IMPROVED MATHEMATICAL RESULTS AND SIMPLIFIED PEDAGOGICAL APPROACHES FOR GRONWALL’S INEQUALITY FOR FRACTIONAL CALCULUS

CHRISTOPHER C. TISDELL

Abstract. Gronwall’s inequality plays an important role in producing new research and in the learning and teaching of differential and integral equations. The purpose of this work is to advance and simplify the current state of knowledge and pedagogical approaches regarding Gronwall’s inequality. In particular: we extend known versions of Gronwall’s inequality for fractional calculus; and we provide simpler and more accessible proofs that can be easily transferred to the classroom. Our work is also timely in the sense that it may be considered as a celebration of the upcoming centenary of the publication of Gronwall’s original results. Thus, we believe this paper is important from mathematical research, pedagogical and historical viewpoints.

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