On the support designs of extremal binary doubly even
self-dual codes

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Abstract. Let $D$ be the support design of the minimum weight of an extremal binary doubly even self-dual $[24m, 12m, 4m + 4]$ code. In this note, we consider the case when $D$ becomes a $t$-design with $t \geq 6$.

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1 Introduction

Let $C$ be an extremal binary doubly even self-dual $[24m, 12m, 4m + 4]$ code. Known examples of $C$ are the extended Golay code $G_{24}$ and the extended quadratic residue code of length 48. It was shown by Zhang [10] that $C$ does not exist if $m \geq 154$. The support of a codeword $c = (c_1, \ldots, c_{24m}) \in C$, $c_i \in \mathbb{F}_2$ is the set of indices of its nonzero coordinates: supp$(c) = \{i : c_i \neq 0\}$. The support design of $C$ for a given nonzero weight $w$ is the design for which the points are the $24m$ coordinate indices, and the blocks are the supports of all codewords of weight $w$. Let $D_w$ be the support design of $C$ for any $w \equiv 0 \pmod{4}$ with $4m + 4 \leq w \leq 24m − (4m + 4)$. Then it is known from the Assmus-Mattson theorem [1] that $D_w$ becomes a 5-design.

One of the most interesting questions around the Assmus-Mattson theorem is the following: “Can any $D_w$ become a 6-design?” Note that no 6-design has yet been obtained by applying the Assmus-Mattson theorem.

In this note, we consider the support design $D(= D_{4m+4})$ of the minimum weight of $C$. By the Assmus-Mattson theorem, $D$ is a 5-$(24m, 4m+4, \binom{5m-2}{m-1})$ design. Suppose that $D$ is a $t$-$(24m, 4m+4, \lambda_t)$ design with $t \geq 6$. It is easily seen that $\lambda_t = \binom{5m-2}{m-1} \binom{4m-5}{t-5} / \binom{24m-5}{t-5}$ is a nonnegative integer. It is known that if $D$ is a 6-design, then it is a 7-design by a strengthening of the Assmus-Mattson theorem [4]. In [2, Theorem 5], Bannai et al. showed that $D$ is not a 6-design if $m$ is $(\leq 153)$ not in the set $\{8, 15, 19, 35, 40, 41, 42, 50, 51, 52, 55, 57, 59, 60, 63, 65, 74, 75, 76, 80, 86, 90, 93, 100, 101, 104, 105, 107, 118, 125, 127, 129, 130, 135, 143, 144, 150, 151\}$. It was also shown in [2] that if $D$ is an 8-design, then $m$ must be in the set $\{8, 42, 63, 75, 130\}$, and $D$ is never a 9-design.

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In this note, we extend a method used in [2]. First, we will prepare it in Section 2. In Section 3, we consider a self-orthogonal 7-design which has parameters equal to those of $D$. Then we will show that there is no self-orthogonal 7-design for some $m$. For the remainder $m$, we consider the support 7- and 8-design of $C$ in Section 4. In summary, our main result is the following theorem.

**Theorem 1.1.** Let $D$ be the support $t$-design of the minimum weight of an extremal binary doubly even self-dual $[24m, 12m, 4m + 4]$ code ($m \leq 153$). If $t \geq 6$, then $D$ is a 7-design and $m$ must be in the set \{15, 52, 55, 57, 59, 60, 63, 90, 93, 104, 105, 107, 118, 125, 127, 135, 143, 151\}, and $D$ is never an 8-design.

# 2 Preparation

A $t$-$(v, k, \lambda)$ design is a pair $D = (X, \mathcal{B})$, where $X$ is a set of points of cardinality $v$, and $\mathcal{B}$ a collection of $k$-element subsets of $X$ called blocks, with the property that any $t$ points are contained in precisely $\lambda$ blocks. It follows that every $i$-subset of points ($i \leq t$) is contained in exactly $\lambda_i = \binom{v-i}{\lambda} / \binom{k-i}{\lambda}$ blocks.

Let $C$ be an extremal binary doubly even self-dual $[24m, 12m, 4m + 4]$ code. Let $D$ be the support design of the minimum weight of $C$. Then $D$ is a 5-$(24m, 4m + 4, \binom{5m-2}{m-1})$ design. Suppose that $D$ is a $t$-$(24m, 4m+4, \lambda_t)$ design with $t \geq 6$, where $\lambda_t = \binom{5m-2}{m-1} \binom{4m-1}{t-5} / \binom{24m-5}{t-5}$. It is known, by a strengthening of the Assmus-Mattson theorem [3], that if $D$ is a 6-design, then it is also a 7-design. Hence $\lambda_6$ and $\lambda_7$ are nonnegative integers. Then, by computations, we have the following lemma:

**Lemma 2.1.** For $1 \leq m \leq 153$, the values $\lambda_6$ and $\lambda_7$ are both nonnegative integers only if $m \in M = \{5, 8, 15, 19, 35, 40, 41, 42, 50, 51, 52, 55, 57, 59, 60, 63, 65, 74, 75, 76, 80, 86, 90, 93, 100, 101, 104, 105, 107, 118, 125, 127, 129, 130, 135, 143, 144, 150, 151\}.

The **Stirling numbers of the second kind** $S(n, k)$ (see [9]) are the number of ways to partition a set of $n$ elements into $k$ nonempty subsets. The Stirling numbers of the second kind can be computed from the sum

$$S(n, k) = \frac{1}{k!} \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^n$$

or the generating function

$$x^n = \sum_{m=0}^{n} S(n, m)(x)_m$$

where $(x)_m = x(x-1) \cdots (x-m+1)$. Special cases include $S(n, 0) = \delta_{n,0}$ and $S(0, k) = \delta_{0,k}$, where $\delta_{n,k}$ is the Kronecker delta.

We can easily obtain a table of the initial Stirling numbers of the second kind:

| $n \backslash k$ | 0   | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   |
|-----------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 1               | 0   | 1   |     |     |     |     |     |     |     |
| 2               | 0   | 1   | 1   |     |     |     |     |     |     |
| 3               | 0   | 1   | 3   | 1   |     |     |     |     |     |
| 4               | 0   | 1   | 7   | 6   | 1   |     |     |     |     |
| 5               | 0   | 1   | 15  | 25  | 10  | 1   |     |     |     |
| 6               | 0   | 1   | 31  | 90  | 65  | 15  | 1   |     |     |
| 7               | 0   | 1   | 63  | 301 | 350 | 140 | 21  | 1   |     |
| 8               | 0   | 1   | 127 | 966 | 1701| 1050| 266 | 28  | 1   |
| $\vdots$        |     |     |     |     |     |     |     |     |     |
For a block $B$ of the design $D$, let $n_i$ be the number of blocks of $B$ that meet $B$ in $i$ points, where $0 \leq i \leq k$. We set $A_s = \sum_{i=0}^{k} (i)_s n_i$ for $0 \leq s \leq t$.

By (1), for $0 \leq t' \leq t$, we have

$$\sum_{i=0}^{k} i' n_i = \sum_{i=0}^{k} \left( \sum_{h=0}^{t'} S(t', h)(i)_h \right) n_i = \sum_{h=0}^{t'} S(t', h)A_h.$$ (2)

The elementary symmetric polynomials in $n$ variables $x_1, x_2, \ldots, x_n$ written $\sigma_{k,n}$ for $k = 0, 1, \ldots, n$, can be defined as $\sigma_{0,n} = 1$ and

$$\sigma_{k,n} = \sigma_k(x_1, x_2, \ldots, x_n) = \sum_{1 \leq w_1 < \cdots < w_k \leq n} x_{w_1} x_{w_2} \cdots x_{w_k}.$$  

Then we have the following lemma.

Lemma 2.2. If $l \leq t$,

$$\sum_{i=0}^{k} (i - x_1)(i - x_2) \cdots (i - x_l) n_i = \sum_{\theta=0}^{l} (-1)^{\theta} \sigma_{\theta,l} \left( \sum_{h=0}^{l-\theta} S(l - \theta, h)A_h \right).$$

Proof.

$$\sum_{i=0}^{k} (i - x_1)(i - x_2) \cdots (i - x_l) n_i$$

$$= \sum_{i=0}^{k} \left\{ i' - \sum_{1 \leq w_1 \leq l} x_{w_1} i^{l-1} + \sum_{1 \leq w_1 < w_2 \leq l} x_{w_1} x_{w_2} i^{l-2} + \cdots + (-1)^{l-1} x_{w_1} x_{w_2} \cdots x_{w_l} \right\} n_i$$

$$= \sum_{\theta=0}^{l} (-1)^{\theta} \sigma_{\theta,l} \left( \sum_{i=0}^{k} i^{l-\theta} n_i \right)$$

by (2),

$$= \sum_{\theta=0}^{l} (-1)^{\theta} \sigma_{\theta,l} \left( \sum_{h=0}^{l-\theta} S(l - \theta, h)A_h \right).$$

$\square$

3 On the nonexistence of some self-orthogonal 7-designs

A $t$-$(v, k, \lambda)$ design is called self-orthogonal if the intersection of any two blocks of the design has the same parity as the block size $k$. Notice that a design obtained from the supports of the minimum weight codewords in a binary doubly even code is self-orthogonal as the blocks have lengths a multiple of 4 and the overlap of supports of any two codewords must have even size.

In this section, let $D' = (X', B')$ be a self-orthogonal 7-$(24m, 4m + 4, \lambda_7)$ design, where $\lambda_7 = \binom{5m-2}{m-1} \binom{4m-1}{4m-5} \binom{4m-2}{4m-6}$ and $m \in M$. For a block $B$ of the design $D'$, let $n_i$ be the number of blocks of $B'$ that meet $B$ in $i$ points, where $0 \leq i \leq 4m + 4$, and $n_i = 0$ if $i$ is odd (since $D'$ is self-orthogonal).
Using the fundamental equation in Koch [6] and also [2, Proof of Theorem 5], we have

$$\sum_{i=0}^{4m+4} \binom{i}{s} n_i = \binom{4m+4}{s} \lambda_s$$

for $0 \leq s \leq t$. Therefore we set

$$A_s = \sum_{i=0}^{4m+4} \binom{i}{s} n_i = (4m+4) \lambda_s$$

for $0 \leq s \leq 7$.

For the design $D'$, we define

$$F(m, 4m + 4; [x_1, x_2, x_3, x_4, x_5, x_6, x_7]) = \sum_{i=0}^{4m+4} (i - x_1)(i - x_2) \cdots (i - x_7)n_i.$$

By Lemma [2,2] we have

$$F(m, 4m + 4; [x_1, x_2, x_3, x_4, x_5, x_6, x_7]) = \sum_{\theta=0}^{7} (-1)^{\theta} \sigma_{\theta,7} \left( \sum_{h=0}^{7-\theta} S(7 - \theta, h)A_h \right)$$

$$= -\sigma_{7,7}A_7 + (\sigma_{6,7} - \sigma_{5,7} + \sigma_{4,7} - \sigma_{3,7} + \sigma_{2,7} - \sigma_{1,7} + 1)A_1$$

$$+ (\sigma_{5,7} + 3\sigma_{4,7} - 7\sigma_{3,7} + 15\sigma_{2,7} - 31\sigma_{1,7} + 63)A_2 + (\sigma_{4,7} - 6\sigma_{3,7} + 25\sigma_{2,7} - 90\sigma_{1,7} + 301)A_3$$

$$+ (\sigma_{3,7} - 10\sigma_{2,7} - 65\sigma_{1,7} + 350)A_4 + (\sigma_{2,7} - 15\sigma_{1,7} + 140)A_5 + (\sigma_{1,7} + 21)A_6 + A_7. \tag{4}$$

Then we have the following theorem.

**Theorem 3.1.** If $m \in \{8, 40, 42, 50, 74, 76, 80, 86, 100, 130, 144, 150\}$, then there is no self-orthogonal 7-$(24m, 4m + 4, \lambda_7)$ design.

**Proof.** By (4), we have

$$F(m, 4m + 4; [0, 2, 4, 6, 8, 10, 12])$$

$$= \sum_{i=0}^{4m+4} i(i - 2)(i - 4)(i - 6)(i - 8)(i - 10)(i - 12)n_i$$

$$= 10395A_1 - 10395A_2 + 4725A_3 - 1260A_4 + 210A_5 - 21A_6 + A_7. \tag{6}$$

By (5), we have

$$n_{14} = \frac{F(m, 4m + 4; [0, 2, 4, 6, 8, 10, 12])}{645120} - 8n_{16} - 36n_{18} - \cdots - \binom{2m+2}{7} n_{4m+4}.$$

By (5) and (6), we compute the values of $\frac{F(m, 4m + 4; [0, 2, 4, 6, 8, 10, 12])}{645120}$ for $m \in M$ by using Magma [3]. Then, if $m \in \{8, 40, 42, 50, 74, 76, 80, 86, 100, 130, 144, 150\}$, we have that $\frac{F(m, 4m + 4; [0, 2, 4, 6, 8, 10, 12])}{645120}$ is not an integer as in Table [1]. Hence $n_{14}$ is not an integer, which is a contradiction. Therefore, there is no self-orthogonal 7-$(24m, 4m + 4, \lambda_7)$ design.

□
Table 1: $F(m, 4m + 4; [0, 2, 4, 6, 8, 10, 12])/645120$

| $m$ | $F(m, 4m + 4; [0, 2, 4, 6, 8, 10, 12])$ |
|-----|----------------------------------------|
| 8   | 1569595833/8                           |
| 40  | 6972267623111828528771787666297086782790166251961/4 |
| 42  | 7413717557642396579773804378982932177616748595565925/2 |
| 50  | 5132258990999864497776773019002155555582981553461213899278199/8 |
| 74  | 482642508676130047547121143238832310639036440390013274122385587 |
| 76  | 17297525223190236190672610693505143217216325470799567243756674 |
| 80  | 273592695616325134916334164382847359509149699382750775301196495 |
| 84  | 5421001331376285920864366065/4 |
| 86  | 74417756725673009993698768020803053889303557285779323981496487527 |
| 100 | 1361289868407320050139095766445811736432305019915914548757138545 |
| 130 | 31897074804355531797221744561023012638883309436534767609796644311 |
| 144 | 10350702724282841278908575614022515865851288631068123683963832873 |
| 150 | 2251791376314593225461257265887240264932451320353248639071269342 |

4 On the nonexistence of some support 7-designs

In this section, let $D$ be the support design of the minimum weight $4m + 4$ of an extremal binary doubly even $[24m, 12m, 4m + 4]$ code $C$. If $D$ is a 7-design, the parameters are $(24m, 4m + 4, \lambda_7)$, where $\lambda_7 = \frac{(5m-2)}{(m-1)} \frac{(4m-1)(4m-2)}{(24m-5)(24m-6)}$, and $m \in M$. By Lemma 2.1 and Theorem 3.1, we consider the remainder $m \in \{5, 15, 19, 35, 41, 51, 52, 55, 57, 59, 60, 63, 65, 75, 90, 93, 101, 104, 105, 107, 118, 125, 127, 129, 135, 143, 151\}$.

Let $C_u$ be the set of all codewords of weight $u$ of $C$, where $4m + 4 \leq u \leq 20m - 4$. Fix $a \in C_u$ and define $n_j^u := |\{c \in C_{4m+4} : |\text{supp}(a) \cap \text{supp}(c)| = j\}|$. Then by using the fundamental equations in [2, Proof of Theorem 5], we have

$$
\sum_{j=0}^{4m+4} \binom{j}{s} n_j^u = \binom{u}{s} \lambda_s
$$

for $s = 0, 1, \ldots, t$. (Note that $n_j^u = 0$ if $j$ is odd, and $n_j^u = 0$ if $j > \frac{u}{2}$.)

These equations have been studied for a $t$-design and some applications, e.g. see Cameron and van Lint [3], Koch [6] and Tonchev [8]. We set $A_s^u = \sum_{j=0}^{4m+4} \binom{j}{s} n_j^u = (u)_s \lambda_s$ for $0 \leq s \leq 7$. 

5
For the design \( D \), we define
\[
F(m, u; [x_1, x_2, x_3, x_4, x_5, x_6, x_7]) = \sum_{j=0}^{4m+4} (j - x_1)(j - x_2) \cdots (j - x_7)n_i^u.
\]

Then as a generalization of equation (4), we have
\[
F(m, u; [x_1, x_2, x_3, x_4, x_5, x_6, x_7]) = \sum_{\theta=0}^{7} (-1)^{\theta} \sigma_{\theta, 7} \left( \sum_{h=0}^{7-\theta} S(7 - \theta, h)A_h^u \right)
\]
\[
= -\sigma_{7, 7} A_7^u + (\sigma_{6, 7} - \sigma_{5, 7} + \sigma_{4, 7} - \sigma_{3, 7} + \sigma_{2, 7} - \sigma_{1, 7} + 1) A_1^u
\]
\[
+ (-\sigma_{5, 7} + 3\sigma_{4, 7} - 7\sigma_{3, 7} + 15\sigma_{2, 7} - 31\sigma_{1, 7} + 63) A_2^u
\]
\[
+ (\sigma_{4, 7} - 6\sigma_{3, 7} + 25\sigma_{2, 7} - 90\sigma_{1, 7} + 301) A_3^u
\]
\[
+ (-\sigma_{3, 7} + 10\sigma_{2, 7} - 65\sigma_{1, 7} + 350) A_4^u
\]
\[
+ (\sigma_{2, 7} - 15\sigma_{1, 7} + 140) A_5^u
\]
\[
+ (-\sigma_{1, 7} + 21) A_6^u + A_7^u.
\]  

Then we give the following theorem. (Note that Theorem 3.1 is a stronger result.)

**Theorem 4.1.** If \( m \in \{5, 19, 35, 41, 51, 65, 75, 101, 129\} \), then \( D \) (the support design of the minimum weight of \( C \)) is not a 7-design.

**Proof.** By (7), we have
\[
F(m, 4m + 8; [0, 2, 4, 6, 8, 10, 12])
\]
\[
= \sum_{j=0}^{4m+4} j(j - 2)(j - 4)(j - 6)(j - 8)(j - 10)(j - 12)n_j^{4m+8}
\]
\[
= 10395 A_1^{4m+8} - 10395 A_2^{4m+8} + 4725 A_3^{4m+8} - 1260 A_4^{4m+8} + 210 A_5^{4m+8} - 21 A_6^{4m+8} + A_7^{4m+8}.
\]

Then we have
\[
n_1^{4m+8} = \frac{F(m, 4m + 8; [0, 2, 4, 6, 8, 10, 12])}{645120} - 8n_1^{4m+8} - 36n_8^{4m+8} - \ldots - \left( \frac{2m + 2}{7} \right) n_{7m+4}^{4m+8}.
\]

If \( m \in \{5, 19, 35, 41, 51, 65, 75, 101, 129\} \), by computation using Magma, we have that \( F(m, 4m + 8; [0, 2, 4, 6, 8, 10, 12]) \) is not an integer as in Table 2. Hence \( n_1^{4m+8} \) is not an integer, which is a contradiction. Therefore, \( D \) is not a 7-design.

Finally, we consider when \( D \) is an 8-design. From Lemma 2.1 if \( \lambda_8 \) is also an integer, we have \( m \in \{8, 42, 63, 75, 130\} \). By Theorem 3.1 and 4.1, we have only the remainder \( m = 63 \). Let \( D'' \) be a self-orthogonal 8-(24m, 4m + 4, \( \lambda_8 \)) design, where \( \lambda_8 = \frac{5m - 2}{4m - 3} \). With \( n_i \) as defined in Section 3, which equals \( n_i^{4m+4} \), we set \( A_s = \sum_{i=0}^{4m+4} (i)_s n_i = (4m + 4) \lambda_s \) for \( 0 \leq s \leq 8 \). For the design \( D'' \), we have
\[
F(m, 4m + 4; [x_1, x_2, x_3, x_4, x_5, x_6, x_8]) = \sum_{i=0}^{4m+4} (i - x_1)(i - x_2) \cdots (i - x_8)n_i
\]
\[
= \sum_{\theta=0}^{8} (-1)^{\theta} \sigma_{\theta, 8} \left( \sum_{h=0}^{8-\theta} S(8 - \theta, h)A_h \right).
\]
Table 2: \( F(m, 4m + 8; [0, 2, 4, 6, 8, 10, 12])/645120 \)

| \( m \) | \( \frac{F(m, 4m + 8; [0, 2, 4, 6, 8, 10, 12])}{645120} \) |
|---|---|
| 5 | 9009/4 |
| 19 | 10290542185356908976248643/8 |
| 35 | 24019252344759794880275676371011296919815805/8 |
| 41 | 122906698177653671012029436288037892461385328646335/4 |
| 51 | 83644964457956799204581597231287964765291012860261529877138985/8 |
| 65 | 72975174207654767982109272917411685745718438510666139598156100263 01949797750545/8 |
| 75 | 71317588499310631419430590525991955846021452139090800919361087401 3324199822838428254310609/4 |
| 101 | 95541360721321819333355415268808168206345704645344007828295378667 33344553036892497209645844917737659658397691862895305/4 |
| 129 | 26230778791143794560883418189575439901696761060680966584544120210 699102593332231931142357704444499518305259605168488333726043587 913132060022892602625/8 |

Then, we have

\[
 n_{16} = \frac{F(m, 4m + 4; [0, 2, 4, 6, 8, 10, 12, 14])}{10321920} - 9n_{18} - 45n_{20} - \cdots - \binom{2m + 2}{8} n_{4m+4}. 
\]

In the case \( m = 63 \), by a computation using Magma, we have

\[
 F(63, 4 \cdot 63 + 4; [0, 2, 4, 6, 8, 10, 12, 14]) \\
 = -16809515472136742134534321134853418244406436165053567105402493489903309445518999/1792. 
\]

Hence \( n_{16} \) is not an integer. Therefore, if \( m = 63 \), there is no self-orthogonal 8-(24m, 4m+4, \lambda_8) design.

Then, for the design \( D \), we have the following theorem.

**Theorem 4.2.** \( D \) is never an 8-design.

Thus the proof of Theorem 1.1 is completed.

By the Assmus-Mattson theorem, the support design of minimum weight of an extremal binary doubly even \([24m + 8, 12m + 4, 4m + 4]\), respectively \([24m + 16, 12m + 8, 4m + 4]\), code is a 3-design, 1-design, respectively. We give the following results by a similar argument to the above.

**Theorem 4.3.** Let \( D_1 \) and \( D_2 \) be the support \( t \)-designs of the minimum weight of an extremal binary doubly even self-dual \([24m + 8, 12m + 4, 4m + 4]\) code \((m \leq 158)\) and \([24m + 16, 12m + 8, 4m + 4]\) code \((m \leq 163)\), respectively.

(1) If \( D_1 \) becomes a 4-design, then \( D_1 \) is a 5-design and \( m \) must be in the set \( \{15, 35, 45, 58, 75, 85, 90, 95, 113, 115, 120, 125\} \). If \( D_1 \) becomes a 6-design, then \( m \) must be in the set \( \{58, 90, 113\} \). If \( D_1 \) becomes a 7-design, then \( m \) must be in the set \( \{58\} \), and \( D_1 \) is never an 8-design.
If $D_2$ becomes a 2-design, then $D_2$ is a 3-design and $m$ must be in the set 
\{5, 10, 20, 23, 25, 35, 44, 45, 50, 55, 60, 70, 72, 75, 79, 80, 85, 93, 95, 110, 118, 120, 121, 123, 
125, 130, 142, 144, 145, 149, 150, 155, 156, 157, 160, 163\}. If $D_2$ becomes a 4-design, then $m$
 must be in the set \{10, 79, 93, 118, 120, 123, 125, 142\}. If $D_2$ becomes a 5-design, then $m$ must
 be in the set \{79, 93, 118, 120, 123, 125, 142\}, and $D_2$ is never a 6-design.

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