Equilibrium for classical zero-point radiation: detailed balance under scattering by a classical charged harmonic oscillator

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Abstract
It has been shown repeatedly over a period of 50 years that the use of relativistic classical physics and the inclusion of classical electromagnetic zero-point radiation leads to the Planck blackbody spectrum for classical radiation equilibrium. However, none of this work involves scattering calculations. In contrast to this work, currently accepted physical theory connects classical physics to only the Rayleigh-Jeans spectrum. Indeed, in the past, it has been shown that a nonlinear classical oscillator (which is necessarily a nonrelativistic scattering system) achieves equilibrium only for the Rayleigh-Jeans spectrum where the random radiation present at the frequency of the second harmonic of the oscillator motion has the same energy per normal mode as the radiation present at the fundamental frequency. Here we continue work emphasizing the importance of relativistic versus nonrelativistic analysis. We consider the scattering of random classical radiation by a charged harmonic oscillator of small but non-zero oscillatory amplitude (which can be considered as a relativistic scattering system) and show that detailed radiation balance holds not only at the fundamental frequency of the oscillator but through the first harmonic corresponding to quadrupole scattering, provided that the radiation energy per normal mode at the first harmonic is double the radiation energy per normal mode at the fundamental frequency. This condition corresponds exactly to the zero-point radiation spectrum which is linear in frequency. It is suggested that for this relativistic scattering system, the detailed balance for zero-point radiation holds not only for the fundamental and first harmonic but extends to all harmonics. Here we have the first example of an explicit relativistic classical scattering calculation; equilibrium corresponds not to the Rayleigh-Jeans spectrum, but rather corresponds to the Lorentz-invariant zero-point radiation spectrum.

1. Introduction

1.1. Aspects missing from the classical physics of 1900
One suspects that the history of physics would be quite different if the physicists at the turn of the 20th century had not been unaware of two crucial ideas of classical physics: 1) the existence of classical electromagnetic zero-point radiation, and 2) the importance of special relativity. Had the earlier physicists included these aspects, they would have extended the explanatory power of classical physics to include blackbody radiation, the decrease of specific heats at low temperature, and the behavior of van der Waals forces. As it happened, these aspects were first described in connection with quantum ideas, and the use of quantum physics has been extended until it is the dominant physical theory of the present day. Only during the past half-century has a small group of physicists gone back to classical physics to note the extensions of that theory made possible by the inclusion of classical zero-point radiation and relativity [1–5].

The problem of the equilibrium spectrum of random radiation, the blackbody radiation problem, provided one of the dilemmas for physics at the turn of the 20th century. The work on blackbody radiation within classical physics has been reviewed recently [6]. Although the work involves a variety of points of view, there is no treatment of the equilibrium spectrum of radiation under scattering by a relativistic classical system. Here we
provide the first relativistic scattering calculation and show that classical zero-point radiation is indeed an equilibrium spectrum.

Relativity imposes strong restrictions on systems. Two relativistic mechanical systems are 1) relativistic point charges in classical electrodynamics, and 2) small harmonic oscillator systems within classical electromagnetism. The first system (that of relativistic point charges in classical electrodynamics) is a familiar relativistic system; the second system involving harmonic oscillators is not obviously relativistic. However, one can imagine a harmonic oscillator system of small amplitude as arising from the classical electromagnetic system where two identical charged particles \( q \) are held at some fixed distance apart, and a third charged particle \( e \) of the same sign is placed between the two charges \( q \) and is allowed to oscillate along the line connecting the two identical charges \( q \). In the approximation of small amplitude of oscillation, the electrostatic potential experienced by the third particle \( e \) becomes a harmonic potential, and the relativistic and nonrelativistic particle motions for the particle \( e \) agree with each other since higher powers of the particle velocity can be ignored. Within this simple harmonic motion of small amplitude (ignoring terms in \( v/c \)), there is no role for the speed of light in vacuum \( c \).

1.2. Radiation-spectrum stability
If this harmonic-oscillator system involving the oscillating charge \( e \) is bathed in random classical electromagnetic radiation, then (using only the dipole approximation for radiation) the harmonic oscillator system comes to equilibrium at an oscillator energy equal to the energy of the radiation normal modes of the same frequency as the oscillator frequency. This energy-balance result connecting a point harmonic-oscillator system with random radiation has been known since Planck’s work at the end of the 19th century [7]. What does not seem to be recognized is that this mechanical system, when treated beyond the dipole approximation, involves detailed radiation balance at the radiation harmonics of the fundamental oscillator frequency provided that the random radiation spectrum corresponds to that of classical electromagnetic zero-point radiation.

Most physicists are satisfied to repeat the erroneous textbook claim [8] that classical physics leads inevitably to the Rayleigh-Jeans spectrum for radiation equilibrium. The truth is far more nuanced [6]. It is indeed true that if one use a nonrelativistic classical theory such as (nonrelativistic) classical statistical mechanics [9] or considers scattering by a nonrelativistic nonlinear dipole oscillator treated in the dipole radiation approximation [10], then one arrives at the Rayleigh-Jeans spectrum. However, if one uses relativistic classical physics and includes classical electromagnetic zero-point radiation, then one arrives at the Planck spectrum for classical radiation equilibrium [5].

In work carried out more than forty years ago, it was shown [10] that the addition of a nonlinear term to a harmonic oscillator led to a nonrelativistic, nonlinear mechanical oscillator which scattered electromagnetic radiation (treated in the radiation-dipole approximation) toward the Rayleigh-Jeans spectrum. Thus the Rayleigh-Jeans spectrum was stable under scattering by this nonrelativistic nonlinear system, but any other spectrum of random classical radiation (including Lorentz-invariant zero-point radiation) was changed by the scattering of this system, and the radiation spectrum was pushed toward the Rayleigh-Jeans spectrum. The crucial aspect is that the scattering system was a nonrelativistic classical system [11].

In the calculation in the present article, we show that a simple harmonic oscillator treated with its radiation multipole moments (not just the electric dipole moment) leaves the spectrum of classical electromagnetic zero-point radiation invariant. Our calculation goes through only the quadrupole order, but there are good reasons to expect the validity of the scattering results to hold for all the multipole moments. Here the crucial aspect is that harmonic oscillator motion in the small-amplitude regime is the same in both relativistic and nonrelativistic physics, whereas the treatment of electromagnetic multipole radiation emission and absorption is fully within the relativistic regime. The spectrum of classical electromagnetic zero-point radiation is invariant under Lorentz transformation, and we should expect that the spectrum will be preserved only by a relativistic classical scattering system. Here we give the first example of a relativistic scattering calculation; indeed it involves equilibrium at the zero-point radiation spectrum.

1.3. Outline of the article
We start out by reviewing the treatment of random radiation within classical physics. We follow this with a review of the exact steady-state behavior of a point harmonic oscillator when located in a bath of random radiation. Then we repeat the calculation in a form involving energy balance for energy absorption and emission, since this is the form which will be used for the quadrupole terms appearing for non-zero amplitude. After demonstrating that the average oscillator amplitude can be obtained correctly from the energy balance at the fundamental frequency of the oscillator, we turn to energy balance at the first harmonic of the oscillator frequency when the oscillator amplitude is non-zero. We first calculate the energy absorbed by the oscillator from the random radiation spectrum during a short time interval \( \tau \), and then we obtain the radiation emission.
We find that energy balance at the first harmonic requires that the radiation per normal mode at the first harmonic should be double the energy per normal mode at the fundamental. This is precisely the statement that the spectrum of random radiation must correspond to the zero-point radiation spectrum which is linear in frequency. Our calculation demonstrates explicitly that classical scattering by this relativistic system does not lead to the Rayleigh-Jeans spectrum but rather to Lorentz-invariant zero-point radiation. We then explain why we expect the detailed balance to extend to all the radiation harmonics of the harmonic-oscillator system. Finally, we comment upon our current understanding of blackbody radiation within classical physics.

2. Set-up for the detailed-balance calculation

2.1. Random classical electromagnetic radiation

We consider a large volume \( V \) containing source-free isotropic random radiation. The random radiation present in the enclosure of volume \( V \) can be given as a sum over plane waves with periodic boundary conditions [12]

\[
E(\mathbf{r}, t) = \sum_{k, \lambda} \tilde{E}(\mathbf{k}, \lambda) \hat{\eta}(\omega) \cos[k \cdot \mathbf{r} - \omega t + \theta(\mathbf{k}, \lambda)]
\]

(1)

\[
B(\mathbf{r}, t) = \sum_{k, \lambda} \hat{k} \times \tilde{E}(\mathbf{k}, \lambda) \hat{\eta}(\omega) \cos[k \cdot \mathbf{r} - \omega t + \theta(\mathbf{k}, \lambda)]
\]

(2)

where the wave vector \( k \) takes the values \( k = \pi l 2\pi/a + \pi m 2\pi/a + \pi n 2\pi/a \) for \( l, m, n \) running over all positive and negative integers, the constant \( a \) is the length of a side of the box for periodic boundary conditions with volume \( V = a^3 \), the random phases \( \theta(\mathbf{k}, \lambda) \) are distributed independently and uniformly [13] over the interval \((0, 2\pi)\), and the amplitude is given by

\[
\hat{\eta}(\omega) = \left( \frac{8\pi \xi(\omega)}{a^3} \right)^{1/2}
\]

(3)

where \( \xi(\omega) \) is the energy per radiation normal mode of frequency \( \omega = ck = c|\mathbf{k}| \). Because our calculations will seem complicated, we will introduce compact notations [14]. The random electric and magnetic fields in equations (1) and (2) will be written compactly as

\[
E(\mathbf{r}, t) = \sum_{\mu, k, \lambda} \tilde{E}(\mathbf{k}, \lambda) \hat{\eta}(\omega) \cos[k \cdot \mathbf{r} - \omega t + \theta(\mathbf{k}, \lambda)]
\]

(4)

\[
B(\mathbf{r}, t) = \sum_{\mu, k, \lambda} \hat{k} \times \tilde{E}(\mathbf{k}, \lambda) \hat{\eta}(\omega) \cos[k \cdot \mathbf{r} - \omega t + \theta(\mathbf{k}, \lambda)]
\]

(5)

where in addition to the sums over \( k \) and \( \lambda \) we include the sum over \( \mu = \pm 1 \),

\[
\sum_{\mu} f(\mu) = f(-1) + f(+1)
\]

(6)

for any function \( f(\mu) \). Here the abbreviated notation writes \( \tilde{E} \) for \( \tilde{E}(\mathbf{k}, \lambda) \hat{\eta}(\omega) \), and

\[
F = \exp[i\mu \mathbf{k} \cdot \mathbf{r}],
\]

(7)

\[
R = \exp[-i\mu \omega t],
\]

(8)

and

\[
A = \exp[i\mu \theta(\mathbf{k}, \lambda)].
\]

(9)

2.2. Dipole oscillator in a harmonic oscillator potential

The charged particle of mass \( m \) and charge \( e \), constrained to move along the \( x \)-axis in a harmonic potential, has the equation of motion (for small amplitude)

\[
m \ddot{x} = -m\omega_0^2 x + m\Gamma \dot{x} + eE_x(0, t)
\]

(10)

where the energy emission and absorption has been evaluated in the point dipole radiation limit, and where \( \Gamma = 2e^2/(3me^2) \). Here the radiation damping term \( m\Gamma \dot{x} \) includes only the electric dipole radiation emission, (and not the damping associated with the electric quadrupole or higher multipole moments), and the driving radiation field \( E(\mathbf{r}, t) \) is evaluated at the center of the oscillator, \( \mathbf{r} = 0 \). This linear equation has been solved many times before [12]. In our compact notation of equation (4), the (real) steady-state solution takes the form

\[
x(t) = \frac{e}{2m} \sum_{\mu, \mathbf{k}, \lambda} e_x \hat{\eta} A \frac{RA}{C}
\]

(11)
where the symbol $C$ stands for
\[ C = -(\mu \omega)^2 + \omega_0^2 - i(\mu \omega)^2. \]  
(12)
The factor $F$ (appearing in equation (7)) does not appear in equation (11) because the driving electric field is evaluated at $r = 0$.

The average value of $x^2$ can be evaluated as
\[
\langle x^2 \rangle = \left( \frac{e}{2m} \right)^2 \sum_{\mu_1, \kappa, h_1} \epsilon_{1, h_1} \frac{R_{A_1}}{C_1} \left( \frac{e}{2m} \sum_{\mu_2, \kappa, \lambda_2} \epsilon_{2, \lambda_2} \frac{R_{A_2}}{C_2} \right)
\]
\[
= \frac{e^2}{2m} \sum_{\mu_1, \kappa, h_1} \sum_{\mu_2, \kappa, \lambda_2} \epsilon_{1, h_1} \epsilon_{2, \lambda_2} \frac{R_{A_1}}{C_1} \frac{R_{A_2}}{C_2} \langle A_1A_2 \rangle
\]  
(13)
Now averaging over the random phases $\theta(k, \lambda)$, we have
\[
\langle A_1A_2 \rangle = \langle \exp[i\mu_1 \theta(k_1, \lambda_1)] \exp[i\mu_2 \theta(k_2, \lambda_2)] \rangle
\]
(14)
where the last expression $\delta_{l(-2)}$ involves a short-hand notation. Now summing over the Kronecker delta terms arising in equation (14), equation (13) becomes
\[
\langle x^2 \rangle = \frac{e^2}{2m} \sum_{\mu_1, \kappa, h_1} \epsilon_{1, h_1} \frac{R_{A_1}}{C_1} \frac{1}{C_2} \langle A_1A_2 \rangle
\]
(15)
since $\hat{\epsilon} (k, \lambda)$ and $\hat{\eta}(\omega)$ are independent of $\mu$, while $R_{A_1}R_{A_2} = 1$, and $C_1C_2 = (\omega - \omega_0^2)^2 + (\Omega\omega)^2$.

Next we assume that the distance $a$ used for the periodic boundary conditions on the radiation is very large so that the sum can be approximated as an integral. Therefore we can write
\[
k_x = \frac{2\pi}{a}, \quad k_y = \frac{2\pi}{a}, \quad k_z = \frac{2\pi}{a},
\]  
(16)
and
\[
\sum_{\lambda} \epsilon_{\lambda} \approx \frac{k^2 - k_x^2}{k^2},
\]  
(17)
and so approximate the sums as
\[
\sum_{\kappa, h} \epsilon_x = \sum_{l,m,n} \sum_{\lambda} \epsilon_x \approx \frac{a}{2\pi} \int d^3k \sum_{\lambda} \epsilon_x = \frac{a^3}{2\pi} \int d^3k \frac{(k_x^2 - k_x^2)}{k^2}.
\]  
(18)
Choosing $x$ as the polar axis, the needed angular integration over $k$ involves
\[
\int d^3 \Omega (1 - \cos^2 \theta) = \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin \theta (1 - \cos^2 \theta) = \frac{8\pi}{3}.
\]  
(19)
Then using equations (3), (12), (18), and (19), we have
\[
\langle x^2 \rangle = \frac{4}{3\pi} \left( \frac{e^2}{2m} \right) \int_0^{2\pi} \frac{d\omega}{c^3} \frac{\omega_0^2 E(\omega)}{a^2} \frac{1}{\pi} \left( \frac{8\pi \mathcal{E}(\omega)}{a^2} \right) \frac{1}{(-\omega^2 + \omega_0^2 + (\Omega\omega)^2)^2}
\]
\[
= \frac{4}{3\pi} \left( \frac{e^2}{2m} \right) \frac{\omega_0^2 E(\omega)}{c^3} \frac{\pi}{2\omega_0} \frac{3mc^3}{2e^2} = \frac{\mathcal{E}(\omega_0)}{m\omega_0^3}
\]
where we have assumed that the integrand is sharply peaked at $\omega_0$ and therefore have substituted $\omega_0$ for every appearance of $\omega$, except where the combination $\omega - \omega_0$ occurs. Also, we have evaluated the integral as
\[
\int_0^{\infty} \frac{d\omega}{(2\omega_0)^2(-\omega + \omega_0)^2 + (\Gamma\omega_0)^2} \approx \int_0^{\infty} \frac{d\omega}{(2\omega_0)^2(-\omega + \omega_0)^2 + (\Gamma\omega_0)^2}
\]
\[
= \frac{\pi}{2\omega_0 \Gamma \omega_0^3} = \frac{\pi}{2\omega_0^3} \left( \frac{3mc^3}{2e^2} \right)
\]  
(20)
using
\[ \int_{-\infty}^{\infty} \frac{dx}{a^2 x^2 + b^2} = \frac{\pi}{ab} \]  
Thus we find
\[ \langle x^2 \rangle = \frac{\mathcal{E}(\omega_0^2)}{m \omega_0^2}. \]  
We notice that the average amplitude of the motion decreases as the mass \( m \) increases for fixed \( \omega_0. \)

### 3. General scattering calculation

The steady-state solution in equation (11) is exact for the linear equation given in equation (10). However, equation (10) is only an approximation to the actual physical situation because it ignores the non-zero amplitude of the oscillator motion. Often one speaks of a 'point' dipole oscillator. The condition for the 'relativistic' compatibility of the motion is that \( |\omega_0 x| \ll c \) so that the charged particle speed is small compared to the speed of light. However, even if this condition is satisfied, equation (10) still ignores the radiation emission and absorption associated with the non-zero excursion of the charged particle. The solution in equation (11) holds for a point dipole oscillator whose excursion is so small that the dipole approximations may be applied consistently and radiation at all the higher harmonics can be ignored. We wish to show that this same expression given in equation (11) holds even when we consider the radiation emission and absorption associated with a finite non-zero excursion of the oscillator where the radiation at the harmonics is treated in detail.

In order to proceed with the more complicated analysis, we first simplify. We drop the radiation damping term (to be treated later) in equation (10), and consider only the driving by the random radiation where we now include the first correction in our approximation, so that
\[ E_x(i\epsilon, t) \approx E_x(0, t) + x[\partial_x E_x(i\epsilon, t)]_{\epsilon' = 0} + \ldots, \]  
giving an equation of motion for the oscillator
\[ \ddot{x} = -\omega_0^2 x + (e/m) \{ E_x(0, t) + x[\partial_x E_x(i\epsilon, t)]_{\epsilon' = 0} + \ldots \}. \]  
We will use this equation to calculate the energy absorbed by the oscillator from the radiation during a short time interval \( \tau \), and then will treat separately the energy emitted by the oscillator into the radiation field.

We write a series expansion for the oscillator displacement
\[ x = x_1 + x_2 + x_3 + \ldots \]  
where the subscript refers to the number of factors of \( \hbar \) which appear in the steady-state expression. Substituting this expansion into equation (24), we can separate the terms depending on the number of powers of \( \hbar \). In first order in \( \hbar \), we have
\[ \ddot{x}_1 = -\omega_0^2 x_1 + (e/m) E_x(0, t), \]  
whereas in second order, we find
\[ \ddot{x}_2 = -\omega_0^2 x_2 + (e/m) x[\partial_x E_x(i\epsilon, t)]_{\epsilon' = 0}. \]  

#### 3.1. Energy balance calculation for the dipole terms

Equation (26) corresponds to the same approximations as in equation (10) except that the radiation damping term has been omitted. Thus equation (26) has no energy-loss mechanism while it is driven by the forcing term. The solution of the differential equation (26) will involve the sum \( x_1 = x_{1c} + x_{1p} \) of a particular solution \( x_{1p} \) of the full equation (26) and a complementary solution \( x_{1c} \) of the homogeneous equation, chosen so as to meet the initial conditions for the displacement and velocity of the oscillator at time \( t = 0 \). We are interested in energy balance, and so we will calculate the average amount of energy delivered to the oscillator by the forcing electromagnetic field during a short time interval \( \tau \), and will then balance this energy against the energy radiated by the oscillator during this same short time interval \( \tau \). Both the amount of energy absorbed and the amount of energy emitted by the oscillator may depend upon the displacement and velocity of the oscillator at time \( t = 0 \). Since we have the solution for the oscillator motion in the dipole approximation, we will take the initial conditions from the full solution given in equation (10). Of course, since we have the full solution for the dipole approximation, we could use this solution directly when calculating the energy absorption and emission in the dipole case. However, since in calculating the quadrupole energy absorption and emission, we will not have the...
exact solution, it seems wise to show the consistency of the basic procedure even though this is redundant in the dipole case.

Provided \( \omega = \omega_0 \), a particular solution of equation (26) corresponds to the steady-state solution obtained from the driving field in equation (1),

\[
x_{1p} = \frac{e}{2m} \sum_{\mu, k, \lambda} \epsilon_x \hbar \frac{RA}{D}
\]

where

\[
D = -(\mu \omega)^2 + \omega_0^2
\]

The summand in equation (28) diverges at \( \omega = \omega_0 \). The complementary solution to equation (26) is a solution to the homogeneous equation, so that

\[
x_{1c} = X \cos \omega_0 t + Y \sin \omega_0 t = \frac{1}{2} \sum_{\mu_3} (X + i\mu_3 Y) R_0
\]

where \( R_0 = \exp[-i\mu_3 \omega_0 t] \). The constants \( X \) and \( Y \) are determined such that at time \( t = 0 \), there is agreement with equation (11), so

\[
x_{1c}(0) = x(0) - x_{1p}(0)
\]

and

\[
\dot{x}_{1c}(0) = \dot{x}(0) - \dot{x}_{1p}(0),
\]

However, the terms \( x(0) \) and \( \dot{x}(0) \) from equation (11) are finite at \( \omega = \omega_0 \) and will not contribute to resonant energy pick-up from the driving electric field. Therefore these terms can be omitted. Thus we need only

\[
x_{1c}(0) \approx -x_{1p}(0) \quad \text{and} \quad \dot{x}_{1c}(0) \approx -\dot{x}_{1p}(0)
\]

from equations (28) and (30), giving

\[
X = -\frac{e}{2m} \sum_{\mu, k, \lambda} \epsilon_x \hbar \frac{A}{D}
\]

and

\[
\omega_0 Y = -\frac{e}{2m} \sum_{\mu, k, \lambda} \epsilon_x \hbar \frac{(-i\mu \omega) A}{\omega_0}. \tag{34}
\]

Then the complementary solution in equation (30) can be written as

\[
x_c = \frac{1}{2} \sum_{\mu_3} \left[ \left( -\frac{e}{2m} \sum_{\mu, k, \lambda} \epsilon_x \hbar \frac{A}{D} \right) + i\mu_3 \left( -\frac{e}{2m} \sum_{\mu, k, \lambda} \epsilon_x \hbar \frac{(-i\mu \omega) A}{\omega_0} \right) \right] R_0
\]

\[
= -\frac{1}{2} \frac{e}{2m} \sum_{\mu, k, \lambda} \epsilon_x \hbar \frac{A}{D} \left( 1 + \frac{\mu_3 \mu \omega}{\omega_0} \right) R_0. \tag{35}
\]

The energy absorbed from the random radiation in the \( x_i \) approximation for the oscillator displacement is

\[
W_i = \int_0^\tau dt \dot{x}_i e E_x(0, t) = \int_0^\tau dt (\ddot{x}_i + \dot{x}_i \dot{e}) e E_x(0, t)
\]

\[
= \int_0^\tau dt \dot{x}_i e E_x(0, t) + \int_0^\tau dt \dot{x}_p e E_x(0, t). \tag{36}
\]

We first consider the last term involving \( \dot{x}_p \) in equation (28) and \( E_x(0, t) \) in equation (4). Taking the average over the random phases, we are dealing with

\[
\left\langle \int_0^\tau dt \dot{x}_p e E_x(0, t) \right\rangle = \left\langle \int_0^\tau dt \frac{e^2}{2m} \sum_{\mu_3, k_3, \lambda_3} \epsilon_{x_3} \hbar_1 \frac{1}{2} \frac{R_1 A_1}{D_1} \left( \sum_{\mu_3, k_3, \lambda_3} \epsilon_{x_3} \hbar_1 \frac{1}{2} \frac{R_1 R_2}{D_2} \langle A_1 A_2 \rangle \right) \right\rangle
\]

\[
= \int_0^\tau \frac{e^2}{2m} \sum_{\mu_3, k_3, \lambda_3} \epsilon_{x_3} \hbar_1 \frac{1}{2} \frac{R_1 R_2}{D_2} \langle A_1 A_2 \rangle \left( \sum_{\mu_3, k_3, \lambda_3} \epsilon_{x_3} \hbar_1 \frac{1}{2} \frac{R_1 R_2}{D_2} \langle A_1 A_2 \rangle \right)
\]

where \( \langle A_1 A_2 \rangle \) is given in equation (14). Now summing over the second set of indices, we have

\[
\left\langle \int_0^\tau dt \dot{x}_p e E_x(0, t) \right\rangle = \int_0^\tau \frac{e^2}{2m} \sum_{\mu_3, k_3, \lambda_3} \epsilon_{x_3} \hbar_1 \frac{1}{2} \frac{R_1 R_2}{D_2} \langle A_1 A_2 \rangle = 0, \tag{37}
\]

since \( \epsilon_{x_3} \) and \( \hbar_1 \) are independent of \( \mu_3 \) while the expression in equation (38) contains exactly one factor of \( \mu_3 = \pm 1 \).
The average over the random phase for the term involving $\chi$ in equation (36) involves equations (35) and (4), giving

$$
\langle W_0 \rangle = \left\langle \int_0^T dt \dot{\chi}, e^{iE_\omega(0, t)} \right\rangle
$$

$$
= \left\langle \int_0^T dt \left\{ -\frac{1}{2} e^2 \sum_{\mu_i, k_0, h_2} \epsilon_{x_1} h_2 \left( 1 + \frac{\mu_0 \mu_1 |\omega_1|}{\omega_0} \right) (-i \mu_0 |\omega_0|) \frac{\partial}{\partial \phi} R_0 \right\} \right\rangle
$$

$$
\times e^{i \left( \sum_{\mu_i, k_0, h_2} \epsilon_{x_1} h_2 \left( R_0 R_2 (A_1 A_2) \right) \right)}
$$

$$
= -\frac{1}{2} \int_0^T dt \frac{e^2}{2m} \sum_{\mu_i, k_0, h_2} \epsilon_{x_1} h_2 \left( 1 + \frac{\mu_0 \mu_1 |\omega_1|}{\omega_0} \right) (-i \mu_0 |\omega_0|) \sum_{\mu_i, k_0, h_2} \epsilon_{x_1} h_2 R_0 R_2 (A_1 A_2)
$$

$$
= -\frac{1}{4} \frac{e^2}{2m} \sum_{\mu_i, k_0, h_2} \epsilon_{x_1} h_2 \left( 1 + \frac{\mu_0 \mu_1 |\omega_1|}{\omega_0} \right) (-i \mu_0 |\omega_0|) \int_0^T dt R_0 (A_1 A_2)
$$

$$
= -\frac{1}{2} \int_0^T dt \frac{e^2}{2m} \sum_{\mu_i, k_0, h_2} \epsilon_{x_1} h_2 \left( 1 + \frac{\mu_0 \mu_1 |\omega_1|}{\omega_0} \right) (-i \mu_0 |\omega_0|) \sum_{\mu_i, k_0, h_2} \epsilon_{x_1} h_2 R_0 R_2 (A_1 A_2)
$$

$$
= -\frac{1}{4} \frac{e^2}{2m} \sum_{\mu_i, k_0, h_2} \epsilon_{x_1} h_2 \left( 1 + \frac{\mu_0 \mu_1 |\omega_1|}{\omega_0} \right) (-i \mu_0 |\omega_0|) \int_0^T dt R_0 (A_1 A_2)
$$

$$
\text{where we have averaged over the random phases as in equation (14) and then summed over } \mu_2, k_2, \text{ and } \lambda_2. \text{ Now we need the integral}
$$

$$
\int_0^T dt R_0 (A_1 A_2) = \int_0^T dt \exp \left[ -i (\mu_0 |\omega_0| - \mu_1 |\omega_1|) t \right]
$$

$$
= 1 - \exp \left[ -i (\mu_0 |\omega_0| - \mu_1 |\omega_1|) T \right].
$$

(41)

Then combining equations (40) and (41), we have

$$
\langle W_0 \rangle = \frac{1}{4} \frac{e^2}{2m} \sum_{\mu_i, k_0, h_2} \epsilon_{x_1} h_2 \left( 1 + \frac{\mu_0 \mu_1 |\omega_1|}{\omega_0} \right) \frac{1}{\omega_0} \exp \left[ -i (\mu_0 |\omega_0| - \mu_1 |\omega_1|) T \right]
$$

$$
\frac{\text{exp}\left[ -i (\mu_0 |\omega_0| - \mu_1 |\omega_1|) T \right]}{(\mu_0 |\omega_0| - \mu_1 |\omega_1|)}.
$$

(42)

Only if $\mu_0 = \mu_1$ will we have a resonant denominator involving $\omega = \omega_0$. Therefore retaining only the resonant terms, we write

$$
\langle W_0 \rangle = \frac{1}{4} \frac{e^2}{2m} \sum_{\mu_i, k_0, h_2} \epsilon_{x_1} h_2 \left( 1 + \frac{\mu_0 \mu_1 |\omega_1|}{\omega_0} \right) \frac{1}{\omega_0} \exp \left[ -i (\mu_0 |\omega_0| - \mu_1 |\omega_1|) T \right]
$$

$$
\frac{\text{exp}\left[ -i (\mu_0 |\omega_0| - \mu_1 |\omega_1|) T \right]}{(\mu_0 |\omega_0| - \mu_1 |\omega_1|)}.
$$

(43)

where we have used

$$
\sum_{\mu_i} \frac{1}{\omega_0 - \omega_1} = \frac{2 \cos(\omega_0 - \omega_1)}{\omega_0 - \omega_1}.
$$

(44)

Now we need to use equations (18) and (19) so as to convert from sums to integrals. Then we have

$$
\langle W_0 \rangle = \frac{1}{2} \frac{e^2}{2m} \left( \frac{a}{2\pi} \right)^3 \int_0^\infty d\omega \frac{8\pi E(\omega)}{3} \left( \frac{8\pi E(\omega)}{3} \right) \frac{1}{\omega_0 - \omega_1}
$$

$$
\approx \frac{1}{2} \frac{e^2}{2m} \left( \frac{a}{2\pi} \right)^3 \frac{1}{c^3} \int_0^\infty d\omega \frac{8\pi E(\omega)}{3} \left( \frac{8\pi E(\omega)}{3} \right) \frac{1}{\omega_0 - \omega_1}
$$

$$
= \frac{1}{2} \frac{e^2}{2m} \left( \frac{a}{2\pi} \right)^3 \frac{1}{c^3} \frac{1}{\omega_0} \frac{8\pi E(\omega_0)}{3} \left( \frac{8\pi E(\omega_0)}{3} \right)
$$

$$
= \frac{2}{3} \frac{e^2}{m} \frac{1}{c^3} \frac{\omega_0^2}{1} \left( \frac{\omega_0}{c^2} \right)^2.
$$

(45)

where we have used [15]

$$
\int_\infty ^\infty dx = \frac{1}{x^2} = \pi \tau.
$$

(47)

The average energy $\langle W_0 \rangle$ radiated by an electric dipole during a time $\tau$ is given by Larmor’s formula

$$
\langle W_0 \rangle = \frac{2}{3} \frac{e^2}{c^3} \frac{\omega_0^2}{(c^2)^2} \tau.
$$

(48)
Comparing the energy pick-up in equation (46) and loss in equation (48), we have
\[
\frac{2}{3} \frac{e^2}{m} \omega_0^4 \left( \frac{\mathcal{E}(\omega_0)}{1} \right) \tau = \frac{2}{3} \frac{e^2}{m} \omega_0^4 (\langle x^2 \rangle \tau).
\]
(49)

or
\[
\langle x^2 \rangle = \frac{\mathcal{E}(\omega_0)}{m \omega_0^2}.
\]
(50)

exactly as found above in equation (22).

3.2. Energy balance for the quadrupole terms

Now we treat the energy balance involving the quadrupole moment when the oscillator behavior is given by the same random motion as was determined using the dipole approximation for radiation. We now consider the terms in equation (24) which involve two factors of \(h\), arising from equation (11) and equation (1),
\[
x = -\omega_0^2 x + \frac{e}{m} \sum_{\mu_1, k_1, \lambda_1} \sum_{\mu_2, k_2, \lambda_2} \epsilon_{x_1} h_{1} e_{x_2} h_{2} (\mu_2 k_{2x}) \frac{R_{1} R_{2}}{C_{1} D_{1+2}} A_{1} A_{2}.
\]
(51)

Once again, we can solve this differential equation in terms of the sum of a particular solution \(x_{2p}\) and a complementary solution \(x_{2c}\) of the homogeneous equation. From the time dependence in \(R_{1}\) and \(R_{2}\), we can obtain the particular solution \(x_{2p}\) as
\[
x_{2p} = \left( \frac{e}{2m} \right)^2 \sum_{\mu_1, k_1, \lambda_1} \sum_{\mu_2, k_2, \lambda_2} \epsilon_{x_1} h_{1} e_{x_2} h_{2} (\mu_2 k_{2x}) \frac{R_{1} R_{2}}{C_{1} D_{1+2}} A_{1} A_{2}.
\]
(52)

where
\[
D_{1+2} = - (\mu_1 \omega_1 + \mu_2 \omega_2)^2 + \omega_0^2.
\]
(53)

The complementary solution involves \(x_{2c}(t) = \mathcal{X} \cos \omega_0 t + \mathcal{Y} \sin \omega_0 t\), when \(x_{2c}(0) = x_{2c}(0) + x_{2p}(0)\) and \(x_{2c}(0) = \dot{x}_{2c}(0) + \dot{x}_{2p}(0)\). Now we are taking the oscillator behavior as unchanged from the radiation dipole behavior so that \(x_{2c}(0) = 0\) and \(\dot{x}_{2c}(0) = 0\), so therefore \(x_{2c}(0) = -x_{2p}\) and \(\dot{x}_{2c}(0) = -\dot{x}_{2p}\). Then noting
\[
x_{2c} = \mathcal{X} \cos \omega_0 t + \mathcal{Y} \sin \omega_0 t = \frac{1}{2} \sum_{\mu_4} (\mathcal{X} + i \mathcal{Y}) R_{0},
\]
(54)

we find
\[
\mathcal{X} = -\left( \frac{e}{2m} \right)^2 \sum_{\mu_4, k_4, \lambda_4} \sum_{\mu_1, k_1, \lambda_1} \sum_{\mu_2, k_2, \lambda_2} \epsilon_{x_1} h_{1} e_{x_2} h_{2} (\mu_2 k_{2x}) \frac{A_{1} A_{2}}{C_{4} D_{1+2}}
\]
(55)

and
\[
\omega_0 \mathcal{Y} = -\left( \frac{e}{2m} \right)^2 \sum_{\mu_4, k_4, \lambda_4} \sum_{\mu_1, k_1, \lambda_1} \sum_{\mu_2, k_2, \lambda_2} \epsilon_{x_1} h_{1} e_{x_2} h_{2} (\mu_2 k_{2x}) \left[ -i (\mu_1 \omega_1 + \mu_2 \omega_2) \right] \frac{A_{1} A_{2}}{C_{4} D_{1+2}}.
\]
(56)

so that
\[
x_{2c} = -\frac{1}{2} \sum_{\mu_4} \left( \frac{e}{2m} \right)^2 \sum_{\mu_1, k_1, \lambda_1} \sum_{\mu_2, k_2, \lambda_2} \epsilon_{x_1} h_{1} e_{x_2} h_{2} (\mu_2 k_{2x}) \left[ 1 + \mu_0 (\mu_1 \omega_1 + \mu_2 \omega_2) \right] \frac{A_{1} A_{2}}{C_{4} D_{1+2}} R_{0}
\]
(57)

and
\[
\dot{x}_{2c} = -\frac{1}{2} \sum_{\mu_4} \left( \frac{e}{2m} \right)^2 \sum_{\mu_1, k_1, \lambda_1} \sum_{\mu_2, k_2, \lambda_2} \epsilon_{x_1} h_{1} e_{x_2} h_{2} (\mu_2 k_{2x}) \left[ 1 + \mu_0 (\mu_1 \omega_1 + \mu_2 \omega_2) \right]
\]
\[
\times \frac{A_{1} A_{2}}{C_{4} D_{1+2}} (-i \mu_0 \omega_0) R_{0}.
\]
(58)
We want to find the average power absorbed by the oscillator in connection with $\dot{x}_2$ during the short time interval $\tau$,

$$\langle W_2 \rangle = \left\langle \int_0^\tau dt \dot{x}_2 e \dot{E}_x (\dot{x}, t) \right\rangle$$

$$= \left\langle \int_0^\tau dt \dot{x}_2 e [E_x (0, t) + x [\partial_x E_x (\dot{x}, t)]_{\dot{t}=0} + ...] \right\rangle$$

$$= \left\langle \int_0^\tau dt \dot{x}_2 e [\partial_x E_x (\dot{x}, t)]_{\dot{t}=0} \right\rangle$$

$$= \left\langle \int_0^\tau dt (\dot{x}_2 + \dot{x}_p) e [\partial_x E_x (\dot{x}, t)]_{\dot{t}=0} \right\rangle$$

$$= \left\langle \int_0^\tau dt (\dot{x}_2 + \dot{x}_p) e [\partial_x E_x (\dot{x}, t)]_{\dot{t}=0} \right\rangle$$

(59)

where we have dropped the term $\left\langle \int_0^\tau dt \dot{x}_2 e [E_x (0, t)] \right\rangle$ because it involves an odd number of random phases $A$ and hence vanishes. Now the term $\left\langle \int_0^\tau dt (\dot{x}_2 + \dot{x}_p) e [\partial_x E_x (\dot{x}, t)]_{\dot{t}=0} \right\rangle$ in equation (59) also vanishes because it involves four sums over $\mu$ but an odd number of factors of $\mu$: one factor of $\mu$ appearing in equation (52), one factor of $\mu$ appearing from the time derivative, and a third factor of $\mu$ appearing from the spatial derivative of the electric field. The calculation is analogous to that for the term $\left\langle \int_0^\tau dt \dot{x}_2 e [E_x (0, t)] \right\rangle$ which vanished in equation (38).

This leaves

$$\langle W_2 \rangle = \left\langle \int_0^\tau dt (\dot{x}_2 + \dot{x}_p) e [\partial_x E_x (\dot{x}, t)]_{\dot{t}=0} \right\rangle$$

(60)

where

$$e [\partial_x E_x (\dot{x}, t)]_{\dot{t}=0} = \epsilon \left( \sum_{\mu \mu \mu \mu} \epsilon_{12} \hbar \frac{R_3 A_3}{C_3} \right) \left[ \sum_{\mu \mu \mu \mu} \epsilon_{24} \frac{\hbar}{2} (i \mu_4 k_{x4}) R_4 A_4 \right]$$

(61)

Introducing equations (38) and (61) into equation (60), we obtain an expression with an odd number of sums over $\mu$ and an odd number of factors of $\mu$,

$$\langle W_2 \rangle = \left\langle \int_0^\tau dt \left( - \frac{1}{2} \right) \sum_{\mu \mu \mu \mu} \epsilon_{12} \hbar \frac{R_3 A_3}{C_3} \left[ \sum_{\mu \mu \mu \mu} \epsilon_{24} \frac{\hbar}{2} (i \mu_4 k_{x4}) R_4 A_4 \right] \right\rangle$$

$$\times \frac{A_1 A_2}{C_1 D_1 + 2} (-i \mu_0 \omega_0) R_0 \left\langle \int_0^\tau dt \left( - \frac{1}{2} \right) \sum_{\mu \mu \mu \mu} \epsilon_{12} \hbar \frac{R_3 A_3}{C_3} \left[ \sum_{\mu \mu \mu \mu} \epsilon_{24} \frac{\hbar}{2} (i \mu_4 k_{x4}) R_4 A_4 \right] \right\rangle$$

(62)

Now averaging over the random phases, we have

$$\langle A_1 A_2 A_3 A_4 \rangle = \delta_{1(2-3)} \delta_{4(2-4)} + \delta_{1(4-3)} \delta_{2(2-4)} + \delta_{1(4-3)} \delta_{2(2-3)}$$

(63)

Resonant behavior will occur only when the random phase $\theta_1$ is matched with $\theta_3$ and the phase $\theta_2$ is matched with $\theta_4$, so that the only term which is resonant and hence of interest is the center term $\delta_{1(2-3)} \delta_{4(2-4)}$ in equation (63). Taking only this center term and summing over the indices labeled 3 and 4, while noting that $\epsilon_{12}$ and $\hbar$ do not depend on $\mu$, while $\mu_2 = 2 - 1$, we have

$$\langle W_2 \rangle = \left( - \frac{1}{4} \right) \sum_{\mu \mu \mu \mu} \epsilon \left( \sum_{\mu \mu \mu \mu} \epsilon_{24} \frac{\hbar}{2} (i \mu_4 k_{x4}) \right)$$

$$\times \left[ 1 + \mu_0 (\mu_1 \omega_1 + \mu_4 \omega_4) \right] \frac{A_1 A_2 A_3 A_4}{C_1 C_3 D_1 + 2} \left( \int_0^\tau dt R_0 R_3 R_4 \right)$$

(64)

Now because the denominator involves $C_1 C_{(-1)} = \left\{ - (\mu_1 \omega_1)^2 + \omega_0^2 - i \Gamma (\mu_1 \omega_1)^3 \right\}$

$$\left\{ - (\mu_1 \omega_1)^2 + \omega_0^2 - i \Gamma (\mu_1 \omega_1)^3 \right\} = (\omega_0^2 + \omega_0^2)^2 + (\Gamma \omega_0^2)^2,$$ while the constant $\Gamma$ is assumed small.
function in the sum over \( k \), is sharply peaked at \( \omega_1 = \omega_0 \). Then we approximate the sum over \( k \) by an integral and set \( \omega_1 = \omega_0 \) everywhere except where the combination \( \omega_1 - \omega_0 \) appears. We extend the lower limit on the integral to \(-\infty\), and so evaluate the sum using equations (18) and (19) as

\[
\sum_{\mu_1, k_2, l_1} \epsilon_1^2 b_1^2 \left[ 1 + \mu_0 \left( \frac{\mu_1 \omega_1 + \mu_2 \omega_2}{\omega_0} \right) \right] \frac{1}{C_i C_i(-1)} \\
\approx \sum_{\mu_1} \left( \frac{a}{2\pi} \right)^3 \int_0^\infty \frac{d\omega_1}{\omega_0^2} \frac{8\pi}{3} \left( \frac{8\pi \mathcal{E} (\omega_0)}{a^2} \right) \left[ 1 + \mu_0 \left( \frac{\mu_1 \omega_0 + \mu_2 \omega_2}{\omega_0} \right) \right] \\
\times \frac{1}{(2\omega_0)^2(-\omega_0 + \omega_0)^2 + (\Gamma_0^2)^2} \\
\approx \sum_{\mu_1} \left( \frac{1}{\pi} \right)^3 \frac{\omega_0^3}{c^3} \frac{8\pi}{3} \left( \pi \mathcal{E} (\omega_0) \right) \left[ 1 + \mu_0 \left( \frac{\mu_1 \omega_0 + \mu_2 \omega_2}{\omega_0} \right) \right] \\
\times \int_{-\infty}^{\infty} dx \frac{1}{(2\omega_0)^2(x^2 + (\Gamma_0^2)^2)} \\
= \sum_{\mu_1} \left( \frac{1}{\pi} \right)^3 \frac{\omega_0^3}{c^3} \frac{8\pi}{3} \left( \pi \mathcal{E} (\omega_0) \right) \left[ 1 + \mu_0 \left( \frac{\mu_1 \omega_0 + \mu_2 \omega_2}{\omega_0} \right) \right].
\]

Now the time integral needed in equation (64) is

\[
\int_0^\tau dt R_0 R_{1(-1)} R_{2(1)} = \int_0^\tau dt \exp\left[ -i\mu_0 \omega_0 t + i\mu_1 \omega_1 t + i\mu_2 \omega_2 t \right] \\
= \exp\left[ -i\mu_0 \omega_0 t + i\mu_1 \omega_1 t + i\mu_2 \omega_2 t \right] - 1,
\]

while the constant \( D_{1,2} \) is

\[
D_{1,2} = [-(\mu_1 \omega_1 + \mu_2 \omega_2)^2 + \omega_0^2].
\]

We will have resonance near \( \omega_2 \approx 2\omega_0 \) only if \( \mu_1 = -\mu_0 = -\mu_2 \). Thus performing the sums over \( \mu_1 \) and \( \mu_2 \), and retaining only the resonant terms, equation (64) becomes

\[
\langle W_2 \rangle \approx \left( \frac{1}{4} \right) \sum_{\mu_0} \epsilon \left( \frac{e}{2m} \right)^3 \sum_{\mu_1, k_2, l_2} \epsilon_1^2 b_1^2 (2k_2^2) \frac{4\mathcal{E}(\omega_0)}{3! \Gamma_c \omega_0^3} \left[ 1 + \mu_0 \left( \frac{\mu_1 \omega_0 + \mu_2 \omega_2}{\omega_0} \right) \right] \\
\times \frac{(-i\mu_0 \omega_0)}{1 - \exp\left[ -i\mu_0 \omega_0 + i\mu_1 \omega_1 + i\mu_2 \omega_2 \right]} \\
\approx \left( \frac{1}{4} \right) \sum_{\mu_0} \epsilon \left( \frac{e}{2m} \right)^3 \sum_{k_2, l_2} \epsilon_1^2 b_1^2 (2k_2^2) \frac{4\mathcal{E}(\omega_0)}{3! \Gamma_c \omega_0^3} \left[ 1 + \mu_0 \left( \frac{\mu_1 \omega_0 + \mu_2 \omega_2}{\omega_0} \right) \right] \\
\times \frac{(-i\mu_0 \omega_0)}{1 - \exp\left[ -i\mu_0 \omega_0 + i\mu_1 \omega_1 + i\mu_2 \omega_2 \right]}.
\]

Next, summing over \( \mu_0 \), we reduce the expression to

\[
\langle W_2 \rangle = \left( \frac{1}{4} \right) \epsilon \left( \frac{e}{2m} \right)^3 \sum_{k_2, l_2} \epsilon_1^2 b_1^2 (2k_2^2) \frac{4\mathcal{E}(\omega_0)}{3! \Gamma_c \omega_0^3} \\
\times \frac{1}{2\omega_0 - \omega_2} \\
\times \frac{2 - 2 \cos(2\omega_0 - \omega_2) \tau}{2\omega_0 - \omega_2}
\]

since

\[
[-(\mu_0 \omega_1 + \mu_0 \omega_2)^2 + \omega_0^2] = -(\omega_0 + \omega_2)^2 + \omega_0^2 = \omega_2(2\omega_0 - \omega_2).
\]

At this point, we convert the sum to an integral using equation (18) and note the angular integration in \( k \)

\[
\int d\Omega (1 - \cos^2 \theta) \cos^2 \theta = \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin \theta (1 - \cos^2 \theta) \cos^2 \theta = \frac{8\pi}{15},
\]
to obtain

$$\langle W_2 \rangle = \left( \frac{1}{4} \right) e \left( \frac{e}{2m} \right)^2 \left\{ \frac{a}{2 \pi} \right\} \int \frac{dk^2}{k^2} \left\{ \frac{8 \pi \xi (\omega_0)}{a^3} \right\}$$

$$\times \frac{4 \xi (\omega_0)}{3(2e^2/3mc^2)} \left( \frac{2 - 2 \cos[(2\omega_0 - \omega_2) \tau]}{(2\omega_0 - \omega_2)^2} \right)$$

$$= \frac{e^2}{8m^2} \left\{ \frac{2}{\pi^2} \int_0^\infty \frac{d\omega_0 \omega_0^2}{\omega_0^2} \frac{8 \pi}{15} \xi (\omega_0) \left[ 1 - \cos[(2\omega_0 - \omega_2) \tau] \right] \right\}.$$

(72)

The expression involves resonance at $\omega_2 = 2\omega_0$, so we approximate all the terms in $\omega_2$ as $2\omega_0$ except those which involve the resonant combination $\omega_2 - 2\omega_0$, and we extend the lower limit of integration to $-\infty$. The needed integral is given in equation (47) so that

$$\langle W_2 \rangle = \frac{e^2}{8m^2} \left\{ \frac{2}{\pi^2} \int_0^\infty \frac{d\omega_0 \omega_0^2}{\omega_0^2} \frac{8 \pi}{15} \xi (\omega_0) \left[ 1 - \cos[(2\omega_0 - \omega_2) \tau] \right] \right\}.$$

(73)

### 3.3. Radiation emitted by the quadrupole moment

In order to obtain the radiation emitted by the quadrupole moment, we consider the radiation emitted by a point charge moving along the x-axis, $r_i(t) = i x_i(t)$. The current density is

$$I_e(r, t) = e \psi \delta^3 [r - i x_i(t)] = e \psi \hat{x}_i(t) \left\{ \delta^3 (r) - x_i(t) \partial_x \delta^3 (r) + \ldots \right\}.$$  (47)

The vector potential due to this current density is

$$A_e (r, t) = \int dt' \int d^3r' \frac{\delta^3 (r - t' - [r - r'/c]/c)}{|r - r'|} I_e (r', t')$$

$$= \int dt' \int d^3r' \frac{\delta^3 (r - t' - [r - r'/c]/c)}{|r - r'|} \{ e \psi \hat{x}_i(t') \partial_x \delta^3 (r') \} + \ldots$$

$$= \int dt' \int d^3r' \frac{\delta^3 (r - t' - [r - r'/c]/c)}{|r - r'|} e \psi \hat{x}_i(t') x_i(t') \partial_x \delta^3 (r') + \ldots$$

$$= \frac{e \psi \hat{x}_i(t - r/c)}{cr} + \int dt' \int d^3r' \partial_x \frac{\delta^3 (r - t' - [r - r'/c]/c)}{c|r - r'|} e \psi \hat{x}_i(t') x_i(t') \delta^3 (r')$$

$$= \frac{e \psi \hat{x}_i(t - r/c)}{cr} - \partial_x \left( \frac{e \psi \hat{x}_i(t - r/c) x_i(t - r/c)}{cr} \right) + \ldots.$$  (75)

The vector potential in equation (75) starts with the dipole term and then includes the quadrupole term. For particle motion along a straight line, there is no magnetic dipole term. Now we are interested in only the radiation from the moving charge $e$, and so we go to the radiation zone. Furthermore, we are concerned with only the quadrupole contribution, which in the radiation zone becomes

$$A_{Q2} (r, t) \approx \frac{e \psi \hat{x}_i(t - r/c) [x_i(r/c)] x_i(t - r/c)}{cr} + \frac{e \psi \hat{x}_i(t - r/c) x_i(t - r/c)}{cr} [x_i(r/c)] \left\{ \frac{3 \hat{x}_i(t - r/c) x_i(t - r/c)}{c^2 r} \right\}.$$  (76)

from the derivative $dr/dx = x/r$. Now we need the magnetic field $B_{Q2} = \nabla \times A_{Q2}$. Then in the radiation zone, we have

$$B_{Q2} \approx -\mathbf{\hat{r}} \times \left\{ e \psi \left[ \frac{\hat{x}_i(t - r/c) [x_i(r/c)] x_i(t - r/c)}{c^2 r} + \frac{3 \hat{x}_i(t - r/c) x_i(t - r/c) [x_i(r/c)]}{c^2 r} \right] \right\}.$$  (77)

Now treating the x-axis as the polar axis, the radiation power emitted per unit solid angle is

$$\frac{dP}{d\Omega} = \frac{c e^2}{4 \pi} \left( \mathbf{\hat{r}} \times \mathbf{\hat{r}} \right)^2 \left[ \frac{\hat{x}_i(t - r/c) [x_i(r/c)] x_i(t - r/c) + 3 \hat{x}_i(t - r/c) x_i(t - r/c) [x_i(r/c)]}{c^2 r} \right]^2$$

$$= \frac{c e^2}{4 \pi} \sin^2 \theta \cos^2 \theta [\hat{x}_i(t - r/c) x_i(t - r/c) + 3 \hat{x}_i(t - r/c) x_i(t - r/c)]^2.$$  (78)
To obtain the total power radiated, we integrate over all solid angles as

$$\int d\Omega \sin^2 \theta \cos^2 \theta = \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin \theta \sin^2 \theta \cos^2 \theta = \frac{8}{15} \pi \tag{79}$$

Then the total power radiated by the quadrupole term is

$$R_0 = \frac{2e^2}{15c^5} \left[ \bar{\chi}_0(t) (t - r/c) \chi_0(t - r/c) + 3 \bar{\chi}_0(t - r/c) \dot{\chi}_0(t - r/c) \right]^2 \tag{80}$$

Inserting the expression for $\chi_0(t)$ given in equation (11), we have

$$\langle P_{\text{Q}} \rangle = \left\{ \frac{2e^2}{15c^5} \left[ \frac{e}{2m} \sum_{\mu_1, k_1, \lambda_1} \epsilon_{x1} b_1 (-i \mu_1 \omega_1)^3 R_{A1} C_1 \left( \frac{e}{2m} \sum_{\mu_2, k_2, \lambda_2} \epsilon_{x2} b_2 \frac{R_{A2}}{C_2} \right) \right] \right\}^2$$

$$+ 3 \left[ \frac{e}{2m} \sum_{\mu_1, k_1, \lambda_1} \epsilon_{x1} b_1 (-i \mu_1 \omega_1)^3 R_{A1} C_1 \left( \frac{e}{2m} \sum_{\mu_2, k_2, \lambda_2} \epsilon_{x2} b_2 \frac{R_{A2}}{C_2} \right) \right] \left[ \frac{e}{2m} \sum_{\mu_3, k_3, \lambda_3} \epsilon_{x3} b_3 (-i \mu_3 \omega_3)^3 R_{A3} C_3 \right]$$

$$\times \left[ \frac{e}{2m} \sum_{\mu_4, k_4, \lambda_4} \epsilon_{x4} b_4 (-i \mu_4 \omega_4)^3 R_{A4} C_4 \right] \tag{81}$$

Upon squaring the square bracket in equation (81), we will be dealing with three terms. The first term from inside the square bracket (corresponding to the square of the first line in equation (81)) is

$$T_1 = \left( \frac{e}{2m} \sum_{\mu_1, k_1, \lambda_1} \epsilon_{x1} b_1 (-i \mu_1 \omega_1)^3 \frac{R_{A1} C_1}{C_2} \right) \left( \frac{e}{2m} \sum_{\mu_2, k_2, \lambda_2} \epsilon_{x2} b_2 \frac{R_{A2}}{C_2} \right)$$

$$\times \left( \frac{e}{2m} \sum_{\mu_3, k_3, \lambda_3} \epsilon_{x3} b_3 (-i \mu_3 \omega_3)^3 \frac{R_{A3} C_3}{C_4} \right) \left( \frac{e}{2m} \sum_{\mu_4, k_4, \lambda_4} \epsilon_{x4} b_4 \frac{R_{A4}}{C_4} \right) \tag{82}$$

$$= \left( \frac{e}{2m} \right)^4 \sum_{\mu_1, k_1, \lambda_1} \sum_{\mu_2, k_2, \lambda_2} \sum_{\mu_3, k_3, \lambda_3} \sum_{\mu_4, k_4, \lambda_4} \epsilon_{x1} \epsilon_{x2} \epsilon_{x3} \epsilon_{x4} b_1 b_2 b_3 b_4$$

$$\times (-i \mu_1 \omega_1)^3 (-i \mu_3 \omega_3)^3 \frac{R_{R1} R_{R3} R_{A1} A_2 A_4}{C_1 C_2 C_3 C_4} \tag{83}$$

When we average over the random phases as in equation (63), this term becomes

$$\langle T_1 \rangle = \left( \frac{e}{2m} \right)^4 \sum_{\mu_1, k_1, \lambda_1} \sum_{\mu_2, k_2, \lambda_2} \sum_{\mu_3, k_3, \lambda_3} \sum_{\mu_4, k_4, \lambda_4} \epsilon_{x1} \epsilon_{x2} \epsilon_{x3} \epsilon_{x4} b_1 b_2 b_3 b_4$$

$$\times (-i \mu_1 \omega_1)^3 (-i \mu_3 \omega_3)^3 \frac{[2(-i \mu_2 \omega_2)^3 + (i \mu_2 \omega_2)^3]}{C_1 C_2 C_3 C_4} \tag{85}$$

since

$$R_{R(-1)} = \exp[-i \mu_1 \omega_1] \exp[i \mu_2 \omega_2] = 1. \tag{86}$$

Now for small charge e, the function $(e^{-i \mu_1 \omega_1})^{-1}$ is sharply peaked at $\omega_1 = \omega_0$. The term involving $(-i \mu_1 \omega_1)^3 (-i \mu_3 \omega_3)^3$ is odd in both $\mu_1$ and $\mu_3$ and therefore will cancel completely. We evaluate the remaining term in the usual way from equations (15)–(22) as

$$\langle T_1 \rangle = \left( \frac{e}{2m} \right)^2 \sum_{\mu_1, k_1, \lambda_1} \sum_{\mu_3, k_3, \lambda_3} \frac{\epsilon_{x1} \epsilon_{x2} \epsilon_{x3} \epsilon_{x4} b_1 b_2 b_3 b_4}{C_1 C_2 C_3 C_4}$$

$$\times \left( \frac{\mathcal{E}(\omega_0) b_n}{m \omega_0^4} \right)^2 = \omega_0^2 \langle x^2 \rangle^2 \tag{87}$$

The cross term in the square bracket in equation (81) is

$$T_2 = 2 \left( \frac{e}{2m} \right)^2 \sum_{\mu_1, k_1, \lambda_1} \sum_{\mu_3, k_3, \lambda_3} \epsilon_{x1} b_1 (-i \mu_1 \omega_1)^3 R_{A1} C_1 \left( \frac{e}{2m} \sum_{\mu_2, k_2, \lambda_2} \epsilon_{x2} b_2 \frac{R_{A2}}{C_2} \right)$$

$$\times \left( \frac{e}{2m} \sum_{\mu_4, k_4, \lambda_4} \epsilon_{x4} b_4 (-i \mu_4 \omega_4)^3 R_{A4} C_4 \right)$$

$$\times \left( \frac{e}{2m} \sum_{\mu_3, k_3, \lambda_3} \epsilon_{x3} b_3 (-i \mu_3 \omega_3)^3 R_{A3} C_3 \right)$$

$$= 6 \left( \frac{e}{2m} \right)^4 \sum_{\mu_1, k_1, \lambda_1} \sum_{\mu_3, k_3, \lambda_3} \sum_{\mu_2, k_2, \lambda_2} \sum_{\mu_4, k_4, \lambda_4} \epsilon_{x1} \epsilon_{x2} \epsilon_{x3} \epsilon_{x4} b_1 b_2 b_3 b_4$$

$$\times (-i \mu_1 \omega_1)^3 (-i \mu_3 \omega_3)^3 (-i \mu_4 \omega_4)^3 \frac{R_{R1} R_{R3} R_{A1} A_2 A_3 A_4}{C_1 C_2 C_3 C_4} \tag{88}$$

$$= 6 \left( \frac{e}{2m} \right)^4 \sum_{\mu_1, k_1, \lambda_1} \sum_{\mu_3, k_3, \lambda_3} \sum_{\mu_2, k_2, \lambda_2} \sum_{\mu_4, k_4, \lambda_4} \epsilon_{x1} \epsilon_{x2} \epsilon_{x3} \epsilon_{x4} b_1 b_2 b_3 b_4$$

$$\times (-i \mu_1 \omega_1)^3 (-i \mu_3 \omega_3)^3 (-i \mu_4 \omega_4)^3 \frac{R_{R1} R_{R3} R_{A1} A_2 A_3 A_4}{C_1 C_2 C_3 C_4} \tag{89}$$

$$= 6 \left( \frac{e}{2m} \right)^4 \sum_{\mu_1, k_1, \lambda_1} \sum_{\mu_3, k_3, \lambda_3} \sum_{\mu_2, k_2, \lambda_2} \sum_{\mu_4, k_4, \lambda_4} \epsilon_{x1} \epsilon_{x2} \epsilon_{x3} \epsilon_{x4} b_1 b_2 b_3 b_4$$

$$\times (-i \mu_1 \omega_1)^3 (-i \mu_3 \omega_3)^3 (-i \mu_4 \omega_4)^3 \frac{R_{R1} R_{R3} R_{A1} A_2 A_3 A_4}{C_1 C_2 C_3 C_4} \tag{90}$$
When we average over the random phases as in equation (63), this term becomes

\[
\langle T2 \rangle = 6 \left( \frac{e}{2m} \right)^4 \sum_{\mu_1, k_1, \mu_2, k_2} \left[ \sum_{\mu_1, k_1, \mu_2, k_2} \epsilon_{x_1}^2 \epsilon_{x_2}^2 \hbar_1^2 \hbar_2^2 (-i\mu_1\omega_1)^3 \right] \times \frac{\omega_1^4}{C_1^{(1)}} \left[ \sum_{\mu_1, k_1, \mu_2, k_2} \epsilon_{x_1}^2 \epsilon_{x_2}^2 \hbar_1^2 \hbar_2^2 (-i\mu_2\omega_2)^3 \right] \frac{\omega_2^2}{C_2^{(2)}}. \tag{91}
\]

Only the last term in equation (91) is even both in the number of sums over \( \mu \) and in the number of factors of \( \mu \), and so does not vanish. We evaluate this term in the usual way as

\[
\langle T2 \rangle = 6 \left( \frac{e}{2m} \right)^2 \sum_{\mu_1, k_1, \mu_2, k_2} \epsilon_{x_1}^2 \hbar_1^2 \frac{\omega_1^4}{C_1^{(1)}} \left[ \left( \frac{e}{2m} \right)^2 \sum_{\mu_1, k_1, \mu_2, k_2} \epsilon_{x_1}^2 \hbar_1^2 \frac{\omega_2^2}{C_2^{(2)}} \right] = 6\omega_0^6 (x^3)^2. \tag{92}
\]

The third term in the square bracket of equation (81) (corresponding to the square of the second line of the equation) is

\[
T3 = 9 \left( \frac{e}{2m} \right)^4 \sum_{\mu_1, k_1, \mu_2, k_2} \epsilon_{x_1}^2 \epsilon_{x_2}^2 \hbar_1^2 \hbar_2^2 (-i\mu_1\omega_1)^3 \frac{R_2 A_2}{C_1} \left[ \left( \frac{e}{2m} \right)^2 \sum_{\mu_1, k_1, \mu_2, k_2} \epsilon_{x_1}^2 \hbar_1^2 (-i\mu_2\omega_2)^3 \frac{R_3 A_3}{C_2} \right] \times \frac{\omega_1^4}{C_1^{(1)}} \left[ \sum_{\mu_1, k_1, \mu_2, k_2} \epsilon_{x_1}^2 \hbar_1^2 (-i\mu_3\omega_3)^3 \frac{R_4 A_4}{C_4} \right]
\]

\[
= 9 \left( \frac{e}{2m} \right)^4 \sum_{\mu_1, k_1, \mu_2, k_2, \mu_3, k_3} \sum_{\mu_1, k_1, \mu_2, k_2, \mu_3, k_3} \epsilon_{x_1} \epsilon_{x_2} \epsilon_{x_3} \epsilon_{x_4} \hbar_1 \hbar_2 \hbar_3 \hbar_4 \times (-i\mu_1\omega_1)^3 (-i\mu_2\omega_2)^3 (-i\mu_3\omega_3)^3 (-i\mu_4\omega_4) \frac{R_1 R_2 R_3 R_4 A_1 A_2 A_3 A_4}{C_1 C_2 C_3 C_4}. \tag{93}
\]

When we average over the random phases as in equation (63), this term becomes

\[
\langle T3 \rangle = 9 \left( \frac{e}{2m} \right)^4 \sum_{\mu_1, k_1, \mu_2, k_2, \mu_3, k_3} \sum_{\mu_1, k_1, \mu_2, k_2, \mu_3, k_3} \epsilon_{x_1} \epsilon_{x_2} \epsilon_{x_3} \epsilon_{x_4} \hbar_1 \hbar_2 \hbar_3 \hbar_4 \times \frac{\omega_1^4}{C_1^{(1)}} \left[ \left( \frac{e}{2m} \right)^2 \sum_{\mu_1, k_1, \mu_2, k_2} \epsilon_{x_1} \epsilon_{x_2} \hbar_1 \hbar_2 \frac{\omega_2^2}{C_2^{(2)}} \right] \frac{\omega_3^2}{C_3^{(3)}} \frac{\omega_4^2}{C_4^{(4)}} = 9\omega_0^6 (x^3)^2. \tag{94}
\]

Only the last term in equation (94) has an even number of terms both in the sums over \( \mu \) and in the number of factors of both \( \mu_1 \) and \( \mu_2 \) and so is nonvanishing. We evaluate this term in the usual way as

\[
\langle T3 \rangle = 9 \left( \frac{e}{2m} \right)^2 \sum_{\mu_1, k_1, \mu_2, k_2} \epsilon_{x_1}^2 \hbar_1^2 \frac{\omega_1^4}{C_1^{(1)}} \left[ \left( \frac{e}{2m} \right)^2 \sum_{\mu_1, k_1, \mu_2, k_2} \epsilon_{x_1}^2 \hbar_1^2 \frac{\omega_2^2}{C_2^{(2)}} \right] \frac{\omega_3^2}{C_3^{(3)}} \frac{\omega_4^2}{C_4^{(4)}} = 9\omega_0^6 (x^3)^2. \tag{95}
\]

Summing the terms in equations (87), (92), and (95), equation (81) becomes

\[
\langle R_0 \rangle = \frac{2 e^2}{15 c^3} \omega_0^6 \frac{\langle \mathcal{E}(\omega_0) \rangle}{m \omega_0^6} (1 + 6 + 9) = \frac{32 e^2}{15 c^3} \omega_0^6 \left( \frac{\langle \mathcal{E}(\omega_0) \rangle}{m \omega_0^6} \right)^2. \tag{96}
\]

Equating the average energy absorbed at frequency \( 2\omega_0 \) in equation (73) with the average energy emitted at \( 2\omega_0 \) from equation (96) during the short time interval \( \tau \), we have

\[
\left( \frac{16}{15} \right) \frac{e^2 (2\omega_0)^2}{m^2 c^3} \mathcal{E}(2\omega_0) \mathcal{E}(\omega_0) \tau = \frac{32 e^2}{15 c^3} \omega_0^6 \left( \frac{\langle \mathcal{E}(\omega_0) \rangle}{m \omega_0^6} \right)^2 \tau. \tag{97}
\]

Energy balance requires

\[
\mathcal{E}(2\omega_0) = 2\mathcal{E}(\omega_0). \tag{98}
\]

This condition states that the spectrum is linear in frequency, which is precisely the character of the Lorentz-invariant spectrum of zero-point radiation.
4. Comments on the physical situation

4.1. Higher multipoles and radiation
The mechanical harmonic-oscillator scattering system which we have considered may be regarded as relativistic provided that \( \omega_0^2(x^2) \ll c^2 \). The usual point-dipole approximation for both the mechanical motion and the interaction with radiation corresponds to the limit \( \langle x^2 \rangle \to 0 \) while \( c^2 \to \infty \) in such a way that the dipole moment squared \( \rho^2 \) is finite \( \rho^2 = c^2 \langle x^2 \rangle = \text{const} \). In the limit \( \langle x^2 \rangle \to 0 \), the mechanical system clearly satisfies the limit of nonrelativistic speed, and all the radiation emission and absorption takes place at the fundamental frequency; there is no radiation interaction at the harmonics since all the higher multipole moments above the dipole moment vanish. In the analysis of the present article, we avoid the limit \( \langle x^2 \rangle \to 0 \). The amplitude of oscillator motion is required to be so small that the speed of the particle is nonrelativistic, but the quadrupole moment is non-vanishing. The quadrupole moment involves two factors of length (not just the one factor of length needed for the dipole moment) and so vanishes in the usual point-dipole limit. Indeed, all the multipole moments above the dipole moment have additional factors of length and so require a non-zero amplitude of motion in order to remain non-zero.

4.2. Role of the constant \( c \)
The mechanical motion of the oscillator involves no factors of the speed of light \( c \) so long as the speed \( v \) of the oscillator is small \( v \ll c \). On the other hand, the radiation energy emitted and the radiation energy absorbed at each harmonic involve the same number of factors of \( c \), so that the condition of radiation balance, harmonic-by-harmonic, gives no role for the ratio \( v/c \). Thus the radiation at the fundamental frequency involves factors of \( c^{-3} \) for both the emitted radiation and the absorbed radiation, as seen in equation (49), while the radiation at the first harmonic involves balancing factors of \( c^{-5} \), as seen in equation (97). Accordingly, as far as an analysis harmonic-by-harmonic is involved, there is no connection between the oscillator speed \( v \) and the radiation speed \( c \). It is only when we sum the series for the radiation emission or absorption that we discover that there is a singularity associated with the radiation interaction when the particle speed \( v \) approaches the speed of light \( c \). [16].

4.3. Adiabatic invariance
In the past, it has been shown that the adiabatic invariance of the point harmonic oscillator fits with the adiabatic invariance of classical electromagnetic zero-point radiation [17]. Thus as the frequency of the oscillator is changed adiabatically, the harmonic oscillator remains in radiation balance with the zero-point radiation. In the earlier work, it was emphasized that the adiabatic invariance in the presence of radiation depended crucially upon the absence of any interaction with radiation harmonics. Based upon the present work, we see that this adiabatic invariance extends beyond the point oscillator out to an oscillator of finite non-zero amplitude. A small oscillator of non-zero excursion has detailed balance with zero-point radiation at higher harmonics. Since the zero-point radiation is invariant under a \( \sigma_{\text{un}} \)-scale transformation [18], the adiabatic invariance under a change of oscillator frequency will continue for a harmonic oscillator of small but non-zero amplitude.

4.4. Limitations on the approximation
The charged particle in a harmonic-oscillator potential which is used in the calculation of this article has a distinct limitation. As described here, the system involves acceleration-based radiation emission, but does not allow any velocity-dependent damping proportional to the random radiation which is present. Thus our description does not allow the treatment of random radiation involving velocity-dependent damping. Now thermal radiation has a preferred inertial frame, and any particle moving relative to this preferred inertial frame will experience a velocity-dependent damping proportional to the thermal radiation which is present [19]. On the other hand, zero-point radiation is Lorentz invariant and so involves no velocity-dependent damping. The mathematical description used in the present article is accurate for zero-point radiation only and does not extend to thermal radiation at non-zero temperature.

5. Discussion of the connection between blackbody radiation and relativity

5.1. Relativistic invariance is strongly restrictive
The basic physical ideas involved in the present calculation are not well known and deserve a broader audience. Many physicists are unaware of the restrictive nature of the requirement that a system should be relativistic. The first three conservation laws of dynamics, associated with symmetries under space translations, time translations, and rotations, involve conservation of linear momentum, energy, and angular momentum; these conservation laws appear in both nonrelativistic and relativistic theories. However, the fourth conservation law associated with Galilean symmetry or relativistic symmetry is quite different between nonrelativistic and
relativistic systems [20]. The conservation law for Galilean invariance of nonrelativistic dynamics essentially repeats the information of the law of conservation of linear momentum and so gives no restrictions on allowed nonrelativistic systems. On the other hand, the conservation law associated with Lorentz invariance has profound limitations on which systems are relativistic. For example, the no-interaction theorem of Currie, Jordan, and Sudarshan [21] states that any relativistic interaction between particles beyond point interactions requires the introduction of a field theory, and relativistic field theories are restricted still further.

5.2. Simplest equilibrium radiation spectra

Within classical physics, there are two spectra for electromagnetic radiation which take particularly simple forms. One of these is the Rayleigh-Jeans spectrum which associates the same energy (taken as $k_B T$) with every radiation normal mode. This spectrum involves one parameter, the energy $k_B T$ per normal mode, and does not distinguish any length or any frequency. This spectrum is associated with nonrelativistic physics and in particular with the equipartition theorem of nonrelativistic statistical mechanics.

The second simple radiation spectrum is that of classical electromagnetic zero-point radiation which has an energy linear in the frequency (energy taken as $\hbar \omega / 2$) with every radiation normal mode. Again, this spectrum involves one parameter $\hbar$ with units corresponding to an angular momentum or to an energy $\times$ time, and does not distinguish any length or frequency. The spectrum is associated with relativity; zero-point radiation is the unique (up to a multiplicative constant) Lorentz-invariant spectrum of random classical radiation, and it takes the same form in every inertial frame.

It is striking that the Planck blackbody radiation spectrum including zero-point radiation is the smoothest possible interpolation between the Rayleigh-Jeans spectrum at low frequencies and the zero-point radiation spectrum at high frequencies [22].

5.3. Scatterers for classical radiation equilibrium

Electromagnetic radiation can not bring itself to equilibrium. Rather, there must be some mechanical scattering system, some ‘black’ particle (a particle which scatters radiation toward the equilibrium spectrum), which enforces the radiation equilibrium. Since radiation equilibrium is determined by a mechanical system, we certainly expect that the nature of the scattering system will influence the radiation equilibrium spectrum. Indeed for classical mechanical systems, nonrelativistic scatterers leave the Rayleigh-Jeans spectrum unchanged, while relativistic scatterers leave the zero-point spectrum unchanged.

The simplest scattering system which allows a transition corresponding to that found for the Planck spectrum (including zero-point radiation) between the Rayleigh-Jeans spectrum at low frequency and the zero-point spectrum at high frequency is that of a charged particle of charge $e$ and mass $m$ in a Coulomb potential where the ratio $m c^2 / k_B T$ provides the mechanical transition parameter matching the radiation transition parameter $\hbar \omega / k_B T$. We notice that for the Coulomb potential, high mechanical mass $m$ is associated with high frequency $\omega$. The Coulomb potential is one of the few mechanical systems where large mass $m$ is associated with high frequency, and small mass $m$ is associated with low frequency.

5.4. Harmonic oscillator scatterers

Although hydrogen-like scatterers involving a Coulomb potential are the relativistic systems which are expected to scatter classical electromagnetic radiation toward the Planck spectrum with zero-point radiation, it seems exceedingly difficult to work with the Coulomb potential as a scatterer. On the other hand, it is vastly easier to treat a charged harmonic oscillator as a scattering system for electromagnetic radiation. Indeed, at the end of the 19th century, Planck calculated the behavior of a charged harmonic oscillator taken in the point-limit when bathed in random classical electromagnetic radiation. Although Planck had initially hoped that the oscillators would serve as ‘black’ particles and determine the spectrum of electromagnetic radiation, it became clear that small (point-limit) harmonic oscillator systems simply acquired an energy which matched the energy of the radiation normal modes at the oscillator frequency. A point harmonic dipole oscillator did not determine the equilibrium radiation spectrum. Acting as a scatterer, the oscillator may change the angular distribution of the radiation, but the point oscillator did not change the frequency spectrum of the random electromagnetic radiation. Planck subsequently turned to statistical mechanics for the harmonic oscillator in an attempt to understand equilibrium for the electromagnetic radiation. And the application of the nonrelativistic equipartition theorem to an oscillator scatterer is still used in physics textbooks as a way of obtaining the Rayleigh-Jeans radiation spectrum [8].

5.5. Extensions for harmonic oscillator scatterers

We wish to avoid the use of statistical mechanics in our exploration of radiation equilibrium, but we would like to use the calculational simplicity of harmonic oscillator systems. Now point oscillator systems are
unsatisfactory because they interact with random radiation at a single frequency. However, there are two natural extensions of the point-limit harmonic oscillator system which will bring the system into contact with the full radiation spectrum. One involves the introduction of a small nonlinear term in the harmonic oscillator potential so as to introduce higher harmonics in the oscillator mechanical motion. The second possible modification is the consideration of all the radiation harmonics associated with a charged particle in a purely harmonic potential but with an amplitude of finite, non-zero excursion.

5.5.1. Nonlinear-oscillator scatterer

The radiation scattering due to an oscillator with a small nonlinear term was considered in 1976 [10]. The introduction of a small nonlinear term in the harmonic-oscillator potential leads to mechanical motion which involves harmonics of the basic oscillator motion so that the particle displacement becomes

\[ x(t) = a_1 \cos(\omega_0 t + \phi_1) + a_2 \cos(2\omega_0 t + \phi_2) + \ldots \]

The ratio \( \frac{a_n}{a_1} \) of the amplitudes \( a_n \) compared to the initial harmonic-oscillator amplitude \( a_1 \) depends upon the arbitrary strength of the nonlinear term in the potential. Although radiation emission and absorption are still treated in the dipole approximation, the presence of the harmonics in the mechanical motion brings the oscillator into contact with not only the radiation at the fundamental oscillator frequency \( \omega_0 \) but also the radiation at the multiples \( n\omega_0 \) of the fundamental frequency \( \omega_0 \). Although the ratios \( \frac{a_n}{a_1} \) may be arbitrary, it turns out that the radiation spectrum which has the same energy per normal mode at every frequency (the Rayleigh-Jeans spectrum) remains unchanged by scattering from this system. Indeed, there have been several calculations, going back to van Vleck’s work of 1924 showing that nonrelativistic nonlinear mechanical scattering systems treated in the dipole limit for their radiation interaction leave the Rayleigh-Jeans spectrum invariant [11].

5.5.2. Finite-amplitude harmonic oscillator

The second possible extension of the small harmonic oscillator scatterer is what is treated in the calculations of the present article. We consider not a change in the harmonic oscillator mechanical potential but rather a calculation of the radiation emitted and absorbed at the radiation harmonics of the fundamental oscillator frequency. The oscillator potential remains unchanged as a harmonic oscillator potential and the free oscillator motion \( x(t) = a_1 \cos(\omega_0 t + \phi_1) \) remains unchanged. However, relativistic classical electrodynamics involves radiation at all the harmonics for any finite non-zero amplitude of oscillation. Thus the relativistic aspects enter not through the mechanical oscillator motion but through the radiation theory. Our analysis calculates the radiation energy balance for the second harmonic corresponding to quadrupole radiation and shows that the spectrum which remains unchanged is the zero-point radiation spectrum. This is the first classical scattering calculation showing explicitly that a relativistic scattering system indeed leaves the relativistically-invariant zero-point radiation spectrum unchanged.

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