Sieve Methods in Random Graph Theory

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Abstract
In this paper, we apply the Turán sieve and the simple sieve developed by R. Murty and the first author to study problems in random graph theory. In particular, we obtain upper and lower bounds on the probability of a graph on $n$ vertices having diameter 2 (or diameter 3 in the case of bipartite graphs) with edge probability $p$ where the edges are chosen independently. An interesting feature revealed in these results is that the Turán sieve and the simple sieve “almost completely” complement each other. As a corollary to our result, we note that the probability of a random graph having diameter 2 approaches 1 as $n \to \infty$ for constant edge probability $p = 1/2$.

Keywords Random graph theory · Probabilistic calculations · Sieve theory · Probabilistic combinatorics

1 Introduction

For the purpose of analyzing the random graphs in this paper, we first introduce two sieves known as the simple sieve and the Turán sieve, which were introduced in [7]. These sieves can be described in terms of a bipartite graph. Let $X$ be a bipartite graph with finite partite sets $A$ and $B$. For $a \in A$ and $b \in B$, we denote by $a \sim b$ if there is
an edge that joins $a$ and $b$. Define

$$\deg b = \#\{a \in A : a \sim b\} \quad \text{and} \quad \omega(a) = \#\{b \in B : a \sim b\}. $$

For $b_1, b_2 \in B$, we define

$$n(b_1, b_2) = \#\{a \in A : a \sim b_1, a \sim b_2\}. $$

In [7], R. Murty and the first author derived an elementary sieve method, called the simple sieve, which states that

$$\#\{a \in A : \omega(a) = 0\} \geq |A| - \sum_{b \in B} \deg b. $$

In the same paper, they also adopted Turán’s proof [9] on Hardy and Ramanujan’s result on the normal order of distinct prime factors of a natural number [3] to prove that

$$\#\{a \in A : \omega(a) = 0\} \leq |A|^2 \cdot \frac{\sum_{b_1, b_2 \in B} n(b_1, b_2)}{(\sum_{b \in B} \deg b)^2} - |A|. $$

The above result is called the Turán sieve.

The combinatorial extension of sieve methods has been investigated before the work of Murty and the first author. For example, Wilson [10] and Chow [2] formulated the combinatorial Selberg sieve. However, since the Möbius function of a lattice is difficult to compute in an abstract setting, the applications of the combinatorial Selberg sieve have been limited. The obstruction is eliminated by the Turán sieve and the simple sieve as they do not use the Möbius function. Hence one can apply them to various combinatorial problems, including graph colouring, Latin squares, tournaments, etc (see [7], [5], [6]). In this paper, we apply both the simple sieve and the Turán sieve to study problems about random graph theory. First, we need the following definition.

**Definition 1** The diameter of a graph $G$ is defined as the maximum number of edges in $G$ that are needed to traverse from one vertex to another in $G$ where we exclude paths that backtrack, detour, and loop.

Let $G(n, p)$ denote the set of all simple graphs on $n$ vertices where each edge is chosen independently with probability $p$. In 1981, Bollobás [1] obtained sharp asymptotic results for the probability of a random graph from $G(n, p)$ vertices having diameter $d$ for any fixed $d \geq 2$ with $n \to \infty$. Here we extend his results and obtain concrete upper and lower bounds on the probability of a random graph from $G(n, p)$ having diameter $d$ where $n$, $p$, and $d$ are fixed. The results of Bollobás’s follow if we let $n \to \infty$. We also study analogous questions for random $k$-partite graphs having diameter $d$ with $k \geq 2$. Although our approaches work for general diameter $d$, to better illustrate the methods, most of this paper will be dedicated to stating and proving our results for diameter 2 or diameter 3 in the case of random bipartite graphs. Here is one of the main theorems of the paper.
Theorem 1  Let \( G(n, p) \) denote the set of all simple graphs on \( n \) vertices where each edge is chosen independently with probability \( p \). Also, let \( P(\text{diam} G(n, p) = 2) \) be the probability of a graph from \( G(n, p) \) having diameter 2. Then

\[
P(\text{diam} G(n, p) = 2) \geq 1 - \frac{n^2(1 - p^2)^{n-2}(1 - p)}{2}
\]

and

\[
P(\text{diam} G(n, p) = 2) \leq \frac{2}{(n - 1)^2(1 - p^2)^{n}(1 - p)} + \frac{8}{n(1 - p^2)^{2}} \left( 1 + \frac{p^3}{(1 - p^2)^{2}} \right)^n.
\]

Corollary 1  Let \( P(\text{diam} G(n, p) = 2) \) be defined as in Theorem 1. If \( p = \frac{1}{2} \), then we have

\[
P(\text{diam} G(n, 1/2) = 2) \geq 1 - \frac{4n^2(3/4)^{n}}{9}.
\]

In the case \( p = \frac{1}{2} \), Gilbert [4] showed that ‘almost all’ graphs are connected. Since a graph with diameter 2 is connected, the above result provides an explicit bound for Gilbert’s result.

In the situation where the edge probability \( p \to 0 \) as \( n \to \infty \), we will show the following corollary.

Corollary 2  Let \( P(\text{diam} G(n, p) = 2) \) be defined as in Theorem 1. Let \( \lim_{n \to \infty} p = 0 \). We have

\[
P(\text{diam} G(n, p) = 2) \geq 1 - (1 + o(1))\frac{n^2}{2}e^{-np^2}
\]

and

\[
P(\text{diam} G(n, p) = 2) \leq (1 + o(1))\left( \frac{2}{n^2}e^{np^2} \right) \left( 1 + 4ne^{-np^2} \right).
\]

Suppose further that

\[
\lim_{n \to \infty} (2 \log n - np^2 - \log 2) = c
\]

for some \( c \in \mathbb{R} \setminus \{0\} \).

1) If \( c > 0 \), we have

\[
P(\text{diam} G(n, p) = 2) \leq (1 + o(1))e^{-c}.
\]

2) If \( c < 0 \), we have

\[
P(\text{diam} G(n, p) = 2) \geq (1 + o(1))(1 - e^c).
\]
We will also study analogous problems for a random directed graph, and a random $k$-partite graph having diameter 2 with $k \geq 3$, or diameter 3 in the case of bipartite graphs. We can also similarly obtain analogous results for directed graphs and Turán graphs and generalise these results for any given diameter $d$. We give the details in a long arxiv note where we prove such results in full generality [8].

As we noted in Corollary 2, the upper bound we obtained through the Turán sieve works effectively for $c > 0$, while the lower bound we obtained through the simple sieve gives a non-trivial result for $c < 0$. It is interesting to see that the Turán sieve and the simple sieve “almost completely” complement each other in this way.

2 Graphs with diameter 2 with the sieves

In this section, we use the Turán sieve and the simple sieve to prove Theorem 1.

**Proof of Theorem 1** For a fixed $n \in \mathbb{N}$, let $G(n, p)$ denote the set of all graphs on $n$ vertices with edge probability $p$, and let $P(diamG(n, p) = 2)$ be the probability of a graph from $G(n, p)$ having diameter 2. Consider the function $g_n : [0, 1] \rightarrow [0, 1]$ defined as $g_n(x) := P(diamG(n, x) = 2)$. There are $2^{n(n-1)/2}$ graphs in total in $G(n, p)$. Let us say $M$ of these have diameter 2 and label these as $G_1, G_2, \ldots, G_M$. For $1 \leq i \leq M$, let $k_i$ denote the number of edges in $G_i$. Then the probability of selecting the graph $G_i$ from $G(n, p)$ according to the edge probability $x$ is $x^{k_i}(1 - x)^{n(n-1)/2 - k_i}$. Therefore,

$$g_n(x) = x^{k_1}(1 - x)^{n(n-1)/2 - k_1} + x^{k_2}(1 - x)^{n(n-1)/2 - k_2} + \cdots + x^{k_M}(1 - x)^{n(n-1)/2 - k_M}. $$

Thus, for each $n \in \mathbb{N}$ the function $g_n$ is continuous. Therefore, we may assume that $p \in \mathbb{Q} \cap (0, 1)$ since $\mathbb{Q} \cap (0, 1)$ is dense in $[0, 1]$.

Let $p = \frac{r}{s}$ where $r = r(n), s = s(n) \in \mathbb{N}$. We let $A$ be the set of all graphs in $G(n, p)$, allowing for a number of duplicates of each possible graph. The reason for this construction of $A$ is to be able to have a uniformly chosen graph in $A$ correctly represent the random graph $G(n, p)$. We accomplish this by letting there be $\binom{n}{2}$ copies of the complete graph, $\binom{n}{2} \left(\frac{s}{r} - 1\right)$ copies of each graph with $\binom{n}{2} - 1$ edges, $\binom{n}{2} \left(\frac{s}{r} - 1\right)^2$ copies of each graph with $\binom{n}{2} - 2$ edges, and so on. By the binomial theorem we have

$$|A| = \sum_{k=0}^{\binom{n}{2}} \binom{\binom{n}{2}}{k} r^k (s - r)^{\binom{n}{2} - k} = s^{\binom{n}{2}}. $$

We let $B$ be all pairs of vertices so $|B| = \binom{n}{2}$. For a graph $a \in A$ and a pair of vertices $b \in B$, we say $a \sim b$ if the pair of vertices $b$ in $a$ do not share a common neighbouring vertex and are not neighbours themselves. Thus, we will have $\omega(a) = 0$ if and only if $a$ is connected with diameter at most 2.

Pick a pair of vertices $b \in B$ and call them $v_1$ and $v_2$. To calculate $\deg b$, we need to calculate the number of graphs in $A$ such that the pair of vertices do not have a
common neighbouring vertex and are not neighbours themselves. Consider first how we construct such a graph. First, for any pair of vertices that do not include either \( v_1 \) and \( v_2 \) we do not care if there is an edge between them or not. We do not, however, want an edge between \( v_1 \) and \( v_2 \) since otherwise \( v_1 \) and \( v_2 \) will be neighbours. For each of the \( n - 2 \) vertices that are not either \( v_1 \) or \( v_2 \), we have a pair of potential edges to take into consideration as well. We know for each of these pairs of edges either one or neither will be in the graph. By the binomial theorem, we have

\[
D(r, s, n) := \deg b = \sum_{k_1=0}^{n-2} \sum_{k_2=0}^{n-2} \binom{n-2}{k_1} \binom{n-2}{k_2} 2^{k_1+k_2} (s - r)^{\binom{n}{2} - k_1 - k_2} \tag{3}
\]

\[
= (s - r)^{\binom{n}{2}} \sum_{k_1=0}^{n-2} \binom{n-2}{k_1} \left(\frac{r}{s - r}\right)^{k_1} \sum_{k_2=0}^{n-2} \binom{n-2}{k_2} \left(\frac{2r}{s - r}\right)^{k_2}
\]

\[
= (s - r)^{\binom{n}{2}} \left(\frac{s}{s - r}\right)^{\binom{n}{2}} \left(\frac{s + r}{s - r}\right)^{n-2}
\]

\[
= (s - r)^{\binom{n}{2}} \left(\frac{s}{s - r}\right)^{\binom{n}{2}} \left(\frac{s^2 - r^2}{(s - r)^2}\right)^{n-2}
\]

\[
= (s - r)^{\binom{n}{2}} \left(\frac{s^2 - r^2}{(s - r)^2}\right)^{n-2} \tag{4}
\]

It follows that

\[
\sum_{b \in B} \deg b = \frac{s^{\binom{n}{2}} n(n - 1)(1 - p^2)^{n-2}(1 - p)}{2}.
\]

By the simple sieve, we obtain

\[
P(diam G(n, p) = 2) \geq 1 - \frac{n(n-1)(1-p^2)^{n-2}(1-p)}{2}
\]

\[
> 1 - \frac{n^2(1-p^2)^{n-2}(1-p)}{2} \tag{5}
\]

since the random graph \( G(n, p) \) is uniform over the set \( A \).

We now try to get an upper bound for \( P(diam G(n, p) = 2) \), in which we need to estimate \( \sum_{b_1, b_2 \in B} n(b_1, b_2) \). In the following, we calculate \( n(b_1, b_2) \), depending on how many vertices \( b_1 \) and \( b_2 \) have in common.

**Case 1** Suppose that \( b_1 \) and \( b_2 \) are two pairs of vertices that have no vertices in common, i.e., \( b_1 \) and \( b_2 \) consist of 4 distinct vertices. For each of \( b_1 \) and \( b_2 \), the probability that the pair of vertices in question are not connected by an edge nor have any common neighbouring vertices is

\[
\frac{D(r, s, n)}{s^{\binom{n}{2}}}
\]
Let \(v_1\) and \(v_2\) be the pair of vertices in \(b_1\) and \(v_3\) and \(v_4\) be the pair of vertices in \(b_2\). To calculate \(n(b_1, b_2)\), we need to calculate the number of graphs in \(A\) such that the each pair of vertices do not have a common neighbouring vertex and are not neighbours themselves.

Consider first how we construct such a graph. First, for any pair of vertices that do not include any of the four vertices \(v_1, v_2, v_3, v_4\) we do not care if there is an edge between them or not. We do not, however, want an edge between \(v_1\) and \(v_2\) and not an edge between \(v_3\) and \(v_4\). Before we consider potential edges involving any other vertices, there are the four potential edges between \(v_1\) and \(v_3\), between \(v_1\) and \(v_4\), between \(v_2\) and \(v_3\), and between \(v_2\) and \(v_4\). Our random graph could have none of these edges in it, exactly one of these edges in it, or exactly two of these edges in it. Notice that it cannot have more than two of these edges in it because then \(v_1\) and \(v_2\) would have a common neighbour vertex between either \(v_3\) or \(v_4\). Also, if we have exactly two of these edges in it, then we have two choices of these two edges: either we have the edges between \(v_1\) and \(v_3\) and between \(v_2\) and \(v_4\) or we have the edges between \(v_1\) and \(v_4\) and between \(v_2\) and \(v_3\).

For each of the \(n - 4\) vertices that are not any of \(v_1, v_2, v_3, v_4\), we have a group of four potential edges to take into consideration as well. The four edges in any specific group can be split into two pairs such that for each pair either one or neither of the edges will be in the graph. Putting this all together, by the binomial theorem, we have

\[
\begin{align*}
n(b_1, b_2) &= \left( (s - r)^4 + 4r(s - r)^3 + 2r^2(s - r)^2 \right) \\
&\quad \cdot \sum_{k_1=0}^{(n-4)\choose 2} \sum_{k_2=0}^{2n-8} \left( \begin{array}{c} (n-4) \\ k_1 \end{array} \right) \left( \begin{array}{c} 2n-8 \\ k_2 \end{array} \right) 2^{k_2} r^{k_1+k_2} (s - r)\binom{c}{2} - k_1 - k_2 - 4 \\
&= (s - r)\binom{c}{2} \left( 1 + \frac{4r}{s - r} + \frac{2r^2}{(s - r)^2} \right) \\
&\quad \cdot \sum_{k_1=0}^{(n-4)\choose 2} \left( \begin{array}{c} (n-4) \\ k_1 \end{array} \right) \left( \frac{r}{s - r} \right)^{k_1} \sum_{k_2=0}^{2n-8} \left( \begin{array}{c} 2n-8 \\ k_2 \end{array} \right) \left( \frac{2r}{s - r} \right)^{k_2} \\
&= (s - r)\binom{c}{2} \left( 1 + \frac{4r}{s - r} + \frac{2r^2}{(s - r)^2} \right) \left( \frac{s + r}{s - r} \right)^{\binom{c}{2} - 2n - 8} \\
&= (s - r)\binom{c}{2} \left( 1 + \frac{4r}{s - r} + \frac{2r^2}{(s - r)^2} \right) \left( \frac{s^2 - r^2}{(s - r)^2} \right)^{2n - 8} \tag{8}
\end{align*}
\]

It follows that

\[ n(b_1, b_2) = s^{(n-4)\choose 2} (s - r)^2 \left( (s - r)^4 + 4r(s - r)^3 + 2r^2(s - r)^2 \right) \left( s^2 - r^2 \right)^{2n - 8} \]

so that

\[
\begin{align*}
n(b_1, b_2) &= \frac{D(r, s, n)^2}{s^{(c/2)}} \cdot \frac{s^4((s - r)^4 + 4r(s - r)^3 + 2r^2(s - r)^2)}{(s^2 - r^2)^4},
\end{align*}
\]
and thus

\[
\sum_{b_1, b_2 \in B, 4 \text{ vertices}} n(b_1, b_2) \leq \binom{n}{2}^2 \frac{D(r, s, n)^2}{s(\ell)} \cdot \frac{p^{-4}((p^{-1} - 1)^4 + 4(p^{-1} - 1)^3 + 2(p^{-1} - 1)^2)}{(p^{-2} - 1)^4} \cdot \binom{n}{2}^2 \frac{D(r, s, n)^2}{s(\ell)} \cdot \left(1 + \frac{4p^3}{(1-p)^2}\right).
\]

**Case 2** Take two pairs of vertices \(b_1 \) and \(b_2\) that have exactly one vertex in common, i.e., \(b_1 \) and \(b_2\) consist of 3 distinct vertices. Let the vertices in \(b_1 \) be \(v_1\) and \(v_2\) and the vertices in \(b_2\) be \(v_2\) and \(v_3\). We can do a similar kind of analysis of edge selection as in Case 1 as follows. First, for any pair of vertices that do not include any of the three vertices \(v_1\), \(v_2\), \(v_3\) we do not care if there is an edge between them or not. We do not, however, want an edge between \(v_1\) and \(v_2\) and not an edge between \(v_2\) and \(v_3\). Given this, however, we do not care if there is an edge between \(v_1\) and \(v_3\).

For each of the \(n - 3\) vertices that are not any of \(v_1\), \(v_2\), \(v_3\), we have a group of three potential edges to take into consideration as well. In each group, either none of the edges appear in the graph, exactly one of the edges appears in the graph, or the edges appear in the graph, exactly one of the edges appears in the graph, or the two edges having \(v_1\) and \(v_3\) as endpoints are in the graph, but the edge with \(v_2\) as an endpoint is not in the graph. Putting this all together, by the binomial theorem, we have

\[
n(b_1, b_2) = (s - r)^2 \sum_{k_1 = 0}^{n - k_2} \binom{n - 3}{k_1} + 1 \sum_{k_2 = 0}^{n - 3} \binom{n - 3}{k_2} \cdot \sum_{k_3 = 0}^{n - 3 - k_2} \binom{n - 3 - k_2}{k_3} 3^{k_2} r^{k_1 + k_2 + 2k_3} (s - r)^{\binom{n - 3}{2} + 3n - 8 - k_1 - k_2 - 2k_3} = (s - r)^2 \binom{n - 3}{2} + 1 \sum_{k_1 = 0}^{n - 3} \binom{n - 3}{k_1} \cdot \sum_{k_2 = 0}^{n - 3 - k_2} \binom{n - 3}{k_2} \cdot 3^{k_2} r^{k_1 + k_2} (s - r)^{\binom{n - 3}{2} + 3n - 8 - k_1 - k_2} \left(1 + \frac{r^2}{(s - r)^2}\right)^{n - 3 - k_2} = (s - r)^2 \left(1 + \frac{r^2}{(s - r)^2} + \frac{3r}{s - r}\right)^{n - 3} \sum_{k_1 = 0}^{n - 3} \binom{n - 3}{1} \left(\frac{n - 3}{2} + 1\right) r^{k_1} (s - r)^{\binom{n - 3}{2} + 3n - 8 - k_1} = (s - r)^2 \left(1 + \frac{r^2}{(s - r)^2} + \frac{3r}{s - r}\right)^{n - 3} (s - r)^{\binom{n - 3}{2} + 3n - 8} \left(\frac{s}{s - r}\right)^{\binom{n - 3}{2} + 1} = (s - r)^2 s^{\binom{n - 3}{2} + 1} (s - r)^3 + r^2 (s - r) + 3r (s - r)^2)^{n - 3} = \frac{D(r, s, n)^2}{s(\ell)} \cdot \frac{s^{n + 1} (s - r)^3 + r^2 (s - r) + 3r (s - r)^2)^{n - 3}}{(s^2 - r^2)^{2n - 4}}.
\]
We deduce
\[
\sum_{b_1, b_2 \in B, \text{ 3 vertices}} n(b_1, b_2) = D(r, s, n)^2 n(n - 1)(n - 2) \cdot \left(1 + \frac{1}{p^{-3 + p^{-2} - p^{-1} - 1}}\right)^{n-3} \\
\leq \frac{D(r, s, n)^2 n(n - 1)(n - 2)}{s^2 (n^2) (1 - p^2)^2} \left(1 + \frac{p^3}{(1 - p)}\right)^{n-3}.
\] (14)

**Case 3** Suppose \(b_1\) and \(b_2\) have two vertices in common. Then the two pairs are identical, and we have
\[
n(b_1, b_2) = \deg b.
\]

It follows that
\[
\sum_{b_1, b_2 \in B, \text{ 2 vertices}} n(b_1, b_2) = \sum_{b \in B} \deg b = \frac{s(n^2)(n - 1)(1 - p^2)^{n-2}(1 - p)}{2}.
\]

Combining Cases 1 – 3, we get
\[
\sum_{b_1, b_2 \in B} n(b_1, b_2) < \left(\frac{n}{2}\right)^2 \frac{D(r, s, n)^2}{s^2(n^2)} \cdot \left(1 + \frac{4p^3}{(1 - p)^2}\right) \\
+ \frac{D(r, s, n)^2 n(n - 1)(n - 2)}{s^2(n^2)(1 - p^2)^2} \left(1 + \frac{p^3}{(1 - p)}\right)^{n-3} \\
+ \frac{s^2(n^2)(n - 1)(1 - p^2)^{n-2}(1 - p)}{2}.
\]

By the Turán sieve, we deduce
\[
P(diamG(n, p) = 2) \\
\leq \frac{2}{n(n - 1)(1 - p^2)^{n-2}(1 - p)} + \frac{4}{n(1 - p^2)^2} \left(1 + \frac{p^3}{(1 - p)}\right)^{n-3} \\
+ \frac{4p^3}{(1 - p)^2}.
\]

Notice that
\[
\frac{p^3}{(1 - p)^2} < \frac{1}{n} \left(1 + n\frac{p^3}{(1 - p)^2}\right) < \frac{1}{n} \left(1 + \frac{p^3}{(1 - p)^2}\right)^n.
\]
It follows that

\[
P(diamG(n, p) = 2) \leq \frac{2}{(n-1)^2(1-p)^n(1-p)} + \frac{8}{n(1-p^2)^2} \left(1 + \frac{p^3}{(1-p)^2}\right)^n. \tag{15}
\]

By (5) and (15) Theorem 1 follows.

We now prove Corollary 2.

**Proof** Using the inequality \(1 - x \leq e^{-x}\), valid for \(0 \leq x \leq 1\), \(x = p^2\) in the lower bound in Theorem 1 gives (1). We can see that the upper bound in Theorem 1 is non-trivial only if \(np^2 < 2 \log n - \log 2\). Then, using the inequality \(e^{-x} \leq 1 - x + \frac{x^2}{2}\), valid for \(0 \leq x \leq 1\), for \(x = p^2\), and the inequality \(1 + x \leq e^x\), valid for \(x \geq 0\), for \(x = \frac{p^3}{(1-p)^2}\), and working out the asymptotics gives in the upper bound in Theorem 1 gives (2).

\(\square\)

**Remark 1** Assume that \(n \geq 100\) and \(p \leq 1/2\). The \(o(1)\) in (1) can be made explicit as \(4p^2\) and the \(o(1)\) in (2) can be made explicit as \(\frac{12(\log n)^2}{n} + \frac{35(\log n)^{3/2}}{n^{1/2}}\).

### 3 Analysis of \(k\)-partite Graphs

Here we apply our analysis to \(k\)-partite graph sets for \(k \geq 3\). First, we present a definition.

**Definition 2** Let \(k \geq 2\). A simple \(k\)-partite graph is an undirected graph whose vertices can be divided into \(k\) sets, such that there are no edges between two vertices in the same set.

We exclude the bipartite case \((k = 2)\) because the only bipartite graph that has diameter 2 is the complete bipartite graph; we analyze that case by itself in the next section.

**Convention 2** For each \(k\)-partite graph, we label the \(k\) partite sets of the graph in a non-decreasing order in terms of the number of vertices each set contains. Thus, the \(i\)th set is a set containing \(n_i\) vertices.

**Theorem 3** Fix \(k \geq 3\) and for each \(n \in \mathbb{N}\), \(n \geq k + 2\), pick \(n_1, n_2, \ldots, n_k \in \mathbb{N}\) such that \(n_1 \leq n_2 \leq \ldots \leq n_k\), \(n_{k-1} \geq 2\), and \(n_1 + n_2 + \cdots + n_k = n\). Let \(n^{(k)} = (n_1, n_2, \ldots, n_k)\) and let \(G(n^{(k)}, p)\) denote the set of all \(k\)-partite graphs with the partite sets having \(n_1, n_2, \ldots, n_k\) vertices respectively where each edge is chosen independently with probability \(p\). Also, let \(P(diamG(n^{(k)}, p) = 2)\) be the probability of a graph from \(G(n^{(k)}, p)\) having diameter 2. Then

\[
P(diamG(n^{(k)}, p) = 2) \geq 1 - \frac{n_k^2(1 - p^2)^{n-n_k}}{2}.
\]
\[
\left(1 + \frac{2n_{k-1}(1 - p^2)^{-nk-1}}{nk} + \frac{2(n-n_k-n_{k-1})(1 - p^2)^{-nk-2}}{n_k}
\right.
\]
\[
+ \frac{2k^2n_{k-1}^2(1 - p^2)^{nk-n_{k-1}-nk-2}}{n_k^2}
\)

and

\[
P(diamG(n^{(k)}, p) = 2)
\leq \frac{2}{n_k(n_k-1)(1-p^2)^{n-n_k}} \left(1 + \frac{2n_{k-1}(1 - p^2)^{-nk-1}(1 - p)}{(n_k-1)} \right)^{-1}
\]
\[
+ \frac{3k^3}{(n_k-1)} \left(1 + \frac{p^3}{(1-p)} \right)^{n-n_k} (1 - p^2)^{-2} + \frac{4p^3}{(1-p)^2}.
\]

**Proof of Theorem 3** As in the proof of Theorem 1, we may assume that \( p \in \mathbb{Q} \cap (0, 1) \) for all \( n \in \mathbb{N} \).

Let \( p = \frac{r}{s} \) where \( r, s \in \mathbb{N} \). As in the proof of Theorem 1, we let \( A \) be the set of all graphs in \( G(n^{(k)}, p) \), allowing for a number of duplicates of each possible graph. As in the proof of Theorem 1 this construction of \( A \) allows us to have a uniformly chosen graph in \( A \) correctly represent the random graph \( G(n^{(k)}, p) \). Since the complete \( k \)-partite graph has \( t := \sum_{1 \leq i < j \leq k} n_in_j \) edges, we have \( r^t \) copies of the complete \( k \)-partite graph and \(|A| = s^t\).

We let \( B \) be all pairs of vertices. Thus, \(|B| = \frac{n(n-1)}{2}\). For a graph \( a \in A \) and a pair of vertices \( b \in B \), we say \( a \sim b \) if the pair of vertices \( b \) in \( a \) do not share a common neighbouring vertex and are not connected by a single edge. Thus, we will have \( \omega(a) = 0 \) if and only if \( a \) is connected with diameter at most 2. Suppose that \( b \in B \) is a pair of vertices that are in the \( i \)th partite set for some \( 1 \leq i \leq k \). Then this pair of vertices has \( n - n_i \) potential common neighbouring vertices. Similarly to how we derived (4) from (3) we again use the binomial theorem on summing all the different possible graphs with their weights to obtain

\[
D(r, s, n, n_i) := \deg b
\]
\[
= (s^2 - p^2)^{n-n_i} s^{t-2n+2n_i}
\]
\[
= (1 - p^2)^{n-n_i} s^{t},
\]

where we replaced \( \binom{n-2}{2} \) with \( t-2n+2n_i \) (the number of potential edges not touching either of the vertices in \( b \)) and \( n-2 \) with \( n-n_i \) (the number of potential common neighbouring vertices in both cases).

Suppose that \( b \in B \) is a pair of vertices with one vertex being in the \( i \)th partite set and the other in the \( j \)th partite set, where \( i < j \). Then this pair of vertices has \( n - n_i - n_j \) potential common neighbouring vertices. Again, similarly to how we derived (4) from (3) we again use the binomial theorem on summing all the different possible graphs with their weights to obtain

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Let \( D(r, s, n, n_i, n_j) : = \deg b \)
\[
= \left( s^2 - r^2 \right)^{n-n_i-n_j} s^{t-2n+2n_i+2n_j} (1 - p) \\
= (1 - p^2)^{n-n_i-n_j} (1 - p)s^t 
\]
since in this case the number of potential edges no touching either of the vertices in \( b \)
is \( t - 2n + 2n_i + 2n_j \) and the number of potential common neighbouring vertices is
\( n - n_i - n_j \).

It follows that
\[
\sum_{b \in B} \deg b \\
= s^t \sum_{i=1}^k \left( \frac{n_i}{2} \right) (1 - p^2)^{n-n_i} + s^t \sum_{1 \leq i < j \leq k} n_i n_j (1 - p^2)^{n-n_i-n_j} (1 - p).
\]

By the simple sieve, we obtain
\[
P(diamG(n^{(k)}, p) = 2) \\
\geq 1 - \frac{n_k^2 (1-p^2)^{n-n_k}}{2} \\
\cdot \left( 1 + \frac{2n_{k-1} (1-p^2)^{n-k-1}}{n_k} + \frac{7k^2 n_{k-1}^2 (1-p^2)^{n-k-1-n-2}}{3n_k^2} \right)
\]
since the random graph \( P(diamG(n^{(k)}, p) = 2) \) is uniform over the set \( A \).

To get an upper bound for \( P(diamG(n^{(k)}, p) = 2) \), we need to estimate \( \sum_{b_1, b_2 \in B} n(b_1, b_2) \). Similar to the proof of Theorem 1, we calculate \( n(b_1, b_2) \) based on the number of vertices \( b_1 \) and \( b_2 \) have in common. Suppose first that \( b_1 \) and \( b_2 \)
have no vertices in common. Then, assuming the respective complete \( k \)-partite graph has four edges between the vertices in \( b_1 \) and the vertices in \( b_2 \), we have
\[
n(b_1, b_2) = \left( (s - r)^4 + 4r(s - r)^3 + 2r^2(s - r)^2 \right) \\
\cdot \sum_{t_1} \sum_{t_2} \sum_{t_3} \left( \frac{t_1}{k_1} \right) \left( \frac{t_2}{k_2} \right) \left( \frac{t_3}{k_3} \right) 2^{k_2+k_3} r^{k_1+k_2+k_3} (s - r)^{t-k_1-k_2-k_3-4},
\]
where \( t_1 \) is the number of potential edges not touching any vertex in \( b_1 \) or \( b_2 \), \( t_2 \) is the number of potential common neighbouring vertices shared by the pair in \( b_1 \), and \( t_3 \) is the number of potential common neighbouring vertices shared by the pair in \( b_2 \). If the respective complete \( k \)-partite graph has exactly three edges between the vertices in \( b_1 \) and the vertices in \( b_2 \), then the factor of \( \left( (s - r)^4 + 4r(s - r)^3 + 2r^2(s - r)^2 \right) \)
is replaced with \( (s - r)^2 + 2r(sr) + r^2 \) with the expression \( (s - r)^{t-k_1-k_2-k_3-4} \)
replaced by \( (s - r)^{t-k_1-k_2-k_3-3} \). If the respective complete \( k \)-partite graph has exactly two edges between the vertices in \( b_1 \) and the vertices in \( b_2 \), then the factor
of \((s-r)^{4} + 4r(s-r)^{3} + 2r^{2}(s-r)^{2}\) is replaced with \((s-r)^{2} + 2r(s-r) + r^{2}\)
with the expression \((s-r)^{t} - k_{1} - k_{2} - k_{3} - 4\) replaced by \((s-r)^{t} - k_{1} - k_{2} - k_{3} - 2\)
and the \(=\) replaced with \(\leq\). If the respective complete \(k\)-partite graph has exactly
one edge between the vertices in \(b_{1}\) and the vertices in \(b_{2}\), then the factor of
\(\left((s-r)^{4} + 4r(s-r)^{3} + 2r^{2}(s-r)^{2}\right)\) is replaced with \((r + s-r)\) with the expres-
sion \((s-r)^{t} - k_{1} - k_{2} - k_{3} - 4\) replaced by \((s-r)^{t} - k_{1} - k_{2} - k_{3} - 1\). If the respective complete
\(k\)-partite graph has no edges between the vertices in \(b_{1}\) and the vertices in \(b_{2}\),
then the factor of \(\left((s-r)^{4} + 4r(s-r)^{3} + 2r^{2}(s-r)^{2}\right)\) is replaced with 1 with the expres-
sion \((s-r)^{t} - k_{1} - k_{2} - k_{3} - 4\) replaced by \((s-r)^{t} - k_{1} - k_{2} - 3\). In all cases, similarly
to how we derived (12) from (7), we can derive

\[
n(b_{1}, b_{2}) \leq \frac{\mathrm{deg} b_{1} \mathrm{deg} b_{2}}{s^{t}} \left(1 + \frac{4p^{3}}{(1-p)^{2}}\right).
\]

Now we consider two pairs of vertices \(b_{1}\) and \(b_{2}\) that have exactly one vertex in
common, i.e., \(b_{1}\) and \(b_{2}\) consist of 3 distinct vertices. First suppose all three vertices
are in the same \(i\)th partite set. Then the equation for \(n(b_{1}, b_{2})\) is the same as in (13)
with the following minor changes. The factor \((s-r)^{2}\) is removed, \(\binom{n}{2} - 3 + 1\) is replaced
with \((t-3)(n-n_{i})\), \(n-3\) is replaced with \(n-n_{i}\), \(n-3-k_{2}\) is replaced with \(n-n_{i}-k_{2}\),
and \(\binom{n}{2} - 3n - 8 - k_{1} - k_{2} - 2k_{3}\) is replaced with \(t - k_{1} - k_{2} - k_{3}\). We can deduce that
in this case

\[
n(b_{1}, b_{2}) \leq \frac{D(r, s, n, n_{i})^{2}}{s^{t}} \left(1 + \frac{1}{p^{-3} + p^{-2} - p^{-1} - 1}\right)^{n-n_{i}}
\]

\[
\leq \frac{D(r, s, n, n_{k})^{2}}{s^{t}} \left(1 + \frac{1}{p^{-3} + p^{-2} - p^{-1} - 1}\right)^{n-n_{k}}
\]

\[
\leq \frac{D(r, s, n, n_{k})^{2}}{s^{t}} \left(1 + \frac{p^{3}}{(1-p)}\right)^{n-n_{k}}
\]

the same way we derived (14) from (13). We now consider the case when the shared
vertex between \(b_{1}\) and \(b_{2}\) is in the \(j\)th partite set and the two other vertices in the pairs
are in the \(i\)th partite set, where \(i \neq j\). For this case, we make the following minor
changes to (13). Remove \(\binom{n}{2} - 3 + 1\) is replaced with \(t - 3n + 3n_{i} + 3n_{j} + 2\), \(n-3\) is replaced with \(n-n_{i}-n_{j}\), \(n-3-k_{2}\) is replaced with \(n-n_{i}-n_{j}-k_{2}\),
and \(\binom{n}{2} - 3n - 8 - k_{1} - k_{2} - 2k_{3}\) is replaced with \(t - k_{1} - k_{2} - k_{3}\). We can deduce that
in this case

\[
n(b_{1}, b_{2}) \leq \frac{D(r, s, n, n_{i}, n_{j})^{2}}{s^{t}} \left(1 + \frac{1}{p^{-3} + p^{-2} - p^{-1} - 1}\right)^{n-n_{i}-n_{j}}
\]

\[
\leq \frac{D(r, s, n, n_{k}, n_{k-1})^{2}}{s^{t}} \left(1 + \frac{1}{p^{-3} + p^{-2} - p^{-1} - 1}\right)^{n-n_{k}-n_{k-1}}
\]

\[
\leq \frac{D(r, s, n, n_{k}, n_{k-1})^{2}}{s^{t}} \left(1 + \frac{p^{3}}{(1-p)}\right)^{n-n_{k}-n_{k-1}}
\]
the same way we derived (14) from (13). We now consider the case when one of the pair of vertices is in the $i$th partite set and the other vertex in the other pair is in the $j$th partite set, where $i \neq j$. For this case, we make the following minor changes to (13). The factor $(s - r)^2$ is replaced with $(s - r)$, $\binom{n-3}{2} + 1$ is replaced with $t - 3n + 3n_i + n_j + 1$, $n - 3$ is replaced with $n - n_i - n_j$, $n - 3 - k_2$ is replaced with $n - n_i - n_j - k_2$, and $(s - r)^{\binom{n-3}{2}+3n-8-k_1-k_2-2k_3}$ is replaced with yet another summation

$$\sum_{k_4=0}^{n_i-1} \sum_{k_5=0}^{n_j-1} \binom{n_i-1}{k_4} \binom{n_j-1}{k_5} 2^{k_4+k_5} (s - r)^{t-k_1-k_2-k_3-k_4-k_5}.$$ 

We can deduce that in this case

$$n(b_1, b_2) \leq \frac{D(r, s, n, n_i)D(r, s, n, n_i, n_j)}{s^l \left(1 - p^2\right)^2} \left(1 + \frac{1}{p^3 + p^2 - p^{-1} - 1}\right)^{n-t-3n_i+n_j-n_i-n_j}$$

the same way we derived (14) from (13). We now consider the case when all three vertices in the two pairs are in three different partite sets, say in the $i$th, $j$th, and $l$th partite sets, where the shared vertex between the pairs is in the $j$th partite set. For this case, we make the following minor changes to (13). Remove $\binom{n-3}{2} + 1$ is replaced with $t - 3n + n_i + n_j + n_l + 4$, $n - 3$ is replaced with $n - n_i - n_j - n_l$, $n - 3 - k_2$ is replaced with $n - n_i - n_j - n_l - k_2$, and $(s - r)^{\binom{n-3}{2}+3n-8-k_1-k_2-2k_3}$ is replaced with yet another double summation

$$\sum_{k_4=0}^{n_i-1} \sum_{k_5=0}^{n_l-1} \binom{n_i-1}{k_4} \binom{n_l-1}{k_5} 2^{k_4+k_5} (s - r)^{t-k_1-k_2-k_3-k_4-k_5}.$$ 

We can deduce that in this case

$$n(b_1, b_2) \leq \frac{D(r, s, n, n_i, n_j)D(r, s, n, n_i, n_j, n_l)}{s^l \left(1 - p^2\right)^2} \left(1 + \frac{1}{p^3 + p^2 - p^{-1} - 1}\right)^{n-t-3n_i+n_j-n_i-n_j-n_l}$$
the same way we derived (14) from (13). Putting it altogether, we get

\[
\sum_{b_1, b_2 \in B} n(b_1, b_2) \\
\leq \sum_{b \in B} \deg b + \left( \frac{\sum_{b \in B} \deg b}{s^t} \right)^2 \left( 1 + \frac{4p^3}{(1 - p)^2} \right) \\
+ \frac{k^3n_k^2(3/4)^{n-n_k}D(r, s, n, n_k, n_k-1)D(r, s, n, n_k, n_k-2)}{s^t(1 - p^2)^2} \left( 1 + \frac{p^3}{(1 - p)} \right)^{n-n_k-n_k-1} \\
+ \frac{k^2n_k^2(3/4)^{n-n_k}D(r, s, n, n_k, n_k-1)}{s^t(1 - p^2)} \left( 1 + \frac{p^3}{(1 - p)} \right)^{n-n_k-n_k-1} \\
+ \frac{kn_2^2(3/4)^{n-n_k}D(r, s, n, n_k)}{s^t(1 - p^2)} \left( 1 + \frac{p^3}{(1 - p)} \right)^{n-n_k}.
\]

Then, by the Turán sieve, we get

\[
P(diamG(n^{(k)}), p) = 2) < \frac{2}{n_k(n_k-1)(1 - p^2)^{n-n_k}} \\
\cdot \left( 1 + \frac{2n_k-1}{(n_k-1)} \right)^{-1} \\
+ \frac{3k^3(1 + \frac{p}{(1-p)} \right)^{n-n_k}(1 - p^2)^{-2}}{(n_k-1)} + \frac{4p^3}{(1 - p)^2}.
\]

This completes the proof of Theorem 3. \qed

By substituting \( p = \frac{1}{2} \), we deduce from Theorem 3 the following.

**Corollary 3** Let \( P(diamG(n^{(k)}), p) = 2) \) be defined as in Theorem 3. If \( p = \frac{1}{2} \), then we have

\[
P(diamG(n^{(k)}), 1/2) = 2) \\
\geq 1 - \frac{n_k^2(3/4)^{n-n_k}}{2} \left( 1 + \frac{2n_k-1}{n_k} \right) + \frac{2(n - n_k - n_k-1)(3/4)^{-n_k-2}}{n_k} \\
+ \frac{2k^2n_k^2(3/4)^{n_k-n_k-1-n_k-2}}{n_k^2}.
\]

In the case when \( p \to 0 \) as \( n \to \infty \), we have the following.

**Corollary 4** Let \( P(diamG(n^{(k)}), p) = 2) \) be defined as in Theorem 3. Let \( \lim_{n \to \infty} p^4(n - n_k) = 0 \). We have
\[ P(diamG(n^k, p) = 2) \geq 1 - \frac{n^2 e^{-p^2(n-n_k)}}{2} \cdot \left( 1 + \frac{2(n - n_k - n_{k-1})e^{p^2 n_{k-2}}}{n_k} + \frac{2n_{k-1} e p^2 n_{k-1}}{n_k} \right) \]

and

\[ P(diamG(n^k, p) = 2) \leq (1 + o(1)) \frac{2e^{p^2 n_{k-1}}}{n_k} \left( 1 + \frac{2n_{k-1} e p^2 n_{k-1}}{n_k} \right)^{-1} \cdot \left( 1 + \frac{3k^2 n^2 k (p^3 - p^2)(n-n_k)}{2(n_{k-1} - 1)} + \frac{3k^2 n_{k-1}(p^3 - p^2)(n-n_k) + p^2 n_{k-1}}{(n_{k-1} - 1)} \right) + \frac{4p^3}{(1-p)^2}. \quad (18) \]

Suppose further that

\[ \lim_{n \to \infty} \left( \log n_{k-2} - \log n + p^2 n_{k-2} \right) = -\infty, \quad (19) \]
\[ \lim_{n \to \infty} \left( 2 \log n + (p^3 - p^2)(n - n_k) - \log n_{k-1} \right) = -\infty, \quad (20) \]
\[ \lim_{n \to \infty} \left( (p^3 - p^2)(n - n_k) + p^2 n_{k-1} + \log n \right) = -\infty, \quad (21) \]

and that

\[ \lim_{n \to \infty} \left( 2 \log n_k - p^2(n - n_k) - \log 2 + \log \left( 1 + \frac{2n_{k-1}}{n_k} e p^2 n_{k-1} \right) \right) = c \]

for some \( c \in \mathbb{R} \).

1) If \( c < 0 \), we have

\[ P(diamG(n^k, p) = 2) \geq 1 - (1 + o(1)) e^c. \]

2) If \( c > 0 \), we have

\[ P(diamG(n^k, p) = 2) \leq (1 + o(1)) e^{-c}. \]
\[
e^{-p^2 (n - n_k)} > (1 - p^2)^{p^{-2} p^2 (n - n_k)} > e^{-p^2 (n - n_k)} (1 - p^2)^{p^2 (n - n_k)}
\]
\[
= e^{-p^2 (n - n_k)} (1 + o(1))
\]
and
\[
e^{p^2 n_k - 1} < (1 - p^2)^{-p^{-2} p^2 n_k - 1} < e^{p^2 n_k - 1} (1 - p^2)^{-p^2 n_k - 1}
\]
\[
= e^{p^2 n_k - 1} (1 + o(1)).
\]
Thus the first term in the upper bound of \( P(diamG(n^{(k)}, p) = 2) \) in Theorem 3 becomes
\[
\frac{2}{n_k(n_k - 1)(1 - p^2)^{p^{-2} p^2 (n - n_k)}}
\]
\[
\cdot \left(1 + \frac{2n_k - 1}{n_k - 1} \frac{2e^{p^2 (n - n_k)}}{n_k} \frac{n - n_k}{p^2 - 1} \right)^{-1}
\]
\[
= (1 + o(1)) \frac{2e^{p^2 (n - n_k)}}{n_k^2} \frac{n - n_k}{p^2 - 1} \left(1 + \frac{2n_k - 1}{n_k} e^{p^2 n_k - 1}\right)^{-1}.
\]  
(23)

For the second term, first note that since \( \lim_{n \to \infty} p^4 (n - n_k) = 0 \), we have
\[
\lim_{n \to \infty} (n - n_k) p^2 \left(\frac{p}{1 - p} - 1\right) - (n - n_k) p^2 (p - 1) = 0.
\]
Thus the second term in the upper bound of \( P(diamG(n^{(k)}, p) = 2) \) in Theorem 3 becomes
\[
\frac{3k^3 e^{p^3 (n - n_k)}}{n_k - 1} \left(1 - p^2\right)^{-2} \left(1 - p^2\right)^{-2} = \frac{3k^3 e^{p^3 (n - n_k)}}{n_k - 1} (1 + o(1)).
\]  
(24)
Combining Eqs. (23) and (24), the upper bound of Corollary 4 follows. Also, by Eqs. (19), (20), and (21), Statements (1) and (2) follow as in the proof of Corollary 2.

**Remark 2** Similar to Remark 1, all \( o(1) \) terms in Corollary 4 can be made explicit.

## 4 Bipartite graphs with diameter 3

Here we analyze bipartite graphs in a similar way to \( k \)-partite graphs, but instead of considering diameter 2, we consider diameter 3 since, except for the complete bipartite graph, all bipartite graphs have diameter at least 3.
Theorem 4 For each $n \in \mathbb{N}$, $n \geq 4$, pick $n_1, n_2 \in \mathbb{N}$ such that $2 \leq n_1 \leq n_2$ and $n_1 + n_2 = n$. Let $G(n_1, n_2, p)$ denote the set of all bipartite graphs with the partite sets having $n_1$ and $n_2$ vertices respectively where each edge is chosen independently with probability $p$. Also, let $P$ denote Theorem 4

For each $n \geq 1$, Graphs and Combinatorics (2023) 39 :39 Page 17 of 23 $P(diamG(n_1, n_2, p)) = 3$) be the probability of a graph from $G(n_1, n_2, p)$ having diameter 3. Then

$$P(diamG(n_1, n_2, p) = 3) \geq 1 - \frac{n_2^2(1 - p^2)^{n_1}}{2} \left(1 + \frac{n_2^2(1 - p^2)^{n_2-n_1}}{n_2^2}\right)$$

and

$$P(diamG(n_1, n_2, p) = 3) \leq \left(\frac{2}{n_2(n_2-1)(1 - p^2)^{n_1}} + \frac{\left(1 + \frac{p^3}{(1-p)^2}\right)^{n_1}}{n_2} \left(8 + \frac{8}{(1-p)}\right) \frac{1}{\left(1 + \frac{n_1(n_1-1)(1 - p^2)^{n_2-n_1}}{n_2(n_2-1)}\right)^{-1}}\right).$$

Proof of Theorem 4 As in the proof of Theorem 1, we may assume that $p \in \mathbb{Q} \cap (0, 1)$ for all $n \in \mathbb{N}$.

Let $p = \frac{r}{s}$ where $r, s \in \mathbb{N}$. As in the proofs of Theorems 1 and 3, we let $A$ be the set of all graphs in $G(n_1, n_2, p)$, allowing for a number of duplicates of each possible graph. As in the proofs of Theorems 1 and 3 this construction of $A$ allows us to have a uniformly chosen graph in $A$ correctly represent the random graph $G(n_1, n_2, p)$. Since the complete bipartite graph has $n_1n_2$ edges, we have $r^{n_1n_2}$ copies of the complete bipartite graph and $|A| = r^{n_1n_2}$.

We let $B$ be the set of all pairs of vertices such that both vertices of a pair occur in the same partite set. Thus, $|B| = \binom{n_1}{2} \binom{n_2}{2}$. For $a \in A$ and $b \in B$, we write $a \sim b$ if the pair of vertices $b$ in the graph $a$ do not share a common neighbouring vertex. Thus, we will have $\omega(a) = 0$ if and only if $a$ is connected with diameter at most 3. For each pair of vertices $b \in B$ in the set containing $n_1$ vertices, there are $n_2$ potential neighbouring vertices to take into account. Similarly to how we derived (4) from (3) we again use the binomial theorem on summing all the different possible graphs with their weights to obtain

$$D(r, s, n, n_1) := \deg b = ((s - r)^2 + 2r(s - r))^{n_2}((s - r) + r)^{n_1n_2-2n_2}.$$ 

For each pair of vertices $b \in B$ in the set containing $n_2$ vertices, the $n_1$ and $n_2$ are switched in the above equality. It follows that

$$\sum_{b \in B} \deg b = \frac{s^{n_1n_2}n_1(n_1-1)(1 - p^2)^{n_2}}{2} + \frac{s^{n_1n_2}n_2(n_2-1)(1 - p^2)^{n_1}}{2}.$$
By the simple sieve, we obtain

\[
P(diamG(n_1, n_2, p) = 3) > 1 - \frac{n_1^2(1 - p^2)^{n_2}}{2} - \frac{n_2^2(1 - p^2)^{n_1}}{2}
= 1 - \frac{n_2^2(1 - p^2)^{n_1}}{2} \left(1 + \frac{n_1^2(1 - p^2)^{n_2-n_1}}{n_2^2}\right)
\]

since the random graph \(P(diamG(n_1, n_2, p) = 3)\) is uniform over the set \(A\).

To get an upper bound for \(P(diamG(n_1, n_2, p) = 3)\), we need to estimate \(\sum_{b_1, b_2 \in B} n(b_1, b_2)\). Similar to the proofs of Theorems 1 and 3, we calculate \(n(b_1, b_2)\) based on the number of vertices \(b_1\) and \(b_2\) have in common. If \(b_1\) and \(b_2\) occur in different partite sets (and hence have no vertices in common), then we have

\[
n(b_1, b_2) = \left(\frac{(n_1-2)(n_2-2)2n-8}{k_1} \right) \sum_{k_2=0}^{k_2=0} \binom{(n_1-2)(n_2-2)}{k_1} \binom{n_1+n_2-4}{k_2} 2^{k_2} (s-r)^{n_1-n_2-k_1-k_2-4}.
\]

Then similarly to how we derived (12) from (7), we can derive

\[
n(b_1, b_2) \leq \frac{\deg b_1 \deg b_2}{s^{n_1n_2}} \left(1 + \frac{4p^3}{(1-p)^2}\right).
\]

Suppose that \(b_1\) and \(b_2\) both occur in the partite set consisting of \(n_1\) vertices and have no vertices in common. Then

\[
n(b_1, b_2) = \sum_{k_1=0}^{k_1=0} \sum_{k_2=0}^{k_2=0} \binom{(n_1-4)n_2}{k_1} \binom{4n_2}{k_2} 2^{k_2} (s-r)^{n_1n_2-k_1-k_2-4}.
\]

Then similarly to how we derived (12) from (7), we can derive

\[
n(b_1, b_2) = \frac{\deg b_1 \deg b_2}{s^{n_1n_2}}
\]

which, by symmetry, is what we also have if \(b_1\) and \(b_2\) both occur in the partite set consisting of \(n_2\) vertices and have no vertices in common.

Now suppose that \(b_1\) and \(b_2\) have exactly one vertex in common. Suppose that they occur in the partite set with \(n_1\) vertices. Then the equation for \(n(b_1, b_2)\) is the same as in (13) with the following minor changes. The factor \((s-r)^2\) is removed, \(\binom{n-3}{2}+1\) is replaced with \(n_1-n_2\), \(n-3\) is replaced with \(n_2\), \(n-3-k_2\) is replaced with \(n_2-k_2\), and \(\binom{n-5}{2}+3n-8-k_1-k_2-2k_3\) is replaced with \(n_1n_2-k_1-k_2-k_3\). We can deduce that in this case
\[ n(b_1, b_2) \leq \frac{D(r, s, n, n_1)^2}{s^t} \left( 1 + \frac{1}{p^{-3} + p^{-2} - p^{-1} - 1} \right)^{n_2} \]

the same way we derived (14) from (13).

Putting it altogether, we get

\[
\sum_{b_1, b_2 \in B} n(b_1, b_2) = \binom{n_1}{2} \binom{n_1 - 2}{2} \frac{D(r, s, n, n_2)^2}{s^{n_1n_2}} + \binom{n_2}{2} \binom{n_2 - 2}{2} \frac{D(r, s, n, n_1)^2}{s^{n_1n_2}} \\
+ 2 \binom{n_1}{2} \binom{n_2}{2} \frac{D(r, s, n, n_1)D(r, s, n, n_2)}{s^{n_1n_2}} \left( 1 + \frac{4p^3}{(1-p)^2} \right)^{n_1} \\
+ \frac{D(r, s, n, n_2)^2n_2(n_2 - 1)(n_2 - 2)}{s^{n_1n_2}} \left( 1 + \frac{p^3}{(1-p)^2} \right)^{n_1} \\
+ \frac{D(r, s, n, n_1)^2n_1(n_1 - 1)(n_1 - 2)}{s^{n_1n_2}} \left( 1 + \frac{p^3}{(1-p)^2} \right)^{n_2} \\
+ \frac{s^{n_1n_2}n_1(n_1 - 1)(1 - p^2)^{n_1}}{2} \left( 1 + \frac{n_1(n_1 - 1)(1 - p^2)^{n_2-n_1}}{n_2(n_2 - 1)} \right). 
\]

Then, by the Turán sieve, we can get

\[
P(diamG(n_1, n_2, p) = 3) \cdot \left( 1 + \frac{n_1(n_1 - 1)(1 - p^2)^{n_2-n_1}}{n_2(n_2 - 1)} \right)^2 \\
< \frac{2}{n_2(n_2 - 1)(1 - p^2)^{n_1}} \left( 1 + \frac{n_1(n_1 - 1)(1 - p^2)^{n_2-n_1}}{n_2(n_2 - 1)} \right) \\
+ \frac{4n_1^3}{n_2^4} \left( 1 + \frac{p^3}{(1-p)^2} \right)^{n_2} \left( 1 - p^2 \right)^{n_2-n_1} \\
+ \frac{8n_1^2 \left( \frac{p^3}{(1-p)^2} \right)}{n_2^2} \left( 1 - p^2 \right)^{n_2-n_1}. 
\]

Notice that

\[
\frac{p^3}{(1-p)^2} < \frac{1}{n_1} \left( 1 + \frac{p^3}{(1-p)^2} \right) < \frac{1}{n_1} \left( 1 + \frac{p^3}{(1-p)^2} \right)^{n_1} 
\]

and

\[
\left( 1 + \frac{p^3}{(1-p)^2} \right) (1 - p^2)^2 < 1.
\]
It follows that
\[
P(diamG(n_1, n_2, p) = 3) = 3\left(1 + \frac{n_1(n_1 - 1)(1 - p^2)^{n_2-n_1}}{n_2(n_2 - 1)}\right)
\]
\[
\leq \frac{2}{n_2(n_2 - 1)(1 - p^2)^{n_1}} + \frac{4n_1^2(1 + \frac{p^3}{(1-p)^2})^{n_2}}{n_2} (1 - p^2)^{2n_2-2n_1}
\]
\[
+ \left(1 + \frac{p^3}{(1-p)^2}\right)^{n_1} \left(4 + \frac{8n_1(1 - p^2)^{n_2-n_1}}{n_2(1-p)}\right)
\]
\[
\leq \frac{2}{n_2(n_2 - 1)(1 - p^2)^{n_1}} + \left(1 + \frac{p^3}{(1-p)^2}\right)^{n_1} \left(8 + \frac{8}{(1-p)}\right)
\]
from which we obtain our upper bound. This completes the proof of Theorem 4. \qed

By substituting in \( p = \frac{1}{2} \), we deduce from Theorem 4 the following.

**Corollary 5** Let \( P(diamG(n_1, n_2, p) = 3) \) be defined as in Theorem 4. If \( p = \frac{1}{2} \), then we have
\[
P(diamG(n_1, n_2, p) = 3) \geq 1 - \frac{n_2^2(3/4)^{n_1}}{2} \left(1 + \frac{n_1^2(3/4)^{n_2-n_1}}{n_2^2}\right)
\]

and
\[
P(diamG(n_1, n_2, p) = 3) \leq \left(\frac{2(4/3)^{n_1}}{n_1(n_2 - 1)} + \frac{24(3/2)^{n_1}}{n_2}\right) \left(1 + \frac{n_1(n_1 - 1)(3/4)^{n_2-n_1}}{n_2(n_2 - 1)}\right)^{-1}.
\]

**Remark 3** The upper bound given for \( P(G(n_1, n_2, p), 1/2) \) in Corollary 5 will in general only be non-trivial when \( n_2 \) much larger than \( n_1 \). For instance, if \( n_1 < \min\left\{\frac{2\log n_2 - \log 8}{\log(4/3)}, \frac{\log n_2 - \log 48}{\log(5/4)}\right\} \), then the upper bound will be less than 1.

In the situation where the edge probability \( p \to 0 \) as \( n \to \infty \), we will show the following.

**Corollary 6** Let \( P(diamG(n_1, n_2, p) = 3) \) be as in Theorem 4. Let \( \lim_{n \to \infty} np^4 = 0 \). We have
\[
P(diamG(n_1, n_2, p) = 3) \geq 1 - \frac{n_2^2 e^{-n_1 p^2}}{2} \left(1 + e^{2\log n_1 - 2\log n_2 - (n_2-n_1)p^2}\right)
\]

and
\[
P(diamG(n_1, n_2, p) = 3) \leq (1 + o(1)) \left(\frac{2}{n_1^2} e^{n_1 p^2}\right) \left(1 + e^{2\log n_1 - 2\log n_2 - (n_2-n_1)p^2}\right)^{-1}
\]
\[
\cdot \left(1 + 8n_2 e^{n_1 p^2(p-1)}\right). \quad (25)
\]
Suppose further that
\[
\lim_{n \to \infty} \left( 2 \log n_1 - 2 \log n_2 - (n_2 - n_1)p^2 \right) = -\infty,
\]
and
\[
\lim_{n \to \infty} \left( 2 \log n_2 - n_1 p^2 - \log 2 \right) = c
\]
for some \( c \in \mathbb{R} \).

1) If \( c < 0 \), we have
\[
P(diamG(n_1, n_2, p) = 3) \geq 1 - (1 + o(1))e^c.
\]

2) If \( c > 0 \), we have
\[
P(diamG(n_1, n_2, p) = 3) \leq (1 + o(1))e^{-c}.
\]

Proof By the upper bound of \( P(diamG(n_1, n_2, p) = 3) \) in Theorem 4, we can get
\[
P(diamG(n_1, n_2, p) = 3) \left( 1 + \frac{n_1(n_1 - 1) \left( 1 - p^2 \right)^{n_2-n_1}}{n_2(n_2 - 1)} \right)
< \frac{2}{n_2(n_2 - 1) \left( 1 - p^2 \right)^{n_1}} + \left( 8 + \frac{8}{1 - p} \right) \left( 1 + \frac{p^3}{(1-p)^2} \right)^{n_1}.
\]
Since \( \lim_{n \to \infty} np^4 = 0 \), we have \( \lim_{n \to \infty} n_1 p^4 = 0 \) and so
\[
\lim_{n \to \infty} n_1 p^2 \left( \frac{p}{(1 - p)^2} - 1 \right) - n_1 p^2 (p - 1) = 0.
\]
Also, since \( p^{-2} \geq 1 \), we have
\[
\left( 1 - p^2 \right)^{n_1} > e^{-n_1 p^2} \left( 1 - p^2 \right)^{n_1 p^2} = e^{-n_1 p^2} (1 - o(1))
\]
and
\[
\left( 1 + \frac{p^3}{(1-p)^2} \right)^{n_1} < \frac{e^{n_1 p^3}}{n_2} = \frac{e^{n_1 p^2}}{n_2^2} \cdot n_2 e^{n_1 p^2 (p/(1-p)^2) - 1}.
\]
Also,
\[
\frac{n_1^2 (1 - p^2)^{n_2 - n_1}}{n_2^2 (1 - p^2)} = e^{2 \log n_1 - 2 \log n_2} (1 - p^2)^{p^{-2} (n_2 - n_1)} p^2 (1 - p^2)^{-1}
\]
\[
> e^{2 \log n_1 - 2 \log n_2 - (n_2 - n_1) p^2} (1 - p^2) (n_2 - n_1) p^2 (1 - p^2)^{-1}.
\]

Since \( \lim_{n \to \infty} n p^4 = 0 \), we have
\[
\lim_{n \to \infty} (1 - p^2)^{(n_2 - n_1) p^2} = 1.
\]

We thus obtain our bounds. Statements (1) and (2) follow as in the proof of Corollary 2.

\section*{Remark 4}
Similar to Remark 1, all \( o(1) \) terms in Corollary 6 can be made explicit.

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