Noncommutative field theory on $\mathbb{R}^3_\lambda$

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• We want to understand the dependence of typical features of NC field theories (for example the mixing UV/IR) on the specific kind of NC.
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- The first non-trivial step is to consider NC spaces with NC parameter which is linear in coordinates (Lie algebra type)

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- I will describe a procedure to explicitly construct many inequivalent star products with such a noncommutativity.
- The easiest one, which is considered here is the one mimicking \(su(2)\) algebra
The noncommutative algebra $\mathbb{R}^3_\lambda$

[Hamou, Lagraa, SheikhJabbari PRD 2002] [GraciaBondia, Lizzi, Marmo, Vitale JHEP 2002]
The noncommutative algebra $\mathbb{R}^3_\lambda$

- It is a subalgebra of the Wick-Voros algebra $\mathbb{R}^4_\theta$, a variation of the Moyal algebra, which exploits the well known realization of three-dimensional Lie algebras as Poisson subalgebras of quadratic-linear functions on $\mathbb{R}^4 \simeq \mathbb{C}^2 \ (i\mathfrak{sp}(4))$
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- $\mathbb{R}^3_\lambda$ is generated by coordinate functions $x^\mu$

$$\pi^*(x^\mu) = \frac{\lambda}{\theta} \bar{z} a e_{ab}^\mu z_b, \quad \mu = 0, .., 3, \quad a, b = 1, 2$$

$\lambda$ constant, real parameter of length dimension;
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$\lambda$ constant, real parameter of length dimension;
- $e^i = \frac{1}{2} \sigma^i, \; i = 1, .., 3 \; e_0 = \frac{1}{2} 1$.
- it is based on the identification of $\mathbb{R}^3$ with $\mathfrak{g}^*$. Here $\mathfrak{g} = \mathfrak{su}(2)$
- Besides being a Poisson subalgebra, it is also a NC subalgebra wrt the Wick-Voros (and Moyal) star product

$$\phi \star \psi (z_a, \bar{z}_a) = \phi (z, \bar{z}) \exp (\theta \overleftarrow{\partial}_{z_a} \overrightarrow{\partial}_{\bar{z}_a}) \psi (z, \bar{z}), \quad a = 1, 2$$

$$[z_a, \bar{z}_b]_* = \theta \delta_{ab}$$
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\phi \star \psi (x) = \exp \left[ \frac{\lambda}{2} \left( \delta_{ij} x_0 + i \epsilon^{k}_{ij} x_k \right) \frac{\partial}{\partial u_i} \frac{\partial}{\partial v_j} \right] \phi(u) \psi(v) |_{u=v=x}
$$

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The noncommutative algebra $\mathbb{R}^3_\lambda$

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$$
\begin{align*}
x_i \ast x_j &= x_i x_j + \frac{\lambda}{2} \left( x_0 \delta_{ij} + i \epsilon_{ij}^k x^k \right) \\
x_0 \ast x_i &= x_i \ast x_0 = x_0 x_i + \frac{\lambda}{2} x_i \\
x_0 \ast x_0 &= x_0 (x_0 + \frac{\lambda}{2}) = \sum_i x_i \ast x_i - \lambda x_0
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$$x_i \ast x_j = x_i x_j + \frac{\lambda}{2} \left( x_0 \delta_{ij} + i \epsilon^k_{ij} x_k \right)$$

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$$[x_i \ast x_j] = i \lambda \epsilon^k_{ij} x_k$$

$x_0 \ast$-commutes with $x_i$ so that we can alternatively define $\mathbb{R}^3_\lambda$ as the $\ast$-commutant of $x_0$; $x_0$ generates the center of the algebra.
The Wick-Voros product

The Wick-Voros product is introduced through a weighted quantization map which, in two dimensions, associates to functions on the complex plane the operator (Berezin quantization)

\[ \hat{\phi} = \hat{\mathcal{W}}_V(\phi) = \frac{1}{(2\pi)^2} \int d^2z \hat{\Omega}(z, \bar{z}) \phi(z, \bar{z}) \]
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where

$$\hat{\Omega}(z, \bar{z}) = \int d^2\eta e^{-(\eta \bar{z} - \bar{\eta} z)} e^{\theta \eta a^\dagger} e^{-\theta \bar{\eta} a}$$

$a, a^\dagger$ are the usual (configuration space) creation and annihilation operators, with commutation relations

$$[a, a^\dagger] = \theta.$$
The inverse map which is the analogue of the Wigner map is represented by:

$$\phi(z, \bar{z}) = \mathcal{W}_V^{-1}(\hat{\phi}) = \langle z | \hat{\phi} | z \rangle$$

with $|z\rangle$ the coherent states defined by $a|z\rangle = z|z\rangle$. 

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The Wick-Voros product is then defined as

\[ \phi \star \psi := \mathcal{W}_V^{-1} \left( \hat{\mathcal{W}}_V(\phi)\hat{\mathcal{W}}_V(\psi) \right) = \langle z | \hat{\phi} \hat{\psi} | z \rangle \]

Unlike the Moyal product

\[ \int \phi \star \psi = \int \psi \star \phi \neq \int \phi \cdot \psi \]
The matrix base
The Wick-Voros matrix base for $\mathbb{R}^4_\theta$

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This is similar to the matrix base introduced for the Moyal product [GraciaBondia, Varilly JMP 1988] In 2-d it is based on the expansion

$$\phi(\bar{z}, z) = \sum_{pq} \tilde{\phi}_{pq} \bar{z}^p z^q, \quad p, q \in \mathbb{N} \quad \tilde{\phi}_{pq} \in \mathbb{C}$$

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Thus we generalize to 4d, $a_a, a_a^\dagger, a = 1, 2$ and use the harmonic oscillator base.
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Thus we generalize to 4d, $a_a, a_a^\dagger, a = 1, 2$ and use the harmonic oscillator base

$$a_1|n_1, n_2\rangle = \sqrt{\theta} \sqrt{n_1}|n_1 - 1, n_2\rangle, \quad a_1^\dagger|n\rangle = \sqrt{\theta} \sqrt{n_1 + 1}|n_1 + 1, n_2\rangle, $$
$$a_2|n_1, n_2\rangle = \sqrt{\theta} \sqrt{n_2}|n_1, n_2 - 1\rangle, \quad a_2^\dagger|n\rangle = \sqrt{\theta} \sqrt{n_2 + 1}|n_1, n_2 + 1\rangle$$
we get

\[ \hat{\phi} = \sum_{P, Q \in \mathbb{N}^2} \phi_{PQ} |P\rangle \langle Q| \quad \phi_{PQ} \in \mathbb{C} \quad |P\rangle := |p_1, p_2\rangle \]

\[ |P\rangle = \frac{a_1^{\dagger p_1} a_2^{\dagger p_2}}{[P! \theta P]^{1/2}} |0\rangle, \quad \forall P = (p_1, p_2) \in \mathbb{N}^2, \]

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thus the matrix base in \( \mathbb{R}^4_{\theta} \)

\[ f_{PQ}(z, \bar{z}) = \langle z_1, z_2 | \hat{f}_{PQ} | z_1, z_2 \rangle = \frac{e^{-\frac{\bar{z}_1 z_1 + \bar{z}_2 z_2}{\theta}}}{\sqrt{P!Q!\theta |P+Q|}} \bar{z}_1^{p_1} \bar{z}_2^{p_2} z_1^{q_1} z_2^{q_2} \]

with \( \hat{f}_{PQ} := |P\rangle \langle Q| \)
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$$f_{PQ}(z,\bar{z}) = \langle z_1,z_2|\hat{f}_{PQ}|z_1,z_2\rangle = \frac{e^{-\frac{\bar{z}_1 z_1 + \bar{z}_2 z_2}{\theta}}}{\sqrt{P!Q!\theta|P+Q|}} z_1^{p_1} \bar{z}_2^{p_2} z_1^{q_1} \bar{z}_2^{q_2}$$

with $\hat{f}_{PQ} := |P\rangle \langle Q|$ and usual nice properties

$$f_{MN} \ast f_{PQ}(z,\bar{z}) = \delta_{NP} f_{MQ}(z,\bar{z})$$

$$\int d^2z_1 d^2z_2 f_{PQ}(z,\bar{z}) = (\pi \theta)^2 \delta_{PQ}$$
The star product becomes a matrix product

$$\phi \star \psi(z, \bar{z}) = \sum \phi_{MN} \psi_{PQ} f_{MN} \star f_{PQ} = \sum \phi_{MP} \psi_{PQ} f_{MQ}$$

and the integral becomes a trace

$$\int \phi \star \psi \star \ldots = (\pi \theta)^2 \text{Tr} \Phi \Psi \ldots$$
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\[ n_1 + n_2 = 2j \quad n_1 - n_2 = 2m \]

with $j(j+1)$ and $m$ eigenvalues of $\hat{X}_i \hat{X}_i$ and $\hat{X}_3$ resp.
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$$| n_1, n_2 \rangle \rightarrow | j + m, j - m \rangle$$

$$\hat{f}_{NP} = | n_1, n_2 \rangle < p_1, p_2 | \rightarrow | j + m, j - m \rangle < \tilde{j} + \tilde{m}, \tilde{j} - \tilde{m} | \equiv \tilde{v}^{j\tilde{j}}_{m\tilde{m}}$$
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$\hat{f}_{NP} = |n_1, n_2><p_1, p_2| \longrightarrow |j + m, j - m><\tilde{j} + \tilde{m}, \tilde{j} - \tilde{m}| \equiv \nu^{\tilde{j}\tilde{m}}_{m\tilde{m}}$

$$f_{NP}(\bar{z}, z) \longrightarrow \nu^{\tilde{j}\tilde{m}}_{m\tilde{m}}(\bar{z}, z)$$
For this to be a base in $\mathbb{R}_\lambda^3$ we impose it to $\star$-commute with $x_0$

$$x_0 \star v^{j\tilde{j}}_{m\tilde{m}}(z, \bar{z}) - v^{j\tilde{j}}_{m\tilde{m}} \star x_0(z, \bar{z}) = \lambda(j - \tilde{j}) v^{j\tilde{j}}_{m\tilde{m}}$$
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This fixes $j = \tilde{j}$. We have then

$$\phi(x_i, x_0) = \sum_j \sum_{m, \tilde{m} = -j} \phi^j_{m\tilde{m}} v^j_{m\tilde{m}}$$

with

$$v^j_{m\tilde{m}} := v^{jj}_{m\tilde{m}} = e^{-\frac{zaza}{\theta}} \frac{z_1^{j+m}z_1^{j+\tilde{m}}z_2^{j-m}z_2^{j-\tilde{m}}}{\sqrt{(j + m)!(j - m)!(j + \tilde{m})!(j - \tilde{m})!}\theta^{4j}}$$
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For this to be a base in $\mathbb{R}_\lambda^3$ we impose it to $\ast$-commute with $x_0$

$$x_0 \ast v_{m\bar{m}}^j(z, \bar{z}) - v_{m\bar{m}}^j \ast x_0(z, \bar{z}) = \lambda (j - \tilde{j}) v_{m\bar{m}}^j$$

This fixes $j = \tilde{j}$. We have then

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The star product acquires the simple form

$$v^j_{m\bar{m}} \ast v_{n\bar{n}}^{\tilde{j}} = \delta^{jj} \delta_{\bar{m}\bar{n}} v^j_{m\bar{m}}$$

$$\int v^j_{m\bar{m}} \ast v_{n\bar{n}}^{\tilde{j}} = \pi^2 \theta^2 \delta^{jj} \delta_{\bar{m}\bar{n}} \delta_{m\bar{n}}.$$
The star product in $\mathbb{R}^3_{\lambda}$ becomes a block-diagonal infinite-matrix product.
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$$
\phi \star \psi = \sum \phi^j_{m_1 \tilde{m}_1} \psi^j_{m_2 \tilde{m}_2} \nu^j_{m_1 \tilde{m}_1} \star \nu^j_{m_2 \tilde{m}_2} = \sum \phi^j_{m_1 \tilde{m}_1} \psi^j_{m_2 \tilde{m}_2} \nu^j_{m_1 \tilde{m}_2} \delta_{\tilde{m}_1 m_2}
$$

$$
= \sum_{j, m_1, \tilde{m}_2} (\Phi^j \cdot \Psi^j)_{m_1 \tilde{m}_2} \nu^j_{m_1 \tilde{m}_2}
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$$
= \sum_{j, m_1, \tilde{m}_2} (\Phi^j \cdot \Psi^j)_{m_1 \tilde{m}_2} \nu^j_{m_1 \tilde{m}_2}
$$

the infinite matrix $\Phi$ gets rearranged into a block-diagonal form, each block being the $(2j + 1) \times (2j + 1)$ matrix

$$
\Phi^j = \{\phi^j_{mn}\}, \quad -j \leq m, n \leq j.
$$
The star product in $\mathbb{R}^3_\lambda$ becomes a block-diagonal infinite-matrix product

$$\phi \star \psi = \sum \phi^j_{m_1 \tilde{m}_1} \psi^j_{m_2 \tilde{m}_2} v^j_{m_1 \tilde{m}_1} \ast v^j_{m_2 \tilde{m}_2} = \sum \phi^j_{m_1 \tilde{m}_1} \psi^j_{m_2 \tilde{m}_2} v^j_{m_1 \tilde{m}_1} \delta_{\tilde{m}_1 m_2}$$

$$= \sum_{j, m_1, \tilde{m}_2} (\Phi^j \cdot \Psi^j)_{m_1 \tilde{m}_2} v^j_{m_1 \tilde{m}_2}$$

the infinite matrix $\Phi$ gets rearranged into a block-diagonal form, each block being the $(2j + 1) \times (2j + 1)$ matrix

$\Phi^j = \{\phi^j_{mn}\}, -j \leq m, n \leq j$.

The integral is defined through the pullback to $\mathbb{R}^4_\theta$

$$\int_{\mathbb{R}^3_\lambda} \phi := \frac{\kappa^3}{\pi^2 \theta^2} \int_{\mathbb{R}^4_\theta} \pi^*(\phi) = \kappa^3 \sum_j \text{Tr}_j \Phi^j$$

with $\text{Tr}_j$ the trace in the $(2j + 1) \times (2j + 1)$ subspace.
Summary of the first part

Patrizia Vitale

Noncommutative field theory on $\mathbb{R}^3$
The algebra $\mathbb{R}_\lambda^3$ with $\star$-product

$$\phi \star \psi(x) = \exp \left[ \frac{\lambda}{2} \left( \delta_{ij} x_0 + i \epsilon_{ijk} x_k \right) \frac{\partial}{\partial u_i} \frac{\partial}{\partial v_j} \right] \phi(u) \psi(v)|_{u=v=x}$$

The matrix base $\nu^j_{m\bar{m}}$
The algebra $\mathbb{R}^3_\lambda$ with $\star$-product

$$\phi \star \psi (x) = \exp \left[ \frac{\lambda}{2} \left( \delta_{ij} x_0 + i \epsilon_{ik} x_k \right) \frac{\partial}{\partial u_i} \frac{\partial}{\partial v_j} \right] \phi(u) \psi(v) |_{u=v=x}$$

- The matrix base $\nu^j_{m\bar{m}}$
- The integral as a trace: $\int \phi \star \psi \star \ldots \star \xi = \kappa^3 \sum_j \text{Tr}_j \Phi^j \Psi^j \ldots \Xi^j$
The scalar action
The Laplacian
All derivations of $\mathbb{R}^3_\lambda$ are inner $D_\mu \rightarrow [\chi_\mu, \cdot]_\star$ ($D_0$ is trivial because $[x_0, f]_\star = 0$ for $f \in \mathbb{R}^3_\lambda$). These generate a dynamics which is "tangent" to the fuzzy spheres of the foliation.
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Indeed, the natural Laplacian operator constructed with inner derivations $\sum_\mu [x_\mu, [x_\mu, \phi]_\star]_\star$, reduces to the usual Laplacian on the fuzzy sphere.
The scalar action

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Indeed, the natural Laplacian operator constructed with inner derivations $\sum_\mu [x_\mu, [x_\mu, \phi]]_\star$, reduces to the usual Laplacian on the fuzzy sphere.

\[ \Delta \phi = \alpha \sum_i D_i^2 \phi + \frac{\beta}{\kappa^4} x_0 \star x_0 \star \phi \]

\[ D_i = \kappa^{-2} [x_i, \cdot]_\star, \quad i = 1, \ldots, 3 \]

\[ \alpha, \beta \text{ real parameters and} \]

\[ x_0 \star \phi = x_0 \phi + \frac{\lambda}{2} x_i \partial_i \phi \]

contains the dilation operator in the radial direction.
With a slight modification the highest derivative term of the Laplacian can be made proportional to the ordinary Laplacian on $\mathbb{R}^3$, for the parameters $\alpha$ and $\beta$ appropriately chosen.

$$
\sum_i [x_i, [x_i, \phi]]_\star = \lambda^2 \left[ x^j \partial_j (x^i \partial_i \phi + x^i \partial_i \phi) \right] - \lambda^2 x_0^2 \partial^2 \phi
$$

$$
x_0 \ast x_0 \ast \phi + \frac{\lambda}{2} x_0 \ast \phi = \frac{\lambda^2}{4} \left[ x^i \partial_i (x^j \partial_j \phi + x^i \partial_i \phi) \right] + \lambda x_0 (x^i \partial_i \phi + \phi) + x_0^2 \phi
$$

With this choice, and $\alpha/\beta = -1/4$, we obtain a term proportional to the ordinary Laplacian, multiplied by $x_0^2$, plus lower derivatives.
The kinetic action is then

\[ S_{\text{kin}}[\phi] = \int \phi \star (\Delta + \mu^2) \phi \]
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As for the potential we consider a quartic interaction but every polynomial interaction can be treated easily

\[ \frac{g}{4!} \int \phi^4 = \frac{\kappa^3 g}{4!} \text{Tr} (\Phi \Phi \Phi \Phi) \]

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As for the potential we consider a quartic interaction but every polynomial interaction can be treated easily

\[ \frac{g}{4!} \int \phi^4 = \frac{k^3 g}{4!} \text{Tr} (\Phi\Phi\Phi\Phi) \]

from which we read the vertex

\[ \mathcal{V} j_1 j_2 j_3 j_4 p_1 \bar{p}_1 ; p_2 \bar{p}_2 ; p_3 \bar{p}_3 ; p_4 \bar{p}_4 = \frac{g}{4!} \delta j_1 j_2 \delta j_2 j_3 \delta j_3 j_4 \delta p_1 p_2 \delta \bar{p}_2 p_3 \delta \bar{p}_3 p_4 \delta \bar{p}_4 p_1 \]
The scalar action
The kinetic action in the matrix base
We express all operators in the matrix base.
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\[ \begin{align*}
  x_+ &= \frac{\lambda}{\theta} \bar{z}_1 z_2 = \lambda \sum_{j,m} \sqrt{(j + m)(j - m + 1)} v_{m,m-1}^j \\
  x_- &= \frac{\lambda}{\theta} \bar{z}_2 z_1 = \lambda \sum_{j,m} \sqrt{(j - m)(j + m + 1)} v_{m,m+1}^j \\
  x_3 &= \frac{\lambda}{2\theta} (\bar{z}_1 z_1 - \bar{z}_2 z_2) = \lambda \sum_{j,m} m v_{m,m}^j \\
  x_0 &= \frac{\lambda}{2\theta} (\bar{z}_1 z_1 + \bar{z}_2 z_2) = \lambda \sum_{j,m} j v_{m,m}^j
\end{align*} \]
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We express all operators in the matrix base

\[ x_+ = \frac{\lambda}{\theta} \tilde{z}_1 z_2 = \lambda \sum_{j,m} \sqrt{(j + m)(j - m + 1)} v_{m,m-1}^j \]

\[ x_- = \frac{\lambda}{\theta} \tilde{z}_2 z_1 = \lambda \sum_{j,m} \sqrt{(j - m)(j + m + 1)} v_{m,m+1}^j \]

\[ x_3 = \frac{\lambda}{2\theta} (\tilde{z}_1 z_1 - \tilde{z}_2 z_2) = \lambda \sum_{j,m} m v_{m,m}^j \]

\[ x_0 = \frac{\lambda}{2\theta} (\tilde{z}_1 z_1 + \tilde{z}_2 z_2) = \lambda \sum_{j,m} j v_{m,m}^j \]

and compute

\[ S_k[\phi] = \kappa^3 \sum \phi_{m_1 \tilde{m}_1}^{j_1} \left( \Delta(\alpha, \beta) + \mu^2 \mathbf{1} \right)_{m_1 \tilde{m}_1; m_2 \tilde{m}_2}^{j_1 j_2} \phi_{m_2 \tilde{m}_2}^{j_2} \]

\[ = \kappa^3 \text{Tr} \left( \Phi(\Delta(\alpha, \beta) + \mu^2 \mathbf{1}) \Phi \right) \]
The scalar action
The kinetic action in the matrix base

with

\[ (\Delta + \mu^2 1)^{j_1 j_2}_{m_1 \tilde{m}_1; m_2 \tilde{m}_2} = \frac{1}{\pi^2 \theta^2} \int \nu_{m_1 \tilde{m}_1}^j \ast (\Delta(\alpha, \beta) + \mu^2 1) \nu_{m_2 \tilde{m}_2}^j \]

\[ = \frac{\chi^2}{\kappa^4} \delta^{j_1 j_2} \{ \delta_{\tilde{m}_1 m_2} \delta_{m_1 \tilde{m}_2} D_{m_2 \tilde{m}_2}^{j_2} - \delta_{\tilde{m}_1, m_2+1} \delta_{m_1, \tilde{m}_2+1} B_{m_2, \tilde{m}_2}^{j_2} \]

\[ - \delta_{\tilde{m}_1, m_2-1} \delta_{m_1, \tilde{m}_2-1} H_{m_2, \tilde{m}_2}^{j_2} \} \]
The scalar action
The kinetic action in the matrix base

with

\[(\Delta + \mu^2 \mathbf{1})_{j_1j_2}^{m_1m_1 \tilde{m}_1;m_2m_2 \tilde{m}_2} = \frac{1}{\pi^2 \theta^2} \int \nu_{m_1 \tilde{m}_1}^{j_1} \ast (\Delta(\alpha, \beta) + \mu^2 \mathbf{1}) \nu_{m_2 \tilde{m}_2}^{j_2} \]

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There are non-diagonal (or non-local, in the language of matrix models) terms.
The scalar action
The kinetic action in the matrix base

with

\[(\Delta + \mu^2 \mathbf{1})_{m_1 \tilde{m}_1; m_2 \tilde{m}_2}^{j_1 j_2} = \frac{1}{\pi^2 \theta^2} \int \mathbf{v}_{m_1 \tilde{m}_1}^{j_1} \ast (\Delta(\alpha, \beta) + \mu^2 \mathbf{1}) \mathbf{v}_{m_2 \tilde{m}_2}^{j_2}\]

\[= \frac{\chi^2}{\kappa^4} \delta_{j_1 j_2} \left\{ \delta \tilde{m}_1 m_2 \delta m_1 \tilde{m}_2 D_{m_2 \tilde{m}_2}^{j_2} - \delta \tilde{m}_1, m_2 + 1 \delta m_1, \tilde{m}_2 + 1 B_{m_2, \tilde{m}_2}^{j_2} \right. \]

\[- \delta \tilde{m}_1, m_2 - 1 \delta m_1, \tilde{m}_2 - 1 H_{m_2, \tilde{m}_2}^{j_2} \}\]

There are **non-diagonal** (or non-local, in the language of matrix models) terms.

Remarks

- In the matrix base the interaction term is diagonal, the kinetic term is not (cfr. Grosse-Wulkenhaar)
The scalar action
The kinetic action in the matrix base

with

\[
\begin{align*}
(\Delta + \mu^2 1)_{j_1j_2}^{m_1\tilde{m}_1; m_2\tilde{m}_2} &= \frac{1}{\pi^2 \theta^2} \int v_{j_1}^{m_1\tilde{m}_1} \ast (\Delta(\alpha, \beta) + \mu^2 1) v_{j_2}^{m_2\tilde{m}_2} \\
&= \frac{\chi^2}{\kappa^4} \delta_{j_1j_2} \left\{ \delta_{\tilde{m}_1m_2} \delta_{m_1\tilde{m}_2} D_{m_2\tilde{m}_2}^{j_2} - \delta_{\tilde{m}_1,m_2+1} \delta_{m_1,\tilde{m}_2+1} B_{m_2,\tilde{m}_2}^{j_2} \
- \delta_{\tilde{m}_1,m_2-1} \delta_{m_1,\tilde{m}_2-1} H_{m_2,\tilde{m}_2}^{j_2} \right\}
\end{align*}
\]

There are non-diagonal (or non-local, in the language of matrix models) terms.

Remarks

- In the matrix base the interaction term is diagonal, the kinetic term is not (cfr. Grosse-Wulkenhaar)
- The action factorizes into an infinite sum of contributions

\[
S[\Phi] = \sum_{j \in \mathbb{N}} S^{(j)}[\Phi]
\]
The scalar action

The propagator

The propagator is defined as
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\[
\sum_{k,l=-j_2}^{j_2} \Delta_{mn;lk} P_{lk;rs}^{j_1j_2 \rightarrow j_1j_3} = \delta_{j_1j_2} \delta_{ms} \delta_{nr}
\]
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The kinetic term may be diagonalized in each subspace at \( j \) fixed.
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The kinetic term may be diagonalized in each subspace at \( j \) fixed. The technique is the same as in [GrosseWulkenhaar]. It uses \( m + l = n + k \) and orthogonal polynomials.
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The kinetic term may be diagonalized in each subspace at $j$ fixed. The technique is the same as in [GrosseWulkenhaar]. It uses $m + l = n + k$ and orthogonal polynomials. It turns out that the polynomials are the dual Hahn polynomials which are proportional to fuzzy spherical harmonics.
The propagator is defined as

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The kinetic term may be diagonalized in each subspace at \( j \) fixed. The technique is the same as in [GrosseWulkenhaar]. It uses \( m + l = n + k \) and orthogonal polynomials.

It turns out that the polynomials are the dual Hahn polynomials which are proportional to fuzzy spherical harmonics.

\[ (P(\alpha, \beta))_{p_1,\tilde{p}_1; p_2,\tilde{p}_2}^{j_1j_2} = \sum_{l=0}^{2j_1} \sum_{k=-l}^{l} \frac{\delta_{j_1j_2}}{(2j_1 + 1)(\frac{\lambda^2}{k^4} \gamma + \mu^2)} (Y_{lk}^{j_1})_{p_1,\tilde{p}_1}(Y_{lk}^{j_2})_{p_2,\tilde{p}_2} \]

with

\[ \gamma = (\alpha l (l+1) + \beta j^2) \]
The scalar action

The propagator

\[(Y^j)_{m\tilde{m}} = <\hat{\psi}^j_{m\tilde{m}}|\hat{Y}^j_{lk}> = \sqrt{2j + 1}(-1)^j\tilde{m}\left(\begin{array}{c|c} j & j \\ \hline m & -\tilde{m} \end{array}\right| l)\]

\[(Y_{lk}^j)_{m\tilde{m}} = (-1)^{-2j} (Y^j_{lk})_{\tilde{m}m}\]
The scalar action

The propagator

\[(\mathcal{Y}_{lk}^j)_{m\tilde{m}} = \langle \hat{\nu}_{m\tilde{m}}^{j} | \hat{Y}_{lk}^j \rangle = \sqrt{2j + 1}(-1)^{j - \tilde{m}} \begin{pmatrix} j & j \\ m & -\tilde{m} \end{pmatrix} \begin{pmatrix} l \\ k \end{pmatrix} \]

\[(\mathcal{Y}_{lk}^{j\dagger})_{m\tilde{m}} = (-1)^{-2j} (\mathcal{Y}_{lk}^j)_{\tilde{m}m} \]

Once we have the propagator and the vertex we can compute correlation functions

(1)
Planar diagram contributing to the 2-point correlation function

\[ A_{p_1\tilde{p}_1; p_2\tilde{p}_2}^{j_1j_2} = \frac{\kappa^4}{\lambda^2} \delta j_1 j_2 \delta p_1 \tilde{p}_2 \delta p_1 \tilde{p}_2 \sum_{l=0}^{2j_1} (-1)^{2j_1} \frac{2l + 1}{(2j_1 + 1)(\gamma(j_1, l; \alpha\beta) + \frac{\kappa^4}{\lambda^2} \mu^2)} \]

which is finite for all j
In the propagating (fuzzy harmonics) base

\[ \tilde{A}_{l_1 k_1; l_2 k_2}^{j_1 j_2 P} = \frac{\kappa^4}{\lambda^2} \delta_{j_1 j_2} \sum_{l=0}^{2j_1} \frac{2l + 1}{\alpha l (l + 1) + \beta j_1^2 + \frac{\kappa^4}{\lambda^2} \mu^2} (-1)^{k_2} \delta_{-k_1 k_2} \delta_{l_1 l_2}. \]

When fixing \( j_1 = j_2 = j \) and \( \beta = 0 \) we retrieve the result for the fuzzy sphere

S. Vaidya, Phys. Lett. B 512, 403 (2001); C. -S. Chu, J. Madore, H. Steinacker, JHEP 0108, 038 (2001)
One-loop calculations

![Diagram](image)

Nonplanar diagram contributing to the two-point function

\[
\mathcal{A}^{j_1 j_3 NP}_{p_1 \tilde{p}_1; p_3 \tilde{p}_3} = \frac{\kappa^4}{\lambda^2} \delta_{j_1 j_3} \sum_{l=0}^{2j_1} \frac{1}{\left( \gamma(j_1, l, \alpha, \beta) + \frac{\kappa^4}{\lambda^2} \mu^2 \right)} \times
\]

\[
\sum_{k} (-1)^{p_1 + \tilde{p}_1} \left( \begin{array}{cc} j_1 & j_1 \\ \tilde{p}_3 & -p_1 \end{array} \right) \left( \begin{array}{cc} j_1 & j_1 \\ p_3 & -\tilde{p}_1 \end{array} \right) \left( \begin{array}{c} l \\ k \end{array} \right)
\]

can be seen to be finite for all values of the indices
In the propagating base

\[
\tilde{A}_{j_1j_2 \; NP}^{l_1k_1; l_2k_2} = \frac{\kappa^4}{\lambda^2} \delta_{j_1j_2} \sum_{l=0}^{2j_1} \frac{(2j_1 + 1)(2l + 1)}{(\alpha l (l + 1) + \beta j_1^2 + \frac{k^4}{\lambda^2} \mu^2)} \times
\]

\[
(-1)^{l_1+l+2j_1-k_1} \delta_{l_1l_2} \delta_{k_1,-k_2} \left\{ \begin{array}{ccc} j_1 & j_1 & l_1 \\ j_1 & j_1 & l \end{array} \right\}
\]

In agreement with

S. Vaidya, Phys. Lett. B 512, 403 (2001); C. -S. Chu, J. Madore, H. Steinacker, JHEP 0108, 038 (2001)

for \( j_1 = j_2, \beta = 0 \)
Conclusions

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We find that it is finite at one-loop. Likely to be finite at all loops.
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Further developments
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- gauge models (in preparation with Antoine Géré and J.-C. Wallet)