Two-time free energy distribution function in the KPZ problem

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I. INTRODUCTION

The problem of directed polymers in a quenched random potential or equivalent problem of the KPZ-equation has describing the growth in time of an interface in the presence of noise have been the subject of intense investigations during past three decades. A few years ago the exact solution for the free energy probability distribution function (PDF) has been found. It was show that this PDF is given by the Tracy-Widom distribution. The two-point free energy distribution function which describes joint statistics of the free energies of the directed polymers coming to two different endpoints has been derived in.

In all these studies, however, the problems were considered in the so called "one-time" situation. For the first time the joint probability distribution function for the free energies of two directed polymers with fixed boundary conditions at two different times has been studied in the paper (in terms of non-rigorous replica Bethe ansatz approach) and in the paper (mathematically rigorous derivation). Unfortunately, the results obtained in these papers are somewhat inconclusive: the final formulas are expressed in terms rather complicated mathematical object whose analytic properties are not known. Moreover, for the moment it is even not clear whether these two results obtained it terms of two different approaches do coincide.

In this paper I am going to derive the joint probability distribution function for somewhat different two time free energy object. It turns out that compared to the previous calculations "more symmetric" structure of this object (see below) makes the corresponding calculations much more simple and the final result can be expressed in the form of a compact formula which could be conditionally called a "two-dimensional" Fredholm determinant with explicit expression for its kernel, eqs. (27)-(29).

The model under consideration is defined in terms of an elastic string $\phi(\tau)$ directed along the $\tau$-axes within an interval $[0, t]$ which passes through a random medium described by a random potential $V(\phi, \tau)$. The energy of a given polymer’s trajectory $\phi(\tau)$ is

$$H[\phi, V] = \int_0^t d\tau \left\{ \frac{1}{2} \left[ \partial_\tau \phi(\tau) \right]^2 + V[\phi(\tau), \tau] \right\};$$

(1)

where the disorder potential $V[\phi, \tau]$ is Gaussian distributed with a zero mean $\overline{V(\phi, \tau)} = 0$ and the $\delta$-correlations $\overline{V(\phi, \tau)V(\phi', \tau')} = u\delta(\tau - \tau')\delta(\phi - \phi')$. The parameter $u$ describes the strength of the disorder. For the fixed boundary conditions, $\phi(0) = 0$, $\phi(t) = x$, the partition function of this model is

$$Z_t(x) = \int_{\phi(0)=0}^{\phi(t)=x} D\phi(\tau) e^{-\beta H[\phi]} = \exp(-\beta F_t(x))$$

(2)

where $\beta$ is the inverse temperature and $F_t(x)$ is the free energy. In the limit $t \to \infty$ the free energy scales as $\beta F_t(x) = \beta f_0 t + \beta x^2/2t + \lambda_1 f(x)$, where $f_0$ is the selfaveraging free energy density, $\lambda_1 = \frac{1}{2}(\beta^3 u^2 t)^{1/3} \propto t^{4/3}$ and $f(x)$ is a random quantity described by the Tracy-Widom distribution. As the first two trivial terms of this free energy can be easily eliminated by simple redefinition of the partition function, they will be omitted in the further calculations, in other words, $Z_t(x) = \exp(-\lambda_1 f(x))$.

Let us consider two partition functions: the first one, $Z_t(0)$, is defined in eq. (2) with the zero boundary conditions, $\phi(0) = \phi(t) = 0$, while the second one, $Z_t, \Delta(0)$, in addition to the above zero boundary conditions contains an
additional constrain $\phi(t - \Delta t) = 0$. In other words, in the second case the polymer trajectory before coming to zero at time $t$ is forced cross the zero at some intermediate time $(t - \Delta t)$.

By definition,

$$Z_t(0) = \int_{-\infty}^{+\infty} dx \, Z_{t-\Delta t}(x) Z^\star_{\Delta t}(x) = \exp(-\lambda_t f)$$

and

$$\tilde{Z}_{t,\Delta t}(0) = Z_{t-\Delta t}(0) Z^\star_{\Delta t}(0) = \exp(-\lambda_t \tilde{f})$$

where $Z^\star_{\Delta t}(x)$ is the partition function of the directed polymer system in which time goes backwards, from $t$ to $(t - \Delta t)$. For technical reasons (for proper regularization of the integration over $x$ at $\pm$ infinities) it is convenient to split the partition function $Z_t(0)$ into two parts, the "left" and the "right" ones:

$$Z_t(0) = \int_0^{+\infty} dx \, Z_{t-\Delta t}(x) Z^\star_{\Delta t}(x) + \int_{-\infty}^0 dy \, Z_{t-\Delta t}(y) Z^\star_{\Delta t}(y)$$

In terms of the free energy definitions (3)-(4) one would like to compute the joint probability distribution function

$$W(f_1, f_2, \Delta) = \lim_{t \to \infty} \text{Prob}[f > f_1; \tilde{f} > f_2]$$

where it is assumed that in the limit $t \to \infty$ the parameter $\Delta \equiv \Delta t/t$ remains finite. In terms of the above partition functions (3)-(4) this quantity can be defined as follows:

$$W(f_1, f_2, \Delta) = \lim_{t \to \infty} \sum_{N=0}^{\infty} \sum_{K=0}^{\infty} \frac{(-1)^{N+K}}{N! K!} \exp(\lambda_t N f_1 + \lambda_t K f_2) \frac{Z_t^N(0) Z^\star_{\Delta t}(0) Z^\star_{\Delta t}(0)}{Z^{\star K}_{\Delta t}(0) Z^{\star K}_{\Delta t}(0)}$$

where $\langle \ldots \rangle$ denotes the averaging over the random potential. Taking into account eq.(5) one gets

$$W(f_1, f_2, \Delta) = \lim_{t \to \infty} \sum_{L,R,K=0}^{\infty} \frac{(-1)^L + R + K}{L! R! K!} \exp(\lambda_t (L + R) f_1 + \lambda_t K f_2) \int_0^0 \ldots \int_0^0 dy_1 \ldots dy_L \int_0^{+\infty} \ldots \int_0^{+\infty} dx_1 \ldots dx_L \times$$

$$\times \left( \prod_{a=1}^L Z_{t-\Delta t}(x_a) Z^\star_{t-\Delta t}(0) \prod_{b=1}^R Z_{t-\Delta t}(y_b) \right) \left( \prod_{a=1}^L Z^\star_{\Delta t}(x_a) Z^\star_{\Delta t}(0) \prod_{b=1}^R Z^\star_{\Delta t}(y_b) \right)$$

Introducing:

$$\Psi(x_1, \ldots, x_N; t) \equiv \frac{Z_t(x_1) Z_t(x_2) \ldots Z_t(x_N)}{Z_t(0) Z_t(0) \ldots Z_t(0)}$$

![FIG. 1: Schematic representation of two directed polymer paths: (1) with $\phi(0) = \phi(t) = 0$, and (2) with $\phi(0) = \phi(t - \Delta t) = \phi(t) = 0$](image-url)
one can easily show that $\Psi(x; t)$ is the wave function of $N$-particle boson system with attractive $\delta$-interaction:

$$
\beta \partial_t \Psi(x; t) = \frac{1}{2} \sum_{a=1}^{N} \partial_{x_a}^2 \Psi(x; t) + \frac{1}{2} \kappa \sum_{a \neq b} \delta(x_a - x_b) \Psi(x; t)
$$

(10)

(where $\kappa = \beta^3 u$) with the initial condition $\Psi(x; 0) = \prod_{a=1}^{N} \delta(x_a)$. The wave function $\Psi(x; t)$ of this quantum problem can be represented in terms of the linear combination of the corresponding eigenfunctions of eq.(10). A generic eigenstate of such system is characterized by $N$ momenta $q_\alpha$ ($\alpha = 1, ..., N$) which split into $M$ ($1 \leq M \leq N$) "clusters" each described by continuous real momenta $\kappa$ with the global constraint $x_a$; $\alpha = 1, ..., M$ and characterized by $n_\alpha$ discrete imaginary "components" (for details see [33–37]):

$$
Q_\alpha \to q_\alpha^n = q_{\alpha} - \frac{i \kappa}{2} (n_\alpha + 1 - 2r) ; \quad (r = 1, ..., n_\alpha ; \quad \alpha = 1, ..., M)
$$

(11)

with the global constraint $\sum_{\alpha=1}^{M} n_\alpha = N$. Explicitly,

$$
\Psi_Q(x) = \sum_{P} \prod_{1 \leq a < b} \left[ 1 + \frac{i \kappa \text{sgn}(x_a - x_b)}{Q_{P_a} - Q_{P_b}} \right] \exp[i \sum_{a=1}^{N} Q_{P_a} x_a]
$$

(12)

where the vector $Q$ denotes the set of all $N$ momenta eq.(11) and the summation goes over $N!$ permutations $P$ of $N$ momenta $Q_\alpha$, over $N$ particles $x_a$. In terms of the above eigenfunctions the solution of eq.(10) can be expressed as follows:

$$
\Psi(x; t) = \frac{1}{N!} \int \mathcal{D}Q |C(Q)|^2 \Psi_Q(x) \Psi_Q(0) \exp(-tE(Q))
$$

(13)

where the symbol $\int \mathcal{D}Q$ denotes the integration over $M$ continuous parameters $\{q_1, ..., q_M\}$, the summations over $M$ integer parameters $\{n_1, ..., n_M\}$ as well as summation over $M = 1, ..., N$. $|C(Q)|^2$ is the normalization factor,

$$
|C(Q)|^2 = \frac{\kappa^N}{\prod_{\alpha=1}^{M} (\kappa n_\alpha)} \prod_{\alpha < \beta} \frac{|q_\alpha - q_\beta - \frac{i \kappa}{2} (n_\alpha - n_\beta)|^2}{\kappa^2}
$$

$$
= \kappa^N \det \left[ \frac{1}{\frac{1}{2} \kappa n_\alpha - iq_\alpha + \frac{\kappa}{2} \kappa n_\beta + iq_\beta} \right]_{\alpha, \beta = 1, ..., M}
$$

(14)

and $E(Q)$ is the eigenvalue (energy) of the eigenstate $\Psi_Q(x)$,

$$
E(Q) = \frac{1}{2 \beta} \sum_{a=1}^{N} \frac{Q_a^2}{2 \beta} = \frac{1}{2 \beta} \sum_{a=1}^{M} n_\alpha q_\alpha^2 - \frac{\kappa^2}{24 \beta} \sum_{a=1}^{M} n_\alpha^3
$$

(15)

In terms of the wave functions (9)-(13) the probability distribution function (8) can be expressed as follows:

$$
W(f_1, f_2, \Delta) = \lim_{t \to \infty} \sum_{L, K, R = 0}^{\infty} \frac{(-1)^{L+K+R}}{L! K! R!} \exp \left[ \lambda_t (L + R) f_1 + \lambda_t K f_2 \right] \times
$$

$$
\times \int_{-\infty}^{0} dx_1 ... dx_L \int_{0}^{+\infty} dy_R ... dy_1 \Psi(x_1, ..., x_L, 0, ..., 0, y_R, ..., y_1; (t - \Delta t)) \Psi^*(x_1, ..., x_L, 0, ..., 0, y_R, ..., y_1; \Delta t)
$$

(16)

where the second (conjugate) wave function represent the "backward" propagation from the time moment $t$ to the previous time moment $t - \Delta t$. Schematically the above expression is represented in Figure 1. The above expression is quite similar to eq.(23) in [1] although here its structure is "more symmetric" which essentially simplify further
FIG. 2: Schematic representation of the directed polymer paths corresponding to eq. (10).

calculations. Repeating the same steps as in the paper [1], in the limit $t \to \infty$ one eventually gets (cf eq.(65) in [1]):

$$W(f_1, f_2, \Delta) = 1 + \sum_{M=1}^{\infty} \frac{(-1)^M}{(M!)^2} \left[ \prod_{\alpha} \int_{-\infty}^{+\infty} \frac{dq_\alpha dp_\alpha}{(2\pi)^2} \int_{-\infty}^{+\infty} dy \ \text{Ai}(y + (1 - \Delta)q_\alpha^2 + \Delta p_\alpha^2 - f_1) \times \right.$$

$$\times \int_{\mathcal{C}} dz_1 d z_2 d z_3 \left( \frac{1}{z_1 z_2 z_3} - \delta(z_1)\delta(z_2)\delta(z_3) \right) J(q_\alpha - p_\alpha, z_1, z_2, z_3) \times \exp\{(z_1 + z_2 + z_3) y + z_3 (f_2 - f_1)\} \right] \times$$

$$\times \det \left[ \frac{1}{z_1 + z_2 + z_3 - i q_\alpha + z_1 \alpha + z_2 \alpha + z_3 \alpha + i q_\alpha} \right]_{\alpha, \alpha'=1, \ldots, M}$$

$$\times \det \left[ \frac{1}{z_1 + z_2 + z_3 + i p_\alpha + z_1 \alpha + z_2 \alpha + z_3 \alpha - i p_\alpha} \right]_{\alpha, \alpha'=1, \ldots, M} \quad (17)$$

where the integration contour $\mathcal{C}$ is shown in figure 2, and

$$J(q - p, z_1, z_2, z_3) = \left( 1 + \frac{z_1}{z_2 + z_3 + \frac{1}{2} i (q - p)^-(1)} \right) \left( 1 + \frac{z_2}{z_1 + z_3 - \frac{1}{2} i (q - p)^+(1)} \right) \quad (18)$$

Here $(q - p)^{(\pm)} \equiv (p - q) \pm i \epsilon$ where the parameter $\epsilon$ has to be set to zero at the end. Determinants in eq.(17) can be represented in the following integral form:

$$\det \left[ \frac{1}{z_\alpha - i q_\alpha + z_\beta + i q_\beta} \right]_{\alpha, \beta=1, \ldots, M} = \sum_{P \in S_M} (-1)^{|P|} \prod_{\alpha=1}^{M} (z_\alpha - i q_\alpha + z_{P_\alpha} + i q_{P_\alpha})^{-1}$$

$$= \prod_{\alpha=1}^{M} \left[ \int_{0}^{\infty} du_\alpha \right] \sum_{P \in S_M} (-1)^{|P|} \prod_{\alpha=1}^{M} \exp\{- (z_\alpha + z_{P_\alpha}) u_\alpha + i (q_\alpha - q_{P_\alpha}) u_\alpha \}$$

$$= \prod_{\alpha=1}^{M} \left[ \int_{0}^{\infty} du_\alpha \right] \sum_{P \in S_M} (-1)^{|P|} \prod_{\alpha=1}^{M} \exp\{- z_\alpha (u_\alpha + u_{P_\alpha}) + i q_\alpha (u_\alpha - u_{P_\alpha}) \} \quad (19)$$
Substituting this representation into eq. (17) and performing integrations over $z_1, z_2$ and $z_3$ one obtains the following result

\[
W(f_1, f_2, \Delta) = 1 + \sum_{M=1}^{\infty} \frac{(-1)^M}{(M!)^2} \sum_{r, r' \in S_M} (-1)^{[p]} \left[ \left( \int_0^\infty du_\alpha dv_\alpha \right) \prod_{\alpha=1}^{M} \int \frac{dq_\alpha dp_\alpha}{(2\pi)^2} \int \frac{dy}{\pi} \text{Ai} \left( y + (1 - \Delta)q_\alpha^2 + \Delta p_\alpha^2 - f_1 + u_\alpha + v_\alpha + u_{p_\alpha} + v_{p_\alpha} \right) \times \exp \left\{ +i q_\alpha (u_\alpha - u_{p_\alpha}) - i p_\alpha (v_\alpha - v_{p_\alpha}) \right\} S[(q_\alpha - p_\alpha); y; (f_2 - f_1)] \right]
\]

where

\[
S[(q - p); y; (f_2 - f_1) = 4\pi \delta(y) \delta(q - p) \theta(f_1 - f_2) + \left( \theta(f_2 - f_1 + y) \theta(f_2 - f_1 - y) - 4\delta(y) \frac{\sin \left[ \frac{\pi}{2} (q - p) (f_2 - f_1) \right]}{(q - p)} \right) \theta(f_2 - f_1)
\]

According to the above equation in the cases: (a) $f_2 < f_1$ and (b) $f_2 > f_1$, one finds two essentially different results for the probability distribution function $W(f_1, f_2, \Delta)$.

(a) $f_2 < f_1$. In this case the distribution function $W(f_1, f_2, \Delta)$ turns out to be independent of $f_2$ and $\Delta$:

\[
W(f_1, f_2, \Delta) \big|_{f_2 < f_1} = W(f_1) = 1 + \sum_{M=1}^{\infty} \frac{(-1)^M}{(M!)^2} \sum_{r, r' \in S_M} (-1)^{[p]} \left[ \left( \int_0^\infty du_\alpha dv_\alpha \right) \prod_{\alpha=1}^{M} \int \frac{dq_\alpha}{\pi} \text{Ai} \left( q_\alpha^2 - f_1 + (u_\alpha + v_{p_\alpha}) + (v_\alpha + u_{p_\alpha}) \right) \exp \left\{ i q_\alpha ((u_\alpha + v_{p_\alpha}) - (v_\alpha + u_{p_\alpha})) \right\} \right]
\]

\[
\times \prod_{\alpha=1}^{M} \left[ \frac{2^{2/3}}{\pi} \text{Ai} \left[ 2^{1/3} (u_\alpha + v_{p_\alpha} - f_1/2) \right] \text{Ai} \left[ 2^{1/3} (u_{p_\alpha} + v_\alpha - f_1/2) \right] \right]
\]

(b) $f_2 > f_1$. This case requires a different treatment and is beyond the scope of this paper.
Redefining \( u_α \to 2^{-1/3} u_α \), \( v_α \to 2^{-1/3} v_α \), and taking into account that

\[
\prod_{α=1}^M \text{Ai}\left[u_{p_α} + v_α - f_1/2^{2/3}\right] = \prod_{α=1}^M \text{Ai}\left[u_α + v_{p_α} - f_1/2^{2/3}\right]
\]

eq \prod_{α=1}^M \text{Ai}\left[u_α + v_{p_α} - f_1/2^{2/3}\right]
\]

we obtain

\[
W(f_1) = 1 + \sum_{M=1}^\infty \frac{(-1)^M}{(M!)^2} \prod_{α=1}^M \left[ \int_0^\infty dv_α \right] \sum_{P, P' \in S_M} (-1)^{|P|} \prod_{α=1}^M \left[ K(v_α - f_1/2^{2/3} \mid v_{p_α} - f_1/2^{2/3}) \right]
\]

where

\[
K(v; v') = \int_0^\infty du \text{Ai}(u + v) \text{Ai}(u + v')
\]

is the Airy kernel. Redefining the permutations, \( P + P' \to P \), we eventually get

\[
W(f_1) = 1 + \sum_{M=1}^\infty \frac{(-1)^M}{(M!)^2} \prod_{α=1}^M \left[ \int_0^\infty dv_α \right] \sum_{P \in S_M} (-1)^{|P|} \prod_{α=1}^M \left[ K(v_α - f_1/2^{2/3} \mid v_{p_α} - f_1/2^{2/3}) \right] = F_2(-f_1/2^{2/3})
\]

Thus we have got the usual Tracy-Widom distribution for the free energy \( f_1 \) of the directed polymer with the zero boundary conditions, as it should be (it is evident that in the limit \( t \to \infty \) with the probability one the free energy \( f_2 \) of the polymer which is forced to pass par the zero at some intermediate time \( (t - \Delta t) \) is larger than the free energy \( f_1 \) of the polymer which is free of this condition).

(b) \( f_2 > f_1 \). In this case the situation becomes more tricky. According to eqs. 20 and 21 we get:

\[
W(f_1, f_2, \Delta)|_{f_2 > f_1} = 1 + \sum_{M=1}^\infty \frac{(-1)^M}{(M!)^2} \prod_{α=1}^M \left[ \int_0^\infty du_α dv_α \right] \sum_{P, P' \in S_M} (-1)^{|P|} \prod_{α=1}^M \left[ K(u_α, v_α; u_{p_α}, v_{p_α}) \left(f_1, f_2, \Delta\right) \right]
\]

where

\[
K(u, v; u', v') \left(f_1, f_2, \Delta\right) = \int_{-(f_2-f_1)}^{f_2-f_1} dy \int_{-\infty}^{+\infty} dq dp \frac{1}{(2\pi)^2} \text{Ai}[y - f_1 + (1 - \Delta) q^2 + \Delta p^2 + u + v + u' + v'] \times \exp\{iq(u - u') - ip(v - v')\} \]

\[
-4 \int_{-\infty}^{+\infty} dq dp \frac{1}{(2\pi)^2} \frac{\sin\left(\frac{1}{2}(q - p)(f_2 - f_1)\right)}{q - p} \text{Ai}[y - f_1 + (1 - \Delta) q^2 + \Delta p^2 + u + v + u' + v'] \times \exp\{iq(u - u') - ip(v - v')\}
\]

The structure in eq. 27 could be conditionally represented as a "two-dimensional" Fredholm determinant:

\[
W \left(f_1, f_2, \Delta\right)|_{f_2 > f_1} = \det[1 - \tilde{K}]
\]

where \( \tilde{K} = K_{\xi, \xi'} \left(f_1, f_2, \Delta\right) \) is the integral operator function of three parameters \( f_1, f_2 \) and \( \Delta \) on the two-dimensional space \( \xi \equiv (u, v) \) with \( (u, v) \in [0, \infty) \) and with the kernel given in eq. 28.

Eqs. 27-29 constitute the central result of this work for the two-time free energy probability distribution function of two directed polymers with the two types of the zero boundary conditions: \( \phi(0) = \phi(t) = 0 \) and \( \phi(0) = \phi(t - \Delta t) = \phi(t) = 0 \) in the thermodynamic limit \( t \to \infty \) such the parameter \( \Delta \equiv \Delta t / t \) remains finite. Analytic properties of the mathematical object defined in Eqs. 27-29 remains to be investigated.

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