Universality and Deviations in Disordered Systems

Giorgio Parisi\(^1\) and Tommaso Rizzo\(^2\)

\(^1\)Dipartimento di Fisica, Università di Roma “La Sapienza”, P.le Aldo Moro 2, 00185 Roma, Italy

\(^2\)“E. Fermi” Center, Via Panisperna 89 A, Compendio Viminale, 00184, Roma, Italy

We compute the probability of positive large deviations of the free energy per spin in mean-field Spin-Glass models. The probability vanishes in the thermodynamic limit as \(P(\Delta f) \propto \exp[-N^2L_2(\Delta f)]\). For the Sherrington-Kirkpatrick model we find \(L_2(\Delta f) = O(\Delta f)^{12/5}\) in good agreement with numerical data and with the assumption that typical small deviations of the free energy scale as \(N^{3/6}\). For the spherical model we find \(L_2(\Delta f) = O(\Delta f)^2\) in agreement with recent findings on the fluctuations of the largest eigenvalue of random Gaussian matrices. The computation is based on a loop expansion in replica space and the non-gaussian behaviour follows in both cases from the fact that the expansion is divergent at all orders. The factors of the leading order terms are obtained resumming appropriately the loop expansion and display universality, pointing to the existence of a single universal distribution describing the small deviations of any model in the full-Replica-Symmetry-Breaking class.

The free energy density of a random system model fluctuates over the disorder with a distribution that in the thermodynamic limit becomes peaked around the typical value \(f_{typ}\). The probability of observing \(O(1)\) deviations (i.e. large deviations) from the typical value are exponentially small. In some simple systems (like the mean field spherical model for spin glasses) the probability of large deviations is highly asymmetric \(1\). In the case of negative deviations \(\Delta f = f - f_{typ}\) we have \(P(\Delta f) \propto \exp[-NL(\Delta f)]\) \(2\); for positive deviations the probability is much smaller: \(P(\Delta f) \propto \exp[-N^2L_2(\Delta f)]\).

This unusual asymmetric behaviour related to the distribution of large deviations is highly universal of a large Gaussian random matrix. The probability of observing large positive deviations scales indeed as \(\exp[-O(N^2)]\) and had been recently computed in \(1\). On the other hand the typical small deviation distribution of the lowest eigenvalue are described by a universal function in the large \(N\)-limit that was computed by Tracy and Widom \(3\) and has since appeared in many apparently unrelated problems (see e.g. \(1\)).

In this letter we show that in the mean-field Sherrington-Kirkpatrick(SK) spin-glass model a similar scaling is present and the probability distribution \(L_2(\Delta f)\) can be computed using the hierarchical replica-symmetry-breaking(RSB) ansatz (the function \(L(\Delta f)\) that is relevant for deviations as computed in \(2\)). Interestingly enough our results also show a great deal of universality suggesting that the small deviations of the free energy of the whole class of full-RSB models are also described by the same universal function at any temperature in the spin-glass phase.

The computation of \(L_2(\Delta f)\) is much more complex than that of \(L(\Delta f)\): indeed the function \(L(\Delta f)\) is infinite for positive \(\Delta f\) \(4\ \ 5\). If we attack the problem in the replica framework it leads to an action of a matrix \(\bar{Q}_{ab}\) of size \(n \times n\) where \(n = \alpha N\) with negative \(\alpha\). Usually fluctuations around the saddle point are negligible (they give a contribution to the total free energy that is proportional to \(n^2\)) however they play a crucial role in this case because the number of elements of the matrix \(\bar{Q}\) is \(O(N^2)\). We have also to face the fact the the saddle point is marginal (i.e. there are zero modes that lead to divergent fluctuations) thus if we proceed naively we obtain a series divergent at all orders. The marginality of mean-field theory is a general well-known feature of full-RSB spin-glass and it stands as the main difficulty to solve many of the open problems in the field. In particular it leads to non-trivial finite-size correction in the SK model for which no systematic computation scheme has been devised up to now \(6\). Besides the fact that no theory can be considered satisfactory if it does not allow to include deviations and corrections, these non-trivial mean-field exponents have been argued to be relevant also for finite-dimensional quantities, notably the stiffness exponent \(7\). Furthermore we do not know how to treat loop corrections to the theory in the spin-glass phase below six dimensions \(8\). Thus the need to tackle the marginal nature of the theory makes the computation as interesting as the problem itself.

In order to compute the so called sample complexity \(L_2(\Delta f) \equiv -\ln(P(\Delta f))/N^2\) we consider the average partition function of \(n = \alpha N\) replicas:

\[
\Phi(\alpha) = \frac{1}{N^2} \ln Z^{1-N},
\]

with \(\alpha\) negative (the bar denotes the average over the disorder). In the usual approach, when we need to compute the typical free energy \((f_{typ})\) the number \((n)\) of replicas goes to zero, but in this case it has to go to \(-\infty\).

The functional \(\Phi(\alpha)\) is the Legendre transform of \(L_2(\Delta f)\). The latter can be obtained as \(-L_2(\Delta f) = \alpha \beta f_{typ} + \alpha \beta \Delta f - \Phi(\alpha)\) where \(\alpha\) is the solution of \(\beta f_{typ} + \beta \Delta f = d\Phi(\alpha)/d\alpha\). The main result of this letter is the evaluation of \(\Phi(\alpha)\) for small \(\alpha\).

Let us now consider what happens in the SK spin-glass model where \(H = \sum_{i,k} J_{i,k} \sigma_i \sigma_k\) with the \(\sigma\)’s being Ising spins and the \(J\)’s being random Gaussian variables with zero average and variance \(1/N\). Through standard manipulations we rewrite \(1\) as an integral of an action depending on a symmetric matrix \(Q_{ab}\) with \(a, b = 1, \ldots, \alpha N\) and \(Q_{aa} = 0\). As the size of the matrix...
is $O(N)$ we cannot simply take the saddle point and we have consider the integral of $O(N^2)$ elements of $Q_{ab}$. In order to perform the integral we find convenient to divide the matrix $\tilde{Q}_{ab}$ in $(\frac{N^2}{\alpha})$ blocks of size $n \times n$, with $n$ some parameter between $1$ and $\alpha N$ that will be eventually sent to zero. The generic matrix element will be written as $Q_{ij}^{ab}$ where the upper indices $i, j = 1, \ldots, N \alpha / n$ label different blocks and the lower indices $a, b = 1, \ldots, n$ label elements inside block $ij$.

The resulting action admits a saddle point with vanishing off-diagonal blocks $Q_{ij}^{ab} = 0$ for all $i \neq j$ and we integrate out the elements of the off-diagonal blocks $i \neq j$ around their zero saddle-point value. After the integration we are left with an integral over the diagonal blocks of the exponential of an $O(N^2)$ action over which we will take the saddle point. We will consider here the simplest situation where the saddle points are such that all the blocks on the diagonal are equal to a given $n \times n$ hierarchical matrix $Q$, i.e. $Q_{ij} = Q \forall i$. At the end the function $\Phi(\alpha)$ will be obtained as the saddle-point value of the following functional over the $n \times n$ diagonal block $Q$:

$$ \Phi(\alpha, Q) = \alpha \beta F[Q] + S[Q, \alpha] \quad (2) $$

where $F[Q]$ is the standard SK free energy functional of a $n \times n$ matrix $Q_{ab}$ that is zero on the diagonal:

$$ F[Q] = - \frac{\beta}{4} + \frac{\beta}{2n} \sum_{a<b} Q_{ab}^2 - \frac{1}{\beta n} \sum_{\{s\}} \exp[\beta^2 \sum_{a<b} Q_{ab} s_a s_b] , \quad (3) $$

and $S[Q, \alpha]$ is the contribution of the fluctuations. Now $S[Q, \alpha]$ is small for small $\alpha$ and it should be treated as a perturbation: in order to obtain the first non trivial term we should compute $S[Q, \alpha]$ at the saddle point of $F[Q]$. This is not an easy task: indeed we find that $S[Q, \alpha]$ can be written as:

$$ S[Q, \alpha] = - \frac{\alpha^2}{N^2} \ln \int \frac{dQ^{ij}_{ab}}{\sqrt{2\pi}} \exp \left[ - \frac{1}{2} \sum_{i<j} \sum_{a<b} (1 - \beta^2 \lambda_a \lambda_b + p)(Q_{ij}^{ab})^2 + \left( - \frac{\alpha}{N} \right)^{\frac{1}{2}} \sum_{i<j<k} \sum_{a<b<c} Q_{ab}^i Q_{bc}^j Q_{ca}^k \lambda_a \lambda_b \lambda_c \right] , \quad (4) $$

where $\lambda_a$ with $a = 1, \ldots, n$ are the eigenvalues of the matrix $P_{ab} \equiv \langle s_a s_b \rangle$ and the averages $\langle \cdot \rangle$ are computed with respect to a single diagonal block. In particular if $Q_{ab}$ extremizes $F[Q]$ we have $Q_{ab} = \langle s_a s_b \rangle$ and therefore $P_{ab} = \delta_{ab} + Q_{ab}$. The above expression is valid at the third order in $Q_{ij}^{ab}$ which is enough to get the first non-linear term in $\Phi(\alpha)$. The diagonal structure above has been obtained performing various manipulations on the original integral. In particular the matrix $P$ enters only though its eigenvalues because the relevant expressions are rotationally invariant at the order considered and the integral in $S[Q, \alpha]$ is invariant under a simultaneous rotation of all the blocks $Q_{ij}^{ab}$. The parameter $p$ has been introduced for later convenience and has to be put to zero eventually (it has the physical meaning of adding a small perturbation on the couplings of replicas in different blocks effectively removing the degeneracy of permutations among them). Note that in the previous expression the off-diagonal indices run from $1$ to $-N/n$ because we have exploited the fact that the function $S[Q, \alpha]$ in the thermodynamic limit is invariant under a rescaling $\{\alpha \rightarrow b \alpha, N \rightarrow b N\}$.

The above expression is suitable for an expansion in powers of $\alpha$. The first $O(\alpha^2)$ term is obtained performing the Gaussian integral:

$$ S[Q, \alpha] = \frac{\alpha^2}{4n^2} \sum_{ab} \ln(1 - \beta^2 \lambda_a \lambda_b + p) + O(\alpha^3) \quad (5) $$

In the limit $\alpha \rightarrow 0$ the extremum of $\Phi[Q, \alpha]$ is given by the extremum of $F[Q]$ with the first $O(\alpha^2)$ correction evaluated on the free solution where $P_{ab} = \delta_{ab} + Q_{ab}$. In the $n \rightarrow 0$ limit the Gaussian correction turns out to be divergent if $p = 0$, this is because the lowest eigenvalue $\lambda_0$ of $P_{ab}$ obeys the following relationship: $\lambda_0 \equiv 1 - \int q(x) dx = 1/\beta$ that holds exactly at all temperatures. The divergence of the $O(\alpha^2)$ correction suggests that the first non-linear term in $\Phi(\alpha)$ has a power smaller than two. We also expect that the series in powers of $\alpha$ will be divergent at all orders and we will have to resum it in some way.

Before going into the next step of the computation, we consider a similar treatment of the spherical model. At zero temperature the energy is minus the largest eigenvalue of a random Gaussian $N \times N$ matrix. The logarithm of the probability that the largest eigenvalue is lower than its typical value is indeed $O(N^2)$ and has been recently computed. These results tell us that in the zero-temperature limit we have: $\Phi(\alpha) = - \alpha \beta - \frac{\alpha^2}{4} \beta + o(\alpha^3/2)$. Repeating the previous procedure we obtain that for the spherical model $\Phi(\alpha)$ can be computed as the saddle-point value of a functional similar to $\Phi(\alpha)$.

The first $O(\alpha)$ term depends on a symmetric $n \times n$ matrix $Q_{ab}$ (not vanishing on the diagonal) and on a parameter $z$ enforcing the spherical constraint:

$$ F[Q, z] = \frac{\beta}{4n} \sum_{ab} Q_{ab}^2 - \frac{z}{\beta} + \frac{1}{2\beta n} \text{Tr} \ln[zI - \beta^2 Q] \quad (6) $$
where $I$ is the identity matrix. The second term is equal (at the third order in $Q^{ij}_{ab}$) to expression (1) provided we have (in matrix notation) $P = (2zI - \beta^2 Q)^{-1}$. The solution of the spherical model in the $\alpha \to 0$ limit is given by a replica-symmetric (RS) $n \times n$ matrix $Q_{ab} \equiv q = 1 - T$ [10] and again the lowest eigenvalue $\lambda_0$ of $P$ turns out to obey the relationship $\lambda_0 = 1/\beta$ leading to a divergent $O(\alpha^2)$ correction.

We now face the task of computing the leading corrections that diverges in the $p \to 0$ limit. This can be done by studying the diagrammatic loop expansion of $S(Q, \alpha)$. We introduce capital indices that corresponds to a couple of upper and lower indices $A = (\alpha)$. Given a graph with vertices of degree three we have to associate to each line in it a couple of different capital indices in such a way that each vertex of the graph corresponds to a term in the action of eq. (1). This limits the number of $K$ free indices and for a graphs with $L$ loops one can prove that $K \leq L + 1$. Valid indexed graphs $G_I$ at four loops are represented in fig. 1.

For a graph $G_I$ with $V$ vertices we have a factor $-(\alpha/N)^{V/2+2}/V!$ times a factor $\beta^2 \lambda_0 \lambda_b / (p + 1 - \beta^2 \lambda_0 \lambda_b)$ where $ab$ are the lower indices on each line. We sum over all the free $K$ upper indices (they can take values $1, \ldots, N/n$) and vertex permutations of the graph. This yields a factor $V!(-N/n)^K$ that has to be divided by an appropriate symmetry factor $M(G_I)$ to avoid overcounting. Relevant graphs are those that yield an $O(N^2)$ contribution, i.e. they satisfy the condition $K = V/2 + 2$ or equivalently $K = L + 1$. Consistently it can be shown that no graph can yield a contribution greater than $O(N^2)$.

Summarizing the sum over the upper indices yields the factor:

$$\frac{(-\alpha)^K (1)^{K+1}}{M(G_I) n^K},$$

that multiplies the result coming from the sum over the lower indices that we analyze now. In order to sum over the lower indices we have to take care of the non trivial structure of the propagator. In particular given an indexed graph each line with lower indices $ab$ in the graph corresponds to a factor $\beta^2 \lambda_0 \lambda_b / (p + 1 - \beta^2 \lambda_0 \lambda_b)$. The resulting object has to be summed over the $K$ free indices (each running from 1 to $n$) leading to:

$$\frac{1}{n^K} \sum_{a_1, \ldots, a_K} \prod_{i=1}^{n} \frac{\beta^2 \lambda_a \lambda_{a_i}}{p + 1 - \beta^2 \lambda_a \lambda_{a_i}}$$

where we have borrowed the factor $n^{-K}$ from (7).

We consider the first RS case: after some algebra we find that at the leading order in $p$ the final result at leading order in $p$ is:

$$\frac{1}{p^{K+1}} \left( \frac{d^K}{dx_1 \ldots dx_K} \prod_{i=1}^{n} \frac{1}{1 - x_a - x_b} \right)_{x=0},$$

where $I$ is the number of lines of the graph.

In the full RSB case it turns out that the dominant divergent contribution comes from shape of the function $q$ around $x = 0$ that we assume to be linear. At the end of the day we obtain that at leading order in $p$ the contribution of the diagram is:

$$\frac{1}{p^{N+1}} \prod_{i=1}^{n} \frac{1}{1 - x_a - x_b}$$

where we used the operator $A_x[g] \equiv \int_0^\infty \frac{1}{2\pi} \frac{da}{a} \frac{d}{dx_a} dx_a$.

As we already noticed a relevant diagram has $K = L+1$ therefore in the cubic theory we have $I = 3K-6$. Rescaling the variable $p$ respectively as $p = (\alpha \beta \sqrt{Q(0)}/z)^{1/3}$, $p = (\alpha \beta q/z)^{1/4}$ and multiplying by the factors coming from (7) we get an expression of $S(Q, \alpha)$ in terms of two functions $f_{RS}(z)$ and $f_{RSB}(z)$. The loop expansion yields the series of the two functions in powers of $z$.

Taking the $z \to \infty$ limit we eventually obtain:

$$S_0 = (\alpha \beta \sqrt{Q(0)})^{12/7} C_{RSB} + o(\alpha)^{12/7}$$

$$S = (\alpha \beta q)^{3/2} C_{RS} + o(\alpha)^{3/2}$$

where $C_{RSB} = \lim_{z \to -\infty} x_{2}\phi_{RSB}(z)$ and $C_{RS} = \lim_{z \to -\infty} z_{2}\phi_{RS}(z)$. At fourth loop order the diagrams are shown in fig. 1 and lead to the following expressions:

$$f_{RSB}(z) = -\frac{\pi}{8} + 0.456 z - 3.278 z^2 + 36.11 z^3 + O(z^4).$$

$$f_{RS}(z) = -\frac{1}{4} + \frac{7}{6} z - 19 z^2 + 443 z^3 + O(z^4).$$

The zero-th order terms comes from a similar treatment of the Gaussian contribution [3]. The problem of extracting the $z \to \infty$ behaviour of functions like $f_{RSB}(z)$ and $f_{RS}(z)$ from their expansions in powers of $z$ has
already appeared in spin-glass literature \cite{12} in a similar context and various resummation methods have been devised. Performing similar analysis on the above series we obtained the following estimates: $C_{RS} = -0.68$, $C_{fRSB} = -0.66$ with a 15% error. The estimate for $C_{RS}$ is in good agreement with the exact result $C_{RS} = -2/3$ quoted above.

In the fRSB case we obtain from \cite{11}:

$$\frac{1}{N^2} \ln P(\Delta f) = -a_+ [\dot{T} \dot{q}(0)]^{-6/5} \Delta f^{12/5} + o(\Delta f^{12/5})$$

(12)

Where $a_+ \equiv (-C_{fRSB})^{-7/5} 35(7/3)^{2/5}/(24/5)^{144}$. The numerical data of Ref. \cite{14} at zero temperature are highly consistent with the $12/5$ scaling, see fig. (2): a linear fit on the data corresponding to $N = 150$ combined with $\lim_{T \to 0} T \dot{q}(0) = 0.743$ \cite{2} leads to an estimate $C_{fRSB} = -0.64(2)$.

Remarkably the previous results display a great deal of universality. Indeed the exponents $12/7$ and $3/2$ and the coefficients $C_{fRSB}$ and $C_{RS}$ depends only on the fRSB or RS structure of the eigenvalues of the matrix $P$. The sole dependence on the actual model and on the temperature being respectively through the parameters $\sqrt{T \dot{q}(0)}$ and $q$. Interestingly enough the same universal behaviour is displayed by the function $L(\Delta f)$, again both for RS and fRSB \cite{2}. For both $L_2(\Delta f)$ and $L(\Delta f)$ we expect the universal behaviour to hold only at the leading order for small $\Delta f$, nevertheless this points towards universality of the corresponding small-deviations distribution. Indeed the reader may have already noticed that the $12/5$ exponent in \cite{12} is fully consistent with the assumptions that the large positive deviations match the far right tail of the distribution of the small-deviations of the free energy density if they scales as $f - f_N = O(N^{1/6})$ \cite{14}, where $f_N$ is the sample average at size $N$. At present the latter hypothesis is widely accepted in the literature after many recent numerical and theoretical investigations (see e.g. \cite{2, 6, 14}) and our findings add further support to it.

We conjecture that the probability distribution $P(\delta)$ of the of the rescaled variable

$$\delta = (f - f_N)N^{5/6}/\sqrt{T \dot{q}(0)}$$

(13)

is of the form $P(\delta) = \exp(-G(\delta))$ where the large $\delta$ behaviour of $G(\delta)$ matches with the behaviour near the origins of the functions $L_2(\Delta f)$ and $L(\Delta f)$, i.e.

$$G(\delta) \simeq a_- |\delta|^{6/5} \quad \text{for} \quad \delta \to -\infty$$

(14)

$$G(\delta) \simeq a_+ |\delta|^{12/5} \quad \text{for} \quad \delta \to +\infty$$

(15)

with $a_- = 1.366$ \cite{2} and $a_+ = 0.36$.

It is natural to assume that for a large class fRSB spin-glass model not only the large $\delta$ behaviour of $G(\delta)$ is independent from of the model, but that the function $G(\delta)$ does not depend on the model for all $\delta$ and it is the same at all temperatures below the critical one. A similar results should be valid for replica symmetric models where the equivalent function $H((f - f_N)N^{2/3}/q)$ is given by Tracy-Widom function \cite{3}.

\[\text{References}\]

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