Novel Adaptive Algorithms for Estimating Betweenness, Coverage and k-path Centralities

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ABSTRACT
An important index widely used to analyze social and information networks is betweenness centrality. In this paper, first given a directed network $G$ and a vertex $r \in V(G)$, we present a novel adaptive algorithm for estimating betweenness score of $r$. Our algorithm first computes two subsets of the vertex set of $G$, called $RF(r)$ and $RT(r)$, that define the sample spaces of the start-points and the end-points of the samples. Then, it adaptively samples from $RF(r)$ and $RT(r)$ and stops as soon as some condition is satisfied. The stopping condition depends on the samples met so far, $|RF(r)|$ and $|RT(r)|$. We show that compared to the well-known existing methods, our algorithm gives a more efficient $(\lambda, \delta)$-approximation.

Then, we propose a novel algorithm for estimating $k$-path centrality of $r$. Our algorithm is based on computing two sets $RF(r)$ and $D(r)$. While $RF(r)$ defines the sample space of the source vertices of the sampled paths, $D(r)$ defines the sample space of the other vertices of the paths. We show that in order to give a $(\lambda, \delta)$-approximation of the $k$-path score of $r$, our algorithm requires considerably less samples. Moreover, it processes each sample faster and with less memory. Finally, we empirically evaluate our proposed algorithms and show their superior performance. Also, we show that they can be used to efficiently compute centrality scores of a set of vertices.

CCS CONCEPTS
• Mathematics of computing → Graph algorithms;

KEYWORDS
Social network analysis, directed graphs, betweenness centrality, coverage centrality, $k$-path centrality, approximate algorithm, adaptive algorithm

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1 INTRODUCTION
Graphs (in particular directed graphs) are a widely used tool for modeling data in different domains, including social networks, information networks, road networks and the world wide web. Centrality is a structural property of vertices or edges in the graph that indicates their importance. For example, it determines the importance of a person within a social network, or a road within a road network. There are several centrality notions in the literature, including betweenness centrality [11], coverage centrality [26] and $k$-path centrality [3].

Although there exist polynomial time algorithms for computing these indices, the algorithms are expensive in practice. However, there are observations that may improve computation of centrality indices in practice. In several applications it is sufficient to compute centrality score of only one or a few vertices. For instance, the index might be computed for only core vertices of communities in social/information networks [24] or for only hubs in communication networks. Another example, discussed in [1, 2], is handling cascading failures. It has been shown that the failure of a vertex with a higher betweenness score may cause greater collapse of the network [23]. Therefore, failed vertices should be recovered in the order of their betweenness scores. This means it is required to compute betweenness scores of only failed vertices, that usually form a very small subset of vertices. Note that these vertices are not necessarily those that have the highest betweenness scores, hence, algorithms that identify the top-$k$ vertices [21] are not applicable. Another example where in a road network it is required to compute betweenness ($k$-path) score of only one vertex, is discussed in [11]. In this paper, we exploit these practical observations to develop more effective algorithms.

In recent years, several approximate algorithms have been proposed in the literature to estimate betweenness/coverage/$k$-path centralities. Some of them are based on sampling shortest paths [6, 21] and some others are based on sampling source vertices (or source-destination vertices) [5, 11]. Very recently, a technique has been proposed that significantly improves the efficiency of the source sampler algorithms in directed graphs. In this technique, for a given vertex $r$, first the set $RF(r)$ of vertices that have a non-zero contribution (dependency score) to the betweenness score of $r$, is computed. Then, this set is used for sampling source vertices [12]. $RF(r)$, which can be computed very efficiently, is usually much smaller than the vertex set of the graph, hence, source vertex sampling can be done more effectively. However, the error bounds presented in [12] are not adaptive and are not of practical convenience (see Section 6).

In the current paper, to estimate betweenness centrality, we further improve this technique by not only restricting the source vertices to $RF(r)$, but also finding a set $RT(r)$ of vertices that can be a destination of a shortest path that passes over $r$. Given a directed graph $G$ and a vertex $r$ of $G$, our algorithm first computes two subsets of the vertex set of $G$, called $RF(r)$ and $RT(r)$. These subsets
can be computed very effectively and define the sample spaces of the start-points and the end-points of the samples (i.e., shortest paths). Then, it adaptively samples from $RF(r)$ and $RT(r)$ and stops as soon as some condition is satisfied. The stopping condition depends on the samples met so far, $|RF(r)|$ and $|RT(r)|$. We theoretically analyze our algorithm and show that in order to estimate betweenness of $r$ with a maximum error $\lambda$ with probability $1-\delta$, it requires considerably less samples than the well-known existing algorithms. In fact, our algorithm tries to collect the advantages of all existing methods. On the one hand, unlike [12], it is adaptive and its error bounds are of practical convenience. On the other hand, unlike algorithms such as [6, 21], it uses the sets $RF(r)$ and $RT(r)$ to prune the search space, that gives significant theoretical and empirical improvements. We also discuss how our algorithm can be revised to compute coverage centrality of $r$.

Then, we propose a novel adaptive algorithm for estimating $k$-path centrality of $r$. Our algorithm is based on computing two sets $RF(r)$ and $D(r)$. While $RF(r)$ defines the sample space of the source vertices of the sampled paths, $D(r)$ defines the sample space of the other vertices of the paths. We show that in order to give a $(\lambda, \delta)$-approximation of the $k$-path score of $r$, our algorithm requires considerably less samples. Moreover, it can process each sampled path faster and with less memory. We also propose a method to determine the number of samples adaptively and based on the samples met so far and $|RF(r)|$. In the end, we evaluate the empirical efficiency of our centrality estimation algorithms over several real-world datasets. We show that in practice, while our betweenness estimation algorithm is usually faster than the well-known existing algorithms, it generates considerably more accurate results. Furthermore, we show that while our algorithm is intuitively designed to estimate betweenness score of only one vertex, it can also be used to effectively compute betweenness scores of a set of vertices. Finally, we show that our algorithm for $k$-path centrality is considerably more accurate than existing methods.

The rest of this paper is organized as follows. In Section 2, we introduce preliminaries and necessary definitions used in the paper. In Section 3, we give an overview on related work. In Section 4, we introduce our betweenness/coverage estimation algorithm and theoretically analyze it. In Section 5, we present our $k$-path centrality estimation algorithm and its analysis. In Section 6, we empirically evaluate our proposed algorithms and show their high efficiency, compared to well-known existing algorithms. Finally, the paper is concluded in Section 7.

2 PRELIMINARIES

We assume that the reader is familiar with basic concepts in graph theory. Throughout the paper, $G$ refers to a directed graph. For simplicity, we assume that $G$ is a connected and loop-free graph without multi-edges. Throughout the paper, we assume that $G$ is an unweighted graph, unless it is explicitly mentioned that $G$ is weighted. $V(G)$ and $E(G)$ refer to the set of vertices and the set of edges of $G$, respectively. For a vertex $v \in V(G)$, the number of head ends adjacent to $v$ is called its in degree and the number of tail ends adjacent to $v$ is called its out degree. For a vertex $v$, $N(v)$ we denote the set of outgoing neighbors of $v$.

A shortest path from $u \in V(G)$ to $v \in V(G)$ is a path whose length is minimum, among all paths from $u$ to $v$. For two vertices $u, v \in V(G)$, if $G$ is unweighted, by $d(u, v)$ we denote the length (the number of edges) of a shortest path connecting $u$ to $v$. If $G$ is weighted, $d(u, v)$ denotes the sum of the weights of the edges of a shortest path connecting $u$ to $v$. By definition, $d(u, u) = 0$. Note that in directed graphs, $d(u, v)$ is not necessarily equal to $d(v, u)$.

The vertex diameter of $G$, denoted by $VD(G)$, is defined as the number of vertices of the longest shortest path of the graph. For $s, t \in V(G)$, $\sigma_s(t)$ denotes the number of shortest paths between $s$ and $t$, and $\sigma_s(t)(v)$ denotes the number of shortest paths between $s$ and $t$ that also pass through $v$. Betweenness centrality of a vertex $r$ is defined as:

$$bc(r) = \frac{1}{|V(G)| \cdot (|V(G)| - 1)} \sum_{s,t \in V(G)} \frac{\sigma_s(t)(r)}{\sigma_s(t)}.$$

Coverage centrality of $r$ is defined as follows [26]:

$$cc(v) = \left\{ \frac{|\{(s, t) \in V(G) \times V(G) : v \text{ is on a shortest path from } s \text{ to } t\}|}{|V(G)| \cdot (|V(G)| - 1)} \right\}.$$

Let $p_{s,t}$ denote a simple path $p$ that starts with vertex $s$ and has $l$ edges. Let also $w_0, u_1, \ldots, u$ denote the vertices in the order they appear in $p_{s,t}$, with $s = w_0$. We define $W(p_{s,t})$ as $|P_{s,t}| = 1 + |N(u_{i-1}) \cup \{s, u_1, u_2, \ldots, u_{i-2}\}|$.

We say $\chi(v \in p_{s,t})$ returns 1 if $v$ lies on $p_{s,t}$, and 0 otherwise. For a vertex $r$ in an unweighted graph $G$, its $k$-path centrality is defined as follows [3]:

$$pc(r) = \frac{1}{|V(G)|} \sum_{s \in V(G)} \sum_{1 \leq l \leq k} \chi\left[ r \in p_{s,t} \right] W(p_{s,t}).$$

The $k$-path centrality of $r$ in a weighted graph is defined in a similar way [3]. We here omit it due to lack of space.

3 RELATED WORK

Brandes [7] introduced an efficient algorithm for computing betweenness centrality of all vertices, which is performed respectively in $O(|V(G)||E(G)|)$ and $O(|V(G)||E(G)| + |V(G)| \log |V(G)|)$ times for unweighted and weighted networks with positive weights. The authors of [9] presented the compression and shattering techniques to improve the efficiency of Brandes’s algorithm. In [14] and [10], the authors respectively studied group betweenness centrality and co-betweenness centrality, two natural extensions of betweenness centrality to sets of vertices. In [8] and [5], the authors proposed approximate algorithms based on selecting some source vertices and computing dependency scores of them on the other vertices in the graph and scaling the results. In the algorithm of Geisberger et.al. [15], the method for aggregating dependency scores changes so that vertices do not profit from being near the selected source vertices. Chehreghani [11] proposed a non-uniform sampler for unbiased estimation of the betweenness score of a vertex. Riondato and Kornaropoulos [21] presented shortest path samplers for estimating betweenness centrality of all vertices or the $k$ vertices that have the highest betweenness scores. Riondato and Upfal [22] introduced the ABRA algorithm and used Rademacher average to determine the number of required samples. Recently, Borassi and Natale [6] presented KADABRA, which is adaptive.

The original definition presented in [3] does not include the normalization part $\frac{1}{|V(G)| \cdot (|V(G)| - 1)}$. Here, due to consistency with the definitions of betweenness and coverage centralities, we use this normalized definition.
and uses bb-BFS to sample shortest paths. Finally, in [12] the authors presented exact and approximate algorithms for computing betweenness centrality of one vertex or a small set of vertices in directed graphs. As discussed earlier, our betweenness estimation algorithm tries to have the advantages of both algorithms presented in [6] and [12]. Yoshida [26] presented an algorithm that uses $\Theta(\log |V(G)|/\lambda^2)$ samples to estimate coverage centrality of a vertex within an additive error $\lambda$ with probability $1 - 1/n^3$. Alahakoon et al. [3] introduced $k$-path centrality of a vertex and proposed the RA-kpath algorithm to estimate it.Mahmoody et al. [19] showed that $k$-path centrality admits a hyper-edge sampler and proposed an algorithm that picks a source vertex uniformly at random, and generates a random simple path of length at most $k$, and outputs the generated simple path as a hyper-edge [19]. The key difference between our algorithm and these two algorithms is that our algorithm restricts the sample spaces of the source vertices and the other vertices of the paths to the sets $\mathcal{RF}$ and $\mathcal{D}$, respectively. This considerably improves the error guarantee and empirical efficiency of our algorithm.

4 BETWEENNESS CENTRALITY

In this section, we present our adaptive approximate algorithm for estimating betweenness centrality of a given vertex $r$ in a directed graph $G$. We start by introducing the sets $\mathcal{RF}(r)$ and $\mathcal{RT}(r)$, that are used to define the sample spaces of start-points and end-points of shortest paths.

**Definition 4.1.** Let $G$ be a directed graph and $r, s \in V(G)$. We say $r$ is reachable from $s$ if there is a (directed) path from $s$ to $r$. The set of vertices that $r$ is reachable from them is denoted by $\mathcal{RF}(r)$.

**Definition 4.2.** Let $G$ be a directed graph and $r, t \in V(G)$. We say $r$ is reachable to $t$ if there is a (directed) path from $r$ to $t$. The set of vertices that $r$ is reachable to them is denoted by $\mathcal{RT}(r)$.

For a given vertex $r \in V(G)$, $\mathcal{RF}(r)$ and $\mathcal{RT}(r)$ can be efficiently computed, using reverse graph.

**Definition 4.3.** Let $G$ be a directed graph. Reverse graph of $G$, denoted by $\overline{G}$, is a directed graph such that: (i) $\overline{V(G)} = V(G)$, and (ii) $(u, v) \in E(\overline{G})$ if and only if $(v, u) \in E(G)$ [12]. To compute $\mathcal{RF}(r)$, we act as follows [12]: (i) first, by flipping the direction of the edges of $G$, $\overline{G}$ is constructed, (ii) then, if $G$ is weighted, the weights of the edges are ignored, (iii) finally, a breadth first search (BFS) or a depth-first search (DFS) on $\overline{G}$ starting from $r$ is performed. All the vertices that are met during the BFS (or DFS), except $r$, are added to $\mathcal{RF}(r)$. To compute $\mathcal{RT}(r)$, we act as follows: (i) if $G$ is weighted, the weights of the edges are ignored, (ii) a breadth-first search (BFS) or a depth-first search (DFS) on $G$ starting from $r$ is performed. All the vertices that are met during the BFS (or DFS), except $r$, are added to $\mathcal{RT}(r)$. Both $\mathcal{RF}(r)$ and $\mathcal{RT}(r)$ can be computed in $O(|E(G)|)$ time, for both unweighted and weighted graphs. Furthermore, we have the following lemma.

**Lemma 4.4.** Given a directed graph $G$ and $r \in V(G)$, the exact betweenness score of $r$ can be computed as follows: $bc(r) = \sum_{s \in \mathcal{RF}(r)} \sum_{t \in \mathcal{RT}(r)} \frac{1}{\binom{|V(G)|}{|V(G)|-1}}$. Let $\sigma_{rt}$ be the value of $\tau$ that Algorithm 1 finds in the end of the iterations done in Lines 8-16. The value of $\sigma_{rt}$, i.e., the stopping condition of the sampling part of Algorithm 1, is determined adaptively and depends on the samples observed so far, $|\mathcal{RF}(r)|$, $|\mathcal{RT}(r)|$. The dependence on $|\mathcal{RF}(r)|$ and $|\mathcal{RT}(r)|$ can be expressed in terms of the parameter $\alpha(r)$ defined as follows: $\alpha(r) = \frac{|\mathcal{RF}(r)|}{|\overline{V(G)}|(|V(G)|-1)}$.

\[ \alpha(r) = \frac{|\mathcal{RF}(r)|}{|\overline{V(G)}|(|V(G)|-1)} \]

Algorithm 1 High level pseudo code of the ABAD algorithm.

1. **Input.** A directed network $G$, a vertex $r \in V(G)$, and real numbers $\lambda, \delta \in (0, 1)$.
2. **Output.** Betweenness score of $r$.
3. if in degree of $r$ is 0 or out degree of $r$ is 0 then
   return 0.
4. end if
5. $c \leftarrow 0$, $\sigma r \leftarrow 0$.
6. $\mathcal{RF} \leftarrow \mathcal{RF}(r)$, $\mathcal{RT} \leftarrow \mathcal{RT}(r)$.
7. while not Stop do
   8. $c \leftarrow 0$.
   9. Pick up $s \in RF$ and $t \in RT$, both uniformly at random.
   10. Pick up a shortest path $\pi$ from $s$ to $t$ uniformly at random.
   11. if $r$ is on $\pi$ then
      12. $c_{tr} \leftarrow |\mathcal{RF}(r)|/|\overline{V(G)}|(|V(G)|-1)$.
   13. end if
   14. $c' \leftarrow c + c_{tr}$, $\tau \leftarrow \tau + 1$.
   15. end while
   16. return $c$.\footnote{ABAD is an abbreviation for Adaptive Betweenness Approximation algorithm for Directed graphs.}
In Theorem 4.8, we discuss the method Stop, that defines the stopping condition. Before that, in Lemmas 4.5 and 4.6, we investigate the expected value and variance of $c_r$’s, that are used by Theorem 4.8.

**Lemma 4.5.** In Algorithm 1, we have $E[c] = bc(r)$.

**Proof.** For each iteration $\tau$ in the loop in Lines 8-16 of Algorithm 1, we have:

$$E[c_\tau] = \sum_{s \in RF} \sum_{r \in RT} \left( \frac{\sigma_s(r)}{\sigma_s[RF||RT]} \right) \frac{|RF||RT|}{|V(G)|(|V(G)| - 1)} = bc(r).$$

Then, $c$ is the average of $c_r$’s. Therefore $E[c] = \frac{\sum_{\tau} E[c_\tau]}{\tau} = bc(r)$.

**Lemma 4.6.** In Algorithm 1, for each $c_\tau$ we have $\text{Var}[c_\tau] = a(r)bc(r) - bc(r)^2$.

**Proof.** We have:

$$\text{Var}[c_\tau] = E[c_\tau^2] - E[c_\tau]^2 = \sum_{s \in RF} \sum_{r \in RT} \frac{\sigma_s(r)}{\sigma_s[RF||RT]} \frac{|RF|^2||RT|^2}{|V(G)|^2(|V(G)| - 1)^2} - bc(r)^2 = a(r)bc(r) - bc(r)^2.$$

□

**Theorem 4.7.** (Theorem 6.1 of [13].) Let $M$ be a constant and $X$ be a martingale, associated with a filter $F$, that satisfies the followings:

(i) $\text{Var}[X_t | F] \leq \sigma_t^2$, for $1 \leq t \leq n$, and (ii) $X_t - X_{t-1} \leq M$, for $1 \leq t \leq n$. We have

$$\Pr[X - \mathbb{E}[X] \geq \lambda] \leq \exp \left( -\frac{\lambda^2}{2 \sum_{\tau=1}^n \sigma_\tau^2 + \frac{4M^2}{3}} \right).$$

(1)

Let $\lambda$ and $\alpha$ be real numbers in $(0, 1)$ and assume that $\omega$ is defined as $\omega = \left( \log_2(VD(G) - 2) + 1 + \log \left( \frac{1}{\tau} \right) \right)$, where the constant $C$ is an universal positive constant and it is estimated to be approximately $0.5$ [18]. By the results in [21], in Algorithm 1 and after $\omega$ samples (i.e., $\tau = \omega$), the estimation error of betweenness score of $r$ will be bounded by $\lambda$, with probability $1 - \delta/2$. For $\tau < \omega$, we have the following theorem.

**Theorem 4.8.** In Algorithm 1, for real values $\delta_1, \delta_2 \in (0, 1)$ and the value $\omega$ defined above, after $\tau < \omega$ iterations of the loop in Lines 8-16, we have

$$\Pr[bc(r) - c \leq -A] \leq \delta_1$$

and

$$\Pr[bc(r) - c \geq B] \leq \delta_2.$$  

(2)

(3)

where

$$A = \frac{1}{\tau} \log \frac{1}{\delta_1} \left( 1 - \frac{\omega \tau(r)}{\tau} + \left( \frac{1}{3} + \frac{\omega \tau(r)}{\tau} \right)^2 + \frac{2 \omega \tau(r)}{\log \frac{1}{\delta_1}} \right),$$

$$B = \frac{1}{\delta_2} \log \frac{1}{\delta_2} \left( 1 + \frac{\omega \tau(r)}{\tau} + \left( \frac{1}{3} + \frac{\omega \tau(r)}{\tau} \right)^2 + \frac{2 \omega \tau(r)}{\log \frac{1}{\delta_2}} \right).$$

□

Mostafa Haghir Chehreghani, Albert Bifet, and Talel Abdessalem

**Proof.** In the following we prove that Equation 2 holds. The correctness of Equation 3 can be proven in a similar way. We define $Y^\tau$ as $\sum_{i=1}^\tau (c_r)$ and martingale $Z^\tau$ as $Y^{\min(\tau, \omega)}$. Using Theorem 4.7, we get:

$$\Pr[Z^\omega - E[Z^\omega] \geq \lambda] = \Pr[\tau c - \tau bc(r) \geq \lambda] \leq \exp \left( -\frac{\lambda^2}{2 \left( \frac{\min(\tau, \omega)}{\tau} \right) \frac{\log \frac{1}{\delta_1}}{\tau}} \right).$$

(4)

Furthermore, Lemma 4.6 yields:

$$\sum_{r=1}^{\min(\tau, \omega)} \text{Var}[c_{\tau}] = \sum_{r=1}^{\min(\tau, \omega)} (a(r)bc(r) - bc(r)^2) \geq \omega \alpha bc(r).$$

If in Equation 4 we use this upper bound on the sum of variances, we get:

$$\Pr[\tau c - \tau bc(r) \geq \lambda] \leq \exp \left( -\frac{\lambda^2}{2 \left( \omega \alpha bc(r) \log \frac{1}{\delta_1} \right)} \right).$$

(5)

Putting the right side of Equation 5 equal to $\delta_1$ yields:

$$\lambda = \frac{1}{3} \log \frac{1}{\delta_1} + \frac{1}{9} \left( \log \frac{1}{\delta_1} \right)^2 + 2 \omega \alpha bc(r) \log \frac{1}{\delta_1}.$$

(6)

Parameter $\lambda$ should not be expressed in terms of $bc(r)$, as it is unknown. Therefore in Equation 6 we should find its value in terms of the other parameters. To do so, we put the obtained value of $\lambda$ into Equation 5 and obtain:

$$\Pr[\tau c - \tau bc(r) \geq \lambda] \leq \exp \left( -\frac{\lambda^2}{2 \left( \omega \alpha bc(r) \log \frac{1}{\delta_1} \right)} \right) \leq \delta_1.$$

(7)

After solving the quadratic equation (with respect to $bc(r)$) of the event inside $\Pr[.]$ of Equation 7 and some simplifications, we get Equation 2.

□

Now the definition of the Stop method can be indicated by Theorem 4.8. This theorem implies that for the defined value of $\omega$, the probability of $\tau = \omega$ and $|bc(r) - c| > \lambda$ is at most $\delta/2$. Moreover, the probability of $\tau < \omega$ and $A > \lambda$ is at most $\delta_1 (\delta/4)$, and the probability of $\tau < \omega$ and $B > \lambda$ is at most $\delta_2 (\delta/4)$. Therefore and using union bounds, in order to have $(\lambda, \delta)$-approximation for the given values of $\lambda$ and $\delta$, in the beginning of each iteration of the loop in Lines 8-16 of Algorithm 1, the current value of the random variable $\tau$ should be either equal to $\omega$, or it should make both terms $A$ of Equation 2 and $B$ of Equation 3 less than or equal to $\lambda$ (with $\delta_1 = \delta_2 = \delta/4$). If these conditions are satisfied, the Stop method returns true and the loop terminates; otherwise, more samples are required, hence, the Stop method returns false.

The main difference between the lower and upper bounds presented in Inequalities 2 and 3 and the lower and upper bounds presented in [6] is that in [6], $\alpha(r)$ is replaced by 1. Since for given

3 Similar to the proof of Theorem 9 of [6], the proof of Theorem 4.8 of the current paper is based on Theorem 6.1 of [13]. The key difference is that in Theorem 9 of [6], the variance of each random variable $c_r$ is $bc(r) - bc(r)^2$, where $bc(r)$ is used as an upper bound. In our theorem, the variance of each random variable $c_r$ is the value presented in Lemma 4.6, where we use $a(r)bc(r)$ as an upper bound.
Novel Adaptive Algorithms for Estimating Betweenness, Coverage and k-path Centralities

values $\lambda$ and $\delta$, $a(r) < 1$ and in most cases $a(r) \ll 1$ (for example, in our extensive experiments reported in Section 6, $a$ is always less than 0.04!), the number of samples (iterations) required by our algorithm is much less than the number of samples required by e.g., KADABRA [6].

Complexity analysis. For unweighted graphs, each iteration of the loop in Lines 8-16 of Algorithm 1 takes $O(|E(G)|)$ time. For weighted graphs with positive weights, it takes $O(|E(G)|+|V(G)| \log |V(G)|)$ time (and for weighted graphs with negative weights, the problem is NP-hard). This is the same as the time complexity of processing each sample by the existing algorithms [26, 21]. In a more precise analysis and when instead of breadth-first search (BFS), balanced bidirectional breadth-first search (bb-BFS) [20] is used to sample a shortest path $\pi$, time complexity of each iteration is improved to $O(b^{d_{\max}})$, where $b$ is the maximum degree of the graph [20]. In Algorithm 1, the number of iterations is determined adaptively and as discussed above, it is considerably less than the number of samples required by the most efficient existing algorithms. Space complexity of our algorithm is $O(|E(G)|$).

Coverage centrality. ABAD can be revised to compute some related indices such as coverage centrality [26] and stress centrality. To compute coverage centrality of $r$, Lines 11-14 of Algorithm 1 are replaced by the following lines:

if $r$ is on a shortest path between $s$ and $t$ then
$$c_r \leftarrow \frac{|RF(r)|}{|V(G)|/|V(G)|-1}$$
end if

In other words, instead of sampling a shortest path $\pi$ between $s$ and $t$, we can check whether $r$ is on some shortest path between them, which can be done by conducting a bb-BFS from $s$ to $t$. If during this procedure $r$ is met, it is on some shortest path from $s$ to $t$; otherwise, it is not. In a way similar to Lemma 4.5 we can show that this method gives an unbiased estimation of coverage centrality of $r$. Moreover, similar to Theorem 4.8 we can define the stopping conditions of estimating coverage centrality.

5 k-PATH CENTRALITY

In this section, we present the APAD algorithm for estimating k-path centrality of a given vertex $r$ in a directed graph $G$. We define the domain of a vertex $r$, denoted by $D(r)$, as $RF(r) \cup \{r\} \cup RT(r)$. Furthermore, we say a path $p$ belong to $D(r)$ and denote it with $p \in D(r)$, iff $V(p) \subseteq D(r)$. It is easy to see that for directed graphs, the first vertex of each path $p_{s,i}$ must belong to $RF(r)$ and all its vertices must belong to $D(r)$. Because, otherwise, $\chi_{r \in p_{s,i}}$ will be zero and hence $p_{s,i}$ will have no contribution to $pc(r)$. This motivates us to present the following equivalent definition of k-path centrality.

$$pc(r) = \frac{1}{k|V(G)|} \sum_{p \in RF(r)} \sum_{1 \leq i \leq k} \sum_{p_{s,i} \in D(r)} \chi_{r \in p_{s,i}} \cdot W(p_{s,i}).$$

(8)

Note that in Equation 8, someone may decide to change the definition of $W(p_{s,i})$ to $I_{r \in D(r)} N(u_{i-1}) \cdot W(p_{s,i})$. The algorithm we propose works with both definitions of $W$.

Algorithm 2 shows the high level pseudo code of the APAD algorithm. It first computes $RF(r)$ and $D(r)$ and stores them respectively in $RF$ and $D$. Then it starts the sampling part where at each iteration, it selects $s$ and $l$ from $RF(r)$ and $\{1, k\}$, respectively, and a path $p_{s,i}$ from the set of all paths that belong to $D(r)$ and start with vertex $s$ and have $l$ edges. The estimation $c_r$ at each iteration $r$ is defined as $c_r = \frac{|RF(r) \cdot W(p_{s,i})|}{|V(G)|/|V(G)|}$ and the final estimation $c_r$ is the average of all $c_r$. While $s$ and $l$ are chosen uniformly at random, $p_{s,i}$ is chosen as follows. First, we initialize $p_{s,i}$ by $s$. Then, at each step $i$, let $u_{i-1}, 1 \leq i \leq l-1$, be the last vertex added to the current $p_{s,i}$. We add to $p_{s,i}$ a vertex $u_i$ chosen uniformly at random from $N(u_{i-1}) \cap D(r) \setminus \{s, u_1, u_2, \ldots, u_{i-2}\}$. This procedure yields the following definition of $c_r$:

$$pc(r) = \sum_{s \in RF} \sum_{1 \leq i \leq k} \sum_{p_{s,i} \in D(r)} \left(\frac{|RF(r) \cdot W(p_{s,i}) \cdot \chi_{r \in p_{s,i}}}{|V(G)|} \cdot P[p_{s,i}] / k|RF(r)|\right)$$

where $P[p_{s,i}]$ is the probability of choosing the path $p_{s,i}. Since c_r = \frac{1}{k|V(G)|}$, its expected value is the same as the expected value of $c_r$. 

D. Then it starts the sampling part where at each iteration, it selects $s$ and $l$ from $RF(r)$ and $\{1, k\}$, respectively, and a path $p_{s,i}$ from the set of all paths that belong to $D(r)$ and start with vertex $s$ and have $l$ edges. The estimation $c_r$ at each iteration $r$ is defined as $c_r = \frac{|RF(r) \cdot W(p_{s,i})|}{|V(G)|/|V(G)|}$ and the final estimation $c_r$ is the average of all $c_r$. While $s$ and $l$ are chosen uniformly at random, $p_{s,i}$ is chosen as follows. First, we initialize $p_{s,i}$ by $s$. Then, at each step $i$, let $u_{i-1}, 1 \leq i \leq l-1$, be the last vertex added to the current $p_{s,i}$. We add to $p_{s,i}$ a vertex $u_i$ chosen uniformly at random from $N(u_{i-1}) \cap D(r) \setminus \{s, u_1, u_2, \ldots, u_{i-2}\}$. This procedure yields the following definition of $c_r$:

Algorithm 2 High level pseudo code of the APAD algorithm

1. Input. A directed network $G$, a vertex $r \in V(G)$, an integer $k$, and real numbers $\lambda, \delta \in (0, 1)$.
2. Output. $k$-path centrality of $r$.
3. if in degree of $r$ is 0 or out degree of $r$ is 0 then
   return 0.
4. end if
5. end if
6. Select $s \in RF$ uniformly at random.
7. Select an integer $l \in [1, k]$ uniformly at random.
8. Select (with probability $P[p_{s,i}]$) a random path $p_{s,i}$ among all paths $p_{s,i} \in D$.
9. if $r$ is on $p_{s,i}$ then
   $c_r \leftarrow \frac{|RF(r) \cdot W(p_{s,i})|}{|V(G)|/|V(G)|}$
end if
10. $c_r \leftarrow c_r + c_r, \tau \leftarrow \tau + 1.$
11. end while
12. $c_r \leftarrow c_r/\tau.$
13. return $c_r.$

Proof. For each iteration $r$ in the loop in Lines 9-18 of Algorithm 2, we have:

$$\mathbb{E}[c_r] = \sum_{s \in RF} \sum_{1 \leq i \leq k} \sum_{p_{s,i} \in D(r)} \left(\frac{|RF(r) \cdot W(p_{s,i}) \cdot \chi_{r \in p_{s,i}}}{|V(G)|} \cdot P[p_{s,i}] / k|RF(r)|\right)$$

where $P[p_{s,i}]$ is the probability of choosing the path $p_{s,i}. Since c_r is the average of $c_r$, its expected value is the same as the expected value of $c_r$. 

In the rest of this section, we derive error bounds for our estimation of $pc(r). Before that, we define $a'(r)$ as the ratio $\frac{|RF(r)|}{|V(G)|}$.

Theorem 5.2. Suppose that in Algorithm 2 the loop in Lines 9-18 is performed for a fixed number of times $\tau$. For large enough values of $\tau$, Algorithm 2 gives a $(\lambda, \delta)$-approximation of $k$-path score of $r$. 
Proof. Our proof is based on Hoeffding inequality [16]. Let $X_1, \ldots, X_T$ be independent random variables bounded by the interval $[a_i, b_i]$. Let also $\mu = \mathbb{E} \left[ \sum_{i=1}^{T} \frac{X_i}{T} \right]$. Hoeffding inequality [16] states that for any $\lambda > 0$, the following holds:

$$\mathbb{P} \left[ \left| \sum_{i=1}^{T} \frac{X_i}{T} - \mu \right| \geq \lambda \right] \leq 2 \exp \left( -2T^2 \lambda^2 / \sum_{i=1}^{T} (b_i - a_i)^2 \right).$$

For each sampled path $p_{k,1}$, using any of the two before mentioned definitions of $W$, the following holds: $\mathbb{P} \left[ p_{k,1} \right] \geq W(p_{k,1})$. This means each $c_r$ is in the interval $[0, \alpha(r)]$. As a result, we can apply Hoeffding inequality on the random variables $c_r$, with $X_i = \epsilon_r$, $T = \tau$, $a_i = 0$ and $b_i = \alpha(r)$. This yields

$$\mathbb{P} \left[ |c_r - pc(r)| \geq \lambda \right] \leq 2 \exp \left( -2\tau^2 \lambda^2 / \alpha(r)^2 \right) = \delta,$$

which means $\tau \geq \frac{\alpha(r)^2 \log \frac{1}{\delta}}{2\lambda^2}$. Algorithm 2 estimates $k$-path score of $r$ within an additive error $\lambda$ with a probability at least $1 - \delta$. \qed

The difference between the error bounds presented in Theorem 5.2 and those that can be obtained for RA-kpath [3] is that on the one hand for the same error guarantee our algorithm requires $\alpha(r)^2$ times less samples. On the other hand, while in our algorithm each iteration (sample) takes $O(k)$ time, in RA-kpath it takes $O(k|V(G)|)$ time. As a result, for the same error guarantee, our algorithm is $O(\frac{|V(G)|}{\alpha(r)^2|D|})$ times faster than RA-kpath. Let $I$ be the set of the edges of $G$ whose both end-points are in $D$. While space complexity of our algorithm is $O(|D| + |I|)$, it is $O(|V(G)| + |E(G)|)$ for RA-kpath. Note that for vertices of real-world graphs, $\alpha$ is usually considerably less than 1 and $|D|$ and $|I|$ are respectively considerably smaller than $|V(G)|$ and $|E(G)|$. Similar results hold for our algorithm against the algorithm of [19], as it uses the same sampling strategy as RA-kpath.

In the last part of this section, similar to the case of betweenness centrality, in Theorem 5.4 we discuss how the stopping condition of Algorithm 2 can be determined adaptively. The proof of this theorem is similar to the proof of Theorem 4.8 (hence, we omit it). There are, however, two key differences between Theorems 4.8 and 5.4. First, as shown in Lemma 5.3, in Theorem 5.4 the variance of each $c_r$ is bounded by $\alpha(r)^2 pc(r)$, where it is $\alpha(r) bc(r)$ in Theorem 4.8. Second, as shown in Theorem 5.2, when the number of samples is $\frac{\alpha(r)^2 \log \frac{1}{\delta}}{2\lambda^2}$ the estimation error of the $k$-path score of $r$ is bounded by $\lambda$, with probability $1 - \delta/2$. We refer to this quantity as $\omega_r$ and use it in Theorem 5.4, as the upper bound, for the required number of samples.

**Lemma 5.3.** In Algorithm 2, for each $c_r$ we have: $\text{Var} \left[ c_r \right] \leq \alpha(r)^2 pc(r)$.

**Proof.** \begin{align*}
\text{Var} \left[ c_r \right] &= \mathbb{E} \left[ c_r^2 \right] - \mathbb{E} \left[ c_r \right]^2 \\
&= \sum_{s \in RF} \sum_{l \in [1,k], \bar{l} \in D} \frac{\left| RF \right|^2 W(p_{s,l})^2 \chi_{r \in p_{s,l}}}{|V(G)|^2} \mathbb{P} \left[ p_{s,l} \right] - pc(r)^2 \\
&\leq \frac{\alpha(r)^2}{k|V(G)|} \sum_{s \in RF} \sum_{l \in [1,k], \bar{l} \in D} (W(p_{s,l})^2 \chi_{r \in p_{s,l}}) - pc(r)^2 \\
&\leq \alpha(r)^2 pc(r) - pc(r)^2 \leq \alpha(r)^2 pc(r). \hspace{1cm} (10)
\end{align*}

Note that Lemma 5.3 holds for both definitions of $W$.

In Algorithm 2, if the number of samples is equal to $\omega_r$, with probability at least $1 - \delta/2$ we have: $|pc(r) - c| < \lambda$. For the case of $\tau < \omega_r$, we present the following theorem.

**Theorem 5.4.** In Algorithm 2, for real values $\delta_1, \delta_2 \in (0,1)$ and the value of $\omega_r$ defined above, after $\tau < \omega_r$ iterations of the loop in Lines 9-18, we have:

$$\mathbb{P} \left[ |pc(r) - c| \leq A' \right] \leq \delta_1 \text{ and } \mathbb{P} \left[ |pc(r) - c| \geq B' \right] \leq \delta_2,$$

where

$$A' = -\frac{1}{\tau} \log \frac{1}{\delta_1} \left( \frac{1}{3} - \frac{\omega \alpha'(r)}{\tau} \right) + \left( \frac{1}{3} - \frac{\omega \alpha'(r)}{\tau} \right)^2 + \frac{2\omega \alpha'(r)}{\log \frac{1}{\delta_1}}$$

$$B' = -\frac{1}{\tau} \log \frac{1}{\delta_1} \left( \frac{1}{3} - \frac{\omega \alpha'(r)}{\tau} \right) + \left( \frac{1}{3} - \frac{\omega \alpha'(r)}{\tau} \right)^2 + \frac{2\omega \alpha'(r)}{\log \frac{1}{\delta_1}}.$$
KADABRA has a parameter $k$ that determines the number of vertices for which we want to estimate the betweenness score. By increasing the value of $k$, KADABRA becomes slower. In our experiments, we set $k$ to 10. BCD is not adaptive and it does not automatically compute the number of samples required for the given values of $\lambda$ and $\delta$. Furthermore, unlike ABAD and KADABRA, BCD is a (source) vertex sampler algorithm, hence, it requires much more time to process each sample. Thus, for BCD we cannot use the same number of samples as ABAD (or KADABRA). In order to have fair comparisons, in our tests we let BCD have as much as samples that it can process within the running time of ABAD. As a result, while ABAD and BCD will have (almost) the same running times, the number of samples of BCD will be much less.

ABAD and BCD need to specify the vertex for which we want to estimate betweenness score. For each dataset, we choose the five vertices that have the highest betweenness scores and run ABAD and BCD for each of them. Choosing the vertices that have the highest betweenness scores has two reasons. First, exact betweenness of the vertices that have a small $\alpha$ (or $|RF|$) can be computed very efficiently [12]. Therefore, using approximate algorithms for them is not very meaningful. In fact by choosing vertices with high betweenness scores we guarantee that the chosen vertices have large enough $\alpha$ (or $|RF|$), so that it makes sense to run approximate algorithms for them. Second, these vertices usually have a higher $\alpha$ than the other vertices. Hence, it is likely that ABAD will estimate betweenness centrality of the other vertices more efficiently. For instance, in web-NotreDame consider those five randomly chosen vertices that are used in the experiments of [12]. These vertices have the following $\alpha$ values: 0.000000001, 0.000011, 0.0000016, 0.00000051 and 0.000017. These values are much smaller than the $\alpha$ of the vertices that have the highest betweenness scores (see Table 1). Therefore, ABAD will have a much better performance for these vertices.

Table 1 reports the experimental results. In each experiment, the bold value shows the one that has the lowest error. In all the experiments, we set $\delta$ to 0.1 for both ABAD and KADABRA. Since the behavior of these algorithms depends on the value of $\lambda$, we run them for three different values of $\lambda$: 0.001, 0.00075 and 0.0005. Over com-amazon, some vertices are not among the vertices for which KADABRA computes the betweenness score, hence, they do not contribute in the reported results. Since the running time of BCD is very close to the running time of ABAD (as we set so), we do not report it in Table 1.

Our results show that ABAD considerably outperforms both KADABRA and BCD. It works always better than KADABRA; furthermore, in most cases (38 out of 42 cases) it is more accurate than BCD. This is due to very low values of $\alpha$ that vertices of real-world graphs have. In our experiments, $\alpha$ is always smaller than 0.04. This considerably restricts the search space and hence, improves the efficiency of ABAD. The vertices tested in our experiments are those that have the highest betweenness scores. This implies that compared to the other vertices, they usually have a high $\alpha$. As a result, the relative efficiency of ABAD with respect to KADABRA for the other vertices of the graphs (that do not have a very high betweenness score) will be even better. It should also be noted that for a vertex $r \in V(G)$, while $\omega$ is computed, the sets $RF(r)$ and $RT(r)$ are computed, too. Therefore, computation of these sets does not impose any extra computational cost on ABAD.

Someone may complain that ABAD (like BCD) estimates betweenness score of only one vertex, whereas KADABRA can estimate betweenness scores of all vertices. However, as mentioned before, in several applications it is sufficient to compute this index for only one vertex or for a few vertices. Even if in an application we require to compute betweenness score of a set of vertices, ABAD can be still useful. In this case, after computing $RF$ and $RT$ of all the vertices in the set in parallel, someone may run the sampling part of Algorithm 1 (Lines 8-16 of the algorithm) for all the vertices in the set in parallel, as was done in the existing algorithms. Note that even if ABAD is run independently for each vertex in the set, in many cases it will outperform KADABRA. For example, consider our experiments where we run ABAD for five vertices that have the highest betweenness scores. In the reported experiments, for each dataset, by simply summing up the running times and the number of samples of five vertices, we can obtain the running time and the number of samples of the independent runs of ABAD for all the five vertices. Even in this case, the number of samples used by ABAD to estimate betweenness centrality of all five vertices is considerably (at least 10 times!) less than the number of samples used by KADABRA. Moreover, in most cases, ABAD is more accurate than KADABRA. Finally, while over p2p-Gnutella31 ABAD takes less time to compute betweenness scores of all five vertices, over the other datasets KADABRA is faster. A reason for the faster performance of KADABRA over com-amazon, com-dblp, email-EuAll and web-NotreDame (despite the fact that it uses much more samples) is that it processes most of these samples in parallel, whereas in the independent runs of ABAD, less samples are processed in parallel. Using considerably less samples by ABAD gives a sign that if ABAD estimates betweenness scores of all vertices of a set in parallel, it might become always faster than KADABRA. Developing a fully parallel version of ABAD that estimates betweenness scores of a set of vertices in parallel is an interesting direction for future work.

### 6.2 k-path centrality

In this section, we present the empirical results of APAD. We compare APAD against RA-kpath [3] that samples a vertex $s \in V(G)$, an integer $l \in [1,k]$ and a random path that starts from $s$ and has $l$ edges. This sampling method has been used by some other work, including [19]. The error bound presented in [3] is not adaptive and has only one parameter $\alpha$ (different from the parameter $\alpha$ used in this paper), instead of two parameters $\lambda$ and $\delta$ that we have for APAD. This makes comparison of the methods slightly challenging. In order to have a fair comparison (regardless of the method used to determine the number of samples for an $(\lambda, \delta)$-approximation), we run each of them for a fixed number of iterations and report the empirical approximation error and running time. APAD requires to
Table 1: Empirical evaluation of ABAD against KADABRA and BCD for five vertices that have the highest betweenness scores. The value of $\delta$ is 0.1 and times are in seconds. Times reported for ABAD include both computing $RT$ and $RF$ and estimating betweenness score.

| Dataset        | $\lambda$ | $\alpha$ | ABAD | KADABRA | BCD |
|---------------|----------|---------|------|---------|-----|
| com-amazon    | 0.001    | 0.000027 | 0.48 | 0.55    | 1.32 |
|               | 0.00075  | 0.000044 | 0.71 | 0.86    | 2.01 |
|               | 0.0005   | 1.39     | 0.41 | 1.04    | 3.65 |
| com-dblp      | 0.001    | 0.0077   | 5.54 | 6.72    | 17.55 |
|               | 0.00075  | 0.0094   | 9.31 | 11.46   | 28.79 |
|               | 0.0005   | 19.07    | 3.41 | 7.46    | 54.46 |
| email-EuAll   | 0.001    | 0.0012   | 1.50 | 1.79    | 1.10 |
|               | 0.00075  | 0.0013   | 2.47 | 2.91    | 2.05 |
|               | 0.0005   | 5.11     | 2.33 | 4.95    | 3.85 |
| p2p-Gnutella31| 0.001    | 0.0021   | 0.91 | 1.25    | 2.80 |
|               | 0.00075  | 0.0033   | 1.51 | 2.15    | 4.47 |
|               | 0.0005   | 2.98     | 1.11 | 2.38    | 8.74 |
| web-NotreDame | 0.001    | 0.2984   | 0.16 | 0.17    | 0.13 |
|               | 0.00075  | 0.3005   | 1.40 | 1.42    | 1.44 |
|               | 0.0005   | 4.42     | 1.16 | 2.31    | 3.85 |

Table 2: Empirical evaluation of APAD against RA-kpath. All times are in seconds. Times reported for APAD include both computing reachable set and estimating $k$-path score.

| Dataset        | #samples | $\alpha'$ | APAD | RA-kpath |
|---------------|---------|---------|------|----------|
| com-amazon    | 50000   | 0.0176  | 0.6202 | 0.6271 |
|               | 100000  | 0.4710  | 1.4945 | 8.7553 |
|               | 500000  | 6.1657  | 6.3111 | 8.5740 |
| com-dblp      | 50000   | 0.2984  | 0.6120 | 1.4015 |
|               | 100000  | 1.4015  | 1.4227 | 8.3598 |
|               | 500000  | 6.0236  | 6.1431 | 6.9700 |
| email-EuAll   | 50000   | 0.3057  | 0.4517 | 0.4570 |
|               | 100000  | 1.0388  | 1.0904 | 3.2181 |
|               | 500000  | 4.3868  | 4.4107 | 3.3479 |
| p2p-Gnutella31| 50000   | 0.2050  | 0.1666 | 0.1715 |
|               | 100000  | 0.7527  | 0.3947 | 0.8433 |
|               | 500000  | 1.5625  | 1.5896 | 6.5869 |
| web-NotreDame | 50000   | 0.3520  | 0.6116 | 0.6152 |
|               | 100000  | 1.2208  | 1.2351 | 3.4717 |
|               | 500000  | 5.9969  | 6.0496 | 2.9064 |

specify the vertex $r$ for which we want to compute $k$-path centrality. In order to have experiments consistent with Section 6.1, in this section we use the same vertices used in the evaluation of ABAD. We repeat each experiment for three times and report the average results.

Table 2 reports the experimental results of $k$-path centrality. In each experiment, the bold value shows the one that has the lowest (average/maximum) error. As can be seen in the table, in most cases APAD shows a better average error and maximum error than RA-kpath. This is particularly more evident over the datasets (e.g., com-amazon) that have a low $\alpha'$ value, which empirically validates our theoretical analysis on the connection between $\alpha'$ and the efficiency of APAD. Note that as before mentioned, RA-kpath computes $k$-path centrality of all vertices, whereas APAD requires to specify the vertex for which we want to compute the centrality score. As a result, for different given vertices, while only one run of RA-kpath is sufficient, we require to have different runs of APAD. Therefore in Table 2, for each dataset and sample size we report
only one time for RA-kpath. In contrast, for each dataset and sample size we have five running times for APAD, where the average and maximum values are reported in Table 2. The times reported for APAD include both computing reachable set and estimating k-path score. These times are only slightly larger than the times reported for RA-kpath. This means that computing the reachable set of a vertex can be done very efficiently and in a time negligible compared to the running time of the whole process.

7 CONCLUSION

In this paper, first we presented a novel adaptive algorithm for estimating betweenness score of a vertex \( r \) in a directed graph \( G \). Our algorithm computes the sets \( \mathcal{RF}(r) \) and \( \mathcal{RT}(r) \), samples from them, and stops as soon as some condition is satisfied. The stopping condition depends on the samples met so far, \( |\mathcal{RF}(r)| \) and \( |\mathcal{RT}(r)| \). We showed that our algorithm gives a more efficient \((\lambda, \delta)\)-approximation than the existing algorithms. Then, we proposed a novel algorithm for estimating k-path centrality of \( r \) and showed that in order to give a \((\lambda, \delta)\)-approximation, it requires considerably less samples. Moreover, it processes each sample faster and with less memory. Finally, we empirically evaluated our centrality estimation algorithms and showed their superior performance.

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