The Köthe dual of mixed Morrey spaces and applications

Houkun Zhang, Jiang Zhou *

Abstract: In this paper, we study the separable and weak convergence of mixed-norm Lebesgue spaces. Furthermore, we prove that the block space $B^{p_0}_{p_1}(\mathbb{R}^n)$ is the Köthe dual of the mixed Morrey space $M^{p_0}_{p_1}(\mathbb{R}^n)$ by the Fatou property of these block spaces. The boundedness of the Hardy–Littlewood maximal function is further obtained on the block space $B^{p_0}_{p_1}(\mathbb{R}^n)$. As applications, the characterizations of $BMO(\mathbb{R}^n)$ via the commutators of the fractional integral operator $I_\alpha$ on mixed Morrey spaces are proved as well as the block space $B^{p_0}_{p_1}(\mathbb{R}^n)$.

Keywords: Köthe dual; Mixed Morrey spaces; Block spaces; $BMO(\mathbb{R}^n)$; Characterization; Fractional integral operators.

1 Introduction

In recent years, since the precise structure of mixed-norm function spaces, the mixed-norm function spaces are widely used in the partial differential equations [3–6]. In 1961, the mixed-norm Lebesgue space $L^{\vec{p}}(\mathbb{R}^n)(0 < \vec{p} \leq \infty)$ were studied by Benedek and Panzone [7]. These spaces are natural generalizations of the classical Lebesgue space $L^p(\mathbb{R}^n)(0 < p \leq \infty)$.

After that, many function spaces with mixed norm were introduced, such as mixed-norm Lorentz spaces [8], mixed-norm Lorentz-Martinciewicz spaces [9], mixed-norm Orlicz spaces [10], anisotropic mixed-norm Hardy spaces [11], mixed-norm Triebel-Lizorkin spaces [12] and weak mixed-norm Lebesgue spaces [13]. More studies can be refereed in [14] and so on.

In 2019, Nogayama [15, 16] introduced mixed Morrey spaces associated with mixed-norm Lebesgue spaces and Morrey spaces, and further proved that the block space $B^{p_0}_{p_1}(\mathbb{R}^n)$ are predual spaces of the mixed Morrey space $M^{p_0}_{p_1}(\mathbb{R}^n)$. In this paper, we will prove that $B^{p_0}_{p_1}(\mathbb{R}^n)$ are also Köthe duals of these mixed Morrey spaces. For more studies and a deeper account of

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*Corresponding author E-mail address: Zhoujiang@xju.edu.cn.
developments about the Köthe dual we may consult [17] and the references therein.

Given $0 < \alpha < n$, for a measurable function $f$ on $\mathbb{R}^n$, the fractional integral operators are defined by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy,$$

and these operators play such a prominent role in real and harmonic analysis [1, 2]. By simple calculations,

$$\int_{\mathbb{R}^n} f(x)I_\alpha g(x)dx = \int_{\mathbb{R}^n} g(x)I_\alpha f(x)dx$$

(1)

can be obtained.

For a locally integrable function $b$ and a measurable function $f$, the commutator of the fractional integral operators is defined by

$$[b, I_\alpha]f(x) := b(x)I_\alpha f(x) - I_\alpha (bf)(x) = \int_{\mathbb{R}^n} \frac{(b(x) - b(y))f(y)}{|x-y|^{n-\alpha}} dy,$$

which was introduced by Chanillo in [18]. For the commutators, a similar formula

$$\int_{\mathbb{R}^n} f(x)[b, I_\alpha]g(x)dx = - \int_{\mathbb{R}^n} g(x)[b, I_\alpha]f(x)dx$$

(2)

can be obtained, where $f, g$ are measurable functions.

Let us recall the classical results. In 1991, Di Fazio and Ragusa given the characterizations of $BMO(\mathbb{R}^n)$ via the boundedness of $[b, I_\alpha]$ on classical Morrey spaces [19]. That is, let $1 < p \leq p_0 < \infty$, $1 < q \leq q_0 < \infty$, $\frac{p}{p_0} = \frac{q}{q_0}$, and $\frac{1}{p_0} - \frac{1}{q_0} = \frac{\alpha}{n}$. Then

$$b \in BMO(\mathbb{R}^n) \Rightarrow [b, I_\alpha] : M_{p_0}^{q_0}(\mathbb{R}^n) \mapsto M_{q_0}^{q_0}(\mathbb{R}^n).$$

Conversely, if $n - \alpha$ is an even integer, then

$$[b, I_\alpha] : M_{p_0}^{q_0}(\mathbb{R}^n) \mapsto M_{q_0}^{q_0}(\mathbb{R}^n) \Rightarrow b \in BMO(\mathbb{R}^n).$$

In 2006, Shirai given another characterizations of $BMO(\mathbb{R}^n)$ via the boundedness of $[b, I_\alpha]$ on classical Morrey spaces [20]. That is

$$b \in BMO(\mathbb{R}^n) \Leftrightarrow \left[ b, I_\alpha \right] : M_{p_0}^{q_0}(\mathbb{R}^n) \mapsto M_{q_0}^{q_0}(\mathbb{R}^n),$$

where $1 < p \leq p_0 < \infty$, $1 < q \leq q_0 < \infty$ and $\frac{1}{p} - \frac{1}{q} = \frac{1}{p_0} - \frac{1}{q_0} = \frac{\alpha}{n}$.

In 2019, Nogayama proved characterizations of $BMO(\mathbb{R}^n)$ via the operator $[b, I_\alpha]$ on mixed Morrey spaces [16]. That is

$$b \in BMO(\mathbb{R}^n) \Leftrightarrow \left[ b, I_\alpha \right] : M_{p_0}^{q_0}(\mathbb{R}^n) \mapsto M_{q_0}^{q_0}(\mathbb{R}^n).$$
where $1 < \frac{1}{p_0} < \frac{\alpha}{n}$, $\frac{n}{p_0} \leq \sum_{i=1}^{n} \frac{1}{p_i}$, $\frac{n}{q_0} \leq \sum_{i=1}^{n} \frac{1}{q_i}$, $\frac{1}{p_0} - \frac{1}{q_0} = \frac{\alpha}{n}$ and $\vec{p} = \frac{\vec{q}}{q_0}$.

In fact, the results of [16] can be regarded as a generation of [19]. In the paper, we generalize the results of [20] on mixed Morrey spaces. We point out that

$$\frac{1}{p} - \frac{\alpha}{n} = \frac{p_0}{p} \left( \frac{1}{p_0} - \frac{\alpha}{p_0 n} \right) \geq \frac{p_0}{p} \left( \frac{1}{p_0} - \frac{\alpha}{n} \right) = p_0 \frac{q_0}{p q_0}.$$  

Thus, it’s easy to see that the results of [19] can be regard as the improved results of [20]. But, for mixed Morrey spaces, we can only prove that

$$\sum_{i=1}^{n} \frac{1}{p_i} - \alpha \geq \sum_{i=1}^{n} \frac{p_0}{p_i} \cdot \left( \frac{1}{p_0} - \frac{\alpha}{n} \right).$$

In other words, the results of [16] are not the improvements of Theorem 6.1. Furthermore, when $\frac{1}{p_0} = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{p_i}$ and $\frac{1}{q_0} = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{q_i}$, the results of Theorem 6.1 are better than [16].

This paper is organized as the following. In Section 2, main definitions are recalled. In Section 3, separability and weak convergence of mixed-norm Lebesgue spaces are also studied. We prove that the Köthe dual of the mixed Morrey space $M_{\vec{p}^0}^p(\mathbb{R}^n)$ are the block space $B_{\vec{p}^0}^p(\mathbb{R}^n)$ by the property of Fatou in Section 4. In Section 5, we prove the bounds for the Hardy–Littlewood maximal function over the block spaces. As an application, the characterizations of $BMO(\mathbb{R}^n)$ is given and the boundedness of $I_\alpha$ and $[b, I_\alpha]$ are also proved in Section 6.

Finally, we make some conventions on notation. Let $\vec{p} = (p_1, \cdots, p_n)$, $\vec{q} = (q_1, \cdots, q_n)$ are $n$-tuples with $1 < p_i, q_i < \infty$, $i = 1, \ldots, n$. $\vec{p} < \vec{q}$ means that $p_i < q_i$ holds, and $\frac{1}{p} + \frac{1}{p'} = 1$ means $\frac{1}{p_i} + \frac{1}{p'_i} = 1$ holds, for each $i = 1, \ldots, n$. The symbol $Q$ denotes the cubes whose edges are parallel to the coordinate axes and $Q(x, r)$ denotes a open cube centered at $x$ of side length $r$. Let $cQ(x, r) = Q(x, cr)$. Denote by the symbol $\mathcal{M}(\mathbb{R}^n)$ the set of all measurable function on $\mathbb{R}^n$. $A \sim B$ means that $A$ is equivalent to $B$. That is $A \leq CB$ and $B \leq CA$, where $C$ is a positive constant. Through all paper, every positive constant $C$ is not necessarily equal.

2 Main definitions

We begin this section with the definition of some maximal functions in the following.

For a locally integrable function $f$, the Hardy–Littlewood maximal operator is defined by, for almost every $x \in \mathbb{R}^n$,

$$M(f)(x) := \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|dy,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ containing $x$, and the sharp maximal
operator is defined by
\[ M^b(f)(x) := \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy, \]
where \( f_Q = \frac{1}{|Q|} \int_Q f \) and the supremum is taken over all cubes \( Q \subset \mathbb{R}^n \) containing \( x \).

The definition of ball (quasi-)Banach function spaces is presented as follows, which were introduced by Sawano et al. [22].

**Definition 2.1.** A (quasi-)Banach space \( X \subset \mathcal{M}(\mathbb{R}^n) \) with (quasi-)norm \( \| \cdot \|_X \) is called a ball (quasi-)Banach function space if

(i) \( |g| \leq |f| \) almost everywhere implies that \( \|g\|_X \leq \|f\|_X \);

(ii) \( 0 \leq f_m \uparrow f \) almost everywhere implies that \( \|f_m\|_X \uparrow \|f\|_X \);

(iii) If \( |Q| < \infty \), then \( \chi_Q \in X \);

(iv) If \( f \geq 0 \) almost everywhere and \( |Q| < \infty \), then
\[ \int_Q f(x) dx \leq C_Q \|f\|_X ; \]
for some positive constants \( C_Q, 0 < c_Q < \infty \), depending on \( Q \) but independent of \( f \).

**Remark 2.1.** If "cube" is replaced by "ball" in the preceding definition, it's also valid. In particular, if we replace any cubes \( Q \) by any measurable sets \( E \) in Definition 2.1, it is (quasi-)Banach function spaces (see [24, Definition 1.1 of Chapter 1]).

The definition of the associate space of a ball (quasi-)Banach function space can be found in [24, chapter 1] as follows.

**Definition 2.2.** For any ball (quasi-)Banach function spaces \( X \), the associate space (also called the Köthe dual) \( X' \) is defined by setting
\[ X' := \left\{ f \in \mathcal{M}(\mathbb{R}^n) : \|f\|_{X'} := \sup_{g \in X, \|g\|_X = 1} \int_{\mathbb{R}^n} |f(x)g(x)| dx < \infty \right\}, \]
where \( \| \cdot \|_{X'} \) is called the associate norm of \( \| \cdot \|_X \).

**Remark 2.2.** (i) In other literatures (for example [31]) the Banach function spaces and the associate spaces are called the Köthe spaces and the Köthe dual respectively. Thus, the associate spaces of ball (quasi-)Banach function spaces are called the Köthe dual of ball (quasi-)Banach function spaces.
(ii) It is easy to prove that
\[
\int_{\mathbb{R}^n} |f(x)g(x)| \, dx \leq \|g\|_X \|f\|_{X'}.
\]

(iii) Due to [32, Lemma 2.6], if $X$ is a ball Banach function space, then
\[
\|f\|_X = \|f\|_{X''},
\]
where $X'' = (X')'$. 

(iv) According to the Definition 2.2 and (iv), it is easy to know that
\[
\|f\|_{X'} = \|\|f\|_X\| = \sup_{\|g\|=1} \int_{\mathbb{R}^n} |f(x)g(x)| \, dx \tag{3}
\]
and
\[
\|f\|_X = \|f\|_{X''} = \|\|f\|_{X'}\| = \|f\|_X. \tag{4}
\]

Furthermore, by (3) and (4), for $f \in X'$ and $g \in X$,
\[
\|f\|_{X'} = \sup_{\|g\|=1} \int_{\mathbb{R}^n} |f(x)g(x)| \, dx = \sup_{\|g\|=1} \left| \int_{\mathbb{R}^n} f(x)g(x) \, dx \right| = \sup_{\|g\|=1} \int_{\mathbb{R}^n} f(x)g(x) \, dx.
\]

Indeed, we have
\[
\int_{\mathbb{R}^n} f(x)g(x) \, dx \leq \left| \int_{\mathbb{R}^n} f(x)g(x) \, dx \right| \leq \int_{\mathbb{R}^n} |f(x)g(x)| \, dx,
\]
and for $h = \text{sgn}(fg)|g|$,
\[
\int_{\mathbb{R}^n} f(x)h(x) \, dx = \left| \int_{\mathbb{R}^n} f(x)h(x) \, dx \right| = \int_{\mathbb{R}^n} |f(x)g(x)| \, dx.
\]

We still recall the notion of the convexity of ball (quasi-)Banach function spaces, which can be found in [22, Definition 2.6].

**Definition 2.3.** Let $X$ be a ball (quasi-)Banach function space and $0 < p < \infty$. The $p$-convexification $X^p$ of $X$ is defined by setting
\[
X^p := \{f \in \mathcal{M}(\mathbb{R}^n) : |f|^p \in X\}
\]
equipped with the (quasi-)norm $\|f\|_{X^p} := \|\|f\|^p\|_X^{\frac{1}{p}}$. 

Obviously, if $X$ is a ball (quasi-)Banach function space, the $X^p$ and $X'$ are also ball (quasi-)Banach function spaces. Now, let us recall $A_p(\mathbb{R}^n)$-weight and weighted Lebesgue spaces.
Definition 2.4. Let $1 < p < \infty$. A weight $\omega$ is said to be of class $A_p(\mathbb{R}^n)$ if

$$ [\omega]_{A_p(\mathbb{R}^n)} := \sup_{Q \subset \mathbb{R}^n} \left( \frac{1}{|Q|} \int_Q \omega(x) dx \right) \left( \frac{1}{|Q|} \int_Q \omega(x) \frac{1}{p} dx \right)^{p-1} < \infty. $$

A weight $\omega$ is said to be of class $A_1(\mathbb{R}^n)$ if

$$ [\omega]_{A_1(\mathbb{R}^n)} := \sup_{Q \subset \mathbb{R}^n} \left( \frac{1}{|Q|} \int_Q \omega(x) dx \right. \left. \|\omega^{-1}\|_{L^{\infty}(\mathbb{R}^n)} \right) < \infty. $$

Define $A_\infty(\mathbb{R}^n) := \bigcup_{1 \leq p < \infty} A_p(\mathbb{R}^n)$. It is well-known that $A_p(\mathbb{R}^n) \subset A_q(\mathbb{R}^n)$ for $1 \leq p \leq q \leq \infty$.

Definition 2.5. Let $0 < p < \infty$ and $\omega \in A_\infty(\mathbb{R}^n)$. The weighted Lebesgue space $L_p^{\omega}(\mathbb{R}^n)$ is defined to be the set of all measurable functions $f$ on $\mathbb{R}^n$ such that

$$ \|f\|_{L_p^{\omega}(\mathbb{R}^n)} := \left( \int_{\mathbb{R}^n} |f(x)|^p \omega(x) dx \right)^{\frac{1}{p}} < \infty. $$

Now, the definitions of mixed-norm Lebesgue spaces are given as follows, which was introduced by Benedek and Panzone [7].

Definition 2.6. Let $1 < \vec{p} < \infty$. The mixed Lebesgue space $L_{\vec{p}}(\mathbb{R}^n)$ is defined by the set of all measurable functions $f$ on $\mathbb{R}^n$, such that

$$ \|f\|_{L_{\vec{p}}(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} \cdots \left( \int_{\mathbb{R}^n} |f(x)|^{p_1} \ dx_1 \right)^{\frac{p_2}{p_1}} \cdots dx_n \right)^{\frac{1}{p_n}} < \infty. $$

Remark 2.3. (i) Note that if $p_1 = p_2 = \cdots = p_n = p$, then $L_{\vec{p}}(\mathbb{R}^n)$ are reduced to the classical Lebesgue space $L^p(\mathbb{R}^n)$, and

$$ \|f\|_{L_{\vec{p}}(\mathbb{R}^n)} = \|f\|_{L^p(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} |f(x)|^p dx \right)^{\frac{1}{p}}. $$

(ii) The mixed-norm Lebesgue space $L_{\vec{p}}(\mathbb{R}^n)$ is a ball Banach function space (see [21]).

In 2019, Nogayama introduced mixed Morrey spaces [15, 16] with combining mixed Lebesgue spaces and Morrey spaces as follows.

Definition 2.7. Let $0 < \vec{p} \leq \infty$ and $0 < p_0 \leq \infty$ satisfy

$$ n \leq \sum_{j=1}^n \frac{1}{p_j}. $$

The mixed Morrey space $M_{\vec{p}}^{p_0}(\mathbb{R}^n)$ is defined as the set of all measurable functions $f$ such that

$$ \|f\|_{M_{\vec{p}}^{p_0}(\mathbb{R}^n)} := \sup \left\{ |Q|^{\frac{1}{p_0} - \frac{1}{p_j}(\sum_{j=1}^n \frac{1}{p_j})} \|f\chi_Q\|_{L_{\vec{p}}(\mathbb{R}^n)} : Q \text{ is a cube in } \mathbb{R}^n \right\} < \infty. $$
Remark 2.4.  
(i) It is obvious that if \( p_1 = p_2 = \cdots = p_n = p \), then \( M_{p_0}^p(\mathbb{R}^n) = M_{p_0}^p(\mathbb{R}^n) \)
and if \( \frac{1}{p_0} = \frac{1}{n} \sum_{j=1}^n \frac{1}{p_j} \), then \( M_{p_0}^{p_0}(\mathbb{R}^n) = L^{p_0}(\mathbb{R}^n) \).

(ii) The Mixed Morrey space \( M_{p_0}^{p_0}(\mathbb{R}^n) \) is a ball Banach function space. In fact, according

\[ \text{to [15, Remark 3.1], mixed Morrey spaces are Banach spaces. Besides, it is easy to prove that:} \]

(a) If \( |g| \leq |f| \) almost everywhere, then
\[
\|g\chi_Q\|_{L^{p_0}(\mathbb{R}^n)} \leq \|f\chi_Q\|_{L^{p_0}(\mathbb{R}^n)},
\]
\[
\|g\chi_Q\|_{M_{p_0}^{p_0}(\mathbb{R}^n)} \leq \|f\chi_Q\|_{M_{p_0}^{p_0}(\mathbb{R}^n)}.
\]

(b) If \( 0 \leq f_m \uparrow f \) almost everywhere, then
\[
\lim_{m \to \infty} \|f_m\|_{M_{p_0}^{p_0}(\mathbb{R}^n)} = \sup_{Q \subset \mathbb{R}^n} |Q|^\frac{1}{p_0} \cdot \frac{1}{\sum_{j=1}^n \frac{1}{p_j}} \lim_{m \to \infty} \|f_m\chi_Q\|_{L^{p_0}(\mathbb{R}^n)}
\]
\[
= \sup_{Q \subset \mathbb{R}^n} |Q|^\frac{1}{p_0} \cdot \frac{1}{\sum_{j=1}^n \frac{1}{p_j}} \|f\chi_Q\|_{L^{p_0}(\mathbb{R}^n)}
\]
\[
= \|f\|_{M_{p_0}^{p_0}(\mathbb{R}^n)};
\]

(c) If \( |Q| < \infty \), then
\[
\|\chi_Q\|_{M_{p_0}^{p_0}(\mathbb{R}^n)} = |Q|^{\frac{1}{p_0}} \cdot \frac{1}{\sum_{j=1}^n \frac{1}{p_j}} < \infty;
\]

(d) If \( |Q| < \infty \) and \( f > \infty \)
\[
\int_Q f(x) dx \leq |Q|^{\frac{1}{\sum_{j=1}^n \frac{1}{p_j}}} \cdot \|f\|_{L^{p_0}(\mathbb{R}^n)}
\]
\[
= |Q|^{\frac{1}{\sum_{j=1}^n \frac{1}{p_j}}} \cdot |Q|^{\frac{1}{p_0} - \frac{1}{n} \sum_{j=1}^n \frac{1}{p_j}} \|f\|_{L^{p_0}(\mathbb{R}^n)}
\]
\[
\leq |Q|^{\frac{1}{\sum_{j=1}^n \frac{1}{p_j}}} \cdot \|f\|_{M_{p_0}^{p_0}(\mathbb{R}^n)}.
\]

But the authors pointed out that mixed Morrey spaces are not Banach function spaces

(see [17, Example 3.3]).

In [16], Nogayama introduced the block space \( B_{p_0}^{p_0}(\mathbb{R}^n) \), which is the predual space of mixed
Morrey space.

Definition 2.8. Let \( 1 < p_0, p < \infty \) and \( \frac{n}{p_0} < \sum_{j=1}^n \frac{1}{p_j} \). A measurable function \( b(x) \) is said to be a \( (p_0', \bar{p}') \)-block if there exists a cube \( Q \) such that
\[
\text{supp } b \subset Q, \quad \|b\|_{L^{p_0'}(\mathbb{R}^n)} \leq |Q|^{\frac{1}{p_0'} - \frac{1}{\sum_{j=1}^n \frac{1}{p_j}}}.
\]
The block space $B_{p^0_0}^{p'}(\mathbb{R}^n)$ is denoted by that a measurable function set of $f = \sum_{i=1}^{\infty} \lambda_i b_i(x)$, where $\{\lambda_i\}_{i=1}^{\infty} \in \ell^1$ and $b_i$ is a $(p'_0, p')$-block for each $i$. The norm $\|f\|_{B_{p^0_0}^{p'}(\mathbb{R}^n)}$ for $f \in B_{p^0_0}^{p'}(\mathbb{R}^n)$ is defined as

$$\|f\|_{B_{p^0_0}^{p'}(\mathbb{R}^n)} = \inf \{\|\{\lambda_i\}_{i=1}^{\infty}\|_{\ell^1} : f = \sum_{i=1}^{\infty} \lambda_i b_i(x), \{\lambda_i\}_{i=1}^{\infty} \in \ell^1, b_i \text{ is a } (p'_0, p')-\text{block for any } i\}.$$  

Let us recall the definition of the $BMO(\mathbb{R}^n)$ space.

**Definition 2.9.** If $b$ is a measurable function on $\mathbb{R}^n$ and satisfies that

$$\|b\|_{BMO(\mathbb{R}^n)} = \sup_{Q \subset \mathbb{R}^n} \frac{1}{|Q|} \int_Q |b(y) - b_Q| dy < \infty,$$

then $b \in BMO(\mathbb{R}^n)$ and $\|b\|_{BMO(\mathbb{R}^n)}$ is the norm of $b$ in $BMO(\mathbb{R}^n)$.

### 3 Weak convergence of mixed-norm Lebesgue spaces

In this section, we discuss the separable and weak convergence of mixed-norm Lebesgue spaces.

**Theorem 3.1.** Let $1 < p < \infty$. Then the mixed-norm Lebesgue space $L^{p}(\mathbb{R}^n)$ is separable space.

**Theorem 3.2.** Let $1 < p < \infty$ and $\{f_k\}_{k=1}^{\infty} \subset L^{p}(\mathbb{R}^n)$. If there exists a positive constant $M$ such that

$$\|f_k\|_{L^{p}(\mathbb{R}^n)} < M,$$

then there exists a subset $\{f_{k_j}\}_{j=1}^{\infty}$ is weak convergence on $L^{p}(\mathbb{R}^n)$.

Before we give our proofs, the following lemma is necessary. According to [21, Proposition 3.8], the following lemma can be obtained.

**Lemma 3.1.** Let $1 < p < \infty$. Then $C_c(\mathbb{R}^n)$ is dense in $L^{p}(\mathbb{R}^n)$, where $C_c(\mathbb{R}^n)$ denoted by the set of continuous functions with compact support.

We denote by $Q_k(k \in \mathbb{Z})$ the collection of cubes in $\mathbb{R}^n$ which are congruent to $[0, 2^{-k})^n$ and vertices lie on the lattice $2^{-k}\mathbb{Z}^n$, that is, $Q_k = \{2^{-k}(i + [0, 1)^n) : i \in \mathbb{Z}^n\}(k \in \mathbb{Z})$. The cubes in $D = \bigcup_{k \in \mathbb{Z}} Q_k$ are called dyadic cubes. Now, the proofs of Theorem 3.1 and Theorem 3.2 can be given.
Proof of Theorem 3.1. From Lemma 3.1, for any \( f \in L^\vec{p}(\mathbb{R}^n) \) and \( \varepsilon < \infty \), there exist \( g \in C_c(\mathbb{R}^n) \) such that
\[
\| f - g \|_{L^\vec{p}(\mathbb{R}^n)} < \varepsilon.
\]
It is easy to know that \( g \) is uniformly continuous. Hence, there exists a sequence of dyadic cube \( \{Q_i\}_{i=1}^N \) and a sequence of rational number \( \{c_i\}_{i=1}^N \) such that
\[
\| g - \sum_{i=1}^N c_i \chi_{Q_i} \|_{L^\vec{p}(\mathbb{R}^n)} < \varepsilon.
\]
We write \( \Gamma \) as a set of simple functions \( \varphi \) and
\[
\varphi(x) = \sum_{i=1}^N c_i \chi_{Q_i},
\]
where \( \{Q_i\}_{i=1}^N \) is a sequence of dyadic cube and \( \{c_i\}_{i=1}^N \) is a sequence of rational numbers. It is obvious that \( \Gamma \) is countable and dense in \( L^\vec{p}(\mathbb{R}^n) \). Thus, \( L^\vec{p}(\mathbb{R}^n) \) is separable.

Proof of Theorem 3.2. According to [7, Theorem 1.a], we know that the dual of \( L^\vec{p}(\mathbb{R}^n) \) is \( L^{\vec{p}'}(\mathbb{R}^n) \), where \( \frac{1}{\vec{p}} + \frac{1}{\vec{p}'} = 1 \) and \( 1 < \vec{p} < \infty \). Hence we only need to prove that there exists a subset \( \{f_{k,j}\}_{j=1}^\infty \) such that for any \( g \in L^{\vec{p}'}(\mathbb{R}^n) \),
\[
\lim_{j \to \infty} \int_{\mathbb{R}^n} f_{k,j}(x)g(x)dx = \int_{\mathbb{R}^n} f(x)g(x)dx,
\]
where \( f \in L^\vec{p}(\mathbb{R}^n) \).

According to Theorem 3.1, we assume that \( \{g_i\}_{i=1}^\infty \) is dense in \( L^{\vec{p}'}(\mathbb{R}^n) \). Write
\[
\mathcal{F}_k(g) = \int_{\mathbb{R}^n} f_k(x)g(x)dx.
\]
Using Hölder’s inequality,
\[
|\mathcal{F}_k(g_i)| \leq M\|g_i\|_{L^{\vec{p}'}(\mathbb{R}^n)}.
\]
According to the boundedness of \( \{\mathcal{F}_k(g_i)\}_{k=1}^\infty \), there exist convergent subsequences \( \{\mathcal{F}_{k,1}(g_1)\}_{k=1}^\infty \). By the same argument, we can find a subsequence \( \{\mathcal{F}_{k,2}(g_2)\}_{k=1}^\infty \) from \( \{\mathcal{F}_{k,1}(g_2)\}_{k=1}^\infty \) such that \( \{\mathcal{F}_{k,2}(g_2)\}_{k=1}^\infty \) is convergence. So for any \( g_{i_0}(i_0 \leq j) \) there exists a convergent subsequences \( \{\mathcal{F}_{k,j}(g_{i_0})\} \). By a diagonal process one can obtain that a subsequence \( \{\mathcal{F}_{j,j}(g_i)\}_{j=1}^\infty \) is convergence for any \( g_i \) and
\[
\mathcal{F}_{j,j}(g) = \int_{\mathbb{R}^n} f_{k,j}(x)g(x)dx := \int_{\mathbb{R}^n} f_{k,j}(x)g(x)dx.
\]
For any \( g \in L^{\vec{p}'}(\mathbb{R}^n) \) and any \( \varepsilon > 0 \), there exists \( g_i \) such that
\[
\| g - g_i \|_{L^{\vec{p}'}(\mathbb{R}^n)} \leq \varepsilon / 2M.
\]

Hence
\[
|\mathcal{F}_{m,m}(g) - \mathcal{F}_{m',m'}(g)| \leq \int_{\mathbb{R}^n} |f_{k_m}(x) - f_{k_{m'}}(x)||g(x)|dx
+ \int_{\mathbb{R}^n} |f_{k_m}(x) - f_{k_{m'}}(x)||g_i(x) - g(x)|dx
\leq \int_{\mathbb{R}^n} |f_{k_m}(x) - f_{k_{m'}}(x)||g_i(x)|dx + \varepsilon.
\]

When \( m \) and \( m' \) are large enough,
\[
|\mathcal{F}_{m,m}(g) - \mathcal{F}_{m',m'}(g)| \leq 2\varepsilon.
\]

Thus \( \{\mathcal{F}_{j,j}(g)\}_{j=1}^{\infty} \) is a Cauchy sequence for any \( g \in L^{\vec{p}'}(\mathbb{R}^n) \). Then let
\[
\mathcal{F}(g) = \lim_{j \to \infty} \mathcal{F}_{j,j}(g)
\]
and \( \mathcal{F}(g) \) be a linear bounded functional on \( L^{\vec{p}'}(\mathbb{R}^n) \). Applying [7, Theorem 1.a], we see that there exist \( f \in L^{\vec{p}}(\mathbb{R}^n) \) such that
\[
\int_{\mathbb{R}^n} f(x)g(x)dx = \mathcal{F}(g) = \lim_{j \to \infty} \mathcal{F}_{j,j}(g) = \lim_{j \to \infty} \int_{\mathbb{R}^n} f_{k_j}(x)g(x)dx.
\]
The proof is completed. \( \square \)

4 The Köthe dual spaces of mixed Morrey spaces

In this section, we study the Fatou property of the block spaces and prove that the block spaces are the Köthe dual spaces of mixed Morrey spaces.

**Theorem 4.1.** Let \( 1 < p_0, \vec{p} < \infty, \frac{n}{p_0} < \sum_{i=1}^{n} \frac{1}{p_i} \). If \( f_k \in B^{\vec{p}_0}_{\vec{p}}(\mathbb{R}^n)(k \in \mathbb{N}) \) are nonnegative functions, \( \|f_k\|_{B^{\vec{p}_0}_{\vec{p}}(\mathbb{R}^n)} \leq 1 \) and \( f_k \uparrow f \) a.e., then
\[
\lim_{k \to \infty} \|f_k\|_{B^{\vec{p}_0}_{\vec{p}}(\mathbb{R}^n)} = \|f\|_{B^{\vec{p}_0}_{\vec{p}}(\mathbb{R}^n)}.
\]

**Theorem 4.2.** Let \( 1 < p_0, \vec{p} < \infty, \frac{n}{p_0} < \sum_{i=1}^{n} \frac{1}{p_i} \). \( B^{\vec{p}_0}_{\vec{p}}(\mathbb{R}^n) \) is the Köthe dual of \( M^{p_0}_{\vec{p}}(\mathbb{R}^n) \).
In fact, a abstract result can be found in [34, Theorem 2.2] for Theorem 4.1. To state our proofs better, we prove Theorem 4.1 by the method from [17] instead of proving the conditions in [34, Theorem 2.2]. We need Lemma 4.1 and Lemma 4.2 (see [16, Theorem 2.7]) before our proof.

**Lemma 4.1.** Let \( 1 < p_0, \bar{p} < \infty, \frac{n}{p_0} < \sum_{i=1}^{n} \frac{1}{p_i} \) and \( f \in B_{p_0}^{p_0'}(\mathbb{R}^n) \). Then \( f \) can be decomposed as

\[
f(x) = \sum_{Q \in D} \lambda_Q b_Q(x),
\]

where \( \lambda_Q \) is a nonnegative number with

\[
\sum_{Q \in D} \lambda_Q \leq 2\|f\|_{B_{p_0}^{p_0'}(\mathbb{R}^n)} \cdot 3^n,
\]

and \( b_Q \) is a \((p_0', \bar{p}')\)-block with \( \text{supp} \ b_Q \subset 3Q \).

**Lemma 4.2.** Let \( 1 < p_0, \bar{p} < \infty \) and \( \frac{n}{p_0} < \sum_{i=1}^{n} \frac{1}{p_i} \). The block space \( B_{p_0}^{p_0'}(\mathbb{R}^n) \) is the predual spaces of mixed Morrey spaces. Furthermore, for any \( f \in M_{p_0}^{p_0}(\mathbb{R}^n) \) and \( g \in B_{p_0}^{p_0'}(\mathbb{R}^n) \),

\[
\|f\|_{M_{p_0}^{p_0}(\mathbb{R}^n)} = \sup \left\{ \int_{\mathbb{R}^n} |f(x)h(x)| \, dx : h \in B_{p_0}^{p_0'}(\mathbb{R}^n), \|h\|_{B_{p_0}^{p_0'}(\mathbb{R}^n)} = 1 \right\}
\]

and

\[
\|g\|_{B_{p_0}^{p_0'}(\mathbb{R}^n)} = \max \left\{ \int_{\mathbb{R}^n} |h(x)g(x)| \, dx : h \in M_{p_0}^{p_0}(\mathbb{R}^n), \|h\|_{M_{p_0}^{p_0}(\mathbb{R}^n)} = 1 \right\}.
\]

Now, let us prove Lemma 4.1, Theorem 4.1 and Theorem 4.2.

**Proof of Lemma 4.1.** For \( f \in B_{p_0}^{p_0'}(\mathbb{R}^n) \), there exist \( \{\lambda_i\}_{i=1}^{\infty} \) and \( \{b_i\}_{i=1}^{\infty} \) such that

\[
f(x) = \sum_{i=1}^{\infty} \lambda_i b_i(x),
\]

where \( \sum_{i=1}^{\infty} |\lambda_i| \leq 2\|f\|_{B_{p_0}^{p_0'}(\mathbb{R}^n)} \) and \( b_i \) is a \((p_0', \bar{p}')\)-block with \( \text{supp} \ b_i \subset Q'_i \). We divide \( \mathbb{N} \) into the disjoint sets \( K(Q) \), \( Q \in D \), as

\[
\mathbb{N} = \bigcup_{Q \in D} K(Q),
\]

and if \( i \in K(Q) \) then \( \text{supp} \ b_i \subset 3Q \) and \( |Q'_i| \geq |Q| \).
Then
\[ f(x) = \sum_{i=1}^{\infty} \lambda_i b_i(x) = \sum_{Q \in \mathcal{D}} \left( \sum_{i \in K(Q)} \lambda_i b_i(x) \right) \]
\[ = \sum_{Q \in \mathcal{D}} \left[ 3^n \sum_{i \in K(Q)} |\lambda_i| \right] \cdot \left[ \left( \sum_{i \in K(Q)} |\lambda_i| \right)^{-1} \sum_{i \in K(Q)} \lambda_i b_i(x) \right] \]
\[ =: \sum_{Q \in \mathcal{D}} \lambda_Q b_Q(x). \]

It is easy to prove that
\[ \sum_{Q \in \mathcal{D}} \lambda_Q = 3^n \sum_{Q \in \mathcal{D}} \sum_{i \in K(Q)} |\lambda_i| \leq 3^n \sum_{i=1}^{\infty} |\lambda_i| \leq 2\|f\|_{B_{p_0'}(\mathbb{R}^n)} \cdot 3^n, \]
and
\[ \|b_Q(x)\|_{L_{p'}(\mathbb{R}^n)} = \left( 3^n \sum_{i \in K(Q)} |\lambda_i| \right)^{-1} \sum_{i \in K(Q)} \lambda_i \|b_i(x)\|_{L_{p'}(\mathbb{R}^n)} \]
\[ \leq \left( 3^n \sum_{i \in K(Q)} |\lambda_i| \right)^{-1} \sum_{i \in K(Q)} |\lambda_i| \|b_i(x)\|_{L_{p'}(\mathbb{R}^n)} \]
\[ \leq \left( 3^n \sum_{i \in K(Q)} |\lambda_i| \right)^{-1} \sum_{i \in K(Q)} |\lambda_i| |Q_i|^{1/p_0} \cdot \frac{1}{p_1^n} \sum_{i=1}^{\infty} \frac{1}{p_i} \]
\[ \leq |Q|^{1/p_0} \cdot \frac{1}{p_1^n} \sum_{i=1}^{\infty} \frac{1}{p_i} \left( 3^n \sum_{i \in K(Q)} |\lambda_i| \right)^{-1} \sum_{i \in K(Q)} |\lambda_i| \]
\[ = |3Q|^{1/p_0} \cdot \frac{1}{p_1^n} \sum_{i=1}^{\infty} \frac{1}{p_i} \cdot 3^n \left( \frac{1}{2^n} \sum_{i=1}^{\infty} \frac{1}{p_i} - \frac{1}{p_0} \right) \]
\[ \leq |3Q|^{1/p_0} \cdot \frac{1}{p_1^n} \sum_{i=1}^{\infty} \frac{1}{p_i}. \]

Hence \( b_Q \) is a \((p_0', p')\)-block with \( \text{supp } b_Q \subset 3Q \). The proof is completed. \( \square \)

**Proof of Theorem 4.1.** For any nonnegative function \( f_k \), Lemma 4.1 yields
\[ f_k(x) = \sum \lambda_{Q,k} b_{Q,k}(x), \]
where \( \lambda_{Q,k} \) is nonnegative number with
\[ \sum_{Q \in \mathcal{D}} \lambda_{Q,k} \leq 2 \cdot 3^n, \quad (5) \]
and $b_{Q,k}$ is a $(p_0, p)$-block with

$$\text{supp } b_{Q,k} \subset 3Q, \quad \|b_{Q,k}\|_{L^{p'}(\mathbb{R}^n)} \leq |3Q|^{\frac{1}{p_0} - \frac{1}{p} \sum_{i=1}^n \frac{1}{p_i}}.$$ 

In view of Theorem 3.2 and diagonalization argument, for any $Q \in \mathcal{D}$, one can obtain that there exist subsequences $\{\lambda_{Q,k_j}\}_{j=1}^\infty$ and $\{b_{Q,k_j}\}_{j=1}^\infty$ such that

$$\lim_{j \to \infty} \int_{\mathbb{R}^n} b_{Q,k_j}(x)g(x)dx = \int_{\mathbb{R}^n} b_Q(x)g(x)dx, \quad \text{for any } g \in L^{p'}(3Q),$$

(6)

$$\lim_{j \to \infty} \lambda_{Q,k_j} = \lambda_Q,$$

(7)

$$f_{k_j}(x) = \sum_{Q \in \mathcal{D}} \lambda_{Q,k_j} b_{Q,k_j}(x).$$

(8)

It is obvious that $\text{supp } b_Q \subset 3Q$. Furthermore, by [7, Theorem 1], there exist $g \in L^{p'}(3Q)$ and $\|g\|_{L^{p'}(3Q)} \leq 1$ such that

$$\|b_Q(x)\|_{L^{p'}(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |b_Q(x)g(x)|dx.$$ 

Hence

$$\|b_Q\|_{L^{p'}(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |b_Q(x)g(x)|dx$$

$$\leq \liminf_{j \to \infty} \int_{\mathbb{R}^n} |b_{Q,k_j}(x)g(x)|dx$$

$$\leq |g\|_{L^{p'}(3Q)} \liminf_{j \to \infty} \|b_{Q,k_j}\|_{L^{p'}(3Q)}$$

$$\leq |3Q|^{\frac{1}{p_0} - \frac{1}{p} \sum_{i=1}^n \frac{1}{p_i}}.$$ 

Thus $b_Q$ is a $(p'_0, p')$-block. Moreover, we conclude that from the Fatou theorem,

$$\sum_{Q \in \mathcal{D}} \lambda_Q \leq \liminf_{j \to \infty} \sum_{Q \in \mathcal{D}} \lambda_{Q,k_j} \leq 2 \cdot 3^n.$$ 

(9)

Then,

$$f_0(x) := \sum_{Q \in \mathcal{D}} \lambda_Q b_Q(x) \in B_{p'_0, p'}(\mathbb{R}^n).$$

Next, let us prove that $f(x) = f_0(x)$ a.e. By the Lebesgue differential theorem, we only need to prove that

$$\lim_{j \to \infty} \int_{Q_0} f_{k_j}(x)dx = \int_{Q_0} f(x)dx = \int_{Q_0} f_0(x)dx$$

holds for any $Q_0 \in \mathcal{D}$. Without loss of generality, let $|Q_0| \leq 1$ and $0 < \varepsilon < 1$. Let

$$\begin{cases}
    \mathcal{D}_1(Q_0) := \{Q \in \mathcal{D} : 3Q \cap Q_0 \neq \emptyset, |3Q| \leq c_1\}, \\
    \mathcal{D}_2(Q_0) := \{Q \in \mathcal{D} : 3Q \cap Q_0 \neq \emptyset, |3Q| \in (c_1, c_2)\}, \\
    \mathcal{D}_3(Q_0) := \{Q \in \mathcal{D} : 3Q \cap Q_0 \neq \emptyset, |3Q| \geq c_2\}.
\end{cases}$$

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where
\[ \frac{1}{c_1 p_0} = \frac{\varepsilon}{12 \cdot 3^n}, \quad \frac{1}{c_2} = \frac{1}{2} \sum_{i=1}^{\infty} \frac{1}{p_i} = \frac{\varepsilon}{12 \cdot 3^n}. \]

For the set \( \mathcal{D}_1(Q_0) \), by (5), (9) and Hölder’s inequality
\[
\sum_{Q \in \mathcal{D}_1(Q_0)} \int_{Q_0} |\lambda_{Q,k_j} b_{Q,k_j}(x) - \lambda_Q b_Q(x)| \, dx
\leq \sum_{Q \in \mathcal{D}_1(Q_0)} \left( \lambda_{Q,k_j} \int_{Q_0} |b_{Q,k_j}(x)| \, dx + \lambda_Q \int_{Q_0} |b_Q(x)| \, dx \right)
\leq \sum_{Q \in \mathcal{D}_1(Q_0)} \left( \lambda_{Q,k_j} + \lambda_Q \right) |3Q|^\frac{1}{p_0}
\leq 2 \cdot 2 \cdot 3^n c_1^{\frac{1}{p_0}} \leq \frac{\varepsilon}{3}.
\]

For the set \( \mathcal{D}_2(Q_0) \), from (5), (9) and Hölder’s inequality on can deduce that
\[
\sum_{Q \in \mathcal{D}_2(Q_0)} \int_{Q_0} |\lambda_{Q,k_j} b_{Q,k_j}(x) - \lambda_Q b_Q(x)| \, dx
\leq \sum_{Q \in \mathcal{D}_2(Q_0)} \left( \lambda_{Q,k_j} \int_{Q_0} |b_{Q,k_j}(x)| \, dx + \lambda_Q \int_{Q_0} |b_Q(x)| \, dx \right)
\leq |Q_0|^{\frac{1}{p_0}} \sum_{i=1}^{\infty} \frac{1}{p_i} \sum_{Q \in \mathcal{D}_2(Q_0)} \left( \lambda_{Q,k_j} + \lambda_Q \right) |3Q|^\frac{1}{p_0}
\leq 2 \cdot 2 \cdot 3^n c_2^{\frac{1}{p_0}} \leq \frac{\varepsilon}{3}.
\]

For the set \( \mathcal{D}_2(Q_0) \), by \(|Q_0| \in (c_1, c_2)\), we know that \( \mathcal{D}_2(Q_0) \) contains the only finite number of dyadic cubes. When \( j \) large enough, applying (6) and (7) we see,
\[
\sum_{Q \in \mathcal{D}_2(Q_0)} \left| \int_{Q_0} \lambda_{Q,k_j} b_{Q,k_j}(x) - \lambda_Q b_Q(x) \, dx \right|
\leq \sum_{Q \in \mathcal{D}_2(Q_0)} |\lambda_{Q,k_j}| \left| \int_{Q_0} b_{Q,k_j}(x) - b_Q(x) \, dx \right|
+ \sum_{Q \in \mathcal{D}_2(Q_0)} |\lambda_{Q,k_j} - \lambda_Q| \left| \int_{Q_0} b_Q(x) - b_Q(x) \, dx \right|
\leq \frac{\varepsilon}{3}.
\]

Hence, \( f_0(x) = f(x) \).

To finish the proof, we need to know that the expressions
\[
\lim_{k \to \infty} \|f_k\|_{B_{p_0}^{p_0'}(\mathbb{R}^n)} = \|f\|_{B_{p_0}^{p_0'}(\mathbb{R}^n)}.
\]
Using Lemma 4.2 and the dominated convergence theorem, we show that there exist \( g \in \mathcal{M}_{\vec{p}}^{\mathcal{P}_0}(\mathbb{R}^n) \) such that

\[
\|f\|_{B_{\vec{p}}'\mathcal{P}_0(\mathbb{R}^n)} = \sup_{\|g\|_{\mathcal{M}_{\vec{p}}^{\mathcal{P}_0}(\mathbb{R}^n)} \leq 1} \int_{\mathbb{R}^n} \lim_{k \to \infty} |f_k(x)g(x)| \, dx
\]

\[
= \lim_{k \to \infty} \sup_{\|g\|_{\mathcal{M}_{\vec{p}}^{\mathcal{P}_0}(\mathbb{R}^n)} \leq 1} \int_{\mathbb{R}^n} |f_k(x)g(x)| \, dx
\]

\[
= \lim_{k \to \infty} \|f_k\|_{B_{\vec{p}}'\mathcal{P}_0(\mathbb{R}^n)}.
\]

The proof is completed. \( \square \)

**Proof of Theorem 4.2.** By Lemma 4.2,

\[
B_{\vec{p}}'\mathcal{P}_0(\mathbb{R}^n) \subset (\mathcal{M}_{\vec{p}}^{\mathcal{P}_0}(\mathbb{R}^n))'.
\]

It suffices to show that if \( f \) satisfies that

\[
\sup \left\{ \int_{\mathbb{R}^n} |f(x)g(x)| \, dx : g \in \mathcal{M}_{\vec{p}}^{\mathcal{P}_0}(\mathbb{R}^n), \|g\|_{\mathcal{M}_{\vec{p}}^{\mathcal{P}_0}(\mathbb{R}^n)} \leq 1 \right\} = M < \infty,
\]

then \( f \in B_{\vec{p}}'\mathcal{P}_0(\mathbb{R}^n) \).

Without loss of generality, let \( f \geq 0 \). For \( k = 1, 2, \cdots \), set \( Q_k = (-k, k)^n \), let

\[
f_k(x) := \min\{f(x)/M, k/M\} \chi_{Q_k}(x).
\]

Notice that

\[
\|f_k\|_{L_{\vec{p}}'(\mathbb{R}^n)} \leq \frac{k}{M} |Q_k|^{1 - \frac{1}{p_0} \sum_{i=1}^n \frac{1}{p_i}} \leq \frac{k}{M} |Q_k|^{1 - \frac{1}{p_0} : |Q_k|^{\frac{1}{p_0} - \frac{1}{p_0} \sum_{i=1}^n \frac{1}{p_i}}.
\]

Hence \( f_k \in B_{\vec{p}}'\mathcal{P}_0(\mathbb{R}^n) \). Lemma 4.2 yields

\[
\|f_k\|_{B_{\vec{p}}'\mathcal{P}_0(\mathbb{R}^n)} = \sup \left\{ \int_{\mathbb{R}^n} |f_k(x)g(x)| \, dx : g \in \mathcal{M}_{\vec{p}}^{\mathcal{P}_0}(\mathbb{R}^n), \|g\|_{\mathcal{M}_{\vec{p}}^{\mathcal{P}_0}(\mathbb{R}^n)} \leq 1 \right\}
\]

\[
\leq \frac{1}{M} \sup \left\{ \int_{\mathbb{R}^n} |f(x)g(x)| \, dx : g \in \mathcal{M}_{\vec{p}}^{\mathcal{P}_0}(\mathbb{R}^n), \|g\|_{\mathcal{M}_{\vec{p}}^{\mathcal{P}_0}(\mathbb{R}^n)} \leq 1 \right\} \leq 1.
\]

By the facts \( f_k \uparrow f/M \) a.e. and Theorem 4.1, we conclude that \( f \in B_{\vec{p}}'\mathcal{P}_0(\mathbb{R}^n) \). \( \square \)
5 The boundedness of the Hardy–Littlewood maximal function on the block spaces $B^p_{p'}(\mathbb{R}^n)$

In this section, we prove the boundedness of the Hardy–Littlewood maximal function on the block space $B^p_{p'}(\mathbb{R}^n)$.

**Theorem 5.1.** Let $1 < p_0, \bar{p} < \infty$ and $\frac{n}{p_0} < \sum_{i=1}^n \frac{1}{p_i}$, $M$ is bounded on $B^p_{p'}(\mathbb{R}^n)$.

Before our proof, let us recall some necessary lemmas. The Lemma 5.1 and Lemma 5.2 can be found in [25, Lemma 3.5] and [26, Lemma 2.2].

**Lemma 5.1.** Let $1 < \bar{p} \leq \infty$. Then there exists a positive constant $C$, depending on $\bar{p}$, such that, for any $f \in \mathcal{L}_{\bar{p}}(\mathbb{R}^n)$,

$$\|Mf\|_{\mathcal{L}_{\bar{p}}(\mathbb{R}^n)} \leq C\|f\|_{\mathcal{L}_{\bar{p}}(\mathbb{R}^n)}.$$

**Lemma 5.2.** If $X$ is a ball Banach function space and $M$ is bounded on $X$, then

$$|Q| \sim \|\chi_Q\|_X \|\chi_Q\|_{X'}.$$

By a similar argument to [23], we show the detailed proof of Theorem 5.1 as follows.

**Proof of Theorem 5.1.** Suppose that $b$ is a $(p_0, \bar{p})$-block and $\text{supp } b \subset Q(x_0, r)$. Let $Q_k = Q(x_0, 2^kr)$, $m_k(x) = \chi_{Q_{k+1} \setminus Q_k}(x)M(b)(x)$ and $m_0(x) = \chi_{Q_1}(x)M(b)(x)$, for $k = 1, 2, \ldots$. Then

$$Mb(x) = \sum_{k=0}^\infty m_k(x),$$

and for $k = 0, 1, 2, \ldots$,

$$\text{supp } m_k \subset Q_{k+1}.$$

Applying Lemma 5.1, we deduce that

$$\|m_0\|_{\mathcal{L}_{\bar{p}}'(\mathbb{R}^n)} \leq \|Mb\|_{\mathcal{L}_{\bar{p}}'(\mathbb{R}^n)} \lesssim \|b\|_{\mathcal{L}_{\bar{p}}'(\mathbb{R}^n)} \leq 2^n \frac{1}{\bar{p}_0} \sum_{i=1}^n \frac{1}{p_i} |Q_1| \frac{1}{p_0} \frac{1}{\bar{p}} \sum_{i=1}^n \frac{1}{p_i}.$$

And using the definition of $M$ and Hölder’s inequality, for any $k \in \mathbb{N}\setminus\{0\}$,

$$|m_k(x)| = \chi_{Q_{k+1} \setminus Q_k}(x)|Mb(x)|$$

$$\lesssim \frac{\chi_{Q_{k+1} \setminus Q_k}(x)}{(2^kr)^n} \int_{Q(x_0, r)} b(y)dy$$

$$\leq \frac{\chi_{Q_{k+1} \setminus Q_k}(x)}{(2^kr)^n} \|b\|_{\mathcal{L}_{\bar{p}}'(\mathbb{R}^n)} \|\chi_{Q(x_0, r)}\|_{\mathcal{L}_{\bar{p}}(\mathbb{R}^n)}.$$
Applying Lemma 5.2, we write
\[
\|m_k\|_{L^{p'}(\mathbb{R}^n)} \lesssim \frac{\|\chi_{Q_{k+1}}\|_{L^{p'}(\mathbb{R}^n)} \cdot \|\chi_{Q_{k+1}}\|_{L^{p}(\mathbb{R}^n)}}{(2^k r)^n} \cdot \|\chi_{Q_{k+1}}\|_{L^{p'}(\mathbb{R}^n)} \cdot \|\chi_{Q_{k+1}}\|_{L^{p}(\mathbb{R}^n)} \cdot \frac{1}{(2^k r)^n} \cdot |Q_{k+1}|^{\frac{1}{p_0} - \frac{1}{n} \sum_{i=1}^{n} \frac{1}{p_i}}.
\]

It is obvious that \(\frac{1}{p_0} > 0\) and
\[
\sum_{k=1}^{\infty} \left(\frac{1}{2^{nk}}\right)^{\frac{1}{p_0}} < C_{n,p_0,\vec{p}} < \infty,
\]
where \(C_{n,p_0,\vec{p}}\) only depends on \(n, p_0\) and \(\vec{p}\). Then, for any \((p_0', \vec{p}')\)-block \(b\),
\[
\|Mb\|_{B^{p_0'}_{\vec{p}'}(\mathbb{R}^n)} \leq C_{n,p_0,\vec{p}}.
\]

If \(f \in B^{p_0'}_{\vec{p}'}(\mathbb{R}^n)\), then there exists a decomposition such that \(f = \sum_{k=1}^{\infty} \lambda_k b_k(x)\) and
\[
\sum_{k=1}^{\infty} |\lambda_k| \leq 2 \|f\|_{B^{p_0'}_{\vec{p}'}(\mathbb{R}^n)}.
\]
Hence
\[
\|M f\|_{B^{p_0'}_{\vec{p}'}(\mathbb{R}^n)} \leq \sum_{k=1}^{\infty} |\lambda_k| \leq 2 \|M b\|_{B^{p_0'}_{\vec{p}'}(\mathbb{R}^n)} \leq C_{n,p_0,\vec{p}} \sum_{k=1}^{\infty} |\lambda_k| \leq 2 C_{n,p_0,\vec{p}} \|f\|_{B^{p_0'}_{\vec{p}'}(\mathbb{R}^n)}.
\]
This completes the proof of Theorem 5.1. \(\square\)

6 Applications

We will prove the boundedness of \([b, I_\alpha]\) on mixed Morrey spaces and block spaces in this section.

**Theorem 6.1.** Let \(0 < \alpha < n\), \(1 < p_0, q_0, \vec{p}, \vec{q} < \infty\),
\[
1 < \vec{p} \leq \vec{q} < \infty \quad \text{and} \quad \alpha = \frac{n}{p_0} - \frac{n}{q_0} = \sum_{i=1}^{n} \frac{1}{p_i} - \sum_{i=1}^{n} \frac{1}{q_i}.
\]
Then, the following conditions are equivalent:
(i) $b \in BMO(\mathbb{R}^n)$.

(ii) $[b, I_\alpha]$ is bounded from $\mathcal{M}_{p_0}^{\mathcal{B}}(\mathbb{R}^n)$ to $\mathcal{M}_{q_0}^{\mathcal{W}}(\mathbb{R}^n)$.

Remark 6.1. Taking $\frac{n}{p_0} = \sum_{i=1}^{n} \frac{1}{p_i}$ and $\frac{n}{q_0} = \sum_{i=1}^{n} \frac{1}{q_i}$, then the result of [16] need the condition

$$p_j \sum_{i=1}^{n} \frac{1}{p_i} = q_j \sum_{i=1}^{n} \frac{1}{q_i} \quad (j = 1, \ldots, n).$$

But Theorem 6.1 does not need this condition.

Theorem 6.2. (i) Let $0 < \alpha < n$, $1 < p_0, q_0, \vec{p}, \vec{q} < \infty$, $\frac{n}{p_0} \leq \sum_{i=1}^{n} \frac{1}{p_i}$ and $\frac{n}{q_0} \leq \sum_{i=1}^{n} \frac{1}{q_i}$. If $I_\alpha$ is bounded from $\mathcal{M}_{p_0}^{\mathcal{B}}(\mathbb{R}^n)$ to $\mathcal{M}_{q_0}^{\mathcal{W}}(\mathbb{R}^n)$, then for any measurable function $f$

$$\|I_\alpha f\|_{\mathcal{M}_{q_0}^{\mathcal{W}}(\mathbb{R}^n)} \leq \|f\|_{\mathcal{M}_{p_0}^{\mathcal{B}}(\mathbb{R}^n)}.$$

(ii) Let $0 < \alpha < n$, $1 < p_0, q_0, \vec{p}, \vec{q} < \infty$, $\frac{n}{p_0} \leq \sum_{i=1}^{n} \frac{1}{p_i}$ and $\frac{n}{q_0} \leq \sum_{i=1}^{n} \frac{1}{q_i}$. If $[b, I_\alpha]$ is bounded from $\mathcal{M}_{p_0}^{\mathcal{B}}(\mathbb{R}^n)$ to $\mathcal{M}_{q_0}^{\mathcal{W}}(\mathbb{R}^n)$, then for any measurable function $f$

$$\|[b, I_\alpha] f\|_{\mathcal{M}_{p_0}^{\mathcal{B}}(\mathbb{R}^n)} \leq \|f\|_{\mathcal{M}_{p_0}^{\mathcal{B}}(\mathbb{R}^n)}.$$

We show some lemmas before our proof. [35, Lemma 4.7] implies Lemma 6.1. The [21, Lemma 2.13] shows Lemma 6.2. Lemma 6.3 shows the sharp maximal theorem on weighted Lebesgue spaces [36, Theorem 3.4.5]. Lemma 6.4 can be referred to [29, Theorem 1.3] and [20, Lemma 4.2]. Note that $f$ is locally integrable function in Lemma 6.4. Lemma 6.5 can be found in [28, Corollary 5.3].

Lemma 6.1. Let $X$ be a ball quasi-Banach function space. If there exists $s \in (1, \infty)$ such that $M$ is bounded on $(X^\frac{1}{s})'$, then there exists $\varepsilon \in (0, 1)$ such that $X$ is continuously embedded into $L_{\omega}^{\varepsilon}(\mathbb{R}^n)$ with $\omega := [M(\chi_{Q[0,1]})]^\varepsilon \in A_1(\mathbb{R}^n)$, namely, there exists a positive constant $C$ such that, for any $f \in X$,

$$\|f\|_{L_{\omega}^{\varepsilon}(\mathbb{R}^n)} \leq C \|f\|_X.$$

Lemma 6.2. Let $X$ be a ball quasi-Banach function spaces and $s \in (0, \infty)$. Let $\mathcal{F}$ be the set of all pairs of nonnegative measurable functions $(F, G)$ such that, for any given $\omega \in A_1(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} (F(x, y))^s \omega(x, y)dxdy \leq C_{(s, [\omega]_{A_1(\mathbb{R}^n)})} \int_{\mathbb{R}^n \times \mathbb{R}^n} (G(x, y))^s \omega(x, y)dxdy.$$

where $C_{(s, [\omega]_{A_1(\mathbb{R}^n)})}$ is a positive constant independent of $(F, G)$, but dependents on $s$ and $A_1(\mathbb{R}^n)$. Assume that there exists a $s_0 \in [s, \infty)$ such that $X^{\frac{1}{s_0}}$ is ball Banach function space and $Mf$ is bounded on $(X^{\frac{1}{s_0}})'$. Then there exists a positive constant $C_0$ such that, for any $(F, G) \in \mathcal{F}$,

$$\|F\|_X \leq C_0 \|G\|_X.$$
Lemma 6.3. Let $1 < s < \infty$, $\omega \in A_s$. Then for any $f \in L^s_\omega(\mathbb{R}^n)$,

$$
\int_{\mathbb{R}^n} (Mf(x))^s \omega(x) dx \leq C \int_{\mathbb{R}^n} \left( M^2 f(x) \right)^s \omega(x) dx
$$

holds.

Remark 6.2. A similar result in [16, Corollary 3.10] requires

$$
Mf \in \mathcal{M}^{s_0}(\mathbb{R}^n)
$$

for some $0 < s_0 < \infty$ and $\bar{s} = (s_1, s_2, \ldots, s_n)$ with $\frac{n}{s_0} < \sum_{i=1}^n \frac{1}{s_i}$. From Lemma 6.1, (10) means

$$
f \in L^s_\omega(\mathbb{R}^n)
$$

for $1 < s < \infty$ and $\omega \in A_1(\mathbb{R}^n)$.

Lemma 6.4. Let $0 < \alpha < n$, $1 < r < \infty$ and $b \in \text{BMO}(\mathbb{R}^n)$. Then for any locally integrable function $f$, there exists a constant $C > 0$ independent of $b$ and $f$ such that

$$
M^2([b, I_\alpha](f))(x) \leq C \|b\|_{\text{BMO}(\mathbb{R}^n)} \left( I_\alpha(|f|)(x) + I_\alpha(|f|^r)(x)^{\frac{1}{r}} \right).
$$

Lemma 6.5. Let $0 < \alpha < n$, $1 < p_0, q_0, \bar{p}, \bar{q} < \infty$, $\frac{n}{p_0} \leq \sum_{i=1}^n \frac{1}{p_i}$ and $\frac{n}{q_0} \leq \sum_{i=1}^n \frac{1}{q_i}$. If

$$
1 < \bar{p} \leq \bar{q} < \infty \quad \text{and} \quad \alpha = \frac{n}{p_0} - \frac{n}{q_0} = \sum_{i=1}^n \frac{1}{p_i} - \sum_{i=1}^n \frac{1}{q_i},
$$

then there exists a positive $C$ such that

$$
\|I_\alpha f\|_{\mathcal{M}^{p_0}_{\bar{p}}(\mathbb{R}^n)} \leq C \|f\|_{\mathcal{M}^{q_0}_{\bar{q}}(\mathbb{R}^n)}.
$$

Lemma 6.6. Let $1 < q_0, \bar{q} < \infty$ and $\frac{n}{q_0} < \sum_{i=1}^n \frac{1}{q_i}$. If $f \in L^s_\omega(\mathbb{R}^n)$ with $1 < s < \infty$ and $\omega \in A_1(\mathbb{R}^n)$, then there exists a positive constant $C$ such that

$$
\|Mf\|_{\mathcal{M}^{q_0}_{\bar{q}}(\mathbb{R}^n)} \leq C \|M^2 f\|_{\mathcal{M}^{q_0}_{\bar{q}}(\mathbb{R}^n)}.
$$

Proof. For any $s \in (0, \infty)$ and $\omega \in A_1(\mathbb{R}^n)$, if $f \in L^s_\omega(\mathbb{R}^n)$, then by Lemma 6.3,

$$
\int_{\mathbb{R}^n} (Mf(x))^s \omega(x) dx \leq C(s, [\omega]_{A_1(\mathbb{R}^n)}) \int_{\mathbb{R}^n} \left( M^2 f(x) \right)^s \omega(x) dx.
$$

Taking $s_0 \in (1, \min\{s, q_0, q_1, q_2, \ldots, q_n\})$, we have

$$
\left( \mathcal{M}^{q_0}_{\bar{q}}(\mathbb{R}^n) \right)^{\frac{1}{s_0}} = \mathcal{M}^{q_0/s_0}_{\bar{q}/s_0}(\mathbb{R}^n)
$$
with $M$ is bounded on $\left(\mathcal{M}_{q/s}^{0/0}((\mathbb{R}^n))^\prime\right) = B_{(q/s),1}((\mathbb{R}^n))$. Thus, by Lemma 6.2,
\[\|Mf\|_{\mathcal{M}_{q}^{0}((\mathbb{R}^n))} \leq C\|M^2f\|_{\mathcal{M}_{q}^{0}((\mathbb{R}^n))},\]
for any $f \in L_{\omega}^{s}((\mathbb{R}^n))$ with $\omega \in A_{1}(\mathbb{R}^n)$ and $0 < s < \infty$.

This completes the proof of Lemma 6.6.

Proof of Theorem 6.1. $(i) \Rightarrow (ii)$. Let $b \in BMO(\mathbb{R}^n)$. Let $f \in \mathcal{M}_{q}^{0}((\mathbb{R}^n))$ and $p = \min\{p_1, p_2, \ldots, p_n\}$. From Hölder’s inequality, we have
\[f \in \mathcal{M}_{q}^{0}((\mathbb{R}^n)).\]
Thus, $[b, I\alpha]f \in \mathcal{M}_{q}^{0}((\mathbb{R}^n))$ with $1/q = 1/p - \frac{\alpha}{n}$. According to Lemma 6.1, there exist $1 < s < \infty$ such that
\[[b, I\alpha]f \in L_{\omega}^{s}((\mathbb{R}^n)),\]
where $\omega = [M(\chi_{Q(0, 1)})]^\epsilon$ with $0 < \epsilon < 1$. Thus, the assumption of Lemma 6.6 is satisfied. We use Lemma 6.6, Lemma 6.4 and Lemma 6.5 to write
\[\|\|[b, I\alpha]f\|_{\mathcal{M}_{q}^{0}((\mathbb{R}^n))} \leq C\|M([b, I\alpha](f))\|_{\mathcal{M}_{q}^{0}((\mathbb{R}^n))}\]
\[\leq C\left\|M\left([b, I\alpha](f)\right)\right\|_{\mathcal{M}_{q}^{0}((\mathbb{R}^n))}\]
\[\leq C\|b\|_{BMO((\mathbb{R}^n))}\left(\|I\alpha(|f|)\|_{\mathcal{M}_{q}^{0}((\mathbb{R}^n))} + \|I\alpha(|f|^r)\|_{\mathcal{M}_{q}^{0}((\mathbb{R}^n))}\right)\]
\[= C\|b\|_{BMO((\mathbb{R}^n))}\left(\|I\alpha(|f|)\|_{\mathcal{M}_{q}^{0}((\mathbb{R}^n))} + \|I\alpha(|f|^r)\|_{\mathcal{M}_{q}^{0}((\mathbb{R}^n))}\right)\]
\[\leq C\|b\|_{BMO((\mathbb{R}^n))}\|f\|_{\mathcal{M}_{q}^{0}((\mathbb{R}^n))}.

$(ii) \Rightarrow (i)$. Assume that $|b, I\alpha|$ is bounded from $\mathcal{M}_{q}^{0}((\mathbb{R}^n))$ to $\mathcal{M}_{q}^{0}((\mathbb{R}^n))$. We use the same method as Janson [30]. Choose $0 \neq z_0 \in \mathbb{R}^n$ such that $0 \not\in Q(z_0, 2\sqrt{n})$. Then for $x \in Q(z_0, 2\sqrt{n})$, $|x|^{-s} \in C^{\infty}(Q(z_0, 2\sqrt{n}))$. Hence, $|x|^{-s}$ can be written as the absolutely convergent Fourier series:
\[|x|^{-s}\chi_{Q(z_0, 2\sqrt{n})}(x) = \sum_{m \in \mathbb{Z}^n} a_m e^{2im\cdot x}\chi_{Q(z_0, 2\sqrt{n})}(x)\]  
(12)
with $\sum_{m \in \mathbb{Z}^n} |a_m| < \infty$.

For any $x_0 \in \mathbb{R}^n$ and $t > 0$, let $Q = Q(x_0, t)$ and $Q_{z_0} = Q(x_0 + z_0t, t)$. Let $s(x) =
\frac{\text{sgn}(\int_Q (b(x) - b(y))dy)}{2|Q|} \cdot \int_Q |b(x) - b_{Q_0}| = \frac{1}{|Q|} \frac{1}{|Q_0|} \int_Q \left| \int_{Q_0} (b(x) - b(y))dy \right| dx 
= \frac{1}{|Q|} \frac{1}{|Q_0|} \int_Q \int_{Q_0} s(x)(b(x) - b(y))dy dx 
= t^{-2n} \int_Q \int_{Q_0} s(x)(b(x) - b(y))dy dx.

If \( x \in Q \) and \( y \in Q_{z_0} \), then \( \frac{y - x}{t^2} \in Q(z_0, 2\sqrt{n}) \). Hence, (12) shows that

\[
\frac{1}{|Q|} \int_Q |b(x) - b_{Q_0}| = t^{-n-\alpha} \sum_{m \in \mathbb{Z}^n} a_m \int_Q \int_{Q_0} s(x)(b(x) - b(y))|x - y|^{\alpha-n} \left( \frac{|x - y|}{t} \right)^{n-\alpha} dy dx
= t^{-n-\alpha} \sum_{m \in \mathbb{Z}^n} a_m \int_Q \int_{Q_0} s(x)(b(x) - b(y))|x - y|^{\alpha-n} e^{-2im \cdot \tilde{\tau}} dy \times e^{2im \cdot \tilde{\tau}} dx 
= t^{-n-\alpha} \sum_{m \in \mathbb{Z}^n} a_m \int_Q [b, I_\alpha](e^{-2im \cdot \tilde{\tau}} \chi_{Q_0})(x) \times s(x) e^{-2im \cdot \tilde{\tau}} dx.

By the definition of Köthe dual spaces,

\[
\frac{1}{|Q|} \int_Q |b(x) - b_{Q_0}| \leq t^{-n-\alpha} \sum_{m \in \mathbb{Z}^n} a_m \left\| [b, I_\alpha](e^{-2im \cdot \tilde{\tau}} \chi_{Q_0}) \right\|_{M_{q'}^0(\mathbb{R}^n)} \left\| s \cdot e^{-2im \cdot \tilde{\tau}} \chi_Q \right\|_{B_{q'}^0(\mathbb{R}^n)}.
\]

It is easy to calculate

\[
\left\| s \cdot e^{-2im \cdot \tilde{\tau}} \chi_Q \right\|_{B_{q'}^0(\mathbb{R}^n)} = \left\| \chi_Q \right\|_{B_{q'}^0(\mathbb{R}^n)} \lesssim t^{n-\frac{\alpha}{90}}.
\]

Hence,

\[
\frac{1}{|Q|} \int_Q |b(x) - b_{Q_0}| = t^{-\frac{\alpha}{90}-\alpha} \sum_{m \in \mathbb{Z}^n} a_m \left\| [b, I_\alpha](e^{-2im \cdot \tilde{\tau}} \chi_{Q_0}) \right\|_{M_{q'}^0(\mathbb{R}^n)}.
\]

According to the hypothesis

\[
\frac{1}{|Q|} \int_Q |b(x) - b_{Q_0}| 
\leq t^{-\frac{\alpha}{90}-\alpha} \sum_{m \in \mathbb{Z}^n} a_m \left\| [b, I_\alpha] \right\|_{M_{p}^{0}(\mathbb{R}^n) \rightarrow M_{q}^{0}(\mathbb{R}^n)} \left\| [b, I_\alpha] \right\|_{M_{p}^{0}(\mathbb{R}^n) \rightarrow M_{q}^{0}(\mathbb{R}^n)} 
= t^{-\frac{\alpha}{90}+\frac{1}{90}} \sum_{m \in \mathbb{Z}^n} a_m \left\| [b, I_\alpha] \right\|_{M_{p}^{0}(\mathbb{R}^n) \rightarrow M_{q}^{0}(\mathbb{R}^n)} 
\leq \sum_{m \in \mathbb{Z}^n} |a_m| \left\| [b, I_\alpha] \right\|_{M_{p}^{0}(\mathbb{R}^n) \rightarrow M_{q}^{0}(\mathbb{R}^n)} \leq C \left\| [b, I_\alpha] \right\|_{L_p(\mathbb{R}^n) \rightarrow L_q(\mathbb{R}^n)}.
\]

Thus, we have

\[
\frac{1}{|Q|} \int_Q |b(x) - b(y)| \leq \frac{2}{|Q|} \int_Q |b(x) - b_{Q_0}| \leq C \left\| [b, I_\alpha] \right\|_{L_p(\mathbb{R}^n) \rightarrow L_q(\mathbb{R}^n)}
\]
This prove \( b \in BMO(\mathbb{R}^n) \).

The proof of Theorem 6.1 is complete. 

**Proof of Theorem 6.2.** (i) Let \( U_{\vec{p}_0}^{p_0} \) denote the unite ball on \( \mathcal{M}_{\vec{p}_0}^{p_0}(\mathbb{R}^n) \). Suppose that \( I_\alpha \) is bounded from \( \mathcal{M}_{\vec{p}_0}^{p_0}(\mathbb{R}^n) \) to \( \mathcal{M}_{\vec{q}_0}^{q_0}(\mathbb{R}^n) \). By Definition 2.2

\[
\| I_\alpha f \|_{B_{\vec{p}'_0}(\mathbb{R}^n)} = \sup_{g \in U_{\vec{p}_0}^{p_0}} \int_{\mathbb{R}^n} |g(x)I_\alpha f(x)| \, dx.
\]

In view of (1), this implies

\[
\| I_\alpha f \|_{B_{\vec{p}'_0}(\mathbb{R}^n)} \leq \sup_{g \in U_{\vec{p}_0}^{p_0}} \int_{\mathbb{R}^n} |f(x)| I_\alpha (|g|)(x) \, dx.
\]

Therefore, by the assumption and (ii) of Remark 2.2,

\[
\| I_\alpha f \|_{B_{\vec{p}'_0}(\mathbb{R}^n)} \leq \| f \|_{B_{\vec{q}'_0}(\mathbb{R}^n)} \sup_{g \in U_{\vec{p}_0}^{p_0}} \| I_\alpha (|g|) \|_{\mathcal{M}_{\vec{q}_0}^{q_0}(\mathbb{R}^n)} \lesssim \| f \|_{B_{\vec{q}'_0}(\mathbb{R}^n)}.
\]

The inverse is similar to the above, so the details are omitted.

(ii) By the same way as (i) and (iv) of Remark 2.2,

\[
\|[b, I_\alpha] f \|_{B_{\vec{p}'_0}(\mathbb{R}^n)} = \sup_{g \in U_{\vec{p}_0}^{p_0}} \left| \int_{\mathbb{R}^n} g(x)[b, I_\alpha] f(x) \, dx \right| = \sup_{g \in U_{\vec{p}_0}^{p_0}} \left| \int_{\mathbb{R}^n} f(x)[b, I_\alpha](g)(x) \, dx \right| \leq \| f \|_{B_{\vec{q}'_0}(\mathbb{R}^n)} \sup_{g \in U_{\vec{p}_0}^{p_0}} \| [b, I_\alpha](g) \|_{\mathcal{M}_{\vec{q}_0}^{q_0}(\mathbb{R}^n)} \lesssim \| f \|_{B_{\vec{q}'_0}(\mathbb{R}^n)}.
\]

The inverse is similar to the preceding argument, so we omit it. The proof of Theorem 6.2 is complete.

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Houkun Zhang
College of Mathematics and System Sciences
Xinjiang University
Urumqi 830046, China
Email address: zanghkmath@163.com

Jiang Zhou
College of Mathematics and System Sciences
Xinjiang University
Urumqi 830046, China
Email address: zhoujiang@xju.edu.cn