Abstract. We introduce the moduli space of quasi-parabolic $SL(2, \mathbb{C})$-Higgs bundles over a compact Riemann surface $\Sigma$ and consider a natural involution, studying its fixed point locus when $\Sigma$ is $\mathbb{CP}^1$ and establishing an identification with a moduli space of null polygons in Minkowski 3-space.

1. Introduction

Moduli spaces of Higgs bundles over a compact Riemann surface $\Sigma$ were introduced by Hitchin in [Hi1] and have innumerable applications in many areas of mathematics and mathematical physics. In particular, Hitchin establishes in [Hi1] a correspondence between isomorphism classes of rank-2 $SL(2, \mathbb{C})$-Higgs bundles over $\Sigma$ with trivial determinant and the moduli space of representations of the fundamental group of $\Sigma$ in $SL(2, \mathbb{C})$. Since then these spaces have been generalized in several different ways.

Simpson in [Si1, Si2] extended these spaces to higher dimensions and, in another direction, to Higgs bundles on punctured Riemann surfaces, introducing the definition of parabolic Higgs bundles. Here he established a correspondence between isomorphism classes of these bundles and representations of the fundamental group of the punctured surface with fixed holonomy around the punctures.

On the other hand, in [Hi2], Hitchin replaced $SL(2, \mathbb{C})$ by any real reductive group $G$, introducing the notion of a $G$-Higgs bundle. In particular, he showed that, when $G^C$ is a complex semisimple Lie group and $G$ is the split real form of $G^C$, the moduli space of representations of the fundamental group of $\Sigma$ in $G$ has a connected component homeomorphic to a Euclidean space. When $G = SL(2, \mathbb{R})$ this component can be identified with the Teichmüller space and is often referred to as Hitchin component. The definition of $G$-Higgs bundles was further extended in [BGM] to include parabolic $G$-Higgs bundles.

In [G-P1] Garcia-Prada considers involutions in the moduli space of $SL(n, \mathbb{C})$-Higgs bundles and studies their fixed point sets. For example, some of the connected components of the sets obtained coincide with the moduli space of $SL(n, \mathbb{C})$-bundles and others are formed by $SL(n, \mathbb{R})$-Higgs bundles. The study of involutions and their fixed point sets was further pursued in [BGH].

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The fixed point sets of involutions on moduli spaces of $G$-Higgs bundles have attracted much attention recently as they are a source of branes [KW]. Indeed, as explained in [HT] and [DP], the moduli space of $G$-Higgs bundles for a semisimple group $G$ and the moduli space of $L^G$-Higgs bundles, for the Langlands dual $L^G$ of $G$, are mirror manifolds in the sense of Strominger, Yau and Zaslow [SYZ] and the fixed point sets of involutions on these spaces are branes, i.e. special subvarieties which can be of one of two types: $A$-branes (which are Lagrangian subvarieties) and $B$-branes (which are complex subvarieties). Since the moduli space of $G$-Higgs bundles is hyperkähler it admits three complex structures, so it is possible to have branes that are simultaneously of type $A$ and $B$ with respect to different complex structures.

The study of involutions on the moduli space of parabolic $SL(2, \mathbb{C})$-Higgs bundles was introduced in [BFGM], where the natural anti-holomorphic involution

\[(E, \Phi) \to (E, -\Phi).\]

was considered. The authors use the isomorphism constructed in [GM] between the moduli space of hyperpolygons (an hyperkähler quotient of $T^*\mathbb{C}^{2n}$) and the moduli space of parabolic $SL(2, \mathbb{C})$-Higgs bundles over $\mathbb{C}P^1$ with trivial underlying vector bundle $E$, to study the fixed point set of (1) (formed by $(B, A, A)$-branes). In particular, it is shown that the non-compact components of this manifold correspond to $SL(2, \mathbb{R})$-representations of the fundamental group of the punctured sphere (cf. [BGM]) and can be identified with moduli spaces of polygons in Minkowski 3-space with edges of fixed Minkowski lengths, with some of the edges lying in future pseudospheres and the others in past pseudospheres. This provided a nice geometrical interpretation of these $(B, A, A)$-branes in terms of moduli spaces of another related problem.

Under Langlands duality the mirrors of these $(B, A, A)$-branes should be $(B, B, B)$-branes. When passing from one type of brane to its mirror, one should expect an intermediate step where one considers a new moduli space of $G$-Higgs bundles: the \textit{moduli space of quasi-parabolic $SL(2, \mathbb{C})$-Higgs bundles over $\mathbb{C}P^1$}. Quasi-parabolic bundles were introduced by Mukai in [M] and by Mehta and Seshadri [MS]. Here we generalize their definition to quasi-parabolic Higgs bundles with an appropriate definition of stability. One should see these spaces as limit spaces of parabolic Higgs bundles where the two parabolic weights at each parabolic point are allowed to coincide.

In this space we consider the anti-holomorphic involution

\[(E, \Phi) \mapsto (E^*, \Phi^t)\]

and give a detailed description of its fixed point set when restricted to the (generic) case when the underlying rank-2 vector bundle over $\mathbb{C}P^1$ is trivial. We obtain $2^n-1-(n+1)$ connected components (where $n$ is the number of quasi-parabolic points) composed of quasi-parabolic $SL(2, \mathbb{C})$-Higgs bundles that admit a direct sum decomposition $E = L_0 \oplus L_1$ where $L_0$ and $L_1$ are trivial bundles over $\mathbb{C}P^1$ and the 1-dimensional flag components of $E$ coincide with the fibers of $L_0$ over some of the quasi-parabolic points and with the fibers of $L_1$ over the others.
To obtain this characterization of the involution fixed point set we establish a correspondence between classes of quasi-parabolic stable Higgs bundles over $\mathbb{CP}^1$ and regular points of a special singular hyperkähler quotient of $T^*\mathbb{C}^{2n}$ by

$$K := \left( SU(2) \times U(1)^n \right) / (\mathbb{Z}/2\mathbb{Z}),$$

here called null hyperpolygons.

We further establish a correspondence between the components of the fixed point set of (2) and moduli spaces of null polygons in Minkowski 3-space, consisting of $SU(1,1)$-equivalence classes of closed polygons in $\mathbb{R}^{2,1}$ (i.e. $\mathbb{R}^3$ equipped with the Minkowski inner product $v \circ w = -x_1x_2 - y_1y_2 + t_1t_2$) with some of the edges in the future light cone and the others in the past. This gives a geometrical interpretation of the $(B,A,A)$-branes obtained in terms of moduli spaces in a related space.

The paper is organized as follows. Null hyperpolygons are defined in Section 2 as the set of regular points of a singular hyperkähler quotient of $T^*\mathbb{C}^{2n}$ by the group $K$ and as a GIT quotient by $K^\mathbb{C}$. In Section 3 we define quasi-parabolic Higgs bundles and establish a diffeomorphism between the moduli space of null hyperpolygons and the moduli space $H^0_n$ of quasi-parabolic $SL(2,\mathbb{C})$-Higgs bundles over $\mathbb{CP}^1$ at a divisor $D$ consisting of $n$ marked points, where the underlying vector bundle is holomorphically trivial. In Section 4 we consider the involution in (2) and study its fixed-point set, showing that it is formed by quasi-parabolic $SL(2,\mathbb{R})$-Higgs bundles. For that we use the diffeomorphism of Section 3. We then define the spaces of null polygons in Minkowski 3-space in Section 5 and, in Section 6, we show that the connected components of the fixed point set of the involution (2) can be identified with null polygon spaces in $\mathbb{R}^{2,1}$. Finally, in Section 7 we study the example of quasi-parabolic $SL(2,\mathbb{C})$-Higgs bundles over $\mathbb{CP}^1$ with four quasi-parabolic points.

2. Null Hyperpolygon spaces

In this section we extend the definition of hyperpolygon spaces introduced in [K].

As usual, let $n$ be a positive integer and let us consider a star-shaped quiver $Q$ with vertex and arrow sets parametrized respectively by $I \cup \{0\}$ and $I$, where

$$I := \{1, \ldots, n\}.$$ 

Moreover, assume that, for each $i \in I$, the corresponding arrow goes from $i$ to 0. The representations of $Q$ with $V_i = \mathbb{C}$ for $i \in I$ and $V_0 = \mathbb{C}^2$ are parametrized by

$$E(Q,V) := \bigoplus_{i \in I} \text{Hom}(V_i, V_0) = \mathbb{C}^{2n}.$$ 

Using the standard diagonal action of $U(2) \times U(1)^n$ on $E(Q,V)$, one obtains an hyper-Hamiltonian action of $U(2) \times U(1)^n$ on the cotangent bundle $T^*E(Q,V) = T^*\mathbb{C}^{2n}$ for the natural hyperkähler structure on $T^*E(Q,V)$ [K]. Note that $T^*E(Q,V)$ can be identified
with the space of representations of the doubled quiver (the quiver with the same set of vertices, where we add an arrow in the opposite direction for every arrow of \( Q \)).

The hyper-Hamiltonian action of \( U(2) \times U(1)^n \) is not effective since every point in \( T^*E(Q,V) \) is fixed by the diagonal circle

\[ \{ (\lambda \cdot \text{Id}_{C^2}, \lambda, \ldots, \lambda) : |\lambda| = 1 \} \subset U(2) \times U(1)^n. \]

Therefore one considers the quotient group

\[ K := \left( U(2) \times U(1)^n \right) / U(1) = \left( SU(2) \times U(1)^n \right) / (\mathbb{Z}/2\mathbb{Z}) , \]

where \( \mathbb{Z}/2\mathbb{Z} \) acts by multiplication of each factor by \(-1\).

Let us consider coordinates \((p,q)\) on \( T^*C^2n \), where \( p = (p_1, \ldots, p_n) \) is the \( n \)-tuple of row vectors

\[ p_i = \begin{pmatrix} a_i \\ b_i \end{pmatrix} \in (C^2)^* \]

and \( q = (q_1, \ldots, q_n) \) is the \( n \)-tuple of column vectors \( q_i = \begin{pmatrix} c_i \\ d_i \end{pmatrix} \in C^2 \). Then, the action of \( K \) on \( T^*C^2n \) is given by

\[ (p, q) \cdot [A; e_1, \ldots, e_n] = \left( (e_1^{-1} p_1 A, \ldots, e_n^{-1} p_n A), (A^{-1} q_1 e_1, \ldots, A^{-1} q_n e_n) \right). \]

It is hyper-Hamiltonian\(^1\) with hyperkähler moment map

\[ \mu_{HK} := \mu_R \oplus \mu_C : T^*C^2n \longrightarrow (\text{su}(2)^* \oplus \mathbb{R}^n) \oplus (\mathfrak{sl}(2, C)^* \oplus (\mathbb{C}^n)^*) , \]

where \( \mu_R \), the real moment map, is

\[ \mu_R(p,q) = \frac{1}{2} \sum_{i=1}^n (q_i q_i^* - p_i^* p_i) + \left( -\frac{1}{2}(|q_1|^2 - |p_1|^2), \ldots, -\frac{1}{2}(|q_n|^2 - |p_n|^2) \right) , \]

with \( ()_0 \) representing the traceless part, and \( \mu_C \), the complex moment map, is given by

\[ \mu_C(p,q) = -\sqrt{-1} \sum_{i=1}^n (q_i p_i)_0 + (\sqrt{-1} p_1 q_1, \ldots, \sqrt{-1} p_n q_n) . \]

Let us consider the set

\[ \mathcal{P}_0^n := \left\{ (p,q) \in \mu_{HK}^{-1}((0,0),(0,0)) : |p_i|^2 + |q_i|^2 \neq 0, \forall i = 1, \ldots, n \right\} . \]

Note that an element \((p,q)\) in \( T^*C^2n \) is in \( \mu_C^{-1}(0,0) \) if and only if

\[ p_i q_i = 0 \quad \text{and} \quad \sum_{i=1}^n (q_i p_i)_0 = 0 , \]

that is, if and only if

\[ a_i c_i + b_i d_i = 0 \]

\(^1\)For the symplectic forms \( \omega_I, \omega_J, \omega_K \) associated to the standard triple of complex structures \( I, J, K \) on \( T^*C^2n \) that satisfy the quaternionic relations.
and
\[ \sum_{i=1}^{n} a_i c_i = 0, \quad \sum_{i=1}^{n} a_i d_i = 0, \quad \sum_{i=1}^{n} b_i c_i = 0. \]

Similarly, \((p, q)\) is in \(\mu^{-1}(0, 0)\) if and only if
\[ |q_i| = |p_i| \quad \text{and} \quad \sum_{i=1}^{n} (q_i q_i^* - p_i p_i^*)_0 = 0, \]
i.e., if and only if
\[ |c_i|^2 + |d_i|^2 = |a_i|^2 + |b_i|^2 \]
and
\[ \sum_{i=1}^{n} |c_i|^2 = \sum_{i=1}^{n} |a_i|^2, \quad \sum_{i=1}^{n} |b_i|^2 = \sum_{i=1}^{n} |d_i|^2, \quad \sum_{i=1}^{n} a_i \bar{b}_i - c_i d_i = 0. \]

**Proposition 2.1.** The group \(K\) acts freely on \(\mathcal{P}_0^n\).

**Proof.** If
\[ (p, q) \cdot [A; e_1, \ldots, e_n] = (p, q) \]
for some \([A; e_1, \ldots, e_n] \in K\) and \((p, q) \in T^*\mathbb{C}^{2n}\), then
\[ e_i^{-1} p_i A = p_i \quad \text{and} \quad A^{-1} q_i e_i = q_i \]
for \(i = 1, \ldots, n\), and so
\[ A p_i^* = e_i p_i^* \quad \text{and} \quad A q_i = e_i q_i \]
for \(i = 1, \ldots, n\).

Since \(q_i \neq 0\) and \(p_i \neq 0\) (as \(|p_i| = |q_i|\) and \(|p_i|^2 + |q_i|^2 \neq 0\) on \(\mathcal{P}_0^n\)), we have that \(q_i\) and \(p_i^*\) are eigenvectors of \(A\) with eigenvalue \(e_i\). If \(A\) has two different eigenvalues \(\lambda\) and \(\lambda^{-1}\), we may assume that
\[ A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} \]
and so, if \(S := \{ i \in I : e_i = \lambda \}\), we have that
\[ q_i = \begin{pmatrix} c_i \\ 0 \end{pmatrix}, \quad p_i = \begin{pmatrix} a_i & 0 \end{pmatrix}, \quad \text{for} \ i \in S, \]
and
\[ q_i = \begin{pmatrix} 0 \\ d_i \end{pmatrix}, \quad p_i = \begin{pmatrix} 0 & b_i \end{pmatrix}, \quad \text{for} \ i \in S^c, \]
for some \(a_i, b_i, c_i, d_i \in \mathbb{C} \setminus \{0\}\).

By moment map conditions (7) and (9), we conclude that \(p_i = q_i = 0\) for every \(i \in I\), which is impossible in \(\mathcal{P}_0^n\).
We conclude that $A \in SU(2)$ has only one eigenvalue and so, since $\det A = 1$, we have $A = \pm \text{Id}$. Moreover, since, for every $i \in I$, we have that $e_i$ is an eigenvalue of $A$ (as $p_i, q_i \neq 0$ for every $i$), we conclude that 

$$[A; e_1, \ldots, e_n] = [\text{Id}; 1, \ldots, 1].$$

\[ \square \]

**Remark 2.1.** The points in $\mu^{-1}_{HK}((0,0),(0,0))$ for which $p_i = q_i = 0$ for $i$ in some subset $S \subset \{1, \ldots, n\}$ are fixed by a subtorus of $K$ of dimension $|S|$. If $(p,q) = (0,0)$ then this point is fixed by $K$.

Since $K$ acts freely on $P_0^n$, we have that 0 is a regular value of the restriction of $\mu_{HK}$ to $P_0^n$ and so $P_0^n$ is a smooth manifold of dimension $5(n-3)$. Moreover, by Proposition 2.1, we obtain the following result.

**Proposition 2.2.** For $n \geq 3$ the hyperkähler quotient

$$X^n_0 := P_0^n/K$$

is a non-empty smooth hyperkähler manifold of dimension $4(n-3)$.

We call $X^n_0$ the space of **null hyperpolygons**.

This space can be described as an algebro-geometric quotient by the complexified group

$$K^C := (\text{SL}(2, \mathbb{C}) \times (\mathbb{C}^*)^n)/\mathbb{Z}/2\mathbb{Z},$$

of $K$. For that, we first need to give a suitable definition of stability.

Given $(p,q) \in T^*\mathbb{C}^{2n}$, a subset $S \subset \{1, \ldots, n\}$ is called **straight** at $(p,q)$ if $q_i$ is proportional to $q_j$ for all $i, j \in S$.

**Definition 2.2.** A point $(p,q) \in T^*\mathbb{C}^{2n}$ is called stable if the following two conditions hold:

1. $q_i, p_i \neq 0$ for all $i \in \{1, \ldots, n\}$
2. the set $\{1, \ldots, n\}$ is not straight at $(p,q)$.

Let $\mu^{-1}_{C1}(0,0)^{st}$ denote the set of points in $\mu^{-1}_{C}(0,0)$ which are stable. Then we have the following result.

**Proposition 2.3.** The group $K^C$ acts freely on $\mu^{-1}_{C1}(0,0)^{st}$.

**Proof.** If

$$(p,q) \cdot [A; e_1, \ldots, e_n] = (p,q)$$

for some $[A; e_1, \ldots, e_n] \in K^C$ and $(p,q) \in \mu^{-1}_{C1}(0,0)^{st}$, then

$$A^T p_i^t = e_i p_i^t$$

and $A q_i = e_i q_i$ for $i = 1, \ldots, n$. Since, by stability condition (i), we have $q_i, p_i \neq 0$ for all $i$, we conclude that $q_i$ is an eigenvector of $A$ with eigenvalue $e_i$ and $p_i^t$ is an eigenvector of $A^T$ also with eigenvalue $e_i$. If $A$ has two different eigenvalues $\lambda$ and $\lambda^{-1}$, we may assume that

$$A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}.$$
and so, if $S := \{i \in I : e_i = \lambda\}$, we have that
\[ q_i = \begin{pmatrix} c_i \\ 0 \end{pmatrix}, \quad p_i = \begin{pmatrix} a_i & 0 \end{pmatrix}, \quad \text{for } i \in S, \]
and
\[ q_i = \begin{pmatrix} 0 \\ d_i \end{pmatrix}, \quad p_i = \begin{pmatrix} 0 & b_i \end{pmatrix}, \quad \text{for } i \in S^c, \]
for some $a_i, b_i, c_i, d_i \in \mathbb{C} \setminus \{0\}$. By moment map conditions (7) and (9), we conclude that $p_i = q_i = 0$ for every $i \in I$, which is impossible by stability condition (i) in Definition 2.2.

If $A$ is not diagonalizable then the eigenspace $U$ of its unique eigenvalue $\lambda$ has dimension 1 and $q_i \in U$ for all $U$, implying that $\{1, \ldots, n\}$ is straight at $(p, q)$, which is impossible by stability condition (ii). We conclude that $A$ is diagonalizable with only one eigenvalue and then, since $\det A = 1$, we have
\[ [A; e_1, \ldots, e_n] = [\text{Id}; 1, \ldots, 1] \in K^C. \]

Before describing $X^n_0$ as a GIT quotient we need the following definition.

**Definition 2.4.** For each $(p, q) \in T^* \mathbb{C}^{2n}$, let $d_{(p,q)} : K^C \to \mathbb{R}$ be the function given by
\begin{equation}
(12) \quad d_{(p,q)}([A; e_1, \ldots, e_n]) = \frac{1}{4} \sum_{i=1}^{n} (|\hat{q}_i|^2 - |q_i|^2 + |\hat{p}_i|^2 - |p_i|^2),
\end{equation}
where $\hat{q}_i := A^{-1}q_i e_i$ and $\hat{p}_i = e_i^{-1}p_i A$.

These functions satisfy a useful property.

**Lemma 2.5.** For each $(p, q) \in T^* \mathbb{C}^{2n}$ let $d_{(p,q)} : K^C \to \mathbb{R}$ be the function in (12). Then
\begin{align*}
d_{(p,q)}([A; e_1, \ldots, e_n] \cdot [B; \tilde{e}_1, \ldots, \tilde{e}_n]) &= d_{(p,q)}([A; e_1, \ldots, e_n]) + d_{(p,q)}([B; \tilde{e}_1, \ldots, \tilde{e}_n])
\end{align*}
for any $[A; e_1, \ldots, e_n], [B; \tilde{e}_1, \ldots, \tilde{e}_n] \in K^C$.

**Proof.**
\begin{align*}
d_{(p,q)}([A; e_1, \ldots, e_n] \cdot [B; \tilde{e}_1, \ldots, \tilde{e}_n]) &= \frac{1}{4} \sum_{i=1}^{n} (|\hat{q}_i|^2 - |q_i|^2 + |\hat{p}_i|^2 - |p_i|^2),
\end{align*}
where $\hat{q}_i := (AB)^{-1}q_i e_i \tilde{e}_i = B^{-1}(A^{-1}q_i e_i)\tilde{e}_i$ and $\hat{p}_i = e_i^{-1}(e_i^{-1}p_i A)B$. Then,
\begin{align*}
d_{(p,q)}([A; e_1, \ldots, e_n] \cdot [B; \tilde{e}_1, \ldots, \tilde{e}_n]) &= \frac{1}{4} \sum_{i=1}^{n} (|B^{-1}(A^{-1}q_i e_i)\tilde{e}_i|^2 - |A^{-1}q_i e_i|^2 + |\tilde{e}_i^{-1}(e_i^{-1}p_i A)B|^2 - |e_i^{-1}p_i A|^2)
+ \frac{1}{4} \sum_{i=1}^{n} (|A^{-1}q_i e_i|^2 - |q_i|^2 + |e_i^{-1}p_i A|^2 - |p_i|^2)
= d_{(p,q)}([A; e_1, \ldots, e_n]) + d_{(p,q)}([B; \tilde{e}_1, \ldots, \tilde{e}_n]).
\end{align*}
By the polar decomposition (a particular case of the Cartan decomposition), we can write every matrix in $SL(2, \mathbb{C})$ as

$$A = e^{\sqrt{-1}S}R$$

with $R \in SU(2)$ and $S \in \mathfrak{su}(2)$ (see, for example, [S] Theorem 6.1)), and every element $[A; e_1, \ldots, e_n] \in K^C \simeq \exp(\sqrt{-1T}) \cdot K$

as

$$[A; e_1, \ldots, e_n] = [e^{\sqrt{-1}\xi_1}; e^{\sqrt{-1}\xi_2}, \ldots, e^{\sqrt{-1}\xi_n}] \cdot [R; \hat{e}_1, \ldots, \hat{e}_n],$$

with $R \in SU(2)$, $\hat{e}_i \in S^1$, $S \in \mathfrak{su}(2)$ and $\xi_i = \sqrt{-1}\xi_i \in \sqrt{-1}\mathbb{R}$, for every $i = 1, \ldots, n$.

Then, using Lemma 2.5, we have

$$d_{(p,q)}([A; e_1, \ldots, e_n]) = d_{(p,q)}([e^{\sqrt{-1}S}; e^{\sqrt{-1}\xi_1}, \ldots, e^{\sqrt{-1}\xi_n}] \cdot [R; \hat{e}_1, \ldots, \hat{e}_n])$$

and

$$d_{(p,q)}([\mu_{\mathbb{R}}(\hat{p}, \hat{q})]; [S, \sqrt{-1}\xi_1, \ldots, \sqrt{-1}\xi_n])$$

with $\hat{q} := (\hat{q}_1, \ldots, \hat{q}_n)$ and $\hat{p} := (\hat{p}_1, \ldots, \hat{p}_n)$, where

$$\hat{q}_i := e^{-\sqrt{-1}tS} q_i e^{-\sqrt{-1}t\xi_i} \quad \text{and} \quad \hat{p}_i := e^{t\xi_i} p_i e^{\sqrt{-1}tS} \quad \text{for} \quad i = 1, \ldots, n;$$

and

$$\frac{\partial^2}{\partial t^2} \left( d_{(p,q)}([e^{\sqrt{-1}S}; e^{-\sqrt{-1}t\xi_1}, \ldots, e^{-\sqrt{-1}t\xi_n}]) \right) \geq 0.$$

**Proof.** Since $S \in \mathfrak{su}(2)$ is diagonalizable and we can take an orthonormal basis of eigenvectors of $S$, we have $S = ADA^{-1}$ with $A \in SU(2)$ and

$$D = \begin{bmatrix} -\sqrt{-1}x & 0 \\ 0 & \sqrt{-1}x \end{bmatrix},$$

for some $x > 0$ (note that $\text{Tr}(S) = 0$).

Then, writing

$$A^{-1}q_i = \begin{pmatrix} c_i \\ d_i \end{pmatrix} \quad \text{and} \quad p_iA = \begin{pmatrix} a_i & b_i \end{pmatrix},$$

with $a_i, b_i, c_i, d_i \in \mathbb{C}$, we obtain, for $[e^{\sqrt{-1}tS}; e^{-\sqrt{-1}t\xi_1}, \ldots, e^{-\sqrt{-1}t\xi_n}] \in \exp(\sqrt{-1}T)$,

$$d_{(p,q)}([e^{\sqrt{-1}tS}; e^{-\sqrt{-1}t\xi_1}, \ldots, e^{-\sqrt{-1}t\xi_n}]) = \frac{1}{4} \sum_{i=1}^n (|\hat{q}_i|^2 - |q_i|^2 + |\hat{p}_i|^2 - |p_i|^2),$$

with

$$\hat{q}_i = A \begin{bmatrix} e^{-tx} & 0 \\ 0 & e^{tx} \end{bmatrix} A^{-1}q_i e^{-\sqrt{-1}t\xi_i} = e^{-\sqrt{-1}t\xi_i}A \begin{bmatrix} e^{-tx}c_i \\ e^{tx}d_i \end{bmatrix}$$

and

$$\hat{p}_i = e^{\sqrt{-1}t\xi_i} p_i A \begin{bmatrix} e^{tx} & 0 \\ 0 & e^{-tx} \end{bmatrix} A^{-1} = e^{\sqrt{-1}t\xi_i} \begin{bmatrix} e^{tx}a_i & e^{-tx}b_i \end{bmatrix} A^{-1}.$$
We conclude that
\[
d_{(p,q)}(e^{t\sqrt{-1}S}; e^{-t\xi_1}, \ldots, e^{-t\xi_n})
\]
\[
\frac{1}{4} \sum_{i=1}^{n} (e^{2t(x+\xi_i)}|a_i|^2 + e^{-2t(x-\xi_i)}|b_i|^2 + e^{-2t(x+\xi_i)}|c_i|^2 + e^{2t(x-\xi_i)}|d_i|^2) + \text{Const.}
\]
(14)

On the other hand, by (13) we have
\[
\mu_{\mathbb{R}}(\hat{p}, \hat{q}) = \frac{1}{2} \sum_{i=1}^{n} (\hat{q}_i \hat{q}_i^* - \hat{p}_i \hat{p}_i^*)_0 \oplus \left(-\frac{1}{2} (|\hat{q}_1|^2 - |\hat{p}_1|^2), \ldots, -\frac{1}{2} (|\hat{q}_n|^2 - |\hat{p}_n|^2)\right),
\]
with
\[
(\hat{q}_i \hat{q}_i^* - \hat{p}_i \hat{p}_i^*)_0 = A \left[ e^{-2t(x+\xi_i)}|c_i|^2 - e^{2t(x+\xi_i)}|a_i|^2, e^{-2t\xi_i}c_i d_i - e^{2t\xi_i}\bar{a}_i b_i, e^{2t(x-\xi_i)}|d_i|^2 - e^{-2t(x-\xi_i)}|b_i|^2 \right] A^{-1}
\]
and
\[
-\frac{1}{2} (|\hat{q}_i|^2 - |\hat{p}_i|^2) = -\frac{1}{2} (e^{-2t(x+\xi_i)}|c_i|^2 + e^{2t(x-\xi_i)}|d_i|^2 - e^{2t(x+\xi_i)}|a_i|^2 - e^{-2t(x-\xi_i)}|b_i|^2).
\]

Hence,
\[
\sqrt{-1} \langle \mu_{\mathbb{R}}(\hat{p}, \hat{q}), (S, \sqrt{-1}\xi_1, \ldots, \sqrt{-1}\xi_n) \rangle = -\frac{1}{2} \sum_{i=1}^{n} ((x + \xi_i)e^{2t(x+\xi_i)}|a_i|^2 - (x - \xi_i)e^{-2t(x-\xi_i)}|b_i|^2 - (x + \xi_i)e^{-2t(x+\xi_i)}|c_i|^2 + (x - \xi_i)e^{2t(x-\xi_i)}|d_i|^2)
\]
\[
= -\frac{d}{dt} \left( d_{(p,q)}([e^{t\sqrt{-1}S}, e^{-t\xi_1}, \ldots, e^{-t\xi_n}]) \right)
\]
and (1) follows.

Differentiating the RHS of (15) we obtain
\[
\frac{d^2}{dt^2} \left( d_{(p,q)}([e^{t\sqrt{-1}S}, e^{-t\xi_1}, \ldots, e^{-t\xi_n}]) \right) = \sum_{i=1}^{n} ((x + \xi_i)^2 e^{2t(x+\xi_i)}|a_i|^2 + (x - \xi_i)^2 e^{-2t(x-\xi_i)}|b_i|^2 + (x + \xi_i)^2 e^{-2t(x+\xi_i)}|c_i|^2 + (x - \xi_i)^2 e^{2t(x-\xi_i)}|d_i|^2).
\]
and (2) follows.

\[\square\]

**Lemma 2.7.** For each \((p,q) \in T^*\mathbb{C}^{2n}\) let \(d_{(p,q)} : K^C \to \mathbb{R}\) be the function in (12). Then

1. \([A; e_1, \ldots, e_n] \in K^C\) is a critical point of \(d_{(p,q)}\) if and only if
   \[(p,q) \cdot [A; e_1, \ldots, e_n] \in \mu_{\mathbb{R}}^{-1}(0,0);\]

2. If \((p,q) \in \mu_{\mathbb{C}}^{-1}(0,0)^s\) then \(d_{(p,q)}\) induces a strictly convex function on \(K^C/K\).

**Proof.** If
\[
[A; e_1, \ldots, e_n] = [e^{\sqrt{-1}S}; e^{\sqrt{-1}\xi_1}, \ldots, e^{\sqrt{-1}\xi_n}] \cdot [R; \hat{e}_1, \ldots, \hat{e}_n] \in K^C,
\]
with $R \in SU(2)$, $\hat{e}_i \in S^1$, $S \in \text{su}(2)$ and $\hat{\xi}_i = \sqrt{-1} \xi_i \in \sqrt{-1} \mathbb{R}$ ($i = 1, \ldots, n$) is a critical point of $d(p,q)$, then

$$
(Dd_{(p,q)})_{[A;e_1,\ldots,e_n]} : T_{[A;e_1,\ldots,e_n]} K^C \to \mathbb{R}
$$

is the zero map. Let

$$
v = \sqrt{-1} (S; \hat{\xi}_1, \ldots, \hat{\xi}_n) \cdot [A; e_1, \ldots, e_n] \in T_{[A;e_1,\ldots,e_n]} K^C
$$

Then,

$$
0 = (Dd_{(p,q)})_{[A;e_1,\ldots,e_n]} (v) = \frac{d}{dt} (d_{(p,q)} \circ c) (1),
$$

where $c : [1 - \varepsilon, 1 + \varepsilon] \to K^C$ is given by

$$
c(t) := [e^{t \sqrt{-1} S}; e^{t \sqrt{-1} \xi_1}, \ldots, e^{t \sqrt{-1} \xi_n}] \cdot [R; \hat{e}_1, \ldots, \hat{e}_n].
$$

(Note that $c(1) = [A; e_1, \ldots, e_n]$ and $\dot{c}(1) = v$.) Then, by Lemma 2.5

$$
0 = \frac{d}{dt} (d_{(p,q)} \circ c) (1) = \frac{d}{dt} (d_{(p,q)}([e^{t \sqrt{-1} S}; e^{t \sqrt{-1} \xi_1}, \ldots, e^{t \sqrt{-1} \xi_n}])),
$$

and, by Lemma 2.6, we have that

$$(p, q) \cdot [e^{\sqrt{-1} S}; e^{\sqrt{-1} \xi_1}, \ldots, e^{\sqrt{-1} \xi_n}] \in \mu^{-1}_R (0, 0).$$

Since $[R; \hat{e}_1, \ldots, \hat{e}_n] \in K$ and $\mu^{-1}_R (0, 0)$ is $K$-invariant, we conclude that

$$(p, q) \cdot [A; e_1, \ldots, e_n] \in \mu^{-1}_R (0, 0).$$

Conversely, if $(p, q) \cdot [A; e_1, \ldots, e_n] \in \mu^{-1}_R (0, 0)$, then, using the decomposition

$$[A; e_1, \ldots, e_n] = [e^{\sqrt{-1} S}; e^{\sqrt{-1} \xi_1}, \ldots, e^{\sqrt{-1} \xi_n}] \cdot [R; \hat{e}_1, \ldots, \hat{e}_n]$$

in (13), we have that

$$(p, q) \cdot [e^{\sqrt{-1} S}; e^{\sqrt{-1} \xi_1}, \ldots, e^{\sqrt{-1} \xi_n}] \in \mu^{-1}_R (0, 0)$$

(as $\mu^{-1}_R (0, 0)$, is $K$-invariant). Moreover, for any

$$v := \sqrt{-1} (X; \sqrt{-1} \hat{x}_1, \ldots, \sqrt{-1} \hat{x}_n) \cdot [A; e_1, \ldots, e_n] \in T_{[A;e_1,\ldots,e_n]} K^C$$

we can take the path $c : [-\varepsilon, \varepsilon] \to K^C$ given by

$$c(t) := [e^{\sqrt{-1} (S+t) X}; e^{\sqrt{-1} (\hat{\xi}_1 + t \hat{x}_1)}, \ldots, e^{\sqrt{-1} (\hat{\xi}_n + t \hat{x}_n)}] \cdot [R; \hat{e}_1, \ldots, \hat{e}_n]$$

(note that $c(0) = [A; e_1, \ldots, e_n]$ and $\dot{c}(0) = v$). Then,

$$
(Dd_{(p,q)})_{[A;e_1,\ldots,e_n]} (v) = \frac{d}{dt} (d_{(p,q)} \circ c) (0)
$$

$$= \frac{d}{dt} \left( d_{(p,q)}([e^{\sqrt{-1} (S+t) X}; e^{\sqrt{-1} (\hat{\xi}_1 + t \hat{x}_1)}, \ldots, e^{\sqrt{-1} (\hat{\xi}_n + t \hat{x}_n)}]) \right) (0)
$$

$$= \frac{d}{dt} \left( d_{(p,q)}([e^{t \sqrt{-1} X}; e^{\sqrt{-1} \xi_1}, \ldots, e^{\sqrt{-1} \xi_n}] \cdot (e^{t \sqrt{-1} \xi_1}; e^{\sqrt{-1} \xi_2}, \ldots, e^{\sqrt{-1} \xi_n})) \right) (0)
$$

since

$$(\hat{p}, \hat{q}) = (p, q) \cdot [e^{\sqrt{-1} S}; e^{\sqrt{-1} \xi_1}, \ldots, e^{\sqrt{-1} \xi_n}] \in \mu^{-1}_R (0, 0),$$

and so $[A; e_1, \ldots, e_n]$ is a critical point of $d_{(p,q)}$. 

By Lemma 2.6 and (16), if \((p, q) \in \mu^{-1}_C(0, 0)^{st}\) then
\[
\frac{d^2}{dt^2} \left( d_{(p,q)}([e^{t\sqrt{-1}S}, e^{-\xi_1}, \ldots, e^{-\xi_n}]) \right) = 0 \iff x = \xi = 0 \text{ for every } i = 1, \ldots, n,
\]
since \(p_i \neq 0\) and \(q_i \neq 0\) for every \(i = 1, \ldots, n\). We conclude that, if \((p, q) \in \mu^{-1}_C(0, 0)^{st}\), we have
\[
\frac{d^2}{dt^2} \left( d_{(p,q)}([e^{t\sqrt{-1}S}, e^{-\xi_1}, \ldots, e^{-\xi_n}]) \right) > 0
\]
for all \([e^{\sqrt{-1}S}; e^{-\xi_1}, \ldots, e^{-\xi_n}] \neq [\text{Id}; 1, \ldots, 1]\), proving (2).

We can now describe \(X^n_0\) as a finite dimensional GIT quotient:

**Theorem 2.8.** Let \(P^n_0\) be the space defined in (3). Then,

(1) \(P^n_0 \subset \mu^{-1}_C(0, 0)^{st}\) and

(2) there exists a natural bijection
\[
\iota : X^n_0 = P^n_0/K \rightarrow \mu^{-1}_C(0, 0)^{st}/K^C.
\]

**Proof.** (1) Let \((p, q) \in P^n_0 \subset \mu^{-1}_C(0, 0, 0)\). By (9), we have that \(p_i, q_i \neq 0\) for all \(i \in \{1, \ldots, n\}\) and so the stability condition (i) in Definition 2.2 holds. If \(\{1, \ldots, n\}\) is straight at \((p, q)\) then, by (7), we can assume that
\[
q_i = \begin{pmatrix} c_i \\ 0 \end{pmatrix}, \quad p_i = \begin{pmatrix} 0 & b_i \end{pmatrix}, \quad \text{for} \quad i = 1, \ldots, n,
\]
with \(b_i, c_i \in C \setminus \{0\}\). Then, by (10) we have
\[
\sum_{i=1}^{n} |c_i|^2 = \sum_{i=1}^{n} |a_i|^2 = 0,
\]
implying that \(p_i = q_i = 0\) for every \(i\), which is impossible. We conclude that condition (ii) in Definition 2.2 is also satisfied.

(2) Let us first see that \(\iota\) is injective. If \(\iota([p, q]_R) = \iota([\hat{p}, \hat{q}]_R)\) for some \([p, q]_R, [\hat{p}, \hat{q}]_R \in X^n_0\) then \([p, q]_C = [\hat{p}, \hat{q}]_C\), implying that
\[
(\hat{p}, \hat{q}) = (p, q) \cdot [A; e_1, \ldots, e_n],
\]
for some \([A; e_1, \ldots, e_n] \in K^C\). Since \((\hat{p}, \hat{q}) \in P^n_0\) we have that \([A; e_1, \ldots, e_n]\) is a critical point of the function \(d_{(p,q)}\) defined in (12) (see Lemma 2.7) and then, as \(d_{(p,q)}\) induces a strictly convex function in \(K^C/K\), we conclude that \([A; e_1, \ldots, e_n] \in K\), implying that \([p, q]_R = [\hat{p}, \hat{q}]_R\).

To see that \(\iota\) is surjective, take any \((p, q) \in \mu^{-1}_C(0, 0)^{st}\). In order to find \([\hat{p}, \hat{q}]_R\) such that \(\iota([\hat{p}, \hat{q}]_R) = [p, q]_C\), it is enough to show that \(d_{(p,q)}\) has a minimum. Indeed, if that is the case and \([A; e_1, \ldots, e_n] \in K^C\) is a minimizer, we know, by Lemma 2.7 that
\[
(\hat{p}, \hat{q}) := (p, q) \cdot [A; e_1, \ldots, e_n] \in \mu^{-1}_R(0, 0)
\]
and then \(\iota([\hat{p}, \hat{q}]_R) = [p, q]_C\). Note that \((\hat{p}, \hat{q}) \in \mu^{-1}_{H^K}((0, 0), (0, 0))\) and, since by stability condition (i) we have \(\hat{p}_i, \hat{q}_i \neq 0\) for every \(i = 1, \ldots, n\), we conclude that \((\hat{p}, \hat{q}) \in P^n_0\).
Now to show that \( d(p,q) \) has a minimum, we know, by Lemma 2.6, that \( d(p,n) \) induces a strictly convex function on \( K^C/K \). Hence, using the polar decomposition of \( K^C \) in (13), it is enough to show that

\[
\lim_{t \to +\infty} d(p,q)\left([e^{t\sqrt{-1}S}; e^{-t\xi_1}, \ldots, e^{-t\xi_n}]\right) = +\infty
\]

for every nonzero \( (S; \xi_1, \ldots, \xi_n) \in \mathfrak{su}(2) \oplus \mathbb{R}^n \).

If \( S = 0 \) then, since \( (0; \xi_1, \ldots, \xi_n) \neq 0 \), there exists some \( i_0 \in \{1, \ldots, n\} \) such that \( \xi_{i_0} \neq 0 \). Moreover, since

\[
d(p,q)\left([e^{t\sqrt{-1}S}; e^{-t\xi_1}, \ldots, e^{-t\xi_n}]\right) = \frac{1}{4} \sum_{i=1}^{n} \left(|e^{t\xi_i} p_i|^2 + |e^{-t\xi_i} q_i|^2\right) + \text{Const},
\]

we have that (17) holds since \( \xi_{i_0}, p_{i_0}, q_{i_0} \neq 0 \).

If \( S \neq 0 \) then, writing \( S = ADA^{-1} \) with \( A \in SU(2) \) and

\[
D = \begin{bmatrix}
-\sqrt{-1}x & 0 \\
0 & \sqrt{-1}x
\end{bmatrix}
\]

for some \( x > 0 \), we have

\[
d(p,q)\left([e^{t\sqrt{-1}S}; e^{-t\xi_1}, \ldots, e^{-t\xi_n}]\right) = \frac{1}{4} \sum_{i=1}^{n} \left(e^{2t(x+\xi_i)}|a_i|^2 + e^{-2t(x-\xi_i)}|b_i|^2 + e^{-2t(x+\xi_i)}|c_i|^2 + e^{2t(x-\xi_i)}|d_i|^2\right) + \text{Const}.
\]

If there exists \( i \in \{1, \ldots, n\} \) such that

\[
(a_i \neq 0 \land \xi_i > -x) \lor (b_i \neq 0 \land \xi_i > x) \lor (c_i \neq 0 \land \xi_i < -x) \lor (d_i \neq 0 \land \xi_i < x)
\]

then (17) holds. Since the case where \( x \geq \xi_i \geq -x \) and \( a_i = d_i = 0 \) is impossible, as we would have

\[
q_i \in \left\langle \left(\begin{array}{c}
1 \\
0
\end{array}\right) \right\rangle \quad \forall i \in \{1, \ldots, n\},
\]

contradicting stability condition \((ii)\) in Definition 2.2 the result follows. (Note that if \( \xi_i > x \) then (18) holds as one of \( a_i, b_i \) is necessarily non zero and similarly, if \( \xi_i < -x \), as one of \( c_i, d_i \) is non zero.)

\[\square\]

From Theorem 2.8 it follows that

\[
X^n_0 = \mu^{-1}_C(0,0)^{\text{st}}/K^C.
\]

We denote the elements in \( \mu^{-1}_C(0,0)^{\text{st}}/K^C \) by \( [p,q]_{\text{st}} \), and by \( [p,q]_R \) the elements in \( \mathcal{P}^\alpha_0/K \), when we need to make an explicit use of one of the two constructions. In all other cases, we will simply write \([p,q]\) for a null hyperpolygon in \( X^n_0 \).
3. Spaces of Quasi-Parabolic Higgs bundles

Let \( \Sigma \) be a compact Riemann surface and consider a set of \( n \) ordered distinguished marked points \( D = \{x_1, \ldots, x_n\} \neq \emptyset \) in \( \Sigma \). A quasi-parabolic vector bundle over \( \Sigma \) at \( D \) consists of a holomorphic vector bundle \( E \) of rank \( r \) over \( \Sigma \) and a collection of flags
\[
E_x := E_{x,1} \supset E_{x,2} \supset \cdots \supset E_{x,s_x} \supset E_{x,s_x+1} = \{0\}
\]
on the fibers \( E_x \) of \( E \) at each marked point \( x \) in \( D \), where, if \( r > 1 \), at least one of the subspaces \( E_{x,2}, \ldots, E_{x,s_x} \) is non-zero. By abuse of notation we also denote by \( E \) the quasi-parabolic bundle. Let \( m_i(x) := \dim E_{x,i}/E_{x,i+1} \) be the multiplicity of the flag of \( E_x \) at \( E_{x,i} \).

**Definition 3.1.** Let \( E \) be a quasi-parabolic bundle over \( \Sigma \) at \( D \) with parabolic structure as in (20). The quasi-parabolic degree of \( E \) is defined as
\[
q\text{-par deg}(E) := \deg(E) + \sum_{x \in D} \sum_{i=1}^{s_x} m_i(x).
\]
Moreover, its quasi-parabolic slope is
\[
q\text{-par } \mu(E) := \frac{q\text{-par deg}(E)}{\text{rank}(E)}.
\]

A quasi-parabolic isomorphism \( \varphi : E \to F \) between two quasi-parabolic bundles at \( D \) with flags of the same type is a bundle isomorphism such that
\[
\varphi(E_{x,i}) \subset F_{x,i} \quad \text{for all } i = 1, \ldots, s_x \text{ and } x \in D.
\]

**Definition 3.2.** Let \( E \) be a vector bundle over \( \Sigma \) with a quasi-parabolic structure at \( D \). A quasi-parabolic bundle \( F \) over \( \Sigma \) at \( D' \subset D \) is a quasi-parabolic subbundle of \( E \) if
- \( F \) is a subbundle of \( E \) in the usual sense and
- for every \( x \in D' \), the flag of \( F_x \) is a nontrivial subflag of \( E_x \).

**Remark 3.3.** In the above definition any subbundle of the underlying vector bundle of a quasi-parabolic bundle is also a quasi-parabolic subbundle since we may assume that \( D' = \emptyset \).

We also introduce a notion of stability.

**Definition 3.4.** A quasi-parabolic bundle \( E \) over \( \Sigma \) at \( D \) is said to be stable if
\[
q\text{-par } \mu(E) > q\text{-par } \mu(L)
\]
for every quasi-parabolic subbundle \( L \) of \( E \).

Let us now restrict to the rank-2 case. A quasi-parabolic bundle of rank-2 over \( \Sigma \) at \( D \) is a rank-2 vector bundle \( E \) over \( \Sigma \) together with a collection of complete flags
\[
E_x := E_{x,1} \supset E_{x,2} \supset \{0\}
\]
on the fibers $E_x$ of each marked point $x \in D$, where $\dim_{\mathbb{C}} E_{x,2} = 1$, for every $x \in D$. Note that
\[ \text{q-par deg}(E) = \deg(E) + 2|D|. \]
The dual $E^*$ of a rank 2-quasi-parabolic bundle $E$ at $D$ is the dual of the holomorphic bundle with the collection of the dual flags at the points of $D$.

If $L$ is a quasi-parabolic line subbundle of $E$ then it has a (trivial) quasi-parabolic structure at the points in the set
\[ S_L := \{ x \in D : L_x \cap E_{x,2} \neq \{0\} \}. \]
Moreover,
\[ \text{q-par deg}(L) = \deg(L) + |S_L|. \]
Then, by Definition 3.4, $E$ is a stable quasi-parabolic bundle if and only if
\[ \deg(E) - 2\deg(L) > -2(|D| - |S_L|) \]
for every quasi-parabolic line subbundle $L$ of $E$.

**Remark 3.5.** If, in addition, $E$ is a trivial vector bundle of rank 2 over $\Sigma$ with a quasi-parabolic structure at $D$, then the corresponding quasi-parabolic bundle is stable if and only if
\[ \deg(L) < |D| - |S_L| \]
for every quasi-parabolic line subbundle $L$.

If $\Sigma = \mathbb{CP}^1$ then $\deg(L) \leq 0$ and we conclude that (23) is always satisfied except when $L$ is the trivial bundle and $|S_L| = |D| = n$. This happens only when all the vector spaces $E_{x,2}$ in (21) are the same for every $x \in D$.

We conclude that all holomorphically trivial rank-2 quasi-parabolic bundles over $\mathbb{CP}^1$ at $D$ are stable except those for which all the 1-dimensional flag elements are equal for all $x \in D$.

Let $K_\Sigma$ denote the holomorphic cotangent bundle of $\Sigma$ and let $\mathcal{O}_\Sigma(D)$ be the line bundle on $\Sigma$ defined by the divisor $D$. A rank-2 quasi-parabolic Higgs bundle over $\Sigma$ at $D$ is a pair $E := (E, \Phi)$, where $E$ is a rank-2 quasi-parabolic vector bundle over $\Sigma$ at $D$ and
\[ \Phi \in H^0(\Sigma, SQParEnd(E) \otimes K_\Sigma(D)) \]
is a Higgs field on $E$. Here $SQParEnd(E)$ is the subsheaf of $End(E)$ formed by strongly quasi-parabolic endomorphisms $\varphi : E \to E$, meaning that
\[ \varphi(E_{x,1}) \subset E_{x,2} \quad \text{and} \quad \varphi(E_{x,2}) = 0, \quad \text{for all } x \in D. \]
Note that $\Phi$ is a meromorphic endomorphism-valued one-form with simple poles along $D$ whose residue at each $x \in D$ is nilpotent with respect to the flag, i.e.,
\[ (\text{Res}_x \Phi)(E_{x,i}) \subset E_{x,i+1} \]
for all $i = 1, 2$ and $x \in D$, where we consider $E_{x,3} = \{0\}$.

The definition of stability extends to quasi-parabolic Higgs bundles in the following way.
**Definition 3.6.** A quasi-parabolic Higgs bundle \( E = (E, \Phi) \) at \( D \) is stable if

1. the residue of the Higgs field \( \Phi \) at each \( x \in D \) is nonzero and
2. for all quasi-parabolic line subbundles \( L \subset E \) which are preserved by \( \Phi \) we have
   \[
   q\text{-par} \mu(E) > q\text{-par} \mu(L).
   \]

If \( G \) is a complex reductive Lie group, a \( G \)-Higgs bundle over \( \Sigma \) is a pair \((P, \Phi)\), where \( P \) is a principal \( G \)-bundle on \( \Sigma \) and \( \Phi \) is a holomorphic section of \( P(\mathfrak{g}) \otimes K \), where \( P(\mathfrak{g}) := P \times_{\text{Ad}} \mathfrak{g} \) is the adjoint bundle associated with \( P \) [HI]. For \( G = \text{GL}(2, \mathbb{C}) \) one obtains classical Higgs bundles. Moreover, for matrix groups, this definition can be restated in terms of vector bundles. In particular, to any principal \( \text{SL}(2, \mathbb{C}) \)-bundle we can associate a rank-2 vector bundle with trivial determinant, using the fundamental 2-dimensional representation of \( \text{SL}(2, \mathbb{C}) \). Conversely, if \( E \) is a rank-2 vector bundle, its frame bundle is a \( \text{GL}(2, \mathbb{C}) \)-principal bundle and, if in addition its determinant is trivial, the structure group of this principal bundle can be reduced to \( \text{SL}(2, \mathbb{C}) \) [HS]. Consequently, we can think of \( \text{SL}(2, \mathbb{C}) \)-Higgs bundles as pairs \((E, \Phi)\), where \( E \) is a rank-2 holomorphic bundle over \( \Sigma \) with trivial determinant, and \( \Phi \) is a traceless holomorphic section of \( \text{End}(E) \otimes K \) [G-P2]. We can further generalize this definition and consider rank-2 quasi-parabolic \( \text{SL}(2, \mathbb{C}) \)-Higgs bundles over \( \Sigma \).

**Definition 3.1.** A rank-2 quasi-parabolic \( \text{SL}(2, \mathbb{C}) \)-Higgs bundle over \( \Sigma \) at \( D \) is a rank-2 quasi-parabolic Higgs bundle \( E = (E, \Phi) \) over \( \Sigma \) at \( D \), where the underlying vector bundle of \( E \) has trivial determinant and the Higgs field \( \Phi \) is traceless.

Let \( \mathcal{H}_0^n \) be the moduli space of stable rank-2 quasi-parabolic \( \text{SL}(2, \mathbb{C}) \)-Higgs bundles over \( \Sigma = \mathbb{CP}^1 \) at \( D \), for which the underlying holomorphic vector bundle is trivial. This space is diffeomorphic to the space of null hyperpolygons \( X_0^n \) in [19]. The correspondence between these two spaces is given by the map

\[
\mathcal{I} : X_0^n \longrightarrow \mathcal{H}_0^n
\]

\[
[p, q]_{st} \longmapsto [E_{(p,q)}, \Phi_{(p,q)}] := [E_{(p,q)}]
\]

where \( E_{(p,q)} \) is the trivial vector bundle \( \mathbb{CP}^1 \times \mathbb{C}^2 \longrightarrow \mathbb{CP}^1 \) with the quasi-parabolic structure given by the flags

\[
\mathbb{C}^2 \supset \langle q_i \rangle \supset \{0\}
\]

over the \( n \) marked points in \( D = \{x_1, \ldots, x_n\} \subset \mathbb{CP}^1 \) and

\[
\Phi_{(p,q)} \in H^0(SQParEnd(E_{(p,q)}) \otimes K_{\mathbb{CP}^1}(D))
\]

is the Higgs field uniquely determined by the residues:

\[
\text{Res}_{x_i} \Phi_{(p,q)} := q_i p_i
\]

for each \( x_i \in D \).

**Theorem 3.2.** The spaces \( X_0^n \) and \( \mathcal{H}_0^n \) are diffeomorphic.
Proof. We first show that the map $\mathcal{I}$ is well-defined: the Higgs field $\Phi_{(p,q)}$ is uniquely defined, the QPHB $E_{(p,q)}$ is stable, and the map $\mathcal{I}$ is independent of the choice of representative in $[p,q]_{st}$. (Note that $\mathcal{I}$ is a continuous algebraic map.)

- Let $[p,q] \in X^n_0$. Since $(p,q) \in \mu^{-1}_C((0,0)$ we have by (21) and (22) that

$$\sum_{i=1}^{n} q_i p_i = \sum_{i=1}^{n} (q_i p_i)_0 = 0.$$ 

Consequently the residues in (25) uniquely determine the meromorphic endomorphism valued 1-form $\Phi_{(p,q)}$ on $\mathbb{C}\mathbb{P}^1$.

- By the stability of $(p,q)$ we have that the set $\{1, \ldots, n\}$ is not straight at $(p,q)$ and then, by Remark 3.5, the quasi-parabolic Higgs bundle is stable.

- To show that $\mathcal{I}$ is independent of the choice of a representative of $[p,q]_{st}$, let $(\tilde{p}, \tilde{q})$ be in the $K^C$-orbit of $(p,q)$ and consider $[E(\tilde{p}, \tilde{q}); \Phi(\tilde{p}, \tilde{q})]$ as before. The Higgs field $\Phi(\tilde{p}, \tilde{q})$ is defined by the residues

$$\text{Res}_{x_i} \Phi(\tilde{p}, \tilde{q}) := \tilde{q}_i \tilde{p}_i = A^{-1} q_i z_i^{-1} p_i A = A^{-1} (q_i p_i) A = A^{-1} \text{Res}_{x_i} \Phi_{(p,q)} A$$

for some $A \in SL(2, \mathbb{C})$ and $z_i \in \mathbb{C}^*$. Moreover, the flags in $E(\tilde{p}, \tilde{q})$ are determined by $\tilde{q}_i = A^{-1} q_i z_i$. Since $q_i z_i \in \langle q_i \rangle$, we have that $A^{-1} \langle q_i \rangle = \langle \tilde{q}_i \rangle$ for every $i \in \{1, \ldots, n\}$ and so $[E(\tilde{p}, \tilde{q}); \Phi(\tilde{p}, \tilde{q})] = [E(p,q), \Phi(p,q)]$.

Let us now see that $\mathcal{I}$ is injective. Let $[p,q]_{st}, [\tilde{p}, \tilde{q}]_{st} \in X^n_0$ be such that

$$\mathcal{I}([p,q]_{st}) = \mathcal{I}([\tilde{p}, \tilde{q}]_{st}).$$

Then there exists $A \in SL(2, \mathbb{C})$ such that $\langle q_i \rangle = A \langle \tilde{q}_i \rangle$ for every $i = 1, \ldots, n$, and so

$$\hat{q} = q \cdot [A; e_1, \ldots, e_n]$$

for some $[A; e_1, \ldots, e_n] \in K^C$. Moreover,

$$\text{Res}_{x_i} \Phi(\tilde{p}, \tilde{q}) = A^{-1} \text{Res}_{x_i} \Phi_{(p,q)} A$$

and so

$$A^{-1} q_i p_i A = \hat{q}_i \hat{p}_i = A^{-1} q_i e_i \hat{p}_i,$$

implying that $q_i p_i A = q_i \hat{p}_i e_i$. Multiplying both sides by $q_i^*$, we obtain

$$q_i^* q_i p_i A = q_i^* q_i \hat{p}_i e_i \Leftrightarrow |q_i|^2 p_i A = |q_i|^2 \hat{p}_i e_i,$$

and, since $|q_i| \neq 0$, we get

$$\hat{p}_i = e_i^{-1} p_i A.$$

We conclude that

$$(\hat{p}, \hat{q}) = (p,q) \cdot [A; e_1, \ldots, e_n]$$

for some $A \in K^C$ and so $[p,q]_{st} = [\hat{p}, \hat{q}]_{st}$.

To show that $\mathcal{I}$ is surjective, let $[E, \Phi] \in \mathcal{H}^n_0$. For each point $x_i \in D$, let $q_i = (c_i, d_i)^t$ be a generator of $E_{x_i,2}$ and, representing the residue of the Higgs field $\Phi$ at $x_i$ by a traceless matrix

$$N_i := \text{Res}_{x_i} \Phi,$$
we take $p_i$ to be
\begin{equation}
    p_i = (a_i, b_i) := \frac{1}{|q_i|^2} \cdot q_i^* N_i.
\end{equation}
Note that $|q_i| \neq 0$ for all $i \in \{1, \ldots, n\}$ since the flags are nontrivial.

The pair $(p, q)$ constructed in this way is in $\mu_c^{-1}(0, 0)$ and is stable:

- Since $N_i$ is nilpotent with respect to the flag, we have that
  \[ p_i q_i = \frac{1}{|q_i|^2} \cdot q_i^* N_i q_i = 0. \]
  Moreover, since $N_i$ is traceless and $N_i q_i = 0$, we get
  \[ q_i p_i = N_i = (q_i p_i)_0, \]
  and then, since the sum of the residues $N_i$ is 0, we obtain
  \[ \sum_{i=1}^{n} (q_i p_i)_0 = 0. \]
  We conclude that $(p, q) \in \mu_c^{-1}(0, 0)$;
- To show that $(p, q)$ is stable, we need to check conditions $(i)$ and $(ii)$ of Definition 2.2. The first one ($q_i \neq 0$ and $p_i \neq 0$ for all $i$) is trivially verified since the flags are complete and, by assumption, the residue of the Higgs field never vanishes on $D$ (note that $q_i p_i = N_i$). For $(ii)$, if $S = \{1, \ldots, n\}$ were straight at $(p, q)$, then all the flag elements $E_x$ for $x \in D$ would be the same complex line $\ell$. We could therefore consider the trivial line bundle $L$ over $\mathbb{CP}^1$ with total space $\mathbb{CP}^1 \times \ell$ which is trivially a quasi-parabolic subbundle of $E$ at $D$ (i.e. $S_L = D$, where $S_L$ is the set defined in (22)). Moreover, it is also preserved by the Higgs field $\Phi$ since
  \[ (\text{Res}_x, \Phi) (\ell) = 0 \]
  and then, by Remark 3.5, the quasi-parabolic Higgs bundle $E$ would not be stable.

It is easy to check that $I([p, q]_{st}) = [E, \Phi]$ and so $I$ is surjective.

4. Involution

The compact real form $\mathfrak{su}(2)$ of $\mathfrak{sl}(2, \mathbb{C})$ is the fixed point set of the anti-linear involution
\[ \iota_1 : \mathfrak{sl}(2, \mathbb{C}) \to \mathfrak{sl}(2, \mathbb{C}) \]
\[ X \mapsto -X^* \]
and the involution
\[ \iota_2 : \mathfrak{su}(2) \to \mathfrak{su}(2) \]
\[ X \mapsto \overline{X} \]
gives the Cartan decomposition
\begin{equation}
    \mathfrak{su}(2) = \mathfrak{k} \oplus \mathfrak{h},
\end{equation}
where
\[ \mathfrak{k} := \{X \in \mathfrak{su}(2) : \iota_2(X) = X\} = \mathfrak{so}(2) \]
and
\[ \mathfrak{h} := \{ X \in \mathfrak{su}(2) : \iota_2(X) = -X \} = \{ 2 \times 2 \text{ traceless, symmetric imaginary matrices} \}. \]

Then, denoting by \( \mathfrak{m} \) the Lie algebra \( \sqrt{-1} \mathfrak{h} \) of traceless symmetric endomorphisms of \( \mathbb{R}^2 \), the split real form \( \mathfrak{t} \oplus \mathfrak{m} \) of \( \mathfrak{sl}(2, \mathbb{C}) \) corresponding to (27) is \( \mathfrak{sl}(2, \mathbb{R}) \). It is the fixed point set of the extension of the involution \( \iota_2 \) to \( \mathfrak{sl}(2, \mathbb{C}) \).

At the group level, the compact real form \( SU(2) \) of \( SL(2, \mathbb{C}) \) is the fixed point set of the anti-holomorphic involution
\[ \iota_1 : SL(2, \mathbb{C}) \to SL(2, \mathbb{C}) \]
\[ X \mapsto (X^*)^{-1} \]
while the anti-holomorphic involution
\[ \iota_2 : SL(2, \mathbb{C}) \to SL(2, \mathbb{C}) \]
\[ X \mapsto \overline{X} \]
has fixed point set the split real form \( SL(2, \mathbb{R}) \) of \( SL(2, \mathbb{C}) \).

The involution \( \theta := \iota_2 \circ \iota_1 \) given by \( \theta(X) = -X^t \) is the Cartan involution of the split real form \( SL(2, \mathbb{R}) \) of \( SL(2, \mathbb{C}) \) (see [OV, Chapter 5]). In the moduli space \( \mathcal{H}^0_n \) of quasi-parabolic Higgs bundles, the map \( \theta \) induces the involution
\[ (E, \Phi) \mapsto (E^*, \Phi^t). \]

Recall that a \( SL(2, \mathbb{R}) \)-Higgs bundle over \( \Sigma \) is a pair \((P, \Phi)\), where \( P \) is a holomorphic principal \( SO(2, \mathbb{C}) \)-bundle over \( \Sigma \) (i.e. a line bundle \( L \)) and \( \Phi \) is a holomorphic section of \( P(\mathfrak{m}^C) \otimes K \), where \( P(\mathfrak{m}^C) := P \times_{\text{Ad} \mathfrak{m}^C} \mathbb{C}^2 = L^2 \oplus L^{-2} \) is the adjoint bundle associated with \( P \) (see for example [Mu]). Indeed, considering the standard complex structure
\[ I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \text{End}(\mathbb{R}^2) \]
we can take the decomposition of \( \mathbb{R}^2 \otimes \mathbb{C} \simeq \mathbb{C}^2 = V_+ \oplus V_- \), where \( V_+ \) and \( V_- \) are the eigenspaces \( V_+ := \langle (1, -\sqrt{-1}) \rangle \) and \( V_- := \langle (1, \sqrt{-1}) \rangle \) of \( I \) (corresponding to the eigenvalues \( \pm \sqrt{-1} \)). Taking the basis \( \{ v_+, v_- \} \) of \( \mathbb{C}^2 \) with
\[ v_+ := (1, -\sqrt{-1}) \in V_+ \quad \text{and} \quad v_- := \frac{1}{2}(1, \sqrt{-1}) \in V_- , \]
a traceless symmetric endomorphism \( \psi \) of \( \mathbb{R}^2 \otimes \mathbb{C} \) for the standard Euclidean structure on \( \mathbb{R}^2 \) (i.e. an element of \( \mathfrak{m}^C \)) can be written in this basis as a matrix
\[ A := \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & -a_{11} \end{pmatrix} \]
with
\[ a_{11} = (a_{11}v_+ + a_{21}v_-) \cdot v_- = v_+ \cdot (a_{12}v_+ - a_{11}v_-) = -a_{11}, \]
since \( \psi(v_+) \cdot v_- = v_+ \cdot \psi(v_-) \). Hence
\[ A = \begin{pmatrix} 0 & a_{12} \\ a_{21} & 0 \end{pmatrix}. \]
Moreover, the adjoint action of $SO(2, \mathbb{C}) \simeq \mathbb{C}^*$ on $\mathfrak{m}^\mathbb{C}$ is given by
\[
\lambda \cdot \psi = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} 0 & a_{12} \\ a_{21} & 0 \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} = \begin{pmatrix} 0 & \lambda^{-2}a_{12} \\ \lambda^2a_{21} & 0 \end{pmatrix}.
\]
Consequently, we can think of an $SL(2, \mathbb{R})$-Higgs bundle over $\Sigma$ as a Higgs bundle $(E, \Phi)$ whose underlying vector bundle is $E = L^2 \oplus L^{-2}$ and the Higgs field $\Phi$ is
\[
\Phi = (\Phi_+, \Phi_-) \in H^0(\Sigma, L^2 \otimes K) \oplus H^0(\Sigma, L^{-2} \otimes K).
\]

Following [BGM] we can again generalize this definition to the quasi-parabolic case.

**Definition 4.1.** A rank-2 quasi-parabolic $SL(2, \mathbb{R})$-Higgs bundle over $\Sigma$ at $D$ is a rank-2 quasi-parabolic Higgs bundle $E = (E, \Phi)$ over $\Sigma$ at $D$, such that

- the underlying vector bundle is $E = L \oplus L^{-1}$ for some line bundle $L$ over $\Sigma$ with quasi-parabolic structure at $D$ given by $E_x := E_{x,1} \supset E_{x,2} \supset \{0\}$, with $E_{x,2} = L_x$ for $x$ in some subset $S_L \subset D$ and $E_{x,2} = L_x^{-1}$ for $x \in D \setminus S_L$;
- the Higgs field is of the form
  \[
  \Phi = \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix} : E \to E \otimes K_{\Sigma}(D)
  \]
  with $\beta = 0$ on the fibers over $S_L$ and $\alpha = 0$ on those of $D \setminus S_L$.

We will see that the fixed-point set of the involution $\Theta : \mathcal{H}_0^n \to \mathcal{H}_0^n$ defined in (28) is the moduli space of rank-2 quasi-parabolic $SL(2, \mathbb{R})$-Higgs bundles over $\mathbb{C}P^1$ at $D$ for which the underlying vector bundle is holomorphically trivial (i.e. those for which the line bundle $L$ in Definition 4.1 is the trivial line bundle over $\mathbb{C}P^1$).

For that we will use the isomorphism in (24) and study the fixed point set of the corresponding involution on the space $X_0^n$ of null hyperpolygons, that is the map
\[
X_0^n \xrightarrow{\iota} X_0^n \\
[p, q] \mapsto [q^t, p^t].
\]
It is holomorphic with respect to the complex structure induced by $I$ and anti-holomorphic with respect to those induced by $J$ and $K$, where $I, J, K$ are the standard hyperkähler complex structures on $T^*\mathbb{C}^{2n}$.

**Remark 4.1.** Note that this map is well-defined. Indeed, if $[\tilde{p}, \tilde{q}] = [p, q] \in X_0^n$, then there exists $[A; e_1, \ldots, e_n] \in \tilde{K}$ such that
\[
(\tilde{p}, \tilde{q}) = (p, q) \cdot [A; e_1, \ldots, e_n]
\]
and so
\[
\tilde{p}^t_i = (e_i^{-1}p_iA)^t = A^tp_i^te_i^{-1} \quad \text{and} \quad \tilde{q}^t_i = (A^{-1}q_ie_i)^t = e_iq_i^tA,
\]
implying that
\[
(\tilde{q}^t, \tilde{p}^t) = (q^t, p^t) \cdot [A^{-1}; e_1^{-1}, \ldots, e_n^{-1}],
\]
and so $[\tilde{q}^t, \tilde{p}^t] = [q^t, p^t]$. Note that, since $A \in SU(2)$, we have

$$(\bar{A})^* \bar{A} = \bar{A^*A} = I$$

and $(\bar{A})^{-1} = A^T$.

Let $\mathcal{S}$ be the collection of subsets $S \subset \{1, \ldots, n\}$ such that $2 \leq |S| \leq n - 2$ and $1 \in S$. For $S \in \mathcal{S}$, consider the sets

$$(30)\quad Z_S := \{[p, q] \in \mathbb{X}_0^n : S \text{ and } S^c \text{ are straight at } (p, q)\}.$$

Then we have the following result.

**Theorem 4.2.** The fixed-point set of the involution in $(29)$ is

$$(X_0^n)^\iota := \bigcup_{S \in \mathcal{S}} Z_S,$$

where $Z_S$ is defined above. Moreover, each $Z_S$ is a non-compact symplectic manifold of dimension $2(n - 3)$ and $(X_0^n)^\iota$ has $2^{n-1} - (n + 1)$ connected components.

**Proof.** Suppose that $[p, q] \in X_0^n$ is a fixed point of $\iota$. Then there exists an element $[A; e_1, \ldots, e_n] \in K \setminus \{Id\}$ such that

$$e_i^{-1}p_iA = q_i^t$$

and $A^{-1}q_ie_i = p_i^t$, for $i = 1, \ldots, n$.

In particular,

$$(31)\quad q_i = Ap_i^te_i^{-1} \quad \text{and} \quad (A\bar{A})^Tp_i^t = p_i^t,$$

and so $p_i^t$ is an eigenvector of $(A\bar{A})^T$ associated to the eigenvalue 1, for all $i = 1, \ldots, n$.

Since $(A\bar{A})^T \in SU(2)$, we conclude that

$$A\bar{A} = \text{Id}.$$

Writing

$$A = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix},$$

with $\alpha, \beta \in \mathbb{C}$ such that $|\alpha|^2 + |\beta|^2 = 1$, we have

$$\text{Id} = A\bar{A} = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \begin{pmatrix} \bar{\alpha} & \bar{\beta} \\ -\beta & \alpha \end{pmatrix} = \begin{pmatrix} |\alpha|^2 - |\beta|^2 & \alpha\bar{\beta} + \alpha\beta \\ -\beta(\bar{\alpha}\beta + \bar{\beta}\alpha) & |\alpha|^2 - |\beta|^2 \end{pmatrix},$$

and so

$$(32)\quad A = \begin{pmatrix} \alpha & k\sqrt{-1} \\alpha \\ k\sqrt{-1} & \bar{\alpha} \end{pmatrix},$$

for some $\alpha \in \mathbb{C}$ and $k \in \mathbb{R}$ such that $|\alpha|^2 + k^2 = 1$.

Assuming, without loss of generality, that

$$p_1 = \begin{pmatrix} 0 & b \end{pmatrix} \quad \text{and} \quad q_1 = \begin{pmatrix} c \\ 0 \end{pmatrix}$$
with \( b, c \in \mathbb{C} \setminus \{0\} \) such that \( |b| = |c| \), we have from (31) and (32) that
\[
\begin{pmatrix}
c \\
0
\end{pmatrix} = \begin{pmatrix}
\frac{\alpha}{k \sqrt{-1}} & k \sqrt{-1} \\
0 & \bar{\alpha}
\end{pmatrix} \begin{pmatrix}
0 \\
b
\end{pmatrix} e_1^{-1}
\]
and so \( \alpha = 0 \), implying that
\[
(33) \quad A = \begin{pmatrix}
0 & \pm \sqrt{-1} \\
\pm \sqrt{-1} & 0
\end{pmatrix}.
\]
Then, writing, as usual,
\[
q_i = \begin{pmatrix}
c_i \\
d_i
\end{pmatrix} \quad \text{and} \quad p_i = \begin{pmatrix}
a_i \\
b_i
\end{pmatrix}, \quad \text{for } i = 2, \ldots, n,
\]
with \((a_i, b_i), (c_i, d_i) \in \mathbb{C}^2 \setminus \{0\}\), we have, from (31),
\[
\begin{pmatrix}
c_i \\
d_i
\end{pmatrix} = \begin{pmatrix}
0 & \pm \sqrt{-1} \\
\pm \sqrt{-1} & 0
\end{pmatrix} \begin{pmatrix}
a_i \\
b_i
\end{pmatrix} e_i^{-1} = \begin{pmatrix}
\pm b_i e_i^{-1} \sqrt{-1} \\
\pm a_i e_i^{-1} \sqrt{-1}
\end{pmatrix}.
\]
Then, by (7), we conclude that \( a_i b_i = 0 \) and so there exists an index set \( S \subset \{1, \ldots, n\} \) with \( 1 \in S \) such that
\[
(34) \quad p_i = \begin{pmatrix}
0 & b_i
\end{pmatrix}, \quad q_i = \begin{pmatrix}
c_i \\
0
\end{pmatrix} \quad \forall i \in S
\]
\[
p_i = \begin{pmatrix}
a_i & 0
\end{pmatrix}, \quad q_i = \begin{pmatrix}
0 \\
d_i
\end{pmatrix} \quad \forall i \in S^c.
\]
Since \( |q_i| = |p_i| \), we conclude that
\[
(35) \quad |c_i| = |b_i| \quad \text{for all } i \in S \quad \text{and} \quad |d_i| = |a_i| \quad \text{for all } i \in S^c.
\]
Moreover, since \( \sum_{i=1}^n (q_i q_i^* - p_i^* p_i) = 0 \), we obtain by (10) that
\[
(36) \quad \sum_{i \in S} |c_i|^2 = \sum_{i \in S^c} |a_i|^2.
\]
On the other hand, since \( \sum_{i=1}^n (q_i p_i) = 0 \), we have by (8) that
\[
(37) \quad \sum_{i \in S} b_i c_i = \sum_{i \in S^c} a_i d_i = 0.
\]
Since by (34) we have \( b_i, c_i \neq 0 \) for all \( i \in S \) (as \( p_i, q_i \neq 0 \) and \( 1 \in S \)), it follows from (37) that \( S \) has cardinality at least two. On the other hand, since \( \{1, \ldots, n\} \) is not straight at \((p, q)\) (by stability condition \((ii)\) in Definition 2.2), we have that \( S^c \neq \varnothing \). Since by (34) we have \( a_i, d_i \neq 0 \) for all \( i \in S \) (as \( p_i, q_i \neq 0 \)), it follows from (37) that \( S^c \) has cardinality at least two. We conclude that \( S \in \mathcal{S} \) and \([p, q] \in Z_S\).

Conversely, given any subset \( S \in \mathcal{S} \), we have that all the points in \( Z_S \) are fixed by \( \iota \). Indeed, if \([p, q] \in Z_S\), then \( S \) and \( S^c \) are straight at \((p, q)\) and so we can assume that
\[
p_i = \begin{pmatrix}
0 & b_i
\end{pmatrix}, \quad q_i = \begin{pmatrix}
c_i \\
0
\end{pmatrix} \quad \forall i \in S
\]
\[
p_i = \begin{pmatrix}
a_i & 0
\end{pmatrix}, \quad q_i = \begin{pmatrix}
0 \\
d_i
\end{pmatrix} \quad \forall i \in S^c.
\]
Then
\[ \iota[p,q] = [q^t, p^t] = [(p,q) \cdot [A; e_1, \ldots, e_n]] = [p,q], \]
with
\[ A = \begin{pmatrix} 0 & \frac{1}{\sqrt{1}} \\ \sqrt{1} & 0 \end{pmatrix} \in SU(2) \]
and
\[ e_i = \frac{b_i}{c_i} \in S^1 \quad \text{if } i \in S \quad \text{and} \quad e_i = \frac{a_i}{d_i} \frac{1}{\sqrt{1}} \quad \text{if } i \in S^c. \]

Finally, since for every subset \( S \subseteq \{1, \ldots, n\} \), we have that either \( 1 \in S \) or \( 1 \in S^c \), the number of subsets in \( S \) is
\[ \frac{1}{2} \sum_{k=2}^{n-2} \binom{n}{k} = 2^{(n-1)} - (n + 1). \]

Using the diffeomorphism in \((24)\) we conclude that the fixed point set of the involution \((28)\) on the moduli space of quasi-parabolic \( SL(2,\mathbb{C}) \)-Higgs bundles \( \mathcal{H}^n_0 \) is formed by the quasi-parabolic \( SL(2,\mathbb{R}) \)-Higgs bundles in \( \mathcal{H}^n_0 \).

**Theorem 4.3.** The fixed-point set of the involution \( \Theta : \mathcal{H}^n_0 \to \mathcal{H}^n_0 \) in \((28)\) on the space of quasi-parabolic \( SL(2,\mathbb{C}) \)-Higgs bundles \( \mathcal{H}^n_0 \) is
\[ \mathcal{H}^n_{0,n} := \bigcup_{1 \in S \subseteq \{1, \ldots, n\}} \mathcal{Z}_S \]
where \( \mathcal{Z}_S \subseteq \mathcal{H}^n_0 \) is formed by the quasi-parabolic \( SL(2,\mathbb{R}) \)-Higgs bundles \( E = (E, \Phi) \in \mathcal{H}^n_0 \) such that

(i) the quasi-parabolic vector bundle \( E \) admits a direct sum decomposition
\[ E = L_0 \oplus L_1, \]
where \( L_0 \) and \( L_1 \) are trivial line bundles over \( \mathbb{CP}^1 \) and the quasi-parabolic structure at \( D \) is given by
\[ E_x := Eq \quad \text{for } x \in \{x_i \in D : i \in S\} \subset D \quad \text{and} \quad E_x = (L_1)_x \quad \text{otherwise}; \]

(1) the Higgs field is of the form
\[ \Phi = \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix} : E \to E \otimes K_{\mathbb{CP}^1}(D) \]
with \( \beta = 0 \) on the fibers over \( x_i \in D \) such that \( i \in S \) and \( \alpha = 0 \) on those over \( x_i \) with \( i \in S^c \). (i.e. it is either upper or lower triangular with respect to the above decomposition, according to whether \( i \) is in \( S \) or in \( S^c \)).

Moreover each \( \mathcal{Z}_S \) is a non-compact manifold of dimension \( 2(n - 3) \) and \( \mathcal{H}^n_{0,n} \) has
\[ 2^{(n-1)} - (n + 1) \]
connected components.
5. Null Polygons in Minkowski 3-space

Let $\mathbb{R}^{2,1}$ be the \textit{Minkowski 3-space}, consisting of $\mathbb{R}^3$ equipped with the signature $(-, -, +)$-inner product given by

$$v \circ w = -v_1 w_1 - v_2 w_2 + v_3 w_3.$$  

The \textit{Minkowski norm} of a vector $v \in \mathbb{R}^{2,1}$ is

$$||v||_{2,1} = \sqrt{|v \circ v|}$$

and the vectors $v \in \mathbb{R}^{2,1}$ are classified according to the sign of $v \circ v$. Those $v$ for which $v \circ v = 0$ are called \textit{light-like} or \textit{null} vectors and form the \textit{light cone}. If $v \circ v > 0$, then $v$ is a \textit{time-like} vector and, if $v \circ v < 0$, it is a \textit{space-like} vector. A null vector is said to be lying in the \textit{future} (respectively \textit{past}) light cone if $v_3 > 0$ (respectively $v_3 < 0$).

The \textit{Minkowski cross product} $\times$ on $\mathbb{R}^{2,1}$ is defined as

$$v \times w := \det \begin{pmatrix} -e_1 & -e_2 & e_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix},$$

for $v = (v_1, v_2, v_3), w = (w_1, w_2, w_3)$ and $e_1, e_2, e_3$ the standard unit vectors in $\mathbb{R}^3$ and the Lie algebra $(\mathbb{R}^3, \times)$ is isomorphic to $\mathfrak{su}(1, 1)$ via the map

$$\begin{pmatrix} x \\ y \\ t \end{pmatrix} \mapsto \frac{1}{2} \begin{pmatrix} -\sqrt{-1}t & x + \sqrt{-1}y \\ x - \sqrt{-1}y & \sqrt{-1}t \end{pmatrix}.$$

Note that $\mathfrak{su}(1, 1)^* \simeq \sqrt{-1} \cdot \mathfrak{su}(1, 1)$ is also identified with $\mathbb{R}^3$.

Under this identification, the Minkowski inner product $\circ$ corresponds to the pairing between $\mathfrak{su}(1, 1)^*$ and $\mathfrak{su}(1, 1)$ given by

$$(A, B) \mapsto 2\sqrt{-1} \cdot \text{trace}(AB),$$

for $A \in \mathfrak{su}(1, 1)^*$ and $B \in \mathfrak{su}(1, 1)$.

The coadjoint action of (the non-compact group) $\text{SU}(1, 1)$ on $\mathfrak{su}(1, 1)^* \simeq \mathbb{R}^{2,1}$ is defined by

$$(A \cdot u) \circ v = u \circ (A^{-1} v A)$$

for every $v \in \mathfrak{su}(1, 1)$, where $A \in \text{SU}(1, 1)$ and $u \in \mathfrak{su}(1, 1)^*$. In particular, if

$$A = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \in \text{SU}(1, 1)$$

(with $\alpha, \beta \in \mathbb{C}$ such that $|\alpha|^2 - |\beta|^2 = 1$), then $A \cdot u$ corresponds to multiplication by the $3 \times 3$-matrix

$$A_{\alpha, \beta} := \begin{pmatrix} \text{Re}(\alpha^2 - \beta^2) & -\text{Im}(\alpha^2 + \beta^2) & -2 \text{Im}(\alpha \beta) \\ \text{Im}(\alpha^2 - \beta^2) & \text{Re}(\alpha^2 + \beta^2) & 2 \text{Re}(\alpha \beta) \\ 2 \text{Im}(\alpha \beta) & 2 \text{Re}(\alpha \beta) & |\alpha|^2 + |\beta|^2 \end{pmatrix}.$$  

The $\text{SU}(1, 1)$ coadjoint orbits in $\mathbb{R}^{2,1}$ are subsets of the surfaces

$$-x^2 - y^2 + t^2 = R, \quad R \in \mathbb{R}.$$
If $R > 0$ one obtains a two-sheeted hyperboloid and each sheet is a coadjoint orbit. If $R < 0$ one obtains a one-sheeted hyperboloid which is also a coadjoint orbit. If $R = 0$ one obtains three distinct coadjoint orbits: the origin and the two components $C^\pm$ of the light cone defined by

$$t = \pm \sqrt{x^2 + y^2}.$$  

We will call $C^\pm$ the future and past light cones.

Each coadjoint orbit of $SU(1,1)$ is a symplectic manifold having an invariant symplectic structure (the Kostant–Kirillov form). Moreover, the action of $SU(1,1)$ is Hamiltonian and the corresponding moment map is the inclusion map to $\mathfrak{su}(1,1)^\ast$.

We will study the geometry of the symplectic quotients of products of several copies of future and past light cones $C^\pm$ with respect to the diagonal $SU(1,1)$-action. For that, let us fix two positive integers $k_1, k_2$ with $k_1 + k_2 = n$. We will consider null polygons in Minkowski 3-space that have the first $k_1$ edges in the past light cone and the last $k_2$ edges in the future light cone. A closed null polygon is one whose sum of the first $k_1$ sides in the past light cone is symmetric to the sum of the last $k_2$ sides in the future light cone. The space of all such closed null polygons can be identified with the zero level set of the moment map

$$\mu : O_1 \times \cdots \times O_n \longrightarrow \mathfrak{su}(1,1)^\ast$$

(39)

for the diagonal $SU(1,1)$–action, where $O_i \cong C^\pm$ is the past light cone if $1 \leq i \leq k_1$, and $O_i \cong C^+$ is the future light cone if $k_1 + 1 \leq i \leq n$, equipped with the Kostant-Kirillov symplectic form on coadjoint orbits.

Let $(O_1 \times \cdots \times O_n)^\text{reg}$ be the set of regular points of $\mu$ in $O_1 \times \cdots \times O_n$. By equivariance, it is invariant under $SU(1,1)$ and it is easy to check that the action of $SU(1,1)$ on $(O_1 \times \cdots \times O_n)^\text{reg}$ is free (cf. Remark 5.1). If

$$\mu^{-1}(0)^\text{reg} := (O_1 \times \cdots \times O_n)^\text{reg} \cap \mu^{-1}(0)$$

is nonempty then it is a smooth submanifold of $(O_1 \times \cdots \times O_n)^\text{reg}$ (by the Implicit Function Theorem). Moreover, if $\omega$ is the symplectic form on $O_1 \times \cdots \times O_n$ and $x \in \mu^{-1}(0)^\text{reg}$, then the kernel of $\omega_x$ on the tangent space

$$T_x(\mu^{-1}(0)^\text{reg}) = \ker(d\mu)_x = (T_x(SU(1,1) \cdot x))^{\omega_x}$$

is

$$T_x(\mu^{-1}(0)^\text{reg}) \cap (T_x(\mu^{-1}(0)^\text{reg}))^{\omega_x} = \{X_x \in T_x(O_1 \times \cdots \times O_n) : X \in \mathfrak{su}(1,1)\}.$$

Hence, $i^*\omega$, where $i : \mu^{-1}(0)^\text{reg} \to O_1 \times \cdots \times O_n$ is the inclusion map, is a closed 2-form on $\mu^{-1}(0)^\text{reg}$ and the leaves of its null foliation are the orbits of $SU(1,1)$. Consequently, the space

$$M_0^{k_1,k_2} := \mu^{-1}(0)^\text{reg}/SU(1,1)$$

is a symplectic manifold.

**Remark 5.1.** A point $u = (u_1, \ldots, u_n) \in O_1 \times \cdots \times O_n$ is regular if and only if its $SU(1,1)$-stabilizer is trivial. Indeed, if it exists $A \in SU(1,1)$ such that $A \neq Id$ and
\[ Au_i = u_i \text{ for } i = 1, \ldots, n \text{ then } \dim(\langle u_1, \ldots, u_n \rangle) = 1 \text{ (meaning that the null polygon lies along a line) and the stabilizer of } u \text{ is 1-dimensional.} \]

To see this, let us consider a matrix \( A_{\alpha,\beta} \) as in (38) fixing a vector \( u \in \mathbb{C}^\pm \). Using a rotation around the \( t \)-axis we can assume, without loss of generality, that \( u = (0,1,\pm 1) \). Then, up to conjugation by a rotation around the \( t \)-axis, we have that \( A_{\alpha,\beta} \) is of the form

\[
A_{\alpha,k\sqrt{\ell}} := \begin{pmatrix} 1 & \pm 2k & -2k \\
\mp 2k & 1 - 2k^2 & \pm 2k^2 \\
-2k & \mp 2k^2 & 1 + 2k^2 \end{pmatrix}
\]

with \( k \in \mathbb{R} \) and so the stabilizer of \( u \) has dimension 1. Moreover, the space of eigenvectors of \( A_{\alpha,k\sqrt{\ell}} \) associated with the eigenvalue 1 is always one-dimensional for every \( k \neq 0 \).

We conclude that for \( n \geq 4 \) and \( k_1, k_2 \geq 2 \) with \( k_1 + k_2 = n \) the space \( M_0^{k_1,k_2} \) is nonempty and its points represent the null polygons that do not lie along a line. Moreover, the spaces \( M_0^{k_1,k_2} \) and \( M_0^{k_2,k_1} \) are symplectomorphic. Note that if \( k_1 = 1 \) or \( k_2 = 1 \) the space is empty since the sum of two noncolinear past null vectors is a past timelike vector and the sum of a past timelike vector with a past null vector is still a past timelike vector.

**Theorem 5.1.** The space \( M_0^{k_1,k_2} \) is non-compact.

**Proof.** Let \([v] = [v_1, \ldots, v_n] \) by a polygon in \( M_0^{k_1,k_2} \), where \([v] = [v_1, \ldots, v_n] \) with \( v_i \in \mathbb{C}^- \) for \( i = 1, \ldots, k_1 \) and \( v_i \in \mathbb{C}^+ \) otherwise. Note that the vectors \( v_1, \ldots, v_{k_1} \) are not all aligned since, by definition, \((v_1, \ldots, v_n) \) is a regular value of the moment map \( \mu \) defined in (39) (cf. Remark 5.1). Therefore, \( w := v_1 + \cdots + v_{k_1} \) is a time-like vector\(^2\) and we can use a rotation around the \( t \)-axis followed by a boost

\[
T_{\phi} := \begin{pmatrix} 1 & 0 & 0 \\
0 & \cosh \phi & \sinh \phi \\
0 & \sinh \phi & \cosh \phi \end{pmatrix} \in SU(1,1)
\]

along the \( y \)-direction to place the vector \( w \) along the \( t \)-axis. Let \( \ell \) be the Minkowski length of \( w \). Note that \( \ell \) never vanishes and can take any value in \((0,\infty)\). Indeed if we take, for instance,

\[
v_1 := (0, m, -m), \quad v_2 = \cdots = v_{k_1} = \left(0, -\frac{m}{k_1 - 1}, -\frac{m}{k_1 - 1}\right)
\]

and

\[
v_{k_1+1} := (0, m, m), \quad v_{k_1+2} = \cdots = v_n = \left(0, -\frac{m}{k_1 - 1}, \frac{m}{k_1 - 1}\right)
\]

with \( m \in \mathbb{N} \), then \( w = v_1 + \cdots + v_{k_1} = (0, 0, -2m) \) and \( \|w\|_{2,1} = 2m \).

We can therefore consider the (continuous) map \( \ell : M_0^{k_1,k_2} \to (0,\infty) \) which for each \([v] \in M_0^{k_1,k_2} \) gives the Minkowski length of the vector \( v_1 + \cdots + v_{k_1} \) and the result follows. Note that \( \ell \) is the moment map of the circle action obtained by rotating the section of the polygon formed by the first \( k_1 \) edges around the \( t \)-axis. This map has no critical values

\(^2\)The sum of two noncolinear past null vectors is a past timelike vector and the sum of a past timelike vector with a past null vector is still a past timelike vector.
so all level sets are diffeomorphic and we obtain a diffeomorphism between $M_{0}^{k_{1},k_{2}}$ and $P \times (0, \infty)$ where $P$ is a level set of $\ell$. 

\[ \square \]

6. Null hyperpolygons, quasi-parabolic $SL(2, \mathbb{R})$-Higgs bundles and null polygons in $\mathbb{R}^{2,1}$

Let $[p, q] \in Z_{S}$ be a fixed point of the involution (29) in the null hyperpolygon space, where $Z_{S}$ is defined in (30) and consider the vectors $u_{i} \in \mathbb{R}^{3}$ given by

$$u_{i} := \frac{1}{2}(p_{i}p_{i} - q_{i}q_{i})_{0} + \frac{\sqrt{(-1)}}{2}(p_{i}q_{i} + q_{i}p_{i})_{0},$$

where we use identifications $su(2)^{*} \cong (\mathbb{R}^{3})^{*} \cong su(1, 1)^{*}$. Assuming, without loss of generality, that

$$p_{i} = (0, b_{i}), \quad q_{i} = \left(\begin{array}{c} c_{i} \\ 0 \end{array}\right)$$

with $|b_{i}| = |c_{i}|$, for $i \in S$ and

$$p_{i} = (a_{i}, 0), \quad q_{i} = \left(\begin{array}{c} 0 \\ d_{i} \end{array}\right)$$

with $|a_{i}| = |d_{i}|$, for $i \in S^{c}$, we have

$$u_{i} = (\text{Re}(b_{i}c_{i}), \text{Im}(b_{i}c_{i}), -|c_{i}|^{2}), \quad \text{for } i \in S$$

and

$$u_{i} = (\text{Re}(a_{i}d_{i}), -\text{Im}(a_{i}d_{i}), |a_{i}|^{2}), \quad \text{for } i \in S^{c}. $$

Note that for $i \in S$ we have

$$u_{i} \circ u_{i} = -|b_{i}c_{i}|^{2} + |c_{i}|^{4} = |c_{i}|^{2}(|c_{i}|^{2} - |b_{i}|^{2}) = 0,$$

and, for $i \in S^{c}$,

$$u_{i} \circ u_{i} = -|a_{i}d_{i}|^{2} + |a_{i}|^{4} = |a_{i}|^{2}(|a_{i}|^{2} - |d_{i}|^{2}) = 0,$$

and so the vectors $u_{i}$ are null vectors in $\mathbb{R}^{2,1}$. By (36) and (37), we have that

$$\sum_{i=1}^{n} u_{i} = 0,$$

and so the vectors $u_{i}$ form a closed null polygon in Minkowski 3-space with $|S|$ sides in the past light cone and $n - |S|$ sides in the future light cone.

Moreover, the vectors $u_{i}$ are not all aligned. Indeed, if the vectors $u_{i}$ were colinear for $i \in S$, there would exist $i_{0} \in S$ and $k_{i} \in \mathbb{R}^{+}$ such that

$$u_{i} = k_{i}u_{i_{0}} \quad \text{for all } i \in S.$$

Then

$$\sum_{i \in S} b_{i}c_{i} = b_{i_{0}}c_{i_{0}} \sum_{i \in S} k_{i} \neq 0,$$
contradicting (37). We conclude that the vectors $u_i$ form a regular null polygon in $\mathbb{R}^{2,1}$ (cf. Remark 5.1).

**Theorem 6.1.** For any $S \subset \{1, \ldots, n\}$, such that $1 \in S$ and $2 \leq |S| \leq n - 2$, the components $Z_S$ and $Z_S$, of the fixed-point sets of the involutions (28) and (29) in $\mathcal{H}_0^1$ and $X^n_0$ respectively, are diffeomorphic to the moduli space

$$M_0^{[S], [S^c]}$$

of closed null polygons in Minkowski 3-space.

**Proof.** Let $S \subset \{1, \ldots, n\}$ be such that $1 \in S$ and $2 \leq |S| \leq n - 2$. After a suitable reshuffling, consider the map $\varphi : Z_S \longrightarrow M_0^{[S], [S^c]}$, defined by (40) and (41). Hence $\varphi([p, q])$ is the element of $M_0^{[S], [S^c]}$ represented by the null polygon in $\mathbb{R}^{2,1}$ whose first $|S|$ sides are the vectors $u_i$ given by (41), and the last $n - |S|$ sides are the vectors $u_i$ given by (41).

Let us first see that $\varphi$ is well defined. For that, consider two representatives $(p, q)$ and $(p', q')$ of the same class $[p, q]$ in $Z_S$. Then there exists $[A; c_1, \ldots, c_n] \in K$ such that

$$e_i^{-1}p_iA = p_i' \quad \text{and} \quad A^{-1}q_i = q_i', \quad i = 1, \ldots, n.$$  

We can assume, without loss of generality, that $p_i = (a_i \ b_i)$, $p'_i = (a'_i \ b'_i)$ with $a_i = a'_i = 0$ for $i \in S$ and $b_i = b'_i = 0$ for $i \in S^c$, while $q_i = (c_i \ d_i)^t$, $q'_i = (c'_i \ d'_i)^t$ with $d_i = d'_i = 0$ for $i \in S$ and $c_i = c'_i = 0$ for $i \in S^c$. Hence,

$$A = \begin{pmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{pmatrix}$$

for some $\alpha = e^{\sqrt{-1}t_0} \in S^1$. Then we have

$$\begin{pmatrix} \text{Re} (b_i c'_i) \\ \text{Im} (b_i c'_i) \\ -|c'_i|^2 \end{pmatrix} = A_{-2\theta_0} \begin{pmatrix} \text{Re} (b_i c_i) \\ \text{Im} (b_i c_i) \\ -|c_i|^2 \end{pmatrix} \quad \text{for } i \in S,$$

and

$$\begin{pmatrix} \text{Re} (a'_i d'_i) \\ -\text{Im} (a'_i d'_i) \\ |a'_i|^2 \end{pmatrix} = A_{-2\theta_0} \begin{pmatrix} \text{Re} (a_i d_i) \\ -\text{Im} (a_i d_i) \\ |a_i|^2 \end{pmatrix} \quad \text{for } i \in S^c,$$

where

$$A_{-2\theta_0} = \begin{pmatrix} \cos 2\theta & \sin 2\theta & 0 \\ -\sin 2\theta & \cos 2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is a rotation around the $t$-axis (an element of $\text{SU}(1, 1)$).

To see that $\varphi$ is injective, let $[p, q], [p', q']$ be two points in $Z_S$ with

$$\varphi([p, q]) = \varphi([p', q']).$$
Then, without loss of generality, we can write
\[ p_i = \begin{pmatrix} 0 & b_i \end{pmatrix}, \quad p_i' = \begin{pmatrix} 0 & b_i' \end{pmatrix} \quad \text{and} \quad q_i = \begin{pmatrix} c_i \\ 0 \end{pmatrix}, \quad q_i' = \begin{pmatrix} c_i' \\ 0 \end{pmatrix}, \]
for \( i = 1, \ldots, |S| \) with
\[
\sum_{i=1}^{|S|} b_i c_i = \sum_{i=1}^{|S|} b_i' c_i' = 0
\]
(cf. (37)) and
\[
p_i = \begin{pmatrix} a_i & 0 \end{pmatrix}, \quad p_i' = \begin{pmatrix} a_i' & 0 \end{pmatrix} \quad \text{and} \quad q_i = \begin{pmatrix} 0 \\ d_i \end{pmatrix}, \quad q_i' = \begin{pmatrix} 0 \\ d_i' \end{pmatrix},
\]
for \( i = |S| + 1, \ldots, n, \) with
\[
\sum_{i=|S|+1}^n a_i d_i = \sum_{i=|S|+1}^n a_i' d_i' = 0,
\]
and there exists \( A_{\alpha,\beta} \in SU(1,1) \) as in (38) such that
\[
\begin{pmatrix} \Re (b_i' c_i') \\ \Im (b_i' c_i') \\ -|c_i'|^2 \end{pmatrix} = A_{\alpha,\beta} \begin{pmatrix} \Re (b_i c_i) \\ \Im (b_i c_i) \\ -|c_i|^2 \end{pmatrix} \quad \text{for } i = 1, \ldots, |S|
\]
and
\[
\begin{pmatrix} \Re (a_i' d_i') \\ -\Im (a_i' d_i') \\ |d_i'|^2 \end{pmatrix} = A_{\alpha,\beta} \begin{pmatrix} \Re (a_i d_i) \\ -\Im (a_i d_i) \\ |d_i|^2 \end{pmatrix} \quad \text{for } i = |S| + 1, \ldots, n.
\]

Then, we have
\[
b_i' c_i' = -2\sqrt{-1}\alpha\beta|c_i|^2 + (\alpha^2 - \beta^2) \Re (b_i c_i) + \sqrt{-1}(\alpha^2 + \beta^2) \Im (b_i c_i),
\]
for \( i = 1, \ldots, |S| \) and so, by (42), we conclude that \( \alpha\beta = 0 \), implying that \( \beta = 0 \) (since \( |\alpha|^2 - |\beta|^2 = 1 \)).

Consequently,
\[
A_{\alpha,\beta} = A_{\alpha,0} = \begin{pmatrix} \Re \alpha^2 & -\Im \alpha^2 & 0 \\ \Im \alpha^2 & \Re \alpha^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{with } |\alpha| = 1,
\]
is a rotation around the t-axis.

Hence
\[
b_i' c_i' = e^{\sqrt{-1}\theta_0} b_i c_i, \quad \text{and} \quad |b_i'|^2 = |c_i'|^2 = |b_i|^2 = |c_i|^2, \quad \text{for } i = 1, \ldots, |S|,
\]
for some \( \theta_0 \in [0, 2\pi) \) and, by (44), we have
\[
a_i' d_i' = e^{-\sqrt{-1}\theta_0} a_i d_i, \quad \text{and} \quad |a_i'|^2 = |d_i'|^2 = |a_i|^2 = |d_i|^2, \quad \text{for } i = |S| + 1, \ldots, n.
\]
Moreover, since
\[ |c'_i| = |c_i| = |b'_i| = |b_i|, \quad \text{for} \quad i = 1, \ldots, |S|, \]
and\[ |a'_i| = |a_i| = |d'_i| = |d_i|, \quad \text{for} \quad i = |S| + 1, \ldots, n, \]
we have\[ c'_i = e^{\sqrt{-1}\gamma_i}c_i, \quad b'_i = e^{\sqrt{-1}(\theta_0 - \gamma_i)}b_i \quad \text{for} \quad i = 1, \ldots, |S| \]
and\[ a'_i = e^{\sqrt{-1}\phi_i}a_i, \quad d'_i = e^{-\sqrt{-1}(\theta_0 + \phi_i)}d_i \quad \text{for} \quad i = 1, \ldots, |S| \]
for some \( \gamma_i, \phi_i \in [0, 2\pi) \).

We conclude that\[ p'_i = p_i A e^{-1} \quad \text{and} \quad q'_i = A^{-1} q_i e_i, \quad i = 1, \ldots, n \]
with \( A = \begin{pmatrix} e^{-\sqrt{-1}\theta_0/2} & 0 \\ 0 & e^{\sqrt{-1}\theta_0/2} \end{pmatrix} \), and\[ e_i := \begin{cases} e^{\sqrt{-1}(\gamma_i - \theta_0/2)}, & \text{if} \ i = 1, \ldots, |S| \\ e^{-\sqrt{-1}(\phi_i + \theta_0/2)}, & \text{if} \ i = |S| + 1, \ldots, n \end{cases} \]
implies that \([p, q] = [p', q']\).

To show that \( \varphi \) is surjective, let us take \([v] \in M^{[S],|S'|}_0\), where \([v] = [v_1, \ldots, v_n]\) with \( v_i \in C^- \) for \( i = 1, \ldots, |S| \) and \( v_i \in C^+ \) otherwise. Note that the vectors \( v_1, \ldots, v_{|S|} \) are not all aligned since, by definition, \((v_1, \ldots, v_n)\) is a regular value of the moment map \( \mu \) defined in \( [39] \) (cf. Remark 5.1). Therefore, \( w := v_1 + \cdots + v_{|S|} \) is a time-like vector and we can use a rotation around the \( t \)-axis followed by a boost along the \( y \)-direction to place the vector \( w \) along the \( t \)-axis. Hence, we can assume that \([v] \) is represented by a polygon with the first \(|S|\) sides past null vectors \((x_i, y_i, -t_i)\) and the last \(n - |S|\) sides future null vectors \((x_i, y_i, t_i)\) with \( t_i = \sqrt{x_i^2 + y_i^2} \), satisfying
\[ \sum_{i=1}^{|S|} x_i = \sum_{i=1}^{|S|} y_i = \sum_{i=|S|+1}^n x_i = \sum_{i=|S|+1}^n y_i = 0. \]

Then \([v] \) is the image of the null hyperpolygon \([p, q]\), with \[ p_i = \begin{pmatrix} 0 & \frac{1}{l_i}(x_i + \sqrt{-1}y_i) \\ \frac{1}{l_i}(x_i - \sqrt{-1}y_i) & 0 \end{pmatrix}, \quad q_i = \begin{pmatrix} l_i \\ 0 \end{pmatrix} \quad \text{for} \ i \in S, \]
and\[ p_i = \begin{pmatrix} 0 & \frac{1}{l_i}(x_i - \sqrt{-1}y_i) \\ \frac{1}{l_i}(x_i + \sqrt{-1}y_i) & 0 \end{pmatrix}, \quad q_i = \begin{pmatrix} 0 \\ l_i \end{pmatrix} \quad \text{for} \ i \in S', \]
with \( l_i = \sqrt{t_i}, \quad i = 1, \ldots, n. \)

Note that \([p, q] \in Z_S \subset X^n_0\), since
\[ \sum_{i \in S} b_i c_i = \sum_{i=1}^{|S|} (x_i + \sqrt{-1}y_i) = 0, \quad \sum_{i \in S'} a_i d_i = \sum_{i=|S|+1}^n (x_i - \sqrt{-1}y_i) = 0 \]
\[|b_i|^2 = t_i = |c_i|^2 \text{ for all } i \in S, \quad |a_i|^2 = t_i = |d_i|^2, \text{ for all } i \in S^c,\]

where, as usual, we write \(p_i = (a_i \ b_i)\) and \(q_i = (c_i \ d_i)^t\), for \(i = 1, \ldots, n\).

Note that clearly \(\varphi\) and its inverse are differentiable and the result follows. \(\square\)

7. An Example

As an example, we consider the case where \(n = 4\).

Let \(\mathcal{H}_0^4\) be the moduli space of quasi-parabolic \(SL(2, \mathbb{C})\)-Higgs bundles \((E, \Phi)\) of rank two over \(\mathbb{CP}^1\) at \(D\) with \(|D| = 4\), where the underlying holomorphic vector bundle \(E\) is trivial and let us consider the space of null hyperpolygons \(X_0^4\).

A subset \(S\) of \(\{1, 2, 3, 4\}\) such that \(|S| \geq 2\) and \(|S^c| \geq 2\) must have exactly two elements.
If, in addition \(1 \in S\), there are exactly three possibilities. Let us denote these sets by \(S_1\), \(S_2\) and \(S_3\), where \(S_j := \{1, 1 + j\}\) for \(j = 1, 2, 3\).

Then the fixed point set of the involution in (29) defined in the space of null hyperpolygons \(X_0^4\) has exactly three connected components \(Z_{S_1}, Z_{S_2}\) and \(Z_{S_3}\) (cf. Theorem 4.2).

By Theorem 6.1 we know that each component \(Z_{S_i}\) is diffeomorphic to \(M_0^{2,2}\), formed by classes of closed polygons in Minkowski 3-space with the first two sides \(u_1, u_2\) in the past light cone and the last two, namely \(u_3\) and \(u_4\), in the future light cone. Let us consider the diagonal vector \(w = u_1 + u_2\) connecting the origin to the third vertex of the polygon. Since the vectors \(u_1\) and \(u_2\) are not aligned, \(w\) is a past time-like vector and we can consider its Minkowski length \(\ell\). Using a rotation around the \(t\)-axis followed by a boost we can assume that \(w\) lies along the \(t\)-axis.

For each value of \(\ell \in (0, \infty)\) we have a circle of possible classes of polygons obtained by rotating the last two sides of the polygon around the diagonal, while fixing the other two. The length \(\ell\) of \(w\) is the moment map for the bending flow obtained by this rotation of the last two sides of the polygon around the diagonal with a constant angular speed while fixing the other two vectors. Hence, \(Z_{S_j}\) is a non-compact toric manifold of dimension 2 with moment map \(\ell\). Moreover, \(\ell\) has no critical values in \((0, \infty)\) so \(Z_{S_j}\) is diffeomorphic to \(\mathbb{C} \setminus \{0\}\).

We conclude that the fixed point sets \((X_0^4)^\iota\) and \(\mathcal{H}_{0,4}^R\) (the space of quasi-parabolic \(SL(2, \mathbb{R})\)-Higgs bundles over \(\mathbb{CP}^1\) with four quasi-parabolic points) of the involutions (29) and (28) have three non-compact components diffeomorphic to \(\mathbb{C} \setminus \{0\}\).

References

[BFGM] I. Biswas, C. Florentino, L. Godinho and A. Mandini, Polygons in Minkowski three space and parabolic Higgs bundles of rank 2 on \(\mathbb{CP}^1\), Transform. Groups 18 (2013), 995–1018.

[BGH] I. Biswas, O. Garcia-Prada and J. Hurtubise, Higgs bundles, branes and Langlands duality, Comm. Math. Phys. 365 (2019), 1005–1018.

[BGM] O. Biquard, O. Garcia-Prada and I. Mundet i Riera, Parabolic Higgs bundles and representations of the fundamental group of a punctured surface into a real group, arXiv:1510.04207 (2015).

[DP] R. Donagi and T. Pantev, Langlands duality for Hitchin systems, Invent. Math. 189 (2012), 653–735.
[F] P. Foth, *Polygons in Minkowski space and the Gelfand-Tseitin method for pseudo-unitary groups*, Jour. Geom. Phy. 58 (2008), 825–832.

[GM] L. Godinho and A. Mandini, *Hyperpolygon spaces and moduli spaces of parabolic Higgs bundles*, Adv. Math. 244 (2013), 465-532.

[G-P1] O. García-Prada, Higgs bundles and surface group representations, *Moduli spaces and vector bundles*, 265–310, *London Math. Soc. Lecture Note Ser.*, 359, Cambridge Univ. Press, Cambridge, (2009).

[G-P2] O. García-Prada, Involutions of the moduli space of $SL(n, \mathbb{C})$-Higgs bundles and real forms, in *Vector Bundles and Low Codimensional Subvarieties: State of the Art and Recent Developments*, Quaderni di Matematica, Editors: G. Casnati, F. Catanese and R. Notari (2007).

[Hi1] N.J. Hitchin, *Self-duality equations on a Riemann surface*, Proc. London Math. Soc. (3) 55 (1987), 59–126.

[Hi2] N.J. Hitchin, *Lie groups and Teichmüller space*, Topology 31 (1992), 449–473.

[HS] S. Helmke and P. Slodowy *Singular elements of affine Kac-Moody groups*, European Congress of Mathematics, 155–172, Eur. Math. Soc., Zürich, 2005.

[HT] T. Hausel and M. Thaddeus, *Mirror symmetry, Langlands duality and the Hitchin system*, Invent. Math. 153 (2003), 197–229.

[K] H. Konno, *On the cohomology ring of the hyperkähler analogue of polygon spaces*, Integrable systems, topology and physics (Tokyo, 2000), 129–149, Contemp. Math., 309, Amer. Math. Soc., Providence, RI, 2002.

[KW] A. Kapustin and E. Witten, *Electric-magnetic duality and the geometric Langlands program*, Commun. Number Th. Phys. 1 (2007), 1–236.

[M] S. Mukai, An introduction to invariants and moduli. *Cambridge Studies in Advanced Mathematics*, 81. Cambridge University Press, Cambridge, 2003.

[Mu] I. Mundet i Riera, *Parabolic Higgs bundles for real reductive Lie groups*, in Geometry and Physics: Volume II: A Festschrift in honour of Nigel Hitchen, Editors: A. Dancer, J. E. Andersen and O. Garcia-Prada, Oxford Scholarship Online (2018).

[MS] V. B. Mehta and C.S. Seshadri, *Moduli of vector bundles on curves with parabolic structures*, Math. Ann. 248 (1980), 205–239.

[OV] A. L. Onishchik and È. B. Vinberg, Lie groups and algebraic groups, Springer Series in Soviet Mathematics, Springer-Verlag, Berlin, (1990).

[S] D. Salamon, Notes on complex Lie groups, preprint (2018).

[Si1] C.T. Simpson, *Constructing variations of hodge structure using Yang-Mills theory and applications to uniformization*, Jour. Amer. Math. Soc. 1 (1988), 867–918.

[Si2] C.T. Simpson, *Harmonic bundles on noncompact curves*, Jour. Amer. Math. Soc. 3 (1990), 713–770.

[SYZ] A. Strominger, S.-T. Yau and E. Zaslow, *Mirror symmetry is T-duality*, Nuclear Phys. B, 479 (1996), 243–259.

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