Comments on “Perspectives on Galactic Dynamics via General Relativity”

D. Vogt*
Instituto de Física Gleb Wataghin, Universidade Estadual de Campinas
13083-970 Campinas, S. P., Brazil

P. S. Letelier†
Departamento de Matemática Aplicada-IMECC, Universidade Estadual
de Campinas 13083-970 Campinas, S. P., Brazil

July 18, 2018

Abstract

In this comment we question some arguments presented in astro-ph/0512048 to refuse the presence of a singular mass surface layer. In particular, incorrect expressions are used for the disk’s surface mass density. We also point out that the procedure of removing the discontinuity on the $z = 0$ plane with a region of continuous density gradient generates other two regions of discontinuities with singular mass surface layers making the model unrealistic.

1 Introduction

In [1] the authors make some comments about their previous work [2] and also reply to certain issues that have been raised by some authors [3, 4, 5] concerning the nature of the matter distribution and the asymptotic behaviour of the spacetime in their model. In particular, [3] pointed out that the use of a $|z|$ function in the solution of the field equations introduces an additional disk on the $z = 0$ plane and later [4] showed that the disk was made of exotic matter. Consequently, [5] examine the question in several ways and present some arguments to dismiss the existence of a physical mass

---

*e-mail: danielvt@ifi.unicamp.br
†e-mail: letelier@ime.unicamp.br
layer present on the $z = 0$ plane. We would like to notice some inconsistencies in their paper. The first is related to the expressions used for the surface density of the singular disk, and is presented in Sec. 2. In Sec. 3 we question the smoothing procedure discussed in Section 4 of [1]. We find that one cannot eliminate disk singularities, moreover one introduces new singularities that make the model unrealistic.

2 The disk’s surface mass density

In [4] we calculate the distributional energy-momentum tensor $Q^a_b$ due to the introduction of an absolute value of $z$ in the metric functions. The resulting non-zero components read [4]

\begin{align}
Q^t_t &= \frac{1}{e^\nu} \left( \frac{NN_z}{r^2} - \nu z \right), \\
Q^\phi_\phi &= -\frac{N_z}{r^2 e^\nu}, \\
Q^t_\phi &= \frac{N_z}{r^2} e^\nu, \\
Q^\phi_t &= -\frac{1}{e^\nu} \left( \frac{NN_z}{r^2} + \nu z \right),
\end{align}

where all quantities are evaluated on $z = 0$. Since $Q^a_b$ is non-diagonal, in order to have obtain the physical variables of the disk, we need to put the energy-momentum tensor in its canonical form. To do that we solve the eigenvalue problem: $Q^a_b \xi^b = \lambda \xi^a$. We thus find

\begin{align}
\lambda_{\pm} &= \frac{T}{2} \pm \frac{\sqrt{D}}{2}, \quad \text{where} \\
T &= Q^t_t + Q^\phi_\phi, \quad D = (Q^t_t - Q^\phi_\phi)^2 + 4Q^t_\phi Q^\phi_t, \quad (3)
\end{align}

and using Eq. (1a)–(1d) result in

\begin{align}
T &= -\frac{2\nu z}{e^\nu}, \quad D = -\frac{4N_z^2}{r^2 e^{2\nu}}. \quad (4)
\end{align}

As the discriminant is always negative, the eigenvalues and corresponding eigenfunctions are complex conjugate. If $V_a$ and $W_a$ denote the timelike and spacelike real eigenvectors, the canonical form of the energy-momentum tensor is

\begin{align}
Q_{ab} = \sigma V_a V_b + p_\phi W_a W_b + \kappa (V_a W_b + W_a V_b), \quad (5)
\end{align}
where $\sigma = T/2$ is the surface density, $p_\phi = -T/2$ denote the azimuthal stresses and $\kappa = \sqrt{-D/2}$ is the heat flow in the azimuthal direction [6]. Thus the expression for the surface density, to order $G^1$, is given by $\sigma = -\nu_z$, or, using the relation $\nu_z = -N_r N_z / r$ we get

$$\sigma = \frac{N_r N_z}{r}.$$  \hspace{1cm} (6)

On the other hand, [1] take following expressions for the surface energy density (Equations (15) and (16) of the paper)

$$\sigma = \frac{NN_z}{r^2} - \nu_z = \frac{NN_z}{r^2} + \frac{N_r N_z}{r},$$  \hspace{1cm} (7)

which they integrate over the surface and compare with the volume integral of their continuous mass density distributions. But Eq. (7) is only the $Q_{tt}$ component Eq. (1a) to order $G^1$. Due to the non-diagonal form of the energy-momentum tensor, the $Q_{tt}$ component solely does not determine the surface density, but there is also a contribution from the $Q_{\phi\phi}$ component Eq. (1d) even to order $G^1$. Thus, the correct expression for $\sigma$ that should be used is Eq. (6). The integration of this equation over the surface would result in half of the value for the mass derived from the volume integral of the continuous mass distribution, since the authors themselves comment on footnote 7 that the two terms in Eq. (7) contribute equally.

It is important to stress that the physical variables of the mass layer on the $z = 0$ plane are not directly given by the principal diagonal terms of the distributional energy-momentum tensor, since the non-diagonal terms are non-negligible. This is further an example of how the non-linearity of General Relativity can manifest even at Newtonian level.

### 3 The smoothing procedure

Another approach used in [1] to examine the presence of a singular mass layer on the symmetry plane was to smooth the solution over an interval that includes the $z = 0$ plane. This was achieved by the choice of $\cosh(k_n z)$ functions to span the symmetry plane in the interval $-z_0 < z < z_0$. For $|z| \geq z_0$, the original solution with exponentials was used. This requires that the functions $N$ and also $N_{,z}$ match at $|z| = z_0$. If this last condition is not satisfied new matter is added. We note that this is also true for the Newtonian gravitational potential, discontinuity of the first derivatives “add matter” whose density can be computed via Poisson’s equation. We argue
that the above mentioned matching cannot be done without adding new matter.

Let us take the function $N$ as follows:

\[
N_1 = -\sum_n C_1 n k_n e^{k_n z} r J_1(k_n r), \quad z \leq -z_0, \quad (8a)
\]

\[
N_2 = -\sum_n C_2 n k_n \cosh(k_n z) r J_1(k_n r), \quad -z_0 < z < z_0, \quad (8b)
\]

\[
N_3 = -\sum_n C_3 n k_n e^{-k_n z} r J_1(k_n r), \quad z \geq z_0, \quad (8c)
\]

and impose the conditions $N_1(-z_0) = N_2(-z_0)$ and $N_{1,z}(-z_0) = N_{2,z}(-z_0)$. Using Eq. (8a)–(8d) we obtain

\[
\sum_n k_n r J_1(k_n r) \left[ C_2 n \cosh(k_n z_0) - C_1 n e^{-k_n z_0} \right] = 0, \quad (9)
\]

\[
\sum_n k_n^2 r J_1(k_n r) \left[ -C_2 n \sinh(k_n z_0) - C_1 n e^{-k_n z_0} \right] = 0. \quad (10)
\]

Since the Bessel functions are linearly independent, the terms in brackets must vanish identically. Subtracting Eq. (9) from Eq. (10) results in $C_2 n e^{k_n z_0} = 0$, which is only satisfied in the real domain if all $C_2 n = 0$. Thus it is not possible to match $N$ and $N_{1,z}$ simultaneously. If we demand continuity of $N_1$, $N_2$ and $N_3$ at $|z| = z_0$ we obtain the following conditions:

\[
C_1 n = C_3 n, \quad \text{and} \quad C_2 n = C_1 n \frac{e^{-k_n z_0}}{\cosh(k_n z_0)}. \quad (11)
\]
On the other hand, demanding continuity of $N_{1,z}$, $N_{2,z}$ and $N_{3,z}$ at $|z| = z_0$ results in

$$C_{1n} = C_{3n}, \quad \text{and} \quad C_{2n} = -C_{1n} \frac{e^{-k_n z_0}}{\sinh(k_n z_0)}.$$  \hfill (12)

Fig. 1(a) sketches what happens if we impose continuity of the functions and Fig. 1(b) if we impose continuity of the derivatives. The only way the functions Eq. (8a)–(8c) and their derivatives could be matched is if an extra set of constants were inserted into Eq. (8b)

$$N_2 = - \sum_n k_n (C_{2n} \cosh(k_n z) + C_{2n}^\text{extra}) r J_1(k_n r),$$  \hfill (13)

but then this would not be a solution of

$$N_{rr} + N_{zz} - \frac{N_r}{r} = 0.$$  \hfill (14)

Assuming conditions Eq. (11) hold, the jump of the derivatives of Eq. (8a)–(8c) with respect to $z$ evaluated at $|z| = z_0$ are given by

$$N_{2,z}(-z_0) - N_{1,z}(-z_0) = N_{3,z}(z_0) - N_{2,z}(z_0) = \sum_n C_{1n} k_n^2 r J_1(k_n r) \cosh(k_n z_0).$$  \hfill (15)

The same kind of discontinuities in the derivatives also appear when a $|z|$ is introduced in the solution. Thus they introduce additional layers of matter now located on the $z = \pm z_0$ planes that makes the solution unrealistic.

D. Vogt thanks CAPES for financial support. P. S. Letelier thanks CNPq and FAPESP for financial support.

**References**

[1] F. I. Cooperstock and S. Tieu, preprint: astro-ph/0512048

[2] F. I. Cooperstock and S. Tieu, preprint: astro-ph/0507619

[3] M. Korzyński, preprint: astro-ph/0508377

[4] D. Vogt and P. S. Letelier, preprint: astro-ph/0510750

[5] D. Garfinkle, preprint: gr-qc/0511082

[6] C. Møller, *The Theory of Relativity*, Oxford University Press, 1972.