On the average sum of the $k$th divisor function over values of quadratic polynomials

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Abstract
Let $F(x) \in \mathbb{Z}[x_1, x_2, \ldots, x_n]$, $n \geq 3$, be an $n$-variable quadratic polynomial with a nonsingular quadratic part. Using the circle method we derive an asymptotic formula for the sum

$$
\sum_{k, F} (X; B) = \sum_{x \in X \cap \mathbb{Z}} \tau_k (F(x));
$$

for $X$ tending to infinity, where $B \subset \mathbb{R}^n$ is an $n$-dimensional box such that $\min_{x \in X} F(x) \geq 0$ for all sufficiently large $X$, and $\tau_k (\cdot)$ is the $k$th divisor function for any integer $k \geq 2$.

Keywords Divisor functions · Quadratic polynomials · Circle method

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1 Introduction

The $k$th divisor function is a generalization of the divisor function $\tau(m) = \sum_{d|m} 1$ which counts the number of ways $m$ can be written as a product of $k$ positive integer numbers. It is defined as
\( \tau_k(m) = \# \left\{ (x_1, x_2, \ldots, x_k) \in \mathbb{Z}^k_+ : m = x_1 x_2 \ldots x_k \right\}, \)

where we assume that \( \tau_k(0) = 0 \). For polynomials \( F(x) \in \mathbb{Z}[x_1, \ldots, x_n] \) consider the sums

\[
T_k(F(x), X) = \sum_{|F(x)| \leq X} \tau_k(|F(x)|).
\]

Understanding the average order of \( \tau_k(m) \), as it ranges over sparse sequences of values taken by polynomials, i.e. of \( T_k(F, X) \), is a problem that has received a lot of attention.

The most studied case is naturally \( k = 2 \). For \( F(x) = F(x_1, x_2) \) a binary irreducible cubic form Greaves [8] showed that there exist real constants \( c_1 > 0 \) and \( c_2 \) depending only on \( F \), such that

\[
T_2(F(x), X) = c_1 X^{2/3} \log X + c_2 X^{2/3} + O_{\varepsilon,F}(X^{9/14+\varepsilon}),
\]

holds for any \( \varepsilon > 0 \) as \( X \to \infty \). If \( F(x_1, x_2) \) is an irreducible quartic form, Daniel [6] proved that

\[
T_2(F(x), X) = c_1 X^{1/2} \log X + O_F(X^{1/2} \log \log X),
\]

where \( c_1 > 0 \) is a constant depending only on \( F \). It seems that \( \deg F = 4 \) is the limit of the current available methods treating divisor sums over binary forms. More related works on the cases \( k = 2 \) and \( n = 2 \) are e.g. de la Bretèche and Browning [4], Browning [3] and Yu [17]. On the other hand, with their paper from 2012 Guo and Zhai [9] reviled the interest toward estimating asymptotically \( T_2(F(x), X) \) for forms in \( n \geq 3 \) variables using the classical circle method. After many other papers extending [9] and dealing with diagonal forms, in a recent work Liu [11] obtained an asymptotic formula for \( T_2(F(x), X) \) for any nonsingular quadratic form \( F \) in \( n \geq 3 \) variables.

For the cases when \( k \geq 3 \) there are only few results in the literature. Friedlander and Iwaniec [7] showed that

\[
\sum_{\substack{n_1^2 + n_2^6 \leq X \\gcd(n_1, n_2) = 1}} \tau_3(n_1^2 + n_2^6) = c X^{2/3} (\log X)^2 + O \left( X^{2/3} (\log X)^{7/4} (\log \log X)^{1/2} \right),
\]

where \( c \) is a constant. Daniel [5, (4.5)] described an asymptotic formula for \( T_k(F(x), X) \) as \( X \to \infty \) for any \( k \geq 2 \) for irreducible binary definite quadratic forms \( F \) and in [5, (4.7)] he proved an asymptotic formula for \( T_3(F(x), X) \) as \( X \to \infty \) for irreducible binary cubic forms \( F \). Sun and Zhang [16], with the help of the circle method, obtained

\[
\sum_{1 \leq m_1, m_2, m_3 \leq X} \tau_3 \left( m_1^2 + m_2^2 + m_3^2 \right)
= c_1 X^3 \log^2 X + c_2 X^3 \log X + c_3 X^3 + O_{\varepsilon}(X^{11/4+\varepsilon}),
\]
where \( c_1, c_2, c_3 \) are constants and \( \varepsilon \) is any positive number. Finally Blomer [2] proved an asymptotic formula for \( T_k(F(x), X) \) for any \( k \geq 2 \), where \( F(x) \) is a form of degree \( k \) in \( n = k - 1 \) variables, coming from incomplete norm form.

In this paper we investigate the average sum of the \( k \)th divisor function over values of quadratic polynomials \( F(x) \), not necessarily homogenous, in \( n \geq 3 \) variables for any \( k \geq 2 \). Every \( n \)-variables quadratic polynomial can be written as

\[
F(x) = x^T Q x + L^T x + N, \tag{1.1}
\]

where \( Q \in \mathbb{Z}^{n \times n} \) is a symmetric matrix, \( L \in \mathbb{Z}^n \) and \( N \in \mathbb{Z} \). Our only additional requirement is that \( Q \) is nonsingular. Let \( B \subset \mathbb{R}^n \) be an \( n \)-dimensional box (i.e. a certain product of intervals) such that \( \min_{x \in X \cap B} F(x) \geq 0 \) for all sufficiently large \( X \), and for each integer \( k \geq 2 \), consider the sum

\[
\Sigma_{k, F}(X; B) = \sum_{x \in X \cap \mathbb{Z}^n} \tau_k(F(x)) \tag{1.2}
\]

as \( X \) tends to infinity. Let us also use the following notation for \( q \in \mathbb{Z}_+ \):

\[
\varrho_F(q) = 1 \frac{q^{n-1}}{q^n-1} \# \{ h \mod q : F(h) \equiv 0 \mod q \}.
\]

Our main result is the following.

**Theorem 1.1** Let \( F(x) \) and \( \Sigma_{k, F}(X; B) \) be defined as in (1.1) and (1.2), respectively, where \( Q \) is a nonsingular matrix and \( n \geq 3 \). Then for any \( \varepsilon > 0 \) there exist real constants \( C_{k,0}(F), C_{k,1}(F), \ldots, \) and \( C_{k,k-1}(F) \), such that for \( X \) tending to infinity we have the asymptotic formula

\[
\Sigma_{k, F}(X; B) = \sum_{r=0}^{k-1} C_{k,r}(F) \int_{XB} (\log F(t))^r dt + O \left( X^{n-\frac{n-2}{n+2}} \min \left( 1, \frac{4}{k+1} \right) + \varepsilon \right),
\]

where the implied constant depends on \( F, k, B \) and \( \varepsilon \), and

\[
C_{k,r}(F) = \frac{1}{r!} \sum_{t=0}^{k-r-1} \frac{1}{t!} \left( \frac{d^t L(s; k, F)}{ds^t} \right)_{|s=1} \text{Res}_{s=1} ((s - 1)^{r+t} \zeta(s))^k.
\]

The function \( L(s; k, F) \) has the Euler product presentation

\[
L(s; k, F) = \prod_p \left( \sum_{\ell \geq 0} \varrho_F(p^\ell) \left( \tau_k(p^\ell) - p^s \tau_k(p^{\ell-1}) \right) \frac{p^\ell}{p^{\ell s}} \right) \left( \frac{1 - p^{-s}}{1 - p^{-1}} \right),
\]

with \( \tau_k(x) := 0 \) for all \( x \notin \mathbb{Z} \), and it is absolutely convergent for all \( \Re(s) > 1/2 \). In particular, the main term has a positive leading coefficient:
C_{k,k-1}(F) = \frac{1}{(k-1)!} \prod_p \left( \sum_{\ell \geq 0} \varrho_F(p^\ell) \tau_{k-1}(p^\ell) \right) \left( 1 - \frac{1}{p} \right)^{k-1} > 0.

First of all, we remark that since $F(x)$ has a nonsingular quadratic part, the set of all zeros of $F(x) = 0$ has a Lebesgue measure 0, so that the logarithm function in the integrals in the main terms is well defined. Note that we provide a formula with $k$ terms, where one can easily see that the main term is of magnitude $X^n (\log X)^{k-1}$ (when $r = k - 1$) and the last secondary term is of magnitude $X^n$ (when $r = 0$). Thus the error term is indeed of a smaller rate.

Using Theorem 1.1, one can get the asymptotic formula for $\Sigma_2, F(X, B)$ in the most studied case of $k = 2$. This recreates and extends the main Theorem of Liu [11] also for nonhomogenous quadratic polynomials, but also provides different expressions for the coefficients. Naturally, they can be also computed explicitly for specific polynomials, a goal we have not pursued in the current paper. Theorem 1.1 also extends the formula [5, (4.5)] of Daniel to quadratic polynomials in more than two variables, further, it elucidates the form of the involved coefficients.

Notations The symbols $\mathbb{Z}_+, \mathbb{Z}$ and $\mathbb{R}$ denote the positive integers, the integers and the real numbers, respectively. $e(z) := e^{2\pi i z}$, $\zeta(s) = \sum_{n \geq 1} n^{-s}$ is the Riemann zeta function, the letter $p$ always denotes a prime. We make use of the $\varepsilon$-convention, which allows us to write $x^\varepsilon \log x \leq c x^\varepsilon$ and $x^{2\varepsilon} \log x^{\varepsilon}$, for example. Furthermore, if not specially specified, all the implied constants of this paper in $O$ and $\ll$ depend on $F$, $k$, $B$ and $\varepsilon$.

2 The proof of Theorem 1.1

2.1 Setting up the circle method

The primary technique used in the proof of the main theorem is the circle method and more precisely its treatment by Pleasants [13]. The recent work on quadratic forms in $n \geq 3$ variables of Liu [11] uses the same circle method techniques, i.e. Weyl differencing, that were already used for general quadratic multivariable polynomials by Pleasants. For the real $X$ from the Definition 1.2 let $L \ll X$ be a positive real parameter which we will choose later in a suitable way, let $a, q \in \mathbb{Z}$, $0 \leq a < q \leq L$ and $\gcd(a, q) = 1$. Then we define the intervals

$$
\mathcal{M}_{a,q}(L) := \left\{ \alpha \in [0, 1] : \left| \alpha - \frac{a}{q} \right| \leq \frac{L}{qX^2} \right\}.
$$

The set of the major arcs is then the union

$$
\mathcal{M}(L) = \bigcup_{0 \leq a < q \leq L, \gcd(a, q) = 1} \mathcal{M}_{a,q}(L), \quad (2.1)
$$

and the set of the minor arcs is the complement $m(L) = [0, 1] \setminus \mathcal{M}(L)$.
We further define the following exponential sums for $\alpha \in \mathbb{R}$:

$$S(\alpha) = \sum_{x \in XB \cap \mathbb{Z}^n} e\left(F(x)\alpha\right)$$

and

$$T(\alpha, Y) = \sum_{0 \leq m \leq Y} \tau_k(m)e(m\alpha).$$

Then, by the well-known identity for $u \in \mathbb{Z}$

$$\int_0^1 e(u\alpha) \, d\alpha = \begin{cases} 1 & \text{if } u = 0, \\ 0 & \text{if } u \neq 0, \end{cases}$$

we have

$$\Sigma_{k, F}(X; B) = \sum_{x \in XB \cap \mathbb{Z}^n} \tau_k(F(x))$$

$$= \int_0^1 S(\alpha)T(-\alpha, C_{F, B}(X)) \, d\alpha$$

$$= \int_{\mathfrak{M}(L)} S(\alpha)T(-\alpha, C_{F, B}(X)) \, d\alpha$$

$$+ \int_{m(L)} S(\alpha)T(-\alpha, C_{F, B}(X)) \, d\alpha$$

$$= I_{\mathfrak{M}(L)} + I_{m(L)},$$

where

$$C_{F, B}(X) := \max_{x \in XB} |F(x)| = X^2 \max_{x \in B} \left| x^T Qx \right| + O(X) \asymp X^2.$$  

We shall prove in Sect. 2.2 that for the contribution from the minor arcs we have

$$I_{m(L)} \ll X^{n+\varepsilon} L^{1-n/2} \quad (2.2)$$

as long as $L \ll X$. Already here we see that we need to require that the number of variables satisfies $n \geq 3$ in order to have an error term of a smaller magnitude than $O(X^n)$. Further, in Sect. 2.3 we will show that

$$I_{\mathfrak{M}(L)} - \sum_{r=0}^{k-1} C_{k, r}(F) \int_{XB} (\log(F(t)))^r \, dt \ll X^{n+\varepsilon} \left( L^{1-n/2} + L^2 X^{-\min\left(1, \frac{4}{r+1}\right)} \right). \quad (2.3)$$
Here for \( r = 0, 1, \ldots, k - 1, \)
\[
C_{k, r}(F) = \sum_{q=1}^{\infty} \beta_{k, r}(q) S_F(q),
\]  
(2.4)
where
\[
S_F(q) = \sum_{\substack{a \in [1, q] \cap \mathbb{Z} \\gcd(a, q) = 1}} q^{-n} \sum_{h \in [1, q] \cap \mathbb{Z}} e \left( \frac{a F(h)}{q} \right),
\]
and the analytic function \( \Phi_k(q, s) \) is defined by Lemma 3.2. We further consider the function
\[
L(s; k, F) = \sum_{q \geq 1} \Phi_k(q, s) S_F(q).
\]  
(2.6)
In Sect. 4.2 we prove that it satisfies
\[
L(s; k, F) = \prod_p \left( \sum_{\ell \geq 0} \mathcal{Q}_F(p^{\ell}) \left( \tau_k(p^{\ell}) - p^{s-1}\tau_k(p^{\ell-1}) \right) \right) \left( \frac{1 - p^{-s})^k}{1 - p^{-1}} \right),
\]  
(2.7)
with \( \tau_k(x) := 0 \) for all \( x \notin \mathbb{Z} \).

Then Theorem 1.1 follows from (2.2), (2.3) and (2.7), after choosing \( L = X^{2 \min\left(1, \frac{1}{\ell+1}\right)} \).

### 2.2 Contribution from the minor arcs

Clearly, if the positive real numbers \( L \) and \( L' \) satisfy \( L \leq L' \), then \( \mathcal{M}(L) \subset \mathcal{M}(L') \), and if \( L \geq X \), then \([0, 1] \subset \mathcal{M}(L)\) follows from Dirichlet’s approximation theorem.

We further define
\[
\mathcal{F}(L) = \mathcal{M}(2L) \setminus \mathcal{M}(L).
\]
Then for a given positive number \( L < X/2 \),
\[
[0, 1] \subset \mathcal{M}(L) \cup \bigcup_{0 \leq j < N} \mathcal{F}(2^j L),
\]
where $N$ is the smallest integer greater than or equal to $(\log(X/L))/\log 2$. Clearly, the set of the small arcs then satisfies

$$m(L) \subset \bigcup_{0 \leq j < N} \mathcal{F}(2^j L). \quad (2.8)$$

To prove the estimate (2.2) over the minor arcs, we would use separate estimates of the two components $S(\alpha)$ and $T(\alpha, X)$ when $\alpha \in \mathcal{F}(L)$. We first state the following result.

**Lemma 2.1** For all positive numbers $L \ll X$,

$$\sup_{\alpha \in \mathcal{F}(L)} |S(\alpha)| \ll X^{n+\varepsilon} L^{-n/2}.$$  

**Proof** This estimate was done by Pleasants [13] even for the range $L \ll X (\log X)^{1/4}$. In the first equation of p. 138 [13] he proves that for $\alpha \in \mathcal{F}(L)$ we have

$$|S(\alpha)| \leq X^n (\log X)^n L^{-r/2},$$

where $r \geq 3$ is the rank of $Q$, and in our case we have assumed that $r = n$.  \hfill \square

We also need the following estimate.

**Lemma 2.2** For all positive numbers $L \ll X$,

$$\int_{\mathcal{F}(L)} \left| T(-\alpha, C_{F,B}(X)) \right| \, d\alpha \ll X^\varepsilon L.$$ 

**Proof** By Cauchy’s inequality, and using the definition of the major arcs (2.1), we have

$$\int_{\mathcal{F}(L)} \left| T(-\alpha, C_{F,B}(X)) \right| \, d\alpha \ll |\mathcal{F}(L)|^{1/2} \left( \int_0^1 \left| T(-\alpha, C_{F,B}(X)) \right|^2 \, d\alpha \right)^{1/2}$$

$$\ll |\mathfrak{M}(2L)|^{1/2} \left( \sum_{1 \leq n \leq C_{F,B}(X)} \tau_k(n) \right)^2 \ll \left( \sum_{1 \leq q \leq L} \frac{2L}{qX^2 \phi(q)} \right)^{1/2} X^{1+\varepsilon} \ll X^{-1+1+\varepsilon} L \ll X^\varepsilon L,$$

where we also applied the well-known bound $\tau_k(n) \ll_k n^\varepsilon$.  \hfill \square

Now the estimate (2.2) over the minor arcs follow from (2.8), Lemmas 2.1 and 2.2, namely

$$I_{m(L)} \ll \sum_{0 \leq j < N} \int_{\mathcal{F}(2^j L)} \left| S(\alpha) T(-\alpha, C_{F,B}(X)) \right| \, d\alpha.$$
\[
\ll \sum_{0 \leq j < N} \sup_{\alpha \in \mathcal{F}(2/L)} |S(\alpha)| \int_{\mathcal{F}(2/L)} |T(-\alpha, C_{F,B}(X))| \, d\alpha
\]
\[
\ll \sum_{0 \leq j < N} X^{n+\varepsilon} (2^j L)^{-n/2} (X^{\varepsilon} 2^j L) \ll X^{n+\varepsilon} L^{1-n/2},
\]
where we used that \( N \ll \log X \).

### 2.3 Contribution from the major arcs

In this subsection we have \( \alpha \in \mathcal{M}_{a,q}(L) \), and we shall write \( \beta = \alpha - a/q \) for the coprime integers \( a \) and \( q \), \( |\beta| \leq L/(qX^2) \) and \( 1 \leq q \leq L \). In order to prove the asymptotic formula (2.3), we need the following statements.

**Lemma 2.3** For \( \alpha \in \mathcal{M}_{a,q}(L) \), and \( \beta = \alpha - a/q \), we have

\[
S(\alpha) = q^{-n} S_F(q, a) \int_{X_B} e \left( F(t) \beta \right) \, dt + O_{B,F} \left( LX^{n-1} \right),
\]

where

\[
S_F(q, a) = \sum_{h \in [1, q] \cap \mathbb{Z}} e \left( \frac{a}{q} F(h) \right).
\]

**Proof** To prove this result we only need to adjust the last equation in the proof of [13, Lemma 8] with the upper bounds \( \beta \leq L/(qX^2) \) and \( q \leq L \). Note that Pleasant does the analysis over a quadratic polynomial with linear coefficients which can depend on \( X \). We are dealing with a quadratic \( F \) with fixed coefficients, which makes the proof even easier. \( \square \)

**Lemma 2.4** Let \( S_F(q, a) \) be defined as in Lemma 2.3. We have

\[
S_F(q, a) \ll_F q^{n/2+\varepsilon},
\]

where the implied constant is independent of \( a \) and \( q \).

**Proof** This is [13, Lemma 10]. \( \square \)

We further need the following two statements. The first one gives a general asymptotic representation of \( T(\alpha, Y) \) and the second one estimates the part of the singular integral coming from the major arcs. The proofs of Lemmas 2.5 and 2.6 will be given in Sects. 3 and 4.1, respectively.

**Lemma 2.5** Let \( Y \asymp X^2 \). We have

\[
T(\alpha, Y) = \sum_{r=0}^{k-1} \beta_{k,r}(q) \int_0^Y (\log u)^r e(u\beta) \, du + O_{k,\varepsilon} \left( LX^{2-\frac{4}{k+1}+\varepsilon} \right).
\]

\( \square \) Springer
where for \( r = 0, 1, \ldots, k - 1 \),

\[
\beta_{k,r}(q) = \frac{1}{r!} \sum_{t=0}^{k-r-1} \frac{1}{t!} \text{Res} \left( (s - 1)^{r+1} \zeta(s)^k \right) \left( \frac{d^t \Phi_k(q, s)}{ds^t} \right)_{s=1} \ll q^{-1+\varepsilon}.
\]

with \( \Phi_k(q, s) \) defined by Lemma 3.2. In particular,

\[
T(\alpha, Y) \ll q^{-1+\varepsilon} X^{2+\varepsilon}.
\]

Lemma 2.6 We have

\[
\int_{|\beta| \leq L/q X^2} d\beta \int_{XB} C_{F,B}(X) \int_{0} e ((F(t) - u)\beta) (\log u)^r \, du
\]

\[
= \int_{XB} (\log F(t))^r \, dt + O \left( \frac{q^{n/2} X^{n+\varepsilon}}{L^{n/2}} \right).
\]

We now prove the asymptotic formula (2.3). Using (2.1) we get

\[
I_{2R(L)} = \int_{2R(L)} S(\alpha) T(-\alpha, C_{F,B}(X)) \, d\alpha
\]

\[
= \sum_{q \leq L} \sum_{0 \leq a < q \atop \gcd(a, q) = 1} \int_{|\beta| \leq L/q X^2} S(\alpha) T(-\alpha, C_{F,B}(X)) \, d\alpha
\]

\[
= \sum_{q \leq L} \sum_{0 \leq a < q \atop \gcd(a, q) = 1} \mathcal{I}_{q,a}.
\]

Since \( 1 \leq q \leq L \ll X \), we have

\[
\mathcal{I}_{q,a} = \int_{|\beta| \leq L/q X^2} S(\alpha) T(-\alpha, C_{F,B}(X)) \, d\beta
\]

\[
= \int_{|\beta| \leq L/q X^2} \left( \frac{S_F(q, a)}{q^n} \int_{XB} e(F(t)\beta) \, dt \right) T(-\alpha, C_{F,B}(X)) \, d\beta
\]

\[
+ O \left( \int_{|\beta| \leq L/q X^2} (LX^{n-1}) q^{-1+\varepsilon} X^{2+\varepsilon} \, d\beta \right)
\]
by Lemma 2.3. Further, by applying Lemmas 2.4, 2.5 and 2.6, we get

\[ I_{q,a} = \sum_{r=0}^{k-1} \frac{S_F(q,a)\beta_{k,r}(q)}{q^n} \int_{|\beta| \leq L/q X^2} \int_X \left( \int e(F(t)\beta) \right) dt \int_0^{C_F,B(X)} (\log u)^r e(-u\beta) du \]

\[ + O \left( \sum_{q > L} q^{-1+\varepsilon} |S_F(q)| X^{n+\varepsilon} \right) \]

\[ = \sum_{r=0}^{k-1} \frac{S_F(q,a)\beta_{k,r}(q)}{q^n} \int_X \left( \int (\log F(t))^r dt + O \left( \frac{X^{n+\varepsilon}}{L^{n/2}} + \frac{X^{n-\frac{4}{k+1}+\varepsilon} L^2}{q^{1+n/2}} + \frac{L^2 X^{n-1+\varepsilon}}{q^2} \right) \right). \]

Recall the notation (2.4) and note that

\[ S_F(q) = \sum_{a \in [1,q] \cap \mathbb{Z}, \gcd(a,q) = 1} q^{-n} S_F(q,a). \]

Then after summing over all \( 1 \leq q \leq L \) and \( 1 \leq a < q, \gcd(a,q) = 1 \), the major arcs \( M(L) \) contribute

\[ I_{M(L)} = \sum_{q \geq 1} S_F(q) \sum_{r=0}^{k-1} \beta_{k,r}(q) \int_X \left( \int (\log F(t))^r dt + O \left( \sum_{q > L} q^{-1+\varepsilon} |S_F(q)| X^{n+\varepsilon} \right) \right) \]

\[ + O \left( X^{n+\varepsilon} L^{1-n/2} + X^{n-\frac{4}{k+1}+\varepsilon} L^2 + L^2 X^{n-1+\varepsilon} \right) \]

\[ = \sum_{r=0}^{k-1} C_{k,r}(F) \int_X \left( \int (\log(F(t)))^r dt + O \left( X^{n+\varepsilon} E \right) \right), \]

with

\[ E = L^{1-n/2} + L^2 \left( X^{-1} + X^{-\frac{4}{k+1}} \right) \ll L^{1-n/2} + L^2 X^{\min\left(\frac{4}{k+1}, 1\right)}. \]

Note that at this step, and at few other places, in order to control the error terms we necessarily have \( n \geq 3 \). This completes the proof of (2.3).

3 The estimates involving the kth divisor function

The usual technique in estimating asymptotically through the circle method average sums similar to \( \Sigma_k F(X,B) \), is the application of nontrivial average estimates of the

\[ \sum_{d \leq X} \Lambda(d) \Lambda(d)^k \]
specific arithmetic function over arithmetic progressions (e.g. [9–11]). Thus in order to prove Lemma 2.5 we first need the following result.

**Lemma 3.1** Let \( h, q \) be integers such that \( 1 \leq h \leq q \) and \( \gcd(h, q) = \delta \). Then for each real number \( x > 1 \), \( q \leq x^{\frac{2}{k+1}} \) and \( \varepsilon > 0 \), we have

\[
A_k(x; h, q) := \sum_{m \leq x \atop \text{m\equiv h \pmod{q}}} \tau_k(m) = M_k(x; h, q) + O_{k, \varepsilon}(x^{1 - \frac{2}{k+1} + \varepsilon}),
\]

where

\[
M_k(x; h, q) = \text{Res}_{s=1} \left( \frac{\zeta(s)^k x^s}{s} f_k(q, \delta, s) \right)
\]

with

\[
f_k(q, \delta, s) = \frac{1}{\varphi(q/\delta)^{\delta s}} \left( \sum_{d|\varphi(q/\delta)} \frac{\mu(d)}{d^s} \right)^k \sum_{d_1d_2...d_k=\delta, t_1|\prod_{j=1}^k d_j} \frac{\mu(t_1) \ldots \mu(t_k)}{(t_1...t_k)^s},
\]

where \( d_1, d_2, \ldots, d_k \) are positive integers and the empty product \( \prod_{j=k+1}^k d_j := 1 \).

**Proof** This lemma is essentially due to Smith [15], and we only adjust it for our purposes. We will extend easily [15, Theorem 3], which covers the case when \( h \) and \( q \) are coprime, to any \( h \) and \( q \). First, Eq. (30) of [15] states that

\[
A_k(x; h, q) = \sum_{d_1d_2...d_k=\delta} \sum_{t_1|\prod_{j=1}^k d_j \atop \gcd(t_1, q/\delta) = 1} \mu(t) A_k \left( \frac{x}{\delta t_1 t_2...t_k}; t_1t_2...t_k \frac{h}{\delta}, \frac{q}{\delta} \right),
\]

where \( d_1, d_2, \ldots, d_r \) are positive integers, \( \mu(t) = \prod_{j=1}^k \mu(t_j) \) and \( \overline{m} \) is the multiplicative inverse of \( m \) modulo \( q \). Then Theorem 3 of [15] states that

\[
A_k(x; h, q) = M_k(x; h, q) + \Delta_k(x; h, q),
\]

where

\[
M_k(x; h, q) = \sum_{d_1d_2...d_k=\delta} \sum_{t_1|\prod_{j=1}^k d_j \atop \gcd(t_1, q/\delta) = 1} \mu(t) \frac{x}{\delta t_1 t_2...t_k} P_k \left( \log \left( \frac{x}{\delta t_1 t_2...t_k} \right), \frac{q}{\delta} \right).
\]
and

\[
\Delta_k(x; h, q) = \sum_{d_1d_2...d_k = \delta} \sum_{\gcd(t_i, q/\delta) = 1, i=1,2,...,k} \mu(t) \left( D_k \left( 0; \frac{t_1}{\delta}...\frac{t_k}{\delta}, \frac{h}{\delta}, \frac{q}{\delta} \right) \right)
\]

\[
+ \sum_{d_1d_2...d_k = \delta} \sum_{\gcd(t_i, q/\delta) = 1, i=1,2,...,k} \mu(t) \left( O \left( \left( \frac{x}{\delta t_1...t_k} \right) ^{k-1} \tau_k \left( \frac{q}{\delta} \log^{k-1}(2x) \right) \right) \right)
\]

Here \( P_k(\log x, q) \) is a polynomial in \( \log x \) of degree \( k - 1 \) and \( D_k(s; h, q) \) is the Dirichlet series corresponding to the sum \( A_k(x; h, q) \). By the definition of \( P_k(\log x, q) \), namely [15, (13)], and the analysis of \( D_k(s; h, q) \) given in particular in [15, (21)], it is easily seen that

\[
x P_k(\log x, q) = \frac{1}{\varphi(q)} \text{Res}_{s=1} \left( \left( \zeta(s) \sum_{d|q} d^{-s} \mu(d) \right) ^{k} \frac{x^s}{s} \right)
\]

Hence

\[
M_k(x; h, q) = \sum_{d_1d_2...d_k = \delta} \sum_{\gcd(t_i, q/\delta) = 1, i=1,2,...,k} \frac{\mu(t_1)...\mu(t_k)}{\varphi(q/\delta)}
\]

\[
\times \text{Res}_{s=1} \left( \left( \zeta(s) \sum_{d|q} \frac{\mu(d)}{d^s} \right) ^{k} \frac{x^s}{(\delta t_1...t_k)^s} \right)
\]

\[
= \text{Res}_{s=1} \left( \zeta(s)^k \frac{x^s}{\varphi(q/\delta)/\delta^s} \left( \sum_{d|q} \frac{\mu(d)}{d^s} \right) ^{k} \sum_{d_1d_2...d_k = \delta} \sum_{\gcd(t_i, q/\delta) = 1, i=1,2,...,k} \frac{\mu(t_1)...\mu(t_k)}{(t_1...t_k)^s} \right)
\]

Thus the main term is

\[
M_k(x; h, q) = \text{Res}_{s=1} \left( \zeta(s)^k \frac{x^s}{s} f_k(q, \delta, s) \right)
\]
where, as defined in the statement of the lemma, we have

\[
f_k(q, \delta, s) = \frac{1}{\varphi(q/\delta)^s} \left( \sum_{d \mid (q/\delta)} \frac{\mu(d)}{d^s} \right)^k \sum_{d_1 \ldots d_k = \delta} \sum_{t_1 \mid (\prod_{i=1}^{k} d_i)} \frac{\mu(t_1) \ldots \mu(t_k)}{(t_1 \ldots t_k)^s}.
\]

Smith [15] conjectured the validity of the estimate \( D_k(0, h, q) \ll q^{k - 1/2 + \varepsilon} \) for any \((q, h) = 1\). This was later affirmed by Matsumoto [12]. Therefore we have the bound

\[
\Delta_k(x; h, q) \ll \sum_{d_1 \ldots d_k = \delta} \sum_{1 \leq h \leq q} \frac{1}{\varphi(q/\delta)} \varphi\left(\frac{q}{\delta}\right)^{k - 1/2 + \varepsilon} + q^\varepsilon \left(\frac{q}{x^{1/2} + \varepsilon}\right)^{k - 1/2 + \varepsilon},
\]

using \( q \leq x^{2/3} \) and \( \tau_k(\delta) \ll \delta^\varepsilon \). This completes the proof of the lemma. \( \square \)

**Lemma 3.2** Let \( q \geq 1 \) be an integer, \( \gcd(a, q) = 1 \) and \( \delta = \gcd(h, q) \). Also let \( f_k(q, \delta, s) \) be defined as in Lemma 3.1. Define

\[
\Phi_{k,a}(q, s) = \sum_{h=1}^{q} e\left(-\frac{ah}{q}\right) f_k(q, \delta, s).
\]

Then \( \Phi_{k,a}(q, s) \) is independent of \( a \) and we may write it as \( \Phi_k(q, s) \). Furthermore, \( \Phi_k(q, s) \) is multiplicative function and

\[
\frac{d^r \Phi_k(q, 1)}{ds^r} \ll_k q^{-1 + \varepsilon}
\]

holds for each integer \( r = 0, 1, \ldots, k - 1 \).

**Proof** First, we have

\[
\Phi_{k,a}(q, s) = \sum_{\delta \mid q} \sum_{1 \leq h \leq q/\delta, \gcd(h, q/\delta) = \delta} e\left(-\frac{ah}{q}\right) f_k(q, \delta, s)
\]

\[
= \sum_{\delta \mid q} f_k(q, \delta, s) \sum_{1 \leq h_1 \leq q/\delta, \gcd(h_1, q/\delta) = 1} e\left(-\frac{ah_1}{q/\delta}\right)
\]

\[
= \sum_{\delta \mid q} c_\delta(a) \cdot f_k(q, q/\delta, s) = \sum_{\delta \mid q} \mu(\delta) f_k(q, q/\delta, s), \quad (3.1)
\]
where \( c_\delta(a) \) is the Ramanujan’s sum and we use the fact that if \( \gcd(a, q/\delta) = \gcd(a, q) = 1 \) then \( c_\delta(a) = \mu(\delta) \). Therefore \( F_{k,a}(q, s) \) is independent on \( a \). Suppose that the positive integers \( q_1 \) and \( q_2 \) are coprime, then

\[
\Phi_k(q_1, s) \Phi_k(q_2, s) = \sum_{\delta_1|q_1} \sum_{\delta_2|q_1/\delta_1} \mu(\delta_1) \mu(\delta_2) f_k(q_1, q_1/\delta_1, s) f_k(q_2, q_2/\delta_2, s) \\
= \sum_{(\delta_1\delta_2)|(q_1q_2)} \mu(\delta_1 \delta_2) f_k(q_1, q_1/\delta_1, s) f_k(q_2, q_2/\delta_2, s),
\]

hence we just need to show that

\[
f_k(q_1, q_1/\delta_1, s) f_k(q_2, q_2/\delta_2, s) = f_k(q_1q_2, q_1q_2/(\delta_1\delta_2), s)
\]

whenever \( \delta_1|q_1 \) and \( \delta_2|q_2 \). For this we use the definition of \( f_k(q, q/\delta, s) \), namely

\[
f_k(q, q/\delta, s) = \frac{\delta^s \varphi(\delta)^s}{\varphi(q)^s} \left( \sum_{d|\delta} \frac{\mu(d)}{d^s} \right)^k \sum_{d_1d_2...d_k=q/\delta} \sum_{t_1|\prod_{j=i+1}^{i+k} d_j} \mu(t_1) \cdots \mu(t_k) \frac{s}{(t_1 \cdots t_k)^s}.
\]

For \( \sigma = \Re(s) \) we obtain

\[
f_k(q, q/\delta, s) \ll \frac{\delta^\sigma \varphi(\delta)^\sigma}{\varphi(q)^\sigma} \prod_{p|\delta} \left( 1 + \frac{1}{p^\sigma} \right)^k \sum_{d_1d_2...d_k=q/\delta} \sum_{t_1|\prod_{j=i+1}^{i+k} d_j} \mu(t_1) \cdots \mu(t_k) \frac{s}{(t_1 \cdots t_k)^s}.
\]

Let us assume that \( s \) lies on a circle with a centre \( s = 1 \), so we can write \( s = 1 + \rho e(\theta) \) with \( \theta \in [0, 1) \) and \( \rho \in (0, 1) \). Then it is easy to see that

\[
f_k(q, q/\delta, s) \ll \frac{\delta^\sigma \varphi(\delta)^\sigma}{\varphi(q)^\sigma} 2^{k\omega(q)} q^e \frac{\delta^\sigma}{\varphi(q)^\sigma}.
\]

Here \( \omega(n) \) is the number of distinct prime factors of \( n \) and we used the well-known fact that \( \omega(n) \ll \frac{\log n}{\log \log n} \) as \( n \to \infty \). Thus we have

\[
\Phi_k(q, s) \ll q^e \sum_{\delta|q} |\mu(\delta)| \frac{\delta^\sigma \varphi(q)^\sigma}{\varphi(q)^\sigma} = q^{-\sigma + e} \prod_{p|q} \left( 1 + \frac{p^\sigma}{p-1} \right)
\]

\[
\ll q^{-\sigma + e} \prod_{p|q} \left( 1 + \frac{p^\sigma}{p} \right).
\]

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On the other hand, when \( \sigma \in (0, 2) \), we have

\[
q^{-\sigma} \prod_{p \mid q} \left(1 + \frac{p^{\sigma}}{p}\right) \ll \begin{cases} q^{-\sigma+\varepsilon} \\ q^{-\sigma+\varepsilon} \prod_{p \mid q} p^{-1+\sigma} \ll q^{-1+\varepsilon} \end{cases} \quad \sigma \in (0, 1); \quad \sigma \in (1, 2).
\]

Therefore for \( \sigma = \Re(s) \), \( 0 < \sigma < 2 \), we get

\[
\Phi_k(q, s) \ll q^{-\min(\sigma, 1)+\varepsilon}.
\]

It is obvious that \( \Phi_k(q, s) \) is analytic for every \( s \in \mathbb{C} \), and for every parameter \( q \) which we consider. Hence one can use Cauchy’s integral formula:

\[
\frac{d^r \Phi_k(q, s)}{ds^r} \bigg|_{s=1} = \frac{r!}{2\pi i} \int_{|\xi-1|=\rho} \frac{\Phi_k(q, \xi)}{(\xi-1)^{r+1}} d\xi \ll \frac{r!}{\rho^r} \max_{\theta \in [0, 1)} |\Phi_k(q, 1 + \rho e(\theta))|,
\]

where \( \rho \in (0, 1) \). Using (3.2) and choosing \( \rho = \varepsilon \), we obtain

\[
\frac{d^r \Phi_k(q, 1)}{ds^r} \ll \frac{r!}{\rho^r} q^{-(1-\rho)+\varepsilon} \ll_{k, \varepsilon} q^{-1+\varepsilon},
\]

as \( q \to \infty \), which completes the proof of the lemma. \( \square \)

Now we can deal with the representation of the sum \( T(\alpha, Y) \).

**Proof of Lemma 2.5** First of all, we pick \( Y \asymp X^2 \). Recall that by Lemma 3.1 for \( q \leq X^{2/(k+1)} \) and \( \beta = \alpha - a/q \) we have

\[
J_k(\alpha, Y) = \sum_{h=1}^{q} e \left( \frac{ah}{q} \right) \sum_{m \equiv h \pmod{q}} \tau_k(m) e(m\beta)
\]

\[
= \sum_{h=1}^{q} e \left( \frac{ah}{q} \right) \int_0^Y e(u\beta) \left( M_k(u; h, q) + O_k(u^{1 - \frac{2}{k+1}+\varepsilon}) \right) du
\]

\[
= \sum_{h=1}^{q} e \left( \frac{ah}{q} \right) \int_0^Y e(u\beta) M'(u; h, q) du + O_k \left( q(1 + |\beta|Y)^{1 - \frac{2}{k+1}+\varepsilon} \right).
\]

Here we also used a summation formula described for example in [9, Lemma 3.7]. It is clear that

\[
\sum_{h=1}^{q} e \left( \frac{ah}{q} \right) M'(u; h, q) = \sum_{h=1}^{q} e \left( \frac{ah}{q} \right) \Res_{s=1} \left( \zeta(s)^k u^{s-1} f_k(q, \delta, s) \right),
\]

\( \square \) Springer
where $\delta = \gcd(q, h)$. This means that

$$T(\alpha, Y) = \int_0^Y e(u\beta) \text{Res}_{s=1} \left( \zeta(s)^k \Phi_k(q, s)u^{s-1} \right) \, du + O\left(q(1 + |\beta|Y)^{1-\frac{2}{r+1}}\right).$$

(3.3)

We now compute $\text{Res}_{s=1} \left( \zeta(s)^k \Phi_k(q, s)u^{s-1} \right)$. The Riemann zeta function has a Laurent series about $s = 1$,

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n \gamma_n}{n!} (s-1)^n,$$

where

$$\gamma_n = \lim_{M \to \infty} \left( \sum_{d=1}^M \frac{\log^n d}{d} - \frac{\log^{n+1} M}{n+1} \right), \quad n \in \mathbb{Z}_{\geq 0}$$

are the Stieltjes constants. Therefore there exist constants

$$\alpha_{k,j} = \text{Res}_{s=1} \left( (s-1)^{j-1} \zeta(s)^k \right), \; j = 1, 2, \ldots, k,$$

and a holomorphic function $h_k(s)$ on $\mathbb{C}$ such that

$$\zeta(s)^k = \sum_{r=1}^{k} \frac{\alpha_{k,r}}{(s-1)^r} + h_k(s).$$

Thus we obtain that

$$\zeta(s)^k u^{s-1} = \sum_{r=1}^{k} \frac{1}{(s-1)^r} \sum_{r_1=0}^{k-r} \alpha_{k,r_1+r} \log^{r_1} u \frac{1}{r_1!} + g_{k,u}(s),$$

for any $u > 0$, where $g_{k,u}(s)$ is a holomorphic function on $\mathbb{C}$ about $s$. The Taylor series for $\Phi_k(q, s)$ at $s = 1$ is

$$\Phi_k(q, s) = \sum_{d=0}^{\infty} \frac{\Phi_k^{(d)}(q, 1)}{d!} (s-1)^d.$$

Therefore the residue of $\zeta(s)^k u^{s-1} \Phi_k(q, s)$ at $s = 1$ is

$$\sum_{d,r \in \mathbb{Z}_+, 1 \leq r \leq k} \frac{\Phi_k^{(d)}(q, 1)}{d!} \sum_{r_1=0}^{k-r} \alpha_{k,r_1+r} \log^{r_1} x \frac{1}{r_1!} = \sum_{r=1}^{k} \frac{\log^{r-1} x}{(r-1)!} \sum_{t=0}^{k-r} \Phi_k^{(t)}(q, 1) \frac{\alpha_{k,r+t}}{t!}. $$
Thus if we define
\[
\beta_{k,r}(q) = \frac{1}{r!} \sum_{t=0}^{k-r-1} \frac{1}{t!} \text{Res}_{s=1} \left( (s-1)^{r+t} \zeta(s)^k \right) \left( \frac{d^r \Phi_k(q,s)}{ds^t} \right)_{s=1}
\]
by Lemma 3.2 we obtain \( \beta_{k,r}(q) \ll q^{-1+\varepsilon} \). Furthermore, the error term in (3.3) is
\[
q(1 + |\beta|Y)^{1-\frac{2}{k+1}+\varepsilon} \ll q(1 + L/q)X^{2-\frac{4}{k+1}+\varepsilon} < LX^{2-\frac{4}{k+1}+\varepsilon}
\]
for \( q \ll L = o \left( X^{\min(1, \frac{4}{k+1})} \right) \), which completes the proof of Lemma 2.5.

4 The singular integral and series

4.1 The singular integral

In this subsection we deal with the singular integral and give a proof of Lemma 2.6. We first prove the following lemmas.

Lemma 4.1 Let \( \beta \in \mathbb{R} \setminus \{0\} \) and \( Y \geq 2 \). We have
\[
\int_0^Y e(-u\beta) (\log u)^r du \ll |\beta|^{-1+\varepsilon}Y^\varepsilon.
\]

Proof We have
\[
\int_0^Y e(-u\beta) (\log u)^r du \ll |\beta|^{-1} \int_0^{Y|\beta|} e(-u\beta/|\beta|) (\log(u/|\beta|))^r du
\]
\[
\ll |\beta|^{-1} \sum_{\ell=0}^r |\log |\beta||^{r-\ell} \int_0^{Y|\beta|} (\log u)^\ell e(-u\beta/|\beta|) du
\]
\[
\ll |\beta|^{-1} \left( Y^\varepsilon |\beta|^\varepsilon + 1 + \sum_{\ell=1}^r Y^{\ell|\beta|} \left| \frac{\log u^{\ell-1}}{u} \right| \right)
\]
\[
\ll |\beta|^{-1+\varepsilon}Y^\varepsilon.
\]
This completes the proof.

Lemma 4.2 Let \( F(t) \) be defined as in (1.1) and \( X \geq 2 \). If \( \beta \in \mathbb{R} \) and \( |\beta| \geq X^{-2} \) then
\[
I_{F,B}(\beta, X) := \int_{X B} e(F(t)\beta) \, dt \ll |\beta|^{-n/2+\varepsilon}.
\]

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Proof First, we notice that from the fact that $Q$ is nonsingular it follows that there exists a transformation, such that

$$\int_{XB} e\left(F(t)\beta\right) \, dt = \int_{XB} e\left((t^T Qt + Lt + N) \beta\right) \, dt$$

$$\ll \left| \int_{XB} e\left((t^T Qt + Lt) \beta\right) \, dt \right| \ll \left| \int_{XB + bF} e\left(y^T Qy\beta\right) \, dy \right|$$

for some $bF \in \mathbb{R}^n$. Here $XB + bF$ is still a box, i.e. a factor of intervals, and we can consider that $B + bF/X$ has a maximal side length smaller than 1. According to [1, Lemma 5.2] of Birch, for a quadratic nonsingular form $G$ and a box $C$ with a maximal side length smaller than 1, we have

$$I_{G,C}(\beta, 1) \ll |\beta|^{-n/2+\varepsilon},$$

where the dependence in this version is uniform on the side length of the box $C$. Indeed, we apply [1, Lemma 5.2] with $K = n/2$, $R = 1$, $d = 2$, after we have noticed that the condition (iii) from [1, Lemma 3.2] is not fulfilled for $k = (K - \varepsilon)\Theta$. Thus [1, Lemma 4.3] holds in our case too, and therefore Lemma 5.2 of Birch applies for our form $Q$. We point out this, since a direct look of the main theorem of Birch implies $n \geq 5$, which is, however, superfluous for [1, Lemma 5.2]. Therefore we have

$$\int_{XB + bF} e\left(y^T Qy\beta\right) \, dy = I_{Q,B+bF/X}(\beta, X) = X^{-n}I_{Q,B+bF/X}(\beta X^{-2}, 1)$$

$$\ll |\beta|^{-n/2+\varepsilon} X^{-2\varepsilon} \ll |\beta|^{-n/2+\varepsilon}.$$

This completes the proof of the lemma. \qed

Proof of Lemma 2.6 Using Lemmas 4.1 and 4.2, we obtain that

$$I_{r,F}(\beta, X) := \int_{XB} dt \int_0^{C_{r,B}(X)} e\left((F(t) - u)\beta\right) (\log u)^r \, du \ll_F |\beta|^{-1-n/2+\varepsilon} X^\varepsilon.$$

This implies that

$$\int_{|\beta| \leq L/qX^2} I_{r,F}(\beta, X) \, d\beta = \int_{\mathbb{R}} I_{r,F}(\beta, X) \, d\beta + O\left(X^\varepsilon (L/qX^2)^{-2}\right). \quad (4.1)$$

Moreover,

$$\int_{\mathbb{R}} I_{r,F}(\beta, X) \, d\beta = \int_{\mathbb{R}} d\beta \int_{XB} \int_0^{C_{r,B}(X)} e\left((F(t) - u)\beta\right) (\log u)^r \, du$$

$$= 2 \int_{\mathbb{R}^+} d\beta \int_0^{C_{r,B}(X)} (\log u)^r \, du \int_{XB} \cos \left[2\pi (u - F(t))\beta\right] \, dt$$
\[ = \frac{1}{\pi} \int_{X} B \, d\beta \int_{\mathbb{R}^+} \frac{C_{F,B}(X)}{\beta} \, d\left( \frac{\sin \left[ 2\pi (u - F(t)) \beta \right]}{\beta} \right) \]

\[ = \frac{1}{\pi} \int_{X} B \, d\beta \int_{0}^{\frac{C_{F,B}(X)}{\beta}} (\log u)^{r} \, d\left( \frac{\sin \left[ 2\pi (u - F(t)) \beta \right]}{\beta} \right) d\beta \]

\[ = \frac{1}{\pi} \int_{X} B \, d\beta \int_{0}^{\frac{C_{F,B}(X)}{\beta}} (\log u)^{r} \, d\left( \frac{\pi}{2} \text{sgn}(u - F(t)) \right), \]

where we have used the fact: \( \int_{0}^{\infty} \frac{\sin(\alpha x)}{x} \, dx = \frac{\pi}{2} \text{sgn}(\alpha) \) and

\[ \text{sgn}(\alpha) := \begin{cases} \frac{\alpha}{|\alpha|} & \alpha \neq 0, \\ 0 & \alpha = 0. \end{cases} \]

By integration by parts we have

\[
\int_{\mathbb{R}} I_{r,F}(\beta, X) \, d\beta = \frac{1}{2} \int_{X} B \, \int_{0}^{\frac{C_{F,B}(X)}{\beta}} (\log u)^{r} \, d(\text{sgn}(u - F(t)))
\]

\[ = \lim_{\epsilon \to 0^+} \int_{X} B \, \left. \left( \frac{dt}{2} \int_{[u - F(t)] \leq \epsilon} (\log u)^{r} \, d(\text{sgn}(u - F(t))) \right) \right|^{F(t) + \epsilon}_{F(t) - \epsilon}
\]

\[ - \int_{[u - F(t)] \leq \epsilon} \, d(\log u)^{r} \]

\[ = \int_{X} B \, \left( \frac{1}{2} \left( 2(\log F(t))^r \, dt + \lim_{\epsilon \to 0^+} O(\epsilon \log^r X) \right) \right) dt
\]

\[ = \int_{X} B \, (\log F(t))^r \, dt. \]

Using (4.1) we get the proof of Lemma 2.6. \( \square \)

### 4.2 The singular series

In this subsection we deal with the singular series, i.e. with the series \( L(s; k, F) \) defined in (2.6), and their presentation stated in Theorem 1.1.
First of all, note that from Lemma 2.4 it follows that $S_F(q) \ll q^{1-n/2+\varepsilon}$ and Lemma 3.2 gives $\frac{d}{ds^r} \Phi_k(q, 1) \ll q^{-1+\varepsilon}$ for any integer $r \in [0, k-1]$. Hence, for any $t = 0, \ldots, k-1$,

$$\left. \frac{d^t L(s; k, F)}{ds^t} \right|_{s=1} = \sum_{q=1}^{\infty} \frac{d^t \Phi_k(q, 1)}{ds^t} S_F(q) \ll \sum_{q=1}^{\infty} q^{-n/2+\varepsilon} \ll 1,$$

as $n \geq 3$. By their definition in Theorem 1.1 this ensures that $C_{k,r}(F), r = 0, \ldots, k-1$, are convergent and indeed well-defined constants.

It is easily seen that $S_F(q)$ defined in (2.5) is real and multiplicative. On the other hand, Lemma 3.2 showed that $\Phi_k(q, s)$ is also multiplicative. Therefore $L(s; k, F) = \sum_{q=1}^{\infty} \Phi_k(q, s) S_F(q)$ has an Euler product representation as follows:

$$L(s; k, F) = \prod_p L_p(s; k, F)$$

with

$$L_p(s; k, F) = 1 + \sum_{m \geq 1} S_F(p^m) \Phi_k(p^m, s)$$

By orthogonality of characters in $\mathbb{Z}/p^m\mathbb{Z}$ for integer $m \geq 1$ it easily follows that

$$Q_F(p^m) = p^{-nm} \sum_{1 \leq a \leq p^m} S_F(p^m, a).$$

Then we have

$$S_F(p^m) = Q_F(p^m) - Q_F(p^{m-1}). \quad (4.2)$$

By the estimate from Lemma 2.4 we get $S_F(p^m) \ll_F (p^m)^{1-n/2+\varepsilon}$ and after telescoping summation of (4.2) we obtain

$$Q_F(p^\ell) - 1 \ll_F \sum_{m=1}^{\ell} (p^m)^{1-n/2+\varepsilon} \ll p^{1-n/2+\varepsilon},$$

where we again used that $n \geq 3$. Then by partial summation, using (4.2) and the estimate (3.2), we have

$$L_p(s; k, F) = \sum_{\ell \geq 0} Q_F(p^\ell) \left( \Phi_k(p^\ell, s) - \Phi_k(p^{\ell+1}, s) \right),$$

where we set $Q_F(1) = \Phi_k(1, s) = 1$.

From (3.1) and the definition of $f_k(q, \delta, s)$ in Lemma 3.1, we see that

\[ \text{Springer} \]
\[
\Phi_k(p^m, s) = f_k(p^m, p^m, s) - f_k(p^m, p^{m-1}, s)
\]

\[
= \frac{1}{p^{ms}} \sum_{d_1d_2...d_{k-1}=p^m} \sum_{t_1 \mid (\prod_{i=1}^{k-1} d_i)} \frac{\mu(t_1) \ldots \mu(t_k)}{(t_1 \ldots t_k)^s} - \frac{(1 - p^{-s})^k}{\varphi(p) p^{(m-1)s}} \tau_k(p^{m-1})
\]

For the first expression above, denote

\[
I_k = \sum_{d_1d_2...d_{k-1}=p^m} \sum_{t_1 \mid (\prod_{i=1}^{k-1} d_i)} \frac{\mu(t_1) \ldots \mu(t_k)}{(t_1 \ldots t_k)^s}.
\]

Then for \(m \geq 1\) and \(k = 2\) we have

\[
I_2 = 1 + m(1 - p^{-s}).
\]

Now using the identities \(\tau_k(p^m) = \sum_{v=0}^{m} \tau_{k-1}(p^{m-v})\), from which it also follows that

\[
\tau_k(p^m) - \tau_{k-1}(p^m) = \tau_k(p^{m-1}), \quad (4.3)
\]

we see that

\[
I_k = \sum_{v=0}^{m} \sum_{d_1d_2...d_{k-1}=p^m} \sum_{t_1 \mid (\prod_{i=1}^{k-1} d_i)} \frac{\mu(t_1) \ldots \mu(t_k)}{(t_1 \ldots t_k)^s} + \sum_{v=1}^{m} \sum_{d_1d_2...d_{k-1}=p^m} \left(1 - \frac{1}{p^s}\right)^{k-1} \tau_{k-1}(p^{m-v})
\]

\[
= I_{k-1} + (1 - p^{-s})^{k-1} \sum_{v=1}^{m} \tau_{k-1}(p^{m-v})
\]

\[
= I_{k-1} + (1 - p^{-s})^{k-1} (\tau_k(p^m) - \tau_{k-1}(p^m))
\]

\[
= 1 + m(1 - p^{-s}) + \sum_{v=3}^{k} (1 - p^{-s})^{v-1} \tau_{v}(p^{m-1})
\]

\[
= \sum_{v=1}^{k} (1 - p^{-s})^{v-1} \tau_{v}(p^{m-1}).
\]
Hence

\[
\Phi_k(p^m, s) = p^{-ms} \left( \sum_{1 \leq v \leq k} (1 - p^{-sv})^{v-1} \tau_v(p^{m-1}) - \tau_k(p^{m-1}) \frac{p^s(1 - p^{-sv})^k}{p - 1} \right).
\]

(4.4)

We now aim to find the value of \( \Phi_k(p^m, s) - \Phi_k(p^{m+1}, s) \) for each nonnegative integer \( m \). When \( m = 0 \) we have

\[
\Phi_k(1, s) - \Phi_k(1, s) = 1 - p^{-s} \left( \sum_{v=1}^{k} (1 - p^{-sv})^{v-1} \tau_v(1) - \tau_k(1) \frac{p^s(1 - p^{-sv})^k}{p - 1} \right) \\
= 1 - p^{-s} \left( \frac{1 - (1 - p^{-sv})^k}{1 - (1 - p^{-sv})} - \frac{p^s(1 - p^{-sv})^k}{p - 1} \right) \\
= (1 - p^{-s})^k \frac{p}{p - 1} = (1 - p^{-1})^{-1}(1 - p^{-s})^k.
\]

If \( f(z) \) is a formal power series, we denote by \( [z^n]f(z) \) the coefficient of \( z^n \) in \( f(z) \). Then for any \( |z| < 1 \) and \( m, v \in \mathbb{Z}_+ \) we have

\[
\tau_v(p^{m-1}) = [z^{m-1}](1 - z)^{-v}.
\]

Since the symbol \( [z^n]f(z) \) has a distributive property, we have

\[
\phi_k(p^m, s) := \frac{1}{p^{ms}} \sum_{v=1}^{k} (1 - p^{-sv})^{v-1} \tau_v(p^{m-1}) \\
= [z^{m-1}] \left( \frac{1}{p^{ms}} \sum_{v=1}^{k} \frac{(1 - p^{-sv})^{v-1}}{(1 - z)^v} \right) \\
= [z^{m-1}] \left( \frac{1}{p^{(m-1)s}} \frac{1}{1 - p^s z} \left( 1 - \frac{(1 - p^{-sv})^k}{(1 - z)^k} \right) \right) \\
= 1 - p^{(1-m)s} (1 - p^{-s})^k [z^{m-1}](1 - p^s z)^{-1}(1 - z)^{-k} \\
= 1 - (1 - p^{-s})^k \sum_{0 \leq \ell < m-1} p^{-s\ell} [z^{\ell}](1 - z)^{-k} \\
= 1 - (1 - p^{-s})^k \left( (1 - p^{-s})^{-k} - \sum_{\ell \geq m} p^{-s\ell} [z^{\ell}](1 - z)^{-k} \right) \\
= (1 - p^{-s})^k \sum_{\ell \geq m} p^{-s\ell} [z^{\ell}](1 - z)^{-k} = (1 - p^{-s})^k \sum_{\ell \geq m} p^{-s\ell} \tau_k(p^\ell).
\]

and then for \( m \geq 1 \) we get

\[
\phi_k(p^m, s) - \phi_k(p^{m+1}, s) = (1 - p^{-s})^k p^{-ms} \tau_k(p^m).
\]
From (4.4) it follows that when \( m \geq 1 \) we have

\[
\Phi_k\left(p^m, s\right) - \Phi_k\left(p^{m+1}, s\right) = \left(\phi_k\left(p^m, s\right) - \frac{1 - p^{-s}}{p^{s-1}(p - 1)} \tau_k\left(p^m\right)\right)
- \left(\phi_k\left(p^{m+1}, s\right) - \frac{1 - p^{-s}}{p^{s(m+1)}(p - 1)} \tau_k\left(p^m\right)\right)
= \left(1 - p^{-s}\right) p^{-ms} \left(\tau_k\left(p^m\right) - \frac{p^s}{p - 1} \left(\tau_k\left(p^{m-1}\right) - \frac{\tau_k\left(p^m\right)}{p^s}\right)\right)
= \left(1 - p^{-s}\right) \frac{p^{-ms}}{1 - p^{-1}} \left(\tau_k\left(p^m\right) - p^{-s-1} \tau_k\left(p^{m-1}\right)\right).
\]

Let \( \sigma := \Re(s) > 0 \). Then according to (3.2) we have \( \Phi_k\left(p^\ell, s\right) \to 0 \), as \( \ell \to 0 \) and \( s \) is fixed. Then after appropriate telescoping summation we can write

\[
L_p\left(s; k, F\right) = 1 + \sum_{\ell \geq 0} \left(Q_F\left(p^\ell\right) - 1\right) \left(\Phi_k\left(p^\ell, s\right) - \Phi_k\left(p^{\ell+1}, s\right)\right)
= 1 + \sum_{\ell \geq 1} O\left(p^{1-n/2+\varepsilon} p^{-\ell} \sigma\left(\tau_k\left(p^\ell\right) + p^{\sigma-1} \tau_k\left(p^{\ell-1}\right)\right)\right).
\]

Let us further assume that \( \sigma > 1/2 \), so that we obtain

\[
L_p\left(s; k, F\right) = 1 + O\left(p^{1-n/2+\varepsilon-\sigma} (1 + p^{\sigma-1})\right) = 1 + O\left(p^{-n/2+\varepsilon} + p^{1-n/2-\sigma+\varepsilon}\right).
\]

Therefore if \( \sigma > \max(1/2, 2-n/2) = 1/2 \), and setting \( \tau_k\left(p^{-1}\right) := 0 \), we have that the Euler product

\[
L(s; k, F) = \prod_p \left(\sum_{\ell \geq 0} Q_F\left(p^\ell\right) \left(\frac{\tau_k\left(p^\ell\right) - p^{-s-1} \tau_k\left(p^{\ell-1}\right)}{p^{s\ell}}\right)\right) \left(\frac{1 - p^{-s}}{1 - p^{-1}}\right)
\]

is absolutely convergent. In particular, by (4.3) we have

\[
L(1; k, F) = \prod_p \left(\sum_{\ell \geq 0} Q_F\left(p^\ell\right) \frac{\tau_{k-1}(p^\ell)}{p^\ell}\right) \left(1 - \frac{1}{p}\right)^{k-1} > 0.
\]

Now from

\[
C_{k,k-1} = \frac{1}{(k - 1)!} L(1; k, F) \text{Res}_{\substack{s=1}} \left((s - 1)^{k-1} \zeta(s)^k\right) = \frac{L(1; k, F)}{(k - 1)!}
\]

we conclude that \( C_{k,k-1} > 0 \), which finalizes the proof of Theorem 1.1.
5 Final remarks

We believe that the application of the circle method in estimating divisor sums over values of quadratic polynomials can be extended also to the sum

$$\Sigma_{k,F}(X;B) := \sum_{x \in X \cap \mathbb{Z}} \tau_k^F(F(x)) .$$

The treatment of the sum $S(\alpha)$ remains the same, and one could use a level of distribution result for the function $\tau_k^F(m)$ given by Rieger (Satz 3 [14]). In this case a separate treatment for $q$ in the middle range $(\log x)^{\lambda} \leq q \leq L \ll X$ might also be required.

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