_X is of type (3, 2), a contradiction. This concludes the proof of Step 1.

Therefore g is of type (2, 0); in particular p := (g ∘ f)(D) = g(Exc(g)) ∈ Y₀ is a point. We apply Rem. 2.20 to g ∘ f: X → Y₀ and D, and get a commutative diagram:

\[ \tilde{X} \xrightarrow{h} \tilde{X}_1 \]

where h is a SQM which factors as a sequence of D-negative flips, and k is an elementary divisorial contraction with exceptional divisor D_i, the transform of D.

Notice that f_1 is an elementary K-negative contraction of type (4, 3), and that Ÿ \雎 D = Ÿ \雎 (f_1 ∘ k)^{-1}(p).

Step 2: when D_X is of type (3, 1) or (3, 0) \mathbb{P}^3, then X_1 is smooth and dim f_1⁻¹(p) = 1, so that Y₀ is smooth at p.
Suppose that $D_X$ is of type $(3, 1)$. Then by Cor. 3.19, $\tilde{X}_1$ is smooth and $k$ is the blow-up of a smooth curve $C \subset \tilde{X}_1$. Moreover $C$ cannot intersect irreducible curves of anticanonical degree 2, and can intersect only finitely many irreducible curves of anticanonical degree 1. Since the image of $\hat{D}$ in $Y_0$ is a point, $C$ is contained in a fiber of $f_1$.

Thus $f_1 : \tilde{X}_1 \to Y_0$ is an elementary $K$-negative contraction of a smooth 4-fold, of type $(4, 3)$. We know that $f_1$ can have isolated 2-dimensional fibers, and that $Y_0$ is smooth outside their images (see Th. 3.1). Moreover the possible 2-dimensional fibers have been classified by Kachi [Kac97] and Andreatta and Wiśniewski [AW97, Prop. 4.3.1]. It is not difficult to see that if $C$ were contained in a 2-dimensional fiber, in any case $C$ should intersect curves of anticanonical degree 2, or infinitely many curves of anticanonical degree 1, which is impossible.

Hence $C$ is contained in a 1-dimensional fiber of $f_1$, $Y_0$ is smooth in $p = f_1(C)$, and $g$ is just the blow-up of $p$.

Suppose that $D_X$ is of type $(3, 0)^Q$. Again by Cor. 3.19 $\tilde{X}_1$ is smooth and $k$ is the blow-up of a point $q \in \tilde{X}_1$. Moreover $q$ cannot belong to irreducible curves of anticanonical degree 1, and can belong at most to finitely many irreducible curves of anticanonical degree 2. Similarly to the previous case, using Th. 3.2 on isolated 2-dimensional fibers of $f_1$, we see that $q$ belongs to a 1-dimensional fiber of $f_1$, so that $p = f_1(q)$ is a smooth point of $Y_0$ and $g$ is just a blow-up.

**Step 3: the case where $D_X$ is of type $(3, 0)^Q$.**

For the rest of the proof, we assume that $D_X$ is of type $(3, 0)^Q$. By Cor. 3.19 we know that $\hat{D}$ is isomorphic to an irreducible quadric, and $q := k(\hat{D}) \in \tilde{X}_1$ is an isolated terminal and factorial singularity. Moreover we have the following properties.

**P1** The point $q$ cannot belong to irreducible curves of anticanonical degree 2, and can belong at most to finitely many irreducible curves of anticanonical degree 1.

**P2** Let $C \subset \tilde{X}_1$ be an irreducible curve such that $q \in C$ and $-K_{\tilde{X}_1} \cdot C = 1$. Then the transform $\hat{C} \subset \hat{X}$ is an exceptional line, and $\hat{D} \cdot \hat{C} = 1$.

**P3** Let $C_1, C_2 \subset \tilde{X}_1$ be distinct irreducible curves such that $-K_{\tilde{X}_1} \cdot C_1 = 1$ and the transform $\hat{C}_2 \subset \hat{X}$ of $C_2$ is an exceptional line. Then either $C_1 \cap C_2 = \emptyset$, or $C_1 \cap C_2 = \{ q \}$.

Indeed (P1) and (P2) follow directly from Cor. 3.19. For (P3), let $\hat{C}_1 \subset \hat{X}$ be the transform of $C_1$. If $q \notin C_1$, then $-K_{\hat{X}} \cdot \hat{C}_1 = 1$, so that $\hat{C}_1 \cap \hat{C}_2 = \emptyset$ by Rem. 3.6 (1), and $C_1 \cap C_2 = \emptyset$. If $q \in C_1$, then $\hat{C}_1$ is an exceptional line by (P2), therefore $\hat{C}_1 \cap \hat{C}_2 = \emptyset$ again by Rem. 3.6 and $C_1 \cap C_2 = \{ q \}$.

**Step 4:** let $T$ be an irreducible component of $f_1^{-1}(p)_{red}$ containing $q$. If $\dim T = 1$, then $-K_{\tilde{X}_1} \cdot T = 1$. If $\dim T = 2$, then $T \cong \mathbb{P}^{r}$, for some $r \geq 0$, and the fibers of the $\mathbb{P}^{1}$-bundle on $T$ have anticanonical degree 1 in $\tilde{X}_1$.

Since $f_1 : \tilde{X}_1 \to Y_0$ is an elementary contraction of type $(4, 3)$, it has fibers of dimension at most 2, and can have at most isolated 2-dimensional fibers. Moreover by Th. 3.1 the general fiber of $f_1$ is a smooth rational curve of anticanonical degree 2.
By degeneration (for instance using the Hilbert scheme), we find a connected curve $C \subset \tilde{X}_1$ containing $q$ and numerically equivalent to a general fiber of $f_1$, so that $C \subseteq f_1^{-1}(p)$. Let $C_0$ be an irreducible component of $C$ containing $q$. We have $-K_{\tilde{X}_1} \cdot C_0 \leq -K_{\tilde{X}_1} \cdot C = 2$, $-K_{\tilde{X}_1} \cdot C_0 > 0$ because $f_1$ is elementary, and $-K_{\tilde{X}_1} \cdot C_0 \in \mathbb{Z}$ because $\tilde{X}_1$ is factorial. Using (P1) we conclude that $-K_{\tilde{X}_1} \cdot C_0 = 1$. Thus if $\dim T = 1$, we have $T = C_0$ and we are done.

If $\dim T = 2$, the possibilities for $(T, (-K_{\tilde{X}_1})|_T)$ are given by Th. 3.2 (i), (ii), or (iii). However (i) is excluded by (P1). In case (ii), again by (P1) $q$ cannot be the vertex of the cone, and $q$ cannot be another point of the cone by (P2) and (P3) (just take the line through $q$ and another line). Thus we are left with (iii), which gives Step 4.

Step 5: the contraction $f_1 \circ k : \tilde{X} \rightarrow Y_0$ is not $K$-negative. If $l_1, \ldots, l_s \subset \tilde{X}$ are the exceptional lines contracted by $f_1 \circ k$, we have $l_1 \equiv \cdots \equiv l_s$, $\tilde{D} \cdot l_j = 1$, $-K_{\tilde{X}_1} \cdot k(l_j) = 1$, and $[l_j]$ belongs to an extremal ray $\sigma$ of $\text{NE}(\tilde{X})$ such that $\text{NE}(f_1 \circ k) = \text{NE}(k) + \sigma$.

We know from Step 4 that $f_1^{-1}(p)$ contains an irreducible curve of anticanonical degree 1 through $q$. By (P3), this gives an exceptional line in $\tilde{X}$ contracted by $f_1 \circ k$, so $f_1 \circ k$ is not $K$-negative. Thus $\text{NE}(f_1 \circ k) = \text{NE}(k) + \sigma$, where $\sigma$ is an extremal ray with $-K_{\tilde{X}'} - \sigma \leq 0$, and by Rem. 3.6 (2) Locus($\sigma$) is a disjoint union of numerically equivalent exceptional lines.

Fix $j \in \{1, \ldots, s\}$. The image $k(l_j) \subset \tilde{X}_1$ is an irreducible curve contained in a fiber of $f_1$, so that $-K_{\tilde{X}_1} \cdot k(l_j) > 0$, while $-K_{\tilde{X}_1} \cdot l_j = -1$. Therefore $l_j \cap \tilde{D} \neq \emptyset$ and $q \in k(l_j)$, in particular $k(l_j) \subseteq f_1^{-1}(p)$.

By Step 4, if $k(l_j)$ is an irreducible component of $f_1^{-1}(p)_{\text{red}}$, then $-K_{\tilde{X}_1} \cdot k(l_j) = 1$. Otherwise, $k(l_j)$ is contained in a 2-dimensional component $T \cong \mathbb{P}^r$ for some $r \geq 0$. By (P3) $k(l_j)$ can intersect the fibers of the $\mathbb{P}^1$-bundle on $T$ only in the point $q$. Therefore $k(l_j)$ is the fiber of the $\mathbb{P}^1$-bundle through $q$ and again $-K_{\tilde{X}_1} \cdot k(l_j) = 1$. We deduce that $\tilde{D} \cdot l_j = 1$ by (P2).

Now notice that $\ker(f_1 \circ k)_* \subseteq \ker(f_1 \circ k)_{\text{red}}$ is 2-dimensional and is generated by $[l_1]$ and $[B]$, where $B$ is a line in the quadric $\tilde{D}$. We have $\tilde{D} \cdot B = -1$, $-K_{\tilde{X}_1} \cdot B = 2$, and $-K_{\tilde{X}_1} \cdot l_1 = -1$. Thus $[\tilde{D}]$ and $[K_{\tilde{X}_1}]$ give linearly independent linear functions on $\ker(f_1 \circ k)_*$, and since $l_1, \ldots, l_s$ have the same intersection with both, we get $l_1 \equiv \cdots \equiv l_s$. Moreover $\sigma$ contains the class of at least one exceptional line, therefore $[l_j] \in \sigma$.

Step 6: we show that $h$ is just one $D$-negative and $K$-negative flip.

First of all notice that $D \subset \tilde{X}$ cannot be isomorphic to a quadric (e.g. because it has a morphism onto $\text{Exc}(g)$), so that $h$ is not an isomorphism. Let’s factor $h$ as $\tilde{X} \twoheadrightarrow \tilde{X}' \twoheadrightarrow \tilde{X}$, where $\tilde{X}'$ is a sequence of $D$-negative flips, and $h''$ is just one $D'$-negative flip, $D' \subset \tilde{X}'$ the transform of $D$. We get a commutative diagram:

$$
\begin{array}{ccc}
\tilde{X} & \xrightarrow{h} & \tilde{X} \\
\downarrow f & & \downarrow k \\
\tilde{X}' & \xrightarrow{h''} & \tilde{X} \\
\end{array}
$$

$$
\begin{array}{ccc}
Y & \xrightarrow{g} & Y_0 \\
\downarrow f_1 & & \downarrow k \\
\tilde{X}_1 & & \\
\end{array}
$$
where \( \varphi \) is a contraction.

Notice that \((h'')^{-1}\) is the flip of a small extremal ray in \( \text{NE}(f_1 \circ k) \). By Step 5 \( \text{NE}(f_1 \circ k) = \text{NE}(k) + \sigma \) and \( k \) is a divisorial contraction, therefore \((h'')^{-1}\) is the flip of \( \sigma \). Since \( K_{\tilde{X}^{'}} \cdot \sigma > 0 \), we see that \( h'' \) is the flip of a \( K \)-negative small extremal ray \( \sigma' \in \text{NE}(\varphi) \). Thus we are left to show that \( h' \) is an isomorphism.

We show that \( \varphi \) is \( K \)-negative. If not, by Rem. 3.6 (2) there exists an exceptional line \( l' \subset \tilde{X}' \) such that \( \varphi(l') = \{ pt \} \). Since \( h'' \) is a \( K \)-negative flip, by Th. 3.3 \( \tilde{X}' \setminus \text{dom}(h'') \) is a union of exceptional planes, and by Rem. 3.6 (3) we get \( l' \subset \text{dom}(h'') \). Therefore the image of \( l' \) in \( \tilde{X} \) is an exceptional line contracted by \( f_1 \circ k \), but whose class is not in \( \sigma \), which contradicts Step 5.

Hence \( \varphi \) is \( K \)-negative, and \( \text{NE}(\varphi) = \sigma' + \tau \) where \( \tau \) is a \( K \)-negative extremal ray.

Suppose by contradiction that \( h' \) is not an isomorphism. Then \( \text{NE}(\varphi) \) must contain the \( D' \)-positive small extremal ray corresponding to the last flip in the factorization of \( h' \). Since \( \text{NE}(\varphi) \) is \( \sigma' + \tau \) and \( D' \cdot \sigma < 0 \), we deduce that \( \tau \) is small, \( D' \cdot \sigma > 0 \), and \( \text{Locus}(\tau) \subset \varphi^{-1}(p) \).

In particular, \( \text{Locus}(\tau) \) is a union of exceptional planes which intersect \( D' \) (see Th. 3.3).

Let \( L \) be one of these exceptional planes.

Since also \( \text{Locus}(\sigma') \) is a union of exceptional planes, and \( \tau \neq \sigma' \), we have \( \dim(L \cap \text{Locus}(\sigma')) \leq 0 \), while \( \dim(L \cap D') \geq 1 \). Thus the transform \( \tilde{L} \subset \tilde{X} \) of \( L \) intersects \( \tilde{D} \), and is contained in \((f_1 \circ k)^{-1}(p)\). Moreover we can find curves in \( \tilde{L} \) having positive intersection with \( \tilde{D} = \text{Exc}(k) \), thus \( \tilde{L} \not\subseteq \tilde{D} \) and \( \dim k(\tilde{L}) = 2 \).

Therefore \( k(\tilde{L}) \) is an irreducible component of \( f_1^{-1}(p)_{\text{red}} \) containing \( q \), and by Step 4 we have \( k(\tilde{L}) \cong \mathbb{P}^r \) for some \( r \geq 0 \). Let \( C_1, C_2 \subset k(\tilde{L}) \) two fibers of the \( \mathbb{P}^1 \)-bundle not containing \( q \). Then their transforms \( \tilde{C}_1, \tilde{C}_2 \subset \tilde{X} \) are disjoint and have anticanonical degree 1, so they do not intersect \( \text{Locus}(\sigma) \) by Rem. 3.6 (1). This yields two disjoint curves in \( L \cong \mathbb{P}^2 \), and we have a contradiction.

**Step 7:** \( f_1^{-1}(p) \) is a one-dimensional reducible fiber of \( f_1 \), and \( s = 2 \).

Since \((g \circ f)^{-1}(p) = D \) and \( h \) is just one flip, we have \((f_1 \circ k)^{-1}(p)_{\text{red}} = \tilde{D} \cup l_1 \cup \cdots \cup l_s \) and \( f_1^{-1}(p)_{\text{red}} = k(l_1) \cup \cdots \cup k(l_s) \). We know from Step 5 that \(-K_{\tilde{X}_1} \cdot k(l_j) = 1\) for every \( j = 1, \ldots, s \), while \(-K_{\tilde{X}_1} \cdot f_1^{-1}(p) = 2 \), so that \( s \leq 2 \).

Consider now the resolution of the flip \( h \) (see Th. 3.3). We get a commutative diagram:

\[
\begin{array}{ccc}
\varphi & \rightarrow & Z \\
\downarrow & & \downarrow \\
\tilde{X} & \longrightarrow & \tilde{X}_1 \\
\downarrow f & & \downarrow k \\
Y & \longrightarrow & \tilde{Y}_0 \\
\end{array}
\]

where \( \varphi \) and \( \psi \) are the blow-ups of the indeterminacy loci of \( h \) and \( h^{-1} \) respectively. We have

\[(f_1 \circ k \circ \psi)^{-1}(p) = (g \circ f \circ \varphi)^{-1}(p) = \varphi^{-1}(D),\]
so that \( f^{-1}_1(p) \) cannot be everywhere non-reduced and \( s = 2 \).

**Step 8: we show the statement.**

We have \( \text{Locus}(\sigma) = l_1 \cup l_2 \) and \( f^{-1}_1(p) = k(l_1) \cup k(l_2) \). By the explicit description of the flip \( h \) (see Th. 3.3), and since \( \hat{D} \cdot l_1 = 1 \), we know that \( D \) is the blow-up of the (possibly singular but irreducible) quadric \( \hat{D} \) in two smooth points. Let \( L_1, L_2 \subset D \) be the exceptional planes; notice that \( L_1 \) and \( L_2 \) lie in the smooth locus of \( D \) and are Cartier divisors in \( D \). Moreover we have \( L_1 \cup L_2 = \text{Locus}(\sigma') \).

Let \( C_{L_1} \subset L_1 \) be lines; we have \( C_{L_1} \equiv C_{L_2} \) and \( D \cdot C_{L_1} = -1 \) because \( \hat{D} \cdot l_1 = 1 \) (see Rem. 3.4). Let moreover \( B \subset \hat{D} \) be a general line and \( B_0 \subset D \) its transform; recall that \( -K_{\tilde{X}} \cdot B = 2 \). Finally let \( F_0 \subset \tilde{X} \) be a general fiber of \( f \), so that \( k(h(F_0)) \) is a general fiber of \( f_1 \). We have:

\[
k(h(F_0)) \equiv 2k(l_1) \text{ in } \tilde{X}_1, \quad h(F_0) \equiv 2l_1 + 2B \text{ in } \tilde{X}, \quad \text{and} \quad F_0 \equiv 2B_0 - 2C_{L_1} \text{ in } \tilde{X}.
\]

Consider now \( f_{|D} : D \to \text{Exc}(g) \). We have \( f(L_1) = \text{Exc}(g) \), and every fiber of \( f_{|D} \) has dimension one (for instance because a 2-dimensional fiber should intersect \( L_1 \) in a curve, which is impossible). Let \( F_D \subset D \) be a fiber of \( f_{|D} \); then \( F_D \equiv F_0 \equiv 2B_0 - 2C_{L_1} \).

If \( i : D \hookrightarrow X \) is the inclusion and \( i_* : \mathcal{N}_1(D) \to \mathcal{N}_1(\tilde{X}) \) the associated push-forward of 1-cycles, we have \( \ker i_* = \mathbb{R}([C_{L_1}] - [C_{L_2}]) \) (because \( \dim \mathcal{N}_1(D) = 3 \) and \( \dim \mathcal{N}_1(D, \tilde{X}) = 2 \) by (4.7)). In particular we get:

\[
F_D \equiv_D 2B_0 - 2C_{L_1} + \lambda(C_{L_1} - C_{L_2}),
\]

where \( \lambda \in \mathbb{R} \) and \( \equiv_D \) denotes numerical equivalence in \( D \). This gives \( (L_1 \cdot F_D)_D = 2 - \lambda \) and \( (L_2 \cdot F_D)_D = \lambda \) (where \( (\cdot)_D \) denotes intersection in \( D \)), so that \( \lambda = 1 \) and \( (L_1 \cdot F_D)_D = 1 \). Therefore \( f_{|L_1} : L_1 \to \text{Exc}(g) \) is an isomorphism, \( \text{Exc}(g) \cong \mathbb{P}^2 \), and \( f(C_{L_1}) \) is a line in \( \text{Exc}(g) \). Moreover \( \text{Exc}(g) \cdot f(C_{L_1}) = D \cdot C_{L_1} = -1 \), hence \( g \) is the blow-up of a smooth point in \( Y_0 \).

---

**Proof of Th. 4.4.** By Cor. 3.8, \( Y \) has at most isolated canonical and factorial singularities, and is a Mori dream space. If \( f \) is regular, then \( \rho_X \leq 11 \) by [Cas08, Cor. 1.2 (iii)].

Suppose that \( Y \) has an elementary rational contraction of fiber type \( g : Y \dasharrow Z \). Then \( g \circ f : X \dasharrow Z \) is a quasi-elementary rational contraction (see Rem. 2.24 and Rem. 2.26), and \( \rho_X - \rho_Z = 2 \). If \( \dim Z \leq 1 \), then \( \rho_Z \leq 1 \) and \( \rho_X \leq 3 \). If instead \( \dim Z = 2 \), Prop. 4.1 yields \( \rho_Z \leq 9 \) and \( \rho_X \leq 11 \).

Therefore we can assume that \( f \) is not regular and \( Y \) has no elementary rational contraction of fiber type; let us also assume that \( \rho_X \geq 6 \).

Let \( h : Y \dasharrow \tilde{Y} \) be a SQM. Then \( h \circ f : X \dasharrow \tilde{Y} \) is an elementary rational contraction (see Rem. 2.34), so that again by Cor. 3.8 \( \tilde{Y} \) has at most isolated canonical and factorial singularities.

We notice that \( h \circ f \) cannot be regular. Indeed \( f \) is not regular over some exceptional plane \( L \subset X \), such that the lines contained in \( L \) have numerical class in some extremal ray \( \sigma \) of \( \text{NE}(X) \). If \( h \circ f \) were a morphism, it would be an elementary contraction of fiber type. In particular we would have \( \text{NE}(h \circ f) \neq \sigma \), so \( h \circ f \) should be finite on \( L \), and
\(\dim(h \circ f)(L) = 2\). Thus \(h^{-1}: \tilde{Y} \rightarrow Y\) should be regular on an open subset of \((h \circ f)(L)\), and \(f\) should be regular on an open subset of \(L\), a contradiction.

Consider an elementary contraction \(g: \tilde{Y} \rightarrow Y_0\). By our assumptions, \(g\) must be birational, therefore Lemmas 4.3 and 4.6 apply. We deduce that either \(g\) is the blow-up of a smooth point of \(Y_0\), or \(\text{Exc}(g)\) a disjoint union of smooth rational curves, lying in the smooth locus of \(\tilde{Y}\), with normal bundle \(\mathcal{O}_E(-1)^{\oplus 2}\). We can choose such a curve \(\tilde{E}\). Therefore we can choose a cone \(\tau\) with normal bundle \(\mathcal{O}_E(-1)^{\oplus 2}\) on an open subset of \((h \circ f)(L)\), whose contraction \(g: \tilde{Y} \rightarrow Y_0\) factors as a sequence of flips of small extremal rays as above, it is not difficult to see that for every irreducible curve \(C \subset Y\) such that \(C \cap \text{dom}(h) \neq \emptyset\), we have \(-K_\tilde{Y} \cdot C = -K_Y \cdot C\), where \(C \subset \tilde{Y}\) is the transform of \(C\).

By our assumptions, there exists a non- movable prime divisor \(E \subset Y\) (otherwise \(\text{Mov}(Y) = \text{Eff}(Y)\) and Cor. 2.25 would yield an elementary rational contraction of fiber type on \(Y\)). Applying Rem. 2.19, we find a SQM \(h_0: Y \rightarrow Y_0\) such that the transform \(E \subset Y_0\) is the exceptional divisor of an elementary divisorial contraction, so that \(E \cong \mathbb{P}^2\) and \(\mathcal{N}_{E/Y_0} \cong \mathcal{O}_{\mathbb{P}^2}(-1)\).

Consider now the contraction \(\varphi: Y \rightarrow T\) defined by \(\text{NE}(Y) \cap K_Y\). We show that \(\varphi\) is birational, \(i.e\). that \(-K_Y\) is big. Since \(h_0\) factors as a sequence of \(K\)-trivial flips, the map \(\tilde{\varphi} := \varphi \circ h_0^{-1}: \tilde{Y}_0 \rightarrow T\) is regular, and \(-K_{\tilde{Y}_0}\) is the pull-back of some ample Cartier divisor on \(T\). In particular \(\tilde{\varphi}\) is finite on \(\tilde{E}\), so that \(\dim(\tilde{\varphi}(\tilde{E})) = 2\). This also shows that \(\varphi\) is generically finite on \(E\).

By contradiction, if \(\varphi\) is of fiber type, then \(T = \tilde{\varphi}(\tilde{E})\) and \(\rho_T = 1\). In particular, \(\mathbb{R}_{\geq 0}[-K_Y] = \varphi^*(\text{Nef}(T))\) is a one-dimensional cone in \(\mathcal{M}_Y\). On the other hand, since \(-K_Y\) is not big, this cone must lie on the boundary of \(\text{Eff}(Y)\), and hence on the boundary of \(\text{Mov}(Y)\). Therefore we can choose a cone \(\tau \in \mathcal{M}_Y\) of dimension \(\rho_T - 1\), containing \(\mathbb{R}_{\geq 0}[-K_Y]\), and lying on the boundary of \(\text{Mov}(Y)\). The corresponding rational contraction \(g_1: Y \rightarrow Y_1\) is an almost Fano variety, and cannot be small (see Ex. 2.6) nor of fiber type (by our assumptions), therefore it is divisorial. On the other hand if \(H \subset \mathcal{N}_Y(Y)\) is the linear span of \(\tau\), we have \([K_Y] \in H = g_1^*(\mathcal{N}_Y(Y))\), and this contradicts our previous description of elementary divisorial rational contractions of \(Y\).

Therefore \(-K_Y\) is nef and big, namely \(Y\) is an almost Fano variety, and \(\varphi\) is birational. Moreover \(\dim(\text{Exc}(\varphi)) \leq 1\), because we have already shown that \(\varphi\) is generically finite on every non-movable prime divisor.

We are going to proceed similarly to the proof of [CJR08, Prop. 2.8]. Let \(\sigma_1, \ldots, \sigma_r\) be the divisorial extremal rays of \(\text{NE}(Y)\), and set \(E_i := \text{Locus}(\sigma_i)\). Then \(E_1, \ldots, E_r\) are pairwise disjoint, so that \(E_i \cdot \sigma_j = 0\) if \(i \neq j\). It is then easy to see that \(\sigma_1 + \cdots + \sigma_r\) is an \(r\)-dimensional face of \(\text{NE}(Y)\), whose contraction \(k: Y \rightarrow Y_r\) is just the blow-up of \(r\) distinct smooth points of \(Y_r\).

\footnote{See e.g. [Cas09] Rem. 4.6 for a similar statement.}
Notice that $Y_r$ has isolated canonical and factorial singularities, and is a Mori dream space by Rem. 2.8. Since $k^*(-K_{Y_r}) = -K_Y + 2(E_1 + \cdots + E_r)$, we see that $-K_{Y_r}$ is nef, and that if $C \subset Y_r$ is an irreducible curve containing some point blown-up by $k$, then $-K_{Y_r} \cdot C \geq 2$. Moreover we have:

$$\rho_X = \rho_{Y_r} + r + 1 \quad \text{and} \quad (-K_{Y_r})^3 = (-K_{Y_r})^3 - 8r,$$

in particular $(-K_{Y_r})^3 \geq (-K_{Y_r})^3 > 0$, so that $-K_{Y_r}$ is big, and $Y_r$ is again almost Fano. It is shown in [Pro03] that $(-K_{Y_r})^3 \leq 72$, which yields $r \leq 8$ and $\rho_X \leq \rho_{Y_r} + 9$.

There exists some extremal ray $\tau$ of $\text{NE}(Y_r)$ with $-K_{Y_r} \cdot \tau > 0$; let $\pi: Y_r \to Z$ be the corresponding contraction. We show that $\dim Z \leq 1$, excluding by contradiction all the other cases. This gives $\rho_{Y_r} \leq 2$ and $\rho_X \leq 11$, and concludes the proof.

Suppose first that $\pi$ is birational. If $\text{Exc}(\pi) \cap k(\text{Exc}(k)) = \emptyset$, we get a $K$-negative, birational extremal ray $\sigma'$ of $\text{NE}(Y)$ different from $\sigma_1, \ldots, \sigma_r$, a contradiction. Therefore $\text{Exc}(\pi)$ must contain some of the points blown-up by $k$.

If $\pi$ is not of type $(2,0)$, then every non-trivial fiber $F$ of $\pi$ has dimension 1, and by [AW97] Cor. 1.15 we have $F \cong \mathbb{P}^1$ and $-K_{Y_r} \cdot F = 1$. In particular, $F$ cannot contain any point blown-up by $k$, so that Exc($\pi$) $\cap k(\text{Exc}(k)) = \emptyset$, a contradiction.

If $\pi$ is of type $(2,0)$, the possibilities for Exc($\pi$) and $(-K_{Y_r})_{|\text{Exc}(\pi)}$ are given by Th. 3.2. We see that the only case where $\text{Exc}(\pi)$ is not covered by curves of anticanonical degree 1 is when $\text{Exc}(\pi) \cong \mathbb{P}^2$ and $(−K_{Y_r})_{|\text{Exc}(\pi)} = \mathcal{O}_{\mathbb{P}^2}(2)$. On the other hand, in this case the transform of $\text{Exc}(\pi)$ in $Y$ would be covered by curves of anticanonical degree zero, which contradicts the fact that Exc($\varphi$) contains no divisors.

Finally, suppose that $\dim Z = 2$. By Th. 3.1 the general fiber of $\pi$ is a smooth rational curve of anticanonical degree 2, therefore $-K_{Y_r} \cdot F = 2$ for every fiber $F$ of $\pi$.

For every $i = 1, \ldots, r$ let $F_i$ be the fiber of $\pi$ through the point $k(E_i)$. Since $k(E_i)$ cannot be contained in curves of anticanonical degree one, $F_i$ must be an integral fiber; let $C_i \subset Y$ be its transform. The formula $k^*(-K_{Y_r}) = -K_Y + 2(E_1 + \cdots + E_r)$ gives:

$$-K_Y \cdot C_i = 0, \quad E_i \cdot C_i = 1, \quad \text{and} \quad E_i \cdot C_j = 0 \quad \text{if} \quad i \neq j;$$

in particular $[C_1], \ldots, [C_r]$ are linearly independent in $\mathcal{N}_1(Y)$.

Consider now the contraction $\pi \circ k: Y \to Z$, and the face $\eta := \text{NE}(\pi \circ k) \cap K_Y^\perp$ of $\text{NE}(Y)$. The unique irreducible curves of anticanonical degree zero contracted by $\pi \circ k$ are $C_1, \ldots, C_r$, therefore $\eta = \mathbb{R}_{\geq 0}[C_1] + \cdots + \mathbb{R}_{\geq 0}[C_r]$ is an $r$-dimensional face of $\text{NE}(Y)$. This implies that each $\mathbb{R}_{\geq 0}[C_i]$ is an extremal ray of $\text{NE}(Y)$, and $C_i$ is a $(-1,-1)$-curve.

We claim that there exists a SQM $Y \dashrightarrow \hat{Y}$ whose indeterminacy locus is exactly $C_1 \cup \cdots \cup C_r$; this can be constructed inductively as follows.

Take a nef divisor $H$ in $Y$ such that $\text{NE}(Y) \cap H^\perp = \eta$, consider the flip $Y \dashrightarrow Y_1$ of $\mathbb{R}_{\geq 0}[C_1]$, and let $C'_1 \subset Y_1$ be the new $(-1,-1)$-curve. Then $[C'_1], [C_2], \ldots, [C_r]$ are linearly independent in $\mathcal{N}_1(Y_1)$, and $H$ yields a nef divisor $H_1$ on $Y_1$ such that $\text{NE}(Y_1) \cap H_1^\perp = \mathbb{R}_{\geq 0}[C'_1] + \mathbb{R}_{\geq 0}[C_2] + \cdots + \mathbb{R}_{\geq 0}[C_r]$. Hence for $i = 2, \ldots, r$ each $\mathbb{R}_{\geq 0}[C_i]$ stays a small extremal ray in $Y_1$. Now we can flip $\mathbb{R}_{\geq 0}[C_2]$, and proceed in the same way.

---

10We still denote by $C_i$ the transform of $C_i$, for $i = 2, \ldots, r$. 38
In the end we get a commutative diagram:

\[
\begin{array}{ccc}
Y & \rightarrow & \hat{Y} \\
\downarrow \kappa & & \downarrow \kappa \\
\hat{Y} & \rightarrow & \hat{Z}
\end{array}
\]

where \( \hat{k} : \hat{Y} \rightarrow Z \) is a contraction.

The transform \( \hat{E}_i \subset \hat{Y} \) of \( E_i \) is isomorphic to \( \mathbb{F}_1 \), and contains a \((-1,-1)\)-curve \( \hat{C}_i \) as the \((-1)\)-curve. If \( G_i \subset \hat{E}_i \) is a fiber of the \( \mathbb{P}^1 \)-bundle, and \( G_0 \subset \hat{Y} \) a general fiber of \( \hat{k} \), it is not difficult to see that \( G_0 \equiv G_i \), so that

\[
\text{NE}(\hat{k}) = \mathbb{R}_{\geq 0}[G_0] + \mathbb{R}_{\geq 0}[\hat{C}_1] + \cdots + \mathbb{R}_{\geq 0}[\hat{C}_r].
\]

Since \( \dim \text{NE}(\hat{k}) = r + 1 \), this implies that \( \mathbb{R}_{\geq 0}[G_0] \) is an extremal ray of \( \text{NE}(\hat{Y}) \), whose contraction is of fiber type. Thus \( Y \) has an elementary rational contraction of fiber type, which contradicts our assumptions, and this concludes the proof. ■

Proof of Th. 1.1. The statement follows from [Cas08] when \( X \) has a regular elementary contraction of fiber type (see the Introduction). The general statement follows from Cor. 3.9, Prop. 4.1, and Th. 4.4. ■

5 Fano 4-folds with \( c_X = 1 \) or \( c_X = 2 \)

In this section we show the following results, which imply Th. 1.3.

**Proposition 5.1.** Let \( X \) be a Fano 4-fold with \( \rho_X \geq 6 \) and \( c_X = 2 \). Then one of the following holds:

(i) \( \rho_X \leq 12 \), and there is a diagram

\[
X \rightarrow X_1 \rightarrow \tilde{X}_1 \rightarrow Y
\]

where \( X_1 \) is a Fano 4-fold, \( h \) is a SQM, \( \tilde{X}_1 \rightarrow Y \) is an elementary contraction and a conic bundle, and \( X \rightarrow X_1 \) is the blow-up of a smooth irreducible surface contained in \( \text{dom}(h) \);

(ii) there exists a Fano 4-fold \( Y \) and \( X \rightarrow Y \) a blow-up of two disjoint smooth irreducible surfaces.

**Proposition 5.2.** Let \( X \) be a Fano 4-fold with \( \rho_X \geq 6 \) and \( c_X = 1 \). Then one of the following holds:

(i) \( \rho_X \leq 11 \) and \( X \) has a SQM \( \tilde{X} \) with an elementary contraction of fiber type \( \tilde{X} \rightarrow Y \) which is a conic bundle;

(ii) \( X \) is obtained by blowing-up a Fano 4-fold \( Y \) in a smooth irreducible surface.
Hence \( \rho^X_D \) which is a Mori program for \(-D\), where \( f_k \) is an elementary contraction of type \((4, 3)\), finite on \( D_k \subset X_k \). Finally \( S \subset X_1 \) is contained in the open subset where the birational map \( X_1 \dashrightarrow X_k \) is an isomorphism.

If \( f_1, \ldots, f_{k-1} \) are all flips, then \( X_1 \dashrightarrow X_k \) is a SQM, and we get \( \rho_{X_1} \leq 11 \) by Th. 4.4. Hence \( \rho_X \leq 12 \) and we have (i).

For the proofs of Prop. 5.1 and 5.2, we need the following property.

**Remark 5.3.** Let \( X \) be a Fano 4-fold with \( c_X \leq 2 \) and \( \rho_X \geq 6 \). Let \( E \subset X \) be a prime divisor which is a smooth \( \mathbb{P}^1 \)-bundle, with fiber \( F \subset E \), such that \( E \cdot F = -1 \). Then \( \mathbb{R}_{\geq 0}[F] \) is an extremal ray of type \((3, 2)\), and it is the unique \( E \)-negative extremal ray of \( \text{NE}(X) \).

**Proof.** Let \( \sigma_1, \ldots, \sigma_h \) be the \( E \)-negative extremal rays of \( \text{NE}(X) \) (notice that \( h \geq 1 \), because \( E \) is not nef). Fix \( i \in \{1, \ldots, h\} \). We have \( \text{Locus}(\sigma_i) \subseteq E \).

If \( \sigma_i \) is of type \((3, 0)\) or \((3, 1)\), then \( \dim \mathcal{N}_1(E, X) \leq 2 \), a contradiction because \( c_X \leq 2 \) and \( \rho_X \geq 6 \). If \( \sigma_i \) is small, then \( \text{Locus}(\sigma_i) \) is a union of exceptional planes (by [Kaw89]), which must intersect every fiber of the \( \mathbb{P}^1 \)-bundle structure on \( E \). This yields \( \dim \mathcal{N}_1(E, X) = 2 \), again a contradiction.

Therefore \( \sigma_i \) is of type \((3, 2)\), \( E = \text{Locus}(\sigma_i) \), and \( (-K_X + E) \cdot \sigma_i = 0 \). This shows that \(-K_X + E \) is nef, and \( \tau := \sigma_1 + \cdots + \sigma_h = (-K_X + E) \cap \text{NE}(X) \) is a face containing \([F]\).

If \( \dim \tau > 1 \), any 2-dimensional face of \( \tau \) yields a contraction of \( X \) onto \( Z \) with \( \rho_X - \rho_Z = 2 \), sending \( E \) to a point or to a curve. This implies that \( \dim \mathcal{N}_1(E, X) \leq 3 \), again a contradiction. Thus \( h = 1 \) and \( \sigma_1 = \mathbb{R}_{\geq 0}[F] \) ■

**Proof of Prop. 5.1.** Let \( D \subset X \) be a prime divisor with \( \text{codim} \mathcal{N}_1(D, X) = 2 \); we apply [Cas11] Prop. 2.5 to \( D \).

Suppose first that we get two disjoint prime divisors \( E_1, E_2 \) which are smooth \( \mathbb{P}^1 \)-bundles, with fibers \( F_i \subset E_i \), such that \( E_1 \cdot F_i = -1 \), \( D \cdot F_i > 0 \), and \([F_i] \notin \mathcal{N}_1(D, X)\), for \( i = 1, 2 \) (that is, \( s = 2 \) in [Cas11] Prop. 2.5).

Fix \( i \in \{1, 2\} \). By Rem. 5.2 \( \mathbb{R}_{\geq 0}[F_i] \) is an extremal ray of type \((3, 2)\), and it is the unique \( E_i \)-negative extremal ray of \( \text{NE}(X) \). If \( F_0 \) is a fiber of the associated contraction, then \( F_0 \cap D \neq \emptyset \) (for \( D \cdot F_i > 0 \)), and \( \dim F_0 \cap D = 0 \) (for \([F_i] \notin \mathcal{N}_1(D, X)\)). Therefore \( \dim F_0 = 1 \), and the ray \( \mathbb{R}_{\geq 0}[F_i] \) is of type \((3, 2)^{sm}\).

This also shows that \(-K_X + E_1 + E_2 \) is nef, and \((-K_X + E_1 + E_2) \cap \text{NE}(X) = \mathbb{R}_{\geq 0}[F_1] + \mathbb{R}_{\geq 0}[F_2] \) is a face of \( \text{NE}(X) \). The associated contraction \( \varphi: X \to Y \) is the smooth blow-up of two disjoint irreducible surfaces. Moreover \( Y \) is Fano, because \( \varphi^*(-K_Y) = -K_X + E_1 + E_2 \), therefore we have (ii).

Suppose now that [Cas11] Prop. 2.5 applied to \( D \) gives just one prime divisor \( E_1 \). As in the previous case, we see that \( E_1 \) is the exceptional divisor of the blow-up \( f_0: X \to X_1 \) of a Fano 4-fold \( X_1 \) along a smooth surface \( S = f_0(E_1) \). Moreover we are in the situation of [Cas11] Lemma 2.8, and we have a sequence:

\[
X = X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \rightarrow \cdots \rightarrow X_{k-1} \xrightarrow{f_{k-1}} X_k \xrightarrow{f_k} Y
\]

which is a Mori program for \(-D\), where \( f_k \) is an elementary contraction of type \((4, 3)\), finite on \( D_k \subset X_k \). Finally \( S \subset X_1 \) is contained in the open subset where the birational map \( X_1 \dashrightarrow X_k \) is an isomorphism.

If \( f_1, \ldots, f_{k-1} \) are all flips, then \( X_1 \dashrightarrow X_k \) is a SQM, and we get \( \rho_{X_1} \leq 11 \) by Th. 4.4. Hence \( \rho_X \leq 12 \) and we have (i).
Suppose now that $f_1, \ldots, f_{k-1}$ are not all flips. Since the map $X_1 \to X_k$ is an isomorphism on $S$, we can replace the sequence above by:

$$X = X_0 \to X_1 \to X_2 \to \cdots \to X_{k-1} \to X_k \to Y,$$

where $g_{k-1}: X_{k-1} \to X_k$ is the blow-up of the image of $S$, and $g_0, \ldots, g_{k-2}$ are not all flips. Notice that the birational map $X \to X'_{k-1}$ gives an isomorphism between $E_1$ and $\text{Exc}(g_{k-1})$.

Let $i \in \{0, \ldots, k-2\}$ be the first index such that $g_i$ is a divisorial contraction. We have:

$$X \xrightarrow{\varphi} X_i \to X_i' \to X_k \to Y,$$

where $\varphi$ is a SQM. Since $\rho_X \geq 6$, Cor. 3.11 applies to $g_i$.

Let $E_2 \subset X$ be the transform of $\text{Exc}(g_i)$, and $p \in E_2$ a point which does not belong to any exceptional plane. Notice that $E_1 \cap E_2 = \emptyset$.

Proceeding as in [Cas11] proof of Lemma 2.8], we construct a curve $C \subset X$ with the following properties:

1. $p \in C$ and $C$ is numerically equivalent to a general fiber $C_0$ of the map $X \to Y$, so that $-K_X \cdot C = 2$ and $E_2 \cdot C = 0$;
2. $C = C' \cup \tilde{F}$, where $\tilde{F}$ is the transform of an integral fiber $F \subset X_k$ of $f_k$, $E_2 \cdot \tilde{F} > 0$, and $\tilde{F} \not\subset E_2$.

Let $\tilde{F}_i \subset X_i$ and $\tilde{F}_{i+1} \subset X_{i+1}'$ be the transforms of $F$. We have $-K_{X_i'} \cdot \tilde{F}_i \leq -K_{X_{i+1}'} \cdot \tilde{F}_{i+1} \leq -K_X \cdot F = 2$ by [Cas09] Lemma 3.8, while $-K_X \cdot \tilde{F} = 1$, therefore by Lemma 3.6 (1) we have two possibilities:

- $(a)$ $-K_{X_i'} \cdot \tilde{F}_i = 2$ and $\tilde{F}$ intersects a unique exceptional plane $L \subset X \setminus \text{dom}(\varphi)$;
- $(b)$ $-K_{X_i'} \cdot \tilde{F}_i = 1$ and $\varphi$ is an isomorphism on $\tilde{F}$.

We assume first that we are in case $(a)$, and show that this gives a contradiction. Since $-K_{X_i'} \cdot \tilde{F}_i = 2 = -K_{X_k} \cdot F$, by [Cas09] Lemma 3.8 the birational map $X_i' \to X_k$ is an isomorphism on $\tilde{F}_i$; recall that the image of $\tilde{F}_i$ in $X_k$ is an integral fiber $F$ of $f_k$. Thus $\tilde{F}_i \cap \text{Exc}(g_i) = \emptyset$ and $\tilde{F}_i$ is a proper fiber of the map $X_i' \to Y$.

On the other hand $\tilde{F} \cap E_2 \neq \emptyset$ by (2), therefore $\tilde{F}$ intersects $E_2$ along the indeterminacy locus of $\varphi$, and we get $\tilde{F} \cap E_2 \subset L$.

In $X_i'$ we have $\varphi(C_0) \equiv \tilde{F}_i$ (recall that $C_0$ is a general fiber of $X \to Y$), hence $C_0 \equiv \tilde{F} + C_L$, where $C_L \subset L$ is a line. But we also have $C_0 \equiv C = \tilde{F} + C'$, so that $C' \equiv C_L$.

This implies that $C'$ is contained in an exceptional plane too. Indeed by taking a general very ample divisor $H \subset X'$, its transform $\tilde{H} \subset X$ is a movable divisor whose base locus is $X \setminus \text{dom}(\varphi)$, and $\tilde{H} \cdot C' = \tilde{H} \cdot C_L < 0$.

On the other hand we have $p \in C \cap E_2 = C' \cup (\tilde{F} \cap E_2)$, so $p$ must belong to some exceptional plane, which contradicts our choice of $p$. 

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Hence we are in case (b). Using (2) we see that \( \tilde{F}_i \cdot \text{Exc}(g_i) = \tilde{F} \cdot E_2 > 0 \), and \( \tilde{F}_i \) is not contained in \( \text{Exc}(g_i) \). Therefore:

\[
\tilde{F}_{i+1} \cap g_i(\text{Exc}(g_i)) \neq \emptyset, \quad \tilde{F}_{i+1} \not\subseteq g_i(\text{Exc}(g_i)), \quad \text{and} \quad -K_{X'_{i+1}} \cdot \tilde{F}_{i+1} \leq 2.
\]

Then Cor. 3.19 yields that \( g_i \) must be of type \((3,2)\), \( \varphi \) gives an isomorphism between \( E_2 \) and \( \text{Exc}(g_i) \), and \( E_2 \) does not contain any exceptional plane.

This implies that \( -K_{X'_{i+1}} \cdot \tilde{F}_{i+1} = 2 = -K_{X_k} \cdot F \), and again by [Cas09, Lemma 3.8] the birational map \( X'_{i+1} \dashrightarrow X_k \) is an isomorphism between \( \tilde{F}_{i+1} \) and \( F \subseteq X_k \), so that \( \tilde{F}_{i+1} \) is a fiber of the map \( X'_{i+1} \dashrightarrow Y \).

Since \( E_2 \) does not contain exceptional planes, the choice of \( p \in E_2 \) was arbitrary. We deduce that \( g_i(\text{Exc}(g_i)) \) is contained in the open subset where the map \( X'_{i+1} \dashrightarrow Y \) is regular and proper.

Finally \( g_i \) cannot have fibers of dimension 2, otherwise the rational map \( X'_{i+1} \dashrightarrow Y \) over an open subset yields a \( K \)-negative local contraction of a smooth variety having a 2-dimensional fiber with a one-dimensional component, which is impossible, see [AW97, Lemma 2.12].

Therefore \( g_i \) is of type \((3,2)^{sm} \), and \( E_2 \) is a smooth \( \mathbb{P}^1 \)-bundle with fiber \( F_2 \subseteq E_2 \) such that \( E_2 \cdot F_2 = -1 \) and \( E_1 \cap E_2 = \emptyset \). Now proceeding as in the first part of the proof we show that we are in (ii).

The proof of Prop. 5.2 is very similar to that of Prop. 5.1.

References

[ADHL10] I. Arzhantsev, U. Derenthal, J. Hausen, and A. Laface, Cox rings, preprint arXiv:1003.4229v2, 2010.

[Ara10] C. Araujo, The cone of pseudo-effective divisors of log varieties after Batyrev, Math. Z. 264 (2010), 179–193.

[AW97] M. Andreatta and J. A. Wiśniewski, A view on contractions of higher dimensional varieties, Algebraic Geometry - Santa Cruz 1995, Proc. Symp. Pure Math., vol. 62, 1997, pp. 153–183.

[Bar10] S. Barkowski, The cone of moving curves of a smooth Fano three- or fourfold, Manuscripta Math. 111 (2010), 305–322.

[Bat99] V. V. Batyrev, On the classification of toric Fano 4-folds, J. Math. Sci. (New York) 94 (1999), 1021–1050.

[BCHM10] C. Birkar, P. Cascini, C. D. Hacon, and J. M. McKernan, Existence of minimal models for varieties of log general type, J. Amer. Math. Soc. 23 (2010), 405–468.

[BDPP04] S. Boucksom, J.-P. Demailly, M. Paun, and T. Peternell, The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension, preprint arxiv:math.AG/0405285, 2004.

[Bel86] M. C. Beltrametti, Contractions of non numerically effective extremal rays in dimension 4, Proceedings of the Conference in Algebraic Geometry (Berlin, 1985), Teubner-Texte Math., vol. 92, 1986, pp. 24–37.

[Bel87] M. C. Beltrametti, On d-folds whose canonical bundle is not numerically effective, according to Mori and Kawamata, Ann. Mat. Pura Appl. (4) 147 (1987), 151–172.
C. Casagrande, Quasi-elementary contractions of Fano manifolds, Compos. Math. 144 (2008), 1429–1460.

C. Casagrande, On Fano manifolds with a birational contraction sending a divisor to a curve, Michigan Math. J. 58 (2009), 783–805.

C. Casagrande, On the Picard number of divisors in Fano manifolds, preprint arXiv:0905.3239v4, 2011, to appear in Ann. Sci. Éc. Norm. Supér.

C. Casagrande, P. Jahnke, and I. Radloff, On the Picard number of almost Fano threefolds with pseudo-index > 1, Internat. J. Math. 19 (2008), 173–191.

H. Conrads, Weighted projective spaces and reflexive simplices, Manuscripta Math. 107 (2000), 215–227.

T. Fujita, Projective varieties of Δ-genus one, Algebraic and Topological Theories (Kinosaki, 1984), 1986, pp. 149–175.

C. Casagrande, Small contractions of four dimensional algebraic manifolds, Math. Ann. 284 (1989), 595–600.

Y. Kawamata, Classification of extremal contractions from smooth fourfolds of (3,1)-type, Proc. Amer. Math. Soc. 127 (1999), 315–321.