QUATERNION LINEAR CANONICAL WAVELET TRANSFORM AND
THE CORRESPONDING UNCERTAINTY INEQUALITIES

AAJAZ A. TEALI

Abstract. The linear canonical wavelet transform has been shown to be a valuable and powerful time-frequency analyzing tool for optics and signal processing. In this article, we propose a novel transform called quaternion linear canonical wavelet transform which is designed to represent two dimensional quaternion-valued signals at different scales, locations and orientations. The proposed transform not only inherits the features of quaternion wavelet transform but also has the capability of signal representation in quaternion linear canonical domain. We investigate the fundamental properties of quaternion linear canonical wavelet transform including Parseval’s formula, energy conservation, inversion formula, and characterization of its range using the machinery of quaternion linear canonical transform and its convolution. We conclude our investigation by deriving an analogue of the classical Heisenberg-Pauli-Weyl uncertainty inequality and the associated logarithmic and local versions for the quaternion linear canonical wavelet transform.

1. Introduction

Wavelet transforms serve as an important and powerful analyzing tool for time-frequency analysis and have been applied in a number of fields including signal processing, image processing, sampling theory, differential and integral equations, quantum mechanics and medicine. However, the signal analysis capability of the wavelet transform is limited in the time-frequency plane as each wavelet component is actually a differently scaled bandpass filter in the frequency domain and hence, it does not serve as an efficient tool for processing the higher dimensional signal whose energy is not well concentrated in the frequency domain. One of the examples of such signal is chirp like signals [1]. Therefore, in order to obtain joint signal representations in both time and frequency domains, the linear canonical wavelet transform (LCWT) has been introduced in the context of time-frequency analysis. The LCWT inherits the excellent mathematical properties of wavelet transform and linear canonical transform along with some fascinating properties of its own. In recent years, this transform has been paid a considerable amount of attention, resulting in many applications in the areas of optics, quantum mechanics, pattern recognition and signal processing [2, 3, 4, 5, 6].

On the other hand, considerable attention has been paid for the representation of signals in quaternion domains as quaternion algebra is the closest in its mathematical properties to the familiar system of the real and complex numbers. The quaternion algebra offers a simple and profound representation of signals wherein several components are to be controlled simultaneously. The development of integral transforms for quaternion valued signals has found numerous applications in 3D computer graphics, aerospace engineering, artificial intelligence and colour image processing. The extension of classical wavelet transform to quaternion algebra has been introduced in [7] and [8], they also demonstrated their various properties. In [12], Traversoni proposed a discrete quaternion wavelet transform, the application of which can be found in [9, 10] and [11].

In the recent years, some authors have generalized the linear canonical transform to quaternion-valued signals, known as the quaternionic linear canonical transform (QLCT).
The QLCT was firstly studied in [13] including prolate spheroidal wave signals and uncertainty principles [14]. Some useful properties of the QLCT such as linearity, reconstruction formula, continuity, boundedness, positivity inversion formula and the uncertainty principle were established in [15, 18, 19, 23]. An application of the QLCT to study of generalized swept-frequency filters was introduced in [17]. Because of the non-commutative property of multiplication of quaternions, there are mainly three various types of 2D quaternion linear canonical transform (QLCTs): two-sided QLCTs, left-sided QLCTs and right-sided QLCTs (refer to [14]). Based on the (two-sided) QLCT [13], the quaternion windowed linear canonical transform of 2D quaternionic signals has been introduced by Wen-Biao Gao and Bing-Zhao Li in [30]. It can reveal the local QLCT-frequency contents and enjoys high concentrations and eliminates the cross term, but it has limitation of having fixed window localization.

The objective of this paper is to eliminate the redundancy of the quaternion wavelet transform (QWT) and quaternion linear canonical transform (QLCT) by introducing the quaternion linear canonical wavelet transform (QLCWT) i.e. the quaternion version of linear canonical wavelet transform. QLCWT generalizes the definition of the QWT in time-QLC-frequency plane by using the modified quaternion linear canonical convolution. The proposed QLCWT not only inherits the features of quaternion wavelet transform but also has the capability of signal representation in quaternion domain. Besides, it has explicit physical interpretation and low complexity. It is hoped that this transform might be useful in three dimensional color field processing, space color video processing, crystallography, aerospace engineering, oil exploration and for the solution of many types of quaternionic differential equations.

The article is organized as follows: We begin in Section 2 by presenting the notation, quaternion algebra and wavelet theory needed to understand and place our results in context. In Section 3, we introduce the concept of quaternion linear canonical wavelet transform and obtain the expected properties of the extended linear canonical wavelet transform including Parseval’s formula, energy conservation, inversion formula, and characterization of its range. The well known Heisenberg-Pauli-Weyl inequality, logarithmic and local uncertainty principle for the quaternionic differential equations.

2. Preliminaries

2.1. Quaternion Algebra. The theory of quaternions was initiated by the Irish mathematician Sir W.R. Hamilton in 1843 and is denoted by $\mathbb{H}$ in his honour. The quaternion algebra provides an extension of the complex number system to an associative non-commutative four-dimensional algebra. The quaternion algebra $\mathbb{H}$ over $\mathbb{R}$ is given by

$$\mathbb{H} = \left\{ f = a_0 + i a_1 + j a_2 + k a_3 : a_0, a_1, a_2, a_3 \in \mathbb{R} \right\},$$

where $i, j, k$ denote the three imaginary units, obeying the Hamilton’s multiplication rules

$$ij = k = -ji, \; jk = i = -kj, \; ki = j = -ik, \; \text{and} \; i^2 = j^2 = k^2 = ijk = -1.$$ 

For quaternions $f_1 = a_0 + i a_1 + j a_2 + k a_3$ and $f_2 = b_0 + i b_1 + j b_2 + k b_3$, the addition is defined componentwise and the multiplication is defined as

$$f_1 f_2 = (a_0 b_0 - a_1 b_1 - a_2 b_2 - a_3 b_3) + i (a_1 b_0 + a_0 b_1 + a_2 b_3 - a_3 b_2) + j (a_0 b_2 + a_2 b_0 + a_3 b_1 - a_1 b_3) + k (a_0 b_3 + a_3 b_0 + a_1 b_2 - a_2 b_1).$$

The conjugate and norm of a quaternion $f = a_0 + i a_1 + j a_2 + k a_3$, are given by $\overline{f} = a_0 - i a_1 - j a_2 - k a_3$ and $\|f\|_{\mathbb{H}} = \sqrt{a_0^2 + a_1^2 + a_2^2 + a_3^2}$, respectively. We also note that an arbitrary quaternion $h$ can be represented by two complex numbers as $f = (a_0 + i a_1) + j (a_2 - i a_3) = f_1 + j f_2$, where $f_1, f_2 \in \mathbb{C}$, and hence, $\overline{f} = \overline{f_1} - j f_2$, with $\overline{f_1}$ denoting the complex conjugate of $f_1$. Moreover, the inner product of any two quaternions $f = f_1 + jf_2$, and $g = g_1 + j g_2$ in $\mathbb{H}$ is defined by
\[ \langle f, g \rangle_{\mathbb{H}} = f \overline{g} = (f_1 \overline{g}_1 + \overline{f}_2 g_2) + j(f_2 \overline{g}_1 - \overline{f}_1 g_2). \]

By virtue of the complex domain representation, a quaternion-valued function \( f : \mathbb{R}^2 \to \mathbb{H} \) can be decomposed as \( f(x) = f_1 + j f_2 \), where \( f_1, f_2 \) are both complex valued functions.

Let us denote \( L^2(\mathbb{R}^2, \mathbb{H}) \), the space of all quaternion valued functions \( f \) satisfying

\[ \|f\|_2 = \left\{ \int_{\mathbb{R}^2} \left( |f_1(x)|^2 + |f_2(x)|^2 \right) dx \right\}^{1/2} < \infty. \]

The norm on \( L^2(\mathbb{R}^2, \mathbb{H}) \) is obtained from the inner product of the quaternion valued functions \( f = f_1 + j f_2 \), and \( g = g_1 + j g_2 \) as

\[ \langle f, g \rangle_2 = \int_{\mathbb{R}^2} \langle f, g \rangle_{\mathbb{H}} \, dx = \int_{\mathbb{R}^2} \left\{ \left( f_1(x) \overline{g}_1(x) + \overline{f}_2(x) g_2(x) \right) + j \left( f_2(x) \overline{g}_1(x) - \overline{f}_1(x) g_2(x) \right) \right\} dx. \]

An easy computation shows that \( L^2(\mathbb{R}^2, \mathbb{H}) \) equipped with above defined inner product is a Hilbert space.

**Definition 2.1.** For any quaternion valued function \( f \in L^1(\mathbb{R}^2, \mathbb{H}) \cap L^2(\mathbb{R}^2, \mathbb{H}) \), the two-sided quaternion Fourier transform (QFT) \( \mathcal{F}_q \) and is given by

\[ \mathcal{F}_q[f(x)](w) = \hat{f}(w) = \int_{\mathbb{R}^2} e^{-ix_1 w_1} f(x) e^{-jx_2 w_2} \, dx, \quad (2.1) \]

where \( x = (x_1, x_2), \) \( w = (w_1, w_2) \) and the quaternion exponential \( e^{-ix_1 w_1} \) and \( e^{-jx_2 w_2} \) are the quaternion Fourier kernels. The corresponding inversion formula is given by

\[ f(x) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{ix_1 w_1} f(x) e^{jx_2 w_2} \, dw. \quad (2.2) \]

### 2.2. Quaternion Linear Canonical Transform.

Due to non-commutativity of quaternion multiplication, there are three types of the quaternion offset linear canonical transform, the left-sided QLCT, the right-sided QLCT, and two-sided QLCT. In this paper, we will extend the theory of two-sided QLCT introduced by [13] as follows:

**Definition 2.2.** Let \( A_s = \begin{bmatrix} a_s & b_s \\ c_s & d_s \end{bmatrix} \), be a matrix parameter such that \( a_s, b_s, c_s, d_s \in \mathbb{R} \) and \( a_s d_s - b_s c_s = 1 \), for \( s = 1, 2 \). The two-sided quaternion linear canonical transform of any quaternion valued function \( f \in L^2(\mathbb{R}^2, \mathbb{H}) \), is given by

\[ \mathcal{L}^{\mathbb{H}}_{A_1, A_2}[f(x)](w) = \left\{ \int_{\mathbb{R}^2} K^i_{A_1}(x_1, w_1) f(x) K^j_{A_2}(x_2, w_2) dx, \quad b_1 b_2 \neq 0 \right\} \]

\[ \sqrt{d_1 d_2 e^{-jx_1 w_1 d_1 w_2}} f(d_1 w_1, d_2 w_2) e^{jx_2 w_2}, \quad b_1 b_2 = 0 \]

where \( x = (x_1, x_2), \) \( w = (w_1, w_2) \) and for \( b_1 b_2 \neq 0 \) ,the quaternion kernels \( K^i_{A_1}(x_1, w_1) \) and \( K^j_{A_2}(x_2, w_2) \) are respectively given by

\[ K^i_{A_1}(x_1, w_1) = \frac{1}{\sqrt{2\pi b_1}} \exp \left\{ \frac{i}{2b_1} \left[ a_1 x_1^2 - 2x_1 w_1 + d_1 w_1^2 - \frac{\pi b_1}{2} \right] \right\} \]

and

\[ K^j_{A_2}(x_2, w_2) = \frac{1}{\sqrt{2\pi b_2}} \exp \left\{ \frac{j}{2b_2} \left[ a_2 x_2^2 - 2x_2 w_2 + d_2 w_2^2 - \frac{\pi b_2}{2} \right] \right\}, \]
The corresponding inversion formula for two-sided QOLCT is given by

\[ f(x) = \int_{\mathbb{R}^2} K_{A_1}^i(x_1, w_1) L_{A_1, A_2}^H [f](w) K_{A_2}^j(x_2, w_2) \, dw. \]  

(2.3)

Two quaternion functions \( f, g \in L^2(\mathbb{R}^2, \mathbb{H}) \) are related to their two-sided QOLCT via the Parseval formula, given as

\[ \left\langle L_{A_1, A_2}^H [f], L_{A_1, A_2}^H [g] \right\rangle_{L^2(\mathbb{R}^2, \mathbb{H})} = \langle f, g \rangle_{\mathbb{H}}. \]  

(2.4)

In particular, \( \|L_{A_1, A_2}^H [f(x)](w)\|_{L^2(\mathbb{R}^2, \mathbb{H})} = \|f\|_{L^2(\mathbb{R}^2, \mathbb{H})} \).

Definition 2.3. The set \( \mathcal{G} = \mathbb{R}^+ \times \mathbb{R}^2 \times SO(2) = \{(a, y, \theta_0); a \in \mathbb{R}^+, y \in \mathbb{R}^2, \theta_0 \in SO(2)\} \) endowed with the operation

\( (a, y, \theta_0) \circ (a', y', \theta_0') = (aa', y + ar_\theta y', \theta_0 + \theta') \),

forms a group called a similitude group on \( \mathbb{R}^2 \) associated with wavelets, where \( SO(2) \) is the special orthogonal group of rotation in \( \mathbb{R}^2 \). The left Haar measures on \( \mathcal{G} \) is given by \( d\eta = da \, dy \, d\theta_0 / a^3 \). For more see [20].

3. Quaternion Linear Canonical Wavelet Transform

In this section, we shall first introduce a new convolution structure for two dimensional linear canonical transform and obtain the corresponding convolution theorem. Then, we shall characterize the admissibility condition in terms of two-sided QLCT and define the quaternion linear canonical wavelet transform in terms of admissible canonical wavelets based on convolution operator in LCT domain and investigate its fundamental properties.

Analogues to Wei et al [21], the generalized convolution theorem for two dimensional linear canonical transform, based on generalized translation is derived as follows.

The \( y \)-generalized translation of signal \( \psi(x) \) is denoted by \( \psi(x \Theta y) \) is given by [26]

\[ \psi(x \Theta y) = \int \rho(w) \Psi(w) K(x, w) K^{-1}(y, w) \, dw, \]  

(3.1)

where \( \Theta \) is argument of function, \( \psi(x \Theta y) \) is generalized delay operator for the generalized translation, \( \rho \) is weight function, \( \Psi(w) \) is transformed function \( \psi(x) \), and \( K(x, w) \) is kernel of transformation. The corresponding generalized translation in two sided QLC-domain is given by

\[
\psi_{A_1}(x \Theta y) = \int_{\mathbb{R}^2} K_{A_1}^{-i}(y_1, w_1) K_{A_1}^i(x_1, w_1) L_{A_1, A_2}^H [\psi(x)](w) K_{A_2}^j(x_2, w_2) K_{A_1}^{-j}(y_2, w_2) \, dw
\]

\[
= \frac{1}{(2\pi)^2 b_1 b_2} \int_{\mathbb{R}^2} \exp \left\{ \frac{i}{2b_1} \left[ a_1 (x_1^2 - y_1^2) - 2(x_1 - y_1)w_1 \right] \right\} L_{A_1, A_2}^H [\psi(x)](w)
\]

\[
\times \exp \left\{ \frac{j}{2b_2} \left[ a_2 (x_2^2 - y_2^2) - 2(x_2 - y_2)w_2 \right] \right\} \, dw
\]

\[
= e^{\frac{ia}{2b_2} (x^2 - y^2)} \left\{ \frac{1}{(2\pi)^2 b_1 b_2} \int_{\mathbb{R}^2} e^{\frac{i(y_1 - x_1)w_1}{b_1}} L_{A_1, A_2}^H [\psi(x)](w) e^{\frac{j(y_2 - x_2)w_2}{b_2}} \, dw \right\} e^{\frac{ia}{2b_2} (x_2^2 - y_2^2)}. \]  

(3.2)
We now introduce a new convolution structure for two dimensional linear canonical transform and obtain the corresponding convolution theorem.

**Definition 3.1.** For any two signals \( f, \psi \in L^2(\mathbb{R}^2, \mathbb{H}) \), we define the generalized convolution operator \( \otimes^A \) by

\[
f \otimes^A \psi (x) = \int_{\mathbb{R}^2} f(y) \psi_A(x) \Theta(y) \, dy,
\]

where \( \psi_A(x) \Theta(y) \) is given by (3.2), and the corresponding convolution theorem is

**Theorem 3.2.** If \( f \otimes^A \psi(x) \) is defined as (3.3) and \( \mathcal{L}_{A_1, A_2}^\mathbb{H}[\psi(x)] \in L^2(\mathbb{R}^2, \mathbb{R}) \) denotes the two dimensional linear canonical transform of a quaternion valued signal \( \psi \). Then we have

\[
\mathcal{L}_{A_1, A_2}^\mathbb{H}[f \otimes^A \psi(x)](w) = \mathcal{L}_{A_1, A_2}^\mathbb{H}[f(x)](w) \cdot \mathcal{L}_{A_1, A_2}^\mathbb{H}[\psi(x)](w).
\]

**Proof.** By definition of QLCT, and well known Fubini Theorem, we have

\[
\begin{align*}
\mathcal{L}_{A_1, A_2}^\mathbb{H}[f \otimes^A \psi(x)](w) & = \int_{\mathbb{R}^2} K_{A_1}^j(x_1, w_1) f \otimes^A \psi(x) K_{A_2}^j(x_2, w_2) \, dx \\
& = \int_{\mathbb{R}^2} K_{A_1}^j(x_1, w_1) \int_{\mathbb{R}^2} f(y) \psi(x) \Theta(y) \, dy \, K_{A_2}^j(x_2, w_2) \, dx \\
& = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} K_{A_1}^i(x_1, w_1) f(y) e^{\frac{\jmath}{2\pi}(x_1^2 - y_1^2)} \times \left\{ \frac{1}{(2\pi)^2b_1b_2} \int_{\mathbb{R}} e^{\frac{i(u_1 - x_1)w_1}{b_1}} \mathcal{L}_{A_1, A_2}^\mathbb{H}[\psi(x)](w) \, dw \right\} e^{\frac{\jmath}{2\pi}(x_2^2 - y_2^2)} K_{A_2}^j(x_2, w_2) \, dx \, dy \\
& = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} K_{A_1}^i(x_1, w_1) f(y) e^{\frac{\jmath}{2\pi}(x_1^2 - y_1^2)} \times \left\{ \frac{1}{b_1b_2} \delta(y_1 - x_1) \cdot \delta(x_2 - y_2) \right\} e^{\frac{\jmath}{2\pi}(x_2^2 - y_2^2)} \mathcal{L}_{A_1, A_2}^\mathbb{H}[\psi(x)](w) K_{A_2}^j(x_2, w_2) \, dx \, dy \\
& = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} K_{A_1}^i(x_1, w_1) f(y) e^{\frac{\jmath}{2\pi}(x_1^2 - y_1^2)} \delta(y_1 - x_1) \cdot \delta(x_2 - y_2) e^{\frac{\jmath}{2\pi}(x_2^2 - y_2^2)} \times \mathcal{L}_{A_1, A_2}^\mathbb{H}[\psi(x)](w) K_{A_2}^j(x_2, w_2) \, dx \, dy \\
& = \int_{\mathbb{R}^2} K_{A_1}^j(x_1, w_1) f(y) \mathcal{L}_{A_1, A_2}^\mathbb{H}[\psi(x)](w) K_{A_2}^j(x_2, w_2) \, dx \\
& = \int_{\mathbb{R}^2} K_{A_1}^j(x_1, w_1) f(y) K_{A_2}^j(x_2, w_2) \, dx \cdot \mathcal{L}_{A_1, A_2}^\mathbb{H}[\psi(x)](w) \\
& = \mathcal{L}_{A_1, A_2}[f(x)](w) \cdot \mathcal{L}_{A_1, A_2}[\psi(x)](w).
\end{align*}
\]

This completes the proof. \( \square \)

Keeping in view the generalized translation in equation (3.2), we have the following definition of quaternion linear canonical wavelet family of \( \psi \in L^2(\mathbb{R}^2, \mathbb{H}) \).
Definition 3.3. For a quaternion signal $\psi \in L^2(\mathbb{R}^2, \mathbb{H})$, we define quaternion linear canonical wavelets as

$$U_{a,y,\theta} \psi(x) = \Psi_{a,y,\theta}^\mathbb{H}(x) = a^{-1} e^{-\frac{ia}{2b_1}(x_1^2-y_1^2)} \psi(r_{-\theta}a^{-1}(x-y)) e^{-\frac{ia}{2b_2}(x_2^2-y_2^2)},$$

(3.5)

where $U_{a,y,\theta} : L^2(\mathbb{R}^2, \mathbb{H}) \rightarrow L^2(\mathbb{H}, \mathbb{H})$ is a unitary operator, $a \in \mathbb{R}^+$, $y \in \mathbb{R}^2$, and $r_{-\theta} \in SO(2)$, with $r_{-\theta} \mathbf{x} = (x_1 \cos \theta + x_2 \sin \theta, -x_1 \sin \theta + x_2 \cos \theta)$, $0 \leq \theta \leq 2\pi$. The order of the terms in (3.5) is fixed because of the non-commutativity of quaternions.

We now prove a lemma which offers us the two-sided quaternion LCT of $\Psi_{a,y,\theta}^\mathbb{H}(x)$, defined in (3.5), in terms of quaternion Fourier transform, which will be fruitful for investigating certain fundamental properties.

Lemma 3.4. The two-sided quaternion LCT of $\Psi_{a,y,\theta}^\mathbb{H}(x)$, defined in (3.5) is given by

$$\mathcal{L}_{A_1,A_2}^\mathbb{H}[\Psi_{a,y,\theta}^\mathbb{H}(x)](w) = \frac{a}{2\pi \sqrt{b_1 b_2}} \exp \left\{ \frac{i}{2b_1} \left[ d_1 w_1^2 - y_1^2 a_1 - 2y_1 w_1 - \frac{\pi b_1}{2} \right] \right\} \mathcal{F}_{\psi} \left( r_{-\theta} \frac{aw}{b} \right) 
\times \exp \left\{ \frac{j}{2b_2} \left[ d_2 w_2^2 + y_2^2 a_2 - 2y_2 w_2 - \frac{\pi b_2}{2} \right] \right\}.$$  

(3.6)

Proof. We have

$$\mathcal{L}_{A_1,A_2}^\mathbb{H}[\Psi_{a,y,\theta}^\mathbb{H}(x)](w) = \frac{1}{2\pi a \sqrt{b_1 b_2}} \int_{\mathbb{R}^2} \exp \left\{ \frac{i}{2b_1} \left[ a_1 x_1^2 - 2x_1 w_1 + d_1 w_1^2 - \frac{\pi b_1}{2} \right] \right\} e^{-\frac{ia}{2b_1}(x_1^2-y_1^2)} \psi(r_{-\theta}a^{-1}(x-y)) e^{-\frac{ia}{2b_2}(x_2^2-y_2^2)} d\mathbf{x} 
\times \exp \left\{ \frac{j}{2b_2} \left[ a_2 y_2^2 + d_2 w_2^2 - \frac{\pi b_2}{2} \right] \right\} 
= \frac{1}{2\pi a \sqrt{b_1 b_2}} \exp \left\{ \frac{i}{2b_1} \left[ d_1 w_1^2 - \frac{\pi b_1}{2} - a_1 y_1^2 \right] \right\} \int_{\mathbb{R}^2} e^{-\frac{i(a_1 x_1 + y_1^2)}{b_1}} \psi(r_{-\theta}x) e^{-\frac{j a y_1^2}{b_2}} a^2 d\mathbf{z} 
\times \exp \left\{ \frac{j}{2b_2} \left[ a_2 y_2^2 + d_2 w_2^2 - \frac{\pi b_2}{2} \right] \right\} 
= \frac{a}{2\pi \sqrt{b_1 b_2}} \exp \left\{ \frac{i}{2b_1} \left[ d_1 w_1^2 - \frac{\pi b_1}{2} - a_1 y_1^2 \right] \right\} \int_{\mathbb{R}^2} e^{-\frac{i(a_1 x_1 + y_1^2)}{b_1}} \psi(r_{-\theta}z) e^{-\frac{j a y_1^2}{b_2}} a^2 d\mathbf{z} 
\times \exp \left\{ \frac{j}{2b_2} \left[ -2y_2 w_2 + a_2 y_2^2 + d_2 w_2^2 - \frac{\pi b_2}{2} \right] \right\}.$$ 

$$= \frac{a}{2\pi \sqrt{b_1 b_2}} \exp \left\{ \frac{i}{2b_1} \left[ d_1 w_1^2 - \frac{\pi b_1}{2} - a_1 y_1^2 \right] \right\} \mathcal{F}_{\psi} \left( r_{-\theta} \frac{aw}{b} \right) 
\times \exp \left\{ \frac{j}{2b_2} \left[ -2y_2 w_2 + a_2 y_2^2 + d_2 w_2^2 - \frac{\pi b_2}{2} \right] \right\}.$$
Remark: From above lemma, we note the following relation

\[
\frac{2\pi \sqrt{b_1 b_2}}{a} L_{A_1, A_2}^H \left[ \Psi_{a, y, \theta}^H (x) \right] (b w) = \exp \left\{ \frac{j}{2 b_1} \left[ d_1 (b_1 w_1)^2 - \frac{\pi b_1 + a_1 y_1^2 - 2y_1 b_1 w_1}{2} \right] \right\} F_q [\psi] (r_{-\theta a w}) \times \exp \left\{ \frac{j}{2 b_2} \left[ -2y_2 b_2 w_2 + a_2 y_2^2 + d_2 (b_2 w_2)^2 - \frac{\pi b_2}{2} \right] \right\} \tag{3.7}
\]

In a sequel, we have the following admissibility condition in terms of two-sided quaternion linear canonical transform.

**Definition 3.5 (Admissible Canonical Quaternion Wavelet).** A quaternion-valued wavelet \( \psi \in L^2 (\mathbb{R}^2, \mathbb{H}) \) is admissible, if

\[
C_\psi = \int_{\mathbb{R}^+} \int_{SO(2)} \left| F_q [\psi] (r_{-\theta a w}) \right|^2 \frac{dad\theta}{a^3} = \int_{\mathbb{R}^+ \times SO(2)} \left| L_{A_1, A_2}^H \left[ \Psi_{a, y, \theta}^H (x) \right] (w) \right|^2 \frac{dad\theta}{a^3} \tag{3.8}
\]

is invertible real constant for a.e. \( w \in \mathbb{R}^2 \).

We are now ready to define the quaternion linear canonical wavelet transform of two dimensional quaternion valued signals.

**Definition 3.6.** The quaternion linear canonical wavelet transform based on 2D-LCT convolution, of a quaternion-valued function \( f \in L^2 (\mathbb{R}^2, \mathbb{H}) \) with respect to an admissible wavelet \( \psi \in L^2 (\mathbb{R}^2, \mathbb{H}) \) is defined by

\[
\mathcal{W}_\psi^H [f] (a, y, \theta) = f \otimes A_s \psi (x) = a^{-1} \int_{\mathbb{R}^2} f (x) e^{\frac{j a \pi}{2} (x_2^2 - y_2^2)} \overline{\psi (r_{-\theta a^{-1}} (x - y))} e^{\frac{j a \pi}{2} (x_1^2 - y_1^2)} dx \tag{3.9}
\]

where \( \Psi_{a, y, \theta}^H \) is given by (3.5).

It is worth to note that the proposed quaternion linear canonical wavelet transform (3.9) boils down to existing quaternion wavelet transform as well as gives birth to some new quaternion wavelet transform which are not yet reported in the open literature:

- For the matrices \( A_s = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \) for \( s = 1, 2 \), the quaternion linear canonical wavelet transform (3.9) boils down to the quaternion wavelet transform given by [10]

  \[
  \mathcal{W}_\psi^H [f] (a, y, \theta) = a^{-1} \int_{\mathbb{R}^2} f (x) \overline{\psi (r_{-\theta a^{-1}} (x - y))} dx. \tag{3.10}
  \]

- For the matrices \( A_s = \begin{bmatrix} 1 & b_s \\ 0 & 1 \end{bmatrix}, b_s \neq 0 \), for \( s = 1, 2 \), we can obtain a new quaternion wavelet transform namely the quaternion Fresnel-Canonical wavelet transform given by

  \[
  \mathcal{W}_\psi^H [f] (a, y, \theta) = a^{-1} \int_{\mathbb{R}^2} f (x) e^{\frac{j a \pi}{2} (x_2^2 - y_2^2)} \overline{\psi (r_{-\theta a^{-1}} (x - y))} e^{\frac{j a \pi}{2} (x_1^2 - y_1^2)} dx. \tag{3.11}
  \]
• For the matrices $A_s = \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix}$ for $s = 1, 2$, the quaternion linear canonical wavelet transform (3.9) reduces to the quaternion fractional wavelet transform given by

$$W_{\psi}^H[f](a, y, \theta) = a^{-1} \int_{\mathbb{R}^2} f(x) e^{-j\cot \alpha (x_2^2 - y_2^2)} \overline{\psi \left( r_\theta a^{-1}(x - y) \right)} e^{j\cot \alpha (x_1^2 - y_1^2)} dx. \quad (3.12)$$

For the demonstration of the quaternion linear canonical wavelet transform (3.9), we shall present an illustrative example.

**Example 3.7.** Consider the two dimensional difference-of-Gaussian wavelets

$$\psi(x) = \lambda^{-2} e^{-\frac{1}{2\lambda^2}(x_1^2 + x_2^2)} - e^{-\frac{1}{2}(x_1^2 + x_2^2)}, \ 0 < \lambda < 1. \quad (3.13)$$

For $\theta = 0$, i.e. $r_\theta = 1$, we have the corresponding quaternion linear canonical wavelet family of $\psi$ as

$$\Psi_{a, y, \theta}(x) = a^{-1} e^{-\frac{ia}{2\lambda^2}(x_1^2 - y_1^2)} \psi(a^{-1}(x - y)) e^{-\frac{ja}{2\lambda^2}(x_2^2 - y_2^2)}$$

$$= a^{-1} e^{-\frac{ia}{2\lambda^2}(x_1^2 - y_1^2)} \left\{ \lambda^{-2} e^{-\frac{(x_1 - y_1)^2 + (x_2 - y_2)^2}{2\lambda^2}} - e^{-\frac{(x_1 - y_1)^2 + (x_2 - y_2)^2}{2\lambda^2}} \right\} e^{-\frac{ja}{2\lambda^2}(x_2^2 - y_2^2)}$$

Then the quaternion linear canonical wavelet transform of the signal

$$f(x) = e^{-\alpha_1 x_1 - \alpha_2 x_2}, \ \alpha_1, \alpha_2 \in \mathbb{R}$$
with respect to above DOG wavelet $\psi$ (as defined in (3.13)) is given by

\[
W^\mathbb{R}_\psi[f](a, y, \theta) = \int_{\mathbb{R}^2} f(x) \Psi^\mathbb{R}_{a, y, \theta}(x) \, dx
\]

\[
= a^{-1} \int_{\mathbb{R}^2} e^{-i a x^2} e^{i a x^2} \frac{y y^2}{2 a^2} \exp \left\{ \frac{y^2}{a^2} \right\} \right\} dR^2
\]

\[
= a^{-1} e^{-\frac{j a y^2}{2 a^2}} \int_{\mathbb{R}^2} e^{-i a x^2} e^{i a x^2} \left\{ \frac{-y y^2}{2 a^2} \exp \left\{ \frac{y^2}{a^2} \right\} \right\} dR^2
\]

\[
= \left(\lambda^2 a\right)^{-1} e^{-\frac{j a y^2}{2 a^2}} \int_{\mathbb{R}^2} e^{-i a x^2} e^{i a x^2} \left\{ \frac{-y y^2}{2 a^2} \exp \left\{ \frac{y^2}{a^2} \right\} \right\} dR^2
\]

\[
= \left(\lambda^2 a\right)^{-1} e^{-\frac{j a y^2}{2 a^2}} \int_{\mathbb{R}^2} e^{-i a x^2} e^{i a x^2} \left\{ \frac{-y y^2}{2 a^2} \exp \left\{ \frac{y^2}{a^2} \right\} \right\} dR^2
\]

\[
= \frac{1}{\lambda^2 a} e^{-\frac{j a y^2}{2 a^2}} e^{-\frac{y^2}{2 a^2}} \int_{\mathbb{R}} \exp \left\{ -x^2 \left\{ \frac{1}{2 a^2} \lambda^2 - \frac{j a x}{2 a^2} \right\} + x^2 \left\{ -\alpha_2 + \frac{y^2}{a^2} \lambda^2 \right\} \right\} dR^2
\]

\[
\times \int_{\mathbb{R}} \exp \left\{ -x^2 \left\{ \frac{1}{2 a^2} \lambda^2 - \frac{j a x}{2 a^2} \right\} + x^2 \left\{ -\alpha_2 + \frac{y^2}{a^2} \lambda^2 \right\} \right\} dR^2
\]

\[
= \frac{1}{\lambda^2 a} \exp \left\{ -\frac{j a y^2}{2 b_2} - \frac{y^2}{2 a^2} \right\} \exp \left\{ \frac{2 a^2 L^2 b_2}{b_2 - j a a^2 \lambda^2} \right\} \exp \left\{ \frac{(y^2 - a^2 \alpha_2 a^2 \lambda^2)^2}{4 b_2 - j a a^2 \lambda^2} \right\}
\]

\[
\times \frac{2 a^2 \lambda^2 b_2}{b_2 - j a a^2 \lambda^2} \exp \left\{ \frac{(y_2 - a^2 \alpha_2 a^2 \lambda^2)^2}{4 b_2 - j a a^2 \lambda^2} \right\} \exp \left\{ \frac{2 a^2 \lambda^2 b_2}{b_2 - j a a^2 \lambda^2} \right\} \exp \left\{ \frac{(y_2 - a^2 \alpha_2 a^2 \lambda^2)^2}{4 b_2 - j a a^2 \lambda^2} \right\}
\]

\[
\times \frac{2 a^2 \lambda^2 b_2}{b_2 - j a a^2 \lambda^2} \exp \left\{ \frac{(y_2 - a^2 \alpha_2 a^2 \lambda^2)^2}{4 b_2 - j a a^2 \lambda^2} \right\} \exp \left\{ \frac{2 a^2 \lambda^2 b_2}{b_2 - j a a^2 \lambda^2} \right\} \exp \left\{ \frac{(y_2 - a^2 \alpha_2 a^2 \lambda^2)^2}{4 b_2 - j a a^2 \lambda^2} \right\}
\]

\[
\times \frac{2 a^2 \lambda^2 b_2}{b_2 - j a a^2 \lambda^2} \exp \left\{ \frac{(y_2 - a^2 \alpha_2 a^2 \lambda^2)^2}{4 b_2 - j a a^2 \lambda^2} \right\} \exp \left\{ \frac{2 a^2 \lambda^2 b_2}{b_2 - j a a^2 \lambda^2} \right\} \exp \left\{ \frac{(y_2 - a^2 \alpha_2 a^2 \lambda^2)^2}{4 b_2 - j a a^2 \lambda^2} \right\}
\]
Proof. Invoking Parseval formula (2.4) for the two-sided QLCT and implementing Lemma 3.4, we have

\[ \langle \mathcal{W}_\phi^H[f], \mathcal{W}_\psi^H[g] \rangle_{L^2(G, \mathbb{H})} = C_\psi(f, g)_{L^2(\mathbb{R}^2, \mathbb{H})} \] (3.13)

where \( G = \mathbb{R}^+ \times \mathbb{R}^2 \times \text{SO}(2, \mathbb{H}) \) is similitude group and \( C_\psi \) is admissibility given by (3.8).

Proof. Invoking Parseval’s Formula. Suppose that \( \psi, \phi \in L^2(\mathbb{R}^2, \mathbb{H}) \) be an admissible wavelets, then for every \( f, g \in L^2(\mathbb{R}^2, \mathbb{H}) \) linear canonical wavelet transform (3.9) satisfies the following properties:

(i) Linearity: \( \mathcal{W}_\phi^H[\alpha_1 f + \alpha_2 g](a, y, \theta) = \alpha_1 \mathcal{W}_\psi^H[f](a, y, \theta) + \alpha_2 \mathcal{W}_\psi^H[g](a, y, \theta) \)

where \( \alpha_1, \alpha_2 \) are quaternion constants in \( \mathbb{H} \)

(ii) Anti-linearity: \( \mathcal{W}_{\alpha_1 \psi + \alpha_2 \phi}[f](a, y, \theta) = \mathcal{W}_\psi[f](a, y, \theta)\overline{\alpha_1} + \mathcal{W}_\phi[g](a, y, \theta)\overline{\alpha_2} \)

(iii) Translation: For any constant \( k \in \mathbb{R}^2 \),

\[ \mathcal{W}_\psi^H[f(a, y, \theta)](a, y, \theta) = e^{\frac{iak^2}{\lambda^2}} \mathcal{W}_\psi^H[e^{-\frac{ia_1 k^2}{\lambda^2}} f(a, y, \theta) e^{\frac{ia_2 k^2}{\lambda^2}}](a, y, \theta) \]

(iv) Scaling: For non-zero constant \( \lambda \),

\[ \mathcal{W}_\psi^H[f(a, y, \theta)](a, y, \theta) = \lambda^{-1} \mathcal{W}_\psi^H[f(\lambda a, \lambda y, \theta)](a, y, \theta) \]

where matrices parameter in R.H.S is \( A_s = [a_s', b_s', c_s, d_s] \) is related to matrices in L.H.S i.e. \( A_s = [a_s, b_s, c_s, d_s] \) by a relation \( \frac{a_s'}{b_s'} = \frac{a_s}{b_s} \lambda^2 \) for \( s = 1, 2 \).

(v) Parity: \( \mathcal{W}_{\phi P \psi}[P f(x)](a, y, \theta) = \mathcal{W}_\psi^H[f(x)](a, -y, \theta) \), where \( P f(x) = f(-x) \).

(vi) Dilation in \( \psi \): \( \mathcal{W}_{\psi \phi}[f(x)](a, y, \theta) = \mathcal{W}_\psi^H[f(x)](a c, y, \theta) \), where \( D_c \phi(x) = \frac{1}{c} \psi \left( \frac{x}{c} \right) \).

Proof. For brevity proof of these properties is omitted.

In our next theorem, we will show that the quaternion linear canonical wavelet transform sets up an isometry from \( L^2(\mathbb{R}^2, \mathbb{H}) \) to \( L^2(\mathbb{R}^+ \times \mathbb{R}^2 \times \text{SO}(2, \mathbb{H})) \).

Theorem 3.9 (Parseval’s Formula). Suppose that \( \psi \in L^2(\mathbb{R}^2, \mathbb{H}) \) be an admissible wavelets, then for every \( f, g \in L^2(\mathbb{R}^2, \mathbb{H}) \), we have
\[
\langle W_{\psi}^{\mathbb{H}}[f], W_{\psi}^{\mathbb{H}}[g] \rangle_{L^2(G, \mathbb{H})} = \int_g W_{\psi}^{\mathbb{H}}[f](a, y, \theta) W_{\psi}^{\mathbb{H}}[g](a, y, \theta) \frac{dadyd\theta}{a^3}
\]
\[
= \int_g \langle f, \Psi_{a,y,\theta}^{\mathbb{H}} \rangle_{L^2(\mathbb{R}^2, \mathbb{H})} \langle g, \Psi_{a,y,\theta}^{\mathbb{H}} \rangle_{L^2(\mathbb{R}^2, \mathbb{H})} \frac{dadyd\theta}{a^3}
\]
\[
= \int_g \left( \mathcal{L}_{A_1, A_2}^H[f], \mathcal{L}_{A_1, A_2}^H[g] \right)_{L^2(\mathbb{R}^2, \mathbb{H})} \frac{dadyd\theta}{a^3}
\]
\[
= \int_g \int_{\mathbb{R}^2} \mathcal{L}_{A_1, A_2}^H[f](w) \mathcal{L}_{A_1, A_2}^H[\Psi_{a,y,\theta}^{\mathbb{H}}(x)](w) dw \int_{\mathbb{R}^2} \mathcal{L}_{A_1, A_2}^H[\Psi_{a,y,\theta}^{\mathbb{H}}(x)](w') \mathcal{L}_{A_1, A_2}^H[g](w') dw' \frac{dadyd\theta}{a^3}
\]
\[
= \frac{a^2}{(2\pi)^2 b_1 b_2} \int_{\mathbb{R}^2} \mathcal{L}_{A_1, A_2}^H[f](w) \exp \left\{ \frac{-j}{2b_2} [d_2 w_2^2 + y_2^2 a_2 - 2y_2 w_2 - \frac{\pi b_2}{2}] \right\}
\]
\[
\times \exp \left\{ \frac{j}{2b_1} [d_1 w_1^2 + y_1^2 a_1 - 2y_1 w_1 - \frac{\pi b_1}{2}] \right\} \frac{dadyd\theta}{a^3}
\]
\[
= \frac{a^2}{(2\pi)^2 b_1 b_2} \int_{\mathbb{R}^2} \mathcal{L}_{A_1, A_2}^H[f](w) \exp \left\{ \frac{-j}{2b_2} [d_2 w_2^2 + y_2^2 a_2 - 2y_2 w_2 - \frac{\pi b_2}{2}] \right\}
\]
\[
\times \exp \left\{ \frac{j}{2b_1} [d_1 w_1^2 + y_1^2 a_1 - 2y_1 w_1 - \frac{\pi b_1}{2}] \right\} \frac{dadyd\theta}{a^3}
\]
where $C_\psi$ is given by (3.8).

This completes the proof of the theorem. $\Box$
Corollary 3.10 (Energy Conservation). For $f = g$, we have the following identity:

$$
\int_{\mathbb{R}^+ \times \mathbb{R}^2 \times \text{SO}(2)} \left\| \mathcal{W}_\psi^H [f](a, y, \theta) \right\|^2_{L_2(\mathbb{H})} \frac{da \, dy \, d\theta}{a^3} = C_\psi \left\| f \right\|^2_{L_2(\mathbb{H})}.
$$

(3.14)

Remark: We see that, except the factor $C_\psi$, the quaternion linear canonical wavelet transform sets up an isometry from $L_2(\mathbb{R}^2, \mathbb{H})$ to $L_2(\mathbb{R}^+ \times \mathbb{R}^2 \times \text{SO}(2), \mathbb{H})$.

The next theorem guarantees the reconstruction of the input quaternion signal from the corresponding quaternion linear canonical wavelet transform.

Theorem 3.11 (Inversion Formula). Suppose that $\psi \in L_2(\mathbb{R}^2, \mathbb{H})$ is an admissible wavelet, then any quaternion signal $f \in L_2(\mathbb{R}^2, \mathbb{H})$ can be reconstructed from the quaternion linear canonical wavelet transform $\mathcal{W}_\psi^H [f](a, y, \theta)$ via the following formula:

$$
f(x) = \frac{1}{C_\psi} \int_{\mathbb{R}^+ \times \mathbb{R}^2 \times \text{SO}(2)} \mathcal{W}_\psi^H [f](a, y, \theta) \Psi_{a,y,\theta}^H(x) \frac{da \, dy \, d\theta}{a^3}, \text{ a.e.}
$$

(3.15)

Proof. For arbitrary $g \in L_2(\mathbb{R}^2, \mathbb{H})$, implication of Theorem 3.9 yields

$$
C_\psi \langle f, g \rangle_{L_2(\mathbb{R}^2, \mathbb{H})} = \int_{\mathcal{G}} \mathcal{W}_\psi^H [f](a, y, \theta) \mathcal{W}_\psi^H [g](a, y, \theta) \frac{da \, dy \, d\theta}{a^3}
$$

$$
= \int_{\mathcal{G}} \mathcal{W}_\psi^H [f](a, y, \theta) \left\langle g, \Psi_{a,y,\theta}^H \right\rangle_{L_2(\mathbb{R}^2, \mathbb{H})} \frac{da \, dy \, d\theta}{a^3}
$$

$$
= \int_{\mathcal{G}} \mathcal{W}_\psi^H [f](a, y, \theta) \int_{\mathbb{R}^2} \Psi_{a,y,\theta}^H \frac{da \, dy \, d\theta}{a^3}
$$

$$
= \left\langle \int_{\mathcal{G}} \mathcal{W}_\psi^H [f](a, y, \theta) \Psi_{a,y,\theta}^H \frac{da \, dy \, d\theta}{a^3}, g \right\rangle_{L_2(\mathbb{R}^2, \mathbb{H})}
$$

where we applied Fobini’s theorem in getting second last equality. Therefore, we have

$$
C_\psi f(x) = \int_{\mathbb{R}^+ \times \mathbb{R}^2 \times \text{SO}(2)} \mathcal{W}_\psi^H [f](a, y, \theta) \Psi_{a,y,\theta}^H(x) \frac{da \, dy \, d\theta}{a^3}, \text{ a.e.}
$$

This completes the proof of theorem.

Theorem 3.12 (Characterization of range). For an admissible wavelet $\psi \in L^2(\mathbb{R}^2, \mathbb{H})$, the range of the quaternion linear canonical wavelet transform $\mathcal{W}_\psi^H$ is a reproducing kernel in $L^2(\mathbb{R}^+ \times \mathbb{R}^2 \times \text{SO}(2), \mathbb{H})$ where kernel is given by

$$
K_\psi(a, y; a', y') = C_\psi^{-1} \left\langle \Psi_{a,y,\theta}^H, \Psi_{a',y',\theta'}^H \right\rangle_{L_2(\mathbb{R}^2, \mathbb{H})}
$$

(3.16)

Moreover,

$$
\left| K_\psi(a, y; a', y') \right| \leq C_\psi^{-1} \left\| \psi \right\|_{L_2(\mathbb{R}^2, \mathbb{H})}, \text{ if } C_\psi > 0.
$$

(3.17)

Proof. Invoking the inversion formula (3.15) in the definition of quaternion linear canonical wavelet transform (3.8), we have
Let \( \Psi \) transform as defined by (3.9). We first prove the following lemma. In this section, we shall establish an analogue of the well-known Heisenberg’s uncertainty inequality yields the classical Heisenberg’s inequality by virtue of Jensen’s inequality. In this way, Heisenberg’s uncertainty principle in harmonic analysis is of central importance in time-frequency analysis as it provides a lower bound for optimal simultaneous resolution in the time and frequency domains (see [28]). This principle has been extended to different time-frequency transforms and several other versions of the uncertainty principle have been investigated from time to time. For instance, Beckner [29] obtained a logarithmic version of the uncertainty principle by using a sharp form of Pitt’s inequality and showed that this version yields the classical Heisenberg’s inequality by virtue of Jensen’s inequality. In this section, we shall establish an analogue of the well-known Heisenberg’s uncertainty inequality and the corresponding logarithmic version for the quaternion linear canonical wavelet transform as defined by (3.9). We first prove the following lemma.

**Lemma 4.1.** Let \( \psi \in L^2(\mathbb{R}^2, \mathbb{H}) \) be an admissible quaternion wavelet, then for every \( f \in L^2(\mathbb{R}^2, \mathbb{H}) \), we have

\[
\int_{\mathcal{G}} \left| w_k \mathcal{F}^\mathbb{H}_{A_1, A_2} \left[ \mathcal{W}_\psi \left[ f \right] \right] (w) \right|^2_{\mathbb{H}_2} \, d\eta = C_\psi \| w_k \mathcal{F}_q [f] (w) \|^2_{L^2(\mathbb{R}^2, \mathbb{H})} \tag{4.1}
\]
Proof. Invoking Parseval formula (3.13) of quaternion linear canonical wavelet transform, we have

$$C_{\psi} \langle f, g \rangle_{L^2(\mathbb{R}^2, \mathbb{H})} = \int_{\mathbb{R}^+} \int_{SO(2)} \int_{\mathbb{R}^2} \mathcal{W}_{\psi}^H[f](a, y, \theta) \mathcal{W}_{\psi}^H[g](a, y, \theta) \frac{dadyd\theta}{a^3}$$

Implementing Plancherel theorem of QFT(two-sided) on L.H.S and Plancherel theorem of QLCT(two-sided) to the $y$-integral on R.H.S of above equation, we obtain

$$C_{\psi} \langle \mathcal{F}_q[f], \mathcal{F}_q[g] \rangle_{L^2(\mathbb{R}^2, \mathbb{H})} = \int_{\mathbb{R}^+} \int_{SO(2)} \left( \mathcal{L}_{A_1, A_2}^H \mathcal{W}_{\psi}^H[f], \mathcal{L}_{A_1, A_2}^H \mathcal{W}_{\psi}^H[g] \right)_{L^2(\mathbb{R}^2, \mathbb{H})} \frac{dadyd\theta}{a^3}$$

Multiplying $|w_k|$ on both sides both sides of above equation, we get

$$C_{\psi} \left<w_k \mathcal{F}_q[f], w_k \mathcal{F}_q[g]\right>_{L^2(\mathbb{R}^2, \mathbb{H})} = \int_{\mathbb{R}^+} \int_{SO(2)} \left< w_k \mathcal{L}_{A_1, A_2}^H \mathcal{W}_{\psi}^H[f], w_k \mathcal{L}_{A_1, A_2}^H \mathcal{W}_{\psi}^H[g] \right>_{L^2(\mathbb{R}^2, \mathbb{H})} \frac{dadyd\theta}{a^3}$$

Finally, for $f = g$, above equation yields

$$C_{\psi} \int_{\mathbb{R}^2} |w_k \mathcal{F}_q[f](w)|^2_{\mathbb{H}} dw = \int_{\mathbb{R}^+} \int_{SO(2)} \int_{\mathbb{R}^2} |w_k \mathcal{L}_{A_1, A_2}^H \mathcal{W}_{\psi}^H[f](w)|^2_{\mathbb{H}} \frac{dadyd\theta}{a^3}$$

This completes the proof of Lemma 4.1. □

We are now ready to establish the Heisenberg-type inequalities for the proposed quaternion linear canonical wavelet transform as defined by (3.9).

**Theorem 4.2.** Let $\psi \in L^2(\mathbb{R}^2, \mathbb{H})$ be an admissible quaternion wavelet, the quaternion linear canonical wavelet transform $\mathcal{W}_{\psi}^H[f](a, y, \theta)$ given by (3.9) satisfies the following uncertainty inequality:

$$\int_{\mathbb{R}^2} |w_k \mathcal{W}_{\psi}^H[f](a, y, \theta)|^2_{\mathbb{H}} dy \cdot \int_{\mathbb{R}^2} |w_k \mathcal{F}_q[f](w)|^2_{\mathbb{H}} dw \geq \frac{b_k}{4} \int_{\mathbb{R}^2} |f(x)|^4_{\mathbb{H}} dx \quad (4.2)$$

**Proof.** Invoking the Heisenberg’s inequality for the quaternion linear canonical transform \cite{14}, we can write

$$\left\{ \int_{\mathbb{R}^2} y_k^2 |f(y)|^2_{\mathbb{H}} dy \right\}^{1/2} \cdot \left\{ \int_{\mathbb{R}^2} w_k^2 |\mathcal{L}_{A_1, A_2}^H[f](w)|^2_{\mathbb{H}} dw \right\}^{1/2} \geq \frac{b_k}{2} \int_{\mathbb{R}^2} |f(x)|^2_{\mathbb{H}} dx.$$

Integrating both sides of the above inequality with respect to measure $\frac{dadyd\theta}{a^3}$, we have

$$\int_{\mathbb{R}^+} \int_{SO(2)} \left\{ \int_{\mathbb{R}^2} y_k^2 |f(y)|^2_{\mathbb{H}} dy \right\}^{1/2} \cdot \left\{ \int_{\mathbb{R}^2} w_k^2 |\mathcal{L}_{A_1, A_2}^H[f](w)|^2_{\mathbb{H}} dw \right\}^{1/2} \frac{da d\theta}{a^3}$$

$$\geq \frac{b_k}{2} \int_{\mathbb{R}^+} \int_{SO(2)} \int_{\mathbb{R}^2} |f(x)|^2_{\mathbb{H}} \frac{da dx d\theta}{a^3}$$

Now by implementing Cauchy-Schwartz inequality, we obtain

$$\left\{ \int_{\mathbb{R}^+} \int_{SO(2)} \int_{\mathbb{R}^2} y_k^2 |f(y)|^2_{\mathbb{H}} dy \frac{da d\theta}{a^3} \right\}^{1/2} \cdot \left\{ \int_{\mathbb{R}^+} \int_{SO(2)} \int_{\mathbb{R}^2} w_k^2 |\mathcal{L}_{A_1, A_2}^H[f](w)|^2_{\mathbb{H}} \frac{dw da d\theta}{a^3} \right\}^{1/2}$$

$$\geq \frac{b_k}{2} \int_{\mathbb{R}^+} \int_{SO(2)} \int_{\mathbb{R}^2} |f(x)|^2_{\mathbb{H}} \frac{da dx d\theta}{a^3}.$$
Considering $\mathcal{W}_{\psi}^H[f](a, y, \theta)$ as a function of $y$ and replacing $f$ by $\mathcal{W}_{\psi}^H[f](a, y, \theta)$ in above equation to obtain

$$\left\{ \int_{\mathbb{R}^+} \int_{SO(2)} \int_{\mathbb{R}^2} y^2 \left| \mathcal{W}_{\psi}^H[f](a, y, \theta) \right|^2_{\mathbb{H}} dy \frac{da \, d\theta}{a^3} \right\}^{1/2} \times \left\{ \int_{\mathbb{R}^+} \int_{SO(2)} \int_{\mathbb{R}^2} w^2 \left| \mathcal{L}_{A_1, A_2} \mathcal{W}_{\psi}^H[f](a, y, \theta) \right|^2_{\mathbb{H}} dw \frac{da \, d\theta}{a^3} \right\}^{1/2} \geq \frac{b_k}{2} \int_{\mathbb{R}^+} \int_{SO(2)} \int_{\mathbb{R}^2} \left| \mathcal{W}_{\psi}^H[f](a, y, \theta) \right|^2_{\mathbb{H}} \frac{da \, dy \, d\theta}{a^3}$$

By virtue of Lemma 4.1, we obtain

$$\left\{ \int_{\mathbb{R}^+} \int_{SO(2)} \int_{\mathbb{R}^2} y^2 \left| \mathcal{W}_{\psi}^H[f](a, y, \theta) \right|^2_{\mathbb{H}} dy \frac{da \, d\theta}{a^3} \right\}^{1/2} \left\{ C_\psi \int_{\mathbb{R}^2} \left| w_k \mathcal{F}_{q}[f](w) \right|^2_{\mathbb{H}} dw \right\}^{1/2} \geq \frac{b_k}{2} C_\psi \left\| f \right\|^2_{L^2(\mathbb{R}^2, \mathbb{H})}$$

Furthermore, on implementing Corollary 3.10 on R.H.S of the above inequality we have

$$\left\{ \int_{\mathbb{R}^+} \int_{SO(2)} \int_{\mathbb{R}^2} y^2 \left| \mathcal{W}_{\psi}^H[f](a, y, \theta) \right|^2_{\mathbb{H}} dy \frac{da \, d\theta}{a^3} \right\}^{1/2} \left\{ C_\psi \int_{\mathbb{R}^2} \left| w_k \mathcal{F}_{q}[f](w) \right|^2_{\mathbb{H}} dw \right\}^{1/2} \geq \frac{b_k}{2} C_\psi \left\| f \right\|^2_{L^2(\mathbb{R}^2, \mathbb{H})}$$

Dividing both sides by $\sqrt{C_\psi}$, we obtain the desired result. \(\Box\)

Before presenting our next result, we have the following definition of space of rapidly decreasing smooth quaternion functions (see [15]).

**Definition 4.3.** For a multi-index $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^+ \times \mathbb{R}^+$, the Schwartz space in $L^2(\mathbb{R}^2, \mathbb{H})$ is defined as

$$\mathcal{S}(\mathbb{R}^2, \mathbb{H}) = \left\{ f \in C^\infty(\mathbb{R}^2, \mathbb{H}); \sup_{t \in \mathbb{R}^2} \left( 1 + |t|^k \right) \left| \frac{\partial^{\alpha_1 + \alpha_2} f(t)}{\partial t_1^{\alpha_1} \partial t_2^{\alpha_2}} \right| < \infty \right\},$$

where $C^\infty(\mathbb{R}^2, \mathbb{H})$ is the set of smooth functions from $\mathbb{R}^2$ to $\mathbb{H}$.

We now establish the logarithmic uncertainty principle for the quaternion linear canonical wavelet transform $\mathcal{W}_{\psi}^H[f]$ as defined by (3.9).

**Theorem 4.4.** For an admissible quaternion wavelet $\psi \in \mathcal{S}(\mathbb{R}^2, \mathbb{H})$ and a signal $f \in \mathcal{S}(\mathbb{R}^2, \mathbb{H})$, the quaternion linear canonical wavelet transform $\mathcal{W}_{\psi}^H[f]$ satisfies the following logarithmic estimate of the uncertainty inequality:

$$\int_{\mathbb{H}} |y| \left| \mathcal{W}_{\psi}^H[g](a, y, \theta) \right|^2_{\mathbb{H}} d\eta + C_\psi \int_{\mathbb{R}^2} |w| \left| \mathcal{F}_{q}[f](w) \right|^2_{\mathbb{H}} dw \geq C_\psi (D + \ln |b|) \int_{\mathbb{R}^2} |f(x)|^2_{\mathbb{H}} dx,$$

where $D = \left( \frac{\Gamma'(1/2)}{\Gamma(1/2)} - \ln \pi \right)$ and $\Gamma$ is a Gamma function.
Proof. For the quaternion-valued function \( f \in L^2(\mathbb{R}^2, \mathbb{H}) \), the time and frequency spreads satisfy the inequality \[ \left( \mathcal{D} + \ln |b| \right) \int_{\mathbb{R}^2} |f(y)|^2 dy \geq \left( \mathcal{D} + \ln |b| \right) \int_{\mathbb{R}^2} |f(y)|^2 dy \]

Replacing \( f(y) \) by \( \mathcal{W}_H^H[f](a, y, \theta) \) in the above inequality, we obtain

\[
\int_{\mathbb{R}^2} \ln |y| \left| \mathcal{W}_H^H[f](a, y, \theta) \right|^2 dy + \int_{\mathbb{R}^2} \ln |w| \left| \mathcal{L}_{A_1 A_2}^H \left[ \mathcal{W}_H^H[f] \right](w) \right|^2 dw \\
\geq \left( \mathcal{D} + \ln |b| \right) \int_{\mathbb{R}^2} \left| \mathcal{W}_H^H[f](a, y, \theta) \right|^2 dy
\]

Integrating above equation with respect to measure \( \frac{dado}{a^3} \), and then applying the Fubini theorem, we obtain

\[
\int_{\mathbb{R}^2} \int_{SO(2)} \int_{\mathbb{R}^2} \ln |y| \left| \mathcal{W}_H^H[f](a, y, \theta) \right|^2 dy \frac{dado}{a^3} + \int_{\mathbb{R}^2} \int_{SO(2)} \int_{\mathbb{R}^2} \ln |w| \left| \mathcal{L}_{A_1 A_2}^H \left[ \mathcal{W}_H^H[f] \right](w) \right|^2 dw \\
\geq \left( \mathcal{D} + \ln |b| \right) \int_{\mathbb{R}^2} \int_{SO(2)} \int_{\mathbb{R}^2} \left| \mathcal{W}_H^H[f](a, y, \theta) \right|^2 dy \frac{dado}{a^3}
\]

Applying Lemma 4.1 for \( w_k^2 = \ln |w_k| \) on L.H.S and Corollary 3.10 on R.H.S, we obtain the desired result

\[
\int_{\mathcal{G}} \ln |y| \left| \mathcal{W}_H^H[g](a, y, \theta) \right|^2 dy + C \psi \int_{\mathbb{R}^2} \ln |w| \left| F_q[f](w) \right|^2 dw \geq C \psi \left( \mathcal{D} + \ln |b| \right) \int_{\mathbb{R}^2} \left| f(x) \right|^2 dx,
\]

This completes the proof of Theorem 4.4. \( \Box \)

In the following, we establish a local type uncertainty principle for quaternion linear canonical wavelet transform \( \mathcal{W}_H^H[f] \) as defined by (3.9).

**Theorem 4.5.** Given an admissible quaternion wavelet \( \psi \in L^2(\mathbb{R}^2, \mathbb{H}) \) and a signal \( f \in L^2(\mathbb{R}^2, \mathbb{H}) \), with \( \| \psi \|^2_{L^2(\mathbb{R}^2, \mathbb{H})} = 1 = \| f \|^2_{L^2(\mathbb{R}^2, \mathbb{H})} \), such that for a measurable set \( E \subset \mathbb{R}^2 \times \mathbb{R}^2 \), \( \epsilon \geq 0 \), and

\[
\int \int_{\mathcal{E}} \left| \mathcal{W}_H^H[f](a, y, \theta) \right|^2 dy dx \geq 1 - \epsilon.
\]

We have \( a(1 - \epsilon) \leq \mu(E) \), where \( \mu(E) \) is Lebesgue measure of \( E \).

**Proof.** From Definition 3.4, we have

\[
\left| \mathcal{W}_H^H[f](a, y, \theta) \right| = a^{-1} \int_{\mathbb{R}^2} f(x) e^{\frac{iab}{2a^2}(x^2-y^2)} \psi\left( r_\theta a^{-1}(x-y) \right) e^{\frac{iab}{2a^2}(x^2-y^2)} dx
\]

\[
\leq \frac{1}{a} \int_{\mathbb{R}^2} |f(x)| \left| \psi\left( r_\theta a^{-1}(x-y) \right) \right| |dx|.
\]

Taking Sup-norm on L.H.S and implementing well known Holders inequality on R.H.S, we obtain

\[
\left\| \mathcal{W}_H^H[f](a, y, \theta) \right\|_{L^\infty(\mathbb{R}^2, \mathbb{H})} \leq \frac{1}{a} \left\| f \right\|_{L^2(\mathbb{R}^2, \mathbb{H})} \left\| \psi \right\|_{L^2(\mathbb{R}^2, \mathbb{H})}
\]

(4.5)
Plugging inequality (4.5) in (4.4), we get

\[
1 - \epsilon \leq \int \int_E \left| \mathcal{W}_\psi^H [f](a, y, \theta) \right|^2 dy \, dx \\
\leq \mu(E) \cdot \left\| \mathcal{W}_\psi^H [f](a, y, \theta) \right\|_{L^\infty(\mathbb{R}^2, \mathbb{H})} \\
\leq \frac{\mu(E)}{a} \left\| f \right\|_{L^2(\mathbb{R}^2, \mathbb{H})} \left\| \psi \right\|_{L^2(\mathbb{R}^2, \mathbb{H})} \\
\leq \frac{\mu(E)}{a}.
\]

This completes the proof of Theorem 4.5.

**Theorem 4.6 (Local uncertainty inequality).** Let \( E \) be a measurable subset of \( \mathbb{R}^2 \times \mathbb{R}^2 \), such that \( 0 < \mu(E) < 1 \), then for every \( f \in L(\mathbb{R}^2, \mathbb{H}) \) and an admissible quaternion wavelet \( \psi \in L(\mathbb{R}^2, \mathbb{H}) \), we have

\[
\left\| f \right\|_{L^2(\mathbb{R}^2, \mathbb{H})} \left\| \psi \right\|_{L^2(\mathbb{R}^2, \mathbb{H})} \leq \frac{1}{\sqrt{1 - \mu(E)}} \left\| \mathcal{W}_\psi^H [f](a, y, \theta) \right\|_{L^2(E, \mathbb{H})} (4.6)
\]

Moreover, for every \( \alpha > 0 \), there exist \( C(\alpha) > 0 \), such that

\[
\left\| f \right\|_{L^2(\mathbb{R}^2, \mathbb{H})} \left\| \psi \right\|_{L^2(\mathbb{R}^2, \mathbb{H})} \leq C(\alpha) \left[ \int \int_{\mathbb{R}^2} \left| (y, x) \right|^{2\alpha} \left| \mathcal{W}_\psi^H [f](a, y, \theta) \right|^2 dy \, dx \right]^{1/2}. (4.7)
\]

**Proof.** By invoking the Corollary 3.10 and Theorem 4.5, we have

\[
\left\| \mathcal{W}_\psi^H [f](a, y, \theta) \right\|_{L^2(\mathbb{R}^2 \times \mathbb{R}^2, \mathbb{H})}^2 = \int \int_{\mathbb{R}^2} \left| \mathcal{W}_\psi^H [f](a, y, \theta) \right|_{\mathbb{H}}^2 dy \, dx \\
= \int \int_E \left| \mathcal{W}_\psi^H [f](a, y, \theta) \right|_{\mathbb{H}}^2 dy \, dx + \int \int_{E^c} \left| \mathcal{W}_\psi^H [f](a, y, \theta) \right|_{\mathbb{H}}^2 dy \, dx \\
\leq \mu(E) \left\| f \right\|_{L^2(\mathbb{R}^2, \mathbb{H})} \left\| \psi \right\|_{L^2(\mathbb{R}^2, \mathbb{H})} + \left\| \mathcal{W}_\psi^H [f](a, y, \theta) \right\|_{L^2(\mathbb{H})}^2
\]

Equivalently,

\[
(1 - \mu(E)) \left\| f \right\|_{L^2(\mathbb{R}^2, \mathbb{H})} \left\| \psi \right\|_{L^2(\mathbb{R}^2, \mathbb{H})} \leq \left\| \mathcal{W}_\psi^H [f](a, y, \theta) \right\|_{L^2(\mathbb{H})}^2
\]

Taking square root on both sides and then dividing both sides by \( \sqrt{1 - \mu(E)} \), we get

\[
\left\| f \right\|_{L^2(\mathbb{R}^2, \mathbb{H})} \left\| \psi \right\|_{L^2(\mathbb{R}^2, \mathbb{H})} \leq \frac{1}{\sqrt{1 - \mu(E)}} \left\| \mathcal{W}_\psi^H [f](a, y, \theta) \right\|_{L^2(\mathbb{H})}
\]

which proves our first assertion.

Now, we fix \( \lambda_0 \in (0, 1] \) small enough such that \( \mu(B_{\lambda_0}) < 1 \), where \( B_{\lambda_0} = \left\{ (y, x) \in \mathbb{R}^2 \times \mathbb{R}^2; |(y, x)| < \lambda_0 \} \), the ball of radius \( \lambda_0 \) centered at origin, we have from inequality (4.6),
\[
\left\| f \right\|_{L^2(\mathbb{R}, \mathbb{H})} \left\| \psi \right\|_{L^2(\mathbb{R}, \mathbb{H})} \leq \frac{1}{\sqrt{1 - \mu(B_{\lambda_0})}} \left[ \int \int_{\left| (y, x) \right| > \lambda_0} \left| \mathcal{W}_\psi^{\mathbb{H}} [f](a, y, \theta) \right|^2 dy dx \right]^{1/2}
\]

\[
\leq \frac{1}{\lambda_0^2 \sqrt{1 - \mu(B_{\lambda_0})}} \left[ \int \int_{\left| (y, x) \right| > \lambda_0} \left| (y, x) \right|^{2\alpha} \left| \mathcal{W}_\psi^{\mathbb{H}} [f](a, y, \theta) \right|^2 dy dx \right]^{1/2}
\]

\[
\leq C(\alpha) \left[ \int \int_{\left| (y, x) \right| > \lambda_0} \left| (y, x) \right|^{2\alpha} \left| \mathcal{W}_\psi^{\mathbb{H}} [f](a, y, \theta) \right|^2 dy dx \right]^{1/2}
\]

where \( C(\alpha) = \frac{1}{\lambda_0^\alpha \sqrt{1 - \mu(B_{\lambda_0})}} \). This completes the proof of Theorem 4.6.

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Department of Mathematics, University of Kashmir, South Campus, Anantnag-192101 Jammu and Kashmir, India.

E-mail address: aajaz.math@gmail.com