Excision theory in the dihedral and reflexive (co)homology of algebras

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Abstract: In this paper, we study an excision theorem of the dihedral and reflexive (co)homology theory of associative algebras. That is, for such an extension, we obtain a six-term exact sequence in the dihedral cohomology. Also, we present and prove the relation between cyclic and dihedral cohomology of algebras and some examples.

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1. Introduction

Hochschild (co)homology is just a theory for associative algebras. Hochschild (1945) introduced simplicial cohomology for algebras over a field and Cartan and Eilenberg (1956) developed algebras over more general rings.

Cyclic (co)homology is a certain (co)homology theory for associative algebras for related branches of mathematics and non-commutative geometry which generalizes the de Rham (co)homology of manifolds. Tsygan (1983) introduced those notions independently for homology and Connes (1985) did it for cohomology. These invariants have many interesting relationships with many older mathematics branches, including the speculation of de Rham, group (co)homology, Hochschild (co)homology, and k-theory.

The hermitian equivalent of cyclic (co)homology is the dihedral (co)homology, independently proposed by Tsygan (1983) and Connes (1985) and proved method in variety of algebras. Dihedral homology of algebras over a field is introduced by Tsygan (1986), which is defined as the homology of the dihedral group in algebra $A$ Hochschild complex.

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PUBLIC INTEREST STATEMENT

The algebraic topology is very useful almost in all areas of applied mathematics that provides solutions to many problems in algebra, topology, differential geometry and quantum field; therefore, it has recently become a subject of interest for many authors in the field of medic, fluid dynamics, physics field and engineering for its applications. In this paper, we have evaluated the important theorem in the algebraic topology is excision theorem. We presented the theory of excision in the dihedral and reflexive (co)homology of algebras and its properties with some proven applications.
The dihedral (co)homology is referred to as (co)homology with group symmetry by Gouda and Alaa (2009). First, two groups of (co)homology theory are considered to exist: discreet and in-discrete. The Hochschild (co)homology of algebra with id in the discrete field is related to Hochschild (1945), Connes (1985) and Tsygan (1983) introduce the first nontrivial (co)homology group. In 1987, the dihedral and reflexive (co)homology of involutive algebra has been studied and in 1989 the remaining (co)homology groups have been studied. Johnson (1972) studied the analog-simplified cohomology of operator algebras. Gouda (2011), Helemskii (1992), and Helenskii (1991) studied the Banach cyclic (co)homology. Gouda (1997) studied the group Banach dihedral cohomology and relationship with cyclic cohomology, Gouda and Alaa (2009) studied the dihedral cohomology groups of some operator algebras. There is no progress calculating operator algebras group symmetry, bisimmetry, and Weil (co)homology, but the cohomology module $k$-module is studied by Gouda (2011).

The first to apply the excision theorem was Penner (2020). It is important for the excision property to study the simplicial trivialities properties for pure algebra and operator algebra. Lykova and Michael (1998) studied the excision property in simplicial cohomology $\mathcal{H}^n(A,A)$ and homology $\mathcal{H}_n(A,A)$ for short exact sequence $0 \to I \to A \to A/I \to 0$.

The notion of $H$-unitality for algebras has been introduced in 1989, and they conducted it to the short exact sequence $0 \to I \to A \to b \to 0$ for the excision in cyclic homology. The bivariant cyclic theory succeeded in the excision of nilpotent extensions because of the theorem by Goodwillie (1985).

The excision theory was developed in 2001 to include $\mathbb{Z}/2$-graded cyclic homology theories based on free extension, but it achieved the Wodzicki's approach. They have studied the Wodzicki's excision theory of simplicial homology and proven it for pure algebra with unital homology in category and they calculated them as an application to the continuous simplicial and cyclic (co)homology by Cortiñas and Valqui (2003).

In our paper, we introduce and study the excision theorem of the reflexive and dihedral (co)homology group of pure algebras. And, we introduce some new proven theorem in the excision theorem of a cyclic homology.

Our work consists of three sections as follows:

In Section 2, we introduce a mathematical review on the definition of Hochschild, cyclic, reflexive, and dihedral (co)homology of algebras.

In Section 3, we discuss some results on Hochschild and cyclic homology achievement of the excision property of $H$-unital algebras, excision of periodic cyclic homology, and excision of cyclic homology.

In Section 4, we provide the proven excision theorem of the dihedral (co)homology for short exact sequence $0 \to I \to A \to A/I \to 0$ as the form $\mathcal{H}D_n(I,A) \cong \mathcal{H}D_{n+1}(A)$ and $\mathcal{H}D^{n-1}(I,A) \cong \mathcal{H}D^n(A)\&\mathcal{H}D^{n-1}(K) \cong \mathcal{H}D^n(A/I).$ And for reflexive (co)homology, it is $\mathcal{H}R_n(I,A) \cong \mathcal{H}D_n(A)\&\mathcal{H}R^{n-1}(I,A) \cong \mathcal{H}R^n(A)$.

So, we prove the relations:

$$
\cdots \to \mathcal{H}R_n(A) \overset{\text{id}}{\longrightarrow} \mathcal{H}C_n(A) \overset{\partial}{\longrightarrow} \mathcal{H}C_{n-2}(A) \overset{\partial}{\longrightarrow} \mathcal{H}R_{n-1}(A) \to \cdots
$$

$$
\cdots \to \mathcal{H}R^n(A) \overset{\text{id}}{\longrightarrow} \mathcal{H}C^n(A) \overset{\partial}{\longrightarrow} \mathcal{H}C^{n+2}(A) \overset{\partial}{\longrightarrow} \mathcal{H}R^{n+1}(A) \to \cdots
$$
The results of the excision theorem of the dihedral cohomology equipped with the results of Intissar (2020) and Kostikov and Romanenkov (2020). Also, our results can introduce this application in a new form.

2. Mathematical review

We begin by briefly recalling the basic definitions concerning homology theory of algebras (the main references are Alaa, 2012; Alaa & Gouda, 2011; Guram & Manuel, 2014; J. Loday, 2013; Noreldeen, 2019).

Suppose that $\mathcal{A}$ is an associative unital algebra over $\mathcal{K}$ ring and $\mathcal{M}$ is bimodule over $\mathcal{A}$ with an involution $*: \mathcal{A} \rightarrow \mathcal{A}; a \rightarrow a^*$ for all $a \in \mathcal{A}$. We define a complex $\mathcal{C}(\mathcal{A}) = (\mathcal{C}_n(\mathcal{A}), b_n)$, since $\mathcal{C}_n(\mathcal{A}) = \mathcal{A}^{\otimes (\mathcal{K} \cup \infty)}$, $\{ \cdot \rightarrow \mathcal{C}_n(\mathcal{A}) \rightarrow \mathcal{C}_{n-1}(\mathcal{A}) \} \geq r$ is the boundary operator:

$$b_n(a_0, a_1, \ldots, a_n) = \sum_{i=0}^{n-1} (-1)^i(a_0, \ldots, a_{i+1}, a_i, \ldots, a_n) + (-1)^n(a_n, a_0, a_1, \ldots, a_{n-1}).$$

It is well known that $b_n b_{n+1} = 0$, and hence $\text{Im}(b_n) \subset \ker(b_n)$. Consider the following complex, called the Hochschild complex,

$$\mathcal{C}(\mathcal{A}, \mathcal{M}) := \cdots \rightarrow \mathcal{M} \otimes \mathcal{A}^{\otimes n} \overset{b}{\rightarrow} \mathcal{M} \otimes \mathcal{A}^{\otimes n-1} \overset{b}{\rightarrow} \cdots \rightarrow \mathcal{M} \otimes \mathcal{A} \overset{b}{\rightarrow} \mathcal{M},$$

and the Hochschild boundary:

$$b : \mathcal{M} \otimes \mathcal{A}^{\otimes n} \rightarrow \mathcal{M} \otimes \mathcal{A}^{\otimes n-1}$$

is the $\mathcal{K}$-linear map given by the formula:

$$b(m, a_1, \ldots, a_n) = (ma_1, a_2, \ldots, a_n) + \sum_{i=1}^{n} (-1)^i(m, a_1, \ldots, a_{i+1}, \ldots, a_n) + (-1)^n(a_n m, a_1, \ldots, a_{n-1}).$$

The following group is called the Hochschild homology of algebra $\mathcal{A}$:

$$\mathcal{H}_n(\mathcal{A}) = (\mathcal{H}_n(\mathcal{C}(\mathcal{A}))) = \frac{\ker(b_n)}{\text{Im}(b_{n+1})},$$

and denoted by $\mathcal{H}_n(\mathcal{A})$. The enveloping of algebra in $\mathcal{A}$ is the tensor product $\mathcal{A}^{\mathcal{E}} = \mathcal{A} \otimes \mathcal{A}\text{opp}$ of $\mathcal{A}$ with its opposite algebra. In the work by Krasauskas et al. (1988), the simplicial (co)homology of $\mathcal{A}$ with coefficients in $\mathcal{M}$ in terms of the functors (Tor) and (Ext) is defined by:

$$\mathcal{H}_n(\mathcal{A}, \mathcal{M}) = (\text{Tor}_n(\mathcal{A}, \mathcal{M}), \mathcal{H}_n(\mathcal{A}, \mathcal{M}), \mathcal{H}_n(\mathcal{A}, \mathcal{M}) = (\text{Ext}_n(\mathcal{A}, \mathcal{M}).$$

We act on the complex $\mathcal{C}(\mathcal{A})$ by the cyclic order group $(n + 1)$ through the cyclic operator $t_n : \mathcal{C}_n(\mathcal{A}) \rightarrow \mathcal{C}_n(\mathcal{A})$ since,

$$t_n(a_0, \ldots, a_{n-1}, a_n) = (-1)^n(a_0, a_0, \ldots, a_{n-1}). \quad (1)$$

The complex $\mathcal{C}_n(\mathcal{A}) = (\mathcal{C}_n(\mathcal{A}), t_n)$ is a sub-complex of $\mathcal{C}(\mathcal{A})$. From the work by Helemskii (1991), the homology of the complex $\mathcal{C}_n(\mathcal{A})$ is called the cyclic homology of algebra $\mathcal{A}$, and denoted by:

$$\mathcal{H}_n(\mathcal{A}) = \mathcal{H}_n(\mathcal{C}_n(\mathcal{A}), t_n) = \mathcal{H}_n \left( \mathcal{C}_n(\mathcal{A}) \bigg/ \mathcal{Z} \right).$$

We act on a complex $\mathcal{C}(\mathcal{A})$ by the reflexive group $\mathbb{Z}/2 = \{-1, +1\}$ of order 2 by means of the reflexive operator $r_n : \mathcal{C}_n(\mathcal{A}) \rightarrow \mathcal{C}_n(\mathcal{A})$ where,

$$r_n(a_0, \ldots, a_{n-1}, a_n) = a(-1)^{(n+1)/2}(a_0, a_1, \ldots, a_n). \quad (2)$$
where \( a = \pm 1, a^2 = 1, (r_n)^2 = 1 \) and \( a^*_n = \text{Im}(a^n) \) under the involution \( * \). If \( \Lambda \) is a category, then another definition of cyclic (co)homology is:

\[
\mathcal{H}_\Lambda^{\alpha}(M) = \text{Tor}^{\Lambda}_{n}(\mathcal{K}_{\Lambda}^{C}, \mathcal{M}), \mathcal{H}^{\alpha}(M) = \text{Ext}^{\alpha}_{\text{gr}_{\Lambda}}(M, \mathcal{K}_{\Lambda}^{C}), n \geq 0.
\]

where each \( \mathcal{K} \)-algebra \( \mathcal{A} \) the cyclic \( \mathcal{K} \)-module.

The complex \( \alpha CR_n(\mathcal{A}) = \frac{\mathcal{C}_n(A)}{\text{Im}(1-r_n)} \) is sub-complex of \( \mathcal{C}_n(A) \). From the work by Alaa (2019), the homology of the complex \( \alpha CR_n(\mathcal{A}) \) is called the reflexive homology of algebra \( \mathcal{A} \), and denoted by:

\[
\alpha \mathcal{H}R_n(\mathcal{A}) = \mathcal{H}_n(\mathcal{CR}^{\ast}(\mathcal{A}), b^{\ast}) = \mathcal{H}_n\left(\frac{\mathcal{C}_n(A)}{\text{Im}(1-r_n)}, b^{\ast}\right).
\]

If we use Equations (1) and (2) together on \( \mathcal{C}(A) \), we have the complex \( \alpha CD_n(\mathcal{A}) = \left(\frac{\mathcal{C}_n(A)}{\text{Im}(1-r_n)}, b^{\ast}\right) \), which is the sub-complex of \( \mathcal{C}_n(A) \).

From the work by Tsygan (1986), the homology of a complex \( \alpha CD_n(\mathcal{A}) \) is called dihedral homology of algebra \( \mathcal{A} \), and denoted by:

\[
\alpha \mathcal{H}D_n = \mathcal{H}_n(\mathcal{CD}^{\ast}(\mathcal{A}), b^{\ast}) = \mathcal{H}_n\left(\frac{\mathcal{C}_n(A)}{\text{Im}(1-r_n)}, b^{\ast}\right).
\]

Another definition of dihedral (co)homology (J-L. Loday, 1998) is

\[
\mathcal{H}D_n(\mathcal{M}) = \text{Tor}^{\text{gr}_{\mathcal{M}}}_{n}(\mathcal{K}^{D}, \mathcal{M}), \mathcal{H}D^{\alpha}(\mathcal{M}) = \text{Ext}^{\alpha}_{\text{gr}_{\mathcal{M}}}(\mathcal{M}, \mathcal{K}^{D}), n \geq 0.
\]

**Definition (2–1):**

Let \( \mathcal{A} \) be \( \mathcal{K} \)-algebra and \( I \)-ideal where \( \mathcal{A} \rightarrow \mathcal{A}/I \) is \( \mathcal{K} \)-split, then there exists the map of the relative homology ((co)homology) for \( \mathcal{A} \) modulo \( I \):

\[
e : \mathcal{H}_n(I) \rightarrow \mathcal{H}_n(\mathcal{A}, I), e : \mathcal{H}^{\alpha}(I) \rightarrow \mathcal{H}^{\alpha}(\mathcal{A}, I).
\]

The ideal \( I \) is said to be excision of simplicial homology (cohomology) if a map is an isomorphism (Cartan & Eilenberg, 1956). Then the sequence:

\[
\cdots \rightarrow \mathcal{H}_n(I) \rightarrow \mathcal{H}_n(\mathcal{A}) \rightarrow \mathcal{H}_n(\mathcal{A}/I) \rightarrow \mathcal{H}_{n-1}(I) \rightarrow \cdots,
\]

\[
\cdots \rightarrow \mathcal{H}^{\alpha}(I) \rightarrow \mathcal{H}^{\alpha}(\mathcal{A}) \rightarrow \mathcal{H}^{\alpha}(\mathcal{A}/I) \rightarrow \mathcal{H}^{\alpha+1}(I) \rightarrow \cdots
\]

is exact.

**Definition (2–2):**

For \( \mathcal{K} \)-split sequence \( \mathcal{A} \rightarrow \mathcal{A}/I \) where \( \mathcal{A} \) be \( \mathcal{K} \)-algebra and \( I \)-ideal, map of relative homology (cohomology) for \( \mathcal{A} \) modulo \( I \) respect is:

\[
j : \mathcal{C}C_n(I) \rightarrow \mathcal{C}C_n(\mathcal{A}/I), j : \mathcal{C}C^{\alpha}(I) \rightarrow \mathcal{C}C^{\alpha}(\mathcal{A}/I).
\]

The excision of the cyclic homology (cohomology) is the ideal \( I \) if the map is an isomorphism (Cartan & Eilenberg, 1956). The sequences

\[
\cdots \rightarrow \mathcal{H}_n(I) \rightarrow \mathcal{H}_n(\mathcal{A}) \rightarrow \mathcal{H}_n(\mathcal{A}/I) \rightarrow \mathcal{H}_{n-1}(I) \rightarrow \cdots,
\]

\[
\cdots \rightarrow \mathcal{H}^{\alpha}(I) \rightarrow \mathcal{H}^{\alpha}(\mathcal{A}) \rightarrow \mathcal{H}^{\alpha}(\mathcal{A}/I) \rightarrow \mathcal{H}^{\alpha+1}(I) \rightarrow \cdots
\]
are exact.

**Theorem (2–3):**

The periodicity exact sequence of the cyclic module \( C \), is

\[
\cdots \rightarrow \mathcal{H}_n(C) \xrightarrow{I} \mathcal{H}_{n-1}(C) \xrightarrow{\delta} \mathcal{H}_{n-2}(C) \xrightarrow{B} \mathcal{H}_{n-1}(C) \rightarrow \cdots
\]

where the map \( I \) is inserted, the simplicial complex for \( C \), becomes bicomplex \( C \). If \( C_n = \mathcal{A}^{op} \), the periodicity exact sequence of the cyclic sequence takes the form (see J-L. Loday, 1998).

\[
\cdots \rightarrow \mathcal{H}_n(\mathcal{A}) \xrightarrow{I} \mathcal{H}_{n-1}(\mathcal{A}) \xrightarrow{\delta} \mathcal{H}_{n-2}(\mathcal{A}) \xrightarrow{B} \mathcal{H}_{n-1}(\mathcal{A}) \rightarrow \cdots
\]

**Corollary (2–4):**

There is a natural long exact sequence for any algebra \( \mathcal{A} \) over the ring \( K \), which contains \( \mathbb{Q} \)

\[
\cdots \rightarrow \mathcal{H}_n(\mathcal{A}) \xrightarrow{I} \mathcal{H}_{n-1}(\mathcal{A}) \xrightarrow{\delta} \mathcal{H}_{n-2}(\mathcal{A}) \xrightarrow{B} \mathcal{H}_{n-1}(\mathcal{A}) \rightarrow \cdots
\]

**Theorem (2–5):**

There are long exact sequences, called exact periodicity sequences of Connes:

\[
\cdots \rightarrow \mathcal{H}^n(\mathcal{A}) \xrightarrow{I} \mathcal{H}^{n+1}(\mathcal{A}) \xrightarrow{\delta} \mathcal{H}^{n+2}(\mathcal{A}) \rightarrow \cdots
\]

\[
\cdots \rightarrow \mathcal{H}^{n+1}(\mathcal{A}) \xrightarrow{I} \mathcal{H}^{n+2}(\mathcal{A}) \xrightarrow{\delta} \mathcal{H}^{n+3}(\mathcal{A}) \rightarrow \cdots
\]

**Definition (2–7):**

Let \( \mathcal{A} \) be an involutive algebra over \( K \). Then dihedral homology of \( \mathcal{A} \) is

\[
\mathcal{H}^D_n(\mathcal{A}) = \mathcal{H}_n(\text{Tot} \mathcal{C}^+ (\mathcal{A}))
\]

and we get

\[
\mathcal{H}_n(\mathcal{A}) = \mathcal{H}^D_n(\mathcal{A}) \oplus \mathcal{H}_n(\mathcal{A})
\]

Connes’ exact periodicity sequence into the direct sum of

\[
\cdots \rightarrow \mathcal{H}^D_n(\mathcal{A}) \rightarrow \mathcal{H}^D_{n-1}(\mathcal{A}) \rightarrow \mathcal{H}^D_{n-2}(\mathcal{A}) \rightarrow \cdots
\]

and

\[
\cdots \rightarrow \mathcal{H}^D_n(\mathcal{A}) \rightarrow \mathcal{H}^D_{n-1}(\mathcal{A}) \rightarrow \mathcal{H}^D_{n-2}(\mathcal{A}) \rightarrow \cdots
\]

**Corollary (2–8):**

Suppose that \( K \) be a field of characteristic zero with a trivial involution. Then

1. \( \mathcal{H}^D_n(\mathcal{K}) = \{ -i \nabla \nabla \cdot | \mathcal{K} \} \)
2. \( \mathcal{H}^D_n(\mathcal{K}) = \{ -i \nabla \nabla \cdot | \mathcal{K} \} \)
3. \( \mathcal{H}^D_n(\mathcal{K}) = \{ K, n=0(\text{mod}4), \text{otherwise} \} \)
4. \( \mathcal{H}^D_n(\mathcal{K}) = \{ K, n=2(\text{mod}4), \text{otherwise} \} \)

In the following section, we will show previous studies of excision theorems in the Hochschild and cyclic (co)homologies of associative algebras. We will also explain some results and examples related to previous studies.
3. Excision in simplicial and cyclic (co)homology

In this part, we introduce some properties and theorems of the Hochschild and cyclic (co)homologies of associative algebras by Buchholtz and Rijke (2019), Quillen (1972), Ralf (2010), Thiel (2006), and Wodzicki (1989). We discuss and study some special theories of excision theorem of simplicial and cyclic (co)homology theory in pure algebras.

**Definition (3–1):**

Let \( A \) be a \( C \)-algebra and \( M \) is right \( A \)-module, \( A \) and \( M \) are unital homologically, if the chain complex \( (A^n, b')_{n \geq 0} \) and \( (M \otimes A^n, b')_{n \geq 0} \) are exact. The same definition is for the left modules. By definition, \( A \) is unital homology algebra if and only if it is unital homology (Krasauskas et al., 1988). Therefore, \( M \) is unital homologically if

\[
\mathcal{H}_n(\mathcal{H}_*(A, M \otimes V)) = 0.
\]

Let abelian category \( C \) be with extensions, then a chain complex is true if its homology vanishes. In this case, \( M \) is unital homologically if

\[
\mathcal{H}_*(A, M \otimes V) = 0.
\]

In general, \( H \)-unital is unrelated to the vanishing of \( \mathcal{H}_*(A, M \otimes V) \) (for more details, see Gouda & Alaa, 2009).

**Lemma (3–2):**

Suppose \( I \rightarrow E \rightarrow Q \) is algebra and \( M \) is unital homology \( J \)-module. Then we find that the \( E \)-module structure is only a structure extended from \( I \)-module structure.

**Theorem (3–3):**

Suppose \( I \rightarrow E \rightarrow Q \) is conflation algebra and \( M \) is \( E, I \)-bi-module. Taking \( M \) is \( E \)-bimodule and \( I \)-module unital homology, and then the map \( \mathcal{H}_*(I, M) \rightarrow \mathcal{H}_*(E, M) \) is quasi-isomorphism. Then we have \( \mathcal{H}_*(I, M \otimes V) \cong \mathcal{H}_*(E, M \otimes V), \forall V \).

**Proof:**

For \( p \in \mathbb{N} \), let \( F_p \) be a complex

\[
\cdots \cdots \rightarrow 0 \rightarrow M \otimes V \overset{b}{\rightarrow} M \otimes E \otimes V \overset{\delta}{\rightarrow} M \otimes E \otimes V \otimes V \overset{b}{\rightarrow} \cdots
\]

\[
\rightarrow M \otimes E^{op} \otimes V \otimes V \overset{\delta}{\rightarrow} M \otimes I \otimes E^{op} \otimes V \overset{b}{\rightarrow} M \otimes I \otimes E^{op} \otimes V \otimes V \overset{b}{\rightarrow} \cdots
\]

With \( M \otimes V \) with zero degree, since \( I \rightarrow E \rightarrow Q \) is pure, and then \( M \otimes E^{op} \otimes V \rightarrow M \otimes I^{op} \rightarrow E^{op} \otimes V \) is inflation \( \forall K, p \geq 0 \). Hence, from Tsygan (1983), the canonical map \( F_p \rightarrow F_{p+1} \) is inflation \( \forall p \). Its cokernel is the chain complex

\[
F_{p+1}/F_p \cong \left( M \otimes I^{op}, b' \right)_{K \geq 0} \otimes \mathbb{Q} \rightarrow E^{op} \otimes V
\]

where \( |p + 1| \) denotes translation by \( p + 1 \). This chain complex is exact because \( M \) is homologically unital as a right \( I \)-module. Since \( F_p \rightarrow F_{p+1} \) is conflation, then the map \( F_p \rightarrow F_{p+1} \) is quasi-isomorphism by Lemma (3–2) and J. Loday (2013). Thus the inclusion \( F_0 \rightarrow F_p \forall p \in \mathbb{N} \). For \( p = 0 \), we get \( F_0 = \mathcal{H}_*(I, M \otimes V) \). In any fixed degree \( n \), we have

\[
(F_p)_n = \mathcal{H}_*(E, M) \otimes \mathbb{Q} \wedge 0
\]

Hence, the canonical map \( \mathcal{H}_n(I, M) \rightarrow \mathcal{H}_n(E, M) \) is a pure quasi-isomorphism.

**Corollary (3–4):**

Consider the pure algebra conflation \( I \rightarrow E \rightarrow Q \) and unital homology \( I \), then

\[
\mathcal{H}_n(I, I) \rightarrow \mathcal{H}_n(E, I) \quad \mathcal{H}_n(I, I) \rightarrow \mathcal{H}_n(I, I)
\]

\[
(I^{op}, b) \rightarrow (I \otimes E^{op}, b) \quad (I^{op}, b) \rightarrow (\Omega(I), b)
\]
are quasi-isomorphisms. The unital extension $I \to E$ leads to a quasi-isomorphisms $\mathcal{H}(I) \to \mathcal{H}_{\mathcal{E}}(I)$ where $E$ is unital. Thus, $\mathcal{H}(E) \to \mathcal{H}_{\mathcal{E}}(I)$ is invertible provided $1$ is projective. Recall that $\Omega^n(I) = I^n \otimes I^\otimes \forall n \geq 1$ and $\Omega^0(I) = I$.

**Theorem (3–5):**

Consider the pure algebra conflation $I \to E \to Q$ and $Q$-bimodule $M$; then, $M$ is $E$-bimodule, where $I$ is unital homology. We get $\mathcal{H}(E) \to \mathcal{H}_Q(M)$ which is quasi-isomorphism. Thus, $\mathcal{H}_Q(M \otimes V) \cong \mathcal{H}_Q(M \otimes V)$ provided $1$ is projective.

**Theorem (3–6):**

Consider the pure algebra conflation $I \to E \to Q$ and $I$ is unital homology and $M_I \to M_E \to M_Q$ is pure conflation of $E$-bimodules. Taking the structure of $E$-bimodule on $M_Q$ come down to the structure of $Q$-bimodule and $M_I$ is unital homology $aE$-module. Then:

$$\mathcal{H}(I, M_I) \to \mathcal{H}(E, M_E) \to \mathcal{H}(Q, M_Q)$$

is sequence, where $I$ is projective. This results in a long accurate natural sequence

$$\cdots \to \mathcal{H}(I, M_I) \to \mathcal{H}(E, M_E) \to \mathcal{H}(Q, M_Q) \to \mathcal{H}(I, M_{I-1}) \to \mathcal{H}(E, M_{E-1})$$

Thus, the following is a pure sequence as well

$$\mathcal{H}(I, M_I) \to \mathcal{H}(E, M_E) \to \mathcal{H}(Q, M_Q)$$

**Proof:**

From theorem (3–4), the map $\mathcal{H}(I, M_I) \to \mathcal{H}(E, M_E)$ is quasi-isomorphism for $M_I$ is unital homology as $I$-module. From theorem (3–5), the map $\mathcal{H}(E, M_E) \to \mathcal{H}(Q, M_Q)$ is quasi-isomorphism since $I$ is unital homology (Ralf, 2010).

The sequence $\mathcal{H}(E, M_E) \to \mathcal{H}(E, M_{E-1}) \to \mathcal{H}(Q, M_Q)$ is pure conflation since $M_E \to M_Q$ is pure. Then the following is a pure sequence as well

$$\mathcal{H}(I, M_I) \to \mathcal{H}(E, M_E) \to \mathcal{H}(Q, M_Q)$$

If we have short exact sequence of algebras with bijective homomorphism in unital homology, we get the long exact sequence in the Hochschild homology theory obtained in the following theorem.

**Theorem (3–7):**

Suppose that $I \to E \to Q$ is pure of $E$-algebras and $I$ is unital homology. Then,

$$\mathcal{H}(I) \to \mathcal{H}(E) \to \mathcal{H}(Q)$$

is a sequence. If $1$ is projective and injective in $C$, then it produces an exact long sequence

$$\cdots \to \mathcal{H}(I) \to \mathcal{H}(E) \to \mathcal{H}(Q) \to \mathcal{H}(I, M_{I-1}) \to \mathcal{H}(E, M_{E-1}) \to \cdots$$

**Theorem (3–8):**

Consider the pure algebra conflation $I \to E \to Q$ and $M$ is $Q$-bimodule and $E$-bimodule. Let $I$ be unital homology and $M$ is injective. So $\mathcal{H}(Q, M) \to \mathcal{H}(E, M)$ is quasi-isomorphism; then, $\mathcal{H}(Q, M) \cong \mathcal{H}(E, M)$.

**Proof:**

Let $F_p \forall p \geq 0$ be the co-chain complex

$$\mathcal{H}(Q, \otimes E, M) \leftarrow \mathcal{H}_3(Q, \otimes E, M) \leftarrow \mathcal{H}_2(Q, \otimes E, M) \leftarrow \mathcal{H}_1(Q, \otimes E, M) \leftarrow \mathcal{H}_0(Q, \otimes E, M) \leftarrow \cdots$$

$$\cdots \leftarrow \mathcal{H}(Q, \otimes E, M) \leftarrow \mathcal{H}_3(Q, \otimes E, M) \leftarrow \mathcal{H}_2(Q, \otimes E, M) \leftarrow \mathcal{H}_1(Q, \otimes E, M) \leftarrow \mathcal{H}_0(Q, \otimes E, M) \leftarrow \cdots$$

$$\cdots \leftarrow \mathcal{H}(Q, \otimes E, M) \leftarrow \mathcal{H}_3(Q, \otimes E, M) \leftarrow \mathcal{H}_2(Q, \otimes E, M) \leftarrow \mathcal{H}_1(Q, \otimes E, M) \leftarrow \mathcal{H}_0(Q, \otimes E, M) \leftarrow \cdots$$

$$\cdots \leftarrow \mathcal{H}(Q, \otimes E, M) \leftarrow \mathcal{H}_3(Q, \otimes E, M) \leftarrow \mathcal{H}_2(Q, \otimes E, M) \leftarrow \mathcal{H}_1(Q, \otimes E, M) \leftarrow \mathcal{H}_0(Q, \otimes E, M) \leftarrow \cdots$$
where $b'$ is the co-boundary map which uses the right $E$-module $Q \otimes E \to Q$ structure and $M$-bimodule structure. For the pure algebra conflation and injective $M$ of $C$, we get

$$F_p \to F_{p+1} \to (\text{Hom}(Q^{op} \otimes I \otimes E^{op} [p + 1], M), (b')^*)$$

(7)

Theorem (3–3) implies that the chain complex $V \otimes (I \otimes E^{op}, b')$ is exact $\forall V$ since $I$ is unital homology, where $M$ is injective and the exact complex in Equation (7). From Theorem (3–2) and Gouda and Alaa (2009), we find that $F_p \to F_{p+1}$ is quasi-isomorphism; then, $F_0 \to F_p \forall p \in \mathbb{N}$. This yields the assertion because

$$F_0 = \mathcal{H}^0(E, M) \cong \mathcal{H}^0(Q, M) \forall p \geq 0$$

Corollary (3–9):

Consider the monoidal category $C$ provided with split extensions class. For $Q$-bimodule $M$, split extensions $I \to E \to \mathbb{Q}$ in $C$ and $(I^{op}, b')$ is exact, then

$$\mathcal{H}^n(E, M) \cong \mathcal{H}^n(Q, M)$$

Theorem (3–10):

For $\bar{C}$, if $I \to E \to \mathbb{Q}$ is split extension. Then the maps $\mathcal{H}(I) \to \mathcal{H}(E) \to \mathcal{H}(Q)$ form the cofiber sequence. Also, the results in cyclic series for $\mathcal{H}P$ and $\mathcal{H}Q$, if $\mathcal{H}(I), \mathcal{H}(E)$ and $\mathcal{H}(Q)$ are injective in $\bar{C}$.

Theorem (3–11):

Let $0 \to I \to A \to A/I \to 0$ be an extension of $K$-algebra, if $I$ is $H$-unital. Then we get

$$\cdots \to \mathcal{H}C_n(A) \to \mathcal{H}C_n(A/I) \to \mathcal{H}C_{n-1}(I) \to \cdots$$

Proof

There is a well-defined functional map for the Hochschild homology $\rho : \mathcal{H}C_n(A) \to \mathcal{H}C_n(A, I)$. In the other hand, it is immediate from the construction of $\mathcal{H}C_n(I)$ that in the framework of non-unital algebras there is a long exact sequence of Connes. From the work by Guram and Manuel (2014), we consider the exact rows commutative diagram:

$$\begin{array}{cccccc}
\cdots & \to & \mathcal{H}C_n(I) & \to & \mathcal{H}C_n(A) & \to \mathcal{H}C_{n-1}(I) & \to \\
\cdots & \to & \mathcal{H}C_n(I) & \to & \mathcal{H}C_n(A) & \to \mathcal{H}C_{n-1}(I) & \to \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \to & \mathcal{H}C_n(I) & \to & \mathcal{H}C_n(A) & \to \mathcal{H}C_{n-1}(I) & \to \cdots
\end{array}$$

We know that $\mathcal{H}C_n(I) \to \mathcal{H}C_n(A, I)$ is an isomorphism when $I$ is $H$-unital which implies that $\mathcal{H}C_n(I) \to \mathcal{H}C_n(A, I)$ is an isomorphism $\forall n, n \in \mathbb{Z}$.

Theorem (3–12):

For $A$ algebra over $K$ and containing $Q$, the following map is an isomorphism

$$p_* : \mathcal{H}C_*(A) \to \mathcal{H}C_*(A)$$

Proof:

We can define the homotopy as

$$h' = 1/(n + 1).id, h'' = -\left(1/(n + 1)\right) \sum_{i=1}^{n} \bar{h}^i$$

which maps from $C_n(A)$ to itself (Cartan & Eilenberg, 1956). One verifies that $h'N + (1 - t)h = id, Nh' + h(1 - t) = id$.

This satisfies that $h_0 = c_0^n(A)$ and the homology of $C_*(A)$ is canonically isomorphic to $C^n_*(A)$.

In the next part, we will show the very important idea of $H$-unitary algebra put by Wodzicki.

Definition (3–13):

Consider the $K$-algebra $I$ and $I$-bimodule $M$. Then, $M$ is $H$-unitary if $(M \otimes I^{op}, b^*) \otimes V$ is
exact. When $\mathcal{M} = \mathcal{I}$, where $\mathcal{I}$ is $H$-unital, taking $\mathcal{M}$ as $\mathcal{I}$-module, we get $\mathcal{M} \otimes \mathcal{I}$ is $H$-unital for $\mathcal{I}$ is $H$-unital.

**Theorem (3–14):**

Suppose that $0 \to \mathcal{I} \to A \to B \to 0$ is a pure extension of $\mathcal{K}$-algebras, $\mathcal{M}$ on $\mathcal{A}$-bimodule and $\mathcal{K}$-module $\mathcal{V}$. For $H$-unital and $I$-bimodule $\mathcal{M}$, then the canonical inclusion $i: (\mathcal{M} \otimes I^{\oplus}, b_{\mathcal{I}}) \otimes \mathcal{V} \to (\mathcal{M} \otimes A^{\oplus}, b_{\mathcal{I}}) \otimes \mathcal{V}$ is quasi-isomorphisms.

**Proof:** see Lykova and Michael (1998).

**Corollary (3–15):**

Suppose that $0 \to \mathcal{I} \to A \to B \to 0$ is an extension of pure $\mathcal{K}$-algebra and the $k$-module $\mathcal{V}$, for $\mathcal{I}$-unital, we have $\alpha : (B \otimes A^{\oplus}, b_{\mathcal{I}}) \otimes \mathcal{V} \to (B \otimes B^{\oplus}, b_{\mathcal{I}}) \otimes \mathcal{V}$.

**Theorem (3–16):**

Suppose that $\mathcal{I}$ is $\mathcal{K}$-algebra. The statements below are equivalent:

(1) $\mathcal{I}$ is $H$-unital,

(2) $\mathcal{I}$ satisfies excision for $\mathcal{H}$-homology,

(3) $\mathcal{I}$ satisfies excision for Hochschild homology,

(4) $\mathcal{I}$ satisfies excision for cyclic homology.

**Proof:**

(1) $\Rightarrow$ (2): Take $0 \to \mathcal{I} \to A \to B \to 0$ is a pure extension of $\mathcal{K}$-algebras, $\mathcal{V}$ and $\mathcal{K}$-module and $\omega : (A \otimes A^{\oplus}, b_{\mathcal{I}}) \otimes \mathcal{V} \to (B \otimes B^{\oplus}, b_{\mathcal{I}}) \otimes \mathcal{V}$ the projection canonical. Taking the diagram following for short exact commutation sequences:

$\begin{align*}
0 & \to (I \otimes A^{\oplus}, b_{\mathcal{I}}) \otimes \mathcal{V} \to (A \otimes A^{\oplus}, b_{\mathcal{I}}) \otimes \mathcal{V} \to (B \otimes B^{\oplus}, b_{\mathcal{I}}) \otimes \mathcal{V} \to 0 \\
0 & \to \ker(\omega) \to (A \otimes A^{\oplus}, b_{\mathcal{I}}) \otimes \mathcal{V} \to (B \otimes B^{\oplus}, b_{\mathcal{I}}) \otimes \mathcal{V} \to 0
\end{align*}$

By corollary (3–15), $\omega_{1}$ is a quasi-isomorphism and also $\mathcal{J}$. Using theorem (3–14), we complete the proof.

(2) $\Rightarrow$ (4): The long exact series

\[ \cdots \to \mathcal{H}_{n-1}(I) \to \mathcal{H}R_{n-1}(I) \to \mathcal{H}R_{n}(I) \to \mathcal{H}R_{n}(I) \to \mathcal{H}R_{n}(I) \to \mathcal{H}R_{n+1}(I) \to \cdots \]

makes this simple consequence.

(3) $\Leftrightarrow$ (4): Immediately follows the exact sequence from the Gysin-connes. (Note: The Gysin sequence is a long, exact sequence not only for differential forms cohomology de Rham but also for integral coefficients cohomology).

(2) $\Rightarrow$ (1): Given $\mathcal{K}$-module $\mathcal{V}$, consider the $\mathcal{K}$-algebra $A = I \otimes \mathcal{V}$ with the product given by $(u, v)(u', v') = (uu', 0)$ and the canonical projection $\omega : (A \otimes A^{\oplus}, b_{\mathcal{I}}) \to (\mathcal{V} \otimes \mathcal{V}^{\oplus}, b_{\mathcal{I}})$.

The complex $\mathcal{V} \otimes (I \otimes I^{\oplus}, b_{\mathcal{I}}) \otimes (I \otimes I^{\oplus}, b_{\mathcal{I}})$ is a simple summand for $\ker(\omega)$ and $\mathcal{I}$ satisfies $\mathcal{H}$-homology excision. Then $\mathcal{V} \otimes (I \otimes I^{\oplus}, b_{\mathcal{I}})$ is exact.

(3) $\Rightarrow$ (1): let $\mathcal{V}$ and $\mathcal{A}$ be as in (2) $\Rightarrow$ (1). $\omega : C_{\ast}(\mathcal{A}) \to C_{\ast}(\mathcal{V})$, the projection canonical and $\mathcal{V}$ is the sub-complex of $\ker(\omega)$ generated by $(a_{0} \otimes \cdots \otimes a_{n}, a'_{0} \otimes \cdots \otimes a'_{n}, \cdots)$ with some $a_{i}$ and some $a'_{i}$. Since $\ker(\omega) = C_{\ast}(\mathcal{I}) \otimes \mathcal{V}$ and $\mathcal{I}$ satisfies Hochschild homology excision, $\mathcal{I}$ is exact. Let $\mathcal{I}$ be not $H$-unital. Taking $x \in \mathcal{V} \otimes I^{\oplus}$ is cycle for $b_{\mathcal{I}}$ that is not a boundary. Of note that $N(x)$ is cycle for $n + 1$ degree in $\mathcal{B}$ that is un-boundary; it is a contradiction with the exactness of $\mathcal{B}$.

In the next section, we will give the main results of this paper. We prove the relations between the cyclic and dihedral (co)homology in algebra, from which we will prove the excision theorems of reflexive and dihedral (co)homology theory as a new result.
4. Excision in the dihedral and reflexive cohomology of algebras

In this part, we introduce the main result in our paper. We prove the relation between the cyclic and dihedral homology of algebras and the relation between the reflexive and dihedral cohomology of algebras in theorems (4–8) and (4–9). We prove the excision property of the dihedral and reflexive cohomology of pure algebras in theorems (4–11) and (4–12). We use references Buchholtz and Rijke (2019), Cortiñas & Valqui (2003), Penner (2020), Quillen (1972) to study the property of excision theorem.

Suppose $\mathcal{A}$ is an involutive algebra over $\mathbb{K}$-field. We denote by $\mathcal{C}^n(\mathcal{A})$ the duality of $n$-chains. We know that the complex $(\mathcal{C}^n(\mathcal{A}), d, n)$ is chain complex, that is $d^2 = 0$, where $d = \sum_{k=1}^{n} (-1)^k \delta^k, n = 1, 2, \ldots, \delta^n : \mathcal{C}^n(\mathcal{A}) \rightarrow \mathcal{C}^{n+1}(\mathcal{A})$.

The operators

$r_n, t_n : \mathcal{C}^n(\mathcal{A}) \rightarrow \mathcal{C}^n(\mathcal{A})$\n
where

$t_n(\lambda)(a_0, a_1, \cdots, a_n) = (-1)^n \lambda(a_n, a_1, \cdots, a_0)$

$r_n(\lambda)(a_0, a_1, \cdots, a_n) = (-1)^{n(n+1)/2} \lambda(a_0, a_n, a_{n-1}, \cdots, a_1)$

where $a_1^\ast$ is the image of the element $a_1$ under an involution $\ast$, we get sub-complex

$\mathcal{C}D^n(\mathcal{A}) = \{ \lambda \in \mathcal{C}^n(\mathcal{A}), t_n \lambda = \lambda, t_n \lambda = a \lambda, \lambda = \pm 1 \}$

of the complex $\mathcal{C}^n(\mathcal{A})$ which is invariant under the operator $\delta$. The cohomology of this complex is called the dihedral cohomology of $\mathcal{A}$ and denoted by $\mathcal{H}D^n(\mathcal{A}) = \pm 1$.

**Theorem (4–1):**

If $\frac{1}{2} \in \mathbb{K}$, then there are the natural isomorphisms:

$\mathcal{H}C_n(\mathcal{A}) \cong \mathcal{H}D_n(\mathcal{A}) \oplus \mathcal{H}D_n(\mathcal{A})$

$\mathcal{H}C^n(\mathcal{A}) \cong \mathcal{H}D^n(\mathcal{A}) \oplus \mathcal{H}D^n(\mathcal{A})$

Then exact sequence

$\cdots \rightarrow \mathcal{H}D_{n+1}(\mathcal{A}) \rightarrow \mathcal{H}C_n(\mathcal{A}) \rightarrow \mathcal{H}D_n(\mathcal{A}) \rightarrow \mathcal{H}D_{n-1}(\mathcal{A}) \rightarrow \cdots$

$\cdots \rightarrow \mathcal{H}C_n(\mathcal{A}) \rightarrow \mathcal{H}D_n(\mathcal{A}) \rightarrow \mathcal{H}C_{n-1}(\mathcal{A}) \rightarrow \cdots$

can put it in a commutative diagram.

Let bicomplex $\mathcal{R}(\mathcal{A}) = \mathcal{R}(\mathcal{A}^\ast)$ with action with the group $\mathbb{Z}/2$ on it. The short sequence of $\mathbb{Z}/2$-complexes:

$0 \rightarrow \mathcal{H}\mathcal{R}(\mathcal{A}) \rightarrow \text{Tot}^+\mathcal{R}(\mathcal{A}) \rightarrow \text{Tot}^+\mathcal{R}(\mathcal{A}) \rightarrow 0$

(8)

The Hochschild complex $\mathcal{C}(\mathcal{A})$ is quasi-isomorphic to the reflexive complex $\mathcal{R}(\mathcal{A})$ (J.-L. Loday, 1998). Suppose that
\[ W^m = \{ K[\mathbb{Z}/2]^{1+\alpha} \to K[\mathbb{Z}/2]^{1+\alpha} \to K[\mathbb{Z}/2]^{1+\alpha} \to \cdots \}, \alpha = \pm 1. \]

We associate with \( W^m \) the exact sequence of \( \mathbb{Z}/2 \)-complexes
\[
0 \to K[\mathbb{Z}/2] \to W^m \to W^{m-1} \to 0
\]
(9)

Take the exact tensor product sequence (8) and (9) over \( k[\mathbb{Z}/2] \) and by applying\n\[ \alpha p(A) = \text{Tot}(\alpha p(A) \otimes_{k[\mathbb{Z}/2]} W^m), \alpha S(A) = \text{Tot}(\alpha p(A) \otimes_{k[\mathbb{Z}/2]} W^m), \]
we obtain the commutative diagram of complexes
\[
\begin{array}{cccccc}
0 & \to & \alpha R(A) & \to & \text{Tot}^\alpha p(A) & \to & \text{Tot}^\alpha p(A)[-2] & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \alpha S(A) & \to & \alpha p(A) & \to & \alpha p(A)[-2] & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \alpha S(A)[-1] & \to & \alpha p(A)[-1] & \to & \alpha p(A)[-3] & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & & 0 & & \\
\end{array}
\]
(10)

The rows and columns are exact (Cartan & Eilenberg, 1956). A quasi-isomorphism \( \alpha R(A) \cong C(A) \) leads to the quasi-isomorphism of the complexes \( \alpha S(A) \cong \alpha p(A) \). Since
\[ \mathcal{H}_s(C(A)) = \mathcal{H}_s(A), \mathcal{H}_s(\alpha S(A)) = \mathcal{H}_s(\mathbb{Z}/2; \alpha R(A)) = \alpha H_s(A), \]
\[ \mathcal{H}_s(\text{Tot}^\alpha p(A)) = \mathcal{H}_s(A), \mathcal{H}_s(\alpha p(A)) = \mathcal{H}_s(\mathbb{Z}/2; \alpha p(A)) = \alpha H^s(A), \]
\[ \mathcal{H}_s(C(A)) = \mathcal{H}_s(A), \mathcal{H}_s(\alpha p(A)) = \mathcal{H}_s(\mathbb{Z}/2; \alpha p(A)) = \alpha H^s(A). \]

Then we obtain an infinite commutative diagram of exact rows and from (10);
\[
\begin{array}{cccccc}
\cdots & \to & \mathcal{H}R_n(A) & \to & \mathcal{H}C_n(A) & \to & \mathcal{H}C_{n-2}(A) & \to & \mathcal{H}C_{n-1}(A) & \to & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \to & \sigma \mathcal{H}R_n(A) & \to & \sigma \mathcal{H}D_{n-2}(A) & \to & \sigma \mathcal{H}D_{n-3}(A) & \to & \sigma \mathcal{H}R_{n-2}(A) & \to & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \to & \sigma \mathcal{H}R_{n-1}(A) & \to & \sigma \mathcal{H}D_{n-3}(A) & \to & \sigma \mathcal{H}D_{n-4}(A) & \to & \sigma \mathcal{H}R_{n-3}(A) & \to & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \to & \mathcal{H}H_n(A) & \to & \mathcal{H}C^n(A) & \to & \mathcal{H}C^{n-2}(A) & \to & \mathcal{H}C^{n-1}(A) & \to & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \to & \sigma \mathcal{H}R^n(A) & \to & \sigma \mathcal{H}D^n(A) & \to & \sigma \mathcal{H}D^{n-2}(A) & \to & \sigma \mathcal{H}R^{n-2}(A) & \to & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \to & \sigma \mathcal{H}R^{n+1}(A) & \to & \sigma \mathcal{H}D^{n+1}(A) & \to & \sigma \mathcal{H}D^{n+2}(A) & \to & \sigma \mathcal{H}R^{n+2}(A) & \to & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \to & \mathcal{H}H^{n+1}(A) & \to & \mathcal{H}C^{n+1}(A) & \to & \mathcal{H}C^{n+2}(A) & \to & \mathcal{H}C^{n+3}(A) & \to & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \to & \sigma \mathcal{H}R^{n+2}(A) & \to & \sigma \mathcal{H}D^{n+2}(A) & \to & \sigma \mathcal{H}D^{n+3}(A) & \to & \sigma \mathcal{H}R^{n+3}(A) & \to & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \to & \sigma \mathcal{H}R^{n+3}(A) & \to & \sigma \mathcal{H}D^{n+3}(A) & \to & \sigma \mathcal{H}D^{n+4}(A) & \to & \sigma \mathcal{H}R^{n+4}(A) & \to & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\end{array}
\]

Example (4–2):

If \( \mathcal{M}_m(A) \) algebra of m-matrices in \( K \)-algebra \( A \) of order \( m \). Then the following isomorphism holds and is called Morita equivalence
\[ \mathcal{H}_s(\mathcal{M}_m(A)) \cong \mathcal{H}_s(A), \mathcal{H}^s(\mathcal{M}_m(A)) \cong \mathcal{H}^s(A). \]
Lykova and Michael (1998) show the same property of cyclic homology. Suppose that $A$ is an involutory associative unital algebra over $K$, and suppose that $M_m(A)$ is algebra of $m$-matrices in $A$. The $M_m(A)$- involution algebra is given by $\mathcal{X} \rightarrow \mathcal{X}'$, $\mathcal{X} = \mathcal{X}_0$, $\mathcal{X}' = (\mathcal{X}_0') \in M_m(A)$.

We take the $K$-module homomorphism $\text{Tr}_n : M_m(A) \otimes (n+1) \rightarrow A \otimes (n+1)$, put $\text{Tr}_n (X^{(0)} \otimes X^{(1)} \otimes \cdots \otimes X^{(n)}) = \sum_{1 \leq i_0 \ldots \leq i_n \leq m} X^{(0)}_{i_0i_0} X^{(1)}_{i_1i_1} \cdots X^{(n)}_{i_ni_n}$

where $X^{(k)}_{ij}$ is the $i$th row and the $j$th column of the matrix $X^{(k)}$.

The collect maps $\text{Tr}_n, n = 0, 1, \cdots$ commutes with the operators $d_n, s_n, t_n, \& r_n, n = 0, 1, \cdots$. The following dihedral homomorphism $k$-module is well defined as $\text{Tr} : M_m(A) \otimes K \rightarrow A \otimes K$.

The dihedral Homology homomorphism is denoted by $\text{Tr} : a \mathcal{H}_n(M_m(A)) \rightarrow a \mathcal{H}_n(A)$, and the dihedral Cohomology homomorphism $\text{Tr} : a \mathcal{H}_n^*(M_m(A)) \rightarrow a \mathcal{H}_n^*(A)$.

**Definition (4–3):**

Let $K$-algebra $A$ be called $H$-unital, for a given algebra $A$, the bar complex $\bar{C}_n(A, I) = (I \otimes C_n(A), 1 \otimes b')$ is a cyclic with $\mathcal{H}^{\text{bar}}_n(A, I) = 0$.

**Definition (4–4):**

An algebra $A$ is called approximately $H$-unital if for each $n,q$ there is a cyclic $b'$ such that: $b'[C_n(A^q)] \supset Ker[b' \circ \iota_n(C_n(A^q))]$ or by means if and only if the complex $b'[C_n(A^q)]$ is a cyclic.

**Lemma (4–5):**

Let $A$ and $I$ be algebras over a field of zero characteristic. Then every extension of $K$-algebras is $0 \rightarrow I \rightarrow A \rightarrow A/I \rightarrow 0$ with kernel $I$. In the dihedral (co)homology we obtain the long exact sequences

\[
\cdots \rightarrow \mathcal{H}D_{n+1}(A) \rightarrow \mathcal{H}D_n(A/I) \rightarrow \mathcal{H}D_n(A) \rightarrow \cdots \tag{11}
\]

\[
\cdots \rightarrow \mathcal{H}D^{n-1}(A) \rightarrow \mathcal{H}D^n(A/I) \rightarrow \mathcal{H}D^n(A) \rightarrow \cdots \tag{12}
\]

with connecting homomorphism $\psi$.

**Proposition (4–6):**

(a) Let $I$ be ideal in the unital algebra. Then, for the multiplication map $\psi : I \otimes I \rightarrow I^2$ and $I$ approximately $H$-unital there exists left $\phi : I^2 \rightarrow I \otimes I$.

(b) Let $\Phi : R \rightarrow I$ be a linear projection on the J such that $\Phi(1) = 0$; then the condition $\phi(x) = 1 \otimes x + b'(1 \otimes \Phi)bA(x)$ defines $R$-linear left $\phi : I \rightarrow R \otimes I$ for the multiplication map $R \otimes I \rightarrow I$ and the restriction of $\phi$ to $I^2$ is a map as in (a).

**Proof:** See Wodzicki (1989).

The long exact sequence of the reflexive (co)homology in algebra is given by the following theorem.
Theorem (4–7):

Let \(0 \to I \to A \to A/I \to 0\) be an extension of \(\mathcal{K}\)-algebra with \(A\) and \(A/I\) unital if \(I\) is \(H\)-unital. For the reflexive (co)homology, we have the long exact sequences

\[
\cdots \to \mathcal{H}R_n(I) \to \mathcal{H}R_n(A) \to \mathcal{H}R_n(A/I) \to \mathcal{H}R_{n-1}(I) \to \cdots
\]

\[
\cdots \to \mathcal{H}R^{n-1}(A/I) \to \mathcal{H}R^{n-1}(A) \to \mathcal{H}R^{n-1}(A/I) \to \mathcal{H}R^n(A/I) \to \cdots
\]

Proof:

There is a well-defined functorial map

\[\mathcal{F}: \mathcal{H}R_n(I) \to \mathcal{H}R_n(A, I), \mathcal{F}: \mathcal{H}R^n(A) \to \mathcal{H}R^n(A, I)\]

It is immediate from the construction of \(\mathcal{H}R_n(I)\) and \(\mathcal{H}C^n(I)\) that in the framework of non-unital algebras there is a long, exact sequence of Connes (1985). Considering the exact rows commutative diagrams:

\[
\begin{array}{ccccccc}
\cdots & \to & \mathcal{H}R_n(I) & \to & \mathcal{H}C_n(I) & \to & \mathcal{H}C_{n-2}(I) & \to & \mathcal{H}R_{n-1}(I) & \to & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \to & \mathcal{H}R_n(I) & \to & \mathcal{H}C_n(I) & \to & \mathcal{H}C_{n-2}(I) & \to & \mathcal{H}R_{n-1}(I) & \to & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \to & \mathcal{H}R^n(I) & \to & \mathcal{H}C^n(I) & \to & \mathcal{H}C^{n+2}(I) & \to & \mathcal{H}R^{n+1}(I) & \to & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \to & \mathcal{H}R^n(I) & \to & \mathcal{H}C^n(I) & \to & \mathcal{H}C^{n+2}(I) & \to & \mathcal{H}R^{n+1}(I) & \to & \cdots \\
\end{array}
\]

The relation between cyclic and reflexive (co)homology of algebra is in the following theorem.

Theorem (4–8):

There is a natural long exact sequence of associative not unital \(\mathcal{K}\)-algebra \(A\) between the cyclic and reflexive (co)homology of \(A\) as

\[
\cdots \to \mathcal{H}R_n(A) \xrightarrow{\delta} \mathcal{H}C_n(A) \xrightarrow{\delta} \mathcal{H}C_{n-2}(A) \xrightarrow{\delta} \mathcal{H}R_{n-1}(A) \to \cdots
\]

\[
\cdots \to \mathcal{H}R^n(A) \xrightarrow{\delta} \mathcal{H}C^n(A) \xrightarrow{\delta} \mathcal{H}C^{n+2}(A) \xrightarrow{\delta} \mathcal{H}R^{n+1}(A) \to \cdots
\]

We get the long exact sequence of the dihedral and cyclic (co)homology of pure algebras and prove it in the following theorems.

Theorem (4–9):

Let \(0 \to I \to A \to A/I \to 0\) be an extension of \(\mathcal{K}\)-algebra with \(A\) and \(A/I\) unital if \(I\) is \(H\)-unital. Then the sequence

\[
\cdots \to \mathcal{H}D_n(I) \to \mathcal{H}D_n(A) \to \mathcal{H}D_n(A/I) \to \mathcal{H}D_{n-1}(I) \to \cdots
\]

\[
\cdots \to \mathcal{H}D^{n-1}(I) \to \mathcal{H}D^{n-1}(A) \to \mathcal{H}D^{n}(A/I) \to \mathcal{H}D^{n}(I) \to \cdots
\]

is long exact sequence.

Proof:
We defined the functorial map 
\[ \partial : \mathcal{H}D_n(I) \rightarrow \mathcal{H}D_n(A, I), \partial : \mathcal{H}D^n(I) \rightarrow \mathcal{H}D^n(A, I) \]

It is immediate from the construction of \( \mathcal{H}C_n(I) \& \mathcal{H}C^n(I) \) that in the framework of non-unital algebras there is a long, exact sequence of Cone’s. Considering the exact rows commutative diagram:

\[ \cdots \rightarrow \mathcal{H}D_n(I) \rightarrow \mathcal{H}R_n(I) \rightarrow \mathcal{H}R_{n-2}(I) \rightarrow \mathcal{H}D_{n-1}(I) \rightarrow \cdots \]

\[ \cdots \rightarrow \mathcal{H}D_n(I) \rightarrow \mathcal{H}R_n(I) \rightarrow \mathcal{H}R_{n-2}(I) \rightarrow \mathcal{H}D_{n-1}(I) \rightarrow \cdots \]

\[ \cdots \rightarrow \mathcal{H}D^n(I) \rightarrow \mathcal{H}R^n(I) \rightarrow \mathcal{H}R^{n+2}(I) \rightarrow \mathcal{H}D^{n+1}(I) \rightarrow \cdots \]

\[ \cdots \rightarrow \mathcal{H}D^n(I) \rightarrow \mathcal{H}R^n(I) \rightarrow \mathcal{H}R^{n+2}(I) \rightarrow \mathcal{H}D^{n+1}(I) \rightarrow \cdots \]

**Theorem (4–10):**

If \( \mathcal{A} \) is associative not unital \( K \)-algebra. There is a long exact sequence of the cyclic and dihedral (co)homology of \( \mathcal{A} \)

\[ \cdots \rightarrow \mathcal{H}D_n(A) \rightarrow \mathcal{H}C_n(A) \rightarrow \mathcal{H}C^n(A) \rightarrow \mathcal{H}D_{n-1}(A) \rightarrow \cdots \]

\[ \cdots \rightarrow \mathcal{H}D^n(A) \rightarrow \mathcal{H}C^n(A) \rightarrow \mathcal{H}C_{n+2}(A) \rightarrow \mathcal{H}D^{n+1}(A) \rightarrow \cdots \]

Now we give and prove the excision theorem of the dihedral and Reflexive (co)homology theory of pure algebras for any short exact sequence.

**Theorem (4–11):**

Suppose an exact short sequence \( 0 \rightarrow I \rightarrow A \rightarrow A/I \) of algebras over a field of zero characteristic, then we have the following six-term sequence in dihedral homology and cohomology

\[ \mathcal{H}D_0(I) \rightarrow \mathcal{H}D_0(I) \rightarrow \mathcal{H}D_0(I) \rightarrow \mathcal{H}D^0(I) \rightarrow \mathcal{H}D^0(I) \rightarrow \mathcal{H}D^0(I) \]

\[ \mathcal{H}D^1(I) \rightarrow \mathcal{H}D^1(I) \rightarrow \mathcal{H}D^1(A) \rightarrow \mathcal{H}D^1(A) \rightarrow \mathcal{H}D^1(A) \rightarrow \mathcal{H}D^1(A) \]

**Proof:** We define the following short exact sequence for the algebra \( \mathcal{A} \)

\[ 0 \rightarrow I.A \rightarrow T.A \rightarrow A \rightarrow 0 \]

where \( T.A \) is the non unital involution tensor algebra over \( \mathcal{A} \), \( I.A \) is the ideal in the unital tensor algebra \( T.A \), then the long exact sequence is

\[ \cdots \rightarrow \mathcal{H}D_{n+1}(I.A) \rightarrow \mathcal{H}D_{n+1}(T.A) \rightarrow \mathcal{H}D_{n+1}(A) \rightarrow \mathcal{H}D_n(T.A) \rightarrow \mathcal{H}D_n(A) \rightarrow \cdots \]

\[ \cdots \rightarrow \mathcal{H}D^{n+1}(I.A) \rightarrow \mathcal{H}D^{n+1}(T.A) \rightarrow \mathcal{H}D^{n+1}(A) \rightarrow \mathcal{H}D^n(T.A) \rightarrow \mathcal{H}D^n(A) \rightarrow \cdots \]

Let \( K \) be the kernel in the exact sequence (Wodzicki, 1989)

\[ 0 \rightarrow K \rightarrow T.A \rightarrow A/I \rightarrow 0 \]

\[ \cdots \rightarrow \mathcal{H}D_{n+1}(K) \rightarrow \mathcal{H}D_{n+1}(T.A) \rightarrow \mathcal{H}D_{n+1}(A/I) \rightarrow \mathcal{H}D_n(K) \rightarrow \mathcal{H}D_n(T.A) \rightarrow \mathcal{H}D_n(A/I) \rightarrow \cdots \]

\[ \cdots \rightarrow \mathcal{H}D^{n+1}(K) \rightarrow \mathcal{H}D^{n+1}(T.A) \rightarrow \mathcal{H}D^{n+1}(A/I) \rightarrow \mathcal{H}D^n(K) \rightarrow \mathcal{H}D^n(T.A) \rightarrow \mathcal{H}D^n(A/I) \rightarrow \cdots \]

where \( K \) the unital tensor algebra \( T.A \) which is free, then \( I.A \) and \( K \) are approximately \( H \)-unital from the long exact sequences for (16) and (17) we have
\[ \mathcal{H}D_n(I,A) \cong \mathcal{H}D_{n-1}(A) \simeq \mathcal{H}D_n(K) \cong \mathcal{H}D_{n-1}(A/I) \]

\[ \mathcal{H}D^{n-1}(I,A) \cong \mathcal{H}D^n(A) \simeq \mathcal{H}D^{n-1}(K) \cong \mathcal{H}D^n(A/I) \]

Let the following short exact sequence (Ralf, 2010)
\[ 0 \rightarrow I \rightarrow A \rightarrow A/I \rightarrow 0 \]
\[ \cdots \rightarrow \mathcal{H}D_{n-1}(I,A) \rightarrow \mathcal{H}D_n(A) \rightarrow \mathcal{H}D_n(K) \rightarrow \mathcal{H}D_n(I) \rightarrow \cdots \]
\[ \cdots \rightarrow \mathcal{H}D^{n-1}(I,A) \rightarrow \mathcal{H}D^{n-1}(A) \rightarrow \mathcal{H}D^n(I) \rightarrow \mathcal{H}D^n(A) \rightarrow \mathcal{H}D^n(I) \rightarrow \cdots \]

From the long exact sequence for this sequence (19), (20) and from Gouda and Alaa (2011), we have the proof of our theorem.

The excision property of the reflexive (co)homology theory of pure algebras is given in the following theorem.

**Theorem (4–12):**
Suppose an exact short sequence \( 0 \rightarrow I \rightarrow A \rightarrow A/I \) of algebras over a field of zero characteristic, then we have the following six-term sequence in dihedral homology and cohomology
\[ \mathcal{H}R_0(I) \leftarrow \mathcal{H}R_0(A) \leftarrow \mathcal{H}R_0(I) \quad \mathcal{H}R^0(I) \leftarrow \mathcal{H}R^0(A) \leftarrow \mathcal{H}R^0(I) \]
\[ \mathcal{H}R^1(I) \leftarrow \mathcal{H}R^1(A) \leftarrow \mathcal{H}R^1(I) \quad \mathcal{H}R^1(I) \leftarrow \mathcal{H}R^1(A) \leftarrow \mathcal{H}R^1(I) \]

**Proof:** By the same manner of theorem (3–11).

**5. Conclusion**
We presented the theory of excision in the dihedral and reflexive (co)homology of algebras for each short exact sequence \( 0 \rightarrow I \rightarrow A \rightarrow A/I \) of algebras over a field of zero in the form
\[ \mathcal{H}D_n(I,A) \cong \mathcal{H}D_{n-1}(A) \quad \mathcal{H}D^{n-1}(I,A) \cong \mathcal{H}D^n(A) \quad \mathcal{H}R_0(I,A) \cong \mathcal{H}R_{n-1}(A) \quad \mathcal{H}R^0(I,A) \cong \mathcal{H}R^n(A). \]
We discussed and proved some theorems in the cyclic and dihedral cohomology of an associative algebra and we studied a new form and new theorems in this area. We can apply this theorem in the operator algebra in the next work and apply this result in our life working as by Intissar (2020) to improve results. Our result can be equipped with the result by Kostikov and Romanenkov (2020) and can improve it by using our result.

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