Type Synthesis and Analysis of a New Class of Bifurcation 3T2Rv Parallel Mechanisms with Variable/invariable Rotational Axes

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ABSTRACT Since the parallel mechanism has many advantages, such as high strength, high precision and so on, many industrial applications require mechanisms with three-dimensional translational and two-dimensional rotational motion (3T2R), 3T2R parallel mechanisms are widely studied. The direction of the first rotation axis is invariant for the existing 3T2R parallel mechanism, and the direction of the second rotation axis is changeable only with the first rotation axis, while the first and second rotation axes themselves are invariant. Based on bifurcation 1Rv parallel mechanisms with a variable/invariable rotation axis, a new 3T2Rv parallel mechanism with two variable/invariable rotational axes is proposed by using finite screw theory. The assembly condition and actuation scheme of this kind of 3T2Rv parallel mechanism are analyzed. The novel bifurcation 3T2Rv parallel mechanism consists of four motion modes: the invariable-invariable axis motion mode, invariable-variable axis motion mode, variable-invariable axis motion mode, and variable-variable axis motion mode. Compared with the traditional fixed axis mechanism, the variable axis mechanism can produce posture changes in other directions in the process of posture adjustment of the mechanism. The 3T2Rv mechanism with two variable axes rotation proposed in this paper can meet the assembly requirements of the third dimensional rotation that the traditional fixed axis 3T2R mechanism can not complete. The existing 3T2R parallel mechanism with two invariable rotation axes is extended to the bifurcation 3T2Rv parallel mechanism with two variable/invariable rotation axes.

INDEX TERMS Variable/invariable rotational axis, 3T2Rv, parallel mechanism, type synthesis, bifurcation motion.

I. INTRODUCTION
Bifurcation parallel mechanism is a kind of parallel mechanism whose degrees of freedom and motion characteristics are changed after passing through some special positions, which are called motion bifurcation points or motion bifurcation singularity points [1-3]. With the rapid development of science and technology, the requirements for multi functions, bearing capacity and precision of the mechanisms are increasing in many modern fields. The bifurcation parallel mechanism has become one of the research hotspots in the field of mechanism and robotics, combining the characteristics of variable degrees of freedom with the change of task of the bifurcation mechanism and the advantages of high load-carrying capacity and high precision of the parallel mechanism. In recent years, the motion bifurcation characteristics of parallel mechanisms have been widely studied. Based on displacement manifold theory, Zeng et al. [4] proposed a synthesis method of generalized parallel mechanisms with motion bifurcation, in which motion bifurcation occurs at singular positions. Jun et al. [5] proposed a class of bifurcation parallel mechanisms with planar and spherical motions. Wang [6] analyzed the bifurcation motion of symmetrical Stewart parallel mechanism. Refaat et al. [7] studied a class of parallel mechanism tools with motion bifurcation. Zhang et al. [8] proposed a new metamorphic mechanism and analyzed its motion bifurcation characteristics and then studied the motion bifurcation characteristics of a special structure 3-PUP parallel mechanism [9]. In recent years, researchers have conducted considerable researches on the type synthesis of 3T2R parallel mechanisms. Based on the theory of linear
transformation and geometric analysis, Benyamin Motefavali et al. [10] proposed a method for the type synthesis of a class of 3T2R parallel mechanisms and classified the mechanisms. Amine et al. [11] analyzed the singularity of 3T2R parallel mechanisms using Grassmann-Cayley algebra and Grassmann geometry. Lin et al. [12] proposed a class of 3T2R parallel mechanisms. The position of the platform of the parallel mechanism is determined by the input of the lower part, while the orientation is mainly determined by the input of the upper part. Chen et al. [13] synthesized a class of fully decoupled 3T2R parallel mechanisms. Jiang et al. [14] synthesized a class of large-angle 3T2R parallel mechanisms. Yang et al. [15] synthesized 3T2R series limbs, which are similar to Exechon motion and applied to synthesize parallel mechanisms. The 3T2R parallel mechanism has two rotation axes. If the rotation axis near the fixed platform is defined as the first rotation axis, then the rotation axis near the moving platform is defined as the second rotation axis. In conclusion, for the existing mechanism with 3T2R motion, the direction of the first rotation axis is invariant, the direction of the second rotation axis is changeable only with the change of the first rotation axis, and the directions of the first and second rotation axes are invariant themselves.

By studying the bifurcation of Schoenflies motion, Li and Hervé [16] synthesized various 3T1R bifurcation parallel mechanisms based on displacement manifold theory. Fang and Tsai [17] proposed a 4-RUC PM capable of three translations and one rotation about an axis with changeable direction in space. In 3T1R bifurcation parallel mechanism [16], the rotation occurs around one of the two specified axes, which is a bifurcation movement from one invariable axis to another. As a result, the mechanism has a variable rotation axis when it passes through the singular position. Inspired by the discovery, Yang Shuofei [18] synthesized a novel 3T1Rv parallel mechanism with a variable axis. However, the direction vectors of the first and second rotation axis in the existing 3T2R parallel mechanism are invariant, unlike the axis of the 3T1Rv parallel mechanism[18] with a variable axis. Inspired by the 3T1Rv parallel mechanism with a variable axis and the bifurcation 3T1R parallel mechanism, a new type of 3T2Rv parallel mechanism with variable/invariable axes is synthesized in this paper.

Because the finite screw is used to describe the finite motion of the mechanism instead of the instantaneous motion, in contrast to the instantaneous screw synthesis method, the need to identify whether the synthesized mechanism is only instantaneous or not is eliminated in finite screw method. And the mechanism studied is characterized by the variable/invariable axes. Therefore, the motion of the mechanism can be completely and accurately described by the finite screw [15, 18], which contains the information of the axis of rotation changing with time. In order to obtain the limb standard form of the finite screw, linearly independent rotational factors and translational factors are added to the end of the finite screw factor of the moving platform. And then through replacing joints and changing the position of joints in the limb standard form, the feasible derived limbs can be obtained through rigorous analytical calculation. By means of the rigorous mathematical derivation, the method is able to ensure the correctness of the branch chain acquired [15]. Then the 3T2Rv parallel mechanism with variable/invariable rotation axes is assembled according to the assembly requirements between the limbs.

The motion of this new bifurcation type of 3T2Rv parallel mechanism consists of four motion modes: invariable-invariable axis motion mode (the direction of the first rotation axis is invariant, the direction of the second rotation axis is changeable only with the first rotation axis, and the direction of its own axis is invariant), variable-invariable axis motion mode (the direction of the first rotation axis is changeable continuously, but the direction of the second rotation axis is changeable only with the first rotation axis, and its own axis is invariant), and variable-variable axis motion mode (the direction of the first rotation axis is changeable constantly, and the direction of the second rotation axis is changeable not only with the first rotation axis but also with itself), variable-invariable axis motion mode (the direction of the first rotation axis is changeable not only with the first rotation axis but also with itself), variable-invariable axis motion mode (the direction of the first rotation axis is changeable continuously, but the direction of the second rotation axis is changeable only with the first rotation axis, and its own axis is invariant), and variable-variable axis motion mode (the direction of the first rotation axis is changeable constantly, and the direction of the second rotation axis is changeable not only with the first rotation axis, but also with its own axis). This motion is similar to the output motion of two 4R spherical mechanisms [19] connected in series to the end link of three translational series limbs. When the spherical 4R mechanism has the bifurcation motion mode of variable/invariable rotation axis, the motion is defined as the bifurcation 3T2Rv motion with variable/invariable rotation axes. Thus, the bifurcation 3T2Rv parallel mechanism with variable/invariable axes has similar practical application as the two series bifurcation spherical 4R mechanisms. Since the spherical 4R mechanism has been applied as a robotic wrist and orienting or flapping device [20-21], the bifurcation 3T2Rv parallel mechanism with variable/invariable axes can be applied to assembly robots. The remainder of this paper is organized as follows. In Section 2, the assembly condition and rotation axis of the bifurcation spherical 4R mechanism with a variable/invariable axis are studied. Based on Section 2, the assembly condition and rotation axes of the bifurcation 2Rv mechanism with variable/invariable rotation axes are studied in Section 3. In Section 4, according to the finite screw method, the feasible limb of the bifurcation 3T2Rv parallel mechanism with variable/invariable rotation axes is obtained. The theory in Section 2 and Section 3 provides a
THEORETICAL BASIS FOR THE STUDY OF ASSEMBLY CONDITIONS AND 
BIREFRACTIVE CHARACTERISTICS OF THE BIREFRACTIVE 3T2Rv PARALLEL MECHANISM WITH VARIABLE/INVARIABLE ROTATION AXES IN SECTION 5. IN ADDITION, SECTION 6 CONTAINS THE CONCLUSIONS OF THIS PAPER.

II. ASSEMBLY CONDITION AND ROTATION AXIS OF 
THE BIREFRACTIVE SPHERICAL 4R MECHANISM WITH A VARIABLE/INVARIABLE AXIS

AS SHOWN IN Fig. 1, the finite screw set of the moving platform of the spherical 4R mechanism can be expressed as Eq. (1).

\[ \{s_{f,PM}\} = \{s_{f,a} \Delta s_{f,a}\} \] (1)

In which \( s_{f,a} = 2\tan \frac{\theta_a}{2}(r_o \times s_a) \), \( s_{f,a} = 2\tan \frac{\theta_a}{2}(r_o \times s_a) \), \( \theta_a \) is the rotation angle of the revolute joint \( R_a \), \( \Delta s_{f,a} \) is the finite screw set of the revolute joint \( R_a \) from pose I to pose II around its axis \( s_a \) (as shown in Fig. 2). \( r_o \) is the vector from the coordinate origin \( O \) to a point on the rotation axis \( s_a \) of the revolute joint \( R_a \) and the rest of the symbols are similar.

The finite screw of a series limb \( R_bR_c \) can be expressed as Eq. (2).

\[ s_{f,ab} = 2\tan \frac{\theta_a}{2}(s_b \times s_b) + 2\tan \frac{\theta_a}{2}(r_0 \times s_a) \] (2)

The finite screw triangular product [22-24] can be expressed as Eq. (3).

\[ s_{f,ab} = \frac{1}{1 - \tan \frac{\theta_a}{2}\tan \frac{\theta_b}{2}}\{s_{fa} + s_{fb} + \frac{1}{2}s_{fa} \times s_{fa} \} \] (3)

Where \( s_{f,ab} \), \( s_{f,a} \), and \( s_{f,a} \) represent the finite screw descriptions of series limb \( R_bR_c \), joint \( R_b \) and joint \( R_c \), respectively. \( s_{f,b} \times s_{f,a} \) is the screw cross product of \( s_{f,b} \) and \( s_{f,a} \), and its corresponding operation is expressed as Eq. (4).

\[ s_{f,a} \times s_{f,a} = \begin{bmatrix} \frac{\theta_a}{2}s_a \times \tan \frac{\theta_a}{2}s_a \\ \frac{\theta_a}{2}s_a \times (\tan \frac{\theta_a}{2}s_a + t_o) \\ \frac{\theta_a}{2}s_a \times (\tan \frac{\theta_a}{2}s_a + t_o) + t_o \times s_a \end{bmatrix} \] (4)

Equation (3) is the screw triangle product. As shown in Fig. 3, based on the closure of the finite screw triangle product, the finite motion of the series limbs of multiple joints can be expressed as Eq. (5).

\[ s_{f,b} \times s_{f,c} = s_{f,a} \Delta s_{f,b} \Delta s_{f,c} \] (5)

Where \( s_{f,a} \), \( s_{f,b} \), and \( s_{f,c} \) represent the finite screw descriptions of the series limb \( R_bR_c \), joint \( R_a \), joint \( R_b \) and joint \( R_c \), respectively.

According to the screw triangle product algorithms, the screw triangle product satisfies the associative law.

\[ (s_{f,a} \Delta s_{f,b}) \Delta s_{f,c} = s_{f,a} \Delta (s_{f,b} \Delta s_{f,c}) \] (6)

FIGURE 1. The finite screw set of the spherical 4R mechanism

FIGURE 2. Sketch of rigid body finite motion

FIGURE 3. Finite motions of a series limb

FIGURE 4. The spherical 4R mechanism at bifurcation position

FIGURE 5. The finite screw triangular product
According to the description of the finite screw of the parallel mechanism, the finite screw of the moving platform of the spherical 4R mechanism can also be written as follows.

$$\{s_{f,4R}\} = \{2\tan \frac{\theta}{2}(r \times s)\}$$  \hspace{1cm} (7)

where $s = (\cos \alpha, \sin \alpha \cos \beta, \sin \alpha \sin \beta)^T$, $s_{3\times 1}$ is the unit direction vector, $r_{3\times 1}$ is the unit position vector of the rotating axis $s$. $(s \times r)_{6\times 1}$ is a six-dimensional column vector composed of two three-dimensional column vectors. $r$ is the position vector from the origin $O$ of the fixed coordinate system to the centre of rotation, as shown in Fig. 4, where $r$ is the zero vector. $s$ is the unit vector of the rotation axis of the output rod, and $\theta$ is the rotation angle of the output rod around axis $s$. The three axes of the moving coordinate system $O'-xyz$ are parallel to the fixed coordinate system $O-XYZ$, $\alpha$ is the angle between $s$ and $x$, and $\beta$ is the angle between the projection of $s$ in the $yz$ plane and the coordinate $y$ axis.

The finite screws of the moving platform of the spherical 4R mechanism can also be written as the intersection of the finite screws of two limbs.

$$\{s_{f,4R}\} = \{s_{f,1l}\} \cap \{s_{f,2l}\}$$  \hspace{1cm} (8)

where $\{s_{f,i}\}$ ($i=1,2$) is the finite screw of the series limb $R_iaR_ib$ and $\{s_{f,1l}\}$ can be represented by the triangular product of the finite screw of revolute joints $R_ia$ and $R_ib$.

$$\{s_{f,1l}\} = \{2\tan \frac{\theta_{ib}}{2}(s_{ib})\Delta 2\tan \frac{\theta_{ia}}{2}(s_{ia})\}$$  \hspace{1cm} (9)

According to Eqs. (2) ~ (9), the finite screw of the axis of the moving platform can be obtained, as shown in Eqs. (10) ~ (12).

\[
\begin{align*}
\mathbf{s}^T (\mathbf{s}_{1a} \times \mathbf{s}_{1b}) \sin^2 (\mathbf{s}_{1a}^T \mathbf{s}_{1b}) &= \frac{(\mathbf{s}^T \mathbf{s}_{2a} - (\cos (\mathbf{s}_{1a}^T \mathbf{s}_{1b})) \mathbf{s}^T \mathbf{s}_{1b})(\mathbf{s}^T \mathbf{s}_{1b} - (\cos (\mathbf{s}_{1a}^T \mathbf{s}_{1b})) \mathbf{s}^T \mathbf{s}_{1a})}{(\mathbf{s}^T \mathbf{s}_{2a} - (\cos (\mathbf{s}_{2a}^T \mathbf{s}_{2b}))) \mathbf{s}^T \mathbf{s}_{2b} - (\cos (\mathbf{s}_{2a}^T \mathbf{s}_{2b})) \mathbf{s}^T \mathbf{s}_{2a} - (\cos (\mathbf{s}_{2a}^T \mathbf{s}_{2b})) \mathbf{s}^T \mathbf{s}_{2b})} - \frac{(\mathbf{s}^T \mathbf{s}_{2b} - (\cos (\mathbf{s}_{2a}^T \mathbf{s}_{2b}))) \mathbf{s}^T \mathbf{s}_{2b} - (\cos (\mathbf{s}_{2a}^T \mathbf{s}_{2b})) \mathbf{s}^T \mathbf{s}_{2b})}{(\mathbf{s}^T \mathbf{s}_{2a} - (\cos (\mathbf{s}_{2a}^T \mathbf{s}_{2b}))) \mathbf{s}^T \mathbf{s}_{2b} - (\cos (\mathbf{s}_{2a}^T \mathbf{s}_{2b})) \mathbf{s}^T \mathbf{s}_{2b})} \\
\mathbf{s} \cdot \mathbf{s} &= \frac{\mathbf{s}^T (\mathbf{s}_{1a} \times \mathbf{s}_{1b})}{\mathbf{s}^T (\mathbf{s}_{1a} - (\cos (\mathbf{s}_{1a}^T \mathbf{s}_{1b})) \mathbf{s}_{1a})} \mathbf{s}_{1a} + \frac{\theta_{ib}}{2} \mathbf{s}_{1b} + \frac{\theta_{ib}}{2} \mathbf{s}_{1a} - (\cos (\mathbf{s}_{1a}^T \mathbf{s}_{1b})) \mathbf{s}_{ia} \\
\tan \frac{\theta_{ia}}{2} \mathbf{s}_{1a} + \frac{\mathbf{s}^T (\mathbf{s}_{1a} \times \mathbf{s}_{ib})}{\mathbf{s}^T (\mathbf{s}_{1a} - (\cos (\mathbf{s}_{1a}^T \mathbf{s}_{ib})) \mathbf{s}_{ib})} \mathbf{s}_{ib} + \frac{\theta_{ia}}{2} \mathbf{s}_{ib} - (\cos (\mathbf{s}_{1a}^T \mathbf{s}_{ib})) \mathbf{s}_{ia} \\
\tan \frac{\theta_{ib}}{2} \mathbf{s}_{2a} + \frac{\mathbf{s}^T (\mathbf{s}_{2a} \times \mathbf{s}_{ib})}{\mathbf{s}^T (\mathbf{s}_{2a} - (\cos (\mathbf{s}_{2a}^T \mathbf{s}_{2b})) \mathbf{s}_{ib})} \mathbf{s}_{ib} + \frac{\theta_{ib}}{2} \mathbf{s}_{ib} - (\cos (\mathbf{s}_{2a}^T \mathbf{s}_{ib})) \mathbf{s}_{ia} \\
\tan \frac{\theta_{ia}}{2} \mathbf{s}_{1a} - (\cos (\mathbf{s}_{1a}^T \mathbf{s}_{ib})) \mathbf{s}_{ib} + \frac{\theta_{ia}}{2} \mathbf{s}_{ib} - (\cos (\mathbf{s}_{1a}^T \mathbf{s}_{ib})) \mathbf{s}_{ia} \\
\tan \frac{\theta_{ib}}{2} \mathbf{s}_{2a} - (\cos (\mathbf{s}_{2a}^T \mathbf{s}_{ib})) \mathbf{s}_{ib} + \frac{\theta_{ib}}{2} \mathbf{s}_{ib} - (\cos (\mathbf{s}_{2a}^T \mathbf{s}_{ib})) \mathbf{s}_{ia} \\
\end{align*}
\]

Eqs. (10) and (12) are the constraint equations of the axis of the moving platform of the spherical 4R mechanism; that is, to say, the two equations are the analytical expressions of the continuous distribution of axes in space and used to derive the assembly conditions of the parallel mechanism with the variable axis. The relationship between the rotation angle $\theta_{ib}$ of the second revolute joint and the output rotation axis $s$ of the spherical 4R mechanism is obtained by Eq. (11), which is used to deduce the motion bifurcation characteristics of the mechanism later.

A. ASSEMBLY CONDITION OF THE SPHERICAL 4R MECHANISM WITH A VARIABLE/INVARIABLE AXIS

When the mechanism is in a singular position, the motion mode of the mechanism may be changed. For the spherical 4R mechanism, when the axes of the four revolute joints are coplanar, the mechanism is in a singular position. According to Eqs. (10) and (12), the conditions of the spherical 4R mechanism with variable/invariable rotation axis bifurcation can be obtained, which means that conditions (1) and (2) are satisfied at the same time.

When the axes of four revolute joints are coplanar, (1) $s_{1a} \neq \pm s_{2a}$, $s_{ib} \neq \pm s_{2b}$, (2) $s_{1a} = \pm s_{2a}$, $s_{ib} = \pm s_{2b}$, $s_{1b} = \pm s_{2b}$, or $s_{1b} = \pm s_{2b}$, $s_{1a} = \pm s_{2a}$.

When the assembly condition of the spherical 4R mechanism does not meet condition (1) completely, that is, when $s_{1a} = \pm s_{2a}$, $s_{ib} = \pm s_{2b}$, Eqs. (10) and (12) become identical, and there is no constraint on the output rotation axis $s$ of the spherical 4R mechanism. Thus, the spherical 4R mechanism increases the uncontrollable degree of freedom, and the output motion of the spherical 4R mechanism degenerates into two rotations. Under this assembly condition, the spherical 4R mechanism has two different motion modes of rotation with an inviable axis.

If only one of two conditions (1) is not satisfied, that is, when $s_{1a} = \pm s_{2a}$, $s_{ib} \neq \pm s_{2b}$ or $s_{1a} \neq \pm s_{2a}$, $s_{ib} = \pm s_{2b}$, according to Eq. (10) and Eq. (12), $s = s_{1a} = \pm s_{2a}$ or
$s = s_{ib} = \pm s_{2b}$, then the spherical 4R mechanism has a motion mode of rotation with an invariable rotation axis.

When the assembly condition of the spherical 4R mechanism does not meet condition (2) completely, that is, when $s_{ia} = \pm s_{2a}, s_{ib} = \pm s_{2a}$, according to Eq. (10) and Eq. (12), $s = s_{ia} = \pm s_{2a}, s = s_{ib} = \pm s_{2a}$, after leaving the singular position, and the motion of the mechanism still has the rotation of a variable rotation axis. Thus, under this assembly condition, the spherical 4R mechanism has a motion mode of rotation with a variable axis and one motion mode of rotation with a variable axis.

When condition (2) is not satisfied, that is, when $s_{ia} \neq \pm s_{2a}, s_{ib} \neq \pm s_{2a}$ or $s_{ia} \neq \pm s_{2a}, s_{ib} \neq \pm s_{2a}$, according to Eq. (10) and Eq. (12), the spherical 4R mechanism has a motion mode of rotation with a variable axis.

According to Eq. (11), the instantaneous motion screw of the moving platform can be obtained by taking the derivative [25] of the finite rotation.

$$s_i = \hat{s}_{j,ab} \mid \theta = 0 = \dot{s}_{ia} + \dot{s}_{ib}$$

(13)

It can be concluded from Eq. (13) that the motion of the output end of the series limb is a combination of the instantaneous motion screws of each joint, and the instantaneous motion screw of the moving platform of the spherical 4R mechanism is the intersection of the instantaneous motion screws of two series limbs. Therefore, the instantaneous rotation axis of the moving platform can be deduced from the instantaneous motion screw.

The instantaneous motion screws of the two limbs of spherical 4R mechanism can be expressed as follows.

$$S_1 = \begin{bmatrix} s_{11}^T; \ 0_{1x3} \end{bmatrix}^T$$

(14)

$$S_2 = \begin{bmatrix} s_{12}^T; \ 0_{1x3} \end{bmatrix}^T$$

(15)

According to the reciprocity between the instantaneous motion screws and constraint screws of the limb, the constraint screws can be expressed as follows.

$$S_1^c = \begin{bmatrix} 1 & 0 & 0; \ 0_{1x3} \end{bmatrix}^T$$

$$S_2^c = \begin{bmatrix} 0 & 1 & 0; \ 0_{1x3} \end{bmatrix}^T$$

$$S_{12}^c = \begin{bmatrix} 0 & 0 & 1; \ 0_{1x3} \end{bmatrix}^T$$

(16)

$$S_1^c \times S_2^c = s_{11}^T \times s_{12}^T$$

(17)

According to the reciprocity between the instantaneous motion screw of the moving platform of the spherical 4R mechanism and all the force screws and couple screws, the instantaneous motion screws can be expressed as Eq. (18).

$$\begin{bmatrix} (s_{11} \times s_{12}) \times (s_{21} \times s_{22}) \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & 0; \ 0_{1x3} \end{bmatrix}^T$$

(18)

That is, the instantaneous rotation axis of the moving platform of the spherical 4R mechanism is expressed as Eq. (19).

$$s = (s_{11} \times s_{12}) \times (s_{21} \times s_{22})$$

(19)

Because $s_{11} \times s_{12}$ and $s_{21} \times s_{22}$ are respectively the normal vectors of the planes formed by the rotation axes of the two revolute joints of the first and second limbs, $(s_{11} \times s_{12}) \times (s_{21} \times s_{22})$ represents the common perpendicular of normal vectors for the two planes formed by the axes of revolute joints $R_{11}, R_{12}$ in the first limb and revolute joints $R_{21}, R_{22}$ in the second limb. That is to say, the intersection of the two planes is the common perpendicular. In this way, the instantaneous rotation axis of the moving platform of the spherical 4R mechanism as shown in Fig. 5 and Fig. 6 is acquired.

In this way, Fig. 5 and Fig. 6 show the bifurcation spherical 4R mechanism in invariable and variable axis motion mode, respectively.
orthogonal to \( s_{11} \), \( s_{22} \) is orthogonal to \( s_{12} \), and the angle between \( s_{21} \) and \( s_{22} \) is \( 0.25\pi \). Then, the rotation axis of the moving platform of the spherical 4R mechanism with variable/invariable rotation axis bifurcation can be obtained, as shown in Fig. 7. \( \theta_2 \) is rotation angle of revolute joint \( R_21 \).

FIGURE 7. Rotation axis of a bifurcation spherical 4R mechanism with a variable/invariable rotation axis

This section is used to derive the instantaneous rotation axis and assembly conditions of 3T2Rv parallel mechanism

\[
\{s_{f,1Rv}\} = \{2\tan(\theta_{d1}/2) (r_s \times s_{b1}) \Delta 2\tan(\theta_{a1}/2) (s_{s1} \times s_{t1})\} 
\cap \{2\tan(\theta_{d2}/2) (s_{s2} \times s_{t2}) \Delta 2\tan(\theta_{a2}/2) (r_s \times s_{b2})\} (20)
\]

The finite screw set of the 1Rv parallel mechanism can be expressed as Eq. (21).

\[
\{s_{f,1Rv}\} = \{2\tan(\theta_{d1}/2) (r_s \times s_{b}) \Delta 2\tan(\theta_{a1}/2) (s_{s} \times s_{t})\} (21)
\]

To obtain the limb standard form, the finite screw linearly independent rotational factors and translational factors are added to the end of the finite screw factor of Eq. (21). The standard forms of 10 limbs with three, four and

III. TYPE SYNTHESIS OF BIFURCATION 3T2Rv PARALLEL MECHANISMS WITH VARIABLE/INVARIABLE AXES

Bifurcation 1Rv parallel mechanism is a one-dimensional rotating parallel mechanism with variable/invariable rotation axis. The type synthesis of the bifurcation 3T2Rv parallel mechanism can be divided into two parts: the type synthesis of the bifurcation 1Rv parallel mechanism and the bifurcation 3T1Rv parallel mechanism. Then, according to the assembly conditions of bifurcation 1Rv and bifurcation 3T1Rv parallel mechanism, the assembly conditions of bifurcation 3T2Rv parallel mechanism with variable/invariable rotation axes can be obtained. The following is the type synthesis of the bifurcation 1Rv and 3T1Rv parallel mechanisms.

A. TYPE SYNTHESIS OF BIFURCATION 1Rv PARALLEL MECHANISMS WITH A VARIABLE/INVARIABLE AXIS

The structure of the bifurcation spherical 4R mechanism is the simplest 1Rv parallel mechanism. Therefore, the finite screw of the bifurcation 1Rv parallel mechanism with a variable/invariable rotation axis can be expressed as the intersection of two limbs (as shown in Fig. 5).

\[
\{s_{f,1Rv}\} = \{2\tan(\theta_{d1}/2) (r_s \times s_{b1}) \Delta 2\tan(\theta_{a1}/2) (s_{s1} \times s_{t1})\} (20)
\]

The finite screw set of the 1Rv parallel mechanism can be expressed as Eq. (21).

\[
\{s_{f,1Rv}\} = \{2\tan(\theta_{d1}/2) (r_s \times s_{b}) \Delta 2\tan(\theta_{a1}/2) (s_{s} \times s_{t})\} (21)
\]
five degree of freedom can be obtained as Eq. (22).

In the limb with two degrees of freedom, \( R_1 R_3 \) and \( R_3 R_a \) can be regarded as equivalent limbs. Therefore, derivatives of limbs with three, four and five degrees of freedom are synthesized. The following is a detailed synthesis of the limb with three degrees of freedom.

1) JOINT REPLACEMENT

The derived limbs are constructed by replacing joints of \( \{s_{f, 1\ell}\}_2 \) in Eq. (22), and then Eq. (27) can be obtained.

According to the finite screw calculation Eqs. (22-24), Eq. (23) can be changed to Eq. (24).

In Eq. (24), \( \exp \) is an exponential function, \( \tilde{S}_a \) is the skew symmetric matrix of vector \( s_a \), \( E_s \) is the three dimensional identity matrix, and the rest of the symbols are similar.

When the radius of the ring tends to infinity, that is \( (r_o - r_i) \to \infty \), the limb represented by Eq. (24) is equivalent to the limb represented by equation \( \{s_{f, 1\ell}\}_2 \), which means that \( R_1 R_3 R_o \) is the equivalent feasible limb of \( P_1 R_1 R_o \).

2) CHANGE IN JOINT POSITION

Since the serial limbs \( R_1 R_3 \) and \( R_3 R_a \) can be regarded as the same limb, only the position between the revolute joints and the prismatic joint in \( \{s_{f, 1\ell}\}_2 \) is changed. By changing the position of the joints in \( \{s_{f, 1\ell}\}_2 \), Eq. (25) is obtained.

According to the finite screw calculation Eqs. [22-24], Eq. (25) can be changed to Eq. (26).

In Eq. (26), \( \tilde{S}_a \) is the skew symmetric matrix of vector \( s_a \).

\[
\begin{align*}
\{s_{f, 1\ell}\}_2 &= \{2\tan \frac{\theta_b}{2} (s_b \times s_b) \Delta \tan \frac{\theta_a}{2} (s_a \times s_a) \Delta 2 \tan \frac{\theta_{1\ell}}{2} (s_{1\ell} \times s_{1\ell}) \}\ (23) \\
\{s_{f, 2\ell}\}_2 &= \{2\tan \frac{\theta_b}{2} (r_o \times s_b) \Delta \tan \frac{\theta_a}{2} (s_a \times s_a) \Delta t(0 \Delta 2 \tan \frac{\theta_a}{2} (s_a \times s_a)) \} (24) \\
\{s_{f, 3\ell}\}_2 &= \{2\tan \frac{\theta_b}{2} (s_a \times s_a) \Delta \tan \frac{\theta_a}{2} (r_o \times s_a) \Delta t(\exp(\theta_1 \tilde{S}_a)) \}\ (25) \\
\{s_{f, 1\ell}\}_3 &= \{2\tan \frac{\theta_b}{2} (s_a \times s_a) \Delta \tan \frac{\theta_a}{2} (s_a \times s_a) \Delta \tan \frac{\theta_a}{2} (s_a \times s_a) \} (26) \\
\{s_{f, 2\ell}\}_3 &= \{2\tan \frac{\theta_b}{2} (s_a \times s_a) \Delta \tan \frac{\theta_a}{2} (s_a \times s_a) \Delta \tan \frac{\theta_a}{2} (s_a \times s_a) \} (27) \\
\{s_{f, 3\ell}\}_3 &= \{2\tan \frac{\theta_b}{2} (r_o \times s_a) \Delta \tan \frac{\theta_a}{2} (r_o \times s_a) \Delta \tan \frac{\theta_a}{2} (r_o \times s_a) \} (28) \\
\{s_{f, 1\ell}\}_4 &= \{2\tan \frac{\theta_b}{2} (s_a \times s_a) \Delta \tan \frac{\theta_a}{2} (s_a \times s_a) \Delta \tan \frac{\theta_a}{2} (s_a \times s_a) \} (29) \\
\{s_{f, 2\ell}\}_4 &= \{2\tan \frac{\theta_b}{2} (s_a \times s_a) \Delta \tan \frac{\theta_a}{2} (s_a \times s_a) \Delta \tan \frac{\theta_a}{2} (s_a \times s_a) \} (30)
\end{align*}
\]

It is known from Eq. (26) that \( R_1 P_1 R_o \) is the equivalent limb of \( P_1 R_1 R_o \). Similarly, it can be concluded that \( R_3 P_1 \) is the equivalent limb of \( P_1 R_1 R_o \).

In the following, the positions of joints are changed, such as the positions of \( R_1 \) and \( R_3 \) in the equation are exchanged, and then Eq. (27) can be obtained.

According to the finite screw calculation formula, Eq. (27) can be changed to Eq. (28).

According to Eq. (28), \( R_1 R_3 R_o \) is a feasible limb.

The positions of joints are changed in \( \{s_{f, 2\ell}\}_3 \) . The following equation can be obtained.

Eq. (29) can be changed into Eq. (30). According to Eq. (30), \( R_3 R_o R_1 \) is a feasible limb.

By using the screw trigonometric product and its algorithm, feasible limbs with three degrees of freedom can be obtained, such as No. 2 and No. 3 in Table 1. Similarly, feasible limbs with four and five degrees of freedom can also be obtained, such as No. 4, No. 5, No. 6, No. 7, No. 8 and No. 9 in Table 1.

In Table 1, the rotation centre of revolute joints \( R_{ao} \) and \( R_{bo} \) passes through point \( o \), while the rotation centre of revolute joint \( R_1 \) in \( R_3 R_o R_{bo} \) does not pass through point \( o \). \( P_{ao} R_o \) indicates that the moving direction of the prismatic joint is perpendicular to the axis of the revolute joint \( R_a \). \( P_1 P_1 P_1 R_1 \) is a series limb of three linearly independent prismatic joints and a revolute joint whose axis does not pass through the rotation centre \( o \); the rest is similar.

In this way, according to Table 1, the parallel mechanism with the 1Rv motion mode can be obtained, as shown in Table 2.
Table 1 Feasible limbs of the 1Rv parallel mechanism

| No. | The limbs standard form | Change in joints position | Joints replacement |
|-----|-------------------------|---------------------------|--------------------|
| 1   | R_oR_o                  |                           | R_o1R_o1R_o2R_o2   |
| 2   | P_oP_oR_o              | R_o1P_oR_o               | R_o1P_oR_oP_o      |
| 3   | R_oR_o                  |                           | R_o1P_oR_oP_o      |
| 4   | P_oP_oP_oR_o           |                           | R_o1P_oP_oP_o      |
| 5   | P_oP_oP_oR_o           |                           | R_o1P_oP_oP_o      |
| 6   | R_oP_oR_o              |                           | R_o1P_oR_o         |
| 7   | P_oP_oP_oR_o           |                           | R_o1P_oP_oP_o      |
| 8   | P_oP_oP_oR_o           |                           | R_o1P_oP_oP_o      |
| 9   | P_oP_oR_o              |                           | R_o1P_oP_o         |
| 10  | R_oR_oR_o              |                           | R_o1R_o           |

Table 2 1Rv parallel mechanism

| NO. | 1Rv mechanism parallel symmetrical limb |
|-----|----------------------------------------|
| 1   | R_oR_o/R_oR_o                         |
| 2   | R_oR_o/R_oR_o                         |
| 3   | R_oR_o/R_oR_o                         |
| 4   | R_oR_o/P_oR_o                         |
| 5   | R_oR_o/P_oR_o                         |
| 6   | R_oR_o/P_oR_o                         |
| 7   | R_oR_o/P_oR_o                         |
| 8   | R_oR_o/P_oR_o                         |
| 9   | R_oR_o/P_oR_o                         |

B. TYPE SYNTHESIS OF BIFURCATION 3T1Rv PARALLEL MECHANISMS WITH A VARIABLE/INVARIABLE AXIS

The finite screw of the bifurcation 3T1Rv parallel mechanism with a variable/invariable axis can be expressed as an intersection of two kinds of feasible limbs.

\[
s_{1,3T1Rv} = \{s_{f,3T1Rv}\}_1 \cap \{s_{f,3T1Rv}\}_2 \tag{31}\]

The finite screw of limb of the bifurcation 3T1Rv parallel mechanism with a variable/invariable axis can be written as a series connection of the 1Rv bifurcation parallel mechanism and the 3T limb.

\[
s_{f,3T1Rv} = \{s_{f,1Rv}\} \Delta x_1(\begin{pmatrix} 0 \\ s_1 \end{pmatrix}) \Delta x_2(\begin{pmatrix} 0 \\ s_2 \end{pmatrix}) \Delta x_1(\begin{pmatrix} 0 \\ s_1 \end{pmatrix}) \tag{32}\]

According to Eq. (32), since the limb standard form of the 3T1Rv parallel mechanism with a variable/invariable rotation axis is already a limb with five degrees of freedom, the linear independent rotation and movement factors are not added to Eq. (32), and only a change in joint position and joint replacement of the standard limb are performed. According to the finite screw algorithm, a feasible limb can be obtained, as shown in Table 3.

According to Eq. (31), when using the feasible limbs in Table 3 to assemble the bifurcation 3T1Rv parallel mechanism with a variable/invariable rotation axis, it is necessary to ensure that the intersection of \(\{s_{f,3T1Rv}\}_i\) generated by the selected / limbs is \(\{s_{f,3T1Rv}\}_i\). It can be seen from Eq. (32) that \(\{s_{f,3T1Rv}\}_i\) generated by each limb should be \(\{s_{f,3T1Rv}\}_i\) or \(\{s_{f,3T1Rv}\}_j\). That is to say, when assembling the bifurcation 3T1Rv parallel mechanism, it is required that m of the / limbs have the same two rotation directions \(s_{1a}\) and \(s_{1b}\) while the remaining /m limbs have the same rotation directions \(s_{2a}\) and \(s_{2b}\). In other words, there should be no more than four directions of revolute joints in all limbs of the bifurcation 3T1Rv parallel mechanism with a variable/invariable rotation axis.

C. TYPE SYNTHESIS OF BIFURCATION 3T2Rv PARALLEL MECHANISMS WITH VARIABLE/INVARIABLE AXES

According to the above synthesis, the bifurcation 3T2Rv parallel mechanism with a variable/invariable rotation axes can be obtained by connecting the bifurcation 1Rv parallel mechanism with variable/invariable rotation axis in series on the moving platform of the bifurcation 3T1Rv parallel mechanism.

\[
s_{f,3T2Rv} = \{s_{f,3T1Rv}\}_i \cap \{s_{f,1Rv}\}_i \tag{33}\]

The 3T1Rv parallel mechanism with a variable/invariable rotation axes is composed of the limbs in Table 3, and the 1Rv parallel mechanism is shown in Table 2.
D. Assembly Condition of the Bifurcation 3T2Rv Parallel Mechanism with Variable/Invariable Axes

According to Eqs. (10) and (12), combined with the assembly conditions of the bifurcation 1Rv parallel mechanism, the assembly conditions of the bifurcation 3T2Rv parallel mechanism with variable/invariable rotation axes can be obtained. That is to say, conditions (3) and (4) are satisfied at the same time. (3) \( s_{1a} \neq \pm s_{2a}, s_{1b} \neq \pm s_{2b} \), \( s_{1c} \neq \pm s_{2c}, s_{id} \neq \pm s_{jd} \); (4) \( s_{1d} = \pm s_{2d}, s_{id} = \pm s_{jd} \), \( s_{1c} = \pm s_{2c}, s_{1d} = \pm s_{2d}, s_{id} = \pm s_{jd} \), or \( s_{id} = \pm s_{jd} \).

The bifurcation 3T2Rv is obtained by arbitrarily selecting the \( m \) and \( n \) limbs in Table 3 (it is required that the \( m \) limbs have the same two rotation directions \( s_{1a} \) and \( s_{1b} \), while the remaining \( n \) limbs have the same rotation directions \( s_{2a} \) and \( s_{2b} \)). The 1Rv mechanism in Table 2 is connected in series on the moving platform of bifurcation 3T1Rv, and then the bifurcation 3T2Rv parallel mechanism with variable/invariable rotation axes can be obtained under the conditions (3) and (4).

E. Actuation Schemes of the Bifurcation 3T2Rv Parallel Mechanism with Variable/Invariable Axes

Since the bifurcation 3T2Rv parallel mechanism with variable/invariable axes is composed of a bifurcation 3T1Rv parallel mechanism and a bifurcation 1Rv parallel mechanism, the actuation schemes consist of two parts.

1) Actuation Schemes of the Bifurcation 3T1Rv Parallel Mechanism

The actuation schemes of the bifurcation 3T1Rv parallel mechanism are as follows:

(1) The actuation joint in each limb should be selected as close to the frame as possible;

(2) The 3-DOF translational motion of the mechanism is generated by three actuation joints, which must be one of the following four cases:

a) P joints whose three directions are not parallel to each other;

b) One R joint and P joints whose directions are not parallel;

c) One P joint and two R joints;

d) Three R joints.

The actuation joint R is one of the parallel joints R, and the actuation joint R in the relative limb should be located at different positions in the limb.

(3) The rotation with a single degree of freedom of the 3T1Rv mechanism is generated by the fourth actuation joint, which can be selected arbitrarily.

2) Actuation Schemes of the Bifurcation 3T2Rv Parallel Mechanism

Because the actuation joint far away from the fixed platform has a bad influence on the dynamic performance of the parallel mechanism, and a 6-DOF limb with an actuation joint is connected in series on the moving platform to control the bifurcation 1Rv parallel mechanism.

A 3T2Rv parallel mechanism is composed of two \( P_{1R_{1}}R_{1}R_{3}b \) limbs in Table 3, a 2-\( R_{2a} \) mechanism in Table 2 and a 6-DOF SPS limb. As shown in Fig. 8 (a), the three revolute joints \( R_{16}, R_{15} \) and \( R_{11} \) of the \( P_{1R_{1}}R_{1}R_{3}bR_{12} \) limb are parallel to each other, and the moving axis of \( P_{1} \) is parallel to the axis of \( R_{16} \), while the three revolute joints \( R_{26}, R_{23} \) and \( R_{21} \) of the \( P_{2R_{2a}}R_{22}R_{22} \) limb are parallel to each other, and the moving axis of \( P_{2} \) is parallel to the axis of \( R_{26} \). The axes of revolute joints \( R_{14}, R_{12}, R_{13} \) and \( R_{14} \) and \( R_{21}, R_{22}, R_{23} \) and \( R_{24} \) satisfy conditions (3) and (4); that is to say, in the initial position and posture, conditions (5) and (6) are satisfied at the same time. (5) \( s_{1a} \neq \pm s_{2a}, \ s_{1b} \neq \pm s_{2b}, \ s_{1c} \neq \pm s_{2c}, \ s_{1d} \neq \pm s_{2d}, \ s_{id} \neq \pm s_{jd} \), \( s_{1c} \neq \pm s_{2c}, \ s_{1d} \neq \pm s_{2d} \).

The joints with arrows in Fig. 8 represent the actuation joints. The synthesized mechanism can be replaced by equivalent joints to obtain more parallel mechanisms. As shown in Fig. 8 (a), \( R_{11}, R_{12} \) and \( R_{21}, R_{22} \) are replaced with Hooke hinges \( U_{1} \) and \( U_{2} \) respectively, and the equivalent mechanism of Fig. 8 (a) can be obtained, as shown in Fig. 8 (b).

It is worth noting that in the mechanism shown in Fig. 13, when the mechanism is in variable/invariable axis motion mode, the five pairs \( P_{1}, R_{16}, P_{2}, R_{26} \) and P with arrows in the Fig. 13 can control the degree of freedom of the mechanism. But when the mechanism is in bifurcation configuration, the instantaneous degree of

| Table 3 Feasible limbs of 3T1Rv parallel mechanism |
|--------------------------------------------------|
| The limb standard form | Change in joint position | Joint replacement |
|------------------------|--------------------------|-------------------|
| \( P_{1}P_{2}P_{3}R_{b} \) | \( P_{1}P_{2}P_{3}P_{4}P_{b} \) | \( R_{1}P_{1}P_{2}P_{3}P_{4}P_{b} \) |
| \( P_{1}P_{2}P_{3}P_{b} \) | \( R_{1}P_{1}P_{2}P_{3}P_{4}P_{b} \) | \( R_{1}P_{1}P_{2}P_{3}P_{4}P_{b} \) |
| \( P_{1}P_{2}P_{3}P_{4}P_{b} \) | \( R_{1}P_{1}P_{2}P_{3}P_{4}P_{b} \) | \( R_{1}P_{1}P_{2}P_{3}P_{4}P_{b} \) |
| \( P_{1}P_{2}P_{3}R_{b} \) | \( P_{1}P_{2}P_{3}P_{4}P_{b} \) | \( R_{1}P_{1}P_{2}P_{3}P_{4}P_{b} \) |
| \( P_{1}P_{2}P_{3}P_{b} \) | \( R_{1}P_{1}P_{2}P_{3}P_{4}P_{b} \) | \( R_{1}P_{1}P_{2}P_{3}P_{4}P_{b} \) |
| \( P_{1}P_{2}P_{3}P_{4}P_{b} \) | \( R_{1}P_{1}P_{2}P_{3}P_{4}P_{b} \) | \( R_{1}P_{1}P_{2}P_{3}P_{4}P_{b} \) |

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freedom will appear. At this time, it is necessary to increase the auxiliary actuation joint to fully control the degree of freedom of the mechanism. That is to say, when the plane composed of the axes of the revolute joints R_{11} and R_{12} is parallel to the plane composed of the axes of the revolute joints R_{21} and R_{22}, the 3T2Rv mechanism will increase an instantaneous degree of freedom. At this bifurcation position, it is necessary to increase the auxiliary actuation joint R_{22} to fully control the degree of freedom of the mechanism. And when the axes of R_{23}R_{24} and the axes of R_{13}R_{14} are coplanar, the 3T2Rv mechanism will also increase an instantaneous degree of freedom. At this bifurcation position, it is necessary to increase the auxiliary actuation joint R_{23} to fully control the degree of freedom of the mechanism.

IV. MOTION MODES OF THE 3T2Rv PARALLEL MECHANISM WITH VARIABLE/INVARIABLE AXES

According to Eq. (11), the output rotation axis of the bifurcation 1Rv mechanism is related to the rotation angle \( \theta_{ib} \) of the revolute joint R_{ib}. Therefore, the bifurcation characteristics of the bifurcation 3T2Rv parallel mechanism with variable/invARIABLE axes can be divided into two cases: \( \theta_{ib}=0 \) and \( \theta_{ib} \neq 0 \).

A. Kinematic modeling of 3T2Rv parallel mechanism

As shown in Fig. 8(b), the rotation axes of universal joint U_{1} and U_{2} close to the moving platform are s_{21} and s_{22} respectively. The moving coordinate system o_{1}x_{1}y_{1}z_{1} is established with the intersection of the axes of the revolute joint R_{11} and the revolute joint R_{23} as the origin o_{1}, o_{2}s_{2} as the x_{1} axis and o_{1}R_{23} as the y_{1} axis, and the moving coordinate system o_{1}x_{1}y_{1}z_{1} is fixed to the link 1. The moving coordinate system o_{1}x_{1}y_{1}z_{1} is established with the intersection point of axes of revolute joint R_{14} R_{24} as origin o, o_{2}s_{2} as the x axis and o_{1}R_{14} as the y axis, and the moving coordinate system o_{1}x_{1}y_{1}z_{1} is fixed to the moving platform. The fixed coordinate system OXYZ is established with the geometric center of the fixed platform as the origin O, the axis direction of the cylindrical pair C_{1b} as the x axis, and the upward direction perpendicular to the plane of the fixed platform as the z axis. Assuming that the rotation angles of cylindrical pairs C_{1} and C_{2} are \( \theta_{1} \) and \( \theta_{2} \) respectively, and the displacement of cylindrical pairs C_{1} and C_{2} are \( q_{1} \) and \( q_{2} \) respectively. The distance between spherical joints S_{1} and S_{2} is \( q_{3} \). The inverse kinematics solution of the parallel mechanism [26, 27] is to solve \( q_{1}, q_{2}, q_{3}, \theta_{1} \) and \( \theta_{2} \), in the case of the given position and orientation of moving platform.

\[
R(\alpha, \beta, \gamma) = \begin{bmatrix}
  c \alpha c \beta & c \alpha s \beta s \gamma - s \alpha c \gamma & c \alpha s \beta c \gamma + s \alpha s \gamma \\
s \alpha c \beta & s \alpha s \beta s \gamma + c \alpha c \gamma & s \alpha s \beta c \gamma - c \alpha s \gamma \\
-s \beta & c \beta s \gamma & c \beta c \gamma
\end{bmatrix}
\] (34)

The moving coordinate system o_{1}x_{1}y_{1}z_{1} adopts Z-Y-X Euler angle, and the rotation matrix is R_{1}. And the moving coordinate system o_{1}x_{1}y_{1}z_{1} adopts Z-Y-X Euler angle, and the rotation matrix is R.

Eq. (34) can also be written as Eq. (35).
\[
\mathbf{R}(\alpha, \beta, \gamma) = \begin{bmatrix} k_{14} & k_{15} & k_{16} \\ k_{24} & k_{25} & k_{26} \\ k_{24} & k_{35} & k_{36} \end{bmatrix}
\]

(35)

\[
\mathbf{R}_1(\phi, \psi) = \begin{bmatrix} s\phi\cos\psi & c\phi & -s\psi \\ s\phi\sin\psi & c\phi & c\psi \\ -s\phi & c\phi & 0 \end{bmatrix}
\]

(36)

Eq. (36) can also be written as Eq. (37)

\[
\mathbf{R}_1(\phi, \psi) = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{11} & k_{22} & k_{23} \\ k_{11} & k_{32} & k_{33} \end{bmatrix}
\]

(37)

where, \( s\alpha = \sin\alpha \), \( c\alpha = \cos\alpha \), others are similar. 

According to the geometric structure of the mechanism, the following parameters can be obtained.

\[
s_{i1} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T, \quad s_{s2} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T, \quad \epsilon_1 = \epsilon_3 = 0.5\pi,
\]

\[
o s_{s2} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T, \quad s_{s3} = \begin{bmatrix} -0.5\sqrt{2} & -0.5\sqrt{2} & 0 \end{bmatrix}^T,
\]

\[
o s_{s4} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T, \quad s_{s5} = \begin{bmatrix} -0.5\sqrt{2} & -0.5\sqrt{2} & 0 \end{bmatrix}^T,
\]

\[
o s_{s6} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T.
\]

The inverse kinematics model of the 3T2Rv parallel mechanism shown in Fig. 9 can be obtained by solving Eq. (38).

\[
\begin{align*}
q_1 & = P_1 \\
q_2 & = 0.5\sqrt{2}(P_4 - P_5) - d - l_{22}\cos\delta_2 \\
q_3 & = \sqrt{(x-k_1)^2 + (y-k_2)^2 + (z-k_3)^2} \\
\theta_1 & = \arcsin\delta_1 - \arctan \frac{P_2 - P_4}{P_3} \\
\theta_2 & = \arcsin \frac{P_3}{\sqrt{2}l_{21}}
\end{align*}
\]

(39)

where, \( k_1 = e + 0.5\sqrt{2}c(k_{14} + k_{15}) \), \( k_2 = k_{31} + k_{32} \),

\[
k_3 = 0.5\sqrt{2}(k_{34} + k_{25}) \), \( P_2 = y + ak_{32} + bk_{33} \), \( P_3 = z + ak_{32} + bk_{33} \), \( k_3 = 0.5\sqrt{2}(k_{34} + k_{35}) \),

\[
P_4 = x - 0.5\sqrt{2}ak_{31} - 0.5\sqrt{2}ak_{32} + bk_{13} \), \( k_4 = k_{11} + k_{12} \),

\[
k_5 = k_{21} + k_{22} \), \( P_5 = y - 0.5\sqrt{2}ak_{21} - 0.5\sqrt{2}ak_{22} + bk_{23} \), \( P_6 = z - 0.5\sqrt{2}ak_{21} - 0.5\sqrt{2}ak_{22} + bk_{33} \), \( P_4 = x + ak_{12} + bk_{13} \),

\[
\delta_2 = \arcsin \left( \frac{P_5}{l_{22}} \right) \cos \left( \arcsin \frac{P_4 - P_5}{\sqrt{2}l_{21}} \right),
\]

\[
\delta_2 = \arcsin \left( \frac{P_5}{l_{22}} \right) \cos \left( \arcsin \frac{P_4 - P_5}{\sqrt{2}l_{21}} \right),
\]

(40)

Since the axes of the revolute joints R_{11} and R_{12}, R_{21} and
\[ R_{22} \text{ satisfy the third equation of Eq. (38), the three parameters } \phi, \varphi \text{ and } \psi \text{ are limited to one-dimensional variables by the two equations of Eq. (40).} \]
\[
\begin{align*}
\sin \phi \cos \varphi + \cos \phi \cos \varphi &= 1 \\
\tan \phi &= \sin \varphi \tan \psi
\end{align*}
\] (40)

Similarly, the axes of the revolute joints \( R_{11} \text{ and } R_{14}, R_{23} \text{ and } R_{24} \text{ satisfy the fourth equation of Eq. (38), the three parameters } \alpha, \beta \text{ and } \gamma \text{ are limited to one-dimensional variables by the two equations of Eq. (41).} \]
\[
\begin{align*}
k_{12} \cos \alpha \cos \beta + k_{22} \sin \alpha \cos \beta - k_{32} \sin \beta &= 0.5 \sqrt{2} \\
m_1 \sin \gamma + m_2 \cos \gamma &= 0
\end{align*}
\] (41)

where, \( m_1 = k_4 \cos \alpha \sin \beta + k_5 \sin \alpha \sin \beta + k_6 \cos \beta \), \( m_2 = -k_4 \sin \alpha + k_5 \cos \alpha \).

The kinematic model of the mechanism shown in Fig. 10 has been established, as shown in Eq. (39).

B. Kinematic simulation of 3T2Rv mechanism

The structural parameters of the mechanism are given as follows: \( a=0.5 \text{ m}, b=-0.18 \text{ m}, c=0.4 \text{ m}, d=0.5 \text{ m}, e=0.8 \text{ m}, l_{11}=l_{12}=0.5 \text{ m}, l_{21}=l_{22}=0.66 \text{ m}. \)

According to the motion function of the moving platform of 3T2Rv parallel mechanism under four motion modes, the kinematic simulation of the mechanism can be obtained through Eq. (39). It should be noted that the motion function of the moving platform needs to meet Eq. (40) and Eq. (41).

1) INVARIABLE-ININVARIABLE AXIS MOTION MODE

As shown in Fig. 8(b), when rotation axis \( s_{11} \) of revolute joint \( R_{11} \) is parallel to rotation axis \( s_{21} \) of revolute joint \( R_{21} \), and rotation axis \( s_{13} \) of revolute joint \( R_{13} \) is parallel to rotation axis \( s_{24} \) of revolute joint \( R_{24} \), \( s_{11} \times s_{21} = 0, s_{13} \times s_{24} = 0 \). When the rotation angles of revolute joints \( R_{12} \) and \( R_{14} \) are zero, \( \theta_{12} = 0, \theta_{14} = 0 \); that is to say, the parallel mechanism shown in Fig. 10 has the motion mode of invariable-invariable rotation axes.

In other words, in the bifurcation position and posture (rotation axis \( s_{11} \) of revolute joint \( R_{11} \) is parallel to rotation axis \( s_{21} \) of revolute joint \( R_{21} \), and rotation axis \( s_{13} \) of revolute joint \( R_{13} \) is parallel to rotation axis \( s_{24} \) of revolute joint \( R_{24} \), revolute joints \( R_{12} \) and \( R_{14} \) are locked, which will keep \( s_{11} \) parallel to \( s_{21} \) and \( s_{13} \) parallel to \( s_{24} \), so that the motion mode of invariable-invariable rotation axes can be realized.

Given that the motion function of the moving platform of 3T2Rv mechanism in the invariable-invariable axis motion mode is formula (42), and the kinematic simulation of the mechanism can be obtained, as shown in Fig. 11 and Fig. 12.

\[
\begin{align*}
x &= \frac{0.5\pi}{6} \sin(\frac{\pi t}{180}) \\
y &= \frac{0.1\pi}{6} \sin(\frac{\pi t}{180}) \\
z &= \frac{0.3\pi}{6} \sin(\frac{\pi t}{180}) + 0.8, 0 < t < 180 \\
\psi &= -\frac{\pi}{3} \sin(\frac{\pi t}{180}) \\
\gamma &= -\frac{\pi}{3} \sin(\frac{\pi t}{180})
\end{align*}
\] (42)

FIGURE 11. Angle displacement of actuation joints in the invariable-invariable axis motion

FIGURE 12. Displacement of actuation joints in the invariable-invariable axis motion

2) INVARIABLE-VARIABLE AXIS MOTION MODE

As shown in Fig. 13, when rotation axis \( s_{11} \) of revolute joint \( R_{11} \) is parallel to rotation axis \( s_{21} \) of revolute joint \( R_{21} \), and rotation axis \( s_{13} \) of revolute joint \( R_{13} \) is not parallel to rotation axis \( s_{24} \) of revolute joint \( R_{24} \), \( s_{11} \times s_{21} = 0, s_{13} \times s_{24} \neq 0 \). When the rotation angle of revolute joint \( R_{12} \) is zero and the rotation angle of \( R_{14} \) is not
zero, $\theta_{12} = 0$, $\theta_{14} \neq 0$; that is to say, the parallel mechanism shown in Fig. 13 has the motion mode of invariable-variable rotation axes. In other words, in the bifurcation position and posture, revolute joint $R_{12}$ is locked, and revolute joint $R_{14}$ is rotated to a certain angle away from the bifurcation position and posture, which will keep $S_{11}$ parallel to $S_{21}$ and $S_{13}$ nonparallel to $S_{24}$, so that the motion mode of invariable-variable rotation axes can be realized.

Given that the motion function of the moving platform of the bifurcation 3T2Rv parallel mechanism with variable/invariable axes in the variable-invariable axis motion mode is formula (44), and the kinematic simulation of the mechanism can be obtained, as shown in Fig. 17 and Fig. 18.

$$\begin{align*}
x &= \frac{0.5\pi}{6} \sin\left(\frac{\pi t}{180}\right) \\
y &= \frac{0.1\pi}{6} \sin\left(\frac{\pi t}{180}\right) \\
z &= \frac{0.3\pi}{6} \sin\left(\frac{\pi t}{180}\right) + 0.4, \quad 0 < t < 180 \\
\phi &= -\frac{\pi}{6} \sin\left(\frac{\pi t}{180}\right) \\
\gamma &= \frac{\pi}{3} \sin\left(\frac{\pi t}{180}\right)
\end{align*}$$

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4) VARIABLE-VARIABLE AXIS MOTION MODE

As shown in Fig. 19, when rotation axis $S_{11}$ of revolute joint $R_{11}$ is not parallel to rotation axis $S_{21}$ of revolute joint $R_{21}$, and rotation axis $S_{13}$ of revolute joint $R_{13}$ is not parallel to rotation axis $S_{24}$ of revolute joint $R_{24}$, $S_{11} \times S_{21} \neq 0$, $S_{13} \times S_{24} \neq 0$. When the rotation angle of revolute joint $R_{12}$ is not zero and the rotation angle of $R_{14}$ is not zero, $\theta_{12} \neq 0$, $\theta_{14} \neq 0$; that is to say, the parallel mechanism shown in Fig. 19 has the motion mode of variable-variable rotation axes.

In other words, in the bifurcation position and posture, revolute joints $R_{12}$ and $R_{14}$ are rotated to a certain angle away from the bifurcation position and posture, which will keep $S_{11}$ nonparallel to $S_{21}$ and $S_{13}$ nonparallel to $S_{24}$, so that the motion mode of variable-variable rotation axes can be realized.

Given that the motion function of the moving platform of 3T2Rv mechanism in the variable-variable axis motion mode is formula (45), and the kinematic simulation of the mechanism can be obtained, as shown in Fig. 20 and Fig. 21.

FIGURE 18. Displacement of actuation joints in the variable-invariable axis motion

FIGURE 19. The 3T2Rv mechanism in the variable-variable axis motion

\[
\begin{align*}
    x &= \frac{0.5 \pi}{6} \sin \left( \frac{\pi t}{180} \right) \\
    y &= \frac{0.1 \pi}{6} \sin \left( \frac{\pi t}{180} \right) \\
    z &= \frac{0.3 \pi}{6} \sin \left( \frac{\pi t}{180} \right) + 0.6 \\
    \varphi &= \frac{\pi}{6} \sin \left( \frac{\pi t}{180} \right) \\
    \beta &= -\frac{\pi}{3} \sin \left( \frac{\pi t}{180} \right)
\end{align*}
\]
assembly is carried out by the 3T2Rv mechanism in invariable-invariable axis motion mode, as shown in Fig. 22. To assemble the cylinder on link 1, the invariable-invariable axes motion mode is only required for the 3T2Rv mechanism to grasp the cylinder, as shown in Fig. 22.

**FIGURE 22. 3T2Rv mechanism grabs workpiece in the invariable-invariable axis motion mode**

**B. CUBOID WORKPIECE TO BE ASSEMBLED**

When the cross section of the workpiece to be assembled is an equilateral triangle, a square, etc., and the rotation angle of the mechanism is small in the third-dimension rotation, the assembly is carried out by the 3T2Rv mechanism in variable-invariable or invariable-variable axis motion mode. The first or second variable axis rotation causes a small angle in the third-dimension rotation to meet the requirements of assembling the workpieces with equilateral triangle or square cross section.

**FIGURE 23. 3T2Rv mechanism grabs workpiece in the variable-invariable axis motion mode**

**VI. CONCLUSIONS**

(1) A new type of bifurcation 3T2Rv parallel mechanism with two variable/invariable axes is proposed based on the theory of finite screws.

(2) The assembly conditions and actuation schemes of this kind of bifurcation 3T2Rv parallel mechanism are analyzed by studying the finite screw expression of the bifurcation spherical 4R mechanism.

(3) The novel bifurcation 3T2Rv parallel mechanism consists of four motion modes: invariable-invariable axis motion mode, variable-invariable axis motion mode, variable-variable axis motion mode, and variable-variable axis motion mode.

The follow-up research can synthesize other parallel mechanisms with two variable axes, such as 1T2Rv, 2T2Rv, etc. And the follow-up research can also analyze the dynamics and other performance of the 3T2Rv parallel mechanism with two variable axes synthesized in this paper.

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