The representation theory of the monoid of all partial functions on a set and related monoids as EI-category algebras

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Abstract

The (ordinary) quiver of an algebra $A$ is a graph that contains information about the algebra’s representations. We give a description of the quiver of $\mathbb{C}PT_n$, the algebra of the monoid of all partial functions on $n$ elements. Our description uses an isomorphism between $\mathbb{C}PT_n$ and the algebra of the epimorphism category, $E_n$, whose objects are the subsets of $\{1, \ldots, n\}$ and morphism are all total epimorphisms. This is an extension of a well known isomorphism of the algebra of $IS_n$ (the monoid of all partial injective maps on $n$ elements) and the algebra of the groupoid of all bijections between subsets of an $n$-element set. The quiver of the category algebra is described using results of Margolis, Steinberg and Li on the quiver of EI-categories. We use the same technique to compute the quiver of other natural transformation monoids. We also show that the algebra $\mathbb{C}PT_n$ has three blocks for $n > 1$ and we give a natural description of the descending Loewy series of $\mathbb{C}PT_n$ in the category form.

Keywords: Monoid algebras, Quivers, EI-categories.

*This paper is part of the author’s PHD thesis, being carried out under the supervision of Prof. Stuart Margolis. The author’s research was supported by Grant No. 2012080 from the United States-Israel Binational Science Foundation (BSF).
1 Introduction

One of the goals of the study of monoid representations is to relate them to the modern representation theory of associative algebras. Given a (finite) monoid $M$, it is of interest to study the properties of its algebra $\mathbb{C}M$ (all representations in this paper are over the field of complex numbers, $\mathbb{C}$). For instance, monoids for which $\mathbb{C}M$ is semisimple are characterized in [4, Chapter 5] and monoids for which $\mathbb{C}M$ is basic are characterized in [1] (along with some other natural types of algebras). Another important invariant of an associative algebra is its ordinary quiver. Saliola [20] described the quiver of left regular bands and Denton, Hivert, Schilling and Thiéry [6] described the quiver of $J$-trivial monoids. These results were generalized by Margolis and Steinberg in [13] where they described the quiver of a class of monoids called rectangular monoids that includes properly all the finite monoids whose algebras are basic. By a description, we mean that they reduced the computation of the quiver to a problem in the representation theory of the maximal subgroups of the monoid. They also characterized [12] regular monoids whose algebras are directed or co-directed with respect to their natural quasihereditary structure. That means that the quiver has only arrows going upwards or downwards with respect to the partial order on vertices induced from the $J$-order of the monoid. Apart from that, not much is currently known about quivers of monoid algebras. The case of classical transformation monoids is clearly of interest. Since the algebra of $\text{IS}_n$ (the symmetric inverse monoid) is semisimple, its quiver has no arrows. However, for the monoids of partial and total functions on $n$ elements, denoted $\text{PT}_n$ and $\text{T}_n$, we get an interesting question. Putcha [16] observed that $\mathbb{C}\text{PT}_n$ is co-directed (if we multiply from right to left), that means that all the arrows in the quiver are going downwards. He [17] also computed the quiver of $\mathbb{C}\text{T}_n$ up to $n = 4$ and partially computed the quiver for $n > 4$. This was enough to deduce that $\mathbb{C}\text{T}_n$ has two blocks for $n > 3$ and that it is not of finite representation type for $n > 4$. We remark that Ponizovski [15] proved that $\mathbb{C}\text{T}_n$ is of finite representation type for $n \leq 3$ and Ringel [18] proved the same for $n = 4$ along with finding relations for the quiver presentation. Recently, Steinberg [25] showed that the quiver of $\mathbb{C}\text{T}_n$ is acyclic and that the global dimension of $\mathbb{C}\text{T}_n$ is $n - 1$.

In this paper we give a full description of the quiver of $\mathbb{C}\text{PT}_n$. It is known [23] that there is an isomorphism between $\mathbb{C}\text{IS}_n$ and the groupoid algebra of the
groupoid of bijections on an $n$-element set. We extend this isomorphism to an isomorphism between $\mathbb{C}\text{PT}_n$ and the algebra of the epimorphism category $E_n$, that is, objects are subsets of $\{1, \ldots, n\}$ and the morphisms are all total epimorphisms. The infinite version of this category was also studied in [21, Chapter 8]. $E_n$ is not a groupoid, but is an EI-category, that is, any endomorphism is an isomorphism. A formula for computing the quiver of skeletal EI-categories was found independently by Margolis and Steinberg [13, Theorem 6.13] and Li [11]. We use their formula to get a description of the quiver by means of representations of the symmetric group. We then use standard tools from the theory of representations of the symmetric group to get a combinatorial description of the number of arrows between any two vertices. We also deduce that the algebra $\mathbb{C}\text{PT}_n$ has three blocks (for $n > 1$) and we give a natural description for the Loewy series of $\mathbb{C}\text{PT}_n$ in the category form. We then compute the quiver of the algebras of other natural transformation monoids related to $\text{PT}_n$: The monoid of all order-preserving partial functions, the monoid of all order-decreasing partial functions, the monoid of all order-decreasing total functions, and the partial Catalan monoid which is the intersection of the first two.

2 Preliminaries

2.1 Representations

Recall that an algebra over a field $K$ is a ring $A$ that is also a vector space over $K$ such that $k(ab) = (ka)b = a(kb)$ for all $k \in K$ and $a, b \in A$. We will consider only unital, finite dimensional $\mathbb{C}$-algebras, that is, $K$ will always be the field of complex numbers and $A$ will be a unital ring that has finite dimension as a $\mathbb{C}$-vector space. A representation of $A$ is a $\mathbb{C}$-algebra homomorphism $\rho : A \to \text{End}_{\mathbb{C}}(V)$ where $V$ is some (finite dimensional) vector space over $\mathbb{C}$. Equivalently we can say that a representation of $A$ is a (finitely generated, left) module over $A$. A non-zero representation $M$ is called irreducible or simple if it does not have proper submodules other than 0. The set of all irreducible representations of $A$ is denoted $\text{Irr} A$. A representation $M$ is called semisimple if it is a direct sum of simple modules. The algebra $A$ is called semisimple if any $A$-module is semisimple. We denote by $\text{Rad} M$ the radical of a module $M$, which is the minimal submodule such that $M / \text{Rad} M$ is a semisimple module. The radical of $A$ is its radical as a left module over itself, which is also the minimal ideal
such that $A/\text{Rad } A$ is a semisimple algebra. It is well known that $\text{Rad } A$ is also the maximal nilpotent ideal of $A$ and the intersection of all maximal ideals of $A$. Clearly $A$ is semisimple if and only if $\text{Rad } A = 0$. The descending Loewy series of a module $M$ is the decreasing sequence of submodules

$$0 \subseteq \ldots \subseteq \text{Rad}^2 M \subseteq \text{Rad } M \subseteq M$$

and the minimal integer $n$ such that $\text{Rad}^n M = 0$ is called the Loewy length of $M$.

Recall that a non-zero idempotent $e \in A$ is called primitive if the existence of idempotents $f, f' \in A$ such that $f + f' = e$ and $ff' = f'f = 0$ implies that $f = e$ and $f' = 0$ (or vice versa). It is known that any irreducible module $N$ is isomorphic to $Ae/\text{Rad}(Ae)$ for some primitive idempotent $e$.

Recall that the ordinary quiver $Q$ of a finite dimensional algebra $A$ is a directed graph defined in the following way: The vertices of $Q$ are in a one-to-one correspondence with the irreducible representations of $A$ (up to isomorphism). If $N_i$ and $N_j$ are irreducible representations of $A$ (identified with two vertices of the quiver) then the number of arrows from $N_i$ to $N_j$ is

$$\dim \text{Ext}^1(N_i, N_j)$$

which is known to be equal to

$$\dim e_j(\text{Rad } A/\text{Rad}^2 A)e_i$$

where $e_i, e_j$ are primitive idempotents corresponding to $N_i$ and $N_j$ (this number is independent of the specific choice of idempotents). Note that an algebra $A$ is semisimple if and only if its quiver has no arrows at all. More about representations of algebras and quivers can be found in [3].

### 2.2 Monoids and monoid representations

Throughout this paper, all monoids are assumed to be finite. Two elements $a, b$ of a finite monoid $M$ are $\mathcal{J}$-equivalent if they generate the same principal ideal, that is

$$a \mathcal{J} b \iff M a M = M b M.$$ 

A group $H$ which is a subsemigroup of $M$ is called a subgroup of $M$ (but $M$ and
do not necessarily have the same unit element). A \( J \)-class is called \textit{regular} if it contains an idempotent. It is clear that any subgroup \( H \subseteq M \) is contained in some regular \( J \)-class. It is well known that in a finite monoid any two maximal subgroups in the same \( J \)-class are isomorphic. We denote by \( T_n \) and \( \text{PT}_n \) the monoids of all total and partial functions on \( n \) elements, respectively. \( \text{IS}_n \) denotes the monoid of all injective partial functions on \( n \) elements. These monoids are fundamental in monoid theory. For instance, [7] is solely devoted to their study. Note that \( \text{PT}_n \) and \( \text{IS}_n \) are partially ordered by containment of relations. For other basics of semigroup theory the reader is referred to [9].

We denote by \( CM \) the \textit{monoid algebra} of \( M \) over \( \mathbb{C} \), which is the \( \mathbb{C} \)-vector space of formal sums \( \{ \sum \alpha_im_i \mid m_i \in M \} \) where the multiplication is induced by the multiplication of \( M \). Since \( M \) is a finite monoid, \( CM \) is an associative, unital and finite dimensional algebra. Clearly \( \dim CM = |M| \). A \( CM \)-module is also called an \( M \)-module. We also denote the set of irreducible \( M \)-representations by \( \text{Irr}M \).

The case where \( M = G \) is a group is of special importance. Maschke’s theorem says that \( CG \) is always a semisimple algebra and it is known that the number of irreducible \( CG \)-modules equals the number of conjugacy classes of \( G \). We denote the trivial representation of any group \( G \) by \( \text{tr}_G \). Recall that if \( V \) is a \( G \)-representations, then \( V^* = \text{Hom}(V, \mathbb{C}) \) is also a \( G \)-representation with operation \((g \cdot \varphi)(v) = \varphi(g^{-1}v)\). The \textit{character} of a representation \( \rho : CG \rightarrow \text{End} V \) is the function \( \chi_\rho : G \rightarrow \mathbb{C} \) defined by \( \chi_\rho(g) = \text{trace} \rho(g) \). It is well known that two \( G \)-representations \( \rho \) and \( \psi \) are isomorphic if and only if \( \chi_\rho = \chi_\psi \). Moreover, if \( S_i, 1 \leq i \leq r \) are the irreducible modules of \( CG \) with characters \( \chi_i, 1 \leq i \leq r \), and the decomposition of a \( CG \)-module \( N \) is

\[
N = \bigoplus_{i=1}^{r} a_iS_i
\]

then

\[
a_i = \langle \chi_N, \chi_i \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_N(g)\overline{\chi_i(g)}.
\]

In order to simplify notation, we sometimes omit the \( \chi \) and write \( N \) also for the character of \( N \).

Let \( H \subseteq G \) be a subgroup of \( G \) and let \( N \) and \( L \) be modules of \( G \) and \( H \), respectively. We denote by \( \text{Res}^G_H N \) and \( \text{Ind}^G_H L \) the \textit{restriction} and \textit{induction}
modules. We also recall the Frobenius reciprocity theorem which states that

$$\langle \text{Ind}^G_H L, N \rangle = \langle N, \text{Res}^G_H L \rangle$$

where here $M, N, \text{Ind}^G_H L$ and $\text{Res}^G_H L$ are the respective characters. Recall that in the special case where $G = S_n$ is the symmetric group, the irreducible representations correspond to partitions of $n$ (or equivalently, to Young diagrams with $n$ boxes). We use the standard notation for partitions, that is, a partition $\alpha_1 + \ldots + \alpha_k = n$ is denoted $[\alpha_1, \ldots, \alpha_k]$ where $\alpha_i \geq \alpha_{i+1}$. Recall that the partition $[n]$ corresponds to the trivial representation and $[1^n] = [1, \ldots, 1]$ corresponds to the sign representation. More about the basics of group representations can be found in [2, Chapters 5-6] and [19].

Returning to the general case of monoid representations. We recall the fundamental theorem of Munn-Ponizovski (see [8, Thm 7] for a modern proof).

**Theorem 2.1** (Munn-Ponizovski). Let $M$ be a monoid and let $H_1, \ldots, H_n$ be representatives of its maximal subgroups, one for every regular $\mathcal{J}$-class. There is a one-to-one correspondence between irreducible representations of $M$ and the irreducible representations of $H_1, \ldots, H_n$.

$$\text{Irr} \mathbb{C}M \leftrightarrow \bigsqcup_{k=1}^n \text{Irr} \mathbb{C}H_k.$$ 

As a result, we get a partial ordering of the irreducible modules. Let $N_1$ and $N_2$ be irreducible $M$-modules, which correspond to the $H_{i_1}$ and $H_{i_2}$-modules $V_1$ and $V_2$, respectively. Then $N_1 \leq N_2$ if $H_{i_1} \leq \mathcal{J} H_{i_2}$. Now, let $Q$ be the quiver of $\mathbb{C}M$. Since the vertices of the quiver are in a one-to-one correspondence with the irreducible representations of $M$ we get a partial ordering of the vertices of $Q$ as well.

2.3 **Categories**

All categories in this paper are finite. Hence we can regard a category $A$ as a set of objects denoted $A^0$, and a set of morphisms denoted $A^1$. If $a, b \in A^0$ then $A(a, b)$ is the set of morphisms from $a$ to $b$. A category is called a groupoid if any morphism is an isomorphism and an EI-category if every endomorphism is an isomorphism. If $A$ is a category, we can define the category algebra $\mathbb{C}A$ consisting of all linear combinations of morphisms with obvious addition and
multiplication. Recall that if \( f \in A(c,d) \) and \( g \in A(a,b) \) are morphisms such that \( b \neq c \) then \( fg = 0 \) in the category algebra. It is well known that groupoid algebras are semisimple \([23\text{ Section 3}]\). In fact, it is not difficult to check that an EI-category algebra \( CA \) is semisimple if and only if \( A \) is a groupoid.

A morphism \( f \in A^1 \) is called irreducible if it is not left or right invertible but whenever \( f = gh \), either \( g \) is left invertible or \( h \) is right invertible. The set of irreducible morphisms from \( a \) to \( b \) is denoted \( \text{IRR} A(a,b) \). Note that if \( A \) is an EI-category, a morphism \( f \) is left or right invertible if and only if it is an isomorphism. Indeed, let \( f \in A(a,b) \) be a morphism such that \( fg = 1_b \) for some \( g \in A(b,a) \). Then \( gf \in A(a,a) \) is an isomorphism so \( f \) is left invertible as well. Hence, \( f \) is irreducible if it is not an isomorphism and whenever \( f = gh \), either \( g \) or \( h \) is an isomorphism. Recall that two categories \( A \) and \( B \) are equivalent if there is a fully faithful and essentially surjective functor \( F : A \to B \). Note that any category \( A \) is equivalent to some skeletal category \( B \) (where skeletal means that no two objects of \( B \) are isomorphic). If \( A \) and \( B \) are equivalent categories then their algebras are Morita equivalent \([20\text{ Proposition 2.2}]\).

### 2.4 Möbius functions

Let \((X, \leq)\) be a finite poset. We view \( \leq \) as a set of ordered pairs. The Möbius function of \( \leq \) is a function \( \mu : \leq \to \mathbb{C} \) that can be defined in the following recursive way:

\[
\mu(x, x) = 1 \\
\mu(x, y) = - \sum_{x \leq z < y} \mu(x, z)
\]

**Theorem 2.2** (Möbius inversion theorem). Let \( V \) be a \( \mathbb{C} \)-vector space and let \( f, g : X \to V \) be functions such that

\[
g(x) = \sum_{y \leq x} f(y)
\]

then

\[
f(x) = \sum_{y \leq x} \mu(y, x)g(y).
\]

More on Möbius functions can be found in \([22\text{ Chapter 3}]\). Important applications of Möbius functions to the representation theory of finite monoids can be found in \([20\text{, }24]\).
3 The quiver of $\mathbb{C} \mathrm{PT}_n$

Let $\mathrm{PT}_n$ be the monoid of all partial functions on $n$ elements. The goal of this section is to describe the quiver of the algebra of $\mathrm{PT}_n$. We remark that in this paper we compose functions from right to left. It is well known that $t \notsim s$ in $\mathrm{PT}_n$ if and only if $\text{rank} s = \text{rank} t$, that is, $|\text{im} t| = |\text{im} s|$. Hence the $\mathcal{J}$-classes of $\mathrm{PT}_n$ are linearly ordered by rank. It is also known that all the $\mathcal{J}$-classes are regular and the maximal subgroup of the $\mathcal{J}$-class of rank $k$ is $S_k$. By theorem 2.1 there is a one-to-one correspondence between $\text{Irr} \mathrm{PT}_n$ and $\bigsqcup_{k=0}^{n} \text{Irr} S_k$. Since the irreducible representations of $S_k$ correspond to Young diagrams with $k$ boxes (or partitions of $k$) it follows that the vertices of the quiver of $\mathbb{C} \mathrm{PT}_n$ correspond to the Young diagrams with $k$ boxes for $0 \leq k \leq n$. For instance, we can identify the vertices of the quiver of $\mathrm{PT}_3$ with the following diagrams:

![Young diagrams]

We have ordered the diagrams according to the partial order on vertices mentioned above. Hence, representations of $S_r$ appear above representations of $S_k$ if $r > k$.

We will now show a way to describe the number of arrows between any two Young diagrams in the quiver. We start by recalling a result from [23]. Denote by $G_n$ the category whose objects are subsets of $\pi = \{1, \ldots, n\}$ and morphisms are in one-to-one correspondence with elements of $\text{IS}_n$. For every $t \in \text{IS}_n$ there is a morphism $G_n(t)$ from $\text{dom} t$ to $\text{im} t$, so multiplication $G_n(s)G_n(t)$ is defined where $\text{im}(t) = \text{dom}(s)$ and the result is $G_n(st)$. In other words, $G_n$ is the
category of all bijections between subsets of an \(n\)-element set. Note that \(G_n\) is a groupoid. Note also that restriction of functions (or containment of relations) is a partial order on \(IS_n\) that turns \(IS_n\) into a partially ordered monoid. We refer to this order as the natural order on \(IS_n\), as it is a special case of the natural ordering of any inverse semigroup (see [5, Section 7.1] or [9, Section 5.2]). The following theorem is [23, Theorem 4.2] for the special case of the symmetric inverse monoid.

**Theorem 3.1.** \(C \text{IS}_n\) is isomorphic to \(C \text{G}_n\). Explicit isomorphisms \(\varphi : C \text{IS}_n \to C \text{G}_n, \psi : C \text{G}_n \to C \text{IS}_n\) are defined (on basis elements) by

\[
\varphi(s) = \sum_{t \leq s} G_n(t)
\]

\[
\psi(G_n(s)) = \sum_{t \leq s} \mu(t, s)t
\]

where \(\leq\) is the standard partial order on \(IS_n\) and \(\mu\) is its Möbius function.

We claim that this isomorphism can be extended into an isomorphism between \(C \text{PT}_n\) and the category of epimorphisms on an \(n\)-element set, which we define now. Denote by \(E_n\) the category whose objects are subsets of \(\pi = \{1, \ldots, n\}\), and whose morphisms are in one-to-one correspondence with the elements of \(\text{PT}_n\). For every \(t \in \text{PT}_n\) there is a morphism \(E_n(t)\) from \(\text{dom}\ t\) to \(\text{im}\ t\), so multiplication \(E_n(s)E_n(t)\) is defined where \(\text{im}(t) = \text{dom}(s)\) and the result is \(E_n(st)\). In other words, \(E_n\) is the category of all total onto functions between subsets of an \(n\)-element set (the “\(E\)” stands for epimorphisms). Note that \(E_n\) is not a groupoid, but it is an EI-category since \(E_n(X, X) \cong S_{|X|}\), where \(X \subseteq \pi\). Furthermore, the groupoid \(G_n\) discussed above is precisely the groupoid of isomorphisms of the category \(E_n\).

**Proposition 3.2.** \(C \text{PT}_n\) is isomorphic to \(C \text{E}_n\). Explicit isomorphisms \(\varphi : C \text{PT}_n \to C \text{E}_n, \psi : C \text{E}_n \to C \text{PT}_n\) are defined (on basis elements) by

\[
\varphi(s) = \sum_{t \leq s} E_n(t)
\]

\[
\psi(E_n(s)) = \sum_{t \leq s} \mu(t, s)t
\]
where \( \leq \) is the natural partial order on \( PT_n \) (containment of relations) and \( \mu \) is its Möbius function.

**Proof.** The proof that \( \varphi \) and \( \psi \) are bijectives is identical to what is done in [23].

\[
\psi(\varphi(s)) = \psi(\sum_{t \leq s} E_n(t)) = \sum_{t \leq s} \psi(E_n(t)) = \sum_{t \leq s} \sum_{u \leq t} \mu(u, t) u = \sum_{u \leq s} u \delta(u, s) = s
\]

and

\[
\varphi \psi(E_n(s)) = \varphi(\sum_{t \leq s} \mu(t, s) t) = \sum_{t \leq s} \mu(t, s) \varphi(t) = E_n(s)
\]

where the last equality follows from the Möbius inversion theorem and the definition of \( \varphi \). Hence, \( \varphi \) and \( \psi \) are bijectives. We now prove that \( \varphi \) is a homomorphism.

Let \( t, s \in PT_n \) we have to show that

\[
\sum_{h \leq ts} E_n(h) = \left( \sum_{t' \leq t} E_n(t') \right) \left( \sum_{s' \leq s} E_n(s') \right). \tag{3.1}
\]

**Case 1.** First assume that \( \text{dom} t = \text{im} s \). It is clear that for any element \( E_n(t's') \) on the right hand side of (3.1), \( t's' \) is less than or equal to \( ts \).

So we have only to show that any \( E_n(h) \) for \( h \leq ts \) appears in the right hand side once. If \( h \leq ts \) one can take \( s' = s|_{\text{dom} h} \) and \( t' = t|_{\text{im} s'} \) and it is clear that \( E_n(t')E_n(s') = E_n(h) \). So \( E_n(h) \) appears in the right hand side. Now assume that \( E_n(t')E_n(s') = E_n(h) \) for some \( t' \leq t \) and \( s' \leq s \) then \( \text{dom} h = \text{dom} t's' = \text{dom} s' \) so \( s' \) has to be \( s|_{\text{dom} h} \) and since \( E_n(t')E_n(s') \neq 0 \) we know that \( t' \) has to be \( t|_{\text{im} s'} \).

So \( E_n(h) \) appears only once.

**Case 2.** \( \text{dom} t \neq \text{im} s \). Choose \( \tilde{s} \leq s \) with maximal domain such that \( \text{im} \tilde{s} \subseteq \text{dom} t \) and define \( \tilde{t} = t|_{\text{im} \tilde{s}} \). It is clear that \( ts = t\tilde{s} \) and \( \text{dom} \tilde{t} = \text{im} \tilde{s} \).

Now,

\[
\sum_{h \leq ts} E_n(h) = \sum_{h \leq t\tilde{s}} E_n(h)
\]
and by case \([\text{I}]\)

\[
\sum_{h \leq \tilde{t}} E_n(h) = \left( \sum_{t' \leq \tilde{t}} E_n(t') \right) \left( \sum_{s' \leq \tilde{s}} E_n(s') \right).
\]

If \(s' \leq s\) but \(s' \notin \tilde{s}\) then \(s' \notin \text{dom} \ t\) so \(E_n(t')E_n(s') = 0\) for any \(t' \leq t\). On the other hand, if \(t' \leq t\) but \(t' \notin \tilde{t}\) \(\text{im} \ s\) so \(E_n(t')E_n(s') = 0\) for any \(s' \leq s\). Hence,

\[
\left( \sum_{t' \leq \tilde{t}} E_n(t') \right) \left( \sum_{s' \leq \tilde{s}} E_n(s') \right) = \left( \sum_{t' \leq t} E_n(t') \right) \left( \sum_{s' \leq s} E_n(s') \right)
\]

and we get the desired equality.

We now want to describe the quiver of the category algebra \(CE_n\) and hence of \(\mathcal{CPT}_n\).

Margolis and Steinberg [13, section 6.3.1] and Li [11] independently described the quiver of skeletal EI-categories and we use their results here. Note that in \(E_n\), all sets of the same cardinality are isomorphic objects. Hence, if we denote by \(SE_n\) the full subcategory of \(E_n\) with the objects \(\mathbb{k} = \{1, \ldots, k\}\) where \(1 \leq k \leq n\) and \(\emptyset = 0\) then \(SE_n\) is equivalent to \(E_n\). This implies that their algebras are Morita equivalent (see [26, Proposition 2.2]) and hence have the same quiver. So we can switch our attention to finding the quiver of \(CS\). Another way to describe \(SE_n\) is as the category with object set \(\{\mathbb{k} | 0 \leq k \leq n\}\) and \(SE_n(\mathbb{k}, \mathbb{r})\) is the set of total onto functions from \(\mathbb{k}\) to \(\mathbb{r}\). We continue to denote the morphism of \(SE_n\) associated to some function \(t\) by \(E_n(t)\). Note that \(SE_n\) is a skeletal EI-category and \(SE_n(\mathbb{k}, \mathbb{k}) \cong S_k\).

**Lemma 3.3.** The irreducible morphisms of \(SE_n\) are precisely the morphisms from \(k+1\) to \(k\). In other words,

\[
\text{IRR } SE_n(\mathbb{r}, \mathbb{k}) = \begin{cases} 
SE_n(\mathbb{r}, \mathbb{k}) & r = k + 1 \\
\emptyset & \text{otherwise}
\end{cases}
\]

**Proof.** It is clear that any morphism from \(k+1\) to \(k\) is irreducible. Now, assume that \(r > k + 1\) and let \(E_n(t) \in SE_n(\mathbb{r}, \mathbb{k})\) be a morphism, that is, \(t\) is a total onto function \(t : \mathbb{r} \to \mathbb{k}\). We can choose distinct \(a, b \in \mathbb{r}\) such that \(t(a) = t(b)\).
and define $s : \tau \to k + 1$ and $h : k + 1 \to k$ by
\[
s(i) = \begin{cases} 
  t(i) & i \neq a \\
  k + 1 & i = a
\end{cases}
\quad \text{and} \quad
h(i) = \begin{cases} 
  i & i \leq k \\
  t(a) & i = k + 1
\end{cases}.
\]

It is clear that $E_n(s)$ and $E_n(h)$ are morphisms in $SE_n$ that are not isomorphisms, but $E_n(t) = E_n(h)E_n(s)$ so $E_n(t)$ is not an irreducible morphism. \hfill \Box

Now we can use the following result, which is precisely [13, Theorem 6.13] and [11, Theorem 4.7] for the case of the field of complex numbers.

**Theorem 3.4.** Let $A$ be a finite skeletal EI-category and denote by $Q$ the quiver of $CA$. Then:

1. The vertex set of $Q$ is $\bigsqcup_{c \in A^0} \text{Irr } A(c, c)$.

2. If $V \in \text{Irr}(A(c, c))$ and $U \in \text{Irr}(A(c', c'))$, then the number of arrows from $V$ to $U$ is the multiplicity of $U \otimes V^*$ as an irreducible constituent in the $A(c', c') \times A(c, c)$-module $\mathbb{C}\text{IRR}(A(c, c'))$. Where the operation on $\mathbb{C}\text{IRR}(A(c, c'))$ is given by $(h, g) * f = hfg^{-1}$.

Applying theorem 3.4 to our case enables us to translate our original question to a problem in the theory of representations of the symmetric group. In our case the endomorphism groups are $S_k$ for $0 \leq k \leq n$ hence the vertex set is $\bigsqcup_{k=0}^n \text{Irr } S_k$. If $V \in \text{Irr}(S_k)$ and $U \in \text{Irr}(S_r)$ are such that $k \neq r + 1$ then there are no arrows from $V$ to $U$ since by lemma 3.3 there are no irreducible morphisms between the corresponding objects in $SE_n$. If $U \in \text{Irr}(S_k)$ and $V \in \text{Irr}(S_{k+1})$ then the number of arrows from $V$ to $U$ is the multiplicity of $U \otimes V^*$ as an irreducible constituent in the $S_k \times S_{k+1}$-module $M$, where $M$ is spanned by all the onto function $f : k + 1 \to k$ and the operation is $(h, g) * f = hfg^{-1}$. Note that $M$ is a permutation module of $S_k \times S_{k+1}$ with basis $X = \{ f : k + 1 \to k | f \text{ is onto} \}$. The action on $X$ is transitive so if we choose any $f \in X$ and denote by $K = \text{Stab}(f)$ its stabilizer then:

$$M = \text{Ind}_{K}^{S_k \times S_{k+1}} \text{tr}_K$$

12
where $\text{tr}_K$ is the trivial module of $K$. Now, choose $f \in X$ to be

$$f(i) = \begin{cases} 
    i & i \leq k \\
    k & i = k + 1 
\end{cases}.$$ 

Let us describe $K$ more explicitly.

**Lemma 3.5.** $K = \{(\sigma,\sigma \tau) \mid \sigma \in S_{k-1}, \ \tau \in S_{\{k,k+1\}}\}$.

**Proof.** Assume $h \in S_k$ and $g \in S_{k+1}$ are such that $hfg = f$. Now, denote $l_1 = g^{-1}(k)$ and $l_2 = g^{-1}(k + 1)$. Since $hfg(l_1) = hfg(l_2)$ and $hfg = f$ then we must have that $l_1 = k$ and $l_2 = k + 1$ or vice versa. In other words $g$ must send $\{k,k+1\}$ onto $\{k,k+1\}$. For $i < k$ if $g(i) = j$ then we must have $h(j) = i$. Now it is clear that there are $\sigma \in S_{k-1}$ and $\tau \in S_{\{k,k+1\}}$ such that $g = \sigma \tau$ and $h = \sigma^{-1}$ (we view $S_{k-1}$ as a subgroup of $S_k$ in the usual way). Since our action is $(h,g) \ast f = hfg^{-1}$ we see that the stabilizer of $f$ is $K = \text{Stab}(f) = \{(\sigma,\sigma \tau) \mid \sigma \in S_{k-1}, \ \tau \in S_{\{k,k+1\}}\} \cong S_{k-1} \times S_2$. \hfill \Box

In the following computations we will write $S_2$ instead of $S_{\{k,k+1\}}$ and regard it as a subgroup of $S_{k+1}$. Also we regard in the usual way $S_{k-1} \times S_2$ and $S_{k-1}$ as subgroups of $S_{k+1}$ and $S_k$ respectively. We also denote by $\text{tr}_2$ the trivial representation of $S_2$.

**Lemma 3.6.** The number of arrows from $V$ to $U$ is the multiplicity of $V$ as an irreducible constituent in the $S_{k+1}$-module $\text{Ind}^{S_{k+1}}_{S_{k-1} \times S_2}(\text{Res}^{S_k}_{S_{k-1}}(U) \otimes \text{tr}_2)$.

**Proof.** The number of arrows from $V$ to $U$ is the multiplicity of $U \otimes V^*$ in $M$ and this number can be expressed by the inner product of characters:

$$\langle U \otimes V^*, \text{Ind}^{S_k \times S_{k+1}}_K \text{tr}_K \rangle$$

(recall that in order to simplify notation, we use the same notation for the
representation and its character). Using Frobenius reciprocity we can see that:

\[
\langle U \otimes V^*, \text{Ind}^{S_{k+1} \times S_k}_{S_k} \text{tr}_K \rangle = \langle \text{Res}^{S_{k+1} \times S_k}_{S_k} (U \otimes V^*), \text{tr}_K \rangle \\
= \frac{1}{|K|} \sum_{(\sigma, \tau) \in K} U \otimes V^*((\sigma, \sigma \tau)) \\
= \frac{1}{|K|} \sum_{(\sigma, \tau) \in S_{k-1} \times S_2} U(\sigma)V^*(\sigma \tau).
\]

Since the characters of $S_n$ are real-valued, $V^*(\sigma \tau) = V(\sigma \tau)$ so this equals:

\[
\frac{1}{|K|} \sum_{(\sigma, \tau) \in S_{k-1} \times S_2} U(\sigma)V(\sigma \tau) = \frac{1}{|K|} \sum_{(\sigma, \tau) \in S_{k-1} \times S_2} V(\sigma \tau)U(\sigma)\text{tr}_2(\tau) \\
= \langle \text{Res}^{S_{k+1}}_{S_k} V, \text{Res}^{S_k}_{S_{k-1}} (U \otimes \text{tr}_2) \rangle.
\]

Again, using Frobenius reciprocity this equals:

\[
\langle V, \text{Ind}^{S_{k+1} \times S_k}_{S_{k-1} \times S_2} (\text{Res}^{S_k}_{S_{k-1}} (U) \otimes \text{tr}_2) \rangle.
\]

The benefit of the description of lemma 3.6 is that the module $\text{Ind}^{S_{k+1} \times S_k}_{S_{k-1} \times S_2} (\text{Res}^{S_k}_{S_{k-1}} (U) \otimes \text{tr}_2)$ has a good combinatorial description using Young diagrams. We will use facts from [10, section 2.8]. Recall that if $\alpha$ is the Young diagram corresponding to $W \in \text{Irr } S_k$ then

\[
\text{Res}^{S_{k-1}}_{S_{k-1}} (W)
\]
is the sum of simple modules that correspond to the diagrams that are obtained from $\alpha$ by removing one box (this is the well known branching rule). Now, if $\alpha$ is the Young diagram that corresponds to $W \in S_{k-1}$ then the module

\[
\text{Ind}^{S_{k+1} \times S_k}_{S_{k-1} \times S_2} (W \otimes \text{tr}_2)
\]
is the sum of simple modules that correspond to the diagrams that are obtained from $\alpha$ by adding two boxes, but not in the same column. This is a special case of Young’s rule.
Hence, if $U$ is an irreducible $S_k$-module that corresponds to a Young diagram $\alpha$ then the $S_{k+1}$-module $\text{Ind}_{S_k \times S_2}^{S_{k+1}} (\text{Res}_{S_k} (U) \otimes \text{tr}_2)$ corresponds to the sum of Young diagrams obtained from $\alpha$ by removing one box and then adding two boxes but not in the same column. A diagram can appear in this summation more than once and we count the diagrams with multiplicity. If $V$ corresponds to a Young diagram $\beta$ then the number of arrows from $V$ to $U$ is the number of times that $\beta$ occur in this summation.

**Example 3.7.** Let $U$ be the standard representation of $S_3$ whose corresponding Young diagram is $\alpha = [2, 1]$:

\[
\begin{array}{c}
\text{\ }
\end{array}
\]

Then the module $\text{Res}_{S_{k-1}} (U)$ corresponds to

\[
\begin{array}{c}
\text{\ }
\end{array} + \begin{array}{c}
\text{\ }
\end{array}
\]

where sum of diagrams means the direct sum of the corresponding simple modules. Now, the module $\text{Ind}_{S_{k-1} \times S_2}^{S_{k+1}} (\text{Res}_{S_k} (U) \otimes \text{tr}_2)$ corresponds to

\[
\begin{array}{c}
\text{\ }
\end{array} + \begin{array}{c}
\text{\ }
\end{array} + \begin{array}{c}
\text{\ }
\end{array} + \begin{array}{c}
\text{\ }
\end{array} + \begin{array}{c}
\text{\ }
\end{array}.
\]

Hence, there are two arrows in the quiver from $\begin{array}{c}
\text{\ }
\end{array}$ to $\begin{array}{c}
\text{\ }
\end{array}$, and one arrow from $\begin{array}{c}
\text{\ }
\end{array}$ and $\begin{array}{c}
\text{\ }
\end{array}$ to $\begin{array}{c}
\text{\ }
\end{array}$. A full drawing of the quiver of $\text{CPT}_4$ is given in the next figure:
In conclusion, we end up with the following theorem:

**Theorem 3.8.** The vertices in the quiver of $\mathbb{C}PT_n$ are in one-to-one correspondence with Young diagrams with $k$ boxes where $0 \leq k \leq n$. If $\alpha \vdash k$, $\beta \vdash r$ are two Young diagrams such that $r \neq k + 1$ then there are no arrows from $\beta$ to $\alpha$. If $r = k + 1$ then there are arrows from $\beta$ to $\alpha$ if we can construct $\beta$ from $\alpha$ by removing one box and then adding two boxes but not in the same column. The number of arrows is the number of different ways that this construction can be carried out.

**Remark 3.9.** Note that up to rank $n - 1$ the quiver of $\mathbb{C}PT_n$ is precisely the quiver of $\mathbb{C}PT_{n-1}$.

4 Other invariants of $\mathbb{C}PT_n$

In this section we use the above results in order to find other important invariants of the algebra $\mathbb{C}PT_n$.

4.1 Connected components of the quiver of $\mathbb{C}PT_n$

**Proposition 4.1.** For every $n > 1$, the quiver of $\mathbb{C}PT_n$ has three connected components, with two isolated components: $\emptyset$ and $[1^n]$ (the sign representation of rank $n$).
Proof. Using theorem 3.8 it is easy to prove the statement by induction. The claim is obvious for \( n = 2 \). For \( n > 2 \), assume that the quiver of \( \mathbb{C}PT_{n-1} \) has three connected components with \([1^{n-1}]\) and \( \emptyset \) being isolated. Now consider the quiver of \( \mathbb{C}PT_n \). Recall that up to rank \( n-1 \) the quiver of \( \mathbb{C}PT_n \) is the quiver of \( \mathbb{C}PT_{n-1} \). Moreover, it is clear that there is no arrow from \([1^n]\) and there exists an arrow from \([3,1^{n-3}]\) to \([1^{n-1}]\). Hence it is left to show that the rank \( n \) vertices (except for \([1^{n-1}]\)) are connected to some rank \( n-1 \) representation (other than \([1^{n-1}]\)). Consider some vertex \( v = [\alpha_1, \ldots, \alpha_k] \) and note that \( \alpha_1 > 1 \) (since \( v \neq [1^n] \)). If \( \alpha_k > 1 \) then there is an arrow from \( v \) to \( u = [\alpha_1, \ldots, \alpha_{k-1}, (\alpha_k-1)] \) because one can remove one box from the last row and add two. The next figure illustrates this case:

\[
\begin{array}{c}
\alpha_1 & u \\
\vdots & \\
\alpha_{k-1} & \\
\alpha_k - 1 & \end{array}
\quad \Rightarrow \quad
\begin{array}{c}
\alpha_1 & v \\
\vdots & \\
\alpha_{k-1} & \\
\alpha_k - 1 & \end{array}
\]

The box removed from \( u \) is marked with "X" and the boxes added to obtain \( v \) are marked with "+". Now, if \( \alpha_k = 1 \) then we can write \( v = [\alpha_1, \ldots, \alpha_{k-1}, 1] \). Let \( l \) be maximal such that \( \alpha_l > 1 \) (such \( l \) exists since \( \alpha_1 > 1 \)). It is easy to observe that there is an arrow from \( v \) to \( u = [\alpha_1, \ldots, \alpha_{k-1}] \). This is because we can remove a box from the \( l \)-th row and add two boxes, one in the \( l \)-th row and one in the last row. The requirement that \( l \) is maximal such that \( \alpha_l > 1 \) ensures that they are not in the same column. This case is illustrated in the next figure:

\[
\begin{array}{c}
\alpha_1 & u \\
\vdots & \\
\alpha_l & \quad X \\
\alpha_{l+1} = 1 & \\
\vdots & \\
\alpha_{k-1} = 1 & \\
\alpha_k = 1 & \end{array}
\quad \Rightarrow \quad
\begin{array}{c}
\alpha_1 & v \\
\vdots & \\
\alpha_l & \\
\alpha_{l+1} = 1 & \quad + \\
\vdots & \\
\alpha_{k-1} = 1 & \\
\alpha_k = 1 & \end{array}
\]

We have marked the removed and added boxes with "X" and "+" respectively as above. This finishes the proof. \( \square \)
4.2 Loewy series of $\mathbb{C} \text{PT}_n$

Now we observe that we can “see” $\text{Rad}^k \mathbb{C}E_n$ inside the category itself. In other words, certain morphisms of the category $E_n$ span $\text{Rad}^k \mathbb{C}E_n$. We start with the case $k = 1$.

We mention that Proposition 4.6 is a similar observation for any EI-category.

**Lemma 4.2.** $\text{Rad} \mathbb{C}E_n = \text{span}\{E_n(t) \mid t \in \text{PT}_n \setminus \text{IS}_n\}$.

*Proof.* Write $R = \text{span}\{E_n(t) \mid t \in \text{PT}_n \setminus \text{IS}_n\}$. It is easy to see that $R$ is a nilpotent ideal hence $R \subseteq \text{Rad} \mathbb{C}E_n$. In addition $\mathbb{C}E_n/R \cong \mathbb{C}G_n$, but $G_n$ is a groupoid and its algebra is semisimple, hence $\text{Rad} \mathbb{C}E_n \subseteq R$ and we are done.

**Lemma 4.3.** $\text{Rad}^k \mathbb{C}E_n = \text{span}\{E_n(t) \mid |\text{dom } t| - |\text{im } t| \geq k\}$.

*Proof.* Clearly $\text{Rad}^k \mathbb{C}E_n \subseteq \text{span}\{E_n(t) \mid |\text{dom } t| - |\text{im } t| \geq k\}$. So it suffices to show the other inclusion. Now, take $t$ such that $|\text{dom } t| - |\text{im } t| \geq k$. It is enough to show that $E_n(t)$ can be written as a product of $k$ elements from $\{E_n(t) \mid |\text{dom } t| - |\text{im } t| \geq 1\}$ which is a basis for $\text{Rad} \mathbb{C}E_n$. This is easily done by induction. The case $k = 1$ is trivial. Now, choose two distinct elements $a$ and $a'$ from $\text{dom } t$ such that $t(a) = t(a')$ and choose $b \notin \text{im } t$. We can write $E_n(t)$ as a product $E_n(t) = E_n(h)E_n(s)$ where

$$s(i) = \begin{cases} t(i) & i \neq a \\ b & i = a \end{cases} \quad h(i) = \begin{cases} i & i \neq b \\ t(a) & i = b \end{cases}. $$

Note that $|\text{dom } h| - |\text{im } h| = 1$ and $|\text{dom } s| - |\text{im } s| \geq k - 1$ so by the induction hypothesis we are done.

**Lemma 4.3** immediately implies the following corollary:

**Corollary 4.4.** The Loewy length of $\mathbb{C} \text{PT}_n$ is $n$.

We can also use lemma 4.3 to get a formula for the dimension of $\text{Rad}^k \mathbb{C} \text{PT}_n$.

Recall that the *Stirling number of the second kind* $S(d, m)$ is the number of ways to partition a set of $d$ objects into $m$ non-empty subsets.
Lemma 4.5. \( \text{dim Rad}^k \mathbb{C} \mathbb{P} \mathbb{T}_n \) equals
\[
\sum_{d=k+1}^{n} \sum_{m=1}^{d-k} \binom{n}{d} \binom{n}{m} S(d,m)m!.
\]

Proof. We have only to count the basis elements given in lemma 4.3. The number of total functions with domain of size \( d \) and image of size \( m \) is
\[
S(d,m)m!
\]
since one has \( S(d,m) \) different ways to partition the domain into \( m \) non-empty subsets and then \( m! \) ways to match the subsets with image elements. There are \( \binom{n}{d} \binom{n}{m} \) ways to choose domain of size \( d \) and image of size \( m \) so all that is left to do is to sum all possible sizes of the domain and image. \( \square \)

5 Quivers of submonoids of \( \mathbb{P} \mathbb{T}_n \) which are order ideals

In this section we apply the method we have used to describe the quiver of \( \mathbb{C} \mathbb{P} \mathbb{T}_n \) in order to find the quiver of the algebra of other well known transformation monoids. All monoids discussed in this section are extensively studied in [7, Chapter 14]. The important observation is the following one: Let \( N \) be a submonoid of \( \mathbb{P} \mathbb{T}_n \) that is also an order ideal, that is, if \( y \in N \) and \( x \leq y \) then \( x \in N \). Let \( D_n \) be the subcategory of \( E_n \) with the same set of objects and morphism set \( \{ E_n(t) \mid t \in N \} \). Note that since \( N \) is an order ideal we have \( \varphi(N) \subseteq \mathbb{C} D_n \) and \( \psi(D_n) \subseteq \mathbb{C} N \). Hence the restriction of \( \varphi \) to \( \mathbb{C} N \) gives an isomorphism with \( \mathbb{C} D_n \). \( D_n \) is also an EI-category so we can again use theorem 3.4 in order to compute the quiver of its algebra.

5.1 Order-preserving partial functions

Let \( \mathbb{P} \mathbb{O}_n \) be the monoid of all order-preserving partial functions on \( \pi \), that is,
\[
\mathbb{P} \mathbb{O}_n = \{ t \in \mathbb{P} \mathbb{T}_n \mid \forall x, y \in \text{dom} t \quad x \leq y \Rightarrow t(x) \leq t(y) \}.
\]
PO\(_n\) is indeed a submonoid of PT\(_n\) and an ideal with respect to inclusion. So we get an isomorphism
\[
\mathbb{C}PO_n \cong \mathbb{C}EO_n
\]
where EO\(_n\) is the subcategory of E\(_n\) with the same set of objects but whose only morphisms are E\(_n\)(t) for t \in PO\(_n\). As before we can take the skeleton of EO\(_n\) which is equivalent to EO\(_n\) hence their algebras have the same quiver. The skeleton will be denotedSEO\(_n\). Its set of objects is \(\{k \mid 0 \leq k \leq n\}\) and SEO\(_n\)(\(\overline{k}, \overline{r}\)) are all the order-preserving functions from \(\overline{k}\) onto \(\overline{r}\). We continue to denote the morphism of SEO\(_n\) associated to the function t by E\(_n\)(t). Similar to lemma 3.3 we have the following lemma.

**Lemma 5.1.**

\[
\text{IRR } \text{SEO}_n(\overline{r}, \overline{k}) = \begin{cases} \text{SEO}_n(\overline{r}, \overline{k}) & r = k + 1 \\ \emptyset & \text{otherwise} \end{cases}
\]

**Proof.** It is clear that any morphism from \(\overline{k} + 1\) to \(\overline{k}\) is irreducible. Now, assume that \(r > k + 1\) and let \(E_n(t) \in \text{SEO}_n(\overline{r}, \overline{k})\) be a morphism, that is, \(t\) is a total onto order-preserving function \(t : \overline{r} \to \overline{k}\). Choose some \(b \in \overline{k}\) whose preimage \(t^{-1}(b)\) contains more than one element and let \(a\) be the maximal element in \(t^{-1}(b)\). Define \(s : \overline{r} \to \overline{k + 1}\) and \(h : \overline{k + 1} \to \overline{k}\) by

\[
s(i) = \begin{cases} t(i) & i < a \\ t(i) + 1 & i \geq a \end{cases} \quad h(i) = \begin{cases} i & i \leq b \\ i - 1 & i > b \end{cases}.
\]

It is clear that \(E_n(s)\) and \(E_n(h)\) are morphisms in EO\(_n\) that are not isomorphisms, but \(E_n(t) = E_n(h)E_n(s)\) so \(E_n(t)\) is not an irreducible morphism.

\(\square\)

Note that the all the endomorphism groups of SEO\(_n\) are trivial and the number of order-preserving functions from \(\overline{k + 1}\) onto \(\overline{k}\) is \(k\). Using theorem 3.4 we can conclude:

**Proposition 5.2.** The vertex set of the quiver of \(\mathbb{C}PO_n\) is \(\{0, \ldots, n\}\). There are \(k\) arrows from \(k + 1\) to \(k\), for \(k = 0, \cdots, n - 1\), and no other arrows.
5.2 Order-decreasing partial and total functions

A function $t \in \text{PT}_n$ is called order-decreasing if $t(x) \leq x$ for every $x \in \text{dom}(t)$.

In this section we want to consider the monoids of all total and partial order-decreasing functions on $\pi$ denoted by $F_n$ and $PF_n$ respectively. We start by proving that these two families are in fact identical.

**Lemma 5.3.** $F_{n+1} \cong PF_n$.

**Proof.** In this proof it will be more convenient to identify $F_{n+1}$ with the set of order-decreasing total functions on $\{0, \ldots, n\}$. Note that any $t \in F_{n+1}$ must satisfy $t(0) = 0$ so we can define $f : F_{n+1} \to PF_n$ by

$$f(t)(i) = \begin{cases} t(i) & t(i) \neq 0 \\ \text{undefined} & t(i) = 0 \end{cases}.$$ 

Conversely, given $s \in PF_n$ we define $g : PF_n \to F_{n+1}$ by

$$g(s)(i) = \begin{cases} s(i) & i \in \text{dom}(s) \\ 0 & \text{otherwise} \end{cases}.$$ 

Clearly $f$ and $g$ are monoid homomorphisms and inverse to each other so we get the required isomorphism. 

One reason for the importance of $F_n$ is the following result. A monoid $M$ is $\mathcal{L}$-trivial (that is, any two distinct elements generate different left ideals) if and only if it is isomorphic to a submonoid of $F_n$ for some $n \in \mathbb{N}$ [13, Theorem 3.6]. In particular, $F_n$ is an $\mathcal{L}$-trivial monoid. We now turn to computing the quivers of $\mathbb{C}PF_n$ and $\mathbb{C}F_n$. Note that $PF_n$ and $F_n$ are not regular monoids, since in a regular $\mathcal{L}$-trivial monoid every element is an idempotent. But this does not prevent us from applying our method. Clearly $PF_n$ is a submonoid of $\text{PT}_n$ which is an order ideal so we have an isomorphism

$$\mathbb{C}PF_n \cong \mathbb{C}EF_n$$

where $EF_n$ is the subcategory of $E_n$ with the same set of objects but whose only morphisms are $E_n(t)$ for $t \in PF_n$.

Note that for any $X \subseteq \pi$, there is only one order-decreasing function $t$ such that $\text{dom} \ t = \text{im} \ t = X$, so every endomorphism group in $EF_n$ is trivial. Moreover, if
$X, Y \subseteq \mathfrak{P}$ where $|X| = |Y|$ and $X \neq Y$ then one of $EF_n(X,Y)$ and $EF_n(Y,X)$ has to be empty. Hence, there are no distinct isomorphic objects in $EF_n$ so $EF_n$ is its own skeleton. Hence, theorem 3.4 implies that the vertices of the quiver of $\mathcal{C}EF_n$ are precisely the objects of $EF_n$ and the morphisms are precisely the irreducible morphisms. All that is left to do is to identify the irreducible morphisms.

In order to do so, we introduce a technical definition. Let $X \subseteq \mathfrak{P}$ and let $j \in X$. Define $j^\prec X$ to be

$$j^\prec X = \max\{x \in \mathfrak{P} \mid x \notin X, x < j\}$$

and if such a maximum does not exist then $j^\prec X = 1$. Note that if $j^\prec X < x < j$ then $x \in X$.

In the following $X$ will always be $\text{dom } t$ so we will usually omit it and write $j^\prec$ instead of $j^\prec X$. Now we can state and prove the following result.

**Lemma 5.4.** $E_n(t)$ is irreducible in $EF_n$ if and only if there exists $j \in \text{dom } t$ such that $t(i) = i$ for any $i \in \text{dom } t \setminus \{j\}$ and $j^\prec \leq t(j)$ where $X = \text{dom } t$.

**Proof.** First assume that $E_n(t)$ is irreducible. Let $j$ be maximal in $\text{dom } t$ such that $t(j) < j$ (such $j$ must exist since $E_n(t)$ is not an isomorphism). If there is another $j' \in \text{dom } t$ such that $t(j') < j'$ then we can define $s, h \in PF_n$ by

$$s(i) = \begin{cases} t(i) & i \in \text{dom } t \setminus \{j\} \\ j & i = j \end{cases} \quad h(i) = \begin{cases} i & i \in \text{im } s \setminus \{j\} \\ t(j) & i = j \end{cases}.$$  

It is easy to observe that $s, h \in PF_n$. We have already seen that the only isomorphisms of $EF_n$ are the identity morphisms. $E_n(h)$ and $E_n(s)$ are not isomorphisms because $s(j') = t(j') < j'$ and $h(j) = t(j) < j$. Since $E_n(h)E_n(s) = E_n(t)$ we get a contradiction. So there is only one $j \in \text{dom } t$ such that $t(j) < j$.

Now assume that $t(j) < j^\prec$. Note that this implies that $j^\prec \notin \text{dom } t$ since $j^\prec \neq 1$. We can define

$$s(i) = \begin{cases} i & i \in \text{dom } t \setminus \{j\} \\ j^\prec & i = j \end{cases} \quad h(i) = \begin{cases} i & i \in \text{im } s \setminus \{j^\prec\} \\ t(j) & i = j^\prec \end{cases}.$$  

Again, $E_n(h)$ and $E_n(s)$ are clearly not isomorphisms. It is easy to see that $s, h \in PF_n$ and $E_n(h)E_n(s) = E_n(t)$ which contradicts the assumption and ends this direction. In the other direction, assume that $t$ is of the required form.
but $E_n(t) = E_n(h)E_n(s)$ where $E_n(h)$ and $E_n(s)$ are not isomorphisms. Since for any $i \in \text{dom } t \setminus \{j\}$ we have $hs(i) = t(i) = i$ we must have that $h(i) = s(i) = i$. Now, since $s$ and $h$ are not the identity on their domains, we must have $j^* \leq t(j) = h(s(j)) < s(j) < j$. But this implies that $s(j) \in \text{dom } t \setminus \{j\}$ hence $hs(j) = s(j) \neq t(j)$, a contradiction. \hfill \Box

The next result now follows immediately.

**Proposition 5.5.** The vertices in the quiver of the algebra $C\mathbb{P}F_n$ are in one-to-one correspondence with subsets of $\mathbb{P}$. For $X,Y \subseteq \mathbb{P}$, the arrows from $X$ to $Y$ are in one-to-one correspondence with onto functions $t: X \rightarrow Y$ for which there exists $j \in X$ such that $t(i) = i$ for $i \in X \setminus \{j\}$ and $j_X \leq t(j) < j$.

Using lemma 5.3 we get a description for the quiver of $F_n$ as well.

**Corollary 5.6.** The vertices in the quiver of the algebra $C\mathbb{F}n$ are in one-to-one correspondence with subsets of $\mathbb{P} - \{0\}$ (where $0 = \emptyset$). For $X,Y \subseteq \mathbb{P} - \{0\}$, the arrows from $X$ to $Y$ are in one-to-one correspondence with onto functions $t: X \rightarrow Y$ for which there exists $j \in X$ such that $t(i) = i$ for $i \in X \setminus \{j\}$ and $j_X \leq t(j) < j$.

### 5.3 Partial Catalan monoid

Define $PC_n$, called the *partial Catalan monoid*, to be the monoid of all partial function on $\mathbb{P}$ which are both order-preserving and order-decreasing. The computation of the quiver of $CPC_n$ is quite similar to that of $C\mathbb{P}F_n$. $PC_n$ is indeed a submonoid of $PT_n$ and an order ideal. We get an isomorphism

$$CPC_n \cong CEC_n$$

where $EC_n$ is the subcategory of $E_n$ with the same set of objects but whose only morphisms are $E_n(t)$ for $t \in PC_n$. Note that $EC_n$ is obtained from $EF_n$ by erasing morphisms so it is clear that it has no isomorphic objects and all the endomorphism groups are trivial. So again we just have to identify the irreducible morphisms.

**Lemma 5.7.** $E_n(t)$ is irreducible in $EC_n$ if and only if there exists $j \in \text{dom } t$ such that $t(i) = i$ for any $i \in \text{dom } t \setminus \{j\}$ and $t(j) = j - 1$. 

23
Proof. Assume that $t$ is of the required form. By lemma 5.4, $E_n(t)$ is irreducible in $EF_n$, so it must be irreducible in $EC_n$ as well. In the other direction, we can prove precisely as in lemma 5.4 that there is a unique $j \in \text{dom } t$ such that $t(j) < j$. Now assume that $t(j) < j - 1$. Note that there is no $k \in \text{dom}(t)$ such that $t(j) < k < j$ because this will imply that $t(j) < k = t(k)$ in contradiction to the fact that $t$ is order-preserving. Define
\[
s(i) = \begin{cases} 
i & i \in \text{dom } t \setminus \{j\} \\
j - 1 & i = j\end{cases}, \quad h(i) = \begin{cases} 
i & i \in \text{im } s \setminus \{j - 1\} \\
t(j) & i = j - 1\end{cases}.
\]
Again, $D_n(h)$ and $D_n(s)$ are clearly not isomorphisms. It is easy to see that $s, h \in \text{PC}_n$ and $E_n(h)E_n(s) = E_n(t)$ which contradicts the assumption and ends the proof. 

We conclude:

**Proposition 5.8.** The vertices in the quiver of the algebra $\mathbb{C}\text{PC}_n$ are in one-to-one correspondence with subsets of $\mathbb{P}$. For $X, Y \subseteq \mathbb{P}$, the arrows from $X$ to $Y$ are in one-to-one correspondence with onto functions $t : X \to Y$ for which there exists $j \in X$ such that $t(i) = i$ for $i \in X \setminus \{j\}$ and $t(j) = j - 1$.

**Acknowledgement:** The author is grateful to the referee for his/her valuable comments and suggestions, and in particular for suggesting lemma 5.3.

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