Abstract

Let $G$ be a graph embedded on a surface $S_\varepsilon$ with Euler genus $\varepsilon > 0$, and let $P \subseteq V(G)$ be a set of vertices mutually at distance at least 4 apart. Suppose all vertices of $G$ have $H(\varepsilon)$-lists and the vertices of $P$ are precolored, where $H(\varepsilon) = \left\lceil \frac{7 + \sqrt{24\varepsilon + 1}}{2} \right\rceil$ is the Heawood number. We show that the coloring of $P$ extends to a list-coloring of $G$ and that the distance bound of 4 is best possible. Our result provides an answer to an analogous question of Albertson about extending a precoloring of a set of mutually distant vertices in a planar graph to a 5-list-coloring of the graph and generalizes a result of Albertson and Hutchinson to list-coloring extensions on surfaces.

Keywords: list-coloring; Heawood number; graphs on surfaces
1 Introduction

For a graph \( G \) the distance between vertices \( x \) and \( y \), denoted \( \text{dist}(x, y) \), is the number of edges in a shortest \( x-y \)-path in \( G \), and we denote by \( \text{dist}(P) \) the least distance between two vertices of \( P \). In [1] M. O. Albertson asked if there is a distance \( d > 0 \) such that every planar graph with a 5-list for each vertex and a set of precolored vertices \( P \) with \( \text{dist}(P) \geq d \) has a list-coloring that is an extension of the precoloring of \( P \). In that paper he proved such a result for 5-coloring with \( d \geq 4 \), answering a question of C. Thomassen. There have been some preliminary answers to Albertson’s question in [4, 8, 11]; initially Tuza and Voigt [17] showed that \( d > 4 \). Kawarabayashi and Mohar [11] have shown that when \( P \) contains \( k \) vertices, there is a function \( d_k > 0 \) that suffices for such list-coloring. Then recently Dvořák, Lidický, Mohar and Postle [9] have announced a complete solution, answering Albertson’s question in the affirmative, independent of the size of \( P \).

Let \( S_\varepsilon \) denote a surface of Euler genus \( \varepsilon > 0 \). Its Heawood number is given by

\[
H(\varepsilon) = \left\lfloor \frac{7 + \sqrt{24\varepsilon + 1}}{2} \right\rfloor
\]

and gives the best possible bound on the chromatic number of \( S_\varepsilon \) except for the Klein bottle whose chromatic number is 6. (For all basic chromatic and topological graph theory results, see [10, 13].) In many instances results for list-coloring graphs on surfaces parallel classic results on surface colorings. Early on it was noted that the Heawood number also gives the list-chromatic number for surfaces; see [10] for history. Also Dirac’s Theorem [7] has been generalized to list-coloring by Böhme, Mohar and Stiebitz for most surfaces; the missing case, \( \varepsilon = 3 \), was completed by Král’ and Škrekovski. This result informs and eases much of our work.

Theorem 1.1 ([5, 12]). If \( G \) embeds on \( S_\varepsilon, \varepsilon > 0 \), then \( G \) can be \( (H(\varepsilon) - 1) \)-list-colored unless \( G \) contains \( K_{H(\varepsilon)} \).

Analogously to Albertson’s question on the plane, we and others (see [11]) ask related list-coloring questions for surfaces. In this paper we ask if there is a distance \( d > 0 \) such that every graph on \( S_\varepsilon, \varepsilon > 0 \), with \( H(\varepsilon) \)-lists on each vertex and a set of precolored vertices \( P \) with \( \text{dist}(P) \geq d \) has a list-coloring that is an extension of the precoloring of \( P \). In [3] Albertson and Hutchinson proved the following result; the main result of this paper generalizes this theorem to list-coloring.
Theorem 1.2 ([3]). For each $\varepsilon > 0$, except possibly for $\varepsilon = 3$, if $G$ embeds on a surface of Euler genus $\varepsilon$ and if $P$ is a set of precolored vertices with $\text{dist}(P) \geq 6$, then the precoloring extends to an $H(\varepsilon)$-coloring of $G$.

Others have studied similar extension questions with $k$-lists on vertices for $k \geq 5$. For example, see [16], Thm. 4.4, for $k \geq 6$ and [11], Thm. 6.1, for $k = 5$; however, in both results the embedded graphs must satisfy constraints depending on the Euler genus and the number of precolored vertices. Our main result is Thm. 1.3, which shows that there is a constant bound on the distance between precolored vertices that ensures list-colorability for all graphs embedded on all surfaces when vertices have $H(\varepsilon)$-lists. It improves on Thm. 1.2 by removing the possible exception for $\varepsilon = 3$, reducing the distance of the precolored vertices from 6 to 4, and broadening the results to list-coloring.

Theorem 1.3. Let $G$ embed on $S_\varepsilon$, $\varepsilon > 0$, and let $P \subset V(G)$ be a set of vertices with $\text{dist}(P) \geq 4$. Then if the vertices of $P$ each have a 1-list and all other vertices have an $H(\varepsilon)$-list, $G$ can be list-colored. The distance bound of 4 is best possible.

When $G$ is embedded on $S_\varepsilon$, let the width [2] denote the length of a shortest noncontractible cycle of $G$; this is also known as edge-width. For list-coloring we have the following corollary of Thms. 1.1 and 1.3.

Corollary 1.4. If $G$ embeds on $S_\varepsilon$, $\varepsilon > 0$, with width at least 4, if the vertices of $P \subset V(G)$ have 1-lists and all other vertices have $H(\varepsilon)$-lists, then $G$ is list-colorable when $\text{dist}(P) \geq 3$. The distance bound of 3 is best possible.

Given that graphs embedded with very large width can be 5-list-colored as proved in [6], it is straightforward to deduce a 6-list-coloring extension result for such graphs. When $G$ embeds on $S_\varepsilon$, $\varepsilon > 0$, with width at least $2^{O(\varepsilon)}$, if a set of vertices $P$ with $\text{dist}(P) \geq 3$ have 1-lists and all others have 6-lists, then after the vertices of $P$ are deleted and the color of each $x \in P$ is deleted from the lists of $x$’s neighbors, the remaining graph has 5-lists, large width, and so is list-colorable. Thus $G$ is list-colorable, but only when embedded with large width whose size increases with the Euler genus of the surface.

A consequence of Thomassen’s proof of 5-list-colorability of planar graphs [15] is that if all vertices of a graph in the plane have 5-lists except that the vertices of one face have 3-lists, then the graph can be list-colored. For surfaces, we offer as a related result another corollary of Thm. 1.3.
Corollary 1.5. If \( G \) embeds on \( S_{\varepsilon}, \varepsilon > 0 \), and contains a set of faces each pair of which is at distance at least two apart, with all vertices on these faces having \( (H(\varepsilon) - 1) \)-lists and all other vertices having \( H(\varepsilon) \)-lists, then \( G \) can be list-colored.

The paper concludes with related questions.

2 Background results on surfaces, Euler genus and the Heawood formula

Let \( S_{\varepsilon} \) denote a surface of Euler genus \( \varepsilon > 0 \). If \( \varepsilon \) is odd, then \( S_{\varepsilon} \) is the nonorientable surface with \( \varepsilon \) crosscaps, but when \( \varepsilon \) is even, \( S_{\varepsilon} \) may be orientable or not. We let \( \mathcal{T} \) denote the torus, the orientable surface of Euler genus 2, and \( K \) the Klein bottle, the nonorientable surface of Euler genus 2.

The Heawood number \( H(\varepsilon) \), defined above, gives the largest \( n \) for which \( K_n \) embeds on a surface \( S_{\varepsilon} \) of Euler genus \( \varepsilon \), as well as the chromatic number of \( S_{\varepsilon} \), except that \( K_6 \) is the largest complete graph embedding on \( K \) and 6 is its chromatic number.

The least Euler genus \( \varepsilon \) for which \( K_n \) embeds on \( S_{\varepsilon} \) is given by the inverse function

\[
\varepsilon = I(n) = \left\lceil \frac{(n - 3)(n - 4)}{6} \right\rceil.
\]

Each \( K_n, n \geq 5 \), of course, has a minimum value of \( \varepsilon > 0 \) for which it embeds on \( S_{\varepsilon} \), called the Euler genus of \( K_n \), but for \( \varepsilon \geq 2 \) more than one surface \( S_{\varepsilon} \) may have the same maximum \( K_n \) that embeds on it. For example, both \( S_5 \) and \( S_6 \) have Heawood number 9 with \( K_9 \) being the largest complete graph embedding on \( K \) and 6 is its chromatic number.

For our results we need to know when \( K_{H(\varepsilon)} \) necessarily has a 2-cell embedding on \( S_{\varepsilon} \). When \( K_n \) embeds on \( S_{\varepsilon} \), but not on \( S_{\varepsilon-1} \), then \( K_n \) necessarily embeds with a 2-cell embedding. When \( K_n \) embeds in addition on \( S_{\varepsilon+1}, \ldots, S_{\varepsilon+i} \) with \( i > 0 \), then it may not have a 2-cell embedding on the latter surfaces. For example, on surfaces \( S_1, \mathcal{T}, S_4, \) and \( S_5 \), the complete graphs
Table 1: Embedding parameters for $K_{H(\varepsilon)}$

| $\varepsilon$ | $H(\varepsilon)$ | $e$ | $f$ | Largest Face | $\varepsilon$ | $H(\varepsilon)$ | $e$ | $f$ | Largest Face |
|---------------|------------------|-----|-----|--------------|---------------|------------------|-----|-----|--------------|
| 1             | 6                | 15  | 10  | 3            | 13            | 12              | 66  | 43  | 6            |
| 2             | 7                | 21  | 14  | 3            | 14            | 12              | 66  | 42  | 9            |
| 3             | 7                | 21  | 13  | 6            | 15            | 13              | 78  | 52  | 3            |
| 4             | 8                | 28  | 18  | 5            | 16            | 13              | 78  | 51  | 6            |
| 5             | 9                | 36  | 24  | 3            | 17            | 13              | 78  | 50  | 9            |
| 6             | 9                | 36  | 23  | 6            | 18            | 13              | 78  | 49  | 12           |
| 7             | 10               | 45  | 30  | 3            | 19            | 14              | 91  | 60  | 5            |
| 8             | 10               | 45  | 29  | 6            | 20            | 14              | 91  | 59  | 8            |
| 9             | 10               | 45  | 28  | 9            | 21            | 14              | 91  | 58  | 11           |
| 10            | 11               | 55  | 36  | 5            | 22            | 15              | 105 | 70  | 3            |
| 11            | 11               | 55  | 35  | 8            | 23            | 15              | 105 | 69  | 6            |
| 12            | 12               | 66  | 44  | 3            | 24            | 15              | 105 | 68  | 9            |

$K_6, K_7, K_8$ and $K_9$ have 2-cell embeddings, respectively, but $K_6, K_7$ and $K_9$ may or may not have 2-cell embeddings on $K$, $S_3$ and $S_6$, respectively.

If $f$ is a face of an embedded graph $G$, let $V(f)$ and $E(f)$ denote the incident vertices and edges of $f$. We say that $V(f) \cup E(f)$ is the boundary of $f$ and that the closure of $f$ is the union of $f$ and its boundary. Each edge of $E(f)$ either lies on another face besides $f$ or it might lie just on $f$. For example, Fig. 1 shows two graphs embedded on the torus, $\mathcal{T}$. In the first graph, edges 2-3 and 4-7 each border two faces, but edges 3-6 and 8-9 each border only one face. The size $s$ of a face $f$ is determined by counting, with multiplicity, the number of edges on its boundary, and we then call $f$ an $s$-region. In other words, when $s_1$ edges of $E(f)$ lie on another face of $G$ besides $f$ and $s_2$ edges lie only on $f$, then we call $f$ an $s$-region where $s = s_1 + 2s_2$. When $f$ is a 2-cell, $E(f)$ forms a single facial walk $W_f$, and the size of the face equals the length of the facial walk, counting multiplicity of repeated edges. Since an $s$-region $f$ may have repeated edges and repeated vertices, we indicate $|V(f)| = t$ by calling $f$ also a $t$-vertex-region where $t \leq s$. Hence the shaded region in the first graph in Fig. 1 is a 13-region and a 9-vertex-region, since two edges and four vertices are repeated; the shaded region in the second graph, with no repeated vertices or edges, is a 13-region...
and a 13-vertex-region.

Figure 1: A 2-cell region in a graph embedded on the torus, $\mathcal{T}$, before and after vertex- and edge-duplication

Here in summary are statistics on 2-cell embeddings of $K_{H(\varepsilon)}$. The patterns presented are visible from Table 1 and are easily derived from Euler’s formula and the function $I(n)$, given above.

**Lemma 2.1.** Let $\varepsilon \geq 1$ and suppose $K_{H(\varepsilon)}$ has a 2-cell embedding on $S_\varepsilon$ (but $S_\varepsilon \neq K_n$). Set $i = \left\lfloor \frac{H(\varepsilon) - 3}{3} \right\rfloor$ so that $H(\varepsilon) = 3i + 3, 3i + 4$ or $3i + 5$ with $i \geq 1$.

1. If $H(\varepsilon) = 3i + 3$, then $\varepsilon = (3i^2 - i)/2, (3i^2 - i + 2)/2, \ldots, (3i^2 + i - 2)/2$. The number of faces of the embedding is given by $f = 3i^2 + 5i + 2, 3i^2 + 5i + 1, \ldots, 3i^2 + 4i + 3$, respectively, and the largest possible face is an $s$-region with $s = 3, 6, \ldots, 3i$, resp.

2. If $H(\varepsilon) = 3i + 4$, then $\varepsilon = (3i^2 + i)/2, (3i^2 + i + 2)/2, \ldots, (3i^2 + 3i)/2$. The number of faces of the embedding is given by $f = 3i^2 + 7i + 4, 3i^2 + 7i + 3, \ldots, 3i^2 + 6i + 4$, respectively, and the largest possible face is an $s$-region with $s = 3, 6, \ldots, 3i + 3$, resp.

3. If $H(\varepsilon) = 3i + 5$, then $\varepsilon = (3i^2 + 3i + 2)/2, (3i^2 + 3i + 4)/2, \ldots, (3i^2 + 5i)/2$. The number of faces of the embedding is given by $f = 3i^2 + 9i + 6, 3i^2 + 9i + 5, \ldots, 3i^2 + 8i + 7$, respectively, and the largest possible face is an $s$-region with $s = 5, 8, \ldots, 3i + 2$, resp.

From the point of view of the genus, given $\varepsilon > 0$, we can determine directly whether or not $K_{H(\varepsilon)}$ necessarily has a 2-cell embedding on $S_\varepsilon$. $K_{H(\varepsilon)}$ necessarily has a 2-cell embedding if and only if $\varepsilon = (3i^2 - i)/2$ or $(3i^2 + i)/2$. 
or \((3i^2 + 3i + 2)/2\) for some value of \(i > 0\). Thus given \(\varepsilon > 0\), we compute \(H(\varepsilon)\) and set \(i = \lfloor H(\varepsilon)/3 \rfloor - 1\) so that \(H(\varepsilon) = 3i + 3, 3i + 4, \) or \(3i + 5\). Then \(K_{H(\varepsilon)}\) necessarily embeds with a 2-cell embedding if \(I(H(\varepsilon)) = \varepsilon\); that is, \(S_{\varepsilon}\) is the genus surface for \(K_{H(\varepsilon)}\).

In the results of Table 1 we do not claim that every 2-cell embedding of \(K_{H(\varepsilon)}\) achieves the maximum face size when that size is greater than three. For example when \(K_{H(\varepsilon)}\) has a largest face being a 5- or 6-region, it might embed as a near-triangulation with one 5- or 6-region, respectively, or it might be a triangulation except for two 4-regions or a triangulation except for a 4- and a 5-region, resp. (An embedding is a near-triangulation if at most one region is not 3-sided.)

We note from Table 1 and Lemma 2.1 that there are some instances of \(\varepsilon\) when \(K_{H(\varepsilon)}\) embeds possibly with an \((H(\varepsilon) - 1)\)-region which might allow for the embedding of two different (not disjoint, but distinct) copies of \(K_{H(\varepsilon)}\) on \(S_{\varepsilon}\), as explained in the next lemma.

Lemma 2.2. Let \(K_{H(\varepsilon)}\) have a 2-cell embedding on \(S_{\varepsilon}\), \(\varepsilon > 0\).

1. The largest possible face in the embedding is an \((H(\varepsilon) - 1)\)-region. If there is an \((H(\varepsilon) - 1)\)-region, there is just one, and the embedding is a near-triangulation.

2. If every face of the embedding is at most an \((H(\varepsilon) - 2)\)-region, then no additional copy of \(K_{H(\varepsilon)}\) can simultaneously embed on \(S_{\varepsilon}\).

3. When \(K_{H(\varepsilon)}\) can embed with an \((H(\varepsilon) - 1)\)-region that is also an \((H(\varepsilon) - 1)\)-vertex-region, then two different copies of \(K_{H(\varepsilon)}\) can embed, by adding a vertex adjacent to all vertices of that region, and then the two complete graphs share a copy of \(K_{H(\varepsilon) - 1}\). Such an embedding is possible only if \(H(\varepsilon) = 3i + 4\) and \(\varepsilon = (3i^2 + 3i)/2\), and the resulting embedding is a triangulation.

We call the latter graph \(DK_{H(\varepsilon)}\); it is also \(K_{H(\varepsilon) + 1} \setminus \{e\}\) for some edge \(e\).

Proof. Suppose that \(K_{H(\varepsilon)}\) has a 2-cell embedding with at least one \(s\)-region where \(s \geq H(\varepsilon) - 1\). Then Euler’s formula plus a count of edges on faces with multiplicities leads to a contradiction to Lemma 2.1 in all cases except when there is precisely one \((H(\varepsilon) - 1)\)-region, \(H(\varepsilon) = 3i + 4\), \(\varepsilon = (3i^2 + 3i)/2\), and all other faces are 3-regions.
Suppose $K_{H(\varepsilon)}$ embeds on $S_\varepsilon$ with every face having at most $H(\varepsilon) - 2$ sides. No two additional vertices in different faces of $K_{H(\varepsilon)}$ can be adjacent. For $2 \leq k \leq 4$, $k$ mutually adjacent, additional vertices cannot form $K_{H(\varepsilon)}$ together with $H(\varepsilon) - k$ vertices on the boundary of a face.

Proofs of remaining parts follow easily from Euler’s Formula and Lemma 2.1.

If $V' \subseteq V(G)$, we denote by $G[V']$ the induced subgraph on the vertices in $V'$; for $E' \subseteq E(G)$, we denote by $G[E']$ the induced subgraph on the edge set $E'$. When $f$ is a face of an embedded $G$, we may also call the subgraph $G[E(f)]$ the boundary of $f$; that is, it may be convenient at times to think of the boundary of a face $f$ as a set $V(f) \cup E(f)$ and at other times as the subgraph $G[E(f)]$.

We restate two very useful corollaries of Thm. 6 in [5]. The first involves a case that is not covered in that theorem, but which follows easily from their proof. If $f$ is the infinite face of a connected plane graph, we call the boundary of $f$ the outer boundary of $G$, and when $G[E(f)]$ is a cycle, we call it the outer cycle. Without loss of generality we may suppose that for a connected plane graph the outer boundary is a cycle.

**Corollary 2.3.** ([5]) Let $G$ be a connected plane graph with outer cycle $C$ that is a $k$-cycle with $k \leq 6$. If every vertex of $G$ has a list of size at least 6, then a precoloring of $C$ extends to all of $G$ unless $k = 6$, there is a vertex in $V(G) \setminus V(C)$ that is adjacent to all vertices of $V(C)$, and its list consists of six colors that appear on the precolored $C$.

Then the results of Thm. 6 in [5] together with Cor. 2.3 give the next corollary.

**Corollary 2.4 ([5]).** Let $G$ be a connected plane graph with outer cycle $C$ that is a $k$-cycle with $3 \leq k \leq 6$. If every vertex of $G$ has a list of size at least $\max(5, k + 1)$, then a precoloring of $C$ extends to all of $G$.

The next lemma is used repeatedly in the proof of Thms. 3.3 and 4.3. It is an extension of the similar result for 5-list-colorings in [5]. The parameters are motivated by the “Largest Face” and $H(\varepsilon)$-list sizes from Table 1.

**Lemma 2.5.** Let $H$ be a connected graph with a 2-cell embedding on $S_\varepsilon$, $\varepsilon > 0$, and let $f$ be a 2-cell $k$-region of $H$, $k \geq 3$. Let $G$ be a plane graph embedded within $f$ and let $G_f$ be a simple, connected graph that consists
of $G$, $H[E(f)]$, and edges joining $V(G)$ and $V(f)$ so that $G_f$ is embedded in the closure of $f$. Let $P = \{v_1, \ldots, v_j\}$ be a subset of $V(G_f)$ satisfying $\text{dist}(P) \geq 3$. Then if every vertex of $G_f$ has an $\ell$-list except that the vertices of $P$ each have a 1-list, every proper precoloring of $H[E(f)]$ extends to a list-coloring of $G_f$ provided that no vertex of $P$ is adjacent to a vertex of $V(f)$ with the same color as its 1-list, and

1. $k = 3$ and $\ell \geq 6$,
2. $k \geq 4$ and $\ell \geq k + 2$, or
3. $k = 6$ or $k \geq 9$, $\ell = k + 1$, and there is no vertex $x$ adjacent to $k + 1$ vertices of $V(f) \cup \{v_i\}$, for some $i = 1, \ldots, j$, with $x$’s list consisting of $\ell = k + 1$ colors that all appear on $V(f) \cup \{v_i\}$.

Proof. Note that $G_f[E(f)] = H[E(f)]$. Also note that the condition $\text{dist}(P) \geq 3$ guarantees that no vertex of $G_f$ is adjacent to more than one $v_i$. For $v_i \in P \setminus V(f)$, we say that we excise $v_i$ if we delete it and delete its color from the list of colors for each neighbor that is not precolored. The proof has three cases that together prove parts 1-3 of the lemma.

Case A. Assume $k = 3$ and $\ell \geq 6$, $4 \leq k \leq 6$ and $\ell \geq k + 2$, or $k = 6$ and $\ell = 7$. In these cases first we excise the vertices of $P \setminus V(f)$ so that every remaining vertex of $G$ has a list of size at least 5 for $k = 3$, of size at least $k + 1$ for $k = 4, 5, 6$, or else of size at least 6 when $k = 6$.

In the following we may need to do some surgery, perhaps repeatedly, on the face $f$ and its boundary, so that we can apply Cor. 2.4. First, more easily, when $f$ is a 2-cell $k$-region on which lies no repeated vertex, then $G_f$ is a plane graph with outer cycle a $k$-cycle, $k \leq 6$. By Cor. 2.4 a precoloring of $G_f[E(f)]$ extends to $G_f \setminus P$ and this coloring extends to all of $G_f$ unless there is a vertex $x$ with a 6-list, adjacent to six vertices of $V(f)$ with the six colors of $x$’s list. If $x$’s list was decreased to a 6-list, $x$ was adjacent to some vertex $v_i$, but this situation is disallowed by hypothesis in part 3.

Otherwise in a traversal of $W_f$ we visit a vertex more than once and may travel along an edge twice. In the former case, each time we revisit a vertex $x$, we can split that vertex in two, into $x_1$ and $x_2$, and similarly divide the edges incident with $x$ so that the face $f$ is expanded to become the new face $f'$, still a $k$-region, and the graph $G_f$ becomes $G_{f'}$ which is naturally embedded in the closure of $f'$ and contains the same adjacencies. Now there is one more vertex in $V(f')$ and the same set of edges $E(f') = E(f)$ on
the boundary and in the boundary subgraph $G_f'[E(f')]$. A precoloring of $G_f[E(f)]$ gives a precoloring of $G_f'[E(f')]$ in which vertices $x_1$ and $x_2$ receive the same color; we call this procedure vertex-duplication. In the latter case, when we revisit an edge $e = (y, y')$, we may visit both of its endpoints twice or one endpoint twice and the other just once. We similarly duplicate the edge $e = (y, y')$ by duplicating one or both of its endpoints and splitting $e$ into two new edges $e_1$ and $e_2$. Then we divide the other edges incident with $e$ so that $G_f$ becomes $G_f'$ which is naturally embedded in the closure of the new face $f'$, still a $k$-region, but now with one or two more vertices in $V(f')$, the same number of edges in $E(f')$ and in $G_f'[E(f')]$, and with one less duplicated edge in $W_f'$. A precoloring of $G_f[E(f)]$ gives a precoloring of $G_f'[E(f')]$ in which duplicated vertices receive the same color; we call this procedure edge-duplication. We note that in both duplications there cannot be a vertex $x$ that is adjacent to both copies of a duplicated vertex (since $G_f$ is a simple graph). As an example, the first graph in Fig. 1 shows a 2-cell face that is a 13-region, in which vertices 3, 6, 7, and 8, are repeated, and edges 3-6 and 8-9 are repeated. Vertex- and edge-duplication produces the second graph, which has a new face that is a 13-region and whose facial walk is a cycle given by 1-8-9-8'-7-2-3-6-5-7'-4-6'-3'-1.

In all cases after vertex- and edge-duplication, the 2-cell $k$-region $f$ becomes a 2-cell $k$-region $f^*$ with no repeated vertex or edge on the outer boundary. $G_f$ has been transformed into a plane graph $G_{f^*}$ with outer cycle, $G_{f^*}[E(f^*)]$, of length $k \leq 6$. The precoloring of $G_f[E(f)]$ has become a precoloring of $G_{f^*}[E(f^*)]$ with duplicated vertices receiving the same color. Then by Cor. 2.4, the precoloring of $G_{f^*}[E(f^*)]$ extends to $G_{f^*} \setminus P$ and so the precoloring of $G_f[E(f)]$ extends to $G_f \setminus P$ and to all of $G_f$ since the exceptional case of part 3 cannot occur. (Since $f^*$ is at most a 6-region and has a duplicated vertex, it is a $t$-vertex-region for some $t < 6$, and there cannot be a vertex adjacent to six vertices of $V(f^*)$.)

Case B. Suppose $k \geq 7$ and $\ell \geq k + 2$ so that in all cases $\ell \geq 9$. For $v \in V(G)$, let $E_f(v)$ denote the set of edges joining $v$ with a vertex of $V(f)$. Suppose there is a vertex $x$ of $V(G)$ that is adjacent to at least $k - 3$ vertices of $V(f)$. If $x = v_i$ for some $i, 1 \leq i \leq j$, then $G_f[E(f) \cup E_f(v_i)]$ can be properly colored by assumption. If $x \neq v_i$ for any $i, 1 \leq i \leq j$, then $x$ is adjacent to either one or no vertex $v_i$, and since $x$ has an $\ell$-list, $\ell \geq k + 2$, the coloring of $G_f[E(f) \cup E_f(v_i)]$ (respectively, $G_f[E(f)]$) extends to $x$. In all cases $G_f[E(f) \cup E_f(x)]$ divides $f$ into regions of size at most 6, and the coloring of $G_f[E(f) \cup E_f(x)]$ extends to the interior of each $s$-region,
3 \leq s \leq 6$, by Case A since interior vertices, other than the $v_i$, have 9-lists.

Otherwise every vertex $x$ in $G$ is adjacent to at most $k - 4$ vertices of $V(f)$. For each such vertex $x$ we delete from its list the colors of $V(f)$ to which it is adjacent. This may reduce the list for $x$ to one of size six or more. Next we excise the vertices of $P$ in $G \setminus V(f)$, resulting in the planar graph $G \setminus P$ with every vertex having a list of size at least five, which can be list-colored by [15]. This list-coloring is compatible with the precoloring of $G_f[E(f)]$ and extends to $P$ and so to all of $G_f$.

Case C. The case of $k = 6, \ell = 7$ was covered in Case A. Suppose that $k \geq 9$ and $\ell = k + 1 \geq 10$. Suppose there is a vertex $x$ of $V(G)$ that is adjacent to at least $k - 4$ vertices of $V(f)$. As before, if $x = v_i$ for some $i, 1 \leq i \leq j$, then $G_f[E(f) \cup E_f(v_i)]$ can be properly colored by assumption. If $x \neq v_i$ for any $i, 1 \leq i \leq j$, then $x$ is adjacent to one or no vertex $v_i$, and the coloring of $G_f[E(f) \cup E_f(v_i)]$ (resp., $G_f[E(f)]$) extends to $x$ in all cases unless (since $\ell = k + 1$) $x$ is adjacent to all vertices of $V(f) \cup \{v_i\}$ for some $i, 1 \leq i \leq j$, and $x$’s list consists of $\ell$ colors all appearing on $V(f) \cup \{v_i\}$. We have disallowed this case. Now $G_f[E(f) \cup E_f(x)]$ forms a graph that consists of triangles and $s$-regions with $s \leq 7$. The coloring of $G_f[E(f) \cup E_f(x)]$ extends to the interior of each region by the previous cases, since $\ell \geq 10$.

Otherwise every vertex $x$ of $G$ is adjacent to at most $k - 5$ vertices of $V(f)$, and we proceed as in the proof of Case B by decreasing the lists of vertices adjacent to $V(f)$ and excising all the $v_i$ to create a planar graph with every vertex having at least a 5-list. The resulting graph is list-colorable with a coloring compatible with that of $G_f[E(f)]$ and extending to $G_f$.

\section{Results on $K_n$ genus surfaces}

Most parts of the proof of the next lemma are clear; these results are used repeatedly in the proof of the main results.

\begin{lemma}
1. Suppose at most one vertex of $K_n$ has a 1-list, at least one vertex has an $n$-list, and the remaining vertices have $(n - 1)$-lists or $n$-lists. Then $K_n$ can be list-colored.

2. If one vertex of $DK_n$ has a 1-list and all other vertices have $n$-lists, then $DK_n$ can be list-colored.

3. If at most six vertices of $DK_n$, $n \geq 7$, have lists of size $n - 1$ and all others have $n$-lists, then $DK_n$ can be list-colored.
\end{lemma}
Proof. We include the proof of part 3. Suppose that one of the two vertices of degree \(n-1\), say \(x\), has an \(n\)-list. Then \(K_n = DK_n \setminus \{x\}\) has at most six vertices with \((n-1)\)-lists and can be list-colored since \(n \geq 7\). This coloring extends to \(x\) which has an \(n\)-list and is adjacent to \(n-1\) vertices of the colored \(K_n\). Otherwise both vertices of degree \(n-1\), say \(x\) and \(y\), have \((n-1)\)-lists, \(L(x)\) and \(L(y)\) respectively. Suppose there is a common color \(c\) in \(L(x)\) and \(L(y)\). Then coloring \(x\) with \(c\) extends to a coloring of \(K_n = DK_n \setminus \{y\}\) after which \(y\) can also be colored with \(c\). Otherwise \(L(x)\) and \(L(y)\) are disjoint. Suppose that when \(DK_n \setminus \{y\}\) is list-colored, the colors on \(K_{n-1} = DK_n \setminus \{x, y\}\) are precisely the \(n-1\) colors of \(L(y)\) so that the coloring does not extend. If there is some vertex \(z\) of \(K_{n-1}\) with an \(n\)-list that contains a color not in \(L(y)\) and different from the color \(c_x\) used on \(x\), we use \(c_x\) on \(z\), freeing up the previous color of \(z\) for \(y\). Otherwise, for every \(z\) with an \(n\)-list, that list equals \(L(y)\) \(\cup\) \(\{c_x\}\). Besides these vertices of \(K_{n-1}\) with prescribed \(n\)-lists, there are at most four others in \(K_{n-1}\) which have \(n-1\) lists. These four vertices might be colored with colors from \(L(x)\), but that still leaves at least one color \(c'_{x} \neq c_{x}\) in \(L(x)\) that has not been used. We change the color of \(x\) to \(c'_{x}\) and the color of one of the \(n\)-list vertices of \(K_{n-1}\) to \(c_{x}\), thus freeing up that vertex’s previous color to be used on \(y\). \(\square\)

Theorem 3.2. Suppose \(G\) embeds on \(S_\varepsilon, \varepsilon > 0\), and does not contain \(K_{H(\varepsilon)}\). Then when every vertex of \(G\) has an \(H(\varepsilon)\)-list except that the \(j\) vertices of \(P = \{v_1, \ldots, v_j\}\), \(j \geq 0\), have \(1\)-lists and \(dist(P) \geq 3\), then \(G\) is list-colorable.

Proof. Let \(G\) embed on \(S_\varepsilon, \varepsilon > 0\), and suppose \(G\) does not contain \(K_{H(\varepsilon)}\). We excise the vertices of \(P = \{v_1, \ldots, v_j\}\), if present, leaving a graph with all vertices having at least \((H(\varepsilon) - 1)\)-lists since \(dist(P) \geq 3\). By [5, 12], the smaller graph can be list-colored, and that list-coloring extends to \(G\). \(\square\)

In particular this result holds for all graphs on the Klein bottle since \(K_7\) does not embed there. The first value not covered by the next theorem is \(\varepsilon = 3\) with \(H(\varepsilon) = 7\).

Theorem 3.3. Suppose \(G\) has a 2-cell embedding on \(S_\varepsilon, \varepsilon > 0\), and contains \(K_{H(\varepsilon)}\). Then when every vertex of \(G\) has an \(H(\varepsilon)\)-list except that the \(j\) vertices of \(P = \{v_1, \ldots, v_j\}\), \(j \geq 0\), have \(1\)-lists, \(G\) is list-colorable provided that \(\varepsilon\) is of the form \(\varepsilon = (3i^2 - i)/2, (3i^2 + i)/2, \) or \((3i^2 + 3i + 2)/2\), for some \(i \geq 1\), and \(dist(P) \geq 4\).
Proof. We know that $K_{H(\varepsilon)}$ necessarily has a 2-cell embedding on $S_\varepsilon$ for $\varepsilon = 1, 4$ as does $K_7$ on $T$. ($K_6$ and $K_7$ may or may not have 2-cell embeddings on $K$ and on $S_3$, respectively.)

The values $\varepsilon = (3i^2 - i)/2, (3i^2 + i)/2, or (3i^2 + 3i + 2)/2$ for some $i \geq 1$ are those for which $K_{H(\varepsilon)}$ necessarily has a 2-cell embedding on $S_\varepsilon$; they give the value of the genus surface of $K_{H(\varepsilon)}$ for each of the modulo 3 classes of $H(\varepsilon)$. Since $\text{dist}(P) \geq 4$, at most one vertex $v_k \in P$ is in or is adjacent to a vertex of $K_{H(\varepsilon)}$ (but not both), and in the latter case $v_k$ is adjacent to at most $H(\varepsilon) - 1$ vertices of the complete graph since $K_{H(\varepsilon)+1}$ does not embed on $S_\varepsilon$. Thus in all cases $K_{H(\varepsilon)} \cup P$ can be list-colored by Lemma 3.1.1. When $\varepsilon = 1, H(\varepsilon) = 6$, and $K_6$ embeds as a triangulation on $S_1$. When $\varepsilon > 1$, if $\varepsilon = (3i^2 - i)/2$ or $(3i^2 + i)/2$, $K_{H(\varepsilon)}$ embeds as a triangulation, and if $\varepsilon = (3i^2 + 3i + 2)/2$, $K_{H(\varepsilon)}$ embeds with the largest face size at most five, and in all cases $H(\varepsilon) \geq 7$. Hence we apply Lemma 2.5 for $\varepsilon \geq 1$ to see that the list-coloring of $K_{H(\varepsilon)}$ extends to the interior of each of its faces and so $G$ is list-colorable.

A similar proof would show that when the orientable surface $S_\varepsilon$ with $\varepsilon$ even is the orientable genus surface for $K_{H(\varepsilon)}$ (i.e., when $\varepsilon$ is even and gives the least Euler genus such that $K_{H(\varepsilon)}$ embeds on orientable $S_\varepsilon$), then for every $G$ with a 2-cell embedding on orientable $S_\varepsilon$ and containing $K_{H(\varepsilon)}$ the same list-coloring result holds. The first corollary of Section 1 also follows easily.

Proof of Cor. 1.4. Suppose $H(\varepsilon) = 3i + 3, i \geq 1$. If $\varepsilon = (3i^2 - i)/2$, then $K_{H(\varepsilon)}$ embeds with $f = (i+1)(3i+2)$ faces by Lemma 2.1.1. $K_{H(\varepsilon)}$ contains $(3i+3)(3i+2)(3i+1)/6$ 3-cycles, more than the number of faces so that $K_{H(\varepsilon)}$ embeds with a noncontractible 3-cycle. Thus in this case $G$ cannot contain $K_{H(\varepsilon)}$ and by Thm. 3.2, $G$ can be list-colored. If $\varepsilon = (3i^2 - i + 2)/2, \ldots, or (3i^2 + i - 2)/2$, then $K_{H(\varepsilon)}$ embeds with fewer than $f = (i+1)(3i+2)$ faces and so the same result holds.

When $H(\varepsilon) = 3i + 4 or 3i + 5, i \geq 1$, an analogous proof shows that $G$ cannot contains $K_{H(\varepsilon)}$ and so is list-colorable.

To see that distance at least 3 is best possible for the precolored vertices, take a vertex $x$ with a $k$-list $L(x)$ and attach $k$ pendant edges to vertices, precolored with each of the colors of $L(x)$. 

\[ \square \]
4 All surfaces

First we explore some topology of surfaces and non-2-cell faces of embedded graphs. Cycles on surfaces (i.e., simple closed curves on the surface), for both orientable and nonorientable surfaces, are of three types: contractible and surface-separating, noncontractible and surface-separating, and noncontractible and surface-nonseparating. (When the meaning is clear, we suppress the prefix “surface.”) A non-2-cell face of an embedded graph must contain a noncontractible surface cycle within its interior. For example, in the second graph in Fig. 1, the shaded region is a 2-cell face, and the unshaded region is a non-2-cell face that contains a noncontractible and nonseparating cycle. (For a more detailed discussion see Chapters 3 and 4 of [13].)

Suppose $f$ is a non-2-cell face of $K_{H(\varepsilon)}$ embedded on $S_\varepsilon$. We repeatedly “cut” along simple noncontractible surface cycles that lie wholly within the face $f$ until the “derived” face or faces become 2-cells. Each “cut” is replaced with one or two disks, creating a new surface, and with each “cut” $K_{H(\varepsilon)}$ stays embedded on a surface $S_{\varepsilon'}$ with $\varepsilon' < \varepsilon$. Below we explain this surface surgery and count the number of newly created faces, called derived faces in the surgery.

**Lemma 4.1.** Suppose $K_{H(\varepsilon)}$ embeds on $S_\varepsilon$, $\varepsilon > 0$. Then the largest possible 2-cell face in the embedding is an $(H(\varepsilon) - 1)$-region.

**Proof.** Suppose the embedded $K_{H(\varepsilon)}$ has a non-2-cell $k$-region $f$; initially there are no derived faces. In $f$ we can find a simple noncontractible cycle $C$, disjoint from its boundary, $V(f) \cup E(f)$. If $C$ is surface-separating, it is necessarily 2-sided. We replace $C$ by two copies of itself, $C$ and $C'$, and insert in each copy a disk, producing surfaces $S(1)$ and $S'(1)$, each with Euler genus that is positive and less than $\varepsilon$. Since $K_{H(\varepsilon)}$ is connected, it is embedded on one of these surfaces, say $S(1)$. The face $f$ of $K_{H(\varepsilon)}$ on $S_\varepsilon$ becomes the derived face $f_1$ of $K_{H(\varepsilon)}$ on $S(1)$ and retains the same set of boundary vertices $V(f_1) = V(f)$ and edges $E(f_1) = E(f)$ so that $f_1$ is also a $k$-region. Initially $f$ is not a derived face, $f_1$ becomes a derived face and the Euler genus decreases by at least 1. If, later on in the process, $f$ is a derived face, then $f_1$ is also a derived face, the number of derived faces does not increase, and the Euler genus decreases by at least 1.

If $C$ is not surface-separating and is 2-sided, we duplicate it and sew in two disks, as above, to create one new surface $S(1)$ of lower and positive Euler genus on which $K_{H(\varepsilon)}$ is embedded. If $C$ was not separating within the
face $f$, then the derived face $f_1$ keeps the same set of boundary vertices and edges as $f$ and remains a $k$-region. As above, the number of derived faces increases by at most 1 and the Euler genus decreases by at least 2. If $C$ was separating within the face $f$, then $f$ splits into two derived faces $f_1$ and $f'_1$. Each vertex of $V(f)$ and each edge of $E(f)$ appears on one of these derived faces or possibly two when it was a repeat on $f$. More precisely, if $f_1$ is a $k_1$-region and $f'_1$ is a $k'_1$-region, then necessarily $k_1 + k'_1 = k$. In this case the Euler genus decreases by 2 and number of derived faces increases by at most 2, increasing by 2 only when the face being cut was an original face of $K_{H(\varepsilon)}$. If $C$ is not surface-separating and is 1-sided, we replace $C$ by a cycle $DC$ of twice the length of $C$ and insert a disk within $DC$, producing a surface $S(1)$ with Euler genus that is less than $\varepsilon$. $K_{H(\varepsilon)}$ remains embedded on $S(1)$, necessarily with positive Euler genus, and the derived face $f_1$ keeps the same boundary vertices and edges as $f$, remaining a $k$-region. Thus the number of derived faces increases by at most 1 and the Euler genus decreases by at least 1.

Now we prove the lemma by induction on the number of non-2-cell faces of the embedded $K_{H(\varepsilon)}$. We know the conclusion holds when there are no non-2-cell faces by Lemma 2.2. Otherwise let $f$ be a non-2-cell $k$-region. We repeatedly cut along simple noncontractible cycles within $f$ and its derived faces, creating surfaces $S(1), S(2), \ldots$ on which $K_{H(\varepsilon)}$ remains embedded. We continue until every derived face of $f$ is a 2-cell. Then $K_{H(\varepsilon)}$ is embedded on, say, $S_{\varepsilon'}$ with $\varepsilon' < \varepsilon$ and has fewer non-2-cell faces. By induction each 2-cell face has size at most $H(\varepsilon) - 1$ and thus every original 2-cell face, which has not been affected by the surgery, also has size at most $H(\varepsilon) - 1$. $\square$

We have purposefully proved more within the previous proof.

**Corollary 4.2.** Suppose $K_{H(\varepsilon)}$ has a non-2-cell embedding on $S_{\varepsilon}$, and suppose that after cutting along noncontractible cycles in non-2-cell faces, $K_{H(\varepsilon)}$ has a 2-cell embedding on $S_{\varepsilon'}$, $\varepsilon' < \varepsilon$. Then the number of faces in the latter embedding that are derived from faces in the original embedding is at most $\varepsilon - \varepsilon'$.

**Proof.** In the previous proof we saw that with some cuts the number of derived faces is increased by at most 1 and the Euler genus is decreased by at least 1; let $c_0$ denote the number of cuts in which there is no increase in the number of derived faces and $c_1$ the number of cuts in which there is an increase of 1 in the number of derived faces. If the increase is always at
most 1, then the result follows. The number of derived faces is increased by 2 precisely when the cutting cycle $C$ within a face $f'$ is 2-sided, is not surface-separating, is separating within $f'$, and $f'$ is an original face of the embedding. In that case the Euler genus is decreased by 2 also; let $c_0$ denote the number of such cuts. Then the decrease in the Euler genus, $\varepsilon - \varepsilon'$ is at least $c_0 + c_1 + 2c_2 \geq c_1 + 2c_2$, which equals the number of derived faces. □

**Theorem 4.3.** Given $\varepsilon > 0$ and $G$ a graph on $n$ vertices that has a 2-cell embedding on $S_\varepsilon$, suppose that $G$ contains $K_{H(\varepsilon)}$. If $P \subset V(G)$ satisfies $\text{dist}(P) \geq 4$, then if the vertices of $P$ each have a 1-list and every other vertex of $G$ has an $H(\varepsilon)$-list, then $G$ can be list-colored.

**Proof.** The proof is by induction on $\varepsilon$ and on $n$. We know the theorem holds for $G$ with a 2-cell embedding on $S_\varepsilon$ for $1 \leq \varepsilon \leq 2$ by Thm. 3.3. Consider graphs with 2-cell embeddings on $S_{\varepsilon^*}$ for $\varepsilon^* \geq 3$. For each such embedded graph, the subgraph $K_{H(\varepsilon^*)}$ inherits an embedding on $S_{\varepsilon^*}$, and $H(\varepsilon^*) \geq 7$.

Since $\text{dist}(P) \geq 4$ we know that at most one vertex of $P$ lies in or is adjacent to a vertex of $K_{H(\varepsilon^*)}$. If there is one, call it $v^*_i$ and if not, ignore reference to $v^*_i$ in the following. By Lemma 3.1.1 we know that $G[V(K_{H(\varepsilon^*)}) \cup \{v^*_i\}]$ can be list-colored since $v^*_i$ is adjacent to at most $H(\varepsilon^*) - 1$ vertices of $K_{H(\varepsilon^*)}$ (because $K_{H(\varepsilon^*)+1}$ does not embed on $S_{\varepsilon^*}$). If $G$ contains a vertex $x$ in neither $V(K_{H(\varepsilon^*)})$ nor $P$, then $G[V(K_{H(\varepsilon^*)}) \cup \{x\}]$ can be list-colored by first coloring $K_{H(\varepsilon^*)}$ and then coloring $x$, which has an $H(\varepsilon^*)$-list and is adjacent to at most $H(\varepsilon^*) - 1$ vertices of $K_{H(\varepsilon^*)}$.

Thus on surface $S_{\varepsilon^*}$ we know the result holds for every graph on $n$ vertices with $n \leq H(\varepsilon^*) + 1$. Let $G$ have $n^*$ vertices, $n^* > H(\varepsilon^*) + 1$, and have a 2-cell embedding on $S_{\varepsilon^*}$.

Let $f$ be a $k$-region in the inherited embedding of $K_{H(\varepsilon^*)}$ with incident vertices $V(f)$ and edges $E(f)$, and let $G_f$ denote the subgraph of $G$ lying in the closure of $f$, $f \cup V(f) \cup E(f)$. Suppose $f$ is a 2-cell face of $K_{H(\varepsilon^*)}$ in whose interior lie vertices of $V(G) \setminus \{V(f) \cup \{v^*_i\}\}$; call these interior vertices $U_f$. Then after deleting the vertices of $U_f$, $G \setminus U_f$ has a 2-cell embedding on $S_{\varepsilon^*}$ with fewer than $n^*$ vertices, contains $K_{H(\varepsilon^*)}$, and contains vertices of $P' \subseteq P$ with $\text{dist}(P') \geq 4$. By induction $G \setminus U_f$ is list-colorable. By Lemma 4.1 $k \leq H(\varepsilon^*) - 1$. We claim that the resulting list-coloring of $G[V(f) \cup \{v^*_i\}]$ extends to $G_f$.

If $k \leq H(\varepsilon^*) - 2$, then the coloring extends by Lemma 2.5.1 and 2.5.2. Otherwise $k = H(\varepsilon^*) - 1$ and the coloring then extends by Lemma 2.5.3, unless there is a vertex $x$ of $G_f$ that has an $H(\varepsilon^*)$-list, is adjacent to $v^*_i$, not
in $V(f)$, and to all vertices of $V(f)$, and its $H(\varepsilon^*)$-list consists of $H(\varepsilon^*)$ colors that appear on its neighbors. Then $G[V(K_{H(\varepsilon^*)}) \cup \{x\}]$ forms $DK_{H(\varepsilon^*)}$, which triangulates $S_{\varepsilon^*}$ and does not contain another vertex of $P$ since $dist(P) \geq 4$. Since $v_i^* \in V$ is adjacent to at most three vertices of $DK_{H(\varepsilon^*)}$ (the vertices of a 3-region), $G[V(DK_{H(\varepsilon^*)}) \cup \{v_i^*\}]$ can be list-colored by Lemma 3.1.3. Then the list-coloring extends to the graph in the interior of each 3-region by Lemma 2.5.1 since $H(\varepsilon^*) \geq 7$.

Thus we can assume that every vertex of $V(G) \setminus \{V(K_{H(\varepsilon^*)}) \cup \{v_i^*\}\}$ lies in a non-2-cell region of the embedding of $K_{H(\varepsilon^*)}$ on $S_{\varepsilon^*}$. We claim there are two vertices of $K_{H(\varepsilon^*)}$ that lie only on its 2-cell faces; we prove that below. One of these might lie in $P$ or be adjacent to $v_i^*$, but the other, say $x^*$, has an $H(\varepsilon^*)$-list and is adjacent only to vertices of $K_{H(\varepsilon^*)}$, precisely $H(\varepsilon^*) - 1$ of these.

In that case we consider $G \setminus \{x^*\}$. If $G \setminus \{x^*\}$ does not contain $K_{H(\varepsilon^*)}$, it can be list-colored by Thm. 3.2. Otherwise $G \setminus \{x^*\}$ does contain $K_{H(\varepsilon^*)}$. $G \setminus \{x^*\}$ might have a 2-cell embedding on $S_{\varepsilon^*}$ or it might not. In the former case, by induction on $n$ it can be list-colored. Suppose that $G \setminus \{x^*\}$ does not have a 2-cell embedding on $S_{\varepsilon^*}$. Then the face $f^*$ that was formed by deleting $x^*$ is the one and only non-2-cell face of that embedding since no other face of $G$ has been changed by the deletion of $x^*$. Then we cut along noncontractible cycles within $f^*$, as described in Lemma 4.1, until every face, derived from $f^*$, is a 2-cell in $G \setminus \{x^*\}$ now embedded on $S_{\varepsilon'}$ with $\varepsilon' < \varepsilon^*$. We have $H(\varepsilon') = H(\varepsilon^*)$ since $G \setminus \{x^*\}$ contains $K_{H(\varepsilon^*)}$. Thus $G \setminus \{x^*\}$ can be list-colored by induction on the Euler genus, and in all cases that coloring extends to $G$ since $x^*$ has a list of size $H(\varepsilon^*)$ which is larger than its degree.

We return to the claim that there are two vertices of $K_{H(\varepsilon^*)}$ that lie only on 2-cell faces of its embedding on $S_{\varepsilon^*}$, given that every vertex of $V(G) \setminus \{V(K_{H(\varepsilon^*)}) \cup \{v_i^*\}\}$ lies in a non-2-cell face of the embedded $K_{H(\varepsilon^*)}$. Since the number of vertices of $G$, $n^*$, is greater than $H(\varepsilon^*) + 1$, there are some non-2-cell faces containing other vertices of $G$. We count the maximum number of vertices of $K_{H(\varepsilon^*)}$ that lie on these non-2-cells to show that number is at most $H(\varepsilon^*) - 2$.

As in Lemma 4.1 we repeatedly cut each non-2-cell face of the embedded $K_{H(\varepsilon^*)}$ until all remaining faces, the original and the derived, are 2-cells; suppose $K_{H(\varepsilon^*)}$ is then embedded on $S_{\varepsilon'}$ with $\varepsilon' < \varepsilon^*$. We know that every vertex originally on a non-2-cell face of $K_{H(\varepsilon^*)}$ is represented on at least one derived face and we show below that the total number of vertices on derived faces is at most $H(\varepsilon^*) - 2$. We also know that $\varepsilon' \geq I(H(\varepsilon^*))$. Let
\( n_1 = \varepsilon' - I(H(\varepsilon^*)), \) which is nonnegative, and \( n_2 = \varepsilon^* - \varepsilon', \) which is positive. The variable \( n_1 \) will determine the face sizes in the 2-cell embedding of \( K_{H(\varepsilon^*)} \) on \( S_{\varepsilon'} \) (see Table 1), and \( n_2 \) will determine the maximum number of derived faces that have been created.

We consider the modulo 3 class of \( H(\varepsilon^*) \), and we begin with the case of \( H(\varepsilon^*) = 3i + 4, i \geq 1 \). We know that \( \varepsilon^* \in \{(3i^2 + i)/2, \ldots, (3i^2 + 3i)/2\} = \{I(3i + 4), \ldots, I(3i + 4) + i\} \) so that \( n_1 + n_2 \leq i \) by Lemma 2.1. By Cor. 4.2 the number of derived faces is at most \( n_2 \). We can determine the possible face sizes of a 2-cell embedding of \( K_{H(\varepsilon^*)} \) on \( S_{\varepsilon'} \) with \( \varepsilon' = I(3i + 4) + n_1 \). A 2-cell embedding on \( S_{I(3i+4)} \) is necessarily a triangulation. A 2-cell embedding on \( S_{I(3i+4)+1} \) consists of triangles except possibly for one 6-region, or triangles plus two faces whose sizes sum to 9, or triangles plus three faces whose sizes sum to 12 (necessarily three 4-regions). More generally when \( \varepsilon' = I(3i + 4) + n_1 \), then the embedding might consist of triangles plus one \((3n_1 + 3)\)-region, or triangles plus two faces whose sizes sum to \( 3n_1 + 6 \), or triangles plus three faces whose sizes sum to \( 3n_1 + 9 \), etc. And if we choose \( n_2 \) faces, all the derived faces, the sum of their sizes can be at most \( 3n_1 + 3n_2 \leq 3i < 3i + 2 = H(\varepsilon^*) - 2 \).

For \( i \geq 1 \), the same calculation holds when \( H(\varepsilon^*) = 3i + 3 \), and when \( H(\varepsilon^*) = 3i + 5 \), a similar count will work. In the latter case we have \( n_1 + n_2 \leq i - 1 \), though the face sizes may be slightly larger. A 2-cell embedding of \( K_{H(\varepsilon^*)} \) on \( S_{I(3i+5)} \) may have triangles plus a 5-region or triangles plus two 4-regions. In general a 2-cell embedding of \( K_{H(\varepsilon^*)} \) on \( S_{I(3i+5)+n_1} \) might have triangles plus one \((3n_1 + 5)\)-region or triangles plus two regions whose sizes sum to \( 3n_1 + 8 \), etc. With \( n_2 \) faces, all the derived faces, their sum of sizes can be at most \( 3n_1 + 3n_2 + 2 \leq 3i - 1 < 3i + 3 = H(\varepsilon^*) - 2 \). \( \square \)

We now complete the proof our main result, Thm. 1.3.

**Proof of Thm. 1.3.** If \( G \) has a non-2-cell embedding on \( S_{\varepsilon} \) that contains \( K_{H(\varepsilon)} \), we can perform surgery on the non-2-cell faces, as we did in the proof of Lemma 4.1 and Thm. 4.3, to obtain a 2-cell embedding of \( G \) on a surface of Euler genus \( \varepsilon' < \varepsilon \) that still contains \( K_{H(\varepsilon)} \), and hence \( H(\varepsilon') = H(\varepsilon) \). We can thus apply Thm. 4.3 to \( G \) on \( S_{\varepsilon'} \). This shows that the result holds for every embedding, 2-cell or non-2-cell, and Thm. 1.3 follows. \( \square \)

The distance bound of 4 in Thms. 1.3 and 4.3 is best possible, for consider \( K_{H(\varepsilon)} \) with a pendant edge attaching a degree-1 vertex to each vertex of \( K_{H(\varepsilon)} \). Give each degree-1 vertex the list \( \{1\} \) and place that vertex in the set.
When every other vertex has an identical $H(\varepsilon)$-list that contains 1, the graph is not list-colorable and $\text{dist}(P) = 3$.

The second corollary of Section 1 now follows easily.

Proof of Cor. 1.5. Let $f_1, \ldots, f_j$ be the faces with vertices with smaller lists. Add a vertex $x_i$ to $f_i$ and make it adjacent to all vertices of $V(f_i)$. Give each $x_i$ a 1-list $\{\alpha\}$ where $\alpha$ appears in no list of a vertex of $G$, and add $\alpha$ to the list of each vertex of $V(f_i)$, now the neighbors of $x_i$. Then $G \cup \{x_1, \ldots, x_j\}$ can be list-colored by Thm. 1.3 since with $P = \{x_1, \ldots, x_j\}$, $\text{dist}(P) \geq 4$, and this coloring is a list-coloring of $G$. \qed

5 Concluding Questions

1. Škrekovski [14] has shown the extension of Dirac’s theorem that if $G$ is embedded on $S_\varepsilon$, $\varepsilon \geq 5$, $\varepsilon \neq 6, 9$, and does not contain $K_{H(\varepsilon)-1}$ or $K_{H(\varepsilon)-4} + C_5$, then $G$ can be ($H(\varepsilon) - 2$)-colored. Is the same true for list-coloring?

2. If $G$ embeds on $S_\varepsilon$ and does not contain one of the two graphs of Question 1, if the vertices of one face have at least ($H(\varepsilon) - 2$)-lists, and if all other vertices have at least $H(\varepsilon)$-lists, can $G$ be list-colored?

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