Serre Relations in the Superintegrable Model

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Abstract. We derive the Serre relations for the generators of the quantum loop algebra \( L(\mathfrak{sl}_2) \) of the superintegrable \( \tau_2 \) model in \( Q \neq 0 \) sectors, thus proving a fundamental conjecture in an earlier paper on the superintegrable chiral Potts model.

PACS numbers: 02.20.Uw, 05.50.+q, 75.10.Hk, 75.10.Jm, 75.10.Pq

AMS classification scheme numbers: 05A30, 20G42, 81R50, 82B20, 82B23

1. Introduction

In 1985 von Gehlen and Rittenberg [1] introduced a special hermitian quantum spin chain with \( N \) states per site, having Ising-like features and generalizing a 3-state model of Howes, Kadanoff and den Nijs [2]. This model was later called superintegrable [3], as the two terms in the hamiltonian generate an Onsager algebra [4] and the Boltzmann weights of the corresponding classical two-dimensional chiral Potts model satisfy star-triangle (Yang–Baxter) relations [5] [6].

A quantum-group theoretical interpretation of the model was first given by Bazhanov and Stroganov [7] when they introduced the \( \tau_2 \) model connecting the integrable chiral Potts model with the six-vertex model. More precisely, a square of four chiral Potts Boltzmann weights [6] is the intertwiner of cyclic representations of the affine quantum group \( U_q(\widehat{\mathfrak{sl}}_2) \) [8], whereas the six-vertex \( R \)-matrix intertwines spin-\( \frac{1}{2} \) highest-weight representations and the \( \tau_2 \) model weights intertwine spin-\( \frac{1}{2} \) and cyclic representations. All this is expressed in a sequence of Yang–Baxter equations involving the intertwiners [7] [9] [10].

Not only do superintegrable chiral Potts models have Ising-like spectra [11] [12], for a periodic chain with spin-shift quantum number \( Q = 0 \) and chain length \( L \) a multiple of \( N \), it has been shown [13] [14] that the eigenspace supports a quantum loop algebra \( L(\mathfrak{sl}_2) \). Furthermore, this loop algebra can be decomposed into \( r \) simple \( \mathfrak{sl}_2 \) algebras, with \( r = m_0 = (N - 1)L/N \) for the ground-state sector [15] [16] [17].

† All equations in [16] are denoted here by prefacing IV to its equation numbers.
We have also worked out the ground-state sector for \( Q \neq 0 \) cases, under the assumption that certain Serre relations hold \([17]\). Even though we have shown that these relations hold when operated on some special vectors (see Appendix B of \([17]\)) and we have also tested them extensively by computer for small systems, a proof has been lacking up to now. In this paper, we shall present the missing proof.

In section 1, we relate the operators used in our paper \([17]\) to generators of \( U_q(\hat{\mathfrak{sl}}_2) \) and operators of the quantum loop algebra \( L(\mathfrak{sl}_2) \) \([13, 14]\). We then use the higher-order quantum Serre relations of Lusztig \([19]\) to derive certain relations in section 2. Next we rewrite these relations in terms of our operators in sections 3 and 4. We can then in section 5 use these relations to prove the Serre relations for the generators used in \([17]\). We end with a brief conclusion in section 6.

2. Relationship between the generators

In our paper \([17]\), the generators \( e_j \) and \( f_j \) are defined in (IV.25), with \( Z \) and \( X \) given in (IV.20). These are different from the usual \( e'_j \) and \( f'_j \) of the quantum group \( U_q(\hat{\mathfrak{sl}}_2) \) \([8]\), but are related by \([13]\)

\[
\begin{align*}
  e'_j &= -q e_j Z_j^{-1/2}, & f'_j &= q Z_j^{-1/2} f_j, & t'_j &= q^{-1} Z_j^{-1}, & \omega &= q^2 = e^{2\pi i / N}.
\end{align*}
\]

Substituting these into the operators \( B_\pm \) and \( C_\pm \) defined on page 368 of \([13]\), and comparing with \( B_1, B_L, C_0 \) and \( C_{L-1} \) defined in (IV.24) and (IV.55), we find

\[
\begin{align*}
  C_+ &= -q^{-L+2} \frac{C_{L-1}}{2} A_L^{-\frac{1}{2}}, & B_+ &= q^{-L+2} A_L^{-\frac{1}{2}} B_1, & A_L &= \prod_{i=1}^{L} Z_i, \\
  C_- &= -q C_{L-1} A_L^{-\frac{1}{2}}, & B_- &= q A_L^{-\frac{1}{2}} B_L.
\end{align*}
\]

From now on, we drop the bars from the \( B \) and \( C \) symbols taken from \([17]\). Defining

\[
\begin{align*}
  C^{(n)}_\pm &= \frac{C_\pm^n}{[n]_q!}, & B^{(n)}_\pm &= \frac{B_\pm^n}{[n]_q!}, & [n]_q! &= \prod_{i=1}^{n} \frac{q^j - q^{-i}}{q - q^{-1}}, \\
  C^{(n)}_\ell &= \frac{C^{(n)}_\ell}{[n]!}, & B^{(n)}_\ell &= \frac{B^{(n)}_\ell}{[n]!}, & [n]! &= \prod_{i=1}^{n} \frac{1 - \omega^i}{1 - \omega}, & \ell &= 0, 1, L - 1, L,
\end{align*}
\]

we find from \([2]\) and \([3]\) the relations

\[
\begin{align*}
  [n]_q! &= q^{-\frac{1}{2}n(n-1)}[n]!, & C^{(n)}_- &= (-1)^n A_L^{-\frac{1}{2}n} C^{(n)}_{L-1}, & B^{(n)}_- &= B^{(n)}_L A_L^{-\frac{1}{2}n}, \\
  C^{(n)}_+ &= (-1)^n q^{n(1-L)} A_L^{-\frac{1}{2}n} C^{(n)}_0, & B^{(n)}_+ &= q^{n(1-L)} B^{(n)}_1 A_L^{-\frac{1}{2}n}.
\end{align*}
\]

3. Higher-order Serre relations

We follow the conventions of Nishino and Deguchi \([13]\) letting

\[
\begin{align*}
  E_0 &= B_+, & E_1 &= C_+, & F_0 &= C_- & F_1 &= B_-,
\end{align*}
\]
so that we may adapt Chapter 7 of Lusztig [19] and define the following function for the cyclic case with $q^{2N} = 1$,

$$f_{i,j,n,m} = f_{n,m} = \sum_{r+s=m}^m (-1)^r q^{r(2n-m+1)} \theta_i^{(r)} \theta_j^{(n)} \theta_i^{(s)}, \quad i, j = 0, 1, \quad j \neq i,$$

(6)

where we may choose $\theta_i = E_i$ or $\theta_i = F_i$. It is shown by Lusztig in Proposition 7.15.(b) [19] that if $m > 2n$, then $f_{n,m} = 0$. For $n = 1$ and $m = 3$, these are the usual quantum Serre relations given in (3.23) through (3.26) of [18].

We follow the steps of Lusztig’s proof. Let us first consider the case $m - 2n \geq N$, so that $f_{n,m-\ell} = 0$ for $\ell \leq N - 1 \leq m - 2n - 1$. Consequently we have

$$g = \sum_{\ell=0}^{N-1} (-1)^\ell q^{(1-m)} f_{n,m-\ell} \theta_i^{(\ell)} = 0.$$ 

(7)

Using (6), we find

$$g = \sum_{\ell=0}^{N-1} \sum_{r+s=m-\ell} (-1)^{\ell+r} q^{(1-m)+r(2n-m+\ell+1)} \theta_i^{(r)} \theta_j^{(n)} \theta_i^{(s)} \theta_i^{(\ell)}$$

$$= \sum_{s=0}^{m-1} c_s \theta_i^{(m-s)} \theta_j^{(n)} \theta_i^{(s)} = 0, \quad r = m - s,$$

(8)

where

$$c_s = \sum_{\ell=0}^{N-1} (-1)^{\ell+m-s} q^{(1-s)+(m-s)(2n-m+1)} \left[ s \atop \ell \right] q, \quad \left[ s \atop \ell \right] q = \frac{[s]!}{[\ell]! [s-\ell]!},$$

(9)

These are exactly the same as in Lusztig. But from now on, we will use the cyclic property as in [18]. We set $s = kN + p$ for $0 \leq k \leq \lfloor m/N \rfloor$, with $0 \leq p \leq N - 1$ if $0 \leq k \leq \lfloor m/N \rfloor - 1$, and $0 \leq p \leq m - N \lfloor m/N \rfloor$ if $k = \lfloor m/N \rfloor$. Using (3.55) of [18], namely

$$\left[ s \atop \ell \right] q = \left[ kN + p \atop \ell \right] q = q^{kN\ell} \left[ p \atop \ell \right] q$$

(10)

we rewrite $c_s$ in (9) as

$$c_{kN+p} = (-1)^{m-kN-p} q^{(m-kN-p)(2n-m+1)} \sum_{\ell=0}^p (-1)^\ell q^{(1-p)} \left[ p \atop \ell \right] q$$

$$= (-1)^{m-kN-p} q^{(m-kN-p)(2n-m+1)} \delta_{p,0},$$

(11)

where 1.3.4 of Lusztig [19], or (3.58) of [18] is used. Substituting this equation into (8), we find

$$(-1)^m q^{m(2n-m+1)} \left[ \theta_i^{(m)} \theta_j^{(n)} \right] + \sum_{k=1}^{\lfloor m/N \rfloor} (-1)^k(N+m-1) \theta_i^{(m-kN)} \theta_j^{(n)} \theta_i^{(kN)} = 0.$$ 

(12)

 Particularly, letting $\theta_i = B_\pm$ and $\theta_j = C_\pm$ and $n = Q$, $m = 2N + Q$, so that $m - 2n = 2N - Q > N$, we find the identity

$$B_\pm^{(2N+Q)} C_\pm^{(Q)} + (-1)^{(N+Q-1)} B_\pm^{(N+Q)} C_\pm^{(Q)} B_\pm^{(N)} + B_\pm^{(Q)} C_\pm^{(Q)} B_\pm^{(2N)} = 0.$$ 

(13)

¶ Use translation $S^- = B^-$, $T^- = B^+$, $S^+ = C^+$, $T^+ = C^-$. 



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Interchanging \(i\) and \(j\), we have
\[
C_{\pm}^{(2N+Q)} B_{\pm}^{(Q)} + (-1)^{(N+Q-1)} C_{\pm}^{(N+Q)} B_{\pm}^{(Q)} C_{\pm}^{(N)} + C_{\pm}^{(Q)} B_{\pm}^{(Q)} C_{\pm}^{(2N)} = 0. \tag{14}
\]

Next we consider the case that \(0 \leq m - 2n \leq N - 1\). Let
\[
g = \sum_{\ell=0}^{m-2n-1} (-1)^{\ell} q^{(1-m)} f_{m-\ell} = \sum_{s=0}^{m} c_s \theta_i^{(m-s)} \theta_j^{(n)} \theta_i^{(s)} = 0, \tag{15}
\]
where
\[
c_s = (-1)^{m-s} q^{(m-s)(2n-m+1)} \sum_{\ell=0}^{m-2n-1} (-1)^{\ell} q^{(1-s)} \left[\frac{s}{\ell}\right]. \tag{16}
\]
Now if we again write \(s = kN + p\), then for \(0 \leq p \leq m - 2n - 1\), \(c_s\) is again summable and is given by (11). However, for \(m - 2n \leq p \leq N - 1\), the sum in (16) is not summable. Nevertheless, since
\[
\theta_i^{(kN+p)} \theta_i^{(N-m+2n)} = \left[\frac{kN + N + p - m + 2n}{N - m + 2n}\right] \theta_i^{(kN+N+p-m+2n)}, \tag{17}
\]
and for \(m - 2n \leq p \leq N - 1\), we have
\[
\left[\frac{kN + N + p - m + 2n}{N - m + 2n}\right] = q^{(N-m+2n)(N-1)} \left[\frac{p - m + 2n}{N - m + 2n}\right] = 0 \tag{18}
\]
we find
\[
\sum_{p=m-2n}^{N-1} c_{kN+p} \theta_i^{(m-kN-p)} \theta_j^{(n)} \theta_i^{(kN+p)} \theta_i^{(N-m+2n)} = 0. \tag{19}
\]
Thus by multiplying \(\theta_i^{(N-m+2n)}\) to \(g\) on the right, we may get rid of the terms involving these unsummable \(c_s\), and find
\[
0 = g \theta_i^{(N-m+2n)} = \sum_{k=0}^{\lfloor m/N \rfloor} c_{kN} \theta_i^{(m-kN)} \theta_j^{(n)} \theta_i^{(kN)} \theta_i^{(N-m+2n)}
\]
\[
= \sum_{k=0}^{\lfloor m/N \rfloor} c_{kN} \theta_i^{(m-kN)} \theta_j^{(n)} \theta_i^{(kN+N-m+2n)} \left[\frac{kN + N - m + 2n}{N - m + 2n}\right] q
\]
\[
= (-1)^{m} q^{m(2n-m+1)} \left\{ \sum_{k=0}^{\lfloor m/N \rfloor} (-1)^k \theta_i^{(m-kN)} \theta_j^{(n)} \theta_i^{(kN+N-m+2n)} \right\}. \tag{20}
\]
If we let \(n = Q\) and \(m = N + Q\), so that \(m - 2n = N - Q > 0\), then (20) becomes
\[
B_{\pm}^{(N+Q)} C_{\pm}^{(Q)} B_{\pm}^{(Q)} = B_{\pm}^{(Q)} C_{\pm}^{(Q)} B_{\pm}^{(N+Q)}, \quad C_{\pm}^{(N+Q)} B_{\pm}^{(Q)} C_{\pm}^{(Q)} = C_{\pm}^{(Q)} B_{\pm}^{(Q)} C_{\pm}^{(N+Q)}. \tag{21}
\]
Now let \(n = N + Q\) and \(m = 3N + Q\). Again we have \(m - 2n = N - Q > 0\), and for such values (20) becomes
\[
B_{\pm}^{(3N+Q)} C_{\pm}^{(N+Q)} B_{\pm}^{(Q)} = B_{\pm}^{(N+Q)} C_{\pm}^{(Q)} B_{\pm}^{(3N+Q)} + B_{\pm}^{(3N+Q)} C_{\pm}^{(Q)} B_{\pm}^{(2N+Q)} - B_{\pm}^{(Q)} C_{\pm}^{(N+Q)} B_{\pm}^{(Q+3N)} = 0, \tag{22}
\]
or
\[
C_{\pm}^{(3N+Q)} B_{\pm}^{(N+Q)} C_{\pm}^{(Q)} = C_{\pm}^{(2N+Q)} B_{\pm}^{(Q)} C_{\pm}^{(N+Q)} + C_{\pm}^{(3N+Q)} B_{\pm}^{(Q)} C_{\pm}^{(2N+Q)} - C_{\pm}^{(Q)} B_{\pm}^{(N+Q)} C_{\pm}^{(Q+3N)} = 0. \tag{23}
\]
If \(Q = 0\), these are the Serre relations given by (3.31) through (3.34) in [18].
4. Alternative form

Substituting (4) into (13) and (14), and using the commutation relations

\[ A_L^{-\frac{1}{2}} C_\ell = q C_\ell A_L^{-\frac{1}{2}}, \quad B_\ell A_L^{-\frac{1}{2}} = q A_L^{-\frac{1}{2}} B_n, \quad \ell = 0, L - 1, \quad n = 1, L, \]  

we find

\[ B_1^{(2N+Q)} C_0^{(Q)} B_1^{(N+Q)} - B_1^{(N+Q)} C_0^{(Q)} B_1^{(N)} + B_1^{(Q)} C_0^{(Q)} B_1^{(2N)} = 0, \]  
\[ C_0^{(2N+Q)} B_1^{(Q)} - C_0^{(N+Q)} B_1^{(Q)} C_0^{(N)} + C_0^{(Q)} B_1^{(Q)} C_0^{(2N)} = 0. \]

Similarly, substituting (4) into (21) and using (24), we obtain

\[ B_1^{(N+Q)} C_0^{(Q)} B_1^{(Q)} = B_1^{(Q)} C_0^{(Q)} B_1^{(N+Q)}, \quad C_0^{(N+Q)} B_1^{(Q)} C_0^{(Q)} = C_0^{(Q)} B_1^{(Q)} C_0^{(N+Q)}. \]

Finally, from (22), (23) together with (4) and (24), we get

\[ B_1^{(3N+Q)} C_0^{(N+Q)} B_1^{(Q)} - B_1^{(2N+Q)} C_0^{(N+Q)} B_1^{(Q)} + B_1^{(Q)} C_0^{(N+Q)} B_1^{(2N)} - B_1^{(Q)} C_0^{(N+Q)} B_1^{(Q+3N)} = 0. \]
\[ C_0^{(3N+Q)} B_1^{(N+Q)} C_0^{(Q)} - C_0^{(2N+Q)} B_1^{(N+Q)} C_0^{(Q)} + C_0^{(Q)} B_1^{(Q+3N)} C_0^{(Q+3N)} = 0. \]

Similar equations hold if we replace \( B_1 \) by \( B_L \) and \( C_0 \) by \( C_{L-1} \).

5. Serre relations for the generators of the loop algebra

We will now prove the Serre relations (IV.90) for the generators given in (IV.88), i.e.,

\[ x_{0,Q}^- = C_0^{(Q)} B_1^{(N+Q)}, \quad x_{0,Q}^+ = C_0^{(N+Q)} B_1^{(Q)} \]

where we have dropped the common constant factors for convenience. We use first the equation on the right and then the one on the left in (21) to find

\[ x_{0,Q}^+ x_{1,Q}^- = \left( C_0^{(N+Q)} B_1^{(Q)} C_0^{(Q)} \right) B_1^{(N+Q)} = C_0^{(Q)} B_1^{(Q)} C_0^{(Q)} B_1^{(N+Q)} = C_0^{(Q)} B_1^{(Q)} B_1^{(N+Q)} = 0. \]

This means

\[ [C_0^{(Q)} B_1^{(Q)}, C_0^{(N+Q)} B_1^{(N+Q)}] = 0. \]

It is easy to verify that

\[ B_1^{(kN+Q)} B_1^{(jN)} = \left[ \frac{kN + jN + Q}{kN + Q} \right] B_1^{(jN+kN+Q)} = \left( \frac{k + j}{k} \right) B_1^{(jN+kN+Q)}. \]

We again use (21) and (33) to find

\[ (x_{1,Q}^-)^2 = C_0^{(Q)} B_1^{(Q)} C_0^{(Q)} B_1^{(Q)} B_1^{(N)} = 2C_0^{(Q)} B_1^{(Q)} C_0^{(Q)} B_1^{(2N+Q)} \]
\[ = C_0^{(Q)} B_1^{(N)} B_1^{(Q)} C_0^{(Q)} B_1^{(Q)} B_1^{(N+Q)} = 2C_0^{(Q)} B_1^{(2N+Q)} C_0^{(Q)} B_1^{(Q)}. \]

As a consequence, we obtain another identity,

\[ C_0^{(Q)} B_1^{(Q)} C_0^{(Q)} B_1^{(2N+Q)} = C_0^{(Q)} B_1^{(2N+Q)} C_0^{(Q)} B_1^{(Q)}. \]
Multiplying (31) and (34) we obtain
\[
x_{0,Q}^+(x_{1,Q}^-)^3 = 2C_0(1)B_1(1)C_0(N+Q)B_1(N+Q)C_1(1)B_0(1)B_1(1)B_1(2N).
\] (37)

Using (32) repeatedly to move operators with higher exponents to the right, and then using (33), we find
\[
x_{0,Q}^+(x_{1,Q}^-)^3 = 6C_0(1)B_1(1)C_0(1)B_1(1)C_0(N+Q)B_1(3N+Q).
\] (38)

Similarly, by repeatedly using (27), and then (32), we also find
\[
(x_{1,Q}^-)x_{0,Q}^+(x_{1,Q}^-)^2 = C_0(1)B_0(N+Q)C_0(N+Q)B_1(N+Q)C_0(1)B_1(N+Q)\]
\[
+ 2C_0(1)B_1(1)C_0(N+Q)B_1(N+Q)C_0(1)B_1(N+Q)B_1(2N+Q).
\] (39)

From (31), (34) and (36), we obtain
\[
(x_{1,Q}^-)^2(x_{0,Q}^+x_{1,Q}^-) = 2C_0(1)B_1(1)[C_0(1)B_1(N+Q)C_0(N+Q)B_1(N+Q)]
\]
\[
+ 2C_0(1)B_1(1)C_0(N+Q)B_1(2N+Q)C_0(N+Q)B_1(N+Q).
\] (40)

Now we use (36) and (33) to get
\[
(x_{1,Q}^-)^3x_{0,Q}^+ = 2C_0(1)B_1(1)[C_0(1)B_1(N+Q)C_0(N+Q)B_1(N+Q)]
\]
\[
+ 6C_0(1)B_1(1)C_0(N+Q)B_1(2N+Q)C_0(N+Q)B_1(N+Q)B_1(2N+Q).
\] (41)

Finally, combining all these, we find
\[
[[[x_{0,Q}^+, x_{1,Q}^-], x_{1,Q}^-], x_{1,Q}^-] = x_{0,Q}^+(x_{1,Q}^-)^3 - 3(x_{1,Q}^-)x_{0,Q}^+(x_{1,Q}^-)^2
\]
\[
+ 3(x_{1,Q}^-)^2(x_{0,Q}^+x_{1,Q}^-) - (x_{1,Q}^-)^3x_{0,Q}^+
\]
\[
= 6C_0(1)B_1(1)C_0(N+Q)B_1(2N+Q)C_0(N+Q)B_1(N+Q)
\]
\[
- B_1(N+Q)C_0(N+Q)B_1(2N+Q)B_1(N+Q) + B_1(3N+Q)C_0(N+Q)B_1(N+Q) = 0,
\] (42)

as seen from (28).

It is straightforward to show that
\[
x_{1,Q}^-(x_{0,Q}^+)^3 = 6C_0(1)B_1(N+Q)C_0(3N+Q)B_1(1)C_0(1)B_1(1)B_1(1)B_1(1).
\] (43)

\[
x_{0,Q}^+(x_{1,Q}^-)^3(x_{0,Q}^+)^2 = 2C_0(N+Q)B_1(N+Q)C_0(2N+Q)B_1(N+Q)\]
\[
+ 2C_0(N+Q)B_1(N+Q)C_0(N+Q)B_1(1)\] (44)

\[
(x_{0,Q}^+)^2x_{1,Q}^-x_{0,Q}^+ = 2C_0(2N+Q)B_1(N+Q)C_0(N+Q)B_1(N+Q)\]
\[
+ 2C_0(2N+Q)B_1(N+Q)C_0(N+Q)B_1(1)\] (45)

\[
(x_{0,Q}^+)^3x_{1,Q}^- = 6C_0(3N+Q)B_1(N+Q)C_0(N+Q)B_1(N+Q)\]
\[
+ 6C_0(3N+Q)B_1(N+Q)C_0(N+Q)B_1(1)\] (46)

Consequently, we use (29) to show that
\[
[[[x_{1,Q}^-, x_{0,Q}^+], x_{0,Q}^+], x_{0,Q}^-] = 0.
\] (47)

We have also defined [20] the generators
\[
\tilde{x}_{0,Q} = B_L(N+Q)C_{L-1}, \quad \tilde{x}_{1,Q}^+ = B_L(N+Q)C_{L-1}.
\] (48)

Since (27), (28) and (20) also hold if we replace $B_1$ by $B_L$ and $C_0$ by $C_{L-1}$, we can follow the same steps to prove
\[
[[[\tilde{x}_{0,Q}, \tilde{x}_{1,Q}^+], \tilde{x}_{1,Q}^-], \tilde{x}_{1,Q}^-] = 0, \quad [[[\tilde{x}_{1,Q}^-], \tilde{x}_{0,Q}^+], \tilde{x}_{0,Q}^-] = 0.
\] (49)
6. Conclusion

The two Serre relations (IV.90) conjectured in [17] have now been proved, see (42) and (47). Two other Serre relations (49) applicable to the quantum subalgebra related to the state $|\bar{\Omega}\rangle$ [17], rather than $|\Omega\rangle$, have also been derived. The Serre relations (32) in [15] are included as the special case $Q = 0$, for which the two subalgebras combine to one quantum loop algebra.

Acknowledgments

This work was supported in part by the National Science Foundation under grant No. PHY-07-58139.

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