The range of once-reinforced random walk in one dimension

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Funding information
This research was supported by the Open access funding enabled and organized by ProjektDEAL.

Abstract
We study once-reinforced random walk on \( \mathbb{Z} \). For this model, we derive limit results on all moments of its range using Tauberian theory.

KEYWORDS
range process, reinforced random walk, scaling limit

1 | INTRODUCTION

Interacting random walks are a class of (mostly non-Markovian) models where the next step of (mostly simple) random walk depends on its path. Some models tend to visit new sites with high probability. Such models arose in chemical physics as a model for long polymer chains, and are discussed in some detail in Chapter 6 of [11]. A particularly prominent example is the self-avoiding walk, which visits every site not more than once. Other models are the myopic (or true) self-avoiding walk, which has higher chances to move to sites it visited less than others. Another class of random walk models—usually referred to as reinforced random walks—visit sites (or walk along edges) more likely they have already seen; see for example, [4, 13].

Here, we study a model which appeared as the hungry random walk in the Physics literature [15] for mimicking chemotaxis. For the model in \( d = 1 \), every site in \( \mathbb{Z} \) contains food, which is eaten up once the walker visits the site. In addition, the walker rather visits sites with food, if it sees a neighboring site containing food (with probability proportional to \( e^{\gamma} \) for some \( \gamma \in \mathbb{R} \)) than going the other direction (with probability proportional to 1). Two models from the probability literature are equivalent to the hungry random walk on \( \mathbb{Z} \): First, the true self-avoiding walk with generalized bond-repulsion, as studied in [17], jumps along an edge \( b \) with probabilities proportional to \( \exp(-\gamma \cdot \ldots

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(number of previous jumps along $b^k$), which equals the hungry random walk in the limit $\kappa \to 0$. Second, the case $\gamma < 0$ is equivalent to once-reinforced random walk or ORRW: here, every edge has initial weight 1, and once the walker goes along an edge, the weight changes to $c > 1$. The walker then chooses its next step according to the edge weights. Apparently, this is the same as the hungry random walker for $\gamma = -\log c < 0$ on $\mathbb{Z}$. Recent literature on the ORRW has focused on recurrence and transience on various graphs (see e.g., [3, 6, 9, 16]). Here, we rather stick to $\mathbb{Z}$ but aim at concrete formulas for the asymptotics of the range of ORRW in Theorem 1. Our analysis is based on a simple decomposition of the inverse of the range process as given in (3). Notably, we cannot compute moments of ORRW itself. At least, we give some heuristics of the variance in Remark 2.5.

In studying the ORRW, we will not restrict ourselves to $c > 1$, but to $c > 0$. A scaling limit of the ORRW in this case was studied in [2, 5, 14]. More precisely, it was shown (see Theorem 1.2 in [5]) that (choose $\alpha = -\beta = 1 - c$) for $1/2 < c < 3/2$, the sequence $X^n = (X_{nt}/\sqrt{n})_{t \geq 0}$ has a limit $Y$ as $n \to \infty$ which solves

$$Y_t = B_t + (1 - c)(\sup\{Y_s : s \leq t\} + \inf\{Y_s : s \leq t\}).$$

(1)

More connections of our results to this equation are discussed in Remark 2.4.

The paper is organized as follows: in the next section, we give our main result, Theorem 1, which gives asymptotics of all moments of the range of the ORRW. Section 3.1 contains some preliminary steps for our proofs. The proof of Theorem 1 is given in Section 3.2.

2 | RESULTS

**Definition 2.1.** Let $c > 0$ and $X = (X_n)_{n=0,1,2,\ldots}$ be the stochastic process with $X_0 = 0$, and for $n = 0, 1, 2, \ldots$, given $X_0, \ldots, X_n$, and setting $M_n := \max_{k \leq n} X_k$ as well as $m_n := \min_{k \leq n} X_k$,

$$X_{n+1} = X_n + 1 \text{ with probability } \begin{cases} \frac{1}{1+c} & \text{if } X_n = M_n, \\ \frac{1}{2} & \text{if } X_n \neq m_n, M_n, \\ \frac{c}{1+c} & \text{if } X_n = m_n \end{cases}$$

and $\mathbb{P}(X_{n+1} = X_n - 1) = 1 - \mathbb{P}(X_{n+1} = X_n + 1)$.

In other words, $X_{n+1} = X_n \pm 1$ with probability proportional to $c$ or 1, if $X$ has or has not visited $X_n \pm 1$ before time $n$. We call $X = (X_n)_{n=0,1,2,\ldots}$ the ORRW on $\mathbb{Z}$ with parameter $c$. Its range by time $n$ is given by

$$R_n := M_n - m_n = M_n + |m_n|.$$  

(2)

Note that only the case $c > 1$ gives a reinforced walk (in the sense that it visits previously seen sites more likely), while the walk has self-avoiding properties for $0 < c < 1$. For $c = 1$, it is just the symmetric Bernoulli walk. Since our proofs work in all cases, we do not distinguish them in the sequel.

The range process $R = (R_n)_{n=0,1,\ldots}$ is a nondecreasing process with jumps of size 1, and is our main object of study. The following ideas are essential to understand our approach. The random time $S_k := \inf\{n : R_n = k\}$ is the first time the ORRW has range $k$ (such that $k \mapsto S_k$ is the generalized inverse of $n \mapsto R_n$), and $T_i := S_{i+1} - S_i$ is the time between $R_n = i$ for the first time and $R_n = i+1$. In order to study
T_i (for i = 1, 2, ...), we note that T_i = 1 with probability 1/(1 + c). Otherwise, the random walk moves within its range (which is i at that time) until it first hits its maximum or minimum, which takes time τ_i, the hitting time of \{-1, i - 1\} of a simple random walk starting in 0. Again, the chance to increase the range is 1/(1 + c), and so on. The number of times the random walk needs a chance of 1/(1 + c) to increase its range is a geometrically distributed random variable Y_i with parameter 1/(1 + c). (Note that Y_i = 0 is possible, that is, we must use the shifted geometrical distribution.) In total, this gives

\[ S_k = 1 + \sum_{i=1}^{k-1} T_i, \quad k = 1, 2, \ldots \] with \( T_i = 1 \) with probability \( \frac{1}{1 + c} \). Otherwise, the random walk moves within its range (which is \( i \) at that time) until it first hits its maximum or minimum, which takes time \( \tau_i \), the hitting time of \( \{-1, i - 1\} \) of a simple random walk starting in 0. Again, the chance to increase the range is \( \frac{1}{1 + c} \), and so on. The number of times the random walk needs a chance of \( \frac{1}{1 + c} \) to increase its range is a geometrically distributed random variable \( Y_i \) with parameter \( \frac{1}{1 + c} \). (Note that \( Y_i = 0 \) is possible, that is, we must use the shifted geometrical distribution.) In total, this gives

\[ S_k = 1 + \sum_{i=1}^{k-1} T_i, \quad k = 1, 2, \ldots \] with

\[ T_i = 1 + \sum_{j=1}^{Y_i} \left( 1 + \tau_j \right), \quad i = 1, 2, \ldots \]

(where we define the empty sum to be 0). Here, \( \tau_i^k, k = 1, 2, \ldots \) are independent and identically distributed as \( \tau_i \) above and also independent from \( Y_i, i = 1, 2, \ldots \). Using (3), we can compute the generating function of \( S_k \) (see Lemma 3.2) and then use \( \mathbb{P}(R_n > k) = \mathbb{P}(S_k < n) \) in order to obtain results on \( R_n \) (see Lemma 3.3 and Proposition 3.4).

We are now ready to formulate our main result, which will be proved in Section 3.2. Throughout, we will write \( a_n \sim b_n \) if \( \frac{a_n}{b_n} \to 1 \) as \( n \to \infty \).

**Theorem 1** (Asymptotic moments of the range). Let \( R_n \) be as in (2) the range of the ORRW with parameter \( c > 0 \). Then,

\[ \mathbb{E}\left[ \left( \frac{R_n}{\sqrt{n}} \right)^\ell \right] \sim \frac{1}{2^{(\ell - 2)/2} \Gamma(\ell/2)} \cdot J_\ell(c), \quad \ell = 1, 2, \ldots \]

with

\[ J_\ell(c) := 2^{\ell c} \int_0^\infty x^{\ell - 1} \left( \frac{e^x}{(e^x + 1)^2} \right)^c \, dx. \]

In particular,

\[ \mathbb{E}\left[ \frac{R_n}{\sqrt{n}} \right] \sim \sqrt{\frac{2}{\pi}} \cdot J_1(c) \quad \text{and} \quad \mathbb{E}\left[ \frac{R_n^2}{n} \right] \sim J_2(c). \]

**Remark 2.2** (The range for \( c = 1 \)). The ORRW with \( c = 1 \) equals the symmetric Bernoulli walk. In this case, several results have been obtained for the range. An early example is [7], who states in his (1.4) that

\[ \mathbb{E}\left[ \frac{R_n}{\sqrt{n}} \right] \sim \sqrt{\frac{8}{\pi}}, \quad \mathbb{E}\left[ \left( \frac{R_n}{\sqrt{n}} \right)^2 \right] \sim 4 \log 2. \]

In this case, we compute

\[ J_1(1) = 4 \int_0^\infty \frac{e^x}{(e^x + 1)^2} \, dx = - \frac{4}{e^x + 1} \bigg|_0^\infty = 2 \]

and Feller’s result for the expectation follows from (4). Moreover, using integration by parts,

\[ J_2(1) = 4 \int_0^\infty x \frac{e^x}{(e^x + 1)^2} \, dx = - \frac{x}{e^x + 1} \bigg|_0^\infty + 4 \int_0^\infty \frac{1}{e^x + 1} \, dx \]

\[ = 4 \int_1^\infty \frac{1}{y - 1} \, dy = 4 \int_1^2 \frac{1}{y} \, dy = 4 \log 2, \]

which gives Feller’s result for the second moment.
In addition to these limiting results, [18, 19] have computed the generating function as well as expectation and variance for $S_k$, given through

$$
\mathbb{E}[S_k] = \binom{k + 1}{2}, \quad \forall [S_k] = \frac{(k - 1)k(k + 1)(k + 2)}{12}.
$$

These results can as well be obtained as follows: Modifying (3) for the case $c = 1$, we can write

$$
S_k = \sum_{i=0}^{k-1} \tau_{i+2},
$$

where $\tau_{i+2}$ is the hitting time of $\{-1, i + 1\}$ of a random walk starting in 0. This holds since the range increases if and only if such a hitting time is observed. We note that $\tau_{i+2}$ is the duration of play of a symmetric Gambler’s ruin starting with 1 and a total of $i + 2$ units. It is a classical result that $\mathbb{E}[\tau_{i+2}] = i + 1$, and $\forall [\tau_{i+2}] = \frac{(i+2)(i+1)i}{3}$ was, for example, derived in [1]. Summing then gives Vallois’ results.

**Remark 2.3 ($J_\ell(c)$ for integer-valued $c$).** If $c$ is an integer, that is, $c = 1, 2, \ldots$, the calculations from (5) and (6) can be generalized and lead to specific expressions for $J_\ell(c)$. We just give the necessary steps for $\ell = 1, 2$, which can then be generalized for larger $\ell$. First, note that a straight-forward calculation shows that, for $c = 1, 2, \ldots$

$$
\frac{d}{dx} \sum_{j=0}^{c-1} \binom{c-1}{j} (-1)^{c-1-j} \frac{1}{j-2c+1} (e^x + 1)^{-j+1} = \left( \frac{e^x}{(e^x + 1)^2} \right)^c.
$$

Hence,

$$
J_1(c) = \sum_{j=0}^{c-1} \binom{c-1}{j} \frac{1}{j-2c+1} 2^{j+1} (-1)^{c-j}.
$$

Next,

$$
\frac{d}{dx} \left( x - \log(e^x + 1) + \sum_{i=1}^{j-1} \frac{1}{i(e^x + 1)^i} \right) = 1 - \frac{e^x}{e^x + 1} \sum_{i=0}^{j-1} \frac{1}{(e^x + 1)^i} = \frac{1}{(e^x + 1)^j},
$$

and therefore, using integration by parts,

$$
J_2(c) = 2^{2c} \int_0^\infty x \left( \frac{e^x}{(e^x + 1)^2} \right)^c dx
$$

$$
= 2^{2c} \sum_{j=0}^{c-1} \binom{c-1}{j} (-1)^{c-j} \frac{1}{j-2c+1} \int_0^\infty \frac{1}{(e^x + 1)^{2c-j-1}} dx
$$

$$
= 2^{2c} \sum_{j=0}^{c-1} \binom{c-1}{j} (-1)^{c-j} \frac{1}{j-2c+1} \left( x - \log(e^x + 1) + \sum_{i=1}^{2c-j-2} \frac{1}{i(e^x + 1)^i} \right) \bigg|_{x=0}^{\infty}
$$

$$
= 2^{2c} \sum_{j=0}^{c-1} \binom{c-1}{j} (-1)^{c-j-1} \frac{1}{2c-j-1} \left( \log(2) - \sum_{i=1}^{2c-j-2} \frac{1}{i(2i)} \right).
$$

For higher $\ell$, more steps using integration by parts are necessary.
Remark 2.4 (Scaling limit of ORRW). Theorem 1 can be understood as results for the scaling limit

\[ X_n / \sqrt{n} \xrightarrow{n \to \infty} Y = (Y_t)_{t \geq 0}, \]

where \( Y \) is given in (1); see [5] for the corresponding limit result. By this, we mean that the range \( \hat{R} \) of \( Y \) satisfies

\[ \mathbb{E}[\hat{R}_t^\ell] = \lim_{n \to \infty} \mathbb{E}[(R_n / \sqrt{n})^\ell] = \frac{1}{2(\ell - 2)/2\Gamma(\ell/2)} \cdot J_\ell(c) \cdot t^{\ell/2}. \]

While the convergence above was only shown for \( 1/2 < c < 3/2 \) in [5], we briefly argue how this converges comes about: Note that \( Y \) solves (1) iff

\[ Y_t - (1 - c)(\sup\{ Y_s : s \leq t\} + |\inf\{ Y_s : s \leq t\}|) \]

is a Brownian motion. (7)

For the ORRW, note that

\[ N_n := X_n - \frac{1 - c}{1 + c} \left( \sum_{k < n} 1_{X_k = M_k} - \sum_{k < n} 1_{X_k = m_k} \right) \]

is a martingale, since (wp = with probability)

\[ N^1_{n+1} - N^1_n = \begin{cases} 
+1 & \text{wp } \frac{1}{2} \text{ if } X_n \neq m_n, M_n, \\
1 - \frac{1 - c}{1 + c} = \frac{2}{1 + c} & \text{wp } \frac{c}{1 + c} \text{ if } X_n = M_n, \\
-1 - \frac{1 - c}{1 + c} = \frac{-2}{1 + c} & \text{wp } \frac{1}{1 + c} \text{ if } X_n = M_n, \\
+1 + \frac{1 - c}{1 + c} = \frac{2}{1 + c} & \text{wp } \frac{c}{1 + c} \text{ if } X_n = m_n, \\
-1 + \frac{1 - c}{1 + c} = \frac{-2c}{1 + c} & \text{wp } \frac{1}{1 + c} \text{ if } X_n = m_n.
\]

For large \( n \), we have that \( \frac{1}{1+c} \sum_{k < nt} 1_{X_k = M_k} \sim M_{nt} \) by the law of large numbers, since every time with \( X_k = M_k \) there is an independent chance of \( 1/(1 + c) \) of increasing \( M \). Moreover, a straight-forward calculation gives that \( N \) has quadratic variation

\[ \langle N \rangle_{nt} = nt - \left( \frac{1 - c}{1 + c} \right)^2 \left( \sum_{k < nt} 1_{X_k = M_k} + \sum_{k < nt} 1_{X_k = m_k} \right) \sim nt - \frac{(1 - c)^2}{1 + c} R_{nt}, \]

where the \( \sim \) follows from \( \frac{1}{1+c} \left( \sum_{k < nt} 1_{X_k = M_k} + \sum_{k < nt} 1_{X_k = m_k} \right) \sim R_n \) by the same argument using the law of large numbers as above. Since \( \mathbb{E}[R_{nt}] = O(\sqrt{nt}) \), as we have shown in Theorem 1, we thus have that the limit of \( \frac{N_n}{\sqrt{n}} \) as \( n \to \infty \) is the same as the limit of

\[ \left( \frac{X_{nt}}{\sqrt{n}} - (1 - c) \left( \frac{M_{nt}}{\sqrt{n}} + \frac{|m_{nt}|}{\sqrt{n}} \right) \right)_{t \geq 0}, \]

which must be a continuous martingale with quadratic variation \( t \) by time \( t \), that is, a Brownian motion. This is enough to conclude that scaling limits of \( X \) satisfy (7).
Remark 2.5 (Towards \( \mathbb{V}[X_n/\sqrt{n}] \)). Although we are able to asymptotically compute all moments of \( R_n/\sqrt{n} \) as in Theorem 1, we are unable to compute asymptotics of (even) moments of \( X_n/\sqrt{n} \). At least, we now give some thoughts and bounds of asymptotics of \( R_n \). We observe that

\[
N_n := X_n^2 - n - 2 \frac{1 - c}{1 + c} \sum_{k < n} |X_k| 1_{X_k \in (m_k, M_k)},
\]

is a mean-zero-martingale, since (note that \( m_k \leq 0 \leq M_k \))

\[
N_{n+1}^1 - N_n^1 = \begin{cases} 
\pm 2X_n & \text{wp } \frac{1}{2} \text{ if } X_n \neq m_n, M_n; \\
2X_n - 2 \frac{1-c}{1+c} X_n = 4X_n \frac{c}{1+c} & \text{wp } \frac{1}{c+1} \text{ if } X_n = M_n; \\
-2X_n - 2 \frac{1-c}{1+c} X_n = -4X_n \frac{1}{1+c} & \text{wp } \frac{c}{1+c} \text{ if } X_n = M_n; \\
4X_n \frac{1}{1+c} & \text{wp } \frac{c}{1+c} \text{ if } X_n = m_n; \\
-4X_n \frac{c}{1+c} & \text{wp } \frac{1}{1+c} \text{ if } X_n = m_n.
\end{cases}
\]

Now, for large \( n \), we have \( M_n = (1 + c) \sum k < n 1_{X_k = M_k} + o(\sqrt{n}) \) by the law of large numbers, hence

\[
2 \sum_{k < n} \mathbb{E}[M_k 1_{X_k = M_k}] = 2(1 + c) \cdot \mathbb{E} \left[ \sum_{\ell < k < n} 1_{X_\ell = M_\ell} 1_{X_k = M_k} \right] + o(\sqrt{n})
\]

\[
= (1 + c) \cdot \mathbb{E} \left[ \left( \sum_{k < n} 1_{X_k = M_k} \right)^2 \right] + O(\sqrt{n}) = \frac{1}{1 + c} M_n^2 + O(\sqrt{n}),
\]

and by symmetry

\[
\mathbb{E}[X_n^2] = n + (1 - c) \mathbb{E}[M_n^2 + m_n^2] + O(\sqrt{n}),
\]

which gives the intuitive result that \( \mathbb{E}[X_n^2] \leq n \) for \( c > 1 \) and \( \mathbb{E}[X_n^2] \geq n \) for \( c < 1 \). Moreover, since \( 0 \leq (M_n - |m_n|)^2 \) implies \( 2M_n|m_n| \leq M_n^2 + |m_n|^2 \),

\[
\frac{R_n^2}{2} = \frac{M_n^2 + 2M_n|m_n| + |m_n|^2}{2} \leq M_n^2 + |m_n|^2 \leq R_n^2,
\]

it also gives the bounds

\[
\left| \frac{1 - c}{2} J_2(c) \right| \lesssim \lim_{n \to \infty} \mathbb{E} \left[ \left( \frac{X_n}{\sqrt{n}} \right)^2 \right] - 1 \lesssim |(1 - c)J_2(c)|. \tag{*}
\]

From Figure 1, we see that the left-hand side (LHS) performs better for \( c > 1 \), while the right-hand side (RHS) is better for \( c < 1 \). The reason is that for \( c > 1 \), the process is more likely to switch between its maximum and minimum, such that \( M_n \approx |m_n| \approx R_n/2 \), leading to \( M_n^2 + |m_n|^2 \approx R_n^2/2 \), while for \( c < 1 \) switching becomes less likely and we rather have that \( R_n \approx M_n \) or \( R_n \approx |m_n| \), which gives \( M_n^2 + |m_n|^2 \approx R_n^2 \).
Simulating the ORRW for varying $c$, we observe $10^5$ independent draws of $X_n/\sqrt{n}$ for $n = 10^5$, and compute the observed variance. We compare this to the two bounds in (*)

3 | PROOF OF THEOREM 1

3.1 | Some preliminaries

Before we come to the proof of Theorem 1, we need some general results. First, in Theorem 2, we recall a classical Tauberian result by Hardy and Littlewood, which will help us to interpret the generating function of $S_k$ from (3). Then, in Lemma 3.1, we recall the generating function of hitting times for a simple symmetric random walk.

**Theorem 2** (A Tauberian result). Let $a_1, a_2, \ldots \geq 0$ such that $\sum_{n=1}^{\infty} a_n x^n$ converges for $|x| < 1$. Suppose that for some $\alpha, A \geq 0$

$$\sum_{n=1}^{\infty} a_n x^n \overset{\sim}{\sim} \frac{A}{(1-x)^{\alpha}}.$$

Then,

$$\sum_{k=1}^{n} a_k \overset{n \to \infty}{\sim} \frac{A}{\Gamma(\alpha + 1)} n^{\alpha}.$$

Moreover, if $\alpha > 1$, and $n \mapsto a_n$ is nondecreasing,

$$a_n \overset{n \to \infty}{\sim} \frac{A \alpha}{\Gamma(\alpha + 1)} n^{\alpha-1}.$$  \hfill (8)

**Proof.** The assertions are classical Tauberian results by Hardy and Littlewood; see for example, Chapter I.7.4 of [10]. Another self-contained proof is given in Proposition 12.5.2 in [12].

The following lemma is rather standard (see e.g., Chapter XIV.4 in [8]), but we provide a proof for completeness.

**Lemma 3.1** (Generating function of hitting times in simple symmetric random walk). Let $Z$ be a simple symmetric random walk with $Z_0 = 0$ and $a, b > 0$. Define $T_k := \inf\{n : Z_n = k\}$, the first
hitting time of \( k \in \mathbb{Z} \), as well as \( T := T_{-a} \wedge T_b \) for \( a, b \geq 0 \). Then,

\[
\mathbb{E}[s^T] = \frac{e^{d_{a,0}} + e^{d_{b,0}}}{1 + e^{d_{(a+b),0}}},
\]

where

\[
d_s := \log \left( \frac{1}{s} \left( 1 + \sqrt{1 - s^2} \right) \right).
\]

(9)

Proof. Recall \( \cosh(r) := \frac{e^{r} + e^{-r}}{2} \) and note that \( \cosh(d_s) = 1/s \). For any \( r \in \mathbb{R} \), using that

\[
\mathbb{E}[e^{rZ_n}] = \mathbb{E}[e^{-rZ_n}] = \prod_{k=1}^n \frac{e^r + e^{-r}}{2} = \cosh(r)^n,
\]

the stochastic process \( (e^{rZ_n}/\mathbb{E}[e^{rZ_n}])_{n=0,1,2,...} \) is a martingale. Therefore, using \( r = \pm d_s \),

\[
M_n := e^{d_{a,0}} \frac{e^{d_{Z_n}}}{\mathbb{E}[e^{d, Z_n}]} + e^{d_{b,0}} \frac{e^{-d, Z_n}}{\mathbb{E}[e^{-d, Z_n}]} = \frac{e^{d_{a,(a+Z_n)} + e^{d,(b-Z_n)}}}{\cosh(d_s)^n}
\]

is a martingale as well. We apply the optional sampling theorem to the bounded martingale \( M_{T\wedge n} \) to obtain

\[
e^{d_{a,0}} + e^{d_{b,0}} = \mathbb{E}[M_0] = \mathbb{E}[M_T] = \mathbb{E}[s^T \left( 1 + e^{d_{(b+a)}} \right)].
\]

From this, we read off the result. \( \blacksquare \)

3.2  |  Proof of Theorem 1

Our analysis of the moments of \( R_n \) will be done via an analysis of the generating function of the random variable \( S_k \), which we defined in (3). We start by computing the generating function of \( S_k \).

Lemma 3.2 (Generating function of \( \tau_i, T_i, \) and \( S_k \)). Fix \( s \in (0, 1) \), recall \( d_s \) from (9) and let

\[
g_s(s) := \frac{e^{d_s} + e^{d_s(1-s)}}{1 + e^{d_s}}, \quad G_s(s) := \frac{s}{1 + c - cgs_s(s)}.
\]

(10)

Then, for \( \tau_i \) and \( T_i \) as in (3), the generating functions are given by

\[
\mathbb{E}[s^{\tau_i}] = g_i(s), \quad \mathbb{E}[s^{T_i}] = G_i(s), \quad i = 2, 3, \ldots.
\]

Moreover, the generating function of \( S_k \) is given by

\[
\mathbb{E}[s^{S_k}] = s \prod_{i=1}^{k-1} G_i(s), \quad k = 1, 2, \ldots
\]

(11)
Proof. The first claim follows directly from Lemma 3.1. For the generating function of $T_i$, note that the generating function of $Y_i \sim \text{geo}(1/(1+c))$ is

$$s \mapsto \frac{1}{c+1} \sum_{k=0}^{\infty} \left( \frac{cs}{c+1} \right)^k = \frac{1}{1+c} \cdot \frac{c+1}{c+1 - cs} = \frac{1}{1+c - cs},$$

hence

$$E[s^{Ti}] = sE[E[s^{\prod_{j=1}^{\infty} Yi}]] = E[(s_{gi}(s))^{Y_i}] = \frac{s}{1+c - cs_{gi}(s)}.$$

The form of the generating function of $S_k$ follows from (3).

Lemma 3.3 (A generating function for moments of $R_n$). Recall $G_x(s)$ from (10). Then, for $s \in (0, 1)$ and $\ell = 0, 1, 2, \ldots$

$$H_{\ell}(s) := \sum_{n=1}^{\infty} s^n \mathbb{E}[R_{n \cdots (R_n + \ell)}] = \frac{\ell + 1}{1-s} \sum_{k=1}^{\infty} k \cdots (k + \ell - 1)s \prod_{i=1}^{k-1} G_i(s).$$

Proof. We will use, for $k, n = 1, 2, \ldots$

$$\mathbb{P}(R_n = k) = \mathbb{P}(R_n < k + 1) - \mathbb{P}(R_n < k) = \mathbb{P}(S_{k+1} > n) - \mathbb{P}(S_k > n) = \mathbb{P}(S_k \leq n) - \mathbb{P}(S_{k+1} \leq n).$$

Then,

$$H_{\ell}(s) = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} s^n k \cdots (k + \ell)(\mathbb{P}(S_k \leq n) - \mathbb{P}(S_{k+1} \leq n))$$

$$= \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} s^n (k \cdots (k + \ell) - (k - 1) \cdots (k + \ell - 1))\mathbb{P}(S_k \leq n)$$

$$= (\ell + 1) \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \sum_{i=1}^{n} s^n k \cdots (k + \ell - 1)\mathbb{P}(S_k = i)$$

$$= (\ell + 1) \sum_{k=1}^{\infty} k \cdots (k + \ell - 1) \sum_{i=1}^{\infty} \mathbb{P}(S_k = i) \sum_{n=i}^{\infty} s^n$$

$$= \frac{\ell + 1}{1-s} \sum_{k=1}^{\infty} k \cdots (k + \ell - 1) \sum_{i=1}^{\infty} \mathbb{P}(S_k = i) s^i$$

and the result follows from (11).

Proposition 3.4 (Asymptotics of $H_{\ell}$). Let $\ell = 0, 1, 2, \ldots$ and $H_{\ell}$ as in Lemma 3.3. Then,

$$H_{\ell}(s) \overset{\text{as}}{\sim} K_{\ell}(1-s)^{-(\ell+\ell)/2},$$

where

$$K_{\ell} = \frac{\ell + 1}{2^{(\ell+1)/2}} \sum_{x=0}^{\infty} x^{\ell} \frac{e^x}{(e^x + 1)^{\ell+1}} dx.$$
Proof. We define

\[ a : x \mapsto \frac{e^x - 1}{e^x + 1} \]

and write for \( g_x(s) \) as in (10), using the definition of \( d_s \),

\[
g_x(s) = \frac{(1 + \sqrt{1 - s^2})/s + e^{d_s x}/(1 + \sqrt{1 - s^2})}{1 + e^{d_s x}}
\]

\[
= \frac{1}{s} \left( 1 - \frac{e^{d_s x} - \sqrt{1 - s^2} - e^{d_s x} s^2/(1 + \sqrt{1 - s^2})}{1 + e^{d_s x}} \right)
\]

\[
= \frac{1}{s} \left( 1 - \frac{e^{d_s x} 1+\sqrt{1-s^2}-s^2}{1+\sqrt{1-s^2}} - \sqrt{1 - s^2} \right)
\]

\[
= \frac{1}{s} \left( 1 - \frac{\sqrt{1 - s^2} e^{d_s x} - 1}{e^{d_s x} + 1} \right) = \frac{1}{s} \left( 1 - a(d_s x) \sqrt{1 - s^2} \right),
\]

as well as

\[
G_x(s) = \frac{s}{1 + ca(d_s x) \sqrt{1 - s^2}}.
\]

In the sequel, we will use \( t := t(s) := 1 - s \) throughout, set

\[
f_t := d_{1-t} = \text{arcosh} \left( \frac{1}{1-t} \right) = \log \left( \frac{1}{1-t} \left( 1 + \sqrt{t(2-t)} \right) \right)
\]

and note that

\[
f_t = - \log(1-t) + \log \left( 1 + \sqrt{t(2-t)} \right) = t + \sqrt{t(2-t)} - \frac{t}{2}(2-t) + O(t^{3/2})
\]

\[
= \sqrt{2t} + O(t^{3/2})
\]

as \( t \to 0 \). Therefore,

\[
g_x(1-t) = \frac{1}{1-t} \left( 1 - a(f_t x) \sqrt{t(2-t)} \right)
\]

\[
= (1 + O(t)) \left( 1 - a(\sqrt{2tx}) \sqrt{2t(1 + O(t))} \right)
\]

\[
= 1 - a(\sqrt{2tx}) \sqrt{2t} + O(t)
\]

and

\[
G_x(1-t) = \frac{1-t}{1 + ca(f_t x) \sqrt{t(2-t)}} = (1-t)(1 - ca(\sqrt{2tx}) \sqrt{2t(1 + O(t))} + O(t))
\]

\[
= 1 - ca(\sqrt{2tx}) \sqrt{2t} + O(t).
\]
The latter can now be used for (note that the empty product arising for $\ell = 0$ is defined to be 1)

$$H_\ell(1-t)^{(3+\ell)/2} = \frac{\ell + 1}{2^{(\ell+1)/2}} \sum_{k=1}^{\infty} \sqrt{2t} \left( \prod_{j=0}^{\ell-1} \sqrt{2t(k+j)} \right) (1-t) \exp \left( \sum_{i=1}^{k-1} \log G_i(1-t) \right)$$

$$= \frac{\ell + 1}{2^{(\ell+1)/2}} \sum_{k=1}^{\infty} \sqrt{2t} \left( \prod_{j=0}^{\ell-1} \sqrt{2t(k+j)} \right) \exp \left( - \sum_{i=1}^{k-1} ca(\sqrt{2ti}) \sqrt{2t} + O(t) \right)$$

$$= \frac{\ell + 1}{2^{(\ell+1)/2}} \int_0^{\infty} x^\ell \exp \left( - \int_0^x ca(y)dy \right) dx \cdot (1+o(1)),$$

where we have used two approximations of integrals with Riemann-sums (with $\sqrt{2t} \approx dx, dy$). The result follows from

$$\int_0^x a(y)dy = \int_1^e \frac{z-1}{z(z+1)}dz = \int_1^e \frac{2}{z+1} - \frac{1}{z}dz = 2 \log(e^x + 1) - 2 \log 2 - x$$

and

$$\exp \left( - c(2 \log(e^x + 1) - 2 \log 2 - x) \right) = 2^c \left( \frac{e^x}{(e^x + 1)^2} \right)^c.$$  

Proof of Theorem 1. We now combine the results of Proposition 3.4 with the Tauberian result from Theorem 2. We obtained in Proposition 3.4 for $a_n = \mathbb{E}[R_n \cdots (R_n + \ell)]$ that

$$H_\ell(s) = \sum_{n=1}^{\infty} a_n s^{(\ell+1)/2} \sim \frac{K_\ell}{(1-s)^{(3+\ell)/2}}$$

and we can apply Theorem 2 with $A = K_\ell$ and $\alpha = (3 + \ell)/2$. In particular, (8) gives that

$$\mathbb{E}[R_n \cdots (R_n + \ell)] \sim \frac{K_\ell(3 + \ell)}{2 \Gamma((5 + \ell)/2)} n^{(1+\ell)/2}.$$

Since $\Gamma(x+1) = x\Gamma(x)$, this implies

$$\mathbb{E}[R_n^{\ell+1}] \sim \frac{1}{2^{(\ell-1)/2} \Gamma((\ell + 1)/2)} \cdot J_{\ell+1}(c) \cdot n^{(\ell+1)/2}, \quad \ell = 0, 1, 2, \ldots$$

with

$$J_\ell(c) := 2^2 \int_0^{\infty} \left( \frac{e^x}{(e^x + 1)^2} \right)^c dx$$

and we are done.

Acknowledgment

We thank Tanja Schilling for introducing us to the model of the hungry random walk. Open access funding enabled and organized by Projekt DEAL.
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How to cite this article: Pfaffelhuber P, Stiefel J. The range of once-reinforced random walk in one dimension. Random Struct Alg. 2021;58:164–175. https://doi.org/10.1002rsa.20948