Quadratic Mass Corrections of Order $\mathcal{O}(\alpha_s^3 m_q^2/s)$ to the Decay Rate of $Z$- and $W$- Bosons

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Abstract

We analytically compute quadratic mass corrections of order $\mathcal{O}(\alpha_s^3 m_q^2/s)$ to the absorptive part of the (non-diagonal) correlator of two axial vector currents. This allows us to find the correction of order $\mathcal{O}(\alpha_s^3 m_q^2/M_W^2)$ to $\Gamma(W \to \text{hadrons})$ as well as similar corrections to $\Gamma(Z \to \text{hadrons})$.

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1 Introduction

Precision measurements of the total and as well as partial Z decay rates have provided one of the most important and, from the theoretical viewpoint, clean determination of the strong coupling constant $\alpha_s$ with a present value of $\alpha_s = 0.1202 \pm 0.0033$ \cite{1}. Theoretical ingredients were the knowledge of QCD corrections to order $\alpha_s^3$ in the limit of massless quarks plus charm and bottom quarks effects (see, e.g. \cite{2} and references therein). These mass corrections which indeed are relevant at the present level of accuracy have been calculated up to the order $\alpha_s^3 m_q^2/s$ for the vector and $\alpha_s^2 m_q^2/s$ for the axial current induced decay rate. In this short note the prediction is extended to include $\alpha_s^3 m_q^2/s$ terms for the (non-singlet part) of the axial current induced rate. At the same time results are obtained for the non-diagonal current correlator with two different masses – a case of relevance e.g. for the W decay rate into charmed and bottom quarks. The same formulae can also be applied to a subclass of corrections which enter single top production in the Drell-Yan like reaction $q\bar{q} \rightarrow t\bar{b}$ far above threshold.

The calculation is based on an approach introduced in Refs. \cite{3,4}. Knowledge of the polarization function to order $\alpha_s^2$, the appropriate anomalous dimensions at order $\alpha_s^3$, combined with the renormalization group equation allows one to predict the corresponding logarithmic terms of order $\alpha_s^3$ and hence the constant terms of the imaginary part. The first of these ingredients has been available since some time \cite{5,6,7,8} while the anomalous dimension can been obtained from Ref. \cite{9} in a straightforward way.

In this short note only the theoretical framework and the analytical results are presented – numerical studies will presented elsewhere.

2 Renormalization Group Analysis

In analogy to the vector case, we take as a starting point the generic vector/axial quark current correlator $\Pi_{\mu\nu}^{V/A}$ which is defined by

$$\Pi_{\mu\nu}^{V/A}(q, m_u, m_d, m, \mu, \alpha_s) = i \int dxe^{iqx} \langle T\left[ j_{\mu}^{V/A}(x)(j_{\nu}^{V/A})^\dagger(0) \right] \rangle$$

$$= g_{\mu\nu} \Pi_{V/A}^{(1)}(Q^2) + q_\mu q_\nu \Pi_{V/A}^{(2)}(Q^2). \quad (1)$$

with $Q^2 = -q^2$, $m_q^2 = \sum f m_f^2$ and $j_{\mu}^{V/A} = \bar{q} \gamma_\mu (\gamma_5) q'$. Here $q$ and $q'$ are just two (generically different) quarks with masses $m_u$ and $m_d$ respectively. Note that the vector and axial correlators are related through

$$\Pi_{A_{\mu\nu}}^{(4)}(q, m_u, m_d, m, \mu, \alpha_s) = \Pi_{V_{\mu\nu}}^{(4)}(q, m_u, -m_d, m, \mu, \alpha_s) \quad (2)$$

The polarization function $\Pi_{V/A}^{(1)}$ and the spectral density $R^{V/A}(s)$ which in turn governs the $Z$ and $W$ decays rate obey the following dispersion relation

$$\Pi_{V/A}^{(1)}(Q^2) = -\frac{1}{12\pi^2} \int_{(m_u+m_d)^2}^{\infty} ds R^{V/A}(s, m_u, m_d, m, \mu, \alpha_s) \frac{s}{s + Q^2} \quad \text{mod sub.} \quad (3)$$

Whereas $R^{V/A}$ as a physical quantity is invariant under renormalization group transformations, the function $\langle T\left[ j_{\mu}^{V/A}(x)(j_{\nu}^{V/A})^\dagger(0) \right] \rangle$ contains some non-integrable singularities.

\[1\]
in the vicinity of the point \( x = 0 \). These cannot be removed by standard quark mass and coupling constant renormalizations, but must be subtracted independently. As a result the relevant renormalization group equation assumes the form \( [2] \)

\[
\mu^2 \frac{d}{d\mu^2} \Pi_{\mu\nu}^{V/A} = (q_\mu q_\nu - g_{\mu\nu} q^2) \gamma_0^\pm(\alpha_s) \frac{1}{16\pi^2} + (m_u + m_d)^2 g_{\mu\nu} \gamma_0^\pm(\alpha_s) \frac{1}{16\pi^2},
\]

where

\[
\mu^2 \frac{d}{d\mu^2} = \mu^2 \frac{\partial}{\partial\mu^2} + \pi \beta(\alpha_s) \frac{\partial}{\partial\alpha_s} + \gamma_m(\alpha_s) \sum_f \bar{m}_f \frac{\partial}{\partial\bar{m}_f}.
\]

Here and below the upper and lower signs give the results for vector and axial vector correlators respectively. From the identity \( (2) \) we infer that both anomalous dimensions \( \gamma_0^\pm \) and \( \gamma_0^\pm \), being not dependent on any masses, also do not depend on the sign. In what follows we will denote

\[
\gamma_0^\pm = \gamma_{VV}^q \quad \text{and} \quad \gamma_0^\pm = \gamma_{VV}^m.
\]

The \( \beta \)-function and the quark mass anomalous dimension \( \gamma_m \) are defined in the usual way

\[
\mu^2 \frac{d}{d\mu^2} \left( \frac{\alpha_s(\mu)}{\pi} \right) = \alpha_s \beta(\alpha_s) \equiv -\sum_{i \geq 0} \beta_i \left( \frac{\alpha_s}{\pi} \right)^{i+1},
\]

\[
\mu^2 \frac{d}{d\mu^2} \bar{m}(\mu) = \bar{m}(\alpha_s) \equiv -\bar{m} \sum_{i \geq 0} \gamma_i^m \left( \frac{\alpha_s}{\pi} \right)^{i+1}.
\]

Their expansion coefficients up to order \( \mathcal{O}(\alpha_s^3) \) are well known \([10, 11, 12, 13]\) and read \((n_f \text{ is the number of quark flavours)}\)

\[
\beta_0 = \left( 11 - \frac{2}{3} n_f \right) / 4, \quad \beta_1 = \left( 102 - \frac{38}{3} n_f \right) / 16,
\]

\[
\beta_2 = \left( \frac{2857}{2} - \frac{5033}{18} n_f + \frac{325}{54} n_f^2 \right) / 64,
\]

\[
\gamma_0^m = 1, \quad \gamma_1^m = \left( \frac{202}{3} - \frac{20}{9} n_f \right) / 16,
\]

\[
\gamma_2^m = \left( 1249 - \frac{2216}{27} + \frac{160}{3} \zeta(3) n_f - \frac{140}{81} n_f^2 \right) / 64.
\]

Another useful and closely related object is the correlator of the (pseudo)scalar quark currents

\[
\Pi^{S/P}(Q^2, m_u, m_d, m, \mu, \alpha_s) = \int e^{iqx} \langle 0 | T \{ j^{S/P}(x)(j^{S/P})^\dagger \}(0) | 0 \rangle.
\]

Scalar and pseudoscalar current correlators are also related in a simple manner:

\[
\Pi^S(Q^2, m_u, m_d, m, \mu) = \Pi^P(Q^2, m_u, -m_d, m, \mu).
\]

For vanishing quark masses scalar and pseudoscalar correlators are, therefore, identical:

\[
\Pi^S = \Pi^P \quad \text{and} \quad \Pi^S = \Pi^P \quad \text{meet the following RG equation}
\]

\[
\left( \mu^2 \frac{d}{d\mu^2} + 2\gamma_m(\alpha_s) \right) \Pi^{S/P} = Q^2 \gamma_q^{SS}(\alpha_s) \frac{1}{16\pi^2}.
\]
The (axial) vector and (pseudo)scalar correlators are connected through a Ward identity \[14\]

\[ q_{\mu}q_{\nu} \Pi_{\mu\nu}^{V/A} = (m_u \mp m_d)^2 \Pi_{V/A}^{S/P} + (m_u \mp m_d) \left( \langle \bar{\psi}_q \psi_q \rangle \mp \langle \bar{\psi}_q' \psi_q' \rangle \right), \tag{13} \]

where the vacuum expectation values on the r.h.s. are understood within the framework of perturbation theory and the minimal subtractions. Equation (13) leads to the following relation between the corresponding anomalous dimensions \[4\]:

\[ \gamma_{VV}^{m} \equiv -\gamma_{SS}^{q}. \tag{14} \]

This relation was used in Ref. \[4\] in order to find the anomalous dimension \(\gamma_{AA}^{m}\) at the \(\alpha_{s}^{2}\) order starting from the results of Ref. \[5\].

In what follows we will be interested in quadratic mass corrections to the polarization operator \(\Pi_{V/A}^{(1)}\) which is convenient to represent in the form \((m = \{m_u, m_d, m\})\):

\[ \Pi_{V/A}^{(1)}(Q^2, m, \mu, \alpha_s) = \frac{3}{16\pi^2} \Pi_{V/A,0}^{(1)}(\frac{\mu^2}{Q^2}, \alpha_s) + \frac{3}{16\pi^2} \Pi_{V/A,2}^{(1)}(\frac{\mu^2}{Q^2}, m, \alpha_s) + \mathcal{O}(m^4). \tag{15} \]

Here the first term on the r.h.s corresponds to the massless limit while the second term stands for quadratic mass corrections. Note that \(\Pi_{V/A,2}^{(1)}\) is a second order polynomial in quark masses: a logarithmic dependence on quark masses may appear starting from \(m^4\) terms only\[4\].

From the RG equation (4) we arrive at the following equation for \(\Pi_{V/A,2}^{(1)}\):

\[ \mu^2 \frac{d}{d\mu^2} \Pi_{V/A,2}^{(1)} = \frac{1}{3} (m_u \mp m_d)^2 \gamma_{VV}^{m}(\alpha_s) \tag{16} \]
or, equivalently, \((L_q = \ln \frac{\mu^2}{Q^2})\)

\[ \frac{\partial}{\partial L_q} \Pi_{V/A,2}^{(1)} = \frac{1}{3} (m_u \mp m_d) \gamma_{VV}^{m} - \left( \beta \alpha_s \frac{\partial}{\partial \alpha_s} + 2 \gamma_{VV}^{m} \right) \Pi_{V/A,2}^{(1)}. \tag{17} \]

The last relation explicitly demonstrates that \(R_{2}^{V/A} - \) the absorptive part of \(\Pi_{V/A,2}^{(1)} - \) depends in order \(\alpha_{s}^{n}\) on the very function \(\Pi_{V/A,2}^{(1)}\) which is multiplied by at least one factor of \(\alpha_{s}\). This means that one needs to know \(\Pi_{V/A,2}^{(1)}\) up to order \(\alpha_{s}^{n-1}\) only to unambiguously reconstruct all \(Q\)-dependent terms in \(\Pi_{V/A,2}^{(1)}\) to \(\alpha_{s}^{n}\), provided, of course, the beta function and anomalous dimensions \(\gamma_{m}\) and \(\gamma_{VV}^{m}\) are known to \(\alpha_{s}^{n}\).

This observation was made first in \[3\] where it was used to find the absorptive part \(R_{2}^{V}\) in order \(\alpha_{s}^{2}\) for the case of the diagonal vector current (that is for the case of \(m_u = m_d\)). In the present paper we will use the results of a recent calculation of \(\gamma_{SS}^{q}\) \[9\] to order \(\alpha_{s}^{3}\) to determine the absorptive part \(R_{2}^{V/A}\) to the same order in the general case of non-diagonal currents.

\[1\]Provided of course that one uses a mass independent renormalization scheme like the \(\overline{\text{MS}}\)-scheme employed in this work.
3 Calculation and results

The result for the function \( \Pi_{V/A,2}^{(2)} \) in the general non-diagonal case to order \( \alpha_s^2 \) was first published in Ref. [5]. On the other hand, the Ward identity [13] expresses the combination \( \Pi_{V/A,2}^{(1)}(Q^2) - \Pi_{V/A,2}^{(2)}(Q^2) \) in terms of the massless polarization operator \( \Pi^S \) known from Refs. [4, 7]. A sum of these two functions leads us to the following result for \( \Pi_{V/A,2}^{(1)} \):

\[
\Pi_{V/A,2}^{(1)} = \frac{m_s^2}{Q^2} \left[ 2 + 2 \ln \frac{\mu^2}{Q^2} \right] + \frac{m_s^2}{Q^2} [-2] \\
+ \frac{m_s^2 \alpha_s}{Q^2 \pi} \left[ \frac{107}{6} - 8 \zeta(3) + \frac{22}{3} \ln \frac{\mu^2}{Q^2} + 2 \ln \frac{2 \mu^2}{Q^2} \right] + \frac{m_s^2 \alpha_s}{Q^2 \pi} \left[ -\frac{16}{3} - 4 \ln \frac{\mu^2}{Q^2} \right] \\
+ \frac{m_s^2 (\alpha_s \pi)^2}{Q^2} \left[ 3241 - 129 \zeta(3) - \frac{1}{2} \zeta(4) + 55 \zeta(5) - \frac{857}{108} n_f + \frac{32}{9} \zeta(3) n_f \right] \\
+ \frac{8221}{72} \ln \frac{\mu^2}{Q^2} - 39 \zeta(3) \ln \frac{\mu^2}{Q^2} - \frac{151}{36} n_f \ln \frac{\mu^2}{Q^2} + \frac{4}{3} \zeta(3) n_f \ln \frac{\mu^2}{Q^2} \\
+ \frac{155}{6} \ln^2 \frac{\mu^2}{Q^2} - \frac{8}{9} n_f \ln^2 \frac{\mu^2}{Q^2} + \frac{19}{6} \ln \frac{3}{Q^2} - \frac{1}{9} n_f \ln \frac{3}{Q^2} \\
+ \frac{13}{9} n_f \ln \frac{\mu^2}{Q^2} - \frac{19}{2} \ln \frac{2 \mu^2}{Q^2} + \frac{1}{3} n_f \ln \frac{2 \mu^2}{Q^2} \\
+ \frac{m_s^2 (\alpha_s \pi)^2}{Q^2} \left[ \frac{128}{9} - \frac{32}{3} \zeta(3) \right].
\]

(18)

Here \( m_- = m_u - m_d \) and \( m_+ = m_u + m_d \), \( Q^2 = -q^2 \), all masses as well as QCD coupling constant \( \alpha_s \) are understood to be taken at a generic value of the 't Hooft mass \( \mu \). All correlators are renormalized within MS-scheme. We have also checked (18) by a direct calculation with the help of the program MINCER [16] written for the symbolic manipulation system FORM [17]. In a particular case of \( m_u = m_d \) Eq. (18) is in agreement with Refs. [15, 8].

Now, as was shown in [3] the anomalous dimension \( \gamma_{m}^{AA} \equiv -\gamma_{q}^{SS} \), and, thus, from the results of [9] we have:

\[
\gamma_{m}^{VV} = -\gamma_{q}^{SS} = 6 \left\{ 1 + \frac{5}{3} \frac{\alpha_s}{\pi} + \frac{\alpha_s}{\pi} \left[ \frac{455}{72} - \frac{1}{2} \zeta(3) - \frac{1}{3} n_f \right] \right. \\
+ \left. \left( \frac{\alpha_s}{\pi} \right)^3 \left[ \frac{157697}{5184} - \frac{1645}{216} \zeta(3) + \frac{15}{8} \zeta(4) + \frac{65}{12} \zeta(5) - \frac{14131}{7776} n_f \right. \\
- \frac{13}{9} \zeta(3) n_f - \frac{11}{12} \zeta(4) n_f - \frac{1625}{11664} n_f^2 + \frac{1}{9} \zeta(3) n_f^2 \right\}.
\]

(19)

At last, integrating eq. (17) we find the spectral density \( R_{2V} \) in general case to order \( \alpha_s^3 \):

\[
R_{2V} = 3 \left\{ \frac{m_s^2}{s} r_{2,V}^+ + \frac{m_s^2}{s} r_{2,V}^- + \frac{m_s^2}{s} r_{2,V}^0 \right\}.
\]

(20)
where the functions $r^V$ are

$$
\begin{align*}
    r_{2,+}^V &= 3 \frac{\alpha_s}{\pi} + \left( \frac{\alpha_s}{\pi} \right)^2 \left[ \frac{253}{8} - \frac{13}{12} n_f + \frac{57}{4} \ln \frac{\mu^2}{s} - \frac{1}{2} n_f \ln \frac{\mu^2}{s} \right] \\
    &+ \left( \frac{\alpha_s}{\pi} \right)^3 \left[ \frac{1261}{2} - \frac{285}{16} \pi^2 + \frac{155}{6} \zeta(3) - \frac{5225}{24} \zeta(5) - \frac{2471}{54} n_f + \frac{17}{12} \pi^2 n_f ight] \\
    &+ \frac{197}{54} \zeta(3) n_f + \frac{1045}{108} \zeta(5) n_f + \frac{125}{216} n^2_f - \frac{1}{36} \pi^2 n^2_f + \frac{4505}{16} \ln \frac{\mu^2}{s} \\
    &- \frac{175}{8} n_f \ln \frac{\mu^2}{s} + \frac{13}{36} n^2_f \ln \frac{\mu^2}{s} + \frac{855}{16} \ln \frac{\mu^2}{s} - \frac{17}{4} n_f \ln \frac{\mu^2}{s} + \frac{1}{12} n^2_f \ln \frac{\mu^2}{s} \\
(21) &
\end{align*}
$$

$$
\begin{align*}
    r_{2,-}^V &= -\frac{3}{2} + \frac{\alpha_s}{\pi} \left[ -\frac{11}{2} \ln \frac{\mu^2}{s} \right] \\
    &+ \left( \frac{\alpha_s}{\pi} \right)^2 \left[ -\frac{8221}{96} + \frac{19}{8} \pi^2 + \frac{117}{4} \zeta(3) + \frac{151}{48} n_f - \frac{1}{12} \pi^2 n_f \\
    &- \zeta(3) n_f - \frac{155}{6} \ln \frac{\mu^2}{s} + \frac{4}{3} n_f \ln \frac{\mu^2}{s} - \frac{57}{8} \ln \frac{\mu^2}{s} + \frac{1}{4} n_f \ln \frac{\mu^2}{s} \right] \\
    &+ \left( \frac{\alpha_s}{\pi} \right)^3 \left[ -\frac{4544045}{3456} + \frac{335}{6} \pi^2 + \frac{118915}{144} \zeta(3) - \frac{635}{2} \zeta(5) + \frac{71621}{648} n_f \\
    &- \frac{209}{48} \pi^2 n_f - \frac{54}{4} \zeta(3) n_f + \frac{55}{4} \zeta(4) n_f + \frac{13171}{7776} n^2_f \\
    &+ \frac{27}{18} \pi^2 n^2_f - \frac{13}{6} \ln \frac{\mu^2}{s} + \frac{285}{16} \ln \frac{\mu^2}{s} + \frac{1755}{8} \zeta(3) \ln \frac{\mu^2}{s} \right] \\
    &+ \frac{8909}{144} n_f \ln \frac{\mu^2}{s} - \frac{17}{12} \pi^2 n_f \ln \frac{\mu^2}{s} - \frac{59}{4} \zeta(3) n_f \ln \frac{\mu^2}{s} - \frac{209}{216} n_f \ln \frac{\mu^2}{s} \\
    &+ \frac{1}{36} \pi^2 n^2_f \ln \frac{\mu^2}{s} + \frac{1}{3} \zeta(3) n^2_f \ln \frac{\mu^2}{s} - \frac{335}{2} \ln \frac{\mu^2}{s} + \frac{209}{16} n_f \ln \frac{\mu^2}{s} \\
    &- \frac{2}{9} n^2_f \ln \frac{\mu^2}{s} - \frac{285}{16} \ln \frac{\mu^2}{s} + \frac{17}{12} n_f \ln \frac{\mu^2}{s} - \frac{1}{36} n^2_f \ln \frac{\mu^2}{s} \right], \\
(22) &
\end{align*}
$$

$$
\begin{align*}
    r_{2,0}^V &= \left( \frac{\alpha_s}{\pi} \right)^3 \left[ -80 + 60 \zeta(3) + \frac{32}{9} n_f - \frac{8}{3} \zeta(3) n_f \right]. \\
(23) &
\end{align*}
$$

The expressions for the hadronic decay rates of the intermediate bosons read:

$$
\begin{align*}
    \Gamma(Z \rightarrow \text{hadrons}) &= \Gamma_0^Z \left[ \sum_f (g_V^f)^2 + (g_A^f)^2 \right] \left( R_0(s) + R_2^V(s, 0, 0, \sqrt{m_b^2 + m_c^2}) \right) \\
    &+ \sum_{f=b,c} (g_V^f)^2 R_2^V(s, m_f, m_f, 0) \\
    &+ \sum_{f=b,c} (g_A^f)^2 R_2^V(s, m_f, m_f, 0), \\
(24) &
\end{align*}
$$
\[ \Gamma(W \rightarrow \text{hadrons}) = \Gamma_0^W \left[ 2 \left( R_0(s) + R_2^V(s, 0, 0, \sqrt{m_b^2 + m_c^2}) \right) \right. \\
\left. + \frac{1}{2} \sum_{i = u, c} |V_{i,j}|^2 \left( R_2^V(s, m_i, m_j, 0) + R_2^A(s, m_i, m_j, 0) \right) \right]. \] (25)

with \( \Gamma_0 = \frac{G_F M_W^3}{6 \pi \sqrt{2}} \), \( g_f^V = I_f^3 - 2 Q_f \sin^2 \theta_w \), \( g_f^A = I_f^3 \) and \( V_{i,j} \) being the CKM matrix.

Here \( R_0(s) \) is the (non-singlet part) of the ratio \( R(s) \) in massless QCD; it was computed to \( \alpha_s^3 \) in [18, 19] and confirmed in [20]; it reads:

\[ R_0(s) = 3 \left\{ 1 + \frac{\alpha_s}{\pi} \right. \]
\[ + \left( \frac{\alpha_s}{\pi} \right)^2 \left[ \frac{365}{24} - \frac{11}{3} \zeta(3) + n_f \left( -\frac{11}{12} + \frac{2}{3} \frac{\zeta(3)}{\pi^2} \right) + \left( -\frac{11}{4} + \frac{1}{6} n_f \right) \ln \frac{s}{\mu^2} \right] \]
\[ + \left( \frac{\alpha_s}{\pi} \right)^3 \left[ \frac{87029}{288} - \frac{121}{48} \zeta(3) + 275 \zeta(5) \right. \]
\[ + n_f \left( -\frac{7847}{216} + \frac{11}{36} \frac{\zeta(3)}{\pi^2} + \frac{262}{9} \zeta(5) - \frac{25}{6} \zeta(3) \right) \]
\[ + \left( -\frac{4321}{48} + \frac{121}{2} \frac{\zeta(3)}{\pi^2} + n_f \left[ \frac{785}{72} - \frac{22}{3} \zeta(3) \right] \right. \]
\[ \left. + n_f^2 \left[ -\frac{151}{162} - \frac{1}{108} \frac{\pi^2}{\zeta(3)} \right] \ln \frac{s}{\mu^2} \right] \}
\] (26)

In deriving Eqs. (24) and (23) we have assumed that (i) the top quark is completely decoupled (the power suppressed corrections to this approximation start from the order \( s \alpha_s^2 \)) and have been studied in Refs. [22, 23, 24]); (ii) all other quarks except for the charmed and bottom ones are massless. Note that for the case of diagonal currents there exist also so-called singlet contributions to \( R(s) \). We will ignore these contributions in what follows as they are absent for the case of non-diagonal currents relevant for the \( W \)-decay (a detailed discussion of the \( Z \)-decay rate including singlet contributions can be found in [24]).

Taking into account the peculiar structure of the general result (20), the last formula can be written in a simpler form, viz.

\[ \Gamma(W \rightarrow \text{hadrons}) = \Gamma_0^W \left[ 2 \left( R_0(s) + R_2^V(s, 0, 0, \sqrt{m_b^2 + m_c^2}) \right) \right. \\
\left. + \left. R_2^V(s, m_{\text{eff}}, 0, 0) \right]. \] (27)

Here

\[ m_{\text{eff}}^2 = \sum_{i = u, c} |V_{i,j}|^2 (m_i^2 + m_j^2) \]

and we have taken into account the fact that

\[ R^V(s, m_i, m_j, 0) + R^A(s, m_i, m_j, 0) = 2 R^V(s, \sqrt{m_i^2 + m_j^2}, 0, 0) = 2 R^A(s, \sqrt{m_i^2 + m_j^2}, 0, 0) \]

6
As a direct consequence of Eqs. (220) we obtain the following expressions for particular functions entering into (24,25)

\[
R^V_{2}(s,m,m,0) = \frac{4m^2}{s^3}3r_{2,+}, \quad (28)
\]

\[
R^A_{2}(s,m,m,0) = \frac{4m^2}{s^3}3r_{2,-}, \quad (29)
\]

\[
R^V_{2}(s,m,0,0) = R^A_{2}(s,m,0,0) = \frac{m^2}{s}3(r_{2,+} + r_{2,-}), \quad (30)
\]

\[
R^V_{2}(s,0,0,m) = R^A_{2}(s,0,0,m) = \frac{m^2}{s}3r_{2,0}. \quad (31)
\]

At last, with \( n_f = 5 \) and \( \mu^2 = s \) the above formulas are simplified to

\[
R^V_{2}(s,m,m,0) = \frac{m^2}{s^3}3 \left\{ 12\frac{\alpha_s}{\pi} + \frac{629\alpha_s^2}{6\pi^2} \left[ \frac{89893}{54} - \frac{1645}{36}\pi^2 + \frac{820}{27}\zeta(3) - \frac{36575}{54}\zeta(5) \right] \right\} \quad (32)
\]

\[
R^A_{2}(s,m,m,0) = \frac{m^2}{s^3}3 \left\{ -6 - 22\frac{\alpha_s}{\pi} + \left( \frac{\alpha_s}{\pi} \right)^2 \left[ -\frac{2237}{8} + \frac{47}{6}\pi^2 + 97\zeta(3) \right] \right\} \quad (33)
\]

\[
R^V_{2}(s,m,0,0) = \frac{m^2}{s^3}3 \left\{ -3 - \frac{5\alpha_s}{2\pi} + \left( \frac{\alpha_s}{\pi} \right)^2 \left[ -\frac{4195}{96} + \frac{47}{24}\pi^2 + \frac{97}{4}\zeta(3) \right] \right\} \quad (34)
\]

or, in the numerical form,

\[
R^V_{2}(s,m,m,0) = \frac{m^2}{s^3}3 \left\{ 12\frac{\alpha_s}{\pi} + 104.833 \left( \frac{\alpha_s}{\pi} \right)^2 + 547.879 \left( \frac{\alpha_s}{\pi} \right)^3 \right\}, \quad (35)
\]

\[
R^A_{2}(s,m,m,0) = \frac{m^2}{s^3}3 \left\{ -6 - 22\frac{\alpha_s}{\pi} - 85.7136 \left( \frac{\alpha_s}{\pi} \right)^2 - 45.7886 \left( \frac{\alpha_s}{\pi} \right)^3 \right\}, \quad (36)
\]

\[
R^V_{2}(s,m,0,0) \equiv R^A_{2}(s,m,0,0) = \frac{m^2}{s^3}3 \left\{ -1.5 - 2.5\frac{\alpha_s}{\pi} + 4.7799 \left( \frac{\alpha_s}{\pi} \right)^2 + 125.523 \left( \frac{\alpha_s}{\pi} \right)^3 \right\}. \quad (37)
\]

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