ON THE $K$-THEORY OF CROSSED PRODUCTS BY AUTOMORPHIC SEMIGROUP ACTIONS

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Abstract. Let $P$ be a semigroup that admits an embedding into a group $G$. Assume that the embedding satisfies the Toeplitz condition of [24] and that the Baum-Connes conjecture holds for $G$. We prove a formula describing the $K$-theory of the reduced crossed product $A \rtimes_r P$ by any automorphic action of $P$. This formula is obtained as a consequence of a result on the $K$-theory of crossed products for special actions of $G$ on totally disconnected spaces. We apply our result to various examples including left Ore semigroups and quasi-lattice ordered semigroups. We also use the results to show that for certain semigroups $P$, including the $ax + b$-semigroup $R \times R^\times$ for a Dedekind domain $R$, the $K$-theory of the left and right regular semigroup C*-algebras $C^*_\lambda(P)$ and $C^*_\rho(P)$ coincide, although the structure of these algebras can be very different.

1. Introduction

A semigroup (or monoid) is a set with an associative multiplication. Recently the authors of this article - in various combinations - have become interested in the study of the C*-algebra $C^*_\lambda(P)$ defined by the left regular representation of a left cancellative semigroup $P$ on the Hilbert space $l^2(P)$. This interest was motivated by the fact that specific semigroups arising from number theory give examples with an intricate, yet tractable, structure. While generalities about semigroup C*-algebras had been studied before by various authors, only little was known about more complicated examples and concerning questions such as nuclearity, $K$-theory, ideal structure etc. The C*-algebra $C^*_\lambda(P)$ contains a natural commutative subalgebra $D$ generated by the range projections of products of the isometries representing the elements of $P$ and their adjoints. These range projections correspond to the “constructible” right ideals in $P$, i.e. to those right ideals that can be constructed from the principal ideals of the form $xP$ by finitely many operations such as intersection etc. The spectrum of $D$ is a totally disconnected space which we denote by $\Omega_P$. Each constructible right ideal in $P$ corresponds to a compact open subset in $\Omega_P$. In [9] we studied the $K$-theory of $C^*_\lambda(P)$ assuming that $P$ satisfies the left Ore condition. This condition provides a systematic way to embed $P$ into an enveloping group $G$ and also allows to dilate actions of $P$ to actions of $G$. In particular the natural action of $P$ on $\Omega_P$ can be dilated to an action of $G$ on a totally disconnected locally compact space $\Omega_{P \subseteq G}$. The C*-algebra $C^*_\lambda(P)$ is then Morita equivalent to the reduced crossed product $C_0(\Omega_{P \subseteq G}) \rtimes_r G$.

2000 Mathematics Subject Classification. Primary 46L05, 46L80; Secondary 20Mxx, 11R04.

Research supported by the Deutsche Forschungsgemeinschaft (SFB 878) and by the ERC through AdG 267079.
In [9] we had then computed the $K$-theory (in fact in a bivariant setting) of this crossed product using a particular feature ("independence", see below) of $\Omega_P$ together with the following "descent to compact subgroups" principle taken from [13, 5].

(DC) Assume that $G$ satisfies the Baum-Connes conjecture with coefficients in the $G$-algebras $A$ and $B$. Let $x$ be a class in $KK^G(A, B)$ which induces, via descent, isomorphisms $K_\ast(A \rtimes H) \cong K_\ast(B \rtimes H)$ for all compact subgroups $H$ of $G$. Then $x$ also induces an isomorphism $K_\ast(A \rtimes G) \cong K_\ast(B \rtimes G)$.

Note that, by [16], the Baum-Connes condition required for $G$ in (DC) holds whenever $G$ is a-T-menable, and hence in particular, if $G$ is amenable.

Using the independence of the set of constructible right ideals in $P$ and principle (DC) we determined in [9] the $K$-theory of $C^\ast_r(P)$ for some prominent semigroups from algebraic number theory. This includes the multiplicative semigroup or the $ax+b$-semigroup for the ring of algebraic integers in a number field or the semigroup of its principal ideals. The answer involved well known concepts from number theory such as the ideal class group and the group of units.

In the present paper we take a new look at the results of [9] from a more general perspective. We start with a general study of group actions on totally disconnected spaces $\Omega$ under an independence condition similar to the one mentioned above.

Roughly speaking, given a totally disconnected $G$-space $\Omega$ we require that one can find a $G$-invariant family $\mathcal{V}$ of compact open subsets of $\Omega$ which generates the set of all compact open sets via finite intersections, unions and difference sets, and which is independent in the sense that no element $U$ of $\mathcal{V}$ can be written as a finite union of elements of $\mathcal{V}$ different from $U$. Let $I = \mathcal{V} \setminus \{\emptyset\}$. We are then able to construct a canonical element $x \in KK^G(C_0(I), C_0(\Omega))$ which satisfies the requirements of (DC).

We are also able to improve the arguments used in [9] to allow for general coefficients. We show that for any action $\alpha : G \to \text{Aut}(A)$ the class $[\text{id}_A] \otimes_x x \in KK^G(A \otimes C_0(I), A \otimes C_0(\Omega))$ will also satisfy the conditions in (DC). Assume, then, that $G$ satisfies the Baum-Connes conjecture with coefficients in $A \otimes C_0(I)$ and $A \otimes C_0(\Omega)$ and denote by $\tau$ resp. $\mu$ the action of $G$ on $\Omega$ resp. $I$. Using the principle (DC), we obtain an isomorphism

$$K_\ast((A \otimes C_0(\Omega)) \rtimes_{\alpha \otimes \tau, r} G) \cong K_\ast((A \otimes C_0(I)) \rtimes_{\alpha \otimes \mu, r} G) \quad (1.1)$$

Moreover, by Green’s imprimitivity theorem the right hand side is in turn isomorphic to the sum, over the $G$-orbits in $I$, of the $K$-theory of the crossed products by the stabilizer groups, i.e. to

$$\bigoplus_{[i] \in G \setminus I} K_\ast(A \rtimes_{\alpha, r} G_i) \quad (1.2)$$

where $G_i$ denotes the stabilizer of $i \in I$.

These results have an independent interest. Most important for us however is again the application to the $K$-theory of semigroup $C^\ast$-algebras and semigroup crossed products. We study semigroup crossed products $A \rtimes_{\alpha, r} P$ in which the semigroup $P$ acts by automorphisms on the $C^\ast$-algebra $A$ in section 4.

In [24] it was shown by the third author that, given independence of the set of constructible right ideals, for our purposes, the left Ore condition for $P$ can be weakened. It suffices to assume that the semigroup $P$ is embedded into a group $G$ and that the inclusion $P \subseteq G$ satisfies the Toeplitz condition introduced in [24].

Under this weaker condition too, the full and reduced $C^\ast$-algebras of $P$ embed as full
corners into full and reduced crossed products by the group $G$. As we will see, there are natural examples of semigroups satisfying the Toeplitz condition but not the left Ore condition. Because of the embedding as a full corner, again the computation of the $K$-theory of a crossed product by $P$ can be reduced to the computation of the $K$-theory of a crossed product by $G$. This crossed product by $G$ is of the form $(A \otimes C_0(\Omega)) \rtimes_{\alpha \otimes \tau, r} G$ considered above, and we can therefore apply formulas (1.1) and (1.2).

We are now in a position to apply our results to explicit classes of semigroups. Consider first a semigroup $P$ which is given as the positive cone in a quasi-lattice ordered group $G$ which satisfies the Baum-Connes conjecture with coefficients. The inclusion $P \subseteq G$ satisfies the Toeplitz condition. For the crossed product of a C*-algebra $A$ by an action $\alpha$ of $P$ by automorphisms, we obtain the striking result

$$K_*(A) \cong K_*(A \rtimes_{\alpha, r} P)$$

i.e. the $K$-theory of the crossed product does not depend on $P$ nor on $\alpha$. This is a far reaching generalization of the well known corresponding result for the action of the Toeplitz algebra by an automorphism on $A$ which in fact was the basis for the proof by Pimsner-Voiculescu of the six term exact sequence for a crossed product by $Z$, [27].

Another important example is the following. Let $R$ be the ring of algebraic integers in a number field (or a more general Dedekind domain). Denote by $R^\times$ its multiplicative semigroup and by $S = R \rtimes R^\times$ its $ax + b$-semigroup. The $K$-theory of $C^*_\lambda(S)$ was determined in [9]. Consider now the opposite semigroup $S^{op}$. Its left regular C*-algebra $C^*_\lambda(S^{op})$ is the right regular C*-algebra $C^*_\rho(S)$ of $S$. We mention that $C^*_\lambda(S)$ and $C^*_\rho(S)$ are very different algebras. For instance, the second algebra admits non-trivial one-dimensional representations while the first one admits only infinite-dimensional representations. Also $S$ satisfies the left Ore condition while $S^{op}$ does not. However, $S^{op}$ satisfies independence and the Toeplitz condition. We can therefore again compute the $K$-theory. Somehow surprisingly, it turns out that $C^*_\lambda(S)$ and $C^*_\rho(S)$ have the same $K$-theory, indeed they are $KK$-equivalent. We also determine the $K$-theory of $C^*_\lambda(S)$ and $C^*_\rho(S)$ for a semidirect product of the form $S = H \rtimes \mathbb{N}$ where $H$ is a group. Again these two C*-algebras are completely different but still have the same $K$-theory.

The paper is organized as follows: After a brief discussion of totally disconnected spaces in §2 we present in §3 our main results on the $K$-theory of crossed products $(A \otimes C_0(\Omega)) \rtimes_r G$. In §4 we deduce our results on the $K$-theory of crossed products $A \rtimes_r P$ by automorphic actions of semigroups and we briefly discuss the consequences for crossed products by the left Ore semigroups studied in §9. Crossed products by quasi-lattice semigroups $P \subseteq G$ are studied in §5. Indeed, the beautiful $K$-theory formula for such crossed products follows from the fact that for quasi-lattice ordered semigroups $P \subseteq G$ the action of $G$ on the set of nonempty constructible left $P$-ideals in $G$ is transitive. We present further examples which show that transitivity of this action is not restricted to this case, and therefore similar $K$-theory formulas can be obtained in more generality. Our results on the left and right regular semigroup C*-algebras $C^*_\lambda(P)$ and $C^*_\rho(P)$ are presented in §6. Finally, in the appendix we discuss some basic constructions in equivariant $KK$-theory of finite dimensional algebras acted upon by compact groups which we need for checking the principle (DC) in §8. These $KK$-results might be known to experts, but seem not to be present in the literature.
2. Preliminaries on totally disconnected spaces

Recall that a locally compact Hausdorff space $\Omega$ is totally disconnected if and only if its topology has a basis of compact open subsets. The corresponding algebras $C_0(\Omega)$ of continuous functions which vanish at infinity are precisely the commutative AF-Algebras. In what follows, if $V \subseteq \Omega$, then $1_V : \Omega \to \mathbb{C}$ denotes the characteristic function of $V$.

**Definition 2.1.** Let $\Omega$ be a totally disconnected locally compact Hausdorff space and let $\mathcal{V}$ be a family of compact open subsets in $\Omega$. Moreover, let $\mathcal{U}_c(\Omega)$ denote the set of all compact open subsets of $\Omega$. Then we say that $\mathcal{V}$ is a generating family of the compact open sets of $\Omega$ if $\mathcal{U}_c(\Omega)$ coincides with the smallest family $\mathcal{U}$ of compact open sets in $\Omega$ which contains $\mathcal{V}$ and which is closed under finite intersections, finite unions, and under taking differences $U \setminus W$ with $U, W \in \mathcal{U}$.

**Lemma 2.2.** Suppose that $\mathcal{V}$ is a family of compact open sets in the totally disconnected space $\Omega$. Then the following are equivalent

1. The set $\{1_V : V \in \mathcal{V}\}$ generates $C_0(\Omega)$ as a C*-algebra.
2. The set $\mathcal{V}$ generates $\mathcal{U}_c(\Omega)$ in the sense of Definition 2.1.

Moreover, if $\mathcal{V}$ is closed under taking finite intersections, then (1) and (2) are equivalent to

3. $\text{span}\{1_V : V \in \mathcal{V}\}$ is a dense subalgebra of $C_0(\Omega)$ containing $\text{span}\{1_U : U \in \mathcal{U}_c(\Omega)\}$.

**Proof.** Let $\mathcal{U}$ be the smallest family of compact open sets in $\Omega$ which contains $\mathcal{V}$ and is closed under finite intersections, finite unions, and taking differences. Since a finite product of characteristic functions is the characteristic function of the finite intersection of the given sets, we may assume without loss of generality that $\mathcal{V}$ is closed under finite intersections. In that case it is easy to see that the algebra generated by $\{1_V : V \in \mathcal{V}\}$ coincides with $\text{span}\{1_V : V \in \mathcal{V}\}$. Since $1_{V_1 \cup V_2} = 1_{V_1} + 1_{V_2} - 1_{V_1 \cap V_2}$ and $1_{V_1 \setminus V_2} = 1_{V_1} - 1_{V_1 \cap V_2}$ we see that this span contains all characteristic functions $1_U$ with $U \in \mathcal{U}$. Thus we may replace $\mathcal{V}$ by $\mathcal{U}$. Note that every function in $\text{span}\{1_U : U \in \mathcal{U}\}$ can be written as a linear combination $\sum_{i=1}^k \lambda_i 1_{U_i}$ in which all $\lambda_i$ are non-zero and in which the $U_i$ are pairwise disjoint.

Suppose now that (1) holds. Then for every compact open set $W$ in $\Omega$ we find a linear combination $\sum_{i=1}^k \lambda_i 1_{U_i}$ with pairwise disjoint $U_1, \ldots, U_k$ in $\mathcal{U}$ and $\lambda_i \neq 0$ such that $\|1_W - \sum_{i=1}^k \lambda_i 1_{U_i}\|_\infty < \frac{1}{2}$. This implies that each set $U_i$ is either a subset of $W$ or $U_i \cap W = \emptyset$. In any case, it follows that $W$ is the union of those $U_i$’s which are contained in $W$. Conversely, if $\mathcal{U} = \mathcal{U}_c(\Omega)$, then every continuous function with compact support can be approximated by locally constant functions with compact supports, which are finite linear combinations of elements in $\{1_U : U \in \mathcal{U}\}$.

**Lemma 2.3.** Suppose that $D$ is a commutative C*-algebra such that $D$ is generated as a C*-algebra by a set of projections $\{e_i : i \in I\} \subseteq D$. Then the Gelfand spectrum $\Omega = \text{Spec}(D)$ of $D$ is totally disconnected and the family of sets $\mathcal{V} = \{e_i^{-1}(\{1\}) : i \in I\}$ is a family of compact open sets in $\Omega$ which generates $\mathcal{U}_c(\Omega)$. Here, for an element $d \in D$, $\hat{d} \in C_0(\Omega)$ denotes the Gelfand-transform of $d$. 

**Acknowledgements:** We are grateful to Marcelo Laca for drawing our attention to [18] and to Mikael Rørdam for pointing out Example 3.20.
Proof. For each finite $F \subseteq I$ let $D_F \subseteq D$ denote the $C^*$-algebra generated by \{e_i : i \in F\}. Then $D_F$ is finite dimensional and $D = \lim_F D_F$. Thus, $D$ is a commutative AF-algebra and therefore $\Omega = \text{Spec}(D)$ is totally disconnected. The second assertion then follows from Lemma \ref{gensets} and the fact that projections $e \in D$ correspond to characteristic functions $1_V \in C_0(\Omega)$ under the Gelfand transform for $V = \hat{e}^{-1}\{1\}$. \hfill \Box

The above lemmas show that it is equivalent to study sets of projections \{e_i : i \in I\} generating a commutative $C^*$-algebra $D$ or sets of compact open subsets of totally disconnected spaces $\Omega$ which generate the compact open sets $U_\emptyset(\Omega)$ in the sense of Definition \ref{def:opensets}. For our $K$-theoretic studies we need generating sets which satisfy a certain independence condition. The following definition is taken from \cite{23} and plays an important rôle in \cite{9} and \cite{24}.

Definition 2.4. Let $J$ be a subset of the power set $\mathcal{P}(Y)$ of a set $Y$. We call $J$ independent, if for every finite family $X, X_1, \ldots, X_k$ of elements in $J$ such that $X = \bigcup_{i=1}^k X_i$, there must be an index $i \in \{1, \ldots, k\}$ such that $X_i = X$.

Making the connection between sets and projections, it makes sense to extend the notion of independence to projections in arbitrary commutative $C^*$-algebras. We need

Lemma 2.5. Suppose that \{e_i : i \in I\} is a set of projections in the commutative $C^*$-algebra $D$. Then for each finite subset $F \subseteq I$ there exists a smallest projection $e \in D$ such that $e_i \leq e$ for all $i \in F$. We then write $e =: \bigvee_{i \in F} e_i$.

Proof. One checks that $\bigvee_{i \in F} e_i = \sum_{\emptyset \neq H \subseteq F} (-1)^{|H|-1} \prod_{i \in H} e_i$. \hfill \Box

Definition 2.6. Suppose that \{e_i : i \in I\} is a set of projections in the commutative $C^*$-algebra $D$. We say that \{e_i : i \in I\} is independent if for all finite sets $F \subseteq I$ and $i_0 \in I$ such that $\bigvee_{i \in F} e_i = e_{i_0}$ it follows that $i_0 \in F$.

Remark 2.7. Let $D$ be a commutative $C^*$-algebra generated by the set of projections \{e_i : i \in I\}. Let $\Omega = \text{Spec}(D)$ denote the Gelfand dual of $D$ and let $V_i := \hat{e}_i^{-1}\{1\}$ for all $i \in I$. Then it is straightforward to check that \{e_i : i \in I\} is independent in the sense of Definition \ref{def:independence} if and only if $\mathcal{V} = \{V_i : i \in I\}$ is independent in the sense of Definition \ref{def:independence}. Conversely, if we start with a family $\mathcal{V}$ of compact open sets in a totally disconnected space $\Omega$, then $\mathcal{V}$ is independent if and only if the set \{1_V : V \in \mathcal{V}\} is an independent set of projections.

The following lemma is obvious, but also follows from \cite{24} Proposition 2.4:

Lemma 2.8. Suppose that \{e_i : i \in I\} is a family of projections in the commutative $C^*$-algebra $D$ which is closed under multiplication up to $0$. Then \{e_i : i \in I\} is independent in the sense of Definition \ref{def:independence} if and only it is linearly independent.

Definition 2.9. Suppose that $\Omega$ is a totally disconnected locally compact Hausdorff space. A family $\mathcal{V}$ of non-empty compact open subsets of $\Omega$ is called a regular basis (for the compact open sets in $\Omega$) if the following are satisfied:

1. $\mathcal{V} \cup \{\emptyset\}$ is closed under finite intersections;
2. $\mathcal{V}$ generates the compact open sets of $\Omega$;
3. $\mathcal{V}$ is independent.
Proposition 2.12. For second countable spaces \( \Omega \) we can prove the existence of a regular basis for the compact open sets of \( \Omega \) is countable.

We have the following countability result for totally disconnected spaces. Recall that a topological space \( \Omega \) is called second countable if and only if the set \( \mathcal{U}_c(\Omega) \) of compact open subsets of \( \Omega \) is second countable.

Lemma 2.10. Let \( \Omega \) be a totally disconnected locally compact Hausdorff space. Then \( \Omega \) is second countable if and only if the set \( \mathcal{U}_c(\Omega) \) of compact open subsets of \( \Omega \) is countable.

Proof. If \( \Omega \) is second countable we can find a countable basis \( \mathcal{U} \) for the topology of \( \Omega \) consisting of compact open subsets of \( \Omega \). But then each compact open subset of \( \Omega \) is a finite union of elements in \( \mathcal{U} \), which shows that \( \mathcal{U}_c(\Omega) \) is countable. The converse is clear.

Remark 2.11. It follows from the above lemma that if \( \Omega \) is a second countable totally disconnected locally compact Hausdorff space, then every regular basis \( \mathcal{V} \) for the compact open sets of \( \Omega \) is countable.

For second countable spaces \( \Omega \) we can prove the existence of a regular basis for the compact open sets in \( \Omega \):

Proposition 2.12. Let \( \Omega \) be a second countable totally disconnected locally compact space. Then there exists a regular basis \( \mathcal{V} \) for the compact open sets of \( \Omega \).

Proof. We first observe that it suffices to consider the case where \( \Omega \) is compact. This follows from the fact that every locally compact totally disconnected space \( \Omega \) can be written as the disjoint union of compact open sets \( \{ \Omega_i : i \in I \} \). Then, if \( \mathcal{V}_i \) is a regular basis for the compact open sets of \( \Omega_i \) for all \( i \in I \), then \( \mathcal{V} = \bigcup_{i \in I} \mathcal{V}_i \) is a regular basis for the compact open sets in \( \Omega \).

So assume from now on that \( \Omega \) is compact. Since \( \Omega \) is second countable, it can be realized as a projective limit \( \Omega = \text{prolim}_{n \in \mathbb{N}} F_n \) for some projective system \( \{ F_n : \varphi_n : F_{n+1} \to F_n \} \) in which all sets \( F_n \) are finite. Recall from the construction of this projective limit that a basis \( \mathcal{U} \) of the topology of \( \Omega \) consisting of compact open sets is given by \( \mathcal{U} = \{ \mu_n^{-1}(x) : n \in \mathbb{N}, x \in F_n \} \), where, for each \( n \in \mathbb{N} \), \( \mu_n : \Omega \to F_n \) denotes the canonical mapping.

In order to construct a regular basis \( \mathcal{V} \) for the compact open sets of \( \Omega \) we first construct bijections \( \psi_n : \{1, \ldots, k_n\} \to F_n \), with \( k_n = |F_n| \), which satisfy the following compatibility condition:

(C) For each \( n \in \mathbb{N} \) let \( m_0 := 0 \) and \( m_l := |\varphi_n^{-1}(\psi_n(\{1, \ldots, l\}))| \) or \( l \in \{1, \ldots, k_n\} \).

We require that \( \varphi_n : F_{n+1} \to F_n \) sends \( \psi_{n+1}(\{m_{l-1} + 1, \ldots, m_l\}) \) to \( \psi_n(l) \) for all \( l \in \{1, \ldots, k_n\} \).

The construction can be done easily by starting with an arbitrary bijection \( \psi_1 : \{1, \ldots, k_1\} \to F_1 \) and then defining the other bijections recursively by obeying condition (C) in each step. Having done this, we may assume as well that \( F_n = \{1, \ldots, k_n\} \) and that \( \varphi_n(\{m_{l-1} + 1, \ldots, m_l\}) = \{l\} \) for each \( 1 \leq l \leq k_n \).

We then define \( \mathcal{V} := \{ V_{n,l} := \mu_n^{-1}(\{1, \ldots, l\}) : n \in \mathbb{N}, 1 \leq l \leq k_n \} \). To see that this is a regular basis for the compact open sets of \( \Omega \) we first observe that each basic open set \( \varphi_n^{-1}(\{l\}) \) can be obtained as a difference of two sets in \( \mathcal{V} \), so it is
clear that $\mathcal{V}$ generates the compact open sets of $\Omega$. To check the other conditions, observe first that condition (C) together with the equation $\varphi_n \circ \mu_{n+1} = \mu_n$ implies that $V_{n,l} = V_{n+1,m}$ for all $n \in \mathbb{N}, 1 \leq l \leq k_n$, with $m_1$ as in (C). By induction, it follows that $V_{n,l} = V_{m,l'}$ for some suitable $1 \leq l' \leq k_m$ whenever, $m \geq n$. So, if finitely many elements $W_1, \ldots, W_r$ in $\mathcal{V}$ are given, we may assume that there exist $n \in \mathbb{N}$ and $1 \leq l_1 \leq l_2 \leq \cdots \leq l_r \leq k_n$ such that $W_i = V_{n,l_i}$ for all $1 \leq i \leq r$. The intersection of these sets then equals $V_{n,l_1}$. This proves that $\mathcal{V}$ is closed under finite intersections. The union of the $W_i$ equals $V_{n,l_r}$, which proves independence. \hfill \square

We close this section with a simple example which illustrates the concept of regular bases for the compact open sets of a totally disconnected space $\Omega$.

Example 2.13. Consider the space $\Omega = \{1,-1\}^\mathbb{Z}$ equipped with the product topology. Then $\Omega$ is homeomorphic to the Cantor space. Recall that the basic open neighborhoods of an element $x = (x_n)_{n \in \mathbb{Z}} \in \Omega$ are given by the sets $W_F(x) := \{y \in \Omega : x_n = y_n \text{ for all } n \in F\}$, where $F$ runs through the finite subsets of $\mathbb{Z}$.

For every finite set $F \subseteq \mathbb{Z}$ (including $\emptyset$) we define $V_F := \{z \in \Omega : z_n = 1 \text{ for all } n \in F\}$ and we let $\mathcal{V}$ denote the family of all such sets $V_F$. Since $V_{F_1} \cap V_{F_2} = V_{F_1 \cup F_2}$ we see that $\mathcal{V}$ is closed under finite intersections. To see that it is independent, observe that for finite sets $F_1, \ldots, F_l$ we have

$$V_{F_1} \cup V_{F_2} \cup \cdots \cup V_{F_l} = \{z \in \Omega : \exists k \in \{1, \ldots, l\} \text{ such that } z_n = 1 \text{ for all } n \in F_k\}$$

which is equal to a set $V_F$ if and only if there exists $i_0 \in \{1, \ldots, k\}$ such that $F = F_{i_0}$ and $F_i \subseteq F$ for all $i \in \{1, \ldots, l\}$. Thus it follows that $\mathcal{V}$ is a regular basis for the compact open sets of $\Omega$ if it generates the compact open sets $\mathcal{U}(\Omega)$ of $\Omega$. For this let $\mathcal{U}$ denote the smallest subset of $\mathcal{U}(\Omega)$ which contains $\mathcal{V}$ and is closed under taking differences, finite intersections and finite unions. It suffices to show that $\mathcal{U}$ contains all basic neighborhoods $W_F(x)$. To see this we first observe that $\Omega = V_\emptyset \in \mathcal{V}$. Then for any fixed $n_0 \in \mathbb{Z}$ the complement $V_{n_0} := \Omega \setminus V_{n_0} = \{z \in \Omega : z_{n_0} = -1\}$ lies in $\mathcal{U}$. For a given finite subset $F$ of $\mathbb{Z}$ and any given $x \in \Omega$ we then have

$$W_F(x) = (\bigcap \{V_n : n \in F, x_n = 1\}) \cap (\bigcap \{V_m : m \in F, x_m = -1\}),$$

so $W_F(x) \in \mathcal{U}$.

3. $K$-THEORY OF CROSSED PRODUCTS BY ACTIONS ON TOTALLY DISCONNECTED SPACES

In this section we extend the ideas of [3] [36] to study the $K$-theory of crossed products of the form $C_\theta(\Omega) \rtimes_\tau, G$ for a continuous action of a second countable locally compact group $G$ on a second countable totally disconnected locally compact space $\Omega$. More generally, we study the $K$-theory of a crossed product $(A \otimes C_0(\Omega)) \rtimes_{\alpha \otimes \tau, G}$ by a diagonal action where $\alpha : G \to \text{Aut}(A)$ is an action of $G$ by $*$-automorphisms on a separable C*-algebra $A$. We will assume that we can find a $G$-invariant regular basis $\mathcal{V}$ for the compact open sets in $\Omega$. Moreover, we will use the assumption that $G$ satisfies the Baum-Connes conjecture for suitable coefficients (see the discussion below).

At the end of this section we will use the $K$-theoretic results of this section to show that a $G$-invariant regular basis for the compact open sets of $\Omega$ cannot always exist (see Examples [5.13] and [5.20] below). But the results in [24] show that such a basis does exist in many interesting situations connected to the study of crossed products by semigroups (e.g., see [5] for explicit examples). Let us give a first positive example:
Example 3.1. Consider the Cantor set $\Omega = \{1, -1\}^\mathbb{Z}$ of Example 2.13. Then $\mathbb{Z}$ acts on $\Omega$ by the shift, i.e., $(m \cdot x)_n := x_{n-m}$ for $m \in \mathbb{Z}$ and $x = (x_n)_{n \in \mathbb{Z}} \in \Omega$. It is then clear that the regular basis $\mathcal{V} = \{V_{F} : F \subseteq \mathbb{Z} \text{ finite}\}$ as constructed in Example 2.13 is $\mathbb{Z}$-invariant.

From now on we assume that $\mathcal{V} = \{V_i : i \in I\}$ is a $G$-invariant regular basis for the compact open sets in $\Omega$. We then may assume without loss of generality that $G$ acts on the index set $I$ via a homomorphism $\mu : G \to S_I$ of $G$ into the group of permutations of $I$ such that $g \cdot V_i = V_{\mu(i)}$ for all $i \in I$ and $g \in G$. Note that it follows from Lemma 2.10 that $I$ is countable (we always assume that the assignment $i \mapsto V_i$ is bijective). In what follows, we equip $I$ with the discrete topology.

Remark 3.2. We should remark that, although $G$ is not assumed to be discrete, the action of $G$ on $I$ is automatically continuous, which just means that the stabilizers $G_i = \{g \in G : gi = i\}$ are open in $G$ for all $i \in I$. This follows from the fact that $G_i$ coincides with the stabilizer $G_{1_{V_i}} = \{g \in G : \tau_g(1_{V_i}) = 1_{V_i}\}$ for the function $1_{V_i}$ under the continuous $\tau$ action of $G$ on $\Omega$. But $G_{1_{V_i}} = \{g \in G : \|\tau_g(1_{V_i}) - 1_{V_i}\|_{\infty} < 1\}$ which is open in $G$.

We are going to construct a class $x \in KK^G(C_0(I), C_0(\Omega))$ which, under some extra condition on $G$ which we explain below, induces via descent an isomorphism

$$K_*(\alpha ) \times_{\alpha \otimes \mu, r} G \cong K_*(\beta ) \times_{\beta \otimes \tau, r} G,$$

and, in good cases, even a $KK$-equivalence between these algebras. The relevant extra conditions are related to the Baum-Connes conjecture for the group $G$, which in case it holds, identifies the $KK$-theory group, but the interested reader is referred to [2] for an introduction to this interesting theory.

The result which is important for us is the following proposition. It is taken from [13], but is based on earlier work in [4, 25, 12], and gives a more detailed formulation of the principle (DC) of the introduction:

Proposition 3.3. Let $A$ and $B$ be $G$-algebras and let $x \in KK^G(A, B)$. Let $j_G(x) \in KK^G(A, B \times_{\beta} G)$ denote the descent of $x$ for the reduced crossed products. For every compact subgroup $H$ of $G$ let

$$\varphi_H : K_*(A \times_H) \to K_*(B \times_H); \quad \varphi_H(y) = y \otimes j_H(\text{res}_H^G(x)).$$

where “$\otimes$” denotes the Kasparov product. Then the following are true:
(1) If $G$ satisfies the Baum-Connes conjecture for $A$ and $B$ and if $\varphi_H$ is an isomorphism for every compact subgroup $H$ of $G$, then $\otimes j_G(x) : K_*(A \rtimes_{\alpha,r} G) \to K_*(B \rtimes_{\beta,r} G)$ is an isomorphism.

(2) If $G$ satisfies the strong Baum-Connes conjecture and if $j_H(\text{res}_H^G(x))$ is a $KK$-equivalence between $A \rtimes_{\alpha} H$ and $B \rtimes_{\beta} H$ for all compact subgroups $H$ of $G$, then $j_G(x)$ is a $KK$-equivalence between $A \rtimes_{\alpha,r} G$ and $B \rtimes_{\beta,r} G$.

The Baum-Connes conjecture with coefficients in arbitrary $G$-algebras admits a counter-example (see [14]). On the other hand, the validity of the conjecture has been checked for many interesting classes of groups. One of the strongest results is given by Higson and Kasparov in [16] where they show that all $a$-$T$-menable groups (this includes all amenable groups and all countably generated free groups) satisfy the strong Baum-Connes conjecture.

For the construction of $x$ we start with homomorphisms $\varphi_i : C \to C_0(\Omega)$ which map $1 \in C$ to the projection $e_i := 1_{V_i} \in C_0(\Omega)$. This gives a class in $KK(C, C_0(\Omega))$. Viewing now this copy of $C$ as the $i$th component of $C_0(I) = \bigoplus_{i \in I} C$ and using the well-known isomorphism

$$KK(C_0(I), C_0(\Omega)) \cong \prod_{i \in I} KK(C, C_0(\Omega))$$

we obtain a class $x \in KK(C_0(I), C_0(\Omega))$. We need to make this class $G$-equivariant. This is a special case of the following general construction:

**Notation 3.4.** Suppose that $C = \bigoplus_{i \in I} C_i$ is a direct sum of $C^*$-algebras $C_i$ and suppose that for all $i \in I$ we have a homomorphism $\varphi_i : C_i \to B$ into some fixed $C^*$-algebra $B$. Then there is a $KK$-class $x \in KK(C, B)$ given by the Kasparov-triple $(E, \varphi, 0)$ with $E = \ell^2(I) \otimes B$ (with grading given by $E_0 = E, E_1 = \{0\}$) equipped with the canonical $B$-valued inner product and with

$$\varphi : C \to K(E) \cong K(\ell^2(I)) \otimes B; \quad \varphi = \bigoplus_{i \in I} \varphi_i.$$  

Alternatively, $x$ is represented by the $*$-homomorphism $\varphi : C \to K(\ell^2(I)) \otimes B$ via the identification $KK(C, K(\ell^2(I)) \otimes B) \cong KK(C, B)$ given by multiplication with the $KK$-class $m_{\ell^2(I)} \otimes \text{id}_B$, where $m_{\ell^2(I)} = [(\ell^2(I), \text{id}_{\ell^2(I)}), 0] \in KK(K(\ell^2(I)), \mathbb{C})$ denotes the class of the canonical Morita equivalence $K(\ell^2(I)) \sim_M \mathbb{C}$.

Suppose, moreover, that $\gamma : G \to \text{Aut}(C), \beta : G \to \text{Aut}(B)$ are actions such that $\gamma$ induces an action $\mu : G \to S_I$ of $G$ on $I$ by permutations and such that $\varphi$ becomes $G$-equivariant with respect to the action $\text{Ad} \mu \otimes \beta$ on $K(\ell^2(I)) \otimes B$. Then the action $\mu \otimes \beta : G \to \text{Aut}(\ell^2(I) \otimes B)$ turns $x$ into a class in $KK_G(C, B)$. (Note that the corresponding class in $KK^G(C, K(\ell^2(I)) \otimes B)$ is just given by the $G$-map $\varphi : C \to K(\ell^2(I)) \otimes B$.)

We use this to construct equivariant $KK$-elements as follows:

**E1** Let $B = C_0(\Omega), C = C_0(I) = \bigoplus_{i \in I} C$ and let $\varphi_i : C \to C_0(\Omega)$ be given by $\varphi_i(1) = e_i = 1_{V_i}$ as above. Then it is straightforward to check that the resulting homomorphism $\varphi = \bigoplus_{i \in I} \varphi_i : C_0(I) \to K(\ell^2(I)) \otimes C_0(\Omega)$ is $G$-equivariant as required in the previous paragraph, and we obtain a class $x \in KK_G(C_0(I), C_0(\Omega))$.

**E2** More generally, let $\alpha : G \to \text{Aut}(A)$ be an action of $G$ on a $C^*$-algebra $A$. Consider the case $B = A \otimes C_0(\Omega)$ and $C = A \otimes C_0(I) \cong \bigoplus_{i \in I} A$, $\psi_i = \text{id}_A \otimes \varphi_i : A \otimes C \to A \otimes C_0(\Omega)$. Then $\psi := \bigoplus_{i \in I} \psi_i : A \otimes C_0(I) \to$
\[ K(\ell^2(I)) \otimes A \otimes C_0(\Omega) \] can be identified with \( \text{id}_A \otimes \varphi \) after applying the flip isomorphism \( K(\ell^2(I)) \otimes A \otimes C_0(\Omega) \cong A \otimes K(\ell^2(I)) \otimes C_0(\Omega) \). Hence the corresponding class in \( KK^G(A \otimes C_0(I), A \otimes C_0(\Omega)) \) coincides with \( \text{id}_A \otimes_{\mathbb{C}} x \) where \( x \in KK^G(C_0(I), C_0(\Omega)) \) is as in (E1).

We need some observations regarding this construction. We start with

**Lemma 3.5.** Let \( H \) be a closed subgroup of \( G \) and suppose that \( C = \bigoplus_{i \in I} C_i \) and \( x \in KK^G(C, B) \) are as in the general construction above. Then for each \( H \)-invariant subset \( J \subseteq I \) let \( C_J := \bigoplus_{j \in J} C_i \subseteq C \) and let \( B_J \subseteq B \) denote the smallest \( H \)-invariant \( C^* \)-subalgebra of \( B \) which contains all images \( \varphi_i(C_i) \), \( i \in J \). Then the above construction applied to \( C_J, B_J \) and \( H \) gives a class

\[
x_J = [(\ell^2(J) \otimes B_J, \varphi_J, 0)] \in KK^H(C_J, B_J),
\]

with \( \varphi_J = \bigoplus_{j \in J} \varphi_i \). Moreover, if \( i^J_J : C_J \rightarrow C \) and \( i^B_J : B_J \rightarrow B \) denote the inclusions, then

\[
[i^C_J] \otimes_{C_J} \text{res}^G_J(x) = x_J \otimes_{B_J} [i^B_J] \in KK^H(C_J, B).
\]

**Proof.** The product \( x_J \otimes_{B_J} [i^B_J] \) is represented by the Kasparov triple \( (\ell^2(J) \otimes B_J) \otimes B_J, B, \varphi_J \otimes 1_B, 0) \cong (\ell^2(J) \otimes B, (i^J_J \otimes i^B_J) \circ \varphi_J, 0) \), while the product \( [i^C_J] \otimes_{C_J} \text{res}^G_J(x) \) is represented by the triple \( (\ell^2(I) \otimes B, (i^J_J \otimes i^B_J) \circ \varphi_J, 0) \), where \( i^J_J : K(\ell^2(J)) \rightarrow K(\ell^2(I)) \) denotes the canonical inclusion. Since both triples differ by the degenerate triple \( (\ell^2(I \setminus J) \otimes B, 0, 0) \), the result follows.

We note that in the alternative picture where we regard \( x \) as an element in \( KK^G(C, K(\ell^2(I)) \otimes B) \) and \( x_J \) as an element in \( KK^H(C_J, K(\ell^2(J)) \otimes B_J) \), the equation of the above lemma translates into the equation

\[
[i^C_J] \otimes_{C_J} \text{res}^G_J(x) = x_J \otimes_{K(\ell^2(J)) \otimes B_J} [i^K_J \otimes i^B_J].
\]

This follows from the equation \( \varphi \circ i^J_J = (i^K_J \otimes i^B_J) \circ \varphi_J \) (which is easily checked on each summand \( C_i \) of \( C_J \)). In the following lemma, \( K_*^H(C) = KK_*^H(C, C) \) denotes \( H \)-equivariant \( K \)-theory for the compact subgroup \( H \) of \( G \). Note that since \( I \) is discrete, it follows from Remark 3.3 that the orbits for the action of \( H \) on \( I \) are automatically finite.

**Lemma 3.6.** Suppose that \( H \subseteq G \) is a compact subgroup and let \( F \) denote the set of all finite \( H \)-invariant subsets of \( I \) ordered by inclusion. Then \( K_*^H(C) = \text{lim}_{J \in F} K_*^H(C_J), K_*^H(B_I) = \text{lim}_{J \in F} K_*^H(B_J) \) and we obtain a commutative diagram

\[
\begin{array}{ccc}
K_*^H(C) & \xrightarrow{\text{lim}_{J \in F} (| \otimes_{C_J} x_J)} & \text{lim}_{J \in F} K_*^H(B_J) \\
\cong & & \cong \\
\downarrow & & \downarrow \\
K_*^H(C) & \xrightarrow{| \otimes_{\text{res}^G_J(x)}|} & K_*^H(B_I).
\end{array}
\]

In particular, if all maps \( | \otimes_{C_J} x_J : K_*^H(C_J) \rightarrow K_*^H(B_J) \) are isomorphisms, the same is true for \( | \otimes_{C_J} \text{res}^G_J(x) : K_*^H(C) \rightarrow K_*^H(B_I) \).

**Proof.** We note first that equivariant \( K \)-theory \( K_*^H(C) \) is continuous for compact groups \( H \), since it can be identified with \( K_*(C \rtimes H) \) via the Green-Julg theorem, and
hence continuity follows from continuity of ordinary $K$-theory. Now the previous lemma implies commutativity of the diagram

$$
\begin{array}{ccc}
K^H_*(C_J) & \xrightarrow{[\cdot] \otimes_{C_J} x_J} & K^H_*(B_J) \\
\downarrow \iota_J^C & & \downarrow \iota_B^F \\
K^H_*(C) & \xrightarrow{[\cdot] \otimes_{Cres} G(x)} & K^H_*(B_I).
\end{array}
$$

and the result then follows from taking limits. \qed

Remark 3.7. Suppose that $H$ is a compact group and $x \in KK^H(C, B)$. Let $j_H(x) \in KK(C \rtimes H, B \rtimes H)$ denote the descent of $x$. Recall that the Green-Julg isomorphism

$$\mu_C^H : K_*^H(C, C) \xrightarrow{\cong} K_*(C \rtimes H)$$

can be described as the composition

$$KK_*^H(C, C) \xrightarrow{\iota} KK_*(C^*(H), C \rtimes H) \xrightarrow{p^*} KK(C, C \rtimes H)$$

where $p : C \to C^*(H)$ sends $1 \in C$ to the projection $1_H \in C(H) \subseteq C^*(H)$ (which is the projection corresponding to the trivial representation $1_H$ of $H$ in the Peter-Weyl decomposition of $C^*(H)$). Then the diagram

$$
\begin{array}{ccc}
K^H_* (C) & \xrightarrow{[\cdot] \otimes x} & K^H_* (B) \\
\downarrow \iota_C^H & & \downarrow \iota_B^H \\
K_* (C \rtimes H) & \xrightarrow{[\cdot] \otimes j_H(x)} & K_* (B \rtimes H)
\end{array}
$$

commutes. This follows from the fact that $j_H$ preserves Kasparov products, and hence

$$\mu_C^H(y) \otimes_{C \rtimes H} j_H(x) = ([p] \otimes_{C^*(H)} j_H(y)) \otimes_{C \rtimes H} j_H(x)$$

$$= [p] \otimes_{C^*(H)} j_H(y \otimes_C x) = \mu_B^H(y \otimes_C x)$$

for all $y \in KK^H(C, C)$. We therefore may replace $H$-equivariant $K$-theory by the $K$-theory of the corresponding crossed products everywhere in the above lemma. In particular, we see that if all maps $[\cdot] \otimes_{C_J} x_J : K_*^H(C_J) \to K_*^H(B_J)$ in that lemma are isomorphisms, then the map

$$[\cdot] \otimes j_H(x) : K_* (C \rtimes H) \to K_* (B \rtimes H)$$

is an isomorphism, too.

We are now coming back to the special situation where the commutative $C^*$-algebra $D = C_0(\Omega)$ is generated by the collection $\{e_i = 1_{V_i} : i \in I\}$ of projections corresponding to the $G$-invariant regular basis $\mathcal{V} = \{V_i : i \in I\}$. Since $\mathcal{V}$ is closed under finite intersections (up to $0$) it follows that the family of projections $\{e_i : i \in I\}$ is invariant under multiplication (up to $0$). Let us consider the case where $I$ is finite:

**Lemma 3.8.** Let $D$ be a commutative $C^*$-algebra generated by a multiplicatively closed (up to $0$) and independent finite family of projections $\{e_i : i \in I\}$. For each $i \in I$ let $e'_i := e_i - \bigvee_{e_j < e_i} e_j$. Then $\{e'_i\}$ is a family of nonzero pairwise orthogonal projections spanning $D$. The transition matrix $\Gamma = (\gamma_{ij})$ determined by the equations $e_j = \sum_{i \in I} \gamma_{ij} e'_i$, is unipotent and therefore invertible over $\mathbb{Z}$.
Proof. Independence shows that $e'_i \neq 0$ for all $i$. For $i \neq j$ we have at least one of both, $e'_i e_j = 0$ or $e_i e'_j = 0$. Both equalities imply $e'_i e'_j = 0$. Since $\dim(D) \leq |I|$, the $e'_i$ linearly span $D$.

If $e'_i \leq e_j$, then $e'_i \leq e_i e_j \leq e_i$, whence $e_i e_j = e_i$ by definition of $e'_i$. This shows that $\gamma_{ij} = 1$ if $e_i \leq e_j$ and $\gamma_{ij} = 0$ otherwise. Thus, for the partial ordering $i \leq j \Leftrightarrow e_i \leq e_j$, the matrix $\Gamma$ is upper triangular with 1’s on the diagonal. Thus $1 - \Gamma$ is nilpotent of order $|I|$ (because there are no strictly increasing sequences of length $\geq |I|$ in $I$). It follows that $\Gamma$ is invertible with inverse $\sum_{n=0}^{|I|} (1 - \Gamma)^n$.

\[
\text{Remark 3.9. Let } C \text{ and } B \text{ be two finite dimensional commutative } C^*\text{-algebras with bases } \{e_1, \ldots, e_n\} \text{ and } \{b_1, \ldots, b_m\} \text{ consisting of pairwise orthogonal projections and equipped with actions of } H \text{ given by permutations of the bases induced by homomorphisms } \mu_C : H \to S_n, \mu_B : H \to S_m. \text{ In the appendix we show that every } H\text{-equivariant matrix } \Gamma \in M(m \times n, \mathbb{Z}) \text{ gives rise to a canonical element}
\]

\[
x_T \in KK^H(C, B)
\]

which by Lemma 7.2 is invertible if and only if $\Gamma$ is invertible. We want to compare this construction with the construction of the element $x_J \in KK^H(C_0(J), D_J)$ in which $J \subseteq I$ is a finite $H$-invariant set and $D_J \subseteq C_0(\Omega)$ is the subalgebra of $C_0(\Omega)$ generated by $\{e_i : i \in J\}$ which we assume to be closed under multiplication up to 0.

Let $\{e_i' : i \in J\}$ be the set of orthogonal projections constructed from $\{e_i : i \in J\}$ as in the above lemma and let $\{c_i : i \in J\}$ be the standard basis of $C_0(J)$. Identifying $J$ with $\{1, \ldots, n\}$ for $n = |J|$, we see from the above lemma that the transition matrix $\Gamma$ for passing from $\{e_i : i \in J\}$ to $\{e_i' : i \in J\}$ is invertible and has only entries 0 or 1. Moreover, the element $x_J \in KK^H(C_0(J), D_J)$ coming from our general construction with $C = C_0(J)$ and $B = D_J$ is given by the Kasparov cycle $[\mathcal{E}_J, \varphi_J, 0]$ with $\mathcal{E}_J = \ell^2(J) \otimes D_J = \bigoplus_{j=1}^{|J|} D_J$ and in which $\varphi_J(c_j)$ acts via the projection $e_j$ in the $j$th component of this direct sum and as 0 in all other components. Thus we may restrict the module to the nondegenerate part $\mathcal{E}_0 := \varphi_J(C_0(J)) \mathcal{E}_J$, which is $\mathcal{E}_0 = \bigoplus_{j=1}^{|J|} e_j D_J$.

On the other hand, the element $x_T \in KK^H(C_0(J), D_J)$ constructed in the appendix from the transition matrix $\Gamma$ and the bases $\{c_1, \ldots, c_n\}$ of $C_0(J)$ and $\{e'_1, \ldots, e'_n\}$ of $D_J$ is given by the Kasparov cycle $[\mathcal{E}, \psi, 0]$ in which $\mathcal{E} = \bigoplus_{j=1}^{|J|} \bigoplus_{i=1}^n (\mathbb{C} \otimes \mathbb{C} e'_i)$ and where $\psi(c_j)$ acts via the projection of the $j$-th summand $\bigoplus_{i=1}^n (\mathbb{C} \otimes \mathbb{C} e'_i)$ of this module. Now, since $e_j = \sum_{i=1}^n \gamma_{ij} e'_i$ and $\gamma_{ij}$ only takes values 0 or 1, one easily checks that $e_j D_J \cong \bigoplus_{i=1}^n \gamma_{ij} e'_i D_J \cong \bigoplus_{i=1}^n (\mathbb{C} \otimes \mathbb{C} e'_i)$ as Hilbert $D_J$-modules and that this isomorphism intertwines $\varphi(c_j)$ with $\psi(e_j)$ for all $1 \leq j \leq n$. This proves $x_J = x_T \in KK^H(C_0(J), D_J)$. In particular, it follows that $x_J$ is invertible!

We are now ready for the main result of this section.

\[
\text{Proposition 3.10. Let } x \in KK^G(C_0(I), C_0(\Omega)) \text{ be as in Notation 3.4 (N1) and let } A \text{ be any } G\text{-algebra. Then for any compact subgroup } H \subseteq G \text{ the restriction}
\]

\[
\text{res}_H^G([\text{id}_A] \otimes_C x) \in KK^H(A \otimes C_0(I), A \otimes C_0(\Omega))
\]

of the class $[\text{id}_A] \otimes_C x \in KK^G(A \otimes C_0(I), A \otimes C_0(\Omega))$ induces isomorphisms

\[
[\cdot] \otimes \text{res}_H^G([\text{id}_A] \otimes_C x) : K^H_*(A \otimes C_0(I)) \xrightarrow{\cong} K_H^G(A \otimes C_0(I))
\]
and, via descent,

\[ [\cdot] \otimes j_H(\text{res}_H^G([\text{id}_A] \otimes x)) : K_*\left((A \otimes C_0(I)) \rtimes H\right) \overset{\sim}{\to} K_*\left((A \otimes C_0(\Omega)) \rtimes H\right). \]

Moreover, if \( H \subseteq G \) is compact such that \( A \rtimes H_i \) lies in the bootstrap class for all stabilizers \( H_i = \{h \in H : hi = i\} \) (this is for example always true if \( A \) is type I) or if \( H = \{e\} \) is the trivial group, then \((A \otimes C_0(I)) \rtimes H \) and \((A \otimes C_0(\Omega)) \rtimes H \) are KK-equivalent.

**Proof.** It follows from Lemma 3.9 (applied to the class \([\text{id}_A] \otimes x\) constructed as in (E2)) that it suffices to show that the corresponding classes \([\text{id}_A] \otimes x, J) \in KK^H(A \otimes C_0(J), A \otimes D_J)\) are invertible for any \( H \)-invariant finite subset \( J \subseteq I \) such that \( \{e_i : i \in J\} \cup \{0\} \) is multiplicatively closed, where \( D_J \subseteq C_0(\Omega) \) is the subalgebra generated by \( \{e_i : i \in J\} \). But this is the case if for all such \( J \subseteq I \) the classes \( x, J) \in KK^H(C_0(J), D_J)\) are invertible, which follows from Remark 3.9 above.

Suppose now that \( A \rtimes H_i \) lies in the bootstrap class for all \( i \in I \). Then for each finite \( H \)-invariant subset \( J \subseteq I \), we have

\[ (A \otimes C_0(J)) \rtimes H = \bigoplus_{[i] \in [H,J]} (A \otimes C_0(H/H_i)) \rtimes H \sim_M \bigoplus_{i \in H_i} A \rtimes H_i, \]

where the Morita equivalence on the right hand side follows from Green’s imprimity theorem [14 Theorem 17]. It follows that \((A \otimes C_0(I)) \rtimes H\) is in the bootstrap class. On the other hand if \( e'_i = e_i - \bigvee_{e_j < e_i} e_j \) for all \( i \in J \) is as in Lemma 3.8 then the morphism \( A \otimes C_0(J) \to A \otimes D_J \) which sends \( a \otimes 1_{\{i\}} \) to \( a \otimes e'_i \) for all \( i \in J \) is an \( H \)-equivariant isomorphism, and hence \((A \otimes C_0(J)) \rtimes H \cong (A \otimes D_J) \rtimes H\). Since the bootstrap class is closed under inductive limits, it follows that \((A \otimes C_0(I)) \rtimes H\) and \((A \otimes C_0(\Omega)) \rtimes H\) both lie in the bootstrap class, and hence satisfy the UCT. Thus the desired KK-equivalence follows from isomorphism in K-theory.

If \( H = \{e\} \) is the trivial subgroup of \( G \), then it follows from the above arguments (with \( A = C \)) that \( x \in KK(C_0(I), C_0(\Omega))\) is a KK-equivalence, from which it follows that \([\text{id}_A] \otimes x\) is a KK-equivalence between \( A \otimes C_0(I)\) and \( A \otimes C_0(\Omega)\).

**Remark 3.11.** The assertion on type I algebras \( A \) in the above proposition follows from the fact that all type I C*-algebras lie in the bootstrap class (e.g. see [1]) and the fact (see [29]) that crossed products of type I C*-algebras by compact groups are type I.

Note that the proposition implies in particular, that for all second countable totally disconnected spaces \( \Omega \) and any choice \( \mathcal{V} = \{V_i : i \in I\} \) of a regular basis for the compact open sets in \( \Omega \) (which exists by Proposition 2.12) the class \( x \in KK(C_0(I), C_0(\Omega))\) constructed above is a KK-equivalence.

Combining Proposition 3.10 with Proposition 3.3 we now get:

**Theorem 3.12.** Suppose that \( G \) satisfies the Baum-Connes conjecture with coefficients in \( A \otimes C_0(I) \) and in \( A \otimes C_0(\Omega) \). Then

\[ [\cdot] \otimes j_G([\text{id}_A] \otimes x) : K_*\left((A \otimes C_0(I)) \rtimes_{\alpha \otimes \mu, r} G\right) \to K_*\left((A \otimes C_0(\Omega)) \rtimes_{\alpha \otimes \tau, r} G\right) \]

is an isomorphism. Moreover, if \( G \) satisfies the strong Baum-Connes conjecture and \( A \rtimes H \) lies in the bootstrap class for all compact subgroups \( H \) of \( G \) which lie in some stabilizer \( G_i \) for the action of \( G \) on \( I \), or if \( G \) satisfies the strong Baum-Connes conjecture and has no non-trivial compact subgroups, then

\[ j_G([\text{id}_A] \otimes x) \in KK\left((A \otimes C_0(I)) \rtimes_{\alpha \otimes \mu, r} G, (A \otimes C_0(\Omega)) \rtimes_{\alpha \otimes \tau, r} G\right) \]
is a $KK$-equivalence.

**Remark 3.13.** If $I$ is a countable discrete $G$-space, then it follows from the results of [3, Theorem 2.5] and [4, Proposition 2.6] and the decomposition of $A \otimes C_0(I)$ as given in (3.1) below, that $G$ satisfies the Baum-Connes conjecture with coefficients in $A \otimes C_0(I)$ if (and only if) all stabilizers $G_i$ for the action of $G$ on $I$ satisfy the conjecture with coefficients in $A$ with respect to the restriction of the given action of $G$ on $A$ to the subgroups $G_i$.

On the other hand, $G$ satisfies the Baum-Connes conjecture with coefficients in $A \otimes C_0(\Omega)$ if and only if the transformation groupoid $\Omega \rtimes G$ satisfies the groupoid version of the Baum-Connes conjecture with coefficients in $A \otimes C_0(\Omega)$ induced by the given $G$-action on $A$. We refer to [30] for the formulation of the Baum-Connes conjecture for groupoids. There it is shown that $\Omega \rtimes G$ satisfies the Baum-Connes conjecture with arbitrary coefficients if the groupoid $\Omega \rtimes G$ is amenable (or, more generally, a-$T$-menable). This situation is much more general than simply assuming that $G$ is amenable or a-$T$-menable. Thus we see that the conditions on the Baum-Connes conjecture in Theorem 3.12 are in particular satisfied for every $G$-algebra $A$ if the following two conditions hold:

1. All stabilizers $G_i$ for the action of $G$ on $I$ are amenable (or a-$T$-menable), and
2. the transformation groupoid $\Omega \rtimes G$ is amenable (or a-$T$-menable).

Since $I$ is discrete, we get a decomposition of $A \otimes C_0(I)$ as a direct sum of $G$-algebras

\[
A \otimes C_0(I) \cong \bigoplus_{[i] \in G \backslash I} A \otimes C_0(G \cdot i) \cong \bigoplus_{[i] \in G \backslash I} A \otimes C_0(G/G_i) \tag{3.1}
\]

where $G_i$ denotes the stabilizer $G_i := \{ g \in G : g \cdot i = i \}$ for $i \in I$ under the $G$-action. We therefore get a decomposition of the reduced crossed products

\[
(A \otimes C_0(I)) \rtimes_{\alpha \otimes \mu, r} G \cong \bigoplus_{[i] \in G \backslash I} (A \otimes C_0(G/G_i)) \rtimes_{\alpha \otimes \mu_i, r} G,
\]

where $\mu_i : G \to \text{Aut}(C_0(G/G_i))$ is the action by left translation. Thus, by a version of Green’s imprimitivity theorem (see [14, Theorem 17] and [21]) we have a canonical Morita equivalence

\[
(A \otimes C_0(G/G_i)) \rtimes_{\alpha \otimes \mu_i, r} G \sim_M A \rtimes_{\alpha, r} G_i.
\]

Therefore, by Morita invariance and continuity of $K$-theory, we obtain a canonical isomorphism

\[
\bigoplus_{[i] \in G \backslash I} K_*(A \rtimes_{\alpha, r} G_i) \cong K_*((A \otimes C_0(I)) \rtimes_{\alpha \otimes \mu, r} G).
\]

Combining this with the isomorphism of Theorem 3.12 we obtain

**Corollary 3.14.** Suppose that $G$, $\Omega$, $\alpha : G \to \text{Aut}(A)$ and $V = \{ V_i : i \in I \}$ are as in Theorem 3.12. Then there is a canonical $KK$-equivalence $y_I \in KK_0(\bigoplus_{[i] \in G \backslash I} A \rtimes_{\alpha, r} G_i, (A \otimes C_0(I)) \rtimes_{\alpha \otimes \mu, r} G)$. Combined with the isomorphism of Theorem 3.12 we get an isomorphism

\[
\bigoplus_{[i] \in G \backslash I} K_*(A \rtimes_{\alpha, r} G_i) \cong K_*((A \otimes C_0(\Omega)) \rtimes_{\alpha \otimes \tau, r} G).
\]
If we have $KK$-equivalence in Theorem 3.14, then the above isomorphism is also induced by a $KK$-equivalence.

In particular, in the special case $A = \mathbb{C}$ we get an isomorphism

$$\bigoplus_{[i] \in G \setminus I} K_*(C^*_r(G_i)) \cong K_*(C_0(\Omega) \rtimes_{\tau,r} G).$$

If $G$ satisfies the strong Baum-Connes conjecture, this isomorphism is obtained from a $KK$-equivalence $\bigoplus_{[i] \in G \setminus I} C^*_r(G_i) \sim_{KK} C_0(\Omega) \rtimes_{\tau,r} G$.

Remark 3.15. We point out that the isomorphism

$$\bigoplus_{[i] \in G \setminus I} K_*(A \rtimes_{\alpha,r} G_i) \cong K_*\left((A \otimes C_0(\Omega)) \rtimes_{\alpha \otimes \mu,r} G\right)$$

of the theorem is induced by a $*$-homomorphism

$$\Psi : \bigoplus_{[i] \in G \setminus I} A \rtimes_{\alpha,r} G_i \to K(\ell^2(I)) \otimes \left((A \otimes C_0(\Omega)) \rtimes_{\alpha \otimes \mu,r} G\right).$$

which can be described as follows. First of all we have a homomorphism

$$\eta_i : A \rtimes_{\alpha,r} G_i \to (A \otimes C_0(\Omega)) \rtimes_{\alpha \otimes \tau,r} G$$

given by the inclusion $A \rtimes_{\alpha,r} G_i \hookrightarrow (A \otimes C_0(\Omega)) \rtimes_{\alpha \otimes \tau,r} G_i$ induced from the $G_i$-equivariant inclusion $A \hookrightarrow A \otimes C_0(\Omega); a \mapsto a \otimes 1_{\Omega}$ followed by the inclusion $(A \otimes C_0(\Omega)) \rtimes_{\alpha \otimes \tau,r} G_i \hookrightarrow (A \otimes C_0(\Omega)) \rtimes_{\alpha \otimes \tau,r} G$. This maps $A \rtimes_{\alpha,r} G_i$ bijectively onto a full corner of $(A \otimes C_0(\Omega)) \rtimes_{\alpha \otimes \tau,r} G$ which establishes Green’s Morita equivalence $A \rtimes_{\alpha,r} G_i \sim_{M} (A \otimes C_0(\Omega)) \rtimes_{\alpha \otimes \tau,r} G$. Using the decomposition

$$(A \otimes C_0(\Omega)) \rtimes_{\alpha \otimes \tau,r} G \cong \bigoplus_{[i] \in G \setminus I} (A \otimes C_0(\Omega)) \rtimes_{\alpha \otimes \tau,r} G,$$

we then obtain a $*$-homomorphism

$$\eta = \bigoplus_{[i] \in G \setminus I} \eta_i : \bigoplus_{[i] \in G \setminus I} A \rtimes_{\alpha,r} G_i \to (A \otimes C_0(\Omega)) \rtimes_{\alpha \otimes \tau,r} G$$

which induces the $KK$-equivalence $y_I$ of Corollary 3.14.

We then recall from our constructions in Notation 3.4 that, after passing from $A \otimes C_0(\Omega)$ to the stabilization $A \otimes K(\ell^2(I)) \otimes C_0(\Omega)$ (with action on $K(\ell^2(I))$) given by $\text{Ad} \mu$, where by abuse of notation we let $\mu$ denote the unitary representation of $G$ on $\ell^2(I)$ induced by the given action $\mu$ of $G$ on $I$, the equivariant $KK$-class $[id_A] \otimes x \in KK^G(A \otimes C_0(\Omega), A \otimes C_0(\Omega))$ represented by the $G$-equivariant $*$-homomorphism $id_A \otimes \varphi : A \otimes C_0(\Omega) \to A \otimes K(\ell^2(I)) \otimes C_0(\Omega)$ with $\varphi = \bigoplus_{i \in I} \varphi_i$ as in Notation 3.4. Recall that $\varphi_i$ sends the element $a \otimes \delta_i \in A \otimes C_0(\Omega)$ to the element $a \otimes p_i \otimes 1_{\Omega}$, where $p_i$ denotes the orthogonal projection onto the $i$th component of $\ell^2(I)$. Thus, the descent of $[id_A] \otimes x$ is represented, up to stabilization, by the $*$-homomorphism

$$\psi := (id_A \otimes \varphi) \rtimes G : (A \otimes C_0(\Omega)) \rtimes_{\alpha \otimes \mu} G \to (A \otimes K(\ell^2(I)) \otimes C_0(\Omega)) \rtimes_{\alpha \otimes \text{Ad} \mu} G.$$
the other hand, following the description of $\Psi$ in the remark, we get $\Psi(\ldots)$. Therefore, the restriction of the isomorphism

$$\pi_i : A \rtimes_{a,r} G_i \to (A \otimes C_0(\Omega)) \rtimes_{a\otimes \tau} G$$

(3.2)

given on a typical element $a \in A$, $g \in G_i$ by $\pi_i(a) = (a \otimes 1_{V_i})u_g$.

**Lemma 3.16.** For each $i \in I$ let $\theta_i : A \rtimes_{a,r} G_i \to \bigoplus_{\{j\} \in G \setminus I} A \rtimes_{a,r} G_j$ denote the canonical inclusion and let $\pi_i$ be as in (3.2) above. Then we have

$$[\pi_i] = [\theta_i] \otimes j_G([\text{id}_A] \otimes \mathcal{C} x) \in KK(A \rtimes_{a,r} G_i, (A \otimes C_0(\Omega)) \rtimes_{a\otimes \tau} G).$$

Therefore, the restriction of the isomorphism

$$\bigoplus_{\{i\} \in G \setminus I} \pi_i : A \rtimes_{a,r} G_i \to K(A \rtimes_{a,r} G_i, (A \otimes C_0(\Omega)) \rtimes_{a\otimes \tau} G)$$

of Corollary 3.14 to the summand $K(A \rtimes_{a,r} G_i)$ is given by

$$(\pi_i)_* : K(A \rtimes_{a,r} G_i) \to K((A \otimes C_0(\Omega)) \rtimes_{a\otimes \tau} G).$$

**Proof.** Let $p_i \in K(\ell^2(I))$ be the projection on the $i$th component of $\ell^2(I)$ and let

$$\theta_i : (A \otimes C_0(\Omega)) \rtimes_{a\otimes \tau} G \to K(\ell^2(I)) \otimes (A \otimes C_0(\Omega)) \rtimes_{a\otimes \tau} G,$$

where $\theta_i(x) = p_i \otimes x$ be the $KK$-equivalence induced by $p_i$ (note that this $KK$-equivalence does not depend on the particular choice of the rank-one projection $p_i \in K(\ell^2(I))$). By Remark 3.17 we have

$$\Psi : \bigoplus_{\{i\} \in G \setminus I} A \rtimes_{a,r} G_i \to K(\ell^2(I)) \otimes (A \otimes C_0(\Omega)) \rtimes_{a\otimes \tau} G$$

as in the remark. So it suffices to show that $\theta_i \circ \pi_i = \Psi \circ j_i$. By construction, on a typical element $au_g \in A \rtimes_{a,r} G_i$ we have $\theta_i(au_g) = p_i \otimes (a \otimes 1_{V_i})u_g$. On the other hand, following the description of $\Psi$ in the remark, we get $\Psi(j_i(au_g)) = p_i \mu_g \otimes (a \otimes 1_{V_i})u_g$. But remember that at this point, $\mu_g$ is the unitary operator acting on $\ell^2(I)$ via the action of $G$ on $I$. Since $g \in G_i$ lies in the stabilizer of $i \in I$, we get $p_i \mu_g = p_i$. Thus $\theta_i(au_g) = \Psi(j_i(au_g))$ for all $a \in A$, $g \in G_i$ and the result follows. 

**Example 3.17.** As a first example we want to study the $K$-theory of the Cantor set $\Omega = \{1, -1\}^\mathbb{Z}$ of Example 3.14 i.e., we consider the action of $\mathbb{Z}$ on $\Omega$ given by the shift $(m \cdot x)_n = x_{n-m}$ for $m \in \mathbb{Z}$ and $x = (x_n)_{n \in \mathbb{Z}} \in \Omega$. Let $\mathcal{F}(\mathbb{Z})$ denote the family of finite subsets of $\mathbb{Z}$ and let $\mathcal{V} = \{V_F : F \in \mathcal{F}(\mathbb{Z})\}$ be the regular basis for the compact open sets in $\Omega$ as constructed in Example 2.14. Then the corresponding action of $\mathbb{Z}$ on $\mathcal{F}(\mathbb{Z})$ is given by translation. It is free on $\mathcal{F}(\mathbb{Z})^* := \mathcal{F}(\mathbb{Z}) \setminus \{\emptyset\}$ and it fixes the empty set. Thus, since $\mathbb{Z}$ satisfies the strong Baum-Connes conjecture, Corollary 3.14 holds.
shows that for any action $\alpha : \mathbb{Z} \to \text{Aut}(A)$, we obtain a $KK$-equivalence between $(A \rtimes_\alpha \mathbb{Z}) \oplus \left( \bigoplus_{[F] \in \mathbb{Z}\setminus \mathcal{F}(\mathbb{Z})^*} A \right)$ and $(A \otimes C(\Omega)) \rtimes_{\alpha \otimes \tau} \mathbb{Z}$. In particular, we obtain an isomorphism

$$K_*(A \rtimes_\alpha \mathbb{Z}) \oplus \left( \bigoplus_{[F] \in \mathbb{Z}\setminus \mathcal{F}(\mathbb{Z})^*} K_*(A) \right) \cong K_*(A \otimes C(\Omega)) \rtimes_{\alpha \otimes \tau} \mathbb{Z}. $$

On the summand $K_*(A)$ corresponding to some $[F] \in \mathbb{Z}\setminus \mathcal{F}(\mathbb{Z})^*$, the isomorphism is induced by the inclusion $a \mapsto a \otimes 1_{V_F} \in A \otimes C(\Omega) \subseteq (A \otimes C(\Omega)) \rtimes_{\alpha \otimes \tau} \mathbb{Z}$. On the summand $A \rtimes_\alpha \mathbb{Z}$ it is given by the descent of the $\mathbb{Z}$-equivariant inclusion $a \mapsto a \otimes 1_{\Omega}$ of $A$ into $A \otimes C(\Omega)$. In the special case where $A = \mathbb{C}$ we obtain a $KK$-equivalence between $C(T) \rtimes_\mathbb{Z} \mathbb{Z}(\mathbb{Z}\setminus \mathcal{F}(\mathbb{Z})^*)$ and $C_0(\Omega) \rtimes_\tau \mathbb{Z}$.

We now present examples of actions for which a $G$-invariant regular basis of the compact open sets of $\Omega$ cannot exist. Our first example shows that there are indeed many $\mathbb{Z}$-actions on the Cantor set, which do not allow a $\mathbb{Z}$-invariant regular basis for the compact open sets.

**Example 3.18.** For any prime $p$ let $\mathbb{Z}_p$ denote the ring of $p$-adic integers. The underlying additive group is totally disconnected and compact and, as a space, is homeomorphic to the Cantor set. Moreover, $\mathbb{Z}$ embeds into $\mathbb{Z}_p$ as a dense subgroup via $n \mapsto n \cdot 1_p$, where $1_p$ denotes the multiplicative unit of $\mathbb{Z}_p$. Consider the translation action $\tau$ of $\mathbb{Z}$ on $\mathbb{Z}_p$ given by $\tau_n(x) = x + n1_p$. This is a minimal action of the type as studied by Riedel in [28]. Let $\chi : \mathbb{Z}_p \to T$ denote the character given by evaluation at $1_p$. Since $1_p$ generates a dense subgroup of $\mathbb{Z}_p$, the character $\chi$ is faithful. The image $G := \chi(\mathbb{Z}_p)$ is the Prüfer $p$-group, i.e., the union of all cyclic subgroups of $T$ with order a power $p^n$ of $p$. Let $\hat{\tau}$ denote the translation action of $\mathbb{Z}_p \cong G$ on $T$. Then there is a canonical isomorphism

$$C(T) \rtimes_{\hat{\tau}} \mathbb{Z}_p \cong C(G) \rtimes_\tau \mathbb{Z},$$

which can be obtained by representing the crossed products faithfully on $L^2(\mathbb{Z}_p \times T)$ and $L^2(\mathbb{Z} \times \mathbb{Z}_p)$ via the canonical regular representations, respectively, and then check that conjugation with the Plancherel isomorphism $\Psi : L^2(\mathbb{Z}_p \times T) \to L^2(\mathbb{Z}_p \times \mathbb{Z}) \cong L^2(\mathbb{Z} \times \mathbb{Z}_p)$ induces the desired isomorphism. Thus we can apply [28, Theorem 3.6] which implies that $K_0(C_0(\mathbb{Z}_p) \rtimes \mathbb{Z})$ is isomorphic to the group $\mathbb{Z}[\frac{1}{p}] = \{ \frac{k}{l} : k \in \mathbb{Z}, l \in \mathbb{N}_0 \} = \{ \frac{k}{l} : k \in \mathbb{Z}, l \in \mathbb{N}_0 \}$ (note that the crossed product in question is also isomorphic to the well known Bunce-Deddens algebra). The abelian group $\mathbb{Z}[\frac{1}{p}]$ is not isomorphic to any direct sum of copies of $\mathbb{Z}$. But if there were a $\mathbb{Z}$-invariant regular basis $V$ for $\mathbb{Z}_p$, Corollary 3.14 would imply that $K_0(C_0(\mathbb{Z}_p) \rtimes \mathbb{Z})$ is isomorphic to a direct sum of copies of $\mathbb{Z}$.

More examples of the above type can be obtained from the results in [28]. For example, the odometer actions described in [20, p. 332] give a big class of $\mathbb{Z}$-actions on the Cantor set for which a $\mathbb{Z}$-invariant regular basis for the compact open sets cannot exist. The following corollary of Theorem 3.12 will be used to give such an example with an action of the free group $F_n$.

**Corollary 3.19.** Let $G$, $\Omega$ be as in Theorem 3.12 (in particular, we assume that $\Omega$ has a $G$-invariant regular basis). In addition, let $G$ be discrete and $K$-amenable in the sense of [21]. Then $K_0(C_0(\Omega) \rtimes_{\tau,G} G)$ contains a copy of $\mathbb{Z}$ as a direct summand.
Corollary 3.14 tells us that $\bigoplus_{i \in G} K_0(C^*_r(G_i)) \cong K_0(C_0(\Omega) \rtimes_r G)$.

Now since $G$ is $K$-amenable, each of the subgroups $G_i$ is also $K$-amenable by [7]. Thus $K_0(C^*_r(G_i)) \cong K_0(C^*(G_i))$ contains a copy of $K_0(C) \cong \mathbb{Z}$ as a direct summand. □

With the help of this corollary, we can now present more examples for which a $G$-invariant regular basis cannot exist. We thank M. Rørdam who drew our attention to the existence of such examples.

Example 3.20. Consider the dynamical system from [24 § 8.2] with the free group $F_n$ acting on the positive part $(\partial F_n)_+$ of its Gromov boundary by left translations. As observed in [24 § 8.2], the crossed product $C_0((\partial F_n)_+) \rtimes_r F_n$ is Morita equivalent to $O_n$, so $K_0(C_0((\partial F_n)_+) \rtimes_r F_n) \cong \mathbb{Z}/(n-1)\mathbb{Z}$. Thus we conclude using the previous corollary that there cannot exist an $F_n$-invariant regular basis for the compact open subsets of $(\partial F_n)_+$.

Question 3.21. Is there an intrinsic characterization for the existence of such invariant regular bases for the compact open subsets in terms of the underlying topological dynamical system?

Remark 3.22. Even if we cannot find a $G$-invariant regular basis there is always the following regularization procedure:

Let $\Omega$ be a totally disconnected, second countable, locally compact $G$-space and consider the $C^*$-algebra $D = C_0(\Omega)$. We can always find a generating family of compact open subsets $V$ of $\Omega$ such that

- $V \cup \{\emptyset\}$ is closed under finite intersections,
- $V$ is $G$-invariant.

One possibility would be $V = \mathcal{U}_c(\Omega)$. More generally, we can start with an arbitrary generating family $V_0$ and let $V$ be the smallest family satisfying the two desired conditions above and containing $V_0$. Of course, in general, $V$ will not be independent. But we can define $D \langle V \rangle$ as the universal $C^*$-algebra $C^*(\{e_V : V \in V\})$ with the set of relations $\mathcal{R}$ given by:

$$e_V = e_V^* = e_{V^2}^2$$

As explained in [24 § 2], the family of projections $\{e_V : V \in V\} \subseteq D \langle V \rangle$ is independent. And by universal property, there is a canonical surjective homomorphism $D \langle V \rangle \to D$ given by $e_V \mapsto 1_V$. Let $D_1$ be the kernel of this surjection. We then obtain a short exact sequence $0 \to D_1 \to D \langle V \rangle \to D \to 0$, and $D_1$ will be $\{0\}$ if and only if the family $V$ we started with was already independent.

In addition, by universal property of $D \langle V \rangle$, every $g \in G$ gives rise to an automorphism of $D \langle V \rangle$ which is determined by $e_V \mapsto e_{gV}$. With this $G$-action on $D \langle V \rangle$, the canonical homomorphism $D \langle V \rangle \to D$ becomes $G$-equivariant. Thus if $G$ is exact, we obtain from the exact sequence above the following exact sequence of the reduced crossed products:

$$0 \to D_1 \rtimes_r G \to D \langle V \rangle \rtimes_r G \to D \rtimes_r G \to 0.$$

We could also dualize and obtain with $\Omega_1 = \text{Spec} D_1$ and $\Omega \langle V \rangle = \text{Spec}(D \langle V \rangle)$ the following exact sequence:

$$0 \to C_0(\Omega_1) \rtimes_r G \to C_0(\Omega \langle V \rangle) \rtimes_r G \to C_0(\Omega) \rtimes_r G \to 0.$$  \hspace{2cm} (3.3)
Since \( \{e_V : V \in \mathcal{V}\} \subseteq D(\mathcal{V}) \) is independent, this family of projections corresponds to a regular basis of \( \Omega(\mathcal{V}) \), so that our method of computing \( K \)-theory applies to the crossed product in the middle of (3.3). The idea would then be to try to use the six term exact sequence in \( K \)-theory for (3.3) to compute \( K \)-theory for \( C_0(\Omega) \rtimes_r G \). This of course means that we have to compute \( K \)-theory for the ideal in (3.3) first. Since \( D_1 = C_0(\Omega_1) \) is again of the same form as \( D = C_0(\Omega) \), we could iterate this regularization process. However, the question is whether in this iteration, we will at some point be able to determine \( K \)-theory for the kernel, i.e. for the analogue of \( D_1 \rtimes_r G \).

4. \( K \)-theory of semigroup crossed products

In this section we want to apply the results of the previous section to the study of the \( K \)-theory of certain semigroup crossed products. Throughout this section we assume that \( P \subseteq G \) is a subsemigroup of the group \( G \) which contains the unit element \( e \in G \). By a right ideal of \( P \) (resp. a right \( P \)-ideal in \( G \)), we mean a subset \( X \) of \( P \) (resp. \( G \)) such that \( XP = X \). For an arbitrary subset \( X \) of \( G \) and for \( g \in G \) we write \( g \cdot X = \{gx : x \in X\} \subseteq G \) for the translate of \( X \) by \( g \). Moreover, if \( X \subseteq P \) and \( p \in P \) we write \( pX := p \cdot X \) and \( p^{-1}X = \{y \in P : py \in X\} = (p^{-1} \cdot X) \cap P \). It is important to observe the difference between the set \( p^{-1} \cdot X \subseteq G \) and the set \( p^{-1}X \subseteq P \) defined above! We recall from [24] the following definition of constructible right ideals in \( P \) and \( G \):

**Definition 4.1.** Let \( P \subseteq G \) be as above. Then the set of constructible right ideals \( J_P \) of \( P \) is defined as the smallest family of subsets of \( P \) which contains the empty set \( \emptyset \) as well as \( P \) and also \( pX, p^{-1}X \) for all \( X \in J_P \) and \( p \in P \). The set of constructible right \( P \)-ideals \( J_{P \subseteq G} \) in \( G \) is the smallest left translation invariant family of subsets \( X \subseteq G \) which contains \( J_P \) and which is closed under taking finite intersections.

As observed in [23] § 3, \( J_P \) is automatically closed under finite intersections.

If \( Y \) is a discrete space and \( X \subseteq Y \) we let \( E_X : \ell^2(Y) \to \ell^2(X) \subseteq \ell^2(Y) \) denote the orthogonal projection, which is given by multiplication with the characteristic function \( 1_X \) of \( X \). If \( J \subseteq P(Y) \), we let

\[
D(J) = C^*(\{E_X : X \in J\}) \subseteq \mathcal{L}(\ell^2(Y))
\]  

(4.1)

denote the commutative \( C^* \)-algebra generated by the projections \( E_X \), \( X \in J \) and we write \( \Omega(J) \) for the Gelfand dual \( \text{Spec}(D(J)) \). Recall from Lemma 2.2 and Lemma 2.3 that \( \Omega(J) \) is totally disconnected and that the family \( \mathcal{V} = \{V_X : X \in J\} \), with \( V_X := E_X^{-1}(\{1\}) \), generates the compact open subsets of \( \Omega(J) \). Moreover, it is clear that the representation \( M : \ell^\infty(Y) \to \mathcal{L}(\ell^2(Y)) \) by multiplication operators \( M(f)\xi = f \cdot \xi \) restricts to an isomorphism between \( C^*(\{1_X : X \in J\}) \subseteq \ell^\infty(Y) \) and \( D(J) \).

If \( P \subseteq G \) is a subsemigroup of a group \( G \), we put \( D_P := D(J_P) \) and \( D_{P \subseteq G} := D(J_{P \subseteq G}) \) where \( D(J_P) \) and \( D(J_{P \subseteq G}) \) are as in (4.1). Similarly, we shall simply write \( \Omega_P \) and \( \Omega_{P \subseteq G} \) for the corresponding totally disconnected spaces \( \Omega(J_P) \) and \( \Omega(J_{P \subseteq G}) \), respectively. Recall that the reduced left semigroup \( C^* \)-algebra \( C^*_r(P) \) is defined as the sub-\( C^* \)-algebra of \( \mathcal{L}(\ell^2(P)) \) which is generated by the isometries \( V_p : \ell^2(P) \to \ell^2(P) \) given by \( V_p \delta_q = \delta_{pq} \), where \( \delta_q \) denotes the Dirac-function at
$q \in P$. For $X \subseteq P$ let $E_X$ denote the orthogonal projection from $\ell^2(P)$ onto $\ell^2(X) \subseteq \ell^2(P)$ as in the above discussion. Then

$$V_p E_X V_p^* = E_{pX} \quad \text{and} \quad V_p^* E_X V_p = E_{p^{-1}X}.$$  

This shows that $C^*_\lambda(P)$ contains all projections $E_X$ with $X \in J_P$, the set of constructible right ideals in $P$. Thus, we see that $D_P \cong C(\Omega_P)$ is a commutative $C^*$-subalgebra of $C^*_\lambda(P)$. On the other hand, since the set $J_{P_{CG}}$ of constructible right $P$-ideals in $G$ is closed under left translation with elements of $G$, the $C^*$-algebra $D_{P_{CG}} = D(J_{P_{CG}}) \subseteq \ell^\infty(G)$ is also invariant under the left translation action $\tau : G \to \text{Aut}(\ell^\infty(G))$. Thus we obtain a well defined action $\tau : G \to \text{Aut}(D_{P_{CG}})$ given on the generators by

$$\tau_g(E_X) = E_{gX}, \quad \forall X \in J_{P_{CG}}.$$  

In what follows we want to compare $C^*_\lambda(P)$ with the reduced crossed product $D_{P_{CG}} \rtimes_{\tau,r} G \cong C_0(\Omega_{P_{CG}}) \rtimes_{\tau,r} G$. Indeed, we want to consider a more general situation in which we start with an action $\alpha : G \to \text{Aut}(A)$ of $G$ on a $C^*$-algebra $A$. Then $\alpha$ restricts to an action of $P$ on $A$ by automorphisms, and we can form the reduced semigroup crossed product $A \rtimes_{\alpha,r} P$ as follows: Assume that $A$ is represented faithfully and nondegenerately on the Hilbert space $\mathcal{H}$. We then obtain a faithful representation $\tilde{\alpha}_P : A \to \mathcal{L}(\mathcal{H} \otimes \ell^2(P))$ by

$$\tilde{\alpha}_P(a)(\xi \otimes e_x) := a^{-1}(a)\xi \otimes e_x \quad \forall \xi \in \mathcal{H}, x \in P. \quad (4.2)$$  

The reduced semigroup crossed product $A \rtimes_{\alpha,r} P$ is then defined as

$$A \rtimes_{\alpha,r} P := C^*\left(\{\tilde{\alpha}_P(a)(1_H \otimes V_p) : a \in A, p \in P\}\right) \subseteq \mathcal{L}(\mathcal{H} \otimes \ell^2(P)) \quad (4.3)$$  

(see [24] for more details).

Let us now recall some results of [24] concerning the question under what conditions on $P \subseteq G$ we can realize $A \rtimes_{\alpha,r} P$ as a full corner of the reduced crossed product $(A \otimes D_{P_{CG}}) \rtimes_{\alpha \otimes \tau,r} G$. We start by recalling [24, Lemma 3.6]:

**Lemma 4.2.** Let $\pi : A \otimes D_{P_{CG}} \to \mathcal{L}(\mathcal{H} \otimes \ell^2(G))$ be the representation defined by

$$\pi(a \otimes d)(\xi \otimes e_x) = a^{-1}(a)\xi \otimes d e_x.$$  

Then $(\pi, 1_H \otimes \lambda_G)$ is a covariant homomorphism of $(A \otimes D_{P_{CG}}, G, \alpha \otimes \tau)$ on $\mathcal{H} \otimes \ell^2(G)$ which induces a faithful representation of the reduced crossed product $(A \otimes D_{P_{CG}}) \rtimes_{\alpha \otimes \tau,r} G$ on $\mathcal{L}(\mathcal{H} \otimes \ell^2(G))$.

Following the notation of [24] we introduce the following

**Notation 4.3.** We let $A \rtimes_{\alpha,r} (P \subseteq G)$ denote the (isomorphic) image of $(A \otimes D_{P_{CG}}) \rtimes_{\alpha \otimes \tau,r} G$ in $\mathcal{L}(\mathcal{H} \otimes \ell^2(G))$ under the representation $\pi \times (1_H \otimes \lambda_G)$ of the above lemma.

Since $P \in J_P \subseteq J_{P_{CG}}$ we have $1_A \otimes E_P \in M(A \otimes D_{P_{CG}})$ which embeds canonically into the multiplier algebra of $(A \otimes D_{P_{CG}}) \rtimes_{\alpha \otimes \tau,r} G$. Extending the representation $\pi \times (1 \otimes \lambda_G)$ of Lemma 4.2 to the multiplier algebra maps $1_A \otimes E_P$ to the projection $1_H \otimes E_P \in \mathcal{L}(\mathcal{H} \otimes \ell^2(G))$. We therefore may consider the corner

$$(1_H \otimes E_P)(A \rtimes_{\alpha,r} (P \subseteq G))(1_H \otimes E_P) \subseteq \mathcal{L}(\mathcal{H} \otimes \ell^2(P))$$  

inside $A \rtimes_{\alpha,r} (P \subseteq G)$.

The following important lemma is the combination of [24, Lemma 3.8] with [24, Lemma 3.9]:
Lemma 4.4. Let $P \subseteq G$ be a subsemigroup of the group $G$. Then for every system $(A, G, \alpha)$ we have that $(1_H \otimes E_P)(A \rtimes_{\alpha,r} (P \subseteq G))(1_H \otimes E_P)$ is a full corner of $A \rtimes_{\alpha,r} P$, and the following are equivalent:

1. $A \rtimes_{\alpha,r} P = (1_H \otimes E_P)(A \rtimes_{\alpha,r} (P \subseteq G))(1_H \otimes E_P)$ for every $C^*$-dynamical system $(A, G, \alpha)$,

2. $C_r^*(P) = E_P C_r^*(P \subseteq G) E_P$, where we set $C_r^*(P \subseteq G) := C \rtimes_{\text{id,r}} (P \subseteq G) \cong D_{P \subseteq G} \rtimes_{\tau,r} G$,

3. For all $g \in G$ we have $E_P \lambda_g E_P \in C_r^*(P)$, and either of these statements implies $E_P D_{P \subseteq G} E_P \subseteq D_P$.

We now recall the definition of the Toeplitz condition from [24]:

Definition 4.5 (cf. [24] Lemma 3.9 and Definition 4.1]). Let $P \subseteq G$ be a subsemigroup of the group $G$. We say that

1. $P \subseteq G$ satisfies the Toeplitz condition if for all $g \in G$ with $E_P \lambda_g E_P \neq 0$, there exist $p_i, q_i$ in $P$ such that $E_P \lambda_g E_P = V_{p_1}^* V_{q_1} \cdots V_{p_n}^* V_{q_n}$,

2. $P \subseteq G$ satisfies the weak Toeplitz condition if the equivalent conditions (1), (2) and (3) of Lemma 4.4 are satisfied, and

3. $P \subseteq G$ satisfies the $K$-theoretic Toeplitz condition if for every system $(A, G, \alpha)$ the inclusion map $i : A \rtimes_{\alpha,r} P \to (1_H \otimes E_P)(A \rtimes_{\alpha,r} (P \subseteq G))(1_H \otimes E_P)$ induces an isomorphism of $K$-theory groups.

It is clear from condition (3) of Lemma 4.4 that the Toeplitz condition implies the weak Toeplitz condition and it is clear from condition (1) of Lemma 4.4 that the weak Toeplitz condition implies the $K$-theoretic Toeplitz condition. The following result of [24] turns out to be extremely useful

Lemma 4.6 (cf [24] Lemma 4.2]). Let $P \subseteq G$ such that the set $\mathcal{J}_P$ of constructible right ideals in $P$ is independent in the sense of Definition 2.4 above and assume that $P \subseteq G$ satisfies the Toeplitz condition. Then the following are true:

1. The set $\mathcal{J}_{P \subseteq G}$ of constructible right $P$-ideals in $G$ is independent.

2. For all $g \in G$ and $X \in \mathcal{J}_P$ we have $g \cdot X \cap P \in \mathcal{J}_P$.

3. $\mathcal{J}_{P \subseteq G} = \{g \cdot X : g \in G, X \in \mathcal{J}_P\}$.

Since the set $\mathcal{J}_{P \subseteq G}$ of constructible right $P$-ideals in $G$ is closed under finite intersections, it follows that the set of projections $\{E_X : X \in \mathcal{J}_{P \subseteq G}\}$ is closed under multiplication. Moreover, since $D_{P \subseteq G}$ is generated by this set of projections, it follows that $\{E_X : X \in \mathcal{J}_{P \subseteq G}\}$ forms a regular basis for $D_{P \subseteq G} \cong C_0(\Omega_{P \subseteq G})$ if and only if $\mathcal{J}_{P \subseteq G}$ is independent in the sense of Definition 2.4 (which implies that $\{E_X : X \in \mathcal{J}_{P \subseteq G}\}$ is independent in the sense of Definition 2.6). Thus, if this is satisfied, we are precisely in the situation of Theorem 3.12 (which we may apply to the totally disconnected space $\Omega_{P \subseteq G}$ and the regular basis $\mathcal{V} = \{V_X : X \in \mathcal{J}_{P \subseteq G}\}$ for the compact open sets of $\Omega_{P \subseteq G}$ with $V_X \subseteq \overline{A_X}(\{1\})$ for $X \in \mathcal{J}_{P \subseteq G}$). As a consequence we get

Theorem 4.7. Let $\mathcal{I}_{P \subseteq G} := \mathcal{J}_{P \subseteq G} \setminus \{0\}$, let $\alpha : G \to \text{Aut}(A)$ be an action of a countable group $G$ on a separable $C^*$-algebra $A$ and assume that the following conditions are satisfied for $P \subseteq G$ and $A$:

1. $P \subseteq G$ satisfies the $K$-theoretic Toeplitz condition;

2. The set $\mathcal{J}_{P \subseteq G}$ of constructible right $P$-ideals in $G$ is independent;
(3) $G$ satisfies the Baum-Connes conjecture with coefficients in $A \otimes C_0(I_{P \subseteq G})$ and $A \otimes D_{P \subseteq G}$.

Then there is a canonical isomorphism

$$\bigoplus_{[X] \in G \setminus I_{P \subseteq G}} K_\ast(A \rtimes_{\alpha,r} G_X) \cong K_\ast(A \rtimes_{\alpha,r} P).$$  \hspace{1cm} (4.4)$$

**Proof.** Conditions (2) and (3) imply that Corollary 3.14 applies to the regular basis of projections $\{E_X : X \in I_{P \subseteq G}\}$ and to the commutative C*-algebra $D_{P \subseteq G} \cong C_0(\Omega_{P \subseteq G})$ generated by this set. Thus the corollary gives a canonical isomorphism

$$\bigoplus_{[X] \in G \setminus I_{P \subseteq G}} K_\ast(A \rtimes_{\alpha,r} G_X) \cong K_\ast((A \otimes D_{P \subseteq G}) \rtimes_{\alpha \otimes r} G) \cong K_\ast(A \rtimes_{\alpha,r} (P \subseteq G)),$$

where the second isomorphism follows from Lemma 4.2. Since $P \subseteq G$ satisfies the K-theoretic Toeplitz condition, we further have

$$K_\ast(A \rtimes_{\alpha,r} P) \cong K_\ast((1_H \otimes E_P)(A \rtimes_{\alpha,r} (P \subseteq G))(1_H \otimes E_P)) \cong K_\ast(A \rtimes_{\alpha,r} (P \subseteq G)),$$

where the second isomorphism follows from the fact that $1_H \otimes E_P$ is a full projection in $M(A \rtimes_{\alpha,r} (P \subseteq G))$. \qed

**Remark 4.8.** Using Lemma 4.6 we see that conditions (1) and (2) in Theorem 4.7 can be replaced by the following (stronger) conditions

(1') $P \subseteq G$ satisfies the Toeplitz condition, and

(2') the set $J_P$ of constructible right ideals in $P$ is independent.

It is often easier to check these conditions rather than conditions (1) and (2) of the theorem.

We should also remark that if $J_{P \subseteq G}$ is independent and $P \subseteq G$ satisfies the Toeplitz condition, then $(A \otimes D_{P \subseteq G}) \rtimes_{\alpha \otimes r} G \cong A \rtimes_{\alpha,r} (P \subseteq G)$ is Morita equivalent, and hence $KK$-equivalent to $A \rtimes_{\alpha,r} P$. Thus if $G$ satisfies the strong Baum-Connes conjecture and if $A \rtimes H$ lies in the bootstrap class for every finite subgroup $H$ of $G$ which stabilizes some ideal $X \in I_{P \subseteq G}$ or if $G$ satisfies the strong Baum-Connes conjecture and has no non-trivial finite subgroups, then we even get a $KK$-equivalence

$$\bigoplus_{[X] \in G \setminus I_{P \subseteq G}} A \rtimes_{\alpha,r} G_X \sim_{KK} A \rtimes_{\alpha,r} P.$$  

In case of trivial coefficients $A = C$, we obtain the following

**Corollary 4.9.** Assume that $P \subseteq G$ satisfies conditions (1), (2) and (3) of Theorem 4.7 for $A = C$. Then there is a canonical isomorphism

$$\bigoplus_{[X] \in G \setminus I_{P \subseteq G}} K_\ast(C^\ast_r(G_X)) \cong K_\ast(C^\ast_r(P)).$$

If, moreover, $G$ satisfies the strong Baum-Connes conjecture, this isomorphism is induced by a $KK$-equivalence.

**Remark 4.10.** Recall that a semigroup $P$ satisfies the left Ore condition if and only if it can be imbedded as a subsemigroup of a group $G$ such that $G = P^{-1}P$. It follows directly from this condition that the inclusion $P \subseteq G$ satisfies the Toeplitz condition. Therefore, if the set $J_P$ of constructible right ideals in $P$ is independent,
the same holds for $J_{P \subseteq G}$ by Lemma 4.6. Thus, if in addition $G$ satisfies the Baum-Connes conjecture for suitable coefficients, the results of the previous section will apply.

This was the situation studied in [9] in which we gave a proof of the above corollary in this situation together with a large number of interesting applications. The results obtained here also allow the study of crossed products by left Ore semigroups by automorphic actions.

Interesting examples of left Ore semigroups are given by semigroups attached to Dedekind domains $R$. Let $R^\times := R \setminus \{0\}$ be its multiplicative semigroup and let $R^s$ denote the group of units in $R$. Consider the semigroups $R^\times$, $R^\times / R^s$ and $R \rtimes R^\times$ as studied in detail in [9]. Let $Q(R)$ denote the quotient field of $R$ and let $Cl_{Q(R)}$ denote the ideal class group of $Q(R)$. For each $\gamma \in Cl_{Q(R)}$ we let $I_\gamma \subseteq Q(R)$ be a representative for $\gamma$ (see [9, §8] for a more detailed discussion).

**Theorem 4.11.** Let $R$ be a Dedekind domain. Then the following are true:

1. For every action $\alpha : R^\times \to Aut(A)$ there is a canonical isomorphism
   \[ K_*(A \rtimes_{\alpha, r} R^\times) \cong \bigoplus_{\gamma \in Cl_{Q(R)}} K_*(A \rtimes_{\alpha, r} R^s). \]

2. For every action $\alpha : R^\times / R^s \to Aut(A)$ there is a canonical isomorphism
   \[ K_*(A \rtimes_{\alpha, r} (R^\times / R^s)) \cong \bigoplus_{\gamma \in Cl_{Q(R)}} K_*(A). \]

3. For every action $\alpha : R \rtimes R^\times \to Aut(A)$ there is a canonical isomorphism
   \[ K_*(A \rtimes_{\alpha, r} (R \rtimes R^\times)) \cong \bigoplus_{\gamma \in Cl_{Q(R)}} K_*(A \rtimes_{\alpha, r} (I_\gamma \rtimes R^s)). \]

All computations necessary for deducing this theorem from Theorem 4.7 have been done in [9] §8]. Note that in all cases of the above theorem, the enveloping groups are amenable, hence satisfy the strong Baum-Connes conjecture. Thus whenever $A$ is type I, the isomorphisms in the above theorem are induced by $KK$-equivalences.

5. **The case of principal constructible ideals and quasi-lattice ordered groups**

In this section we discuss a situation which is particularly nice for our purposes. Assume that $P$ is a subsemigroup of a group $G$ such that all constructible right $P$-ideals in $G$ are principal, i.e. $J_{P \subseteq G} = \{ g \cdot P : g \in G \} \cup \{ \emptyset \}$. As observed in [24] §8.1, it follows that $P \subseteq G$ is Toeplitz. Moreover, another consequence is that $J_P = \{ pP : p \in P \} \cup \{ \emptyset \}$ so that $J_P$ is clearly independent. Conversely, if all constructible ideals of $P$ are principal, i.e. if $J_P = \{ pP : p \in P \} \cup \{ \emptyset \}$, and if $P \subseteq G$ is Toeplitz, then $J_{P \subseteq G} = \{ g \cdot P : g \in G \} \cup \{ \emptyset \}$. This is a consequence of [24] Lemma 4.2. Therefore, we may apply our general K-theoretic result to this situation. Since the stabilizer $G_P$ of $P \in I_{P \subseteq G}$ is equal to the group $P^*$ of invertible elements in $P$, we see that the left hand side of the isomorphism (4.4) equals $K_*(A \rtimes_{\alpha, r} P^*)$.

Recall from [4.3] the construction of the crossed product $A \rtimes_{\alpha, r} P$. Let $\iota_A = \widetilde{\alpha_P} : A \to A \rtimes_{\alpha, r} P$ be as in (4.2) and let $\iota_{P^*} : P^* \to U(\ell^2(P))$ given by $\iota_{P^*}(p) = V_p$, where we recall that for all $p \in P$ we have $V_p \epsilon_x = \epsilon_{px}$, $x \in P$. Then $(\iota_A, \iota_{P^*})$ is
4.2. We need to show that this diagram commutes. Following the definitions we see that
with notations as in (4.2) and (4.3). On the other side we have

\[ \pi \]\n
Thus \( \iota_A \times \iota_{P^*} \) factors through a faithful \( * \)-homomorphism

\[ \iota_{A \times r, P^*} : A \rtimes_{\alpha, r} P^* \hookrightarrow A \rtimes_{\alpha, r} P. \] (5.1)

**Theorem 5.1.** Suppose that \( \mathcal{F}_{P < G} = \{ g \cdot P : g \in G \} \cup \{ \emptyset \} \). Let \( G \) act on a C*-algebra \( A \) by \( \alpha \) such that \( G \) satisfies the Baum-Connes conjecture for \( A \otimes D_{P \trianglelefteq G} \) with respect to the diagonal action and that the group of invertible elements \( P^* \) in \( P \) satisfies the Baum-Connes conjecture for \( A \). Then the homomorphism \( \iota_{A \times r, P^*} : A \rtimes_{\alpha, r} P^* \rightarrow A \rtimes_{\alpha, r} P \) induces an isomorphism in K-theory

\[ K_s(A \rtimes_{\alpha, r} P^*) \cong K_s(A \rtimes_{\alpha, r} P). \]

If, moreover, \( A \rtimes_{\alpha, r} H \) is in the bootstrap class for every finite subgroup of \( P^* \) and if \( G \) satisfies the strong Baum-Connes conjecture, then \( \iota_{A \times r, P^*} \) is a KK-equivalence.

**Proof.** Note first that \( g \mapsto g \cdot P \) induces a bijection \( G/P^* \cong \mathcal{I}_{P \trianglelefteq G} \) and hence it follows from [3, Theorem 2.6] that \( G \) satisfies the Baum-Connes conjecture for \( A \otimes C_0(\mathcal{I}_{P \trianglelefteq G}) \) if and only if \( P^* \) satisfies the conjecture for \( A \). It follows that the conditions of Theorem 4.1 are satisfied and that the left hand side of the isomorphism (4.1) equals \( K_s(A \rtimes_{\alpha, r} P^*) \). Hence Theorem 4.7 implies the desired result as soon as we have checked that the resulting isomorphism (or KK-equivalence)

\[ \Phi : K_s(A \rtimes_{\alpha, r} P^*) \xrightarrow{\cong} K_s(A \rtimes_{\alpha, r} P) \]

is implemented by the inclusion \( \iota_{A \times r, P^*} : A \rtimes_{\alpha, r} P^* \rightarrow A \rtimes_{\alpha, r} P \).

For this recall that by Lemma 4.10 the isomorphism \( \Phi \) is obtained by the KK-class \( [\pi_P] \otimes [\mu]^{-1} \in KK_0(A \rtimes_{\alpha, r} P^*, A \rtimes_{\alpha, r} P) \) with \( \pi_P : A \rtimes_{\alpha, r} P^* \rightarrow (A \otimes D_{P \trianglelefteq G}) \rtimes_{\alpha \otimes r, r} G \) given by \( \pi_P(au_g) = (a \otimes E_P)u_g \) and where \( \mu : A \rtimes_{\alpha, r} P \rightarrow A \rtimes_{\alpha, r} (P \subseteq G) \cong (A \otimes D_{P \trianglelefteq G}) \rtimes_{\alpha \otimes r, r} G \) denotes the realization of \( A \rtimes_{\alpha, r} P \) as the full corner

\[ (1_H \otimes E_P)(A \rtimes_{\alpha, r} (P \subseteq G)) \subseteq A \rtimes_{\alpha, r} (P \subseteq G) \cong (A \otimes D_{P \trianglelefteq G}) \rtimes_{\alpha \otimes r, r} G. \]

Consider the diagram

\[ \begin{array}{ccc}
A \rtimes_{\alpha, r} P^* & \xrightarrow{\pi_P} & (A \otimes D_{P \trianglelefteq G}) \rtimes_{\alpha \otimes r, r} G \\
\iota_{A \times r, P^*} & \downarrow \cong & \pi \times (1 \otimes \lambda) \\
A \rtimes_{\alpha, r} P & \xrightarrow{\mu} & A \rtimes_{\alpha, r} (P \subseteq G)
\end{array} \]

where the isomorphism \( \pi \times (1 \otimes \lambda) \) in the right vertical arrow is described in Lemma 4.2. We need to show that this diagram commutes. Following the definitions we see that

\[ \iota_{A \times r, P^*}(au_g) = \alpha_P(a)(1 \otimes V_g) \in \mathcal{B}(\mathcal{H} \otimes \ell^2(P)) \]

with notations as in (4.2) and (4.3). On the other side we have

\[ \pi \times (1_H \otimes \lambda)(\pi_P(au_g)) = \pi \times (1_H \otimes \lambda)(a \otimes E_P)u_g = \pi(a \otimes E_P)(1_H \otimes \lambda_g). \]
By Lemma 4.2 we get for $\xi \in \mathcal{H}$ and $\epsilon_x \in \ell^2(G)$:
\[
\pi(a \otimes E_P)(1_{\mathcal{H}} \otimes \lambda_g)(\xi \otimes \epsilon_x) = \alpha_{(g\xi)_r}(a) \xi \otimes E_P \epsilon_x
\]
\[
= \left\{ \begin{array}{ll}
0 & \text{if } x \notin P \\
\overline{\alpha}_P(a)(1_{\mathcal{H}} \otimes V_g)(\xi \otimes \epsilon_x) & \text{if } x \in P
\end{array} \right.
\]
which gives the desired result.

As a special case, we can treat quasi-lattice ordered groups. Recall from [26] that $P \subseteq G$ is called quasi-lattice ordered if the following conditions are satisfied:

(QL0) $P \cap P^{-1} = \{ e \}$;

(QL1) for all $g \in G$ the intersection $P \cap (g \cdot P)$ is either empty or of the form $pP$ for some $p \in P$.

Condition (QL2) from [26] is automatically satisfied as was observed in [6]. (QL1) implies that $J_{P \subseteq G} = \{ g \cdot P : g \in G \} \cup \{ \emptyset \}$. So we are in the situation that all constructible right $P$-ideals in $G$ are principal. The Toeplitz condition is shown in [21] §8.1. Hence, since (QL0) implies $P^* = \{ 1 \}$ we obtain from Theorem 5.1

Theorem 5.2. Suppose that $P \subseteq G$ is quasi-lattice ordered as defined above. Let $\alpha : G \to \text{Aut}(A)$ be a $G$-action on a $C^*$-algebra $A$ such that $G$ satisfies the Baum-Connes conjecture for $A \otimes D_{P \subseteq G}$ with respect to the diagonal action. Then the canonical inclusion $\iota_A : A \to A \rtimes_{\alpha,r} P$ induces an isomorphism
\[
K_*(A) \cong K_*(A \rtimes_{\alpha,r} P).
\]

If, moreover, $G$ satisfies the strong Baum-Connes conjecture and if $G$ is torsion-free or $A$ lies in the bootstrap class, then $\iota_A$ is a $K$-$K$-equivalence.

Remark 5.3. The easiest example of a quasi-lattice semigroup is the case $\mathbb{N} \subseteq \mathbb{Z}$ where $\mathbb{N}$ denotes the non-negative integers. If $\alpha : \mathbb{Z} \to \text{Aut}(A)$ is an action by automorphisms, then the crossed product $A \rtimes_{\alpha,r} \mathbb{N}$ coincides with the Toeplitz algebra $\mathcal{T} = \mathcal{T}(A)$ as constructed by Pimsner and Voiculescu in [27]. Indeed, the main work in proving the famous six-term sequence for computing the $K$-theory of $A \rtimes_{\alpha} \mathbb{N}$ as given in [27] Theorem 2.4 is to show that the canonical embedding $\iota_A : A \to A \rtimes_{\alpha,r} \mathbb{N} = \mathcal{T}(A)$ induces an isomorphism in $K$-theory. The above theorem gives a very general version of this important result of Pimsner and Voiculescu.

We should also point out that for positive cones $P$ in certain quasi-lattice ordered groups $G$ (right-angled Artin groups of a special type) and for the trivial coefficient $A = \mathbb{C}$, the result $K_*(\mathbb{C}) \cong K_*(C^*_\lambda(P))$ was already obtained in [18] Theorem 3.3 and Proposition 3.4.

We proceed by constructing natural examples of subsemigroups of groups which satisfy $J_{P \subseteq G} = \{ g \cdot P : g \in G \} \cup \{ \emptyset \}$ without being quasi-lattice ordered:

Let $R$ be a principal ideal domain and $M_n^\times(R) := \{ p \in M_n(R) : \det(p) \neq 0 \}$.

Lemma 5.4. The constructible right ideals of $P = M_n^\times(R)$ are principal.

Proof. We want to show that $J_P = \{ pP : p \in P \} \cup \{ \emptyset \}$. The only thing which we have to prove is that for every $p,q \in P$, the right ideal $q^{-1}pP$ is also of the form $rP$ for some $r \in P$.

Let $\tilde{q} \in P$ satisfy $qq = \tilde{q}q = \det(q) \cdot 1_n$ ($1_n$ is the identity matrix). Then $q^{-1}pP = (\tilde{q}q)^{-1}(\tilde{q}p)P = (\det(q) \cdot 1_n)^{-1}(\tilde{q}p)P = (\det(q) \cdot 1_n)^{-1}((\tilde{q}pP) \cap \det(q) \cdot P)$. Now
consider the Smith normal form of $\tilde{q}p$, i.e. find $u$, $v$ in $SL_n(R) \subseteq P$ such that $u\tilde{q}pv$ is diagonal, $u\tilde{q}pv = \text{diag}(\alpha_1, \ldots, \alpha_n)$. Thus
\begin{align*}
(qP) \cap \det(q) \cdot P &= (u^{-1}\text{diag}(\alpha_1, \ldots, \alpha_n)v^{-1}P) \cap (\det(q) \cdot P) \\
&= u^{-1}(\text{diag}(\alpha_1, \ldots, \alpha_n)P \cap \det(q) \cdot P) \\
&= u^{-1}\text{diag}(\text{lcm}(\alpha_1, \det(q)), \ldots, \text{lcm}(\alpha_n, \det(q)))P.
\end{align*}
Therefore $q^{-1}P$ can be written as
\begin{align*}
(\det(q) \cdot 1_n)^{-1}((\tilde{q}pP) \cap \det(q) \cdot P) \\
&= (\det(q) \cdot 1_n)^{-1}u^{-1}\text{diag}(\text{lcm}(\alpha_1, \det(q)), \ldots, \text{lcm}(\alpha_n, \det(q)))P \\
&= u^{-1}\text{diag}(\det(q)^{-1}\text{lcm}(\alpha_1, \det(q)), \ldots, \det(q)^{-1}\text{lcm}(\alpha_n, \det(q)))P.
\end{align*}

Set $r := u^{-1}\text{diag}(\det(q)^{-1}\text{lcm}(\alpha_1, \det(q)), \ldots, \det(q)^{-1}\text{lcm}(\alpha_n, \det(q)))$, and we arrive at $q^{-1}P = rP$. □

Going through the proof, it becomes clear that our argument applies whenever $P$ is a subsemigroup of $M_2^\alpha(R)$ such that
\begin{itemize}
  \item $SL_n(R) \subseteq P$,
  \item for every $q$ in $P$, the element $\tilde{q} \in M_2^\alpha(R)$ uniquely determined by $q\tilde{q} = \det(q) \cdot 1_n$ also lies in $P$,
  \item whenever a diagonal matrix $\text{diag}(\alpha_1, \ldots, \alpha_n)$ lies in $P$, then for every $q$ in $P$, the diagonal matrix $\text{diag}(\det(q)^{-1}\text{lcm}(\alpha_1, \det(q)), \ldots, \det(q)^{-1}\text{lcm}(\alpha_n, \det(q)))$ also lies in $P$.
\end{itemize}

The second condition implies that $P$ is left Ore. Thus $P \subseteq P^{-1}P =: G$ is Toeplitz. As we have shown that all constructible ideals of $P$ are principal, it follows from our discussions above that $J_P \subseteq G = \{g \cdot P : g \in G\} \cup \{\emptyset\}$.

In general, for such semigroups, our conditions concerning the Baum-Connes conjecture are very difficult to verify. But at least for $n = 2$ we get:

**Theorem 5.5.** Suppose that $R$ is a principal ideal domain with field of fractions $K$ and let $P \subseteq M_2^\alpha(R)$ be a subsemigroup satisfying the above conditions. Let $G = P^{-1}P \subseteq GL_2(K)$. Then for every action $\alpha : G \to \text{Aut}(A)$ the inclusion $\iota_{A\rtimes_{\alpha,r} P^*} : A \rtimes_{\alpha,r} P^* \to A \rtimes_{\alpha,r} P$ induces an isomorphism $K_*(A \rtimes_{\alpha,r} P^*) \cong K_*(A \rtimes_{\alpha,r} P)$.

Moreover, if $A \rtimes_{\alpha} F$ satisfies the UCT for every finite subgroup $F$ of $P^*$ (which is true if $A$ is type I), then $\iota_{A\rtimes_{\alpha,r} P^*}$ induces a KK-equivalence.

**Proof.** Since $G$ is a countable subgroup of $GL(2, K)$ it is a-T-menable by [15, Theorem 4]. Thus it follows from [16] that $G$ satisfies the strong Baum-Connes conjecture and the proof follows from Theorem 5.1 and Remark 4.8. □

6. The Left and Right Regular C*-Algebra for a Semidirect Product

Let $S$ be a cancellative semigroup. In this section we are interested not only in the left regular C*-algebra $C^*_L(S)$, but also in the right regular C*-algebra $C^*_R(S)$ generated by the right regular representation $\rho$ of $S$ on $\ell^2(S)$. Since $C^*_R(S)$ is obviously isomorphic to the left regular C*-algebra of the opposite semigroup, we might formulate the corresponding arguments in terms of the left regular representation of
the opposite semigroup. However it will be more convenient to work directly with the right regular representation. We will be especially interested in comparing the $K$-theory for the right and left regular $C^*$-algebras.

6.1. Ideal independence and Toeplitz condition for the right regular $C^*$-algebra of a semidirect product semigroup. Assume that the semigroup $S$ is embedded into a group $\bar{S}$ and let $E$ denote the orthogonal projection of $\ell^2(\bar{S})$ onto $\ell^2(S)$. Denote by $\rho, \bar{\rho}$ the right regular representations of $S$ and $\bar{S}$, respectively. We say that $S \subseteq \bar{S}$ satisfies the right Toeplitz condition if every operator $E \bar{\rho}(t)E \neq 0$ with $t \in \bar{S}$ can be written as a finite product of elements of the form $\rho(s), s \in S$ and their adjoints. Of course, this is just saying that the embedding of the opposite semigroups $S^{\text{op}} \subseteq \bar{S}^{\text{op}}$ satisfies the ordinary Toeplitz condition. By a constructible left ideal in $S$ we mean a left ideal $I$ such that the opposite ideal $I^{\text{op}}$ is a constructible right ideal in $S^{\text{op}}$.

Let now $P$ be a semigroup with unit which acts (on the left) by injective endomorphisms on the group $H$. We can form the semidirect product $S = H \rtimes P$. The elements of $H \rtimes P$ are pairs $(h, p), h \in H, p \in P$ and the multiplication rule is given by $(h_1, p_1)(h_2, p_2) = (h_1 p_1(h_2), p_1 p_2)$. Note that $H \times P$ is left or right cancellative if and only if $P$ is.

**Proposition 6.1.1.** The left ideals in $S$ are exactly the subsets of the form $H \times I$ where $I$ is a left ideal in $P$. The constructible left ideals in $S$ are exactly those ideals $H \times I$ where $I$ is a constructible left ideal in $P$.

**Proof.** The subsets of the given form are obviously left ideals. Conversely, assume that $K$ is a left ideal in $S$. Then $(x, q) \in K$ implies that $(H, q) \subseteq K$ and $(y, Pq) \subseteq K$ for all $y \in H$. Thus $K$ is as claimed. Moreover, if $K = H \times I$ is a left ideal and $(h, p) \in H \times P$, then $K(h, p) = H \times Ip$ and $K(h, p)^{-1} = H \times Ip^{-1}$. This shows the assertion concerning the constructible left ideals. \hfill \Box

**Corollary 6.1.2.** If the constructible left ideals of $P$ form an independent family, then the same is true for the constructible left ideals of $S$.

**Proof.** Obvious from Proposition 6.1.1. \hfill \Box

Assume now that $P$ satisfies the left Ore condition so that $P$ can be embedded into a group $\bar{P}$ for which $\bar{P} = P^{-1} P$.

Moreover then $\bar{P}$ can be written as $\lim_{\longrightarrow p \in P} P$, i.e. as the limit of the inductive system $(L_p)_{p \in \bar{P}}$ where $L_p = P$ and the map $L_p \rightarrow L_q$ is given by multiplication by $q$. We set $\bar{H} = P^{-1} H = \lim_{\longrightarrow p \in P} H$ with the analogous inductive limit.

**Proposition 6.1.3.** The semigroup $S$ has a natural embedding into the group $\bar{S} = \bar{H} \rtimes \bar{P}$.

We denote by $\rho$ the right regular representation of $S$ on $\ell^2(S)$ and by $\bar{\rho}$ the right regular representation of $\bar{S}$ on $\ell^2(\bar{S})$. As above $E$ denotes the orthogonal projection $\ell^2(S) \rightarrow \ell^2(S)$. We consider $P, \bar{P}$ and $H, \bar{H}$ as subsemigroups of $S, \bar{S}$.

**Lemma 6.1.4.** Let $(g, z)$ be an element of $\bar{S}$, $g \in \bar{H}, z \in \bar{P}$. Then $E\bar{\rho}((g, z))E = E\bar{\rho}(z)E\bar{\rho}(g)E$. 


Proof. Both operators evaluated on an element $\xi_{(a,x)}$ of the standard orthonormal basis of $\ell^2(S)$ give $\xi_{(ax(h),x)}$ if $x$ is such that $ax(h) \in H$, $xg \in P$, and give 0 otherwise.

Lemma 6.1.5. Let $z \in P$, $h \in H$ and $g = z(h) \in \hat{H}$. Assume that

$$\{x \in P : xz \in P\} = \{x \in P : xz(h) \in H\}. \quad (6.1)$$

Then $E\bar{\rho}(g)E = E\bar{\rho}(z)^*E\rho(h)E\bar{\rho}(z)E$.

Proof. Let $\xi_{(a,x)}$, $a \in H$, $x \in P$ be an element of the standard orthonormal basis in $\ell^2(S)$. We have

$$E\bar{\rho}(z)^*E\rho(h)E\bar{\rho}(z)\xi_{(a,x)} = \begin{cases} E\bar{\rho}(z)^*E\rho(h)\xi_{(a,xz(h),x)} & \text{if } xz \in P, \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand

$$E\bar{\rho}(g)\xi_{(a,x)} = E\bar{\rho}(z(h))\xi_{(a,x)} = \begin{cases} \xi_{(axz(h),x)} & \text{if } xz(h) \in H, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 6.1.6. Assume that $P$ satisfies the right Toeplitz condition. Then for each $z \in P$ the operator $E\bar{\rho}(z)E$ is a product of operators of the form $\rho(p)$ or $\rho(p)^*$, $p \in P$.

Proof. Let $\rho_0$, $\bar{\rho}_0$ denote the right regular representation of $P$, $\hat{P}$ on $\ell^2(P)$, $\ell^2(P)$, respectively, and let $E_0$ be the orthogonal projection of $\ell^2(\hat{P})$ onto $\ell^2(P)$. By assumption, there are $p_i, q_i \in P$ such that $E_0\rho_0(z)E_0$ is a product of the form $\rho_0(p_1)\rho_0(q_1)^* \cdots \rho_0(p_n)\rho_0(q_n)^*$. The Hilbert space $\ell^2(S)$ has the following filtration by subspaces

$$\ell^2(S) = \ell^2(H \rtimes P) \subseteq L = \ell^2(H \rtimes \hat{P}) \subseteq \ell^2(\hat{H} \rtimes \hat{P})$$

On the subspace $L \cong \ell^2(H) \otimes \ell^2(\hat{P})$, the operator $\bar{\rho}(z)$ acts like $1 \otimes \bar{\rho}_0(z)$. Similarly $E$ acts like $1 \otimes E_0$ on $L$. Thus, $E\bar{\rho}(z)E$ acts like

$$1 \otimes \rho_0(p_1)\rho_0(q_1)^* \cdots \rho_0(p_n)\rho_0(q_n)^*$$

and therefore $E\bar{\rho}(z)E = \rho(p_1)\rho(q_1)^* \cdots \rho(p_n)\rho(q_n)^*$.

Proposition 6.1.7. Assume that $P$ satisfies the right Toeplitz condition and that the following condition is satisfied:

$$\text{for every } \bar{h} \in \hat{H}, \text{ there exists } z \in \tilde{P} \text{ and } h \in H \text{ such that} \quad (6.2)$$

$$\bar{h} = z(h) \text{ and } P \cap Pz^{-1} = \{x \in P : x(\bar{h}) \in H\}.$$ 

Then $S \subseteq \tilde{S}$ satisfies the right Toeplitz condition.

Proof. This follows by combining Lemmas 6.1.4, 6.1.5, 6.1.6. In fact, let $(g, w)$ be an element of $\tilde{S}$. Then by Lemma 6.1.3 we have $E\bar{\rho}(g, w)E = E\bar{\rho}(w)E\rho(g)E$. By Lemma 6.1.6 $E\bar{\rho}(w)E$ is a product of operators of the form $\rho(p)$ or $\rho(p)^*$, $p \in P$. Let finally $z \in \tilde{P}$ and $h \in H$ such that $g = z(h)$ and such that

$$\{x \in P : xz \in P\} = \{x \in P : xz(h) \in H\}$$

(this is equivalent to $P \cap Pz^{-1} = \{x \in P : x(g) \in H\}$). Then, by Lemma 6.1.5

$$E\bar{\rho}(g)E = E\bar{\rho}(z)^*E\rho(h)E\bar{\rho}(z)E.$$ 

Moreover, by Lemma 6.1.6 $E\bar{\rho}(z)E$ and $E\bar{\rho}(z)^*E$ are also products of the desired form.
6.2. K-theory for the right regular C*-algebra of a semidirect product. Let us now compute the K-theory of $C^*_ρ(H \rtimes P)$. We apply our general K-theoretic result switching from right actions of $H \rtimes P$ to left actions of $(H \rtimes P)^{op}$. To do so, we need to compute the orbits of the (right) $H \rtimes P$-action on the family of constructible left $H \rtimes P$-ideals in $H \rtimes P$. If $H \rtimes P \subseteq H \rtimes P$ is right Toeplitz and has independent constructible left ideals, then by Lemma 6.1.10 applied to $(H \rtimes P)^{op}$, we know that every constructible left $H \rtimes P$-ideal in $H \rtimes P$ is in the orbit of a constructible left ideal of $H \rtimes P$. Thus it suffices to consider constructible left ideals of $H \rtimes P$. They are of the form $H \times X$ for some constructible left ideal $X$ of $P$ (see Proposition 6.1.11). For every such non-empty $X$, the stabilizer $\{g \in H \rtimes P : (H \times X) \cdot g = H \times X\}$ is given by

$$S(X) := X^{-1}(H) \rtimes (\chi P)$$

where $X^{-1}(H) = \{h \in H : x(h) \in H\}$ for all $x \in X$ and $\chi P = \{p \in P : Xp = X\}$. Therefore, combining Corollary 6.1.2, Proposition 6.1.7 and Theorem 1.7, we obtain

**Corollary 6.2.1.** Let $\alpha : (\bar{H} \rtimes \bar{P})^{op} \to \text{Aut}(A)$ be an action of $(\bar{H} \rtimes \bar{P})^{op}$ on a C*-algebra $A$. Let $X$ be a set of constructible left ideals of $H \rtimes P$ such that $\{H \times X : X \in X\}$ is a complete system of representatives for the $H \times P$-orbits of the family of non-empty constructible left $H \rtimes P$-ideals in $H \rtimes P$. Assume that $P$ has independent constructible left ideals, that the action of $P$ on $H$ satisfies the condition in Proposition 6.1.7 and that $(\bar{H} \rtimes \bar{P})^{op}$ satisfies the Baum-Connes conjecture with coefficients. Then we have a canonical isomorphism

$$K_\ast(A \rtimes_{\alpha,r} (H \rtimes P)^{op}) \cong \bigoplus_{X \in X} K_\ast(A \rtimes_{\alpha,r} S(X)^{op}).$$

In particular, we obtain the following K-theoretic formula for the right regular C*-algebra $C^*_ρ(P)$:

$$K_\ast(C^*_ρ(H \rtimes P)) \cong \bigoplus_{X \in X} K_\ast(C^*_ρ(S(X))).$$

6.3. Right and left C*-algebras for semidirect products by $\mathbb{N}$. To deduce the right Toeplitz condition for $H \times P \subseteq H \rtimes P$, we used in Proposition 6.1.7 condition (6.2) which says: for every $\bar{h} \in \bar{H}$ there exists $z \in \bar{P}$ and $\bar{h} \in \bar{H}$ such that $\bar{h} = z(\bar{h})$ and $P \cap Pz^{-1} = \{x \in P : x(\bar{h}) \in H\}$. Note that the set $\{x \in P : x(\bar{h}) \in H\}$ is always a left ideal of $P$. Moreover, we have:

**Lemma 6.3.1.** (6.2) holds if for every $\bar{h} \in \bar{H}$, the left ideal $\{x \in P : x(\bar{h}) \in H\}$ is principal, i.e. of the form $Pp$ for some $p \in P$.

**Proof.** If $\{x \in P : x(\bar{h}) \in H\} = Pp$, then $p$ itself satisfies $p(\bar{h}) \in H$. Thus there exists $\bar{h} \in \bar{H}$ such that $\bar{h} = p^{-1}(\bar{h})$, and setting $z = p^{-1}$, we see that (6.2) is satisfied. \qed

In general, it is not clear which left ideals of $P$ arise as sets of the form $\{x \in P : x(\bar{h}) \in H\}$ for some $\bar{h} \in \bar{H}$. So in general, we can only deduce the following

**Corollary 6.3.2.** Assume that all non-empty left ideals of $P$ are principal. Then for every action of $P$ on some group $H$, condition (6.2) holds true. In particular, (6.2) holds for every $\mathbb{N}$-action.
Lemma 6.3.3. Assume that the (additive) semigroup \( \mathbb{N} \) acts by injective endomorphisms \( \alpha_n \) on the group \( H \). The set of constructible right ideals in \( H \times \mathbb{N} \) coincides with the set of principal ideals. The principal right ideals are exactly the subsets of the form \( h\alpha_n(H) \times \{ n \} \) with \( h \in H/\alpha_n(H) \), where \( (n) = \{ n+k : k \in \mathbb{N} \} \) denotes the principal ideal generated by \( n \) in \( \mathbb{N} \).

Proof. The principal right ideals in \( H \times \mathbb{N} \) are of the form \( (h,n)(H \times \mathbb{N}) = h\alpha_n(H) \times (n) \). We show that the set of principal ideals is closed under the operation \( I \mapsto (g,k)^{-1}I \). One easily checks that
\[
(g,k)^{-1}(h\alpha_n(H) \times (n)) = \begin{cases} H \times \mathbb{N} & \text{if } g^{-1}h \in \alpha_n(H), \\ \emptyset & \text{otherwise,} \end{cases}
\]
and
\[
(g,k)^{-1}(h\alpha_n(H) \times (n)) = \begin{cases} a\alpha_{n-k}(H) \times (n-k) & \text{if } g^{-1}h = \alpha_k(a), a \in H, \\ \emptyset & \text{if } g^{-1}h \notin \alpha_k(H), \end{cases}
\]
depending if \( k \geq n \) in the first case or \( k \leq n \) in the second one. \( \square \)

Theorem 6.3.4. Let \( \mathbb{N} \) act by injective endomorphisms on the group \( H \) and assume that the enveloping group \( \bar{H} \rtimes \mathbb{N} = \bar{H} \rtimes \mathbb{Z} \) satisfies the Baum-Connes conjecture with coefficients. Let \( C^*_\rho(H \rtimes \mathbb{N}) \) and \( C^*_\lambda(H \rtimes \mathbb{N}) \) denote the right and left regular C*-algebra of \( H \rtimes \mathbb{N} \), respectively. Then
\begin{enumerate}
\item \( K_*(C^*_\rho(H \rtimes \mathbb{N})) \cong K_*(C^*_\lambda(H)) \).
\item \( K_*(C^*_\rho(H \rtimes \mathbb{N})) \cong K_*(C^*_\lambda(H)) \).
\end{enumerate}
In particular \( C^*_\rho(H \rtimes \mathbb{N}) \) and \( C^*_\lambda(H \rtimes \mathbb{N}) \) have the same K-theory.

Proof. (a) is an immediate consequence of the description of the constructible left ideals of \( H \times \mathbb{N} \) in Proposition 6.1.1 together with the formula for \( K_* \) in Corollary 6.2.4.

(b) The set \( J \) of constructible right ideals in \( H \times \mathbb{N} \) is described in Lemma 6.3.3. It is obviously independent. Every ideal in \( J \) has full orbit under the action of \( H \rtimes \mathbb{Z} \) and the stabilizer group of the ideal \( H \times \mathbb{N} \) is \( H \). Thus the assertion follows from Theorem 4.7. \( \square \)

It is by no means obvious that \( C^*_\rho(H \rtimes \mathbb{N}) \) and \( C^*_\lambda(H \rtimes \mathbb{N}) \) should have the same K-theory. In fact, as C*-algebras they look very different from each other. For instance in the situation of the following remark (\( H \) abelian and \( H/\varphi(H) \) finite) \( C^*_\rho(H \rtimes \mathbb{N}) \) admits non-trivial one-dimensional representations (see [23]) while every non-zero quotient of \( C^*_\lambda(H \rtimes \mathbb{N}) \) contains a non-trivial isometry and therefore \( C^*_\lambda(H \rtimes \mathbb{N}) \) admits no non-trivial finite-dimensional representations.

Remark 6.3.5. Consider the special case where \( H \) is abelian and \( \mathbb{N} \) acts via the injective endomorphism \( \varphi \) on \( H \). Assume also that \( H/\varphi(H) \) is finite and that \( \bigcap_{n \geq 0} \varphi^n(H) = \{ 0 \} \). Then the left regular C*-algebra \( C^*_\lambda(H \rtimes \mathbb{N}) \) has the algebra \( \mathfrak{A}[\varphi] \) studied in [11] as natural quotient. In [11], it was shown that the K-theory of \( \mathfrak{A}[\varphi] \) is determined by a six term exact sequence of the form
\[
K_*C^*(H) \xrightarrow{1-b(\varphi)} K_*C^*(H) \xrightarrow{\partial} K_*\mathfrak{A}[\varphi]
\]
On the other hand, we know by Theorem 6.3.4 that $K_\ast(C_\rho^\ast(H \rtimes \mathbb{N})) \cong K_\ast(C^\ast(H))$.

It can be shown that the long exact sequence associated with the extension

$$0 \to \text{Ker } \pi \to C^\ast(H \rtimes \mathbb{N}) \xrightarrow{\pi} \mathfrak{A}[\varphi] \to 0$$

is exactly the exact sequence in (6.3).

### 6.4. Right and left C*-algebras for $ax + b$-semigroups of Dedekind rings.

Recall that a Dedekind ring is a noetherian integrally closed integral domain in which every non-zero prime ideal is maximal. The prime example of a Dedekind ring is the ring of algebraic integers in a number field. If $R$ is a Dedekind ring, its $ax + b$-semigroup is, by definition, the semidirect product $R \rtimes \mathbb{R}$ where $R^\times = R \setminus \{0\}$ denotes the multiplicative semigroup of the ring and $R$ (by abuse of notation) its additive group.

**Proposition 6.4.1.** Let $R$ be a Dedekind domain with field of fractions $Q$. Then the inclusion of $ax + b$-semigroups $R \rtimes R^\times \subseteq Q \rtimes Q^\times$ satisfies the right Toeplitz condition.

**Proof.** We apply Proposition 6.1.7 with $H = R$, $P = R^\times$. Since the inclusion $R^\times \subseteq Q^\times$ satisfies the left Toeplitz condition, by commutativity it also satisfies the right condition. Moreover, given $0 \neq \bar{h} \in Q$, choose $h = 1$ and $z = \bar{h}$. These elements obviously have the properties required in Proposition 6.1.7. □

**Theorem 6.4.2.** Let $R^\ast$ be the group of units in $R$ and choose for every ideal class $\gamma \in Cl_{Q(R)}$ an ideal $I_\gamma$ of $R$ which represents $\gamma$. The $K$-theory of the right regular C*-algebra $C_\rho^\ast(R \rtimes R^\times)$ is given by the formula

$$K_\ast(C_\rho^\ast(R \rtimes R^\times)) \cong \bigoplus_{\gamma \in Cl_{Q(R)}} K_\ast(C_\lambda^\ast(I_\gamma^{-1} \rtimes R^\times)).$$

Here we use the notation, familiar from number theory,

$$I_\gamma^{-1} = \{x \in Q : xy \in R, \forall y \in I_\gamma\}$$

for the fractional ideal $I_\gamma^{-1}$ in the quotient field $Q$ of $R$ (it satisfies $I_\gamma^{-1}I_\gamma = R$).

**Proof.** By Proposition 6.1.4, the constructible left ideals of $R \rtimes R^\times$ are in bijection with the constructible ideals of $R^\times$. These ideals are exactly of the form $I^\times$ where $I$ is a ring ideal in $R$, see [9]. The orbits under the action of the enveloping group of $R^\times$ are labeled by the elements $\gamma$ of the class group $Cl_{Q(R)}$. According to the discussion before Corollary 6.2.1 the stabilizer group for such an element $\gamma$ is $I_\gamma^{-1} \rtimes R^\times$. The assertion now follows from Corollary 6.2.1. □

In particular, comparing with the result obtained in [9], we see that the left and right regular C*-algebras of $R \rtimes R^\times$ are KK-equivalent (they both are KK-equivalent to $\bigoplus_\gamma C_\lambda^\ast(I_\gamma^{-1} \rtimes R^\times)$). Again, this is by no means obvious since $C_\rho^\ast(R \rtimes R^\times)$ and $C_\ast(R \rtimes R^\times)$ are quite different (again the first algebra admits non-trivial one-dimensional representations while by [8] the second one admits a largest ideal (which contains any other non-trivial ideal) with a simple quotient (the ring C*-algebra of [10])).
6.5. Wreath products. We here discuss another important class of specific semidirect products, so-called wreath products. For this we take a left Ore semigroup $P$, a group $\Gamma$ with unit $e$ and form

$$\Gamma^P = \bigoplus_{x \in P} \Gamma = \{ f : P \to \Gamma : f(x) = e \text{ for almost all } x \in P \}.$$ 

$P$ acts on $\Gamma^P$ by shifting from the left, i.e. $p(f)(x) = e$ if $x \notin pP$ and $p(f)(x) = f(p^{-1}x)$ if $x \in pP$. Let $\Gamma \wr P = \Gamma^P \rtimes P$ be the semidirect product attached to this action of $P$ on $\Gamma^P$. The semigroup $\Gamma \wr P$ is in a canonical way a subsemigroup of $\Gamma \wr (P \cap P^{-1}P)$.

We first consider the left regular representation. Let $J_P$, $J_{\Gamma \wr P}$ be the families of constructible right ideals in $P$, $\Gamma \wr P$, respectively. Then we have

**Lemma 6.5.1.** $J_{\Gamma \wr P} = \{(f \cdot (\Gamma^X)) \times X : f \in \Gamma^P, X \in J_P\}$.

**Proof.** It is clear that the right hand side contains $\emptyset$, $\Gamma \wr P$ and that it is closed under left multiplication. Moreover, given $f \in \Gamma^P$, $X \in J_P$ and $(h, p) \in \Gamma \wr P$, either $(h, p)^{-1}((f \cdot (\Gamma^X)) \times X)$ is empty or there is $\tilde{f} \in \Gamma^P$ with $hp(f) \in f \cdot (\Gamma^X)$. In the latter case, it is immediate that

$$(h, p)^{-1}((f \cdot (\Gamma^X)) \times X) = (\tilde{f} \cdot (\Gamma^X)) \times p^{-1}X.$$

\[\square\]

**Corollary 6.5.2.** If $J_P$ is independent, then $J_{\Gamma \wr P}$ is independent.

**Proof.** Assume that we have

$$(f \cdot (\Gamma^X)) \times X = \bigcup_{i=1}^n (f_i \cdot (\Gamma^X_i)) \times X_i$$

for some $f, f_1, \ldots, f_n$ in $\Gamma^P$ and $X, X_1, \ldots, X_n$ in $J_P$. Projecting down onto the $P$-coordinate, we see that $X = \bigcup_{i=1}^n X_i$. Hence by independence of $J_P$, we must have $X = X_i$ for some $i$. Therefore, $\Gamma^X = \Gamma^X_i$. But because $f \cdot (\Gamma^X)$ and $f_i \cdot (\Gamma^X_i)$ are either equal or disjoint, we deduce that $(f \cdot (\Gamma^X)) \times X = (f_i \cdot (\Gamma^X_i)) \times X_i$. \[\square\]

We now turn to the right regular representation. In the particular situation of the action of $P$ on $\Gamma^P$, we can say a bit more about condition [6.2] in Proposition [6.17]. Namely, take $f \in \Gamma^P$ with support $\{x_1, \ldots, x_n\}$, i.e. $f(x) = e$ whenever $x \in P \setminus \{x_1, \ldots, x_n\}$ and $f(x_i) \neq e$ for all $1 \leq i \leq n$. Then

$$\{p \in P : p(f) \in \Gamma^P\} = P \cap \bigcap_{i=1}^n Px_i^{-1}.$$ 

Therefore, the ideals which arise as sets of the form $\{p \in P : p(f) \in \Gamma^P\}$ are precisely the constructible left ideals of $P$ if $P \subseteq \bar{P}$ is assumed to be right Toeplitz (see [24, Lemma 4.2]). So by Lemma [6.3.1] and Proposition [6.17] we deduce

**Corollary 6.5.3.** If $P \subseteq \bar{P}$ is right Toeplitz and all the constructible left ideals of $P$ are principal, then $\Gamma \wr P \subseteq \Gamma \wr \bar{P}$ is right Toeplitz.
As a particular example, take \( \Gamma = \mathbb{Z}/2\mathbb{Z} \) and \( P = \mathbb{N} \). Then the enveloping group of \( \Gamma \wr P \) is the classical lamplighter group \( (\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z} \). To compute K-theory, we can simply apply Theorem 6.3.3 and we obtain

\[
K_\ast(C_\ast'((\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{N})) \cong K_\ast(\bigotimes_{i=1}^\infty C_\ast(\mathbb{Z}/2\mathbb{Z})) \cong K_\ast(C_\rho(\mathbb{Z}/2\mathbb{Z} \wr \mathbb{N})).
\]

7. Appendix: A remark on equivariant K-theory for finite dimensional commutative C*-algebras

Suppose that \( C \) and \( B \) are finite dimensional commutative C*-algebras, i.e., there exist positive integers \( n \) and \( m \) such that \( C \cong \mathbb{C}^n \) and \( B \cong \mathbb{C}^m \) and we may choose bases of pairwise orthogonal projections \( \{c_1, \ldots, c_n\} \) and \( \{b_1, \ldots, b_m\} \) of \( C \) and \( B \). Recall that by the UCT-theorem we have isomorphisms

\[
KK(C, B) \cong \text{Hom}(K_0(C), K_0(B)) \cong M(m \times n, \mathbb{Z})
\]

where the first one is given by sending a class \( x \in KK(C, B) \) to the associated homomorphism \( [\cdot] \otimes_C x : K_0(C) \to K_0(B) \) and the second one is given by describing this map with respect to the canonical generators \( \{[c_1], \ldots, [c_n]\} \) and \( \{[b_1], \ldots, [b_m]\} \) of \( K_0(C) \) and \( K_0(B) \), respectively, i.e., the matrix \( \Gamma = (\gamma_{ij}) \) corresponding to \( x \) is determined by

\[
[c_j] \otimes_C x = \sum_{i=1}^m \gamma_{ij} [b_i]
\]

for all \( 1 \leq j \leq n \).

Let us describe how we may construct for a given matrix \( \Gamma \in M(m \times n, \mathbb{Z}) \) the corresponding class \( x_\Gamma \in KK(C, B) \). For this we first decompose \( \Gamma \) as the difference \( \Gamma = \Gamma^+ - \Gamma^- \) where \( \Gamma^+ \) is the matrix built out of \( \Gamma \) by replacing all negative entries by 0 and \( \Gamma^- := \Gamma^+ - \Gamma \). We then construct a graded Kasparov module \( \mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^- \) with

\[
\mathcal{E}^+ = \bigotimes_{j=1}^n \left( \bigoplus_{i=1}^m \left( \mathbb{C}^\gamma_{ij} \otimes B_i \right) \right) \quad \text{and} \quad \mathcal{E}^- = \bigotimes_{j=1}^n \left( \bigoplus_{i=1}^m \left( \mathbb{C}^\gamma_{ij} \otimes B_i \right) \right)
\]

equipped with the canonical \( B \)-valued inner products, where \( B_i = \mathbb{C}b_i \subseteq B \) denotes the ideal generated by \( b_i \). For each \( 1 \leq j \leq n \) let \( p_j^+ \in \mathcal{K}(\mathcal{E}^+) \) denote the orthogonal projection on the \( j \)th summand \( \bigoplus_{i=1}^m (\mathbb{C}^\gamma_{ij} \otimes B_i) \) of \( \mathcal{E}^+ \), and, similarly, we let \( p_j^- \in \mathcal{K}(\mathcal{E}^-) \) denote the orthogonal projection onto the \( j \)th summand \( \bigoplus_{i=1}^m (\mathbb{C}^\gamma_{ij} \otimes B_i) \) of \( \mathcal{E}^- \). We then define a homomorphism

\[
\varphi^+ : C \to \mathcal{K}(\mathcal{E}^+) ; \quad \varphi^+ \left( \sum_{j=1}^n \lambda_k c_j \right) = \sum_{i=1}^n \lambda_j p_j^+
\]

and, in a similar way, we define the homomorphism \( \varphi^- : C \to \mathcal{K}(\mathcal{E}^-) \). Then one easily checks that

\[
x_\Gamma = [(\mathcal{E}^+ \oplus \mathcal{E}^- , \varphi^+ \oplus \varphi^- , 0)] \in KK(C, B)
\]

is the class corresponding to \( \Gamma \in M(m \times n, \mathbb{Z}) \).

Suppose now that \( G \) is a locally compact group which acts on \( C \) and \( B \) via permutations of the bases \( \{c_1, \ldots, c_n\} \) and \( \{b_1, \ldots, b_m\} \), respectively. Let \( \mu_C : G \to S_n \) and \( \mu_B : G \to S_m \) denote the corresponding homomorphisms into the permutation
groups $S_n$ and $S_m$, respectively. We shall often simply write $g \cdot j$ (resp. $g \cdot i$) for $\mu_C(g)(j)$ (resp. $\mu_B(g)(i)$). We note that these actions will always factor through actions of some finite quotient $G/N$ of $G$, so that in the following discussion one could assume as well that $G$ is finite.

For $x \in KK^G(C, B)$, the corresponding element in $\text{Hom}(K_0(C), K_0(B))$ is equivariant with respect to the actions of $G$ on $K_0(C)$ and $K_0(B)$ induced by the given actions on $C$ and $B$, respectively. This implies that the corresponding matrix $\Gamma \in M(m \times n, \mathbb{Z})$ satisfies the relation $\Gamma \circ \mu_C(g) = \mu_B(g) \circ \Gamma$ for all $g \in G$. This easily translates to the condition $\gamma_{g \cdot i, g \cdot j} = \gamma_{ij}$ for each entry $\gamma_{ij}$ of $\Gamma$. It therefore follows that the same relations hold for $\Gamma^+$ and $\Gamma^-$ and we may define an action $\mu_\mathcal{E}: G \to \text{Aut}(\mathcal{E}^+/\mathcal{E}^-)$ by

$$\mu_\mathcal{E}(g) \left( \bigoplus_{j=1}^n \left( \bigoplus_{i=1}^m v_{ij} \otimes b_i \right) \right) = \bigoplus_{j=1}^n \left( \bigoplus_{i=1}^m v_{g^{-1} \cdot i, g^{-1} \cdot j} \otimes b_i \right).$$

Let us check that $\varphi = (\varphi^+, \varphi^-): C \to \mathcal{K}(\mathcal{E}^+ \oplus \mathcal{E}^-)$ is $G$-equivariant. For this let $c_i$ be a fixed basis element of $C$. We want to compare $\varphi^+(\mu_C(g)(c_i))$ with $\mu_\mathcal{E}(g)\varphi^+(c_i)\mu_\mathcal{E}(g)^{-1}$ and we do this by computing what both operators do to the $(i, j)$-th summand $C^{\gamma_j} \otimes B_i$ of $\mathcal{E}^+$. First of all, the projection $\varphi^+(\mu_C(g)(c_i)) = p^g_{ij}$ fixes the element $v_{ij} \otimes b_i$ if $j = g \cdot l$ and sends it to 0 if $j \neq g \cdot l$. In order to compute $\mu_\mathcal{E}(g)\varphi^+(c_i)\mu_\mathcal{E}(g)^{-1}(v_{ij} \otimes b_i)$ we first observe that $\mu_\mathcal{E}(g^{-1})$ moves $v_{ij} \otimes b_i$ to the element $v_{ij} \otimes b_{g^{-1} \cdot i}$ at the $(g^{-1} \cdot i, g^{-1} \cdot j)$-th place. Then $\varphi^+(c_i) = p^g_{ij}$ will fix this element if $l = g^{-1} \cdot j$ (i.e. $j = g \cdot l$) and will send it to 0 else. Finally $\mu_\mathcal{E}(g)$ will move $v_{ij} \otimes b_{g^{-1} \cdot i}$ to the element $v_{ij} \otimes b_i$ at the $(i, j)$-th place if $j = g \cdot l$. This shows the desired result. The same computation yields equivariance of $\varphi^-.$

**Remark 7.1.** We could have constructed the same element by any other decomposition $\Gamma = \Gamma^+ - \Gamma^-$ as long as both matrices $\Gamma^+, \Gamma^-$ only have positive integer entries and satisfy the relations $\tilde{\gamma}_{g \cdot i, g \cdot j}^{+/\mathcal{E}} = \gamma_{ij}$. In fact, if we do the construction with the help of such an alternative decomposition to obtain a class $\tilde{x}_\Gamma = [(\mathcal{E}^+ \oplus \mathcal{E}^-, \varphi^+ \oplus \varphi^-, 0)]$, then the difference $\tilde{x}_\Gamma - x_\Gamma$ is represented by the Kasparov triple $[(\mathcal{F}^+ \oplus \mathcal{F}^-, \psi^+ \oplus \psi^-), 0]$ with

$$\mathcal{F}^+ = \tilde{\mathcal{E}}^+ \oplus \mathcal{E}^-, \mathcal{F}^- = \tilde{\mathcal{E}}^- \oplus \mathcal{E}^+ \text{ and } \psi^+ = \tilde{\varphi}^+ \oplus \varphi^-, \psi^- = \varphi^- \oplus \tilde{\varphi}^+. $$

Using the equation $\Gamma^+ + \tilde{\Gamma}^- = \Gamma^- + \tilde{\Gamma}^+$, one checks that

$$\mathcal{F}^+ = \mathcal{F}^- = \bigoplus_{j=1}^n \left( \bigoplus_{i=1}^m (C^{\tilde{\gamma}_j^+} \otimes B_i) \right) \text{ and } \psi^+ = \psi^-,$$

which implies that the triple $(\mathcal{F}^+ \oplus \mathcal{F}^-, \psi^+ \oplus \psi^-, 0)$ is operator homotopic to the degenerate triple $(\mathcal{F}^+ \oplus \mathcal{F}^-, \psi^+ \oplus \psi^-, (\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}))$ via $t \mapsto t \left( \begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix} \right)$.

We shall need

**Lemma 7.2.** Suppose that $C = \mathbb{C}^n, B = \mathbb{C}^m, A = \mathbb{C}^k$ are equipped with actions of the locally compact group $G$ given by homomorphisms $\mu_C: G \to S_n, \mu_B: G \to S_m, \text{ and } \mu_A: G \to S_k$. Let $\Gamma \in M(m \times n, \mathbb{Z})$ and $\Lambda \in M(n \times k, \mathbb{Z})$ such that

$$\Gamma \circ \mu_C(g) = \mu_B(g) \circ \Gamma \quad \text{and} \quad \Lambda \circ \mu_B(g) = \mu_A(g) \circ \Lambda$$
for all $g \in G$. Then $x^G_{\Gamma} \otimes_B x^G_A = x^G_{A \cdot \Gamma}$ in $KK^G(C, A)$. In particular, if $m = n$ and $\Gamma \in \text{GL}(n, \mathbb{Z})$, then $x^G_{\Gamma} \in KK^G(C, B)$ is invertible with inverse given by the class $x^{G, -1}_{\Gamma} \in KK^G(B, C)$.

Proof. Let $(\mathcal{E}^+ \oplus \mathcal{E}^-, \varphi^+ \oplus \varphi^-, 0)$ and $(\mathcal{F}^+ \oplus \mathcal{F}^-, \psi^+ \oplus \psi^-, 0)$ denote the corresponding Kasparov triples as constructed above from $\Gamma$ and $\Lambda$. Then the product $x^G_{\Gamma} \otimes_B x^G_A$ is represented by the triple $(\mathcal{G}^+ \oplus \mathcal{G}^-, \mu^+ \oplus \mu^-, 0)$ with

$$
\mathcal{G}^+ = (\mathcal{E}^+ \otimes_B \mathcal{F}^+) \oplus (\mathcal{E}^- \otimes_B \mathcal{F}^-) \quad \text{and} \quad \mathcal{G}^- = (\mathcal{E}^+ \otimes_B \mathcal{F}^-) \oplus (\mathcal{E}^- \otimes \mathcal{F}^+)
$$

and with $\mu^+ = (\varphi^+ \otimes 1_{\mathcal{F}^+}) \oplus (\varphi^- \otimes 1_{\mathcal{F}^-})$ and $\mu^- = (\varphi^+ \otimes 1_{\mathcal{F}^-}) \oplus (\varphi^- \otimes 1_{\mathcal{F}^+})$. Of course, these modules decompose into summands of the form $(\mathcal{C}_{\gamma/i} \otimes_B B_i) \otimes_B (\mathcal{C}_{\lambda/i} \otimes A_i)$, where $A_i = \mathcal{C}_{\Lambda} a_i$ for $a_i$ an element in a given basis $\{a_1, \ldots, a_k\}$ of pairwise orthogonal projections of $A$, and we now compute these summands: Since $b_i^2 = b_i$, the balanced tensor product $(\mathcal{C}_{\gamma/i} \otimes_B B_i) \otimes_B (\mathcal{C}_{\lambda/i} \otimes A_i)$ is generated by elementary tensors $(v_{ij} \otimes b_i) \otimes (w_{lr} \otimes a_r)$ modulo the relation

$$(v_{ij} \otimes b_i) \otimes (w_{lr} \otimes a_r) = (v_{ij} \otimes b_i) \otimes \psi^{i/r} (b_i)(w_{lr} \otimes a_l)$$

which forces the element to be zero if $r \neq i$, and which is always satisfied if $r = i$. Thus we see that

$$(\mathcal{C}_{\gamma/i} \otimes B_i) \otimes_B (\mathcal{C}_{\lambda/i} \otimes A_i) \cong \begin{cases} 
\mathcal{C}_{\gamma/i} \otimes A_i & \text{if } i = r, \\
0 & \text{if } i \neq r.
\end{cases}$$

Moreover, the projections $\mu^{i/-}(c_i)$ will fix these spaces if and only if $i = j$ and will send them to 0 otherwise. Summing up over $i$ and using Remark 7.1 then shows that $[(\mathcal{G}^+ \oplus \mathcal{G}^-; \mu^+ \oplus \mu^-; 0)]$ equals $x^G_{A \cdot \Gamma}$.

\[\square\]

References

[1] B. Blackadar, K-theory for operator algebras. Second edition. Mathematical Sciences Research Institute Publications 5, Cambridge University Press, Cambridge, 1998.

[2] P. Baum, A. Connes and N. Higson, Classifying space for proper actions and K-theory of group C*-algebras, Contemporary Mathematics 167 (1994), 241-291.

[3] J. Chabert and S. Echterhoff, Permanence properties of the Baum-Connes conjecture. Doc. Math. 6 (2001), 127-183.

[4] J. Chabert, S. Echterhoff and R. Nest, The Connes-Kasparov conjecture for almost connected groups and for linear p-adic groups, Publ. Math. Inst. Hautes Études Sci. No. 97 (2003), 239-278.

[5] J. Chabert, S. Echterhoff and H. Oyono-Oyono, Going-down functors, the K"unneth formula, and the Baum-Connes conjecture. Geom. Funct. Anal. 14 (2004), 491-528.

[6] J. Crisp and M. Laca, On the Toeplitz algebras of right-angled and finite-type Artin groups, J. Austral. Math. Soc. 72 (2002), 223-245.

[7] J. Cuntz, K-theoretic amenability for discrete groups, J. Reine Angew. Math. 344 (1983), 180–195.

[8] J. Cuntz, C. Deninger and M. Laca, C*-algebras of Toeplitz type associated with algebraic number fields, arXiv:1105.5359v2, to appear in Math. Ann.

[9] J. Cuntz, S. Echterhoff and X. Li, K-theory for semigroup C*-algebras, arXiv:1201.4680v1.

[10] J. Cuntz and X. Li, The regular C*-algebra of an integral domain, Clay Mathematics Proceedings 11, 149–170, 2010.

[11] J. Cuntz and A. Vershik, C*-algebras associated with endomorphisms and polymorphisms of compact abelian groups, arXiv:1202.5960v2.
[12] S. Echterhoff, W. Lück, N. C. Phillips and S. Walters, The structure of crossed products of irrational rotation algebras by finite subgroups of $SL_2(\mathbb{Z})$, J. reine angew. Math 639 (2010), 173–221.

[13] S. Echterhoff, R. Nest and H. Oyono-Oyono, Fibrations with noncommutative fibres, Journal of Noncommutative Geometry 3 (2009), 377–417.

[14] P. Green, The local structure of twisted covariance algebras, Acta. Math. 140 (1978), 191–250.

[15] E. Guentner, N. Higson and S. Weinberger, The Novikov conjecture for linear groups, Publ. Math. Inst. Hautes Études Sci. 101 (2005), 243–268.

[16] N. Higson and G. Kasparov, $E$-theory and $KK$-theory for groups which act properly and isometrically on Hilbert space, Invent. Math. 144 (2001), 23–74.

[17] N. Higson, V. Lafforgue and G. Skandalis, Counterexamples to the Baum-Connes conjecture, Geom. Funct. Anal. 12 (2002), no. 2, 330–354.

[18] N. Ivanov, The $K$-theory of Toeplitz $C^*$-algebras of right-angled Artin groups, Trans. Amer. Math. Soc. 362 (2010), no. 11, 6003–6027.

[19] G. Kasparov, Equivariant $KK$-theory and the Novikov conjecture, Invent. Math. 91 (1988), 147–201.

[20] I. Putnam, The $C^*$-algebras associated with minimal homeomorphisms of the Cantor set, Pacific J. Math. 136 no. 2 (1989), 329–353.

[21] J. Quigg and J. Spielberg, Regularity and hyporegularity in $C^*$-dynamical systems. Houston J. Math. 18 (1992), 139–152.

[22] M. Laca, From endomorphisms to automorphisms and back: dilations and full corners, J. London Math. Soc. 61 (2000), 893–904.

[23] X. Li, Semigroup $C^*$-algebras and amenability of semigroups, arXiv:1105.5539v2, to appear in J. Functional Analysis.

[24] X. Li, Nuclearity of semigroup $C^*$-algebras and the connection to amenability, arXiv:1203.0021v2.

[25] R. Meyer and R. Nest, The Baum-Connes conjecture via localisation of categories, Topology 45 (2006), no. 2, 209–259.

[26] A. Nica, $C^*$-algebras generated by isometries and Wiener-Hopf operators, J. Operator Theory 27 (1992), 17–52.

[27] M. Pimsner and D. Voiculescu, Exact sequences for $K$-groups and $Ext$-groups of certain cross-product $C^*$-algebras, J. Operator Theory 4 (1980) 93–118.

[28] N. Riedel, Classification of the $C^*$-algebras associated with minimal rotations. Pacific J. Math. 101 no. 1 (1982), 153–161.

[29] M. Takesaki, Covariant representations of $C^*$-algebras and their locally compact automorphism groups, Acta Math. 119 (1967), 273–303.

[30] J.-L. Tu, La conjecture de Baum-Connes pour les feuilletages moyennable, K-theory 17 (1999), 215-264.