Erratum: “Effect of Electrical Resistivity on the Damping of Slow Sausage Modes” (2020, ApJ, 897, 120)

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1. Introduction

In the published article we studied slow sausage modes in a cylindrical flux tube in linearized resistive magnetohydrodynamics (MHD) using cylindrical coordinates (r, \( \varphi \), z). We assumed that electrical resistivity is small, such that the diffusion of the background magnetic field could be neglected and the approximation of a sharp boundary separating two homogeneous plasmas inside and outside the tube could be made. The perturbed MHD quantities \( f_i \) were taken of the form

\[
f_i = \tilde{f}_i(r) \exp \{i(k_c z - \omega t)\}.
\]

The bar is dropped next for simplicity. We then derived the solution of the compression \( R(r) \):

\[
R(r) = \begin{cases} 
C_1 J_0(\kappa_- r) + C_3 J_0(\kappa_+ r) & \text{if } r < a \\
C_2 H_0(\kappa_- r) + C_4 H_0(\kappa_+ r) & \text{if } r > a'
\end{cases}
\]

where \( C_1, C_2, C_3, \) and \( C_4 \) are constants; \( J_0 \) is the complex-valued Bessel function of the first kind of order 0; \( H_0 \) stands for either \( H_0^{(1)} \) or \( H_0^{(2)} \) depending on the argument of the function (the amplitude of the oscillations having to decrease as \( r \to \infty \), corresponding to a wave propagating energy outwardly from the cylinder); and

\[
\kappa_{\pm} = \frac{1}{2A} \left( -k_c^2 A \pm \sqrt{k_c^4 A^2 - 2k_c^2 A + 4m_2^2 A + 1} + 1 \right),
\]

with \( A = \frac{\omega_p^2}{(k_c^2 \nu^2 - \omega^2) \nu^2} \) and \( m_2^2 = \frac{(k_c^2 \nu^2 - \omega^2) (k_c^2 \nu^2 - \omega^2) - \nu^2 \omega^2}{(k_c^2 \nu^2 - \omega^2) \nu^2 + \omega^2} \), with \( \nu = \frac{\nu_0 \nu_1}{\nu_0^2 + \nu_1^2} \). From \( R \), all the other MHD quantities could be derived.

To derive the dispersion relation of the modes, four boundary conditions were needed at the interface between the two plasmas. The boundary conditions that were used are the following:

\[
[f] = 0,
\]

\[
[P_1] = 0,
\]

\[
[n \cdot B] = 0,
\]

\[
[n \times E] = v_i B,
\]

with \( \xi \), the radial component of the plasma displacement, \( P_1 \) the perturbed total pressure, \( B \) the magnetic field, \( E \) the electric field, \( v_i \) the component of the plasma velocity along its normal, and \( n \) the unit normal to the boundary. Here \([f] \) denotes the jump of a quantity \( f \) at the boundary.

2. Corrections

2.1. Correction 1

The fourth boundary condition, Equation (7), is actually not correct for our model. The correct form of this boundary condition in this case is

\[
[n \times E] = 0.
\]

However, this is equivalent to Equation (6) for time-varying \( B \) and thus cannot be used as an independent fourth boundary condition. Instead, the following condition needs to be used, as it must hold at any position in a resistive plasma (Roberts 1967):

\[
[n \times B] = 0.
\]

As in our case we have \( n = 1 \), and \( B = B_{1z} \mathbf{1}_z + B_1 \mathbf{1}_r \), Equation (9) becomes \([B_z] = 0\), with \( B_z \) the \( z \)-component of \( B \). However, we note that our approximation of the interface remaining discontinuous entails that there remains a nonvanishing jump in the background
magnetic field $B_0$ instead of a continuous smoothing of $B_0$ in a narrow diffusion layer around the initial interface. Equation (9) is hence not fulfilled by the background magnetic field $B_0$, similarly to the case of an ideal plasma (in which case the left-hand side of Equation (9) defines $\mu_0 J$, with $J$ the surface current on the interface). Only the perturbed magnetic field will thus satisfy Equation (9), yielding

$$[B_{1c}]=0. \tag{10}$$

The dispersion relation obtained by imposing the boundary conditions (4), (5), (6), and (9) is then altered (with respect to the one described in the published article, namely, Equation (18) therein) and becomes

$$\begin{align*}
\kappa_+ \cdot \kappa_+ J_0(\kappa_+ a) H_0(\kappa_- a) H_1(\kappa_+ a) J_1(\kappa_- a) F_1(\omega) \\
- \kappa_+ \cdot \kappa_- J_0(\kappa_+ a) H_1(\kappa_- a) H_0(\kappa_+ a) J_1(\kappa_- a) F_2(\omega) \\
- \kappa_+ \cdot \kappa_- J_1(\kappa_- a) H_0(\kappa_+ a) H_0(\kappa_+ a) J_0(\kappa_- a) \\
\cdot B_{0z} \cdot B_{1zc} \cdot \eta^2 \cdot \rho_0 \cdot \omega^2 \cdot \beta_{\mu_0}^2 A_\mu^2(k_{e+1}^2 - k_{e-1}^2)(k_{e-1}^2 - k_{e+1}^2)(k_{e-1}^2 - k_{e+1}^2) \cdot \beta_{\mu_0}^2 A_\mu^2(\kappa_+^2 v_A^2 - \omega^2) \\
- \kappa_+ \cdot \kappa_+ J_0(\kappa_- a) H_1(\kappa_- a) H_0(\kappa_+ a) J_1(\kappa_- a) \\
\cdot B_{0z} \cdot B_{1zc} \cdot \eta^2 \cdot \rho_0 \cdot \omega^2 \cdot \beta_{\mu_0}^2 A_\mu^2(k_{e+1}^2 - k_{e-1}^2)(k_{e-1}^2 - k_{e+1}^2)(k_{e-1}^2 - k_{e+1}^2) \cdot \beta_{\mu_0}^2 A_\mu^2(\kappa_-^2 v_A^2 - \omega^2) \\
- \kappa_+ \cdot \kappa_- J_1(\kappa_- a) H_0(\kappa_+ a) H_0(\kappa_+ a) J_0(\kappa_- a) F_3(\omega) \\
+ \kappa_+ \cdot \kappa_- J_1(\kappa_- a) H_1(\kappa_- a) H_0(\kappa_+ a) J_0(\kappa_- a) F_4(\omega) = 0. \tag{11}\end{align*}$$

Here $F_1$, $F_2$, $F_3$, and $F_4$ are very long expressions in $\omega$. Equation (11) should thus replace Equation (18) in the published article. We call the left-hand side of Equation (11) the dispersion function.

To find the ideal limit (i.e., $\eta \to 0$) of this dispersion relation, the asymptotic approximations for large arguments of the Bessel and Hankel functions $J_0$, $J_1$, $H_0^{(1)}$, $H_0^{(2)}$, $H_1$, and $H_1^{(2)}$ (see Abramowitz & Stegun 1965, Chapter 9) can be used for the functions having $\kappa_+ a$ as an argument in Equation (11), as indeed $|\kappa_+| \to \infty$ when $\eta \to 0$. From the asymptotic form of $J_n$ for large arguments it can be seen that, for a complex nonreal $\tilde{z}$ for which arg$(\tilde{z})$ remains constant while $|\tilde{z}| \to \infty$, we have

$$J_n(\tilde{z}) \approx \frac{1}{\sqrt{2\pi \tilde{z}}} e^{i\left(\tilde{z} - \frac{\pi}{2} - \frac{\arg(\tilde{z})}{\sqrt{2}}\right)} \tag{12}$$

for $-\pi < \arg(\tilde{z}) < 0$,

$$J_n(\tilde{z}) \approx \frac{1}{\sqrt{2\pi \tilde{z}}} e^{-i\left(\tilde{z} - \frac{\pi}{2} - \frac{\arg(\tilde{z})}{\sqrt{2}}\right)} \tag{13}$$

for $0 < \arg(\tilde{z}) < \pi$.

Furthermore, the asymptotic forms of the Hankel functions for $|z| \to \infty$ are given by

$$H_n^{(1)}(\tilde{z}) \approx \frac{2}{\pi \tilde{z}} e^{i\left(\tilde{z} - \frac{\pi}{2} - \frac{\arg(\tilde{z})}{\sqrt{2}}\right)} \tag{14}$$

for $-\pi < \arg(\tilde{z}) < 2\pi$,

$$H_n^{(2)}(\tilde{z}) \approx \frac{2}{\pi \tilde{z}} e^{-i\left(\tilde{z} - \frac{\pi}{2} - \frac{\arg(\tilde{z})}{\sqrt{2}}\right)} \tag{15}$$

for $-2\pi < \arg(\tilde{z}) < \pi$.

The frequencies of trapped modes in ideal MHD are real because they are not damped. Therefore, if we are interested in the trapped slow surface mode in the photospheric conditions used in Section 4.2 of the published article (and taken from Figure 3 in Edwin & Roberts 1983), for example, we have $\omega_{\kappa e} < \omega < \omega_{\kappa i}$. Using the approximations (18) and (19) for $\kappa_-$ and $\kappa_+$ given in the next section of this erratum, we find that $\arg(\kappa_+ a) = \pi/4$ and $\arg(\kappa_- a) = -\pi/4$. Hence, for this mode we use Equations (12) and (14).

As $\eta \to 0$, the first term on the left-hand side of Equation (11) is $O(\eta)$, whereas the second, third, fourth, and fifth terms are $O(\sqrt{\eta})$. The last term is 0 because $F_4$ is identically 0 in this limit. For surface modes this leads to the following limit of the dispersion relation
with $\tau_{\pm} = \left(\frac{k^2 v^2 + \omega^2}{\nu_{\pm, \infty}}\right)^{1/2}$. This relation is almost the dispersion relation from ideal MHD given, for example, by Equation (8) in Edwin & Roberts (1983), but there is a discrepancy due to the model not being entirely physical. Indeed, there remains a jump in the background magnetic field at $r = a$ over time such that the boundary condition (9) is not fulfilled by the background. In the special case where $B_{\text{azi}} = B_{0z}$ the model is physical, as there is no diffusion of the background magnetic field, and condition (9) is fulfilled by the background. We see that in that case Equation (16) reverts exactly to the ideal dispersion relation.

The plots resulting from the corrected dispersion relation (11) and corresponding to Figures 2 and 3 of the published article are very similar to those, and our qualitative discussion of them in Section 4 remains entirely valid. The correct plots resulting from the dispersion relation (11) are shown in Figures 2 and 3.

### 2.2. Correction 2

A second minor correction concerns the ideal limit discussed in Section 3.2 in the published article and is required owing to the fact that we never gave an explicit definition of $m$. It should first be clarified what is meant by "—". We can take as the definition for $\sqrt{\xi}$ of a number $\xi \in \mathbb{C}$ the principal value of the square root of $\xi$, defined as the solution $w \in \mathbb{C}$ to $\xi = w^2$ with $-\frac{\pi}{2} < \text{Arg}(w) \leq \frac{\pi}{2}$. Then, $m$ is defined by

$$m = \sqrt{\frac{(k^2 v^2 - \omega^2)(k^2 v^2 - \omega^2)}{(k^2 v^2 - \omega^2)(v^2 + s^2)}}.$$  

(17)
In Section 3.2 we mentioned that for \( \eta \to 0 \) we have \( \kappa_- \to \sqrt{-m^2} \). However, it should be

\[
\kappa_- \to \pm \sqrt{-m^2}.
\]  

(18)

The plus sign must be used except if \( \arg(\kappa^2) \to -\pi \) as \( \eta \to 0 \). In that case the argument of the square root function \( \sqrt{-m^2} \) in Equation (18) lies on the other branch of the multivalued square root function with respect to the argument of the external \( \sqrt{-m^2} \) in the true expression of \( \kappa_- \) in Equation (3). For completeness we note that

\[
\kappa_+ \approx \pm \sqrt{(k_+^2 - \omega^2)(\nu^2 + \nu'^2)}
\]

(19)

for \( \eta \to 0 \), which is used in order to find the ideal limit of dispersion relation (11) in the previous section.

As a result, the expression for \( R(r) \) in Equation (2) becomes

\[
R(r) = \begin{cases} 
C_1 J_0(\pm \sqrt{-m^2} r) & \text{if } r < a \\
C_2 H_0(\pm \sqrt{-m^2} r) & \text{if } r > a 
\end{cases}
\]

(20)

in the limit \( \eta \to 0 \), where \( C_1 \) and \( C_2 \) are constants. This can be rewritten in the standard form found in Equations (6) and (7) in Edwin & Roberts (1983). Indeed, for \( \xi \in \mathbb{C} \) we have \( I_n(\xi) = (-i)^n J_n(i\xi) \), as well as \( K_n(\xi) = \frac{\Gamma(\frac{1}{2})}{\sqrt{\pi}} H^{(1)}_{n+\frac{1}{2}}(i\xi) \) when \(-\pi < \arg(\xi) \leq \frac{\pi}{2}\) and \( K_n(\xi) = -\frac{\Gamma(\frac{1}{2})}{\sqrt{\pi}} H^{(2)}_{n+\frac{1}{2}}(i\xi) \) when \(-\frac{\pi}{2} < \arg(\xi) \leq \pi\). For trapped modes, \( m^2 > 0 \) and hence \( \sqrt{-m^2} = im_\pi \). Therefore, if the solution (20) contains \( H_0(\sqrt{-m^2} r) \) in the part \( r > a \), the Hankel function \( H_0^{(1)} \) must be used and we get \( H_0^{(1)}(im_\pi r) = 2/(i\pi) K_0(m_\pi r) \).

If Equation (20) contains \( H_0(-\sqrt{-m^2} r) \) in the part \( r > a \), the Hankel function \( H_0^{(2)} \) must be used and we get \( H_0^{(2)}(-im_\pi r) = -2/(i\pi) K_0(m_\pi r) \). Furthermore, \( J_0 \) is an even function, and hence \( J_0(\pm \sqrt{-m^2} r) \) can always be replaced with \( J_0(\sqrt{-m^2} r) \) in the part \( r < a \) of the solution, which can be rewritten as \( I_0(m_\pi r) \) if \( m^2 > 0 \) (i.e., for a surface mode). Hence, we retrieve the ideal solution in its form given by Edwin & Roberts (1983).

### 2.3. Correction 3

In Section 5 of the published article we investigated the long-wavelength limit of the dispersion relation. Whereas the method employed in that section was correct, the new dispersion relation found here does not allow us to calculate in advance the expression
of the frequency $\omega$ for the limit $k_\tau a \to 0$, rendering the method followed in the published article useless. It turns out to be actually quite complicated to calculate the long-wavelength limit of this dispersion relation for slow modes owing to their having a frequency close to $\omega_{ci}$.

As mentioned in the published article, it can be shown that $\kappa_- \to 0$ when $k_\tau a \to 0$. However, as $\kappa_+ \to \infty$ when $\eta \to 0$, the derivation of a long-wavelength limiting form of the dispersion relation using truncated Taylor series like in Section 5 of the published article is only applicable for extremely small values of $k_\tau a$. These are so small that they are not relevant for realistic conditions of slow modes in photospheric pores, as the small value of resistivity keeps $\kappa_+ a$ several orders of magnitude higher than them. For observations of slow sausage modes in pore conditions (such as discussed, e.g., by Grant et al. 2015; Moreels et al. 2015; Freij et al. 2016) we indeed have $\kappa_+ a \gg 1$. The slow-mode oscillations observed in photospheric pores do not have a long wavelength, but even for values of $k_\tau a$ typically considered in the long-wavelength limit in coronal loops (say, $10^{-2} \leq k_\tau a \ll 1$) the condition $\kappa_+ a \gg 1$ is still valid. It therefore does not seem useful to further attempt to find an analytical expression for the dispersion relation in the long-wavelength limit with $\eta \neq 0$, and the results presented in Section 5 in the published article are best ignored.

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