On the failure of the Hörmander multiplier theorem in a limiting case

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Abstract. We discuss the Hörmander multiplier theorem for $L^p$ boundedness of Fourier multipliers in which the multiplier belongs to a fractional Sobolev space with smoothness $s$. We show that this theorem does not hold in the limiting case $|1/p - 1/2| = s/n$.

1. Introduction

Let $m$ be a bounded function on $\mathbb{R}^n$. We define the associated linear operator

$$T_m(f)(x) = \int_{\mathbb{R}^n} \hat{f}(\xi) m(\xi) e^{2\pi i x \cdot \xi} \, d\xi, \quad x \in \mathbb{R}^n,$$

where $f$ is a Schwartz function on $\mathbb{R}^n$ and $\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} \, dx$ is the Fourier transform of $f$. The problem of characterizing the class of functions $m$ for which the operator $T_m$ admits a bounded extension from $L^p(\mathbb{R}^n)$ to itself for all $1 < p < \infty$ is one of the principal questions in harmonic analysis. We say that $m$ is an $L^p$ Fourier multiplier if the above mentioned property is satisfied. While it is a straightforward consequence of Plancherel’s identity that all bounded functions are $L^2$ Fourier multipliers, the structure of the set of $L^p$ Fourier multipliers for $p \neq 2$ turns out to be significantly more complicated.

A classical theorem of Mikhlin [10] asserts that if the condition

$$|\partial^\alpha m(\xi)| \leq C_\alpha |\xi|^{-|\alpha|}, \quad \xi \neq 0,$$

(1.1)

is satisfied for all multiindices $\alpha$ with size $|\alpha| \leq [n/2] + 1$, then $T_m$ admits a bounded extension from $L^p(\mathbb{R}^n)$ to itself for all $1 < p < \infty$. A subsequent result by Hörmander [9] showed that the pointwise estimate (1.1) can be replaced by a weaker Sobolev-type condition:

$$\sup_{R>0} R^{-n+2|\alpha|} \int_{\{\xi \in \mathbb{R}^n : R<|\xi|<2R\}} |\partial^\alpha m(\xi)|^2 \, d\xi < \infty.$$

(1.2)

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Although theorems of Mikhlin and Hörmander admit a variety of applications, their substantial limitation stems from the fact that they can only be applied to functions which are $L^p$ Fourier multipliers for all values of $p \in (1, \infty)$. One can overcome this difficulty using an interpolation argument as in Calderón and Torchinsky [1] or Connett and Schwartz [2], [3]; the conclusion is, roughly speaking, that the closer $p$ is to 2, the fewer derivatives are needed in conditions (1.1) or (1.2).

To be able to formulate things precisely, let us now recall the notion of fractional Sobolev spaces. For $s > 0$ we denote by $(I - \Delta)^{s/2}$ the operator given on the Fourier transform side by multiplication by $(1 + 4\pi^2|\xi|^2)^{s/2}$. If $1 < r < \infty$, then the norm in the fractional Sobolev space $L^r_s$ is defined by

$$\|f\|_{L^r_s} = \|(I - \Delta)^{s/2} f\|_{L^r}.$$  

The version of the Mikhlin–Hörmander multiplier theorem due to Calderón and Torchinsky ([1], Theorem 4.7) says that inequality (1.3)

$$\|T_m f\|_{L^p} \leq C \sup_{D \in \mathbb{Z}} \|\phi(\xi) m(2^D \xi)\|_{L^r_s} \|f\|_{L^p}$$

holds provided that

$$\left|\frac{1}{p} - \frac{1}{2}\right| < \frac{s}{n}.$$  

Here, $\phi$ stands for a smooth function on $\mathbb{R}^n$ supported in the set $\{\xi \in \mathbb{R}^n : 1/2 < |\xi| < 2\}$ and satisfying $\sum_{D \in \mathbb{Z}} \phi(2^D \cdot) = 1$. Additionally, it was pointed out in [4] that the equality $|1/p - 1/2| = 1/r$ is not essential for (1.3) to be true, and (1.4) can thus be replaced by the couple of inequalities

$$\left|\frac{1}{p} - \frac{1}{2}\right| \leq \frac{s}{n}, \quad \frac{1}{r} \leq \frac{s}{n}.$$  

Let us notice that the latter inequality in (1.5) is dictated by the embedding of $L^r_s$ into the space of essentially bounded functions. Related to this we also mention that the Sobolev-type condition in (1.3) can be further weakened by replacing the Sobolev space $L^r_s$ with $L^{n/s}$, see [7].

Let us now discuss the sharpness of the first condition in (1.5). It is well known that if inequality (1.3) holds, then we necessarily have $|1/p - 1/2| \leq s/n$, see [8], [17], [11], [12] and [4]. On the critical line $|1/p - 1/2| = s/n$, there are positive endpoint results by Seeger [13], [14], [15]. In particular, it is shown in [15] that inequality (1.3) holds when $|1/p - 1/2| = s/n$ and $r > n/s$ if the Sobolev space $L^r_s$ is replaced by the Besov space $B^s_{1,r}$, defined by

$$\|f\|_{B^s_{1,r}} = \sum_{k=0}^{\infty} 2^{ks} \|(\varphi_k \hat{f})\|_{L^r}.$$  

Here, $\varphi_0$ stands for a Schwartz function on $\mathbb{R}^n$ such that $\varphi_0(x) = 1$ if $|x| \leq 1$ and $\varphi_0(x) = 0$ if $|x| \geq 3/2$, and $\varphi_k(x) = \varphi_0(2^{-k}x) - \varphi_0(2^{1-k}x)$ for $k \in \mathbb{N}$.  


We recall that $B_{s}^{1,r}$ is embedded into $L^{r}$, thanks to the equivalence
\[
\|f\|_{L^{r}} \approx \left( \sum_{k=0}^{\infty} 2^{2ks} |(\varphi_{k}\hat{f})(2^{k})^{\gamma}|^{2} \right)^{1/2}
\]
and to embeddings between sequence spaces.

In this note we show that Hörmander’s condition involving the Sobolev space $L^{r}$ fails to guarantee $L^{p}$ boundedness of $T_{m}$ in the limiting case $|1/p - 1/2| = s/n$. In fact, we can even include the more general Lorentz–Sobolev spaces $L^{r_{1},r_{2}}$ in our discussion, providing thus a negative answer to the open problem A.2 raised in Appendix A of the recent paper [16]. We recall that the Lorentz–Sobolev space $L^{r_{1},r_{2}}$ is defined as
\[
\|f\|_{L^{r_{1},r_{2}}} = \|(I - \Delta)^{s/2} f\|_{L^{r_{1},r_{2}}},
\]
where
\[
\|g\|_{L^{r_{1},r_{2}}} = \begin{cases} \left( \int_{0}^{\infty} t^{r_{2}/r_{1} - 1}(g^{*}(t))^{r_{2}} dt \right)^{1/r_{2}} & \text{if } 1 < r_{1} < \infty \text{ and } 1 \leq r_{2} < \infty, \\
\sup_{t>0} t^{1/r_{1}} g^{*}(t) & \text{if } 1 < r_{1} < \infty \text{ and } r_{2} = \infty.
\end{cases}
\]
Here,
\[
g^{*}(t) = \inf\{\lambda > 0 : \|\{x \in \mathbb{R}^{n} : |g(x)| > \lambda\}\| \leq t\}, \quad t > 0,
\]
stands for the nonincreasing rearrangement of $g$.

Our result has the following form.

**Theorem 1.** Let $1 < p < \infty$, $p \neq 2$, and let $s > 0$ be such that
\[
|1/p - 1/2| = s/n.
\]
Assume that $1 < r_{1} < \infty$, $1 \leq r_{2} < \infty$ and $\phi$ is a smooth function on $\mathbb{R}^{n}$ supported in the set $\{\xi \in \mathbb{R}^{n} : 1/2 < |\xi| < 2\}$. Then there is no finite constant $C$ such that the inequality
\[
\|T_{m}f\|_{L^{p}} \leq C \sup_{D \in \mathbb{Z}} \|\phi(\xi) m(2^{D} \xi)\|_{L^{r_{1},r_{2}}} \|f\|_{L^{p}}
\]
holds for all $m$ and $f$.

The proof of Theorem 1 uses the randomization technique in the spirit of [18], Chapter 4, which has been further developed in [4] and [5].

**2. Proof of Theorem 1**

Let $s > 0$ and let $\Psi$ be a non-identically vanishing Schwartz function on $\mathbb{R}^{n}$ supported in the set $\{\xi \in \mathbb{R}^{n} : |\xi| < 1/2\}$. Then for any fixed integer $K$ and for any $t \in [0,1]$, we define
\[
m_{t}(\xi) = \sum_{N=1}^{K} \sum_{k \in \mathbb{N}^{n} : N2^{N} < |k| < (N+1/2)2^{N}} a_{N,k}(t) \Psi(2^{N} \xi - k),
\]
where \( a_{N,k}(t) \) denotes the sequence of Rademacher functions indexed by the elements of the countable set \( \mathbb{N} \times \mathbb{N}^n \), and \( c_N = 2^{-N^s}N^{-s} \).

**Lemma 2.** Let \( 1 < r_1 < \infty, \ 1 \leq r_2 \leq \infty \), and let \( \phi \) be as in Theorem 1. Then

\[
\sup_{D \in \mathbb{Z}} \| \phi(\xi)m(2^D\xi)\|_{L^{r_1}_{\nu}r_2} \leq C,
\]

with \( C \) independent of \( t \) and \( K \).

**Proof.** We first observe that it is enough to consider the case when \( r_1 = r_2 \); the general case then follows by real interpolation. For simplicity of notation, we write \( r = r_1 \) in what follows.

One can verify that \( \phi(\xi)m(2^D\xi) = 0 \) for all \( \xi \) if \( D < -1 \). We can thus assume that \( D \geq -1 \). For any such \( D \), we denote

\[
A_D = \left\{ \xi \in \mathbb{R}^n : \frac{1}{2} - \frac{1}{4 \cdot 2^D} < |\xi| < 2 + \frac{3}{4 \cdot 2^D} \right\}.
\]

Using the version of the Kato–Ponce inequality from [6], we get

\[
(2.1) \quad \| \phi(\xi)m(2^D\xi)\|_{L^r_{\nu}} = \|(I - \Delta)^{s/2}[\phi(\xi)m(2^D\xi)]\|_{L^r_{\nu}}
\]

\[
= \|(I - \Delta)^{s/2} [\phi(\xi)\chi_{A_D}(\xi)m(2^D\xi)]\|_{L^r_{\nu}}
\]

\[
\leq \|(I - \Delta)^{s/2} [\phi(\xi)]\|_{L^\infty} \| \chi_{A_D}(\xi)m(2^D\xi)\|_{L^r_{\nu}}
\]

\[
+ \| \phi(\xi)\|_{L^\infty} \|(I - \Delta)^{s/2} [\chi_{A_D}(\xi)m(2^D\xi)]\|_{L^r_{\nu}}
\]

\[
\leq \| \chi_{A_D}(\xi)m(2^D\xi)\|_{L^r_{\nu}} + \| (-\Delta)^{s/2} [\chi_{A_D}(\xi)m(2^D\xi)]\|_{L^r_{\nu}},
\]

since \( \phi \) is a Schwartz function.

An elementary calculation yields that the last two terms in (2.1) are bounded by a constant independent of \( D, t \) and \( K \). Indeed, using support properties of \( \Psi \) and the definition of the set \( A_D \), we deduce that

\[
(2.2) \quad \chi_{A_D}(\xi)m(2^D\xi)
\]

\[
= \min_{N = \max(1,2D+1)} \sum_{k \in \mathbb{N}^n : N2^N < |k| < (N+1)/2^N} c_{NA_N,k}(t) \Psi(2^{N+D}\xi - k).
\]

Since \( |c_N| \leq 1 \) for all \( N \) and the functions \( \Psi(2^{N+D}\xi - k) \) have pairwise disjoint supports in \( N \) and \( k \) (for the fixed \( D \)), we obtain

\[
(2.3) \quad \| \chi_{A_D}(\xi)m(2^D\xi)\|_{L^r} \leq C(n,r,\Psi).
\]

Further, we denote \( \Phi = (-\Delta)^{s/2}\Psi \) and observe that

\[
(-\Delta)^{s/2} [\chi_{A_D}(\cdot)m(2^D\xi)](\xi)
\]

\[
= \min_{N = \max(1,2D+1)} \sum_{k \in \mathbb{N}^n : N2^N < |k| < (N+1)/2^N} a_{N,k}(t) \Phi(2^{N+D}\xi - k).
\]
Let $\alpha > n + n/r$ be an integer. Since $\Phi$ is a Schwartz function, we have
$$|\Phi(\xi)| \lesssim (1 + |\xi|)^{-\alpha}.$$  
This yields
$$|(-\Delta)^{s/2} [\chi_{A_D} (\cdot) m_t(2^D \cdot)](\xi)|$$
$$\lesssim \sum_{N = \max(1, 2^{D-1})}^{\min(K, 2^{D+1})} \sum_{k \in \mathbb{N}^n : N 2^N < |k| < (N+1/2) 2^N} (1 + |2^N + D \xi - k|)^{-\alpha}$$
$$\approx \sum_{N = \max(1, 2^{D-1})}^{\min(K, 2^{D+1})} \int_{\{z \in \mathbb{R}^n : N 2^N < |z + 2^N D \xi| < (N+1/2) 2^N\}} (1 + |z|)^{-\alpha} dz.$$  
Now, if $z \in \mathbb{R}^n$ satisfies $N 2^N < |z + 2^N D \xi| < (N+1/2) 2^N$ then it can be verified that $|z| > 2^N$ holds for all but three values of $N$ (the exceptional $N$’s are those close to $2^D |\xi|$). In addition, if $|\xi|$ is large (say, $|\xi| \geq 6$) and $N \leq 2^{D+1}$ then $|z| > 2^{N-2} |\xi|$. This yields
$$|(-\Delta)^{s/2} [\chi_{A_D} (\cdot) m_t(2^D \cdot)](\xi)| \leq C(n, \alpha, s, \Psi), \quad \xi \in \mathbb{R}^n,$$
and
$$|(-\Delta)^{s/2} [\chi_{A_D} (\cdot) m_t(2^D \cdot)](\xi)| \leq C(n, \alpha, s, \Psi) |\xi|^{n-\alpha}, \quad |\xi| \geq 6.$$
A combination of estimates (2.4) and (2.5) then implies
$$\|(-\Delta)^{s/2} [\chi_{A_D} \cdot m_t(2^D \cdot)]\|_{L^r} \leq C(n, r, s, \Psi).$$  
Finally, combining estimates (2.1), (2.3) and (2.6), we obtain the desired conclusion.  

Proof of Theorem 1. We may assume, without loss of generality, that $p < 2$; the result in the case $p > 2$ will then follow by duality.

Let $t$, $K$ and $m_t$ be as described at the beginning of this section, and let $\varphi$ be a Schwartz function such that $\varphi(\xi) = 1$ if $|\xi| \leq 2$. Define a function $f$ via its Fourier transform by $\hat{f}(\xi) = \varphi(\xi/K)$. Then $\hat{f}(\xi) = 1$ if $|\xi| \leq 2K$. It is straightforward to verify that $m_t(\xi)$ is supported in the set $|\xi| < 2K$. Therefore, we have
$$m_t(\xi) \hat{f}(\xi) = m_t(\xi),$$  
and so
$$T_{m_t} f(x) = \sum_{N = 1}^{K} c_N \sum_{k \in \mathbb{N}^n : N 2^N < |k| < (N+1/2) 2^N} a_{N,k}(t) 2^{-nN} (f^{-1} \Psi) \left(\frac{x}{2^N}\right) e^{2\pi i x \cdot \frac{k}{2^N}}.$$
By Fubini’s theorem and Khintchine’s inequality, we obtain
\[
\int_0^1 \|T_m f(x)\|_{L^p}^p \, dt = \int_{\mathbb{R}^n} \int_0^1 |T_m f(x)|^p \, dt \, dx
\]
\[
\approx \int_{\mathbb{R}^n} \left( \sum_{K=1}^{K} \sum_{k \in \mathbb{N}^n : \ 2^{N-1} < |k| < 2^{N+1} 2^N} c_N^2 2^{-2nN} \left( \mathcal{F}^{-1} \Psi \left( \frac{x}{2^N} \right) \right)^2 \right)^{p/2} \, dx
\]
\[
\approx \sum_{N=1}^{K} c_N^2 N^{n-1} 2^{-nN} \left( \mathcal{F}^{-1} \Psi \left( \frac{x}{2^N} \right) \right)^2 \, dx.
\]

Let $A > 0$ be such that $\mathcal{F}^{-1} \Psi$ does not vanish in $\{ y \in \mathbb{R}^n : A \leq |y| < 2A \}$. Then
\[
\int_0^1 \|T_m f(x)\|_{L^p}^p \, dt
\]
\[
\approx \sum_{N=1}^{K} c_N^2 N^{n-1} 2^{-nN} \left( \mathcal{F}^{-1} \Psi \left( \frac{y}{2^N} \right) \right)^p \, dy
\]
\[
\approx \sum_{N=1}^{K} c_N^2 N^{n-1} 2^{-nN} \left( \mathcal{F}^{-1} \Psi \right)^p \, dy
\]
\[
\approx \sum_{N=1}^{K} c_N^2 N^{n-1} 2^{-nN} \left( \mathcal{F}^{-1} \Psi \right)^p \, dy
\]
\[
= K \sum_{N=1}^{K} N^{p-n} p-2
\]

where the last equality follows from (1.6). We observe that $np - n - p/2 > -1$ as
\[
\frac{p}{1} > \frac{n - 1}{n - 1/2}.
\]
Thus,
\[
(2.7) \quad \int_0^1 \|T_m f(x)\|_{L^p}^p \, dt \approx K^{n}p-n-p/2+1.
\]

Let us now estimate the $L^p$-norm of $f$. Since $f(x) = K^n(\mathcal{F}^{-1} \varphi)(Kx)$, we obtain
\[
(2.8) \quad \|f\|_{L^p}^p = K^n \int_{\mathbb{R}^n} |(\mathcal{F}^{-1} \varphi)(Kx)|^p \, dx
\]
\[
= K^{n}p-n \int_{\mathbb{R}^n} |(\mathcal{F}^{-1} \varphi)(y)|^p \, dy \approx K^{n}p-n.
\]
Assume that inequality (1.7) is satisfied. Then, applying (1.7) with $m = m_t$, integrating with respect to $t$ and using Lemma 2, we get

$$
\int_0^1 \|T_{m_t} f(x)\|_{L^p}^p \, dt \leq C \|f\|_{L^p}^p,
$$

which implies, via (2.7) and (2.8), that

$$K^{np-n-p/2+1} \leq C K^{np-n},$$

or, equivalently,

$$(2.9) \quad K^{1-p/2} \leq C.$$

As $p < 2$, we have $\lim_{K \to \infty} K^{1-p/2} = \infty$, which contradicts (2.9). The proof is complete.

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References

[1] Calderón, A. P. and Torchinsky, A.: Parabolic maximal functions associated with a distribution. II. *Advances in Math.* 24 (1977), no. 2, 101–171.

[2] Connett, W. C. and Schwartz, A. L.: The theory of ultraspherical multipliers. *Mem. Amer. Math. Soc.* 9 (1977), no. 183, iv+92 pp.

[3] Connett, W. C. and Schwartz, A. L.: A remark about Calderón’s upper s method of interpolation. In *Interpolation spaces and allied topics in analysis (Lund, 1983)*, 48–53. Lecture Notes in Math. 1070, Springer, Berlin, 1984.

[4] Grafakos, L., He, D., Honzík, P. and Nguyen, H. V.: The Hörmander multiplier theorem I: The linear case revisited. *Illinois J. Math.* 61 (2017), no. 1-2, 25–35.

[5] Grafakos, L., He, D. and Slavíková, L.: $L^2 \times L^2 \to L^1$ boundedness criteria. To appear in *Math. Ann.* Doi: https://doi.org/10.1007/s00208-018-1794-5, 2019.

[6] Grafakos, L. and Oh, S.: The Kato–Ponce inequality. *Comm. Partial Differential Equations* 39 (2014), no. 6, 1128–1157.

[7] Grafakos, L. and Slavíková, L.: A sharp version of the Hörmander multiplier theorem. *Int. Math. Res. Not. IMRN* 2019, no. 15, 4764–4783.

[8] Hirschman, I. I., Jr.: On multiplier transformations. *Duke Math. J.* 26 (1959), 221–242.

[9] Hörmander, L.: Estimates for translation invariant operators in $L^p$ spaces. *Acta Math.* 104 (1960), 93–140.

[10] Mikhlin, S. G.: On the multipliers of Fourier integrals. (Russian) *Dokl. Akad. Nauk SSSR (N.S.)* 109 (1956), 701–703.

[11] Miyachi, A.: On some Fourier multipliers for $H^p(\mathbb{R}^n)$. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* 27 (1980), no. 1, 157–179.
[12] Miyachi, A. and Tomita, N.: Minimal smoothness conditions for bilinear Fourier multipliers. Rev. Mat. Iberoam. 29 (2013), no. 2, 495–530.

[13] Seeger, A.: A limit case of the Hörmander multiplier theorem. Monatsh. Math. 105 (1988), no. 2, 151–160.

[14] Seeger, A.: Estimates near $L^1$ for Fourier multipliers and maximal functions. Arch. Math. (Basel) 53 (1989), no. 2, 188–193.

[15] Seeger, A.: Remarks on singular convolution operators. Studia Math. 97 (1990), no. 2, 91–114.

[16] Seeger, A. and Trebels, W.: Embeddings for spaces of Lorentz–Sobolev type. Math. Ann. 373 (2019), no. 3-4, 1017–1056.

[17] Wainger, S.: Special trigonometric series in $k$-dimensions. Mem. Amer. Math. Soc. 59 (1965), 102 pp.

[18] Wolff, T. H.: Lectures on harmonic analysis. University Lecture Series 29, Amer. Math. Soc., Providence, RI, 2003.

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