**Abstract**

We consider excitons in a quantum dot as \(q\)-deformed systems. The interaction of some excitonic systems with one cavity mode is considered. The dynamics of the system is obtained by diagonalizing the total Hamiltonian, and the emission spectrum of a quantum dot is derived.

The physical consequences of a \(q\)-deformed exciton on the emission spectrum of a quantum dot are given. It is shown that when the exciton system deviates from Bose statistics, the emission spectra will become multi-peak. With our investigation we try to find the origin of the \(q\)-deformation of the exciton. The optical response of excitons, which is affected by the nonlinear nature of \(q\)-deformed systems, up to the second order of approximation is calculated and the absorption spectrum of the system is given.

(Some figures in this article are in colour only in the electronic version)

**1. Introduction**

An exciton is an elementary excitation of a semiconductor which consists of a pair of two correlated fermions, the electron and the hole. Analogous to the hydrogen atom, it is characterized by a binding energy \(E_b\) and a Bohr radius \(a_B\). Because an exciton is composed of two fermions, it is a composite boson. Particularly, in a bulk semiconductor, when the excitation of a system is dilute, i.e. \(n_{ex}a_B^3 \ll 1\), where \(n_{ex}\) is the exciton density, a bosonic description of the system is convenient [1]. Also Bose–Einstein condensation of excitons, which is an essential characteristic of boson systems, has been considered theoretically [2]. When the density of excitons increases, the above condition is violated. In this situation the statistics of excitons deviates from Bose statistics.

In low-dimensional semiconductor systems such as a quantum well (QW), quantum wire or quantum dot (QD), due to the small dimensions and loss of translational symmetry, exciton excitation differs from the exciton in bulk materials [3]. In semiconductor nanostructures the size of the system strongly affects the exciton properties. For example, in the case of the quantum well it is shown that [4], if the well width is larger (smaller) than the Bohr radius of the exciton, the spectrum of the quantum well has properties similar to the situation in which excitons are bosons (fermions). Hence, the size of the system directly affects the quantum statistics of the excitons in that system. Recently similar results have been obtained for QD [5]. In [5] the effects of different statistics of excitons on the emission spectra of a QD are investigated, and the origin of the different statistics of excitons is considered. The same results have also been obtained for the quantum well. If the size of the QD is smaller (larger) than the exciton Bohr radius, excitons behave like fermions (bosons). Real statistics of excitons in the interaction is considered in [6] and references therein. As mentioned before in the high-density regime, exciton statistics deviates from Bose statistics. This is due to the increase of mutual forces between the excitations of the system and then the Pauli exclusion principle plays a dominant role [7]. The appearance of Bose statistics of the exciton–biexciton system and Pauli exclusion effects in a superlattice have been considered experimentally [8].

Bosons and fermions are the only two kinds of particles realized in nature. The conditions mentioned for excitons (in one regime they are like bosons and in another like fermions) are properties of a special kind of statistics called intermediate statistics [9]. Bose and Fermi statistics are two limiting cases of this statistics. The properties of this statistics have been considered by many authors [10–12]. Operator realization of intermediate statistics is similar to \(q\)-deformed operators [13]. Bosonic \(q\)-deformed operators [14] are a generalization of the Heisenberg algebra obtained by introducing a deformation.
parameter \( q \). Deviation of this parameter from 1 shows the deviation of the algebra from the Heisenberg algebra. It is shown that it is possible to describe correlated fermion pairs with \( q \)-deformed bosons [15]. Therefore, it is reasonable to consider an exciton system as a \( q \)-deformed system. We assume that the creation and annihilation operators of excitons obey a \( q \)-deformed algebra. A \( q \)-deformed description of Frenkel excitons has been considered recently [16].

The algebra generated by \( q \)-deformed operators is given by

\[
\left[ b_q, b_q^\dagger \right] = \frac{b_q b_q^\dagger - q^{-1} b_q^\dagger b_q}{\Delta(q - q^{-1})}, \quad \left[ \hat{a}, b_q \right] = b_q^\dagger, \quad \left[ \hat{a}, b_q^\dagger \right] = -b_q,
\]

where \( \hat{a} = b^\dagger b \) is the usual particle number operator. A representation of this algebra is given in [17]. In the case of excitons, the \( q \)-parameter can depend on excitation number and the physical size of system.

In this paper, we consider the interaction of light with a QD embedded in a microcavity. By considering excitons in QD as \( q \)-deformed bosons (the case of \( q \)-deformed fermions is straightforward) we study the emission spectrum of the system. As is clear, the first commutator in (1) explicitly depends on the number of excitons. Hence, this is a system in which light interacts with a nonlinear active medium. Therefore, we shall obtain the linear and nonlinear responses of a \( q \)-deformed exciton system. Knowledge of the interaction of light with a nonlinear medium (\( q \)-deformed excitons) and its optical response is important for the interpretation of experimental results [31]. On the other hand, we compare the obtained results with some experimental ones and in this manner we investigate the physical origin of \( q \)-deformation of excitons. In section 2, we derive the spectrum of a QD when one exciton mode interacts with a single-mode cavity field. In section 3, we consider the interaction of two exciton modes with a single cavity mode. In section 4, the nonlinear response of a QD is derived up to the second order of approximation. Finally, we summarize our conclusions in section 5.

2. Model Hamiltonian

We consider a QD embedded in a microcavity which interacts with a single-mode cavity field. We assume that the excitations in the QD have an intermediate statistics [5], and their creation and annihilation operators obey a \( q \)-deformed algebra. We can express the \( q \)-deformed operators in terms of ordinary boson operators by the following maps:

\[
b_q = \sqrt{\frac{\Delta(q - q^{-1})}{\Delta(q - q^{-1}) - 1}} b_q^\dagger, \quad b_q^\dagger = \sqrt{\frac{\Delta(q - q^{-1})}{\Delta(q - q^{-1}) - 1}} b_q^\dagger,
\]

where \( b \) and \( b^\dagger \) are the ordinary boson operators and \( \hat{a} = b^\dagger b \). The ordinary commutator of \( q \)-deformed exciton operators is

\[
\left[ b_q, b_q^\dagger \right] = \frac{q}{q + 1} \left[ q^a + q^{-a+1} \right] = k(\hat{a}).
\]

Deviation of this commutator from an ordinary boson algebra \( (b, b^\dagger) = 1 \) relates to deviation of the \( q \)-parameter from 1. It is clear that this generalized commutator depends on the number of excitations. It seems that by using this algebra we can consider some nonlinear phenomena in the system related to the population of excitons. For example, biexciton effects can be considered in this manner as an effective approach. Therefore the deformation parameter \( q \) can represent some physical parameters such as the ratio of the size of the system to the Bohr radius of exciton. Interaction of the QD with the single-mode cavity field in the rotating wave approximation can be described by the following Hamiltonian:

\[
\hat{H} = \hbar \omega \hat{a}^\dagger \hat{a} + \hbar \kappa a_q b_q^\dagger + \hbar g (\hat{a} b_q^\dagger + \hat{a}^\dagger b_q),
\]

where \( \hat{a} \) and \( \hat{a}^\dagger \) are the creation and annihilation operators of the cavity field and \( [\hat{a}, \hat{a}^\dagger] = \delta_{ij} \). We shall consider a phenomenological damping for the system which relates to both subsystems: photons and excitons. As is clear from the Hamiltonian (4), the exciton number is not a constant of motion. Because of the dependence of the exciton operator, \( b_q \), on the exciton number, the resulting equations of motion become a nontrivial set of coupled equations. On the other hand, since the total number of excitations (excitons and photons) is conserved we can diagonalize the Hamiltonian in the subspace of a definite excitation. To consider this dynamics we propose an approach based on diagonalization of the Hamiltonian by using the polariton transformation [18]. This procedure depends on some unitary transformations which diagonalize the model Hamiltonian. As is usual in this procedure [19], new operators have the same commutation relation as the original operators (free operators). Here, there are two distinct sets of operators, the cavity mode operators which obey the usual boson commutation relation and exciton operators that are \( q \)-deformed bosons. Therefore, with the presence of these two different statistics, mixed operators (polariton operators) do not have specific statistics. They can be considered as ordinary boson operators or \( q \)-deformed operators. We consider both situations and we study the physical results associated with each situation in the resonance fluorescence spectrum of a QD.

2.1. Boson polaritons

In order to solve the dynamical system, we perform the following transformation:

\[
\hat{\tilde{p}}_k = \kappa_k \hat{b}_q + v_k \hat{a}.
\]

Due to the presence of the \( q \)-deformed operator \( \hat{b}_q \), we call this transformation a polariton-like transformation. As mentioned before, \( \hat{b}_q \) depends on the number of excitons explicitly and this causes the Hopfield coefficients \( \kappa_k \) and \( v_k \) to depend on the number of excitons. Hence, the transformation (5) can be considered as a nonlinear polariton transformation. This kind of transformation has been considered recently for the case of the Bogoliubov transformation [20, 21]. We assume that the polariton-like operators obey the usual boson commutation relations

\[
[\hat{p}_k, \hat{p}_{k'}^\dagger] = \delta_{kk'} \Rightarrow [\hat{\tilde{p}}_k, \hat{\tilde{p}}_{k'}^\dagger] = |\kappa_k|^2 k(\hat{a}) + |v_k|^2 = 1,
\]

where the operator-valued function \( k(\hat{a}) \) was introduced by equation (3). We choose unknown coefficients \( \kappa_k \) and \( v_k \) so that the Hamiltonian (4) becomes diagonal in terms of the
polariton-like operators
\[ \hat{H} = \hbar \sum_k \Omega_k \hat{p}_k \hat{\rho}_k, \]
where \( \Omega_k \) is the polariton spectrum and \( k \) refers to different polariton branches. By taking into account a phenomenological damping for exciton and photon systems separately, the unknown coefficients \( u_k, v_k \) satisfy the following set of equations:
\[
\begin{align*}
[\omega_{ek}k(\hbar) - \Omega_k - i\gamma_{ek}]u_k + v_k g &= 0, \\
u_k g k(\hbar) + (\omega - \Omega_k - i\gamma_{ek})v_k &= 0.
\end{align*}
\]
In this set of equations, \( \gamma_{ek} \) and \( \gamma_{ph} \) are the exciton and photon damping constants, respectively. From these equations the polariton spectrum can be obtained as
\[
\Omega_k = \frac{\omega_{ek}k(\hbar) + \omega - i(\gamma_{ek} + \gamma_{ph})}{2} \pm \frac{1}{2}\sqrt{[\omega_{ek}k(\hbar) - \omega - i(\gamma_{ek} - \gamma_{ph})]^2 + 4\gamma_{ph}^2 k(\hbar)}.
\]
It is apparent that the \( q \)-deformed description of excitons causes the splitting between these energy eigenvalues to be increased in comparison to the bosonic description of excitons. Using the set of equations (8) and the polariton spectrum (9) we find the coefficients for two polariton branches
\[
\begin{align*}
u_k &= -\frac{\omega_{ek}k(\hbar) - i(\gamma_{ek} - \gamma_{ph})}{\sqrt{[\omega_{ek}k(\hbar) - \omega - 2\Omega_k + \omega_{ek}k(\hbar) - i(\gamma_{ek} + \gamma_{ph})]^2 + 4\gamma_{ph}^2 k(\hbar)}}, \\
u_k &= -\frac{\omega_{ek}k(\hbar) - i(\gamma_{ek} - \gamma_{ph})}{\sqrt{[\omega_{ek}k(\hbar) - \omega - 2\Omega_k + \omega_{ek}k(\hbar) - i(\gamma_{ek} + \gamma_{ph})]^2 + 4\gamma_{ph}^2 k(\hbar)}}.
\end{align*}
\]
By employing these coefficients all necessary parameters for the polariton Hamiltonian are determined.

Now we can consider the dynamics of the polariton operators. The time evolution of the polariton operators is governed by the polariton Hamiltonian (7)
\[ \hat{p}_k = -\frac{1}{\hbar} [\hat{p}_k, \hat{H}] = -i\Omega_k \hat{p}_k. \]
Let us consider damping effects by taking into account a phenomenological damping term and noise operator in the dynamical equations of the polariton operators. Hence, the time evolution of the polariton operators is given by
\[ \hat{p}_k = -i\Omega_k \hat{p}_k - \Gamma_k \hat{p}_k + \hat{F}_\rho_k(t). \]
where \( \hat{F}_\rho_k(t) \) is the Langevin noise operator which depends on the reservoir variables and \( \Gamma_k \) is the damping constant of the \( k \)th polariton branch given by \( \Gamma_k = \frac{\gamma_{ph}}{2\Omega_k}. \) Correlation functions of the noise operators determine the physical properties of the system. The Langevin noise operators are such that their expectation values \( \langle \hat{F}_k \rangle \) vanish, but their second-order moments do not [22]. They are intimately linked with the global dissipation and in a Markovian environment they take the form
\[ \langle \hat{F}_\rho_k(t) \hat{F}_\rho_k(t') \rangle = 2\Gamma_k \delta(t - t'). \]
Neglecting the phonon effects by decreasing the temperature, other sources of damping like spontaneous recombination of exciton and photon loss are considered as Markovian processes. It follows, on solving equation (12), that
\[ \hat{p}_k(t) = \hat{p}_k(0) e^{-(i\Omega_k - \Gamma_k)t} + \int_0^t e^{-i(\Omega_k - \Gamma_k)(t - \tau)} \hat{F}_\rho_k(\tau) d\tau. \]
In this equation, we set the initial time equal to zero.

The power spectrum of the scattered light for statistical stationary fields is given by [23]
\[ S(r, \omega) = \frac{A(r)}{\pi} \text{Re} \int_0^\infty (\hat{E}^-(r, \tau) \hat{E}^+(r, t + \tau)) e^{i\omega \tau} d\tau, \]
where \( \hat{E}^\pm \) are the positive and negative frequency parts of the electric field operator. Expressing the cavity field operators in terms of the creation and annihilation operators we have
\[ S(r, \omega) = \frac{A(r)}{\pi} \text{Re} \int_0^\infty (\hat{a}^+(0) \hat{a}(\tau)) e^{i\omega \tau} d\tau. \]
Here, we set \( t = 0 \) and \( A(r) \) depends on the mode function of the cavity field.

Now we can express the field and exciton creation and annihilation operators in terms of the polariton ones:
\[ \hat{a} = v_1^+ \hat{p}_1 + v_2^+ \hat{p}_2, \quad \hat{b}_0 = k(\hbar)(u_1^+ \hat{p}_1 + u_2^+ \hat{p}_2), \]
and at time \( t \) we have
\[ \hat{a}(t) = v_1^+ \hat{p}_1(t) + v_2^+ \hat{p}_2(t). \]
Now to calculate the resonance fluorescence spectrum we have to determine the initial state of the system. We assume at \( t = 0 \) that the cavity field is in a coherent state \( |\alpha\rangle \), and the exciton subsystem is in its vacuum state. Under this condition, by using equation (14) the resonance fluorescence spectrum is obtained as
\[ S(r, \omega) = \frac{A(r)|\alpha|^2}{\pi} \times \left[ |v_1|^2 \frac{\Gamma_1}{(\omega - \Omega_1)^2 + \Gamma_1^2} + |v_2|^2 \frac{\Gamma_2}{(\omega - \Omega_2)^2 + \Gamma_2^2} \right]. \]
values of the deformation $q$. Material parameters are chosen as $\omega = 1.75$ eV, $\omega_{\text{ex}} = 1.75$ eV, $g = 200 \mu$ eV, $\gamma_{\text{ex}} = 20 \mu$ eV, $\gamma_{\text{ph}} = 40 \mu$ eV [24], $n = 1$ and $|q|^2 = 9$. As is clear when $q = 1$, the spectrum has a similar variation to the experimental results [24]. This figure shows that when $q = 1$ (nondeformed case) the power spectrum of the fluorescence light is a double peak centred at $\omega = \Omega_1$ and $\omega = \Omega_2$. By increasing the deviation of $q$ from 1, it is apparent from the different plots in this figure that the splitting between the two peaks increases and the height of one of the peaks decreases. This result has been reported in resonance fluorescence of excitons when the biexcitonic interaction is taken into account. It has been shown [4] that biexcitonic effects are a redshift of the transition frequencies, emergence of sidebands due to the switch-on forbidden transitions and asymmetry of the emission spectrum. The binding energy of a biexciton in a QD causes a shift in the spectrum of the system. In the present model the splitting of the spectrum (Rabi splitting) depends on the $q$-parameter. Hence, changing this parameter affects the spectrum. Then as a reason of deviation of excitons from the ideal Bose system we can consider the Coulomb interaction between them. On the other hand, $q$-deformed exciton operators depend on the total number of excitons, and biexciton interaction occurs when there is more than one exciton. This similarity gives the clue that the $q$-deformation can be considered as an effective approach to take into account the biexciton effects. As mentioned before, the $q$-parameter can depend on the size of the sample. The plotted resonance fluorescence spectrum in figure 1 makes clear some differences of optical properties of different size QD. For large values of $q$, compared with 1, the spectrum will reduce to one peak. This case is characteristic of the weak coupling regime.

2.2. q-deformed polaritons

In this subsection, we assume that the polariton operators are $q$-deformed operators. According to the $q$-deformed nature of the exciton system we assume the following algebra for the polariton operators:

$$\left[ \hat{b}_k, \hat{b}_k^\dagger \right] = \hat{b}_k \hat{b}_k^\dagger - s^{-1} \hat{b}_k^\dagger \hat{b}_k = \hat{s} \hat{b}_k,$$  \hspace{1cm} (20)

where $s$ denotes the deformation parameter corresponding to the polariton system and $\hat{b}_k$ numbers the operator for the $k$th polariton branch. The ordinary commutator for these operators is

$$\left[ \hat{b}_k, \hat{b}_k^\dagger \right] = |v_k|^2 \hat{k}(\hat{\mu}) + |v_k|^2 = \frac{s}{s + 1} [s \hat{b}_k^\dagger + s^{-1} \hat{b}_k] = M(\hat{b}_k).$$  \hspace{1cm} (21)

Using the same approach as the previous subsection we obtain the following set of equations for the coefficients of transformation:

$$\left( \omega_{\text{ex}} \hat{k}(\hat{\mu}) - i \gamma_{\text{ex}} - \Omega_k^2 M(n_k) \right) \mu_k + v_k g = 0,$$

$$u_k g \hat{k}(\hat{\mu}) + [\omega - i \gamma_{\text{ph}} - \Omega_k^2 M(n_k)] v_k = 0.$$  \hspace{1cm} (22)

From this set of equations we derive the deformed polariton spectrum as

$$\Omega_k^2 = \frac{\omega_{\text{ex}} \hat{k}(\hat{\mu}) + \omega - i (\gamma_{\text{ex}} + \gamma_{\text{ph}})}{2M(n_k)}$$

$$\pm \sqrt{[\omega_{\text{ex}} \hat{k}(\hat{\mu}) - \omega - i (\gamma_{\text{ex}} + \gamma_{\text{ph}})]^2 + 4g^2 \hat{k}(\hat{\mu})},$$  \hspace{1cm} (23)

and the transformation coefficients read as

$$u_k = \frac{\sqrt{M(n_k) [\omega - i \gamma_{\text{ph}} - \Omega_k^2 M(n_k)]}}{k(\hat{\mu}) [\omega - 2\Omega_k^2 M(n_k) + \omega_{\text{ex}} \hat{k}(\hat{\mu}) - i (\gamma_{\text{ex}} + \gamma_{\text{ph}})]},$$

$$v_k = \frac{\sqrt{M(n_k) [\omega_{\text{ex}} \hat{k}(\hat{\mu}) - i \gamma_{\text{ex}} - \Omega_k^2 M(n_k)]}}{\omega - 2\Omega_k^2 M(n_k) + \omega_{\text{ex}} \hat{k}(\hat{\mu}) - i (\gamma_{\text{ex}} + \gamma_{\text{ph}})}.$$  \hspace{1cm} (24)

By determining all the variables, the polariton Hamiltonian (diagonal Hamiltonian) will be determined. By applying the same procedure as before we derive the resonance fluorescence spectrum in this case as follows:

$$S(r, \omega) = \frac{A(r)|\alpha|^2 (|v_1|^2 + |v_2|^2)}{\pi} \times \sum_{i=1,2} |v_i|^2 \frac{1}{(\omega - \Omega_k^2 M(n_k))^2 + \Gamma_i^2}.$$  \hspace{1cm} (25)

Figure 2 shows the plot of $S(\omega)$ versus $\omega$. Parameters are chosen as $\omega = 1.75$ eV, $\omega_{\text{ex}} = 1.75$ eV, $g = 200 \mu$ eV, $\gamma_{\text{ex}} = 20 \mu$ eV, $\gamma_{\text{ph}} = 40 \mu$ eV, $n = 1$ and $|q|^2 = 9$. In all figures we have $q = 1$. The solid line corresponds to case $s = 1$. For the dotted line we have $s = 1.007$ and for the dashed line $s = 1.01$.

3. Interaction of light with two exciton modes

We now consider the interaction of one cavity mode with a QD when two exciton modes are coupled to the field mode. As before, we assume that the exciton system is expressed by the $q$-deformed operators. The total Hamiltonian of the system under consideration can be written as follows:

$$H = h c a^\dagger a + \sum_{i=1,2} \omega_{\text{ex}} \hat{b}_i^\dagger \hat{b}_i + \hbar g \sum_{i=1,2} (\hat{\alpha} \hat{b}_i^\dagger + \hat{a}^\dagger \hat{b}_i).$$  \hspace{1cm} (26)

We assume both excitons have the same coupling constant with the cavity mode. We solve this system as before by diagonalizing the Hamiltonian. For this purpose we perform the following transformation:

$$\hat{p}_k = u_k \hat{b}_{\alpha} + v_k \hat{a}.$$  \hspace{1cm} (27)
We consider the situation in which the polariton operators obey the nondeformed Bose statistics

\[
\hat{p}_i, \hat{p}_j^\dagger = |u_i|^2\hat{k}(\hat{n}_1) + |v_j|^2\hat{k}(\hat{n}_2) + |v_1|^2 = 1,
\]

where \( \hat{n}_i \) (\( i = 1, 2 \)) represents the number operator for each excitonic mode. As is clear in this case there are three polariton branches. Assuming the transformation (27) diagonalizes the Hamiltonian (26), this polariton Hamiltonian takes the following form:

\[
\hat{H} = \hbar \sum_k \Omega_k \hat{p}_k^\dagger \hat{p}_k,
\]

where summation is over all polariton branches. The following equation determines the polariton spectrum:

\[
(c - \Omega_k)[(d - \Omega_k)(\omega - i\gamma_\text{ph} - \Omega_k) - g^2 k(\hat{n}_1)]
- g^2 k(\hat{n}_2)(d - \Omega_k) = 0,
\]

where \( c = \omega_{\text{exc}} k(\hat{n}_1) - i\gamma_\text{exc} \) and \( d = \omega_{\text{exc}} k(\hat{n}_2) - i\gamma_\text{exc} \). By deriving the polariton spectrum the transformation parameters are obtained as

\[
u_k = \frac{g}{A} (c - \Omega_k)/(d - \Omega_k) - g^2 k(\hat{n}_2),
\]

\[
x_k = \frac{g^2 k(\hat{n}_1)}{A},
\]

\[
y_k = \frac{-g}{A} (c - \Omega_k)/(d - \Omega_k) - g^2 k(\hat{n}_2),
\]

where the parameter \( A \) is given by

\[
A = [g^2 k(\hat{n}_1) + (c - \Omega_k)]^2[(d - \Omega_k)(\omega - i\gamma_\text{ph} - \Omega_k) - g^2 k(\hat{n}_2)]^2 + g^6 k^2(\hat{n}_1)k(\hat{n}_2)^2.
\]

In this manner, all the parameters which appear in the polariton Hamiltonian are determined. By repeating the approach of the previous section the resonance fluorescence spectrum of a system with different initial conditions can be determined. If we assume at \( t = 0 \) the cavity mode is in the coherent state \( |\alpha\rangle \) and the QD is in the vacuum state \( |0\rangle \), the resonance fluorescence spectrum is given by

\[
S(r, \omega) = \frac{|\alpha|^2 A(r)}{\pi} \sum_k \frac{|v_k|^2 \Gamma_k}{\Gamma_k^2 + (\omega - \Omega_k)^2}.
\]

To show the complex structure (multi-peak structure) of this spectrum figure 3 presents the spectra on a logarithmic scale.

In previous sections, we considered some physical results of a q-deformed description of excitons. The q-deformed description can serve as a nonlinear description of excitons. It is well known that different kinds of nonlinearity in an exciton system lead to different orders of nonlinear response of the system [26, 27]. Therefore, we try to obtain the optical response of a driven quantum dot, where its optical excitations are considered as q-deformed systems. For this purpose we will calculate the coherent absorption of a QD in this regime. In this section, we neglect all damping effects and we consider the Hamiltonian of the system as follows:

\[
\hat{H} = \hbar \omega_\text{d} \hat{a}_\text{d}^\dagger \hat{a}_\text{d} + \hbar \omega_{\text{exc}} \hat{b}_\text{q}^\dagger \hat{b}_\text{q} + \hbar g (\hat{a}_\text{d}^\dagger \hat{a}_\text{d} + \hat{a}_\text{d} \hat{a}_\text{d}^\dagger).
\]

In the electron picture, the induced dipole moment by transition of an electron is described by \( \hat{\mu} = \hat{a}_\text{d}^\dagger \hat{a}_\text{d} + \hat{a}_\text{d} \hat{a}_\text{d}^\dagger \) [28]. The operator \( \hat{a}_\text{d}^\dagger \hat{a}_\text{d} \) is the creation (annihilation) operator for an electron in the valence band (level in the case of QD), and \( \hat{a}_\text{d} \hat{a}_\text{d}^\dagger \) is the creation (annihilation) operator for an electron in the conduction band. Hence, creation of an exciton is denoted by \( \hat{a}_\text{d}^\dagger \hat{a}_\text{d} = \hat{b}_\text{q}^\dagger \). Therefore, we can write the dipole operator of QD as \( \hat{\mu} = \hat{b}_\text{q}^\dagger \hat{b}_\text{q} \). The macroscopic polarization is the expectation value of the polarization operator.

The optical response function represents the reaction of the system to an external classic field \( F(t) \) coupled to the variables of the system [29], i.e., the dipole operator. Hence, we consider an external field as a pump source and we treat the reaction of the QD to it. Then the total Hamiltonian of the system is given by

\[
\hat{H} = \hbar \omega_\text{d} \hat{a}_\text{d}^\dagger \hat{a}_\text{d} + \hbar \omega_{\text{exc}} \hat{b}_\text{q}^\dagger \hat{b}_\text{q} + \hbar g (\hat{a}_\text{d}^\dagger \hat{a}_\text{d} + \hat{a}_\text{d} \hat{a}_\text{d}^\dagger)
- [\bar{d}_\text{ec} \cdot \vec{E}(t) \hat{b}_\text{q} + \bar{d}_\text{ec} \cdot \vec{E}(t) \hat{b}_\text{q}^\dagger],
\]

where \( \bar{d}_\text{ec} \) denotes the dipole matrix element. The Hamiltonian in the interaction picture has the form

\[
\hat{H}_\text{int} = \hbar g [(\hat{a}_\text{d}^\dagger \hat{a}_\text{d} + \hat{a}_\text{d} \hat{a}_\text{d}^\dagger) \hat{b}_\text{q} + \hat{a}_\text{d}^\dagger \hat{a}_\text{d} \hat{b}_\text{q}^\dagger]
- [\bar{d}_\text{ec} \cdot \vec{E}(t) e^{i\omega_{\text{exc}}t} \hat{b}_\text{q} + \bar{d}_\text{ec} \cdot \vec{E}(t) e^{i\omega_{\text{exc}}t} \hat{b}_\text{q}^\dagger].
\]
the time-dependent density matrix of excitons is given by
\[ \rho(t) = \rho(t_0)U(t,t_0)^{-1} \]
where \[ U(t,t_0) = \mathcal{T} \int_{t_0}^{t} \exp \left[ -\frac{i}{\hbar} \int_{t_0}^{t'} \hat{H}_{\text{int}}(t') \, dt' \right] \] is the timeordered evolution operator and \[ \rho(t_0) \] is the total density matrix of the system at initial time. We assume that the quantum field and the exciton system are both in the vacuum state. Therefore, the time-dependent density matrix of excitons is given by
\[ \hat{\rho}_{\text{exc}}(t) = \sum_n |n\rangle \hat{U}(t,t_0) |0\rangle_{\text{exc}} \langle 0| \langle 0\rangle_{\text{exc}} \hat{U}(t,t_0)^\dagger |n\rangle, \]
where summation is carried over the field state and the matrix elements of the time evolution operator are in the basis of field states. By using the Feynman disentanglement theorem \[ \hat{H}_{\text{int}} = \hat{H}_{\text{1}}(t) + \hat{H}_{\text{2}}(t), \]
we can write Hamiltonian in \[ \hat{H}_{\text{1}}(t) = \frac{\hbar g}{2} [\hat{a} \hat{b}_q^\dagger e^{-i[\omega_{ \text{exc}}, \hat{q}]} + \hat{a}^\dagger e^{i[\omega_{ \text{exc}}, \hat{q}]}, \]
\[ \hat{H}_{\text{2}}(t) = -i [\hat{a}_{\text{cv}} \cdot \hat{E}(t) e^{-i\omega_{ \text{exc}}(\hat{b}_q^\dagger + \hat{a}^\dagger \cdot \hat{E}(t) \hat{b}_q^\dagger e^{i\omega_{ \text{exc}}(\hat{b}_q^\dagger + \hat{a}^\dagger)}]. \]
As is clear, \[ \hat{H}_{\text{2}}(t) \] depends only on the exciton operators. The time evolution operator can be written as
\[ \hat{U}(t,t_0) = \mathcal{T} \int_{t_0}^{t} (\hat{H}_{\text{1}}(t') + \hat{H}_{\text{2}}(t')) \, dt' \]
\[ \times \exp \left[ -\frac{i}{\hbar} \int_{t_0}^{t} \hat{H}_{\text{2}}(s) \, ds \right]. \]
In this equation we use the Feynman notation \[ \hat{H}_{\text{int}} = \hat{H}_{\text{1}}(t) + \hat{H}_{\text{2}}(t). \]
These two exponential terms are not disentangled from each other. They are correlated, and in doing integration, we have to take into account the ordering of the operators. In the calculation of the matrix element of this operator in the basis of field states, the second exponential can be considered as an ordinary c-number function of \( t' \), because it is independent of the field operators:
\[ \langle i | \hat{U}(t,t_0) | j \rangle = \langle i | \mathcal{T} \int_{t_0}^{t} (\hat{H}_{\text{1}}(t') + \hat{H}_{\text{2}}(t')) \, dt' \]
\[ \times \exp \left[ -\frac{i}{\hbar} \int_{t_0}^{t} \hat{H}_{\text{2}}(s) \, ds \right] | j \rangle. \]
On the other hand, we consider all the exciton operators in \[ \hat{H}_{\text{1}}(t) \] as ordinary c-number functions, and we can write
\[ \langle i | \hat{U}(t,t_0) | j \rangle = \langle i | \mathcal{T} \int_{t_0}^{t} (\hat{H}_{\text{1}}(t') + \hat{H}_{\text{2}}(t')) \, dt' \]
\[ \times \exp \left[ -\frac{i}{\hbar} \int_{t_0}^{t} \hat{H}_{\text{2}}(s) \, ds \right] | j \rangle. \]
where \( B(t) \) is an ordinary function corresponding to the exciton operators. As is clear, this matrix element is a function of the exciton operators. The influence of the exciton system is completely contained in this operator functional and factored term in \[ (42). \] By using the Feynman theorem, the above matrix element can be written as
\[ \langle i | \exp \left[ -\frac{i}{\hbar} \int_{t_0}^{t} \hat{a}_q^\dagger e^{i\omega_{ \text{exc}} B(t') \, dt'} \right] \]
\[ \times \exp \left[ -\frac{i}{\hbar} \int_{t_0}^{t} \hat{a}_q^\dagger e^{i\omega_{ \text{exc}} B(t') \, dt'} \right] | j \rangle, \]
where \[ \hat{a}_q^\dagger = \hat{V}^{-1}(t) \hat{a}_q \hat{V}(t), \] and
\[ \hat{V}(t) = \exp \left[ -\frac{i}{\hbar} \int_{t_0}^{t} B(t') e^{i\omega_{ \text{exc}} \, dt'} \right]. \]
In this manner, the density matrix of the exciton system takes the following form:
\[ \hat{\rho}_{\text{exc}}(t) = \sum_n \frac{1}{n!} S_1 (\hat{a}_q, \hat{b}_q^\dagger) |0\rangle_{\text{exc}} \langle 0| S_2 (\hat{a}_q, \hat{b}_q^\dagger). \]
where
\[ S_1 (\hat{a}_q, \hat{b}_q^\dagger) = -g_{\omega_{ \text{exc}} \hat{a}_q^\dagger e^{-i[\omega_{ \text{exc}}, \hat{q}]} + \hat{a}_q^\dagger e^{i[\omega_{ \text{exc}}, \hat{q}]}, \]
\[ S_2 (\hat{a}_q, \hat{b}_q^\dagger) = -g_{\omega_{ \text{exc}} \hat{b}_q^\dagger e^{-i[\omega_{ \text{exc}}, \hat{q}]} + \hat{b}_q e^{i[\omega_{ \text{exc}}, \hat{q}]}, \]
\[ L(\hat{\rho}_{\text{exc}}) = \hat{b}_q^\dagger \hat{b}_q e^{-i[\omega_{ \text{exc}}, \hat{q}]}, \]
\[ f (\hat{b}_q, \hat{b}_q^\dagger) = \hat{b}_q^\dagger \hat{b}_q e^{-i[\omega_{ \text{exc}}, \hat{q}]}, \]
Figure 4. Plots of the absorption spectrum versus ω. We consider the 1s-exciton and the corresponding parameters are chosen as g = 200 μeV and αex = 1574 meV. Solid plot corresponds to nondeformed case q = 1. For the dotted line q = 1.01 and for the dashed line q = 0.99.

\[
\times \int \int \int \int \int d\vec{v}_c \cdot \vec{E}(t') \hat{d}_{\vec{v}_c} \cdot \vec{E}(r) \hat{d}_{\vec{v}_c} \cdot \vec{E}(s) \\
\times e^{i\alpha h_0(1-r-kn+1)'} dt' dr ds \\
+ \frac{i}{2h^3} \sqrt{f_q(n)h_0(n)!} e^{-\frac{i}{h}L(0)} \sqrt{f_q(n)!f_q(n)} \\
\times \int \int \int \int d\vec{v}_c \cdot \vec{E}(t') \hat{d}_{\vec{v}_c} \cdot \vec{E}(r) \hat{d}_{\vec{v}_c} \cdot \vec{E}(s) \\
\times e^{i\alpha \omega_{(k Ln)(r-s)+1}} dt' dr ds \\
- \frac{i}{2h^3} \sqrt{f_q(n+1)!f_q(n+1)!} e^{-\frac{i}{h}L(1)} \\
\times \hat{E}(s) \hat{d}_{\vec{v}_c} \cdot \vec{E}(t') e^{-i\alpha h_0(1-r-kn+1)'} dt' dr ds \\
- \frac{i}{2h^3} \sqrt{f_q(n)h_0(n+1)!} e^{-\frac{i}{h}L(1)} \\
\times \sqrt{f_q(n+1)!f_q(n+1)!} \int \int \int d\vec{v}_c \cdot \vec{E}(r) \hat{d}_{\vec{v}_c} \cdot \vec{E}(s) \hat{d}_{\vec{v}_c} \cdot \vec{E}(t') \\
\times e^{i\alpha \omega_{(k Ln)(r-s)+1}} dt' dr ds, \tag{49}
\]

where \( h_j(n) = \left( \frac{-1}{j!} \right)^{j-1/h_0(1-r-kn+1)'} e^{\alpha h_0(n+j'-r-kn+1)'} \), \( j = 1, 0 \) and \( f_q(n) = \sqrt{\frac{d - q - r}{q - q}} \). This equation shows that in the considered conditions the second-order response function is equal to zero. Now we can calculate the linear and nonlinear electric susceptibilities of the exciton system from this equation. Generalized linear and nonlinear absorption spectra of this system are shown in figures 4–6 for different values of q-parameter. In these plots, the 1s-exciton is considered. In these figures we choose h = e = 1, g = 200 μeV and αex = 1574 meV. Figure 4 shows the plots of the linear absorption spectra and figure 6 shows the plots of the nonlinear spectra. On the other hand, the three-dimensional plot of linear absorption coefficients is given in figure 5. It is clear that changes of q-parameter strongly affect absorption spectra of the system. These figures show that in the presence of q-deformation the absorption of the probe beam shows a complex structure: a multiple-like absorption pattern appears with one strong peak and some side bands. The presence of these side bands is a signature of the optical generation of a nonlinear exciton (a q-deformed exciton). The negative part of the absorption spectrum demonstrates gain of the probe beam. Due to the resonance interaction of a pump with the exciton transition, the gain effect comes from the coherent energy exchange between the pump and probe beams through the QD nonlinearity. The obtained absorption spectra are very similar to the experimental results [31]. In [31] the absorption spectra of a driven charged QD are derived experimentally. A charged QD is a nonlinear medium and is similar to our model. Therefore it can be considered as an experimental test of our model.

Figure 5. 3D plots of the absorption spectrum versus ω and the deformation parameter q. The physical parameters are the same as in figure 4.

Figure 6. Plots of the nonlinear absorption spectrum versus ω. We consider 1s-exciton and the corresponding parameters are chosen as g = 200 μeV and αex = 1574 meV. The solid plot corresponds to the nondeformed case q = 1. For the dotted plot q = 1.01. For the dashed plot q = 0.99.

5. Conclusion

The q-deformed description of excitons in a QD and its physical consequences were considered. We showed that
increasing the $q$-parameter will lead to increase of splitting between peaks in the spectrum and asymmetry of the spectrum. Similar effects were observed when biexciton effects were taken into account. In experiments with QD it is shown \[24\] that the same results are obtained at different temperatures. Then we can associate this physical parameter as a source of $q$-deformation. The temperature dependence of the emission energy of the system can be attributed to the change in the refractive index of its active medium with temperature. We have derived the optical response of a QD with a $q$-deformed exciton. As mentioned before, a $q$-deformed description of excitons will lead to the dependence of the optical response on the $q$ parameter. Hence, due to the wide range of $q$ parameter and its effects on the optical response we can consider some parameters like temperature and interaction between the excitons which affect the optical response of QD as sources of $q$-deformation of excitons. As mentioned, the relation of quantum statistics of excitons in the QD and the size of the QD has been considered. Then we can consider the ratio of exciton Bohr radius to the dimension of the system and exciton population as two main sources of $q$-deformation. The $q$-deformed operator depends on the total number of associated particles of the system. Therefore, we can interpret the $q$-deformed operator as an operator which consists of effects of other excitations of the system implicitly. Then it is reasonable to consider this description as an effective description which takes into account some nonlinearities in the exciton system. As we saw in the case of the interaction of light with two excitons, when $q = 1$ this system showed a two-peak spectrum. By increasing the deviation of exciton from Bose statistics, the spectrum becomes multi-peak. Due to the nonlinear nature of the $q$-deformed exciton we showed that different orders of the nonlinear response function of this system can be calculated. From coincidence of the obtained results and experimental results, we can conclude that a $q$-deformed description of excitons can be a considerable model for excitons. By comparing the results obtained in this paper with experimental ones we can investigate the origin of this description of excitons. As pointed out, the ratio of the system dimension to the Bohr radius of exciton is one of the sources of deviation of excitons from usual bosons. The obtained results are very similar to the effects of the exciton–exciton interaction \[4, 32\] which relates to the exciton population and biexciton binding energy. On the other hand, it is shown that \[1\] the exciton density is another source of the deviation from ordinary bosons. To sum up, we attribute the origin of $q$-deformation of the excitons to their density, their mutual interactions, confinement size and other parameters which cause fluctuation of optical response of the system. A $q$-deformed description of an active medium causes the optical properties of system to depend on the $q$-parameter. Then, it seems that the physical effects of the parameters which affect the optical properties of the active medium (like refractive index) can be described by the present approach. The $q$-parameter can be considered as a variation parameter whose values can be obtained from comparison of theoretical and experimental results.

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