Continuum random combs and scale-dependent spectral dimension

Max R Atkin, Georgios Giasemidis and John F Wheater

Rudolf Peierls Centre for Theoretical Physics, 1 Keble Road, Oxford OX1 3NP, UK

E-mail: m.atkin1@physics.ox.ac.uk, g.giasemidis1@physics.ox.ac.uk and j.wheater1@physics.ox.ac.uk

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Abstract
Numerical computations have suggested that in causal dynamical triangulation models of quantum gravity (CDT) the effective dimension of spacetime in the ultraviolet (UV) is lower than in the infrared (IR). In this paper we develop a simple model based on the previous work on random combs, which share some of the properties of CDT, in which this effect can be shown to occur analytically. We construct a definition for short- and long-distance spectral dimensions and show that the random comb models exhibit scale-dependent spectral dimension defined in this way. We also observe that a hierarchy of apparent spectral dimensions may be obtained in the cross-over region between UV and IR regimes for suitable choices of the continuum variables. Our main result is valid for a wide class of tooth length distributions thereby extending previous work on random combs by Durhuus et al.

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1. Introduction

This work was motivated by observations made in some formulations of quantum gravity which we will explain shortly. However, the question as to whether it is possible to define consistently a spectral dimension which depends on the scale at which a system is probed by a random walk is of more general interest. In this paper, we will construct a definition for such a spectral dimension and show that there are models which do indeed exhibit scale-dependent spectral dimension defined in this way.

The demonstration by ’t Hooft and Veltman that quantized general relativity is perturbatively non-renormalizable in four dimensions [1] led to the search for non-perturbative formulations of quantum gravity and there are now several approaches to this problem. It was first advocated by Weinberg [2] that there might be a non-trivial ultraviolet fixed point and this has been pursued in continuum calculations by a number of authors [10, 11]. There is
now quite a lot of evidence for such a fixed point although it is not conclusive; precisely because GR is perturbatively non-renormalizable it is necessary to make some assumptions or ultimately uncontrolled approximations in these calculations. An alternative approach within the fixed-point philosophy is to discretize spacetime and to look for a critical point or line where a continuum limit may be taken to recover continuum gravity.

Early attempts based on the so-called Euclidean quantum gravity model (see for instance [6]) did not lead to a continuum limit in four dimensions but the situation improved with the introduction of the causal dynamical triangulation model (CDT) by Ambjørn and Loll in 1998 [4]. This defines the gravitational path integral as a sum over discretized geometries (see [5] for a recent review). In contrast with the earlier Euclidean quantum gravity model, the CDT approach takes account of the Lorentzian nature of the path integral by building in a well-defined temporal structure from the start. As an example of the success of the CDT approach, numerical simulations have shown that in four-dimensional CDT a large scale structure in the form of a four-dimensional de Sitter universe emerges purely from quantum fluctuations [7, 8]. This is a highly non-trivial result keeping in mind that one is dealing with a background-independent formulation. Other approaches to quantum gravity include string theory, loop quantum gravity and causal sets but it is the results on the nature of spacetime obtained using the CDT and fixed-point calculations that particularly concern us here.

To discuss the nature of a quantum spacetime at any distance scale requires a quantitative measure which is universal; that is to say that it can be defined in any of the models we are interested in and is insensitive to cut-off scale physics while conveying information about the longer distance structure. The simplest such characterizations are various definitions of dimension of which the most familiar is the Hausdorff dimension. The Hausdorff dimension $d_h$ is defined provided the volume $V(R)$ of a ball of radius $R$ takes the form

$$ V(R) \sim R^{d_h} $$

if $R$ is large enough. An alternative probe of structure comes from the behaviour of unbiased random walks (equivalently diffusion) in whatever ensemble of spacetimes is being considered. The probability $P(t)$ that the walk returns to its starting point after time $t$ provides one of the simplest probes of the nature of spacetime in quantum gravity models. The spectral dimension $d_s$ is defined if

$$ P(t) \sim t^{-d_s/2} $$

For which values of $t$ this should be true is a subtlety with which we will be concerned in this paper. For random walks on discretized graphs where the walker is allowed to hop from one vertex to a neighbour at each time step, behaviour of the form (2) is expected at $t \to \infty$ i.e. when the walk is much longer than the short distance cut-off scale. On the other hand in the continuum the classic picture is that (2) describes the behaviour as $t \to 0$. Of course these are not, at least in principle, incompatible as they can be related by a rescaling.

An unexpected result from the numerical simulation of CDTs is that the spectral dimension apparently varies from 4 at large scales to 2 at small scales [9]. Very recently similar results for this phenomenon of dimensional reduction have also been observed by other approaches such as the exact renormalization group [10, 11], Horava–Lifshitz gravity [12, 13], and in three-dimensional CDT [14] and some further implications are discussed in [15]. Such behaviour is not a priori implausible in quantum gravity because there is a dimensionful parameter, namely Newton’s constant or equivalently the Planck length. Studying the spectral dimension at different distance scales raises questions of definition and in the case of numerical simulations, discretization problems. In a numerical simulation, the largest available distance scale is determined by what will fit in the computer and short distance scales are often not much greater than the ultraviolet cut-off, or discretization scale. Ideally there should be a hierarchy
in which the long distance scale is much greater than the short distance scale which is in turn much greater than the cut-off. In this paper, we develop a simple model based on previous work on random combs [3]. These are a family of simple geometrical models which share some of the properties of the CDT model; instead of an ensemble of triangulations we have an ensemble of graphs consisting of an infinite spine with teeth of identically independently distributed length hanging off (we define these graphs precisely in section 2). It was shown in [3] that the spectral dimension is determined by the probability distribution for the length of the teeth. In this paper, we show that it is possible to extend the work of [3] by taking a continuum limit thus ensuring that the cut-off scale is much shorter than all physical distance scales. We find that the spectral dimension is 1 if we take the physical distance explored by the random walk to 0 and there exist a number of continuum limits in which the long distance spectral dimension differs from its short distance counterpart. As a by-product of this work we also extend some of the proofs given in [3] to a wider class of probability distributions.

This paper is organized as follows. In section 2, we briefly review some known results for combs and their spectral dimension and then explain how in principle these can be extended to show different spectral dimensions at long- and short-distance scales. In section 3, we introduce a simple model which we prove does in fact exhibit a spectral dimension that is different in the UV and IR. This model forms the basis of all later generalizations. In section 4, we generalize the results of section 3 to combs in which teeth of any length may appear with a probability governed by a power law. In section 5, we examine the possibility of intermediate scales in which the spectral dimension differs from both its UV and IR values. In section 6, we analyse the case of a comb in which the tooth lengths are controlled by an arbitrary probability distribution and show that continuum limits exist in which the short distance spectral dimension is 1 while the long distance spectral dimension can assume values in one-to-one correspondence with the positions of the real poles of the Dirichlet series generating function for the probability distribution. We then show how these techniques can be used to extend the results of [3]. In section 7, we discuss our results and possible directions for future work. Some technical matters are contained in the appendices.

2. Combs and walks

In this section, we review some basic facts about random combs and random walks. As much as possible we use the same notation and conventions as [3] and refer to that paper for proofs omitted here.

2.1. Definitions

We use the definition of a comb given in [3]. Consider the nonnegative integers regarded as a graph, which we denote $N_{\infty}$, so that $n$ has the neighbours $n \pm 1$ except for 0 which only has 1 as a neighbour. Furthermore, let $N_\ell$ be the integers $0, 1, \ldots, \ell$ regarded as a graph so that each integer $n \in N_\ell$ has two neighbours $n \pm 1$ except for 0 and $\ell$ which only have one neighbour, 1 and $\ell - 1$, respectively. A comb $C$ is an infinite rooted tree graph with a special subgraph $S$ called the spine which is isomorphic to $N_{\infty}$ with the root at 0. At each vertex of $S$, except the root 0, one of the graphs $N_\ell$ or $N_{\infty}$ is attached there. We adopt the convention that these linear graphs which are glued to the spine are attached at their endpoint 0. The linear graphs attached to the spine are called the teeth of the comb, see figure 1. We will find it convenient to say that a vertex on the spine with no tooth has a tooth of length 0. We will denote by $T_n$ the tooth attached to the vertex $n$ on $S$, and by $C_k$ the comb obtained by removing the links $(0, 1), \ldots, (k - 1, k)$, the teeth $T_1, \ldots, T_k$ and relabelling the remaining vertices
on the spine in the obvious way. The number of nearest neighbours of a vertex \( v \) will be denoted \( \sigma(v) \).

It is convenient to give names to some special combs which occur frequently. We denote by \( C = \ast \) the full comb in which every vertex on the spine is attached to an infinite tooth, and by \( C = \infty \) the empty comb in which the spine has no teeth (so an infinite tooth is itself an example of \( C = \infty \)).

Now let \( C \) denote the collection of all combs and define a probability measure \( \nu \) on \( C \) by letting the length of the teeth be identically and independently distributed by the measure \( \mu \). We will refer to the set \( C \) equipped with the probability measure \( \nu \) as a random comb.

Measurable subsets \( A \) of \( C \) are called events and \( \nu(A) \) is the probability of the event \( A \). The measure of the set of combs \( A \) with teeth at \( n_1, n_2, \ldots, n_k \) having lengths \( \ell_1, \ell_2, \ldots, \ell_k \) is

\[
\nu(A) = \prod_{j=1}^{k} \mu(\ell_j) .
\]

For any \( \nu \)-integrable function \( F \) defined on \( C \), we define the expectation value

\[
\langle F(C) \rangle = \int F(C) \, d\nu .
\]

We will often use the shorthand \( \bar{F} \) for \( \langle F(C) \rangle \).

2.2. Random walks

We consider a simple random walk on the comb \( C \) and count the time \( t \) in integer steps. At each time step the walker moves from its present location at vertex \( v \) to one of the neighbours of \( v \) chosen with equal probabilities \( \sigma(v)^{-1} \). Unless otherwise stated the walker always starts at the root at time \( t = 0 \).

The generating function for the probability \( p_C(t) \) that the walker is at the root at time \( t \), having left it at \( t = 0 \), is defined by

\[
Q_C(x) = \sum_{t=0}^{\infty} (1-x)^{t/2} p_C(t) .
\]

and we denote by \( P_C(x) \) the corresponding generating function for the probability that the walker returns to the root for the first time, excluding the trivial walk of length 0. Since walks returning to the root can be decomposed into walks returning for the first, second, etc time, we have

\[
Q_C(x) = \frac{1}{1 - P_C(x)} .
\]
It is convenient to consider contributions to $P_C(x)$ and $Q_C(x)$ from walks which are restricted. Let $P_C^{(n)}(x)$ denote the contribution to $P_C(x)$ from walks whose maximal distance along the spine from the root is $n$ and define

$$P_C^{(<n)}(x) = \sum_{k=0}^{n-1} P_C^{(k)}(x)$$

which is the contribution from all walks which do not reach the point $n$ on the spine. Similarly we define

$$P_C^{(>n)}(x) = \sum_{k=n}^{\infty} P_C^{(k)}(x).$$

Clearly $P_C(x)$ can be recovered from $P_C^{(n)}(x)$ by setting $n \to \infty$. We define the corresponding restricted contributions to $Q_C(x)$ in the same way. By decomposing walks contributing to $P_C^{(<n)}(x)$ into a step to 1, walks returning to 1 without visiting the root, and finally a step back to the root, it is straightforward to show that

$$P_C^{(<n)}(x) = \frac{1 - x}{3 - P_{T_k}(x) - P_C^{(<n-1)}(x)},$$

where we have adopted the convention that for the empty tooth, $T = \emptyset$,

$$P_{\emptyset}(x) = 1.$$  

Relation (9) can be used to compute the generating function explicitly for any comb with a simple periodic structure and we list some standard results in appendix A. There are a number of elementary lemmas which characterize the dependence of $P_C(x)$ on the length of the teeth and the spacing between them [3]. We state them here in a slightly generalized form which is useful for our subsequent manipulations.

**Lemma 1.** The function $P_C^{(<n)}(x)$ is a monotonic increasing function of $P_{T_k}(x)$ and $P_C^{(<n-k)}(x)$ for any $n > k \geq 1$.

**Lemma 2.** $P_C^{(<n)}(x)$ is a decreasing function of the length, $\ell_k$, of the tooth $T_k$ for any $n > k \geq 1$.

**Lemma 3.** Let $C'$ be the comb obtained from $C$ by swapping the teeth $T_k$ and $T_{k+1}$, $k < n - 1$. Then $P_C^{(<n)}(x) > P_C'^{(<n)}(x)$ if and only if $P_{T_k}(x) > P_{T_{k+1}}(x)$.

The proofs use (9) and follow those given in [3] for the case $n = \infty$.

An important corollary, valid for any comb, of these lemmas is that

$$x^{-\frac{1}{4}} \leq Q_C(x) \leq x^{-\frac{1}{2}},$$

which we will refer to as the trivial upper and lower bounds on $Q_C(x)$. The result follows from lemma 2 with $n = \infty$, which gives

$$P_*(x) \leq P_C(x) \leq P_\infty(x),$$

and the explicit expressions for $P_*(x)$ and $P_\infty(x)$ given in appendix A.
2.3. Two-point functions

Two-point correlation functions on the comb correspond to the probability of a walk beginning at the root being at a particular vertex on the spine at time \(t\). In particular, let \(p_C(t; n)\) denote the probability that a random walk that starts at the root at time zero is at the vertex \(n\) on the spine at time \(t\) having not visited the root in the intervening period. We will refer to the generating function for these probabilities as the two-point function, \(G_C(x; n)\), and define it by

\[
G_C(x; n) = \sum_{t=1}^{\infty} (1 - x)^{t/2} p_C(t; n).
\]

(13)

\(G_C(x; n)\) may be expressed as

\[
G_C(x; n) = \sigma(n)(1 - x)^{-n/2} \prod_{k=0}^{n-1} P_C^k(x)
\]

(14)

which may be used in conjunction with lemma 2 to obtain the bounds

\[
\frac{G_{*}(x; n)}{3} \leq \frac{G_C(x; n)}{\sigma(n)} \leq \frac{G_{\infty}(x; n)}{2}.
\]

(15)

Now let \(r_C(t; n)\) denote the probability that a random walk that starts at the root at time zero is at the vertex \(n\) on the spine for the first time at time \(t\) having not visited the root in the intervening time. We define the modified two-point function, \(G_C^0(x; n)\), by

\[
G_C^0(x; n) = \sum_{t=1}^{\infty} (1 - x)^{t/2} r_C(t; n)
\]

(16)

and note the following lemmas.

**Lemma 4.** The contribution \(P_C^{(>N)}(x)\) to \(P_C(x)\) from walks whose maximal distance from the root is \(N\) or greater satisfies

\[
P_C^{(>N-1)}(x) \leq 3x^{-1/2}G_C^0(x; N)^2.
\]

(17)

The proof is given in section 2.4 of [3].

**Lemma 5.** The modified two-point function satisfies

\[
G_{*}^{(0)}(x; n) \leq G_C^{(0)}(x; n) \leq G_{\infty}^{(0)}(x; n).
\]

(18)

To prove this note that

\[
G_C^{(0)}(x; n) = \frac{(1 - x)^{-(n-2)/2}}{\sigma(n-1)} \prod_{k=0}^{n-2} P_C^{(<n-k)}(x)
\]

(19)

and use lemma 2.
2.4. Spectral dimension and the continuum limit

The spectral dimension of random combs was studied in [3]. In this work, the probability distributions $\mu(\ell)$ for the length $\ell$ of a tooth were chosen to be fixed sets of numbers so the teeth at adjacent sites on the spine are not only independent but generally show large fluctuations relative to each other. In these circumstances, the spectral dimension, $d_s$, describes the large $t$ dependence of the return probability $p_C(t)$ and is given by

$$d_s = -2 \lim_{t \to \infty} \frac{\log(p_C(t))}{\log t}$$

if the limit exists. In fact, it is much more convenient to deal with generating functions and by a Tauberian theorem [17] we expect that if (20) holds then, as $x \to 0$,

$$Q_C(x) \sim x^{-1 + d_s/2},$$

where by $f(x) \sim g(x)$ we mean that

$$cg(x) \leq f(x) \leq c'g(x), \quad 0 < x < x_0,$$

where $c$, $c'$ and $x_0$ are positive constants. Property (21) was adopted in [3] as the definition of spectral dimension, assuming it exists. Heuristically the spectral dimension characterizes certain aspects of the long distance structure of a graph as observed by a walker who goes on a very long walk and hence probes that structure. The spectral dimension of an ensemble average is defined in the same way, simply replacing $p_C(t)$ and $Q_C(x)$ by their respective expectation values.

In this paper, we study the possibility of different spectral dimensions on different distance scales. To do this we have to generalize our definition from (20) or (21) and introduce at least one characteristic distance scale $L \gg 1$ into the probabilities $\mu(\ell)$ which determine the structure of the comb. We then assign the value $a$ to the distance between adjacent vertices in the graph and take the limit $a \to 0$ and $L \to \infty$ in such a way that the scaled combs have a finite characteristic distance scale; we will refer to this limit as the ‘continuum’ limit and quantities which exist in this limit as continuum quantities. Walks much longer than $L$ will probe different structure from walks much shorter than $L$ but nonetheless both can be very long in units of the underlying cut-off scale $a$.

In the following sections, we will denote the dependence of a function on a number of variables $L_i, i = 1, \ldots, N$, by $L_i$ passed as one of the function arguments. Given a random comb ensemble specified by $\mu(\ell; L_i)$ and the corresponding $\bar{Q}(x; L)$, we define

$$\hat{Q}(\xi; \lambda_i) = \lim_{a \to 0} a^{\Delta_\mu} \hat{Q}(a^{\Delta_\mu} \lambda_i; a^{-\Delta_\mu} L_i),$$

where the scaling dimensions $\Delta_\mu$ and $\Delta_\lambda$ are chosen to ensure a non-trivial limit and the combinations $\xi \lambda_i$ are dimensionless. $\hat{Q}$ can be used to define the spectral dimension at short and long distances.

In the following discussion, we assume for simplicity that there is just one scale $L$ and that the spectral dimension in the sense of (20) exists for $\langle p_C(t) \rangle$ which implies that there exists a constant $t_0$ such that $\langle p_C(t + 1) \rangle < \langle p_C(t) \rangle, t > t_0$. Note that

$$\sum_{t=0}^{T} \langle p_C(t) \rangle (1 - x)^{t/2} = \hat{Q}(x; L) - \sum_{t=T+1}^{\infty} \langle p_C(t) \rangle (1 - x)^{t/2}$$

$$= \hat{Q}(x; L) - (1 - x)^{(T+1)/2} \sum_{t=0}^{\infty} \langle p_C(t + T + 1) \rangle (1 - x)^{t/2}$$

$$> \hat{Q}(x; L) - (1 - x)^{(T+1)/2} \sum_{t=0}^{\infty} \langle p_C(t) \rangle (1 - x)^{t/2}. \quad (24)$$
Now choose
\[ T = \left\lfloor a^{-1} \frac{1}{\xi \log (1 + \frac{1}{\xi})} \right\rfloor - 1 \] (25)
and set \( x = a \xi \) and \( L = a^{-\Delta} \lambda \Delta \) in (24) to get
\[ a^{\Delta_r} \tilde{Q}(a \xi; a^{-\Delta} \lambda \Delta) (1 - \exp(-\xi \lambda)) < a^{\Delta_r} \sum_{t=0}^{T} \langle p_C(t) \rangle (1 - (1 - \xi a)^t)^{1/2} < a^{\Delta_r} \tilde{Q}(a \xi; a^{-\Delta} \lambda \Delta). \] (26)
Provided that the limit in (23) exists, we see that the behaviour of \( \tilde{Q}(\xi; \lambda) \) as \( \xi \to \infty \) characterizes the properties of walks of continuum time duration less than
\[ \lim_{\xi \to \infty} \frac{1}{\xi \log (1 + \frac{1}{\xi})} = \lambda, \] (27)
and we define the spectral dimension \( d_0^s \) at short distances by
\[ d_0^s = 2 \left( 1 + \lim_{\xi \to \infty} \frac{\log (\tilde{Q}(\xi; \lambda))}{\log \xi} \right), \] (28)
provided this limit exists.
We can define the spectral dimension at long distances in a similar way. First note that by (11)
\[ \sqrt{T} \geq \sum_{t=0}^{\infty} \langle p_C(t) \rangle \left( 1 - \frac{1}{T} \right)^{t/2} > \sum_{t=0}^{T} \langle p_C(t) \rangle \left( 1 - \frac{1}{T} \right)^{t/2} > \left( 1 - \frac{1}{T} \right)^T \sum_{t=0}^{T} \langle p_C(t) \rangle \] (29)
so that
\[ \tilde{Q}(x; L) - \sqrt{T} \left( 1 - \frac{1}{T} \right)^{-T} < \sum_{t=T+1}^{\infty} \langle p_C(t) \rangle (1 - x)^{t/2} < \tilde{Q}(x; L). \] (30)
This time letting \( T = [a^{-1} \xi^{-1} \log (1 + \xi \lambda)] - 1 \) we get
\[ a^{\Delta_r} \tilde{Q}(a \xi; a^{-\Delta} \lambda \Delta) - e^{\sqrt{T} \xi^{-1} \log (1 + \xi \lambda)} < a^{\Delta_r} \sum_{t=T+1}^{\infty} \langle p_C(t) \rangle (1 - (1 - \xi a)^t)^{1/2} \]
\[ < a^{\Delta_r} \tilde{Q}(a \xi; a^{-\Delta} \lambda \Delta). \] (31)
Provided that the limit in (23) exists and that \( \tilde{Q}(\xi; \lambda) \) diverges as \( \xi \to 0 \) we see that its behaviour characterizes the properties of walks of continuum time duration greater than
\[ \lim_{\xi \to 0} \xi^{-1} \log (1 + \xi \lambda) = \lambda. \] We then define the spectral dimension \( d_\infty^s \) at long distances to be
\[ d_\infty^s = 2 \left( 1 + \lim_{\xi \to 0} \frac{\log (\tilde{Q}(\xi; \lambda))}{\log \xi} \right), \] (32)
provided this limit exists.
It is by no means obvious that there are graph ensembles for which the limits (23) followed by (28) and (32) exist. However, in the rest of this paper, we will show for comb ensembles of increasing generality that this is indeed the case. Clearly at the very least any such ensemble must have a characteristic distance scale \( \lambda \) that survives the continuum limit otherwise such behaviour is impossible. In all the examples given in this paper, it turns out that the exponent \( \Delta_\mu = \frac{1}{2} \).
3. A simple comb

We now introduce a random comb whose spectral dimension differs on long and short length scales and thus illustrates that the behaviour described in section 2.4 can actually occur. This comb is defined by the measure

$$\mu(\ell; L) = \begin{cases} 
1 - \frac{1}{L}, & \ell = 0, \\
\frac{1}{L}, & \ell = \infty, \\
0, & \text{otherwise}.
\end{cases}$$  \hspace{1cm} (33)

This random comb has infinite teeth and they occur with an average separation of $L$. Intuitively we would expect that if a random walker did not move further than a distance of order $L$ from its starting position it would not see the teeth and therefore would measure a spectral dimension of 1. If however it were allowed to explore the entire comb, it would see something roughly equivalent to a full comb and so feel a much larger spectral dimension. To prove this intuition correct we proceed by computing upper and lower bounds for $\bar{Q}$ which are uniform in $L$ and for $0 < x < x_0$, where the constant $x_0$ is equal to 1 unless otherwise stated, and then take the continuum limit to obtain bounds for $\tilde{Q}$.

With complete generality we may obtain a lower bound on $\bar{Q}(x)$ by the use of Jensen’s inequality which takes the form

Lemma 6. Let $\bar{P}_T(x; L_i)$ be the mean first return probability generating the function of the teeth of the comb defined by $\mu(\ell; L_i)$; then,

$$\bar{Q}(x; L) \geq (1 + x - \bar{P}_T(x; L_i))^{-\frac{1}{2}}.$$  \hspace{1cm} (34)

The proof is given in [3]. For the comb (33), we have

$$\bar{P}_T(x; L) = 1 - \frac{1}{L} (1 - P_{\infty}(x)) = 1 - \frac{\sqrt{x}}{L}$$  \hspace{1cm} (35)

which implies

$$\bar{Q}(x; L) \geq \left( \frac{\sqrt{x}}{L} + x \right)^{-\frac{1}{2}}.$$  \hspace{1cm} (36)

Letting $x = a \xi$ and $L = a^{-\frac{1}{2}} \lambda^\frac{1}{2}$ gives

$$\bar{Q}(\xi; \lambda) = \lim_{a \to 0} a^\frac{1}{2} \bar{Q}(a \xi; a^{-\frac{1}{2}} \lambda^\frac{1}{2}) \geq \xi^{-\frac{1}{2}} (\nu^{-\frac{1}{2}} + 1)^{-\frac{1}{2}},$$  \hspace{1cm} (37)

where we have introduced the dimensionless variable $\nu = \xi \lambda$.

To find an upper bound on $\tilde{Q}(x; L)$, we follow [3] and use lemmas 1, 2 and 3 to compare a typical comb in the ensemble with the comb consisting of a finite number of infinite teeth at regular intervals. First we define the event

$$A(D, k) = \{C : D_i \leq D : i = 0, \ldots, k\},$$  \hspace{1cm} (38)

where $D_i$ is the distance between the $i$ and $i + 1$ teeth and then write

$$\tilde{Q}(x; L) = \int_C Q_C(x; L) \, dv = \int_{C \setminus A(D, k)} Q_C(x; L) \, dv + \int_{A(D, k)} Q_C(x; L) \, dv.$$  \hspace{1cm} (39)

Since the $D_i$ are independently distributed

$$\nu(A(D, k)) = (1 - (1 - 1/L)^D)^k.$$  \hspace{1cm} (40)
Consider a comb \( C \in A(D, k) \); then, by lemmas 1, 2 and 3,
\[
P_C(x; L) \leq P_C(x),
\]
where \( C' \) is the comb obtained by removing all teeth beyond the \( k \) tooth and moving the remaining teeth so that the spacing between each is \( D \). Now we can write
\[
P_C(x) = P_{C'}^{(\leq Dk)}(x) + P_{C'}^{(> Dk - 1)}(x).
\]
Since the walks contributing to \( P_{C'}^{(\leq Dk)}(x) \) do not go beyond the last tooth we have
\[
P_{C'}^{(\leq Dk)}(x) \leq P_{*D}(x),
\]
where \( *D \) denotes the comb consisting of infinite teeth regularly spaced and separated by a distance \( D \). Using (43), lemmas 4 and 5, we have
\[
P_C(x) \leq P_{*D}(x) + 3x^{-\frac{1}{2}}G(0)_{\infty}(x; Dk).
\]
Taking the continuum limit of (46) and using the results of appendix A then gives
\[
\tilde{Q}(\xi; \lambda) \leq \xi^{-\frac{1}{2}}F(\xi\lambda),
\]
where
\[
F(v) = \begin{cases} 
1 + o(v^{-1}), & v \to \infty, \\
\sqrt{\frac{1}{v^2} + o(v^2)}, & v \to 0.
\end{cases}
\]
It follows from (28), (32), (37) and (48) that
\[
d_0^0 = 1, \quad d_\infty^0 = \frac{3}{2}.
\]

4. Combs with power law measures

We now consider slightly more general combs in which the measure on the teeth is a power law of the form
\[
\mu(\ell; L) = \begin{cases} 
1 - \frac{1}{L}, & \ell = 0, \\
\frac{1}{L} C_\alpha \ell^{-\alpha}, & \ell > 0,
\end{cases}
\]
where \( C_\alpha \) is a normalization constant and as before \( L \) plays the role of a distance scale. We consider laws in the range \( 2 > \alpha > 1 \) as it is known that for \( \alpha \geq 2 \) the comb has spectral dimension \( d_s = 1 \) in the sense of (21) [3] and therefore it is not possible to get a spectral dimension deviating from 1 on any scale.

To compute a lower bound on the return probability generating function for the above distribution we apply lemma 6 and reduce the problem to computing an upper bound on \( 1 - \tilde{P}_T(x) \). The first return generating function \( P_T(x) \) for a tooth of length \( \ell \) is recorded in
bounding \( \tanh(u) \) above by the function \( f(u) = u \) for \( u < 1 \) and \( f(u) = 1 \) for \( u \geq 1 \) gives

\[
1 - \bar{P}_T(x; L_i) \leq \sqrt{x} \left( \sum_{\ell=1}^{[m_\infty(x)\ell]} \mu(\ell; L_i) \right) \]
\[
\leq \sqrt{x} - m_\infty(x) \sqrt{x} \int_0^{1/m_\infty(x)} \left( \sum_{\ell=0}^{[u]} \mu(\ell; L_i) \right) du.
\]

To obtain the second inequality, we have applied the Abel summation formula. We therefore have

Lemma 7. For a random comb defined by the measure \( \mu \),

\[
1 - \bar{P}_T(x; L_i) \leq \sqrt{x} - m_\infty(x) \sqrt{x} \int_0^{1/m_\infty(x)} \chi(u; L_i) du
\]

where the cumulative probability function \( \chi(u; L_i) \) is defined by \( \chi(u; L_i) = \sum_{\ell=0}^{[u]} \mu(\ell; L_i) \).

We will see shortly that all behaviour of the spectral dimension of the continuum comb is encoded in the asymptotic expansion of \( \chi(u; L) \) as \( u \) goes to infinity. In the present case, \( \chi(u; L) \) is trivially related to the partial sum of the Riemann \( \zeta \)-function whose leading asymptotic behaviour is well known and we find

\[
\chi(u; L) = 1 - C_\alpha \frac{u^{1-\alpha}}{L} + \delta(u),
\]

where

\[
|\delta(u)| < \frac{c}{L u^a}, \quad u \geq 2.
\]

It follows that for \( x < x_0 \), where \( m_\infty(x_0) = \frac{1}{2} \),

\[
1 - \bar{P}_T(x; L_i) \leq m_\infty(x) \sqrt{x} \left( \frac{b_1}{L} m_\infty(x)^{a-2} + \frac{b_2}{L} m_\infty(x)^{a-1} + \frac{b_3}{L} \right).
\]

with \( b_{1,2,3} \) being constants depending only on \( \alpha \) and \( b_1 > 0 \). Choosing \( L = a^{-\Delta^\prime} \lambda^{\Delta^\prime} \) with \( \Delta^\prime = 1 - \alpha/2 \) yields a lower bound on the continuum return generating function

\[
\hat{Q}(\xi, \lambda) \geq \xi^{-1/2}(1 + b_1 (\xi \lambda)^{-(1-\alpha/2)})^{-1/2}.
\]

To obtain a comparable upper bound we need

Lemma 8. For any random comb and positive integers \( H, D \) and \( k \), the return probability generating function is bounded above by

\[
\hat{Q}(x; L_i) \leq x^{-1/2} (1 - (1 - p)^D)^k + Q_U(x) (1 - (1 - p)^D)^k,
\]

where

\[
p = \sum_{\ell=H+1}^{\infty} \mu(\ell; L_i),
\]

\[
Q_U(x) = \left[ 1 - P_{H+D}(x) - 3 x^{-1/2} G_\infty^{(0)}(x; Dk)^2 \right]^{-1}.
\]

1 In this particular case, we could in fact compute \( 1 - \bar{P}_T(x) \) exactly by the Abel summation formula. However, the bound we use is good enough to give the desired result with the advantage that the calculation can be done with elementary functions.

2 In general, it is not obvious that this asymptotic expansion exists due to the discontinuous nature of \( \chi \). We will address this issue later when we consider generic measures.
and $P_{H,D}$ is the first return probability generating function for the comb with teeth of length $H + 1$ equally spaced at intervals of $D$.

The proof is a slight modification of the upper bound argument used in section 3. First define a long tooth to be one whose length is greater than $H$; then, the probability that a tooth at a particular vertex is long is

$$p = \sum_{\ell = H + 1}^{\infty} \mu(\ell; L_i). \quad (60)$$

Define the event

$$A(D, k) = \{ C : D_i \leq D : i = 0, ..., k \} \quad (61)$$

where now $D_i$ is the distance between the $i$ and $i + 1$ long teeth so that

$$\bar{Q}(x; L_i) = \int_{C} Q_C(x; L_i) d\nu = \int_{C / A(D,k)} Q_C(x; L_i) d\nu + \int_{A(D,k)} Q_C(x; L_i) d\nu. \quad (62)$$

Since the $D_i$ are independently distributed

$$\nu(A(D, k)) = (1 - (1 - p)^D)^k. \quad (63)$$

Now use lemmas 2 and 3 in turn to note that for

$$P_{C / A(D,k)}(x, L) \leq P_{C}(x, L) \quad (64)$$

where $C'$ is the comb in which all teeth but the first $k$ long teeth have been removed and the remaining long teeth have been arranged so that they have length $H$ and a constant inter-tooth distance $D$. By the same arguments as we used in section 3 to get (44) we obtain the bound

$$P_{C / A(D,k)}(x, L) \leq P_{H,\star,D} + 3x^{-1/2}G^{(0)}(x, Dk)^2. \quad (65)$$

Lemma 8 then follows from (11), (63) and (65). We now specialize to the power law measure (51) and set $H = \tilde{H}$, $D = \tilde{D}$ and $k = \tilde{k}$, where

$$\tilde{H} = x^{-1/2} \tilde{D} = (\Delta + 1)\frac{\alpha - 1}{c_{\alpha}} x^{\Delta - 1/2} L |\log x|^{1/\Delta'} \tilde{k} = (xL)^{1/\Delta'}$$

(66)

Using lemma 8, the scaling expressions for $P_{H,\star,D}$ and $G^{(0)}$ given in (A.7), and taking the continuum limit, gives, after a substantial amount of algebra,

$$\tilde{Q}(\xi ; \lambda) \leq \xi^{-1/2} F(\xi \lambda), \quad (67)$$

where

$$F(v) = \begin{cases} 1 + O(v^{-1}), & v \to \infty, \\ c_1 v^{1/2-a/4} \sqrt{|\log v^2|} + O(v^{1/4}), & v \to 0. \end{cases} \quad (68)$$

The main result of this section is

**Theorem 9.** The comb with the power law measure (51) for the tooth length has

$$d_s^0 = 1, \quad d_s^\infty = 2 - \frac{a}{2}. \quad (69)$$

The result follows immediately from (28), (32), (57) and (67).
5. Multiple scales

Given the results for the power law distribution, it is natural to investigate the behaviour for a random comb that has a hierarchy of length scales. The easiest way to achieve such a comb is through a double power law distribution,

\[ \mu(\ell; L_i) = \begin{cases} 
  1 - L_1^{-\alpha_1} - L_2^{-\alpha_2}, & \ell = 0, \\
  \frac{1}{L_1} C_1 l^{-\alpha_1} + \frac{1}{L_2} C_2 l^{-\alpha_2}, & \ell > 0.
\end{cases} \tag{70} \]

We may assume without loss of generality that the length scales \( L_i \) scale in the continuum limit to lengths \( \lambda_i \) such that \( \lambda_1 < \lambda_2 \) and that \( 1 < \alpha_i < 2 \).

Following the procedure of previous sections, a lower bound on \( \tilde{Q}(\xi; \lambda_i) \) is obtained by using lemma 6 and noting that \( \chi(x; L_i) \) for this comb is essentially the sum of the cumulative probability functions for each power law. This gives

\[ 1 - \tilde{P}_T(x; L_i) \leq m_\infty(x) \sqrt{\sum_{i=1}^2 \left( \frac{b_{1i}}{L_i} m_\infty(x)^{\alpha_1-2} + \frac{b_{2i}}{L_i} m_\infty(x)^{\alpha_2-1} + \frac{b_{3i}}{L_i} \right)}. \]

Choosing \( L_i \) to scale like \( L_i = a^{-\Delta_i} \lambda_i^{\Delta_i} \), where \( \Delta_i = 1 - \alpha_i/2 \), gives a bound on the continuum return generating function of

\[ \xi^{-1/2} (c_0 + c_1 (\xi \lambda_1)^{-(1-\alpha_1/2)} + c_2 (\xi \lambda_2)^{-(1-\alpha_2/2)})^{-1/2} \leq \tilde{Q}(\xi). \tag{72} \]

An upper bound on \( \tilde{Q}(\xi) \) is obtained by the application of lemma 8 in which we set \( H = [\tilde{D}] \), \( D = [\tilde{D}] \) and \( k = [\tilde{k}] \), where

\[ \tilde{H} = x^{-1/2} \]

\[ \tilde{D} = \beta x^{-1/2} G(x L_1^{1/\Delta_1}, x L_2^{1/\Delta_2})^{-1} \log x L_1^{1/\Delta_1} \]

\[ \tilde{k} = G(x L_1^{1/\Delta_1}, x L_2^{1/\Delta_2}) \]

and for convenience we have introduced the function

\[ G(v_1, v_2) = \frac{C_1}{\alpha_1 - 1} v_1^{-\Delta_1} + \frac{C_2}{\alpha_2 - 1} v_2^{-\Delta_2}. \tag{74} \]

Using lemma 8 and the scaling expressions in appendix A then gives

\[ \tilde{Q}(x; \lambda_i) \leq \xi^{-1/2} \left[ 1 - \left( 1 - v_1^{-s} \right)^{G} \right] \]

\[ + \frac{(1 - v_1^{-s})^{G-1}}{3 \cosh^2(|\log v_1^{s}|(1 - 1/G)) + \gamma + \sqrt{\gamma^2 + 1 + 2 \gamma \coth(|\log v_1^{s}|/G)}} \] \tag{75} \]

where \( v_1 = \xi \lambda_i \), \( s = \text{sgn}(\log v_1) \), \( \gamma = \tanh(1) \) and we have suppressed the arguments of \( G(v_1, v_2) \) in order to maintain readability.

We can now examine (72) and (75) to see what they tell us about the behaviour of \( \tilde{Q}(\xi; \lambda_i) \) on various length scales.

- When \( \xi \gg \lambda_i^{-1} \) both upper and lower bounds of \( \tilde{Q}(\xi; \lambda_i) \) are dominated by the \( \xi^{-1/2} \) behaviour, so taking the \( \xi \to \infty \) limit leads to \( d_i^\infty = 1 \) as in the previous sections.
- If \( \alpha_1 < \alpha_2 \) then when \( \xi \ll \lambda_i^{-1} \), both upper and lower bounds of \( \tilde{Q}(\xi; \lambda_i) \) are dominated by the \( \xi^{-\alpha_1/4} \) behaviour so taking the \( \xi \to 0 \) limit leads to \( d_i^\infty = 2 - \alpha_1/2 \). There is no regime in which \( \alpha_2 \) controls the behaviour.
If $\alpha_2 < \alpha_1$ then when $\xi \ll \Lambda_1^{-1}$ where

$$\Lambda_1^{-1} = \lambda_1^{(2-\alpha_1)/(\alpha_1-\alpha_2)} \lambda_2^{(2-\alpha_2)/(\alpha_2-\alpha_1)}$$

both upper and lower bounds of $\tilde{Q}(\xi; \lambda_i)$ are dominated by the $\xi^{-\alpha_i/4}$ behaviour so taking the $\xi \to 0$ limit leads to $d_0^\infty = 2 - \alpha_2/2$. However, there is an intermediate regime $\Lambda_1^{-1} \ll \xi \ll \lambda_1^{-1}$ where the $\xi^{-\alpha_1/4}$ behaviour dominates and $\tilde{Q}(\xi; \lambda_i)$ lies in the envelope given by

$$c_1 \xi^{-\alpha_1/4} < \tilde{Q}(\xi; \lambda_i) < c_2 \xi^{-\alpha_1/4} \sqrt{|\log \xi^\beta|},$$

where the upper and lower bounds will have corrections suppressed by powers of $\xi \lambda_2$ and the upper bound will also have corrections of order $\xi^\beta$. Both $\lambda_2$ and $\beta$ may be chosen to make the corrections arbitrarily small in this scale range. The system therefore appears to have spectral dimension $\delta_S = 2 - \alpha_1/2$ in this regime. This is a fairly weak statement because $\tilde{Q}(\xi)$ could in principle exhibit a wide variety of behaviours between its upper and lower bounds; this region is just a part of the crossover regime from $d_0^\infty$ to $d_0^\infty$. However, as we are free to chose $\lambda_2$ to be as large as we like compared to $\lambda_1$, this regime can exist over a scale range of arbitrarily large size. We therefore can force the leading behaviour of $\tilde{Q}(\xi; \lambda_i)$ in this range to be as close to a power law with exponent $\delta_S = 2 - \alpha_1/2$ as we like. This is what might be observed, for example, in a numerical simulation; if the difference between the scales $\lambda_1$ and $\lambda_2$ is large, then there will be a substantial range of walk lengths in which the data will indicate a spectral dimension of $\delta_S$. We will refer to a spectral dimension that appears in this weaker way as an apparent spectral dimension and denote it by $\delta_S$ rather than $d_S$.

6. Generic distributions

So far we have considered combs in which the distribution of tooth lengths has been governed by power laws or double power laws. In this section, we extend the results of the previous sections to the case where the form of the tooth length distribution is left arbitrary. The most general situation is that the measure on the combs is a continuous function of some parameters $w_i$. The continuum limit of such a comb is obtained in the usual way but with parameters $w_i$ scaling in a non-trivial way: $w_i = w_i + a^d \omega_i$. Given a random comb with such a measure, we would like to know how many distinct continuum limits exist and for each compute how the spectral dimension depends on the length scale.

The approach we adopt here closely mimics the arguments of the preceding sections; indeed the main complication is technical. As we have seen the properties of the continuum comb are controlled by the asymptotic expansion of $\chi(u)$ as $u$ goes to infinity. The main difference in the generic case is that we may arrange matters so that the scaling dimensions of the coefficients in the asymptotic expansion are such that sub-leading terms appear in the continuum. For the generic case we obviously have no way of knowing the full asymptotic expansion of $\chi(u; w_i)$. However, we will see that for a large class of measures, the form of the asymptotic expansion is encoded in the asymptotic expansion of a particular generating function for $\mu(\ell; w_i)$.

Our first task is to introduce this generating function and relate it to the asymptotic expansion of $\chi(u; w_i)$. To this end we introduce the notion of a smoothed sum [16]

$$\chi_{\pm}(u; w_i) = \sum_{\ell=0}^{\infty} \mu(\ell; w_i) \eta_{\pm}(\ell/u) = \mu(0; w_i) + \sum_{\ell=1}^{\infty} \mu(\ell; w_i) \eta_{\pm}(\ell/u) \equiv \mu(0) + \chi_{\pm}^{(1)}(u; w_i),$$

(78)
Figure 2. An illustration of the strip $\Sigma_1$ in which $D_\mu(s; w_i)$ satisfies property (1) given in the text. The poles of $D_\mu(s; w_i)$ are denoted by crosses and the horizontal strips in which the growth condition holds are indicated by the dark grey regions.

where $\eta_\pm$ is the smooth cut-off function introduced in appendix B and $u$ controls where the cut-off occurs. Such smoothed sums are related to $\chi(u; w_i)$ by

$$
\chi_-(u; w_i) \leq \chi(u; w_i) \leq \chi_+(u; w_i).
$$

The reason for introducing the smoothed sums is that we may use powerful techniques from the complex analysis to compute their asymptotic expansion (see e.g. [17]). The generating function to which the asymptotic expansion of the smoothed sums is related is the Dirichlet series generating function of $\mu(\ell; w_i)$:

$$
D_\mu(s; w_i) = \sum_{\ell=1}^{\infty} \frac{\mu(\ell; w_i)}{\ell^s}.
$$

We now introduce a number of results and notations.

- For a strip in the complex plane $\Sigma(b, a) \equiv \{z : a < \text{Re}[z] < b\}$ where $b > a$, we say $D_\mu(s; w_i)$ has slow growth in $\Sigma(b, a)$ if for all $s \in \Sigma$ we have $D_\mu(s; w_i) \sim O(s^r)$ for some $r > 0$ as $\text{Im}[s] \to \pm \infty$. We say $D_\mu(s; w_i)$ has weak slow growth if the above property only holds for a countable number of horizontal regions across the strip. See figure 2.

- Define $S_\pm$ to be the set containing the triples $(-\sigma + i\tau, -k, -r_\pm)$ such that the Laurent expansion of $D_\mu(s; w_i)M[\Psi_{\pm\epsilon}](s + 1)/s$ about the point $-\sigma + i\tau$ contains the term $r_\pm/(s+\sigma-i\tau)^k$ where $k > 0$ and $-\sigma+i\tau \neq 0$. Here, $M$ denotes the Mellin transformation and $\Psi_{\pm\epsilon}$ is introduced in appendix B. We will often find it useful to refer to the elements of $S_\pm$ using an index $j$ and denote the $j$th element of $S_\pm$ by $(-\sigma_j + i\tau_j, -k_j, -r_{j, \pm})$. Note that since $M[\Psi_{\pm\epsilon}](s + 1)$ is analytic in $\Sigma$ the positions of the poles are determined only by $D_\mu(s; w_i)$. 

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• Define the indexing sets $J_R$ and $J_C$, such that if $j \in J_R$ then $\tau_j = 0$ whereas if $j \in J_C$ then $\tau_j \neq 0$ and for both $-\sigma_j + i \tau_j \in \Sigma$, i.e. they index the poles in $\Sigma$ which lie on the real line and off the real line, respectively.

• For a Dirichlet series with positive coefficients, the abscissa of absolute and conditional convergence coincides. Furthermore, since $\sum_{j=0}^{\infty} \mu(\xi; w_i) = 1$, the abscissa of convergence is less than zero.

• Landau’s theorem: for a Dirichlet series with positive coefficients, there exists a pole at the abscissa of convergence. A corollary of this is that the coefficient of the most singular term in the Laurent expansion about the abscissa of convergence is positive and furthermore it is the rightmost pole of the Dirichlet series in the complex plane.

If $D_\mu(s; w_i)$ is of slow growth in $\Sigma(a, b)$ and no pole occurs to the right of $\Sigma(a, b)$, then by using expression (C.10) the asymptotic expansion of $\chi(\pm u; w_i)$ is

$$\chi(\pm u; w_i) = \mu(0; w_i) + \frac{1}{2\pi i} \oint_C u^j \frac{D_\mu(s; w_i) M[\Psi_{\pm \epsilon}](s + 1)}{s} ds - R(u; w_i)$$

$$= 1 - \sum_{j \in J_R} r_{j, \pm}(w_i) (\log u)^{k_j - 1} u^{-\sigma_j} - \frac{1}{2} \sum_{j \in J_C} (r_{j, +}(w_i) u^{i\tau_j} + r_{j, -}(w_i) u^{-i\tau_j}) (\log u)^{k_j - 1} u^{-\sigma_j} - R(u; w_i),$$

where $C$ is the rectangular contour introduced in appendix C, $r_{j, \pm}$ are the coefficients of the Laurent expansions of $D_\mu(s; w_i) M[\Psi_{\pm \epsilon}](s + 1)/s$, and $R(u)$ is a remainder function which satisfies $R(u) \sim u^{-N}$, where $N > \sigma_j$, as $u$ goes to infinity. Note that the difference between the coefficients $r_{j, +}$ and $r_{j, -}$ is of order $\epsilon$, the parameter introduced in appendix B, which can be taken to be arbitrarily small and so $r_{j, +}$ and $r_{j, -}$ are for all purposes equal.

We are now in a position to prove the main result of this section. Note that in the following, we require that the continuum comb is such that $\Delta_\mu = 1/2$ and so has spectral dimension 1 on the smallest scales. It seems very likely that this restriction could be lifted but we will not pursue such a generalization here.

**Theorem 10.** For a comb in which the teeth are distributed according to the measure $\mu$ if $D_\mu$ has slow growth in the strip $\Sigma = \{z : -1 < \text{Re}[z] \leq 0\}$ and has no poles on its line of convergence besides at the abscissa, then continuum limits of the comb exist with $d_1^0 = 1$ and $d_2^\infty$ taking values in the set $\{\frac{3-\sigma_j}{2} : 0 < \sigma_j < 1, j \in J_R\}$.

**Proof.** The proof proceeds in much the same way as the proofs in the previous sections; we first derive upper and lower bounds on $Q_C$ and then use these bounds to deduce the behaviour of $\tilde{Q}_C$ in different scale ranges.

We begin with the lower bound which by Jensen’s inequality amounts to finding an upper bound on $1 - \tilde{P}_T$. Note that (34) implies that the scaling dimension of $1 - \tilde{P}_T$ must be greater than or equal to 1 in order for $\Delta_\mu = 1/2$ and only contributes to the continuum limit if it has scaling dimension 1. From lemma 7 we have

$$1 - \tilde{P}_T(x; w_i) \leq m_\infty(x) \sqrt{\lambda} \int_0^{\frac{\pi}{\lambda}} (1 - \chi(\pm(u; w_i))) du$$

$$\leq \sqrt{\lambda} m_\infty(x) C(K; w_i) + m_\infty(x) \sqrt{\lambda} \int_K^{\frac{\pi}{\lambda}} (1 - \chi(\pm(u; w_i))) du,$$

\[ (82) \]

Since we have assumed that there are no poles to the right of $\Sigma$ we may choose $c = b$, where $c$ is the constant appearing in the definition of the contour.
where \( C(K) = \int_0^K (1 - \chi_(u; w_1)) du \) is a constant independent of \( x \). It is important to recall that since \( 0 \leq \chi_(u; w_1) \leq 1 \) for all values of \( u \) and \( w_1, \ldots, w_M \), in particular when \( w_1, \ldots, w_M \) assume their critical values, the scaling dimensions of the coefficients \( r_j, \pm \) appearing in (81) must be positive as otherwise \( \chi_(u; w_1) \) would diverge if we were to set \( w_1, \ldots, w_M \) to their critical values. Upon performing the integration in (82) we will find that a given \( r_j, \pm \) now is the coefficient for a number of terms of increasing scaling dimension. Since we require the scaling dimension of \( 1 - \tilde{P}_T \) to be greater than or equal to 1, only the term with the smallest scaling dimension can appear in the continuum limit and we will drop any term that does not appear in the continuum limit. If we now substitute (81) into (82) we find

\[
1 - \tilde{P}_T(x; w_i) \leq \sqrt{x}m_\infty(x)C(K; w_i) + \sqrt{x} \sum_{j \in j_R} \left( \frac{r_j,-(w_i)}{(k_j - 1)!} \right) \left( -\log m_\infty \right)^{\delta_j} \mu_j
\]

where the remainder term \( R \) disappears since it cannot appear in the scaling limit. We now suppose we may choose the critical values and scaling dimensions of the parameters \( w_i \) such that \( r_j, \pm \) has a scaling form that can be written as

\[
r_j,\pm = \left( \frac{a}{\lambda_j,\pm} \right)^{\theta_j} \left( -\frac{1}{2} \log a \right)^{\theta_j},
\]

where \( \theta_j, \tilde{\theta}_j \) and \( \lambda_j,\pm \) are constants and we have included the factor of \( \frac{1}{2} \) for later convenience. From (83) one can see that since we require \( \Delta_\mu = 1/2 \) the only terms which appear in the continuum limit are those for which \( \theta_j = (1 - \sigma_j)/2 \) and \( \tilde{\theta}_j = -(k_j - 1) \) and so it is useful to define a restricted indexing set

\[
\tilde{J}_R = \{ j \in J_R : \theta_j = (1 - \sigma_j)/2 \text{ and } \tilde{\theta}_j = -(k_j - 1) \}
\]

and an equivalent one \( \tilde{J}_C \) for \( J_C \). The continuum limit is thus

\[
1 - \tilde{P}_T(x; w_i) \leq aC \xi + \sum_{j \in j_R} \frac{a}{(k_j - 1)!} \xi \left( \xi \lambda_j,\pm \right)^{1-\sigma_j}
\]

\[
+ \sum_{j \in \tilde{J}_C} \frac{a}{(k_j - 1)!} \left( \text{Re}[\phi_j] \cos \left( \frac{T_j}{2} \log(a\xi) \right) + \text{Im}[\phi_j] \sin \left( \frac{T_j}{2} \log(a\xi) \right) \right)
\]

\[
\times \xi \left( \xi \lambda_j,\pm \right)^{1-\sigma_j}.
\]

Before deriving an upper bound on \( \tilde{Q}_C \), we must analyse the consequences of any of the oscillatory terms in the second sum appearing in the continuum limit (i.e. if \( \tilde{J}_C \) is non-empty). In the case of the double power law measure, we saw that one could obtain the intermediate behaviour in which an effective spectral dimension was measured that differed from the UV and IR spectral dimensions. A similar phenomenon occurs in the generic case when the various length scales of the continuum limit are well separated. If we scale the coefficients of the oscillatory terms such that these terms appear in the continuum limit, then we are led to an inconsistency by the following argument.

(1) Let the term associated with an oscillatory term have index \( j_0 \). Consider choosing the scaling form of \( x \) and \( r_j,\pm \) such that \( \xi |\lambda_j,\pm| \ll 1 \) but \( \xi |\lambda_j,\pm| \gg 1 \) if \( |\lambda_j,\pm| \gg |\lambda_j,\pm| \).
(2) If we were to take the scaling limit in such a scenario, then the term that dominates the size of $1 - \bar{P}_T$ is the one associated with the length scale $|\lambda_{j_{1,\pm}}|$. This term will be oscillating around a mean of zero and hence must be going negative infinitely often as we approach the continuum.

(3) Since $\bar{P}_T$ is a probability generating function, it cannot be negative, hence showing we have an unphysical limit.

The cause of this behaviour is that we declared by fiat that the parameters $w_i$ must be scaled to give (84); however, the parameters must also satisfy the constraints $\mu(\ell; w_i) > 0$ and $\sum_{i=0}^{\infty} \mu(\ell; w_i) = 1$ for all values of the parameters and we have not ensured that these constraints are compatible with (84). It is sufficient for our purposes to understand that the oscillatory terms prevent the continuum limit being taken for certain walk lengths so they are certainly unphysical; we, therefore, must scale them so that they disappear in the continuum.

We therefore have as a lower bound on $Q_c$,

$$\hat{Q}(\xi; \omega) \geq \xi^{-1/2} \left( 1 + \delta_{AC(\xi_k,0)}C + \sum_{j \in J_R} \frac{1}{(k_j - 1)(1 - \sigma_j)^{1 - \epsilon_j}} \right)^{-1/2}.$$  \hspace{1cm} (87)

We now consider the upper bound. Without loss of generality we may arrange that the indexing set $J_R$ has the property that if $j_1 < j_2$ then $\lambda_{j_1,\pm} < \lambda_{j_2,\pm}$. Furthermore, define $H(x) = [\bar{H}(x)] = [x^{-1/2}]$, $D(x) = [\bar{D}(x)] = \left[ \frac{2\beta}{1 - \sigma_I}(1 - \chi(\bar{H}(x)))^{-1} \log \left( \frac{r_I x^{2 + 1 - \epsilon_I}}{\frac{1}{2} \log x \gamma \chi(\bar{H}(x))} \right) \right]$, $k(x) = [\bar{k}(x)] = [x^{-1/2}(1 - \chi(\bar{H}(x)))]$, where $I = \inf \bar{J}_R$, i.e. $\lambda_{I,\pm}$ is the smallest length scale with a spectral dimension differing from 1. Some modified versions of the above quantities will also be needed:

$$D_{\pm}(x) = [\bar{D}_{\pm}(x)] = \left[ \frac{2\beta}{1 - \sigma_I}(1 - \chi_{\pm}(\bar{H}(x)))^{-1} \log \left( \frac{r_I x^{2 + 1 - \epsilon_I}}{\frac{1}{2} \log x \gamma \chi(\bar{H}(x))} \right) \right],$$  \hspace{1cm} (88)

$$k_{\pm}(x) = [\bar{k}_{\pm}(x)] = [x^{-1/2}(1 - \chi_{\pm}(\bar{H}(x)))].$$

It is clear that $\bar{D}_-(x) \leq \bar{D}(x) \leq \bar{D}_+(x)$ and $\bar{k}_-(x) \leq \bar{k}(x) \leq \bar{k}_+(x)$ which together with lemma 8 allows us to conclude that the continuum limit is

$$\hat{Q}(\xi; \omega) \leq \xi^{-1/2} \left[ 1 - e^{G_{-\log(1-\epsilon_I^{\gamma^{\delta}})} + \frac{\epsilon^{G_{-1}\log(1-\epsilon_I^{\gamma^{\delta}})}}{3 \coth^2(\log \epsilon_I^{\gamma^{\delta}}/|1/G_{+}|) - \sqrt{\gamma^2 + 1 + 2\gamma \coth(\log \epsilon_I^{\gamma^{\delta}}/|1/G_{+}|)}}} \right] \hspace{1cm} (90)$$

where we have defined $G_{\pm}(\xi, \lambda_1, \ldots) = \sum_{j \in J_R} \frac{\epsilon^{1-\epsilon_I}}{(k_j - 1)!}$ and arranged that no oscillatory terms appear.

We are now in a position to prove theorem 10 by analysing the behaviour of (87) and (90) for various walk lengths. We first note that both the upper and lower bounds are controlled by the relative sizes of the quantities $\epsilon_I^{1+\epsilon_I/2}$. In particular, the largest of these quantities, $\Upsilon = \sup_{j \in J_R} \epsilon_I^{1+\epsilon_I/2}$, will determine the behaviour in a particular scale range; if we suppose $\Upsilon = \epsilon_I^{1+\epsilon_I/2}$ for some $j$ then the leading contribution to both the upper and lower bounds will be proportional to $\xi^{-(1+\epsilon_I)/4}$. Using this we find the following behaviour.
• On very short scales corresponding to \( \xi \gg \lambda_j^{-1} \) for all \( j \) the lower bound of \( \hat{Q}(\xi) \) is dominated by the \( \xi^{-\frac{1}{2}} \) behaviour, which together with the trivial upper bound means that taking the \( \xi \to \infty \) limit leads to \( d_s^0 = 1 \) as in the previous sections.

• On very long scales corresponding to \( \xi \ll \lambda_j^{-1} \) for all \( j \) there will exist a length scale \( \Lambda \) such that for \( 0 < \xi < \Lambda^{-1} \), \( \Upsilon = \nu_j^{-1+\sigma_j/2} \) where \( \sigma_j \leq \sigma_j \) for all \( j \). Explicitly \( \Lambda \) will be given by

\[
\Lambda = \sup \left\{ \frac{\lambda_j^{-2(\sigma_j-\sigma))/\lambda_j^{-2(\sigma_j-\sigma)}}{\lambda_j^{-2(\sigma_j-\sigma)/\lambda_j^{-2(\sigma_j-\sigma)}}} : j \in \tilde{J}_R \right\}.
\]

If we take the limit \( \xi \to 0 \) we obtain \( d_s^\infty = (3 - \sigma_j)/2 \).

We see that the spectral dimension increases monotonically on successive length scales and the spectral dimension has measured values given by \( d_s = (3 - \sigma_{j(J)})/2 \) therefore proving theorem 10.

It is interesting to note that a constraint on the crossover behaviour exists for generic measures much as it did for the case of the double power law measure. In particular, on intermediate scales, there exists \( J \in \tilde{J}_R \) such that \( \xi \ll \lambda_j^{-1} \) for \( j \leq J \) and \( \xi \gg \lambda_j^{-1} \) for \( j > J \). Hence, \( \nu_j^{-1+\sigma_j/2} \ll 1 \) if \( j > J \) and so \( \Upsilon = \sup[\nu_j^{-1+\sigma_j/2} : j \in \tilde{J}_R, j \leq J] \). This means that there will exist a length scale \( \Lambda_J \) such that for \( \lambda_{j+1}^{-1} < \xi < \Lambda_j^{-1} \), \( \Upsilon = \nu_{j+1}^{-1+\sigma_{j+1}/2} \) where we have defined \( \hat{j}(J) \) implicitly by \( \sigma_{j(J)} \leq \sigma_j \) for all \( j \leq J \). Of course this does not ensure that \( \lambda_{j+1}^{-1} < \Lambda_j^{-1} \); indeed, if the length scales \( \lambda_j \) are not sufficiently separated this may not be true and we would not have a scale range in which this term dominated. The expression for \( \Lambda_J \) is

\[
\Lambda_J = \sup \left\{ \frac{\lambda_j^{-2(\sigma_j-\sigma)/\lambda_{j(J)}^{-2(\sigma_j-\sigma)/\lambda_{j(J)}^{-2(\sigma_j-\sigma)}}}}{\lambda_j^{-2(\sigma_j-\sigma)/\lambda_{j(J)}^{-2(\sigma_j-\sigma)}}} : j \in \tilde{J}_R, j \leq J \right\}
\]

and so we may choose \( \lambda_{j+1} \) independent of \( \Lambda_J \) thereby allowing the scale range over which this behaviour exists to be arbitrarily large. This would result in an apparent spectral dimension of \( \delta_s = (3 - \sigma_{j(J)})/2 \).

Finally, an interesting application of the techniques used to prove theorem 10 is that they allow one to analyse a wider class of combs than the class for which the results of [3] are valid. In particular, it was proven in [3] that for a random comb the spectral dimension is \( d_s = (3 - \gamma_0)/2 \) where \( \gamma_0 = \sup[\gamma : I_\gamma < \infty] \) and \( I_\gamma = \sum_{\ell=0}^{\infty} \mu(\ell)\ell^\gamma \). This was proved subject to the assumption that there exists \( d > 0 \) such that

\[
\sum_{\ell=0}^{\infty} \mu(\ell) \sim x^d
\]

as \( x \) goes to zero.

Given the results in this section, we see that \( -\gamma_0 \) may be interpreted as the abscissa of convergence for the Dirichlet series generating function. Furthermore, it is clear from our results that there are distributions where the assumption (93) does not hold and that we may use the techniques we have developed to analyse these cases. Recalling that for a random comb we may compute the spectral dimension using relation (21), we must perform a similar analysis to that done for the continuum comb but now only scaling \( x \) to zero.

Due to Landau’s theorem, there always exists a pole at the abscissa. Suppose it is of order \( k \) and consider \( 1 - \chi_{k}(u) \).
We have demonstrated that there exist models in which a scale-dependent spectral dimension can be shown to exist analytically. That we could do this was important as up to now all the evidence for scale-dependent spectral dimension in CDT has been numerical in nature and therefore open to the criticism that it might be due to discretization effects or other numerical artefacts. Furthermore, numerical results do not provide any understanding of the mechanism causing the reduction in dimensionality and so we hope the work begun in this paper may be extended to shed some light on this question. This is not an unreasonable expectation as the techniques used to compute the spectral dimensions of random combs [3] have been extended to the computation of the spectral dimensions of random trees [18]. Such random trees are closely related to two-dimensional CDT via the bijection described in [19] where it was used to bound the spectral dimension of the spacetime arising in such models.

7. Conclusions

We have demonstrated that there exist models in which a scale-dependent spectral dimension can be shown to exist analytically. That we could do this was important as up to now all the evidence for scale-dependent spectral dimension in CDT has been numerical in nature and therefore open to the criticism that it might be due to discretization effects or other numerical artefacts. Furthermore, numerical results do not provide any understanding of the mechanism causing the reduction in dimensionality and so we hope the work begun in this paper may be extended to shed some light on this question. This is not an unreasonable expectation as the techniques used to compute the spectral dimensions of random combs [3] have been extended to the computation of the spectral dimensions of random trees [18]. Such random trees are closely related to two-dimensional CDT via the bijection described in [19] where it was used to bound the spectral dimension of the spacetime arising in such models.
One might ask how closely related a random comb is to an actual model of quantum gravity. In models of pure quantum gravity in two dimensions, the only observable is the spatial volume of the universe. For the case of both CDT and random trees, one may show that the evolution of the spatial volume with time is generated by a Hamiltonian. A crucial difference for the random combs is that the growth of a comb is not a Markovian process in the same sense as the growth of a CDT; the vertices at height $h$ from the root are not related to those at height $h-1$ by a local transfer matrix so a Hamiltonian formulation of the evolution of a comb-like universe is not possible. An extension of the work described in this paper, using the techniques of [18], to the case of random trees would be very interesting as this would constitute an example of dimensional reduction for a model which does admit a Hamiltonian formulation.

We have given in theorem 10 a reasonably complete classification of what behaviours one can find on a continuum comb and indeed it is likely that the cases not covered by the theorem do not have a well-defined spectral dimension. The behaviour for the combs covered by theorem 10 is fairly rich as there exists a cross-over region between the UV and IR behaviours in which a hierarchy of apparent spectral dimensions exists. The proof of the bijection between two-dimensional CDT and trees in [19] shows that any ensemble of critical Galton–Watson trees is in bijection with a CDT-like theory. It is only when the random tree has the uniform measure that we obtain precisely CDT on the other side of the bijection. This opens the intriguing possibility of constructing a variation of CDT, constructed from an ensemble of random trees with a measure that differs from the uniform measure and with a dependence on a length scale. This length scale could then be scaled while taking a continuum limit, as we have done in the work here. Such a model would likely have intermediate length scales with apparent spectral dimensions different from the UV and IR values. We hope to pursue these possibilities in future work.

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Appendix A. Standard generating functions

We record here a number of standard results for generating functions for random walks on combs; the details of their calculation are given in [3].

On the empty comb $C = \infty$, we have

$$P_{\infty}(x) = 1 - \sqrt{x},$$  \hspace{1cm} (A.1)

$$P^{n}_{\infty}(x) = (1 - x) \frac{(1 + \sqrt{x})^{n-1} - (1 - \sqrt{x})^{n-1}}{(1 + \sqrt{x})^{n} - (1 - \sqrt{x})^{n}},$$  \hspace{1cm} (A.2)

$$G_{\infty}^{(0)}(x; n) = (1 - x)^{n/2} \frac{2\sqrt{x}}{(1 + \sqrt{x})^{n} - (1 - \sqrt{x})^{n}}.$$  \hspace{1cm} (A.3)

Note that we can promote $n$ to being a continuous positive semi-definite real variable in these expressions; $G_{\infty}^{(0)}(x; n)$ is then a strictly decreasing function of $n$ and $P_{\infty}^{n}(x)$ a strictly
increasing function of \( n \). The finite line segment of length \( \ell \) has

\[
P_\ell(x) = 1 - \sqrt{x} \frac{(1 + \sqrt{x})^\ell - (1 - \sqrt{x})^\ell}{(1 + \sqrt{x})^\ell + (1 - \sqrt{x})^\ell} \quad \text{(A.4)}
\]

which is sometimes convenient to write as

\[
P_\ell(x) = \sqrt{x} \tanh (m_\infty(x) \ell) \quad \text{(A.5)}
\]

where

\[
m_\infty(x) = \frac{1}{2} \log \frac{1 + \sqrt{x}}{1 - \sqrt{x}}. \quad \text{(A.6)}
\]

Again, \( \ell \) can be promoted to a continuous positive semi-definite real variable of which \( P_\ell(x) \) is a strictly increasing function. The first return probability generating function for the comb with teeth of length \( \ell + 1 \) equally spaced at intervals of \( n \) is given by

\[
P_{\ell,n}(x) = \frac{3 - P_\ell(x)}{2} - \frac{1}{2} \left[ (3 - P_\ell(x) - 2 P_\infty(x))^2 - 4 G_\infty^{(0)}(x; n)^2 \right]^{\frac{1}{2}} \quad \text{(A.7)}
\]

and \( P_{\ell,n}(x) \) is obtained by setting \( \ell = \infty \) in this formula. \( P_{\ell,n}(x) \) is a strictly decreasing function of \( \ell \) and increasing function of \( n \), viewed as continuous positive semi-definite real variables.

We also need the scaling limits of some of these quantities. They are

\[
\lim_{a \to 0} a^{-\frac{1}{2}} G_\infty^{(0)}(a \xi; a^{-\frac{1}{2}} \nu) = \xi^{-\frac{1}{2}} \text{cosech}(\nu \xi^{-\frac{1}{2}}) \quad \text{(A.8)}
\]

and

\[
\lim_{a \to 0} a^{-\frac{1}{2}} \left( 1 - P \left( a^{-\frac{1}{2}} \rho, a^{-\frac{1}{2}} \nu \right) \right) = -\frac{1}{2} \xi^{-\frac{1}{2}} \tanh(\rho \xi^{-\frac{1}{2}}) + \frac{1}{2} \xi^{-\frac{1}{2}} \left[ 4 + 4 \tanh \rho \xi^{-\frac{1}{2}} \coth \nu \xi^{-\frac{1}{2}} + \tanh^2 \rho \xi^{-\frac{1}{2}} \right]. \quad \text{(A.9)}
\]

### Appendix B. Bump functions

A function \( \psi : \mathbb{R} \to \mathbb{R} \) is a bump function if \( \psi \) is smooth and has compact support. We will now prove some properties concerning the Mellin transformation of a bump function \( \psi \), \( \mathcal{M}[\psi](s) \), which has support on \([a, b]\) where \( b > a > 0 \).

**Lemma 11.** The critical strip of the Mellin transform of the \( n \)th derivative of \( \psi \), \( \psi^{(n)} \), is \( \mathbb{C} \) for all \( n \).

**Proof.** Recall that the Mellin transform is defined by \( \mathcal{M}[\psi](s) = \int_0^\infty \psi(x)x^{s-1} \, dx \). Since \( \psi \) has compact support, we have

\[
\mathcal{M}[\psi^{(n)}](s) = \int_a^b \psi^{(n)}(x)x^{s-1} \, dx \quad \text{(B.1)}
\]

and since \( \psi \) is smooth, \( |\psi^{(n)}| \) is bounded on \([a, b]\) by some constant \( K \), so

\[
|\mathcal{M}[\psi^{(n)}](s)| \leq K \int_a^b x^{s-1} \, dx \quad \text{(B.2)}
\]

and the RHS is finite for all \( s \in \mathbb{C} \) since \( b > a > 0 \). This also shows that \( \mathcal{M}[\psi^{(n)}](s) \) is holomorphic for all \( s \).
Lemma 12. Given $n \in \mathbb{Z}^+$, $|\mathcal{M}[\psi](\sigma + iD)| \leq \frac{1}{D^n} \mathcal{M}[|\psi^{(n)}|](\sigma + n)$ for all $s \in \mathbb{C}$.

Proof. Recall from the previous lemma that the critical strip of the Mellin transform of $\psi$ and its derivatives coincides with $\mathbb{C}$. We can therefore use the integral representation of the Mellin transform to prove statements valid for all $s \in \mathbb{C}$. On integration by parts

$$\mathcal{M}[\psi^{(n)}](s) = -\int_a^b \psi^{(n+1)}(x) \frac{x^s}{s} dx = -\frac{1}{s} \mathcal{M}[\psi^{(n+1)}](s + 1)$$  \hspace{1cm} (B.3)

and therefore,

$$\mathcal{M}[\psi](s) = \frac{(-1)^n}{\prod_{k=0}^{n-1} (s + k)} \int_a^b \psi^{(n)}(x) x^{s+n} dx = \frac{(-1)^n}{\prod_{k=0}^{n-1} (s + k)} \mathcal{M}[\psi^{(n)}](s + n).$$  \hspace{1cm} (B.4)

Hence,

$$|\mathcal{M}[\psi](\sigma + iT)| \leq \frac{1}{D^n} \int_a^b |\psi^{(n)}(x)| x^\sigma dx = \frac{1}{D^n} \mathcal{M}[|\psi^{(n)}|](\sigma + n).$$  \hspace{1cm} (B.5)

Given a bump function $\Psi_\pm$, which is always positive has support $[1, 1 \pm \epsilon]$ and is scaled such that its integral is 1, we define the cut-off function to be

$$\eta_\pm(x) = 1 - \int_{-\infty}^{x} \Psi_\pm(x) dx.$$  \hspace{1cm} (B.6)

Lemma 13. The critical strip of $\mathcal{M}[\eta_\pm](s)$ is given by $\text{Re}[s] > 0$. The analytic continuation of $\mathcal{M}[\eta_\pm](s)$ to all $s$ is given by

$$\mathcal{M}[\eta_\pm](s) = \frac{1}{s} \mathcal{M}[\Psi_\pm](s + 1).$$  \hspace{1cm} (B.7)

Proof. The analytic continuation of $\mathcal{M}[\eta_\pm](s)$ may be obtained by applying integration by parts to the Mellin transform of $\eta_\pm$ and recalling by lemma (11) that $\mathcal{M}[\Psi_\pm](s)$ is holomorphic everywhere.

Appendix C. Asymptotic series and Dirichlet series

Starting with

$$S_\pm(y) = \sum_{\ell=0}^{\infty} \mu(\ell) \eta_\pm(\ell y) = \mu(0) + \sum_{\ell=1}^{\infty} \mu(\ell) \eta_\pm(\ell y) \equiv \mu(0) + S_\pm^{(1)}(y)$$  \hspace{1cm} (C.1)

where $\eta_\pm$ is the smooth cut-off function introduced in appendix B and $y$ controls where the cut-off occurs, we take the Mellin transform of $S_\pm^{(1)}(y)$ with respect to $y$:

$$\mathcal{M}[S_\pm^{(1)}](s) = \int_{0}^{\infty} S_\pm^{(1)}(y) y^{s-1} dy = \mathcal{D}_\mu(s) \mathcal{M}[\eta_\pm](s) = \mathcal{D}_\mu(s) \mathcal{M}[\Psi_\pm](s + 1)/s$$  \hspace{1cm} (C.2)

where $\mathcal{D}_\mu(s)$ is the Dirichlet series associated with the measure

$$\mathcal{D}_\mu(s) = \sum_{\ell=1}^{\infty} \frac{\mu(\ell)}{\ell^s}.$$  \hspace{1cm} (C.3)
Furthermore the remainder term

\[ S_\pm(y) = \mu(0) + \frac{1}{2\pi i} \oint_{C} y^{-s} D_\mu(s) \frac{\mathcal{M}[\Psi_{\pm \epsilon}]}{s} \mathrm{d}s. \]  

(C.4)

This may be computed by rewriting the above as

\[ S_\pm(y) = \mu(0) + \frac{1}{2\pi i} \oint_{C} y^{-s} D_\mu(s) \frac{\mathcal{M}[\Psi_{\pm \epsilon}]}{s} \mathrm{d}s - R(y) \]  

(C.5)

where the contour C is the rectangle composed of the points \( \{c - i\infty, c + i\infty, -N + i\infty, -N - i\infty\} \) with \( N > 0 \), \( c \) such that the contour is the right of all poles and

\[ R(y) = \int_{-N - i\infty}^{-N + i\infty} y^{-s} D_\mu(s) \frac{\mathcal{M}[\Psi_{\pm \epsilon}]}{s} \mathrm{d}s. \]  

(C.6)

If \( D_\mu \) has slow growth in the strip \( -N \leq \mathrm{Re}[z] \leq 0 \), then due to lemma (12), which shows that \( \mathcal{M}[\Psi_{\pm \epsilon}] \) decays faster than any polynomial as \( t \) goes to infinity, the contributions from integrating along the contours \( c + i\infty \) to \( -N + i\infty \) and from \( c - i\infty \) to \( -N - i\infty \) are zero. Furthermore the remainder term \( R(y) \) satisfies

\[ |R(y)| \leq \frac{y^N}{N} \int_{-N - i\infty}^{-N + i\infty} |D_\mu(s)||\mathcal{M}[\Psi_{\pm \epsilon}](s + 1)| \mathrm{d}s \]  

(C.7)

and so will only contribute terms of order \( y^N \) to \( S_\pm(y) \). We therefore have

\[ S_\pm(y) = \mu(0) + \frac{1}{2\pi i} \oint_{C} y^{-s} D_\mu(s) \frac{\mathcal{M}[\Psi_{\pm \epsilon}]}{s} \mathrm{d}s \]  

\[ = 1 + \sum_{s_i \in S/\Sigma} \mathrm{res} \left[ \frac{D_\mu(s) \mathcal{M}[\Psi_{\pm \epsilon}]}{s} y^{-s}; s = s_i \right] - R(y) \]  

(C.8)

(C.9)

where \( S \) is the set of positions of the poles of \( D_\mu \) and we have used the fact that \( D_\mu(0) = 1 - \mu(0) \). Finally, define \( \chi_\pm(u) = \sum_{l=0}^{\infty} \mu(l) \eta_{\pm}(l/u) \). By relating this function to \( S_\pm(y) \) we may write

\[ \chi_\pm(u) = \mu(0) + \frac{1}{2\pi i} \oint_{C} u^s D_\mu(s) \frac{\mathcal{M}[\Psi_{\pm \epsilon}]}{s} \mathrm{d}s \]  

\[ = 1 + \sum_{s_i \in S/\Sigma} \mathrm{res} \left[ \frac{D_\mu(s) \mathcal{M}[\Psi_{\pm \epsilon}]}{s} u^{s}; s = s_i \right] - R(y). \]  

(C.10)

(C.11)

References

[1] ’t Hooft G and Veltman M J G 1974 One loop divergencies in the theory of gravitation Ann. Poincaré Phys. Theor. A 20 69–94
[2] Weinberg S 1980 Ultraviolet divergences in quantum theories of gravitation General Relativity ed S Hawking and W Israel pp 790–831
[3] Durhuus B, Jonsson T and Wheater J F 2006 Random walks on combs J. Phys. A: Math. Gen. 39 1009–38
[4] Ambjørn J and Loll R 1998 Non-perturbative Lorentzian quantum gravity, causality and topology change Nucl. Phys. B 536 407–34 (arXiv:hep-th/9805108)
[5] Ambjørn J, Jurkiewicz J and Loll R 2009 Quantum gravity as sum over spacetimes arXiv:0906.3947 [gr-qc]
[6] Ambjørn J, Durhuus B and Jonsson T 1997 Quantum Geometry: A Statistical Field Theory Approach (Cambridge: Cambridge University Press)
[7] Ambjørn J, Jurkiewicz J and Loll R 2004 Emergence of a 4D world from causal quantum gravity Phys. Rev. Lett. 93 131301 (arXiv:hep-th/0404156)

[8] Ambjørn J, Goerlich A, Jurkiewicz J and Loll R 2008 Planckian birth of the quantum de Sitter universe Phys. Rev. Lett. 100 091304 (arXiv:0712.2485 [hep-th])

[9] Ambjørn J, Jurkiewicz J and Loll R 2005 Spectral dimension of the universe Phys. Rev. Lett. 95 171301 (arXiv:hep-th/0505113)

[10] Litim D F 2004 Fixed points of quantum gravity Phys. Rev. Lett. 92 201301 (arXiv:hep-th/0312114)

[11] Lauscher O and Reuter M 2005 Fractal spacetime structure in asymptotically safe gravity J. High Energy Phys. JHEP10(2005)050 (arXiv:hep-th/0508202)

[12] Horava P 2009 Quantum gravity at a Lifshitz point Phys. Rev. D 79 084008 (arXiv:0901.3775 [hep-th])

[13] Horava P 2009 Spectral dimension of the universe in quantum gravity at a Lifshitz point Phys. Rev. Lett. 102 161301 (arXiv:0901.3775 [hep-th])

[14] Benedetti D and Henson J 2009 Spectral geometry as a probe of quantum spacetime Phys. Rev. D 20 124036 (arXiv:0911.0401 [hep-th])

[15] Carlip S 2010 The small scale structure of space-time arXiv:1009.1136 [hep-th]

[16] Tao T The Euler–Maclaurin formula, Bernoulli numbers, the zeta function, and real-variable analytic continuation http://terrytao.wordpress.com/tag/euler-summation-formula/

[17] Flajolet P, Gourdon X and Dumas P 1995 Mellin transforms and asymptotics: harmonic sums Theor. Comput. Sci. 144 3–58

[18] Durhuus B, Jonsson T and Wheater J F 2007 The spectral dimension of generic trees J. Stat. Phys. 128 1237–60 (arXiv:math-ph/0607020)

[19] Durhuus B, Jonsson T and Wheater J F 2007 On the spectral dimension of causal triangulations J. Stat. Phys. 139 859–81 (arXiv:0908.3643 [math-ph])