Excursions away from a regular point for one-dimensional symmetric Lévy processes without Gaussian part

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Abstract

The characteristic measure of excursions away from a regular point is studied for a class of symmetric Lévy processes without Gaussian part. It is proved that the harmonic transform of the killed process enjoys Feller property. The result is applied to prove extremeness of the excursion measure and to prove several sample path behaviors of the excursion and the $h$-path processes.

Keywords: excursion theory; Lévy process; Feller property; extreme points
Mathematics Subject Classifications (2000): 60G51; 60J50; 60G17

1 Introduction

Itô [19] has proved that the point process of excursions away from a regular point for a strong Markov process is Poisson. Its characteristic measure will be simply called the excursion measure. Itô’s theorem shows that, for a given minimal process, there is a one-to-one correspondence between an excursion measure and a strong Markov extension of the minimal process. In the same paper, he established the integral representation formula on the convex set of the normalized excursion measures of strong Markov extensions of the minimal process.

In the present paper, we study several properties of the excursion measure and the $h$-path process for one-dimensional symmetric Lévy processes. Under certain assumptions which imply no Gaussian part, we prove extremeness of the excursion measure and several sample path behaviors of the excursion and the $h$-path processes. Our study is motivated by a recent study of Yano–Yano–Yor [31] about penalization problems for one-dimensional symmetric $\alpha$-stable processes of index $1 < \alpha \leq 2$.

Let $\{(X_t : t \geq 0), (\mathcal{F}_t : t \geq 0), (P_x : x \in \mathbb{R})\}$ denote the canonical representation of the one-dimensional Brownian motion. Let $n^B$ stand for the Brownian excursion measure.

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Then the measure $n^B$ is represented as
\[ n^B = \frac{1}{2}n^+ + \frac{1}{2}n^- \quad (1.1) \]
where $n^+$ (resp. $n^-$) stands for the excursion measure for (resp. the negative of) the reflecting Brownian motion. We remark that the formula (1.1) is a special case of Itô’s integral representation formula ([19] Theorem 7.1); see also [4, Section V.6]): Let $E$ denote the convex cone of the excursion measures for non-trivial strong Markov extensions of the killed process. Let $E_1$ denote the convex subset of $E$ whose elements $\mu$ are normalized in the sense that $\mu[1-e^{-\zeta}] = 1$ where $\zeta$ stands for the lifetime. Then any given element $\mu_0$ of the set $E_1$ is represented as
\[ \mu_0(\cdot) = \int_{\text{ex}(E_1)} \mu(\cdot) \Pi(d\mu) \]
for some probability measure $\Pi$ on the set $\text{ex}(E_1)$ of extreme points of $E_1$. We say that $n \in E$ is an extreme direction if $n$ is proportional to an extreme point of $E_1$, i.e.,
\[ \frac{n(\cdot)}{n[1-e^{-\zeta}]} \in \text{ex}(E_1). \]

Let $\{P^+_x : x \geq 0\}$ (resp. $\{P^-_x : x \leq 0\}$) denote the law of (resp. the negative of) the three-dimensional Bessel process. Let $P^{+_,-}$ denote the law of the symmetrized three-dimensional Bessel process, i.e., $P^{+_,-} = P^+_x$ for $x > 0$, $P^{+_,-} = P^-_x$ for $x < 0$ and $P^{+_,-}_0 = \frac{1}{2}P^+_0 + \frac{1}{2}P^-_0$. The process $\{X_t, P^{+_,-}_x\}$ is the harmonic transform or $h$-path process with respect to the harmonic function $h(x) = |x|$ of the killed Brownian motion $\{P^0_x : x \in \mathbb{R} \setminus \{0\}\}$ in the sense that $dP^{+_,-}_t|_{\mathcal{F}_t} = \frac{|X_t|}{|x|} dP^0_t|_{\mathcal{F}_t}$ for any $x \in \mathbb{R} \setminus \{0\}$. The law $P^{+_,-}_0$ is related to the excursion measure $n^B$ in the following Imhof relation (see, e.g., [23, Exercise XII.4.18]):
\[ dP^{+_,-}_t|_{\mathcal{F}_t} = \frac{|X_t|}{n^B(|X_t|)} dn^B|_{\mathcal{F}_t}, \quad t > 0. \]

Here we remark that the law $\frac{1_{(t>0)}}{n^B(\cdot>0)} |X_t| dn^B|_{\mathcal{F}_t}$ is nothing but the law of the Brownian meander. Moreover, the excursion measure $n^B$ admits the following lifetime disintegration formula (see [18] Section III.4.3 and [23] Theorem XII.4.2):
\[ n^B(\cdot) = \int_0^\infty P^{+_,-}_0(\cdot|X_{t-} = 0) \frac{dt}{\sqrt{2\pi t^3}}. \quad (1.2) \]

In other words, under $n^B$, the law of the lifetime $\zeta$ is $dt/\sqrt{2\pi t^3}$ and the conditional law of the excursion process given $\zeta = t$ is $P^b_0(\cdot|X_{t-} = 0)$.

We point out the following three facts:
(i) The excursion measure $n^B$, being represented as (1.1), is not an extreme direction;
(ii) The germ $\sigma$-field $\mathcal{F}_{0+} = \cap_{s \geq 0} \mathcal{F}_s$ is not trivial under $n^B$; in fact, the set $A = \{\exists t > 0$ such that $\forall s \leq t, X_s \geq 0\}$ belongs to $\mathcal{F}_{0+}$, but neither $A$ nor $A^c$ is $n^B$-null;
The semigroup \(\{T^+_t : t \geq 0\}\) corresponding to the \(h\)-path process does not enjoy Feller property; in fact, for a positive continuous function \(f\) with compact support in \([0, \infty)\), we have

\[
\lim_{x \to 0^+} T^+_t f(x) = P_0^+[f(X_t)] > 0 \quad \text{while} \quad \lim_{x \to 0^-} T^+_t f(x) = 0.
\]

Main theorems

Let us state the main theorems of the present paper. Let \(\{P_x : x \in \mathbb{R}\}\) denote the law of a one-dimensional Lévy process. We assume that the following conditions are satisfied:

(A0) The process is symmetric;
(A1) The origin is regular for itself;
(A2) The process is not a compound Poisson.

The set of these conditions (A0)-(A2) will be denoted simply by (A). Then it holds (see Section 3.1) that the Lévy–Khintchine exponent

\[
\theta(\lambda) = v\lambda^2 + 2\int_{(0,\infty)} (1 - \cos \lambda x) \nu(dx), \quad \lambda \in \mathbb{R}
\]

with the Gaussian coefficient \(v\) and the Lévy measure \(\nu(dx)\) satisfies

\[
\int_0^\infty \min\{\lambda^2, 1\} \frac{1}{\theta(\lambda)} d\lambda < \infty.
\]

Hence there exists a continuous density \(u_q(x)\) of the resolvent kernel given by

\[
u(x) = \frac{1}{\pi} \int_0^\infty \cos \lambda x \frac{1}{q + \theta(\lambda)} d\lambda
\]

and, in addition, the following function is well-defined (see also [26, Lemma 1]):

\[
h(x) = \lim_{q \to 0^+} \left\{u_q(0) - u_q(x)\right\} = \frac{1}{\pi} \int_0^\infty \frac{1 - \cos \lambda x}{\theta(\lambda)} d\lambda.
\]

Remark that the process is recurrent or transient according as

\[
\kappa = \lim_{q \to 0^+} \frac{1}{u_q(0)} = \left\{\frac{1}{\pi} \int_0^\infty \frac{1}{\theta(\lambda)} d\lambda \right\}^{-1}
\]

is zero or positive (see Section 3.2).

Let \(T_{[x]}\) denote the first hitting time of \(x \in \mathbb{R}\). Let \(L(t, x)\) denote the local time process. We denote by \(\{P^0_x : x \in \mathbb{R}\}\) the law of the process killed upon hitting the origin, which we simply call the killed process in short. We denote by \(n\) the excursion measure.

**Theorem 1.1.** Suppose that the condition (A) is satisfied. Then the function \(h(x)\) is invariant excessive with respect to the killed process, i.e.,

\[
P^0_x[h(X_t)] = h(x), \quad t > 0, \ x \in \mathbb{R} \setminus \{0\}.
\]
Remark. The function $h(x)$ above is harmonic in the sense of [7, Definition 4.3.2] where the Laplacian is replaced by the generator.

**Theorem 1.2.** Suppose that the condition (A) is satisfied. Then

$$n[h(X_t)] = 1, \quad t > 0.$$  

Theorems 1.1 and 1.2 will be proved in Section 4.

We introduce the $h$-path process $\{P^h_x : x \in \mathbb{R}\}$ as the law on the canonical space such that, for any $t > 0$,

$$dP^h_x | F_t = h(X_t) dP^0_x | F_t, \quad x \in \mathbb{R} \setminus \{0\}, \quad (1.7)$$

$$dP^h_0 | F_t = h(X_t) dP^0_0 | F_t, \quad x = 0. \quad (1.8)$$

Note that such a family of probability laws on $\mathbb{D}$ exists uniquely, because the identities (1.7) and (1.8) induce a consistent family of probability laws on $\mathbb{D}$ by the Markov properties of the killed process and the excursion process. Let us denote the corresponding semigroup by $\{T^h_t : t \geq 0\}$.

**Theorem 1.3** (Lifetime disintegration formula). Suppose that the condition (A) is satisfied. Then there exists a positive completely monotone function $\rho(t)$ such that

$$n(\cdot) = \int_0^\infty P^h_0(\cdot | X_{t-} = 0) \rho(t) dt + \kappa P^h_0(\cdot). \quad (1.9)$$

In other words,

(i-a) $n(\zeta \in dt) = \rho(t) dt$ on $(0, \infty)$;

(i-b) $n(\cdot | \zeta = t) = P^h_0(\cdot | X_{t-} = 0)$ for $0 < t < \infty$;

(ii-a) $n(\zeta = \infty) = \kappa$;

(ii-b) $n(\cdot | \zeta = \infty) = P^h_0(\cdot)$.

Theorem 1.3 will be proved in Section 5.2. We must remark that Getoor–Sharpe [16, Theorem 7.6] have proved that the excursion measure for quite general Markov processes admits lifetime disintegration formula with a certain bridge process as its conditional distribution. (Note that $P^{0,t,0}$ in [16] corresponds to our $P^h_x(\cdot | X_{t-} = 0)$, $\eta(t, 0, 0)$ to $\rho(t)$, $q^*(t, 0, x)$ to $\rho(t, x)$, and $q(t, x, y)$ to $P^0(x, y)$.) Theorem 1.3 asserts that, in this particular case, the conditional distribution is given by the bridge process of the $h$-path process. In the same way as Theorem 1.3, we may prove that, for any $x \in \mathbb{R} \setminus \{0\},$

$$P^0_x(\cdot) = \int_0^\infty P^h_x(\cdot | X_{t-} = 0) P^0_x(\zeta \in dt) + \kappa h(x) P^h_x(\cdot). \quad (1.10)$$

In particular, we have $P^0_x(\cdot | \zeta = t) = P^h_x(\cdot | X_{t-} = 0)$.

Although it seems superflous, we need the following extra assumption for some technical reason:

**(T)** The function $\theta(\lambda)$ is non-decreasing in $\lambda > \lambda_0$ for some $\lambda_0 > 0.$
The following theorem asserts that the \( h \)-path process is transient.

**Theorem 1.4.** Suppose that the conditions (A) and (T) are satisfied. Then

\[
P_x^h \left( \lim_{t \to \infty} |X_t| = \infty \right) = 1, \quad x \in \mathbb{R}.
\]

Theorem 1.4 will be proved in Section 6.3.

We need the following assumption:

\[(B) \quad \lim_{x \to 0} \frac{x}{h(x)} = 0.\]

From the assumption (B) it follows that the Gaussian coefficient \( v \) is zero (see Lemma 7.1). Now let us state our main theorem.

**Theorem 1.5.** Suppose that the conditions (A), (B) and (T) are satisfied. Then the semigroup \( \{T^h_t : t \geq 0 \} \) enjoys Feller property.

As applications of Theorem 1.5, we obtain

**Corollary 1.1 (Extremeness property).** Suppose that the conditions (A), (B) and (T) are satisfied. Then the excursion measure \( n \) is an extreme direction.

**Corollary 1.2 (Oscillatory entrance property).** Suppose that the conditions (A), (B) and (T) are satisfied. Then the excursion process enters oscillatingly, i.e.,

\[n \left( \{ \exists \{ t_n \} \text{ with } t_n \searrow 0 \text{ such that } \forall n, X_{t_n}X_{t_{n+1}} < 0 \}^c \right) = 0.\]

The following results are concerned about sample path behaviors of the \( h \)-path processes.

**Corollary 1.3 (Oscillatory entrance property).** Suppose that the conditions (A), (B) and (T) are satisfied. Then the \( h \)-path process enters oscillatingly, i.e.,

\[P_0^h \left( \exists \{ t_n \} \text{ with } t_n \searrow 0 \text{ such that } \forall n, X_{t_n}X_{t_{n+1}} < 0 \right) = 1.\]

**Corollary 1.4 (Oscillatory property in the long time).** Suppose that the process \( \{(X_t, (P_x))\} \) is a symmetric stable process of index \( 1 < \alpha < 2 \). Then

\[P_0^h \left( \limsup_{t \to \infty} X_t = \limsup_{t \to \infty} (-X_t) = \infty \right) = 1.\]

Theorem 1.5 and Corollaries 1.1, 1.2, 1.3 and 1.4 will be proved in Section 7. Here we briefly sketch how to prove Corollary 1.1, provided that Theorem 1.5 is proved, as follows:

The semigroup \( \{T^h_t : t \geq 0 \} \) enjoys Feller property

\[\text{Prop. 2.1} \quad \Rightarrow \quad \text{The germ } \sigma \text{-field } F_{0+} \text{ is trivial under } P_0^h \]

\[\text{eq. 1.8} \quad \Leftrightarrow \quad \text{The germ } \sigma \text{-field } F_{0+} \text{ is trivial under } n \]

\[\text{Thm. 2.4} \quad \Leftrightarrow \quad \text{The excursion measure } n \text{ is an extreme direction.}\]
Example 1.1. When it is a stable process, the process satisfies the condition (A) if and only if it is a symmetric $\alpha$-stable process of index $1 < \alpha \leq 2$. Up to multiplicative constant, we have

$$\theta(\lambda) = |\lambda|^{\alpha} \quad \text{for some } 1 < \alpha \leq 2.$$ 

Hence the condition (T) is automatically satisfied. The harmonic function is given by

$$h(x) = C(\alpha)|x|^{\alpha-1}, \quad x \in \mathbb{R}$$

where

$$C(\alpha) = \frac{1}{\pi} \int_{0}^{\infty} \frac{1 - \cos \lambda}{\lambda^\alpha} d\lambda = \begin{cases} \frac{\Gamma(2-\alpha)}{\pi \alpha} \sin \frac{\alpha \pi}{2} & \text{for } 1 < \alpha < 2, \\ \frac{1}{2} & \text{for } \alpha = 2. \end{cases}$$

The density $\rho(t)$ and the constant $\kappa$ in the formula (1.9) are given by

$$\rho(t) = \frac{(\alpha - 1)\pi}{\Gamma(1 - 1/\alpha)\Gamma(1/\alpha)^2} t^{1/\alpha-2} \quad \text{and} \quad \kappa = 0.$$ 

Note that the condition (B) is satisfied if and only if $1 < \alpha < 2$.

The organization of the present paper is as follows. In Section 2, we recall several preliminary facts about one-dimensional Lévy processes in general settings. In Section 3 we recall several preliminary facts assuming that the process is symmetric. In Section 4 we prove harmonicity of $h(x)$ and study its properties. In Section 5 we prove the lifetime disintegration formula for the excursion measure. In Section 6 we prove several lemmas for later use. The transience of the $h$-path process will be proved there. Section 7 is devoted to the proof of Feller property of the semigroup corresponding to the $h$-path process. The extremeness property of the excursion measure and the sample path behaviors of the excursion and the $h$-path processes will be proved in this section.

Remarks

Remark. Contrary to that for diffusion processes, boundary problem for Markov processes with jumps is extremely difficult because of non-locality. We must remark that Chen–Fukushima–Ying [6, Sections 4 and 5] (see also Fukushima–Tanaka [14]) have proved under quite a general assumption that there exists a unique extension of the minimal process which conserves a given weak duality. Thanks to this striking result, we know that, at least in the settings of the present paper, the symmetric extension of the minimal process is unique.

Remark. Based on a kind of Martin boundary argument (see Lemma 6.4), Ikeda–Watanabe [17] (see also Takada [28]) have studied sample path behaviors before hitting the origin for (possibly non-symmetric) one-dimensional Lévy processes. With the help of their results, we can give another proof of a special case of Corollary 1.2 without using Feller property of the $h$-path process (see Section 7.4).
Remark. Several aspects of excursion measures have been extensively studied for Brownian motions (see, e.g., [18] Section III.4.3 and [23] Chap. XII), for diffusion processes (see, e.g., [25]; see also [13]), for reflected Lévy processes (see, e.g., [2] Chap.VI) and for spectrally one-sided Lévy processes (see, e.g., [2, Chap.VII]). In particular, the lifetime disintegration formula for one-dimensional diffusion processes can be found in Pitman–Yor [21, 22] and Yano [29].

For symmetric Lévy processes, however, we cannot find any literature about the excursion measure, except general theories and Fitzsimmons–Getoor [10] (see also Yano–Yano [30]) who have studied the law of the time spent on the positive side by the conditional process \( \{ (X_t : 0 \leq t \leq T), n(\cdot|\zeta = T) \} \). Note that Theorem 1.3 asserts that the conditional law is given by the bridge \( P^h_0(\cdot|X_T = 0) \) of the \( h \)-path process.

Remark. In order to study regularity of the local time \( L(t,x) \), Barlow [1] (see also [2, Section V.3]) has introduced the following function:

\[
h_B(x) = P_x[L(T_{\{0\}},x)] = \lim_{q \to 0^+} \left\{ u_q(0) - \frac{u_q(x)u_q(-x)}{u_q(0)} \right\}.
\]

The function \( h_B \) is related to our \( h \) as follows:

\[
h_B(x) = 2h(x) - \kappa h(x)^2.
\]

In particular, \( h_B = 2h \) in the recurrent case.

Remark. Salminen–Yor [26] have obtained the following Tanaka formula:

\[
h(X_t - x) = h(x) + N_t^x + c_1 L(t,x)
\]

where \( N_t^x \) is a \( P_0 \)-martingale and \( c_1 \) is some constant.

Remark. The \( h \)-path process \( (X_\cdot, P^h_\cdot) \) is considered to be the process \( (X_\cdot, P_\cdot) \) conditioned never to hit the origin. In fact, in the case of symmetric \( \alpha \)-stable process of index \( 1 < \alpha \leq 2 \), Yano–Yano–Yor [31] proved that

\[
\lim_{t \to \infty} P_x[Z_s|T_{\{0\}} > t] = P^h_x[Z_s]
\]

for all non-negative \( \mathcal{F}_s \)-measurable functional \( Z_s \).

2 Preliminary facts: general case

Let \( \mathbb{D} \) denote the set of càdlàg paths \( w : [0, \infty) \to \mathbb{R} \cup \{\Delta\} \) such that \( w(t) = \Delta \) for all \( t \geq \zeta(w) \) where

\[
\zeta(w) = \inf\{t \geq 0 : w(t) = \Delta\}.
\]

Here the topology of \( \mathbb{R} \cup \{\Delta\} \) is the one-point compactification of \( \mathbb{R} \). The point \( \Delta \) is called the cemetery and \( \zeta(w) \) is called the lifetime of a path \( w \in \mathbb{D} \). The space \( \mathbb{D} \) is equipped with Skorokhod topology. Let \( \mathcal{B}_{+b}(\mathbb{R}) \) denote the set of measurable functions which are
non-negative or bounded. For \( f \in B_{+,b}(\mathbb{R}) \), we define \( \| f \| = \sup_{x \in \mathbb{R}} |f(x)| \). Let \( C_0(\mathbb{R}) \) denote the set of continuous functions which vanish at infinity, i.e., \( \lim_{|x| \to \infty} f(x) = 0 \).

Let \( (X_t : t \geq 0) \) denote the coordinate process: \( X_t(w) = w(t) \), \( t \geq 0 \). Let \( (\mathcal{F}_t : t \geq 0) \) denote the natural filtration: \( \mathcal{F}_t = \sigma(X_s : s \leq t) \). Let \( (P_x : x \in \mathbb{R}) \) be the law of a one-dimensional Lévy process on the canonical space \( \mathbb{D} \). Throughout this section, we do not suppose that \( \{ (X_t), (P_x) \} \) is symmetric. The corresponding semigroup and resolvent operator will be denoted by

\[
T_t f(x) = P_x[f(X_t)] \quad \text{and} \quad U_q f(x) = \int_0^\infty e^{-qt} T_t f(x) dt, \quad (2.1)
\]

respectively. It is well-known that

\[
P_0[e^{i\lambda X_t}] = e^{t\psi(\lambda)}, \quad \lambda \in \mathbb{R}
\]

where the Lévy–Khintchine exponent \( \psi(\lambda) \) is given by

\[
\psi(\lambda) = -v\lambda^2 + ia\lambda + \int_{\mathbb{R}} \left( e^{i\lambda x} - 1 - \frac{i\lambda x}{1 + x^2} \right) \nu(dx), \quad \lambda \in \mathbb{R}
\]

for some \( v \geq 0 \), \( a \in \mathbb{R} \) and some positive Radon measure \( \nu \) on \( \mathbb{R} \) such that

\[
\int_{\mathbb{R}} \min\{x^2, 1\} \nu(dx) < \infty.
\]

Set

\[
\theta(\lambda) = -\Re \psi(\lambda) = v\lambda^2 + \int_{\mathbb{R}} (1 - \cos \lambda x) \nu(dx), \quad \lambda \in \mathbb{R}.
\]

### 2.1 Germ triviality

It is obvious that the semigroup \( \{ T_t : t \geq 0 \} \) enjoys the Feller property; in particular,

\[
T_t C_0(\mathbb{R}) \subset C_0(\mathbb{R}), \quad t \geq 0. \quad (2.2)
\]

**Proposition 2.1** (Blumenthal [3]). For any \( x \in \mathbb{R} \), the germ \( \sigma \)-field \( \mathcal{F}_{0+} \) is trivial under \( P_x \).

The following proof says that *Feller property implies germ triviality.*

**Proof.** Let \( A \in \mathcal{F}_{0+}, f \in C_0(\mathbb{R}) \) and \( t, \varepsilon > 0 \). By the Markov property, we have

\[
P_x[1_A f(X_{t+\varepsilon})] = P_x[1_A T_{t} f(X_\varepsilon)].
\]

Now let \( \varepsilon \) tend to 0+. By the continuity of \( T_t f \), by right-continuity of paths and by the dominated convergence theorem, we obtain

\[
P_x[1_A f(X_t)] = P_x(A) T_t f(x) = P_x(A) P_x[f(X_t)].
\]

The rest of the proof is a standard argument. \( \square \)
2.2 The condition for the origin to be regular for itself

Recall the following conditions:

(A1) The origin is regular for itself;

(A2) The process is not a compound Poisson.

Theorem 2.1 (Kesten [20] and Bretagnolle [5]). The conditions (A1)-(A2) are satisfied if and only if

\[ \int_{\mathbb{R}} \text{Re} \left( \frac{1}{q - \psi(\lambda)} \right) d\lambda < \infty, \quad q > 0 \]  

(2.3)

and

either \( v > 0 \) or \( \int_{(-1,1)} |x| \nu(dx) = \infty \).  

(2.4)

In what follows, we suppose that the conditions (A1)-(A2) are satisfied. By Fourier inversion, we see that the function

\[ p_t(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \text{Re} e^{-i\lambda x + t\psi(\lambda)} d\lambda, \quad t > 0, \ x \in \mathbb{R} \]  

(2.5)

is a continuous density of the transition probability:

\[ P_x(X_t \in A) = \int_A p_t(y - x) dy \]

for \( t > 0, \ x \in \mathbb{R} \) and \( A \in \mathcal{B}(\mathbb{R}) \). The Laplace transform of \( p_t(x) \):

\[ u_q(x) = \int_0^\infty e^{-qt} p_t(x) dt = \frac{1}{2\pi} \int_{\mathbb{R}} \text{Re} \left( \frac{e^{-i\lambda x}}{q - \psi(\lambda)} \right) d\lambda, \quad q > 0, \ x \in \mathbb{R} \]  

(2.6)

is a continuous density of the resolvent kernel:

\[ P_x \left[ \int_0^\infty e^{-qt} 1_A(X_t) dt \right] = \int_A u_q(y - x) dy \]

for \( q > 0, \ x \in \mathbb{R} \) and \( A \in \mathcal{B}(\mathbb{R}) \). We note that the resolvent equation

\[ U_q - U_r + (q - r)U_q U_r = 0 \]

implies that

\[ \int u_q(y - x) u_r(z - y) dy = \frac{1}{q - r} \{ u_r(z - x) - u_q(z - x) \} \]  

(2.7)

for all \( q, r > 0 \) with \( q \neq r \) and \( z, x \in \mathbb{R} \).

Theorem 2.2. Suppose that the conditions (A1)-(A2) are satisfied. Then

\[ P_x \left[ e^{-qt(0)} \right] = \frac{u_q(-x)}{u_q(0)}, \quad x \in \mathbb{R}, \ q > 0. \]  

(2.8)

Theorem 2.2 can be proved via Hunt’s switching identity, which is based on the duality between \((X_t)\) and \((-X_t)\). The proof can be found, e.g., in [21 pp. 64], and so we omit it.
2.3 Killed process

We define the killed process as

\[
X^0_t = \begin{cases} 
X_t & t < T_{\{0\}}, \\
\Delta & t \geq T_{\{0\}}.
\end{cases}
\]  

(2.9)

The laws on the space \( \mathbb{D} \) of the killed process \((X^0_t : t \geq 0)\) under \( \{P_x : x \in \mathbb{R} \setminus \{0\}\} \) will be denoted by \( \{P^0_x : x \in \mathbb{R} \setminus \{0\}\} \). The corresponding semigroup and resolvent operator will be denoted by \( T^0_t \) and \( U^0_q \), respectively, in the same way as (2.1).

Set

\[
p^0_t(x, y) = p_t(y-x) - \int_0^t p_{t-s}(y)P_x(T_{\{0\}} \in ds)
\]  

(2.10)

and

\[
u^0_q(x, y) = \int_0^\infty e^{-qt}p^0_t(x, y)dt = u_q(y-x) - \frac{u_q(-x)u_q(y)}{u_q(0)}
\]  

(2.11)

for \( t, q > 0 \) and \( x, y \in \mathbb{R} \setminus \{0\} \). Note that the second identity in (2.11) follows from Theorem 2.2. Then \( p^0_t(x, y) \) is the continuous density of the transition probability for the killed process and \( u^0_q(x, y) \) is that of the resolvent.

2.4 Excursion

Since every point in \( \mathbb{R} \) is regular for itself, the process admits the local time \( L(t, x) \) (see, e.g., [2, Chap. V]): \( L(t, x) \) is a measurable process such that

(i) for any \( x \in \mathbb{R} \), \( t \mapsto L(t, x) \) is continuous \( P_0 \)-almost surely (see, e.g., [2] Proposition V.1.2). For joint continuity, see, e.g., [2] Theorem V.3.15);

(ii) for \( P_0 \)-almost all path, \( \int_0^t 1_A(X_s)ds = \int_A L(t, x)dx \) for all \( t \geq 0 \) and \( A \in \mathcal{B}(\mathbb{R}) \).

We simply write \( L_t = L(t, 0) \). Denote its right-continuous inverse by \( \tau(l) \):

\[\tau(l) = \inf \{ t > 0 : L(t, 0) > l \}.\]

Then the process \((\tau(l) : l \geq 0)\) is a subordinator such that

\[P_0[e^{-\tau(l)}] = e^{-l/u_q(0)}, \quad q > 0, \ l \geq 0\]

(see, e.g., [2] pp. 131]).

Now we can apply Itô’s excursion theory ([19]; see also [2] and [4] for details). We adopt the same notations as in [31] Section 3. We denote its characteristic measure by \( \mathbf{n} \) and call it the excursion measure. The measure \( \mathbf{n} \) has its support on the set of excursions away from the origin:

\[
\mathbb{E} = \{ e \in \mathbb{D} : 0 < \zeta(e) \leq \infty \} \cap \{ e \in \mathbb{D} : e(t) \neq 0 \text{ for } 0 < t < \zeta(e) \}.
\]

We call an element \( e \) of \( \mathbb{E} \) an excursion path. If \( \zeta(e) = \infty \), then such an excursion path \( e \in \mathbb{E} \) is called a final excursion. Note that the measure \( \mathbf{n} \) is \( \sigma \)-finite on \( \mathcal{F}_t \) for any \( t > 0 \); in fact, \( \mathbf{n}(\zeta > t) < \infty \) and \( \{ \zeta > t \} \in \mathcal{F}_t \).
Theorem 2.3 (Markov property of \( n \)). For any \( t > 0 \) and for any non-negative \( \mathcal{F}_t \)-measurable functional \( Z_t \) and any non-negative functional \( F \) on \( \mathbb{D} \), it holds that

\[
n[Z_t F(X_{t+})] = \int n[Z_t; X_t \in dx] P^0_x[F(X)].
\]

For the proof of Theorem 2.3, see [19, Theorem 6.3] and also [23, Theorem XII.4.1].

The following theorem asserts that extremeness of the excursion measure is equivalent to germ triviality (see also Itô [19, Theorem 7.1]).

**Theorem 2.4.** The excursion measure \( n \) is an extreme direction if and only if the germ \( \sigma \)-field \( \mathcal{F}_{0+} \) is trivial under \( n \), i.e.,

for any \( A \in \mathcal{F}_{0+} \), it holds either that \( n(A) = 0 \) or that \( n(A^c) = 0 \).

Although it is rather obvious, we give the proof for convenience of the reader.

**Proof.** Suppose that \( \mathcal{F}_{0+} \) is trivial under \( n \) and that \( m \) is absolutely continuous with respect to \( n \). We denote by \( D \) the Radon–Nikodym density on \( \mathcal{F}_\infty \): \( dm = Ddn \) on \( \mathcal{F}_\infty \), and by \( D_t \) that on \( \mathcal{F}_t \): \( dm = D_t dn \) on \( \mathcal{F}_t \). Then, by the Markov property of \( m \) and \( n \), we have

\[
m[Z_t F(X_{t+})] = m[Z_t P^0_{X_t}[F(X)]] = n[D_t Z_t P^0_{X_t}[F(X)]] = n[D_t Z_t F(X_{t+})].
\]

Hence we have \( D = D_t \) \( n \)-almost everywhere, which implies that \( D \) is \( \mathcal{F}_{0+} \)-measurable \( n \)-almost everywhere. Now we see that \( D \) is constant \( n \)-almost everywhere. This proves that \( n \) is an extreme direction.

Suppose that \( \mathcal{F}_{0+} \) is not trivial under \( n \). Then there exists \( A \in \mathcal{F}_{0+} \) such that \( n(A) \neq 0 \) and \( n(A^c) \neq 0 \). The measure \( n \) is decomposed into the sum

\[
dn = 1_A dn + 1_{A^c} dn,
\]

where the measures \( 1_A dn \) and \( 1_{A^c} dn \) are mutually singular and they are elements of \( \mathcal{E} \) (see [24, Theorem 2]). Thus we see that \( n \) is not an extreme direction. The proof is complete.

2.5 Duality

Let \( \hat{P}_x \) denote the law of \((-X_t)\) under \( P_{-x} \). Then the following duality holds:

\[
\int f(x) P_x[g(X_t)] dx = \int \hat{P}_x[f(X_t)] g(x) dx
\]

for any non-negative measurable functions \( f \) and \( g \). Since Theorem 2.2 is valid also for the dual process \( \{(X_t), (\hat{P}_x)\} \), we have

\[
\hat{P}_x \left[ e^{-qT_{10}} \right] = \frac{u_q(x)}{u_q(0)}, \quad x \in \mathbb{R}, \ q > 0.
\]

(2.12)

The following theorem is due to Chen–Fukushima–Ying [6, Eq. (2.8)] and Fitzsimmons–Getoor [11, eq. (3.22)], where the theorem has been proved in quite general settings.
Theorem 2.5. Suppose that the conditions (A1)-(A2) are satisfied. Then, for any non-negative measurable function \( f \), it holds that
\[
\int_0^\infty e^{-qt} n[f(X_t)]dt = \int f(x) \hat{P}_x [e^{-qT}] dx \tag{2.13}
\]
For the proof of Theorem 2.5, see also [31, Theorem 3.3].

Corollary 2.1. Suppose that the conditions (A1)-(A2) are satisfied. Then
\[
\int_0^\infty e^{-qt} n(\zeta > t)dt = \frac{1}{qu_q(0)}, \quad q > 0. \tag{2.14}
\]
Proof. Taking \( f(x) \equiv 1 \) in the identity (2.13), we obtain the desired identity.

The following theorem relates the entrance law density with the hitting time density.

Theorem 2.6. Suppose that the conditions (A1)-(A2) are satisfied. Then there exists a bi-measurable function \( \rho(t, x) \) which is at the same time a space density of the entrance law
\[
n(X_t \in dx) = \rho(t, x)dx \tag{2.15}
\]
and a time density of the law of the first hitting time for the dual process
\[
\hat{P}_x(T_{[0]} \in dt; T_{[0]} < \infty) = \rho(t, x)dt. \tag{2.16}
\]
That is,
\[
\rho(t, x) = \frac{n(X_t \in dx)}{dx} = \frac{\hat{P}_x(T_{[0]} \in dt; T_{[0]} < \infty)}{dt}.
\]
For the proof of Theorem 2.6, see [31, Theorem 3.5].

Corollary 2.2. Suppose that the conditions (A1)-(A2) are satisfied. Then it holds that
\[
p_0^0(x, y) = p_t(y - x) - \{p_0(0) * \rho(-, -x) * \rho(-, y)\}(t)
\]
where \( f * g \) stands for the convolution of \( f \) and \( g \): \( f * g(t) = \int_0^t f(t - s)g(s)ds \).

Proof. The identity (2.11) can be written as
\[
u_q^0(x, y) = u_q(y - x) - u_q(0) \cdot \frac{u_q(-x)}{u_q(0)} \cdot \frac{u_q(y)}{u_q(0)}.
\]
Since we have
\[
\int_0^\infty e^{-qt} \rho(t, x)dt = \frac{u_q(x)}{u_q(0)}, \quad q > 0, \ x \in \mathbb{R},
\]
we obtain the desired result.
2.6 Continuous entrance property

**Theorem 2.7.** Suppose that the conditions (A1)-(A2) are satisfied. Then

\[ n(\{X_0 = 0\}^c) = 0. \]  
(2.17)

We must be careful in this property (2.17); in fact, it is not necessarily satisfied by general Markov processes with jumps, while it is obviously satisfied by diffusion processes.

**Proof.** It is well-known (see, e.g., [27, Theorem 6.31.5]) that any function \( f \) of class \( C^2 \) with compact support belongs to the domain of the generator and

\[
\lim_{t \to 0^+} \frac{P_t[f(X_t)] - f(x)}{t} = \nu f''(x) + af'(x) + \int_{\mathbb{R}} \left\{ f(x + y) - f(x) - \frac{y}{1 + y^2} f'(x) \right\} \nu(dy)
\]

where \( \nu, a \) and \( \nu \) have been introduced in (1.3). Suppose that \( f \) is non-negative and its support is contained in \( \mathbb{R} \setminus \{0\} \). Then we have

\[
\lim_{t \to 0^+} \frac{1}{t} P_0[f(X_t)] = \int_{\mathbb{R}} f(y) \nu(dy).
\]

This implies that

\[
C := \sup_{t > 0} \frac{1}{t} P_0[f(X_t)] < \infty.
\]

Now we have

\[
q^2 \int u_q(x) f(x) dx = q^2 \int_0^\infty e^{-qt} P_0[f(X_t)] dt \leq C, \quad q > 0.
\]

Thus, by (2.13), we have

\[
q \int_0^\infty e^{-qt} n[f(X_t)] dt = \frac{1}{qu_q(0)} \cdot q^2 \int u_q(x) f(x) dx \leq \frac{C}{qu_q(0)},
\]

which converges to 0 as \( q \to \infty \). This implies that

\[
\liminf_{t \to 0^+} n[f(X_t)] = 0.
\]

By Fatou’s lemma, we obtain

\[
n[f(X_0)] \leq \liminf_{t \to 0^+} n[f(X_t)] = 0.
\]

This proves \( n(\{X_0 = 0\}^c) = 0 \).
3 Preliminary facts: symmetric case

In what follows, we suppose that the following condition is satisfied:

(A0) The process is symmetric, i.e., \( P_x = \hat{P}_x, x \in \mathbb{R} \).

Then we see that

\[
-\psi(\lambda) = \theta(\lambda) = v\lambda^2 + 2 \int_{(0, \infty)} (1 - \cos \lambda x) \nu(dx), \quad \lambda \in \mathbb{R}
\]

where the corresponding Lévy measure is also symmetric: \( \nu(-dx) = \nu(dx) \).

3.1 The Kesten–Bretagnolle condition in the symmetric case

Lemma 3.1. Suppose that the condition (A0) is satisfied. Then the conditions (A1)-(A2) are satisfied if and only if

\[
\int_0^\infty \frac{1}{1 + \theta(\lambda)} d\lambda < \infty, \quad (3.1)
\]

and

\[
\theta(\lambda) \to \infty \quad \text{as} \quad \lambda \to \infty. \quad (3.2)
\]

In this case, moreover, it holds that

\[
\int_0^\infty \min\{\lambda^2, 1\} \frac{d\lambda}{\theta(\lambda)} < \infty. \quad (3.3)
\]

Proof. (i) Suppose that the conditions (A1)-(A2) are satisfied. Then (2.3) of Theorem 2.1 implies that (3.1) holds. Note that

\[
e^{-\theta(\lambda)} = \int e^{i\lambda x} p_1(x) dx, \quad \lambda \in \mathbb{R}.
\]

Since \( p_1(x) \geq 0 \) and \( \int p_1(x) dx < \infty \), we may apply the Riemann–Lebesgue theorem to obtain \( e^{-\theta(\lambda)} \to 0 \) as \( \lambda \to \infty \), which implies (3.2).

Suppose that \( \theta(\lambda_0) = 0 \) for some \( \lambda_0 \in \mathbb{R} \setminus \{0\} \). Then we have \( P_0[e^{i\lambda_0 X_t}] = 1 \), which implies that \( P_x(X_t = 0) = 1 \) for all \( t \geq 0 \). This contradicts the assumption (A2). Hence we obtain

\[
\theta(\lambda) \neq 0, \quad \lambda \in \mathbb{R} \setminus \{0\}. \quad (3.4)
\]

It holds that

\[
\frac{\theta(\lambda)}{\lambda^2} \geq v + 2 \int_{(0,1)} \frac{1 - \cos \lambda x}{(\lambda x)^2} x^2 \nu(dx)
\]

\[
\to v + \int_{(0,1)} x^2 \nu(dx) \quad \text{as} \quad \lambda \to 0 \quad (3.5)
\]
by the dominated convergence theorem. By (2.4) of Theorem 2.1, we see that the limit in (3.5) is positive. Hence we see that there exists a positive constant $C$ such that $\theta(\lambda) \geq C\lambda^2$ for small $\lambda$. Combining it with (3.1), (3.2) and (3.4), we obtain (3.3).

(ii) Suppose that (3.1) and (3.2) are satisfied. Suppose also that (2.4) is not satisfied, i.e., that $v = 0$ and $\int_{(0,1)} x \nu(dx) < \infty$. Then we have

$$\frac{\theta(\lambda)}{\lambda} \leq \int_{(0,1)} \frac{1 - \cos \lambda x}{\lambda x} x \nu(dx) + \frac{4}{\lambda} \nu([1, \infty))$$

by the dominated convergence theorem. Hence we have $\int_{\lambda_0}^{\infty} \frac{d\lambda}{\theta(\lambda)} = \infty$ for any $\lambda_0 > 0$, which contradicts (3.1) and (3.2). The proof is now complete. 

In what follows, we suppose that the condition (A) is satisfied, i.e., that all of the conditions (A0)-(A2) are satisfied. Then we have

$$p_t(x) = \frac{1}{\pi} \int_{0}^{\infty} \frac{1}{\lambda} \left( \cos \lambda x \right) e^{-\theta(\lambda)} d\lambda$$

and

$$u_q(x) = \frac{1}{\pi} \int_{0}^{\infty} \frac{\cos \lambda x}{q + \theta(\lambda)} d\lambda.$$ 

Moreover, with the help of Lemma 3.1 the function

$$h(x) = \lim_{q \to 0^+} \left\{ u_q(0) - u_q(x) \right\} = \frac{1}{\pi} \int_{0}^{\infty} \frac{1 - \cos \lambda x}{\theta(\lambda)} d\lambda$$

is well-defined.

### 3.2 Recurrence and transience

Recall that

$$\kappa = \lim_{q \to 0^+} \frac{1}{u_q(0)} = \left\{ \frac{1}{\pi} \int_{0}^{\infty} \frac{1}{\theta(\lambda)} d\lambda \right\}^{-1} \in [0, \infty).$$

We note that $\int_{1}^{\infty} \frac{1}{\theta(\lambda)} d\lambda < \infty$ by Lemma 3.1. Now the following equivalence relations are well-known (see, e.g., [2, Section I.4]):

(i) $\kappa = 0$, or $\int_{0}^{1} \frac{1}{\theta(\lambda)} d\lambda = \infty$, if and only if the process is recurrent, i.e.,

$$P_x(T_{(-r,r)} < \infty) = 1, \quad x \in \mathbb{R}, \ r > 0$$

where $T_{(-r,r)}$ stands for the first hitting time of the interval $(-r,r)$;

(ii) $\kappa > 0$, or $\int_{0}^{1} \frac{1}{\theta(\lambda)} d\lambda < \infty$, if and only if the process is transient, i.e.,

$$P_x \left( \lim_{t \to \infty} |X_t| = \infty \right) = 1, \quad x \in \mathbb{R}.$$
**Theorem 3.1.** Suppose that the condition (A) is satisfied. Then
(i) If the process is recurrent, then
\[ P_x(T_{\{0\}} < \infty) = 1, \quad x \in \mathbb{R}; \]
(ii) If the process is transient, then
\[ P_x(T_{\{0\}} = \infty) = \kappa h(x), \quad x \in \mathbb{R}. \]

*Proof.* On one hand, we have
\[ \lim_{q \to 0^+} P_x \left[ e^{-qT_{\{0\}}} \right] = P_x(T_{\{0\}} < \infty). \]
On the other hand, we have
\[ P_x \left[ e^{-qT_{\{0\}}} \right] = \frac{u_q(x)}{u_q(0)} = 1 - \frac{u_q(0) - u_q(x)}{u_q(0)} q \to 0^+ 1 - \kappa h(x). \]
This proves the claims (i) and (ii) at the same time. \( \square \)

### 3.3 The distribution of the lifetime

**Theorem 3.2.** Suppose that the condition (A) is satisfied. Then there exists a completely monotone density \( \rho(t) \) such that
\[ n(\zeta \in dt) = \rho(t) dt \quad \text{on } (0, \infty) \quad (3.7) \]
and that
\[ n(\zeta = \infty) = \kappa \quad (3.8) \]

*Proof.* By (2.14), we obtain
\[ n(\zeta = \infty) = \lim_{q \to 0^+} \frac{1}{u_q(0)} = \kappa. \]

Let us introduce the following positive Borel measure \( \sigma \) on \([0, \infty)\):
\[ \sigma(A) = \frac{1}{\pi} \int_0^\infty 1_A(\theta(\lambda)) d\lambda, \quad A \in \mathcal{B}([0, \infty)). \]
Then we have
\[ \int_{[0, \infty)} \frac{1}{1 + \xi} \sigma(d\xi) = \frac{1}{\pi} \int_0^\infty \frac{1}{1 + \theta(\lambda)} d\lambda < \infty. \]
In particular, we see that \( \sigma \) is a Radon measure. Since we have
\[ u_q(0) = \int_{[0, \infty)} \frac{1}{q + \xi} \sigma(d\xi), \quad q > 0, \]

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it is well-known (see, e.g., [8, Chapter II]) that there exists another Radon measure $\sigma^*$ on $[0, \infty)$ such that

$$\int_{[0,\infty)} \frac{1}{1 + \xi} \sigma^*(d\xi) < \infty$$

and

$$\frac{1}{qu_q(0)} = \int_{[0,\infty)} \frac{1}{q + \xi} \sigma^*(d\xi), \quad q > 0. \quad (3.9)$$

Combining (2.14) and (3.9), we obtain

$$n(\zeta > t) = \int_{[0,\infty)} e^{-t\xi} \sigma^*(d\xi), \quad t > 0. \quad (3.10)$$

Therefore we conclude that $n(\zeta \in dt) = \rho(t)dt$ where

$$\rho(t) = \int_{[0,\infty)} e^{-t\xi} \sigma^*(d\xi), \quad t > 0. \quad (3.11)$$

It is obvious by definition that the function $\rho(t)$ is completely monotone. Now the proof is complete.

\[ \square \] \[ \square \]

### 3.4 Another expression of $\rho(t, x)$

The following theorem gives another expression of the density $\rho(t, x)$:

**Theorem 3.3.** Suppose that the condition (A) is satisfied. Then

$$\rho(t, x) = \frac{1}{t} \int_0^t \left\{ R(t-s)p_s(x) + (t-s)R'(t-s)p_s(x) + R(t-s)\frac{d}{ds}p_s(x) \right\} ds$$

where $R(t) = n(\zeta > t) = \int_{[0,\infty)} e^{-t\xi} \sigma^*(d\xi)$.

Theorem 3.3 may be proved in the same way as [30, Proposition 4.2], so we omit the proof.

### 4 Harmonic function for the killed process

Set

$$h_q(x) = u_q(0) - u_q(x) = \frac{1}{\pi} \int_0^\infty \frac{1 - \cos \lambda x}{q + \theta(\lambda)} d\lambda, \quad q > 0, \quad x \in \mathbb{R}. \quad (4.1)$$

We note that $h_q(x) \geq 0$ and that, for each $x \in \mathbb{R}$, the function $q \mapsto h_q(x)$ increases as $q > 0$ decreases. Recall that

$$h(x) = \lim_{q \to 0^+} h_q(x) = \frac{1}{\pi} \int_0^\infty \frac{1 - \cos \lambda x}{\theta(\lambda)} d\lambda, \quad x \in \mathbb{R}.$$
4.1 Harmonicity of $h(x)$

To prove that the function $h(x)$ is harmonic, we need the following

**Lemma 4.1.** Suppose that the condition (A) is satisfied. Then it holds that

$$\int u_q(y - x)h(y)dy = \frac{h(x) + u_q(x)}{q}, \quad q > 0, \quad x \in \mathbb{R}, \quad (4.2)$$

and, consequently, that

$$\int p_t(y - x)h(y)dy = h(x) + \int_0^t p_s(x)ds, \quad t > 0, \quad x \in \mathbb{R}. \quad (4.3)$$

**Proof.** Note that

$$\int u_q(y - x)h(y)dy = \lim_{r \to 0^+} \int u_q(y - x)h_r(y)dy$$

by the monotone convergence theorem. Let $r$ be such that $0 < r < q$. Then

$$\int u_q(y - x)h_r(y)dy = \int u_q(y - x)\{u_r(0) - u_r(y)\}dy$$

$$= \frac{u_r(0)}{q} - \int u_q(y - x)u_r(-y)dy$$

where we used the symmetry: $u_r(y) = u_r(-y)$. By the resolvent equation (2.7) with $z = 0$, the last equation becomes

$$\frac{u_r(0)}{q} - \frac{1}{q - r}\{u_r(x) - u_q(x)\} = \frac{h_r(x)}{q} - \frac{ru_r(x)}{q(q - r)} + \frac{u_q(x)}{q - r}.$$

Letting $r \to 0^+$, we have $h_r(x) \to h(x)$ and $ru_r(x) \to 0$, and hence we see that the last equation tends to the right hand side of (4.2). Now (4.3) is obvious, and the proof is complete. □

Now we proceed to the proof of Theorem 1.1.

**Proof of Theorem 1.1** In order to prove the assertion, it suffices to show that

$$qU_0^q h = h, \quad q > 0.$$  \hspace{1cm} (1.4)

By definition of $U_0^q$, we have

$$U_0^q h(x) = \int u_0^q(x, y)h(y)dy$$

$$= \int u_q(y - x)h(y)dy - \frac{u_q(x)}{u_q(0)} \int u_q(y)h(y)dy.$$  \hspace{1cm} (1.5)

By Lemma 4.1, we obtain

$$U_0^q h(x) = \frac{h(x) + u_q(x)}{q} - \frac{u_q(x)}{u_q(0)} \cdot \frac{u_q(0)}{q} = \frac{h(x)}{q}.$$  \hspace{1cm} (1.6)
This completes the proof.

Now let us prove Theorem 1.2.

**Proof of Theorem 1.2** By Lemma 4.1, we have

$$\int h(y) \frac{u_q(y)}{u_q(0)} \, dy = \frac{1}{q}, \quad q > 0.$$  \hfill (4.4)

By (2.15), (2.16) and (2.8), we see that the identity (4.4) is equivalent to

$$\int_0^{\infty} e^{-qt} n[h(X_t)] \, dt = \frac{1}{q}, \quad q > 0.$$  \hfill (4.5)

Therefore we obtain the desired identity.

4.2 Several properties of $h(x)$

Let us study several properties of $h(x)$.

**Lemma 4.2.** Suppose that the condition (A) is satisfied. Then the following assertions hold:

(i) $h(x)$ is continuous;

(ii) $h(0) = 0$;

(iii) $h(x) > 0$ for all $x \in \mathbb{R} \setminus \{0\}$;

(iv) $\lim_{|x| \to \infty} h(x) = \frac{1}{\kappa} = \frac{1}{\pi} \int_0^{\infty} \frac{1}{\theta(\lambda)} \, d\lambda \in (0, \infty]$.

**Proof.** The assertion (i) is obvious by the dominated convergence theorem. The assertions (ii) and (iii) are obvious by definition. Let us prove the assertion (iv).

**Transient case.** Since $\int_0^{\infty} \frac{1}{\theta(\lambda)} \, d\lambda < \infty$, we may apply the Riemann–Lebesgue theorem to obtain

$$\lim_{|x| \to \infty} \int_0^{\infty} \frac{\cos \lambda x}{\theta(\lambda)} \, d\lambda = 0.$$

This proves that $\lim_{|x| \to \infty} h(x) = 1/\kappa$.

**Recurrent case.** We have $\int_0^{\infty} \frac{1}{\theta(\lambda)} \, d\lambda = \infty$. Let $\varepsilon > 0$. Since $\int_0^{\infty} \frac{1}{\theta(\lambda)} \, d\lambda < \infty$, we may apply the Riemann–Lebesgue theorem to obtain

$$\lim_{|x| \to \infty} \int_{\varepsilon}^{\infty} \frac{\cos \lambda x}{\theta(\lambda)} \, d\lambda = 0.$$  

Hence we obtain

$$\liminf_{|x| \to \infty} \pi h(x) \geq \int_{\varepsilon}^{\infty} \frac{1}{\theta(\lambda)} \, d\lambda.$$

Letting $\varepsilon \to 0^+$, we obtain $\lim_{|x| \to \infty} h(x) = \infty$.

**Lemma 4.3.** Suppose that the condition (A) is satisfied. Then

$$\sup_{|x| < 1} \frac{|x|}{h(x)} < \infty.$$
Proof. By (1.3), we have
\[ \theta(\lambda) \leq v\lambda^2 + \lambda^2 \int_{(0,1)} x^2 \nu(dx) + 4\nu([1, \infty)), \quad \lambda \in \mathbb{R}. \]

Now we see that there exists a constant \( C \) such that
\[ \theta(\lambda) \leq C\lambda^2, \quad \lambda > 1. \]

Hence we obtain
\[ \pi h(x) \geq \int_1^\infty \frac{1 - \cos \lambda x}{\theta(\lambda)} d\lambda \geq \frac{1}{C} \int_1^\infty \frac{1 - \cos \lambda x}{\lambda^2} d\lambda = \frac{|x|}{C} \int_{|x|}^\infty \frac{1 - \cos \lambda}{\lambda^2} d\lambda. \]

For \( |x| < 1 \), we obtain
\[ \pi h(x) \geq \frac{|x|}{C} \int_1^\infty \frac{1 - \cos \lambda}{\lambda^2} d\lambda. \]

Since \( \int_1^\infty \frac{1 - \cos \lambda}{\lambda^2} d\lambda > 0 \), we complete the proof. \( \square \)

Lemma 4.4. Suppose that the condition (A) is satisfied. Then, for any fixed \( q > 0 \),
\[ \lim_{x \to 0} \frac{h(x) - h_q(x)}{h(x)} = 0. \]  

(4.6)

Proof. By definitions, we have
\[ h(x) - h_q(x) = \frac{1}{\pi} \int_0^\infty \frac{1 - \cos \lambda x}{\theta(\lambda)} \frac{q}{q + \theta(\lambda)} d\lambda \geq 0. \]  

(4.7)

Let \( \varepsilon > 0 \) be arbitrary. Since \( \theta(\lambda) \to \infty \) as \( \lambda \to \infty \) (Lemma 3.1), there exists a constant \( L > 0 \) such that \( \frac{q}{q + \theta(\lambda)} < \varepsilon \) for all \( \lambda > L \). Now we have
\[ h(x) - h_q(x) \leq \frac{1}{\pi} \int_0^L \frac{1 - \cos \lambda x}{\theta(\lambda)} d\lambda + \varepsilon \int_L^\infty \frac{1 - \cos \lambda x}{\theta(\lambda)} d\lambda \]
\[ \leq \frac{x^2}{2\pi} \int_0^L \frac{\lambda^2}{\theta(\lambda)} d\lambda + \varepsilon h(x) \]

for all \( x \in \mathbb{R} \). Hence, by Lemma 4.3, we obtain
\[ \limsup_{x \to 0} \frac{h(x) - h_q(x)}{h(x)} \leq \varepsilon. \]

Since \( \varepsilon > 0 \) is arbitrary, we obtain (4.6). \( \square \)
4.3 Green function for the killed process

The function

\[ u_0^0(x, y) = \lim_{q \to 0^+} u_q^0(x, y), \quad (x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\} \]

is called the Green function for the killed process. The following formula can also be found in Eisenbaum–Kaspi–Marcus–Rosen–Shi [9, Theorem 6.1]:

**Lemma 4.5.** Suppose that the condition (A) is satisfied. Then

\[ u_0^0(x, y) = h(x) + h(y) - h(y - x) - \kappa h(x)h(y), \quad x, y \in \mathbb{R} \setminus \{0\}. \quad (4.8) \]

**Proof.** Noting that

\[ u_q^0(x, y) = h_q(x) + h_q(y) - h_q(y - x) - \frac{h_q(x)h_q(y)}{u_q(0)}, \quad (4.9) \]

we obtain the desired result by letting \( q \to 0^+ \). \qed

5 The \( h \)-path process of the killed process

Recall that the \( h \)-path process of the killed process has been introduced as

\[ dP^h_x|_{\mathcal{F}_t} = \begin{cases} 
\frac{h(X_t)}{h(x)} dP^0_x|_{\mathcal{F}_t}, & x \in \mathbb{R} \setminus \{0\}, \\
\frac{h(X_t)}{h(y)} d\mathcal{N}|_{\mathcal{F}_t}, & x = 0.
\end{cases} \]

It is obvious by definition that \( P^h_x(T_{\{0\}} = +\infty) = 1 \) for any \( x \in \mathbb{R} \). For \( t > 0 \), we define

\[ p^h_t(x, y) = \frac{p^0_t(x, y)}{h(x)h(y)}, \quad x, y \in \mathbb{R} \setminus \{0\} \]

and

\[ p^h_t(x, 0) = p^h_t(0, x) = \rho(t, x), \quad x \in \mathbb{R} \setminus \{0\}. \]

Then \( p^h_t(x, y) \) is a density of the transition probability of the process \( \{X_t : t \geq 0, (P^h_x : x \in \mathbb{R})\} \) with respect to the symmetrizing measure \( h(y)^2 dy \):

\[ P^h_x(X_t \in A) = \int_A p^h_t(x, y) h(y)^2 dy \]

for \( t > 0, x \in \mathbb{R} \) and \( A \in \mathcal{B}(\mathbb{R}) \). For \( q \geq 0 \), we define

\[ u^h_q(x, y) = \frac{u^0_q(x, y)}{h(x)h(y)}, \quad x, y \in \mathbb{R} \setminus \{0\} \]
and
\[ u_q^h(x, 0) = u_q^h(0, x) = \frac{u_q(x)}{h(x)u_q(0)}, \quad x \in \mathbb{R} \setminus \{0\}. \]

Then \( u_q^h(x, y) \) is a density of the resolvent kernel:
\[
P_x^h \left[ \int_0^\infty e^{-qt}1_A(X_t)dt \right] = \int_A u_q^h(x, y)h(y)^2dy
\]
for \( q \geq 0, x \in \mathbb{R} \) and \( A \in \mathcal{B}(\mathbb{R}) \).

### 5.1 Chapman–Kolmogorov identities

We define
\[ p_t^h(0, 0) = \rho(t), \quad t > 0 \]
where the function \( \rho(t) \) has been introduced in (3.11). Then the formulae (2.14) and (3.8) imply that
\[
\int_0^\infty (1 - e^{-qt}) p_t^h(0, 0)dt = \frac{1}{u_q(0)} - \kappa, \quad q > 0.
\]

**Proposition 5.1.** Suppose that the condition (A) is satisfied. Then the following Chapman–Kolmogorov identities hold:
\[
\int p^h_s(x, y)p^h_t(y, z)h(y)^2dy = p^h_{s+t}(x, z), \quad s, t > 0, \quad x, z \in \mathbb{R}.
\]

**Proof.** The identity (5.2) is immediate in the case where \( x, z \in \mathbb{R} \setminus \{0\} \) by the Markov property of the killed process, and in the case where \( xz = 0 \) except where \( x = z = 0 \) by the Markov property of the excursion measure. Hence it suffices to prove the identity (5.2) in the case where \( x = z = 0 \).

Let \( q, r > 0 \) with \( q \neq r \). On the one hand, letting \( x = z = 0 \) in the resolvent equation (2.7) and using the symmetry, we have
\[
\int u_q(y)u_r(y)dy = \frac{1}{q - r} \{ u_r(0) - u_q(0) \}.
\]

Dividing both sides by \( u_q(0)u_r(0) \), we obtain
\[
\int u_q^h(0, y)u_r^h(y, 0)h(y)^2dy = \frac{1}{q - r} \left\{ \frac{1}{u_q(0)} - \frac{1}{u_r(0)} \right\}.
\]

On the other hand, we compute the double Laplace transform of \( p^h_{s+t}(0, 0) \) as
\[
\int_0^\infty ds \int_0^\infty dt e^{-qs-rt} p^h_{s+t}(0, 0)
\]
\[
= \frac{1}{q - r} \int_0^\infty (e^{-ru} - e^{-qu}) p^h_u(0, 0)du
\]
\[
= \frac{1}{q - r} \left\{ \int_0^\infty (1 - e^{-qu}) p^h_u(0, 0)du - \int_0^\infty (1 - e^{-ru}) p^h_u(0, 0)du \right\}
\]
\[
= \frac{1}{q - r} \left\{ \frac{1}{u_q(0)} - \frac{1}{u_r(0)} \right\}.
\]
where we used the identity (5.1). This shows that the right hand side of (5.3) is the double Laplace transform of $p_{s+t}(0,0)$. Since the left hand side of (5.3) is the double Laplace transform of $\int p_{s}^{h}(0,y)p_{t}^{h}(y,0)h(y)^{2}dy$, we obtain the desired identity. 

\[5.2\] Lifetime disintegration formula

Let us define the law of the bridge process

$$P_{0,0;\cdot}^{h}(\cdot):=P_{0}^{h}(\cdot|X_{t-}=0), \quad t>0$$

as the unique probability measure carried on the set \{ $X_{t-}=0, \zeta=t$ \} such that

$$dP_{0,0;\cdot}|_{\mathcal{F}_{s}}=\frac{p_{t-s}^{h}(X_{s},0)}{p_{t}^{h}(0,0)}dP_{0}|_{\mathcal{F}_{s}}, \quad s \in (0,t).$$

Then the process \{(X_{s}: 0 \leq s \leq t), P_{0,0;\cdot}^{h}\} is a time-inhomogeneous Markov process with its transition probability given by

$$P_{0,0;\cdot}^{h}(X_{v} \in \mathcal{B}|X_{u}=a) = \frac{p_{v-u}^{h}(a,b)p_{t-v}^{h}(b,0)}{p_{t-u}^{h}(a,0)}h(b)^{2}db, \quad 0 \leq u \leq v \leq t, \ a,b \in \mathbb{R}.$$ 

We call the process the bridge of the process \{(X_{s}: s \geq 0), P_{0}^{h}\} given $X_{t-}=0$. For further properties of the bridge process, see Fitzsimmons–Pitman–Yor [12].

Now we are in a position to prove Theorem 1.3.

**Proof of Theorem 1.3**

We consider a cylinder set $\Gamma$ of $\mathcal{F}$ of the form

$$\Gamma = \{X_{t_{1}} \in A_{1}, \ldots, X_{t_{n}} \in A_{n}\} \quad (5.5)$$

for some sequence $0 < t_{1} < t_{2} < \cdots < t_{n} < \infty$ and some sets $A_{1}, \ldots, A_{n} \in \mathcal{B}(\mathbb{R} \setminus \{0\})$.

Let $t > t_{n}$. By the Markov property of the excursion measure, we have

$$n(\Gamma \cap \{t < \zeta < \infty\}) = n[1_{\Gamma}P_{X_{t_{n}}}(t-t_{n}<T_{\{0\}}<\infty)].$$

By the definition of $P_{0}^{h}$, we have

$$n(\Gamma \cap \{t < \zeta < \infty\}) = P_{0}^{h}[1_{\Gamma}P_{X_{t_{n}}}(t-t_{n}<T_{\{0\}}<\infty)]h(X_{t_{n}}).$$

Note that, by Theorem 2.6, we have

$$P_{y}(t-t_{n}<T_{\{0\}}<\infty) = \int_{t}^{\infty}\rho(s-t_{n},y)ds.$$ 

Hence we obtain

$$n(\Gamma \cap \{t < \zeta < \infty\}) = \int_{t}^{\infty}dsP_{0}^{h}[1_{\Gamma}\frac{\rho(s-t_{n},X_{t_{n}})}{h(X_{t_{n}})}]$$

$$= \int_{t}^{\infty}dsP_{0}^{h}[1_{\Gamma}\frac{p_{s-t_{n}}^{h}(X_{t_{n}},0)}{p_{t}^{h}(0,0)}]$$

$$= \int_{t}^{\infty}dsP_{s}^{h}(0,0)P_{0}^{h}[1_{\Gamma}\frac{p_{s-t_{n}}^{h}(X_{t_{n}},0)}{p_{t}^{h}(0,0)}]$$

$$= \int_{t}^{\infty}dsP_{s}^{h}(0,0)P_{0}^{h}(\Gamma|X_{t_{n}}=0).$$
Letting $t \to t_{n+}$, we have

$$n(\Gamma \cap \{\zeta < \infty\}) = \int_0^\infty P^h_0(\Gamma | X_{s-} = 0) \rho^h_s(0,0) ds,$$

(5.6)

since $\Gamma \subset \{\zeta > t_n\}$ and $P^h_0(\zeta = s | X_{s-} = 0) = 1$.

Note that, by Theorem 3.1 we have

$$P^h_y(T_\{0\} = \infty) = \kappa h(y)$$

whichever the process is recurrent or transient. Hence we obtain

$$n(\Gamma \cap \{\zeta = \infty\}) = \kappa P^h_0(\Gamma).$$

(5.7)

Therefore, combining (5.6) and (5.7), we obtain

$$n(\Gamma) = \int_0^\infty P^h_0(\Gamma | X_{s-} = 0) \rho^h_s(0,0) ds + \kappa P^h_0(\Gamma).$$

Noting that the set of all cylinder sets $\Gamma$ of the form (5.5) generate the whole $\sigma$-field, we obtain the desired result.

5.3 Green function for the $h$-path process

The function

$$u^h_0(x,y) = \lim_{q \to 0^+} u^h_q(x,y), \quad (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$$

is called the Green function for the $h$-path process.

**Lemma 5.1.** Suppose that the condition (A) is satisfied. Then

$$u^h_0(x,y) = \frac{1}{h(x)h(y)} \{ h(x) + h(y) - h(y-x) \} - \kappa, \quad x,y \in \mathbb{R} \setminus \{0\},$$

and that

$$u^h_0(x,0) = u^h_0(0,x) = \frac{1}{h(x)} - \kappa, \quad x \in \mathbb{R} \setminus \{0\}.$$

**Proof.** The first identity is obvious from Lemma 4.5. If $x \in \mathbb{R} \setminus \{0\}$, then we have

$$h(x)u^h_q(x,0) = \frac{u_q(0) - h_q(x)}{u_q(0)},$$

(5.8)

and hence we obtain the second identity.

6 Key lemmas

We need several lemmas for later use.
6.1 Regularity of $h_q(x)$

**Lemma 6.1.** Suppose that the conditions (A) and (T) are satisfied. For $q \geq 0$, set

$$
\Theta_q(d\eta) = \frac{1}{\pi(q + \theta(\eta))^2} d\theta(\eta) \quad \text{on } (\lambda_0, \infty).
$$

Then

$$
\int_{(\lambda_0, \infty)} \eta \Theta_q(d\eta) < \infty, \quad q \geq 0.
$$

**Proof.** Since the function $\lambda \mapsto \theta(\lambda)$ increases as $\lambda > 0$ increases (the assumption (T)) and since $\theta(\lambda) \to \infty$ as $\lambda \to \infty$ (Lemma 3.1), we see that $\Theta_q(d\eta)$ is well-defined as a positive Borel measure on $(\lambda_0, \infty)$. Since

$$
\int_{(\lambda_0, \infty)} \eta \Theta_q(d\eta) = \int_{(\lambda_0, \infty)} \Theta_q(d\eta) \left\{ \lambda_0 + \int_{\lambda_0}^{\eta} d\lambda \right\}
$$

$$
= \frac{\lambda_0}{\pi(q + \theta(\lambda_0))} + \int_{\lambda_0}^{\infty} d\lambda \int_{(\lambda, \infty)} \Theta_q(d\eta)
$$

$$
= \frac{\lambda_0}{\pi(q + \theta(\lambda_0))} + \frac{1}{\pi} \int_{\lambda_0}^{\infty} \frac{1}{q + \theta(\lambda)} d\lambda < \infty,
$$

we complete the proof. \qed

Set $h_0 = h$. Then we have

$$
\Theta_q(d\eta) = \frac{1}{\pi(q + \theta(\eta))^2} d\theta(\eta) \quad \text{on } (\lambda_0, \infty).
$$

**Lemma 6.2.** Suppose that the conditions (A) and (T) are satisfied. Let $q \geq 0$ be fixed. Then the following statements hold:

(i) There exists a constant $C_q$ such that, for any $x, y$ such that $0 < 2|x| < |y|,

$$
\left| \frac{h_q(y - x) - h_q(y)}{x} \right| \leq C_q \left( |y| + \frac{1}{|y|} \right);
$$

(ii) For any $y \in \mathbb{R} \setminus \{0\}$, it holds that

$$
\lim_{x \to 0} \frac{h_q(y + x) + h_q(y - x) - 2h_q(y)}{x} = 0;
$$

(iii) For any $\varepsilon > 0$, it holds that

$$
\lim_{x \to 0} \frac{1}{x} \int_{-\varepsilon}^{\varepsilon} \left\{ h_q(y + x) + h_q(y - x) - 2h_q(y) \right\} dy = 0.
$$
Proof. (i) For \( v \in \mathbb{R} \), we split \( h_q(v) \) into the sum of

\[
h_q^{(1)}(v) = \frac{1}{\pi} \int_0^{\lambda_0} \frac{1 - \cos \lambda v}{q + \theta(\lambda)} \, d\lambda \quad \text{and} \quad h_q^{(2)}(v) = \frac{1}{\pi} \int_{\lambda_0}^{\infty} \frac{1 - \cos \lambda v}{q + \theta(\lambda)} \, d\lambda.
\]

We note that

\[
\frac{h_q^{(1)}(y - x) - h_q^{(1)}(y)}{x} = \int_0^{\lambda_0} A_{\lambda,y}^{(1)}(x) \frac{\lambda^2}{q + \theta(\lambda)} \, d\lambda
\]

where

\[
A_{\lambda,y}^{(1)}(x) = \frac{\cos(\lambda y) - \cos(\lambda(y - x))}{\pi \lambda^2 x}.
\]

Since we have

\[
\left| A_{\lambda,y}^{(1)}(x) \right| = \frac{1}{\pi \lambda^2 |x|} \left| \int_0^x (-\lambda) \sin(\lambda(y - v)) \, dv \right| \leq \frac{2|y|}{\pi},
\]

we obtain

\[
\left| \frac{h_q^{(1)}(y - x) - h_q^{(1)}(y)}{x} \right| \leq \frac{2|y|}{\pi} \int_0^{\lambda_0} \frac{\lambda^2}{\theta(\lambda)} \, d\lambda.
\]

Define a function \( \varphi \) as \( \varphi(0) = 1 \) and

\[
\varphi(v) = \frac{\sin v}{v}, \quad v \neq 0.
\]

Since \( \varphi'(v) = \frac{\cos v}{v} - \frac{\sin v}{v^2} \) and \( |\varphi'(v)| \leq 2/|v| \) for all \( v \neq 0 \), we have

\[
\left| \frac{\varphi(\lambda(y - x))}{x} - \varphi(\lambda y) \right| = \left| \frac{1}{x} \int_0^x (-\lambda) \varphi'(\lambda(y - v)) \, dv \right| \leq \frac{4}{|y|}, \quad 0 < 2|x| < |y|
\]

for all \( \lambda > 0 \). Using Fubini’s theorem and integrating by parts, we have

\[
h_q^{(2)}(v) = \int_{(\lambda_0, \infty)} \left\{ \int_{\lambda_0}^{\eta} (1 - \cos \lambda v) \, d\lambda \right\} \Theta_q(d\eta)
\]

\[
= \int_{(\lambda_0, \infty)} \{ \eta - \lambda_0 - \varphi(\eta v) \eta + \varphi(\lambda_0 v) \lambda_0 \} \Theta_q(d\eta)
\]

for all \( v \in \mathbb{R} \). Thus we see that

\[
\frac{h_q^{(2)}(y - x) - h_q^{(2)}(y)}{x} = \int_{(\lambda_0, \infty)} A_{\eta,y}^{(2)}(x) \eta \Theta_q(d\eta)
\]

where

\[
A_{\eta,y}^{(2)}(x) = \frac{\varphi(\eta(y - x))}{x} - \varphi(\eta y) + \frac{\lambda_0}{\eta} \cdot \frac{\varphi(\lambda_0(y - x))}{x} - \varphi(\lambda_0 y).}
\]
By (6.6), we have

$$|A_{n,y}^{(2)}(x)| \leq \frac{8}{|y|}, \quad 0 < 2|x| < |y|$$

and hence, we obtain

$$\left| \frac{h_q^{(2)}(y-x) - h_q^{(2)}(y)}{x} \right| \leq \frac{8}{|y|} \int_{(\lambda_0, \infty)} \eta \Theta_q(d\eta), \quad 0 < 2|x| < |y|.$$ This implies (6.2).

(ii) Note that

$$\frac{h_q^{(1)}(y+x) + h_q^{(1)}(y-x) - 2h_q^{(1)}(y)}{x} = \int_0^{\lambda_0} \left\{ A_{\lambda,y}^{(1)}(x) - A_{\lambda,y}^{(1)}(-x) \right\} \frac{\lambda^2}{q + \theta(\lambda)} d\lambda$$

and that

$$\frac{h_q^{(2)}(y+x) + h_q^{(2)}(y-x) - 2h_q^{(2)}(y)}{x} = \int_{(\lambda_0, \infty)} \left\{ A_{n,y}^{(2)}(x) - A_{n,y}^{(2)}(-x) \right\} \eta \Theta_q(d\eta).$$

Since we have

$$\lim_{x \to 0} \left\{ A_{\lambda,y}^{(i)}(x) - A_{\lambda,y}^{(i)}(-x) \right\} = 0, \quad i = 1, 2,$$

we obtain (6.3) by the dominated convergence theorem.

(iii) Since the estimate (6.5) is valid for all $x \neq 0$ and $y \in \mathbb{R}$, we may apply the dominated convergence theorem to obtain

$$\lim_{x \to 0} \frac{1}{x} \int_{-\varepsilon}^{\varepsilon} \left\{ h_q^{(1)}(y+x) + h_q^{(1)}(y-x) - 2h_q^{(1)}(y) \right\} dy = 0. \quad (6.7)$$

Note that

$$\int_{-\varepsilon}^{\varepsilon} \left\{ \varphi(\lambda(y+x)) + \varphi(\lambda(y-x)) - 2\varphi(\lambda y) \right\} dy$$

$$= \int_{-\varepsilon}^{\varepsilon} dy \int_0^x (-\lambda) \left\{ \varphi'(\lambda(y-v)) - \varphi'(\lambda(y+v)) \right\} dv$$

$$= 2 \int_0^x \left\{ \varphi(\lambda(\varepsilon - v)) - \varphi(\lambda(\varepsilon + v)) \right\} dv$$

for all $\lambda > 0$. Thus we have

$$\int_{-\varepsilon}^{\varepsilon} \left\{ A_{n,y}^{(2)}(x) - A_{n,y}^{(2)}(-x) \right\} dy = -\frac{2}{x} \int_0^x \left\{ \varphi(\eta(\varepsilon - v)) - \varphi(\eta(\varepsilon + v)) \right\} dv$$

$$+ \frac{2\lambda_0}{\eta x} \int_0^x \left\{ \varphi(\lambda_0(\varepsilon - v)) - \varphi(\lambda_0(\varepsilon + v)) \right\} dv. \quad (6.8)$$
Since \( \varphi(v) \) is bounded in \( v \in \mathbb{R} \), we see that the integral of the left hand side of (6.8) is bounded in \( \eta > \lambda_0 \) and \( x \neq 0 \) and it converges to 0 as \( x \to 0 \). Since we have
\[
\frac{1}{x} \int_{-\varepsilon}^{\varepsilon} \{ h_q^{(2)}(y + x) + h_q^{(2)}(y - x) - 2h_q^{(2)}(y) \} \, dy = \int_{(\lambda_0, \infty)} \eta \Theta_q(d\eta) \int_{-\varepsilon}^{\varepsilon} \{ A_{n,y}^{(2)}(x) - A_{n,y}^{(2)}(-x) \} \, dy,
\]
we see that this integral converges to 0 as \( x \to 0 \) by the dominated convergence theorem. Therefore we obtain (6.4), which completes the proof.

6.2 Limiting properties of the resolvent densities

Define
\[
p_h^{s,h}(x, y) = p_t^{h}(x, y) + p_t^{h}(-x, y) = p_t^{h}(x, y) + p_t^{h}(x, -y)
\]
for \( t > 0, x, y \in \mathbb{R} \), and
\[
u_h^{s,h}(x, y) = u_t^{h}(x, y) + u_t^{h}(-x, y) = u_t^{h}(x, y) + u_t^{h}(x, -y)
\]
for \( q \geq 0, x, y \in \mathbb{R} \). Then \( p_h^{s,h}(x, y) \) and \( u_h^{s,h}(x, y) \), respectively, are densities of the transition probability and the resolvent kernel, respectively, of the process \( \{(|X_t| : t \geq 0), (P^h_x : x \in [0, \infty))\} \) with respect to the symmetrizing measure \( h(y)^2dy \):
\[
P^h_x(|X_t| \in A) = \int_A p_t^{h,s}(x, y) h(y)^2dy
\]
for all \( t > 0, x \in [0, \infty) \) and \( A \in \mathcal{B}((0, \infty)) \), and
\[
P^h_x \left[ \int_0^\infty e^{-qt}1_A(|X_t|) \, dt \right] = \int_A u_t^{h,s}(x, y) h(y)^2dy
\]
for all \( q \geq 0, x \in [0, \infty) \) and \( A \in \mathcal{B}((0, \infty)) \).

**Lemma 6.3.** Suppose that the conditions (A) and (T) are satisfied. Then the following assertions hold:

(i) For any \( q \geq 0 \) and \( y \in \mathbb{R} \setminus \{0\} \),
\[
\lim_{x \to 0} u_t^{h,s}(x, y) = u_t^{h,s}(0, y) = 2u_t^{h}(0, y);
\]

(ii) For any \( q > 0 \), it holds that
\[
\lim_{\varepsilon \to 0^+} \limsup_{x \to 0} \int_{-\varepsilon}^{\varepsilon} u_t^{h}(x, y) h(y)dy = 0.
\]
Proof. (i) Let $x, y \in \mathbb{R} \setminus \{0\}$. If $q > 0$, we combine (1.9) with (3.8) to cancel $u_q(0)$, and then we have

$$u_q^h(x, y) = u_q^h(0, y) \cdot \frac{h_q(x)}{h(x)} - \frac{x}{h(x)h(y)} h_q(y - x) - h_q(y).$$  \hfill(6.11)

Then the identity (6.11) is still valid for all $q \geq 0$. Hence we have

$$u_q^{h,s}(x, y) = 2u_q^h(0, y) \cdot \frac{h_q(x)}{h(x)} - \frac{x}{h(x)h(y)} h_q(y + x) + h_q(y - x) - 2h_q(y).$$  \hfill(6.12)

By Lemmas 4.3, 4.4 and 6.2, we obtain (6.9).

(ii) Integrating both sides of (6.12) with respect to $h(y)dy$, we have

$$0 \leq 2\int_{-\varepsilon}^{\varepsilon} u_q^h(x, y)h(y)dy = \int_{-\varepsilon}^{\varepsilon} u_q^{h,s}(x, y)h(y)dy$$

$$\leq \frac{2h_q(x)}{h(x)} \int_{-\varepsilon}^{\varepsilon} u_q^h(0, y)h(y)dy + \frac{x}{h(x)} \int_{-\varepsilon}^{\varepsilon} dy \frac{u_q(y + x) + u_q(y - x) - 2u_q(y)}{x}$$

By Lemmas 4.3, 4.4 and 6.2, we obtain

$$\limsup_{x \to 0} \int_{-\varepsilon}^{\varepsilon} u_q^h(x, y)h(y)dy \leq \int_{-\varepsilon}^{\varepsilon} u_q^h(0, y)h(y)dy.$$  

Since

$$\int u_q^h(0, y)h(y)dy = \frac{1}{u_q(0)} \int u_q(y)dy = \frac{1}{qu_q(0)} < \infty,$$

we obtain (6.10). \hfill\Box

**Lemma 6.4.** Suppose that the conditions (A), (B) and (T) are satisfied. Let $q \geq 0$ be fixed. Then it holds that

$$\lim_{x \to 0} u_q^h(x, y) = u_q^h(0, y), \quad y \in \mathbb{R} \setminus \{0\}. \quad (6.13)$$

Consequently, it holds that

$$\lim_{z \to 0} \frac{u_q^h(z, x)}{u_q^h(z, y)} = \frac{u_q^h(0, x)h(x)}{u_q^h(0, y)h(y)}, \quad x, y \in \mathbb{R} \setminus \{0\}. \quad (6.14)$$

**Proof.** Let $q \geq 0$ and $y \in \mathbb{R} \setminus \{0\}$ be fixed. Recall the identity (6.11):

$$u_q^h(x, y) = u_q^h(0, y) \cdot \frac{h_q(x)}{h(x)} - \frac{x}{h(x)h(y)} h_q(y - x) - h_q(y).$$

By Lemma 4.4, (i) of Lemma 6.2 and the assumption (B), we obtain (6.13).

Let $q \geq 0$ and $x, y \in \mathbb{R} \setminus \{0\}$ be fixed. Then we obtain

$$\lim_{z \to 0} \frac{u_q^h(z, x)}{u_q^h(z, y)} = \lim_{z \to 0} \frac{u_q^h(z, x)h(x)}{u_q^h(z, y)h(y)} = \frac{u_q^h(0, x)h(x)}{u_q^h(0, y)h(y)},$$

which proves (6.14). \hfill\Box
6.3 Transience of the $h$-path process

Let us prove Theorem 6.4.

Proof of Theorem 6.4 By a well-known theorem (see, e.g., [7, Theorem 3.7.2]), it suffices to prove the following:

(i) The function

$$[0, \infty) \ni x \mapsto \int_K u_0^{h,s}(x, y) h(y)^2 dy$$

is lower-semicontinuous for any compact set $K$ of $[0, \infty)$;

(ii) There exists a nearly Borel function $f$ which is positive almost everywhere such that

$$0 < \int_0^\infty f(y) u_0^{h,s}(x, y) h(y)^2 dy < \infty.$$  \hfill (6.15)

Recall that

$$u_0^{h,s}(x, y) = \frac{2}{h(x)h(y)} \left\{ h(x) + h(y) - \frac{h(x - y) + h(x + y)}{2} \right\} - 2\kappa.$$

The claim (i) is obvious by (i) of Lemma 6.3 and by Fatou’s lemma. The claim (ii) is also obvious; in fact, we may take $f(y) = \min\{1, y^{-2}h(y)^{-2}\}$, which is a continuous function. Now the proof is complete.

6.4 The excursion measure of hitting a single point

Before closing this section, we give the following formula about the excursion measure of hitting a single point.

Theorem 6.1. Suppose that the conditions (A), (B) and (T) are satisfied. Let $a \in \mathbb{R} \setminus \{0\}$. Then it holds that

$$n(T_{\left\{ a \right\}} < \zeta) = \frac{1 - \kappa h(a)}{h_B(a)}$$

where $h_B(a) = 2h(a) - \kappa h(a)^2$.

Proof. Let $x \in \mathbb{R}$ and $b \in \mathbb{R} \setminus \{a\}$. In our settings of symmetric Lévy processes, Getoor’s formula [15, Theorem 6.5] leads to

$$P_x(T_{\left\{ a \right\}} < T_{\left\{ b \right\}}) = \frac{h(a - b) - h(a - x) + h(b - x) - \kappa h(b - x)h(a - b)}{h_B(a - b)}.$$ 

Letting $b = 0$ and using the symmetry $h(-x) \equiv h(x)$, we have

$$P_x(T_{\left\{ a \right\}} < T_{\left\{ 0 \right\}}) = \frac{h(a) - h(a - x) + h(x) - \kappa h(x)h(a)}{h_B(a)}.$$
Let $\varepsilon > 0$. By the Markov property, we have

$$n(\varepsilon < T_{\{a\}} < \zeta) = n \left[ P_{x}^{0}(T_{\{a\}} < \zeta; \varepsilon < T_{\{a\}} \land \zeta) \right] = \frac{1}{h_{B}(a)} \left[ h(a) - h(a - X_{\varepsilon}) + h(X_{\varepsilon}) - \kappa h(X_{\varepsilon}) h(a); \varepsilon < T_{\{a\}} \land \zeta \right]$$

$$= \frac{1}{h_{B}(a)} P_{x}^{0} \left[ \frac{X_{\varepsilon}}{h(X_{\varepsilon})} \cdot \frac{h(a) - h(a - X_{\varepsilon})}{h(X_{\varepsilon})} + 1 - \kappa h(a); \varepsilon < T_{\{a\}} \right]. \quad (6.16)$$

Now we let $\varepsilon \to 0^{+}$. On the one hand, using the assumption (B), Lemma 4.3 and (i) of Lemma 6.2, we apply the dominated convergence theorem to see that the quantity (6.17) converges to

$$\frac{1 - \kappa h(a)}{h_B(a)}.$$ 

On the other hand, using monotone convergence theorem, we see that the quantity (6.16) converges to $n(T_{\{a\}} < \zeta)$, we obtain the desired result. \[\square\]

7 Feller property of the $h$-path process

Define

$$T_{t}^{h} f(x) = P_{x}^{h}[f(X_{t})], \quad t \geq 0, f \in B_{+, b}(\mathbb{R}).$$

Then the Markov property implies that the family $\{T_{t}^{h} : t \geq 0\}$ forms a transition semigroup:

(T1) $T_{t+s}^{h} = T_{t}^{h} T_{s}^{h}$ for all $t, s \geq 0$;
(T2) $T_{0}^{h}$ equals the identity;
(T3) $0 \leq f \leq 1$ implies that $0 \leq T_{t}^{h} f \leq 1$ for all $t \geq 0$.

Note that (T3) implies the contraction property:

(T4) $\|T_{t}^{h} f\| \leq \|f\|$ for $t \geq 0$ and $f \in B_{+, b}(\mathbb{R})$.

We write the corresponding resolvent operator as

$$U_{q}^{h} f(x) = \int_{0}^{\infty} e^{-qt} T_{t}^{h} f(x) dt, \quad q > 0, f \in B_{+, b}(\mathbb{R}). \quad (7.1)$$

Then it is immediate that the family $\{U_{q}^{h} : q > 0\}$ satisfies the following properties:

(U1) $U_{q}^{h} - U_{r}^{h} + (q - r) U_{q}^{h} U_{r}^{h} = 0$ for $q, r > 0$;
(U2) $0 \leq f \leq 1$ implies that $0 \leq q U_{q}^{h} f \leq 1$ for $q > 0$.

Note that (U2) implies the contraction property:

(U3) $\|q U_{q}^{h} f\| \leq \|f\|$ for $q > 0$ and $f \in B_{+, b}(\mathbb{R})$.

Lemma 7.1. Suppose that the condition (A) is satisfied. Then the condition (B), i.e.,

$$\lim_{x \to 0} \frac{x}{h(x)} = 0,$$

implies $v = 0$ in (1.3).
Proof. Suppose that $v > 0$. Then we have $\theta(\lambda) \geq v\lambda^2$. Hence we obtain

$$h(x) \leq \frac{1}{\pi} \int_0^\infty \frac{1 - \cos Ax}{v\lambda^2} d\lambda = \frac{|x|}{v\pi} C(2) = \frac{|x|}{2v\pi}.$$

This prevents the condition (B). \qed

Recall that the Feller property of the semigroup $\{T^h_t : t \geq 0\}$ is stated precisely as follows:

(F1) $T^h_t C_0(\mathbb{R}) \subset C_0(\mathbb{R})$ for all $t \geq 0$;

(F2) $\|T^h_t f - f\| \to 0$ as $t \to 0+$ for all $f \in C_0(\mathbb{R})$.

In order to prove Theorem 1.5 we shall prove the following

**Proposition 7.1.** Suppose that the conditions (A), (B) and (T) are satisfied. Then the following statements hold:

(i) $T^h_t f(x) \to f(x)$ as $t \to 0+$ for all $x \in \mathbb{R}$ and $f \in C_0(\mathbb{R})$.

(ii) $U^h_q C_0(\mathbb{R}) \subset C_0(\mathbb{R})$ for each $q > 0$.

The proof of Proposition 7.1 will be given in Section 7.1. To deduce Theorem 1.5 from Proposition 7.1 is a kind of general argument, and so we omit it. See [23, Proposition III.2.4] for details.

### 7.1 Feller property of the resolvent of the $h$-path process

Now we are in a position to prove Proposition 7.1.

**Proof of Proposition 7.1** (i) It is obvious since $T^h_t f(x) = P^h_x [f(X_t)]$ and $P^h_x$ is a probability measure on the càdlàg space $\mathbb{D}$.

(ii) By the contraction property (U3), it suffices to show that $U^h_q C_c(\mathbb{R}) \subset C_0(\mathbb{R})$ where $C_c(\mathbb{R})$ stands for the class of continuous functions $\mathbb{R} \to \mathbb{R}$ with compact supports. Let $f \in C_c(\mathbb{R})$ be fixed and let us prove that $U^h_q f \in C_0(\mathbb{R})$.

Recall that, for $x \in \mathbb{R} \setminus \{0\}$,

$$U^h_q f(x) = \int f(y) u^h_q(x, y) h(y) dy = \frac{1}{h(x)} \int f(y) u^0_q(x, y) h(y) dy. \quad (7.2)$$

Since $f$ has compact support, and the functions $h$ and $f$ are continuous and $u^h_q$ is continuous outside the origin, it is obvious that the function $U^h_q f(x)$ is continuous at $x \in \mathbb{R} \setminus \{0\}$.

By (7.2), we have

$$|U^h_q f(x)| \leq \frac{1}{h(x)} U^0_q f(x) \sup_{y \in \text{Supp}(f)} |h(y)|.$$

As $|x| \to \infty$, we have $1/h(x) \to \kappa < \infty$ ((iv) of Lemma 4.2) and $U^0_q f(x) \to 0$. Since $h$ is continuous ((i) of Lemma 4.2), we see that $U^h_q f(x)$ vanishes at infinity.

Let us prove that the function $U^h_q f(x)$ is continuous at $x = 0$. Let $\varepsilon > 0$. Then, by Lemma 6.4 and by the dominated convergence theorem, we obtain

$$\lim_{x \to 0} \int_{|x| > \varepsilon} f(y) u^h_q(x, y) h(y)^2 dy = \int_{|x| > \varepsilon} f(y) u^h_q(0, y) h(y)^2 dy.$$
We estimate the integral on the interval $[-\varepsilon, \varepsilon]$ as
\[
\left| \int_{-\varepsilon}^{\varepsilon} f(y) u_h^q(x, y) h(y)^2 dy \right| \leq \|fh\| \int_{-\varepsilon}^{\varepsilon} u_h^q(x, y) h(y) dy.
\]
Note that the right hand side coincide with
\[
\|fh\| \int_{-\varepsilon}^{\varepsilon} u_h^q(x, y) h(y) dy
\]
by the symmetry $h(-y) = h(y)$. Hence, by Lemma 6.3 we obtain
\[
\lim_{\varepsilon \to 0^+} \limsup_{x \to 0} \left| \int_{-\varepsilon}^{\varepsilon} f(y) u_h^q(x, y) h(y)^2 dy \right| = 0.
\]
Therefore we conclude that $U_q^h f(x) \to U_q^h f(0)$ as $x \to 0$, which completes the proof. 

7.2 Extremeness property

Let us proceed to prove Corollary 1.1.

Proof of Corollary 1.1 By the Feller property of the semigroup $\{T^h_t : t \geq 0\}$, we can prove, in the same way as Proposition 2.1, that the germ $\sigma$-field $\mathcal{F}_{0+}$ is trivial under $P_0^h$. Since $n$ is mutually absolutely continuous with respect to $P_0^h$, we see that the germ $\sigma$-field $\mathcal{F}_{0+}$ is trivial also under $n$. Hence, by Theorem 2.4, we conclude that $n$ is an extreme direction. The proof is now complete.

7.3 Sample path behaviors

Let us prove Corollaries 1.2 and 1.3.

Proof of Corollaries 1.2 and 1.3 Set
\[
\Omega_0^+ = \{ \exists t_0 > 0 \text{ such that } \forall t < t_0, X_t \geq 0 \},
\]
\[
\Omega_0^- = \{ \exists t_0 > 0 \text{ such that } \forall t < t_0, X_t \leq 0 \}
\]
and
\[
\Omega_0^{+-} = \{ \exists \{t_n\} \text{ with } t_n \searrow 0 \text{ such that } \forall n, X_{t_n} X_{t_{n+1}} < 0 \}.
\]
Then the space $\mathbb{D}$ is decomposed into the disjoint union:
\[
\mathbb{D} = \Omega_0^+ \cup \Omega_0^- \cup \Omega_0^{+-}.
\]
Moreover, it is obvious that the three sets $\Omega_0^+$, $\Omega_0^-$ and $\Omega_0^{+-}$ are all elements of $\mathcal{F}_{0+}$. Since $\mathcal{F}_{0+}$ is trivial under $P_0^h$, we see that only one of the three probabilities $P_0^h(\Omega_0^+)$, $P_0^h(\Omega_0^-)$ and $P_0^h(\Omega_0^{+-})$ is one and the other two are zero. By the symmetry: $P_0^h(X \in \cdot) = P_0^h(-X \in \cdot)$, we see that $P_0^h(\Omega_0^+)$ and $P_0^h(\Omega_0^-)$ coincide, which turn out to be zero. Therefore we conclude that $P_0^h(\Omega_0^{+-}) = 1$. This also proves that $n((\Omega_0^{+-})^c) = 0$, which completes the proof.
Now we prove Corollary 1.4.

Proof of Corollary 1.4

Set

$$\Omega^+_\infty = \{ \exists t_0 > 0 \text{ such that } \forall t > t_0, X_t \geq 0 \},$$
$$\Omega^-\infty = \{ \exists t_0 > 0 \text{ such that } \forall t > t_0, X_t \leq 0 \}$$

and

$$\Omega^{+,-}_\infty = \{ \exists\{t_n\} \text{ with } t_n \not\to \infty \text{ such that } \forall n, X_{t_n}X_{t_{n+1}} < 0 \}.$$  

Then the space $\mathbb{D}$ is decomposed into the disjoint union:

$$\mathbb{D} = \Omega^+_\infty \cup \Omega^-\infty \cup \Omega^{+,-}_\infty.$$

Since we have

$$P_0^h (\forall t > 0, X_t \neq 0) = 1$$

by the local equivalence between $P_0^h$ and $n$, we see that

$$\Omega^+_\infty \cup \Omega^-\infty = \bigcup_{n=1}^{\infty} \{ \forall t > n, X_nX_t > 0 \} \text{ under } P_0^h.$$  

Suppose that the process $\{(X_t), (P_x)\}$ is a symmetric stable process of index $1 < \alpha < 2$. Then, from the original process $\{(X_t), (P_x)\}$, its $h$-path process $\{(X_t), (P_0^h)\}$ inherits the scaling property: for any fixed $c > 0$,

$$\left(c^{-1/\alpha}X_{ct} : t \geq 0\right) \overset{\text{law}}{=} (X_t : t \geq 0) \text{ under } P_0^h.$$  

This implies that the probability

$$P_0^h (\forall t > s, X_sX_t > 0)$$

for fixed $s > 0$ does not depend on the choice of $s > 0$. Hence we obtain

$$P_0^h (\Omega^+_\infty \cup \Omega^-\infty) = \lim_{n \to \infty} P_0^h (\forall t > n, X_nX_t > 0)$$
$$= \lim_{n \to \infty} P_0^h (\forall t > 1/n, X_{1/n}X_t > 0)$$
$$= \lim_{n \to \infty} P_0^h (X_t \text{ have the same sign for all } t > 0),$$

which proves to be zero by Corollary 1.3. Hence we conclude that $P_0^h(\Omega^{+,-}_\infty) = 1$. By the transience of the $h$-path process $\{(X_t), P_0^h\}$ (Theorem 1.4), we have

$$\Omega^{+,-}_\infty = \left\{ \limsup_{t \to \infty} X_t = \limsup_{t \to \infty} (-X_t) = \infty \right\} \text{ under } P_0^h.$$  

Therefore the proof is complete.  

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7.4 Remark on a connection with a result of Ikeda–Watanabe

Finally we make a remark on a connection with a result of Ikeda–Watanabe [17]. Set

\[ \Omega_i^{+,-} = \{ \exists \{ t_n \} \text{ with } t_n \not\in T_0 \text{ such that } \forall n, X_{t_n}X_{t_{n+1}} < 0 \} . \]

**Theorem 7.1** (Theorem 3.3 of [17]). Suppose that, for any fixed \( q > 0 \),

\[
0 < \lim \inf_{\varepsilon \to 0^+} \frac{u_q(0) - u_q(-\varepsilon)}{u_q(0) - u_q(\varepsilon)} \leq \lim \sup_{\varepsilon \to 0^+} \frac{u_q(0) - u_q(-\varepsilon)}{u_q(0) - u_q(\varepsilon)} < \infty \tag{7.5}
\]

and that

\[
\lim_{\varepsilon \to 0} \frac{u_q(x) - u_q(x + \varepsilon)}{u_q(0) - u_q(\varepsilon)} = 0, \quad x \in \mathbb{R} \setminus \{0\}. \tag{7.6}
\]

Then it holds that

\[
P_x(\Omega_i^{+,-} | T_0 < \infty) = 1, \quad x \in \mathbb{R} \setminus \{0\}. \tag{7.7}
\]

Suppose that the conditions (A), (B) and (T) are satisfied. Then, by the symmetry \( u_q(x) = u_q(-x) \), we see that the assumption (7.5) is satisfied. Since

\[
\frac{u_q(x) - u_q(x + \varepsilon)}{u_q(0) - u_q(\varepsilon)} = \varepsilon \cdot \frac{h(\varepsilon)}{h_q(\varepsilon)} \cdot \frac{h_q(x + \varepsilon) - h_q(x)}{\varepsilon},
\]

we see, by Lemma 4.4, (i) of Lemma 6.2 and the assumption (B), that the assumption (7.6) is also satisfied. Hence we may apply Theorem 7.1 to obtain (7.7). The formula (7.7) implies that

\[
\mathbf{n}(\Omega_i^{+,-} \cap \{ \zeta < \infty \}) = 0. \tag{7.8}
\]

Through time reversal property (see [14, Lemma 4.1] and [6, Lemma 5.2]) of excursion paths with finite lifetime, the formula (7.3) implies that

\[
\mathbf{n}(\Omega_0^{+,-} \cap \{ \zeta < \infty \}) = 0.
\]

This is a special case of Corollary 1.2.

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