CHARACTERIZING SEMIGROUPS $X$ WITH COMMUTATIVE EXTENSIONS

$\varphi(X)$, $\lambda(X)$, $N_2(X)$, $\nu(X)$

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Abstract. We characterize semigroups $X$ whose semigroups of filters $\varphi(X)$, maximal linked systems $\lambda(X)$, linked upfamilies $N_2(X)$, and upfamilies $\nu(X)$ are commutative.

1. Introduction

In this paper we investigate the algebraic structure of various extensions of an inverse semigroup $X$ and detect semigroups whose extensions $\varphi(X)$, $\lambda(X)$, $N_2(X)$, $\nu(X)$ and their subsemigroups $\varphi^*(X)$, $\lambda^*(X)$, $N_2^*(X)$, $\nu^*(X)$ are commutative.

The thorough study of various extensions of semigroups was started in [9] and continued in [1]–[6]. The largest among these extensions is the semigroup $\nu(X)$ of all upfamilies on $X$.

A family $\mathcal{F}$ of subsets of a set $X$ is called an upfamily if $\emptyset \notin \mathcal{F} \neq \emptyset$ and for each set $F \in \mathcal{F}$ any subset $E \supseteq F$ of $X$ belongs to $\mathcal{F}$. Each family $\mathcal{F}$ of non-empty subsets of $X$ generates the upfamily

$$ \langle F : F \in \mathcal{F} \rangle = \{ E \subseteq X : \exists F \in \mathcal{F} \ F \subseteq E \} $$

we denote the transversal of $\mathcal{F}$. By [8], $(\mathcal{F}^\perp)^\perp = \mathcal{F}$, so

$$ \perp : \nu(X) \to \nu(X), \quad \perp : \mathcal{F} \mapsto \mathcal{F}^\perp, $$

is an involution on $\nu(X)$. For a subset $S \subseteq \nu(X)$ we put $S^\perp = \{ F^\perp : F \in S \} \subseteq \nu(X)$.

The compact Hausdorff space $\nu(X)$ contains many other important extensions of $X$ as closed subspaces. In particular, it contains the spaces $N_2(X)$ of linked upfamilies, $\lambda(X)$ of maximal linked upfamilies, $\varphi(X)$ of filters, and $\beta(X)$ of ultrafilters on $X$; see [8]. Let us recall that an upfamily $\mathcal{F} \in \nu(X)$ is called

- linked if $A \cap B \neq \emptyset$ for any sets $A, B \in \mathcal{F}$;
- maximal linked if $\mathcal{F} = \mathcal{F}^\perp$ for any linked upfamily $\mathcal{F}^\perp \in \nu(X)$ that contains $\mathcal{F}$;
- a filter if $A \cap B \in \mathcal{F}$ for any $A, B \in \mathcal{F}$;
- an ultrafilter if $\mathcal{F} = \mathcal{F}^\perp$ for any filter $\mathcal{F}^\perp \in \nu(X)$ that contains $\mathcal{F}$.

The family $\beta(X)$ of all ultrafilters on $X$ is called the Stone-Čech extension and the family $\lambda(X)$ of all maximal linked upfamilies is called the superextension of $X$, see [14] and [16]. It can be shown that $\lambda(X) = \{ \mathcal{F} \in \nu(X) : \mathcal{F}^\perp = \mathcal{F} \}$, so $\lambda(X)^\perp = \lambda(X)$ and $\beta(X)^\perp = \beta(X)$. The arrows in the following diagram denote the identity inclusions between various extensions of a set $X$.

\[
\begin{array}{ccc}
\varphi(X)^\perp & \rightarrow & N_2(X)^\perp \\
\uparrow & & \uparrow \\
X & \rightarrow & \beta(X) \\
\downarrow & & \downarrow \\
\varphi(X) & \rightarrow & N_2(X) \\
\end{array}
\]

\[\rightarrow\]

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We say that an upfamily $\mathcal{U} \in \nu(X)$ is finitely supported if $\mathcal{U} = \langle F_1, \ldots, F_n \rangle$ for some non-empty finite subsets $F_1, \ldots, F_n \subset X$. By $\nu^*(X)$ we denote the subspace of $\nu(X)$ consisting of finitely supported upfamilies on $X$. Let

$$\varphi^*(X) = \varphi(X) \cap \nu^*(X), \quad \lambda^*(X) = \lambda(X) \cap \nu^*(X), \quad N_2^*(X) = N_2(X) \cap \nu^*(X).$$

Since each finitely supported ultrafilter is principal, the set $\beta^*(X) = \beta(X) \cap \nu^*(X)$ coincides with $X$ (identified with the subspace of all principal ultrafilters in $\nu(X)$). The embedding relations between these spaces of finitely supported upfamilies are indicated in the following diagram:

$$\begin{array}{ccc}
\varphi^*(X) \downarrow & \twoheadrightarrow & N_2^*(X) \downarrow \\
X \twoheadrightarrow & \beta^*(X) & \twoheadrightarrow \lambda^*(X) \twoheadrightarrow \nu^*(X) \\
& \varphi^*(X) \downarrow & \nu^*(X) \downarrow
\end{array}$$

Any map $f : X \to Y$ induces a continuous map

$$vf : \nu(X) \to \nu(Y), \quad vf : F \mapsto \{A \subset Y : f^{-1}(A) \in F\}.$$

By \([8]\), $vf(F^\perp) = (vf(F))^\perp$ and $vf(\beta(X)) \subset \beta(Y)$, $vf(\lambda(X)) \subset \lambda(Y)$, $vf(\varphi(X)) \subset \varphi(Y)$, $vf(N_2(X)) \subset N_2(Y)$, and $vf(\nu^*(X)) \subset \nu^*(Y)$. If the map $f$ is injective, then $vf$ is a topological embedding, which allows us to identify the extensions $\beta(X)$, $\lambda(X)$, $\varphi(X)$, $N_2(X)$ and $\nu(X)$ with corresponding closed subspaces in $\beta(Y)$, $\lambda(Y)$, $\varphi(Y)$, $N_2(Y)$ and $\nu(Y)$, respectively.

If $*: X \times X \to X$, $*: (x, y) \mapsto xy$, is a binary operation on $X$, then there is an obvious way of extending this operation onto the space $\nu(X)$. Just put

$$A \circledast B = \langle A \ast B : A \in A, \ B \in B \rangle$$

where $A \ast B = \{ab : a \in A, \ b \in B\}$ is the pointwise product of sets $A, B \subset X$. The upfamily $A \circledast B$ will be called the \textit{pointwise product} of the upfamilies $A, B$. It is clear that this extension $\circledast : \nu(X) \times \nu(X) \to \nu(X)$ of the operation $\ast : X \times X \to X$ is associative and commutative if $\ast$ is so. For an associative binary operation $\ast$ on $X$, its extension $\nu(X)$ endowed with the operation $\circledast$ of pointwise product becomes a semigroup, containing the subspaces $\varphi(X)$, $\varphi^*(X)$, $N_2(X)$, and $N_2^*(X)$ as subsemigroups. However, the subspaces $\beta(X)$ and $\lambda(X)$ are not subsemigroups in $(\nu(X), \circledast)$. To make $\beta(X)$ a semigroup extension of $X$, Ellis \([7]\) suggested a less obvious extension of an (associative) binary operation $\ast : X \times X \to X$ to an (associative) binary operation on $\beta(X)$ letting

$$\beta(X) \circledast \beta(X) = \bigcup_{a \in A} \bigcup_{B \in B} \{B_a : \{B_a\}_{a \in A} \subset B\}$$

for ultrafilters $A, B \in \beta(X)$. It is clear that $A \circledast B \subset A \ast B$ but in general $A \circledast B \neq A \ast B$.

In \([9]\) it was observed that the formula \((1)\) determines an extension of the operation $\ast$ to an (associative) binary operation $\circledast : \nu(X) \times \nu(X) \to \nu(X)$ on the extension $\nu(X)$ of $X$. So, for each semigroup $(X, \ast)$, its extension $\nu(X)$ endowed with the extended operation $\circledast$ is a semigroup, containing the subspaces $\beta(X)$, $\varphi(X)$, $\lambda(X)$, $N_2(X)$ as closed subsemigroups. Moreover, since $\varphi^*(X)$ is a subsemigroup of $\nu(X)$, the subspaces $X = \beta^*(X)$, $\varphi^*(X)$, $\lambda^*(X)$, $N_2^*(X)$ also are subsemigroups of $\nu(X)$. Algebraic and topological properties of these semigroups have been studied in \([9, 11–6]\). In particular, in \([6]\) and \([3]\) we studied properties of extensions of groups, while \([11]\) and \([12]\) were devoted to extensions of semilattices and inverse semigroups, respectively. In contrast to the operation $\circledast$, the extended operation $\circledast$ on the semigroup $\nu(X)$ and its subsemigroups rarely is commutative. For example, by \([3]\) a group $X$ has commutative superextension $\lambda(X)$ if and only if $X$ is a group of cardinality $|X| \leq 4$. According to \([11]\), a semilattice $X$ has commutative extension $\nu(X)$ if and only if $X$ is finite and linearly ordered.

Let $X$ be a semigroup. A subsemigroup $S \subset \nu(X)$ is defined to be \textit{supercommutative} if

$$A \ast B = A \circledast B = B \circledast A = B \ast A$$

for any upfamilies $A, B \in S$. It is clear that each supercommutative subsemigroup $S \subset \nu(X)$ is commutative. The converse is not true as we shall see in Section \([12]\).

In this paper we study the commutativity and supercommutativity of the semigroups $\nu(X)$, $N_2(X)$, $\lambda(X)$, $\varphi(X)$, $\beta(X)$, $\nu^*(X)$, $N_2^*(X)$, $\lambda^*(X)$, $\varphi^*(X)$ and characterize semigroups $X$ whose various extensions are
commutative or supercommutative. In the preliminary Sections \([2, 3]\) we shall analyze the structure of periodic commutative semigroups and projective extensions of semigroup. Section \(5\) is devoted to square-linear semigroups which will play a crucial role in Sections \(7\) and \(8\) devoted to the study of commutativity and supercommutativity of the semigroups \(\nu(X), \nu^* (X), N_2(X)\) and \(N_2^*(X)\). In Section \(6\) we characterize semigroups \(X\) with (super)commutative extensions \(\beta(X), \varphi(X), \varphi^*(X)\), and in Section \(9\) we detect semigroups with commutative extensions \(\lambda(X)\) and \(\lambda^*(X)\). In Section \(10\) we study the structure of semigroups \(X\) whose superextension \(\lambda(X)\) is supercommutative.

2. The Structure of Periodic Commutative Semigroups

In this section we recall some known information on the structure of periodic commutative semigroups. A semigroup \(S\) is called periodic if each element \(x \in S\) generates a finite semigroup \(\{x^k\}_{k \in \mathbb{N}}\). A semigroup \(S\) generated by a single element \(x\) is called monogenic. If a monogenic semigroup is infinite, then it is isomorphic to the additive semigroup \(\mathbb{N}\) of positive integers. A finite monogenic semigroup \(S = \{x^k\}_{k \in \mathbb{N}}\) also has simple structure (cf. [13] \S 1.2): there are positive integer numbers \(n < m\) such that

- \(S = \{x, \ldots, x^{m-1}\}, m = |S| + 1\) and \(x^n = x^m\);
- \(C_x = \{x^n, \ldots, x^{m-1}\}\) is a cyclic subgroup of \(S\);
- the cyclic subgroup \(C_x\) coincides with the minimal ideal of \(S\);
- the neutral element \(e_x\) of the group \(C_x\) is a unique idempotent of \(S\) and the cyclic group \(C_x\) is generated by the element \(xe_x\).

Such monogenic semigroups will be denoted by \(\langle x \mid x^n = x^m \rangle\).

For a semigroup \(S\) let

\[ E(S) = \{e \in S : ee = e\} \]

be the idempotent part of \(S\). For each idempotent \(e \in E(S)\) let

\[ H_e = \{x \in S : \exists y \in S \ xyx = x, \ yxy = y, \ xy = e = yx\} \]

be the maximal subgroup of \(S\) containing the idempotent \(e\). The union

\[ C(S) = \bigcup_{e \in E(S)} H_e \]

of all (maximal) subgroups of \(S\) is called the Clifford part of \(S\). The Clifford part \(C(S)\) is contained in the regular part

\[ R(S) = \{x \in S : x \in xSx\} \]

of \(S\). If a semigroup \(S\) is commutative, then \(R(S) = C(S)\) and the subsets \(E(S)\) and \(R(S) = C(S)\) are subsemigroups of \(S\).

If a semigroup \(S\) is periodic, then for each element \(x \in S\) the monogenic semigroup \(\{x^k\}_{k \in \mathbb{N}}\) contains a unique idempotent \(e_x\). So, we can consider the map

\[ c_x : S \to E(S), \ e_x : x \mapsto e_x, \]

which projects the semigroup \(S\) onto its idempotent part \(E(S)\). The map

\[ c_x : S \to C(S), \ c_x : x \mapsto e_x \cdot x, \]

projects the semigroup \(S\) onto its Clifford part. If a periodic semigroup \(S\) is commutative, then the projections \(e_x : S \to E(S)\) and \(c_x : S \to C(S)\) are semigroup homomorphisms. In this case, for every idempotent \(e \in E(S)\), \(S_e = \{x \in S : e_x = e\}\) is a subsemigroup of \(S\) with a unique idempotent \(e\). So, the semigroup \(S\) decomposes into the disjoint union \(S = \bigcup_{e \in E(S)} S_e\) of semigroups \(S_e\) parametrized by idempotents \(e \in E(S)\).

3. Projection Extensions of Semigroups

A semigroup \(X\) is called a projection extension of a subsemigroup \(Z \subseteq X\) if there is a function \(\pi : X \to Z\) (called the projection of \(X\) onto \(Z\)) such that

- \(\pi(z) = z\) for each \(z \in Z\);
- \(x \cdot y = \pi(x) \cdot \pi(y)\) in \(Z\) for all \(x, y \in X\).

It follows from \(\pi(xy) = xy = \pi(x) \cdot \pi(y)\) that the projection \(\pi : X \to Z\) necessarily is a homomorphism of \(X\) onto its subsemigroup \(Z\).

If a map \(\pi : X \to Z\) of semigroups \(X\) and \(Z\) is a homomorphism, then by [3] the map \(\nu \pi : \nu(X) \to \nu(Z)\) is a homomorphism too. So, we have the following statement.
Proposition 3.1. If a semigroup $X$ is a projection extension of a subsemigroup $Z \subseteq X$, then the projection $\pi : X \to Z$ induces a homomorphism $\psi \pi : \psi(X) \to \psi(Z)$ witnessing that the semigroup $\psi(X)$ is a projection extension of the subsemigroup $\psi(Z)$.

Corollary 3.2. Assume that a semigroup $X$ is a projection extension of a subsemigroup $Z \subseteq X$, $\pi : X \to Z$ is a projection of $X$ onto $Z$. Then each subgroup $S \subseteq \psi(X)$ with $\psi(S) \subseteq S$ is a projection extension of the subsemigroup $\psi(S) = S \cap \psi(Z)$. Consequently, the semigroup $S$ is (super)commutative if and only if so is its subsemigroup $S \cap \psi(Z)$.

Corollary 3.3. Assume that a semigroup $X$ is a projection extension of a subsemigroup $Z \subseteq X$, and $x \in \{\nu, \psi, \lambda, \beta\}$. The extension $\psi(Z)$ of $X$ is (super)commutative if and only if the extension $\psi(Z)$ of the subsemigroup $Z$ is (super)commutative.

4. Semi-complete digraphs

In this section we recall some information on digraphs. In the next section this information will be used for describing the structure of square-linear semigroups.

By an directed graph (briefly, a digraph) we shall understand a pair $(X, \Delta)$ consisting of a set $X$ and a subset $\Delta \subseteq X \times X$. Elements $x \in X$ are called vertices and ordered pairs $(x, y) \in \Delta$ called edges of the digraph $(X, \Delta)$. An edge $(x, y) \in \Delta$ is called pure if $y \neq x$. A digraph $(X, \Delta)$ is called complete if $\Delta = X \times X$ and semi-complete if $\Delta \cup \Delta^{-1} = X \times X$, where $\Delta^{-1} = \{(y, x) : (x, y) \in \Delta\}$.

A sequence $x_0, \ldots, x_n$ of vertices of a digraph $(X, \Delta)$ is called a (pure) cycle of length $n$ if $x_0 = x_n$ and for every $i < n$ the pair $(x_i, x_{i+1})$ is a (pure) edge of the digraph $(X, \Delta)$. A cycle $x_0, x_1, \ldots, x_n$ in a digraph $(X, D)$ is called bipartite if the number $n$ is even and for each indices $i, j \in \{1, \ldots, n\}$ with odd difference $i - j$ we get $(x_i, y_j) \notin \Delta \cap \Delta^{-1}$. Bipartite cycles can be equivalently defined as cycles $x_0, x_1, x_2, y_1, x_3, y_2, \ldots, y_n, x_n$ such that $(x_i, y_j) \notin \Delta \cap \Delta^{-1}$ for any $1 \leq i, j \leq n$.

It is easy to see that a cycle of length 4 is bipartite if and only if it is pure.

Lemma 4.1. A semi-complete digraph $(X, \Delta)$ contains a pure cycle of length 4 if and only if it contains a bipartite cycle.

Proof. Let $x_0, x_1, \ldots, x_n$ be a bipartite cycle in the digraph of the smallest possible length $n$. The length $n$ is even and cannot be equal to 2 as otherwise $(x_1, x_2) = (x_1, x_0) \in \Delta \cap \Delta^{-1}$. So, $n \geq 4$.

We claim that $n = 4$. Assume conversely that $n > 4$ and consider the pair $(x_0, x_3)$. Since the cycle is bipartite and the digraph $(X, \Delta)$ is semi-complete, either $(x_0, x_3)$ or $(x_3, x_0)$ is a pure edge of the digraph. If $(x_3, x_0) \in \Delta$, then $x_0, x_1, x_2, x_3, x_0$ is a bipartite (and pure) cycle of length 4 in $(X, \Delta)$. If $(x_0, x_3) \in \Delta$, then $x_0, x_3, x_3, \ldots, x_n$ is a bipartite cycle of length $n - 2 \geq 4$ in $(X, \Delta)$, which contradicts the minimality of $n$. \hfill \Box

5. Square-linear semigroups

A semigroup $S$ is called linear if $xy \in \{x, y\}$ for any elements $x, y \in S$. It follows that each element $x$ of a linear semigroup is an idempotent. So, linear semigroups are bands, i.e., semigroups of idempotents. Commutative bands are called semilattices. So, each linear commutative semigroup is a semilattice. Each semilattice $E$ is endowed with a partial order $\leq$ defined by $x \leq y$ if $xy = x$.

A semigroup $S$ is called square-linear if $xy \in \{x^2, y^2\}$ for all elements $x, y \in S$.

Proposition 5.1. Let $S$ be a square-linear commutative semigroup and $x, y, z \in S$ be any elements. Then

1. $S$ is periodic and $x^3 = x^4 = e_x \in E(S)$;
2. the idempotent part $E(S)$ of $S$ is a linear semilattice;
3. the Clifford part $C(S)$ of $S$ coincides with $E(S)$;
4. $xy = e_x e_y$ if $x^2, y^2 \in E(S)$;
5. $xyz = e_x e_y e_z$;
6. If $x^2 \notin E(S)$, then $e_x$ is the largest element of the semilattice $E(S)$.

Proof. 1. It follows from $x^3 = x \cdot x^2 \in \{x^2, x^3\}$ that $x^3 = x^4 = e_x$ and hence $x^3 = x^n$ for all $n \geq 3$. So, the monogenic semigroup $\{x^n\}_{n \in \mathbb{N}} = \{x, x^2, x^3\}$ is finite and hence $S$ is periodic.

2. If $x, y$ are idempotents, then $xy \in \{x^2, y^2\} = \{x, y\}$ implies that the semilattice $E(S)$ is linear.

3. The identity $x^3 = x^4$ implies that each subgroup of $S$ is trivial and hence $C(S) = E(S)$.

4. If $x^2, y^2 \in E(S)$, then $x^3 = x^2$ and hence $x^2 = x^4 = x^3 = e_x$. Then $xy \in \{x^2, y^2\} = \{e_x, e_y\}$ implies that $xy$ is an idempotent and hence $xy = e_{xy} = e_x \cdot e_y$ as the projection $e_{xy} : S \to E(S)$ is a homomorphism.

5. First we show that $xyz \in E(S)$. Since $S$ is square-linear, we get $xy \in \{x^2, y^2\}$. We lose no generality assuming that $xy = x^2$. Now consider the product $xz \in \{x^2, z^2\}$. If $xz = x^2$, then $xyz = x^2 z = x(xz) = x^3 \in E(S)$.
E(S). If \( xx = z^2 \), then \( xyz = x^2z = xz = xz^2 = xzz = z^2z = z^3 \in E(S) \). Since the projection \( e_x : S \to E(S) \) is a homomorphism, we conclude that \( xyz = e_x e_y e_z \).

6. Assume that \( x^2 \notin E(S) \) but the idempotent \( e_x \) is not maximal in the linear semilattice \( E(S) \). Then there is an idempotent \( e \in E(S) \) such that \( ee_x = e_x \neq e \). It follows that \( xe \notin \{x^2, e^2\} \). We claim that \( xe \neq e \).

Assuming that \( xe = e \), we conclude that \( xee = ee = e \). On the other hand, the preceding item guarantees that \( xee = e_x e_x e_x = ee \neq e \). So, \( xe = x^2 \notin E(S) \), which contradicts \( xe = xee \in E(S) \). \( \square \)

Each square-linear semigroup \( S \) endowed with the set of directed edges

\[ \Delta = \{(x, y) \in S \times S : xy = x^2 \} \]

becomes a semicomplete digraph. In fact, the algebraic structure of a square-linear semigroup \( S \) is completely determined by its digraph structure \( \Delta \) and the duplication map \( S \to S \), \( x \mapsto xy \). The semigroup operation \( S \times S \to S \), \( (x, y) \mapsto xy \), can be recovered from \( \Delta \) and the duplication map by the formula

\[ xy = \begin{cases} x^2 & \text{if } (x, y) \in \Delta, \\ y^2 & \text{if } (y, x) \in \Delta. \end{cases} \]

6. Commutativity of the semigroups \( \beta(X), \varphi(X) \) and \( \varphi^*(X) \)

The following characterization was proved in Theorem 4.27 of [12].

**Theorem 6.1.** For a commutative semigroup \( X \) the following conditions are equivalent:

1. the semigroup \( \beta(X) \) is commutative;
2. \( \{a_n b_n : k, n \in \omega, \ k < n \} \cap \{b_n a_n : k, n \in \omega, \ k < n \} \neq \emptyset \) for any sequences \( (a_n)_{n \in \omega} \) and \( (b_n)_{n \in \omega} \) in \( X \).

**Corollary 6.2.** If the semigroup \( \beta(X) \) is commutative, then

1. for each square-linear subsemigroup \( S \subset X \) the set \( \{x^2 : x \in S \} \) is finite;
2. each subsemigroup of \( X \) contains a finite ideal;
3. each monogenic subsemigroup of \( X \) is finite.

**Proof.** Assume that the semigroup \( \beta(X) \) is commutative.

1. Assume that \( X \) contains a square-linear subsemigroup \( S \subset X \) with infinite subset \( \{x^2 : x \in S \} \). Then there is a sequence \( \{x_n\}_{n \in \omega} \) in \( S \) such that \( x_n^2 \neq x_m^2 \) for any \( n \neq m \). Define a 2-coloring \( \chi : [\omega]^2 \to \{0, 1\} \) of the set \( [\omega]^2 = \{(n, m) \in \omega^2 : n < m \} \) letting

\[ \chi(n, m) = \begin{cases} 0 & \text{if } x_n x_m = x_n^2 \\ 1 & \text{if } x_n x_m = x_m^2. \end{cases} \]

By Ramsey’s Theorem [15] (see also [10] Theorem 5), there is an infinite subset \( \Omega \subset \omega \) and a color \( c \in \{0, 1\} \) such that \( \chi(n, m) = c \) for any pair \( (n, m) \in [\omega]^2 \cap \Omega^2 \). Let \( \Omega = \{k_n : n \in \omega \} \) be the increasing enumeration of the set \( \Omega \). Then for the sequences \( a_n = x_{k_n} \) and \( b_n = x_{k_{n+1}} \), \( n \in \omega \), we get

\[ \{a_k b_n : k < n \} \subset \{a_k^2 : k \in \omega \} \cap \{b_k^2 : k \in \omega \} = \{x_{k_{n+1}} : n \in \omega \} \cap \{x_{k_{n+1}}^2 : n \in \omega \} = \emptyset, \]

which implies that the semigroup \( \beta(X) \) is not commutative according to Theorem 6.1.

2. Let \( S \) be an infinite subsemigroup of \( X \). Then the semigroup \( \beta(S) \subset \beta(X) \) is commutative and hence contains at most one minimal left ideal. In this case Corollary 2.23 of [11] guarantees that the semigroup \( S \) contains a finite ideal.

3. By the preceding item, for every \( x \in X \) the monogenic semigroup \( \{x^k\}_{k \in \mathbb{N}} \) contains a finite ideal and hence is finite. \( \square \)

**Theorem 6.3.** For a commutative semigroup \( X \) and the semigroup \( \varphi(X) \) of filters on \( X \) the following conditions are equivalent:

1. \( \varphi(X) \) is commutative;
2. \( \varphi(X) \) is supercommutative;
3. \( \{a_k b_n : k \leq n \} \cap \{b_k a_{n+1} : n \in \omega \} \neq \emptyset \) for any sequences \( (a_n)_{n \in \omega} \) and \( (b_n)_{n \in \omega} \) in \( X \).

**Proof.** We shall prove the implications (2) \( \Rightarrow \) (1) \( \Rightarrow \) (3) \( \Rightarrow \) (2). The implication (2) \( \Rightarrow \) (1) is trivial.

(1) \( \Rightarrow \) (3) Assume that the semigroup \( \varphi(X) \) is commutative and take any sequences \( (a_n)_{n \in \omega} \) and \( (b_n)_{n \in \omega} \) in \( X \). Consider the filter \( \mathcal{A} = \{A\} \) generated by the set \( A = \{a_n : n \in \omega \} \) and the filter \( \mathcal{B} = \{B \subset X : \exists n \forall m \geq n \ b_m \in B\} \). It follows that the set \( C = \{a_k b_n : k \leq n \} \) belongs to the product \( \mathcal{A} \ast \mathcal{B} \). Since the semigroup \( \varphi(X) \) is commutative, \( C \in \mathcal{A} \ast \mathcal{B} = \mathcal{B} \ast \mathcal{A} \) and hence there is a set \( B \in \mathcal{B} \) such that \( BA \subset C \). By the definition of
the filter $\mathcal{B}$, the set $B$ contains some element $b_m$. Then $b_m a_{m+1} = BA = AB \subset C$ and hence the intersection $\{a_k b_n\}_{k \leq n} \cap \{b_n a_{n+1}\}_{n \in \omega} \supset b_m a_{m+1}$ is not empty.

(3) $\Rightarrow$ (2) Assume that $A \ast B \neq A \oplus B$ for some filters $A, B \in \varphi(X)$. Then $A \ast B \not\subset A \oplus B$ and some set $C \subset A \ast B$ does not belong to the filter $A \oplus B$. This means that $A \ast B \not\subset C$ for any sets $A \in A$ and $B \in B$. We lose no generality assuming that the set $C$ is of the basic form $C = \bigcup_{a \in A} a \ast B_a$ for some set $A \in A$ and family $(B_a)_{a \in A} \subset B^A$. Pick any point $a_0 \in A$ and consider the set $B_0 = B_{a_0} \in B$. Since $A \ast B_0 \not\subset C$, there are points $b_0 \in B_0$ and $a_1 \in A$ such that $a_1 b_0 \not\in C$. Now consider the set $B_1 = B_0 \cap B_{a_1}$. Since $A \ast B_1 \not\subset C$, there are points $b_1 \in B_1$ and $a_2 \in A$ such that $a_2 b_1 \not\in C$. Proceeding by induction, for every $n \in \omega$ we shall construct two sequences of points $(a_n)_{n \in \omega}$ and $(b_n)_{n \in \omega}$ in $X$ such that

$$
\begin{align*}
(1) & \quad a_n \in A; \\
(2) & \quad b_n \in \bigcap_{i=0}^{n} B_{a_i}; \\
(3) & \quad a_{n+1} b_n \not\in C
\end{align*}
$$

for every $n \in \omega$.

Observe that for each $i \leq n$ we get $a_i b_n \in a_i B_{a_i} \subset C$ and hence $\{a_k b_n\}_{k \leq n} \cap \{b_n a_{n+1}\}_{n \in \omega} \subset C \cap (X \setminus C) = \emptyset$.

Proposition 6.4. For each commutative semigroup $X$ the semigroup $\varphi^*(X)$ is supercommutative. Moreover, $A \ast B = A \oplus B$ for each $A \in v^*(X), B \in \varphi(X)$.

Proof. It is sufficient to prove that $A \ast B \subset A \oplus B$ for each $A \in v^*(X), B \in \varphi(X)$. Let $C \in A \ast B$. We lose no generality assuming that the set $C$ is of the basic form $C = \bigcup_{a \in A} a \ast B_a$ for some finite set $A \in A$ and family $(B_a)_{a \in A} \subset B^A$. Since the set $A$ is finite, by definition of a filter, the intersection $\bigcap_{a \in A} B_a$ is nonempty and belongs to $B$. Hence $C \supset \bigcup_{a \in A} a \ast \left(\bigcap_{a \in A} B_a\right) = A \ast \left(\bigcap_{a \in A} B_a\right) \in A \oplus B$.

Problem 6.5. Characterize semigroups $X$ whose Stone-Čech extension $\beta(X)$ is supercommutative.

7. The (super)commutativity of the semigroups $v(X)$ and $v^*(X)$

In this section we shall characterize semigroups $X$ whose extensions $v^*(X)$ and $v(X)$ are commutative or supercommutative. The characterization will be given in terms of square-linear semigroups $X$ endowed with the digraph structure

$$
\Delta = \{(x, y) \in X \times X : xy = x^2\}.
$$

Theorem 7.1. For a commutative semigroup $X$ the following conditions are equivalent:

1. the semigroup $v^*(X)$ is commutative;
2. $v^*(X)$ is supercommutative;
3. $A \ast B \supseteq B \ast A$ for any filters $A, B \in \varphi^*(X) \subset v^*(X)$;
4. $A \ast B = A \oplus B$ for any upfamilies $A \in v^*(X)$ and $B \in v(X)$;
5. $(xu, yv) \cap \{xy, vu\} \neq \emptyset$ for any points $x, y, u, v \in X$;
6. $X$ is a square-linear semigroup whose digraph $(X, \Delta)$ contains no bipartite cycles.

Proof. We shall prove the implications $(4) \Rightarrow (2) \Rightarrow (1) \Rightarrow (3) \Rightarrow (5) \Rightarrow (6)$ $(4) \Rightarrow (4)$ among which the implications $(4) \Rightarrow (2) \Rightarrow (1)$ $(3)$ are trivial.

(3) $\Rightarrow$ (5) Assume that $\{xu, yv\} \cap \{xy, vu\} = \emptyset$ for some points $x, y, u, v \in X$, and consider the filters $A = \{\{x, y\}\}$ and $B = \{\{u, v\}\}$, which belong to the semigroup $\varphi^*(X)$. It is easy to see that $B^A = \{\{u\}, \{v\}\}$. Observe that $\{xu, yv\} \in A \ast B^A$ and $B^A \ast A = \{\{ux, uy\}, \{vx, vy\}\}$. Since $\{xu, yv\} \notin \{\{ux, uy\}, \{vx, vy\}\}$, we conclude that $A \ast B^A \neq B^A \ast A$.

(5) $\Rightarrow$ (6) To show that the semigroup $X$ is square-linear, take any two points $a, b \in X$, and put $x = v = a$ and $y = u = b$. Then $\{ab\} = \{xu, yv\} \subset \{xy, vu\} = \{a^2, b^2\}$, which means that the semigroup $X$ is square-linear. Next, we show that its digraph $(X, \Delta)$ contains no bipartite cycle. Assuming the converse and applying Lemma 4.1, we conclude that $X$ contains a pure cycle $x_0, x_1, x_2, x_3, x_4$ of length 4. For every $0 \leq i \leq 3$ the inclusion $(x_i, x_{i+1}) \in \Delta \setminus \Delta^{-1}$ implies $x_i x_{i+1} = x_i^2 \neq x_{i+1}^2$. Since $x_4 = x_0$, we get $x_4 x_1 = x_0 x_1 = x_1^2 \neq x_1^2$. Then for the points $x = x_1, y = x_3, u = x_2, v = x_4$, we get

$$
\{xu, yv\} \cap \{uy, vx\} = \{x_1 x_3, x_3 x_4\} \cap \{x_2 x_3, x_3 x_4\} = \{x_1^2, x_3^2\} \cap \{x_2^2, x_4^2\} = \emptyset.
$$

So, the condition (4) does not hold.

(6) $\Rightarrow$ (4) Assume that the subgroup $X$ is square-linear, but $A \ast B \neq A \oplus B$ for some upfamilies $A \in v^*(X)$ and $B \in v(X)$. Then $A \ast B \not\subset A \oplus B$ and hence $C \notin A \oplus B$ for some set $C \in A \ast B$. We lose no generality assuming that $C$ is of the basic form $C = \bigcup_{a \in A} a \ast B_a$ for some set $A \in A$ and sets $B_a \in B, a \in A$. Since $A \in v^*(X)$, we can assume that the set $A$ is finite.
Taking into account that \( C \not\in A \oplus B \), we conclude that \( A \ast B_a \not\subset C \) for each \( a \in A \). Choose any element \( a_0 \in A \). By induction, for every \( k \in \omega \) we shall choose points \( b_k \in B_{a_k} \) and \( a_{k+1} \in A \) with \( a_{k+1} \ast b_k \not\in C \) as follows. Assume that for some \( k \in \omega \) a point \( a_k \in A \) has been constructed. Consider the set \( B_{a_k} \ast A = A \ast B_{a_k} \not\subset C \) and find two points \( a_{k+1} \in A \) and \( b_k \in B_{a_k} \) such that \( b_k a_{k+1} \not\in C \).

Since the set \( A \supseteq \{a_k\}_{k \in \omega} \) is finite, for some point \( a \in A \) the set \( \Omega = \{k \in \omega : a_k = a\} \) is infinite. Fix any three numbers \( p, q, r \in \Omega \) such that \( 1 < p < p + 1 < q < q + 1 < r \). Since \( X \) is a square-linear semigroup, \( a_q b_q \in \{a^2_q, b^2_q\} \).

Now consider two cases.

(i) \( a_q b_q = b^2_q \). In this case we shall show that \( (b_{q+i}, a_{q+i}) \in \Delta \) and \( (a_{q+i+1}, b_{q+i}) \in \Delta \) for every \( i \in \omega \). This will be proved by induction on \( i \in \omega \). If \( i = 0 \), then the inclusion \( (b_q, a_q) \in \Delta \) follows from the equality \( a_q b_q = a^2_q \). Assume that for some \( i \in \omega \) we have proved that \( (b_{q+i}, a_{q+i}) \in \Delta \), which is equivalent to \( a_{q+i} b_{q+i} = b^2_{q+i} \). It follows from \( b^2_{q+i} = a_{q+i} b_{q+i} \neq b_{q+i} a_{q+i+1} \in \{b^2_{q+i}, a^2_{q+i+1}\} \) that \( b_{q+i} a_{q+i+1} = a^2_{q+i+1} \) and hence \( (a_{q+i+1}, b_{q+i}) \in \Delta \). Taking into account that \( (b^2_{q+i+1}, a^2_{q+i+1}) \in \Delta \), we see that \( a_{q+i+1} b_{q+i+1} = b^2_{q+i+1} \) and \( (b_{q+i+1}, a_{q+i+1}) \in \Delta \), which completes the inductive step.

Taking into account that \( \{b^2_{q+i+1}\}_{i \in \omega} = \{a^2_{q+i+1}\}_{i \in \omega} \subseteq \{a_k b_k\}_{k \in \omega} \subseteq C \) and \( \{a^2_{q+i+1}\}_{i \in \omega} = \{b_{q+i} a_{q+i+1}\}_{i \in \omega} \subseteq \{b_k a_{k+1}\}_{k \in \omega} \subseteq X \setminus C \), we conclude that \( \{b^2_{q+i+1}\}_{i \in \omega} \subseteq \{a^2_{q+i+1}\}_{i \in \omega} = \emptyset \), which implies that \( (b_{q+i}, a_{q+i+1}) \notin \Delta \setminus \Delta^{-1} \) for every \( i, j \in \omega \).

Now we see that \( a_r, b_{r-1}, a_{r-1}, \ldots, b_q, a_q \) is a bipartite cycle in the digraph \( (X, \Delta) \).

(ii) \( a_q b_q = a^2_q \). In this case we shall show that \( (a_{q-i}, b_{q-i}) \in \Delta \) and \( (b_{q-i+1}, a_{q-i}) \in \Delta \) for every \( 0 \leq i < q \). This will be proved by induction on \( i < q \). If \( i = 0 \), then the inclusion \( (a_q, b_q) \in \Delta \) follows from the equality \( a_q b_q = a^2_q \). Assume that for some non-negative number \( i < q - 1 \) we have proved that \( (a_{q-i}, b_{q-i}) \in \Delta \), which is equivalent to \( a_{q-i} b_{q-i} = a^2_{q-i} \). It follows from \( a^2_{q-i} = a_{q-i} b_{q-i} \neq b_{q-i-1} a_{q-i} \in \{a^2_{q-i+1}, a^2_{q-i-1}\} \) that \( a_{q-i-1} b_{q-i-1} = b^2_{q-i-1} \) and hence \( (a_{q-i-1}, b_{q-i-1}) \in \Delta \). Taking into account that \( b^2_{q-i-1} = b_{q-i-1} a_{q-i-1} \neq a_{q-i-1} b_{q-i-1} \in \{a^2_{q-i-1}, b^2_{q-i-1}\} \), we see that \( a_{q-i-1} b_{q-i-1} = a^2_{q-i-1} \) and \( (a_{q-i-1}, b_{q-i-1}) \in \Delta \), which completes the inductive step.

Taking into account that \( \{a^2_{q-i-1}\}_{i \in \omega} = \{a_k b_k\}_{k \in \omega} \subseteq C \) and \( \{b^2_{q-i-1}\}_{i \in \omega} = \{b_{q-i-1} a_{q-i}\}_{i \in \omega} \subseteq \{b_k a_{k+1}\}_{k \in \omega} \subseteq X \setminus C \), we conclude that \( \{a^2_{q-i-1}\}_{i \in \omega} \subseteq \{b^2_{q-i-1}\}_{i \in \omega} = \emptyset \), which implies that \( (b_{q-i-1}, a_{q-j}) \notin \Delta \setminus \Delta^{-1} \) for every \( 0 \leq i, j < q \).

Now we see that \( a_p, b_p, a_{p+1}, b_{p+1}, \ldots, a_{q-1}, b_{q-1}, a_q \) is a bipartite cycle in the digraph \( (X, \Delta) \).}

**Theorem 7.2.** For a commutative semigroup \( X \) the following conditions are equivalent:

1. the semigroup \( v(X) \) is commutative;
2. \( v(X) \) is supercommutative;
3. the semigroups \( v^*(X) \) and \( \beta(X) \) are commutative;
4. \( A \ast B^+ = B^+ \ast A \) for any filters \( A, B \in \mathcal{F}(X) \);
5. \( \{a_n b_n\}_{n \in \omega} \cap \{b_n a_{n+1}\}_{n \in \omega} \neq \emptyset \) for any sequences \( (a_n)_{n \in \omega} \) and \( (b_n)_{n \in \omega} \) in \( X \).

**Proof.** We shall prove the implications \( (2) \Rightarrow (1) \Rightarrow (4) \Rightarrow (5) \Rightarrow (2) \) and \( (1) \Rightarrow (3) \Rightarrow (5) \).

The implications \( (2) \Rightarrow (1) \Rightarrow (4) \) are trivial.

(4) \( \Rightarrow (5) \) Assume that there are sequences \( A = \{a_n\}_{n \in \omega} \) and \( B = \{b_n\}_{n \in \omega} \) in \( X \) such that \( \{a_n b_n\}_{n \in \omega} \cap \{b_n a_{n+1}\}_{n \in \omega} = \emptyset \). Consider the filters \( A = \langle A \rangle \) and \( B = \langle B \rangle \). It follows that \( \{b_n\} \in B^+ = \{C \subseteq X : C \cap B \neq \emptyset\} \) for every \( n \in \omega \). Assume that \( A \ast B^+ = B^+ \ast A \).

Since \( \{a_n b_n\}_{n \in \omega} \in A \ast B^+ = B^+ \ast A \), there is \( k \in \omega \) such that \( b_k \ast A \supset \{a_n b_n\}_{n \in \omega} \), which is not possible as \( b_k a_{k+1} \notin \{a_n b_n\}_{n \in \omega} \). So, \( A \ast B^+ \neq B^+ \ast A \).

(5) \( \Rightarrow (2) \) Assume that \( A \ast B \neq A \oplus B \) for some upfamilies \( A, B \in v(X) \). Then \( A \ast B \not\subseteq A \oplus B \) and hence \( C \not\subseteq A \oplus B \) for some set \( C \in A \ast B \). We lose no generality assuming that \( C \) is of basic form \( C = \bigcup_{a \in A} a B_a \) for some set \( A \in \mathcal{A} \) and sets \( B_a \in \mathcal{B}, a \in A \).

Taking into account that \( C \not\subseteq A \oplus B \), we conclude that \( B_a \ast A = A \ast B_a \not\subseteq C \) for each \( a \in A \). Choose any elements \( a_0 \in A \). By induction, for every \( k \in \omega \) we can choose points \( b_k \in B_{a_k} \) and \( a_{k+1} \in A \) such that \( b_k a_{k+1} \not\in C \). Then the sequences \( (a_n)_{n \in \omega} \) and \( (b_n)_{n \in \omega} \) have the required property \( \{a_n b_n\}_{n \in \omega} \cap \{b_n a_{n+1}\}_{n \in \omega} \subseteq C \cap (X \setminus C) = \emptyset \), which shows that \( (5) \) does not hold.

The implication \( (1) \Rightarrow (3) \) is trivial.
(3) ⇒ (5). Assume that the semigroups $\beta(X)$ and $\nu^*(X)$ are commutative but \( \{a_n b_n\}_{n \in \omega} \cap \{b_n a_n + 1\}_{n \in \omega} = \emptyset \) for some sequences \( (a_n)_{n \in \omega} \) and \( (b_n)_{n \in \omega} \). By Theorem 7.1, the semigroup $X$ is square-linear and its digraph $(X, \Delta)$ contains no bipartite cycles.

Two cases are possible.

(i) $a_n b_n \neq b_n^2$ for all $n \in \omega$, and then $a_n b_n = a_n^2$ for all $n \in \omega$. Then for each $n \in \omega$ we get \( \{b_n^2, a_n^2\}_{n \in \omega} \cap \{a_n b_n, a_n a_n + 1\}_{n \in \omega} = b_n a_n + 1 = a_n^2 \). If for every $i < j$ we get $a_i b_j = a_i^2$ and $b_i a_j = b_j^2$, then \( \{a_i b_j\}_{i < j} \cap \{b_i a_j\}_{i < j} = \emptyset \) and the semigroup $\beta(X)$ is not commutative by Theorem 6.1. So, there are numbers $i < j$ such that $a_i b_j \neq a_i^2$ or $b_i a_j \neq b_j^2$.

If $a_i b_j \neq a_i^2$, then $a_i b_j = b_j^2$, and $a_i, b_i, a_{i+1}, b_{i+1}, \ldots, a_j, b_j, a_i$ if a bipartite cycle in the digraph $(X, \Delta)$, which is not possible.

If $b_i a_j \neq b_j^2$, then $b_i a_j = a_j^2$, and then $b_i a_{i+1}, b_{i+1}, \ldots, b_{j-1}, a_j, b_i$ is a bipartite cycle in the digraph $(X, \Delta)$, which is not possible.

(ii) $a_n b_m = b_m^2$ for some $m \in \omega$. Repeating the argument of the proof of the implication (5) ⇒ (3) of Theorem 7.1 we can check that for every $i \in \omega$ $a_{m+i} b_{m+i} = b_{m+i}^2 \neq a_{m+i+1} = b_{m+i} a_{m+i+1}$ and hence \( \{b_m^2\}_{i \in \omega} \cap \{a_{m+i+1}\}_{i \in \omega} \subset \{a_k b_k\}_{k \in \omega} \cap \{b_k a_{k+1}\}_{k \in \omega} = \emptyset \). If for every $i < j$ we get $a_{m+i} b_{m+j} = b_{m+j}^2$ and $b_{m+i} a_{m+j} = a_{m+j}^2$, then $a_{m+i} b_{m+j} \cap \{b_{m+i} a_{m+j}\}_{i < j} = \emptyset$ and the semigroup $\beta(X)$ is not commutative by Theorem 6.1. So, there are numbers $i < j$ such that $a_{m+i} b_{m+j} \neq b_{m+j}^2$ or $b_{m+i} a_{m+j} \neq a_{m+j}^2$.

If $a_{m+i} b_{m+j} \neq b_{m+j}^2$, then $a_{m+i} b_{m+j} = a_{m+j}^2$, and $a_{m+i}, b_{m+j}, a_{m+j}, \ldots, b_{m+i}, a_{m+i}$ is a bipartite cycle in the digraph $(X, \Delta)$, which is not possible.

If $b_{m+i} a_{m+j} \neq a_{m+j}^2$, then $b_{m+i} a_{m+j} = b_{m+i}^2$, and $b_{m+i}, a_{m+j}, \ldots, b_{m+i+1}, a_{m+i+1}, b_{m+i+1}$ is a bipartite cycle in the digraph $(X, \Delta)$, which is a contradiction. \( \square \)

8. (Super)Commutativity of semigroups $N_2^*(X)$ and $N_2^*(X)$

In this section we develop semigroups with (super) commutative extensions $N_2(X)$ or $N_2^*(X)$.

Theorem 8.1. For a commutative semigroup $X$ the following conditions are equivalent:

1. the semigroup $N_2^*(X)$ is commutative;
2. $N_2^*(X)$ is supercommutative;
3. \( \{xu, yv\} \cap \{xy, yw, xw, yw\} = \emptyset \) for any points $x, y, u, v, w \in X$;
4. $A \ast B = A \ast B$ for any upfamilies $A \in N_2^*(X)$ and $B \in N_2(X)$;
5. $A \ast B = B \ast A$ for any $A \in \nu^*(X)$ and $B \in N_2^*(X)$;
6. Either $X$ is a square-linear semigroup whose digraph $(X, \Delta)$ contains no bipartite cycles or else $X$ contains a 2-element subgroup $H$ such that $x^2 \in H$ and $xy = x^3 y^3$ for each points $x, y \in X$.

Proof. We shall prove the implications (4) ⇒ (2) ⇒ (1) ⇒ (5) ⇒ (3) ⇒ (6) ⇒ (4) among which (4) ⇒ (2) ⇒ (1) ⇒ (5) are trivial.

To prove that (5) ⇒ (3), assume that $\{xu, yv\} \cap \{xy, yu, xw, yw\} = \emptyset$ for some points $x, y, u, v, w \in X$. Consider the filter $A = \langle \{x, y\} \rangle$ and the linked upfamily $B = \langle \{u, w\}, \{v, w\} \rangle$. By (5), $A \ast B = B \ast A$. Observe that the set $\{xu, xy, yu, yw, xu, yw\} = x \cdot \{v, w\} \cup y \cdot \{u, w\}$ belongs to the upfamily $A \ast B = B \ast A$. Then either $\{u, w\} \cdot \{x, y\} \subset \{xy, xu, yu, yw\}$ or $\{v, w\} \cdot \{x, y\} \subset \{xy, xu, yu, yw\}$. None of the inclusions is possible as $xu, yv \notin \{xy, xu, yu, yw\}$.

(3) ⇒ (6) If the semigroup $\nu^*(X)$ is commutative, then by Theorem 7.1, $X$ is a square-linear semigroup whose digraph $(X, \Delta)$ contains no bipartite cycles. So, we assume that the semigroup $\nu^*(X)$ is not commutative. Given any element $a \in X$, put $x = v = a, y = u = a^2$, and $w = a^3$. Then the condition (3) implies $xu = yv = a^2 \subseteq \{xu, yu, xw, yw\} = \{a^2, a^4, a^5\}$, which yields $a^2 = a^5$ for each $a \in X$. So, the semigroup $X$ is periodic and its set of idempotents $E = \{e \in X : e^2 = e\}$ is not empty. We claim that the semilattice $E$ is linear. Assuming the converse, find two idempotents $x, y \in E$ with $xy \notin \{x^2, y^2\}$ and put $u = v = y, w = xy$. Then $\{xu, yv\} \cap \{xy, xu, yu, yw\} = \{x^2, y^2\} \cap \{xy\} = \emptyset$, which contradicts the condition (3).

Next, we show that the semilattice $E$ has the smallest element. Assume the opposite. Since the semigroup $\nu^*(X)$ is not commutative, Theorem 7.1 yields four points $x, y, u, v \in X$ such that $\{xu, yv\} \cap \{xy, yu\} = \emptyset$. Consider the projection $e_x : X \rightarrow E, e_x : x \mapsto e_x$ of $X$ onto its idempotent band. Since the linear semilattice $E$ does not have the smallest element, there is an idempotent $w \in E$ such that $we_x = w 
eq e_x$ and $we_y = w 
eq e_y$. It follows that $ex_x = e_x \cdot e_x = w \neq e_x$ and hence $xu \neq xu$. By analogy we can prove that $\{xu, yv\} \cap \{xy, yu, xw, yw\} = \emptyset$, which implies $\{xu, yv\} \cap \{xy, xu, yu, yw\} = \emptyset$ and contradicts (3).

Therefore, the semilattice $E$ has the smallest element, which will be denoted by $e$. We claim that the maximal group $H_e$ containing this idempotent is not trivial. It follows from $\{xu, yv\} \cap \{xy, yu\} = \emptyset$ and $\{xu, yv\} \cap \{xy, xu, xw, yw\} = \emptyset \neq \{xy, xu, yu, yw\}$ that the set $\{xe, ye\}$ contains two elements and lies
Theorem 8.2. The points

It suffices to prove the implications (2).

Proof. The projection in the maximal subgroup \( H_e \) of the idempotent \( e \). So, the group \( H_e \) is not trivial. The equality \( a^3 = a^5 \) holding for each element \( a \in X \) implies that \( a^2 = e \) for each element \( a \) of the group \( H_e \). We claim that \( |H_e| = 2 \). In the other case, we could find three pairwise distinct points \( a, b, ab \in H_e \setminus \{ e \} \). Put \( x = u = a, y = v = b, \) and \( w = e \). Then \( \{ xu, yv \} \cap \{ xv, yu, xw, yw \} = \{ e \} \cap \{ ab, a, b \} = \emptyset \), which contradicts (3).

So, \( H_e = \{ e, h \} \) for some element \( h \in H_e \). Next, we show that \( e \) is the unique element of the semilattice \( E \). Assume that \( E \) contains some idempotent \( f \neq e \) and consider the points \( x = f, y = h, u = e, v = h, w = f \). Observe that \( \{ xu, yv \} \cap \{ xv, yu, xw, yw \} = \{ fe, h^2 \} \cap \{ fh, he, ff, hf \} = \{ e, e \} \cap \{ h, f \} = \emptyset \), which contradicts (3).

Next, we check that \( a^2 \in H_e \) for each \( a \in X \). Assume conversely that \( a^2 \notin H_e \). It follows from \( a^3 = a^5 \) that \( a^4 \) is an idempotent which coincides with \( e \) and hence \( a^3 \in H_e \). If \( a^3 = e \), then we can consider the points \( x = a, y = h, u = a^2, v = h \) and \( w = a \). Then \( \{ xu, yv \} \cap \{ xv, yu, xw, yw \} = \{ a^3, ha^2, a^2, ha \} = \{ e \} \cap \{ h, a^2 \} = \emptyset \), which contradicts (2). So, \( a^3 = h \) and then \( a^{2^i + 1} = h \) and \( a^{2^i + 2} = e \) for all \( i \in \mathbb{N} \). Consider the points \( x = a, y = a^2, u = a^3, v = a^2, \) and \( w = a \). Then \( \{ xu, yv \} \cap \{ xv, yu, xw, yw \} = \{ a^4 \} \cap \{ a^3, a^5, a^2, a^3 \} = \emptyset \), which contradicts (3).

Finally, we show that \( ab \in H_e \) for any points \( a, b \in X \). Assuming that \( ab \notin H_e \), for some \( a, b \in X \), consider the points \( x = a, y = b, u = b, v = a, \) and \( w = e \). Then \( \{ xu, yv \} \cap \{ xv, yu, xw, yw \} = \{ ab \} \cap \{ a^2, b^2, ae, be \} \subset \{ ab \} \cap H_e = \emptyset \), which contradicts (2). So, \( ab \in H_e \), and then \( ab = (ab)^3 = a^3b^3 \).

(6) \( \Rightarrow \) (4) If \( X \) is a square-linear semigroup whose digraph \( (X, \Delta) \) contains no bipartite cycle, then by Theorem 7.1 \( A \ast B = A \circ B \) for any upfamilies \( A \in \upsilon(X) \) and \( B \in \upsilon(X) \). Now assume that \( X \) contains a two-element subgroup \( H \subset X \) such that \( x^3 = H \) and \( xy = x^3y^3 \) for any points \( x, y \in X \). This means that for the projection \( \pi : X \to H, \pi : x \mapsto x^3 \), the semigroup \( X \) is a projection extension of the subgroup \( H \). Then the semigroup \( N_2(X) \) is a projection extension of the subsemigroup \( N_2(H) \). Since \( |H| = 2 \), by Proposition 6.4 the semigroup \( N_2(H) = \varphi^\ast(H) \) is supercommutative and hence for any linked upfamilies \( A, B \in N_2(X) \) we get

\[ A \ast B = \upsilon\pi(A) \ast \upsilon\pi(B) = \upsilon\pi(A) \circ \upsilon\pi(B) = A \circ B. \]

Theorem 8.2. For a semigroup \( X \) the following conditions are equivalent:

1. the semigroup \( N_2(X) \) is commutative;
2. \( N_2(X) \) is supercommutative;
3. the semigroups \( N_2^i(X) \) and \( \beta(X) \) are commutative;
4. \( A \ast B = A \circ \circ B \) for any upfamilies \( A \in \varphi(X) \) and \( B \in N_2(X) \);
5. for every sequence \( (a_i)_{i \in \omega} \in X^\omega \) and symmetric matrix \( (b_{ij})_{i,j \in \omega} \subset X^{\omega \times \omega} \) we get
\[ \{ a_i \cdot b_{ij} \}_{1 \leq i \leq \omega} \cap \{ b_{ij} \cdot a_{i+1} \}_{1 \leq i \leq \omega} \neq \emptyset \].
6. either the semigroup \( \upsilon(X) \) is commutative or else \( X \) contains a 2-element subgroup \( H \) such that \( x^3 \in H \) and \( xy = x^3y^3 \) for each points \( x, y \in X \).

Proof. It suffices to prove the implications (2) \( \Rightarrow \) (1) \( \Rightarrow \) (3) \( \Rightarrow \) (6) \( \Rightarrow \) (2) and (2) \( \Rightarrow \) (4) \( \Rightarrow \) (5) \( \Rightarrow \) (2). In fact, the implications (2) \( \Rightarrow \) (1) \( \Rightarrow \) (3) and (2) \( \Rightarrow \) (4) are trivial.

(3) \( \Rightarrow \) (6) Assume that the semigroups \( N_2^i(X) \) and \( \beta(X) \) are commutative but the semigroup \( \upsilon(X) \) is not commutative. By Theorem 7.2 the semigroup \( \upsilon(X) \) is not commutative. Combining Theorems 7.1 and 8.1 we conclude that \( X \) contains a 2-element subgroup \( H \) such that \( x^3 \in H \) and \( xy = x^3y^3 \) for each points \( x, y \in X \).

(6) \( \Rightarrow \) (2) If \( \upsilon(X) \) is commutative, then by Theorem 7.2 it is supercommutative and so is its subsemigroup \( N_2(X) \). If \( X \) contains a 2-element subgroup \( H \) such that \( x^3 \in H \) and \( xy = x^3y^3 \) for each points \( x, y \in X \), then for the projection \( \pi : X \to H, \pi : x \mapsto x^3 \), the semigroup \( X \) is a projection extension of the subgroup \( H \). By Proposition 4.1 the semigroup \( N_2(X) \) is a projection extension of the subsemigroup \( N_2(H) \). Since \( |H| = 2 \), the semigroup \( N_2(H) = \varphi^\ast(H) \) is supercommutative by Proposition 6.4. Being a projection extension of the supercommutative semigroup \( N_2(H) \), the semigroup \( N_2(X) \) is supercommutative by Corollary 5.3.

(4) \( \Rightarrow \) (5) Assume that for some sequence \( (a_i)_{i \in \omega} \in X^\omega \) and some symmetric matrix \( (b_{ij})_{i,j \in \omega} \subset X^{\omega \times \omega} \) we get \( \{ a_i \cdot b_{ij} \}_{1 \leq i \leq \omega} \cap \{ b_{ij} \cdot a_{i+1} \}_{1 \leq i \leq \omega} = \emptyset \). Consider the filter \( A = (A) \in \varphi(X) \subset N_2(X) \) generated by the set \( A = \{ a_i \}_{i \in \omega} \) and the linked system \( B \) generated by the family \( \{ B_i \}_{i \in \omega} \) of sets \( B_i = \{ b_{ij} \}_{i \in \omega} \). Observe that the set \( C = \{ a_i b_{ij} \}_{i,j \in \omega} \) belongs to \( A \ast B \). Assuming that \( A \ast B = A \circ \circ B \), we would find a number \( i \in \omega \) such that \( A \ast B_i \subset C \), which is not possible as \( a_{i+1} b_{ii} \notin C \).

(5) \( \Rightarrow \) (2) Assume that \( A \ast B \) is not supercommutative, we could find two linked upfamilies \( A, B \in N_2(X) \) such that \( A \ast B \not\subset A \circ \circ B \). Then for some set \( A \in A \) and a family \( (B_a)_{a \in A} \in B^A \), we get \( \bigcup_{a \in A} \varphi B_a \not\subset A \circ \circ B \). It follows that for every \( a \in A \) the product \( A \ast B_a \) is not contained in the set \( C = \bigcup_{a \in A} a \ast B_a \), which allows us to construct inductively two sequences of points \( (a_i)_{i \in \omega} \subset A^\omega \) and \( (b_i)_{i \in \omega} \in X^\omega \) such that \( b_i \in B_{a_i} \) and \( a_{i+1} b_i \not\in C \) for every \( i \in \omega \). For every numbers \( i < j \) put \( b_i = b_i \) and let \( b_{ij} = b_{ji} \) be some point of the
Theorem 9.1. For a commutative semigroup \(X\) the following conditions are equivalent:

1. the semigroup \(\lambda(X)\) is commutative;
2. for any symmetric matrices \((a_{ij})_{i,j \in \omega}, (b_{ij})_{i,j \in \omega} \in X^{\omega \times \omega}\) we get \(\{a_{ii} \cdot b_{jj}\}_{i,j \in \omega} \cap \{b_{ii} \cdot a_{jj}\}_{i,j \in \omega} \neq \emptyset\).

Proof. (1) \(\Rightarrow\) (2) Assuming that the semigroup \(\lambda(X)\) is not commutative, find two maximal linked systems \(A, B \in \lambda(X)\) such that \(A \ast B \neq B \ast A\). The maximal linked upfamilies \(A \ast B\) and \(B \ast A\) are distinct and hence contain two disjoint sets \(C \in A \ast B\) and \(C' \in B \ast A\). For these sets there are sets \(A \in B, B \in B\) and families of sets \((B_a)_{a \in A} \in B^A, (A_b)_{b \in B} \in A^B\) such that \(\bigcup_{A \in A} aB_a \subset C\) and \(\bigcup_{B \in B} bA_a \subset C'\).

By induction we can construct two sequences \(\{a_{ii}\}_{i \in \omega} \subset A\) and \(\{b_{ii}\}_{i \in \omega} \subset B\) such that \(b_{ii} \in B \cap B_{a_{ii}}\) and \(a_{i+1,i+1} \in A \cap A_{b_{ii}}\) for every \(i \in \omega\). Since the upfamilies \(B\) and \(A\) are linked, for every numbers \(i < j\) we can choose points \(b_j \in B_{a_{ii}} \cap B_{a_{ij}}\) and \(a_{i+1,j+1} \in A_{b_{ii}} \cap A_{b_{ij}}\), and put and \(b_j = b_j\) and \(a_{i+1,j+1} = a_{i+1,j+1}\). Also put \(a_0 = a_0 = a_00\) for all \(i \in \omega\). In such way we have defined two symmetric matrices \((a_{ij})_{i,j \in \omega}\) and \((b_{ij})_{i,j \in \omega}\) with coefficients in the semigroup \(X\). Observe that for each \(i, j \in \omega\) we get \(a_{ii} \ast b_{jj} \in a_{ii} \ast B_{a_{jj}} \subset C\) and \(b_{ii} \ast a_{i+1,j} \in b_{ii} \ast A_{a_{i+1,j}} \subset C'\), which implies that the sets \(\{a_{ii} \ast b_{jj}\}_{i,j \in \omega}\) and \(\{b_{ii} \ast a_{i+1,j}\}_{i,j \in \omega}\) are disjoint.

(2) \(\Rightarrow\) (1) Assume that there are two symmetric matrices \((a_{ij})_{i,j \in \omega}, (b_{ij})_{i,j \in \omega} \in X^{\omega \times \omega}\) such that the sets \(\{a_{ii} \cdot b_{jj}\}_{i,j \in \omega}\) and \(\{b_{ii} \cdot a_{jj}\}_{i,j \in \omega}\) are disjoint. Consider the sets \(A = \{a_{ii}\}_{i \in \omega}\) and \(A_1 = \{a_{ij}\}_{i,j \in \omega}\) which form a linear system \(\{A_{ii} \in A\}_{i = 0}^{\omega}\) can be enlarged to a maximal linked system \(A\). On the other hand, the sets \(B = \{b_{ii}\}_{i \in \omega}\) and \(B_1 = \{b_{ij}\}_{i,j \in \omega}\) form a linked upfamily, which can be enlarged to a maximal linked upfamily \(B\). We claim that \(A \ast B \neq B \ast A\). This follows from the fact that the maximal linked upfamilies \(A \ast B\) and \(B \ast A\) contains the disjoint sets

\[
\{a_{ii} \cdot b_{jj}\}_{i,j \in \omega} = \bigcup_{a_{ii} \in A} a_{ii}B_{a_{ii}} \subset A \ast B
\]

and

\[
\{b_{ii} \cdot a_{i+1,j}\}_{i,j \in \omega} = \bigcup_{b_{ii} \in B} b_{ii}A_{b_{ii}} \subset B \ast A.
\]

Therefore the semigroup \(A\) is not commutative.

For a set \(X\) consider the subset

\[
\lambda^*_X(X) = \{A \in \lambda(X) : \exists Y \subset X \text{ such that } |Y| \leq 3 \text{ and } A \in \lambda(Y) \subset \lambda(X)\} \subset \lambda^*(X).
\]

Theorem 9.2. For a commutative semigroup \(X\) the following conditions are equivalent:

1. the semigroup \(\lambda^*(X)\) is commutative;
2. any two maximal linked systems \(A, B \in \lambda^*_X(X)\) commute;
3. any two maximal linked systems \(A \in \lambda^*(X)\) and \(B \in \lambda(X)\) commute;
4. for any elements \(a, b, c, x, y, z \in X\) the sets \(\{ax, ay, cy, cz\}\) and \(\{xc, xb, za, zb\}\) are not disjoint;
5. for any elements \(x_0, x_1, x_2, x_3, x_4, x_5\) the sets \(\{x_1 x_2, x_2 x_3, x_3 x_4, x_4 x_5\}\) and \(\{x_1 x_2, x_2 x_3, x_3 x_4, x_4 x_5, x_5 x_0\}\) are not disjoint.

Proof. It suffices to prove the implications (3) \(\Rightarrow\) (1) \(\Rightarrow\) (2) \(\Rightarrow\) (4) \(\Leftrightarrow\) (5) \(\Rightarrow\) (1). In fact, the implications (3) \(\Rightarrow\) (1) \(\Rightarrow\) (2) are trivial while the equivalence (4) \(\Leftrightarrow\) (5) follows from the observation that for any points \(b = x_0, x = x_1, a = x_2, y = x_3, c = x_4, z = x_5\) in \(X\) we get

\[
\{ax, ay, cy, cz\} \cap \{xc, xb, za, zb\} = \{x_1 x_2, x_2 x_3, x_3 x_4, x_4 x_5\} \cap \{x_1 x_4, x_4 x_5, x_5 x_0\}.
\]

(2) \(\Rightarrow\) (4) Assume that for some elements \(a, b, c, x, y, z \in X\) the sets \(\{ax, ay, cy, cz\}\) and \(\{xc, xb, za, zb\}\) are disjoint. Consider the maximal linked systems \(A = \{A \subset X : |A \cap \{a, b, c\}| \geq 2\} \text{ and } \lambda = \{A \subset X : |A \cap \{x, y, z\}| \geq 2\}\) and observe that \(A, \lambda \in \lambda^*_X(X)\) and the products \(A \ast \lambda\) and \(\lambda \ast A\) are distinct since they contain disjoint sets

\[
a\{x, y\} \cup c\{y, z\} \in A \ast \lambda\) and \(\lambda \ast A\text{, and } x\{a, b\} \cup z\{a, b\} \in \lambda \ast A\text{ are disjoint. Hence, we can additionally assume that the set } A \text{ is finite.}
\]
By analogy with the proof of Theorem 9.1, construct inductively two sequences \((a_i)_{i \in \omega} \in A^\omega\) and \((b_i)_{i \in \omega} \in B^\omega\) such that \(b_i \in B \cap B_n\) and \(a_{i+1} \in A \cap A_n\). Since the set \(A\) is finite, there are two numbers \(k, m\) such that \(0 < k < m - 1\) and \(a_k = a_m\).

Let \(n = m - k \geq 2\) and consider the group \(\mathbb{Z}_n = \{0, 1, \ldots, n - 1\}\) endowed with the group operation of addition modulo \(n\), which will be denoted by the symbol \(\oplus\). So, \(1 \oplus (n - 1) = 0\). For each \(i \in \mathbb{Z}_n\) let \(a_{i+k} = b_{i+k}\). For every numbers \(i < j\) in \(\mathbb{Z}_n\) choose points \(b_{ij} = b_{ji} \in B_{ai} \cap B_{aj}\) and \(a_{ij} = a_{ji} \in A_{b_{ij}} \cap A_{b_{ji}}\), where \(i', j' \in \mathbb{Z}_n\) are unique numbers such that \(i = i' \oplus 1\) and \(j = j' \oplus 1\). It follows that \(a_{ii}b_{ij} \in a_{ii}B_{ai} \subset U_{AB}\) and \(b_{ii}a_{i+1,j} \in b_{ii}A_{b_{ii}} \in U_{BA}\). So,

\[
\{a_{ii} * b_{ij}\}_{i,j \in \mathbb{Z}_n} \cap \{b_{ii} * a_{i+1,j}\}_{i,j \in \mathbb{Z}_n} \subset U_{AB} \cap U_{BA} = \emptyset.
\]

By induction on \(i \in \mathbb{Z}_n\) we shall prove that \(a_{00} * b_{ii} \in U_{AB}\). This is trivial for \(i = 0\). Assume that for some positive number \(i < n - 1\) we have proved that \(a_{00} * b_{ii} \in U_{AB}\). Let \(x_0 = a_{i+1,i+2}, x_1 = b_{ii}, x_2 = a_{00}, x_3 = b_{0,i+1}, x_4 = a_{ii+1,i+1}, x_5 = b_{i+1,i+1}\).

It follows that

\[
\{x_1x_2, x_2x_3, x_3x_4, x_4x_5\} = \{b_{ii} * a_{00}, a_{00} * b_{0,i+1}, b_{0,i+1} * a_{ii+1,i+1}, a_{ii+1,i+1} * b_{i+1,i+1}\} \subset U_{AB} \cup a_{00} * B_{b_{ii}} \cup b_{0,i+1} * A_{b_{ii+1,i+1}} \cup a_{ii+1,i+1} * b_{i+1,i+1} \subset U_{AB}.
\]

On the other hand,

\[
\{x_0x_1, x_0x_5, x_1x_4\} = \{a_{i+1,i+2} * b_{ii}, a_{i+1,i+2} * b_{i+1,i+1}, b_{ii} * a_{ii+1,i+1}\} \subset b_{ii}A_{b_{ii+1,i+1}} \cup b_{i+1,i+1} * A_{b_{ii+1,i+1}} \in U_{BA}.
\]

Then \(\{x_1x_2, x_2x_3, x_3x_4, x_4x_5\} \cap \{x_0x_1, x_0x_5, x_1x_4\} \subset U_{AB} \cap U_{BA} = \emptyset\). By the condition (4), the intersection

\[
\{x_1x_2, x_2x_3, x_3x_4, x_4x_5\} \cap \{x_0x_1, x_0x_5, x_1x_4, x_2x_5\}
\]

is not empty, which implies that \(a_{00} * b_{ii+1,i+1} = x_2x_5 \in \{x_1x_2, x_2x_3, x_3x_4, x_4x_5\} \subset U_{AB}\). After completing the inductive construction, we conclude that \(a_{00} * b_{n-1,n-1} \in U_{AB}\) which is impossible as

\[
a_{00} * b_{n-1,n-1} = a_k * b_{k,n} = a_m * b_{m-1} = b_{m-1} * a_m \in U_{BA}.
\]

\[\square\]

We shall apply Theorem 9.2 to detecting monogenic semigroups that have commutative superextensions.

**Theorem 9.3.** For a monogenic semigroup \(X = \{x^k\}_{k \in \mathbb{N}}\) the following conditions are equivalent

1. \(\lambda(X)\) is commutative;
2. \(\lambda^*(X)\) is commutative;
3. \(x^n = x^m\) for some \((n,m) \in \{(1,2), (1,3), (2,3), (1,4), (2,4), (3,4), (1,5), (2,5), (3,5), (4,5), (2,6)\}\).

**Proof.** We shall prove the implications (3) \(\Rightarrow\) (1) \(\Rightarrow\) (2) \(\Rightarrow\) (3), among which the implication (1) \(\Rightarrow\) (2) is trivial.

(3) \(\Rightarrow\) (1) Assume that \(x^n = x^m\) for some

\[(n,m) \in \{(1,2), (1,3), (2,3), (1,4), (2,4), (3,4), (1,5), (2,5), (3,5), (4,5), (2,6)\}.
\]

If \((n,m) \in \{(1,2), (1,3), (1,4), (1,5)\}\) then \(X\) is isomorphic to a cyclic group of order \(\leq 4\) and \(\lambda(X)\) is commutative by Theorem 5.1 of [6].

If \((n,m) = (2,3)\), then the semigroup \(\lambda(X) = X\) is commutative.

If \((n,m) \in \{(2,4), (3,4)\}\), then \(|X| = 3\) and \(\lambda(X) = X \cup \{\Delta\}\) where \(\Delta = \{A \subset X : |A| \geq 2\}\). Taking into account that \(xy = yx \) and \(x \cdot x = x \cdot \Delta\) for all \(x, y \in X\), we see that the semigroup \(\lambda(X)\) is commutative.

If \((n,m) = (2,5)\), then \(x^n = x^a\) for every \(a \in X\) and hence \(X = \{x, x^2, x^3, x^4\}\) is a projective extension of the cyclic subgroup \(\{x^2, x^3, x^4\}\). In this case the commutativity of \(\lambda(X)\) follows from the commutativity of \(\lambda(C_4)\) according to Proposition 3.3.

By analogy, for \((n,m) = (2,6)\) the commutativity of the semigroup \(\lambda(X)\) follows from the commutativity of the semigroup \(\lambda(C_6)\).

Now consider the case \((n,m) = (3,5)\). In this case \(X = \{x, x^2, x^3, x^4\}\) and the semigroup \(\lambda(X)\) contains 12 elements:

\[k = \langle \{x^k\}\rangle,
\]

\[\Delta_k = \{\{A \subset X : |A| = 2, x^k \notin A\}\}
\]

and

\[\Delta_k = \{\{X \setminus \{x^k\} : A \subset X, |A| = 2, x^k \notin A\}\}\],

where \(k \in \{1, 2, 3, 4\}\). The following Cayley table of multiplication in the semigroup \(\lambda(X)\) implies the commutativity of \(\lambda(X)\):
In the final case \((n, m) = (4, 5)\), the product of any two nonprincipal maximal linked upfamilies is equal to the principal ultrafilter \(\langle \{x^2\} \rangle\), which implies that the semigroup \(\lambda(X)\) is commutative.

\((2) \Rightarrow (3)\) Let \(X = \{x^k\}_{k \in \mathbb{N}}\) be a monogenic semigroup with commutative extension \(\lambda^\bullet(X)\). If \(|X| \leq 4\), then \(x^n = x^m\) for some \((n, m) \in \{(1, 2), (1, 3), (2, 3), (1, 4), (2, 4), (3, 4), (1, 5), (2, 5), (3, 5), (4, 5)\}\). If \(x^6 = x^2\), then we are done. So, we assume that \(x^6 \neq x^2\) and \(|X| \geq 5\). In this case the elements \(x, x^2, x^3, x^4, x^5\) are pairwise distinct.

We claim that \(x^7 \in \{x^3, x^4\}\). In the opposite case we can put \(x_0 = x^4, x_1 = x^3, x_2 = x, x_3 = x^2, x_4 = x^2, x_5 = x\) and observe that

\[
\{x_1x_2, x_2x_3, x_3x_4, x_4x_5\} \cap \{x_1x_4, x_2x_5, x_0x_1, x_0x_5\} = \{x^3, x^4\} \cap \{x^2, x^5, x^7\} = \emptyset,
\]

which implies that the semigroup \(\lambda^\bullet(X)\) is not commutative according to Theorem 9.2. This contradiction shows that \(x^7 \in \{x^3, x^4\}\) and hence the monogenic semigroup \(X\) is finite.

If \(x^7 = x^3\), then we can put \(x_0 = x^3, x_1 = x_2 = x, x_3 = x^3, x_4 = x_5 = x^2\) and observe that

\[
\{x_1x_2, x_2x_3, x_3x_4, x_4x_5\} \cap \{x_1x_4, x_2x_5, x_0x_1, x_0x_5\} = \{x^2, x^4, x^5\} \cap \{x^3, x^6, x^7\} = \emptyset
\]

since \(x^6 \neq x^2\). By Theorem 9.2 the semigroup \(\lambda^\bullet(X)\) is not commutative.

If \(x^7 = x^4\), then we can put \(x_0 = x_1 = x, x_2 = x^4, x_3 = x^4, x_4 = x_5 = x^2\) and observe that

\[
\{x_1x_2, x_2x_3, x_3x_4, x_4x_5\} \cap \{x_1x_4, x_2x_5, x_0x_1, x_0x_5\} = \{x^5, x^7, x^4\} \cap \{x^3, x^6, x^2, x^3\} = \emptyset
\]

which implies that the semigroup \(\lambda^\bullet(X)\) is not commutative according to Theorem 9.2.

Now we establish some structural properties of semigroups \(X\) having commutative superextensions \(\lambda(X)\).

A semigroup \(X\) is called a 0-bouquet of its subsemigroups \(X_\alpha, \alpha \in I\), if

- \(X = \bigcup_{\alpha \in I} X_\alpha\);
- \(X\) has two-sided zero 0;
- \(X_\alpha \cap X_\beta = X_\alpha \star X_\beta = \{0\}\) for any distinct indices \(\alpha, \beta \in I\).

In this case we write \(X = \bigvee_{\alpha \in I} X_\alpha\).

**Proposition 9.4.** Assume that a semigroup \(X = \bigvee_{\alpha \in I} X_\alpha\) is a 0-bouquet of its subsemigroups \(X_\alpha, \alpha \in I\). The superextension \(\lambda(X)\) is commutative if and only if for each \(\alpha \in I\) the semigroup \(\lambda(X_\alpha)\) is commutative.

**Proof.** The “only if” part is trivial. To prove the “if” part, assume that the semigroup \(\lambda(X)\) is not commutative. By Theorem 9.1 there are two symmetric matrices \((a_{ij})_{i,j \in \omega}\) and \((b_{ij})_{i,j \in \omega}\) with the coefficients in \(X\) such that the sets \(A = \{a_{ij} * b_{ij}\}_{i,j \in \omega}\) and \(B = \{b_{ij} * a_{i+1,j}\}_{i,j \in \omega}\) are disjoint. Then \(0 \notin A\) or \(0 \notin B\).

First assume that \(0 \notin A\). Find an index \(\alpha \in I\) such that \(a_{00} \in X_\alpha\). It follows from \(0 \notin \{a_{00}b_{0j}\}_{j \in \omega}\) that \(b_{0j} \in X_\alpha\) for all \(j \in \omega\). Observe that for every \(i \in \omega\) we get \(a_{0i}b_{0i} = a_{0i}b_{00} \neq 0\) and hence \(a_{0i} \in X_\alpha\). Finally, for each \(i, j \in \omega\), the inequality \(a_{ij}b_{ij} \neq 0\) implies that \(b_{ij} \in X_\alpha\). So, \(\{a_{ii}\}_{i \in \omega} \cup \{b_{ij}\}_{i,j \in A} \subset X_\alpha\). Now for every \(i, j \in \omega\) put

\[
a'_{ij} = \begin{cases} a_{ij} & \text{if } a_{ij} \in X_\alpha, \\
0 & \text{otherwise}
\end{cases}
\]

and observe that \((a'_{ij})_{i,j \in \omega}\) is a symmetric matrix with coefficients in \(X_\alpha\). It follows that \((a'_{ij}b_{ij})_{i,j \in \omega} = \{a_{ii}b_{ii}\}_{i,j \in \omega} = A\) and \((b_{ij}a_{i+1,j})_{i,j \in \omega} \subset (B \cap X_\alpha) \cup \{0\}\). Since \(A \cap (B \cup \{0\}) = \emptyset\), Theorem 9.1 implies that the semigroup \(\lambda(X_\alpha)\) is not commutative.

By analogy, we can treat the case \(0 \notin B\). In this case there is \(\alpha \in I\) such that \(\{a_{ii}\}_{i \in \omega} \cup \{a_{i+1,j}\}_{i,j \in \omega} \subset X_\alpha \setminus \{0\}\). Changing the element \(a_{00}\) by 0, if necessary, we get \(\{a_{ij}\}_{i,j \in \omega} \subset X_\alpha\). Now for every \(i, j \in \omega\) put

\[
b'_{ij} = \begin{cases} b_{ij} & \text{if } b_{ij} \in X_\alpha, \\
0 & \text{otherwise}
\end{cases}
\]
Observe that \((a_{ij})_{i,j \in \omega} \text{ and } (b_{ij})_{i,j \in \omega}\) are symmetric matrices with coefficients in \(X\) such that \(\{a_{ij}b_{ij}\}_{i,j \in \omega} \subseteq A \cup \{0\}\) and \(\{a_{ij}a_{i+1,j}\}_{i,j \in \omega} = (a_{i}a_{i+1})_{j \in \omega} = B\). Since \((A \cup \{0\}) \cap B = \emptyset\), Theorem 9.1 implies that the semigroup \(\lambda(X_n)\) is not commutative. 

Now we detect regular semigroups \(X\) whose superextensions \(\lambda(X)\) are commutative.

In the following theorem for a natural number \(n \in \mathbb{N}\) by

\[ C_n = \{z \in \mathbb{C} : z^n = 1\} \]

we denote the cyclic group of order \(n\) and by

\[ L_n = \{0, \ldots, n - 1\} \]

the linear semilattice endowed with the operation of minimum.

For two semigroups \((X, \ast)\) and \((Y, \star)\) by \(X \sqcup Y\) we denote the semigroup \(X \times \{0\} \cup Y \times \{1\}\) endowed with the semigroup operation

\[(a, i) \circ (b, j) = \begin{cases} 
(a \ast b, 0) & \text{if } i = 0 \text{ and } j = 0, \\
(a, 0) & \text{if } i = 0 \text{ and } j = 1, \\
(b, 0) & \text{if } i = 1 \text{ and } j = 0, \\
(a \ast b, 1) & \text{if } i = 1 \text{ and } j = 1.
\end{cases}\]

The semigroup \(X \sqcup Y\) is called the \textit{ordered union} of the semigroups \(X\) and \(Y\). For example, the ordered union \(L_1 \sqcup C_2\) is isomorphic to the multiplicative semigroup \(\{-1, 0, 1\}\).

**Theorem 9.5.** The superextension \(\lambda(X)\) of a regular semigroup \(X\) is commutative if and only if one of the following conditions holds:

- \(X\) is isomorphic to one of the semigroups: \(C_2, C_3, C_4, C_2 \times C_2, C_2 \times L_2, L_1 \sqcup C_2, C_2 \sqcup L_n\) for some \(n \in \mathbb{N}\);
- \(X = \bigvee_{\alpha \in A} X_{\alpha}\) for some subsemigroups \(X_{\alpha}, \alpha \in A, \) isomorphic to \(L_1 \sqcup C_2\) or \(L_n\) for \(n \in \mathbb{N}\).

**Proof.** To prove the “if” part, assume that a semigroup \(X\) satisfies conditions (1) or (2). If \(X\) is isomorphic to one of the groups \(C_2, C_3, C_4, C_2 \times C_2\), then its superextension \(\lambda(X)\) is commutative according to Theorem 5.1 of [6]. If \(X\) is isomorphic to \(C_2 \times L_2\) or \(C_2 \sqcup L_n\) for some \(n \in \mathbb{N}\), then \(\lambda(X)\) is commutative by Theorem 1.1 of [5].

Next, assume that \(X = \bigvee_{\alpha \in A} X_{\alpha}\) is a 0-bouquet of its subsemigroups \(X_{\alpha}, \alpha \in A, \) isomorphic to \(L_1 \sqcup C_2\) or \(L_n, n \in \mathbb{N}\). By Theorem 1.1 of [5], the superextension of the semigroups \(L_1 \sqcup C_2\) and \(L_n, n \in \mathbb{N}\), are commutative. Consequently, for every \(\alpha \in X_{\alpha}\) the superextension \(\lambda(X_{\alpha})\) is commutative and by Proposition 9.4 the superextension \(\lambda(X)\) is commutative too. This completes the proof of the “if” part. The prove the “only if” part we shall use the following:

**Lemma 9.6.** The superextension \(\lambda(X)\) of a semigroup \(X\) is not commutative if \(X\) is isomorphic to one of the semigroups:

- (1) \(L_1 \sqcup C_n\) for \(n \geq 3\);
- (2) \(C_n \sqcup L_1\) for \(n \geq 3\);
- (3) \(L_1 \sqcup C_2 \sqcup L_1\);
- (4) \(L_2 \sqcup C_2\);
- (5) \((C_2 \times C_2) \sqcup L_1\);
- (6) \(L_1 \sqcup (C_2 \times C_2)\);
- (7) \(C_2 \sqcup C_2\).

**Proof.** If \(X = L_1 \sqcup C_n = \{e_1\} \cup \{a^n\}_{n \geq 1}\) for some \(n \geq 3\), then the maximal linked upfamilies \(\square = \{e_1, a^n\}, \{e_1, a\}, \{e_1, a^{-1}\}, \{a^n, a, a^{-1}\}\) and \(\Delta = \{\{a^n, a\}, \{a^n, a^{-1}\}, \{a, a^{-1}\}\}\) do not commute, since \(\{e_1, a^0\} = a^0 \{e_1, a^0\} \cup a\{e_1, a^{-1}\} \in \square \ast \square\) while \(\{e_1, a^0\} \notin \square \ast \Delta\).

2. If \(X = C_n \sqcup L_1 = \{a^n\}_{n \geq 0} \cup \{e_2\}\) for some \(n \geq 3\), then the maximal linked upfamilies \(\square = \{\{a^2, a\}, \{a^2, a^0\}, \{a^2, e_2\}, \{a^2, e_2\}\}\) and \(\Delta = \{\{a^2, e_2\}, \{a^2, a^2\}, \{e_2, a^2\}\}\) do not commute, since \(\{a^2, e_2\} = a^2 \{a^2, e_2\} \cup e_2\{a^2, e_2\} \in \square \ast \square\) while \(\{e_2, a^2\} \notin \triangle \ast \square\).

3. If \(X = L_1 \sqcup C_2 \sqcup L_1 = \{e_1\} \cup \{e_2, a\} \cup \{e_3\}\) where \(a \neq a^2 = e_2\), then the maximal linked upfamilies \(\square = \{e_1, e_3\}, \{e_2, e_3\}, \{e_1, e_2, a\}\) and \(\square = \{\{a, e_1\}, \{a, e_2\}, \{a, e_3\}, \{e_1, e_2, e_3\}\}\) do not commute, since \(\{e_1, e_2\} = e_1\{e_1, e_2, e_3\} \cup e_2\{e_1, e_2, e_3\} \cup a\{e_1, e_1\} \in \square \ast \square\) while \(\{e_1, e_2\} \notin \square \ast \square\).

4. If \(X = L_2 \sqcup C_2 = \{e_1, e_2\} \cup \{e_3, a\}\) where \(a \neq a^2 = e_3\), then the maximal linked upfamilies \(\square = \{e_1, e_2\}, \{e_1, e_3\}, \{e_1, a\}, \{e_2, e_3\}\) and \(\square = \{\{e_2, a\}, \{e_2, e_3\}, \{a, e_3\}\}\) do not commute, since \(\{e_2, e_3\} = e_2\{e_2, e_3\} \cup e_3\{e_2, e_3\} \cup a\{e_2, e_2\} \in \square \ast \square\) while \(\{e_2, e_3\} \notin \square \ast \square\).
5. If $X = (C_2 \times C_2) \sqcup \{e_2\}$ where $C_2 \times C_2 = \{e_1, a, b, ab\}$ and $a^2 = b^2 = (ab)^2 = e_1$, then the maximal linked upfamilies $\square = \{(a, b), (a, e_1), (a, e_2), (e_1, 2, b)\}$ and $\Delta = \{(e_1, e_2), (e_1, a), (e_2, a)\}$ do not commute, since $\{(a, e_2) = \langle e_1, e_2 \rangle \cup \{e_1, 2, a\} \in \square \ast \Delta$ and $\{(a, e_2) \notin \square \ast \Delta$.

6. If $X = \{e_1\} \sqcup (C_2 \times C_2)$ where $C_2 \times C_2 = \{e_2, a, b, ab\}$ and $a^2 = b^2 = (ab)^2 = e_2$, then the maximal linked upfamilies $\square = \{(e_1, e_2), (e_1, a), (e_1, b), (e_2, a, b)\}$ and $\Delta = \{(e_2, a), (e_2, b), (a, b)\}$ do not commute, since $\langle e_1, e_2 \rangle = e_2 \langle e_1, e_2 \rangle \cup \{a, e_1, a\} \in \Delta \ast \square$ and $\langle e_1, e_2 \rangle \notin \square \ast \Delta$.

7. Finally assume that $X = C_2 \sqcup C_2 = \{e_1, a_1\} \sqcup \{e_2, a_2\}$ where $e_1 < e_2$ are idempotents of $X$, $a_1^2 = e_1$, $a_2^2 = e_2$, and $e_1 \ast a_2 = e_1$. In this case the maximal linked upfamilies

$$\square_e = \{(e_1, a_1), (e_1, a_2), (e_1, e_2), (a_1, a_2, e_2)\} \text{ and } \square_a = \{(a_1, e_1), (a_1, e_2), (a_1, a_2), (e_1, a_2, e_2)\}$$

do not commute as $\langle e_1, e_2 \rangle = e_1 \langle e_1, e_2 \rangle \cup \{e_2, e_1, e_2\} \cup \{a_1, e_2, a_2\} \in \square_a \ast \square_e$ while $\langle e_1, e_2 \rangle \notin \square_e \ast \square_a$. \qed

Now we are ready to prove the "only if" part of Theorem 9.5. Assume that the superextension $\lambda(X)$ is commutative. In this case the regular semigroup $X$ is commutative and consequently $X$ is a Clifford inverse semigroup. By Theorem 5.1 of [6], the commutativity of $\lambda(X)$ implies that each subgroup of $X$ has cardinality $\leq 4$. By Theorem 2.7 [5], the idempotent band $E(X) = \{x \in X : xx = x\}$ of $X$ is a 0-bouquet of finite linear semilattices.

First we assume that $E(X)$ is a finite linear semilattice, which can be written as $E(X) = \{e_1, \ldots, e_n\}$ for some idempotents $e_1 < \cdots < e_n$. For every $i \in \{1, \ldots, n\}$ by $H_{e_i}$ we denote the maximal subgroup of $X$ containing the idempotent $e_i$. As we have shown the group $H_{e_i}$ has cardinality $|H_{e_i}| \leq 4$.

If $n = 1$, then the Clifford inverse semigroup $X$ coincides with the group $H_{e_1}$ and hence is isomorphic to $C_1 = L_1, C_2, C_3, C_4$ or $C_2 \times C_2$.

So, we assume that $n \geq 2$. Lemma 9.6(2,5) implies that for every $i < n$ the maximal subgroup $H_{e_i}$ has cardinality $|H_{e_i}| \leq 2$. For the maximal idempotent $e_n$ of $E(X)$ the complement $I = X \setminus H_{e_n}$ is an ideal in $X$. So, we can consider the quotient semigroup $X/I$, which is isomorphic to $L_1 \sqcup H_{e_n}$. The commutativity of $\lambda(X)$ implies the commutativity of the semigroup $\lambda(X/I)$. Now Lemma 9.6(1,6) implies that $|H_{e_n}| \leq 2$.

If $|E(X)| \geq 3$, then for any $1 < i < n$, the maximal subgroup $H_{e_i}$ is trivial according to Lemma 9.6(3) and then for the maximal idempotent $e_n$, the subgroup $H_{e_n}$ is trivial according to Lemma 9.6(4). Therefore, all maximal groups $H_{e_i}$, $1 < i \leq n$, are trivial. If the group $H_{e_n}$ is trivial, then $X = E(X)$ is isomorphic to the linear semilattice $L_n$. If $H_{e_n}$ is not trivial, then $H_{e_n}$ is isomorphic to $C_2$ and $X$ is isomorphic to $C_2 \sqcup L_{n-1}$.

It remains to consider the case $|E(X)| = 2$. In this case the groups $H_{e_1}, H_{e_2}$ have cardinality $\leq 2$ and then $X$ is isomorphic to $L_2, C_2 \sqcup L_1$, $L_1 \sqcup C_2$, $C_2 \times L_2$ or $C_2 \sqcup C_2$. However the case $X \cong C_2 \sqcup C_2$ is excluded by Lemma 9.6(7). This completes the proof of the case of linear semilattice $E(X)$.

Now we consider the case of non-linear semilattice $E(X)$. Write $E(X)$ as a 0-bouquet $E(X) = \bigvee_{a \in A} E_a$ of finite linear semilattices $E_a$. Let $e_0$ be the minimal idempotent of the semilattice $E(X)$. Since $E(X)$ is not linear, there are two idempotents $e_1, e_2 \in E(X) \setminus \{e_0\}$ such that $e_1 e_2 = e_0$. We claim that the maximal subgroup $H_{e_0}$ containing the idempotent $e_0$ is trivial. It follows from the "linear" case, that $|H_{e_0}| \leq 2$. Assuming that $H_{e_0}$ is not trivial, write $H_{e_0} = \{a, e_0\}$ and consider the maximal linked upfamilies $\Delta_0 = \{(a, e_1), \langle e_1, e_2 \rangle, \langle a, e_2 \rangle\}$ and $\Delta_a = \{(e_0, e_1), \langle e_1, e_2 \rangle, \langle e_2, e_0, e_2 \rangle\}$ which do not commute since $\Delta_0 \ast \Delta_a = \{\{e_0\} \neq \{\{e_0\} \neq \{\{a\} \neq \Delta_0 \ast \Delta_0\}$.

Consequently, the maximal subgroup $H_{e_0}$ is trivial and hence for every $\alpha \in A$ the subsemigroup $X_\alpha = \bigcup_{e \in E_a} H_e$ is isomorphic to $L_1 \sqcup C_2$ or $L_n$, $n \in \mathbb{N}$, by the preceding "linear" case. \qed

Theorems 9.3 and 9.5 imply:

**Corollary 9.7.** If a semigroup $X$ has commutative superextension $\lambda(X)$, then

1. for each $x \in X$ there is a pair $(n, m) \in \{(2, 5), (2, 6), (3, 5), (4, 5)\}$ such that $x^n = x^m$;
2. the idempotent semilattice $E(X) = \{x \in X : xx = x\}$ of $X$ is a 0-bouquet of finite linear semilattices; (3) the regular part $R(X) = \{x \in X : xx = x\}$ of $X$ is isomorphic to one of the following semigroups:
   - $L_1, C_2, C_3, C_4, C_2 \times C_2, C_2 \times L_2, C_2 \bigcup L_n$ for some $n \in \mathbb{N}$;
   - a 0-bouquet $\bigvee_{\alpha \in A} X_\alpha$ of subsemigroups $X_\alpha$, $\alpha \in I$, isomorphic to $L_1 \sqcup C_2$ or $L_n$ for $n \geq 2$.

10. **The supercommutativity of the superextensions $\lambda(X)$**

By Theorems 6.3, 7.1, 7.2, 8.1, 8.2 for any semigroup $X$, the semigroups $\lambda(X)$, $\varphi(X)$, $\varphi(X)$, $N_2(X)$, $N_2(X)$ are supercommutative if and only if they are commutative. In contrast, the supercommutativity of the superextension $\lambda(X)$ is not equivalent to its commutativity.

**Theorem 10.1.** For a monogenic semigroup $X = \{x^k\}_{k \in \mathbb{N}}$ the following conditions are equivalent:

1. the semigroup $\lambda(X)$ is supercommutative;
Proof. We shall prove the implications $(3) \Rightarrow (1) \Rightarrow (2) \Rightarrow (3)$ among which the implication $(1) \Rightarrow (2)$ is trivial.

$(3) \Rightarrow (1)$. Assume that $x^n = x^m$ for some $(n, m) \in \{(1, 2), (1, 3), (2, 3), (2, 4), (3, 4), (4, 5)\}$. For $(n, m) \in \{(1, 2), (1, 3), (2, 3)\}$ the monogenic semigroup $X$ has cardinality $|X| \leq 2$ and then the semigroup $\lambda(X) = X$ is supercommutative.

If $(n, m) = (2, 4)$, then the monogenic semigroup $X$ has cardinality $|X| = 3$ and for the unique non-principal maximal linked system $\Delta = \{A \subset X : |A| \geq 2\}$ in $\lambda(X)$ the product $\Delta \odot \Delta$ is equal to the principal ultrafilter $\langle x^2 \rangle = \Delta \ast \Delta$, which implies that the semigroup $\lambda(X)$ is supercommutative.

If $(n, m) \in \{(3, 4), (4, 5)\}$, then any two nonprincipal maximal linked systems $A, B$ contain sets $A \subseteq A, B \subseteq B$ such that $x \notin A, x \notin B$. Then $AB$ is a singleton, which implies $A \odot B = A \ast B$. Consequently, the semigroup $\lambda(X)$ is supercommutative.

$(2) \Rightarrow (3)$ Assume that for a monogenic semigroup $X = \{x^k\}_{k \in \mathbb{N}}$ the superextension $\lambda^*(X)$ is supercommutative. Then it is commutative and by Theorem 9.3, $x^n = x^m$ for some pair $(n, m) \in \{(1, 2), (1, 3), (2, 3), (1, 4), (2, 4), (3, 4), (3, 5), (2, 5), (3, 5), (4, 5), (2, 6)\}$.

We claim that $|m - n| \leq 2$. In the opposite case $X$ contains a cyclic subgroup $C$ of cardinality $|C| \geq 3$. The subgroup $C$ contains an element $x \in C$ such that the points $x^{-1}, x^0, x^1$ are pairwise distinct. Then for the maximal linked system $\Delta = \{\{x^{-1}, x^0\}, \{x^0, x^1\}, \{x^{-1}, x^1\}\} \in \lambda^*(X)$ the product

$$\Delta \odot \Delta = \langle \{x^{-2}, x^{-1}, x^0\}, \{x^{-1}, x^0, x^1\}, \{x^0, x^1, x^2\}\rangle$$

does not belong to $\lambda(C)$, which implies that $\Delta \odot \Delta \neq \Delta \ast \Delta$ and contradicts the supercommutativity of $\lambda(X)$. So, $(m - n) \leq 2$, which implies that $(n, m) \in \{(1, 2), (1, 3), (2, 3), (2, 4), (3, 4), (3, 5), (4, 5)\}$. It remains to exclude the case $(n, m) = (3, 5)$. In this case $X = \{x, x^2, x^3, x^4\}$ and for the maximal linked upfamilies

$$\square = \{\{x^2, x^3, x^4\}, \{x, x^2\}, \{x, x^3\}, \{x, x^4\}\}$$

and $\Delta = \langle \{x^2, x^3, x^4\}\rangle$ we get

$$\square \odot \Delta = \langle \{x^2, x^4\}, \{x^3, x^4\}\rangle \neq \square \ast \Delta,$$

which contradicts the supercommutativity of the semigroup $\lambda(X)$. $\square$

In the following theorem by $V_3$ we denote the semilattice $\{0, 1\}^2 \setminus \{(1, 1)\}$ endowed with the operation of coordinatewise minimum. Observe that a semilattice $X$ is isomorphic to $V_3$ if and only if $|X| = 3$ and $X$ is not linear.

**Theorem 10.2.** The superextension $\lambda(X)$ of a regular semigroup $X$ is supercommutative if and only if $X$ is isomorphic to one of the semigroups: $C_2$, $L_1 \sqcup C_2$, $V_3$ or $L_n$ for $n \in \mathbb{N}$.

**Proof.** First we prove the “if” part of the theorem. If $X = C_2$, then its superextension $\lambda(X) = X$ is supercommutative as all maximal linked upfamilies on $X$ are principal ultrafilters.

If $X = L_1 \sqcup C_2$, then $\lambda(X)$ is supercommutative since for the unique non-principal maximal linked system $\Delta = \{A \subset X : |A| \geq 2\}$ we get $\Delta \odot \Delta = \Delta = \Delta \ast \Delta$.

If $X = V_3$, then $\lambda(X)$ is supercommutative since for the unique non-principal maximal linked system $\Delta = \{A \subset X : |A| \geq 2\}$ the products $\Delta \odot \Delta = \langle \min V_3 \rangle = \Delta \ast \Delta$ coincide with the principal ultrafilter generated by the minimal element $(0, 0) = \min V_3$ of the semilattice $V_3$.

If $X = L_n$ for some $n \in \mathbb{N}$, then the supercommutativity of the semigroup $\lambda(X)$ follows from Theorem 2.5 of [14].

To prove the “only if” part, assume that $X$ is a regular semigroup with supercommutative superextension $\lambda(X)$. First observe that every subgroup $G$ of $X$ has cardinality $|G| \leq 2$. In the opposite case the group $G$ contains an element $x \in X$ such that $|\{x^1, x^0, x^{-1}\}| = 3$ where $x^0$ is the idempotent of the group $G$. Then for the maximal linked system $\Delta = \langle \{x^{-1}, x^0\}, \{x^0, x^1\}, \{x^{-1}, x^1\}\rangle$ the product $\Delta \odot \Delta = \langle \{x^{-2}, x^{-1}, x^0\}, \{x^{-1}, x^0, x^1\}, \{x^0, x^1, x^2\}\rangle$ does not belong to $\lambda(X)$ and hence is not equal to $\Delta \ast \Delta$. This contradiction shows that all subgroups of $X$ have cardinality $\leq 2$. This fact combined with Theorem 5.5 yields that $X$ is isomorphic to one of the semigroups:

- $L_1, C_2, C_2 \sqcup L_n$ for some $n \in \mathbb{N}$;
- a 0-bouquet $\bigvee_{\alpha \in I} X_\alpha$ of subsemigroups $X_\alpha$, $\alpha \in I$, isomorphic to $L_1 \sqcup C_2$ or $L_n$ for $n \geq 2$.

It remains to exclude the semigroups from this list, whose superextensions are not supercommutative.

If $X = C_2 \sqcup L_n$, then $X$ contains the semigroup $C_2 \sqcup L_1 = \{e_1, a\} \cup \{e_2\}$ where $a^2 = e_1 \neq a$ and $e_1 < e_2$ are idempotents. In this case for the maximal linked system $\Delta = \langle \{a, e_1\}, \{e_1, e_2\}, \{a, e_2\}\rangle$ we get $\Delta \odot \Delta = \langle \{a, e_1\}, \{e_1, e_2\}\rangle \notin \lambda(X)$ and hence $\Delta \odot \Delta \neq \Delta \ast \Delta$, which means that $\lambda(X)$ is not supercommutative.
If $X = L_1 \sqcup C_2 = \{e_1\} \cup \{e_2, a\}$ where $a^2 = e_2 > e_1$, then for the maximal linked system $\triangle = \langle\{e_1, e_2\}, \{e_1, e_3\}, \{e_2, a, e_2\}\rangle$ we get $\triangle \oplus \triangle = \langle\{e_1, e_2\}, \{e_2, a\}\rangle \notin \lambda(X)$ and hence $\triangle \oplus \triangle \neq \triangle \ast \triangle$, which means that $\lambda(X)$ is not supercommutative.

It remains to consider the case when $X = \bigcup_{\alpha \in I} X_{\alpha}$ is a 0-bouquet of subsemigroups $X_{\alpha}, \alpha \in I$, isomorphic to $L_n$ for $n \geq 2$. If $|I| = 1$, then $X$ is isomorphic to $L_n$ for some $n \geq 2$ and $\lambda(X)$ is supercommutative according to the “if” part.

If $|I| = 2$, then $X = X_i \uplus X_j$ for some non-trivial linear subsemilattices $X_i, X_j \subset X$ such that $X_i \cap X_j = (\min X_i)$. If $|X_i| = |X_j| = 2$, then the semilattice $X$ is isomorphic to the semilattice $V_3$ and its superextension $\lambda(X)$ is supercommutative as proved in the “if” part. So, we assume that $|X_i| \geq 3$ or $|X_j| \geq 3$. We loss no generality assuming that $|X_i| \geq 3$. Then we can find idempotents $e_0 < e_1 < e_2$ in $X_i$ and $e_3 \in X_j \setminus X_i$ such that $e_1 e_3 = e_2 e_3 = e_0 = \min X$. In this case for the maximal linked system $\triangle = \langle\{e_1, e_2\}, \{e_1, e_3\}, \{e_2, e_3\}\rangle$ the product $\triangle \oplus \triangle = \langle\{e_0, e_1\}, \{e_1, e_2\}\rangle \notin \lambda(X)$ and hence $\triangle \oplus \triangle \neq \triangle \ast \triangle$, which means that $\lambda(X)$ is not supercommutative.

If $|I| \geq 3$, then the semigroup $X$ contains a 4-element semilattic $V_4 = \{e_0, e_1, e_2, e_3\}$ where $e_i e_j = e_0 = \min X$ for any distinct number $i, j \in \{1, 2, 3\}$. In this we can consider the maximal linked system $\triangle = \langle\{e_1, e_2\}, \{e_1, e_3\}, \{e_2, e_3\}\rangle \in \lambda(V_4) \subset \lambda(X)$ and observe that $\triangle \oplus \triangle = \langle\{e_0, e_1\}, \{e_0, e_2\}, \{e_0, e_3\}\rangle \notin \lambda(X)$. Consequently, $\triangle \oplus \triangle \neq \triangle \ast \triangle$ and the semigroup $\lambda(X)$ is not supercommutative. \hfill \square

Theorems 10.11 and 10.2 imply:

**Corollary 10.3.** If a semigroup $X$ has supercommutative superextension $\lambda(X)$, then

1. for each $x \in X$ we get $x^4 \in \{x^2, x^3\}$;
2. the regular part $R(X) = \{x \in X : x \in xXX\}$ of $X$ is isomorphic to $C_2, L_1 \sqcup C_2, V_3$ or $L_n$ for some $n \in \mathbb{N}$.

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