NEW GENERAL INTEGRAL INEQUALITIES FOR SOME
GA-CONVEX AND QUASI-GEOMETRICALLY CONVEX
FUNCTIONS VIA FRACTIONAL INTEGRALS

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Abstract. In this paper, the author introduces the concept of the quasi-
metrically convex and defines a new identity for fractional integrals. By us-
ning of this identity, author obtains new estimates on generalization of Hadamard,
Ostrowski and Simpson type inequalities for GA-s-convex, quasi-geometrically
convex and (s, m)-GA-convex functions via Riemann Liouville fractional in-
1. Introduction

Let real function \( f \) be defined on some nonempty interval \( I \) of real line \( \mathbb{R} \). The
function \( f \) is said to be convex on \( I \) if inequality

\[
f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y)
\]

holds for all \( x, y \in I \) and \( t \in [0, 1] \).

Following inequalities are well known in the literature as Hermite-Hadamard
inequality, Ostrowski inequality and Simpson inequality respectively:

Theorem 1. Let \( f : I \subseteq \mathbb{R} \rightarrow \mathbb{R} \) be a convex function defined on the interval \( I \) of
real numbers and \( a, b \in I \) with \( a < b \). The following double inequality holds

\[
f \left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}.
\]

Theorem 2. Let \( f : I \subseteq \mathbb{R} \rightarrow \mathbb{R} \) be a mapping differentiable in \( I^\circ \), the interior of
\( I \), and let \( a, b \in I^\circ \) with \( a < b \). If \( |f'(x)| \leq M \), \( x \in [a, b] \), then we the following
inequality holds

\[
\left| f(x) - \frac{1}{b - a} \int_a^b f(t)dt \right| \leq \frac{M}{b - a} \left[ \frac{(x - a)^2 + (b - x)^2}{2} \right]
\]

for all \( x \in [a, b] \). The constant \( \frac{1}{4} \) is the best possible in the sense that it cannot be
replaced by a smaller one.

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son type inequality, GA-s-convex function, quasi-geometrically convex function, (s, m)-GA-convex
function.
**Theorem 3.** Let \( f : [a, b] \to \mathbb{R} \) be a four times continuously differentiable mapping on \((a, b)\) and \( \|f^{(4)}\|_\infty = \sup_{x \in (a, b)} |f^{(4)}(x)| < \infty \). Then the following inequality holds:

\[
\left| \frac{1}{3} \left[ \frac{f(a) + f(b)}{2} + 2f\left(\frac{a + b}{2}\right) \right] - \frac{1}{b - a} \int_a^b f(x)dx \right| \leq \frac{1}{2880} \|f^{(4)}\|_\infty (b - a)^4.
\]

The following definitions are well known in the literature.

**Definition 1 (III III).** A function \( f : I \subseteq (0, \infty) \to \mathbb{R} \) is said to be GA-convex (geometric-arithmetically convex) if

\[ f(x^t y^{1-t}) \leq tf(x) + (1-t)f(y) \]

for all \( x, y \in I \) and \( t \in [0, 1] \).

**Definition 2 (21).** For \( s \in (0, 1] \), a function \( f : I \subseteq (0, \infty) \to \mathbb{R} \) is said to be GA-s-convex (geometric-arithmetically s-convex) if

\[ f(x^t y^{1-t}) \leq t^s f(x) + (1-t)^s f(y) \]

for all \( x, y \in I \) and \( t \in [0, 1] \).

It can be easily seen that for \( s = 1 \), GA-s-convexity reduces to GA-convexity.

**Definition 3 (3).** Let \( f : (0, b] \to \mathbb{R}, b > 0, \) and \( (s, m) \in (0, 1]^2 \). If

\[ f(x^t y^{m(1-t)}) \leq t^s f(x) + m(1-t)^s f(y) \]

for all \( x, y \in (0, b] \) and \( t \in [0, 1] \), then \( f \) is said to be a \((s, m)\)-GA-convex function.

**Definition 4 (III III).** A function \( f : I \subseteq (0, \infty) \to (0, \infty) \) is said to be GG-convex (called in [22] geometrically convex function) if

\[ f(x^t y^{1-t}) \leq f(x)^t f(y)^{1-t} \]

for all \( x, y \in I \) and \( t \in [0, 1] \).

We give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used throughout this paper.

**Definition 5.** Let \( f \in L[a, b] \). The Riemann-Liouville integrals \( J_{a+}^\alpha f \) and \( J_{b-}^\alpha f \) of order \( \alpha > 0 \) with \( a \geq 0 \) are defined by

\[
J_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \; x > a
\]

and

\[
J_{b-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \; x < b
\]

respectively, where \( \Gamma(\alpha) \) is the Gamma function defined by \( \Gamma(\alpha) = \int_0^\infty e^{-t}t^{\alpha-1}dt \) and \( J_{a+}^0 f(x) = J_{b-}^0 f(x) = f(x) \).
In the case of $\alpha = 1$, the fractional integral reduces to the classical integral. Properties concerning this operator can be found [4, 9, 14].

In recent years, many authors have studied errors estimations for Hermite-Hadamard, Ostrowski and Simpson inequalities; for refinements, counterparts, generalization see [1, 2, 3, 5, 6, 7, 12, 13, 15, 16, 17, 18, 19, 20]. In this paper, new identity for fractional integrals have been defined. By using of this identity, we obtained a generalization of Hadamard, Ostrowski and Simpson type inequalities for GA-$s$-convex, quasi-geometrically convex, $(s, m)$-convex functions via Riemann Liouville fractional integral.

2. Generalized integral inequalities for some GA-convex functions via fractional integrals

Let $f : I \subseteq (0, \infty) \to \mathbb{R}$ be a differentiable function on $I^\circ$, the interior of $I$, throughout this section we will take

$$I_f(x, \lambda, \alpha, a, b) = (1 - \lambda) \left[ \ln^\alpha \frac{x}{a} + \ln^\alpha \frac{b}{x} \right] f(x) + \lambda \left[ f(a) \ln^\alpha \frac{x}{a} + f(b) \ln^\alpha \frac{b}{x} \right]$$

$$- \Gamma(\alpha + 1) \left[ J^\alpha_{(\ln x)^-} (f \circ \exp) (\ln a) + J^\alpha_{(\ln x)^+} (f \circ \exp) (\ln b) \right]$$

where $a, b \in I$ with $a < b$, $x \in [a, b]$, $\lambda \in [0, 1]$, $\alpha > 0$ and $\Gamma$ is Euler Gamma function. In order to prove our main results we need the following identity.

**Theorem 4.** Let $f : I \subseteq (0, \infty) \to \mathbb{R}$ be a function such that $f \in L[a, b]$, where $a, b \in I$ with $a < b$. If $f$ is a GA-convex function on $[a, b]$, then the following inequalities for fractional integrals hold:

$$f \left( \sqrt{ab} \right) \leq \frac{\Gamma(\alpha + 1)}{2 \ln \frac{b}{a}} \left\{ J^\alpha_{(\ln a)^-} (f \circ \exp) (\ln b) + J^\alpha_{(\ln b)^+} (f \circ \exp) (\ln a) \right\} \leq \frac{f(a) + f(b)}{2}$$

with $\alpha > 0$.

**Proof.** Since $f$ is a GA-convex function on $[a, b]$, we have for all $x, y \in [a, b]$ (with $t = 1/2$ in the inequality (1.1))

$$f \left( \sqrt{xy} \right) \leq \frac{f(x) + f(y)}{2}.$$

Choosing $x = a^{t}b^{1-t}$, $y = b^{t}a^{1-t}$, we get

$$f \left( \sqrt{ab} \right) \leq \frac{f(a^{t}b^{1-t}) + f(b^{t}a^{1-t})}{2}$$

(2.2)
Multiplying both sides of (2.2) by $t^{\alpha-1}$, then integrating the resulting inequality with respect to $t$ over $[0,1]$, we obtain

$$f\left(\sqrt{ab}\right) \leq \frac{\alpha}{2} \left\{ \int_0^1 f(a^t b^{1-t}) dt + \int_0^1 f(b^t a^{1-t}) dt \right\}$$

$$= \frac{\alpha}{2} \left\{ \int_a^b \left( \frac{\ln b - \ln u}{\ln b - \ln a} \right)^{\alpha-1} f(u) \frac{du}{u \ln \frac{u}{a}} + \int_0^1 \left( \frac{\ln u - \ln a}{\ln b - \ln a} \right)^{\alpha-1} f(u) \frac{du}{u \ln \frac{u}{a}} \right\}$$

$$= \frac{\alpha}{2 (\ln \frac{b}{a})^\alpha} \left\{ \int_{\ln a}^{\ln b} (\ln b - t)^{\alpha-1} f(e^t) dt + \int_{\ln a}^{\ln b} (t - \ln a)^{\alpha-1} f(e^t) dt \right\}$$

$$= \frac{\Gamma(\alpha + 1)}{2 (\ln \frac{b}{a})^\alpha} \left\{ J_1^{\alpha}(f \circ \exp)(\ln b) + J_1^{\alpha}(f \circ \exp)(\ln a) \right\}$$

and the first inequality is proved.

For the proof of the second inequality in (2.1) we first note that if $f$ is a convex function, then, for $t \in [0,1]$, it yields

$$f(a^t b^{1-t}) \leq tf(a) + (1 - t)f(b)$$

and

$$f(b^t a^{1-t}) \leq tf(b) + (1 - t)f(a).$$

By adding these inequalities we have

$$f(a^t b^{1-t}) + f(b^t a^{1-t}) \leq f(a) + f(b).$$

Then multiplying both sides of (2.3) by $t^{\alpha-1}$, and integrating the resulting inequality with respect to $t$ over $[0,1]$, we obtain

$$\int_0^1 f(a^t b^{1-t}) t^{\alpha-1} dt + \int_0^1 f(b^t a^{1-t}) t^{\alpha-1} dt \leq [f(a) + f(b)] \int_0^1 t^{\alpha-1} dt$$

i.e.

$$\frac{\Gamma(\alpha + 1)}{(\ln \frac{b}{a})^\alpha} \left\{ J_1^{\alpha}(f \circ \exp)(\ln b) + J_1^{\alpha}(f \circ \exp)(\ln a) \right\} \leq f(a) + f(b).$$

The proof is completed. \hfill \Box

**Lemma 1.** Let $f : I \subseteq (0, \infty) \rightarrow \mathbb{R}$ be a differentiable function on $I^\circ$ such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. Then for all $x \in [a,b]$, $\lambda \in [0,1]$ and $\alpha > 0$ we have:

$$I_f(x, \lambda, \alpha, a, b) = a \left( \frac{x}{a} \right)^{\alpha+1} \int_0^1 (t^\alpha - \lambda) \left( \frac{x}{a} \right)^t f' \left( x^t a^{1-t} \right) dt$$

$$- b \left( \frac{b}{x} \right)^{\alpha+1} \int_0^1 (t^\alpha - \lambda) \left( \frac{x}{b} \right)^t f' \left( x^t b^{1-t} \right) dt.$$
Proof. By integration by parts and twice changing the variable, for \( x \neq a \) we can state

\[
(2.5) \quad a \ln \frac{x}{a} \int_0^1 (t^\alpha - \lambda) \left( \frac{x}{a} \right)^t f' \left( t^\alpha a^{1-t} \right) dt
\]

\[
= \int_0^1 (t^\alpha - \lambda) df \left( t^\alpha a^{1-t} \right)
\]

\[
= (t^\alpha - \lambda) f \left( t^\alpha a^{1-t} \right) \bigg|_0^1 - \frac{\alpha}{(\ln x/a)^\alpha} \int_a^x (\ln u - \ln a)^{\alpha-1} \frac{f(u)}{u} du
\]

\[
= (1 - \lambda) f(x) + \lambda f(a) - \frac{\alpha}{(\ln x/a)^\alpha} \int_{\ln a}^{\ln x} (t - \ln a)^{\alpha-1} f(e^t) dt
\]

\[
= (1 - \lambda) f(x) + \lambda f(a) - \frac{\Gamma (\alpha+1)}{(\ln x/a)^\alpha} f_{(\ln a)-}^a (f \circ \exp) (\ln a)
\]

and for \( x \neq b \) similarly we get

\[
(2.6) \quad -b \ln \frac{b}{x} \int_0^1 (t^\alpha - \lambda) \left( \frac{x}{b} \right)^t f' \left( t^\alpha b^{1-t} \right) dt
\]

\[
= \int_0^1 (t^\alpha - \lambda) df \left( t^\alpha b^{1-t} \right)
\]

\[
= (t^\alpha - \lambda) f \left( t^\alpha b^{1-t} \right) \bigg|_0^1 - \frac{\alpha}{(\ln x/b)^\alpha} \int_x^b (\ln b - \ln u)^{\alpha-1} \frac{f(u)}{u} du
\]

\[
= (1 - \lambda) f(x) + \lambda f(b) - \frac{\alpha}{(\ln x/b)^\alpha} \int_{\ln x}^{\ln b} (t - \ln b)^{\alpha-1} f(e^t) dt
\]

\[
= (1 - \lambda) f(x) + \lambda f(b) - \frac{\Gamma (\alpha+1)}{(\ln x/b)^\alpha} J_{(\ln x)+}^b (f \circ \exp) (\ln b)
\]

Multiplying both sides of (2.5) and (2.6) by \((\ln x/a)^\alpha\) and \((\ln x/b)^\alpha\), respectively, and adding the resulting identities we obtain the desired result. For \( x = a \) and \( x = b \) the identities

\[
I_f (a, \lambda, \alpha; a, b) = b \left( \ln \frac{b}{a} \right)^{\alpha+1} \int_0^1 (t^\alpha - \lambda) \left( \frac{a}{b} \right)^t f' \left( t^\alpha b^{1-t} \right) dt,
\]

and

\[
I_f (b, \lambda, \alpha; a, b) = a \left( \ln \frac{b}{a} \right)^{\alpha+1} \int_0^1 (t^\alpha - \lambda) \left( \frac{b}{a} \right)^t f' \left( t^\alpha a^{1-t} \right),
\]

can be proved respectively easily by performing an integration by parts in the integrals from the right side and changing the variable. \( \square \)
2.1. For GA-s-convex functions.

**Theorem 5.** Let $f : I \subset (0, \infty) \to \mathbb{R}$ be a differentiable function on $I^\circ$ such that $f' \in L[a, b]$, where $a, b \in I^\circ$ with $a < b$. If $|f'|^q$ is GA-s-convex on $[a, b]$ in the second sense for some fixed $q \geq 1$, $x \in [a, b]$, $\lambda \in [0, 1]$ and $\alpha > 0$ then the following inequality for fractional integrals holds

$$
|I_f (x, \alpha, a, b)| \\
\leq (2A_1)^{\frac{1}{q}} (\alpha, \lambda) \left\{ a \left( \ln \frac{x}{a} \right)^{\alpha + 1} \left( |f' (x)|^q A_2 (x, \alpha, \lambda, s, q) + |f' (a)|^q A_3 (x, \alpha, \lambda, s, q) \right)^{\frac{1}{q}} \\
+ b \left( \ln \frac{b}{x} \right)^{\alpha + 1} \left( |f' (x)|^q A_4 (x, \alpha, \lambda, s, q) + |f' (b)|^q A_5 (x, \alpha, \lambda, s, q) \right)^{\frac{1}{q}} \right\},
$$

where

$$
A_1 (\alpha, \lambda) = \frac{2\alpha \lambda^{\alpha + \frac{1}{q}} + 1}{\alpha + 1} - \lambda,
$$

$$
A_2 (x, \alpha, \lambda, s, q) = \int_0^1 |t^\alpha - \lambda| \left( \frac{x}{a} \right)^{qt} t^s dt,
$$

$$
A_3 (x, \alpha, \lambda, s, q) = \int_0^1 |t^\alpha - \lambda| \left( \frac{x}{a} \right)^{qt} (1 - t)^s dt,
$$

$$
A_4 (x, \alpha, \lambda, s, q) = \int_0^1 |t^\alpha - \lambda| \left( \frac{x}{b} \right)^{qt} t^s dt,
$$

$$
A_5 (x, \alpha, \lambda, s, q) = \int_0^1 |t^\alpha - \lambda| \left( \frac{x}{b} \right)^{qt} (1 - t)^s dt.
$$

**Proof.** From Lemma 1, property of the modulus and using the power-mean inequality we have

$$
|I_f (x, \alpha, a, b)| \leq a \left( \ln \frac{x}{a} \right)^{\alpha + 1} \int_0^1 |t^\alpha - \lambda| \left( \frac{x}{a} \right)^{qt} |f' (x^\alpha a^{-1 - t})| dt \\
+ b \left( \ln \frac{b}{x} \right)^{\alpha + 1} \int_0^1 |t^\alpha - \lambda| \left( \frac{x}{b} \right)^{qt} |f' (x^\alpha b^{1 - t})| dt
$$

$$
\leq a \left( \ln \frac{x}{a} \right)^{\alpha + 1} \left( \int_0^1 |t^\alpha - \lambda| dt \right)^{1 - \frac{1}{q}} \left( \int_0^1 |t^\alpha - \lambda| \left( \frac{x}{a} \right)^{qt} |f' (x^\alpha a^{-1 - t})|^q dt \right)^{\frac{1}{q}} \\
+ b \left( \ln \frac{b}{x} \right)^{\alpha + 1} \left( \int_0^1 |t^\alpha - \lambda| dt \right)^{1 - \frac{1}{q}} \left( \int_0^1 |t^\alpha - \lambda| \left( \frac{x}{b} \right)^{qt} |f' (x^\alpha b^{1 - t})|^q dt \right)^{\frac{1}{q}}.
$$

(2.8)
Since $|f'|^q$ is GA-$s$-convex on $[a, b]$ we get
\[
\int_0^1 |t^\alpha - \lambda| \left( \frac{x}{a} \right)^{qt} |f'(x^t a^{1-t})|^q \, dt \leq \int_0^1 |t^\alpha - \lambda| \left( \frac{x}{a} \right)^{qt} (t^s |f'(x)|^q + (1-t)^s |f'(a)|^q) \, dt
\]
(2.9)
\[
= |f'(x)|^q A_2 (x, \alpha, \lambda, s, q) + |f'(a)|^q A_3 (x, \alpha, \lambda, s, q),
\]
\[
\int_0^1 |t^\alpha - \lambda| \left( \frac{x}{b} \right)^{qt} |f'(x^t b^{1-t})|^q \, dt \leq \int_0^1 |t^\alpha - \lambda| \left( \frac{x}{b} \right)^{qt} (t^s |f'(x)|^q + (1-t)^s |f'(b)|^q) \, dt
\]
(2.10)
\[
= |f'(x)|^q A_4 (x, \alpha, \lambda, s, q) + |f'(b)|^q A_5 (x, \alpha, \lambda, s, q),
\]
by simple computation
\[
\int_0^1 |t^\alpha - \lambda| \, dt = \int_0^1 (\lambda - t^\alpha) \, dt + \int_0^1 (t^\alpha - \lambda) \, dt
\]
(2.11)
\[
= \frac{2\alpha \lambda^{1+\frac{1}{q}} + 1}{\alpha + 1} - \lambda.
\]
Hence, if we use (2.9), (2.10) and (2.11) in (2.8), we obtain the desired result. This completes the proof. \(\square\)

**Corollary 1.** Under the assumptions of Theorem 4 with $s = 1$, the inequality (2.7) reduced to the following inequality
\[
|I_f(x, \lambda, \alpha, a, b)| \leq A_1^{-\frac{1}{q}} (\alpha, \lambda) \left\{ a \left( \ln \frac{x}{a} \right)^{\alpha+1} \left( |f'(x)|^q A_2 (x, \alpha, \lambda, 1, q) + |f'(a)|^q A_3 (x, \alpha, \lambda, 1, q) \right)^{\frac{1}{q}} + b \left( \ln \frac{b}{x} \right)^{\alpha+1} \left( |f'(x)|^q A_4 (x, \alpha, \lambda, 1, q) + |f'(b)|^q A_5 (x, \alpha, \lambda, 1, q) \right)^{\frac{1}{q}} \right\}.
\]

**Corollary 2.** Under the assumptions of Theorem 5 with $s = 1$ and $\alpha = 1$, the inequality (2.7) reduced to the following inequality
\[
|I_f(x, \lambda, \alpha, a, b)| = \ln \frac{b}{a} (1-\lambda) f(x) + \lambda \left[ f(a) \ln \frac{x}{a} + f(b) \ln \frac{b}{x} \right] - \alpha \int_a^b \frac{f(u)}{u} \, du
\]
\[
\leq A_1^{-\frac{1}{q}} (1, \lambda) \left\{ a \left( \ln \frac{x}{a} \right)^2 \left( |f'(x)|^q A_2 (x, 1, \lambda, 1, q) + |f'(a)|^q A_3 (x, 1, \lambda, 1, q) \right)^{\frac{1}{q}} + b \left( \ln \frac{b}{x} \right)^2 \left( |f'(x)|^q A_4 (x, 1, \lambda, 1, q) + |f'(b)|^q A_5 (x, 1, \lambda, 1, q) \right)^{\frac{1}{q}} \right\}.
\]

**Corollary 3.** Under the assumptions of Theorem 6 with $q = 1$, the inequality (2.7) reduced to the following inequality
\[
|I_f(x, \lambda, \alpha, a, b)| \leq \left\{ a \left( \ln \frac{x}{a} \right)^{\alpha+1} \left( |f'(x)| A_2 (x, \alpha, \lambda, s, 1) + |f'(a)| A_3 (x, \alpha, \lambda, s, 1) \right) + b \left( \ln \frac{b}{x} \right)^{\alpha+1} \left( |f'(x)| A_4 (x, \alpha, \lambda, s, 1) + |f'(b)| A_5 (x, \alpha, \lambda, s, 1) \right) \right\}.
\]
Corollary 4. Under the assumptions of Theorem 3 with \( x = \sqrt{a}b \), \( \lambda = \frac{1}{3} \), from the inequality (2.7) we get the following Simpson type inequality for fractional integrals

\[
\left| 2^{\alpha - 1} \left( \ln \frac{b}{a} \right)^{-\alpha} \int f \left( \sqrt{ab}, \frac{1}{3}, \alpha, a, b \right) \right| \\
= \frac{1}{6} \left[ f(a) + 4f \left( \sqrt{ab} \right) + f(b) \right] - \frac{2^{\alpha - 1} \Gamma (\alpha + 1)}{(\ln \frac{b}{a})^\alpha} \left[ J^\alpha_{(\ln \sqrt{ab})} - (f \circ \exp) (\ln a) + J^\alpha_{(\ln \sqrt{ab})} (f \circ \exp) (\ln b) \right] \\
\leq \frac{\ln \frac{b}{a}}{4} A_1^{1-\frac{a}{\alpha}} \left( \alpha, \frac{1}{3} \right) \left\{ \left[ f' \left( \sqrt{ab} \right) \right]^q A_2 \left( \sqrt{ab}, \frac{1}{3}, s, q \right) + |f'(a)|^q A_3 \left( \sqrt{ab}, \frac{1}{3}, s, q \right) \right\}^\frac{1}{q} \\
+ b \left[ f' \left( \sqrt{ab} \right) \right]^q A_4 \left( \sqrt{ab}, \frac{1}{3}, s, q \right) + |f'(b)|^q A_5 \left( \sqrt{ab}, \frac{1}{3}, s, q \right) \right\}^\frac{1}{q}.
\]

Corollary 5. Under the assumptions of Theorem 3 with \( x = \sqrt{a}b \), \( \lambda = 0 \), from the inequality (2.7) we get

\[
\left| 2^{\alpha - 1} \left( \ln \frac{b}{a} \right)^{-\alpha} \int f \left( \sqrt{ab}, 0, \alpha, a, b \right) \right| \\
= \left[ f \left( \sqrt{ab} \right) - \frac{2^{\alpha - 1} \Gamma (\alpha + 1)}{(\ln \frac{b}{a})^\alpha} \left[ J^\alpha_{(\ln \sqrt{ab})} - (f \circ \exp) (\ln a) + J^\alpha_{(\ln \sqrt{ab})} (f \circ \exp) (\ln b) \right] \right] \\
\leq \frac{\ln \frac{b}{a}}{4} \left( \frac{1}{\alpha + 1} \right)^{1-\frac{a}{\alpha}} \left\{ a \left[ f' \left( \sqrt{ab} \right) \right]^q A_2 \left( \sqrt{ab}, 0, s, q \right) + |f'(a)|^q A_3 \left( \sqrt{ab}, 0, s, q \right) \right\}^\frac{1}{q} \\
+ b \left[ f' \left( \sqrt{ab} \right) \right]^q A_4 \left( \sqrt{ab}, 0, s, q \right) + |f'(b)|^q A_5 \left( \sqrt{ab}, 0, s, q \right) \right\}^\frac{1}{q}.
\]

Corollary 6. Under the assumptions of Theorem 3 with \( x = \sqrt{a}b \) and \( \lambda = 1 \), from the inequality (2.7) we get

\[
\left| 2^{\alpha - 1} \left( \ln \frac{b}{a} \right)^{-\alpha} \int f \left( \sqrt{ab}, 1, \alpha, a, b \right) \right| \\
= \frac{f(a) + f(b)}{2} - \frac{2^{\alpha - 1} \Gamma (\alpha + 1)}{\left( \ln \frac{b}{a} \right)^\alpha} \left[ J^\alpha_{(\ln \sqrt{ab})} - (f \circ \exp) (\ln a) + J^\alpha_{(\ln \sqrt{ab})} (f \circ \exp) (\ln b) \right] \\
\leq \frac{\ln \frac{b}{a}}{4} \left( \frac{\alpha}{\alpha + 1} \right)^{1-\frac{a}{\alpha}} \left\{ a \left[ f' \left( \sqrt{ab} \right) \right]^q A_2 \left( \sqrt{ab}, 1, s, q \right) + |f'(a)|^q A_3 \left( \sqrt{ab}, 1, s, q \right) \right\}^\frac{1}{q} \\
+ b \left[ f' \left( \sqrt{ab} \right) \right]^q A_4 \left( \sqrt{ab}, 1, s, q \right) + |f'(b)|^q A_5 \left( \sqrt{ab}, 1, s, q \right) \right\}^\frac{1}{q}.
\]

Corollary 7. Let the assumptions of Theorem 3 hold. If \(|f'(x)| \leq M\) for all \( x \in [a, b] \) and \( \lambda = 0 \), then from the inequality (2.7) we get the following Ostrowski
type inequality for fractional integrals
\[
\left| \left( \ln \frac{x}{a} \right)^\alpha + \left( \ln \frac{b}{x} \right)^\alpha \right| f(x) - \Gamma(\alpha + 1) \left[ J_{(ln x)}^\alpha (f \circ \exp) (\ln a) + J_{(ln x)}^\alpha (f \circ \exp) (\ln b) \right]
\leq M \left( \frac{1}{\alpha + 1} \right)^{1 - \frac{1}{q}} \left\{ a \left( \ln \frac{x}{a} \right)^\alpha [A_2(x, \alpha, 0, s, q) + A_3(x, \alpha, 0, s, q)]^{\frac{1}{q}} + b \left( \ln \frac{b}{x} \right)^\alpha [A_4(x, \alpha, 0, s, q) + A_5(x, \alpha, 0, s, q)]^{\frac{1}{q}} \right\}
\]
for each \( x \in [a, b] \).

2.2. For quasi-geometrically convex functions.

**Definition 6.** A function \( f : I \subseteq (0, \infty) \rightarrow \mathbb{R} \) is said to be quasi-geometrically convex on \( I \) if
\[
f\left( x^t y^{1-t} \right) \leq \sup \{ f(x), f(y) \},
\]
for any \( x, y \in I \) and \( t \in [0, 1] \).

Clearly, any GA-convex and geometrically convex functions are quasi-geometrically convex functions. Furthermore, there exist quasi-convex functions which are neither GA-convex nor geometrically convex. In that context, we point out an elementary example. The function \( f : (0, 4] \rightarrow \mathbb{R} \),
\[
f(x) = \begin{cases} 
1, & x \in (0, 1] \\
(x - 2)^2, & x \in [1, 4]
\end{cases}
\]
is neither GA-convex nor geometrically convex on \((0, 4]\) but it is a quasi-geometrically convex function on \((0, 4]\).

**Theorem 6.** Let \( f : I \subseteq (0, \infty) \rightarrow \mathbb{R} \) be a differentiable function on \( I^\circ \) such that \( f' \in L[a, b] \), where \( a, b \in I^\circ \) with \( a < b \). If \( |f'|^q \) is quasi-geometrically convex on \([a, b]\) for some fixed \( q \geq 1 \), \( x \in [a, b] \), \( \lambda \in [0, 1] \) and \( \alpha > 0 \) then the following inequality for fractional integrals holds
\[
|I_f(x, \lambda, \alpha, a, b)| \leq \left( A_1^{-\frac{1}{q}} (\alpha, \lambda) \left\{ a \left( \ln \frac{x}{a} \right)^{\alpha + 1} (\sup \{|f'(x)|^q, |f'(a)|^q\})^{\frac{1}{q}} B_1^\frac{1}{q} (x, \alpha, \lambda, s) + b \left( \ln \frac{b}{x} \right)^{\alpha + 1} (\sup \{|f'(x)|^q, |f'(b)|^q\})^{\frac{1}{q}} B_2^\frac{1}{q} (x, \alpha, \lambda, s) \right\} \right)
\]
where
\[
A_1(\alpha, \lambda) = \frac{2\alpha \lambda^{1+\frac{1}{q}} + 1}{\alpha + 1} - \lambda,
\]
\[
B_1(x, \alpha, \lambda, q) = \int_0^1 |t^\alpha - \lambda| \left( \frac{x}{a} \right)^{qt} dt
\]
\[
B_2(x, \alpha, \lambda, q) = \int_0^1 |t^\alpha - \lambda| \left( \frac{x}{b} \right)^{qt} dt
\]
\textbf{Proof.} We proceed similarly as in the proof Theorem 5. Since $|f'|^q$ is quasi-geometrically convex on $[a, b]$, for all $t \in [0, 1]$

$$|f' (x^t a^{1-t})|^q \leq \sup \{|f' (x)|^q, |f' (a)|^q\}$$

and

$$|f' (x^t b^{1-t})|^q \leq \sup \{|f' (x)|^q, |f' (b)|^q\}.$$ 

Hence, from the inequality (2.8) we get

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we proceed similarly as in the proof Theorem 6. Since $f'$ is quasi-geometrically convex on $[a, b]$, for all $t \in [0, 1]$

$$|f' (x^t a^{1-t})|^q \leq \sup \{|f' (x)|^q, |f' (a)|^q\}$$

and

$$|f' (x^t b^{1-t})|^q \leq \sup \{|f' (x)|^q, |f' (b)|^q\}.$$ 

Hence, from the inequality (2.8) we get

$$\begin{align*}
\leq \ a \left( \ln \frac{x}{a} \right)^{\alpha + 1} \left( \int_0^1 |t^\alpha - \lambda| \, dt \right)^{1 - \frac{q}{\alpha}} 
& \quad \left( \int_0^1 |t^\alpha - \lambda| \left( \frac{x}{a} \right)^q \sup \{ |f' (x)|^q, |f' (a)|^q \} \, dt \right)^{\frac{q}{\alpha}} 
+ \ b \left( \ln \frac{b}{x} \right)^{\alpha + 1} \left( \int_0^1 |t^\alpha - \lambda| \left( \frac{x}{b} \right)^q \sup \{ |f' (x)|^q, |f' (b)|^q \} \, dt \right)^{\frac{q}{\alpha}} 
\end{align*}$$

$$|I_f (x, \lambda, \alpha, a, b)| \leq \left( \int_0^1 |t^\alpha - \lambda| \, dt \right)^{1 - \frac{q}{\alpha}}$$

$$\times \left\{ \ a \left( \ln \frac{x}{a} \right)^{\alpha + 1} \left( \sup \{ |f' (x)|^q, |f' (a)|^q \} \right)^{\frac{q}{\alpha}} \left( \int_0^1 |t^\alpha - \lambda| \left( \frac{x}{a} \right)^q \, dt \right)^{\frac{q}{\alpha}} 
+ \ b \left( \ln \frac{b}{x} \right)^{\alpha + 1} \left( \sup \{ |f' (x)|^q, |f' (b)|^q \} \right)^{\frac{q}{\alpha}} \left( \int_0^1 |t^\alpha - \lambda| \left( \frac{x}{b} \right)^q \, dt \right)^{\frac{q}{\alpha}} \right\}$$

$$\leq \ A_1^{\alpha + 1} (x, \lambda) \left\{ a \left( \ln \frac{x}{a} \right)^{\alpha + 1} \left( \sup \{ |f' (x)|^q, |f' (a)|^q \} \right)^{\frac{q}{\alpha}} B_1^\frac{q}{\alpha} (x, \alpha, \lambda, q) 
+ \ b \left( \ln \frac{b}{x} \right)^{\alpha + 1} \left( \sup \{ |f' (x)|^q, |f' (b)|^q \} \right)^{\frac{q}{\alpha}} B_2^\frac{q}{\alpha} (x, \alpha, \lambda, q) \right\}$$

which completes the proof.

\textbf{Corollary 8.} Under the assumptions of Theorem 14 with $q = 1$, the inequality (2.12) reduced to the following inequality

$$|I_f (x, \lambda, \alpha, a, b)| \leq \left\{ \ a \left( \ln \frac{x}{a} \right)^{\alpha + 1} B_1 (x, \alpha, \lambda, 1) \sup \{ |f' (x)|, |f' (a)| \} 
+ \ b \left( \ln \frac{b}{x} \right)^{\alpha + 1} B_2 (x, \alpha, \lambda, 1) \sup \{ |f' (x)|, |f' (b)| \} \right\}.$$
Corollary 9. Under the assumptions of Theorem 6 with \( x = \sqrt{ab} \), \( \lambda = \frac{1}{3} \), from the inequality (2.12) we get the following Simpson type inequality or fractional integrals

\[
\left| \frac{1}{6} \left[ f(a) + 4f\left(\sqrt{ab}\right) + f(b) \right] - \frac{2^{\alpha-1}\Gamma(\alpha + 1)}{(\ln \frac{b}{a})^\alpha} \left[ J_\alpha^a (f \circ \exp) (\ln a) + J_\alpha^b (f \circ \exp) (\ln b) \right] \right|
\leq \frac{\ln \frac{b}{a}}{4} \left\{ \left( \frac{1}{\alpha + 1} \right)^{1 - \frac{\alpha}{3}} \left[ a \left[ \sup \left\{ \left| f' \left( \sqrt{ab} \right) \right|, \left| f' (a) \right| \right] \right] + b \left[ \sup \left\{ \left| f' \left( \sqrt{ab} \right) \right|, \left| f' (b) \right| \right] \right] \right\}^{\frac{1}{3}} \left( \sqrt{ab}, \frac{1}{3}, q \right)
\]

Corollary 10. Under the assumptions of Theorem 6 with \( x = \sqrt{ab} \), \( \lambda = 0 \), from the inequality (2.12) we get the following midpoint type inequality or fractional integrals

\[
\left| f\left(\sqrt{ab}\right) - \frac{2^{\alpha-1}\Gamma(\alpha + 1)}{(\ln \frac{b}{a})^\alpha} \left[ J_\alpha^a (f \circ \exp) (\ln a) + J_\alpha^b (f \circ \exp) (\ln b) \right] \right|
\leq \frac{\ln \frac{b}{a}}{4} \left\{ \left( \frac{1}{\alpha + 1} \right)^{1 - \frac{\alpha}{3}} \left[ a \left[ \sup \left\{ \left| f' \left( \sqrt{ab} \right) \right|, \left| f' (a) \right| \right] \right] + b \left[ \sup \left\{ \left| f' \left( \sqrt{ab} \right) \right|, \left| f' (b) \right| \right] \right] \right\}^{\frac{1}{3}} \left( \sqrt{ab}, \frac{1}{3}, q \right)
\]

Corollary 11. Under the assumptions of Theorem 6 with \( x = \sqrt{ab} \), \( \lambda = 1 \), from the inequality (2.12) we get the following trapezoid type inequality or fractional integrals

\[
\left| \frac{f(a) + f(b)}{2} - \frac{2^{\alpha-1}\Gamma(\alpha + 1)}{(\ln \frac{b}{a})^\alpha} \left[ J_\alpha^a (f \circ \exp) (\ln a) + J_\alpha^b (f \circ \exp) (\ln b) \right] \right|
\leq \frac{\ln \frac{b}{a}}{4} \left\{ \left( \frac{1}{\alpha + 1} \right)^{1 - \frac{\alpha}{3}} \left[ a \left[ \sup \left\{ \left| f' \left( \sqrt{ab} \right) \right|, \left| f' (a) \right| \right] \right] + b \left[ \sup \left\{ \left| f' \left( \sqrt{ab} \right) \right|, \left| f' (b) \right| \right] \right] \right\}^{\frac{1}{3}} \left( \sqrt{ab}, 1, q \right)
\]

\times \left\{ \left[ a \left[ \sup \left\{ \left| f' \left( \sqrt{ab} \right) \right|, \left| f' (a) \right| \right] \right] + b \left[ \sup \left\{ \left| f' \left( \sqrt{ab} \right) \right|, \left| f' (b) \right| \right] \right] \right\}^{\frac{1}{3}} \left( \sqrt{ab}, x, 0, q \right)
\]

Corollary 12. Let the assumptions of Theorem 6 hold. If \( |f'(x)| \leq M \) for all \( x \in [a, b] \) and \( \lambda = 0 \), then from the inequality (2.12) we get the following Ostrowski type inequality or fractional integrals

\[
\left| \left( \left( \frac{\ln \frac{b}{a}}{a} \right)^\alpha + \left( \frac{\ln \frac{b}{x}}{x} \right)^\alpha \right) f(x) - \Gamma(\alpha + 1) \left[ J_\alpha^a (f \circ \exp) (\ln a) + J_\alpha^b (f \circ \exp) (\ln b) \right] \right|
\leq \frac{M}{(\alpha + 1)^{1 - \frac{\alpha}{3}}} \left[ a \left( \frac{\ln \frac{b}{a}}{a} \right)^{\alpha + 1} B_{1/3}^1 (x, x, 0, q) + b \left( \frac{\ln \frac{b}{x}}{x} \right)^{\alpha + 1} B_{2/3}^2 (x, x, 0, q) \right]
\]

2.3. For \((s, m)\)-convex functions. Similarly lemma \( \Pi \) can we proved the following lemma.

Lemma 2. Let \( f : I \subseteq (0, \infty) \rightarrow \mathbb{R} \) be a differentiable function on \( I^c \) such that \( f' \in L[a^m, b^m] \), where \( a^m, b \in I \) with \( a < b \). Then for all \( x \in [a, b], \lambda \in [0, 1] \) and
of the modulus and using the power-mean inequality we have

\[ I_f(x^m, \lambda, \alpha, a^m, b^m) = m^{\alpha+1} a^m \left( \ln \frac{a}{x} \right)^{\alpha+1} \int_0^1 (t^\alpha - \lambda) \left( \frac{x}{t^\alpha} \right)^{mt} f' \left( x^{mt} a^m(1-t) \right) dt \]

\[ + m^{\alpha+1} b^m \left( \ln \frac{b}{x} \right)^{\alpha+1} \int_0^1 (t^\alpha - \lambda) \left( \frac{x}{t^\alpha} \right)^{mt} f' \left( x^{mt} b^m(1-t) \right) dt. \]

**Theorem 7.** Let \( f : I \subseteq (0, \infty) \to \mathbb{R} \) be a differentiable function on \( I^\circ \) such that \( f' \in L[a^m, b^m] \), where \( a^m, b \in I^\circ \) with \( a < b \). If \( |f'|^q \) is \((s, m)\)-convex on \([a^m, b]\) for some fixed \( q \geq 1, x \in [a, b], \lambda \in [0, 1] \) and \( \alpha > 0 \) then the following inequality for fractional integrals holds

\[ |I_f(x^m, \lambda, \alpha, a^m, b^m)| \leq m^{\alpha+1} A_1(\alpha, \lambda)^{1-\frac{1}{q}} \]

\[ \times \left\{ a^m \left( \ln \frac{x}{a} \right)^{\alpha+1} \left( |f'(x^m)|^q C_1(x, \alpha, \lambda, q, m, s) + m |f'(a)|^q C_2(x, \alpha, \lambda, q, m, s) \right)^{\frac{1}{q}} \]

\[ + b^m \left( \ln \frac{x}{b} \right)^{\alpha+1} \left( |f'(x^m)|^q C_3(x, \alpha, \lambda, q, m, s) + m |f'(b)|^q C_4(x, \alpha, \lambda, q, m, s) \right)^{\frac{1}{q}} \]

where

\[ A_1(\alpha, \lambda) = \frac{2\alpha^{1+\frac{1}{q}} + 1}{\alpha + 1} - \lambda, \]

\[ C_1(x, \alpha, \lambda, q, m, s) = \int_0^1 |t^\alpha - \lambda| \left( \frac{x}{t^\alpha} \right)^{qmt} t^s dt, \]

\[ C_2(x, \alpha, \lambda, q, m, s) = \int_0^1 |t^\alpha - \lambda| \left( \frac{x}{t^\alpha} \right)^{qmt} (1 - t^s) dt, \]

\[ C_3(x, \alpha, \lambda, q, m, s) = \int_0^1 |t^\alpha - \lambda| \left( \frac{x}{b} \right)^{qmt} t^s dt, \]

\[ C_4(x, \alpha, \lambda, q, m, s) = \int_0^1 |t^\alpha - \lambda| \left( \frac{x}{b} \right)^{qmt} (1 - t^s) dt. \]

**Proof.** We proceed similarly as in the proof Theorem 5. From Lemma 2 property of the modulus and using the power-mean inequality we have

\[ I_f(x^m, \lambda, \alpha, a^m, b^m) = m^{\alpha+1} a^m \left( \ln \frac{a}{x} \right)^{\alpha+1} \int_0^1 (t^\alpha - \lambda) \left( \frac{x}{t^\alpha} \right)^{mt} f' \left( x^{mt} a^m(1-t) \right) dt \]

\[ + m^{\alpha+1} b^m \left( \ln \frac{b}{x} \right)^{\alpha+1} \int_0^1 (t^\alpha - \lambda) \left( \frac{x}{t^\alpha} \right)^{mt} f' \left( x^{mt} b^m(1-t) \right) dt. \]

\[ |I_f(x^m, \lambda, \alpha, a^m, b^m)| \leq m^{\alpha+1} \left( \int_0^1 |t^\alpha - \lambda| dt \right)^{1-\frac{1}{q}} \]
Corollary 14. Under the assumptions of Theorem 7 with (2.13) reduced to the following inequality (2.16)

\[ \int_0^1 |t^\alpha - \lambda| \left( \frac{\ln x}{a} \right)^{qmt} \left| f' \left( x^{mt} a^{m(1-t)} \right) \right|^q dt \]

Since \( |f'|^q \) is \((s, m)\)-convex on \([a^m, b^m]\), for all \( t \in [0, 1] \)

(2.15) \[ |f' \left( x^{mt} a^{m(1-t)} \right)|^q \leq t^s |f'(x^m)|^q + m (1-t^s) |f'(a)|^q, \]

(2.16) \[ |f' \left( x^{mt} b^{m(1-t)} \right)|^q \leq t^s |f'(x^m)|^q + m (1-t^s) |f'(b)|^q. \]

If we use (2.14), (2.16) and (2.11) in (2.14), we obtain the desired result. This completes the proof. \(\square\)

Corollary 13. Under the assumptions of Theorem 7 with \(q = 1\), the inequality (2.13) reduced to the following inequality

\[ I_f(x^m, \lambda, \alpha, a^m, b^m) \leq m^{\alpha+1} \]

\[ \times \left\{ a^m \left( \ln \frac{x}{a} \right)^{\alpha+1} \left( |f'(x^m)| C_1(x, \alpha, 1, m, s) + m |f'(a)| C_2(x, \alpha, 1, m, s) \right) + b^m \left( \ln \frac{b}{x} \right)^{\alpha+1} \left( |f'(x^m)| C_3(x, \alpha, 1, m, s) + m |f'(b)| C_4(x, \alpha, 1, m, s) \right) \right\} \]

Corollary 14. Under the assumptions of Theorem 7 with \( x = \sqrt{ab}, \lambda = \frac{1}{3} \) from the inequality (2.13) we get the following Simpson type inequality or fractional integrals

\[ 2^{\alpha-1} \left( m \ln \frac{b}{a} \right)^{-\alpha} I_f \left( \left( \sqrt{ab} \right)^m, \frac{1}{3}, \alpha, a^m, b^m \right) \]

\[ = \frac{1}{6} \left[ f(a^m) + 4 f \left( \left( \sqrt{ab} \right)^m \right) + f(b^m) \right] - \frac{2^{\alpha-1} \Gamma(\alpha + 1)}{(m \ln \frac{b}{a})^\alpha} \]

\[ \times \left[ J^m_{(m \ln \sqrt{ab})} (f \circ \exp) (m \ln a) + J^m_{(m \ln \sqrt{ab})} (f \circ \exp) (m \ln b) \right] \]

\[ \leq \frac{m \ln \frac{b}{a}}{4} A_1^{\frac{1}{\alpha}} \left( \alpha, \frac{1}{3} \right) \]

\[ \times \left\{ \left( \left| f' \left( \left( \sqrt{ab} \right)^m \right) \right|^q \right| C_1 \left( \sqrt{ab}, \alpha, \frac{1}{3}, q, m, s \right) + m |f'(a)|^q C_2 \left( \sqrt{ab}, \alpha, \frac{1}{3}, q, m, s \right) \right)^{\frac{1}{\alpha}} \]

\[ + \left( \left| f' \left( \left( \sqrt{ab} \right)^m \right) \right|^q \right| C_3 \left( \sqrt{ab}, \alpha, \frac{1}{3}, q, m, s \right) + m |f'(b)|^q C_4 \left( \sqrt{ab}, \alpha, \frac{1}{3}, q, m, s \right) \right)^{\frac{1}{\alpha}} \right\}. \]
Corollary 15. Under the assumptions of Theorem 7 with $x = \sqrt{ab}$, $\lambda = 0$, from the inequality (2.13) we get the following inequality
\[
2^{\alpha-1} \left( m \ln \frac{b}{a} \right)^{-\alpha} I_f \left( \left( \sqrt{ab} \right)^m , 0, \alpha, a^m, b^m \right) \\
= \left| f \left( \left( \sqrt{ab} \right)^m \right) - \frac{2^{\alpha-1} \Gamma (\alpha + 1)}{\left( m \ln \frac{b}{a} \right)^\alpha} \left( J^a_{(m \ln \sqrt{ab})} - (f \circ \exp) (m \ln a) + J^a_{(m \ln \sqrt{ab})} - (f \circ \exp) (m \ln b) \right) \right| \\
\leq \frac{m \ln \frac{b}{a}}{4} \left( \frac{1}{\alpha+1} \right)^{\frac{1}{\alpha}} \left\{ \left| f' \left( \left( \sqrt{ab} \right)^m \right) \right|^q C_1 \left( \sqrt{ab}, \alpha, 0, q, m, s \right) + m \left| f' \left( a \right) \right|^q C_2 \left( \sqrt{ab}, \alpha, 0, q, m, s \right) \right\}^{\frac{1}{q}} \\
+ \left| f' \left( \left( \sqrt{ab} \right)^m \right) \right|^q C_3 \left( \sqrt{ab}, \alpha, 0, q, m, s \right) + m \left| f' \left( b \right) \right|^q C_4 \left( \sqrt{ab}, \alpha, 1, q, m, s \right) \right\}^{\frac{1}{q}}.
\]

Corollary 16. Under the assumptions of Theorem 7 with $x = \sqrt{ab}$, $\lambda = 1$, from the inequality (2.13) we get the following inequality
\[
2^{\alpha-1} \left( m \ln \frac{b}{a} \right)^{-\alpha} I_f \left( \left( \sqrt{ab} \right)^m , 1, \alpha, a^m, b^m \right) \\
= \left| \frac{f(a^m) + f(b^m)}{2} - \frac{2^{\alpha-1} \Gamma (\alpha + 1)}{\left( m \ln \frac{b}{a} \right)^\alpha} \left( J^a_{(m \ln \sqrt{ab})} - (f \circ \exp) (m \ln a) + J^a_{(m \ln \sqrt{ab})} - (f \circ \exp) (m \ln b) \right) \right| \\
\leq \frac{m \ln \frac{b}{a}}{4} \left( \frac{\alpha}{\alpha+1} \right)^{\frac{1}{\alpha}} \left\{ \left| f' \left( \left( \sqrt{ab} \right)^m \right) \right|^q C_1 \left( \sqrt{ab}, \alpha, 1, q, m, s \right) + m \left| f' \left( a \right) \right|^q C_2 \left( \sqrt{ab}, \alpha, 1, q, m, s \right) \right\}^{\frac{1}{q}} \\
+ \left| f' \left( \left( \sqrt{ab} \right)^m \right) \right|^q C_3 \left( \sqrt{ab}, \alpha, 1, q, m, s \right) + m \left| f' \left( b \right) \right|^q C_4 \left( \sqrt{ab}, \alpha, 1, q, m, s \right) \right\}^{\frac{1}{q}}.
\]

Corollary 17. Let the assumptions of Theorem 7 hold. If $|f' (u)| \leq M$ for all $u \in [a^m, b^m]$ and $\lambda = 0$, then from the inequality (2.13) we get the following Ostrowski type inequality or fractional integrals
\[
\left| \left( \ln \frac{x}{a} \right)^\alpha + \left( \ln \frac{b}{x} \right)^\alpha \right| f(x^m) - \frac{\Gamma (\alpha + 1)}{m^\alpha} \left( J^{m \ln x}_{(m \ln \sqrt{ab})} - (f \circ \exp) (m \ln a) + J^{m \ln x}_{(m \ln \sqrt{ab})} - (f \circ \exp) (m \ln b) \right) \right| \\
\leq \frac{mM}{(\alpha+1)^{\frac{1}{\alpha}}} \left\{ a^m \left( \ln \frac{x}{a} \right)^{\alpha+1} (C_1 (x, \alpha, 0, q, m, s) + mC_2 (x, \alpha, 0, q, m, s)) \right\}^{\frac{1}{\alpha}} \\
+ b^m \left( \ln \frac{b}{x} \right)^{\alpha+1} (C_3 (x, \alpha, \lambda, q, m, s) + mC_4 (x, \alpha, \lambda, q, m, s)) \right\}^{\frac{1}{\alpha}}.
\]

for each $x \in [a, b]$.

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