THE CUP PRODUCT OF BROOKS QUASIMORPHISMS

MICHELLE BUCHER AND NICOLAS MONOD

ABSTRACT. We prove the vanishing of the cup product of the bounded cohomology classes associated to any two Brooks quasimorphisms on the free group. This is a consequence of the vanishing of the square of a universal class for tree automorphism groups.

1. INTRODUCTION

Although bounded cohomology found a great variety of applications, it remains so mysterious that even for a (non-abelian) free group $F$ of finite rank, we do not know much about it.

More precisely, beyond the trivial case of $H^1_b(F, R) = 0$, it is known that both $H^2_b(F, R)$ and $H^3_b(F, R)$ are infinite-dimensional. However, $H^{n\geq 4}_b(F, R)$ remains completely unknown; in particular, we do not know whether $H^4_b(F, R)$ vanishes or not.

The first infinite family of non-trivial classes in $H^2_b(F, R)$ are provided by Brooks quasimorphisms [1] (anticipated by Johnson [7, 2.8] and Rhemtulla [12]); we recall their definition. Pick any reduced word $w$ in a choice of free generators for $F$ and consider the counting function $f_w : F \to R$ defined on $g \in F$ by

$$f_w(g) = \#\{\text{occurrences of } w \text{ in } g\} - \#\{\text{occurrences of } w \text{ in } g^{-1}\}.$$

If $w$ is reduced to one letter (or trivial), then $f_w$ is a homomorphism. In all other cases, $f_w$ is a quasimorphism and defines a non-trivial class $\beta_w \in H^2_b(F, R)$ unless $w$ is conjugated to a power of a letter. The space spanned by all these $\beta_w$ is infinite-dimensional [5] and is dense in $H^2_b(F, R)$ for a suitable topology of pointwise convergence [4, 5.7]. (Following Brooks, we allow overlaps when counting occurrences, whilst other authors do not; see [5, p. 251] for the density in our setting.)

The aim of this note is to show that the cup product of any two elements in this dense sub-space vanishes in $H^4_b(F, R)$.

**Theorem 1.** Let $\beta_w, \beta_{w'} \in H^2_b(F, R)$ be the bounded cohomology classes associated to two Brooks quasimorphisms on $F$.

Then $\beta_w \smile \beta_{w'} = 0$ in $H^4_b(F, R)$.

We were informed by N. Heuer that he independently obtained a similar result [6] by methods completely different from ours.

We can give a rather transparent proof of Theorem 1 by realizing bounded cohomology with the aligned chains that we introduced in [2]. This simplifies the combinatorics and allows us to exhibit a natural explicit coboundary for the cup product.

Moreover, we can carry out this task at once for all $w, w'$ simultaneously — by working instead with the universal class $[\omega]$ that we now proceed to define (similar constructions were considered in [10 §2], in [11 7.11] and in [3 §9]).
Let \( T = (V, E) \) be a locally finite tree with Serre’s conventions, which means in particular that an element of \( E \) represents an oriented edge and that \( E \) is endowed with a fixed-point-free involution \( e \mapsto \overline{e} \) reversing the orientation. We denote by \( P \) the set of paths, namely sequences \( p = (e_1, \ldots, e_n) \) of successive edges \( e_i \in E \) without backtracking. The reverse path is \( \overline{p} = (\overline{e_n}, \ldots, \overline{e_1}) \) and \( n \) is the length of \( p \). Given two vertices \( x, y \) we denote by \([x, y]\) the path connecting them. The path module \( R_{alt}[P] \) is the \( \text{Aut}(T) \)-module of all elements of the free vector space \( R[P] \) that change sign when replacing a path by its reverse. We define an \( \text{Aut}(T) \)-equivariant map \( \theta : V^2 \to R_{alt}[P] \) by setting
\[
\theta(x_0, x_1)(p) = \pm 1
\]
if \( p \) (respectively \( \overline{p} \)) is contained as a sub-path in \([x_0, x_1]\), and 0 in all other cases. We define
\[
\omega = d\theta : V^3 \to R_{alt}[P]
\]
as the coboundary of \( \theta \). We recall here that \( d \) will always be the usual alternating sum of the maps omitting the individual variables; we refer to the preliminaries below for explicit values of \( \omega \).

In order to view \( \omega \) as a cocycle in bounded cohomology, we need to specify a norm on \( R_{alt}[P] \); of course, \( \theta \) should be unbounded for this norm since otherwise the class of \( \omega \) would be trivial. The specific norm is however not too relevant; one property we want is that, when restricted to the free vector space on the set of paths of length \( n \), it is equivalent to the \( \ell^1 \)-norm \( \| \cdot \|_{n, 1} \). One explicit choice is the norm \( \| \cdot \|_{\text{path}} = \sum_{n \geq 1} \frac{1}{n!} \| \cdot \|_{n, 1} \) whose normalisation factor \( 1/n! \) is an arbitrary way to ensure uniform boundedness statements in the proofs.

Furthermore, we write \( \mathcal{P} \) for the completion of \( R_{alt}[P] \). Indeed, even though our arguments will be explicit and finitary, the general tools of continuous bounded cohomology work best with Banach spaces.

A choice of free generators for the free group \( F \) determines an embedding of \( F \) into the automorphism group of the corresponding tree \( T \). We view \( \omega \) as a cocycle for the continuous bounded cohomology \( H^*_{cb} \) of the locally compact group \( \text{Aut}(T) \).

Moreover, every path in \( T \) is labelled by a reduced word in \( F \). Thus, given a reduced word \( w \), we can define an \( F \)-invariant bounded linear form \( \lambda_w \) on \( R_{alt}[P] \), hence also on \( \mathcal{P} \), by specifying its values on individual paths as follows:
\[
\lambda_w(p) = \begin{cases} 
1 & \text{if } w \text{ labels } p, \\
-1 & \text{if } w \text{ labels } \overline{p}, \\
0 & \text{otherwise.}
\end{cases}
\]

This definition ensures that if \( g \in F \) labels \([x_0, x_1]\), then
\[
\lambda_w \circ \theta(x_0, x_1) = f_w(g).
\]
Therefore, we deduce immediately the following relation between the universal class \([\omega]\) and individual quasimorphisms.

**Proposition 2.** Let \( \beta_w \in H^2_b(F, R) \) be the bounded cohomology class associated to a Brooks quasimorphism on \( F \) for the chosen generators. Then \( \beta_w \) is the image of the class of \( \omega \) under the map
\[
H^2_{cb}(\text{Aut}(T), \mathcal{P}) \xrightarrow{\text{rest}} H^2_b(F, \mathcal{P}) \xrightarrow{(\lambda_w)} H^2_b(F, R),
\]
where the first arrow is the restriction map and the second is induced by \( \lambda_w \). \( \square \)
The cup product of two elements of $H^2_{cb}(\text{Aut}(T), \mathcal{P})$ is a class in $H^1_{cb}$ with values in the tensor product module $\mathcal{P} \otimes \mathcal{P}$, which we can also (projectively) complete to $\hat{\mathcal{P}} \otimes \hat{\mathcal{P}}$ (see the preliminaries for the norm). The naturality of the cup product now implies:

**Corollary 3.** Given two reduced words $w$ and $w'$, we keep all the above notation.

Then $[\omega] \sim [\omega]$, viewed as a class with coefficients in $\hat{\mathcal{P}} \otimes \hat{\mathcal{P}}$, is mapped to $\beta_w \sim \beta_{w'}$

$$H^1_{cb}(\text{Aut}(T), \hat{\mathcal{P}} \otimes \hat{\mathcal{P}}) \longrightarrow H^1_{cb}(F, \mathbb{R})$$

under the restriction followed by $(\lambda_w \otimes \lambda_{w'})_*$. \hfill \Box

In view of Corollary 3, Theorem 1 is now an immediate consequence of the following vanishing result for the square of the universal class $[\omega]$.

**Theorem 4.** Let $T$ be a locally finite tree.

Then the class of $\omega \sim \omega$ vanishes in $H^4_{cb}(\text{Aut}(T), \mathcal{P} \otimes \mathcal{P})$.

The remainder of this note is devoted to the proof of Theorem 4.

2. Preliminaries

The cup-product of bounded cocycles ranging in $R_{\text{all}}[P]$ or $\mathcal{P}$ is trivially bounded for any cross-norm on the tensor product with underlying norm $\| \cdot \|_{\text{path}}$ on each factor. For cross-norms, we refer to [3]. We shall choose the projective cross-norm $\| \cdot \|_{\pi}$ and denote by $\mathcal{P} \otimes \mathcal{P}$ the corresponding completion. Since this is the largest cross-norm, the vanishing result of Theorem 4 with respect to $\| \cdot \|_{\pi}$ implies the corresponding vanishing for any other cross-norm.

We say that a path $p$ is **carried** by a path $q$, and write $p \subseteq q$, if either $p$ or $\overline{p}$ is contained in $q$ as a sub-path. We attach a sign $\pm$ to these two cases, referred to as the **orientation** of $p$ relative to $q$. We define the **interior** $\text{Int}(p) \subseteq V$ of a path $p$ to consist of all the vertices of the path except its two extremities.

Recall that any three vertices $x_0, x_1, x_2 \in V$ determine a **center** $c \in V$ characterized as the unique common vertex of all $[x_i, x_j]$. Given a path $p$, the definition of $\omega$ now shows that $\omega(x)(p) = \pm 1$ when $p$ is carried by some $[x_i, x_j]$ and $c \in \text{Int}(p)$, and that $\omega(x)(p)$ vanishes otherwise.

A path can contain at most $n - 1$ sub-paths of length $n$ containing a given vertex in their interior. Therefore, considering all three configurations and two orientations, we can bound the norm of $\omega$ by

$$\|\omega(x)\|_{\text{path}} \leq 3 \cdot 2 \cdot \sum_{n \geq 1} \frac{1}{n!} (n - 1) = 6,$$

witnessing that $\omega$ is indeed uniformly bounded.

Recall that a $(q + 1)$-tuple $(x_0, \ldots, x_q) \in V^{q+1}$ is **aligned** if the vertices $x_0, \ldots, x_q$ are contained in some geodesic segment of $T$. This tuple is furthermore said to be **coherent** if these $q + 1$ vertices are distinct and in increasing order for one of the two linear orders induced on $\{x_0, \ldots, x_q\}$ by any such segment. We denote by $V^{q+1}_{\text{coh}} \subseteq V^{q+1}$ the set of coherent aligned tuples.

Below, we shall be particularly interested in the above description of $\omega(x)$ specialized to coherent triples $x \in V^3_{\text{coh}}$. In that case, $\omega(x)(p) = \pm 1$ if $x_1 \in \text{Int}(p)$ and $p$ is carried by
[x_0, x_2], with the sign given by the orientation of p relative to [x_0, x_2], and vanishes in all other cases.

\[ p \quad \Rightarrow \quad \omega = 1. \]

3. The coherent resolution

Let \( E \) be any isometric Banach \( \text{Aut}(T) \)-module and recall that \( H^q_{\text{cb}}(\text{Aut}(T), E) \) can be computed with the (non-augmented) complex \( \ell^\infty(V^{q+1}, E)^{\text{Aut}(T)} \) of \( \text{Aut}(T) \)-equivariant elements of the resolution

\[
\begin{align*}
(i) & \quad 0 \rightarrow E \rightarrow \ell^\infty(V, E) \rightarrow \ell^\infty(V^2, E) \rightarrow \ell^\infty(V^3, E) \rightarrow \cdots
\end{align*}
\]

(see e.g. [2] 4.5.2). There is a natural restriction map to the complex \( \ell^\infty(V^{q+1}, E) \) on coherent tuples, but we warn the reader that the latter is not a resolution of \( E \).

Recall that an element of \( \ell^\infty(V^{q+1}, E) \) is called alternating if any permutation \( \sigma \) of the variables corresponds to the multiplication by the signature \( \text{sign}(\sigma) \). We denote by \( \tau_q \) the permutation of \( \{0, \ldots, q\} \) that reverses the order and observe that its signature is \( (-1)^{\lfloor \frac{q}{2} \rfloor} \), where \( \lfloor \cdot \rfloor \) denotes the integer part. Consider the \( \text{Aut}(T) \)-equivariant involution \( \hat{\tau}_q \) of \( \ell^\infty(V^{q+1}, E) \) defined by \( \hat{\tau}_q(\alpha)(x) = \text{sign}(\tau_q)\alpha(x^{\tau_q}) \). Being an involution, it induces an eigenspace decomposition

\[
\ell^\infty(V^{q+1}, E) = \ell^\infty_+(V^{q+1}, E) \oplus \ell^\infty_-(V^{q+1}, E)
\]

which is preserved by \( \text{Aut}(T) \). Although \( \ell^\infty_+(V^{q+1}, E) \) is not a resolution, we have:

**Proposition 5.** The sub-complex

\[
(ii) \quad 0 \rightarrow E \rightarrow \ell^\infty_+(V, E) \rightarrow \ell^\infty_+(V^2, E) \rightarrow \ell^\infty_+(V^3, E) \rightarrow \cdots
\]

is a resolution. Moreover, the map

\[
A_q \circ \text{rest} : \ell^\infty(V^{q+1}, E) \rightarrow \ell^\infty_+(V^{q+1}, E)
\]

from (i) to (ii) obtained by restriction followed by the projection \( A_q = (\hat{\tau}_q + \text{Id})/2 \) yields an isomorphism between \( H^q_{\text{cb}}(\text{Aut}(T), E) \) and the cohomology of the complex

\[
(iii) \quad 0 \rightarrow \ell^\infty_+(V^{1, E})^{\text{Aut}(T)} \rightarrow \ell^\infty_+(V^{2, E})^{\text{Aut}(T)} \rightarrow \ell^\infty_+(V^{3, E})^{\text{Aut}(T)} \rightarrow \cdots
\]

**Proof.** Following [2], we denote by \( \ell^\infty_-(V^{q+1}, E) \) the sub-space of alternating maps defined on aligned tuples. The restriction to coherent tuples thus induces an isomorphism

\[
\ell^\infty_-(V^{q+1}, E) \cong \ell^\infty_+(V^{q+1}, E).
\]

Therefore, the first statement is simply a reformulation of Corollary 8 of [2]. Moreover, as observed there, the modules \( \ell^\infty_+(V^{q+1}, E) \) are relatively injective in the sense of bounded cohomology because the \( \text{Aut}(T) \)-action on the set of aligned tuples is proper, see [9] 4.5.2]. More precisely, \( \ell^\infty_+(V^{q+1}, E) \) is a direct factor of the larger space without the alternation condition, to which [9] 4.5.2 applies, and one concludes as in [9] 7.4.5 by an alternation map.
A direct computation using the relation \( \text{sign}(\tau_q) \cdot \text{sign}(\tau_{q+1}) = (-1)^{q+1} \) shows that \( \hat{\tau}_q \) is a chain map. In particular, \( \hat{\tau}_q \) automatically preserves the decomposition \( \ell_\infty(V^{q+1}_{\text{coh}}, E) \) and \( A_q \) is a chain map as well. Now the second statement follows by general cohomological principles (see e.g. §7.2 in [9]). □

4. A PRIMITIVE FOR THE SQUARE OF \( \omega \) ON COHERENT TUPLES

We define an \( \text{Aut}(T) \)-equivariant map

\[
B : V^4_{\text{coh}} \longrightarrow R_{\text{alt}}[P] \otimes R_{\text{alt}}[P]
\]

by setting, for any coherent 4-tuple \( x \) and any paths \( p_1, p_2 \in P, \)

\[
B(x)(p_1, p_2) = \pm 1
\]

whenever all the following hold:

- both \( p_1 \) and \( p_2 \) are carried by \([x_0, x_3] \),
- the interior of \( p_1 \) and of \( p_2 \) are disjoint,
- \( x_i \in \text{Int}(p_i) \) for each \( i = 1, 2 \).

In that case, the sign \( \pm 1 \) is the product of the orientations of \( p_1 \) and of \( p_2 \) relative to \([x_0, x_3] \).

All this is perhaps much more intuitive in a picture, drawn for two of the four orientation possibilities:

- \[
\begin{array}{cccccc}
\longrightarrow & p_1 & \longrightarrow & p_2 \\
& x_0 & x_1 & x_2 & x_3
\end{array}
\]

\[\Rightarrow B = 1.\]

- \[
\begin{array}{cccccc}
\longrightarrow & p_1 & \longrightarrow & p_2 \\
& x_0 & x_1 & x_2 & x_3
\end{array}
\]

\[\Rightarrow B = -1.\]

In all other cases, we set \( B(x)(p_1, p_2) = 0 \).

**Proposition 6.** We have \( dB(x) = \omega \bowtie \omega(x) \) for every coherent 5-tuple \( x \).

**Proof.** Let \( p_1, p_2 \in P \). By definition,

\[
\omega \bowtie \omega(x)(p_1, p_2) = \omega(x_0, x_1, x_2)(p_1) \cdot \omega(x_2, x_3, x_4)(p_2).
\]

Thus, \( \omega \bowtie \omega(x)(p_1, p_2) \neq 0 \) if and only if all the following hold:

(iv) \[
\begin{cases}
  x_1 \in \text{Int}(p_1) \text{ and } p_1 \sqsubseteq [x_0, x_2], \\
  x_3 \in \text{Int}(p_2) \text{ and } p_2 \sqsubseteq [x_2, x_4].
\end{cases}
\]

As for \( dB \), we observe that \( dB(x)(p_1, p_2) = 0 \) unless possibly

(v) \[
\begin{cases}
  p_1, p_2 \text{ have disjoint interior and are carried by } [x_0, x_4], \\
  x_1 \text{ or } x_2 \in \text{Int}(p_1), \\
  x_2 \text{ or } x_3 \in \text{Int}(p_2).
\end{cases}
\]

In the case when Conditions (iv) are not satisfied, Conditions (v) are not either; therefore in that case \( dB \) and \( \omega \bowtie \omega \) agree since they both vanish.
Suppose now that Conditions (v) are satisfied. By symmetry, we can assume that the orientation of \( p_1 \) and \( p_2 \) are compatible with the orientation of \([x_0, x_4]\) (and hence of \([x_0, x_3]\) and of \([x_1, x_4]\)). Since \( p_1 \) and \( p_2 \) have disjoint interior, \( x_2 \) is contained in at most one of \( \text{Int}(p_1) \) or \( \text{Int}(p_2) \); we can suppose that it is not contained in \( \text{Int}(p_1) \), the other case being completely analogous. We have now three cases:

**First case:** \( x_1 \in \text{Int}(p_1), x_2 \notin \text{Int}(p_1) \cup \text{Int}(p_2) \) and \( x_3 \in \text{Int}(p_2) \).

The value of \( \omega \sim \omega(x)(p_1, p_2) \) is \(+1\), while the only non-zero summand in

\[
dB(x)(p_1, p_2) = \sum_{i=0}^{4} (-1)^i B(\ldots, \hat{x}_i, \ldots)
\]

is the one for \( i = 2 \), which is indeed also \(+1\).

**Second case:** \( x_1 \in \text{Int}(p_1) \) and \( x_2, x_3 \in \text{Int}(p_2) \).

Condition (iv) is not satisfied and hence \( \omega \sim \omega \) vanishes. As for \( dB \), only the summands for \( i = 2 \) and \( i = 3 \) are non-zero and cancel out to give \( dB(x)(p_1, p_2) = 0 \).

**Third case:** \( x_1 \in \text{Int}(p_1), x_2 \in \text{Int}(p_2) \) and \( x_3 \notin \text{Int}(p_2) \).

Again, condition (iv) is not satisfied and \( \omega \sim \omega \) vanishes. As for \( dB \), only the summands for \( i = 3 \) and \( i = 4 \) are non-zero and cancel out to give \( dB(x)(p_1, p_2) = 0 \). □

5. Proof of Theorem 4

We first verify that the primitive \( B \) is bounded.

**Lemma 7.** The map \( B \) is uniformly bounded on \( V^4_{\text{coh}} \) with respect to the projective norm \( \| \cdot \|_\pi \).

**Proof.** Fix \( x \in V^4_{\text{coh}} \) and consider abusively any path \( p_i \) as an element of \( \mathbb{R}[P] \). By definition of the projective cross-norm, we can bound \( \| B(x) \|_\pi \) by \( \sum(\|p_1\|_{\text{path}} \cdot \|p_2\|_{\text{path}}) \), where the sum runs over all pairs \((p_1, p_2)\) on which \( B(x) \) does not vanish. Arguing as in our estimate for the norm of \( \omega \), we have at most \( 2(n_1 - 1)(n_2 - 1) \) such pairs whenever we fix the length \( n_i \) of
each $p_i$. Since on the other hand we have $\|p_i\|_{\text{path}} = 1/n_i!$, we conclude that $B(x)$ has norm at most
\[
\sum_{n_1, n_2} \frac{2(n_1 - 1)(n_2 - 1)}{n_1! n_2!} = 2 \left( \sum_n \frac{n - 1}{n!} \right)^2 = 2.
\]
\[\square\]

At this point, we conclude that $A_3(B)$ belongs to $\ell_\infty^c(V^1_{\text{coh}}, \mathcal{P})$. Since $A_*$ is a chain map (as pointed out in the proof of Proposition 5), we deduce from Proposition 6 that we have $A_4(\omega \sim \omega) = dA_3(B)$. Now Proposition 5 implies that the class of $\omega \sim \omega$ vanishes, completing the proof of Theorem 4.
\[\square\]

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University of Geneva, 1211 Geneva 4, Switzerland  
E-mail address: michelle.bucher-karlsson@unige.ch

EPFL, SB-MATH-EGG, 1015 Lausanne, Switzerland  
E-mail address: nicolas.monod@epfl.ch