Localization and the connection between $U_q(\mathfrak{so}(3))$ and $U_{\tilde{q}}(\mathfrak{osp}(1|2))$

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Abstract. In [1] we constructed homomorphisms from the quantized enveloping algebras of the Euclidean and Euclidean super Lie algebras, $U_q(\mathfrak{iso}(2))$ and $U_{\tilde{q}}(\tilde{\mathfrak{iso}}(2))$, onto their images in localizations of $U_q(\mathfrak{sl}(2))$ and $U_{\tilde{q}}(\mathfrak{osp}(1|2))$, respectively, and, conversely, we described homomorphisms of $U_q(\mathfrak{sl}(2))$ and $U_{\tilde{q}}(\mathfrak{osp}(1|2))$ into localizations of $U_q(\mathfrak{iso}(2))$ and $U_{\tilde{q}}(\tilde{\mathfrak{iso}}(2))$. Herewith we use these homomorphisms to obtain a new relationship between $U_{\tilde{q}}(\mathfrak{osp}(1|2))$ and $U_q(\mathfrak{so}(3))$.

1. Introduction
Localization, or formation of quotients, is a powerful tool in mathematics with many applications. It can be used to relate different algebraic structures which share some common similarities. An important example of this is the Gelfand-Kirillov conjecture [2] which asserts that, for the universal enveloping algebra of a Lie algebra over an algebraically closed field, its quotient field is isomorphic to some skew field extension of a Weyl algebra over a transcendental extension of the base field.

Another related example, which is of importance to physics, is given in [3], where they demonstrate a $*$-isomorphism between commutative algebraic extensions of the Lie fields of $SO(1,4)$ and the Poincaré group. In [4] we used this isomorphism of Lie fields to construct representations of the Poincaré Lie algebra out of unitary representations of $SO(1,4)$. Analogous result for $SO(2,3)$ and the Poincaré Lie algebra are established in [5] except that the isomorphism of Lie fields is no longer a $*$-isomorphism. From this result and a similar study of the other real forms of $SO(5)$ and the much easier to analyze lower dimensional cases, it is clear that similar such isomorphisms of Lie fields of $G = SO(p,q)$ and associated semidirect products of the form $SO(p-1,q) \times_s V$ and $SO(p,q-1) \times_s V$ exists for all $SO(p,q)$ groups with $p + q \leq 5$.

In [1] we generalized the ideas of the previous paragraph to quantized Lie algebras and to quantized Lie superalgebras. Specifically, we considered $U_q(\mathfrak{sl}(2))$ and $U_q(\mathfrak{osp}(1,2))$, and their relationship to $U_q(\mathfrak{iso}(2))$ and $U_{\tilde{q}}(\tilde{\mathfrak{iso}}(2))$, respectively. Here $U_q(\mathfrak{iso}(2))$ is the $q$ deformed universal enveloping algebra of the Euclidean group in the plane which, if we neglect its Hopf algebra structure, is isomorphic to $U(\mathfrak{iso}(2))$, the universal enveloping algebra of the Euclidean group [6]. Similarly, $U_{\tilde{q}}(\tilde{\mathfrak{iso}}(2))$ is the $\tilde{q}$ deformed universal enveloping algebra of the Euclidean Lie superalgebra, which is likewise isomorphic to $U(\tilde{\mathfrak{iso}}(2))$, again, only as an isomorphism of enveloping algebras, not as a Hopf algebra isomorphism. $U_q(\mathfrak{sl}(2))$ and $U_q(\mathfrak{osp}(1,2))$ are the
standard Drinfeld-Jimbo $q$ deformations of the universal enveloping algebras of $\mathfrak{sl}(2)$ and its supersymmetric counterpart, $\mathfrak{osp}(1, 2)$, respectively. In [1] we constructed homomorphisms from $U_q(\mathfrak{sl}(2))$ and $U_q(\mathfrak{iso}(2))$ onto their images into extensions of localizations of $U_q(\mathfrak{iso}(2))$ and $U_q(\mathfrak{sl}(2))$, respectively, and we used these homomorphism to obtain results on the relationship between representations of the respective algebras. There we also constructed homomorphisms from $U_q(\mathfrak{osp}(1|2))$ and $U_q(\mathfrak{iso}(2))$ onto their images into extensions of localizations of $U_q(\mathfrak{iso}(2))$ and $U_q(\mathfrak{osp}(1|2))$, respectively, and we used these homomorphism to obtain new representations of $U_q(\mathfrak{osp}(1|2))$.

Here we use results about $U_q(\mathfrak{osp}(1|2))$ and $U_q(\mathfrak{sl}(2))$ (more or less $U_q(\mathfrak{so}(3))$), mentioned in the previous paragraph with an aim towards obtaining a better understanding of the results in [7] for $U_q(\mathfrak{osp}(1|2n))$ and $U_q(\mathfrak{so}(2n+1))$, something which we have succeeded in demonstrating completely and quite clearly in this paper for the $n=1$ case. For an alternative approach to understanding the relationship between $U_q(\mathfrak{osp}(1|2n))$ and $U_q(\mathfrak{so}(2n+1))$ see [8].

The existence of a one-to-one correspondence between the finite dimensional representations of $\mathfrak{osp}(1|2n)$ and $\mathfrak{so}(2n+1)$, except in the case of what they call spinorial representations, has been known for some time [9]. For quantum algebras, this is the result of [7], namely, a connection between $U_q(\mathfrak{osp}(1|2n))$ and $U_{-q}(\mathfrak{so}(2n+1))$, which only holds for non-spinorial representations. Work of Aizawa et. al in [10] shows that the even-dimensional representations, for which the said isomorphism is not applicable, still possess properties reflecting this isomorphism. In this paper we specialize to the $n=1$ case and construct a mapping from $U_q(\mathfrak{osp}(1|2))$ into a localization of $U_q(\mathfrak{so}(3))$ which can be used to obtain a homomorphism of a quotient of $U_q(\mathfrak{osp}(1|2))$ into its image in a localization of $U_q(\mathfrak{so}(3))$. Since we work at the level of abstract (quantized enveloping) algebras and their localizations our results for $n=1$ are more general that those in [7]. In particular, they are applicable not only to finite dimensional representations but also to infinite dimensional ones. Furthermore, it is possible that our results could help to explain the phenomenon in [10] to which we just alluded.

Remarks on notation and definitions: We follow the conventions about notation established in [1]. Except for elements of the Cartan subalgebras for which we always use bold faced letters, quantities made out of elements of $U_q(\mathfrak{iso}(2))$ and $U_q(\mathfrak{sl}(2))$ and of their localizations are usually denoted with bold faced letters and we use plain faced letters to denote elements of $U_q(\mathfrak{so}(2n+1))$ and $U_q(\mathfrak{osp}(1|2n))$ and their localizations. Elements of Lie superalgebras are always denoted with tildes placed over the letters.

Our definitions of $U_q(\mathfrak{so}(2n+1))$ and $U_q(\mathfrak{osp}(1|2n))$ differ slightly from those given in [7]. For the Drinfeld-Jimbo $U_q(\mathfrak{so}(2n+1))$ we take the standard definition found in [11], which uses $q = q^{d_{i}}$ where $d_{i} := (\alpha_{i}, \alpha_{i})/2$ with $\alpha_{i}$ being the $i$th simple root. Presumably the two definitions can be related to one another by a simple rescaling of some of the generators.

2. $U_q(\mathfrak{so}(2n+1))$ and $U_q(\mathfrak{osp}(1|2n))$

We recall the $q$-deformation $U_q(\mathfrak{so}(2n+1)) = U_q(\mathfrak{so}(2n+1, C))$ is defined as follows. Let $(\ , )$ be the standard inner product on $\mathbb{R}^n$. The root system of $U_q(\mathfrak{so}(2n+1))$ is defined in terms of the standard orthonormal unit vectors, $\epsilon_1, \ldots, \epsilon_n$, which form a basis of $\mathbb{R}^n$ and satisfy $(\epsilon_i, \epsilon_j) = \delta_{ij}$, $i, j = 1, \ldots, n$. For the simple root system we have:

$$\Pi = \{\alpha_1, \ldots, \alpha_n\}$$

$$\alpha_j = \epsilon_j - \epsilon_{j+1}, \quad j = 1, \ldots, n-1; \quad \alpha_n = \epsilon_n.$$
The Cartan matrix of $U_q(\mathfrak{so}(2n+1))$ is the $n \times n$ matrix $(a_{jk}) = 2(\alpha_j, \alpha_k)/\langle \alpha_j, \alpha_k \rangle$. Explicitly we have:

$$A = (a_{ij}) = \begin{pmatrix}
2 & -1 & 0 & \ldots & 0 & 0 & 0 \\
-1 & 2 & -1 & \ldots & 0 & 0 & 0 \\
& \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
& & 0 & 0 & \ldots & 2 & -1 & 0 \\
& & & 0 & 0 & \ldots & -1 & 2 & -2 \\
& & & & \ldots & 0 & -1 & 2 & \ldots \\
& & & & & \ldots & 0 & \ldots & \ldots \\
\end{pmatrix}$$

We introduce the Cartan-Weyl basis with $H_i$, $i = 1, \ldots, n$, being the basis of the Cartan subalgebra $\mathcal{H}$ associated with the simple roots and $X_i^\pm$, $i = 1, \ldots, n$, being the simple root vectors. Let $q \in \mathbb{C}$ and set $K_i = q_i^{H_i}$ with $q_i = (q)^{d_i}$ where $d_i := (\alpha_i, \alpha_i)/2$ and $q_i$ such that $q_i^2 \neq 1$ for $i = 1, 2, \ldots, n$. Further let $[x]_{q_i} = \frac{q_i-x}{q_i-1}$. We then define $U_q(\mathfrak{so}(2n+1))$ as the unital associative complex algebra with generators $K_i, K_i^{-1}$ and $X_j^\pm$ ($1 \leq i \leq n$) and defining relations [11]:

$$K_iK_j = K_jK_i, K_iK_i^{-1} = 1, K_iX_j^\pm K_i^{-1} = q_i^{\pm a_{ij}}X_j^\pm \tag{2.1a}$$

$$[X_i^+, X_j^-] = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}} \tag{2.1b}$$

$$\sum_{r=0}^{1-a_{ij}} (-1)^r \frac{[1-a_{ij}]_{q_i}}{[1-a_{ij} - r]_{q_i}} (X_i^\pm)^{1-a_{ij} - r} X_j^\pm (X_i^\pm)^r = 0, (i \neq j). \tag{2.1c}$$

Expressing the first two of the above equations in terms of the $H_i$ and explicitly writing out the last set of equations, we have that Eqs. (2.1a), (2.1b) and (2.1c) are (formally) equivalent to

$$[H_i, H_j] = 0, [H_i, X_j^\pm] = \pm a_{ij} X_j^\pm, \tag{2.1a'}$$

$$[X_i^+, X_j^-] = \delta_{ij} [H_i]_{q_i}, \tag{2.1b'}$$

$$\begin{cases}
(X_i^\pm)^2 X_j^\pm - [2]_{q_i} X_i^\pm X_j^\pm + X_i^\pm (X_j^\pm)^2 = 0 & \text{for } i \pm 1 \neq n \\
(X_n^\pm)^3 X_{n-1}^\pm - [3]_{q_i}^1 (X_n^\pm)^2 X_{n-1}^\pm + [3]_{q_i}^1 (X_n^\pm)^2 X_{n-1}^\pm (X_n^\pm)^2 - X_{n-1}^\pm (X_n^\pm)^3 = 0 \\
X_j^\pm X_j^\pm - X_j^\pm X_j^\pm = 0 & \text{for } j \neq i \pm 1. 
\end{cases} \tag{2.1c'}$$

We note that the $\pm$ in the superscripts are not correlated with the $\pm$ in the subscripts. We do not bother to write down the definitions of the Cartan-Weyl generators corresponding to the non-simple roots, since we do not need to make use of them in this paper.

We now state the definition of $U_q(\mathfrak{osp}(1|2n))$. We again introduce $\epsilon_1, \ldots, \epsilon_n$, $(\epsilon_i, \epsilon_j) = \delta_{ij}$, $i, j = 1, \ldots, n$ and the root system of $U_q(\mathfrak{osp}(1|2n))$ is given by

$$\Delta_0 = \{+\epsilon_i \pm \epsilon_j, -\epsilon_i \pm \epsilon_j, 1 \leq i < j \leq n, \pm 2\epsilon_i, 1 \leq i \leq n\}, \quad \text{(even roots)}$$

$$\Delta_1 = \{\pm \epsilon_1, 1 \leq i \leq n\}, \quad \text{(odd roots)}.$$
with
\[ \alpha_j = \epsilon_j - \epsilon_{j+1}, \quad j = 1, \ldots, n-1; \alpha_n = \epsilon_n. \]
The \( \alpha_i \) for \( i < n \) are even and the root \( \alpha_n \) is odd. We note that the even root system \( \Delta_0 \) is the root system of \( U_q(\mathfrak{sp}(2n)) \) i.e. that of \( U_q(C_n) \) [12]. (Note that \( \mathfrak{sp}(2n) \) is also denoted by \( \mathfrak{sp}(n) \) in mathematical literature.) We introduce the Cartan-Weyl basis with \( \hat{H}_i, \ i = 1, \ldots, n, \) being the basis of the Cartan subalgebra \( \hat{H} \) associated with the simple roots and \( \hat{X}^\pm_i, \ i = 1, \ldots, n \) being the simple root vectors. We let
\[
A = (a_{ij}) = \begin{pmatrix}
2 & -1 & 0 & \cdots & 0 & 0 & 0 \\
-1 & 2 & -1 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 2 & -1 & 0 \\
0 & 0 & 0 & \cdots & -1 & 2 & -1 \\
0 & 0 & 0 & \cdots & 0 & -2 & 2
\end{pmatrix}
\]
denote the Cartan matrix of \( U_q(\mathfrak{sp}(2n)) \) and consider its decomposition \( A = A^dA^s \) with \( A^d = \text{diag}(1, \ldots, 1, 2) \) and \( A^s = (a^s_{ij}) \). Except for its last row, the matrix for \( A^s \) is the same as the matrix for \( A \). The last row of \( A^s \) is obtained from the matrix for \( A \) by dividing all terms in the last row by \( \frac{1}{2} \).

Let \( \tilde{K}_i = \tilde{q}_i\hat{H}_i \) with \( \tilde{q}_i = (\tilde{q})^d_i \) where, as before, \( d_i := (\alpha_i, \alpha_i)/2 \) and \( \tilde{q}_i \) such that \( \tilde{q}_i^2 \neq 1 \) for \( i = 1, 2, \ldots, n \). We can now define \( U_q(\mathfrak{osp}(1|2n)) \) as the unital associative complex algebra with generators \( \tilde{K}_i, \tilde{K}_i^{-1} \) and \( \hat{X}^\pm_i \) \( (1 \leq i \leq n) \) and relations [13] [7]:
\[
\tilde{K}_i\tilde{K}_j = \tilde{K}_j\tilde{K}_i, \quad \tilde{K}_i\tilde{K}_j^{-1} = \tilde{K}_j^{-1}\tilde{K}_i = 1, \quad \tilde{K}_i\hat{X}^\pm_j\tilde{K}_i^{-1} = q_i^{\pm a_{ij}}\hat{X}^\pm_j
\]
\[
[\hat{X}^+_i, \hat{X}^-_j] = \delta_{ij}\frac{\tilde{K}_i - \tilde{K}_i^{-1}}{\tilde{q}_i - \tilde{q}_j^*} = \delta_{ij}[\hat{H}_i][\tilde{q}_i], \quad (2.2b')
\]
\[
(\hat{X}^\pm_{i+1})^2\hat{X}^\pm_i - [2][\hat{X}^\pm_{i+1}\hat{X}^\pm_i]\hat{X}^\pm_{i+1} + \hat{X}^\pm_i(\hat{X}^\pm_{i+1})^2 = 0 \quad \text{for } i \pm 1 \neq n
\]
\[
(\hat{X}^\pm_{n-1})^3\hat{X}^\pm_n + [3](-\tilde{q})^3(\hat{X}^\pm_n)^2\hat{X}^\pm_{n-1} + [3](-\tilde{q})^2\hat{X}^\pm_n\hat{X}^\pm_{n-1}(\hat{X}^\pm_n)^2 + \hat{X}^\pm_{n-1}(\hat{X}^\pm_n)^3 = 0 \quad (2.2c')
\]
\[
\hat{X}^\pm_i\hat{X}^\pm_j - \hat{X}^\pm_j\hat{X}^\pm_i = 0 \quad \text{for } j \neq i \pm 1.
\]
Just as in the \( U_q(\mathfrak{so}(2n+1)) \) case considered above, the \( \pm \) in the superscripts are not correlated with the \( \pm \) in the subscripts. Furthermore, Eq. (2.2a) is, as for \( U_q(\mathfrak{so}(2n+1)) \), formally equivalent to the following:
\[
[H_i, H_j] = 0, \quad [H_i, \hat{X}^+_j] = \pm a^s_{ij}\hat{X}^+_j. \quad (2.2a')
\]
The \( \mathbb{Z}_2 \) grading on \( U_q(\mathfrak{osp}(1|2n)) \) is that induced from the following grading on the generators:
\[ \text{deg} \hat{X}^+_n = 1 \quad \text{and deg} a = 0 \quad \text{for } a \text{ being any one of the other generators of } U_q(\mathfrak{osp}(1|2n)). \]
3. The Algebras $U_q(\mathfrak{so}(2))$ and $U_{\tilde{q}}(\mathfrak{so}(2))$

The quantized universal enveloping algebra $U_q(\mathfrak{so}(2))$ is the complex, unital associative algebra with generators $A$, $B$, $K$, $K^{-1}$ satisfying the relations $KK^{-1} = K^{-1}K = 1$ and

$$KBK^{-1} = q^2 B, \quad KCK^{-1} = q^{-2} C$$

$$BC = CB.$$  \hfill (3.1) \hfill (3.2)

We have equivalently, with $K = q^H$, generators $H$, $P^+ = B$ and $P^- = C$ and relations

$$[H, P^\pm] = \pm 2P^\pm,$$  \hfill (3.1')

$$[P^+, P^-] = 0.$$  \hfill (3.2')

The center $\mathfrak{Z}(U_q(\mathfrak{so}(2)))$ of $U_q(\mathfrak{so}(2))$ is generated by

$$Y^2 = P^+P^- = P^-P^+.$$  \hfill (3.3)

Note that algebra defined by Eqs. (3.1') and (3.2') is the same as the enveloping algebra of $\mathfrak{so}(2)$, the classical Euclidean Lie algebra of two dimensional Euclidean space; however, as a Hopf algebra it is different [6]. $U_q(\mathfrak{so}(2))$ can be obtained from $U_q(\mathfrak{so}(3))$ by a contraction process analogous to that defined by Segal [14] and Inönü and Wigner [15] for Lie algebras, namely, replace $K$, $E$, $F$ by $K$, $\rho E$, $\rho F$ and let $\rho \downarrow 0$ [16].

The quantized universal enveloping algebra of the Euclidean Lie superalgebra, $U_q(\mathfrak{iso}(2))$, is defined exactly as in the previous paragraph except with anticommutator instead of commutator in Eq. (3.2') and tildes placed over the $P^\pm$, $K$ and $H$. Thus, in terms of the generators $H$ and $\tilde{P}^\pm$, we have the relations

$$[\tilde{H}, \tilde{P}^\pm] = \pm 2\tilde{P}^\pm,$$  \hfill (3.4)

$$\{\tilde{P}^+, \tilde{P}^-\} = 0.$$  \hfill (3.5)

Let $\tilde{Y}^2$ be the element of $\mathfrak{Z}(U_q(\mathfrak{iso}(2)))$ given by

$$\tilde{Y}^2 = -i(-1)^{\tilde{H}} \tilde{P}^+\tilde{P}^- = i(-1)^{\tilde{H}} \tilde{P}^-\tilde{P}^+.$$  \hfill (3.6)

In order to give a precise meaning to expressions like $(-1)^{\tilde{H}} = e^{i\pi\tilde{H}}$ which occurs in this equation and other similar expressions which occur frequently throughout the rest of the paper, we refer the reader to [17]. There they describe how to construct extensions of enveloping algebras which accommodate such quantities as $e^{i\pi\tilde{H}}$ defined by infinite series (cf. [17]). To keep things simple we do not make a distinction between enveloping algebras and necessary such extensions for incorporating such formal series expansions.

4. Localizations of $U_q(\mathfrak{so}(3))$ and $U_q(\mathfrak{iso}(2))$

In this section and the next we recall our results in [1] on localizations of $U_q(\mathfrak{so}(3))$ and $U_q(\mathfrak{osp}(1|2))$ and associated homomorphisms of $U_q(\mathfrak{so}(3))$ and $U_q(\mathfrak{osp}(1|2))$ into various localizations. In the next remaining sections, we make the substitutions $q \rightarrow q^2$ and $\tilde{q} \rightarrow \tilde{q}^2$, i.e. replace $q$ in section 2 everywhere by $q^2$ etc. We also use a slightly different definition for $q$ numbers, namely, $[x]_q = \frac{(q^{1/2} - q^{-1/2})}{(q^{1/2} - q^{-1/2})}$ in order to conform with that used in [1]. (The change amounts simply to the following: $[x]_q^{1/2}$ in section 2 = $[x]_q$ in this section and the next two, and similarly with $\tilde{q}$ replacing $q$.) Let $X^\pm$ and $H$ denote the generators of $U_q(\mathfrak{so}(3))$, the relations for
which are given by Eqs. (2.1a') and (2.1b') for \( n = 1 \) with subscripts on the generators omitted, i.e.

\[
[H, X^\pm] = \pm 2X^\pm
\]  

\[
[X^+, X^-] = [H]_{q^2}
\]  

For \( q \) not a root of unity, \( \mathfrak{Z}(U_q(\mathfrak{so}(3))) \) is generated by

\[
\Delta_q = X^+X^- + \left( \frac{H - 1}{2} \right)_{q^2}^2 - \frac{1}{4} \cdot I = X^-X^+ + \left( \frac{H + 1}{2} \right)_{q^2}^2 - \frac{1}{4} \cdot I .
\]  

For a description of \( \mathfrak{Z}(U_q(\mathfrak{so}(3))) \) when \( q \) is a root of unity we refer the reader to [11].

Now define

\[
X^\pm = \left\{ \pm \frac{1}{Y} \left[ \frac{H \pm 1}{2} \right]_{q^2} + I \right\}, \quad P^\pm = P^\pm \left\{ \pm \frac{1}{Y} \left[ \frac{H \pm 1}{2} \right]_{q^2} + I \right\}
\]  

(4.3\pm)

where \( Y \) is a solution of the algebraic equation \( Y^2 - P^+P^- = 0 \) in \( U_q(\mathfrak{so}(2)) \) and is assumed to commute with all elements of \( U_q(\mathfrak{so}(2)) \). Let \( \tau(X^\pm) = X^\pm \) and \( \tau(H) = H \). This defines a mapping \( \tau \) from \( U_q(\mathfrak{so}(3)) \) into a commutative algebraic extension of the localization of \( U_q(\mathfrak{so}(2)) \) with denominators consisting of powers of \( Y \).

**Proposition 1** The map \( \tau \) defines a homomorphism from \( U_q(\mathfrak{sl}(2)) \) onto its image. In particular the \( X^\pm \) defined by Eqs. (4.3\pm) together with \( H \) satisfy the relations, Eqs. (4.1a) and (4.1b), of the generators of \( U_q(\mathfrak{so}(3)) \). Furthermore, let \( \Delta_q \) be defined by Eq. (4.2) but with \( X^\pm \) replacing \( X^\pm \). Then \( \Delta_q = Y^2 - \frac{1}{4} \cdot I \).

For a proof of this result we refer the reader to [1].

Now let \( Y \) be such that it commutes with all elements of \( U_q(\mathfrak{sl}(2)) \) and satisfies the equation

\[
Y^2 = \Delta_q + \frac{1}{4} \cdot I .
\]  

(4.4)

Let \( P^\pm = (D_L^+)^{-1}X^\pm = X^\pm(D_L^+)^{-1} \) with \( D_L^+ = \left( \pm \frac{1}{Y} \left[ \frac{H + 1}{2} \right]_{q^2} + I \right) \) and \( D_R^+ = \left( \pm \frac{1}{Y} \left[ \frac{H + 1}{2} \right]_{q^2} + I \right) \) and define \( \tau' \) by \( \tau'(P^\pm) = P^\pm \) and \( \tau'(H) = H \).

**Proposition 2** \( \tau' \) extends by linearity to a homomorphism from \( U_q(\mathfrak{so}(2)) \) into a localization of \( U_q(\mathfrak{sl}(2)) \). In particular, \( \tau'(P^\pm) \) and \( \tau'(H) \) satisfy the commutation relations, Eqs. (3.1') and (3.2'), of \( U_q(\mathfrak{so}(2)) \) and, furthermore, \( P^+P^- = Y^2 \).

For the proof of this proposition we again refer to [1].

5. **Localizations of \( U_q(\mathfrak{osp}(1|2)) \) and \( U_q(\mathfrak{so}(2)) \)**

The definition of the \( \bar{q} \)-deformation \( U_q(\mathfrak{osp}(1|2)) \) of the enveloping algebra of \( \mathfrak{osp}(1|2) \) is obtained from Eqs. (2.1a'), (2.1b') by specializing to \( n = 1 \). Just as for the \( U_q(\mathfrak{so}(3)) \) treated in the previous section, we omit the subscripts on the generators, and for \( U_q(\mathfrak{osp}(1|2)) \) we have generators \( \bar{H}, \bar{X}^\pm \) with relations \( \{ \cdot, \cdot \} \) denote anticommutator):

\[
[\bar{H}, \bar{X}^\pm] = \pm \bar{X}^\pm ,
\]  

(5.1a)

\[
\{ \bar{X}^+, \bar{X}^- \} = [\bar{H}]_{q^2} .
\]  

(5.1b)

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1 In other words we consider a commutative algebraic extension \( \hat{U}_q(\mathfrak{so}(2)) \) of \( U_q(\mathfrak{so}(2)) \) defined as follows: \( \hat{U}_q(\mathfrak{so}(2)) = \{ a + bY | a, b \in U_q(\mathfrak{so}(2)) \} , \{ Y, a \} = 0 \forall a \in U_q(\mathfrak{so}(2)), Y^2 - P^+P^- = 0. \)
The Casimir operator of $U_q(\mathfrak{osp}(1|2))$ is $\tilde{\Delta}_q = \tilde{S}_q^2 + 2 \cdot I$ with [1]

$$S_q = \frac{q^{1/2}k - q^{-1/2}k^{-1}}{q - q^{-1}} - (q^{1/2} + q^{-1/2}) \tilde{X}^- \tilde{X}^+ = [\tilde{H} + \frac{1}{2}, \tilde{q}]_{\tilde{q}} - [2]_q \tilde{X}^- \tilde{X}^+ =$$

$$= -\frac{q^{-1/2}k - q^{1/2}k^{-1}}{(q - q^{-1})} + (q^{1/2} + q^{-1/2}) \tilde{X}^+ \tilde{X}^- = -[\tilde{H} - \frac{1}{2}, q]_q + [2]_q \tilde{X}^+ \tilde{X}^-.$$

We now describe homomorphisms associated with $U_q(\tilde{\mathfrak{iso}}(2))$ and $U_q(\mathfrak{iso}(2))$. These will be used in the next section in order to relate $U_q(\mathfrak{so}(2n + 1))$ and $U_q(\mathfrak{osp}(1|2n))$ for $n = 1$. Define $\tau_0$ by $\tau_0(H) = -2\tilde{H}$, $\tau_0(P^+) = -\tilde{P}^+$, $\tau_0(P^-) = e^{-i\pi\tilde{H}}\tilde{P}^+$ and $\tilde{\tau}_0$ by $\tilde{\tau}_0(H) = -\frac{1}{2}H$, $\tilde{\tau}_0(\tilde{P}^+) = -\tilde{P}^+$, $\tilde{\tau}_0(\tilde{P}^-) = e^{-\frac{i\pi}{2}\tilde{H}}\tilde{P}^-$. **Proposition 3** $\tau_0$ and $\tilde{\tau}_0$ define Lie algebra and Lie superalgebra homomorphisms from $U_q(\mathfrak{iso}(2))$ and $U_q(\tilde{\mathfrak{iso}}(2))$ onto their images in $U_q(\mathfrak{iso}(2))$ and $U_q(\tilde{\mathfrak{iso}}(2))$, respectively.

In order to prove this result we extend $\tau_0$ and $\tilde{\tau}_0$ by linearity to $\mathfrak{iso}(2)$ and $\tilde{\mathfrak{iso}}(2)$, respectively. We verify the respective commutation relations and then lift the mappings to the enveloping algebras.

Next we describe homomorphisms of $U_q(\mathfrak{osp}(1|2))$ and $U_q(\tilde{\mathfrak{iso}}(2))$ into extensions of localizations of $U_q(\tilde{\mathfrak{iso}}(2))$ and $U_q(\mathfrak{osp}(1|2))$, respectively. Let

$$\tilde{X}^\pm = \left(\frac{1}{Y} \sqrt{e^{i\pi \tilde{H}} \left[\tilde{H} + \frac{1}{2}\right]_{\tilde{q}^2} + \sqrt{\pm I}} \right) \tilde{P}^\pm = \frac{\tilde{P}^\pm}{\sqrt{[2]_q}} = \left(\frac{1}{Y} \sqrt{e^{i\pi \tilde{H}} \left[\tilde{H} + \frac{1}{2}\right]_{\tilde{q}^2} + \sqrt{\pm I}} \right)$$

where $I$ is the identity in $U_q(\tilde{\mathfrak{iso}}(2))$.

**Proposition 4** If $\tilde{Y}$ is such that it commutes with all elements of $U_q(\tilde{\mathfrak{iso}}(2))$ and satisfies Eq. (3.6), then Eqs. (5.3±) define a homomorphism $\tilde{\tau}_q$ from $U_q(\mathfrak{osp}(1|2))$ into a localization of $U_q(\tilde{\mathfrak{iso}}(2))$, with $\tilde{\tau}_q(\tilde{X}^\pm) = \tilde{X}^\pm$ and $\tilde{\tau}_q(\tilde{H}) = \tilde{H}$. Furthermore, let $\tilde{S}_q$ be defined by Eqn. (5.2) but with $\tilde{X}^\pm$ replacing $\tilde{X}^\pm$, then $\tilde{S}_q = -e^{-i\pi\tilde{H}}\tilde{Y}^2$.

Now let $\tilde{P}^\pm = (\tilde{D}_{R}^\pm)^{-1} \tilde{X}^\pm = \tilde{X}^\pm (\tilde{D}_{R}^\pm)^{-1}$ with $\sqrt{[2]_q} \tilde{D}_{R}^\pm = \frac{\pm i}{Y} \sqrt{e^{i\pi \tilde{H}} \left[\tilde{H} + \frac{1}{2}\right]_{\tilde{q}^2} + \sqrt{\pm I}}$ and

$\sqrt{[2]_q} \tilde{D}_{R} = \frac{\pm i}{Y} \sqrt{e^{i\pi \tilde{H}} \left[\tilde{H} + \frac{1}{2}\right]_{\tilde{q}^2} + \sqrt{\pm I}}$ where now $I$ is the identity in $U_q(\mathfrak{osp}(1|2))$.

**Proposition 5** Let $\tilde{\tau}_q(\tilde{P}^\pm) = \tilde{P}^\pm$ and $\tilde{\tau}_q^t(\tilde{H}) = \tilde{H}$. If $\tilde{Y}$ is such that it commutes with all elements of $U_q(\mathfrak{osp}(1|2))$ and satisfies

$$\tilde{Y}^2 + e^{i\pi \tilde{H}} \tilde{S}_q = 0,$$

then $\tilde{\tau}_q$ extends to a homomorphism of $\tilde{\mathfrak{iso}}(2)$ onto its image in a localization of $U_q(\mathfrak{osp}(1|2))$. In particular $\tilde{P}^\pm$ and $\tilde{H}$ satisfy the commutation relations (3.4) and (3.5) of $U(\mathfrak{iso}(2))$ and, furthermore, $\tilde{P}^+ \tilde{P}^- = i e^{-i\pi \tilde{H}} \tilde{Y}^2$.

We refer the reader to [1] for proofs of these propositions.

6. A mapping of $U_q(\mathfrak{osp}(1|2))$ into a localization of $U_q(\mathfrak{so}(3))$ and the connection between $U_q(\mathfrak{so}(2n + 1))$ and $U_q(\mathfrak{osp}(1|2n))$

We now make use of the homomorphisms described in the above propositions to construct a mapping from $U_q(\mathfrak{osp}(1|2))$ into a localization of $U_q(\mathfrak{so}(3))$ and determine conditions for this.
mapping to be a homomorphism. Having established this we can construct representations of $U_q(\mathfrak{osp}(1|2))$ out of representations of $U_q(\mathfrak{so}(3))$, provided the representation of $U_q(\mathfrak{so}(3))$ lifts to a representation of its localization into which $U_q(\mathfrak{osp}(1|2))$ is embedded. Necessary conditions for $U_q(\mathfrak{so}(3))$ representations to lift to representations of their localizations are stated in Lemma 1 of [1], and it is easy to see that there are nontrivial such representations of $U_q(\mathfrak{so}(3))$ for which the conditions of the Lemma are satisfied, for example, extensions of infinitesimally unitarizable principle series representations of $U_q(\mathfrak{so}(2,1))$ to its complexification, $U_q(\mathfrak{so}(3))$ (cf. [18]).

We define the mapping $\pi$ from $U_q(\mathfrak{osp}(1|2))$ into a localization of $U_q(\mathfrak{so}(3))$ using the composition of homomorphisms $\tau' \circ \tau_0 \circ \tilde{\tau}$. (For simplicity, we set $\tilde{\tau} = \tilde{\tau}_q$, where $\tilde{\tau}_q$ is defined in Proposition 4.) We have:

$$\pi(\tilde{H}) = (\tau' \circ \tau_0 \circ \tilde{\tau})(H) , \quad (6.1)$$

$$\pi(\tilde{X}^\pm) = (\tau' \circ \tau_0 \circ \tilde{\tau})(X^\pm). \quad (6.2 \pm)$$

Using the definitions of $\tilde{\tau}$, $\tau_0$ and $\tau'$ given in the previous two sections, we are led to the following:

$$\pi(\tilde{H}) = -\frac{1}{2} H , \quad (6.3)$$

$$\pi(\tilde{X}^\pm) = \pm \frac{1}{\sqrt{2|\tilde{q}|}} (-1)^{-\frac{H}{2} + \frac{H}{4} + \frac{H}{4}} \left( \frac{1}{\sqrt{\frac{1}{2} \left[ \frac{H + 1}{2} \right]_{|\tilde{q}|^2} + \sqrt{\pm I}} \right) X^\pm$$

$$= - \frac{1}{\sqrt{2|\tilde{q}|}} X^\pm (-1)^{-\frac{H}{4} + \frac{H}{4}} \left( \pm \frac{1}{\sqrt{\frac{1}{2} \left[ \frac{H + 1}{2} \right]_{|\tilde{q}|^2} + \sqrt{\pm I}} \right) \quad (6.4 \pm)$$

and

$$\tilde{Y}^2 = i(-1)^{-H} Y^2 . \quad (6.5)$$

From calculations using these equations and with the help of some identities for $q$ numbers given in [19] we obtain

$$[\pi(\tilde{H}), \pi(\tilde{X}^\pm)] = \pm \pi(\tilde{X}^\pm) \quad (6.6 \pm)$$

$$\{\pi(\tilde{X}^+), \pi(\tilde{X}^-)\} = (-1)^{H} [\pi(\tilde{H})]_{|\tilde{q}|^2} . \quad (6.7)$$

In order for $\pi$ to define a homomorphism we clearly must have

$$(-1)^{H} = I . \quad (6.8)$$

In other words, we must consider a quotient algebra for which this additional condition, along with the defining relations (5.1a) and (5.2b) of $U_q(\mathfrak{osp}(1|2))$ hold true. We have thus recovered, for the special case of $n = 1$, the main criterion established in [7] in order that $U_q(\mathfrak{osp}(1|2n))$ be related to $U_q(\mathfrak{so}(2n + 1))$. The homomorphism defined by Eqs. (6.3) and (6.4±) permits us to relate representations of the two algebras, $U_q(\mathfrak{osp}(1|2n))$ and $U_q(\mathfrak{so}(2n + 1))$, in which representations the actions of $\pi(\tilde{X}^\pm)$ and $\pi(\tilde{H})$ are well-defined. We have already mentioned in the introduction to this section a class of representations for which this is the case. In addition, Eqs. (6.4±) are also applicable for studying the relationship between highest weight representations of $U_q(\mathfrak{osp}(1|2))$ and $U_q(\mathfrak{so}(3))$ even though the actions of $\left\{ \pm \frac{1}{\sqrt{\frac{1}{2} \left[ \frac{H + 1}{2} \right]_{|\tilde{q}|^2} + \sqrt{\pm I}} \right\}^{-1}$ and $\left\{ \pm \frac{1}{\sqrt{\frac{1}{2} \left[ \frac{H + 1}{2} \right]_{|\tilde{q}|^2} + \sqrt{\pm I}} \right\}^{-1}$ in such representations will not be well-defined as
operators on the representation spaces (cf. [18]); nevertheless, the operators representing their ratios will still be well-defined for appropriate choices of $q$ and $\tilde{q}$. We leave it to the reader to work out the straightforward details, in order to convince himself of this.

Notice that we did not need to impose any relationship between $q$ and $\tilde{q}$ in order to establish the above homomorphism. In the general case ($n$ arbitrary) studied in [7], it turns out that $q^2 = -\tilde{q}^2$ (in notation adopted in sections 4 and 5). The reason for this can be understood, at least partially, by comparing the $q$-Serre relations, Eqs. (2.1$c'$) and (2.2$c'$), which for the $n = 1$ case are vacuous.

Finally, it would be interesting to try to generalize our results to arbitrary $n$ by combining them with a variant of the mapping from $U_q(\mathfrak{so}(2n + 1))$ to $U_q(\mathfrak{osp}(1|2n))$ which is given in [7]. A variant of it is necessary in order to account for the above noted difference in the definitions given in [7] with the standard definitions of the $q$ deformed Drinfeld-Jimbo enveloping algebras. This might provide us with even more insight into understanding the connection between $U_q(\mathfrak{so}(2n + 1))$ and $U_q(\mathfrak{osp}(1|2n))$ for arbitrary $n$.

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