Integration over Tropical Plane Curves and Ultradiscretization

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Abstract

In this article we study holomorphic integrals on tropical plane curves in view of ultradiscretization. We prove that the lattice integrals over tropical curves can be obtained as a certain limit of complex integrals over Riemann surfaces.

1 Introduction

A tropical curve is a kind of algebraic curve defined over the tropical semifield \( T = \mathbb{R} \cup \{ \infty \} \) equipped with the min-plus operations:

\[
“x + y” = \min\{x, y\}, \quad “xy” = x + y.
\]

The geometry over tropical curves was introduced by several authors [8, 11]. Among these works, the theory of integration over tropical curves was introduced by Mikhalkin and Zharkov in [9]. According to their work, a holomorphic differential on a tropical curve is defined as a global section of the real cotangent sheaf (Definition 4.1 [9]). Using the concept of tropical differentials, they derive the definition of a tropical holomorphic integral.

As one of the applications of tropical geometry, a number of authors have tried to solve problems concerning integrable systems or dynamical systems by using the method of tropical geometry [4, 6].

The bridge between integrable systems and tropical geometry is the method of ultradiscretization. Ultradiscretization is a kind of limiting procedure, which is usually described as \(- \lim_{\varepsilon \to 0^+} \varepsilon \log \cdot \). The two formulae

\[
- \lim_{\varepsilon \to 0^+} \varepsilon \log (e^{-a/\varepsilon} \cdot e^{-b/\varepsilon}) = a + b, \quad - \lim_{\varepsilon \to 0^+} \varepsilon \log (e^{-a/\varepsilon} + e^{-b/\varepsilon}) = \min [a, b], \quad (1.1)
\]

(a, b ∈ \( \mathbb{R} \)) are fundamental. Through ultradiscretization, we translate objects over \( \mathbb{C} \) into the min-plus algebra.

In this paper we study the tropicalization (or ultradiscretization) of holomorphic integrals over complex plane curves. Loosely speaking, the question we wish to answer
is: “Why is it that tropical integrals are able to tell us something about the behaviour of complex integrals?”

Many researchers have studied the relationship between analytic curves and tropical curves. Katz, Markwig and Markwig [7] studied the $j$-invariant of cubic curves and its tropicalization. For the genus zero and genus one cases, Speyer [12] proved the existence of an analytic curve tropicalization of which coincides with a given tropical curve in any ambient space. Helm and Katz [3] discussed the relationship between the tropical curve and the monodromy action on the Hodge structure.

In order to make use of the established results for hypersurfaces (for example Viro’s approximation theorem [5, 13]), we restrict ourselves to tropical integral calculus over plane curves instead of considering more general tropical curves which have been studied by many researchers. The main theorem (Theorem 4.3.1) gives us the exact relation between them. (Figure 1).

Figure 1: Theorem 4.3.1 shows us the direct relation between two kinds of “integrals”.

Note: Throughout this paper, $\varepsilon$ is a small real parameter $0 < \varepsilon < 1$. The symbol $e$ denotes the real number $e^{-1/\varepsilon}$. A formal Puiseux series with respect to $e$ is a formal sum of the form $\sum_{i=-n}^{\infty} a_i e^{i/d}$, where $a_i$ is a complex number and $d$ is a positive integer. If a formal Puiseux series converges for $\varepsilon$ sufficiently small, it is called convergent Puiseux series. $K$ denotes the field of convergent Puiseux series. The field $K$ has the standard non-archimedean valuation $\text{val}: K \to \mathbb{Q} \cup \{+\infty\}$. (val(0) = +\infty). In this paper, we assume that all the Puiseux series considered are convergent, unless otherwise stated.

2 Approximation of hypersurfaces

2.1 PL-polynomials and tropical hypersurfaces

The purpose of this section is to give a brief review of the method for the approximation of hypersurfaces of algebraic tori given in [13, §6].

Let $C$ be the complex number field and $C^n$ be the complex $n$-space. Throughout this paper, $U$ denotes the unit circle $\{x \in C | |x| = 1\}$, and $\mathbb{C}R^n$ denotes the algebraic torus $\{(x_1,x_2,\ldots,x_n) | x_1x_2\cdots x_n \neq 0\}$.

For a small positive parameter $0 < \varepsilon < 1$, define the maps $l(\varepsilon) : \mathbb{C}R^n \to \mathbb{R}^n$ and $a : \mathbb{C}R^n \to U^n (:= U \times \cdots \times U)$ by the formulae

$$l(\varepsilon)(x_1,\ldots,x_n) = (-\varepsilon \log |x_1|,\ldots,-\varepsilon \log |x_n|), \quad a(x_1,\ldots,x_n) = \left(\frac{x_1}{|x_1|},\ldots,\frac{x_n}{|x_n|}\right).$$
It is clear that the map $la(\varepsilon) : \mathbb{C}R^n \rightarrow \mathbb{R}^n \times U^n$ defined by $x \mapsto (l(\varepsilon)(x), a(x))$ is a diffeomorphism for any $\varepsilon$.

For $w \in \mathbb{R}$ and $\varepsilon > 0$, denote by $Q_{w,\varepsilon}$ the transformation $\mathbb{C}R^n \rightarrow \mathbb{C}R^n$ defined by

$$Q_{w,\varepsilon}(x_1, \ldots, x_n) = (e^{-w_1/\varepsilon}x_1, \ldots, e^{-w_n/\varepsilon}x_n), \quad \text{where } w = (w_1, \ldots, w_n).$$

We abbreviate the symbol $Q_{w,\varepsilon}$ as $Q_w$ if there is no confusion.

Let $T_w : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be the translation $x \mapsto x + w$. By definition, we can derive the relation

$$la(\varepsilon) \circ Q_w \circ la(\varepsilon)^{-1} = T_w \times \text{id}_{U^n}. \quad (2.1)$$

Our main object is an algebraic hypersurface of $\mathbb{C}R^n$ defined by a Laurent polynomial, which is an element of the ring $\mathbb{C}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$. Denote by $V_{\mathbb{C}R^n}(f)$ the algebraic set in $\mathbb{C}R^n$ defined by the Laurent polynomial $f(x_1, \ldots, x_n)$, and let $V_W(f) := V_{\mathbb{C}R^n}(f) \cap W$ for a subset $W \subset \mathbb{C}R^n$.

For $w = (w_1, \ldots, w_n) \in \mathbb{Z}^n$ and ordered $n$ variables $x = (x_1, \ldots, x_n)$, we abbreviate the monomial $x_1^{w_1} \cdots x_n^{w_n}$ as $x^w$. Let $\{V_{\mathbb{C}R^n}(f_\varepsilon)\}_\varepsilon$ be an one-parameter family of algebraic hypersurfaces, where $\varepsilon$ is a positive real parameter and $f_\varepsilon = f_\varepsilon(x_1, \ldots, x_n)$ is a Laurent polynomial with coefficients depending on $\varepsilon$. In this paper, we mainly consider the polynomials $f_\varepsilon$ of the form:

$$f_\varepsilon(x) = \sum_{w \in \mathbb{Z}^n} a_w(\varepsilon)x^w, \quad x = (x_1, \ldots, x_n), \quad a_w(\varepsilon) \in K,$$

where $a_w(\varepsilon) \equiv 0$ except for finitely many $w \in \mathbb{Z}^n$. We call this type of polynomial a parameterised L-polynomial, or pL-polynomial.

Define a tropical polynomial $\text{Val} \,(X; f_\varepsilon)$ associated with $f_\varepsilon$ by the formula

$$\text{Val}(X; f_\varepsilon) := \min_{w \in \mathbb{Z}^n} [\text{val}(a_w) + w_1 X_1 + \cdots + w_n X_n], \quad X = (X_1, \ldots, X_n).$$

A tropical hypersurface defined by $f_\varepsilon$ is a subset of $\mathbb{R}^n$ defined by

$$\left\{ P = (A_1, \ldots, A_n) \in \mathbb{R}^n \mid \text{the function } \text{Val} \,(X; f_\varepsilon) \text{ is not smooth at } X = P \right\}. \quad (2.2)$$

(For explicit examples for $n = 2$, see Section 3.1). We denote this tropical hypersurface by $TV_{\mathbb{R}^n}(f_\varepsilon)$.

Let $a = \{(X, \text{Val}(X; f_\varepsilon)) \mid X \in \mathbb{R}^n\} \subset \mathbb{R}^n \times \mathbb{R}$ be the graph of $\text{Val}(X; f_\varepsilon)$. Clearly, $a$ is the skeleton of an $(n+1)$-dimensional convex (unbounded) polytope. The tropical hypersurface $TV_{\mathbb{R}^n}(f_\varepsilon)$ is the image of the collection of $(n-1)$-faces in $a$ by the natural projection $\mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$.

### 2.2 Canonical expressions of pL-polynomials

We define $||f|| := \max_w |a_w|$ for a Laurent polynomial $f(x_1, \ldots, x_n) = \sum_w a_w x^w$.

Let $P = (A_1, \ldots, A_n)$ be a point in $\mathbb{R}^n$. There exist finitely many integer vectors $w^{(1)}, \ldots, w^{(n)} \in \mathbb{Z}^n$ such that the equations $\text{Val}(P; f_\varepsilon) = \text{val}(a_{w^{(i)}}) + w^{(i)}_1 A_1 + \cdots + w^{(i)}_n A_n$.
... + w_n^{(i)} A_n, (i = 1, 2, ..., α) hold. Let Θ(P) be a set of these vectors. Define the polynomial ˜f_ε by the formula ˜f_ε = \sum_{w \in Θ(P)} a_w(ε) x^w.

**Remark 2.2.1** Because Θ(P) = \{w\} implies the formula \text{Val}(X; f_ε) = \text{val}(a_w) + w_1 X_1 + ... + w_n X_n around X = P, the number of elements of Θ(P) is always greater than one for P ∈ TV_{R^+}(f_ε).

We note that the following two easy lemmas:

**Lemma 2.2.1** Let O = (0, ..., 0) be the origin of R^n. Then,

\text{Val}(P; f_ε) = \text{Val}(O; f_ε ◦ Q_P), \quad \text{where } P = (A_1, ..., A_n).

It is clear by the definition of \text{Val}(P; f_ε).

**Lemma 2.2.2** \((f_ε ◦ Q_P)^O = ˜f_ε ◦ Q_P.

Let \( f_ε = \sum_w a_w(ε) x^w \) and \( P = (A_1, ..., A_n) \). Then,

\((f_ε ◦ Q_P)^O = \sum_w a_w e^{A_1 w_1 + ... + A_n w_n x^w} = \sum_{w \in Θ(P)} a_w e^{A_1 w_1 + ... + A_n w_n x^w} = ˜f_ε ◦ Q_P,

where \( \check{\{ w | \text{Val}(a_w) + A_1 w_1 + ... + A_n w_n = \text{Val}(O; f_ε ◦ Q_P) \} } \).

Next we consider the decomposition:

\[ f_ε = \left( \sum_{w \in Θ(O)} + \sum_{w \notin Θ(O)} \right)(a_w(ε) x^w) = f^{O} + \sum_{w \notin Θ(O)} a_w(ε) x^w. \]

By definition, the elements of the set Θ(O) satisfy the following relation: \( w ∈ Θ(O), v \notin Θ(O) ⇒ \text{Val}(O; f_ε) = \text{val}(a_w) < \text{val}(a_v) \). Therefore the pL-polynomial \( e^{-\text{Val}(O; f_ε)} f_ε \) can be decomposed as \( e^{-\text{Val}(O; f_ε)} f_ε = f_1 + f_2 \), where

\[ f_1 = e^{-\text{Val}(O; f_ε)} f^{O} = \sum_w b(ε) x^w \text{ s.t. } \text{val}(b(ε)) = 0, \]

and \( f_2 = \sum_w b'(ε) x^w \text{ s.t. } \text{val}(b'(ε)) > 0. \)

Seeing the facts that i) \( \text{val}(a(ε)) = 0 ⇔ \lim_{ε→0^+} a(ε) ∈ \mathbb{C} \setminus \{0\} \), ii) \( \text{val}(a(ε)) > 0 ⇔ \lim_{ε→0^+} a(ε) = 0 \) for \( a(ε) ∈ K \), we can decompose uniquely the pL-polynomial \( e^{-\text{Val}(O; f_ε)} f_ε \) as:

\[ e^{-\text{Val}(O; f_ε)} f_ε = f^{O} + Δ(f_ε), \]

where \( f^{O} \) is the Laurent polynomial \( \lim_{ε→0^+} e^{-\text{Val}(O; f_ε)} f^{O} \) and \( ||Δ(f_ε)|| \) becomes to zero when \( ε → 0^+ \). Of course, \( f^{O} \) does not depend on \( ε \).

We can derive a more general formula soon:

**Proposition 2.3** Let \( f_ε \) be a pL-polynomial and \( P \) be a point in R^n. Then the pL-polynomial \( e^{-\text{Val}(P; f_ε)} f_ε ◦ Q_P \) is uniquely decomposed as:

\[ e^{-\text{Val}(P; f_ε)} f_ε ◦ Q_P = f^{P} + Δ(f_ε ◦ Q_P), \]

where \( f^{P} \) is the Laurent polynomial \( \lim_{ε→0^+} e^{-\text{Val}(P; f_ε)} f^{P} ◦ Q_P \) and \( ||Δ(f_ε ◦ Q_P)|| \) becomes zero when \( ε → 0^+ \).
To prove this, it is sufficient to substitute \( f_\varepsilon \rightarrow f_\varepsilon \circ Q_P \) to (2.3) and to use Lemmas 2.2.1 and 2.2.2.

Hearafter we denote \( R^P(\varepsilon) := e^{-\text{Val}(P;\varepsilon)} f_\varepsilon \circ Q_P \) and \( \Delta^P := \Delta(f_\varepsilon \circ Q_P) \) for simplicity. Then (2.3) is expressed as \( R^P(\varepsilon) = f^P + \Delta^P \). We call this expression the canonical expression of \( f_\varepsilon \) at \( P \). The Laurent polynomial \( f^P \) is considered as the ‘main part’ of \( R^P(\varepsilon) \). We call it the \( P \)-truncation of \( f_\varepsilon \). Note that \( R^P(\varepsilon) \) is continuous with respect to \( \varepsilon \) and \( P \), but \( f^P \) and \( \Delta^P \) are continuous with respect to \( \varepsilon \) only.

### 2.4 Approximation theorem (local version)

Let \( M \) be a smooth submanifold of a smooth manifold \( X \). A tubular neighbourhood of \( M \) in \( X \) is a submanifold \( N \subset X \) such that (i) \( M \subset \text{Int} N \), (ii) there exists a smooth retraction \( p : N \rightarrow M \) such that \( p^{-1}(x) \) is diffeomorphic to the \((\dim X - \dim M)\)-dimensional ball for any \( x \in M \). If \( X \) is equipped with a metric, a tubular neighbourhood \( N \) of \( M \) is called a tubular \( \mu \)-neighbourhood when any fibre \( p^{-1}(x) \) is contained in a ball of radius \( \mu \) centred at \( x \).

Now we introduce a flat metric into the space \( R^n \times U^n \). Let \( d_{R^n} \) be the distance function over \( R^n \) defined by Euclidean metric and \( d_{U^n} \) be the distance function over \( U^n \) defined by the standard flat metric of torus \( U \). Define the distance function \( \text{dist}_{(\varepsilon)} \) over \( R^n \times U^n \) by the formula \( \text{dist}_{(\varepsilon)} := \varepsilon^{-1} d_{R^n} + d_{U^n} \). When a tubular neighbourhood contained in \( R^n \times U^n \) is also a tubular \( \mu \)-neighbourhood with respect to \( \text{dist}_{(\varepsilon)} \), it is called a tubular \((\mu, \varepsilon)\)-neighbourhood.

**Remark 2.4.1** Readers might suspect that it would be unnecessary to define such a complicated distance \( \text{dist}_{(\varepsilon)} \). In fact, it is enough to consider the simpler distance function \( d_{R^n} + d_{U^n} \) in order to prove the approximation theorem 2.6.3. However, this is an essential procedure for the method of approximation. See Remark 2.6.1.

The main role of tubular neighbourhoods is to formalise the approximation of hypersurfaces of \( R^n \times U^n \). For later arguments, it is convenient to consider some special class of tubular neighbourhoods. The tubular neighbourhood \( p : N \rightarrow M \) is called normal if (i) any fibre \( p^{-1}(x) \) consists of segments of geodesics, (ii) any fibre \( p^{-1}(x) \) intersects with \( M \) orthogonally at \( x \). Note that two normal tubular neighbourhoods \( p : N \rightarrow M \) and \( p' : N' \rightarrow M \) coincides with each other on \( N \cap N' \).

Denote the subset \( \{ x = (x_1, \ldots, x_n) \in CR^n | 1/r < |x_i| < r, \forall i \} \) by \( D(r) \) for a positive number \( r > 1 \). Let \( \Lambda \) be a finite subset of \( Z^n \), and \( f = \sum_{w \in \Lambda} a_w x^w \) and \( g = \sum_{w \in \Lambda} \beta_w x^w \) be Laurent polynomials. Define \( F := f + g \). We consider the behaviour of two algebraic sets \( V_{CR^n}(F) \) and \( V_{CR^n}(f) \) when \( ||g|| \) goes to zero without changing \( f \). Assume that \( V_{CR^n}(f) \) is a smooth hypersurface of \( CR^n \). Standard arguments based on the Implicit Function Theorem give us the following lemma:

**Lemma 2.4.1** Fix \( r > 1 \) arbitrarily. Then, for arbitrary \( \mu_0 > 0 \), there exists a positive number \( \delta_0 \) such that:

\[
||g|| < \delta_0 \Rightarrow \left( V_{D(r)}(f) \text{ is a smooth section of a normal tubular } \mu_0 \text{-neighbourhood } N \rightarrow V_{D(r)}(F). \right)
\]
We call \( f_x \) non-singular if there exists a positive number \( \delta > 0 \) such that \( \varepsilon \in (0, \delta) \Rightarrow V_{CR^x}(f_x) \) is non-singular, and we call \( f_x \) totally non-singular if, for all \( P \in TV_{R^n}(f_x), f^P \) is non-singular (with the usual meaning).

**Remark 2.4.2** Totally non-singularity does not imply non-singularity.

Recall that \( la(\varepsilon) : CR^n \rightarrow R^n \times U^n \) is a diffeomorphism between two topological spaces. The following theorem is an essential part of the method of approximation.

**Theorem 2.4.2 (Approximation theorem at the origin)** Assume that \( f_x \) is a non-singular and totally non-singular \( pL \)-polynomial, and that the origin \( O = (0, \ldots, 0) \) of \( R^n \) is contained in \( TV_{R^n}(f_x) \). Let \( R^O f_x = f^O + \Delta^O \) be the canonical expression of \( f_x \) at \( O \). Then for arbitrary \( \mu > 0 \), there exists a positive number \( \delta > 0 \) such that

\[
||\Delta^O|| < \delta \Rightarrow
\text{there exists an open neighbourhood } W_\varepsilon \text{ of } O \in R^n \text{ such that } \\
\{W_\varepsilon \times U^n\} \cap \{la(\varepsilon)(V_{CR^x}(f_x))\} \text{ is a smooth section of } \\
a tubular } \mu, \varepsilon \text{-neighbourhood } N \rightarrow \{W_\varepsilon \times U^n\} \cap \{la(\varepsilon)(V_{CR^x}(f_x))\}. 
\]

Moreover, we can assume that the preimage of this tubular neighbourhood by \( la(\varepsilon) \) is normal.

Fix a positive number \( r > 1 \) arbitrarily and let \( \mu_0 := (2\sqrt{nr})^{-1}. \mu \). To begin with, note that we can derive soon the following relation by direct calculations:

\[
\text{dist}_{CR^n}(\alpha, \beta) < \mu_0, \quad \alpha, \beta \in D(r) \Rightarrow \text{dist}(la(\varepsilon)(\alpha), la(\varepsilon)(\beta)) < \mu.
\]

By Lemma 2.4.1 there exists a small number \( \delta > 0 \) such that \( ||\Delta^O|| < \delta \Rightarrow V_{D(r)}(f^O) \) is a smooth section of a normal tubular \( \mu_0 \)-neighbourhood \( N \rightarrow V_{D(r)}(f_x) \).

Therefore, it is sufficient to define \( W_\varepsilon := l(\varepsilon)(D(r)) \) and \( N := la(\varepsilon)(N) \). Clearly, we have \( W_\varepsilon \supseteq O \).

The slight extension of Theorem 2.4.2 can be proved:

**Corollary 2.5 (Approximation theorem (local version))** Assume \( f_x \) is a \( pL \)-polynomial satisfying the same condition stated in Theorem 2.4.2. Let \( P \) be a point in \( TV_{R^n}(f_x) \) and \( R^P f_x = f^P + \Delta^P \) be the canonical expression of \( f_x \) at \( P \). Then for arbitrary \( \mu > 0 \), there exists a positive number \( \delta > 0 \) such that

\[
||\Delta^P|| < \delta \Rightarrow
\text{there exists an open neighbourhood } W_\varepsilon \text{ of } P \in R^n \text{ such that } \\
\{W_\varepsilon \times U^n\} \cap \{(T_P \times id_{U^n}) \circ la(\varepsilon)(V_{CR^x}(f^P))\} \text{ is a smooth section of } \\
a tubular } \mu, \varepsilon \text{-neighbourhood } N \rightarrow \{W_\varepsilon \times U^n\} \cap \{la(\varepsilon)(V_{CR^x}(f_x))\}. 
\]

Moreover, if \( f^P = f^Q \ (P, Q \in TV_{R^n}(f_x)) \), we can take same \( \delta \) for these two points.

Let \( \tilde{T} := T_P \times id_{U^n} \). To prove the first statement, it is sufficient to to apply Theorem 2.4.2 to the canonical expression \( R^P(f_x) = f^P + \Delta^P \) and to use the following equation:

\[
\tilde{T} \circ la(\varepsilon)(V(R^P f_x)) = \tilde{T} \circ la(\varepsilon)(V(f_x \circ Q_P)) = \tilde{T} \circ la(\varepsilon) \circ Q_P^{-1}(V(f_x)) = la(\varepsilon)(V(f_x)).
\]

The second statement follows from the fact that \( \tilde{T} \) does not change the metric.
2.6 Approximation theorem (global version)

Theorem 2.4.2 and Corollary 2.5 show us the existence of a small region in which two varieties $V_{\mathbb{C}R^n}(f_{\varepsilon})$ and $V_{\mathbb{C}R^n}(f_P)$ are ‘similar’. We extend this region by gluing tubular neighbourhoods.

As mentioned above, the tropical hypersurface $TV_{\mathbb{R}^n}(f_{\varepsilon})$ is an union of finitely many (not necessarily bounded) $(n-1)$-faces. This tropical hypersurface is an image of $(n-1)$-faces of a convex $(n+1)$-polytope by the projection $\mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$. Therefore, naturally $TV_{\mathbb{R}^n}(f_{\varepsilon})$ has a cell decomposition. We express this decomposition as $TV_{\mathbb{R}^n}(f_{\varepsilon}) = \bigcup_{\lambda \in \Lambda} X_\lambda$ formally. The index set $\Lambda$ is finite.

Lemma 2.6.1 For $P_1, P_2 \in X_\lambda$, the truncations $f^{P_1}$ and $f^{P_2}$ coincide with each other.

Let $a \in \mathbb{R}^n \times \mathbb{R}$ be the graph of $\text{Val}(X; f_{\varepsilon})$ and $p : a \sim \mathbb{R}^n$ be a restriction of the projection $\mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$ to $a$.

Assume $f^{P_1} \neq f^{P_2}$. This means $\Theta(P_1) \neq \Theta(P_2)$. We can assume $w = (w_1, \ldots, w_n)$ $\in$ $\Theta(P_1)$ and $w \notin \Theta(P_2)$. It follows that $p^{-1}(P_1)$ is contained in the face $a \cap \{\text{Val}(X; f_{\varepsilon}) = \text{val}((a_w) + w_1X_1 + \cdots + w_nX_n)\}$, and that $p^{-1}(P_2)$ is not. Then $P_1$ and $P_2$ are not contained in a same cell.

We say that a smooth submanifold $V_1 \subset X$ is $(\mu, \varepsilon)$-approximated by another smooth submanifold $V_2 \subset X$ around a point $P \in X$ if there exists an open neighbourhood $W \ni P$ such that $W \cap V_2$ is a section of a tubular $(\mu, \varepsilon)$-neighbourhood $N \rightarrow W \cap V_1$. By Corollary 2.5 and Lemma 2.6.1 there exists positive $\delta$ (that does not depend on $P$ because $\Lambda$ is finite!) such that

$$||\Delta^P|| < \delta \Rightarrow \text{la}(\varepsilon)(V(f_{\varepsilon})) \text{ is } (\mu, \varepsilon)-\text{approximated by } (T_P \times \text{id}) \circ \text{la}(\varepsilon)(V(f^P)) \text{ around } P.$$

Lemma 2.6.2 There exists a positive number $\zeta$ such that $\varepsilon \in (0, \zeta) \Rightarrow ||\Delta^P|| < \delta$, $\forall P \in TV_{\mathbb{R}^n}(f_{\varepsilon})$.

Fix a small $0 < \varepsilon < 1$. (Then $e = e^{-1/\varepsilon} < 1$). It is sufficient to prove that $\{||\Delta^P|| \mid P \in X_\lambda\}$ has an upper bound for each $X_\lambda$. (Note that $\Lambda$ is finite). Clearly, $||\Delta^P||$ is continuous with respect to $P \in X_\lambda$.

(i) When the closure of $X_\lambda$ is compact, it is sufficient to confirm the fact that $\lim_{P \rightarrow \partial X_\lambda} ||\Delta^P||$ is finite.

(ii) When $X_\lambda$ is unbounded, we should consider the behaviour of $||\Delta^P||$ when $|P| \rightarrow \infty$. The pL-polynomial $\Delta^P$ is of the form

$$\sum_w e^{\text{val}(a_w) + w_1P_1 + \cdots + w_nP_n - \text{Val}(P; f_{\varepsilon})} x^w,$$

where $w$ runs over all the element of $\mathbb{Z}^n$ such that $\text{val}(a_w) + w_1P_1 + \cdots + w_nP_n - \text{Val}(P; f_{\varepsilon}) > 0$. Therefore, when $P$ goes infinity along $X_\lambda$, the function $\text{val}(a_w) + w_1P_1 + \cdots + w_nP_n - \text{Val}(P; f_{\varepsilon})$ should grow to infinity or be constant. Then the limit $\lim_{|P| \rightarrow \infty} ||\Delta^P||$ should be finite.

Recall that two normal tubular neighbourhoods (or their images by $\text{la}(\varepsilon)$) coincides with each other on their intersection. By gluing these local tubular neighbourhoods, we obtain a global tubular neighbourhood which gives us the approximation theorem:
Theorem 2.6.3 (Approximation theorem (global version)) Let $f_\varepsilon$ be a non-singular and totally non-singular pL-polynomial. Denote the canonical expression of $f_\varepsilon$ at $P$ by $\mathbb{R}^P f_\varepsilon = f^P + \Delta^P$. Then for arbitrary $\mu > 0$, there exists a positive number $\zeta > 0$ such that

$$\varepsilon \in (0, \zeta) \Rightarrow$$

\[
\begin{align*}
\text{There exists a tubular } (\mu, \varepsilon)\text{-neighbourhood } N &\rightarrow la(\varepsilon)(V_{\mathbb{C}\mathbb{R}^n}(f_\varepsilon)) \text{ such that} \\
(T_P \times \text{id}_{\mathbb{C}^n}) \circ la(\varepsilon)(V_{\mathbb{C}\mathbb{R}^n}(f^P)) &\text{ is a smooth section of it around } P.
\end{align*}
\]

Remark 2.6.1 For two distinct points $\alpha, \beta$ of $V(f^P)$, the distance

$$\text{dist}(\varepsilon)(la(\varepsilon)(\alpha), la(\varepsilon)(\beta))$$

does not depend on $\varepsilon$. (Recall the definition of $la(\varepsilon)$). This fact is the reason we defined the seemingly complicated distance $\text{dist}(\varepsilon)$. If this distance were not independent of $\varepsilon$, Theorem 2.6.3 would not give us any approximation of hypersurfaces.

2.7 Surjectivity theorem

In this section, we introduce the surjectivity theorem without proof. For details, the reader should consult the following references: Einsiedler, M. Kapranov, M. and Lind, D. [1]; Payne, S. [10].

Let $f_\varepsilon(x)$ be a pL-polynomial in $n$ valuables $x_1, \ldots, x_n$ and $\mathbb{R}^P f_\varepsilon = f^P + \Delta^P$ be its canonical expression at a point $P = (A_1, \ldots, A_n)$ in $TV_{\mathbb{C}\mathbb{R}^n}(f_\varepsilon)$. Then we have the following theorem.

Theorem 2.7.1 (Surjectivity theorem) Let $p = (p_1, \ldots, p_n) \in V_{\mathbb{C}\mathbb{R}^n}(f^P)$. Then there exist $n$ Puiseux series $\tilde{p}_1 = p_1 e^{A^1} + p_1' e^{A'^1} + p_1'' e^{A''} + \cdots$, $\tilde{p}_n = p_n e^{A^n} + p_n' e^{A'^n} + p_n'' e^{A''} + \cdots$ such that $(\tilde{p}_1, \ldots, \tilde{p}_n) \in V_{\mathbb{C}\mathbb{R}^n}(f_\varepsilon)$.

This theorem states information about pointwise convergence of $V_{\mathbb{C}\mathbb{R}^n}(f_\varepsilon)$. Briefly, the approximation theorem 2.6.3 deals with global information of $V_{\mathbb{C}\mathbb{R}^n}(f_\varepsilon)$ and the surjectivity theorem 2.7.1 deals with local information.

3 Plane Curves over $K$ and Tropical Curves

In the rest of this paper, we consider the two-dimensional case. In this case the varieties $V_{\mathbb{C}\mathbb{R}^2}(f_\varepsilon)$ and $V_{\mathbb{R}^2}(f^P)$ are complex curves (or Riemannian surfaces) contained in the algebraic torus $\mathbb{C}^2$. We assume $V_{\mathbb{C}\mathbb{R}^2}(f_\varepsilon)$ is non-singular and totally non-singular unless otherwise stated. Now we denote $V_{\mathbb{C}\mathbb{R}^2}(f_\varepsilon)$, $V_{\mathbb{R}^2}(f^P)$, $TV_{\mathbb{R}^2}(f_\varepsilon)$, $\ldots$ etc. by $V(f_\varepsilon)$, $V(f^P)$, $TV(f_\varepsilon)$, $\ldots$ etc. for simplicity.

Recall that $K$ is the Puiseux series field with the standard non-archimedean valuation $\text{val}$. Define the multiplicative group $R^\times := \{x \in K \mid \text{val}(x) = 0\}$. Any element $x$ of $R^\times$ tends to a finite non-zero complex number when $\varepsilon$ tends to zero. Denote the limit by $\text{top}(x)$. Let

$$f_\varepsilon(x, y) = \sum_{i=0}^{N} a_i(x) y^{N-i} = a_0(x) y^N + a_1(x) y^{N-1} + \cdots + a_N(x)$$

(3.1)
be a polynomial over $K$. As $K$ is algebraically closed, $a_i$ ($i = 0, 1, \ldots, N$) has the expression:

$$a_i(x) = c_i e^{A_i x^{m_i}} \prod_{j=1}^{d_i} (x - u_{i,j} e^{B_{i,j}}), \quad c_i, u_{i,j} \in \mathbb{R}^\times, \quad A_i, B_{i,j} \in \mathbb{Q}, \quad m_i \in \mathbb{N}. \quad (3.2)$$

Define the algebraic curve $C_e := V(f_e)$. By changing variables $x \mapsto x e^{-R}$ and $y \mapsto y e^{-R'}$ ($R, R' > 0$), we can assume $A_i, B_{i,j} > 0$ without loss of generality. In the present paper, we investigate algebraic curves over $K$ which satisfy the following genericness condition:

**Genericness condition.** The numbers $\text{top}(u_{i,j}) \forall i, j$ are all distinct.

For the proof of our main theorem 4.3.1 we will impose a slightly stricter conditions on the curve. We discuss these conditions in the appendix.

### 3.1 Examples

We first give some examples of plane curves over $K$ and their tropicalization. These examples show us how to approximately reconstruct the plane curve from its tropicalization.

#### 3.1.1 Example (I)

Let $C_e$ be the curve defined by the pL-polynomial

$$f_e(x, y) = (x + e)^2 + (x + e^2)(x + e^3)y + e^8 = 0. \quad (3.3)$$

The tropicalization of $C_e$ is

$$\text{Trop}(f_e) := \left\{ (X, Y) \in \mathbb{R}^2 \mid \min \left\{ X + 2Y, 2Y + 1, 2X + Y, X + Y + 2, Y + 5, 8 \right\} \right. \text{ is not smooth.} \right\}. \quad (3.4)$$

Let us denote this variety by $\text{Trop C}$ simply. Figure 2 shows $\text{Trop C}$. $\text{Trop C}$ has four vertices $\alpha = (1, 1), \beta = (2, 4), \gamma = (2, 3)$ and $\delta = (2.5, 3.5)$, and has one closed loop. We use the term “edge” only when it means a segment of finite length. Edges of infinite length shall be called leaves hereafter. The genus of $\text{Trop C}$ is the number of independent closed cycles over $\text{Trop C}$. In this case, genus($\text{Trop C}$) = 1.

The canonical expression of $f_e$ at $\alpha$ is $R^\alpha f_e = (x^2 + y^2 + x^2y) + (e x y + e^2 y + e^3 y + e^8)$. By the approximation theorem 2.6.3, the curve $\text{la}(\varepsilon)(C_e)$ is approximated by the translation of $\text{la}(\varepsilon)(V(f^\alpha)) = \text{la}(\varepsilon)(V(x^2 + y^2 + x^2y)) = \text{la}(\varepsilon)(V_{CR}^\infty(x + y + x^2))$ around $\alpha$. Similarly, around the points $\beta, \gamma, \delta \in \text{Trop C}$, $\text{la}(\varepsilon)(C_e)$ is approximated by $\text{la}(\varepsilon)(V(x^2y + xy + 1)), \text{la}(\varepsilon)(V(y + x^2 + x))$ and $\text{la}(\varepsilon)(V(y^2 + xy + 1))$ respectively.

Figure 3 shows four ‘local’ Riemannian surfaces $V(f^\alpha), V(f^\beta), V(f^\gamma)$ and $V(f^\delta)$.

According to the approximation theorem, we can approximately draw the form of the Riemannian surface $C_e$ for small $\varepsilon > 0$. Gluing the local data in Figure 3 along $\text{Trop C}$ (Figure 2), we can sketch $\text{la}(\varepsilon)(C_e)$ (or $C_e$ which is diffeomorphic to $\text{la}(\varepsilon)(C_e)$) as in Figure 4. Four small spheres, four long cylinders and five horns make up the figure. For later arguments, we take the completion of $C_e$ that is a compact Riemannian surface.
Figure 2: The variety $\text{Trop}C$. It consists of four vertices, four edges and four leaves. The genus of $\text{Trop}C$ is one.

Figure 3: ‘Local’ Riemannian surfaces. All of genus 0. The points $(x, y)$ satisfying $x = \infty, 0$ or $y = \infty, 0$ are described in the figure.

Figure 4: A sketch of (the completion of) $C_\varepsilon$ consisting of four spheres, four cylinders and five horns.
3.1.2 Example (II)

Let us consider
\[ C_\varepsilon : y^3 + (x + e^4)y^2 + e^2(x + e)(x + 2e)y + e^{10} = 0 \]  \hspace{1cm} (3.5)
and its tropicalization:
\[ \text{Trop} C_\varepsilon : \min \left[ 3Y, X + 2Y, 2Y + 4, 2X + Y + 2, \frac{1}{X + Y + 3, Y + 4, 10} \right] \text{ is not smooth.} \]

Trop \( C_\varepsilon \) has three vertices \( \alpha = (1, 6), \beta = (1, 3) \) and \( \gamma = (2, 2) \). The truncations associated with these points are: \( f_\alpha = (x + 1)(x + 2)y + 1, f_\beta = xy^2 + (x + 1)(x + 2) \) and \( f_\gamma = y^3 + xy^2 + 2y \) respectively. Figure 5 displays an approximate sketch of (the completion of) \( C_\varepsilon \). Although \( C_\varepsilon \) is of genus one, the genus of Trop \( C_\varepsilon \) is zero. This difference comes from the edge which connects \( \alpha \) and \( \beta \) in Trop \( C_\varepsilon \). Two long cylinders are associated with this one edge.

![Diagram of the sketch of \( C_\varepsilon \).](image)

Figure 5: The sketch of \( C_\varepsilon \).

In both examples above, the shapes of Riemannian surfaces \( V(f^P) \) (\( P \) is a vertex of Trop \( C_\varepsilon \)) are essential for drawing \( C_\varepsilon \). We denote these Riemannian surfaces by Riemannian sub-surfaces.
In example (II), there exists an edge associated to two cylinders. This property reflects the fact that: For \( P = (1, Y) \in \text{Trop} C \) \((3 < Y < 6)\), the variety \( V(f^P) = V((x+1)(x+2)) = V(x+1) \prod V(x+2) \) consists of two irreducible components. In such a case, we call the edge \( \alpha \partial \) of multiplicity 2.

Here we remark on the behaviour of sub-surfaces. It may happen that a sub-surface is reducible or of genus more than 0. In these cases, the genus of \( \text{Trop} C \) becomes inferior to the genus of \( C_{\tau} \).

The following definition is given for Section 4.

**Definition 3.1.1** A curve over \( K \) is called non-degenerate if all of its Riemannian sub-surfaces are irreducible and of genus 0.

### 3.2 Multiplicity of tropical edges

In this section, we define the vertical thickness, the horizontal thickness and the multiplicity of tropical edges.

#### 3.2.1 Vertical and horizontal thickness

The ‘Val’ function associated with the pL-polynomial \( a_i(x) \) \((3.2)\) is expressed as

\[
\text{Val} (X; a_i) = A_i + m_i X + \sum_{j=1}^{d_i} \min [X, B_{i,j}], \quad (i = 0, 1, \ldots, N). \tag{3.6}
\]

Let \( F_i(X) := \text{Val} (X; f_z) \). Then the defining condition of \( \text{Trop} C \) is expressed as

\[
\text{Trop} C : \quad \min_{i=0, \ldots, N}[F_i(X) + (N - i)Y] \quad \text{is not smooth.} \tag{3.7}
\]

Of course, we have \( \text{Val} (X, Y; f_z) = \min_{i=0, \ldots, N}[F_i(X) + (N - i)Y] \). Define the domain \( \mathcal{D}_i \subset \mathbb{R}^2 \) \((i = 0, 1, \ldots, N)\) by

\[
\mathcal{D}_i := \{ (X, Y) \mid \text{Val} (X, Y; f_z) = F_i(X) + (N - i)Y \}.
\]

The domains \( \mathcal{D}_i \) separate the plane \( \mathbb{R}^2 \) into at most \( N + 1 \) pieces: \( \mathbb{R}^2 = \bigcup_{i=0}^{N} \mathcal{D}_i \). If \( i < j \), \((x, y_i) \in \mathcal{D}_i \) and \((x, y_j) \in \mathcal{D}_j \) then \( y_i \leq y_j \). \((\ast)\) From the definition of \( \mathcal{D}_i \) and \( \mathcal{D}_j \), we have \( F_i(x) + (N - i)y_i \leq F_j(x) + (N - j)y_j \) and \( F_j(x) + (N - j)y_j \leq F_i(x) + (N - i)y_i \) if \((j - i)y_i \leq (j - i)y_j \). It follows that \((j - i)y_i \leq (j - i)y_j \) \((\ast)\). Let us define the piecewise linear function \( \mathcal{N}_i(X) \) \((i = 1, 2, \ldots, N)\) defined by the relation \( \mathcal{N}_i(X) = \min_{j \geq i} \{ Y \mid (X, Y) \in \mathcal{D}_i \} \). Note that \( \mathcal{N}_{i+1}(X) \geq \mathcal{N}_i(X) \) for all \( X \).

Using the function \( \mathcal{N}_i(X) \), we obtain another expression of \( \text{Trop} C \). We formally regard \( \mathcal{N}_{N+1}(X) := +\infty \) and \( \mathcal{N}_0(X) := -\infty \) for any \( X \).

**Proposition 3.3** Let \( L_{i,j} \) be the vertical edge which connects

\[
(B_{i,j}, \mathcal{N}_i(B_{i,j})) \quad \text{and} \quad (B_{i,j}, \mathcal{N}_{i+1}(B_{i,j})).
\]

Then, the set \( \left( \bigcup_{i=1}^{N} \{ Y = \mathcal{N}_i(X) \} \right) \cup \left( \bigcup_{i,j} L_{i,j} \right) \) coincides with \( \text{Trop} C \).

**Remark 3.3.1** If \( \mathcal{N}_i(B_{i,j}) = \mathcal{N}_{i+1}(B_{i,j}) \), then \( L_{i,j} = \{ \text{a point} \} \).
Let $G_i := \{(X, Y) \in \mathbb{R}^2 \mid Y = N_i(X)\}$. Because it is obvious, by definition, that $\text{Trop} C \supset \left( \bigcup_{i=1}^N G_i \right)$, it is sufficient to prove $\text{Trop} C \setminus \left( \bigcup_{i=1}^N G_i \right) = \left( \bigcup_{i,j} L_{i,j} \right)$.

Choose a connected component $O$ of $\mathbb{R}^2 \setminus \left( \bigcup_{i=1}^N G_i \right)$. The domain $O$ is contained in $\mathcal{D}_i$ for some $i$, where $\mathcal{D}_i$ is the set of inner points of $\mathcal{D}_i$. For any point $(X, Y)$ in $O$, it follows that $\text{Val}(X, Y; f_x) = F_i(X) + (N-i)Y$. Then, a point $(X, Y) \in \text{Trop} C \cap O$ must be a point at which $F_i(X)$ is not smooth. Because the function $F_i(X)$ is not smooth if $X = B_{i,j}$, we conclude that $\text{Trop} C \setminus \left( \bigcup_{i=1}^N G_i \right)$ consists of the sets $\{X = B_{i,j}\} \cap O = L_{i,j}$.

By use of Proposition 3.3, we introduce the vertical thickness of edges.

**Definition 3.3.1** Let $E \subset \text{Trop} C$ be an edge. We call the number $\sharp \{i \mid E \subset G_i\}$ vertical thickness of $E$. In other words, the vertical thickness of an edge $E$ is a difference of the maximum element and the minimum element of the set

$$\{w_2 \in \{0, \ldots, N\} \mid \text{Val}(X, Y; f_x) = \text{val}(a_w) + w_1X + w_2Y, \ \forall (X, Y) \in E\}.$$ 

For example, the vertical thickness of a vertical edge is 0.

**Lemma 3.3.1** The vertical thickness of $E$ equals to the degree of the projection

$$V(f^P) \to \mathbb{C} \setminus \{0\}; \ (x, y) \mapsto x, \quad P \in \text{Int} E.$$ 

First note that the truncation $f^P \ (P \in \text{Int} E)$ is determined uniquely by Lemma 2.6.1. The Laurent polynomial $f^P$ is of the form $f^P = \sum c_w x^{w_1} y^{w_2}$, where $w = (w_1, w_2)$ runs over the set $\{w \mid \text{Val}(X, Y; f_x) = \text{val}(a_w) + w_1X + w_2Y, \ \forall (X, Y) \in E\}$ and $c_w$ is a non-zero complex number for any $w$. Because the degree of the projection $x : V(f^P) \to \mathbb{C} \setminus \{0\}$ equals to the difference between the maximum degree and the minimum degree w.r.t. $y$ consisted in $f^P$, the desired result is obtained. 

Let $C^T \subset V(f^P(y, x))$ be the curve obtained from $C^P$ by switching the $x$ and $y$ coordinates. For an edge $E \subset \text{Trop} C$, we define the horizontal thickness of $E$ by the vertical thickness of $E^T \subset \text{Trop} C^T$ which is the image of $E$ by the morphism $(X, Y) \mapsto (Y, X)$. For example, the horizontal thickness of horizontal edges is 0.

3.3.1 Multiplicity

As in example (I) in Section 3.1 above, it may happen that more than one cylinders (or horns) are associated with one edge (or one leaf).

**Definition 3.3.2** Let $E \subset \text{Trop} C$ be an edge (resp. a leaf). The multiplicity of $E$ is the number of cylinders (resp. horns) associated with $E$.

Let $E \subset \text{Trop} C$ be an edge (resp. a leaf) of multiplicity $m$, and $P$ be a point in $\text{Int} E$. By the approximation theorem 2.6.3 the variety $V(f^P)$ must be decomposed into $m$ irreducible components: $V(f^P) = V(f^P_1) \cdots \bigcup V(f^P_m)$, where
We often regard the edge $E$ as the union of distinguished $m$ edges: $E = E_1 \sqcup \cdots \sqcup E_m$ (See Figure 5 in Section 4), where each $E_i$ corresponds to $V(f_i^P)$ ($P \in \text{Int} E$) and is taken to be of multiplicity one.

Define the vertical thickness of $E_i$ as the degree of the projection: $V(f_i^P) \to \mathbb{C} \setminus \{0\}$; $(x, y) \mapsto x$. Denote the vertical thickness of $E_i$ by $q_i$. Naturally, $q_1 + \cdots + q_m$ equals to the vertical thickness of $E$. Similarly, we can define the horizontal thickness of $E_i$ as the vertical thickness of $E^T_i$.

The following definition is given for the next section.

**Definition 3.3.3** Let $L_{i,j}$ be the vertical edge which connects $(B_{i,j}, N_i(B_{i,j}))$ and $(B_{i,j}, N_{i+1}(B_{i,j}))$. The ceiling of $L_{i,j}$ is the set $G_{i+1} = \{(X, Y) | Y = N_{i+1}(X)\}$ and the floor of $L_{i,j}$ is the set $G_i = \{(X, Y) | Y = N_i(X)\}$.

### 3.4 Regularity of tropical curves

Let $P = (X_0, Y_0)$ be a point in $\text{Trop} C = TV(f_\varepsilon)$, where $f_\varepsilon = \sum_{w=(w_1, w_2) \in \mathbb{Z}^2} a_w x^{w_1} y^{w_2}$.

Considering the set $\Theta(P) = \{w \in \mathbb{Z}^2 | \text{Val} (X_0, Y_0; f_\varepsilon) = \text{Val} (a_w) + w_1 X_0 + w_2 Y_0\}$ (Section 2.2), we have $\sharp \Theta(P) \geq 2$ (Remark 2.2.1). More precisely, the number of elements of $\Theta(P)$ satisfies the following inequalities:

\begin{align}
\text{i) } & P \text{ is an inner point of some edge of } \text{Trop} C \Rightarrow \sharp \Theta(P) \geq 2, \\
\text{ii) } & P \text{ is a vertex of } \text{Trop} C \Rightarrow \sharp \Theta(P) \geq 3.
\end{align}

These inequalities reflect the fact that the intersection of (generic) two planes is a line and the intersection of (generic) three planes is a point in $\mathbb{R}^3$.

In the present paper, we often assume some genericness condition on the defining polynomial of $\text{Trop} C$.

**Definition 3.4.1** The tropical plane curve $TV(f_\varepsilon)$ is regular if the equalities in (3.8) hold.

### 4 Integration Theory

The integration theory over tropical curves was first introduced in [9]. Hereafter we will show that the ultradiscrete limit of holomorphic integrals over $C_\varepsilon$ coincides with the holomorphic integral over $\text{Trop} C$ (for $C_\varepsilon$ of some type).

#### 4.1 Definition of the holomorphic integral over tropical curves

In this section, we give a brief introduction to integration theory over tropical curves, following [4, 9].

We first equip $\text{Trop} C$ with the structure of a metric graph. Let $E$ be an edge of $\text{Trop} C$. $E$ has the expression $E = \{(X_0, Y_0) + t(u, v) | 0 \leq t \leq \ell\}$, $(u, v \in \mathbb{Z})$. It can be assumed that $u$ and $v$ are coprime without loss of generality. We call the vector $(u, v)$ the primitive vector of $E$. We define a tropical length $\ell_T$ of $E$ by $\ell_T(E) := \ell$. With this length the tropical curve $\text{Trop} C$ becomes a metric graph.
The metric on Trop $C$ defines a symmetric bilinear form $\ell_T(\cdot, \cdot)$ on the space of paths in Trop $C$. For this, we define $\ell_T(\Gamma, \Gamma) := \ell_T(\Gamma)$ for non-self-intersecting path $\Gamma$, and extend it to any pairs of paths bilinearly. Figure 6 shows an example of $\ell_T(\cdot, \cdot)$. Note that the number $|\ell_T(\Gamma_1, \Gamma_2)|$ equals the tropical length $\ell_T(\Gamma_1 \cap \Gamma_2)$. This bilinear form gives the tropical length of intersection of two paths up to sign.

Figure 6: Example of a metric graph. We have $\ell_T(\Gamma_1, \Gamma_1) = \ell_1 + \ell_3 + \ell_5 + \ell_6$, $\ell_T(\Gamma_2, \Gamma_2) = \ell_2 + \ell_4 + \ell_6 + \ell_7$ and $\ell_T(\Gamma_1, \Gamma_2) = \ell_T(\Gamma_2, \Gamma_1) = -\ell_6$.

Let $g$ be the genus of Trop $C$ and choose a homology basis $T_{\beta_1}, \ldots, T_{\beta_g} \in H_1(\text{Trop } C; \mathbb{Z})$.

A tropical period matrix $B_T$ is the $g \times g$ matrix defined by $B_T := (\ell_T(T_{\beta_i}, T_{\beta_j}))_{i,j}$.

Since $\ell_T$ is non-degenerate, $B_T$ is symmetric and positive definite.

Here, we note the relation between the tropical length, the multiplicity and the vertical thickness of the edge in Trop $C$. Let

$$E := \{((1-t)X_0 + tX_1, (1-t)Y_0 + tY_1) \mid 0 \leq t \leq 1, X_0 \preceq X_1\}$$

be a non-vertical edge of vertical thickness $q$ and of horizontal thickness $w$. From the definition of vertical thickness and horizontal thickness, it follows that $E$ is part of the line defined by the equation: $aX + bY + c = (a + w)X + (b \pm q)Y + c'$ (see Definition 3.3.1).

**Lemma 4.1.1** Let $\xi := \gcd(q, w)$. Then the tropical length of $E$ is expressed as:

$$\ell_T(E) = \frac{\xi}{q}(X_1 - X_0).$$

We can assume $(X_0, Y_0) = (0, 0)$ by translation. Then we obtain $wX_1 = \pm qY_1$. Let $\eta := \gcd(X_1, Y_1)$ and

$$\begin{cases} X_1 = x\eta, \\ Y_1 = y\eta \end{cases}, \quad \begin{cases} q = \mu\xi, \\ w = \nu\xi \end{cases}.$$

Then, we conclude $\mu = x$ and $\nu = y$ by elementary arguments. From the definition of the tropical length we obtain

$$\ell_T(E) = \gcd(X_1, Y_1) = \eta = \frac{X_1}{x} = \frac{\xi}{q}X_1.$$
Next, we introduce affine transformations of tropical curves. Let \( T \) be a tropical curve in \( \mathbb{R}^2 \). For a \( 2 \times 2 \) matrix \( \theta = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \), such that \( \alpha \delta - \beta \gamma = 1 \), we have a new set
\[
\mathcal{U} := \{ \theta(X,Y)^T \in \mathbb{R}^2 \mid (X,Y) \in T \}.
\]
For the complex curve \( \{ f_\varepsilon(x,y) = 0 \} \), this transformation associates with the translation \( x \mapsto x^\delta y^{-\beta}, y \mapsto x^{-\gamma} y^\alpha \), which is invertible and holomorphic. In particular, \( \mathcal{U} \) is also a tropical curve associated with \( f_\varepsilon(x^\delta y^{-\beta}, x^{-\gamma} y^\alpha) = 0 \).

Concerning such affine translations, we have the following fundamental result.

**Proposition 4.2** Let \( \theta \in M_2(\mathbb{Z}) \) be a \( 2 \times 2 \) matrix with \( \det \theta = 1 \). Then

(i) The length of an edge is \( \theta \)-invariant.

(ii) For an edge \( L \in \mathbb{R}^2 \), there exists \( \theta \in M_2(\mathbb{Z}) \) such that \( \theta \cdot L \) is vertical.

(i) Let \( (u,v) \) be a primitive vector of the edge \( L \in \mathbb{R}^2 \). It is sufficient to prove that the image \( \theta(u,v)^T = (\alpha u + \beta v, \gamma u + \delta v) \) is also primitive. For this, we have only to prove \( \gcd(\alpha u + \beta v, \gamma u + \delta v) = 1 \). This can be proved by elementary methods and we omit the proof.

(ii) For the primitive vector \( (u,v) \) of \( L \), it is enough to define \( \theta = \begin{pmatrix} v & -u \\ w & z \end{pmatrix} \) such that \( vz + wu = 1 \).

**Remark 4.2.1** The vertical and the horizontal thickness of an edge depend on the coordinate functions \( X \) and \( Y \).

### 4.3 Main theorem

Now we proceed to integration theory over \( C_\varepsilon = V(f_\varepsilon) \). In order to make the problem easier, we deal only with the case where:

i) \( f_\varepsilon \) is non-singular and totally non-singular (Section [2],

ii) \( C_\varepsilon \) is non-degenerate,

iii) \( \text{Trop} \ C \) is regular.

Conditions i)–iii) and the *genericness condition* (Section [3]) lead to the following properties (see Appendix):

iv) for each edge \( E \), \( m = \gcd(q, w) \), where \( m, q, w \) are respectively the multiplicity, the vertical thickness and the horizontal thickness of \( E \),

v) for each edge \( E = E_1 \sqcup \cdots \sqcup E_m \), \( q_1 = \cdots = q_m \), \( w_1 = \cdots = w_m \), where \( q_i, w_i \) are the vertical thickness and the horizontal thickness of \( E_i \).
Remark 4.3.1 The curves $C$ introduced in Examples (I) and (II) in Section 3.1 satisfy these conditions.

We say that $C_\varepsilon$ has a good tropicalization if $C_\varepsilon$ and $\text{Trop} C$ satisfy the genericness condition in Section 3 and conditions i)–iii) above.

Remark 4.3.2 The conditions i) and ii) are necessary conditions to construct the integration theory. The condition iii) is required for simplicity of the calculations. (The author is not sure whether the condition iii) can be omitted.)

By the approximation theorem 2.6.3 there exists small $\zeta > 0$ such that all $C_\varepsilon$ ($\varepsilon \in (0, \zeta)$) are homotopic. Hereafter $\varepsilon$ denotes a small real number which satisfies $0 < \varepsilon < \zeta$ unless otherwise is stated.

Let $g$ be the genus of $C_\varepsilon$. Define homology cycles $\alpha_1, \alpha_2, \ldots, \alpha_g \in H_1(C_\varepsilon; \mathbb{Z})$ as in Figure 7. Any cycle is associated with a long cylinder connects two sub-surfaces. Next define the homology cycles $\beta_1, \beta_2, \ldots, \beta_g \in H_1(C_\varepsilon; \mathbb{Z})$ such that the intersection index $\alpha_i \circ \beta_j$ is $\delta_{i,j}$, in a canonical way. We assume the cycles $\alpha_i = \alpha_i(\varepsilon)$ and $\beta_j = \beta_j(\varepsilon)$ are continuous with respect to $\varepsilon$. Denote the normalised holomorphic differentials over $C_\varepsilon$ by $\omega_1, \omega_2, \ldots, \omega_g$ ($\int_{\alpha_i} \omega_j = \delta_{i,j}$). The period matrix $B_\varepsilon$ of the Riemannian surface $C_\varepsilon$ is the $g \times g$ matrix defined by $B_\varepsilon := (\int_{\beta_i} \omega_j)_{i,j}$.

Figure 7: An example of the definition of $\alpha_1, \ldots, \alpha_g; \beta_1, \ldots, \beta_g$ in $H^1(C; \mathbb{Z})$. We always consider that an $\alpha$-cycle surrounds a cylinder.

Now we state the main theorem of this paper. Take a homology basis $T_{\beta_1}, \ldots, T_{\beta_g} \in H_1(\text{Trop} C; \mathbb{Z})$ associated with a homology basis $\beta_1, \ldots, \beta_g \in H_1(C; \mathbb{Z})$. If more than one cylinder can be associated with one tropical edge, we say that hidden edges and cycles exist here (see Figure 8). When we define homology cycles in
If there exist an edge with multiplicity $m > 1$, we regard this edge as the union of $m$ edges of multiplicity one. $H_1(\text{Trop } C; \mathbb{Z})$, these edges must be distinguished. Recall that we regard an edge $E$ of multiplicity $m$ as the union: $E = E_1 \sqcup E_2 \sqcup \cdots \sqcup E_m$. Let $B_T := (t_T(T_{\beta_i}, T_{\beta_j}))_{i,j}$ be a period matrix of $\text{Trop } C$.

**Theorem 4.3.1** If $C_\varepsilon$ has a good tropicalization, then

$$B_\varepsilon \sim -\frac{1}{2\pi i \varepsilon} B_T \quad (\varepsilon \to 0).$$

The rest of this paper is devoted to the proof of this theorem.

**Preliminaries for integral calculus**

In this section, we study integral calculus over $V(f_\varepsilon)$ and the asymptotic behaviour of integrals when $\varepsilon$ tends to zero. For a pL-polynomial $f_\varepsilon$, let us define the variety $\tilde{V}(f) := \{(x, y, \varepsilon) \in \mathbb{C} \times \mathbb{R} | f_\varepsilon(x, y) = 0\}$. Denote the natural embedding $\mathbb{C} \to \mathbb{C} \times \mathbb{R}$; $(x, y) \mapsto (x, y, \varepsilon)$ by $j_\varepsilon$. Naturally it follows that $j_\varepsilon^{-1}(\tilde{V}(f)) = V(f_\varepsilon)$.

Let $\mathcal{U} \subset \tilde{V}(f)$ be a simply connected domain and $\omega_\varepsilon$ be a 1-form over $\mathcal{U}$ such that i) $\omega_\varepsilon$ is a holomorphic differential over $j_\varepsilon^{-1}(\mathcal{U}) = \mathcal{U} \cap V(f_\varepsilon)$, and ii) $\omega_\varepsilon$ is continuous with respect with $\varepsilon$. By elementary arguments in complex analysis, we can prove the existence of a primitive function $\Omega_\varepsilon$ of $\omega_\varepsilon$. The integration of $\omega_\varepsilon$ along a smooth path $[0, 1] \to j_\varepsilon^{-1}(\mathcal{U})$; $\theta \mapsto \gamma_\varepsilon(\theta)$ is defined by the formula $\int_{\gamma_\varepsilon} \omega_\varepsilon := \Omega_\varepsilon(\gamma_\varepsilon(1)) - \Omega_\varepsilon(\gamma_\varepsilon(0))$.

Let $(0, 1) \times [0, 1] \to \tilde{V}(f)$; $(\varepsilon, \theta) \mapsto \gamma_\varepsilon(\theta)$ be a smooth map such that $\gamma_\varepsilon(\theta) \in V(f_\varepsilon)$ for all $\varepsilon$ and $\theta$. Our aim in this section is to evaluate the asymptotic behaviour of the value $\int_{\gamma_\varepsilon} \omega_\varepsilon$ when $\varepsilon$ goes to zero.

Due to the surjectivity theorem, more detailed information can be added to the definition of a path on $\tilde{V}(f)$. Let $v_1 = (X_1, Y_1)$ and $v_2 = (X_2, Y_2)$ be two points in $\text{Trop } C = TV(f_\varepsilon)$ and $R^{v_i} f_\varepsilon = f^{v_i} + \Delta^{v_i}$ ($i = 1, 2$) be the canonical expressions of $f_\varepsilon$ at $v_i$. 

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Consider a path $\gamma' : [0, 1] \to V(f^{\varepsilon})$ on the variety $V(f^{\varepsilon})$. By the surjectivity theorem, there exists the smooth map $(0, 1) \times [0, 1] \to \tilde{V}(f) : (\varepsilon, \theta) \mapsto \gamma(\theta)$ such that

$$
\gamma_{\varepsilon}(\theta) = (x_{\varepsilon}(\theta) e^{X_1}, y_{\varepsilon}(\theta) e^{Y_1}) \in V(f_{\varepsilon}), \quad \text{and} \quad \lim_{\varepsilon \to 0^+} (x_{\varepsilon}(\theta), y_{\varepsilon}(\theta)) \to (x', y'), \quad (\forall \theta),
$$

where $x_{\varepsilon}(\theta), y_{\varepsilon}(\theta) \in \mathbb{R}^x$ for any $\theta$. We often abbreviate the above notation as $\gamma(\theta) = (x e^{X_1}, y e^{Y_1})$, $x, y \in \mathbb{R}^x$ if there is no chance of confusion.

Similarly, if $\tilde{V}(f)$ is connected, for two points $(x_1, y_1) \in V(f^{\varepsilon_1})$ and $(x_2, y_2) \in V(f^{\varepsilon_2})$, there exists a smooth map $(0, 1) \times [0, 1] \to \tilde{V}(f) : (\varepsilon, \theta) \mapsto \gamma_{\varepsilon}(\theta)$ such that

$$
\gamma_{\varepsilon}(0) = (x_1 e^{X_1}, y_1 e^{Y_1}), \quad \gamma_{\varepsilon}(1) = (x_2 e^{X_2}, y_2 e^{Y_2}),
$$

and $\lim_{\varepsilon \to 0^+} (x_{\varepsilon}, y_{\varepsilon}) \to (x_i, y_i)$. We often use the notation $\int_{\gamma(0)}^{\gamma(1)} \omega$ instead of $\int_{\gamma_{\varepsilon}} \omega$ if the meaning is clear.

### 4.3.1 Approximation of integral calculus

In the rest of this paper, $b_{i, j}$ denotes $\int_{\beta_{i, j}} \omega_i$.

Let $\mathcal{S}$ be the set of 1-forms over $\tilde{V}(f_{\varepsilon})$ such that i) $\omega_{\varepsilon} \in \mathcal{S}$ is a meromorphic differential over $j_{\varepsilon}^{-1}(\tilde{V}(f)) = V(f_{\varepsilon})$, ii) $\omega_{\varepsilon} \in \mathcal{S}$ is continuous with respect to $\varepsilon$.

Define the subsets $\mathcal{M}$ and $\mathcal{F}$ by the formulae

$$
\mathcal{M} := \{ \omega_{\varepsilon} \in \mathcal{S} \mid \int_{\beta_{i, j}} \omega_{\varepsilon} = \sum_{i, j = 1}^g c_{i, j}(\varepsilon) b_{i, j}, \text{ where } -\lim_{\varepsilon \to 0^+} \varepsilon \log c_{i, j} > 0, \ \forall i, j, k \},
$$

$$
\mathcal{F} := \{ \omega_{\varepsilon} \in \mathcal{S} \mid \lim_{\varepsilon \to 0^+} |\int_{\beta_{i, j}} \omega_{\varepsilon}| < +\infty, \ \forall i \}.
$$

It is clear that they are $\mathbb{R}^x$-vector spaces.

**Remark 4.3.3** If the main theorem 4.3.1 is true, is follows that $\mathcal{M} \subset \mathcal{F}$.

We say a differential over Riemannian surface is of the first kind if it has no singularity, of the second kind if it has poles without residue and of the third kind if it has poles with non-zero residue. For the proof of the main theorem, we start with differentials of the third kind. Let $P_+, P_- \in \text{Crit}(f)$. A smooth curve $C_\varepsilon$ has the normalised differential of the third kind $\omega_{P_+ - P_-} = \omega_{P_+ - P_-}(\varepsilon)$, possessing simple poles with residue $+1/(2\pi)$ at $P_+$ and $-1/(2\pi)$ at $P_-$, and holomorphic over $C_\varepsilon \setminus \{ P_+, P_- \}$ satisfying $\int_{C_\varepsilon} \omega_{P_+ - P_-} = 0$ ($i = 1, 2, \ldots, g$). Generally, for $n$ points $P_1, \ldots, P_n \in C_\varepsilon$ and complex numbers $c_1 + \cdots + c_n = 0$, there is a unique normalised differential $\omega_{c_1 P_1 + \cdots + c_n P_n}$, with residue $c_i/(2\pi i)$ at $P_i$ ($i = 1, \ldots, n$).

Recall that a point in $\{ x = \infty, 0 \} \cup \{ y = \infty, 0 \}$ is associated with a leaf in $\text{Trop} C$. (cf. Section 3.1). To $\gamma' : [0, 1] \to V(f^{\varepsilon})$ is continuous with respect to $\varepsilon$. A smooth curve $C_{\varepsilon}$ has the normalised differential of the third kind $\omega_{P_+ - P_-} = \omega_{P_+ - P_-}(\varepsilon)$, possessing simple poles with residue $+1/(2\pi)$ at $P_+$ and $-1/(2\pi)$ at $P_-$, and holomorphic over $C_{\varepsilon} \setminus \{ P_+, P_- \}$ satisfying $\int_{C_{\varepsilon}} \omega_{P_+ - P_-} = 0$ ($i = 1, 2, \ldots, g$). Generally, for $n$ points $P_1, \ldots, P_n \in C_{\varepsilon}$ and complex numbers $c_1 + \cdots + c_n = 0$, there is a unique normalised differential $\omega_{c_1 P_1 + \cdots + c_n P_n}$, with residue $c_i/(2\pi i)$ at $P_i$ ($i = 1, \ldots, n$).

Recall that a point in $\{ x = \infty, 0 \} \cup \{ y = \infty, 0 \}$ is associated with a leaf in $\text{Trop} C$. (cf. Section 3.1).

**Lemma 4.3.2** Let $P_+, P_- \in C_{\varepsilon}$ are two points in $\{ x = \infty \} \cup \{ x = 0 \} \cup \{ y = \infty \} \cup \{ y = 0 \}$ which are associated with the same leaf in $\text{Trop} C$. Then it follows that $\omega_{P_+ - P_-} \in \mathcal{M} + \mathcal{F}$.

Let $L \subset \text{Trop} C$ be the leaf which includes $P_+$. By rotation, $L$ can be assumed to be vertical tending to $Y = +\infty : L = \{ (B, Y) \mid Y \geq \mathcal{N}(B) \}$, where $\mathcal{N}(X)$ is a tropical function as defined in Section 4. Hereafter, we denote this function by $\mathcal{N}(X)$. 

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Let \( f_\varepsilon(x, y) = \sum_{i=0}^{N} a_i(x) y^{N-i}, a_N(x) = e^{A x^m} \prod_{j=1}^{d} (x - u_j e^{B_j}), (c, u_j \in \mathbb{R}, m \in \mathbb{N}, A, B_j \in \mathbb{Q}_{>0}) \) be the defining polynomial of \( C_\varepsilon \). By the above assumption the \( y \)-coordinate of \( P_\pm \) equals 0. Then the \( x \)-coordinate of \( P_\pm \) can be written as \( x(P_+) = u_{k_1} e^B, x(P_-) = u_{k_2} e^B \) for some \( k_1, k_2 \in \{1, 2, \ldots, d\} \) such that \( B = B_{k_1} = B_{k_2} \).

Now we proceed to integration calculus. Consider the polynomial \( \phi(x) \) defined by

\[
\frac{1}{2\pi i} \left\{ \frac{1}{x - u_{k_2} e^B} - \frac{1}{x - u_{k_1} e^B} \right\} = \frac{\phi(x)}{a_N(x)}.
\]

Define the new differential \( \omega_i \) by

\[
\omega_i := \frac{\phi(x) \, dx}{y \, f_y(x, y)},
\]

where \( f_y(x, y) := \partial_y f_x(x, y) = \sum_{i=0}^{N-1} (N - i) a_i(x) y^{N-i-1} \).

The singularity of \( \omega_i \) must be contained in \( \{x = \infty\} \cup \{y = \infty, 0\} \). (: \( dx/f_y \) is always holomorphic over smooth plane curves).

The following sublemmas describes the behaviour of the differential \( \omega_i \).

**Sublemma 1** The distribution of residues of \( \omega_i \) is given as follows:

i) \( \text{Res}_{P_+} \omega_i = \frac{1}{2\pi i} + o(e^0) \),

ii) \( \text{Res}_{P_-} \omega_i = -\frac{1}{2\pi i} + o(e^0) \),

iii) \( \text{Res}_P \omega_i = o(e^0), P \neq P_\pm \).

i)–ii) Let \( v \in \text{Trop} C \) be the vertex which is at the foot of \( L \) and let \( \Omega \) be the sub-surface associated with \( v \). We take small cycles \( \gamma_\pm \subset \Omega \) which loop around \( P_\pm \) anti-clockwise and which satisfy

\[
(x, y) \in \gamma_\pm \Rightarrow x = re^B, y = se^{N(B)}, \exists r, s \in \mathbb{R}^\times.
\]

Denote the vertical thickness of the floor of \( L \) by \( q' \). On \( \gamma_\pm \subset \Omega \), the dominant terms of \( f_\varepsilon(x, y) = \sum a_i(x) y^{N-i} \) are \( a_N \) and \( a_{N-q'} y^{q'} \). Then the dominant term of \( f_y(x, y) = \sum (N - i) a_i(x) y^{N-i-1} \) is \( q'a_N y^{q'-1} \). Hence, on \( \gamma_\pm \), one has

\[
y f_y \sim q'a_{N-q'} y^{q'} = -q'a_N + \cdots.
\]

For the second equation in \((4.3)\), we used \( f_x(x, y) = 0 \). Then, we obtain

\[
\int_{\gamma_\pm} \omega_i \sim \int_{\gamma_\pm} \frac{\phi(x) \, dx}{-q'a_N(x)}.
\]

We claim that the integral on the right hand side takes the value \( \mp 1 \). To prove this, we recall the relation

\[
(x, y) \in \Omega \Rightarrow 0 = f_\varepsilon(x, y) = a_{N-q'} y^{q'} + a_N + o(e^{q' N(B)} + \text{val}(f_{N-q'})).
\]

Hence \( q' \) of the zeros of \( a_N(x) \) satisfy the equation \( (x, y) = (u_{k_1} e^B, o(e^{N(B)})) \), which implies these points \( (x, y) \) are in the horn containing \( P_+ \). (There also exist \( q' \) points satisfying \( x = u_{k_2} e^B, y = o(e^{N(B)}) \) in the horn containing \( P_- \).) The circles \( \gamma_+ \) and \( \gamma_- \) encircle these \( q' \) points respectively. (Figure \( \text{\ref{fig:1}} \). By definition of \( \phi(x) \), the residues of \( -\phi(x)/(q'a_N(x)) \) equal \pm 1/q' at each pole. Therefore, by the residue theorem, we obtain

\[
\int_{\gamma_+} \omega_i = 1 + o(e^0), \quad \int_{\gamma_-} \omega_i = -1 + o(e^0).
\]
(iii) Let \( P \in (\{x = \infty, 0\} \cup \{y = \infty, 0\}) \setminus \{P_+, P_\-\} \). Defining a circle \( \gamma \) in the appropriate sub-surface, we can assume that \( \gamma \) surrounds one horn containing \( P \). Let \( L' \) be a leaf and \( q'' \) be the vertical thickness of \( L' \). The leaf \( L' : (a) \) contained in \( G_N = \{Y = \mathcal{N}(X)\} \) and \( \gamma \) does not surround any pole of \( \omega \) or (b) contained in \( \{Y < \mathcal{N}(X)\} \). In case (b), the dominant term of \( f_\varepsilon \) is neither \( a_N \) nor \( a_N - q''y'' \), and so there exists positive \( \delta \) such that \( |a_N| < e^\delta |f_\varepsilon| \). Therefore we have

\[
\int_{\gamma} \omega_\varepsilon \sim \begin{cases} \int \{ \phi(x)/(-q''a_N(x)) \} \, dx & (L' \subset G_N) \\ o(e^0) & \text{(otherwise)} \end{cases},
\]

which yields \( \int_{\gamma} \omega_\varepsilon = o(e^0) \) (cf. (4.2)).

\[\blacksquare\]

**Sublemma 2**

i) \( \int_{\alpha_i} \omega_\varepsilon = o(e^0) \) \( \forall i, \)

ii) \( \omega_\varepsilon \in \mathcal{F}. \)

i) Let \( E \) be the edge associated with \( \alpha_i \). The edge \( E : (a) \) contained in \( G_N = \{Y = \mathcal{N}(X)\} \) and \( \alpha_i \) does not surround any pole of \( \omega_\varepsilon \) or (b) contained in \( \{Y < \mathcal{N}(X)\} \). In each case, the Residue theorem and (4.7) immediately leads to the evaluation:

\[\int_{\alpha_i} \omega_\varepsilon = o(e^0).\]

ii) Let us decompose \( \beta_i = \beta_i(\varepsilon) \) into finitely many parts. Denote one of these portions by \( \gamma_\varepsilon(\theta) \). It is sufficient to prove the finiteness of the limit of integral for arbitrary simply connected region \( \mathcal{U} \in \tilde{V}(f) \) and arbitrary path \( \gamma_\varepsilon(\theta) : (0, 1) \times [0, 1] \to \mathcal{U} \). Moreover, we can assume \( j^{-1}(\mathcal{U}) \) is contained in some cylinder. Let \( E \) be the associated edge. If \( E \not\subset G_N \), it follows that \( \int_{\gamma_\varepsilon} \omega_\varepsilon = o(e^0) \) by (4.7). Let \( E \subset G_N \). Denote the vertical thickness of \( E \) by \( q \). Let \( x = r_1e^{X_1} \) be the \( x \)-coordinate of \( \gamma_\varepsilon(0) \).
and \( x = r_2 e^{X_2} \) the \( x \)-coordinate of \( \gamma_{t}(1) \). Then,

\[
\int_{\gamma_t} \omega = \int_{r_1 e^{X_1}} \{ \phi(x)/(-q a_N(x)) \} \, dx + o(e^0)
\]

\[
= (-2q\pi i)^{-1} \int_{r_1 e^{X_1}} \left\{ (x - u_{k_1} e^B)^{-1} - (x - u_{k_2} e^B)^{-1} \right\} \, dx + o(e^0)
\]

\[
= (-2q\pi i)^{-1} \log \left\{ \frac{r_2 e^{X_2} - u_{k_1} e^B}{r_1 e^{X_1} - u_{k_1} e^B} \right\} + o(e^0)
\]

\[
= (-2q\pi i)^{-1} \log \left\{ \frac{r_2 e^{X_2} - u_{k_2} e^B}{r_1 e^{X_1} - u_{k_2} e^B} \right\} + o(e^0).
\]

Recalling that \( e = e^{-1/\varepsilon} \), we conclude that the expression in the last line converges to the finite number

\[
(-2q\pi i)^{-1}, (-2q\pi i)^{-1} \log (u_{k_1}/u_{k_2}), \text{ or } (-2q\pi i)^{-1} \log (u_{k_2}/u_{k_1})
\]

when \( \varepsilon \to 0^+ \). \[\blacksquare\]

In the final step of the proof of Lemma 4.3.2 we use the Riemann bilinear relation \[\text{[2]}:\]

\[
\sum_{i=1}^{g} (A'_i B_i - A_i B'_i) = 2\pi i \cdot \sum_j \text{Res}_p \left( \omega^{(3)} \right) \cdot \int_{\beta_j} \omega^{(1)} \tag{4.8}
\]

\( A_i = \int_{\alpha_i} \omega^{(3)}, \quad B_i = \int_{\beta_i} \omega^{(3)}, \quad A'_i = \int_{\alpha_i} \omega^{(1)}, \quad B'_i = \int_{\beta_i} \omega^{(1)} \)

\( \omega^{(3)} \) is of the third kind. \( \omega^{(1)} \) is of the first kind.

Applying this formula for \( \omega_i \) (of third kind) and \( \omega_i \) (of first kind) \( (i = 1, \ldots, g) \), we obtain

\[
\int_{\beta_i} \omega_i - o(e^0) \cdot \sum_{i=1}^{g} (\int_{\beta_i} \omega_i) = (1 + o(e^0))(\int_{\beta_i} \omega_i).
\]  

(4.9)
due to Sublemma 1 and 2 (\( : A'_i = \delta_{i,j}, A_i = o(e^0) \)). On the other hand, applying the formula for \( \omega_{p_+ - p_-} \) and \( \omega_i \), we obtain

\[
\int_{\beta_i} \omega_{p_+ - p_-} = \int_{P_{p_+}} \omega_i.
\]

(4.10)

Thus, we derive

\[
\int_{\beta_i} (\omega_i - 1 + o(e^0)) \cdot \omega_{p_+ - p_-} = o(e^0) \times \sum_{i=1}^{g} (\int_{\beta_i} \omega_i) \quad (\forall i),
\]

which implies \( \omega_i - 1 + o(e^0) \cdot \omega_{p_+ - p_-} \in \mathcal{M} \). This relation can be rewritten as \( (1 + o(e^0))\omega_{p_+ - p_-} \in \mathcal{M} + \mathcal{F} \), or \( \omega_{p_+ - p_-} \in \mathcal{M} + \mathcal{F} \). \[\blacksquare\]

### 4.3.2 Differentials associated with edges

Next we consider \textbf{differentials associated with edges}. Let \( E' \subset \text{Trop} C \) be an edge of multiplicity \( m \). By rotation, it can be assumed that \( E' \) is vertical without loss of generality. Denote the horizontal thickness of the vertical edge \( E' \) by \( w \).

Let \( E^* = \{(B,(1-t)Y_0 + tY_1) | 0 \leq t \leq 1, Y_0 < Y_1 \} \) and

\[
\{\text{the ceiling of } E'\} \subset G_{l+1} = \{Y = N_{l+1}(X)\},
\]

\[
\{\text{the floor of } E'\} \subset G_l = \{Y = N_l(X)\}.
\]

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By definition, it follows that $Y_1 = \mathcal{N}_{j+1}(B)$ and $Y_0 = \mathcal{N}_j(B)$: Because $E'$ is of finite length, it follows that $1 \leq I \leq N - 1$. The defining polynomial $f_z(x,y)$ of $C_\varepsilon$ is of the form

$$f_z(x,y) = \sum_{i=0}^{N} a_i(x)y^{N-i}, \quad a_I(x) = ce^{A}x^n \prod_{j} (x - u_j e^{B_j}),$$

where $c, u_j \in \mathbb{R}^\times, A, B_j \in \mathbb{Q}_{>0}$, $n \in \mathbb{N}$. We rewrite the polynomial $a_I(x)$ as

$$a_I(x) = ce^{A}x^n \prod_{j=1}^{w} (x - u_j e^{B_j}) \cdot \prod_{j>\omega} (x - u_j e^{B_j}),$$

where $w$ is the horizontal thickness of $E'$ and $B_j \neq B (j > w)$. Let $E' = E_1 \cdots E_w$ be the decomposition into edges of multiplicity one. Note that the horizontal thickness of $E_j$ equals 1.

We define the differential associated with the edge $E \equiv E_1$ by:

$$\omega_E := \frac{\phi(x) \cdot y^{N-I-1}}{\int f_y} \, dx, \quad \frac{\phi(x)}{a_I(x)} := \frac{-1}{2\pi i (x - u_1 e^{B})}. \quad (4.11)$$

Let $(x, y) = (re^X, se^Y)$ be a point on $C_\varepsilon$. Assume $(X, Y) \in G_f = \{Y = \mathcal{N}_j(X)\}$ on the edge of vertical thickness $q$. If $J = I, I + 1$, we can use the estimation

$$yf_y = \begin{cases} (N-I+q) a_{I-q} y^{N-I+q} + (N-I) a_I y^{N-I} + \cdots & \text{if } (X, Y) \in G_I \\ (N-I-q) a_{I-q} y^{N-I-q} + (N-I) a_I y^{N-I} + \cdots & \text{if } (X, Y) \in G_I+1. \end{cases} \quad (4.11)$$

This implies that

$$\omega_E \sim \begin{cases} \{\phi(x)/q a_I\} \, dx & \text{if } (X, Y) \in G_I \\ \{\phi(x)/q a_I\} \, dx & \text{if } (X, Y) \in G_I+1. \end{cases} \quad (4.12)$$

If $J \neq I, I + 1, a_I y^{N-I}$ cannot be a dominant term of $yf_y$, and it follows that

$$\omega_E = o(x^0) \, dx \quad \text{if } (X, Y) \in G_J (J \neq I, I + 1). \quad (4.13)$$

Next we study the estimation of $\omega_E$ on vertical edges. Let $M \subset \mathrm{Trop} C$ be a vertical edge whose ceiling is contained in $G_{J+1}$ and whose floor is contained in $G_J$. Let us rewrite the expression of $\omega_E$ into

$$\omega_E = -\phi(x)y^{N-I-1}\frac{dy}{f_x}, \quad \frac{\phi(x)}{a_{J}} = \frac{-1}{2\pi i} \cdot ce^{A}x^n \prod_{j \neq 1} (x - u_j e^{B_j}).$$

($\therefore \frac{dy}{f_x} = \frac{-\phi}{a_{J}}$ for smooth curve $C_\varepsilon$).

Due to the definition of $M$, the dominant term of $f_z(x,y)$ on

$$\{(x, y) | (\mathrm{val} (x), \mathrm{val} (y)) \in M\}$$

is $a_J y^{N-J}$. We claim that the dominant term of $f_x$ is $a'_J y^{N-J}$, where $a'_J(x) := \frac{d}{dx} \phi(x)$ and $f_x = \sum_i a'_i y^{N-i}$.

**Lemma 4.3.3** The dominant term of $f_x$ on $M$ is $a'_J y^{N-J}$.  

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We consider the difference \( \text{val}(a_i) - \text{val}(a_i') \) \((1 \leq i \leq N)\). Let

\[
M = \{(B_0, (1-t)Y_2 + tY_3) \mid 0 \leq t \leq 1, Y_2 < Y_3\},
\]

and \( a_i = ce^{A_i x^{n_i} - x} \prod_j (x - u_{i,j}e^{B_{i,j}}) \). Then, \( a_i' \) is of the form:

\[
a'_i = cn_i e^{A_i x^{n_i} - x} \prod_j (x - u_{i,j}e^{B_{i,j}}) + ce^{A_i x^{n_i}} \sum_k \prod_{j \neq k} (x - u_{i,j}e^{B_{i,j}}).
\]

From this expression we see that

\[
\text{val}(a_i(x)) - \text{val}(a'_i(x)) \leq \text{val}(x). \tag{4.14}
\]

On the other hand, if \( i = J \), \( a_J(x) \) is of the form

\[
a_J = c_J e^{A_J x^{n_J} - x} \prod_j (x - u_{J,j}e^{B_{J,j}}), \quad \#\{j \mid B_0 = B_{J,j} \} > 0.
\]

Let \( \Lambda := \{j \mid B_0 = B_{J,j}\} \). We can assume \( \Lambda = \{1, 2, \ldots, w'\} \) by exchanging the indices if necessary. Again we consider the derivative \( a'_J \). Among the factors

\[
x - u_{J,1}e^{B_0}, \quad x - u_{J,2}e^{B_0}, \quad \ldots, \quad x - u_{J,w}e^{B_0},
\]

\((x - u_{J,k}e^{B_0})\) is the only one that becomes very small when we take \( x = u_{J,k}e^{B_0} + o(e^{B_0}) \) because of the genericness condition in Section 3. Thus, for fixed \( k \in \Lambda \), it follows that

\[
x = u_{J,k}e^{B_0} + o(e^{B_0}) \Rightarrow a'_J(x) \sim c_J e^{A_J x^{n_J}} \prod_{j \neq k} (x - u_{J,j}e^{B_{J,j}}) \tag{4.15}
\]

which implies

\[
\text{val}(a_J(x)) - \text{val}(a'_J(x)) = B_0 = \text{val}(x). \tag{4.16}
\]

Therefore, (4.14) and (4.16) give rise to

\[
(\text{val}(x), \text{val}(y)) \in M \Rightarrow \begin{cases} x = u_{J,k}e^{B_0} + o(e^{B_0}) & (\exists k) \\ \text{val}(a_J y^{N-J}) \leq \text{val}(a_J y^{N-i}) \end{cases}
\]

\[
\Rightarrow \text{val}(a'_J y^{N-J}) - \text{val}(a'_J y^{N-i}) 
\quad \leq -\text{val}(x) + \text{val}(a'_J y^{N-J}) + \text{val}(x) - \text{val}(a'_J y^{N-i}) \leq 0
\]

for each \( i = 0, 1, \ldots, N \). In particular, we can conclude that \( f_x \sim a'_J y^{N-J} \) on \( M \).}

Recall that the floor of \( E \) is contained in \( G_I \), and that the floor of \( M \) is contained in \( G_J \). From the explicit form of \( a'_J \) given in (4.15) and the definition of \( \phi(x) \) (4.11), we obtain

\[
-\phi(x) y^{N-J}/f_x |_{(\text{val}(x), \text{val}(y)) \in M} = \begin{cases} (2\pi)^{-1} + o(e^\theta) & (J = I \text{ and } x = u_1 e^B + o(e^B)) \\ o(e^\theta) & (J = I \text{ and } x = u_1 e^B + o(e^B), j > 1) \\ o(e^\theta) & (J \neq I) \end{cases}
\]

\[24\]
These relations lead to the following:

\[ \omega_E \sim \begin{cases} 
(2\pi i)^{-1} (dy/y) & \text{val}(x), \text{val}(y) \in E \\
o(\epsilon_0) dx & \text{val}(x), \text{val}(y) \in M \neq E
\end{cases} \quad (4.17) \]

The three equations (4.12), (4.13) and (4.17) gives us the singularities of \( \omega_E \). Let \( L_1 \) be the leftmost leaf of \( G_I \) and \( L_2 \) be the leftmost leaf of \( G_{I+1} \). Denote the vertical thickness of \( L_i \) \((i = 1, 2)\) by \( q_i \) and the multiplicity by \( m_i \), respectively. Let us consider the decomposition \( L_i = L_i, 1 \sqcup L_i, 2 \sqcup \cdots \sqcup L_i, m_i \) and denote the vertical thickness of \( L_i, j \) by \( q_{i, j} \) \((q_i = q_{i,1} + \cdots + q_{i,m_i})\), the horn associated with \( L_i, j \) by \( \Sigma_{i, j} \), the vertex that is the end point of \( L_i \) by \( v_i \) and the sphere associated with \( v_i \) by \( \Omega_i \).

The set \( \left\{ \{ x = \infty, 0 \} \cup \{ y = \infty, 0 \} \right\} \cap \Sigma_{i, j} \) has only one element, which we denote by \( P_{i, j} \). Consider cycles \( \gamma_{i, j} \in \Omega_i \) which loop around the point \( P_{i, j} \) anti-clockwise. By (4.12) it then follows that

\[ \int_{\gamma_{1, j}} \omega_E = +(q_{1, j}/q_1) + o(\epsilon_0), \quad (4.18) \]
\[ \int_{\gamma_{2, j}} \omega_E = -(q_{2, j}/q_2) + o(\epsilon_0). \quad (4.19) \]

Hence, we obtain the following:

**Proposition 4.4** The differential \( \omega_E \) has a pole with residue \(+q_{1, j}/(2q_1\pi i)\) at \( P_{1, j} \) and a pole with residue \(-q_{2, j}/(2q_2\pi i)\) at \( P_{2, j} \).

Let \( \gamma \in H_1(C_{\epsilon}; \mathbb{Z}) \) be a cycle which loops a cylinder \( \Sigma \). We fix the direction of \( \gamma \) as Figure 10. The integral \( \int_{\gamma} \omega_E \) takes various values depending on the position of \( \Sigma \) in \( C_{\epsilon} \). Let \( M \) be a tropical edge associated with a cylinder \( \Sigma \).

(i) The case \( M = E \).

When one runs around a cylinder \( \Sigma \), the \( y \)-coordinate runs around the origin. Then, by (4.17), we derive

\[ \int_{\gamma} \omega_E = \oint_{0+} (2\pi i)^{-1} \frac{dy}{y} + o(\epsilon_0) = \int_0^{2\pi} (2\pi i)^{-1} i d\theta + o(\epsilon_0) \]
\[ = 1 + o(\epsilon_0). \quad (4.20) \]

(ii) In the case that \( M \) is vertical and \( M \neq E \), it follows that \( \int_{\gamma} \omega_E = o(\epsilon_0) \).
(iii) When $M \subset G_I \setminus G_{I+1}$, we consider the edge $M' = M_1 \amalg M_2 \amalg \cdots \amalg M_m$ ($M = M_1$) and denote the vertical thickness of $M_i$ by $q_i$ ($q_i = q/m$). From (4.12) and (4.12) it follows that

$$\int_\gamma \omega_E = \begin{cases} 
q_i/q + o(e^0) & (M \subset \{X < B\}) \\
q_i/q + o(e^0) & (M \subset \{X > B\}) 
\end{cases}.$$  

(4.21)

We used the assumption that $C$ has a good tropicalization for the first equality.

(iv) When $M \subset G_{I+1} \setminus G_I$, one has that

$$\int_\gamma \omega_E = \begin{cases} 
1/m + o(e^0) & (M \subset \{X < B\}) \\
1/m + o(e^0) & (M \subset \{X > B\}) 
\end{cases}.$$  

(4.22)

(v) When $M \subset G_i$ ($i \neq I, I+1$), one has that $\int_\gamma \omega_E = o(e^0)$.

The remaining case is the degenerate case: $G_I \cap G_{I+1} \neq \emptyset$. (Figure 11)

(vi) The case $M \subset G_I \cap G_{I+1}$.

Since $\alpha_I y^{N-I}$ is not a dominant term in $f_y$, it follows that

$$\int_\gamma \omega_E = o(e^0).$$  

(4.23)

Figure 11: Two subsets $G_I = \{Y = \mathcal{N}_I(X)\}$ and $G_{I+1} = \{Y = \mathcal{N}_{I+1}(X)\}$ may have intersection. The edge $M$ is in the intersection.

Now we proceed for the $\beta$-cycle of $\omega_E$. For this, we first calculate the integrals along the cylinders. Let $\Sigma \subset C_t$ be a cylinder. Take a path $\rho$ which runs along $\Sigma$. We calculate the integral $\int_\rho \omega_E$ by using (4.12) and (4.17).

**Lemma 4.4.1** Let $\mathcal{I} := \int_\rho \omega_E$. ($\rho$ runs along $\Sigma$).

1. If $\Sigma$ is associated with $E = \{(B,(1-t)Y_0 + tY_1)| 0 \leq t \leq 1, Y_0 < Y_1\}$, then

$$\mathcal{I} = -(2\pi i \varepsilon)^{-1}(Y_1 - Y_0) + o(e^0).$$

2. If $\Sigma$ is associated with a vertical edge except $E$, $\mathcal{I} = o(e^0)$.

3. If $\Sigma$ is associated with a non-vertical edge $L$ of multiplicity $m$ and of vertical thickness $q$ in $G_I$:

$$L = \{(1-t)X_0 + tX_1, (1-t)Y_0 + tY_1)| 0 \leq t \leq 1, X_0 < X_1, Y_0 < Y_1\} \subset G_I,$$

then

$$\mathcal{I} = -(2q\pi i \varepsilon)^{-1}(\min[B, X_1] - \min[B, X_0]) + o(e^0).$$
4. If $\Sigma$ is associated with a non-vertical edge $L$ of multiplicity $m$ and of vertical thickness $q$ in $G_{I+1}$:
\[ L = \{(1-t)X_0 + tX_1, (1-t)Y_0 + tY_1) \mid 0 \leq t \leq 1, X_0 < X_1, Y_0 < Y_1 \} \subset G_{I+1}, \]
then $I = +(2q\pi\varepsilon)^{-1}(\min B, X_1 - \min B, X_0) + o(e^0)$.

5. If $\Sigma$ is associated with a non-vertical edge in $G_J$ ($J \neq I, I + 1$), then $I = o(e^0)$.

6. If $\Sigma$ is associated with an edge in $G_I \cap G_{I+1}$, then $I = o(e^0)$.

1. From (4.17) it follows that
\[ I = (2\pi)^{-1} \cdot \int_{s_0}^{s_1} e^{x_1} \frac{(dy/y)}{(2\pi i)} = (2\pi i)^{-1} \log \{(s_1/s_0)e^{y_1-y_0}\} \]

\[ = -(2\pi i)^{-1}(y_1 - y_0) + \cdots. \]

2. This can be obtained from (4.17).

3. From (4.12),
\[ I = (2\pi i)^{-1} \cdot \int_{r_0}^{r_1} e^{x} \{1/q(x - u_{j_0}e^{B})\} dx \]

\[ = (2q\pi i)^{-1} \log \{(r_1e^{x_1} - u_{j_0}e^{B})/(r_0e^{x_0} - u_{j_0}e^{B})\} \]
\[ = -(2q\pi i)^{-1}(\min [B, X_1] - \min [B, X_0]) + \cdots. \]

4. This can be obtained in the same way as the case 3.

5. This follows from (4.13).

6. This follows from (4.23).

The result of Lemma 4.4.1 is easily understood by means of the tropical bilinear form. Let $\Gamma_E \subset \text{Trop C}$ be a path with direction defined by the following route:
\[ (X = -\infty) \xrightarrow{\text{on} G_I} (X = B, Y = Y_0) \xrightarrow{\text{on} E} (X = B, Y = Y_1) \xrightarrow{\text{on} G_{I+1}} (X = -\infty). \]
Using Lemma 4.4.1 we can restate the claim of Lemma 4.4.1 as:
\[ I = \int_{\rho} \omega_E = -(2\pi i)^{-1} \cdot m_L^{-1} \cdot \ell_T(L, \Gamma_E), \] (4.24)
where $m_L$ is the multiplicity of $L$. Recall that the bilinear form $\ell_T(\cdot, \cdot)$ gives the tropical length of intersection up to sign.

In fact, the equations (4.20, 4.23) can be rewritten using the intersection number.
Let $\gamma \subset C_\varepsilon$ be a closed path surrounding some cylinder and $E \subset \text{Trop C}$ be a directed edge. Denote the cylinder associated with $E$ by $\Sigma_E$. And define the intersection number $(\gamma \circ E)$ by
\[ (\gamma \circ E) := \begin{cases} +1 & (\gamma \text{ surrounds the cylinder } \Sigma_E \text{ by positive direction}) \\ -1 & (\gamma \text{ surrounds the cylinder } \Sigma_E \text{ by negative direction}) \\ 0 & (\text{else}) \end{cases}. \]
(Cf. figure 12).
When $\gamma \subset C_\varepsilon$ is a closed path loops a cylinder $\Sigma$, it follows that
\[ \int_{\gamma} \omega_E = m^{-1}(\gamma \circ \Gamma_E) + o(e^0), \] (4.25)
where $m$ is the multiplicity of the edge associated with $\Sigma$.  

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The definition of the intersection number \((\gamma \circ E)\).

**Poles without residue**

The differential \(\omega_E\) also has poles without residue and we can neglect the influence of these poles. In fact, by (4.20–4.23), we can show that
\[
\int_{\gamma} x^k \omega_E = o(e^0) \quad (k \leq -1, \gamma \subset \{\text{val}(x) \ll 0\}),
\]
which implies
\[
\omega_E = \left(\frac{c_n}{z^n} + \cdots + \frac{c_1}{z} + c_0 + c_1 z + \cdots\right) dz \quad \text{at } z \in \{x = \infty\},
\]
\[
\Rightarrow c_k = o(e^0) \quad (k < -1).
\]

### 4.4.1 A modified differential

Let \(E = E_1 \sqcup \cdots \sqcup E_m\) be an edge of multiplicity \(m\). Define the differential \(\upsilon := \omega_{E_i} - \omega_{E_j}\), where \(\omega_{E_i}\) is the differential associated with the edge \(E_i\). Then, \(\upsilon\) satisfies the following:

(i) \(\upsilon\) has no pole with residue,

(ii) For a closed path \(\gamma\) surrounding a cylinder \(\Sigma\) in \(C_\epsilon\), one has that
\[
\int_{\gamma} \upsilon = (\gamma \circ (E_i - E_j)) + o(e^0).
\]

(iii) For a path \(\rho \subset C_\epsilon\) which runs along a cylinder \(\Sigma\), it follows that
\[
\int_{\rho} \upsilon = -(2\pi i\epsilon)^{-1} \cdot \ell_T(L, E_i - E_j),
\]
where \(L\) is the edge which is associated with \(\Sigma\) and which is of multiplicity one.

Now we define a new differential which is a modification of \(\omega_E\). Let \(\Gamma_E = E \sqcup E^{(1)} \sqcup E^{(2)} \sqcup \cdots \sqcup E^{(n)}\) be the decomposition into edges. We decompose each \(E^{(i)}\) into edges of multiplicity one: \(E^{(i)} = E^{(i)}_1 \sqcup \cdots \sqcup E^{(i)}_{m_i}\), where \(m_i\) is the multiplicity of \(E^{(i)}\).

Now we define a new modified differential associated with \(E\). Let
\[
\tilde{\omega}_E := \omega_E + \upsilon^{(1)} + \upsilon^{(2)} + \cdots + \upsilon^{(n)},
\]
\[
\upsilon^{(i)} = m_i^{-1} \left\{ (\omega_{E^{(i)}_1} - \omega_{E^{(i)}_2}) + (\omega_{E^{(i)}_2} - \omega_{E^{(i)}_3}) + \cdots + (\omega_{E^{(i)}_1} - \omega_{E^{(i)}_{m_i}}) \right\}.
\]

For the path \(\tilde{\Gamma}_E\) which is defined by \(\tilde{\Gamma}_E := E \sqcup E^{(1)}_1 \sqcup E^{(1)}_2 \sqcup \cdots \sqcup E^{(1)}_{m_i}\), we can rewrite the equation (4.24) as
\[
\int_{\rho} \tilde{\omega}_E = -(2\pi i\epsilon)^{-1} \ell_T(L, \tilde{\Gamma}_E), \quad (\rho \text{ runs along } L),
\]
and we also rewrite the equation (4.25) as
\[
\int_{\gamma} \tilde{\omega}_E = (\gamma \circ \tilde{\Gamma}_E) + o(e^0).
\]
4.4.2 Proof of the theorem

We have finished all the preparations necessary to complete the proof of the main theorem. Let $X$ be a set of edges contained in $\text{Trop} C$. Denote the free additive abelian group which is generated by the elements of $X$ by $\mathbb{Z}X$.

For a closed path $\Gamma$ contained in $\text{Trop} C$, choose edges $E_1, \ldots, E_n; F_1, \ldots, F_m$ of multiplicity 1 such that $\tilde{\Gamma}E_1 + \cdots + \tilde{\Gamma}E_n - \tilde{\Gamma}F_1 - \cdots - \tilde{\Gamma}F_m = \Gamma \in \mathbb{Z}X$. Define the differential

$$\omega'_E := \tilde{\omega}E_1 + \cdots + \tilde{\omega}E_n - \tilde{\omega}F_1 - \cdots - \tilde{\omega}F_n.$$ 

By Proposition 4.4, (4.26) and (4.27), $\omega'_E$ has the following properties:

(i) $\omega'_E$ has singularities in $\{x = \infty, 0\} \cup \{y = \infty, 0\}$.

(ii) Let $P_1, \ldots, P_q$ be points in $\{x = \infty, 0\} \cup \{y = \infty, 0\}$ and suppose that these are associated with the same leaf of $\text{Trop} C$. Then, $\sum_{i=1}^{q} \text{Res}_{P_i}(\omega'_E) = 0$.

(iii) Let $\alpha$ be a closed path surrounding a cylinder $\Sigma$ which is associated with the edge $E \subset \text{Trop} C$. Then,

$$\int_{\alpha} \omega'_E = (\alpha \circ \Gamma) + o(e^0). \quad (4.28)$$

For a leaf $\Gamma_{\infty}$ of infinite length in $\text{Trop} C$, we define the differential $\omega_{\Gamma_{\infty}}$ by:

$$\omega_{\Gamma_{\infty}} := \omega_{c_1P_1 + \cdots + c_nP_n}, \quad \left( \text{P_i is a point in the horns associated with } \Gamma_{\infty} \right).$$

s.t. $c_i = \text{Res}_{P_i}(\omega'_E)$ is not 0.

(Recall $'\omega_{c_1P_1 + \cdots + c_nP_n}'$ is the normalised differential of the third kind.) Summing these differentials of the third kind for all edges of infinite length:

$$\omega_{\infty} := \sum_{|\Gamma_{\infty}| = \infty} \omega_{\Gamma_{\infty}}.$$ 

Then, the new differential $\omega_\Gamma := \omega'_E - \omega_{\infty}$ satisfies: i) $\int_{\alpha} \omega_\Gamma = \int_{\alpha} \omega'_E$ and ii) $\omega_\Gamma$ has no singularity with non-zero residue ($\omega_\Gamma$ is of the second kind). Moreover, by adding the normalised differentials of the second kind, we can assume $\omega_\Gamma$ is of first kind. (Recall that we can neglect the poles without residue.)

**Lemma 4.4.2** $\omega'_E - \omega_\Gamma \in \mathcal{M} + \mathcal{F}$.

Due to Lemma 4.3.2, it follows that $\omega_{\Gamma_{\infty}} \in \mathcal{M} + \mathcal{F}$ for each leaf $\Gamma_{\infty}$. Because $\mathcal{M}$ and $\mathcal{F}$ are vector spaces, the required result is obtained soon. \[\blacksquare\]

**Proof of the theorem.** Let $\Gamma = T_{\beta_i}$, that is the closed path on $\text{Trop} C$ associated with the $\beta$-cycle $\beta_i$ on $C_\epsilon$. (See Section 4.3) By (4.28), it follows that

$$\int_{\alpha_j} \omega_{T_{\beta_i}} = \int_{\alpha_j} \omega'_{T_{\beta_i}} = \begin{cases} 1 + o(e^0) & (i = j) \\ o(e^0) & (i \neq j) \end{cases}.$$ 

(4.29)

Let $\omega_j$ be the $j$-th normalised holomorphic differential (Section 4.3). Clearly, (4.29) means

$$\omega_j = \omega_{T_{\beta_i}} \cdot (1 + o(e^0)), \quad \forall j.$$
or equivalently \( \omega_j - \omega_{T\beta_j} \in \mathcal{M} \). Due to Lemma 4.4.2 we obtain

\[
\omega_i - \omega'_{T\beta_i} \in \mathcal{M} + \mathcal{F}.
\] (4.30)

Consider \( g \times g \) matrices \( \mathbf{B} = (\int_{\beta_j} \omega_i)_{i,j} \) and \( \mathbf{B}' = (\int_{\beta_j} \omega'_{T\beta_i})_{i,j} \). Equation (4.30) can be rewritten as

\[
B - B' = o(e^0)B + B^\dagger \lim_{\varepsilon \to 0^+} |B^\dagger| < \infty,
\] (4.31)

or

\[
\{I + o(e^0) \cdot \Delta\} \cdot B - B' = B^\dagger,
\] (4.32)

where \( I \) is the identity matrix and \( \Delta \) is a \( g \times g \) matrix. On the other hand, from (4.24) one concludes that \( \mathbf{B}' \) tends to infinity when \( \varepsilon \to 0^+ \). We thus obtain

\[
B \sim B' \quad (\varepsilon \to 0^+),
\] (4.33)

by taking a limit \( \varepsilon \to 0^+ \) of (4.32).

To conclude the proof of theorem, it is sufficient to prove that

\[
\int_{\beta_j} \omega'_{T\beta_i} = -(2\pi \ii\varepsilon)^{-1} \cdot \ell(T_{\beta_j}, T_{\beta_i}),
\]

which is a mere linear combination of copies of (4.26). 

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A Genericness Condition

In this paper, we introduced some conditions on \( C_\varepsilon \) and Trop\( C \) to make the problem easier. Let \( f_\varepsilon(x, y) = \sum a_i(x)y^{N-i} \) be the defining polynomial of \( C_\varepsilon \), where

\[
a_i(x) = c_\varepsilon e^{A_i} x^{m_i} \prod_{j=1}^{d_i} (x - u_{i,j} e^{B_{i,j}}), \quad c_\varepsilon, u_{i,j} \in \mathbb{R}, \quad A_i, B_{i,j} \in \mathbb{Q}, \quad m_i \in \mathbb{N}.
\]

Let \( \theta \in SL_2(\mathbb{Z}) \) be a rotation of Trop\( C \). The translation \( \theta \) naturally acts on \( C_\varepsilon \) by \( x \mapsto x^\delta y^{-\beta}; y \mapsto x^{-\gamma} y^\alpha \). Define the new polynomial

\[
f_\theta(x, y) = f_\varepsilon(x^\delta y^{-\beta}, x^{-\gamma} y^\alpha) = \sum a_i^\theta(x)y^{N'-i}.
\]

To be precise, we assumed three conditions:

Genericness condition. For fixed \( \theta \in SL_2(\mathbb{Z}) \), \( \text{top}(u_{i,j}^\theta) \in \mathbb{C} \setminus \{0\} \) \((\forall i,j)\) are all distinct.

Condition I. For each edge \( E \), \( m = g.c.d.(q, w) \), where \( m, q, w \) respectively are the multiplicity, the vertical thickness and the horizontal thickness of \( E \).
Condition II. For each edge $E = E_1 \sqcup \cdots \sqcup E_m$, $q_1 = \cdots = q_m$, $w_1 = \cdots = w_m$, where $q_i, w_i$ respectively are the vertical thickness and the horizontal thickness of $E$.

The following relation exists between these conditions.

**Proposition A.1** Genericness condition $\Rightarrow$ Condition II $\Rightarrow$ Condition I.

(Genericness cond. $\Rightarrow$ Cond. II) We first prove the case when $E = E_1 \sqcup \cdots \sqcup E_m$ is vertical. Then it is clear that $q_1 = \cdots = q_m = 0$. The defining equation of vertical edge $E$ is of the form $(a + w)X + bY + c = aX + bY + c'$. When we substitute $(x, y) = (re^X, se^Y), (X, Y) \in E$ into $f_z(x, y)$, the polynomial $a_{N-b}(x)y^b$ is dominant. Moreover, we can derive the relation $\text{top}(r - u_{i,j_1})(r - u_{i,j_2}) \cdots (r - u_{i,j_w}) = 0, \quad j = N - b, j_k \in \{j \mid B_{i,j} = (c - c')/w\}$.

By the genericness condition, this equation implies that $w$ distinct cylinders in $C$ are associated with the edge $E$. Then $m = w$, which implies $w_1 = w_2 = \cdots = w_m = 1$.

In the general case, we consider $\theta \in SL_2(\mathbb{Z})$ such that $\theta \cdot E = (\theta \cdot E_1) \sqcup \cdots \sqcup (\theta \cdot E_m)$ is vertical. In fact, the vertical thickness and the horizontal thickness of $E_i$ satisfy the relation

$$q_i = \alpha q_i^\theta + \beta w_i^\theta, \quad q_i = \gamma q_i^\theta + \delta w_i^\theta,$$

where $q_i^\theta$ and $w_i^\theta$ are the vertical thickness and the horizontal thickness of $\theta \cdot E_i$. This equation implies $q_1 = \cdots = q_m$ and $w_1 = \cdots = w_m$.

(Cond. II $\Rightarrow$ Cond. I) From the equation $w_1 = \cdots = w_m = 1$ for the vertical edge $E = E_1 \sqcup \cdots \sqcup E_m$ we conclude that it is enough to prove that $g.c.d.(q, w)$ is invariant under the rotation $\theta$. This fact follows immediately from (A.1).

Due to the above proposition, we can claim that the only essential assumption for our arguments is the genericness condition.

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