ON WHITTAKER MODULES OVER A CLASS OF ALGEBRAS SIMILAR TO U(sl₂)

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Abstract. Motivated by the study of invariant rings of finite groups on the first Weyl algebras $A₁(\mathbb{C})$ and finding interesting families of new noetherian rings, a class of algebras similar to $U(sl₂)$ were introduced and studied by Smith in [13]. Since the introduction of these algebras, research efforts have been focused on understanding their weight modules, and many important results were already obtained in [13] and [6]. But it seems that not much has been done on the part of nonweight modules. In this note, we generalize Kostant’s results in [5] on the Whittaker model for the universal enveloping algebras $U(g)$ of finite dimensional semisimple Lie algebras $g$ to Smith’s algebras. As a result, a complete classification of irreducible Whittaker modules (which are definitely infinite dimensional) for Smith’s algebras is obtained, and the submodule structure of any Whittaker module is also explicitly described.

Introduction

Motivated by the study of the invariant rings of finite groups on the first Weyl algebra $A₁$ and other important things, an interesting class of algebras $R(f)$ similar to $U(sl₂)$ were introduced and studied by Smith in [13]. Each algebra $R(f)$ is generated by three generators $E, F, H$ subject to the relations $EF - FE = f(H)$, $HE - EH = E$, and $HF - FH = -F$, where $f$ is a polynomial in $H$. These algebras serve as a subclass of Witten’s 7-parameter deformations of $U(sl₂)$ as studied in [6]. As their name indicates, these algebras share a lot of similar properties with $U(sl₂)$. The ring theoretic properties and the highest weight modules were first investigated in detail in [13]. These algebras are somewhat commutative noetherian domain, and have the GK-dimension 3 ([13]). The center $Z(R)$ of $R(f)$ is also proved to be isomorphic to the polynomial ring in one variable. The primitive ideals are classified by Smith ([13]). Furthermore, a similar theory of highest weight modules and the category $𝒪$ is also constructed for these algebras by Smith ([13]). In particular, for some special parameters $f$, all finite dimensional representations of $R(f)$ are semisimple. For more details, we refer the reader to [13]. These algebras have also been further studied in [4] and [9] from the points of views of both ring theoretic properties and representation theory.

Since the introduction of these algebras, a lot of research efforts have been focused on trying to understand their weight modules ([13], [6]). But it seems to us that not much has been done for the part of nonweight modules. So it might be useful to present some specific constructions for nonweight irreducible modules over these algebras. In this paper, we are able to work out such a possibility by generalizing
Kostant’s results on the Whittaker model for the universal enveloping algebras $U(\mathfrak{g})$ of finite dimensional semisimple Lie algebras $\mathfrak{g}$ to Smith’s algebras $R(f)$. As an application, we obtain a complete classification of all irreducible Whittaker modules, and the submodule structure of any Whittaker module is also completely determined.

The initial investigation of the Whittaker model and hence Whittaker modules for semisimple Lie algebras was started by Kostant in the seminal paper [5]. The study of Whittaker modules is closely related to the Whittaker equations and has nice applications in the theory of Toda lattice. For a nonsingular character of the nilpotent subalgebra $\mathfrak{n}^+$ of $\mathfrak{g}$, Kostant introduced the Whittaker model of the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$ for finite dimensional semisimple Lie algebras $\mathfrak{g}$. Whittaker model was used to study the structure of Whittaker modules over $U(\mathfrak{g})$ and several important structure theorems were proved by Kontant for Whittaker modules in [5]. Note that Whittaker modules are very similar to Verma modules. But Whittaker modules have a special feature in that they are irreducible if and only if they admit a central character, which is is not always the case for Verma modules. The Whittaker model was later on generalized and studied for singular characters of $\mathfrak{n}^+$ by Lnych in his Ph.D. thesis [7]. Other similar works on this subject also appeared in [9] and [10]. As a matter of fact, Verma modules and Whittaker modules are two extreme cases of generalized Whittaker modules ([7], [9] and [10]). Furthermore, generalized Whittaker modules are mapped to holonomic $D$--modules on the flag variety of $\mathfrak{g}$ via the Beilinson-Bernstein localization ([2]). Based on this observation, a geometric study of Whittaker modules for finite dimensional semisimple Lie algebras was carried out in [9] and [10].

In addition, a quantum analogue of the Whittaker model has been constructed by Sevoastyanov for the topological version $U_h(\mathfrak{g})$ of quantized enveloping algebras by using their realizations via Coxeter elements in [14]. The major difficulty of a direct generalization of Kostant’s results to the quantized case lies in the fact that there is no nonsingular character for the positive part of the quantized enveloping algebras because of the quantized Serre relations. To resolve this issue, he has to turn to the topological version of quantized enveloping algebras which has different realizations admitting nonsingular characters for the positive part. In the case of $\mathfrak{g} = sl_2$, the situation is slightly different, since the quantized Serre relations are vacuum. Thus a direct generalization of Kostant’s approach should work. And this has recently been worked out by Ondrus in [11]. We have to admit that it is just a pure luck that a similar pattern works for Smith’s algebras.

Now let us mention a bit about the organization of this paper. In Section 1, we recall the definition of Smith’s algebras and some basic results on their properties. In Section 2, we construct the Whittaker model of the center $Z$ of $R(f)$, and classify all irreducible Whittaker modules. In Section 3, we investigate the submodule structure of any Whittaker module. Throughout this paper, the base field will be assumed to be C, though the results hold over any algebraically closed field of characteristic zero.

1. Algebras similar to $U(sl_2)$

In this section, we recall the definition and basic properties of the algebras $R(f)$ similar to $U(sl_2)$ as introduced by Smith in [13].
Definition 1.1. (See [13]) Let \( f \) be a polynomial in \( H \), the algebra \( R(f) \) is defined to be the \( \mathbb{C} \)-algebra generated by \( E, F, H \) subject to the following relations:

\[
EF - FE = f(H), \quad HE - EH = E, \quad HF - FH = -F.
\]

and \( R(f) \) is called an algebra similar to \( U(sl_2) \). We will sometimes denote it by \( R \) in short.

Proposition 1.1. (See [13]) \( R(f) \) has \( GK \)-dimension 3.

\[ \square \]

Proposition 1.2. (See [13]) \( R(f) \) is a somewhat commutative algebra.

\[ \square \]

Corollary 1.1. If \( V \) is a simple module, then every element of \( \text{End}_{R(f)}(V) \) is a scalar.

Proof: This follows from Quillen’s Lemma and the fact \( R \) is a somewhat commutative algebra ([13]). For more detail, we refer the reader to [13]. \[ \square \]

Let \( R(E) \) denote the subalgebra of \( R(f) \) generated by \( E \), and \( R(F, H) \) the subalgebra generated by \( F, H \). Then we have

Proposition 1.3. \( R(F, H) \) is isomorphic to the enveloping algebra of the two dimensional nonabelian Lie algebra.

Proof: The proof is obvious. \[ \square \]

From [13], we have the following fact:

Proposition 1.4. For any polynomial \( f(H) \), there exist another polynomial \( u(H) \) such that \( f = \frac{1}{2}\Delta(u) = \frac{1}{2}(u(H + 1) - u(H)) \).

\[ \square \]

In addition, \( R(f) \) has a Casimir element \( \Omega \) which is defined as \( \Omega = 2FE + u(H + 1) \) and a simple calculation shows the following:

Proposition 1.5. (See [13]) The center \( Z(R) \) of \( R \) is a polynomial ring generated by one variable \( \Omega = 2FE + u(H + 1) \) over \( \mathbb{C} \), where \( u \) is the polynomial such that \( f(H) = 1/2\Delta(u) \).

\[ \square \]

2. The Whittaker Model for the center \( Z(R) \) of \( R(f) \)

In this section, we work out the Whittaker model for the center \( Z(R(f)) \) of \( R(f) \), and use it to study Whittaker modules over \( R(f) \). We obtain similar results as in [5]. In fact, We will closely follow the formulation in [5] with some slight modifications.

Definition 2.1. An algebra homomorphism \( \eta : R(E) \to \mathbb{C} \) is called a nonsingular character of \( R(E) \) if \( \eta(E) \neq 0 \).

Definition 2.2. Let \( V \) be an \( R \)-module, a vector \( v \in V \) is called a Whittaker vector of type \( \eta \) if \( E \) acts on it through a nonsingular character \( \eta \), i.e., \( Ev = \eta(E)v \). If \( V = Rv \), then we call \( V \) a Whittaker module of type \( \eta \), and \( v \) is called a cyclic Whittaker vector of type \( \eta \).
From now on, we fix such a nonsingular character \( \eta \) of \( R(E) \). The following proposition follows from the triangular decomposition of \( R \) in [13]:

**Proposition 2.1.** \( R \) is isomorphic to \( R(F, H) \otimes R(E) \) as vector spaces and \( R \) is a free module over \( R(E) \).

Let \( \eta \): \( R(E) \rightarrow \mathbb{C} \) be the fixed nonsingular character of \( R(E) \), and we denote by \( R_{\eta}(E) \) the kernel of the character \( \eta \). Then we have

**Proposition 2.2.** \( R(E) = \mathbb{C} \oplus R_{\eta}(E) \). Thus \( R \cong R(F, H) \oplus R R_{\eta}(E) \).

**Proof:** Since \( R(E) = \mathbb{C} \oplus R_{\eta}(E) \) and \( R = R(F, H) \otimes (\mathbb{C} \oplus R_{\eta}(E)) \), so \( R \cong R(F, H) \oplus R R_{\eta}(E) \).

Now we can define a projector \( \pi : R \rightarrow R(F, H) \) from \( R \) into \( R(F, H) \) by taking the \( R(F, H) \) component of any \( u \in R \). We denote the image \( \pi(u) \) of \( u \) by \( u^\eta \) for short.

**Lemma 2.1.** If \( v \in Z(R) \), then we have \( u^\eta v^\eta = (uv)^\eta \).

**Proof:** Let \( u,v \in Z(R) \), then we have

\[
uv - u^\eta v^\eta = (u-u^\eta)v + u^\eta(v-v^\eta) = v(u-u^\eta) + u^\eta(v-v^\eta) \in RR_{\eta}(E).
\]

So \( (uv)^\eta = u^\eta v^\eta \).

**Proposition 2.3.** \( \pi : Z(R) \rightarrow R(F, H) \) is an algebra isomorphism of \( Z(R) \) onto its image \( W(F, H) \) in \( R(F, H) \).

**Proof:** It follows from that above lemma that \( \pi \) is a homomorphism of algebras. Since \( Z(R) = \mathbb{C}[\mathcal{O}] \) and \( \pi(\mathcal{O}) = 2\eta(E)F + u(H + 1) \) which is not zero in \( W(F, H) \), so \( \pi \) is a bijection. Hence \( \pi \) is an algebra isomorphism from \( Z(R) \) onto its image \( W(F, H) \) in \( R(F, H) \).

**Lemma 2.2.** If \( u^\eta = u \), then we have \( u^\eta v^\eta = (uv)^\eta \) for any \( v \in R \).

**Proof:** \( uv - u^\eta v^\eta = (u-u^\eta)v + u^\eta(v-v^\eta) = u^\eta(v-v^\eta) \in RR_{\eta}(E) \). So we have \( u^\eta v^\eta = (uv)^\eta \) for any \( v \in R \).

Let \( \hat{A} \) be the subalgebra of \( R \) generated by \( H \). Now we introduce a new gradation on \( R \) by setting \( \text{deg}(H) = 1, \text{deg}(E) = \text{deg}(F) = \text{deg}(f) + 1 \). This gradation is motivated by the so called \( x_0 \)-gradation suggested by Kazhdan (see [3]) for the universal enveloping algebras \( U(g) \) of semisimple Lie algebras \( g \). Let us denote \( \text{deg}(f) \) by \( d \). We can define a filtration of \( R(F, H) \) as follows:

\[
R_{(k)}(F, H) = \bigoplus_{i(d+1)+j \leq k} R_{i,j}(F, H)
\]

where \( R_{i,j}(F, H) \) is the vector space spanned by \( F^iH^j \). Since \( W(F, H) \) is a subalgebra of \( R(F, H) \), it inherits a filtration from \( R(F, H) \). In addition, \( \hat{A} \) has the natural gradation with \( \text{deg}(H) = 1 \). Let us put \( W(F, H)_q = \mathbb{C} - \text{spann}\{1, \Omega^\eta, \cdots, (\Omega^\eta)^q\} \), then we have the following:

**Theorem 2.1.** \( R(F, H) \) is free as a right module over \( W(F, H) \). And the multiplication induces an isomorphism

\[
\hat{A} \otimes W(F, H) \rightarrow R(F, H)
\]
as right \( W(F, H) \)-modules. In particular, we have the following

\[
\bigoplus_{p+q(d+1)=k} \hat{A}_p \otimes W(F, H)_q \cong R(F, H)_{(k)}
\]
\textbf{Proof:} First of all, $\hat{A} \times \text{W}(F, H) \rightarrow \text{R}(F, H)$ is bilinear. So by the universal property of the tensor product, there is a linear map from $\hat{A} \otimes \text{W}(F, H)$ into $\text{R}(F, H)$ defined by multiplication. It is easy to see this map is a homomorphism of right $\text{W}(F, H)$-modules and surjective as well. Now we show that it is injective. Suppose $\sum_{i=0}^{n} a_i H^i \left( \sum_{j=0}^{m} b_j (2\eta(E)) F + u(H + 1) \right) = 0$ with $b_m \neq 0$. Then we have $\sum_{i=0}^{n} a_i H^i b_m (2\eta(E)) = m F^m + g(H, F) = 0$, where the $F$-degree of $g$ is less than $m$ by direct computations. But $H^i F^j$ are part of the basis of $\text{R}(H, F)$ as a vector space, hence $\sum_{i=0}^{n} a_i H^i = 0$. Thus, the theorem has been proved.

Let $Y_\eta$ be the left $\text{R}$-module defined by $Y_\eta = \text{R} \otimes \mathbb{C}_\eta$ where $\mathbb{C}_\eta$ is the $1$-dimensional $\text{R}(E)$-module defined by $\eta$. It is easy to see that $Y_\eta = \text{R}/\text{R} \eta(E)$ is a Whittaker module with a cyclic vector $1_\eta = 1 \otimes 1$. And we have a quotient map

$$\text{R} \rightarrow Y_\eta$$

If $u \in \text{R}$, then $u^\eta$ is the unique element in $\text{R}(F, H)$ such that $u 1_\eta = u^\eta 1_\eta$. As in [5], we define the $\eta$-reduced action of $\text{R}(E)$ on $\text{R}(F, H)$ such that $\text{R}(F, H)$ is an $\text{R}(E)$ module in the following way:

$$x \bullet v = (xu)^\eta - \eta(x) v$$

where $x \in \text{R}(E)$, $v \in \text{R}(F, H)$.

\textbf{Lemma 2.3.} Let $u \in \text{R}(F, H)$ and $x \in \text{R}(E)$, then we have

$$x \bullet u^\eta = [x, u]^\eta$$

\textbf{Proof:} $[x, u] 1_\eta = (xu - ux) 1_\eta = (xu - \eta(x) u) 1_\eta$. Hence

$$[x, u]^\eta = (xu)^\eta - \eta(x) u^\eta = (xu)^\eta - \eta(x)u^\eta = x \bullet u^\eta$$

\hfill \Box

\textbf{Lemma 2.4.} Let $x \in \text{R}(E)$, $u \in \text{R}(F, H)$, and $v \in \text{W}(E, F)$, then we have

$$x \bullet (uv) = (x \bullet u)v.$$

\textbf{Proof:} Let $v = w^\eta$ for some $w \in \text{Z}(\text{R})$, then $uv = uw^\eta = u^\eta w^\eta = (uw)^\eta$. Thus

$$x \bullet (uv) = x \bullet (uw)^\eta = [x, uw]^\eta = (x, u) w^\eta = [x, u]^\eta w^\eta = [x, u] v = (x \bullet u)v = (x \bullet u)v.$$

\hfill \Box

Let $V$ be an $\text{R}$-module and $R_V$ be the annihilator of $V$ in $\text{R}$. Then $R_V$ defines a central ideal $Z_V \subset \text{Z}$ by setting $Z_V = R_V \cap \text{Z}$. Suppose that $V$ is a Whittaker module with a cyclic Whittaker vector $w$, we denote by $R_w$ the annihilator of $w$ in $\text{R}$. It is obvious that $R_R \eta(E) + RZ_V \subset R_w$. In the next theorem, we show that the reversed inclusion holds.

First of all, we need a lemma:

\textbf{Lemma 2.5.} Let $X = \{ v \in \text{R}(F, H) \mid (x \bullet v)w = 0, x \in \text{R}(E) \}$. Then

$$X = (\hat{A} \otimes \text{W}_V(F, H)) + \text{W}(F, H)$$

where $W_V(F, H) = (Z_V)^\eta$. In fact, $R_w(F, H) \subset X$ and

$$R_w(F, H) = \hat{A} \otimes W_V(F, H)$$

where $R_w(F, H) = R_w \cap \text{R}(F, H)$. 

Proof: Let us denote by $Y = \tilde{A} \otimes W_V(F, H) + W(F, H)$. Thus we need to verify $X = Y$. Let $v \in W(F, H)$, then $v = u^\eta$ for some $u \in Z(R)$. So $x \cdot v = x \cdot u^\eta = (xu^\eta)^\eta - \eta(x)u^\eta = \eta(x)^\eta - \eta(x)^\eta = (xu^\eta)^\eta - \eta(x)^\eta = xu^\eta - \eta(x)^\eta = 0$. So $W(F, H) \subset X$. Let $u \in Z_V$ and $v \in R(F, H)$, then for any $x \in R(E)$ we have $x \cdot (vu^\eta) = (x \cdot v)u^\eta$. Since $u \in Z_V$, then $u^\eta \in R_w$. Thus we have $vu^\eta \in X$, hence $\tilde{A} \otimes R_V(F, H) \subset X$. This proves $Y \subset X$. Let $A_i$ be the one dimensional subspace of $R(H)$ spanned by $H^\eta$ and $W_V(F, H)$ be the complement of $W_V(F, H)$ in $W(F, H)$. Set $M_i = A_i \otimes W_V(F, H)$, then we have the following:

$$W(F, H) = M \oplus Y$$

where $M = \sum_{i \geq 1} M_i$. We show that $M \cap X = 0$. Let $M[k] = \sum_{1 \leq i \leq k} M_i$, then $M[k]$ are a filtration of $M$. Suppose $k$ is the smallest integer such that $X \cap M[k] \neq 0$ and $0 \neq y \in X \cap M[k]$. Then we have $y = \sum_{1 \leq i \leq k} y_i$ where $y_i \in \tilde{A} \otimes W(V(F, H))$. Now we have $0 \neq E \cdot y_i \in A_{i-1} \otimes W_V(F, H)$ for $i \geq 1$. Hence we have $E \cdot y \in M[k-1]$. This is a contradiction. So we have $X \cap M = 0$. Now we prove that $R_w(F, H) \subset X$. Let $u \in R_w(F, H)$ and $x \in R(E)$, then we have $xuw = 0$ and $uxw = \eta xuw = 0$. Thus $x \cdot u = [x, u]^\eta$. Since $u \in R_w(F, H)$, then $x \cdot u = [x, u]^\eta$. Thus $x \cdot u \in R_w(F, H)$. So $u \in X$ by the definition of $X$. Now we prove the above statement.

In fact, $W_V(F, H) = (Z^\eta_V)$ and $W_V(F, H) \subset R_w(F, H)$, then we can uniquely write $v = u^\eta$ for $u \in Z(R)$. Then $uv = 0$ implies $uw = 0$ and hence $u \in Z(R) \cap R_w(F, H)$. Since $V$ is generated cyclically by $w$, we have proved the above statement.

Obviously, we have $R(E, F, H)Z_V \subset R_w(E, F, H)$. Thus we have $\tilde{A} \otimes W_V(F, H) \subset R_w(F, H)$, hence we have $R_w(F, H) = \tilde{A} \otimes W_V(F, H)$. □

Theorem 2.2. Let $V$ be a Whittaker module admitting a cyclic Whittaker vector $w$, then we have

$$R_w = RZ_V + RR^\eta(E).$$

Proof: It is obvious that $RZ_V + RR^\eta(E) \subset R_w(E, F, H)$, we show that $u \in R(E, F, H)Z_V + R(E, F, H)R^\eta(E)$. Let $v = u^\eta$, then it suffices to show that $v \in \tilde{A} \otimes W_V(F, H)$. But $v \in R_w(F, H) = \tilde{A} \otimes W_V(F, H)$. □

Theorem 2.3. Let $V$ be any Whittaker module for $R$, then the correspondence

$$V \rightarrow Z_V$$

sets up a bijection between the set of all equivalence classes of Whittaker modules and the set of ideals of $Z(R)$.

Proof: Let $V_i$, $i = 1, 2$ be two Whittaker modules. If $Z_{V_i} = Z_{V_2}$, then clearly $V_1$ is equivalent to $V_2$ by the above Theorem. Now let $Z_*$ be an ideal of $Z(R)$ and let $L = RZ_* + R^\eta(E)$. Then $V = R/L$ is a Whittaker module with a cyclic Whittaker vector $w = \tilde{1}$. Obviously, we have $R_w = L$. So that $L = R_w = RZ_* + RR^\eta(E)$. This implies that $\eta(L) = (Z_*)^\eta = \eta(R_w) = (ZV)^\eta$. Since $\eta$ is injective, thus $Z_V = Z_*$. Hence we have completed the proof.

Theorem 2.4. Let $V$ be an $R$–module. Then $V$ is a Whittaker module if and only if

$$V \cong R \otimes_{Z \otimes R(E)} (Z/Z_*)^\eta$$
Theorem 2.5. Let $\text{Ann}_{R\otimes R(F)}(1_*) = R(E)Z_0 + Z(R)R_\eta(E)$. The annihilator of $w = 1 \otimes 1$, is $R_w = R(E,F,H)Z_0 + R(E,F,H)R_\eta(E)$. Then the result follows from the last theorem.

Proof: If $1_*$ is the image of 1 in $Z/Z_0$, then $Ann_{Z(Z \otimes R(F))}(1_*) = R(E)Z_0 + Z(R)R_\eta(E)$. The annihilator of $w = 1 \otimes 1$, is $R_w = R(E,F,H)Z_0 + R(E,F,H)R_\eta(E)$. Then the result follows from the last theorem.

Theorem 2.5. Let $V$ be an $R-$module with a cyclic Whittaker vector $w \in V$. Then any $v \in V$ is a Whittaker vector if and only if $v = uw$ for some $w \in Z(R)$.

Proof: If $v = uw$ for some $u \in Z(R)$, then obviously $v$ is a Whittaker vector. Now let $v = uw$ for some $u \in R$ be a Whittaker vector of $V$. Then $v = u^\theta w$ by the definition of Whittaker module. So that we can assume that $u \in R(F,H)$. If $x \in R(E)$, we have $xvw = \eta(x)uw$ and $uxw = \eta(x)uw$. Thus we have $[x,u]w = 0$ and hence $[x,u]^\theta w = 0$. But we have $x \cdot u = [x,u]^\theta$. Thus we have $u \in X$. We can now write $u = u_1 + u_2$ where $u_1 \in R(F,H)$, and $u_2 \in W(F,H)$. Then $u_1w = 0$. Thus $u_2w = v$. But now $u_2 = u_3^\theta$ where $u_3 \in Z(R)$. So we have $v = u_3w$, which proves the theorem.

Now let $V$ be a Whittaker module and $End_R(V)$ be the endomorphism ring of $V$ as an $R-$module. Then we can define the following homomorphism of algebras defined by the action of $Z(R)$ on $V$:

$$\pi_V : Z \longrightarrow End_R(V)$$

It is clear that $Z(R)/Z(V) = \pi_V(Z(R)) \subset End_R(V)$. In fact the next theorem says that this inclusion is equal as well.

Theorem 2.6. Assume that $V$ is a whittaker module. Then $End_R(V) \cong Z/Z_V$. In particular $End_R(V)$ is commutative.

Proof: Let $w \in V$ be a cyclic Whittaker vector. If $\alpha \in End_R(V)$, then we have $\alpha(w) = uw$ for some $u \in Z(R)$ by the above theorem. Thus we have $\alpha(vw) = uvw = uuv = u\alpha(v)$. Hence $\alpha = \pi_V(u)$, which proves the theorem.

Now we are going to construct some Whittaker modules. Let $\xi : Z(R) \longrightarrow \mathbb{C}$ be a central character. For any given central character $\xi$, $Z_\xi = Ker(\xi) \subset Z(R)$ is a maximal ideal of $Z(R)$. Since $\mathbb{C}$ is an algebraically closed field, then $Z_\xi = (\Omega - \eta_\xi)$. Given any central character $\xi$, let $C_{\xi,\eta}$ be the one dimensional $Z \otimes R(E) -$module defined by $uvy = \xi(u)\eta(v)y$ for any $u \in Z$ and $v \in R(E)$. Let $Y_{\xi,\eta} = R(E,F,H) \otimes_{Z \otimes R(E)} C_{\xi,\eta}$

It is easy to see that $Y_{\xi,\eta}$ is a Whittaker module of type $\eta$ and admits an infinitesimal central character $\xi$. Since $\mathbb{R}$ is almost commutative, so by Quillen’s lemma, we know every irreducible representation has an infinitesimal central character. As studied in [32], we know $R$ has a similar Verma module theory. In fact, Verma modules also fall into the category of Whittaker modules if we allow the trivial central character on $R(E)$. Namely, we have

$$M_\lambda = R \otimes_{R(E,H)} \mathbb{C}_\lambda$$

where $R(H)$ acts on $\mathbb{C}_\lambda$ through $\lambda$ and $R(E)$ acts trivially on $\mathbb{C}_\lambda$. $M_\lambda$ admits an infinitesimal character with $\xi = \xi(\lambda)$. It is well-known that Verma modules may not be necessarily irreducible, even though they have infinitesimal central characters. While Whittaker modules are on the other extreme as shown in the next theorem:
Theorem 2.7. Let $V$ be a Whittaker module for $R$. Then the following conditions are equivalent.

(1) $V$ is irreducible.

(2) $V$ admits a central character.

(3) $Z_V$ is a maximal ideal.

(4) The space of Whittaker vectors of $V$ is one-dimensional.

(5) All nonzero Whittaker vectors of $V$ are cyclic.

(6) The centralizer $\text{End}_R(V)$ is reduced to the constants $\mathbb{C}$.

(7) $V$ is isomorphic to $Y_{\xi,\eta}$ for some central character $\xi$.

Proof: It is easy to see that (2) – (7) are equivalent to each other by using the above theorems we have just proved. We also know (1) implies (2). To complete the proof, it suffices to show that (4) implies irreducibility. To this true, we show that any submodule $V'$ of $V$ contains a nonzero Whittaker vector, which closes the proof. Let $v \in V$, we recall that the reduced $\eta$–action is defined as follows:

$$x \cdot v = xv - \eta(x)v$$

for any $x \in R(E,F,H)$. If $u \in R$ and $x \in R(E)$, then we have $x \cdot (uv) = xuv - \eta(x)uv = [x,u]v + uxv - \eta(x)uv$. Since $uxv = \eta(x)uv$, thus we have the following:

$$x \cdot (uv) = [x,u]v$$

Now let

$$R(E,F,H) \longrightarrow V$$

be the morphism from $R(E,F,H)$ into $V$ by mapping $u \in R$ to $u \cdot w$. Then this map is a homomorphism of the $R(E)$–module $R$ under the adjoint action of $R(E)$ into the $R(E)$–module $V$ under $\eta$–reduced action. Note the adjoint actin of $R(E)$ on $R$ is locally finite. Let $0 \neq v \in V'$ and write $v = uw$ for $u \in R$. Let $R_0$ be the $R(E)$–submodule of $R$ generated by $u$. Then the submodule $R_0 \subset R$ is finite dimensional. Thus the image $V_0$ of $R_0$ inside $V$ is finite dimensional. And $R(E)$ is the enveloping algebra of the one dimensional Lie algebra generated by $E$, which acts nilpotently on $V_0$ via the reduced action. Since we have $v \in V_0$, then $V_0 \subset V'$. So by Engel’s Theorem, we have $x \cdot v_0 = 0$ for some $0 \neq v_0 \in V_0$ for all $x \in R(E)$. So $v_0$ is a Whittaker vector.

It is easy to prove the following theorems, we refer the reader to [5] for details about their proofs:

Theorem 2.8. Let $V$ be an $R$–module which admits an infinitesimal character. Assume that $w \in V$ is a Whittaker vector. Then the submodule $Rw \subset V$ is irreducible.

Theorem 2.9. Let $V_1, V_2$ be any two irreducible $R$–modules with the same infinitesimal character. If $V_1$ and $V_2$ contain Whittaker vectors, then these vectors are
unique up to scalars. Furthermore, $V_1$ and $V_2$ are isomorphic to each other as $R$-modules.

In fact, we have the following description about the basis of any irreducible Whittaker module $(V, w)$, where $w \in V$ is a cyclic Whittaker vector.

**Theorem 2.10.** Let $(V, w)$ be an irreducible Whittaker module with a Whittaker vector $w$, then $V$ has a $\mathbb{C}$-vector space basis consisting of elements $\{H^iw \mid i \geq 0\}$.

**Proof:** Since $w$ is a cyclic Whittaker vector of the Whittaker module $(V, w)$, then we have $V = Rw$. Since $R = R(F, H) \otimes_{R(E)} \mathbb{C} \eta$, then we have $V = R(F, H)w$. Since $(V, w)$ is irreducible, then $(V, w)$ has a central infinitesimal character. Thus we have $\Omega w = \lambda(\Omega)w$. Now $\Omega w = (2\eta(E)F + u(H + 1))w$. Hence the action of $F$ on $(V, w)$ is uniquely determined by the action of $H$ on $(V, w)$. Thus the theorem follows.

3. The submodule structure of a Whittaker module $(V, w)$

In this section, we spell out the details about the structure of submodules of a Whittaker module $(V, w)$. We have a clean description of all submodules through the geometry of the affine line $\mathbb{A}^1$. Throughout this section, we will fix a Whittaker module $V$ of type $\eta$ and a cyclic Whittaker vector $w$ of $V$. Our argument is more or less the same as the one in [11].

**Lemma 3.1.** Let $Z(R) = \mathbb{C}[\Omega]$ be the center of $R$, then any maximal ideal of $Z(R)$ is of the form $(\Omega - a)$ for some $a \in \mathbb{C}$.

**Proof:** This fact follows from the assumption that $\mathbb{C}$ is algebraically closed field and Hilbert Nullstenllenzuts Theorem.

Let $Z_V$ be the annihilator of $V$ in $Z(R)$, then $Z_V = (f(\Omega))$ for some polynomial $f(\Omega) \in Z(R)$. Suppose that $f = \prod_{i=1,2,\cdots,m} f^n_i$, for some irreducible polynomials $f_i$. Then we have the following:

**Proposition 3.1.** $V_i = R \prod_{j \neq i} f^n_j w$ are indecomposable submodules of $V$. In particular, we have $V = V_1 \oplus \cdots \oplus V_m$ as a direct sum of submodules.

**Proof:** It is easy to verify that $V_i$ are submodules. Now we show each $V_i$ is indecomposable. Suppose not, we can assume without loss of generality that $V_i = W_1 \oplus W_2$. Note that $Z_{V_i} = Z_{W_1} \cap Z_{W_2}$. Since $Z(R)$ is a principal ideal domain, hence $Z_{V_i} = (g_i(\Omega))$. Thus we have $g_i \mid f^n_i$. This implies that the decomposition is not a direct sum. Therefore, $V_i$ are all indecomposable. The decomposition follows from the Chinese Reminder Theorem.

**Proposition 3.2.** Let $(V, w)$ be a Whittaker module and $Z_V =< f^n >$ where $f$ is an irreducible polynomial in $\mathbb{C}[\Omega]$. Let $V_i = Rf^i w, i = 0, \cdots n$ and $S_i = V_i / V_{i+1}, i = 0, \cdots, n - 1$. Then $S_i, i = 0, \cdots, n - 1$ are irreducible Whittaker modules of the same type $\eta$ and form a composition series of $V$. In particular, $V$ is of finite length.

**Proof:** The proof follows from the fact that $Z_{S_i} =< f^i >$ for all $i$. 

Remark 3.1. $V$ being of finite length is an analogue of the classical situation. In deed, in the classical case, Whittaker modules of $U(\mathfrak{g})$ are mapped to holonomic $D-$modules on the flag variety of $\mathfrak{g}$ by the Beilinson-Bernstein localization ([2]), and therefore are of finite length ([9] and [10]).

With the same assumption, we have the following

Corollary 3.1. $V$ has a unique maximal submodule $V_1$.

Proof: This is obvious, since the only maximal ideal of $Z_V$ is $<f>$.  

Based on the above propositions, the irreducibility and indecomposib ility are reduced to the investigation of $Z_V$. One has that $V$ is irreducible if and only if $Z_V$ is a maximal ideal. And $V$ is indecomposable if and only if $Z_V$ is primary. The following proposition is just a refinement of the submodule struture of $(V, w)$.

Proposition 3.3. Suppose $(V, w)$ is an indecomposible Whittaker module with $Z_V =<f^n>$, then any submodule $V' \subset V$ is of the form:

$$V' = Rf^i w$$

for some $i \in \{0, \cdots , n\}$.  

Now we are going to investigate the submodule structure of any Whittaker module $(V, w)$ with a nontrivial central annihilator $Z_V$. First of all, we recall some notations from [5]. Let $V' \subset V$ be any submodule of $V$, we define an ideal of $Z$ as follows:

$$Z(V') = \{x \in Z \mid xV' \subset V'\}$$

We may call $Z(V')$ the normalizer of $V'$ in $Z$. Conversely, for any ideal $J \subset Z$ containing $Z_V$, $JV \subset V$ is a submodule of $V$.

Theorem 3.1. Let $(V, w)$ be a Whittaker module with $Z_V \neq 0$. Then there is a one-to-one correspondence between the set of all submodules of $V$ and the set of all ideals of $Z$ containing $Z_V$ given by the maps $V' \rightarrow Z(V')$ and $J \rightarrow JV$. These maps are inverse to each other.

Proof: The proof is straightforward.  

Now we have a description of the basis of any Whittaker module $(V, w)$ as follows.

Proposition 3.4. Let $(V, w)$ be a Whittaker module and suppose that $Z_V =< f(\Omega) >$ where $f \neq 0$ is a monic polynomial of degree $n$. Then $B = \{F^i H^j w \mid 0 \neq i \leq n-1, j \in \mathbb{Z}_{\geq 0}\}$ is a basis of $(V, w)$. If $f = 0$, then $B = \{F^i H^j w \mid i, j \in \mathbb{Z}_{\geq 0}\}$ is a basis of $(V, w)$.

Proof: The proof is easy.  

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