The Ablowitz-Ladik system on a graph

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Abstract

This paper presents an approach to study initial-boundary value (IBV) problems for integrable nonlinear differential-difference equations (DDEs) posed on a graph. As an illustrative example, we consider the Ablowitz–Ladik system posed on a graph that is constituted by $N$ semi-infinite lattices (edges) connected through some boundary conditions. We first show that analyzing this problem is equivalent to analyzing a certain matrix IBV problem; then we employ the unified transform method (UTM) to analyze this matrix IBV problem. We also compare our results with some previously known studies. In particular, we show that the inverse scattering method (ISM) for the integrable DDEs on the set of integers can be recovered from the UTM applied to our $N = 2$ graph problem as a particular case, and the non-local reductions of integrable DDEs can be obtained as local reductions from our results.

Keywords: Ablowitz–Ladik system, inverse scattering method, unified transform method, initial-boundary value problem
Mathematics Subject Classification numbers: 37K10, 37K15

1. Introduction

In recent years, the subject of nonlinear evolution equations on graphs has attracted increasing attention due to its rich mathematical structures as well as wide physical applications; see for example [1–14] and references therein. In comparison with the problem on the full-line or the problem on the half-line, the problem on a graph is more complicated and is not a fully developed subject yet. However the study on this subject is currently fast growing; see for example [2–6] for recent developments regarding the integrable nonlinear Schrödinger (NLS) equation on various simple graphs.

In this paper, we show how to analyze initial-boundary value (IBV) problems for integrable nonlinear differential-difference equations (DDEs) on a graph by using the unified transform
method (UTM) [19]. The illustrative example we choose is the Ablowitz–Ladik (AL) system [33–35]:

$$
\begin{align*}
\frac{idn}{dt} + q_{n+1} - 2q_n + q_{n-1} - p_n q_n (q_{n+1} + q_{n-1}) &= 0, \\
\frac{idn}{dt} - p_{n+1} + 2p_n - p_{n-1} + p_n q_n (p_{n+1} + p_{n-1}) &= 0,
\end{align*}
$$

(1.1)

where $q_n = q(n, t)$ and $p_n = p(n, t)$ are complex functions. We consider this integrable DDE on a graph $\mathcal{G}$ that is made of $N \geq 1$ semi-infinite lattices (edges) connected through some boundary conditions at $n = 0$ and at $n = -1$.

In order to analyze such a problem, we follow the following idea from the paper [37] about the problem of integrable partial differential equations (PDEs) on a star graph: mapping the problem on a graph to a matrix IBV problem and then extending the UTM for analyzing IBV problems in the scalar case to the one in the matrix case. The present paper provides a discrete analogue of the main results of papers [37] and [38] by Caudrelier.

We note that the UTM, introduced by Fokas [15, 19] for analyzing IBV problems of integrable PDEs, provides an important generalization of the inverse scattering method (ISM). The UTM has been implemented to analyze IBV problems for both integrable PDEs and integrable DDEs; see for example [15–26] for this method on integrable PDEs and see [27–31] for this method on integrable DDEs.

The main results derived in the present paper are stated in proposition 1 and theorem 1. Proposition 1 implies that the analysis of the AL lattice system (1.1) on the graph $\mathcal{G}$ is equivalent to the analysis of a certain matrix IBV problem. Thus we can analyze the AL lattice system on the graph $\mathcal{G}$ by extending the UTM for integrable DDEs in the scalar case to the one in the matrix case. Theorem 1 shows that the solution of the AL lattice system on the graph $\mathcal{G}$ can be expressed in terms of the solution of an appropriate matrix Riemann–Hilbert (RH) problem. Moreover, we compare our results with some previously known studies for integrable DDEs and illustrate how our results embrace these studies as a particular case. In particular, we show in detail that the standard ISM for the AL system on the set of integers (see [35]) can be recovered as a special case of the UTM applied to our $N = 2$ graph problem; see theorem 2. We also show that both the integrable discrete NLS (IDNLS) equation (see [35]) and the non-local IDNLS equation (see [43]) can be obtained as standard local reductions of our matrix AL system on the set of non-negative integers; see proposition 3. Thus, in addition to the ISM for the IDNLS equation, the ISM for the non-local IDNLS equation (see [43]) can be also recovered from our results; see proposition 4.

The paper is organized as follows: in section 2, we introduce the problem of the AL system on a graph and then we formulate this problem into a certain matrix IBV problem. In section 3, we implement the UTM to analyze the matrix IBV problem formulated in section 2. In section 4, we compare our results with some previously known studies and show how these previous studies can be recovered from our results. We further discuss our results in section 5.

2. Problem formulation

2.1. AL system on a graph

We consider the AL lattice system (1.1) on the graph $\mathcal{G}$. Recall that the graph $\mathcal{G}$ is the union of $N$ edges: each edge is made of semi-infinite lattice $\mathbb{N}_0$, the set of non-negative integers, namely $\mathbb{N}_0 = \{0, 1, 2, \cdots\}$; these edges are connected to each other through some boundary conditions.
We introduce $N$-copies of the AL lattice system (1.1) for functions $\{q^\alpha(n,t), p^\alpha(n,t)\}$, $\alpha = 1, 2, \cdots, N$. The AL lattice system on the graph $\mathcal{G}$ is equivalent to a system of $N$ AL systems such that each $\{q^\alpha(n,t), p^\alpha(n,t)\}$ lives on edge $\alpha$, is a pair of functions of $n \in \mathbb{N}_0$ and $0 < t < T$, and the edges meet one another through some boundary conditions at $n = 0$ and at $n = -1$. Therefore the problem reads, for $\alpha = 1, 2, \cdots, N$:

\[
\begin{align*}
&i \frac{dq^\alpha_n}{dt} + g^\alpha_{n+1} - 2q^\alpha_n + q^\alpha_{n-1} - p^\alpha_n q^\alpha_n (q^\alpha_{n+1} + q^\alpha_{n-1}) = 0, \quad n \in \mathbb{N}_0, \quad 0 < t < T, \\
&i \frac{dp^\alpha_n}{dt} - p^\alpha_{n+1} + 2p^\alpha_n - p^\alpha_{n-1} + p^\alpha_n q^\alpha_n (p^\alpha_{n+1} + p^\alpha_{n-1}) = 0, \quad n \in \mathbb{N}_0, \quad 0 < t < T, \\
&q^\alpha(n,0) = q^\alpha_0(n), \quad p^\alpha(n,0) = p^\alpha_0(n), \\
&q^\alpha(-1,t) = g^\alpha_{-1}(t), \quad q^\alpha(0,t) = g^\alpha_{0}(t), \quad p^\alpha(-1,t) = h^\alpha_{-1}(t), \quad p^\alpha(0,t) = h^\alpha_{0}(t),
\end{align*}
\]

where $q^\alpha_0(n)$, $p^\alpha_0(n)$ denote the initial data, and $g^\alpha_{j}(t)$, $h^\alpha_{j}(t)$, $j = -1, 0$, denote the boundary values. For each $\alpha$, equation (2.1) is the discrete compatibility condition

\[
\frac{d\alpha(n+1,z)}{dt} = \frac{d\alpha(n,z)}{dt} \text{ of the following linear systems (called the Lax pair) [30, 36]}
\]

\[
\begin{align*}
\begin{cases}
\mu^\alpha(n,t,z) = \mathcal{V}^\alpha(n,t,z)\mu^\alpha(n,t,z) = \mathcal{U}^\alpha(n,t)\mu^\alpha(n,t,z)
\end{cases}
\end{align*}
\]

where $\mu^\alpha(n,t,z)$ is a $2 \times 2$ matrix,

\[
\begin{align*}
&f^\alpha(n,t) = \sqrt{1 - q^\alpha(n,t)p^\alpha(n,t)}, \quad \omega(z) = \frac{1}{2}(z - \bar{z}^{-1})^2, \\
&\mathcal{Z} = \begin{pmatrix} z & 0 \\ 0 & \bar{z}^{-1} \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\
&\mathcal{U}^\alpha(n,t) = \begin{pmatrix} 0 & q^\alpha(n,t) \\ p^\alpha(n,t) & 0 \end{pmatrix}, \\
&\mathcal{V}^\alpha(n,t,z) = i \left( \mathcal{U}^\alpha(n-1,t)\mathcal{Z} - \mathcal{U}^\alpha(n,t)\mathcal{Z}^{-1} - \frac{1}{2}(\mathcal{U}^\alpha(n-1,t)\mathcal{U}^\alpha(n-1,t) + \mathcal{U}^\alpha(n-1,t)\mathcal{U}^\alpha(n,t)) \right) \sigma_3,
\end{align*}
\]

and the symbol $\mathcal{Z}\mu^\alpha(n,t,z)$ stands for $\mathcal{Z}\mu^\alpha(n,t,z)\mathcal{Z}^{-1}$.

We emphasize that, as the problem in continuous case [37], in order to describe a genuine graph problem we should consider (2.1) and (2.2) for all $\alpha$ as a whole rather than $N$ disconnected problems. Indeed, in situations of interest for problems on a graph, only special combinations of $g^\alpha_{-1}(t)$, $g^\alpha_{0}(t)$, $h^\alpha_{-1}(t)$ and $h^\alpha_{0}(t)$ are supposed to be given, for example, the linearizable boundary conditions and the integrable defect boundary conditions describing nontrivial connections between the semi-infinite lattices of a discrete graph (see section 4 for details in the following). In order to present a general framework to implement the UTM to the lattice graph problem we will not focus on the connections between the edges of the graph in this and the next sections; we will return to these nontrivial connections on some examples and show how some previously known studies fit within our framework in section 4.

We also note that there is another deeper reason that one should consider the IBV problem (2.1) and (2.2) as a whole: Similarly to the problems in continuous case (the integrable PDEs on a star graph) [2, 3], only appropriate boundary conditions can make an integrable DDE
on a lattice graph a well-posed problem. In this paper, we will concentrate on the presentation of our method and not on the rigorous analysis (for example, the choice of appropriate functional spaces) related to well-posedness for the IBV problems for the AL system. These important questions are beyond the scope of the present paper. For our purposes, we assume in the following that all initial data has sufficient decay as \( n \to \infty \) and all boundary values are of appropriate smoothness and sufficient decay.

2.2. Mapping the problem on the graph to a certain matrix IBV problem

We have formulated the AL lattice system (1.1) on the graph \( \mathcal{G} \) into the IBV problem (2.1) and (2.2). We next show that analyzing such an IBV problem is equal to analyzing a certain matrix IBV problem.

We introduce the \( N \times N \) diagonal-matrices

\[
Q(n,t) = \text{diag} \left( q^1(n,t), \ldots, q^N(n,t) \right), \tag{2.5a}
\]

\[
P(n,t) = \text{diag} \left( p^1(n,t), \ldots, p^N(n,t) \right), \tag{2.5b}
\]

\[
\mathcal{F}(n,t) = \text{diag} \left( f^1(n,t), \ldots, f^N(n,t) \right). \tag{2.5c}
\]

We note that (2.1) is equivalent to the matrix valued AL system

\[
i \frac{\partial Q}{\partial t} + Q_{n+1} - 2Q_n + Q_{n-1} - P_n Q_n (Q_{n+1} + Q_{n-1}) = 0, \quad n \in \mathbb{N}_0, \quad 0 < t < T,
\]

\[
i \frac{\partial P}{\partial t} - P_{n+1} + 2P_n - P_{n-1} + P_n Q_n (P_{n+1} + P_{n-1}) = 0, \quad n \in \mathbb{N}_0, \quad 0 < t < T,
\]

where \( Q_n = Q(n,t) \) and \( P_n = P(n,t) \) are chosen to be the diagonal-matrices in the form of (2.5a) and (2.5b). Moreover, the initial and boundary condition (2.2) is equivalent to

\[
Q(n,0) = Q_0(n), \quad P(n,0) = P_0(n),
\]

\[
Q(-1,t) = G_{-1}(t), \quad Q(0,t) = G_0(t),
\]

\[
P(-1,t) = H_{-1}(t), \quad P(0,t) = H_0(t),
\]

where

\[
Q_0(n) = \text{diag} \left( q_0^1(n), \ldots, q_0^N(n) \right),
\]

\[
P_0(n) = \text{diag} \left( p_0^1(n), \ldots, p_0^N(n) \right),
\]

\[
G_i(t) = \text{diag} \left( g_i^1(t), \ldots, g_i^N(t) \right), \quad i = -1, 0.
\]

\[
H_i(t) = \text{diag} \left( h_i^1(t), \ldots, h_i^N(t) \right), \quad i = -1, 0.
\]

The matrix AL system (2.6) is the discrete compatibility condition of the following linear systems

\[
F(n,t) \mu(n+1,t,z) - \hat{Z} \mu(n,t,z) = U(n,t) \mu(n,t,z) Z^{-1}, \tag{2.9a}
\]

\[
\mu_i(n,t,z) - i \omega(z) [\Sigma_3, \mu(n,t,z)] = V(n,t,z) \mu(n,t,z), \tag{2.9b}
\]
where \( \mu(n,t,z) \) is a \( 2N \times 2N \) matrix,
\[
\Sigma_3 = \begin{pmatrix}
I_N & 0 \\
0 & -I_N
\end{pmatrix},
\]
\[
Z = \begin{pmatrix}
zI_N & 0 \\
0 & z^{-1}I_N
\end{pmatrix},
\]
\[
F(n,t) = \begin{pmatrix}
F(n,t) & 0 \\
0 & F(n,t)
\end{pmatrix},
\]
\[
U(n,t) = \begin{pmatrix}
0 & Q(n,t) \\
P(n,t) & 0
\end{pmatrix},
\]
\[
V(n,t,z) = i \left( U(n-1,t)Z - U(n,t)Z^{-1} - \frac{1}{2} (U(n,t)U(n-1,t) + U(n-1,t)U(n,t)) \right) \Sigma_3,
\]
and the symbol \( \hat{Z} \mu(n,t,z) \) stands for \( Z\mu(n,t,z)Z^{-1} \).

We need to introduce some notations. Denote by \( \mathbb{M}_m \) the algebra of \( m \times m \) matrices over \( \mathbb{C} \). For \( M \in \mathbb{M}_{2N} \), we write \( M = \begin{pmatrix}
M^{11} & M^{12} \\
M^{21} & M^{22}
\end{pmatrix} \), where the blocks \( M^{jk} \) and \( j, k = 1,2 \) are \( N \times N \) matrices. We write \( M_d = \begin{pmatrix}
M_{d1}^{11} & M_{d1}^{12} \\
M_{d2}^{21} & M_{d2}^{22}
\end{pmatrix} \) and \( M_o = \begin{pmatrix}
M_{o1}^{11} & M_{o1}^{12} \\
M_{o2}^{21} & M_{o2}^{22}
\end{pmatrix} \), where \( M_{d1}^{jk} \) and \( M_{o1}^{jk} \) denote the diagonal and the off-diagonal parts of \( M^{jk} \), respectively. We denote \( \mathbb{M}_d = \{ M_d, M \in \mathbb{M}_{2N} \} \) and \( \mathbb{M}_o = \{ M_o, M \in \mathbb{M}_{2N} \} \) the corresponding sets. Consider the isomorphism
\[
\theta : \prod_{j=1}^{N} \mathbb{M}_2 \to \mathbb{M}_d
\]
\[
(M^1, \cdots, M^N) \mapsto M = \sum_{j=1}^{N} M^j \otimes E_{jj},
\]
where \( \{ E_{jk} \}_{j,k=1}^{N} \) is the canonical basis of \( \mathbb{M}_N \), and the algebra structure of \( \prod_{j=1}^{N} \mathbb{M}_2 \) is defined by the pointwise operations. We find the following result [37]:

**Lemma 1.** \( \mathbb{M}_d \) and \( \mathbb{M}_o \) are vector subspaces of \( \mathbb{M}_{2N} \) and the direct sum decomposition \( \mathbb{M}_{2N} = \mathbb{M}_d \oplus \mathbb{M}_o \) holds. Moreover, \( \mathbb{M}_d \) is a subalgebra of \( \mathbb{M}_{2N} \), which is isomorphic to the direct product \( \prod_{j=1}^{N} \mathbb{M}_2 \) as algebras.

In analogy with the problem in continuum case [37], the key observation is that the fundamental solution of (2.9) with an appropriate normalization, is an \( \mathbb{M}_d \)-valued function of \( n,t,z \) in the domain where it is defined. More precisely, we have

**Proposition 1.** Let \( \mu(n,t,z) \) be the fundamental solution of (2.9) with normalization \( \mu(n_0,t_0,z) = I_{2N} \) at a fixed point \( (n_0,t_0) \in \mathbb{N}_0 \times \mathbb{R}^+ \). Then \( \mu(n,t,z) \in \mathbb{M}_d \) in the domain wherever it is defined.

**Proof.** This conclusion is deduced by using lemma 1 and the linearity of equations (2.9) for \( \mu(n,t,z) \).

We recall that all the ingredients required for the implementation of the UTM can be derived from the fundamental solutions of the associated Lax pair by algebraic manipulations;
see for example [19] and [30] for details. We therefore deduce from proposition 1 that the implementation of the UTM to the matrix AL system (2.6) and thus to the AL system on the graph \( \mathcal{G} \) can be entirely formulated in \( \mathbb{M}_d \).

3. The unified transformation method for the AL system on a graph

In the above section, we have mapped the problem of the AL lattice system on the graph \( \mathcal{G} \) to a \( \mathbb{M}_d \)-valued matrix IBV problem. In this section, we show how to analyze such a \( \mathbb{M}_d \)-valued functions; this fact will become clear in section 3.1.2.

In the above section, we have mapped the problem of the AL lattice system on the graph \( \mathcal{G} \) to a \( \mathbb{M}_d \)-valued matrix IBV problem via the UTM. The detailed derivations regarding the implementation of the UTM in the present matrix case can be obtained easily via a similar manner as presented in the scalar case [27, 30]. For the economy of presentation, here we will skip several details on these derivations and we will only present the main and essential steps about the implementation of the UTM. We refer the reader to section 4 of [30] for more details on these derivations.

3.1. Direct part of the UTM

3.1.1. The eigenfunctions. We define three different \( 2N \times 2N \) matrix-valued eigenfunctions \( \{ \mu_j(n, z, t) \}_1^3 \) as simultaneous solutions of the linear systems (2.9). These three eigenfunctions are normalized respectively at \( (n, t) = (0, 0) \), at \( (n, t) = (\infty, t) \), and at \( (n, t) = (0, T) \). They are given by:

\[
\begin{align*}
\mu_1(n, t, z) &= C(n, t)C^{-1}(0, t) \left( I_{2N} + Z^n \int_0^t e^{i\omega(t-t')\Sigma_3} (V\mu_1(0, t', z)) \, dt' \right) \\
&\quad + C(n, t)Z^{-1} \sum_{m=0}^{n-1} C^{-1}(m, t)\bar{Z}^{n-m}(U(m, t)\mu_1(m, t, z)), \\
\mu_2(n, t, z) &= C(n, t) \left( I_{2N} - Z^{-1} \sum_{m=n}^{\infty} C^{-1}(m, t)\bar{Z}^{n-m}(U(m, t)\mu_2(m, t, z)) \right), \quad (3.1) \\
\mu_3(n, t, z) &= C(n, t)C^{-1}(0, t) \left( I_{2N} - Z^n \int_t^\infty e^{i\omega(t-t')\Sigma_3} (V\mu_3(0, t', z)) \, dt' \right) \\
&\quad + C(n, t)Z^{-1} \sum_{m=0}^{n-1} C^{-1}(m, t)\bar{Z}^{n-m}(U(m, t)\mu_3(m, t, z)),
\end{align*}
\]

where \( I_{2N} \) denotes the \( 2N \times 2N \) identity matrix, and

\[
C(n, t) = \prod_{m=n}^{\infty} F(m, t), \quad C(-\infty) = \lim_{n \to -\infty} C(n, t), \quad (3.2)
\]

and the symbol \( e^{\Sigma_3} \) acts on a \( 2N \times 2N \) matrix \( A \) as follows

\[
e^{\Sigma_3}A = e^{\Sigma_3}Ae^{-\Sigma_3}. \quad (3.3)
\]

Note that proposition 1 implies that \( \{ \mu_j(n, z, t) \}_1^3 \) are \( \mathbb{M}_d \)-valued functions. Thus the spectral functions corresponding to \( \{ \mu_j(n, z, t) \}_1^3 \) are also \( \mathbb{M}_d \)-valued functions; this fact will become clear in section 3.1.2.

We introduce the following domains for the AL system (see figure 1):

\[
D_{in} = \left\{ z \in \mathbb{C} \mid |z| < 1 \right\}, \quad D_{out} = \left\{ z \in \mathbb{C} \mid |z| > 1 \right\}, \quad (3.4a)
\]
\(D_+ = \{ z \in \mathbb{C} \mid \text{Im}(\omega(z)) > 0 \}\), \(D_- = \{ z \in \mathbb{C} \mid \text{Im}(\omega(z)) < 0 \}\), \(\text{(3.4b)}\)

\(D_{+\text{in}} = \{ z \in \mathbb{C} \mid |z| < 1, \arg z \in \left(\frac{\pi}{2}, \pi \right) \cup \left(\frac{3\pi}{2}, 2\pi \right) \}\), \(\text{(3.4c)}\)

\(D_{-\text{in}} = \{ z \in \mathbb{C} \mid |z| < 1, \arg z \in \left(0, \frac{\pi}{2} \right) \cup \left(\pi, \frac{3\pi}{2} \right) \}\), \(\text{(3.4d)}\)

\(D_{+\text{out}} = \{ z \in \mathbb{C} \mid |z| > 1, \arg z \in \left(0, \frac{\pi}{2} \right) \cup \left(\pi, \frac{3\pi}{2} \right) \}\), \(\text{(3.4e)}\)

\(D_{-\text{out}} = \{ z \in \mathbb{C} \mid |z| > 1, \arg z \in \left(\frac{\pi}{2}, \pi \right) \cup \left(\frac{3\pi}{2}, 2\pi \right) \}\), \(\text{(3.4f)}\)

and we denote by \(\bar{D}_{\text{in}}, \bar{D}_{\text{out}}, \bar{D}_{\pm\text{in}}, \bar{D}_{\pm\text{out}}\) the closure of these domains. We will use the notations:

\[
\mu_j(n, t, z) = (\mu_j^L(n, t, z) \mid \mu_j^R(n, t, z)), \quad j = 1, 2, 3, 
\]

where \(\mu_j^L(n, t, z)\) is the \(2N \times N\) left block and \(\mu_j^R(n, t, z)\) is the \(2N \times N\) right block of the matrix \(\mu_j(n, t, z)\). From (3.1), we deduce that

- \(\mu_1(n, t, z)\) is analytic for \(z \in \mathbb{C} \setminus \{0\}\); \(\mu_1^L(n, t, z)\) is continuous and bounded for \(z \in \bar{D}_{-\text{out}}\), and \(\mu_1^R(n, t, z)\) is continuous and bounded for \(z \in \bar{D}_{+\text{in}}\).
- \(\mu_2^L(n, t, z)\) is analytic for \(D_{\text{in}}\) and it is continuous and bounded for \(z \in \bar{D}_{\text{in}}\), and \(\mu_2^R(n, t, z)\) is analytic for \(D_{\text{out}}\) and it is continuous and bounded for \(z \in \bar{D}_{\text{out}}\).
- \(\mu_3(n, t, z)\) is analytic for \(z \in \mathbb{C} \setminus \{0\}\) (in the case of finite \(T\)). If \(T = \infty\), \(\mu_3^L(n, t, z)\) is analytic for \(z \in \bar{D}_{+\text{out}}\), and \(\mu_3^R(n, t, z)\) is analytic for \(z \in \bar{D}_{-\text{in}}\), \(\mu_3^L(n, t, z)\) is continuous and bounded for \(z \in \bar{D}_{+\text{out}}\), and \(\mu_3^R(n, t, z)\) is continuous and bounded for \(z \in \bar{D}_{-\text{in}}\).
3.1.2. The spectral functions. The matrices \( \{\mu_j(n, t, z)\} \) are related:
\[
\mu_2(n, t, z) = \mu_1(n, t, z) \hat{Z}^a e^{i\omega(z) tE_2} s(z), \tag{3.6a}
\]
\[
\mu_3(n, t, z) = \mu_1(n, t, z) \hat{Z}^a e^{i\omega(z) tE_3} S(z). \tag{3.6b}
\]
Evaluating equation (3.6a) at \( n = 0 \) and \( t = 0 \), and evaluating equation (3.6b) at \( n = 0 \) and \( t = T \), we obtain
\[
s(z) = \mu_2(0, 0, z), \tag{3.7a}
\]
\[
S^{-1}(z) = e^{-i\omega(z) T E_2} \mu_1(0, T, z). \tag{3.7b}
\]
Proposition 1 and the formulae (3.7) imply that \( \{a(z), b(z), \hat{a}(z), \hat{b}(z)\} \) and \( \{A(z), B(z), \hat{A}(z) and \hat{B}(z)\} \) are all \( N \times N \) diagonal-matrix valued functions (called spectral functions), and they have the following properties:

- the entries of \( a(z) \) and \( b(z) \) are analytic for \( |z| > 1 \) and continuous and bounded for \( |z| \geq 1 \).
- the entries of \( \hat{a}(z) \) and \( \hat{b}(z) \) are analytic for \( |z| < 1 \) and continuous and bounded for \( |z| \leq 1 \).
- the entries of \( A(z) \) and \( B(z) \) are analytic for \( z \in \mathbb{C} \setminus \{0\} \) and continuous and bounded for \( z \in D_- \) (in the case of finite \( T \)). If \( T = \infty \), the functions \( A(z) \) and \( B(z) \) are defined and analytic for \( z \in D_- \).
- the entries of \( \hat{A}(z), \hat{B}(z) \) are analytic for \( z \in \mathbb{C} \setminus \{0\} \) and continuous and bounded for \( z \in D_+ \) (in the case of finite \( T \)). If \( T = \infty \), the functions \( \hat{A}(z) \) and \( \hat{B}(z) \) are defined and analytic for \( z \in D_+ \).

The Lax pair (2.9) implies that
\[
a(z)\hat{a}(z) - b(z)\hat{b}(z) = I_N, \tag{3.9a}
\]
\[
A(z)\hat{A}(z) - B(z)\hat{B}(z) = I_N. \tag{3.9b}
\]
These identities immediately yield
\[
s^{-1}(z) = \left( \begin{array}{cc} a(z) & -b(z) \\ \hat{b}(z) & \hat{a}(z) \end{array} \right), \quad S^{-1}(z) = \left( \begin{array}{cc} A(z) & -B(z) \\ \hat{B}(z) & \hat{A}(z) \end{array} \right). \tag{3.10}
\]

3.1.3. The symmetry properties. In analogy to the scalar case [27, 30], from equations (3.1), (3.7) and (3.8), we find the following symmetry relations regarding the spectral functions:
\[
a(-z) = a(z), \quad b(-z) = -b(z), \quad \hat{a}(-z) = \hat{a}(z), \quad \hat{b}(-z) = \hat{b}(z),
\]
\[
A(-z) = A(z), \quad B(-z) = -B(z), \quad \hat{A}(-z) = \hat{A}(z), \quad \hat{B}(-z) = -\hat{B}(z). \tag{3.11}
\]
Let us introduce the following functions:
\[
\gamma(z) = b(z)a^{-1}(z), \quad \bar{\gamma}(z) = \bar{b}(z)a^{-1}(z), \\
R(z) = B(z)\tilde{A}^{-1}(z), \quad \bar{R}(z) = \bar{B}(z)\tilde{A}^{-1}(z), \\
\tilde{d}(z) = \tilde{a}(z)\bar{A}(z) - \bar{b}(z)\bar{B}(z), \quad d(z) = a(z)\bar{A}(z) - b(z)\bar{B}(z), \\
\Gamma(z) = \bar{B}(z)a^{-1}(z)d^{-1}(z), \quad \bar{\Gamma}(z) = B(z)\bar{a}^{-1}(z)d\bar{d}^{-1}(z).
\] (3.12)

It follows from (3.11) that
\[
\gamma(-z) = -\gamma(z), \quad \bar{\gamma}(-z) = -\bar{\gamma}(z), \\
d(-z) = d(z), \quad \bar{d}(-z) = \bar{d}(z), \\
\Gamma(-z) = -\Gamma(z), \quad \bar{\Gamma}(-z) = -\bar{\Gamma}(z).
\] (3.13)

### 3.1.4 A Riemann–Hilbert problem

We define \( M_{n,t,z} = M_+(n,t,z) \), for \( z \in \mathbb{D}_+ \), and \( M_{n,t,z} = M_-(n,t,z) \), for \( z \in \mathbb{D}_- \), where \( M_\pm(n,t,z) \) are defined by
\[
M_+(n,t,z) = \begin{cases}
C^{-1}(n,t) \left( \mu_1^L(n,t,z), \left( I_2 \otimes \tilde{a}(z) \right)^{-1} \mu_1^R(n,t,z) \right), & z \in \mathbb{D}_{\text{in}}, \\
C^{-1}(n,t) \left( \left( I_2 \otimes d(z) \right)^{-1} \mu_1^L(n,t,z), \mu_1^R(n,t,z) \right), & z \in \mathbb{D}_{\text{out}}.
\end{cases}
\]
\[
M_-(n,t,z) = \begin{cases}
C^{-1}(n,t) \left( \mu_1^L(n,t,z), \left( I_2 \otimes \tilde{a}(z) \right)^{-1} \mu_1^R(n,t,z) \right), & z \in \mathbb{D}_{\text{in}}, \\
C^{-1}(n,t) \left( \left( I_2 \otimes a(z) \right)^{-1} \mu_1^L(n,t,z), \mu_1^R(n,t,z) \right), & z \in \mathbb{D}_{\text{out}}.
\end{cases}
\] (3.14)

From (3.1) and (3.14) one can deduce that
\[
M(n,t,z) = I_{2N} + U(n - 1,t)Z^{-1} + \begin{pmatrix} O_{N \times N}(z^{-2}, \text{even}) & O_{N \times N}(z^3, \text{odd}) \\ O_{N \times N}(z^{-3}, \text{odd}) & O_{N \times N}(z^2, \text{even}) \end{pmatrix}, \quad z \to (\infty, 0). 
\] (3.15)

Here the notation \( O_{N \times N}(z^2, \text{even}) \) (\( O_{N \times N}(z^{-2}, \text{even}) \)) indicates that the remaining terms are \( N \times N \) diagonal-matrices with even powers of \( z \) (\( z^{-1} \)) in diagonal-elements; the notation \( O_{N \times N}(z^3, \text{odd}) \) (\( O_{N \times N}(z^{-3}, \text{odd}) \)) indicates that the remaining terms are \( N \times N \) diagonal-matrices with odd powers of \( z \) (\( z^{-1} \)) in diagonal-elements; the symbol \( z \to (\infty, 0) \) means \( z \to \infty \) for the \( 2N \times N \) left block, while \( z \to 0 \) for the \( 2N \times N \) right block of the \( 2N \times 2N \) matrix \( M(n,t,z) \).

Using (3.6), we can formulate (3.14) into the following \( 2N \times 2N \) matrix RH problem
\[
M_-(n,t,z) = M_+(n,t,z)J(n,t,z), \quad z \in L = L_1 \cup L_2 \cup L_3 \cup L_4.
\] (3.16)

where \( J(n,t,z) = J_j(n,t,z) \) for \( z \in L_j, j = 1, 2, 3, 4, \) and the jump matrices are defined by
\[
J_1(n,t,z) = \tilde{Z}^n e^{i\omega(z)\tilde{g}_1}, \begin{pmatrix} I_N & \Gamma(z) \\ 0 & I_N \end{pmatrix}, \quad z \in L_1,
\]
\[
J_2(n,t,z) = \tilde{Z}^n e^{i\omega(z)\tilde{g}_2}, \begin{pmatrix} I_N & \Gamma(z) \\ 0 & -\Gamma(z) \end{pmatrix}, \quad z \in L_2,
\]
\[
J_3(n,t,z) = \tilde{Z}^n e^{i\omega(z)\tilde{g}_3}, \begin{pmatrix} I_N - \gamma(z)\tilde{\gamma}(z) & \gamma(z) \\ -\tilde{\gamma}(z) & I_N \end{pmatrix}, \quad z \in L_3,
\]
\[
J_4(n,t,z) = \tilde{Z}^n e^{i\omega(z)\tilde{g}_4}, \begin{pmatrix} I_N - \gamma(z)\tilde{\gamma}(z) & \gamma(z) \\ -\tilde{\gamma}(z) & I_N \end{pmatrix}, \quad z \in L_4.
\] (3.17)

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and the contours \( L_j, j = 1, 2, 3, 4 \), are defined by (see figure 2):
\[
L_1 = D_{\text{in}} \cap D_{\text{out}}, \quad L_2 = D_{\text{in}} \cap D_{\text{out}}, \quad L_3 = D_{\text{out}} \cap D_{\text{in}}, \quad L_4 = D_{\text{out}} \cap D_{\text{out}}.
\]  
\[ (3.18) \]

The blocks (12) and (21) of (3.15) induce the following expressions:
\[
Q(n, t) = \lim_{z \to 0} (z^{-1} M(n + 1, t, z))^12, \\
P(n, t) = \lim_{z \to \infty} (z M(n + 1, t, z))^{21},
\]  
\[ (3.19) \]

where the indexes ‘12’ and ‘21’ denote the blocks (12) and (21) in the natural decomposition of a matrix in \( M_d \).

If the spectral functions \( a(z), \tilde{a}(z), d(z) \) and \( \tilde{d}(z) \) can have zeros, then \( M(n, t, z) \) is a meromorphic function of \( z \). In this case, the RH problem (3.16) is singular, thus, one has to consider the corresponding residue relations. The relations (3.11) and (3.13) imply that \( a(z), \tilde{a}(z), d(z) \) and \( \tilde{d}(z) \) are even functions of \( z \). Thus, their zeros always appear in opposite pairs. We assume that

- for each \( \alpha \in \{1, \ldots, N\} \), \( a^\alpha(z) \) has at most \( 2K^\alpha \) simple zeros \( \{z_j^{\alpha}\}_{j=1}^{2K^\alpha} \), \( K^\alpha = K_2^\alpha + K_3^\alpha \), where \( z_j^{\alpha} \in D_{\text{out}}, j = 1, 2, \ldots, 2K_1^\alpha, \) such that \( \alpha_j^{\alpha} = -\alpha_j^{\alpha}, j = 1, \ldots, K_1^\alpha; z_j^{\alpha} \in D_{\text{out}}, j = 2K_1^\alpha + 1, \ldots, 2K_1^\alpha + 2K_2^\alpha, \) such that \( \alpha_j^{\alpha} = -\alpha_j^{\alpha}, j = 2K_1^\alpha + 1, \ldots, 2K_1^\alpha + 2K_2^\alpha \).

- For each \( \alpha \in \{1, \ldots, N\} \), \( a^\alpha(z) \) has at most \( 2K^\alpha \) simple zeros \( \{\tilde{z}_j^{\alpha}\}_{j=1}^{2K^\alpha} \), \( K^\alpha = K_1^\alpha + K_2^\alpha \), where \( \tilde{z}_j^{\alpha} \in D_{\text{in}}, j = 1, \ldots, 2K_1^\alpha, \) such that \( \tilde{z}_j^{\alpha} = -\tilde{z}_j^{\alpha}, j = 1, \ldots, K_1^\alpha; \tilde{z}_j^{\alpha} \in D_{\text{in}}, j = 2K_1^\alpha + 1, \ldots, 2K_1^\alpha + 2K_2^\alpha, \) such that \( \tilde{z}_j^{\alpha} = -\tilde{z}_j^{\alpha}, j = 2K_1^\alpha + 1, \ldots, 2K_1^\alpha + 2K_2^\alpha \).

- For each \( \alpha \in \{1, \ldots, N\} \), \( a^\alpha(z) \) has at most \( 2\Lambda^\alpha \) simple zeros \( \{\lambda_j^{\alpha}\}_{j=1}^{2\Lambda^\alpha} \) for \( z \in D_{\text{out}}, \) such that \( \lambda_j^{\alpha} = -\lambda_j^{\alpha}, j = 1, \ldots, \Lambda^\alpha \).

- For each \( \alpha \in \{1, \ldots, N\} \), \( d^\alpha(z) \) has at most \( 2\Lambda^\alpha \) simple zeros \( \{\lambda_j^{\alpha}\}_{j=1}^{2\Lambda^\alpha} \) for \( z \in D_{\text{in}}, \) such that \( \lambda_j^{\alpha} = -\lambda_j^{\alpha}, j = 1, \ldots, \Lambda^\alpha \).

- None of the zeros of \( a^\alpha(z) \) for \( z \in D_{\text{in}} \) coincide with a zero of \( d^\alpha(z) \).

- None of the zeros of \( \tilde{a}^\alpha(z) \) for \( z \in D_{\text{in}} \) coincide with a zero of \( \tilde{d}^\alpha(z) \).

For conciseness, we drop the \((n, t)\) dependence and denote by \( M^\alpha(z) \) and \( M_{\alpha}(z) \) the first and last \( N \) columns of \( M_{\alpha}(n, t, z) \). With similar calculations as used for the AL system in the scalar case [27, 30], we obtain the following residues conditions
\[
\text{Res}_{\alpha=1} M^\alpha(z) = (z_j^{\alpha})^{-2n} e^{-2\omega(z_j^{\alpha})} M^\alpha(z_j^{\alpha}) b^{-1}(z_j^{\alpha}) \text{Res}_{\alpha=1} a^{-1}(z), j = 1, \ldots, 2K^\alpha, \\
\text{Res}_{\alpha=1} M^\alpha(z) = (z_j^{\alpha})^{-2n} e^{2\omega(z_j^{\alpha})} M^\alpha(z_j^{\alpha}) b^{-1}(z_j^{\alpha}) \text{Res}_{\alpha=1} \tilde{a}^{-1}(z), j = 1, \ldots, 2K^\alpha, \\
\text{Res}_{\alpha=1} M^\alpha(z) = (\lambda_j^{\alpha})^{-2n} e^{-2\mu(z_j^{\alpha})} M^\alpha(\lambda_j^{\alpha}) b(\lambda_j^{\alpha}) a^{-1}(\lambda_j^{\alpha}) \text{Res}_{\alpha=1} \tilde{d}^{-1}(z), j = 1, \ldots, 2\Lambda^\alpha, \\
\text{Res}_{\alpha=1} M^\alpha(z) = (\lambda_j^{\alpha})^{-2n} e^{2\mu(z_j^{\alpha})} M^\alpha(\lambda_j^{\alpha}) b(\lambda_j^{\alpha}) a^{-1}(\lambda_j^{\alpha}) \text{Res}_{\alpha=1} \tilde{d}^{-1}(z), j = 1, \ldots, 2\Lambda^\alpha, 
\]  
\[ (3.20) \]

where \( \text{Res}_{\alpha=1} a^{-1}(z) \) is the diagonal matrix whose only non-zero entries are for those \( \beta \)‘s, \( \beta = 1, \ldots, N \), such that \( z = z_j^{\alpha} \) is a zero of \( a^\beta(z) \), in which case the element reads \( 1/a^\beta(z_j^{\alpha}) \), and similarly for notations \( \text{Res}_{\alpha=1} \tilde{a}^{-1}(z) \), \( \text{Res}_{\alpha=1} \tilde{d}^{-1}(z) \) and \( \text{Res}_{\alpha=1} \tilde{d}^{-1}(z) \).

3.1.5. The global relation. Using (3.7b) in (3.6a) with \( n = 0, t = T \), we obtain
\[
\mu_2(0, T, z) = e^{i\omega(z)\Sigma_3}(5^{-1}(z)s(z)).
\]  
\[ (3.21) \]
Using the definition of $\mu_2(n, t, z)$ (the second equation of (3.1)) in the above formula, we obtain

$$S^{-1}(z) s(z) = C(0, T) - e^{-i\omega(z)Te^z} G(z, T),$$

where

$$G(z, t) = C(0, t) Z^{-1} \sum_{m=0}^{\infty} C^{-1}(m, t) \hat{Z}^{-m} \hat{U}(m, t) \mu_2(m, t, z).$$

(3.23)

The (12)-block and (21)-block of equation (3.22) yield the following global relation

$$A(z) b(z) - B(z) a(z) = -e^{-2i\omega(z)T} G^{12}(z, T), \quad |z| > 1,$$

(3.24a)

$$\hat{A}(z) \hat{b}(z) - \hat{B}(z) \hat{a}(z) = -e^{2i\omega(z)T} G^{21}(z, T), \quad |z| < 1,$$

(3.24b)

where $G^{12}(z, T)$ and $G^{21}(z, T)$ are (12)-block and (21)-block of the matrix $G(z, T)$. As $T = \infty$, the global relation (3.24) becomes

$$A(z) b(z) - B(z) a(z) = 0, \quad z \in \mathbb{D}_{\text{out}},$$

$$\hat{A}(z) \hat{b}(z) - \hat{B}(z) \hat{a}(z) = 0, \quad z \in \mathbb{D}_{\text{in}}.$$  

(3.25)

3.2. The inverse part of the UTM

Consider the initial data $Q_0(n)$ and $P_0(n)$, and the boundary values $G_l(t)$ and $H^l(t)$, $l = -1, 0$; see (2.8). Denote by $l^1(N_0)$ the space of sequences $\{a_n\}_{n \in N_0}$, such that $\sum_{n=0}^{\infty} |a_n| < \infty$. Denote by $U_0(n)$ the matrix $U(n, 0)$, in which $Q(n, 0)$ and $P(n, 0)$ are replaced respectively by $Q_0(n)$ and $P_0(n)$. Denote by $F_0(n)$ the matrix $F(n, 0)$, in which $F(n, 0)$ is replaced by $F_0(n) = \text{diag} \left( \sqrt{1 - q_0^l(n)} p_0^l(n), \ldots, \sqrt{1 - q_0^n(n)} p_0^n(n) \right)$. Denote by $W(t)$ the matrix
$U(l,t)$, $l = -1, 0$, in which $Q(l,t)$ and $P(l,t)$ are replaced respectively by $G(t)$ and $H(t)$; denote

$$
V_0(t, z) = i \left( W_{-1}(t) Z - W_0(t) Z^{-1} - \frac{1}{2} (W_0(t) W_{-1}(t) + W_{-1}(t) W_0(t)) \right) \Sigma_3. \quad (3.26)
$$

Motivated by section 3.1, we define the spectral functions as follows.

**Definition 1.** Given $Q_0(n), P_0(n)$ such that the entries of them belong to $l^1(\mathbb{N}_0)$, define the $2N \times 2N$ matrix valued function $\phi(n, z)$ by the unique solution of

$$
F_0(n) \phi(n + 1, z) - \dot{Z} \phi(n, z) = U_0(n) \phi(n, z) Z^{-1},
$$

$$
\lim_{n \to \infty} \phi(n, z) = I_{2N}. \quad (3.27)
$$

Furthermore, we define the $N \times N$ diagonal-matrix valued spectral functions $a(z), b(z), \tilde{a}(z)$ and $\tilde{b}(z)$ by

$$
a(z) = \phi^{22}(0, z), \quad b(z) = \phi^{12}(0, z), \quad |z| \geq 1,
$$

$$
\tilde{a}(z) = \phi^{11}(0, z), \quad \tilde{b}(z) = \phi^{21}(0, z), \quad |z| \leq 1, \quad (3.28)
$$

where the indexes ‘11’, ‘12’, ‘21’ and ‘22’ denote respectively the blocks (11), (12), (21) and (22) in the natural decomposition of $2N \times 2N$ matrices in $\mathbb{M}_d$.

**Definition 2.** Given $G(t), H(t), l = -1, 0$, such that the entries of them are smooth functions for $0 < t < T$, define the $2N \times 2N$ matrix valued function $\varphi(t, z)$ by the unique solution of

$$
\varphi(t, z) - i \omega(z) [\Sigma_3, \varphi(t, z)] = V_0(t, z) \varphi(t, z),
$$

$$
\varphi(0, z) = I_{2N}. \quad (3.29)
$$

Furthermore, we define the $N \times N$ diagonal-matrix valued spectral functions $A(z), B(z), \tilde{A}(z)$ and $\tilde{B}(z)$ by

$$
A(z) = \varphi^{11}(T, z), \quad B(z) = -e^{-2i\omega(z)T} \varphi^{12}(T, z), \quad z \in \mathbb{C} \setminus \{0\},
$$

$$
\tilde{A}(z) = \varphi^{22}(T, z), \quad \tilde{B}(z) = -e^{2i\omega(z)T} \varphi^{21}(T, z), \quad z \in \mathbb{C} \setminus \{0\}, \quad (3.30)
$$

where, as before, the indexes ‘11’, ‘12’, ‘21’ and ‘22’ denote respectively the blocks (11), (12), (21) and (22) in the natural decomposition of $2N \times 2N$ matrices in $\mathbb{M}_d$.

If $T = \infty$, we assume that the entries of $G(t), H(t)$ and $l = -1, 0$ belong to the Schwartz class, and we use an alternative definition of the spectral functions $\{A(z), B(z), \tilde{A}(z) and \tilde{B}(z)\}$ based on the solution $\mu_3(0, t, z)$; namely we let

$$
A(z) = \eta^{22}(0, z), \quad B(z) = \eta^{12}(0, z), \quad \text{Im} \langle \omega(z) \rangle \leq 0,
$$

$$
\tilde{A}(z) = \eta^{11}(0, z), \quad \tilde{B}(z) = \eta^{21}(0, z), \quad \text{Im} \langle \omega(z) \rangle \geq 0, \quad (3.31)
$$
where \( \eta(t, z) = \begin{pmatrix} \eta^{11}(t, z) & \eta^{12}(t, z) \\ \eta^{21}(t, z) & \eta^{22}(t, z) \end{pmatrix} \) is the unique solution of

\[
\eta(t, z) - i\omega(z)[\Sigma_3, \eta(t, z)] = V_0(t, z)\eta(t, z),
\]

\[
\lim_{t \to \infty} \eta(t, z) = I_{2N}.
\]

(3.32)

The main result for the AL equation on the graph \( \mathcal{G} \) is the following:

**Theorem 1.** Let spectral functions \( \{a(z), b(z), \tilde{a}(z), \tilde{b}(z)\} \), corresponding to \( \{Q_0(n), P_0(n)\} \), be defined according to definition 1. Suppose that \( \{G_i(t), H_i(t)\} \), \( l = -1, 0 \) exist, such that the spectral functions \( \{A(z), B(z), \tilde{A}(z), \tilde{B}(z)\} \) defined by definition 2, satisfy the global relation (3.24). Define \( M(n, t, z) \) as the solution of the following \( 2N \times 2N \) matrix RH problem:

- \( M(n, t, z) \) is sectionally meromorphic for \( z \in \mathbb{C} \setminus L \), where \( L = L_1 \cup L_2 \cup L_3 \cup L_4 \), and \( \{L_j\}_4 \) are defined by (3.18); see figure 2 for these contours.

- \( M(n, t, z) = M_n(n, t, z)J(n, t, z) \quad \text{for} \quad z \in L, \)

\[
M(n, t, z) = \begin{pmatrix} O_{N \times N}(z^{-1}, \text{even}) & O_{N \times N}(z, \text{odd}) \\ O_{N \times N}(z^{-1}, \text{odd}) & O_{N \times N}(z^2, \text{even}) \end{pmatrix}, \quad z \to (\infty, 0).
\]

(3.34)

- \( M(n, t, z) \) satisfies residue conditions (3.20).

Then \( M(n, t, z) \) exists and is unique. Furthermore, let

\[
Q(n, t) = \lim_{z \to 0}(z^{-1}M(n + 1, t, z))^{12}, \quad P(n, t) = \lim_{z \to \infty}(zM(n + 1, t, z))^{21}
\]

(3.35)

then \( Q(n, t), P(n, t) \) solves the matrix AL system (2.6) as well as satisfies initial-boundary value conditions (2.7).

**Proof.** Similarly with the problem in the continuum case [37], we transform the proof of this theorem in the matrix case to the proof of N-copies of the analogous theorem in the scalar case. We consider the isomorphism \( \theta \) defined by (2.11). For this \( \theta \), we denote by \( \{M^1(n, t, z), \ldots, M^N(n, t, z)\} \) the preimage of \( M(n, t, z) \). Then each \( M^\alpha(n, t, z) \) is defined as the solution of the \( 2 \times 2 \) RH problem that takes the same form as the above one but formulated by the spectral functions \( \{a^\alpha(z), b^\alpha(z), \tilde{a}^\alpha(z), \tilde{b}^\alpha(z)\} \) corresponding to \( \{q^\alpha_0(n), p^\alpha_0(n)\} \), and by the spectral functions \( \{A^\alpha(z), B^\alpha(z), \tilde{A}^\alpha(z), \tilde{B}^\alpha(z)\} \) corresponding to \( \{g^\alpha_1(t), g^\alpha_0(t), h^\alpha_1(t), h^\alpha_0(t)\} \). Note that the formula (3.35) is equivalent to setting \( q^\alpha(n, t) = \lim_{z \to 0}(z^{-1}M^\alpha(n + 1, t, z))^{12} \), \( p^\alpha(n, t) = \lim_{z \to \infty}(zM^\alpha(n + 1, t, z))^{21} \) for \( \alpha = 1, \ldots, N \). For the latter scalar case, it follows from [30] that each \( M^\alpha(n, t, z) \) exists uniquely, and \( q^\alpha(n, t), p^\alpha(n, t) \) solves the AL equation with the initial-boundary conditions:

- \( q^\alpha(n, 0) = q^\alpha_0(n) \), \( q^\alpha(-1, t) = g^\alpha_1(t) \), \( q^\alpha(0, t) = g^\alpha_0(t) \), \( p^\alpha(-1, t) = h^\alpha_1(t) \), \( p^\alpha(0, t) = h^\alpha_0(t) \).

Hence, \( Q(n, t), P(n, t) \) defined by (3.35), solves the AL equation (2.6) and satisfies the initial-boundary conditions (2.7).
3.3. Analysis of the global relation

For the matrix version of AL equation (2.6), the matrix-valued unknown boundary values \( G_\alpha(t) \), \( H_\alpha(t) \) can be characterized as follows.

**Proposition 2.** Denote by \( \partial D^\pm_{\text{in}} \) and \( \partial D^\pm_{\text{out}} \) the oriented boundaries of \( D^\pm_{\text{in}} \) and \( D^\pm_{\text{out}} \), such that \( D^\pm_{\text{in}} \) and \( D^\pm_{\text{out}} \) lie in the left-hand side of the increasing direction. Let \( \varphi(t, z) \) be defined by (3.29); let \( \varphi^{11}(t, z) \), \( \varphi^{12}(t, z) \), \( \varphi^{21}(t, z) \), \( \varphi^{22}(t, z) \) be the blocks (11), (12), (21), (22) in the natural decomposition of \( \varphi(t, z) \). The unknown boundary values \( G_\alpha(t) \), \( H_\alpha(t) \) associated with the AL equation (1.1) are given by

\[
\begin{align*}
G_0(t) + G_{-1}(t) &= \frac{1}{\pi i} \left[ \int_{\partial D^+_{\text{in}}} \psi(t, z) dz + \mathcal{E}_1(t) \int_{\partial D^+_{\text{out}}} e^{2i\omega(z) b(z)} a^{-1}(z) \varphi^{11}(t, z) dz \right], \\
H_0(t) + H_{-1}(t) &= \frac{1}{\pi i} \left[ \int_{\partial D^-_{\text{in}}} \tilde{\psi}(t, z) dz + \mathcal{E}_1(t) \int_{\partial D^-_{\text{out}}} e^{-2i\omega(z) \tilde{b}(z)} a^{-1}(z^{-1}) \varphi^{22}(t, z^{-1}) dz \right],
\end{align*}
\]

where

\[
\psi(t, z) = z^{-2} \left( \mathcal{E}_1^{-1}(t) \varphi^{12}(t, z) - \mathcal{E}_1(t) \varphi^{12}(t, z^{-1}) \right),
\]

\[
\tilde{\psi}(t, z) = z^{-2} \left( \mathcal{E}_1^{-1}(t) \varphi^{21}(t, z) - \mathcal{E}_1(t) \varphi^{21}(t, z^{-1}) \right),
\]

\[
\mathcal{E}_1(t) = \text{diag} \left( e^\frac{1}{2} \hat{L}^\alpha(\varphi_0^0(\xi)), e^{-\frac{1}{2} \hat{L}^\alpha(\varphi_0^0(\xi))}, \ldots, e^\frac{1}{2} \hat{L}^\alpha(\varphi_N^0(\xi)), e^{-\frac{1}{2} \hat{L}^\alpha(\varphi_N^0(\xi))} \right).
\]

**Proof.** By employing lemma 1 and proposition 1, we can map the proof of the formulae (3.36) for \( G_0(t) \), \( H_0(t) \) in matrix version to the proof of the analogous formulae for each \( g^\alpha_0(t) \), \( h^\alpha_0(t) \), \( \alpha = 1, \ldots, N \), in scalar version. For \( \alpha = 1, \ldots, N \), the analogous formulae for each \( g^\alpha_0(t) \), \( h^\alpha_0(t) \) in scalar version can be derived by the analysis of the global relation and by the use of certain asymptotic considerations of the eigenfunctions; for details see section 5.2 of [30].

4. Comparison with previous results

In the above two sections, by an algebraic formulation and then by implementing the UTM to the resulting matrix IBV problem, we have presented a framework to analyze the problem of the AL system (1.1) on the graph \( G \). For the scalar case \( N = 1 \), we immediately recover the results for the problem of AL system on the set of non-negative integers [30]. For \( N \geq 2 \), as mentioned earlier, for a genuine graph, the \( N \) edges are not separated, instead, they are connected to each other through appropriate boundary conditions. It is the goal of this section to discuss some nontrivial connections on examples and show how some previously known studies fit within our framework.

4.1. Recovering the problem on the integers

In this subsection, we show that the standard ISM for the AL system on the set of integers can be recovered as a special case of UTM applied to our \( N = 2 \) graph problem with the two edges of the graph connected through the following boundary conditions

\[
\begin{align*}
G_0(t) &= \sigma G_{-1}(t) \sigma, \\
H_0(t) &= \sigma H_{-1}(t) \sigma,
\end{align*}
\]

(4.1)
where

\[
\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad G_i(t) = \begin{pmatrix} g^1_i(t) & 0 \\ 0 & g^2_i(t) \end{pmatrix}, \quad H_i(t) = \begin{pmatrix} h^1_i(t) & 0 \\ 0 & h^2_i(t) \end{pmatrix}, \quad l = -1, 0. \tag{4.2}
\]

### 4.1.1. ISM on the integers.

In order to compare the ISM on the integers with our \( N = 2 \) matrix UTM on the non-negative integers, it is very convenient to put the two problems in the same size. More precisely, we will use a redundant \( 4 \times 4 \) Lax pair formulation instead of the standard \( 2 \times 2 \) one for the implementation of ISM for the AL equation.

We consider the following \( 4 \times 4 \) Lax pair formulation

\[
F^{\text{int}}(n, t)\mu^{\text{int}}(n + 1, t, z) - Z\mu^{\text{int}}(n, t, z) = U^{\text{int}}(n, t)\mu^{\text{int}}(n, t, z)Z^{-1}, \tag{4.3a}
\]

\[
\mu^{\text{int}}_z(n, t, z) - i\omega(z)[\Sigma_3, \mu^{\text{int}}(n, t, z)] = V^{\text{int}}(n, t, z)\mu^{\text{int}}(n, t, z), \tag{4.3b}
\]

where \( \mu^{\text{int}}(n, t, z) \) is a \( 4 \times 4 \) matrix, and

\[
F^{\text{int}}(n, t) = \text{diag} \left( f(n, t), f(-n - 1, t), f(n, t), f(-n - 1, t) \right),
\]

\[
Z = \begin{pmatrix} zI_2 & 0 \\ 0 & z^{-1}I_2 \end{pmatrix}, \quad \Sigma_3 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}.
\]

\[
U^{\text{int}}(n, t) = \begin{pmatrix} 0 & Q^{\text{int}}(n, t) \\ p^{\text{int}}(n, t) & 0 \end{pmatrix},
\]

\[
V^{\text{int}}(n, t, z) = \frac{1}{2} \left( U^{\text{int}}(n - 1, t)Z - U^{\text{int}}(n, t)Z^{-1} - U^{\text{int}}(n, t)U^{\text{int}}(n - 1, t) - U^{\text{int}}(n - 1, t)U^{\text{int}}(n, t) \right) \Sigma_3, \tag{4.4}
\]

with

\[
f(n, t) = \sqrt{1 - q(n, t)p(n, t)},
\]

\[
Q^{\text{int}}(n, t) = \begin{pmatrix} q(n, t) & 0 \\ 0 & q(-n - 1, t) \end{pmatrix},
\]

\[
p^{\text{int}}(n, t) = \begin{pmatrix} p(n, t) & 0 \\ 0 & p(-n - 1, t) \end{pmatrix}. \tag{4.5}
\]

The compatibility condition of (4.3) gives rise to the following \( 2 \times 2 \) matrix AL equation

\[
i\frac{dQ^{\text{int}}_n}{dt} + Q^{\text{int}}_{n+1} - 2Q^{\text{int}}_n + Q^{\text{int}}_{n-1} - \frac{Q^{\text{int}}_{n+1} + Q^{\text{int}}_{n-1}}{2} = 0,
\]

\[
i\frac{dP^{\text{int}}_n}{dt} - P^{\text{int}}_{n+1} + 2P^{\text{int}}_n - P^{\text{int}}_{n-1} + \frac{P^{\text{int}}_{n+1} + P^{\text{int}}_{n-1}}{2} = 0, \tag{4.6}
\]

where \( Q^{\text{int}}_n = Q^{\text{int}}(n, t) \) and \( P^{\text{int}}_n = P^{\text{int}}(n, t) \). It is clear that reconstructing \( q(n, t), p(n, t) \) is equivalent to reconstructing \( Q^{\text{int}}(n, t), P^{\text{int}}(n, t) \). The standard ISM for AL equation in the case of \( 2 \times 2 \) Lax pair formulation [30, 35] can be easily generalized to the above (redundant) \( 4 \times 4 \) case. Here we only present the essential results; for details see [30, 35].

We need to define two particular solutions of equation (4.3a) with normalizations as \( n \to \mp \infty \). They are given by the following summation equations:
Furthermore, we define
\[
\mu_{\text{int}}^-(n, t, z) = C_{\text{int}}(n, t) \left( (C_{\text{int}}(-\infty))^{-1} + Z^{-1} \sum_{m=-\infty}^{n-1} (C_{\text{int}}(m, t))^{-1} Z^{m-m} (Q_{\text{int}}(m, t) \mu_{\text{int}}^-(m, t, z)) \right),
\]
(4.7a)
\[
\mu_{\text{int}}^+(n, t, z) = C_{\text{int}}(n, t) \left( I_4 - Z^{-1} \sum_{m=n}^{\infty} (C_{\text{int}}(m, t))^{-1} Z^{m-m} (Q_{\text{int}}(m, t) \mu_{\text{int}}^+(m, t, z)) \right).
\]
(4.7b)

These two solutions are related:
\[
\mu_{\text{int}}^+(n, t, z) = \mu_{\text{int}}^-(n, t, z) Z^n e^{i\omega(z)\Sigma_1} s_{\text{int}}(z), \quad |z| = 1,
\]
(4.8)
where
\[
s_{\text{int}}(z) = \begin{pmatrix} \tilde{a}_{\text{int}}(z) & b_{\text{int}}(z) \\ \tilde{b}_{\text{int}}(z) & a_{\text{int}}(z) \end{pmatrix},
\]
(4.9)
with \(a_{\text{int}}(z), b_{\text{int}}(z), \tilde{a}_{\text{int}}(z), \tilde{b}_{\text{int}}(z)\) being 2 \times 2 diagonal-matrices. Evaluating equation (4.8) as \(n \to -\infty\) and at \(t = 0\), we find
\[
s_{\text{int}}(z) = \lim_{n \to -\infty} \tilde{Z}^{-n} \mu_{\text{int}}^+(n, 0, z).
\]
(4.10)

Define \(M_{\text{int}}^+(n, t, z)\) as the unique solution of the following RH problem:

\[ \item M_{\text{int}}^+(n, t, z) \text{ is sectionally meromorphic for } |z| > 1 \text{ and } |z| < 1. \]
\[ \item M_{\text{int}}^+(n, t, z) = M_{\text{int}}^+(n, t, z) J_{\text{int}}(n, t, z), \quad |z| = 1,
\]
(4.11)

where \(M_{\text{int}}^+(n, t, z) = M_{\text{int}}^+(n, t, z)\) for \(|z| \leq 1\), \(M_{\text{int}}^+(n, t, z) = M_{\text{int}}^+(n, t, z)\) for \(|z| > 1\), and \(J_{\text{int}}(n, t, z)\) is defined via the spectral functions \(\{a_{\text{int}}(z), b_{\text{int}}(z), \tilde{a}_{\text{int}}(z), \tilde{b}_{\text{int}}(z)\}\) by
\[
J_{\text{int}}(n, t, z) = Z^n e^{i\omega(z)\Sigma_1} \begin{pmatrix} I_2 & -\gamma_{\text{int}}(z) \\ \tilde{\gamma}_{\text{int}}(z) & I_2 + \gamma_{\text{int}}(z)\tilde{\gamma}_{\text{int}}(z) \end{pmatrix},
\]
(4.12)
with
\[
\gamma_{\text{int}}(z) = b_{\text{int}}(z) (\tilde{a}_{\text{int}}(z))^{-1}, \quad \tilde{\gamma}_{\text{int}}(z) = \tilde{b}_{\text{int}}(z) (a_{\text{int}}(z))^{-1}.
\]
(4.13)

\[ \item M_{\text{int}}^+(n, t, z) = I_4 + \begin{pmatrix} O_{2 \times 2}(z^{-2}, \text{even}) & O_{2 \times 2}(z, \text{odd}) \\ O_{2 \times 2}(z^{-1}, \text{odd}) & O_{2 \times 2}(z^2, \text{even}) \end{pmatrix}, \quad z \to (\infty, 0).
\]
(4.14)

\[ \item M_{\text{int}}^+(n, t, z) \text{ satisfies appropriate residue conditions}^1. \]

Furthermore, we define \(Q_{\text{int}}(n, t), P_{\text{int}}(n, t)\) by
\[
Q_{\text{int}}(n, t) = \lim_{z \to 0} (z^{1/2} M_{\text{int}}^+(n + 1, t, z))^{1/2}, \quad P_{\text{int}}(n, t) = \lim_{z \to \infty} (zM_{\text{int}}(n + 1, t, z))^{1/2}.
\]
(4.15)

Then \(Q_{\text{int}}(n, t), P_{\text{int}}(n, t)\) solves the AL system (4.6) with the initial condition

\[ \text{For conciseness, here we do not discuss details on the poles of } M_{\text{int}}^+(n, t, z) \text{ and the associated residue conditions; one can refer to section 3 for details on such results for the AL system on a general graph.} \]
\[ Q^{\text{int}}(n,0) = \begin{pmatrix} q_0(n) & 0 \\ 0 & q_0(-n-1) \end{pmatrix}, \]
\[ P^{\text{int}}(n,0) = \begin{pmatrix} p_0(n) & 0 \\ 0 & p_0(-n-1) \end{pmatrix}. \] (4.16)

4.1.2. ISM as a special case of UTM. Equipped with the above results, we are now able to demonstrate that the problem on the set of integers is a special case of our \( N = 2 \) graph problem. We choose the initial data of the \( N = 2 \) graph problem as
\[ q_1^0(n) = q_0(n), \quad q_2^0(n) = q_0(-n-1), \quad n \geq 0, \]
\[ p_1^0(n) = p_0(n), \quad p_2^0(n) = p_0(-n-1), \quad n \geq 0, \] (4.17)
and assume the two edges of the graph are connected via boundary conditions in the form of (4.1). The point is that we must eliminate involvement of boundary conditions in the reconstruction formula in our \( N = 2 \) graph case, since only the initial condition plays a role in the reconstruction formula in the case of ISM.

**Lemma 2.** In the case of initial-boundary conditions satisfying (4.1) and (4.17), the associated spectral matrices \( s(z), S(z) \) and \( s^{\text{int}}(z) \) satisfy the following relations
\[ s(z) = \Sigma s(\frac{1}{z}) \Sigma s^{\text{int}}(z), \] (4.18a)
\[ S(z) = \Sigma S(\frac{1}{z}) \Sigma, \] (4.18b)
where
\[ \Sigma = \sigma_3 \otimes \sigma, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \] (4.19)

**Proof.** Using symmetry relations
\[ U^{\text{int}}(n-1,t) = -\Sigma U^{\text{int}}(-n,t) \Sigma, \]
\[ F^{\text{int}}(n-1,t) = \Sigma F^{\text{int}}(-n,t) \Sigma, \] (4.20)
we can deduce that, if \( \mu^{\text{int}}(n,t,z) \) solves (4.3a), then so does \( \Sigma \mu^{\text{int}}(-n,t,\frac{1}{z}) \Sigma \). This aspect and the uniqueness of normalized solutions implies that
\[ \mu^{\text{int}}_-(n,t,z) = \Sigma \mu^{\text{int}}_+(-n,t,\frac{1}{z}) \Sigma. \] (4.21)

Using (4.21) in (4.8), we find
\[ \mu^{\text{int}}_+(n,t,z) = \Sigma \mu^{\text{int}}_+(n,t,\frac{1}{z}) \Sigma Z^{n} e^{-\omega(z)t} \Sigma s^{\text{int}}(z), \quad |z| = 1. \] (4.22)

For \( n \geq 0 \), using (4.17) we find that \( \mu_2(n,0,z) \) and \( \mu^{\text{int}}_+(n,0,z) \) satisfy the same equation and have the same normalization as \( n \to \infty \). Thus, for \( n \geq 0 \), we have
\[ \mu_2(n, 0, z) = \mu_n^{\text{int}}(n, 0, z). \]

Evaluating equation (4.23) at \( n = 0 \) and using \( \mu_2(0, 0, z) = s(z) \), we obtain
\[ s(z) = \mu_n^{\text{int}}(0, 0, z). \] (4.24)

Evaluating equation (4.22) at \( n = t = 0 \) and using (4.24), we find (4.18a). We now turn to the proof of the symmetry (4.18a). Using (4.1) we find that \( \mu_1(0, t, z) \) and \( \Sigma \mu_1(0, t, \frac{1}{z}) \Sigma \) satisfy the same equation and have the same normalization as \( t \to 0 \). Thus, we have
\[ \mu_1(0, t, z) = \Sigma \mu_1(0, t, \frac{1}{z}) \Sigma. \] (4.25)

Similarly, we find
\[ \mu_3(0, t, z) = \Sigma \mu_3(0, t, \frac{1}{z}) \Sigma. \] (4.26)

Evaluating at \( n = 0 \) for the following formula
\[ \mu_3(n, t, z) = \mu_1(n, t, z) \tilde{Z}\tilde{e}^{\text{int}}(z) e_{n, t} S(z), \] (4.27)
we obtain
\[ \mu_3(0, t, z) = \mu_1(0, t, z) \tilde{e}^{\text{int}}(z) e_{n, t} S(z). \] (4.28)

Letting \( z \to \frac{1}{z} \) in (4.28) and using the symmetries (4.25) and (4.26), we obtain
\[ \mu_3(0, t, z) = \mu_1(0, t, z) \tilde{e}^{\text{int}}(z) e_{n, t} S(z). \] (4.29)

Equations (4.28) and (4.29) yield (4.18a). \( \square \)

**Theorem 2.** Consider the RH problem defined in theorem 1 subject to \( N = 2 \) and particular initial-boundary values satisfying (4.1) and (4.17). Denote by \( M_{\text{red}}(n, t, z) \) the unique solution of this RH problem and let \( Q_{\text{red}}(n, t) \), \( P_{\text{red}}(n, t) \) be defined by (3.35) but with \( M(n, t, z) \) replaced by \( M_{\text{red}}(n, t, z) \). Then
\[ Q_{\text{int}}(n, t) = Q_{\text{red}}(n, t), \quad P_{\text{int}}(n, t) = P_{\text{red}}(n, t), \quad n \in \mathbb{N}_0. \] (4.30)

**Proof.** Define
\[
\tilde{M}_{\text{red}}(n, t, z) = \begin{cases} 
M_{\text{red}}(n, t, z), & z \in D_{-\text{in}} \cup D_{+\text{out}}, \\
M_{\text{red}}(n, t, z)J_1(n, t, z), & z \in D_{+\text{in}}, \\
M_{\text{red}}(n, t, z)J_3^{-1}(n, t, z), & z \in D_{-\text{out}}.
\end{cases}
\] (4.31)

We find that \( \tilde{M}_{\text{red}}(n, t, z) \) only has a jump across the unit circle:
\[ \tilde{M}_{\text{red}}(n, t, z) = \tilde{M}_{+} J_2(n, t, z), \quad |z| = 1, \] (4.32)
where

\[
J_2(n, t, z) = \tilde{Z}^0 e^{i\omega(z) \tilde{E}_1} \left( I_2 \begin{pmatrix} I_2 & \tilde{\Gamma}(z) - \gamma(z) \\ \gamma(z) - \Gamma(z) & I_2 - (\gamma(z) - \Gamma(z)) (\gamma(z) - \tilde{\Gamma}(z)) \end{pmatrix} \right).
\]

(4.33)

We can deduce that

\[
\tilde{\gamma}(z) - \Gamma(z) = \tilde{\gamma}^{int}(z),
\]

(4.34a)

\[
\gamma(z) - \tilde{\Gamma}(z) = \gamma^{int}(z).
\]

(4.34b)

Indeed, the symmetry (4.18a) implies \(\tilde{B}(z)\tilde{A}^{-1}(z) = -\sigma\tilde{B}(\frac{1}{z})\tilde{A}^{-1}(\frac{1}{z})\). Using the global relation (3.25), we obtain \(B(z)A^{-1}(z) = -\sigma b(\frac{1}{z})\tilde{a}^{-1}(\frac{1}{z})\sigma\). Using this relation we can write

\[
\Gamma(z) = -a^{-1}(z)\sigma \tilde{b}(\frac{1}{z})\sigma \left( a(z)\sigma a(\frac{1}{z})\sigma + b(z)\sigma b(\frac{1}{z})\sigma \right)^{-1}.
\]

(4.35)

Using (4.35) and the identity \(a(z)\tilde{a}(z) - b(z)\tilde{b}(z) = I_2\), we can write

\[
\tilde{\gamma}(z) - \Gamma(z) = \left( \tilde{b}(z)\sigma \tilde{a}(\frac{1}{z})\sigma + a(z)\sigma \tilde{b}(\frac{1}{z})\sigma \right) \left( a(z)\sigma a(\frac{1}{z})\sigma + b(z)\sigma b(\frac{1}{z})\sigma \right)^{-1}.
\]

(4.36)

The symmetry (4.18a) implies

\[
\tilde{b}(z)\sigma \tilde{a}(\frac{1}{z})\sigma + a(z)\sigma \tilde{b}(\frac{1}{z})\sigma = \tilde{b}^{int}(z),
\]

(4.37)

\[
a(z)\sigma \tilde{a}(\frac{1}{z})\sigma + b(z)\sigma \tilde{b}(\frac{1}{z})\sigma = a^{int}(z).
\]

Inserting (4.37) into (4.36), we find (4.34a). Similar calculations yield (4.34b). It follows from (4.34) that

\[
J_2(n, t, z) = J^{int}(n, t, z).
\]

(4.38)

Moreover, (4.31) implies that the normalization (4.14) also holds for \(\tilde{M}^{red}(n, t, z)\). Therefore, \(\tilde{M}^{red}(n, t, z)\) and \(M^{int}(n, t, z)\) exactly satisfy the same RH problem. Thus

\[
\tilde{M}^{red}(n, t, z) = M^{int}(n, t, z), \quad n \in \mathbb{N}_0.
\]

(4.39)

Finally, using

\[
\lim_{z \to 0} (z^{-1} \tilde{M}^{red}(n, t, z))^{12} = \lim_{z \to 0} (z^{-1} M^{red}(n, t, z))^{12},
\]

\[
\lim_{z \to \infty} (z \tilde{M}^{red}(n, t, z))^{21} = \lim_{z \to \infty} (z M^{red}(n, t, z))^{21},
\]

(4.40)

we find (4.30).

The formula (4.30) implies that the problem on the set of integers can be recovered from our \(N = 2\) matrix problem on the set of non-negative integers as a special case by the following very intuitive formula.
\[ q(n,t) = \theta(n)q_1(n,t) + \theta(-n-1)q_2(-n-1,t), \quad n \in \mathbb{Z}, \quad (4.41a) \]
\[ p(n,t) = \theta(n)p_1(n,t) + \theta(-n-1)p_2(-n-1,t), \quad n \in \mathbb{Z}, \quad (4.41b) \]

where \( \theta(n) \) is defined by the following Heaviside theta function
\[ \theta(n) = \begin{cases} 1, & n \geq 0, \\ 0, & n < 0. \end{cases} \]

**Remark 1.** As the problem in continuous case [38], in order to discuss the reduction of the problem on \( N = 2 \) graph to the problem on the line, we used a redundant \( 4 \times 4 \) Lax pair formulation instead of the standard \( 2 \times 2 \) one for the implementation of ISM for the AL equation. It seems that such a treatment is not the most concise way to approach the IST. However, this treatment enables us to show precisely that the ISM for an integrable DDE on the set of integers is just a special case of the (matrix) UTM on the set of non-negative integers. This result provides a converse of the following well-known statement: the UTM for an integrable DDE with linearizable boundary conditions can be formulated into a special case of the ISM on the set of integers [29, 32]. On the other hand, this treatment enables us to view the new non-local reductions of integrable DDEs as standard local reductions of our matrix DDEs and thus the ISM for the non-local DDEs fits naturally within our framework; see the following section 4.2 for details.

### 4.2. The IDNLS and non-local IDNLS equations as standard local reductions

In the case of the continuum problem, Caudrelier has established in [38] that both the classical NLS equation and the non-local NLS equation [40–42] can be obtained as standard local reductions of a matrix AKNS system with an appropriate boundary condition. Here we show this is also the case for the discrete problem: both the IDNLS equation [35] and the non-local IDNLS equation [43] can be obtained as standard local reductions of our matrix AL system on the non-negative integers with the boundary values satisfying (4.1). Thus the ISM for the IDNLS equation and especially for the non-local IDNLS equation fit naturally within our results on the graph problem.

#### 4.2.1. Reductions in the potentials

It can be checked directly that the matrix AL system (2.6) in the case \( N = 2 \) admits the following two local reductions between the potentials:
\[ P(n,t) = \nu Q_1(n,t), \quad \nu = \pm 1, \quad n \in \mathbb{N}_0, \quad (4.42a) \]
\[ P(n,t) = \nu \sigma Q_1(n,t) \sigma, \quad \nu = \pm 1, \quad n \in \mathbb{N}_0, \quad (4.42b) \]

where \( Q_1(n,t) \) denotes the Hermitian of \( Q(n,t) \), and \( \sigma \) is defined by (4.2). We note that, as in the continuum case [38], the reductions (4.42) admit an interpretation in the reduction group theory [39]: they arise as two different representations of a local \( \mathbb{Z}_2 \) reduction imposing on the \( 4 \times 4 \) Lax pair of the matrix AL equation (2.6) in the case of \( N = 2 \).

We first consider the reduction (4.42a). In this case, the matrix AL system (2.6) in the case \( N = 2 \) becomes
\[ i \frac{dQ_n}{dr} + Q_{n+1} - 2Q_n + Q_{n-1} - \nu Q_1(n) (Q_{n+1} + Q_{n-1}) = 0. \]
The above equation holds for \( n \in \mathbb{N}_0 \) and \( t > 0 \), since we deal with an IBV problem for the lattice equation on the non-negative integers. Note that we require that the reduction is compatible with the boundary condition (4.1), thus we can apply (4.41a), namely \( Q_n = \text{diag}(q_n, q_{-n-1}), n \in \mathbb{N}_0 \), to (4.43). This in turn yields

\[
\begin{align*}
\frac{d q_{n+1}}{d t} + 2 q_n + q_{n-1} - \nu |q_n|^2 (q_{n+1} + q_{n-1}) &= 0, \quad n \in \mathbb{N}_0, \quad (4.44a) \\
\frac{d q_{n-1}}{d t} + 2 q_{n-1} + q_{n} - \nu |q_{n-1}|^2 (q_{n-2} + q_{n}) &= 0, \quad n \in \mathbb{N}_0. \quad (4.44b)
\end{align*}
\]

Equations (4.44a) and (4.44b) can be combined into

\[
\frac{d q_n}{d t} + q_{n+1} - 2 q_n + q_{n-1} - \nu |q_n|^2 (q_{n+1} + q_{n-1}) = 0, \quad n \in \mathbb{Z}, \quad (4.45)
\]

which is nothing but the IDNLS equation [35] on the integers. Next we consider the reduction (4.42b). In this case, the matrix AL system (2.6) in the case \( N = 2 \) becomes

\[
\frac{d Q_{n+1}}{d t} + 2 Q_n + Q_{n-1} - \nu \sigma Q_n^* \sigma Q_n (Q_{n+1} + Q_{n-1}) = 0, \quad (4.46)
\]

which holds for \( n \in \mathbb{N}_0 \) and \( t > 0 \) as before. By applying (4.41a), namely \( Q_n = \text{diag}(q_n, q_{-n-1}), n \in \mathbb{N}_0 \), to (4.46), we obtain

\[
\begin{align*}
\frac{d q_{n+1}}{d t} + q_{n+1} - 2 q_n + q_{n-1} - \nu q_n q_{n-1}^* (q_{n+1} + q_{n-1}) &= 0, \quad n \in \mathbb{N}_0, \quad (4.47a) \\
\frac{d q_{n-1}}{d t} + q_{n-1} - 2 q_{n-1} + q_{n} - \nu q_{n-1} q_n^* (q_{n-2} + q_{n}) &= 0, \quad n \in \mathbb{N}_0. \quad (4.47b)
\end{align*}
\]

Equations (4.47a) and (4.47b) can be combined into the following non-local IDNLS equation on the integers:

\[
\frac{d q_n}{d t} + q_{n+1} - 2 q_n + q_{n-1} - \nu q_n q_{n-1}^* (q_{n+1} + q_{n-1}) = 0, \quad n \in \mathbb{Z}. \quad (4.48)
\]

In summary, we find

**Proposition 3.** The IDNLS equation (4.45) and the non-local IDNLS equation (4.48) can be obtained respectively as standard local reductions (4.42a) and (4.42b) of the matrix AL system (2.6) in the case of \( N = 2 \) and the boundary condition satisfying (4.1).

**Remark 2.** The non-local IDNLS equation (4.48) is slightly different from the following non-local IDNLS equation presented in [43]:

\[
\frac{d q_n}{d t} + q_{n+1} - 2 q_n + q_{n-1} - \nu q_n q_{n-1}^* (q_{n+1} + q_{n-1}) = 0. \quad (4.49)
\]

The non-local term appearing in the nonlinear terms of our equation (4.48) is \( q^*(-n - 1, t) \), while the non-local term appearing in the nonlinear terms of equation (4.49) is \( q^*(-n, t) \). We note that, in general, the non-local term can be taken as \( q^*(-n - n_0, t) \), and the resulting non-local IDNLS equation reads...
\[ i \frac{dq_n}{dt} + q_{n+1} - 2q_n + q_{n-1} - \nu q_n q_{-n-n_0} (q_{n+1} + q_{n-1}) = 0, \tag{4.50} \]

where \( n_0 \) is an arbitrary fixed integer. Indeed, the AL system (1.1) admits a general non-local reduction \( p(n, t) = \nu q^*(-n-n_0, t) \), \( \nu = \pm 1 \). Using this reduction, we immediately find the above general non-local IDNLS equation (4.50).

4.2.2. Reduction symmetries in the spectral functions. The reductions in the potentials induce important symmetries in the spectral functions. For the non-local IDNLS equation, it was shown in [43] that the resulting symmetries of the spectral functions are very different from those of classical IDNLS equation. We will show in the following that these two different symmetries associated with the two equations also appear naturally from the reductions (4.42) imposed on the matrix AL system (2.6) in the case \( N = 2 \).

We first write the symmetry reductions (4.42) as
\[ U(n, t) = -DU^\dagger(n, t)D, \tag{4.51} \]
where, for the IDNLS equation (4.45),
\[ D = \begin{pmatrix} -\nu I_2 & 0 \\ 0 & I_2 \end{pmatrix}, \tag{4.52} \]
while for the non-local IDNLS equation (4.48),
\[ D = \begin{pmatrix} -\nu \sigma & 0 \\ 0 & \sigma \end{pmatrix}. \tag{4.53} \]

**Lemma 3.** Under the reduction (4.51), the associated spectral matrices \( s(z) \) and \( S(z) \) satisfy symmetry relations:
\[ s^{-1}(z) = Ds^\dagger(\frac{1}{z^*})D, \tag{4.54a} \]
\[ S^{-1}(z) = DS^\dagger(\frac{1}{z^*})D. \tag{4.54b} \]

**Proof.** It can be checked directly that under the reduction (4.51), if \( \mu(n, t, z) \) solves (2.9) in the case \( N = 2 \), then \( \mu^{-1}(n, t, z) \) and \( D\mu^\dagger(n, t, \frac{1}{z^*})D \) satisfy the same equations
\[ \Phi(n + 1, t, z)F(n, t) - \hat{Z}\Phi(n, t, z) = -Z\Phi(n, t, z)U(n, t), \tag{4.55a} \]
\[ \Phi_t(n, t, z) - i\omega(z)[\Sigma_3, \Phi(n, t, z)] = -\Phi(n, t, z)V(n, t, z). \tag{4.55b} \]
Applying this to \( \mu_2(n, 0, z) \) and using the uniqueness of a normalized solution, we find
\[ \mu_2^{-1}(n, 0, z) = D\mu_2^\dagger(n, 0, \frac{1}{z^*})D. \tag{4.56} \]
Applying this to \( \mu_3(0, t, z) \) and using the uniqueness of a normalized solution, we find
\[ \mu_3^{-1}(0, t, z) = D\mu_3^\dagger(0, t, \frac{1}{z^*})D. \tag{4.57} \]
Evaluating equation (4.56) at \( n = 0 \), we find (4.54). Evaluating equation (4.57) at \( t = 0 \), we find (4.54).

**Proposition 4.** The matrix of spectral functions of the IDNLS equation (4.45) is of the generic form

\[
\mathbf{s}_{\text{IDNLS}}(z) = \begin{pmatrix}
(\alpha(z))^* & \beta(z) \\
\nu(\beta(z)) & \alpha(z)
\end{pmatrix},
\]

(4.58)

The matrix of spectral functions of the non-local IDNLS equation (4.48) is of the generic form

\[
\mathbf{s}_{\text{NIDNLS}}(z) = \begin{pmatrix}
\tilde{\alpha}(z) & \beta(z) \\
\nu(\beta(z)) & \alpha(z)
\end{pmatrix},
\]

(4.59)

where \( \alpha(z) \) and \( \tilde{\alpha}(z) \) satisfy extra symmetries

\[
\alpha(z) = (\alpha(z^*))^*, \\
\tilde{\alpha}(z) = (\tilde{\alpha}(z^*))^*.
\]

(4.60)

**Proof.** From (4.18a), we find

\[
(\mathbf{s}^{\text{int}}(z))^{-1} = \Sigma \mathbf{s}^{\text{int}}(z) \Sigma.
\]

(4.61)

The formula (4.54) implies

\[
(\mathbf{s}^{\text{int}}(z))^{-1} = \mathbf{D} \left( \mathbf{s}^{\text{int}}(\frac{1}{z^*}) \right) \mathbf{D}.
\]

(4.62)

For \( \mathbf{D} \) being defined by (4.52), from (4.61) and (4.62) we find

\[
\mathbf{s}^{\text{int}}(z) = \begin{pmatrix}
(\alpha(z))^* & 0 & 0 & 0 \\
0 & \alpha(z) & \beta(z) & 0 \\
0 & \beta(z^*) & \alpha(z) & 0 \\
0 & 0 & 0 & (\alpha(z^*))^* \\
\end{pmatrix}.
\]

(4.63)

In view of the structure of (4.5), from the above redundant \( 4 \times 4 \) matrix we extract the following \( 2 \times 2 \) matrix for spectral functions of the IDNLS equation (4.45):

\[
\mathbf{s}_{\text{IDNLS}}(z) = \begin{pmatrix}
(\alpha(z))^* & \beta(z) \\
\nu(\beta(z)) & \alpha(z)
\end{pmatrix}.
\]

(4.64)

For \( \mathbf{D} \) being defined by (4.53), from (4.61) and (4.62) we find

\[
\mathbf{s}^{\text{int}}(z) = \begin{pmatrix}
\tilde{\alpha}(z) & 0 & 0 & 0 \\
0 & \tilde{\alpha}(z) & \beta(z) & 0 \\
0 & \beta(z^*) & \alpha(z) & 0 \\
0 & 0 & 0 & (\tilde{\alpha}(z^*))^* \\
\end{pmatrix}.
\]

(4.65)
with
\[ a_{\text{int}}(z) = (a_{\text{int}}(z^*))^*, \]
\[ \tilde{a}_{\text{int}}(z) = (\tilde{a}_{\text{int}}(z^*))^*. \]

(4.66)

In view of the structure of (4.5), from the above redundant $4 \times 4$ matrix we extract the following $2 \times 2$ matrix for spectral functions of the non-local IDNLS equation (4.48):
\[ s_{\text{IDNLS}}(z) = \begin{pmatrix} \tilde{a}_{\text{int}}(z) & b_{\text{int}}(z) \\ \nu (b_{\text{int}}(z^*))^* & a_{\text{int}}(z) \end{pmatrix}, \]

(4.67)

where $a_{\text{int}}(z)$ and $\tilde{a}_{\text{int}}(z)$ satisfy extra symmetries (4.60).

4.3. AL lattice system with an integrable defect

In classical integrable field theories, a defect for an integrable system in space can be viewed as an internal boundary condition on the fields and their time and space derivatives at a given point (defect location) [47–50]. A crucial observation was that the integrability of a defect system could survive if one defines defect boundary conditions for the system as Bäcklund transformations (BTs) frozen at the defect location [50]. Using this observation, a generating function for the defect contributions to the conserved quantities was explicitly constructed in [51] for integrable PDEs associated with the AKNS spectral problem, and furthermore the integrability of the defect system was studied in [52]. However, for the integrable DDEs, to our knowledge, the analogous result for the AL lattice system (1.1) has not been reported in the literature. We will present this important result in appendix.

Here we note that the problem on the line with a defect fits within the framework of the problem on a simple graph. Indeed, for the problem in the continuum case, the line with a defect at a fixed point can be seen as a graph with two half-lines attached to a vertex, and the associated defect condition can be interpreted as a boundary condition representing a connection between the two edges of the graph [37]. For the problem in the discrete case, by freezing at $n = 0$ the BT (A.9) of the AL system derived in appendix, we find the following defect boundary condition
\[ c_1 (E(t))^2 g_0^1(t) - c_2 g_0^2(t) = c_4 (E(t))^2 g_{-1}^2(t) - c_3 g_{-1}^1(t), \]
\[ c_1 (E(t))^2 h_{-1}^2(t) - c_2 h_{-1}^1(t) = c_4 (E(t))^2 h_0^2(t) - c_3 h_0^1(t), \]
\[ c_1 (E(t))^2 g_0^1(t) - c_2 g_0^2(t) \]
\[ = i \left\{ \left( c_2 + c_4 (E(t))^2 \right) (g_0^1(t) - g_{-1}^2(t)) - \left( c_3 + c_1 (E(t))^2 \right) \left( g_0^1(t) - g_{-1}^2(t) \right) \right\}, \]
\[ c_4 (E(t))^2 h_{0}^1(t) - c_3 h_{0}^2(t) \]
\[ = i \left\{ \left( c_3 + c_1 (E(t))^2 \right) \left( h_{-1}^1(t) - h_{0}^2(t) \right) + \left( c_2 + c_4 (E(t))^2 \right) \left( h_{0}^1(t) - h_{-1}^1(t) \right) \right\}, \]
\[ + c_4 (E(t))^2 h_{0}^1(t) \left( g_0^1(t) h_{1}^1(t) + g_{-1}^1(t) h_{0}^2(t) \right) - c_3 h_{0}^2(t) \left( h_{0}^1(t) g_{1}^1(t) + g_{-1}^1(t) h_{1}^2(t) \right). \]

(4.68)
where the dot denotes the derivative with respect to time, $c_1$, $c_2$, $c_3$ and $c_4$ are arbitrary constants, and

$$E(t) = \exp \left\{ -\frac{i}{2} \int_0^t \left( g_0^2(\tau)h_{-1}^2(\tau) - g_0(\tau)h_{-1}^1(\tau) - h_0^1(\tau)g_{-1}^2(\tau) + h_0(\tau)g_{-1}^1(\tau) \right) d\tau \right\}.$$  

In our present context, this defect boundary condition represents a nontrivial connection between two semi-infinite lattices of the $N = 2$ graph. In particular, for the IDNLS equation (4.45), the additional symmetry $p(n,t) = \nu q^*(n,t)$ reduces the above defect condition to

$$c_1 (E(t))^2 g_0(t) - c_2 g_0^3(t) = c_1^* (E(t))^2 g_{-1}^2(t) - c_2^* g_{-1}^1(t),$$

$$c_1 (E(t))^2 g_0^3(t) - c_2 g_0^3(t)$$

$$= i \left\{ \left( c_2 + c_1^* (E(t))^2 \right) (g_0^2(t) - g_{-1}^2(t)) - \left( c_2^* + c_1 (E(t))^2 \right) (g_0^1(t) - g_{-1}^1(t)) \right\}$$

$$- c_1 (E(t))^2 g_0^3(t) (g_{-1}^1(t)h_0^0(t) + g_0^1(t)h_{-1}^0(t)) + c_2 g_0^3(t) (g_0^1(t)h_{-1}^0(t) + h_0^0(t)g_{-1}^1(t)) \right\},$$

where $c_1$ and $c_2$ are arbitrary constants, and

$$E(t) = \exp \left\{ -\nu \int_0^t \mathrm{Im} \left( g_0^2(\tau) (g_{-1}^2(\tau))^* - g_0^1(\tau) (g_{-1}^1(\tau))^* \right) d\tau \right\}.$$  

The defect conditions (4.68) and (4.69) look very complicated, however they emerge naturally from the BT of AL system. This implies that specific soliton solutions may be constructed explicitly. For the problem in the continuous case, this was done for the NLS equation in [50] by direct ansatz on the one and two soliton solutions. The analogous issue for the present discrete case will be studied in the future. Here we only point out that, using the resulting solutions with $t = 0$ and $x = 0$ as the initial and boundary data, our approach enables us to exactly compute the spectral functions and to implement the inverse part of the method explicitly.

### 4.4. Linearizable boundary conditions for $N \geq 2$

For the problem in the scalar case, a particular class of boundary conditions called linearizable exists [19], in which case the unknown boundary values can be eliminated from the reconstruction formulae of the solutions. By generalizing the definition of linearizable boundary conditions from the scalar case [19, 30] to the present matrix case, we obtain that the boundary values $G_{-1}(t)$ and $G_0(t)$ and $H_{-1}(t)$ and $H_0(t)$ are linearizable if there exists a nonsingular $2N \times 2N$ matrix $K(z)$ such that

$$K(z) (i\omega(z)\Sigma_3 + V(0,t,z)) = \left( i\omega(z)\Sigma_3 + V(0,t,\frac{1}{z}) \right) K(z),$$

where $V(n,t,z)$ is given by (2.10). Let us consider the simplest case where $K(z)$ is independent of $z$. We write $K$ in the form

$$K = \begin{pmatrix} K^{11} & K^{12} \\ K^{21} & K^{22} \end{pmatrix},$$

where the blocks $K^j$ and $j,k = 1,2$ are $N \times N$ matrices. By substituting (4.71) into (4.70) and matching the coefficients of $z$ on both sides of the resulting equation, we find that $K^{12} = K^{21} = 0$ and the following boundary conditions
\( G_0(t)K^{22} + K^{11}G_{-1}(t) = 0, \quad G_{-1}(t)K^{22} + K^{11}G_0(t) = 0, \)
\( H_0(t)K^{11} + K^{22}H_{-1}(t) = 0, \quad H_{-1}(t)K^{11} + K^{22}H_0(t) = 0. \) \( (4.72) \)

For example, taking \( K^{11} = -K^{22} = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix}, \) \( (4.73) \)
we obtain the boundary conditions
\( g^0_j(t) = g^{N+1-j}_{-1}(t), \quad h^0_j(t) = h^{N+1-j}_{-1}(t), \quad j = 1, \cdots, N. \) \( (4.74) \)

In particular, the boundary conditions \( (4.1) \) correspond to \( (4.74) \) in the case \( N = 2. \)

We note that, for the linearizable boundary conditions, the associated spectral functions possess important symmetry properties. Using these symmetry properties together with the global relation, it is possible to eliminate the unknown boundary values from the reconstruction formulae of the solutions; see [19, 30] for details. For the present discrete case, the symmetry properties in the spectral functions are given by
\( K(z)S(z)e^{-i\omega(z)T\Sigma_3} = S(1/z)e^{-i\omega(z)T\Sigma_3}K(z), \) \( (4.75) \)
where \( S(z) \) is defined by \( (3.7) \) and \( (3.8) \). Indeed, the symmetry property \( (4.70) \) induces the following symmetry property for \( \mu_1(0, t, z) \) (recall that \( \mu_1(n, t, z) \) is defined by \( (3.1) \)):
\( K(z)\mu_1(0, t, z)e^{i\omega(z)T\Sigma_3} = \mu_1(0, t, z)e^{i\omega(z)T\Sigma_3}K(z). \) \( (4.76) \)
By using the definition of \( S(z) \) (see the formula \( (3.7b) \)), we obtain the symmetry properties \( (4.75) \).

5. Conclusions and discussions

We have presented an approach to analyze IBV problems for integrable DDEs on a graph that is composed of \( N \) semi-infinite lattices (edges). As in the continuous case, we first formulated the problem on the graph into a certain matrix IBV problem. Then we analyzed such a matrix IBV problem by extending the UTM for integrable DDEs in the scalar case to the one in the matrix case. We also discussed the boundary conditions describing nontrivial connections between the edges of the graph on some examples and demonstrated how some previously known studies fit within our framework. These boundary conditions include the linearizable boundary conditions and the integrable defect boundary conditions. In particular, for the linearizable boundary conditions in the case \( N = 2, \) we showed in detail that how our results reproduce the standard ISM on the set of integers as a particular case and how a non-local reduction of an integrable DDE can be obtained as a standard local reduction from our results.

We end this work with a brief discussion on the so-called Kirchhoff boundary conditions [3, 53, 54]. These kind of boundary conditions, in addition to the linearizable and the integrable defect boundary conditions, describe an interesting nontrivial connection between the edges of a graph. For the NLS equation in the continuous case, the Kirchhoff boundary conditions are described by
\[ q^1(0,t) = q^2(0,t) = \cdots = q^N(0,t), \]
\[ \sum_{j=1}^{N} \partial_t q^j(0,t) = 0. \]  
(5.1)

By a direct discretization of the boundary conditions (5.1), we guess that the analogous boundary conditions in the present discrete case are
\[ g_1^0(t) = g_2^0(t) = \cdots = g_N^0(t), \]
\[ h_1^0(t) = h_2^0(t) = \cdots = h_N^0(t), \]
\[ \sum_{j=1}^{N} \left( g_j^0(t) - g_{j-1}^0(t) \right) = 0, \]
\[ \sum_{j=1}^{N} \left( h_j^0(t) - h_{j-1}^0(t) \right) = 0. \]  
(5.2)

For the problem in the continuous case, some impressive results have been presented for the Kirchhoff boundary value problems. For example, the global well-posedness and the behaviour of solitary wave solution of the NLS equation with Kirchhoff boundary conditions on the \( N = 3 \) star-graph (called \( Y \)-junction in the literature) were studied recently in [3]. However, the extensions of the results in the continuous case to the present discrete case are still open problems. We hope that these topics will stimulate further work.

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Appendix. AL lattice system with an integrable defect

We now derive a defect condition for the AL system that can preserve the integrability of the system. This result provides a discrete analogue of the integrable defect condition for the integrable PDEs [51]. We first derive a BT for AL system. Then, by fixing the BT at \( n = 0 \), we obtain the desired integrable defect boundary conditions. Finally, we construct explicitly a generating function for the infinite number of conserved quantities for the defect AL system.

A.1. Bäcklund transformations for the AL system

Let \( q^1(n,t), p^1(n,t) \) and \( q^2(n,t), p^2(n,t) \) be two solutions of the AL system (1.1) for \( n \in \mathbb{Z} \); let \( \Phi^j(n,t,z) = \left( \Phi_1^j(n,t,z), \Phi_2^j(n,t,z) \right)^T, j = 1, 2 \), be the corresponding eigenfunctions that satisfy the following auxiliary problems, for \( j = 1, 2 \),
\[ \Phi^j(n+1,t,z) = \mathcal{W}_j(n,t,z)\Phi^j(n,t,z), \]  
(\( A.1a \))
\[ \Phi^j_t(n,t,z) = \mathcal{T}_j(n,t,z)\Phi^j(n,t,z), \]  
(\( A.1b \))
where
\[ \mathcal{W}_j(n,t,z) = \frac{1}{f^j(n,t)} (Z + U^j(n,t)). \]  
(\( A.2a \))
with $Z, f(n,t), U(n,t), V(n,t), j = 1, 2,$ being defined by (2.4). Suppose that $\Phi^1(n,t,z)$ and $\Phi^2(n,t,z)$ are related by gauge transformation

$$\Phi^2(n,t,z) = D(n,t,z)\Phi^1(n,t,z),$$

where $D(n,t,z)$ is a $2 \times 2$ matrix. The matrix $D(n,t,z)$ satisfies the following equations

$$D(n + 1,t,z)W^1(n,t,z) = W^2(n,t,z)D(n,t,z),$$

$$D_1(n,t,z) = T^2(n,t,z)D(n,t,z) - D(n,t,z)T^1(n,t,z).$$

We look for $D(n,t,z)$ in the form of

$$D(n,t,z) = zD_2(n,t) + \frac{1}{z} D_1(n,t) + D_0(n,t),$$

where the $2 \times 2$ matrices $D_0(n,t), D_1(n,t)$ and $D_2(n,t)$ are dependent on $q^j(n,t), p^j(n,t), j = 1, 2,$ but independent on $z$. By substituting (A.5) into (A.4) and equating the coefficients of powers of $z$, we find

**Lemma A.1.** Let a solution of (A.4a) be in the form of (A.5). Then

$$D_2 = \begin{pmatrix} d^{11}_2(n,t) & 0 \\ 0 & d^{22}_2(n,t) \end{pmatrix}, \quad D_1 = \begin{pmatrix} d^{11}_1(n,t) & 0 \\ 0 & d^{22}_1(n,t) \end{pmatrix}, \quad D_0 = \begin{pmatrix} 0 & d^{02}_0(n,t) \\ d^{10}_0(n,t) & 0 \end{pmatrix},$$

where, for all $n \in \mathbb{Z},$

$$d^{11}_2(n,t) = c_1E(t)\Delta(n,t), \quad d^{22}_2(n,t) = \frac{c_2}{E(t)\Delta(n,t)},$$

$$d^{11}_1(n,t) = \frac{c_3}{E(t)\Delta(n,t)}, \quad d^{22}_1(n,t) = c_4E(t)\Delta(n,t),$$

$$d^{02}_0(n,t) = d^{11}_2(n,t)q^1(n,t) - d^{22}_2(n,t)q^2(n,t) = d^{22}_1(n,t)q^1(n-1,t) - d^{11}_1(n,t)q^2(n-1,t),$$

$$d^{12}_1(n,t) = d^{11}_1(n,t)p^1(n-1,t) - d^{22}_2(n,t)p^1(n-1,t) = d^{22}_2(n,t)p^1(n,t) - d^{11}_1(n,t)p^2(n,t),$$

with $c_1, c_2, c_3$ and $c_4$ being arbitrary constants, $\Delta(0,t) = 1,$ and

$$\Delta(n,t) = \prod_{m=-n}^{n-1} \frac{1 - q^1(m,t)p^1(m,t)}{1 - q^2(m,t)p^2(m,t)}, \quad \Delta(-n,t) = \prod_{m=-n}^{-1} \frac{1 - q^2(m,t)p^2(m,t)}{1 - q^1(m,t)p^1(m,t)}, \quad n \geq 1,$$

$$E(t) = e^{(1/2)\int_{-\infty}^{t} \frac{q^1(\tau)q^2(\tau)p^1(-\tau)p^2(-\tau)-q^2(\tau)q^2(\tau)p^1(-\tau)p^2(-\tau)+p^1(\tau)p^2(\tau)q^2(-\tau)\,d\tau}.$$  

**Lemma A.2 (BT for the AL system).** Let $q^j(n,t), p^j(n,t)$ and $q^2(n,t), p^2(n,t)$ be two solutions of the AL system (1.1); let the corresponding eigenfunctions be related by gauge transformation (A.3) with $D(n,t,z)$ being given by lemma A.1. Then $q^j(n,t), p^j(n,t)$ and $q^2(n,t), p^2(n,t)$ are related by the following difference and differential equations
\[d_1^{i}(n,t)q_1(n,t) - d_2^{i}(n,t)q_1(n,t) = d_1^{i}(n,t)q_2(n-1,t) - d_1^{i}(n,t)q_1(n-1,t),\]
\[d_2^{i}(n,t)p_1(n,t) - d_1^{i}(n,t)p_2(n,t) = d_2^{i}(n,t)p_2(n-1,t) - d_2^{i}(n,t)p_1(n-1,t),\]
\[d_1^{i}(n,t)q_2(n,t) - d_2^{i}(n,t)q_1(n,t) = i\left\{ (d_2^{i}(n,t) + d_2^{j}(n,t)) (q_1(n,t) - q_2(n-1,t)) - (d_1^{i}(n,t) + d_1^{j}(n,t)) (q_1(n,t) - q_1(n-1,t)) \right.\]
\[- \left. - d_1^{i}(n,t)q_1(n,t) (q_1(n-1,t) p_1(n,t) + q_2(n,t) p_2(n-1,t)) \right.\]
\[\left. + d_2^{i}(n,t)q_2(n,t) (q_1(n,t) p_1(n-1,t) + p_2(n,t) q_2(n-1,t)) \right\},\]
\[= i\left\{ (d_2^{i}(n,t) + d_2^{j}(n,t)) (p_1(n-1,t) - p_1(n,t)) + (d_2^{i}(n,t) + d_2^{j}(n,t)) (p_1(n,t) - p_1(n-1,t)) \right.\]
\[\left. + d_2^{i}(n,t)q_1(n,t) (q_1(n,t) p_1(n-1,t) + p_2(n,t) q_2(n-1,t)) \right.\]
\[\left. - d_1^{i}(n,t)p_1(n,t) (p_1(n,t) q_1(n-1,t) + q_2(n,t) p_2(n-1,t)) \right\},\]  
\tag{A.9}

where \(d_k^{ij}(n,t), j, k = 1, 2\) are given by (A.7).

**Proof.** By substituting (A.5) and (A.6) into (A.4a), we find that the off-diagonal entries of the resulting equation of (A.4a) yield the first two equations of (A.9), while the off-diagonal entries of the resulting equation of (A.4b) yield the last two equations of (A.9).

**Remark A.3.** In the above lemma, we derived both the \(n\)-part (the first two equations of (A.9)) and the \(t\)-part (the last two equations of (A.9)) of the BT for the AL system. We note that the \(n\)-part of the BT presented here takes a different form than the one previously presented in [29], since we employed a different normalisation of the Lax pair. In [29], the authors showed that the discrete version of the Robin boundary conditions for the AL system can be derived via imposing an appropriate mirror symmetry on the \(n\)-part of the BT they derived. Here we note that it is also the case for the BT (A.9). Indeed, by imposing the mirror symmetry

\[q_1(n,t) = q_1(-n-1,t) \equiv q(-n-1,t), \quad p_1(n,t) = -p_1(-n-1,t) \equiv -p(-n-1,t), \]  
\tag{A.10}

on the \(n\)-part of the BT (A.9), we find the following boundary conditions,

\[(c_1 - c_4)q(0,t) - (c_2 - c_3)q(-1,t) = 0, \quad (c_1 + c_4) p(0,t) + (c_2 + c_3) p(-1,t) = 0,\]
\tag{A.11}

which is just the Robin boundary conditions for the AL system on the half-line. We also point out that the \(t\)-part of the BT (A.9) together with the mirror symmetry (A.10) yields the following time-dependent boundary conditions

\[c_1 \frac{dq(0,t)}{dt} - c_2 \frac{dq(-1,t)}{dt} = -i(c_1 + c_2 + c_3 + c_4) (q(0,t) - q(-1,t)),\]
\[c_4 \frac{dp(0,t)}{dt} + c_3 \frac{dp(-1,t)}{dt} = -i(c_1 - c_2 + c_3 - c_4) (p(0,t) - p(-1,t)).\]  
\tag{A.12}

To our knowledge, the above boundary conditions have not been reported in the literature (they differ from the time-dependent boundary conditions derived very recently in [55]).

**A.2. Defect conditions as frozen Bäcklund transformations**

As in the continuum case, we define defect conditions for the AL system (1.1) as the BT (A.9) ‘frozen’ at \(n = n_0\), the position of the defect. More precisely, for a fixed point \(n_0 \in \mathbb{Z}\)
we suppose \(q^1(n, t), p^1(n, t)\) satisfies the AL lattice system (1.1) for \(n < n_0; q^2(n, t), p^2(n, t)\) satisfies the AL system (1.1) for \(n > n_0;\) at \(n = n_0,\) they are connected by the relations (A.9). For the associated auxiliary problem, we suppose the system (A.1a) with \(j = 1\) exists for \(n < n_0,\) while the one with \(j = 2\) exists for \(n > n_0,\) and the two systems are connected by the relations (A.3) at \(n = n_0.\) Following the terminology of the analogous problem for integrable PDEs [47–52], we refer to the matrix \(\mathcal{D}(n_0, t, z)\) as the defect matrix and the relations (A.9) at \(n = n_0\) as defect conditions.

We note that such defect conditions preserve the integrability of the AL system in the sense that an infinite set of conserved quantities exists. Indeed, we find the following conclusion.

**Proposition A.4.** Let \(\Omega^j = \Phi^j(n, z) \Phi^j(n, z)^{-1}, \ j = 1, 2,\) and let

\[
I(z) = \left. I_{\text{left}}(z) + I_{\text{right}}(z) + I_{\text{defect}}(z) \right|_{n=n_0},
\]

where

\[
I_{\text{left}}(z) = \sum_{n=-\infty}^{n_0-1} \ln \left( (f^1(n, t))^{-1} (1 + z^{-1} q^1(n, t) \Omega^1) \right),
\]

\[
I_{\text{right}}(z) = \sum_{n=n_0}^{\infty} \ln \left( (f^2(n, t))^{-1} (1 + z^{-1} q^2(n, t) \Omega^2) \right),
\]

\[
I_{\text{defect}}(z) = \ln \left( \mathcal{D}^{11} + \mathcal{D}^{12} \Omega^1 \right) \bigg|_{n=n_0},
\]

and \(\mathcal{D}^i\) and \(i, j = 1, 2\) are the \(ij\)-entries of the defect matrix \(\mathcal{D}(n, t, z).\) Then

\[
I_j(z) = 0.
\]

**Proof.** From (A.1a), we find

\[
(S - 1) \ln \Phi^j(n, t, z) = \ln \left( (f^j(n, t))^{-1} (z + q^j(n, t) \Omega^j) \right), \ j = 1, 2,
\]

\[
(S - 1) \ln \Phi^j(n, t, z) = (S - 1) \left( T^j_{11} + T^j_{12} \Omega^j \right), \ j = 1, 2,
\]

where \(S\) denotes the shift operator, i.e. \(S f(n) = f(n + 1),\) and the quantities \(T^j_{kl}\) and \(k, l = 1, 2\) denote \(kl\)-entries of the matrix \(T_j(n, t, z).\) Then, \((S - 1) \ln \Phi^j(n, t, z) = (S - 1) \ln \Phi^j(n, t, z),\) yields

\[
\ln \left( (f^j(n, t))^{-1} (z + q^j(n, t) \Omega^j) \right) = (S - 1) \left( T^j_{11} + T^j_{12} \Omega^j \right), \ n < n_0,
\]

\[
(S - 1) \ln \Phi^j(n, t, z) = (S - 1) \left( T^j_{21} + T^j_{22} \Omega^2 \right), \ n > n_0.
\]
Using the above relations and the rapid decay of \(q^j(n, t)\) and \(p^j(n, t)\), \(j = 1, 2\), we obtain

\[
\left( t^\text{left}_{\text{bulk}}(z) + t^\text{right}_{\text{bulk}}(z) \right)_t = \left( T_1^{11} + T_1^{12}\Omega^1 - T_2^{11} - T_2^{12}\Omega^2 \right) I_{n=n_0}.
\] (A.18)

Next, we show that the contribution of the defect to the conserved quantities cancels out the right-hand side of (A.18). From (A.3), we find

\[
\Omega^2 = (D^{21} + D^{22}\Omega^1) \left( D^{11} + D^{12}\Omega^1 \right)^{-1}.
\] (A.19)

Using (A.19) and using (A.4b) at \(n = n_0\) to eliminate \(T_2^{11}\) and \(T_2^{12}\), we can write the right-hand side of (A.18) as

\[
- \left( D^{11} + D^{12}\Omega^1 \right) \left( D^{11} + D^{12}\Omega^1 \right)^{-1} \bigg|_{n=n_0}.
\] (A.20)

Equation (A.1b) implies the Ricatti equation

\[
\Omega^1_t = T_1^{21} - 2T_1^{11}\Omega^1 - T_1^{12}\left( \Omega^1 \right)^2.
\] (A.21)

Using this, the expression (A.20) becomes

\[
- \left( D^{11} + D^{12}\Omega^1 \right) \left( D^{11} + D^{12}\Omega^1 \right)^{-1} \bigg|_{n=n_0}.
\] (A.22)

From (A.14c), we find that \(\partial t I_{\text{defect}}(z)\) cancels out (A.22). Thus (A.15) holds.

Proposition A.4 implies that the formula (A.13) provides a generating function for the infinite number of conserved quantities for the AL system with a defect at \(n = n_0\). We are able to recursively derive the explicit forms of these conserved quantities by expanding \(\Omega^j(n, t, z)\) and \(j = 1, 2\), in terms of negative powers of \(z\). Indeed, from (A.1a) we find that \(\Omega^j\) satisfies the following difference equation, for \(j = 1, 2\),

\[
q^j(n, t)\Omega^j(n + 1, t, z) + \Omega^j(n + 1, t, z) + z \Omega^j(n + 1, t, z) - z^{-1}\Omega^j(n, t, z) - p^j(n, t) = 0.
\] (A.23)

We expand \(\Omega^j\) in terms of negative powers of \(z\) as

\[
\Omega^j = \sum_{k=1}^{\infty} \Omega^j_k(n, t)z^{-k}, \quad j = 1, 2.
\] (A.24)

By inserting (A.24) into (A.23) and by equating the coefficients of powers of \(z\), we arrive at

\[
\Omega^j_{2k}(n, t) = 0, \quad k \geq 1.
\] (A.25a)

\[
\Omega^j_1(n, t) = p^j(n - 1, t), \quad \Omega^j_2(n, t) = p^j(n - 2, t) \left( 1 - p^j(n - 1, t)q^j(n - 1) \right),
\] (A.25b)

\[
\Omega^j_{2k+1}(n, t) = \Omega^j_{2k-1}(n - 1, t) - q^j(n - 1) \sum_{l+m=2k} \Omega^j_l(n - 1, t) \Omega^j_m(n, t), \quad k \geq 1.
\] (A.25c)

Substituting (A.24) and (A.25a) into (A.13) and (A.14a), we finally obtain the infinite set of conserved quantities for the defect system order by order.
Remark A.5. We note that different aspects regarding integrable boundary conditions and Bäcklund transformations have been studied in [44–46].

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