Fixed points of rational type contractions in
G-metric spaces

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Abstract: We establish three major fixed-point theorems for functions satisfying
generalized rational type almost contraction conditions. Firstly we consider the case
of a single mapping, secondly we look at the case of a triplet of mappings and we
conclude by the case of a family of mappings. The theorems we present generalize
similar results already obtained by Abbas, Rhoades, Gaba, and others. The operators
we consider are all of the weakly Picard type.

1. Introduction and preliminaries

Recently, applications of G-metric spaces, in the fields like optimization theory, differential and inte-
gral equations, have been discovered and this has generated a lot of interest for these type of spaces
(see Mustafa, Obiedat, & Awawdeh, 2008; Mustafa & Sims, 2006; Shoaib, Arshad, & Kazmi, 2017).
Their relevance is no more to be demonstrated as it has been extensively discussed in the literature.
In this paper, we prove three main fixed point results in that setting. We propose generalizations
which ensure existence results for fixed points, and to this goal we investigate the character of the
sequence of iterates \( \{T^n x\}_{n=0}^{\infty} \) (resp. \( \{T_i(x_{i-1})\}_{i=0}^{\infty} \)) where \( T: X \to X \) (resp. \( T_i: X \to X \)) is (resp. are) the
map (resp. maps) under consideration, \( x \in X \) and \( X \) a complete G-metric space. More precisely, we
consider mappings that satisfy a rational type almost contraction and the results we present are
comparable to previous ones already obtained in Gaba (2017). The paper is divided in two major sec-
tions, a first section which gives an introduction and some preliminaries and a second section which
deals with the statements of results. The second section contains three subsections of which the
first two present proofs making use of classical arguments (already used in Gaba, 2017), and of
which the third one presents a result based on \( \alpha \)-series, see Sihag et al. (2014). The elementary facts
about G-metric spaces can be found in Gaba (2017), Mustafa and Sims (2006) and the references
therein. We give here a summary of these prerequisites.

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about G-metric spaces can be found in Gaba (2017), Mustafa and Sims (2006) and the references
therein. We give here a summary of these prerequisites.
Definition 1.1 (see [Mustafa & Sims, 2006, Definition 3]) Let $X$ be a nonempty set, and let the function $G : X \times X \times X \to [0, \infty)$ satisfy the following properties:

(G1) $G(x, y, z) = 0$ if $x = y = z$ whenever $x, y, z \in X$;
(G2) $G(x, x, y) > 0$ whenever $x, y \in X$ with $x \neq y$;
(G3) $G(x, x, y) \leq G(x, y, z)$ whenever $x, y, z \in X$ with $z \neq y$;
(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \ldots$ (symmetry in all three variables);
(G5) for any points $x, y, z, a \in X$

\[ G(x, y, z) \leq [G(x, a, a) + G(a, y, z)]. \]

Then $(X, G)$ is called a $G$-metric space.

Definition 1.2 (see [Mustafa & Sims, 2006]) Let $(X, G)$ be a $G$-metric space, and let $(x_n)_{n \geq 1}$ be a sequence of points of $X$, therefore, we say that the sequence $(x_n)_{n \geq 1}$ is $G$-convergent to $x \in X$ if $\lim_{n, m \to \infty} G(x_n, x_m) = 0$, that is, for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x_n, x_m) < \epsilon$, for all $n, m \geq N$. We call $x$ the limit of the sequence and write $x_n \to x$ or $\lim_{n \to \infty} x_n = x$.

Proposition 1.3 (Compare [Mustafa & Sims, 2006, Proposition 6]) Let $(X, G)$ be a $G$-metric space. Define on $X$ the metric $d_g$ by $d_g(x, y) = G(x, y, y) + G(x, x, y)$ whenever $x, y \in X$. Then for a sequence $(x_n)_{n \geq 1} \subseteq X$, the following are equivalent

(i) $(x_n)$ is $G$-convergent to $x \in X$.
(ii) $\lim_{n, m \to \infty} G(x_n, x_m) = 0$.
(iii) $\lim_{n \to \infty} d_g(x_n, x) = 0$.
(iv) $\lim_{n \to \infty} G(x_n, x_n, x) = 0$.
(v) $\lim_{n \to \infty} G(x_n, x, x) = 0$.

Definition 1.4 (See Mustafa & Sims, 2006) Let $(X, G)$ be a $G$-metric space. A sequence $(x_n)_{n \geq 1}$ is called a $G$-Cauchy sequence if for any $\epsilon > 0$, there is $N \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \epsilon$ for all $n, m, l \geq N$, that is $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to +\infty$.

Proposition 1.5 (Compare [Mustafa & Sims, 2006, Proposition 9])

In a $G$-metric space $(X, G)$, the following are equivalent

(i) The sequence $(x_n)_{n \geq 1} \subseteq X$ is $G$-Cauchy.
(ii) For each $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \epsilon$ for all $m, n \geq N$.

Definition 1.6 (Compare [Mustafa & Sims, 2006, Definition 4]) A $G$-metric space $(X, G)$ is said to be symmetric if

$G(x, y, z) = G(x, y, x)$ for all $x, y \in X$.

Definition 1.7 (Compare [Mustafa & Sims, 2006, Definition 9]) A $G$-metric space $(X, G)$ is $G$-complete if every $G$-Cauchy sequence of elements of $(X, G)$ is $G$-convergent in $(X, G)$.

Theorem 1.8 (see Mustafa & Sims, 2006) A $G$-metric $G$ on a $G$-metric space $(X, G)$ is continuous on its three variables.

We conclude this introductory part with:
Definition 1.9 (Compare [Sihag et al., 2014, Definition 2.1]) For a sequence \((a_n)_{n=1}^{\infty}\) of nonnegative real numbers, the series \(\sum_{n=1}^{\infty} a_n\) is an \(\alpha\)-series if there exist \(0 < \lambda < 1\) and \(n(\lambda) \in \mathbb{N}\) such that

\[
\sum_{i=1}^{L} a_i \leq \lambda L \text{ for each } L \geq n(\lambda).
\]

2. The results

This section on our main results begins with the case of a single map.

2.1. Single maps

THEOREM 2.1 Let \((X, G)\) be a symmetric \(G\)-complete \(G\)-metric space and \(T\) be a mapping from \(X\) to itself. Suppose that \(T\) satisfies the following condition:

\[
G(Tx, Ty, Tz) \leq \left( \frac{aG(Tx, y, z) + bG(x, Ty, z) + cG(x, y, Tz)}{(b + c)G(x, Ty, Ty) + bG(y, Ty, Ty) + cG(Tz, Tz, Tz) + 1} \right) G(x, y, z),
\]

(2.1)

for all \(x, y, z \in X\), where \(a, b, c\) are non-negative reals. Then

(a) \(T\) has at least one fixed point \(\xi \in X\);
(b) for any \(x \in X\), the sequence \((T^nx)_{n=1}^{\infty}\) \(G\)-converges to a fixed point of \(T\);
(c) if \(\xi, x \in X\) are two distinct fixed points, then

\[
G(\xi, x, x) = G(\xi, x, x) \geq \frac{1}{a + b + c}.
\]

Proof. We imitate the steps of the proof of [Gaba, 2017 Theorem 2.1].

Let \(x_0 \in X\) be arbitrary and construct the sequence \((x_n)_{n=1}^{\infty}\) such that \(x_{n+1} = Tx_n\). Moreover, we may assume, without loss of generality that \(x_n \neq x_m\) for \(n \neq m\).

For the triplet \((x_n, x_{n+1}, x_{n+2})\), and by setting \(d_n = G(x_n, x_{n+1}, x_{n+2})\), we have:

\[
0 < d_n = G(x_n, x_{n+1}, x_{n+2}) = G(Tx_{n-1}, Tx_n, Tx_n) \leq \frac{aG(x_n, x_{n+1}, x_{n+2}) + bG(x_{n+1}, x_{n+2}, x_{n+3}) + cG(x_{n-1}, x_n, x_{n+1})}{(b + c)d_{n-1} + bd_n + cd_{n+1} + 1} d_{n-1} \leq \frac{(b + c)d_{n-1} + (b + c)d_n}{(b + c)d_{n-1} + (b + c)d_n + 1} d_{n-1},
\]

since

\[
G(x_{n-1}, x_{n+1}, x_{n+2}) \leq G(x_{n-1}, x_n, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+3}) = G(x_{n-1}, x_n, x_{n+1}) + G(x_{n-1}, x_{n+1}, x_{n+2}).
\]

If we set

\[
a_n = \frac{(b + c)d_{n-1} + (b + c)d_n}{(b + c)d_{n-1} + (b + c)d_n + 1},
\]

we get, iteratively

\[
d_n \leq a_n d_{n-1} \leq a_n a_{n-1} d_{n-2} \leq \cdots \leq a_n a_{n-1} \cdots a_2 d_0.
\]

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Claim: The sequence \((a_n)_{n \geq 1}\) is a non-increasing sequence of non-negative reals.

Indeed, since we have

\[(b + c)d_{n-1} + (b + c)d_n \leq (b + c)d_{n-1} + (b + c)d_n + 1,
\]

it is very clear that for any natural number \(n \in \mathbb{N}\), \(0 \leq a_n < 1\), and so \(d_n < d_{n-1}\). We then have the following consecutive equivalences:

\[
d_n \leq d_{n-1} \iff d_n + d_{n+1} \leq d_{n-1} + d_n \iff 1 + \frac{1}{(b + c)d_{n-1} + (b + c)d_n} \leq 1 + \frac{1}{(b + c)d_n + (b + c)d_{n+1}} \iff \frac{1}{d_n} \leq \frac{1}{d_{n+1}}.
\]

Hence

\[a_n \cdots a_1 = a_1^n \to 0 \text{ as } n \to \infty.\]

Therefore

\[\lim_{n \to \infty} a_n \cdots a_1 = 0,
\]

hence

\[\lim_{n \to \infty} d_n = 0.
\]

For any \(m, n \in \mathbb{N}\), \(m > n\), since we have

\[G(x_n, x_m, x_m) \leq \sum_{i=0}^{m-n} G(x_{n+i}, x_{n+i+1}, x_{n+i+1}),\]

the above translates to

\[G(x_n, x_m, x_m) \leq \sum_{i=0}^{m-n} d_{n+i},
\]

and we obtain

\[G(x_n, x_m, x_m) \leq \sum_{i=0}^{m-n} (a_n \cdots a_1) d_i.
\]

Put \(b_k = a_k \cdots a_1\) and observe that

\[\lim_{k \to \infty} \frac{b_{k+1}}{b_k} = \lim_{k \to \infty} a_k = 0 \text{ since } a_k = \frac{(b + c)d_{k-1} + (b + c)d_k}{(b + c)d_{k-1} + (b + c)d_k + 1} \text{ and } \lim_{k \to \infty} d_k = 0.
\]

Hence

\[\sum_{k=0}^{\infty} b_k < \infty,
\]

therefore

\[\sum_{i=0}^{m-n} (a_n \cdots a_1) \to 0 \text{ as } m \to \infty.
\]

In other words, \((x_n)_{n \geq 1}\) is a \(G\)-Cauchy sequence so \(G\)-converges to some \(\xi \in X\).
Claim: $\xi$ is a fixed point of $T$.

For the triplet $(x_{n+1}, T\xi, T\xi)$ in (2.1), we get

$$G(x_{n+1}, T\xi, T\xi) \leq \left( \frac{(b + c)G(\xi, T\xi, \xi)}{(b + c)d_n + (b + c)G(\xi, T\xi, \xi) + 1} \right) G(x_n, \xi, \xi). \tag{2.2}$$

On taking the limit on both sides of (2.2), and using the fact that the function $G$ is continuous, we have

$$G(\xi, T\xi, T\xi) \leq \left( \frac{(b + c)G(\xi, T\xi, \xi)}{(b + c)G(\xi, T\xi, \xi) + 1} \right) G(\xi, \xi, \xi),$$

i.e. $G(\xi, T\xi, T\xi) = 0$, thus $T\xi = \xi$.

If $\kappa$ is a fixed point of $T$ with $\kappa \neq \xi$, then

$$G(\xi, \kappa, \kappa) = G(T\xi, Tk, Tx) \leq |aG(\xi, \kappa, \kappa) + (b + c)G(\xi, \kappa, \kappa)G(\xi, \kappa, \kappa) \leq (a + b + c)G(\xi, \kappa, \kappa)^2.$$

Therefore,

$$G(\xi, \kappa, \kappa) = G(\xi, \xi, \kappa) \geq \frac{1}{a + b + c}. \tag{2.3}$$

The following two corollaries, particular cases of Theorem 2.1, are of interest for us, due to our previous work in Gaba (2017).

**Corollary 2.2** Let $(X, G)$ be a symmetric $G$-complete $G$-metric space and $T$ be a mapping from $X$ to itself. Suppose that $T$ satisfies the following condition:

$$G(Tx, Ty, Tz) \leq \left( \frac{G(x, y, z) + \frac{1}{2}(G(x, Ty, z) + G(x, y, Tz))}{G(x, Tz, Tx) + \frac{1}{2}(G(y, Ty, Ty) + G(z, Tz, Tz)) + 1} \right) G(x, y, z),$$

for all $x, y, z \in X$. Then

(a) $T$ has at least one fixed point $\xi \in X$;

(b) for any $x \in X$, the sequence $(T^n x)_{n \geq 1}$ $G$-converges to a fixed point;

(c) if $\xi, \kappa \in X$ are two distinct fixed points, then

$$G(\xi, \kappa, \kappa) = G(\xi, \xi, \xi) \geq \frac{1}{2}. \tag{2.4}$$

**Proof** Apply Theorem 2.1 with $a = 1, b = c = \frac{1}{2}$. \hfill $\Box$

**Example 2.3** (Compare [Gaba, 2017, Example 2.2])

Let $X = \left\{ 0, \frac{1}{2}, 1 \right\}$ and let $G: X^3 \to [0, \infty)$ be defined by

$$G(0, 1, 1) = 6 = G(1, 0, 0), \ G\left(0, \frac{1}{2}, \frac{1}{2}\right) = 4 = G\left(\frac{1}{2}, 0, 0\right)$$

$$G\left(1, \frac{1}{2}, 1\right) = 5 = G\left(1, \frac{1}{2}, \frac{1}{2}\right), \ G\left(0, \frac{1}{2}, 1\right) = \frac{15}{2}$$

$$G(x, x, x) = 0 \ \forall x \in X.$$
\((X, G)\) is a symmetric \(G\)-complete \(G\)-metric space.

Let \(T : X \to X\) be defined by \(T(0) = 0\), \(T(\frac{1}{2}) = \frac{1}{2}\), \(T(1) = 0\).

\[
G\left(T0, T\left(\frac{1}{2}\right), T\left(\frac{1}{2}\right)\right) = G\left(0, \frac{1}{2}, \frac{1}{2}\right) = 4; G(T0, T1, T1) = G(0, 0, 0) = 0;
\]
\[
G\left(T\left(\frac{1}{2}\right), T1, T1\right) = G\left(\frac{1}{2}, 0, 0\right) = 4; G\left(T0, T\left(\frac{1}{2}\right), T1\right) = G\left(0, \frac{1}{2}, 0\right) = 4.
\]

We have

\[
4 = G\left(T0, T\left(\frac{1}{2}\right), T\left(\frac{1}{2}\right)\right) = G\left(0, \frac{1}{2}, \frac{1}{2}\right)
\]
\[
\leq G(T0, \frac{1}{2}, \frac{1}{2}) + \frac{1}{2} G(0, T\left(\frac{1}{2}\right), \frac{1}{2}) + \frac{1}{2} G(0, \frac{1}{2}, T\left(\frac{1}{2}\right))
\]
\[
\leq G(0, 0, 0) + G\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) + 1
\]
\[
= \frac{4 + 2 + 2}{1} = 4 = 32.
\]

Again,

\[
0 = G(T0, T1, T1) = G(0, 0, 0)
\]
\[
\leq G(T0, 1, 1) + \frac{1}{2} G(0, T1, 1) + \frac{1}{2} G(0, 1, T1)
\]
\[
\leq G(0, 0, 0) + G(1, T1, T1) + 1
\]
\[
= \frac{6 + 3 + 3}{6}.
\]

Also, \(4 = G\left(T\left(\frac{1}{2}\right), T1, T1\right) = G\left(\frac{1}{2}, 0, 0\right)\)
\[
\leq G\left(T\left(\frac{1}{2}\right), 1, 1\right) + \frac{1}{2} G\left(T1, T1, 1\right) + \frac{1}{2} G\left(\frac{1}{2}, 1, T1\right)
\]
\[
\leq G\left(\frac{1}{2}, T\left(\frac{1}{2}\right), T\left(\frac{1}{2}\right)\right) + G(1, T1, T1) + 1
\]
\[
\times G\left(\frac{1}{2}, 1, 1\right)
\]
\[
= \frac{5 + \frac{15}{2}}{2} = 5.
\]

Finally,
\[ 4 = G\left(T_0, T\left(\frac{1}{2}\right), T_1\right) = G\left(0, \frac{1}{2}, 0\right) \]
\[ \leq \frac{G(T_0, \frac{1}{2}, 1) + \frac{1}{2}G(0, T\left(\frac{1}{2}\right), 1) + \frac{1}{2}G(0, \frac{1}{2}, T_1)}{G(0, T_0, T_0) + \frac{1}{2}G\left(\frac{1}{2}, T\left(\frac{1}{2}\right), T\left(\frac{1}{2}\right)\right) + \frac{1}{2}G(1, T_1, T_1) + 1} \times G\left(0, \frac{1}{2}, 1\right) \]
\[ = \frac{G(0, \frac{1}{2}, 1) + \frac{1}{2}G(0, \frac{1}{2}, 1) + \frac{1}{2}G(0, \frac{1}{2}, 0) + \frac{1}{2}G(0, 0, 0) + \frac{1}{2}G\left(\frac{1}{2}, \frac{1}{2}\right) + \frac{1}{2}G(1, 0, 0) + 1}{G(0, 0, 0) + \frac{1}{2}G\left(\frac{1}{2}, \frac{1}{2}\right) + \frac{1}{2}G(1, 0, 0) + 1} \times G\left(0, \frac{1}{2}, 1\right) \]
\[ = \frac{15 + 4\times\frac{1}{2} \times 15}{7}. \]

Therefore \( T \) satisfies all the conditions of Theorem 2.2. Also, \( T \) has two distinct fixed points \( \{0, \frac{1}{2}\} \) and
\[ G\left(0, \frac{1}{2}, \frac{1}{2}\right) = G\left(\frac{1}{2}, 0, 0\right) = 4 \geq \frac{1}{a + b + c} = \frac{1}{2}. \]

**Corollary 2.4.** (Compare [Gaba, 2017, Theorem 2.1]) Let \((X, G)\) be a symmetric G-complete G-metric space and \( T \) be a mapping from \( X \) to itself. Suppose that \( T \) satisfies the following condition:
\[ G(T_x, T_y, T_z) \leq \left( \frac{G(T_x, y, z) + G(x, T_y, z) + G(x, y, T_z)}{2G(x, T_x, T_x) + G(y, T_y, T_y) + G(z, T_z, T_z) + 1} \right) G(x, y, z), \] (2.4)
for all \( x, y, z \in X \). Then

(a) \( T \) has at least one fixed point \( x \in X \);
(b) for any \( x \in X \), the sequence \((T^n x)_{n \geq 1}\) G-converges to a fixed point;
(c) if \( x, k \in X \) are two distinct fixed points, then
\[ G(x, k, k) = G(x, x, x) \geq \frac{1}{3}. \]

**Proof**

Apply Theorem 2.1 with \( a = 1, b = c = 1 \).

The previous results naturally extend if we consider a partially ordered complex valued G-metric space. Moreover, one can replace the non-negative real constants \( a, b, c \) by non-negative real valued functions.

We can define a partial order \( \leq \) on the set \( \mathbb{C} \) of complex numbers by setting, for any \( z_1, z_2 \in \mathbb{C} \),
\[ z_1 \leq z_2 \iff \text{Re}(z_1) \leq \text{Re}(z_2) \text{ and } \text{Im}(z_1) \leq \text{Im}(z_2) \iff z_2 \geq z_1. \]

Moreover, on partial ordered G-metric space, the convergence of a sequence is interpreted in the canonical way, i.e. for a sequence \((x_n)_{n \geq 1} \leq (X, G, \leq)\) where \((X, G, \leq)\) is a partial ordered complex valued G-metric space,
\[ (x_n)_{n \geq 1} \text{ G-converges to } c \iff \forall c \in \mathbb{C}, \text{ with } 0 \leq c, \exists n_0 \in \mathbb{N} : \forall n > n_0 \ G(x^n, x, c) \leq c. \]

Similarly for G-Cauchy sequences. Furthermore, a self mapping \( T \) defined on a partial ordered G-metric space \((X, G, \leq)\) is nondecreasing if \( T x \leq T y \) whenever \( x \leq y \), for \( x, y \in X \).

We then state the result:
Theorem 2.5 Let $(X, G, ≤) \text{ be a symmetric, } G\text{-complete, complex valued } G\text{-metric space. Assume that if } x_n \text{ is a nondecreasing sequence of elements of } X \text{ such that } x_n \text{ } G\text{-converges to } x^\ast, \text{ then } x_n ≤ x^\ast \text{ for all } n \in \mathbb{N}. \text{ Let } T: X \to X \text{ be a nondecreasing mapping such that:}

\[
G(Tx, Ty, Tz) ≤ \left( \frac{\alpha G(Tx, y, z) + b G(x, Ty, z) + c G(x, y, Tz)}{(b + c) G(x, x, Tx) + b G(y, Ty, y) + c G(z, z, Tz) + 1} \right) G(x, y, z),
\]

(2.5)

for all } x ≤ y ≤ z ∈ X \text{ where } α := a(x, y, z), b := b(x, y, z), c := c(x, y, z) \text{ are non-negative real valued functions.}

If there exists } x_0 ∈ X \text{ with } x_0 ≤ Tx_0 \text{ then}

(i) } T \text{ has at least one fixed point } ξ ∈ X;

(ii) \text{ for any } x ∈ X, \text{ the sequence } (T^nx_0)_{n≥1} \text{ G-converges to a fixed point;}

(iii) \text{ if } ξ, κ ∈ X \text{ are two distinct fixed points, then}

\[
G(ξ, κ, κ) = G(ξ, ξ, κ) ≥ \frac{1}{α + b + c}.
\]

Proof Following the steps of the proof of Theorem 2.1, it is very easy to see that the sequence of iterates } T^nx_0, n = 1, 2, \ldots, \text{ is nondecreasing and } G\text{-converges to some } ξ ∈ X. \text{ Therefore } x_n ≤ ξ \text{ for all } n \in \mathbb{N}. \text{ Now applying (2.5) to the triplet } (x_{n+1}, Tξ, Tξ) \text{ we have:}

\[
G(x_{n+1}, Tξ, Tξ) = G(Tx_n, Tξ, Tξ) ≤ \left( \frac{\alpha G(x_{n+1}, ξ, ξ) + (b + c) G(ξ, Tξ, ξ)}{(b + c) G(x_n, x_n, x_{n+1}) + (b + c) G(ξ, Tξ, Tξ) + 1} \right) G(x_n, ξ, ξ).
\]

Now taking the limit as } n → ∞, \text{ and using the fact that the function } G \text{ is continuous, we have:}

\[
G(ξ, Tξ, Tξ) ≤ \left( \frac{(b + c) G(ξ, Tξ, ξ)}{(b + c) G(ξ, Tξ, Tξ) + 1} \right) G(ξ, ξ, ξ),
\]

i.e. } G(ξ, Tξ, Tξ) = 0, \text{ thus } Tξ = ξ.

If } κ \text{ is a fixed point of } T \text{ with } κ ≠ ξ, \text{ then}

\[
G(ξ, κ, κ) = G(Tξ, Tk, Tk) ≤ \left| \alpha G(ξ, κ, κ) + (b + c) G(ξ, κ, κ) \right| G(ξ, κ, κ)
\]

\[
≤ (α + b + c)(G(ξ, κ, κ))^2.
\]

Therefore,

\[
G(ξ, κ, κ) = G(ξ, ξ, κ) ≥ \frac{1}{α + b + c}.
\]

Another variant of Theorem 2.1 goes as follows:

Theorem 2.6 Let } (X, G) \text{ be a symmetric } G\text{-complete } G\text{-metric space and } T \text{ be a mapping from } X \text{ to itself. Suppose that } T \text{ satisfies the following condition:}

\[
G(Tx, Ty, Tz) ≤ K(x, y, z) G(x, y, z),
\]

(2.6)
for all \(x, y, z \in X\), where \(a = a(x, y, z)\), \(b = b(x, y, z)\), \(c = c(x, y, z)\) are non-negative real valued functions and

\[
K(x, y, z) = \frac{a.G(x, Ty, Tz) + b.G(Tx, Ty, Tz) + c.G(Ty, Ty, Tz)}{a.G(x, Ty, Tz) + \left(\frac{a}{2} + b\right).G(y, Ty, Ty) + \left(\frac{a}{2} + c\right).G(z, Tz, Tz) + 1}.
\]

Then \(T\) has at least one fixed point \(\xi \in X\).

**Remark 2.7** In general, the self mapping \(T\) in Theorem 2.6 (as well as in Theorem 2.1) is a weakly\(^4\) Picard operator. Moreover, the reader can convince him/her-self that if \(\xi\) and \(\kappa\) are fixed points of \(T\) in \(X\), a lower bound can be found for \(G(\xi, \xi, \kappa) = G(\xi, x, \kappa)\) (see point (c) in Theorem 2.1). Furthermore, Theorem 2.6 can be expressed in a setting of a partially ordered complex valued \(G\)-metric space.

We conclude this subsection by proving the following result, which presents a reverse rational type contraction. Actually, this mapping can be classified as an expansion type mapping.

**Theorem 2.8** Let \((X, G)\) be a symmetric \(G\)-complete \(G\)-metric space and \(T\) be an onto self mapping on \(X\). Suppose that \(T, P, Q\) satisfy the following condition:

\[
G(Tx, Ty, Tz) \geq A(x, y, z) G(x, y, z),
\]

for all \(x, y, z \in X, x \neq y\), where \(a, b, c\) are non-negative reals and

\[
A(x, y, z) = \frac{a.G(x, Ty, Tz) + \left(\frac{a}{2} + b\right).G(y, Ty, Ty) + \left(\frac{a}{2} + c\right).G(z, Tz, Tz) + 1}{a.G(x, Ty, Tz) + b.G(Tx, Ty, Tz) + c.G(Ty, Ty, Tz)}.
\]

Then \(T\) has at least one fixed point \(\xi \in X\).

**Proof** Let \(Tx = Ty\), then

\[0 = G(Tx, Ty, Ty) \geq A(x, y, z) G(x, y, y).\]

Hence \(G(x, y, y) = 0\), which implies that \(x = y\). So \(T\) is injective and invertible.

If \(H\) is the inverse mapping of \(T\), then for \(x, y, z \in X, x \neq y\), we have

\[
G(x, y, z) = G(T(Hx), T(Hy), T(Hz)) \geq A(Hx, Hy, Hz) G(Hx, Hy, Hz).
\]

Hence for all \(x, y, z \in X, x \neq y\)

\[
G(Hx, Hy, Hz) \leq \frac{1}{A(Hx, Hy, Hz)} G(x, y, z).
\]

From Theorem 2.6, the inverse mapping \(H\) has a fixed point \(u \in X\), i.e. \(Hu = u\). But \(u = T(H(u)) = T(u)\). Thus \(u\) is also a fixed point of \(T\). \(\Box\)

In the next subsection, we consider the case of a triplet of functions and we state an analogue of Theorem 2.1.

### 2.2. Triples of maps

**Theorem 2.9** Let \((X, G)\) be a symmetric \(G\)-complete \(G\)-metric space and \(T, P, Q\) be three self mappings on \(X\). Suppose that \(T, P, Q\) satisfy the following condition:
\[ G(Tx, Py, Qz) \leq \left( \frac{aG(Tx, y, z) + bG(x, Py, z) + cG(x, y, Qz)}{(b + c)G(x, Tx, Tx) + (a + b + 2c)G(y, Py, Py) + cG(z, Qz, Qz) + 1} \right) G(x, y, z), \tag{2.8} \]

for all \( x, y, z \in X \) where \( a = a(x, y, z), b = b(x, y, z), c = c(x, y, z) \) are non-negative functions. Then \( T, P \) and \( Q \) have a common fixed point, i.e. \( \exists u \in X \) such that \( Tu = Pu = Qu = u \).

**Proof** For any initial point \( x_0 \in X \), we construct the sequence \( (x_n)_{n \geq 1} \) by setting \( x_{3n+1} = Tx_{3n}, x_{3n+2} = Px_{3n+1}, x_{3n+3} = Qx_{3n+2}, n \geq 0 \).

Without loss of generality, assume that \( x_n \neq x_m \) for \( n \neq m \).

Plugging in \( (x_{3n+1}, x_{3n+2}, x_{3n+3}) = (Tx_{3n}, Px_{3n+1}, Qx_{3n+2}) \) in (2.8), we have:

\[ G(x_{3n+1}, x_{3n+2}, x_{3n+3}) = G(Tx_{3n}, Px_{3n+1}, Qx_{3n+2}) \leq H_n G(x_{3n}, x_{3n+1}, x_{3n+2}), \]

where

\[ H_n = \frac{aG(x_{3n}, x_{3n+1}, x_{3n+2}) + bG(x_{3n}, x_{3n+2}, x_{3n+3}) + cG(x_{3n}, x_{3n+1}, x_{3n+3})}{(b + c)G(x_{3n}, x_{3n+1}, x_{3n+1}) + (a + b + 2c)G(x_{3n+1}, x_{3n+2}, x_{3n+2}) + cG(x_{3n+2}, x_{3n+3}, x_{3n+3}) + 1}. \]

Each of the term in the numerator of \( H_n \) can be bounded as follows:

\[
\begin{align*}
G(x_{3n+1}, x_{3n+2}, x_{3n+3}) &= G(x_{3n+1}, x_{3n+2}, x_{3n+2}) \\
G(x_{3n}, x_{3n+1}, x_{3n+2}) &\leq G(x_{3n}, x_{3n+1}, x_{3n+1}) + G(x_{3n+1}, x_{3n+2}, x_{3n+2}) \\
G(x_{3n}, x_{3n+1}, x_{3n+3}) &\leq G(x_{3n}, x_{3n+1}, x_{3n+1}) + G(x_{3n+1}, x_{3n+2}, x_{3n+3}) \\
&\quad + G(x_{3n+1}, x_{3n+2}, x_{3n+2}) \\
&\quad + G(x_{3n+1}, x_{3n+2}, x_{3n+3}) \\
&\quad + G(x_{3n+2}, x_{3n+2}, x_{3n+3}) + G(x_{3n+2}, x_{3n+3}, x_{3n+3}).
\end{align*}
\]

By setting \( d_n = G(x_{3n}, x_{3n+1}, x_{3n+1}) \), \( H_n \) is bounded as

\[ H_n \leq \frac{aG_{3n+1} + bG_{3n+1} + cG_{3n+1}}{d_{3n+1} + d_{3n+2} + 1}, \]

i.e.

\[ H_n \leq \frac{(b + c)d_{3n} + (a + b + 2c)d_{3n+1} + cG_{3n+2}}{(b + c)d_{3n} + (a + b + 2c)d_{3n+1} + 1}. \]

So if we denote

\[ a_n := \frac{(b + c)d_{3n} + (a + b + 2c)d_{3n+1} + cG_{3n+2}}{(b + c)d_{3n} + (a + b + 2c)d_{3n+1} + 1}, \]

plugging in \( (x_{3n+1}, x_{3n+2}, x_{3n+3}) = (Tx_{3n}, Px_{3n+1}, Qx_{3n+2}) \) in (2.8) implies

\[ G(x_{3n+1}, x_{3n+2}, x_{3n+3}) \leq a_n G(x_{3n}, x_{3n+1}, x_{3n+2}). \]

If we inspire ourselves from the proof of Theorem 2.1, one can easily establish that the sequence \( (d_{3n})_{n \geq 1} \) is a non-increasing sequence of non-negative real numbers and that for any natural number \( n \in \mathbb{N}, 0 \leq a_n < 1 \). Moreover, it is readily seen that \( (x_n)_{n \geq 1} \) is a \( G \)-Cauchy sequence so \( G \)-converges to some \( \xi \in X \).
Let \((x, y, z)\) in (2.8), we get
\[
G(T_\xi, P_\xi, Q_\xi) \leq \gamma G(\xi, x, \xi) = 0,
\]
where \(\gamma \geq 0\) can easily be recovered from (2.8). Hence
\[
G(T_\xi, P_\xi, Q_\xi) = 0 \quad \text{i.e.} \quad T_\xi = P_\xi = Q_\xi.
\]
Again, from (2.8), we can write that:
\[
\begin{align*}
G(T_\xi, x_{3n+2}, x_{3n+3}) &= G(T_\xi, Px_{3n+1}, Qx_{3n+2}) \leq \gamma_1 G(\xi, x_{3n+1}, x_{3n+2}), \\
G(x_{3n+1}, P_\xi, x_{3n+2}) &= G(Tx_{3n}, P_\xi, Qx_{3n+2}) \leq \gamma_2 G(x_{3n}, \xi, x_{3n+1}), \\
G(x_{3n+1}, Q_\xi, P_\xi, x_{3n+2}) &= G(Tx_{3n}, Px_{3n+1}, Q_\xi) \leq \gamma_3 G(x_{3n}, x_{3n+1}, \xi),
\end{align*}
\]
where \(\gamma_1, \gamma_2\) and \(\gamma_3\) can easily be recovered from (2.8).

Since
\[
\lim_{n \to \infty} G(\xi, x_{3n+1}, x_{3n+2}) = \lim_{n \to \infty} G(x_{3n}, \xi, x_{3n+1}) = \lim_{n \to \infty} G(x_{3n}, x_{3n+1}, \xi) = G(\xi, x, \xi)^2 = 0,
\]
the relations (2.9), (2.10) and (2.11) respectively give that \(G(T_\xi, \xi, \xi) = 0\), \(G(\xi, P_\xi, \xi) = 0\) and \(G(\xi, \xi, Q_\xi) = 0\), i.e.
\[
T_\xi = P_\xi = Q_\xi = \xi.
\]
This completes the proof. □

Remark 2.10 The reader can convince him/her self that if we replace the condition (2.8) by
\[
G(Tx, Py, Qz) \leq \left(\frac{a G(x, Py, Qz) + b G(Tx, y, Qz) + c G(Tx, Py, z)}{d G(x, Tx, Ty) + e G(y, Py, Py) + f G(z, Qz, Qz) + 1}\right) G(x, y, z),
\]
where the non-negative functions \(a, b, c, d, e, f\) are well chosen, then \(P, Q\) and \(T\) have a common fixed point.

We conclude this article with the case of a family of mappings.

2.3. Families of maps

Here, in this last subsection of the manuscript, we consider the case of a family of functions and we state an analogue of Theorem 2.9.

We make use of the following special class \(\Phi\) of homogeneous functions. Let \(\Phi\) be the class of continuous, non-decreasing, sub-additive and homogeneous functions \(F: [0, \infty) \to [0, \infty)\) such that \(F^{-1}(0) = \{0\}\) and \(F(1) \leq 1\).

Theorem 2.11 Let \((X, G)\) be a symmetric G-complete G-metric space and \((T_n)\) be a family of self mappings on \(X\) such that
\[
F(G(T_n, x, y, z)) \leq F\left(\Delta_i \left(\begin{array}{c}
\alpha_i G(T_i, x, y, z) + \alpha_i G(x, T_i, y, z) + \alpha_i G(x, y, T_i, z) \\
(\alpha_i + \alpha_i) T_i^s + (\alpha_i + \alpha_i + 2 \alpha_i) T_i^r + \alpha_i T_i^s + 1
\end{array}\right)\right) F(G(x, y, z)),
\]
where \(\Delta_i := G(x, T_i, x, T_i)\) and \(\alpha_i := G(x, y, z)\) are non-negative functions, the constants \(\alpha_i\) are such that 0 \(\leq \alpha_i \leq 1\), \(i, j, k = 1, 2, \ldots\), and some \(T \in \Phi\) homogeneous with degree \(s\).
If
\[ \sum_{i=1}^{\infty} (\Delta x_{ij})^2. \]
is an \( a \)-series, then \( (Tn) \) has a common fixed point in \( X \).

**Proof**  For any \( x_0 \in X \), we construct the sequence \((x_n)_{n \geq 0}\) by setting \( x_n = T_n(x_{n-1}), n = 1, 2, \ldots \). We may assume without loss of generality that \( x_n \neq x_m \) for all \( n \neq m \in \mathbb{N} \). We observe that, by setting \( d_i = G(x_i, x_{i+1}, x_{i+1}), i \geq 1 \), and plugging in the triplet \((x_i, x_{i+1}, x_{i+1})\) we have

\[
F(G(x_i, x_{i+1}, x_{i+2})) = F(T_n x_{i-1}, T_n x_{i+1}, T_n x_{i+1})
\leq (\Delta x_{ij})^2 F(a_i) F(G(x_{i-1}, x_{i+1}, x_{i+1})),
\]

where

\[
a_i = \frac{(a_{i+1} + a_{i+2})d_{i+1} + (a_i + a_{i+1} + 2a_{i+2})d_i + a_{i+2}d_{i+1}}{(a_{i+1} + a_{i+2})d_{i+1} + (a_i + a_{i+1} + 2a_{i+2})d_i + a_{i+2}d_{i+1} + 1}.
\]

When we write the above for the triplet \((x_i, x_j, x_j)\), we obtain

\[
F(G(x_i, x_j, x_j)) \leq (\Delta x_{ij})^2 F(a_i) F(G(x_0, x_1, x_2)).
\]

Also we get

\[
F(G(x_j, x_j, x_j)) \leq (\Delta x_{ij})^2 F(a_i) F(G(x_i, x_1, x_2))
\leq (\Delta x_{ij})^2 (\Delta x_{ij})^2 F(a_i) F(G(x_0, x_1, x_2)).
\]

Hence, we derive, iteratively, that

\[
F(G(x_i, x_m, x_{m+2})) \leq \left( \prod_{n=1}^{m} \prod_{i=1}^{\infty} (\Delta x_{ij})^2 \right) \left[ \prod_{i=1}^{n} F(a_i) \right] F(G(x_0, x_1, x_2)).
\]

Therefore, for all \( l > m > n > 2 \), since

\[
G(x_n, x_m, x_j) \leq G(x_n, x_{m+1}, x_{m+1}) + G(x_{m+1}, x_{n+2}, x_{n+2}) + \ldots + G(x_{l-1}, x_{l-1}, x_{l-1})
\leq G(x_n, x_{m+1}, x_{m+1}) + G(x_{m+1}, x_{n+2}, x_{n+2}) + \ldots + G(x_{l-1}, x_{l-1}, x_{l-1}),
\]

using the fact that \( F \) is sub-additive, we write

\[
F(G(x_n, x_m, x_j)) \leq \left( \prod_{i=1}^{l} \prod_{i=1}^{\infty} (\Delta x_{ij})^2 \right) \left[ \prod_{i=1}^{n} F(a_i) \right] + \ldots + \left( \prod_{i=1}^{l-2} \prod_{i=1}^{\infty} (\Delta x_{ij})^2 \right) \left[ \prod_{i=1}^{l-2} F(a_i) \right] F(G(x_0, x_1, x_2))
\]

\[
= \left( \sum_{k=0}^{l-2} \prod_{i=1}^{\infty} (\Delta x_{ij})^2 \right) \left[ \prod_{i=1}^{n} F(a_i) \right] F(G(x_0, x_1, x_2))
\]

\[
= \left( \sum_{k=0}^{l-2} \prod_{i=1}^{\infty} (\Delta x_{ij})^2 \right) \left[ \prod_{i=1}^{n} F(a_i) \right] F(G(x_0, x_1, x_2)).
\]
We already know that the sequence \((a_n)_{n \in \mathbb{N}}\) is a sequence of non-negative reals and that for any natural number \(n \in \mathbb{N}, 0 \leq a_n < 1\). Therefore for any natural number \(n \in \mathbb{N}, k(a_n) \leq 1\). Hence
\[
\left[ \prod_{i=1}^{k} F(a_i) \right] \leq 1.
\]

Now, let \(\lambda\) and \(n(\lambda)\) as in Definition 1.9, then for \(n \geq n(\lambda)\) and using the fact that the geometric mean of non-negative real numbers is at most their arithmetic mean, it follows that
\[
F(G(x_n, x_m, x_l)) \leq \left( \frac{1-\lambda}{\lambda} \right)^k F(G(x_0, x_1, x_2))
\]
As \(n \rightarrow \infty\), we deduce that \(G(x_n, x_m, x_l) \rightarrow 0\). Thus \((x_n)_{n \geq 1}\) is a \(G\)-Cauchy sequence and since \(X\) is complete there exists \(x^* \in X\) such that \((x_n)_{n \geq 1}\) \(G\)-converges to \(x^*\).

Furthermore, for any \(i \geq 1\)
\[
F(G(T^i \xi, x_{i+1}, x_{i+2})) \leq \gamma \cdot F(G(\xi, x_i, x_{i+1})),
\]
for some \(\gamma \geq 0\). Now taking the limit using the fact that the function \(G\) is continuous, we obtain
\[
F(G(T^i \xi, x^*, x^*)) = 0, \quad \text{i.e.} \ T^i x^* = x^* \text{ for any } i \geq 1.
\]

This terminates the proof.

**Funding**
The author received no direct funding for this research.

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**Citation information**
Cite this article as: Fixed points of rational type contractions in \(G\)-metric spaces, Yaë Ulrich Gaba, Cogent Mathematics & Statistics (2018), 5: 1444904.

**Notes**
1. This means that the sequence of iterates
\[ T^n x_0, \quad n = 1, 2, ... \]
for any initial point \(x_0\) converges to a fixed and this fixed point might surely not be unique.
2. See Theorem 1.8

3. The function \(F(x) = \sqrt{x}\) is an example of such function.

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