Polarization Profiles of Scattered Emission Lines.

I. General Formalism for Optically Thin Rayleigh Scattering

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**Abstract**

A general theoretical framework is developed for interpreting spectropolarimetric observations of optically thin emission line scattering from small dust particles. Spatially integrated and spatially resolved line profiles of both scattered intensity and polarization are calculated analytically from a variety of simple kinematic models. These calculations will provide a foundation for further studies of emission line scattering from dust and electrons in such diverse astrophysical environments as Herbig-Haro objects, symbiotic stars, starburst galaxies and active galactic nuclei.

Subject headings: ISM: dust, extinction—ISM: jets and outflows—ISM: reflection nebulae—polarization

1 Introduction

The Doppler shifts of optical emission lines which have been scattered by surrounding dust and electrons can provide useful information about the kinematics, geometry, and physical conditions of astrophysical flows. In principle, the scatterers can provide views of the line-emitting gas from different directions, allowing the 3-dimensional velocity of the emitting gas to be determined and revealing sources which are obscured from direct view. Unfortunately, the interpretation of these scattered emission lines is not straightforward since, in general, the geometry of the scattering and the velocity of the scatterers is unknown. If spectropolarimetric observations are available, then determination of the relative orientation and velocity of the source and the scatterers becomes more plausible, since the scattered light will be partially polarized to a degree dependent on the angle of scattering and on the details of the scattering process.

In this paper, a variety of very simple models of the scattering geometry and kinematics are investigated. Polarimetric line profiles are calculated (both spatially integrated and spatially resolved) for the cases of Rayleigh scattering by small dust particles, where the scattering phase function has a particularly simple form and it is feasible to derive analytic expressions for the scattered line profiles.

Previous work on the problem of the scattering of emission lines by dust or electrons has tended to involve detailed numerical radiative transfer modelling, often using Monte Carlo codes. Examples of the use of such techniques can be found applied to, e.g., electron scattering in Wolf-Rayet winds (Hillier 1991) or scattering in circumstellar dust shells (Lefèvre 1992; Meaburn, Walsh & Wolstencroft 1992). While such approaches have obvious merits, they are limited in their applicability, often needing to be specially tailored for each object, and it is difficult to draw general conclusions from them. A simpler approach is taken by Wood, Brown & Fox (1993), who model electron scattering in the circumstellar disks of Be stars, but, even here, numerical techniques are employed. In the present paper, numerical calculations are eschewed for the most part and scattered line profiles of intensity and polarization are calculated analytically. The price that must be paid in order to make this approach tractable is to restrict one’s attention to optically thin Rayleigh scattering of narrow emission lines which originate from a point source. The chief advantage of this approach is that it makes it possible to investigate a wide range of models for the geometry and kinematics of the scatterers and to develop a broad understanding of the ways in which these affect the line profiles. This understanding can then provide a firm foundation for further investigation into the effects of different phase functions (dust), thermal broadening (electrons), an extended source, absorption, multiple scattering and other complications. This development will be carried out in further papers of this series and detailed applications of the resultant techniques will be made to particular cases such as upstream dust scattering in Herbig-Haro objects (Noriega-Crespo, Calvet & Böhm 1991), dust scattering in the jet of the symbiotic star R Aquarii (Solf 1992), dust scattering in the superwinds of starburst galaxies (Scarrott, Eaton & Axon 1991) and electron scattering in active galactic nuclei (Miller, Goodrich & Matthews 1991). Some preliminary results in these areas can be found in Henney (1992).

2 Rayleigh Scattering

Arbitrarily polarized light is fully described by means of the four-component Stokes vector (Stokes 1852) $\mathbf{I} = [I, Q, U, V]^T$, which can be functionally defined in terms of measurements of the light intensity as follows: $I$ is the total intensity of the light; $Q$ is the difference between the intensities after the light has passed through a linear dichroic plate with its transmitting axis aligned respectively perpendicular and parallel to a reference direction; $U$ is the difference between the intensities with the same plate aligned at ±45° to the reference direction, and $V$ is the difference between the intensities after the light has passed through a right and left handed circular dichroic plate respectively. Unpolarized light has $Q = U = V = 0$ and fully polarized light has $I' = Q^2 + U^2 + V^2$. Partially polarized light has a degree of polarization

$$p = (Q^2 + U^2 + V^2)^{1/2}/I$$

(1)
and an angle of polarization $\chi$ (with respect to the reference direction in the definition of $Q$ and $U$ above) given by

$$\tan 2\chi = U/Q.$$  \hspace{1cm} (2)

Note that, since one must choose a reference direction corresponding to $\chi = 0$, there is a certain arbitrariness in the specification of $Q$ and $U$. In astronomy this reference direction is conventionally taken to be North (or sometimes, the galactic plane). The effect on the Stokes vector of a rotation of the reference direction by an angle $\phi$ is to multiply it by the rotation matrix

$$M(\phi) = \begin{bmatrix} 1 & 0 & 0 & 0 \\
0 & \cos 2\phi & \sin 2\phi & 0 \\
0 & -\sin 2\phi & \cos 2\phi & 0 \\
0 & 0 & 0 & 1 \end{bmatrix}.$$  \hspace{1cm} (3)

In the following discussion, it is imagined that light is emitted from a source at the origin and scattered by a single scatterer at $[x, y, z]$ toward an observer who is in the far field and is looking down the negative $z$-axis (see Figure 1). When the light is scattered, the components of the scattered Stokes vector are linearly related to those of the incident Stokes vector. Hence, the scattering cross-section can be written as a matrix. One way of representing the differential cross-section per solid angle $\Omega$ is then

$$\sigma_\Omega = \frac{3\sigma_0}{16\pi}K,$$  \hspace{1cm} (4)

where $\sigma_0$ is the mean scattering cross-section and the matrix $K$ (the scattering kernel; Pomraning 1973) contains the direction-dependent scattering information. The matrix $\sigma_\Omega$ is normalized so that

$$\int_{4\pi} \sigma_\Omega I d\Omega = \sigma_0 I$$  \hspace{1cm} (5)

for any Stokes vector $I$. If $I_0$ is the Stokes vector of the unscattered light at the source (in the direction of the scatterer), then, by considering the solid angle at the source subtended by the scatterer, the scattered Stokes vector will be

$$I_S = \sigma_\Omega I_0/R^2,$$  \hspace{1cm} (6)

where $R^2 = x^2 + y^2 + z^2$. If the scatterers are much smaller than the wavelength of the light, then, so long as both are specified with respect to the scattering plane, the Stokes vector of the scattered light is related to that of the incident light via the Rayleigh scattering matrix (Chan-
Hence the scattered light is always partially polarized for scattering angles of 90°. In this section, three simple classes of problems are considered as examples of Rayleigh scattering: first, that of a stationary dust cloud scattering light from a moving unpolarized source at its center; second, that of an outflowing dusty wind; and third, that of a free-falling dusty inflow, the last two both scattering light from a stationary unpolarized central source. In the first instance, the scattering cloud/wind/infall is assumed to be spherically symmetric and the light source is assumed to be a point, although the first of these assumptions is subsequently relaxed for the first two cases.

Considering the spatially integrated light first, the three models are equivalent and it is evident on symmetry grounds that the scattered light will have no net polarization. This can be verified by taking the scattered intensity from a single scatterer (eq. [3]) and integrating it over all the scatterers in the cloud. If the number density of scatterers is \( n(R) \) and the cloud radius is \( R_c \) then this yields

\[
I_s = \frac{3\sigma_0 I_0}{16\pi} \int_{R_c} n(R) \frac{X(\Theta, \psi) dV = \tau I_0}{R^2} = \tau I_0
\]

where \( \tau = \sigma_0 \int_0^{R_c} n dR \) is the scattering optical depth to the source. Of course, since \( \tau \ll 1 \), the scattered light makes a negligible contribution to the total intensity unless the direct light from the source is obscured.

### 3.1 Spatially Integrated Scattered Line Profiles

#### 3.1.1 Moving Source in a Stationary Scattering Cloud

If it is now supposed that the source of light is moving with a velocity \( \mathbf{u}_0 \) with respect to the (stationary) dust cloud and that the light it emits is monochromatic (of frequency \( \nu_0 \)), then the scattered light will be Doppler shifted by \( \Delta \nu = (\nu_0/c) \mathbf{R} \cdot \mathbf{u}_0 \), where \( \mathbf{R} \) is the unit vector in the direction of the scatterer from the source. One can define a dimensionless velocity shift (frequency shift) of the scattered light as \( v = -\mathbf{R} \cdot \mathbf{u}_0/\nu_0 \equiv -\cos \gamma \). This is illustrated in Figure 2 in which the \( x \)-axis is taken to be along the projection of \( \mathbf{u}_0 \) onto the plane of the sky and the angle between \( \mathbf{u}_0 \) and the \( x \)-axis is denoted by \( \alpha \). It is apparent that all scatterers that lie on the surface of a cone of opening half-angle \( \gamma \) “see” the same Doppler shift \( v = -\cos \gamma \) in the light from the source. This will be called the “isovelocity cone” and \( \xi \) will signify the angle.

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1 In keeping with established convention, red shifts will be treated as positive.
around this cone, measured from a direction parallel to the $y$-axis (see Figure 2).

The spatially integrated scattered line shape can best be determined by considering the relationship between the two angular co-ordinate frames $\{\gamma, \xi\}$ and $\{\Theta, \psi\}$. Equating Cartesian components of a unit vector in the two co-ordinate frames gives

$$
\begin{align*}
\mathbf{r} &= \sin \Theta \sin \psi = \sin \gamma \sin \xi \sin \alpha + \cos \gamma \cos \alpha \quad (14a) \\
\mathbf{r} &= \sin \Theta \cos \psi = \sin \gamma \cos \xi \quad (14b) \\
\mathbf{z} &= \cos \Theta = \sin \gamma \sin \xi \cos \alpha - \cos \gamma \sin \alpha \quad (14c)
\end{align*}
$$

so that, using equation (10), the scattering phase function becomes

$$
X(\gamma, \xi) = \frac{3}{4} \left[ \begin{array}{c} a_0 + a_1 \sin \xi + a_2 \sin^2 \xi \\ b_0 + b_1 \sin \xi + b_2 \sin^2 \xi + b_3 \cos^2 \xi \\ c_1 \sin \xi + c_4 \sin \xi \cos \xi \end{array} \right] (15)
$$

where

$$
\begin{align*}
a_0 &= \sin^2 \alpha \cos^2 \gamma + 1; & a_1 &= -\frac{1}{2} \sin 2\alpha \sin 2\gamma; & a_2 &= \cos^2 \alpha \sin^2 \gamma; \\
b_0 &= \cos^2 \alpha \cos^2 \gamma; & b_1 &= \frac{1}{2} \sin 2\alpha \sin 2\gamma; & b_2 &= \sin^2 \alpha \sin^2 \gamma; \\
b_3 &= -\sin^2 \gamma; & c_1 &= -\cos \alpha \sin 2\gamma; & c_4 &= -2 \sin \alpha \sin^2 \gamma.
\end{align*}
$$

To find the scattered intensity in a velocity range $v_1$ to $v_2$ it is necessary to integrate equation (9) over all the dust between the two isovelocity cones corresponding to $v_1$ and $v_2$. Hence,

$$
I(v_1 \rightarrow v_2) = \frac{\tau I_0}{4\pi} \int_{\cos^{-1} v_2}^{\cos^{-1} v_1} \int_{\xi = 0}^{2\pi} X(\gamma, \xi) \sin \gamma \, d\gamma \, d\xi.
$$

(16)

Since $d\tau = \sin \gamma \, d\gamma$, the specific intensity (per unit velocity range) is then

$$
I_v = \left| \frac{dI}{dv} \right| = \frac{\tau I_0}{4\pi} \int_0^{2\pi} X(v, \xi) \, d\xi = \frac{3}{8} \tau I_0 \left[ \begin{array}{c} a_0 + \frac{1}{2}a_2 \\ b_0 + \frac{1}{2}b_2 + \frac{1}{4}b_3 \\ 0 \end{array} \right],
$$

(17)

so that

$$
\begin{align*}
I_v &= \frac{3}{16} \tau I_0 \left\{ 2 \left(1 + v^2\right) + \left(1 - 3v^2\right) \cos^2 \alpha \right\}, \\
Q_v &= \frac{3}{16} \tau I_0 \left(1 - 3v^2\right) \cos^2 \alpha, \\
U_v &= 0.
\end{align*}
$$

(18)

The polarization of the scattered light is then

$$
p_v = \frac{1 - 3v^2 \cos^2 \alpha}{2 \left(1 + v^2\right) + \left(1 - 3v^2\right) \cos^2 \alpha},
$$

(19)
Figure 3: Line profiles of spatially integrated light, Rayleigh scattered by a stationary spherical dust cloud that surrounds a moving source, for various inclinations $\alpha$ of the source velocity to the plane of the sky. Solid line is specific intensity and dashed line is degree of polarization.

while the angle of polarization $\chi$ is zero (perpendicular to the projected source velocity) when $Q_v$ is positive, or 90° (parallel to the projected source velocity) when $Q_v$ is negative. These quantities are plotted in Figure 3 for various values of $\alpha$.

It should be noted that equation (19) is more general than equations (18a) and (18b) and applies to any distribution of scatterers that has cylindrical symmetry about the direction of the source velocity. This is because the number density $n(R, \gamma, \xi)$ cancels out when taking the ratio of $Q$ and $I$. (Cylindrical symmetry is still required, since it is assumed when performing the $\xi$ integral in eq. [17].)

To illustrate how the intensity profile is modified if the scattering cloud is not spherical, the case of prolate and oblate ellipsoidal clouds will be considered.

Ellipsoidal Scattering Clouds: If the scattering cloud is non-spherical, then the chance of a source photon being scattered is dependent on its initial direction. For a prolate ellipsoid of eccentricity $e$, with symmetry axis aligned with the source velocity, this can be translated into a direction dependent optical depth of

$$
\frac{\tau(\gamma)}{\tau_\star} = \left( \frac{1 - e^2 \sin^2 \alpha}{1 - e^2 \cos^2 \gamma} \right)^{1/2},
$$

where $\tau_\star$ is the optical depth along the line of sight from the observer (corresponding to $\gamma = \pi/2 - \alpha$). Hence, the intensity of the scattered light is given by

$$
I_v = \frac{3}{8} \tau_\star I_0 \left( \frac{1 - e^2 \sin^2 \alpha}{1 - e^2 v^2} \right)^{1/2} \times \left\{ 2 \left(1 + v^2 \right) + (1 - 3 v^2) \cos^2 \alpha \right\}.
$$

A similar treatment for an oblate ellipsoid yields

$$
I_v = \frac{3}{8} \tau_\star I_0 \left( \frac{1 - e^2 \sin^2 \alpha}{1 - e^2 v^2} \right)^{1/2} \times \left\{ 2 \left(1 + v^2 \right) + (1 - 3 v^2) \cos^2 \alpha \right\}.
$$

The resultant line shapes are plotted for various values of $e$ and $\alpha$ in Figures 4 and 5. The polarization is not shown since it is the same as for a spherical cloud. It is apparent that prolate ellipsoids produce line shapes in which most of the intensity is concentrated toward $v = \pm 1$ whereas oblate ellipsoids produce line shapes strongly peaked at $v = 0$. The reason for this can be seen if one considers the distributions of scatterers in the two cases: in prolate clouds, most of the scatterers are concentrated along the direction of the source velocity, either “upstream” or “downstream”, while in oblate clouds they are concentrated toward the plane perpendicular to the source velocity and hence see little Doppler shift. It will be noticed

\[\text{Figure 4: Same as Figure 3 but for prolate spheroidal scattering clouds. Different line types correspond to different eccentricities, as indicated in the key.}\]
that the integrated intensity (with respect to $v$) is not the same in each case. This is because the intensities are normalized with respect to $\tau I_0$, where $\tau$ is the scattering optical depth to the source along the line of sight. Generally, $\tau$ is not equal to the angle-averaged optical depth (weighted by the phase function) $\bar{\tau}$, although it is this latter that determines the integrated scattered intensity. For example, a prolate ellipsoid at $\alpha = 0$ or an oblate ellipsoid at $\alpha = \pi/2$ would each have $\tau < \bar{\tau}$.

### 3.1.2 Constant Velocity Scattering Wind

If the scattering wind is supposed to flow radially outward at constant speed $u_w$, then the Doppler frequency shift in the scattered light will be $\Delta \nu = (v_0/c)u_w(1 + \hat{R} \cdot \hat{z})$, where $\hat{z}$ is the unit vector along the $z$-axis (from the source toward the observer). In the same manner as for the moving source model, a dimensionless velocity shift $v = 1 + \hat{R} \cdot \hat{z}$ is introduced which again defines isovelocity cones. In this model, however, the opening half-angle of the cone is the scattering angle $\Theta$ and this is related to the velocity shift by $\cos \Theta = v − 1$. Determination of the scattered line profile is hence far simpler than in the preceding section, giving

$$I_v = \frac{\tau I_0}{4\pi} \left[ \frac{v^2 - 2v + 2}{8\tau I_0} \right]. \quad (23)$$

The scattered line is unpolarized because the isovelocity cones project onto the plane of the sky as circles. This line shape is illustrated in Figure 5 and it can be seen that it is identical to the moving source model with $\alpha = \pi/2$.

**Outflowing Bicone:** The outflow axis of the wind is taken to make an angle $\alpha$ with the plane of the sky and the wind density is assumed to be independent of direction within the two cones of opening half-angle $\delta_c$ about this axis, and zero outside the cones. These scattering cones will intersect a given isovelocity cone along 0, 2 or 4 radii with position angles $(\Psi_1 \ldots \Psi_4)$ given by

$$\sin \Psi_1 = \sin \Psi_2 = \frac{\cos \delta_c - \cos \theta \sin \alpha}{\sin \theta \cos \alpha},$$

$$\sin \Psi_3 = \sin \Psi_4 = \frac{-\cos \delta_c - \cos \theta \sin \alpha}{\sin \theta \cos \alpha}, \quad (24)$$

where $\theta = \cos^{-1} |v - 1|$. The scattered Stokes vector is therefore given by

$$I_v = \frac{\tau I_0}{4\pi} \left\{ \int_{\Psi_1}^{\Psi_2} X(v, \Psi) d\Psi + \int_{\Psi_3}^{\Psi_4} X(v, \Psi) d\Psi \right\}, \quad (25)$$

which can be integrated to give the following:

$$I_v = \frac{3}{16\pi} \tau I_0 (v^2 - 2v + 1) F_1 \quad (26a)$$

$$Q_v = \frac{3}{16\pi} \tau I_0 (2v - v^2) F_2 \quad (26b)$$

$$U_v = 0, \quad (26c)$$

**Figure 5:** Same as Figure 4 but for oblate spheroidal scattering clouds.

**Figure 6:** Spatially integrated line profile from a constant-velocity spherically symmetric wind of scatterers.
where

\[ F_1 = \begin{cases} 
2\pi & \text{if } \alpha - \theta \geq \pi/2 - \delta \\
2\pi - 2(\Psi_1 - \Psi_3) & \text{if } \theta - \alpha \geq \pi/2 - \delta \\
0 & \{ \begin{array}{l} \text{if } \alpha + \theta > \pi/2 + \delta \\ \text{or } \alpha + \theta < \pi/2 - \delta \end{array} \} \\
\pi - 2\Psi_1 & \text{otherwise} 
\end{cases} \]  \quad (27a)

\[ F_2 = \begin{cases} 
0 & \text{if } \alpha - \theta \geq \pi/2 - \delta \\
\sin 2\Psi_1 - \sin 2\Psi_3 & \text{if } \theta - \alpha \geq \pi/2 - \delta \\
0 & \{ \begin{array}{l} \text{if } \alpha + \theta > \pi/2 + \delta \\ \text{or } \alpha + \theta < \pi/2 - \delta \end{array} \} \\
\sin 2\Psi_1 & \text{otherwise} 
\end{cases} \]  \quad (27b)

In Figures 7 and 8, examples of these line shapes are given for narrow (\(\delta_c = 0.2\) radians) and wide (\(\delta_c = 1.0\) radians) cones.

**Outflowing Disk:** If the wind is confined to a thin equatorial disk, of angular half-thickness \(\delta_d\) then the scattered line profiles can be calculated by subtracting the result for a biconical wind from that of a spherical one so that \(\delta_d = \pi/2 - \delta_c\). Neglecting terms higher than quadratic in \(\delta_d\) (thin disk) then gives

\[
I_v = \frac{3}{8} \tau \delta_d I_0 \frac{v^2 - 2v + 2}{(2v - \sin^2 \alpha)^{1/2}} 
\]

\[
Q_v = -\frac{3}{8} \tau \delta_d I_0 \frac{(1 + \sin^2 \alpha)(2v - v^2 - 2\sin^2 \alpha)}{\cos^2 \alpha (2v - \sin^2 \alpha)^{1/2}}, \quad (28a)
\]

\[
Q_v = -\frac{3}{8} \tau \delta_d I_0 \frac{(1 + \sin^2 \alpha)(2v - v^2 - 2\sin^2 \alpha)}{\cos^2 \alpha (2v - \sin^2 \alpha)^{1/2}}, \quad (28b)
\]

where \(\alpha\) is now the angle between the normal to the disk and the plane of the sky. Examples of these line shapes are given in Figure 9.

**3.1.3 Free-falling Inflow**

If a dusty flow is assumed to fall radially inwards from rest at infinity towards a gravitating body of mass \(M_\star\), then, neglecting any deceleration, the infall velocity will be given by \(u_{in} = (R_0/R)^{1/2} u_0\) where \(u_0 = (2GM_\star/R_0)^{1/2}\) is the velocity at the inner cut off radius \(R_0\) (which may be identified with the grain destruction radius). The inflow is supposed to have an outer radius \(R_c\) and mass conservation dictates that the dust number density have the form \(n = \eta_0 (R_0/R)^{3/2}\). The Doppler frequency shift of the scattered light is \(\Delta \nu = -\langle \nu/c \rangle u_{in} (1 + \hat{R} \cdot \hat{z})\), in a similar manner to the outflowing wind but with the opposite sign.

Figure 7: Integrated spectra of intensity (solid line) and degree of polarization (dashed line) for a scattering biconical wind with cone opening half-angle \(\delta_c = 0.2\) and at various inclinations \(\alpha\) of the cone axis to the plane of the sky.
Figure 8: Integrated spectra of intensity (solid line) and degree of polarization (dashed line) for a scattering biconical wind with cone opening half-angle $\delta_c = 1.0$ and at various inclinations $\alpha$ of the cone axis to the plane of the sky.

Figure 9: Integrated spectra of intensity (solid line) and degree of polarization (dashed line) for a scattering wind in the form of a thin disk at various angles $\alpha$ between the normal to the disk and the plane of the sky. Note that the intensity diverges at $v = 1 \pm \cos \alpha$ and so the values at these points are a function of the velocity resolution used in the plots.
The isovelocity scattering surfaces for a free-falling inflow of dust are surfaces of revolution formed by rotating the above curves about the line of sight to the center of the flow. The curves are marked with the dimensionless Doppler shift induced in emission lines scattered by dust lying on them. In this example, the ratio of inner to outer dust radius is $\epsilon_0 = 0.2$. As can be seen, the isovelocity surfaces are closed for $v < -2\epsilon_0^{1/2} \simeq -0.89$.

In this case however the dust velocity is not constant, so the dimensionless velocity shift is dependent on the dust radius:

$$v = -(\epsilon_0/R)^{1/2}(1 + \cos \Theta),$$

where $R = R/R_c$ and $\epsilon_0 = R_0/R_c$. This means that the isovelocity surfaces are no longer cones but have the more complex shapes shown in Figure 11. By integrating the scattered intensity over these isovelocity surfaces, it is possible to determine the spatially integrated scattered line profile of the infall as

$$I_v = \frac{3}{8(1-\epsilon_0^{1/2})} \times \begin{cases} 
G_1 & \text{if } v < -2\epsilon_0^{1/2} \\
G_2 & \text{if } v \geq -2\epsilon_0^{1/2}
\end{cases}$$

where

$$G_1 = \ln \epsilon_0 - 2v (\epsilon_0^{-1/2} - 1) - \frac{1}{2}v^2 (\epsilon_0^{-1} - 1)$$

$$G_2 = 2\ln \left(-\frac{1}{2}v\right) + 2 + 2v + \frac{1}{2}v^2.$$

Note that the scattered line profiles are unpolarized because the isovelocity surfaces are circularly symmetric from the point of view of the observer. Example line profiles are shown in Figure 11 for various values of $\epsilon_0$. It can be seen that for high $\epsilon_0$ (corresponding to a small spread in dust velocities with radius) the profile is similar to that from the constant velocity wind but with the sign of the Doppler shift reversed. As $\epsilon_0$ is decreased there is a greater proportion of slow-moving dust and so the line profile becomes increasingly skewed toward $v = 0$.

### 3.2 Spatially Resolved Scattered Line Profiles: Position-Velocity Diagrams

In this section, an analytic approach is used to calculate the Stokes intensities of the Rayleigh scattered light resolved both spatially and in velocity for two of the classes of models presented above.

#### 3.2.1 Moving Source in a Stationary Scattering Cloud

For the purposes of this section, the scattering cloud will be taken to be spherical and homogeneous. The observer is looking down the negative $z$-axis (as in Figure 2) and the position of a scatterer in the cloud is characterized by dimensionless coordinates $[\bar{x}, \bar{y}, \bar{z}]$ where $\bar{x} = x/R_c$ etc. Straightforward geometry then shows that the dimensionless Doppler shift of light scattered at $[\bar{x}, \bar{y}, \bar{z}]$ is

$$v = \frac{\bar{x} \cos \alpha + \bar{z} \sin \alpha}{(\bar{x}^2 + \bar{y}^2 + \bar{z}^2)^{1/2}}.$$

Figure 10: The isovelocity scattering surfaces for a free-falling inflow of dust are surfaces of revolution formed by rotating the above curves about the line of sight to the center of the flow. The curves are marked with the dimensionless Doppler shift induced in emission lines scattered by dust lying on them. In this example, the ratio of inner to outer dust radius is $\epsilon_0 = 0.2$. As can be seen, the isovelocity surfaces are closed for $v < -2\epsilon_0^{1/2} \simeq -0.89$.

Figure 11: Spatially integrated scattered line profiles from a spherically symmetric free-falling inflow with ratios of inner to outer flow radius $\epsilon_0 = 0.05, 0.25$ and 0.5.
Hence, solving this equation for \( \tilde{z} \), a line of sight \([\tilde{x}, \tilde{y}]\) intersects a given isovelocity cone \( v \) in zero, one or two places given by

\[
\{ \tilde{z}_a, \tilde{z}_b \} = \left\{ \tilde{x} \sin \alpha \cos \alpha \pm v \left[ \tilde{x}^2 \left( 1 - v^2 \right) + \tilde{y}^2 \left( \sin^2 \alpha - v^2 \right) \right]^{1/2} \right\} \times \left( v^2 - \sin^2 \alpha \right)^{-1}
\]

so long as \( v \) satisfies

\[
\tilde{x} \cos \alpha \pm \left[ 1 - \left( \tilde{x}^2 + \tilde{y}^2 \right) \right]^{1/2} < |v| < \left( 1 - \tilde{y}^2 \cos^2 \alpha \right)^{1/2},
\]

where the positive sign in the equation and inequalities applies to \( \tilde{z}_a \) and the negative sign to \( \tilde{z}_b \). The left-hand side of equation (33) reflects the fact that most lines of sight will not intersect the source velocity vector, while the right-hand side is the result of the finite size of the scattering cloud. The scattered line profile along a line of sight can then be calculated as

\[
I_v = \left| \frac{dI}{dv} \right| = \frac{\tau F_0}{4 \pi \theta_c^2} \left( \sum_{i=a,b} X(\tilde{x}, \tilde{y}, \tilde{z}) \right) \left( \frac{d\tilde{z}_i}{dv} \right)
\]

where \( F_0 \) is the flux of the source and \( \theta_c \) is the apparent angular radius of the scattering cloud. The phase function in Cartesian coordinates can be written as

\[
X(\tilde{x}, \tilde{y}, \tilde{z}) = \frac{3}{4(\tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2)} \left[ \tilde{x}^2 + \tilde{y}^2 + 2\tilde{z}^2 \right]
\]

and from equation (34) it follows that

\[
\left| \frac{d\tilde{z}}{dv} \right| = \frac{(\tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2)^{3/2}}{(\tilde{x}^2 + \tilde{y}^2) \sin \alpha - \tilde{x} \tilde{z} \cos \alpha}.
\]

It is then necessary to eliminate \( \tilde{z} \) from equation (34) using equation (32). For the general case this leads to very complicated expressions which are best calculated by computer but results for two special cases are presented here.

**Source Velocity in Plane of the Sky (\( \alpha = 0 \)):** If the dimensionless velocity shift \( v \) lies between \( \tilde{x} \) and \( \tilde{x} / (\tilde{x}^2 + \tilde{y}^2)^{3/2} \), then

\[
I_v = \frac{3\pi F_0 \left[ \tilde{x}^2 \left( 2 - v^2 \right) - \tilde{y}^2 \tilde{y}^2 \right]}{8 \pi \theta_c^2 \tilde{x} \left[ \tilde{x}^2 \left( 1 - v^2 \right) - \tilde{y}^2 \tilde{y}^2 \right]^{3/2}},
\]
\[
Q_v = \frac{3\pi F_0 v^2 \left( \tilde{x}^2 - \tilde{y}^2 \right)}{8 \pi \theta_c^2 \tilde{x} \left[ \tilde{x}^2 \left( 1 - v^2 \right) - \tilde{y}^2 \tilde{y}^2 \right]^{3/2}}.
\]

**Source Velocity Along Line of Sight (\( \alpha = \pi/2 \)):** For \( \alpha = \pi/2 \) the scattering geometry has circular symmetry from the point of view of the observer so it is convenient to introduce a dimensionless impact parameter \( r = (\tilde{x}^2 + \tilde{y}^2)^{1/2} \). Then, if \(|v| \leq (1 - r^2)^{1/2} \),

\[
I_v = \frac{3\pi F_0 (1 + v^2)}{16 \pi \theta_c^2 r (1 - v^2)^{1/2}},
\]
\[
Q_v = \frac{3\pi F_0 (\tilde{x}^2 - \tilde{y}^2)(1 - v^2)^{1/2}}{16 \pi \theta_c^2 r^3},
\]
\[
U_v = \frac{-6\pi F_0 \tilde{x} \tilde{y} (1 - v^2)^{1/2}}{16 \pi \theta_c^2 r^3},
\]

otherwise \( I_v = Q_v = U_v = 0 \). The degree and angle of polarization are given by

\[
p_v = \frac{\tilde{y}^2 - \tilde{y}^2 (\tilde{x}^2 + \tilde{y}^2)}{2 \tilde{x}^2 v^2 (\tilde{x}^2 + \tilde{y}^2)},
\]
\[
\chi = \frac{\pi}{2} + \tan^{-1} \left( \frac{\tilde{y}}{\tilde{x}} \right).
\]

Note that \( \chi \) is independent of \( v \), which is true for all values of \( \alpha \).

**Example Spectra and Position-Velocity Diagrams:** In Figure 12 the positions on the scattering cloud of six apertures used in constructing example spectra, three slits and three point apertures, are shown. In Figures 13 to 15, the spectra and position-velocity diagrams from these apertures are shown for different values of \( \alpha \). These spectra include the broadening effects of the frequency profile of the source emission line and of atmospheric seeing. Both are assumed to be Gaussian in form; the source profile is taken to have a FWHM of 0.16\( \mu_0 \) and the seeing profile to have a FWHM of 0.11\( \theta_c \).
3.2.2 Constant Velocity Scattering Wind

**Spherically symmetric wind:** For a constant velocity wind (assuming a constant dust-gas ratio), mass conservation requires the dust number density to be of the form

\[ n(R) = n_0 \left( \frac{R_0}{R} \right)^2 = \frac{n_0 \epsilon_0^2}{r^2 + z^2} \]  

where \( n_0 \) is the number density at an inner cut-off radius \( R_0 = \epsilon_0 R_c \). The scattering optical depth to the source is then given by

\[ \tau = \sigma_0 n_0 R_c \epsilon_0 (1 - \epsilon_0) \]  

so that the line profiles are given by

\[ I_v = \frac{\tau F_0}{4 \pi \theta_c^2} \frac{\epsilon_0}{1 - \epsilon_0} \left( \frac{d \xi}{dv} \right) \frac{1}{(r^2 + \xi^2)^{1/2}} \]  

Note that in this instance a line of sight can only intersect an isovelocity cone in, at most, one place and that the situation is very similar to the \( \alpha = \pi/2 \) case of § 3.2.1. Hence, if \( (1 - (r/\epsilon_0)^2)^{1/2} \leq |1 - v| \leq (1 - r^2)^{1/2} \) then

\[ I_v = \frac{3 \tau F_0 \epsilon_0 (2v - v^2)^{1/2}(v^2 - 2v + 2)}{16 \pi \theta_c^2 (1 - \epsilon_0)^{3/2}} \]  

\[ \rho_v = \frac{2v - v^2}{v^2 - 2v + 2} \]  

Figure 13: Doppler- and seeing-broadened position-velocity diagrams for a stationary cloud, scattering an emission line originating in a moving source. Aperture A: centrally placed slit, aligned with the source velocity. Dimensionless Doppler width \( w = 0.1 \). Dimensionless seeing width \( a_s = 0.066 \). Intensity contours are logarithmic, with successive contours corresponding to a ratio of \( 2^{1/2} \). Value of lowest contour is given on each diagram in units of \( \tau I_0 \) and is \( 2^{-6} \) times the peak value. Also shown are vectors whose lengths are proportional to the degree of polarization and whose orientation shows the position angle of the polarization.
Figure 14: Same as Figure 13 but for Aperture B: offset slit, parallel to the source velocity. Lowest contour is $2^{-1.5}$ times the peak value.

Figure 15: Same as Figure 13 but for Aperture C: offset slit, perpendicular to the source velocity. Lowest contour is $2^{-1.5}$ times the peak value.
Figure 16: Doppler- and seeing-broadened spectra for a stationary cloud, scattering an emission line originating in a moving source. Aperture D: centered on the source. Dimensionless Doppler width \( w = 0.1 \). Dimensionless seeing width \( a_s = 0.066 \).

Figure 17: Same as Figure 16 but for Aperture E: centered upstream of the source.

Figure 18: Same as Figure 16 but for Aperture F: centered to the side of the source.

otherwise \( I_v = 0 \) (\( \chi \) is still given by eq. [38b]). As in the \( \alpha = \pi/2 \) case of § 3.2.1, variation in the line profiles with position is chiefly caused by the conditions for intersection between the line of sight and isovelocity cone.

**Outflowing Bicone:** With a conical wind, the intensity and polarization profiles will still be given by equation (44) but the range of allowed velocities for the scattered light from a given line of sight is subject to the additional condition that the line of sight must intersect the cone of scatterers. To simplify matters, only lines of sight that intersect the cone symmetry axis (\( \bar{y} = 0 \)) are considered. In this case, the conditions that there be scattered flux at velocity \( v \) from a line of sight \( \bar{x} \) are given in Table 1, where all three conditions must be satisfied using either entirely the left terms in curly brackets or entirely the right. The position velocity diagrams so obtained (corresponding to a slit placed along the projected axis of the conical outflow) are illustrated in Figures 19 and 20 for both a narrow cone and a wide cone at various inclinations to the plane of the sky.

**Outflowing Disk:** Since the disk is assumed thin, the spatially resolved scattered line profile at each position on the disk will just be a single spike. Hence, the position-velocity diagram for a slit placed along the projected minor axis of the disk (\( \bar{y} = 0 \)) will be a ridge at \( v = 1 - \cos \alpha \) for negative \( \bar{x} \) and a ridge at \( v = 1 + \cos \alpha \) for positive \( \bar{x} \), both with a degree of polarization \( \sin^2 \alpha / (1 + \cos^2 \alpha) \), whereas for a slit along the projected major axis of the disk (\( \bar{x} = 0 \))
the position-velocity diagram is merely a ridge at \( v = 0 \) with 100\% polarization.

4 Discussion

The calculations presented in this paper have been selected because they are amenable to analytic treatment and hence represent extreme idealizations of situations likely to be encountered in astrophysical objects. Nevertheless, they demonstrate in a simple fashion the type of scattered line shapes that will be produced in different situations.

4.1 Moving Source

The moving source model (§§3.1.1 and 3.2.1) is applicable to any case in which a line-emitting plasma is moving with respect to a dusty environment, such as Herbig-Haro objects. This is discussed in much greater detail in Henney, Raga & Axon (1994) and Henney, Axon & Raga (1994), Papers II and III of this series. It may also furnish an alternative explanation to that proffered in Solf (1992) for the enormous line widths observed in knots of the collimated outflow from R Aquarii.

It is found that the spatially integrated line shapes from these models show scattered wings extending a distance equal to the source velocity \( u_0 \) to both sides of the rest frequency of the line (Figure 3). These wings are polarized to a degree dependent on the inclination of the source velocity to the line of sight, being highest when the source is moving in the plane of the sky and zero if the source is moving directly toward or away from the observer. Of course, the Doppler shift of the direct light from the source means that a source moving toward the observer would be seen to have an unpolarized red wing of width \( 2u_0 \) and a source moving away from the observer would have a similar unpolarized blue wing, while a source moving in the plane of the sky would have polarized red and blue wings. In this latter case, the polarization is highest at the extremes of the wings, drops to zero and then rises again toward line center (although in the center dilution by the intrinsic light will reduce the observed polarization substantially). Departure from spherical symmetry of the scattering cloud (Figures 4 and 5) causes the blue- and red-shifted scattered light to become more (less) intense than the unshifted light if the cloud is prolate (oblate).

Of the results for spatially resolved line shapes that are presented in §3.2.1, it is perhaps those for Aperture A of Figure 12 (corresponding to a narrow slit placed along a diameter of the cloud, parallel to the direction of motion of the source) that are the most interesting (Figure 13). For the source velocity in the plane of the sky, these show a characteristic “double triangle” morphology to the position-velocity diagram of the scattered light. The scattered light is blue-shifted to the right (upstream) of the source and red-shifted to the left (downstream). The polarization is highest for the scattered light that undergoes the largest Doppler shift (red or blue) since it is the dust directly in front of or behind the source that scatters light through 90\(^\circ\), leading to maximum polarization (Eq. [11]).

As the angle \( \alpha \) between the source direction and the plane of the sky increases, some upstream dust appears to the left of the source and some downstream dust to the right. Eventually, when the source is moving directly toward or away from the observer, this leads to the situation illustrated in the bottom-right panel of Figure 13 in which the

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**Note.**—See text for explanation of the symbols.
Figure 19: Scattered position velocity diagrams from a slit placed along the axis of a narrow outflowing bicone of opening half-angle $\delta_c = 0.2$ radians at various inclinations $\alpha$ to the plane of the sky.

Figure 20: Same as Figure 19 but for a wide outflowing bicone of opening half-angle $\delta_c = 1.0$ radians.
position-velocity diagram is symmetrical about $x = 0$. In this latter case, it is the scattered light which undergoes no Doppler shift which has the highest polarization since the dust in the plane of the sky is now directly to the sides of the motion of the source. Note that, for both $\alpha = 0$ and $\alpha = \pi / 2$, the area of the position-velocity diagram in which the direct light from the source will lie ($x = 0, v = 0$ for $\alpha = 0, v = \pm 1$ for $\alpha = \pm \pi / 2$) is well away from the area of highest polarization so there will be little dilution of the scattered light there. Note also that, unlike in the spatially integrated case, the scattered light is polarized for all inclinations of the source velocity.

### 4.2 Scattering Wind

These models (§§ 3.1.2 and 3.2.2) are applicable to any case in which chromospheric emission lines are scattered from a stellar wind. This may be dust scattering, as is possible in evolved late-type stars such as Mira variables (Anandarao, Pottasch & Vaidya 1993; Lefèvre 1992), or electron scattering, e. g. in Be stars (Wood, Brown & Fox 1993), B[e] supergiants (Boyd & Marlborough 1991), or Wolf-Rayet stars (Schulte-Ladbeck et al. 1992). They are also relevant to the scattering by dust and electrons in outflows from active galactic nuclei (Miller & Goodrich 1990).

Of course, for electron scattering the thermal broadening of the electron velocity distribution will significantly modify the results presented here for all but the highest Mach number winds (the thermal speed of the electrons will exceed the bulk wind velocity so long as the Mach number $M \approx 2$), so that in such cases the single scattering model presented here is not strictly applicable. These issues will be addressed in a forthcoming paper.

One obvious result is that some asymmetry of the wind is necessary for the integrated scattered line profile to be polarized. For a spherical wind (Figure 3), the scattering merely produces an unpolarized red wing, extending up to twice the wind velocity $u_w$. If the wind is concentrated toward the polar directions (Figures 5 and 6), then, for outflows in the plane of the sky, one finds a polarized red bump at a redshift of $u_w$, whose width depends on the degree of collimation of the wind. As the outflow axis is rotated toward the observer (increasing $\alpha$), the bump splits into two, which move apart to redshifts of zero and $2u_w$ as $\alpha \to \pi / 2$ and whose polarizations diminish to zero.

For equatorially enhanced winds (approximated as a disk; Figure 4), the intensity profiles of the scattered lines for a given inclination of the symmetry axis $\alpha$ can be seen to be qualitatively similar to the profiles for a narrow cone (Figure 7) with inclination $\pi / 2 - \alpha$, as would be expected. The polarization profiles do not follow this pattern, however, and show a maximum at a red shift of $u_w$ and a general decrease in polarization as the disk changes from an edge-on to a face-on orientation.

The spatially resolved position-velocity diagrams are not illustrated for the spherically symmetric wind but they are very similar to those for the moving source model with $\alpha = \pi / 2$ (bottom-right panels of Figures 13–15), except for three differences:

1. The Doppler shifts will range from 0 to $2u_w$ instead of from $-u_w$ to $u_w$.

2. In the spectrogram from Aperture A, there will be an elliptical “hole” in the center because of the inner cut-off to the density distribution.

3. The scattered brightness will fall off more rapidly with distance from the center of the cloud because of the inverse-square density distribution in the wind.

With the conical wind (Figures 19 and 20), the position-velocity diagrams are the same as for the spherical wind but with certain portions masked out. The sizes and positions of the non-empty regions of position-velocity space depend on the opening angle and orientation of the cones.

### 4.3 Scattering Inflow

The free-falling inflow model (§ 3.1.3) is applicable to the scattering haloes of young stellar objects, which often show density profiles indicative of an infalling envelope (Weintraub et al. 1992).

The spatially integrated line shapes (Figure 11) can be seen to depend sensitively on the value assumed for the ratio of inner to outer dust radius $\epsilon_0$ (see discussion after eq. 29). It should be remembered that the dimensionless Doppler shift $v$ is scaled to the infall velocity at the inner cut-off radius, which is proportional to $\epsilon_0^{-1 / 2}$, so that the velocity scales of Figure 11 are different if all the inflows are assumed to have the same outer radius and central mass.

Calculations of the spatially resolved line profiles from this model are not presented here because they are too involved for the simple analytic approach used in this paper. In addition, it is likely that many young stellar objects have optically thick haloes (Whitney & Hartmann 1993) and in such cases the single scattering model presented here is not strictly applicable. These issues will be addressed in Paper III, in which Monte Carlo simulations of multiple scattering will be made and where the effects of a non-zero angular momentum for the infall will be treated.
5 Summary

This paper has presented a small collection of line profiles (both intensity and polarization) that result from the optically thin Rayleigh scattering of an emission line by dust in its environment. The calculations have all been performed analytically and this has necessarily restricted the complexity of the models employed. Nonetheless, they form a basis for understanding the ways in which the geometric and kinematic relations between the line source and the scatterers determine the scattered line profiles.

Although, the calculations have been performed for the optically thin case, for one of the kinematic models considered (moving source in a stationary cloud) large optical depths will make very little difference to the scattered line profiles. This is because there is no relative motion between the scatterers themselves and hence the only Doppler shift is that induced by the first scattering. Of course, for the wind and infall models, the optically thick case will be quite different, with much more extended wings to the profiles due to multiple scattering. However, the polarization will also be reduced compared with the single scattering case so that the optically thin models are perhaps more likely to be relevant to sources with observable polarization changes across their line profiles.

One effect that has been ignored in this paper and that can have a significant effect on the line profiles is that of a non-Rayleigh scattering phase function for the dust. This is probably important at optical wavelengths, where the scattering phase function can be quite forward-peaked unless the dust grains are smaller than usual. This issue will considered in depth in Paper III.

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