Total and Inverse Domination Numbers of Certain Graphs

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Abstract- For any graph $G$ having vertex set $V(G)$ then the subset set $D \subseteq V(G)$ is known as a dominating set if every single vertex of $G$ not belonging to $D$ is adjoining to not less than one vertex in $D$. The domination number $\gamma(G)$ is the minimum number of elements contained in a minimum dominating set $D$ of $G$. Any subset $D$ in $V(G)$ is known as total dominating set if each and every vertex of $V$ in $G$ is adjoining to not less than one vertex of $D$. The set which contains minimum number of elements among all total dominating set is the minimum total dominating set and its cardinality denoted as total domination number $\gamma_t(G)$. The inverse dominating set $D'$ is defined as that $D$ is a minimum dominating set of $G$, if there exist an another dominating set say $D'$ in $V - D$ corresponding to $D$ and its cardinality is the inverse domination number $\gamma'(G)$. In this paper we give the total and inverse domination numbers of certain graphs.

Keywords: dominating set, total domination number, inverse dominating set and inverse domination number.

1. Introduction

A graph or a network is a mathematical model comprising of points called vertices or nodes and lines called edges or links which join certain pair of points. The number of vertices incident at a point is called the degree of that vertex and is denoted by $deg(v)$. The notation $\Delta(G)$ and $\delta(G)$ denoted the maximum and minimum degrees of a graph $G$ respectively. A vertex $v \in V$ is called pendant vertex if $deg(v) = 1$. An edge of a graph is said to be pendant edge if one of its vertices is a pendant vertex [1].

Around 1960 the study of dominating sets in Graph theory began. According to Hedetneimi and Laskar (1990)[2], the problems in domination were studied around 1950’s onwards, but its rate of research much more increased from the year 1970. In 1958 the concept of domination number in graphs known as “coefficient of external stability” was defined by Berge, later it should be called as dominating set by Ore in 1962[4]. The notation $\gamma(G)$ for the domination number of graph $G$ was used by Cockayane and Hedetneimi[5] in 1977. Also they gave an interesting survey results of time about dominating sets in graphs [6],[7].

Any subset $D$ in $V(G)$ is known as total dominating set if each and every vertex of $V$ in $G$ is adjoining to not less than one vertex of $D$. The set which contains minimum number of elements among all total dominating set is the minimum total dominating set and its cardinality denoted as total
domination number $\gamma_t(G)$. This concept was introduced by Cockayne et al. [8] which was very much used in fault tolerance.

The inverse dominating set $D'$ is defined as that $D$ is a minimum dominating set of $G$, if there exist another dominating set say $D'$ in $V - D$ corresponding to $D$. Among all inverse dominating set there is a set with minimum number of elements is called minimum inverse dominating set and its cardinality denoted as the inverse domination number $\gamma'(G)$ which was introduced by Kulli and Sigarkanthi [9] and it was studied by several graph theorists.

In rest of the sections we give the total and inverse domination numbers for Lollipop graph $L_{m,n}$, Fly graph $F(m,n)$ and Jellyfish graph $J(r,s)$.

2. Definitions

Definition 2.1: A lollipop graph is the graph obtained from a complete graph $K_m$ with one vertex is attached by a path of length $n$ and it is denoted by $L_{m,n}$.

Definition 2.2[10]: A shell graph $S_n$ obtained by taking $(n - 3)$ concurrent chords in a cycle $C_n$. All the chords are coincident in one vertex is called the apex vertex.

Definition 2.3[10]: Two bow graphs with their apexes merged is called a double shell.

Definition 2.4[10]: A fly graph is a bow graph with exactly two pendant edges at the apex vertex.

Definition 2.55[10]: The jellyfish graph $J(r,s)$ is from a $4-$cycle $uvwux$, by joining $v$ and $x$ with an edge and appending $r$ pendant edges to $u$ and $s$ pendant edges to $w$.

Definition 2.6: The floor function of a real number $x$ is the greatest integer less than or equal to $x$ and it is denoted by $\lfloor x \rfloor$. Suppose that $n \leq x < n + 1$, where $n$ is an integer, then $\lfloor x \rfloor = n$.

Definition 2.7: The roof or ceiling function of a real number $x$ is the greatest integer less than or equal to $x$ and it is denoted by $\lceil x \rceil$. Suppose that $n - 1 < x \leq n$, where $n$ is an integer, then $\lceil x \rceil = n$.

3. Preliminary Results

Theorem 3.1[11]: If the graph $G$ has $p$ vertices and maximum degree $\Delta$, then $\gamma(G) \geq p/\Delta + 1$.

Theorem 3.2[8]: Let $G$ be a connected graph with vertices $p \geq 3$, then $\gamma_t(G) \leq 2p/3$.

Theorem 3.3[12]: For Path $P_n$, $n \geq 2$ then $\gamma_t(P_n) = \lfloor n/3 \rfloor$.

Theorem 3.4[13]: For Path $P_n$, $n \geq 2$ then

$$\gamma_t(P_n) = \begin{cases} \frac{n}{2}, & \text{if } n \equiv 0 \pmod{4} \\ \lfloor \frac{n}{2} \rfloor + 1, & \text{otherwise} \end{cases}$$

Theorem 3.5[8]: For any graph $G$ with $p$ vertices and no isolates, then $\gamma_t(G) \leq p - \Delta(G) + 1$. 

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Theorem 3.6[9]: For Path $P_n$, $n \geq 2$ then

$$\gamma'(P_n) = \begin{cases} \left\lceil \frac{n}{3} \right\rceil + 1, & \text{if } n \equiv 0 \pmod{3} \\ \left\lfloor \frac{n}{3} \right\rfloor, & \text{otherwise} \end{cases}$$

4. Main Results

Theorem 4.1: Let $G$ be Lollipop graph $L_{m,n}$ then

$$\gamma_t(L_{m,n}) = 2 + \begin{cases} \frac{n-2}{2}, & \text{if } n-2 \equiv 0 \pmod{4} \\ \left\lfloor \frac{n-2}{2} \right\rfloor + 1, & \text{otherwise} \end{cases}$$

Proof: Assume $G$ be a Lollipop graph on $m+n$ nodes and $\left\lceil \frac{m(m-1)}{2} \right\rceil + n$ lines where $m \geq 2, n \geq 1$. In $G$ there is no isolated vertex. We know that for a complete graph $K_m$ the domination number is always one.

For minimizing the total dominating set we choose two consecutive vertices $v_1$ and $u_1$ from $G$. Here $v_1$ dominates $v_2, v_3, \ldots, v_m, u_1$ and $u_1$ dominates $u_2$, therefore we need to find out the total dominating set for path $P_{n-2}$. From the Theorem 3.4 we know that for path $P_{n-2}$ is

$$\gamma_t(P_{n-2}) = \begin{cases} \frac{n-2}{2}, & \text{if } n-2 \equiv 0 \pmod{4} \\ \left\lfloor \frac{n-2}{2} \right\rfloor + 1, & \text{otherwise} \end{cases}$$

Thus for Lollipop graph $L_{m,n}$ the total domination number is

$$\gamma_t(L_{m,n}) = 2 + \begin{cases} \frac{n-2}{2}, & \text{if } n-2 \equiv 0 \pmod{4} \\ \left\lfloor \frac{n-2}{2} \right\rfloor + 1, & \text{otherwise} \end{cases}$$

Hence the proof.

Theorem 4.2: Let $G$ be Lollipop graph $L_{m,n}$ then the inverse domination number is
\[ \gamma'(L_{m,n}) = 1 + \begin{cases} \left\lfloor \frac{n}{3} \right\rfloor + 1, & \text{if } n \equiv 0 \pmod{3} \\ \left\lfloor \frac{n}{3} \right\rfloor, & \text{otherwise} \end{cases} \]

**Proof:** Let \( G \) be Lollipop graph on \( m + n \) vertices and \( \left\lfloor \frac{m(m-1)}{2} \right\rfloor + n \) edges where \( m \geq 2, n \geq 1 \). In \( G \) there is no isolated vertex. Also we know that for complete graph \( K_m \) the inverse domination number is one. Suppose \( v_1 \) is the dominating set for \( K_m \), we need to choose any one of the vertex from \( v_2, v_3 \ldots v_m \) to dominate \( K_m \) and it is clear that \( \gamma'(K_m) = 1 \). Then we need to choose inverse dominating set for path \( P_n \) of vertices \( u_1, u_2, u_3, \ldots u_n \). From the Theorem 3.6 we know that,

\[ \gamma'(P_n) = \begin{cases} \left\lfloor \frac{n}{3} \right\rfloor + 1, & \text{if } n \equiv 0 \pmod{3} \\ \left\lfloor \frac{n}{3} \right\rfloor, & \text{otherwise} \end{cases} \]

Therefore we have,

\[ \gamma'(L_{m,n}) = \gamma'(K_m) + \gamma'(P_n) \]

Hence it is clear that the inverse domination number for Lollipop graph \( L_{m,n} \) is

\[ \gamma'(L_{m,n}) = 1 + \begin{cases} \left\lfloor \frac{n}{3} \right\rfloor + 1, & \text{if } n \equiv 0 \pmod{3} \\ \left\lfloor \frac{n}{3} \right\rfloor, & \text{otherwise} \end{cases} \]

Hence the Proof.

**Theorem 4.3:** Let \( G \) be Fly graph \( F(m,n) \). Then the total and inverse domination numbers are

\[ \gamma_t(F(m,n)) = 2 \]

\[ \gamma'(F(m,n)) = 2 + \left\lfloor \frac{m}{3} \right\rfloor + \left\lfloor \frac{n}{3} \right\rfloor \]

**Proof:** Assume the graph \( G \) as a fly graph having shells of orders \( m \) and \( n \) excluding the apex. After including one apex and two pendant vertices in \( G \), the number of vertices of \( F(m,n) \) is \( m + n + 3 \) with \( 2(m + n) \) edges. The vertex \( u \) denoted as the apex of the fly graph and the vertices from top to bottom in the left and the right wing are denoted as follows \( v_1, v_2, v_3 \ldots v_m \) and \( u_1, u_2, u_3 \ldots u_n \) respectively. Clearly \( G \) has two pendant vertices which are denoted as \( v \) and \( w \).

**Figure 2.** Fly Graph \( F(m,n) \)
The domination number for $F(m, n)$ is 1 since the apex vertex $u$ is the minimum dominating set of $G$. Then by choosing any one adjacent vertex of $u$ from the vertices of left or right wing gives the total domination number of $G$ i.e., $\gamma_t(F(m, n)) = 2$.

For finding the inverse domination number of $(m, n)$, we need to choose the vertices other than the apex vertex $u$ from $G$ i.e (to choose vertices from left and right wing of $G$). The vertices $v_1, v_2, v_3 \ldots v_m$ in left wing induce a path $P_m$ and the vertices $u_1, u_2, u_3 \ldots u_n$ in right wing induce a path $P_n$. By using the Theorem 3.3 we have $\gamma(P_m) = \left\lceil \frac{m}{3} \right\rceil$ and $\gamma(P_n) = \left\lceil \frac{n}{3} \right\rceil$. By adding the above two domination number with pendent vertices $v$ and $w$ gives the inverse domination number for $F(m, n)$.

Therefore the inverse domination number for $F(m, n)$ is

$$\gamma'(F(m, n)) = 2 + \left\lceil \frac{m}{3} \right\rceil + \left\lceil \frac{n}{3} \right\rceil.$$  

Hence the proof.

**Theorem 4.4:** Let $G$ be Jellyfish graph $J(r, s)$ then the total and inverse domination numbers are

$$\gamma_t(J(r, s)) = 3$$

$$\gamma'(J(r, s)) = r + s + 1$$

**Proof:** Assume the graph $G$ be a jellyfish graph with $r + s + 4$ nodes and $r + s + 5$ lines where $r, s \geq 3$.

Since the vertex $u$ and $w$ are the minimum dominating set and domination number is 2, we need to choose either the vertex $v$ or $x$ to get the total dominating set for Jellyfish Graph. Thus

$$\gamma_t(J(r, s)) = 3.$$  

For finding the inverse domination number, we have to choose the pendant vertices $u_1, u_2, u_3 \ldots u_r$ and $w_1, w_2, w_3 \ldots w_s$ also choosing either the vertex $v$ or $x$.

Thus we have $\gamma'(J(r, s)) = r + s + 1$. Hence the proof.
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