Notes on $J$-Holomorphic Maps

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Abstract

These notes present a systematic treatment of local properties of $J$-holomorphic maps and of Gromov’s convergence for sequences of such maps, specifying the assumptions needed for all statements. In particular, only one auxiliary statement depends on the manifold being symplectic. The content of these notes roughly corresponds to Chapters 2 and 4 of McDuff-Salamon’s book on the subject.

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1 Introduction

Gromov’s introduction [6] of pseudoholomorphic curves techniques into symplectic topology has revolutionized this field and led to its numerous connections with algebraic geometry. The ideas put forward in [6] have been further elucidated and developed in [11, 15, 16, 10] and in many other works. The most comprehensive introduction to the subject of pseudoholomorphic curves and Gromov’s convergence for sequences of $J$-holomorphic maps. The present notes contain an alternative systematic exposition of these two topics with generally sharper specification of the assumptions needed for each statement. Chapter 3 and Sections 6.2 and 6.3 in [12] concern the third fundamental building block of the subject, transversality for $J$-holomorphic maps. A more streamlined and general treatment of this topic is the concern of [19].

The present notes build on the lecture notes on $J$-holomorphic maps written for the class the author taught at Stony Brook University in Spring 2014. The lectures themselves were based on the handwritten notes he made while studying [11] back in graduate school and were also influenced by the more thorough exposition of the same topics in [12]. The author would like to thank D. McDuff and D. Salamon for the time and care taken in preparing and updating these books, the students in the Spring 2014 class for their participation that guided the preparation of the original version of the present notes, and X. Chen for thoughtful comments during the revision process.

1.1 Stable maps

A (smooth) Riemann surface (without boundary) is a pair $(\Sigma, j)$ consisting of a smooth two-dimensional manifold $\Sigma$ (without boundary) and a complex structure $j$ in the fibers of $T\Sigma$. A nodal Riemann surface is a pair $(\Sigma, j)$ obtained from a Riemann surface $(\tilde{\Sigma}, j)$ by identifying pairs of distinct points of $\tilde{\Sigma}$ in a discrete subset $S_{\Sigma}$ (with no point identified with more than one other point); see the left-hand sides of Figures 1 and 2. The pair $(\tilde{\Sigma}, j)$ is called the normalization of $(\Sigma, j)$; the images of the points of $S_{\Sigma}$ in $\Sigma$ are called the nodes of $\Sigma$. We denote their complement in $\Sigma$ by $\Sigma^*$.

An irreducible component of $(\Sigma, j)$ is the image of a topological component of $\tilde{\Sigma}$ in $\Sigma$. Let

$$a(\Sigma) = \frac{2 - \chi(\tilde{\Sigma}) + |S_{\Sigma}|}{2},$$

where $\chi(\tilde{\Sigma})$ is the Euler characteristic of $\tilde{\Sigma}$, be the (arithmetic) genus of $\Sigma$. An equivalence between Riemann surfaces $(\Sigma, j)$ and $(\Sigma', j')$ is a homeomorphism $h : \Sigma \to \Sigma'$ induced by a biholomorphic map $\tilde{h}$ from $(\tilde{\Sigma}, j)$ to $(\tilde{\Sigma}', j')$. We denote by $\text{Aut}(\Sigma, j)$ the group of automorphisms, i.e. self-equivalences, of a Riemann surface $(\Sigma, j)$.

Let $(X, J)$ be an almost complex manifold. If $(\Sigma, j)$ is a Riemann surface, a smooth map $u : \Sigma \to X$ is called $J$-holomorphic map if

$$du \circ j = J \circ du : T\Sigma \to u^*TX.$$ 

A $J$-holomorphic map from a nodal Riemann surface $(\Sigma, j)$ is a tuple $(\Sigma, j, u)$, where $u : \Sigma \to X$ is a continuous map induced by a $J$-holomorphic map $\tilde{u} : \tilde{\Sigma} \to X$; see Figures 1 and 2. An equivalence between $J$-holomorphic maps $(\Sigma, j, u)$ and $(\Sigma', j', u')$ is an equivalence

$$h : (\Sigma, j) \to (\Sigma', j').$$
between the underlying Riemann surfaces such that \( u = u' \circ h \). We denote by \( \text{Aut}(\Sigma, j, u) \) the group of automorphisms, i.e. self-equivalences, of a J-holomorphic map \((\Sigma, j, u)\). A J-holomorphic map \((\Sigma, j, u)\) is called **stable** if \((\Sigma, j)\) is compact and \( \text{Aut}(\Sigma, j, u) \) is a finite group.

The Riemann surface \((\Sigma, j)\) on the left-hand side of Figure 1 is obtained by identifying the marked points of two copies of a smooth elliptic curve \((\Sigma_0, j_0, z_1^*)\), i.e. a torus with a complex structure and a marked point. The Riemann surface \((\Sigma_0, j_0)\) with the marked point \( z_1^* \) is biholomorphic to \( \mathbb{C}/\Lambda \) with the marked point 0 for some lattice \( \Lambda \subset \mathbb{C} \) and thus has an automorphism of order 2 that preserves \( z_1^* \) (it is induced by the map \( z \mapsto -z \) on \( \mathbb{C} \)). This is the only non-trivial automorphism of \((\Sigma_0, j_0)\) preserving \( z_1^* \) if \( j_0 \) is generic; in special cases, the group of such automorphisms is either \( \mathbb{Z}_4 \) or \( \mathbb{Z}_6 \). Each automorphism of \((\Sigma_0, j_0)\) preserving \( z_1^* \) gives rise to an automorphism of \((\Sigma, j)\) fixing one of the irreducible components. There is also an automorphism of \((\Sigma, j)\) which interchanges the two irreducible components of \( \Sigma \). Since it does not commute with the automorphisms preserving one of the components, \( \text{Aut}(\Sigma, j, u) \approx D_4 \) in most cases and contains \( D_4 \) in the special cases. If \( u: \Sigma \to \Sigma_0 \) is the identity on each irreducible component, \((\Sigma, j, u)\) is a stable J-holomorphic map; the interchange of the two irreducible components is then the only non-trivial automorphism of \((\Sigma, j, u)\). The J-holomorphic maps \( u: \Sigma \to \Sigma_0 \) obtained by sending either or both irreducible components of \( \Sigma \) to \( z_1^* \) instead are also stable, but have different automorphism groups. If \((\Sigma_0, j_0)\) were taken to be the Riemann sphere \( \mathbb{P}^1 \), the J-holomorphic map \( u: \Sigma \to \Sigma_0 \) restricting to the identity on each copy of \( \Sigma_0 \) would still be stable. However, a map \( u: \Sigma \to \Sigma_0 \) sending either copy of \( \Sigma_0 \) to \( z_1^* \) would not be stable, since the group of automorphisms of \( \mathbb{P}^1 \) fixing a point is a complex two-dimensional submanifold of \( \text{PSL}_2 \).

Let \((\Sigma, j)\) be a compact connected Riemann surface of genus \( g \). If \( g \geq 2 \), then \( \text{Aut}(\Sigma, j) \) is a finite group. If \( g = 1 \), then \( \text{Aut}(\Sigma, j) \) is an infinite group, but its subgroup fixing any point is finite. If \( g = 0 \), then the subgroup of \( \text{Aut}(\Sigma, j) \) fixing any pair of points is infinite, but the subgroup fixing any triple of points is trivial. If in addition \((X, J)\) is an almost complex manifold and \( u: \Sigma \to X \) is a non-constant J-holomorphic map, then the subgroup of \( \text{Aut}(\Sigma, j) \) consisting of the automorphisms such that \( u = u \circ h \) is finite; this is an immediate consequence of Corollary \( \text{[52]} \). If \( (\Sigma, j) \) is a compact nodal Riemann surface, a J-holomorphic map \((\Sigma, j, u)\) is thus stable if and only if

- every genus 1 topological component of the normalization \( \tilde{\Sigma} \) of \( \Sigma \) such that \( u \) restricts to a constant map on its image in \( \Sigma \) contains at least 1 element of \( S_\Sigma \) and
- every genus 0 topological component of \( \tilde{\Sigma} \) such that \( u \) restricts to a constant map on its image in \( \Sigma \) contains at least 3 elements of \( S_\Sigma \).
1.2 Gromov's topology

Given a Riemann surface \((\Sigma, j)\), a Riemannian metric \(g\) on a smooth manifold \(X\) determines the energy \(E_g(f)\) for every smooth map \(f : \Sigma \rightarrow X\); see \((2.5)\) and \((2.6)\). The fundamental insight in \([6]\) that laid the foundations for the pseudoholomorphic curves techniques in symplectic topology and for the moduli spaces of stable maps and related curve-parametrizing objects in algebraic geometry is that a sequence of stable \(J\)-holomorphic maps \((\Sigma, j, u_i)\) into a compact almost complex manifold \((X, J)\) with

\[
\liminf_{i \rightarrow \infty} \left( |\pi_0(\Sigma_i)| + a(\Sigma_i) + E_g(u_i) \right) < \infty
\]  

(1.1)

has a subsequence converging in a suitable sense to another stable \(J\)-holomorphic map.

The notion of Gromov's convergence of a sequence of stable \(J\)-holomorphic maps \((\Sigma, j, u_i)\) to another stable \(J\)-holomorphic map \((\Sigma_\infty, j_\infty, u_\infty)\) comes down to

(GC1) \(|\pi_0(\Sigma_i)| = |\pi_0(\Sigma_\infty)|\) and \(a(\Sigma_i) = a(\Sigma_\infty)\) for all \(i\) large,

(GC2) \((\Sigma_\infty, j_\infty)\) is at least as singular as \((\Sigma_i, j_i)\) for all \(i\) large,

(GC3) the energy is preserved, i.e. \(E_g(u_i) \rightarrow E_g(u_\infty)\) as \(i \rightarrow \infty\), and

(GC4) \(u_i\) converges to \(u_\infty\) uniformly in the \(C^\infty\)-topology on compact subsets of \(\Sigma_\infty\).

Most applications of the pseudoholomorphic curves techniques in symplectic topology involve \(J\)-holomorphic maps from the Riemann sphere \(\mathbb{P}^1\). This is a special case of the situation when the complex structures \(j_i\) on the domains \(\Sigma_i\) of \(u_i\) are fixed. The condition \([\text{GC4}]\) can then be formally stated in a way clearly indicative of the rescaling procedure of \([6]\).

**Definition 1.1** (Gromov's Compactness I). Let \((X, J)\) be an almost complex manifold with Riemannian metric \(g\) and \((\Sigma, j)\) be a compact Riemann surface. A sequence \((\Sigma, j, u_i)\) of stable \(J\)-holomorphic maps converges to a stable \(J\)-holomorphic map \((\Sigma_\infty, j_\infty, u_\infty)\) if

1. \((\Sigma_\infty, j_\infty)\) is obtained from \((\Sigma, j)\) by identifying a point on each of \(\ell\) trees of Riemann spheres \(\mathbb{P}^1\), for some \(\ell \in \mathbb{Z}^\geq 0\), with distinct points \(z^*_1, \ldots, z^*_\ell \in \Sigma\),

2. \(E_g(u_\infty) = \lim_{i \rightarrow \infty} E_g(u_i)\),

3. there exist \(h_i \in \text{Aut}(\Sigma, j)\) with \(i \in \mathbb{Z}^+\) such that \(u_i \circ h_i\) converges to \(u_\infty\) uniformly in the \(C^\infty\)-topology on compact subsets of \(\Sigma - \{z^*_1, \ldots, z^*_\ell\}\),

4. for each \(z^*_1, \ldots, z^*_\ell \in \Sigma \subset \Sigma_\infty\) and all \(i \in \mathbb{Z}^+\) sufficiently large, there exist a neighborhood \(U_j \subset \Sigma\) of \(z^*_j\), an open subset \(U_{j,i} \subset \mathbb{C}\), and a biholomorphic map \(\psi_{j;i} : U_{j,i} \rightarrow U_j\) such that
   
   \[ U_i \subset U_{i+1} \text{ and } U = \bigcup_{i=1}^{\ell} U_{j,i} \text{ for every } j = 1, \ldots, \ell, \]
   \[ u_i \circ h_i \circ \psi_{j;i} \text{ converges to } u_\infty \text{ uniformly in the } C^\infty\text{-topology on compact subsets of the complement of the nodes } \infty, w^*_{j,1}, \ldots, w^*_{j,k_j}, \text{ in the sphere } \mathbb{P}^1_j \text{ attached at } z^*_j \in \Sigma, \]
   \[ \text{condition } (4) \text{ applies with } \Sigma, (z^*_1, \ldots, z^*_\ell), \text{ and } u_i \circ h_i \text{ replaced by } \mathbb{P}^1, (w^*_1, \ldots, w^*_{j,k_j}), \text{ and } u_i \circ h_i \circ \psi_{j;i}, \text{ respectively, for each } j = 1, \ldots, \ell. \]
Figure 2: Gromov’s limit of a sequence of $J$-holomorphic maps $u_i: \Sigma \rightarrow X$

An example of a possible limiting map with $\ell = 2$ trees of spheres is shown in Figure 2. The recursive condition (1) in Definition 1.1 is equivalent to the Rescaling axiom in [12, Definition 5.2.1] on sequences of automorphisms $\phi^\lambda_i$ of $\mathbb{P}^1$; they correspond to compositions of the maps $\psi^j$ associated with different irreducible components of $\Sigma_{\infty}$. The single energy condition (2) in Definition 1.1 is replaced in [12, Definition 5.2.1] by multiple conditions of the Energy axiom. These multiple conditions are equivalent to (2) if the other three axioms in [12, Definition 5.2.1] are satisfied.

**Theorem 1.2** (Gromov’s Compactness I). Let $(X, J)$ be a compact almost complex manifold with Riemannian metric $g$, $(\Sigma, j)$ be a compact Riemann surface, and $u_i: \Sigma \rightarrow X$ be a sequence of non-constant $J$-holomorphic maps. If $\lim \inf E_g(u_i) < \infty$, then the sequence $(\Sigma, j, u_i)$ contains a subsequence converging to some stable $J$-holomorphic map $(\Sigma_{\infty}, j_{\infty}, u_{\infty})$ in the sense of Definition 1.1.

This theorem is established in Section 5.3 by assembling together a number of geometric statements obtained earlier in these notes. In Section 5.4, we relate the convergence notion of Definition 1.1 in the case of holomorphic maps from $\mathbb{C}P^1$ to $\mathbb{C}P^n$, which can always be represented by $(n+1)$-tuples of homogeneous polynomials in two variables, to the behavior of the linear factors of the associated polynomials.

The convergence notion of Definition 1.1 can be equivalently reformulated in terms of deformations of the limiting domain $(\Sigma_{\infty}, j_{\infty})$ so that it readily extends to sequences of stable $J$-holomorphic maps with varying complex structures $j_i$ on the domains $\Sigma_i$. This was formally done in the algebraic geometry category by [4], several years after this perspective had been introduced into the field informally, and adapted to the almost complex category by [10]. We summarize this perspective below.

Let $(\Sigma, j)$ be a nodal Riemann surface. A flat family of deformations of $(\Sigma, j)$ is a holomorphic map $\pi: \mathcal{U} \rightarrow \Delta$, where $\mathcal{U}$ is a complex manifold and $\Delta \subset \mathbb{C}^N$ is a neighborhood of 0, such that

- $\pi^{-1}(\lambda)$ is a nodal Riemann surface for each $\lambda \in \mathbb{C}^n$ and $\pi^{-1}(0) = (\Sigma, j)$,
- $\pi$ is a submersion outside of the nodes of the fibers of $\pi$,
- for every $\lambda^* \equiv (\lambda^*_1, \ldots, \lambda^*_N) \in \Delta$ and every node $z^* \in \pi^{-1}(\lambda^*)$, there exist $i \in \{1, \ldots, N\}$ with $\lambda_i = 0$, neighborhoods $\Delta_{\lambda^*}$ of $\lambda^*$ in $\Delta$ and $\mathcal{U}_{z^*}$ of $z^*$ in $\mathcal{U}$, and a holomorphic map
  $$\Psi: \mathcal{U}_{z^*} \rightarrow \left\{(x_1, \ldots, x_N, x, y) \in \Delta_{\lambda^*} \times \mathbb{C}^2 : xy = \lambda_i \right\}$$

such that $\Psi$ is a homeomorphism onto a neighborhood of $(\lambda^*, 0, 0)$ and the composition of $\Psi$ with the projection to $\Delta_{\lambda^*}$ equals $\pi|_{\mathcal{U}_{z^*}}$.  

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Figure 3: A complex-geometric presentation of a flat family of deformations of \((\Sigma_\infty, j_\infty) = \pi^{-1}(0)\) and a differential-geometric presentation of the domains of the maps \(u_i\) in Definition 1.3.

If \(\pi: U \longrightarrow \Delta\) is a flat family of deformations of \((\Sigma, i)\) and \(\Sigma\) is compact, there exists a neighborhood \(U^* \subset U\) of \(\Sigma^* \subset \pi^{-1}(0)\) such that

\[
\pi_{|U^*} : U^* \longrightarrow \Delta_0 \equiv \pi(U^*) \subset \Delta
\]

is a trivializable \(\Sigma^*\)-fiber bundle in the smooth category. For each \(\lambda \in \Delta_0\), let

\[
\psi_\lambda : \Sigma^* \longrightarrow \pi^{-1}(\lambda) \cap U^*
\]

be the corresponding smooth identification. If \(\lambda_i \in \Delta\) is a sequence converging to \(0 \in \Delta\) and \(u_i : \pi^{-1}(\lambda_i) \longrightarrow X\) is a sequence of continuous maps that are smooth on the complements of the nodes of \(\pi^{-1}(\lambda_i)\), we say that the sequence \(u_i\) converges to a smooth map \(u : \Sigma^* \longrightarrow X\) u.c.s. if the sequence of maps

\[
u_i \circ \psi_{\lambda_i} : \Sigma^* \longrightarrow X
\]

converges uniformly in the \(C^\infty\)-topology on compact subsets of \(\Sigma^*\). This notion is independent of the choices of \(U^*\) and trivialization of \(\pi_{|U^*}\).

**Definition 1.3** (Gromov’s Convergence II). Let \((X, J)\) be an almost complex manifold with Riemannian metric \(g\). A sequence \((\Sigma_i, j_i, u_i)\) of stable \(J\)-holomorphic maps converges to a stable \(J\)-holomorphic map \((\Sigma_\infty, j_\infty, u_\infty)\) if \(E_g(u_\infty) \longrightarrow E_g(u_i)\) as \(i \longrightarrow \infty\) and there exist

(a) a flat family of deformations \(\pi: U \longrightarrow \Delta\) of \((\Sigma_\infty, j_\infty)\),

(b) a sequence \(\lambda_i \in \Delta\) converging to \(0 \in \Delta\), and

(c) equivalences \(h_i : \pi^{-1}(\lambda_i) \longrightarrow (\Sigma_i, j_i)\)

such that \(u_i \circ h_i\) converges to \(u_\infty|_{\Sigma_\infty}\) u.c.s.

By the compactness of \(\Sigma_\infty\), the notion of convergence of Definition 1.3 is independent of the choice of metric \(g\) on \(X\). It is illustrated in Figure 3. If the Riemann surfaces \((\Sigma_i, j_i)\) are smooth, the limiting Riemann surface \((\Sigma_\infty, j_\infty)\) is obtained by pinching some disjoint embedded circles in the smooth two-dimensional manifold \(\Sigma\) underlying these Riemann surfaces.
If \((\Sigma_i, i_i) = (\Sigma, j)\) for all \(i\) as in Definition [1.3] only contractible circles are pinched to produce \(\Sigma_\infty\); it then consists of \(\Sigma\) with trees of spheres attached. The family \(\pi: \mathcal{U} \rightarrow \Delta\) is obtained by starting with the family
\[
\pi_0: \mathcal{U}_0 \equiv \mathbb{C} \times \Sigma \rightarrow \mathbb{C},
\]
then blowing up \(\mathcal{U}_0\) at a point of \(\{0\} \times \Sigma\) to obtain a family \(\pi_1: \mathcal{U}_1 \rightarrow \mathbb{C}\) with the central fiber \(\Sigma_1 \equiv \pi_1^{-1}(0)\) consisting of \(\Sigma\) with \(\mathbb{P}^1\) attached, then blowing up a smooth point of \(\Sigma_1\), and so on. The number of blowups involved is precisely the number of nodes of \(\Sigma_\infty\), i.e. four in the case of Figure 2 and two in the case of Figure 3. The pinched annuli on the right-hand side of Figure 3 correspond to \(\phi_\alpha(B_\delta(z_\alpha\beta)) \cup \phi_\beta(B_\delta(z_\beta\alpha))\) in the notation of [12, Chapters 4,5].

With the setup of Definition [1.3] let \(B_\delta(z^*) \subset \mathcal{U}\) denote the ball of radius \(\delta \in \mathbb{R}^+\) around a point \(z^* \in \mathcal{U}\) with respect to some metric on \(\mathcal{U}\). Then,
\[
\lim_{\delta \rightarrow 0} \lim_{i \rightarrow \infty} \text{diam}_g(u_i(h_i(\pi^{-1}(\lambda_i) \cap B_\delta(z^*)))) = 0 \quad \forall \ z^* \in \Sigma_\infty. \tag{1.2}
\]
This is immediate from the last condition in Definition [1.3] if \(z^* \in \Sigma^{*}_\infty\). If \(z^* \in \Sigma^{*}_\infty - \Sigma^{*}_\infty\) is a node of \(\Sigma_\infty\), (1.2) is a consequence of both convergence conditions of Definition [1.3] and the maps \(u_i\) being \(J\)-holomorphic. It is a reflection of the fact that bubbling or any other kind of erratic \(C^0\)-behavior of a sequence of \(J\)-holomorphic maps requires a nonzero amount of energy in the limit, but the two convergence conditions of Definition [1.3] ensure that all limiting energy is absorbed by \(u(\Sigma^{*}_\infty\) and thus none is left for bubbling around the nodes of \(\Sigma_\infty\). An immediate implication of (1.2) is that \(u_i(h_i(\pi^{-1}(\lambda_i) \cap B_\delta(z^*)))\) is contained in a geodesic ball around \(u_\infty(z^*)\) in \(X\). Thus,
\[
u_{i*}[\Sigma_i] = u_{\infty*}[\Sigma_\infty] \in H_2(X; \mathbb{Z})
\]
for all \(i \in \mathbb{Z}^+\) sufficiently large. If \(\Sigma_\infty\) is a tree of spheres (and thus so is each \(\Sigma_i\)), then \(u_i\) with \(i\) sufficiently large lies in the equivalence class in \(\pi_2(X)\) determined by \(u_\infty\) for the same reason.

**Theorem 1.4** (Gromov’s Compactness II). Let \((X, J)\) be a compact almost complex manifold with Riemannian metric \(g\) and \((\Sigma_i, j_i, u_i)\) be a sequence of \(\mathcal{S}\)-stable \(J\)-holomorphic maps into a compact almost complex manifold \((X, J)\). If it satisfies (1.7), then it contains a subsequence converging to some stable \(J\)-holomorphic map \((\Sigma_\infty, j_\infty, u_\infty)\) in the sense of Definition [1.3].

This theorem is obtained by combining the compactness of the Deligne-Mumford moduli spaces \(\mathcal{M}_{1,1}\) of stable (possibly) nodal elliptic curves and \(\mathcal{M}_g\) of stable nodal genus \(g \geq 2\) curves with the proof of Theorem [1.2] in Section [5.3]. One first establishes Theorem [1.4] under the assumption that each \((\Sigma_i, j_i)\) is a smooth connected Riemann surface of genus \(g \geq 1\) (the \(g = 0\) case is treated by Theorem [1.2]). If \(g = 1\), we add a marked point to each domain \((\Sigma_i, j_i)\) and take a subsequence converging in \(\mathcal{M}_{1,1}\) to the equivalence class of some stable nodal elliptic curve \((\Sigma'_\infty, j'_\infty, z'_\infty)\). If \(g \geq 2\), we take a subsequence of \((\Sigma_i, j_i)\) converging in \(\mathcal{M}_g\) to the equivalence class of some stable nodal genus \(g\) curve \((\Sigma'_\infty, j'_\infty)\). This ensures the existence of a flat family of deformations \(\pi': \mathcal{U}' \rightarrow \Delta'\) of \((\Sigma'_\infty, j'_\infty)\), of a sequence \(\lambda' \in \Delta'\) converging to 0 in \(\Delta'\), and of equivalences \(h_i: \pi'^{-1}(\lambda') \rightarrow (\Sigma_i, j_i)\). The associated neighborhood \(\mathcal{U}^{**}\) of \(\Sigma_\infty^{**}\) in \(\mathcal{U}'\) can be chosen so that \(\pi'^{-1}(\lambda) - \mathcal{U}^{**}\) consists of finitely many circles for every \(\lambda' \in \Delta'\) sufficiently small. The complement of the image of the associated identifications
\[
\psi'_{\lambda}: \Sigma_\infty^{**} \rightarrow \pi'^{-1}(\lambda) \cap \mathcal{U}^{**}
\]
in \(\pi'^{-1}(\lambda)\) has the same property.
One then applies the construction in the proof of Theorem 1.2 to the sequence of $J$-holomorphic maps
\[ u_i \circ h'_i : \Sigma'^{\infty}_\infty \to X \]
to obtain a $J$-holomorphic map $\tilde{u}'_{\infty}$ from the normalization $\tilde{\Sigma}'_{\infty}$ of $\Sigma_{\infty}$ and finitely $J$-holomorphic maps from trees of $P^1$. Each of these trees will have one or two special points that are associated with points of $\tilde{\Sigma}'_{\infty}$ (the latter happens if bubbling occurs at a preimage of a node of $\Sigma'^{\infty}_{\infty}$ in $\tilde{\Sigma}'_{\infty}$). Identifying these trees with the corresponding points of $\tilde{\Sigma}'_{\infty}$ as in the proof of Theorem 1.2, we obtain a $J$-holomorphic map $(\Sigma_{\infty}, j_{\infty}, u_{\infty})$ satisfying the requirements of Definition 1.3. It is necessarily stable if $g \geq 2$, or $\Sigma'_{\infty}$ is smooth, or $\Sigma_{\infty}$ contains a separating node. Otherwise, the identifications $h'_i$ may first need to be reparametrized to ensure that either the limiting map $\tilde{u}'_{\infty}$ is not constant or the sequence $u_i \circ h_i$ produces a bubble at least one smooth point of $\tilde{\Sigma}'_{\infty}$.

A $k$-marked Riemann surface is a tuple $(\Sigma, j, z_1, \ldots, z_k)$ such that $(\Sigma, j)$ is a Riemann surface and $z_1, \ldots, z_k \in \Sigma^*$ are distinct points. If $(X, J)$ is an almost complex manifold, a $k$-marked $J$-holomorphic map into $X$ is a tuple $(\Sigma, j, z_1, \ldots, z_k, u)$, where $(\Sigma, j, z_1, \ldots, z_k)$ is $k$-marked Riemann surface and $(\Sigma, j, u)$ is a $J$-holomorphic map into $X$. The degree of such a map is the homology class
\[ A = u_*[\Sigma] \in H_2(X; \mathbb{Z}). \]
The notions of equivalence, stability, and convergence as in Definition 1.3 and the above convergence argument for smooth domains $(\Sigma_i, j_i)$ readily extend to $k$-marked $J$-holomorphic maps. The general case of Theorem 1.4, including its extension to stable marked maps, is then obtained by

- passing to a subsequence of $(\Sigma_i, j_i, u_i)$ with the same topological structure of the domain,
- viewing it as a sequence of tuples of $J$-holomorphic maps with smooth domains with an additional marked point for each preimage of the nodes in the normalization, and
- applying the conclusion of the above argument to each component of the tuple.

1.3 Moduli spaces

The natural extension of Definition 1.3 to marked $J$-holomorphic maps topologizes the moduli space $\overline{M}_{g,k}(X, A; J)$ of equivalence classes of stable degree $A$ $k$-marked genus $g$ $J$-holomorphic maps into $X$ for each $A \in H_2(X; \mathbb{Z})$. The evaluation maps
\[ \text{ev}_i : \overline{M}_{g,k}(X, A; J) \to X, \quad (\Sigma, j, z_1, \ldots, z_k, u) \mapsto u(z_i), \]
are continuous with respect to this topology. If $2g + k \geq 3$, there is a continuous map
\[ f : \overline{M}_{g,k}(X, A; J) \to \overline{M}_{g,k} \]
to the Deligne-Mumford moduli space of stable $k$-marked genus $g$ nodal curves obtained by forgetting the map $u$ and then contracting the unstable components of the domain.

There is a continuous map
\[ f_{k+1} : \overline{M}_{g,k+1}(X, A; J) \to \overline{M}_{g,k}(X, A; J) \]
(1.3)
obtained by forgetting the last marked point \( z_{k+1} \) and then contracting the components of the domain to stabilize the resulting \( k \)-marked \( J \)-holomorphic map. For each \( i = 1, \ldots, k \), this fibration has a natural continuous section

\[
s_i : \overline{M}_{g,k}(X, A; J) \to \overline{M}_{g,k+1}(X, A; J)
\]

described as follows. For a \( k \)-marked nodal Riemann surface \((\Sigma, j, z_1, \ldots, z_k)\), let \((\Sigma', j', z_1, \ldots, z_{k+1})\) be the \((k+1)\)-marked nodal Riemann surface so that \((\Sigma', j')\) consists of \((\Sigma, j)\) with \( P_1 \) attached at \( z_i, z'_1, z'_i \in P_1 \), and \( z'_j = z_j \in \Sigma \) for all \( j = 1, \ldots, k \) different from \( k \); see Figure 4. We define

\[
s_i([\Sigma, j, z_1, \ldots, z_k, u]) = [\Sigma', j', z'_1, \ldots, z'_{k+1}, u'],
\]

with \((\Sigma', j', z'_1, \ldots, z'_{k+1})\) as described and \( u' \) extending \( u \) over the extra \( P_1 \) by the constant map with value \( u(z_i) \). The pullback

\[
L_i \to \overline{M}_{g,k}(X, A; J)
\]

of the vertical tangent line bundle of (1.3) by \( s_i \) is called the universal tangent line bundle at the \( i \)-th marked point. Let \( \psi_i = c_1(L_i^*) \) be the \( i \)-th descendant class.

A remarkable property of Gromov’s topology which lies behind most of its applications is that the moduli space \( \overline{M}_{g,k}(X, A; J) \) is Hausdorff and has a particularly nice deformation-obstruction theory. In the algebraic-geometry category, the latter is known as a perfect two-term deformation-obstruction theory. In the almost complex category, this is reflected in the existence of an atlas of finite-dimensional approximations in the terminology of [10] or of an atlas of Kuranishi charts in the terminology of [10].

If \((X, J)\) is an almost complex manifold and \( J \) is tamed by a symplectic form \( \omega \), then the energy \( E_g(u) \) of degree \( A \) \( J \)-holomorphic map \( u \) with respect to the metric \( g \) determined by \( J \) and \( \omega \) is \( \omega(A) \); see (2.7). In particular, it is the same for all elements of the moduli space \( \overline{M}_{g,k}(X, A; J) \). If in addition \( X \) is compact, then Theorem 1.4 implies that this moduli space is also compact. Combining this with the remarkable property of the previous paragraph, the constructions of [1] [2] [10] [3] endow \( \overline{M}_{g,k}(X, A; J) \) with a virtual fundamental class. It depends only on \( \omega \), in a suitable sense, and not an almost complex structure \( J \) tamed by \( \omega \). This class in turn gives rise to Gromov-Witten invariants of \((X, \omega)\):

\[
\langle \tau_{\alpha_1}^{a_1} \cdots \tau_{\alpha_k}^{a_k} \rangle^X_{g,A} \equiv \langle (\psi_1^{a_1} c_1^{a_1} \alpha_1) \cdots (\psi_k^{a_k} c_1^{a_k} \alpha_k), [\overline{M}_{g,k}(X, A; J)]^{vir} \rangle \in \mathbb{Q}
\]

for all \( a_i \in \mathbb{Z}^\geq 0 \) and \( \alpha_i \in H^*(X; \mathbb{Q}) \).
2 Preliminaries

An outline of these notes with an informal description of the key statements appears in Section 2.1; Figure 5 indicates primary connections between these statements. Sections 2.2 introduces the most frequently used notation and terminology and makes some basic observations.

2.1 Overview of the main statements

The main technical statement of Section 3 of these notes and of Chapter 2 in [12] is the Carleman Similarity Principle; see Proposition 3.1. It yields a number of geometric conclusions about the local behavior of a $J$-holomorphic map $u: \Sigma \to X$ from a Riemann surface $(\Sigma, j)$ into an almost complex manifold $(X, J)$. For example, for every $z \in \Sigma$ contained in a component of $\Sigma$ on which $u$ is not constant, the $\ell$-th derivative of $u$ at $z$ in a chart around $u(z)$ does not vanish for some $\ell \in \mathbb{Z}^+$; see Corollary 3.3. We denote by $\text{ord}_z u \in \mathbb{Z}^+$ the minimum of such integers $\ell$ and call it the order of $u$ at $z$. If $u$ is constant on the component of $\Sigma$ on containing $z$, we set $\text{ord}_z u = 0$. A point $z \in u$ is singular, i.e. $d_z u = 0$, if and only if $\text{ord}_z u \neq 1$.

If $u$ is not constant on every connected component of $\Sigma$, the singular points of $u$ and the preimages of a point $x \in X$ are discrete subsets of $\Sigma$; see Corollary 3.4. In the case $\Sigma$ is compact, the second statement of Corollary 3.4 implies that

$$\text{ord}_x u \equiv \sum_{z \in u^{-1}(x)} \text{ord}_z u \in \mathbb{Z}^+ \quad \forall x \in X; \quad (2.1)$$

we call this number the order of $u$ at $x$. If $x \notin \text{Im}(u)$, then $\text{ord}_x u = 0$. By Corollary 3.11, the number (2.1) is seen by the behavior of the energy (2.5) of $u$ and its restrictions to open subsets of $\Sigma$. This observation underpins the Monotonicity Lemma for $J$-holomorphic maps, which bounds below the energy required to “escape” from a small ball in $X$; see Proposition 3.12.

The main technical statement of Section 4 of these notes and of Chapter 4 in [12] is the Mean Value Inequality. It bounds the pointwise differentials $d_z u$ of a $J$-holomorphic map $u$ from $(\Sigma, j)$ into $(X, J)$ of sufficiently small energy $E_2(u)$ by $E_2(u)$, i.e. by the $L^2$-norm of $du$, from above and immediately yields a bound on the energy of non-constant $J$-holomorphic maps from $S^2$ into $(X, J)$ from below; see Proposition 4.1 and Corollary 4.2 respectively. The Mean Value Inequality also implies that the energy of a $J$-holomorphic map $u$ from a cylinder $[-R, R] \times S^1$ carried by $[-R+T, R-T] \times S^1$ and the diameter of the image of this middle segment decay at least exponentially with $T$, provided the overall energy of $u$ is sufficiently small. As shown in the proof of Proposition 5.5 this technical implication ensures that the energy is preserved under Gromov’s convergence and the resulting bubbles connect.

Another important implication of Proposition 4.1 is that a continuous map from a Riemann surface $(\Sigma, j)$ into an almost complex manifold $(X, J)$ which is holomorphic outside of a discrete collection of points and has bounded energy is in fact holomorphic on all of $\Sigma$; see Proposition 4.8. This conclusion plays a central role in the proof of Lemma 5.4. Theorem 1.2 is deduced from Lemma 5.4 and Proposition 5.5 in Section 5.3.

Combined with Proposition 3.1 and some of its corollaries, Proposition 4.1 implies that every non-constant $J$-holomorphic map from a compact Riemann surface $(\Sigma, j)$ factors through a somewhere
injective \( J \)-holomorphic map from a compact Riemann surface \((\Sigma', j')\); see Proposition 4.11. The proof of this statement with \( X \) compact appears in Chapter 2 of [12], but uses the Removal Singularities Theorem proved in Chapter 4 of [12].

### 2.2 Notation and terminology

Let \((\Sigma, j)\) be a Riemann surface, \(V\) be a vector bundle over \(\Sigma\), and

\[
\mu, \eta \in \Gamma(\Sigma; T^*\Sigma \otimes_R V) \quad \text{and} \quad g \in \Gamma(\Sigma; T^*\Sigma \otimes^2 \otimes_R V).
\]

For a local coordinate \(z = s + it\), define

\[
g(\mu \otimes \eta) = (g(\mu(\partial_s), \eta(\partial_s)) + g(\mu(\partial_t), \eta(\partial_t))) \, ds \wedge dt, \\
g(\mu \wedge \eta) = (g(\mu(\partial_s), \eta(\partial_t)) - g(\mu(\partial_t), \eta(\partial_s))) \, ds \wedge dt. \tag{2.2}
\]

By a direct computation, the 2-forms \(g(\mu \otimes \eta)\) and \(g(\mu \wedge \eta)\) are independent of the choice of local coordinate \(z = s + it\). Thus, \eqref{2.2} determines global 2-forms on \(\Sigma\) (which depend on the choice of \(j\)).

We denote by \(i\) the standard complex structure on \(\mathbb{C}\) and by \(J_{\mathbb{C}^n}\) the standard complex structures on \(\mathbb{C}^n\) and \(T\mathbb{C}^n\). For an almost complex structure \(J\) and a 2-form \(\omega\) on a manifold \(X\), we define a 2-tensor and a 2-form on \(X\) by

\[
g_J(v, v') = \frac{1}{2}(\omega(v, Jv') - \omega(Jv, v')) \quad \forall v, v' \in T_x X, \ x \in X. \tag{2.3}
\]

\[
\omega_J(v, v') = \frac{1}{2}(\omega(Jv, Jv') - \omega(v, v'))
\]

We note that

\[
g_J(v, v) + g_J(v', v') = 2\omega(v, v') + g_J(v + Jv', v + Jv') + 2\omega_J(v, v') \quad \forall v, v' \in T_x X, \ x \in X. \tag{2.4}
\]
The 2-form $\omega$ tames $J$ if $g(v,v) > 0$ for all $v \in TX$ nonzero; in such a case, $\omega$ is nondegenerate and $g_J$ is a metric. The almost complex structure $J$ is $\omega$-compatible if $\omega$ tames $J$ and $\omega_J = 0$.

Let $X$ be a manifold, $(\Sigma, j)$ be a Riemann surface, and $f : \Sigma \to X$ be a smooth map. We denote the pullbacks of a 2-tensor $g$ and a 2-form $\omega$ on $X$ to the vector bundle $f^*TX$ over $\Sigma$ also by $g$ and $\omega$. If $g$ is a Riemannian metric on $X$ and $U \subseteq \Sigma$ is an open subset, let

$$E_g(f) \equiv \frac{1}{2} \int_\Sigma g(df \otimes df) \in [0, \infty] \quad \text{and} \quad E_g(f; U) \equiv E_g(f|_U)$$

be the energy of $f$ and of its restriction to $U$. By the first equation in (2.2),

$$E_g(f) = \frac{1}{2} \int_\Sigma |df|^2_{g; g}$$

is the square of the $L^2$-norm of $df$ with respect to the metric $g$ on $X$ and a metric $g_\Sigma$ compatible with $j$. In particular, the right-hand side of (2.6) depends on the metric $g$ on $X$ and on the complex structure $j$ on $\Sigma$, but not the metric $g_\Sigma$ on $\Sigma$ compatible with $j$.

Let $J$ be an almost complex structure on a manifold $X$ and $(\Sigma, j)$ be a Riemann surface. For a smooth map $u : \Sigma \to X$, define

$$\bar{\partial}_J u = \frac{1}{2} (df + J \circ df \circ j) \in \Gamma(\Sigma; (T^*\Sigma, j)^{0,1} \otimes_\mathbb{C} f^*(TX, J)).$$

If $\omega$ is a 2-form on $X$ taming $J$ and $u : \Sigma \to X$ is $J$-holomorphic, then

$$E_{g_J}(f) = \int_\Sigma (f^*\omega + 2g_J(\bar{\partial}_J f \otimes_\mathbb{C} \bar{\partial}_J f) + f^*\omega_J)$$

by (2.5) and (2.4). If $J$ is $\omega$-compatible, the last term above vanishes. A smooth map $u : \Sigma \to X$ is $J$-holomorphic if $\bar{\partial}_J u = 0$. For such a map, the last two terms in (2.7) vanish.

For each $R \in \mathbb{R}^+$, denote by $B_R \subset \mathbb{C}$ the open ball of radius $R$ around the origin and let

$$B^*_R = B_R - \{0\}.$$

If in addition $(X, g)$ is a Riemannian manifold and $x \in X$, let $B^*_\delta(x) \subset X$ be the ball of radius $\delta$ around $x$ in $X$ with respect to the metric $g$.

Let $(X, J)$ be an almost complex manifold and $(\Sigma, j)$ be a Riemann surface. A smooth map $u : \Sigma \to X$ is called

- somewhere injective if there exists $z \in \Sigma$ such that $u^{-1}(u(z)) = \{z\}$ and $d_z u \neq 0$,
- multiply covered if $u = u' \circ h$ for some smooth connected orientable surface $\Sigma'$, branched cover $h : \Sigma \to \Sigma'$ of degree different from $\pm 1$, and a smooth map $u' : \Sigma' \to X$,
- simple if it is not multiply covered.

By Proposition 4.11, every $J$-holomorphic map from a compact Riemann surface is simple if and only if it is somewhere injective (the if implication is trivial).
3 Local Properties

We begin by studying local properties of $J$-holomorphic maps $u$ from Riemann surfaces $(\Sigma, j)$ into almost complex manifolds $(X, J)$ that resemble standard properties of holomorphic maps. None of the statements in Section 3 depend on $X$ being compact; very few depend on $\Sigma$ being compact.

3.1 Carleman Similarity Principle

Carleman Similarity Principle, i.e. Proposition 3.1 below, is a local description of solutions of a nonlinear differential equation which generalizes the equation $\bar{\partial}_J u = 0$. It states that such solutions look similar to holomorphic maps and implies that they exhibit many local properties one would expect of holomorphic maps.

**Proposition 3.1** (Carleman Similarity Principle, [2, Theorem 2.2]). Suppose $n \in \mathbb{Z}^+$, $p, \epsilon \in \mathbb{R}^+$ with $p > 2$, $J \in L^1_1(B^c; \text{End}_{\mathbb{R}} \mathbb{C}^n)$, $C \in L^p(B^c; \text{End}_{\mathbb{R}} \mathbb{C}^n)$, and $u \in L^p(B^c; \mathbb{C}^n)$ are such that

\[ u(0) = 0, \quad J(z)^2 = -\text{Id}_{\mathbb{C}^n}, \quad u_s(z) + J(z)u_t(z) + C(z)u(z) = 0 \quad \forall \quad z = s + it \in B^c. \tag{3.1} \]

Then, there exist $\delta \in (0, \epsilon)$, $\Phi \in L^1_1(B^\delta; \text{GL}_{2n} \mathbb{R})$, and a $J_{\mathbb{C}^n}$-holomorphic map $\sigma: B^\delta \rightarrow \mathbb{C}^n$ such that

\[ \sigma(0) = 0, \quad J(z)\Phi(z) = \Phi(z)J_{\mathbb{C}^n}, \quad u(z) = \Phi(z)\sigma(z) \quad \forall \quad z \in B^\delta. \tag{3.2} \]

By the Sobolev Embedding Theorem [18, Corollary 4.3], the assumption $p > 2$ implies that $u$ is a continuous function. In particular, all equations in (3.1) and in (3.2) make sense. This assumption also implies that the left-hand sides of the third equation in (3.1) and of the second equation in (3.2) and the right-hand side of the third equations in (3.2) lie in $L^p_1$.

**Example 3.2.** Let $\mathfrak{c}: \mathbb{C} \rightarrow \mathbb{C}$ denote the usual conjugation. Define

\[ \tilde{J}(z_1, z_2) = \begin{pmatrix} i & 0 \\ -2is_1 \mathfrak{c} & i \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ s_1 \mathfrak{c} & 1 \end{pmatrix} J_{\mathbb{C}^2} \begin{pmatrix} 1 & 0 \\ s_1 \mathfrak{c} & 1 \end{pmatrix}^{-1} : \mathbb{C}^2 \rightarrow \mathbb{C}^2 \quad \forall \quad z_i = s_i + it_i, \]

\[ u: \mathbb{C} \rightarrow \mathbb{C}^2, \quad u(s + it) = (z, s^2). \]

Thus, $\tilde{J}$ is an almost complex structure on $\mathbb{C}^2$ and $u$ is a $\tilde{J}$-holomorphic map, i.e. it satisfies the last condition in (3.1) with $J(z) = \tilde{J}(u(z))$ and $C(z) = 0$. The functions

\[ \sigma: \mathbb{C} \rightarrow \mathbb{C}^2, \quad \sigma(z) = (z, 0), \quad \Phi: \mathbb{C} \rightarrow \text{GL}_{4\mathbb{R}}, \quad \Phi(s + it) = \begin{pmatrix} 1 & \frac{1}{s \mathfrak{c} + i \mathfrak{s}} & 0 \\ 0 & 1 \end{pmatrix}, \]

satisfy (3.2).

**Corollary 3.3.** Let $n, p, \epsilon, J, C$, and $u$ be as in Proposition 3.1. If in addition $J_0 = J_{\mathbb{C}^n}$ and $u$ does not vanish to infinite order 0, then there exist $\ell \in \mathbb{Z}^+$ and $\alpha \in \mathbb{C}^n - 0$ such that

\[ \lim_{z \rightarrow 0} \frac{u(z) - \alpha z^\ell}{z^t} = 0. \]

**Proof.** This follows from (3.2) and from the existence of such $\ell$ and $\alpha$ for $\sigma$. \qed
Corollary 3.4. Suppose \((X, J)\) is an almost complex manifold, \((\Sigma, j)\) is a Riemann surface, and \(u: \Sigma \rightarrow X\) is a \(J\)-holomorphic map. If \(u\) is not constant on every connected component of \(\Sigma\), then the subset
\[ u^{-1}\left(\{(u(z): \ z \in \Sigma, \ \operatorname{d}z u = 0\}\}\right) \subset \Sigma \]
is discrete. If in addition \(x \in X\), the subset \(u^{-1}(x) \subset \Sigma\) is also discrete.

Proof. The first and third equations in (3.2) immediately imply the second claim (but not the first, since \(\Phi\) may not be in \(C^1\)). The first claim follows from Corollary 3.3 and Taylor’s formula for \(u\) (as well as from Corollary 3.6).

Before establishing the full statement of Proposition 3.1, we consider a special case.

Lemma 3.5. Suppose \(n \in \mathbb{Z}^+\) and \(p, \epsilon \in \mathbb{R}^+\) are as in Proposition 3.3. \(A \in L^p(B; \operatorname{End}_C^n)\), and \(u \in L^1_n(B; \mathbb{C}^n)\) are such that
\[ u(0) = 0, \quad u_s + J_{\mathbb{C}^n} u_t(z) + A(z)u(z) = 0 \quad \forall \ z = s + it \in B_\epsilon. \tag{3.3} \]
Then, there exist \(\delta \in (0, \epsilon)\), \(\Phi \in L^p_1(B; \text{GL}_n \mathbb{C})\), a \(J_{\mathbb{C}^n}\)-holomorphic map \(\sigma: B_\delta \rightarrow \mathbb{C}^n\) such that
\[ \sigma(0) = 0, \quad \Phi(0) = \text{Id}_{\mathbb{C}^n}, \quad u(z) = \Phi(z)\sigma(z) \quad \forall \ z \in B_\delta. \tag{3.4} \]

Proof. For each \(\delta \in [0, \epsilon]\), we define
\[ A_\delta \in L^p(S^2; \operatorname{End}_C^n) \quad \text{by} \quad A_\delta(z) = \begin{cases} A(z), & \text{if } z \in B_\delta; \\ 0, & \text{otherwise}; \end{cases} \]
\[ D_\delta: L^p(S^2; \operatorname{End}_C^n) \rightarrow L^p(S^2; (T^*S^2)^{0,1} \otimes \operatorname{End}_C^n) \quad \text{by} \quad D_\delta \Theta = (\Theta_s + J_{\mathbb{C}^n} \Theta_t + A_\delta \Theta) \, d\bar{z}. \]
Since the cokernel of \(D_0 = 2\bar{\partial}\) is isomorphic \(H^1(S^2; \mathbb{C}) \otimes \operatorname{End}_C^n\), \(D_0\) is surjective and the homomorphism
\[ \tilde{D}_0: L^p_1(S^2; \operatorname{End}_C^n) \rightarrow L^p(S^2; (T^*S^2)^{0,1} \otimes \operatorname{End}_C^n) \oplus \operatorname{End}_C^n, \quad \Theta \rightarrow (D_0 \Theta, \Theta(0)), \]
is an isomorphism. Since
\[ \|D_\delta \Theta - D_0 \Theta\|_{L^p} \leq \|A_\delta\|_{L^p} \|\Theta\|_{C^0} \leq C \|A_\delta\|_{L^p} \|\Theta\|_{L^p} \quad \forall \ \Theta \in L^p_1(S^2; \operatorname{End}_C^n) \]
and \(\|A_\delta\|_{L^p} \rightarrow 0\) as \(\delta \rightarrow 0\), the homomorphism
\[ \tilde{D}_\delta: L^p_1(S^2; \operatorname{End}_C^n) \rightarrow L^p(S^2; (T^*S^2)^{0,1} \otimes \operatorname{End}_C^n) \oplus \operatorname{End}_C^n, \quad \Theta \rightarrow (D_\delta \Theta, \Theta(0)), \]
is also an isomorphism for \(\delta > 0\) sufficient small. Let \(\Theta_\delta = D_\delta^{-1}(0, \text{Id}_{\mathbb{C}^n})\). Since \(D_\delta\) is an isomorphism,
\[ \|\Theta_\delta - \text{Id}_{\mathbb{C}^n}\|_{C^0} \leq C \|\Theta_\delta - \text{Id}_{\mathbb{C}^n}\|_{L^1} \leq C' \|D_\delta(\Theta_\delta - \text{Id}_{\mathbb{C}^n})\|_{L^p} = C' \|A_\delta\|_{L^p}. \]
Since \(\|A_\delta\|_{L^p} \rightarrow 0\) as \(\delta \rightarrow 0\), \(\Theta_\delta \in L^p_1(B; \text{GL}_n \mathbb{C})\). By (3.3) and \(D_\delta \Theta_\delta = 0\), the function \(\sigma = \Theta_\delta^{-1} u\) satisfies
\[ \sigma(0) = 0, \quad \sigma_s + J_{\mathbb{C}^n} \sigma_t = 0 \quad \forall \ z \in B_\delta, \]
i.e. \(\sigma\) is \(J_{\mathbb{C}^n}\)-holomorphic, as required. \(\square\)
**Proof of Proposition 3.1.** (1) Since \( B_{\epsilon} \) is contractible, the complex vector bundles \( u^*(T\mathbb{C}^n, J_{\mathbb{C}^n}) \) and \( u^*(T\mathbb{C}^n, J) \) over \( B_{\epsilon} \) are isomorphic. Thus, there exists
\[
\Psi \in L^1(B_{\epsilon}; \text{GL}_{2n}\mathbb{R}) \quad \text{s.t.} \quad J(z)\Psi(z) = \Psi(z)J_{\mathbb{C}^n} \quad \forall \, z \in B_{\epsilon}.
\]
Let \( v = \Psi^{-1}u \). By the assumptions on \( u \), \( v \in L^1(B_{\epsilon}; \mathbb{C}^n) \) and
\[
v(0) = 0, \quad v_s(z) + J_{\mathbb{C}^n}v_t(z) + C(z)v(z) = 0 \quad \forall \, z = s + it \in B_{\epsilon}, \tag{3.5}
\]
where \( C = \Psi^{-1} \cdot (\Psi_s + J\Psi_t + C\Psi) \in L^p(B_{\epsilon}; \text{End}_{\mathbb{R}}\mathbb{C}^n) \).

Thus, we have reduced the problem to the case \( J = J_{\mathbb{C}^n} \).

(2) Let \( \tilde{C} = \frac{1}{2}(\tilde{C} \mp J_{\mathbb{C}^n} C) \) be the \( \mathbb{C} \)-linear and \( \mathbb{C} \)-antilinear parts of \( \tilde{C} \), i.e. \( \tilde{C}^\pm J_{\mathbb{C}^n} = \pm J_{\mathbb{C}^n} \tilde{C}^\pm \).

With \( \langle \cdot, \cdot \rangle \) denoting the Hermitian inner-product on \( \mathbb{C}^n \) which is \( \mathbb{C} \)-antilinear in the second input, define
\[
D \in L^\infty(B_{\epsilon}; \text{End}_{\mathbb{R}}\mathbb{C}^n), \quad D(z)w = \begin{cases} |v(z)|^{-2}\langle v(z), w \rangle v(z), & \text{if } v(z) \neq 0; \\ 0, & \text{otherwise}; \end{cases} \quad A = \tilde{C}^+ + \tilde{C}^- D.
\]

Since \( DJ_{\mathbb{C}^n} = -J_{\mathbb{C}^n}D \) and \( Dv = v \), \( A \in L^p(B_{\epsilon}; \text{End}_{\mathbb{C}}\mathbb{C}^n) \) and \( Av = \tilde{C}v \). Thus, by (3.5),
\[
v_s + J_{\mathbb{C}^n}v_t + Av = 0.
\]

The claim now follows from Lemma 3.5. \( \square \)

**Corollary 3.6.** Suppose \( n \in \mathbb{Z}^+ \), \( \epsilon \in \mathbb{R}^+ \), \( J \) is a smooth almost complex structure on \( \mathbb{C}^n \) with \( J_0 = J_{\mathbb{C}^n} \), and \( u : B_{\epsilon} \to \mathbb{C}^n \) is a \( J \)-holomorphic map with \( u(0) = 0 \). Then, there exist \( \delta \in (0, \epsilon) \), \( C \in \mathbb{R}^+ \), \( \Phi \in C^0(B_{\delta}; \text{GL}_{2n}\mathbb{R}) \), and a \( \mathbb{C}^n \)-holomorphic map \( \sigma : B_{\delta} \to \mathbb{C}^n \) such that \( \Phi \) is smooth on \( B_{\delta}^* \),
\[
\sigma(0) = 0, \quad \Phi(0) = \text{Id}_{\mathbb{C}^n}, \quad J(u(z))\Phi(z) = \Phi(z)J_{\mathbb{C}^n}, \quad u(z) = \Phi(z)\sigma(z), \quad |d_z\Phi| \leq C \forall \, z \in B_{\delta}^*.
\]

**Proof.** We can assume that \( u \) is not identically 0 on some neighborhood of \( 0 \in B_{\epsilon} \). Similarly to (1) in the proof of Proposition 3.1, there exists
\[
\Psi \in C^\infty(\mathbb{C}^n; \text{GL}_{2n}\mathbb{R}) \quad \text{s.t.} \quad \Psi(0) = \text{Id}_{\mathbb{C}^n}, \quad J(x)\Psi(x) = \Psi(x)J_{\mathbb{C}^n} \quad \forall \, x \in \mathbb{C}^n.
\]

Let \( v(z) = (u(z))^{-1}u(z) \). By Corollary 3.3, we can choose complex linear coordinates on \( \mathbb{C}^n \) so that
\[
v(z) = (f(z), g(z))h(z) \in \mathbb{C} \oplus \mathbb{C}^{n-1} \quad \forall \, z \in B_{\epsilon}.
\]
for some \( \epsilon' \in (0, \epsilon) \), holomorphic function \( h \) on \( B_{\epsilon'} \) with \( h(0) = 0 \), and continuous functions \( f \) and \( g \) on \( B_{\epsilon'} \) with \( f(0) = 1 \) and \( g(0) = 0 \). By Lemma 3.7 below applied with \( f \) above and with each component of \( g \) separately, there exists \( \delta \in (0, \epsilon') \) so that the function
\[
\Phi : B_{\delta} \to \text{GL}_{2n}\mathbb{R}, \quad \Phi(z) = \Psi(u(z)) \begin{pmatrix} f(z) & 0 \\ g(z) & 1 \end{pmatrix},
\]
is continuous on \( B_{\delta} \) and smooth on \( B_{\delta} - 0 \) with \( |d_z\Phi| \) uniformly bounded on \( B_{\delta} - 0 \). Taking \( \sigma(z) = (h(z), 0) \), we conclude the proof. \( \square \)
Lemma 3.7. Suppose $\epsilon \in \mathbb{R}^+$, and $f, h: B_\epsilon \rightarrow \mathbb{C}$ are continuous functions such that $h$ is holomorphic, $h(z) \neq 0$ for some $z \in B_\epsilon$, and the function
\[ B_\epsilon \rightarrow \mathbb{C}, \quad z \rightarrow f(z)h(z), \] (3.6)
is smooth. Then there exist $\delta \in (0, \epsilon)$ and $C \in \mathbb{R}^+$ such that $f$ is differentiable on $B_\epsilon - 0$ and
\[ |dz f| \leq C \quad \forall \ z \in B_\delta - 0. \] (3.7)

Proof. After a holomorphic change of coordinate on $B_{2\delta} \subset B_\epsilon$, we can assume that $h(z) = z^\ell$ for some $\ell \in \mathbb{Z} \geq 0$. Define
\[ g: B_{2\delta} \rightarrow \mathbb{C}, \quad g(z) = f(z)z^\ell - f(0)z^\ell. \]
By Taylor’s Theorem and the smoothness of the function (3.6), there exists $C > 0$ such that the smooth function $g$ satisfies
\[ |g(z)| \leq C|z|^{\ell+1} \quad \forall \ z \in B_\delta. \]
Dividing $g$ by $z^\ell$, we thus obtain (3.7).

Remark 3.8. Corollary 3.6 refines the conclusion of Proposition 3.1 for $J$-holomorphic maps. In contrast to the output $(\Phi, \sigma)$ of Proposition 3.1, the output of Corollary 3.6 does not depend continuously on the input $u$ with respect to the $L^p$-norms. This makes Corollary 3.6 less suitable for applications in settings involving families of $J$-holomorphic maps.

3.2 Local structure of $J$-holomorphic maps

We now obtain three corollaries from Proposition 3.1. They underpin important geometric statements established later in these notes, such as Propositions 3.12 and 4.11 and Lemma 5.4.

Corollary 3.9 (Unique Continuation). Suppose $(X, J)$ is an almost complex manifold, $(\Sigma, j)$ is a connected Riemann surface, and
\[ u, u': (\Sigma, j) \rightarrow (X, J) \]
are $J$-holomorphic maps. If $u_0$ and $u'_0$ agree to infinite order at $z_0 \in \Sigma$, then $u' = u'$.

Proof. Since the subset of the points of $\Sigma$ at which $u$ and $u'$ agree is closed to infinite order, it is enough to show that $u = u'$ on some neighborhood of $z_0$. By the continuity of $u$, we can assume that $X = \mathbb{C}^n$, $\Sigma = B_1$, $z_0 = 0$, and $u(0), u'(0) = 0$. Let
\[ w = u' - u: B_\epsilon \rightarrow \mathbb{C}^n. \]
Since $J$ is $C^1$,
\[ J(x+y) = J(x) + \int_0^1 \frac{dJ(x+ty)}{dt} dt = J(x) + \sum_{i=1}^n y_i \int_0^1 \frac{\partial J}{\partial y_i} |_{x+ty} dt. \] (3.8)

Since $u$ and $u'$ are $J$-holomorphic, (3.8) implies that
\[ \partial_s w + J(u(z)) \partial_t w + C(z)w(z) = 0, \quad \text{where} \quad C \in L^p(B_1; \text{End}_\mathbb{R}\mathbb{C}^n), \]
\[ C(z)y = \sum_{i=1}^n y_i \left( \int_0^1 \frac{\partial J}{\partial y_i} \bigg|_{v(z)+tw(z)} dt \right) \partial_t w |_z. \]
By Proposition 3.1, there thus exist $\delta \in (0,1)$, $\Phi \in L^p_1(B_\delta; \text{GL}_{2n}\mathbb{R})$, and holomorphic map $\bar{w}: B_\delta \rightarrow \mathbb{C}^n$ such that

$$w(z) = \Phi(z)\bar{w}(z) \quad \forall \ z \in B_\delta.$$ 

Since $w$ vanishes to infinite order at 0, it follows that $\bar{w}(z)=0$ for all $z\in B_\delta$ (otherwise, $w$ would satisfy the conclusion of Corollary 3.3) and thus $w(z)=0$ for all $z\in B_\delta$. 

\[\square\]

**Corollary 3.10.** Suppose $(X, J)$ is an almost complex manifold,

$$u, u': (\Sigma, i), (\Sigma', j') \rightarrow (X, J)$$

are $J$-holomorphic maps, $z_0 \in \Sigma$ is such that $d_{z_0} u \neq 0$, and $z_0' \in \Sigma'$ is such that $u'(z_0') = u(z_0)$. If there exist sequences $z_i \in \Sigma - z_0$ and $z_i' \in \Sigma' - z_0'$ such that

$$\lim_{i \rightarrow \infty} z_i = z_0, \quad \lim_{i \rightarrow \infty} z_i' = z_0', \quad \text{and} \quad u(z_i) = u'(z_i) \quad \forall \ i \in \mathbb{Z}^+,$$

then there exists a holomorphic map $\sigma: U' \rightarrow \Sigma$ from a neighborhood of $z_0'$ in $\Sigma'$ such that $\sigma(z_0') = z_0$ and $u'|_{U'} = u \circ \sigma$.

**Proof.** It can be assumed that $(\Sigma, j, z_0), (\Sigma', j', z_0') = (B_1, i_0, 0)$, where $B_1 \subset \mathbb{C}$ is the unit ball with the standard complex structure. Since $d_{z_0} u \neq 0$ and $u$ is $J$-holomorphic, $u$ is an embedding near $0 \in B_1$ and so is a slice in a coordinate system. Thus, we can assume that

$$u, u' \equiv (v, w): (B_1, 0) \rightarrow (\mathbb{C} \times \mathbb{C}^{n-1}, 0), \quad u(z) = (z, 0) \in \mathbb{C} \times \mathbb{C}^{n-1},$$

and $u, u'$ are $J$-holomorphic with respect to some almost complex structure

$$J(x, y) = \begin{pmatrix} J_{11}(x, y) & J_{12}(x, y) \\ J_{21}(x, y) & J_{22}(x, y) \end{pmatrix}: \mathbb{C} \times \mathbb{C}^{n-1} \rightarrow \mathbb{C} \times \mathbb{C}^{n-1}, \quad (x, y) \in \mathbb{C} \times \mathbb{C}^{n-1}.$$ 

Since $J$ is $C^1$,

$$J_{ij}(x, y) = J_{ij}(x, 0) + \int_0^1 \frac{dJ_{ij}(x, ty)}{dt} dt = J_{ij}(x, 0) + \sum_{i=1}^{n-1} y_i \int_0^1 \left. \frac{\partial J_{ij}}{\partial y_i} \right|_{(x, ty)} dt. \quad (3.9)$$

Since $u$ is $J$-holomorphic,

$$J_{21}(x, 0) = 0, \quad J_{22}(x, 0)^2 = -\text{Id} \quad \forall \ x \in B_1 \subset \mathbb{C}. \quad (3.10)$$

Since $u'$ is $J$-holomorphic,

$$\partial_x w + J_{22}(v(z), w(z)) \partial_t w + J_{21}(v(z), w(z)) \partial_t v = 0.$$ 

Combining this with (3.9) and the first equation in (3.10), we find that

$$\partial_x w + J_{22}(v(z), 0) \partial_t w + C(z) w(z) = 0, \quad \text{where} \quad C \in L^p(B_1; \text{End}_{\mathbb{R}} \mathbb{C}^{n-1}),$$

$$C(z)y = \sum_{i=1}^{n-1} y_i \left( \left( \int_0^1 \frac{\partial J_{22}}{\partial y_i} (v(z), tu(z)) dt \right) \partial_t w |_z + \left( \int_0^1 \frac{\partial J_{21}}{\partial y_i} (v(z), tu(z)) dt \right) \partial_t v |_z \right).$$

By Proposition 3.1 and the second identity in (3.10), there thus exist $\delta \in (0,1)$, $\Phi \in L^p_1(B_\delta; \text{GL}_{2n-2}\mathbb{R})$, and holomorphic map $\bar{w}: B_\delta \rightarrow \mathbb{C}^{n-1}$ such that

$$w(z) = \Phi(z)\bar{w}(z) \quad \forall \ z \in B_\delta.$$ 

Since $u'(z_i') = u(z_i)$, $\bar{w}(z_i') = 0$ for all $i \in \mathbb{Z}^+$. Since $z_i' \rightarrow 0$ and $z_i' \neq 0$, it follows that $w = 0$. This implies the claim with $U' = B_\delta$ and $\sigma = v$. 

\[\square\]
Corollary 3.11. Let \((X, J)\) be an almost complex manifold with a Riemannian metric \(g\) and \(x \in X\) be such that \(g\) is compatible with \(J\) at \(x\). If \(u: \Sigma \rightarrow X\) is a \(J\)-holomorphic map from a compact Riemann surface with boundary, then

\[
\lim_{\delta \to 0} \frac{E_g(u; u^{-1}(B_\delta^g(x)))}{\pi \delta^2} = \text{ord}_x u.
\]

Proof. By the continuity of \(u\), we can assume that \(X = \mathbb{C}^n\), \(J\) agrees with the standard complex structure \(J_{\mathbb{C}^n}\) at the origin, \(g\) agrees with the standard metric \(g_{\mathbb{C}^n}\) at the origin, \(\Sigma = \overline{B_R}\) for some \(R \in \mathbb{R}^+\), and \(u(0) = 0\). In particular, there exists \(C \geq 1\) such that

\[
|J_x - J_{\mathbb{C}^n}| \leq C|x|, \quad |g_x - g_{\mathbb{C}^n}| \leq C|x| \quad \forall\ x \in \mathbb{C}^n \text{ s.t. } |x| \leq 1,
\]

(3.11)

where \(|\cdot|\) denotes the usual norm of \(x\) (i.e. the distance to the origin with respect to \(g_{\mathbb{C}^n}\)).

Let \(\ell \equiv \text{ord}_0 u\) and \(\alpha \in \mathbb{C}^{n-1} - 0\) be as in Corollary 3.3 where \(0 \in B_R\) is the origin in the domain of \(u\). Thus, there exist \(\epsilon \in (0, 1)\) and \(C \in \mathbb{R}^+\) such that

\[
u(z) = \alpha \cdot (z^\ell + f(z)), \quad |f(z)| \leq C|z|^\ell + 1 \quad \forall\ z \in B_\epsilon.
\]

(3.12)

Let \(z = s + it\) as before. By (3.12), there exists \(C \in \mathbb{R}^+\) such that

\[
u_s(z) = \alpha \cdot (\ell z^{\ell - 1} + f_s(z)), \quad \nu_u(z) = \alpha \cdot (\ell i z^{\ell - 1} + f_t(z)), \quad |f_s(z)|, |f_t(z)| \leq C|z|^\ell \quad \forall\ z \in B_\epsilon.
\]

(3.13)

We can also assume that the three constants \(C\) in (3.11), (3.12), and (3.13) are the same, \(C \geq 1\),

\[
C_\alpha \epsilon = (C + C|\alpha| + C^2|\alpha|) \epsilon \leq 1,
\]

and \(|\nu(z)| \leq 1\) for all \(z \in B_\epsilon\). By (3.11), (3.13),

\[
\frac{|u(z)|_g}{|\alpha||z|^\ell} - 1, \quad \frac{|u_s(z)|_g}{|\alpha||z|^{\ell - 1}} - 1, \quad \frac{|u_t(z)|_g}{|\alpha||z|^{\ell - 1}} - 1 \leq C|z| + C|\alpha||z|^\ell + C^2|\alpha||z|^\ell + 1
\]

\[
\leq C_\alpha |z| \quad \forall\ z \in B_\epsilon,
\]

(3.14)

where \(|\cdot|_g\) denotes the distance to the origin in \(\mathbb{C}^n\) with respect to the metric \(g\) and the corresponding norm on \(T\mathbb{C}^n\).

Given \(r \in (0, 1)\), let \(\delta_r \in (0, \epsilon)\) be such that

\[
C_\alpha \left(\frac{2\delta_r}{(1-r)|\alpha|}\right)^{1/\ell} \leq r.
\]

(3.15)

For any \(\delta \in [0, \delta_r]\), (3.14) and (3.15) give

\[
|z| \leq \left(\frac{\delta}{(1+r)|\alpha|}\right)^{1/\ell} \quad \Rightarrow \quad u(z) \in B_{\delta(r)}^g(0),
\]

\[
u(z) \in B_{\delta(r)}^g(0) \quad \Rightarrow \quad |z| \leq \left(\frac{\delta}{(1-r)|\alpha|}\right)^{1/\ell},
\]

\[
|z| \leq \left(\frac{\delta}{(1-r)|\alpha|}\right)^{1/\ell} \quad \Rightarrow \quad 1 - r \leq \frac{|u_s(z)|_g}{|\alpha||z|^{\ell - 1}}, \quad \frac{|u_t(z)|_g}{|\alpha||z|^{\ell - 1}} \leq 1 + r.
\]
Combining these, we obtain

\[
\int_{|z| \leq \left( \frac{\delta}{1+r|z|} \right)^{\frac{1}{\ell}}} (1-r)^2 (|\alpha| |z|^{\ell-1})^2 \leq \frac{1}{2} \int_{u^{-1}(B_\delta^g(0))} (|u_x|^2 + |u_t|^2) \\
\leq \int_{|z| \leq \left( \frac{\delta}{1+r|z|} \right)^{\frac{1}{\ell}}} (1+r)^2 (|\alpha| |z|^{\ell-1})^2.
\]

Evaluating the outer integrals, we find that

\[
\left( \frac{1-r}{1+r} \right)^2 \ell \pi \delta^2 \leq E_g \left( u; u^{-1}(B_\delta^g(0)) \right) \leq \left( \frac{1+r}{1-r} \right)^2 \ell \pi \delta^2.
\]

These inequalities hold for all \( r \in (0, 1) \) and \( \delta \in (0, \delta_r) \); the claim is obtained by sending \( r \to 0 \).

### 3.3 The Monotonicity Lemma

Proposition 3.12 below is a key step in the continuity part of the proof of the Removal of Singularity Proposition 5.1. The precise nature of the lower energy bound on the right hand-side of (3.16) does not matter, as long as it is positive for \( \delta > 0 \).

**Proposition 3.12** (Monotonicity Lemma). If \((X, J)\) is an almost complex manifold and \(g\) is a Riemannian metric on \(X\) compatible with \(J\), there exists a continuous function \(C_{g,J} : X \to \mathbb{R}^+\) with the following property. If \((\Sigma, j)\) is a compact Riemann surface with boundary, \(u : \Sigma \to X\) is a \(J\)-holomorphic map, \(x \in X\), and \(\delta \in \mathbb{R}^+\) are such that \(u(\partial \Sigma) \cap B_\delta^g(x) = \emptyset\), then

\[
E_g(u) \geq (\text{ord}_x u) \frac{\pi \delta^2}{(1+C_{g,J}(x)\delta)^4}.
\]

(3.16)

If \(\omega(\cdot, \cdot) \equiv g(J(\cdot, \cdot))\) is a symplectic form on \(X\), then the above fraction can be replaced by the product \(\pi \delta^2 e^{-C_{g,J}(x)\delta^2}\).

According to this proposition, “completely getting out” of the ball \(B_\delta(x)\) via a \(J\)-holomorphic map requires an energy bounded below by a little less than \(\pi \delta^2\). Thus, the \(L_1^2\)-norm of a \(J\)-holomorphic map \(u\) exerts some control over the \(C^0\)-norm of \(u\). If \(p > 2\), the \(L_p^1\)-norm of any smooth map \(f\) from a two-dimensional manifold controls the \(C^0\)-norm of \(f\). However, this is not the case of the \(L_2^1\)-norm, as illustrated by the example of [12, Lemma 10.4.1]: the function

\[
f_\epsilon : \mathbb{R}^2 \to [0, 1], \quad f_\epsilon(z) = \begin{cases} 1, & \text{if } |z| \leq \epsilon; \\
\ln |z| / \ln \epsilon, & \text{if } \epsilon \leq |z| \leq 1; \\
0, & \text{if } |z| \geq 1;
\end{cases}
\]

with any \(\epsilon \in (0, 1)\) is continuous and satisfies

\[
\int_{\mathbb{R}^2} |df_\epsilon|^2_g = -\frac{2\pi}{\ln \epsilon}.
\]

It is arbitrarily close in the \(L_1^2\)-norm to a smooth function \(\bar{f}_\epsilon\). Thus, it is possible to “completely get out” of \(B_\delta^g(x)\) using a smooth function with arbitrarily small energy (\(\bar{f}_\delta\) does this for the ball \(B_1(1)\) in \(\mathbb{R}\)).

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By (2.7), the holomorphic maps are the local minima of the functional
\[ C^\infty(\Sigma; X) \to \mathbb{R}, \quad f \to E_g(f) - \int_X f^*\omega_J, \]
for every compact Riemann surface \((\Sigma, i)\) without boundary. This fact underlines Lemma 3.16, the key ingredient in the proof of the Monotonicity Lemma. Lemma 3.16 implies that the ratio of \(E_g(u; u^{-1}(B_2^g(x)))\) and the fraction on the right-hand side \((3.16)\) is a non-decreasing function of \(\delta\), as long as \(u(\partial\Sigma) \cap B_2^g(x) = \emptyset\). By Corollary 3.11, this ratio approaches \(\text{ord}_x u\) as \(\delta\) approaches 0. These two statements imply Proposition 3.12.

We first make some general Riemannian geometry observations. Let \((X, g)\) be a Riemannian manifold. Denote by \(\exp: \mathcal{W}_g \to X\), the exponential map from a neighborhood of \(X\) in \(TX\) with respect to the Levi-Civita connection \(\nabla\) of \(g\). For each \(v \in TX\), we denote by
\[ \gamma_v: [0,1] \to X, \quad \gamma_v(\tau) = \exp_x(\tau v), \]
the geodesic with \(\gamma'_v(0) = v\). Let
\[ r_g: X \to \mathbb{R}^+ \quad \text{and} \quad d_g: X \times X \to \mathbb{R}^\geq 0 \]
be the injectivity radius of \(\exp\) and the distance function. For each \(x \in X\), define
\[ \zeta_x \in \Gamma(B_{r_g(x)}^g(x); TX) \quad \text{by} \quad \exp_y(\zeta_x(y)) = x, \quad g(\zeta_x(y), \zeta_x(y)) < r_g(x)^2 \quad \forall y \in B_{r_g(x)}^g(x). \]

**Lemma 3.13.** Let \((X, g)\) be a Riemannian manifold and \(x \in X\). If \(\alpha: (-\epsilon, \epsilon) \to X\) is a smooth curve such that \(\alpha(0) \in B_{r_g(x)}^g(x)\), then
\[ \frac{1}{2} \frac{d}{d\tau} d_g(x, \alpha(\tau))^2 \bigg|_{\tau = 0} = -g(\alpha'(0), \zeta_x(\alpha(0))). \]

**Proof.** If \(\beta(\tau) = \exp_x^{-1} \alpha(\tau)\), then
\[ \frac{1}{2} \frac{d}{d\tau} d_g(x, \alpha(\tau))^2 \bigg|_{\tau = 0} = \frac{1}{2} \frac{d}{d\tau} |\beta(\tau)|^2 \bigg|_{\tau = 0} = g(\beta'(0), \beta(0)). \]

By Gauss’s Lemma,
\[ g(\beta'(0), \beta(0)) = g(\{d_{\beta(0)} \exp_x\}(\beta'(0)), \{d_{\beta(0)} \exp_x\}(\beta(0))) = g(\alpha'(0), -\zeta_x(\alpha(0))). \]

This establishes the claim. \(\square\)

**Lemma 3.14.** If \((X, g)\) is a Riemannian manifold, there exists a continuous function \(C_g: X \to \mathbb{R}^+\) with the following property. If \(x \in X\), \(v \in T_xX\) with \(|v|_g < \frac{1}{2} r_g(x)\), and \(\tau \to J(\tau)\) is a Jacobi vector field along the geodesic \(\gamma_v\) with \(J(0) = 0\), then
\[ |J'(1) - J(1)|_g \leq C_g(x)|v|_g^2 |J(1)|_g. \]

**Proof.** Let \(R_g\) be the Riemann curvature tensor of \(g\) and \(f(\tau) = |\tau J'(\tau) - J(\tau)|_g\). Then, \(f(0) = 0\) and
\[ f(\tau)f'(\tau) = \frac{1}{2} \frac{d}{d\tau} f(\tau)^2 = g(\tau J''(\tau), \tau J'(\tau) - J(\tau)) = \tau g(R(\gamma'_v(\tau), J(\tau))\gamma'_v(\tau), \tau J'(\tau) - J(\tau)) \leq C_g(x)|v|_g^2 |J(\tau)|_g \tau f(\tau). \]
If $C_g$ is sufficiently large, then $|J(\tau)|_g \leq C_g(x)|J(1)|_g$. Thus,
\[
f(\tau)f'(\tau) \leq C_g(x)|v^2_{\rho}|J_v(\tau)|_g f(\tau) \leq C_g(x)^2|v^2_{\rho}|J(1)|_g f(\tau), \quad f'(\tau) \leq C_g(x)^2|v^2_{\rho}|J(1)|_g \tau.
\]
The claim follows from the last inequality. \hfill \Box

**Corollary 3.15.** If $(X, g)$ is a Riemannian manifold, there exists a continuous function $C_g: X \to \mathbb{R}^+$ with the following property. If $x \in X$, then
\[
|\nabla w_\xi|_y + w|_g \leq C_g(x)d_g(x, y)^2|w|_g \quad \forall \ w \in T_yX, \ y \in B^g_{r_g(x)/2}(x).
\]

**Proof.** Let $\tau \mapsto u(s, \tau)$ be a family of geodesics such that
\[
u(s, 0) = x, \quad u(0, 1) = y, \quad \frac{d}{ds}u(s, 1) \bigg|_{s=0} = w.
\]
Since $\tau \mapsto u(s, \tau)$ is a geodesic,
\[
\frac{d}{d\tau} u(s, \tau) \bigg|_{\tau=1} = \{d_{u_{\tau}(s, 0)} \exp_x\}(u_{\tau}(s, 0)) = -\xi_x(u(s, 1)),
\]
\[
\frac{D}{d\tau} \frac{du(s, \tau)}{ds} \bigg|_{(s, \tau)=(0, 1)} = -\nabla_w \xi_x|_y.
\]
Furthermore, $J(\tau) \equiv \frac{du(s, \tau)}{ds} \bigg|_{s=0}$ is a Jacobi vector field along the geodesic $\tau \mapsto u(0, \tau)$ with
\[
J(0) = 0, \quad J(1) = w, \quad J'(1) = \frac{D}{d\tau} \frac{du(s, \tau)}{ds} \bigg|_{(s, \tau)=(0, 1)} = -\nabla_w \xi_x|_y.
\]
Thus, the claim follows from Lemma 3.14. \hfill \Box

**Lemma 3.16.** Suppose $(X, \omega)$ is a symplectic manifold, $J$ is an almost complex structure on $X$ tamed by $\omega$, and $\nabla$ is the Levi-Civita connection of the metric $g_J$. If $(\Sigma, j)$ is a compact Riemann surface with boundary and $u: \Sigma \to X$ is a $J$-holomorphic map, then
\[
\int_{\Sigma} g_J(du \otimes j \nabla \xi) = \int_{\Sigma} \left(u^* \{\nabla \xi \omega_J\} + \omega_J(du \wedge j \nabla \xi)\right) \quad \forall \ \xi \in \Gamma(\Sigma; u^*TX) \text{ s.t. } \xi|_{\partial \Sigma} = 0.
\]

**Proof.** For $\tau \in \mathbb{R}$ sufficiently close to 0, define
\[
u_r: \Sigma \to X, \quad \nu_r(z) = \exp_{u(z)}(\tau \xi(z)).
\]
Since $\xi|_{\partial \Sigma} = 0, \nu_r|_{\partial \Sigma} = u|_{\partial \Sigma}$. Denote by $\hat{\Sigma}$ the closed oriented surface obtained by gluing two copies of $\Sigma$ along the common boundary and reversing the orientation on the second copy. Let
\[
h_\nu: \hat{\Sigma} \to X
\]
be the map restricting to $u_\tau$ on the first copy of $\Sigma$ and to $u$ on the second.

By (2.7),
\[
E(\tau) \equiv E_{g_J}(u_\tau) - \int_{\Sigma} u_\tau^*\omega - E_{g_J}(u) = \int_{\Sigma} \hat{u}_\nu^*\omega + 2 \int_{\Sigma} g_J(\hat{\partial}u_\nu \otimes j \hat{\partial}u_\tau) \geq 0 \quad \forall \tau.
\]
Since \( \omega \) is closed and \( \hat{u}_s \) represents the zero class in \( H_2(X; \mathbb{Z}) \), the first integral on the right-hand side above vanishes. Thus, the function \( \tau \mapsto E(\tau) \) is minimized at \( \tau = 0 \) (when it equals 0) and so

\[
0 = E'(0) = \frac{d}{d\tau} \left( E_{g_J}(u_\tau) - \int u_\tau \omega_J \right) \bigg|_{\tau=0} = \frac{d}{d\tau} \left( \frac{1}{2} \int_S g_J(du_\tau \otimes dt) - \int u_\tau^* \omega_J \right) \bigg|_{\tau=0};
\]  

(3.17)

the last equality above uses the definition of \( E(u_\tau) \) in (2.5).

Let \( z = s+it \) be a local coordinate on \((\Sigma, i)\). Since \( \nabla \) is torsion-free,

\[
\frac{D}{dt}(u_\tau) \bigg|_{\tau=0} \equiv \frac{D}{dt} \frac{du_\tau}{ds} \bigg|_{\tau=0} = \frac{D}{ds} \frac{du_\tau}{dt} \bigg|_{\tau=0} = \frac{D}{ds} \xi \equiv \nabla \xi, \quad \frac{D}{dt}(u_\tau) \bigg|_{\tau=0} = \nabla \xi.
\]

Since \( \nabla \) is also \( g \)-compatible,

\[
\frac{1}{2} \frac{d}{dt} g_J(du_\tau \otimes dt) \bigg|_{\tau=0} = \left( g_J \left( u_s, \frac{d}{dt}(u_\tau) \bigg|_{\tau=0} \right) + g_J \left( u_t, \frac{d}{dt}(u_\tau) \bigg|_{\tau=0} \right) \right) ds \wedge dt = g_J(u_s, \nabla \xi) + g_J(u_t, \nabla \xi) = g_J(du \otimes \nabla \xi),
\]

\[
\frac{d}{dt} u_\tau^* \omega_J \bigg|_{\tau=0} = \left( \nabla \omega_J \right)(u_s, u_t) + \omega_J \left( \frac{D}{dt}(u_\tau) \bigg|_{\tau=0} , u_t \right) + \omega_J \left( u_s, \frac{D}{dt}(u_\tau) \bigg|_{\tau=0} \right) ds \wedge dt = u^* \left( \nabla \omega_J \right) + \omega_J(du \wedge \nabla \xi).
\]

Combining this with (3.17), we obtain the claim. \( \square \)

**Proof of Proposition 3.12** Let \( \delta_g : X \to \mathbb{R}^+ \) be a continuous function such that for every \( x \in X \) there exists a symplectic form \( \omega_x \) on \( B^2_{\delta_g(x)}(x) \) so that \( J \) is tamed by \( \omega_x \) on \( B^2_{\delta_g(x)}(x) \) and compatible with \( \omega_x \) at \( x \). We assume that \( 2\delta_g(x) \leq r_g(x) \) for every \( x \in X \). It is sufficient to establish the proposition for each \( x \in X \) and each \( \delta \leq \delta_g(x) \) under the assumption that the metric \( g \) is determined by \( J \) and \( \omega_x \) on \( B^2_{\delta_g(x)}(x) \).

Choose a \( C^\infty \)-function \( \eta : \mathbb{R} \to [0, 1] \) such that

\[
\eta(\tau) = \begin{cases} 
1, & \text{if } \tau \leq \frac{1}{2}; \\
0, & \text{if } \tau \geq 1;
\end{cases} \quad \eta'(\tau) \leq 0. \tag{3.18}
\]

For a compact Riemann surface with boundary \((\Sigma, j)\), a smooth map \( u : \Sigma \to X, x \in X \), and \( \delta \in \mathbb{R}^+ \), define

\[
\eta_{u, x, \delta} \in C^\infty(\Sigma; \mathbb{R}), \quad \eta_{u, x, \delta}(z) = \eta \left( \frac{d_g(x, u(z))}{\delta} \right),
\]

\[
E_{u, x, \eta}(\delta) = \frac{1}{2} \int_\Sigma \eta_{u, x, \delta}(z) g(du \otimes dt), \quad E_{u, x}(\delta) = E_g(u; u^{-1}(B^2_{\delta}(x)));
\]

We show in the remainder of this proof that there exists a continuous function \( C_{g, J} : X \to \mathbb{R}^+ \) such that

\[
- \delta E'_{u, x, \eta}(\delta) + 2E_{u, x, \eta}(\delta) \leq 2C_{g, J}(x) \delta E_{u, x, \eta}(\delta) + C_{g, J}(x) \delta^2 E'_{u, x, \eta}(\delta) \tag{3.19}
\]

(3.19)
for every compact Riemann surface with boundary \((\Sigma,i)\), \(J\)-holomorphic map \(u:\Sigma \rightarrow X\), and \(\delta \in (0, \delta_g(x))\) such that \(u(\partial \Sigma) \cap B_\delta^2(x) = \emptyset\). This inequality is equivalent to

\[
\left( E_{u,x,\eta}(\delta) \right) \left( 1 + C_{g,J}(x) \delta \right)^4 \geq 0.
\]

By Lebesgue’s Dominated Convergence Theorem, \(E_{u,x,\eta}(\delta)\) approaches \(E_{u,x}(\delta)\) from below as \(\eta\) approaches the characteristic function \(\chi_{(-\infty,1)}\) of \((-\infty,1)\). Thus, the function

\[
\delta \rightarrow E_{u,x}(\delta) \left( 1 + C_{g,J}(x) \delta \right)^4
\]

is non-decreasing as long as \(u(\partial \Sigma) \cap B_\delta^2(x) = \emptyset\). By Corollary 3.11,

\[
\lim_{\delta \to 0} \left( E_{u,x}(\delta) \left( 1 + C_{g,J}(x) \delta \right)^4 \right) = \lim_{\delta \to 0} \frac{E_{u,x}(\delta)}{\delta^2} = (\text{ord}_x u) \pi.
\]

This implies the first claim.

Fix \(x \in X\). We note that

\[
E'_{u,x,\eta}(\delta) = -\frac{1}{2} \int_{\Sigma} \eta' \left( \frac{d_g(x,u(z))}{\delta} \right) \frac{1}{\delta} \frac{d_g(x,u(z))}{d_g(x,u(z))} \frac{d(x,u(z))}{\delta} g \left( (du \otimes 1) \frac{d(x,u(z))}{\delta} \right).
\]

(3.20)

For a compact Riemann surface with boundary \((\Sigma,i)\), a smooth map \(u:\Sigma \rightarrow X\), and \(\delta \in (0, \delta_g(x))\), let

\[
\xi_{u,x,\delta} \in \Gamma(\Sigma; u^*TX), \quad \xi_{u,x,\delta}(z) = -\eta_{u,x,\delta}(z) \xi_x(u(z));
\]

the vanishing assumption in (3.18) implies that \(\xi_{u,x,\delta}\) is well-defined. If \(u(\partial \Sigma) \cap B_\delta^2(x) = \emptyset\), then \(\xi_{u,x,\delta}|_{\partial \Sigma} = 0\). By Lemma 3.13,

\[
\nabla \xi_{u,x,\delta}|_z = \eta' \left( \frac{d_g(x,u(z))}{\delta} \right) \frac{1}{\delta} \frac{d_g(x,u(z))}{d_g(x,u(z))} g \left( (du \otimes 1) \frac{d(x,u(z))}{\delta} \right) \nabla \xi_x u + \eta_{u,x,\delta}(z) \nabla \xi_x \cdot \nabla z.
\]

(3.21)

Along with Corollary 3.15, (3.20), and the last assumption in (3.18), this implies that

\[
\int_{\Sigma} d_g(x,u(z)) g \left( (du \otimes 1) \nabla \xi_{u,x,\delta} \right) \leq 2 \delta^2 E_{u,x,\eta}(\delta) + 2 \left( 1 + C_g(x) \delta \right) \delta E_{u,x,\eta}(\delta).
\]

(3.22)

By the \(\omega_x\)-compatibility assumption on \(J\) at \(x\), there exists a continuous function \(C: X \rightarrow \mathbb{R}^+\) such that

\[
\int_{\Sigma} \left| (\omega_x)_J (du \wedge \nabla \xi_{u,x,\delta}) \right| \leq C(x) \left( \int_{\Sigma} d_g(x,u(z)) g \left( (du \otimes 1) \nabla \xi_{u,x,\delta} \right) \right)
\]

for all \(u\) and \(\delta\) as above. Along with this, Lemma 3.16 implies that there exists a continuous function \(C: X \rightarrow \mathbb{R}^+\) such that

\[
\left| \int_{\Sigma} g \left( (du \otimes 1) \nabla \xi_{u,x,\delta} \right) \right| \leq C(x) \left( \int_{\Sigma} \left| g \left( (du \otimes 1) \xi_{u,x,\delta} \right) + d_g(x,u(z)) g \left( (du \otimes 1) \nabla \xi_{u,x,\delta} \right) \right| \right)
\]

for every compact Riemann surface with boundary \((\Sigma,i)\), \(J\)-holomorphic map \(u:\Sigma \rightarrow X\), and \(\delta \in (0, \delta_g(x))\) such that \(u(\partial \Sigma) \cap B_\delta^2(x) = \emptyset\). Combining this with (3.22), we conclude that there exists a continuous function \(C: X \rightarrow \mathbb{R}^+\) such that

\[
\left| \int_{\Sigma} g \left( (du \otimes 1) \nabla \xi_{u,x,\delta} \right) \right| \leq C(x) \left( \delta E_{u,x,\eta}(\delta) + \delta^2 E_{u,x,\eta}(\delta) \right)
\]

(3.23)
for all $u$ and $\delta$ as above.

Suppose $(\Sigma, j)$ is a compact Riemann surface with boundary, $u : \Sigma \to X$ is a smooth map, and $\delta \in (0, \delta_g(x))$. Let $z = s + it$ be a coordinate on $(\Sigma, j)$. By (3.21),

$$
g(u_s, \nabla_s \xi_{u, x, \delta}) = \eta' \left( \frac{d_g(x, u(z))}{\delta} \right) \frac{1}{\delta d_g(x, u(z))} g(u_s, \zeta_x(u(z)))^2 + \eta_{u, x, \delta}(z) g(u_s, \nabla_s(-\zeta_x)|z).
$$

By Corollary 3.15,

$$
|u_s|^2 \leq g(u_s, \nabla_s(-\zeta_x)|z) + C_g(x)d_g(x, u(z))^2|u_s|^2 \quad \forall z \in u^{-1}(B_{\delta_g(x)}(x)).
$$

If $u$ is $J$-holomorphic, then $|u_s| = |u_t|$, $\langle u_s, u_t \rangle = 0$, and

$$
\frac{1}{2}(|u_s|^2 + |u_t|^2) d_g(x, u(z))^2 = |u_s|^2|\zeta_x(u(z))|^2 \geq g(u_s, \zeta_x(u(z)))^2 + g(u_t, \zeta_x(u(z)))^2.
$$

Since $\eta' \leq 0$, (3.24) and (3.26) give

$$
\frac{1}{2}\eta' \left( \frac{d_g(x, u(z))}{\delta} \right) \frac{d_g(x, u(z))}{\delta} \left( |u_s|^2 + |u_t|^2 + \eta_{u, x, \delta}(z)(|u_s|^2 + |u_t|^2) \right) \\
\leq g(u_s, \nabla_s \xi_{u, x, \delta}) + g(u_t, \nabla_t \xi_{u, x, \delta}) + C_g(x)\eta_{u, x, \delta}(z)d_g(x, u(z))^2(|u_s|^2 + |u_t|^2).
$$

Along with (3.20), this implies that

$$
-\delta E'_{u, x, \eta} (\delta) + 2E_{u, x, \eta} (\delta) \leq \int \Sigma g(du \otimes \nabla \xi_{u, x, \delta}) + 2C_g(x)\delta^2 E_{u, x, \eta} (\delta)
$$

for every compact Riemann surface with boundary $(\Sigma, j)$, $J$-holomorphic map $u : \Sigma \to X$, and $\delta \in (0, \delta_g(x))$. Combining this inequality with (3.28), we obtain (3.19).

Suppose $\omega \equiv g(J \cdot , \cdot)$ is a symplectic form on $X$. By Lemma 3.10, the left-hand side of (3.23) then vanishes. From (3.28), we thus obtain

$$
-\delta E'_{u, x, \eta} (\delta) + 2E_{u, x, \eta} (\delta) \leq 2C_{g, J}(x)\delta^2 E_{u, x, \eta} (\delta).
$$

The reasoning below (3.19) now yields the second claim.

\[\square\]

4 \hspace{1em} Mean Value Inequality and applications

We now move to properties of $J$-holomorphic maps $u$ from Riemann surfaces $(\Sigma, j)$ into almost complex manifolds $(X, J)$ that are of a more global nature. They generally concern the distribution of the energy of such a map over its domain and are consequences of the Mean Value Inequality for $J$-holomorphic maps. These fairly technical properties lead to geometric conclusions such as Propositions 4.3 and 5.1.
4.1 Statement and proof

According to Cauchy’s Integral Formula, a holomorphic map \( u: B_R \to \mathbb{C}^n \) satisfies

\[
u'(0) = \frac{1}{2\pi i} \oint_{|z|=r} \frac{u(z)}{z^2} \, dz \quad \forall r \in (0, R).
\]

This immediately implies that a bounded holomorphic function defined on all of \( \mathbb{C} \) is constant. The Mean Value Inequality of Proposition 4.1 bounds the norms of the differentials of \( J \)-holomorphic maps of sufficiently small energy away from the boundary of the domain “uniformly” by their \( L^2 \)-norms. In general, one would not expect the value of a function to be bounded by its integral. The Mean Value Inequality implies that a \( J \)-holomorphic map which is defined on all of \( \mathbb{C} \) and has sufficiently small energy is in fact constant; see Corollary 4.2.

**Proposition 4.1** (Mean Value Inequality). If \((X, J)\) is an almost complex manifold and \( g \) is a Riemannian metric on \( X \) compatible with \( J \), there exists a continuous function \( h_{J,g}: X \times \mathbb{R} \to \mathbb{R}^+ \) with the following property. If \( u: B_R \to X \) is a \( J \)-holomorphic map such that

\[ u(B_R) \subset B^C_x(x) \quad \text{and} \quad E_g(u) < h_{J,g}(x, r) \]

for some \( x \in X \) and \( r \in \mathbb{R} \), then

\[
|d_0 u|^2 < \frac{16}{\pi R^2} E_g(u). \tag{4.1}
\]

**Proof.** Let \( \phi(z) = \frac{1}{2} |d_z u|^2 \). By Lemma 4.7 below, \( \Delta \phi \geq -A_{J,g} \phi^2 \) with \( A_{J,g}: X \times \mathbb{R} \to \mathbb{R}^+ \) determined by \( (X, J, g) \). The claim with \( h_{J,g} = \pi/8A_{J,g} \) thus follows from Proposition 4.6. \( \square \)

**Corollary 4.2** (Lower Energy Bound). If \((X, J)\) is a compact almost complex manifold and \( g \) is a Riemannian metric on \( X \), then there exists \( h_{J,g} \in \mathbb{R}^+ \) such that \( E_g(u) \geq h_{J,g} \) for every non-constant \( J \)-holomorphic map \( u: S^2 \to X \).

**Proof.** By the compactness of \( X \), we can assume that \( g \) is compatible with \( J \). Let \( h_{J,g} > 0 \) be the minimal value of the function \( h_{J,g} \) in the statement of Proposition 4.1 on the compact space \( X \times [0, \text{diam}_g(X)] \). If \( u: S^2 \to X \) is \( J \)-holomorphic map with \( E_g(u) < h_{J,g} \),

\[
|d_z u|^2 < \frac{16}{\pi R^2} E_g(u) \leq \frac{16}{\pi R^2} E_g(u) \quad \forall z \in \mathbb{C}, \ R \in \mathbb{R}^+
\]

by Proposition 4.1 since \( B_R(z) \subset \mathbb{C} \) as Riemann surfaces. Thus, \( d_z u = 0 \) for all \( z \in \mathbb{C} \), and so \( u \) is constant. \( \square \)

If \( \phi: U \to \mathbb{R} \) is a \( C^2 \)-function on an open subset of \( \mathbb{R}^2 \), let

\[
\Delta \phi = \frac{\partial^2 \phi}{\partial s^2} + \frac{\partial^2 \phi}{\partial t^2} = \phi_{ss} + \phi_{tt}
\]

denote the Laplacian of \( \phi \).

**Exercise 4.3.** Show that in the polar coordinates \((r, \theta)\) on \( \mathbb{R}^2 \),

\[
\Delta \phi = \phi_{rr} + r^{-1} \phi_r + r^{-2} \phi_{\theta\theta}. \tag{4.2}
\]

**Lemma 4.4.** If \( \phi: \overline{B_R} \to \mathbb{R} \) is \( C^2 \), then

\[
2\pi R \phi(0) = -R \int_{(r, \theta) \in B_R} (\ln R - \ln r) \Delta \phi + \int_{\partial B_R} \phi. \tag{4.3}
\]
Proof. By Stokes’ Theorem applied to $\phi d\theta$ on $B_R - B_\delta$,

$$
\oint_{\partial B_R} \phi d\theta - \oint_{\partial B_\delta} \phi d\theta = \int_{B_R - B_\delta} \phi_r dr \wedge d\theta = \int_0^{2\pi} \int_{\delta}^R (r\phi_r) r^{-1} dr d\theta \\
= \int_0^{2\pi} (\ln R - \ln \delta) \phi_r(\delta, \theta) d\theta + \int_0^{2\pi} \int_{\delta}^R (\ln R - \ln r) (\phi_{rr} + r^{-1} \phi_r) r dr d\theta ;
$$

the last equality above is obtained by applying integration by parts to the functions $\ln r - \ln R$ and $r\phi_r$. Sending $\delta \to 0$ and using (4.2), we obtain

$$
\frac{1}{R} \int_{\partial B_R} \phi - 2\pi \phi(0) = 0 + \int_{(r, \theta) \in B_R} (\ln R - \ln r) \Delta \phi ,
$$

which is equivalent to (4.3). \hfill \Box

**Corollary 4.5.** If $\phi : B_R \to \mathbb{R}$ is $C^2$ and $\Delta \phi \geq -C$ for some $C \in \mathbb{R}^+$, then

$$
\phi(0) \leq \frac{1}{8} CR^2 + \frac{1}{\pi R^2} \int_{B_R} \phi .
$$

Proof. By (4.3),

$$
2\pi r \phi(0) \leq Cr \int_0^{2\pi} \int_0^r (\ln r - \ln \rho) d\rho d\theta + \int_{\partial B_r} \phi = Cr \cdot 2\pi \cdot \frac{r^2}{4} + \int_{\partial B_r} \phi \quad \forall r \in (0, R).
$$

Integrating the above in $r \in (0, R)$, we obtain

$$
2\pi \phi(0) \cdot \frac{R^2}{2} \leq 2\pi C \cdot \frac{R^4}{16} + \int_{B_R} \phi .
$$

This inequality is equivalent to (4.4). \hfill \Box

**Proposition 4.6.** If $\phi : B_R \to \mathbb{R}^+ \geq 0$ is $C^2$ and there exists $A \in \mathbb{R}^+$ such that $\Delta \phi \geq -A \phi^2$ and

$$
\int_{B_R} \phi < \frac{\pi}{8A} ,
$$

then

$$
\phi(0) \leq \frac{8}{\pi R^2} \int_{B_R} \phi .
$$

Proof. Replacing $A$ by $\tilde{A} = R^2 A$ and $\phi$ by

$$
\tilde{\phi} : B_1 \to \mathbb{R}, \quad \tilde{\phi}(z) = \phi(Rz) ,
$$

we can assume that $R = 1$, as well as that $\phi$ is defined on $B_1$.

1. Define

$$
f : [0, 1) \to \mathbb{R}^+ \geq 0 \quad \text{by} \quad f(r) = (1-r)^2 \max_{B_r} \phi .
$$

In particular, $f(0) = \phi(0)$ and $f(1) = 0$. Choose $r^* \in [0, 1)$ and $z^* \in B_{r^*}$ such that

$$
f(r^*) = \sup_{B_{r^*}} f \quad \text{and} \quad \phi(z^*) = \sup_{B_{r^*}} \phi \equiv c^* .
$$
Let $\delta = \frac{1}{2}(1-r^*) > 0$; see Figure 6. Thus,

$$\sup_{B_\delta(z^*)} \phi \leq \sup_{B_{r^*+\delta}} \phi = \frac{f(r^*+\delta)}{(1-(r^*+\delta))^2} \leq \frac{f(r^*)}{(1-r^*)^2} = 4\phi(z^*) = 4c^*.$$ 

In particular, $\Delta \phi \geq -A\phi^2 \geq -16Ac^2$ on $B_\delta(z^*)$.

(2) Using Corollary 4.5, we thus find that

$$c^* = \phi(z^*) \leq \frac{1}{8} \cdot 16Ac^2 \cdot \rho^2 + \frac{1}{\pi \rho^2} \int_{B_\rho(z^*)} \phi \leq 2Ac^2\rho^2 + \frac{1}{\pi \rho^2} \int_{B_1} \phi \quad \forall \rho \in [0,\delta]. \quad (4.6)$$

If $2Ac^2\delta^2 \leq \frac{1}{2}$, the $\rho = \delta$ case of the above inequality gives

$$\frac{1}{2}c^* \leq \frac{1}{\pi \delta^2} \int_{B_1} \phi, \quad \phi(0) = f(0) \leq f(r^*) = 4c^* \cdot \delta^2 \leq \frac{8}{\pi} \int_{B_1} \phi,$$

as claimed. If $2Ac^2\delta^2 \geq \frac{1}{2}$, $\rho \equiv (4Ac^*)^{-\frac{1}{2}} \leq \delta$ and (4.6) gives

$$c^* \leq 2A\rho^2 \cdot \frac{1}{4Ac^*} + \frac{4Ac^*}{\pi} \int_{B_1} \phi.$$

Thus, $\frac{\pi}{8A} \leq \int_{B_1} \phi$, contrary to the assumption. \qed

**Lemma 4.7.** If $(X,J)$ is an almost complex manifold and $g$ is a Riemannian metric on $X$ compatible with $J$, there exists a continuous function $A_{J,g}: X \times \mathbb{R} \rightarrow \mathbb{R}^+$ with the following property. If $\Omega \subset \mathbb{C}$ is an open subset, $u: \Omega \rightarrow X$ is a $J$-holomorphic map, and $u(\Omega) \subset B^g_\rho(x)$ for some $x \in X$ and $r \in \mathbb{R}$, then the function $\phi(z) \equiv \frac{1}{2}|dz|_g$ satisfies $\Delta \phi \geq -A_{J,g}(x,r)\phi^2$.

**Proof.** Let $z = s + it$ be the standard coordinate on $\mathbb{C}$. Denote by $u_s$ and $u_t$ the $s$ and $t$-partials of $u$, respectively. Since $u$ is $J$-holomorphic, i.e. $u_u = -Ju_t$, and $g$ is $J$-compatible, i.e. $g(J\cdot,J\cdot) = g(\cdot,\cdot)$, $|u_s|_g^2 = |u_t|_g^2$. Since the Levi-Civita connection $\nabla$ of $g$ is $g$-compatible and torsion-free,

$$\frac{1}{2} \frac{d^2}{dt^2}|u_s|_g^2 = |\nabla_t u_s|_g^2 + \langle \nabla_t \nabla_t u_s, u_t \rangle_g = |\nabla_t u_s|_g^2 + \langle \nabla_t \nabla_s u_t, u_s \rangle_g. \quad (4.7)$$

Similarly,

$$\frac{1}{2} \frac{d^2}{ds^2}|u_t|_g^2 = |\nabla_s u_t|_g^2 + \langle \nabla_s \nabla_t u_s, u_t \rangle_g. \quad (4.8)$$
Since \( u_s = -Ju_t \),
\[
\langle \nabla_s \nabla_t u_s, u_t \rangle_g = -\langle \nabla_s \nabla_t (Ju_t), u_t \rangle_g \\
= -\langle J \nabla_s \nabla_t u_t, u_t \rangle_g - \langle (\nabla_s J) \nabla_t u_t, u_t \rangle_g - \langle \nabla_s ((\nabla_t J) u_t), u_t \rangle_g \\
= -\langle \nabla_s \nabla_t u_s, u_t \rangle_g - \langle (\nabla_s J) \nabla_t u_t, u_t \rangle_g - \langle \nabla_s ((\nabla_t J) u_t), u_t \rangle_g.
\]
(4.9)

Putting (4.7)–(4.9), we find that
\[
\frac{1}{2} \Delta \phi = |\nabla_t u_s|^2 + |\nabla_s u_t|^2 + \langle R_g(u_t, u_s)u_t, u_s \rangle_g - \langle (\nabla_s J) \nabla_t u_t, u_t \rangle_g - \langle \nabla_s ((\nabla_t J) u_t), u_t \rangle_g,
\]
where \( R_g \) is the curvature tensor of the connection \( \nabla \). Since \( u(\Omega) \subset B^\beta_\Omega(x) \),
\[
\langle R_g(u_t, u_s)u_t, u_s \rangle_g \leq C_g(x, r)|u_s|^2|u_t|^2, \\
|\langle (\nabla_s J) \nabla_t u_t, u_t \rangle_g| \leq C_J,g(x, r)|u_s||u_t||\nabla_t(Ju_t)|_g \leq C_J,g(x, r)|u_s||u_t||u_s|_g + |\nabla_t u_s|_g, \\
\leq (C_J,g(x, r) + C_J,g(x, r)^2)|u_s|^2|u_t|^2 + |\nabla_t u_s|^2, \\
|\langle \nabla_s ((\nabla_t J) u_t), u_t \rangle_g| \leq C_J,g(x, r)|u_t|^2|u_s|_g|u_t|_g + |\nabla_s u_t|_g, \\
\leq C_J,g(x, r)|u_s|^3|u_t|_g^3 + C_J,g(x, r)^2|u_t|^4 + |\nabla_s u_t|^2.
\]
(4.10)

Combining (4.10) and (4.11), we find that
\[
\frac{1}{2} \Delta \phi \geq -C(x, r)(|u_s|^2|u_t|^2 + |u_s||u_t|^3 + |u_t|^4) \geq -3C(x, r)\phi^2,
\]
as claimed.

\section{4.2 Regularity of J-holomorphic maps}

By Cauchy’s Integral Formula, a continuous extension of a holomorphic map \( u: B^*_R \rightarrow \mathbb{C}^n \) over the origin is necessarily holomorphic. By Proposition 4.8 below, the same is the case for a J-holomorphic map \( u: B^*_R \rightarrow X \) of bounded energy.

**Proposition 4.8.** Let \((X, J)\) be an almost complex manifold and \( g \) be a Riemannian metric on \( X \). If \( R \in \mathbb{R}^+ \) and \( u: B_R \rightarrow X \) is a continuous map such that \( u|_{B^*_R} \) is a J-holomorphic map and \( E_g(u; B^*_R) < \infty \), then \( u \) is smooth and J-holomorphic on \( B_R \).

For a smooth loop \( \gamma: S^1 \rightarrow X \), define
\[
\gamma'(\theta) = \frac{d}{d\theta} \gamma(e^{i\theta}) \in T_{\gamma(e^{i\theta})}X \quad \text{and} \quad \ell_g(\gamma) = \int_0^{2\pi} |\gamma'(\theta)|_g d\theta \in \mathbb{R} \geq 0
\]
to be the velocity of \( \gamma \) and the length of \( \gamma \), respectively.

**Lemma 4.9** (Isoperimetric Inequality). Let \((X, J, g), R, \) and \( u \) be as in Proposition 4.8 and
\[
\gamma_r: S^1 \rightarrow X, \quad \gamma_r(e^{i\theta}) = u(re^{i\theta}) \quad \forall r \in (0, R).
\]
There exist \( \delta \in (0, R) \) and \( C \in \mathbb{R}^+ \) such that
\[
E_g(u; B^*_r) \leq C\ell_g(\gamma_r)^2 \quad \forall r \in (0, \delta).
\]
(4.12)
We can assume that the metric \( g \) for some \( C \).

By Proposition 4.1 and the finiteness assumption on \( E \),

\[ E: (0, R) \to \mathbb{R}, \quad E(r) = E_g(u; B_r^g). \]

We can assume that the metric \( g \) is determined by \( J \) and \( \omega_0 \) on \( B_{\delta_0}^g(x_0) \).

For a smooth loop \( \gamma: S^1 \to B_{\delta_0}^g(x_0) \), define

\[ \xi_\gamma: S^1 \to T_{x_0}X \quad \text{by} \quad \exp_{x_0} \xi_\gamma(e^{i\theta}) = \gamma(e^{i\theta}), \quad |\xi_\gamma(e^{i\theta})| < \delta_0, \]

\[ f_\gamma: B_1 \to X, \quad f_\gamma(r e^{i\theta}) = \exp_{x_0}(r \xi_\gamma(e^{i\theta})). \]

In particular,

\[ |\partial_r f_\gamma(r e^{i\theta})|_g = |\xi_\gamma(e^{i\theta})|_g \leq \ell_g(\gamma)/2, \quad |r^{-1} \partial_\theta f_\gamma(r e^{i\theta})|_g = |d_r \xi_\gamma(e^{i\theta})(\xi_\gamma'(\theta))|_g \leq C|\gamma'(\theta)|_g \]

for some \( C \in \mathbb{R}^+ \) determined by \( x_0 \). Thus,

\[ \left| \int_{B_1} f_\gamma^* \omega \right| \leq C \int_0^{2\pi} |\partial_r f_\gamma(r e^{i\theta})|_g |r^{-1} \partial_\theta f_\gamma(r e^{i\theta})|_g r \, dr \, d\theta \]

\[ \leq C' \ell_g(\gamma) \int_0^{2\pi} |\gamma'(\theta)|_g r \, dr \, d\theta = \frac{1}{2} C' \ell_g(\gamma)^2 \]

for some \( C, C' \in \mathbb{R}^+ \) determined by \( x_0 \) and \( \omega_0 \).

By Proposition 4.11 and the finiteness assumption on \( E(u; B_{\delta_0}^g) \), there exists \( \delta \in (0, R/2) \) such that

\[ |\gamma'_r(\theta)|_g^2 = |\partial_\theta u(\theta e^{i\theta})|_g^2 \leq \frac{32}{\pi} E(2r) \quad \forall \ r \in (0, \delta), \]

\[ \ell_g(\gamma_r)^2 = 128\pi E(2r) \quad \forall \ r \in (0, \delta). \]

By the continuity of \( u \), we can assume that \( u(B_{2\delta}) \subset B_{\delta_0}^g(x_0) \). For \( r \in (0, \delta) \) and \( \rho \in (0, r) \), define

\[ F_{\rho r}: S^2 \to X \]

to be the map obtained from \( u|_{B_{\rho} - B_{\rho}} \) by attaching disks to the boundary components \( \partial B_{\rho} \) and \( \partial B_{\rho} \) and letting \( F_{\rho r} \) be given by \( f_{\gamma_r} \) and \( f_{\gamma_r} \) on these two disks, respectively; see Figure 7. Since \( F_{\rho r} \) is homotopic to a constant map and \( \omega_0 \) is closed,

\[ 0 = \int_{S^2} F_{\rho r}^* \omega_0 = E_g(u; B_r - B_{\rho}) + \int_{B_1} f_{\gamma_r}^* \omega_0 - \int_{B_1} f_{\gamma_r}^* \omega_0 . \]
Combining this with (4.13) and (4.15), we obtain
\[ E_g(u; B_r - B_\rho) \leq C \ell_g(\gamma_r)^2 + CE(2\rho) \]  
(4.16)
for some \( C \in \mathbb{R}^+ \) independent of \( r \) and \( \rho \) as above. Since \( E_g(u; B_R^*) < 0, \ E(2\rho) \to 0 \) as \( \rho \to 0 \).

Taking the limit of (4.16) as \( \rho \to 0 \), we thus obtain (4.12).

**Corollary 4.10.** If \((X, J, g), R, \) and \( u \) are as in Proposition 4.8, there exist \( \delta \in (0, R) \) and \( C, \mu \in \mathbb{R}^+ \) such that
\[ |d_{re^\theta} u|_g \leq Cr^{-1} \quad \forall r \in (0, \delta). \]  
(4.17)

**Proof.** Let \( \gamma_r, \delta, C, \) and \( E(r) \) be as in the statement and proof of Lemma 4.9. Thus,
\[ E(r) \equiv \frac{1}{2} \int_0^{2\pi} \int_0^r |d_{re^\theta} u|^2_g d\rho d\theta \leq C \ell_g(\gamma_r)^2 = \frac{1}{2} Cr^2 \left( \int_0^{2\pi} |d_{re^\theta} u|^2_g d\theta \right)^2 \leq C \pi r^2 \int_0^{2\pi} |d_{re^\theta} u|^2_g d\theta = 2C \pi r E'(r) \quad \forall r \in (0, \delta). \]

This implies that
\[ \left( r^{-1/2C \pi E(r)} \right) \left[ \frac{d}{dr} \right] \left( r^{-1/2C \pi E(r)} \right) \geq 0, \quad E(r) \leq \delta^{-1/2C \pi E(\delta)} \cdot r^{1/2C \pi} \equiv C' r^{2\mu} \quad \forall r \in (0, \delta). \]

Combining this with (4.14), we obtain (4.17) with \( \delta \) replaced by \( \delta/2 \). \( \square \)

**Proof of Proposition 4.8.** With \( \mu \) as in Corollary 4.10 let \( p \in \mathbb{R}^+ \) be such that \( p > 2 \) and \( (1-\mu)p < 2 \). In particular,
\[ u|_{B_R/2} \in L_1^p(B_R/2; X), \quad \partial_J u|_{B_R/2} = 0 \in L^p(B_R/2; X). \]

By elliptic regularity, this implies that \( u \) is smooth; see [12, Theorem B.4.1]. By the continuity of \( \partial_J u, \) \( u \) is then \( J \)-holomorphic on all of \( B_R \). \( \square \)

### 4.3 Global structure of \( J \)-holomorphic maps

We next combine the local statement of Proposition 3.1 and some of its implications with the regularity statement of Proposition 4.8 to obtain a global description of \( J \)-holomorphic maps.

**Proposition 4.11.** Let \((X, J)\) be an almost complex manifold, \((\Sigma, i)\) be a compact Riemann surface, \( u : \Sigma \to X \) be a \( J \)-holomorphic map. If \( u \) is simple, then \( u \) is somewhere injective and all limit points of the set
\[ \{ z \in \Sigma : |u^{-1}(u(z))| > 1 \} \]  
(4.18)
are critical points of \( u \).

Suppose \((X, J)\) is an almost complex manifold, \((\Sigma, i)\) is a Riemann surface, and \( u : \Sigma \to X \) is a \( J \)-holomorphic map. Let
\[ \Sigma^*_u = \Sigma - u^{-1}(u(\{ z \in \Sigma : d_z u = 0 \})) \]  
(4.19)
be the preimage of the regular values of \( u \) and
\[ R^*_u \subset \Sigma^*_u \times \Sigma^*_u \]
be the subset of pairs \((z, z')\) such that there exists a diffeomorphism \(\varphi_{z'z} : U_z \to U_{z'}\) between neighborhoods of \(z\) and \(z'\) in \(\Sigma\) satisfying
\[
\varphi_{z'z}(z) = z' \quad \text{and} \quad u|_{U_z} = u \circ \varphi_{z'z}.
\]
(4.20)

Denote by \(R_u \subset \Sigma \times \Sigma\) the closure of \(R_u^*\).

It is immediate that \(R_u^*\) is an equivalence relation on \(\Sigma\) and \(u(z) = u(z')\) whenever \((z, z') \in R_u^*\).
Thus, \(R_u\) is also a reflexive and symmetric relation and \(u(z) = u(z')\) whenever \((z, z') \in R_u\). By Lemma 4.14 below, \(R_u\) is transitive as well. We denote this equivalence relation by \(\sim_u\). Let
\[
h_u : \Sigma \to \Sigma/\sim_u \quad \text{and} \quad u' : \Sigma' \to X
\]
be the quotient map and the continuous map induced by \(u\), respectively. In particular,
\[
u = u' \circ h_u : \Sigma \to X.
\]

In the case \(\Sigma\) is compact, we will show that \(\Sigma'\) inherits a Riemann surface structure \(j'\) from \(j\) so that the maps \(h_u\) and \(u'\) are \(j'-\) and \(J\)-holomorphic, respectively. If the degree of \(h\) is 1, we will show that all limit points of the set (4.18) are critical points of \(u\).

**Lemma 4.12.** Suppose \((X, J)\) is an almost complex manifold, \(R \in \mathbb{R}^+\), and \(u : B_R \to X\) is a non-constant \(J\)-holomorphic map such that \(d_z u \neq 0\) for all \(z \in B_R^J\). Then there exist \(m \in \mathbb{Z}^+\) and a neighborhood \(U_0\) of 0 in \(B_R\) such that
\[
h_u : U_0 \cap B_R^J \to h_u(U_0 \cap B_R^J) \subset B_R'
\]
is a covering projection of degree \(m\).

**Proof.** By the continuity of \(u\), we can assume that \(X = \mathbb{C}^n\), \(u(0) = 0\), and \(J_0 = J_{\mathbb{C}^n}\). As shown in the proof of Corollary 3.13, there exist \(\epsilon \in (0, R)\) and \(\delta \in (0, \epsilon/2)\) such that
\[
U_0 \equiv u^{-1}(u(B_\delta)) \cap B_\epsilon \subset B_{2\delta}.
\]
By Proposition 3.11 and the compactness of \(B_{2\delta} \subset B_R\), the number
\[
m(z) \equiv |h_u^{-1}(h_u(z)) \cap U_0|
\]
is finite for every \(z \in U_0 \cap B_R^J\).

Suppose \(z_i \in B_R^J\) and \(z'_i \in U_0\) are sequences such that \(z_i\) converges to some \(z_0 \in B_R^J\) with \(z_i \neq z_0\) for all \(i\) and \(h_u(z_i) = h_u(z'_i)\) for all \(i\). Passing to a subsequence, we can assume that \(z'_i\) converges to some \(z'_0 \in B_{2\delta}\). By the continuity of \(u\), \(u(z'_0) = u(z_0)\) and so \(z'_0 \in U_0\). Corollary 3.10 then implies that \(h_u(z'_0) = h_u(z_0)\). Since \(B_R^J\) is connected, this implies that the number \(m(z)\) is independent of \(z \in U_0 \cap B_R^J\); we denote it by \(m\).

Suppose \(z \in U_0 \cap B_R^J\) and
\[
h_u^{-1}(h_u(z)) \cap U_0 = \{z_1, \ldots, z_m\}.
\]
Let \(\varphi_i : U_1 \to U_i\) for \(i = 1, \ldots, m\) be diffeomorphisms between neighborhoods of \(z_1\) and \(z_i\) in \(U_0 \cap B_R^J\) such that
\[
\varphi_i(z_1) = z_i, \quad u = u \circ \varphi_i \quad \forall i, \quad U_i \cap U_j = \emptyset \quad \forall i \neq j.
\]

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and \( u: U_1 \rightarrow X \) is injective. Then \( h_u(U_1) \subset B'_R \) is an open neighborhood of \( h_u(z) \),

\[
h_u^{-1}(h_u(U_1)) \cap U_0 = \bigcup_{i=1}^{m} U_i,
\]

and \( h_u: U_i \rightarrow h_u(U_1) \) is a homeomorphism. Thus, (4.22) is a covering projection of degree \( m \). \( \square \)

**Lemma 4.13.** Suppose \((X,J), R, \) and \( u \) are as in Lemma 4.12. Then there exists a neighborhood \( U_0 \) of 0 in \( B_R \) such that

\[
\Psi_0: h_u(U_0) \rightarrow \mathbb{C}, \quad h_u(z) = \prod_{z' \in h_u^{-1}(h_u(z)) \cap U_0} z', \quad (4.23)
\]

is a homeomorphism from an open neighborhood of \( h_u(0) \) in \( B'_R \) to an open neighborhood of 0 in \( \mathbb{C} \) and \( \Psi_0 \circ h_u|_{U_0} \) is a holomorphic map.

**Proof.** By Lemma 4.12 there exists a neighborhood \( U_0 \) of 0 in \( B_R \) so that (4.22) is a covering projection of some degree \( m \in \mathbb{Z}^+ \). Since the restriction of \( u \) to \( B'_R \) is a \( J \)-holomorphic immersion, the diffeomorphisms \( \varphi_i \) as in the proof of Lemma 4.12 are holomorphic. Thus, the map

\[
\Psi_0 \circ h_u|_{U_0 \cap B'_R} : U_0 \cap B'_R \rightarrow \mathbb{C}, \quad z \mapsto \prod_{z' \in h_u^{-1}(h_u(z)) \cap U_0} z'
\]

is holomorphic. Since it is also bounded, it extends to a holomorphic map

\[
\overline{\Psi}_0: U_0 \rightarrow \mathbb{C}.
\]

This extension is non-constant and vanishes at 0.

After possibly shrinking \( U_0 \), we can assume that there exist \( k \in \mathbb{Z}^+ \) and \( C \in \mathbb{R}^+ \) such that

\[
C^{-k}|z|^k \leq |\overline{\Psi}_0(z)| \leq C^k|z|^k \quad \forall z \in U_0. \quad (4.24)
\]

Since \( \overline{\Psi}_0(z') = \overline{\Psi}_0(z) \) for all \( z' \in h_u^{-1}(h_u(z)) \cap U_0 \), it follows that

\[
C^{-2}|z| \leq |z'| \leq C^2|z| \quad \forall z' \in h_u^{-1}(h_u(z)) \cap U_0, \quad z \in U_0,
\]

\[
C^{-2m}|z|^m \leq |\overline{\Psi}_0(z)| \leq C^{2m}|z|^m \quad \forall z \in U_0.
\]

Along with (4.24), the last estimate implies that \( k = m \) and that \( \overline{\Psi}_0 \) has a zero of order precisely \( m \) at \( z = 0 \). Thus, shrinking \( \delta \) in the proof of Lemma 4.12 if necessary, we can assume that \( \Phi_0 \) is \( m:1 \) over \( \overline{U}_0 \cap B'_R \). This implies that the map (4.23) and its extension over the closure of \( h_u(U_0) \) in \( B'_R \) are continuous and injective. Since the closure of \( h_u(U_0) \) is compact and \( \mathbb{C} \) is Hausdorff, we conclude that (4.23) is a homeomorphism onto an open subset of \( \mathbb{C} \). \( \square \)

**Lemma 4.14.** Suppose \((X,J), (\Sigma, i), \) and \( u \) are as in Proposition 4.11 and \((x,y) \in \mathcal{R}_u \). For every neighborhood \( U_x \) of \( x \) in \( \Sigma \), the image of the projection

\[
\mathcal{R}_u \cap (U_x \times X) \rightarrow X
\]

to the second component contains a neighborhood \( U_y \) of \( y \) in \( \Sigma \).
Proof. By Corollary 3.4, the last set in (4.19) is finite. By the same reasoning as in the last part of the proof of Lemma 4.12,

\[ h_u : \Sigma^*_u \rightarrow h_u(\Sigma^*_u) \subset \Sigma' \]  

is a local homeomorphism. Since \( u(z) = u(z') \) for all \((z, z') \in \mathcal{R}_u^*\), the definition of \( \Sigma^*_u \) thus implies that (4.25) is a finite-degree covering projection over each topological component of \( h_u(\Sigma^*_u) \). Since the complement of finitely many points in a connected Riemann surface is connected, the degree of this covering over a point \( h_u(z) \) depends only on the topological component of \( \Sigma \) containing \( z \). For any point \( z \in \Sigma \), not necessarily in \( \Sigma^*_u \), we denote this degree by \( d(z) \).

By Corollary 3.4 the set

\[ S \equiv u^{-1}(u(x)) \subset \Sigma \]

is finite. Let \( W \subset X \) be a neighborhood of \( u(x) \) such that the topological components \( \Sigma_s \) of \( u^{-1}(W) \) containing the points \( s \in S \) are pairwise disjoint (if \( U \) is a union of disjoint balls around the points of \( S \), then

\[ W \equiv X - u(\Sigma - U) \]

works). By Lemma 4.12 for each \( s \in S \) there exists a neighborhood \( U'_s \) of \( s \) in \( \Sigma_s \) such that

\[ h_u : U'_s - \{s\} \rightarrow h_u(U'_s - \{s\}) \subset \Sigma' \]

is a covering projection of some degree \( m_s \in \mathbb{Z}^+ \); we can assume that \( U'_s \subset U_x \). Along with the compactness of \( \Sigma \), the former implies that

\[
|h_u^{-1}(u(y')) \cap U'_s| \in \{0, m_s\} \quad \forall y' \in U'_s \cap \Sigma^*_u, \quad s, s' \in S,
\]

\[
\sum_{s \in S} |h_u^{-1}(u(y')) \cap U'_s| = d(s') \quad \forall y' \in U'_s \cap \Sigma^*_u, \quad s' \in S. \tag{4.26}
\]

Define

\[ \mathcal{P}_y(S) = \{ S' \subset S : \sum_{s \in S'} m_s = d(y) \}. \]

Let \( U''_y \subset U'_y \) be a connected neighborhood of \( y \). For each \( S' \in \mathcal{P}_y(S) \), define

\[ U''_{y, S'} = \{ y' \in U''_y \cap \Sigma^*_u : \{ s \in S : h_u^{-1}(u(y')) \cap U'_s \neq \emptyset \} = S' \}. \]

By (4.26), these sets partition \( U''_y \cap \Sigma^*_u \). Since each set

\[ \{ y' \in U''_y \cap \Sigma^*_u : h_u^{-1}(u(y')) \cap U'_s \neq \emptyset \} \]

is open, (4.26) also implies that each set \( U''_{y, S'} \) is open. Since the set \( U''_y \cap \Sigma^*_u \) is connected, it follows that \( U''_y = U''_{y, S_y} \) for some \( S_y \in \mathcal{P}_y(S) \). Since \((x, y) \in \mathcal{R}_u, \ x \in S_y \). Thus, the image of the projection

\[ \mathcal{R}_u \cap (U'_x \times X) \rightarrow X \]

to the second component contains the neighborhood \( U''_y \) of \( y \) in \( \Sigma \).

\[ \square \]

Corollary 4.15. Suppose \((X, J), (\Sigma, j)\), and \( u \) are as in Proposition 4.11. The quotient map \( h_u \) in (4.21) is open and closed.
Proof. The openness of $h_u$ is immediate from Lemma 4.14. Suppose $A \subset \Sigma$ is a closed subset and $y_i \in h_u^{-1}(h_u(A))$ is a sequence converging to some $y \in \Sigma$. Let $x_i \in A$ be such that $h_u(x_i) = h_u(y_i)$. Passing to a subsequence, we can assume that the sequence $x_i$ converges to some $x \in A$. Since $\Sigma - \Sigma^*_u$ consists of isolated points, we can also assume that $y_i \in \Sigma^*_u$ and so $(x_i, y_i) \in R^*_u$. Thus, $(x, y) \in R_u$ and so $y \in h_u^{-1}(h_u(A))$. We conclude that $h_u$ is a closed map. \hfill \Box

Proof of Proposition 4.11. Let $\Sigma'$, $h_u$, and $u'$ be as in (4.21). By the second statement in Corollary 4.15 and [13, Lemma 73.3], $\Sigma'$ is a Hausdorff topological space. Fix a Riemannian metric $g$ on $X$. For $(z, z') \in R_u^*$ with $z \neq z'$, the neighborhoods $U_z$ and $U_{z'}$ as in (4.20) can be chosen so that they are disjoint and $u|_{U_z}$ is an embedding. Since $u$ is $J$-holomorphic, $\varphi_{z'z}$ is then a biholomorphic map and $h_u|_{U_z}$ is a homeomorphism onto $h_u(U_z) \subset \Sigma'$. Thus, the Riemann surface structure $j$ on $\Sigma$ determines a Riemann surface structure $j'$ on $h_u(\Sigma_u^*)$ so that $h_u|_{\Sigma_u^*}$ is a holomorphic covering projection of $h_u(\Sigma_u^*)$ and $u'|_{h_u(\Sigma_u^*)}$ is a $J$-holomorphic map with

$$E_g(u'; h_u(\Sigma_u^*)) \leq E_g(u). \quad (4.27)$$

By Corollary 4.1, $\Sigma_u^* - h_u(\Sigma_u^*)$ consists of finitely many points. By the first statement in Corollary 4.1 and by Lemma 4.13, $j'$ extends over these points to a Riemann surface structure on $\Sigma'$; we denote the extension also by $j'$. Since the continuous map $h_u$ is $j'$-holomorphic outside of the finitely many points of $\Sigma - \Sigma^*_u$, it is holomorphic everywhere. Since the continuous map $u'$ is $J$-holomorphic on $h_u(\Sigma_u^*)$, (4.27) and Proposition 4.10 imply that it is $J$-holomorphic everywhere.

Suppose $z \in \Sigma$ and $z_i, z_i' \in \Sigma$ with $i \in \mathbb{Z}^+$ are such that

$$d_z u \neq 0, \quad z_i \neq z_i', \quad u(z_i) = u(z_i') \quad \forall i, \quad \lim_{i \to \infty} z_i = z.$$

Passing to a subsequence, we can assume that the sequence $z_i'$ converges to some point $z' \in \Sigma$ with $u(z') = u(z)$. Since the restriction of $u$ to a neighborhood of $z$ is an embedding, $z' \neq z$. By Corollary 3.10 there exists a diffeomorphism $\varphi_{z'z}$ as in (4.20). Thus, $h_u(z) = h_u(z')$, the map $h_u$ is not injective, and $u$ is not simple. \hfill \Box

4.4 Energy bound on long cylinders

Proposition 4.16 and Corollary 4.17 below concern $J$-holomorphic maps from long cylinders. Their substance is that most of the energy and variation of such maps are concentrated near the ends. These technical statements are used to obtain important geometric conclusions in Sections 5.2 and 5.3.

Proposition 4.16. If $(X, J)$ is an almost complex manifold and $g$ is a Riemannian metric on $X$, then there exist continuous functions $\delta_{J,g}, h_{J,g}, C_{J,g} : X \to \mathbb{R}^+$ with the following properties. If $u : [-R, R] \times S^1 \to X$ is a $J$-holomorphic map such that $\text{Im} u \subset B^3_{\delta_{J,g}(u(0,1))}(u(0,1))$, then

$$E_g(u([-R+T, R-T] \times S^1)) \leq C_{J,g}(u(1,0))e^{-T}E_g(u) \quad \forall \ T \geq 0. \quad (4.28)$$

If in addition $E_g(u) < h_{J,g}(u(0,1))$, then

$$\text{diam}_g(u([-R+T, R-T] \times S^1)) \leq C_{J,g}(u(1,0))e^{-T/2}\sqrt{E_g(u)} \quad \forall \ T \geq 1. \quad (4.29)$$
Corollary 4.17. If \((X, J)\) is a compact almost complex manifold and \(g\) is a Riemannian metric on \(X\), there exist \(h_{J,g}, C_{J,g} \in \mathbb{R}^+\) with the following property. If \(u : [-R, R] \times S^1 \rightarrow X\) is a \(J\)-holomorphic map such that \(E_g(u) < h_{J,g}\), then
\[
E_g(u; [-R+T, R-T] \times S^1) \leq C_{J,g}e^{-T}E_g(u) \quad \forall \, T \geq 1,
\]
\[
diam_g(u([-R+T, R-T] \times S^1)) \leq C_{J,g}e^{-T/2}E_g(u) \quad \forall \, T \geq 2.
\]
As an example, the energy of the injective map
\[
[-R, R] \times S^1 \rightarrow \mathbb{C}, \quad (s, \theta) \mapsto se^{i\theta},
\]
is the area of its image, i.e. \(\pi(e^{2R} - e^{-2R})\). Thus, the exponent \(e^{-T}\) in (4.28) can be replaced by \(e^{-2T}\) in this case. The proof of Proposition 4.16 shows that in general the exponent can be taken to be \(e^{-\mu T}\) with \(\mu\) arbitrarily close to 2, but at the cost of increasing \(C_{J,g}\) and reducing \(\delta_{J,g}\).

Lemma 4.18 (Poincare Inequality). If \(f : S^1 \rightarrow \mathbb{R}^n\) is a smooth function such that \(\int_0^{2\pi} f(\theta) d\theta = 0\), then
\[
\int_0^{2\pi} |f(\theta)|^2 d\theta \leq \int_0^{2\pi} |f'(\theta)|^2 d\theta.
\]
Proof: We can write \(f(\theta) = \sum_{k=\pm \infty} a_k e^{i k \theta}\). Since \(\int_0^{2\pi} f(\theta) d\theta = 0, a_0 = 0\). Thus,
\[
\int_0^{2\pi} |f(\theta)|^2 d\theta = \sum_{k=\pm \infty} |a_k|^2 \leq \sum_{k=\pm \infty} |k a_k|^2 = \int_0^{2\pi} |f'(\theta)|^2 d\theta.
\]

Proof of Proposition 4.16. It is sufficient to establish the first statement under the assumption that \((X, g)\) is \(\mathbb{C}^n\) with the standard Riemannian metric, \(J\) agrees with the standard complex structure \(J_{\mathbb{C}^n}\) at \(0 \in \mathbb{C}^n\), and \(u(0, 1) = 0\). Let
\[
\bar{\partial} u = \frac{1}{2} (u_t + J_{\mathbb{C}^n} u_\theta).
\]
By our assumptions, there exist \(\delta', C > 0\) (dependent on \(u(0, 1)\)) such that
\[
|\bar{\partial} z u| \leq C \delta |dz u| \quad \forall \, z \in u^{-1}(B_{\delta}(0)), \, \delta \leq \delta'. \tag{4.30}
\]
Write \(u = f + i g\), with \(f, g\) taking values in \(\mathbb{R}^n\) and assume that \(\text{Im} u \subset B_{\delta}(0)\). By (2.4),
\[
|du|^2 = 4|\bar{\partial} u|^2 + 2d(f \cdot dg).
\]
Combining this with (4.30) and Stokes’ Theorem, we obtain
\[
\int_{[-t, t] \times S^1} |du|^2 \leq 4C^2 \delta^2 \int_{[-t, t] \times S^1} |dz u|^2 + 2 \int_{(t) \times S^1} f \cdot g_\theta \, d\theta - 2 \int_{(-t) \times S^1} f \cdot g_\theta \, d\theta. \tag{4.31}
\]
Let \(\tilde{f} = f - \frac{1}{2\pi} \int_0^{2\pi} f \, d\theta\). By Hölder’s inequality and Lemma 4.18,
\[
\left| \int_{\{\pm t\} \times S^1} f \cdot g_\theta \, d\theta \right| = \int_{\{\pm t\} \times S^1} \bar{\tilde{f}} g_\theta \, d\theta \leq \left( \int_{\{\pm t\} \times S^1} |\bar{\tilde{f}}|^2 d\theta \right)^{\frac{1}{2}} \left( \int_{\{\pm t\} \times S^1} |g_\theta|^2 d\theta \right)^{\frac{1}{2}} \leq \frac{1}{2} \int_{\{\pm t\} \times S^1} |u_\theta|^2 d\theta. \tag{4.32}
\]
Since
\[3|u_\theta|^2 = 2|u_\theta|^2 + |u_t - 2\bar{\partial}u|^2 \leq 2|du|^2 + 8|\bar{\partial}u|^2,\]
the inequalities \([4.30], [4.32]\) give
\[
(1-4C^2\delta^2)\int_{[-t,t] \times S^1} |du|^2 \leq \frac{2}{3} (1+4C^2\delta^2) \left( \int_{\{t\} \times S^1} |du|^2 + \int_{\{-t\} \times S^1} |du|^2 \right).
\]
Thus, the function
\[\varepsilon(T) \equiv E_g(u; [-R+T, R-T]) \equiv \frac{1}{2} \int_{[-R+T, R-T] \times S^1} |du|^2 d\theta dt\]
satisfies \(\varepsilon(T) \leq -\varepsilon'(T)\) for all \(T \in [-R, R]\), if \(\delta\) is sufficiently small (depending on \(C\)). This implies \([4.28]\).

Let \(h_{J,g}(x) = (x, \delta_{J,g}(x))\), with \(h_{J,g}(\cdot, \cdot)\) as in Proposition \([4.1]\) and \(\delta_{J,g}(\cdot)\) as provided by the previous paragraph. Suppose \(u\) also satisfies the last condition in Proposition \([4.16]\). By Proposition \([4.1]\) and \([4.28]\),
\[|d_{(t,\theta)}u| \leq 3\sqrt{E_g(u; [-|t|-1, |t|+1] \times S^1)} \leq 3\sqrt{C_{J,g}(u(0,1))e^{(1+|t|-R)/2}E_g(u)}\]
for all \(t \in [-R+1, R-1]\) and \(\theta \in S^1\). Thus, for all \(t_1, t_2 \in [-R+T, R-T]\) with \(T \geq 1\) and \(\theta_1, \theta_2 \in S^1\),
\[d_g(u(t_1, \theta_1), u(t_2, \theta_2)) \leq 3\sqrt{C_{J,g}(u(0,1))E_g(u)} \left( \pi e^{(1+|t|-R)/2} + \int_{t_1}^{t_2} e^{(1+|t|-R)/2} dt \right) \leq (3\pi+12)\sqrt{C_{J,g}(u(0,1))e^{(1-T)/2}E_g(u)}.
\]
This establishes \([4.29]\). \(\square\)

**Lemma 4.19.** If \((X, J)\) is a compact almost complex manifold and \(g\) is a Riemannian metric on \(X\), there exists a continuous function \(\epsilon_{J,g}: \mathbb{R}^+ \rightarrow \mathbb{R}^+\) with the following property. If \(\delta \in \mathbb{R}^+\) and \(u: (-R, R) \times S^1 \rightarrow X\) is a \(J\)-holomorphic map with \(E_g(u) < \epsilon_{J,g}(\delta)\), then
\[\text{diam}_{g}(u([-R+1, R-1] \times S^1)) \leq \delta.\]

**Proof.** By Proposition \([3.12]\) and the compactness of \(X\), there exists \(c_{J,g} \in \mathbb{R}^+\) with the following property. If \((\Sigma, J)\) is a compact connected Riemann surface with boundary, \(u: \Sigma \rightarrow X\) is a non-constant \(J\)-holomorphic map, \(x \in X\), and \(\delta \in \mathbb{R}^+\) are such that \(u(\partial \Sigma) \cap B^g_\delta(x) = \emptyset\), then
\[E_g(u) \geq c_{J,g}\delta^2.\] (4.33)

Let \(h_{J,g} > 0\) be the minimal value of the function \(h_{J,g}\) in the statement of Proposition \([4.1]\) on the compact space \(X \times [0, \text{diam}_{g}(X)]\).

Suppose \(u: (-R, R) \times S^1 \rightarrow X\) is a \(J\)-holomorphic map with \(E_g(u) < h_{J,g}\) and
\[\delta_u \equiv \text{diam}_{g}(u([-R+1, R-1] \times S^1)) > 32\sqrt{E_g(u)}.
\]
By the first condition on $u$,

$$\left|d_z u\right|_g^2 \leq \frac{16}{\pi} E_g(u) \quad \forall \ z \in [-R+1, R-1] \times S^1,$$

$$\text{diam}_g(u(r \times S^1)) \leq 8\sqrt{E_g(u)} \quad \forall \ r \in [-R+1, R-1]. \quad (4.34)$$

Let $r_-, r_0, r_+ \in [-R+1, R-1]$ and $\theta_-, \theta_0, \theta_+ \in S^1$ be such that

$$r_- < r_0 < r_+, \quad d_g(u(r_0, \theta_0), u(r_\pm, \theta_\pm)) \geq \frac{1}{2}\delta_u.$$

By (4.34), we can apply (4.33) with

$$\Sigma = [r_-, r_+] \times S^1, \quad x = u(r_0, \theta_0), \quad \delta = \frac{1}{4}\delta_u,$$

and $u$ replaced by its restriction to $\Sigma$. We conclude that

$$E_g(u) \geq \frac{c_{J,g}}{16} \delta_u^2.$$

It follows that the function

$$\epsilon_{J,g}: \mathbb{R}^+ \rightarrow \mathbb{R}^+, \quad \epsilon_{J,g}(\delta) = \min\left(h_{J,g}, \frac{\delta^2}{32^2}, \frac{c_{J,g}}{16} \delta^2\right),$$

has the desired property. \qed

**Proof of Corollary 4.17.** Let $\delta \in \mathbb{R}^+$ be the minimum of the function $\delta_{J,g}$ in Proposition 4.16 and $\epsilon_{J,g}(\cdot)$ be as in Lemma 4.19. Take $C_{J,g}$ to be the maximum of the function $C_{J,g}$ in Proposition 4.16 times $e$ and $h_{J,g} \in \mathbb{R}^+$ to be smaller than the minimum of the function $h_{J,g}$ in Proposition 4.16 and the number $\epsilon_{J,g}(\delta)$. \qed

# 5 Limiting behavior of $J$-holomorphic maps

This section studies the limiting behavior of sequences of $J$-holomorphic maps from Riemann surfaces into a compact almost complex manifold $(X, J)$. The compactness of $X$ plays an essential role in the statements below, in contrast to nearly all statements in Sections 3 and 4.

## 5.1 Removal of Singularity

By Cauchy’s Integral Formula, a bounded holomorphic map $u: B^*_R \rightarrow \mathbb{C}^n$ extends over the origin. By Proposition 5.1 below, the same is the case for a $J$-holomorphic map $u: B^*_R \rightarrow X$ of bounded energy if $X$ is compact.

**Proposition 5.1** (Removal of Singularity). Let $(X, J)$ be a compact almost complex manifold and $u: B^*_R \rightarrow X$ be a $J$-holomorphic map. If the energy $E_g(u)$ of $u$, with respect to any metric $g$ on $X$, is finite, then $u$ extends to a $J$-holomorphic map $\tilde{u}: B_R \rightarrow X$.

A basic example of a holomorphic function $u: \mathbb{C}^* \rightarrow \mathbb{C}$ that does not extend over the origin $0 \in \mathbb{C}$ is $z \mapsto 1/z$. The energy of $u|_{B^*_R}$ with respect to the standard metric on $\mathbb{C}$ is given by

$$E(u; B^*_R) = \frac{1}{2} \int_{B_R} |du|^2 = \int_{B_R} \frac{1}{|z|^2} = \int_0^{2\pi} \int_0^R r^{-1} dr d\theta < \infty.$$
The above integral would have been finite if $|du|^2$ were replaced by $|du|^{2-\epsilon}$ for any $\epsilon > 0$. This observation illustrates the crucial role played by the energy in the theory of $J$-holomorphic maps.

By Cauchy’s Integral Formula, the conclusion of Proposition 5.1 holds if $J$ is an integrable almost complex structure and $u(B_\delta^*)$ is contained in a complex coordinate chart for some $\delta \in (0, R)$. We will use the Monotonicity Lemma to show that the latter is the case if the energy of $u$ is finite; the integrability of $J$ turns out to be irrelevant here.

**Proof of Proposition 5.1.** In light of Proposition 4.8, it is sufficient to show that $u$ extends continuously over the origin. Let $c_{J,g}, h_{J,g} \in \mathbb{R}^+$ be as in the proof of Lemma 4.19. We can assume that $R = 1$ and $u$ is non-constant. Define

$$v: \mathbb{R}^- \times \mathbb{S}^1 \to X, \quad v(r, e^{i\theta}) = u(e^{r+i\theta}).$$

This map is $J$-holomorphic and satisfies $E_g(v) = E_g(u) < \infty$.

Since $E_g(u) < \infty$,

$$\lim_{r \to -\infty} E_g(v; (-\infty, r) \times \mathbb{S}^1) = \lim_{r \to -\infty} E_g(u; B_c^*) = 0. \quad (5.1)$$

In particular, there exists $R \in \mathbb{R}^-$ such that

$$E_g(v; (-\infty, r) \times \mathbb{S}^1) < h_{J,g} \quad \forall r < R.$$

By Proposition 4.11 and our choice of $h_{J,g}$, this implies that

$$|dzv|_g^2 \leq \frac{16}{\pi} E_g(v; (-\infty, r+1) \times \mathbb{S}^1) \quad \forall z \in (-\infty, r) \times \mathbb{S}^1, \quad r < R - 1,$$

$$\text{diam}_g(v(\{r\} \times \mathbb{S}^1)) \leq 4\sqrt{\pi} E_g(v; (-\infty, r+1) \times \mathbb{S}^1) \quad \forall r < R - 1.$$

Combining the last bound with (5.1), we obtain

$$\lim_{r \to -\infty} \text{diam}_g(v(\{r\} \times \mathbb{S}^1)) = 0.$$

Thus, it remains to show that $\lim_{r \to -\infty} v(r, 1)$ exists.

Since $X$ is compact, every sequence in $X$ has a convergent subsequence. Suppose there exist

$$\delta \in \mathbb{R}^+, \quad x, y \in X, \quad i_k, r_k \in \mathbb{R}^- \quad \text{s.t.} \quad d_g(x, y) > 3\delta, \quad r_{k+1} < i_k < r_k,$$

$$v(\{i_k\} \times \mathbb{S}^1) \subset B_\delta(x), \quad v(\{r_k\} \times \mathbb{S}^1) \subset B_\delta(y).$$

We thus can apply (4.33) with $\Sigma$, $x$, and $u$ replaced by

$$\Sigma_k \equiv [r_{k+1}, r_k] \times \mathbb{S}^1, \quad x_k \equiv u(i_k, 1), \quad \text{and} \quad v_k \equiv v|_{\Sigma_k},$$

respectively. We conclude that

$$E_g(v) \geq \sum_k E_g(v; \Sigma_k) = \sum_k E_g(v_k) \geq \sum_k c_{J,g} \delta^2 = \infty.$$

However, this contradicts the assumption that $E_g(v) = E_g(u) < \infty$. \qed
The limit exists and is nonzero.

For every sequence \( B \) compact subsets of \( X \), the resulting bubbles connect, and their number is finite.

Lemma 5.2. Suppose \((X, J)\) is an almost complex manifold with a Riemannian metric \( g \) and \( u_i : B_1 \rightarrow X \) is a sequence of \( J \)-holomorphic maps converging uniformly in the \( C^\infty \)-topology on compact subsets of \( B_1^* \) to a \( J \)-holomorphic map \( u : B_1 \rightarrow X \) such that the limit

\[
m(\delta) \equiv \lim_{\delta \rightarrow 0} \lim_{i \rightarrow \infty} E_g(u_i; B_\delta)
\]

exists and is nonzero.

1. The limit \( m(\delta) \) exists and is a continuous, non-decreasing function of \( \delta \).
2. For every sequence \( z_i \in B_1 \) converging to 0, \( \lim_{i \rightarrow \infty} E_g(u_i; B_\delta(z_i)) = m(\delta) \).
3. For every sequence \( z_i \in B_1 \) converging to 0, \( \mu \in (0, m) \), and \( i \in \mathbb{Z}^+ \) sufficiently large, there exists a unique \( \delta_i(\mu) \in (0, 1 - |z_i|) \) such that \( E_g(u_i; B_{\delta_i(\mu)}(z_i)) = \mu \). Furthermore,

\[
\lim_{\delta \rightarrow 0} \lim_{i \rightarrow \infty} E_g(u_i; B_\delta(z_i) - B_{\delta_i(\mu)}(z_i)) = m - \mu.
\]

Proof. (1) Since \( du_i \) converges uniformly to \( du \) on compact subsets of \( B_1^* \),

\[
m(\delta) \equiv \lim_{i \rightarrow \infty} E_g(u_i; B_\delta) = \lim_{\delta' \rightarrow 0} \lim_{i \rightarrow \infty} E_g(u_i; B_{\delta'}) + \lim_{\delta' \rightarrow 0} \lim_{i \rightarrow \infty} E_g(u_i; B_\delta - B_{\delta'}) = m + \lim_{\delta' \rightarrow 0} E_g(u; B_\delta - B_{\delta'}) = m + E_g(u; B_\delta).
\]

Since \( E_g(u; B_\delta) \) is a continuous, non-decreasing function of \( \delta \), so is \( m(\delta) \).

(2) For all \( \delta, \delta' \in \mathbb{R}^+ \) and \( z_i \in B_{\delta'}, B_{\delta - \delta'} \subset B_\delta(z_i) \subset B_{\delta + \delta'} \). Thus,

\[
E_g(u_i; B_{\delta - \delta'}) \leq E_g(u_i; B_\delta(z_i)) \leq E_g(u_i; B_{\delta + \delta'})
\]

for all \( i \in \mathbb{Z}^+ \) sufficiently large and

\[
\lim_{\delta' \rightarrow 0} m(\delta - \delta') \leq \lim_{\delta' \rightarrow 0} \lim_{i \rightarrow \infty} E_g(u_i; B_\delta(z_i)) \leq \lim_{\delta' \rightarrow 0} m(\delta + \delta') \quad \forall \delta' \in \mathbb{R}^+.
\]

The claim now follows from [1].
By (2) (1) and (5.2),
\[
\lim_{i \to \infty} E_g(u_i; B_{\delta}(z_i)) = m(\delta) \geq m.
\]
Thus, there exists \( i(\mu) \in \mathbb{Z}^+ \) such that
\[
E_g(u_i; B_{\delta}(z_i)) > \mu \quad \forall i \geq i(\mu).
\]
Since \( E_g(u_i; B_{\delta}(z_i)) \) is a continuous, increasing function of \( \delta \) which vanishes at \( \delta = 0 \), for every \( i \geq i(\mu) \) there exists a unique \( \delta_i(\mu) \in (0, \delta) \) such that \( E_g(u_i; B_{\delta_i(\mu)}(z_i)) = \mu \).

By (2) (1) and (5.2),
\[
\lim_{R \to \infty} \lim_{\delta \to 0} \lim_{i \to \infty} E_g(u_i; B_{R\delta}(z_i)) = \lim_{R \to \infty} \lim_{\delta \to 0} m(R\delta) = \lim_{R \to \infty} m = m.
\]
Combining this with the definition of \( \delta_i(\mu) \), we obtain (5.3).

**Corollary 5.3.** If \((X, J)\) is a compact almost complex manifold with a Riemannian metric \( g \), then there exists \( h_{J,g} \in \mathbb{R}^+ \) with the following properties. If \( u_i : B_1 \to X \) is a sequence of \( J \)-holomorphic maps converging uniformly in the \( C^\infty \)-topology on compact subsets of \( B_1^* \) to a \( J \)-holomorphic map \( u : B_1 \to X \) such that
\[
\lim_{i \to \infty} \max_{B_{1/2}} |du_i|_g = \infty
\]
and the limit (5.2) exists, then

(1) \( m \geq h_{J,g} \);

(2) for every sequence \( z_i \in B_\delta \) converging to 0 and \( \mu \in (m - h_{J,g}, m) \), the numbers \( \delta_i(\mu) \in (0, 1 - |z_i|) \) of Lemma 5.2/3 satisfy
\[
\lim_{R \to \infty} \lim_{\delta \to 0} \lim_{i \to \infty} E_g(u_i; B_{R\delta_i(\mu)}(z_i)) = m, \tag{5.4}
\]
\[
\lim_{R \to \infty} \lim_{\delta \to 0} \lim_{i \to \infty} \text{diam}_g(u_i(B_\delta(z_i) - B_{R\delta_i(\mu)}(z_i))) = 0. \tag{5.5}
\]

**Proof.** Let \( h_{J,g} \) be the smaller of the constants \( h_{J,g} \) in Corollaries 4.2 and 4.17. Let \( u_i, u, \) and \( m \) be as in the statement of the corollary.

(1) For each \( i \in \mathbb{Z}^+ \), let
\[
M_i = \max_{B_{1/2}} |dz_i|_g \in \mathbb{R}^+
\]
and \( z_i \in \overline{B_{1/2}} \) be such that \( |dz_i|_g = M_i \). Since \( M_i \to \infty \) as \( i \to \infty \) and \( u_i \) converges uniformly in the \( C^\infty \)-topology on compact subsets of \( B_1^* \) to \( u \), \( z_i \to 0 \). For \( i \in \mathbb{Z}^+ \) such that \( |z_i| + 1/\sqrt{M_i} < 1/2 \), define
\[
v_i : B_{\sqrt{M_i}} \to X, \quad v_i(z) = u_i(z + z/M_i).
\]
Thus, \( v_i \) is a \( J \)-holomorphic map with
\[
|dv_i|_g = |dz_i|_g = 1, \quad E_g(v_i) = E_g(u_i; B_1/\sqrt{M_i}(z_i)) \leq E_g(u_i; B_{z_i+1/\sqrt{M_i}}). \tag{5.6}
\]
By the first statement in (5.6) and the ellipticity of the \( \bar{\partial} \)-operator, a subsequence of \( v_i \) converges uniformly in the \( C^\infty \)-topology on compact subsets of \( \mathbb{C} \) to a non-constant \( J \)-holomorphic map \( v: \mathbb{C} \to X \). By the second statement in (5.6) and Lemma 5.3(1)

\[
E_g(v) \leq \limsup_{i \to \infty} E_g(u_i; B_1/\sqrt{\pi \ell}(z_i)) \leq \lim_{\delta \to 0} \lim_{i \to \infty} E_g(u_i; B_{\delta}) = m. \tag{5.7}
\]

By Proposition 5.1, \( v \) thus extends to a \( J \)-holomorphic map \( \tilde{v}: \mathbb{P}^1 \to X \). By Corollary 4.2

\[
E_g(v) = E_g(\tilde{v}) \geq h_{J,g}.
\]

Combining this with (5.7), we obtain the first claim.

By the first two statements in Lemma 5.2 and (5.2),

\[
\lim_{\delta \to 0} \lim_{i \to \infty} E_g(u_i; B_\delta(z_i)) = \lim_{\delta \to 0} m(\delta) = m. \tag{5.8}
\]

After passing to a subsequence of \( u_i \), we can thus assume that there exists a sequence \( \delta_i \to 0 \) such that

\[
\lim_{i \to \infty} E_g(u_i; B_{\delta_i}(z_i)) = m. \tag{5.9}
\]

Since \( \delta_i \to 0 \), (5.8) and (5.9) imply that

\[
\lim_{R \to \infty} \lim_{i \to \infty} E_g(u_i; B_{R\delta_i}(z_i)) = m. \tag{5.10}
\]

Suppose \( \mu \in (m-h_{J,g}, m) \). By (5.10) and the definition of \( \delta_i(\mu) \),

\[
\lim_{R \to \infty} \lim_{i \to \infty} E_g(u; B_{R\delta_i}(z_i)-B_{\delta_i}(z_i)) = m-\mu < h_{J,g}.
\]

Thus, Corollary 4.17 applies with \((R,T)\) replaced by \((\frac{1}{2} \ln(R\delta_i/\delta_i(\mu)), \ln R)\) and \( u \) replaced by the \( J \)-holomorphic map

\[
v: (-R, R) \times S^1 \to X, \quad v(r, e^{i\theta}) = u(z_i + \sqrt{R\delta_i(\mu)} e^{r+iy}).
\]

By the first statement of Corollary 4.17

\[
E_g(u; B_{\delta_i}(z_i)) - E_g(u; B_{R\delta_i}(z_i)) = E_g(u; B_{\delta_i}(z_i) - B_{R\delta_i}(z_i)) \leq \frac{C_{J,g}}{R} E_g(u)
\]

for all \( i \) sufficiently large (depending on \( R \)); see Figure 9. Combining this with (5.9), we obtain (5.4).

It remains to establish (5.5). By (5.3), for all \( R > 0 \) and sufficiently small \( \delta > 0 \) (depending on \( R \)) there exists \( i(R, \delta) \in \mathbb{Z}^+ \) such that

\[
E_g(u_i; B_{R\delta}(z_i) - B_{\delta_i}(z_i)) < h_{J,g} \quad \forall \ i > i(R, \delta).
\]

Thus, Corollary 4.17 applies with \((R,T)\) replaced by \((\frac{1}{2} \ln(R\delta/\delta_i(\mu)), \ln R)\) and \( u \) replaced by the \( J \)-holomorphic map

\[
v: (-R, R) \times S^1 \to X, \quad v(r, e^{i\theta}) = u(z_i + \sqrt{R\delta_i(\mu)} e^{r+iy}).
\]

By the second statement of Corollary 4.17

\[
\text{diam}_g(u_i(B_{\delta}(z_i) - B_{R\delta_i}(z_i))) \leq \frac{C_{J,g}}{\sqrt{R}} h_{J,g} \quad \forall \ i > i(R, \delta).
\]

This gives (5.5).
Lemma 5.4. If \((X,J)\) is a compact almost complex manifold with a Riemannian metric \(g\), then there exists a function \(N: \mathbb{R} \rightarrow \mathbb{Z}\) with the following property. If \((\Sigma,i)\) is compact Riemann surface, \(S_0 \subset \Sigma\) is a finite subset, and \(u_i: U_i \rightarrow X\) is a sequence of \(J\)-holomorphic maps from open subsets of \(\Sigma\) with

\[
U_i \subset U_{i+1}, \quad \Sigma - S_0 = \bigcup_{i=1}^{\infty} U_i, \quad \text{and} \quad E \equiv \liminf_{i \to \infty} E_g(u_i) < \infty, \tag{5.11}
\]

then there exist a subset \(S \subset \Sigma\) with \(|S| \leq N(E) + |S_0|\) and a subsequence of \(u_i\) converging uniformly in the \(C^\infty\)-topology on compact subsets of \(\Sigma - S\) to a \(J\)-holomorphic map \(u: \Sigma \rightarrow X\).

Proof. Let \(h_{J,g}\) be the minimal value of the function provided by Proposition 4.1. For \(E \in \mathbb{R}^+\), let \(N(E) \in \mathbb{Z}^{\geq 0}\) be the smallest integer such that \(E \leq N(E) h_{J,g}\).

Let \(\Sigma, S_0, u_i, \) and \(E\) be as in the statement of the lemma and \(N = N(E) + |S_0|\). Fix a Riemannian metric \(g_{\Sigma}\) on \(\Sigma\). For \(z \in \Sigma\) and \(\delta \in \Sigma\), let \(B_\delta(z) \subset \Sigma\) denote the ball of radius \(\delta\) around \(z\). By Proposition 4.1 there exists \(C \in \mathbb{R}^+\) with the following property. If \(u: \Sigma \rightarrow X\) is a \(J\)-holomorphic map, \(z \in \Sigma\), and \(\delta \in \mathbb{R}^+\), then

\[
E_g(u; B_\delta(z)) < h_{J,g} \quad \implies \quad |d_z u|_g \leq C/\delta^2. \tag{5.12}
\]

For every pair \(i,j \in \mathbb{Z}^+\), let \(\{z_{ij}^k\}_{k=1}^N\) be a subset of points of \(\Sigma\) containing \(S_0\) such that

\[
z \in \Sigma_{ij}^* \equiv \Sigma - \bigcup_{k=1}^N B_{2/3}(z_{ij}^k) \quad \implies \quad E_g(u_i; B_{1/3}(z) \cap U_i) < h_{J,g}. \tag{5.13}
\]

By (5.12) and (5.13),

\[
|d_z u_i|_g \leq Cj^2 \quad \forall z \in \Sigma_{ij}^* \text{ s.t. } B_{1/3}(z) \subset U_i. \tag{5.14}
\]

After passing to a subsequence of \(\{u_i\}\), we can assume that the sequence \(E_g(u_i)\) converges to \(E\) and that the sequence \(\{z_{ij}^k\}_{k \in \mathbb{Z}^+}^{N}\) converges to some \(z_j^k \in \Sigma\) for every \(k = 1, \ldots, N\) and \(j \in \mathbb{Z}^+\). Along with (5.14) and the first two assumptions in (5.11), this implies that

\[
\limsup_{i \to \infty} |d_z u_i|_g \leq C j^2 \quad \forall z \in \Sigma_{ij}^* \tag{5.15}
\]

After passing to another subsequence of \(\{u_i\}\), we can assume that the sequence \(\{z_j^k\}_{j \in \mathbb{Z}^+}^{N}\) converges to some \(z^k \in \Sigma\) for every \(k = 1, \ldots, N\).
By (5.15) and the ellipticity of the $\bar{\partial}$-operator, a subsequence of $u_i$ converges uniformly in the $C^\infty$-topology on compact subsets of $\Sigma_i^*$ to a $J$-holomorphic map $v_1: \Sigma_i^* \to X$. By (5.15) and the ellipticity of the $\bar{\partial}$-operator, a subsequence of this subsequence in turn converges uniformly in the $C^\infty$-topology on compact subsets of $\Sigma_j^*$ to a $J$-holomorphic map $v_j: \Sigma_j^* \to X$ for every $j \in \mathbb{Z}^+$. The limiting maps satisfy

$$v_j|_{\Sigma_j \cap \Sigma_j^*} = v_j'|_{\Sigma_j \cap \Sigma_j^*}, \quad \forall j, j' \in \mathbb{Z}^+.$$ 

Thus, the map

$$u: \Sigma^* \equiv \{z^k\} \to X, \quad u(z) = v_j(z) \quad \forall z \in \Sigma_j^*,$$

is well-defined and $J$-holomorphic.

By construction, the final subsequence of $u_i$ converges uniformly in the $C^\infty$-topology on compact subsets of $\Sigma^*$ to $u$. This implies that

$$E_g(u) \leq \liminf_{i \to \infty} E_g(u_i) = E.$$ 

By Proposition 5.1 $u$ thus extends to a $J$-holomorphic map $\Sigma \to X$. \hfill $\square$

### 5.3 Gromov’s convergence

We next show that a sequence of maps as in Corollary 5.3 gives rise to a continuous map from a tree of spheres attached at $0 \in B_1$, i.e. a connected union of spheres that have a distinguished, base component and no loops; the distinguished component will be attached at $\infty \in S^2$ to $0 \in B_1$.

The combinatorial structure of such a tree is described by a finite rooted linearly ordered set, i.e. a partially ordered set $(I, \prec)$ such that

- (RS1) there is a minimal element (root) $i_0 \in I$, i.e. $i_0 \prec h$ for every $h \in I - \{i_0\}$, and

- (RS2) for all $h_1, h_2, i \in I$ with $h_1, h_2 \prec i$, either $h_1 = h_2$, or $h_1 \prec h_2$, or $h_2 \prec h_1$.

For each $i \in I - \{i_0\}$, let $p(i) \in I$ denote the immediate predecessor of $i$, i.e. $p(i) \in I$ such that $h \prec p(i) \prec i$ for all $h \in I - \{p(i)\}$ such that $h \prec i$. Such $p(i) \in I$ exists by [RS2] and is unique. In the first diagram in Figure 10, the vertices (dots) represent the elements of a rooted linearly ordered set $(I, \prec)$ and the edges run from $i \in I - \{i_0\}$ down to $p(i)$.

Given a finite rooted linearly ordered set $(I, \prec)$ with minimal element $i_0$ and a function

$$z: I - \{i_0\} \to \mathbb{C}, \quad i \mapsto z_i, \quad \text{s.t.} \quad (p(i_1), z_{i_1}) \neq (p(i_2), z_{i_2}) \quad \forall i_1, i_2 \in I - \{i_0\}, \quad i_1 \neq i_2,$$

let

$$\Sigma = \left( \bigsqcup_{i \in I} \{i\} \times S^2 \right)/\sim, \quad (i, \infty) \sim (p(i), z_i) \quad \forall i \in I - \{i_0\};$$

see the second diagram in Figure 10. Thus, the tree of spheres $\Sigma$ is obtained by attaching $\infty$ in the sphere indexed by $i$ to $z_i$ in the sphere indexed by $p(i)$. The last condition in (5.16) insures that $\Sigma$ is a nodal Riemann surface, i.e. each non-smooth point (node) has only two local branches (pieces homeomorphic to $\mathbb{C}$).
Figure 10: A rooted linearly ordered set and an associated tree of spheres

**Proposition 5.5.** Let \((X, J)\) be a compact almost complex manifold with a Riemannian metric \(g\) and \(u_i : B_1 \to X\) be a sequence of \(J\)-holomorphic maps converging uniformly in the \(C^\infty\)-topology on compact subsets of \(B_1^*\) to a \(J\)-holomorphic map \(u : B_1 \to X\). If the limit
\[
m \equiv \lim_{\delta \to 0} \lim_{i \to \infty} E_g(u_i ; B_\delta) \tag{5.17}
\]
exists and is nonzero, then there exist
\begin{enumerate}
  \item a nodal Riemann surface \((\Sigma_{\infty}, j_{\infty})\) consisting of \(B_1\) with a tree of Riemann spheres \(P^1\) attached at \(0 \in B_1\),
  \item a \(J\)-holomorphic map \(u_{\infty} : \Sigma_{\infty} \to X\),
  \item a subsequence of \(\{u_i\}\) still denoted by \(\{u_i\}\), and
  \item a biholomorphic map \(\psi_i : U_i \to B_1/2\), where \(U_i \subset \mathbb{C}\) is an open subset,
\end{enumerate}
such that
\begin{enumerate}
  \item \(E_g(u_{\infty} ; \Sigma_{\infty} - B_1) = m\), \(U_i \subset U_{i+1}\), and \(\mathbb{C} = \bigcup_{i=1}^{\infty} U_i\),
  \item \(u_i \circ \psi_i\) converges to \(u_{\infty}\) uniformly in the \(C^\infty\)-topology on compact subsets of the complement of the nodes \(\infty, w_1^*, \ldots, w_k^*\) in the sphere \(P^1_0\) attached at \(0 \in B_1\),
  \item if \(u_{\infty}|_{P^1_0}\) is constant, \(P^1_0\) contains at least three nodes of \(\Sigma_{\infty}\);
  \item \(u_i \circ \psi_i\) applies with \(B_1\), \(\{\{u_i\}, 0\}\), and \(m\) replaced by a neighborhood of \(w_r^*\) in \(\mathbb{C}\), \(\{u_i \circ \psi_i\}, w_r^*\), and
    \[
m'_r \equiv \lim_{\delta \to 0} \lim_{i \to \infty} E_g(u_i \circ \psi_i ; B_\delta(w_r^*)) \tag{5.18}
    \]
    respectively, for each \(r = 1, \ldots, k\).
\end{enumerate}

**Proof.** Let \(h_{J, g}\) be the smaller of the numbers \(h_{J, g}\) in Corollaries 4.2 and 5.3. In particular, \(m \geq h_{J, g}\).

For each \(i \in \mathbb{Z}^+\) sufficiently large, choose \(z_i \in \overline{B_{1/2}}\) so that
\[
\max_{z \in B_{1/2}} |du_i|_g = |dz_i u_i|_g. \tag{5.19}
\]
Since \( u_i \) converges uniformly in the \( C^\infty \)-topology on compact subsets of \( B_1^* \) to \( u, z_i \rightarrow 0 \) as \( i \rightarrow \infty \). Thus, \( B_{1/2}(z_i) \subset B_1 \) for all \( i \in \mathbb{Z}^+ \) sufficiently large. By Lemma 5.4 \( \delta \) for all \( i \in \mathbb{Z}^+ \) sufficiently large there exists \( \delta_i \in (0, 1/2) \) such that

\[
E_g(u_i; B_{\delta_i}(z_i)) = m - \frac{h_{J,g}}{2}.
\]

Define

\[
\psi_i : U_i \equiv \{ w \in \mathbb{C} : z_i + \delta_i w \in B_{1/2} \} \rightarrow B_{1/2} \quad \text{by} \quad \psi_i(w) = z_i + \delta_i w.
\]

Since \( \delta_i \rightarrow 0 \), the second property in (11a) holds.

For each \( i \in \mathbb{Z}^+ \) sufficiently large, let

\[
v_i = u_i \circ \psi_i : U_i \rightarrow X.
\]

Since \( u_i \) is \( J \)-holomorphic and \( \psi_i \) is biholomorphic onto its image, \( v_i \) is a \( J \)-holomorphic map with \( E_g(v_i) = E_g(u_i; B_{1/2}) \). Along with Lemma 5.2 (2), this implies that

\[
\lim_{i \rightarrow \infty} E_g(v_i) = m(1/2) < \infty.
\]

By Lemma 5.4, there thus exist a finite collection \( w_1^*, \ldots, w_k^* \in \mathbb{C} \) of distinct points and a subsequence of \( \{u_i\} \), still denoted by \( \{u_i\} \), such that \( v_i \) converges uniformly in the \( C^\infty \)-topology on compact subsets of \( \mathbb{P}^1 - \{\infty, w_1^*, \ldots, w_k^* \} \) to a \( J \)-holomorphic map \( u : \mathbb{P}^1 \rightarrow X \). In particular, (1b) holds and \( |dv_i|_g \) is uniformly bounded on compact subsets of \( \mathbb{P}^1 - \{\infty, w_1^*, \ldots, w_k^* \} \). We can also assume that the limit (5.18) exists for every \( r = 1, \ldots, k \). We note that

\[
E_g(v) + \sum_{r=1}^k m'_r = \lim_{R \rightarrow \infty} \lim_{\delta \rightarrow 0} \lim_{i \rightarrow \infty} E_g(v_i; B_R - \bigcup_{r=1}^k B_{\delta}(w_r^*)) + \sum_{r=1}^k \lim_{\delta \rightarrow 0} \lim_{i \rightarrow \infty} E_g(v_i; B_{\delta}(w_r^*)) = \lim_{R \rightarrow \infty} \lim_{i \rightarrow \infty} E_g(v_i, B_R) = \lim_{R \rightarrow \infty} \lim_{i \rightarrow \infty} E_g(u_i, B_{R\delta_i}(z_i)) = m;
\]

the last equality holds by 5.4.

Let \( \delta_0 \in \mathbb{R}^+ \) be such that the balls \( B_{\delta_0}(w_r^*) \) are pairwise disjoint. If

\[
\limsup_{i \rightarrow \infty} \max_{\overline{B_{\delta_0}(w_r^*)}} |dv_i| < \infty
\]

for some \( r \), then \( \{v_i\} \) converges uniformly in the \( C^\infty \)-topology on \( \overline{B_{\delta_0}(w_r^*)} \) to \( v \) by the ellipticity of the \( \partial \)-operator. Thus, we can assume that

\[
\lim_{i \rightarrow \infty} \sup_{\overline{B_{\delta_0}(w_r^*)}} |dv_i| = \infty
\]

for every \( r = 1, \ldots, k \). In light of Corollary 5.3 (1) \( m'_r \geq h_{J,g} \).

We next show that \( u(0) = v(\infty) \), i.e. that the bubble \( (\mathbb{P}^1_0, v) \) connects to \( (B_1, u) \) at \( z = 0 \). Note that

\[
d_g(u(0), v(\infty)) = \lim_{R \rightarrow \infty} \lim_{\delta \rightarrow 0} d_g(u(\delta), v(R)) = \lim_{R \rightarrow \infty} \lim_{\delta \rightarrow 0} \lim_{i \rightarrow \infty} d_g(u_i(z_i + \delta), v_i(R))
\]

\[
= \lim_{R \rightarrow \infty} \lim_{\delta \rightarrow 0} \lim_{i \rightarrow \infty} d_g(u_i(z_i + \delta), u_i(z_i + R\delta_i)) \leq \lim_{R \rightarrow \infty} \lim_{\delta \rightarrow 0} \lim_{i \rightarrow \infty} \text{diam}_g(u_i(B_{\delta}(z_i) - B_{R\delta_i}(z_i))).
\]

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Suppose \( v : \mathbb{P}^1 \to X \) is a constant map. By (5.21), \( k \geq 1 \) and so there exists \( w^* \in \mathbb{C} \) such that \( |d_w v_i| \to \infty \) as \( i \to \infty \). By (5.19) and the definition of \( \psi_i \), \( |d_0 v_i| \geq |d_w v_i| \) for all \( w \in \mathbb{C} \) contained in the domain of \( v_i \) and so \( |d_0 v_i| \to \infty \) as \( i \to \infty \). By (5.18) and (5.20),

\[
m' \equiv \lim_{\delta \to 0} \lim_{i \to \infty} E_g(v_i) \leq \lim_{i \to \infty} E_g(v_i; B_1) = |d_0 v_i; B_{\delta_i}(z_i)| = m - \frac{h}{2} < m,
\]

and so \( k \geq 2 \), as claimed in (4c). Since the amount of energy of \( v_i \) contained in \( \mathbb{C} - B_1 \) approaches \( h_{j,g}/2 \), as illustrated in Figure 11, there must be in particular a bubble point \( w^*_r \) with \( |w^*_r| = 1 \), though this is not material.

The above establishes Proposition 5.5 whenever \( k = 0 \) by taking

\[
u_\infty|_{B^1} = u \quad \text{and} \quad u_\infty|_{\mathbb{P}^1} = v.
\]

Since \( m'_r \geq h_{j,g} \) for every \( r, k = 0 \) if \( m < 2h_{j,g} \). If \( k \geq 1 \), \( m'_r \leq m - h_{j,g} \) by (5.21) because \( E_g(v) \geq h_{j,g} \) if \( v \) is not constant by Corollary 4.2 and \( k \geq 2 \) otherwise by the above. Thus, by induction on \( [m/h_{j,g}] \in \mathbb{Z}^+ \), we can assume that Proposition 5.5 holds when applied to \( \{v_i\} \) on \( B_{\delta_{i}}(w^*_r) \subset \mathbb{C} \) with \( r = 1, \ldots, k \). This yields a tree \( \Sigma \) of Riemann spheres \( \mathbb{P} \) with a distinguished smooth point \( \infty \) and a \( J \)-holomorphic map \( v_r: \Sigma \to X \) such \( v_r(\infty) = v(w^*_r) \) and \( E_g(v_r) = m'_r \). Combining the last equality with (5.21), we obtain

\[
E_g(v) + \sum_{r=1}^{k} E_g(v_r) = m.
\]

Identifying \( \infty \) in the base sphere of each \( \Sigma_r \) with \( w^*_r \in \mathbb{P}^1 \), which has been already attached to \( 0 \in B_{\delta}^1 \), we obtain a \( J \)-holomorphic map \( u_\infty: \Sigma_\infty \to X \) with the desired properties; see Figure 12.

\textbf{Proof of Theorem 1.2.} Fix a Riemannian metric \( g_\Sigma \) on \( \Sigma \). For \( z \in \Sigma \) and \( \delta \in \Sigma \), let \( B_{\delta}(z) \subset \Sigma \) denote the ball of radius \( \delta \) around \( z \).
By Lemma [5.4] there exist a finite collection \(z^*_1, \ldots, z^*_\ell \in \Sigma\) of distinct points and a subsequence of \(\{u_i\}\), still denoted by \(\{u_i\}\), such that \(u_i\) converges uniformly in the \(C^\infty\)-topology on compact subsets of \(\Sigma - \{z^*_1, \ldots, z^*_\ell\}\) to a \(J\)-holomorphic map \(u: \Sigma \rightarrow X\). In particular, \(|du_i|_g\) is uniformly bounded on compact subsets of \(\Sigma - \{z^*_1, \ldots, z^*_\ell\}\). We can also assume that the limit

\[
\lim_{\delta \to 0} \lim_{i \to \infty} E_g(u_i; B_\delta(z^*_j)) = \lim_{i \to \infty} E_g(u_i).
\]

(5.22)

Let \(\delta_0 \in \mathbb{R}^+\) be such that the balls \(B_{\delta_0}(z^*_j)\) are pairwise disjoint. If

\[
\limsup_{i \to \infty} \max_{\mathbb{B}_{\delta_0}(z^*_j)} |du_i| < \infty
\]

for some \(j\), then \(\{u_i\}\) converges uniformly in the \(C^\infty\)-topology on \(\mathbb{B}_{\delta_0}(z^*_j)\) to \(u\) by the ellipticity of the \(\bar{\partial}\)-operator. Thus, we can assume that

\[
\lim_{i \to \infty} \sup_{\mathbb{B}_{\delta_0}(z^*_j)} |du_i| = \infty
\]

for every \(j = 1, \ldots, \ell\).

For each \(j = 1, \ldots, \ell\), Proposition [5.5] provides a tree \(\Sigma_j\) of Riemann spheres \(\mathbb{P}^1\) with a distinguished smooth point \(\infty\) and a \(J\)-holomorphic map \(v_j: \Sigma_j \rightarrow X\) such \(v_j(\infty) = v(w^*_j)\) and \(E_g(v_j) = m_j\). Combining the last equality with (5.22), we obtain

\[
E_g(v) + \sum_{j=1}^\ell E_g(v_j) = \lim_{i \to \infty} E_g(u_i).
\]

Identifying the distinguished point \(\infty\) of each \(\Sigma_j\) with \(z^*_j \in \Sigma\), we obtain a Riemann surface \((\Sigma_\infty, j_\infty)\) and a \(J\)-holomorphic map \(u_\infty: \Sigma_\infty \rightarrow X\) with the desired properties.
If $\Sigma = \mathbb{P}^1$ and the limit map $u$ above is constant, then $\ell \geq 1$ by (5.22). Suppose $\ell \in \{1, 2\}$. Let

$$M_i = \sup_{B_{\delta_0}(z_1^i)} |du_i|$$

and parametrize $\mathbb{P}^1$ so that $z_1^* = 0$. Define

$$h_i : \mathbb{P}^1 \rightarrow \mathbb{P}^1, \quad h_i(z) = z + z/M_i,$$

and apply the preceding argument with $u_i$ replaced by $u_i \circ h_i$. By the proof of Corollary 5.3(1), the limiting map $u|_\Sigma$ is then non-constant and $(\Sigma^\infty, j^\infty, u)$ is a stable $J$-holomorphic map.

### 5.4 A example

We now give an example illustrating Gromov’s convergence in a classical setting.

Let $n \in \mathbb{Z}^+$, with $n \geq 2$, and $\mathbb{P}^{n-1} = \mathbb{C}P^{n-1}$. Denote by $\ell$ the positive generator of $H_2(\mathbb{P}^{n-1}; \mathbb{Z}) \approx \mathbb{Z}$, i.e. the homology class represented by the standard $\mathbb{P}^1 \subset \mathbb{P}^{n-1}$. A degree $d$ map $f : \mathbb{P}^1 \rightarrow \mathbb{P}^{n-1}$ is a continuous map such that $f^*[\mathbb{P}^1] = d\ell$. A holomorphic degree $d$ map $f : \mathbb{P}^1 \rightarrow \mathbb{P}^{n-1}$ is given by

$$[u, v] \rightarrow [R_1(u, v), \ldots, R_n(u, v)]$$

for some degree $d$ homogeneous polynomials $R_1, \ldots, R_d$ on $\mathbb{C}^2$ without a common linear factor. Since the tuple $(\lambda R_1, \ldots, \lambda R_n)$ determines the same map as $(R_1, \ldots, R_n)$ for any $\lambda \in \mathbb{C}^*$, the space of degree $d$ holomorphic maps $f : \mathbb{P}^1 \rightarrow \mathbb{P}^{n-1}$ is a dense open subset of

$$X_{n,d} = (\text{Sym}^d \mathbb{C}^2 - \{0\})/\mathbb{C}^* \approx \mathbb{P}((d+1)n-1).$$

Suppose $f_k : \mathbb{P}^1 \rightarrow \mathbb{P}^{n-1}$ is a sequence of holomorphic degree $d \geq 1$ maps and

$$R_k = [R_{k;1}, \ldots, R_{k;n}] \in X_{n,d}$$

are the associated equivalence classes of $n$-tuples of homogeneous polynomials without a common linear factor. Passing to a subsequence, we can assume that $[R_k]$ converges to some

$$R = [(v_1 u - u_1 v)^{d_1} \ldots (v_m u - u_m v)^{d_m} S_1, \ldots, (v_1 u - u_1 v)^{d_1} \ldots (v_m u - u_m v)^{d_m} S_n] \in X_{n,d},$$

with $d_1, \ldots, d_m \in \mathbb{Z}^+$ and homogeneous polynomials

$$S \equiv [S_1, \ldots, S_n] \in X_{n,d_0}$$

without a common linear factor and with $d_0 \in \mathbb{Z}^\geq 0$. By (5.23),

$$d_0 + d_1 + \ldots + d_m = d.$$

Rescaling $(R_{k;1}, \ldots, R_{k;n})$, we can assume that

$$\lim_{k \rightarrow \infty} R_{k;i} = (v_1 u - u_1 v)^{d_1} \ldots (v_m u - u_m v)^{d_m} S_i \quad \forall i = 1, \ldots, n.$$  (5.24)
Suppose \( z_0 \in \mathbb{C} - \{ u_1/v_1, \ldots, u_m/v_m \} \). Since the polynomials \( S_1, \ldots, S_n \) do not have a common linear factor, \( S_{i_0}(z_0, 1) \neq 0 \) for some \( i_0 = 1, \ldots, n \). This implies that \( R_{k;i_0}(z_0, 1) \neq 0 \) for all \( k \) large enough and so
\[
\lim_{k \to \infty} \frac{R_{k;i}(z, 1)}{R_{k;i_0}(z, 1)} = \lim_{k \to \infty} \frac{R_{k;i}(z, 1)}{R_{k;i_0}(z, 1)} = \frac{(v_1 z - u_1)^{d_1} \cdots (v_m z - u_m)^{d_m} S_i(z, 1)}{(v_1 z - u_1)^{d_1} \cdots (v_m z - u_m)^{d_m} S_{i_0}(z, 1)} = \frac{S_i(z, 1)}{S_{i_0}(z, 1)}
\]
for all \( i = 1, \ldots, n \) and \( z \) close to \( z_0 \). Furthermore, the convergence is uniform on a neighborhood of \( z_0 \). Thus, the sequence \( f_k \) \( C^\infty \)-converges on compact subsets of \( \mathbb{P}^1 - \{ [u_1, v_1], \ldots, [u_m, v_m] \} \) to the holomorphic degree \( d_0 \) map \( g : \mathbb{P}^1 \to \mathbb{P}^{n-1} \) determined by \( S \).

Let \( \omega \) be the Fubini-Study symplectic form on \( \mathbb{P}^{n-1} \) normalized so that \( \langle \omega, \ell \rangle = 1 \) and \( E(\cdot) \) be the energy of maps into \( \mathbb{P}^{n-1} \) with respect to the associated Riemannian metric. For each \( \delta > 0 \) and \( j = 1, \ldots, m \), denote by \( B_\delta([u_j, v_j]) \) the ball of radius \( \delta \) around \([u_j, v_j]\) in \( \mathbb{P}^1 \) and let
\[
\mathbb{P}^1_\delta = \mathbb{P}^1 - \bigcup_{j=1}^m B_\delta([u_j, v_j]).
\]
For each \( j = 1, \ldots, m \), let
\[
m_{[u_j, v_j]}(\{f_k\}) = \lim_{\delta \to 0} \lim_{k \to \infty} E(f_k; B_\delta([u_j, v_j])) \in \mathbb{R}^{\geq 0}
\]
be the energy sinking into the bubble point \([u_j, v_j]\). By Theorem 1.2 the number \( m_{[u_j, v_j]}(\{f_k\}) \) is the value of \( \omega \) on some element of \( H_2(\mathbb{P}^{n-1}; \mathbb{Z}) \), i.e. an integer. Below we show that \( m_{[u_j, v_j]}(\{f_k\}) = d_j \).

Since the sequence \( f_k \) \( C^\infty \)-converges to the degree \( d_0 \) map \( g : \mathbb{P}^1 \to \mathbb{P}^{n-1} \) on compact subsets of \( \mathbb{P}^1 - \{ [u_1, v_1], \ldots, [u_m, v_m] \} \),
\[
d_0 = \langle \omega, d_0 \ell \rangle = E(g) = \lim_{\delta \to 0} E_g(g; \mathbb{P}^1_\delta) = \lim_{\delta \to 0} \lim_{k \to \infty} E(f_k; \mathbb{P}^1_\delta).
\]
Thus,
\[
\sum_{j=1}^m m_{[u_j, v_j]}(\{f_k\}) = \sum_{j=1}^m \lim_{\delta \to 0} \lim_{k \to \infty} E(f_k; B_\delta([u_j, v_j])) = \lim_{\delta \to 0} \lim_{k \to \infty} E(f_k; \bigcup_{j=1}^m B_\delta([u_j, v_j]))
\]
\[
= \lim_{\delta \to 0} \lim_{k \to \infty} (E_g(f_k) - E_g(f_k; \mathbb{P}^1_\delta)) = d - d_0 = d_1 + \ldots + d_m.
\]
In particular, \( m_{[u_j, v_j]}(\{f_k\}) = d_j \) if \( m = 1 \), no matter what the “residual” tuple of polynomials \( S \) is. We use this below to establish this energy identity for \( m > 1 \) as well.

By (5.24), for all \( k \in \mathbb{Z}^+ \) sufficiently large there exist \( \lambda_{k;i;j;p} \in \mathbb{C} \) with \( i = 1, \ldots, n \), \( j = 1, \ldots, m \), and \( p = 1, \ldots, d_j \) and tuples
\[
S_k \equiv [S_{k;1}, \ldots, S_{k;n}] \in \mathcal{X}_{n;d_0}
\]
of polynomials without a common linear factor such that
\[
\lim_{k \to \infty} S_k = S, \quad \lim_{k \to \infty} \lambda_{k;i;j;p} = 1 \quad \forall i, j, p,
\]
\[
R_{k;i}(u, v) = \prod_{j=1}^m \prod_{p=1}^{d_j} (v_j u - \lambda_{k;i;j;p} u_j v) \cdot S_{k;i}(u, v) \quad \forall k, i.
\]
For each \( j_0 = 1, \ldots, m \), let
\[
T_{j_0} \equiv [T_{j_0}; \ldots, T_{j_0};] \in \mathcal{X}_{n:d-d_{j_0}}
\]
be a tuple of polynomials without a common linear factor. If in addition, \( i = 1, \ldots, n \), \( \epsilon \in \mathbb{R} \), and \( k \in \mathbb{Z}^+ \), let
\[
S_{i:j_0;\epsilon}(u, v) \equiv \prod_{j \neq j_0}^m (v_j u - u_j v)^{d_j} \cdot S_i(u, v) + \epsilon T_{j_0};(u, v), \quad i = 1, \ldots, n,
\]
\[
R_{k;i:j_0;\epsilon}(u, v) \equiv R_{k;i}(u, v) + \epsilon \prod_{p=1}^{d_{j_0}} (v_{j_0} u - \lambda_{k;i;j_0;p} u_{j_0} v) \cdot T_{j_0};(u, v), \quad i = 1, \ldots, n.
\]

The polynomials within each tuple \((S_{i:j_0;\epsilon})_{i=1}^n\) and \((R_{k;i:j_0;\epsilon})_{i=1}^n\) have no common linear factor for all \( \epsilon \in \mathbb{R}^+ \) sufficiently small and \( k \) sufficiently large (with the conditions on \( \epsilon \) and \( k \) mutually independent). We denote by
\[
\begin{align*}
&f_{k;j_0;\epsilon}: \mathbb{P}^1 \longrightarrow \mathbb{P}^{n-1} \\
&\text{the holomorphic degree } d \text{ map determined by the tuple}
\end{align*}
\]
\[
R_{k;j_0;\epsilon} \equiv [R_{k;1;j_0;\epsilon}, \ldots, R_{k;n;j_0;\epsilon}].
\]

Since
\[
\lim_{k \to \infty} R_{k;j_0;\epsilon} = [(v_1 u - u_1 v)^{d_{j_0}} S_{1;j_0;\epsilon}, \ldots, (v_1 u - u_1 v)^{d_{j_0}} S_{n;j_0;\epsilon}] \in \mathcal{X}_{n:d}
\]
and the polynomials \( S_{1;j_0;\epsilon}, \ldots, S_{n;j_0;\epsilon} \) have no linear factor in common,
\[
\lim_{\delta \to 0} \lim_{k \to \infty} E(f_{k;j_0;\epsilon}; B_{\delta}([u_{j_0}, v_{j_0}])) \equiv m_{u_{j_0}, v_{j_0}}(\{f_{k;j_0;\epsilon}\}) = d_{j_0} \quad (5.25)
\]
by the \( m = 1 \) case established above.

For \( \delta \in \mathbb{R}^+ \) sufficiently small, \( \epsilon \in \mathbb{R}^+ \) sufficiently small, and \( k \) sufficiently large,
\[
\prod_{j \neq j_0}^m \prod_{p=1}^{d_j} (v_j u - \lambda_{k;i;j_0;p} u_j v) \cdot S_{k;i}(u, v) \neq 0 \quad \forall [u, v] \in B_{\delta}([u_{j_0}, v_{j_0}]).
\]

Thus, the ratios
\[
\frac{R_{k;i;j_0;\epsilon}(u, v)}{R_{k;i}(u, v)} = 1 + \epsilon \frac{\prod_{j \neq j_0}^{d_j} (v_j u - \lambda_{k;i;j_0;p} u_j v) \cdot S_{k;i}(u, v)}{\prod_{j \neq j_0}^{d_j} (v_j u - \lambda_{k;i;j_0;p} u_j v) \cdot S_{k;i}(u, v)}
\]
cconverge uniformly to 1 on \( B_{\delta}([u_{j_0}, v_{j_0}]) \) as \( \epsilon \to 0 \). Thus, there exists \( k^* \in \mathbb{Z}^+ \) such that
\[
\limsup_{\epsilon \to 0} \sup_{k \geq k^*, \delta \in B_{\delta}([u_{j_0}, v_{j_0}])} \left| \frac{|dz f_{k;j_0;\epsilon}|}{|dz f_k|} - 1 \right| = 0.
\]

It follows that
\[
 \begin{align*}
 m_{u_{j_0}, v_{j_0}}(\{f_k\}) \equiv \lim_{\delta \to 0} \lim_{k \to \infty} E(f_k; B_{\delta}([u_{j_0}, v_{j_0}])) = \lim_{\delta \to 0} \lim_{k \to \infty} E(f_{k;j_0;\epsilon}; B_{\delta}([u_{j_0}, v_{j_0}])) \\
= \lim_{\epsilon \to 0} \lim_{\delta \to 0} \lim_{k \to \infty} E(f_{k;j_0;\epsilon}; B_{\delta}([u_{j_0}, v_{j_0}])) = \lim_{\epsilon \to 0} d_{j_0} = d_{j_0};
\end{align*}
\]
the second-to-last equality above holds by (5.25).

Suppose that either \( d_0 \geq 1 \) or \( m \geq 3 \). Otherwise, the maps \( f_k \) can be reparametrized so that \( d_0 \neq 0 \); see the last paragraph of the proof of Theorem 1.2 at the end of Section 5.3. By Theorem 1.2 and the above, a subsequence of \( \{f_k\} \) converges to the equivalence class of a holomorphic degree \( d_0 \) map \( f: \Sigma \to \mathbb{P}^{n-1} \), where \( \Sigma \) is a nodal Riemann surface consisting of the component \( \Sigma_0 = \mathbb{P}^1 \) corresponding to the original \( \mathbb{P}^1 \) and finitely many trees of \( \mathbb{P}^1 \)s coming off from \( \Sigma_0 \). The maps on the components in the trees are defined only up reparametrization of the domain. By the above, \( f|_{\Sigma_0} \) is the map \( g \) determined by the “relatively prime part” \( S \) of the limit \( R \) of the tuples of polynomials. The trees are attached at the roots \([u_j, v_j]\) of the common linear factors \( v_j u - u_j v \) of the polynomials in \( R \); the degree of the restriction of \( f \) to each tree is the power of the multiplicity \( d_j \) of the corresponding common linear factor.

The same reasoning as above applies to the sequence of maps

\[
(id_{\mathbb{P}^1}, f_k): \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^{n-1},
\]

but the condition that either \( d_0 \geq 1 \) or \( m \geq 3 \) is no longer necessary for the analogue of the conclusion in the previous paragraph. This implies that the map

\[
\mathcal{M}_{0,0}(\mathbb{P}^1 \times \mathbb{P}^{n-1}, (1, d)) \to \mathcal{X}_{n,d}, \quad [f, g] \to [g \circ f^{-1}],
\]

from the subspace of \( \mathcal{M}_{0,0}(\mathbb{P}^1 \times \mathbb{P}^{n-1}, (1, d)) \) corresponding to maps from \( \mathbb{P}^1 \) extends to a continuous surjective map

\[
\overline{\mathcal{M}}_{0,0}(\mathbb{P}^1 \times \mathbb{P}^{n-1}, (1, d)) \to \mathcal{X}_{n,d}.
\]

In particular, Gromov’s moduli spaces refine classical compactifications of spaces of holomorphic maps \( \mathbb{P}^1 \to \mathbb{P}^{n-1} \). On the other hand, the former are defined for arbitrary almost Kahler manifolds, which makes them naturally suited for applying topological methods. The right-hand side of (5.26) is known as the linear sigma model in the Mirror Symmetry literature. The morphism (5.26) plays a prominent role in the proof of mirror symmetry for the genus 0 Gromov-Witten invariants in [5] and [8]; see [7, Section 30.2].

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