Global Low-Energy Weak Solution and Large-Time Behavior for the Compressible Flow of Liquid Crystals

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Abstract

We consider the weak solution of the simplified Ericksen-Leslie system modeling compressible nematic liquid crystal flows in $\mathbb{R}^3$. When the initial data is small in $L^2$ and initial density is positive and essentially bounded, we first prove the existence of a global weak solution in $\mathbb{R}^3$. The large-time behavior of a global weak solution is also established.

1 Introduction

We consider the following hydrodynamic system modeling the flow of nematic liquid crystal materials [2,7,20],

\begin{align}
\rho_t + \nabla \cdot (\rho u) &= 0, \\
\rho u_t + \rho u \cdot \nabla u + \nabla P(\rho) &= \mu \Delta u + \lambda \nabla \text{div} u - \nabla d \cdot \Delta d, \\
\partial_t d + u \cdot \nabla d &= \Delta d + |\nabla d|^2 d,
\end{align}

for $(t, x) \in [0, +\infty) \times \mathbb{R}^3$. Here $\rho$, $u = (u^1, u^2, u^3)^t$ and $P$ denote the density, the velocity, and the pressure respectively. $d = (d^1, d^2, d^3)^t$ is the unit-vector ($|d| = 1$) on the sphere $S^2 \subset \mathbb{R}^3$ representing the macroscopic molecular orientation of the liquid crystal materials. $\mu$ and $\lambda$ are positive viscosity constants, and $\text{div}$ and $\Delta$ are the usual spatial divergence and Laplace operators.

The above system (1.1) is a simplified version of the Ericksen-Leslie model for the hydrodynamics of nematic liquid crystals. The mathematical analysis of the incompressible liquid crystal flows was initiated by Lin and Liu in [21,22]. In [29,30], Wang proved the global existence of strong solutions in whole space under some small conditions. In dimension two, the global existence and uniqueness of weak solutions were
studied in [10,11,18,23,31] and reference therein. On the other hand, the global existence of weak solutions to incompressible liquid crystal in $\mathbb{R}^3$ is still an outstanding open question.

When the fluid is allowed to be compressible, the Ericksen-Leslie system becomes more complicated. To our knowledge, there seems very few analytic works available yet. The local-in-time strong solutions to the initial value or initial boundary value problem of system (1.1) with nonnegative initial density were studied in [3,13]. Based on [16,17], the blow up criterion of strong solutions were obtained in [13,14]. The global existence and uniqueness of strong solution in critical space were studied in [12]. Motivated by [15], when the initial data were sufficiently smooth and were suitably small in some energy-norm, the global well-posedness of classical solutions were proved in [19]. Especially global weak solutions in two dimensions was established in [6] under some small condition or geometric angle condition.

Our aim in this paper is to establish the global existence of low-energy weak solutions of system (1.1), with the following initial conditions:

$$\begin{align*}
(\rho(\cdot, 0), u(\cdot, 0), d(\cdot, 0)) &= (\rho_0, u_0, d_0),
\end{align*}$$

where $\rho_0$ is bounded above and below away from zero, $|d_0| = 1$, $u_0, \nabla d_0 \in L^p(\mathbb{R}^3)$ for some $p > 6$, and modulo constants, $(\rho_0, u_0, d_0)$ is small in $L^2(\mathbb{R}^3)$. Thus the total initial energy is small, but no other smallness or regularity conditions are imposed.

When the direction field $d$ does not appear, (1.1) reduces to the compressible Navier-Stokes equations. The global classical solutions were first obtained by Matsumura-Nishida [25,26] for initial data close to a non-vacuum equilibrium in $H^3(\mathbb{R}^3)$. In particular, the theory requires that the solution has small oscillations from a uniform non-vacuum state so that the density is strictly away from the vacuum and the gradient of the density remains bounded uniformly in time. Later, Hoff [8,9] studied the problem for discontinuous initial data. For the existence of solutions for arbitrary data, the major breakthrough is due to Lions [24] (see also Feireisl [5]), where he obtains global existence of weak solutions-defined as solutions with finite energy. Suen and Hoff [28] adopted Hoff’s techniques to obtain global existence of low-energy weak solutions for the magnetohydrodynamics. In this paper, we shall study the Cauchy problem (1.1)-(1.2) for liquid crystals and establish the global existence and large time behavior of low-energy weak solutions. However, compared with the compressible Navier-Stokes equations, some new difficulties arise due to the additional presence of the liquid crystal directional field. Especially, the super critical nonlinearity $|\nabla d|^2 d$ in the transported heat flow of harmonic map equation (1.1c) and the strong coupling nonlinearity $\nabla d \cdot \Delta d$ in the momentum equations (1.1b) will cause serious difficulties in the proofs of the time-independent global energy estimates.

To state the main results in a precise way, we first introduce some notations and conventions which will be used throughout the paper. For a given unit vector $n \in S^2$ and a positive integer $m$, we denote

$$H^m_n(\mathbb{R}^3; S^2) := \{d : d - n \in H^m(\mathbb{R}^3), |d| = 1 \ a.e. \ in \ \mathbb{R}^3\}.$$
We use the usual notation for Hölder seminorms: for $v: \mathbb{R}^3 \rightarrow \mathbb{R}^m$ and $\alpha \in (0, 1]$,

$$\langle v \rangle^\alpha = \sup_{x_1, x_2 \in \mathbb{R}^3 \atop x_1 \neq x_2} \frac{|v(x_2) - v(x_1)|}{|x_2 - x_1|^\alpha};$$

and for $v: Q \subseteq \mathbb{R}^3 \times [0, \infty) \rightarrow \mathbb{R}^3$ and $\alpha_1, \alpha_2 \in (0, 1]$,

$$\langle v \rangle_{Q}^{\alpha_1, \alpha_2} = \sup_{(x_1, t_1), (x_2, t_2) \in Q \atop (x_1, t_1) \neq (x_2, t_2)} \frac{|v(x_2, t_2) - v(x_1, t_1)|}{|x_2 - x_1|^\alpha_1 + |t_2 - t_1|^\alpha_2}.$$

If $X$ is a Banach space we will abbreviate $X^3$ by $X$ when convenient. Finally if $I \subset [0, \infty)$ is an interval, $C^1(I; X)$ will be the elements $v \in C(I; X)$ such that the distribution derivative $v_t \in \mathcal{D}'(int I; \mathbb{R}^3)$ is realized as an element of $C(I; X)$.

As it was pointed out in [8], the effective viscous flux plays an important role in the mathematical theory of compressible fluid dynamics. More precisely, let $F$ and $\omega$ be the effective flux and vorticity defined by

$$F \equiv (\mu + \lambda) \text{div} u - (P(\rho) - P(\bar{\rho})) \quad \text{and} \quad \omega \equiv \nabla \times u. \quad (1.3)$$

It is not hard to check that

$$\nabla d \Delta d = \text{div}(\nabla d \otimes \nabla d) - \nabla \frac{|
abla d|^2}{2}.$$

So, it follows from (1.1b) that

$$\Delta F = \text{div}(\rho \dot{u} + \text{div}(\nabla d \otimes \nabla d) - \nabla \frac{|
abla d|^2}{2}), \quad \mu \Delta \omega = \nabla \times (\rho \dot{u} + \text{div}(\nabla d \otimes \nabla d)), \quad (1.4)$$

where “·” denotes the material derivative, i.e.,

$$\dot{f} := \partial_t f + u \cdot \nabla f.$$

Now we give a precise formulation of our results. First concerning the pressure $P$, we focus our interest on the case of isentropic flows and assume that

$$P(\rho) = a \rho^\gamma \quad \text{with} \quad a > 0, \gamma \geq 1. \quad (1.5)$$

Next we fix a positive reference density $\bar{\rho}$ and then choose positive bounding densities $\underline{\rho}$ and $\bar{\rho}$ satisfying

$$\underline{\rho} < \min\{\bar{\rho}, \rho'\} \quad \text{and} \quad \max\{\bar{\rho}, \rho''\} < \bar{\rho}, \quad (1.6)$$

and finally we define a positive number $\delta$ by

$$\delta = \min\{\min\{\bar{\rho}, \rho'\} - \underline{\rho}, \bar{\rho} - \max\{\bar{\rho}, \rho''\}, \frac{1}{2}(\bar{\rho} - \underline{\rho})\}. \quad (1.7)$$
(Notice that $\delta$ need not be “small” in the usual sense.) Concerning the diffusion coefficients $\mu$ and $\lambda$ we assume that

$$0 \leq \lambda < \frac{3 + \sqrt{21}}{6} \mu. \quad (1.8)$$

It follows that

$$\frac{1}{4} \mu (p - 2) - \frac{[\frac{1}{4} \lambda (p - 2)]^2}{\frac{1}{2} \mu + \lambda} > 0 \quad (1.9)$$

for $p = 6$ and consequently for some $p > 6$, which we now fix.

Concerning the initial data $(\rho_0, u_0, d_0)$, we assume there is a positive number $N$, which may be arbitrary large, and a positive number $b < \delta$ such that

$$\|u_0\|_{L^p} + \|
abla d_0\|_{L^p} \leq N \quad (1.10)$$

and

$$\rho + b < \text{ess inf} \rho_0 \leq \text{ess sup} \rho_0 < \bar{\rho} - b. \quad (1.11)$$

We assume also that

$$d_0 - n \in H^1_n([0, \infty); S^2) \quad (1.12)$$

and write

$$C_0 \triangleq \int_{\mathbb{R}^3} \left( \frac{1}{2} \rho_0 |u_0|^2 + G(\rho_0) + \frac{1}{2} |\nabla d_0|^2 \right) dx, \quad (1.13)$$

where $G(\rho)$ is the potential energy density defined by

$$G(\rho) = \rho \int_0^\rho \frac{P(s) - P(\bar{\rho})}{s^2} ds. \quad (1.14)$$

It is clear that there exist two positive constant $c_1, c_2$ only depending on $\rho, \bar{\rho}$, and $\bar{\rho}$

$$c_1(\rho, \bar{\rho}, \bar{\rho})(\rho - \bar{\rho})^2 \leq G(\rho) \leq c_2(\rho, \bar{\rho}, \bar{\rho})(\rho - \bar{\rho})^2. \quad (1.15)$$

Weak solutions of (1.1)-(1.2) are defined in a usual way.

**Definition 1.1.** A pair of functions $(\rho, u, H)$ is said to be a weak solution of (1.1)-(1.2) provided that $(\rho - \bar{\rho}, \rho u) \in C([0, \infty); H^{-1}(\mathbb{R}^3))$, $d - n \in C([0, \infty); L^2(\mathbb{R}^3))$, $u, \nabla d \in L^\infty([0, \infty); L^2(\mathbb{R}^3))$, $\nabla u \in L^2((0, \infty); L^2(\mathbb{R}^3))$, and $|d(\cdot, t)| = 1$ a.e. in $\mathbb{R}^3$ for $t \geq 0$. Moreover, the following identities hold for any test function $\psi \in \mathcal{D}(\mathbb{R}^3 \times (t_1, t_2))$ with $t_2 \geq t_1 \geq 0$ and $j = 1, 2, 3$:

$$\int_{t_1}^{t_2} \int_{\mathbb{R}^3} \rho \psi(x, t) dx dt = \int_{t_1}^{t_2} \int_{\mathbb{R}^3} (\rho \psi_t + \rho u \cdot \nabla \psi) dx dt,$$

$$\int_{t_1}^{t_2} \int_{\mathbb{R}^3} \rho u^j \psi(x, t) dx dt = \int_{t_1}^{t_2} \int_{\mathbb{R}^3} (\mu \nabla u^j \cdot \nabla \psi + \lambda (\text{div} u) \psi_x) dx dt$$

$$= \int_{t_1}^{t_2} \int_{\mathbb{R}^3} (\rho u^j \psi_t + \rho u^j u \cdot \nabla \psi + P(\rho) \psi_x - \frac{1}{2} |\nabla d|_2^2 \psi_j + d_x^j \nabla d \cdot \nabla \psi) dx dt,$$
and
\[\int_{\mathbb{R}^3}(d^i - n^2)(x, t)\psi(x, t)dx|_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_{\mathbb{R}^3} \nabla d^i \nabla \psi dx dt = \int_{t_1}^{t_2} \int_{\mathbb{R}^3}((d^i - n^2)\psi_t - u \cdot \nabla d^i \psi + |\nabla d|^2 d^i \psi) dx dt.\]

Our main results are formulated as the following theorem.

**Theorem 1.2.** Assume that the system parameters in (1.1) satisfy the conditions (1.5)-(1.9) and let positive numbers \(N\) and \(b < \delta\) be given. Then there are positive constants \(\varepsilon, C, \text{ and } \theta\) depending on the parameters and assumptions in (1.5)-(1.9), on \(N\), and on a positive lower bound for \(b\), such that, if initial data \((\rho_0, u_0, d_0)\) are given satisfying (1.10)-(1.14) with
\[C_0 < \varepsilon,\] (1.16)
then there is a solution \((\rho, u, d)\) to (1.1)-(1.2) in the sense of the definition 1.1. Moreover, the solution satisfies the following:
\[\rho - \bar{\rho}, \rho u \in C([0, \infty); H^{-1}(\mathbb{R}^3)),\] (1.17)
\[d - n \in C([0, \infty); L^2(\mathbb{R}^3)),\] (1.18)
\[\nabla u \in L^2([0, \infty); \mathbb{R}^3),\] (1.19)
\[u(\cdot, t), \nabla d(\cdot, t) \in H^1(\mathbb{R}^3), \ t > 0\] (1.20)
\[F(\cdot, t), \omega(\cdot, t) \in H^1(\mathbb{R}^3), \ t > 0\] (1.21)
\[\langle u \rangle \frac{1}{\mathbb{R}^3} \frac{1}{|\tau, \infty)}, \langle \nabla d \rangle \frac{1}{\mathbb{R}^3} \frac{1}{|\tau, \infty) \leq C(\tau)C_0^b,\] (1.22)
where \(C(\tau)\) may depend additionally on a positive lower bound for \(\tau\),
\[\rho \leq \rho(x, t) \leq \bar{\rho} \ a.e. \ on \ \mathbb{R}^3 \times [0, \infty),\] (1.23)
and
\[\sup_{t \geq 0} \int_{\mathbb{R}^3} [|ho - \bar{\rho}|^2 + |u|^2 + \frac{1}{2} |\nabla d|^2 + \sigma(|\nabla u|^2 + |\nabla^2 d|^2) + \sigma^5(|F|^2 + |\nabla \omega|^2)] dx + \int_0^\infty \int_{\mathbb{R}^3} [|
abla u|^2 + |\Delta d| + |\nabla d|^2|d|^2 + \sigma(|\dot{u}|^2 + |\nabla d_t|^2 + |\nabla \omega|^2)] dx ds \leq CC_0^b,\] (1.24)
where \(\sigma(t) = \min\{1, t\}\). Moreover we also have the following large-time behavior:
\[\lim_{t \to \infty} (\|\rho - \bar{\rho}\|_{L^1(\mathbb{R}^3)} + \|u\|_{W^{1, r}(\mathbb{R}^3)} + \|\nabla d\|_{W^{1, r}(\mathbb{R}^3)}) = 0,\] (1.25)
holds for \(l \in (2, \infty), r \in (2, 6)\).

The rest of this paper is devoted to prove Theorem 1.2. In Section 2, we collect some useful inequalities and basic results. In Section 3, we derive the time-independent energy estimates of the solution. The key pointwise upper and lower bound of the density are established in Section 4. Finally, the proof of the main results will be done in Section 5.
2 Preliminaries

In this section, we state some auxiliary lemmas, which will be frequently used in the sequel. We start with the well-known Gagliardo-Nirenberg inequality (see, for instance, [1,32]).

Lemma 2.1. First, given \( r \in [2, 6] \) there is a constant \( C(r) \) such that for \( f \in H^1(\mathbb{R}^3) \),
\[
\|f\|_{L^r(\mathbb{R}^3)} \leq C(r) \|f\|_{L^2(\mathbb{R}^3)}^{(6-r)/2r} \|
abla f\|_{L^2(\mathbb{R}^3)}^{(3r-6)/2r}.
\] (2.1)

Next, for any \( r \in (3, \infty) \) and \( q > 1 \), there is a constant \( C(r, q) \) such that for \( f \in L^q(\mathbb{R}^3) \cap W^{1,r}(\mathbb{R}^3) \),
\[
\|f\|_{L^\infty(\mathbb{R}^3)} \leq C(r, q) \|f\|_{L^q(\mathbb{R}^3)}^{q(r-3)/(3r+q(r-3))} \|
abla f\|_{L^r(\mathbb{R}^3)}^{3r/(3r+q(r-3))},
\] (2.2)

and
\[
\langle f \rangle_{\mathbb{R}^3} \leq C(r) \|
abla f\|_{L^r(\mathbb{R}^3)},
\] (2.3)
where \( \alpha = 1 - \frac{4}{r} \).

The next lemma is due to Hoff [8], which will be used to prove the uniform (in time) bound of density.

Lemma 2.2. If \( \Gamma \) is the fundamental solution for the Laplace operator in \( \mathbb{R}^3 \), then given \( p_1 \in [1, 3) \) and \( p_2 \in (3, \infty] \), there is a constant \( C = C(n, p_1, p_2) \) such that
\[
\|\Gamma_{x,j} \ast f\|_{L^\infty(\mathbb{R}^3)} \leq C(n, p_1, p_2) \|f\|_{L^{p_1}(\mathbb{R}^3)} + \|f\|_{L^{p_2}(\mathbb{R}^3)}.
\] (2.4)

Finally, we need the local-in-time existence theorem of (1.1)-(1.2). Using the mollifier technique, the local solutions which can be proved rigorously by the standard method of Matsumura and Nishida [25,26].

Proposition 2.3. Assume that the initial data \((\rho_0, u_0, d_0)\) satisfies
\[
(\rho_0 - \bar{\rho}, u_0) \in H^3(\mathbb{R}^3) \quad d_0 \in H^4_n(\mathbb{R}^3; \mathbb{S}^2) \quad \text{and} \quad \inf_{x \in \mathbb{R}^3} (\rho_0(x)) > 0.
\] (2.5)

Then there exists a positive time \( T_0 \), which may depend on \( \inf_{x \in \mathbb{R}^3} (\rho_0(x)) \), such that the Cauchy problem (1.1)-(1.2) has a unique smooth solution \((\rho, u, d)\) on \( \mathbb{R}^3 \times [0, T_0] \) satisfying
\[
\rho(x, t) > 0 \text{ for all } x \in \mathbb{R}^3, t \in [0, T_0],
\] (2.6)
\[
\rho - \bar{\rho} \in C([0, T_0]; H^3) \cap C^1([0, T_0]; H^2),
\] (2.7)
\[
u \in C([0, T_0]; H^3) \cap C^1([0, T_0]; H^1),
\] (2.8)
\[ d - n \in C([0, T_0]; H^4_n(\mathbb{R}^3; S^2)) \cap C^1([0, T_0]; H^2_n(\mathbb{R}^3; S^2)). \] (2.9)

In view of Lemma 2.1 and the classical estimates of elliptic system, we have

**Lemma 2.4.** Let \((\rho, u, d)\) be as in Proposition 2.3, then there exists a generic positive constant \(C\), depending only on \(\mu\) and \(\lambda\) and \(p\), such that for \(p \in (1, \infty)\)

\[
\|\nabla F\|_{L^p(\mathbb{R}^3)} + \|\nabla \sigma\|_{L^p(\mathbb{R}^3)} \leq C(\|\rho \dot{u}\|_{L^p(\mathbb{R}^3)} + \|\nabla d \cdot \Delta d\|_{L^p(\mathbb{R}^3)}), \tag{2.10}
\]

\[
\|\nabla u\|_{L^p(\mathbb{R}^3)} \leq C(\|F\|_{L^p(\mathbb{R}^3)} + \|\omega\|_{L^p(\mathbb{R}^3)} + \|P(\sigma) - P(\tilde{\rho})\|_{L^p(\mathbb{R}^3)}), \tag{2.11}
\]

where \(F\) and \(\omega\) are defined in (1.4).

**Proof.** An application of the \(L^p\)-estimate of elliptic systems to (1.4) gives (2.10). On the other hand, since \(-\Delta u = -\nabla \text{div} u + \nabla \times \omega\), it holds that

\[
\nabla u = -\nabla (-\Delta)^{-1} \nabla \text{div} u + \nabla (-\Delta)^{-1} \nabla \times \omega,
\]

which, combined with the Marcinkiewicz multiplier theorem (Stein [20,p.96]), we arrive at

\[
\|\nabla u\|_{L^p(\mathbb{R}^3)} \leq C(\|\text{div} u\|_{L^p(\mathbb{R}^3)} + \|\omega\|_{L^p(\mathbb{R}^3)} \\
\leq C(\|F\|_{L^p(\mathbb{R}^3)} + \|\omega\|_{L^p(\mathbb{R}^3)} + \|P(\sigma) - P(\tilde{\rho})\|_{L^p(\mathbb{R}^3)}).
\]

Thus the proof the lemma is completed.

### 3. A priori estimates

This section is devoted to establish a number of a priori bounds for local-in-time smooth solutions, corresponding roughly to (1.24). Those are rather long and technical. We have therefore omitted those which are identical to or nearly identical to arguments given elsewhere in the literature of whose details we regard as routine. On the other hand, we have endeavored to describe the flow of the arguments in such a way that the diligent reader can reconstruct the details without undue difficulty.

Let \(T > 0\) be fixed and assume that \((\rho, u, d)\) is a smooth solution of (1.1)-(1.2). We define a functional \(A(t)\) for a given such solution that

\[
A(t) = \sup_{0 \leq s \leq t} \int_{\mathbb{R}^3}[\sigma(|\nabla u|^2 + |\nabla^2 d|^2) + \sigma^5(|\dot{u}|^2 + |\nabla \omega|^2 + |\nabla d|^2)]dx \\
+ \int_0^t \int_{\mathbb{R}^3}[\sigma(|\dot{u}|^2 + |\nabla d|^2 + |\nabla \omega|^2) + \sigma^5(|\dot{u}|^2 + |\nabla^2 d|^2)]dxds, \tag{3.1}
\]

where \(\sigma(t) = \min\{1, t\}\), and we obtain the following a priori bound for \(A(t)\) under the assumptions that the initial energy \(C_0\) in (1.13) is small enough and that the density remains bounded above and below away from zero:

**Proposition 3.1.** Assume that the system parameters in (1.1) satisfy the conditions in (1.5)-(1.8) and let positive numbers \(N\) and \(b < \delta\) be given. Assume \((\rho, u, d)\)
is a solution of (1.1) on $\mathbb{R}^3 \times [0, T]$ in the sense of Proposition 2.3 with initial data $(\rho_0, u_0) \in H^3(\mathbb{R}^3)$ and $d_0 \in H^4_{\text{loc}}(\mathbb{R}^3; \mathbb{S}^2)$ satisfying (1.10)-(1.13), then there are positive constants $\varepsilon, M,$ and $\theta$ depending on the parameters and assumptions in (1.5)-(1.8), on $N$, and a positive lower bound for $b$, such that if $C_0 < \varepsilon$ and

$$\rho \leq \rho(x, t) \leq \bar{\rho} \text{ on } \mathbb{R}^3 \times [0, T],$$

then

$$A(T) \leq MC_0^\theta.$$

The proof will be given in a sequence of lemmas in which we estimate a number of auxiliary functionals. To describe these we first recall the definition of (1.9) of $p$, which is an open condition, and which therefore allows us to choose $q \in [6, \min\{p, 12\})$ which also satisfies (1.9). Then for given $(\rho, u, d)$ we define

$$\bar{A}(t) = \sup_{1 \leq s \leq t} \int_{\mathbb{R}^3} (|\nabla u|^2 + |\nabla d|^2 + |\dot{u}|^2 + |\nabla \omega|^2 + |\nabla d_1|^2)dx$$

$$+ \int_0^s \int_{\mathbb{R}^3} (|\dot{u}|^2 + |\nabla d_1|^2 + |\nabla \omega|^2 + |\nabla \dot{u}|^2 + |\nabla^2 d|^2)dxds,$$

$$B_q(t) = \sup_{0 \leq s \leq t} \int_{\mathbb{R}^3} (|u|^q + |\nabla d|^q)dx + \int_0^q \int_{\mathbb{R}^3} (|u|^{q-2} |\nabla u|^2 + |\nabla d|^{q-2} |\nabla d|^2)dxds$$

$$+ \int_0^q \int_{\mathbb{R}^3} (|u|^{q-4} |\nabla(|u|^2)|^2 + |\nabla d|^{q-4} |\nabla(|\nabla d|^2)|^2)dxds,$$

$$D(t) = \sup_{0 \leq s \leq t} \int_{\mathbb{R}^3} \sigma^5 (|\nabla d|^2 |\nabla^2 d|^2 + |u|^2 |\nabla^2 d|^2 + |\nabla u|^2 |\nabla d|^2)dx,$$

$$\bar{D}(t) = \sup_{1 \leq s \leq t} \int_{\mathbb{R}^3} (|\nabla d|^2 |\nabla^2 d|^2 + |u|^2 |\nabla^2 d|^2 + |\nabla u|^2 |\nabla d|^2)dx,$$

$$E(t) = \int_0^t \int_{\mathbb{R}^3} [\sigma^2 (|\nabla u|^2 + |\nabla^2 d|^3) + \sigma^5 (|\nabla u|^4 + |\nabla^2 d|^4)]dxds$$

$$+ \sum_{1 \leq k_1, j_1 \leq 3} \int_0^t \int_{\mathbb{R}^3} \sigma u^{j_1}_{k_1} u^{j_2}_{k_2} u^{j_3}_{k_3} dxds,$$

and

$$\bar{E}(t) = \int_1^t \int_{\mathbb{R}^3} (|\nabla u|^3 + |\nabla d|^2 |\nabla^2 d|^2 + |\nabla d|^4 + |\nabla^2 d|^4)dxds.$$

It will be seen that the assumed regularity (2.6)-(2.9) suffices to justify the estimates that follow. We begin with the following $L^2$ energy estimate.

**Lemma 3.2.** Assume that the hypotheses and notations of Proposition 3.1 are in force. Then

$$\sup_{0 \leq t \leq T} \int_{\mathbb{R}^3} (|\rho - \bar{\rho}|^2 + |u|^2 + |\nabla d|^2)dx$$

$$+ \int_0^T \int_{\mathbb{R}^3} (|\nabla u|^2 + |\nabla d|^2 |\nabla^2 d|^2)dxdt \leq MC_0. \quad (3.2)$$

**Proof.** Multiplying (1.1b) by $u$ and integrating over $\mathbb{R}^3$, we have

$$\frac{d}{dt} \int_{\mathbb{R}^3} \frac{1}{2} \rho u^2 dx + \int_{\mathbb{R}^3} \nabla P(\rho) u dx + \int_{\mathbb{R}^3} (\mu |\nabla u|^2 + \lambda |\nabla u|^2) dx = - \int_{\mathbb{R}^3} u \cdot \nabla d \cdot \Delta d. \quad (3.3)$$
Proof. For $0 < t \leq 1$ and the fact that $T > 1$ and if $\triangle d > 0$, then

\begin{align*}
\sup_{0<s\leq t} \sigma \int_{\mathbb{R}^3} (|\nabla u|^2 + |\nabla^2 d|^2) dx + \int_0^t \int_{\mathbb{R}^3} \sigma (|\dot{u}|^2 + |\nabla d_t|^2) dx ds & \\
\leq M [C_0 + C_0^{\frac{q-4}{2}} B_q^{\frac{2}{q-2}} + C_0^{\frac{q-6}{2}} B_q^{\frac{4}{q-2}} + E],
\end{align*}

(3.6)

and if $T > 1$ and $1 \leq t \leq T$, then

\begin{align*}
\sup_{1 \leq s \leq t} \int_{\mathbb{R}^3} (|\nabla u|^2 + |\nabla^2 d|^2) dx + \int_0^t \int_{\mathbb{R}^3} (|\dot{u}|^2 + |\nabla d_t|^2) dx ds & \\
\leq M [C_0 + C_0^{\frac{q}{2}} A^{\frac{1}{2}} + E] + A(1).
\end{align*}

(3.7)

Proof. For $0 \leq t \leq 1$ and $T$, multiplying the equation (1.1b) by $\sigma \dot{u}$ and integrating over $\mathbb{R}^3 \times [0, t]$, we have

\begin{align*}
\sup_{0<s\leq t} \sigma \int_{\mathbb{R}^3} |\nabla u|^2 dx + \int_0^t \int_{\mathbb{R}^3} |\dot{u}|^2 dx ds & \\
\leq M \{ C_0 + \int_0^t \int_{\mathbb{R}^3} \sigma [\dot{u}(div(\nabla d \otimes \nabla d) - \frac{1}{2} \nabla |\nabla d|^2) dx ds] + E \}.
\end{align*}

(3.8)

Differentiating (1.1c) with respect to $x$, we have

\begin{equation}
\nabla d_t - \Delta \nabla d = \nabla (|\nabla d|^2 d - u \cdot \nabla d).
\end{equation}

(3.9)

Multiplying the above equation by $\sigma \nabla d_t$ and integrating over $\mathbb{R}^3 \times [0, t]$, we have

\begin{align*}
\frac{1}{2} \sigma \int_{\mathbb{R}^3} |\Delta d|^2 dx + \int_0^t \int_{\mathbb{R}^3} \sigma |\nabla d_t|^2 dx ds & \\
= \frac{1}{2} \int_0^t \int_{\mathbb{R}^3} \sigma |\Delta d|^2 dx ds + \int_0^t \int_{\mathbb{R}^3} \sigma \nabla d_t \nabla (|\nabla d|^2 d - u \cdot \nabla d) dx ds.
\end{align*}
Adding this to (3.8) and combining with Cauchy’s inequality we then get

\[ \sup_{0 \leq s \leq t} \sigma \int_{\mathbb{R}^3} (|\nabla u|^2 + |\Delta d|^2)dx + \int_0^t \int_{\mathbb{R}^3} \sigma(|u|^2 + |\nabla d_t|^2)dxds \leq M\{C_0 + E + \int_0^t \int_{\mathbb{R}^3} \sigma'(|\Delta d|^2)dxds + \int_0^t \int_{\mathbb{R}^3} \sigma(|\nabla^2 d|^2|\nabla d|^2 + |\nabla u|^2|\nabla d|^2 + |\nabla^2 d|^2|u|^2)|dxds \} \]  

(3.10)

The right terms can be estimated as follows:

\[ \int_0^t \int_{\mathbb{R}^3} \sigma'(|\Delta d|^2)dxds \leq M \int_0^{t\uparrow} \int_{\mathbb{R}^3} |\Delta d + |\nabla d|^2d|^2 + |\nabla d|^4dxds \leq M[C_0 + (\int_0^{t\uparrow} \int_{\mathbb{R}^3} |\nabla d|^2dxds)^{\frac{2}{\nu-2}} \int_0^{t\uparrow} \int_{\mathbb{R}^3} |\nabla d|^2dxds]^{\frac{\nu-2}{2}} \leq M(C_0 + C_0^{\frac{2}{\nu-2}} B_{q^{-}}). \]

\[ \int_0^t \int_{\mathbb{R}^3} \sigma(|\nabla u|^2|\nabla d|^2)dxds \leq M(\int_0^t \int_{\mathbb{R}^3} \sigma^2|\nabla u|^3dxds)^{\frac{2}{\nu}} (\int_0^t \int_{\mathbb{R}^3} |\nabla d|^6dxds)^{\frac{1}{\nu}} \leq M[H + (\int_0^t \int_{\mathbb{R}^3} |\nabla d|^2dxds)^{\frac{2}{\nu-2}} (\int_0^t \int_{\mathbb{R}^3} |\nabla d|^4dxds)^{\frac{2}{\nu-2}}] \leq M(C_0 + C_0^{\frac{2}{\nu-2}} B_{q^{-}}). \]

The other two terms in the integral on the right side of (3.10) are bounded in a similar way, and (3.6) follows.

For 1 \leq t \leq T, as in (3.10), we have

\[ \sup_{1 \leq s \leq t} \int_{\mathbb{R}^3} (|\nabla u|^2 + |\Delta d|^2)dx + \int_1^t \int_{\mathbb{R}^3} (|\dot{u}|^2 + |\nabla d|^2)dxds \leq M\{C_0 + E + \int_1^t \int_{\mathbb{R}^3} (|\nabla^2 d|^2|\nabla d|^2 + |\nabla u|^2|\nabla d|^2 + |\nabla^2 d|^2|u|^2)|dxds \} + A(1). \]  

(3.11)

Using the fact that

\[ |\nabla d|^2 = -d \cdot \Delta d \hspace{1em} (since \hspace{0.5em} |d| = 1), \]  

(3.12)

the right terms can be bounded as follows:

\[ \int_1^t \int_{\mathbb{R}^3} |\nabla u|^2|\nabla d|^2dxds \leq M \int_1^t \int_{\mathbb{R}^3} |\nabla u|^3 + |\nabla d|^6dxds \leq M \int_1^t \int_{\mathbb{R}^3} |\nabla u|^3 + |\nabla d|^2|\nabla^2 d|^2dxds \leq M\bar{E} \]

\[ \int_1^t \int_{\mathbb{R}^3} |\nabla^2 d|^2|u|^2dxds \leq \int_1^t \int_{\mathbb{R}^3} |\nabla^2 d|^4 + |u|^4dxds \leq \bar{E} + \int_1^t (\int_{\mathbb{R}^3} |u|^2dx)^{\frac{1}{2}} (\int_{\mathbb{R}^3} |\nabla u|^2dx)^{\frac{1}{2}}ds \leq M(E + C_0^{\frac{2}{\nu-2}} A_{\frac{\nu}{2}}). \]

Taking the above results into (3.11), then (3.7) follows. Thus the proof of lemma is completed.

Next we derive preliminary bounds for \( \dot{u} \) and \( \nabla d_t \) in \( L^\infty([0,T]; L^2(\mathbb{R}^3)) \).
Lemma 3.4. Assume that the hypotheses and notations of Proposition 3.1 are in force. Then for $0 < t \leq 1 \land T$,

$$
\sup_{0<s\leq t} \sigma^5 \int_{\mathbb{R}^3} (|\dot{u}|^2 + |
abla d_t|^2)dx + \int_0^t \int_{\mathbb{R}^3} \sigma^5 (|\nabla \dot{u}|^2 + |
abla^2 d_t|^2)dxds \leq M[C_0 + E + C_0^\frac{3\zeta-1}{4} B_q^\frac{2}{1-\zeta} + C_0^\frac{3\zeta-5}{4} B_q^\frac{2}{1-\zeta}] (3.13)
$$

and if $T > 1$ and $1 \leq t \leq T$, then

$$
\sup_{1\leq s\leq t} \int_{\mathbb{R}^3} (|\dot{u}|^2 + |
abla d_t|^2)dx + \int_0^t \int_{\mathbb{R}^3} (|\nabla \dot{u}|^2 + |
abla^2 d_t|^2)dxds \leq M\{C_0 + C_0\tilde{A}\tilde{E} + C_0^{\frac{2(\zeta-3)}{4}} B_q^{\frac{2}{1-\zeta}} \tilde{A} + \tilde{E}\} + A(1). (3.14)
$$

Proof. By the definition of material derivative, we can write (1.2) as follows,

$$
\rho \dot{u} + \nabla(P(\rho)) = \mu \Delta u + \lambda \nabla \text{div}u - \nabla d \cdot \nabla d. \quad (3.15)
$$

Differentiation (3.15) with respect to $t$ and using (1.1), we have

$$
\rho \dot{u} + \rho u \cdot \nabla \dot{u} + \nabla(P(\rho)_t) + (\nabla d \cdot \Delta d)_t = \mu \Delta \dot{u} + \lambda \nabla \text{div}u - [\mu \Delta (u \cdot \nabla u) + \lambda \nabla \text{div}(u \cdot \nabla u)]
$$

$$
+ \text{div}[(\mu \Delta u + \lambda \nabla \text{div}u) \otimes u - \nabla P(\rho) \otimes u - (\nabla d \cdot \Delta d) \otimes u]. \quad (3.16)
$$

Multiplying (3.16) by $\sigma^5 \dot{u}$, and integrating over $\mathbb{R}^3 \times [0, t]$, we obtain that for $0 \leq t \leq 1 \land T$,

$$
\sup_{0<s\leq t} \sigma^5 \int_{\mathbb{R}^3} |\dot{u}|^2 dx + \int_0^t \int_{\mathbb{R}^3} \sigma^5 |\nabla \dot{u}|^2 dxds \leq M[C_0 + E + C_0^\frac{3\zeta-1}{4} B_q^\frac{2}{1-\zeta} + C_0^\frac{3\zeta-5}{4} B_q^\frac{2}{1-\zeta}]
$$

$$
+ \int_0^t \int_{\mathbb{R}^3} \sigma^5 |\nabla d|^2(|\nabla d_t|^2 + |u|^2|\nabla^2 d|^2)dxds]. (3.17)
$$

Next we differentiate (3.9) with respect to $t$, multiply by $\sigma^5 \nabla d_t$ and integrate over $\mathbb{R}^3 \times [0, t]$ to obtain

$$
\frac{1}{2} \sigma^5 \int_{\mathbb{R}^3} |\nabla d_t|^2 dx + \int_0^t \int_{\mathbb{R}^3} \sigma^5 |\nabla^2 d_t|^2 dxds
$$

$$
= \frac{1}{2} \int_0^t \int_{\mathbb{R}^3} \sigma^4 \sigma' |\nabla d_t|^2 dxds + \int_0^t \int_{\mathbb{R}^3} \sigma^5 \nabla(|\nabla d|^2 - u \cdot \nabla d)_t \nabla d_t dxds.
$$

Adding this to (3.17), integrating by parts, using Cauchy’s inequality, we then have

$$
\sup_{0<s\leq t} \sigma^5 \int_{\mathbb{R}^3} |\dot{u}|^2 + |\nabla d|^2 dx + \int_0^t \int_{\mathbb{R}^3} \sigma^5 (|\nabla \dot{u}|^2 + |\nabla^2 d|^2) dxds \leq M[C_0 + E + C_0^\frac{3\zeta-1}{4} B_q^\frac{2}{1-\zeta} + C_0^\frac{3\zeta-5}{4} B_q^\frac{2}{1-\zeta}]
$$

$$
+ \int_0^t \int_{\mathbb{R}^3} \sigma^5 |\nabla d|^2 |\dot{u}|^2 + \int_0^t \int_{\mathbb{R}^3} \sigma^5 (|\nabla^2 d|^2) dxds. (3.18)
$$
By (2.2), the terms on right side can be bounded by

\[
\int_0^t \int_{\mathbb{R}^3} \sigma^5 |\nabla d|^2 |u|^2 |\nabla u|^2 dxds \\
\leq \int_0^t \int_{\mathbb{R}^3} \sigma^5 (|\nabla d|^8 + |u|^8 + |\nabla u|^4) dxds \\
\leq E + \sup_{1 \leq s \leq t} \| (u, \nabla d)(s) \|_{L^4(\mathbb{R}^3)}^4 \int_0^t \sigma^5 \| (u, \nabla d)(s) \|_{L^\infty(\mathbb{R}^3)}^4 ds \\
\leq E + M(C_0^{q-2} B_q^{2-q})[E + C_0^{q-2} B_q^{2-q}].
\]

By (1.1c) and (3.12), we have

\[
\int_0^t \int_{\mathbb{R}^3} \sigma^5 |\nabla d|^4 |d_t|^2 dxds \\
\leq M \int_0^t \int_{\mathbb{R}^3} \sigma^5 (|\nabla d|^4 |\nabla \nabla d|^2 + |\nabla d|^4 |u|^2 |\nabla d|^2 + |\nabla d|^8) dxds \\
\leq M \int_0^t \int_{\mathbb{R}^3} \sigma^5 (|\nabla d|^4 |\nabla \nabla d|^2 + |\nabla d|^4 |u|^2 |\nabla d|^2) dxds \\
\leq ME + M(C_0^{q-2} B_q^{2-q})[E + C_0^{q-2} B_q^{2-q}].
\]

The last term on the right side in (3.18) can be bounded by

\[
\int_0^t \int_{\mathbb{R}^3} \sigma^5 |\nabla d|^2 |\dot{u}|^2 dxds \\
\leq (\int_0^t \int_{\mathbb{R}^3} |\nabla d|^4 dxds)^{\frac{1}{2}} (\int_0^t \int_{\mathbb{R}^3} \sigma^{\frac{15}{2}} |\dot{u}|^3 dxds)^{\frac{1}{2}} \\
\leq (C_0^{q-6} B_q^{2-q})^{\frac{1}{2}} (\int_0^t \int_{\mathbb{R}^3} \sigma^{\frac{15}{2}} \| \dot{u} \|_{L^2(\mathbb{R}^3)}^2 \| \nabla \dot{u} \|_{L^2(\mathbb{R}^3)}^2 dxds)^{\frac{1}{2}} \\
\leq (C_0^{q-6} B_q^{2-q})^{\frac{1}{2}} A.
\]

The other integrals on the right side of (3.18) are bounded in a similar way, and (3.13) follows.

For \(1 \leq t \leq T\), as in (3.18), we have

\[
\sup_{1 \leq s \leq t} \int_{\mathbb{R}^3} |\dot{u}|^2 + |\nabla d|^2 dx + \int_0^t \int_{\mathbb{R}^3} (|\nabla u|^2 + |\nabla d|^2) dxds \\
\leq M \left[ \int_0^1 \int_{\mathbb{R}^3} |\nabla d|^2 |u|^2 (|\nabla u|^2 + |\nabla d|^2) + |\nabla d|^4 |d_t|^2 dxds \\
+ \int_0^t \int_{\mathbb{R}^3} (|\nabla d|^2 |\dot{u}|^2 + |\nabla d|^2 |\nabla d_t|^2 + |\nabla d_t|^2 |u|^2) dxds \\
+ C_0 + \bar{E} \right] + A(1). \tag{3.19}
\]

The terms on the right can be bounded by

\[
\int_0^t \int_{\mathbb{R}^3} |\nabla d|^2 |u|^2 |\nabla u|^2 dxds \\
\leq \int_0^t \int_{\mathbb{R}^3} (|\nabla d|^8 + |u|^8 + |\nabla u|^4) dxds \\
\leq \bar{E} + \sup_{1 \leq s \leq t} \| (u, \nabla d)(s) \|_{L^4(\mathbb{R}^3)}^{\frac{10}{3}} \int_1^t \| (u, \nabla d)(s) \|_{L^\infty(\mathbb{R}^3)}^{\frac{10}{3}} ds \\
\leq \bar{E} + \sup_{1 \leq s \leq t} \| (u, \nabla d)(s) \|_{L^2(\mathbb{R}^3)}^{\frac{4}{3}} \| \nabla (u, \nabla d)(s) \|_{L^2(\mathbb{R}^3)}^{\frac{2}{3}} \\
\times \int_0^t \| (u, \nabla d)(s) \|_{L^2(\mathbb{R}^3)}^{\frac{7}{3}} \| \nabla (u, \nabla d)(s) \|_{L^2(\mathbb{R}^3)}^{\frac{1}{3}} ds \\
\leq \bar{E} + MC_0 \bar{E}.
\]
By (1.1c) and (3.12), we have
\[
\int_0^t \int_{\mathbb{R}^3} |\nabla d|^4 |d_t|^2 \, dx \, ds \\
\leq M \int_0^t \int_{\mathbb{R}^3} (|\nabla d|^4 |\nabla^2 d|^2 + |\nabla d|^4 |u|^2 |\nabla d|^2 + |\nabla d|^8) \, dx \, ds \\
\leq M \int_0^t \int_{\mathbb{R}^3} (|\nabla d|^4 |\nabla^2 d|^2 + |\nabla^2 d|^2 |u|^2 |\nabla d|^2) \, dx \, ds \\
\leq M \hat{E} + M \hat{C}_0 \hat{A}.
\]
The last term on the right side in (3.19) can be bounded by
\[
\int_0^t \int_{\mathbb{R}^3} |\nabla d|^2 |\dot{u}|^2 \, dx \, ds \\
\leq \int_0^t (\int_{\mathbb{R}^3} |\nabla d|^3 dx)^{\frac{2}{3}} (\int_{\mathbb{R}^3} |\dot{u}|^6 dx)^{\frac{1}{3}} \, ds \\
\leq \sup_{0 \leq s \leq t} \int_{\mathbb{R}^3} |\nabla d|^3 dx \left( \int_0^t \int_{\mathbb{R}^3} |\nabla d|^2 dx \right)^{\frac{1}{3}} \int_0^t \int_{\mathbb{R}^3} |\nabla d|^2 dx \\
\leq C_0^{\frac{2(3-q)}{2q}} B_q^{\frac{2}{2-q}} \hat{A}.
\]
The other integrals on the right side of (3.19) are bounded in a similar way, and (3.14) follows. Thus the proof of lemma is completed.

Next we derive a number of auxiliary estimates needed to close the bounds in the previous two lemmas. We begin with a bound for the vorticity \( \omega \).

**Lemma 3.5.** Assume that the hypotheses and notations of Proposition 3.1 are in force. Then for \( 0 < t \leq 1 \wedge T \),
\[
\sup_{0 < s \leq t} \int_{\mathbb{R}^3} \sigma^5 (|\nabla F|^2 + |\nabla \omega|^2) dx + \int_0^t \int_{\mathbb{R}^3} \sigma (|\nabla F|^2 + |\nabla \omega|^2) dx \, ds \\
\leq M (D + E^2 C_0^{\frac{3q}{3q-2}} B_q^{\frac{1}{3q-2}}) + \sup_{0 < s \leq t} \int_{\mathbb{R}^3} \sigma^5 |\dot{u}|^2 dx + \int_0^t \int_{\mathbb{R}^3} \sigma |\dot{u}|^2 dx \, ds, \tag{3.20}
\]
and if \( T > 1 \) and \( 1 \leq t \leq T \), then
\[
\sup_{1 \leq s \leq t} \int_{\mathbb{R}^3} (|\nabla F|^2 + |\nabla \omega|^2) dx + \int_1^t \int_{\mathbb{R}^3} (|\nabla F|^2 + |\nabla \omega|^2) dx \, ds \\
\leq M (D + E + \sup_{1 \leq s \leq t} \int_{\mathbb{R}^3} |\dot{u}|^2 dx + \int_1^t \int_{\mathbb{R}^3} |\dot{u}|^2 dx \, ds). \tag{3.21}
\]

**Proof.** By (2.10) and the definition of \( D, E \), we can easily get (3.20) and (3.21). The proof of lemma is completed.

Next we derive an estimate for the functional \( B_q \).

**Lemma 3.6.** Assume that the hypotheses and notations of Proposition 3.1 are in force. Then for any \( 0 < t \leq T \),
\[
B_q \leq M (C_0^{\frac{p-q}{p-q-2}} N^{\frac{q-2}{p-q-2}} + C_0^{\frac{2+3}{3q-2}} B_q^{\frac{3q-11}{3q-2}} + C_0^{\frac{q-3}{q-2}} B_q^{\frac{3q-5}{q-2}}). \tag{3.22}
\]
Proof. We multiply (1.1b) by \(|u|^{q-2}u\) and integrate over \(\mathbb{R}^3 \times (0, t)\) to obtain that

\[
q^{-1} \int_{\mathbb{R}^3} \rho |u|^q dx \big|_0^t + \int_0^t \int_{\mathbb{R}^3} \mu |u|^{q-2} |\nabla u|^2 dx ds \\
+ \int_0^t \int_{\mathbb{R}^3} \left[ \frac{\mu}{17} (q-2)^2 |u|^{q-4} |\nabla (|u|^2)|^2 + \lambda |u|^{q-2} (\text{div} u)^2 \right] dx ds \\
= \int_0^t \int_{\mathbb{R}^3} \left[ (P(\rho) - P(\rho)) \text{div}(|u|^{q-2} u) - |u|^{q-2} u \nabla \cdot \Delta d \right] dx ds \\
- \int_0^t \int_{\mathbb{R}^3} \frac{1}{2} \lambda (q-2) |u|^{-4} (\text{div} u) u \cdot \nabla (|u|^2) dx ds.
\]

(3.23)

For any \(\eta > 0\),

\[
| - \int_0^t \int_{\mathbb{R}^3} \frac{1}{2} \lambda (q-2) |u|^{-4} (\text{div} u) u \cdot \nabla (|u|^2) dx ds | \\
\leq \frac{1}{2} \lambda (q-2) \int_0^t \int_{\mathbb{R}^3} |u|^{-2} (\text{div} u) |u|^{q-2} |\nabla (|u|^2)| dx ds \\
\leq \frac{1}{4} \lambda (q-2) \eta \int_0^t \int_{\mathbb{R}^3} |u|^{-2} (\text{div} u)^2 dx ds + \eta^{-1} \int_0^t \int_{\mathbb{R}^3} |u|^{-4} |\nabla (|u|^2)|^2 dx ds,
\]

so if we choose

\[
\frac{1}{4} \lambda (q-2) \eta = \beta \mu + \lambda
\]

for a positive \(\beta\) to be determined, then the term in question will be bounded by

\[
3 \beta \mu \int_0^t \int_{\mathbb{R}^3} |u|^{-2} |\nabla u|^2 dx ds + \lambda \int_0^t \int_{\mathbb{R}^3} |u|^{-2} (\text{div} u)^2 dx ds \\
+ \left[ \frac{2 \lambda (q-1)^2}{\beta \mu + \lambda} \right] \int_0^t \int_{\mathbb{R}^3} |u|^{-4} |\nabla (|u|^2)|^2 dx ds.
\]

Substituting this into (3.23), we then get

\[
q^{-1} \int_{\mathbb{R}^3} \rho |u|^q dx \big|_0^t + \mu (1 - 3 \beta) \int_0^t \int_{\mathbb{R}^3} |u|^{-2} |\nabla u|^2 dx ds \\
+ \left[ \frac{2 \lambda (q-1)^2}{\beta \mu + \lambda} \right] \int_0^t \int_{\mathbb{R}^3} |u|^{-4} |\nabla (|u|^2)|^2 dx ds \\
\leq \left| \int_0^t \int_{\mathbb{R}^3} (P(\rho) - P(\rho)) \text{div}(|u|^{q-2} u) dx ds \right| \\
+ \left| \int_0^t \int_{\mathbb{R}^3} |u|^{-2} u \nabla \cdot \Delta d dx ds \right|.
\]

Recall that \(q \in [6, p]\), thus (1.9) holds with \(p\) replaced by \(q\), and this is the condition that brackets on the left here is positive when \(\beta = \frac{1}{3}\). It follows this term is positive for some \(\beta \in (0, \frac{1}{3})\), which we now fix. It then follows that

\[
q^{-1} \int_{\mathbb{R}^3} \rho |u|^q dx \big|_0^t + \int_0^t \int_{\mathbb{R}^3} |u|^{-2} |\nabla u|^2 dx ds + \int_0^t \int_{\mathbb{R}^3} |u|^{-4} |\nabla (|u|^2)|^2 dx ds \\
\leq M \left| \int_0^t \int_{\mathbb{R}^3} (P(\rho) - P(\rho)) \text{div}(|u|^{q-2} u) dx ds \right| + \left| \int_0^t \int_{\mathbb{R}^3} |u|^{-2} u \nabla \cdot \Delta d dx ds \right|.
\]

(3.24)

We multiply (3.9) by \(|\nabla d|^{q-2} \nabla d| d|^2 \nabla d|^2| dx ds \) and integrate over \(\mathbb{R}^3 \times (0, t)\) to obtain that

\[
q^{-1} \int_{\mathbb{R}^3} |\nabla d|^q dx \big|_0^t + \int_0^t \int_{\mathbb{R}^3} |\nabla d|^{q-2} |\nabla^2 d|^2 dx ds \\
+ \int_0^t \int_{\mathbb{R}^3} \left[ (q-2) |\nabla d|^{q-4} \nabla (|\nabla d|^2)|^2 dx ds \\
= \int_0^t \int_{\mathbb{R}^3} |\nabla d|^{q-2} \nabla d |\nabla (|\nabla d|^2|^2 d - u \cdot \nabla d) dx ds.
\]

(3.25)
and integrating by parts, we have

\[ I_3 \leq \int_0^t \int_{\mathbb{R}^3} |u|^{q-2} |\nabla u|^2 \, dx \, ds + \int_0^t \int_{\mathbb{R}^3} |u|^{q-2} |\nabla| \nabla (|u|)|^2 dx \, ds \]

\[ \leq \frac{B_q}{q} \left[ \int_0^t \int_{\mathbb{R}^3} |\nabla d|^{q+2} dx \, ds \right]^{\frac{1}{2}} + \left( \int_0^t \int_{\mathbb{R}^3} |u|^{q+2} dx \, ds \right)^{\frac{1}{2}} \]

\[ \leq MB_q \left[ \int_0^t \int_{\mathbb{R}^3} |\nabla d|^{q+2} dx \, ds \right]^{\frac{1}{2}} + \left( \int_0^t \int_{\mathbb{R}^3} |u|^{q+2} dx \, ds \right)^{\frac{1}{2}} \]

\[ \leq MB_q \left[ \int_0^t \int_{\mathbb{R}^3} |\nabla d|^{q+2} dx \, ds \right]^{\frac{1}{2}} + \left( \int_0^t \int_{\mathbb{R}^3} |u|^{q+2} dx \, ds \right)^{\frac{1}{2}} \]

\[ \leq MB_q \left[ \int_0^t \int_{\mathbb{R}^3} |\nabla d|^{q+2} dx \, ds \right]^{\frac{1}{2}} + \left( \int_0^t \int_{\mathbb{R}^3} |u|^{q+2} dx \, ds \right)^{\frac{1}{2}} \]

Substituting these results into (3.26) gives (3.22). Thus the proof of lemma is completed.

Next we derive a bound for the functional $D$ and $\bar{D}$. 
Lemma 3.7. Assume that the hypotheses and notations of Proposition 3.1 are in force. Then for $0 < t \leq 1 \wedge T$,

$$D \leq M [C_0^{\frac{q}{q-1}} B_q^\frac{1}{q} A + A^2 + A^5],$$

(3.27)

and if $T > 1$ and $1 \leq t \leq T$, then

$$\tilde{D} \leq M [C_0^\frac{1}{2} \tilde{A}^2 + \tilde{A}^2 + \tilde{A}_5].$$

(3.28)

Proof. We give the proof of (3.27), that of (3.28) being similar. By Lemma 2.1, we have

$$\int_{\mathbb{R}^3} \sigma^5 |\nabla^2 d|^2 |\nabla d|^2 \, dx \leq \sigma^4 \|\nabla d(\cdot, t)\|_\infty^2 [\sigma^2 |\nabla^2 d(\cdot, t)\|_{L^2(\mathbb{R}^3)}^2]
\leq MA [\sigma^4 \|\nabla d(\cdot, t)\|_{L^1(\mathbb{R}^3)}^2 + \sigma^4 \|\nabla^2 d(\cdot, t)\|_{L^1(\mathbb{R}^3)}^2]$$

$$\leq MA [C_0^{\frac{q}{q-1}} B_q^\frac{1}{q} + \|\sigma^\frac{1}{2} \nabla^2 d(\cdot, t)\|_{L^2(\mathbb{R}^3)}^\frac{3}{2} \|\sigma^\frac{1}{2} \nabla^3 d(\cdot, t)\|_{L^2(\mathbb{R}^3)}]$$

$$\leq MA [C_0^{\frac{q}{q-1}} B_q^\frac{1}{q} + A^\frac{1}{2} \|\sigma^\frac{1}{2} \nabla^3 d(\cdot, t)\|_{L^2(\mathbb{R}^3)}^\frac{1}{2}].$$

From (3.9) and (3.12), we have

$$\|\sigma^\frac{1}{2} \nabla^3 d(\cdot, t)\|_{L^2(\mathbb{R}^3)}^2 \leq M \int_{\mathbb{R}^3} \sigma^5 (|\nabla d|^2 + |u|^2 |\nabla^2 d|^2 + |\nabla u|^2 |\nabla d|^2 + |\nabla^2 d|^2 |\nabla d|^2) \, dx \leq M (A + D),$$

so that

$$\int_{\mathbb{R}^3} \sigma^5 |\nabla^2 d|^2 |\nabla d|^2 \, dx \leq MA [C_0^{\frac{q}{q-1}} B_q^\frac{1}{q} + A^\frac{1}{2} (A + D)^\frac{1}{2}].$$

The other terms included in $D$ are estimated in exactly the same way, and (3.27) follows. The proof of lemma is completed.

The following lemma contains the required bound for the pressure term in (2.11), which has been proved in Hoff [8, Lemma 3.3].

Lemma 3.8. Assume that the hypotheses and notations of Proposition 3.1 are in force. Then it holds

$$\int_0^t \int_{\mathbb{R}^3} \sigma^5 |\rho - \bar{\rho}|^4 \, dx \, ds \leq M [C_0 + \int_0^t \int_{\mathbb{R}^3} \sigma^5 |F|^4 \, dx \, ds].$$

(3.29)

We can now obtain the required estimates for the functional $E$ and $E$.

Lemma 3.9. Assume that the hypotheses and notations of Proposition 3.1 are in force. Then there are polynomials $\varphi_1$ and $\varphi_2$ whose degrees and coefficients depend on the same $M$ quantities as $M$ in the statement of Proposition 3.1 such that: for $0 < t \leq 1 \wedge T$

$$E \leq M [\varphi_1 (C_0) + \varphi_2 (A + B_q)],$$

(3.30)
and if $T > 1$ and $1 \leq t \leq T$, then

$$E \leq M[\varphi_1(C_0 + A(1) + B_q(1)) + \varphi_2(A + B_q)].$$  \hspace{1cm} (3.31)

The polynomial $\varphi_1$ contains no constant term and the monomials in $\varphi_2$ all have degrees strictly greater than 1.

**Proof.** Since the term $|\sum_{1 \leq k_1, j, m \leq 3} \int_0^t \int_{\mathbb{R}^3} \sigma u^{j_1} u^{j_2} u^{j_3} dxds|$ has been bounded exactly in Hoff [8]. So here we just bound the other terms for simplicity.

First for $0 < t \leq 1 \land T$, from (2.1), (3.9) and (3.12) we have

$$\int_0^t \int_{\mathbb{R}^3} \sigma^\frac{2}{3} |\nabla^2 d| dxds$$

and

$$\int_0^t \int_{\mathbb{R}^3} \sigma^5 |\nabla^2 d| dxds$$

From Lemma 2.4, Lemma 3.2, Lemma 3.5 and the definition of $F, \omega$, we have

$$\int_0^t \int_{\mathbb{R}^3} \sigma^5 |\nabla u|^3 dxds$$

Thus combining the above results and Lemma (3.7), we yields (3.30).
Now for $1 \leq t \leq T$, if we take $q = 4$ in (3.25) and integrate by parts to obtain that
\[ \int_{\mathbb{R}^3} |\nabla d|^4 dx + \int_t^t \int_{\mathbb{R}^3} |\nabla d|^2 |\Delta d|^2 dx ds + \int_t^t \int_{\mathbb{R}^3} |\nabla(|\nabla d|^2)|^2 dx ds \leq M\left(\int_{\mathbb{R}^3} |\nabla d|^4 dx \right)^2 + \int_t^t \int_{\mathbb{R}^3} |\Delta d| |\nabla d| |\nabla d|^2 dx ds + \int_t^t \int_{\mathbb{R}^3} |\nabla d|^2 |\nabla d|^2 dx ds \]
\[ \leq M\left[ C_0^{\frac{4q}{q-4}} B_q(1)^{\frac{2}{q-2}} + \int_t^t \int_{\mathbb{R}^3} |\Delta d| + |\nabla d|^2 |\nabla d|^2 dx ds \right. \\
+ \left. \int_t^t \int_{\mathbb{R}^3} |\nabla d|^2 |\nabla d|^2 dx ds \right]^{\frac{q}{2}} \left( \int_0^t \int_{\mathbb{R}^3} |\nabla u|^2 dx ds \right)^{\frac{q}{2}}. \]

Combining the results of Lemmas 3.2-3.9, we have the following bound for $A + B_q$.

**Lemma 3.10.** Assume that the hypotheses and notations of Proposition 3.1 are in force. Then there are polynomials $\varphi_1$ and $\varphi_2$ as described in Lemma 3.9 such that for $0 < t \leq 1$ and $T$,
\[ A + B_q \leq M[\varphi_1(C_0) + \varphi_2(A + B_q)], \]
and if $T > 1$ and $1 \leq t \leq T$, then
\[
\bar{A} + B_q \leq M[\varphi_1(C_0 + A(1) + B_q(1)) + \varphi_2(\bar{A} + B_q)].
\] (3.33)

**Proof of Proposition 3.1.** Proposition now follows immediately from the bounds (3.32) and (3.33) and the fact that the functions $A, \bar{A}, B_q$ are continuous in time.

### 4 Pointwise bounds for the density

In this section we derive pointwise bounds for the density $\rho$, bounds which are independent both of time and of initial smoothness. This will then close the estimates of Proposition 2.1 to give an uncontingent estimate for the functional $A$ defined in (3.1).

We begin with two auxiliary lemmas. The first lemma is a maximum-principle arguments applied integral curves of the velocity field, which has been proved in Hoff [8].

**Lemma 4.1** Let $(\rho, u, d)$ be as in Proposition 3.1 and suppose that $0 < c_1 \leq \rho \leq c_2$ on $\mathbb{R}^3 \times [0, T]$. Fix $t_0 \geq 0$ and define the particle trajectories $x: [0, \infty) \times \mathbb{R}^3 \to \mathbb{R}^3$ by
\[
\begin{cases}
\dot{x}(t, y) = u(x(t, y), t), \\
x(t_0, y) = y.
\end{cases}
\]

Then there is a constant $C$ depending only on $c_1$ and $c_2$ such that if $g \in L^1(\mathbb{R}^3)$ is nonnegative and $t \in [0, T]$, then each of the integrals $\int_{\mathbb{R}^3} g(x(t, y))dy$ and $\int_{\mathbb{R}^3} g(x)dx$ is bounded by $C$ times the other.

Next we derive a result relating the Hölder-continuity of $u(\cdot, t)$ to various norms appearing in the definition (2.1) of the functional $A$.

**Lemma 4.2.** Let $(\rho, u, d)$ be as in Proposition 3.1. Then for $\alpha \in (0, \frac{1}{2}]$ and $t \in (0, T]$, we have
\[
\langle u(\cdot, t) \rangle^\alpha \leq M[[\nabla u(\cdot, t)]^\frac{1-2\alpha}{L^2(\mathbb{R}^3)} \|\nabla \omega(\cdot, t)\|^\frac{1+2\alpha}{L^2(\mathbb{R}^3)} + (C_0 + \|\nabla u(\cdot, t)\|^2_{L^2(\mathbb{R}^3)})^{\frac{1-2\alpha}{2}}
\times ([\Delta u(\cdot, t)]^\frac{2}{L^2(\mathbb{R}^3)} + ([\nabla \cdot d] \Delta d(\cdot, t)]^\frac{2}{L^2(\mathbb{R}^3)})^{\frac{1+2\alpha}{4}} + C_0^{\frac{1-2\alpha}{1-\alpha}}] \tag{4.1}
\]

**Proof.** Let $\alpha \in (0, \frac{1}{2}]$ and define $r \in (3, 6]$ by $r = \frac{3}{1-\alpha}$. Then by (2.3) and (2.11), we have
\[
\langle u(\cdot, t) \rangle^\alpha \leq M[[F(\cdot, t)]^\frac{6-r}{L^r(\mathbb{R}^3)} \|\nabla \omega(\cdot, t)\|^\frac{2r-6}{L^2(\mathbb{R}^3)} + \|\rho - \bar{\rho}\|(\cdot, t)]^\frac{r}{L^r(\mathbb{R}^3)} \tag{4.2}
\]

By (2.1), we obtain
\[
\|\nabla(\cdot, t)\|_{L^r(\mathbb{R}^3)} \leq M([\|\omega(\cdot, t)\|^\frac{6-r}{L^r(\mathbb{R}^3)} \|\nabla \omega(\cdot, t)\|^\frac{2r-6}{L^2(\mathbb{R}^3)}) \leq M[\|\nabla u(\cdot, t)\|^\frac{1-2\alpha}{L^2(\mathbb{R}^3)} \|\nabla \omega(\cdot, t)\|^\frac{1+2\alpha}{L^2(\mathbb{R}^3)}]
\]
and
\[\|F(\cdot,t)\|_{L^r(\mathbb{R}^3)} \leq M(\|F(\cdot,t)\|_{L^2(\mathbb{R}^3)} \|\nabla F(\cdot,t)\|_{L^2(\mathbb{R}^3)})\]
\[\leq M(\|\rho - \check{\rho}\|_{L^2(\mathbb{R}^3)} + \|\nabla u(\cdot,t)\|_{L^2(\mathbb{R}^3)})^{\frac{1}{\alpha}}\times(\|\dot{u}(\cdot,t)\|_{L^2(\mathbb{R}^3)} + \|\nabla d \cdot \nabla d(\cdot,t)\|_{L^2(\mathbb{R}^3)})^{\frac{\alpha}{\alpha - 2}}\]
\[\leq M(\|C_0 + \|\nabla u(\cdot,t)\|_{L^2(\mathbb{R}^3)}\|_{L^2(\mathbb{R}^3)})^{\frac{1}{\alpha}}\times(\|\dot{u}(\cdot,t)\|_{L^2(\mathbb{R}^3)} + \|\nabla d \cdot \nabla d(\cdot,t)\|_{L^2(\mathbb{R}^3)})^{\frac{\alpha}{\alpha - 2}}\].

Putting the above results into (4.2) yields (4.1). Thus the proof the lemmas is completed.

Now we derive the upper and lower pointwise bounds for the density.

**Proposition 4.3.** Assume that the system parameters in (1.1) satisfy the conditions (1.5)-(1.8) and let positive numbers \(N\) and \(b \leq \delta\) be given. Assume \((\rho, u, d)\) is a solution of (1.1) on \(\mathbb{R}^3 \times [0,T]\) in the sense of Proposition 2.3 with initial data \((\rho_0, u_0) \in H^3(\mathbb{R}^3)\) and \(d_0 \in H^1(\mathbb{R}^3; S^2)\) satisfying (1.10)-(1.13). Then there are positive constants \(\varepsilon, M,\) and \(\theta\) depending on the parameters and assumptions in (1.5)-(1.8), on \(N,\) and a positive lower bound for \(b,\) such that, if \(C_0 < \varepsilon\) and \(\rho(x,t) > 0\) on \(\mathbb{R}^3 \times [0,T],\) then in fact
\[\rho \leq \rho \leq \check{\rho} \quad \text{on} \quad \mathbb{R}^3 \times [0,T],\] (4.3)
and
\[A(T) \leq MC_0^\theta.\] (4.4)

**Proof.** First we choose positive numbers \(\kappa\) and \(\kappa'\) satisfying
\[\rho < \kappa < \check{\rho} + b < \check{\rho} - b < \kappa' < \check{\rho}.\]

Recall that \(\rho_0\) takes values in \([\rho + b, \check{\rho} - b]\), so that \(\rho \in [\rho, \check{\rho}]\) on \(\mathbb{R}^3 \times [0,\tau]\) for some positive \(\tau\) by the time regularity (2.7). It then follows from Proposition 3.1 that \(A(\tau) \leq MC_0^\theta,\) where \(M\) is now fixed. We shall that if \(C_0\) is further restricted, then in fact that \(\kappa < \rho < \kappa'\) on all of \(\mathbb{R}^3 \times [0,T],\) and therefore that \(A(T) \leq MC_0^\theta\) as well. We shall prove the required upper bound, the proof of the lower bound being similar.

For \(y \in \mathbb{R}^3\) and define the corresponding particle path \(x(t)\) by
\[
\begin{cases}
\dot{x}(t,y) = u(x(t,y),t), \\
x(t_0,y) = y.
\end{cases}
\]

Suppose that there is a time \(t_1 \leq \tau\) such that \(\rho(x(t_1),t_1) = \kappa'.\) We may take \(t_1\) minimal and then choose \(t_0 < t_1\) maximal such that \(\rho(x(t_0),t_0) = \check{\rho} - b.\) Thus \(\rho(x(t),t) \in [\check{\rho} - b, \kappa']\) for \(t \in [t_0,t_1]\). We divide into two steps:

**Step 1.** \(t_0 < t_1 \leq T \wedge 1\)

We have from the definition (1.3) of \(F\) and the mass equation that
\[(\mu + \lambda)\frac{d}{dt}[\log \rho(x(t),t) - \log \check{\rho}] + P(\rho(x(t),t)) - P(\check{\rho}) = -F(x(t),t).\]
Integrating from $t_0$ to $t_1$ and abbreviating $\rho(x(t), t)$ by $\rho(t)$, etc., we then obtain

$$(\mu + \lambda) \log \rho(s)|_{t_0}^{t_1} + \int_{t_0}^{t_1} [P(s) - P(\tilde{\rho})] ds = -\int_{t_0}^{t_1} F(s) ds.$$  \hfill (4.5)

We shall show that

$$\int_{t_0}^{t_1} F(s) ds \leq \tilde{M}C_0^\theta$$  \hfill (4.6)

for a constant $\tilde{M}$ which depends on the same quantities as the $M$ from Proposition 3.1 (which has been fixed). If so, then from (3.5), we have

$$(\mu + \lambda) [\log \kappa' - \log(\tilde{\rho} - \bar{b})] \leq -\int_{t_0}^{t_1} [P(s) - P(\tilde{\rho})] ds + \tilde{M}C_0^\theta \leq \tilde{M}C_0^\theta.$$  \hfill (4.7)

where the last inequality holds because $\rho(t)$ takes values in $[\tilde{\rho} - \bar{b}, \kappa'] \subset [\max\{\tilde{\rho}, \rho''\}, \tilde{\rho}]$, and $P$ is increasing on $[\tilde{\rho}, \tilde{\rho}]$. But (4.7) cannot hold if $C_0$ is small depending on $M, \kappa'$, and $\tilde{\rho} - \bar{b}$. Stipulating the smallness condition, we therefore conclude that there is no time $t_1$ such that $\rho(t_1) = \rho(x(t_1), t_1) = \kappa'$. Since $y \in \mathbb{R}^3$ was arbitrary, it follows that $\rho < \kappa'$ on $\mathbb{R}^3 \times [0, \tau]$, as claimed. The proof that $\rho > \kappa$ is similar.

To prove (4.6) we let $\Gamma$ be the fundamental solution of the Laplace operator in $\mathbb{R}^3$ and apply (1.4) to write

$$\int_{t_0}^{t_1} F(s) ds = \int_{t_0}^{t_1} \int_{\mathbb{R}^3} (\nabla_x \Gamma(x(s) - y)) \rho \dot{u}(y, s) dy ds$$

$$+ \int_{t_0}^{t_1} \int_{\mathbb{R}^3} (\nabla_x \Gamma(x(s) - y)) (\nabla d \cdot \Delta d)(y, s) dy ds.$$  \hfill (4.8)

By Lemma 4.2, the first integral on the right side of (4.8) is bounded exactly as in Lemma 4.2 of Hoff [8]:

$$\int_{t_0}^{t_1} \int_{\mathbb{R}^3} (\nabla_x \Gamma(x(s) - y)) \rho \dot{u}(y, s) dy ds$$

$$\leq \|\nabla \Gamma * (\rho u)(\cdot, t_1)\|_{L^{\infty}(\mathbb{R}^3)} + \|\nabla \Gamma * (\rho u)(\cdot, t_2)\|_{L^{\infty}(\mathbb{R}^3)}$$

$$+ \int_{t_0}^{t} \int_{\mathbb{R}^3} \Gamma_{x,y}(x(s) - y)[u^k(x(s), s) - u^k(y, s)](\rho u)^2(y, s) dy ds$$

$$\leq MC_0^\theta + \tilde{M}C_0^\theta \int_{t_0}^{t} \langle u(\cdot, s) \rangle^\alpha ds$$

$$\leq MC_0^\theta + \tilde{M}C_0^\theta \int_{t_0}^{t} \langle u(\cdot, s) \rangle^\alpha ds$$

$$\leq \tilde{M}C_0^\theta + \tilde{M}C_0^\theta \left\|\int_{t_0}^{t} s^{-\frac{2q}{3+q}} ds\right\|^\frac{1}{2} \int_{t_0}^{t} \left\|\nabla u(\cdot, t)\right\|_{L^2(\mathbb{R}^3)} ds \left\|\int_{t_0}^{t} s^{-\frac{2q}{3+q}} ds\right\|^\frac{1}{2}$$

$$\times \left(\int_{t_0}^{t_1} s^2 \|\Delta u(\cdot, t)\|_{L^2(\mathbb{R}^3)} ds \right)^\frac{1}{3+q}$$

$$\leq MC_0^\theta,$$

if $\alpha < \frac{1}{6}$. Note that (3.22) holds for $q = 6$, thus if $2 < r < \frac{3q}{q+3}$, by (2.4), the second integral on the right side of (4.8) can be bounded in as

$$\int_{t_0}^{t_1} \int_{\mathbb{R}^3} (\nabla_x \Gamma(x(s) - y)) (\nabla d \cdot \Delta d)(y, s) dy ds$$

$$\leq \tilde{M} \int_{t_0}^{t} \left\|\nabla d \Delta d\right\|_{L^2(\mathbb{R}^3)} ds + \left\|\nabla d \Delta d\right\|_{L^{\infty}(\mathbb{R}^3)} ds$$

$$\leq \tilde{M} \int_{t_0}^{t} \left\|\nabla d \Delta d\right\|_{L^2(\mathbb{R}^3)} ds + \left\|\nabla d \Delta d\right\|_{L^{\infty}(\mathbb{R}^3)} ds$$

$$\leq \tilde{M} C_0^\theta.$$
Thus the proof of (4.6) is completed.

**Step 2.** $1 \leq t_0 < t_1$.

Again by the mass equation and the definition (1.3) of $F$, 
\[
\frac{d}{dt}(\rho(t) - \bar{\rho}) + (\mu + \lambda)^{-1} \rho(t)(P(t) - \bar{P}) = (\mu + \lambda)^{-1} \rho(t)F(t).
\]

Multiplying by $(\rho(t) - \bar{\rho})$ we get 
\[
\frac{1}{2}(\rho(t) - \bar{\rho})^2 + (\mu + \lambda)^{-1} f(t) \rho(t)(\rho(t) - \bar{\rho})^2 = -(\mu + \lambda)^{-1} \rho(t)(\rho(t) - \bar{\rho})F(t), \quad (4.9)
\]
where 
\[
f(t) = (P(t) - \bar{P})(\rho(t) - \bar{\rho})^{-1}.
\]

Since $f(t) \geq 0$ on $[t_0, t_1]$, thus integrating (4.9) over $[t_0, t_1]$, we arrive at 
\[
|\kappa' - \bar{\rho}|^2 + |\rho - b - \bar{\rho}|^2 \leq \bar{M} \int_{t_0}^{t_1} \|F(\cdot, s)\|_{L^\infty(\mathbb{R}^3)}^2 ds. \quad (4.10)
\]

So that if we show that 
\[
\int_{t_0}^{t_1} \|F(\cdot, s)\|_{L^\infty(\mathbb{R}^3)}^2 ds \leq MC_0^g. \quad (4.11)
\]
Then as in Step 1, (4.10) cannot hold if $C_0$ is sufficiently small. Since $y \in \mathbb{R}^3$ was arbitrary, it follows that $\rho < \kappa'$ on $\mathbb{R}^3 \times [0, \tau]$, as claimed.

To prove (4.11) we apply (1.4) and (2.4) to get 
\[
\int_{t_0}^{t_1} \|F(\cdot, s)\|_{L^\infty(\mathbb{R}^3)}^2 ds \leq \int_{t_0}^{t_1} \|\bar{\varphi}(\cdot, s)\|_{L^2(\mathbb{R}^3)}^2 ds + \int_{t_0}^{t_1} \|(\nabla d \cdot \Delta d)(\cdot, s)\|_{L^2(\mathbb{R}^3)}^2 ds
\]
\[
+ \int_{t_0}^{t_1} \|\bar{\varphi}(\cdot, s)\|_{L^4(\mathbb{R}^3)}^2 ds + \int_{t_0}^{t_1} \|(\nabla d \cdot \Delta d)(\cdot, s)\|_{L^4(\mathbb{R}^3)}^2 ds
\]
\[
\leq MC_0^g + \int_{t_0}^{t_1} \|\bar{\varphi}(\cdot, s)\|_{L^4(\mathbb{R}^3)}^2 ds + \int_{t_0}^{t_1} \|(\nabla d \cdot \Delta d)(\cdot, s)\|_{L^4(\mathbb{R}^3)}^2 ds.
\]

The terms integral on the right side above can be bounded as 
\[
\int_{t_0}^{t_1} \|\bar{\varphi}(\cdot, s)\|_{L^4(\mathbb{R}^3)}^2 ds \leq \int_{t_0}^{t_1} \|\bar{\varphi}(\cdot, s)\|_{L^2(\mathbb{R}^3)}^2 \|\nabla \bar{\varphi}(\cdot, s)\|_{L^2(\mathbb{R}^3)}^2 ds
\]
\[
\leq \left( \int_{t_0}^{t_1} \|\bar{\varphi}(\cdot, s)\|_{L^2(\mathbb{R}^3)}^2 ds \right)^{\frac{1}{2}} \left( \int_{t_0}^{t_1} \|\nabla \bar{\varphi}(\cdot, s)\|_{L^2(\mathbb{R}^3)}^2 ds \right)^{\frac{1}{2}}
\]
\[
\leq MC_0^g
\]
and 
\[
\int_{t_0}^{t_1} \|(\nabla d \cdot \Delta d)(\cdot, s)\|_{L^4(\mathbb{R}^3)}^2 ds
\]
\[
\leq \int_{t_0}^{t_1} \|(\nabla d \Delta d)(\cdot, s)\|_{L^2(\mathbb{R}^3)}^2 \|\nabla (\nabla d \cdot \Delta d)(\cdot, s)\|_{L^2(\mathbb{R}^3)}^2 ds
\]
\[
\leq \left( \int_{t_0}^{t_1} \|(\nabla d \Delta d)(\cdot, s)\|_{L^2(\mathbb{R}^3)}^2 ds \right)^{\frac{1}{2}} \left( \int_{t_0}^{t_1} \|\nabla (\nabla d \cdot \Delta d)(\cdot, s)\|_{L^2(\mathbb{R}^3)}^2 ds \right)^{\frac{1}{2}}
\]
\[
\leq MC_0^g,
\]
where the last inequality follows from Proposition 3.1. Thus (4.11) is proved. The proof of Proposition is completed.
5 Proof of Theorem 1.2

In this section, we prove Theorem 1.2 by constructing weak solutions as limits of smooth solutions. So, we first prove the global-in-time existence of smooth solutions with smooth initial data which is strictly away vacuum and is only of small energy.

**Proposition 5.1.** Assume that \((\rho_0, u_0, d_0)\) satisfy (2.5). Then for any \(0 < T < \infty\), there exists a unique smooth solution \((\rho, u, d)\) of (1.1)-(1.13) on \(\mathbb{R}^3 \times [0, T]\) satisfying (2.6)-(2.9) with \(T_0\) being replaced by \(T\), provided the initial energy \(C_0\) satisfies the smallness condition (1.16) with \(\varepsilon > 0\) being the same one as in Proposition 3.1 and Proposition 4.3.

**Proof.** The standard local existence result (Proposition 2.3) shows that the Cauchy problem (1.1)-(1.2) admits a unique local smooth solution \((\rho, u, d)\) on \(\mathbb{R}^3 \times [0, T_0]\). In view of Lemma 3.2 and Proposition 4.3, we have

\[
A(T_0) + \sup_{0 \leq t \leq T_0} \int_{\mathbb{R}^3} (|\rho - \bar{\rho}|^2 + |u|^2 + |\nabla d|^2) dx + \int_0^{T_0} \int_{\mathbb{R}^3} (|\nabla u|^2 + |\triangle d + |\nabla d|^2|^2) dx dt \leq MC_0,
\]

and

\[
\rho \leq \rho \leq \bar{\rho} \quad \text{on} \quad \mathbb{R}^3 \times [0, T_0].
\]

Then the standard arguments based on the local existence results together with the a priori bounds (5.1)-(5.2), we deduce that \((\rho, u, d)\) is in fact the unique smooth solution of (1.1)-(1.13) on \(\mathbb{R}^3 \times [0, T]\) for any \(0 < T < \infty\).

With the help of Proposition 5.1, we are in a position to prove Theorem 1.2.

**Proof of Theorem 1.2.** For any map \(d_0 \in H_0^1(\mathbb{R}^3; \mathbb{S}^2)\), there exists \(d_0^m \in H_0^1(\mathbb{R}^3; \mathbb{S}^2)\) such that

\[
\lim_{m \to \infty} \|d_0^m - d_0\|_{H^1(\mathbb{R}^3)} = 0.
\]

Let

\[
\rho_0^m = J_\frac{1}{m} * \rho_0, \quad u_0^m = J_\frac{1}{m} * u_0,
\]

where \(J = J_m(x)\) is the standard mollifier. Then \((\rho_0^m, u_0^m) \in H^3(\mathbb{R}^3)\) and \(d_0^m - n \in H_n^1(\mathbb{R}^3; \mathbb{S}^2)\) and the initial norm for \((\rho_0^m, u_0^m, \nabla d_0^m)\) (i.e., the right side of (1.13) with \((\rho_0, u_0, \nabla d_0)\) replaced by \((\rho_0^m, u_0^m, \nabla d_0^m)\)) is bounded by \(C_0\). The above Proposition can be applied to obtain a global smooth solution \((\rho^m, u^m, d^m)\) of (1.1)-(1.13) satisfying (3.2), (4.3) and (4.4) for all \(t > 0\) uniformly in \(m\).

In view of (2.3) and (2.11), we see from Sobolev embedding theorem that

\[
\langle u_m(t), t \rangle^\frac{1}{2} \leq C \|\nabla u_m\|_{L^6(\mathbb{R}^3)} \leq C (\|F_m\|_{L^6(\mathbb{R}^3)} + \|\omega_m\|_{L^6(\mathbb{R}^3)} + \|P_m - \bar{P}\|_{L^6(\mathbb{R}^3)} + C(\tau)(1 + \|\tilde{u}_m\|_{L^2(\mathbb{R}^3)} + \|\nabla d_m \cdot \triangle d_m\|_{L^2(\mathbb{R}^3)}) \leq C(\tau), \quad t \geq \tau > 0,
\]

\[
(5.3)
\]
where $F^m, \omega^m$ and $P^m$ are the functions $F, \omega$ and $P$ with $(\rho, u, d)$ being replaced by $(\rho^m, u^m, d^m)$.

In addition to (5.3), we also have

$$|u^m(x, t) - \frac{1}{BR(x)} \int_{BR(x)} u^m(y, t)dy| \leq C(\tau)R^\frac{\alpha}{2},$$

and hence, for $0 < \tau \leq t_1 \leq t_2$

$$|u^m(x, t_2) - u^m(x, t_1)| \leq \frac{1}{|BR(x)|} \int_{t_1}^{t_2} |u^m(y, t)|dydt + C(\tau)R^\frac{\alpha}{2} \leq CR^{-\frac{\alpha}{2}}|t_2 - t_1|^\frac{3}{2} + C(\tau)R^\frac{\alpha}{2} \leq C(\tau)R^{-\frac{\alpha}{2}}|t_2 - t_1|^\frac{3}{2} + CR^\frac{\alpha}{2}.$$

Taking $R = |t_2 - t_1|^{\frac{1}{4}}$ in (5.4), we get

$$|u^m(x, t_2) - u^m(x, t_1)| \leq C(\tau)|t_2 - t_1|^\frac{3}{8}, \quad 0 < \tau \leq t_1 \leq t_2 < \infty. \quad (5.5)$$

The same estimates in (5.3) and (5.5) also hold for $d$ and $\nabla d$. Thus, we have proved that $\{u^m\}, \{d^m\}$ and $\{\nabla d^m\}$ are uniform Hölder continuity away from $t = 0$. As a result, it follows from Ascoli-Arzela theorem that

$$u^m \longrightarrow u, \quad d^m \longrightarrow d \text{ uniformly on compact sets in } \mathbb{R}^3 \times (0, \infty). \quad (5.6)$$

Moreover, by argument in [24] (see also [5]), we know that

$$\rho^m \longrightarrow \rho \text{ strongly in } L^p(\mathbb{R}^3 \times (0, \infty)), \quad \forall p \in [2, \infty). \quad (5.7)$$

Therefore, passing to the limit as $m \longrightarrow \infty$ by (5.6) and (5.7) we obtain the limited $$(\rho, u, d)$$ which is indeed a weak solution of (1.1)-(1.13) in the sense of Definition 1.1 and satisfies (1.17)-(1.24).

Next we derive the large-time behavior of $(\rho, u, d)$ in (1.25). This can be done as the ones in [4], however, for completeness we sketch the proof here. We first deduce from the mass equation that

$$(P(\rho) - \tilde{P})_t + u \cdot \nabla (P(\rho) - \tilde{P}) + \gamma P(\rho)div u = 0.$$ 

Multiplying the above equation by $4(P(\rho) - \tilde{P})^3$ and integrating it over $\mathbb{R}^3$, we get that

$$\frac{d}{dt} \|P(\rho) - \tilde{P}\|_{L^4(\mathbb{R}^3)}^4 = \int_{\mathbb{R}^3} (|P(\rho) - \tilde{P}|^4)|div u - 3\gamma P(\rho)(P(\rho) - \tilde{P})^3|div u|dx,$$

which, together (3.29) shows that

$$\int_1^\infty \frac{d}{dt} \|P(\rho) - \tilde{P}\|_{L^4(\mathbb{R}^3)}^4 dt \leq C(1 + \int_1^\infty |F|^4_{L^4(\mathbb{R}^3)}ds) \int_1^\infty |\nabla u|^2_{L^2(\mathbb{R}^3)}ds \leq C.$$
As a result, we have
\[ \| P(\rho) - \tilde{P} \|_{L^4(\mathbb{R}^3)} \to 0 \text{ as } t \to \infty. \]
This, together with (3.2) and the uniform lower and upper bound of density, shows that
\[ \lim_{t \to \infty} \| \rho - \tilde{\rho} \|_{L^l(\mathbb{R}^3)} = 0 \quad (5.8) \]
holds for any \( l \in (2, \infty) \).

Following the argument in [4], we take a sequence
\[ u^m(t, x) := u(t + m, x), \]
for all integer \( m \), and \((x, t) \in \mathbb{R}^3 \times [1, 2] \). Then from (1.24), we have
\[ \lim_{m \to \infty} \int_0^1 \| \nabla u^m \|_{L^2(\mathbb{R}^3)} = 0. \]
From (1.24) again, we have
\[ \| u^m \|_{H^1(\mathbb{R}^3)} \leq C \text{ uniformly for } t, m. \]
Thus we arrive at
\[ \lim_{m \to \infty} \| u^m \|_{L^2(\mathbb{R}^3)} = 0 \text{ uniformly for } t. \]
That means
\[ \lim_{t \to \infty} \| u(t) \|_{L^2(\mathbb{R}^3)} = 0. \quad (5.9) \]
For \( t \geq 1 \), from (2.10) and (2.11), we obtain that
\[ \| \nabla u(t) \|_{L^6(\mathbb{R}^3)} \leq C(\| F(t) \|_{L^6(\mathbb{R}^3)} + \| \omega(t) \|_{L^6(\mathbb{R}^3)} + \| (P(\rho) - \tilde{P})(t) \|_{L^6(\mathbb{R}^3)}) \leq C(1 + \| \nabla F(t) \|_{L^2(\mathbb{R}^3)} + \| \nabla \omega(t) \|_{L^2(\mathbb{R}^3)} + \| (P^\delta)(t) \|_{L^2(\mathbb{R}^3)}) \leq C(1 + \| \dot{u}(t) \|_{L^2(\mathbb{R}^3)} + \| (\nabla d \Delta d)(t) \|_{L^2(\mathbb{R}^3)}) \quad (5.10) \]
Combining (1.24) (5.9) and (5.10), we have
\[ \lim_{t \to \infty} \| u \|_{W^{1, r}(\mathbb{R}^3)} = 0, \quad (5.11) \]
holds for \( r \in (2, 6) \).

Similarly, we have
\[ \lim_{t \to \infty} \| \nabla d \|_{W^{1, r}(\mathbb{R}^3)} = 0, \quad (5.12) \]
holds for \( r \in (2, 6) \). Putting (5.8), (5.11) and (5.12) together gives (1.25). Thus the proof of Theorem 1.2 is completed.

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