DETERMINATION OF A WAVE FIELD IN A LATERALLY INHOMOGENEOUS MEDIUM FROM BOUNDARY DATA

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The Cauchy problem is studied for a perturbed wave equation in a half-plane with data given on a part of the space-time boundary. The equation in consideration describes a wave process in a laterally inhomogeneous medium. A reconstruction algorithm is proposed, which is applicable to the problem of determining a nonstationary wave field from boundary data arising in geophysics.

Bibliography: 13 titles.

Dedicated to Mikhail I. Belishev on the occasion of his jubilee

1. Introduction

Let \( u(x, y, t) \) be a solution to the following hyperbolic equation:

\[
\frac{\partial^2 u}{\partial t^2} - \Delta u + qu = 0,
\]

(1)

\((x, y) \in \mathbb{R} \times [0, \infty)\) (the half-plane), \( t \in \mathbb{R} \), and let satisfy the conditions

\[
u|_{y=0} = f, \quad \partial_y u|_{y=0} = g.
\]

(2)

In the present paper, the problem of determination of the solution \( u \) from the Cauchy data \( f, g \) is investigated. The following particular case is considered: the coefficient \( q \) in Eq. (1) is assumed to be a function of \( x \). Thus Eq. (1) is a model of a nonstationary wave process in a laterally inhomogeneous medium. We will obtain a relation, which allows us to determine \( u \) at any given point \((x_0, y_0, t_0)\) from the values of the functions \( f, g \) on a certain bounded set of points \((x, t)\), which depends on \((x_0, y_0, t_0)\). We will show that this relation applies in the case where Eq. (1) is satisfied in a cylinder \( \Omega \times \mathbb{R} \), where \( \Omega \) is a subset of the half-plane. The intersection of the boundary \( \partial \Omega \) with the boundary of the half-plane is assumed to be nonempty, and the Cauchy data \( f, g \) are assumed to be given on some set the spatial projection of which lies in this intersection.

The Cauchy problem (1), (2) is ill-posed in the sense of Hadamard [1]. However, the solution \( u \) is uniquely determined in some part of the space-time cylinder dependent on the set on which \( f, g \) are given. This follows from the unique continuation property across a noncharacteristic surface for Eq. (1). This property is established for various types of linear partial differential equations [1,2]. The most complete results concerning the hyperbolic equations were obtained in [3]. In particular, these results imply the unique continuation property for Eq. (1), assuming that \( q \) is a sufficiently smooth function of \( x, y \).

In the case \( q \equiv 0 \), a number of algorithms for solving the problem in consideration are known. One of the pioneering results is that of R. Courant concerning the ultrahyperbolic equations in the half-space (see [4]). The Cauchy problem for the wave equation in a domain was considered in [5,6]. The inversion formula for the problem in the three-dimensional half-space obtained in [7] allows determining a solution to the wave equation from the Cauchy data given on a certain unbounded subset of the space-time boundary.

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Problems of determination of solutions to hyperbolic equations from boundary data arise in geophysics [8], photoacoustic tomography [9], tsunami source identification problems [10], and coefficient inverse problems [11].

2. On the Schrödinger operator on the line

To solve the Cauchy problem (1), (2), we will deal with the generalized eigenfunction expansion of the solution $u$ associated with the Schrödinger operator $L = -\partial_x^2 + q(x)$ on the line. In the case $q \equiv 0$, this expansion is identical to the Fourier transform in $x$

$$\hat{u}(k, y, t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ikx} u(x, y, t) \, dx,$$

which satisfies the relations

$$\partial_t^2 \hat{u} - \partial_y^2 \hat{u} + k^2 \hat{u} = 0, \quad \hat{u}|_{y=0} = \hat{f}, \quad \partial_y \hat{u}|_{y=0} = \hat{g}.$$

For arbitrary $q$, we will obtain the same Cauchy problem with a spectral parameter. We will solve this problem and apply the inverse transform in $k$, which will result in an expression for $u$ involving the data $f, g$. After that we will show that this expression involves the Cauchy data on a bounded set.

We will need some facts on the Schrödinger operator [12]. The latter is well defined under some restrictions on the potential $q$. We will consider the Schrödinger operator only with a $C^\infty$-smooth compactly supported real-valued potential $q$. In this case, the operator $L$ defined on functions from $L^2(\mathbb{R})$ that have the second derivative from $L^2(\mathbb{R})$ is a self-adjoint operator in $L^2(\mathbb{R})$. The operator $L$ has the absolutely continuous spectrum consisting of finite number of negative simple eigenvalues, which will be denoted by $\{-\kappa_l^2\}_{l=1}^M$, $\kappa_l > 0$.

Let $A$ be a sufficiently large number such that supp $q \subset [-A, A]$. Any solution to the Schrödinger equation

$$-\partial_x^2 \varphi + q \varphi = k^2 \varphi \tag{3}$$

equals a linear combination of the exponentials $e^{\pm ikx}$ if $|x| > A$:

$$\varphi(x) = \alpha_+ e^{ikx} + \alpha_- e^{-ikx}, \quad x > A, \tag{4}$$

$$\varphi(x) = \beta_+ e^{ikx} + \beta_- e^{-ikx}, \quad x < -A. \tag{5}$$

For real nonzero $k$, we define the function $\varphi_1(x, k)$ as the solution to Eq. (3) on the whole line that satisfies (4), (5) with $\alpha_- = 1/\sqrt{2\pi}$, $\beta_+ = 0$. Next we define the function $\varphi_2(x, k)$ as the solution to Eq. (3) that satisfies (4), (5) with $\alpha_- = 0$, $\beta_+ = 1/\sqrt{2\pi}$. For $k \in \mathbb{R} \setminus \{0\}$, the pair of the functions $\varphi_{1,2}(x, k)$ (as well as $\varphi_{1,2}(x, -k)$) forms a basis in the space of the generalized eigenfunctions corresponding to the point $k^2$ of the absolutely continuous spectrum. Further we will consider $\varphi_{1,2}(x, k)$ only for $k > 0$.

Along with $\varphi_{1,2}(x, k)$, we introduce the function $\varphi_0(x, k)$ defined for $x \in \mathbb{R}$ and $k$ from the finite set of imaginary numbers $\{i\kappa_l\}_{l=1}^M$ related to the discrete spectrum. For $k = i\kappa_l$, we define $\varphi_0(\cdot, i\kappa_l)$ as the normed eigenfunction of $L$ corresponding to the eigenvalue $-\kappa_l^2$. The normed eigenfunction is determined uniquely up to a factor of the form $e^{i\alpha}$, which is chosen arbitrarily.
An arbitrary function \( \psi \in L_2(\mathbb{R}) \) assumes the eigenfunction expansion \( \hat{\psi} = (\hat{\psi}_0, \hat{\psi}_1, \hat{\psi}_2) \) associated with the operator \( L \),

\[
\hat{\psi}_j(k) = \int_{\mathbb{R}} \psi(x) \varphi_j(x, k)^* \, dx, \quad j = 0, 1, 2
\]  

(henceforth * means the complex conjugation), where the functions \( \hat{\psi}_{1,2}(k) \) are defined for \( k > 0 \), while \( \hat{\psi}_{0}(k) \) is defined for \( k \in \{i\varepsilon_l\}_{l=1}^M \). This expansion is a unitary transformation acting in the following spaces

\[
L_2(\mathbb{R}) \to \mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2, \quad \mathcal{H}_0 = \mathbb{C}^M, \quad \mathcal{H}_1 = \mathcal{H}_2 = L_2(\mathbb{R}_+).
\]  

The inverse transformation has the form

\[
\psi(x) = \sum_{j=1,2} \int_{\mathbb{R}_+} \hat{\psi}_j(k) \varphi_j(x, k) \, dk + \sum_{l=1}^M \hat{\psi}_0(i\varepsilon_l) \varphi_0(x, i\varepsilon_l).
\]  

The diagonal representation of the operator \( L \) is based on the eigenfunction expansion. Namely, if \( \psi \) belongs to the domain of definition of the operator \( L \) and \( \tau = L\psi \), then

\[
\hat{\tau}_j(k) = k^2 \hat{\psi}_j(k), \quad j = 0, 1, 2.
\]  

It can be seen from the definitions given above that in the relations of the form (9), the variable \( k \) can be any positive number for \( j = 1, 2 \), whereas \( k \in \{i\varepsilon_l\}_{l=1}^M \) for \( j = 0 \).

For a function \( \Phi(\lambda) \) of real variable, the function of the operator \( \Phi(L) \) can be described by the relation

\[
\hat{\tau}_j(k) = \Phi(k^2) \hat{\psi}_j(k), \quad j = 0, 1, 2,
\]  

where \( \tau = \Phi(L)\psi \). We will also use the following representation for \( \Phi(L) \) in the case where \( \Phi(\lambda) \) is a complex-analytic function

\[
\Phi(L) = \frac{1}{2\pi i} \int_{\Gamma} \Phi(\lambda)(L - \lambda I)^{-1} \, d\lambda,
\]  

where the integral is taken over an appropriate clockwise oriented contour \( \Gamma \) that encircles the spectrum of the operator \( L \). The resolvent \( (L - \lambda I)^{-1} \) in this representation is an integral operator the Schwartz kernel of which equals the Green’s function \( G_\lambda(x_0, x) \) (which is continuous) of Eq. (3) for \( \lambda = k^2 \). Further we will use the estimate (see, e.g., [12])

\[
|G_{k^2}(x_0, x)| \leq C(q) e^{-\text{Im} k |x-x_0|/|k|}
\]  

valid for any \( k \in \mathbb{C} \) such that \( \text{Im} k \geq 1 + \max\{\varepsilon_l\}_{l=1}^M \).

### 3. Determination of the solution \( u \)

In this section, we will describe the scheme of solving the Cauchy problem considered, assuming that \( q \) is compactly supported. We will also assume that the solution \( u(x, y, t) \) to Eq. (1) (and so the Cauchy data \( f, g \)) is compactly supported in \( x \). Namely, for any \( T > 0 \), the restriction of \( u \) to the set \( \{0 \leq y \leq T, |t| \leq T\} \) is compactly supported. Note that such solutions do exist since one can take a solution to an initial boundary value problem for Eq. (1) with compactly supported initial data (recall that the given equation describes waves propagating with finite velocity). In Sec. 5, these restrictions on the supports of \( q \) and \( u \) will be eliminated.
Applying the transformation (6) to Eq. (1), we obtain \((j = 0, 1, 2)\)

\[
0 = \int_{\mathbb{R}} \varphi_j(x, k)^* (\partial_t^2 - \partial_y^2 + \mathcal{L})u(x, y, t) \, dx = (\partial_t^2 - \partial_y^2 + k^2) \widehat{u}_j(k, y, t).
\]

Here we switched the order of differentiation with respect to \(y, t\) and integration, which is justified since the function \(u\) is smooth, \(u(\cdot, y, t)\) is compactly supported, and \(\varphi_j\) is bounded in \(x\) for any fixed \(k\). We also applied relation (9). Treating relations (2) in a similar way, we arrive at the following Cauchy problem for \(\widehat{u}_j(k, y, t), j = 0, 1, 2:\)

\[
\partial_t^2 \widehat{u}_j - \partial_y^2 \widehat{u}_j + k^2 \widehat{u}_j = 0, \quad \widehat{u}_j|_{y=0} = \widehat{f}_j, \quad \partial_y \widehat{u}_j|_{y=0} = \widehat{g}_j. \tag{13}
\]

The fundamental solution to this problem has the form

\[
\theta(y - |t|) r(k, y, t), \quad r(k, y, t) = \frac{1}{2} J_0 \left(ik \sqrt{y^2 - t^2}\right)
\]

(\(\theta\) is the Heaviside function, \(J_0\) is the Bessel function of the first kind of order 0). This can be verified directly or derived from [4, Chap. V]. Further we will need estimates of the functions \(r(k, y, t)\) and \(\partial_y r(k, y, t)\) for large (generally, complex) \(k\). To obtain these estimates, we use the following representation of the Bessel function [4]:

\[
J_0(\zeta) = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} e^{i \zeta \sin s} \, ds,
\]

which means that

\[
r(k, y, t) = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} e^{ik \sqrt{y^2 - t^2} \sin s} \, ds = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} \text{ch} \left(k \sqrt{y^2 - t^2} \cdot \sin s\right) \, ds,
\]

\[
\partial_y r(k, y, t) = \frac{ky}{2\pi} \int_{-\pi/2}^{\pi/2} \text{sh} \left(k \sqrt{y^2 - t^2} \cdot \sin s\right) \frac{\sin s \, ds}{\sqrt{y^2 - t^2}}
\]

(note that \(\partial_y r(k, y, t)\) is regular at \(|t| = y\)). These relations imply the desired estimates for arbitrary \(k \in \mathbb{C}\):

\[
|r(k, y, t)| \leq \frac{1}{2} e^{k'|t|\sqrt{y^2 - t^2}} , \quad |\partial_y r(k, y, t)| \leq Ck^2 y e^{k'|t|\sqrt{y^2 - t^2}} , \tag{14}
\]

where \(k' = \text{Re}k\). In the second estimate, we used the inequality \(|\zeta^{-1} \text{sink}\zeta| \leq Ce^{\text{Re}\zeta}|\zeta|\). Note that these estimates are far from being optimal. However, they will suffice for our purposes.

Now write the solution to the problem (13) in terms of the fundamental solution

\[
\widehat{u}_j(k, y_0, t_0) = \frac{1}{2} \left( \widehat{f}_j(k, t_0 + y_0) + \widehat{f}_j(k, t_0 - y_0) \right) + \int_{|t-t_0|\leq y_0} \left[ \partial_{y_0} r(k, y_0, t - t_0) \widehat{f}_j(k, t) + r(k, y_0, t - t_0) \widehat{g}_j(k, t) \right] \, dt.
\]
Multiplying both sides by \( e^{-hk^2} \), \( h > 0 \), we obtain
\[
e^{-hk^2}\tilde{u}_j(k, y_0, t_0) = \frac{1}{2} e^{-hk^2} \left( \tilde{f}_j(k, t_0 + y_0) + \tilde{f}_j(k, t_0 - y_0) \right)
+ \int_{|t-t_0| \leq y_0} e^{-hk^2} \left[ \partial_{y_0} r(k, y_0, t - t_0) \tilde{f}_j(k, t) + r(k, y_0, t - t_0) \tilde{g}_j(k, t) \right] dt.
\]
For fixed \( y_0, t_0 \), the function \( \tilde{u}_j(k, y_0, t_0) \) of the variable \( k \) belongs to \( \mathcal{H}_j \) (see (7)), hence both sides of the last relation belong to \( \mathcal{H}_j \) as well. Since the index \( j \) takes all possible values \( 0,1,2 \), both sides of the relation can be regarded as elements of \( \mathcal{H} \). In view of estimates (14), the regularizing factor \( e^{-hk^2} \) forces the integrand on the right-hand side to decay rapidly for large \( k \). Therefore the integrand can be considered as an element of the space \( \mathcal{H} \) that depends continuously on \( t \), whereas the integral itself can be regarded as that of a continuous \( \mathcal{H} \)-valued function of \( t \). Now apply the inverse transformation (8) to both sides of the relation. This transformation applied to the last term on the right-hand side can be passed under the integral sign. In view of (10), we obtain the following relation in \( L_2(\mathbb{R}) \)
\[
e^{-h\mathcal{L}} u(\cdot, y_0, t_0) = \frac{1}{2} e^{-h\mathcal{L}} (f(\cdot, t_0 + y_0) + f(\cdot, t_0 - y_0))
+ \int_{|t-t_0| \leq y_0} \left[ \Phi^D_h(\mathcal{L}; y_0, t - t_0) f(\cdot, t) + \Phi^N_h(\mathcal{L}; y_0, t - t_0) g(\cdot, t) \right] dt. \tag{15}
\]
Here
\[
\Phi^D_h(\lambda; y, t) = e^{-h\lambda} \partial_y r\left(\sqrt{\lambda}, y, t\right), \quad \Phi^N_h(\lambda; y, t) = e^{-h\lambda} r\left(\sqrt{\lambda}, y, t\right).
\]
On both sides of (15), the functions of the operator \( \mathcal{L} \) act on the elements of \( L_2(\mathbb{R}) \). Note that the corresponding operators are bounded in \( L_2(\mathbb{R}) \), since the functions \( e^{-h\lambda}, \Phi^D_h(\lambda; y, t), \) and \( \Phi^N_h(\lambda; y, t) \) of \( \lambda \) are bounded on the spectrum of \( \mathcal{L} \).

Now we pass to the limit in relation (15) as \( h \to 0 \). For \( \psi \in L_2(\mathbb{R}) \), the function \( e^{-h\mathcal{L}} \psi \) converges to \( \psi \) in \( L_2(\mathbb{R}) \) as \( h \to 0 \). Applying this to \( \psi = u(\cdot, y_0, t_0) \) and \( \psi = f(\cdot, t_0 \pm y_0) \), we obtain
\[
u(\cdot, y_0, t_0) = \frac{1}{2} (f(\cdot, t_0 + y_0) + f(\cdot, t_0 - y_0))
+ \lim_{h \to 0} \int_{|t-t_0| \leq y_0} \left[ \Phi^D_h(\mathcal{L}; y_0, t - t_0) f(\cdot, t) + \Phi^N_h(\mathcal{L}; y_0, t - t_0) g(\cdot, t) \right] dt \tag{17}
\]
(the limit is understood in the sense of \( L_2(\mathbb{R}) \)).

Relation (17) allows determining the solution \( u \) from the Cauchy data \( f, g \). This relation is local in \( t \) since for fixed \( y_0, t_0 \), the function \( u(\cdot, y_0, t_0) \) is determined from the Cauchy data given for \( t \) ranging over the bounded interval \( |t-t_0| \leq y_0 \). In Sec. 4, we will show that relation (17) can be localized both in \( x \) and \( t \) in the sense that it is possible to determine \( u \) at a fixed point \( (x_0, y_0, t_0) \) from the values of the functions \( f, g \) on a bounded set dependent on \( (x_0, y_0, t_0) \). However, we first verify that relation (17) is a pointwise equality. To do this, we consider once more the terms of the form \( e^{-h\mathcal{L}} \psi \) in relation (15). Observe that \( e^{-h\mathcal{L}} \psi = \Psi(\cdot, h) \), where \( \Psi \) is the solution to the parabolic equation
\[
\partial_h \Psi = \Delta \Psi - q \Psi,
\]
in which \( h \) plays the role of time, with the initial data \( \Psi|_{h=0} = \psi \). Therefore for any \( h > 0 \), the function \( e^{-h\mathcal{L}} \psi \) is continuous in \( \mathbb{R} \). Moreover, in the context of (15), the initial function
ψ is smooth and has compact support, which implies that
\[ (e^{-hL} \psi)(x_0) = \Psi(x_0, h) \to \psi(x_0), \quad h \to 0 \]
for any \( x_0 \in \mathbb{R} \) (see, e.g., [13, Chap. IV]). From these observations, it follows that the left-hand side and the first term on the right-hand side of (15) are continuous functions in \( \mathbb{R} \), having pointwise limits as \( h \to 0 \). Hence the same is true for the integral on the right-hand side of (15). Relation (17) now reads as follows:
\[
u(x_0, y_0, t_0) = \frac{1}{2} (f(x_0, t_0 + y_0) + f(x_0, t_0 - y_0)) + \lim_{h \to 0} \left\{ \int_{|t-t_0| \leq y_0} \left[ \Phi^D_h(\mathcal{L}; y_0, t - t_0) f(\cdot, t) + \Phi^N_h(\mathcal{L}; y_0, t - t_0) g(\cdot, t) \right] dt \right\} \quad (17')
\]
for any \( x_0, t_0 \in \mathbb{R} \), \( y_0 > 0 \). The expression in the figure brackets is the value of the integral (the latter is understood as a function in \( \mathbb{R} \)) at \( x_0 \).

4. LOCALIZATION OF RELATION (17')

Now we investigate the functions of the operator \( \mathcal{L} \) occurring on the right-hand side of (17'). According to (11), we have (\( |t| \leq y_0 \))
\[
\Phi^N_h(\mathcal{L}; y_0, t) = \frac{1}{2\pi i} \int_{\Gamma} \Phi^N_h(\lambda; y_0, t)(\mathcal{L} - \lambda I)^{-1} d\lambda.
\]
Make a substitution \( k = \sqrt{\lambda} \) in the integral, assuming that \( \text{Im} \, k > 0 \) on the contour of integration. After a deformation of the contour, we obtain
\[
\Phi^N_h(\mathcal{L}; y_0, t) = \frac{1}{\pi i} \int_{-\infty + ic/h}^{+\infty + ic/h} \Phi^N_h(k^2; y_0, t)(\mathcal{L} - k^2 I)^{-1} k dk.
\]
Here \( c, h \) are any positive numbers such that \( c/h > z_l \), \( l = 1, \ldots, M \). For the Schwartz kernel \( K^N_h(x_0, x; y_0, t) \) of the operator \( \Phi^N_h(\mathcal{L}; y_0, t) \), this relation implies
\[
K^N_h(x_0, x; y_0, t) = \frac{1}{\pi i} \int_{-\infty + ic/h}^{+\infty + ic/h} \Phi^N_h(k^2; y_0, t) G_{k^2}(x_0, x) k dk
\]
(recall that \( G_{k^2}(x_0, x) \) is the Green’s function of Eq. (3) for \( \lambda = k^2 \)). The function \( \Phi^N_h \) decays rapidly for large \( k \). Taking also into account that the kernel \( G_{k^2}(x_0, x) \) is bounded on the contour of integration, which follows from (12), and is continuous in \( x_0, x \), the kernel \( K^N_h \) is continuous in \( x_0, x \) as well.

Making a substitution in the integral in the previously obtained formula for \( K^N_h \) and using the definition (16), for \( |t| \leq y_0 \) we obtain the relation
\[
K^N_h(x_0, x; y_0, t) = \frac{1}{\pi i} \int_{-\infty + ic}^{+\infty + ic} e^{-k^2/h} r(k/h, y_0, t) G_{(k/h)^2}(x_0, x) h^{-2} k dk
\]
\[
(18)
\]
For any fixed \( c \) and sufficiently small \( h \), the absolute value of the integrand is estimated using (12) and (14) by the expression (up to the factor \( C(q) \))
\[
h^{-1} e^{-c|x-x_0|/h} e^{(|k| |z-\text{Re}k^2|/h)} = h^{-1} e^{-c|x-x_0|/h} e^{(|k| |z-k^2+c^2|/h)},
\]
where $z = \sqrt{y_0^2 - t^2}$. Since $e^{k'}z/h < e^{k'\frac{z}{h}} + e^{-k'\frac{z}{h}}$, the resulting expression is majorized by
\[
h^{-1}e^{-c|x-x_0|+z^2/4c^2/h}\sum_{\pm}e^{-(k'\pm z/2)^2/h}.
\]
Now taking $c = |x-x_0|/2$ and integrating with respect to $k'$, we deduce that the absolute value of the integral (18) does not exceed
\[
Ch^{-1/2}e^{(y_0^2-t^2-(x-x_0)^2)/(4h)}.
\]
Now it can be seen that in the case $(x-x_0)^2 + t^2 > y_0^2$, the integral (18) tends to zero as $h \to 0$. Thus
\[
\lim_{h \to 0} K_h^N(x_0, x; y_0, t) = 0, \quad (x-x_0)^2 + t^2 > y_0^2,
\]
where the limit is uniform with respect to $x$, $y_0$, $t$ belonging to any compact subset of
\[
\{(x,y_0,t) \mid (x-x_0)^2 + t^2 > y_0^2, \quad |t| \leq y_0\}.
\]
For the kernel $K_h^D(x_0, x; y_0, t)$ of the operator $\Phi_h^D(L; y_0, t)$, the assertion analogous to (19) holds true.

It is convenient to introduce the notation
\[
K_h = (K_h^D, K_h^N), \quad F = \begin{pmatrix} f \\ g \end{pmatrix}.
\]
In this notation, relation (19) and its counterpart for $K_h^D$ takes the form
\[
\lim_{h \to 0} K_h(x_0, x; y_0, t) = 0, \quad (x-x_0)^2 + t^2 > y_0^2,
\]
where the limit is uniform in the sense specified in (19).

Now we write relation (17’ in terms of $K_h$:
\[
u(x_0, y_0, t_0) = \frac{1}{2} (f(x_0, t_0 + y_0) + f(x_0, t_0 - y_0))
+ \lim_{h \to 0} \int_{|t-t_0| \leq y_0} dt \int \mathbb{R} K_h(x_0, x; y_0, t-t_0)F(x, t) \, dx.
\]
Using relation (19), in which the limit is uniform on compact sets, and our assumption that the functions $u$, $f$, $g$ are compactly supported in $x$, we can replace the integral with respect to $x$ over $\mathbb{R}$ by that over the interval $|x-x_0| \leq y_0 + \varepsilon, \varepsilon > 0$. Thus we arrive at the relation
\[
u(x_0, y_0, t_0) = \frac{1}{2} (f(x_0, t_0 + y_0) + f(x_0, t_0 - y_0))
+ \lim_{h \to 0} \int_{|t-t_0| \leq y_0} dt \int_{|x-x_0| \leq y_0 + \varepsilon} K_h(x_0, x; y_0, t-t_0)F(x, t) \, dx,
\]
which is a localized version of relation (17’), since $u$ is determined from the Cauchy data on the bounded set.

The derivation of (20) from (17’) relies on the fact that the kernel $K_h$ tends to zero as $h \to 0$ if $(x, t)$ lies outside a certain bounded set. It should be noted, however, that if $(x, t)$ belongs to this set, then $K_h$ generally grows exponentially as $h \to 0$. In the case $q \equiv 0$, this follows from the quite explicit formula for this kernel obtained in [5]. Due to the growth of the integrand in (20), the limit of the integral generally does not exist if $f$, $g$ are arbitrary smooth functions, which do not correspond to any Cauchy data. Therefore if the data are known with some error, it is necessary to approximate this limit with the value of the integral computed for some positive $h$. 

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5. Generalization of relation (20)

In this section, we will show that the assumption that Eq. (1) is satisfied in the entire half-plane is not necessary for the validity of relation (20). We will consider the case where the solution $u(x,y,t)$ is defined for $(x,y) \in \Omega$, $t \in \mathbb{R}$, where $\Omega$ is a bounded relatively open subset of the half-plane $\{y \geq 0\}$. We will assume that the intersection of the boundary $\partial \Omega$ with the line $\{y = 0\}$ is nonempty. The Cauchy data will be assumed to be given on the set the spatial projection of which lies in this intersection.

Equation (1) makes sense, provided that the coefficient $q(x)$ is defined on a sufficiently large interval dependent on $\Omega$. We will assume that $q(x)$ has a smooth extension to $\mathbb{R}$. Clearly, this extension can be chosen so that it have compact support. In this case, the Schrödinger operator $L$ is well defined, as well as the kernel $K_h$ occurring on the right-hand side of (20). The following theorem imply, in particular, that the right-hand side in (20) does not depend on the choice of the extension of the potential $q$.

**Theorem 1.** Suppose that a $C^\infty$-smooth function $u(x,y,t)$ satisfies Eq. (1) in the cylinder $\Omega \times \mathbb{R}$. The coefficient $q$ in the equation is assumed to be a $C^\infty$-smooth compactly supported real-valued function of $x \in \mathbb{R}$. Let $(x_0,y_0) \in \Omega$, and let the closed set

$$\{(x,y) \mid |x - x_0| \leq y_0 - y, y \geq 0\}$$

be a subset of $\Omega$ (see Fig. 1). Then for any sufficiently small positive $\varepsilon$, relation (20) holds true, where $f, g$ are the Cauchy data determined by relations (2).

![Fig. 1. The set $\Omega$ and the set (21) (the hatched region).](image)

To prove the theorem, we choose a smooth function $\chi(x,y)$ in the half-plane $\{y \geq 0\}$ that has the support in $\Omega$ and is equal to unity in a neighbourhood of the set (21). Put $\tilde{u}(x,y,t) = \chi(x,y)u(x,y,t)$ if $(x,y) \in \Omega$, and $\tilde{u}(x,y,t) = 0$ otherwise. Thus we obtain a smooth function defined for $(x,y)$ from the entire half-plane and for all $t \in \mathbb{R}$. We have

$$\partial_t^2 \tilde{u} - \Delta \tilde{u} + q \tilde{u} = \rho, \quad \tilde{u}|_{y=0} = \tilde{f}, \quad \partial_y \tilde{u}|_{y=0} = \tilde{g},$$

(22)

where

$$\rho = -2\partial_x \chi \partial_x u - u \Delta \chi, \quad \tilde{f} = \chi f, \quad \tilde{g} = \chi g + f \partial_y \chi$$

(in the last two equalities, $\chi$ and $\partial_y \chi$ denote the restrictions of the corresponding functions to $\{y = 0\}$).

The Cauchy problem (22) can be treated by the argument exposed in Secs. 3, 4 with the only modification concerning the presence of $\rho$ on the right-hand side of the wave equation. This results in the relation
\[
\tilde{u}(x_0, y_0, t_0) = \frac{1}{2} \left( \tilde{f}(x_0, t_0 + y_0) + \tilde{f}(x_0, t_0 - y_0) \right)
\]

\[ + \lim_{h \to 0} \left\{ \int_{|t-t_0| \leq y_0} dt \int_{|x-x_0| \leq y_0 + \varepsilon} K_h(x_0, x; y_0, t - t_0) \tilde{F}(x, t) \, dx \right. \]

\[ + \int_{0}^{y_0} dy \int_{|t-t_0| \leq y_0 - y} dt \int_{\mathbb{R}} K_h^N(x_0, x; y_0 - y, t - t_0) \rho(x, y, t) \, dx \right\}, \tag{23}
\]

which is a substitute for (20). Here \( \tilde{F} = (\tilde{f}, \tilde{g})^T \), \( \varepsilon \) is an arbitrary positive number. By virtue of relation (19), the integral over \( \mathbb{R} \) in the last term in figure brackets can be replaced by that over any neighborhood of the interval \( |x - x_0| \leq y_0 - y \). After this modification, the multiple integral with respect to \( x, y, t \) depends only on the values of the function \( \rho \) at points \( (x, y, t) \) such that \( (x, y) \) belongs to the corresponding neighborhood of the set (21). However, if this neighborhood is chosen sufficiently small, then we have \( \chi = 1 \), which implies \( \rho = 0 \). Thus the term involving \( \rho \) in (23) can be dropped. Similarly, \( \tilde{F} \) can be replaced by \( F \) in the first term in figure brackets, provided that the boundary \( \partial \Omega \) contains the interval \( [x_0 - y_0 - \varepsilon, x_0 + y_0 + \varepsilon] \) (i.e., \( \varepsilon \) is sufficiently small).

It remains to observe that \( \tilde{f}(x_0, t_0 \pm y_0) = f(x_0, t_0 \pm y_0) \) and \( \tilde{u}(x_0, y_0, t_0) = u(x_0, y_0, t_0) \), which is due to the equality \( \chi = 1 \) being valid at the corresponding points. Thus we arrive at relation (20).

The theorem established here means, in particular, that in the problem on the half-plane considered in Secs. 3, 4, it is possible to eliminate the restriction that the solution \( u \) and the potential \( q \) are compactly supported in \( x \). Indeed, for any given \( x_0, y_0 \), one can choose a bounded domain \( \Omega \) containing the set (21) and an arbitrary smooth compactly supported potential \( \tilde{q} \) such that \( \tilde{q}(x) = q(x) \) whenever \( (x, y) \in \Omega \). Then, according to Theorem 1, the value \( u(x_0, y_0, t_0) \) can be found using relation (20), in which the potential \( q \) should be replaced by \( \tilde{q} \).

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