Phase-space descriptions of operators and the Wigner distribution in quantum mechanics I. A Dirac inspired view

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Abstract. Drawing inspiration from Dirac’s work on functions of non commuting observables, we develop a fresh approach to phase space descriptions of operators and the Wigner distribution in quantum mechanics. The construction presented here is marked by its economy, naturalness and more importantly, by its potential for extensions and generalisations to situations where the underlying configuration space is non Cartesian.
1. Introduction

The development of classical phase space methods to describe quantum systems whose
kinematics is governed by Cartesian variables has its origin in two important and
independent ideas. The first, due to Weyl [1], is the setting up of an unambiguous
rule or convention that maps each (real) classical dynamical variable, a classical phase-
space function, into a corresponding (hermitian) operator for the quantum system in a
linear manner. The second, due to Wigner [2], is the definition of a (real) phase-space
distribution function representing each (pure or mixed) quantum state in a complete
manner. Later work [3] clarified that these two rules or definitions are exactly inverses of
one another, as a result of which quantum mechanical expectation values can be written
in a classical-looking form as phase-space integrals.

In an interesting paper developing the analogies between classical and quantum
mechanics, Dirac [4] discussed the general problem of expressing a quantum dynamical
variable, an operator, as a function of the basic complete and irreducible set of operators
of the quantum system. The latter forms a non-commutative set, leading to the concept
of ordering rules in forming functions of them. In this context, Dirac used a description
of an operator by a collection of mixed matrix elements, the rows and columns labelled
by two different orthonormal bases for the Hilbert space of the system.

Phase space distributions have played a significant role in optics, particularly in
unifying radiometry and radiative transfer with the theory of partial coherence. These
activities were triggered by the pioneering work of Walther on radiometry and that of
Wolf on radiative transfer. Walther introduced two definitions for the radiance function.
The first definition [5] is analogous to the Wigner distribution and, indeed, the reader’s
attention was drawn by the author to this fact. Walther’s second definition [6], which
has since been used in hundreds of radiometry papers may be seen, in retrospect, to
be analogous to the Dirac inspired view of phase space distributions developed here.
Interestingly, this remark applies equally well to Wolf’s expression for the specific
intensity [7], having been inspired by the second definition of Walther’s.

The purpose of the present paper is to show that one can arrive at the Weyl-
Wigner formalism and results starting from Dirac’s ideas and following a physically
well-motivated and at the same time an extremely elementary and transparent series of
steps. This illuminates the use of phase-space language for quantum mechanics from
an alternative perspective. The properties of Wigner distributions have been studied
in considerable detail by many authors. It has been shown for instance, that if one
asks for a phase-space description of a quantum state obeying a small number of very
reasonable conditions, then the Wigner distribution is the unique answer. In important
work by several authors [8] the Weyl-Wigner formalism has been shown to be one of
several different possibilities all based on the use of phase-space methods. In the light of
this, an alternative line of argument leading directly and very easily to the same answer
may be of interest.

Among the important properties of the Wigner distribution corresponding to a
given quantum state is that one recovers the two quantum probability distributions in position or in momentum space by integrating over the momentum or the position variable respectively, in the Wigner distribution. This is called the recovery of the marginals. In our treatment this too becomes immediately evident upon inspection and completely transparent, no calculations being needed at all. The value of expressing things in this way is that it may suggest possible generalizations to other quantum-mechanical situations where the basic variables and commutation relations are not of the Cartesian Heisenberg type. We have in mind for instance the case of finite-dimensional quantum systems [9], quantum mechanics on Lie groups [10], non-commutative geometric quantum schemes, deformed or $q$-quantum kinematics etc. In all these cases the presentation of the very well known and familiar case in a very concise and transparent manner may help suggest interesting modifications to be studied.

A brief summary of the present work is as follows. After a quick review of the familiar phase space descriptions of operators in quantum mechanics in Section 2, in Section 3 we examine the question as to how the trace of products of two operators can be expressed as phase-space integrals in terms of their phase space representatives in such a way that the inherent symmetry of the trace operation is manifest. We show that this can be done at the expense of introducing a kernel. We investigate the properties of this kernel in detail and show how by defining a square root of the kernel in a special way one is naturally lead to the definition of Wigner distribution and associated phase point operators. Some consequences of the Wigner-Weyl correspondence are examined in Section 4, followed by a discussion on the recovery of the marginal distributions in Section 5. Section 6 contains our concluding remarks and further outlook.

2. Elementary Phase-Space Descriptions of Operators

We consider a one-dimensional quantum system whose basic operators are a hermitian Cartesian pair $\hat{q}, \hat{p}$ obeying the Heisenberg commutation relation:

$$[\hat{q}, \hat{p}] = i\hbar. \quad (1)$$

For corresponding classical phase-space variables, as well as particular classical values or quantum eigenvalues, we use $q, q', q'', \ldots, p, p', p'', \ldots$. The continuum-normalized eigenstates $|q\rangle, |p\rangle$ of $\hat{q}$ and $\hat{p}$ obey as usual:

$$\langle q'|q\rangle = \delta (q - q'), \quad \langle p|p'\rangle = \delta (p - p'), \quad \langle q|p\rangle = (2\pi\hbar)^{-1/2} \exp (iqp/\hbar). \quad (2)$$

From the completeness relations:

$$\int_{-\infty}^{+\infty} dq |q\rangle \langle q| = \int_{-\infty}^{+\infty} dp |p\rangle \langle p| = \mathbb{I}, \quad (3)$$

we obtain immediately the two operator statements:

$$\delta (\hat{q} - q) = |q\rangle \langle q|, \quad \delta (\hat{p} - p) = |p\rangle \langle p|. \quad (4)$$
Consider now a quantum-mechanical operator $\hat{A}$. It can certainly be completely described by its position-space matrix elements $\langle q'|\hat{A}|q\rangle$ which constitute in general a non-local kernel (in case $\hat{A}$ is unitary, $\langle q'|\hat{A}|q\rangle$ is the overlap between eigenstates of the “old” position operator $\hat{q}$ and a “new” one: $\hat{q}' = \hat{A}\hat{q}\hat{A}^{-1}$, and then the kernel of $\hat{A}$ would be the exponential of $(i$ times) the analogue of the classical generating function of a canonical transformation of the type “$q - Q$” \(\Pi\)). The kernel corresponding to $\hat{A}^\dagger$ is:

$$\langle q'|\hat{A}^\dagger|q\rangle = \langle q|\hat{A}|q'\rangle^*.$$  \hspace{1cm} (5)

To move towards a description of $\hat{A}$ at a classical phase-space level it is natural to consider, in the spirit of Dirac \(\Pi\), the mixed matrix element $\langle q|\hat{A}|p\rangle$ which, regarded as a function of the phase-space variables $q$ and $p$, certainly also describes $\hat{A}$ completely. For later convenience we include a non-vanishing plane wave factor and define the “left” phase-space representative of $\hat{A}$ as the function:

$$A_l(q,p) = \langle q|\hat{A}|p\rangle \langle p|q\rangle = Tr \{\hat{A}|p\rangle\langle p|q\rangle\langle q|\}$$

$$= Tr \{\hat{A}\delta (\hat{p} - p) \delta (\hat{q} - q)\}$$

$$= (2\pi\hbar)^{-1/2} \langle q|\hat{A}|p\rangle \exp(-iqp/\hbar).$$ \hspace{1cm} (6)

Here, of course, $\langle p|q\rangle$ is the kernel of the unitary operator corresponding to Fourier transformation, which interchanges $\hat{q}$ and $\hat{p}$. It is interesting to note that in Dirac’s treatment \(\Pi\) $A_l(q,p)$ is regarded essentially as the ratio $\langle q|\hat{A}|p\rangle / \langle q|p\rangle$, which determines the form of $\hat{A}$ as a function of $\hat{q}$ and $\hat{p}$ in standard ordered form, i.e. $\hat{q}$ to the left of $\hat{p}$.

Even if $\hat{A}$ is hermitian, $A_l(q,p)$ is in general complex. However we do have, as is particularly obvious from the second line in Eq.(6):

$$\int dpA_l(q,p) = \langle q|\hat{A}|q\rangle, \quad \int dqA_l(q,p) = \langle p|\hat{A}|p\rangle,$$

$$Tr \{\hat{A}\} = \int\int dqdpA_l(q,p).$$ \hspace{1cm} (7)

As an alternative to the above, the “right” phase-space representative of $\hat{A}$ is given by:

$$A_r(q,p) = \langle p|\hat{A}|q\rangle \langle q|p\rangle = Tr \{\hat{A}|q\rangle\langle q|p\rangle\langle p|\}$$

$$= Tr \{\hat{A}\delta (\hat{q} - q) \delta (\hat{p} - p)\}$$

$$= (2\pi\hbar)^{-1/2} \langle p|\hat{A}|q\rangle \exp(iqp/\hbar).$$ \hspace{1cm} (8)

This is related to expressing $\hat{A}$ in anti-standard form, $\hat{p}$ to the left of $\hat{q}$, and we have again:

$$\int dpA_r(q,p) = \langle q|\hat{A}|q\rangle, \quad \int dqA_r(q,p) = \langle p|\hat{A}|p\rangle,$$

$$Tr \{\hat{A}\} = \int\int dqdpA_r(q,p).$$ \hspace{1cm} (9)
Thus we have two equally elementary phase-space descriptions of the operator \( \hat{A} \) on the same footing, with the roles of coordinate and momentum interchanged to go from one to the other. As noted above, even in the hermitian case in general both \( A_l (q, p) \) and \( A_r (q, p) \) are complex. More generally under hermitian conjugation we have:

\[
\hat{B} =: \hat{A}^\dagger \Rightarrow B_r (q, p) = A_l (q, p)^* ,
\]

so in the hermitian case we have:

\[
\hat{A}^\dagger = \hat{A} \Rightarrow A_r (q, p) = A_l (q, p)^* .
\]

We can now ask if we can pass in a natural way to a third phase-space description of \( \hat{A} \) standing exactly “midway” between \( A_l (q, p) \) and \( A_r (q, p) \), thus treating \( \hat{q} \) and \( \hat{p} \) symmetrically. This is achieved in the next Section.

3. Operator Product Traces and Passage to the Weyl-Wigner Description

Consider two generally non-commuting operators \( \hat{A} \) and \( \hat{B} \). The trace of their product is symmetric under their interchange and can be expressed in two ways using classical phase space:

\[
Tr \{ \hat{A} \hat{B} \} = \int dqdp \langle q | \hat{A} | p \rangle \langle p | \hat{B} | q \rangle = 2\pi \hbar \int dqdp A_l (q, p) B_r (q, p) = 2\pi \hbar \int dqdp A_r (q, p) B_l (q, p) .
\]

The last line follows from the previous one by symmetry under interchange of \( \hat{A} \) and \( \hat{B} \). However in each of these two phase-space integrals the manifest symmetry in \( \hat{A} \) and \( \hat{B} \) is lacking. One can ask if such symmetry can be restored while continuing to work with phase-space quantities. Towards this end we begin by first expressing \( \langle \hat{A} \hat{B} \rangle_l (q, p) \) entirely in terms of \( A_l (q', p') \) and \( B_l (q'', p'') \):

\[
\left( \hat{A} \hat{B} \right)_l (q, p) = \langle q | \hat{A} \hat{B} | p \rangle \langle p | q \rangle = \int dq' dp' \langle q | \hat{A} | p' \rangle \langle p' | \hat{B} | q' \rangle \langle q' | p \rangle \langle p | q \rangle = \int dq' dp' A_l (q, p') K_l (q, p' ; q', p) B_l (q', p) ,
\]

\[
K_l (q, p' ; q', p) = (2\pi \hbar)^2 \langle q | p' \rangle \langle p' | q' \rangle \langle q' | p \rangle \langle p | q \rangle = \exp \{ i (q - q') (p' - p) / \hbar \} .
\]

The first line in the definition of \( K_l \) following from: \( \langle q | \hat{A} | p' \rangle = A_l (q, p') / \langle p' | q \rangle \) and from: \( 1 / \langle p' | q \rangle = 2\pi \hbar \langle p | p' \rangle \). The non-local convolution involved in expressing \( \langle \hat{A} \hat{B} \rangle_l \) in terms of \( A_l \) and \( B_l \) is an indication already of the general situation since we are dealing with mixed matrix elements; it is a forerunner of the Moyal or “star” product when the transition to the Weyl-Wigner description of operators is completed. We may also note that, aside from the continuum normalization of the \( \hat{q} \) and \( \hat{p} \) eigenvectors, the

\[\footnote{Thus, as is well known, from the point of view of trace calculations the standard and anti-standard orderings are dual to one another.}\]
kernel $K_l$ is a four-vertex Bargmann invariant [12]. Hence its phase, which is the area of the phase-space rectangle with vertices $(q, p), (q, p'), (q', p)$ and $(q', p')$ is a geometric phase [13]. Now combining Eq.(13) with Eq.(17) and relabelling some variables for convenience, we get $Tr\{\hat{A}\hat{B}\}$ entirely in terms of left representatives:

$$\text{Tr}\{\hat{A}\hat{B}\} = \int \int \int dq dp dq' dp' A_l(q, p) K_l(q, p; q', p') B_l(q', p'). \quad (14)$$

The kernel $K_l(q, p; q', p')$ is explicitly symmetric under: $(q, p) \leftrightarrow (q', p')$, so we have a classical phase-space expression for $\text{Tr}\{\hat{A}\hat{B}\}$ manifestly symmetric in $\hat{A}$ and $\hat{B}$, but at the cost of a kernel. In addition to symmetry, this kernel possesses two important properties: it is invariant under phase-space translations as it depends only on the differences $q - q'$, $p - p'$; and it satisfies the “marginals” equations:

$$\int dp' K_l(q, p; q', p') = 2\pi\hbar \delta(q - q'),$$

$$\int dq' K_l(q, p; q', p') = 2\pi\hbar \delta(p - p'). \quad (15)$$

The most natural question is to ask if this kernel can in some sense be “transformed away” while maintaining manifest symmetry in $\hat{A}$ and $\hat{B}$. This can be done if we can express it as the “square” or the convolution of some more elementary kernel, say in the form:

$$K_l(q, p; q', p') = \int \int dq'' dp'' \xi(q'', p''; q, p) \xi(q'', p''; q', p'). \quad (16)$$

From the known properties of $K_l(q, p; q', p')$ we can demand that $\xi(q, p; q', p')$ too be a symmetric function of its (pairs of) arguments; be invariant under phase-space translations and so depend only on the differences $q - q'$, $p - p'$; and possess the “marginals” property:

$$\int dp' \xi(q, p; q', p') = \sqrt{2\pi\hbar} \delta(q - q'),$$

$$\int dq' \xi(q, p; q', p') = \sqrt{2\pi\hbar} \delta(p - p'). \quad (17)$$

Easy calculation shows that the expression:

$$\xi(q, p; q', p') = \sqrt{\frac{2}{\pi\hbar}} \exp \left\{ 2i \frac{(q - q')(p - p')}{\hbar} \right\} \quad (18)$$

obeys all the conditions imposed above. If we use this in Eq.(14) and associate one factor of $\xi$ each with $A_l$ and $B_l$ we arrive at the simpler expression:

$$\text{Tr}\{\hat{A}\hat{B}\} = \frac{1}{2\pi\hbar} \int dq dp A(q, p) B(q, p), \quad (19)$$

where $A(q, p)$ arises from $A_l(q, p)$ via:

$$A(q, p) = \sqrt{2\pi\hbar} \int \int dq' dp' \xi(q, p; q', p') A_l(q', p'). \quad (20)$$

It is interesting that while as a function of four phase-space variables $\xi$ is the pointwise square of $K_l$, as a kernel it is the square root of $K_l$.\footnote{\textit{Phase-space descriptions of operators and the Wigner distribution I}}
and similarly for $B(q, p)$. We have thus achieved, by a two-step procedure, our objective of expressing $\text{Tr}\{\hat{A}\hat{B}\}$ as a manifestly symmetric classical phase-space integral, with one phase-space function each representing $\hat{A}$ and $\hat{B}$ and with no additional kernel. One can now see that in the case $\hat{A}^\dagger = \hat{A}, \hat{B}^\dagger = \hat{B}$, since $\text{Tr}\{\hat{A}\hat{B}\}$ is real and $\hat{A}$ and $\hat{B}$ can be chosen independently, $A(q, p)$ and $B(q, p)$ must be individually real:

$$\hat{A} = \hat{A}^\dagger \Rightarrow A(q, p) = A(q, p)^*.$$  \hfill (21)

Expression (20) for $A(q, p)$ is indeed the Weyl-Wigner representative of $\hat{A}$ in phase-space form. With elementary manipulations we can express it in the equivalent forms [14]:

$$A(q, p) = 2 \int \int dq' dp' A_t(q', p') \exp \left\{ 2i (q - q') (p - p') / \hbar \right\}$$ \hfill (22)

or:

$$A(q, p) = 2\pi \hbar \text{Tr} \left\{ \hat{A} \hat{W}(q, p) \right\}, \hfill (24)$$

where:

$$\hat{W}(q, p) = \frac{1}{2\pi \hbar} \int \int dq' \frac{1}{2} q' \langle q - \frac{1}{2} q' \rangle \exp \left\{ ipq' / \hbar \right\} \hfill (25)$$

Representing this “phase-point operator” as:

$$\hat{W}(q, p) = \int dq' dq'' |q'\rangle \langle q'| \hat{W}(q, p) |q''\rangle \langle q''|,$$ \hfill (26)

the matrix elements are given by:

$$\langle q'| \hat{W}(q, p) |q''\rangle = \frac{1}{\pi \hbar} \delta (q' + q'' - 2q) \exp \left\{ ip (q' - q'') / \hbar \right\}. \hfill (27)$$

It is evident from the first line in Eq. (25) that $\hat{W}(q, p)$ is hermitian as well as that:

$$\text{Tr} \left\{ \hat{W}(q, p) \right\} = \frac{1}{2\pi \hbar}. \hfill (28)$$

The Weyl correspondence makes use (see below) of the “Weyl operators” $\exp \left\{ i (x\hat{p} - k\hat{q}) / \hbar \right\}$ that are labelled by the phase-space points $x$ and $k$. Then, one can prove that:

$$\hat{W}(q, p) = \int \int \frac{dx dk}{(2\pi \hbar)^2} \exp \left\{ -i (xp - kq) \right\} \exp \left\{ i (x\hat{p} - k\hat{q}) / \hbar \right\}, \hfill (29)$$

i.e. that the phase-point operators are the symplectic Fourier transforms of the Weyl operators. Indeed, a straightforward calculation shows that:

$$\langle q'| \exp \left\{ i (x\hat{p} - k\hat{q}) / \hbar \right\} |q''\rangle = \delta (x + q' - q'') \exp \left\{ -i k \frac{q' + q''}{2\hbar} \right\} \hfill (30)$$

and using this result to evaluate the matrix elements of both sides of Eq. (29) one obtains back Eq. (27).

We look at some more properties of the operators $\hat{W}(q, p)$ in the next Section.

\[†\] The factor in Eq. (14) is chosen so that $\hat{A}$ and $A(q, p)$ have the same physical dimension.
4. Some Consequences of the Weyl-Wigner Correspondence

It is illuminating to compute the phase-space functions representing the operators \( \delta (\hat{q} - q) \delta (\hat{p} - p), \hat{W}(q,p) \) and \( \delta (\hat{p} - p) \delta (\hat{q} - q) \) occurring in Eqs.(13), (18) and (25) under the Weyl-Wigner correspondence. This helps us also understand the structure of the kernel \( \xi(q,p;q',p') \) better. It is well known that the Weyl correspondence takes elementary classical exponentials to elementary operator exponentials according to:

\[
\exp \{i(\sigma q - \tau p)\} \rightarrow \exp \{i(\sigma \hat{q} - \tau \hat{p})\}
\]

and then extends this to general functions \( A(q,p) \) by linearity. For the operators mentioned above we find using Eqs.(19) and (20):

\[
\hat{A} = \delta (\hat{q} - q') \delta (\hat{p} - p') \Rightarrow A_t(q,p) = \frac{1}{2\pi\hbar} \delta (q - q') \delta (p - p')
\]

\[
\Rightarrow A(q,p) = \frac{1}{\pi\hbar} \exp \left\{2i(q - q')(p - p')/\hbar\right\},
\]

\[
\hat{A} = \delta (\hat{p} - p') \delta (\hat{q} - q') \Rightarrow A(q,p) = \frac{1}{\pi\hbar} \exp \left\{-2i(q - q')(p - p')/\hbar\right\},
\]

the second result following from the first by hermitian conjugation, and we see that, up to a factor, the latter is just the kernel \( \xi(q,p;q',p') \) of Eq.(18). For \( \hat{W}(q',p') \) we find from Eq.(25):

\[
\hat{A} = \hat{W}(q',p') \Rightarrow A_t(q,p) = \frac{2}{(2\pi\hbar)^2} \exp \{ -2i(q - q')(p - p')/\hbar \}
\]

\[
\Rightarrow A(q,p) = \delta (q - q') \delta (p - p').
\]

At first sight, it is not easy to recognize that the operators \( \hat{W}(q',p') \) stand "midway" between \( \delta (\hat{q} - q') \delta (\hat{p} - p') \) and \( \delta (\hat{p} - p') \delta (\hat{q} - q') \), in the sense of treating \( \hat{q} \) and \( \hat{p} \) symmetrically, or that the classical representative \( \delta (q - q') \delta (p - p') \) of \( \hat{W}(q',p') \) also stands "midway" as a phase-space function between the two functions \( (1/\pi\hbar) \exp \{ \pm 2i(q - q')(p - p')/\hbar \} \). However this is actually so, as can be appreciated by looking at the Fourier transforms with respect to \( q' \) and \( p' \). Classically we have:

\[
\frac{1}{(2\pi)^2} \int \int d\sigma d\tau \exp \{i[\sigma(q - q') - \tau(p - p')]/\hbar\} \times \{\exp(\pm i\hbar\sigma\tau/2) \text{ or } 1\}
\]

\[
= \frac{1}{\pi\hbar} \exp \{ \pm 2i(q - q')(p - p')/\hbar \} \text{ or } \delta(q - q') \delta(p - p')
\]

and the Weyl map then preserves these "relative positions" among the corresponding operators.

Two other known properties of the \( \hat{W}(q,p) \) are immediately read off from the relations assembled above, with no need for any calculations. Since \( \delta(q - q') \delta(p - p') \) is real, \( \hat{W}(q',p') \) is hermitian, as we have already noted, and from Eqs.(19) and (34) they are trace orthonormal in the continuum sense:

\[
\text{Tr} \left\{ \hat{W}(q,p) \hat{W}(q',p') \right\} = \frac{1}{2\pi\hbar} \int \int dq'' dp'' \delta(q - q'') \delta(p - p'') \delta(q' - q'') \delta(p' - p'')
\]

\[
= \frac{1}{2\pi\hbar} \delta(q - q') \delta(p - p').
\]
Thus the inverse of Eq. (24) reads:
\[
\hat{A} = \int dq dp A(q, p) \hat{W}(q, p).
\]  

(37)

It is known [15] that the operators \( \hat{W}(q, p) \) obey the following interesting anticommutation relations with \( \hat{q} \) and \( \hat{p} \):
\[
\frac{1}{2} \{ \hat{q}, \hat{W}(q, p) \} = q \hat{W}(q, p), \quad \frac{1}{2} \{ \hat{p}, \hat{W}(q, p) \} = p \hat{W}(q, p).
\]  

(38)

These are operator versions of corresponding immediately obvious classical relations stemming from: \( q \delta(q - q') = q' \delta(q - q') \). If we set \( q = p = 0 \) in Eq. (38) we see that \( \hat{W}(0, 0) \) anticommutes with both \( \hat{q} \) and \( \hat{p} \):
\[
\hat{q} \hat{W}(0, 0) = -\hat{W}(0, 0) \hat{q}, \quad \hat{p} \hat{W}(0, 0) = -\hat{W}(0, 0) \hat{p}.
\]  

(39)

In turn this means that \( \hat{W}(0, 0)^2 \) commutes with both \( \hat{q} \) and \( \hat{p} \), so it must be a multiple of the unit operator. We can easily convince ourselves that \( \hat{W}(0, 0) \) is nonzero, so we must be dealing here with a nonzero multiple, which means that \( \hat{W}(0, 0) \) has an inverse. The existence of the inverse together with the first of Eqs. (39) leads at once to: \( \hat{W}(0, 0) \hat{q} \hat{W}(0, 0)^{-1} = -\hat{q} \) and similarly for \( \hat{p} \), so the upshot is that \( \hat{W}(0, 0) \) is a multiple of the parity operator. Now, e.g. in the \( \{|q\rangle\} \) basis, the matrix elements of the parity operator \( \hat{P} \) are given by (cfr. Eq. (2)):
\[
\langle q | \hat{P} | q' \rangle = \langle q | -q' \rangle = \delta(q + q')
\]  

(40)

and then, in this basis:
\[
Tr \{ \hat{P} \} = \int dq \langle q | \hat{P} | q \rangle = \int dq \delta(2q) = \frac{1}{2}.
\]  

(41)

Taking then traces and using Eq. (28) fixes the proportionality factor and we find:
\[
\hat{W}(0, 0) = \frac{1}{\pi \hbar} \hat{P}.
\]  

(42)

Summarizing, one finds the Weyl correspondence:
\[
A(q, p) = \delta(q) \delta(p) \Rightarrow \hat{A} = \hat{W}(0, 0) = \frac{1}{\pi \hbar} \hat{P}.
\]  

(43)

**Remark.** We would like to stress that what we mean here by “trace” of an operator is defined as the sum (or integral) of the diagonal matrix elements of the operator with respect to a *given* basis, and that the *only* bases that we are using here are the \( \{|q\rangle\} \) and \( \{|p\rangle\} \) bases, in each of which, e.g. and consistently: \( Tr \{ \hat{P} \} = 1/2 \). This does not mean that we are claiming that \( \hat{P} \) is a trace-class operator, which would imply much stronger requirements that the parity operator is not likely to meet.

One can carry the previous analysis one step further. The unitary operator:
\[
\hat{U}(q, p) =: \exp \{ i [p \hat{q} - q \hat{p}] / \hbar \}
\]  

(44)

acts as a phase-space displacement operator. Indeed:
\[
\hat{U}(q, p) \hat{q} \hat{U}(q, p)^\dagger = \hat{q} - q, \quad \hat{U}(q, p) \hat{p} \hat{U}(q, p)^\dagger = \hat{p} - p
\]  

(45)
and hence, for any (analytic) operator $\hat{O} = \hat{O}(\hat{q}, \hat{p})$:

$$
\hat{U}(q, p) \hat{O}(\hat{q}, \hat{p}) \hat{U}(q, p)^\dagger = \hat{O}(\hat{q} - q, \hat{p} - p).
$$

(46)

Taking then: $\hat{O}(\hat{q}, \hat{p}) = \hat{W}(0, 0)$ from Eq.(29) one obtains immediately:

$$
\hat{U}(q, p) \hat{W}(0, 0) \hat{U}(q, p)^\dagger = \hat{W}(q, p)
$$

(47)

and hence, from Eq.(43):

$$
\hat{W}(q, p) = \frac{1}{\pi \hbar} \hat{P} \hat{U}(q, p)^\dagger.
$$

(48)

Thus the phase-point operators are just the parity operation with respect to general phase-space points. This leads to the at first sight unexpected operator property:

$$
\hat{W}(q, p)^2 = \frac{1}{(\pi \hbar)^2} \mathbb{I},
$$

(49)

which means that the eigenvalues of $\hat{W}(q, p)$ are $\pm 1/\pi \hbar$. This will be used in the next Section.

The purpose of this discussion was to show that this otherwise rather unexpected fact is an immediate consequence of the relevant operator relations given above.

5. Recovery of Marginal Distributions

We have mentioned in Section 1 that an important property of the Wigner distribution is that upon partial integration over either $p$ or $q$ the complementary quantum-mechanical probability distribution emerges. In this Section we show how this happens practically automatically or manifestly if we use the relations given in Section 3.

The general Weyl association is as given in Eqs.(20), (25). For the density operator $\hat{\rho}$ representing a (pure or mixed) quantum state, we use a different normalization and define the Wigner distribution by:

$$
\rho(q, p) = T r \{ \hat{\rho} \hat{W}(q, p) \},
$$

(50)

so that the general expression for the expectation value of $\hat{A}$ in state $\hat{\rho}$ has the form:

$$
T r \{ \hat{\rho} \hat{A} \} = \int dq dp A_{l,r}(q, p) \rho(q, p).
$$

(51)

We notice at this point that Eq.(49) concerning the spectrum of $\hat{W}(q, p)$ has the following implication:

$$
|\rho(q, p)| \leq \frac{1}{\pi \hbar}.
$$

(52)

This known property of Wigner distributions is usually obtained by using the Cauchy-Schwartz inequality, so it is interesting to see it emerging here by a much more elementary argument. Now we turn to the marginals.

For a general operator $\hat{A}$ we have seen in Eqs.(7) and (9) that:

$$
\int dp A_{l,r}(q, p) = \langle q | \hat{A} | q \rangle, \quad \int dq A_{l,r}(q, p) = \langle p | \hat{A} | p \rangle.
$$

(53)
At the next step, for the kernel $\xi(q, p; q', p')$ of Eq. (16) we have the properties in Eq. (17). Combining the above two pairs of equations with the passage (20) from $A_l(q, p)$ to the Weyl representative $A(q, p)$ of $\hat{A}$, it is immediately seen that:

$$\int dp A(q, p) = 2\pi\hbar \langle q|\hat{A}|q\rangle, \quad \int dq A(q, p) = 2\pi\hbar \langle p|\hat{A}|p\rangle. \quad (54)$$

Essentially no work has to be done to get these results. In the case of the density operator $\hat{\rho}$ we omit the factor $2\pi\hbar$, so we recover the marginal probability distributions as:

$$\int dp \rho(q, p) = \langle q|\hat{\rho}|q\rangle, \quad \int dq \rho(q, p) = \langle p|\hat{\rho}|p\rangle. \quad (55)$$

We can also obtain operator forms of such statements for $\hat{W}(q, p)$. Knowing that the Weyl map is linear, that $\hat{W}(q', p')$ corresponds to $\delta(q - q')\delta(p - p')$, and that the operators corresponding to $\delta(q - q')$ and $\delta(p - p')$ are $\delta(q - q')$ and $\delta(p - p')$, we have:

$$\int dp' \delta(q - q') \delta(p - p') \equiv \int dp' \hat{W}(q', p') = \delta(q - q') = |q\rangle\langle q'|, \quad (56)$$

$$\int dq' \delta(q - q') \delta(p - p') \equiv \int dq' \hat{W}(q', p') = \delta(p - p') = |p\rangle\langle p'|. \quad (57)$$

The purpose of showing the recovery of marginals in this manner is again to emphasize how elementary and transparent the derivations are.

6. Concluding Remarks

To conclude, we have shown how by expressing the trace of product of two operators in terms of their phase space representatives and some fairly elementary steps one is naturally led to the concept of the Wigner distribution. Crucial to this construction is a kernel with the structure of a Bargmann invariant and its square root endowed with certain desirable properties. A noteworthy feature of the approach developed here is its economy - no auxiliary constructs are required at all and above all the facility with which it lends itself to application to non Cartesian situations such as finite state quantum systems which are of particular relevance to quantum computation and quantum information processing. That this is indeed the case will be demonstrated in a companion paper.

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