Galilei invariant theories.
III. Wave equations for massless fields

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Abstract

Galilei invariant equations for massless fields are obtained via contractions of relativistic wave equations. It is shown that the collection of non-equivalent Galilei-invariant wave equations for massless fields with spin equal 1 and 0 is very broad and describes many physically consistent systems. In particular, there exist a huge number of such equations for massless fields which correspond to various contractions of representations of the Lorentz group to those of the Galilei one.

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1 Introduction

It was observed by Le Bellac and Lévy-Leblond [1] already in 1973 that the non-relativistic limit of the Maxwell equations is not defined in a unique way. According to Le Bellac and Lévy-Leblond (p. 218), the term "non-relativistic" means "in agreement with the principle of Galilean relativity". Moreover, they claimed that there exist two Galilei invariant theories of electromagnetism which can be obtained by appropriate limiting procedures starting with the Maxwell theory.

The combination of word "Galilean electromagnetism" itself introduced in [1] looked rather strange since it is pretty well known that electromagnetic phenomena are in perfect accordance with the Einstein relativity principle. However physicists are always interested when non-relativistic approximations are adequate, which makes the results of paper [1] quite popular. The importance of such result is emphasized by the fact that the correct definition of non-relativistic limit is by no means a simple problem in general and in the case of theories of massless fields in particular, see, for example, [2].

Analyzing the contents of main impact journals in theoretical and mathematical physics one finds that the interest of researches in Galilean aspects of electrodynamics belongs to evergreen subjects. Various approaches to Galilei invariant theories which were discussed briefly in [3]. An effective approach to Galilean electromagnetism was used in papers [4], [5] in which Maxwell equations were generalized in to (1+4)-dimensional Minkowski space and then reduced to Galilei invariant equations. Such reduction is based on the fact that the Galilei group is a subgroup of the generalized Poincaré group (i.e., of the group of motions of the flat (1+4)-dimensional Minkowski space). For reduction of representations of the group $P(1, 4)$ to those of the Galilei group see refs. [6] and [7].

Nevertheless, the Galilean electromagnetism still contains many unsolved problems, for example, the question of complete description of all possible Galilean theories for vector and scalar massless fields. And exactly solution of this problem is the main issue of the present paper.

In paper [3] indecomposable representations of homogeneous Galilei group $HG(1, 3)$ were derived, namely all those which when restricted to representations of the rotation subgroup of the group $HG(1, 3)$, are decomposed to spin 0 and spin 1 representations. Moreover, their connection with representations of the Lorentz group via the Inönü-Wigner contractions [8] were studied in [3] and [9]. These results open a way to complete classification of the wave equations describing the related fields both massive and massless ones.

In the present paper we use our knowledge of indecomposable represen-
tations of the homogeneous Galilei algebra $hg(1,3)$ to figure out the Galilei invariant equations for vector and scalar massless fields. We shall show that in contrast to the corresponding relativistic equations for which there are only two possibilities – the Maxwell equations and equations for the longitudinal massless field, the number of possible Galilean equations is very huge. Among them there are equations with more component and less component fields then in the Maxwell equation.

These results can be clearly interpreted in terms of representation and contraction theories. As it was proved in papers [3] and [9], there is a large variety of possible contractions of representations of the Lorentz group to those of Galilei one, and, consequently many non-equivalent Galilean massless fields. In the following sections we use these results of [3], [9] to describe connections of relativistic and Galilean formulations of theories for massless fields. These connections appear to be rather non-trivial: in particular, completely decoupled relativistic systems could be contracted to coupled Galilean ones.

In Sections 2 and 3 we present some results of paper [3] related to the classification of indecomposable representations of the homogeneous Galilei group and contractions of the related representations of Lorentz group. These results are used in Sections 4 and 5 to classify all non-equivalent Galilei-invariant equations for massless fields. Then the classification results are summarized and discussed in Section 6.

## 2 Indecomposable representations of the homogeneous Galilei group

The Galilei group $G(1,3)$ consists of the following transformations of temporal and spatial variables:

\[
\begin{align*}
t &\rightarrow t' = t + a, \\
x &\rightarrow x' = Rx + vt + b,
\end{align*}
\]

where $a, b$ and $v$ are real parameters of time translation, space translations and pure Galilei transformations respectively, and $R$ is a rotation matrix.

The homogeneous Galilei group $HG(1,3)$ is a subgroup of $G(1,3)$ leaving invariant a point $x = (0,0,0)$ at time $t = 0$. It is formed by space rotations and pure Galilei transformations, i.e., by transformations (1) with $a = b \equiv 0$.

The Lie algebra $hg(1,3)$ of the homogeneous Galilei group includes six basis elements, i.e., three generators $S_a, a = 1, 2, 3$ of the rotation subgroup and three generators $\eta_a$ of Galilean boosts. These basis elements should
satisfy the following commutation relations

\[
\begin{align*}
[S_a, S_b] &= i\varepsilon_{abc} S_c, \\
[\eta_a, S_b] &= i\varepsilon_{abc} \eta_c, \\
[\eta_a, \eta_b] &= 0.
\end{align*}
\] (2)

All indecomposable representations of \(HG(1,3)\) which when restricted to the subgroup of rotations are decomposed to direct sums of vector and scalar representations, has been found in paper [3]. These indecomposable representations are labeled by triplets of numbers \(n, m, \lambda\) and denoted as \(D(m, n, \lambda)\). The labeling numbers take the values

\[-1 \leq (n - m) \leq 2, \; n \leq 3,
\]

\[
\lambda = \begin{cases} 
0 & \text{if } m = 0, \\
1 & \text{if } m = 2 \text{ or } n - m = 2, \\
0, 1 & \text{if } m = 1, n \neq 3.
\end{cases}
\] (3)

In accordance with (3) there exist ten non-equivalent indecomposable representations \(D(m, n, \lambda)\). The spaces of representation \(D(m, n, \lambda)\) can include three types of rotational scalars \(A, B, C\) and five types of vectors \(R, U, W, K, N\) whose transformation laws with respect to the Galilei boost are:

\[
\begin{align*}
A &\rightarrow A' = A, \\
B &\rightarrow B' = B + v \cdot R, \\
C &\rightarrow C' = C + v \cdot U + \frac{1}{2}v^2 A, \\
R &\rightarrow R' = R, \\
U &\rightarrow U' = U + vA, \\
W &\rightarrow W' = W + v \times R, \\
K &\rightarrow K' = K + v \times R + vA, \\
N &\rightarrow N' = N + v \times W + vB + v(v \cdot R) - \frac{1}{2}v^2 R
\end{align*}
\] (4)

where \(v\) is a vector whose components are parameters of the Galilei boost, \(v \cdot R\) and \(v \times R\) are scalar and vector products of vectors \(v\) and \(R\).

The carrier spaces of indecomposable representations of group \(hg(1,3)\) include such sets of scalars \(A, B, C\) and vectors \(R, U, W, K, N\) which transform between themselves w.r.t. the transformations (4) but cannon be split
to a direct sum of invariant subspaces. There exist exactly ten such sets:

\[
\begin{align*}
\{A\} & \iff D(0,1,0), \\
\{R\} & \iff D(1,0,0), \\
\{B, R\} & \iff D(1,1,0), \\
\{A, U\} & \iff D(1,1,1), \\
\{A, U, C\} & \iff D(1,2,1), \\
\{W, R\} & \iff D(2,0,0), \\
\{R, W, B\} & \iff D(2,1,0), \\
\{A, K, R\} & \iff D(2,1,1), \\
\{A, B, K, R\} & \iff D(2,2,1), \\
\{A, N, W, R\} & \iff D(3,1,1).
\end{align*}
\]

Thus in contrary to the relativistic case where are only three Lorentz covariant quantities which transforms as vectors or scalars under rotations, i.e., the relativistic four-vector, antisymmetric tensor of second order and scalar, there are ten indecomposable sets of Galilean vectors and scalars which we enumerate in equation (5).

3 Constructions of representations of the Lorentz algebra

It is well known that the Galilei algebra can be obtained from the Poincaré algebra by a limiting procedure called "the Inönü-Wigner contraction" [8]. Representations of these algebras also can be connected by this kind of contraction. However, this connection is more complicated in two reasons. First, the contraction of a non-trivial representation of the Lorentz algebra yields to the representation of the homogeneous Galilei algebra which the generators of Galilei boost are represented trivially, so to obtain a non-trivial representation it is necessary to apply additionally a similarity transformation which depends on the contraction parameter in a tricky way. Second, to obtain an indecomposable representations of \( h g(1,3) \), it is necessary in general to start with completely reducible representations of the Lie algebra of Lorentz group.

In paper [3] representations of the Lorentz group which can be contracted to representations \( D(m, n, \lambda) \) of the Galilei group were found and the related contractions where specified. Here we present only a part of the results from [3] which will be used in what follows.

Let us start with the representation \( D(\frac{1}{2}, \frac{1}{2}) \) of the Lie algebra \( so(1,3) \) of the Lorentz group, whose carrier space is formed by four-vectors. Basis of
this representation is given by $4 \times 4$ matrices of the following form:

$$S_{ab} = \varepsilon_{abc} \begin{pmatrix} s_c & 0_{3 \times 1} \\ 0_{1 \times 3} & 0 \end{pmatrix}, \quad S_{0a} = \begin{pmatrix} 0_{3 \times 3} & -k_a^\dagger \\ k_a & 0 \end{pmatrix}. \quad (6)$$

Here $s_a$ are matrices of spin one with elements $(s_a)_{bc} = i \varepsilon_{abc}$, $k_a$ are $1 \times 3$ matrices of the form

$$k_1 = (i, 0, 0), \quad k_2 = (0, i, 0), \quad k_3 = (0, 0, i). \quad (7)$$

The Inönü-Wigner contraction consists of transformation to a new basis

$$S_{ab} \to S'_{ab}, \quad S_{0a} \to \varepsilon S'_{0a}$$

followed by a similarity transformation of all basis elements $S_{\mu\nu} \to S'_{\mu\nu} = VS_{\mu\nu}V^{-1}$ with a matrix $V$ depending on contracting parameter $\varepsilon$. Moreover, $V$ should depend on $\varepsilon$ in a tricky way, such that all the transformed generators $S'_{ab}$ and $\varepsilon S'_{0a}$ are kept non-trivial and non-singular when $\varepsilon \to 0$. \[8\]

There exist two matrices $V$ for representation (6), namely

$$V_1 = \begin{pmatrix} \varepsilon I_{3 \times 3} & 0_{3 \times 1} \\ 0_{1 \times 3} & 1 \end{pmatrix}, \quad \text{(8)}$$

and

$$V_2 = \begin{pmatrix} I_{3 \times 3} & 0_{3 \times 1} \\ 0_{1 \times 3} & \varepsilon \end{pmatrix}. \quad \text{(9)}$$

Using $[8]$ we obtain

$$S'_{ab} = V_1 S_{ab} V_1^{-1} = S_{ab}, \quad S'_{0a} = \varepsilon V_1 S_{0a} V_1^{-1} = \begin{pmatrix} 0_{3 \times 3} & -\varepsilon^2 k_a^\dagger \\ k_a & 0 \end{pmatrix}. \quad (10)$$

Then, passing $\varepsilon$ to zero, we come to the following matrices

$$S_a = \frac{1}{2} \varepsilon_{abc} S_{bc} = \begin{pmatrix} s_a & 0_{3 \times 1} \\ 0_{1 \times 3} & 0 \end{pmatrix}, \quad \eta_a = \lim_{\varepsilon \to 0} S'_{0a} |_{\varepsilon=0} = \begin{pmatrix} 0_{3 \times 3} & 0_{3 \times 1} \\ k_a & 0 \end{pmatrix}. \quad (11)$$

Analogously, using matrix $V_2$ instead of $V_1$ we obtain

$$S_a = \begin{pmatrix} s_a & 0_{3 \times 1} \\ 0_{1 \times 3} & 0 \end{pmatrix}, \quad \eta_a = \begin{pmatrix} 0_{3 \times 3} & -k_a^\dagger \\ k_a & 0 \end{pmatrix}. \quad (12)$$

Matrices (11) and (12) satisfy commutation relations (2), i.e., they realize representations of algebra $hg(1,3)$. They are generators of indecomposable
representations $D(1,1,0)$ and $D(1,1,1)$ of the homogeneous Galilei group respectively. Indeed, denoting vectors from the related representation spaces as

$$
\Psi = \begin{pmatrix} R_1 \\ R_2 \\ R_3 \\ B \end{pmatrix} \quad \text{for } D(1,1,0) \quad \text{and} \quad \tilde{\Psi} = \begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ A \end{pmatrix} \quad \text{for } D(1,1,1)
$$

and using the transformation laws (11) for $A$, $B$, $R = \text{column}(R_1, R_2, R_3)$ and $U = \text{column}(U_1, U_2, U_3)$ we easily find the corresponding Galilei boost generators $\eta_a$ in the forms (11) and (12). As far as rotation generators $S_a$ are concerned they are direct sums of matrices of spin one (which are responsible for transformations of 3-vectors $R$ and $U$) and zero matrices (which keep scalars $A$ and $B$ invariant).

To obtain five-dimension representation $D(1,2,1)$ we are supposed to start with a direct sum of representations $D(\frac{1}{2}, \frac{1}{2})$ and $D(0,0)$ of the Lorentz group. The corresponding generators of algebra $so(1,3)$ have the form

$$
\hat{S}_{\mu\nu} = \begin{pmatrix} S_{\mu\nu} \\ . \\ . \\ 0 \end{pmatrix},
$$

where $S_{\mu\nu}$ are matrices (6) and the dots denote zero matrices of an appropriate dimension. The corresponding matrix of similarity transformation can be written as:

$$
V_3 = \begin{pmatrix} I_{3 \times 3} & 0_{3 \times 1} & 0_{3 \times 1} \\ 0_{1 \times 3} & \frac{1}{2} \varepsilon & \frac{1}{2} \varepsilon \\ 0_{1 \times 3} & -\varepsilon^{-1} & \varepsilon^{-1} \end{pmatrix}, \quad U_3^{-1} = \begin{pmatrix} I_{3 \times 3} & 0_{3 \times 1} & 0_{3 \times 1} \\ 0_{1 \times 3} & \varepsilon^{-1} & \frac{1}{2} \varepsilon \\ 0_{1 \times 3} & \varepsilon^{-1} & \varepsilon^{-1} \end{pmatrix}.
$$

As a result we obtain the following basis elements of representation $D(1,2,1)$ of algebra $hg(1,3)$:

$$
S_a = \begin{pmatrix} s_a \\ 0_{1 \times 3} \\ 0_{1 \times 3} \\ 0_{3 \times 1} \end{pmatrix}, \quad \eta_a = \begin{pmatrix} 0_{3 \times 3} & k_a^\dagger & 0_{3 \times 1} \\ 0_{1 \times 3} & 0 & 0 \end{pmatrix}.
$$

Matrices $\eta_a$ (15) generate transformations (11) of components of vector-function $\hat{\Psi} = \text{column}(U, A, C)$ so that the relation

$$
\hat{\Psi} \rightarrow \hat{\Psi}' = \exp(i \eta \cdot v) \hat{\Psi}
$$

being written componentwise, simply coincides with the corresponding transformation properties written in (11).

In paper [3] was shown how all the other representations $D(m,n,\lambda)$ of homogeneous Galilei algebra can be obtained via special contractions of appropriate representations of Lorentz algebra. Here we have presented only such contractions which will be used later on.
4 Galilean massless fields

To construct Galilean massless equations it is possible to use the same approach which we have used in [10] to derive equations for the massive fields. However we prefer to apply another technique which consists in contractions of the relativistic wave equations.

4.1 Galilean limits of Maxwell’s equations

According to the analysis made by Lévy-Leblond and Le Bellac in 1967 [11], [12] (see also [12], [13]) there are two Galilean limits of Maxwell’s equations.

In the so called ”magnetic” Galilean limit we receive a pre-Maxwellian electromagnetism. The corresponding equations for a magnetic field $H$ and electric field $E$ take the form

\[ \nabla \times E_m - \frac{\partial H_m}{\partial t} = 0, \quad \nabla \cdot E_m = \epsilon j_0^m, \]
\[ \nabla \times H_m = \epsilon j_m, \quad \nabla \cdot H_m = 0 \]  \hspace{1cm} (16)

where $j = (j_0^m, j_m)$ is an electric current and $\epsilon$ is electric charge.

Equations (16) are invariant with respect to the Galilei transformations provided vectors $H_m, E_m$ and electric current $j$ cotransform as

\[ H_m \rightarrow H_m, \quad E_m \rightarrow E_m - v \times H_m, \]
\[ j_m \rightarrow j_m, \quad j_0^m \rightarrow j_0^m + v \cdot j_m. \]  \hspace{1cm} (17)

Introducing Galilean vector-potential $A = (A^0, A)$ such as

\[ H_m = \nabla \times A, \quad E_m = -\frac{\partial A}{\partial t} - \nabla A^0 \]  \hspace{1cm} (18)

we conclude that the related transformation lows for $A$ have the following form:

\[ A^0 \rightarrow A^0 + v \cdot A, \quad A \rightarrow A. \]  \hspace{1cm} (19)

The other, ”electric” Galilean limit of Maxwell’s equations looks as

\[ \nabla \times H_e + \frac{\partial E_e}{\partial t} = \epsilon j_e, \quad \nabla \cdot E_e = \epsilon j_e^4, \]
\[ \nabla \times E_e = 0, \quad \nabla \cdot H_e = 0, \]  \hspace{1cm} (20)

whilst the Galilean transformation lows have the following form

\[ H_e \rightarrow H_e + v \times E_e, \quad E_e \rightarrow E_e, \]
\[ j_e \rightarrow j_e + v j_e^4, \quad j_e^4 \rightarrow j_e^4. \]  \hspace{1cm} (21)

7
Vectors $H_e$ and $E_e$ can be expressed via vector-potentials as follows

$$H_e = \nabla \times A, \quad E_e = -\nabla A^4$$

(22)

and so the corresponding Galilei transformations for the vector-potential are:

$$A^4 \to A^4, \quad A \to A + \mathbf{v} A^4;$$

(23)

Till this point we just presented the results of the Lévy-Leblond analysis of two possible Galilean limits of Maxwell’s equations. Let us note that these results admit clear interpretation using the representation theory. Indeed, in accordance with the results presented in Section 3 there exist exactly two non-equivalent representations of the homogeneous Galilei group the carrier spaces of which are four-vectors – the representations $D_1(1, 1, 0)$ and $D_1(1, 1, 1)$. In other words there are exactly two non-equivalent Galilei transformations for four-vector-potentials and currents, which are given explicitly by equations (19), (17) and (23), (21). Equations for massless fields invariant with respect to these transformations are given by relations (16) and (20) respectively.

Both representations, i.e., $D_1(1, 1, 0)$ and $D_1(1, 1, 1)$, can be obtained via contractions of the representation $D(1/2, 1/2)$ of the Lorentz group whose carrier space is formed by relativistic four-vectors. The related contraction matrices are given explicitly by relations (8) and (9). Each of these contractions generates the Galilean limit of Maxwell’s equations, and in this way we obtain easily the systems (16) and (20). In the following section we will obtain equations (16) and (20) via contraction of a more general system of relativistic equations for massless fields.

### 4.2 Extended Galilean electromagnetism

In accordance with our analysis of vector field representations of the Galilei group there exists the only representation, namely, $D_1(1, 2, 1)$ whose carrier space is formed by five-vectors. Such five-vectors appears naturally in many Galilean models, especially in those ones which are constructed via reduction approach [3], i.e., starting with models invariant with respect to the extended Poincaré group $P(1, 4)$ and making reduction to its Galilean subgroup.

As mentioned in [3], there is a formal possibility to introduce such five component gauge fields which join and extend the magnetic and electric Galilean limits of relativistic four–vector potentials. However the physical meanings of the related theories was not clear. Moreover, Maxwell’s electrodynamics can be contracted either to the magnetic (16) or to the electric
limit (20), and it is generally accepted to think that it is impossible to formulate a consistent theory including both ‘electric’ and ‘magnetic’ types of Galilean gauge fields, see, e.g., [5].

In contrary to [5], we shall show that it is possible to join the ‘electric’ and ‘magnetic’ Galilean gauge fields since the Galilean five-vector potential appears naturally via contraction of a relativistic theory. Rather surprisingly, the related relativistic equations can be decoupled to two non-interacting subsystems whereas its contracted counterpart appears to be coupled. This is in accordance with the observation presented in [3] that some indecomposable representations of the homogeneous Galilei group appears via contractions of the completely reducible representations of the Lorentz group.

Let us start with relativistic equations for the vector-potential $A^\mu$

$$p^\mu p_\mu A^\nu = -ej^\nu$$

in the Lorentz gauge

$$p_\mu A^\mu = 0 \text{ or } p_0 A^0 = p \cdot A.$$  (25)

Consider also the inhomogeneous d’Alembert equation for a relativistic scalar field which we denote as $A^4$:

$$p^\mu p_\mu A^4 = ej^4.$$  (26)

Introducing the related vectors of the field strengths in the standard form

$$H = \nabla \times A, \quad E = -\frac{\partial A}{\partial x_0} - \nabla A^0, \quad F = \nabla A^4, \quad F^0 = \frac{\partial A^4}{\partial x_0}$$  (27)

we come to Maxwell’s equations for $E$ and $H$:

$$\nabla \times E - \frac{\partial H}{\partial x_0} = 0, \quad \nabla \cdot H = 0,$$

$$\nabla \times H + \frac{\partial E}{\partial x_0} = ej, \quad \nabla \cdot E = ej^0$$  (28)

and the following equations for $F$ and $F^0$

$$\frac{\partial F^0}{\partial x_0} + \nabla \cdot F = ej^4,$$

$$\nabla \times F = 0, \quad \frac{\partial F}{\partial x_0} = \nabla F^0.$$  (29)
Surely the system of equations (28) and (29) is completely decoupled. Rather surprisingly its Galilean counterpart which will be obtained using the Inönü-Wigner contraction appears to be coupled. This contraction can be made directly for equations (28), (29) but we prefer a more simple way with using the potential equations (24).

The system of equations (24)-(26) is a decoupled system of relativistic equations for the five component function

\[ \hat{A} = \text{column}(A^1, A^2, A^3, A^0, A^4) = \text{column}(A, A^0, A^4). \]  

Moreover, the components \((A^1, A^2, A^3, A^0)\) transform as four-vector and \(A^4\) transforms as a scalar, so the generators \(S_{\mu\nu}\) of the Lorentz group defined on \(A\) are direct sums of the four-vector generators \(\hat{S}_{\mu\nu}\) and zero matrices given by equation (13). In other words, such generators realize the direct sum \(D(\frac{1}{2}, \frac{1}{2}) \oplus D(0, 0)\) of representations of algebra \(so(1, 3)\).

In accordance with results of paper [3] the completely reduced representation of the Lie algebra of Lorentz group whose basis elements are given by equation (13) can be contracted either to direct sum of indecomposable representations of the the Galilei algebra \(hg(1, 3)\) or to indecomposable representation \(D_1(1, 2, 1)\) this algebra.

Let us consider the first possibility, i.e., the contraction to the indecomposable representation. Such contraction is presented in Section 3, see equations (13)-(15) here.

Let us demonstrate that this contraction reduces the decoupled relativistic system (24), (26) to a system of coupled equations invariant with respect to the Galilei group. Indeed denoting \(A' = U_3 A = (A', A^0, A^4)\) and \(j' = U_3 j = (j' j^0, j'^4)\) and taking into account that the relativistic variable \(x_0\) is related to non-relativistic variable \(t\) as \(x_0 = ct\), we come to the following system of equations for the transformed quantities:

\[ p^2 A'^k = -e j'^k, \quad i \frac{\partial A'^4}{\partial t} = p \cdot A'. \]  

(31)

Generators of Galilei group for vectors \(A'\) and \(j'\) are given by equation (15), so under the Galilei transformation (11) they cotransform in accordance with the representation \(D_1(1, 2, 1)\), i.e.,

\[ A^0 \rightarrow A^0 + v \cdot A + \frac{v^2}{2} A^4, \quad A \rightarrow A + v A^4, \quad A^4 \rightarrow A^4, \]  

(32)

and

\[ j^4 \rightarrow j^4, \quad j \rightarrow j + v j^4, \quad j^0 \rightarrow j^0 + v \cdot j + \frac{1}{2} v^2 j^4. \]  

(33)
Of course transformations (11), (32) and (33) keep the system (31) invariant. In accordance with (31) the related field strengths (compare with (27))

\[ W = \nabla \times A', \quad N = -\frac{\partial A'}{\partial t} - \nabla A^0, \quad R = \nabla A^4, \quad B = \frac{\partial A^4}{\partial t} \]  

(34)
satisfy the following equations

\[ J_0 \equiv \nabla \cdot N - \frac{\partial}{\partial t} B - e j^0 = 0, \]
\[ J \equiv \nabla \times W + \nabla B - e j = 0, \]
\[ J^4 \equiv \nabla \cdot R - e j^4 = 0, \]
\[ N' \equiv \frac{\partial}{\partial t} W + \nabla \times N = 0, \]
\[ N^4 \equiv \frac{\partial}{\partial t} R - \nabla B = 0, \]
\[ R \equiv \nabla \times R = 0, \]
\[ B \equiv \nabla \cdot W = 0. \]

(35)

Like (31), equations (35) are covariant with respect to the Galilei group. Moreover, the Galilei transformations for fields \( R, W, N, B \) and current \( j \) are given by equations (4) and (33) respectively. In other words, these fields and the current \( j \) form carrier spaces of representation \( D_1(3,1,1) \) and \( D_1(1,2,1) \) of algebra \( hg(1,3) \) correspondingly.

In contrast with the decoupled relativistic system of equations (28) and (29) its Galilean counterpart (35) appears to be a coupled system of equations for vectors \( R, W, N \) and scalar \( B \).

The system of equations (35) was obtained in paper [4] by reduction of generalized Maxwell equations invariant with respect to extended Poincaré group \( P(1,4) \) including one time variable and four spatial variables. We prove that this system is nothing but the contracted version of system (28), (29) including the ordinary Maxwell equations and equations for four-gradient of scalar potential. In other words, the system of Galilei invariant equations (35) admits a clear physical interpretation as a non-relativistic limit of the system of familiar equations (28) and (29).

4.3 Reduced Galilean electromagnetism

In contrast with the relativistic case the Galilei invariant approach makes it possible to reduce the number of field variables. For example, considering the magnetic limit (16) of the Maxwell equations it is possible to restrict ourselves to the case \( \mathbf{H}_m = 0 \) in as much as this condition is invariant with respect to the Galilei transformations (17). Notice that in the relativistic theory such condition can be imposed only in a particular frame of references and will be affected by the Lorentz transformation.
In the mentioned sense equations (35) are reducible too. They are defined on the most extended multiplet of vector and spinor fields which is a carrier space of indecomposable representation of the homogeneous Galilei group. The related representation $D_1(3, 1, 1)$ is indecomposable but reducible, i.e., includes subspaces invariant with respect to the Galilei group. This makes it possible to reduce the number of dependent components of equations (35) without violating its Galilei invariance.

In this section we consider systematically all possible Galilei invariant constraints which can be imposed on solutions of equations (35) and present the related reduced versions of Galilean electromagnetism.

In accordance with (4) and (33) the vector $\mathbf{R}$ and the fourth component $j^4$ of current form invariant subspaces with respect to Galilei transformations. Thus we can impose the Galilei-invariant conditions

$$\mathbf{R} = 0 \quad \text{or} \quad \nabla A^4 = 0, \quad j^4 = 0$$

and reduce the system (35) to the following one

$$\frac{\partial}{\partial t} \mathbf{H} + \nabla \times \mathbf{E} = 0,$$
$$\nabla \times \mathbf{H} = e_1, \quad \nabla \cdot \mathbf{H} = 0,$$
$$\nabla \cdot \mathbf{E} = \frac{\partial}{\partial t} S + ej^0,$$
$$\nabla S = 0$$

(37)

where we denote $\mathbf{H} = W|_{\mathbf{R}=0}$, $\mathbf{E} = N|_{\mathbf{R}=0}$, $S = B|_{\mathbf{R}=0}$.

The vectors $\mathbf{E}$, $\mathbf{H}$ and scalar $S$ belong to a carrier space of the representation $D_1(2, 1, 1)$. Their Galilei transformation laws look as

$$\mathbf{E} \to \mathbf{E} + v \times \mathbf{H} + v S,$$
$$\mathbf{H} \to \mathbf{H}, \quad S \to S.$$

(38)

In accordance with (38) $S$ belongs to invariant subspace of Galilean transformations, so we can impose one more Galilei invariant condition

$$S = 0 \quad \text{or} \quad \frac{\partial A^4}{\partial t} = 0.$$

(39)

As a result we come to equations (16), i.e., to the magnetic limit of Maxwell’s equations. Thus equations (16) are nothing but the system of equations (35) with the additional Galilei invariant constrains (36) and (39).

Galilei transformations for solutions of equations (16) are given by equations (17). Again we recognize the invariant subspace spanned on vectors $\mathbf{H}_m$, and so it is possible to impose the invariant condition

$$\mathbf{H}_m = 0 \quad \text{or} \quad A = \nabla \varphi, \quad j_m = 0$$

(40)
where $\varphi$ is a solution of the Laplace equation. As a result we come to the following system

$$\nabla \times \mathbf{E} = 0, \quad \nabla \cdot \mathbf{E} = e\rho$$

(41)

where we denote $\mathbf{E} = E_m|_{\mathbf{H}_m=0}$ and $\rho = j^0_m|_{\mathbf{H}_m=0}$.

Equation (41) is still Galilei invariant, moreover, both $\mathbf{E}$ and $\rho$ are not changing under the Galilei transformations. The related potential $A$ is constrained by conditions (36), (39) and (40). Moreover, up to gauge transformations it is possible to set in (40) $A = 0$.

### 4.4 Other reductions

Equations (37), (16) and (41) exhaust all Galilei invariant systems which can be obtained starting with (35) and imposing additional constraints which reduce the number of dependent variables. To find other Galilei invariant equations for massless vector fields we use the observation that the Galilei transformations for equations (35) have the form given by relations (4) and (33) if we change $N \rightarrow \mathcal{N}$, $W \rightarrow \mathcal{W}$, \ldots here. Thus the subsystem of equations (35) obtained by excluding equations $N = 0$ and $J^0 = 0$: \begin{equation}
\mathcal{J} \equiv \nabla \times W + \frac{\partial}{\partial t} R - ej = 0, \\
\mathcal{J}^4 \equiv \nabla \cdot R - ej^4 = 0, \\
\mathcal{W} \equiv \frac{\partial}{\partial t} R - \nabla B = 0, \\
\mathcal{R} \equiv \nabla \times R = 0, \\
B \equiv \nabla \cdot W = 0
\end{equation}

(42)

is Galilei covariant too and does not include dependent variables $N$ and $j^0$. The Galilean transformations for $W, R, B$ and $j, j^4$ are still given by equations (4) and (33).

Following the analogous reasonings it is possible to exclude from (42) equations $\mathcal{J} = 0$ and $B = 0$ and obtain the system

$$\nabla \cdot R - ej^4 = 0, \\
\frac{\partial}{\partial t} R - \nabla B = 0, \\
\nabla \times R = 0$$

(43)

which includes only two vector and two scalar variables. The related potential without loss of generality reduces to the only variable $A^4$.

The other way to reduce system (42) is to exclude the equation $\mathcal{W} = 0$. As a result we come to the electric limit for the Maxwell equation (20) for $W = H_e$ and $R = E_e$.
Thus in addition to (35), (37), (16) and (41) we have three more Galilei invariant systems given by equations (42), (43) and (20). These equations admit additional reductions by imposing Galilei invariant constrains to their solutions.

Considering (20) we easily find a possible invariant conditions $E_e = 0$, $j_e^4 = 0$ which reduce it to the following equations

$$\nabla \times \mathbf{H} = \epsilon j, \quad \nabla \cdot \mathbf{H} = 0$$

(44)

where we denote $\mathbf{H} = H_e|E_e = 0$.

Let us return to system (42). This system can be reduced by imposing the Galilei invariant conditions $R = 0, j^4 = 0$ to the following form:

$$\nabla \times \mathbf{H} - \epsilon j = 0,$$

$$\nabla \cdot \mathbf{H} = 0, \quad \nabla S = 0$$

(45)

where we change the notations $W \to \mathbf{H}$ and $B \to S$. The related potential have the form $A = (A^4, 0, A)$, moreover, $A^4$ should satisfy the condition $\nabla A^4 = 0$.

We see that in contrast with the relativistic theory there exist a big variety of equations for massless vector fields invariant with respect to the Galilei group. The list of such equations is given by formulae (16), (20), (35), (37), (41)–(45).

5 Discussion

The revision of classical results [1] related to Galilean electromagnetism made in the present paper appears to be successful. Our knowledge of indecomposable representations of homogeneous Galilei group defined in vector and scalar fields [3] made it possible to complete the results of Le Bellac and Lévy-Leblond [1] and present an extended class of Galilei-invariant equations for massless vector fields. Among them are decoupled systems of first order equations which include the same number of components as the Maxwell equations and equations with other numbers of components as well. The most extended system includes ten components while the most reduced one has three components only.

It is necessary to stress that the majority of obtained equations admit clear physical interpretations. Thus equation (41) and (44) are applied in electro- and magnetostatics respectively. The very procedure of deducing of Galilei invariant equations for vector fields used in the present paper makes
their interpretation to be rather straightforward, since any obtained equation has its relativistic counterpart.

We see that the number of Galilean wave equations for massless vector fields is rather extended, and so there are many possibilities to describe interaction of non-relativistic charged particles with external gauge fields. Some of these possibilities have been discussed in [3] and [10], see also [7], [14] and [4], [5]. Starting with the found equations and using the list of functional invariants for Galilean vector fields presented in [9] it is easy to construct nonlinear models invariant with respect to the Galilei group, including its supersymmetric extensions.

Invariance of any particular found equation with respect to the Galilei transformations can be verified by direct calculation. The main result presented in this paper is the completed description of all such equations.

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