Invariant subalgebras of von Neumann algebras arising from negatively curved groups

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Abstract

Using an interplay between geometric methods in group theory and soft von Neumann algebraic techniques we prove that for any icc, acylindrically hyperbolic group \( \Gamma \) its von Neumann algebra \( L(\Gamma) \) satisfies the so-called ISR property: any von Neumann subalgebra \( N \subseteq L(\Gamma) \) that is normalized by all group elements in \( \Gamma \) is of the form \( N = L(\Sigma) \) for a normal subgroup \( \Sigma \rhd \Gamma \). In particular, this applies to all groups \( \Gamma \) in each of the following classes: all icc (relatively) hyperbolic groups, most mapping class groups of surfaces, all outer automorphisms of free groups with at least three generators, most graph product groups arising from simple graphs without visual splitting, etc. This result answers positively an open question of Amrutam and Jiang from [AJ22].

In the second part of the paper we obtain similar results for factors associated with groups that admit nontrivial (quasi)cohomology valued into various natural representations. In particular, we establish the ISR property for all icc, nonamenable groups that have positive first \( L^2 \)-Betti number and contain an infinite amenable subgroup.

1 Introduction

Given a locally compact group \( G \), and a continuous homomorphism of \( G \) into the group of \( * \)-automorphisms of an operator algebra \( \mathcal{A} \), say \( \Phi : G \to \text{Aut}(\mathcal{A}) \), one associates a triple \((\mathcal{A}, \Phi, G)\) called a covariant system. The notion of a covariant system is a generalization of Murray and von Neumann’s well-known group measure space construction [MvN36]. Due to its connection with the mathematical formulation of quantum field theory and statistical mechanics, covariant systems have been well studied over the years, see [BR79, DKR66, Tak67, JQ91].

The question of Galois correspondence for covariant group systems was initiated by Takesaki and Tatsuuma in [TT71], where they established that for a locally compact group \( G \) there is a one-to-one correspondence between closed normal subgroups of \( G \) and invariant von Neumann subalgebras of \( L^\infty(G) \). In [JQ91], Jorgensen and Quan further developed the study of Galois correspondence for covariant group systems associated with a locally compact group \( G \). They established that there is a one-to-one correspondence between \( G \)-invariant \( * \)-subalgebras of \( L^1(G) \) and the lattice of normal subgroups of \( G \), see [JQ91, Theorem 4.7]. They also established a one-to-one correspondence between invariant Hopf \( C^* \)-subalgebras of \( C^*_r(G) \) and the lattice of normal subgroups of \( G \), see [JQ91, Theorem 5.10].

A natural analogue of the aforementioned Galois correspondence results in the setting of discrete groups is to study the structure of invariant von Neumann subalgebras of the associated group von
Neumann algebra. This study was undertaken by two of the authors of this paper in [CD19], where the structure of \( \Gamma \)-invariant factors inside \( L(\Gamma) \), for \( \Gamma \) an icc discrete group, was investigated. In particular, it was established that [CD19, Theorem 3.15] for any \( \Gamma \)-invariant \( II_1 \) factor \( N \subset L(\Gamma) \) there exists a normal subgroup \( \Lambda \triangleleft \Gamma \), with \( N \subseteq L(\Lambda) \subseteq N \vee N' \cap L(\Gamma) \). In particular, \( \Gamma \)-invariant irreducible subfactors of \( L(\Gamma) \) can only arise from normal subgroups of \( \Gamma \).

Around the same time, motivated by Peterson’s seminal work [Pe14] on noncommutative Margulis normal subgroup theorem via character rigidity and noncommutative Poisson boundaries (see also [CP13, CP17, DP21]), Alekseev and Brugger [AB21] studied \( \Gamma \)-invariant subfactors of \( L(\Gamma) \), where \( \Gamma \) is an irreducible lattice in a simple higher rank Lie-group, and established that any \( \Gamma \)-invariant subfactor of \( L(\Gamma) \) has finite Jones index. This result was established by adapting Peterson’s techniques, though as mentioned in [AB21], the result can also be derived by combining [CD19, Theorem 3.15], with Margulis’ normal subgroup Theorem.

Motivated by the results in [Pe14, AB21, CD19], Kalantar and Panagopoulos [KP21] showed that any \( \Gamma \)-invariant von Neumann subalgebras of \( L(\Gamma) \) arise as the group von Neumann algebra of a normal subgroup, with \( \Gamma \) being an irreducible lattice in a connected semi-simple Lie group \( G \) with trivial center, no nontrivial compact factors, and such that all simple factors of \( G \) are higher rank. This result as well as [CD19] further inspired Amrutam and Jiang to study invariant von Neumann subalgebras of \( L(\Gamma) \) for other classes of groups complementary to lattices in higher rank groups [AJ22]. Specifically they established the ISR property for \( L(\Gamma) \), whenever \( \Gamma \) is a nonamenable torsion free group that is either hyperbolic, or has positive first \( L^2 \)-Betti number and satisfies Peterson-Thom’s condition (\( * \)) in [PT11]. In their investigation they also raised the question whether the ISR property holds true for all factors associated to all icc hyperbolic groups or, more generally, all icc acylindrically hyperbolic groups [Osi16]. These two questions were precisely the impetus of our investigation.

However, despite all these results, there are still many remarkable classes of groups for which it was not known whether their corresponding group factors satisfy the ISR property. Making use of Theorem 3.1 and the aforementioned strategy, in this paper we investigate \( \Gamma \)-invariant subalgebras of factors \( L(\Gamma) \) arising from natural classes of “negatively curved” groups \( \Gamma \) intensively studied in the geometric and representation theory of groups.

In the first part of the paper, using a combinatorial condition that relies on geometric word analysis and \( n \)-qons inequalities for hyperbolically embedded subgroups from [Osi07, DGO17] in conjunction with a generic von Neumann algebraic argument (see also Theorem 3.1), we establish the ISR property for the von Neumann algebras of all icc, acylindrically hyperbolic groups, [Osi16].

**Theorem A.** Let \( \Gamma \) be any icc, acylindrically hyperbolic group. Then for any von Neumann subalgebra \( N \subseteq L(\Gamma) \) satisfying \( \Gamma \subset N_{L(\Gamma)}(N) \), one can find a normal subgroup \( \Sigma \triangleleft \Gamma \) such that \( N = L(\Sigma) \).

This result generalizes several of the prior results, ([AJ22, Theorem 1.2], [CD19, Theorem 3.15, Corollary 3.17]) and it also answers positively the second aforementioned question of Amrutam-Jiang, (see [AJ22, Question 5.1]). We notice that Theorem A applies to a vast category of geometric groups including all icc hyperbolic groups, all icc relative hyperbolic groups, most mapping class groups, all outer automorphism groups of free groups with at least three generators, all nontrivial graph product groups whose underlying graph does not admit a visual splitting, etc. To this end, we emphasize that our approach for Theorem A is very different in essence from the deformation/rigidity theory methods employed in [CD19], using more generic von Neumann algebraic technique (Theorems 3.1
and 4.1) in combination to very strong group theoretic properties of the hyperbolically embedded subgroups of these groups (Theorem 2.8). We mention in passing that since icc acylindrically hyperbolic groups $\Gamma$ do not have nontrivial normal amenable subgroups, Theorem A yields that $L(\Gamma)$ also does not have nontrivial amenable von Neumann subalgebras that is normalized by $\Gamma$.

In the second part of the paper, we shift our perspective and study $\Gamma$-invariant subalgebras of $L(\Gamma)$ via methods similar to the ones used in [CD19]. As in that case, the negative curvature information we heavily exploit, is the existence of unbounded (quasi)cocycles on $\Gamma$ that are valued into its left regular representation. Even though our techniques are largely based on the analysis of (quasi)-cocycles and arrays maps developed within the deformation/rigidity theory framework [Pet09, Pet09b, CP10, Si10, Va10b, CS11, CSU11, CSU13, CKP15], we are able to refine some of them and as a consequence we obtained more general results when compared to [CD19, Theorem 3.16]. Our result is the following

**Theorem B.** Let $\Gamma$ be an icc group satisfying one of the following conditions:

1) $\Gamma$ admits an unbounded, non-proper 1-cocycle into a mixing representation;

2) $\Gamma$ is exact, torsion free and admits an unbounded quasi-cocycle into a weakly-$\ell^2$ mixing representation;

For any von Neumann subalgebra $N \subseteq L(\Gamma)$ satisfying $\Gamma \subseteq N L(\Gamma)$, one can find a normal subgroup $\Sigma \triangleleft \Gamma$ such that $N = L(\Sigma)$.

Appealing to [T09, Lemma 3., Theorem 3.4] and [PT11, Theorem 2.6], item 1) of Theorem B further implies that for every icc, non-amenable groups $\Gamma$ which has positive first $L^2$-Betti number and contains an infinite amenable subgroup, $L(\Gamma)$ satisfies the ISR property; this generalizes [AJ22, Theorem 1.3] by completely removing the assumption that $\Gamma$ needs to satisfy the Atiyah conjecture and replacing the torsion free assumption with a weaker condition. In connection to this we believe that the ISR conditions holds for all icc non-amenable groups with positive first $L^2$-Betti number but we do not have a proof in this generality. To pursue a new technique in this direction we believe it would be instrumental to tackle the torsion group constructed in [Osi09].

We also point out that the conclusion of Theorem B still holds true for all nonamenable, $\Gamma$-invariant subalgebras $N$ under slightly different and in some regards more general assumptions: a) any exact group $\Gamma$ that admits an unbounded quasi-cocycle into a weakly-$\ell^2$ mixing representation; b) any exact group $\Gamma$ that admits a proper array into a weakly-$\ell^2$ representation. For details we refer the reader to Theorem 5.1.

We believe that the ISR property of $L(\Gamma)$ is in fact more intimately connected with a specific algebraic structure of $\Gamma$ that is an implicit manifestation of the aforementioned negative curvature condition. Namely we believe that $L(\Gamma)$ satisfies the ISR property whenever $\Gamma$ has trivial amenable radical. More precisely we conjecture the following

**Conjecture C.** Let $\Gamma$ be an icc group. If $A \subset L(\Gamma)$ is a diffuse abelian von Neumann subalgebra such that $\Gamma \subset NL(\Gamma)(A)$ then one can find an amenable normal subgroup $\Sigma < \Gamma$ such that $A \subseteq L(\Sigma)$. 

3
2 Preliminaries

2.1 Notations and terminology

Throughout the paper, we write $K \subset I$ to mean that $K$ is a finite subset of $I$. Given a set $I$ we will denote by $|I|$ its cardinality. If $G$ is a group and $K, L \subset G$ are subsets we will denote by $KL = \{gh | g \in K, h \in L\}$ and

$$\langle K, L \rangle^{2m} = KKL \cdots KL,$$

where there are $m$ copies of $KL$ on the right-hand side.

2.2 Arrays and quasicocycles on groups

The notion of arrays was introduced by the first author and T. Sinclair in [CS11], and further studied in [CSU11, CSU13, CKP15] as a conceptual tool for understanding strong solidity of $\text{II}_1$ factors. Indeed, [CS11, Theorem A] shows that the group von Neumann algebra of an icc, hyperbolic group is strongly solid. The notion of arrays generalizes that of a cocycle associated with an orthogonal group representation. In fact, any quasicocycle (see below for definition) provides examples of arrays. In this section, we recall briefly the notion of arrays. In the next section we will briefly recall the notion of Gaussian dilation of $\mathcal{L}(\Gamma)$. Our main interest in the study of arrays and Gaussian dilation lies in understanding the structure of $\Gamma$-invariant subalgebras of $\mathcal{L}(\Gamma)$. In [CD19] Gaussian deformations were used to understand the structure of invariant subfactors of the group von Neumann algebras associated with “negatively curved groups”.

Throughout this section $\Gamma$ will denote a countable discrete group, and $\pi : \Gamma \to O(\mathcal{H})$ will denote an orthogonal representation of $\Gamma$ into some Hilbert space $\mathcal{H}$. The representation $\pi$ is called mixing if $\langle \pi_g(\xi), \eta \rangle \to 0$ as $g \to \infty$, for all $\xi, \eta \in \mathcal{H}$. The representation $\pi$ is called weakly-$\ell^2$ if it is weakly contained in the left regular representation $\ell^2_\Gamma$. Recall that a map $q : \Gamma \to \mathcal{H}$ is called a quasicocycle for $\pi$, if there exists some constant $D > 0$ such that $\|q(gh) - \pi_g(q(h)) - q(g)\| \leq D$ for all $g, h \in \Gamma$. The defect of a quasicocycle, denoted by $D(q)$, is the infimum over all such $D$. A quasicocycle with defect 0 is just a 1-cocycle with coefficients in $\pi$. If a quasicocycle satisfies $q(g) = -\pi_g(q(g^{-1}))$ for all $g \in \Gamma$ then we say that the quasicocycle is anti-symmetric. As noted by Thom in [T09, Section 5], for any quasicocycle $q$, the quasicocycle defined by $\tilde{q}(g) = \frac{1}{2}(q(g) - \pi_{g^{-1}}q(g))$ is anti-symmetric, and a bounded distance away from $q$. Hence, without loss of generality, we shall assume henceforth that any quasicocycle under discussion is anti-symmetric. We denote the space of all unbounded anti-symmetric quasicocycles associated with the representation $\pi$ by $QH^1_{\text{as}}(\Gamma, \pi)$.

Many remarkable groups studied in geometric or representation group theory admit quasicocycles in their left regular representations. For example whenever $\Gamma$ is an acylindrically hyperbolic group [Osi16] we have that $QH^1_{\text{as}}(\Gamma, \ell^2\Gamma) \neq 0$, [HO13]. These cover all non-elementary (relatively) hyperbolic groups, most mapping class groups of finite genus surfaces, all outer automorphism groups of free groups with at least two generators, all graph product groups that do not admit direct product decompositions, etc.

A map $q : \Gamma \to \mathcal{H}$ is called an array into $\mathcal{H}$ if it satisfies the following bounded equivariance condition:

$$\sup_{h \in \Gamma} \|q(ghk) - \pi_g(q(h))\| = C(g, k) < \infty \text{ for all } g, k \in \Gamma.$$
Clearly, quasicocycles are examples of arrays.

Finally, an array (or quasicocycle) \( q \) is called proper in for every \( C > 0 \) the ball \( B_C = \{ g \in \Gamma : \|q(g)\| \leq C \} \) is finite.

### 2.3 Gaussian von Neumann algebras and their deformations.

As in the previous section, \( \Gamma \) denotes a countable, discrete group, and \( \pi : \Gamma \to \mathcal{O}(\mathcal{H}) \) denotes an orthogonal representation. Following the treatment in [PS12], the orthogonal representation \( \pi \), via the Gaussian construction, gives rise to a nonatomic standard probability measure space \( (X, \mu) \) such that \( L^\infty(X, \mu) \) is generated by a family of unitaries \( \{ \omega(\xi) : \xi \in \mathcal{H} \} \) (Gaussian random variables) satisfying the following relations:

- **a)** \( \omega(0) = 1, \omega(\xi_1 + \xi_2) = \omega(\xi_1)\omega(\xi_2), \omega(\xi)^* = \omega(-\xi) \) for all \( \xi, \xi_1, \xi_2 \in \mathcal{H} \).
- **b)** \( \tau(\omega(\xi)) = e^{-\|\xi\|^2} \), where \( \tau \) denotes the trace on \( L^\infty(X, \mu) \) obtained by integration.

Furthermore, there exists a probability measure preserving action of \( \Gamma \) on \( (L^\infty(X, \mu), \tau) \) such that the induced action \( \pi_\Gamma : L^\infty(X, \mu) \to L^\infty(X, \mu) \) also satisfies \( \pi_\Gamma(\omega(\xi)) = \omega(\pi_\Gamma(\xi)) \) for all \( g \in \Gamma \), and \( \xi \in \mathcal{H} \). The action \( \pi_\Gamma : (X, \mu) \to (X, \mu) \) is called the Gaussian action associated with \( \pi \).

Denote by \( M = L(\Gamma) \) the group von Neumann algebra of \( \Gamma \). The Gaussian dilation of \( M \) is defined as the crossed product von Neumann algebra \( \tilde{M} = L^\infty(X, \mu) \rtimes \Gamma \). We also denote by \( e_M \) the orthogonal projection from \( L^2(\tilde{M}) \) onto \( L^2(M) \).

Let \( q : \Gamma \to \mathcal{H} \) be an array associated with \( \pi \). As in [Si10, CS11, CSU11, CKP15], we construct a deformation arising from \( q \) via exponentiation as follows. For each \( t \in \mathbb{R} \), let \( V_t \in \mathcal{U}(L^2(X, \mu) \otimes \ell^2(\Gamma)) \) be defined by

\[
V_t(\xi \otimes \delta_g) = \omega(tq(g))\xi \otimes \delta_g \text{ for all } \xi \in L^2(X, \mu), \text{ and } g \in \Gamma.
\]

This procedure is referred to as the Gaussian deformation associated with \( q \).

For future use, we record some properties of the Gaussian deformation. The reader may consult [CS11, CKP15] for the proofs.

**Theorem 2.1.** Let \( \Gamma, q \) and \( \pi \) be as above, and let \( V_t \) denote the Gaussian deformations constructed above. Then the following holds:

- **a)** (Transversality) \( V_t \) is a strongly continuous one-parameter group of unitaries satisfying the following transversality property: For each \( t \in \mathbb{R} \), and each \( \eta \in L^2(M) \), we have

\[
\|e_M^t \circ V_t(\eta)\|_2^2 \leq \|\eta - V_t(\eta)\|_2^2 \leq 2\|e_M^t \circ V_t(\eta)\|_2^2.
\]

- **b)** (Asymptotic bimodularity) For each \( x, y \in C^*_\pi(\Gamma) \) we have

\[
\lim_{t \to 0} \left( \sup_{\eta \in L^2(M)_1} \|xV_t(\eta)y - V_t(x\eta y)\|_2 \right) = 0.
\]

- **c)** ([CKP15, Proposition 6.4]) Let \( F \subset M \) be a finite subset. Denote by \( cu(F) = \{ \sum_i \mu_i x_i : x_i \in F, \mu_i \in \mathbb{C}, |\mu_i| \leq 1 \} \). Let \( X \subset (M)_1 \) be a set such that \( e_M^t \circ V_t \to 0 \) uniformly on \( X \). Then \( e_M^t \circ V_t \to 0 \) uniformly on \( cu(F) \cdot X \cdot cu(F) \).

- **d)** (Spectral gap argument [Po06, CS11]) if \( N \subseteq L(\Gamma) \) has no amenable direct, \( \pi \) is weakly \( \ell^2 \), and \( \Gamma \) is exact then \( e_M^t \circ V_t \to 0 \) uniformly on \( (N' \cap L(\Gamma))_1 \).
e) ([CS11]) For $C > 0$ denote by $\mathcal{B}_C = \{ g \in \Gamma : \|q(g)\| \leq C \}$. Let $P_{\mathcal{B}_C}$ denote the orthogonal projection onto the Hilbert subspace spanned by $\mathcal{B}_C$ inside $l^2(\Gamma)$. Let $A \subseteq L(\Gamma)$ be a von Neumann subalgebra, such that $V_t \to \text{Id}$ uniformly on $(A)_1$. Then for any $\varepsilon > 0$ we can find $C > 0$ such that $\|a - P_{\mathcal{B}_C}(a)\|_2 < \varepsilon$ for all $a \in (A)_1$.

For further use we also record the following technical result.

**Theorem 2.2.** Let $A \subseteq L(\Gamma)$ be a von Neumann subalgebra normalized by $\Gamma$ and assume that $V_t \to \text{Id}$ uniformly on $(A)_1$. Then $A$ is not diffuse.

**Proof.** Assume by contradiction that $A$ is diffuse.

Also, let $D$ be the defect of the quasicocycle $q$. Fix $\varepsilon > 0$. Since $V_t \to \text{Id}$ uniformly on $(A)_1$ from part e) in the prior result one can find $C > 0$ such that

$$\|a - P_{\mathcal{B}_C}(a)\|_2 < \varepsilon \text{ for all } a \in (A)_1. \tag{2.1}$$

As $q$ is unbounded, there exists $g \in \Gamma \setminus \mathcal{B}_{2C+2D}$. By [CSU13, Theorem 3.1], there exists $K \subseteq \mathcal{B}_C$; a finite subset, such that

$$g(\mathcal{B}_C \setminus K) \cap (\mathcal{B}_C \setminus K) = \emptyset. \tag{2.2}$$

As $A$ is diffuse, there is $a \in \mathcal{U}(A)$ such that $\|P_K(a)\|_2 < \varepsilon$, and $\|P_K(u_gau_g^*)\|_2 < \varepsilon$. Combining these with inequality (2.1) we get

$$\|a - P_{\mathcal{B}_C \setminus K}(a)\|_2 = \|a - P_{\mathcal{B}_C} \circ P_{\Gamma \setminus K}(a)\|_2 = \|a - P_{\mathcal{B}_C}(a - P_K(a))\|_2 \leq \|a - P_{\mathcal{B}_C}(a)\|_2 + \|P_{\mathcal{B}_C} \circ P_K(a)\|_2 \leq \varepsilon + \|P_K(a)\|_2 \leq 2\varepsilon. \tag{2.3}$$

Similarly, we have

$$\|u_gau_g^* - P_{\mathcal{B}_C \setminus K}(u_gau_g^*)\| \leq 2\varepsilon. \tag{2.4}$$

Then basic estimates together with inequalities (2.4)-(2.3) and equation (2.2) show that

$$1 = \|u_g\|^2 = |\langle u_g, u_g \rangle - |\langle u_gau_g^*u_g, u_g \rangle| \leq \|P_{\mathcal{B}_C \setminus K}(u_gau_g^*)u_g, u_g\| + \|P_{\mathcal{B}_C \setminus K}(u_gau_g^*) - u_gau_g^*\|_2$$

$$\leq \|P_{\mathcal{B}_C \setminus K}(u_gau_g^*)u_g, u_g\| + \|P_{\mathcal{B}_C \setminus K}(u_gau_g^*) - u_gau_g^*\|_2 + \|P_{\mathcal{B}_C \setminus K}(u_gau_g^*)\|_2\|P_{\mathcal{B}_C \setminus K}(a) - a\|$$

$$\leq 0 + 4\varepsilon,$$

which is a contradiction for $\varepsilon < \frac{1}{4}$. \hspace{1cm} \Box

### 2.4 Acylindrically hyperbolic groups

The notion of an acylindrically hyperbolic group, introduced by Osin [Osi16] as a generalization of non-elementary hyperbolic and relatively hyperbolic groups, is defined using the notion of an acylindrical action, which was introduced in [Sel92] for actions on trees, and in [Bow08] for actions on general metric spaces.
Definition 2.3. Let $\Gamma$ be a group acting on a metric space $(S, d)$ by isometries. We say the action is acylindrical if for every $D > 0$, there exists $R, N > 0$ such that if two points $x, y \in S$ satisfy $d(x, y) > R$, then
$$|\{ g \in G \mid d(x, gx) < D \text{ and } d(y, gy) < D \}| < N.$$ 

Definition 2.4. A group $\Gamma$ is acylindrically hyperbolic if it admits a non-elementary acylindrical action on a Gromov hyperbolic space $S$.

In the above, $S$ is Gromov hyperbolic if it is a geodesic metric space (i.e., every two points can be connected by a geodesic) and there exists a constant $\delta \geq 0$ such that whenever we have a triangle in $S$ with geodesic sides $a, b, c$, the side $c$ is contained in the $\delta$-neighborhood of $a \cup b$ and similarly for $a$ and $b$; the action of $\Gamma$ on $S$ is non-elementary if the limit set of $\Gamma$ on the Gromov boundary of $S$ has at least 3 points, which is equivalent to that $\Gamma$ has unbounded orbits and is not virtually cyclic [Osi16, Theorem 1.1].

Acylindrically hyperbolic groups can be equivalently defined using the notion of a hyperbolically embedded subgroup, which we recall below.

Throughout this paper, we think of graphs as metric spaces. Given a graph $S$, we think of every edge of $S$ to have length 1 and the distance between two points $x, y \in S$ is the length of a shortest path between $x, y$.

Now, let $\Gamma$ be a group with a subgroup $C$. Fix a subset $X \subseteq \Gamma$ such that $X \cup C$ generates $\Gamma$ as a group. Consider the Cayley graph $\text{Cay}(\Gamma, X \sqcup C)$. Here, for technical purposes we use disjoint union instead of union. We emphasize that we do allow non-trivial intersection between $X$ and $C$, and when this happens we will have multiple edges between certain pairs of vertices in $\text{Cay}(\Gamma, X \sqcup C)$.

Note also that the Cayley graph $\text{Cay}(C, C)$ can be identified with a subgraph of $\text{Cay}(\Gamma, X \sqcup C)$, i.e., the subgraph whose vertices and edges are all labeled by $C$.

An edge path $p \subseteq \text{Cay}(\Gamma, X \sqcup C)$ is called $C$-admissible if $p$ does not contain edges of $\text{Cay}(C, C)$. Note that we allow a $C$-admissible path $p$ to contain vertices of $\text{Cay}(C, C)$; using admissible paths we can define a relative metric $\hat{d}_C$ on $C$: for every pair of elements $c_1, c_2 \in C$, $\hat{d}_C(c_1, c_2)$ equals the length of a shortest $C$-admissible path between the vertices of $\text{Cay}(\Gamma, X \sqcup C)$ labeled by $c_1, c_2$, if such a path exists; or $\infty$ otherwise. The arithmetic laws of $[0, \infty)$ extend naturally to $[0, \infty]$ and it is easy to verify that $\hat{d}_C : C \times C \to [0, \infty]$ defines a metric on $C$. If $p$ is an edge path of $\text{Cay}(\Gamma, X \sqcup C)$ whose edges are labeled by elements of $C$, then we let $\hat{\ell}_C(p) = \hat{d}_C(1, c_1^{-1}c_2)$, where $c_1$ (resp. $c_2$) is the labeled of the initial (resp. terminal) vertex of $p$.

Definition 2.5. We say that a subgroup $C$ hyperbolically embeds into a group $\Gamma$ with respect to a set $X \subseteq \Gamma$, denoted as $C \hookrightarrow^h (\Gamma, X)$, if the following hold.

(i) $X \cup C$ generates $\Gamma$ as a group.

(ii) $\text{Cay}(\Gamma, X \sqcup C)$ is Gromov hyperbolic.

(iii) The metric $\hat{d}_C$ is locally finite, i.e., every ball of finite radius contains only finitely many elements.

We say that $C$ hyperbolically embeds into $\Gamma$ if there exists a set $X \subseteq \Gamma$ such that (i), (ii) and (iii) hold.
The following is the main result of [Osi16].

**Theorem 2.6** (Osin, 2016). A group $\Gamma$ is acylindrically hyperbolic if and only if $\Gamma$ has a proper infinite hyperbolically embedded subgroup.

Now suppose we have groups $\Gamma \geq C$ and a subset $X \subseteq \Gamma$ such that $C \hookrightarrow h(\Gamma, X)$. Let $p$ be a path in $\text{Cay}(\Gamma, X \sqcup C)$. A $C$-subpath $q$ of $p$ is a subpath of $p$ all of whose edges are labeled by elements of $C$ (if $p$ is a cycle then we allow $q$ to be a subpath of a cyclic shift of $p$). Also, $q$ is called a $C$-component if $q$ is a $C$-subpath and $q$ is not properly contained in any other $C$-subpath. Two $C$-components $q_1, q_2$ of $p$ are connected if there is an edge path $q_3$ all of whose edges are labeled by elements of $C$ such that $q_3$ connects a vertex of $q_1$ to a vertex $q_2$. If a $C$-component $q$ is not connected to any other $C$-components, then we say $q$ is an isolated $C$-component. A useful property of isolated $C$-components is that, in a geodesic polygon, the total $\ell_C$-length of the isolated $C$-components is bounded uniformly in the number of sides of the polygon. The following is a simplified version of [DGO17, Proposition 4.14], which generalizes [Osi07, Proposition 3.2].

**Proposition 2.7** (Dahmani–Guirardel–Osin, 2017). There exists $D > 0$ such that the following holds. Let $p$ be an $n$-gon in $\text{Cay}(\Gamma, X \sqcup C)$ with geodesic sides and let $I$ be the subset of sides consisting of isolated $C$-components of $p$. Then

$$\sum_{q \in I} \ell_C(q) \leq Dn.$$  

**Theorem 2.8.** Let $\Gamma$ be a group with a hyperbolically embedded subgroup $C$. Then for every $K \subseteq \Gamma \setminus C$, there exists $L \subseteq C$ such that for all $m,n \in \mathbb{N}^+$,

$$\langle (C \setminus L), K \rangle^{2m} \cap \langle K, (C \setminus L) \rangle^{2n} = \emptyset.$$  

(2.5)

**Proof.** Fix $X \subseteq \Gamma$ such that $C \hookrightarrow h(\Gamma, X)$. For each $k \in K$, there exists a geodesic word $w_k$ over $(X \cup X^{-1}) \cup C$ representing $k$ such that the following holds.

- Let $u_k$ (resp. $v_k$) be the maximal initial (resp. terminal) segment of $w_k$ labeled by a word over $C$. Write $w_k$ as the concatenation of $u_k, w_k', v_k$. Then no non-trivial initial or terminal segment of $w_k'$ can represent an element of $C$.

We note that $w_k' \neq \emptyset$ for all $k$ as $K \cap C = \emptyset$. Let $R = \{u_k, u_k^{-1}, v_k, v_k^{-1} \mid k \in K\}$, let $D$ be the constant given by Proposition 2.7 and let $L \subseteq C$ be the finite set consisting of elements $c \in C$ such that there exists $r_1, r_2 \in R$ with

$$\hat{d}_C(1, r_1 cr_2) \leq 4D.$$  

(2.6)

Note that $L = L^{-1}$.

**Claim 2.9.** There do not exist $c_1, \ldots, c_{m+n} \in C \setminus L$ and $k_1, \ldots, k_{m+n} \in K \cup K^{-1}$ such that

$$\prod_{i=1}^{m+n} c_ik_i = 1.$$  

**Proof of Claim 2.9.** Suppose for the contrary that such a set of $c_i, k_i$ exists. For simplicity, we will assume that $k_i \in K$ for all $1 \leq i \leq m + n$. The general case can be treated in the same way.
Figure 1: The case $m = n = 2$: the geodesic 8-gon $p$ with two possibilities of the path $t$

For $1 \leq i \leq m + n$, let $c'_i \in C$ be the element represented by the word $v_{k_{i-1}}c_iu_{k_i}$ (subscript modulo $m + n$). Consider the path $p \subseteq \text{Cay}(\Gamma, X \sqcup C)$ labeled by the word $\prod_{i=1}^{m+n} c'_i w'_{k_i}$. For $1 \leq i \leq m + n$, let $p_i$ (resp. $q_i$) be the subpath of $p$ labeled by the word $c'_i$ (resp. $w'_{k_i}$). Then $p$ is a $(2m + 2n)$-gon with geodesic sides $p_i, q_i$; here we used that $w'_{k_i} \neq \emptyset$.

For all $i$, the sides $p_i$ are $C$-components of $p$, and inequality (2.6) implies that $\hat{\ell}_C(p_i) > 4D$. So Proposition 2.7 implies that the $C$-components $p_i$ cannot all be isolated in $p$. Up to relabeling the subpaths $p_i, q_i$, we may assume:

- $p_1$ is connected to either $p_k$ or some $C$-component of $q_{k-1}$ for some $k$. In the former case, let $s = p_k$ and in the latter case, let $s$ be the corresponding $C$-component of $q_{k-1}$. Let $t$ be an edge labeled by an element of $C$ connecting the terminal vertex of $p_1$ to the initial vertex of $s$, let $q'_{k-1}$ be the segment of $q_{k-1}$ from the initial vertex of $q_{k-1}$ to the initial vertex of $s$, and let $q$ be the polygon formed by $q_1, p_2, q_2, \cdots, q_{k-1}, q'_{k-1}$ and $t$. Then $p_2, \cdots, p_{k-1}$ are isolated $C$-components of $q$ (if some $p_i, i \in \{2, 3, \cdots, k-1\}$, is not isolated, simply rename $p_i$ to be $p_1$).

We refer to Figure 1 for an illustration of the case $m = n = 2$.

If $k = 2$, then some non-trivial initial segment of $w'_1$ represents an element of $C$, a contradiction. If $k > 2$, then $q$ is a $(2k - 2)$-gon with geodesic sides. $q$ has $k - 2$ isolated $C$-components $p_2, \cdots, p_{k-1}$. Inequality (2.6) and Proposition 2.7 then implies

$$4D(k-2) < (2k-2)D,$$

which contradicts the assumption $k > 2$.

Equation (2.5) is an immediate consequence of the above claim.

2.5 Popa’s intertwining techniques

More than fifteen years ago, S. Popa introduced in [Po03, Theorem 2.1 and Corollary 2.3] a powerful analytic criterion for identifying intertwiners between arbitrary subalgebras of tracial von Neumann algebras. This is known as Popa’s intertwining-by-bimodules technique in the current literature. Popa’s intertwining-by-bimodules technique has played a key role in the classification of von Neumann algebras program via Popa’s deformation/rigidity theory.
Theorem 2.10. [Po03] Let \((\mathcal{M}, \tau)\) be a separable tracial von Neumann algebra and let \(\mathcal{P}, \mathcal{Q} \subseteq \mathcal{M}\) be (not necessarily unital) von Neumann subalgebras. Then the following are equivalent:

1. There exist \(p \in \mathcal{P}(\mathcal{P}), q \in \mathcal{P}(\mathcal{Q})\), a \(*\)-homomorphism \(\theta : p\mathcal{P}p \to q\mathcal{Q}q\) and a partial isometry \(0 \neq v \in q\mathcal{M}p\) such that \(\theta(x)v = vx\), for all \(x \in p\mathcal{P}p\).

2. For any group \(G \subset \mathcal{U}(\mathcal{P})\) such that \(G'' = \mathcal{P}\) there is no sequence \((u_n)_n \subset G\) satisfying \(\|E_{\mathcal{Q}}(xu_ny)\|_2 \to 0\), for all \(x, y \in \mathcal{M}\).

If one of the two equivalent conditions from Theorem 2.10 holds then we say that \(a \text{ corner of } \mathcal{P}\) embeds into \(\mathcal{Q}\) inside \(\mathcal{M}\), and write \(\mathcal{P} \prec_{\mathcal{M}} \mathcal{Q}\). If we moreover have that \(\mathcal{P}p' \prec_{\mathcal{M}} \mathcal{Q}\), for any projection \(0 \neq p' \in \mathcal{P}' \cap 1\mathcal{P}1\mathcal{P}\) (equivalently, for any projection \(0 \neq p' \in \mathcal{Z}(\mathcal{P}' \cap 1\mathcal{P}1\mathcal{P}))\), then we write \(\mathcal{P} \prec_{\mathcal{M}}' \mathcal{Q}\). We refer the readers to the survey papers [Po07, Va10a, Io18] for recent progress in von Neumann algebras using deformation/rigidity theory.

For future use, we recall a special case of [HPV10, Corollary 7], that generalizes [PV08, Theorem 6.16].

Proposition 2.11 (Houdayer-Popa-Vaes). Let \(\Gamma \rhd (\mathcal{A}, \tau)\) be a trace preserving action, and let \(M = \mathcal{A} \rtimes \Gamma\). Let \(B \subseteq M\) be a regular von Neumann algebra, and let \(\Sigma < \Gamma\) be a subgroup such that \(B \preceq_M \mathcal{A} \rtimes \Sigma\). Then, \(B \preceq_M \mathcal{A} \rtimes (\cap_g \Sigma g^{-1})\) for all \(g_1, \ldots, g_n \in \Gamma\).

3 Some general results on \(\Gamma\)-invariant subalgebras of \(L(\Gamma)\)

In this section we collect together a few general facts concerning \(\Gamma\)-invariant subalgebras of \(L(\Gamma)\).

The first result, is closely related to [CD19, Theorem 3.15], and we include it here just for completeness as it will be used in the sequel.

Theorem 3.1. Let \(\Gamma\) be an icc group, and \(B \subseteq L(\Gamma)\) be a subfactor such that \(\Gamma \subseteq N_{L(\Gamma)}(B)\). Then, there exists a normal subgroup \(\Sigma \lhd \Gamma\) such that \(B \vee (B' \cap L(\Sigma)) = L(\Sigma)\). Moreover, one can find collections of unitaries \(\{v_g\}_{g \in \Sigma} \subset \mathcal{U}(B)\) and \(\{w_g\}_{g \in \Sigma} \subset \mathcal{U}(B' \cap L(\Sigma))\) and a 2-cocycle \(c : \Sigma \times \Sigma \to \mathbb{T}\) such that for all \(g, h \in \Sigma\) the following relations hold:

\[
\begin{align*}
    u_g &= v_g w_g \\
    v_g v_h &= c_{g,h} v_{gh} \\
    w_g w_h &= \overline{c_{g,h}} w_{gh}
\end{align*}
\]

Proof. Using [CD19, Theorem 3.15], one can find a normal subgroup \(\Sigma \lhd \Gamma\) such that \(B \subseteq L(\Sigma) \subseteq B \vee B' \cap L(\Gamma)\). For the reader’s convenience we recall next the argument from [CD19]. Consider the set \(S = \{g \in \Gamma : E_B(u_g) \neq 0\}\) and notice it satisfies \(S = S^{-1}\) and \(1 \in S\). Fix \(g \in S\) and consider the inner automorphism of \(L(\Gamma)\) given by \(\alpha = \text{ad}(u_g)\). Note that \(\alpha\) restricts to a \(*\)-automorphism of \(B\).

Thus we have that \(\alpha(x) E_B(u_g) = E_B(u_g)x\) for all \(x \in B\). This implies \(b_g := E_B(u_g)u_g^* \in B' \cap L(\Gamma)\). From the choice of \(g\) we have \(b_g \neq 0\). Thus we have \(E_B(u_g) = b_g u_g\). Hence, \(E_B(u_g) E_B(u_g)^* = b_g b_g^*\).

Therefore, \(b_g b_g^* \in B' \cap B = \mathbb{C}\), which implies \(b_g b_g^* = \tau(b_g b_g^*)\). Hence, by normalizing if necessary, we can find a unitary \(c_g \in \mathcal{U}(B)\) such that \(E_B(u_g) = \|b_g\|_2 c_g\). As \(E_B(u_g) = b_g u_g\), we get \(u_g = \|b_g\|_2 b_g c_g\).

Note that \(\|b_g\|_2 b_g^* \in \mathcal{U}(B' \cap L(\Gamma))\). So \(u_g \in \mathcal{U}(B) \mathcal{U}(B' \cap L(\Gamma)) \subseteq B \vee B' \cap L(\Gamma)\).

Next let \(\Sigma\) be the set of all \(g \in \Gamma\) such that

\[
    u_g = v_g w_g
\]

(3.4)
where $v_g \in U(B)$, and $w_g \in U(B' \cap L(\Gamma))$. We notice that $\Sigma$ is in fact a normal subgroup of $L(\Gamma)$ containing the set $S$, and hence $B \subseteq L(\Sigma)$. Since $L(\Sigma) \subseteq B \vee (B' \cap L(\Gamma))$ by construction, we get $B \subseteq L(\Sigma) \subseteq B \vee B' \cap L(\Gamma)$. Furthermore, since $B$ is a factor, using Ge’s tensor splitting theorem [Ge96], we get $L(\Sigma) = B \vee (B' \cap L(\Sigma))$.

Fix $g, h \in \Sigma$. Using (3.4) we have $v_{gh}w_{gh} = u_{gh} = u_gu_h = v_gw_gv_hw_h$ which further implies $v^*_g v_g v_h = w_{gh}w^*_k w^*_g$. Notice the left-hand side of this equation belongs to $B$ while the right-hand side belongs to $B' \cap L(\Sigma)$. Since $B$ is a factor, we necessarily have that

$$v^*_g v_g v_h = w_{gh}w^*_k w^*_g = c_{g,h} \in \mathbb{C}. \quad (3.5)$$

Moreover, this equation also entails that $c_{g,h} = \tau(v^*_g v_g v_h)$ and $|c_{g,h}| = \|v^*_g v_g v_h\|_2 = 1$.

In conclusion, the $\mathbb{T}$-valued 2-cocycle relations stated in the statement are satisfied.

For further use we also record the following result.

**Proposition 3.2.** Let $M = L(\Gamma)$ be a II$_1$ factor. Then any finite dimensional subspace $A \subseteq M$ that is invariant under the conjugation action by $\Gamma$ satisfies $A = \mathbb{C}1$.

**Proof.** Assume by contradiction $A \neq \mathbb{C}1$. Consider $B = \{x - \tau(x)1 : x \in A\} \subseteq M$ and note it is a nontrivial finite dimensional subspace that is invariant under conjugating by elements of $\Gamma$. Pick $b_1, \ldots, b_k \subset B$, where $k \geq 1$, a (finite) orthonormal basis with respect to the dot product induced by the trace of $M$. Thus for every $g \in \Gamma$ and $i = 1, \ldots, k$ one can find scalars $\alpha(g,i) \in \mathbb{C}$ with $\sup_{g,i} |\alpha(b, i)| \leq 1$ so that

$$u_g b_1 u_g^{-1} = \sigma_g (b_1) = \sum_{i=1}^k \alpha(g,i)b_i. \quad (3.6)$$

Fix $\varepsilon > 0$. Using basic $\| \cdot \|_2$-approximations there is a finite set $F_\varepsilon \subset \Gamma \setminus \{1\}$ such that for every $i = 1, \ldots, k$ there is $b'_i \in M$ satisfying the following properties:

$$\|b_i - b'_i\|_2 \leq \varepsilon \quad (3.7)$$
$$\text{its support } \sup(b'_i) \subseteq F_\varepsilon. \quad (3.8)$$

As $\Gamma$ is icc one can find $g \in \Gamma$ such that $gF_\varepsilon g^{-1} \cap F_\varepsilon = \emptyset$; in particular, by (3.8) we have $\langle u_g b'_i u_{g^{-1}}, b'_i \rangle = 0$, for all $i = 1, \ldots, k$. This combined with (3.6), (3.7) and Cauchy-Schwartz inequality show that

$$1 = \|b_1\|_2^2 = \|u_g b_1 u_{g^{-1}}\|_2^2 = \langle u_g b_1 u_{g^{-1}}, u_g b_1 u_{g^{-1}} \rangle = \sum_{i=1}^k \alpha(g,i) \langle u_g b_1 u_{g^{-1}}, b_i \rangle \leq \sum_{i=1}^k |\langle u_g b_1 u_{g^{-1}}, b_i \rangle| \leq \varepsilon k + \varepsilon(1 + \varepsilon)k + \sum_{i=1}^k |\langle u_g b'_i u_{g^{-1}}, b'_i \rangle| \quad (3.9)$$

$$= (2\varepsilon + \varepsilon^2)k.$$

This however leads to a contradiction when $\varepsilon < (3k)^{-1}$. \qed
Corollary 3.3. Let $M = L(\Gamma)$ be a $\Pi_2$ factor. Then for a von Neumann subalgebra $A \subseteq M$ satisfying $\Gamma \subseteq \mathcal{N}_M(A)$ we have that $\mathcal{A}$ is either trival, or diffuse.

Proof. Let $\mathcal{Z}(A)$ be the unique projection such that $A\mathcal{Z}$ is completely atomic, and $A(1 - \mathcal{Z})$ is diffuse. Assume $A$ is not diffuse, so $\mathcal{Z} \neq 0$. Hence, we can find a family of nontrivial minimal projections $\mathcal{P} := \{\mathcal{Z}_i : i \in I\} \subseteq \mathcal{P}(\mathcal{Z}(A))$ and finite dimensional von Neumann algebras $M_i \cong M_{n_i}(\mathbb{C})$ for some $n_i$ such that $A\mathcal{Z} = \bigoplus_i M_i \mathcal{Z}_i$. For each $g \in \Gamma$, we denote by $\alpha_g$ the $*$-automorphism of $M$ given by $\alpha_g = \text{ad}(u_g)$. Note that $\alpha_g(A\mathcal{Z}) = A\mathcal{Z}_j$ for all $g \in \Gamma$. Moreover, each $\alpha_g$ leaves the center $\bigoplus_i \mathcal{C}_i \mathcal{Z}_i$ invariant. Thus $\alpha_g$ leaves the set $\mathcal{P}$ invariant and hence $\Gamma$ acts on $\mathcal{P}$ by $g \cdot \mathcal{Z}_i = \alpha_g(\mathcal{Z}_i)$. As $\alpha_g$ is trace preserving, and $M$ is a finite factor, there exists a finite subset $\mathcal{P}_0 := \{\mathcal{Z}_1, \ldots, \mathcal{Z}_n\} \subseteq \mathcal{P}$ which is $\Gamma$ invariant. In particular $\text{span}\mathcal{P}_0 \subset M$ is a finite dimensional $\Gamma$-invariant subspace and by Proposition 3.2 we get that $\mathcal{Z}_i = 1$ and hence $A = M_1$. Thus $M_1$ is a $\Gamma$ invariant finite dimensional von Neumann algebra and once again Proposition 3.2 implies that $A = C_1$. 

We notice that if $A \subset L(\Gamma)$ is any $\Gamma$-invariant von Neumann subalgebra then so is its center $\mathcal{Z} = \mathcal{Z}(A)$. Thus the prior results altogether imply that to understand the structure of $\Gamma$-invariant subalgebras it is imperative to look at the diffuse abelian case. In [AJ22, Example 3.5] were presented situations of abelian von Neumann subalgebras that do not arise from subgroups. For example if we take any nontrivial wreath product $\Gamma = \Sigma \wr A$ where $\Sigma$ is abelian then for any nontrivial von Neumann subalgebra $B \subset L(\Sigma)$ the infinite tensor product $A := \overline{\bigotimes} \mathbb{A} \subset L(\Sigma^{(A)}) \subset L(\Gamma)$ is obviously a $\Gamma$-invariant von Neumann subalgebra which may not arise from any subgroup of $\Gamma$. One can construct more examples of the following type. Let $\Sigma < \Gamma$ be a normal inclusion where $\Gamma$ is icc and the finite conjugacy radical of $\Sigma$ is infinite and nonabelian. Then the center $A := \mathcal{Z}(L(\Sigma)) \subset L(\Sigma) \subset L(\Gamma)$ is an $\Gamma$-invariant von Neumann subalgebra that does not arise from a subgroup.

4 Proof of Theorem A

First we show that the presence of combinatorial relations of the type (2.5) in a group $\Gamma$ is an obstruction to the existence of diffuse abelian $\Gamma$-invariant subalgebra in $L(\Gamma)$. Similar analysis was used to great effect in other classification aspects in von Neumann algebras via deformation/rigidity theory, notably in [Po04, IPP05] and more recently [Cios21a].

Theorem 4.1. Let $\Gamma$ be a group for which there exists an infinite weakly malnormal subgroup $C < \Gamma$ satisfying the following property: for every finite subset $F \subset \Gamma \setminus C$ there exists a finite subset $K \subset C$ such that

$$F(C \setminus K)F(C \setminus K) \cap (C \setminus K)F(C \setminus K)F = \emptyset.$$ 

(4.1) Assume there are commuting von Neumann subalgebras $A_1, A_2 \subseteq L(\Gamma)$ such that $\Gamma \subset \mathcal{N}_{L(\Gamma)}(A_1) \cap \mathcal{N}_{L(\Gamma)}(A_2)$. Then either $A_1$ or $A_2$ is atomic.

Proof. Assume for the sake of contradiction that both $A_1$, $A_2$ are diffuse. Henceforth we denote by $M = L(\Gamma)$.

Next we briefly argue that $A_j \nsubseteq L(C)$ for $j = 1, 2$. Indeed, as $A_j$ is regular in $M$, if $A_j \nsubseteq L(C)$, by Proposition 2.11 we would have that $A_j \nsubseteq L(\bigcap_{i=1}^n g_i C g_i^{-1})$ for all $g_1, \ldots, g_n \in \Gamma$. However, since $C$ is weakly malnormal there exist $g_1, \ldots, g_k \in \Gamma \setminus C$ such that $\bigcap_{i=1}^n g_i C g_i^{-1} | < \infty$. Using this we further get that $A_j$ has a nontrivial atomic corner, which contradicts that $A_j$ is diffuse.

We now prove the following
Claim 4.2. For every \( j = 1, 2 \) and \( \varepsilon > 0 \) there exists \( a_j \in U(A_j) \), a finite set \( F^j_\varepsilon \subset \Gamma \setminus C \), and an element \( a_\varepsilon \in M \) supported on \( F^j_\varepsilon \) such that \( \|a_\varepsilon\|_2 \leq 2 \), and \( \|a_j - a_\varepsilon\|_2 < \varepsilon \).

Proof of Claim 4.2: Fix \( j = 1, 2 \) and \( \varepsilon > 0 \). As \( A_j \not\subset M \) \( L(C) \), by Theorem 2.10 there is \( a_j \in U(A) \) such that

\[
\|E_{L(C)}(a_j)\|_2 < \frac{\varepsilon}{3}.
\]

By Kaplansky’s Density Theorem, there exists \( b_j \in M \) supported on a finite subset \( G_j \subset \Gamma \) such that

\[
\|a_j - b_j\|_2 < \frac{\varepsilon}{3} \quad \text{and} \quad \|b_j\|_\infty \leq 1.
\]

Now, let \( a_\varepsilon^j = b_j - E_{L(C)}(b_j) \), and note that \( a_\varepsilon^j \) is supported on the finite set \( F^j_\varepsilon := G_j \setminus C \). Moreover we can see that

\[
\|a_\varepsilon^j\|_\infty = \|b_j - E_{L(C)}(b_j)\|_\infty \leq \|b_j\|_\infty + \|E_{L(C)}(b_j)\|_\infty \leq 2.
\]

Finally, using triangle inequality together with other basic estimates and (4.2)-(4.3) we see that

\[
\|a_j - a_\varepsilon^j\|_2 = \|a_j - (b_j - E_{L(C)}(b_j))\|_2 \leq \|a_j - b_j\|_2 + \|E_{L(C)}(b_j)\|_2 \\
\leq \|a_j - b_j\|_2 + \|E_{L(C)}(a_j)\|_2 + \|E_{L(C)}(a_j - b_j)\|_2 \\
\leq 2\|a_j - b_j\|_2 + \|E_{L(C)}(a_j)\|_2 \leq 3 \cdot \frac{\varepsilon}{3} = \varepsilon,
\]

which finishes the proof of the claim.

Fix \( j = 1, 2 \) and \( \varepsilon > 0 \), let \( a_j \in U(A_j) \), \( F^j_\varepsilon \subset \Gamma \setminus C \) and \( a_\varepsilon^j \) as in the statement of Claim 4.2. Let \( F_\varepsilon = F^1_\varepsilon \cup F^2_\varepsilon \). Since \( A_1 \) and \( A_2 \) are commuting \( \Gamma \) invariant von Neumann subalgebras, we have \( u_g a_1 u_g^* a_2 = u_2 u_g a_1 u_g^* \) for all \( g \in \Gamma \). Thus using basic estimates we get

\[
1 = \|u_g a_1 u_g^* a_2\|_2^2 = |\langle u_g a_1 u_g^* a_2, u_g a_1 u_g^* a_2 \rangle| = |\langle u_g a_1 u_g^* a_2, u_2 u_g a_1 u_g^* \rangle| \\
\leq |\langle u_g (a_1 - a_\varepsilon^j) u_g^* a_2, a_2 u_g a_1 u_g^* \rangle| + |\langle u_g a_1 u_g^* a_2, a_2 u_g a_1 u_g^* \rangle| \leq \|a_1 - a_\varepsilon^j\|_2 + \|a_\varepsilon^j\|_2 \\
\leq \|a_1\|_\infty \|a_2 - a_\varepsilon^j\|_2 + \|u_g a_1 u_g^* a_2, a_2 u_g a_1 u_g^*\| \\
\leq \varepsilon(1 + \|a_1\|_\infty + \|a_2\|_\infty + \|a_\varepsilon^j\|_\infty + \|a_1 u_g a_1 u_g^*\| + \|a_2 u_g a_1 u_g^*\| + \|u_g a_1 u_g^* a_2, a_2 u_g a_1 u_g^*\| \\
\leq \varepsilon(1 + \|a_1\|_\infty + \|a_2\|_\infty + \|a_\varepsilon^j\|_\infty + \|a_2\|_\infty + \|a_\varepsilon^j\|_\infty + \|a_1 u_g a_1 u_g^*\| + \|a_2 u_g a_1 u_g^*\| + \|u_g a_1 u_g^* a_2, a_2 u_g a_1 u_g^*\|).
\]

Since \( \|a_\varepsilon^j\|_\infty \leq 2 \) this further implies that for all \( g \in \Gamma \) we have

\[
1 \leq 15\varepsilon + |\langle u_g a_1 u_g^* a_2, a_2 u_g a_1 u_g^* \rangle|.
\]

(4.4)

Now, let \( K \subseteq C \) be the finite subset corresponding to \( F_\varepsilon \) that satisfies condition (4.1). Since \( C \) is infinite, there exists \( g \in C \setminus K \) such that \( g^{-1} \in C \setminus K \), and therefore by condition (4.1) we have

\[
F_\varepsilon(C \setminus K)F_\varepsilon(C \setminus K) \cap (C \setminus K)F_\varepsilon(C \setminus K)F_\varepsilon = \emptyset.
\]

(4.5)

However, this implies that \( \langle u_g a_1 u_g^* a_2, a_2 u_g a_1 u_g^* \rangle = 0 \). Using this in inequality (4.4) we get \( 1 \leq 15\varepsilon \), which is a contradiction for \( \varepsilon \) sufficiently small. This finishes the proof. \( \square \)
Proof of Theorem A. By Theorem 4.1 we get that $Z(B)$ is atomic. By Corollary 3.3 we get that $Z(B) = \mathbb{C}$, and hence $B$ is a factor. Using Theorem Theorem 3.1 one can find a normal subgroup $\Sigma \vartriangleleft \Gamma$ such that $B \vartriangleleft (B' \cap L(\Sigma)) = L(\Sigma)$. As $\Sigma \vartriangleleft \Gamma$ is normal it follows that its FC-center $\Sigma^{fc} \vartriangleleft \Gamma$ is an amenable normal subgroup. Since $\Gamma$ is icc acylindrically hyperbolic it follows that $\Sigma^{fc} = 1$. In particular, $\Sigma$ is an icc group and hence $B$ and $B' \cap L(\Sigma)$ are factors. Thus using Theorem 4.1 we get that either $B \cong M_n(\mathbb{C})$ or $B' \cap L(\Sigma) \cong M_n(\mathbb{C})$ for some $n \in \mathbb{N}$. Moreover, both $B$ and $B' \cap L(\Sigma)$ are invariant under the conjugacy action of $\Gamma$. However, as $\Gamma$ is icc, Proposition 3.2 implies that $n = 1$ which yields the desired conclusion. □

5 Proof of Theorem B

In this section we will investigate $\Gamma$-invariant subalgebras in $\Pi_1$ factors associated with groups satisfying representations-valued cohomological type of negative curvature. Specifically, we will be using deformation/rigidity arguments tailored to the analysis of arrays/quasicocycles from [Pet09, Pet09b, Si10, CP10, Va10b, CS11, CSU11, CSU13, CKP15] to prove Theorem B and other related results.

Our first theorem shows that in such group factors if there are $\Gamma$-invariant von Neumann subalgebras that may not be implemented by normal subgroups they have to be necessarily amenable and have diffuse centers.

Theorem 5.1. Let $\Gamma$ be an icc exact group satisfying one of the following conditions:

1) $\Gamma$ admits an unbounded quasi-cocycle into an weakly-$\ell^2$, mixing representation;

2) $\Gamma$ admits a proper array into a weakly-$\ell^2$ representation.

Assume $N \subseteq L(\Gamma)$ is a nonamenable von Neumann subalgebra, such that $\Gamma \subseteq N_{\mathcal{L}(\Gamma)}(N)$. Then, there exists a normal subgroup $\Sigma \vartriangleleft \Gamma$ such that $N = L(\Gamma)$.

Proof. First we show $N$ is a factor. Let $Z = Z(N)$ be the center of $N$. As $u_g N u^*_g = N$ for all $g \in \Gamma$, we also have $u_g Z u^*_g = Z$, for all $g \in \Gamma$.

By Corollary 3.3, we may now assume by contradiction that $Z$ is diffuse. Let $q : \Gamma \to \mathcal{H}_\pi$ be an unbounded quasi-cocycle satisfying condition 1) or a proper array satisfying condition 2). Let $M = L(\Gamma)$. Following Section 2.3 let $M \subseteq \hat{M} = L^\infty(X^\pi) \rtimes \Gamma$ be the Gaussian extension and let $V_t : L^2(M) \to L^2(\hat{M})$ be the Gaussian deformation corresponding to $q$. Let $e_M : \hat{M} \to M$ be the orthogonal projection.

Since $N$ is nonamenable, there is $0 \neq p \in Z$ such that $Np$ has no amenable direct summand. Thus applying a version of Popa’s spectral gap argument (see part d) in Theorem 2.1) we get

$$\lim_{t \to 0} \left( \sup_{a \in (Z)_1} \| e_M^t \circ V_t(\alpha p) \|_2 \right) = 0.$$  \hspace{1cm} (5.1)

Equation (5.1) and [CKP15, Proposition 6.4] imply that for every $g \in \Gamma$ we have $\lim_{t \to 0} \left( \sup_{a \in (Z)_1} \| e_M^t \circ V_t(u_g ap u^*_g) \|_2 \right) = 0$. Since $u_g(Z)_1 u^*_g = (Z)_1$, we further have

$$\lim_{t \to 0} \left( \sup_{a \in (Z)_1} \| e_M^t \circ V_t(au_g pu^*_g) \|_2 \right) = 0, \text{ for all } g \in \Gamma.$$  \hspace{1cm} (5.2)
Next we prove the following

**Claim 5.2.** \( \lim_{t \to 0} \left( \sup_{a \in (Z)_1} \| e^{\tau}_M \circ V_t(a) \|_2 \right) = 0. \)

**Proof of Claim 5.2.** Fix \( \varepsilon > 0 \). As \( L(\Gamma) \) is a factor, by a standard convexity argument one can find \( g_1, \ldots, g_n \in \Gamma \), and \( \mu_1, \ldots, \mu_n > 0 \), with \( \sum_{i=1}^n \mu_i = 1 \), such that

\[
\| \sum_{i=1}^n \mu_i u_{g_i} p_{g_i}^* - \tau(p)1 \|_2 < \frac{\varepsilon \tau(p)}{2}.
\]

(5.3)

By (5.2) one can find \( t_\varepsilon > 0 \) such that for all \( 0 < |t| \leq t_\varepsilon \) we have

\[
\| e^{\tau}_M \circ V_t(a u_{g_i} p_{g_i}^*) \|_2 < \frac{\varepsilon \tau(p)}{2}, \text{ for all } i \in \{1, \ldots, n\}, a \in (Z)_1.
\]

(5.4)

Fix \( a \in (Z)_1 \) and \( 0 < |t| \leq t_\varepsilon \). Then triangle inequality, and equations (5.3), (5.4) show that

\[
\tau(p) \| e^{\tau}_M \circ V_t(a) \|_2 = \| e^{\tau}_M \circ V_t(a \tau(p)) \|_2 \leq \frac{\varepsilon \tau(p)}{2} + \| e^{\tau}_M \circ V_t(a(\sum_{i=1}^n \mu_i u_{g_i} p_{g_i}^*)) \|_2
\]

\[
\leq \frac{\varepsilon \tau(p)}{2} + \sum_{i=1}^n \mu_i \| e^{\tau}_M \circ V_t(a u_{g_i} p_{g_i}^*) \|_2 \leq \frac{\varepsilon \tau(p)}{2} + \frac{\varepsilon \tau(p)}{2} = \varepsilon \tau(p).
\]

Hence, \( \| e^{\tau}_M \circ V_t(a) \|_2 \leq \varepsilon \). As \( a \in (Z)_1 \), \( 0 < |t| \leq t_\varepsilon \) were arbitrary, the claim follows.

Notice that the prior claim and transversality property imply that \( V_t \to I \) as \( t \to 0 \) uniformly on \((Z)_1\). Thus if we where on case 1) Theorem 2.2 already leads to a contradiction. So assume we are in case 2).

Let \( C > 0 \). We denote by \( \mathfrak{B}_C = \{ g \in \Gamma : \| q(g) \| \leq C \} \), and by \( \mathcal{P}_{\mathfrak{B}_C} \) the orthogonal projection onto the Hilbert subspace of \( \mathfrak{B}_C \) inside \( \ell^2(\Gamma) \). Since \( V_t \to I \) as \( t \to 0 \) uniformly on \((Z)_1\), by item e) in Theorem 2.1, for every \( \varepsilon > 0 \), one can find \( C > 0 \) such that

\[
\| a - \mathcal{P}_{\mathfrak{B}_C}(a) \|_2 \leq \varepsilon, \text{ for all } a \in (Z)_1.
\]

(5.5)

Since \( q \) is proper, \( \mathfrak{B}_C \) are finite, and the prior inequality already contradicts that \( Z \) is diffuse.

Using Theorem 3.1 one can find \( \Sigma < \Gamma \) a normal subgroup such that \( N \vee (N' \cap L(\Sigma)) = L(\Sigma) \). As \( N \) has no amenable direct summand then Popa’s spectral gap argument implies that

\[
\lim_{t \to 0} \sup_{a \in (N' \cap L(\Sigma))_1} \| e^{\tau}_M \circ V_t(a) \|_2 = 0.
\]

By transversality, this implies that \( V_t \to I \), as \( t \to 0 \), uniformly on \((N' \cap L(\Sigma))_1\). Since \( N' \cap L(\Sigma) \) is \( \Gamma \)-invariant, Theorem 2.2 implies that \( N' \cap L(\Sigma) \) is not diffuse. Thus it admits a nontrivial atomic corner that is also invariant under the conjugation action by \( \Gamma \). Then using Corollary 3.3 we get that \( N' \cap L(\Gamma) = \mathbb{C}1 \) and hence \( N = L(\Sigma) \), as desired.

In view of the prior result and Theorem 3.1 we are left to analyze amenable \( \Gamma \)-invariant subalgebras and more specifically the case of the abelian ones. Before proceeding to this part of the proof we need to record the following lemma which is also of independent interest.
Lemma 5.3. Let $A \subseteq L(\Gamma) =: M$ be an abelian von Neumann subalgebra that is $\Gamma$-invariant. Let $g \in \Gamma$ such that $E_A(u_g) \neq 0$. If we denote by $\Omega = vC_\Gamma((g)) \leq \Gamma$ one can find a projection $0 \neq z_g \in A' \cap Z(L(\Omega))$ such that $Az_g \subseteq L(\Omega)$.

Proof. Fix $g \in \Gamma$ with $E_A(u_g) \neq 0$. We denote by $\alpha_g$ the automorphism of $M$ given by $\text{ad}(u_g)$. Note that $\alpha_g$ restricts to an automorphism of $A$. We thus have $\alpha_g(x)u_g = u_gx$ for all $x \in A$. Applying the conditional expectation $E_A$ we also get $\alpha_g(x)E_A(u_g) = E_A(u_g)x$ for all $x \in A$. These two relations combined imply that $E_A(u_g)u_g^* \in A' \cap M$. Let $e_g \in A' \cap M$ be such that $E_A(u_g) = e_gu_g$. As $E_A(u_g) \neq 0$, we have $e_g \neq 0$. Also notice that $E_A(e_g) = e_g e_g^* = A e_g A = 0$. Also notice that $E_A(e_g) = e_g e_g^* = E_A(u_g)E_A(u_g^*) \geq 0$. We denote by $f = \sup(E_A(e_g)) = \sup(e_g e_g^*) = \sup(e_g^*) \in A$ the support projection. Since $e_gu_g \in A \subseteq A' \cap M$ then $ae_g u_g = e_g u_g a$ for all $a \in A$. Therefore $\alpha_g = \alpha_g(a)e_g$ for all $a \in A$. This implies that $e e_g e_g^* = \alpha_g(a) e_g$ and hence $af = \alpha_g(a)f$ for all $a \in A$: in particular, $\alpha_g$ is identity on $Af$. This further entails that $Af \subseteq L((g))' \cap L(\Gamma) \subseteq L(vC_\Gamma((g)))$.

To this end denote by $\Omega := vC_\Gamma((g)) \leq \Gamma$. As $Af \subseteq L(\Omega)$, we get that $u_h Af u_h^* \subseteq L(h\Omega h^{-1}) = L(\Omega)$, for all $h \in \Omega$. As $A$ is $\Omega$-invariant, this further yields $Au_h f u_h^* \subseteq L(\Omega)$ for all $h \in \Omega$. Hence, letting $z_g = \vee_{h \in \Omega} u_h f u_h^* \in Z(L(\Omega))$ we further get $Az_g \subseteq L(\Omega)$.

With these preparations at hand we are ready to derive the proof of Theorem B under the assumptions of item 2).

Theorem 5.4. Let $\Gamma$ be a torsion free, exact group that admits an unbounded quasi-cocycle into a mixing, weakly-$\ell^2$ representation. Then $L(\Gamma)$ satisfies the ISR property.

Proof. Let $N \subseteq L(\Gamma)$ be a $\Gamma$-invariant von Neumann subalgebra. Let $A = Z(N)$ and note it is $\Gamma$-invariant as well. Next we show that $A = C1$. Suppose by contradiction $A \neq C1$. By Corollary 3.3 $A$ must be diffuse. Let $1 \neq \Sigma < \Gamma$ be the (infinite) smallest normal subgroup of $\Gamma$ such that $A \subseteq L(\Sigma)$. The existence of $\Sigma$ is guaranteed by Zorn’s lemma together with the fact that $A$ is normalized by $\Gamma$.

Observe that [T09, Lemma 3.3, Theorem 3.4] imply that $\Sigma$ is nonamenable. Next we briefly argue that $\Sigma$ is icc. Using the assumptions and [CKP15, Theorem 7.1] we must have that $FC(\Sigma)$ is finite. Moreover, since $\Sigma \vartriangleleft \Gamma$ it follows that $FC(\Sigma) \vartriangleleft \Gamma$ is also normal. As $\Gamma$ is icc we conclude that $FC(\Sigma) = 1$ and hence $\Sigma$ is icc. In conclusion $\Sigma < \Gamma$ is a normal icc subgroup. Therefore, if $q$ is an unbounded quasicocycle into a mixing weakly-$\ell^2$ representation of $\Gamma$ then its restriction $q|_{\Sigma}$ to $\Sigma$ is also an unbounded quasicocycle into a mixing, weakly-$\ell^2$ representation of $\Sigma$ (see [CKP15, Proposition 4.4 d])). Therefore, using the prior paragraph we can assume without any loss of generality that $A \not\subseteq L(\Sigma)$ for any proper subgroup $\Sigma < \Gamma$.

Since $A \neq C1$ then one can find $1 \neq g \in \Gamma$ such that if we denote by $\Omega = vC_\Gamma((g))$ then one can find $0 \neq z \in A' \cap Z(L(\Omega))$ such that $Az \subseteq L(\Omega)$. As $\Gamma$ is torsion free then $(g) < \Gamma$ is an infinite cyclic subgroup. As $\Omega = vC_\Gamma((g))$, from definitions there is an increasing sequence of finitely generated subgroups $\cdots \Omega_n \leq \Omega_{n+1} \leq \cdots \leq \Omega$ such that $\bigcup_n \Omega_n = \Omega$ and the centralizers $C_{(g)}(\Omega_n) \leq (g)$ has finite index for all $n$. Altogether these imply that $L(\Omega)$ has property Gamma of Murray and von Neumann. Now assume that $\Omega$ is non-amenable. Then using [CSU13, Theorem 3.1] we must have that the quasi-cocycle $q$ is bounded on $\Omega$. In particular, the corresponding deformation $e_M^1 \circ V_t \to 0$ on the unit ball of $L(\Omega)$. Thus $e_M^1 \circ V_t \to 0$ on the unit ball of $Af$. Then arguing exactly as in the proof of Theorem 3.3, we reach a contradiction. Hence, $\Omega$ must be amenable. As $Az \subseteq L(\Omega)$, and as $\Gamma$ normalizes $A$, we get $A \lesssim_{L(\Gamma)} L(\Omega)$. Using Proposition 2.11 this further implies that
Thus $N$ is a factor. If it is nonamenable the conclusion follows from Theorem 5.1. So assume $N$ is amenable. By Theorem 3.1 one can find a normal subgroup $\Sigma \vartriangleleft \Gamma$ such that $N \cap (N' \cap L(\Sigma)) = L(\Sigma)$. Since $\Gamma$ does not have amenable normal subgroup, it follows that the FC-center of $\Sigma$ is trivial and hence $\Sigma$ is icc. Moreover, we have that $\Sigma$ is a non-amenable. Recall that two commuting amenable subalgebras generate an amenable subalgebra in a $\Pi_1$ factor. Thus $N' \cap L(\Sigma)$ is nonamenable and by Theorem 5.1, $N' \cap L(\Sigma) = L(\Omega)$ for some icc non-amenable subgroup $\Omega < \Sigma$. Using factoriality we also have that $N = L(\Sigma) \cap (N' \cap L(\Sigma))' = L(\Sigma) \cap L(\Omega)' \subseteq L(vC_\Sigma(\Omega))$. However, since $vC_\Sigma(\Omega)$ is the FC-center of $\Omega$ and $\Omega$ is icc we get $vC_\Sigma(\Omega) \cap \Omega = 1$ and hence $P = C_1$. In conclusion, $N = L(vC_\Sigma(\Omega))$; in fact, it is easy to see we have $vC_\Sigma(\Omega) = C_\Sigma(\Omega)$. In particular, this implies the desired conclusion.

\[ \square \]

**Remark:** The argument in the last paragraph of the above proof in fact demonstrates that there are no $\Gamma$-invariant amenable subfactors inside $L(\Gamma)$. Indeed, as $\Gamma$ is $C^*$-simple, and $\Sigma \vartriangleleft \Gamma$, $\Sigma$ is $C^*$-simple. Hence so are $\Omega$ and $C_\Sigma(\Omega)$. As $N$ is amenable, this forces $C_\Sigma(\Omega)$ to be amenable, which in turn forces $C_\Sigma(\Omega)$ to be trivial, as the amenable radical of $\Sigma$ is trivial.

Now we derive the proof of Theorem B under the assumptions of item 1). Our approach follows closely the general strategy developed in the proof of [CP10, Lemma 3.1, Theorem 3.2] involving analysis of unbounded derivations (see also [Va10b, Theorem 4.1] for the version of this analysis in the context of Gaussian deformations) and we include all the details just for the reader’s convenience.

**Theorem 5.5.** Let $\Gamma$ be any icc group that admits an unbounded, non-proper, 1-cocycle into a mixing representation; in particular, $\Gamma$ can be any non-amenable group $\Gamma$ that has positive first $L^2$-Betti number and admits an infinite amenable subgroup (e.g. when $\Gamma$ is torsion free). Then $L(\Gamma)$ satisfies the ISR property.

**Proof.** Let $M = L(\Gamma)$. Let $A = Z(\Gamma)$ be its center and notice it is $\Gamma$-invariant as well. Assume, by contradiction that $A$ is a diffuse.

Since $q$ is nonproper, we can find $C > 0$ such that $\mathcal{B}_C := \{ g \in \Gamma : \|q(g)\| \leq C \}$ is infinite. Fix $(g_n)_n \subset \mathcal{B}_C$ an infinite set of group elements. Let $\omega$ be a nonprincipal ultrafilter, and let $E_M : M^\omega \to M$ denote the unique trace preserving conditional from the ultrapower $M^\omega$ onto $M$. We denote by $u^\omega = (u_{g_n})_n \in \mathcal{U}(M^\omega)$.

Just as in [CP10], we continue by splitting the proof into two cases that we analyze separately.

**Case I:** There exists $a \in \mathcal{U}(A)$ such that $E_M(u^\omega au^\omega^*) = \lim_{n \to \omega} u_{g_n} au^*_{g_n} = 0$.

Let $(c_n) = u_{g_n} au^*_{g_n} \in \mathcal{U}(A)$. Fix $b \in \mathcal{U}(A)$. Using that $c_n b = bc_n$ for all $n$ and then $V_t(x y) = V_t(x)V_t(y)$ for all $x, y$ basic estimates show that for all $n$ we have

\[
\|e_M^* \circ V_t(b)\|_2^2 = |\langle c_n e_M^* \circ V_t(b), c_n e_M^* \circ V_t(b) \rangle| \\
\leq |\langle c_n e_M^* \circ V_t(b), e_M^* \circ V_t(c_n b) \rangle| + \| c_n V_t(b) - V_t(c_n b) \|_2 \\
= |\langle c_n e_M^* \circ V_t(b)c_n^*, e_M^* \circ V_t(b) \rangle| + \| c_n V_t(b) - V_t(c_n b) \|_2 + \| V_t(b)c_n - V_t(bc_n) \|_2 \\
\leq |\langle c_n e_M^* \circ V_t(b)c_n^*, e_M^* \circ V_t(b) \rangle| + 2\| c_n - V_t(c_n) \|_2
\]

(5.6)
Since $g_n \in \mathfrak{B}_C$ for all $n$ and since $V_t \to \text{Id}$ pointwise then the same argument from the proof of part e) in Theorem 2.1 shows that $\sup_n \|V_t(c_n) - c_n\|_2 \to 0$. Moreover, since $\lim_{n \to \omega} \|E_M(c_n)\|_2 = 0$, mixingness of the representation (see for instance [CP10, Lemma 2.5] or the earlier results [PS12, Pet09b]) shows that $\lim_n \|c_n e^\perp_M \circ V_t(b) c_n^* e^\perp_M \circ V_t(b)\| = 0$. Using these and taking limit over $n$ in equation (5.6) and we get that $e^\perp_M \circ V_t \to 0$ uniformly on $(A)_1$. This however leads to a contradiction using the last part of the proof of Theorem 5.1.

**Case II:** There are $1 > C > 0$, $a \in \mathcal{U}(A)$ so that $\|E_M(u^*a^\omega a^\omega^*)\|_2 = \lim_{n \to \omega} u g_n \omega g_n^* \|_2 = C > 0$.

Let $c_n = u g_n \omega g_n^* \in \mathcal{U}(A) \omega$ and denote by $c^\omega_n = (c_n)_n = u^\omega \omega^\omega a^\omega \omega^\omega \in A^\omega$. As in the proof of [CP10, Lemma 3.1, Case II], let $u \in \mathcal{U}(A' \cap M)$ such that $E_M(c^\omega_n) = u E_M(c^\omega_n) \in A' \cap M$. Also, let

$$p \in \mathcal{P}(A' \cap M) \omega$$

be the spectral projection of $|E_M(c^\omega_n)|$ corresponding to $[0, 1 - \frac{C}{2}]$. Then we have that $\|E_M(c^\omega_n)\| \cdot (1 - p) \geq 1 - \frac{2}{C} \|p\|$. Arguing as in proof of [CP10, Lemma 3.1, Case II] this further implies $\|p\| \geq 1 - \frac{1}{2} \|1 - p\| \geq 1 - \frac{2}{C} E_M(c^\omega_n) \|_2 \geq 1 - \frac{2}{C} (1 - C) = \frac{C}{2}$. Hence, $\tau(p) = \|p\| \geq \frac{C}{2}$; in particular $p \neq 0$. Moreover, if $y := E_M(u^*c^\omega_n) = |E_M(c^\omega_n)| \in M$, one can also check that

$$\|yp\|_{\infty} \leq 1 - \frac{C}{2}. \quad (5.7)$$

Let $s_n := u^*c_n - y$ for all $n$.

Now fix $b \in \mathcal{U}(A)$. Using $c_n b = bc_n$ for all $n$ together with basic calculations as in the previous case and (5.7) we get

$$\|pe^\perp_M \circ V_t(b)\|^2 \leq \langle u^* c_n p e^\perp_M \circ V_t(b), u^* c_n p e^\perp_M \circ V_t(b) \rangle$$

$$\leq \langle s_n p e^\perp_M \circ V_t(b), u^* c_n p e^\perp_M \circ V_t(b) \rangle + \langle y p e^\perp_M \circ V_t(b), u^* c_n p e^\perp_M \circ V_t(b) \rangle$$

$$\leq \langle u s_n p e^\perp_M \circ V_t(b), p e^\perp_M \circ V_t(c_n b) \rangle + \|c_n V_t(b) - V_t(c_n b)\|_2 + \langle y p e^\perp_M \circ V_t(b), u^* c_n p e^\perp_M \circ V_t(b) \rangle$$

$$\leq \langle u s_n p e^\perp_M \circ V_t(b) c_n^*, p e^\perp_M \circ V_t(b) \rangle + \|c_n V_t(b) - V_t(c_n b)\|_2 + \|y p e^\perp_M \circ V_t(b), u^* c_n p e^\perp_M \circ V_t(b) \rangle$$

$$\leq \langle u s_n p e^\perp_M \circ V_t(b) c_n^*, p e^\perp_M \circ V_t(b) \rangle + 2 \|c_n - V_t(c_n)\|_2 + \|y p e^\perp_M \circ V_t(b), u^* c_n p e^\perp_M \circ V_t(b) \rangle$$

$$\leq \langle u s_n p e^\perp_M \circ V_t(b) c_n^*, p e^\perp_M \circ V_t(b) \rangle + 2 \|c_n - V_t(c_n)\|_2 + \|y p e^\perp_M \circ V_t(b), u^* c_n p e^\perp_M \circ V_t(b) \rangle$$

This further shows that

$$\|pe^\perp_M \circ V_t(b)\|^2 \leq \frac{2}{C} \langle u s_n p e^\perp_M \circ V_t(b) c_n^*, p e^\perp_M \circ V_t(b) \rangle + \langle y p e^\perp_M \circ V_t(b), s_n p e^\perp_M \circ V_t(b) \rangle + 2 \|c_n - V_t(c_n)\|_2$$

Once again we have $\sup_n \|V_t(c_n) - c_n\|_2 \to 0$, as $t \to 0$. Since $\lim_n \|E_M(u s_n p)\|_2 = \lim_n \|E_M(s_n p)\|_2 = 0$ the mixingness of the representation shows that $\lim_n \langle u s_n p e^\perp_M \circ V_t(b) c_n^*, p e^\perp_M \circ V_t(b) \rangle = 0$ and
\[ \lim_n \| \langle g p e_M^I \circ V_t(b), s_n p e_M^I \circ V_t(b) \rangle \| = 0. \] 
Thus taking the limit over \( n \) in equation (5.8) we get that
\[
\lim_{t \to 0} \left( \sup_{b \in \mathcal{U}(A)} \| p e_M^I \circ V_t(b) \|_2 \right) = 0.
\]
Since \( V_t(p) V_t(b) = V_t(pb) \) for all \( b \) and \( \| p - V_t(p) \| \to 0 \) as \( t \to 0 \) this further implies that
\[
\lim_{t \to 0} \left( \sup_{b \in \mathcal{U}(A)} \| e_M^I \circ V_t(pb) \|_2 \right) = 0.
\]
Proceeding as in the proof of Claim 5.2 this further implies that \( \lim_{t \to 0} \left( \sup_{a \in (N)_1} \| e_M^I \circ V_t(a) \|_2 \right) = 0. \) Finally, using the last part of the proof of Theorem 5.1, this leads to a contradiction.

Thus \( N \) is a factor and by Theorem 3.1 one can find a normal subgroup \( \Sigma \triangleleft \Gamma \) such that \( N \vee (N' \cap L(\Sigma)) = L(\Sigma) \). Since \( \Gamma \) does not have amenable normal subgroup, it follows that the FC-center of \( \Sigma \) is trivial and hence \( \Sigma \) is icc. Moreover, we have that \( \Sigma \) is a non-amenable and hence at least \( N \) or \( P = N' \cap L(\Sigma) \) is nonamenable factor. Assume \( P \) is nonamenable. Popa’s spectral gap argument implies that
\[
\lim_{t \to 0} \left( \sup_{a \in (N)_1} \| e_M^I \circ V_t(a) \|_2 \right) = 0.
\]
By transversality, this implies that \( V_t \to I \), as \( t \to 0 \), uniformly on \( (N)_1 \). Since \( N \) is \( \Gamma \)-invariant, Theorem 2.2 implies that \( N \) is not diffuse. Thus it admits a nontrivial atomic corner that is also invariant under the conjugation action by \( \Gamma \). Then using Corollary 3.3 we get that \( N = C_1 \), as desired. The other case also implies through a similar argument that \( N' \cap L(\Sigma) = C_1 \) and hence \( N = L(\Sigma) \), which concludes the proof.

5.1 Applications to invariant subalgebras of reduced group \( C^* \)-agebras

In this subsection we collect together several immediate consequences of our main results to the study of invariant subalgebras or reduced group \( C^* \)-algebras. To derive our results we first notice the following elementary result.

**Lemma 5.6.** Let \( \Sigma < \Gamma \) be a countable discrete groups. If \( A \subseteq C^*_r(\Gamma) \) is a \( C^* \)-subalgebra such that \( A \subseteq L(\Sigma) \), then \( A \subseteq C^*_r(\Sigma) \).

**Proof.** Fix \( a \in A \) and let \( a_n \in \mathbb{C}[\Gamma] \) be a sequence such that \( \| a - a_n \|_\infty \to 0 \). Since \( A \subseteq L(\Sigma) \), applying \( E_{L(\Sigma)} \) we obtain
\[
\| a - E_{L(\Sigma)}(a_n) \|_\infty = \| E_{L(\Sigma)}(a - a_n) \|_\infty \leq \| a - a_n \|_\infty \to 0.
\]
As \( a_n \in \mathbb{C}[\Gamma] \), clearly \( E_{L(\Sigma)}(a_n) \in C^*_r(\Sigma) \) and hence \( a \in C^*_r(\Sigma) \).

**Corollary 5.7.** Let \( \Gamma \) be any icc group that satisfies any of the hypotheses of Theorems A, B, or Theorem 5.4. Let \( A \subseteq C^*_r(\Gamma) \) be any \( \Gamma \)-invariant \( C^* \)-subalgebra. Then one can find a normal subgroup \( \Sigma < \Gamma \) such that \( A \subseteq C^*_r(\Sigma) \) and \( A'' = L(\Sigma) \).
Finally, since our claim. Since this holds for all $\varepsilon > 0$ and
from the proofs of the Theorems A, B, and 5.4 it follows that the center $Z(L(\Sigma)) = \mathbb{C}$. Therefore, $A' \cap C^*_r(\Sigma) \subseteq A' \cap L(\Sigma) = Z(L(\Sigma)) = \mathbb{C}$.

Next we claim that $\theta$ is an automorphism. Notice that whenever $\Sigma$ is torsion free this is satisfied if instead of $\Sigma$ amenable we require $\Sigma$ to have trivial amenable radical. More precisely, we conjecture the following

Conjecture 5.9. Let $\Gamma$ be an icc group and let $A \subseteq C^*_r(\Gamma)$ be any $\Gamma$-invariant $C^*$-subalgebra with expectation. Then one can find a normal subgroup $\Sigma < \Gamma$ such that $A = L(\Sigma)$.

Proof. By Corollary 5.7, there exists $\Sigma < \Gamma$ such that $A \subseteq C^*_r(\Sigma)$, and $A' = L(\Sigma)$. Moreover, $A' \cap C^*_r(\Sigma) \subseteq A' \cap L(\Sigma) = Z(L(\Sigma)) = \mathbb{C}$.

Let $E : C^*_r(\Gamma) \to A$ be a conditional expectation. (We are not assuming $E$ is trace preserving).

Notice that $E$ restricts to a conditional expectation $E : C^*_r(\Sigma) \to A$. Fix $g \in \Sigma$ and consider the automorphism $\theta(x) = \text{ad}(u_g)$ of $A$. As $\Sigma$ normalizes $A$, we get that $\theta(x)u_g = u_gx$ for all $x \in A$.

Hence, $u_g^*E(u_g)x = xu_g^*E(u_g)$ for all $x \in A$, which implies that $u_g^*E(u_g) \in A' \cap C^*_r(\Sigma) = \mathbb{C}$. Thus,

$$E(u_g) = c_gu_g, \quad \text{where } c_g \in \mathbb{C}. \quad (5.9)$$

In particular, this yields that whenever $c_g \neq 0$, then $u_g \in A$.

Let $S = \{ g \in \Sigma : E(u_g) \neq 0 \}$ and denote by $\Sigma_0 := \langle S \rangle < \Gamma$, the subgroup generated by $S$. By the previous paragraph, we get that $C^*_r(\Sigma_0) \subseteq A \subseteq C^*_r(\Sigma)$. In particular, we have $\Sigma_0 \leq \Sigma$.

Next we claim that $A = C^*_r(\Sigma_0)$. Towards this fix $\varepsilon > 0$ and $a \in A$. As $\mathbb{C}[\Sigma]$ is norm dense in $C^*_r(\Sigma)$, we can find a finite set $F \subset \Sigma$ such that $\|a - \sum_{g \in F} a_gu_g\| < \varepsilon$. As $E$ is norm-decreasing, this estimate together with (5.9) imply that $\varepsilon > \|a - \sum_{g \in F} a_gu_g\| \geq \|E(a - \sum_{g \in F} a_gu_g)\| = \|a - \sum_{g \in F \cap \Sigma_0} a_gc_gu_g\|$. Since this holds for all $\varepsilon > 0$ and $a \in A$ it follows that $\mathbb{C}[\Sigma_0]$ is norm dense in $A$, which establishes our claim.

Finally, since $A'' = L(\Sigma)$ and $C^*_r(\Sigma_0)'' = L(\Sigma_0)$, we get $\Sigma_0 = \Sigma$, thereby finishing the proof. \qed

5.2 Final remarks and open problems

We believe that the ISR property of $L(\Gamma)$ is a condition that is more intrinsic to a specific algebraic structure of $\Gamma$ which happens to be implicit in both cases, when $\Gamma$ is a lattice in a higher rank Lie group or when it satisfies a negative curvature property. Namely, we believe $L(\Gamma)$ satisfies the ISR property whenever $\Gamma$ has trivial amenable radical. More precisely, we conjecture the following

Conjecture 5.10. Let $\Gamma$ be an icc group and let $A \subseteq L(\Gamma)$ be a diffuse abelian von Neumann subalgebra such that $N_{L(\Gamma)}(A) \subseteq \Gamma$. Then one can find an amenable normal subgroup $\Sigma < \Gamma$ such that $A = L(\Sigma)$.

Establishing this conjecture in its full generality seems difficult at this time. However, we propose to investigate the following, seemingly easier intermediate conjecture which is already hinted by our previous results in the case torsion free groups $\Gamma$.

Conjecture 5.10. Let $\Gamma$ be an icc group and let $A \subseteq L(\Gamma)$ be a diffuse abelian von Neumann subalgebra such that $N_{L(\Gamma)}(A) \subseteq \Gamma$. Then one can find an amenable, $\gamma$-normal subgroup $\Sigma < \Gamma$ and a nonzero projection $z \in A' \cap Z(L(\Sigma))$ such that $Az \subseteq L(\Sigma)$.

Notice that when $\Gamma$ is torsion free this is satisfied if instead of $\Sigma$ amenable we require $\Sigma$ to have infinite FC-radical (and hence inner amenable, etc).
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References

[AB21] V. Alekseev, R. Brugger, A rigidity result for normalized subfactors, J. Operator Theory, 86 (2021) no. 1, 3–15.

[BR79] O. Bratteli, D.W. Robinson, Operator Algebras and Quantum Statistical Mechanics I, Springer-Verlag, 1979.

[AJ22] T. Amrutam, Y. Jiang, On invariant von Neumann subalgebras rigidity property, https://arxiv.org/abs/2205.10700

[BIP18] R. Boutonnet, A. Ioana, J. Peterson, Properly proximal groups and their von Neumann algebras, Ann. Sci. Éc. Norm. Super. Volume 54, fascicule 2 (2021), 445–482.

[Bow08] B. H. Bowditch, Tight geodesics in the curve complex, Invent. Math. 171 (2008), no. 2, 281–300.

[CD19] I. Chifan, S. Das, Rigidity results for von Neumann algebras arising from mixing extensions of profinite actions of groups on probability spaces, Math. Ann., 378 (2020), 907–950.

[CK15] I. Chifan, Y. Kida, $OE$ and $W^*$ superrigidity results for actions by surface braid groups, Proc. Lond. Math. Soc. 111 (2015), no. 6, 1431–1470.

[CIK13] I. Chifan, A. Ioana, Y. Kida, $W^*$-superrigidity for arbitrary actions of central quotients of braid groups Math. Ann., 361 (2015), 563–582.

[CKP15] I. Chifan, Y. Kida, S. Pant, Primeness results for von Neumann algebras associated with surface braid groups, Int. Math. Res. Not. 16 (2016), 4807–4848.

[CP10] I. Chifan, J. Peterson, Some unique group measure space decomposition results, Duke Math. J. 162 (2013), no. 11, 1923–1966.

[CIOS21a] I. Chifan, A. Ioana, D. Osin, B. Sun, Wreath-like product groups and rigidity of their von Neumann algebras, Preprint 2021, arXiv:2111.04708.

[CIOS21b] I. Chifan, A. Ioana, D. Osin, B. Sun, Uncountable families of $W^*$ and $C^*$-superrigid Kazhdan groups, Preprint 2021.

[CS11] I. Chifan, T. Sinclair, On the structural theory of $II_1$ factors of negatively curved groups, Ann. Sci. Éc. Norm. Supér. 46 (2013), 1–33.

[CSU11] I. Chifan, T. Sinclair, B. Udrea, On the structural theory of $II_1$ factors of negatively curved groups, II. Actions by product groups, Adv. Math. 245 (2013), 208–236.

[CSU13] I. Chifan, T. Sinclair, B. Udrea, Inner amenability for groups and central sequences in factors, Ergodic Theory Dynam. Systems 36 (2016), 1106–1029.

[CP13] D. Creutz, J. Peterson, Character rigidity for lattices and commensurators, Preprint arXiv: 1311.4513

[CP17] D. Creutz, J. Peterson, Stabilizers of ergodic actions of lattices and commensurators, Trans. Amer. Math. Soc. 369, no. 6, 4119–4166 (2017)

[DGO17] F. Dahmani, V. Guirardel, D. Osin, Hyperbolically embedded subgroups and rotating families in groups acting on hyperbolic spaces, Memoirs of the Amer. Math. Soc., 245 (2017), no. 1156.

[DP21] S. Das, J. Peterson, Poisson boundaries of $II_1$ factors, accepted to Compos. Math., arxiv:2009.11787

[DKR66] S. Doplicher, D. Kastler, D. Robinson, Covariance algebras in field theory and statistical mechanics, Comm. Math. Phys. 3 (1966), 1–28.

[Ge96] L. Ge, On maximal injective subalgebras of factors, Adv. Math., 118 (1996), 34–70.

[GK96] L. Ge, R. Kadison, On tensor products of von Neumann algebras, Invent. Math., 123 (1996), 453–466.
C. Houdayer, S. Popa, S. Vaes, A class of groups for which every action is W∗-superrigid, Groups Geom. Dyn. 7 (2013), 577–590.

M. Hull, D. Osin, Induced quasicocycles on groups with hyperbolically embedded subgroups, Algebr. Geom. Topol. 13 (2013), no. 5, 2635–2665.

A. Ioana, J. Peterson, S. Popa, Amalgamated free products of weakly rigid factors and calculation of their symmetry groups. Acta Math. 200 (2008), 85–153.

A. Ioana, Rigidity for von Neumann algebras, Proceedings of the International Congress of Mathematicians-Rio de Janeiro 2018. Vol. III. Invited lectures, 1639–1672, World Sci. Publ., Hackensack, NJ, 2018.

P. Jorgensen, X. Quan, Covariance group C∗-algebras and Galois correspondence, Int. J. Math., Vol. 2, No. 6 (1991), 673–699.

M. Kalantar, N. Panagopoulos, A noncommutative normal subgroup theorem for lattices of semisimple Lie groups, https://arxiv.org/abs/2108.02928v1

F.J. Murray, J. von Neumann, On rings of operators, Ann. of Math. 37 (1936), 116–229.

D. Osin, Acylindrically hyperbolic groups, Trans. Amer. Math. Soc. 368 (2016), no. 2, 851-888.

D. Osin, Peripheral fillings of relatively hyperbolic groups, Invent. Math. 167 (2007), no. 2, 295–326.

D. Osin, L2-Betti numbers and non-unitarizable groups without free subgroups, Int. Math. Res. Not. 2009, no. 22, 4220–4231.

N. Ozawa, S. Popa, On a class of II1 factors with at most one Cartan subalgebra, Ann. of Math. 172 (2010), 713–749.

N. Ozawa, S. Popa, On a class of II1 factors with at most one Cartan subalgebra II, Amer. J. Math. 132 (2010), no. 3, 841–866.

J. Peterson, Character rigidity for lattices in higher-rank groups, preprint, www.math.vanderbilt.edu/peters10/ rigidity.pdf (2014).

J. Peterson, L2-rigidity in von Neumann algebras, Invent. Math. 175 (2009), no. 2, 417–433.

J. Peterson, Examples of group actions which are virtually W∗-superrigid, Preprint 2009, arXiv:1002.1745.

J. Peterson, T. Sinclair, On cocycle superrigidity for Gaussian actions, Ergodic Theory Dynam. Systems 32 (2012), no. 1, 249–272.

J. Peterson, A. Thom, Group cocycles and the ring of affiliated operators, Invent. Math. 185 (2011), no. 3, 561–592.

S. Popa, On a class of type II1 factors with Betti numbers invariants, Ann. of Math. 163 (2006), 809-899.

S. Popa, Strong rigidity of II1 factors arising from malleable actions of w-rigid groups I, Invent. Math. 165 (2006), 369–408.

S. Popa, Strong rigidity of II1 factors arising from malleable actions of w-rigid groups II, Invent. Math. 165 (2006), 409–451.

S. Popa, Deformation and rigidity for group actions and von Neumann algebras, International Congress of Mathematicians. Vol. I, 445–477, Eur. Math. Soc., Zürich, 2007.

S. Popa, On the superrigidity of malleable actions with spectral gap, J. Amer. Math. Soc. 21 (2008), 981–1000.

S. Popa, S. Vaes, Strong rigidity of generalized Bernoulli actions and computations of their symmetry groups, Adv. Math., 217 (2008), 833–872.

S. Popa, S. Vaes, Unique Cartan decomposition for II1 factors arising from arbitrary actions of free groups, Acta Math. 212 (2014), 141–198.

S. Popa, S. Vaes, Unique Cartan decomposition for II1 factors arising from arbitrary actions of hyperbolic groups, J. Reine Angew. Math. 694 (2014), 215–239.

Z. Sela, Acylindrical accessibility for groups, Invent. Math. 129 (1997), no. 3, 527–565.

T. Sinclair, Strong solidity of group factors from lattices in SO(n,1) and SU(n,1), J. Funct. Anal. 260 (2011), no. 11, 3209–3221.
[Tak67] M. Takesaki, *Covariant representations of C*-algebras and their locally compact automorphism groups*, Acta Math. **119** (1967), 272–303.

[TT71] M. Takesaki and N. Tatsuuma, *Duality and subgroups*, Ann. of Math. **93** (1971), 344–364.

[T09] A. Thom, *Low degree bounded cohomology invariants and negatively curved groups*, Groups, Geom. Dyn. **3** (2009), 343–358.

[Va10a] S. Vaes, *Rigidity for von Neumann algebras and their invariants*, Proceedings of the International Congress of Mathematicians (Hyderabad, India, 2010) Vol III, Hindustan Book Agency, 2010, 1624–1650.

[Va10b] S. Vaes, *One-cohomology and the uniqueness of the group measure space decomposition of a II₁ factor*, Math. Ann. **355** (2013), 661–696.