Convergence of three-dimensional loop-erased random walk in the natural
parametrization

Xinyi Li ∗  Daisuke Shiraishi †

December 6, 2018

Abstract

In this work we consider loop-erased random walk (LERW) and its scaling limit in three dimen-
sions, and prove that 3D LERW parametrized by renormalized length converges to its scaling limit
parametrized by some suitable measure with respect to the uniform convergence topology in the
lattice size scaling limit. Our result improves the previous work [7] of Gady Kozma which shows
that the rescaled trace of 3D LERW converges weakly to a random compact set with respect to the
Hausdorff distance.

Contents

1 Introduction 2
  1.1 Introduction and main results ........................................................................... 2
  1.2 Some words about the proof ............................................................................. 5
    1.2.1 Proof of Theorem 1.1 ................................................................................. 5
    1.2.2 Proof of Theorem 1.3 assuming Theorem 1.2 ........................................... 7

2 Notation and background 9
  2.1 General notation .............................................................................................. 9
  2.2 Estimates on Green’s functions ......................................................................... 10
  2.3 Some metric spaces .......................................................................................... 11
  2.4 Weak convergence of probability measures ..................................................... 11
  2.5 Loop-erased random walk ................................................................................. 12
    2.5.1 Definition ................................................................................................... 12
    2.5.2 Laplacian random walk and domain Markov property ............................... 12
    2.5.3 Length of 3D LERW .................................................................................. 13
    2.5.4 Escape probability and one-point function estimate .................................... 13
    2.5.5 Scaling limit of 3D LERW ........................................................................ 14
    2.5.6 Quasi loops ................................................................................................ 14
  2.6 Uniform spanning tree ...................................................................................... 14
  2.7 Coupling .......................................................................................................... 15
  2.8 Separation lemma ............................................................................................. 16

3 Setup and preliminary estimates for the key $L^2$-estimate (1.3) .................. 16

4 Proof of the key estimate (1.3) ........................................................................... 19
  4.1 Statement and a sketch of the proof ................................................................. 19
  4.2 Asymptotic independence of LERW .............................................................. 24
  4.3 Localization of box crossings ......................................................................... 26
  4.4 Decomposition and conditioning of paths ...................................................... 31
  4.5 Decoupling non-intersection events ............................................................... 33
  4.6 The point-crossing probability ...................................................................... 38
  4.7 Proof of (4.7) ................................................................................................. 39

∗ Department of Mathematics, The University of Chicago
† Graduate School of Informatics, Kyoto University
1 Introduction

1.1 Introduction and main results

Understanding the relationship between lattice-based probabilistic models and their limiting processes in $\mathbb{R}^d$ is one of the fundamental problems in probability theory. The archetypal (and quite well studied) example is, of course, Donsker’s invariance principle which states that the scaling limit of simple random walk (SRW) is Brownian motion. However, many models arising from statistical physics are essentially more complicated than SRW, due to strong interaction of the process with its past. For instance, for random curves with strong self-repulsion (in other words, the curve is not allowed to visit its past), even the existence of the scaling limit is not trivial at all.

For models with strong self-repulsion, the most interesting cases are $d = 2, 3$. In one dimension, the conditioning on self-repulsion forces the curve to be a straight line and nothing interesting happens. In four or more dimensions, loosely speaking, conditioning on self-avoidance does not have a strong effect on the behavior of the curve. Although this does not follow directly from the fact that Brownian motion is a simple curve in $d \geq 4$, nevertheless, in high dimensions, as a general strategy one can analyze rigorously curves with self-repulsion by comparing it with Brownian motion.

In this article, we will focus on a particular example, the loop-erased random walk (LERW), which is the random simple path obtained through erasing all loops chronologically from an SRW path. We refer readers to Section 2.5 for an introduction to the model and a repository of tools needed in this work.

LERW was originally introduced by Greg Lawler in 1980. Since then, it has been studied extensively both in mathematics and physics literature. Indeed, LERW has a strong connection with other models in statistical physics, especially the uniform spanning tree (UST) which arises in statistical physics in conjunction with the Potts model. Let us also mention that one can interpret LERW not only as the loop-erasure of the SRW but also as a Laplacian random walk (see Section 2.5 for details). Loosely speaking, conditioning the LERW $\gamma$ up to $k$-th step, the transition probability for the next step is given by the solution of a discrete Dirichlet problem on the complement of $\gamma[0, k]$. Although these models are also interesting on a general graph, in this article we shall restrict our attention to LERW in $\mathbb{Z}^d$ (and mainly with $d = 3$).

As we have already pointed out, LERW for $d \geq 4$ was already well studied - it can be proved without resorting to lace expansion that in this case the scaling limit of LERW is Brownian motion (see Chapter 7 of [8]).

The case of $d = 2$ is more difficult, yet also already well understood. Schramm first conjectured in [20] that LERW has a conformally invariant scaling limit, which is characterized by Schramm-Loewner evolution (SLE) with parameter 2. This conjecture was subsequently confirmed in [13]. Since then, there have been substantial progresses on this subject, and thanks to our knowledge on SLE, one can prove very fine results on LERW in two dimensions, in particular convergence in the natural parametrization, see [14].
What about $d = 3$? Unfortunately, relatively little is known compared with other dimensions. The crucial reason is that we do not have nice tools like SLE to describe the scaling limit of 3D LERW directly. Hence, understanding this limit will most likely remain a hard task for a long time. That said, as the study of planar LERW first motivated the discovery of SLE, which fueled the breakthroughs in studies of other planar lattices models from statistical mechanics, understanding the LERW scaling limit and seeking its comprehensive description is, we believe, also vital to the study of other models from statistical physics in three dimensions.

We now turn to the main results in this paper. Let us start by briefly explaining notation as well as some known results on the scaling limit for 3D LERW here (see Section 2 for any missing definition).

Let $S^{(n)} = (S^{(n)}(k))_{k \geq 0}$ be the SRW on $2^{-n} \mathbb{Z}^3$ started at the origin. We write $T^{(n)}$ for the first time that $S^{(n)}$ exits from $\mathbb{D} := \{ x \in \mathbb{R}^3 : |x| < 1 \}$ the unit open ball centered at the origin. We then write $\gamma_n := \text{LE}(S^{(n)}[0, T^{(n)}])$ for the loop-erasure of $S^{(n)}$ up to the first exit time $T^{(n)}$.

Let $(\mathcal{H}(\mathbb{B}), d_{\text{Haus}})$ be the space of all non-empty compact subsets in $\mathbb{B}$ endowed with the Hausdorff metric $d_{\text{Haus}}$. Regarding $(\gamma_n)_{n \geq 0}$ as random elements of the metric space $(\mathcal{H}(\mathbb{B}), d_{\text{Haus}})$, Gady Kozma proves in [7] that there exists a random compact set $\mathcal{K}$ that $(\gamma_n)_{n \geq 0}$ converges weakly to as $n \to \infty$ with respect to the Hausdorff metric $d_{\text{Haus}}$. In fact, he shows that $(\gamma_n)_{n \geq 0}$ is a Cauchy sequence with respect to the Prokhorov metric (see Section 2.4 for the Prokhorov metric). It is also proved that with probability one, $\mathcal{K}$ is a simple curve (see [21]) and the Hausdorff dimension of $\mathcal{K}$ is equal to $\beta \in (1, \frac{5}{4}]$ (see [22]) where $\beta$ is some deterministic constant (see Section 2.5.3 for more discussion on the exponent $\beta$).

In short, the purpose of this article is to study the scaling limit with respect to a topology stronger than the Hausdorff metric. Our choice of the topology is the supremum distance $\rho$ defined as follows: letting $\lambda_1 : [0, t_1] \to \mathbb{B}$ ($i = 1, 2$) be two continuous curves of duration $t_i$, their distance is defined as

$$\rho(\lambda_1, \lambda_2) = |t_1 - t_2| + \max_{0 \leq s \leq 1} |\lambda_1(st_1) - \lambda_2(st_2)|. \tag{1.1}$$

To deal with the scaling limit with respect to the metric $\rho$, we consider the time rescaled LERW $\eta_n$ defined by

$$\eta_n(t) = \gamma_n(2^{3n}t) \quad \text{for} \quad 0 \leq t \leq M_n/2^{3n},$$

where $M_n$ stands for the length (number of lattice steps) of $\gamma_n$ (we let the walk traverse each edge in unit time and assume linear interpolation of $\gamma_n$ here so that $\eta_n$ becomes a continuous curve). It is already proved in [10] that this choice of the time scaling factor is correct in the sense that $M_n/2^{3n}$ is tight (see Section 2.5.3 for known results of $M_n$).

When we study $\eta_n$ and its limit with respect to the distance $\rho$, the first crucial issue is to give a suitable time parametrization for the scaling limit $\mathcal{K}$. With this in mind, we begin with the following random measure $\mu_n$ in $\mathbb{B}$ defined by

$$\mu_n = 2^{-3n} \sum_{x \in \gamma_n \cap 2^{-n} \mathbb{Z}^3} \delta_x,$$

where $\delta_x$ is the Dirac measure at $x$. Note that for each point $x$ lying on the curve $\eta_n$, we can compute the exact time that $\eta_n$ passes through $x$ by measuring the weight of the sub-path of $\gamma_n$ between the origin and $x$ via the measure $\mu_n$. In this way, the curve $\eta_n$ is obtained by parametrizing $\gamma_n$ by the measure $\mu_n$. Thus, it is natural to consider the limiting measure of $\mu_n$ and parametrize $\mathcal{K}$ by this measure. As in the discrete we give equal weight to each lattice hit by $\gamma_n$ (or equivalently, traverse $\eta_n$ in a constant speed), we call this kind of parametrization the natural parametrization of $\mathcal{K}$.

The first main result guarantees the existence of the limit of $\mu_n$. Let $\mathcal{M}(\mathbb{B})$ be the space of all finite measures on $\mathbb{B}$ endowed with the weak convergence topology.

**Theorem 1.1.** As $n \to \infty$, the sequence of the joint law $(\gamma_n, \mu_n)$ converges weakly to some $(\mathcal{K}, \mu)$ with respect to the product topology of $\mathcal{H}(\mathbb{B})$ and $\mathcal{M}(\mathbb{B})$ where $\mathcal{K}$ is Kozma’s scaling limit. Furthermore, the limit measure $\mu$ is a measurable function of $\mathcal{K}$.

In order to parametrize $\mathcal{K}$ via the measure $\mu$, we need the following basic properties of $\mu$. For a point $x \in \mathcal{K}$, let $\mathcal{K}_x$ be the simple curve on $\mathcal{K}$ between the origin and $x$ (recall that $\mathcal{K}$ is a simple curve almost surely, see Section 2.5.5).

**Theorem 1.2.** With probability one, the support of the measure $\mu$ coincides with $\mathcal{K}$. Moreover, it follows that with probability one, for each $x \in \mathcal{K}$

$$\lim_{y \to x} \mu_y(K_y) = \mu(K_x).$$
We now parametrize $K$ through the measure $\mu$. By Theorem 1.2, it follows that for each $t \in [0, \mu(K)]$, there exists a unique point $x_t \in K$ satisfying $t = \mu(K_{x_t})$. Define $\eta(t) = x_t$ for $t \in [0, \mu(K)]$. It also follows from Theorem 1.2 that $\eta$ is a random continuous curve whose time duration is $\mu(K)$. The next theorem gives the desired convergence in $\rho$-metric.

**Theorem 1.3.** As $n \to \infty$, $\eta_n$ converges weakly to $\eta$ with respect to the metric $\rho$.

As a corollary of these theorems, we can also deal with the scaling limit of the infinite loop-erased random walk (ILERW) as follows. Recall that $S(n)$ stands for SRW on $2^{-n}\mathbb{Z}^3$ started at the origin. Since $S(n)$ is transient, the loop-erasure of $S(n)[0, \infty)$ is well-defined. We then write $\gamma_n^\infty = \text{LE}(S(n)[0, \infty))$ for the ILERW on $2^{-n}\mathbb{Z}^3$. We also consider the time-rescaled version defined by

$$\eta_n^\infty(t) = \gamma_n^\infty(2^{\alpha_n}t) \quad \text{for } t \geq 0,$$

where again we assume the linear interpolation of $\gamma_n^\infty$ so that $\eta_n^\infty$ becomes a random element of $(\mathcal{C}, \chi)$, the metric space of continuous curves defined on $[0, \infty)$ (see Section 2.3 for the space $\mathcal{C}$), equipped with the metric $\chi$

$$\chi(\lambda_1, \lambda_2) = \sum_{k=1}^{\infty} 2^{-k} \max_{0 \leq t \leq k} \left\{ |\lambda_1(t) - \lambda_2(t)|, 1 \right\},$$

for two continuous curves $\lambda_1, \lambda_2 \in \mathcal{C}$. The next theorem confirms the existence of the scaling limit of ILERW with respect to the metric $\chi$.

**Theorem 1.4.** There exists a random continuous curve $\eta^\infty \in \mathcal{C}$ such that as $n \to \infty$, $\eta_n^\infty$ converges weakly to $\eta^\infty$ with respect to the metric $\chi$.

We now give some comments on our results and briefly discuss some related open questions. See also Remark 8.9 for comments on the scaling limit of ILERW.

**Remark 1.5.** 1) To our best knowledge, LERW is the only model with self-repulsion for which we can prove the existence of the scaling limit with respect to the supremum distance for any dimensions. Concerning the uniform convergence of LERW, see Chapter 7 of [8] for $d \geq 4$ and [10] for $d = 2$.

2) It is worth noting that so far we still do not have a nice description of the scaling limit $\eta$ or $\eta^\infty$, despite the convergence results we obtain in this work. In [21], we conjecture that $\eta$ is the unique random continuous simple curve such that we obtain Brownian motion after adding loops of Brownian loop soup appropriately (see Conjecture 1.3, ibid., for the precise form). Unfortunately, the present paper does not give any progress for this conjecture. Hence it remains a big challenge to give a “good” description for $\eta$.

3) As the setup of ILERW is in some sense more natural than that of LERW in a domain, we hope that showing the existence of $\eta^\infty$ would facilitate studies of 3D LERW and related subjects. For instance, results established in this work will be useful in the study of 3D uniform spanning trees in the forthcoming paper [10].

4) One may wonder if it is possible to give a direct description of $\mu$ above. As the case of two dimensions, the scaling limit SLE$_2$ is parametrized by its Minkowski content in the natural parametrization (see [14]), it is very natural to conjecture that in three dimensions, it is also possible to identify $\mu$ with the Minkowski content of $\eta$. We will be addressing this problem in our future works. See Remark 6.3 for more discussions.

The organization of the paper is as follows. In Section 1.2 we will give a sketch of proofs for the main theorems. We will introduce some notions and explain background which will be used in this article in Section 2. A crucial part of this work is the second moment estimate 1.3 on the occupation measure and the number of boxes hit by the LERW. The precise setup and some preliminary estimates will be given in Section 3. The key estimate 1.3 will be proved in Section 4 (see Proposition 4.1). Using this $L^2$-estimate, for each box $B \subset \mathbb{D}$, we will give an $L^2$-approximation of $\mu_n(B)$ by some measurable quantity with respect to $\mathcal{K}$ in Section 5 (see Proposition 5.6). Then Theorem 1.1 will be proved in Section 6 (Theorem 6.2). In Section 7, we will first show that the time rescaled LERW $\eta_n$ is tight with respect to the supremum distance $\rho$, see Proposition 7.1. Next, we will prove Theorem 1.2 and Theorem 1.3 in Section 7 (see Proposition 7.7 and Proposition 7.8 for Theorem 1.2 and Theorem 1.3, resp). Finally, we will study the ILERW in Section 8 and prove Theorem 1.4 in Theorem 8.8.

**Acknowledgements:** The authors are grateful to Greg Lawler for numerous helpful and inspiring discussions. X.L. wishes to thank Kyoto University for its warm hospitality during his visit, when part of
this work was conceived. DS wishes to thank The University of Chicago for its warm hospitality during his visit, when part of this work was conceived. The authors would also thank the generous support from the National Science Foundation via the grant DMS-1806979 for the conference “Random Conformal Geometry and Related Fields”, where part of this work was accomplished.

1.2 Some words about the proof

In this subsection, we will give the sketch of the proof of Theorems 1.1 and 1.3 and will skip Theorems 1.2

and 1.4 as they are neither surprising nor a difficult part of this paper. For Theorem 1.3, let us mention that the distribution of the initial part of \( \gamma \) and 1.4 as they are neither surprising nor a difficult part of this paper. For Theorem 1.4, let us mention

Geometry and Related Fields”, where part of this work was accomplished.

1.2.1 Proof of Theorem 1.1

Theorem 1.1 bears similar spirit to Theorem 1.1 of [11]. The proof is carried out through the following steps.

- By Corollary 1.3 of [10], it follows that \( E(\mu_n(B)) = 2^{-\beta n} E(M_n) \) is uniformly bounded in \( n \). This implies that the sequence of the joint law \((\gamma_n, \mu_n)\) is tight with respect to the product topology of \( \mathcal{H}(\mathbb{D}) \) and \( \mathcal{M}(\mathbb{D}) \). Thus, we can find a subsequence \( n_k \) and some measure \( \mu \) such that \((\gamma_{n_k}, \mu_{n_k})\) converges weakly to \((K, \mu)\). (We write \( n = n_k \) for simplicity.)

- We want to show that \( \mu = f(K) \) where \( f \) is a deterministic function which does not depend on the choice of the subsequence, which implies Theorem 1.1

- To this end, take a box \( B \subset \mathbb{D} \) with \( \text{dist}(0 \cup \partial \mathbb{D}, B) > 0 \). In order to prove the claim above, the crucial step is to show that \( \mu(B) \) is measurable with respect to \( K \). We will explain how to prove the measurability of \( \mu(B) \) in the rest of this subsection.

- Take \( \epsilon > 0 \) with \( 2^{-n} \ll \epsilon \). the reader should regard \( \epsilon \) as a mesoscopic scale quantity while the mesh size \( 2^{-n} \) is in the microscopic scale. Indeed, in the end we will let \( 2^{-n} \rightarrow 0 \) first, and then let \( \epsilon \rightarrow 0 \). Decompose \( B \) into \( \epsilon \)-boxes \( B_1, \cdots, B_N \).

- Let \( X_i \) be the number of points in \( B \cap 2^{-n} \mathbb{Z}^3 \) hit by \( \gamma_n \). Let \( Y_i = 1\{\gamma_n \cap B_i \neq \emptyset\} \) be the indicator function of the event that \( \gamma_n \) hits \( B_i \). Note that \( \mu_n(B) = 2^{-\beta n} \sum_i X_i \). We also remark that \( Y_i = 0 \) implies \( X_i = 0 \) by definition.

- We can almost regard the indicator function \( Y_i \) as a “measurable” function of \( K \). Why? What we should keep in mind is that by Skorokhod’s representation theorem we can define \( \{(\gamma_n, \mu_n)\}_{n \geq 0} \) and \((K, \mu)\) in the same probability space such that \((\gamma_n, \mu_n)\) converges to \((K, \mu)\) with respect to the topology of \( \mathcal{H}(\mathbb{D}) \otimes \mathcal{M}(\mathbb{D}) \) almost surely. We will show in Section 5 that with probability one if \( K \) hits \( B_i \) then \( K \) actually enters into the interior of \( B_i \). This implies that with high probability \( Y_i \) is equal to \( Z_i = 1\{K \cap B_i \neq \emptyset\} \) for sufficiently large \( n \), see Corollary 5.3 for this.

- With this in mind, we want to write \( X_i \) in terms of \( Y_i \). More precisely, we want to “predict” \( X_i \) by \( a_0 Y_i \) with some deterministic \( a_0 \) that depend only on \( n \) and \( k \). Taking the expectation in the both sides of \( X_i \approx a_0 Y_i \), we see that this idea should go through if \( E(X_i | \gamma_n \cap B_i \neq \emptyset) \) approximately does not depend on \( i \). We will explain this crucial observation below.

- Condition that \( \gamma_n \) hits \( B_i \). We then decompose \( \gamma_n \) into three parts The first part \( \gamma^1 \) stands for the sub-path between the origin and the first time that \( \gamma_n \) hits \( B_i \). The third part \( \gamma^3 \) stands for the sub-path of \( \gamma_n \) between the last exit point from \( B_i \) and the endpoint. Let \( \gamma^2 \) be the rest of \( \gamma_n \). We write \( x \in \partial_i B_i \) (resp. \( y \in \partial_i B_i \)) for the starting (resp. the end) point of \( \gamma^2 \).

We then condition \( \gamma^1 \) and \( \gamma^3 \). The domain Markov property for LERW dictates that the conditional distribution of \( \gamma^2 \) is given by the loop-erasure of the “bridge” between \( x \) and \( y \) conditioned that the bridge does not intersect with \( \gamma^1 \cup \gamma^3 \) (see (2.13) for the domain Markov property).
Let us turn back to the occupation measure \( \mu_n \), for otherwise there is a big chance for it to hit \( \gamma^1 \cup \gamma^3 \). From this, it follows that there exists some \( r_\epsilon > 0 \) (depending only on \( \epsilon \) with \( \epsilon \ll r_\epsilon \ll 1 \)) such that the distribution of \( \gamma^2 \) depends only on \( (\gamma^1 \cup \gamma^3) \cap B(x, r_\epsilon) \) approximately. Hence, in order to compute the conditional expectation \( E(X_i \mid \gamma_n \cap B_i \neq \emptyset) \), the information we need is only the shape of \( \gamma_n \cap B(x, r_\epsilon) \), the “local behavior” of LERW around \( B_i \).

So the problem boils down to the following: take two “typical” points \( a \) and \( b \) lying on \( \gamma_n \), compare the shape of \( \gamma_n \cap B(a, r_\epsilon) \) and \( \gamma_n \cap B(b, r_\epsilon) \) and prove that the distribution of them are very similar. The similarity of \( \gamma_n \cap B(a, r_\epsilon) \) and \( \gamma_n \cap B(b, r_\epsilon) \) can be proved by some coupling argument established in [11] and [16] (see Section 2.7 for the coupling argument). Roughly speaking, we can find a suitable \( \widehat{r}_\epsilon > 0 \) (depending only on \( \epsilon \) with \( r_\epsilon \ll \widehat{r}_\epsilon \ll 1 \)) such that whatever happens outside of \( B(a, \widehat{r}_\epsilon) \) for \( \gamma_n \), has almost no impact on the distribution of \( \gamma_n \cap B(a, r_\epsilon) \). In other words, the distribution of \( \gamma_n \cap B(a, r_\epsilon) \) is almost the same as some kind of invariant measure, much like the “local part” of the infinite two-sided LERW from [11]. This implies that \( E(X_i \mid \gamma_n \cap B_i \neq \emptyset) \) approximately does not depend on \( i \).

In view of the discussion above, we define \( \alpha_0 \) as follows. We write \( x_0 = (1/2, 0, 0) \) and \( B_0 \) for the \( \epsilon \)-box centered at \( x_0 \). Let \( \alpha_0 = E(X_0 \mid \gamma_n \cap B_0 \neq \emptyset) \) be the corresponding conditional expectation for the “reference” box \( B_0 \), where \( X_0 \) stands for the number of points \( B_0 \cap 2^{-n}Z^3 \) hit by \( \gamma_n \). Under this notation, the discussion above can be rephrased as the following “one-point” estimate.

\[
\alpha_0 \simeq E(X_i \mid \gamma_n \cap B_i \neq \emptyset) \quad \text{for each } i.
\]

However, in order to ensure the convergence of \( \mu_n \), we need a little more than a one-point estimate. In fact, in Proposition 1.1 we will prove the following \( L^2 \)-estimate:

\[
E\left[\left( \sum_i X_i - \alpha_0 \sum_i Y_i \right)^2 \right] \leq \xi(n, \epsilon) \left( E\left[ \sum_i X_i \right] \right)^2, \tag{1.3}
\]

where \( \xi(n, \epsilon) \) converges to zero when \( 2^{-n} \) tends to zero first and then \( \epsilon \) tends to zero subsequently. The inequality (1.3) ensures that \( \sum_i X_i \) can be well approximated by \( \sum_i \alpha_0 Y_i \). The reader should regard the inequality (1.3) as a law of large numbers type estimate. Of course, \( X_i \) is not i.i.d. for our system. However, as explained above, it still enjoys asymptotic stationarity so that \( \alpha_0 \simeq E(X_i \mid \gamma_n \cap B_i \neq \emptyset) \) for each \( i \). Similar coupling argument as above also gives an approximate independence for the system.

Although (1.3) is quite intuitive, its actual proof is unfortunately quite long. We will give its proof in Sections 3 and 4. A sketch will be also given at the beginning of Section 4.

Let us turn back to the occupation measure \( \mu_n(B) = 2^{-6n} \sum_i X_i \). We first mention that one-point function estimates derived in [16] tell that the expected number of the lattice points hit by \( \gamma_n \) in \( B \) is comparable to \( 2^{3n} \). Thus, dividing both sides of (1.3) by \( 2^{3n} \), we see that the \( L^2 \)-distance between \( \mu_n(B) \) and \( \alpha_0 2^{-3n} \sum_i Y_i \) is bounded above by \( \xi(n, \epsilon) \).
• We now apply a sharp one-point function estimate obtained in Theorem 1.1 of [16] (see [22] for
the precise form of this estimate) to conclude Theorem 1.1. In Section 5, we will prove in Proposition
5.5 that there exists a universal constant $c_0$ such that
\[ a_0 2^{-\beta n} = [1 + \tilde{\xi}(n, \epsilon)] c_0 e^\delta, \]
where $\tilde{\xi}(n, \epsilon)$ tends to zero in the same way as $\xi(n, \epsilon)$ (i.e. when $2^{-n}$ tends to zero first and then
$\epsilon$ tends to zero subsequently). Combining these estimates, we will prove in Proposition 5.6 that
$L^2$-distance between $\mu_n(B)$ and $c_0 e^\delta \sum Y_i$ goes to zero in the same way as above. Note that
the choice of this universal constant $c_0$ is crucial. Since the quantity $c_0 e^\delta \sum Y_i$ is almost “measurable”
with respect to $\mathcal{K}$, we can conclude that this is also the case for $\mu(B)$.

1.2.2 Proof of Theorem 1.3 assuming Theorem 1.2

Here we will explain step by step how to obtain Theorem 1.3 assuming Theorem 1.2.

• The first crucial step is to show the tightness of $\eta_n$ with respect to the supremum distance $\rho$ which
will be carried out in Proposition 7.1. It follows from Theorem 1.4 of [22] and Corollary 1.3 of [16]
that taking $A$ sufficiently large, the time duration of $\eta_n$ denoted by $t_n = 2^{-\beta n} M_n$ lies in the interval
$[1/\lambda, \lambda]$ with high probability uniformly in $n$. So the tightness boils down to the equicontinuity of
$\eta_n$ in the following sense. Let $\epsilon, \delta > 0$. Define the event $A_{n, \epsilon, \delta}$ by
\[ A_{n, \epsilon, \delta} = \{ \max_{t, s \in [0, t_n], |t - s| \leq \delta} |\eta_n(t) - \eta_n(s)| > \epsilon \}. \]
We want to show that for each $\epsilon > 0$, the probability of this event $A_{n, \epsilon, \delta}$ goes to zero uniformly in
in $n$ as $\delta$ tends to zero.

• Take $0 \leq s < t \leq t_n$ with $t - s \leq \delta$. Remind that we wish to estimate the distance between $\eta_n(s)$
and $\eta_n(t)$. Hence, we begin with an easy case that $s = 0$. In this case, we can use exponential tail
lower bounds on $M_n$ derived in [22] (see Theorem 1.4 of [22]) to conclude that
\[ \lim_{\delta \to 0} \sup_{n \geq 1} P \left( \max_{0 \leq t \leq \delta} |\eta_n(t)| > \delta^{\frac{\epsilon}{2}} \right) = 0. \]

Since $\eta_n(s+\cdot)$ does not have the same distribution as $\eta_n(\cdot)$, unfortunately we cannot apply Theorem
1.4 of [22] directly to show the distance between $\eta_n(s)$ and $\eta_n(t)$ is small for general $s < t$ with
$t - s \leq \delta$. To deal with this issue, we use the (wired) uniform spanning tree (UST). The precise
definition of the UST will be given in Section 2.6.

• Let $D_n = \mathbb{D} \cap 2^{-n} \mathbb{Z}^3$ be the set of lattice points in $\mathbb{D}$. Write $U_n$ for the wired UST on $D_n$.
For each $x, y \in D_n$, we denote the unique path lying in $U_n$ connecting $x$ and $y$ (resp. $\partial D_n$) by $\gamma_n^{x, y}$
(resp. $\gamma_n^{\partial D_n}$). Then it follows from Wilson’s algorithm (see Section 2.6 for this) that $\gamma_n^{x, y}$
has the same distribution as $\gamma_n$. Denote the graph distance on $U_n$ by $d_{4n}$.

• Fix $\epsilon > 0$. Suppose that the event $A_{n, \epsilon, \delta}$ occurs. This implies that there exist $0 \leq s < t \leq M_n$ with
t $- s \leq \delta^{2 \beta n}$ such that the distance between $\gamma_n(s)$ and $\gamma_n(t)$ is greater than $\epsilon$. We write $x = \gamma_n(s)$
y and $y = \gamma_n(t)$.

• Now we introduce a “$\delta$-net” $F_3$ as follows. Fix a (deterministic) set of lattice points $F_3 = \{x_i\} \subset D_n$
such that every point in $D_n$ lies in $B(x_i, \delta)$ for some $i$ and the number of points in $F_3$ is comparable
to $\delta^{-3}$.

• To construct $U_n$, we perform Wilson’s algorithm in the following way.
  (i) Let $U_n^0 = \gamma_n$ be the LERW connecting the origin and $\partial D_n$.
  (ii) Consider the SRW $R^1$ started at $x_1$ until it hits $U_n^0 \cup \partial D_n$. We write $U_n^1$ for the union of $U_n^0$
and the loop-erasure of $R^1$. Next, start the SRW $R^2$ from $x_2$ until it hits $U_n^1 \cup \partial D_n$. Denote the
union of $U_n^1$ and the loop-erasure of $R^2$ by $U_n^2$. Continue this procedure until all points in $F_3$ are
included in the tree. Let $U_n'$ be the output tree.
  (iii) Perform Wilson’s algorithm for points in $D_n \setminus U_n'$ to obtain $U_n$.  

7
• Theorem 3.1 of [21] proves that $\gamma_n$ is a “hittable” set with high probability in the sense that if we consider another SRW whose starting point is close to $\gamma_n$, it is likely for this SRW to hit $\gamma_n$ immediately (see (2.24) for the precise statement of Theorem 3.1 of [21]). Using this fact in step (ii) of Wilson’s algorithm performed above, we can show that with high probability there exist a universal constant $a > 0$ and a point $z \in F_3$ such that the following three conditions hold:

(I) $\gamma_n \cap B(x, \delta^a) = \emptyset$;  
(II) $d_{D_n}(x, z) \leq \delta^a 2^{3n}$;  
(III) $y \in \gamma_n \cap \partial D_n$.

Figure 2: Location of $z$ relative to $x$ and $y$ on the tree.

Note that we cannot take $z$ as one of the nearest point from $x$ among $F_3$ because of some control of union bounds, but we can still take it within distance $\delta^a$ of $x$.

• Therefore, on the event $A_{n, \epsilon, \delta}$, there exists a point $z \in F_3$ such that $d_{D_n}(z, y) \leq (\delta^a + \delta) 2^{3n}$ and $|z - y| \geq \epsilon - \delta^a$. This implies that the first time that $\gamma_n^\epsilon$ exits from $B(z, \epsilon - \delta^a)$ is smaller than $(\delta^a + \delta) 2^{3n}$. However, the distribution of $\gamma_n^\epsilon$ is same as LE($R^2$) where $R^2$ stands for the SRW started from $z$ and stopped at $\partial D_n$. Consequently, if we write $T_z$ for the first time that LE($R^2$) exits from $B(z, \epsilon - \delta^a)$, we have

$$P(A_{n, \epsilon, \delta}) \leq \sum_{z \in F_3} P(T_z \leq (\delta^a + \delta) 2^{3n}) + \text{(small error)}.$$  

Thus, we can now apply exponential tail lower bounds derived in Theorem 1.4 of [22]. Taking $\delta$ sufficiently small so that $\delta^a \ll \epsilon$, Theorem 1.4 of [22] tells that $P(T_z \leq (\delta^a + \delta) 2^{3n})$ decays like $\exp(-\delta^{-b})$ for some universal constant $b > 0$. Thus, the sum in the right hand side of (1.6) is bounded above by $\delta^{-3} \exp(-\delta^{-b})$. This gives the desired equicontinuity of $\eta_n$.

• From the tightness of $\eta_n$ with respect to the distance $\rho$, we can find a subsequence $\{n_k\}$ such that $(\eta_{n_k}, \mu_{n_k})$ converges weakly to some $\zeta$ with respect to the topology of the space of continuous curves endowed with the supremum distance $\rho$ (denote this space by $C(D)$) and $M(D)$. What we want to show is that $\zeta$ has the same distribution as $\eta$ of Theorem 1.3. In order to do it, as the first step, we will show in Proposition 7.3 that the continuous curve $\zeta$ is injective almost surely.

• We use Skorokhod’s representation theorem again to couple $(\eta_{n_k}, \mu_{n_k})$ and $(\zeta, \nu)$ in the same probability space such that $(\eta_{n_k}, \mu_{n_k})$ converges to $(\zeta, \nu)$ with respect to the topology of $C(D) \otimes M(D)$ almost surely. Let $\hat{\zeta} = \{\zeta(t) : 0 \leq t \leq \tau\}$ be the range of $\zeta$ which is a random element of $H(D)$ (here $\tau$ stands for the time duration of $\zeta$). We remind that $\hat{\zeta}$, $\nu$ has the same distribution as $(K, \mu)$ of Theorem 1.1. Write $\eta^*$ for the curve obtained by parametrizing $\hat{\zeta}$ via the measure $\nu$, which is an element of $C(D)$ by Theorem 1.2. Since $\eta^*$ is a measurable function of $\hat{\zeta}$, $\nu$, we see that the curve $\eta^*$ has same distribution as $\eta$ of Theorem 1.3. Consequently, it suffices to show that $\zeta = \eta^*$ almost surely in the coupling above.

• We will show the identity $\zeta = \eta^*$ holds with probability one in Proposition 7.11 by contradiction. So suppose that with positive probability there exist $t$ and $t'$ with $t \neq t'$ such that $\zeta(t) = \eta^*(t')$. Then there are two cases that we need to consider: Case 1. $t > t'$ and Case 2. $t < t'$.

• Case 1 is easy. Suppose that $t > t'$. Let $A = \zeta[0, t]$ be the range of $\zeta$ up to time $t$. Denote the set of points within distance $\epsilon$ from $A$ ($\epsilon$-neighborhood of $A$) by $A_\epsilon$. The monotonicity of the measure $\nu$ implies that $\nu(A_\epsilon)$ goes to $t'$ as $\epsilon$ tends to zero. Take $\epsilon > 0$ sufficiently small so that $\nu(A_\epsilon) < (t + t')/2$. However, since $\mu_{n_k}$ converges to $\nu$, we can show that for $k$ sufficiently large, $\mu_{n_k}(A_\epsilon) < (3t + t')/4$. On the other hand, the uniform convergence of $\eta_k$ to $\zeta$ implies that taking
We start with set-theoretical notations. For $d \geq 1$, we write $\mathbb{Z}^d$ and $\mathbb{R}^d$ for $d$-dimensional integer lattice and Euclidean space respectively. In this article, unless otherwise mentioned, we will consider $d = 3$ only. For most of the time, we will also focus on the rescaled lattice $N^{-1}\mathbb{Z}^d$ with dyadic mesh size $N^{-1} = 2^{-n}$ for some integer $n$. For a subset $A \subset \mathbb{R}^d$, we denote its boundary by $\partial A$. Let $\overline{A} = A \cup \partial A$ be the closure of $A$. For two subset $A, B \subset \mathbb{R}^d$, we write $\text{dist}(A, B) = \inf_{x \in A, y \in B} |x - y|$ for the distance between $A$ and $B$. If $A = \{x\}$, we write $\text{dist}(\{x\}, B) = \text{dist}(x, B)$.

For a subset $A \subset N^{-1}\mathbb{Z}^d$, we let $\partial A = \{x \in N^{-1}\mathbb{Z}^d \mid \exists y \in A \text{ such that } |x - y| = \frac{1}{n}\}$ be the outer boundary of $A$. We write $\text{in } A = \{x \in A \mid \exists y \notin A \text{ such that } |x - y| = \frac{1}{n}\}$ for the inner boundary of $A$. Here $|\cdot|$ stands for the Euclid distance in $\mathbb{R}^d$. We write $\mathbb{D} = \{x \in \mathbb{R}^d \mid |x| < 1\}$ for the unit open ball in $\mathbb{R}^d$ centered at the origin. Denote its closure by $\overline{\mathbb{D}}$.

For a subset $A \subset \mathbb{R}^d$, we denote its boundary by $\partial A$. Let $\overline{A} = A \cup \partial A$ be the closure of $A$. For two subset $A, B \subset \mathbb{R}^d$, we write $\text{dist}(\overline{A}, \overline{B}) = \inf_{x \in A, y \in B} |x - y|$ for the distance between $A$ and $B$. If $A = \{x\}$, we write $\text{dist}(\{x\}, \overline{B}) = \text{dist}(x, \overline{B})$.

For $x \in N^{-1}\mathbb{Z}^d$ and $r > 0$, we set $B(x, r) = \{y \in N^{-1}\mathbb{Z}^d \mid |x - y| < r\}$ for the (discrete) ball of radius $r$ centered at $x$. We write $B(r) = B(0, r)$ when $x = 0$.

We now turn to paths and random walks. To fit in the setup of this work, we will also use $N^{-1}\mathbb{Z}^d$ instead of $\mathbb{Z}^d$ for the underlying graph.
A path $\lambda = [\lambda(0), \lambda(1), \ldots, \lambda(m)] \subset N^{-1}Z^d$ is a sequence of points lying on $N^{-1}Z^d$ satisfying $|\lambda(i - 1) - \lambda(i)| = \frac{1}{2}$ for each $1 \leq i \leq m$. We call $m$ the length of $\lambda$ denoted by $\text{len}(\lambda)$. For $\lambda$ a path of length $m$, we write $\lambda^R = [\lambda(m), \lambda(m - 1), \ldots, \lambda(0)]$ for the reversal of $\lambda$. If $\lambda(i) \neq \lambda(j)$ for any $i \neq j$, we call $\lambda$ a simple path. For two paths $\lambda = [\lambda(0), \lambda(1), \ldots, \lambda(m)] \subset N^{-1}Z^d$ and $\lambda' = [\lambda'(0), \lambda'(1), \ldots, \lambda'(m')] \subset N^{-1}Z^d$ with $\lambda(m) = \lambda'(0)$, let $\lambda \oplus \lambda' = [\lambda(0), \lambda(1), \ldots, \lambda(m), \lambda'(1), \ldots, \lambda'(m')]$ be their concatenation.

Let write $S = S^{(N)} = (S^{(N)}(k))_{k \geq 0}$ for a simple random walk in $N^{-1}Z^d$. We often omit the subscript $(N)$. We write $P^x$ and $E^x$ for its probability law and the expectation when it starts from a point $x$. We omit the subscript if $x = 0$. Let $T_{x,r} = \inf\{k \geq 0 \mid S(k) \notin B(x, r)\}$ stand for the first time that the SRW $S$ exits from $B(x, r)$. Here we use the convention that $\inf \emptyset = +\infty$. Also, we set $T_r = T_{0,r}$ for the case that $x = 0$.

We end this subsection with conventions on asymptotics and constants. For two sequences $a_n$ and $b_n$, we write
\begin{itemize}
  \item $a_n \leq b_n$ if $\exists c > 0$ such that $ca_n \leq b_n$ for all $n$;
  \item $a_n \sim b_n$ if $\lim_{n \to \infty} a_n / b_n = 1$;
  \item $a_n \asymp b_n$ if $\log a_n \sim \log b_n$.
\end{itemize}
For two functions $f(x)$ and $g(x)$, we write $g(x) = O(f(x))$ if $g(x) \leq cf(x)$ for some universal constant $c > 0$. If we wish to imply the constant $c$ may depend on another quantity, say $\epsilon$, we write $O_{\epsilon}(f(x))$. Similarly, we write $g(x) = o(f(x))$ if $g(x) / f(x) \to 0$. For $a \in \mathbb{R}$, we write $\lfloor a \rfloor$ for the largest integer less than or equal to $a$. For clarity of notations, we will not use the notation $\lfloor \cdot \rfloor$ for the floor function and avoid using $\lceil \cdot \rceil$ throughout this article.

We use $c, C, c_1, c_2$ to denote arbitrary positive constants which may change from line to line. If a constant depends on some other quantity, this will be made explicit. For example, if $c$ depends on $\epsilon$, we write $c_\epsilon$.

### 2.2 Estimates on Green’s functions

In this subsection, we will introduce some useful estimates on Green’s functions, which will be used many times in this paper. Note that as for most of the time we will be working on $N^{-1}Z^3$, the results introduced here will appear scaled in the main text.

Let $S$ be the SRW on $\mathbb{Z}^3$. Let $1 \leq m < n$. Write $A = \{x \in \mathbb{Z}^3 \mid m \leq |x| \leq n\}$. Set $\tau = \inf\{k \geq 0 \mid S(k) \in \partial A\}$. Then it is proved in Proposition 1.5.10 of [8] that for all $x \in A$,
\begin{equation}
P^x(|S_\tau| \leq m) = \frac{|x|^{-1} - n^{-1} + O(m^{-2})}{m^{-1} - n^{-1}}. \tag{2.1}
\end{equation}

We often want to consider the case that $m = 1$ and $|x|$ is large. In that case, the estimate (2.1) is not useful because the error term $O(m^{-2})$ is much bigger than $|x|^{-1}$. To deal with this issue, we introduce the Green’s function $G : \mathbb{Z}^3 \otimes \mathbb{Z}^3 \to [0, \infty)$ by
\begin{equation}
G(x, y) = E^x(\sum_{j=0}^{\infty} 1\{S(j) = y\}) \text{ for } x, y \in \mathbb{Z}^3. \tag{2.2}
\end{equation}

Since $S$ is transient, $G(x, y) < \infty$ for each $x, y \in \mathbb{Z}^3$. We write $G(x) = G(0, x)$. Theorem 1.5.4 of [8] shows that there exists a universal constant $a > 0$ such that
\begin{equation}
G(x) = a|x|^{-1} + O(|x|^{-2}) \text{ as } |x| \to \infty. \tag{2.3}
\end{equation}

Suppose $m = 1$. Note that $|S_\tau| \leq m \Leftrightarrow S_\tau = 0$ in this case. By considering a bounded martingale $M_j = G(S(j \land \tau))$, we have
\begin{equation}
P^x(S_\tau = 0) = \frac{a|x|^{-1} - an^{-1} + O(|x|^{-2})}{G(0) - an^{-1}}. \tag{2.4}
\end{equation}

We now consider a domain $A \subset \mathbb{Z}^d$. For $x, y \in \mathbb{Z}^d$, we write $G_A(x, y)$ for the Green’s function on $A$. More precisely, letting $\tau = \{k \geq 0 \mid S(k) \in \partial A\}$, we have
\begin{equation}
G_A(x, y) = E^x(\sum_{k=0}^{\tau} 1\{S(k) = y\}). \tag{2.5}
\end{equation}
Let \( C_n = \{ x \in \mathbb{Z}^3 \mid |x| \leq n \} \) be the discrete ball of radius \( n \). Recall that \( G_{C_n}(\cdot, \cdot) \) is Green’s function on \( C_n \) defined as in (2.5). We will need the following lemma in Section 3.

**Lemma 2.1.** There exists a universal constant \( C < \infty \) such that for all \( 0 < \epsilon_1 < \epsilon_2 < 10^{-2} \) and \( n \geq 1 \) with \( \epsilon_1 < \epsilon_2^2 \) and \( 1/n < \epsilon_1^2 \),
\[
\left| \frac{G_{C_n}(x,y)}{G_{C_n}(x',y')} - 1 \right| \leq C \epsilon_1^{3/4} \quad \text{and as a special case,} \quad \left| \frac{G_{C_n}(0,x)}{G_{C_n}(0,x')} - 1 \right| \leq C \epsilon_1^{3/4},
\]
as long as four points \( x, x', y, y' \in C_n \) satisfy the following conditions:

(i) \( \text{dist}(x, \partial C_n) \geq \epsilon_2 n \) and \( \text{dist}(y, \partial C_n) \geq \epsilon_2 n \),
(ii) \( |x - y| \geq \epsilon_2 n \) and \( |x - y| \leq 10 \min \{ n - |x|, n - |y| \} \),
(iii) \( |x - x'| \leq \epsilon_1 n \) and \( |y - y'| \leq \epsilon_1 n \).

**Proof.** We first note that it suffices to work with \( (x, y) \) and \( (x, y') \). Observe that
\[
G_{C_n}(x,y) = G(x,y) - H_{C_n}(x,y) \quad \text{where} \quad H_{C_n}(x,y) := \sum_{z \in \partial C_n} P_x(S_{\tau_z} = z) G(z,y),
\]
and a similar decomposition exists for \( G_{C_n}(x,y') \). By (2.5),
\[
\left| \frac{G(x,y)}{G(x,y')} - 1 \right| \leq O \left( \frac{\epsilon_1}{\epsilon_2} \right) \quad \text{and} \quad \left| \frac{G(z,y)}{G(z,y')} - 1 \right| \leq O \left( \frac{\epsilon_1}{\epsilon_2} \right) \quad \text{uniformly for all} \quad z \in \partial C_n.
\]
Hence, \( |H_{C_n}(x,y)/H_{C_n}(x,y') - 1| \leq O(\epsilon_1/\epsilon_2) \) as well. Also, by Harnack principle and the assumption on the location of \( x, y \), we see that \( G_{C_n}(x,y) \geq \epsilon G(x,y) \) for some universal constant \( \epsilon > 0 \). The claim follows as an easy corollary.

### 2.3 Some metric spaces

We let \( (\mathcal{H}(\overline{D}), d_{\text{Haus}}) \) be the space of all non-empty compact subsets of \( \overline{D} \) endowed with the Hausdorff metric
\[
d_{\text{Haus}}(A,B) := \max \left\{ \sup_{a \in A} \inf_{b \in B} |a-b|, \sup_{b \in B} \inf_{a \in A} |a-b| \right\} \quad \text{for} \quad A, B \in \mathcal{H}(\overline{D}). \tag{2.7}
\]
It is well known that \( (\mathcal{H}(\overline{D}), d_{\text{Haus}}) \) is a compact, complete metric space (see [5] for example).

We set \( (\mathcal{C}(\overline{D}), \rho) \) for the space of continuous curves \( \lambda : [0,t_\lambda] \rightarrow \overline{D} \) with the time duration \( t_\lambda \geq 0 \), where the metric \( \rho \) is defined by
\[
\rho(\lambda_1, \lambda_2) = |t_1 - t_2| + \max_{0 \leq t \leq t_1} |\lambda_1(t) - \lambda_2(t_2)| \quad \text{for} \quad \lambda_1, \lambda_2 \in \mathcal{C}(\overline{D}). \tag{2.8}
\]
It is easy to show that \( (\mathcal{C}(\overline{D}), \rho) \) is a separable metric space (see Section 2.4 of [6]).

We write \( (\mathcal{C}, \chi) \) for the space of continuous curves \( \lambda : [0,\infty) \rightarrow \mathbb{R}^d \), where the metric \( \chi \) is defined by
\[
\chi(\lambda_1, \lambda_2) = \sum_{k=1}^{\infty} 2^{-k} \max_{0 \leq t \leq k} \min \left\{ |\lambda_1(t) - \lambda_2(t)|, 1 \right\} \quad \text{for} \quad \lambda_1, \lambda_2 \in \mathcal{C}. \tag{2.9}
\]
It is easy to check that \( (\mathcal{C}, \chi) \) is a complete, separable metric space (see Section 2.4 of [6]).

Finally, we let \( \mathcal{M}(\overline{D}) \) the space of all finite measures on \( \overline{D} \) equipped with the topology of the weak convergence. The space \( \mathcal{M}(\overline{D}) \) is complete, metrizable and separable (cf. [2], [19]).

### 2.4 Weak convergence of probability measures

We will briefly recall basic facts on the weak convergence of probability measures here. See Chapter 3 of [3] for the details and proofs.

Let \( (M, d) \) be a metric space with its Borel sigma algebra \( B(M) \). We denote the space of all probability measures on \( (M, B(M)) \) by \( \mathcal{P}(M) \), where the space \( \mathcal{P}(M) \) is equipped with the topology of the weak
convergence. For a subset $A \subset M$ and $\epsilon > 0$, we write $A_\epsilon = \{x \in M \mid \exists y \in A \text{ such that } d(x, y) < \epsilon\}$ for the $\epsilon$-neighborhood of $A$. The Prokhorov metric $\pi : \mathcal{P}(M)^2 \to [0, 1]$ is defined by

$$\pi(\mu, \nu) = \inf \left\{ \epsilon > 0 \mid \mu(A) \leq \nu(A_\epsilon) + \epsilon \text{ and } \nu(A) \leq \mu(A_\epsilon) + \epsilon \text{ for all } A \in \mathcal{B}(M) \right\}. \quad (2.10)$$

It is well known that if $(M, d)$ is separable, convergence of measures in the Prokhorov metric is equivalent to weak convergence of measures. Therefore, $\pi$ is a metrization of the topology of weak convergence on $\mathcal{P}(M)$.

It is also well known that if $(M, d)$ is a compact metric space, the metric space $(\mathcal{P}(M), \pi)$ is compact.

Finally, if $(M, d)$ is a complete, separable metric space, the metric space $(\mathcal{P}(M), \pi)$ is complete.

### 2.5 Loop-erased random walk

#### 2.5.1 Definition

Take a path $\lambda = [\lambda(0), \lambda(1), \cdots, \lambda(m)] \subset N^{-1}Z^d$ with $\text{len}(\lambda) < \infty$. We define its loop-erasure $\text{LE}(\lambda)$ in the following way. Let $t_0 = \max\{k \leq m \mid \lambda(k) = \lambda(0)\}$ and for each $i \geq 1$, write $t_i = \max\{k \leq m \mid \lambda(k) = \lambda(t_{i-1} + 1)\}$. Set $l = \min\{i \mid \lambda(t_i) = \lambda(m)\}$. Then, $\text{LE}(\lambda)$ is defined through

$$\text{LE}(\lambda) = [\lambda(t_0), \lambda(t_1), \cdots, \lambda(t_l)]. \quad (2.11)$$

Note that $\text{LE}(\lambda) \subset \lambda$ is a simple path satisfying that $\text{LE}(\lambda)(0) = \lambda(0)$ and $\text{LE}(\lambda)(l) = \lambda(m)$.

Let $S$ be the SRW on $N^{-1}Z^d$ and write $T < \infty$ for some (random or non-random) time. We call such loop-erasure $\text{LE}(S[0, T])$ a loop-erased random walk (LERW). If $d \geq 3$, since SRW is transient, we can consider the loop-erasure of $S[0, \infty)$. In this case, we call $\text{LE}(S[0, \infty))$ the infinite loop-erased random walk (ILERW).

#### 2.5.2 Laplacian random walk and domain Markov property

One can also interpret LERW as the so called Laplacian random walk.

Take a subset $A \subset N^{-1}Z^d$ with $x \in A$. Let $S$ be the SRW on $N^{-1}Z^d$ started at $x$. Suppose that $\tau_A = \inf\{k \geq 0 \mid S(k) \notin A\}$ is finite almost surely. Write $\gamma = \text{LE} (S[0, \tau_A])$ for the LERW started at $x$ and stopped at $\partial A$.

Fix a simple path $\lambda = [\lambda(0), \lambda(1), \cdots, \lambda(m)] \subset A$ with $\lambda(0) = x$. We are interested in the following transition probability of $\gamma$:

$$p(\lambda, y) := P(\gamma(m + 1) = y \mid \gamma[0, m] = \lambda),$$

where $y \notin \lambda$ is a point adjacent to $\lambda$. To compute this transition probability, we let $f$ be the unique (discrete) harmonic function on $A \setminus \lambda$ such that $f(\lambda(i)) = 0$ for all $0 \leq i \leq m$ and $f(z) = 1$ for all $z \in \partial A$.

Then it follows (see Proposition 7.3.1 of [8]) that

$$p(\lambda, y) = \sum_{z \in N^{-1}Z^d, |z - \lambda(m)| = 1} \frac{f(y)}{f(z)}. \quad (2.12)$$

Therefore, the transition probability is well defined at each step of this walk, thanks to the unique solution of the discrete Dirichlet problem discussed above.

The domain Markov property for LERW follows immediately from (2.12). Let $R$ be the random walk started at $\lambda(m)$ conditioned that $R[1, T_A] \cap \lambda = \emptyset$, where $T_A$ stands for the first time that $R$ exits from $A$. Take a simple path $\lambda' = [\lambda'(0), \lambda'(1), \cdots, \lambda'(r)]$ satisfying that $\lambda'(0) = \lambda(m)$, $[\lambda'(0), \lambda'(1), \cdots, \lambda'(r-1)] \subset A$ and $\lambda'(r) \in \partial A$.

The domain Markov property of $\gamma$ (see Proposition 7.3.1 of [8]) states that

$$P(\gamma[m, \text{len}(\gamma)] = \lambda' \mid \gamma[0, m] = \lambda) = P(\text{LE}(R[0, T_A]) = \lambda'). \quad (2.13)$$

Namely, conditioned on $\gamma[0, m] = \lambda$, the distribution of the rest of $\gamma$ is given by the loop-erasure of $R$. 

12
2.5.3 Length of 3D LERW

In Section 2.5.3, Section 2.5.5 and Section 2.5.6 below, we will focus on LERW in three dimensions.

Let $S$ be the SRW on $\mathbb{Z}^3$ started at the origin. We write $T^{(n)} = \inf\{k \geq 0 \mid S(k) \notin \mathbb{D}\}$ for the first time that $S$ exits from the unit ball. Set $\gamma_n = \text{LE}(S[0,T^{(n)}])$ for the LERW with mesh size $2^{-n}$. We let $M_n := \text{len}(\gamma_n)$ be the length of the LERW. It is proved in [10] that there exist universal constants $c, c', \epsilon > 0$ such that for all $n \geq 1$,

$$c2^{(1+\epsilon)n} \leq E(M_n) \leq c'2^{2\beta n}.$$  \hspace{1cm} (2.14)

The result above is subsequently proved in [22] that there exists a constant $\beta \in (1, \frac{5}{3}]$ such that for all $n \geq 1$

$$E(M_n) = 2^{(\beta + o(1))n} \text{ as } n \to \infty.$$  \hspace{1cm} (2.15)

The constant $\beta$ is called the growth exponent of the LERW in three dimensions, which is also sometimes colloquially referred to as the “dimension” of 3D LERW. It is also shown in [22] that there exist $c_1, c_2, \delta > 0$ such that for all $n \geq 1$ and $b \geq 1$,

$$P\left(M_n/E(M_n) \in \left[1/b, b\right]\right) \geq 1 - c_1 \exp \left(-c_2 b^\delta\right).$$  \hspace{1cm} (2.16)

Recently, the authors establish sharp estimates for the probability that $\gamma_n$ hits a given point in [16] (see (2.22) for the one-point estimate). In particular, Corollary 1.3 of [16] shows that

$$E(M_n) \asymp 2^{\beta n}.$$  \hspace{1cm} (2.17)

Combining this with (2.16), we see that $2^{-\beta n} M_n$ is tight.

2.5.4 Escape probability and one-point function estimate

Throughout this subsection, we consider two independent SRW’s $S^1$ and $S^2$ on $\mathbb{Z}^3$ started at the origin. Let $T^i_n$ be the first time that $S^i$ exits from $B(n)$. In [16], the following non-intersection probabilities are considered. For $m \leq n$, we define

$$\text{Es}(n) := P\left(\text{LE}(S^1[0,T^1_n]) \cap S^2[1,T^2_n] = \emptyset\right),$$  \hspace{1cm} (2.18)

$$\text{Es}(m, n) := P\left(\text{LE}(S^1[0,T^1_n]) \cap S^2[1,T^2_n] = \emptyset\right),$$  \hspace{1cm} (2.19)

where $t = \text{len}(\text{LE}(S^1[0,T^1_n]))$ stands for the length of the LERW and

$$s = \max \left\{k \leq t \mid \text{LE}(S^1[0,T^1_n])(k) \notin B(m)\right\}$$

denotes the last time that the LERW exits from $B(m)$. As a shorthand, later in this work, when we are working on the rescaled lattice $N^{-1}\mathbb{Z}^3$, we will also sometimes write $\text{Es}(a,b)$ instead of $\text{Es}(aN,bN)$.

It is proved in Theorem 1.2 of [16] that there exist universal constants $c, \delta > 0$ such that for all $n \geq 1$,

$$\text{Es}(2^n) = c2^{-2(2-\beta)n} \left(1 + O(2^{-\delta n})\right),$$  \hspace{1cm} (2.20)

where $\beta$ is the constant as in (2.15). Furthermore, in Corollary 1.3 of [16], it is proved that for all $1 \leq m \leq n$

$$\text{Es}(n) \asymp n^{-2(2-\beta)}, \quad \text{Es}(m, n) \asymp \left(\frac{n}{m}\right)^{-2(2-\beta)}.$$  \hspace{1cm} (2.21)

In parallel to these results, the following sharp one-point function estimate was also obtained in [16]. Take a point $x \in \mathbb{D}\setminus\{0\}$. We set $x_n$ for the one of the nearest point from $2^n x$ among $\mathbb{Z}^3$. Then Theorem 1.1 of [16] shows that there exist absolute constants $c, \delta > 0$ and a constant $c_\delta > 0$ which depends only on $x \in \mathbb{D}\setminus\{0\}$ such that for all $n \geq 1$ and $x \in \mathbb{D}\setminus\{0\}$

$$P\left(x_n \in \text{LE}(S^1[0,T^1_{2^n}]\right)) = c_\delta 2^{-3(2-\beta)n} \left(1 + O(d_x^{-\delta}2^{-\delta n})\right) \text{ (as } n \to \infty),$$  \hspace{1cm} (2.22)

where $d_x = \min\{|x|, 1-|x|\}$.
2.5.5 Scaling limit of 3D LERW

Recall that the metric space \((\overline{H(D)}, d_{Haus})\) was defined as in (2.7). We can regard the LERW \(\gamma_n\) as a random element of \(H(\overline{D})\). Kozma (see [7]) proves that \(\gamma_n\) (or the probability measure on \(H(\overline{D})\) induced by \(\gamma_n\), more precisely) is a Cauchy sequence with respect to the Prokhorov metric. Since \(H(\overline{D})\) is a compact, complete metric space, this implies that there exists some random \(K \in H(\overline{D})\) such that \(\gamma_n\) converges weakly to \(K\) with respect to the metric \(d_{Haus}\).

Some topological properties of \(K\) are studied in [21]. We recall that \(\gamma \in H(\overline{D})\) is a simple curve if \(\gamma\) is homeomorphic to the interval \([0, 1]\). For a simple curve \(\gamma \in H(\overline{D})\), we write \(\gamma^s\) and \(\gamma^e\) for its end points. Let

\[
\Gamma := \{ \gamma \in H(\overline{D}) \mid \gamma\text{ is a simple curve satisfying } \gamma^s = 0 \text{ and } \gamma \cap \partial \overline{D} = \{ \gamma^e \} \}.
\]

Theorem 1.2 of [21] shows that \(K \in \Gamma\) almost surely.

In [23], it is proved that the Hausdorff dimension of \(K\) is equal to \(\beta\) almost surely.

2.5.6 Quasi loops

Take a discrete path \(\lambda \subset 2^{-n} \mathbb{Z}^3\). Let \(R\) be the SRW on \(2^{-n} \mathbb{Z}^3\). Suppose that the distance between \(R(0)\) and \(\lambda\) is much smaller than the diameter of \(\lambda\). Does \(R\) intersect with \(\lambda\) before exiting a large ball centered at \(R(0)\) with high probability? Unfortunately, the answer is “no” in three dimensions. For example, if we take \(\lambda\) as a straight line, it is unlikely for \(R\) to hit \(\lambda\). The reader should recall that Brownian motion does not hit a straight line in three dimensions.

As we discussed in Section 2.5.3, the “dimension” of LERW is equal to \(\beta > 1\). Thus, if \(\lambda\) in the paragraph above is the trace of LERW, we expect that it is very likely for \(R\) to intersect \(\lambda\) immediately. Theorem 3.1 of [21] confirms this intuition. In fact, there exist universal constants \(\delta, c > 0\) such that for all \(\epsilon > 0\) and \(n \geq 1\),

\[
P\left[ \text{For all } x \in D_n \text{ with dist}(x, \gamma_n) \leq \epsilon^2, P_x^R \left( R[0, T^R_{x, \sqrt{\epsilon}}] \cap \gamma_n = \emptyset \right) \leq \epsilon^\delta \right] \geq 1 - ce, \tag{2.24}
\]

where

- \(D_n = \{ x \in 2^{-n} \mathbb{Z}^3 \mid x \in D \}\) is the set of lattice points lying in \(D\);
- \(R\) is a SRW which is independent of \(\gamma_n\).

Thanks to the estimate (2.24), we can prove that LERW has no “quasi-loops” with high probability in the following sense. Let \(\lambda \subset 2^{-n} \mathbb{Z}^3\) be a path. Take \(0 < s < r\). We say \(\lambda\) has a \((s, r)\)-quasi-loop at \(x \in 2^{-n} \mathbb{Z}^3\) if there exist \(k \leq l\) such that \(\lambda(k), \lambda(l) \in B(x, s)\) and \(\lambda[k, l] \not\subset B(x, r)\). Let \(QL(s, r; \lambda)\) be the set of all such \(x\)’s. It is shown in Theorem 6.1 of [21] that there exist universal constants \(M, a, c > 0\) such that for all \(\epsilon > 0\) and \(n \geq 1\),

\[
P\left( QL(\epsilon^M, \sqrt{\epsilon}; \gamma_n) \neq \emptyset \right) \leq ce^a. \tag{2.25}
\]

2.6 Uniform spanning tree

In this subsection, we will discuss uniform spanning trees (UST), which is closely related to LERW. Although the results stated in this section hold for more general graph, because we want to keep the same notation as in Section 2.5.3, we will only consider uniform spanning trees on \(2^{-n} \mathbb{Z}^3\) here.

Recall that \(\gamma_n\) stands for the LERW on \(2^{-n} \mathbb{Z}^3\) defined in Section 2.5.3. We let \(D_n = \{ x \in 2^{-n} \mathbb{Z}^3 \mid x \in D \}\) stands for the set of lattice points lying in \(D\). Note that the endpoint of \(\gamma_n\) lies on \(\partial D_n\). We now view \(\partial D_n\) as a single point and with slight abuse of notation still denote it by \(\partial D_n\) and write \(G_n\) for the induced graph. We call a spanning tree on \(G_n\) a \emph{wired} spanning tree.

The wired uniform spanning tree (UST) in \(G_n\) denoted by \(U_n\) is a random graph obtained by choosing uniformly random among all wired spanning trees on \(D_n \cup \partial D_n\).

In order to generate \(U_n\), we conduct the following Wilson’s algorithm (see [24]):

- Take an arbitrary ordering of \(D_n = \{ x_1, x_2, \ldots, x_m \}\);
- Consider the SRW \(R^1\) started at \(x_1\) until it first hits \(\partial D_n\). Let \(U^1 = \text{LE}(R^1)\) be its loop-erasure;
- For \(i \geq 1\), consider another SRW \(R^{i+1}\) started at \(x_{i+1}\) until it first hits \(U^i \cup \partial D_n\). Let \(U^{i+1} = U^i \cup \text{LE}(R^{i+1})\);
• Continue this procedure until all points in $D_n$ are included in the tree.

It is proved in [24] that the final output tree in the algorithm above has the same distribution as $U_n$ for any ordering of $D_n$. As a corollary, if we write $\gamma^n_{0, \partial}$ for the unique path in $U_n$ between the origin and $\partial D_n$, then the LERW $\gamma_n$ has the same distribution as $\gamma^n_{0, \partial}$.

See Chapter 9 of [12] for more discussions on the uniform wired spanning tree.

2.7 Coupling

In this section, we will briefly introduce some results obtained by a coupling argument in [11] and [16]. These results will be used in Section 4.

We first consider a pair of LERW and SRW in three dimensions. Take $k \geq 1$. Define

$$\Lambda_k := \{ (\gamma, \lambda) \mid (\gamma, \lambda) \text{ satisfies the conditions (i), (ii) and (iii)} \},$$

where

(i) $\gamma$ is a simple path in $\mathbb{Z}^3$ satisfying that $\gamma[0, \text{len}(\gamma) - 1] \subset B(k)$ and $\gamma(\text{len}(\gamma)) \notin \partial B(k)$.

(ii) $\lambda$ is a path in $\mathbb{Z}^3$ satisfying that $\lambda[0, \text{len}(\lambda) - 1] \subset B(k)$ and $\lambda(\text{len}(\lambda)) \notin \partial B(k)$.

(iii) If $\gamma \cap \lambda = \{ (\gamma(0) \cap \lambda(0)) \}$ and $\lambda$ does not pass through $\lambda(0)$ after time 1, i.e., $\lambda(0) \notin \lambda[1, \text{len}(\lambda)]$.

Note that, if $\gamma(0) = \lambda(0)$, the condition (iii) is equivalent to $\gamma[0, \text{len}(\gamma) \cap \lambda[1, \text{len}(\lambda)] = \emptyset$ and $\gamma[1, \text{len}(\gamma)] \cap \lambda[0, \text{len}(\lambda)] = \emptyset$.

Let $\Lambda^t_{\gamma, \lambda}$. Write $x = \gamma(\text{len}(\gamma))$ and $y = \lambda(\text{len}(\lambda))$ for their endpoints. Let

- $R_1^\gamma$ be the random walk on $\mathbb{Z}^3$ started at $x$ and conditioned that $R_1^\gamma[1, \infty) \cap (\gamma \cup \Theta_1) = \emptyset$;

- $R_2^\lambda$ be the SRW started at $y$ conditioned after time 0.

Note that by the domain Markov property of LERW (see (2.13) for this), $\text{LE}(\gamma)$ has the same distribution as the infinite LERW $\gamma^\infty$ started at $\gamma(0)$ and conditioned that $\gamma^\infty[0, \text{len}(\gamma)] = \gamma$. By strong Markov property of SRW, $R_2^\lambda$ has the same distribution as the SRW $X$ started at $\lambda(0)$, and conditioned to avoid $\lambda(0)$ after time 0 and that $X[0, \text{len}(\lambda)] = \lambda$. Thus, in this sense, we can regard $\Lambda_k$ as the set of “initial configurations” of infinite LERW and SRW which do not intersect each other.

Let $l > k$. Write $T_l^1$ for the first time that $\text{LE}(R_1^\gamma)$ exits from $B(l)$. Similarly, set $T_l^2$ for the first time that $R_2^\lambda$ exits from $B(l)$. Let

$$A_{\gamma, \lambda} = \{ \gamma \oplus \text{LE}(R_1^\gamma)[0, T_l^1], \lambda \oplus R_2^\lambda[0, T_l^2] \} \in \Gamma_l$$

be the event that $\gamma \oplus \text{LE}(R_1^\gamma)[0, T_l^1]$ does not intersect with $\lambda \oplus R_2^\lambda[0, T_l^2]$.

We are interested in the (conditional) distribution of $(\text{LE}(R_1^\gamma), R_2^\lambda)$ conditioned on the event $A_{\gamma, \lambda}$. Of course, this conditional distribution depends on the initial configuration $(\gamma, \lambda)$. However, after a long travel both LERW and SRW gradually “forget” their initial configuration in the following sense. Take $m > l > k$. Let

$$\mu^{k,l,m}_{\gamma, \lambda}(\eta^1, \eta^2) := P(\text{LE}(R_1^\gamma)[T_l^1, T_m^1] = \eta^1, R_2^\lambda[T_l^2, T_m^2] = \eta^2 \mid A_{\gamma, \lambda}^m)$$

be the probability measure on $\Lambda_m$ which is induced by $(\text{LE}(R_1^\gamma)[T_l^1, T_m^1], R_2^\lambda[T_l^2, T_m^2])$ conditioned on the event $A_{\gamma, \lambda}^m$.

In [16], it is shown that there exist universal constants $c, \delta > 0$ such that for all $k < l < m$ and any $(\gamma, \lambda), (\gamma', \lambda') \in \Lambda_k$,

$$\| \mu^{k,l,m}_{\gamma, \lambda} - \mu^{k,l,m}_{\gamma', \lambda'} \| \leq c \left( \frac{k}{m} \right)^\delta,$$

where $\| \cdot \|$ stands for the total variation distance.

We now give a generalization of the coupling result introduced above which will be useful in this work. More precisely, as in Section 4 we will be estimate box-crossing probabilities, we must consider a more complicated scenario here, where part of the initial configuration of the LERW and SRW are some “fossilized” sets instead of paths, and these fossilized sets are not necessarily non-intersecting.

We now consider a new collection of pairs of paths with initial configurations. Let

$$\overline{\Lambda}_k := \{ (\gamma, \lambda, \Theta_1, \Theta_2) \mid (\gamma, \lambda, \Theta_1, \Theta_2) \text{ satisfies the conditions (i), (ii), (iii) and (iv)} \},$$

where (i)(ii)(iii) are the same as (i)(ii)(iii) below (2.26) and
(iv) $\Theta_\gamma$ and $\Theta_\lambda$ are subsets of $B(k)$, such that
$$
\gamma \cap \Theta_\gamma = \{\gamma(0)\}; \quad \lambda \cap \Theta_\lambda = \{\lambda(0)\}.
$$

Note that $\Theta_\gamma \cap \Theta_\lambda$ is not necessarily empty and if $\Theta_\gamma = \gamma(0)$ and $\Theta_\lambda = \lambda(0)$, then the setup is reduced to that of $\Lambda_k$ in (2.26).

We now modify the definition of $R^1$ and $R^2$ above by letting them avoid $\gamma \cup \Theta_\gamma$ instead of $\gamma$, and $\Theta_\lambda$ instead of $\gamma$, respectively. We then define in the same manner the event $A_{(\gamma,\lambda,\Theta_\gamma,\Theta_\lambda)}^m$ and the probability measure $\mu_{(\gamma,\lambda,\Theta_\gamma,\Theta_\lambda)}(\eta_1,\eta_2)$ on $\Gamma_m$ (note that for this measure we can safely “forget” the past, including the “fossilized” sets). Then, as we will see below, by a modification of the argument in [16], one can show that there exist universal constants $c, \delta > 0$ such that for all $k < l < m$ and any $(\gamma_0, \lambda_0) \in \Lambda_k$, and $(\gamma, \lambda, \Theta_\gamma, \Theta_\lambda) \in \Gamma_k$,
$$
\| \mu_{(\gamma, \lambda, \Theta_\gamma, \Theta_\lambda)}^k - \mu_{(\gamma, \lambda, \Theta_\gamma, \Theta_\lambda)}^{k, l, m} \| \leq c \left( \frac{k}{l} \right)^\delta.
$$

(2.31)

We now briefly explain how to modify the arguments from [16] to obtain (2.31). Recall that, loosely speaking, in [16] the argument goes as follows: we start with the coupling of two pairs of independent LERW’s tilted by a term involving loop measures from [11] where this tilting term is in the form of (4.7) of [16] or $Q_k$ in Section 2.1 of [11], then attach loops to one of the LERW’s to recover an SRW, and finally show that the pair of LERW and SRW obtained above has the law of (2.28).

We then observe that

- While defining the tilting term in (a) above, the initial configuration need not necessarily be a path;
- As pointed out in Remark 4.4 of [16], when we tilt the law of a pair of LERW paths which are not completely non-intersecting, we can exclude the loops that pass through both paths;
- As pointed out in the comments above Remark 4.8 of [16], the separation lemma and the coupling arguments are sufficiently stable, hence the same coupling process remains valid for this very general setup.

Finally we point out a remarkable observation obtained from these coupling results above. Let $S$ be the SRW in $\mathbb{Z}^d$ started at the origin and let $\gamma^\infty = LE(S[0, \infty))$ be the infinite LERW. Take a point $x \in \gamma^\infty$ which is far from the origin. Take $k \ll l \ll |x|$. The estimate (2.29) roughly says that whatever happens in the inside of $B(x, k)$ for the infinite LERW, there is no influence on $\gamma^\infty$ outside of $B(x, l)$. This gives a certain kind of asymptotic independence for the LERW. This observation will be extensively incorporated in Section 3.

2.8 Separation lemma

Here we will use the same notations as in Section 2.7. Take $k \geq 1$ and a pair of paths $(\gamma, \lambda) \in \Lambda_k$ (see (2.26) for $\Lambda_k$). For each $l > k$, we recall that the non-intersection event $A_{(\gamma, \lambda)}^l$ was defined as in (2.27). The separation lemma (Theorem 6.5 of [22]) roughly states that the LERW $LE(R^1_k)$ and SRW $R^2_k$ conditioned on the event $A_{(\gamma, \lambda)}^{2k}$ have a good chance of being reasonably far apart even if the end point of $\gamma$ is very close to that of $\lambda$. To be more precise, let
$$
D_l(\gamma, \lambda) = l^{-1} \min \left\{ \text{dist} \left( \gamma(\text{len}(\gamma)), \lambda \right), \text{dist} \left( \lambda(\text{len}(\lambda)), \gamma \right) \right\},
$$

(2.32)

for $(\gamma, \lambda) \in \Lambda_l$. Then Theorem 6.5 of [22] proves that there exists a universal constant $c > 0$ such that for all $k \geq 1$ and all $(\gamma, \lambda) \in \Lambda_k$
$$
P \left\{ D_{2k} \left( \gamma \oplus LE(R^1_k)[0, T_{2k}], \lambda \oplus R^2_k[0, T_{2k}] \right) \geq c \right\} \geq A_{(\gamma, \lambda)}^{2k} \geq c.
$$

(2.33)

This estimate will be used many times in the present article.

3 Setup and preliminary estimates for the key $L^2$-estimate (1.3)

In this section, we will introduce the setup for the crucial $L^2$-estimate (1.3), and give a few preliminary up-to-constant second moment estimates on the number of cubes hit by LERW.
Let \( k \geq 1 \) be an integer. Take a cube \( B \) of side length \( 2^{-k} \) which is lying in \( \mathbb{D} = \{ x \in \mathbb{R}^3 \mid |x| < 1 \} \). We assume that \( \text{dist}(B, \{0\} \cup \partial \mathbb{D}) > 2^{-k} \). Write \( \pi \) for the center of \( B \).

We divide \( B \) into smaller cubes \( B_1, B_2, \ldots, B_{k_m} \) of side length \( 2^{-k^4} \) so that the number of the smaller cubes satisfies \( k_m = 2^{k(k^4-k)} \). Let \( x_i \) be the center of \( B_i \). We write \( B_i' \) for the cube of radius \( 3 \cdot 2^{-k^4} \) centered at \( x_i \). Throughout this section and the next section, we write

\[
\epsilon = 6 \cdot 2^{-k^4} \text{ for side length of } B_i' ;
\]

\[
r = \min(|\pi|, 1 - |\pi|) \text{ for the distance between } \pi \text{ and } 0 \cup \partial \mathbb{D}.
\]

We consider a loop-erased random walk on \( 2^{-n} \mathbb{Z}^3 \). Assume that \( n \) is large enough so that \( 2^{-n} \ll 2^{-k^4} \). Let \( S \) be the simple random walk on \( 2^{-n} \mathbb{Z}^3 \) started at the origin and let \( T \) be the first time that \( S \) exits from \( \mathbb{D} \). Let \( \gamma \) be the loop-erasure of \( S[0,T] \).

Let \( X_i \) be the number of points in \( B_i \) hit by \( \gamma \). Note that since \( \gamma \) is a simple path, \( X_i \) is also the time \( \gamma \) spends in \( B_i \). Then, let \( Y_i \) be the indicator function of the event that \( \gamma \) hits \( B_i' \). We set

\[
X = \sum_{i=1}^{k_m} X_i
\]

for the total number of points in \( B \) hit by \( \gamma \) and set

\[
Y = \sum_{i=1}^{k_m} Y_i
\]

for the number of cubes among \( B_1, \ldots, B_{k_m} \) hit by \( \gamma \).

As discussed in Section 1.2.1, we are going to use \( Y \) to predict \( X \) in the form of an \( L^2 \)-estimate (see 1.3) as well as Proposition 4.1. To this end, we need to give some preliminary up-to-constant bounds for various second moment estimates.

We start with the upper bound.

**Proposition 3.1.** There exists universal constant \( C > 0 \) and constant \( N_k \) depending only on \( k \) such that for all \( k \geq 1, n \geq N_k \), and \( B \) satisfying \( \text{dist}(B, \{0\} \cup \partial \mathbb{D}) > 2^{-k} \), it follows that for any \( i, j = 1, \ldots, k_m \)

\[
P(Y_i = Y_j = 1) \leq C \left( \frac{\epsilon}{r} \right)^{3-\beta} \left( \frac{\epsilon}{l} \right)^{3-\beta} \quad \text{if } B \subset \frac{2}{3} \mathbb{D};
\]

\[
P(Y_i = Y_j = 1) \leq C \left( \frac{\epsilon}{l} \right)^{3-\beta} \left( \frac{\epsilon}{r} \right)^{3-\beta} \quad \text{if } B \subset \mathbb{D} \setminus \frac{1}{3} \mathbb{D},
\]

where \( \beta \) is the exponent as in (2.15).

**Proof.** This proposition follows easily from Theorem 3.1.1 of [23]. We first assume that \( B \subset \frac{2}{3} \mathbb{D} \). Then by (3.72) in the proof of Theorem 3.1.1 of [23], it follows that there exists universal constant \( C > 0 \) and constant \( N_k \) depending only on \( k \) such that for all \( k \geq 1, n \geq N_k \)

\[
P(Y_i = Y_j = 1) \leq C \left( \frac{\epsilon}{r} \right)^{3-\beta} \left( \frac{\epsilon}{l} \right)^{3-\beta} \text{Es}(\epsilon^{2n} r^{2n}) \text{Es}(\epsilon^{2n} l^{2n})
\]

where \( \text{Es}(\cdot, \cdot) \) is defined as in (2.15). However, by (2.20), we see that the RHS of the inequality above is bounded above by \( C(\epsilon/r)^{2-\beta} (\epsilon/l)^{2-\beta} \), which gives (3.3).

Next, assume \( B \subset \mathbb{D} \setminus \frac{1}{3} \mathbb{D} \). Then by (3.72) in the proof of Theorem 3.1.1 of [23] again, we see that there exists universal constant \( C > 0 \) and constant \( N_k \) depending only on \( k \) such that for all \( k \geq 1, n \geq N_k \)

\[
P(Y_i = Y_j = 1) \leq C \text{Es}(\epsilon^{2n} r^{2n}) \text{Es}(\epsilon^{2n} l^{2n}).
\]

Then (3.4) again follows from (2.20).

Proposition 3.1 gives the following second moment estimate on \( Y \). Note that since in Section 4, we will only be dealing with the case of \( B \subset \frac{2}{3} \mathbb{D} \), we will also only give bounds in this case.

**Proposition 3.2.** Let \( N_k \) be the constant as in Proposition 3.1. Then there exists a universal constant \( C > 0 \) such that for all \( k \geq 1, n \geq N_k \), and \( B \subset \frac{2}{3} \mathbb{D} \) satisfying \( \text{dist}(B, \{0\}) > 2^{-k} \), it follows that

\[
E(Y^2) \leq C \epsilon^{-2\beta} l^{2\beta},
\]

where \( \beta \) is the constant as in (2.15).
Proof. By the definition of $\text{ES}(. , \cdot)$, it is not difficult to show that there exists $C > 0$, such that for $i = 1, \ldots , m_k$, 
\[ E(Y_i) \leq C \left( \frac{\epsilon}{7} \right)^{3-\beta}. \] (3.6)

Then by (3.3) and the observation that $r > 2^{-k}$, we have 
\[ E(Y^2) = \sum_{i=1}^{k_m} E(Y_i) + \sum_{1 \leq i \neq j \leq k_m} E(Y_i Y_j) \leq C \left( \frac{\epsilon}{r} \right)^{\beta} + C \sum_{i=1}^{k_m} \sum_{l=1}^{4r} \left( \frac{\epsilon}{7} \right)^{3-\beta} \left( \frac{\epsilon}{7} \right)^{3-\beta} \leq C \left( \frac{\epsilon}{r} \right)^{2\beta} \]
which completes the proof. 

We now also give a lower bound on the $L^2$-estimate. We will also only focus on the case of $B \subseteq \overline{{\mathbb{D}}}^3$ as explained above Proposition 3.2.

Proposition 3.3. Let $N_k$ be the constant as in Proposition 3.1. Then there exists universal constants $c_1, c_2 > 0$ such that for all $k \geq 1$, $n \geq N_k$ and $B \subseteq \overline{{\mathbb{D}}}^3$ satisfying $\text{dist}(B, \{0\}) > 2^{-k}$, it follows that 
\[ c_1 \left( \frac{\epsilon}{r} \right)^{3-\beta} \left( \frac{\epsilon}{7} \right)^{3-\beta} \leq \text{P}(0 \xrightarrow{\epsilon} B'_1 \xrightarrow{\epsilon} B'_2) \leq c_2 \left( \frac{\epsilon}{r} \right)^{3-\beta} \left( \frac{\epsilon}{7} \right)^{3-\beta}. \] (3.7)

Proof. As the upper bound is already given in Proposition 3.1, we only show the lower bound.

Decomposing the path of $S$ at $T^i$ and $T^j$ as in the proof of Theorem 3.1.1 of [23], the probability $P(0 \xrightarrow{\epsilon} B'_1 \xrightarrow{\epsilon} B'_2)$ can be bounded from below by 
\[ \sum_{y \in \partial B'_1} \sum_{z \in \partial B'_2} G_{\mathcal{D}}(0, y) G_{\mathcal{D}}(y, z) P(F_{y, z}), \] (3.8)

where for each $y \in \partial B'_1$ and $z \in \partial B'_2$ the event $F_{y, z}$ is defined in (3.10) as follows. Here as a slight abuse of notation, we write 
\[ G_{\mathcal{D}} := G_{\mathcal{D} \cap 2^n \mathbb{Z}^3} \] (3.9)

for short.

We introduce three independent random walks $Y^1, Y^2$ and $Y^3$ such that
- $Y^1$ starts from $y$ and is conditioned that it hits the origin at time $\tau_0$ before exiting $\mathbb{D}$; 
  \[ u = \text{len}(\text{LE}(Y^1[0, \tau_0])), \ u' = \max\{t \leq u | \text{LE}(Y^1[0, \tau_0])(t) \in B'_1\}; \]
- $Y^2$ starts from $y$ and is conditioned that it hits $z$ at time $\tau_2$ before exiting $\mathbb{D}$;
- $Y^3$ is the simple random walk started at $z$ stopped at exiting $\mathbb{D}$.

Then,
\[ F_{y, z} := A \cap B \cap C \cap D \cap E, \] (3.10)

where
- $A := \{ u' \text{ is smaller than the first time that } \text{LE}(Y^1[0, \tau_0]) \text{ exits from } B(x_i, \frac{\epsilon}{2}) \}$;
- $B := \{ \text{LE}(Y^1[0, \tau_0])[u', u] \cap (Y^2[1, \sigma_2] \cup Y^3[0, T]) = \emptyset \}$;
- $C := \{ Y^2[1, \sigma_2] \cap B'_1 = \emptyset \} \cap \{ Y^3[1, T] \cap (B(x_i, \frac{\epsilon}{2}) \cup B'_j) = \emptyset \}$;
- $D := \{ \exists k \in [T_{x_i}, T_{x_i} + \frac{\epsilon}{2}] \text{ such that } (Y^2[0, k] \cup B(x_i, \frac{\epsilon}{2})) \cap Y^2[k + 1, \sigma_2] = \emptyset \}$;
- $E := \{ \text{LE}(Y^2[1, \sigma_2]) \cap Y^3[0, T] = \emptyset \}$.

It follows from the separation lemma (See Section 2.8, Proposition 4.6 of [17], Lemma 4.3 and Lemma 6.9 of [22] that
\[ P(F_{y, z}) \geq c (\epsilon 2^n)^{-2} \text{ES}(\epsilon 2^n, t2^n) \text{ES}(\epsilon 2^n, r2^n) \approx (\epsilon 2^n)^{-2} \left( \frac{\epsilon}{7} \right)^{2-\beta} \left( \frac{\epsilon}{7} \right)^{2-\beta}. \]

Since the number of points in $\partial B'_1 \cap 2^{-n} \mathbb{Z}^3$ is comparable to $(\epsilon 2^n)^2$, using the fact that $G(0, y) \approx (r2^n)^{-1}$ and $G(y, z) \approx (l2^n)^{-1}$, it follows that
\[ \sum_{y \in \partial B'_1} \sum_{z \in \partial B'_2} G(0, y) G(y, z) P(F_{y, z}) \geq c (\epsilon 2^n)^4 (r2^n)^{-1} (l2^n)^{-1} (\epsilon 2^n)^{-2} \left( \frac{\epsilon}{7} \right)^{2-\beta} \left( \frac{\epsilon}{7} \right)^{2-\beta} = \alpha (\epsilon 2^n)^{-6} - \beta - 3 \beta - 3, \]
which completes the proof. 

\[ \square \]
We end this section by giving the following remark on other up-to-constant $L^2$-bounds.

**Remark 3.4.** By similar arguments, we can also derive up-to-constant bounds for two-point functions or hybrid point-box-crossing probabilities which will be discussed in Section 4 (similarly defined as point-crossing and box-crossing probabilities in (4.22) and (4.23), see also (3.12) below).

For example, under the same setup as Prop 3.3, for $v \in B'_1$ and $w \in B'_1$, there exist universal constants $c_1, c_2 > 0$ and a constant $N_k < \infty$ depending only on $k$ such that for all $k \geq 1$ and $n \geq N_k$,

$$c_1 \left( \frac{2^{-n}}{r} \right)^{3-\beta} \left( \frac{\tau}{r} \right)^{3-\beta} \leq P(0 \xrightarrow{\gamma} v \xrightarrow{\gamma} w) \leq c_2 \left( \frac{2^{-n}}{r} \right)^{3-\beta} \left( \frac{\tau}{r} \right)^{3-\beta};$$

$$c_1 \left( \frac{2^{-n}}{r} \right)^{3-\beta} \left( \frac{\epsilon}{\tau} \right)^{3-\beta} \leq P(0 \xrightarrow{\gamma} v \xrightarrow{\gamma} B'_1), \quad P(0 \xrightarrow{\gamma} B'_1 \xrightarrow{\gamma} w) \leq c_2 \left( \frac{2^{-n}}{r} \right)^{3-\beta} \left( \frac{\epsilon}{\tau} \right)^{3-\beta}. \quad (3.12)$$

Note that (4.11) is a corollary of (3.7), (3.11) and (3.12) and the second claim of (4.17) is a corollary of (3.11). See also Remarks 4.18 for more discussions related to hybrid point-box-crossing probabilities.

4 Proof of the key estimate (1.3)

In this section we will be focusing on the key estimate (1.3), whose precise statement is given in Proposition 4.1. We will first give a sketch of the proof in Section 4.1. Intermediate steps of this proof will be laid out in subsequent sections. We will conclude the proof in Section 4.7.

4.1 Statement and a sketch of the proof

We first recall all notations introduced at the beginning Section 3. In order to state the $L^2$-estimate, we need to consider a reference cube. Let $B_0$ be the cube of side length $2^{-k}$ centered at $x_0 = (\frac{1}{2}, 0, 0)$. We write $B'_0$ for the cube of radius $3 \cdot 2^{-k}$ centered at $x_0$. We set $X_0$ for the number of points in $B_0$ which is passed through by $\gamma$. Let $Y_0$ be the indicator function of the event that $\gamma$ hits $B'_0$. Let

$$\alpha_0 = \alpha_0(n, k) := E\left( X_0 \mid Y_0 = 1 \right). \quad (4.1)$$

We are now ready to state the key estimate (1.3) in its precise form.

**Proposition 4.1.** There exist universal constants $c > 0$, $C < \infty$ and a constant $N_k$ depending only on $k$ such that for all $k \geq 1$, $n \geq N_k$ and $B$ satisfying $\text{dist}(B, \{0\} \cup \partial D) > 2^{-k}$, we have

$$E\left( (X - \alpha_0 Y)^2 \right) \leq C 2^{-ck^2} [E(X)]^2. \quad (4.2)$$

Since its proof is quite long, in the rest of this subsection, we will give an outline of the proof. Note that without loss of generality, throughout this section we will assume

$$B \subset \frac{2}{3} D, \quad (4.3)$$

for the other case $B \subset D \setminus \frac{1}{3} D$ can be dealt with similarly with only a few differences on the up-to-constant bounds, see also Proposition 3.4.

1. The LHS of (4.2) can be rewritten as

$$E\left( (X - \alpha_0 Y)^2 \right) = \sum_{1 \leq i, j \leq k_m} E(Y_i Y_j) E\left( (X_i - \alpha_0)(X_j - \alpha_0) \mid Y_i = Y_j = 1 \right). \quad (4.4)$$

2. To deal with the sum in the RHS of (4.4), we will need to distinguish the typical case where $i$ and $j$ are distant from each other, from the atypical case where $i$ and $j$ are very close to each other, whose contribution towards the RHS of (4.4) is negligible. To be more precise, we say that the pair $(i, j)$ is **good**, if $|x_i - x_j| \geq 2^{-k^2}$, and **bad** otherwise. \quad (4.5)
3. Note that
\[
E\left((X_i - \alpha_0)(X_j - \alpha_0) \mid Y_i = Y_j = 1\right) = E\left(X_i X_j \mid Y_i = Y_j = 1\right) - \alpha_0 E\left(X_i \mid Y_i = Y_j = 1\right) - \alpha_0 E\left(X_j \mid Y_i = Y_j = 1\right) + \alpha_0^2. \tag{4.6}
\]

A central result in this section is that if \((i, j)\) is good,
\[
E\left(X_i X_j \mid Y_i = Y_j = 1\right) = \left(1 + O(2^{k^2})\right)\alpha_0^2; \tag{4.7}
\]
\[
E\left(X_i \mid Y_i = Y_j = 1\right) = \left(1 + O(2^{k^2})\right)\alpha_0; \tag{4.8}
\]
\[
E\left(X_j \mid Y_i = Y_j = 1\right) = \left(1 + O(2^{k^2})\right)\alpha_0, \tag{4.9}
\]
which implies that
\[
E\left((X_i - \alpha_0)(X_j - \alpha_0) \mid Y_i = Y_j = 1\right) = \alpha_0^2 O(2^{k^2}). \tag{4.10}
\]

The complete proof of (4.7) will be given in Section 4.7 (see also Paragraph 8 below for explanations on a key equation (4.26) in the program). Explanations on necessary modifications for proving (4.8) and (4.9) will be given in Remark 4.18.

4. For the case that \((i, j)\) is bad, it is not difficult to show (see Remark 3.4) that
\[
E\left(X_i X_j \mid Y_i = Y_j = 1\right) \leq C\alpha_0^2; \quad E\left(X_i \mid Y_i = Y_j = 1\right), E\left(X_j \mid Y_i = Y_j = 1\right) \leq C\alpha_0 \tag{4.11}
\]
for some universal constant \(C < \infty\).

5. Therefore, we have
\[
E\left((X - \alpha_0 Y)^2\right) = \sum_{1 \leq i, j \leq k_m} E(Y_i Y_j) E\left((X_i - \alpha_0)(X_j - \alpha_0) \mid Y_i = Y_j = 1\right) \\
= \sum_{(i, j) \text{ good}} E(Y_i Y_j) E\left((X_i - \alpha_0)(X_j - \alpha_0) \mid Y_i = Y_j = 1\right) \\
+ \sum_{(i, j) \text{ bad}} E(Y_i Y_j) E\left((X_i - \alpha_0)(X_j - \alpha_0) \mid Y_i = Y_j = 1\right) \\
\leq C 2^{-k^2} \sum_{(i, j) \text{ good}} E(Y_i Y_j) \alpha_0^2 + C \sum_{(i, j) \text{ bad}} E(Y_i Y_j) \alpha_0^2. \tag{4.12}
\]

6. We now bound \(E(Y_i Y_j)\) for bad \((i, j)\)'s. As a direct consequence of (4.3.7), we have
\[
\sum_{(i, j) \text{ bad}} E(Y_i Y_j) \leq C 2^{-ck^2} \sum_{(i, j) \text{ good}} E(Y_i Y_j), \tag{4.13}
\]
for some universal constants \(0 < c, C < \infty\) (this is a corollary of Proposition 3.3). Thus, we have
\[
E\left((X - \alpha_0 Y)^2\right) \leq C 2^{-ck^2} \sum_{(i, j) \text{ good}} E(Y_i Y_j) \alpha_0^2. \tag{4.14}
\]

However, by (4.7), we see that
\[
\sum_{(i, j) \text{ good}} E(Y_i Y_j) \alpha_0^2 \leq C \sum_{(i, j) \text{ good}} E(X_i X_j) \leq C \sum_{1 \leq i, j \leq k_m} E(X_i X_j) \tag{4.15}
\]
Consequently, we have
\[
E\left((X - \alpha_0 Y)^2\right) \leq C 2^{-ck^2} \sum_{1 \leq i, j \leq k_m} E(X_i X_j) = C 2^{-ck^2} E(X^2). \tag{4.16}
\]
7. It is not difficult to show that
\[ E(X) \geq c2^{-3k}2^{\beta n} \quad \text{and} \quad E(X^2) \asymp E(X)\left(2^{-k} \cdot 2^n\right)\beta. \tag{4.17} \]
In fact, the first claim follows from Theorem 1.1 of \[16\] and the second claim follows from \[3.11\]. See also Remark \[3.4\]. Thus, we have
\[ E(X^2) \leq C2^{(3-\beta)k} \left(E(X)\right)^2, \tag{4.18} \]
which gives
\[ E\left((X - \alpha_0 Y)^2\right) \leq C2^{-ck^2+(3-\beta)k} \left(E(X)\right)^2 \leq C2^{-c'k^2} \left(E(X)\right)^2. \tag{4.19} \]
This gives \[4.2\].

8. Therefore, the crucial step is to establish \[4.7\], \[4.8\] and \[4.9\]. Here we only explain the sketch of the proof for \[4.7\]. Assume that \((i,j)\) is good. Note that
\[ E\left(X_i X_j \mid Y_i = Y_j = 1\right) = \frac{E\left(X_i X_j\right)}{P\left(Y_i = Y_j = 1\right)} = \sum_{v \in B_i, w \in B_j} \frac{P\left(v, w \in \gamma\right)}{P\left(\gamma \cap B_i' = \emptyset, \gamma \cap B_j' = \emptyset\right)}. \tag{4.20} \]

9. We now deal with the fraction in the summand of the RHS of \[4.20\]. We continue assuming that \((i,j)\) is good. Note that
\[ \frac{P\left(v, w \in \gamma\right)}{P\left(\gamma \cap B_i' = \emptyset, \gamma \cap B_j' = \emptyset\right)} = \frac{P\left(0 \xrightarrow{\gamma} v \xrightarrow{\gamma} w\right) + P\left(0 \xrightarrow{\gamma} w \xrightarrow{\gamma} v\right)}{P\left(0 \xrightarrow{\gamma} B_i' \xrightarrow{\gamma} B_j'\right) + P\left(0 \xrightarrow{\gamma} B_j' \xrightarrow{\gamma} B_i'\right)}, \tag{4.21} \]
where we let
\[ \left\{0 \xrightarrow{\gamma} x \xrightarrow{\gamma} y\right\} := \left\{\gamma \text{ first hits } x \text{ and then hits } y\right\} \quad \text{and} \tag{4.22} \]
\[ \left\{0 \xrightarrow{\gamma} A \xrightarrow{\gamma} A'\right\} := \left\{\gamma \text{ first hits } A \text{ and then hits } A'\right\}. \tag{4.23} \]

for points \(x \neq y\) and for sets \(A, A'\) with \(A \cap A' = \emptyset\). We will show that
\[ |B_0|^2P\left(0 \xrightarrow{\gamma} v \xrightarrow{\gamma} w\right) = \alpha_0^2P\left(0 \xrightarrow{\gamma} B_i' \xrightarrow{\gamma} B_j'\right) \left(1 + O(2^{-k^2})\right) \tag{4.24} \]
\[ |B_0|^2P\left(0 \xrightarrow{\gamma} w \xrightarrow{\gamma} v\right) = \alpha_0^2P\left(0 \xrightarrow{\gamma} B_j' \xrightarrow{\gamma} B_i'\right) \left(1 + O(2^{-k^2})\right), \tag{4.25} \]
which gives \[4.7\].

**Remark 4.2.** Note that the argument above gives equivalently the following estimates: for all \(v \in B_i\) and \(w \in B_j\), we have
\[ E\left(X_i X_j \mid Y_i = Y_j = 1\right) = \left(1 + O(2^{-k^2})\right) \frac{|B_0|^2 P\left(v, w \in \gamma\right)}{P\left(\gamma \cap B_i' = \emptyset, \gamma \cap B_j' = \emptyset\right)}, \tag{4.26} \]
where \(|B_0|\) stands for the cardinality of \(B_0\), and for all \(v, v' \in B_i\) and \(w, w' \in B_j\),
\[ P\left(v, w \in \gamma\right) = \left(1 + O(2^{-k^2})\right) P\left(v', w' \in \gamma\right). \tag{4.27} \]

10. We will explain how to derive \[4.24\]. The equation \[4.25\] can be proved similarly. We begin the derivation by giving explicit representations for the point-crossing and box-crossing probabilities in \[4.24\] via non-intersecting probabilities of LERW and SRW.

We first re-express the box-crossing probability in \[4.24\], which is the harder one of the two. We introduce the following notation:
• Fix $y \in \partial B'_i$ and $z \in \partial B'_j$. Write
  
  – $X^1$ for the random walk started at $y$ and conditioned to hit the origin before leaving $\mathbb{D}$;
  – $X^2$ for the random walk started at $z$ and conditioned to hit $y$ before leaving $\mathbb{D}$;
  – $X^3$ for the simple random walk started at $z$.

Assume that $X^1, X^2$ and $X^3$ are independent and write $P_{1,2,3}^{y,z,z}$ for the law of $(X^1, X^2, X^3)$.

• Let $\tau_0 = \min\{t \geq 0 \mid X^1(t) = 0\}$ be the first time that $X^1$ hits the origin.
• Let $\sigma_y = \min\{t \geq 0 \mid X^2(t) = y\}$ be the first time that $X^2$ hits $y$.
• Let $T = \min\{k \geq 0 \mid X^3(k) \notin \mathbb{D}\}$ be the first time that $X^3$ leaves $\mathbb{D}$.

• For a path $\lambda = [\lambda(0), \lambda(1), \cdots, \lambda(m)]$, let $(\lambda)^R = [\lambda(m), \lambda(m-1), \cdots, \lambda(0)]$ be its time reversal.

• Write $\gamma^1 = LE(\mathbb{X}^1[0,\tau_0])$ for the loop-erasure of $X^1$ and $w_1 = \text{len}(\gamma^1)$ for the length of $\gamma^1$.

• Let $\sigma := \max\{t \mid X^2(t) \in \partial B(X_i, 2^{-k^3})\}$. Write $\gamma^2 = LE(\mathbb{X}^2[0,\sigma])$ be the loop-erasure of $X^2[0,\sigma]$. Let $w_2 = \text{len}(\gamma^2)$ be the length of $\gamma^2$.

• Let $\rho_1 = \inf\{k \geq 0 \mid \gamma^1(k) \notin B(x_1, 2^{-k^3+k^2})\}$ be the first time that $\gamma^1$ exits the ball of radius $2^{-k^3+k^2}$ centered at $x_1$ and let $v_1 = \max\{k \leq \rho_1 \mid \gamma(k) \in B'_i\}$ be the last time (up to $\rho_1$) that $\gamma^1$ exits $B'_i$. We define $\rho_2$ and $v_2$ similarly with $\gamma^1$, $x_i$ and $B'_i$ replaced by $\gamma^2$, $x_j$ and $B'_j$ respectively.

As a convention, for two quantities $p$ and $q$ which depend on $k$ (and implicitly $n \gg k$), we write $p \asymp q$ if $p = (1 + O(2^{-k^3}))q$ where the constant for $O$ is universal.

With some effort, we are able to show that (see 3.5 for convention on the use of $G_D$)

$$P\left(0 \xrightarrow{\gamma^1} B'_i \xrightarrow{\gamma^2} B'_j\right) \simeq \sum_{y \in \partial B'_i} \sum_{z \in \partial B'_j} G_D(0, y)G_D(y, z)P_{1,2,3}^{y,z,z}(\mathcal{H}),$$

(4.28)

where the event $\mathcal{H}$ is defined by

$$\mathcal{H} := \left\{ \gamma^1[1, v_1] \cap (X^2[0, T]) = 0; \quad X^2[0, T] \cap B'_i = \emptyset; \quad X^3[0, T] \cap (B'_i \cup B'_j) = \emptyset \right\}. \quad (4.29)$$

11. For any two paths $\lambda$ and $\lambda'$ with $\lambda(0) \in B'_i$ and $\lambda'(0) \in B'_j$,

• write $U_i, \cdots$ for the first time that $\lambda$ exits $B(x_i, 2^{-m})$;
• write $U_j, \cdots$ for the first time that $\lambda'$ exits $B(x_j, 2^{-m})$;
• in this subsection only, write $a = U_{i,-k^2}$, $b = U_{i,-10k^3}$, $e = U_{j,-k^2}$ and $f = U_{j,-10k^3}$ for short.

Note that

$$\text{dist}\left(B(x_i, 2^{-k^2}), B(x_j, 2^{-k^2})\right) \geq \frac{1}{2} 2^{-k^2},$$

since $(i,j)$ is good. Now define the event $\mathcal{G}$ by

$$\mathcal{G} := \mathcal{G}^1 \cap \mathcal{G}^2,$$

(4.30)

where

$$\mathcal{G}^1 = \left\{ \gamma^1[1, a] \cap (X^2)^R[1, a] = \emptyset, \, (X^2)^R[1, a] \cap B'_i = \emptyset \right\}$$

and

$$\mathcal{G}^2 = \left\{ \gamma^2[1, e] \cap X^3[1, e] = \emptyset, \, X^3[1, e] \cap B'_j = \emptyset \right\}.$$

Note that $\mathcal{H} \subset \mathcal{G}$. Thus, we have

$$P_{1,2,3}^{y,z,z}(\mathcal{H}) = P_{1,2,3}^{y,z,z}(\mathcal{H} \mid \mathcal{G})P_{1,2,3}^{y,z,z}(\mathcal{G}).$$

(4.31)

12. We consider next the two-point function $P(0 \xrightarrow{\gamma} v \xrightarrow{\gamma} w)$ in 4.24. We start by introducing the following notation:
• Let $Y^1$ be the random walk started at $v$ conditioned to hit the origin before leaving $D$;
• Let $Y^2$ be the random walk started at $w$ conditioned to hit $v$ before leaving $D$;
• Let $Y^3$ be the simple random walk started at $w$.
• Assume that $Y^1, Y^2, Y^3$ are independent. We define $\tau_0, \sigma_v$ and $T$ similarly as in Paragraph 10. Write $P_{1,2,3}^{v,w}$ for the law of $(Y^1, Y^2, Y^3)$.
• Write $\eta^1 = \text{LE}(Y^1[0, \tau_0])$ for the loop-erasure of $Y^1$ and $t^1 = \text{len}(\eta^1)$ for the length of $\eta^1$.
• Write $\eta^2 = (\text{LE}(Y^2[0, \sigma_v]))^R$ for the time reversal of the loop-erasure of $Y^2$ and $t^2 = \text{len}(\eta^2)$ for the length of $\eta^2$.

Now define the event $\tilde{H}$ by
\[
\tilde{H} = \left\{ \eta^1[0, t^1] \cap (Y^2[0, \sigma_v - 1] \cup Y^3[0, T]) = \emptyset, \ \eta^2[0, t^2] \cap Y^3[1, T] = \emptyset \right\}.
\] (Compare this with $\mathcal{G}$ defined as in (4.1).) Then Proposition 8.1 of [22] shows that
\[
P\left( 0 \xrightarrow{\tilde{H}} v \xrightarrow{\tilde{H}} w \right) = G_D(0, x_v)G_D(v, w)P_{1,2,3}^{v,w} \left( \tilde{H} \right).
\] (4.33)

13. We define $\tilde{G}$ by
\[
\tilde{G} = \left\{ \eta^1[0, a] \cap Y^2[1, a] = \emptyset, \ \eta^2[0, c] \cap Y^3[1, c] = \emptyset \right\}.
\] (4.34)

Compare this with $\mathcal{H}$ defined as in (4.30). Similarly, we can decompose $\tilde{G}$ as follows. Let
\[
\tilde{G}^1 = \left\{ \eta^1[0, a] \cap Y^2[1, a] = \emptyset \right\}, \quad \tilde{G}^2 = \left\{ \eta^2[0, c] \cap Y^3[1, c] = \emptyset \right\},
\] then $\tilde{G} = \tilde{G}^1 \cap \tilde{G}^2$. Note that $\tilde{H} \subset \tilde{G}$ so that
\[
P_{1,2,3}^{v,w} \left( \tilde{H} \right) = P_{1,2,3}^{v,w} \left( \tilde{H} \mid \tilde{G} \right) P_{1,2,3}^{v,w} \left( \tilde{G} \right).
\] (4.36)

Therefore, combining (4.28), (4.31), (4.33) and (4.36), and Green function estimates from Lemma 2.1, we have
\[
P\left( 0 \xrightarrow{\tilde{G}} v \xrightarrow{\tilde{G}} w \right) \approx \sum_{y \in \partial B' \cap \partial B'_j} G_D(0, y)G_D(v, z)P_{1,2,3}^{v,z} \left( \tilde{H} \right)
\] (4.37)

14. The same argument as in the proof of Corollary 2.7 of [16] shows that there exists a deterministic real-valued function $f$ of a pair of four paths such that for all $y \in \partial B'_j$ and $z \in \partial B'_j$,
\[
P_{1,2,3}^{v,w} \left( \tilde{H} \mid \tilde{G} \right) \approx E_{1,2,3}^{v,w} \left\{ f \left( \eta^1[b, a], Y^2[b, a], \eta^2[f, c], Y^3[f, c], f, c \right) \right\} \left( \tilde{G} \right),
\] (4.38)
\[
P_{1,2,3}^{v,z} \left( \tilde{H} \mid \tilde{G} \right) \approx E_{1,2,3}^{v,z} \left\{ f \left( \gamma^1[b, a], (X^2)[b, a], \gamma^2[f, c], X^3[f, c], f, c \right) \right\} \left( \tilde{G} \right),
\] (4.39)

where $a, b, c, f$ are defined as in Paragraph 11. Therefore, if we write $\mu_{v,w}^{v,z}$ for the probability measure induced by $(\eta^1[b, a], (Y^2)[b, a], \eta^2[f, c], Y^3[f, c], f, c)$ conditioned on $\tilde{G}$ and if we set $\mu_{v,z}^{v,z}$ for the probability measure induced by $(\gamma^1[b, a], (X^2)[b, a], \gamma^2[f, c], X^3[f, c], f, c)$ conditioned on $\tilde{G}$, the equation (4.39) can be written as
\[
P_{1,2,3}^{v,w} \left( \tilde{H} \mid \tilde{G} \right) \approx \mu_{v,w}^{v,z} \left\{ f \left( \eta^1[b, a], Y^2[b, a], \eta^2[f, c], Y^3[f, c], f, c \right) \right\},
\] (4.40)
\[
P_{1,2,3}^{v,z} \left( \tilde{H} \mid \tilde{G} \right) \approx \mu_{v,z}^{v,z} \left\{ f \left( \gamma^1[b, a], (X^2)[b, a], \gamma^2[f, c], X^3[f, c], f, c \right) \right\}.
\] (4.41)
15. We write

- $\mathfrak{P}^w$ for the probability measure induced by $(\gamma^1[b,a], (X^2)^R[b,a])$ conditioned on $\mathcal{G}^1$;
- $\mathfrak{P}^z$ for the probability measure induced by $(\gamma^2[f,e], X^3[f,e])$ conditioned on $\mathcal{G}^2$;
- $\mathfrak{P}^w$ for the probability measure induced by $(\eta^1[b,a], (Y^2)^R[b,a])$ conditioned on $\tilde{\mathcal{G}}^1$;
- $\mathfrak{P}^w$ for the probability measure induced by $(\eta^2[f,e], Y^3[f,e])$ conditioned on $\tilde{\mathcal{G}}^2$.

By Proposition 4.6 of [17], we see that $Y^2[0,a]$ and $\eta^2[0,e]$ is almost independent. Namely, for any pair of paths $(\lambda, \lambda')$, we have

$$P\left((Y^2)^R[0,a], \eta^2[0,e] = (\lambda, \lambda')\right) = \left(1 + O(2^{-k^3})\right) P\left((Y^2)^R[0,a] = \lambda\right)P\left(\eta^2[0,e] = \lambda'\right),$$

(4.42)

where $O(2^{-k^3})$ does not depends on $(\lambda, \lambda')$. This gives that for any pair of paths $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$, we have

$$\mu^{v,w,w}(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = \mathfrak{P}^w(\lambda_1, \lambda_2)\mathfrak{P}^w(\lambda_3, \lambda_4) \left(1 + O(2^{-k^3})\right).$$

(4.43)

Here again $O(2^{-k^3})$ does not depends on $(\lambda_1, \lambda_2, \lambda_3, \lambda_4).$ Similarly, using Proposition 4.6 of [17] again, we have

$$\nu^{v,z,z}(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = \mathfrak{P}^w(\lambda_1, \lambda_2)\mathfrak{P}^z(\lambda_3, \lambda_4) \left(1 + O(2^{-k^3})\right),$$

(4.44)

uniformly for a pair of paths $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$.

16. Now we use a coupling result established in Section 3 of [16] (see Section 2.7 for more details), which gives that

$$||\mathfrak{P}^w - \mathfrak{P}^z|| \leq c \cdot 2^{-ck^3}; \quad ||\mathfrak{P}^w - \mathfrak{P}^w|| \leq c \cdot 2^{-ck^3}. $$

(4.45)

Combining these estimates with (4.43) and (4.44), we see that

$$\|\mu^{v,w,w} - \nu^{v,z,z}\| \leq O(2^{-k^3}),$$

(4.46)

where $O(2^{-k^3})$ does not depends on $y, z$ or $v, w$. This gives that

$$\mu^{v,w,w}\left\{ f\left(\eta^1[b,a], (Y^2)^R[b,a], \eta^2[f,e], Y^3[f,e]\right) \right\} \simeq \nu^{v,z,z}\left\{ f\left(\gamma^1[b,a], (X^2)^R[b,a], \gamma^2[f,e], X^3[f,e]\right) \right\}. $$

(4.47)

Here again $\simeq$ does not depends on $y, z$ or $v, w$. Therefore, combining this with (4.37), (4.40) and (4.41), we have

$$P\left(0 \xrightarrow{v} v \xrightarrow{w}\right) \simeq \sum_{y \in \partial B'_i, z \in \partial B'_j} P^{y,z,z}_{1,2,3}\left(H\right) \sum_{y \in \partial B'_i, z \in \partial B'_j} P^{y,z,z}_{1,2,3}\left(H\right) \simeq \sum_{y \in \partial B'_i, z \in \partial B'_j} P^{y,z,z}_{1,2,3}\left(H\right) \simeq \frac{P\left(0 \xrightarrow{v} x_0\right) \cdot P\left(0 \xrightarrow{w} x_0\right)}{P\left(0 \xrightarrow{v} B'_i\right) \cdot P\left(0 \xrightarrow{w} B'_j\right)}. $$

(4.48)

By definition of $\alpha_0$ (see (4.41)), we see that

$$\alpha_0 \simeq |B_0| \cdot \frac{P\left(0 \xrightarrow{v} x_0\right)}{P\left(0 \xrightarrow{v} B'_0\right)},$$

(4.49)

Therefore, we see that

$$\frac{P\left(0 \xrightarrow{v} v \xrightarrow{w}\right)}{P\left(0 \xrightarrow{v} B'_i \xrightarrow{w} B'_j\right)} \simeq \frac{\alpha_0^2}{|B_0|^2},$$

(4.50)

which gives (4.24). As explained in Paragraph 9, this implies (4.7). Consequently, as explained in Paragraph 3, along with (4.8) and (4.9) which can be proved similarly, we thus finish the proof of (4.2).
4.2 Asymptotic independence of LERW

Throughout this section, we will need to apply the asymptotic independence of LERW many times. Hence we summarize it here in the following form.

**Lemma 4.3.** Fix $v, w \in 2^{-n}Z^3 \cap D$, write $l = |v - w|$ and pick $\epsilon \in (0, \frac{1}{2})$. For a path $\lambda$ started at $w$ and ended at $v$, let $\sigma_v$ be its total length, write $\tilde{t} := \max\{t \leq \sigma_v : \lambda(t) \in \partial B(v, \epsilon l)\}$ for the last time that $\lambda$ passes through $\partial B(v, \epsilon l)$ and write $\tilde{u}$ be the first time that $\lambda$ hits $\partial B(w, \epsilon l)$. Let $R$ be a random walk conditioned to hit $v$ before exiting $D$. Then for any paths $\eta^1$ and $\eta^2$, it follows that

$$P\left(R[\tilde{t}, \sigma_v] = \eta^1, \ LE(R[0, \sigma_v])[0, \tilde{u}] = \eta^2\right) = P\left(R[\tilde{t}, \sigma_v] = \eta^1\right)P\left(LE(R[0, \sigma_v])[0, \tilde{u}] = \eta^2\right)(1 + O(\sqrt{\epsilon})).$$

**(4.51)**

**Proof.** Notice that since $\epsilon \in (0, \frac{1}{2})$, we have $B(v, \sqrt{\epsilon l}) \cap B(w, \sqrt{\epsilon l}) = \emptyset$.

Let $b = \eta^2(\text{len}(\eta^2))$ be the endpoint of $\eta^2$. Also we write $\tilde{s}$ be the last time (up to $\sigma_v$) that $R$ visits $b$. Then we see that $\tilde{s} < \tilde{t}$ since $B(v, \epsilon l) \cap B(w, \epsilon l) = \emptyset$. With this in mind, we decompose the LHS of (4.51) through the value of $\tilde{s}$, and obtain that

$$P^w\left(LE(R[0, \sigma_v])[0, \tilde{u}] = \eta^2, \ R[\tilde{t}, \sigma_v] = \eta^1\right) = P^w\left(LE(R[0, \tilde{s}])[0, \tilde{u}] = \eta^2, \ \tilde{s} < \tilde{t}, \ R[\tilde{s} + 1, \sigma_v] \cap \eta^2 = \emptyset, \ R[\tilde{t}, \sigma_v] = \eta^1\right)$$

$$= \sum_{m=0}^{\infty} P^w\left(LE(R[0, m])[0, \tilde{u}] = \eta^2, \ R[m + 1, \sigma_v] \cap \eta^2 = \emptyset, \ R[\tilde{t}, \sigma_v] = \eta^1\right)$$

$$= \sum_{m=0}^{\infty} P^w\left(LE(R[0, m])[0, \tilde{u}] = \eta^2\right)P^b\left(R[1, \sigma_v] \cap \eta^2 = \emptyset, \ R[\tilde{t}, \sigma_v] = \eta^1\right)$$

$$= P^w\left(LE(R[0, \sigma_v])[0, \tilde{u}] = \eta^2\right)P^b\left(R[\tilde{t}, \sigma_v] = \eta^1 \mid R[1, \sigma_v] \cap \eta^2 = \emptyset\right).$$

Next, we will replace the event $\{R[1, \sigma_v] \cap \eta^2 = \emptyset\}$ by the event $\{R[1, T_{w, \sqrt{\epsilon l}}] \cap \eta^2 = \emptyset\}$ in the following way (here $T_{w, \sqrt{\epsilon l}}$ stands for the first time that $R$ exits from $B(w, \sqrt{\epsilon l})$). To do it, note that

$$P^b\left(R[1, T_{w, \sqrt{\epsilon l}}] \cap \eta^2 = \emptyset, \ R[T_{w, \sqrt{\epsilon l}}, \sigma_v] \cap \eta^2 \neq \emptyset\right) \leq C\sqrt{\epsilon} P^b\left(R[1, T_{w, \sqrt{\epsilon l}}] \cap \eta^2 = \emptyset\right)$$

and that

$$P^b\left(R[\tilde{t}, \sigma_v] = \eta^1, \ R[1, T_{w, \sqrt{\epsilon l}}] \cap \eta^2 = \emptyset, \ R[T_{w, \sqrt{\epsilon l}}, \sigma_v] \cap \eta^2 \neq \emptyset\right) \leq C\sqrt{\epsilon} P^b\left(R[1, T_{w, \sqrt{\epsilon l}}] \cap \eta^2 = \emptyset\right) P^b\left(R[\tilde{t}, \sigma_v] = \eta^1\right),$$

since the event $\{R[T_{w, \sqrt{\epsilon l}}, \sigma_v] \cap \eta^2 \neq \emptyset\}$ implies that $R$ returns to $B(w, \epsilon l)$ after the time $T_{w, \sqrt{\epsilon l}}$. This gives that

$$P^b\left(R[\tilde{t}, \sigma_v] = \eta^1 \mid R[1, \sigma_v] \cap \eta^2 = \emptyset\right) = P^b\left(R[\tilde{t}, \sigma_v] = \eta^1 \mid R[1, T_{w, \sqrt{\epsilon l}}] \cap \eta^2 = \emptyset\right)(1 + O(\sqrt{\epsilon})).$$

But by Harnack principle, we have

$$P^{v_1}\left(R[\tilde{t}, \sigma_v] = \eta^1\right) = P^{v_2}\left(R[\tilde{t}, \sigma_v] = \eta^1\right)(1 + O(\sqrt{\epsilon})), \text{ for all } v_1, v_2 \in B(w, \sqrt{\epsilon l}).$$

Consequently, it follows that

$$P^b\left(R[\tilde{t}, \sigma_v] = \eta^1 \mid R[1, T_{w, \sqrt{\epsilon l}}] \cap \eta^2 = \emptyset\right) = P^{v_1}\left(R[\tilde{t}, \sigma_v] = \eta^1\right)(1 + O(\sqrt{\epsilon})),$$

which completes the proof.

**Remark 4.4.** Similar decomposition of a path with asymptotic independence as in Lemma 4.3 can be found in Proposition 4.6 of [17].
4.3 Localization of box crossings

Starting from this section, we will deal with \((4.24)\). As the analysis of box-crossing probabilities is more involved than that for point-crossing probabilities, we will deal with the RHS of \((4.24)\) first, and postpone the analysis of LHS until Section 4.6. The main result of this subsection is Proposition 4.5 which rewrites the box-crossing probability into a form that is easier to analyze. This proposition corresponds to \((4.28)\) in the outline.

We now remind the setup and some notations. We take a cube \(B \subset \mathbb{D}\) of side length \(2^{-k}\) satisfying \(\text{dist}(\{0\} \cup \partial \mathbb{D}, B) > 2^{-k}\). We first assume \(B \subset \frac{1}{2} \mathbb{D}\) (the argument in this section also works for the other case similarly). We recall that \(B\) is partitioned into \(k_m = 2^{3(k^3-k)}\) cubes \(B_1, B_2, \ldots, B_{k_m}\) of side length \(2^{-k^3}\). We denote the center of the cube \(B_i\) by \(x_i\). The cube \(B'_i\) stands for the cube of radius \(3 \cdot 2^{-k^3}\) centered at \(x_i\).

Let \(S[0,T]\) be the SRW with mesh size \(2^{-n}\) started from the origin, stopped at exiting \(\mathbb{D}\) and write \(\gamma = \text{LE}\{S[0,T]\}\) for its loop-erasure. We also recall that \(X_i\) stands for the number of points in \(B_i\) hit by \(\gamma\) and that \(Y_i\) stands for the indicator function of the event that \(\gamma\) hits \(B'_i\). As discussed at the beginning of Section 4.1, we also consider a reference cube \(B_0\) of side length \(2^{-k^3}\) centered at \(x_0 = (1/2, 0, 0)\). For this reference cube, we define \(X_0\) and \(Y_0\) similarly. Then the parameter \(a_0\) was defined by the expectation of \(X_0\) conditioned on \(\{Y_0 = 1\}\).

Pick \(i, j \in \{1, 2, \ldots, m_k\}\) and keep them fixed throughout this subsection. From now on until the end of Section 4.6, unless otherwise specified, we assume \((i, j)\) is good.

Throughout this section, we set
\[
l = |x_i - x_j|; \quad r = |x_i| \quad \text{(in the case of } B \not\subset \frac{1}{2} \mathbb{D}, \text{we set } r = 1 - |x_i|); \quad \epsilon = 2^{-k^3}.
\]

We now state the main result of this subsection.

Let \(X^1, X^2, X^3\) be independent RW’s started at \(y, z, z\) respectively such that \(X^1\) is conditioned to hit 0 before leaving \(\mathbb{D}\) and \(X^2\) is conditioned to hit \(y\) before leaving \(\mathbb{D}\). We then write \(\tau_0, \sigma_y\) for the duration of \(X^1\) and \(X^2\) respectively. We also stop \(X^3\) at first exiting \(\mathbb{D}\) and write \(T\) for its duration. We write
\[
s := T_{x_i, 2^{-k^3+k^3}}(X^2); \quad \sigma := \max \{t | X^2(t) \in \partial B(X_i, 2^{-k^3})\}.
\]

We then consider \(\gamma^1 = \text{LE}(X^1[0, \tau_0])\) and \(\gamma^2 = \text{LE}(X^2[0, \sigma])\) and write
\[
w_1 = \text{len}(\gamma^1) \quad \text{and} \quad w_2 = \text{len}(\gamma^2)
\]
for the length of \(\gamma^1\) and \(\gamma^2\) respectively.

Let
\[
\rho_1 = T_{x_i, 2^{-k^3+k^3}}(\gamma^1); \quad v_1 = \max \{t \leq \rho_1 | \gamma^1(t) \in B'_i\}
\]
and
\[
\rho_2 = T_{x_i, 2^{-k^3+k^3}}(\gamma^2); \quad v_2 = \max \{t \leq \rho_2 | \gamma^2(t) \in B'_j\}.
\]

**Proposition 4.5.** For good \((i, j)\), one has
\[
P\left(0 \xrightarrow{\gamma} B'_i \xrightarrow{\gamma} B'_j\right) = \sum_{y \in \partial B'_i} \sum_{z \in \partial B'_j} G_{\mathbb{D}}(0, y) G_{\mathbb{D}}(y, z) I_{1,2,3}(H) \left(1 + O(2^{-c k^2})\right),
\]
where
\[
H := \left\{ \begin{array}{l}
\gamma_1^1[v_1, w_1] \cap (X^2[s, \sigma_y] \cup X^3[0, T]) = \emptyset; \\
X^2[s, \sigma_y] \setminus 1 \cap B'_i = \emptyset; \\
X^3[1, T] \cap (B'_i \cup B'_j) = \emptyset
\end{array} \right\}.
\]

Before diving into the proof, which will be postponed till the end of this subsection, we first point out some observations. Let \(T^i\) (resp. \(T^j\)) be the last time (up to \(T\)) that the simple random walk \(S\) visits \(B'_i\) (resp. \(B'_j\)). By definition of the loop-erasing procedure, it follows that
\[
\gamma \cap B'_i \neq \emptyset \iff \text{LE}(S[0, T^i])[0, u_i] \cap S[T^i + 1, T] = \emptyset
\]
where \(u_i\) stands for the first time that \(\text{LE}(S[0, T^i])\) hits \(B'_i\). Thus, it follows that
\[
\left\{ \gamma \cap B'_i \neq \emptyset, \gamma \cap B'_j \neq \emptyset \right\} \iff \left\{ \begin{array}{l}
\text{LE}(X[0, T^i])[0, u_i] \cap S[T^i + 1, T] = \emptyset, \\
\text{LE}(S[0, T^j])[0, u_j] \cap S[T^j + 1, T] = \emptyset
\end{array} \right\}.
\]
Note that although we assume no information on the order of $T^i$ and $T^j$ in \((4.58)\), in order to deal with the event $\{0 \xrightarrow{\gamma} B'_i \xrightarrow{\gamma} B'_j\}$, we do need to discuss whether $T^i < T^j$. In fact, we will show below in Proposition 4.6 that conditioned on $\{0 \xrightarrow{\gamma} B'_i \xrightarrow{\gamma} B'_j\}$, it is very unlikely that $T^j < T^i$.

**Proposition 4.6.** There exist a universal constant and an integer $N_k \geq 1$ depending only on $k$, s.t. for all $k \geq 1$ and $n \geq N_k$, 

$$
P \left( 0 \xrightarrow{\gamma} B'_i \xrightarrow{\gamma} B'_j \text{ and } T^j < T^i \right) \leq C 2^{-k^4 + k^2} P \left( 0 \xrightarrow{\gamma} B'_i \xrightarrow{\gamma} B'_j \right).
$$

\((4.59)\)

**Proof.** Suppose that both $\{0 \xrightarrow{\gamma} B'_i \xrightarrow{\gamma} B'_j\}$ and $\{T^j < T^i\}$ happens. Writing 

$$m := \min \left\{ t \geq 0 \mid \text{LE}(S[0,T^j])(t) \in B'_j \right\},$$

we claim that 

$$\text{LE}(S[0,T^j])[0,m] \cap B'_i \neq \emptyset.$$ 

\((4.60)\)

To see this, assume the contrary. Since $\gamma$ hits $B'_j$, we have $\text{LE}(S[0,T^j])[0,m] \cap S[T^j,T] = \emptyset$, which in turn implies that $\gamma(0,m] = \text{LE}(S[0,T^j])[0,m]$. Therefore, $\gamma(0,m] \cap B'_i = \emptyset$ but $\gamma(0,m] \cap B'_j \neq \emptyset$. This contradicts with our assumption that $0 \xrightarrow{\gamma} B'_i \xrightarrow{\gamma} B'_j$. So we have \((4.60)\) as desired.

However, assuming $T^j < T^i$, we see that $S[T^j,T] \cap B'_i \neq \emptyset$. Thus, we have 

$$P \left( 0 \xrightarrow{\gamma} B'_i \xrightarrow{\gamma} B'_j \text{ and } T^j < T^i \right) \leq P \left( H \right),$$

\((4.61)\)

where 

$$H = \left\{ \text{LE}(S[0,T^j])[0,m] \cap S[T^j,T] = \emptyset, \text{LE}(S[0,T^j])[0,m] \cap B'_i \neq \emptyset, S[T^j,T] \cap B'_i \neq \emptyset \right\}.$$ 

As in the proof of Theorem. 3.1.1 of \([23]\), we rewrite the probability in the RHS of \((4.61)\) for the inequality above in terms of three independent SRW’s $S^1$, $S^2$, $S^3$, started from $0$, $y \in \partial B'_i$ and $z \in \partial B'_j$, resp., s.t. $S_1$ and $S^2$ are conditioned to hit $y$ and $z$ resp. before exiting $D$. Let $\sigma_y, \sigma_z$ and $T$ be the duration of $S^1$, $S^2$ and $S^3$ respectively. Then it follows that 

$$P[H] = \sum_{y \in \partial B'_i} \sum_{z \in \partial B'_j} P_{1,2,3}^{0,y,z}(T^{y,z})$$

where 

$$T^{y,z} := \left\{ \begin{aligned}
\text{LE}(S^1[0,\sigma_y])[0,t_1] \cap S^2[1,\sigma_z] = \emptyset; & \quad S^2[1,\sigma_z] \cap B'_i = \emptyset; \quad S^3[1,T] \cap B'_i = \emptyset \\
\text{LE}(S^1[0,\sigma_y] \oplus S^2[0,\sigma_z])[0,t_2] \cap S^3[1,1] = \emptyset; & \quad S^3[0,T] \cap B'_i \neq \emptyset \\
\text{LE}(S^1[0,\sigma_y] \oplus S^2[0,\sigma_z])[0,t_3] \cap S^3[\zeta,T] = \emptyset
\end{aligned} \right\}.$$ 

here $t_1 := \inf \{ t \geq 0 \mid \text{LE}(S^1[0,\sigma_y])(t) \in B'_i \}$ and $t_2 := \inf \{ t \geq 0 \mid \text{LE}(S^1[0,\sigma_y] \oplus S^2[0,\sigma_z])(t) \in B'_j \}$. We also write

$$t_3 = \min \left\{ m \geq 0 \mid \text{LE}(S^1[0,\sigma_y] \oplus S^2[0,\sigma_z])(m) \in \partial B(x_i,4L) \right\}, \quad \xi := T_{x_i,1/3}(S^3),$$

$$\chi = \min \{ t \geq \xi \mid S^3(t) \in B'_i \}, \quad \zeta = \min \{ t \geq \chi \mid S^3(t) \in \partial B(x_i,4L) \}.$$ 

Then, it is easy to see that 

$$T^{y,z} \subset \left\{ \begin{aligned}
\text{LE}(S^1[0,\sigma_y])[0,t_1] \cap S^2[1,\sigma_z] = \emptyset; & \quad S^2[1,\sigma_z] \cap B'_i = \emptyset; \quad S^3[1,\xi] \cap B'_j = \emptyset \\
\text{LE}(S^1[0,\sigma_y] \oplus S^2[0,\sigma_z])[0,t_2] \cap S^3[1,1] = \emptyset; & \quad S^3[\xi,T] \cap B'_i \neq \emptyset \\
\text{LE}(S^1[0,\sigma_y] \oplus S^2[0,\sigma_z])[0,t_3] \cap S^3[\zeta,T] = \emptyset
\end{aligned} \right\},$$ 

By Harnack principle, there exists some $c > 0$ such that 

$$P_{1,2,3}^{0,y,z}(T^{y,z}) \leq c \times \frac{\xi}{T} E_{1,2,3}^{0,y,z} [p(S^1, S^2)],$$

where 

$$p(S^1, S^2) := q(S^1, S^2) \text{LE}(S^1[0,\sigma_y])[0,t_1] \cap S^2[1,\sigma_z] = \emptyset \quad S^2[1,\sigma_z] \cap B'_i = \emptyset \quad \times S^3[1,\xi] \cap \text{LE}(S^1[0,\sigma_y] \oplus S^2[0,\sigma_z])[0,t_2] = \emptyset \quad S^3[\xi,T] \cap \partial B'_i = \emptyset$$

and 

$$q(S^1, S^2) := P^S_{3} \left[ \text{LE}(S^1[0,\sigma_y] \oplus S^2[0,\sigma_z])[0,t_3] \cap S^3[\zeta,T] = \emptyset \right].$$

27
Note that \( q \) is a function of \( S^1 \) and \( S^2 \). However, by (3.70) of [23], we see that
\[
E_{1,2,3}^{y_0, y_1, y_2}[p(\alpha^1, \alpha^2)] \leq C \frac{1}{\epsilon \cdot 2^n} \text{Es}(\epsilon 2^n, I_2^n) \frac{1}{l_2^n} \text{Es}(\epsilon 2^n, I_2^n) \text{Es}(2^n, \epsilon l_2^n) \frac{1}{2^n} \frac{1}{\epsilon 2^n}.
\]
Thus, summing over \( y \in \partial, B'_j \) and \( z \in \partial, B'_j \), we have
\[
P\left(0 \xrightarrow{\gamma} B'_i \xrightarrow{\gamma} B'_j \text{ and } T^i < T^j\right) \leq C \frac{\epsilon}{7} \left(\frac{\epsilon}{7}\right)^{3-\beta} \frac{\epsilon}{7}.
\]
Therefore, by Proposition 3.3 we have
\[
P\left(0 \xrightarrow{\gamma} B'_i \xrightarrow{\gamma} B'_j \text{ and } T^i < T^j\right) \leq P\left(0 \xrightarrow{\gamma} B'_i \xrightarrow{\gamma} B'_j\right) \frac{\epsilon}{7}.
\]
Since \( l \geq 2^{-k^2} \) and \( \epsilon = 2^{-k^4} \), we obtain (4.59) as desired. \( \square \)

By (3.7) and (4.59),
\[
P\left(0 \xrightarrow{\gamma} B'_i \xrightarrow{\gamma} B'_j\right) = P\left(0 \xrightarrow{\gamma} B'_i \xrightarrow{\gamma} B'_j \text{ and } T^i < T^j\right)(1 + O(2^{-k^4+k^2})).
\]
Thus, we only need to deal with
\[
P\left(0 \xrightarrow{\gamma} B'_i \xrightarrow{\gamma} B'_j \text{ and } T^i < T^j\right)
\]
We now write
\[
u_i = \min \{ t \geq 0 \mid \text{LE}(S[0, T^i])(t) \in B'_i \} \text{ and } t_i = \text{len}(\text{LE}(S[0, T^i])),
\]
and define \( \nu_j \) and \( t_j \) similarly.

The next lemma shows that conditioned on \( \{0 \xrightarrow{\gamma} B'_i \xrightarrow{\gamma} B'_j\} \), with high probability,
\[
\text{LE}(S[0, T^i])|_{\nu_i, t_i} \subseteq B(x_i, 2^{-k^4+k^2}).
\]
In other words, with high probability, the behavior of the LEW when it passes \( B_i \), is very “local”.

**Lemma 4.7.** There exist \( c, c' > 0 \), such that
\[
P(I) \leq c2^{-c'k^2} \cdot P\left(0 \xrightarrow{\gamma} B'_i \xrightarrow{\gamma} B'_j\right),
\]
where
\[
I := \left\{ 0 \xrightarrow{\gamma} B'_i \xrightarrow{\gamma} B'_j, T^i < T^j, \text{LE}(S[0, T^i])|_{\nu_i, t_i} \not\subseteq B(x_i, 2^{-k^4+k^2}) \right\}
\]

*Proof.* By (3.69) of [23], it follows that
\[
P[I] \leq C \sum_{r=k^2}^{\infty} 2^{-\delta r} \left(\frac{\epsilon}{7}\right)^{3-\beta} \left(\frac{\epsilon}{7}\right)^{3-\beta} \leq C \cdot 2^{-4k^2} P\left(0 \xrightarrow{\gamma} B'_i \xrightarrow{\gamma} B'_j \text{ and } T^i < T^j\right)
\]
for some \( \delta > 0 \) and \( C < \infty \). \( \square \)

Our next goal is to “decouple” the path between the origin and \( B'_i \) and that between \( B'_i \) and \( B'_j \). In other words, we would like to show that in the events we consider, roughly speaking, \( \text{LE}(S[0, T^i]) \) can be replaced by \( \text{LE}(S[0, T^i]) \) and \( \text{LE}(S[T^i, T^j]) \). We write \( \tilde{\nu}_j := \min \{ t \geq 0 \mid \text{LE}(S[T^i, T^j])(t) \in B'_j \} \).

**Lemma 4.8.** There exists \( c > 0 \), such that
\[
P\left(0 \xrightarrow{\gamma} B'_i \xrightarrow{\gamma} B'_j \text{ and } T^i < T^j\right) = P(J) (1 + O(2^{-c'k^2})),
\]
where
\[
J := \left\{ T^i < T^j, \text{LE}(S[0, T^i])|_{0, \nu_i} \cap S[T^i + 1, T] = \emptyset; \text{LE}(S[T^i, T^j])|_{0, \tilde{\nu}_j} \cap S[T^j + 1, T] = \emptyset \right\}.
\]
Proof. We only prove the $\leq$ direction. The opposition direction can be proved similarly. By Lemma 4.7
\[
P\left( 0 \not\to B_i' \not\to B_j' \text{ and } T^i < T^j \right) = P(J_0) (1 + O(2^{-ck^2})), \tag{4.66}
\]
where
\[
J_0 := \left\{ 0 \not\to B_i' \not\to B_j', T^i < T^j, \{u_i, t_i\} \subset B(x_i, 2^{-k^4 + k^2}) \right\}.
\]
Thus, in order to prove the $\leq$ direction of (4.64), it suffices to show
\[
P(J_0) \leq P(J) (1 + O(2^{-ck^2})). \tag{4.67}
\]
Let
\[
\chi_1 = \max \left\{ t \geq T^i \mid S(t) \subset B(x_i, 2^{-k^4 + k^2}) \right\} \quad \text{and} \quad \chi_2 = \max \left\{ t \geq T^j \mid S(t) \subset B(x_i, 2^{-k^4 + 2k^2}) \right\}.
\]
Then, similar argument as in the proof of Lemma 4.7 shows that
\[
P\left( 0 \not\to B_i' \not\to B_j' \text{ and } T^i < T^j, S[T^i, \chi_1] \not\subset B(x_i, 2^{-k^4 + 2k^2}) \right) \leq c2^{-ck^2} P\left( 0 \not\to B_i' \not\to B_j' \text{ and } T^i < T^j \right),
\]
and
\[
P\left( 0 \not\to B_i' \not\to B_j' \text{ and } T^i < T^j, S[T^i, \chi_2] \not\subset B(x_i, 2^{-k^4 + 3k^2}) \right) \leq c2^{-ck^2} P\left( 0 \not\to B_i' \not\to B_j' \text{ and } T^i < T^j \right).
\]
Thus, we only need to show that
\[
P(J_1) \leq P(J) (1 + O(2^{-ck^2})), \tag{4.68}
\]
where
\[
J_1 := \left\{ 0 \not\to B_i' \not\to B_j' \text{ and } T^i < T^j; \quad \{u_i, t_i\} \subset B(x_i, 2^{-k^4 + k^2}); \quad S[T^i, \chi_1] \subset B(x_i, 2^{-k^4 + 2k^2}); \quad S[T^i, \chi_2] \subset B(x_i, 2^{-k^4 + 3k^2}) \right\}.
\]
If $J_1$ occurs, then it follows that
\[
\{0, u_j\} \cap B(x_i, 2^{-k^4 + 3k^2})^c \subset L(E) \{0, 0_u\} \cap B(x_i, 2^{-k^4 + 3k^2})^c \subset L(E) \{0, 0_u\}.
\]
Therefore, we have
\[
J_1 \subset G := \left\{ T^i < T^j; \quad \{u_i, t_i\} \subset B(x_i, 2^{-k^4 + k^2}); \quad \{0, u_j\} \cap B(x_i, 2^{-k^4 + 3k^2})^c \subset \emptyset; \quad \{0, u_j\} \cap B(x_i, 2^{-k^4 + 3k^2})^c \subset \emptyset \right\} \tag{4.69}
\]
Thus for some $\delta > 0$, we have
\[
P(0 \not\to B_i' \not\to B_j' \text{ and } T^i < T^j) \leq (1 + O(2^{-\delta k^2})) P(G). \tag{4.70}
\]
By definition of $G$ and $J$, it follows that
\[
P(G) \leq P(G, \{0, \{S[T^i, T^j] \cap B(x_i, 2^{-k^4 + 3k^2}) \neq \emptyset\} \} + P(J). \tag{4.71}
\]
However, by Lemma 4.6 we see that
\[
P(0 \not\to B_i' \not\to B_j' \text{ and } T^i < T^j) \leq (1 + O(2^{-\delta k^2})) P(G). \tag{4.72}
\]
Combining (4.66), (4.70), (4.71) and (4.72), we obtain (4.67) as desired. \qed

Recall the definition of $u_i$ and $t_i$ in (4.63). We now define
\[
s_i := \max \{ t \mid L(E) \{0, 0_u\} \cap \partial B(x_i, 2^{-k^4 + k^2}) \} \quad \text{and} \quad \tilde{u}_i := \min \{ t \geq s_i \mid L(E) \{0, 0_u\} \cap \partial B(x_i, 2^{-k^4 + k^2}) \neq \emptyset \}
\]
and define $s_j, \tilde{u}_j$ similarly. Note that $u_i \leq \tilde{u}_i$ and that $u_i < \tilde{u}_i$ if and only if $L(E) \{0, 0_u\} \cap \partial B(x_i, 2^{-k^4 + k^2}) \neq \emptyset$. The same observation works for $u_j$ and $\tilde{u}_j$ in the same fashion. Since LERW has almost no “quasi-loops”, we see that $u_i = \tilde{u}_i$ and $u_j = \tilde{u}_j$ with high probability. The next lemma gives a quantitative statement of the above observation.

29
Lemma 4.9. Remind the definition of $J$ at (4.65). There exists $c > 0$ such that

$$P[J] = P[K](1 + O(2^{-ck^2})),$$

where

$$K := \{ T^i < T^j, \ LE(S[0, T^i])[0, \tilde{u}_i] \cap S[T^i, T] = \emptyset, \ LE(S[T^i, T^j])[0, \tilde{u}_j] \cap S[T^j, T] = \emptyset \}.$$

Proof. By arguments similar to Proposition 3.3 we have that

$$P[J] \leq \left(\frac{\epsilon}{r}\right)^{3-\beta} \left(\frac{\epsilon}{l}\right)^{3-\beta}$$

and

$$P[K] \leq \left(\frac{\epsilon}{r}\right)^{3-\beta} \left(\frac{\epsilon}{l}\right)^{3-\beta}.$$

Therefore to prove this lemma, it suffices to show that

$$P[J, u_i \neq \tilde{u}_i] \leq C2^{-ck^2}P(J); \quad P[J, u_j \neq \tilde{u}_j] \leq C2^{-ck^2}P(J),$$

and

$$P[K, u_i \neq \tilde{u}_i] \leq C2^{-ck^2}P(K); \quad P[K, u_j \neq \tilde{u}_j] \leq C2^{-ck^2}P(K).$$

As their proofs are very similar, we will only prove the first inequality of (4.70). Observe that

$$J \cap \{ u_i \neq \tilde{u}_i \} \implies \left\{ \text{LE}(S[0, T^i])[u_i, t_i] \cap \partial B(x_i, 2^{-k^2}) \neq \emptyset \right\}.$$

Hence by an argument similar to the proof of Lemma 4.4, one can show that

$$P[J, u_i \neq \tilde{u}_i] \leq c2^{-k^2}\epsilon \cdot \text{Es}(\epsilon 2^{-k^2}l^2) \cdot \text{Es}(\epsilon 2^{k^2}l^2) \cdot \frac{\epsilon}{r} \cdot \frac{\epsilon}{l}$$

$$\leq C \cdot \frac{\epsilon}{r} \cdot \frac{\epsilon}{l} \cdot \left(\frac{\epsilon}{r}\right)^{2-k^2} \cdot \left(\frac{\epsilon}{l}\right)^{2-k^2} = c \cdot \left(\frac{\epsilon}{r}\right)^{3-\beta} \cdot \left(\frac{\epsilon}{l}\right)^{3-\beta} 2^{-(\beta-1)k^2}.$$

Since $\beta > 1$, if we write $c = \beta - 1 > 0$, we obtain the first inequality of (4.70) as desired. $\square$

Up to now, we have changed the box-crossing event into $K$ and proved that

$$P\left(0 \xrightarrow{\gamma} B'_1 \xrightarrow{\gamma} B'_2\right) = P(K)(1 + O(2^{-ck^2})).$$

Out next goal is to replace $S[T^i, T^j]$ by $S[T^i, \Sigma]$ where

$$\Sigma := \inf \{ t \geq T^i \mid S(t) \in \partial B(x_j, 2^{-k^3+k^2}) \},$$

and to show that this replacement will not change the probability very much.

Lemma 4.10. There exists $c > 0$ such that

$$P(K) = P(L) \left(1 + O(2^{-ck^2})\right),$$

where

$$L := \left\{ T^i < T^j; \ LE(S[0, T^i])[0, \tilde{u}_i] \cap (S[T^i + 1, \Sigma] \cup S[T^j, T]) = \emptyset; \ LE(S[T^i, T^j])[0, \tilde{u}_j] \cap S[T^j + 1, T] = \emptyset \right\}.$$

Proof. Note that by definition of $L$, $K \subset L$. Hence to prove (4.80), it suffices to obtain an upper bound for the probability of

$$M := L \setminus K = \left\{ T^i < T^j \right.$$ \(LE(S[0, T^i])[0, \tilde{u}_i] \cap (S[T^i + 1, \Sigma] \cup S[T^j, T]) = \emptyset; \ LE(S[0, T^j])[0, \tilde{u}_j] \cap S[T^j + 1, T] = \emptyset \right\}.$$

Recall that $l = |x_i - x_j|$. We first deal with $M_1 := M \cap \left\{ \text{LE}(S[0, T^i])[0, \tilde{u}_i] \cap B(x_j, 2^{-k^2}l) \neq \emptyset \right\}$. Similar argument as in the proof of Lemma 4.6 gives that:

$$P[M_1] \leq C2^{-k^2} \cdot l \cdot \text{Es}(\epsilon 2^{-k^2}l) \cdot \frac{\epsilon}{r} \cdot l \cdot \text{Es}(\epsilon 2^k l) .$$

$$\leq C' \left(\frac{\epsilon}{r}\right)^{3-\beta} \left(\frac{\epsilon}{l}\right)^{3-\beta} 2^{(\beta-3)k^2} \leq C'\left(\frac{\epsilon}{r}\right)^{3-\beta} \left(\frac{\epsilon}{l}\right)^{3-\beta} 2^{(\beta-3)k^2} P[K].$$
Therefore, we only need to estimate the probability of \( M_2 := M \cap \{ \text{dist}(\text{LE}(S[0, T^i])|0, \hat{u}_i), x_j) \geq 2^{-k^2}l \} \).

With this in mind, for \( q = 0, 1, \ldots, k^2 \), let
\[
M^q := \left\{ 2^{-k^2+q} \cdot l < \text{dist}(\text{LE}(S[0, T^i])|0, \hat{u}_i), x_j) \leq 2^{-k^2+q+1} \cdot l \right\}.
\]

Observe that, suppose \( M \cap M^q \) occurs, since \( \text{LE}(S[0, T^i])|0, \hat{u}_i) \cap S[\Sigma, T^j] \neq \emptyset \) and \( \text{LE}(S[0, T^i])|0, \hat{u}_i) \subset B(x_j, 2^{-k^2+q} \cdot l)^c \), it follows that
\[
S[\Sigma, T^j] \cap \partial B(x_j, 2^{-k^2+q} \cdot l) \neq \emptyset.
\]

Therefore,
\[
(M \cap M^q) \subset B^q := \begin{cases}
T^i < T^j, \\
\text{LE}(S[0, T^i])|0, \hat{u}_i) \cap B(x_j, 2^{-k^2} \cdot l \cdot 2^{q+1}) \neq \emptyset, \\
\text{LE}(S[0, T^i])|0, \hat{u}_i) \cap (S[T^i+1, \Sigma] \cup S[T^j, T]) = \emptyset, \\
S[\Sigma, T^j] \cap B(x_j, 2^{-k^2} \cdot l \cdot 2^q) \neq \emptyset, \\
\text{LE}(S[T^i, T^j])|0, \hat{u}_j) \cap S[T^j+1, T] = \emptyset
\end{cases}
\]

must occur. As in the proof of Lemma 4.6 (note that in this case we need to truncate according to \( \text{dist}(x_j, S[\Sigma, Y]) \), where \( Y \) is the first time after \( \Sigma \) that \( S \) hits \( \partial B(x_j, 2^{-k^2+q} \cdot l) \)), we have
\[
P(B^q) \leq C \cdot \frac{2^{-k^2+q} \cdot l}{\epsilon} \cdot \text{Es}(2^{-k^2+q} \cdot l, l, \epsilon) \cdot \frac{2^{-k^3+k^2}}{2^{-k^2+q} \cdot l} \cdot \text{Es}(\epsilon, 2^{-k^3+k^2}) \text{Es}(l, r).
\]

After simplification, we know that for some \( c, C', > 0 \),
\[
P(B^q) \leq C \cdot \left( \frac{\epsilon}{r} \right)^{2-\beta} \left( \frac{\epsilon}{l} \right)^{2-\beta} 2^{c(k^3+q)} \leq C'2^{-c(k^3+q)} P[K].
\]

Hence
\[
P[M] \leq P[M_1] + P[M_2] \leq C2^{(\beta-3)k^2} P[K] + \sum_{q=0}^{k^2} C'2^{-c(k^3+q)} P[K] \leq C'2^{-c(k^3+q)} P[K].
\]

(4.83)

This finishes the proof of (4.80). \( \square \)

Before proving Proposition 4.3, we need to perform one last operation. Let \( \Xi \) be the first time after \( T^i \) the walk \( S \) hits \( \partial B(x_j, 2^{-k^4}) \) and define
\[
N := \left\{ T^i < T^j; \begin{array}{l}
\text{LE}(S[0, T^i])|0, \hat{u}_i) \cap (S[T^i+1, \Sigma] \cup S[T^j, T]) = \emptyset; \\
\text{LE}(S[\Xi, T^j])|0, \hat{u}_j) \cap S[T^j+1, T] = \emptyset
\end{array} \right\}.
\]

By an argument very similar to Lemma 4.10, we can show that for some \( c > 0 \),
\[
P[L] = P[N](1 + O(2^{-c(k^2)})).
\]

(4.84)

We are now finally ready to prove Proposition 4.3.

**Proof of Proposition 4.3.** The claim follows by (4.80), (4.84) and a last-exit decomposition argument on the event \( L \), similar to (3.19) of [23]. \( \square \)

### 4.4 Decomposition and conditioning of paths

In this subsection, we are going to analyze the event \( \mathcal{H} \) through conditioning on the behavior of the paths in the vicinity of \( B_t \) and \( B_j \). As discussed at the beginning of Section 4.3, throughout this section we continue assuming that \((i, j)\) is good.

For any two paths \( \lambda \) and \( \lambda' \) with \( \lambda(0) \in B_t' \) and \( \lambda'(0) \in B_j' \), we write \( U_{i, -m} \) for the first time that \( \lambda \) exits \( B(x_i, 2^{-m}) \) and write \( U_{j, -m} \) for the first time that \( \lambda' \) exits \( B(x_j, 2^{-m}) \).
With Proposition 4.5 in mind, we now take \( y \in \partial_i B'_i \) and \( z \in \partial_i B'_j \), and define the following events (remind the definition of \( \gamma^1 \) and \( \gamma^2 \) below (4.52))

\[
G^1 = \left\{ \gamma^1[v_1, v'_1] \cap X^2[\sigma, \sigma_y - 1] = \emptyset, X^2[\sigma, \sigma_y - 1] \cap B'_i = \emptyset \right\}
\]

and

\[
G^2 = \left\{ \gamma^2[v_2, v'_2] \cap X^2[1, t_3] = \emptyset, X^2[1, t_3] \cap B'_j = \emptyset \right\},
\]

where \( v_1, v_2 \) were defined in (4.53) and (4.54), \( t_3 := U_{j,-k^5}(X^3) \), and

\[
v'_1 := U_{i,-k^5}(\gamma^1), \quad v'_2 := U_{j,-k^5}(\gamma^2).
\]

Note that by definition, \( v_1 \leq \rho_1 \leq v'_1 \) and \( v_2 \leq \rho_2 \leq v'_2 \). We observe that by Lemma 4.3,

\[
P(G^1 \cap G^2) = P(G^1)P(G^2)(1 + O(2^{-k^2})).
\]

We now let

- \( \Delta_i \) be the set of paths \((a, b)\) satisfying that
  \[
P(\{ \gamma^1[0, v'_1], X^2[\sigma, \sigma_y] \} \cap G^1) > 0;
  \]

- \( \Delta_j \) be the set of paths \((c, d)\) satisfying that
  \[
P(\{ \gamma^2[0, v'_2], X^3[0, t_3] \} \cap G^2) > 0;
  \]

- \( G_{a,b} = \{ \{ \text{LE}(X^1[0, \tau_0]) \} [0, v'_1], X^2[\sigma, \sigma_y] \} = (a, b) \};

- \( G_{c,d} = \{ \{ \text{LE}(X^2[0, \bar{\sigma}]) \} [0, t'_2], X^3[0, t_3] \} = (c, d) \}.

Similar to (4.87), we also have

\[
P(G_{a,b} \cap G_{c,d}) = P(G_{a,b})P(G_{c,d})(1 + O(2^{-k^2})).
\]

Then, it follows that

\[
P(H \mid G^1 \cap G^2) = \sum_{(a, b) \in \Delta_i} \sum_{(c, d) \in \Delta_j} P(H \mid G_{a,b} \cap G_{c,d}) \times \frac{P[G_{a,b} \cap G_{c,d}]}{P[G^1 \cap G^2]} \times \frac{P[H \mid G_{a,b} \cap G_{c,d}]}{P[H \mid G_{a,b} \cap G_{c,d}]} (1 + O(2^{-k^2})).
\]

We conclude this subsection by giving some notations which will be useful in the subsequent subsections where we are going to show that (loosely speaking) \( P(H \mid G_{a,b} \cap G_{c,d}) \) is a function of the “end part” of \((a, b, c, d)\).

- Define
  \[
  \Gamma_i = \left\{ (a, b) : \text{simple path, } \text{len}(a) = u^a; a[0, a^a - 1] \subset B(x_i, 2^{-k^3}), \text{ } a(u^a) \subset \partial B(x_i, 2^{-k^3}) \right\};
  \]
  and define \( \Gamma_j, u^c, u^d \) similarly by replacing \( a, b, i \) by \( c, d, j \) respectively in the definition above.

- For \((a, b) \in \Gamma_i \) and \((c, d) \in \Gamma_j \), let
  \[
  \pi(a, b, c, d) = (a[U_{i,-10k^3}, u^a], b[U_{i,-10k^3}, u^b], c[U_{j,-10k^3}, u^c], d[U_{j,-10k^3}, u^d])
  \]
  be the end part of \((a, b, c, d)\). Without loss of generality, we also define \( \pi(a, b) \) and \( \pi(c, d) \) similarly.

- For \((a, b) \in \Gamma_i \) and \((c, d) \in \Gamma_j \) with \( x_{\mathfrak{R}} \equiv a, b, c, d \) the respective ending point these paths, let \( R^a, R^c, R^b \) and \( R^d \) be independent random walks such that
- \( R^{a,0} \) starts from \( x_a \) and is conditioned to hit the origin before hitting \( a \) and \( \partial \mathbb{D} \);
- \( R^{c,b} \) starts from \( x_c \) and is conditioned to hit \( x_b \) before leaving \( \mathbb{D} \) and that \( R^{c,b}[1, \hat{\tau}] \cap c = \emptyset \), where \( \hat{\tau} \) is the last (up to the first exit from \( \mathbb{D} \)) time that \( R^{c,b} \) hits \( x_b \);
- \( R^d \) starts from \( x_d \).

- For \((a, b) \in \Gamma_i\), let
  \[
  s^a = \begin{cases} 
  \text{The last time } a \text{ passes } B'_i, & \text{if } a(0) \in B(x_i, -k^4 + k^3) \text{ and } a \text{ passes } B'_i \text{ before } U_{i,2-k^4+k^3}; \\
  0 & \text{otherwise.}
  \end{cases}
  \]

  We define \( s^c \) similarly with \( a, i \) replaced by \( c, j \) respectively.

  - Define
    \[
    f(a, b, c, d) = P \left( \left( [a[s^a, u^a] \cup \operatorname{LE}(R^{a,0}[0, \tau_0]) \cap (R^{c,b}[\tilde{\tau}, \hat{\tau}] \cup b \cup R^d[0, T] \cup d) = \emptyset, \right) \right) \right),
    \]
    where \( \tilde{\tau} \) stands for the last time (up to \( \hat{\tau} \)) that \( R^{c,b} \) hits \( \partial B(x_j, 2^{-k^3+k^2}) \).

  By the domain Markov property and the strong Markov property, for \((a, b) \in \Delta_i\) and \((c, d) \in \Delta_j\), we have
  \[
  P(\mathcal{H} | \mathcal{G}_{a,b} \cap \mathcal{G}_{c,d}) = f(a, b, c, d).
  \]

\section{Decoupling non-intersection events}

In this subsection, we are going to derive Proposition \ref{prop:decoupling}, which rewrites \( P(\mathcal{H}) \) into a “decoupled” form that facilitates the comparison with the corresponding rewritten form of point-crossing probabilities given in Proposition \ref{prop:rewritten}. As an intermediate step, we will show in Lemma \ref{lem:decoupling} that (in the notation of last subsection) for a typical \((a, b) \in \Delta_i\) and \((c, d) \in \Delta_j\), \( f(a, b, c, d) \) almost only depends on \( \pi(a, b, c, d) \). Remind that throughout this subsection we still assume that \( (i, j) \) is good.

We start with the definition of a function to measure the “typicality” of \((a, b, c, d)\). In the notation of last subsection, let

\[
\hat{\tau}\] is the last time (up to \( \hat{\tau} \)) that \( R^{c,b} \) hits \( \partial B(x_i, 2^{-k^3+1}) \).

One may compare our definition here with the definition of \( h \) for the one-point estimate case in in Section 3.3 of \[16\]. See the discussion in Remark 3.6, ibid., for the significance of such functions.

\begin{lemma}
It follows that
\[
f(a, b, c, d) \asymp h(a, b, c, d) \operatorname{Es}(2^{-k^3}, r) \operatorname{Es}(2^{-k^3}, l).
\]
\end{lemma}

\begin{proof}
Take \((a, b) \in \Gamma_i\) and \((c, d) \in \Gamma_j\). We will first show that for some \( c > 0, \)
\[
f(a, b, c, d) \leq c \cdot h(a, b, c, d) \operatorname{Es}(2^{-k^3}, r) \operatorname{Es}(2^{-k^3}, l).
\]

We define some random times as follows:
\begin{itemize}
  \item \( \eta^1 := \max \{ t | \operatorname{LE}(R^{a,0}[0, \tau_0])(t) \in \partial B(x_i, 4l) \} \);
  \item \( \eta^2 := \min \{ t | \operatorname{LE}(R^{a,0}[0, \tau_0])(t) \in \partial B(x_i, l/3) \} \);
  \item \( \eta^3 := \max \{ t \leq \hat{\tau} | R^{c,b}(t) \in \partial B(x_i, l/3) \} \);
\end{itemize}
• \( \eta^4 := \min \left \{ t \mid LE(R^{c,b}[0, \tau])(t) \in \partial B(x_j, l/3) \right \} \);

• \( \eta^5 := \min \left \{ t \mid R^d(t) \in \partial B(x_j, l/3) \right \} \).

Let

\[
\mathcal{N} := \left\{ \begin{array}{l}
(a[s^a, u^a] \cup LE(R^{a,0}[0, \tau_0])[0, \eta^2]) \cap (R^{c,b}[\eta^3, \tau] \cup b) = \emptyset \\
(LE(R^{c,b}[0, \tau])[0, \eta^1] \cup c[s^c, u^c]) \cap (R^d[0, \eta^5] \cup d) = \emptyset \\
LE(R^{a,0}[0, \tau_0])[\eta^1, \tau_0] \cap R^d[\eta^5, T] = \emptyset
\end{array} \right\}.
\]

Then by definition of \( f(a, b, c, d) \), we have

\[
f(a, b, c, d) \leq P(\mathcal{N}).
\]

However, since \( LE(R^{a,0}[0, \tau_0])[0, \eta^2] \) and \( LE(R^{a,0}[0, \tau_0])[\eta^1, \tau_0] \) are independent up to constant, by strong Markov property and the Harnack principle, it follows that

\[
P(\mathcal{N}) \leq c \cdot Es(l, r)P\left( \mathcal{L}^1 \right) \cdot P\left( \mathcal{L}^2 \right),
\]

where

\[
\mathcal{L}^1 := \left\{ \begin{array}{l}
(a[s^a, u^a] \cup LE(R^{a,0}[0, \tau_0])[0, \eta^2]) \\
\cap (R^{c,b}[\eta^3, \tau] \cup b) = \emptyset
\end{array} \right\}
\]

and

\[
\mathcal{L}^2 := \left\{ \begin{array}{l}
(LE(R^{c,b}[0, \tau])[0, \eta^1] \cup c[s^c, u^c]) \\
\cap (R^d[0, \eta^5] \cup d) = \emptyset
\end{array} \right\}
\]

(4.94)

The same argument for Proposition 3.5 of [16] implies that

\[
P(\mathcal{L}^1) \asymp P\left( \mathcal{Z}^1 \right) \times Es(2^{-k^3}, l), \quad \text{and} \quad P(\mathcal{L}^2) \asymp P\left( \mathcal{Z}^2 \right) \times Es(2^{-k^3}, l).
\]

where

\[
\mathcal{Z}^1 := \left\{ (a[s^a, u^a] \cup LE(R^{a,0}[0, \tau_0])[0, U_{i,-k^3+1}]) \cap (R^{c,b}[\tau, \tau] \cup b) = \emptyset \right\}
\]

and

\[
\mathcal{Z}^2 := \left\{ (LE(R^{c,b}[0, \tau])[0, \eta^1] \cup c[s^c, u^c]) \cap (R^d[0, U_{j,-k^3+1}] \cup d) = \emptyset \right\}.
\]

By Lemma 4.3 again, it follows that

\[
h(a, b, c, d) \asymp P\left( \mathcal{Z}^1 \right) \times P\left( \mathcal{Z}^2 \right).
\]

Thus, we have

\[
f(a, b, c, d) \leq c \cdot Es(l, r)(Es(2^{-k^3}, l))^2 \times h(a, b, c, d) \asymp Es(2^{-k^3}, r)Es(2^{-k^3}, l)h(a, b, c, d),
\]

which gives the desired upper bound.

We now turn to the lower bound. Remind the definition of \( \eta^1 \sim \eta^5 \) at the beginning of the proof. A key observation is the following separation lemma. Writing

\[
\gamma^1 := LE(R^{a,0}[0, \tau_0])[0, \eta^2]; \quad \lambda^1 := R^{c,b}[\eta^3, \tau]; \quad \gamma^2 := R^{c,b}[0, \tau][0, \eta^4]; \quad \lambda^2 = R^d[0, \eta^5],
\]

we have:

\[
P[\left( \gamma^1, \lambda^1 \right) \text{ are well-separated}] P[\left( \gamma^2, \lambda^2 \right) \text{ are well-separated}] \geq c.
\]

Since the events \( \mathcal{L}^1 \) and \( \mathcal{L}^2 \) are independent up to constants, we see that

\[
P\left( \mathcal{L}^1 \right) \times P\left( \mathcal{L}^2 \right) \geq c.
\]

34
Thus, letting
\[ M := \mathcal{L}^1 \cap \mathcal{L}^2 \cap \{ (\gamma^1, \lambda^1), (\gamma^2, \lambda^2) \text{ are well-separated} \}, \]
(4.95)
A standard technique as in Proposition 3.5 of [16] shows that conditioned on \( M \), the probability of the event considered in the definition of \( f \) (see (4.91)) happens is bounded by \( c \cdot \text{Es}(l, r) \), thus,
\[ f(a, b, c, d) \geq c \text{Es}(l, r) P(M) \geq c \text{Es}(l, r) P(\mathcal{L}^1) P(\mathcal{L}^2). \]
But we have already proved that
\[ P(\mathcal{L}^1) P(\mathcal{L}^2) \approx h(a, b, c, d) \left( \text{Es}(2^{-k^3} l) \right)^2. \]
We thus finish the proof.

Recall that \( \mathcal{L}^1 \) and \( \mathcal{L}^2 \) are defined in (4.94). Let
\[ h^1(a, b) = P(\mathcal{L}^1) \text{ and } h^2(c, d) = P(\mathcal{L}^2). \]
Then we have the next lemma.

**Lemma 4.12.** Suppose that \((a, b) \in \Gamma_i \) and \((c, d) \in \Gamma_j \) satisfy
\[ h^1(\pi(a, b)) \geq 2^{-k^3} \text{ and } h^2(\pi(c, d)) \geq 2^{-k^3}, \]
(4.96)
then it follows that
\[ |f(a, b, c, d) - f(\pi(a, b, c, d))| \leq C 2^{-k^3} f(\pi(a, b, c, d)). \]

**Proof.** Take \((a, b) \in \Gamma_i \) and \((c, d) \in \Gamma_j \). Recall that \( x_\bullet \) stands for the ending point of \( \bullet \) for \( \bullet = a, b, c, d \). Let \( R^1, R^2, R^3 \) be independent RW’s such that
- \( R^1 \): starts from \( x_a \) and is conditioned to hit \( 0 \) before leaving \( \mathbb{D} \);
- \( R^2 \): starts from \( x_c \) and is conditioned to hit \( x_b \) before leaving \( \mathbb{D} \);
- \( R^3 \): starts from \( x_d \).
Write \( \sigma^2 \) for the last time (up to \( T \)) that \( R^2 \) hits \( x_b \) and \( \tilde{\sigma}^2 \) for the last time (up to \( \sigma^2 \)) that \( R^2 \) hits \( \partial B(x, 2^{-k^3+k^2}) \).
Let \((a', b', c', d') = \pi(a, b, c, d)\) be the end part of \((a, b, c, d)\). Note that
\[ (x_a, x_b, x_c, x_d) = (x_{a'}, x_{b'}, x_{c'}, x_{d'}). \]

Define
\[
H_1 = \left\{ R^1[1, \tau_0] \cap a = \emptyset, R^2[1, \sigma^2] \cap c = \emptyset \right\}
\]
\[
\left\{ (a[s^a, w^a] \cup \text{LE}(R^1[0, \tau_0])) \cap (R^2[\tilde{\sigma}^2, \sigma^2] \cup b \cup R^3[0, T] \cup d) = \emptyset \right\}
\]
\[ \left\{ \text{LE}(R^2[0, \sigma^2]) \cup c[s^c, u^c] \cap (R^3[0, T] \cup d) = \emptyset \right\}, \]
and
\[ H_2 = \{ R^1[1, \tau_0] \cap a = \emptyset, R^2[1, \sigma^2] \cap c = \emptyset \} \]

We then define \( H'_1 \) and \( H'_2 \) through replacing \( a, b, c, d \) by \( a', b', c', d' \) respectively in the definition of \( H_1 \) and \( H_2 \). Since \( \bullet \subset \bullet' \) and \( [s^\bullet, u^\bullet] \subset [s'^\bullet, u'^\bullet] \) for \( \bullet = a, b, c, d \), it follows that
\[ H_1 \subset H'_1 \text{ and } H_2 \subset H'_2. \]

We will first show that \( P(H_2) \) is close to \( P(H'_2) \). Note that
\[ P(H'_2) - P(H_2) \leq P(H'_2, R^1[1, \tau_0] \cap a \neq \emptyset) + P(H'_2, R^2[1, \sigma^2] \cap c \neq \emptyset). \]
However, by Proposition 6.1.1 of [22], it follows that
\[ P(H'_2, R^1[1, \tau_0] \cap a \neq \emptyset) \leq c 2^{-9k^3} P(H_2) \text{ and } P(H'_2, R^2[1, \sigma_0] \cap c \neq \emptyset) \leq c 2^{-9k^3} P(H_2). \]
Thus,
\[ P(H_2) = P(H'_2)(1 + O(2^{-9k^3})). \]
Next we will compare \( P(H_1, H_2) \) and \( P(H'_1, H'_2) \). Note that \( P(H_1, H_2) \leq P(H'_1, H'_2) \). On the other hand, by Prop. 6.1.1 of [22] again, we have

\[
P(H'_1, H'_2) - P(H_1, H_2) \\
\leq P \left( H'_2, \left( R^1[0, \tau_0] \cup R^2[0, \sigma^2] \cup R^3[0, T] \right) \right) \cap B \left( \{x_i, x_j\}, 2^{-10k^3} \right) \neq 0 \leq c \cdot 2^{-9k^3} P(H_2).\]

By Lemma 4.11 we have

\[
f(\pi(a, b, c, d)) \sim h^1(\pi(a, b)) \cdot h^2(\pi(c, d)) \cdot \tilde{H}(2^{-k^3}, r) \cdot \tilde{H}(2^{-k^3}, l).
\]

By the assumption (4.96), it follows that

\[
f(\pi(a, b, c, d)) \geq c \cdot 2^{-k^3} \cdot \tilde{H}(2^{-k^3}, r) \cdot \tilde{H}(2^{-k^3}, l) \geq c2^{-4k^3}.
\]

Combining definitions of \( f \), this implies that

\[
f(\pi(a, b, c, d)) = \frac{P(H'_1, H'_2)}{P(H_2)} \geq 2^{-4k^3},
\]

i.e.,

\[
P(H'_1, H'_2) \geq O(2^{-4k^3}) P(H_2).
\]

Therefore,

\[
P(H_1, H_2) - P(H'_1, H'_2) \leq c \cdot 2^{-5k^3} \cdot P(H'_1, H'_2).
\]

Thus,

\[
f(a, b, c, d) = \frac{P(H_1, H_2)}{P(H_2)} = \frac{P(H'_1, H'_2)}{P(H_2)} \left( 1 + O(2^{-5k^3}) \right) = f(\pi(a, b, c, d)) \left( 1 + O(2^{-5k^3}) \right) = (4.97)
\]

which completes the proof. \( \square \)

By (4.89) and (4.92), we have showed that

\[
P(H|G^1 \cap G^2) = \sum_{(a,b) \in \Gamma_1} \sum_{(c,d) \in \Gamma_2} f(a, b, c, d) \frac{P(G_{a,b})}{P(G^1)} \frac{P(G_{c,d})}{P(G^2)} \left( 1 + O(2^{-c^k}) \right).
\]

- Let \( \gamma_{x}^y, S_0^2, \gamma_x^y \) and \( S_0^2 \) be independent ILERW’s and SRW’s started at \( y \) and \( z \) respectively. We now define the following events (compare them with the definition of \( G^1 \) and \( G^2 \) at (4.85) and (4.86)):

\[
G^1_\infty = \{ \gamma_{x}^y[v_1, U_{i,-k}] \cap S_0^2[1, U_{i,-k}] = \emptyset, S_0^2[1, U_{i,-k}] \cap B'_i = \emptyset \};
\]

\[
G^2_\infty = \{ \gamma_{x}^y[\varepsilon_1, U_{j,-k}] \cap S_0^2[1, U_{j,-k}] = \emptyset, S_0^2[1, U_{j,-k}] \cap B'_j = \emptyset \},
\]

where \( v_1 \) (as a slight abuse of notation, compare this with \( v_1 \) defined for \( \gamma_1 \), see (4.53)) is the last time before \( U_{i,-k^4+k^4,1} \) that \( \gamma_{x}^y \) hits \( \partial B'_i \), and \( v_2 \) is defined similarly.

- We then define \( \nu^\theta(\cdot, \cdot) \) for the probability measure induced by \( (\gamma_{x}^y[0, U_{i,-k}], S_0^2[0, U_{i,-k}]) \) conditioned on the event \( G^1 \), and define \( \nu^\varphi(\cdot, \cdot) \) similarly with conditioning on \( G^2 \).

- In the above notation, the distribution of \( (LE(X^1[0, \tau_0]), [0, v_1], X^2[\Omega, \sigma^2]) \) is close enough to that of \( (\gamma_0^y[0, U_{i,-k}], S_0^2[0, U_{i,-k}]) \), (and similarly the distribution of \( (LE(X^2[0, \tau], [0, v_2], X^3[0, \tau])) \) is close enough to that of \( (\gamma_0^y[0, U_{j,-k}], S_0^2[0, U_{j,-k}]) \), we are able to replace \( P(G_{a,b})/P(G^1) \) and \( P(G_{c,d})/P(G^2) \) by \( \nu^\theta \) and \( \nu^\varphi \) respectively, and hence (4.98) can be rewritten as follows:

\[
P(H|G^1 \cap G^2) = \sum_{(a,b) \in \Gamma_1} \sum_{(c,d) \in \Gamma_2} f(a, b, c, d) \nu^\theta(a, b) \nu^\varphi(c, d) \left( 1 + O(2^{-c^k}) \right).
\]

As an application of Lemma 4.12 we have the following lemma.

**Lemma 4.13.** Remind the definition of \( \Delta_i \) and \( \Delta_j \) below (4.100). It follows that

\[
P(H|G^1 \cap G^2) = \sum_{(a,b) \in \Delta_i} \sum_{(c,d) \in \Delta_j} f(\pi(a, b, c, d)) \nu^\theta(\pi(a, b)) \nu^\varphi(\pi(c, d)) \left( 1 + O(2^{-c^k}) \right).
\]

36
Proof. By the separation lemma, it follows that

\[ \nu^\mathcal{G}(\{(a, b) \in \Delta_i \mid (a, b) \text{ is well-separated}\}) \geq c. \]

and

\[ \nu^\mathcal{G}(\{(c, d) \in \Delta_j \mid (c, d) \text{ is well-separated}\}) \geq c. \]

If \((a, b)\) and \((c, d)\) are well-separated, it is easy to see that

\[ h^1(\pi(a, b)) \geq c \text{ and } h^2(\pi(c, d)) \geq c. \]

Thus, by Lemma 4.11,

\[ f(\pi(a, b, c, d)) \geq c \cdot \text{Es}(2^{-k^3}, l)\text{Es}(2^{-k^3}, r), \]

if \((a, b)\) and \((c, d)\) are well-separated.

Therefore we have

\[
\sum_{(a, b) \in \Delta_i} \sum_{(c, d) \in \Delta_j} f(\pi(a, b, c, d)) \nu^\mathcal{G}(a, b) \nu^\mathcal{G}(c, d) \leq \text{Es}(2^{-k^3}, l)\text{Es}(2^{-k^3}, r). \tag{4.102}
\]

Set \(C_i = \{(a, b) \in \Delta_i \mid h^1(\pi(a, b)) \geq 2^{-k^3}\}\) and \(C_j = \{(c, d) \in \Delta_j \mid h^2(\pi(c, d)) \geq 2^{-k^3}\}\). Then, by Lemma 4.11

\[
\sum_{(a, b) \in C_i \text{ or } (c, d) \notin C_j} f(\pi(a, b, c, d)) \nu^\mathcal{G}(a, b) \nu^\mathcal{G}(c, d) \leq C2^{-k^3}\text{Es}(2^{-k^3}, r)\text{Es}(2^{-k^3}, l). \tag{4.103}
\]

Hence,

\[
P(\mathcal{H}|\mathcal{G}^1 \cap \mathcal{G}^2) \overset{4.101}{=} (1 + O(2^{-ck^2})) \sum_{(a, b) \in C_i \text{ or } (c, d) \notin C_j} f(\pi(a, b, c, d)) \nu^\mathcal{G}(a, b) \nu^\mathcal{G}(c, d)
\]

\[
+ (1 + O(2^{-ck^2})) \sum_{(a, b) \notin C_i \text{ or } (c, d) \in C_j} f(\pi(a, b, c, d)) \nu^\mathcal{G}(a, b) \nu^\mathcal{G}(c, d)
\]

\[
\overset{4.103}{=} (1 + O(2^{-ck^2})) \sum_{(a, b) \in \Delta_i \text{ or } (c, d) \notin \Delta_j} f(\pi(a, b, c, d)) \nu^\mathcal{G}(a, b) \nu^\mathcal{G}(c, d)
\]

\[
+ C2^{-k^3}\text{Es}(2^{-k^3}, r)\text{Es}(2^{-k^3}, l)
\]

\[
\overset{4.102}{=} (1 + O(2^{-ck^2})) \sum_{(a, b) \in \Delta_i \text{ or } (c, d) \notin \Delta_j} f(\pi(a, b, c, d)) \nu^\mathcal{G}(a, b) \nu^\mathcal{G}(c, d).
\]

which gives the lemma. \(\square\)

Once we obtain 4.13 modifying Corollary 3.11 or Proposition 5.6 of [16], we have the following corollary. Let

\[
\Gamma_i := \{(\pi, \beta) \mid (\pi, \beta) \in \Gamma_i, \pi(0), \beta(0) \in \partial B(x_i, 2^{-10k^3})\}, \text{ and define } \Gamma_j \text{ similarly.} \tag{4.104}
\]

Then, we write \(\nu^\mathcal{H}(\cdot, \cdot)\) for the probability measure on \(\Gamma_i\) induced by \(\pi(\gamma_{\mathcal{H}}[0, U_{i-k^3}], S_{y_2}[0, U_{i-k^3}])\) conditioned on the event \(\mathcal{G}^1\) and define \(\nu^\mathcal{H}(\cdot, \cdot)\) as a probability measure on \(\Gamma_j\) similarly.

Corollary 4.14.

\[
P(\mathcal{H}|\mathcal{G}^1 \cap \mathcal{G}^2) = (1 + O(2^{-ck^2})) \sum_{(\pi, \beta) \in \Gamma_i \text{ or } (\pi, \beta) \in \Gamma_j} f(\pi, \beta, \tau, \alpha) \nu^\mathcal{H}(\pi, \beta) \nu^\mathcal{H}(\tau, \alpha).
\]

We are now ready to state the main result of this section.

Proposition 4.15.

\[
P(0 \rightarrow B' \rightarrow B' \rightarrow B')
\]

\[
= \sum_{y \in \partial B'} \sum_{z \in \partial B'} \sum_{(\pi, \beta) \in \Gamma_i \text{ or } (\pi, \beta) \in \Gamma_j} f(\pi, \beta, \tau, \alpha) \nu^\mathcal{H}(\pi, \beta) \nu^\mathcal{H}(\tau, \alpha) P(\mathcal{G}^1) P(\mathcal{G}^2) (1 + O(2^{-ck^2})) \tag{4.105}
\]
Proof. By Proposition 4.5 and Corollary 4.14 we have
\[
P(0 \xrightarrow{w} B_i \xrightarrow{w} B_j) = \sum_{y \in \partial B_i \cap \partial B_j} \sum_{z \in \partial B_i \cap \partial B_j} G(0,y)G(y,z) \times P(H)(1 + O(2^{-ck^2}))
\]
\[
= \sum_{y \in \partial B_i \cap \partial B_j} G(0,y)P(H | \mathcal{G}, \mathcal{G}^2)P(\mathcal{G}^2)(1 + O(2^{-ck^2}))
\]
\[
= \sum_{y \in \partial B_i \cap \partial B_j} G(0,y)\sum_{(\pi,\beta) \in \mathcal{T}_i} \sum_{(\pi,\delta) \in \mathcal{T}_j} f(\pi,\beta,\delta)\pi^w(\pi,\delta)P(\mathcal{G}^2)(1 + O(2^{-ck^2})).
\]
This finishes the proof. \(\square\)

4.6 The point-crossing probability

In this subsection, we are going to deal with the two-point crossing probability. As the situation is very similar to that of box-crossing probabilities, we will skip the proof but only give analogues of Propositions 4.5 and 4.15 which will be directly quoted in the proof of 4.7. As in previous subsections, we still assume that \((i,j)\) is good throughout this subsection.

We start with the decomposition of paths. As discussed in Paragraph 12 of the proof sketch,
\[
P(0 \xrightarrow{w} v \xrightarrow{w} w) = G(0,v)G(v,w)P(F^w),
\] (4.106)
where
- \(Y^1\): is a RW started from \(v\), conditioned to hit 0 before leaving \(D\);
- \(Y^2\): is a RW started from \(w\), conditioned to hit \(v\) before leaving \(D\);
- \(Y^3\): is a SRW started from \(v\);
- \(F^w := \{\text{LE}(Y^1[0, \tau_0] \cap (Y^2[1, \tau_v - 1] \cup Y^3[0, T]) = 0, \text{LE}(Y^2[0, \tau_v] \cap Y^3[1, T] = 0)\}.

Similar to the event \(H\) defined in (4.106), we can replace the event \(F^w\) by
\[
\tilde{H} := \{\text{LE}(Y^1[0, \tau_0]) \cap (Y^2[\hat{s}, \tau_v - 1] \cup Y^3[0, T]) = 0, \text{LE}(Y^2[0, \hat{s}]) \cap Y^3[1, T] = 0\}.
\] (4.107)
where (compare this to (4.52))
\[
\hat{s} := T_{x_i,2^{-k^3}+k^2}(Y^2); \quad \hat{s} := \text{last time (up to } \tau_v) \text{ that } Y^2 \text{ hits } \partial B(x_i, 2^{-k^3}).
\]
The claim above is summarized in the following proposition, which can be proved in a similar way as Proposition 4.5

**Proposition 4.16.** For any good \((i,j)\) and any \(v \in B_i\) and \(w \in B_j\),
\[
P(0 \xrightarrow{w} v \xrightarrow{w} w) = G(0,v)G(v,w)P(\tilde{H})(1 + O(2^{-ck^2})).
\] (4.108)

Then, as analogue of (4.85) and (4.86), we define
\[
\tilde{G}^1 := \{\text{LE}(Y^1[0, \tau_0]) \cap Y^2[\hat{s}, \tau_v - 1] = 0\}; \quad \tilde{G}^2 := \{\text{LE}(Y^1[0, \tau_0]) \cap Y^3[1, \tau_{\hat{s} - 1}] = 0\},
\] (4.109)
and as analogue of (4.99) and (4.100) we also define
\[
\tilde{G}_{\infty}^1 := \{\gamma_{\infty}^1[0, U_{i, -k^3}] \cap S^2_{\infty}[1, U_{i, -k^3}] = 0\}; \quad \tilde{G}_{\infty}^2 := \{\gamma_{\infty}^2[0, U_{j, -k^3}] \cap S^w_{\infty}[1, U_{j, -k^3}] = 0\},
\] (4.110)
where \(\gamma_{\infty}^1, \gamma_{\infty}^2\) is an infinite LERW started at \(v\) (resp.) and \(S^w_{\infty}(S^w_{\infty})\) is the independent SRW started at \(v\) (resp.).

Recall the definition of \(\pi_{1}, \pi_{2}\) in (4.108) and the definition of \(\pi\) in (4.90). We let \(\pi^w(\cdot, \cdot)\) be the probability measure on \(\mathcal{T}_i\) induced by \(\pi(\gamma_{\infty}^1[0, U_{i, -k^3}], S^2_{\infty}[0, U_{i, -k^3}])\) conditioned on \(\tilde{G}_{\infty}^1\), and define \(\pi^w(\cdot, \cdot)\) similarly. As an analogue of Proposition 4.15 we have the following decomposition of the point-crossing probability.

**Proposition 4.17.** For any good \((i,j)\) and any \(v \in B_i\) and \(w \in B_j\),
\[
P(0 \xrightarrow{w} v \xrightarrow{w} w) = G(0,v)G(v,w)P(\tilde{G}^1)P(\tilde{G}^2) \sum_{(\pi,\beta) \in \pi_{1}} \sum_{(\pi,\delta) \in \pi_{2}} f(\pi,\beta,\delta)\pi^w(\pi,\beta)\pi^w(\pi,\delta)(1 + O(2^{-ck^2})).
\] (4.111)
4.7 Proof of (4.7)

Proof of (4.7). Comparing Propositions 4.15 and 4.17, we see that in order to prove (4.7), it suffices to compare \( \mathcal{P} \) with \( \mathcal{P}^v \) for \( v \in B_i, y \in \partial B'_i \), and \( \mathcal{P}^w \) with \( \mathcal{P}^z \) for \( w \in B_j, z \in \partial B'_j \). Remind that \( x, x_j \) are the centers of \( B_i \) and \( B_j \) respectively. We observe that by (2.29), we have

\[
\| \mathcal{P}^v - \mathcal{P}^{x_j} \| \leq c: 2^{-ck^3}, \quad \| \mathcal{P}^w - \mathcal{P}^{x_j} \| \leq c: 2^{-ck^3},
\]

(4.112)

and by (2.31), we have

\[
\| \mathcal{P}^w - \mathcal{P}^{x_j} \| \leq c: 2^{-ck^3}, \quad \| \mathcal{P}^{z_j} - \mathcal{P}^{x_j} \| \leq c: 2^{-ck^3},
\]

(4.113)

(note that in this case, in the notation of Section 4.5, esp. (4.99) and (4.100), the quadruple \((\gamma, \lambda, \Theta, \Theta)\) in (2.30) and (2.31) should be taken as \((\gamma_\infty [v_1, U_{i-k+k'}, \gamma_\infty^w [0, v_1], B'_i) \) and \((\gamma_\infty [v_2, U_{j-k+k'}, \gamma_\infty^w [0, v_2], B'_j) \)

respectively). By (4.112) and the observation that \( \mathcal{P}^* \) are probability measures, we have

\[
\left| \sum_{(\pi, \delta) \in \Gamma_i} \sum_{(\pi, \delta) \in \Gamma_j} f(\pi, \bar{\pi}, \bar{\delta}) \mathcal{P}^v (\pi, \bar{\pi}) \mathcal{P}^w (\pi, \bar{\delta}) - \sum_{(\pi, \delta) \in \Gamma_i} \sum_{(\pi, \delta) \in \Gamma_j} f(\pi, \bar{\pi}, \bar{\delta}) \mathcal{P}^{x_i} (\pi, \bar{\pi}) \mathcal{P}^{x_j} (\pi, \bar{\delta}) \right|
\leq C2^{-ck^3} \max_{(\pi, \bar{\pi}) \in \Gamma_i, (\pi, \bar{\pi}) \in \Gamma_j} f(\pi, \bar{\pi}, \bar{\delta}) \leq C2^{-ck^3} \sum_{(\pi, \delta) \in \Gamma_i} \sum_{(\pi, \delta) \in \Gamma_j} f(\pi, \bar{\pi}, \bar{\delta}) \mathcal{P}^{x_i} (\pi, \bar{\pi}) \mathcal{P}^{x_j} (\pi, \bar{\delta}),
\]

where in (*) we applied the observation that by separation lemma (see Section 2.8),

\[
\max_{(\pi, \bar{\pi}) \in \Gamma_i, (\pi, \bar{\pi}) \in \Gamma_j} f(\pi, \bar{\pi}, \bar{\delta}) = \text{Es}(2^{-k^3}, l) \times \text{Es}(2^{-k^3}, r),
\]

and

\[
\sum_{(\pi, \delta) \in \Gamma_i} \sum_{(\pi, \delta) \in \Gamma_j} f(\pi, \bar{\pi}, \bar{\delta}) \mathcal{P}^{x_i} (\pi, \bar{\pi}) \mathcal{P}^{x_j} (\pi, \bar{\delta}) \leq \text{Es}(2^{-k^3}, l) \text{Es}(2^{-k^3}, r).
\]

This gives that

\[
\sum_{(\pi, \delta) \in \Gamma_i} \sum_{(\pi, \delta) \in \Gamma_j} f(\pi, \bar{\pi}, \bar{\delta}) \mathcal{P}^v (\pi, \bar{\pi}) \mathcal{P}^w (\pi, \bar{\delta}) = (1 + O(2^{-ck^3})) \sum_{(\pi, \delta) \in \Gamma_i} \sum_{(\pi, \delta) \in \Gamma_j} f(\pi, \bar{\pi}, \bar{\delta}) \mathcal{P}^{x_i} (\pi, \bar{\pi}) \mathcal{P}^{x_j} (\pi, \bar{\delta}).
\]

Similarly, by (4.113) we have

\[
\sum_{(\pi, \delta) \in \Gamma_i} \sum_{(\pi, \delta) \in \Gamma_j} f(\pi, \bar{\pi}, \bar{\delta}) \mathcal{P}^w (\pi, \bar{\pi}) \mathcal{P}^z (\pi, \bar{\delta}) = (1 + O(2^{-ck^3})) \sum_{(\pi, \delta) \in \Gamma_i} \sum_{(\pi, \delta) \in \Gamma_j} f(\pi, \bar{\pi}, \bar{\delta}) \mathcal{P}^{x_i} (\pi, \bar{\pi}) \mathcal{P}^{x_j} (\pi, \bar{\delta}).
\]

By (2.6),

\[
G_2 (0, v) G_2 (v, w) = (1 + O(2^{-ck^3})) G_2 (0, y) G_2 (y, z).
\]

Therefore, we have

\[
\frac{P \left( 0 \xrightarrow{z} v \xrightarrow{z} w \right)}{P \left( 0 \xrightarrow{z} B'_i \xrightarrow{z} B'_j \right)}
\]

\[
(1 + O(2^{-ck^3})) \times \frac{P(\tilde{G}^1) P(\tilde{G}^2) \sum_{(\pi, \delta) \in \Gamma_i} \sum_{(\pi, \delta) \in \Gamma_j} f(\pi, \bar{\pi}, \bar{\delta}) \mathcal{P}^{x_i} (\pi, \bar{\pi}) \mathcal{P}^{x_j} (\pi, \bar{\delta})}{\sum_{y \in \partial B'_i} P(\tilde{G}^1) \sum_{z \in \partial B'_j} P(\tilde{G}^2) \sum_{(\pi, \delta) \in \Gamma_i} \sum_{(\pi, \delta) \in \Gamma_j} f(\pi, \bar{\pi}, \bar{\delta}) \mathcal{P}^{x_i} (\pi, \bar{\pi}) \mathcal{P}^{x_j} (\pi, \bar{\delta})}
\]

\[
= (1 + O(2^{-ck^3})) \times \frac{P(\tilde{G}^1) P(\tilde{G}^2)}{\sum_{y \in \partial B'_i} P(\tilde{G}^2) \sum_{z \in \partial B'_j} P(\tilde{G}^2)}.
\]

Remind the definition of \( G^1 \) and \( G^2 \) in (4.99) and (4.100) and of \( G^1 \) and \( G^2 \) in (4.110). By Proposition 4.6 of [17], we have

\[
\frac{P \left( 0 \xrightarrow{z} v \xrightarrow{z} w \right)}{P \left( 0 \xrightarrow{z} B'_i \xrightarrow{z} B'_j \right)} = (1 + O(2^{-ck^3})) \times \frac{P(\tilde{G}^1) P(\tilde{G}^2)}{\sum_{y \in \partial B'_i} P(\tilde{G}^1) \sum_{z \in \partial B'_j} P(\tilde{G}^2)}.
\]
However, by translation invariance (here \(v_0 = v - x_i + x_0\) and \(\tilde{G}_{1,v_0}^1\) stands for an event defined in a similar way as \(\tilde{G}_{1,x_0}^1\) but started from \(v_0\) instead)

\[
\frac{P(\tilde{G}_{1,x_0}^1)}{\sum_{y \in \partial B_{x_0}^1} P(\tilde{G}_{1,x_0}^1)} = \frac{P(\tilde{G}_{1,v_0}^1)}{\sum_{y \in \partial B_{v_0}^1} P(\tilde{G}_{1,v_0}^1)} \quad (4.114)
\]

It is not difficult to show that RHS of \((4.114)\) is equal to \(\sum_{v \in B_i} P(v) \cdot \sum_{w \in B_j} P(w) \cdot \sum_{\gamma \in B_j} P(\gamma) = P(\gamma \cap B_i \neq \emptyset) \cdot P(\gamma \cap B_j \neq \emptyset) \cdot (1 + O(2^{-ck^2})) \), and hence, \(E[X_i X_j \mid Y_i = Y_j = 1] = a_0^2 (1 + O(2^{-ck^2}))\), which is the desired estimate!

**Remark 4.18.** We now briefly give the modifications needed for proving \((4.8)\) and \((4.9)\). Without loss of generality we focus on \((4.8)\). Similar to \((4.21)\) and \((4.24)\), one can show that

\[
E \left( X_i \mid Y_i = Y_j = 1 \right) = (1 + O(2^{-k^2})) \cdot \frac{P \left( x_i \xrightarrow{\gamma} x_i \right) + P \left( x_i \xrightarrow{\gamma} x_i \right)}{P \left( x_i \xrightarrow{\gamma} x_i \right) + P \left( x_i \xrightarrow{\gamma} x_i \right)},
\]

with \(\{ x_i \xrightarrow{\gamma} x_i \} \) and \(\{ x_i \xrightarrow{\gamma} x_i \} \) are defined in a way similar to \((4.22)\) and \((4.23)\). Then, one can repeat the arguments in the proof of \((4.7)\) to deal with the hybrid point-box-crossing probabilities, where the same decomposition, localization and coupling techniques can be applied, and show that

\[
|B_0| P \left( x_i \xrightarrow{\gamma} x_i \right) = a_0 \left( 1 + O(2^{-k^2}) \right) \quad \text{and} \quad |B_0| P \left( x_i \xrightarrow{\gamma} x_i \right) = a_0 \left( 1 + O(2^{-k^2}) \right),
\]

which implies \((4.8)\).

### 5 \(L^2\)-approximation of \(\mu_n\)

In this section, we are going finalize our preparatory works for the proof of Theorem 1.1 by giving the final form of the key \(L^2\)-estimate. More precisely, after obtaining an accurate asymptotics of \(\alpha_0\) defined in \((4.1)\) in Proposition 5.5, we will rewrite Proposition 4.1 as Proposition 5.6. We remark that Proposition 5.6 is an analog of Proposition 4.11 of \([4]\).
5.1 A preparatory result

We now briefly review notations and the setup. Let $S$ be the SRW on $2^{-n}\mathbb{Z}^3$ started at the origin and $T$ be the first time that $S$ exits from $\mathbb{D}$. We write $\gamma_n = \text{LE}(S[0,T])$ for the LERW. We define a random measure $\mu_n$ by

$$\mu_n := \sum_{x \in \gamma_n \cap 2^{-n}\mathbb{Z}^3} \delta_x 2^{-\beta n},$$  \hspace{1cm} (5.1)

where $\delta_x$ stands for the Dirac measure at $x$ and the constant $\beta$ is defined as in (2.15). As we discussed in Section 1.2.1, for each box $B \subset \mathbb{D}$, we want to approximate $\mu_n(B)$ by some measurable quantity with respect to the scaling limit $\mathcal{K}$.

With this in mind, take a box $B \subset \mathbb{D}$ with $\text{dist}(B, \{0\} \cup \partial \mathbb{D}) > 0$ and divide it into smaller boxes $B_1, B_2, \ldots, B_N$ as we did in Section 4. Write $X_i$ for the number of lattice points in $B_i$ hit by the LERW and let $Y_i$ be the indicator function of the event that the LERW hits some enlargement of $B_i$ denoted by $B'_i$. In Section 4 we have already proved that $\sum_i X_i$ can be well approximated by $\alpha_0 \sum_i Y_i$ with appropriate choice of $\alpha_0$. However, what we really want to control is $\mu_n(B) = 2^{-\beta n} \sum_i X_i$.

To obtain the approximation of $\mu_n(B)$, the first step is to show that the indicator function $Y_i$ can be approximated by the indicator function of the event that $\mathcal{K}$ hits $B'_i$. Since $\gamma_n$ converges weakly to $\mathcal{K}$ as $n \to \infty$ with respect to the Hausdorff distance (see Section 2.3 for the Hausdorff distance), by Skorokhod’s representation theorem, we can define $\{\gamma_n\}_{n \geq 1}$ and $\mathcal{K}$ on the same probability space so that

$$\lim_{n \to \infty} d_{\text{Haus}}(\gamma_n, \mathcal{K}) = 0$$ \hspace{1cm} (5.2)

almost surely. Throughout this section, we assume this coupling of loop-erased random walks $\{\gamma_n\}_{n \geq 1}$ and its scaling limit $\mathcal{K}$. In this coupling, we will prove in Corollary 5.3 that for sufficiently large $n$,

$$\gamma_n \text{ hits } B'_i \iff \mathcal{K} \text{ hits } B'_i \quad \text{for each } i \text{ with high probability.}$$ \hspace{1cm} (5.3)

In this sense, the indicator function $Y_i$ as above is well approximated by the indicator function of the event that $\mathcal{K}$ hits $B'_i$.

In order to show the relation (5.3), we need to prove that if we condition $\gamma_n$ to hit $B'_i$, then with high (conditional) probability $\gamma_n$ actually hits a smaller box contained in $B'_i$; see Proposition 5.1 for the precise statement.

Let $x \in \mathbb{D}$ and $k \geq 1$. Set $B$ for the cube of radius $3 \cdot 2^{-k^4}$ centered at $x$. Suppose that $\text{dist}(B, \{0\} \cup \partial \mathbb{D}) > 2^{-k}$. Define $r = r_k = 2^{-2^{11}}$ for $k \geq 1$. We set $\hat{B}$ and $\tilde{B}$ for the cubes of radius $3 \cdot 2^{-k^4} - r_k$ and $3 \cdot 2^{-k^4} + r_k$ centered at $x$ respectively. Then we have the following proposition.

**Proposition 5.1.** Let $x, B, r_k$ and $\tilde{B}$ be as above. Then there exists a universal constant $c$ such that for all $n$

$$P\left(\gamma_n \cap B \neq \emptyset \text{ and } \gamma_n \cap \hat{B} = \emptyset\right) \leq c2^{-k^{10}},$$ \hspace{1cm} (5.4)

and similarly,

$$P\left(\gamma_n \cap B'' \neq \emptyset \text{ and } \gamma_n \cap \hat{B} = \emptyset\right) \leq c2^{-k^{10}}.$$ \hspace{1cm} (5.5)

**Proof.** As two claims are very similar, we will only prove (5.4).

Let $\tau$ be the first time that $\gamma$ hits $B$. We will first prove that $\gamma(\tau)$ is not close to an edge of $B$. Write $l = l_k = 6 \cdot 2^{-k^4}$ for the side length of $B$. Taking $a_1, a_2$ and $a_3$ appropriately, we can write $B = \prod_{i=1}^3 [a_i, a_i + l]$. We set $e_1, e_2, \ldots, e_{12}$ for the twelve edges of $B$ and write $\tilde{e} = \bigcup_{i=1}^{12} e_i$ for its union. Let

$$F = \{y \mid \text{dist}(y, \tilde{e}) \leq 2r\}$$

be the set of points which is close to $\tilde{e}$ the union of edges of $B$. Suppose that $\gamma \cap F \neq \emptyset$. Then $S[0,T]$ must hit $F$. By a capacity estimate, see Proposition 6.4.1 of [12], it follows that there exists a universal constant $c$ such that

$$P\left(\gamma \cap F \neq \emptyset\right) \leq ck^4 2^{-k^{11}}.$$  

Therefore, if we write $F_1, F_2, \ldots, F_6$ for the faces of $B$, then we have

$$P\left(\gamma \cap B \neq \emptyset \text{ and } \gamma \cap \hat{B} = \emptyset\right) \leq c k^4 2^{-k^{11}} + \sum_{i=1}^6 P\left(\gamma \cap B \neq \emptyset, \gamma \cap \hat{B} = \emptyset, \gamma(\tau) \in F_i, \gamma \cap F = \emptyset\right).$$
So it suffices to show that

$$P\left( \gamma \cap B \neq \emptyset, \gamma \cap \widehat{B} = \emptyset, \gamma(\tau) \in F_1, \gamma \cap F = \emptyset \right) \leq c2^{-k^{10}} \quad (5.6)$$

for each $i = 1, 2, \cdots, 6$. We will only prove the inequality above for $i = 1$. Without loss of generality, we can assume $F_1 = [a_1, a_3 + \ell] \times [a_2, a_2 + \ell] \times \{a_3\}$.

Let $\widehat{B} = [a_1, a_1 + \ell] \times [a_2, a_2 + \ell] \times [a_3 + r, a_3 + r + \ell]$ be a cube obtained by translating $B$ with distance $r$ in the direction of $z$-axis. We also write $\widehat{\gamma} = \gamma + (0, 0, r)$ for the simple path obtained by translating $\gamma$ with distance $r$ in the direction of $z$-axis. Let $\tau$ be the first time that $\widehat{\gamma}$ hits $B$.

Suppose that $\gamma \cap B \neq \emptyset$, $\gamma(\tau) \in F_1$ and $\gamma \cap F = \emptyset$. Then clearly $\widehat{\gamma}$ hits $\widehat{B}$. We write $t$ for the first time that $\widehat{\gamma}$ hits $\widehat{B}$. By definition, we have $t = \tau$. We claim that $\widehat{\gamma}(\tau)$ the location of $\widehat{\gamma}$ at its first hitting of $B$ is lying on $F_1$. To see it, we first note that

$$\text{dist}(\widehat{\gamma}(\tau), F_1) \leq r$$

since otherwise $\widehat{\gamma}$ would have another first hitting point of $\widehat{B}$ which is different from $\widehat{\gamma}(t)$. However, if $\widehat{\gamma}(\tau) \notin F_1$, then $\gamma \cap F = \emptyset$ which leads a contradiction. Therefore, $\widehat{\gamma}(\tau) \in F_1$. Consequently, we have

$$\gamma \cap B \neq \emptyset, \gamma(\tau) \in F_1 \text{ and } \gamma \cap F = \emptyset \Rightarrow \widehat{\gamma} \cap \widehat{B} \neq \emptyset \text{ and } \widehat{\gamma}(\tau) \in F_1.$$  

Therefore, we have

$$P\left( \widehat{\gamma} \cap \widehat{B} \neq \emptyset, \widehat{\gamma}(\tau) \in F_1 \right) \geq P\left( \gamma \cap B \neq \emptyset, \gamma(\tau) \in F_1 \right) - c_0k^{4}2^{-k^{11}}, \quad (5.7)$$

for some universal constant $c_0 > 0$. We also note that it is not difficult to show that

$$P\left( \gamma \cap B \neq \emptyset, \gamma(\tau) \in F_1 \right) \geq c2^{-3k^4},$$

for some universal constant $c > 0$.

Next we will compare

$$P\left( \widehat{\gamma} \cap \widehat{B} \neq \emptyset, \widehat{\gamma}(\tau) \in F_1 \right) \text{ with } P\left( \gamma \cap \widehat{B} \neq \emptyset, \gamma(\tau) \in F_1 \right)$$

via a simple coupling argument. Let $w = (0, 0, r)$. We write $\tilde{S}$ for the simple random walk on $2^{-n}\mathbb{Z}^3$ started at $w$ and set $\tilde{T}$ for the first time that $\tilde{S}$ exits from $\mathbb{D} + w$. Then $\tilde{\gamma}$ has the same distribution as that of $\gamma^* = \text{LE}(\tilde{S}[0, T])$. We write $T'$ for the first time that $\tilde{S}$ exits from $\mathbb{D}$ as well. Let $\gamma' = \text{LE}(\tilde{S}[0, T])$. We first compare

$$P\left( \widehat{\gamma} \cap \widehat{B} \neq \emptyset, \widehat{\gamma}(\tau) \in F_1 \right) = P\left( \gamma^* \cap \widehat{B} \neq \emptyset, \gamma^*(\tau^*) \in F_1 \right)$$

with

$$P\left( \gamma' \cap \widehat{B} \neq \emptyset, \gamma'(\tau') \in F_1 \right),$$

where $\tau^*$ (resp. $\tau'$) stands for the first time that $\gamma^*$ (resp. $\gamma'$) hits $B$. To do this, let

$$G_1 = \left\{ \gamma^* \cap \widehat{B} \neq \emptyset, \gamma^*(\tau^*) \in F_1 \right\} \text{ and } G_2 = \left\{ \gamma' \cap \widehat{B} \neq \emptyset, \gamma'(\tau') \in F_1 \right\}.$$  

It is not difficult to see that

$$P\left( \text{diam}(\tilde{S}[\tilde{T} \land T, \tilde{T} \lor T]) \geq r^{3/2} \right) \leq cr^2.$$  

Suppose that $G_3 := \left\{ \text{diam}(\tilde{S}[\tilde{T} \land T, \tilde{T} \lor T]) \leq r^{3/2} \right\}$ occurs. Then we see that $\gamma^*[0, u] = \gamma'[0, u']$, where $u$ (resp. $u'$) stands for the first time that $\gamma^*$ (resp. $\gamma'$) exits from $\{y \in \mathbb{R}^3 \mid |y| < 1 - 2\sqrt{r}\}$. Therefore, if $G_1 \cap G_2 \cap G_3$ or $G_1^c \cap G_2 \cap G_3$ occur, then $\tilde{S}[\tilde{v}, \tilde{T} \lor T] \cap B \neq \emptyset$ where $v$ stands for the first time that $\tilde{S}$ exits from $\{y \in \mathbb{R}^3 \mid |y| < 1 - 2\sqrt{r}\}$. However, since $\text{dist}(B, \{0\} \cup \partial \mathbb{D}) > 2^{-k}$, it is easy to see that

$$P\left( \tilde{S}[\tilde{v}, \tilde{T} \lor T] \cap B \neq \emptyset \right) \leq c2^k\sqrt{r}.$$  

Consequently, we see that

$$|P\left( \widehat{\gamma} \cap \widehat{B} \neq \emptyset, \widehat{\gamma}(\tau) \in F_1 \right) - P\left( \gamma' \cap \widehat{B} \neq \emptyset, \gamma'(\tau') \in F_1 \right)| \leq c2^k\sqrt{r}. \quad (5.8)$$
We now compare
\[ P\left( \gamma' \cap \tilde{B} \neq \emptyset, \gamma'(\tau') \in F_1 \right) \quad \text{with} \quad P\left( \gamma \cap \tilde{B} \neq \emptyset, \gamma(\tau) \in F_1 \right). \]

Recall that \( \gamma \) is the loop-erasure of \( S[0, T] \) where \( S \) starts from the origin while \( \gamma' \) is the loop-erasure of \( \tilde{S}[0, T] \) where \( \tilde{S} \) starts from \( w = (0, 0, r) \). We also recall that \( \tau \) (resp. \( \tau' \)) stands for the first time that \( \gamma \) (resp. \( \gamma' \)) hits \( B \). Although the starting point of \( S \) is different from that of \( \tilde{S} \), we can define them on the same probability space such that
\[ P(H) \geq 1 - c\gamma^{\frac{3}{2}}, \]
where
\[ H = \left\{ \tilde{S}(k + t_1) = S(k + t_2) \text{ for all } k \geq 0 \right\} \]
Here \( t_1 \) (resp. \( t_2 \)) stands for the first time that \( S \) (resp. \( \tilde{S} \)) exits from \( r^{\frac{3}{2}}D \). We assume this coupling of \( S \) and \( \tilde{S} \). Then they have a cut point with high probability in the following sense. Let \( t'_1 \) (resp. \( t'_2 \)) stand for the first time that \( S \) (resp. \( \tilde{S} \)) exits from \( r^{\frac{3}{2}}D \). We call \( k \) a cut time for \( S \) (resp. \( \tilde{S} \)) if the following three conditions hold:
- \( t_1 \leq k \leq t'_1 \) (resp. \( t_2 \leq k \leq t'_2 \)),
- \( S[k + 1, T] \cap (S[t_1, k] \cup r^{\frac{3}{2}}D) = \emptyset \) (resp. \( \tilde{S}[k + 1, T] \cap (\tilde{S}[t_2, k] \cup r^{\frac{3}{2}}D) = \emptyset \)),
- \( S(k) \in r^{\frac{3}{2}}D \setminus r^{\frac{3}{2}}D \) (resp. \( \tilde{S}(k) \in r^{\frac{3}{2}}D \setminus r^{\frac{3}{2}}D \)).

Let \( H' \) be the event that \( S \) has a cut time. Then as in the proof of Theorem 1.2 of [9], one can show that
\[ P(H') \geq 1 - c2^{-c'2^{k+11}} \]
for some universal constants \( c, c' > 0 \).

Now suppose that \( H \cap H' \) occurs. We write \( k + t_1 \) (with \( k \geq 0 \)) for a cut time of \( S \). Then it is easy to see that \( k + t_2 \) is a cut time for \( \tilde{S} \). Furthermore, by definition of the loop-erasing procedure, one can see that
\[
\begin{align*}
\gamma &= \text{LE}(S[0, T]) = \text{LE}(S[0, k + t_1]) \oplus \text{LE}(S[k + t_1 + 1, T]), \\
\gamma' &= \text{LE}(\tilde{S}[0, T]) = \text{LE}(\tilde{S}[0, k + t_2]) \oplus \text{LE}(\tilde{S}[k + t_2 + 1, T]).
\end{align*}
\]
Note that the event \( H \) ensures that \( \text{LE}(S[k + t_1 + 1, T]) = \text{LE}(\tilde{S}[k + t_2 + 1, T]) \). Also we mention that \( \text{LE}(S[k + t_1 + 1, T]) \subset r^{\frac{3}{2}}D \) by definition of the cut time. In particular, on the event \( H \cap H' \), it follows that the event
\[ H'' := \left\{ \gamma(k + v_1) = \gamma'(k + v_2) \text{ for all } k \geq 0 \right\} \]
occurs. Here \( v_1 \) (resp. \( v_2 \)) stands for the first time that \( \gamma \) (resp. \( \gamma' \)) exits from \( r^{\frac{3}{2}}D \).

Since \( \text{dist}(B, \{0\} \cup \partial D) > 2^{-k} \), we see that
\[ H'' \cap \left\{ \gamma \cap \tilde{B} \neq \emptyset, \gamma(\tau) \in F_1 \right\} \Leftrightarrow H'' \cap \left\{ \gamma' \cap \tilde{B} \neq \emptyset, \gamma'(\tau') \in F_1 \right\} \]
Thus, we have
\[
P\left( \gamma \cap \tilde{B} \neq \emptyset, \gamma(\tau) \in F_1 \right) = P\left( \gamma \cap \tilde{B} \neq \emptyset, \gamma(\tau) \in F_1, H''\right) + P\left( \gamma \cap \tilde{B} \neq \emptyset, \gamma(\tau) \in F_1, (H'')^c \right)
= P\left( \gamma' \cap \tilde{B} \neq \emptyset, \gamma'(\tau') \in F_1, H''\right) + O\left(2^{-c2^{k+11}}\right)
= P\left( \gamma' \cap \tilde{B} \neq \emptyset, \gamma'(\tau') \in F_1\right) + O\left(2^{-c2^{k+11}}\right), \tag{5.9}
\]
for some universal constant \( c > 0 \).

Combining (5.8) with (5.9), we see that
\[
\left| P\left( \gamma \cap \tilde{B} \neq \emptyset, \gamma(\tau) \in F_1 \right) - P\left( \gamma \cap \tilde{B} \neq \emptyset, \gamma(\tau) \in F_1 \right) \right| \leq C2^{-c2^{k+11}}, \tag{5.10}
\]

43
for some universal constants $c, C > 0$. By (5.7), we have
\[
P\left(\gamma \cap \tilde{B} \neq \emptyset, \gamma(\tau) \in F_1\right)
= P\left(\gamma \cap B \neq \emptyset, \gamma(\tau) \in F_1\right) - P\left(\gamma \cap B = \emptyset, \gamma(\tau) \in F_1\right)
\leq P\left(\tilde{\gamma} \cap \tilde{B} \neq \emptyset, \tilde{\gamma}(\tau) \in F_1\right) + c_0 k^4 2^{-k^{11}} - P\left(\gamma \cap B \neq \emptyset, \gamma \cap \tilde{B} = \emptyset, \gamma(\tau) \in F_1\right).
\]
Thus, using (5.10), we see that
\[
P\left(\gamma \cap B \neq \emptyset, \gamma \cap \tilde{B} = \emptyset, \gamma(\tau) \in F_1\right)
\leq P\left(\tilde{\gamma} \cap \tilde{B} \neq \emptyset, \tilde{\gamma}(\tau) \in F_1\right) - P\left(\gamma \cap \tilde{B} \neq \emptyset, \gamma(\tau) \in F_1\right) + c_0 k^4 2^{-k^{11}}
\leq C 2^{-k^{10}}
\]
which gives (5.6) and we finish the proof.

\begin{remark}
A similar coupling of two LERW’s with different initial configurations can also be found in Section 2.6 of [11].
\end{remark}

Recall that we fix some cube $B$ of radius $3 \cdot 2^{-k^4}$ centered at $x \in \mathbb{D}$ satisfying $\text{dist}(B, \{0\} \cup \partial \mathbb{D}) > 2^{-k}$. Also we recall that $\tilde{B}$ and $\hat{B}$ stands for the cube of radius $3 \cdot 2^{-k^4} - r_k$ centered at $x$ and that $r = r_k = 2^{-2k^{11}}$.

We assume that $\{\gamma_n\}_{n \geq 1}$ and $\mathcal{K}$ are coupled such that (5.2) holds. Let $\tilde{Y}_n$ (resp. $\hat{Z}$) be the indicator function of the event that $\gamma_n \cap B \neq \emptyset$ (resp. $\mathcal{K} \cap B \neq \emptyset$). Then we have the following corollary.

\begin{corollary}
With the notation above, it follows that there exists $N = N_k$ depending on $k$ such that for all $n \geq N_k$
\[
P\left(\tilde{Y}_n = \hat{Z}\right) \geq 1 - c 2^{-k^{10}}
\]
for some universal constant $c > 0$.
\end{corollary}

\begin{proof}
We recall that $(\gamma_n)_{n \geq 1}$ and $\mathcal{K}$ are coupled such that (5.2) holds. Therefore, it follows that there exists $N = N_k$ depending on $k$ such that for all $n \geq N_k$,
\[
P\left(d_{\text{Haus}}(\gamma_n, \mathcal{K}) \geq \frac{1}{2} \cdot r_k\right) \leq 2^{-k^{10}}.
\]
(5.12)

Suppose that $\tilde{Y}_n \neq \hat{Z}$. Then either $A_1 := \{\tilde{Y}_n = 1, \hat{Z} = 0\}$ or $A_2 := \{\tilde{Y}_n = 0, \hat{Z} = 1\}$ occur. Note that
\[
P(A_1) \leq 2^{-k^{10}} + P\left(d_{\text{Haus}}(\gamma_n, \mathcal{K}) < \frac{1}{2} \cdot r_k, A_1\right).
\]
So suppose that $d_{\text{Haus}}(\gamma_n, \mathcal{K}) < \frac{1}{2} \cdot r_k$ and $A_1$ hold. If $\gamma_n$ hits $\hat{B}$, then $\gamma \cap B \neq \emptyset$. Thus, we see that $d_{\text{Haus}}(\gamma_n, \mathcal{K}) < \frac{1}{2} \cdot r_k$ and $A_1$ imply that $\gamma_n \cap \hat{B} = \emptyset$. Therefore, by (5.4), we have
\[
P(A_1) \leq c 2^{-k^{10}}.
\]

Next we will estimate $P(A_2)$. It follows that
\[
P(A_2) \leq 2^{-k^{10}} + P\left(d_{\text{Haus}}(\gamma_n, \mathcal{K}) < \frac{1}{2} \cdot r_k, A_2\right).
\]
Suppose that $d_{\text{Haus}}(\gamma_n, \mathcal{K}) < \frac{1}{2} \cdot r_k$ and $A_2$ hold. Then $\gamma_n$ hits $\hat{B}$ but it does not hit $B$. Therefore, by (5.5), we see that
\[
P(A_2) \leq c 2^{-k^{10}},
\]
which completes the proof.
\end{proof}
5.2 $L^2$-approximation

With the boxes $B_i$’s from Sections 3 and 4 in mind, we now consider the cubes $Q_1, Q_2, \cdots, Q_{L_k}$, so that each $Q_i$ is of radius $2^{-k^i}$, centered in $2^{-k^i+1}Z^3$ and having non-empty intersection with $\mathbb{D}$. Note that $L_k \simeq 2^{4k}$. We write

$$I_k = \left\{ 1 \leq i \leq L_k \mid \text{dist}(Q_i, \{0\} \cup \partial \mathbb{D}) > 2^{-k} \right\}$$

for a set of subscripts for “typical” cubes. For each $i \in I_k$, we let $Z^i$ be the indicator function of the event that $\gamma_n \cap Q_i \neq \emptyset$ (resp. $K \cap Q_i \neq \emptyset$). Since $2^{-k^{i0}}2^{4k} \leq c2^{-k}$, we have the following corollary:

**Corollary 5.4.** With the notation above, it follows that there exists $N = N_k$ depending on $k$ such that for all $n \geq N_k$

$$P\left(Z^i_n = Z^i \; \text{for all} \; i \in I_k\right) \geq 1 - c2^{-kn} \tag{5.13}$$

for some universal constant $c > 0$.

We recall that $B_0$ stands for the “reference” cube of side length $2^{-k^4}$ centered at $x_0 = \left(\frac{1}{2}, 0, 0\right)$. We also recall that $B_0'$ for the cube of radius $3 \cdot 2^{-k^4}$ centered at $x_0$ and that $\alpha_0$ is defined as in (4.1). Let

$$W_k := 1\{K \cap B_0' \neq \emptyset\} \tag{5.14}$$

be the indicator function of the event that $K$ hits $D_3^+$. Then we have the following proposition.

**Proposition 5.5.** There exists universal constant $c > 0$, and constants $\zeta(k)$, $N_k$ depending only on $k \geq 1$ such that for all $k \geq 1$ and $n \geq N_k$,

$$\frac{\alpha_0(n, k)}{2^{3n}} = \left(1 + O(2^{-ck^3})\right)\zeta(k), \tag{5.15}$$

where the constant $\beta$ is defined as in (2.15) and the constant used in the notation $O$ is universal.

**Proof.** By definition of $\alpha_0$, we see that

$$\frac{\alpha_0}{2^{3n}} = \frac{\sum_{\gamma_n \in B_0}P(x \in \gamma_n)}{2^{3n}P(Y_0 = 1)} \tag{5.16}$$

We will first show that

$$P(x \in \gamma_n) = \left(1 + O(2^{-ck^3})\right)P(x_0 \in \gamma_n), \tag{5.17}$$

uniformly in $x \in B_0$. To do it, we let $X$ be the random walk conditioned to hit the origin before leaving $\mathbb{D}$ and let $Y$ be the simple random walk. Then by Lemma 5.1 of [16], we see that

$$P(x \in \gamma_n) = G_{B(2^n)}(0, 2^n x)P^{x,x} \left(\text{LE}(X[0, \tau_0]) \cap Y[1, T] = \emptyset\right) \tag{5.18}$$

where $G_{B(r)}(\cdot, \cdot)$ stands for Green’s function for the simple random walk on $B(r) \cap Z^3$. As in (5.5) of [16], we see that

$$G_{B(2^n)}(0, 2^n x) = G_{B(2^n)}(0, 2^n x_0) \left(1 + O(2^{-ck^3})\right),$$

uniformly in $x \in B_0$. Thus we need to compare the probability in the RHS of (5.18). For that purpose, let $R^1$ and $R^2$ be the independent simple random walks. Then by definition,

$$P^{x,x} \left(\text{LE}(X[0, \tau_0]) \cap Y[1, T] = \emptyset\right) = \frac{P^{x,x} \left(\text{LE}(R^1[0, \tau^1_0]) \cap R^2[1, T^2] = \emptyset, \; \tau^1_0 < T^1\right)}{P^x(\tau^1_0 < T^1)}$$

where $\tau^1_0$ stands for the first time that $R^1$ hits the origin and $T^i$ is the first time that $R^i$ exits from $\mathbb{D}$. However, by (5.7) of [16], it follows that

$$P^x(\tau_0^1 < T^1) = P^{x_0}(\tau_0^1 < T^1)(1 + O(2^{-ck^3})),$$

uniformly in $x \in B_0$. Therefore, in order to prove (5.17), it suffices to show that

$$P^{x,x} \left(\text{LE}(R^1[0, \tau^1_0]) \cap R^2[1, T^2] = \emptyset, \; \tau^1_0 < T^1\right)$$

$$= P^{x_0,x_0} \left(\text{LE}(R^1[0, \tau^1_0]) \cap R^2[1, T^2] = \emptyset, \; \tau^1_0 < T^1\right)(1 + O(2^{-ck^3})), \tag{5.19}$$

(5.15)
uniformly in \( x \in B_0 \). To show this, we will follow Lemma 5.3 of [16]. Let \( T_i^r \) be the first time that \( R^i \) hits \( \partial B(r) \). A similar argument in the proof of Lemma 5.3 of [16] gives that
\[
P^{x_0, x_0} \left( \text{LE}(R^1[0, \tau_0^1]) \cap R^2[1, T^2] = \emptyset, \ \tau_0^1 < T^1 \right)
= P^{x_0, x_0} \left( \text{LE}(R^1[0, \tau_0^1]) \cap R^2[1, T^2] = \emptyset, \ \tau_0^1 < T^1 \right) \left( 1 + O(2^{-ck^3}) \right),
\]
for some \( c > 0 \). Namely, the loop-erasure of the end part of \( R^1 \) is not important to compute the probability in the LHS of (5.20) (see the proof of Lemma 5.3 of [16] for the details). On the other hand, it follows from Proposition 1.5.10 of [8] that for all \( z \in \partial B(2^{-k^3}) \)
\[
P^z(\tau_0^1 < T^1) = \tilde{c}2^{k^3}2^{-n} \left( 1 + O(2^{-k^3}) \right),
\]
for some universal constant \( \tilde{c} > 0 \). Using this and the strong Markov property, we have
\[
P^{x_0, x_0} \left( \text{LE}(R^1[0, T_{2 \rightarrow k^3}]) \cap R^2[1, T^2] = \emptyset, \ \tau_0^1 < T^1 \right)
= E^{x_0, x_0} \left[ 1 \left( T_{2 \rightarrow k^3} < T^1, \ \text{LE}(R^1[0, T_{2 \rightarrow k^3}]) \cap R^2[1, T^2] = \emptyset \right) P^{R^1}(T_{2 \rightarrow k^3}^1) \right] \left( \tau_0^1 < T^1 \right)
= \tilde{c}2^{k^3}2^{-n} P^{x_0, x_0} \left( T_{2 \rightarrow k^3} < T^1, \ \text{LE}(R^1[0, T_{2 \rightarrow k^3}]) \cap R^2[1, T^2] = \emptyset \right) \left( 1 + O(2^{-k^3}) \right).
\]
Let \( y = x - x_0 \). Note that \(|y| \leq 3 \cdot 2^{-k^3} \). Define
\[
\hat{B}(2^{-k^3}) = B(2^{-k^3}) + y, \\
\hat{D} = D + y.
\]
We write
- \( \hat{T}_{2 \rightarrow k^3}^1 \) for the first time that \( R^1 \) hits \( \partial \hat{B}(2^{-k^3}) \);
- \( \hat{T}^i \) for the first time that \( R^i \) exits from \( \hat{D} \).

By the translation invariance, we see that
\[
P^{x_0, x_0} \left( T_{2 \rightarrow k^3}^1 < T^1, \ \text{LE}(R^1[0, T_{2 \rightarrow k^3}]) \cap R^2[1, T^2] = \emptyset \right)
= P^{x, x} \left( \hat{T}_{2 \rightarrow k^3}^1 < T^1, \ \text{LE}(R^1[0, \hat{T}_{2 \rightarrow k^3}]) \cap R^2[1, \hat{T}^2] = \emptyset \right).
\]
By Lemma 5.3 of [16] again, we see that
\[
P^{x, x} \left( \text{LE}(R^1[0, \tau_0^1]) \cap R^2[1, T^2] = \emptyset, \ \tau_0^1 < \hat{T}^1 \right)
= P^{x, x} \left( \text{LE}(R^1[0, \hat{T}_{2 \rightarrow k^3}]) \cap R^2[1, \hat{T}^2] = \emptyset, \ \tau_0^1 < \hat{T}^1 \right) \left( 1 + O(2^{-ck^3}) \right),
\]
for some \( c > 0 \). Using the strong Markov property and Proposition 1.5.10 of [8] as above, it follows that
\[
P^{x, x} \left( \text{LE}(R^1[0, \tau_0^1]) \cap R^2[1, T^2] = \emptyset, \ \tau_0^1 < \hat{T}^1 \right)
= \tilde{c}2^{k^3}2^{-n} P^{x, x} \left( \hat{T}_{2 \rightarrow k^3}^1 < \hat{T}^1, \ \text{LE}(R^1[0, \hat{T}_{2 \rightarrow k^3}]) \cap R^2[1, \hat{T}^2] = \emptyset \right) \left( 1 + O(2^{-k^3}) \right),
\]
where \( \tilde{c} \) is the constant as in (5.21). Consequently, we have
\[
P^{x_0, x_0} \left( \text{LE}(R^1[0, \tau_0^1]) \cap R^2[1, T^2] = \emptyset, \ \tau_0^1 < T^1 \right)
= P^{x_0, x_0} \left( \text{LE}(R^1[0, T_{2 \rightarrow k^3}]) \cap R^2[1, T^2] = \emptyset, \ \tau_0^1 < T^1 \right) \left( 1 + O(2^{-ck^3}) \right)
= \tilde{c}2^{k^3}2^{-n} P^{x_0, x_0} \left( T_{2 \rightarrow k^3} < T^1, \ \text{LE}(R^1[0, T_{2 \rightarrow k^3}]) \cap R^2[1, T^2] = \emptyset \right) \left( 1 + O(2^{-ck^3}) \right)
= \tilde{c}2^{k^3}2^{-n} \left( \tilde{c}2^{k^3}2^{-n} \right)^{-1} P^{x, x} \left( \text{LE}(R^1[0, \hat{T}_{2 \rightarrow k^3}]) \cap R^2[1, \hat{T}^2] = \emptyset, \ \tau_0^1 < \hat{T}^1 \right) \left( 1 + O(2^{-ck^3}) \right)
= P^{x, x} \left( \text{LE}(R^1[0, \tau_0^1]) \cap R^2[1, T^2] = \emptyset, \ \tau_0^1 < T^1 \right) \left( 1 + O(2^{-ck^3}) \right),
\]
which gives $5.19$. Thus, we have $5.17$.

Returning to the equation $5.16$, we have

\[
\frac{\alpha_0}{2^\gamma} = \frac{|B_0| P(x_0 \in \gamma_n)}{2^\gamma P(Y_0 = 1)} \left( 1 + O(2^{-ck^3}) \right)
\]

\[
= \frac{2^{-2ck^4}2^n P(x_0 \in \gamma_n)}{2^\gamma P(Y_0 = 1)} \left( 1 + O(2^{-ck^3}) \right),
\]

where $|B_0|$ stands for the number of points in $B_0 \cap 2^{-n}Z^3$.

By $2.22$, it follows that there exists universal constants $c_1 > 0$ and $\delta > 0$ such that

\[
P(x_0 \in \gamma_n) = c_1 2^{-(3-\beta)n} \left( 1 + O(2^{-\delta n}) \right).
\]

Changing $N_k$ if needed, we see that for all $n \geq N_k$

\[
P(x_0 \in \gamma_n) = c_1 2^{-(3-\beta)n} \left( 1 + O(2^{-ck^3}) \right).
\]

Thus, we have

\[
\frac{\alpha_0}{2^\gamma} = \frac{c_1 2^{-2ck^4}}{P(Y_0 = 1)} \left( 1 + O(2^{-ck^3}) \right).
\]

On the other hand, by Corollary $5.3$, it follows that for all $n \geq N_k$

\[
\left| P(Y_0 = 1) - P(W_k = 1) \right| \leq c2^{-k^{10}},
\]

which gives

\[
P(Y_0 = 1) = \left( 1 + O(2^{-ck^3}) \right) P(W_k = 1).
\]

Letting

\[
\zeta(k) := \frac{c_1 2^{-2ck^4}}{P(W_k = 1)},
\]

we obtain the claim $5.15$ as desired.

Take a cube $B$ of side length $2^{-k}$ which is lying in $D = \{ x \in \mathbb{R}^3 \mid |x| < 1 \}$. We assume that $\text{dist}(B, \{0\} \cup \partial D) > 2^{-k}$. We divide $B$ into smaller cubes $B_1, B_2, \ldots, B_{k_m}$ of side length $2^{-k}$ so that $k_m = 2^{3(k^2-k)}$. Let $x_i$ be the center of $B_i$. We write $B_i'$ for the cube of radius $3 \cdot 2^{-k}$ centered at $x_i$.

We recall that $X_1$ stands for the number of points in $B_1$ which is passed through by $\gamma_n$ and that $Y_1$ stands for the indicator function of the event that $\gamma_n$ hits $B_1'$. We also recall that $X = X_1 + X_2 + \cdots + X_{k_m}$ stands for the number of points in $B$ hit by $\gamma_n$ and that $Y = Y_1 + Y_2 + \cdots + Y_{k_m}$ stands for the number of cubes $B_i'$ hit by $\gamma_n$. We also recall that the random measure $\mu_n$ was defined by $5.1$.

We are ready to show the following proposition.

**Proposition 5.6.** With the notation above, it follows that there exist universal constant $c, C, c^* > 0$ and a constant $N_k$ depending only on $k \geq 1$ such that for all $k \geq 1$ and $n \geq N_k$

\[
E \left[ \left( \mu_n(B) - \zeta(k)Y \right)^2 \right] \leq C 2^{-ck^2}.
\]

**Proof.** Note that

\[
\mu_n(B) = 2^{-\beta n} X.
\]

Thus, by Propositions $4.1$ and $5.5$ we have

\[
\| \mu_n(B) - \zeta(k)Y \|_2 \leq \| 2^{-\beta n} X - \alpha_0 2^{-\beta n} Y \|_2 + \| \alpha_0 2^{-\beta n} Y - \zeta(k)Y \|_2
\]

\[
= E \left( (X - \alpha_0 Y)^2 \right)^{\frac{1}{2}} 2^{-\beta n} + \left| \alpha_0 2^{-\beta n} \zeta(k) \right| E(Y^2)^{\frac{1}{2}}
\]

\[
\leq C 2^{-ck^2} E(X) 2^{-\beta n} + \zeta(k) E(Y^2)^{\frac{1}{2}} C 2^{-ck^2} \leq C 2^{-ck^2},
\]

where in the last inequality we used Corollary 1.3 of $16$ and Proposition $5.2$ to show that

\[
E(X) 2^{-\beta n} \leq C \text{ and } \zeta(k) E(Y^2)^{\frac{1}{2}} \leq C.
\]

This gives the proposition.

\[\square\]
6 Proof of Theorem 1.1

We recall that the random measure $\mu_n$ is defined as in (5.1). The goal of this section is to prove Theorem 1.1, which states that the joint law of $(\gamma_n, \mu_n)$ converges weakly to some $(K, \mu)$ with respect to the topology of $\mathcal{H}(\mathbb{D}) \otimes \mathcal{M}(\mathbb{D})$ where $K$ stands for Kozma’s scaling limit of the LERW (see Section 2.3 for $\mathcal{H}(\mathbb{D})$ and $\mathcal{M}(\mathbb{D})$).

To achieve this, we will first prove the tightness of $\{(\gamma_n, \mu_n)\}_{n \geq 1}$ in Section 6.1. Then we will establish the desired convergence of $(\gamma_n, \mu_n)$ in Theorem 6.2.

6.1 Tightness

Recall that $\gamma_n = \text{LE}(S[0, T])$ stands for the loop-erasure of $S[0, T]$ where $S$ is the SRW on $2^{-n} \mathbb{Z}^3$ started at the origin and $T$ stands for the first time that $S$ exits from the unit ball $\mathbb{D}$. We also recall the random measure $\mu_n$ is defined as in (5.1).

The next proposition shows that $\{(\gamma_n, \mu_n)\}_{n \geq 1}$ is tight.

Proposition 6.1. The family of variables $\{(\gamma_n, \mu_n)\}_{n \geq 1}$ is tight with respect to the product topology of $(\mathcal{H}(\mathbb{D}), d_{\text{Haus}})$ and the topology of the weak convergence on $\mathcal{M}(\mathbb{D})$.

Proof. Since $\gamma_n$ converges weakly with respect to the Hausdorff distance, the family of variables $\{\gamma_n\}_{n \geq 1}$ is obviously tight. Thus, it is enough to show that $(\mu_n)_{n \geq 1}$ is tight in the topology of the weak convergence on $\mathcal{M}(\mathbb{D})$. To prove this tightness of $(\mu_n)_{n \geq 1}$, it suffices to show

$$\sup_{n \geq 1} E \left( \mu_n(\mathbb{D}) \right) < \infty. \quad (6.1)$$

Note that

$$\mu_n(\mathbb{D}) = 2^{-\beta n} \text{len}(\gamma_n).$$

Therefore, the bound (6.1) follows from Corollary 1.3 of [16]. \qed

6.2 Characterization of subsequential limit

In Section 6.1, we proved that $\{(\gamma_n, \mu_n)\}_{n \geq 1}$ is tight. We also know that $\gamma_n$ converges weakly to $K$ as $n \to \infty$ with respect to the Hausdorff distance. Therefore, we can find a subsequence $\{n_l\}_{l \geq 1}$ and a limiting random measure $\mu^*$ such that

$$(\gamma_{n_l}, \mu_{n_l}) \xrightarrow{d} (K, \mu^*). \quad (6.2)$$

In this subsection, we will show that $\mu^*$ does not depend on the choice of the subsequence $\{n_l\}_{l \geq 1}$, i.e., we will prove that there exists a unique possible choice for the measure $\mu^*$. In order to prove this, it is enough to show that $\mu^*$ is a measurable function of $K$. This is summarized in the following theorem, which restates Theorem 1.1.

Theorem 6.2. The family of random variables $\{(\gamma_n, \mu_n)\}_{n \geq 1}$ converges weakly to some $(K, \mu)$ in the product topology of $(\mathcal{H}(\mathbb{D}), d_{\text{Haus}})$ and the topology of the weak convergence on $\mathcal{M}(\mathbb{D})$. Here $K$ is Kozma’s scaling limit of $\gamma_n$ and $\mu$ is a measurable function of $K$.

Proof. Recall that we can find a subsequence $\{n_l\}_{l \geq 1}$ and a limiting random measure $\mu^*$ such that (6.2) holds. It suffices to show that the measure $\mu^*$ is a measurable function of $K$, Kozma’s scaling limit of $\gamma_n$.

By Skorokhod’s representation theorem, we can define $\{(\gamma_{n_l}, \mu_{n_l})\}_{l \geq 1}$ and $(K, \mu^*)$ on the same probability space such that

$$\gamma_{n_l} \to K \text{ almost surely with respect to the topology of } (\mathcal{H}(\mathbb{D}), d_{\text{Haus}});$$

$$\mu_{n_l} \to \mu^* \text{ almost surely with respect to the topology of the weak convergence on } \mathcal{M}(\mathbb{D}). \quad (6.4)$$

Let $(\Phi(\overline{\mathbb{D}}), \| \cdot \|_{\infty})$ be the space of real-valued continuous functions on $\overline{\mathbb{D}}$ endowed with the uniform norm $\| \cdot \|_{\infty}$. We take a countable basis $\{\phi_j\}_{j \geq 1}$ of the space $(\Phi(\overline{\mathbb{D}}), \| \cdot \|_{\infty})$. We may assume that $\| \phi_j \|_{\infty} \leq 1$ for all $j \geq 1$.

Fix $j \geq 1$. We will show that $\mu^*(\phi_j)$ is a measurable function of $K$. Recall that $\| \phi_j \|_{\infty} \leq 1$. Since $\phi_j$ is a continuous function on $\overline{\mathbb{D}}$, for each $m \geq 1$ there exists $k_m \in \mathbb{N}$ such that

$$\max \left\{ |\phi_j(x) - \phi_j(y)| \mid x, y \in \overline{\mathbb{D}}, |x - y| \leq 4 \cdot 2^{-k_m} \right\} \leq 2^{-m}. \quad (6.5)$$
We may assume that $k_m \geq m$ and $k_m < k_{m+1}$ for all $m \geq 1$. When there is no confusion, we will write $k$ instead of $k_m$ for simplicity. With the definition of $B$ at the beginning of Section 3 in mind, we consider a covering of $\mathbb{B}$ by cubes $E_1, E_2, \cdots, E_{L_k}$ of side length $2^{-k_m}$ with centers in $2^{-k_m+1} \mathbb{Z}^3$. Note that the number of cubes $L_k$ satisfies $L_k \geq 2^{3k_m}$. We set
\begin{equation*}
I_k = \left\{ 1 \leq i \leq L_k \mid \text{dist}(E_i, \{0\} \cup \partial \mathbb{B}) > 2^{-k_m} \right\}
\end{equation*}
for the set of indices of “typical” cubes. Let
\begin{equation*}
H_k = \bigcup_{i \in I_k} E_i
\end{equation*}
be the union of typical cubes. We also write
\begin{equation*}
I_k^c = \left\{ 1 \leq i \leq L_k \mid i \notin I_k \right\}
\end{equation*}
for the complement of $I_k$. With the definition of $B$’s at the beginning of Section 3 in mind, for each $i \in L_k$, we also decompose $E_i$ into smaller cubes $D_1^i, D_2^i, \cdots, D_{R_k}^i$ of side length $2^{-m}$. Notice that the number of such cubes $R_k$ satisfies $R_k \geq 2^{3(k_m-k_m)}$ and that $R_k$ does not depend on $i$ thanks to the translation invariance.

Let $N_{k_m}$ be the integer as in Corollary 5.3 where we replace $k$ by $k_m$. From now on, we write $n = n_l$ and assume $n \geq N_{k_m}$. We will first compare
\begin{equation*}
\mu_n(\phi_j) = \int_{\mathbb{B}} \phi_j d\mu_n \quad \text{with} \quad J_1 := \int_{H_k} \phi_j d\mu_n.
\end{equation*}
Note that
\begin{equation*}
\left| \mu_n(\phi_j) - J_1 \right| \leq \int_{\mathbb{B} \setminus H_k} \phi_j d\mu_n \leq \mu_n \left( \mathbb{B} \setminus H_k \right).
\end{equation*}
However, it is not difficult to see that
\begin{equation*}
E \left[ \mu_n \left( \mathbb{B} \setminus H_k \right) \right]^2 \leq C 2^{-2k_m}
\end{equation*}
for some universal constant $C < \infty$ (see the proof of Theorem 8.1.4 of [22] for this). This implies that
\begin{equation}
\| \mu_n(\phi_j) - J_1 \|_2 \leq C 2^{-k_m}
\end{equation}
for some universal constant $C < \infty$. Here $\| \cdot \|_2$ stands for the $L^2$-norm.

Note that
\begin{equation*}
J_1 = \sum_{i \in I_k} \int_{E_i} \phi_j d\mu_n.
\end{equation*}
Let $x_i$ be the center of $B_i$. We next compare $J_1$ with the Riemann sum
\begin{equation}
J_2 := \sum_{i \in I_k} \phi_j(x_i) \mu_n(E_i).
\end{equation}
Since $|x - y| \leq 3 \cdot 2^{-k_m}$ for all $x, y \in E_i$ and for any $i \in I_k$, using (6.5), we see that
\begin{equation*}
|J_1 - J_2| \leq 2^{-m} \mu_n(\mathbb{B}).
\end{equation*}
It follows from Theorem 8.1.6 of [22] and Corollary 1.3 of [16] that
\begin{equation*}
\sup_n E \left[ \mu_n(\mathbb{B}) \right]^2 < \infty.
\end{equation*}
Thus, we have
\begin{equation}
\| J_1 - J_2 \|_2 \leq C 2^{-m}
\end{equation}
for some universal constant $C < \infty$.

We now replace $J_2$ by some macroscopic quantity $J_3$ as follows. Take $i \in I_k$. We recall that the cube $E_i$ is decomposed into $R_k$ cubes $D_1^i, D_2^i, \cdots, D_{R_k}^i$ of side length $2^{-k_m}$. We let $x_i^j$ be the center of $D_i^j$ for
Thus, by Hölder’s inequality, we have

\[ Y_q^i := 1 \left\{ \gamma_n \cap D_q^{i,+} \neq \emptyset \right\} \quad (6.10) \]

be the indicator function of the event that \( \gamma_n \) hits \( D_q^{i,+} \). We also let

\[ U_i := \sum_{q=1}^{R_k} Y_q^i \quad (6.11) \]

be the number of smaller cubes \( D_1^i, D_2^i, \ldots, D_{R_k}^i \) hit by \( \gamma_n \). Note that if \( B \) in Section 3 coincides with \( E_i \), then, \( Y \) defined in (3.2) coincides with \( U_i \) here.

Recall our choice of a reference cube at the beginning of Section 4.1, where we picked \( x_0 = (\frac{1}{2}, 0, 0) \) and the cube \( B_0^i \) of radius \( 3 \cdot 2^{-k^m} \) centered at \( x_0 \). Then by Proposition 5.6 we see that for each \( i \in I_k \)

\[ E \left[ \mu_n(E_i) - \zeta(k_m)U_i \right]^2 \leq C2^{-ck^m}, \quad (6.12) \]

where \( c, C \) are universal constants as in Proposition 5.4 and \( \zeta(\cdot) \) is defined in (5.24). With this in mind, we let

\[ J_3 := \sum_{i \in I_k} \phi_j(x_i)\zeta(k_m)U_i. \quad (6.13) \]

Then it follows from (6.12) that

\[ \| J_2 - J_3 \|_2 \leq \sum_{i \in I_k} Z_i \| z \|_2 \leq C2^{-ck^m} L_k \leq C2^{-ck^m} 2^{3k_m} \leq C2^{-c'k^m} \quad (6.14) \]

where

\[ Z_i := \left| \mu_n(E_i) - \zeta(k_m)U_i \right|. \]

Finally, we replace \( J_3 \) by \( J_4 \) a measurable quantity with respect to the scaling limit \( K \) as follows. Recall that \( \{ (\gamma_n, \mu_n) \} \) and \( (K, \mu^*) \) are coupled so that (6.3) and (6.4) hold. (We also recall that for simplicity we write \( n_i = n \).) The indicator functions \( Y_q^i \) and \( U_i \) are defined as in (6.10) and (6.11). With this in mind, we write

\[ \tilde{W}_q^i := 1 \left\{ K \cap D_q^{i,+} \neq \emptyset \right\} \quad (6.15) \]

for the indicator function of the event that \( K \) hits \( D_q^{i,+} \). We also let

\[ \tilde{W}_i := \sum_{q=1}^{R_k} \tilde{W}_q^i \quad (6.16) \]

be the number of smaller cubes \( D_1^i, D_2^i, \ldots, D_{R_k}^i \) hit by \( K \). Now we define

\[ J_4 = J_4^{i,k} := \sum_{i \in I_k} \phi_j(x_i)\zeta(k_m)\tilde{W}_i. \quad (6.17) \]

Note that \( J_4 \) is a measurable function of \( K \) depending on \( j \) and \( k \) (we recall that \( j \) is a subscript for \( \phi_j \) the basis of the space \( (\Phi(\mathbb{D}), \| \cdot \|_\infty) \)). By Corollary 5.5 in the above coupling, if \( n \geq N_{k_m} \), we have

\[ P \left( Y_q^i = \tilde{W}_q^i \text{ for all } i \in I_k \text{ and } q = 1, 2, \ldots, R_k \right) \geq 1 - c2^{-k^m} \quad (6.18) \]

for some universal constant \( c \). This implies that

\[ P(F) \leq c2^{-k^m}, \quad \text{with} \quad F := \left\{ Y_i \neq \tilde{W}_i \text{ for some } i \in I_k \right\}. \quad (6.19) \]

Thus, by Hölder’s inequality, we have

\[ \| J_3 - J_4 \|_2 \leq \left( \| J_3 \|_3 + \| J_4 \|_3 \right) \cdot P(F)^{\frac{1}{k}} \leq c2^{-c'k^m} \left( \| J_3 \|_3 + \| J_4 \|_3 \right). \quad (6.20) \]
Here $\| \cdot \|_3$ stands for the $L^3$-norm. However, note that

$$|J_3| = \left| \sum_{i \in I_k} \phi_j(x_i)\zeta(k_m)Y_i \right| \leq C \sum_{i \in I_k} \zeta(k_m)R_k$$

$$\leq C\| I_k \|_3 \zeta(k_m)R_k \leq C L_k R_k \zeta(k_m) \leq C 2^{3k_m} 2^{3(\beta-k_m)} 2^{-3k_m} \leq C 2^{k_m},$$

where we used that fact that $\beta \in (1, \frac{3}{2}]$ (see Section 2.5.3) and that

$$P(W_k = 1) \leq 2^{-(3-\beta)k_m}$$

in the inequalities above (see Lemma 7.1 of [21] and (3.79) of [23] for the equation (6.21)). Similarly, we have

$$|J_4| \leq C 2^{k_m}.$$  

Thus, we have

$$\| J_3 \|_3 + \| J_4 \|_3 \leq C 2^{k_m}.$$  

Combining this with (6.20), it follows that

$$\| J_3 - J_4 \|_2 \leq C 2^{-c m}.$$  

Consequently, it follows that for all $m \geq 1$ and $n_l \geq N_{k_m}$

$$\| \mu_{n_l} (\phi_j) - J_4 \|_2 \leq C 2^{-m}.$$  

We recall that $J_4 = J_4^{j,k}$ is a measurable function of $K$ which depends on $j$ and $m$ but does not depend on $n_l$. Therefore, the inequality of (6.23) implies that the sequence of measurable variables $\{ J_4^{j,k_m} \}_{m \geq 1}$ is a Cauchy sequence in $L^2$ So it has an $L^2$-limit. We write

$$\mu (\phi_j) := \lim_{m \to \infty} J_4^{j,k_m} \text{ in } L^2.$$  

Note that the limit $\mu (\phi_j)$ is of course a measurable function of $K$. The inequality (6.23) also gives that

$$\| \mu (\phi_j) - J_4^{j,k_m} \|_2 \leq C 2^{-m}.$$  

Thus, the triangle inequality gives that

$$\| \mu (\phi_j) - \mu_{n_l} (\phi_j) \|_2 \leq C 2^{-m}.$$  

for all $n_l \geq N_{k_m}$. Thus, we have

$$\lim_{l \to \infty} \| \mu (\phi_j) - \mu_{n_l} (\phi_j) \|_2 = 0.$$  

On the other hand, using a uniform control on $E[\mu_{n_l} B]_3$ (see Theorem 8.1.6 of [22] and Corollary 1.3 of [16]) and the fact that $\mu_{n_l}$ converges weakly to $\mu^*$ almost surely by (6.4), we see that

$$\lim_{l \to \infty} \| \mu^* (\phi_j) - \mu_{n_l} (\phi_j) \|_2 = 0.$$  

Combining this with (6.27), it follows that almost surely for each $j \geq 1$,

$$\mu^* (\phi_j) = \mu (\phi_j).$$  

This shows that $\mu^* (\phi_j)$ is a measurable function of $K$. This implies that $\mu^*$ is also measurable with respect to $K$. Thus, the measure $\mu^*$ can be characterized uniquely. Letting $\mu = \mu^*$, we thus finish the proof.

**Remark 6.3.** The measure $\mu$ we constructed here can be regarded as, with a slight abuse of terminology, the “box-counting” content of $K$. In other words, it counts the number of small cubes crossed by $K$ appropriately rescaled with respect to the size of the cubes. As discussed in Remark 1.3, we may wonder if $\mu$ is actually equivalent to some more natural fractal measure (e.g., Minkowski content as in the case of two dimensions) on $K$, which, if confirmed, would allow us to investigate the properties of the scaling limit of 3D LERW in a scale-free context.
7 Convergence in the natural parametrization

In this section, we finally prove Theorem 1.3, the weak convergence of the time-rescaled LERW with respect to the supremum distance (see Section 4 for the precise way of the time-rescaling). We regard the time-rescaled $\gamma_\eta$ (with a linear interpolation) as a random element of $\mathcal{C}(\mathbb{D})$, the space of continuous curves in $\mathbb{D}$ equipped with the uniform distance $\rho$ (see Section 2.3 for $(\mathcal{C}(\mathbb{D}), \rho)$). The goal of this section is to prove that the time-rescaled $\gamma_\eta$ converges weakly to a random process $\eta$ with respect to the metric $\rho$, which will be done in Theorem 7.10. Here $\eta$ is a random curve obtained by parametrizing Kozma’s scaling limit $\mathcal{K}$ using the measure $\mu$ as in Theorem 6.2.

To do this, we first show that $\gamma_\eta$ is tight with respect to the metric $\rho$ in Section 7.1 using some ideas from the study of uniform spanning trees. We will next establish some basic properties of the measure $\rho$ in Section 7.3. Those properties of $\rho$ will be used to parametrize $\mathcal{K}$. Finally, we will show that $\gamma_\eta$ converges weakly to $\eta$ with respect to the metric $\rho$ in Section 7.4.

7.1 Tightness with respect to the metric $\rho$

We first recall the metric space $\mathcal{C}(\mathbb{D})$ defined as in Section 2.3. We also recall that $\mathcal{S}$ is the simple random walk on $2^{-n}\mathbb{Z}^3$ started at the origin and that $T$ stands for the first time that $\mathcal{S}$ exits from $\mathbb{D}$. Write $\gamma_\eta = \text{LE}(\mathcal{S}[0, T])$ for the loop-erasure of $\mathcal{S}[0, T]$. We set $t_\eta = \text{len}(\eta)$ for the length (number of lattice steps) for $\gamma_\eta$. We know that the expectation of $t_\eta$ is comparable to $2^{\beta n}$ (see Section 2.5.3 for this). By definition, $\gamma_\eta$ is a sequence of points in $2^{-n}\mathbb{Z}^3$. However, through linear interpolation, we can regard

$$
\gamma_\eta : [0, t_\eta] \to \mathbb{R}^3
$$

as a random element of the metric space $(\mathcal{C}(\mathbb{D}), \rho)$. We then rescale the time parametrization and write

$$
\eta_\eta(t) = \gamma_\eta \left( 2^{\beta n} t \right) \text{ for } t \in [0, 2^{-\beta n} t_\eta].
$$

(7.1)

Note that $\eta_\eta$ is also a random element of the metric space $(\mathcal{C}(\mathbb{D}), \rho)$. The purpose of this subsection is to show that $\eta_\eta$ is tight with respect to the distance $\rho$ in the next proposition.

Proposition 7.1. The sequence of variables $\{\eta_\eta\}_{\eta \geq 1}$ is tight with respect to the metric $\rho$.

Before we prove this proposition, we need some preparatory results which shows that with high probability, the path $\gamma_\eta$ does not behave too wildly.

Take $\lambda > 1$. Recall that $D_\lambda = \{x \in 2^{-n}\mathbb{Z}^3 \mid |x| < 1\}$. We first claim that $\gamma_\eta$ is “hittable” for a simple random walk in the following sense. Let $R$ be a simple random walk on $2^{-n}\mathbb{Z}^3$ which is independent of $\mathcal{S}$. For $\delta > 0$, we define the event $H_\delta$ by

$$
H_\delta := \left\{ P^R_0(R \cap \gamma_\eta = \emptyset) \leq \lambda^{-\delta} \text{ for any } x \in D_\lambda \text{ with dist}(x, \gamma_\eta) \leq \lambda^{-1} \right\},
$$

where $P^R_0$ stands for the probability law of $R$ assuming $R(0) = x$ and

- $T_{v,r} = \inf \left\{ k \geq 0 \mid |R(k) - v| \geq r \right\}$: first exit time from $B(v, r)$ for $R$,
- $R = R[0, T_{x,\lambda^{-\frac{1}{2}}}]$: path of $R$ until its first exit from $B(x, \lambda^{-\frac{1}{2}})$.

Note that $H_\delta$ stands for the event that $R$ is likely to hit $\gamma_\eta$ whenever the starting point of $R$ is close to $\gamma_\eta$. Then by Theorem 3.1 of [21], it follows that there exist $\delta > 0$ and $C < \infty$ such that for all $\lambda \geq 1$ and $n \geq 1$

$$
P(H_\delta) \geq 1 - C\lambda^{-\frac{1}{2}}.
$$

(7.2)

We next claim that $\gamma_\eta$ has no “quasi-loops” with high probability in the following sense. For $\eta > 0$, we define the event $G_\eta$ by

$$
G_\eta := \left\{ |x - y| \geq \lambda^{-\frac{1}{c\eta}} \text{ for all } x, y \in \gamma_\eta \text{ with diam}(\gamma_\eta) \geq \lambda^{-\eta} \right\},
$$

where $\gamma_\eta$ stands for a path lying in $\gamma_\eta$ connecting $x, y \in \gamma_\eta$. Then, by Theorem 6.1 of [21], it follows that there exist $\eta > 0, c > 0$ and $C < \infty$ such that for all $\lambda \geq 1$ and $n \geq 1$

$$
P(G_\eta) \geq 1 - C\lambda^{-c\eta}.
$$

(7.3)
We also claim that we can cover $\gamma_n$ with not too many small balls with high probability. Take $\delta \in (0,1)$ and $\eta \in (0,1)$ so that (7.2) and (7.3) hold. Let

$$r := \lambda^{-\frac{d+\eta}{2}}. \quad (7.4)$$

We define a sequence of random times $\{t_k\}$ as follows. Let $t_0 = 0$ and

$$t_k := \inf \left\{ t_{k-1} \leq j \leq \text{len}(\gamma_n) \mid |\gamma_n(j) - \gamma_n(t_{k-1})| \geq r \right\}$$

for $k \geq 1$. Here we set $\inf \emptyset = \infty$. We write

$$N := \min \left\{ k \mid \text{dist}(\gamma_n(t_k), \partial D) \leq 3r \right\}$$

for the first $k$ that $\gamma_n(t_k)$ reaches within distance $3r$ from $\partial D$. We will give a bound on $N$. Let

$$F := \left\{ N \leq r^{-4} \right\}.$$ 

By comparing the length of the sequence of $r$-balls for the simple random walk, it is easy to show that there exist $c > 0$ and $C < \infty$ such that

$$P(F) \geq 1 - Ce^{-\frac{c}{r}}. \quad (7.5)$$

For $k = 1, 2, \cdots, N$, we write

$$y_k = \gamma_n(t_k) \quad \text{for the centre of the $k$-th ball of radius $r$.} \quad (7.6)$$

We define the event $F'$ by

$$F' := \left\{ \text{diam}\left(\gamma_n[t_N, \text{len}(\gamma_n)]\right) \leq r^{\frac{1}{2}} \right\}.$$ 

Again, by comparing the diameter of the corresponding end part of the simple random walk path, it is easy to see that there exists $C < \infty$ such that

$$P(F') \geq 1 - Cr^{\frac{1}{2}}. \quad (7.7)$$

We now summarize our discussions above in the lemma below.

**Lemma 7.2.** Setting $A := H_\delta \cap G_\eta \cap F \cap F'$, we have

$$P(A) \geq 1 - C\lambda^{-c\delta\eta} \quad (7.8)$$

for some $c > 0$ and $C < \infty$. Here $\delta$ comes from (7.2).

Now we condition $\gamma_n$ on the “good” event $A = H_\delta \cap G_\eta \cap F \cap F'$. It will be useful to regard $\gamma_n$ as a deterministic path satisfying $A$ for a while. Note that on the event $A$, it follows that for all $1 \leq k \leq N$

$$B(y_k, \lambda^{-\frac{\delta}{2}}) \cap \left( \gamma_n[0, t_{k-1}] \cup \gamma_n[t_{k+1}, \text{len}(\gamma_n)] \right) = \emptyset \quad (7.9)$$

since $\gamma_n$ has no quasi-loops.

We now claim that if we consider the wired uniform spanning tree $U_n$ on $D_n \cup \partial D_n$ (see Section 2.6 for the definition and related concepts) constructed through Wilson’s algorithm with $\gamma_n$ being sampled first, and pick a dense test set, then the sub-tree of $U_n$ connecting this test set and $\partial D_n$ should behave nicely with high probability, which (as we will prove later) implies that $\gamma_n$ is equicontinuous.

We now fix a $\lambda^{-1}$-net $\{x_1, x_2, \cdots, x_{m_3}\} \subset D_n$ so that $m_3 \asymp \lambda^3$ and

$$D \subset \bigcup_{i=1}^{m_3} B(x_i, \lambda^{-1}).$$

Note that for all $y_k$'s defined in (7.6), we can find $x_k$ from this net such that $|x_k - y_k| \leq \lambda^{-1}$ for each $k = 1, 2, \cdots, N$. Consider a simple random walk $R^k$ started at $x_k$ on $2^{-n}Z^3$ and we let $R^k$ run until it hits $\gamma_n$. We write $u_k$ for the first time that $R^k$ hits $\gamma_n$. Write

$$R^k := R^k[0, u_k], \quad L^k := \text{LE}(R^k) \text{ and } t_k := \text{len}(L^k) \text{ for the number of steps for } L^k. \quad (7.10)$$
By $H_\delta$, it follows that $R^k \subset B(x_k, \lambda^{-\frac{1}{2}})$ with probability at least $1 - \lambda^{-\delta}$. However, by $F$, we have $N \leq r^{-4}$. Thus, taking sum for $k = 1, 2, \cdots, N$, we see that with probability at least $1 - \lambda^{-\frac{1}{2}}$ for all $k = 1, 2, \cdots, N$, we have $R^k \subset B(x_k, \lambda^{-\frac{1}{2}})$. Furthermore, observe that (7.9) ensures that the end point of $R^k$ must lie on $\gamma_n[t_{k-1}, t_{k+1}]$. We denote this end point by $z_k = R^k(u_k)$. Then, 

$$P[J] > 1 - \lambda^{-\frac{1}{4}},$$

where $J := \{R^k \subset B(x_k, \lambda^{-\frac{1}{2}}) \text{ and } z_k \in \gamma_n[t_{k-1}, t_{k+1}] \text{ for all } k = 1, 2, \cdots, N\}$. (7.11)

Next we will show that with high probability $l_k$ is bounded above by $\lambda^{-\frac{1}{2}}2^{3n}$. To see this, suppose that $l_k \geq \lambda^{-\frac{1}{2}}2^{3n}$. Recall that $r$ is defined in (7.3). For each $x \in D_n$, we write $\gamma^x$ for the unique path on the uniform spanning tree $U_n$ connecting $x$ and $\partial D_n$. Using the notation as above, we see that $\gamma^x_k = L^k \cap \gamma^x_k$. Since

$$L^k \subset B(x_k, \lambda^{-\frac{1}{2}})$$

by (7.11), writing $s_x$ for the first exit time from $B(x, 2\lambda^{-\frac{1}{2}})$ for $\gamma^x$ where $x \in D_n$ with $\text{dist}(x, \partial D) \geq 2r$, then it follows that $s_{x_k} > l_k$. Namely, this first exit point $\gamma^x_k(s_{x_k})$ lies on $\gamma^x_k$. Consequently, we have $s_{x_k} > \lambda^{-\frac{1}{2}}2^{3n}$. Therefore, letting

$$J' := \{l_k \leq \lambda^{-\frac{1}{2}}2^{3n} \text{ for all } k = 1, 2, \cdots, N\},$$

we see that

$$(J')^c \cap J \subset \{s_{x_i} > \lambda^{-\frac{1}{2}}2^{3n} \text{ for some } i = 1, 2, \cdots, m_\Lambda \text{ with } \text{dist}(x_i, \partial D) \geq 2r\}.$$  

(7.13)

However, it follows from Theorem 8.1.6 of [22] that for each $x_i$ with $\text{dist}(x_i, \partial D) \geq 2r$,

$$P\left(s_{x_i} > \lambda^{-\frac{1}{2}}2^{3n}\right) \leq Ce^{-c\lambda},$$

for some $c > 0$ and $C < \infty$. This shows that for some $c > 0$ and $C' < \infty$

$$P(J \cap J') \geq 1 - C'e^{-c\lambda}.$$  

(7.14)

Finally, for each $x \in D_n$ with $\text{dist}(x, \partial D) \geq 2r$, we write $s'_x$ for the first exit time from $B(x, 2\lambda^{-\frac{1}{2}})$ for $\gamma^x$. Then by Theorem 8.2.6 of [22], it follows that for each $x \in D_n$ with $\text{dist}(x, \partial D) \geq 2r$,

$$P\left(s'_x \leq r^{3}2^{3n}\right) \leq C \exp\{-cr^{-\rho}\}$$

for some $C < \infty$, $c > 0$ and $\rho > 0$. Therefore, if we define the event $J''$ by

$$J'' := \{s'_x \geq r^{3}2^{3n} \text{ for all } i = 1, 2, \cdots, m_\Lambda \text{ with } \text{dist}(x_i, \partial D) \geq 2r\},$$

(7.15)

then we have

$$P(J'') \geq 1 - C\lambda^3 \exp\{-cr^{-\rho}\}. $$

(7.16)

We now summarize our discussions above in the lemma below.

**Lemma 7.3.** Setting $B := J \cap J' \cap J''$. We have

$$P(B|A) \geq 1 - C\lambda^{-\frac{4}{3}},$$

(7.17)

where $C > 0$ is universal and $\delta$ comes from (7.2).

We are now ready to prove the tightness.

**Proof of Proposition 7.7.** Recall that the time duration of $\eta_n$ is equal to $2^{-\beta n}t_n$ where $t_n = \text{len}(\gamma_n)$ is the number of lattice steps of the LERW. Since

$$\lim_{\lambda \to \infty} \sup_{n \geq 1} P\left(2^{-\beta n}t_n \notin [1/\lambda, \lambda]\right) = 0$$

(see (2.16) and (2.17) for this), it suffices to prove the equicontinuity of $\eta_n$. We now define a “bad” event

$$W := \{\exists x, x' \in \gamma_n \text{ such that } |x - x'| \geq r^{\frac{1}{2}} \text{ and } \text{len}(\gamma_n^{x,x'}) \leq \frac{1}{3} \cdot r^{3}2^{3n}\}$$

(7.18)
where we recall that \( \gamma_{x,x'}^t \) stands for the path lying on \( \gamma_n \) which connects \( x \) and \( x' \). Suppose that \( A \cap B \) occurs. Then it follows that \( W \) cannot occur. Why? To see this, without loss of generality, we may assume \( x' \in \gamma^t \). Then it is not difficult to see that there exists \( k = 1, 2, \ldots, N - 1 \) such that \( \{y_{k-1}, y_k, y_{k+1}\} \subseteq \gamma_{x,x'}^t \). Writing \( \tau \) for the first time that \( \gamma^t \) exits from \( B(x, \frac{1}{2}) \), it follows that \( \gamma^t = L^k \oplus \gamma^t_k \) and that \( \gamma^t \{0, \tau\} \subseteq \gamma_{x,x'}^t \). Thus, if \( A \cap B \cap W \) occurs, we have \( z_{n,k} \leq \lambda^{-\frac{k}{2}2^{5n}} + \frac{1}{2} \cdot r^{3/2}2^{5n} < \frac{3}{2} \cdot r^{3/2}2^{5n} \) which contradicts \( J'' \). Thus, \( A \cap B \cap W = \emptyset \). Therefore, by (7.8) and (7.17),

\[
P(W) \leq P(A^c) + P(A \cap B^c) \leq C \lambda^{-cd_\eta}.
\]

This shows that \( \{\eta_n\} \) is equicontinuous with high probability. So we finish the proof. \( \square \)

### 7.2 Guideline to parametrization

In this subsection, we will give a guideline to the parametrization works we will be carrying out in Sections 7.2 and 7.3.

We first make a convention here: to distinguish elements between \( C(\overline{\Omega}) \) and \( H(\overline{\Omega}) \), we will write \( \hat{\lambda} = \{\lambda(t) \mid 0 \leq t \leq t_n\} \) for the range of \( \lambda \) if \( \lambda \in C(\overline{\Omega}) \). Note that naturally \( \hat{\lambda} \in H(\overline{\Omega}) \).

By Proposition 7.1, we can find a subsequence \( \{n_l\}_{l \geq 1} \) of \( \lambda \in \mathcal{M}(\overline{\Omega}) \) and a random element \( \nu \) of \( \mathcal{M}(\overline{\Omega}) \) so that

\[
(\eta_{n_l}, \mu_{n_l}) \overset{d}{\to} (\zeta, \nu) \quad (l \to \infty),
\]

with respect to the product topology of \( (C(\overline{\Omega}), \rho) \) and the topology of the weak convergence on \( \mathcal{M}(\overline{\Omega}) \). By Skorokhod’s representation theorem, we can couple \( \{\eta_{n_l}, \mu_{n_l}\}_{l \geq 1} \) and \( (\zeta, \nu) \) in the same probability space such that

\[
\lim_{l \to \infty} \rho(\eta_{n_l}, \zeta) = 0 \text{ a.s., and } \mu_{n_l} \to \nu \text{ w.r.t the topology of weak convergence on } \mathcal{M}(\overline{\Omega}).
\]

We have not yet proved that \( \zeta \) does not depend on the subsequence \( \{n_l\}_{l \geq 1} \) (this is what we want to show). However, by Theorem 6.2 we have already proved that

\[
(\hat{\eta}_{n_l}, \hat{\mu}_{n_l}) \overset{d}{\to} (K, \mu) \quad (l \to \infty),
\]

with respect to the topology of \( H(\overline{\Omega}) \otimes \mathcal{M}(\overline{\Omega}) \). This gives that

\[
(\hat{\zeta}, \nu) \overset{d}{\to} (K, \mu)
\]

as a random element of the product space \( H(\overline{\Omega}) \times \mathcal{M}(\overline{\Omega}) \).

In this subsection, we will show the following two basic properties of \( \nu \) from (7.20):

(a) The support of \( \nu \) satisfies

\[
\text{supp}(\nu) = \hat{\zeta}
\]

almost surely. In particular, it follows that almost surely

\[
\nu(\hat{\zeta}_x) < \nu(\hat{\zeta}_y)
\]

as long as two points \( x, y \in \hat{\zeta} \) satisfy \( \hat{\zeta}_x \subset \hat{\zeta}_y \). Here \( \hat{\zeta}_x \) stands for the range of the curve between the origin and \( x \) lying in \( \hat{\zeta} \), \( \hat{\zeta} \) for \( x \in \hat{\zeta} \) (thus, we regard \( \hat{\zeta}_x \) as an element of \( H(\overline{\Omega}) \)). Note that by (7.27), for each \( x \in \hat{\zeta} \), \( \hat{\zeta}_x \subset \hat{\zeta} \) is (the range of) the simple curve whose starting point is the origin and endpoint is \( x \).

(b) It follows that almost surely for each \( x \in \hat{\zeta} \)

\[
\lim_{|z-y| \to 0} \nu(z-y) = \nu(\hat{\zeta}_x).
\]

Once we obtain the properties of \( \nu \) as above, we can parametrize \( \hat{\zeta} \) using the measure \( \nu \). Notice that for each \( t \in [0, \nu(\hat{\zeta})] \), there is a unique point \( x_t \in \hat{\zeta} \) satisfying that \( \nu(\hat{\zeta}_{x_t}) = t \). Therefore, we define

\[
\eta^* (t) := x_t \quad \text{for } t \in [0, \nu(\hat{\zeta})].
\]
Then we see that $\eta$ is a random element of $\mathcal{C}(\overline{B})$ which is a measurable function of $(\zeta, \nu)$.

We remark that since $(\zeta, \nu)$ has the same distribution as that of $(\mathcal{K}, \mu)$ as in (7.23), the properties (a) and (b) also hold when we replace $(\zeta, \nu)$ by $(\mathcal{K}, \mu)$ in the statements. In particular, it follows that for each $t \in [0, \mu(\mathcal{K})]$, there is a unique point $y_t \in \mathcal{K}$ satisfying that $\mu(\mathcal{K}_y) = t$. Here $\mathcal{K}_y$ stands for the trajectory of the curve from the origin to $x$ lying in $\mathcal{K}$ for $x \in \mathcal{K}$. Therefore, we can define

$$\eta(t) := y_t \quad \text{for } t \in [0, \mu(\mathcal{K})],$$

(7.28)

which is a random element of $\mathcal{C}(\overline{B})$ and measurable with respect to $(\mathcal{K}, \mu)$. Thus, by (7.23), we have

$$\eta^* \overset{d}{=} \eta \quad \text{and} \quad (\eta^*, \nu) \overset{d}{=} (\eta, \mu).$$

(7.29)

In Section 7.4, we will show that (7.24) and (7.26) and then show (7.30).

Thus, in order to show (7.32), it is enough to establish (a), (b) (i.e., (7.24) and (7.26)) and then show (7.30) in Section 7.4 (see Proposition 7.7 and Proposition 7.8).

### 7.3 Some properties of $\mu$

In this subsection we will show several basic properties of $\mu$. Since the distribution of $\nu$ coincides with that of $\mu$ (see (7.23) for this), in order to obtain these properties, it suffices to establish the corresponding properties for $\nu$.

**Proposition 7.4.** Take a box $B \subset \mathbb{D}$ with $0 \notin \partial B$. It follows that $\nu(\partial B) = 0$ almost surely. Furthermore, we have $\nu(\partial \mathbb{D}) = 0$ with probability one.

**Proof.** Take a box $B \subset \mathbb{D}$ with $0 \notin \partial B$. Let $r_0 = \text{dist}(0, \partial B) > 0$. We assume the coupling of $\{(\eta_n, \mu_n)\}_{n \geq 1}$ and $(\zeta, \nu)$ on which (7.21) holds throughout the proof of the proposition. We will prove $\nu(\partial B) = 0$ almost surely by contradiction. So suppose that

$$P(\nu(\partial B) > 0) \geq c > 0.$$  

(7.33)

Then it follows that there exists $\delta > 0$ such that

$$P(\nu(\partial B) \geq \delta) \geq \frac{c}{2}.$$  

(7.34)

For $\epsilon \in (0, \frac{r_0}{3})$, we let $G_\epsilon = \{x \in \mathbb{R}^3 \mid \text{dist}(x, \partial B) < \epsilon\}$. Note that $G_\epsilon$ is an open set with $0 \notin G_\epsilon$. Since $\partial B \subset G_\epsilon$, by (7.34), we see that

$$P(\nu(G_\epsilon) \geq \delta) \geq \frac{c}{2}.$$  

(7.35)

for all $\epsilon \in (0, \frac{r_0}{3})$. Write $N_{\epsilon, n}$ for the number of points in $G_\epsilon \cap 2^{-n}\mathbb{Z}^3$ hit by $\gamma_n$. Since $0 \notin G_\epsilon$, it follows from Proposition 8.2 of [22] that there exists $C < \infty$ such that for all $x \in G_\epsilon \cap 2^{-n}\mathbb{Z}^3$,

$$P(x \in \gamma_n) \leq C 2^{-(3-\beta)n},$$

where the constant $C$ depends only on $r_0$ but does not depend on $\epsilon, x$ and $n$ (note that $r_0$ is fixed). Thus, we have

$$E(N_{\epsilon, n}) \leq C 2^{-3n} = C 2^{\beta n}.$$
This implies that $E(\mu_n(G_\epsilon)) \leq C\epsilon$. By Markov’s inequality, we have

$$P(\mu_n(G_\epsilon) \geq \frac{\delta}{2}) \leq \frac{C\epsilon}{\delta}.$$  

Now take $\epsilon > 0$ sufficiently small so that $\frac{C\epsilon}{\delta} < \frac{\epsilon}{4}$. For this $\epsilon$, we have

$$P(\mu_n(G_\epsilon) < \frac{\delta}{2}) \geq 1 - \frac{c}{4},$$

for all $n \geq 1$. Thus, by the reverse Fatou lemma it follows that

$$P(\mu_n(G_\epsilon) < \frac{\delta}{2} \text{ for infinitely many } l) \geq 1 - \frac{c}{4}. \quad (7.36)$$

Suppose that both events $\{\mu_n(G_\epsilon) < \frac{\delta}{2} \text{ for infinitely many } l\}$ and $\{\nu(G_\epsilon) > \delta\}$ occur simultaneously (this is possible by (7.35) and (7.36)). Since $\mu_n$ converges weakly to $\nu$ and $G_\epsilon$ is open, it follows that

$$\nu(G_\epsilon) \leq \liminf_{l \to \infty} \mu_n(G_\epsilon) < \frac{\delta}{2},$$

which contradicts $\nu(G_\epsilon) \geq \delta$. Thus, we see that $\nu(\partial B) = 0$ almost surely which is the first claim of the proposition.

As for the second claim, we can prove it similarly because the number of point in $\gamma_n$ which is close to $\partial D$ is small enough compared with $2^{3n}$. So we finish the proof. \(\square\)

**Remark 7.5.** Thanks to Proposition 7.4, considering the coupling in the proof of the proposition, it follows that for any box $B \subset D$ with $0 \notin \partial B$,

$$\lim_{l \to \infty} \mu_n(B) = \nu(B) \quad (7.37)$$

almost surely. From this, if we consider a collection of boxes $B_1, B_2, \cdots$ with $0 \notin \partial B_i$ for each $i \geq 1$, writing $B' = B_1 \cup B_2 \cup \cdots$, it follows that

$$\lim_{l \to \infty} \mu_n(B') = \nu(B') \quad \text{almost surely.} \quad (7.38)$$

Furthermore, we have

$$\lim_{l \to \infty} \mu_n(D) = \nu(D) \quad \text{almost surely.} \quad (7.39)$$

Next we will deal with the support of the measure $\nu$. We start with a definition.

**Definition 7.6.** Given $\epsilon \in (0, 1)$, we decompose $\mathbb{R}^3$ into a collection of closed boxes $B_1, B_2, \cdots$ with side length $\epsilon$ with centers in $\frac{1}{\epsilon} \mathbb{Z}^3$. We then use $B_1, B_2, \cdots, B_M$ to denote the “minimal” covering of $\overline{D}$ and specify that $0 \in B_1$. Note that $M = M_\epsilon \asymp \epsilon^{-3}$.

**Proposition 7.7.** It follows that

$$\text{supp}(\nu) = \hat{\zeta} \quad \text{almost surely.} \quad (7.40)$$

**Proof.** As in the proof of Proposition 7.4 we continue assuming the coupling $\{(\eta_n, \mu_n)\}_{n \geq 1}$ and $(\zeta, \nu)$ on which (7.21) holds.

We will first prove that $\text{supp}(\nu) \subset \hat{\zeta}$ almost surely. To show that, it suffices to prove that with probability one $\nu(F) = 0$ for each closed set $F \subset \overline{D} \setminus \hat{\zeta}$. So take a closed subset $F \subset \overline{D} \setminus \hat{\zeta}$. Taking $\epsilon > 0$ sufficiently small, we see that $F_\epsilon := F + B(\epsilon)$ also has a positive distance from $\hat{\zeta}$. Here $F + B(\epsilon)$ stands for a set of points whose distance from $F$ is $< \epsilon$. Since $\lim_{\epsilon \to 0} \rho(\eta_n, \zeta) = 0$, it follows that $F_\epsilon \cap \eta_n = \emptyset$ for sufficiently large $l$, which implies that $\mu_n(F_\epsilon) = 0$. Since $F_\epsilon$ is open, it follows that $\nu(F_\epsilon) = 0$, which gives $\nu(F) = 0$. Thus, we see that $\text{supp}(\nu) \subset \hat{\zeta}$ almost surely.

We will next prove that $\text{supp}(\nu) = \hat{\zeta}$ almost surely. For a point $x \in \hat{\zeta}$, we write $\hat{\zeta}_x$ for the trace of the sub-path of $\hat{\zeta}$ between the origin and $x$. Note that $\hat{\zeta}_x$ is a random element of $\mathcal{H}(\mathbb{D})$ which is a simple path for each $x \in \hat{\zeta}$ (see Section 2.5.5 for this fact). In order to prove $\text{supp}(\nu) = \hat{\zeta}$, it suffices to show that for each $x, y \in \zeta$ with $\hat{\zeta}_x \subseteq \zeta_y$, we have $\nu(\hat{\zeta}_y \setminus \hat{\zeta}_x) > 0$. 

57
We will show this by contradiction. Let
\[ D_0 = \sup \left\{ |x - y| \mid x, y \in \hat{\zeta} \text{ with } \hat{\zeta}_x \subseteq \hat{\zeta}_y, \nu(\hat{\zeta}_y \setminus \hat{\zeta}_x) = 0 \right\}. \]
Suppose that there exist \( x, y \in \hat{\zeta} \) with \( \hat{\zeta}_x \subseteq \hat{\zeta}_y \) such that \( \nu(\hat{\zeta}_y \setminus \hat{\zeta}_x) = 0 \). Then we have \( D_0 > 0 \). So suppose that
\[ P(D_0 > 0) \geq c_0 > 0. \] (7.41)
Taking \( \delta > 0 \) sufficiently small, we have
\[ P(D_0 > \delta) \geq \frac{c_0}{2}. \] (7.42)
Recall that for \( r \in (0, 1) \), we defined the event \( W \) as in (7.18). We also recall the inequality (7.19) which gives that
\[ P(W) \leq C_1 r^{c_2} \]
for some universal constants \( C_1, c_2 > 0 \), which further implies that \( \{ \eta_n \}_{n \geq 1} \) is equicontinuous. Take \( r_0 > 0 \) sufficiently small so that
\[ r_0^3 \leq \frac{\delta}{8} \text{ and } C_1 r_0^{c_2} \leq \frac{c_0}{8}. \] (7.43)
With this choice of \( r_0 \), by (7.19), if we write
\[ V_n := W^c = \left\{ \forall x, x' \in \gamma_n \text{ with } \text{len}(\gamma_n) \geq \frac{r_0^3}{3}, \text{ we have } |x - x'| < r_0^2 \right\}, \]
it follows that
\[ P(V_n) \geq 1 - \frac{c_0}{8} \]
for all \( n \geq 1 \). By the reverse Fatou lemma, we see that
\[ P(\text{\( V_n \) occurs for infinitely many } l) \geq 1 - \frac{c_0}{8}. \] (7.44)
Take \( \epsilon \in (0, 1) \). Remind the definition of minimal covering of \( \hat{D} \) from Definition 7.6. We also define
\[ I_\epsilon := \left\{ 1 \leq i \leq m \mid B_i \cap \hat{\zeta} \neq \emptyset \right\} \text{ and } J_\epsilon := \left\{ 1 \leq i \leq m \mid B_i \cap B_j \neq \emptyset \text{ for some } j \in I_\epsilon \right\}. \]
Note that
\[ B_\epsilon := \bigcup_{i \in I_\epsilon} B_i \]
is a covering of \( \hat{\zeta} \) with \( B_\epsilon \setminus \hat{\zeta} \) as \( \epsilon \to 0 \). We also mention that the set of points within distance \( \frac{\epsilon}{2} \) from \( \hat{\zeta} \) is contained in \( B_\epsilon \).
Suppose that both of the events \( \{ D_0 \geq \delta \} \) and \( \{ V_{n_i} \text{ occurs for infinitely many } l \} \) happen (we assume that \( V_{n_{i,j}} \text{ occurs for } j = 1, 2, \cdots \)). These two events can occur simultaneously because of (7.42) and (7.44).
Assuming these two events, there exist \( x, y \in \hat{\zeta} \) with \( \hat{\zeta}_x \subseteq \hat{\zeta}_y \) such that \( \nu(\hat{\zeta}_y \setminus \hat{\zeta}_x) = 0 \) and \( |x - y| \geq \frac{2\delta}{3} \). Here \( \hat{\zeta}_{x,y} \) stands for the trace of the sub-path of \( \eta \) between \( x \) and \( y \) which is a closed set, i.e., \( \hat{\zeta}_{x,y} = (\hat{\zeta}_y \setminus \hat{\zeta}_x) \cup \{x\} \). Fixing such two points \( x, y \in \hat{\zeta} \), we define
\[ I_{\epsilon}^{x,y} := \left\{ 1 \leq i \leq m \mid B_i \cap \hat{\zeta}_{x,y} \neq \emptyset \right\} \text{ and } J_{\epsilon}^{x,y} := \left\{ 1 \leq i \leq m \mid B_i \cap B_j \neq \emptyset \text{ for some } j \in I_{\epsilon}^{x,y} \right\}. \]
Let \( B_{\epsilon}^{x,y} := \bigcup_{i \in J_{\epsilon}^{x,y}} B_i \) be a covering of \( \hat{\zeta}_{x,y} \). Since \( \nu(\hat{\zeta}_{x,y}) = 0 \) and \( B_{\epsilon}^{x,y} \setminus \hat{\zeta}_{x,y} \) as \( \epsilon \to 0 \), using the monotone convergence theorem, we have \( \nu(B_{\epsilon}^{x,y}) \to 0 \) as \( \epsilon \to 0 \). Thus, we can take \( \epsilon_0 \in (0, \frac{\delta}{100}) \) sufficiently small so that \( \nu(B_{\epsilon}^{x,y}) \leq \frac{\epsilon_0^3}{100} \). Since \( \mu_{n_{i,j}} \) converges weakly to \( \nu \) and \( \nu(\partial B_{\epsilon}^{x,y}) = 0 \) by Proposition 7.4, taking \( j \) sufficiently large, it follows that
\[ \mu_{n_{i,j}}(B_{\epsilon}^{x,y}) \leq \frac{\epsilon_0^3}{5}. \] (7.45)
Note that $B_{t_{y}}^{v}$ contains the set of points within distance $\frac{\epsilon}{2}$ from $\hat{\zeta}_{x,y}$. We also mention that $\zeta : [0,t_{\zeta}] \to \hat{\zeta}$ is a continuous curve which is moving on $\hat{\zeta}$. At this moment, we do not know whether it is a bijection (we will show it later though). However, since $\zeta$ is continuous and $\hat{\zeta}_{x} \subseteq \hat{\zeta}_{y}$, if we write

$$t_{z}^{\zeta} = \inf\{t \mid \zeta(t) = z\}$$

for $z \in \hat{\zeta}$, then we have $t_{z}^{\zeta} < t_{y}^{\zeta}$. So we can define

$$\sigma = \max\{t \leq t_{y}^{\zeta} \mid \zeta(t) = x\}.$$

It follows from an easy topological consideration that $\zeta[\sigma,t_{y}^{\zeta}] = \hat{\zeta}_{x,y}$. Therefore, we have $\zeta[\sigma,t_{y}^{\zeta}] \subseteq B_{t_{y}}^{v}$. Since $\lim_{j \to \infty} \rho(\eta_{n_{j}},\zeta) = 0$, taking $j$ sufficiently large, it follows that there exist $s < s'$ such that the following conditions hold:

- $\eta_{n_{j}}[s,s'] \subseteq B_{t_{y}}^{v}$.
- $|\eta_{n_{j}}(s) - x| \leq \frac{\epsilon_{0}}{2}$, $|\eta_{n_{j}}(s') - y| \leq \frac{\epsilon_{0}}{2}$.

Note that it follows from the triangle inequality, $|x - y| \geq \frac{2\delta}{3}$ and $\epsilon_{0} < \frac{\delta}{100}$ that

$$|\eta_{n_{j}}(s) - \eta_{n_{j}}(s')| \geq |x - y| - \frac{\epsilon_{0}}{2} \geq \frac{\delta}{2} > \frac{\delta}{100}.$$

Since $V_{n_{j}}$ occurs, writing $x_{1} = \eta_{n_{j}}(s)$ and $x_{2} = \eta_{n_{j}}(s')$, we have

$$\text{len}(\gamma_{n_{j}}^{x_{1},x_{2}}) > \frac{\epsilon_{0}2^{3n_{j}}}{3}.$$

Consequently, since $\gamma_{n_{j}}^{x_{1},x_{2}} \subseteq B_{t_{y}}^{v}$, we have

$$\mu_{n_{j}}(B_{t_{y}}^{v}) > \frac{\epsilon_{0}3}{3}, \quad (7.46)$$

which contradicts $\langle 7.45 \rangle$. Thus, we conclude that $\text{supp}(\nu) = \hat{\zeta}$ almost surely and finish the proof. \qed

Recall that $\hat{\zeta}_{x}$ stands for the trace of the sub-path of $\eta$ between the origin and $x \in \hat{\zeta}$. The next proposition shows that $\nu(\hat{\zeta}_{x})$ is a continuous function in $x \in \hat{\zeta}$.

**Proposition 7.8.** It follows that with probability one, for each $x \in \hat{\zeta}$, we have

$$\lim_{\nu \in \hat{\zeta} \atop (x \to x_{0})} \nu(\hat{\zeta}_{y}) = \nu(\hat{\zeta}_{y}). \quad (7.47)$$

**Proof.** Take $x \in \hat{\zeta}$. Suppose that a sequence $\{y_{m}\}_{m \geq 1} \subset \hat{\zeta}$ satisfies

$$\hat{\zeta}_{y_{m+1}} \subseteq \hat{\zeta}_{y_{m}} \quad \text{for all } m \geq 1$$

and $y_{m} \to x$ as $m \to \infty$. Since $\hat{\zeta}_{y_{m}} \searrow \hat{\zeta}_{x}$, it follows from the monotone convergence theorem that $\nu(\hat{\zeta}_{y_{m}}) \to \nu(\hat{\zeta}_{x})$ as $m \to \infty$.

Now suppose that a sequence $\{y_{m}\}_{m \geq 1} \subset \hat{\zeta}$ satisfies

$$\hat{\zeta}_{y_{m+1}} \supseteq \hat{\zeta}_{y_{m+1}} \quad \text{for all } m \geq 1$$

and $y_{m} \to x$ as $m \to \infty$. It follows that $\hat{\zeta}_{y_{m}} \nearrow (\hat{\zeta}_{x} \setminus \{x\})$ as $m \to \infty$. Therefore, in order to prove this proposition, it suffices to show that with probability one,

$$\nu(\{x\}) = 0 \quad \text{for all } x \in \hat{\zeta}. \quad (7.48)$$

We will show $\langle 7.48 \rangle$ by contradiction. To do it, let

$$L_{0} := \sup\{\nu(\{x\}) \mid x \in \hat{\zeta}\}$$

59
so that (7.48) is equivalent to the equation $L_0 = 0$. So suppose that
\[ P(L_0 > 0) \geq c_0 > 0 \] (7.49)
for some positive constant $c_0$. Then by taking $\delta > 0$ sufficiently small, we have
\[ P(L_0 > \delta) \geq \frac{c_0}{2}. \] (7.50)

Again we assume the coupling of $\{(\eta_n, \mu_n)\}_{l \geq 1}$ and $(\zeta, \nu)$ on which (7.21) holds. Take $\epsilon \in (0, 1)$ and consider the minimal covering of $\mathbb{B}$ from Definition 7.6. Let $X_i^{n, \epsilon}$ be the number of points in $B_i$ hit by $\gamma_n$.

Remind that $B_1$ is the box containing the origin. Then it follows from Theorem 8.4 and Proposition 8.5 of [22] that
\[ E(X_1^{n, \epsilon}) \approx (\epsilon 2^n)^{\beta}. \]
Furthermore, by Theorem 8.6 of [22], for all $\kappa > 0$ we have
\[ P\left( X_i^{n, \epsilon} > \kappa (\epsilon 2^n)^{\beta} \right) \leq Ce^{-c \kappa} \]
for some universal constants $0 < c, C < \infty$.

What about other boxes? Imitating the proof of Theorem 8.4 of [22], we can show that for all $1 \leq i \leq M$ and $p \geq 1$
\[ E\left\{ (X_i^{n, \epsilon})^p \right\} \leq C_0 p! (\epsilon 2^n)^{p \beta} \]
for some universal constant $C_0 < \infty$. So letting $c_1 = \frac{1}{3C_0}$, we have
\[ E\left\{ \exp \left( \frac{c_1 X_i^{n, \epsilon}}{\epsilon 2^{n/2}} \right) \right\} < \infty, \]
for all $1 \leq i \leq M$. Therefore, it follows from Markov’s inequality that for all $\kappa > 0$ and $1 \leq i \leq M$
\[ P\left( X_i^{n, \epsilon} > \kappa (\epsilon 2^n)^{\beta} \right) \leq Ce^{-c \kappa} \]
for some universal constants $0 < c, C < \infty$. Since $\beta > 1$ (see Section 2.5.3 for this), this implies that
\[ P\left( X_i^{n, \epsilon} > \sqrt{\epsilon 2^{3 \beta n}} \right) \leq Ce^{-c \sqrt{\epsilon}}. \]
Taking the union bounds for $1 \leq i \leq M$ (recall that $M \asymp \epsilon^{-3}$), we see that
\[ P\left( X_i^{n, \epsilon} > \sqrt{\epsilon 2^{3 \beta n}} \text{ for some } i = 1, 2, \ldots, M \right) \leq Ce^{-c \sqrt{\epsilon}}. \] (7.51)

Now we take $\epsilon_0$ sufficiently small so that
\[ \sqrt{\epsilon_0} < \frac{\delta}{10} \text{ and } Ce^{-c_0 \sqrt{\epsilon_0}} < \frac{c_0}{10}. \]

With this choice of $\epsilon_0$, if we write
\[ H_n := \left\{ X_i^{n, \epsilon_0} \leq \sqrt{\epsilon_0 2^{3 \beta n}} \text{ for all } i = 1, 2, \ldots, M \right\}, \]
it follows that
\[ P\left( H_n \text{ occurs for infinitely many } l \right) \geq 1 - \frac{c_0}{10}. \] (7.52)
Combining this with (7.50), we can assume that the following two events happen simultaneously:
\begin{itemize}
  \item $L_0 > \delta$;
  \item $H_{n_j}$ occurs for $j = 1, 2, \ldots$.
\end{itemize}
Since $L_0 > \delta$, there exists $x \in \zeta$ such that $\nu(\{x\}) \geq \frac{\delta}{4}$. Considering a box $B_i$ with $x \in B_i$, this implies that there exists $i \in \{1, 2, \ldots, M_{n_0}\}$ such that $\nu(B_i) \geq \frac{\delta}{4}$. Since $\mu_{n_j}$ converges weakly to $\nu$ and $\nu(\partial B_i) = 0$ by Proposition [7.4] it follows that

$$\mu_{n_j}(B_i) \geq \frac{\delta}{4}$$  \hfill (7.53)

for sufficiently large $j$. On the other hand, since $H_{n_j}$ occurs, we have $X_{i}^{n_j, \epsilon_0} \leq \sqrt{\epsilon_0} 2^{\beta n_j}$, which implies that

$$\mu_{n_j}(B_i) \leq \sqrt{\epsilon_0} < \frac{\delta}{10},$$  \hfill (7.54)

which contradicts (7.53). Consequently, it follows that with probability one $\nu(\{x\}) = 0$ for all $x \in \zeta$ and we finish the proof.

Before finishing this subsection, we will show that the (subsequential) scaling limit $\zeta : [0, t_\zeta] \to \zeta$ is a bijection.

**Proposition 7.9.** Suppose that $\eta_{n_j}$ converges weakly to $\zeta$ as $l \to \infty$ with respect to the metric $\rho$. Then $\zeta : [0, t_\zeta] \to \zeta$ is a bijection almost surely.

**Proof.** We assume that $\{\eta_{n_j}\}_{j \geq 1}$ and $\eta$ are coupled in the same probability space such that $\lim_{l \to \infty} \rho(\eta_{n_j}, \zeta) = 0$ almost surely.

Since $\zeta = \{\zeta(t) \mid t \in [0, t_\zeta]\}$, it suffices to show that $\zeta : [0, t_\zeta] \to \zeta$ is an injection almost surely. We will prove it by contradiction. Let

$$R_0 := \sup \{t - t' \mid 0 \leq t, t' \leq t_\zeta, \zeta(t) = \zeta(t')\}$$

so that $\zeta : [0, t_\zeta] \to \zeta$ is an injection if and only if $R_0 = 0$. So suppose that

$$P(R_0 > 0) \geq c_1 > 0$$

for some positive constant $c_1$. Then we can find $\delta > 0$ such that

$$P(R_0 \geq \delta) \geq \frac{c_1}{2}.$$  \hfill (7.55)

Take $\epsilon \in (0, 1)$ and consider the minimal covering of $\mathbb{B}$ from Definition [7.6]. Recall that $X_{i}^{n, \epsilon}$ is the number of points in $B_i$ hit by $\gamma_n$ which is defined in the proof of Proposition [7.8]. By (7.54), we have

$$P(V_\alpha) \geq 1 - C_1 e^{-\frac{\alpha}{300}}$$

where $V_\alpha := V_{n, \epsilon} := \{X_{i}^{n, \epsilon} \leq \sqrt{\epsilon} 2^{\beta n} \text{ for all } i = 1, 2, \ldots, M\}$

for some universal constants $0 < c_2, C_1 < \infty$.

We will next give a control on the probability that quasi-loops of certain size appear in the discrete (see Section [2.5.6] for definitions of quasi-loops). By Theorem 6.1 of [21], it follows that there exist $M < \infty$, $a > 0$ and $C_2 < \infty$ such that for all $n \geq 1$ and $\epsilon > 0$

$$P\left(\text{QL}(e^{M}, \epsilon; \eta_n) \neq \emptyset\right) \leq C_2 e^{a}.$$  \hfill (7.57)

We take $\epsilon = \epsilon_1 \in (0, 1)$ sufficiently small so that

$$\sqrt{\epsilon_1} \leq \frac{\delta}{300}, \quad C_1 e^{-\frac{\epsilon_1}{300}} \leq \frac{\epsilon_1}{300} \quad \text{and} \quad C_2 e^{\alpha} \leq \frac{\epsilon_1}{300}.$$  \hfill (7.56)

where the constants $c_2, C_1$ are as in (7.56) and the constants $a, C_2$ are coming from (7.57). With this choice of $\epsilon_1$, it follows from the reverse Fatou lemma that

$$P\left(V_{n_j} \cap W_{n_j} \text{ occurs for infinitely many } l\right) \geq 1 - \frac{c_1}{150}.$$  \hfill (7.58)

Combining this with (7.55), we can assume that the following two events happen simultaneously:

- $R_0 \geq \delta$;
- $V_{n_j} \cap W_{n_j}$ occurs for $j = 1, 2, \ldots.$
Since \( R_0 \geq \delta \), we can find \( 0 \leq t < t' \leq t_\xi \) with \( t' - t > \frac{\delta}{2} \) such that \( \zeta(t) = \zeta(t') \). Since \( \lim_{j \to \infty} \rho(\eta_{ni_j}, \zeta) = 0 \), taking \( j \) sufficiently large, we can find \( 0 \leq u < u' \leq t_{ni_j} \) such that the following conditions hold:

- \( u' - u > \frac{\delta}{2} \);
- \( |\eta_{ni_j}(u) - \zeta(t)| \leq \frac{\epsilon}{500} \) and \( |\eta_{ni_j}(u') - \zeta(t')| \leq \frac{\epsilon}{500} \).

By the fact that \( \zeta(t) = \zeta(t') \), the second condition ensures that

\[
|\eta_{ni_j}(u) - \eta_{ni_j}(u')| \leq \frac{\epsilon M}{50}.
\]

(7.59)

Let \( B_q \) be the box of side length \( \epsilon_1 \) which contains \( \eta_{ni_j}(u) \). Also let \( B_{q_1}, B_{q_2}, \ldots, B_{q_{27}} \) be the set of boxes of side length \( \epsilon_1 \) satisfying that \( B_{q_k} \cap B_q \neq \emptyset \) for each \( k = 1, 2, \ldots, 27 \). Since the event \( V_{ni_j} \) occurs, it follows that

\[
\eta_{ni_j}[u, u'] \not\subseteq \bigcup_{k=1}^{27} B_{q_k}.
\]

(7.60)

Why does \( 7.60 \) hold? Because if \( \eta_{ni_j}[u, u'] \subseteq \bigcup_{k=1}^{27} B_{q_k} \), this implies that

\[
X_{q_1}^{ni_j, \epsilon_1} + X_{q_2}^{ni_j, \epsilon_1} + \cdots + X_{q_{27}}^{ni_j, \epsilon_1} \geq (u' - u)2^3n_j - 2 \geq \frac{\delta}{8} \cdot 2^3n_j.
\]

From this, we see that there exists \( k \in \{1, 2, \ldots, 27\} \) such that

\[
X_{q_k}^{ni_j, \epsilon_1} \geq \frac{\delta}{216} \cdot 2^3n_j > \sqrt{\epsilon_1}2^3n_j,
\]

which contradicts \( V_{ni_j} \). Thus, (7.60) holds. However, combining (7.59) with (7.60), we see that \( \eta_{ni_j} \) has an \((\frac{M}{50}, \epsilon_1)\)-quasi-loop at \( \eta_{ni_j}(u) \). Therefore,

\[
\text{QL}(\frac{M}{50}, \epsilon_1; \eta_{ni_j}) \neq \emptyset.
\]

But this contradicts \( W_{ni_j} \). So we finish the proof.

\[\square\]

### 7.4 Weak convergence with respect to the supremum distance

Let us summarize our current standing point. We continue assuming the coupling of \( \{(\eta_{ni}, \mu_{ni})\}_{i \geq 1} \) and \( (\zeta, \nu) \) on which (7.21) holds. By Propositions 7.7 and 7.8 we see that with probability one \( \nu(\zeta_x) \) is strictly monotone increasing in \( x \in \zeta \) and \( \nu(\zeta_x) \) is continuous in \( x \in \zeta \). Therefore, we may consider the reparametrized curve \( \eta^* \in C(\overline{\mathbb{D}}) \) via the measure \( \nu \) as in (7.27). We also mention that \( \eta^* \) is a measurable function of \( (\zeta, \nu) \).

We will show in Proposition 7.11 that under this coupling

\[
\lim_{t \to \infty} \rho(\eta_{ni}, \eta^*) = 0 \text{ almost surely.}
\]

(7.61)

This gives that

\[
(\eta_{ni}, \mu_{ni}) \to (\eta^*, \nu) \text{ almost surely,}
\]

(7.62)

which implies (7.30) and consequently (7.32) as discussed in Section 7.3.

Once we establish (7.62), we can recall the fact that \( \eta^* \) is a measurable function of \( (\hat{\eta}, \nu) \) in the same way as \( \eta \) is with respect to \( (\mathcal{K}, \mu) \) and the fact that \( (\hat{\eta}, \nu) \overset{d}{=} (\mathcal{K}, \mu) \) (see (7.28) for this) to conclude that \( \zeta \) does not depend on the subsequence \( \{\eta_{ni}\}_{i \geq 1} \), and obtain Theorem 1.3, which we restate below.

**Theorem 7.10.** Recall that \( \eta_n \) is defined as in (7.11) and that \( \zeta \) is defined as in (7.28). It follows that

\[
\eta_n \overset{d}{\to} \eta^* \text{ (as } n \to \infty),
\]

(7.63)

with respect to the topology of \( (C(\overline{\mathbb{D}}), \rho) \).
As discussed at the beginning of this subsection, now it suffices to prove (7.61), which is restated in the following proposition.

**Proposition 7.11.** Assuming the coupling discussed above, it follows that
\[ \zeta = \eta^* \text{ almost surely.} \] (7.64)

Note that (7.64) implies (7.61).

**Proof.** We need to prove that with probability one
\[ t \zeta = t \eta^* \text{ and } \zeta(t) = \eta^*(t) \text{ for all } t \in [0, t_\zeta]. \] (7.65)

We will first prove that \( t \zeta = t \eta^* \) almost surely. Note that since \( \lim_{l \to \infty} \rho(\eta_{n_l}, \zeta) = 0 \), it follows that \( t_{n_l} \to t_\zeta \) as \( l \to \infty \). But by definition, we have
\[ t_{n_l} = 2^{-\beta_{n_l}} \text{len}(\gamma_{n_l}). \]
On the other hand, it follows that
\[ \mu_{n_l}(D) = 2^{-\beta_{n_l}} \text{len}(\gamma_{n_l}), \]
which implies that \( \mu_{n_l}(D) \to t_\zeta \) as \( l \to \infty \). We know that \( \mu_{n_l} \) converges to \( \nu \) with respect to the topology of weak convergence on \( M(D) \). Combining this with Proposition 7.4, it follows that \( \mu_{n_l}(D) \to \nu(D) \).

Thus, we have with probability one
\[ t \zeta = \nu(D) = \nu(\hat{\eta}^*) = t \eta^*, \]
where we used Proposition 7.7 in the second equation.

From now on we write
\[ T := t \zeta = t \eta^*. \]

We will next prove that \( \zeta(t) = \eta^*(t) \) for all \( t \in [0, T] \) (7.66) by contradiction. We now let
\[ Z_0 := \sup \left\{ |t - t'| \left| t, t' \in [0, T], \zeta(t) = \eta^*(t') \right. \right\}. \] (7.67)

Then it follows from Proposition 7.9 that
\[ Z_0 = 0 \iff (7.66). \] (7.68)

With this in mind, suppose that
\[ P(Z_0 > 0) \geq c_3 > 0 \] (7.69)
for some positive constant \( c_3 \). We can find \( \delta > 0 \) such that
\[ P(Z_0 > \delta) \geq \frac{c_3}{2}. \] (7.70)

We will need to consider the following event:
\[ U_{n,r} := \left\{ \forall x, x' \in \gamma_{n_l} \text{ with } \text{len}(\gamma_{n_l}^x x') \leq \frac{r \beta_{n_l}}{3}, \text{ we have } |x - x'| < r^\frac{1}{2} \right\} \]
(note that \( U_{n,r} \) is the complement of \( W \) from (7.18)), and also \( V_{n,\epsilon} \) and \( W_{n,\epsilon} \) defined in (7.56) and (7.57) respectively as
\[ V_{n,\epsilon} = \left\{ X^n_{i,\epsilon} \leq \sqrt{2} \beta_{n_l} \text{ for all } i = 1, 2, \ldots, M \right\}; \]
\[ W_{n,\epsilon} = \left\{ \text{QL}(e^M, \epsilon; \eta_{n_l}) = \emptyset \right\}. \]

We already know that there exist \( c_4 > 0, C_4 \in [1, \infty) \) and \( M \in [1, \infty) \) such that for all \( r, \epsilon \in (0,1) \) and all \( n \geq 1 \),
\[ P(U_{n,r} \cap V_{n,\epsilon} \cap W_{n,\epsilon}) \geq 1 - C_4 r^{c_4} - C_4 \epsilon^{-C_4} - C_4 \epsilon^{c_4}. \]
(see (7.19), (7.56) and (7.57) for this). We take $\epsilon_0 > 0$ sufficiently small so that

$$\sqrt{\epsilon_0} < \frac{\delta}{3000}, \quad C_4 \epsilon_0 < \frac{c_3}{3000} \quad \text{and} \quad C_4 e^{-\frac{c_4}{\epsilon_0}} < \frac{c_3}{3000}.$$  

With this choice of $\epsilon_0$, if we write $r_0 = 100^{-3} \epsilon_0 M$ and set

$$U_n' = U_{n,r_0}, \quad V_n' = V_{n,\epsilon_0} \quad \text{and} \quad W_n' = W_{n,\epsilon_0},$$  

we have $P(U_n' \cap V_n' \cap W_n') \geq 1 - \frac{\epsilon_0}{1000}$ for all $l \geq 1$. Thus, we have

$$P\left(U_n' \cap V_n' \cap W_n' \text{ occurs for infinitely many } l \right) \geq 1 - \frac{c_3}{1000}. \quad (7.72)$$

Therefore, we may assume that the following two events happen simultaneously:

- $Z_0 > \delta$
- $U_{n_{ij}}' \cap V_{n_{ij}}' \cap W_{n_{ij}}'$ occurs for $j = 1, 2, \cdots$

Since $Z_0 > \delta$, we can find $t, t' \in [0, T]$ such that $|t - t'| > \frac{\delta}{2}$ and $\zeta(t) = \eta^*(t')$. There are two cases as follows.

**Case 1:** $t - t' > \frac{\delta}{2}$

This case is easy. Letting $x = \zeta(t) = \eta^*(t')$ and $y = \zeta(t')$, we write

$$A := \tilde{\zeta}_y = \zeta[0, t] = \eta^*[0, t'], \quad (7.73)$$

and then define $A_\epsilon$ as follows. Recall the minimal covering of $\mathcal{D}$ from Definition 7.6. Let $I_A = \{1 \leq i \leq M \mid A \cap B_i \neq \emptyset\}$ and let $J_A = \{1 \leq j \leq M \mid B_j \subset B_i \text{ for some } i \in I_A\}$. Define

$$A_\epsilon = \bigcup_{i \in J_A} B_i. \quad (7.74)$$

Note that $A_\epsilon \subset A$ as $\epsilon \to 0$. We also mention that $\nu(A) = t'$. Therefore, by the monotone convergence theorem, taking $\epsilon_1$ sufficiently small, we have

$$\nu(A_{\epsilon_1}) \leq t' + \frac{\delta}{10}. \quad (7.75)$$

Moreover, since $\mu_{n_t}$ converges weakly to $\nu$, by Proposition 7.4 and taking $t$ sufficiently large, it follows that

$$\mu_{n_t}(A_{\epsilon_1}) \leq t' + \frac{\delta}{5}. \quad (7.75)$$

On the other hand, since $\rho(\eta_{n_t}, \zeta) \to 0$ as $l \to \infty$, taking $l$ sufficiently large, we have

$$\eta_{n_t}[0, t] \subset A_{\epsilon_1},$$

which implies that $\mu_{n_t}(A_{\epsilon_1}) \geq t > t' + \frac{\delta}{2}$. This contradicts (7.75). Therefore, we also get a contradiction for the first case.

**Case 2:** $t' - t > \frac{\delta}{2}$

Recall the definition of $A$ in (7.73). By Proposition 7.9 it follows that in Case 2,

$$A \subset A' := \zeta[0, t'] = \tilde{\zeta}_y.$$

Since $\lim_{j \to \infty} \rho(\eta_{n_{t_j}}, \zeta) = 0$, taking $j$ sufficiently large, we have

$$\rho(\eta_{n_{t_j}}, \zeta) < r_0.$$

From this, we can find $0 \leq u < u' \leq t_{n_{t_j}}$ such that the following conditions hold:

- $0 < u - t < r_0$ and $0 < t' - u' < r_0$;
\[ |\eta_{n_1j}(u) - \zeta(t)| \leq r_0^4 \] and \[ |\eta_{n_1j}(u') - \zeta(t')| \leq r_0^4. \]

Imitating the proof of (7.60) (note that we assume \( V'_{nj} \) occurs), we see that \( \eta_{nj}[u,u'] \not\subset B(x,\epsilon_0) \).

However, by \( W'_{nj} \), we see that \( \text{dist}(\eta_{nj}(u'),A) \geq \epsilon_0^2 \), because otherwise \( \eta_{nj} \) has an \( (\epsilon_0^2,\epsilon_0) \)-quasi-loop. With this in mind, let \( z = \eta_{nj}(u') \) and let \( \tilde{u} = \max \{ s \leq u' \mid \eta_{nj}(s) \in \partial B(z,2r_0^4) \} \).

Since \( 2r_0^4 = 50^{-1}\epsilon_0^2 \) and \( \text{dist}(z,A) \geq \epsilon_0^2 \), we see that \( \text{dist}(B(z,2r_0^4),A) \geq \epsilon_0^3 \).

We also mention that it follows from \( U_{nj} \) that \( u' - \tilde{u} > r_0^3 \).

Consequently, we see that

\[ \# \left( \{ \gamma_{nj}(k) \mid 0 \leq k \leq u'2^{\beta_{nj}} \} \cap A' \right) \leq \left( u' - \frac{r_0^3}{3} \right)2^{\beta_{nj}}, \tag{7.76} \]

where \( A' \) is defined as follows. We consider the minimal covering of \( D \) from Definition 7.6 with \( \epsilon = r_0 \) and denote the boxes by \( B'_1, B'_2, \cdots \). We still write \( \epsilon_0 = r_0^{-3} \). Let \( I' = \{ 1 \leq i \leq M \mid A \cap B'_i \neq \emptyset \} \) and let \( J' = \{ 1 \leq j \leq M \mid B'_j \cap B'_i \text{ for some } i \in I' \} \). Define

\[ A' = \bigcup_{i \in J'} B'_i \tag{7.77} \]

Since \( A = \eta^*[0,t'] \), it follows that \( \nu(A) = t' \). Because \( A \subset A' \), we have \( \nu(A') \geq t' \). Furthermore, because we know that \( \mu_{nj} \) converges weakly to \( \nu \), it follows from Proposition 7.4 that taking \( j \) sufficiently large,

\[ \mu_{nj}(A') \geq t' - \frac{r_0^3}{10}, \tag{7.78} \]

which implies that

\[ \# \left( \{ \gamma_{nj}(k) \mid 0 \leq k \leq \text{len}(\gamma_{nj}) \} \cap A' \right) \geq \left( t' - \frac{r_0^3}{10} \right)2^{\beta_{nj}}. \tag{7.79} \]

Combining this with (7.76), since \( u' < t' \), it follows that

\[ \eta_{nj}[u',t_{nj}] \cap A' \neq \emptyset. \tag{7.80} \]

But this contradicts \( W''_{nj} \) because (7.80) ensures that \( \eta_{nj} \) has a quasi-loop. So we get a contradiction for the second case.

This finishes the proof of Proposition 7.11. \( \square \)

8 The scaling limit of infinite loop-erased random walk

8.1 Notations and proof structure

In this section, we will consider the infinite loop-erased random walk and show that it has a scaling limit in natural parametrization as stated in Theorem 1.4.
We will first introduce some notation. Recall that in Section 2.3 we have defined \( \mathcal{C} \) the space of all continuous curves \( \lambda : [0, \infty) \to \mathbb{R}^3 \) satisfying \( \lambda(0) = 0 \) and \( \lim_{t \to \infty} \text{diam}(\lambda[0, t]) = \infty \) and a metric \( \chi \) on the space \( \mathcal{C} \) through
\[
\chi(\lambda_1, \lambda_2) = \sum_{m=1}^{\infty} 2^{-m} \min \left\{ \rho(\lambda_1^{(m)}, \lambda_2^{(m)}), 1 \right\}
\]
for \( \lambda_i \in \mathcal{C} \), \( m \geq 1 \) is defined as follows:
- We write \( B = \{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid \max\{ |x_1|, |x_2|, |x_3| \} < 1 \} \) for the open box with side length 2 centered at the origin;
- For \( m \geq 1 \) (not necessarily an integer) and \( \lambda \in \mathcal{C} \), we write \( \tau^m_\lambda = \tau_\lambda(m) \) for the first time that \( \lambda \) exits from \( B_m := mB \) and let
\[
\lambda^{(m)} : t \in [0, \tau^m_\lambda] \mapsto \lambda(t) \in \mathbb{R}^3
\]
be the truncation of \( \lambda \) up to \( \tau^m_\lambda \);
- For \( m \geq 1 \), let \( \mathcal{C}^{(m)} \) be the space of continuous curves \( \lambda : [0, t^\lambda] \to \mathbb{R}^3 \) where \( t^\lambda \in [0, \infty) \) stands for the time duration of \( \lambda \). Note that \( \lambda^{(m)} \in \mathcal{C}^{(m)} \) for \( \lambda \in \mathcal{C} \). We also mention that \( t^\lambda = \tau^m_\lambda \) for \( \lambda \in \mathcal{C}^{(m)} \), i.e., the time duration of \( \lambda \) coincides with the first time that \( \lambda \) exits from \( B_m \).

It is easy to check that \( \chi \) is a metric on \( \mathcal{C} \) and that under the metric \( \chi \), the space \( \mathcal{C} \) is separable.

We now turn to ILERW. Let \( S \) be the simple random walk on \( 2^{-n} \mathbb{Z}^3 \) started at the origin. Since it is transient, we can consider the loop-erasure of \( S[0, \infty) \). We then call the random simple path
\[
\gamma_n^\infty = \text{LE}(S[0, \infty))
\]
the infinite loop-erased random walk (ILERW), Note that almost surely, \( \lim_{k \to \infty} |\gamma_n^\infty(k)| = \infty \). Through linear interpolation, we can regard \( \gamma_n^\infty \) as a random element of \( \mathcal{C} \), which we assume throughout this section. We also write
\[
\eta_n^\infty(t) = \gamma_n^\infty \left( 2^{\beta_n} t \right) \quad \text{for } t \geq 0
\]
for the rescaled process which is also a random element of \( \mathcal{C} \).

We now briefly introduce the structure of this section.
- As mentioned above, the goal of this section is Theorem 1.4 which states that there exists a random variable \( \gamma^\infty \) taking values in \( \mathcal{C} \) such that the sequence \( \{ \gamma_n^\infty \}_{n \geq 1} \) converges weakly to \( \gamma^\infty \) with respect to the metric \( \chi \).
- In Section 8.2 we will prove a preliminary estimate on the first exit time from boxes for LERW, see Proposition 8.1 for the precise statement.
- In Section 8.3 we will first prove the existence of “local” scaling limit in Proposition 8.4 for truncated LERW’s. For some regularity reasons, we need to truncate with a box instead of a ball. Without too much effort, this convergence result above can be transferred to ILERW. We will show in Proposition 8.7 that similarly truncated ILERW also has a scaling limit.
- Finally, at the end of this section, we will show that the law of these truncated scaling limits are compatible as we increase the size of the aforementioned neighborhood, which implies that a global limit can be constructed. This will be wrapped in the proof of Theorem 8.8 which restates Theorem 1.4.

### 8.2 An estimate on the first exit time for LERW

This subsection is dedicated to the proof of Proposition 8.1.

We first recall some notations. Take \( m \geq 1 \) (not necessarily an integer). We recall that \( S \) is the simple random walk on \( 2^{-n} \mathbb{Z}^3 \). Let \( T^{(m)}_n \) be the first time that \( S \) exits from \( \mathbb{D}_m := m \mathbb{D} \). We write \( \gamma_n^{(m)} = \text{LE}(S[0, T^{(m)}_n]) \) for the loop-erasure of \( S[0, T^{(m)}_n] \) (assuming linear interpolation). Let \( \tau^{(m)}_n = \text{len}(\gamma^{(m)}_n) \) be the length of \( \gamma^{(m)}_n \) which is equal to the first time that \( \gamma^{(m)}_n \) exits from \( \mathbb{D}_m \). Also define
\[
\eta^{(m)}_n(t) = \gamma^{(m)}_n \left( 2^{\beta_n} t \right) \quad \text{for } t \in [0, 2^{-\beta_n} \tau^{(m)}_n].
\]
Writing \( t_n^{(m)} = 2^{-\beta n} r_n^{(m)} \) for the time duration of \( \eta_n^{(m)} \), we can regard
\[
\eta_n^{(m)} : [0, t_n^{(m)}] \to \mathbb{B}_m
\] (8.3)
as a random continuous curve.

Now take \( r \geq 1 \) which is also not necessarily an integer. Suppose that \( 5r < m \). Then \( \eta_n^{(m)} \) must hit \( \partial \mathbb{B}_r \) before time \( t_n^{(m)} \). Thus we can define
\[
t_n^{(m)} = \inf \{ t \geq 0 \mid \eta_n^{(m)}(t) \in \partial \mathbb{B}_r \}.
\] (8.4)

The main result of this subsection is the following proposition, which states that with high probability the first exit time \( t_n^{(m)} \) is close to \( t_n^{r,\delta} \), when \( \delta \) is sufficiently small.

**Proposition 8.1.** For any \( \epsilon \in (0, \frac{1}{2}) \), there exists \( \delta > 0 \) such that for all \( n \geq 1 \), \( m \geq 1 \) and \( r \geq 1 \) with \( 5r < m \) we have
\[
P\left( t_n^{(m)} - t_n^{(m)} \leq \epsilon \right) \geq 1 - \epsilon.
\] (8.5)

As the proof bears some similarity with Proposition 5.1, we will start with the definition of some unwanted "bad" events \( F_1 \) through \( F_5 \) and control the probability they happen.

Take \( m \geq 1 \) and \( r \geq 1 \) with \( 5r < m \). For \( k \geq 1 \), we write
\[
\delta = \delta_k = 2^{-2^{k+1}}.
\]

Let \( A = \mathbb{B}_{r+6 \cdot 2^{-k+4}} \setminus \mathbb{B}_r \). Then we can find a collection of cubes \( \{ B_i \}_{i=1}^{N_k} \) of radius \( 3 \cdot 2^{-k+4} \) satisfying the following conditions:

- \( N_k \leq 10 \cdot 2^{2k} \);
- \( \bigcup_{i=1}^{N_k} B_i = A \);
- Each face of \( B_i \) is parallel to \( x \)-\( y \) or \( y \)-\( z \) or \( z \)-\( x \) plane for all \( 1 \leq i \leq N_k \).

Let \( x_i \) be the center of the cube \( B_i \). Note that \( x_i \in \partial \mathbb{B}_{r+3 \cdot 2^{-k+4}} \) for all \( 1 \leq i \leq N_k \). Thus \( \text{dist}(B_i, \{0\} \cup \partial \mathbb{B}_m) \geq r \). We write \( B_i' \) for the cube of radius \( 3 \cdot 2^{-k+4} - \delta_k \) centered at \( x_i \) whose face is parallel to \( x \)-\( y \) or \( y \)-\( z \) or \( z \)-\( x \) plane.

Proposition 5.1 says that if \( \eta^{(m)}_n \) hits a box \( B_i \), then with very high probability it will also hit \( B_i' \), the concentric box with slightly smaller size. More precisely, there exists a universal constant \( c \) such that for all \( n \), \( k \) and \( 1 \leq i \leq N_k \),
\[
P\left( \eta_n^{(m)} \text{ hits } B_i \text{ but } \eta_n^{(m)} \text{ does not hit } B_i' \right) \leq c2^{-k+10}.
\]

Taking sum for \( 1 \leq i \leq N_k \), it follows that
\[
P(F_1) \leq c2^{-k^2} \quad \text{where } F_1 := \bigcup_{i=1}^{N_k} \left\{ \eta_n^{(m)} \text{ hits } B_i \text{ but } \eta_n^{(m)} \text{ does not hit } B_i' \right\}.
\]

We then note that by Theorem 6.1 of [21] it follows that there exist universal constants \( M < \infty \) and \( C < \infty \) such that for all \( n \) and \( k \)
\[
P(F_2) \leq C2^{-\frac{k^4}{4}} \quad \text{where } F_2 := \left\{ \text{QL} \left( 100 \cdot 2^{-k+4} , 2^{-\frac{k^4}{2}}; \eta_n^{(m)} \right) \neq \emptyset \right\}
\] (8.6)
see Section 2.5.6 for QL(s, r; \lambda) a set of quasi-loops. Namely, \( \eta_n^{(m)} \) has no quasi-loops with high probability.

We now define an event which intuitively says that "after exiting \( \mathbb{B}_r \), the LERW wanders for a long distance but has not exited from \( \mathbb{B}_{r+4} \) yet". Let
\[
u = \inf \left\{ t \geq t_n^{(m)} \mid |\eta_n^{(m)}(t) - \eta_n^{(m)}(t_n^{(m)})| \geq 2^{-\frac{k^4}{4}} \right\}.
\]

Now we define the event \( F_3 \) as follows:
\[
F_3 := \left\{ \eta_n^{(m)} \left( t_n^{(m)}, \nu \right) \cap \partial \mathbb{B}_{r+\delta_k} = \emptyset \right\}.
\] (8.7)

Namely, \( F_3 \) stands for the event that \( \eta_n^{(m)} \) does not intersect with \( \mathbb{B}_{r+\delta_k} \) from \( t_n^{(m)} \) to \( \nu \).
Lemma 8.2. It follows that
\[ P(F_3) \leq C2^{-\frac{4}{3^3}}. \]  
(8.8)

Proof. Note that
\[ P(F_3) = P(F_3 \cap (F_1 \cup F_2)) + P(F_3 \cap F_1^c \cap F_2^c) \leq C2^{-\frac{4}{3^3}} + P(F_3 \cap F_1^c \cap F_2^c). \]

So suppose that \( F_3 \cap F_1^c \cap F_2^c \) occurs. There is a cube \( B_i \) such that
\[ \eta_n^{(m)}(t_n^{(m)}) \in \partial B_i. \]

Notice that \( B_i' \subset B_i^{c+\delta_k} \). Since \( F_3 \) occurs, we see that
\[ \eta_{n,r}^{(m)}[t_n^{(m)}, u] \cap B_i^{c+\delta_k} = \emptyset, \]
which implies that
\[ \eta_{n,r}^{(m)}[t_n^{(m)}, u] \cap B_i' = \emptyset. \]

On the other hand, since \( \eta_0^{(m)} \) hits \( B_i' \) and \( F_{\infty}^2 \) occurs, it follows that \( \eta_0^{(m)} \) must hit \( B_i' \). Notice that \( \eta_0^{(m)}[0, t_n^{(m)}] \cap B_i' = \emptyset \) since \( t_n^{(m)} \) is the first time that \( \eta_0^{(m)} \) hits \( B_i' \). Thus, it follows that \( \eta_n^{(m)}[u,t_n^{(m)}] \cap B_i' \neq \emptyset \). By definition of \( u \), the diameter of \( \eta_n^{(m)}[t_n^{(m)}, u] \) is bigger than \( 2^{-\frac{4}{3^3}} \). Thus, \( \eta_n^{(m)} \) has a \((100 \cdot 2^{-\frac{4}{3^3}}, 2^{-\frac{4}{3^3}})\)-quasi-loop. This contradicts with \( F_{\infty}^2 \). Therefore, we can conclude that \( F_3 \cap F_1^c \cap F_2^c = \emptyset \). The claim (8.8) thus follows.

Next we will decompose \( \mathbb{D}_m \) into a collection of cubes \( \{\hat{B}_i\}_{i=1}^{L_k} \) satisfying the following:

- The side length of \( \hat{B}_i \) is equal to \( \epsilon_k = 2^{-\frac{4}{3^3}} \) for each \( 1 \leq i \leq L_k \) (recall that the constant \( M \) is coming from (8.6)).
- The number of cubes \( L_k \) is comparable to \( m^3 \epsilon_k^{-3} \).
- \( \mathbb{D}_m \subset \bigcup_{i=1}^{L_k} \hat{B}_i \).

We write \( X_i^{n,\epsilon_k} \) for the number of points in \( \hat{B}_i \cap 2^{-n} \mathbb{Z}^3 \) hit by \( \eta_0^{(m)} \). Letting
\[ F_4 := \left\{ X_i^{n,\epsilon_k} > \sqrt{k}2^{-3n} \text{ for some } i = 1, 2, \ldots, L_k \right\}, \]
representing an event that roughly says \textit{the LERW hits ‘too many’ points in some box}, it follows from the Markov inequality estimate (7.51) that
\[ P(F_4) \leq Cm^3 e^{-\frac{3}{n}} \leq Ce^{-\frac{3}{n}}, \]
where we used the fact that \( m \) is a constant in the last inequality.

Finally, we define the eventual “bad” event \( F_5 \) by
\[ F_5 := \left\{ t_n^{(m)} - t_n^{(m)} \geq 30 \cdot \sqrt{\epsilon_k} \right\}. \]  
(8.9)

Then we have the following lemma.

Lemma 8.3.
\[ P(F_5) \leq C \epsilon_k. \]  
(8.10)

Proof. Note that
\[ P(F_5) = P(F_5 \cap (F_3 \cup F_1)) + P(F_5 \cap F_1^c \cap F_3^c) \leq C \epsilon_k + P(F_5 \cap F_1^c \cap F_3^c). \]  
(8.11)

So suppose that \( F_5 \cap F_1^c \cap F_3^c \) occurs. We can find a cube \( \hat{B}_i \) satisfying \( \eta_0^{(m)}(t_n^{(m)}) \in \hat{B}_i \). We write \( \hat{B}_{i_0}, \hat{B}_{i_1}, \ldots, \hat{B}_{i_{26}} \) for a set of cubes with \( \hat{B}_{i_q} \cap \hat{B}_{i_q} \) for \( q = 0, 1, \ldots, 26 \). Since the event \( F_5 \) occurs, we see that
\[ \eta_n^{(m)}[t_n^{(m)}, t_n^{(m)}] \subset \bigcup_{q=0}^{26} \hat{B}_{i_q}. \]

On the other hand, since the event \( F_5 \) occurs, the number of points in \( 2^{-n} \mathbb{Z}^3 \) hit by \( \eta_n^{(m)}[t_n^{(m)}, t_n^{(m)} + \delta_k] \) is bounded below by \( 30 \cdot \sqrt{\epsilon_k}2^{3n} \). This implies that there exists some \( 0 \leq q \leq 26 \) such that \( X_n^{n,\epsilon_k} > \frac{10}{2} \cdot \sqrt{\epsilon_k}2^{3n} \). But this contradicts with \( F_5^2 \). Therefore, we conclude that \( F_5 \cap F_3^c \cap F_1^c = \emptyset \). The inequality (8.10) then follows from (8.11). □
Proof of Proposition 8.4. Take $\epsilon \in (0, \frac{1}{2})$. We can take $k$ sufficiently large so that $30 \cdot \sqrt{\epsilon/k} < \epsilon$ and $Ck < \epsilon$ where the constant $C$ is coming from (8.10). For this choice of $k$, we let $\delta := \delta_k = 2^{-\frac{k+1}{11}}$. Then, it follows from (8.10) that
\[
P\left(\tau_{n,r+\delta}^{(m)} - \tau_{n,r}^{(m)} \leq \epsilon\right) \geq 1 - \epsilon,
\]
which finishes the proof.

\[
\]

8.3 Existence of the scaling limit

Recall the discussion at the end of Section 8.1. Take $m \geq 1$ and $r \geq 1$ with $5r < m$. Recall that $\eta_{n,r}^{(m)}$ is defined as in (8.3). Also we recall that $\tau_{n,r}^{(m)}$ stands for the first time that $\eta_{n,r}^{(m)}$ exits from $B_r$ (see (8.4)). If we let
\[
\eta_{n,r}^{(m)} : t \in [0, \tau_{n,r}^{(m)}] \mapsto \eta_{n,r}^{(m)}(t) \in \mathbb{D}_r
\]
be the sub-path of $\eta_{n,r}^{(m)}$ truncated up to the time $\tau_{n,r}^{(m)}$, we see that $\eta_{n,r}^{(m)}$ is a random element of the metric space $(C(r), \rho)$ where $C(r)$ is defined below (8.1) and the metric $\rho$ is defined in (2.8). The next proposition shows that $\eta_{n,r}^{(m)}$ converges weakly as $n \to \infty$.

Proposition 8.4. Take $m \geq 1$ and $r \geq 1$ with $5r < m$. There exists a random element $\eta_{\infty,r}^{(m)}$ of $(C(r), \rho)$ such that $\eta_{n,r}^{(m)}$ converges weakly to $\eta_{\infty,r}^{(m)}$ as $n \to \infty$ with respect to the metric $\rho$.

Before we proceed to the proof, we will first introduce some notation.

We start with a metric space $(D^{(m)}, \rho)$ where $D^{(m)}$ is the space of continuous curves $\lambda : [0, t^\lambda] \to \mathbb{D}_m$ where $t^\lambda \in [0, \infty)$ stands for the time duration of $\lambda$ and $\rho$ is the uniform metric defined in (2.8). Note that $\eta_{n,r}^{(m)}$ is a random element of the metric space $(D^{(m)}, \rho)$ for each $n$.

By an easy modification of Theorem 7.10 we have the following proposition.

Proposition 8.5. Under the setup above, there exists a random element $\eta_{\infty}^{(m)}$ of the metric space $(D^{(m)}, \rho)$ such that $\eta_{n}^{(m)}$ converges weakly to $\eta_{\infty}^{(m)}$ with respect to the metric $\rho$.

We recall that $\tau_{n}^{(m)}$ stands for the time duration of $\eta_{n}^{(m)}$ which is equal to the first time that $\eta_{n}^{(m)}$ hits $\partial D_m$ (see (8.3) for this). We write $\tau_{\infty}^{(m)}$ for the time duration of $\eta_{\infty}^{(m)}$. By Theorem 1.2 of [21], it follows that with probability one
\[
\eta_{\infty}^{(m)} : [0, \tau_{\infty}^{(m)}] \to \mathbb{D}_m
\]
is a simple path satisfying
\[
\eta_{\infty}^{(m)}(0) = 0, \eta_{\infty}^{(m)}[0, \tau_{\infty}^{(m)}] \subset \mathbb{D}_m \text{ and } \eta_{\infty}^{(m)}(\tau_{\infty}^{(m)}) \in \partial \mathbb{D}_m.
\] (8.13)

Namely, the time $\tau_{\infty}^{(m)}$ coincides with the first time that $\eta_{\infty}^{(m)}$ hits $\partial \mathbb{D}_m$.

By Skorokhod’s representation theorem, we can couple $\{\eta_{n}^{(m)}\}_{n \geq 1}$ and $\eta_{\infty}^{(m)}$ in the same probability space such that with probability one
\[
\lim_{n \to \infty} \rho(\eta_{n}^{(m)}, \eta_{\infty}^{(m)}) = 0
\] (8.14)

From now on till the end of the proof for Proposition 8.4 we always will assume this coupling of $\{\eta_{n}^{(m)}\}_{n \geq 1}$ and $\eta_{\infty}^{(m)}$.

Since $\eta_{\infty}^{(m)}$ satisfies (8.13), we may define
\[
\tau_{\infty}^{(m)} := \inf\{t \geq 0 \mid \eta_{\infty}^{(m)}(t) \in \partial \mathbb{D}_r\}
\]

so that $\tau_{\infty,r}^{(m)} < \tau_{\infty}^{(m)} < \infty$. We write
\[
\eta_{\infty,r}^{(m)} : t \in [0, \tau_{\infty,r}^{(m)}] \mapsto \eta_{\infty,r}^{(m)}(t) \in \mathbb{D}_r
\] (8.15)

for the sub-path of $\eta_{\infty}^{(m)}$ truncated up to the time $\tau_{\infty,r}^{(m)}$, which is a random element of $(C^r, \rho)$. We will show that in the coupling above, $\eta_{n,r}^{(m)}$ converges to $\eta_{\infty,r}^{(m)}$ in probability as $n \to \infty$ with respect to the metric $\rho$, which implies that $\eta_{n,r}^{(m)}$ converges weakly to $\eta_{\infty,r}^{(m)}$.

To do this, we will first show the following lemma.
Lemma 8.6. In the notation above,
\[
\lim_{n \to \infty} |\eta_{n,r}^{(m)} - \eta_{\infty,r}^{(m)}| = 0 \text{ in probability.} \tag{8.16}
\]

Proof. To show \[8.21\], it suffices to define some “good” events \( F_1, F_2 \) and \( F_3 \) below, and show that 1) these good events happen with high probability; 2) restricted on these events \( |t_{n,r}^{(m)} - \eta_{\infty,r}^{(m)}| \) is very small.

Let \( \alpha_1, \alpha_2 \in (0, \frac{1}{2}) \). We write \( a := (\alpha_1, \alpha_2) \). Since \( \eta_{\infty}^{(m)} \in (0, \infty) \) almost surely, we can find a constant \( c_a \in (0, 1) \) depending only on \( a \) such that
\[
P(F_1) \geq 1 - a, \tag{8.17}
\]
where the event \( F_1 \) is defined by
\[
F_1 := \left\{ c_a < \eta_{\infty}^{(m)} < \frac{1}{c_a} \right\}.
\]

Also it follows from Proposition \[8.1\] that there exists a constant \( \delta_a \in (0, a) \) depending only on \( a \) such that for all \( n \geq 1 \),
\[
P(F_2) \geq 1 - a \tag{8.18}
\]
where the event \( F_2 \) is defined by
\[
F_2 := \left\{ t_{n,r}^{(m)} - \eta_{\infty}^{(m)} \leq a, t_{n,r}^{(m)} - \eta_{\infty}^{(m)} \leq a \right\}.
\]

By \[8.14\], we see that there exists a constant \( N_a \) depending only on \( a \) such that for all \( n \geq N_a \)
\[
P(F_3) \geq 1 - a \tag{8.19}
\]
where the event \( F_3 \) is defined by
\[
F_3 := \left\{ \rho(t_{n,r}^{(m)}, \eta_{\infty}^{(m)}) \leq c_a^2 \delta_a \right\}.
\]

Now suppose that the event \( F_1 \cap F_2 \cap F_3 \) occurs. By \( F_3 \), it follows that
\[
|t_{n,r}^{(m)} - \eta_{\infty}^{(m)}| + \sup_{0 \leq s \leq 1} |t_{n,r}^{(m)}(t_{n,r}^{(m)} s) - \eta_{\infty}^{(m)}(t_{n,r}^{(m)} s)| \leq c_a^2 \delta_a. \tag{8.20}
\]

We let
\[
s_0 := \frac{t_{\infty,r}^{(m)}}{t_{n,r}^{(m)}},
\]
so that \( s_0 \in (0, 1) \) and \( t_{\infty,r}^{(m)} s_0 = t_{n,r}^{(m)} \). Note that \( \eta_{\infty}^{(m)}(t_{\infty,r}^{(m)} s_0) = \eta_{\infty}^{(m)}(t_{n,r}^{(m)} s_0) \in \partial B_r \). Also, we mention that \( \eta_{n}^{(m)}(t_{n,r}^{(m)} s_0) \in B_{r - c_a^2 \delta_a} \) because of \[8.20\]. This implies that
\[
t_{n,r-c_a^2 \delta_a} \leq t_{n,r}^{(m)} s_0.
\]

On the other hand, since \( \eta_{\infty}^{(m)}[0, t_{\infty,r}^{(m)} s_0] = \eta_{\infty}^{(m)}[0, t_{n,r}^{(m)} s_0] \subset \mathbb{F}_r \), by \[8.20\] again, we see that
\[
\eta_{\infty}^{(m)}[0, t_{n,r}^{(m)} s_0] \subset \mathbb{F}_{r + c_a^2 \delta_a},
\]
which implies that
\[
t_{n,r}^{(m)} s_0 \leq t_{n,r+c_a^2 \delta_a}^{(m)}.
\]

Combining these estimates with \( c_a \in (0, 1) \), we have
\[
t_{n,r}^{(m)} s_0 \leq t_{n,r}^{(m)} s_0 \leq t_{n,r+c_a^2 \delta_a}^{(m)}.
\]

From this and \( F_2 \), it follows that
\[
|t_{n,r}^{(m)} s_0 - t_{n,r}^{(m)}| \leq a
\]
However, by \( F_1 \) and \[8.20\], it follows that
\[
\left| \frac{t_{n,r}^{(m)}}{t_{\infty,r}^{(m)}} - 1 \right| \leq \frac{c_a^2 \delta_a}{t_{\infty,r}^{(m)}} \leq c_a \delta_a.
\]
This gives
\[ |\hat{t}_n^{(m)}s_0 - \hat{t}_r^{(m)}| = |\hat{t}_n^{(m)}\frac{\hat{f}_n^{(m)}}{\hat{t}_\infty^{(m)}} - \hat{t}_\infty^{(m)}| \leq \hat{t}_\infty^{(m)}c_\delta \leq \delta \leq a. \]

Consequently, on the event \( F^1 \cap F^2 \cap F^3 \), we have
\[ |\hat{t}_n^{(m)} - \hat{t}_\infty^{(m)}| \leq 2a. \]
Therefore, for all \( n \geq N_a \)
\[ P\left(|\hat{t}_n^{(m)} - \hat{t}_\infty^{(m)}| \leq 2a\right) \geq 1 - 3a, \]
which implies that for all \( n \geq N_a \)
\[ P\left(|\hat{t}_n^{(m)} - \hat{t}_\infty^{(m)}| \leq a_1\right) \geq 1 - a_2. \tag{8.21} \]

Since \( a_1 \) and \( a_2 \) are arbitrary positive numbers, we see that \( \hat{t}_n^{(m)} \) converges to \( \hat{t}_\infty^{(m)} \) as \( n \to \infty \) in probability. This finishes the proof of (8.16).

We now return to the proof of Proposition 8.4.

**Proof of Proposition 8.4.** Take \( m \geq 1 \) and \( r \geq 1 \) with \( 5r < m \). Recall that
\[ \rho(\eta_n^{(m)}, \eta_{\infty}^{(m)}) = |\hat{t}_n^{(m)} - \hat{t}_\infty^{(m)}| + \max_{0 \leq s \leq 1} |\eta_n^{(m)}(\hat{t}_n^{(m)}s) - \eta_{\infty}^{(m)}(\hat{t}_\infty^{(m)}s)|. \tag{8.22} \]

Since we have obtained a bound on the first term in the RHS of (8.22), we will now work on a bound on the second term.

Take \( a_1, a_2 \in (0, \frac{1}{2}) \). Again we let \( a = (a_1a_2)^{100} \). Since we already establish (8.21), it suffices to estimate the second term in the RHS of (8.22). To do it, we first note that since \( \{\eta_n^{(m)}\}_{n \geq 1} \) is a convergent sequence with respect to the metric \( \rho \), it is equicontinuous in the following sense: there exists \( u_a \in (0, a) \) depending only on \( a \) such that for all \( n \geq 1 \)
\[ P(F_n^4) \geq 1 - a \tag{8.23} \]
where the event \( F_n^4 \) is defined by
\[ F_n^4 := \left\{ \max_{0 \leq s, t \leq \hat{t}_\infty^{(m)}} \left| \eta_n^{(m)}(s) - \eta_n^{(m)}(t) \right| \leq a \right\}. \]

We write \( \tilde{a} = u_a \). It follows from (8.17), (8.18) and (8.19) that there exist constants \( c_{\tilde{a}} \in (0, 1) \), \( \delta_{\tilde{a}} \in (0, \tilde{a}) \) and \( N_{\tilde{a}} \in \mathbb{N} \) depending only on \( \tilde{a} \) such that if we let
\[ \tilde{F}_1 := \left\{ c_{\tilde{a}} < \hat{t}_\infty^{(m)} < \frac{1}{c_{\tilde{a}}} \right\}; \tilde{F}_2 := \left\{ \hat{t}_n^{(m)}r - \hat{t}_\infty^{(m)} \leq \tilde{a}, \hat{t}_n^{(m)}r - \hat{t}_\infty^{(m)} \leq \tilde{a} \right\}; \tilde{F}_3 := \left\{ \rho(\eta_n^{(m)}, \eta_{\infty}^{(m)}) \leq c_{\tilde{a}}^2\delta_{\tilde{a}} \right\}, \]
then we have
\[ P(\tilde{F}_1) \geq 1 - \tilde{a}; \quad P(\tilde{F}_2) \geq 1 - \tilde{a} \text{ for all } n \geq 1; \quad P(\tilde{F}_3) \geq 1 - \tilde{a} \text{ for all } n \geq N_{\tilde{a}}. \]

We already showed that on the event \( \tilde{F}_1 \cap \tilde{F}_2 \cap \tilde{F}_3 \) we have
\[ \left| \frac{\hat{f}_n^{(m)}}{\hat{t}_\infty^{(m)}} - 1 \right| \leq c_{\tilde{a}}\delta_{\tilde{a}} \text{ and } |\hat{t}_n^{(m)} - \hat{t}_\infty^{(m)}| \leq 2\tilde{a}. \tag{8.24} \]

With this in mind, suppose that the event \( \tilde{F}_1 \cap \tilde{F}_2 \cap \tilde{F}_3 \cap F_4^4 \) occurs. Take \( s_1 \in [0, 1] \). Define
\[ \tilde{s}_1 := s_1 \cdot \frac{\hat{f}_n^{(m)}}{\hat{t}_\infty^{(m)}} \]
so that \( 0 \leq \tilde{s}_1 < 1 \) and \( \hat{t}_n^{(m)}s_1 = \hat{t}_\infty^{(m)}s_1 \). Thus, it follows from \( \tilde{F}_3 \) that
\[ |\eta_n^{(m)}(\hat{t}_n^{(m)}s_1) - \eta_{\infty}^{(m)}(\hat{t}_\infty^{(m)}s_1)| \leq c_{\tilde{a}}^2\delta_{\tilde{a}}. \]

71
Notice that $\eta^{(m)}$ \( (t^{(m)}_n s_1) \) = $\eta^{(m)}$ \( (s_1 t^{(m)}_n) \) = $\eta^{(m)}$ \( (s_1 t^{(m)}_n) \). Therefore,

$$\left| \eta^{(m)}_n \left( t^{(m)}_n s_1 \right) - \eta^{(m)}_{\infty, r} \left( s_1 t^{(m)}_n \right) \right| \leq c^3 d^2 \bar{a} < a.$$

So we want to compare $\eta^{(m)}_n \left( t^{(m)}_n s_1 \right)$ with $\eta^{(m)}_{\infty, r} \left( s_1 t^{(m)}_n \right)$. For this, we mention that by (8.24) and $\hat{F}_1$ we have

$$\left| \eta^{(m)}_n \left( t^{(m)}_n s_1 \right) - \eta^{(m)}_{\infty, r} \left( s_1 t^{(m)}_n \right) \right| \leq c^3 d^2 \bar{a}. \quad \text{(8.24)}$$

Notice that $\eta^{(m)}_n \left( s_1 t^{(m)}_n \right)$ = $\eta^{(m)}_{\infty, r} \left( s_1 t^{(m)}_n \right)$. Therefore, on the event $\hat{F}_1 \cap \hat{F}_2 \cap \hat{F}_3 \cap F_4$, it follows that

$$\left| \eta^{(m)}_n \left( t^{(m)}_n \right) - t^{(m)}_{\infty, r} \right| \leq 2\bar{a} \quad \text{and} \quad \left| \eta^{(m)}_n \left( s_1 t^{(m)}_n \right) - \eta^{(m)}_{\infty, r} \left( s_1 t^{(m)}_n \right) \right| \leq 2a.$$

Recall that $s_1 \in [0, 1]$ is arbitrary. So this implies that on the event $\hat{F}_1 \cap \hat{F}_2 \cap \hat{F}_3 \cap F_4$, we have

$$\rho(\eta^{(m)}_n, \eta^{(m)}_{\infty, r}) \leq 4a.$$

Thus, for all $n \geq N_\bar{a}$ we see that

$$P(\rho(\eta^{(m)}_n, \eta^{(m)}_{\infty, r}) \leq 4a) \geq 1 - 4a.$$

Since $\bar{a} = u_a$ depends only on $a$, $N_\bar{a}$ also depends only on $a$. Therefore, this implies that $\eta^{(m)}_{\infty, r}$ converges to $\eta^{(m)}_{\infty, r}$ as $n \to \infty$ in probability. So we finish the proof.

We now switch back to the setup of ILER. We recall that $\gamma^\infty$ stands for the infinite loop-erased random walk in mesh $2^{-n} Z^3$ (assuming linear interpolation) and that the rescaled process $\eta^\infty$ is defined as in (8.2). Take $r \geq 1$. We write $T^\infty_{n, r}$ for the first time that $\eta^\infty_n$ exits from $B_r$. Let

$$\eta^\infty_{n, r} : t \in [0, T^\infty_{n, r}] \mapsto \eta^\infty_n(t) \in \mathbb{B}_r \quad \text{(8.25)}$$

be the sub-path of $\eta^\infty_n$ truncated up to the time $T^\infty_{n, r}$, which is a random element of the metric space $(C^{(r)}, \rho)$. The next proposition proves that $\eta^\infty_{n, r}$ converges weakly as $n \to \infty$ for each $r \geq 1$.

**Proposition 8.7.** Take $r \geq 1$. There exists a random element $\eta^\infty_{r, \infty}$ of the space $(C^{(r)}, \rho)$ such that $\eta^\infty_{n, r}$ converges weakly to $\eta^\infty_{r, \infty}$ as $n \to \infty$ with respect to the uniform metric $\rho$.

**Proof.** Take $r \geq 1$. We first mention that $(C^{(r)}, \rho)$ is a separable metric space. We let $M^{(r)}$ be the space of all probability measures on $(C^{(r)}, \rho)$ endowed with its Borel sigma algebra $B(C^{(r)})$. It is well known that the topology of weak convergence on $M^{(r)}$ is metrizable by the Prokhorov metric $\pi$ defined as in (2.10) (see Section 2.4 for this fact).

With this in mind, we let $\mu^\infty_{n, r}$ be the probability measure induced by $\eta^\infty_{n, r}$. It suffices to show that the sequence $\{\mu^\infty_{n, r}\}_{n \geq 1}$ converges with respect to the metric $\pi$. To do it, take $\epsilon \in (0, \frac{1}{10})$ and let

$$m = \frac{1}{\epsilon}.$$

Recall that $\gamma^{(m)}_n$ stands for the loop-erasure of $S[0, T^{(m)}_n]$ where $T^{(m)}_n$ is the first time that $S$ exits form $D_m$. Also we recall that $\eta^{(m)}_n$ and $\eta^{(m)}_{n, r}$ are defined as in (8.3) and (8.12), respectively. It follows from Corollary 4.5 of (17) that for each curve $\lambda \in C^{(r)}$ and all $n \geq 1$,

$$P(\eta^{(m)}_{n, r} = \lambda) = P(\eta^\infty_{n, r} = \lambda) \left(1 + O\left(\frac{1}{m}\right)\right). \quad \text{(8.26)}$$
Let \( \mu_{n,r}^{(mr)} \) be the probability measure induced by \( \eta_{n,r}^{(mr)} \), then we have
\[
\pi\left(\mu_{n,r}^{(mr)}, \mu_{n,r}\right) \leq C\epsilon. \tag{8.27}
\]

On the other hand, By Proposition 8.4, we see that \( \{\mu_{n,r}^{(mr)}\}_{n \geq 1} \) is a convergent sequence with respect to the metric \( \pi \). Therefore, there exists a probability measure \( \nu_{r,m} \) on \((C^1), \rho\) such that
\[
\lim_{n \to \infty} \pi\left(\mu_{n,r}^{(mr)}, \nu_{r,m}\right) = 0 \tag{8.28}
\]

What we want to show is that the sequence \( \{\mu_n^{(mr)}\}_{n \geq 1} \) converges with respect to the uniform metric \( \pi \). However, an easy modification of Proposition 7.1 shows that the sequence \( \{\mu_n^{(mr)}\}_{n \geq 1} \) is tight with respect to the metric \( \rho \). Thus, taking two arbitrary sub-sequences \( \{\mu_n^{(mr)}\}_{n \geq 1} \) and \( \{\mu_{n,k}^{(mr)}\}_{k \geq 1} \), we can find suitable sub-subsequences \( \{\mu_n^{(mr)}\}_{j \geq 1} \) and \( \{\mu_{n,k}^{(mr)}\}_{j \geq 1} \) of them such that both \( \{\mu_n^{(mr)}\}_{j \geq 1} \) and \( \{\mu_{n,k}^{(mr)}\}_{j \geq 1} \) converge with respect to the metric \( \pi \). It suffices to prove that the limit measure of \( \{\mu_n^{(mr)}\}_{j \geq 1} \) coincides with that of \( \{\mu_{n,k}^{(mr)}\}_{j \geq 1} \). To do it, we write \( \sigma_j := \mu_{n,k}^{(mr)} \) and \( \sigma_j^2 := \mu_{n,k}^{(mr)} \). Also, we let \( \sigma_j \) be the limit measure of \( \sigma_j \). We want to show \( \sigma_j = \sigma_j^2 \). Recalling (8.28), we see that if we take \( j \) sufficiently large,
\[
\pi\left(\sigma_j, \sigma_j^2\right) \leq \epsilon, \pi\left(\nu_{r,m}, \mu_{n,k}^{(mr)}\right) \leq \epsilon, \pi\left(\mu_{n,k}^{(mr)}, \nu_{r,m}\right) \leq \epsilon \text{ and } \pi\left(\sigma_j, \sigma_j^2\right) \leq \epsilon. \tag{8.29}
\]

It follows from (8.27) that
\[
\pi\left(\sigma_j, \mu_{n,k}^{(mr)}\right) \leq C\epsilon \text{ and } \pi\left(\sigma_j^2, \mu_{n,k}^{(mr)}\right) \leq C\epsilon, \tag{8.30}
\]

for all \( j \geq 1 \). Combining (8.29) with (8.30), by taking \( j \) sufficiently large, we have
\[
\pi\left(\sigma_j, \sigma_j^2\right) \leq \pi\left(\sigma_j, \sigma_j^2\right) + \pi\left(\sigma_j, \mu_{n,k}^{(mr)}\right) + \pi\left(\mu_{n,k}^{(mr)}, \nu_{r,m}\right) + \pi\left(\nu_{r,m}, \mu_{n,k}^{(mr)}\right) + \pi\left(\mu_{n,k}^{(mr)}, \sigma_j^2\right) + \pi\left(\sigma_j^2, \sigma_j^2\right) \leq (2C + 4)\epsilon.
\]

Since the constant \( C \) is universal and \( \epsilon \in (0, \frac{1}{m}) \) is arbitrary, this implies that \( \sigma_j = \sigma_j^2 \). So we finish the proof.

Now we are ready to prove the existence of a global scaling limit. The following theorem restates Theorem 1.4. Recall that \( \eta_n^\infty \) stands for the rescaled infinite loop-erased random walk defined as in (8.2). We also recall that the metric space \((C, \chi)\) is defined as in (2.9).

**Theorem 8.8.** Then there exists a random element \( \eta^\infty \) of the space \((C, \chi)\) such that \( \eta_n^\infty \) converges weakly to \( \eta^\infty \) as \( n \to \infty \) with respect to the metric \( \chi \).

**Proof.** Recall that \( \eta_n^\infty \) is defined as in (8.25) and \( \mu_n^{(mr)} \) stands for the probability measure induced by \( \eta_n^{(mr)} \). From Proposition 8.7, for each \( r \), there exists a random element \( \eta^{\infty,r} \) of \((C^r), \rho\) such that \( \eta_n^{(mr)} \) converges weakly to \( \eta^{\infty,r} \) as \( n \to \infty \). So if we let \( \mu^{\infty,r} \) be the probability measure induced by \( \eta^{\infty,r} \), then \( \mu_n^{(mr)} \) converges to \( \mu^{\infty,r} \) as \( n \to \infty \) with respect to the Prokhorov metric \( \pi \). It is clear that the sequence of the measures \( \{\mu_n^{(mr)}\}_{r \geq 1} \) satisfies the suitable consistency condition. Therefore, we can see that there exists a random element \( \eta^{\infty} \) of the space \((C, \chi)\) such that the distribution of the truncated curve \( \eta^{\infty} [0, \tau_n^\infty] \) coincides with that of \( \eta^{\infty,r} \) for each \( r \geq 1 \). Here \( \tau_n^\infty \) stands for the first time that \( \eta^{\infty} \) exits from \( B_r \). Therefore, we can conclude that \( \eta^\infty \) converges weakly to \( \eta^\infty \) as \( n \to \infty \) with respect to the metric \( \chi \).

**Remark 8.9.** It would be also very interesting to ask what properties \( \eta^{\infty} \) satisfy. For instance, it is very natural to expect the law of \( \eta^{\infty} \) is translation- and scale-invariant, which, if confirmed, would give an explicit form of the one-point function for 3D ILERW defined similarly as the \( c_x \) in (2.22).

**References**

[1] O. Angel, D. Croydon, S. Hernandez Torres and D. Shiraishi. Scaling limit of the uniform spanning tree and the associated random walk in three dimensions. In preparation.

[2] P. Billingsley. *Convergence of Probability Measures*. John Wiley and Sons, 1999.

[3] S. N. Ethier and T. G. Kurtz. *Markov processes*. John Wiley and Sons, 1986.
[4] C. Garban, G. Pete and O. Schramm. Pivotal, cluster, and interface measures for critical planar percolation. *J. Amer. Math. Soc.*, 26:939-1024, 2013.

[5] J. Henrikson. Completeness and total boundedness of the Hausdorff metric. *MIT Undergraduate Journal of Mathematics*, 69-80. 1999.

[6] I. Karatzas and S. Shreve. *Brownian motion and stochastic calculus*, 2nd corrected ed. Springer, 1998.

[7] G. Kozma. The scaling limit of loop-erased random walk in three dimensions. *Acta Math.*, 199(1):29-152, 2007.

[8] G. Lawler. *Intersections of random walks*. Birkhauser, 1991.

[9] G. Lawler. Cut Times for Simple Random Walk. *Electron. J. Probab.*, 1(13):1-24, 1996.

[10] G. Lawler. Loop-erased random walk. *Perplexing problems in probability: Festschrift in honor of Harry Kesten* (M. Bramson and R. T. Durrett, eds.), Progr. Probab., 44, Birkhauser, 1999.

[11] G. Lawler. The infinite two-sided loop-erased random walk. Preprint, available at [arXiv:1802.06667](https://arxiv.org/abs/1802.06667).

[12] G. Lawler and V. Limic. *Random Walk: A Modern Introduction*. Cambridge University Press, 2010.

[13] G. Lawler, O. Schramm and W. Werner. Conformal invariance of planar loop-erased random walks and uniform spanning trees. *Ann. Probab.*, 32(1B):939-995, 2004.

[14] G. Lawler and F. Viklund. Convergence of radial loop-erased random walk in the natural parametrization. Preprint, available at [arXiv:1703.03729](https://arxiv.org/abs/1703.03729).

[15] T. Lindvall. *Lectures on the Coupling Method*. Wiley series in Probability and Mathematical Statistics, John Wiley and Sons, 1992.

[16] X. Li and D. Shiraishi. One-point function estimates for loop-erased random walk in three dimensions. Preprint, available at [arXiv:1807.00541](https://arxiv.org/abs/1807.00541).

[17] R. Masson. The growth exponent for planar loop-erased random walk. *Electron. J. Probab.*, 14(36):1012-1073, 2009.

[18] P. Moerters and Y. Peres. *Brownian Motion*. Cambridge Series in Statistical and Probabilistic Mathematics, Cambridge University Press, 2010.

[19] K. R. Parthasarathy. *Probability Measures on Metric Spaces*. Academic Press, 1967.

[20] O. Schramm. Scaling limits of loop-erased random walks and uniform spanning trees. *Isr. J. Math.*, 118(1):221-288, 2000.

[21] A. Sapozhnikov and D. Shiraishi. Brownian motion, simple paths, and loops. *Prob. Theory Relat. Fields*, 172(34):615662, 2018.

[22] D. Shiraishi. Growth exponent for loop-erased random walk in three dimensions. *Ann. Probab.*, 46(2):687-774, 2018.

[23] D. Shiraishi. Hausdorff dimension of the scaling limit of loop-erased random walk in three dimensions. To appear in *Ann. I. H. Poincaré*, also available at [arXiv:1604.08091](https://arxiv.org/abs/1604.08091).

[24] D. B. Wilson. Generating random spanning trees more quickly than the cover time. *Proceedings of the twenty-eighth annual ACM symposium on Theory of computing*, pp. 296-303, 1996.