A lower bound for Heilbronn's triangle-problem

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Abstract

A conjecture by H. A. Heilbronn (before 1950) stated that if \( n \) points are placed on the convex domain of unit area at a plane (no three points on a straight line), some of the triangles defined by these points always have an area not larger than \( O(n^{-2}) \), irrespective to the placing of the points.

Komlós, Pintz and Szemerédi \[K\] disproved this in 1981 by showing that \( n \) points can be placed on the convex domain of unit area so that any of the triangle’s area is not less than \( O(n^{-2}\log n) \).

In this paper we are going to prove the following:

**Theorem 1:** There is a constant \( n_1 \in \mathbb{R}_+ \) and a construction of a Set of \( n_0 \) points within the unit circle of the plane where all the triangles defined by the points of this Set have an area \( A \geq c n_0^{3/2}\log^{-7/2} n_0 \) if \( n_0 > n_1 \). Here \( c \leq 0.25588... \)

Introduction

Conventions

All quantities in this paper are real.

Small Latin letters are used for positive integers, except for \( e \), the base of the natural logarithm. Another exception is \( c \), which denotes a positive constant, not necessarily the same from case to case. The letters \( p, q \) and \( r \) are reserved to denote radii of the circles examined. The expressions of \( p \)-circle and \( p \)-polygon are often used. These mean a circle with radius \( p \), and a regular \( p \)-sided polygon written into that circle, respectively.

Capital letters either denote points or real numbers, except \( N \), which is an integer. The letter \( A \) is reserved for the area, of course, it is variable from case to case. \( H \) is the height over the base side of a triangle. \( P, Q \) and \( R \) denote some vertices of the polygons. \( O \) stands for the Origo, the centre of the circles. \( O() \) is the Ordo in the sense of Bachmann - Landau, i.e. the magnitudes of the remnants that are added to the main term, getting the value of the expression.

The Greek letter \( \lambda \) is used for an angle, the \( \Delta \) means triangle, while \( \pi \) retains its well-known numeric value. As usual, \( \pi() \) is standing for the number of primes from 2 to the argument. The \# symbol is used for cardinality, i.e. it denotes the “number of”.

The word “about” is often used in this paper. It means that there is no exact equality, for instance a count of elements is estimated by an approximating quantity instead of using the exact integer. Nevertheless, when the relative difference between the proper value and the estimated one is smaller than any positive constant, we may use the word “about”. When it is worth calculating the remnants, we use the \( O() \) notations instead of the “about”.

The “disk” holds all the circles we have, but none of smaller or larger circles.

A straight line is denoted by naming two points on it, e.g., \( PQ \) line. The word “line” is always there. The distance between points \( P \) and \( Q \) is denoted by \( PQ \), in that case the word “line” is omitted.
The initial Set

Let \( N \) denote a large positive integer. Let \( N \) and \( p \in \mathbb{P} \), \( p \in (N - [N/\log N], N - 2) \). Take the \( p \)-sided regular polygon on the plane, written into a circle with radius \( p \) around the Origo. The word “circle”, from now on, means one of these concentric circles, unless it is noted to be different from them.

We rotate the polygons around the Origo, each of them with a different small angle \( cs/(tN) \), where \( t \) is the radius of the circle containing the polygon and \( s \equiv t^2 \pmod{N} \) where \( s \) has the smallest positive number satisfying the congruence. Here \( 0 < c \leq 1 \) is a constant, having the same value for all polygons. All the vertices of all these polygons are the initial elements of our Set.

**Theorem 2:** If \( p < q < r \) then the area \( A \) of the \( \Delta \) formed from the \( P \), \( Q \) and \( R \) vertices (each on their circle) closest to a half-axis is \( A \geq c/N \).

**Proof of Theorem 2:**

First, we show that two different polygons are turned with different angles. For this we may assume that the different circles have radii \( p < q \).

Clearly, if \( u \equiv p^2 \) and \( v \equiv q^2 \pmod{N} \) then \( u \neq v \) since \( |v - u| \equiv |q^2 - p^2| = (q - p) \cdot (q + p) \not\equiv 0 \pmod{N} \) because \( 0 < q - p < N \) and \( N < q + p < 2N \) due to \( \min(p, q) > N/2 \), \( \max(p, q) < N \). But \( u \neq v \) then \( u/p \neq v/q \) because \( uq = vp \) is impossible due to \( uq \equiv p^2q \not\equiv q^2p \equiv vp \pmod{N} \) since \( pq \not\equiv 0 \pmod{N} \) and \( p \neq q \). Thus, the angles of rotations are all different. Therefore, they are still different when we apply any value for \( 0 < c << 1 \), the same factor for all the polygons. If \( c \) is positive, but much less than 1, then the rotation by the angle of \( cs/(tN) \) is, indeed, minor compared to the \( 2\pi/t \), the central angle between the polygon’s neighbouring vertices.

Secondly, for instance, take the half-axis \( +X \). The ordinate of the vertex closest to the axis on any circle is not much smaller than the minute \( cs/N \). (The difference is about \( O(N^{-3}) \)). Bear in mind that any power of the radius \( \pmod{N} \) is smaller than \( N \).

Hence, if \( A \) means the area of the \( \Delta \), then \( 2A = \begin{vmatrix} 1 & p & cu/N \\ 1 & q & cv/N \\ 1 & r & cw/N \end{vmatrix} = c/N \cdot \begin{vmatrix} 1 & p & p^2 \\ 1 & q & q^2 \\ 1 & r & r^2 \end{vmatrix} \pmod{N} = c/N \cdot |(p - q)(q - r)(r - p)| \pmod{N} \).

Because \( |(p - q)(q - r)(r - p)| \pmod{N} \) can be any integer \( > 0 \) and \( < N \), so, \( A \geq c/N \).

Thirdly, turning the entire system by \( \pm 90 \) degrees, we can prove **Theorem 2** for the \( Y \) axis, etc.

By this, **Theorem 2** is proven.

The above means that there is a \( c \in \mathbb{R} \) that \( A \geq c/N \) for \( \Delta \) that have vertices of the Set, nearest to the half-axes.

The small rotation was introduced for the axes, or near to them, so these shall have no vertices leading to a \( \Delta \) with area “too small”.

It is possible to prove that no three points of the Set are collinear, but it is not necessary because the \( \Delta \) of collinear vertices would have zero area, and in this case, we drop one of the vertices, i.e., eliminate the \( \Delta \) that have a too small area.

**The method of this paper**
Our method is as follows: Estimate the $Z \in \mathbb{R}_+$ threshold that governs the acceptance or rejection of a $\Delta$. We might drop a vertex of the $\Delta$ from the Set if the $\Delta$'s area is smaller than $Z$.

Consider a straight line through two different points of the Set, $P$ and $Q$. If this line is further away from the nearest $R$ points (different from $P$ and $Q$) of the Set than the threshold, then we keep the point $R$ in the Set. Otherwise, this $R$ point might be dropped, i.e., the $\Delta$ that has an area too small might be eliminated. After doing this procedure for all $PQ$ lines and to them nearby $R$ points, we estimate the number of lost points. We are going to see that the number of points we need to drop can be fewer than all the points in the initial Set. The length of $Z$ threshold might be varied from case to case. $\text{Min}(Z)$ for the entire disk means that we certainly delete the $\Delta$s that have smaller area than $\text{Min}(Z)$. It obviously means that all the $\Delta$s remaining in the system have greater area than $\text{Min}(Z)$ for the disk.

It is important to remark that we cannot say that a certain point will be kept as an element of the final Set because we don’t know which points will be dropped. Some of the points that are retained in a certain phase might be eliminated in a later phase of the dropping procedure. We can, however, say that we drop many of the initial points but not all. The final cardinality of the Set is relevant, not the elements that are retained.

The cardinality of the initial Set

According to Pierre Dusart [D], simplified a bit, $\pi(x) = x/\log x + x/\log^2 x + O(x/\log^3 x)$, if $x$ is large enough.

Using the above, it is easy to verify that $\pi(N(1 - 1/\log N)) = N/\log N + O(N/\log^3 N)$, if $N$ is large enough. Therefore, the number of polygons, i.e., the number of primes in the interval is $\pi(N - 2) - \pi(N - N/\log N) = N/\log^2 N + O(N/\log^3 N)$. Each polygon has at least $[N - N/\log N]$ vertices and at most $N - 2$ vertices, therefore the cardinality of the initial Set is at least and at most $n = [N^2/\log^2 N + O(N^2/\log^3 N)]$ where the Ordo bears different constants for the upper and lower estimates. Thus, the cardinality of the initial Set is $n = N^2/\log^2 N$.

Choice of the names for the vertices

Let the three vertices of the $PQR \Delta$ be placed on two or three polygons. Call the radii of the circles on what $P$, $Q$ and $R$ vertices reside $p$, $q$ and $r$. We consider the $PQ$ line. Let $PQ$ be the “base” of the $PQR \Delta$.

The hit points and the exclusion zones

The two points where the $PQ$ lines cut into the $r$ circle we call hit points. In fact, we assume that the hit points are uniformly distributed on the $r$ circle, also between neighbouring $R$ vertices.

The $2Z$ long interval around an $r$-vertex is called an exclusion zone, or just zone, because we might exclude from the Set the $R$ vertex where the hit-point is falling into that zone for some $P$ and $Q$ points. The exclusion zone is served to ignore the $PQR \Delta$s that have too small an area. Less $Z$ means fewer deletions.

On the assumed uniform distribution of the hit points

The $r$ can be any of the radii (except $p$ and/or $q$), thus the $PQ$ lines do not “know” the location of the $r$-circles because large primes’ distance from each other are not predictable. There is no reason to have more or fewer hit points on one arc of the $r$-circle than another arc of the same size on the same circle. Therefore, we may assume that the hit points on the $r$-circle are uniformly distributed.
Although we cannot prove that the hit points (of any possible \( P \) and/or \( Q \)) are uniformly distributed in the \( 2\pi \) long arc between neighbouring \( R \) vertices, it may still be assumed.

The exclusion zone’s concept is based upon the assumed uniform distribution of the \( PQ \) lines’ hit points on the \( r \)-circles.

Since the zone’s length is \( 2Z \), a single \( PQ \) line’s one of two hit points has \( Z/\pi \) probability to fall into the zone around the nearest \( r \)-vertex.

**Prove Theorem 1**

To prove **Theorem 1**, we need two **Lemmas**. The first one says that if the hit point’s distance from the nearest vertex is at least \( Z \), then the area of a \( PQR \) \( \Delta \) is also greater than or equal to \( Z \), whatever the positive \( Z \) might be (\( Z \ll \pi \)). **Lemma 2** shows for two or three circles that the threshold \( Z = c \sqrt{N^2/\log N} \) is suitable.

**Hit points outside the zone**

**Lemma 1**: Let \( W \) be a hit point of the \( PQ \) line on the \( r \) circle, where the \( P \) and \( Q \) points are on the \( p \) and \( q \) circles, respectively (both points belong to the Set). If the \( W \) point has the distance from the nearest \( R \) vertex on the \( r \) circle is greater than a threshold \( Z \), then the area \( A \) of the \( PQR \) \( \Delta \) is \( A > Z \).

**Proof of Lemma 1**

Name the circles on which the \( \Delta \)'s vertices reside, \( p \), \( q \) and \( r \), in increasing order of the radii. Look at the retained \( R \) points. For these \( Z < WR \), where \( W \) is a hit point. See Diagram 1 below!

![Diagram 1](image)

\( W \) is on the \( r \) circle, while \( R \) is on the \( T \) line, perpendicular to the \( WO \) line. If \( N \) is great and \( RW < N^{1/4} \) then the distance of \( R \) from the \( r \) circle is very short (less than \( \frac{1}{2}N^{3/2} + O(N^2) \)), so we make a small error when placing \( R \) onto the \( T \) line. Besides, in our case \( RW < N^{1/4} \).

Let \( \lambda \) denote the smallest angle between the \( PQ \) line and the \( OW \) line. \( R' \) denotes the \( r \) circle’s and the \( PQ \) line’s shared point that is closest to \( P \). Of course, it may be \( W \), as well. Thus, \( OR' = OW \) and \( H = RW \cdot \cos \lambda \).

The \( p \) and \( q \) circles cut the \( OR' \) line at the \( P' \) and \( Q' \) points, where \( Q'R' < P'R' < R'V \). Hence, \( R'V = PR' \cdot \cos \lambda > 2 \), so \( PR' \geq 2/cos \lambda \), because \( P \) and \( Q \) are on different circles and \( \min(|q - p|) \geq 2 \).
As we seen, \( H > Z \cdot \cos \lambda \). Finally, \( A = \frac{1}{2} PR^2 - H > \frac{1}{2} \cdot 2 / \cos \lambda \cdot Z \cdot \cos \lambda \). Therefore \( A > Z \). This is valid for the \( PR^2 \) base. But \( PQ \cos \lambda \geq 2 \), too, so it is valid for the \( PQ \) base, as well.

By that, **Lemma 1** is proven. The Lemma is valid for \( p = q < r \) and \( p < q = r \) cases, too. For \( p = q < r \) the \( R' \) point is \( W \); for \( p < q = r \) the \( R' \) point is \( Q \). The proof is similar, when \( R \) is on the other side of the \( PQ \) line.

**How large can the Z threshold be?**

The principles

No matter how big or small \( Z \) is, \( A \geq Z \) when the hit point of the \( PQ \) line on the \( r \) circle have a distance from the nearest \( R \) vertex that exceeds \( Z \), according to **Lemma 1**. Otherwise, the \( PQR \Delta \) might be eliminated. Hence, for the deletion we shall use the same \( \min(Z) \) for all the cases. One should be careful to limit the number of drops, i.e., not to drop the whole Set. We shall be sure that at least a quarter of the initial Set remains at the end of the dropping procedure.

We fix the \( Q \) point, temporarily. Now we look at the \( PQ \) lines for all possible \( P \). We collect the \( PQ \) bases into two classes (the first of these has several subclasses according to the increasing length of \( PQ \)). Each subclass will limit the \#R, i.e., the number of \( r \)-vertices that can be close to that given \( PQ \) line. Knowing that the probability of the hit point falling into the exclusion zone around the nearest \( R \), is less than \( Z/\pi \), we can estimate the number of points that might be eliminated.

**Lemma 2**: For the case of three vertices on two or three polygons, \( Z = c \cdot N^{-1}/\sqrt{\log N} \).

**Proof of Lemma 2**

Let \( Q \) be any vertex of the Set, thus \( \#Q \leq N^2/\log^2 N \) overall, i.e., \( \#Q \) is (over)estimated to be at most as big as the Set’s initial cardinality. Let us estimate \( \#P \) and by that the number of \( PQ \) lines. We use this moment also to estimate \( \#R \) for the \( R \) points near that \( PQ \) line. After those estimates, we let \( Q \) run through all the points of the Set.

For a fixed \( Q \), the \( \#P \) can be (over)estimated as the half area of a \( U \) circle (a special circle, not concentric with the other circles), centred in \( Q \) and having radius of \( \max(PQ) \). The reason for using half of the circle is \( p < q \).

We create many subclasses for the length of the bases, namely
\[
N^{2 + 3/8 \cdot 1/m} < PQ < N^{10 + 9j/16 \cdot 1/m} \quad (1 \leq j \leq \lceil 16m/9 \rceil + 1)
\]

It is important that we deal with these intervals in their increasing length, starting with the smallest \( j \) and ending with the largest one.

The first interval starts at \( N^{3/8 - 1/m} < 2 \), if \( m \) is large enough, e.g., \( m \geq \lceil \log N \rceil \). The 2 should be the absolute minimum of \( PQ \). The following intervals overlap, so that each interval starts before the previous one ends, since \( (2 + 3j)/8 < (10 + 9(j - 1))/16 \) for any \( j > 1 \).

Let us estimate \( \#P! \)

Denote \( G = \min(P,Q) \) within a subclass. Consider a grid made by quadrats, each having an edge \( G \) long. The \#nodes of the grid positioned in the half of circle \( V \) around centre \( Q \) with radius \( \max(PQ) \) is an (over)estimate for \( \#P \) belongs to a fixed \( Q \). If there were more points to be counted, those would be placed inside some squares, hence, two points would be closer to each other than \( G \). This would be a contradiction with \( G = \min(P,Q) \). Former subclasses have dealt with this. So, the area of the half circle \( V \) is divided by the \( G^2 \) area of the smallest square in the grid, and this could give us an
over-estimate for the number of nodes of the grid. (The node at the upper right would stand for the square.)

Note, that there is well-known waste at the edge of the half circle. It is at most about \( O((#P)\frac{3}{2}) \) according to C. F. Gauss (Circle-problem) \([G]\). This waste is marginal; thus, we can disregard it.

\[
#P_j < \frac{\pi}{2} \left( \max(PQ) \right)^2 / (\min(PQ))^2 = \frac{\pi}{2} N^{10 + 9j/8} \cdot 1/m / \sqrt{10 + 6j/8} \cdot 1/m = \frac{\pi}{2} N^{\frac{3}{2} + j/8} \cdot 1/m.
\]

Now, we look at the \( R \) points near to the \( PQ \) line. We know that their number is at most about \( \max(L) / G \) in this subclass, because the divisor is the minimum of \( PQ \). Two \( R \) points that are nearer to each other than \( G \), belong to previous subclasses.

The length of \( L \) is an overestimate of the useful part of the \( PQ \) line that consists of one or two sections. \( R_j \) cannot be outside the largest circle, nor can it be closer to the origin than the smallest circle’s radius. Hence, the length of \( L \), or the joint length of the two sections will not exceed the length of the longest continuous section, i.e., twice the length of the tangent from a point on the outermost circle drawn to the innermost circle. So, \( \max(L) \) is about \( 2\sqrt{2} \cdot \sqrt{\log N + O(N^{3/2}/N)} \) (See the Appendix). Ultimately, \( \#R_j \leq 2\sqrt{2} \cdot \sqrt{\log N} \cdot 1/N \cdot 10^{2 + 3j/8} \cdot 1/m \).

The overall number of \( \Delta \)s with base of \( PQ \) on the \( PQ \) lines which have very small area for this subclass is \( \#P_j \cdot \#Q \cdot \#R_j < (\frac{\pi}{2} N^{10 + j + 1}/8 \cdot 1/m) \cdot (N^{3/2} / n^2)N \cdot (2\sqrt{2} \cdot \sqrt{\log N}) = \pi\sqrt{2} \cdot N^{3 - \frac{3}{2}/m} / \sqrt{\log N} \).

Let \( m \) be large, say, \( m = [\log N] \). Please observe that \( N^{1/m} = e^{\log N / [\log N]} = e + O(\log 1/N) \).

How many \( \Delta \)s might be rejected due to their area too small? There are at most about \( \#P_j \cdot \#Q \cdot \#R_j \leq \pi\sqrt{2} \cdot N^{3} / \sqrt{\log N} \) points as candidate for deletion in this subclass of \( P_j \). Since there is no \( j \) on the right-hand-side, we might assume that the same \( Z \) serves for all the subclasses. Overall, we might drop \( \frac{3}{4} N^{3} / \sqrt{\log N} \) points from the Set by treating for \([16/9 \cdot \log N + 1]\) subclasses, so there are at most about \( 16\pi\sqrt{2} e/9 \cdot N^{3} / \sqrt{\log N} \) points that will be deleted with \( Z / \pi \) probability. Therefore, we drop \( 16\pi\sqrt{2} e/9 \cdot N^{3} / \sqrt{\log N} \cdot Z / \pi = \frac{3}{4} N^{3} / \sqrt{\log N} \) points. The \( \frac{3}{4} \) can be changed to any other number less than 1 because we do not want to eliminate all the points. We choose \( \frac{3}{4} \) for convenience. Hence, \( Z = c_2 N^{3} / \sqrt{\log N} \) where \( c_2 = 27 / (64\pi\sqrt{2} e) \).

In cases when \( p = q < r \) or \( p < q = r \), \( Z \) cannot be smaller than \( c_2 N^{3} / \sqrt{\log N} \) because the \( P, Q \) and \( R \) points are essentially on three circles, even if one circle resembles to another one.

For the vertices resting on two or three circles, \( Z = c_2 N^{3} / \sqrt{\log N} \). By this, \textbf{Lemma 2} is proven.

\textbf{Summary}

After dropping at most \( \frac{3}{4} N^{3} / \sqrt{\log N} \) points of the original cardinality, the Set shall retain at least a quarter of its original points. (The \( \Delta \)s in \textbf{Theorem 2} have greater area than \( c/N \), because \( c \) can be chosen such. Hence, we do not remove any points from the Set for these.)

In the following the subscript index refers to the number of circles that give home to the \( P \), \( Q \) and \( R \) vertices of the \( \Delta \).

Trivially, \( Z_1 = A \geq 8\pi^{2} N^{-1} + O(N^{-1}\log N) \). Due to \textbf{Lemma 2} \( Z_2 = Z_3 = c_2 N^{3} / \sqrt{\log N} \). From these we conclude that \( Z = c_2 N^{3} / \sqrt{\log N} \), if \( N \) is great enough. So, after the deletions, constructed from the remaining points the \( \Delta \)s area \( A \) is certainly greater than \( Z = c_2 N^{3} / \sqrt{\log N} \).

\textbf{Mapping onto the unit circle as domain}
The Set’s starting cardinality was about \( n \approx N^2/\log^2 N \). After the drop this became \( n \geq \frac{1}{4} N^2/\log^2 N \).

One can verify that the inverse of this final cardinality is \( N \leq n^{1/2} \log n + O(n^{1/2} \log^3 n) \). Please see that \( \log N = \frac{1}{2} \log n + O(\log \log n) \).

The disk-ring domain has been used throughout this paper. Now we project our result onto the unit circle, i.e., one that has radius 1. We have seen that for the \( \Delta \) having the smallest area \( A \) constructed from the remaining points of the Set has \( A \geq Z = c_2 N^{1/2} \log N \approx \sqrt{2c_2} n^{1/2} \log^{3/2} n \). For the mapping we must divide the disk’s area by \( N^2 \leq n \log 2n + O(n) \). Therefore, on the unit circle, \( A \geq c n^{3/2} \log^{7/2} n + O(n^{3/2} \log^{11/2} n) \), if \( n \) is large enough.

By this, the **Theorem 1** is proven.

The unit circle has an area of \( \pi \). To satisfy the prescription of the original conjecture, i.e., the convex domain of area 1, one shall change the value of the constant factor.

Reference

[K] Komlós, J.; Pintz, J.; Szemerédi, E. (1982), "A lower bound for Heilbronn's problem", *Journal of the London Mathematical Society*, 25 (1): 13–24, doi:10.1112/jlms/s2-25.1.13, MR 0645860.

[D] P. Dusart (2010) "Estimates of Some Functions Over Primes without R.H." (PDF). arxiv.org. https://arxiv.org/PS_cache/arxiv/pdf/1002/1002.0442v1.pdf

[G] C. F. Gauss *Werke*, 2, Göttingen (1863) pp. 269–291

**Appendix: on max(l)**

Let \( a, b, X \in \mathbb{R}_+, \ a < b \), and let \( X \leq a \). See Diagram 2 below.

Trivially, \( a^2 - X^2 < b^2 - X^2 \). It is also clear that \( a^2 - X^2 < \sqrt{(b^2 - X^2)(a^2 - X^2)} \). From this follows \( a^2 + b^2 - 2X^2 - 2\sqrt{(a^2 - X^2)(b^2 - X^2)} < b^2 - a^2 \). Thus, \( \sqrt{(b^2 - X^2)} - \sqrt{(a^2 - X^2)} < \sqrt{(b^2 - a^2)} \). This means that \( FM \leq JK \) when \( 0 < X \leq a \). If \( a < X \leq b \), then \( JK \) is obviously greater than \( BE \).