Pricing with Variance Gamma Information

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In the information-based pricing framework of Brody, Hughston & Macrina, the market filtration \(\{\mathcal{F}_t\}_{t \geq 0}\) is generated by an information process \(\{\xi_t\}_{t \geq 0}\) defined in such a way that at some fixed time \(T\) an \(\mathcal{F}_T\)-measurable random variable \(X_T\) is “revealed”. A cash flow \(H_T\) is taken to depend on the market factor \(X_T\), and one considers the valuation of a financial asset that delivers \(H_T\) at \(T\). The value \(S_t\) of the asset at any time \(t \in [0, T)\) is the discounted conditional expectation of \(H_T\) with respect to \(\mathcal{F}_t\), where the expectation is under the risk neutral measure and the interest rate is constant. Then \(S_T = H_T\), and \(S_t = 0\) for \(t \geq T\). In the general situation one has a countable number of cash flows, and each cash flow can depend on a vector of market factors, each associated with an information process. In the present work, we construct a new class of models for the market filtration based on the variance-gamma process. The information process is obtained by subordinating a particular type of Brownian random bridge with a gamma process. The filtration is taken to be generated by the information process together with the gamma bridge associated with the gamma subordinator. We show that the resulting extended information process has the Markov property and hence can be used to price a variety of different financial assets, several examples of which are discussed in detail.

Key words: Information-based asset pricing, Lévy processes, gamma processes, variance gamma processes, Brownian bridges, gamma bridges, nonlinear filtering.

I. INTRODUCTION

The theory of information-based asset pricing put forward by Brody, Hughston & Macrina [3, 4, 17] is concerned with the determination of the price processes of financial assets from first principles. The simplest version of the model is as follows. We fix a probability space \((\Omega, \mathcal{F}, \mathbb{P})\). An asset delivers a single random cash flow \(H_T\) at some specified time \(T > 0\), where time \(0\) denotes the present. The cash flow is a function of a random variable \(X_T\), which we can think of as a “market factor” that is in some sense revealed at time \(T\). In the general situation there will be many factors and many cash flows, but for the present we assume that there is a single factor \(X_T : \Omega \rightarrow \mathbb{R}\) such that the sole cash flow at time \(T\) is given by \(H_T = h(X_T)\) for some Borel function \(h : \mathbb{R} \rightarrow \mathbb{R}^+\). For simplicity we assume that interest rates are constant and that \(\mathbb{P}\) is the risk neutral measure. We require that \(H_T\) should be integrable. Under these assumptions, the value of the asset at time \(0\) is given by

\[
S_0 = e^{-rT} \mathbb{E}[h(X_T)],
\]

where \(\mathbb{E}\) denotes expectation under \(\mathbb{P}\) and \(r\) is the short rate. Since the single “dividend” is paid at time \(T\), the value of the asset at any time \(t \geq 0\) is of the form

\[
S_t = e^{-r(T-t)} \mathbb{I}_{\{t < T\}} \mathbb{E} \left[ h(X_T) \mid \mathcal{F}_t \right],
\]

where \(\mathbb{I}_{\{t < T\}}\) is the indicator function of the event \(\{t < T\}\).
where $\{\mathcal{F}_t\}_{t \geq 0}$ is the market filtration. The task now is to model the filtration, and this will be done explicitly. The idea is that the filtration should contain partial or “noisy” information about the market factor $X_T$, and hence also the impending cash flow, in such a way that $X_T$ is $\mathcal{F}_T$-measurable. This can be achieved by allowing $\{\mathcal{F}_t\}$ to be generated by a so-called information process $\{\xi_t\}_{t \geq 0}$ having the property that for each value of $t$ such that $t \geq T$ the random variable $\xi_t$ is $\mathcal{F}_{X_T}$-measurable.

In previous work on information-based asset pricing, models have been constructed using Brownian bridge information processes [3, 4, 6, 11, 15, 17, 22], gamma bridge information processes [5], Lévy random bridge information processes [12–14], and Markov bridge information processes [18]. In what follows we present a new model for the market filtration, based on the variance-gamma process. The idea is to create a two-parameter family of information processes associated with the random market factor $X_T$. One of the parameters is the information flow-rate $\sigma$. The other is an intrinsic parameter $m$ associated with the variance gamma process. In the limit as $m$ tends to infinity, the variance-gamma information process reduces to the type of Brownian bridge information process considered by Brody, Hughston & Macrina [3, 4, 17].

The plan of the paper is as follows. In Section II we recall properties of the gamma process, introducing the so-called scale parameter $\kappa > 0$ and shape parameter $m > 0$. A standard gamma subordinator is defined to be a gamma process with $\kappa = 1/m$. The mean at time $t$ of a standard gamma subordinator is $t$. In Theorem 1 we prove that an increase in the shape parameter $m$ results in a transfer of weight from the Lévy measure of any interval $[c, d]$ in the space of jump size to the Lévy measure of any interval $[a, b]$ such that $b - a = d - c$ and $c > a$. Thus, roughly speaking, an increase in $m$ results in an increase in the rate at which small jumps occur relative to the rate at which large jumps occur. In Section III we recall properties of the variance-gamma process and the gamma bridge, and in Definition 1 we introduce the so-called normalized variance-gamma bridge. In Lemmas 1, 2, 3, and 4 we work out various properties of the normalized variance-gamma bridge. Then in Theorem 2 we show that the normalized variance-gamma bridge and the associated gamma bridge are jointly Markov, a property that turns out to be useful in what follows thereafter. In Section IV at Definition 2, we introduce the so-called variance-gamma information process. The information process carries noisy information about the value of a market factor $X_T$ that will be revealed to the market at time $T$, where the noise is represented by the normalized variance-gamma bridge. In Lemma 4 we present a formula that relates the values of the information process at different times, and by use of that we establish in Theorem 3 that the information process and the associated gamma bridge are jointly Markov. In Section V we consider a market where the filtration is generated by a variance gamma information process along with the associated gamma bridge. Then in Theorem 4 we present a general formula for the price process of a financial asset that at time $T$ pays a single dividend given by a function $h(X_T)$ of the market factor. In particular, the a priori distribution of the market factor can be quite arbitrary, specified by a probability measure $F_{X_T}(dx)$ on $\mathbb{R}$, and the only requirement being that $h(X_T)$ should be integrable. Finally, in Section VI we present a number of examples, based on various choices of the distribution for the market factor and various choices for the payoff function, the results being summarized in Propositions 1, 2, 3, and 4. We conclude with a few comments on calibration.
II. GAMMA SUBORDINATORS

We begin with some remarks about the gamma process. Let us as usual write $\mathbb{R}^+$ for the non-negative real numbers. Let $\kappa$ and $m$ be strictly positive constants. A continuous random variable $G : \Omega \to \mathbb{R}^+$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ will be said to have a gamma distribution with scale parameter $\kappa$ and shape parameter $m$ if

$$P[G \in dx] = \mathbb{I}_{\{x > 0\}} \frac{1}{\Gamma[m]} \kappa^{-m} x^{m-1} e^{-x/\kappa} dx,$$

where

$$\Gamma[a] = \int_0^\infty x^{a-1} e^{-x} dx$$

denotes the standard gamma function for $a > 0$, and we recall the relation $\Gamma[a+1] = a \Gamma[a]$.

A calculation shows that $E[G] = \kappa m$, and $\text{Var}[G] = \kappa^2 m$. There exists a two-parameter family of gamma processes of the form $\Gamma : \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$ on $(\Omega, \mathcal{F}, \mathbb{P})$. By a gamma process with scale parameter $\kappa$ and shape parameter $m$ we mean a Lévy process $\{\Gamma_t\}_{t \geq 0}$ such that for each $t > 0$ the random variable $\Gamma_t$ is gamma distributed with

$$P[\Gamma_t \in dx] = \mathbb{I}_{\{x > 0\}} \frac{1}{\Gamma[mt]} \kappa^{-mt} x^{mt-1} e^{-x/\kappa} dx.$$

If we write $(a)_0 = 1$ and $(a)_k = a(a+1)(a+2) \cdots (a+k-1)$ for the so-called Pochhammer symbol, we find that $E[\Gamma_t] = \kappa t$ and $\text{Var}[\Gamma_t] = \nu^2 t$, where $\mu = \kappa m$ and $\nu^2 = \kappa^2 m$, or equivalently $m = \mu^2/\nu^2$, and $\kappa = \nu^2/\mu$. The Lévy exponent for such a process is given for $\alpha < 1$ by

$$\psi_t(\alpha) = \frac{1}{t} \log E[\exp(\alpha \Gamma_t)] = -m \log (1 - \kappa \alpha),$$

and for the corresponding Lévy measure we have

$$\nu_t(dx) = \mathbb{I}_{\{x > 0\}} m x^{-1} e^{-x/\kappa} dx.$$

One can then check that the Lévy-Khinchine relation

$$\psi_t(\alpha) = \int_\mathbb{R} (e^{\alpha x} - 1 - \mathbb{I}_{\{|x| < 1\}} \alpha x) \nu_t(dx) + p \alpha$$

holds for an appropriate choice of $p$ (see, for example, [16], Lemma 1.7).

By a standard gamma subordinator we mean a gamma process $\{\gamma_t\}_{t \geq 0}$ for which $\kappa = 1/m$. This implies that $E[\gamma_t] = t$ and $\text{Var}[\gamma_t] = m^{-1} t$. The standard gamma subordinators thus constitute a one-parameter family of processes labelled by $m$. An interpretation of the parameter $m$ is given by the following.

**Theorem 1.** Let $\{\gamma_t\}_{t \geq 0}$ be a gamma subordinator with parameter $m$. Let $\nu_m[a,b]$ be the Lévy measure of the interval $[a,b]$ for $0 < a < b$. Then for any interval $[c,d]$ such that $c > a$ and $d - c = b - a$ the ratio

$$\frac{\nu_m[a,b]}{\nu_m[c,d]}$$

is (i) strictly greater than one, and (ii) strictly increasing as a function of $m$. 
Proof. We begin by establishing (i). By definition we have
\[ \nu_m[a, b] = \int_a^b m \frac{1}{x} e^{-mx} \, dx. \] (10)

Let \( \delta = c - a > 0 \) and note that the integrand in the right hand side of (10) is a decreasing function of the variable of integration. This allows one to conclude that
\[ \nu_m[a + \delta, b + \delta] = \int_{a + \delta}^{b + \delta} m \frac{1}{x} e^{-mx} \, dx < \int_a^b m \frac{1}{x} e^{-mx} \, dx, \] (11)
and hence
\[ 0 < \nu_m[c, d] < \nu_m[a, b]. \] (12)

We proceed to establish (ii). A calculation shows that
\[ \nu_m[a, b] = m \left( E_1[ma] - E_1[mb] \right), \] (13)
where \( E_1(z) \) is defined for \( z > 0 \) by
\[ E_1(z) = \int_z^\infty \frac{e^{-x}}{x} \, dx. \] (14)

See [H], section 5.1.1. Next, we compute the derivative of the quotient in (9), which gives
\[ \frac{\partial}{\partial m} \left( \frac{\nu_m[a, b]}{\nu_m[c, d]} \right) = \frac{1}{m \left( E_1[mc] - E_1[md] \right)} e^{-ma} \left( 1 - e^{-m\Delta} \right) \left( \frac{\nu_m[a, b]}{\nu_m[c, d]} - e^{m(c-a)} \right), \] (15)
where \( \Delta = d - c = b - a \). We see that
\[ \frac{1}{m \left( E_1[mc] - E_1[md] \right)} e^{-ma} \left( 1 - e^{-m\Delta} \right) > 0, \] (16)
which implies that the sign of the derivative in (15) is the same as that of
\[ \frac{\nu_m[a, b]}{\nu_m[c, d]} - e^{m(c-a)}. \] (17)

Finally, we observe that
\[ \int_0^{\Delta m} \frac{e^{-u}}{u + am} \, du > \int_0^{\Delta m} \frac{e^{-u}}{u + cm} \, du, \] (18)
which implies that
\[ e^{ma} \int_{am}^{bm} \frac{e^{-x}}{x} \, dx > e^{mc} \int_{cm}^{dm} \frac{e^{-x}}{x} \, dx, \] (19)
or equivalently,
\[ \frac{\nu_m[a, b]}{\nu_m[c, d]} > e^{m(c-a)}. \] (20)

It follows then from (15) that the ratio (9) is strictly increasing as a function of the parameter \( m \), and that completes the proof. \( \square \)

We see that the effect of an increase in the value of \( m \) is to transfer weight from the Lévy measure of any interval \([c, d] \subset \mathbb{R}^+\) to any earlier (possibly overlapping) interval \([a, b] \subset \mathbb{R}^+\) of the same length. The Lévy measure of any such interval is the rate of arrival of jumps for which the jump size lies in the given interval.
III. NORMALIZED VARIANCE-GAMMA BRIDGE

Let us fix a standard Brownian motion \( \{ W_t \}_{t \geq 0} \) on \( (\Omega, \mathcal{F}, \mathbb{P}) \) and an independent standard gamma subordinator \( \{ \gamma_t \}_{t \geq 0} \) with parameter \( m \). By a standard variance-gamma process with parameter \( m \) we mean a time-changed Brownian motion \( \{ V_t \}_{t \geq 0} \) of the form

\[
V_t = W_{\gamma_t}.
\]  

(21)

It is straightforward to check that \( \{ V_t \} \) is itself a Lévy process, with Lévy exponent

\[
\psi_V(\alpha) = -m \log \left( 1 - \frac{\alpha^2}{2m} \right).
\]  

(22)

Properties of the variance-gamma process, and financial models based on it, have been investigated extensively in [9, 19–21] and many other works.

The other object we require going forward is the gamma bridge [5, 10, 23]. Let \( \{ \gamma_t \} \) be a standard gamma subordinator with parameter \( m \). For fixed \( T > 0 \) the process \( \{ \gamma_{tT} \}_{t \geq 0} \) defined by

\[
\gamma_{tT} = \frac{\gamma_t}{\gamma_T}
\]  

(23)

for \( 0 \leq t \leq T \) and \( \gamma_{tT} = 1 \) for \( t > T \) will be called a standard gamma bridge, with parameter \( m \), over the interval \([0, T]\). One can check that \( \gamma_{tT} \) has a beta distribution. In particular, one finds that its density is given by

\[
\mathbb{P}[\gamma_{tT} \in dy] = 1_{\{0 < y < 1\}} \frac{y^{mt-1}(1-y)^{m(T-t)-1}}{B[mt, m(T-t)]} dy,
\]  

(24)

where

\[
B[a,b] = \frac{\Gamma[a] \Gamma[b]}{\Gamma[a+b]}.
\]  

(25)

It follows then by use of the integral formula

\[
B[a,b] = \int_0^1 y^{a-1}(1-y)^{b-1} dy
\]  

(26)

that for all \( n \in \mathbb{N} \) we have

\[
\mathbb{E}[\gamma_{tT}^n] = \frac{B[mt+n, m(T-t)]}{B[mt, m(T-t)]},
\]  

(27)

and hence

\[
\mathbb{E}[\gamma_{tT}^n] = \frac{(mt)^n}{(mT)^n}.
\]  

(28)

Accordingly, one has \( \mathbb{E}[\gamma_{tT}] = t/T \) and \( \mathbb{E}[\gamma_{tT}^2] = t(mt+1)/T(mT+1) \), and therefore

\[
\text{Var}[\gamma_{tT}] = \frac{t(T-t)}{T^2(1+mT)}.
\]  

(29)

One observes, in particular, that the expectation of \( \gamma_{tT} \) does not depend on \( m \), whereas the variance of \( \gamma_{tT} \) decreases as \( m \) increases.
Lemma 1. For fixed $T > 0$, the process $\{\Gamma_{tT}\}_{t \geq 0}$ defined by
\[
\Gamma_{tT} = \gamma T^{-\frac{1}{2}} (W_{\gamma} - \gamma_{tT} W_{tT})
\] (30)
for $0 \leq t \leq T$ and $\Gamma_{tT} = 0$ for $t > T$ will be called a normalized variance gamma bridge.

We proceed to work out various properties of this process. We observe that $\mathbb{E}[\Gamma_{tT}] = 0$, that $\text{Var}[\Gamma_{tT} | \gamma_t, \gamma_T] = \gamma_{tT} (1 - \gamma_{tT})$, and hence that
\[
\text{Var}[\Gamma_{tT}] = \frac{mt (T - t)}{T (1 + mT)}.
\] (31)

Now, recall that the gamma process and the associated gamma bridge have the property that $\gamma_{st}$ and $\gamma_u$ are independent for $0 \leq s \leq t \leq u$ and $t > 0$. It follows that $\gamma_{st}$ and $\gamma_{uv}$ are independent for $0 \leq s \leq t \leq u \leq v$ and $t > 0$. Furthermore, we have:

Lemma 1. If $0 \leq s \leq t \leq u$ and $t > 0$ then $\gamma_{st}$ and $\gamma_u$ are independent.

Proof. Using the tower property, we find that
\[
F_{\Gamma_{st} \gamma_u}(x, y) = \mathbb{E} \left[1_{\{\Gamma_{st} \leq x\}} 1_{\{\gamma_u \leq y\}} \right] \\
= \mathbb{E} \left[ \mathbb{E} \left[1_{\{\Gamma_{st} \leq x\}} 1_{\{\gamma_u \leq y\}} | \gamma_s, \gamma_t, \gamma_u \right] \right] \\
= \mathbb{E} \left[1_{\{\gamma_u \leq y\}} \mathbb{E} \left[1_{\{\Gamma_{st} \leq x\}} | \gamma_s, \gamma_t, \gamma_u \right] \right] \\
= \mathbb{E} \left[1_{\{\gamma_u \leq y\}} \mathbb{E} \left[N \left(x \left((1 - \gamma_{st}) (\gamma_{st})^{-\frac{1}{2}}\right)\right)\right] \right] \\
= \mathbb{E} \left[N \left(y \left((1 - \gamma_{uv}) (\gamma_{uv})^{-\frac{1}{2}}\right)\right)\right],
\] (32)
where the last line follows from the independence of $\gamma_{st}$ and $\gamma_u$, and $N(\cdot)$ denotes the standard normal distribution function. \qed

As an immediate consequence, we also have

Lemma 2. If $0 \leq s \leq t \leq u \leq v$ and $t > 0$ then $\gamma_{st}$ and $\gamma_{uv}$ are independent.

Further, we have:

Lemma 3. If $0 \leq s \leq t \leq u \leq v$ and $t > 0$ then $\Gamma_{st}$ and $\Gamma_{uv}$ are independent.

Proof. We recall that the Brownian bridge $\{\beta_{tT}\}_{0 \leq t \leq T}$ defined by
\[
\beta_{tT} = W_t - \frac{t}{T} W_T
\] (33)
for $0 \leq t \leq T$ and $\beta_{tT} = 0$ for $t > T$ is Gaussian with $\mathbb{E}[\beta_{tT}] = 0$, $\text{Var}[\beta_{tT}] = t (T - t)/T$, and $\text{Cov}[\beta_{sT}, \beta_{tT}] = s(T - t)/T$ for $0 \leq s \leq t \leq T$. Using the tower property we find that
\[
F_{\Gamma_{st} \Gamma_{uv}}(x, y) = \mathbb{E} \left[1_{\{\Gamma_{st} \leq x\}} 1_{\{\Gamma_{uv} \leq y\}} \right] \\
= \mathbb{E} \left[ \mathbb{E} \left[1_{\{\Gamma_{st} \leq x\}} 1_{\{\Gamma_{uv} \leq y\}} | \gamma_s, \gamma_t, \gamma_u, \gamma_v \right] \right] \\
= \mathbb{E} \left[ \mathbb{E} \left[1_{\{\Gamma_{st} \leq x\}} | \gamma_s, \gamma_t, \gamma_u, \gamma_v \right] \mathbb{E} \left[1_{\{\Gamma_{uv} \leq y\}} | \gamma_s, \gamma_t, \gamma_u, \gamma_v \right] \right] \\
= \mathbb{E} \left[N \left(x \left((1 - \gamma_{st}) (\gamma_{st})^{-\frac{1}{2}}\right)\right)\right] \mathbb{E} \left[N \left(y \left((1 - \gamma_{uv}) (\gamma_{uv})^{-\frac{1}{2}}\right)\right)\right],
\] (34)
and that concludes the proof. \qed
Lemma 4. If \(0 \leq s \leq t \leq u\) and \(t > 0\) then \(\Gamma_{su} = (\gamma_{tu})^{\frac{1}{2}} \Gamma_{st} + \gamma_{st} \Gamma_{tu}\).

This follows from a straightforward calculation. Then we obtain

Theorem 2. The processes \(\{\Gamma_{tT}\}_{0 \leq t \leq T}\) and \(\{\gamma_{tT}\}_{0 \leq t \leq T}\) are jointly Markov.

Proof. To establish the Markov property it suffices to show that for any bounded measurable function \(\phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}\), any \(n \in \mathbb{N}\), and any \(0 \leq t_n \leq t_{n-1} \leq \ldots \leq t_1 \leq t \leq T\), we have

\[
E[\phi(\Gamma_{tT}, \gamma_{tT}) | \Gamma_{t_1T}, \gamma_{t_1T}, \Gamma_{t_2T}, \gamma_{t_2T}, \ldots, \Gamma_{t_nT}, \gamma_{t_nT}] = E[\phi(\Gamma_{tT}, \gamma_{tT}) | \Gamma_{t_1T}, \gamma_{t_1T}] .
\]  
(35)

We present the proof for \(n = 2\). Thus we need to show that

\[
E[\phi(\Gamma_{tT}, \gamma_{tT}) | \Gamma_{t_1T}, \gamma_{t_1T}, \Gamma_{t_2T}, \gamma_{t_2T}] = E[\phi(\Gamma_{tT}, \gamma_{tT}) | \Gamma_{t_1T}, \gamma_{t_1T}] .
\]  
(36)

We remark that as a consequence of Lemma 4 we have

\[
E[\phi(\Gamma_{tT}, \gamma_{tT}) | \Gamma_{t_1T}, \gamma_{t_1T}, \Gamma_{t_2T}, \gamma_{t_2T}] = E[\phi(\Gamma_{tT}, \gamma_{tT}) | \Gamma_{t_1T}, \gamma_{t_1T}, \Gamma_{t_2T}, \gamma_{t_2T}] .
\]  
(37)

Therefore, it suffices to show that

\[
E[\phi(\Gamma_{tT}, \gamma_{tT}) | \Gamma_{t_1T}, \gamma_{t_1T}, \Gamma_{t_2T}, \gamma_{t_2T}] = E[\phi(\Gamma_{tT}, \gamma_{tT}) | \Gamma_{t_1T}, \gamma_{t_1T}] .
\]  
(38)

Let us write

\[
f_{\Gamma_{tT}, \gamma_{tT}, \Gamma_{t_1T}, \gamma_{t_1T}, \Gamma_{t_2T}, \gamma_{t_2T}}(x, y, a, b, c, d)
\]  
(39)

for the joint density of the random variables \(\Gamma_{tT}, \gamma_{tT}, \Gamma_{t_1T}, \gamma_{t_1T}, \Gamma_{t_2T}, \gamma_{t_2T}\). Then for the conditional density of the \(\Gamma_{tT}\) and \(\gamma_{tT}\) given \(\Gamma_{t_1T} = a, \gamma_{t_1T} = b, \Gamma_{t_2T} = c, \gamma_{t_2T} = d\) we have

\[
g_{\Gamma_{tT}, \gamma_{tT}}(x, y, a, b, c, d) = \frac{f_{\Gamma_{tT}, \gamma_{tT}, \Gamma_{t_1T}, \gamma_{t_1T}, \Gamma_{t_2T}, \gamma_{t_2T}}(x, y, a, b, c, d)}{f_{\Gamma_{t_1T}, \gamma_{t_1T}, \Gamma_{t_2T}, \gamma_{t_2T}}(a, b, c, d)} .
\]  
(40)

Thus,

\[
E[\phi(\Gamma_{tT}, \gamma_{tT}) | \Gamma_{t_1T}, \gamma_{t_1T}, \Gamma_{t_2T}, \gamma_{t_2T}] = \int_{\mathbb{R}} \int_{\mathbb{R}} \phi(x, y) g_{\Gamma_{tT}, \gamma_{tT}}(x, y, \Gamma_{t_1T}, \gamma_{t_1T}, \Gamma_{t_2T}, \gamma_{t_2T}) \, dx \, dy .
\]  
(41)

Similarly,

\[
E[\phi(\Gamma_{tT}, \gamma_{tT}) | \Gamma_{t_1T}, \gamma_{t_1T}] = \int_{\mathbb{R}} \int_{\mathbb{R}} \phi(x, y) g_{\Gamma_{tT}, \gamma_{tT}}(x, y, \Gamma_{t_1T}, \gamma_{t_1T}) \, dx \, dy .
\]  
(42)

We shall show that

\[
g_{\Gamma_{tT}, \gamma_{tT}}(x, y, \Gamma_{t_1T}, \gamma_{t_1T}, \Gamma_{t_2T}, \gamma_{t_2T}) = g_{\Gamma_{tT}, \gamma_{tT}}(x, y, \Gamma_{t_1T}, \gamma_{t_1T}) .
\]  
(43)
Writing

\[ F_{\Gamma_T, \gamma_T, \Gamma_1 T, \gamma_1 T, \Gamma_2 T, \gamma_2 T} (x, y, a, b, c, d) \],

\[
= \mathbb{E} \left[ \mathbb{1}_{\{\Gamma_T < x\}} \mathbb{1}_{\{\gamma_T < y\}} \mathbb{1}_{\{\Gamma_1 T < a\}} \mathbb{1}_{\{\gamma_1 T < b\}} \mathbb{1}_{\{\Gamma_2 T < c\}} \mathbb{1}_{\{\gamma_2 T < d\}} \right]
\]

(44)

for the joint distribution function, we see that

\[
F_{\Gamma_T, \gamma_T, \Gamma_1 T, \gamma_1 T, \Gamma_2 T, \gamma_2 T} (x, y, a, b, c, d)
\]

\[
= \mathbb{E} \left[ \mathbb{1}_{\{\Gamma_T < x\}} \mathbb{1}_{\{\gamma_T < y\}} \mathbb{1}_{\{\gamma_1 T < b\}} \mathbb{1}_{\{\gamma_2 T < d\}} \right]
\]

(45)

where the last step follows by virtue of Lemma [3]. Thus we have

\[
F_{\Gamma T, \gamma_T, \Gamma_1 T, \gamma_1 T, \Gamma_2 T, \gamma_2 T} (x, y, a, b, c, d)
\]

\[
= \mathbb{E} \left[ \mathbb{1}_{\{\Gamma_T < x\}} \mathbb{1}_{\{\gamma_T < y\}} \mathbb{1}_{\{\gamma_1 T < b\}} \mathbb{1}_{\{\gamma_2 T < d\}} \right]
\]

\[
\times N \left( \frac{c}{\sqrt{(1 - \gamma_2 T) (\gamma_2 T)}} \right) \mathbb{1}_{\{\gamma_2 T < d\}}
\]

(46)

where the next to last step follows by virtue of Lemma [2]. Similarly,

\[
F_{\Gamma_1 T, \gamma_1 T, \Gamma_2 T, \gamma_2 T} (a, b, c, d)
\]

\[
= \mathbb{E} \left[ \mathbb{1}_{\{\gamma_1 T < b\}} \mathbb{1}_{\{\gamma_2 T < d\}} \right]
\]

(47)

and hence

\[
F_{\Gamma_{1 T}, \gamma_{1 T}, \Gamma_{2 T}, \gamma_{2 T}} (a, b, c, d)
\]

\[
= \mathbb{E} \left[ \mathbb{1}_{\{\gamma_1 T < b\}} \mathbb{1}_{\{\gamma_2 T < d\}} \right]
\]

(48)
Finally, we invoke Lemma 2, Lemma 3, and Theorem 2 to conclude that

\[ f_{\Gamma t \tau, \gamma t \tau, \Gamma w, \gamma w, \Gamma Z, \gamma Z}(x, y, a, b, c, d) = f_{\Gamma t \tau, \gamma t \tau}(x, y, a, b) \times f_{\Gamma w, \gamma w}(c, d), \]

and

\[ f_{\Gamma t \tau, \gamma t \tau, \Gamma Z, \gamma Z}(a, b, c, d) = f_{\Gamma t \tau, \gamma t \tau}(a, b) \times f_{\Gamma Z, \gamma Z}(c, d), \]

and the theorem follows.

\[ \square \]

IV. VARIANCE GAMMA INFORMATION

Fix \( T > 0 \) and let \( \{\Gamma_{t\tau}\} \) be a normalized variance gamma bridge, as defined by (30). Let \( \{\gamma_{t\tau}\} \) be the associated gamma bridge defined by (23). Let \( X_T \) be a random variable and assume that \( X_T, \{\gamma_t\}_{t \geq 0} \) and \( \{W_t\}_{t \geq 0} \) are independent. Then we are led to the following:

**Definition 2.** By a variance-gamma information process carrying the market factor \( X_T \) we mean a process \( \{\xi_t\}_{t \geq 0} \) that takes the form

\[ \xi_t = \Gamma_{t\tau} + \sigma \gamma_{t\tau} X_T \]

for \( 0 \leq t \leq T \) and \( \xi_t = \sigma X_T \) for \( t > T \), where \( \sigma \) is a positive constant.

The market filtration is assumed to be the standard augmented filtration generated jointly by \( \{\xi_t\} \) and \( \{\gamma_{t\tau}\} \). An elementary calculation gives

**Lemma 5.** If \( 0 \leq s \leq t \leq T \) and \( t > 0 \) then

\[ \xi_s = \Gamma_{st} (\gamma_{t\tau})^{\frac{1}{2}} + \xi_t \gamma_{st}. \]

Then we are led to the following result required for the valuation of assets.

**Theorem 3.** The processes \( \{\xi_t\}_{0 \leq t \leq T} \) and \( \{\gamma_{t\tau}\}_{0 \leq t \leq T} \) are jointly Markov.

**Proof.** It suffices to show that for any \( n \in \mathbb{N} \) and any times \( 0 < t_1 < t_2 < \cdots < t_n \) we have

\[ \mathbb{E} \left[ \phi(\xi_t, \gamma_{t\tau}) \mid \xi_{t_1}, \xi_{t_2}, \ldots, \xi_{t_n}, \gamma_{t_1}, \gamma_{t_2}, \ldots, \gamma_{t_n} \right] = \mathbb{E} \left[ \phi(\xi_t, \gamma_{t\tau}) \mid \xi_{t_1}, \gamma_{t_1} \right]. \]

We present the proof for \( n = 2 \). Thus, we propose to show that

\[ \mathbb{E} \left[ \phi(\xi_t, \gamma_{t\tau}) \mid \xi_{t_1}, \xi_{t_2}, \gamma_{t_1}, \gamma_{t_2} \right] = \mathbb{E} \left[ \phi(\xi_t, \gamma_{t\tau}) \mid \xi_{t_1}, \gamma_{t_1} \right]. \]

By Lemma 5, we have

\[ \mathbb{E} \left[ \phi(\xi_t, \gamma_{t\tau}) \mid \xi_{t_1}, \xi_{t_2}, \gamma_{t_1}, \gamma_{t_2}, \right] = \mathbb{E} \left[ \phi(\xi_t, \gamma_{t\tau}) \mid \xi_{t_1}, \xi_{t_2}, \gamma_{t_1}, \gamma_{t_2}, \right]. \]

Finally, we invoke Lemma 2, Lemma 3 and Theorem 2 to conclude that

\[ \mathbb{E} \left[ \phi(\xi_t, \gamma_{t\tau}) \mid \xi_{t_1}, \xi_{t_2}, \gamma_{t_1}, \gamma_{t_2}, \right] = \mathbb{E} \left[ \phi(\Gamma_{t\tau} + \gamma_{t\tau} \sigma X_T, \gamma_{t\tau}) \mid \Gamma_{t_1\tau} + \gamma_{t_1} \sigma X_T, \gamma_{t_1} \right]. \]

The generalization to \( n > 2 \) is straightforward.
V. INFORMATION BASED PRICING

Now we are in a position to consider the valuation of a financial asset in the setting just discussed. One recalls that $\mathbb{P}$ is understood to be the risk-neutral measure and that the interest rate is constant. The payoff of the asset at time $T$ is taken to be an integrable random variable of the form $h(X_T)$ for some Borel function $h$, where $X_T$ is the information revealed at $T$. The filtration is generated jointly by the variance-gamma information process $\{\xi_t\}$ and the associated gamma bridge $\{\gamma_{tT}\}$. The value of the asset at time $t \in [0, T)$ is then given by the general expression (2), which on account of Theorem 3 reduces in the present context to

$$S_t = e^{-r(T-t)} \mathbb{E}[h(X_T) \mid \xi_t, \gamma_{tT}].$$

(57)

Our goal now is to work out this expectation explicitly. Let us write $F_{X_T}$ for the a priori distribution function of $X_T$. Thus $F_{X_T} : x \in \mathbb{R} \to F_{X_T}(x) \in [0, 1]$ and we have

$$F_{X_T}(x) = \mathbb{P}(X_T \leq x).$$

(58)

Occasionally, it will be convenient typographically to write $F_{X_T}^{(x)}$ in place of $F_{X_T}(x)$, and similarly for other distribution functions. To proceed with the calculation of the conditional expectation of $h(X_T)$, we require the following:

**Lemma 6.** Let $X$ be a random variable with distribution $\{F_X(x)\}_{x \in \mathbb{R}}$ and let $Y$ be a continuous random variable with distribution $\{F_Y(y)\}_{y \in \mathbb{R}}$ and density $\{f_Y(y)\}_{y \in \mathbb{R}}$. Then for all $y \in \mathbb{R}$ for which $f_Y(y) \neq 0$ we have

$$F_{X \mid Y = y}^{(x)} = \frac{\int_{u \in (-\infty, x]} f_Y^{(y)}(y) \, dF_X^{(u)}}{\int_{u \in (-\infty, \infty)} f_Y^{(y)}(y) \, dF_X^{(u)}},$$

(59)

**Proof.** For any two random variables $X$ and $Y$ it holds that

$$\mathbb{P}(X \leq x, Y \leq y) = \mathbb{E}\left[ \mathbf{1}_{\{X \leq x\}} \mathbf{1}_{\{Y \leq y\}} \right] = \mathbb{E}\left[ \mathbb{E}\left[ \mathbf{1}_{\{X \leq x\}} \mid Y \right] \mathbf{1}_{\{Y \leq y\}} \right] = \mathbb{E}\left[ F_{X \mid Y}^{(x)} \mathbf{1}_{\{Y \leq y\}} \right]$$

(60)

and hence

$$\mathbb{P}(X \leq x, Y \leq y) = \int_{u \in (-\infty, y]} F_{X \mid Y = u}^{(x)} \, dF_Y^{(u)}.$$

(61)

Here we have used the fact that for each $x \in \mathbb{R}$ there exists a Borel measurable function $P_x : y \in \mathbb{R} \to P_x(y) \in [0, 1]$ such that $\mathbb{E}\left[ \mathbf{1}_{\{X \leq x\}} \mid Y \right] = P_x(Y)$. Then for each $y \in \mathbb{R}$ we define

$$F_{X \mid Y = y}^{(x)} = P_x(y).$$

(62)

By symmetry, we also have

$$\mathbb{P}(X \leq x, Y \leq y) = \int_{u \in (-\infty, x]} F_{Y \mid X = u}^{(y)} \, dF_X^{(u)}.$$

(63)
from which it follows that
\[ \int_{u \in (-\infty, x]} F_{Y|X=u}^{(y)} \, dF_X^{(u)} = \int_{v \in (-\infty, y]} F_{X|Y=v}^{(x)} \, dF_Y^{(v)}. \] (64)

Now consider the measure \( F_{X|Y=y} (dx) \) on \( (\mathbb{R}, \mathcal{B}) \) defined for each \( y \in \mathbb{R} \) by setting
\[ F_{X|Y=y} (A) = \mathbb{E} \left[ \mathbb{1}_{\{X \in A\}} | Y = y \right] \] (65)
for any \( A \in \mathcal{B} \). Then \( F_{X|Y=y} (dx) \) is absolutely continuous with respect to \( F_X (dx) \). In particular, suppose that \( F_X (B) = 0 \) for some \( B \in \mathcal{B} \). Now, \( F_{X|Y=y} (B) = \mathbb{E} \left[ \mathbb{1}_{\{X \in B\}} | Y = y \right] \). But if \( \mathbb{E} \left[ \mathbb{1}_{\{X \in B\}} \right] = 0 \), then \( \mathbb{E} \left[ \mathbb{E} \left[ \mathbb{1}_{\{X \in B\}} | Y \right] \right] = 0 \), and hence \( \mathbb{E} \left[ \mathbb{1}_{\{X \in B\}} | Y = y \right] = 0 \), and therefore \( \mathbb{E} \left[ \mathbb{1}_{\{X \in B\}} | Y = y \right] = 0 \). Thus \( F_{X|Y=y} (B) \) vanishes for any \( B \in \mathcal{B} \) for which \( F_X (B) \) vanishes. It follows by the Radon-Nikodym theorem that for each \( y \in \mathbb{R} \) there exists a density \( \{g_y(x)\}_{x \in \mathbb{R}} \) such that
\[ F_{X|Y=y}^{(x)} = \int_{u \in (-\infty, x]} g_y(u) \, dF_X^{(u)}. \] (66)

Note that \( \{g_y(x)\} \) is determined uniquely apart from its values on \( F_X \)-null sets. Inserting (66) into (64) we obtain
\[ \int_{u \in (-\infty, x]} F_{Y|X=u}^{(y)} \, dF_X^{(u)} = \int_{v \in (-\infty, y]} \int_{u \in (-\infty, x]} g_v(u) \, dF_X^{(u)} \, dF_Y^{(v)}, \] (67)
and thus by Fubini’s theorem we have
\[ \int_{u \in (-\infty, x]} F_{Y|X=u}^{(y)} \, dF_X^{(u)} = \int_{v \in (-\infty, y]} \int_{u \in (-\infty, x]} g_v(u) \, dF_Y^{(v)} \, dF_X^{(u)}. \] (68)

It follows then that \( \{F_{Y|X=x}^{(y)}\}_{x \in \mathbb{R}} \) is determined uniquely apart from its values on \( F_X \)-null sets, and we have
\[ F_{Y|X=x}^{(y)} = \int_{v \in (-\infty, y]} g_v(x) \, dF_Y^{(v)}. \] (69)

Now suppose that \( Y \) is a continuous random variable. Then its distribution function \( \{F_Y^{(y)}\}_{y \in \mathbb{R}} \) is absolutely continuous and admits a density \( \{f_Y^{(y)}\}_{y \in \mathbb{R}} \). In that case, (69) can be written in the form
\[ F_{Y|X=x}^{(y)} = \int_{v \in (-\infty, y]} g_v(x) \, f_Y^{(v)} \, dv, \] (70)
from which it follows that for each value of \( x \) the conditional distribution function \( \{F_{Y|X=x}^{(y)}\}_{y \in \mathbb{R}} \) is absolutely continuous and admits a density \( \{f_{Y|X=x}^{(y)}\}_{y \in \mathbb{R}} \) such that
\[ f_{Y|X=x}^{(y)} = g_y(x) \, f_Y^{(y)}. \] (71)
The desired result (59) then follows from (66) and (71) if we observe that
\[ f_Y^{(y)} = \int_{u \in (-\infty, \infty)} f_{Y|X=x}^{(y)} \, dF_X^{(u)}, \] (72)
and that concludes the proof.
Armed with Lemma 6, we are in a position to work out the conditional expectation that leads to the asset price, and we obtain the following:

**Theorem 4.** The variance-gamma information-based price of a financial asset with payoff \( h(X_T) \) at time \( T \) is given for \( t < T \) by

\[
S_t = e^{-r(T-t)} \int_{x \in \mathbb{R}} h(x) \frac{e^{(\sigma \xi_t x - \frac{1}{2} \sigma^2 x^2 \gamma_T) (1-\gamma_T)^{-1}}}{\int_{y \in \mathbb{R}} e^{(\sigma \xi_t y - \frac{1}{2} \sigma^2 y^2 \gamma_T) (1-\gamma_T)^{-1}}} dF_{X_T}^{(x)}.
\]  

(73)

**Proof.** To calculate the conditional expectation of \( h(X_T) \), we observe that

\[
\mathbb{E} [h(X_T) \mid \xi_t, \gamma_{tT}] = \mathbb{E} \left[ \mathbb{E} [h(X_T) \mid \xi_t, \gamma_{tT}, \gamma_T] \mid \xi_t, \gamma_{tT} \right],
\]

by the tower property, where the inner expectation takes the form

\[
\mathbb{E} [h(X_T) \mid \xi_t = \xi, \gamma_{tT} = b, \gamma_T = g] = \int_{x \in \mathbb{R}} h(x) dF_{X_T \mid \xi_t = \xi, \gamma_{tT} = b, \gamma_T = g}^{(x)}.
\]

(75)

Here by Lemma 6 the conditional distribution function is

\[
F_{X_T \mid \xi_t = \xi, \gamma_{tT} = b, \gamma_T = g}^{(x)} = \frac{\int_{u \in (-\infty, x]} \int_{\xi_t \mid \gamma_{tT} = b, \gamma_T = g} f^{(\xi)}(\xi) \int f^{(\xi)}(\xi) \mid \gamma_{tT} = b, \gamma_T = g \frac{dF_{X_T}}{dF_{X_T}} \mid \gamma_{tT} = b, \gamma_T = g}{\int_{u \in (-\infty, x]} \int f^{(\xi)}(\xi) \mid \gamma_{tT} = b, \gamma_T = g \frac{dF_{X_T}}{dF_{X_T}} \mid \gamma_{tT} = b, \gamma_T = g}
\]

\[
= \frac{\int_{u \in (-\infty, x]} \int f^{(\xi)}(\xi) \mid \gamma_{tT} = b, \gamma_T = g \frac{dF_{X_T}}{dF_{X_T}} \mid \gamma_{tT} = b, \gamma_T = g}{\int_{u \in (-\infty, x]} \int f^{(\xi)}(\xi) \mid \gamma_{tT} = b, \gamma_T = g \frac{dF_{X_T}}{dF_{X_T}} \mid \gamma_{tT} = b, \gamma_T = g}
\]

\[
= \frac{\int_{u \in (-\infty, x]} e^{(\sigma \xi u - \frac{1}{2} \sigma^2 u^2 b) (1-b)^{-1}} dF_{X_T}^{(u)}}{\int_{u \in (-\infty, x]} e^{(\sigma \xi u - \frac{1}{2} \sigma^2 u^2 b) (1-b)^{-1}} dF_{X_T}^{(u)}}.
\]

(76)

Therefore, the inner expectation in equation (74) is given by

\[
\mathbb{E} [h(X_T) \mid \xi_t, \gamma_{tT}, \gamma_T] = \int_{x \in \mathbb{R}} h(x) \frac{e^{(\sigma \xi_t x - \frac{1}{2} \sigma^2 x^2 \gamma_T) (1-\gamma_T)^{-1}}}{\int_{y \in \mathbb{R}} e^{(\sigma \xi_t y - \frac{1}{2} \sigma^2 y^2 \gamma_T) (1-\gamma_T)^{-1}}} dF_{X_T}^{(x)}.
\]

(77)

Now, by the tower property of conditional expectation we know that

\[
\mathbb{E} [h(X_T) \mid \xi_t, \gamma_{tT}] = \mathbb{E} \left[ \mathbb{E} [h(X_T) \mid \xi_t, \gamma_{tT}, \gamma_T] \mid \xi_t, \gamma_{tT} \right]
\]

(78)

But the right hand side of (77) depends only on \( \xi_t \) and \( \gamma_{tT} \). It follows immediately that

\[
\mathbb{E} [h(X_T) \mid \xi_t, \gamma_{tT}] = \int_{x \in \mathbb{R}} h(x) \frac{e^{(\sigma \xi_t x - \frac{1}{2} \sigma^2 x^2 \gamma_T) (1-\gamma_T)^{-1}}}{\int_{y \in \mathbb{R}} e^{(\sigma \xi_t y - \frac{1}{2} \sigma^2 y^2 \gamma_T) (1-\gamma_T)^{-1}}} dF_{X_T}^{(x)},
\]

(79)

which translates into equation (73), and that concludes the proof. \( \square \)
VI. EXAMPLES

In conclusion, we present examples of variance-gamma information pricing for specific choices of (a) the payoff function $h : \mathbb{R} \rightarrow \mathbb{R}^+$ and (b) the distribution of the market factor $X_T$.

**Example 1.** We begin with the simplest case, which is that of a unit-principal credit-risky bond without recovery. We set $h(x) = x$, with $\mathbb{P}(X_T = 0) = p_0$ and $\mathbb{P}(X_T = 1) = p_1$, where $p_0 + p_1 = 1$. Thus, we have

$$F_{X_T}(x) = p_0 \delta_0(x) + p_1 \delta_1(x),$$  \hspace{1cm} (80)

where

$$\delta_a(x) = \int_{y \in (-\infty, x]} \delta_a(dy),$$  \hspace{1cm} (81)

and $\delta_a(dx)$ denotes the Dirac measure concentrated at the point $a$, and we are led to the following:

**Proposition 1.** The variance-gamma information-based price of a unit-principal credit-risky discount bond with no recovery is given by

$$S_t = e^{-r(T-t)} \frac{p_1 \, e^{\left(\sigma \xi_t - \frac{1}{2} \sigma^2 \gamma_T \right) (1-\gamma_T)^{-1}}}{p_0 + p_1 \, e^{\left(\sigma \xi_t - \frac{1}{2} \sigma^2 \gamma_T \right) (1-\gamma_T)^{-1}}},$$  \hspace{1cm} (82)

Figure 1: Credit-risky bonds with no recovery. The panels on the left show simulations of trajectories of the variance gamma information process, and the panels on the right show simulations of the corresponding price trajectories. Prices are quoted as percentages of the principal, and the interest rate is taken to be zero. From top to bottom, we show trajectories having $\sigma = 1, 2$, respectively. We take $p_0 = 0.4$ for the probability of default and $p_1 = 0.6$ for the probability of no default. The value of $m$ is 100 in all cases. Fifteen simulated trajectories are shown in each panel.
Now let ω ∈ Ω denote the outcome of chance. By use of equation (52) one can check rather directly that if $X_T(\omega) = 1$, then $\lim_{t \to T} S_t = 1$, whereas if $X_T(\omega) = 0$, then $\lim_{t \to T} S_t = 0$. More explicitly, we find that

$$S_t \bigg|_{X_T(\omega) = 0} = e^{-r(T-t)} \frac{p_1 \exp \left[ \sigma \left( \gamma_T^{-1/2} (W_{\gamma_T} - \gamma_{\tau_T} W_{\gamma_T}) - \frac{1}{2} \sigma \gamma_{\tau_T} \right) (1 - \gamma_{\tau_T})^{-1} \right]}{p_0 + p_1 \exp \left[ \sigma \left( \gamma_T^{-1/2} (W_{\gamma_T} - \gamma_{\tau_T} W_{\gamma_T}) - \frac{1}{2} \sigma \gamma_{\tau_T} \right) (1 - \gamma_{\tau_T})^{-1} \right]} \quad (83)$$

whereas

$$S_t \bigg|_{X_T(\omega) = 1} = e^{-r(T-t)} \frac{p_1 \exp \left[ \sigma \left( \gamma_T^{-1/2} (W_{\gamma_T} - \gamma_{\tau_T} W_{\gamma_T}) + \frac{1}{2} \sigma \gamma_{\tau_T} \right) (1 - \gamma_{\tau_T})^{-1} \right]}{p_0 + p_1 \exp \left[ \sigma \left( \gamma_T^{-1/2} (W_{\gamma_T} - \gamma_{\tau_T} W_{\gamma_T}) + \frac{1}{2} \sigma \gamma_{\tau_T} \right) (1 - \gamma_{\tau_T})^{-1} \right]} \quad (84)$$

and the claimed limiting behaviour of the asset price follows by inspection.

![Figure 2: Credit-risky bonds with no recovery. From top to bottom we show trajectories having σ = 3, 4, respectively. The other parameters and assumptions are the same as in Figure 1](image)

**Example 2.** As a somewhat more sophisticated version of the previous example, we consider the case of a defaultable bond with random recovery. We shall work out the case where $h(x) = x$ and the market factor $X_T$ takes the value $c$ with probability $p_1$ and $X_T$ is uniformly distributed over the interval $[a, b]$ with probability $p_0$, where $0 \leq a < b \leq c$. Thus, for the probability measure of $X_T$ we have

$$F_{X_T}(dx) = p_0 \mathbb{1}_{\{a \leq x < b\}} \, dx + p_1 \delta_c(dx), \quad (85)$$

and for the distribution function we obtain

$$F_{X_T}(x) = p_0 \mathbb{1}_{\{a \leq x < b\}} + \mathbb{1}_{\{x \geq c\}} \quad (86)$$

The bond price at time $t$ is then obtained by working out the expression

$$S_t = e^{-r(T-t)} \frac{p_0 \int_a^b x e^{(\sigma \xi_T - \frac{1}{2} \sigma^2 x^2 \gamma_T)(1-\gamma_T)^{-1}} \, dx + p_1 c e^{(\sigma \xi_T - \frac{1}{2} \sigma^2 \gamma_T)(1-\gamma_T)^{-1}}}{p_0 \int_a^b e^{(\sigma \xi_T - \frac{1}{2} \sigma^2 x^2 \gamma_T)(1-\gamma_T)^{-1}} \, dx + p_1 e^{(\sigma \xi_T - \frac{1}{2} \sigma^2 \gamma_T)(1-\gamma_T)^{-1}}} \quad (87)$$
and it should be evident that one can obtain a closed-form solution. To work this out in
detail, it will be convenient to have an expression for the incomplete first moment of a
normally-distributed random variable with mean $\mu$ and variance $\nu^2$. Thus we set

$$N_1(x, \mu, \nu) = \frac{1}{\sqrt{2\pi\nu^2}} \int_{-\infty}^x y \exp \left( -\frac{1}{2} \frac{(y - \mu)^2}{\nu^2} \right) \, dy,$$  

(88)

and for convenience we also set

$$N_0(x, \mu, \nu) = \frac{1}{\sqrt{2\pi\nu^2}} \int_{-\infty}^x \exp \left( -\frac{1}{2} \frac{(y - \mu)^2}{\nu^2} \right) \, dy.$$  

(89)

Then we have

$$N_1(x, \mu, \nu) = \mu N \left( \frac{x - \mu}{\nu} \right) - \frac{\nu}{\sqrt{2\pi}} \exp \left( -\frac{1}{2} \frac{(x - \mu)^2}{\nu^2} \right),$$  

(90)

and of course

$$N_0(x, \mu, \nu) = N \left( \frac{x - \mu}{\nu} \right),$$  

(91)

where $N(\cdot)$ is the standard normal distribution function. We also set

$$f(x, \mu, \nu) = \frac{1}{\sqrt{2\pi\nu}} \exp \left( -\frac{1}{2} \frac{(x - \mu)^2}{\nu^2} \right).$$  

(92)

Finally, we obtain

**Proposition 2.** The variance-gamma information-based price of a defaultable discount bond
with a uniformly-distributed fraction of the principal paid on recovery is given by

$$S_t = e^{-r(T-t)} \frac{p_0 \left( N_1(b, \mu, \nu) - N_1(a, \mu, \nu) \right) + p_1 c f(c, \mu, \nu)}{p_0 \left( N_0(b, \mu, \nu) - N_0(a, \mu, \nu) \right) + p_1 f(c, \mu, \nu)},$$  

(93)

where

$$\mu = \frac{1}{\sigma} \xi_t, \quad \nu = \frac{1}{\sigma} \sqrt{\frac{1 - \gamma_{\xi T}}{\gamma_{\xi T}}}. \quad (94)$$

**Example 3.** Next we consider the case when the payoff of an asset at time $T$ is log-normally
distributed. This will hold if $h(x) = e^x$ and $X_T \sim \text{Normal}(\mu, \nu^2)$. It will be convenient to
look at the slightly more general payoff obtained by setting $h(x) = e^{q x}$ with $q \in \mathbb{R}$. If we
recall the identity

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \exp \left( -\frac{1}{2} A x^2 + B x \right) \, dx = \frac{1}{\sqrt{A}} \exp \left( \frac{1}{2} \frac{B^2}{A} \right),$$  

(95)

which holds for $A > 0$ and $B \in \mathbb{R}$, a calculation gives

$$I_t(q) := \int_{-\infty}^\infty e^{q x} \frac{1}{\sqrt{2\pi\nu}} \exp \left[ -\frac{1}{2} \frac{(x - \mu)^2}{\nu^2} + \frac{1}{1 - \gamma_{\xi T}} \left( \sigma \xi_t x - \frac{1}{2} \sigma^2 x^2 \gamma_{\xi T} \right) \right] \, dx$$

$$= \frac{1}{\nu \sqrt{A_t}} \exp \left( \frac{1}{2} \frac{B_t^2}{A_t} - C \right),$$  

(96)
where
\[ A_t = \frac{1 - \gamma_t + \nu^2 \sigma^2 \gamma_t}{\nu^2 (1 - \gamma_t)} , \quad B_t = q + \frac{\mu}{\nu^2} + \frac{\sigma \xi_t}{1 - \gamma_t} , \quad C = \frac{1}{2} \frac{\mu^2}{\nu^2} . \] (97)

For \( q = 1 \), the price is thus given in accordance with Theorem by
\[ S_t = e^{-r(T-t)} \frac{I_t(1)}{I_t(0)} . \] (98)

Then clearly we have
\[ S_0 = e^{-rT} \exp \left( \mu + \frac{1}{2} \nu^2 \right) , \] (99)

and a calculation leads to the following:

**Proposition 3.** The variance-gamma information-based price of a financial asset with a log-normally distributed payoff such that \( \log (S_T) \sim \text{Normal}(\mu, \nu^2) \) is given for \( t \in (0, T) \) by
\[ S_t = e^{r t} S_0 \exp \left[ \frac{\nu^2 \sigma^2 \gamma_t (1 - \gamma_t)^{-1}}{1 + \nu^2 \sigma^2 \gamma_t (1 - \gamma_t)^{-1}} \left( \frac{1}{\sigma \gamma_t} \xi_t - \mu - \frac{1}{2} q \nu^2 \right) \right] . \] (100)

More generally, one can consider the case of a so-called power-payoff derivative for which
\[ H_T = (S_T)^q , \] (101)

where \( S_T = \lim_{t \to T} S_t \) is the payoff of the asset priced above in Proposition 3. See [2] for aspects of the theory of power-payoff derivatives. In the present case if we write
\[ C_t = e^{-r(T-t)} \mathbb{E}_t [(S_T)^q] \] (102)

for the value of the power-payoff derivative at time \( t \), we find that
\[ C_t = e^{r t} C_0 \exp \left[ \frac{\nu^2 \sigma^2 \gamma_t (1 - \gamma_t)^{-1}}{1 + \nu^2 \sigma^2 \gamma_t (1 - \gamma_t)^{-1}} \left( \frac{q}{\sigma \gamma_t} \xi_t - q \mu - \frac{1}{2} q^2 \nu^2 \right) \right] , \] (103)

where
\[ C_0 = e^{-rT} \exp \left[ q \mu + \frac{1}{2} q^2 \nu^2 \right] . \] (104)
Figure 3: Log-normal payoff. The panels on the left show simulations of the trajectories of the information process, whereas the panels on the right show simulations of the corresponding price process trajectories. From the top to bottom, we show trajectories having $\sigma = 1, 2$, respectively. The value for $m$ is 100. We take $\mu = 0, \nu = 1$, and show 15 simulated trajectories in each panel.

Figure 4: Log-normal payoff. From the top row to the bottom, we show trajectories having $\sigma = 3, 4$, respectively. The other parameters and assumptions are the same as those in Figure 3.

**Example 4.** Next we consider the case where the payoff is exponentially distributed. We let $X_T \sim \exp(\lambda)$, so $\mathbb{P}[X_T \in dx] = \lambda e^{-\lambda x} \, dx$, and take $h(x) = x$. A calculation shows that

$$
\int_0^\infty x \exp \left[ -\lambda x + \left( \sigma \xi_t x - \frac{1}{2} \sigma^2 x^2 \gamma_{\xi T} \right) (1 - \gamma_{\xi T}^{-1}) \right] \, dx = \frac{\mu - N(0, \mu, \nu)}{f(0, \mu, \nu)},
$$

where we set

$$
\mu = \frac{1}{\sigma} \frac{\xi_t}{\gamma_{\xi T}} - \frac{\lambda}{\sigma^2} \frac{1 - \gamma_{\xi T}}{\gamma_{\xi T}}, \quad \nu = \frac{1}{\sigma} \sqrt{\frac{1 - \gamma_{\xi T}}{\gamma_{\xi T}}},
$$

(105)
and
\[
\int_0^\infty \exp \left[ -\lambda x + \left( \sigma \xi_t x - \frac{1}{2} \sigma^2 x^2 \gamma_T \right) (1 - \gamma_T)^{-1} \right] dx = \frac{1 - N_0(0, \mu, \nu)}{f(0, \mu, \nu)}. \tag{107}
\]

As a consequence we obtain:

**Proposition 4.** The variance-gamma information-based price of a financial asset with an exponentially distributed payoff is given by

\[
S_t = \frac{\mu - N_1(0, \mu, \nu)}{1 - N_0(0, \mu, \nu)}. \tag{108}
\]

In conclusion, we remark that the variance-gamma information model introduced in this paper can be calibrated to market data as follows. The distribution of the random variable \(X_T\) can be inferred by observing the current prices of derivatives for which the payoff is

\[
H_T = e^{rT} \mathbb{1}_{X_T \leq K} \tag{109}
\]

for \(K \in \mathbb{R}\). The information flow-rate parameter \(\sigma\) and the underlying shape parameter \(m\) can then be inferred from option prices.

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