On the relation of Voevodsky’s algebraic cobordism to Quillen’s $K$-theory

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September 26, 2007

Abstract

Quillen’s algebraic $K$-theory is reconstructed via Voevodsky’s algebraic cobordism. More precisely, for a ground field $k$ the algebraic cobordism $\mathbf{P}^1$-spectrum $MGL$ of Voevodsky is considered as a commutative $\mathbf{P}^1$-ring spectrum. Setting $MGL^i = \oplus_{p-2q=i} MGL^{p,q}$ we regard the bigraded theory $MGL^{p,q}$ as just a graded theory. There is a unique ring morphism $\phi: MGL^0(k) \to \mathbb{Z}$ which sends the class $[X]_{MGL}$ of a smooth projective $k$-variety $X$ to the Euler characteristic $\chi(X, \mathcal{O}_X)$ of the structure sheaf $\mathcal{O}_X$. Our main result states that there is a canonical grade preserving isomorphism of ring cohomology theories

$$\varphi: MGL^*(X,U) \otimes MGL^0(k) \mathbb{Z} \cong KTT_*(X,U) = K'_*(X-U)$$

on the category $\text{SmOp}/k$ in the sense of [6], where $KTT_*$ is Thomason-Trobaugh $K$-theory and $K'_*$ is Quillen’s $K'$-theory. In particular, the left hand side is a ring cohomology theory. Moreover both theories are oriented in the sense of [6] and $\varphi$ respects the orientations. The result is an algebraic version of a theorem due to Conner and Floyd. That theorem reconstructs complex $K$-theory via complex cobordism [1].

1 Introduction

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where BGL is the Voevodsky K-theory $\mathbb{P}^1$-spectrum representing $(2,1)$-periodic algebraic K-theory. To describe the second step recall that $\text{BGL}^* = K_{TT}^*[\beta, \beta^{-1}]$ for an element $\beta \in \text{BGL}^{2,1}(k) \subseteq \text{BGL}^0(k)$. Taking the quotient of both sides of the last isomorphism modulo the ideal generated by the element $(\beta + 1)$ we get the required isomorphism $\tilde{\varphi}$.

Here are a few words on the construction of $\tilde{\varphi}$. The $\mathbb{P}^1$-spectrum $\text{MGL}$ is defined as $(S^0, \text{Th}(\mathcal{J}(1)), \text{Th}(\mathcal{J}(2)), \ldots)$, where $\mathcal{J}(i)$ is the tautological vector bundle over the Grassmannian $\text{Gr}(i, \infty)$ and $\text{Th}(\mathcal{J}(i))$ is its Thom space. There is a monoidal structure on $\text{MGL}$ described in [5]. The obvious morphism

$$(*, \text{Th}(\mathcal{J}(1)), \text{Th}(\mathcal{J}(1)) \wedge \mathbb{P}^1, \ldots) \to (S^0, \text{Th}(\mathcal{J}(1)), \text{Th}(\mathcal{J}(2)), \ldots)$$

defines an element $\text{th}_{\text{MGL}} \in \text{MGL}^{2,1}(\text{Th}(\mathcal{J}(1)))$, which is a tautological orientation of $\text{MGL}$. It turns out that the pair $(\text{MGL}, \text{th}_{\text{MGL}})$ is universal among all pairs $(E, \text{th})$ with a commutative ring $\mathbb{P}^1$-spectrum $E$ and a Thom orientation $\text{th} \in E^{2,1}(\text{Th}(\mathcal{J}(1)))$. The last assertion means that there exists a unique monoidal morphism

$$\varphi_{\text{th}} : \text{MGL} \to E$$

in the motivic stable homotopy category $\text{SH}(k)$ taking the class $\text{th}_{\text{MGL}}$ to $\text{th}$ (see [11] or [5, Thm. 2.3.1] for precise formulations). That morphism $\varphi_{\text{th}}$ gives rise to a functor transformation

$$\tilde{\varphi}_{\text{th}} : \text{MGL}^*(X, U) \otimes_{\text{MGL}^0(k)} E^0(k) \to E^*(X, U).$$

The pair $(\text{BGL}, \text{th}_{\text{MGL}})$, as it is described in [4] or in 2.2.3, induces the morphism $\tilde{\varphi}$ mentioned in (1).

There exist pairs $(E, \text{th}^E)$ such that the induced monoidal morphism $\tilde{\varphi}_{\text{th}}$ is not an isomorphism. For example one can take $E = \mathbb{H} \mathbb{Z}$ with its canonical orientation, as mentioned in [2, Introduction]. However for the pair $(\text{BGL}, \text{th}_{\text{MGL}})$ it is an isomorphism. A section $s : K_{TT}^0 \to \text{MGL}^{0,0}$ of the natural transformation $\varphi : \text{MGL}^{0,0} \to \text{BGL}^{0,0} = K_{TT}^0$ is crucial for the proof of the main result. This section is constructed in Section 3.1.

### 2 Recollection

Our main result relates Voevodsky’s algebraic cobordism theory $\text{MGL}^{*,*}$ to Quillen’s $K'$-theory. We analyze these theories via their representing objects in the motivic stable homotopy category $\text{SH}(S)$. Consider [4, Appendix] for the basic terminology, notation, constructions, definitions, results on motivic homotopy theory. Nevertheless, here is a short summary.
2.1 Motivic homotopy theory

Let $S$ be a Noetherian separated finite-dimensional scheme $S$. One may think of $S$ being the spectrum of a field or the integers. A motivic space over $S$ is a functor

$$A: SmOp/S \to sSet$$

(see [4, Appendix]). The category of motivic spaces over $S$ is denoted $\mathbf{M}(S)$. This definition of a motivic space is different from the one considered by Morel and Voevodsky in [3] – they consider only those simplicial presheaves which are sheaves in the Nisnevich topology on $Sm/S$. With our definition the Thomason-Trobaugh $K$-theory functor obtained by using big vector bundles is a motivic space on the nose. It is not a simplicial Nisnevich sheaf. This is the reason why we prefer to work with the above notion of “space”.

We write $H^\text{cm}_\bullet(S)$ for the pointed motivic homotopy category and $SH^\text{cm}(S)$ for the stable motivic homotopy category over $S$ as constructed in [4, A.3.9, A.5.6]. By [4, A.3.11 resp. A.5.6] there are canonical equivalences to $H_\bullet(S)$ of [3] resp. $SH(S)$ of [12]. Both $H^\text{cm}_\bullet(S)$ and $SH^\text{cm}(S)$ are equipped with closed symmetric monoidal structures such that the $P^1$-suspension spectrum functor is a strict symmetric monoidal functor

$$\Sigma^\infty_{P^1}: H^\text{cm}_\bullet(S) \to SH^\text{cm}(S).$$

Here $P^1$ is considered as a motivic space pointed by $\infty \in P^1$. The symmetric monoidal structure $(\wedge, I_S = \Sigma^\infty_{P^1} S_+)$ on the homotopy category $SH^\text{cm}(S)$ is constructed on the model category level by employing the category $\mathbf{MSS}(S)$ of symmetric $P^1$-spectra. This symmetric monoidal category satisfies the properties required by Theorem 5.6 of Voevodsky congress talk [12]. From now on we will usually omit the superscript $(-)^\text{cm}$.

Every $P^1$-spectrum $E$ represents a cohomology theory on the category of pointed motivic spaces. Namely, for a pointed motivic space $(A, a)$ set

$$E^{p,q}(A, a) = \text{Hom}_{SH(S)}(\Sigma^\infty_{P^1}(A, a), \Sigma^{p,q}(E))$$

and

$$E^{*,*}(A, a) = \bigoplus_{p,q} E^{p,q}(A, a).$$

This definition extends to motivic spaces via the functor $A \mapsto A_+$ which adds a disjoint basepoint. That is, for a non-pointed motivic space $A$ set $E^{p,q}(A) = E^{p,q}(A_+, +)$ and $E^{*,*}(A) = \bigoplus_{p,q} E^{p,q}(A)$.

Every $X \in Sm/S$ defines a representable motivic space – constant in the simplicial direction – taking an smooth $S$-scheme $U$ to $\text{Hom}_{Sm/S}(U, X)$. It is not possible in general to choose a basepoint for representable motivic spaces.
So we regard $S$-smooth varieties as motivic spaces (non-pointed) and set

$$E^{p,q}(X) = E^{p,q}(X_+, +).$$

Given a $\mathbb{P}^1$-spectrum $E$ we will reduce the double grading on the cohomology theory $E^{*,*}$ to a grading by defining $E^m = \oplus_{m=p-2q} E^{p,q}$ and $E^* = \oplus_mE^m$. We often write $E^*(k)$ for $E^*(\text{Spec}(k))$ below.

A $\mathbb{P}^1$-ring spectrum is a monoid $(E, \mu, e)$ in $(\text{SH}(S), \wedge, I_S)$. A commutative $\mathbb{P}^1$-ring spectrum is a commutative monoid $(E, \mu, e)$ in $(\text{SH}(S), \wedge, 1)$.

The cohomology theory $E^*$ defined by a $\mathbb{P}^1$-ring spectrum is a ring cohomology theory. The cohomology theory $E^*$ defined by a commutative $\mathbb{P}^1$-ring spectrum is a ring cohomology theory, however it is not necessarily graded commutative. The cohomology theory $E^*$ defined by an oriented commutative $\mathbb{P}^1$-ring spectrum (to be defined below) is a graded commutative ring cohomology theory by [9].

Occasionally a $\mathbb{P}^1$-ring spectrum $(E, \mu, e)$ might have a model $(E', \mu', e')$ which is a symmetric $\mathbb{P}^1$-ring spectrum, that is, a symmetric $\mathbb{P}^1$-spectrum $E'$ equipped with a strict multiplication $\mu': E' \wedge E' \to E'$ which is strictly associative and strictly unital for the unit $e': \Sigma_{\mathbb{P}^1}(S_+) \to E'$. This is the case for the algebraic cobordism $\mathbb{P}^1$-ring spectrum $MGL$, as described below. Such a model for the algebraic $K$-theory $\mathbb{P}^1$-ring spectrum $BGL$ is currently not known to us.

For the rest of the paper let $k$ be a field and $S = \text{Spec}(k)$. Usually $S$ will be replaced by $k$ in the notation. We work in this text with the algebraic cobordism $\mathbb{P}^1$-spectrum $MGL$ and the algebraic $K$-theory $\mathbb{P}^1$-spectrum $BGL$ as described in [4, Defn. 1.2.4] and [5, Sect. 2.1] respectively. The spectrum $MGL$ is a commutative ring $\mathbb{P}^1$-spectrum by that construction. The spectrum $BGL$ is equipped with a structure of a commutative $\mathbb{P}^1$-ring spectrum as explained in [4, Thm. 2.1.1].

The $\mathbb{P}^1$-spectrum $BGL$ has the following underlying sequence of pointed motivic sequence: $(\mathcal{K}_0, \mathcal{K}_1, \mathcal{K}_2, \ldots)$. Here are the main properties of $BGL$ that will be used below:

(i) $\mathcal{K}_i = \mathcal{K}_0$ for all $i$ and $\mathcal{K}_0$ is weakly equivalent to the Thomason-Trobaugh $K$-theory motivic space $\mathbb{K}^{TT}$ [4, Sect. 1.2].

(ii) $BGL$ is an $\Omega_{\mathbb{P}^1}$-spectrum; in particular, for any pointed motivic space $A$ one has $BGL_{0,0}(A) = \text{Hom}_{\mathbb{H}_*}(A, \mathcal{K}_0)$.

(iii) There is a motivic weak equivalence $w: \mathbb{Z} \times \text{Gr} \to \mathcal{K}_0$ which classifies the tautological element $\xi_\infty = \tau_\infty - \infty \in \mathbb{K}_0^{TT}(\mathbb{Z} \times \text{Gr})$ [4, Sect. 1.2].
Another property of BGL is that it represents algebraic $K$-theory as a ring cohomology theory. Let $K^T_*$ be Thomason-Trobaugh $K$-theory functor \[10\]. The morphism $\Sigma^\infty \mathcal{K}_0 \to \text{BGL}$ adjoint to the identity map $\text{id} : \mathcal{K}_0 \to \mathcal{K}_0$ defines an isomorphism

\[ Ad : K^T_* \to \text{BGL}^{*,0} \]

of cohomology theories on the category $\text{SmOp}/k$ in the sense of \[6\]. This isomorphism respects the product structures by \[4, \text{Cor. 2.2.4}\].

An invertible Bott element $\beta \in \text{BGL}^{2,1}(\text{Spec}(k))$ is constructed in \[4, \text{Section 1.3}\]. For every pointed motivic space $A$ the morphism

\[ \text{BGL}^{*,0}(A) \otimes \text{BGL}^0(\text{Spec}(k)) \to \text{BGL}^{*,*}(A) \] (2)

given by $a \otimes \beta \mapsto a \cup \beta$ is a ring isomorphism by \[4, \text{Sect. 1.3}\]. Furthermore \[\text{BGL}^0(\text{Spec}(k)) = \mathbb{Z}[\beta, \beta^{-1}]\] is the ring of Laurent polynomials on the Bott element $\beta$. To say the same in a different way,

\[ \text{BGL}^{*,0}(A)[\beta, \beta^{-1}] \cong \text{BGL}^{*,*}(A). \] (3)

The special case $A = X/(X \setminus Z)$ where $X$ is a smooth $k$-variety and $Z \subset X$ is a closed subset implies the following result \[4, \text{Cor. 1.3.6}\].

**Corollary 2.1.1.** Let $X$ be a smooth $k$-scheme, $Z$ a closed subset of $X$ and $U = X \setminus Z$ its open complement. Then there are isomorphisms

\[ K^T_{*,Z}(X)[\beta, \beta^{-1}] \cong \text{BGL}^{*,*}(X/U) = \text{BGL}^*(X/U) \] (4)

\[ K^T_{*,Z}(X) \cong \text{BGL}^{*,*}(X/U)/(\beta + 1)\text{BGL}^{*,*}(X/U) \] (5)

of ring cohomology theories on $\text{SmOp}/k$ in the sense of \[6\].

We refer to \[5\] for a construction of the commutative $\mathbb{P}^1$-ring spectrum $\text{MGL}$. For the purposes of the work to be presented we will need to know only two properties of $\text{MGL}$, which we refer to as Quillen universality and BGL-cellularity (see Section 3.1 below).

### 2.2 Oriented commutative ring spectra

Following Adams and Morel we define an orientation of a commutative $\mathbb{P}^1$-ring spectrum. However we prefer to use Thom classes instead of Chern classes. Consider the pointed motivic space $\mathbb{P}^\infty = \text{colim}_{n \geq 0} \mathbb{P}^n$ having base point $g_1 : S = \mathbb{P}^0 \hookrightarrow \mathbb{P}^\infty$.

The tautological “vector bundle” $\mathcal{I}(1) = \mathcal{O}_{\mathbb{P}^\infty}(-1)$ is also known as the Hopf bundle. It has zero section $z : \mathbb{P}^\infty \hookrightarrow \mathcal{I}(1)$. The fiber over the point
$g_1 \in \mathbb{P}^\infty$ is $A^1$. For a vector bundle $V$ over a smooth $k$-scheme $X$, with zero section $z : X \hookrightarrow V$, let the Thom space $\text{Th}(V)$ of $V$ be the Nisnevich sheaf associated to the presheaf $Y \mapsto V(Y) / (V \setminus z(X))(Y)$ on the Nisnevich site $\text{Sm}/k$. In particular $\text{Th}(V)$ is a pointed motivic space in the sense of [4, Defn. A.1.1]. It coincides with Voevodsky’s Thom space [12, p. 422], since $\text{Th}(V)$ is already a Nisnevich sheaf. The Thom space of the Hopf bundle is then defined as the colimit $\text{Th}(\mathcal{T}(1)) = \text{colim}_{n \geq 0} \text{Th}(\mathcal{O}_{\mathbb{P}^n}(-1))$. Abbreviate $T = \text{Th}(A^1_S)$.

Let $E$ be a commutative $\mathbb{P}^1$-ring spectrum. The unit gives rise to an element $1 \in E^{0,0}(\text{Spec}(k)_+)$. Applying the $\mathbb{P}^1$-suspension isomorphism to that element we get an element $\Sigma_{\mathbb{P}^1}(1) \in E^{2,1}(\mathbb{P}^1, \infty)$. The canonical covering of $\mathbb{P}^1$ defines motivic weak equivalences

$$\mathbb{P}^1 \sim \mathbb{P}^1 / A^1 \leftarrow A^1 / A^1 \setminus \{0\} = T$$

of pointed motivic spaces inducing isomorphisms

$$E(\mathbb{P}^1, \infty) \leftarrow E(A^1 / A^1 \setminus \{0\}) \rightarrow E(T).$$

Let $\Sigma_T(1)$ be the image of $\Sigma_{\mathbb{P}^1}(1)$ in $E^{2,1}(T)$.

**Definition 2.2.1.** Let $E$ be a commutative ring $\mathbb{P}^1$-spectrum. A Thom orientation of $E$ is an element $th \in E^{2,1}(\text{Th}(\mathcal{T}(1)))$ such that its restriction to the Thom space of the fibre over the distinguished point coincides with the element $\Sigma_T(1) \in E^{2,1}(T)$. A Chern orientation of $E$ is an element $c \in E^{2,1}(\mathbb{P}^\infty)$ such that $c|_{\mathbb{P}^1} = -\Sigma_{\mathbb{P}^1}(1)$. An orientation of $E$ is either a Thom orientation or a Chern orientation. One says that a Thom orientation $th$ of $E$ coincides with a Chern orientation $c$ of $E$ provided that $c = z^*(th)$ or equivalently the element $th$ coincides with the one $th(\mathcal{O}(-1))$ given by (7) below.

**Remark 2.2.2.** The element $th$ should be regarded as the Thom class of the tautological line bundle $\mathcal{T}(1) = \mathcal{O}(-1)$ over $\mathbb{P}^\infty$. The element $c$ should be regarded as the Chern class of the tautological line bundle $\mathcal{T}(1) = \mathcal{O}(-1)$ over $\mathbb{P}^\infty$.

**Example 2.2.3.** The following orientations given right below are relevant for our work. Here $\text{MGL}$ denotes the $\mathbb{P}^1$-ring spectrum representing algebraic cobordism obtained in [5, Defn 2.1.1] and $\text{BGL}$ denotes the $\mathbb{P}^1$-ring spectrum representing algebraic $K$-theory constructed in [4, Theorem 2.2.1].

- Let $u_1 : \Sigma_{\mathbb{P}^1}(\text{Th}(\mathcal{T}(1)))(-1) \rightarrow \text{MGL}$ be the canonical map of $\mathbb{P}^1$-spectra. Set $th^{\text{MGL}} = u_1 \in \text{MGL}^{2,1}(\text{Th}(\mathcal{T}(1)))$. Since $th^{\text{MGL}}|_{\text{Th}(1)} = \Sigma_{\mathbb{P}^1}(1)$ in $\text{MGL}^{2,1}(\text{Th}(1))$, the class $th^{\text{MGL}}$ is an orientation of $\text{MGL}$. 

Set \( c^{BGL} = (-\beta) \cup ([\mathcal{O}] - [\mathcal{O}(1)]) \in BGL^{2,1}(\mathbf{P}^\infty) \). The relation (11) from [4] shows that the class \( c \) is an orientation of BGL. Let \( th^{BGL} \) be the equivalent Thom orientation.

### 3 Oriented cohomology theories

Let \((E, c)\) be an oriented commutative \(\mathbf{P}^1\)-ring spectrum (See 2.2.1). In this section we compute the \(E\)-cohomology of infinite Grassmannians. The results are the expected ones – see Theorem 3.0.6.

The oriented \(\mathbf{P}^1\)-ring spectrum \((E, c)\) defines an oriented cohomology theory on \(Sm\mathcal{O}p/k\) in the sense of [6, Defn. 3.1] as follows. The restriction of the functor \(E^{*,*}\) to the category \(Sm\mathcal{O}p/k\) is a ring cohomology theory. By [6, Th. 3.35] it remains to construct a Chern structure on \(E^{*,*}\) in the sense of [6, Defn. 3.2]. Let \(H_\bullet(k)\) be the homotopy category of pointed motivic spaces over \(k\). The functor isomorphism \(Hom_{H_\bullet(k)}(-, \mathbf{P}^\infty) \rightarrow \text{Pic}(-)\) on the category \(Sm/k\) provided by [3, Thm. 4.3.8] sends the class of the identity map \(\mathbf{P}^\infty \rightarrow \mathbf{P}^\infty\) to the class of the tautological line bundle \(\mathcal{O}(-1)\) over \(\mathbf{P}^\infty\). For a line bundle \(L\) over \(X \in Sm/k\) let \([L]\) be the class of \(L\) in the group \(\text{Pic}(X)\). Let \(f_L : X \rightarrow \mathbf{P}^\infty\) be the morphism in \(H_\bullet(k)\) corresponding to the class \([L]\) under the functor isomorphism above. For a line bundle \(L\) over \(X \in Sm/k\) set \(c(L) = f_L^*(c) \in E^{2,1}(X)\). Clearly, \(c(\mathcal{O}(-1)) = c\).

The assignment \(L/X \mapsto c(L)\) is a Chern structure on \(E^{*,*}\) in the sense of [6]. In particular, \((BGL, c^{BGL})\) defines an oriented ring cohomology theory on \(Sm\mathcal{O}p/k\).

Given this Chern structure, one obtains a theory of Thom classes \(V/X \mapsto th(V) \in E^{2\text{rank}(V),\text{rank}(V)}(\text{Th}_X(V))\) on the cohomology theory \(E^{*,*}\) in the sense of [6, Defn. 3.1] as follows. There is a unique theory of Chern classes \(V \mapsto c_i(V) \in E^{2i,i}(X)\) such that for every line bundle \(L\) on \(X\) one has \(c_1(L) = c(L)\). For a rank \(r\) vector bundle \(V\) over \(X\) consider the vector bundle \(W := \mathbf{1} \oplus V\) and the associated projective vector bundle \(\mathbf{P}(W)\) of lines in \(W\). Set

\[
\tilde{th}(V) = c_r(p^*(V) \otimes \mathcal{O}_{\mathbf{P}(W)}(1)) \in E^{2r,r}(\mathbf{P}(W)). \tag{6}
\]

It follows from [6, Cor. 3.18] that the support extension map

\[
E^{2r,r}(\mathbf{P}(W)/(\mathbf{P}(W) \setminus \mathbf{P}(1))) \rightarrow E^{2r,r}(\mathbf{P}(W))
\]

is injective and \(\tilde{th}(E) \in E^{2r,r}(\mathbf{P}(W)/(\mathbf{P}(W) \setminus \mathbf{P}(1)))\). Set

\[
th(E) = j^*(\tilde{th}(E)) \in E^{2r,r}(\text{Th}_X(V)), \tag{7}
\]
where \( j : \text{Th}_X(V) \to \mathbb{P}(W)/(\mathbb{P}(W) \setminus \mathbb{P}(1)) \) is the canonical motivic weak equivalence of pointed motivic spaces induced by the open embedding \( V \hookrightarrow \mathbb{P}(W) \). The assignment \( V/X \mapsto \text{th}(V) \) is a theory of Thom classes on the cohomology theory \( E^{\ast\ast}|_{\text{sm}0/p/k} \) by the proof of [6, Thm. 3.35]. So the Thom classes are natural, multiplicative and satisfy the following Thom isomorphism property.

**Theorem 3.0.4.** For a rank \( r \) vector bundle \( p : V \to X \) on \( X \in \text{Sm}/k \) with zero section \( z : X \hookrightarrow V \), the map

\[
- \cup \text{th}(V) : E^{\ast\ast}(X) \to E^{\ast\ast+2r,\ast+r}(V/(V \setminus z(X)))
\]

is an isomorphism of two-sided \( E^{\ast\ast}(X) \)-modules, where \(- \cup \text{th}(V)\) is written for the composition map \((- \cup \text{th}(V)) \circ p^{\ast}\).

**Proof.** See [6, Defn. 3.32.(4)]. \( \square \)

Let \( \text{Gr}(n, n+m) \) be the Grassmann scheme of \( n \)-dimensional linear subspaces of \( \mathbb{A}_S^{n+m} \). The closed embedding \( \mathbb{A}^{n+m} = \mathbb{A}^{n+m} \times \{0\} \hookrightarrow \mathbb{A}^{n+m+1} \) defines a closed embedding

\[
\text{Gr}(n, n+m) \hookrightarrow \text{Gr}(n, n+m+1).
\]

The tautological vector bundle is denoted \( \mathcal{J}(n, n+m) \to \text{Gr}(n, n+m) \). The closed embedding (8) is covered by a map of vector bundles \( \mathcal{J}(n, n+m) \to \mathcal{J}(n, n+m+1) \). Let \( \text{Gr}(n) = \text{colim}_{m \geq 0} \text{Gr}(n, n+m), \mathcal{J}(n) = \text{colim}_{m \geq 0} \mathcal{J}(n, n+m) \) and \( \text{Th}(\mathcal{J}(n)) = \text{colim}_{m \geq 0} \text{Th}(\mathcal{J}(n, n+m)) \). These colimits are taken in the category of motivic spaces over \( S \).

**Remark 3.0.5.** It is not difficult to prove that \( E^{\ast\ast}(\text{Gr}(n, n+m)) \) is multiplicatively generated by the Chern classes \( c_i(\mathcal{J}(n, n+m)) \) of the tautological vector bundle \( \mathcal{J}(n, n+m) \). This proves the surjectivity of the pull-back maps \( E^{\ast\ast}(\text{Gr}(n, n+m+1)) \to E^{\ast\ast}(\text{Gr}(n, n+m)) \) and shows that the canonical map \( E^{\ast\ast}(\text{Gr}(n)) \to \varprojlim E^{\ast\ast}(\text{Gr}(n, n+m)) \) is an isomorphism. Thus for each \( i \) there exists a unique element \( c_i = c_i(\mathcal{J}(n)) \in E^{2i,i}(\text{Gr}(n)) \) which for each \( m \) restricts to the element \( c_i(\mathcal{J}(n, n+m)) \) under the obvious pull-back map.

**Theorem 3.0.6.** Let \( c_i = c_i(\mathcal{J}(n)) \in E^{2i,i}(\text{Gr}(n)) \) be the element defined just above. Then

\[
E^{\ast\ast}(\text{Gr}(n)) = E^{\ast\ast}(k)[[c_1, c_2, \ldots, c_n]]
\]

is the formal power series on the \( c_i \)'s. The inclusion \( i : \text{Gr}(n) \hookrightarrow \text{Gr}(n+1) \) satisfies \( i^{\ast}(c_m) = c_m \) for \( m < n+1 \) and \( i^{\ast}(c_{n+1}) = 0 \).

**Proof.** See [5, Thm. 2.0.6]. \( \square \)
**Remark 3.0.7.** For a smooth variety $X$ and a vector bundle $E$ over $X$ the class $c_1(E)$ is additive with respect to short exact sequences of vector bundles. Thus it defines a homomorphism $K_0^{TT}(X) \to E^{2,1}(X)$. Moreover that homomorphism is natural in $X$. For an element $\alpha \in K_0^{TT}(X)$ we will write $c_1(\alpha)$ for the image of $\alpha$. Now take the space $\text{Gr}(n)$ and recall that the map $\mathbb{K}_0^{TT}(\text{Gr}(n)) \to \lim \rightarrow \mathbb{K}_0^{TT}(\text{Gr}(n,n + m))$ is an isomorphism by [4, Sect. 1.2] and the map $E^*(\text{Gr}(n)) \to \lim \rightarrow E^*(\text{Gr}(n,n + m))$ is an isomorphism by Remark 3.0.5. Thus we have a homomorphism $\mathbb{K}_0^{TT}(\text{Gr}(n)) \to E^{2,1}(\text{Gr}(n))$. In the same way we may get a homomorphism $\mathbb{K}_0^{TT}(\mathbb{Z} \times \text{Gr}) \to E^{2,1}(\mathbb{Z} \times \text{Gr})$, where $\text{Gr} = \colim_{n \geq 0} \text{Gr}(n)$. For an element $\alpha \in \mathbb{K}_0^{TT}(\mathbb{Z} \times \text{Gr})$ we will write $c_1(\alpha)$ for its image in $E^{2,1}(\mathbb{Z} \times \text{Gr})$.

### 3.1 Universality and cellularity

The main result of this Section is Theorem 3.1.4.

**Definition 3.1.1 (Universality Property).** Let $(U, u)$ be an oriented commutative ring $\mathbb{P}^1$-spectrum over $S$. We say that $(U, u)$ is *Quillen universal* if for every commutative ring $\mathbb{P}^1$-spectrum $E$ over $S$ the assignment $\varphi \mapsto \varphi(u) \in U^{2,1}((\text{Th}(\mathbb{F}(1)))$ identifies the set of monoid homomorphisms $\varphi: U \to E$ in the motivic stable homotopy category $\text{SH}(S)$ with the set of orientations of $E$.

**Remark 3.1.2.** The Universality Theorem ([11] or [5]) implies that the $\mathbb{P}^1$-spectrum $\text{MGL}$, equipped with its canonical orientation $\text{th}^\text{MGL}$ from 2.2.3, is Quillen universal.

If $E$ is a commutative $\mathbb{P}^1$-ring spectrum over $k$ and $A$ is a pointed motivic space over $k$, $E^*(A)$ is an $E^0(k)$-module in a natural way. A monoid homomorphism $\phi: E_1 \to E_2$ induces an $E^0(k)$-module homomorphism $E^*_1(A) \to E^*_2(A)$. In particular, if $(U, u)$ is a Quillen universal oriented commutative ring $\mathbb{P}^1$-spectrum over $k$ and $(E, \text{th})$ is an oriented commutative ring $\mathbb{P}^1$-spectrum over $k$, the monoid homomorphism $\varphi: U \to E$ in $\text{SH}(k)$ induces the homomorphism $\varphi_A: U^*(A) \otimes U^0(k) E^0(k) \to E^*(A)$.
and in particular the homomorphism
\[ \varphi^0_A : U^0(A) \otimes_{U^0(k)} E^0(k) \to E^0(A). \] (12)
Both are natural in \( A \).

From now on we will insert \((BGL, \text{th})_0\) for \( (E, \text{th}) \) (see Example 2.2.3).

Set \( \bar{U}^*(A) = U^*(A) \otimes_{U^0(k)} BGL^0(k) \) and \( \bar{U}^0(A) U^0(A) \otimes_{U^0(k)} BGL^0(k) \).

**Definition 3.1.3** (Weakly BGL-cellular). Let \((U, u)\) be a Quillen universal \( P_1 \)-ring spectrum, and let \( \tilde{\varphi}^0_A : U^0(A) \otimes_{U^0(k)} BGL^0(k) \to BGL^0(A) \) be the homomorphism induced by the orientation \( \text{th} \) on \( BGL \) (see Example 2.2.3). Then \((U, u)\) is called weakly BGL-cellular if there exists an integer \( N \) such that the map \( \varphi^0_{U_n} \) is an isomorphism for \( n \geq N \), where \( U_n \) is the \( n \)-th term of the \( P_1 \)-spectrum \( U \).

A pointed motivic space \( A \) is called small if the covariant functor
\[ \text{Hom}_{\text{SH}(S)}(\Sigma^\infty_{P_1}, A, -) \]
on \( \text{SH}(S) \) commutes with arbitrary coproducts.

**Theorem 3.1.4.** Let \((U, u)\) be a Quillen universal oriented commutative \( P_1 \)-ring spectrum over a field \( k \). Suppose \((U, u)\) is weakly BGL-cellular. Then the homomorphism \( \tilde{\varphi}_A \) is an isomorphism for all small pointed motivic spaces \( A \).

**Proof.** The proof consists of several steps. Our first aim is to prove that the homomorphisms \( \tilde{\varphi}^0_A \) are isomorphisms, where \( A \) is a small pointed motivic space. First we construct a section of the natural transformation
\[ \varphi^{0,0} : U^{0,0} \to BGL^{0,0} \]
of functors on the category of small pointed motivic spaces. Recall that for every oriented commutative \( P_1 \)-ring spectrum \((E, \text{th})\) the ring cohomology theory \( E^{n,*}|_{\text{SmOp}/k} \) is an oriented cohomology theory on the category \( \text{SmOp}/k \) (see Section 3). Let \( \mathbb{F}_{E, \text{th}} \) be the induced commutative formal group law over the ring \( E^0(k) \). Let \( \Omega \) be the complex cobordism ring and let \( l_{E, \text{th}} : \Omega \to E^0(k) \) be the unique ring homomorphism sending the universal formal group \( \mathbb{F}_\Omega \) to \( \mathbb{F}_{E, \text{th}} \).

Since \( c_1 \in E^{2,1} \) the coefficients \( a_{ij} \) of the formal group law \( \mathbb{F}_{E, \text{th}} \) are in \( E^{-2(i+j-1), -(i+j-1)}(k) \). Thus the homomorphism \( l_{E, \text{th}} \) is grade preserving, that is, it takes \( \Omega^{2i} \) into \( E^{2i,i}(k) \) for any \( i \).

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Set
\[ [P^n]_E = l_{E,th}([\mathbb{C}P^n]) \in E^{-2n,-n}(k), \tag{13} \]
where \([\mathbb{C}P^n]\) is the class of the complex projective space \(\mathbb{C}P^n\) in \(\Omega\). Although the class \([P^n]_E\) depends on the orientation class \(th\), we use the notation \([P^n]_E\) instead. If \((E',th')\) is another oriented commutative \(\mathbf{P}^1\)-ring spectrum and \(\psi: E \to E'\) is a monoid homomorphism in the category \(\text{SH}(S)\) which preserves orientation classes, then it sends the formal group law \(F_{E,th}\) to \(F_{E',th'}\). In particular \(\psi([P^n]_E) = [P^n]_{E'}\). Applying this observation to the monoid homomorphism \(\psi\) one obtains
\[ \varphi([P^1]_U) = [P^1]_{\text{BGL}}. \]

To compute \([P^1]_{\text{BGL}}\) recall that the coefficient at \(XY\) in the formal group law \(F_\Omega\) coincides with the class \(-[\mathbb{C}P^1]\) in \(\Omega\). The formal group law \(F_{\text{BGL}}\) coincides with \(X + Y + \beta^{-1}XY\), since \(c_{\text{BGL}}(L) = ([1] - [L^\vee])(-\beta)\). Thus the equality
\[ [P^1]_{\text{BGL}} = -\beta^{-1} \]
holds. We are ready to construct a section.

Let \(\text{Gr}(n)\) be the pointed motivic space described right above Theorem 3.0.6. Set \(\text{Gr} = \colim_{n \geq 0} \text{Gr}(n)\). Consider the unique element \(\infty - \tau_\infty^\vee \in \mathbb{K}^T_0(\mathbb{Z} \times \text{Gr})\) such that for each integer \(m\) its restriction to the subspace \(\{m\} \times \text{Gr}(n,2n) \subseteq \mathbb{Z} \times \text{Gr}\) coincides with the element \(-m + n - [\tau_\infty^\vee] \in K_0^T(\text{Gr}(n,2n))\) (compare with description of the element \(\xi_\infty = \tau_\infty - \infty\) in [4, Sect. 1.2]). Let \(c^U_1(\infty - \tau_\infty^\vee)\) be the image of \(\infty - \tau_\infty^\vee\) in \(E^{2,1}(\mathbb{Z} \times \text{Gr})\) as described in Remark 3.0.7. Consider the map
\[ s: \Sigma^\infty_{\mathbf{P}_1}(\mathbb{Z} \times \text{Gr}) \to U \tag{14} \]
in the stable homotopy category category \(\text{SH}(S)\) given by the element
\[ c^U_1(\infty - \tau_\infty^\vee) \cup [P^1]_U \in U^{0,0}(\mathbb{Z} \times \text{Gr}). \]

**Claim 3.1.5.** One has \(\varphi(c^U_1(\infty - \tau_\infty^\vee) \cup [P^1]_U) = \tau_\infty - \infty \in \text{BGL}^{0,0}(\mathbb{Z} \times \text{Gr})\).

In fact,
\[
\varphi(c^U_1(\infty - \tau_\infty^\vee) \cup [P^1]_U) = \quad c^\text{BGL}_1(\infty - \tau_\infty^\vee) \cup [P^1]_{\text{BGL}} \\
= (\infty - \tau_\infty) \cup \beta \cup (-\beta^{-1}) \\
= \tau_\infty - \infty
\]

Claim 3.1.5 shows that the composition
\[ \varphi \circ s: \Sigma^\infty_{\mathbf{P}_1}(\mathbb{Z} \times \text{Gr}) \to \text{BGL} \]
coincedes with the adjoint of the motivic weak equivalence \( w : \mathbb{Z} \times \text{Gr} \to \mathcal{X}_0 \) from Section 2.

This fact together with properties \((ii)\) and \((iii)\) of BGL mentioned in Section 2 shows that for every pointed motivic space \( A \) the map

\[
s_A : \text{BGL}^{0,0}(A) = [A, \mathcal{X}_0] = [A, \mathbb{Z} \times \text{Gr}] \to [\Sigma^\infty_{\text{BGL}}(A), U] = U^{0,0}(A)
\]

is a section of the map \( \varphi^0_A : U^{0,0}(A) \to \text{BGL}^{0,0}(A) \). Moreover, the section \( s_A \) is natural in \( A \). Let

\[
\overline{s}_A : \text{BGL}^0 \to \overline{U}^0
\]

and recall that \( \overline{s}_A : \overline{U}^*(A) \to \text{BGL}^*(A) \) is the induced \( \text{BGL}^0(k) \)-module homomorphism. To extend the section \( s \) to a section \( s^0 : \text{BGL} \to \overline{U}^0 \) of the natural transformation \( \varphi^0 : \overline{U}^0 \to \text{BGL}^0 \) of functors on pointed motivic spaces. Note that

\[
\text{BGL}^0 = \text{BGL}^{0,0}[\beta, \beta^{-1}]
\]

for the Bott element \( \beta \in \text{BGL}^{2,1}(k) \) (see (3)). Thus for every pointed motivic space \( A \), every homogeneous element \( \alpha \in \text{BGL}^0(A) \) can be presented in a unique way in the form \( a \cup \beta^i \) with \( a \in \text{BGL}^{0,0}(A) \). Define

\[
\overline{s}_A^0 : \text{BGL}^0 \to \overline{U}^0
\]

by

\[
s_A(a \cup \beta^i) = s_A(a) \otimes \beta^i \in U^0(A), \quad a \in \text{BGL}^{0,0}(A).
\]

It is immediate that \( s_A^0 \) is natural in \( A \). The computation

\[
\overline{\varphi}_A^0(s^0(a \cup \beta^i)) = \overline{\varphi}_A^0(s(a) \otimes \beta^i) = \varphi^0(s(a)) \cup \beta^i = a \cup \beta^i
\]

proves the following

**Claim 3.1.6.** The map \( s_A^0 \) is a section of \( \overline{\varphi}_A^0 \).

If for a pointed motivic space \( A \) the map \( \overline{\varphi}_A^0 \) is an isomorphism, then \( s_A^0 \) is an isomorphism inverse to \( \overline{\varphi}_A^0 \). In particular, one has \( \overline{s}_A^0 \circ \overline{\varphi}_A^0 = \text{id} \).

The homomorphism \( \overline{\varphi}_A^0 \) is an isomorphism for the pointed motivic spaces \( U_n \) with \( n \geq N \), since \( U \) is weakly BGL-cellular. The class \([u_n] \in U^{2n,n}(U_n, *)\) of the canonical morphism \( u_n : \Sigma^\infty_{\text{BGL}}U_n(-n) \to U \) then satisfies the following relation:

\[
(\overline{s}_U^0 \circ \overline{\varphi}_U^0)([u_n]) = [u_n] \otimes 1 \in \overline{U}^0(U_n).
\]
Now we are ready to check that \( \varphi^0_A \) is an isomorphism for all small pointed motivic spaces. Recall that for a small pointed motivic space \( A \) there is a canonical isomorphism of the form

\[
U^{2,i}(A) = \colim_n [\Sigma^{2n,n}(A), U_{i+n}]_{H_*} \tag{17}
\]

where \( \Sigma^{2n,n} = \Sigma^n_{P_1} \). This isomorphism implies that for every element \( a \in U^{2,i}(A) \) there exists an integer \( n \geq 0 \) such that \( \Sigma^{2n,n}(a) = f^*([u_n]) \) for an appropriate map \( f: \Sigma^{2n,n}(A) \to U_{i+n} \) in the homotopy category \( H_* \). Here \( \Sigma^{2n,n} \) is the \( n \)-fold \( \Sigma \)-suspension of \( A \).

The surjectivity of \( \varphi^0_A \) is clear, since \( s^0_\cdot \) is its section. It remains to check the injectivity of \( \varphi^0_A \). Take a homogeneous element \( \alpha \in U^{2,i}(A) \subseteq U^0(A) \) such that \( \varphi^0_A(\alpha) = 0 \). It has the form \( \alpha = a \otimes \beta^m \) for a homogeneous element \( a \in U^0(A) \). Since the element \( \beta \) is invertible in \( \text{BGL}^*(k) \), one concludes \( \varphi^0_A(\alpha) = 0 \).

Choose an integer \( n \geq 0 \) such that \( \Sigma^{2n,n}(a) = f^*([u_n]) \). The map \( \varphi \) of \( P^1 \)-spectra respects the suspension isomorphisms. Thus \( \varphi^{\Sigma^{2n,n}_A}(\Sigma^{2n,n}(a)) = \Sigma^{2n,n}(\varphi_A(a)) = 0 \) and \( (s^0_\Sigma^{2n,n}_A \circ \varphi^{\Sigma^{2n,n}_A})(\Sigma^{2n,n}(a)) = 0 \) too. The chain of relations in \( U^0(\Sigma^{2n,n}A) \) given by

\[
0 = (s^0_{\Sigma^{2n,n}_A} \circ \varphi^{\Sigma^{2n,n}_A})(\Sigma^{2n,n}(a)) = (s^0_{\Sigma^{2n,n}_A} \circ \varphi^{\Sigma^{2n,n}_A})(f^*([u_n])) = f^*((s^0_{U_{i+n}} \circ \varphi_{U_{i+n}})([u_n])) = f^*([u_n] \otimes 1) = f^*([u_n] \otimes 1) = \Sigma^{2n,n}(a) \otimes 1
\]

implies that \( \Sigma^{2n,n}(a \otimes 1) = \Sigma^{2n,n}(a) \otimes 1 = 0 \). Because the \( n \)-fold suspension map

\[
\Sigma^{2n,n} : U^0(A) \to U^0(\Sigma^{2n,n}A)
\]

is an isomorphism, \( a \otimes 1 = 0 \) in \( U^0(A) = U^0(\Sigma^{2n,n}A) \). This proves the injectivity and hence the bijectivity of \( \varphi^0_A \) for all small motivic spaces.

To prove that \( \varphi_A \) is an isomorphism for all small motivic spaces we will use the fact that \( \varphi_A \) respects the \( P^1 \)-suspension isomorphisms. For every integer \( i \in \mathbb{Z} \) choose an integer \( n \geq 0 \) with \( n \geq i \). Then for a pointed motivic space \( A \) one may form the suspension \( G^m_{\text{ad}} \wedge S^{n-i} \wedge A = S^n \wedge S^{n-i,0} \wedge A \) in the category of pointed motivic spaces, which supplies the commutative diagram

\[
\begin{array}{ccc}
\text{BGL}^i(A) & \xrightarrow{\Sigma^{2n,n}} & \text{BGL}^i(S^{2n,n} \wedge A) \\
\varphi^0_A & \uparrow & \varphi^0_{\Sigma^{2n,n} \wedge A} \\
U^i(A) & \xrightarrow{\Sigma^{2n,n}} & U^i(S^{2n,n} \wedge A)
\end{array}
\]

\[
\begin{array}{ccc}
\text{BGL}^0(S^n \wedge S^{n-i,0} \wedge A) & \xleftarrow{\Sigma^{2n,n}} & \text{BGL}^0(S^n \wedge S^{n-i,0} \wedge A) \\
\varphi^0_{S^n \wedge S^{n-i,0}} & \downarrow & \varphi^0_{S^n \wedge S^{n-i,0}} \\
U^0(S^n \wedge S^{n-i,0} \wedge A) & \xleftarrow{\Sigma^{2n,n}} & U^0(S^n \wedge S^{n-i,0} \wedge A)
\end{array}
\]
with the suspension isomorphisms $\Sigma^{2n,n} = \Sigma^n_{\mathbb{P}^1}$ and $\Sigma^{k,0}$. The map $\varphi^0_B$ is an isomorphism for $B$ a small pointed motivic space, hence so is $\varphi^A_A$. We proved that the map $\varphi_A$ is an isomorphism for $A$ being small.

3.2 The BGL-cellularity of algebraic cobordism

**Theorem 3.2.1.** The oriented commutative ring $\mathbb{P}^1$-spectrum $(\text{MGL}, \text{th}^{\text{MGL}})$ from Example 2.2.3 is weakly BGL-cellular.

**Proof.** We will prove that the homomorphism $\varphi^0_A$ is an isomorphism for $A$ being one of the pointed motivic spaces $\text{Spec}(k)_+, \mathbb{P}^\infty_+, \text{Gr}(n)_+$ and $\text{Th}(\mathcal{T}_n) = \text{MGL}_n$.

The map $\varphi^0_k$ is an isomorphism, since it is the identity map. The case $n = 1$ of Theorem 3.0.6 implies that $\text{MGL}^0(\mathbb{P}^\infty) = \text{MGL}^0(k)[[c^\text{MGL}]]$, whence

$$\text{MGL}^0(\mathbb{P}^\infty) = \text{MGL}^0(k)[[c^\text{MGL}]]$$

(the formal power series on the first Chern class $c^\text{MGL}$ of the tautological line bundle $\mathcal{O}(-1)$). The same holds for BGL. Namely

$$\text{BGL}^0(\mathbb{P}^\infty) = \text{BGL}^0(k)[[c^\text{BGL}]].$$

By its definition the morphism $\varphi$ takes the orientation class $\text{th}^{\text{MGL}}$ to the orientation class $\text{th}^K$ and so it preserves the first Chern class. Whence the map $\varphi^0_{\mathbb{P}^\infty}$ coincides with the map of formal power series induced by the isomorphism $\varphi^0_k$ of coefficients rings. Hence $\varphi^0_{\mathbb{P}^\infty}$ is an isomorphism as well.

Consider now $A = \text{Gr}(n)_+$. By Theorem 3.0.6 its MGL-cohomology ring is the ring of formal power series on the Chern classes of the tautological bundle $\mathcal{T}_n$ over the coefficient ring $\text{MGL}^*(k)$. The same holds for the BGL-cohomology ring. As observed above, the map $\varphi$ preserves the first Chern class, thus it preserves all Chern classes. Thus $\varphi^0_{\text{Gr}(n)}$ is an isomorphism.

Now consider $A = \text{Th}(\mathcal{T}_n)$. The morphism $\varphi$ respects Thom classes (see (6) and (7)). The vertical arrows in the commutative diagram

$$
\begin{array}{ccc}
\text{MGL}^0((\text{Th}(\mathcal{T}_n), *)) & \xrightarrow{\varphi^0_{\text{Th}(\mathcal{T}_n), *}} & \text{BGL}^0(\text{Th}(\mathcal{T}_n)) \\
\text{MGL}^0(\text{Gr}(n)) & \xrightarrow{\varphi^0_{\text{Gr}(n)}} & \text{BGL}^0(\text{Gr}(n))
\end{array}
$$

are isomorphisms induced by the Thom isomorphism 3.0.4. The map $\varphi^0_{\text{Gr}(n)}$ is an isomorphism by the preceding case, thus $\varphi^0_{\text{Th}(\mathcal{T}_n), *}$ is an isomorphism.

$\square$
3.3 The main result

Let $k$ be a field and $S = \text{Spec}(k)$. By Theorem [5, Theorem 2.2.1] and Example 2.2.3 there exists a unique monoid homomorphism

$$
\varphi : \text{MGL} \to \text{BGL}
$$

in $\text{SH}(S)$ such that $\varphi (\text{th}^{\text{MGL}}) = \text{th}^{K}$. It induces a natural transformation

$$
\bar{\varphi}_{A} : \text{MGL}^{*}(A) = \text{MGL}^{*}(A) \otimes_{\text{MGL}^{0}(k)} \text{BGL}^{0}(k) \to \text{BGL}^{*}(A).
$$

Theorem 3.3.1. The homomorphism

$$
\bar{\varphi}_{A} : \text{MGL}^{*}(A) \otimes_{\text{MGL}^{0}(k)} \text{BGL}^{0}(k) \to \text{BGL}^{*}(A)
$$

is an isomorphism for all small pointed motivic spaces.

Proof. In fact, $(\text{MGL}, \text{th}^{\text{MGL}})$ is Quillen universal by [11] and [5], and weakly BGL-cellular by Theorem 3.2.1. Theorem 3.1.4 completes the proof. \qed

Remark 3.3.2. There is an unpublished result due to Morel and Hopkins, which states that there is a canonical isomorphism of the form

$$
\text{MGL}^{*,*}(X) \otimes_{L} \mathbb{Z}[\beta, \beta^{-1}] \to \text{BGL}^{*,*}(X)
$$

where $L$ denotes the Lazard ring carrying the universal formal group law. If the canonical homomorphism $L \to \text{MGL}^{0}(k)$ is an isomorphism, Theorem 3.3.1 implies their result.

Let $X$ be a smooth $k$-scheme and $Z \subseteq X$ a closed subset, with open complement $U \subseteq X$. Consider the small pointed motivic space $X/U$ and take the quotients of both sides of the isomorphism (19) modulo the principal ideal generated by the element $1 \otimes (\beta + 1)$. Corollary 2.1.1 then implies that the natural transformation

$$
\bar{\varphi}_{X/U} : \text{MGL}^{*}(X/U) \otimes_{\text{MGL}^{0}(k)} \mathbb{Z} \to \text{KTT}^{*,*}(X)
$$

is an isomorphism, where $\text{KTT}^{*,*}(X)$ are the Thomason-Trobaugh $K$-groups with supports. This family of isomorphisms shows that the functor

$$
(X, U) \mapsto \text{MGL}^{*}(X/U) \otimes_{\text{MGL}^{0}(k)} \mathbb{Z}
$$

is a ring cohomology theory in the sense of [6]. This implies the first part of our main result.
Theorem 3.3.3 (Main Theorem). Let \( X \in \text{Sm}/k \) and \( Z \subseteq X \) be a closed subset.

- The family of isomorphisms
  \[
  \tilde{\varphi}_{X/(X-Z)} : \text{MGL}^* (X/(X-Z)) \otimes_{\text{MGL}^0 (k)} Z \to K_{-*}^{TT} (X)
  \]
  (21)
  form an isomorphism \( \tilde{\varphi} \) of ring cohomology theories on \( \text{Sm}\mathcal{O}_p/k \).

- The natural isomorphism \( \tilde{\varphi} \) respects orientations provided that \( \text{MGL}^* \) and \( K_{-*}^{TT} \) are considered as oriented cohomology theories in the sense of [6] with orientations given by the Thom class \( th_{\text{MGL}} \otimes 1 \) from 2.2.3 and the Chern structure \( L/X \mapsto [O] - [L^{-1}] \). In particular, the composition
  \[
  \xymatrix{
  \text{MGL}^0 (k) \ar[r]^{a \mapsto a \otimes 1} & \text{MGL}^0 (k) \otimes Z \ar[r]^{b \otimes c \mapsto \varphi (b) \cdot c} & Z
  }
  \]
  sends the class \([X] \in \text{MGL}^0 (X)\) of a smooth projective \( k \)-variety \( X \) to the Euler characteristic \( \chi (X, \mathcal{O}_X) \) of the structure sheaf \( \mathcal{O}_X \).

Proof. The first part is already proven. To prove the second part, consider the orientations \( th_{\text{MGL}} \) and \( th_K \) from 2.2.3. Note that by the very definition of \( \varphi \) it sends \( th_{\text{MGL}} \) to \( th_K \). Thus it respects the Chern structures on \( \text{MGL}^* \) and \( \text{BGL}^* \) described in Section 3.

The quotient map \( \text{BGL}^* \to K_{-*}^{TT} \) takes the Bott element \( \beta \) to \((-1)\). Thus it takes the Chern structure on \( \text{BGL}^* \) to the Chern structure on \( K_{-*}^{TT} \) given by \( L/X \mapsto [0] - [L^{-1}] \in K_0 (X) \). This shows that \( \tilde{\varphi} : \text{MGL}^* (-) \otimes_{\text{MGL}^0 (k)} Z^* \to K_{-*}^{TT} \) respects the orientations described in the Theorem 3.3.3.

Let \( f \mapsto f_{\text{MGL}} \) resp. \( f \mapsto f_K \) be the integrations on \( \text{MGL}^* \) resp. \( K_{-*}^{TT} \) given by these Chern structures via Theorem [8, Thm. 4.1.4]. By Theorem [7, Thm. 1.1.10] the composition \( \text{MGL}^* \to \text{BGL}^* \to K_{-*}^{TT} \) respects the integrations on \( \text{MGL}^* \) and \( K_{-*}^{TT} \) since it preserves the Chern structures. In particular, given a smooth projective \( S \)-scheme \( f : X \to \text{Spec}(k) \), the diagram

\[
\xymatrix{
\text{MGL}^0 (X) \otimes_{\text{MGL}^0 (k)} Z \ar[r]^\tilde{\varphi} & K_{0}^{TT} (X) \\
\text{MGL}^0 (X) \otimes_{\text{MGL}^0 (k)} Z \ar[r]^\tilde{\varphi} & K_{0}^{TT} (k) \\
\ar[u]^{f_{\text{MGL}}} & \ar[u]^{f_K} 
}
\]

commutes where \( f_{\text{MGL}} \) and \( f_K \) are the push-forward maps for \( \text{MGL}^* \) and \( K_{-*}^{TT} \) respectively. The integration \( f \mapsto f_K \) on \( K_{-*}^{TT} \) respecting the Chern structure \( L \mapsto [0] - [L^{-1}] \) coincides with the one given by the higher direct images by Theorem [7, Thm. 1.1.11]. The last one sends the class \([V] \in \text{MGL}^0 (V)\) to \([V] \in K_{-*}^{TT} (V)\) via \( \tilde{\varphi} \).
$K_0(X)$ of a vector bundle $V$ over a smooth projective variety $X$ to the Euler characteristic $\chi(X, V)$ of the sheaf $V$ of sections of $V$.

Recall that for an oriented cohomology theory $A$ with a Chern structure $L \mapsto c(L)$ and for a smooth projective variety $f: X \to \text{Spec}(k)$ its class $[X]_A \in A(\text{Spec}(k))$ is defined as $f_A(1)$. Here $f_A: A(X) \to A(\text{Spec}(k))$ is the push-forward morphism respecting the Chern structure (see [8, Thm. 4.1.4]). The notation $f_A$ is misleading, since $f_A$ depends on the Chern structure as well. Taking the element $1 \in \text{MGL}^{0,0}(X)$ and using the commutativity of the very last diagram we see that

$$\bar{\varphi}([X]_{\text{MGL}} \otimes 1) = \chi(X, O_X).$$

Whence the Theorem. \hfill \square

4 Acknowledgements

The work was supported by the SFB 701 at the Universität Bielefeld, the RTN-Network HPRN-CT-2002-00287, the grants RFFI 03-01-00633a and INTAS-05-100008-8118, and the Fields Institute for Research in Mathematical Science.

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