Quantum bit threads of MERA tensor network in large $c$ limit

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The Ryu-Takayanagi (RT) formula is a crucial concept in current theory of gauge-gravity duality and emergent phenomena of geometry. Recent reinterpretation of this formula in terms of a set of “bit threads” is an interesting effort in understanding holography. In this paper, we investigate a quantum generalization of the “bit threads” based on tensor network, with particular interests in the multi-scale entanglement renormalization ansatz (MERA). We demonstrate that, in the large $c$ limit, isometries of the MERA can be regarded as “sources” (or “sinks”) of the information flow, which extensively modifies the original picture of the bit threads by introducing a new variable $\rho$: density of the isometries. In this modified picture of information flow, the isometries can be viewed as generators of the flow, which is consistent with the fact that isometries are generators of dilation. The strong subadditivity and related properties of the entanglement entropy are also obtained in this new picture.

I. INTRODUCTION

One of the most important developments in AdS/CFT correspondence in the past few years is the discovery of the Ryu-Takayanagi (RT) entanglement entropy formula[1]. This formula states that entanglement entropy of a subregion $A$ of a $d + 1$ dimensional CFT on the boundary of $d + 2$ dimensional AdS is proportional to the area of a certain codimension-two extremal surface in the bulk:

$$S_A = \frac{\text{area}(m(A))}{4G_N}, \quad (1)$$

where $m(A)$ is the minimal bulk surface in AdS time slice, which is homologous to $A$, i.e., $m(A) \sim A$. This formula, connecting two important concepts in different fields, suggests some deep relations between quantum gravity and quantum information. Recent progress clearly shows that the RT formula plays a central role in understanding the emergence of spacetime.

When exploring the conceptual implications of the RT formula, however, it was firstly noticed by Freedman and Headrick in [2] that there are some subtleties of the formula. For instance, there is a strangely discontinuous transition of the bulk minimal surface under continuous deformations of $A$. To remove these subtleties, they invoked the notion of “flow” which is defined as a divergenceless norm-bounded vector. It turns out that, with the help of the max flow-min cut (MF/MC) principle, this “flow” interpretation of the RT formula is more reasonable: the discontinuous jump disappears and there is more transparent information-theoretic meaning of the properties of the entanglement entropy. In construction of the flow picture of RT formula, the MF/MC theorem plays a crucial role. It roughly states that in some idealized limit, the transport capacity of a classical network is equal to a measure of what needs to be cut to totally sever the network.

The above picture, however, is logically incomplete, considering the whole picture is built on classical theory of network. More precisely, it seems weird that quantum states (or qubits) are transported by a classical network. More reasonable picture should be replaced by a quantum flow network which is tensor network as will see below. In this sense, the above “flow” picture of the RT formula is a semi-classical approximation of some unknown quantum (and fundamental) formulations.

For this sake let us move to a tensor network description of quantum physics. Recent study of entanglement in strongly coupled many-body systems has developed a set of real-space renormalization group methods such as the tensor network state representation. In the past few years it has been extensively studied in statistical physics and condensed matter physics. A tensor network description of wavefunctions of a quantum many-body system has a merit to tremendously reduce the number of parameters (from exponential to polynomial) needed in the computation. This makes it a very efficient representation of the wavefunction of the system. In addition, tensor network representation allows an easy way to visualize the entanglement structure, and the area law of the entanglement entropy is inherent in the tensor network. More attractive property comes from connections between tensor network and the AdS/CFT correspondence[4], which was first pointed out by Swingle in [5], where he noticed that the renormalization direction along the graph can be viewed as an emergent (discrete) radial dimension of the AdS space. From this perspective, the holography stems from physics at different energy scales and the AdS geometry can be emerged from QFTs[6]. As to the holographic entanglement entropy, the tensor network-based RT formula can be interpreted as sum over all d.o.f of

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neighbor sites from UV to IR.

Based on these considerations, one natural question is this: what is the “flow” picture of the tensor network-based RT formula? In this paper we mainly pay attention to the answer to this question. It turns out that the solution needs the quantum MF/MC(QMF/QMC) theorem which is found recently in [7, 8]. This theorem, which is quantum analogy of the MF/MC for tensor networks, states that the quantum max-flow of a tensor network is no bigger than the quantum min-cut of the network. Particularly, for some tensor networks such as MERA, the quantum max-flow is equal to the quantum min-cut, in the large central charge c limit. Based on this theorem and information-theoretic considerations, we define a new variable \( \rho \) which can be interpreted as the density of the tensor networks under question. Physically integral of \( \rho \) over a region can be viewed as the source (or sink) of the tensor network and plays a significant role in the flow description of the RT formula. We shows that it determines the structure of the tensor network on the basis of a fixed Lorentzian manifold \((M, g)\). More precisely, \( dV_{\text{network}} = \rho(x)\sqrt{g}dV \) of this tensor network. In addition, from information point of view a tensor network is a quantum circuit that maps a reference state to a target state by the network and quantum gates(tensors). In this language, \( \rho \) is the density of compression or decompression of quantum bits through reducing or expanding the dimensions of Hilbert space. Specifically, MERA tensor network where \( \rho =\text{constant} \) has \( dS_2 \) geometry, if we regard MERA as kinematic space of an AdS \(_3 \) timeslice as first pointed out in [9]. In this case, our formulation suggests a naive picture that the evolution of our universe can be regarded as a huge and complex quantum circuits and inflation is a progress that decompresses and entangles quantum bits continuously.

The organization of this paper is the following. In section 2 we first give a brief review on Freedman-Headrick’s proposal of bit threads and holography, followed by a brief introduction of the QMF/QMC theorem. To find out the relation between the QMF/QMC theorem and the RT formula in MERA tensor network, in this section we also have reviewed the MERA on kinematic space. In section 3 we give a information-theoretic interpretation of the MERA, with particular interests in the information-theoretic meaning of the isometry. In section 4 we propose our “flow” language of a tensor network with the help of QMF/QMC theorem. Several physical key points of the picture are discussed in section 5. In last section we draw our main conclusions and discussions.

II. BACKGROUND SETUP

A. Bit threads and holography

The RT formula (1) can be reinterpreted as a “max-flow” from max-flow/min-cut(MF/MC) theorem in Riemannian geometry as firstly explored in [2]. To understand this point, we define a divergenceless vector field \( v \) as a “flow” satisfying the following two properties[2]:

\[
|v| \leq C, \quad \nabla v = 0,
\]

where \( C \) is a positive constant. Then a flux of \( v \) that through an oriented manifold surface \( m \sim A \) can be defined as an integral:

\[
\int_{m(A)} v := \int_{m(A)} \sqrt{h} n_{\mu} v^\mu,
\]

where \( h \) is the determinant of the induced metric on \( m \) and \( n_{\mu} \) is the unit normal vector. The maximal flux should be bounded by a bottleneck

\[
\int_{A} v = \int_{m(A)} v \leq C \int_{m(A)} \sqrt{h} = C \text{ area}(m).
\]

This inequality is staturated by \( C \) where \( n_{\mu} v^\mu = C \) holds. This indicates that a flow reaches its maximum if and only if \( n_{\mu} v^\mu = C \) holds, and the maximal flux is equal to the minimal area multiplying a constant:

\[
\max_{v} \int_{A} v = C \min_{m \sim A} \text{ area}(m).
\]

It is necessary to introduce two extensions of the theorem. Firstly, when we change \( A \) continuously the maximal flow \( v(A) \) also varies continuously. Secondly, consider two disjoint regions \( A \) and \( B \) of the boundary, in general we cannot find a flow which maximizes the flux through \( A \) and \( B \) simultaneously, i.e.

\[
\int_{A} v + \int_{B} v \leq C \text{area}(m(AB)) \quad < C \text{ area}(m(A)) + C \text{ area}(m(B)).
\]

We call this “nesting” property.

Now return to holography. One can replace the minimal area with the maximal flow and the RT formula can be rewritten in the following way:

\[
S(A) = \max_{v} \int_{A} v,
\]

where \( C = 1/(4G_N) \). Recall that a magnetic field is visualized as field lines in common. Similarly, these flow
lines \( v \) can be regarded as oriented “bit threads” from boundary to bulk. The upper bound \( 1/(4G_N) \) of flow can be interpreted as this: the bit threads cannot be tighter than one per 4 Planck areas in \( 1/N \) effects. Then a thread which emanates from boundary region \( A \) should be viewed as one independent bit of information carrying out of \( A \). From this point of view the maximal number of independent information is the entanglement entropy \( S(A) \).

From this “flow” language, we can obtain the conditional entropy and mutual information. Let \( v(A; B) \) denote the flow which not only maximizes the flux through \( A \) but also maximizes the flux through \( AB \), i.e. \( v(A) \) or \( v(A, B) \). Then the conditional entropy \( H(A|B) := S(AB) - S(B) \) can be rewritten as an expression in terms of flows [2]. Consider two regions case, the conditional entropy can be written as

\[
H(A|B) = \int_{AB} v(B; A) - \int_B v(B; A) \quad (10)
\]

\[
= \int_A v(B; A). \quad (11)
\]

At the same time, flux through \( A \) reaches its minima. From this, one can also write down the mutual information \( I(A : C) := S(A) - H(A|C) \). Without loss of generality, one can choose the entropy of \( A \): \( \int_A v(A; C) \), then \( I(A : C) \) can be rewritten as

\[
I(A : B) = \int_A v(A; C) - \int_B v(C; A) \quad (12)
\]

\[
= \int_A (v(A; C) - v(C; A)). \quad (13)
\]

This is the flux which can be shifted between \( A \) and \( C \). Similarly, in three regions case, we can also write down the conditional mutual information \( I(A : C|B) := S(AB) + S(BC) - S(ABC) - S(B) = H(A|C) - H(A|BC) \). Without loss of generality, one can choose \( H(A|B) = \int_A v(B, A; C) \) and \( H(A|BC) = \int_A v(B, C; A) \), in this way \( I(A : C|B) \) can be expressed as:

\[
I(A : C|B) = \int_A v(B, A; C) - \int_A v(B, C; A) \quad (14)
\]

\[
= \int_A (v(B, A; C) - v(B, C; A)). \quad (15)
\]

B. Quantum Max-flow/Min-cut

The quantum max-flow min-cut(QMF/QMC) conjecture was first presented in [7]. Then in [8] Cui et al. showed that this conjecture does not hold in general, but under some given conditions it is valid. There are two versions of this conjecture, we first review it by mean of introducing the first version.

Tensor network can be regarded as a graph \( G(\tilde{V}, E) \) which is undirected, meanwhile has a set of inputs \( S \) and a set of outputs \( T \). \( \tilde{V} \) is a disjoint partition \( \tilde{V} = S \cup T \) of \( V \). A set of flows \( a = \{a \}_{\tilde{e} \in \tilde{E}} \) can be interpreted as this: the bit threads cannot be more than one per 4 Planck areas in \( 1/N \) effects. Then a thread which emanates from boundary region \( A \) should be viewed as one independent bit of information carrying out of \( A \). From this point of view the maximal number of independent information is the entanglement entropy \( S(A) \).

Before that, the definition of “cut” must be stated. If there exists a partition \( \tilde{V} = \tilde{S} \cup \tilde{T} \) so that \( \tilde{S} \subseteq \tilde{S}, \tilde{T} \subseteq \tilde{T} \), then a cut \( A \) is a set that \( A = \{\{u, v\} \in E : u \in \tilde{S}, v \in \tilde{T}\} \). Intuitively, removing the edges in \( C \) will lead to disconnect path from \( S \) to \( T \).

**Definition 1** (Quantum Min-cut). The quantum min-cut QMC\((G,a)\) is the minimum value of product of capacities over all edge cut sets, i.e.

\[
QMC(G,a) := \min_A \prod_{\tilde{e} \in \tilde{C}} a_{\tilde{e}}. \quad (17)
\]

There is a linear map \( \beta(G,a;T) \in V_S \otimes V_T = \text{Hom}(V_S,V_T) \) from inputs to outputs: \( V_S \rightarrow V_T \) acting on the inputs state:

\[
\beta(G,a;T)|i_1,\ldots,i_{|S|}S| := \sum_{j_1,\ldots,j_{|T|}} C_{i_1,\ldots,i_{|S|}; j_1,\ldots,j_{|T|}} \times |j_1,\ldots,j_{|T|}T|. \quad (18)
\]
It is obvious that in this basis the matrix $C$ is exactly the $\beta(G,a;T)$. Then one can define the quantum max-flow as following:

**Definition 2** (Quantum Max-flow). For over all tensor assignments, there exists a maximal value of the rank of map $\beta(G,a;T)$ and we define this maximal value as the quantum max-flow:

$$QMF(G,a) := \max_T \text{rank}(\beta(G,a;T)).$$

Cui et al. [8] stated that $QMF(G,a)$ is not equal to $QMC(G,a)$ in general. In fact the $QMF(G,a)$ is always no larger than $QMC(G,a)$ in a given finite graph $G$: $QMF(G,a) \leq QMC(G,a)$. The equality holds in a special case that can be considered as a weak QMF/QMC:

**Theorem 1** (Quantum Max-flow Min-cut theorem). For a given graph $G(V,E)$, if the capacity $a$ of each edge is a power of $d$, where $d > 0$ is an integer, then

$$QMF(G,a) = QMC(G,a).$$

Now let us turn to the entanglement entropy between inputs and outputs of tensor network and see its relation with the QMF and QMC. The Hilbert space of a pure state (17) is $H = V_S \otimes V_T$ and one can be considered as a weak QMF/QMC. For a given graph $G(V,E)$, if the capacity $a$ of each edge is a power of $d$, where $d > 0$ is an integer, then $QMF(G,a) = QMC(G,a)$. The equality holds in a special case that can be considered as a weak QMF/QMC:

**Theorem 2**. For a given graph $G(V,E)$, if the capacity $a$ of each edge is a power of $d$, where $d > 0$ is an integer, then

$$MEE(G,a) \leq \log QMC(G,a) = \log QMF(G,a).$$

The second version of QMF/QMC conjecture is more restricted. The vertices of the same type have to be assigned the same tensor. More specifically, we put an ordering $O$ to the ends of the edges incident to each vertex and define a valence type $B_v$ of a vertex $v$ to be the sequence $(a_{e(v,1)},...,a_{e(v,d_v)})$, where $a_{e(v,d_v)}$ is dimensions of the Hilbert space of edge $e(v,d_v)$. Let $B(G,a,O)$ denote the set of valence type of vertices of graph $G$. Now the vertices with the same valence type have to be assigned the same tensor $T = \{T_B : B \in B(G,a,O)\}$.

From above we also obtain a linear map, which denoted by $\beta(G,a,O;T)$ in $Hom(V_S,V_T)$. The definition of quantum max-flow $QMF(G,a,O)$ for this second version is the maximum rank of $\beta(G,a,O;T)$, similar to the first version. The difference is here: Conditions stated in Theorem 1 are insufficient to guarantee the equality $QMF = QMC$, even when the Hilbert space dimension is same in each edge in the graph. We still need some additional conditions and we will discuss it latter.

### C. MERA on kinematic space

For the following convenience let us give a brief review on kinematic space which was firstly formulated in [9, 10]. Given a hyperbolic plane $\mathbb{H}^2$ which is a time slice of pure $AdS_3$ spacetime:

$$ds^2 = d\rho^2 + \sinh^2 \rho d\theta^2.$$  

The equation of a space-like geodesic that anchors on boundary points is:

$$\tanh \rho \cos(\theta - \theta) = \cos \alpha,$$

where $(\theta, \alpha)$ are parameters which label a oriented geodesic as shown in FIG.3. Space of all these geodesics $(\theta, \alpha)$ forms a 2-dimensional manifold which is called kinematic space. A geodesic can be described by a point $(\theta, \alpha)$ in the kinematic space. Crofton’s formula in $\mathbb{H}^2$ states that the length of a curve $\gamma$ can be measured by the number of the geodesics which intersect it, i.e.:

$$\text{length of } \gamma = \frac{1}{4} \int_K n(g,\gamma) Dg,$$

where $n(g,\gamma)$ is the number of geodesics intersect $\gamma$ and $Dg$ is the measure on the kinematic space

$$Dg = \frac{\partial^2 S(u,v)}{\partial u \partial v} du dv.$$

If the curve is a geodesic with two ends $u$ and $v$ anchoring on boundary then $S(u,v)$ is the length of the geodesic. We have used a light-cone coordinate on the kinematic space:

$$u = \theta - \alpha \quad \text{and} \quad v = \theta + \alpha.$$
Eq. (26) is also the line element of the kinematic space multiplied by some coefficient
\[ ds_{\text{kinematic}}^2 = \frac{\partial^2 S_{\text{ent}}(u,v)}{\partial u \partial v} du dv. \tag{28} \]
where we have replaced \( S \) by \( S_{\text{ent}} \) (the entanglement entropy) because of the RT formula. Therefore the entanglement entropy can be represented by a volume in kinematic space.

\[ S_{\text{ent}} = \frac{\text{length of } \gamma}{4G} = \frac{1}{4} \int_K n(g,\gamma) \frac{\partial^2 S_{\text{ent}}(u,v)}{\partial u \partial v} du dv. \tag{29} \]

Czech et al.\cite{Czech} argued that, considering the auxiliary causal structure of MERA, it is the vacuum kinematic space instead of the \( AdS_3 \) time slice that should be viewed as the corresponding geometry of the MERA. So the kinematic space becomes an intermediary in the \( AdS/CFT \).

One of the key points of this argument is the casual structure of the MERA. It turns out that this makes it more natural to match such a network with a Lorentzian manifold, as first mentioned in \cite{Hartman}. To proceed, let us consider exclusive causal cone for part of lattices. Boundary of this causal cone is called as causal cut of the MERA network as shown by the red line in FIG.4(a). The method to calculate the entanglement entropy – given a holographic interest in MERA in \cite{Czech} – is just counting the number of edges cut by the causal cut in this network, with each edge assigning a weight \( \log \chi \), where \( \chi \) is the Hilbert dimension of edges. In the same footing, one can also calculate the conditional mutual information by counting the number of edges. Recall that conditional mutual information is defined as following:

\[ I(A:C|B) = S(AB) + S(BC) - S(ABC) - S(B) \tag{30} \]

It means that the conditional mutual information can be obtained by counting the number of edges which is the net reduction of edges through a causal diamond from bottom up as shown in FIG.4(b). One may find that this is the same as counting the number of isometries living in this diamond, because every isometry has two input edges and only one output edge from bottom up. In other words, every isometry soaks up an edge so that counting the number of it is precisely equal to counting the decrease in the number of edges. It shows that the conditional mutual information is proportional to the number of the isometries in causal diamond, and is also proportional to the volume of this diamond which can be easily seen from (28).

Based on these observations, connections between conditional mutual information and volume in kinematic space can be built. Czech et al.\cite{Czech} adopted conditional mutual information as a definition of volume in MERA,

\[ D(\text{isometries}) = I(A:C|B). \tag{31} \]

This formula evaluates the amount of the volume after compressing the state living on its past edges. In this vacuum MERA, the ‘density of compression’ is proportional to the number of isometry. More explicitly, the metric of a discrete tensor network is given by

\[ ds_{\text{network}} = I(\Delta u, \Delta v|B) \xrightarrow{\text{MERA}} (\# \text{ of isometries})\Delta u \Delta v, \]

and this is the metric of kinematic space, also the volume element in kinematic space.

III. A INFORMATION-THEORETIC LANDSCAPE

As stated in the previous section, we prefer to treat MERA as a discrete kinematic space rather than the original slice of \( AdS \) space. This statement is based on the following two observations: (a) they share the same causal structure and (b) regarding entanglement as “flux” through causal cut, which equally counts the number of lines on causal cut has more natural interpretation in kinematic space\cite{Czech}. As a consequence, bit threads in \( AdS \) time slice should have an information-theoretic interpretation on kinematic space. To see this, we first recall flows in \( AdS \) time slice. A flow \( v \) is a vector field and one can define a set of integral curves of \( v \) whose transverse density equals \(|v|\). Each flow line, the so-called “bit threads”, is an oriented thread connecting two different points on the boundary. For example, given a time slice of \( AdS \), we can split boundary into two parts \( A \) and \( A^c \), then the information (flow) shared between \( A \) and \( A^c \) is

\[ I(A:A^c) = 2S(A). \tag{33} \]

A thread between \( A \) and \( A^c \) connects two points on \( A \) and \( A^c \), of which one is the start point of thread (belongs to \( A \)) and the other is the end point (belongs to \( A^c \)). These two end points can be mapped to a point in the kinematic space, which is denoted by \((u,v)\). We therefor have a correspondence between original space and kinematic space, as sketched in FIG.5.

In the previous section we have mentioned that one of important properties of the bit threads is \(|v|\leq 1/(4G_N)\), which means that one cannot contain more than \(1/(4G_N)\) information in unit area. The minimal surface \( m(A) \) is the place where flow density reaches its maximum, i.e, \(|v| = 1/(4G_N)\). This indicates that we have \(1/4G_N \) bits
information per unit area on the surface. From (33), we can think $A$ and $A^c$ share $2 \times 1/(4G_N)$ bits information per unit areas of minimal surface, or equivalently, $[u - \Delta u, u]$ and $[v, v + \Delta v]$ shared $2 \times 1/(4G_N)$. Mapping this “area” (actually is a geodesic length in 2d) to kinematic space yields a “volume” (an area in 2d) of some region as shown in FIG 5. This implies that $2 \times 1/(4G_N)$ bits information in minimal surface correspond to $2 \times 1/(4G_N)$ bits information in $\Delta u \Delta v$ in kinematic space, which is equivalent to count the number of isometries in this domain.

\[
I(\text{number of shared information})_{AdS} = I(\text{number of isometries})_{\text{kinematic}} \left|\Delta u \Delta v\right|
\]

This point can be also verified from the causal structure point of view. The shared information between $[u - \Delta u, u]$ and $[v, v + \Delta v]$ can be naively regarded as some non-intersect geodesics included in the tube $[u - \Delta u, u] [v, v + \Delta v]$, which are time-like between $[u, v]$ and $[u - \Delta u, v + \Delta v]$. In kinematic space, these are isometries in a causal diamond between $(u, v)$ and $(u - \Delta u, v + \Delta v)$, which, as expected, is the region where the conditional mutual information $I([u - \Delta u, u]: [v, v + \Delta v] | [u, v])$ is calculated.

Above analysis strongly suggests us to regard the isometries as “sources” (or “sinks”) of information. Indeed, if we deem the information flow $1/4G_N$ from up to down in MERA through isometries, then these isometries decompress it. The decomposition results in $2 \times 1/(4G_N)$ bits information, which is the conditional mutual information between $[u - \Delta u, u]$ and $[v, v + \Delta v]$. And the isometries play roles in sharing the information in these two intervals on the boundary as shown in right hand side of FIG.5. That is to say, isometries provide a local “density of compression(or decompression)” of network and such MERA network can be regarded as an iterative compression algorithm which maps the density matrix of a interval to a compressed state on causal cut.

Now let us turn to the entangler of the tensor network. Above we have already known that the isometry acts like a decompression(or compression) into information, as well as the key factor for the space-time structure. The entangler, however, does not contribute to the flow in the domain. In MERA, entangler plays a role of creating new entanglement between neighboring sites. So in this quantum channel, the entangler plays the same role as in MERA. They mix the information and create the entanglement between sites. From space-time point of view, these entanglers entangle the space. Geometrically, as one decreases the entanglement between the degrees of freedom for two region, the distance between points increases[12]. In other words, entangler plays a role in “gluing” the space together that we cannot ignore it.

IV. TOWARDS A TENSOR NETWORK/FLOW CORRESPONDENCE

A. QMF=QMC implies a isometric tensor

In this subsection we consider the QMF=QMC case. Under this condition, the tensor between a causal cut and boundary is an isometric tensor which is defined in [13]. We set the causal cut as the inputs $S$ of network and the corresponding boundary region as the outputs $T$ of network. Now identifying $V_S$ with $V_T$ using the chosen basis in $V_S$, one can determines an $\alpha(G, a; T)$. We denote the basis of $V_S, V_T$ as $|i\rangle_S, |j\rangle_T$ and let matrix of $\beta(G, a; T)$ be $C$ under this basis, we have

\[
\alpha(G, a; T) = \sum_{i,j} C_{ji} |j\rangle_T \langle i|_S.
\]

In other words, $\beta$ is a tensor which map from causal cut to boundary:

\[
\beta : |i\rangle_S \mapsto \sum_j C_{ji} |j\rangle_T
\]

We assumed that $S$ is a minimal cut and $\dim(S) \leq \dim(T)$. If QMF/QMC conjecture hold in this tensor network QMF=QMC. Then after an appropriate ordering of the basis elements in $V_S$ and $V_T$, the map $\beta$ have such simple form [8]:

\[
\begin{pmatrix}
1 \\
\vdots \\
0
\end{pmatrix}
\]

Let $M = QMC(G, a)$ is the dimension of inputs, The upper block matrix is $M \times M$. So we have

\[
\sum_j \beta_{ij}^\dagger \beta_{ji} = \delta_{ij}.
\]
that implies the bipartition of network have maximal entanglement between two parts of this network (\(|S|\) is the number of cut legs and \(a\) is the Hilbert dimension of each edge.). In the next subsection our model is considered in the QMF=QMC case and more interpretation about this will be showed in section VA.

**B. General setup**

We start by explaining a tensor network in a “flow” language that is convenient for our discussion. Suppose there is a flow through a boundary, which can be denoted by \(f^\mu\). Its flux \(\mathcal{F}\) through a boundary region \(A\) is obtained by integration \(f^\mu\) over this region:

\[
\mathcal{F} = \int_A f := \int_A \sqrt{|h|} n_\mu f^\mu. \tag{39}
\]

where \(h\) is the determinant of the induced metric on \(A\)(Actually because of the IR divergence we should choose a IR cut-off surface \(A_{\infty}\), but for simplification we still use \(A\) to denote it). It is obvious that it satisfies the additivity

\[
\int_A f + \int_B f = \int_{AB} f. \tag{40}
\]

Now let us consider a cut \(C_A \sim A\) to be an oriented codimension-one submanifold in network which is homologous to \(A\). As claimed in the last section, isometries in the tensor network may plays a role of source (or sink), therefore, in general \(\int_A f \neq \int_{C_A} f\). Instead, one extra term which describes the contributions from tensors should be added

\[
\int_{C_A} f = \int_A f + \int_{D_A} \rho. \tag{41}
\]

where \(\int_{D_A} \rho := \int_{D_A} \rho \sqrt{-g} d\omega\) and \(g\) is the determinant of the metric on this Lorentzian manifold. \(D_A\) is the region enclosed in \(A \cup C_A\). \(\rho\) can be viewed as density of tensors and should satisfy the following two properties:

\[
|\rho| \leq \rho_M, \tag{42}
\]

\[
\nabla_\mu f^\mu = -\rho, \tag{43}
\]

where \(\rho_M\) is a positive constant. The first constraint implies finiteness of the density. From (41), it is straightforward to get

\[
0 = -\int_{AB} f - \int_{BC} f + \int_{ABC} f + \int_B f = \int_{D_A} \rho + \int_{D_B} \rho - \int_{D_{ABC}} \rho - \int_{D_B} \rho + \int_{C_B} f - \int_{C_{ABC}} f + \int_{C_B} f = \int_D \rho + \int_{\partial D} f. \tag{44}
\]

where \(D\) is the region \(D_{ABC} + D_B - D_{AB} - D_{BC}\). That means for an arbitrary region in a network the Gauss’ theorem is always tenable. The second term of last equality in (44) is the flux and can be denoted by \(D\).

\[
\mathcal{D} := \int_{\partial D} f = -\int_{D} \rho. \tag{45}
\]

This implies that a flow is incoming from the bottom up of the causal diamond. Meanwhile, the constraint \(|\rho| \leq \rho_M\) implies the flux is bounded by

\[
|\mathcal{D}| \leq \rho_M \int_D \sqrt{|g|} d\omega = \rho_M V_D. \tag{46}
\]

So what does the flux stand for in this picture? It turns out that it is more reasonable if we regard the flux \(\int_{C_A} f\) as the logarithm of the rank of \(\beta(G,a;T)\) \[14\]. In other words, we have treated edges on \(A\) as inputs and edges on \(C_A\) as outputs as shown in FIG.6(a), and the flux is given by

\[
\int_{C_A} f := \log \{\text{rank} \ \beta(D_A)\} = \int_{D_A} \rho + \int_{A} f. \tag{47}
\]

Assume we have chosen a tensor assignment that makes the rank \(\beta(D_A)\) maximal. From QMF/QMC conjecture one can obtain immediately that:

\[
\int_A f = -\int_{D_A} \rho + \int_{C_A} f = -\int_{D_A} \rho + \max \log \{\text{rank} \ \beta(D_A)\} \leq -\int_{D_A} \rho + \log \ \text{QMC}(D_A). \tag{48}
\]

The second equality holds when the QMF/QMC theorem is satisfied.

Now let us consider a case where all the cuts \((C_{AB}, C_{BC}, C_{ABC} \text{ and } C_B)\) in (44) are causal cuts, then it determines a causal domain \(D\). One would define coordinates \((u, v)\) on this network and choose \(A\) and \(B\) arbitrary as
Quantum max-flow/min-cut theorem [8] states that for a tensor network whose Hilbert space dimension of every edge is a power of an integer \(\chi\), then QMF=QMC. Hence, in the following considerations, we pay attention to tensor networks where each edge’s capacity is a power of integer. In this case, the Hilbert space dimensions of edges associated with a tensor of degree \(m\) are given, respectively, by \(\chi^{d_1}, \chi^{d_2}, \cdots, \chi^{d_m}\), where \(d_1, d_2, \cdots, d_m\) are all non-negative integers. We can map this graph to a graph of degree \((d_1 + d_2 + \cdots + d_m)\) such that the Hilbert space dimension of each edge is \(\chi\). For instance, consider a simple case \(T_{ijk}\), where \(m=3\), i.e., two input edges and one output edge. The output edge has \(\chi^2\) capacity and the rest edges have \(\chi\) respectively, see FIG.7(a). Suppose we have a graph that has two parallel edges connecting \(a\) and \(b\). Clearly there is a one-to-one mapping between left side and right side in FIG.7(a), which preserves the rank and each tensor \(T_{ijk}\) in Hilbert space \(\mathbb{C}^{C^{n^2}}\) can be reshaped as \(T_{ijk,k_2}\). After decomposition the capacity of each edge of the tensor becomes \(\chi\). In a word, any tensor where capacity of each edge is power (maybe different) of integer can be reshaped to a tensor where the capacity of each edge has the same power of the integer.

**C. QMF/QMC give the density of compression in MERA**

Now we are on the point of thinking the potential applications of QMF/QMC to the MERA tensor network. In this case, one thing should be careful. Generically, given a tensor network, we have the freedom to assign tensors in network. For MERA, however, it is usually homogeneous. We usually put the same isometries and entanglers everywhere. If this is the case, instead of the first version of QMF/QMC, we should adopt the second version of the QMF/QMC. This leads to a problem: we can not make sure whether there exits a type of tensor which satisfies \(QMF=QMC\). Fortunately, this equality holds asymptotically under specific conditions, as what we will talk in VB.

Now let us suppose that all edges of the MERA are associated with the same Hilbert space dimension \(\chi\), and \(QMF=QMC\) is satisfied when we consider a large \(c\) limit as will be discussed in VB. We now consider an exclusive causal cone of region \(A\) as a sub network \(D_A\) of the whole network. The edges living on UV cutoff of the network \(D_A\) are set to be inputs, while the edges emanating from the exclusive causal cone are regarded as outputs (see FIG.6(a) for detail). For MERA network, it is obvious that the edges which are cut by causal cone form a minimum cut set of the network \(D_A\) because the number of edges living on a space-like cut is always more than the one which lives on a causal cut. For simplification, we assume that the number of output edges is \(k\) so that the dimension of that Hilbert space (also QMC of this sub network) is equal to \(\chi^k\). Then one can simply obtain the quantum max-flow of \(D_A\) as \(\chi^k\) due to the QMF/QMC theorem.

From [8], we know that the entanglement entropy between inputs and outputs for this network reaches its maximum \(MEE(D_A) = \log QMC(D_A) = \log \chi^k = k \log \chi\). Clearly, it shows that the entanglement entropy is equivalent to counting the number of edges cut by the causal cut where the weight of each edge is just \(\log \chi\). From these we can argue that the entanglement entropy can be regarded as a “flux” through the causal diamond of an oriented Lorentzian manifold as pointed out in [9] and the flux is determined by QMC\((D_A)\). The maximum flux of a edge is equal to \(\log \chi\). In other words, given an arbitrary edge of a network the corresponding flux has an upper bound \(\log \chi\). That is to say, if a network satisfies the QMF/QMC theorem everywhere then the flux of each edge would achieve its maximum value. Recalling our argument (44) and (48) in II.A, the equality in (48) holds because of the QMF/QMC theorem and one can obtain the flux of a causal diamond \(D\) by (44)

\[
D = -\left(\int_{D_{AB}} \rho + \int_{D_{BC}} \rho - \int_{D_{ABC}} \rho - \int_{D_B} \rho\right)
= -\rho\left(\int_{D_A} + \int_{D_{BC}} - \int_{D_{ABC}} - \int_{D_B}\right)
= -\rho V_D
= \log \frac{QMC(D_{AB}) \cdot QMC(D_{BC})}{QMC(D_{ABC}) \cdot QMC(D_B)}.
\]

In the second line \(\rho\) is taken out from integral because MERA has same isometries everywhere so it is independent of \((u,v)\). One should be able to notice that in MERA the causal cut is also the min-cut of \(D_A\). The last line of (50) can be expressed as \((k_{AB} + k_{BC} - k_{ABC} - k_B) \cdot \log \chi\), where \(k_{AB}\) is the number of min-cuts of \(D_{AB}\) and so on.

Obviously, canceling incoming and outgoing flow will yield flux \(D\). In [9] the authors claimed that the number of remaining edges is equal to the number of isometries inside the diamond we consider. Eq. (50) shows that
\( D = \rho V_D \) when the QMF/QMC theorem holds. Recalling the fact that the flux \( D \) is exactly proportional to the number of isometries inside the diamond since the flux of each edge is at most \( \log \chi \). We thus call \( \rho \) the density of isometries. It implies from (50) that the number of isometries is directly proportional to volume. This recovers the argument given in [9]. In that article, the density of compression of a compression network is proportional to the number density of isometries for a vacuum MERA. Meanwhile the volume of the causal diamond stands for some conditional information for corresponding regions \( A, B \) and \( C \) on the boundary as shown in FIG.7(b). Based on these observations, the authors claimed that \( D(\text{isometries}) = I(A, C|B) \), which is a relation between the number of isometries and corresponding volume. This statement is consistent with our argument given above (50), namely, as the QMF/QMC theorem holds for a given network, \( \rho V_D \) then can be interpreted as the density of compression. From this point of view, physically, \( \rho \) can be viewed as density of isometries or equivalently density of compression.

From FIG.7(b), it is obvious that a causal diamond contains numbers of tilted chessboards, each of them corresponds to an isometry. This implies that the volume of every minimum chessboard (or unit chessboard) is same because it contains only one isometry. This property turns out to be the key for the tensor network to have the geometry of \( ds_2 \). This can be also obtained from (50) when \( D \) is an infinitesimal causal diamond. Indeed, it follows from [8, 9] and (50) that

\[
dV = D = |\rho|V_D = I(A, C|B) = 4 \log \chi \cdot \frac{dudv}{(v - u)^2}.
\]

The last equality we will obtain in the coming discussion. One can find directly that

\[
|\rho| = 2 \log \chi.
\]

### D. Holographic entanglement entropy

In this subsection we return to discuss the holographic entanglement entropy, in the framework of our flow language. Before doing that, we should claim two important properties of this flow which are useful in our following discussion. Consider a tensor network which includes coarse-grainings (or isometries), firstly we assume that the Hilbert space dimension of each edge is equal. For such a network, going along renormalization flow each step of coarse-graining will reduce the number of edges. That means in a causal domain \( D \) the lower cut number is always greater than the upper cut number. We assume that the flow runs along the RG flow direction. The first property is that the flux of this region \( \oint_{\partial D} f \) is always nonnegative, from (44) it educes

\[
\int_D \rho \leq 0.
\]

The second property is deduced from the first property apparently. Suppose we have two regions \( A, B \) of the boundary, then the flux \( D_{AB} \) is always greater than the sum of \( D_A \) and \( D_B \):

\[
\int_{D_{AB}} \rho \leq \int_{D_A} \rho + \int_{D_B} \rho.
\]

Generalization to more than two regions is the same. This inequality implies that there exist some densities of compression of \( D_{AB} \) that are not included in \( D_A \) and \( D_B \). Actually these densities provide the conditional mutual information between \( A \) and \( B \) on boundary as shown in [9].

For a given flow, it is easy to check from the second property that

\[
\int_{C_{AB}} f = \int_{AB} f - \int_{D_{AB}} \rho \\
\leq \int_A f - \int_{D_A} \rho + \int_B f - \int_{D_B} \rho \\
= \int_{C_A} f + \int_{C_B} f.
\]

Now return to the MERA network which exhibits these two properties properly. Then the RT formula can be represented by

\[
S(A) = \max \int_{C_A} f,
\]

which is the maximum flux through the causal cut \( C_A \). We can also rewrite it as \( \int_{C_A} f(A) \). Then by using the second property (54), and choosing a flow \( f(AB) \) which maximizes the flux through \( AB \), we have

\[
S(A) + S(B) \geq \int_{C_A} f(AB) + \int_{C_B} f(AB) \\
\geq \int_{C_{AB}} f(AB) = S(AB).
\]

This is nothing but the subadditivity of the entanglement entropy. The first property (53) also implies this property, considering the mutual information \( I(A : B) := S(A) + S(B) - S(AB) = \int_D \rho \geq 0 \), where \( D = D_{AB} - D_A - D_B \). Similarly, for the case including three regions, we choose a flow \( f(ABC) \) which maximizes the flux through \( A, B, AB, BC \) and \( ABC \) simultaneously. Eq.(53) leads to the strong subadditivity of the entanglement entropy

\[
I(A : C|B) = \int_{C_{ABC}} f(A, B, C) + \int_{C_B} f(A, B, C) \\
- \int_{C_B} f(A, B, C) - \int_{C_{ABC}} f(A, B, C) \\
= S(AB) + S(BC) - S(B) - S(ABC) \\
= - \int_D \rho \geq 0.
\]
Moreover, if we define a region $A'$ as a copy of $A$, then the two properties show

$$S(AB) + S(A) \geq \int_{AB} f(B, A', A) + \int_{D_{AB}} \rho$$

$$+ \int_{A'} f(B, A', A) + \int_{D_{A'}} \rho$$

$$\geq \int_{A'AB} f(B, A', A) + \int_{D_{A'AB}} \rho$$

$$\geq \int_{D_{A}} \rho + \int_{B} f(B, A', A) = S(B)$$

(59)

This is the Araki-Lieb inequality.

V. INTERPRETATION

In this section we try to give more details about the quantum bit threads model from physical point of view, such as the auxiliary space-time and the role of the central charge $c$ of boundary quantum system. We will see because of $\rho$ = constant, the 2D space-time structure constructing by a coarse-graining tensor network is a $dS_2$, which is the same as kinematic space. We can also obtain the relation of Hilbert dimension $\chi$ and central charge $c$ of boundary theory. On the other hand, We will talk about the role of central charge $c$ in QMF/QMC and space-time.

A. The auxiliary space-time

The kinematic space of an $AdS_3$ time slice, which is equivalent to an auxiliary $dS_2$ according to the first law of entanglement entropy[15, 16], can be constructed by the conditional mutual information of a boundary system.

It follows from (50) that the conditional mutual information can be written as $\log_2 \chi (k_{AB} + k_{BC} - k_{ABC} - k_B)$, where $k_{AB}, k_{BC} \cdots$ are, respectively, the numbers of edges cut by $C_{AB}, C_{BC}$ and so on. For such a coarse-graining tensor network, a region with length $l_{AB}$ satisfies $l_{AB} \cdot e^{-k_{AB}/2} \sim 1$, namely,

$$\chi \approx 2 \log l_{AB}.$$  

(60)

Consider the case that regions $A : (u - du, u), B : (u, v)$ and $C : (v, v + dv)$ construct a volume element on $(u, v)$ in the kinematic space, one can obtain

$$dV_{network} \approx 2 \log \chi \left( \log \frac{(v - u + du)(v - u + dv)}{(v - u + dv + du)(v - u)} \right)$$

$$= 2 \log \chi \cdot \frac{du dv}{(v - u)^2} + O(dudv),$$

(61)

which is the conditional mutual information $I(A : C|B) = \partial_u \partial_v S_{\text{ent}}(u, v) du dv$. Comparing it with the entanglement entropy of the boundary interval $(u, v)$, i.e, $S_{\text{ent}} = (c/3) \log (v - u)/\epsilon$, we have

$$\log \chi \approx \frac{c}{6}. $$

(62)

One thing deserves emphasis. The above discussion is applied in the planar coordinates of $dS_2$. Its topology is a plane $\mathbb{R}^1 \times \mathbb{R}^1$ rather than a cylinder $S^1 \times \mathbb{R}$. A plane implies that its cylinder circumference is much larger than the interval, $\Sigma \gg (v - u)$ (actually it is infinite). In other words, if we consider a lattice model on the boundary, the number of sites in $(v - u)$ is much less than those in its complement $(v - u)^c$. If we do not distinguish the direction of geodesics in $AdS_3$ time slice, this kinematic space only covers a half of the full $dS_2$, which is the planar patch $\mathcal{O}^+$ (or $\mathcal{O}^-$) [17, 18].

However, one should be able to note that the present flow model is a toy model in the sense that we have assumed that the edges are maximally entangled[19]. In other words, flows in each edge are same and reach their maxima $\log \chi$ simultaneously. This is a strict constraint and only works for some special tensor networks, such as prefect tensor[13] or random tensor[20]. Recent attempt in constructing a continuous tensor network based on path-integral optimizations[21, 22] may have clues to overcome this difficulty. MERA as a effective simulation for the ground state of real CFT, the entanglement entropy shouldn’t be maximal. However, one can control the degree of entanglement in MERA for approximating the real CFT [23].

If the tensor network is MERA, we know the entanglement entropy of the state with $l_0$ sites have upper bound

$$S_{\text{MERA}}(l_0) \leq 4 f(k) \log k \log \chi,$$

(63)

where the interval is larger than the lattice spacing i.e. $l_0 \gg 1$. $\chi$ is the the Hilbert dimension of each edge, $k$ is the number of sites in a block to be coarse-grained and $f(k)$ is a function of $k$ with $f(k) \leq k - 1$. $\log \chi$ is the maximum entanglement entropy of a single edge when we trace out the rest of the MERA. It’s instructive to introduce a parameter $\eta \in [0, 1]$ to describes the degree of entanglement [23]. In other words, the average entanglement entropy per edge in MERA is $\eta \log \chi$. Then one can write the entanglement entropy as

$$S_{\text{MERA}}(l_0) = 4 f(k) \log k \log \chi \cdot \eta \log \chi.$$  

(64)

Recall the entanglement entropy of CFT $S(l_0) = (c/3) \log l_0$, then the MERA entropy (64) gives a central charge

$$c = \frac{3L}{2G} = 12 \eta f(k) \frac{\log \chi}{\log k/ \log k}.$$  

(65)

Because each edge have the same average entanglement entropy $\eta \log \chi$ we still can count the number cut by causal cut for obtaining the entanglement of region $l_0$. The auxiliary space-time given by MERA is still a de Sitter, but with relation between $\chi$ and central charge $c$ (65) different with (62).
B. The large c limit

Central charge $c$ is a measure of the number of degrees of freedom. In a strong coupling limit of field theory, when the number of degree of freedom are very large $c \sim N^2 \gg 1$ the string interaction becomes weak and we just consider the classical string limit [24, 25]. In other words, in such a limit one can discuss something about the dynamics of quantum CFT by studying a dual semi-classical gravitational physics in space-time.

In (50) we take out $\rho$ from the integral because we have restricted all the isometries tensor in tensor network in a same type. This corresponds to the second version of QMF/QMC conjecture since we loss the freedom to assign tensors. In this circumstances the QMF/QMC conjecture is not always valid even when each edge has the same capacity, just like the MERA. What we have is a bound for QMF: $QMF(G,\alpha) \leq QMC(G,\alpha)$

Fortunately, recent work from Hastings [26] proved that this conjecture is “asymptotically” true in the limit as $\chi \to \infty$. That is to say, the ratio of the QMF to the QMC converges to 1 as $\chi$ tends to infinity. We write $QMF(G,\chi,O)$ to denote the QMF for a given graph $G$ with ordering $O$ and capacity $\chi$ in every edge. Ref. [26] showed that

$$QMF(G,\chi,O) = QMC(G,\chi) \cdot (1 - \mathcal{O}(1)).$$

The higher-order term $\mathcal{O}(1)$ we consider as a asymptotic function of $\frac{1}{\chi}$, which may also depend on $G$ and $O$. From (62) this implies the QMF/QMC conjecture is asymptotically true in a large central charge limit. The entanglement entropy $S(A) = \int_{C_A} f$ is asymptotically equal to log $QMC$ and we can just count the number of cut legs. We therefor have a dual classical, at least semi-classical gravitational theory in the auxiliary $dS_2$ space-time. Things can be simplified in such large $c$ limit and computations of entanglement entropy can be made by a holographic map to a volume in auxiliary space.

C. Conserved charge and symmetry

The Gauss’s theorem (45) and analogy of electromagnetic field strongly suggest there is a symmetric origin of $\rho$. More precisely, just like the conserved charge of electromagnetic field is the result of local $U(1)$ gauge invariance, $\rho$ defined here should be related to some conserved charges of som symmetry. Indeed, the fact that $\rho$ is density of isometries which denote the scale transformation invariance of a quantum system on the boundary indicates that $\rho$ should be a result of scale invariance. It is this reason that $\rho$ can be also called as information charge density. To see this more explicitly, let us give a brief review on continuous MERA (cMERA).

CMERA was explored firstly in [27] which is a continuous version of MERA network in order to extend the tensor network description of field theory [28]. Recent progress showed that it is possible to describe the complexity of a free QFT by using cMERA [29–31]. Consider an initial state , e.g., a reference state $|\Omega\rangle$ which is a trivial state in the sense that there is no entanglement between any region in real space and therefore it is also called as IR state. The intermediate state $|\psi\rangle$ of the system can be obtained by acting a unitary transformation $U(s,s_{1R})$ on $|\Omega\rangle$:

$$|\Psi\rangle = U(s,s_{1R})|\Omega\rangle.$$

The unitary operator is given by

$$U(s_1,s_2) = T \exp \left[ -i \int_{s_2}^{s_1} (K(s) + L) ds \right],$$

where parameter $s$ represents layers of the MERA, and, as well it has the meaning of length scale in emergent geometry. $L$ is the generator of non-relativistic dilations which plays a role of rescaling, and $K$ is the entangler which creates new correlated degrees of freedom nontrivially only in $k \ll \Lambda$ with $\Lambda$ a UV cutoff. In a 1+1 massless free boson CFT with Hamiltonian $H = (1/2) \int dx : [\pi(x)^2 + (\partial \phi(x))^2]$, they are given by [27]

$$L = \frac{1}{2} \int dk \left[ \pi(-k)(k\partial_k + \frac{1}{2})\phi(k) + h.c. \right],$$

$$K = \frac{1}{2} \int dk g(k) |\pi(-k)\phi(k) + h.c.|,$$

where the optimized function $g(k)$ smoothly approaches 1/2 in large $k$ and 0 in small $k$, i.e.,

$$g(k) \sim \begin{cases}
1/2, & |k| \ll \Lambda, \\
0, & |k| \gg \Lambda.
\end{cases}$$

With $g(k)$ in $|k| \ll \Lambda$, one can show that the operator $L + K$ acts as a generator of dilations $D$, which is the relativistic scaling operator $L’$

$$L’ = D = \frac{1}{2} \int dk \left[ \pi(-k)(k\partial_k + 1)\phi(k) + h.c. \right].$$

As discussed above, entangler $K$ plays an important role of entanglement in our space-time construction. One should not ignore its contribution to the conserved charge. That means the conserved charge is results of scale invariance generated by relativistic scaling generator $L’ = L + K$. This confirms our previous statement that $\rho$ should be a result of some symmetry and clearly, it is the scale invariance that leads to the definition of this quantity.

VI. CONCLUSIONS AND DISCUSSIONS

By making use of the QMF/QMC theorem developed recently in tensor network, we have proposed a tensor network/flow correspondence, which is a quantum generalization of the flow description of the RT formula in [2]. Based on information-theoretic considerations, we
suggest that for MERA we need to introduce a new variable \( \rho \), which is interpreted as the density of the tensor networks. Physically, this term is viewed as the source (or sink) of the tensor networks thus plays a significant role in the flow description of the RT formula. Its value is not only closely related to the geometric structure of the tensor network (equivalently, the metric of emergent spacetimes from the emergent point of view), but also the density of compression or decompression of quantum bits through reducing or expanding the dimension of Hilbert space, which implies a naive picture that the evolution of our universe can be regarded as a huge and complex quantum circuits and inflation is a progress that decompressing and entangling quantum bits continuously.

Let us close with several possible future directions.

A. Connections with tensor network/cosmology correspondence

Things will become more interesting if we consider a local density \( \rho(t, x) \) which depends on position and time in background manifold. That means the corresponding tensor networks have more different types of vertices which are assigned with different tensors. In general, different tensors have different capacities of compression or decompression, and thus generate different amounts of flow. Therefore, the emergent space-times from this network have different geometry from \( dS_2 \).

On the other hand, current cosmological observations support an accelerating universe. The tensor network/geometry correspondence, in principle, provides us a way to discuss the potential tensor network/cosmology correspondence (analogy of \( dS/CFT \) correspondence [32, 33]). For simplicity, let us consider a 2D toy model with metric

\[
ds^2 = -dt^2 + a^2(t)dx^2. \tag{73}
\]

As usual, let us assume tensor network is on the kinematic space, whose metric is (51) and \( v - u = t \). We assume the homogeneity of space-time so that the \( \rho(t) \) only depends on \( t \). One can then write down the general form of this network metric,

\[
ds_{TN}^2 = \frac{l^2 \rho(t)}{t^2}(-dt^2 + dx^2), \tag{74}
\]

where parameter \( l \) have length dimension. Comparing this with (73) and letting \( a(t) = -(dt/d\tau)^{-1} \), we obtain \( l^2 \rho(t) = a^2(t) t^2 \) and

\[
\tau = -\int_0^t \frac{\sqrt{t^2 \rho(t)\bar{v}^3}}{t'} dt'. \tag{75}
\]

As a consequence, with this transformation \( \tau = \tau(t) \) one can obtain the Friedmann-Robertson-Walker (FRW) metric (73), and \( a(\tau) \) becomes the scale factor. So in principle, one can construct a tensor network corresponding to this \( \rho(t) \) to describe an accelerating universe. From tensor network point of view of this toy model, we start from a trivial state \( |0 \rangle \) containing no entanglement. As we run the tensor network, the circuit decompress and entangle degrees of freedom by quantum gates, which is a unitary transformation on Hilbert space (as shown in [34, 35]).

B. Connections with causal sets theory

Causal sets theory is one of the ways toward quantum gravity and was first proposed in [36]. It says that our space-time is discrete and is constructed by some events and their causal relations. These events and their causal relations can be described by a partially ordering set and hence is called as causal set. One can use these elements to build a Lorentzian manifold approximately. The way is the following: one embeds a causal set \( C \) into a manifold \( (M, g) \) faithfully so that the volume of a region is proportional to the number of the elements in this region (FIG.9). It turns out that this causal set can recover this Lorentzian manifold approximately. Faithful embedding means that the elements distributed in a volume \( V \) region follow a Poisson distribution

\[
P(k) = \frac{(\delta V)^k e^{-\delta V}}{k!}, \tag{76}
\]

which is called sprinkling. \( \delta \) is the sprinkling density. From this random distribution, one is aware of the fluctuations of the volume \( \sqrt{\delta V} \) in the region under consideration. These fluctuations are non-trivial because they would provide some non-localities in our space-time, and they occur because of the discrete structure of space-time and can be regarded as some quantum fluctuations which may provide us some meaningful physical results. They can be suppressed when the volume \( V \) become very large \( \sqrt{\delta V}/\delta V = 1/\sqrt{\delta V} \to 0 \). From (51) and (62) one can naively think this is nothing else but the large \( c \) limit.

In other words, if we consider the large \( c \) limit in boundary theory, then the higher-order quantum corrections of emerged space-time can be suppressed faithfully, so that one can use the volume to calculate leading order of
entanglement entropy which has no higher-order corrections. More details are out of the scope of this paper and we leave it for future works\cite{37}.

C. QMF/QMC in quantum complexity and deep learning

Recently the quantum Hamiltonian complexity has been made rich connections with physical system. Physicists concern about some properties of local Hamiltonian in condensed system, such as the ground state properties or entanglement properties.

In \cite{8} it was shown that the quantum max-flow is related to the so-called quantum satisfiability problem, $QSAT$, which is defined as the quantum version of $k-SAT$ in \cite{38}. It was found that in some specific cases the problem of $QSAT$ and QMF are equivalent. So they give a conjecture similar to one in $QSAT$ that the QMF/QMC conjecture holds when the Hilbert space dimensions of edges become very huge.

$$\lim_{\chi \to \infty} QMF(G, \chi) = QMC(G, \chi). \quad (77)$$

This conjecture was proved in \cite{26}. We have shown that for integer $\chi$, the max-flow/min-cut in classical network is always valid. This, however, fails for a tensor network: the quantum version of max-flow/min-cut conjecture does not hold in general except for large $\chi$. So this result (77) indicates that a quantum phenomenon ($QMF \neq QMC$) disappears in the large system limit. This agrees with the argument in the last subsection: in the large $c$ limit, the quantum fluctuations of emergent space-time can be effectively suppressed. So naively, the quantum Hamiltonian complexity may closely relate to how many quantum corrections we should consider in Einstein’s gravity. Ituitively, this is because the quantum corrections will increase the difficulties of the computation. Machine learning is also related to tensor network through renormalization in condensed matter physics\cite{39} and possibly have significant holographic meanings\cite{40}\cite{41}. It was shown in \cite{42} that the quantum max-flow provides a non-trivial measure of the ability of tensor network to model correlations in a so-called deep convolutional network. All of these imply a very deep meaning of our understanding about emergent gravity.

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Detailed explanation will be made in Section V B.

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