ON THE ABUNDANCE THEOREM FOR NUMERICALLY TRIVIAL CANONICAL DIVISORS IN POSITIVE CHARACTERISTIC

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ABSTRACT. In this paper, we prove the abundance theorem for numerically trivial canonical divisors on strongly $F$-regular varieties, assuming that the geometric generic fibers of the Albanese morphisms are strongly $F$-regular.

1. Introduction

The abundance conjecture predicts that if a variety is minimal then its canonical divisor is semi-ample. This conjecture is one of the most important problems in the minimal model theory. In characteristic zero, the conjecture has been verified when the variety has only canonical singularities and the canonical divisor is numerically trivial by Kawamata [10, Theorem 8.2]. This has been generalized to klt pairs by Nakayama [20, V.4.8 Theorem], and to lc pairs by Campana–Koziarz–Păun [1, Theorem 0.1], Gongyo [6, Theorem 1.2] and Kawamata [11, Theorem 1].

In this paper we prove the theorem below, which can be viewed as a positive characteristic analog of Nakayama’s theorem mentioned above.

Theorem 1.1. Let $X$ be a normal projective variety over an algebraically closed field of characteristic $p > 0$. Let $\Delta$ be an effective $\mathbb{Q}$-Weil divisor on $X$ such that $(X, \Delta)$ is strongly $F$-regular. Assume that $m(K_X + \Delta)$ is Cartier for an integer $m > 0$ not divisible by $p$, and that $K_X + \Delta$ is numerically trivial. Let $\alpha : X \to A$ be the Albanese morphism of $X$ and let $X_\eta$ denote the geometric generic fiber of $\alpha$ over its image. If $(X_\eta, \Delta|_{X_\eta})$ is strongly $F$-regular, then $K_X + \Delta$ is $\mathbb{Q}$-linearly trivial.

Here, strong $F$-regularity is a class of singularities defined only in positive characteristic, which is closely related to klt singularities in characteristic zero.

We give a brief explanation of the proof of Theorem 1.1. For simplicity, we suppose that $\Delta = 0$ and $K_X$ is an algebraically trivial Cartier divisor. Then $K_X \sim \alpha^* L$ for an algebraically trivial Cartier divisor $L$ on $A$. Our goal is to show that $L \sim_{\mathbb{Q}} 0$. By the assumption on the geometric generic fiber of $\alpha$, we can apply [4, Theorem 1.1] and [23, Theorems 4.1 and 9.1], so we see that $\alpha$ is a flat surjective morphism whose every fiber is integral and strongly $F$-regular. Let $B$ be a sufficiently ample Cartier divisor on $X$. Let $r$ be the rank of $\alpha_* \mathcal{O}_X(B)$. We consider the $r$-th fiber product $Y := X \times_A \cdots \times_A X$ of $X$ over $A$. Let $f : Y \to A$ be the natural projection. Applying the argument due to Viehweg [26, Proof of Theorem 6.22], we get an...
effective $f$-ample Cartier divisor $C$ on $Y$ such that $\det(f^*\mathcal{O}_Y(C)) \cong \mathcal{O}_A$. Using [2, Theorem 6.10 (2)], we see that $f^*\mathcal{O}_Y(C)$ is a numerically flat vector bundle. Then by Theorem [2,2], we find unipotent vector bundles $U_1, \ldots, U_l$ and algebraically trivial line bundles $N_1, \ldots, N_l$ such that

$$f^*\mathcal{O}_Y(C) \cong \bigoplus_{i=1}^l U_i \otimes N_i.$$  

Applying an argument similar to [5, Proof of Theorem 3.2], we get some $\mu \in \mathbb{Z}_{>0}$ such that for every $M \in M(\mu)$, where

$$M(\mu) := \left\{ \left( \bigotimes_{i=1}^l N_i^{n_i} \right) \left| 0 \leq n_i \in \mathbb{Z} \text{ with } \sum_{i=1}^l n_i = \mu \right. \right\},$$

we have $M \otimes \mathcal{O}_A(\nu L) \in M(\mu)$ for infinitely many $\nu \in \mathbb{Z}_{>0}$. Since $M(\mu)$ is a finite set, we conclude that $L \sim \mathbb{Q} 0$.

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2. Preliminaries

2.1. Notation and conventions. We work over an algebraically closed field $k$ of characteristic $p > 0$. A variety is an integral separated $k$-scheme of finite type. Let $\varphi : S \rightarrow T$ be a morphism of schemes and let $U$ be a $T$-scheme. Let $S_U$ and $\varphi_U : S_U \rightarrow U$ denote the fiber product $S \times_T U$ of $S$ and $U$ over $T$ and its second projection, respectively. For an $\mathcal{O}_S$-module $\mathcal{G}$, its pullback to $S_U$ is denoted by $\mathcal{G}_U$.

We use the same notation for a $\mathbb{Q}$-divisor if its pullback is well-defined. Let $X$ be an $\mathbb{F}_p$-scheme. The Frobenius morphism of $X$ is denoted by $F_X : X \rightarrow X$. We denote the source of $F_X^e$ by $X^e$. Let $f : X \rightarrow Y$ be a morphism of $\mathbb{F}_p$-schemes. When we regard $X$ and $Y$ as the sources of $F_X^e$ and $F_Y^e$, respectively, $f$ is denoted by $f^{(e)} : X^e \rightarrow Y^e$. We define $e$-th relative Frobenius morphism of $f$ to be $F_{X/Y}^{(e)} := (F_X^e, f^{(e)}) : X^e \rightarrow X \times_Y Y^e = X_{Y^e}$.

2.2. Numerically flat vector bundles on abelian varieties. We recall properties of numerically flat vector bundles on abelian varieties.

**Definition 2.1.** Let $\mathcal{E}$ be a vector bundle on a projective variety $V$. We say that $\mathcal{E}$ is numerically flat if $\mathcal{E}$ and its dual $\mathcal{E}^\vee$ are nef.

One can easily check that the numerical flatness of $\mathcal{E}$ is equivalent to both $\mathcal{E}$ and $(\det(\mathcal{E}))^\vee$ are nef.

**Definition 2.2.** Let $\mathcal{E}$ be a vector bundle on an abelian variety $A$. We say that $\mathcal{E}$ is homogeneous if $t_a^*\mathcal{E} \cong \mathcal{E}$ for every closed point $a \in A$, where $t_a$ is the translation map of $A$ by $a$. 
**Definition 2.3.** Let \( \mathcal{E} \) be a vector bundle on an abelian variety \( A \). We say that \( \mathcal{E} \) is **unipotent** if \( \mathcal{E} \) is an iterated extension of \( \mathcal{O}_A \).

**Theorem 2.4.** Let \( A \) be an abelian variety. Let \( \mathcal{E} \) be a vector bundle on \( A \). Then the following are equivalent:

1. \( \mathcal{E} \) is numerically flat;
2. \( \mathcal{E} \) is an iterated extension of algebraically trivial line bundles;
3. \( \mathcal{E} \) is homogeneous vector bundle;
4. \( \mathcal{E} \cong \bigoplus_i \mathcal{U}_i \otimes \mathcal{L}_i \), where each \( \mathcal{L}_i \) is an algebraically trivial line bundle and each \( \mathcal{U}_i \) is a unipotent vector bundle.

**Proof.** (1) \( \Rightarrow \) (2): This follows from [15, §6]. (2) \( \Rightarrow \) (4): This is easily proved by induction on the rank. (4) \( \Rightarrow \) (1): This is obvious. (3) \( \Rightarrow \) (4): This has been proved by [18, Theorem 2.3]. (4) \( \Rightarrow \) (3): This has been shown in [19, Theorem 4.17]. \( \Box \)

### 3. Albanese morphisms of varieties with numerically trivial canonical divisor

In this section, we study the Albanese morphisms of varieties with numerically trivial canonical divisor.

**Lemma 3.1.** Let \( V \) be a projective variety over \( k \). Let \( \mathcal{L} \) be a numerically trivial line bundle on \( V \). Let \( R \) be a finitely generated \( \mathbb{F}_p \)-algebra over which the model \( V_R \) of \( V \) and \( \mathcal{L}_R \) of \( \mathcal{L} \) can be defined. Then there exists a dense open subset \( S \) of \( \text{Spec } R \) such that \( \mathcal{L}_\mu \) is numerically trivial for every closed point \( \mu \in S \).

**Proof.** Let \( S \subseteq \text{Spec } R \) be a dense open subset such that the morphism \( V_R \to \text{Spec } R \) is projective over \( S \). Put \( V_S := (V_R)_S \). Let \( \mathcal{H}_S \) be a line bundle on \( V_S \) that is ample over \( S \), and set \( \mathcal{L} := \mathcal{H}_S|_V \). By [13, Chapter I, §4, Proposition 3], we have that \( \mathcal{L} \) is numerically trivial if and only if \( (\mathcal{L} \cdot \mathcal{H}^{\dim V-1}) = (\mathcal{L}^2 \cdot \mathcal{H}^{\dim V-2}) = 0 \). Since the intersection number is independent of the choice of fiber, the lemma follows. \( \Box \)

**Definition 3.2** ([24, Definition 3.1]). Let \( X \) be an affine normal variety and let \( \Delta \) be an effective \( \mathbb{Q} \)-Weil divisor on \( X \). We say that the pair \( (X, \Delta) \) is **strongly \( F \)-regular** if for every effective divisor \( D \) on \( X \), there exists an \( e \in \mathbb{Z}_{>0} \) such that the composite

\[
\mathcal{O}_X \xrightarrow{F_X^e} F_X^e \mathcal{O}_X \xrightarrow{i} F_X^e \mathcal{O}_X([(p^e - 1)\Delta + D])
\]

splits. Here, \( i \) is the natural inclusion.

Let \( X \) be a normal variety and let \( \Delta \) be an effective \( \mathbb{Q} \)-Weil divisor on \( X \). We say that \( (X, \Delta) \) is **strongly \( F \)-regular** if there exists an affine open cover \( \{U_i\} \) of \( X \) such that \( (U_i, |\Delta|_{U_i}) \) is strongly \( F \)-regular for each \( i \). When \( (X, 0) \) is strongly \( F \)-regular, we simply say that \( X \) is **strongly \( F \)-regular**.

Recall if \( X \) is strongly \( F \)-regular, then \( X \) is Cohen–Macaulay ([9, (6.27) Proposition]).

**Proposition 3.3.** Let \( X \) be a Cohen–Macaulay normal projective variety and let \( \Delta \) be an effective \( \mathbb{Q} \)-Weil divisor on \( X \). Assume that \( m(K_X + \Delta) \) is a numerically trivial Cartier divisor for an integer \( m > 0 \) not divisible by \( p \). Let \( \alpha : X \to A \) be the
Albanese morphism of \(X\), and let \(X_{\bar{\pi}}\) denote the geometric generic fiber of \(\alpha\) over its image. If \((X_{\bar{\pi}}, \Delta|_{X_{\bar{\pi}}})\) is strongly \(F\)-regular, then

1. \(\alpha\) is a flat surjective morphism,
2. \(\text{Supp} \Delta\) does not contain any component of any closed fiber of \(f\), and
3. for every closed fiber \(x_{\gamma}\), the pair \((X_{x_{\gamma}}, \Delta|_{X_{x_{\gamma}}})\) is strongly \(F\)-regular.

Proof. By [1], Theorem 1.1], the morphism \(\alpha\) is surjective and \(X_{\bar{\pi}}\) is integral. By [23, Theorem 4.1], \(\alpha\) is equi-dimensional, so \(\alpha\) is flat, since \(X\) is Cohen–Macaulay [17, p.179, Corollary]. Thus (1) holds. Furthermore, (2) follows from [23, Proposition 4.2 (2)]. We prove (3). Fix a closed point \(y \in A\). We show that \(X_{y}\) is integral. Let \(z \in A\) be a general closed point. Applying [2 Theorème 12.2.4], we may assume that \(X_{z}\) is integral. Let \(R\) be a finitely generated \(k\)-algebra over which the models \(\alpha_{R}: X_{R} \rightarrow A_{R}, \Delta_{R}, y_{R}\) and \(z_{R}\) of \(\alpha: X \rightarrow A, \Delta, y\) and \(z\) can be defined, respectively. Then, by Lemma [3,1] there is a dense open subset \(S \subseteq \text{Spec} \, R\) such that \(K_{X_{\mu}} + \Delta_{\mu}\) is a numerically trivial \(\mathbb{Q}\)-Cartier divisor for every \(\mu \in S\). Since the residue field \(\kappa(\mu)\) of \(\mu\) is finite, we have \(\kappa(\mu) \subseteq \overline{\mathbb{F}_{r}}\). Therefore, thanks to [23, Theorem 9.1], we see that \(((X_{\mu})_{y_{\mu}}, (\Delta_{\mu})_{y_{\mu}}) \cong ((X_{\mu})_{z_{\mu}}, (\Delta_{\mu})_{z_{\mu}})\). Since \(((X_{\mu})_{z_{\mu}}, (\Delta_{\mu})_{z_{\mu}})\) is the reduction of \((X_{z}, \Delta|_{X_{z}})\) over \(\mu\), we may assume that \((X_{\mu})_{z_{\mu}}\) is geometrically integral, using [4, Théorème 12.2.4]. Thus \((X_{\mu})_{y_{\mu}}\) is geometrically integral, and so \(X_{y}\) is integral by [4, Théorème 12.2.4] again. Applying [22, Theorem B] and an argument similar to the above, we can prove that \((X_{y}, \Delta|_{X_{y}})\) is strongly \(F\)-regular. \(\Box\)

4. PROOF OF THE MAIN THEOREM

In this section, we prove the main theorem in this paper. We first show the following three lemmas that are used in the proof of the main theorem.

Lemma 4.1. Let \(V\) and \(W\) be normal varieties, and let \(\Gamma\) and \(\Delta\) be effective \(\mathbb{Q}\)-Weil divisors on \(V\) and \(W\), respectively. Assume that \((V, \Gamma)\) and \((W, \Delta)\) are strongly \(F\)-regular. Then the pair \((V \times_k W, \text{pr}_1^* \Gamma + \text{pr}_2^* \Delta)\) is also strongly \(F\)-regular, where \(\text{pr}_i\) is the \(i\)-th projection.

Note that since \(\text{pr}_i\) is flat, we can define the pullback \(\text{pr}_i^* D\) of any \(\mathbb{Q}\)-divisor \(D\).

Proof. We assume that \(V\) and \(W\) are affine. Let \(D\) (resp. \(E\)) be an effective Cartier divisor on \(V\) (resp. \(W\)) such that \((V_D, \Gamma_D)\) (resp. \((W_E, \Delta_E)\)) is log smooth, where \(V_D := V \setminus \text{Supp} \, D\) and \(\Gamma_D := \Gamma|_{V_D}\) (resp. \(W_E := W \setminus \text{Supp} \, E\) and \(\Delta_E := \Delta|_{W_E}\)). Here, by log smooth we mean that \(V_D\) is smooth and \(\Gamma_D\) has simple normal crossing support. Set \(C := \text{pr}_1^* D + \text{pr}_2^* E\) and \(\Theta := \text{pr}_1^* \Gamma + \text{pr}_2^* \Delta\). Then \((V \times_k W \setminus \text{Supp} \, C, \Theta|_{V \times_k W \setminus \text{Supp} \, C}) = (V_D \times_k W_E, (\text{pr}_1|_{V_D})^* \Gamma_D + (\text{pr}_2|_{W_E})^* \Delta_E)\) is log smooth. Since

\[\mathcal{O}_V \rightarrow F_{V, *}^{e} \mathcal{O}_V ([(p^e - 1) \Gamma + D])\]

and

\[\mathcal{O}_W \rightarrow F_{W, *}^{e} \mathcal{O}_W ([(p^e - 1) \Delta + E])\]

split for some \(e > 0\), the morphism \(\mathcal{O}_{V \times W} \rightarrow F_{V \times W, *} \mathcal{O}_{V \times W} ([(p^e - 1) \Theta + C])\) splits. Hence, the assertion follows from [24, Theorem 3.9]. \(\Box\)
Lemma 4.2. Let \( f : V \to W \) be a surjective morphism between projective varieties. Let \( A \) be an \( f \)-ample Cartier divisor on \( V \) such that the natural morphism \( f^* f_* \mathcal{O}_V(A) \to \mathcal{O}_V(A) \) is surjective. Let \( \mathcal{F} \) be a coherent sheaf on \( V \). Then there exists an \( m_0 \in \mathbb{Z}_{>0} \) such that the multiplication morphism
\[
f_*(\mathcal{F} \otimes \mathcal{O}_V(mA + N)) \otimes f_* \mathcal{O}_V(nA) \to f_*(\mathcal{F} \otimes \mathcal{O}_V((m + n)A + N))
\]
is surjective for each \( m, n \in \mathbb{Z}_{>0} \) with \( m \geq m_0 \) and every \( f \)-nef Cartier divisor \( N \) on \( V \).

Proof. By the relative version of Castelnuovo–Mumford regularity [16, Example 1.8.24], it is enough to show that \( \mathcal{F} \otimes \mathcal{O}_V(mA + N) \) is \( 0 \)-regular with respect to \( A \) and \( f \). This follows from the relative Fujita vanishing [12, Theorem 1.5]. \( \square \)

Lemma 4.3. Let \( \mathcal{E} \) be a vector bundle on a projective variety \( V \). Let \( H \) be a Cartier divisor on \( V \). If \( \mathcal{O}_V(H) \otimes F^e_\mathcal{E} \) is nef for infinitely many \( e \geq 1 \), then \( \mathcal{E} \) is nef.

Proof. Set \( P := \mathbb{P}(\mathcal{E}) := \text{Proj}(\bigoplus_{m \geq 0} S^m(\mathcal{E})) \). Let \( \pi : P \to V \) be the natural projection. Then we have the following commutative diagram:

\[
\begin{array}{ccccccccc}
P & \xrightarrow{\pi^e} & \mathbb{P}(F^e_\mathcal{E}) & \xrightarrow{F^e_{P/V}} & P_{V^e} & \xrightarrow{w^e} & P & \xrightarrow{\pi} & V \\
& & & & \downarrow & & & & \\
& & & & \pi_{V^e} & & & & \\
& & & & \pi_{V^e} & & & & \\
\end{array}
\]

Let \( T \) be a Cartier divisor on \( P \) such that \( \mathcal{O}_P(T) \cong \mathcal{O}_P(1) \). By the assumption, \( \mathcal{O}_{P_{V^e}}(\pi_{V^e}^* H) \otimes \mathcal{O}_{P_{V^e}}(1) \) is nef, so
\[
p^e(p^{-e}\pi^* H + T) = \pi^* H + p^e T = F^e_{P/V}(\pi_{V^e}^* H + w^e T)
\]
is nef, and hence \( p^{-e}\pi^* H + T \) is nef. Note that \( \mathcal{O}_{P_{V^e}}(1) \cong w^e \mathcal{O}_P(1) \). Since this holds for infinitely many \( e \) by the assumption, we conclude that \( T \) is nef. \( \square \)

Definition 4.4. Let \( \mathcal{G} \) be a coherent sheaf on a projective variety \( V \). Then we set
\[
\mathbb{L}(\mathcal{G}) := \{ \mathcal{N} \in \text{Pic}^r(V) | \text{there is a non-zero morphism } \mathcal{G} \to \mathcal{N} \}.
\]

Recall that \( \text{Pic}^r(V) \) is the set of numerically trivial line bundles on \( V \).

Lemma 4.5. Let \( \mathcal{E} \) be a nef vector bundle on a smooth projective variety \( V \). Then \( \mathbb{L}(\mathcal{E}) \) is a finite set.

Proof. Let \( H \) be an ample divisor on \( V \). We set the slope \( \mu(\mathcal{F}) \) of a non-zero torsion-free coherent sheaf \( \mathcal{F} \) as
\[
\mu(\mathcal{F}) := \frac{c_1(\mathcal{F}) \cdot H^{n-1}}{\text{rank}(\mathcal{F})}.
\]
Let \( 0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_n = \mathcal{E} \) be the Harder–Narasimhan filtration of \( \mathcal{E} \). Put \( \mathcal{E}^r := \mathcal{E}/\mathcal{E}_{r-1} \). We first prove that \( \mathbb{L}(\mathcal{E}) \subseteq \mathbb{L}(\mathcal{E}^r) \). Take an \( \mathcal{N} \in \mathbb{L}(\mathcal{E}) \). We show that \( \mathcal{N} \in \mathbb{L}(\mathcal{E}^r) \). By the definition of \( \mathbb{L}(\mathcal{E}) \), there is a non-zero morphism \( \mathcal{E} \to \mathcal{N} \). Let \( \iota \)
be the largest integer in \( \{0, 1, \ldots, n - 1\} \) such that the induced morphism \( \mathcal{E}_i \to \mathcal{N} \) is zero. Consider the induced non-zero morphism \( \mathcal{E}_{i+1}/\mathcal{E}_i \to \mathcal{N} \). Since \( \mathcal{E}_{i+1}/\mathcal{E}_i \) is \( \mu \)-semistable, we have

\[
0 \leq \mu(\mathcal{E}') \leq \mu(\mathcal{E}_{i+1}/\mathcal{E}_i) \leq \mu(\mathcal{N}) = 0.
\]

Note that the first inequality follows from the nefness of \( \mathcal{E} \). Hence we get \( \mu(\mathcal{E}') = \mu(\mathcal{E}_{i+1}/\mathcal{E}_i) = 0 \), which means that \( i = n - 1 \), so we find a non-zero morphism \( \mathcal{E}' \to \mathcal{N} \), i.e., \( \mathcal{N} \in \mathbb{L}(\mathcal{E}') \). Next, we prove that \( \mathbb{L}(\mathcal{E}') \) is a finite set, by the induction on the rank. When \( \text{rank}(\mathcal{E}') = 1 \), a non-zero morphism \( \mathcal{E}' \to \mathcal{N} \) to an \( \mathcal{N} \in \text{Pic}^r(V) \) induces the non-zero morphism \( \mathcal{E}' \to \mathcal{N} \) between line bundles, which is an isomorphism, as \( \mu(\mathcal{E}') = \mu(\mathcal{N}) = 0 \). Thus \( \mathbb{L}(\mathcal{E}') = \{\mathcal{E}'\} \). We next deal with the case when \( \text{rank}(\mathcal{E}') \geq 2 \). Take \( \mathcal{N}_1, \mathcal{N}_2 \in \mathbb{L}(\mathcal{E}') \) with \( \mathcal{N}_1 \not\cong \mathcal{N}_2 \). Let \( \varphi_i : \mathcal{E}' \to \mathcal{N}_i \) denote the given non-zero morphisms for \( i = 1, 2 \). We show that the morphism \( \text{Ker}(\varphi_1) \to \text{Ker}(\varphi_2) \) induced by \( \varphi_2 \) is non-zero. If this holds, then \( \mathcal{N}_2 \in \mathbb{L}(\text{Ker}(\varphi_1)) \), which means that \( \mathbb{L}(\mathcal{E}') \subseteq \mathbb{L}(\text{Ker}(\varphi_1)) \cup \{\mathcal{N}_1\} \), so the assertion follows from the induction hypothesis.

Suppose that \( \alpha = 0 \). Then we obtain a non-zero morphism \( \text{Im}(\varphi_1) \to \mathcal{N}_2 \). By the \( \mu \)-semistability of \( \mathcal{E}' \), we have

\[
0 \leq \mu(\mathcal{E}') \leq \mu(\text{Im}(\varphi_1)) \leq \mu(\mathcal{N}_2) = 0,
\]

so \( \mu(\text{Im}(\varphi_1)) = \mu(\mathcal{N}_2) \), and hence \( (\text{Im}(\varphi_1)) \cong \mathcal{N}_2 \). However, by an argument similar to the above, we get \( (\text{Im}(\varphi_1)) \cong \mathcal{N}_1 \), so \( \mathcal{N}_1 \cong (\text{Im}(\varphi_1)) \cong \mathcal{N}_2 \), which contradicts the choice of \( \mathcal{N}_1 \) and \( \mathcal{N}_2 \). Thus we conclude that \( \alpha \neq 0 \). \( \square \)

**Proof of Theorem 1.1.** By Proposition 3.3, \( \alpha : X \to A \) is a flat surjective morphism with strongly \( F \)-regular closed fibers. Let \( s \) be the Cartier index of \( K_X + \Delta \). Note that \( p \nmid s \). By [14, COROLLARY 6.17], there is a numerically trivial \( \mathbb{Q} \)-Cartier divisor \( L \) on \( A \) such that \( K_X + \Delta \sim_{\mathbb{Q}} \alpha^s L \). Let \( t \) be the smallest positive integer such that \( t(K_X + \Delta) \) and \( tL \) are Cartier and \( t(K_X + \Delta) \sim_{\mathbb{Z}} t\alpha^s L \). Then \( s|t \).

Let \( I := \{i \in \mathbb{Z}_{>0} | 0 \leq i < t \text{ and } s|i\} \).

**Step 1.** Let \( B \) be an ample Cartier divisor on \( X \). Replacing \( B \) by \( lB \) for \( l \gg 0 \), we may assume that the following conditions hold:

(b1) For every closed point \( a \in A \), the natural morphism

\[
(\alpha_*\mathcal{O}_X(B)) \otimes k(a) \to H^0(X_a; \mathcal{O}_{X_a}(B))
\]

is an isomorphism. This follows from the proof of [3, Lemma 3.5] and Grauert’s theorem (see [3, III, Corollary 12.9]).

(b2) For each \( m \in \mathbb{Z}_{>0} \) and each \( i \in I \), the sheaf \( \alpha_*\mathcal{O}_X(-i(K_X + \Delta) + mB) \) is locally free. This follows from the ampleness of \( B \) and the flatness of \( f \).

(b3) For each \( m, n \in \mathbb{Z}_{>0} \) and every \( \alpha \)-nef Cartier divisor \( N \) on \( X \), the multiplication morphism

\[
\alpha_*\mathcal{O}_X(mB + N) \otimes \alpha_*\mathcal{O}_X(nB) \to \alpha_*\mathcal{O}_X((m + n)B + N)
\]

is surjective (by Lemma [4,2]).
(b4) For each $e \in \mathbb{Z}_{>0}$ and every $f$-nef Cartier divisor $N$, the morphism

\[ \alpha^e_*(\mathcal{O}_X((1 - p^e)(K_X + \Delta) + p^e(B + N))) \]

\[ \psi_{(X/A, \Delta)}(B_{A^e} + N_{A^e}) \rightarrow \alpha_{A^e*}\mathcal{O}_{X_{A^e}}(B_{A^e} + N_{A^e}) \cong F_{A^e}^* \alpha_*\mathcal{O}_X(B + N) \]

is surjective, where

\[ \psi_{(X/A, \Delta)}(B_{A^e} + N_{A^e}) := \alpha_{A^e*}\left( \phi_{(X/A, \Delta)}^e(B_{A^e} + N_{A^e}) \right). \]

For the construction of the morphism

\[ \phi_{(X/A, \Delta)}^e(B_{A^e} + N_{A^e}) : F_{X/A}^e \mathcal{O}_X((1 - p^e)(K_X + \Delta) + p^e(B + N)) \rightarrow \mathcal{O}_{X_{A^e}}(B_{A^e} + N_{A^e}), \]

see [21 §3] or [22 §3]. This condition follows from [3, Lemma 3.7]. Note that, by Proposition 3.3, $\alpha$ is a flat morphism whose every closed fiber is strongly $F$-regular.

**Step 2.** Let $r > 0$ be the rank of $\alpha_*\mathcal{O}_X(B)$. We consider the $r$-th fiber product

\[ Y := X \times_A \cdots \times_A X \]

of $X$ over $A$. Since $\alpha$ is flat, so is each projection $pr_i$, and hence we can take the pullback $pr_i^*D$ of every $\mathbb{Q}$-Weil divisor $D$ on $X$. Set $\Gamma := \sum_{i=1}^r pr_i^*\Delta$. Then we get

\[ t(K_Y + \Gamma) = t(K_{Y/A} + \Gamma) \sim \sum_{i=1}^r pr_i^*t(K_{X/A} + \Delta) \sim \sum_{i=1}^r pr_i^* \alpha^*L = rtf^*L. \]

Here, $f : Y \rightarrow A$ is the natural projection. Since $(X_a, \Delta|_{X_a})$ is strongly $F$-regular for all closed fibers $X_a$, the pair $(Y, \Gamma|_{Y_a})$ is also strongly $F$-regular by Lemma 4.1.

Put $B' := \sum_{i=1}^r pr_i^*B$. Since $\alpha : X \rightarrow A$ is flat, we have

\[ f_*\mathcal{O}_Y(B') = f_* \left( \bigotimes_{i=1}^r pr_i^*\mathcal{O}_X(B) \right) \cong \bigotimes_{i=1}^r \alpha_*\mathcal{O}_X(B). \]

Since $r$ is the rank of $\alpha_*\mathcal{O}_X(B)$, we get the morphism

\[ \varphi : \det(\alpha_*\mathcal{O}_X(B)) \rightarrow \bigotimes_{i=1}^r \alpha_*\mathcal{O}_X(B) \]

that is locally defined as

\[ x_1 \land \cdots \land x_r \mapsto \sum_{\sigma \in \mathfrak{S}_r} \text{sign}(\sigma) \cdot x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(r)}, \]

where $\mathfrak{S}_r$ is the symmetric group of degree $r$. Put $\mathcal{H} := \det(\alpha_*\mathcal{O}_X(B))$. Let $H$ be a Cartier divisor on $A$ with $\mathcal{O}_A(H) \cong \mathcal{H}$. Then $\varphi$ induces the morphism

\[ \mathcal{O}_A \rightarrow \mathcal{H}^{-1} \otimes \left( \bigotimes_{j=1}^r \alpha_*\mathcal{O}_X(B) \right) \cong \mathcal{H}^{-1} \otimes f_*\mathcal{O}_Y(B') \cong f_*\mathcal{O}_Y(B' - f^*H). \]

Therefore, there is an effective $f$-ample Cartier divisor $C \sim B' - f^*H$ on $Y$. 

Suppose that $Y_a \subseteq \text{Supp } C$. Then $c$ is the zero-map. Let $V \subseteq A$ be a neighborhood of $a$ such that $(\alpha_* \mathcal{O}_X(B))|_V$ is free, and let $x_1, \ldots, x_r$ be a basis of that. Then

$$0 = \sum_{\sigma \in S_r} \text{sign}(\sigma) \cdot \prod_{i=1}^r \text{pr}_i^*(x_{\sigma(i)}|_{X_a}) \otimes \prod_{i=1}^r \text{pr}_i^*(x_{\sigma(r)}|_{X_a}),$$

which means that

$$\{\prod_{i=1}^r \text{pr}_i^*(x_{\sigma(1)}|_{X_a}) \otimes \cdots \otimes \text{pr}_i^*(x_{\sigma(r)}|_{X_a})\}_{\sigma \in S_r} \subset H^0(Y_a, \mathcal{O}_{Y_a}(B')) \cong \bigotimes_{i=1}^r H^0(X_a, \mathcal{O}_{X_a}(B))$$

is linearly dependent, and so

$$\{x_1|_{X_a}, \ldots, x_r|_{X_a}\} \subset H^0(X_a, \mathcal{O}_{X_a}(B))$$

is also linearly dependent. This contradicts that the natural morphism

$$(\alpha_* \mathcal{O}_X(B))|_V \otimes k(a) \to H^0(X_a, \mathcal{O}_{X_a}(B))$$

is an isomorphism by (b1).

**Step 4.** We prove that $\det(f_* \mathcal{O}_Y(C)) \cong \mathcal{O}_A$. This follows from the following calculation:

$$\det(f_* \mathcal{O}_Y(C)) \cong \det(f_* \mathcal{O}_Y(B' - f^* H))$$

$$\cong \det(H^{-1} \otimes f_* \mathcal{O}_Y(B'))$$

$$\cong \det\left(H^{-1} \otimes \bigotimes_{i=1}^r \alpha_* \mathcal{O}_Y(B)\right)$$

$$\cong H^{-r} \otimes \det\left(\bigotimes_{i=1}^r \alpha_* \mathcal{O}_Y(B)\right) = H^{-r} \otimes H^r \cong \mathcal{O}_A.$$
By (b2), the sheaf $f_\ast \mathcal{O}_Y(mC)$ is a vector bundle. Since $L$ is numerically trivial, we see that $f_\ast \mathcal{O}_Y(mC)$ is weakly positive over $A$, so it is a nef vector bundle (cf. [25, Definition und Lemma 1.10]).

**Step 6.** We prove that $f_\ast \mathcal{O}_Y(-i(K_Y + \Gamma) + C)$ is a nef vector bundle for each $i \in I$, using Lemma [4.3]. For each $a \in \mathbb{Z}_{>0}$, we have the multiplication morphism

$$
\mu_a : \bigotimes^a \alpha_\ast \mathcal{O}_X(mB) \rightarrow \alpha_\ast \mathcal{O}_X(amB),
$$

which is surjective by (b3). Take $e \in \mathbb{Z}_{>0}$ so that $(1 - p^e)(K_Y + \Gamma)$ is Cartier (i.e., $s | (p^e - 1)$). Let $a, b$ be integers such that $p^e = am + b$ and $0 \leq b < m$. Fix $i \in I$. We consider the following sequence of morphisms:

$$
f_\ast \mathcal{O}_Y((1 - p^e - ip^e)(K_Y + \Gamma) + bC) \otimes \Bigg( \bigotimes^a f_\ast \mathcal{O}_Y(mC) \Bigg)
$$

$$
\cong \mathcal{H}^{-p^e} \otimes f_\ast \mathcal{O}_Y((1 - p^e - ip^e)(K_Y + \Gamma) + bB') \otimes \Bigg( \bigotimes^a f_\ast \mathcal{O}_Y(mB') \Bigg)
$$

$$
\cong \mathcal{H}^{-p^e} \otimes \left( \bigotimes^r \left( \alpha_\ast \mathcal{O}_X((1 - p^e - ip^e)(K_Y + \Delta) + bB) \otimes \left( \bigotimes^a \alpha_\ast \mathcal{O}_X(mB) \right) \right) \right)
$$

$$
\xrightarrow{\mu_1} \mathcal{H}^{-p^e} \otimes \left( \bigotimes^r \left( \alpha_\ast \mathcal{O}_X((1 - p^e - ip^e)(K_Y + \Delta) + bB) \otimes \alpha_\ast \mathcal{O}_X(amB) \right) \right)
$$

$$
\xrightarrow{\mu_2} \mathcal{H}^{-p^e} \otimes \left( \bigotimes^r \left( \alpha_\ast \mathcal{O}_X((1 - p^e - ip^e)(K_Y + \Delta) + p^eB) \right) \right)
$$

$$
\cong \mathcal{H}^{-p^e} \otimes (f_\ast \mathcal{O}_Y((1 - p^e - ip^e)(K_Y + \Gamma) + p^eB') + (1 - p^e - ip^e)(K_Y + \Gamma) + p^eC).
$$

Here, $\mu_1$ is induced from $\mu_a$, so $\mu_1$ is surjective. Also, $\mu_2$ is induced from the multiplication map, so it is surjective by (b3), where note that $-(K_Y + \Delta)$ is nef.

By (b4), we get the surjection

$$
(2) \quad f_\ast \mathcal{O}_Y((1 - p^e - ip^e)(K_Y + \Gamma) + p^eC') \twoheadrightarrow F_A^e f_\ast \mathcal{O}_Y(-i(K_Y + \Gamma) + C).
$$

Combining this with the above sequence of morphisms, we obtain the surjection

$$
(3) \quad f_\ast \mathcal{O}_Y((1 - p^e - ip^e)(K_Y + \Gamma) + bC) \otimes \Bigg( \bigotimes^a f_\ast \mathcal{O}_Y(mC) \Bigg)
$$

$$
\twoheadrightarrow F_A^e f_\ast \mathcal{O}_Y(-i(K_Y + \Gamma) + C).
$$

For each $i \in I$, let $c_i, d_i$ be integers such that $ip^e + p^e - 1 = c_i t + d_i$ and $0 \leq d_i < t$. We may assume that $e \gg 0$ so that $c_i > 0$ for each $i \in I$. Then

$$
f_\ast \mathcal{O}_Y((1 - p^e - ip^e)(K_Y + \Gamma) + bC) \cong \mathcal{O}_A(-c_i rtL) \otimes f_\ast \mathcal{O}_Y(-d_i(K_Y + \Gamma) + bC).
$$
Note that $s|d_0$, since $s|(p^e - 1)$ and $s|t$. Put
\[ G := \bigoplus_{0 \leq b < m, 0 \leq d < t, s|d} f_\ast \mathcal{O}_Y(-d(K_Y + \Gamma) + bC). \]

Let $\mathcal{K}$ be an ample line bundle such that $G \otimes \mathcal{K}$ is globally generated. We have the following sequence of surjections:
\[ O_A(-c_i s,tL) \otimes \left( H^0(A, G \otimes \mathcal{K} \otimes_{k} \mathcal{O}_A) \otimes \left( \bigotimes_{a} f_\ast \mathcal{O}_Y(mC) \right) \right) \]
\[ \rightarrow O_A(-c_i s,tL) \otimes G \otimes \mathcal{K} \otimes \left( \bigotimes_{a} f_\ast \mathcal{O}_Y(mC) \right) \]
\[ \rightarrow \mathcal{K} \otimes O_A(-c_i s,tL) \otimes f_\ast \mathcal{O}_Y(-d_i(K_Y + \Gamma) + bC) \otimes \left( \bigotimes_{a} f_\ast \mathcal{O}_Y(mC) \right) \]
\[ \cong \mathcal{K} \otimes f_\ast \mathcal{O}_Y((1 - p^e - ip^e)(K_Y + \Gamma) + bC) \otimes \left( \bigotimes_{a} f_\ast \mathcal{O}_Y(mC) \right) \]
\[ \rightarrow \mathcal{K} \otimes F_A^{e_+} f_\ast \mathcal{O}_Y(-i(K_Y + \Gamma) + C) \]

Here, the second (resp. fourth) morphism comes from the definition of $G$ (resp. morphism (3)). Since $f_\ast \mathcal{O}_Y(mC)$ is a nef vector bundle, the source of the composite of the above morphisms is nef, so the target $\mathcal{K} \otimes F_A^{e_+} f_\ast \mathcal{O}_Y(-i(K_Y + \Gamma) + C)$ is also nef as the composite is surjective. Then Lemma 4.3 implies that $f_\ast \mathcal{O}_Y(-i(K_Y + \Gamma) + C)$ is nef.

**Step 7.** We prove the assertion. By Step 5, the sheaf
\[ \mathcal{F} := \bigoplus_{i \in I} f_\ast \mathcal{O}_Y(-i(K_Y + \Gamma) + C) \]
is a nef vector bundle, so we see from Lemma 4.5 that $\mathbb{L}(\mathcal{F})$ is a finite set. Pick $\mathcal{N} \in \mathbb{L}(\mathcal{F})$. Then there is a non-zero morphism $f_\ast \mathcal{O}_Y(-i(K_Y + \Gamma) + C) \rightarrow \mathcal{N}$ for some $i \in I$. We consider the following morphisms that are generically surjective:
\[ (5) \quad O_A(-c_i s,tL) \otimes \mathcal{F} \otimes \left( \bigotimes_{a} f_\ast \mathcal{O}_Y(C) \right) \xrightarrow{\gamma} F_A^{e_+} f_\ast \mathcal{O}_Y(-i(K_Y + \Gamma) + C) \rightarrow \mathcal{N}^{p^e}. \]

Here, $\gamma$ is constructed by the same construction as that in Step 6. Let $\mathcal{E}$ denote $f_\ast \mathcal{O}_Y(C)$. In Step 4 (resp. Step 6), we proved that $\det(\mathcal{E}) = O_A$ (resp. $\mathcal{E}$ is nef). Hence we see that $\mathcal{E}$ is numerically flat, so Theorem 2.4 tells us that there are algebraically trivial line bundles $\mathcal{N}_1, \ldots, \mathcal{N}_l$ and unipotent vector bundles $\mathcal{U}_1, \ldots, \mathcal{U}_l$ such that
\[ (6) \quad \mathcal{E} \cong \bigoplus_{j=1}^{l} \mathcal{U}_j \otimes \mathcal{N}_j. \]

Then one can easily check that
\[ (i) \quad \mathbb{L}(\mathcal{E}) = \{ \mathcal{N}_1, \ldots, \mathcal{N}_l \}, \]
(ii) there is a filtration

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_{v-1} \subset \mathcal{E}_v = \mathcal{E}$$

such that $\mathcal{E}_{j+1}/\mathcal{E}_j \in \mathbb{L}(\mathcal{E})$ for each $j = 0, 1, \ldots, v - 1$.

Hence, from (ii) and morphism (5), we obtain the non-zero morphism

$$\mathcal{O}_A(-c_i rtL) \otimes \mathcal{F} \otimes \mathcal{P} \to \mathcal{N}^{p^e}, \tag{7}$$

where $\mathcal{P}$ is an element of the set $\mathcal{M}^{(p^e-1)}$ that is defined as follows: for each $\mu \in \mathbb{Z}_{>0}$, we define $\mathcal{M}^{(\mu)}$ by

$$\mathcal{M}^{(\mu)} := \left\{ \lambda \prod_{j=1}^{\lambda} \mathcal{M}_j \right\} \mathcal{M}_j \in \mathbb{L}(\mathcal{F}) \text{ and } 0 \leq n_j \in \mathbb{Z} \text{ with } \sum_{j=1}^{\lambda} n_j = \mu \right\}.$$

Note that $\mathbb{L}(\mathcal{E}) \subseteq \mathbb{L}(\mathcal{F})$. From morphism (7), we obtain that

$$\mathcal{Q} := \mathcal{O}_A(c_i rtL) \otimes \mathcal{N}^{p^e} \otimes \mathcal{P}^{-1} \in \mathbb{L}(\mathcal{F}),$$

which means that

$$\mathcal{N}^{p^e} \otimes \mathcal{O}_A(c_i rtL) \cong \mathcal{P} \otimes \mathcal{Q} \in \mathcal{M}^{(p^e)}.$$

Thus, we see that for each $\mathcal{N} \in \mathbb{L}(\mathcal{F})$, there is $i \in I$ such that

$$\mathcal{N}^{p^e} \otimes \mathcal{O}_A(c_i rtL) \in \mathcal{M}^{(p^e)}.$$

Set $\lambda := |\mathbb{L}(\mathcal{F})|$ and $\mu := (p^e - 1)\lambda + 1$. Take $\mathcal{M} \in \mathcal{M}^{(\mu)}$. Then, we obtain from the pigeonhole principle that there is $\mathcal{N} \in \mathbb{L}(\mathcal{F})$ with $\mathcal{M} \otimes \mathcal{N}^{-p^e} \in \mathcal{M}^{(\mu-p^e)}$, and so

$$\mathcal{M} \otimes \mathcal{O}_A(c_i rtL) \cong \mathcal{M} \otimes \mathcal{N}^{-p^e} \otimes \mathcal{N}^{p^e} \otimes \mathcal{O}_A(c_i rtL) \in \mathcal{M}^{(\mu)}$$

for some $c_i$. We replace $\mathcal{M}$ with $\mathcal{M} \otimes \mathcal{O}_A(c_i rtL)$ and repeat the same argument as above, then we find an $i_2 \in I$ such that

$$\mathcal{M} \otimes \mathcal{O}_A((c_i + c_{i_2})rtL) \in \mathcal{M}^{(\mu)}.$$

Recall that $c_i > 0$ for each $i \in I$ by the choice of $e$. Repeating this argument again and again, we get that the set

$$\mathcal{C} := \{ c \in \mathbb{Z}_{>0} \mid \mathcal{M} \otimes \mathcal{O}_A(crtL) \in \mathcal{M}^{(\mu)} \}$$

is an infinite set. If $L \not\sim \mathcal{Q} 0$, then $\{ \mathcal{O}_A(crtL) \mid c \in \mathcal{C} \}$ is an infinite set, so

$$\{ \mathcal{M} \otimes \mathcal{O}_A(crtL) \mid c \in \mathcal{C} \}$$

is also an infinite set, but this set is contained in $\mathcal{M}^{(\mu)}$, which contradicts the finiteness of $\mathcal{M}^{(\mu)}$. Thus, we conclude that $L \sim \mathcal{Q} 0$. \qed
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