CATEGORIFYING INVOLUTIVE BIALGEBRAS AND INVOLUTIVE HOPF ALGEBRAS

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Abstract. The category of involutive non-commutative sets encodes the structure of an associative algebra with involution over a commutative ground ring. We prove that the categories of involutive bialgebras and involutive Hopf algebras are equivalent to categories of algebras over a PROP constructed from the category of involutive non-commutative sets.

Introduction

The categorification of algebras over a unital commutative ring \( k \) to algebras over a PROP was first introduced by Markl in order to study the deformation theory of algebras [Mar96]. In that paper he defined PROPs, in terms of generators and relations, whose categories of algebras are equivalent to the category of associative algebras, the category of commutative algebras and the category of bialgebras over \( k \) [Mar96, Examples 2.5, 2.6 and 2.7].

Pirashvili [Pir02] gave an explicit description of a PROP that categorified associative algebras, commutative algebras and bialgebras. This PROP is constructed from the category of non-commutative sets, introduced by Feigin and Tsygan [FT87, A10], using the generalized Quillen \( Q \)-construction of Fiedorowicz and Loday [FL91, 2.5].

In this paper we introduce the PROP of involutive non-commutative sets, denoted \( IF(as) \). Following the argument of Pirashvili we categorify involutive bialgebras over a commutative ring to algebras over a PROP constructed from \( IF(as) \) via the generalized \( Q \)-construction. As a corollary we show that a subcategory of algebras over this PROP is equivalent to the category of involutive Hopf algebras.

The paper is organized as follows. In Section 1 we recall the definitions of involutive bialgebras and involutive Hopf algebras over a commutative ring, together with some examples of interest.

In Section 2 we introduce the PROP of involutive non-commutative sets, \( IF(as) \). We prove that the category of algebras over this PROP is equivalent to the category of involutive \( k \)-algebras. We also prove that the category of \( IF(as)^{op} \)-algebras is equivalent to the category of involutive \( k \)-coalgebras.

In Section 3 we recall the definition of a double category and the notion of a Mackey functor for a double category. We construct a double category, denoted \( IF(as)_2 \), from the category of involutive non-commutative sets. We prove that the structure of an involutive bialgebra can be encoded as a Mackey functor from \( IF(as)_2 \) to the category of \( k \)-modules.

In Section 4 we recall the generalized \( Q \)-construction. We obtain a PROP, denoted \( Q \), by applying the \( Q \)-construction to the double category \( IF(as)_2 \). We prove that the category
of $Q$-algebras is equivalent to the category of involutive bialgebras. We deduce an equivalence of categories between a subcategory of $Q$-algebras and the category of involutive Hopf algebras as a corollary.

1. **Algebras, Coalgebras, Bialgebras and Hopf Algebras**

We recall the notions of involutive bialgebra and involutive Hopf algebra over a commutative ring $k$.

**Definition 1.1.** For a $k$-module $M$ the *twist map* $\tau : M \otimes M \rightarrow M \otimes M$ is determined by $m_1 \otimes m_2 \mapsto m_2 \otimes m_1$.

1.1. **Algebras.**

**Definition 1.2.** A $k$-algebra is a $k$-module $A$ together with

- a $k$-bilinear map $\mu : A \otimes A \rightarrow A$ called *multiplication* and
- a $k$-linear map $\eta : k \rightarrow A$ called the *unit*

satisfying the conditions

$$\mu \circ (\mu \otimes \text{id}_A) = \mu \circ (\text{id}_A \otimes \eta) \quad \text{and} \quad \mu \circ (\eta \otimes \text{id}_A) = \text{id}_A = \mu \circ (\text{id}_A \otimes \eta)$$

under the canonical isomorphisms $k \otimes A \cong A \cong A \otimes k$.

**Definition 1.3.** An *involution* on a $k$-algebra $A$ is an anti-homomorphism of algebras $j : A \rightarrow A$ which squares to the identity. That is, a $k$-linear map $j$ such that

$$j^2 = \text{id}_A, \quad j \circ \eta = \eta \quad \text{and} \quad j \circ \mu = \mu \circ (j \otimes j) \circ \tau,$$

where $\tau$ is the twist map of Definition 1.1. A $k$-algebra equipped with an involution is called *involutive*.

Let $\mathbf{IAlg}$ denote the category of involutive $k$-algebras and involution-preserving algebra morphisms.

1.2. **Coalgebras.**

**Definition 1.4.** A $k$-coalgebra is a $k$-module $C$ together with

- a $k$-linear map $\Delta : C \rightarrow C \otimes C$ called the *comultiplication* and
- a $k$-linear map $\varepsilon : C \rightarrow k$ called the *counit*

satisfying the conditions

$$(\text{id}_C \otimes \Delta) \circ \Delta = (\Delta \otimes \text{id}_C) \circ \Delta \quad \text{and} \quad (\text{id}_C \otimes \varepsilon) \circ \Delta = \text{id}_C = (\varepsilon \otimes \text{id}_C) \circ \Delta$$

under the canonical isomorphisms $C \otimes k \cong C \cong k \otimes C$.

**Definition 1.5.** An *involution* on a $k$-coalgebra $C$ is an anti-homomorphism of coalgebras $j : C \rightarrow C$ which squares to the identity. That is, a $k$-linear map $j$ such that

$$j^2 = \text{id}_C, \quad \varepsilon \circ j = \varepsilon \quad \text{and} \quad \Delta \circ j = (j \otimes j) \circ \tau \circ \Delta = \tau \circ (j \otimes j) \circ \Delta,$$

where $\tau$ is the twist map of Definition 1.1. A $k$-coalgebra equipped with an involution is called *involutive*.

Let $\mathbf{ICoAlg}$ denote the category of involutive $k$-coalgebras and involution-preserving coalgebra morphisms.
1.3. Bialgebras.

**Definition 1.6.** Let $A$ be a $k$-module with both the structure of a $k$-algebra and a $k$-coalgebra. We call $A$ a **$k$-bialgebra** if the relations
\begin{itemize}
  \item $\Delta \circ \mu = (\mu \otimes \mu) \circ (id_A \otimes \tau \otimes id_A) \circ (\Delta \otimes \Delta)$,
  \item $\varepsilon \circ \eta = id_k$, $\varepsilon \circ \mu = \varepsilon \otimes \varepsilon$ and $\Delta \circ \eta = \eta \otimes \eta$
\end{itemize}
hold under the the canonical isomorphism $k \otimes k \cong k$.

**Definition 1.7.** An **involution** on a $k$-bialgebra $A$ is an antihomomorphism of bialgebras $j: A \rightarrow A$ which squares to the identity. That is, a $k$-linear map $j$ such that $j^2 = id_A$ and $j$ is simultaneously a $k$-algebra involution and a $k$-coalgebra involution with respect to the underlying structures. A $k$-bialgebra equipped with an involution is called **involutive**. The category of involutive bialgebras and involution-preserving morphisms is denoted by $\text{IBiAlg}$.

1.4. Hopf algebras.

**Definition 1.8.** A $k$-bialgebra $A$ is called a **Hopf algebra** if it is equipped with a $k$-linear map $S: A \rightarrow A$, called the **antipode**, such that the relation $\mu \circ (S \otimes id_A) \circ \Delta = \eta \circ \varepsilon = \mu \circ (id_A \otimes S) \circ \Delta$ holds.

Note that the antipode of a Hopf algebra is an antihomomorphism of the underlying bialgebra.

**Definition 1.9.** A Hopf algebra is called **involutive** if the antipode squares to the identity. That is, if the antipode is an involution for the underlying bialgebra. The category of involutive Hopf algebras and involution-preserving Hopf algebra morphisms is denoted $\text{IHopf}$.

**Example 1.10.** If a Hopf algebra is commutative or cocommutative then it is involutive ([Swe69, Proposition 4.0.1 6]).

The next three examples are specific instances of this fact, all being cocommutative.

**Example 1.11.** Let $G$ be a group. The **group algebra** $k[G]$ is an involutive Hopf algebra.

**Example 1.12.** Let $M$ be a $k$-module. The **tensor algebra** $T(M)$, the **symmetric algebra** $S(M)$ and the **exterior algebra** $E(M)$ are involutive Hopf algebras.

**Example 1.13.** If $k$ is a field and $\mathfrak{g}$ is a Lie algebra over $k$ then the **universal enveloping algebra** $U(\mathfrak{g})$ is an involutive Hopf algebra.

**Example 1.14.** If a Hopf algebra is cosemisimple over a field of characteristic zero then it is involutive ([LRS88, Theorem 4]).

2. THE PROP $\mathcal{I}F(as)$

We define the PROP of **involutive non-commutative sets**, denoted $\mathcal{I}F(as)$, and prove that the category of $\mathcal{I}F(as)$-algebras is equivalent to the category of involutive $k$-algebras. The opposite category of a PROP is also a PROP and we use this to prove that the category of $\mathcal{I}F(as)^{op}$-algebras is equivalent to the category of involutive $k$-coalgebras. The extra structure of $\mathcal{I}F(as)$ over the PROP $F(as)$ encodes an involution compatible with (co)associative (co)multiplication. We also define the PROP $\mathcal{I}F$ which encodes an involution compatible with (co)commutative (co)multiplication.
2.1. PROPs.

**Definition 2.1.** For \( n \geq 1 \) we define \( \underline{n} \) to be the set \( \{1, \ldots, n\} \). We define \( \underline{0} = \emptyset \).

**Definition 2.2.** A PROP is a symmetric strict monoidal category \((\mathbf{C}, \otimes, I)\) such that, firstly, the objects of \( \mathbf{C} \) are the natural numbers identified with the finite sets \( \underline{n} = \{1, \ldots, n\} \) for \( n \geq 1 \) together with \( \underline{0} = \emptyset = I \) and, secondly, the tensor product is given by addition. That is, \( \underline{m} \otimes \underline{n} = \underline{m+n} \).

**Definition 2.3.** Let \( \mathbf{C} \) be a PROP. A \( \mathbf{C} \)-algebra is a symmetric strict monoidal functor from \( \mathbf{C} \rightarrow \mathbf{kMod} \), where \( \mathbf{kMod} \) is the category of \( \mathbf{k} \)-modules.

**Definition 2.4.** Let \( \mathbf{C} \) be a PROP. We denote the category of \( \mathbf{C} \)-algebras and natural transformations by \( \text{Alg}_{\mathbf{C}} \).

2.2. The PROPs \( \mathcal{IF} (\text{as}) \) and \( \mathcal{IF} \). We define the PROP of involutive, non-commutative sets, \( \mathcal{IF} (\text{as}) \). It will have as objects the sets \( \underline{n} \) of Definition 2.1 for \( n \geq 0 \). An element \( f \in \text{Hom}_{\mathcal{IF} (\text{as})} (\underline{n}, \underline{m}) \) will be a map of sets such that the preimage of each singleton \( i \in \underline{m} \) is a totally ordered set such that each element comes adorned with a superscript label from the group \( C_2 = \langle t | t^2 = 1 \rangle \). Note that for \( m \geq 1 \), the set \( \text{Hom}_{\mathcal{IF} (\text{as})} (\underline{0}, \underline{m}) \) will be the singleton set consisting of the unique set map \( \emptyset \rightarrow \underline{m} \) and \( \text{Hom}_{\mathcal{IF} (\text{as})} (\underline{m}, \underline{0}) \) will be the empty set.

**Remark 2.5.** Henceforth we will say that a morphism in \( \mathcal{IF} (\text{as}) \) is a map of sets together with a labelled, ordered set for each preimage. In particular, note that we will use preimage to mean preimage of a singleton.

**Example 2.6.** Let \( f \in \text{Hom}_{\mathcal{IF} (\text{as})} (\underline{5}, \underline{4}) \) have underlying map of sets

\[
\begin{array}{ccc}
1 & 2 & 3 \\
1 & 2 & 3 & 4 \\
1 & 2 & 3 & 4 & 5 \\
\end{array}
\]

with the following labelled, ordered sets as preimages:

\[
f^{-1}(1) = \{2^1\}, \quad f^{-1}(2) = \{1^t\}, \quad f^{-1}(3) = \{4^t < 5^1\} \quad \text{and} \quad f^{-1}(4) = \{3^t\}.
\]

We will denote composition in \( \mathcal{IF} (\text{as}) \) by \( \bullet \) in order to distinguish from the composition of maps of sets. In particular, we use \( \circ \) for two morphisms in \( \mathcal{IF} (\text{as}) \) if we are referring to the composite of the underlying maps of sets. In order to ease notation we have chosen not to introduce notation for the forgetful functor \( \mathcal{IF} (\text{as}) \rightarrow \text{Set} \).

Let \( f_1 \in \text{Hom}_{\mathcal{IF} (\text{as})} (\underline{n}, \underline{m}) \) and \( f_2 \in \text{Hom}_{\mathcal{IF} (\text{as})} (\underline{m}, \underline{l}) \).

In order to define the composite \( f_2 \bullet f_1 \in \text{Hom}_{\mathcal{IF} (\text{as})} (\underline{n}, \underline{l}) \) we must provide a map of sets and describe the labelled total orderings on each of the preimages.

As a map of sets, \( f_2 \bullet f_1 \) is the composite of the underlying map of sets \( f_2 \circ f_1 \).

In order to specify a labelled, ordered set for the preimage of each singleton in \( \underline{l} \) under the composite we first make a definition.

**Definition 2.7.** We define an action of \( C_2 \), which will be denoted by a superscript, on finite, ordered sets with \( C_2 \)-labels by

\[
\left\{ j_1^{\alpha_1} < \cdots < j_r^{\alpha_r} \right\}^t = \left\{ j_r^{t\alpha_r} < \cdots < j_1^{t\alpha_1} \right\}.
\]

That is, we invert the ordering and multiply each label by \( t \in C_2 \).
Definition 2.8. Let $f_1 \in \text{Hom}_{\mathcal{IF}(as)}(\underline{n}, \underline{m})$ and $f_2 \in \text{Hom}_{\mathcal{IF}(as)}(\underline{m}, \underline{l})$. We define $f_2 \circ f_1 \in \text{Hom}_{\mathcal{IF}(as)}(\underline{n}, \underline{l})$ to have underlying map of sets $f_2 \circ f_1$. We define the labelled totally ordered set $(f_2 \circ f_1)^{-1}(i)$ to be the ordered disjoint union of labelled, ordered sets

$$\prod_{j \in f_2^{-1}(i)} f_1^{-1}(j)^{\alpha_j}.$$ 

Definition 2.9. The category of involutive, non-commutative sets, $\mathcal{IF}(as)$, has as objects the sets $\underline{n}$ of Definition 2.1 for $n \geq 0$. An element of $\text{Hom}_{\mathcal{IF}(as)}(\underline{n}, \underline{m})$ is a map of sets with a total ordering on each preimage such that each element of the domain comes adorned with a superscript label from the group $C_2$. Composition of morphisms is as defined in Definition 2.8.

Remark 2.10. For each $m \geq 1$, the set $\text{Hom}_{\mathcal{IF}(as)}(\underline{0}, \underline{m})$ is the singleton set consisting of the unique set map $\emptyset \to \underline{m}$ and $\text{Hom}_{\mathcal{IF}(as)}(\underline{m}, \underline{0})$ is the empty set.

Remark 2.11. A variant of the category $\mathcal{IF}(as)$ was first introduced in the author’s thesis [Gra19, Part V]. That variant of the category is shown to be isomorphic to $D$, the category associated to the hyperoctahedral crossed simplicial group of Fiedorowicz and Loday [FL91, Section 3].

Remark 2.12. Recall the PROP of non-commutative sets, $\mathcal{F}(as)$, from [Pir02, Section 3]. That is, the category whose objects are the sets $\underline{n}$ for $n \geq 0$ and whose morphisms are maps of sets with a total ordering on the preimage of each singleton in the codomain. We observe that $\mathcal{F}(as)$ is isomorphic to the subcategory of $\mathcal{IF}(as)$ which contains only the morphisms for which every label is $1 \in C_2$.

Definition 2.13. We define the fundamental morphisms $m$, $u$, $i$ and $\tau$ of $\mathcal{IF}(as)$ as follows.

- Let $m \in \text{Hom}_{\mathcal{IF}(as)}(\underline{2}, \underline{1})$ be defined by $m^{-1}(1) = \{1^1 < 2^1\}$,
- let $u$ be the unique morphism in $\text{Hom}_{\mathcal{IF}(as)}(\underline{0}, \underline{1})$,
- let $i \in \text{Hom}_{\mathcal{IF}(as)}(\underline{1}, \underline{1})$ be defined by $i^{-1}(1) = \{1^1\}$ and
- let $\tau_1 \in \text{Hom}_{\mathcal{IF}(as)}(\underline{2}, \underline{2})$ be defined by $\tau_1^{-1}(1) = \{2^1\}$ and $\tau_1^{-1}(2) = \{1^1\}$.

Remark 2.14. The morphism $m$ will encode the multiplication and comultiplication in a bialgebra, the morphism $u$ will encode the unit and counit and the morphism $i$ will encode the involution.

Remark 2.15. We observe that for $n \geq 1$, the unique morphism in $\text{Hom}_{\mathcal{IF}(as)}(\underline{0}, \underline{n})$ is the $n$-fold disjoint union of the fundamental morphism $u$. We therefore denote the unique morphism in $\text{Hom}_{\mathcal{IF}(as)}(\underline{0}, \underline{n})$ by $u^n$.

Definition 2.16. Let $\mathcal{IF}$ be the category whose objects are the sets $\underline{n}$ of Definition 2.1 for $n \geq 0$. A morphism in $\mathcal{IF}$ is a map of sets such that the elements of the preimage of each singleton in the codomain comes adorned with a label from $C_2$. Composition is given by composition of set maps and multiplication of labels.

Definition 2.17. Let $\mathcal{C} = \mathcal{IF}(as)$ or $\mathcal{IF}$. A morphism $f \in \mathcal{C}$ is called a surjection if the underlying map of sets is a surjection. An elementary surjection is a surjection $f : \underline{n} \to \underline{m}$ such that $n - m \leq 1$.

2.3. $\mathcal{IF}(as)$-algebras. We prove that there is an equivalence between the category of $\mathcal{IF}(as)$-algebras and the category of involutive $k$-algebras.
Definition 2.18. Let $A$ be an involutive, associative $k$-algebra with unit. We define a
symmetric strict monoidal functor $L_\ast(A) : I\mathcal{F}(as) \to k\text{Mod}$ on objects by

$$L_\ast(A)(n) = \begin{cases} k & n = 0, \\ A^{\otimes n} & n \neq 0. \end{cases}$$

For $f \in \text{Hom}_{I\mathcal{F}(as)}(n, m)$ where $n > 0$, $L_\ast(A)(f) : A^{\otimes n} \to A^{\otimes m}$ is determined by

$$a_1 \otimes \cdots \otimes a_n \mapsto \left( \prod_{i \in f^{-1}(1)} a_i^{\alpha_i} \right) \otimes \cdots \otimes \left( \prod_{i \in f^{-1}(m)} a_i^{\alpha_i} \right)$$

where $\prod^{\leq}$ is the ordered product with respect to the total orderings of the preimage data of $f$, each $\alpha_i$ is the label on $i$ in the preimage of $f$ and

$$a^\alpha = \begin{cases} a & \alpha = 1 \\ j(a) & \alpha = t. \end{cases}$$

An empty product is defined to be the multiplicative identity $1_A$.

For the unique morphism $u^n \in \text{Hom}_{I\mathcal{F}(as)}(\{0\}, n)$, $L_\ast(A)(u^n) : k \to A^{\otimes n}$ is determined by

$$\lambda \mapsto \eta(\lambda) \otimes \cdots \otimes \eta(\lambda).$$

We call the functor $L_\ast(A)(-)$ the Loday functor.

Definition 2.19. Let

$$L_\ast(-) : I\text{Alg} \to \text{Alg}_{I\mathcal{F}(as)}$$

be the functor which on objects assigns the Loday functor $L_\ast(A)$ to an involutive $k$-algebra
with unit, $A$.

For $f \in \text{Hom}_{I\text{Alg}}(A, B)$, $L_\ast(f)$ is the natural transformation determined by applying $f$ to each tensor factor.

Lemma 2.20. Let $F \in \text{Alg}_{I\mathcal{F}(as)}$. The $k$-module $F(1)$ has the structure of an involutive $k$-algebra with unit.

Proof. Let $A = F(1)$. Recall the fundamental morphisms of $I\mathcal{F}(as)$ from Definition 2.13.
We define $\mu = F(m)$, $\eta = F(u)$ and $j = F(i)$. Since $F$ is a symmetric strict monoidal functor we have morphisms $\mu : A \otimes A \to A$, $\eta : k \to A$ and $j : A \to A$. We note that $F(\tau_1) = \tau$, the twist map of Definition 1.1. We claim that the morphisms $\mu$, $\eta$ and $j$ endow $A$ with the structure of an involutive $k$-algebra.

Firstly, the morphisms $m \bullet (m \otimes \text{id}_A)$ and $m \bullet (\text{id}_A \otimes m)$ are equal in $\text{Hom}_{I\mathcal{F}(as)}(\{2\}, \{1\})$ and so $\mu \circ (\mu \otimes \text{id}_A) = \mu \circ (\text{id}_A \otimes \mu)$.

Secondly, the morphisms $m \bullet (u \otimes \text{id}_A)$ and $m \bullet (\text{id}_A \otimes u)$ are both equal to the identity morphism in $\text{Hom}_{I\mathcal{F}(as)}(\{1\}, \{1\})$ and so $\mu \circ (\eta \otimes \text{id}_A) = \text{id}_A = \mu \circ (\text{id}_A \otimes \eta)$.

We note that $t^2 = \text{id}_A$ and $i \circ u = u$ in $I\mathcal{F}(as)$ so $j^2 = \text{id}_A$ and $j \circ \eta = \eta$.

The composites $i \bullet m$, $m \bullet (i \otimes i) \bullet \tau_1$ and $m \bullet \tau_1 \bullet (i \otimes i)$ are all equal to the morphism $g \in \text{Hom}_{I\mathcal{F}(as)}(\{2\}, \{1\})$ defined by $g^{-1}(1) = \{2^l < 1^l\}$. Therefore, applying the symmetric strict monoidal functor $F$ we obtain the equalities $j \circ \mu = \mu \circ \tau \circ (j \otimes j) = \mu \circ (j \otimes j) \circ \tau$ as required. 

□
Definition 2.21. Let \( \text{ev}_1 : \text{Alg}_{I\mathcal{F}(as)} \to I\text{Alg} \) be the functor defined on objects by

\[
F \mapsto F(1).
\]

Given a natural transformation \( \Theta : F \Rightarrow G \) of \( I\mathcal{F}(as) \)-algebras, we define

\[
\text{ev}_1(\Theta) = \Theta(1) : F(1) \to G(1).
\]

Proposition 2.22. There is an equivalence of categories

\[
I\text{Alg} \simeq \text{Alg}_{I\mathcal{F}(as)}.
\]

Proof. The composite \( \text{ev}_1 \circ L_\star(-) \) is equal to the identity functor on the category \( I\text{Alg} \).

Since \( F \) is symmetric strict monoidal there are isomorphisms of \( k \)-modules

\[
F(n) \cong F(1)^{\otimes n} = L_\star(\text{ev}_1(F))(n),
\]

natural in \( n \). That is, the composite \( L_\star \circ \text{ev}_1 \) is naturally isomorphic to the identity functor on the category \( \text{Alg}_{I\mathcal{F}(as)} \). \( \square \)

2.4. \( I\mathcal{F}(as)^{op} \)-algebras.

Definition 2.23. Let \( C \in I\text{CoAlg} \). We define a symmetric strict monoidal functor

\[
L_\star(C)(-): I\mathcal{F}(as)^{op} \to k\text{Mod}
\]

on objects by

\[
L_\star(C)(n) = \begin{cases} k & n = 0, \\ C^{\otimes n} & n \neq 0. \end{cases}
\]

For \( f \in \text{Hom}_{I\mathcal{F}(as)}(n,m) \) where \( n > 0 \), the morphism \( L_\star(C)(f): A^{\otimes m} \to A^{\otimes n} \) is determined by

\[
a_1 \otimes \cdots \otimes a_m \mapsto b_1^{\alpha_1} \otimes \cdots \otimes b_n^{\alpha_n}
\]

where \( b_i = a_{f(i)} \), \( \alpha_i \) is the label on \( i \) in the preimage data of \( f \) and

\[
\begin{aligned}
\alpha = 1 & \quad \text{if } \alpha = 1, \\
\alpha = t. & \quad \text{if } \alpha = t.
\end{aligned}
\]

For the unique morphism \( u^n \in \text{Hom}_{I\mathcal{F}(as)}(1,n) \), the morphism \( L_\star(C)(u^n): A^{\otimes n} \to k \) is determined by

\[
a_1 \otimes \cdots \otimes a_n \mapsto \varepsilon(a_1) \cdots \varepsilon(a_n).
\]

We call this functor the \textit{coLoday functor}.

Definition 2.24. We define the functor

\[
L^\star(-): I\text{CoAlg} \to \text{Alg}_{I\mathcal{F}(as)^{op}}
\]

be the functor which on objects assigns the coLoday functor \( L^\star(C) \) to an involutive \( k \)-coalgebra with counit, \( C \).

Given \( f \in \text{Hom}_{I\text{CoAlg}}(A,B) \) the natural transformation \( L^\star(f) \) is determined by applying \( f \) to each tensor factor.

Lemma 2.25. Let \( G \in \text{Alg}_{I\mathcal{F}(as)^{op}} \). The \( k \)-module \( G(1) \) has the structure of a \( k \)-coalgebra with counit and involution.

Proof. Let \( C = G(1) \). Recall the fundamental morphisms of Definition 2.13. We define

\[
\Delta = G(m), \varepsilon = G(u) \quad \text{and} \quad j = G(i).
\]

The morphisms \( \Delta, \varepsilon \) and \( j \) endow \( C \) with the structure of an involutive \( k \)-algebra. One verifies the necessary relations similarly to Lemma 2.20. \( \square \)
Definition 2.26. Let $ev^1 : \text{Alg}_{\mathcal{IF}(as)^{op}} \to \text{ICoAlg}$ be the functor defined on objects by

\[ ev^1(G) = G(1). \]

Given a natural transformation $\Theta : G \Rightarrow H$ of $\mathcal{IF}(as)^{op}$-algebras we define

\[ ev^1(\Theta) = \Theta(1) : G(1) \to H(1). \]

Proposition 2.27. There is an equivalence of categories

\[ \text{ICoAlg} \simeq \text{Alg}_{\mathcal{IF}(as)^{op}}. \]

Proof. The proof is similar to Proposition 2.22 with the functors $\mathcal{L}^*(-)$ and $ev^1$. □

Remark 2.28. One can show similarly that the category of $\mathcal{IF}$-algebras is equivalent to the category of commutative $k$-algebras with involution and the category of $\mathcal{IF}^{op}$-algebras is equivalent to the category of cocommutative $k$-coalgebras with involution.

3. Double Categories and Mackey Functors

We construct a double category from $\mathcal{IF}(as)$ and prove that involutive bialgebras correspond to Mackey functors from this double category into the category of $k$-modules. As a corollary we show that involutive Hopf algebras correspond to such Mackey functors that satisfy a simple condition on the fundamental morphisms of $\mathcal{IF}(as)$. We construct a double category from $\mathcal{IF}$ analogously. We also define two double categories that combine the structure of $\mathcal{IF}(as)$ and $\mathcal{IF}$ which will encode commutativity and cocommutativity.

3.1. Double categories. Recall from [FL91, Section 2.1] that a small double category $C$ consists of a set of objects, a set of horizontal morphisms, a set of vertical morphisms and a set of bimorphisms subject to natural composition identities.

Definition 3.1. Let $C$ be a double category. We denote the category of objects and horizontal morphisms by $C_h$. We denote the category of objects and vertical morphisms by $C_v$.

3.2. The double category $\mathcal{IF}(as)_2$.

Definition 3.2. The double category $\mathcal{IF}(as)_2$ has as objects the objects of $\mathcal{IF}(as)$. Furthermore, the sets of horizontal and vertical morphisms in $\mathcal{IF}(as)_2$ are both equal to the set of all morphisms in $\mathcal{IF}(as)$. A bimorphism in $\mathcal{IF}(as)_2$ is a not necessarily commutative square

\[
\begin{array}{ccc}
  n & \xrightarrow{f_1} & p \\
  \varphi_1 & & \varphi \\
  m & \xrightarrow{f} & q
\end{array}
\]

of morphisms in $\mathcal{IF}(as)$ such that

- the underlying diagram of finite sets is a pullback square,
- for all $x \in m$ the map $\varphi_1^{-1}(x) \to \varphi^{-1}(f(x))$ induced by $f_1$ is an isomorphism of labelled, ordered sets and
- for all $y \in p$ the map $f_1^{-1}(y) \to f^{-1}(\varphi(y))$ induced by $\varphi_1$ is an isomorphism of labelled ordered sets.

Remark 3.3. The composition laws of a double category can be verified using the fact that the composite of pullback squares is itself a pullback square and using the composition rule for morphisms in $\mathcal{IF}(as)$ described in Definition 2.8.
Definition 3.4. Let $B_1$, $B_2$, $B_3$, $B_4$ and $I$ denote the bimorphisms

\[
\begin{array}{ccccccc}
1 & \xrightarrow{n^{12}} & 2 & m & u^{12} & 2 & 1 \\
2 & \xrightarrow{m} & 1 & m & id_0 & 1 & 1 \\
\end{array}
\]

respectively in $IF_2$. The morphism $\tau_{2,3}$ is transposition $(2 3)$ with the label $1 \in C_2$ for each preimage. We call $B_1$, $B_2$, $B_3$, $B_4$ and $I$ the fundamental bimorphisms of $IF_2$.

Remark 3.5. The fundamental bimorphisms encode the compatibility conditions of an involutive bialgebra. The notation is chosen such that the bimorphisms $B_1$ to $B_4$ encode the compatibility conditions of a bialgebra and $I$ encodes the compatibility condition of an involution.

Definition 3.6. A bimorphism

\[
\begin{array}{ccc}
m & \xrightarrow{f_1} & p \\
\varphi_1 & \downarrow & \varphi \\
m & \xrightarrow{f} & q \\
\end{array}
\]

in $IF_2$ is called an elementary bimorphism if both $f$ and $\varphi$ are elementary surjections in $IF$, in the sense of Definition 2.17.

Remark 3.7. Every elementary surjection in $IF$ is either a bijection or can be written as the composite of a bijection followed by an order-preserving map. Since $IF$ is a symmetric strict monoidal category, it follows that every elementary bimorphism in $IF_2$ can be constructed from elementary bimorphisms such that at least one of $f$ and $\varphi$ is a bijection and the fundamental bimorphism $B_1$ using only the disjoint union and composition.

Definition 3.8. The double category $IF_2$ is defined similarly to $IF_2$; the objects are those of $IF$, the sets of horizontal and vertical morphisms are the set of morphisms in $IF$ and the bimorphisms are defined similarly to the bimorphisms of $IF_2$.

Definition 3.9. The double category $V$ has as objects the objects of $IF$. The set of vertical morphisms is the set of morphisms in $IF$. The set of horizontal morphisms is the set of morphisms in $IF$. The bimorphisms are defined similarly to those of $IF_2$ except that the horizontal morphisms are now in $IF$.

The double category $H$ is defined similarly; the set of horizontal morphisms is the set of morphisms in $IF$, the set of vertical morphisms is the set of morphisms in $IF$ and the bimorphisms are defined similarly to those of $IF_2$ except that the vertical morphisms are in $IF$.

3.3. Mackey functors.

Definition 3.10. A Janus functor from a double category $C$ to $kMod$ consists of

- a functor $F_\ast \in \text{Fun}(C_h, kMod)$ and
- a functor $F^\ast \in \text{Fun}(C^op, kMod)$

such that for each object $C \in C$, $F_\ast(C) = F^\ast(C)$. We write $F(C)$ for ease of notation.

Definition 3.11. A Mackey functor $F = (F_\ast, F^\ast)$ from a double category $C$ to $kMod$ is a Janus functor such that given a bimorphism
in $C$, the equality
\[ F^*(\varphi)F_*(f) = F_*(f_1)F^*(\varphi_1) \]
holds.

**Definition 3.12.** We denote by $M$ the category of Mackey functors $\mathcal{IF}(\text{as})_2 \rightarrow \mathbf{kMod}$ and natural transformations.

**Definition 3.13.** A Mackey functor $F \in M$ is said to satisfy the dagger condition if it satisfies the relation
\[ F_*(m \cdot (i \uplus \text{id}_1)) F^*(m) = F_*(u)F^*(u) = F_*(m \cdot (\text{id}_1 \uplus i)) F^*(m) \]
on the fundamental morphisms $m$, $u$ and $i$ in $\mathcal{IF}(\text{as})$. We call such an object a Mackey $\dagger$-functor.

We denote the subcategory of $M$ consisting of Mackey $\dagger$-functors and natural transformations by $M_\dagger$.

**Lemma 3.14.** Let $C = \mathcal{IF}(\text{as})_2$, $\mathcal{IF}_2$, $\mathcal{V}$ or $\mathcal{H}$. A Janus functor $F : C \rightarrow \mathbf{kMod}$ is a Mackey functor if and only if

1. for any bijection $g : n \rightarrow n$ one has $F^*(g)F_*(g) = \text{id}_{F(n)}$ and
2. for any elementary bimorphism
\[ \begin{array}{ccc}
\mathbb{N} & \xrightarrow{f_1} & \mathbb{N} \\
\varphi_1 \downarrow & & \downarrow \varphi \\
\mathbb{M} & \xrightarrow{f} & \mathbb{N}
\end{array} \]

one has
\[ F^*(\varphi)F_*(f) = F_*(f_1)F^*(\varphi_1). \]

**Proof.** One can show that the results of [BDFP01, Section 9], up to and including Lemma 9.4 can be restated in terms of these double categories. \(\square\)

Recall the Loday functor of Definition 2.18 and the coLoday functor of Definition 2.23.

**Theorem 3.15.** Let $M$ be a $k$-module that is equipped simultaneously with the structure of an involutive $k$-algebra and an involutive $k$-coalgebra. Then $M$ is an involutive bialgebra if and only if
\[ \mathcal{L}(M) = (\mathcal{L}_*(M), \mathcal{L}^*(M)) : \mathcal{IF}(\text{as})_2 \rightarrow \mathbf{kMod} \]
is a Mackey functor.

**Proof.** Suppose $\mathcal{L}(M)$ is a Mackey functor. Applying $\mathcal{L}(M)$ to the fundamental bimorphisms $B_1$ to $B_4$ we observe that the conditions of Definition 3.14 are satisfied so $M$ has the structure of a bialgebra. Furthermore, applying $\mathcal{L}(M)$ to the fundamental bimorphism $I$ we see that $\mathcal{L}^*(i)\mathcal{L}_*(i) = \text{id}_M$. By the uniqueness of inverses we see that $\mathcal{L}_*(i) = \mathcal{L}^*(i)$ and we have the structure of an involutive bialgebra.

Conversely, suppose that $M$ has the structure of an involutive bialgebra. Condition I of Lemma 3.14 holds for the Janus functor $\mathcal{L}(M)$. It follows from the proof of [BDFP01, ...]}
Lemma 9.1] that Condition 2 of Lemma 3.14 is satisfied for all elementary bimorphisms where at least one of $f$ and $\varphi$ is a bijection.

Since $M$ has the structure of a bialgebra, Condition 4 is also satisfied for the fundamental bimorphism $B_1$. By Remark 3.7 and the fact that $\mathcal{L}(M)$ and $\mathcal{L}^*(M)$ are symmetric strict monoidal functors, Condition 2 of Lemma 3.14 is satisfied for all elementary bimorphisms in $\mathcal{I}\mathcal{F}(as)_2$ and $\mathcal{L}(M)$ is a Mackey functor as required. □

Remark 3.16. If in addition $M$ is commutative (resp. cocommutative) one can restate the theorem in terms of the double category $\mathcal{V}$ (resp. $\mathcal{H}$). If $M$ is both commutative and cocommutative one can restate the theorem in terms of the double category $\mathcal{I}\mathcal{F}$.

Corollary 3.17. Let $M$ be a $k$-module that is equipped simultaneously with the structure of an involutive $k$-algebra and an involutive $k$-coalgebra. Then $M$ is an involutive Hopf algebra if and only if $\mathcal{L}(M)$ is a Mackey ⨁-functor.

Proof. By Theorem 3.15, $M$ is an involutive bialgebra if and only if $\mathcal{L}(M)$ is a Mackey functor. The extra condition here is precisely the relation requiring the involution to be an antipode in a Hopf algebra. □

4. The Main Result

We define category $\mathcal{Q}$ by applying Fiedorowicz and Loday’s generalization of Quillen’s $Q$-construction [FL91, Section 2.5] to the double category $\mathcal{I}\mathcal{F}(as)_2$. We prove that the category of $\mathcal{Q}$-algebras is equivalent to the category of involutive bialgebras.

4.1. The generalized $Q$-construction. The generalized $Q$-construction assigns a category to a double category satisfying the star condition. This construction is analogous to starting with the bicategory of spans [Ben67, 2.6] in a category with pullbacks $C$, and forming the category whose objects are those of $C$ and whose morphisms are equivalence classes of spans.

Definition 4.1. A double category $C$ is said to satisfy the star condition if a horizontal morphism $f: T \to V$ and a vertical morphism $\varphi: S \to V$ with the same codomain determine a unique bimorphism in $C$.

For the remainder of this subsection, let $C = \mathcal{I}\mathcal{F}(as)_2, \mathcal{I}\mathcal{F}_2, \mathcal{V}$ or $\mathcal{H}$.

Remark 4.2. Given a horizontal morphism $f: m \to q$ and a vertical morphism $\varphi: p \to q$ in $C$ we determine a unique bimorphism by first taking the pullback of the underlying maps of sets. The resulting maps have a unique lift to the category $C$ where the preimage data is induced from $f$ and $\varphi$ using the conditions on bimorphisms. Hence $C$ satisfies the star condition.

Remark 4.3. The categories $C_h$ and $C_v$ have the same isomorphisms, namely the morphisms whose underlying map of sets is a bijection.

Definition 4.4. Let

$$\begin{array}{c}
\varphi \quad \varphi_1
\end{array}
\quad
\begin{array}{c}
p \quad f \quad m
\end{array}
\quad
\begin{array}{c}
p \quad f_1 \quad m
\end{array}$$

be two diagrams in $C$. We consider $\varphi$ and $\varphi_1$ as vertical morphisms in $C$. Similarly we consider $f$ and $f_1$ as horizontal morphisms in $C$. We say two such diagrams are equivalent if there exists an isomorphism $h$ such that the diagram

11
commutes.

**Definition 4.5.** Let $QC$ be the category whose objects are the objects of $C$. An element of $\text{Hom}_C(n, m)$ is an equivalence class of diagrams

$$
\begin{array}{c}
\downarrow \\
\phi \\
\end{array}
\begin{array}{c}
p \\
\downarrow h \\
m \\
\end{array}
\begin{array}{c}
\phi_1 \\
p \\
\downarrow f_1 \\
m \\
\end{array}
\begin{array}{c}
f \\
m \\
\end{array}

\text{Composition is defined via the star condition. Given composable morphisms}

$$
\begin{array}{c}
n \\
\phi \\
p \\
\downarrow f \\
m \\
\end{array}
\begin{array}{c}
f \\
m \\
\end{array}
\text{and}
\begin{array}{c}
m \\
\psi \\
p \\
\downarrow g \\
l \\
\end{array}
\begin{array}{c}
g \\
l \\
\end{array}

\text{we have a diagram}

$$
\begin{array}{c}
r \\
\psi_1 \\
p \\
\downarrow \phi \\
n \\
\phi_1 \\
p \\
\downarrow g \\
f \\
\downarrow m \\
\psi \\
\downarrow l \\
\end{array}

\text{by the star condition. We therefore define the composite to be}

$$
\begin{array}{c}
n \\
\psi \circ \phi \\
p \\
\downarrow f_1 \\
r \\
\end{array}
\begin{array}{c}
l \\
\end{array}
$$

**Remark 4.6.** The category $QC$ is a PROP under the disjoint union.

**Definition 4.7.** Let $f \in \text{Hom}_C(n, m)$. We define $i_*(f) \in \text{Hom}_{QC}(n, m)$ to be the morphism

$$
\begin{array}{c}
n \\
\downarrow \text{id}_n \\
p \\
\downarrow f \\
m \\
\end{array}
\begin{array}{c}
f \\
m \\
\end{array}
\begin{array}{c}
n \\
\downarrow \text{id}_n \\
p \\
\downarrow f_1 \\
r \\
\end{array}
\begin{array}{c}
r \\
\end{array}
$$

Let $g \in \text{Hom}_C(m, n)$. We define $i^*(g) \in \text{Hom}_{QC}(m, n)$ to be the morphism

$$
\begin{array}{c}
m \\
\downarrow g \\
p \\
\downarrow \text{id}_m \\
n \\
\end{array}
\begin{array}{c}
n \\
\end{array}
$$

We denote the resulting morphisms of PROPs by

$$
i_* : C_h \rightarrow QC \quad \text{and} \quad i^* : C_{op} \rightarrow QC.$$
Definition 4.10. We denote the subcategory of $\text{Alg}_Q$ consisting of $Q$-†-algebras and natural transformations by $\text{Alg}_{Q^\dagger}$.

Lemma 4.11. The category of Mackey functors $M$ of Definition 3.12 is equivalent to the category of functors $Q \to \mathbf{kMod}$.

Proof. The proof is analogous [Pir02, Lemma 5.1].

Remark 4.12. It follows from the proof of [Pir02, Lemma 5.1] that the equivalence preserves symmetric strict monoidal functors and preserves the dagger condition.

Theorem 4.13. There is an equivalence of categories

$$\text{Alg}_Q \simeq \text{IBiAlg}$$

between the category of $Q$-algebras and the category of involutive bialgebras.

Proof. Suppose $M$ is an involutive bialgebra. By Theorem 3.15, this corresponds to the Mackey functor $\mathcal{L}(M)$, which is symmetric strict monoidal. By Lemma 4.11, this is equivalent to a functor $\mathcal{L}(M): Q \to \mathbf{kMod}$, which is still symmetric strict monoidal by Remark 4.12. This functor is therefore a $Q$-algebra.

Conversely, let $F$ be a $Q$-algebra and set $M = F(\mathbf{1})$. The functor $F \circ i_\ast$ is an $\mathcal{I}\mathcal{F}(as)$-algebra and the functor $F \circ i^\ast$ is an $\mathcal{I}\mathcal{F}(as)^{op}$-algebra. By Propositions 2.22 and 2.27, there are natural transformations of functors $F \circ i_\ast \cong \mathcal{L}_\ast(M)$ and $F \circ i^\ast \cong \mathcal{L}^\ast(M)$ and $M$ has the structure of both an involutive algebra and an involutive coalgebra. By Lemma 4.11, the functor $(\mathcal{L}_\ast(M), \mathcal{L}^\ast(M)) \cong (F \circ i_\ast, F \circ i^\ast)$ is a Mackey functor. Therefore, by Theorem 3.15, $M$ has the structure of an involutive bialgebra as required.

Corollary 4.14. There is an equivalence of categories

$$\text{Alg}_{Q^\dagger} \simeq \text{IHopf}$$

between the category of $Q$-†-algebras and the category of involutive Hopf algebras.

Proof. The proof is analogous to that of Theorem 4.13 using Corollary 3.17 and the fact that the equivalence of Lemma 4.11 preserves the dagger condition.

Corollary 4.15. Recall the double categories $\mathcal{V}$, $\mathcal{H}$ and $\mathcal{I}\mathcal{F}_2$.

1. The category of $Q(\mathcal{V})$-algebras is equivalent to the category of commutative bialgebras with involution. Furthermore $Q(\mathcal{V})$ is isomorphic to the category of finitely generated free monoids with involution, $\text{IMon}$.

2. The category of $Q(\mathcal{H})$-algebras is equivalent to the category of cocommutative bialgebras with involution. Furthermore $Q(\mathcal{H})$ is isomorphic to the category $\text{IMon}^{op}$.

3. The category of $Q(\mathcal{I}\mathcal{F}_2)$-algebras is equivalent to the category of commutative and cocommutative bialgebras with involution.

Proof. The proof is analogous to [Pir02, Theorem 5.2].

Remark 4.16. The author’s attempts to categorify general Hopf algebras in this manner have so far proved unsuccessful. Whilst having a candidate for a suitable PROP, it seems that the notion of an antipode does not encode nicely as a bimorphism in a double category in the same was as an involution does.
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