Numerical modeling of the 2015 Illapel Tsunami source by the inversion of DART buoys data

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Abstract. The problem of the tsunami source reconstruction from the remote measurements of an arriving waves in the deep-water tsunamiimeters is considered as an inverse problem of mathematical physics in the class of the ill-posed problems. The latter determines the inevitable instability of the numerical solution. We propose a method providing more control on the numerical instability of the solution. The DART buoys data allows one to apply the linear shallow-water theory for the wave propagation and the r-solution method for the tsunami waveform inversion. The applied approach is independent from the earthquake source characteristics, because the only observed tsunami waveforms and a roughly estimated tsunami source area are used. A good coincidence of observed and computed tsunami waveforms was a criterion of the quality of the inversion. In this paper we study the 16 September 2015 Illapel (Chile) event and conduct a reconstruction of a tsunami source.

1. Introduction

The mega thrust earthquakes quite often cause large tsunamis that may inflict a severe loss and pain to the population of the coastal communities. The development of the Tsunami Warning Systems is likely to be the most constructive way in the mitigation of the destructive effect of potential tsunamis. The key point in the state-of-the-art tsunami forecasting is the numerical simulation using a reliable tsunami source.

Among the mathematical approaches based on the inversion of the near-field water-level data one should mention the widespread methods based on Green’s functions technique (GFT) combined with seismic data [Satake, 1989, 2013; Tanioka et al.,1996; Baba et al.,2009; Mulia et al.,2016], the least square inversion method combined with the GFT and the optimization approach first proposed by C. Pires and P.M.A. Miranda (2001). In the work [Percival et al.2011] source coefficients were estimated by inversion of DART data accompanied with seismic data analysis. The minimal residuals method, when the source shape is estimated by adjustment of model sources to match observed waveforms is successfully used in NOAA (USA) [Piatanesi et al. 2001].

It is well known that the ill-posed problems undergo strong constraints on the mathematical apparatus employed for its solution. The proposed approach is based on the least squares and the truncated singular value decomposition (SVD) techniques. The applied inversion method was first proposed in [14] and
has been basically described in previous papers [15], [16]. To overcome the potential instability of the numerical solution caused generally by the noisiness of real data, the $r$-solution method is used. It consists in projecting the exact solution on the stable subspace which is a linear hull of $r$ first right singular vectors. The subspace, where the solution is sought for, is chosen by analyzing the properties of a singular spectrum of the direct problem operator, determined by the observation system and bathymetry. Moreover, an observation system for retrieving the initial tsunami waveform can be optimally arranged by using this approach with selecting the «most» informative set of the tsunami wave recorders used. The latter, in turn, contributes to the essential improve of the inversion result and provides the possibility to calculate the tsunami waveforms at the points not included in the observation system.

2. The method
In the present paper, the tsunami wave propagation is considered within the scope of the linear shallow-water theory. Then, the wave run-up is not considered here. The curvature of the Earth is neglected. We assume that tsunami excited by an abrupt vertical dislocation of the bottom and water surface repeats the bottom in the tsunami source area. The function of the water surface elevation relative to the mean sea level is also the solution of the linear shallow-water equations:

$$
\eta_{tt} + g \nabla \cdot (h \vec{V}) = 0 \\
\vec{V}_t + g \nabla \eta = 0
$$

completed by following initial conditions:

$$
\eta|_{t=0} = \varphi(x, y), \quad \vec{V}_t|_{t=0} = 0
$$

and the boundary conditions:

- on the solid boundary
  $$
  \vec{V} \cdot \vec{n} = 0
  $$

- on the open boundary
  $$
  -c\vec{V} \cdot \vec{n} - \eta_{tt} + \frac{c^2 \partial^2 \eta}{2 \partial t^2} = 0
  $$

In the above equations, vector $\vec{V} = (v_x, v_y)$ is the horizontal fluid velocity vector whose $x$- and $y$-components are, respectively, $v_x$ and $v_y$, $h(x, y)$ is the water depth, $g$ is the gravity acceleration, $c(x, y) = \sqrt{gh(x, y)}$ is the wave phase velocity; $\vec{n}$ and $\vec{t}$ are the unit vectors, outwardly normally directed to the boundary and tangential directed, respectively, $\varphi(x, y)$ is the initial water displacement defined in the tsunami source area $\Omega$: $\{(x, y) \in [0, l_1] \times [0, l_2]\}$.

The inverse problem is to infer the unknown initial water displacement $\varphi(x, y)$ as output data, while the observed tsunami waveforms as input data are assumed to be known on a set of points $R = \{(x_p, y_p)\}, p = 1 \ldots P$:

$$
\eta(x_p, y_p, t) = \eta_0(x_p, y_p, t), \quad (x_p, y_p) \in R
$$

An unknown initial tsunami waveform (below called tsunami source) is represented as a part of a spatial harmonic series $\{\varphi_{mn}(x, y)\}$ in the source area $\Omega$:

$$
\varphi(x, y) = \sum_{m=1}^{M} \sum_{n=1}^{N} c_{mn} \varphi_{mn}, \quad \varphi_{mn} = \sin \frac{m\pi}{l_1} x \cdot \sin \frac{n\pi}{l_2} y
$$
with the unknown coefficients \(\vec{c} = \{c_m\}\). In our case, the inverse problem data are the observed waveforms (marigrams) \(\vec{\eta}_0\) known on a set of points \((x_p, y_p), p = 1 \ldots P\) and at time instants \(t_j, j = 1 \ldots T\). Thus the vector \(\vec{\eta}_0\) is expressed as follows:

\[
\vec{\eta}_0 = A\vec{c},
\]

where \(A\) is a matrix whose columns consist of computed waveforms for every spatial harmonic \(\varphi_{mn}(x, y)\) used as initial condition to the direct problem (1) – (4). As a result of the computations, the so-called \(r\)-solution is a projection of the exact solution onto a linear hull of the first right singular vectors \(\{\vec{e}_j\}, j = 1 \ldots r\) corresponding to the largest singular values of the matrix \(A\):

\[
\varphi^{[r]}(x, y) = \sum_{j=1}^{r} \alpha_j \cdot \vec{e}_j,
\]

where \(\alpha_j\) are the left and the right singular vectors of the matrix \(A\) and \(s_j\) are its singular values. As in all the inverse seismic problems, the numerical solutions obtained by mathematical methods become unstable due to the presence of the noise \(\vec{e}\) in real data. In other words, one can assume \(\vec{\eta}_0\) as \(\vec{\eta}_0 + \vec{e}\). Hence, \(\alpha_j = \frac{(\eta_0 + \vec{e}, \vec{e}_j)}{s_j}\) and \(\frac{s_j}{(\vec{e}, \vec{e}_j)} \to 0\), when \(j \to \infty\).

The location of tsunami waveforms recorders affects the choice of the value \(r\) by such a way: the better is the configuration of the observational system the longer is a weakly decreasing part of the spectrum. The number \(r\) is taken to be much smaller than a minimal dimension of the matrix. It is natural that as \(r\) increases, the amount of information on the \(r\)-solution obtained also increases, while the stability decreases.

Finally, \(r\) is determined by the behavior of a singular spectrum which is determined by the observational system and bathymetry.

3. The simulation of the Chile Tsunami of September 16, 2015

The 2015 Illapel earthquake occurred at 22:54:32 (UT) on 16 September, at 31.573°S, 72.674°W at a depth of 22.4 km (according to the United States Geological Survey). This earthquake as well as the associated tsunami have been studied by many researchers from various aspects allowing them to design moderately different models of the earthquake source based on the various geophysical data: near-field seismograms, teleseismic waveform and back projections, GPS and InSAR data, and tsunami waveforms [11]. There has also been determined the tsunami source area, approximately, 70 km to the NW of the epicenter by the method of the backward ray tracing of the observed tsunamis [3]. The distribution of tsunami heights was assumed to be consistent with the slip distribution, their maxima varying from 4 m to 12 m. In this study, the inversion method described above was applied to the 2015 Illapel event to reconstruct the tsunami source by the inversion of the tsunami waveforms recorded only by the Deep-Ocean Assessment and Reporting of Tsunamis (DART) buoys.

The simulation domain is the water part of the rectangle \(\Pi = \{(x; y) : 102^\circ W \leq x \leq 70^\circ W; 12^\circ N \leq y \leq 38^\circ S\}\). It is covered with a 1-minute grid of 1921×3001 points to follow the GEBCO bathymetry (http://www.gebco.net/). The source area is assumed to be \(\Omega = \{71.6^\circ W \leq x \leq 73^\circ W; 30^\circ S \leq y \leq 32.5^\circ S\}\). It is covered with a grid of 77×151 points. The DART data provided by the National Oceanic and the Atmospheric Administration (NOAA) of the United States (http://www.ndbc.noaa.gov/dart.shtml). At the coastal boundaries of the calculation domain, the full reflection conditions, as well as at the open sea boundaries, the wave permeability conditions are formulated. The time step \(\Delta t = 4\) sec, and the length of each marigram is \(N_t = 1001\), beginning with the first wave arrival at the corresponding recorder. Figure 1 shows the calculation domain with five deep-water recorders DART buoys numbered counterclockwise and marked by magenta circles: 1–32402; 2–32401; 3–32412; 4–32411; 5–43413.
The original FORTRAN code is used to simulate the tsunami propagation. The code is based on a finite difference algorithm using an explicit-implicit difference scheme constructed with a four-point stencil on a uniform rectangular staggered grid [5] for solving problem (1) – (4). The finite-difference scheme is of the second-order spatial approximation and of the first-order time approximation. The decomposition of the right singular vectors based on the spatial harmonics can be represented in the form:

$$
\vec{v}_j = \sum_{m=1}^{M} \sum_{n=1}^{N} \beta_{mn} \varphi_{mn}(x, y).
$$

Figure 2 shows 64 right singular vectors decomposed to the spatial harmonics in the case of the 16 September 2015 Illapel Chile Tsunami: the coefficients $\beta_{mn}$ are represented on the axis $Oz$; there are indices of the right singular vectors and the indices of the harmonics on the horizontal axes. The harmonics $\{\varphi_k\} = \{\varphi_{mn}\}$ are enumerated in the following order: $k = n + (m - 1) \times N$, $m = 1 \ldots M$ (in longitude), $n = 1 \ldots N$ (in latitude). Number 15 corresponds to a maximum value of the number of the spatial harmonics in latitude and in longitude and is determined by the size of a spatial grid and the time step.

Figure 2. The right singular vectors in terms of spatial harmonics: the indices of harmonics $\{\varphi_k\} = \{\varphi_{mn}\}$, $k = n + (m - 1) \times N$, $m = 1 \ldots M$ (in longitude), $n = 1 \ldots N$ (in latitude); on the axis $Z$: the values of components of the singular vectors.

Figure 3. The cosines of the angles between each spatial harmonic and the subspace of the right singular vectors correspond to value $r = 37$; on the vertical axis: the values of cosines.
It is clear from Figure 2 that approximately 40 first right singular vectors contain the components corresponding to both the low- and the high-frequency harmonics. Using the singular vectors with large indices will result in the “dispersion” of the form of a source into numerous high-frequency sea surface oscillations. On the contrary, when one uses a small number of the first singular vectors to represent the solution, the high-frequency harmonics are poorly represented for this set of singular vectors, and as a result, one obtains a fairly “rude” form of the source with low extreme values. The behavior of the right singular vectors presented in Figure 2 with respect to the spatial harmonics remains structurally the same even if the dimension of the full space changes. The structure of the right singular vectors is predetermined by the location of the observation system and the bottom topography in the simulation domain.

How many spatial harmonics are reasonable? The value of the angle between the vector of each harmonic $\varphi_k$ and the subspace of a quasi-solution will determine the contribution of each harmonic to the solution obtained. The cosines of the angles between the subspace corresponding to $r = 37$ and the spatial harmonics $\varphi_k, k = 1 \ldots 225 (M = 15; N = 15)$ were computed. In Figure 3, there are indices of the harmonics by longitude and by latitude on the horizontal axes, the vertical axis corresponds to the cosine values. It is clear from Figure 3 that the harmonics with the indices exceeding 8 in latitude and in longitude have bigger angles, i.e. the smaller cosine values meaning that the corresponding harmonics are poorly represented within the truncated subspace and, thus, their contribution to the $r$-solution will be small, whereas numerical artifacts would be large.

In the course of the numerical experiments, the following fact was found out: each harmonic is precisely reconstructed (99.8%) only on the space based on a full set of the right singular vectors (for the case of noiseless data). Moreover, when truncated subspaces are used for the reconstruction, a series of additional harmonics (mostly, high-frequency ones) appear including a noticeable distortion of the amplitudes. A decrease in the parameter $r$ has a greater effect on the higher-frequency harmonics. The higher is the frequency, the larger are the artifacts. These facts contribute to the understanding that the artifacts in the solution are inevitable and are caused by the ill-posed nature of the problem.

Hence, one can conclude that for the case under consideration the optimal number of the spatial harmonics varies from 49 to 64 ($M=7; 8; N=7; 8$).

The results of the tsunami source recovery for the Illapel Chile Tsunami based on the optimal parameters, revealed in the course of the above discussion, are represented in Figure 4 with following designation: the black dashed line indicates to the tsunami source area inferred by the back tracing method according to [3]; the yellow dashed line indicates, approximately, to the axis of the trench; the red circle indicates to the earthquake epicenter.

![Figure 4](image-url)

**Figure 4.** The results of recovering the Illapel Chile Tsunami Source. Configuration v123: (a) $\varphi_{max}=12.01m, \varphi_{min}=-7.46m, r=38$; (b) $\varphi_{max}=10.3m, \varphi_{min}=-8.8m, r=36$. Configuration v12345: (c) $\varphi_{max}=9.01m, \varphi_{min}=-6.25m, r=36$
The results of the numerical experiments conducted are in good agreement with the results obtained from seismic data by other researchers [11]. The main uplift was placed near to the point (31°S, 72.31°W). As in other studies [3], [13], [4], it has been determined that the tsunami area is located, approximately, 70 km to the NW of the epicenter that coincides with the center of a fault slip [11]. In addition, the extreme values vary from 9 m in the positive and to 6.3 m in the negative displacement and are comparable with those obtained in the above-mentioned studies. However, the artifacts did not prevent us from getting a good matching in the observed and in the computed marigrams, which is a criterion of the quality of the inversion in this study. In addition, the location of the main uplift was reconstructed with a reasonable agreement with the ones obtained by other researchers. Figure 5 represents the computed marigrams and the observed ones at each point of observational system. We must emphasize that by using the tsunami waveforms from recorders 1,2,3 only, we have also obtained a good matching of the marigrams at points 4-32411, 5-43413.

**Figure 5.** A comparison of the observed tsunami waveforms (black lines) with the calculated ones in configurations: v123, M=8, N=8, r=36 (blue lines); M=7, N=7, r=38 (green lines); v12345, M=8, N=8, r=36 (red lines). The labels on the horizontal axis indicate to the time (in minutes) after the earthquake origin time; numbers above the graphs correspond to the DART buoy ID

4. **Conclusion**
The method proposed suppresses the instability due to the ill-posed nature of the problem. The characteristic features of the $r$-solution obtained are largely determined by the underlying bathymetry and the observation system. Moreover, using the SVD truncation causes the appearance of an inevitable artifacts in the solution due to the use of the non-optimal spatial discretization of an unknown function. Based on the numerical experiments, one can conclude that the method presented provides a good matching in the observed and in the calculated tsunami waveforms. Implementation of the methodology proposed to the 16 September 2015 Chile tsunami has provided the reliable tsunami source reconstruction

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