Functional Calculus on BMO-type Spaces of Bourgain, Brezis and Mironescu

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Abstract A nonlinear superposition operator $T_g$ related to a Borel measurable function $g : \mathbb{C} \to \mathbb{C}$ is defined via $T_g(f) := g \circ f$ for any complex-valued function $f$ on $\mathbb{R}^n$. This article is devoted to investigating the mapping properties of $T_g$ on a new BMO type space recently introduced by Bourgain, Brezis and Mironescu [J. Eur. Math. Soc. (JEMS) 17 (2015), 2083-2101], as well as its VMO and CMO type subspaces. Some sufficient and necessary conditions for the inclusion result and the continuity property of $T_g$ on these spaces are obtained.

1 Introduction

Recently, Bourgain, Brezis and Mironescu [21] introduced a new BMO type space $B$ on the unit cube, which is large enough to include the BMO space, the space $BV$ of functions of bounded variation and the Sobolev space $W^{1/p,p}$ with $p \in (1, \infty)$ as its special cases, and meanwhile it is also small enough to ensure that any integer-valued element belonging to its VMO type subspace $B_0$ is necessarily constant. This implication property

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f \in X \text{ being integer-valued} \implies f = \text{constant almost everywhere}
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of a space $X$ is known before to be true for the VMO space and the Sobolev space $W^{1/p,p}$ with $p \in [1, \infty)$, which are both subspaces of $B_0$. Later in [1], Ambrosio, Bourgain, Brezis and Figalli further found an interesting connection between the BMO type space and the notion of perimeter of sets. Indeed, via a global version of the norm of the new BMO type space, they found a new characterization of perimeter of sets independent of the theory of distributions.

In view of these remarkable applications of new BMO type spaces in analysis and geometry, it would be interesting to explore more properties or characterizations of these spaces. The main aim of this article is to clarify the mapping properties of the nonlinear superposition operator on these new BMO type spaces. Recall that a superposition operator $T_g$ (also called Nemytskij operator) related to a Borel measurable function $g : \mathbb{C} \to \mathbb{C}$ is given by

\begin{equation}
T_g(f) := g \circ f \quad \text{for any complex-valued function } f.
\end{equation}

This nonlinear operator $T_g$ appears frequently in various branches of mathematics and it plays a crucial role in nonlinear analysis as well as its applications to ordinary or partial differential equations, physics and engineering; see, for example, [4, 23, 24, 31] for some of its recent applications.
The study of the behavior of superposition operators on function spaces has a long history. Some early works on the behavior of superposition operators on Sobolev spaces can be found in Marcus and Mizel [28, 29, 30]. In [2], Appell and Zabrejko studied superposition operators on Lebesgue, Orlicz and Hölder spaces. During the last three decades, several important progresses on the study of superposition operators have been made on function spaces with fractional-order of smoothness (such as Sobolev spaces, Hölder-Zygmund spaces, Besov spaces and Triebel-Lizorkin spaces), due to Bourdaud and Sickel et al. For example, we refer the reader to [5, 16, 6, 10, 32, 34, 35] for Sobolev spaces, to [33, 7, 8, 11, 32, 36, 37, 17, 19, 20, 18] for Besov and Triebel-Lizorkin spaces, to [12] for Hölder-Zygmund spaces and to [14, 15] for spaces of functions of bounded $p$-variation; see also [3] for more historical information. The study of the superposition operators on classical BMO-type spaces can be found in [27, 22, 25, 9, 13]. Of particular importance to us is the article [13] of Bourdaud, Lanza de Cristoforis and Sickel, which provides a nearly complete picture on the mapping properties of superposition operators on BMO and its subspaces VMO and CMO on $\mathbb{R}^n$. Based on these, it is natural to study the behavior of the superposition operators on the aforementioned new BMO type space $B$ introduced in [21, 1].

To state the main results of this article, we begin with some basic notation and notions. For any $r \in (0, \infty)$ and $a \in \mathbb{R}^n$, let $Q_r(a) := Q(a, r)$ denote the open cube centered at $a$ with side length $r$. An open cube with side length $r$ is called an $r$-cube. Given a cube $Q \subset \mathbb{R}^n$ and a complex-valued locally integrable function $f$ defined on $\mathbb{R}^n$, we let

$$M(f, Q) := \int_Q |f(x) - f_Q| \, dx,$$

where

$$f_Q := \frac{1}{|Q|} \int_Q f(x) \, dx.$$

Let $Q_0 := (0, 1)^n$ be the unit open cube of $\mathbb{R}^n$. Denote by $L^1(Q_0)$ the set of all complex-valued measurable functions $f$ on $\mathbb{R}^n$ such that $\int_{Q_0} |f(x)| \, dx$ is finite. For any $f \in L^1(Q_0)$ and $\varepsilon \in (0, 1)$, let

$$[f]_{\varepsilon, Q_0} := \sup_{\mathcal{F}_\varepsilon} \left\{ e^{\varepsilon^{-1}} \sum_{j \in J} M(f, Q(\varepsilon a_j)) \right\},$$

where the supremum is taken over all collections $\mathcal{F}_\varepsilon := \{Q(\varepsilon a_j)\}_{j \in J}$ of mutually disjoint $\varepsilon$-cubes in $Q_0$ with sides parallel to the coordinate axes of $\mathbb{R}^n$ and cardinality $\#\mathcal{F}_\varepsilon = \#J \leq 1/\varepsilon^{n-1}$. Here and hereafter, for any set $E$, we use $\#E$ to denote its cardinality. The BMO type space $B(Q_0)$ is defined as the collection of all $f \in L^1(Q_0)$ such that

$$\sup_{0 < \varepsilon < 1} [f]_{\varepsilon, Q_0} < \infty.$$

For any $f \in B(Q_0)$, we define the corresponding norm

$$\|f\|_{B(Q_0)} := \int_{Q_0} |f(x)| \, dx + \sup_{0 < \varepsilon < 1} [f]_{\varepsilon, Q_0}.$$

We point out that this BMO type space $B(Q_0)$, denoted originally by $B$ in [21], was equipped with the norm $\|f\|_B := \sup_{0 < \varepsilon < 1} [f]_{\varepsilon, Q_0}$ therein, which makes $B$ into a Banach space modulo the space.
of constant functions. Since the operator $T_g$ is not defined on the quotient space, we use the norm $\| \cdot \|_{B(Q_0)}$ instead of $\| \cdot \|_B$ throughout this article.

Recall that the classical space $\text{BMO}(Q_0)$ is defined to be the set of all complex-valued locally integrable functions on $Q_0$ such that

$$\|f\|_{\text{BMO}(Q_0)} := \sup_{Q \subset Q_0} M(f, Q) < \infty,$$

where the supremum is taken over all cubes $Q$ in $Q_0$. It is obvious that the space $\text{BMO}(Q_0)$ is a subspace of $B(Q_0)$. Moreover, it was pointed out in [21, p. 2084] that, when $n = 1$, $\text{BMO}(Q_0) = B(Q_0)$, while when $n > 1$, $\text{BMO}(Q_0)$ is strictly smaller than $B(Q_0)$.

Let $B_c(Q_0)$ be the closure of the set $C_c^\infty(Q_0)$ in $B(Q_0)$, and $B_0(Q_0)$ the set of all $f \in B(Q_0)$ such that

$$\limsup_{\epsilon \to 0} [f]_{\epsilon, Q_0} = 0$$

or, equivalently,

$$\limsup_{\delta \to 0; 0 < \epsilon < \delta} [f]_{\epsilon, Q_0} = 0.$$

It is easy to show that $B_c(\mathbb{R}^n) \hookrightarrow B_0(\mathbb{R}^n)$ and $B_c(Q_0) \hookrightarrow B_0(Q_0)$. Here and hereafter, for any two vector space $X$ and $Y$, the symbol $X \subset Y$ only means that $X$ is a subset of $Y$, and $X \hookrightarrow Y$ means that not only $X \subset Y$ but also the embedding from $X$ into $Y$ is continuous. It is also easy to see that $\text{VMO}(Q_0) \subset B_0(Q_0)$ and $\text{CMO}(Q_0) \subset B_c(Q_0)$, where

$$\text{VMO}(Q_0) := \left\{ f \in \text{BMO}(Q_0) : \lim_{\epsilon \to 0} \sup_{Q \subset Q_0, \ell(Q) \leq \epsilon} M(f, Q) = 0 \right\}$$

and $\text{CMO}(Q_0)$ denotes the closure of $C_c^\infty(Q_0)$ in $\text{BMO}(Q_0)$. Here and hereafter, for any cube $Q$, we use $\ell(Q)$ to denote its side length.

We also consider an analogous global version of $B(Q_0)$. Given a complex-valued locally integrable function $f$ on $\mathbb{R}^n$ and $\epsilon \in (0, 1)$, define

$$[f]_\epsilon := \sup_{\mathcal{F}_\epsilon} \left\{ \epsilon^{n-1} \sum_{j \in J} M(f, Q_\epsilon(a_j)) \right\},$$

where the supremum is now taken over all collections $\mathcal{F}_\epsilon := \{Q_\epsilon(a_j)\}_{j \in J}$ of mutually disjoint $\epsilon$-cubes in $\mathbb{R}^n$ with sides parallel to the coordinate axes and cardinality $\#\mathcal{F}_\epsilon = \#J \leq 1/\epsilon^{n-1}$. Denote by $B(\mathbb{R}^n)$ the space of all complex-valued functions $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ such that

$$\|f\|_{B(\mathbb{R}^n)} := \sup_{|Q| = 1} \int_Q |f(x)| \, dx + \sup_{0 < \epsilon < 1} [f]_\epsilon < \infty,$$

where the first supremum is taken over all 1-cubes $Q$ in $\mathbb{R}^n$ with sides parallel to the coordinate axes. By this definition, it is easy to see that $B(\mathbb{R}^n)$ is translation invariant.

Here, it should be mentioned that the limit when $\epsilon \to 0$ of an isotropic variant $I_\epsilon(f)$ of $[f]_\epsilon$, defined via removing the restriction “sides parallel to the coordinate axes” from the definition of $[f]_\epsilon$, was used in [1] to give a new characterization of the perimeter of sets, independent of the
Theorem 1.2. The following five statements are equivalent:

1. \( T \) is in \( BMO \) when it shares the same inherent regularity as \( BMO \). 
2. \( T \) is in \( BMO \) (resp. BMO) in \( [13, \text{Theorem 1}] \), which ensures the inclusion \( T(B^n) \subset B(B^n) \) here is same as that for \( T(BMO(B^n)) \subset BMO(B^n) \) and \( T_{\ell}(\text{smo}(B^n)) \subset \text{smo}(B^n) \) in [13, Theorem 1]. This phenomenon in some sense implies that the space \( B \) shares the same inherent regularity as \( BMO \), though the space \( B \) is strictly bigger than \( BMO \) when \( n > 1 \). Based on this observation, we can also know that the condition for \( T(B_0(B^n)) \subset B_0(B^n) \) in Theorem 1.2 below is same as that for \( T(BVMO(B^n)) \subset VMO(B^n) \) and \( T_{\ell}(\text{vmo}(B^n)) \subset \text{vmo}(B^n) \) in [13, Theorem 2].

Theorem 1.1. The following five statements are equivalent:

1. \( \sup_{x,y \in \mathbb{C}} (1 + |x - y|)^{-1} |g(x) - g(y)| < \infty \);
2. \( T_g(B(\mathbb{R}^n)) \subset B(\mathbb{R}^n) \);
3. \( T_g(B_c(\mathbb{R}^n)) \subset B(\mathbb{R}^n) \);
4. \( T_g(B(Q_0)) \subset B(Q_0) \);
5. \( T_g(B_c(Q_0)) \subset B(Q_0) \).

Moreover, if any of the above holds true, then \( T_g \) maps bounded subsets of \( B(\mathbb{R}^n) \) (resp. \( B(Q_0) \)) to bounded subsets of \( B(\mathbb{R}^n) \) (resp. \( B(Q_0) \)).
(i) $g$ is uniformly continuous;
(ii) $T_g(B_0(\mathbb{R}^n)) \subset B_0(\mathbb{R}^n)$;
(iii) $T_g(B_c(\mathbb{R}^n)) \subset B_0(\mathbb{R}^n)$;
(iv) $T_g(B_0(Q_0)) \subset B_0(Q_0)$;
(v) $T_g(B_c(Q_0)) \subset B_0(Q_0)$.

Moreover, if any of the above holds true, then $T_g$ maps bounded subsets of $B_0(\mathbb{R}^n)$ (resp. $B_0(Q_0)$) to bounded subsets of $B_0(\mathbb{R}^n)$ (resp. $B_0(Q_0)$).

When the target spaces become $B_c(\mathbb{R}^n)$ or $B_c(Q_0)$, we have the following result.

**Theorem 1.3.** (a) $T_g(B_c(\mathbb{R}^n)) \subset B_c(\mathbb{R}^n)$ if and only if $g$ is uniformly continuous and $g(0) = 0$.
(b) $T_g(B_c(Q_0)) \subset B_c(Q_0)$ if and only if $g$ is uniformly continuous.

We point out that the condition for $T_g(B_c(\mathbb{R}^n)) \subset B_c(\mathbb{R}^n)$ in Theorem 1.3 is same as that for $T_g(cm(\mathbb{R}^n)) \subset cm(\mathbb{R}^n)$ in [13, Corollary 1].

One key tool to prove Theorem 1.3 is the continuity of $T_g$ at $f \in B_0(\mathbb{R}^n)$ (resp. $B_0(Q_0)$) as a map from $B(\mathbb{R}^n)$ (resp. $B(Q_0)$) to itself, whenever $g$ is uniformly continuous (see Proposition 4.4 below). This continuity result, together with Theorems 1.2 and 1.3, also easily implies the following theorem on the continuity of $T_g$.

**Theorem 1.4.** (a) The following are equivalent:
   (i) $g$ is uniformly continuous;
   (ii) $T_g$ is continuous from $B_0(\mathbb{R}^n)$ to $B_0(\mathbb{R}^n)$;
   (iii) $T_g$ is continuous from $B_0(Q_0)$ to $B_0(Q_0)$;
   (iv) $T_g$ is continuous from $B_c(Q_0)$ to $B_c(Q_0)$.
(b) $T_g$ is continuous from $B_c(\mathbb{R}^n)$ to $B_c(\mathbb{R}^n)$ if and only if $g$ is uniformly continuous and $g(0) = 0$.

When the target space is $B(\mathbb{R}^n)$, the uniformly continuity of $g$ is no longer enough to ensure the continuity of $T_g$. Indeed, we have the following conclusion.

**Theorem 1.5.** The operator $T_g$ is continuous from $B(\mathbb{R}^n)$ to $B(\mathbb{R}^n)$ if and only if $g$ is $\mathbb{R}$-affine, that is, $g(z)$ is of form $\alpha z + \beta$ for some complex numbers $\alpha$ and $\beta$ and for any $z \in \mathbb{C}$.

The organization of this article is as follows. As preparatory works for proving main theorems, in Section 2, we establish a grouping lemma (see Lemma 2.1) which provides a suitable way to enlarge and grouping cubes in order to fit the definition of B spaces. A consequent application of Lemma 2.1 is given in Proposition 2.3, in which we obtain some uniformly estimates of integral averages for functions in $B(\mathbb{R}^n)$ and $B(Q_0)$. Using these results in Section 2, we give the proof of Theorem 1.1 in Section 3, by first establishing several auxiliary lemmas, including a result about the pointwise multipliers on the BMO-type spaces. The proofs of Theorems 1.2 and 1.3 and the proof of Theorem 1.5 are presented, respectively, in Section 4 and Section 5. Here we
point out that, since the structure of B spaces are more complicated than BMO, compared with the arguments in [13] for the classical BMO spaces, the proofs given in this article are sometimes much more subtle and sophisticated (see, for example, Lemma 2.1 and Proposition 2.3).

Throughout this article, let \( \mathbb{N} := \{1, 2, \ldots\} \) and \( \mathbb{Z} := \{0, \pm 1, \ldots\} \). We use \( C \) to denote a positive constant that is independent of the main parameters involved but whose value may differ from line to line. Sometimes we use \( C(\alpha, \beta, \ldots) \) to indicate that a constant \( C \) depends on the given parameters \( \alpha, \beta, \ldots \). If \( f \leq Cg \), we then write \( f \lesssim g \), and if \( f \leq g \leq f \), we then write \( f \sim g \). For any \( s \in \mathbb{R} \), denote by \( \lfloor s \rfloor \) the largest integer not greater than \( s \). For any cube \( Q \) in \( \mathbb{R}^n \), the notation \( \ell(Q) \) denotes the side length of \( Q \). For any \( \lambda \in (0, \infty) \) and any cube \( Q \) in \( \mathbb{R}^n \), denote by \( \lambda Q \) the cube with the same center as that of \( Q \) but of side length \( \lambda \ell(Q) \). Also, for any set \( E \), we use \#\( E \) to denote its cardinality.

## 2 A grouping lemma

Let us begin with the following grouping lemma. For any \( j \in \mathbb{Z} \) and \( k \in \mathbb{Z} \), let \( Q_{jk} \) denote the dyadic cube \( 2^{-j}([0, 1]^n + k) \). Denote by \( Q \) the collection of all dyadic cubes and \( Q_j := \{Q_{jk}\}_{k \in \mathbb{Z}} \).

**Lemma 2.1.** Let \( k_0 \in \mathbb{N} \) and \( k_0 \geq 2 \).

(a) Let \( \{Q_i\}_{i \in J} \) be a family of mutually disjoint open \( 2^{-k_0} \)-cubes in \( \mathbb{R}^n \) with \( \#J \leq 2^{k_0(n-1)} \). For each \( i \in J \), let \( Q_i := 2Q_i \), which is of side length \( 2^{1-k_0} \) and \( Q_i \) is contained in \( Q \). Then there exists a positive integer \( N = N(n) \leq 2^n \) such that the cubes \( \{Q_i\}_{i \in J} \) enjoy the following properties:

(i) \( J = J^1 \cup \cdots \cup J^N \);  
(ii) for every \( 1 \leq j \leq N \), the cubes \( \{Q_i\}_{i \in J^j} \) are mutually disjoint; 
(iii) for every \( 1 \leq j \leq N \), the cardinality \( \#J^j \leq \#J/2^{n-1} \).

(b) Let \( \{Q_i\}_{i \in J} \) be a family of mutually disjoint dyadic cubes in \( Q_0 \) with side length \( 2^{-k_0} \) and \( \#J \leq 2^{k_0(n-1)} \). For each \( i \in J \), let \( Q_i \) be the unique dyadic cube with side length \( 2^{1-k_0} \) contained in \( Q_0 \) such that \( Q_i \) is contained in \( Q \). Then there exists a positive integer \( N = N(n) \leq 2^n \) such that the items (i)-(iii) in (a) remain true.

**Proof.** First we show (a). Since all \( Q_i \) are open, we know that any point in \( \mathbb{R}^n \) can be covered by at most \( 2^n \) elements from \( \{Q_i\}_{i \in J} \), due to the non-overlapping property of \( \{Q_i\}_{i \in J} \). With this observation, the grouping procedure can be done as follows. Put the index \( i = 1 \) in \( J_1 \). If \( Q_3 \) does not intersect \( Q_1 \) and \( \#J_1 < \#J/2^{n-1} \), then we put the index \( i = 2 \) in \( J_1 \); otherwise we put the index \( i = 2 \) in \( J_2 \). Next, we look at \( Q_3 \) and consider three cases:

- If \( Q_3 \) does not intersect \( Q_1 \) and \( \#J_1 < \#J/2^{n-1} \), then put the index \( i = 3 \) in \( J_1 \).
- If \( Q_1 \) intersects \( Q_3 \) or \( \#J_1 = [\#J/2^{n-1}] \), but \( Q_3 \) does not intersect \( Q_2 \) and \( \#J_2 < [\#J/2^{n-1}] \), then put the index \( i = 3 \) in \( J_2 \).
- If \( Q_2 \) intersects \( Q_1 \) or \( \#J_1 = [\#J/2^{n-1}] \), and \( Q_3 \) intersects \( Q_2 \) or \( \#J_2 = [\#J/2^{n-1}] \), then put the index \( i = 3 \) in \( J_3 \).
Continuing the above procedure, we can divide \( \{ \tilde{Q}_i \}_{i \in J} \) into at most \( N (\leq 2^n) \) groups, \( \{ \tilde{Q}_i \}_{i \in J_1}, \ldots, \{ \tilde{Q}_i \}_{i \in J_N} \), so that each group is a collection of mutually disjoint cubes with cardinality not more than \( \#J/2^{n-1} \).

Now, we show (b). By the geometric properties of dyadic cubes, we know that, if \( Q_i \) is a dyadic cube contained in \( Q_0 \) with side length \( \leq 1/2 \), then the unique dyadic cube \( \tilde{Q}_i \) containing \( Q \) with side length \( 2\ell(Q_i) \) is contained in \( \tilde{Q}_0 \). In this case, when \( i \neq j \), it might happen that \( \tilde{Q}_i = \tilde{Q}_j \). Also, a dyadic cube \( \tilde{Q}_i \) can serve as the 2-times dyadic extension of at most \( 2^n \) dyadic cubes in \( \{Q_i\}_{i \in J} \).

Based on these observations, following the same grouping procedure as in (a), we immediately obtain the desired conclusion of Lemma 2.1(b). This finishes the proof of Lemma 2.1.

Observe that the supremum over \( \epsilon \in (0, 1) \) in \( \| \cdot \|_{B(Q_0)} \) and \( \| \cdot \|_{B(\mathbb{R}^n)} \) can be equivalently taken over \( \{2^{-k} : k \in \mathbb{N}\} \).

**Lemma 2.2.** There exists a positive constant \( C := C_{(n)} \) such that

\[
C^{-1} \|f\|_{B(\mathbb{R}^n)} \leq \sup_{Q \in \mathcal{Q}} f(\mathbb{R}^n) = \int_Q |f(x)| \, dx + \sup_{k \in \mathbb{N}} \|f\|_{L^2(Q_{2^{-k}})} \leq \|f\|_{B(\mathbb{R}^n)}, \quad \forall f \in B(\mathbb{R}^n)
\]

and

\[
C^{-1} \|f\|_{B(Q_0)} \leq \int_{Q_0} |f(x)| \, dx + \sup_{k \in \mathbb{N}} \|f\|_{L^2(Q_{2^{-k}})} \leq \|f\|_{B(Q_0)}, \quad \forall f \in B(Q_0).
\]

**Proof.** By similarity, we only consider \( \| \cdot \|_{B(Q_0)} \). Since the second inequality is trivial, we only prove the first one.

Let \( f \in B(Q_0) \). If \( \epsilon \in (0, 1/2) \), then there exists \( k \in \mathbb{N} \) such that \( 2^{-k-1} < \epsilon \leq 2^{-k} \). For any \( \epsilon \)-cube \( Q_\epsilon \) in \( Q_0 \), there exists a \( 2^{-k} \)-cube \( Q \subset Q_0 \) containing \( Q_\epsilon \). Thus,

\[
M(f, Q_\epsilon) \leq 2^n M(f, f_Q) + |Q_\epsilon - f_Q| \leq 2^{n+1} M(f, f_Q).
\]

If \( \epsilon \in (1/2, 1) \), then

\[
M(f, Q_\epsilon) \leq 2 \int_{Q_\epsilon} |f(x)| \, dx \leq 2^{n+1} \int_{Q_0} |f(x)| \, dx.
\]

Thus, \( \|f\|_{Q_0} \leq \sup_{k \in \mathbb{N}} \|f\|_{L^2(Q_{2^{-k}})} + \int_{Q_0} |f(x)| \, dx \). This finishes the proof of Lemma 2.2.

Applying Lemmas 2.1 and 2.2, we have the following estimates of functions from \( B(\mathbb{R}^n) \) or \( B(Q_0) \).

**Proposition 2.3.** There exists a positive constant \( C \), depending only on \( n \), such that the following assertions are true:

(i) for any \( f \in B(\mathbb{R}^n) \) and \( k_0 \in \mathbb{N} \),

\[
2^{-k_0 n} \sum_{j \in J_0} \int_{Q_{2^{-k_0}}(a_j)} |f| \leq C \|f\|_{B(\mathbb{R}^n)},
\]

where \( \{Q_{2^{-k_0}}(a_j)\}_{j \in J_0} \) are any mutually disjoint \( 2^{-k_0} \)-cubes in \( \mathbb{R}^n \) with sides parallel to the coordinate axes and \( \# J_0 \leq 2^{k_0(n-1)} \).
(ii) for any \( f \in B(Q_0) \) and \( k_0 \in \mathbb{N} \),

\[
2^{-k_0n} \sum_{j \in J_0} \int_{Q_{2^{-k_0}(a_j)}} |f| \leq C \|f\|_{B(Q_0)},
\]

where \( \{Q_{2^{-k_0}(a_j)}\}_{j \in J_0} \) are any mutually disjoint \( 2^{-k_0} \)-cubes in \( Q_0 \) with sides parallel to the coordinate axes and \( |J_0| \leq 2^{k_0(n-1)} \).

**Proof.** First, we show (i). If \( k_0 = 1 \), then \( |J_0| \leq 2^{n-1} \) and hence

\[
2^{-k_0n} \sum_{j \in J_0} \int_{Q_{2^{-k_0}(a_j)}} |f| \leq \sup_{|Q|=1} \int_Q |f| \leq \|f\|_{B(\mathbb{R}^n)}.
\]

Below we assume that \( k_0 \geq 2 \). Since \( |J_0| \leq 2^{k_0(n-1)} \), from Lemma 2.1(a), it follows that there exist 2-times extensions of the cubes \( \{Q_{2^{-k_0}(a_j)}\}_{j \in J_0} \), denoted by \( \{Q_{2^{-k_0+1}(a_j,j)}\}_{j \in J_1} \), so that the set \( \{Q_{2^{-k_0+1}(a_j,j)}\}_{j \in J_1} \) can be divided into \( N_1 = 2^n \) subgroups, where \( J_0 = J_1 \cup \cdots \cup J_1^{N_1} \). Moreover, for each \( i \in \{1, \ldots, N_1\} \), the cubes \( \{Q_{2^{-k_0+1}(a_j,j)}\}_{j \in J_1^i} \) are mutually disjoint and \( |J_1^i| \leq \|J_0/2^{n-1} \leq 2^{(k_0-1)(n-1)} \).

If \( k_0 - 1 \geq 2 \), we repeat the above procedure for the each group \( \{Q_{2^{-k_0+2}(a_j,j)}\}_{j \in J_2^i} \) with \( i \in \{1, \ldots, N_1\} \), and determine a desired collection \( \{Q_{2^{-k_0+2}(a_j,j)}\}_{j \in J_2^i} \) of \( 2^{-k_0+2} \)-cubes, where \( J_2^i = J_1^j \).

Moreover, by Lemma 2.1(a), we know that the set \( J_2^i \) can be divided into \( N_2^i = 2^n \) subgroups, denoted by \( \{J_2^i, \ldots, J_2^{i,2^{N_2^i}}\} \), such that the cubes \( \{Q_{2^{-k_0+2}(a_j,j)}\}_{j \in J_2^i} \) for each \( \ell \in \{1, \ldots, N_2^i\} \) are mutually disjoint and \( |J_2^i| \leq \|J_1^i/2^{n-1} \leq 2^{(k_0-2)(n-1)} \). Write \( J_2 := \bigcup_{i=1}^{N_1} \bigcup_{\ell=1}^{N_2^i} J_2^i \). Again we have \( J_2 = J_1 = J_0 \).

Iteratively, we can find sets \( \{J_1, J_2, \ldots, J_{k_0-1}\} \) of indices, having the following properties: for any \( m \in \{1, \ldots, k_0 - 1\} \),

(P-a) \( J_{k_0-1} = \cdots = J_1 = J_0 \);

(P-b) each \( J_m \) can be written as

\[
J_m = \bigcup_{i_1=1}^{N_1} \bigcup_{i_2=1}^{N_{i1}} \cdots \bigcup_{i_m=1}^{N_{i_1,i_2,\ldots,i_{m-1}}} J_{i_1,i_2,\ldots,i_m}^{i_1,i_2,\ldots,i_m}
\]

with every \( \# J_{i_1,i_2,\ldots,i_m}^{i_1,i_2,\ldots,i_m} \leq 2^{(k_0-m)(n-1)} \);

(P-c) for each \( a_{j,m-1} \) with \( j \in J_{m-1}^{i_1,i_2,\ldots,i_{m-1}} \subset J_{m-1} \), there exist \( i_m \in \{1, \ldots, N_{i_1,i_2,\ldots,i_{m-1}}\} \) and some point \( a_{j,m} \) with \( j' \in J_{m-1}^{i_1,i_2,\ldots,i_{m-1}} \) such that

\[
Q_{2^{-k_0+m-1}(a_{j,m-1})} \subset Q_{2^{-k_0+m-1}(a_{j',m})}.
\]

(P-d) the cubes in \( \{Q_{2^{-k_0+m}(a_{j,m})}\}_{j \in J_{m-1}^{i_1,i_2,\ldots,i_m}} \) are mutually disjoint.
Therefore, for each point $a_j$ with $j \in J_0$, there exists a sequence of points,
\[
\{a_{j_1,1}, a_{j_2,2}, \ldots, a_{j_{k_0-1},k_0-1}\},
\]
such that $j_i \in J_i$ for any $i \in \{1, \ldots, k_0-1\}$ and
\[
Q_{2^{-k_0}}(a_j) \subset Q_{2^{-k_0+1}}(a_{j_{k_0-1}}) \subset \cdots \subset Q_{2^{-1}}(a_{j_{k_0-1},k_0-1}).
\]
Thus,
\[
\mathcal{F} Q_{2^{-k_0}}(a_j) |f| \leq \sum_{i=1}^{k_0-2} |f_{Q_{2^{-k_0+1}}(a_{j_{k_0-1}})} - f_{Q_{2^{-k_0+2}}(a_{j_{k_0-1}})}| + |f_{Q_{2^{-1}}(a_{j_{k_0-1}})}| + \sup_{|Q|=1} \int |f|.
\]
If $k_0 = 2$, then the middle term in the above summation on $i \in \{1, \ldots, k_0-2\}$ disappears. From the above formula and $J_{k_0-1} = \cdots = J_1 = J_0$, we deduce that
\[
2^{-k_0} \sum_{J \in J_0} \mathcal{F} Q_{2^{-k_0}}(a_j) |f| \leq 2^{-k_0} \sum_{J \in J_1} \mathcal{F} Q_{2^{-k_0+1}}(a_{j_1}) |f - f_{Q_{2^{-k_0+1}}(a_{j_1})}|
\]
\[+ 2^{-k_0} \sum_{i=1}^{k_0-2} \sum_{J \in J_{i+1}} \mathcal{F} Q_{2^{-k_0+1}}(a_{j_{i+1}}) |f - f_{Q_{2^{-k_0+2}}(a_{j_{i+1}})}|
\]
\[+ \#J_0 2^{-k_0} \sup_{|Q|=1} \int |f| =: Z_1 + Z_2 + Z_3.
\]
Using $J_1 = \bigcup_{i=1}^{N_1} J_{i}$, $\#J_{i} \leq 2^{(k_0-1)(n-1)}$ and Lemma 2.2, we have
\[
Z_1 = 2^{-k_0} \sum_{i=1}^{N_1} \sum_{J \in J_{i}} \mathcal{F} Q_{2^{-k_0+1}}(a_{j_1}) |f - f_{Q_{2^{-k_0+1}}(a_{j_1})}|
\]
\[\leq 2^{-k_0} \sum_{i=1}^{N_1} 2^{(k_0-1)(n-1)} |f|_{2^{-k_0+1}} \leq \sup_{k \in \mathbb{N}} |f|_{2^{-k}} \leq \|f\|_{B^{(R^n)}}.
\]
By the above property (P-b) and Lemma 2.2, we obtain
\[
Z_2 = 2^{-k_0} \sum_{m=2}^{k_0-1} \sum_{J \in J_m} \mathcal{F} Q_{2^{-k_0+m}}(a_{j_m}) |f - f_{Q_{2^{-k_0+m}}(a_{j_m})}|
\]
\[= 2^{-k_0} \sum_{m=2}^{k_0-1} \sum_{N_{N_{J_{m}}}} \sum_{i_{m}=1}^{N_{N_{J_{m}}}} \sum_{J \in J_{m}} \mathcal{F} Q_{2^{-k_0+m}}(a_{j_{m}}) |f - f_{Q_{2^{-k_0+m}}(a_{j_{m}})}|
\]
Finally, from \( \#J_0 \leq 2^{k_0(n-1)} \), it follows easily that

\[
Z_3 \leq \sup_{Q \in \mathcal{D}} \int_Q |f| \leq \|f\|_{B(\mathbb{R}^n)}.
\]

Combining the estimations of \( Z_1 \) through \( Z_3 \), we obtain (2.1). This finishes the proof of (i).

Now we prove (ii). For any \( j \in J_0 \), since \( Q_{2^{-k_0}a_j} \) is a dyadic cube with side length \( 2^{-k_0} \) in \( Q_0 \). We write these dyadic cubes as

\[
\{ Q_{k_0,1}(a_j^1), \ldots, Q_{k_0,N_j}(a_j^{N_j}) \},
\]

where \( N_j \) depends on \( a_j \) and \( N_j \leq 2^n \). Then

\[
\int_{Q_{2^{-k_0}a_j}} |f| \leq \sum_{i=1}^{N_j} \int_{Q_{k_0,i}(a_j^i)} |f|.
\]

By the mutually disjointness of \( \{ Q_{2^{-k_0}a_j} \} \) and the geometric properties of dyadic cubes, we know that a dyadic cube of side length \( 2^{-k_0} \) can intersect at most \( 2^n \) cubes from \( \{ Q_{2^{-k_0}a_j} \} \), which implies that the same dyadic cube can appear at most \( 2^n \) times in the family

\[
\{ Q_{k_0,i}(a_j^i) : j \in J_0, i \in \{1, \ldots, N_j\} \}.
\]

Therefore, the set \( \{ Q_{k_0,i}(a_j^i) : j \in J_0, i \in \{1, \ldots, N_j\} \} \) can be decomposed into \( 2^n \) subgroups

\[
\{ Q_i \}_{i \in J_1}, \{ Q_i \}_{i \in J_2}, \ldots, \{ Q_i \}_{i \in J_{2^n}}
\]

of dyadic cubes with side length \( 2^{-k_0} \) in \( Q_0 \), where, for any \( k \in \{1, \ldots, 2^n\}, \#J_k \leq 2^{k_0(n-1)} \) and \( \{ Q_i \}_{i \in J_k} \) are mutually disjoint. Then

\[
\sum_{j \in J_0} \int_{Q_{2^{-k_0}a_j}} |f| \leq 2^n \sum_{k=1}^{2^n} \sum_{i \in J_k} \int_{Q_i} |f|.
\]

For each \( k \in \{1, \ldots, 2^n\} \), by an argument similar to that used in the proof of (i), with Lemma 2.1(a) used therein replaced by Lemma 2.1(b), we conclude that

\[
2^{-k_0n} \sum_{i \in J_k} \int_{Q_i} |f| \leq \|f\|_{B(Q_0)},
\]

whence

\[
2^{-k_0n} \sum_{j \in J_0} \int_{Q_{2^{-k_0}a_j}} |f| \leq 2^n \sum_{k=1}^{2^n} \|f\|_{B(Q_0)} \leq \|f\|_{B(\mathbb{R}^n)}.
\]

This proves (2.2), which completes the proof of (ii) and hence of Proposition 2.3.
3 Proof of Theorem 1.1

The aim of this section is to prove Theorem 1.1. To this end, we first establish some technical lemmas. Given a quasi-Banach space $X$ equipped with a quasi-norm $\| \cdot \|_X$, we recall that a function $h$ defined on $\mathbb{R}^n$ is called a pointwise multiplier on $X$ if there exists a positive constant $C$ such that $\|hf\|_X \leq C\|f\|_X$ for any $f \in X$. Applying Proposition 2.3, we have the following results on the pointwise multipliers of $B(\mathbb{R}^n)$ and $B(Q_0)$. Recall that $C^1_0(\mathbb{R}^n)$ denotes the set of all continuously differentiable functions with compact support on $\mathbb{R}^n$ and $C^1_0(Q_0)$ set of all continuously differentiable functions with compact support on $Q_0$.

**Proposition 3.1.**  
(i) The elements in $C^1_0(\mathbb{R}^n)$ are pointwise multipliers on $B(\mathbb{R}^n)$.

(ii) The elements in $C^1_0(Q_0)$ are pointwise multipliers on $B(Q_0)$.

**Proof.** First, let us prove (i). Fix $\phi \in C^1_0(\mathbb{R}^n)$. It suffices to show that, for any $f \in B(\mathbb{R}^n)$,

$$
\|\phi f\|_{B(\mathbb{R}^n)} \leq \|\phi\|_{L^\infty(\mathbb{R}^n)} + \|\nabla \phi\|_{L^\infty(\mathbb{R}^n)} \|f\|_{B(\mathbb{R}^n)}.
$$

Obviously, for any cube $Q$ with $|Q| = 1$,

$$
\frac{1}{|Q|} \int_Q |f(x)| \phi(x) \, dx \leq \|\phi\|_{L^\infty(\mathbb{R}^n)} \int_Q |f(x)| \, dx \leq \|\phi\|_{L^\infty(\mathbb{R}^n)} \|f\|_{B(\mathbb{R}^n)}.
$$

Next, let $k_0 \in \mathbb{N}$ and $\mathcal{F}_{2^{-k_0}} := \{Q_{2^{-k_0}}(a_j)\}_{j \in J_0}$ be a collection of mutually disjoint $2^{-k_0}$-cubes in $\mathbb{R}^n$ with $\#J_0 \leq 2^{k_0(n-1)}$. Then, for any $j \in J_0$,

$$
\frac{1}{|Q_{2^{-k_0}}(a_j)|} \int_{Q_{2^{-k_0}}(a_j)} |\phi - (f\phi)_{Q_{2^{-k_0}}(a_j)}| \\
\leq \frac{1}{|Q_{2^{-k_0}}(a_j)|} \int_{Q_{2^{-k_0}}(a_j)} |(f - f_{Q_{2^{-k_0}}(a_j)})\phi| + \int_{Q_{2^{-k_0}}(a_j)} |f_{Q_{2^{-k_0}}(a_j)}\phi - (f\phi)_{Q_{2^{-k_0}}(a_j)}| \\
\leq \|\phi\|_{L^\infty(\mathbb{R}^n)} \int_{Q_{2^{-k_0}}(a_j)} |f - f_{Q_{2^{-k_0}}(a_j)}| + \frac{\sqrt{n}}{2} 2^{-k_0}\|\nabla \phi\|_{L^\infty(\mathbb{R}^n)} \int_{Q_{2^{-k_0}}(a_j)} |f|.
$$

Obviously,

$$
2^{-k_0(n-1)} \sum_{j \in J_0} \frac{1}{|Q_{2^{-k_0}}(a_j)|} \int_{Q_{2^{-k_0}}(a_j)} |f - f_{Q_{2^{-k_0}}(a_j)}| \leq |f|_{2^{-k_0}} \leq \|f\|_{B(\mathbb{R}^n)}.
$$

Meanwhile, Proposition 2.3(i) gives us that

$$
2^{-k_0(n-1)} 2^{-k_0} \sum_{j \in J_0} \int_{Q_{2^{-k_0}}(a_j)} |f| \leq \|f\|_{B(\mathbb{R}^n)}.
$$

Thus,

$$
2^{-k_0(n-1)} \int_{Q_{2^{-k_0}}(a_j)} |f\phi - (f\phi)_{Q_{2^{-k_0}}(a_j)}| \leq \|\phi\|_{L^\infty(\mathbb{R}^n)} + \|\nabla \phi\|_{L^\infty(\mathbb{R}^n)} \|f\|_{B(\mathbb{R}^n)}.
$$
Taking supremum over all \( k_0 \in \mathbb{N} \) in both sides of the above inequality, we obtain
\[
\sup_{k \in \mathbb{N}}[f \phi]_{2^k} \leq [\|\phi\|_{L^\infty(\mathbb{R}^n)} + \|\nabla \phi\|_{L^\infty(\mathbb{R}^n)}] \|f\|_{B(\mathbb{R}^n)},
\]
which combined with Lemma 2.2(i) implies (3.1). This finishes the proof of (i).

To prove (ii), we fix \( \phi \in C_c^1(\mathcal{Q}_0) \). It is a trivial fact that
\[
\int_{\mathcal{Q}} |f(x)\phi(x)| \, dx \leq \|\phi\|_{L^\infty(\mathcal{Q})} \int_{\mathcal{Q}} |f(x)| \, dx, \quad \forall \, f \in B(\mathcal{Q}_0).
\]
Similarly to the proof of (i), we use Proposition 2.3(ii) to deduce that
\[
\sup_{k \in \mathbb{N}}[f \phi]_{2^k, \mathcal{Q}_0} \leq [\|\phi\|_{L^\infty(\mathbb{R}^n)} + \|\nabla \phi\|_{L^\infty(\mathbb{R}^n)}] \|f\|_{B(\mathcal{Q}_0)}.
\]
This, combined with Lemma 2.2(ii), implies that
\[
\|\phi\|_{B(\mathcal{Q}_0)} \leq [\|\phi\|_{L^\infty(\mathbb{R}^n)} + \|\nabla \phi\|_{L^\infty(\mathbb{R}^n)}] \|f\|_{B(\mathcal{Q}_0)}, \quad \forall \, f \in B(\mathcal{Q}_0),
\]
which completes the proof of (ii) and hence of Proposition 3.1. \( \square \)

**Lemma 3.2.** For any \( \lambda \in [1, \infty) \), there exists a positive constant \( C \), depending only on \( n \), such that
\[
\|f(\lambda \cdot)\|_{B(\mathbb{R}^n)} \leq C\|f\|_{B(\mathbb{R}^n)}, \quad \forall \, f \in B(\mathbb{R}^n).
\]

**Proof.** For any cube \( \mathcal{Q} \), write \( \mathcal{Q}^\lambda := \{\lambda x : x \in \mathcal{Q}\} \). Notice that \( \mathcal{Q}^\lambda \) is a cube with the same center as that of \( \mathcal{Q} \) but of side length \( \lambda \ell(\mathcal{Q}) \). Let \( L \geq 0 \) be the unique integer such that \( 2^{L-1} < \lambda \leq 2^L \).

Observe that, when \( |\mathcal{Q}| = 1 \), there exist \( 2^{Ln} \) cubes \( \{\mathcal{Q}_1, \ldots, \mathcal{Q}_{2^{Ln}}\} \) with side length 1 so that \( \mathcal{Q}^\lambda \subset \bigcup_{i=1}^{2^{Ln}} \mathcal{Q}_i \), and hence
\[
\int_{\mathcal{Q}} |f(\lambda x)| \, dx = \int_{Q^\lambda} |f(x)| \, dx \leq \frac{1}{\lambda^n} \sum_{i=1}^{2^{Ln}} \int_{\mathcal{Q}_i} |f(x)| \, dx \leq 2^n \sup_{|\mathcal{Q}|=1} \int_{\mathcal{Q}} |f(y)| \, dy,
\]
which implies that
\[
\sup_{|\mathcal{Q}|=1} \int_{\mathcal{Q}} |f(\lambda x)| \, dx \leq 2^n\|f\|_{B(\mathbb{R}^n)}.
\]
Next, let \( \epsilon \in (0, 1) \) and \( \mathcal{F}_\epsilon = \{Q_{\epsilon, (a_j)}\}_{j \in J} \) be a collection of mutually disjoint \( \epsilon \)-cubes in \( \mathbb{R}^n \) with \( \#J \leq \epsilon^{1-n} \).

Then
\[
\epsilon^{n-1} \sum_{j \in J} \int_{Q_{\epsilon, (a_j)}} |f(\lambda x) - (f(\lambda \cdot))_{Q_{\epsilon, (a_j)}}| \, dx \leq \epsilon^{n-1} \sum_{j \in J} \int_{Q_{\epsilon, (a_j, \lambda)}} |f(x) - f(y)| \, dx \, dy.
\]
It \( \epsilon \lambda \geq 1 \), similarly to the previous argument, we find that
\[
\epsilon^{n-1} \sum_{j \in J} \int_{Q_{\epsilon, (a_j)}} |f(\lambda x) - (f(\lambda \cdot))_{Q_{\epsilon, (a_j)}}| \, dx \leq 2 \epsilon^{n-1} \sum_{j \in J} \int_{Q_{\epsilon, (a_j, \lambda)}} |f(x)| \, dx
\]
\[
\leq 2 \epsilon^{n-1} \sum_{j \in J} 2^n \sup_{|\mathcal{Q}|=1} \int_{\mathcal{Q}} |f(y)| \, dy \leq 2^{n+1}\|f\|_{B(\mathbb{R}^n)}.
\]
If $\varepsilon \lambda < 1$, noticing that \{Q_{\varepsilon, j}(a_j, l_j)\}_{j \in J}$ are also mutually disjoint, we separate $J$ as the union of \{J_1, \ldots, J_{2^{(n-1)}}\} with each $\# J_i \leq (\varepsilon \lambda)^{1-n}$ for $i \in \{1, \ldots, 2^{(n-1)}\}$, and we then have

$$
eq \sum_{j \in J} \int_{Q_{\varepsilon, j}(a_j)} |f(\lambda x) - (f(\lambda x))_0| \, dx = \sum_{j \in J} \int_{Q_{\varepsilon, j}(a_j)} |f(x) - f_{Q_{\varepsilon, j}(a_j)}| \, dx$$

$$\leq \lambda^{1-n} \sum_{j \in J} |f(x)| \leq 2^{n-1} \|f\|_{B^o(R^n)}.$$

This finishes the proof of Lemma 3.2.

**Lemma 3.3.**

(i) If $T_g[B_c(Q_0)] \subset B(\mathbb{R}^n)$ and $g(0) = 0$, then there exists a cube $Q \subset \mathbb{R}^n$ and positive constants $C_1$ and $C_2$ such that $\|g \circ f\|_{B^o(R^n)} \leq C_2$ for any $f \in B_c(\mathbb{R}^n)$ with $\text{supp} \ f \subset Q$ and $\|f\|_{B^o(R^n)} \leq C_1$.

(ii) The conclusion in (i) is true for $B(Q_0)$; that is, if $T_g[B_c(Q_0)] \subset B(Q_0)$ and $g(0) = 0$, then there exists a cube $Q \subset Q_0$ and positive constants $C_1$ and $C_2$ such that $\|g \circ f\|_{B^o(Q_0)} \leq C_2$ for any $f \in B_c(\mathbb{R}^n)$ with $\text{supp} \ f \subset Q$ and $\|f\|_{B^o(Q_0)} \leq C_1$.

**Proof.** By similarity, we only prove (i). We argue by contradiction. Assume that the conclusion (i) of this lemma is false, that is, for any cube $Q \subset \mathbb{R}^n$ and any positive constants $C_1$ and $C_2$, there exists $f \in B_c(\mathbb{R}^n)$ with $\text{supp} \ f \subset Q$ and $\|f\|_{B^o(R^n)} \leq C_1$ such that $\|g \circ f\|_{B^o(R^n)} > C_2$.

Let \{Q_j\}_{j \in \mathbb{N}} be a sequence of mutually disjoint cubes contained in $\mathbb{R}^n$. Pick a sequence \{\phi_j\}_{j \in \mathbb{N}} \subset C_c^\infty(\mathbb{R}^n)$ so that, for any $j \in \mathbb{N}$, $\phi_j \equiv 1$ on $\frac{1}{4} Q_j$ and $\phi_j \equiv 0$ out of $Q_j$. For any $j \in \mathbb{N}$, by Proposition 3.1, there exists a positive number $\gamma_j$ such that

$$\|\phi_j h\|_{B^o(R^n)} \leq \gamma_j \|h\|_{B^o(R^n)}, \quad \forall \ h \in B(\mathbb{R}^n).$$

Fix $j \in \mathbb{N}$. If we take $C_1 = 2^{-j}$ and $C_2 = \gamma_j$, then there exists $f_j \in B_c(\mathbb{R}^n)$ with $\text{supp} \ f_j \subset Q_j$ and $\|f_j\|_{B^o(R^n)} \leq 2^{-j}$ such that $\|g \circ f_j\|_{B^o(R^n)} > \gamma_j$. Define $f := \sum_{j \in \mathbb{N}} f_j$, which converges in $B(\mathbb{R}^n)$. Indeed, $f \in B_c(\mathbb{R}^n)$ and $\|f\|_{B^o(R^n)} \leq \sum_{j \in \mathbb{N}} \|f_j\|_{B^o(R^n)} \leq 1$, which also implies that $f(x) = \sum_{j \in \mathbb{N}} f_j(x) \text{ holds true almost everywhere.}$ This further implies that

$$f(x) = \begin{cases} f_j(x) & \text{for almost every } x \in \frac{1}{4} Q_j; \\ 0 & \text{for almost every } x \in Q_j \setminus \left(\frac{1}{4} Q_j\right). \end{cases}$$

Further, from $g(0) = 0$, we deduce that $(g \circ f) \phi_j = g \circ f_j$ holds true almost everywhere. By the assumption $T_g[B_c(\mathbb{R}^n)] \subset B(\mathbb{R}^n)$, we know that $g \circ f \in B(\mathbb{R}^n)$. However, it follows from (3.2) that

$$j \gamma < \|g \circ f\|_{B^o(R^n)} = \|g \circ f\|_{B^o(R^n)} \leq \gamma_j \|g \circ f\|_{B^o(R^n)},$$

that is, $\|g \circ f\|_{B^o(R^n)} > j$ for any $j \in \mathbb{Z}$, which is a contradiction. This finishes the proof of (i) and hence of Lemma 3.3.

**Lemma 3.4.** For any integer $j \geq 3$, there exists a non-negative function $\theta \in C_c^\infty(\mathbb{R}^n)$ such that $\theta(x) = 1$ if $|x| \leq \frac{1}{2}$, $\theta(x) = 0$ if $|x| \geq \frac{1}{2}$, $0 \leq \theta \leq 1$ and $\|\theta\|_{B^o(R^n)} \leq C \log_2 j^{-1}$ for some positive constant $\tilde{C}$ independent of $j$ and $\theta$. 

\]
Proof. The proof is similar to that of [13, Lemma 8]. Indeed, we only need to replace the definition of \( \theta_j \) in [13, p. 535] by
\[
\theta_j(x) = \frac{u(\log_2(2|x|))}{\log_2 j}, \quad \forall x \in \mathbb{R}^n,
\]
where \( u \) is a smooth function on \( \mathbb{R} \) with \( 0 \leq u \leq 1, \ u \equiv 1 \) on \( (-\infty, -1] \) and \( u \equiv 0 \) on \( [0, \infty) \). The remainder of the proof is the same as that of [13, Lemma 8], which completes the proof of Lemma 3.4. \( \Box \)

We also need the following conclusion, which is inspired by [8] and [13, Lemma 2].

**Lemma 3.5.**
(i) Assume that there exist positive constants \( c_1, c_2 \) and \( c_3 \in [0, \infty) \) and a cube \( K \subset \mathbb{R}^n \) such that
\[
\sup_{e \in (0, c_2)} [g \circ f]_e = \sup_{e \in (0, c_2)} \left\{ \frac{\epsilon^{n-1}}{2} \sum_{j \in J} M(g \circ f, Q_e(a_j)) \right\} \leq c_3
\]
for any function \( f \in C_c^\infty(\mathbb{R}^n) \) with \( \text{supp } f \subset K \) and \( \|f\|_{L(\mathbb{R}^n)} \leq c_1 \), where the supremum is taken over a collection \( \mathcal{F}_e := \{Q_e(a_j)\}_{j \in J} \) of mutually disjoint \( \epsilon \)-cubes in \( \mathbb{R}^n \) with cardinality \( \#\mathcal{F}_e = \#J \leq 1/\epsilon^{n-1} \). Then there exists a positive constant \( m \), independent of \( g \) and \( f \), such that
\[
\sup_{e \in (0, c_2)} \{\|g(a) - g(b)\}: a, b \in \mathbb{C}, |a - b| \leq mc_1 \} \leq 4^{n+1}c_3.
\]
(ii) The corresponding conclusion of (i) for \( B(Q_0) \) is also true; namely, one can replace all \( \mathbb{R}^n \) in (i) by \( Q_0 \) and \( [g \circ f]_e \) by \( [g \circ f]_{Q_0} \), and (3.4) remains true.

**Proof.** First, let us prove (i). Noticing that the supremum in (3.3) and (3.4) are invariant after modulus of constants. Without loss of generality, we may assume that \( g(0) = 0 \) (otherwise we may use \( \tilde{g} := g - g(0) \) instead of \( g \)).

Observe that the norm \( \|f\|_{L(\mathbb{R}^n)} \) and the term in the left-hand side of (3.3) are translation invariant. This, together with Lemma 3.2, implies that we can assume that \( K = Q_0 \) via replacing \( c_1 \) and \( c_2 \) by \( \alpha_1c_1 \) and \( \alpha_2c_2 \) for some positive constants \( \alpha_1 \) and \( \alpha_2 \) depending only on \( K \). Let \( a, b \in \mathbb{C} \) satisfy
\[
|a - b| \leq \frac{\alpha_1c_1}{6}.
\]

With \( \tilde{C} \) as in Lemma 3.4, we pick an integer \( j \geq 3 \) so that
\[
2^{-j} < \alpha_2c_2 \quad \text{and} \quad \frac{1}{\log_2 j} < \frac{\alpha_1c_1}{2\tilde{C}(|a| + 1)}.
\]

We also assume that \( j \) is chosen large enough so that the ball \( B(\overline{0}, \frac{1}{j}) \) contains more than \( 2^{j(n-1)} \) disjoint \( 2^{-j} \)-cubes. Applying Lemma 3.4 with a translation, we know that there exists a function \( \theta \in C_c^\infty(\mathbb{R}^n) \) such that \( \text{supp } \theta \subset Q_0, \theta \equiv 1 \) on \( B((\frac{1}{2}, \ldots, \frac{1}{2}), \frac{1}{j}) \), and
\[
|a| \|\theta\|_{bmo(\mathbb{R}^n)} \leq \frac{\tilde{C}|a|}{\log_2 j} < \frac{\alpha_1c_1}{2}.
\]
Notice that $\text{bmo}(\mathbb{R}^n) \subset B(\mathbb{R}^n)$ with continuous embedding. We also have

$$\tag{3.6} |a||\theta| |B(\mathbb{R}^n) \leq |a||\theta| |\text{bmo}(\mathbb{R}^n) < \frac{\alpha_1 c_1}{2}. \quad (3.6)$$

By the choice of $j$ above, we know that the ball $B((\frac{1}{2}, \ldots, \frac{1}{2}), \frac{1}{2})$ contains more than $2^{j(n-1)}$ disjoint $2^{-j}$-cubes. We select $2^{j(n-1)}$ such cubes, denoted by $\{Q_1, \ldots, Q_{2^{j(n-1)}}\}$. Then $\theta \equiv 1$ on all such $Q_i$ with $i \in \{1, \ldots, 2^{j(n-1)}\}$. Denote by $k_i$ the lower-left-corner point of $\frac{1}{2}Q_i$. Then $\frac{1}{2}Q_i = k_i + (0, 2^{-j-2})^n$.

Choose $\phi \in C_c^\infty(\mathbb{R}^n)$ satisfying $\text{supp} \phi \subset Q_0$, $\phi \equiv 1$ on $(1/4, 1/2)^n$ and $0 \leq \phi \leq 1$. For any $x \in \mathbb{R}^n$, we define

$$\quad f(x) := (b-a) \sum_{i=1}^{2^{j(n-1)}} \phi(2^{j+1}(x-k_i)) + a\theta(x).$$

Then $f \in C_c^\infty(\mathbb{R}^n)$, $\text{supp} \ f \subset Q_0$ and for each $i \in \{1, \ldots, 2^{j(n-1)}\}$, $f \equiv b$ on $(\frac{1}{4}Q_i) \setminus (\frac{1}{8}Q_i)$ and $f \equiv a$ on $Q_i \setminus (\frac{1}{8}Q_i)$. Consequently, $g \circ f \equiv g(b)$ on $(\frac{1}{4}Q_i) \setminus (\frac{1}{8}Q_i)$ and $g \circ f \equiv g(a)$ on $Q_i \setminus (\frac{1}{8}Q_i)$. Moreover, since $\text{supp} \ (\phi(2^{j+1}(\cdot - k_i)) \subset Q_i$ and $\{(Q_i)_{i=1}^{2^{j(n-1)}}$ are mutually disjoint, from (3.6) and (3.5), it follows that

$$\quad \|f\|_{B(\mathbb{R}^n)} \leq |b-a| \left\| \sum_{i=1}^{2^{j(n-1)}} \phi(2^{j+1}(\cdot - k_i)) \right\|_{B(\mathbb{R}^n)} + |a||\theta|_{B(\mathbb{R}^n)}$$

$$\quad \leq 3|b-a| \left\| \sum_{i=1}^{2^{j(n-1)}} \phi(2^{j+1}(\cdot - k_i)) \right\|_{L^\infty(\mathbb{R}^n)} + |a||\theta|_{B(\mathbb{R}^n)} < 3|b-a| + \frac{\alpha_1 c_1}{2} < \alpha_1 c_1.$$

Further, by the above discussion, (3.3) and the fact $g(0) = 0$, we conclude that

$$\quad |g(b) - g(a)|$$

$$\leq 2^{-j(n-1)} \sum_{i=1}^{2^{j(n-1)}} \left| \int_{(\frac{1}{4}Q_i) \setminus (\frac{1}{8}Q_i)} g \circ f(x) \, dx - \int_{Q_i \setminus (\frac{1}{8}Q_i)} g \circ f(x) \, dx \right|$$

$$\leq 2^{-j(n-1)} \sum_{i=1}^{2^{j(n-1)}} \left| \int_{(\frac{1}{4}Q_i) \setminus (\frac{1}{8}Q_i)} |g \circ f(x) - (g \circ f)_{Q_i}| \, dx + \int_{Q_i \setminus (\frac{1}{8}Q_i)} |g \circ f(x) - (g \circ f)_{Q_i}| \, dx \right|$$

$$\leq 2^{-j(n-1)} \sum_{i=1}^{2^{j(n-1)}} \left[ \int_{(\frac{1}{4}Q_i) \setminus (\frac{1}{8}Q_i)} \frac{1}{4^{n-1} - 2n} + \frac{1}{8^{-n} - 1} \right] \int_{Q_i} |g \circ f(x) - (g \circ f)_{Q_i}| \, dx$$

$$\leq 4^{n+1} 2^{-j(n-1)} \sum_{i=1}^{2^{j(n-1)}} M(g \circ f, Q_i) \leq 4^{n+1} c_3.$$ 

This proves the desired conclusion of (i) with $m = 1/6$.

The proof of (ii) is similar to that of (i), so we omit the details here. This finishes the proof of Lemma 3.5. \hfill \Box
The following equivalent descriptions of Theorem 1.1(i) from [13, Proposition 1] is necessary for the proof of Theorem 1.1. A function \( g : \mathbb{C} \to \mathbb{C} \) is called Lipschitz continuous if
\[
\text{Lip}(g) := \sup_{x,y \in \mathbb{C}, x \neq y} \frac{|g(x) - g(y)|}{|x - y|} < \infty.
\]

**Lemma 3.6.** The following are equivalent:

(a) \( \sup_{x,y \in \mathbb{C}} (1 + |x - y|)^{-1} |g(x) - g(y)| < \infty; \)

(b) there exist positive constants \( \alpha \) and \( C \) such that \( |g(x) - g(y)| \leq C \) for any \( x, y \in \mathbb{C} \) satisfying \( |x - y| \leq \alpha; \)

(c) \( g \) is a sum of a bounded Borel measurable function and a Lipschitz continuous function.

We are now ready to prove Theorem 1.1.

**Proof of Theorem 1.1.** Obviously, we have (ii) \( \Rightarrow \) (iii) and (iv) \( \Rightarrow \) (v). Next we show (i) \( \Rightarrow \) (ii) and (i) \( \Rightarrow \) (iv). By Lemma 3.6, we can separately consider the case when \( g \) is bounded and the case when \( g \) is Lipschitz continuous.

If \( g \) is bounded, then \( g \circ f \) is bounded. Since \( L^\infty(\mathbb{R}^n) \hookrightarrow B(\mathbb{R}^n) \) and \( L^\infty(Q_0) \hookrightarrow B(Q_0) \), it easily follows that \( T_g(B(\mathbb{R}^n)) \subset B(\mathbb{R}^n) \) and \( T_g(B(Q_0)) \subset B(Q_0) \).

If \( g \) is Lipschitz continuous, then, for any cube \( Q \), we have
\[
\int_Q |g \circ f(x) - (g \circ f)(0)| \, dx \leq \int_Q \int_Q |g \circ f(x) - g \circ f(y)| \, dy \, dx \leq 2 \text{Lip}(g) \int_Q |f(x) - f(0)| \, dx
\]
and
\[
\int_Q |g \circ f(x)| \, dx \leq \int_Q |g \circ f(x) - g(0)| \, dx + |g(0)| \leq \text{Lip}(g) \int_Q |f(x)| \, dx + |g(0)|,
\]
which imply that
\[
\|g \circ f\|_{B(\mathbb{R}^n)} \leq 2 \text{Lip}(g)\|f\|_{B(\mathbb{R}^n)} + |g(0)|
\]
and
\[
\|g \circ f\|_{B(Q_0)} \leq 2 \text{Lip}(g)\|f\|_{B(Q_0)} + |g(0)|.
\]
Thus, \( T_g(B(\mathbb{R}^n)) \subset B(\mathbb{R}^n) \) and \( T_g(B(Q_0)) \subset B(Q_0) \). These prove (i) \( \Rightarrow \) (ii) and (i) \( \Rightarrow \) (iv).

Finally, assume that (iii) or (v) holds true. Via a subtracting \( g(0) \) if necessary, we may also assume that \( g(0) = 0 \). Then, by Lemmas 3.3 and 3.5, we conclude that \( g \) satisfies Lemma 3.6(b), and hence (i) holds true. This proves (iii) \( \Rightarrow \) (i) and (v) \( \Rightarrow \) (i), and then finishes the proof of Theorem 1.1. \( \square \)

As an immediate consequence of Theorem 1.1, we have the following result.

**Corollary 3.7.** The following statements are equivalent:

(i) \( \sup_{x,y \in \mathbb{C}} (1 + |x - y|)^{-1} |g(x) - g(y)| < \infty; \)

(ii) \( T_g(B_0(\mathbb{R}^n)) \subset B(\mathbb{R}^n); \)

(iii) \( T_g(B(0,Q_0)) \subset B(Q_0). \)
4 Proofs of Theorems 1.2 and 1.3

One key tool to prove Theorem 1.2 is the following two lemmas, inspired by [13, Lemma 3].

Lemma 4.1. If \( T_\varphi(B_c(\mathbb{R}^n)) \subset B_0(\mathbb{R}^n) \) and \( g(0) = 0 \), then for any \( \epsilon \in (0, 1) \), there exists a cube \( P \subset Q_0 \) and two positive constants \( c_1 \) and \( c_2 \) such that

\[
\sup_{\delta \in (0, c_2]} \sup_{Q \in F_\delta} \delta^{n-1} \sum_{Q \in F_\delta} \int_Q |g \circ f - (g \circ f)_Q| \leq \epsilon
\]

for any \( f \in C_c^\infty(\mathbb{R}^n) \) with \( \text{supp} \, f \subset P \) and \( \| f \|_{L^\infty(\mathbb{R}^n)} \leq c_1 \), where the second supremum is taken over all collections \( F_\delta \) of disjoint \( \delta \)-cubes with \( \#F_\delta \leq \delta^{1-n} \).

Proof. We argue by contradiction. Assume that there exists a positive \( \epsilon_0 \) such that, for any cube \( P \subset Q_0 \) and any pair \((c_1, c_2)\) of positive numbers, there exist a function \( f \in C_c^\infty(\mathbb{R}^n) \) supported in the cube \( P \) and satisfying \( \| f \|_{L^\infty(\mathbb{R}^n)} \leq c_1 \), and a collection \( F_\delta \) of disjoint \( \delta \)-cubes with \( \delta \leq c_2 \) and \( \#F_\delta \leq \delta^{1-n} \) such that

\[
\delta^{n-1} \sum_{Q \in F_\delta} \int_Q |g \circ f - (g \circ f)_Q| \geq \epsilon_0.
\]

For any integer \( j \geq 9 \), consider the cube

\[
P_j := (0, 2^{-1}(1 + j)^{-2})^n + \frac{1}{j}(1, \ldots, 1).
\]

Then \( P_j \subset (2^{-j}, 1 - 2^{-j})^n \subset Q_0 \) for all \( j \geq 9 \). Moreover, \( P_j \cap P_i = \emptyset \) whenever \( i \neq j \) and \( i, j \geq 9 \). Pick \( \phi \in C_c^\infty(\mathbb{R}^n) \) with \( \text{supp} \, \phi \subset Q_0 \), \( 0 \leq \phi \leq 1 \) and \( \phi \equiv 1 \) on \( \frac{1}{9}Q_0 \). Define \( \phi_j(x) := \phi(2(j+1)^2(x - cp_j)) \) for any \( j \geq 9 \) and \( x \in \mathbb{R}^n \), where \( cp_j := \frac{1}{j}(1, \ldots, 1) \) is the center of the cube \( P_j \). Then \supp \( \phi_j \subset P_j \), \( \supp \, \phi_j \equiv 1 \) on \( \frac{1}{9}P_j \) and

\[
\|\nabla \phi_j\|_{L^\infty(\mathbb{R}^n)} = 2(j+1)^2\|\nabla \phi\|_{L^\infty(\mathbb{R}^n)}.
\]

By the above contradiction assumption, for each \( j \geq 2 \), there exist \( f_j \in C_c^\infty(\mathbb{R}^n) \) supported in the cube \( \frac{1}{9}P_j \) and satisfying \( \| f_j \|_{L^\infty(\mathbb{R}^n)} \leq 2^{-j} \), as well as a collection \( F_{\delta_j} := \{Q_{j,\ell}\}_\ell \) of disjoint \( \delta_j \)-cubes with \( \delta_j \leq 2^{-j} \) and \( \#F_{\delta_j} \leq \delta_j^{1-n} \), such that

\[
\delta_j^{n-1} \sum_{Q_{j,\ell} \in F_{\delta_j}} \int_{Q_{j,\ell}} |g \circ f_j - (g \circ f_j)_{Q_{j,\ell}}| \geq \epsilon_0.
\]

Since \( g(0) = 0 \) and \( \supp \, f_j \subset \frac{1}{9}P_j \), we may assume that \( Q_{j,\ell} \cap P_j \neq \emptyset \) for any \( Q_{j,\ell} \in F_{\delta_j} \). Such an assumption implies that those \( Q_{j,\ell} \) are close to \( P_j \). Meanwhile, notice that the side length of each \( Q_{j,\ell} \) is far less than that of \( P_j \). Consequently, we find that each \( Q_{j,\ell} \subset Q_0 \) and that \( Q_{j,\ell} \cap Q_{k} = \emptyset \) for any \( i \) and \( k \) whenever \( j \neq \ell \) and \( j, \ell \geq 9 \).

Define \( f := \sum_{j=9}^{\infty} f_j \). Then \( f \in C_c^\infty(\mathbb{R}^n) \subset B_c(\mathbb{R}^n) \), and hence \( g \circ f \in B_0(\mathbb{R}^n) \). For any \( j \geq 9 \), by \( g(0) = 0 \), \( \supp \, f_j \subset \frac{1}{9}P_j \), \( \phi_j \equiv 1 \) on \( \frac{1}{9}P_j \) and \( f(x) = f_j(x) \) for almost every \( x \in P_j \), we have

\[
(g \circ f)\phi_j = g \circ f_j \quad \text{for almost every } x \in \mathbb{R}^n.
\]
Thus,

\begin{equation}
(4.1) \quad \epsilon_0 \leq \delta_j^{n-1} \sum_{Q_j \in \mathcal{F}_n} \int_{Q_j} |g \circ f_j - (g \circ f_j)_{Q_j}| \\
= \delta_j^{n-1} \sum_{Q_j \in \mathcal{F}_n} \int_{Q_j} |(g \circ f) \phi_j - ((g \circ f) \phi_j)_{Q_j}| \\
\leq \delta_j^{n-1} \sum_{Q_j \in \mathcal{F}_n} \left[ \|\phi_j\|_{L^\infty(\mathbb{R}^n)} \int_{Q_j} |g \circ f - (g \circ f)_{Q_j}| + \frac{\sqrt{n}}{2} \delta_j \|\nabla \phi_j\|_{L^\infty(\mathbb{R}^n)} \int_{Q_j} |g \circ f| \right] \\
\leq \delta_j^{n-1} \sum_{Q_j \in \mathcal{F}_n} \left[ \|\phi_j\|_{L^\infty(\mathbb{R}^n)} \int_{Q_j} |g \circ f - (g \circ f)_{Q_j}| + \sqrt{n} \delta_j (j+1)^2 \|\nabla \phi\|_{L^\infty(\mathbb{R}^n)} \int_{Q_j} |g \circ f| \right] \\
\leq \delta_j^{n-1} \sum_{Q_j \in \mathcal{F}_n} \int_{Q_j} |g \circ f - (g \circ f)_{Q_j}| + \sqrt{n} \|\nabla \phi\|_{L^\infty(\mathbb{R}^n)} \|g \circ f\|_{L^\infty(Q_0)} 2^{-j}(j+1)^2.
\end{equation}

Notice that $T_g(\mathcal{B}_c(\mathbb{R}^n)) \subset \mathcal{B}_0(\mathbb{R}^n)$ implies that $T_g(\mathcal{B}_c(\mathbb{R}^n)) \subset \mathcal{B}(\mathbb{R}^n)$. Thus, by Theorem 1.1 and Lemma 3.6, $g$ can be written as the sum of a bounded Borel measurable function and a Lipschitz continuous function, both take a bounded set in $\mathbb{C}$ to a bounded set. From this observation and the fact that $f \in C_c^\infty(Q_0)$, we deduce that $\|g \circ f\|_{L^\infty(Q_0)}$ is finite. Then, by taking $j$ large enough in (4.1), we conclude that

$$\frac{\epsilon_0}{2} \leq \delta_j^{n-1} \sum_{Q_j \in \mathcal{F}_n} \int_{Q_j} |g \circ f - (g \circ f)_{Q_j}|.$$

This is a contradiction to the fact $g \circ f \in \mathcal{B}_0(\mathbb{R}^n)$, as desired. This finishes the proof of Lemma 4.1. \hfill \Box

An argument similar to that used in the proof of Lemma 4.1 gives its following counterpart, which is also used in the proof of Theorem 1.2; we omit the details.

**Lemma 4.2.** If $T_g(\mathcal{B}_c(Q_0)) \subset \mathcal{B}_0(Q_0)$ and $g(0) = 0$, then, for any $\epsilon \in (0, 1)$, there exist a cube $P \subset Q_0$ and two positive constants $c_1$ and $c_2$ such that

$$\sup_{\delta \in (0, c_2]} \sup_{\mathcal{F}_\delta} \delta_j^{n-1} \sum_{Q_j \in \mathcal{F}_\delta} \int_{Q_j} |g \circ f - (g \circ f)_{Q_j}| \leq \epsilon$$

for any $f \in C_c^\infty(Q_0)$ with $\text{supp} \ f \subset P$, $\|f\|_{L^\infty(Q_0)} \leq c_1$, where the second supremum is taken over all collections $\mathcal{F}_\delta$ of disjoint $\delta$-cubes in $Q_0$ with $\# \mathcal{F}_\delta \leq \delta^{1-n}$.

To prove Theorems 1.2 and 1.3, we also need the following well-known fact on the relation between uniformly continuous functions and modulus of continuity (see [26, Chapter 2, Section 6]). Recall that a function $w : [0, \infty) \to [0, \infty)$ is called a *modulus of continuity* of a function $g$ provided that

$$|g(x) - g(y)| \leq w(|x - y|), \quad \forall \ x, y \in \mathbb{C} \quad \text{and} \quad \lim_{t \to 0} w(t) = 0.$$
Lemma 4.3. If a function $g$ is uniformly continuous, then it has concave increasing modulus of continuity.

Proof of Theorem 1.2. Observe that (ii) $\implies$ (iii) and (iv) $\implies$ (v) are trivial.

Next we show (i) $\implies$ (ii). Let $g$ be a uniformly continuous function on $\mathbb{C}$, and $w$ its related concave increasing modulus of continuity, whose existence is due to Lemma 4.3. For any $f \in B(\mathbb{R}^n)$, we have

$$
\sup_{|Q|=1} \int_Q |g \circ f| \leq \sup_{|Q|=1} \int_Q w(|f(x)|) \, dx + |g(0)| \leq w\left(\sup_{|Q|=1} \int_Q |f|\right) + |g(0)| < \infty.
$$

For any $f \in B(\mathbb{R}^n)$ and any collection of $(\epsilon)$-cubes $Q$ in $\mathbb{R}^n$ with $\#F_\epsilon \leq \epsilon^{-n}$, by the Jensen inequality, we find that

$$
e^n-1 \sum_{Q \in F_\epsilon} \int_Q |g \circ f - (g \circ f)_Q| \leq e^n-1 \sum_{Q \in F_\epsilon} \int_Q \int_Q |g \circ f(x) - g \circ f(y)| \, dx \, dy
$$

$$
\leq e^n-1 \sum_{Q \in F_\epsilon} \int_Q \int_Q w(|f(x) - f(y)|) \, dx \, dy
$$

$$
\leq w\left(e^n-1 \sum_{Q \in F_\epsilon} \int_Q \int_Q |f(x) - f(y)| \, dx \, dy\right)
$$

$$
\leq w\left(2e^n-1 \sum_{Q \in F_\epsilon} \int_Q |f - f_Q|\right).
$$

From this, it follows that, when $f \in B_0(\mathbb{R}^n)$,

$$
\lim_{\delta \to 0} \sup_{x \in (0,\delta)} |g \circ f| \leq \lim_{\delta \to 0} w\left(2 \sup_{x \in (0,\delta)} |f|\right) = 0.
$$

This proves that $T_g(B_0(\mathbb{R}^n)) \subset B_0(\mathbb{R}^n)$ and hence (i) $\implies$ (ii). The proof of (i) $\implies$ (iv) is similar, and we omit its details.

Finally, we consider (iii) $\implies$ (i) and (v) $\implies$ (i). Without loss of generality, we may assume $g(0) = 0$, by possibly subtracting $g(0)$. If $T_g(B_c(\mathbb{R}^n)) \subset B_0(\mathbb{R}^n)$ (resp. $T_g(B_c(Q_0)) \subset B_0(Q_0)$), then the uniformly continuity of $g$ in (i) follows from Lemmas 3.5 and 4.1 (resp. 4.2). This finishes the proof of Theorem 1.2.

To show Theorem 1.3, we need Theorem 1.2 and the following result on the continuity of $T_g$.

Proposition 4.4. If $g$ is uniformly continuous, then $T_g$ is continuous at $f \in B_0(\mathbb{R}^n)$ (resp. $B_0(Q_0)$) as a map from $B(\mathbb{R}^n)$ (resp. $B(Q_0)$) to itself.

The proof of Proposition 4.4 replies on the following conclusion from [13, Lemma 4].

Lemma 4.5. Assume that $g$ has a concave increasing modulus of continuity $w$ satisfying $w(t) \to 0$ as $t \to 0$. Then, for any locally integrable functions $f$ and $h$, and any cube $Q$,

$$
\int_Q |g \circ (f + h) - g \circ f - (g \circ f + h) - g \circ f_Q|
$$

is uniformly continuous.
\[ \leq \min \left\{ 2w \left( 2 \int_Q |f - f_Q| \right) + w \left( 2 \int_Q |h - h_Q| \right) , \ 2w \left( \int_Q |h| \right) \right\}. \]

By Lemma 4.5, the proof of Proposition 4.4 is similar to that of [13, Proposition 2], and we give some details here for completeness.

**Proof of Proposition 4.4.** Due to similarity, we only consider the case when \( f \in B_0(\mathbb{R}^n) \). Let \( \delta \in (0, 1) \) and \( w \) be a related concave increasing modulus of continuity of \( g \). Define

\[ M_\delta := \sup_{\epsilon \in (0, \delta)} |f|_\epsilon. \]

Then, for any \( \epsilon > 0 \), there exists a \( \delta \in (0, 1/2) \) such that \( w(2M_\delta) < \epsilon \), due to \( f \in B_0(\mathbb{R}^n) \) and \( \lim_{t \to 0} w(t) = 0 \). By \( \lim_{t \to 0} w(t) = 0 \) again, we can also take \( \eta > 0 \) such that \( w(\eta/\delta^n) < \epsilon \).

Assume now \( h \in B(\mathbb{R}^n) \) satisfying that \( ||h||_{B(\mathbb{R}^n)} \leq \eta \). Then, for any collection \( F_\epsilon \) of disjoint \( \epsilon \)-cubes, by Lemma 4.5 and the Jensen inequality, we find that, when \( \epsilon \in (0, \delta) \),

\[ I_\epsilon := e^{n-1} \sum_{Q \in F_\epsilon} \int_Q |g \circ (f + h) - g \circ f - (g \circ (f + h) - g \circ f)_{Q}| \]

\[ \leq 2w \left( 2e^{n-1} \sum_{Q \in F_\epsilon} \int_Q |f - f_Q| \right) + w \left( 2e^{n-1} \sum_{Q \in F_\epsilon} \int_Q |h - h_Q| \right) \]

\[ \leq 2w(2M_\delta) + w(2||h||_{B(\mathbb{R}^n)}) < 2\epsilon + w(2\eta) < 2\epsilon + w(\eta/\delta^n) < 3\epsilon, \]

while when \( \epsilon \in (\delta, 1) \),

\[ I_\epsilon \leq 2e^{n-1} \sum_{Q \in F_\epsilon} w \left( \int_Q |h| \right) \leq 2e^{n-1} \sum_{Q \in F_\epsilon} w(\delta^n||h||_{B(\mathbb{R}^n)}) < 2\epsilon. \]

Furthermore, for any cube \( Q \) with \( |Q| = 1 \), by Lemma 4.3 and the Jensen inequality, we have

\[ \int_Q |g \circ (f + h) - g \circ f| \leq \int_Q w(|h|) \leq w \left( \int_Q |h| \right) \leq w(||h||_{B(\mathbb{R}^n)}) \leq w(\eta) < \epsilon. \]

Altogether, we conclude that \( ||T_g(f + h) - T_g f||_{B(\mathbb{R}^n)} \to 0 \) as \( ||h||_{B(\mathbb{R}^n)} \to 0 \), as desired. This finishes the proof of Proposition 4.4. \( \Box \)

Now we use Proposition 4.4 to show Theorem 1.3.

**Proof of Theorem 1.3.** Let us first prove (a). If \( T_g(B_c(\mathbb{R}^n)) \subset B_c(\mathbb{R}^n) \), then \( T_g(B_c(\mathbb{R}^n)) \subset B_0(\mathbb{R}^n) \) and hence \( g \) is uniformly continuous, due to Theorem 1.2. On the other hand, \( T_g(B_c(\mathbb{R}^n)) \subset B_c(\mathbb{R}^n) \) also implies that \( g(0) = T_g(0) \in B_c(\mathbb{R}^n) \). Notice that a constant function belonging to \( B_c(\mathbb{R}^n) \) must be zero. Thus, we have \( g(0) = 0 \).

Conversely, we assume that \( g \) is uniformly continuous and \( g(0) = 0 \). By Theorem 1.2 and Proposition 4.4, we know that \( T_g \) is continuous from \( B_c(\mathbb{R}^n) \) to \( B_0(\mathbb{R}^n) \). Moreover, when \( f \in C_0^\infty(\mathbb{R}^n) \), the condition \( g(0) = 0 \) ensures that \( g \circ f \) is a continuous function with compact support, and hence it is a uniform limit of a sequence of functions in \( C_0^\infty(\mathbb{R}^n) \). This implies that \( g \circ f \in \)
\( B_c(\mathbb{R}^n) \) whenever \( f \in C_c^\infty(\mathbb{R}^n) \). Combining these observations, we conclude that \( T_g(B_c(\mathbb{R}^n)) \subset B_c(\mathbb{R}^n) \).

The proof of (b) is almost the same as that of (a); the only difference is that we need to show that any constant function \( C \) belongs to \( B(\mathbb{R}) \). Assume that \( T_g(\mathbb{R}) \subset B(\mathbb{R}) \). It suffices to prove that, for any \( \varepsilon \in (0, 1) \), there exists a \( \phi \in C_c^\infty(\mathbb{R}) \) such that \( \| C - \phi \|_{L^1(\mathbb{R})} < \varepsilon \). To see this, without loss of generality, we may assume that \( C > 0 \). Pick \( \delta > 0 \) such that \( 1 - (1 - 2\delta)^n < \varepsilon \delta \). Then we choose a smooth function \( \phi \) such that \( \text{supp} \phi \subset (1 - \delta)Q_0 \), \( 0 \leq \phi \leq C \), \( \phi \equiv C \) on \( (1 - 2\delta)Q_0 \), and it is easy to see that

\[
\| C - \phi \|_{L^1(Q_0)} = \| C - \phi \|_{L^1((1-\delta)Q_0)} \leq 2C[1 - (1 - 2\delta)^n] < \varepsilon.
\]

This finishes the proof of Theorem 1.3.

\( \Box \)

## 5 Proof of Theorem 1.5

To prove Theorem 1.5, we begin with the following proposition.

**Proposition 5.1.** Assume that \( T_g \) is continuous at the constant function zero as a map from the space \( (C_c^\infty(\mathbb{R}^n), \| \cdot \|_{B(\mathbb{R}^n)}) \) to \( B(\mathbb{R}^n) \), namely, for any \( h \in C_c^\infty(\mathbb{R}^n) \),

\[
\| T_g(h) - T_g(0) \|_{B(\mathbb{R}^n)} \to 0 \quad \text{as} \quad \| h \|_{B(\mathbb{R}^n)} \to 0.
\]

Then \( g \) is uniformly continuous.

**Proof.** Notice that constant functions belong to \( B(\mathbb{R}^n) \) and \( T_g(0) = g(0) \). Thus, without loss of generality, we may assume that \( g(0) = 0 \). Then the condition of this proposition implies that, for any \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that \( \| T_g(h) \|_{B(\mathbb{R}^n)} < \varepsilon \) for any \( h \in C_c^\infty(\mathbb{R}^n) \) with \( \| h \|_{B(\mathbb{R}^n)} < \delta \). The uniformly continuity of \( g \) is then an immediate consequence of Lemma 3.5, which completes the proof of Proposition 5.1.

**Proof of Theorem 1.5.** Assume that \( T_g \) is continuous from \( B(\mathbb{R}^n) \) to \( B(\mathbb{R}^n) \). Since constant functions belong to \( B(\mathbb{R}^n) \), without loss of generality, we may assume that \( g(0) = 0 \). By the above proposition, we know that \( g \) is uniformly continuous.

Next we show that \( g \) is \( \mathbb{R} \)-affine. To this end, for any \( k \in \mathbb{Z}^n \), we consider the cube \( Q_{0,k} := [0, 1)^n + k \) in \( \mathbb{R}^n \) and denote by \( c_{0,k} \) the center of \( Q_{0,k} \). Let \( \overline{Q}_{0,k} \) be the sub-dyadic cube of \( Q_{0,k} \) with side length \( \frac{1}{2} \), which is located in the “lower left corner” of \( Q_{0,k} \), that is, \( \overline{Q}_{0,k} = \frac{1}{2}Q_{0,k} - (\frac{1}{4}, \ldots, \frac{1}{4}) \).

We let \( \eta \) be the characteristic function of the set \( \bigcup_{k \in \mathbb{Z}^n} \overline{Q}_{0,k} \).

For any \( j \geq 3 \), by Lemma 3.4, we select a non-negative function \( \theta_j \in C_c^\infty(\mathbb{R}^n) \) such that \( \theta_j \equiv 1 \) on \( [-\frac{1}{j}, \frac{1}{j}]^n \), \( \theta_j \equiv 0 \) outside \( [-\frac{1}{2}, \frac{1}{2}]^n \), \( 0 \leq \theta \leq 1 \) and \( \| \theta_j \|_{\text{BMO}(\mathbb{R}^n)} \leq [\log_2 j]^{-1} \). Now we fix a large number \( M \in \mathbb{N} \), for example, \( M \geq 10^{10} \). Let \( k_1 \) be the origin of \( \mathbb{R}^n \) and define \( k_i := k_{i-1} + M(1, \ldots, 1) \) for any \( i \in \{2, \ldots, 2^{j(n-1)}\} \). Define

\[
\varphi_j(x) := \sum_{i=1}^{2^{j(n-1)}} \theta_j(x - k_i), \quad \forall x \in \mathbb{R}^n.
\]

Obviously, \( \varphi_j \in C_c^\infty(\mathbb{R}^n) \), \( \varphi_j \equiv 1 \) on \( k_i + [-\frac{1}{2}, \frac{1}{2}]^n \) for each \( i \in \{1, \ldots, 2^{j(n-1)}\} \) and \( \varphi_j \) vanishes outside \( \bigcup_{i=1}^{2^{j(n-1)}} (k_i + [-\frac{1}{2}, \frac{1}{2}]^n) \).
For any cube $Q$ satisfying $|Q| \leq 1$ and $Q \cap \text{supp} \varphi_j \neq \emptyset$, there exists a unique $i \in \{1, \ldots, 2^{j(n-1)}\}$ such that

$$Q \cap \left(k_i + \left[\frac{1}{2}, \frac{1}{2}\right]\right) \neq \emptyset,$$

and hence

$$\int_Q \varphi_j(x) \, dx = \int_Q \theta_j(x - k_i) \, dx = \int_{k_i + Q} \theta_j(x) \, dx.$$

This implies that

$$\|\varphi\|_{B(\mathbb{R}^n)} \leq \sup_{|Q|=1} \int_Q \varphi_j(x) \, dx + \sup_{|Q|\leq1} \int_Q |\varphi_j(x) - \varphi_j| \, dx \leq \|\theta_j\|_{\text{BMO}(\mathbb{R}^n)} \to 0 \quad \text{as} \quad j \to \infty.$$

For $i \in \{1, \ldots, 2^{j(n-1)}\}$, let $R_i := k_i + (-2^{-j-1}, 2^{-j-1})$. By the definition of $\eta$, it is easy to see that, for any two complex numbers $\alpha$ and $\beta$,

$$\int_{R_i} |g \circ (\beta \eta + \alpha \varphi_j)(x) - g \circ (\beta \eta)(x) - (g \circ (\beta \eta + \alpha \varphi_j) - g \circ (\beta \eta))_{R_i}| \, dx$$

$$\geq \frac{1}{|R_i|} \int_{k_i + (-2,2)^n} |g \circ (\beta \eta + \alpha \varphi_j)(x) - g \circ (\beta \eta)(x) - 2^{-n}[g(\beta + \alpha) - g(\beta)] - (1 - 2^{-n})g(\alpha)| \, dx$$

$$= 2^{-n}[g(\beta + \alpha) - g(\beta)] - 2^{-n}[g(\beta + \alpha) - g(\beta)] - (1 - 2^{-n})g(\alpha)$$

Consequently,

$$\|g \circ (\beta \eta + \alpha \varphi_j) - g \circ (\beta \eta)\|_{B(\mathbb{R}^n)}$$

$$\geq 2^{-j(n-1)} \sum_{j=1}^{2^{j(n-1)}} \int_{R_i} |g \circ (\beta \eta + \alpha \varphi_j)(x) - g \circ (\beta \eta)(x) - (g \circ (\beta \eta + \alpha \varphi_j) - g \circ (\beta \eta))_{R_i}| \, dx$$

$$\geq 2^{-n}(1 - 2^{-n})|g(\beta + \alpha) - g(\beta) - g(\alpha)|.$$

Letting $j \to \infty$, using the continuity of $T_g$ from $B(\mathbb{R}^n)$ to $B(\mathbb{R}^n)$, we conclude that $g(\beta + \alpha) = g(\beta) + g(\alpha)$ for any two complex numbers $\alpha$ and $\beta$. From this and the continuity of $g$, together with a standard argument, we deduce that $g$ is $\mathbb{R}$-affine. This finishes the proof of Theorem 1.5. \qed

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