ON THE DISTRIBUTION OF THE EICHLER–SHIMURA MAP AND THE
ZEROES OF PERIOD POLYNOMIALS

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Abstract. In this paper we determine the limiting distribution of the Eichler–Shimura map or equivalently the limiting joint distribution of the coefficients of the period polynomials associated to a fixed cusp form (as the element of the group varies). The limiting distribution is shown to be a product of two independent distributions, one of which is connected to the additive twist of the cuspidal $L$-function. Furthermore we determine the asymptotic behavior of the zeroes of the period polynomials in the same limit. We use the method of moments and the main ingredients in the proofs are additive twists of $L$-functions and bounds for both individual and sums of Kloosterman sums.

1. Introduction

Let $S_k(\Gamma_0(N))$ denote the space of cusp forms of even weight $k \geq 4$ and level $N$. The Eichler–Shimura map defines an $\mathbb{R}$-linear isomorphism between $S_k(\Gamma_0(N))$ and a parabolic cohomology group introduced by Eichler. In this paper we determine the asymptotic distribution of the image of a fixed cusp form under this map or equivalently the asymptotic joint distribution of the coefficients of the period polynomials of a fixed cusp form. We also determine an asymptotic expression for the zeroes of the period polynomials of a fixed cusp form, supplementing recent work of Jin, Ma, Ono, and Soundararajan [8], see also [4]. For $k = 2$ the period polynomials degenerate to constants and are known as modular symbols introduced by Birch and Manin. Petridis and Risager showed that modular symbols are asymptotically normally distributed [11], [12]. From a cohomological point of view the period polynomials are the natural generalization of modular symbols, but in this paper we show however that for $k \geq 4$ the coefficients of the period polynomials are not asymptotically normal in any sense.

To be more precise; to each cusp form $f \in S_k(\Gamma_0(N))$ and each

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N),$$

Eichler–Shimura associate [14, Chapter 8] the following $(k - 1)$-dimensional real vector

$$u_f(a/c) = u_f(\gamma) = \begin{pmatrix} \text{Re} \int_{\gamma \infty}^{\infty} f(z)dz, \text{Re} \int_{\gamma \infty}^{\infty} f(z)zdz, \ldots, \text{Re} \int_{\gamma \infty}^{\infty} f(z)z^{k-2}dz \end{pmatrix}^T \in \mathbb{R}^{k-1},$$

where $T$ denotes matrix transpose. The map $u_f : \Gamma \to \mathbb{R}^{k-1}$ can be shown to satisfy a 1-cocycle relation and an additional parabolic condition, which we will make precise.
Remark. The distribution is just the pullback by this continuous function of the Lebesgue measure on the critical values are just rational values of a continuous function and consequently the product space $[0, 1] \times [0, 1]$.

Theorem 1.1. Let $f \in S_k(\Gamma_0(N))$ be a cusp form of even weight $k \geq 4$ and level $N$. Then we have for any fixed subset $\Omega \subset \mathbb{R}^{k-1}$ that

$$
\frac{\# \{a/c \in T_{\leq 1,c} \mid (2\pi/e(k-1))^{k-2} u_f(a/c) \in \Omega \} \varphi(c)}{\mu([0,1] \times [0,1])} \to \mu([0,1] \times [0,1]) \circ F^{-1}(\Omega),
$$

as $c \to \infty$ with $c \equiv 0 \pmod{N}$. Here $\mu([0,1] \times [0,1])$ is the standard Lebesgue measure on the product space $[0, 1] \times [0, 1]$ and

$$
F : [0, 1] \times [0, 1] \to \mathbb{R}^{k-1},
$$

is given by

$$
F(x, y) := \text{Im} L(f \otimes e(x), k - 1) \left( y^{k-2}, \ldots, y, 1 \right)^T.
$$

Remark 1.2. As was noted in [1, Section 1.4.1] the individual distribution of the critical values of $L(f \otimes e(\gamma \infty), s)$ for $s \neq k/2$ are not that interesting since for $\Re s > (k + 1)/2$ the critical values are just rational values of a continuous function and consequently the distribution is just the pullback by this continuous function of the Lebesgue measure on $[0, 1]$.
the circle (and similarly for \( \Re s < (k + 1)/2 \) using the functional equation). In order to handle the distribution of the Eichler–Shimura map (or equivalently the coefficients of period polynomials) we however need to control the dependence between the different critical values of \( L(f \otimes e(\gamma \infty), s) \) and maps of the type \( \gamma \mapsto (\gamma \infty)^j \). In the end, the specific shape of the limiting distribution amounts to the non-trivial cancellation in sum of Kloosterman sums with uniformity in the frequencies and thus non-trivial input is needed.

**Remark 1.3.** Given an orthogonal basis \( f_1, \ldots, f_d \) for \( S_k(\Gamma_0(N)) \), we can also compute the joint distribution of

\[
\mathbf{u}_k := (u_{f_1}, \ldots, u_{f_d})^T \in \mathbb{R}^{d(k-1)},
\]

when appropriately normalized, with a similar proof. We have however restricted the exposition to a single cusp form \( f \) for notational simplicity. For the complete orthogonal basis the result is that the random variables defined from \( \frac{(2\pi/c)^{k-2}}{\Gamma(k-1)} \mathbf{u}_k \) converge in distribution to the random variable

\[
F_k(Y, Z),
\]

where \( Y, Z \) are two independent and uniformly distributed random variables on \([0, 1)\) and \( F_k : [0, 1) \times [0, 1) \to \mathbb{R}^{d(k-1)} \) is given by

\[
F_k(y, z) := \left( \Im L(f_1 \otimes e(y), k - 1) z^{k-2}, \ldots, \Im L(f_1 \otimes e(y), k - 1), \ldots, \Im L(f_d \otimes e(y), k - 1) \right)^T \in \mathbb{R}^{d(k-1)}.
\]

In particular it is worth noticing that \( u_{f_i}(\gamma) \) and \( u_{f_j}(\gamma) \) for \( i \neq j \) are highly dependent as opposed to the case \( k = 2 \) (see [9 Section 7.2]).

**Remark 1.4.** If \( f \in S_k(\Gamma_0(N)) \) then it follows from work of Jin, Ma, Ono and Soundararajan [8 Theorem 1.2] that for \( k \geq 6 \) the polynomials \( r_{f,s}(\sqrt{N}X) \) converge coefficient for coefficient to \( X^{k-2} - 1 \) as \( N \to \infty \).

**Remark 1.5.** The author [9] and independently Bettin and Drappeau [11] (for level 1) have considered the distribution of central values of additive twists of \( L \)-functions of cusp forms of arbitrary even weight and showed that they are normally distributed. As was also noted in [9 Remark 3.4] the coefficients of the period polynomial can be expressed as a linear combination of critical values of additive twists (including the central value). However the left-most critical value at \( s = 1 \) will be the dominating term, which is why we see that the distribution degenerates (and in particular is not normal).

### 1.2. Results for general \( \Gamma \)

We also obtain results for general cofinite, discrete subgroups \( \Gamma \) of \( \text{PSL}_2(\mathbb{R}) \) with a cusp at infinity of width 1 (see [17 Chapter 2] for details) when taking an extra average. To state our results, we introduce the following set;

\[
T_{\leq 1} = T_{\leq 1, \Gamma} := \{ r = \gamma \infty \in \mathbb{R} \mid \gamma \in \Gamma, 0 \leq r < 1 \}.
\]

This is a slight modification of the set \( T = T_\Gamma \) defined in [12], which parametrizes the double coset \( \Gamma_\infty \backslash \Gamma \). In this paper we need to choose a representative, since \( u_f(\gamma) \) is not invariant under the action of \( \Gamma_\infty \) from the left. One would get similar results by choosing different representatives.

Using the argument in the proof of [12 Proposition 2.2], we see that to any \( r \in T_{\leq 1} \) there is
a unique \( \gamma \in \Gamma / \Gamma_\infty \) with lower left entry \( c > 0 \) such that \( r = \gamma \infty \) and we define \( c(r) := c \). Following \([12]\) we define the following set:

\[
T_{\leq 1}(X) := \{ r \in T_{\leq 1} \mid c(r) \leq X \}.
\]

Given a general cusp form \( f \in S_k(\Gamma) \) we can similarly define the additive twists of the associated \( L \)-function, which satisfy the analogous properties as we explain in Section 2.3 below. In this setting our result is the following.

**Theorem 1.6.** Let \( f \in S_k(\Gamma) \) be a cusp form of even weight \( k \geq 4 \). Then we have for any fixed subset \( \Omega \subset \mathbb{R}^{k-1} \) that

\[
\frac{\# \{ r \in T_{\leq 1}(X) \mid \frac{(2\pi/c(r))^{k-2}}{\Gamma(\frac{k-2}{2})} u_f(r) \in \Omega \}}{\# T_{\leq 1}(X)} \rightarrow \mu_{(0,1) \times (0,1)}(F^{-1}(\Omega)),
\]

as \( X \rightarrow \infty \), with \( \mu_{(0,1) \times (0,1)} \) and \( F : [0,1) \times [0,1) \rightarrow \mathbb{R}^{k-1} \) as in Theorem \([12]\).

1.3. **Zeroes of period polynomials.** The vector \( u_f \) encodes the periods of \( f \in S_k(\Gamma) \), which were introduced in a slightly different setting by M. Eichler in his study of parabolic cohomology \([5]\). He defined the period polynomials associated to \( f \) as

\[
r_{f,\gamma}(X) := \frac{1}{(k-1)!} \int_{\gamma^{-1} \infty}^\infty f(z)(z - X)^{k-2} dz,
\]

where \( \gamma \in \Gamma \). Note that the entries of \( u_f(\gamma^{-1}) \) are the real parts of the coefficients of this polynomial (up to a scaling by factorials). The Eichler–Shimura isomorphism can also be described intrinsically and naturally in terms of period polynomials as was done in \([10]\). Our results can be interpreted as determining the joint distribution of the coefficients of the period polynomials (with \( \gamma \) replaced by \( \gamma^{-1} \)). Recently there has been a lot of study in the analytic properties of period polynomials, especially the location of the zeroes of \( r_{f,S} \), where

\[
S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},
\]

(see \([4]\) for a complete list of references). The results of this paper should be seen more in relation with these results rather than with those of Petridis and Risager \([12]\).

For \( f \in S_k(\Gamma_0(N)) \) a primitive new form of weight \( k \geq 6 \), we can also asymptotically determine the zeroes of \( r_{f,a/c} \) as \( c \rightarrow \infty \). The assumptions on \( f \) are made in order to ensure that \( L(f \otimes \epsilon(x), k - 1) \) is non-zero for all \( x \in \mathbb{R} \).

**Theorem 1.7.** Let \( f \in S_k(\Gamma_0(N)) \) be a primitive new form of even weight \( k \geq 6 \) and level \( N \). Then \( r_{f,\gamma} \) is a polynomial of degree \( k - 2 \) for any \( \gamma \in \Gamma_0(N) \). Furthermore all zeroes \( x_0 \) of \( r_{f,\gamma} \) satisfy

\[
x_0 = d/c + O_k\left(\left\lfloor d/c \right\rfloor + 1\right)^{\frac{(k-3)}{(k-2)} c^{-2/(k-2)}}
\]

where \( (c, d) \) is the bottom row of \( \gamma \) and \(-d/c = \gamma^{-1} \infty\).

**Remark 1.8.** Analogously Jin, Ma, Ono and Soundararajan \([8]\, \text{Theorem 1.2}\) building on works of others (see \([4]\)) determined the zeroes of \( r_{f,S} \) as either the weight \( k \) or level \( N \) tend to infinity. In their case the zeroes satisfy a version of the Riemann Hypothesis, of which no analogue seems to exist in our setting.
I would like to express my gratitude to Dorian Goldfeld and Columbia University for their hospitality and to my advisor Morten Risager and Riccardo Pengo for valuable suggestions.

2. Preliminaries and Background

In this section we will introduce some background on respectively the Eichler–Shimura isomorphism, bounds on sums of Kloosterman sums and finally additive twists of modular $L$-functions.

2.1. Background on the Eichler–Shimura isomorphism. The purpose of this section is to show how $u_f$ (equivalently the periods of $f$) appears "in nature". We will argue that from a cohomological point of view, $u_f$ defines the natural higher weight analogue of modular symbols. We will follow the exposition in [14, Chapter 8] below.

Let $G$ be any group and let $X$ be a left $\mathbb{Z}[G]$-module. Then one can define cohomology groups;

$$H^i(G, X) := Z^i(G, X)/B^i(G, X),$$

consisting of a quotient of certain maps

$$u: G \times \ldots \times G \to X,$$

corresponding to a specific choice of injective resolution.

In particular for $i = 1$ we have the following explicit description

$$Z^1(G, X) = \{ u: G \to X \mid u(g_1g_2) = u(g_1) + g_1u(g_2), \forall g_1, g_2 \in G \},$$

$$B^1(G, X) = \{ v: G \to X \mid \exists x_v \in X \text{ such that } v(g) = (g - 1)x_v, \forall g \in G \}.$$

Now fix a subset $P \subset G$ and consider

$$Z^1_P(G, X) := \{ u \in Z^1(G, X) \mid u(p) \in (p - 1)X, \forall p \in P \},$$

which we note still contains the boundaries $B^1(G, X)$. From this we define the first $P$-cohomology group as;

$$H^1_P(G, X) := Z^1_P(G, X)/B^1(G, X).$$

In our case we consider $G = \Gamma$ a discrete, co-finite subgroup of $\text{PSL}_2(\mathbb{R})$. We now have the canonical action of $\Gamma$ on $\mathbb{R}^2$ given by

$$\gamma(x, y)^T := (ax + by, cx + dy)^T.$$

This can be extended to a unique action of $\Gamma$ on $\mathbb{R}^n$ characterized by the property

$$\gamma((x, y)^n)^T = ((\gamma(x, y))^n)^T,$$

where $(x, y)^n := (x^{n-1}, x^{n-2}y, \ldots, y^{n-1})$.

We denote by $X$ the set $\mathbb{R}^{k-1}$ with the associated $\mathbb{Z}[\Gamma]$-module structure. Finally we let $P$ be the set of parabolic elements of $\Gamma$ and form Eichler’s parabolic cohomology group $H^1_P(\Gamma, X)$. We note that parabolic cohomology groups carry a natural Hecke action.

Now given a cusp form $f \in S_k(\Gamma)$ of weight $k$, we can define a map $u_f : \Gamma \to \mathbb{R}^{k-1}$ by sending $f$ to its periods;

$$u_f(\gamma) := \left( \text{Re} \int_{\gamma \infty}^\infty f(z)dz, \text{Re} \int_{\gamma \infty}^\infty f(z)zdz, \ldots, \text{Re} \int_{\gamma \infty}^\infty f(z)z^{k-2}dz \right),$$
and it can be shown that \( u_f \in Z^1_p(\Gamma, X) \). The main theorem of Eichler–Shimura \([14]\) Theorem 8.4] is now that the \( \mathbb{R} \)-linear map

\[
f \mapsto \{ \text{cohomology class of } u_f \} \in H^1_p(G, X)
\]
gives an isomorphism between \( S_k(\Gamma) \) and \( H^1_p(G, X) \), which carries a natural action of the Hecke algebra (see the seminal paper \([2]\) for a purely algebraic proof of these facts).

In fact one can define modular symbols associated to \( S_k(\Gamma) \) for all weights \( k \) \([15]\) Section 1.2], and show that the parabolic cohomology groups \( H^1_p(G, X) \) are isomorphic to the cuspidal modular symbols (see \([15]\) Theorem 5.2.1] for details).

2.2. Spectral bounds of sums of Kloosterman sums. An important ingredient when proving our main results is the cancellation in Kloosterman sums. For arithmetic subgroups we have very strong bounds for individual Kloosterman sums from Weil’s work on the Riemann Hypothesis over finite fields, but for general Fuchsian groups of the first kind, we only have non-trivial bounds when we average over the moduli.

To be more precise let \( \Gamma \) be a co-finite, discrete subgroup of \( \text{PSL}_2(\mathbb{R}) \) with a cusp at infinity of width 1. Then we define the Kloosterman sum with frequencies \( m, n \) and modulus \( c \) as:

\[
S(m, n; c) := \sum_{\gamma \in \Gamma_{\infty} \backslash \Gamma} e \left( \frac{d}{m} + \frac{a}{c} \right)
\]

where \( c \) is the lower-left entry of some matrix \( \gamma \in \Gamma \). It can be shown that

\[
\# \left\{ \gamma \in \Gamma_{\infty} \backslash \Gamma \mid 0 \leq c \leq X \right\} \ll X^2,
\]

which yields the following trivial bound

\[
S(m, n; c) \ll c^2,
\]

uniformly in \( m, n \), see \([7]\) Proposition 2.8]. If \( \Gamma = \Gamma_0(N) \) is a Hecke congruence group, we can do much better by Weil’s bound;

\[
|S(m, n; c)| \leq d(c)c^{1/2}(m, n, c)^{1/2}.
\]

The point is now that if we average over the moduli \( c \), we can also detect cancelation in Kloosterman sums for general \( \Gamma \).

The most powerful tools for obtaining bounds for sums of Kloosterman sums come from the spectral theory of automorphic forms following an approach initiated by Selberg. We refer to \([7]\) for a comprehensive background on the spectral theory of automorphic forms.

In this approach the spectrum of the automorphic Laplacian \( \Delta = \Delta_{\Gamma} \) plays a prominent role. It can be shown that \( \Delta \) is a non-negative unbounded operator with \( \lambda = 0 \) as an eigenvalue corresponding to the constant function. Furthermore the famous Selberg conjecture predicts that for congruence subgroups \( \Gamma_0(N) \) the first non-zero eigenvalue is \( \geq 1/4 \). It is known that there exists non-congruence subgroups \( \Gamma \) such there \( \Delta_{\Gamma} \) has non-zero eigenvalues arbitrarily close to 0.

For \( n = 0 \) the Kloosterman sum is a generalization of the classical Ramanujan sum and the \( m \)th Fourier coefficient of the Eisenstein series;

\[
E(z, s) = E_{\Gamma}(z, s) = \sum_{\gamma \in \Gamma_{\infty} \backslash \Gamma} \text{Im}(\gamma z)^s
\]
is exactly
\[
\Gamma(s)\zeta(2s)^{-1}\sum_{c > 0} \frac{S(m, 0; c)}{c^{2s}}.
\]
Recall that by the general theory of Eisenstein series due to Selberg, \(E(z, s)\) has its rightmost pole at \(s = 1\), which is a simple pole with residue \(\text{vol}(\Gamma)^{-1}\). All the other finitely many poles in \(1/2 < \Re s < 1\) are also simple and the residues are eigenfunctions for \(\Delta\). Combining this with standard complex analysis one gets

\[
\#T_{\leq 1}(X) = \frac{X^2}{\text{vol}(\Gamma)} + O(X^{2-\delta_\Gamma}),
\]
for some \(\delta_\Gamma > 0\) depending on the spectral gap for \(\Gamma\).

Furthermore since the pole at \(s = 1\) of the Eisenstein series has constant residue, it follows that for \(m \neq 0\) the Dirichlet series

\[
\sum_c S(m, 0; c) c^{2s},
\]
where the sum is over lower-left entries of matrices in \(\Gamma\), has analytic continuation to \(\Re s > \Re s_1 \geq 1/2\) where \(\lambda_1 = s_1(1 - s_1)\) is the smallest non-zero eigenvalue. From this one easily proves

\[
\sum_{c \leq X} S(m, 0; c) \ll_{\Gamma} |m|^{1/2}X^{2-\delta_\Gamma},
\]
for some \(\delta_\Gamma > 0\) (see [12, (3.6)]).

For \(mn \neq 0\) the corresponding Dirichlet series

\[
\sum_c S(m, n; c) c^{2s},
\]
shows up in the Fourier coefficients of the Poincaré series

\[
P_m(z, s) = \sum_{\gamma \in \Gamma \cap \Gamma} e(m\gamma z)(\text{Im }\gamma z)^s,
\]
as was brilliantly used by Goldfeld and Sarnak in [6] to obtain bounds on sums of Kloosterman sums. Using analytic properties of the resolvent of \(\Delta_\Gamma\), they show that \(P_m(z, s)\) has meromorphic continuation with possible poles only at the spectrum of \(\Delta_\Gamma\) and from this they obtain bounds for sums of Kloosterman sums. For our applications the dependence on \(m, n\) is essential, but this dependence is not clear from the statement of their theorem [6 Theorem 2]. However using [6] Remark 1] one can adapt their arguments to deduce the bound

\[
\sum_{c \leq X} S(n, m; c) \ll_{\Gamma} mnX^{2-\delta_\Gamma},
\]
for some \(\delta_\Gamma > 0\) depending on the spectral gap of \(\Gamma\). We will omit the details.

### 2.3. Additive twists

The idea behind the proofs of the main theorems is to relate the periods of \(f \in S_k(\Gamma)\) to critical values of additive twists of the \(L\)-function of \(f\). The additive twists are defined as

\[
L(f \otimes e(r), s) := \sum_{n \geq 1} \frac{a_f(n)e(nr)}{n^s},
\]
where \( r \in \mathbb{R} \) and \( e(x) = e^{2\pi i x} \) which a priori converges for \( \text{Re} \ s > (k + 1)/2 \) by Hecke’s bound \((12)\). If \( r \) corresponds to a cusp of \( \Gamma \) then \( L(f \otimes e(r), s) \) satisfies analytic continuation by the integral representation:

\[
L(f \otimes e(r), s) = \frac{(2\pi)^s}{\Gamma(s)} \int_0^\infty f(r + iy)y^s \frac{dy}{y}.
\]

Furthermore if \( r = a/c = \gamma_{\infty} \) with

\[
\gamma = \begin{pmatrix} a \\ c \\ b \\ d \end{pmatrix} \in \Gamma,
\]

the completed \( L \)-function satisfies the following functional equation

\[
\Lambda(f \otimes e(a/c), s) := \Gamma(s) \left( \frac{c}{2\pi} \right)^s L(f \otimes e(a/c), s) = (-1)^k \Lambda(f \otimes e(-d/c), k - s),
\]

where \(-d/c = \pi = \gamma_{-\infty}^{-1}.\)

The relation between the periods of \( f \) and additive twists is given by the following.

**Lemma 2.1.** Let \( l \in \mathbb{Z}_{\geq 0} \) be a non-negative integer. Then we have

\[
\int_{\gamma_{\infty}}^\infty f(z) z^j dz = \sum_{j=0}^l \binom{l}{j} (a/c)^{l-j} (-2\pi i)^{-j-1} \Gamma(j+1) L(f \otimes e(a/c), j+1),
\]

where \( a/c = \gamma_{\infty}. \)

**Proof.** By a straightforward computation we have

\[
\int_{\gamma_{\infty}}^\infty f(z) z^j dz = i \int_0^\infty f(a/c + it)(a/c + it)^l dt
\]

\[
= \sum_{j=0}^l (a/c)^{l-j} j^{j+1} \Gamma(j+1) L(f \otimes e(a/c), j+1),
\]

as wanted. \( \square \)

It turns out that the dominating term for all of these periods will be the left-most critical value;

\[
L(f \otimes e(a/c), 1),
\]

which is hinted to by the following proposition.

**Proposition 2.2.** Let

\[
\gamma = \begin{pmatrix} a \\ c \\ b \\ d \end{pmatrix} \in \Gamma.
\]

Then we have

(i) \( L(f \otimes e(a/c), \sigma) \ll 1 \) for \( \sigma \geq k/2 + 1, \)

(ii) \( L(f \otimes e(a/c), k/2) \ll c^\sigma, \)

(iii) \( L(f \otimes e(a/c), \sigma) \ll c^{k-2\sigma} \) for \( \sigma \leq k/2 - 1. \)
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Proof. Case (i) For \( \sigma \geq k/2 + 1 \) we get by Hecke’s bound [1.2.3] the following uniform bound:

\[
L(f \otimes e(a/c), \sigma) \ll \sum_{n \geq 1} \frac{|a_f(n)|}{n^\sigma} \leq \sum_{n \geq 1} \frac{|a_f(n)|}{n^{k/2+1}} < \infty,
\]

which is independent of \( a/c \) and \( \sigma \).

Case (ii) The bound on the central value was proved by the author [9, Corollary 5.5].

Case (iii) Finally for \( \sigma \leq k/2 - 1 \), we get by the functional equation (2.5) the following:

\[
L(f \otimes e(a/c), \sigma) = \frac{\Gamma(k - \sigma)(2\pi)^{-k+\sigma}}{\Gamma(\sigma)(2\pi)^{-\sigma}} e^{k-2\sigma} L(f \otimes e(-d/c), k - \sigma),
\]

and since \( k - \sigma \geq k/2 + 1 \) the result follows from (i). Observe that we avoid the poles of the \( \Gamma \)-function in the numerator.

\[\square\]

3. ON THE ZEROES OF THE PERIOD POLYNOMIALS

In this section we will apply the bounds in Proposition 2.2 to determine the asymptotic behavior of the zeroes of the period polynomials associated to a fixed cusp form as the denominator of the cusp varies.

Let \( f \in S_k(\Gamma_0(N)) \) be a fixed primitive form of even weight \( k \geq 6 \). Consider the period polynomials associated to \( f \):

\[
r_{f,\gamma}(X) = \frac{1}{(k-1)!} \int_{i \gamma - i \infty}^\infty f(z)(z - X)^{k-2} dz = b_{f,k-2}(\gamma) X^{k-2} + \ldots + b_{f,0}(\gamma),
\]

where \( \gamma \in \Gamma \). We have the following bound on the Fourier coefficients of \( f \) due to Deligne [3]:

\[
|a_f(n)| \leq d(n)n^{(k-1)/2},
\]

which implies that

\[
\sum_{n \geq 2} \frac{|a_f(n)|}{n^{k-1}} \leq \sum_{n \geq 2} \frac{d(n)}{n^{(k-1)/2}} = \zeta((k-1)/2)^2 - 1 \leq \zeta(5/2)^2 - 1 = 0.799... < 1.
\]

This shows that \( L(f \otimes e(x), k - 1) \) is bounded both from above and away from zero uniformly in \( x \in \mathbb{R} \). Thus \( r_{f,\gamma} \) is actually a polynomial of degree \( k - 2 \) and it makes sense to define

\[
\tilde{b}_i(\gamma) = \tilde{b}_{f,i}(\gamma) := b_{f,i}(\gamma)/b_{f,k-2}(\gamma), \quad i = 0, \ldots, k - 2.
\]

We can now prove the promised asymptotic expression for the zeroes of \( r_{f,\gamma} \) as \( c \to \infty \).

Proof of Theorem 1.2. Let \( \gamma = -1/\infty = -d/c \). First of all by using (2.6) and the bounds from Proposition 2.3 we conclude the following:

\[
(3.1) \quad \tilde{b}_i(\gamma) = \binom{k-2}{i} \rho^{-k-2-i} + O_k(|\rho|^{k-3-i} c^{-2}),
\]

which in particular implies \( \tilde{b}_i(\gamma) \ll_k |\rho|^{k-2-i} \).

Now we will show that any zero \( x_0 \) of \( r_{f,\gamma} \) is bounded by \( O_k(|\rho|) \). So assume that a zero \( x_0 \) of \( r_{f,\gamma} \) satisfies \( |x_0| \geq |\rho| \). Then using (3.1), we get the bound

\[
|x_0|^{k-2} = |\tilde{b}_{k-3}(\gamma)x_0^{k-3} - \ldots - \tilde{b}_0(\gamma)| \ll_k |\rho||x_0|^{k-3},
\]
which implies \( x_0 \ll_k |\tau| \) as wanted.

Now combining \( x_0 \ll_k |\tau| \) with (3.1), we conclude for any root \( x_0 \) of \( r_f, \gamma \) we have that
\[
0 = x_0^{-k-2} + \tilde{b}_{k-3}(\gamma)x_0^{-k-3} + \ldots + \tilde{b}_0(\gamma) = (x_0 + \tau)^{-k-2} + O_k((1 + |x_0|)^{k-3}c^{-2}),
\]
which implies that \( |x_0 + \tau| \ll_k (1 + |\tau|)^{(k-3)/(k-2)}c^{-2/(k-2)} \) as wanted. \( \square \)

If we restrict to \( \gamma \in \Gamma_\infty \setminus \Gamma_0(N) \) such that \( \tau = \gamma^{-1} \in T_{\leq 1} \) (with notation as in (1.5)) we conclude that the zeroes of \( r_{f,\gamma} \) satisfy
\[
x_0 = -\tau + O_k(c^{-2/(k-2)}).
\]
In particular when, say, \( \tau \gg 1/\log c \), then \( -\tau \) is the main term above. As \( c \to \infty \) it is clear that \( \tau \in T_{\leq 1} \) satisfies \( \tau \gg 1/\log c \) with probability one. We will omit the details but from the above one can easily deduce the following.

**Corollary 3.1.** Let \( f \in \mathcal{S}_k(\Gamma_0(N)) \) be cusp form of weight \( k \geq 6 \) and level \( N \). Then we have for any fixed subset \( \Omega \subset \mathbb{R}^{k-1} \) that
\[
\#\{\gamma^{-1} \in T_{\leq 1} \mid (\tilde{b}_{f,0}(\gamma), \ldots, \tilde{b}_{f,k-2}(\gamma))^T \in \Omega\}
\]
\[
\overset{\phi(c)}{\to} \mu_{(0,1)}(\tilde{F}^{-1}(\Omega)),
\]
as \( c \to \infty \) with \( c \equiv 0 \mod N \), where \( \mu_{(0,1)} \) is the Lebesgue measure on \([0,1)\) and \( \tilde{F} : [0,1) \to \mathbb{R}^{k-1} \) is given by
\[
\tilde{F}(x) = \left( x^{k-2}, \frac{k-2}{k-3}x^{k-3}, \ldots, \frac{k-2}{1}x, 1 \right)^T.
\]

4. On the distribution of the Eichler–Shimura map

In this section we will prove Theorem 1.1 and Theorem 1.6 using the method of moments. More precisely this is done by firstly computing all the moments of the random variable \( u_f \) on respectively \( T_{\leq 1} \) and \( T_{\leq 1}(X) \) and then applying a result from probability theory due to Fréchet–Shohat to determine the limiting distribution.

4.1. Computation of the moments of \( u_f \). We will actually compute all the complex moments and then deduce the real ones by taking linear combinations.

To state our results we let (as above)
\[
f(z) = \sum_{n \geq 1} a_f(n)q^n;
\]
be the Fourier expansion of a cusp form \( f \in \mathcal{S}_k(\Gamma) \). Then we define the following Dirichlet series for \( \alpha, \beta \in \mathbb{Z}_{\geq 0} \);
\[
L_{f,\alpha,\beta}(s) := \sum_{n_1, \ldots, n_\alpha, n_\beta > 0} \frac{a_f(n_1) \cdots a_f(n_\alpha)\overline{a_f(n_{\alpha+1})} \cdots \overline{a_f(n_{\alpha+\beta})}}{(n_1 \cdots n_{\alpha+\beta})^s}
\]
\[
= \int_0^1 L(f \otimes e(x), s) x^{\alpha,\beta} dx,
\]
which converges absolutely for \( \text{Re} s > (k + 1)/2 \) by Hecke’s bound (1.2), where we use the notation \( z^{\alpha,\beta} = z^\alpha \overline{z}^\beta \).

For \( \Gamma = \Gamma_0(N) \) a Hecke congruence group, we get the following moments, where all implied constants might depend on \( f \).
Theorem 4.1. Let \( f \in S_k(\Gamma_0(N)) \) be a cusp form of even weight \( k \geq 4 \). Then for any non-negative integers:
\[
\alpha_0, \ldots, \alpha_{k-2}, \beta_0, \ldots, \beta_{k-2},
\]
not all zero and \( c \equiv 0 \mod (N) \), we have that
\[
\frac{1}{\varphi(c)} \sum_{0 \leq a < c, j = 0}^{k-2} \prod_{(a,c)=1}^{\infty} \left( \frac{(2\pi/c)^{k-2}}{\Gamma(k-1)i} \int_{a/c}^\infty f(z)z^{j}dz \right)^{\alpha_j,\beta_j}
= \frac{L_{f,\alpha,\beta}(k-1)}{1 + \sum_{j=0}^{k-2} j \cdot (\alpha_j + \beta_j)} + O_{\varepsilon,\alpha,\beta}(c^{-1/6+\varepsilon}),
\]
(4.1)
where \( \alpha = \alpha_0 + \ldots + \alpha_{k-2} \) and \( \beta = \beta_0 + \ldots + \beta_{k-2} \).

For general \( \Gamma \) we have to take an extra average in order to calculate the moments.

Theorem 4.2. Let \( f \in S_k(\Gamma) \) be a cusp form of even weight \( k \geq 4 \). Then for any non-negative integers:
\[
\alpha_0, \ldots, \alpha_{k-2}, \beta_0, \ldots, \beta_{k-2},
\]
not all zero, we have that
\[
\frac{1}{\#T_{\leq 1}(X)} \sum_{r \in T_{\leq 1}(X)} \prod_{j=0}^{k-2} \left( \frac{(2\pi/c(r))^k}{\Gamma(k-1)i} \int_{r}^\infty f(z)z^{j}dz \right)^{\alpha_j,\beta_j}
= \frac{L_{f,\alpha,\beta}(k-1)}{1 + \sum_{j=0}^{k-2} j \cdot (\alpha_j + \beta_j)} + O_{\alpha,\beta}(X^{-\delta_\Gamma}),
\]
for some \( \delta_\Gamma > 0 \) depending on the spectral gap of \( \Gamma \), where \( \alpha = \alpha_0 + \ldots + \alpha_{k-2} \) and \( \beta = \beta_0 + \ldots + \beta_{k-2} \).

Observe that these moments are exactly what we expect from the statements of Theorem 1.1 and Theorem 1.6.

Proof of Theorem 4.1 and Theorem 4.2. In the following all implied constants may depend on \( f, \alpha \) and \( \beta \). In view of (2.5) we can express the periods of \( f \) as a linear combination of critical values of the additive twists \( L(f \otimes e(r), s) \) and by the functional equation, we have the equality
\[
L(f \otimes e(r), 1) = e(r)^{k-2} \frac{\Gamma(k-1)}{(2\pi)^{k-2}} L(f \otimes e(s), k-1)
\]
with \( r = \gamma \infty \) and \( \tau = \gamma^{-1} \infty \). Using Proposition 2.2, this implies that
\[
\prod_{j=0}^{k-2} \left( \frac{(2\pi/c(r))^{k-2}}{\Gamma(k-1)i} \int_{r}^\infty f(z)z^{j}dz \right)^{\alpha_j,\beta_j}
= L(f \otimes e(s), k-1)^{\alpha,\beta} + O(c(r)^{-2})
\]
(4.3)
where \( z^{\alpha,\beta} = z^{\alpha,\beta} \) and
\[
N = N(\alpha_1, \ldots, \alpha_{k-2}, \beta_1, \ldots, \beta_{k-2}) := \sum_{j=0}^{k-2} j \cdot (\alpha_j + \beta_j).
\]

In order to deal with the term \( r^N \), we apply a standard smooth approximation. So let
$\varphi : \mathbb{R} \to \mathbb{R}$ be a smooth function with compact support in $(0, 1)$ such that $\int_0^1 \varphi(x)dx = 1$. Then we define the following approximation to the Dirac measure at $x = 0$;

$$\varphi_\delta(x) := \delta^{-1}\varphi(x/\delta),$$

where $\delta > 0$ is some small constant to be chosen. We think of $\varphi_\delta$ as a function on the circle $S^1$ by extending its values on $[0, 1)$ periodically, where we use the model $[0, 1]/(0 \sim 1)$ for $S^1$.

Associated to the periodic functions $h_j : S^1 \to \mathbb{R}$ defined by $h_j(x) = x^j$ for $x \in [0, 1)$, we define the following smooth approximation;

$$h_{j,\delta} := h_j * \varphi_\delta,$$

where $*$ denotes the (additive) convolution product on $S^1$. The convolution $h_{j,\delta}$ satisfies the following standard properties;

$$\hat{h}_{j,\delta}(l) \ll_A \frac{1}{(\delta(1 + |l|))^A}, \quad \hat{h}_{j,\delta}(0) = \hat{h}_j(0) = \frac{1}{j+1},$$

where $A > 0$ and $\hat{h}_{j,\delta}$ denotes the Fourier transform on $S^1$. And furthermore

$$h_{j,\delta}(x) = h_j(x) + O(\delta),$$

for $\delta \leq x < 1$. This estimate fails for $0 \leq x \leq \delta$, but it is standard to show that the contribution from $r \in T_{\leq 1}(X)$ (respectively $r \in T_{\leq 1,c}$) with $r < \delta$ is negligible and will not affect the error term\footnote{More precisely it is obvious that $\{r \in T_{\leq 1,c} \mid r < \delta\} \ll \delta c$ and by using the cancellation in Kloosterman sums one can also show that $\{r \in T_{\leq 1}(X) \mid r < \delta\} \ll \delta X^2$.}. So we can replace $r^N$ by the approximation $h_{N,\delta}$ at the cost of changing the error term in (4.3) to

$$O(\delta + c(r)^{-2}).$$

Finally we replace $h_{N,\delta}$ by its Fourier expansion to arrive at the following expression for the main term;

$$L(f \otimes e(\overline{r}), k-1)^{\alpha,\beta} h_{N,\delta}(r)$$

$$= \sum_{l \in \mathbb{Z}} \hat{h}_{N,\delta}(l) e(lr) L(f \otimes e(\overline{r}), k-1)^{\alpha,\beta}$$

$$= \sum_{l \in \mathbb{Z}} \hat{h}_{N,\delta}(l) \sum_{n_1,\ldots,n_{\alpha + \beta} > 0,} a_f(n_1) \cdots a_f(n_{\alpha}) a_f(n_{\alpha+1}) \cdots a_f(n_{\alpha+\beta})$$

$$\times e(lr + \overline{r}(n_1 + \ldots + n_\alpha - n_{\alpha+1} - \ldots - n_{\alpha+\beta})),
\tag{4.5}$$

using that $L(f \otimes e(\overline{r}), k-1)$ is absolutely convergent and so is the Fourier expansion of $h_{N,\delta}$ in view of (4.4).

Now the case where $\Gamma = \Gamma_0(N)$ is a Hecke congruence group, we average (4.3) over $r \in T_{\leq 1,c}$. Since all of the $r$-dependence is in the exponential, we see the Kloosterman sums entering the picture. The main contribution comes from the diagonal terms corresponding to $l = 0$ and $n_1 + \ldots + n_\alpha = n_{\alpha+1} + \ldots + n_{\alpha+\beta}$, which contribute

$$L_{f,\alpha,\beta}(k-1) \hat{h}_{N,\delta}(0) = L_{f,\alpha,\beta}(k-1) \frac{1}{N + 1}. 
\tag{4.6}$$
In order to handle the off-diagonal contributions, we apply Weil’s bound \(2.2\), which bounds the off-diagonal terms by the following:

\[
\ll \frac{d(c)^{1/2}}{\varphi(c)} \sum_{l \neq 0} \sum_{n_1, \ldots, n_{\alpha + \beta}} \left| h_{N, \delta}(l) \right| \frac{|a_f(n_1) \cdots a_f(n_{\alpha + \beta})|}{(n_1 \cdots n_{\alpha + \beta})^{k-1}} \left( l, c, \sum_{i=1}^{\alpha} n_i - \sum_{j=\alpha+1}^{\beta} n_j \right)^{1/2}
\]

\[
\ll_{\epsilon, \alpha, \beta} \frac{c^{1/2+\epsilon}}{\varphi(c)} \left( \sum_{l \neq 0} \left| h_{N, \delta}(l) \right| \right) \left( \sum_{n_1, \ldots, n_{\alpha + \beta}} \frac{|a_f(n_1) \cdots a_f(n_{\alpha + \beta})|}{(n_1 \cdots n_{\alpha + \beta})^{k-1}} \max(n_1, \ldots, n_{\alpha + \beta})^{1/2-\epsilon} \right)
\]

\[
\ll_{\epsilon, \alpha, \beta} \frac{c^{1/2+\epsilon}}{\varphi(c)} \sum_{l \neq 0} \left| h_{N, \delta}(l) \right|
\]

using Hecke’s bound \(1.2\) to show finiteness of the sum over \(n_1, \ldots, n_{\alpha + \beta}\). Combining the above with the fact that \(\varphi(c) \gg c^{1-\epsilon}\), we arrive at the following:

\[
\frac{1}{\varphi(c)} \sum_{0 \leq a < c} \prod_{j=0}^{k-2} \left( \frac{2\pi/c}{\Gamma(k-1)} \int_{a/c}^{\infty} f(z) z^{j} dz \right)^{\alpha \cdot \beta} = L_{f, \alpha, \beta}(k-1) h_{N, \delta}(0) + O_{\epsilon} \left( \delta + c^{-2} + c^{-1/2+\epsilon} \sum_{l \neq 0} \left| h_{N, \delta}(l) \right| \right).
\]

Next we apply \(4.4\) with \(A = 2 + \epsilon\) to ensure convergence of the sum \(\sum_{l \neq 0} \left| h_{N, \delta}(l) \right|\) and arrive at the following error term;

\[
O_{\epsilon} (\delta + c^{-2} + c^{-1/2+\epsilon} \delta^{-2-\epsilon})
\]

Finally we choose \(\delta = c^{-1/6}\) to balance the error terms.

The argument for general \(\Gamma\) is similar, only now we average \(4.3\) over \(r \in T_{\leq 1}(X)\). Again the main contribution is given by \(4.6\).

When dealing with the off-diagonal contribution, we first of all have to trivially bound the terms in \(4.3\) with \(\min(n_1, \ldots, n_{\alpha + \beta}) > X^{\delta_1}\) for some \(\delta_1 > 0\) to be chosen appropriately. This is necessary since the dependence on the frequencies in \(2.4\) is not as strong as in Weil’s bound (actually this extra step is only needed when \(k = 4\)).

Now using the trivial bound for the exponentials, this truncation yields

\[
\frac{1}{\# T_{\leq 1}(X)} \sum_{r \in T_{\leq 1}(X)} L(f \otimes e(\overline{H}), k-1) h_{N, \delta}(r)
\]

\[
= \sum_{\hat{h}_{N, \delta}(l)} \sum_{0 < n_1, \ldots, n_{\alpha + \beta} < X^{\delta_1}} \frac{a_f(n_1) \cdots a_f(n_{\alpha}) a_f(n_{\alpha + 1}) \cdots a_f(n_{\alpha + \beta})}{(n_1 \cdots n_{\alpha + \beta})^{k-1}}
\]

\[
\times \frac{1}{\# T_{\leq 1}(X)} \sum_{r \in T_{\leq 1}(X)} e(\xi(r + \overline{H}(n_1 + \ldots + n_{\alpha} - n_{\alpha + 1} - \ldots - n_{\alpha + \beta}))) + O(X^{-\delta_1(k-3)/2}).
\]
Now we apply the bound for sums of Kloosterman sums \([2.4]\) which yields the following bound for the remaining off-diagonal contribution from \([4.8]\):

\[
\ll_{\alpha, \beta} X^{-\delta_1} \left( \sum_{l} |\hat{h}_{N, \delta}(l)| \cdot |l| \right) \times \sum_{0 < n_1, \ldots, n_{\alpha+\beta} < X^{\delta_1}} \frac{|a_f(n_1) \cdots a_f(n_{\alpha+\beta})|}{(n_1 \cdots n_{\alpha+\beta})^{k-1}} \max(n_1, \ldots, n_{\alpha+\beta})
\]

\[
\ll_{\alpha, \beta} X^{-\delta_1} \left( \sum_{l} |\hat{h}_{N, \delta}(l)| \cdot |l| \right) \max(1, X^{-\delta_1(k-5)/2}) ,
\]

using also that \(# T_{\leq 1}(X) \gg X^2\) by \([2.3]\).

Now we apply \([4.4]\) with \(A = 3 + \varepsilon\) to ensure finiteness of the first sum above and then choose \(\delta\) and \(\delta_1\) to balance the error terms. This yields a power savings, which we will not make explicit. This finishes this case as well. \(\square\)

4.2. **Determining the limiting distribution.** In order to conclude the proofs of Theorem 1.1 and Theorem 1.6 we need to setup our problem in a probability theoretical framework.

Let \(f \in S_k(\Gamma)\) be as above and consider the following normalization of the periods of \(f\):

\[
\tilde{u}_{f,i}(r) := \frac{(2\pi / c(r))^k - r}{\Gamma(k - 1)} u_{f,i}(r), \quad i = 0, \ldots, k - 2 ,
\]

where \(r = \gamma \infty\) with \(\gamma \in \Gamma\). According to whether \(\Gamma\) is a congruence subgroup or not, we consider for each \(c \equiv 0 \pmod{N}\) (respectively \(X > 0\)) the following random variable:

\[
\tilde{u}_f := (\tilde{u}_{f,0}, \ldots, \tilde{u}_{f,k-2}),
\]

defined on the outcome space \(T_{\leq 1,c}\) (respectively \(T_{\leq 1}(X)\)) endowed with the discrete \(\sigma\)-algebra and the uniform measure. Then one can easily check (by taking linear combinations of the complex moments) that Theorem 4.1 (respectively Theorem 4.2) implies that as \(c \to \infty\) (respectively \(X \to \infty\)), the moments of the random variable \(\tilde{u}_f\) converge to those of the random variable

\[
F(Y, Z),
\]

where \(Y, Z\) are two independent random variables uniformly distributed with respect to the Lebesgue measure on \([0, 1)\) and \(F : [0, 1] \times [0, 1) \to \mathbb{R}^k\) is given (as in Theorem 4.1) by

\[
F(y, z) = \text{Im} L(f \otimes e(y), k - 1)(z^{k-2}, \ldots, z, 1)^T .
\]

In order to conclude that the random variables associated with \(\tilde{u}_f\) converge in distribution to \(F(Y, Z)\) as \(c \to \infty\) (respectively \(X \to \infty\)), we will combine three results from probability theory due to Fréchet–Shohat, Cramér–Wold and Carleman. A similar but slightly simpler argument was carried out in [9, Section 5.3].

**Proof of Theorem 1.1 and Theorem 1.6.** Given a sequence of 1-dimensional random variables \((X_n')_{n \geq 1}\) such that all moments exist and converge as \(n \to \infty\) to the moments of some other random variable \(Y'\) then it follows from the Fréchet–Shohat Theorem [13, page 17] that if \(Y'\) is uniquely determined by its moments then the random variables \((X_n')_{n \geq 1}\) converge in distribution to \(Y'\).

Our random variables are however multidimensional so we have to combine the Fréchet–Shohat Theorem with a result of Cramér and Wold [13, page 18], which says that if \((X_n')_{n \geq 1}\) is a sequence of \((d + 1)\)-dimensional random variables:

\[
X_n' = (X_{n,0}', \ldots, X_{n,d}'),
\]
and \( Y' = (Y_0', \ldots, Y_d') \) is a \((d + 1)\)-dimensional random variable such that
\[
t_0 X'_{n,0} + \ldots + t_d X'_{n,d}
\]
converge in distribution as \( n \to \infty \) to
\[
t_0 Y_0' + \ldots + t_d Y_d'
\]
for any \((d + 1)\)-tuple \((t_0, \ldots, t_d) \in \mathbb{R}^{d+1} \), then \( X'_n \) converges in distribution to \( Y' \) as \( n \to \infty \).

Thus by combining Fréchet–Shohat and Cramér–Wold with our calculation of the moments in Theorem 4.1 (respectively Theorem 4.2), it is enough to show that for any (say non-trivial) linear combination, the following random variable:
\[
(4.9) \quad t_0 \text{Im} L(f \otimes e(Y), k - 1) + t_1 \text{Im} L(f \otimes e(Y), k - 1)Z + \ldots + t_{k-2} \text{Im} L(f \otimes e(Y), k - 1)Z^{k-2}
\]
is uniquely determined by its moments. By a condition due to Carleman (see (4.10) below), this boils down to showing that the moments are sufficiently bounded from above, which is clear in our case since \( Z \) is bounded by 1 and
\[
|\text{Im} L(f \otimes e(Y), k - 1)| \leq \sum_{n \geq 1} \frac{|a_f(n)|}{n^{k-1}} < \infty,
\]
both with probability one. To sum up and be precise; if we denote by \( \alpha_{2m} \) the \( 2m \)’th moment of (4.9), then we have
\[
(4.10) \quad \sum_{m \geq 1} \alpha_{2m}^{-1/2m} \geq \sum_{m \geq 1} \left( c(t_0, \ldots, t_{k-2}) \sum_{n \geq 1} \frac{|a_f(n)|}{n^{k-1}} \right)^{2m}^{-1/2m} = \infty,
\]
where \( c(t_0, \ldots, t_{k-2}) \) is a certain constant depending on \( t_0, \ldots, t_{k-2} \). Thus it follows from the Carleman condition [13, page 46] that the random variable (4.9) is uniquely determined by its moments. Thus we conclude the proof of Theorem 1.1 and Theorem 1.6 using the results of Fréchet–Shohat and Cramér–Wold mentioned above.

\[\square\]

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