Differential privacy offers formal quantitative guarantees for algorithms over datasets, but it assumes attackers that know and can influence all but one record in the database. This assumption often vastly overapproximates the attackers’ actual strength, resulting in unnecessarily poor utility. Recent work has made significant steps towards privacy in the presence of partial background knowledge, which can model a realistic attacker’s uncertainty. Prior work, however, has definitional problems for correlated data and does not precisely characterize the underlying attacker model. We propose a practical criterion to prevent problems due to correlations, and we show how to characterize attackers with limited influence or only partial background knowledge over the dataset. We use these foundations to analyze practical scenarios: we significantly improve known results about the privacy of counting queries under partial knowledge, and we show that thresholding can provide formal guarantees against such weak attackers, even with little entropy in the data. These results allow us to draw novel links between \( k \)-anonymity and differential privacy under partial knowledge. Finally, we prove composition results on differential privacy with partial knowledge, which quantifies the privacy leakage of complex mechanisms.

Our work provides a basis for formally quantifying the privacy of many widely-used mechanisms, e.g. publishing the result of surveys, elections or referendums, and releasing usage statistics of online services.

**Abstract**

Differential privacy offers formal quantitative guarantees for algorithms over datasets, but it assumes attackers that know and can influence all but one record in the database. This assumption often vastly overapproximates the attackers’ actual strength, resulting in unnecessarily poor utility. Recent work has made significant steps towards privacy in the presence of partial background knowledge, which can model a realistic attacker’s uncertainty. Prior work, however, has definitional problems for correlated data and does not precisely characterize the underlying attacker model. We propose a practical criterion to prevent problems due to correlations, and we show how to characterize attackers with limited influence or only partial background knowledge over the dataset. We use these foundations to analyze practical scenarios: we significantly improve known results about the privacy of counting queries under partial knowledge, and we show that thresholding can provide formal guarantees against such weak attackers, even with little entropy in the data. These results allow us to draw novel links between \( k \)-anonymity and differential privacy under partial knowledge. Finally, we prove composition results on differential privacy with partial knowledge, which quantifies the privacy leakage of complex mechanisms.

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1 Introduction

Differential privacy (DP) is an established privacy notion for algorithms over datasets. DP models strong attackers that can not only learn, but can even influence, all but one element from the input dataset. While this strong attacker model over-approximates realistic attackers, it can also lead to overly cautious choices of noise parameters, unnecessarily deteriorating the algorithms’ accuracy. Relaxing the assumptions about attackers’ background knowledge and their influence on the data set can lead to smaller noise parameters and, in turn, to more accurate results.

Consider a national referendum where more than 10 million people vote on a Yes/No question. Do the exact results of this referendum reveal information about individuals? It is very reasonable to assume that no realistic attacker has background knowledge of more than 99% of all votes. The remaining uncertainty of 1% of the data (100k data points) leads to a significant uncertainty that can, if properly quantified, show that no attacker can use the results of the referendum to determine how a given individual voted: the referendum results are private, even if no noise was added to them.

In some scenarios, one cannot exclude that the attacker can influence the entire dataset. But there are many natural scenarios in which the risk of an attacker injecting a large number of data points into the dataset is negligible: censuses, phone polls, or elections are natural examples. In those cases, it is appropriate to consider privacy guarantees that model weaker attackers without influence over the dataset, but with some background knowledge. While the precise estimation of an attacker’s capabilities may be difficult, a more balanced privacy analysis should characterize the privacy leakage against attackers with varying degrees of background knowledge and influence over the dataset.

As we show in this paper, while there is a rich body of prior work on this topic, it fails to account for data with correlations, it does not make the attacker model explicit and precise, and it leaves the following questions unanswered. What are appropriate notions of privacy for scenarios where the attacker has partial knowledge about a dataset that contains correlations? How to define notions of privacy under partial knowledge that cleanly delineate between attackers who only have some background knowledge and attackers who can influence the data? And can we use these notions in practical contexts, without making risky assumptions on the data, on how many computations are performed on the data, or on the adversary’s capabilities?

1.1 Approach and contributions

In this work, we provide a theoretical foundation for answering these questions, which we apply to common use cases. Our formalism solves the problems that previous definitions have when the data contains correlations, and it clearly delineates between attackers that only have some background knowledge and attackers that can influence the data. We build on these foundations to further analyze common noiseless mechanisms and prove strong and intuitive results about their privacy. Our main contributions are as follows.

First, we show that existing notions of privacy under partial knowledge break down when the data contains correlations, allowing very revealing mechanisms to be mistakenly considered private. We propose a practical criterion and approach to fix this class of issues.

Second, we show that there are two distinct ways to model partial knowledge, depending on whether the attacker can only learn some properties of the data or can modify the data. We define two corresponding notions of privacy under partial knowledge: active partial knowledge, where the attacker can influence the dataset, and passive, where the attacker is unable to influence the dataset. We show that these notions have natural properties, and prove that they are equivalent for a large class of common mechanisms and assumptions over the data. Moreover, we show that the active partial knowledge assumption can be used to alleviate the challenge of precisely estimating the dataset distribution.

Third, putting these results to work, our results provide a formal account of the privacy of common kinds of queries. We show that counting queries under partial knowledge can provide privacy, with

---

1Here, we assume the absence of malicious insiders who could break privacy anyway by leaking the entire dataset.
significantly lower bounds than those previously given. We also show that thresholding—solely returning the output if it is larger than a given threshold—can render counting queries private against passive attackers, even if the input distribution does not have enough enough entropy for the previous result to apply.

Fourth, we prove bounds for the sequential composition of noiseless mechanisms. This allows us to quantify the privacy leakage of multiple mechanisms with the same input, or of a mechanism repeated over time.

1.2 Related work
Among the numerous variants of differential privacy (DP) [DP20], two main variants model adversaries with partial background knowledge, using indistinguishability: noiseless privacy [Dua09, BBG+11], and distributional DP (DDP) [BGKS13]. This work discusses the shortcomings of DDP in Section 2 and in Section 3 we use the formalism of noiseless privacy to define active and passive partial knowledge DP.

Other variants, which also model adversaries with partial background knowledge, are not based on indistinguishability, but directly constrain the posterior knowledge of an attacker as a function of their prior knowledge. Among those are adversarial privacy [RHMS09], membership privacy [LQS+13], and aposteriori noiseless privacy [BBG+11]. It is straightforward to adapt the examples given in this paper to show that these definitions suffer from the same flaws as noiseless privacy when data has correlations. These definitions also also do not delineate between passive and active attackers. Because of space constraints, we do not study them in detail.

Several other definitions have been proposed. Pufferfish privacy [KM12] can be seen as a generalization of noiseless privacy, and similarly, coupled-worlds privacy [BGKS13] (and its inference-based variant) generalizes distributional differential privacy: instead of protecting individual tuples, they protect arbitrary sensitive properties of the data. It is straightforward to generalize our results to the more generic frameworks.

1.3 Background on existing definitions
We first recall the original definitions of \((\varepsilon, \delta)\)-indistinguishability and \((\varepsilon, \delta)\)-differential privacy.

\textbf{Definition 1} \((\varepsilon, \delta)\)-indistinguishability [DMNS06]. Two random variables \(A\) and \(B\) are \((\varepsilon, \delta)\)-indistinguishable if for all measurable sets \(X\) of possible events:

\[
\begin{align*}
\mathbb{P}[A \in X] & \leq e^{\varepsilon} \cdot \mathbb{P}[B \in X] + \delta \\
\mathbb{P}[B \in X] & \leq e^{\varepsilon} \cdot \mathbb{P}[A \in X] + \delta.
\end{align*}
\]

We denote this as \(A \approx_{\varepsilon, \delta} B\). If \(\delta = 0\), we call this \(\varepsilon\)-indistinguishability, and denote it by \(\approx_{\varepsilon}\) (c.f. Table 1).

In all the following, \(\mathcal{D}\) designates the set of possible databases. A database \(D\) is a family of records: \(D = (D(i))_{i \leq n}\), where each \(D(i)\) is in a fixed set \(\mathcal{T}\) of possible records, and \(n\) is the size of the set \(D\). We only consider databases of fixed size \(n\), and usually omit the range of database indices \(i\). Mechanisms, typically denoted \(\mathcal{M}\), take databases as input, and output some value in an output space \(\mathcal{O}\).

\textbf{Definition 2} \((\varepsilon, \delta)\)-differential privacy [DMNS06]. A privacy mechanism \(\mathcal{M}\) satisfies \((\varepsilon, \delta)\)-differential privacy (DP) if for any databases \(D_1\) and \(D_2\) that differ only on the data of one record, \(\mathcal{M}(D_1) \approx_{\varepsilon, \delta} \mathcal{M}(D_2)\). If \(\delta = 0\), we call this \(\varepsilon\)-differential privacy.

2 Correlated data
When data is correlated, dependencies create problems for privacy definitions that assume an attacker with partial knowledge. To illustrate this, we recall two previously introduced definitions that model this situation differently: noiseless privacy (NP) and distributional differential privacy (DDP). We show that both definitions have undesirable consequences when data is correlated, and use DDP as a starting point to solve this problem in two steps. First, we modify DDP and introduce a new definition, causal differential privacy (CDP), to prevent its most direct problems. Second, we propose a criterion that encompasses many use-cases but avoids known issues with correlated data, and makes CDP equivalent to NP. This allows us to cleanly define a rigorous notion of DP with partial knowledge, which allows for many practical use cases, but avoids the known issues with correlated data.
will be close to − of the record values is not NP: for all indices  

Example 1. Assume \[BGKS13\] argue that this definition is too strong. The following example illustrates their argument.

much. NP attempts to capture this intuition for an attacker with partial knowledge, but Bassily et

outputs \[n\] conditioned on the event "\[M\]

Here, the notation \(\P_{\theta}[\hat{B},D(i) = a] \neq 0\) and \(\P_{\theta}[\hat{B},D(i) = b] \neq 0\) (we call this condition “\(\hat{B}\) is compatible with \(D(i) = a\) and \(D(i) = b\)”).

\[\mathcal{M}(D)_{\theta,\hat{B},D(i)=a} \approx_{\varepsilon,\delta} \mathcal{M}(D)_{\theta,\hat{B},D(i)=b}. \tag{1}\]

Here, the notation \(\mathcal{M}(D)_{\theta,\hat{B},D(i)=a}\) refers to the random variable defined by \(\mathcal{M}(D)\), where \(D \sim \theta\), conditioned on the event “\(\hat{B} = \hat{B}\) and \(D(i) = a\)”.

The original intuition behind DP states that changing one record must not change the output too much. NP attempts to capture this intuition for an attacker with partial knowledge, but Bassily et al. \[BGKS13\] argue that this definition is too strong. The following example illustrates their argument.

Example 1. Assume \(\theta\) has a global parameter \(\mu\) that is either +1 or -1 with equal probabilities, and outputs \(n\) normally distributed records with mean \(\mu\) and a small standard deviation. Releasing the average of the record values is not NP: for all indices \(i\), \(\mathcal{M}(D)_{\theta,D(i)=1}\) will be close to 1 and \(\mathcal{M}(D)_{\theta,D(i)=-1}\) will be close to −1, so the two distributions are very distinguishable. This happens even though the impact

| \(\varepsilon,\delta\) | \(\varepsilon,\delta\)-indistinguishable |
| Set of possible records |
| \(D\) | Set of possible databases |
| \(D(i)\) | Single record in \(D\) |
| \(D_{-i}\) | Database \(D\) with \(D(i)\) removed |
| \(D_{\rightarrow b}\) | Database \(D\) with \(D(i)\) replaced by value \(b\) |
| \(n\) | Size of the database |
| \(B\) | Set of auxiliary information/partial knowledge |
| \(B\) | Partial (or “background”) knowledge |
| \(\theta\) | Probability distribution on \(D\) or \(D \times B\) |
| \(\Theta\) | Set of probability distributions |
| \(\P_{\theta}\) | Abbreviation of \(\P_{(D,B) \rightarrow \theta}\) |
| \(\hat{B}\) | Observation of \(B\); abbreviation of \(B = \hat{B}\) |
| \(\hat{D}\) | Observation of \(D\); abbreviation of \(D = \hat{D}\) |
| \(O\) | Output space of mechanisms |
| \(\mathcal{M}\) | Mechanism |
| \(X|E\) | Random variable \(X\) conditioned on event \(E\) |

Table 1: Notations used in this paper.

Note that there has been substantial debates about the impact of correlations on the guarantees that DP provides. The debates are summarized in \[TSD17\], where the authors suggest a possible resolution: interpreting DP as a causal property. In this section, we continue this line of work in the context of partial knowledge. In particular, we show that modifying the definition in the same way as the “causal variants” of \[TSD17\] is not sufficient to solve all issues created by the presence of correlations in the data, when the attacker only has partial knowledge.

For simplicity, in this section, we only consider the case with \(\delta = 0\). We re-introduce \(\delta > 0\) in Section 3.
of a single record in the database is low: once \( \mu \) is fixed, the random choice of \( i \) is unlikely to have a large effect on the global average.

This example shows a definitional problem. If the attacker previously knows \( \mu \), revealing \( M(D) \) does not give much additional information on the target \( D(i) \): an attacker with less initial knowledge is considered more powerful. We study this in detail in Section 2.3.

The authors propose an alternative definition to fix this problem: \((\Theta, \varepsilon)\)-distributional differential privacy. It requires that \( M \) can be simulated by another mechanism \( \text{Sim} \), that does not have access to the sensitive property. The intuition is as follows: if \( M(D) \) is close to \( \text{Sim}(D_{-i}) \) for some simulator \( \text{Sim} \), then \( M \) cannot leak “too much” about the value of \( D(i) \).

**Definition 4** ((\( \Theta, \varepsilon \))-distributional differential privacy [BGKS13]). Given a family \( \Theta \) of probability distributions on \( D \times B \), a mechanism \( M \) satisfies \((\Theta, \varepsilon)\)-distributional differential privacy \( ((\Theta, \varepsilon)\text{-DDP}) \) if there is a simulator \( \text{Sim} \) such as for all probability distributions \( \theta \in \Theta \), all \( B \in B \), all \( i \), and all \( a \in T \) such that \( \hat{B} \) is compatible with \( D(i) = a \):

\[
M(D)\|_{\theta, \hat{B}, D(i) = a} \approx_{\varepsilon} \text{Sim}(D_{-i})\|_{\theta, \hat{B}, D(i) = a},
\]

where \( D_{-i} \) is the database \( D \) from which the record \( i \) has been removed.

The distribution \( \theta \) and the mechanism \( M = \text{avg} \) from Example 1 satisfy this definition: the simulator can be defined as simply running \( M \) on \( D_{-i} \), possibly after adding +1 or −1 depending on the other records.

### 2.2 Distributional differential privacy under correlations

This critique of NP is similar to the critique of the associative view of DP in [TSD17]. But the proposed fix has a flaw: Sim can use strong dependencies in the data to artificially satisfy the definition. In the following example, the values of different records are strongly correlated, and the simulator *cheats* by using these correlations: consequently, the identity function is considered private!

**Example 2.** Let \( \theta \) output \( n \) duplicate records: for all \( i < n \), \( D(2i) \) is picked from some probability distribution \( R \), and \( D(2i + 1) = D(2i) \). Then the identity function \( \text{Id} \), which simply outputs its input without any noise, is \((\{\theta\}, 0)\text{-DDP}! \) Indeed, the simulator can simply replace the missing record by its duplicate and output the entire database: \( \text{Id}(D) \) is exactly the same as \( \text{Sim}(D_{-i}) \).

Here, the dependency relationships are “extreme”, as each record is duplicated. But even when records are less strongly correlated, the problem is still present. In fact, the more dependencies are in the data, the more accurately the simulator can simulate the missing record, and the more “private” the mechanism is (since \( \varepsilon \) gets lower): a more powerful adversary, who can exploit dependencies in the data, is considered weaker by the definition. This is clearly undesirable.

How can we formalize an adversary that cannot “cheat” using dependencies in the data? We propose one possible option: using the same technique as the causal variants of DP described in [TSD17], we simply change the target record after the distribution is generated.

**Definition 5** ((\( \Theta, \varepsilon \))-causal differential privacy). Given a family \( \Theta \) of probability distributions on \( D \times B \), a mechanism \( M \) satisfies \((\Theta, \varepsilon)\)-causal differential privacy \( ((\Theta, \varepsilon)\text{-CDP}) \) if for all probability distributions \( \theta \in \Theta \), all \( i \), all \( a, b \in T \), and all \( \hat{B} \in B \) compatible with \( D(i) = a \) and \( D(i) = b \):

\[
M(D)\|_{\theta, \hat{B}, D(i) = a} \approx_{\varepsilon} M(D_{-i+b})\|_{\theta, \hat{B}, D(i) = a},
\]

where \( D_{-i+b} \) is the database \( D \), where the \( i \)-th record has been replaced by \( b \).

CDP still captures DP’s intuition: the change in one data point should not influence the output of the mechanism too much. However, the change happens after the influence of the dependencies in the data. This version is strictly stronger than the original version: if a mechanism \( M \) is \((\Theta, \varepsilon)\text{-CDP} \), then it is also \((\Theta, \varepsilon)\text{-DDP} \). Indeed, the simulator \( \text{Sim} \) can always replace the missing record with an arbitrary value \( b \) and return \( M(D_{-i+b}) \).

In [BGKS13], the authors also introduce an *inference-based* version of DDP. We can as easily adapt CDP to this different formalization.
Definition 6 ((Θ,ε)-inference-based causal differential privacy). Given a family Θ of probability distributions on $D \times B$, a mechanism $M$ satisfies $(\Theta, \varepsilon)$-inference-based causal differential privacy ($(\Theta, \varepsilon)$-IBCDP) if for all probability distributions $\theta \in \Theta$, for all indices $i$, all $b \in B$, all $t \in O$, and all $\hat{B} \in B$, compatible with $M(D) = t$ and $M(D_{i \rightarrow b}) = t$:

$$D(i|\theta, \hat{B}, M(D) = t) \approx_{\varepsilon} D(i|\theta, \hat{B}, M(D_{i \rightarrow b}) = t)$$

where $D_{i \rightarrow b}$ is the database $D$, where the record $i$ has been replaced by $b$.

Note that this definition is equivalent to the indistinguishability-based version (Definition 5) up to a change in parameters.

Proposition 1. $(\Theta, \varepsilon)$-CDP implies $(\Theta, 2\varepsilon)$-IBCDP, and $(\Theta, \varepsilon)$-IBCDP implies $(\Theta, 2\varepsilon)$-CDP.

Proof. The first implication can be proven in the same way as Theorem 1 in [BCKSI13], replacing $Sim(D_{-i})$ by $M(D_{i \rightarrow b})$. For the second implication, suppose that a mechanism $M$ is $(\Theta, \varepsilon)$-IBCDP, and assume that the attacker has no background knowledge. Consider an index $i$, two possible record values $a, b \in T$, and one possible output value $t \in O$. Bayes’ rule gives us:

$$\frac{P[M(D_{i \rightarrow b}) = t | D(i) = a]}{P[M(D) = t | D(i) = a]} = \frac{P[D(i) = a | M(D_{i \rightarrow b}) = t]}{P[D(i) = a | M(D) = t]} \cdot \frac{P[M(D) = t]}{P[M(D_{i \rightarrow b}) = t]}$$

The first term is between $e^{-\varepsilon}$ and $e^{\varepsilon}$ since $M$ is $(\Theta, \varepsilon)$-IBCDP. We only need to show that the second term is also between $e^{-\varepsilon}$ and $e^{\varepsilon}$ to conclude the proof. Notice that when $D(i) = b$, we have $D_{i \rightarrow b} = D$. Thus:

$$1 = \frac{P[M(D_{i \rightarrow b}) = t | D(i) = b]}{P[M(D) = t | D(i) = b]} = \frac{P[D(i) = b | M(D_{i \rightarrow b}) = t]}{P[D(i) = b | M(D) = t]} \cdot \frac{P[M(D) = t]}{P[M(D_{i \rightarrow b}) = t]}$$

Again, the first term is between $e^{-\varepsilon}$ and $e^{\varepsilon}$ since $M$ is $(\Theta, \varepsilon)$-IBCDP. Since multiplying it with the second term gives 1, the second term is also between $e^{-\varepsilon}$ and $e^{\varepsilon}$. If the attacker has background knowledge, all the probabilities above are conditioned by $\hat{B}$, and the same reasoning holds.

In the more general case where the attacker does have some partial knowledge, all probabilities above are conditioned by the value of this partial knowledge, and the same reasoning holds.

This equivalence is only true in the context of this section, where $\delta = 0$; we explain later why it fails when $\delta > 0$.

Does Example 3 satisfy CDP? It depends: if $b$ can take arbitrarily large values, $\text{avg}(D_{i \rightarrow b})$ can be arbitrarily distinguishable from $\text{avg}(D)$. Otherwise, $b$ can only have a bounded influence on the average and $\{(\theta), \varepsilon\}$-CDP can hold for some $\varepsilon$. In other words, when using the fixed version of the definition, whether a given mechanism is CDP depends on the sensitivity of the mechanism. This is a good thing: it suggests that it captures the same intuition as DP.

Example 3 shows that CDP is not stronger than NP. Is the reverse true? In Example 4, we show that this is not the case.

Example 3. Consider the same $\theta$ as for Example 1; it depends on a global parameter, $\mu$, which is either $+1$ or $-1$ with equal probabilities, and each of the $n$ records is normally distributed with mean $\mu$ and a small standard deviation $\sigma$. Let $M$ be the algorithm that counts outliers: it computes the average $\tilde{\mu}$ of all data points, and returns the number of records outside $[\tilde{\mu} - 5\sigma, \tilde{\mu} + 5\sigma]$. As we saw before, conditioning $\theta$ on a value of $D(i)$ is approximately equivalent to fixing $\mu$: the number of outliers is going to be the same no matter what (0 with high probability). However, if we first condition $\theta$ on $D(i) = a$, and then change this record into $b$, we can choose $b$ so that this record becomes an outlier; and make it 1 with high probability. Thus, this mechanism is NP but not CDP.

Even though Example 3 shows that NP does not imply CDP, it is natural to think that in many cases, if you change one data point $i$ as well as all data points correlated with it, it will have a bigger influence on the algorithm that if you only change $i$ without modifying the rest of the data. In Example 4, we show that even for a simple data dependencies and mechanisms, we can find counterexamples to this intuition.

Example 4. Consider a probability distribution $\theta$ that outputs 2n records, such as for all $i < n, D(2i)$ is picked from some probability distribution $R$ with values in $N$, and $D(2i + 1) = D(2i)$. Then the mechanism that sums all records might be in NP, but cannot be in CDP. Indeed, if $D \sim \theta$, then $\sum_i D(i)$ will always be even, but changing one record without modifying its duplicate can make the sum odd.
This last example that finding a special case where NP implies CDP is likely difficult. There is, however, a special case where both are equivalent: the absence of dependencies in the data. If changing one record does not influence other records, then NP and CDP are equivalent. This result is similar to Corollary 2 in [BGKS13], but is simpler and without the change in parameters.

Proposition 2. Let $\Theta$ be a family of probability distributions such that for all $\theta \in \Theta$ and all $\hat{B} \in \mathcal{B}$, the random variables $D(i)_{\theta, \hat{B}}$ are mutually independent. Then a mechanism $M$ is $(\Theta, \varepsilon)$-NP iff it is $(\Theta, \varepsilon)$-CDP.

Proof. Under these conditions, $D(i)_{\theta, \hat{B}, D(i) = a}$ is exactly $D(i)_{\theta, \hat{B}, D(i) = b}$, so $M(D)_{\theta, \hat{B}, D(i) = b}$ is the same as $M(D_{i \rightarrow b})_{\theta, \hat{B}, D(i) = a}$. The statement follows.

This natural property, combined with the better behavior of CDP in scenarios like Example 2 might seem like CDP is a better alternative to NP, when one wants to capture an attacker with partial knowledge, under the causal interpretation of differential privacy. However, even with this fix, when records are not independent, CDP is not always safe to use. We present an example from Adam Smith (personal correspondence, 2018-09-28) showing that a slightly modified version of the identity function can still be CDP if records are strongly correlated.

Example 5. Let $\theta$ output 3a triplicated records: for all $i < n$, $D_{3i}$ is picked from some probability distribution $R$, and $D(3i + 1) = D(3i + 2) = D(3i)$. Let $M$ be a mechanism that “corrects” a modified record: if there is a record value $x$ appearing only once, and a value $y$ appearing only twice, then $M$ changes the record $x$ to $y$; then $M$ always outputs the entire database. It is easy to check that $M$ is $(\{\theta\}, 0)$-CDP.

This example is more artificial than Example 2 as the mechanism itself “cheats” to use dependencies in the data. Nonetheless, it shows that some mechanisms that leak the full database can be CDP. Thus, using CDP as a privacy measure of a given mechanism is dangerous if no information about the mechanism is known. We do not know whether more natural mechanisms could lead to similar counterexamples, for certain classes of probability distributions; but it is clear that simply applying the same technique as the causal variants of [TSD17] is insufficient to solve entirely the problems with correlations under partial knowledge.

2.3 Imposing an additional criterion on the definition

In this section, we propose a criterion that the distribution $\theta$ must satisfy before NP can be used. We argue that when this criterion is not satisfied, NP is not a good measure of the privacy of a mechanism. But when it is satisfied, we obtain natural properties that are false in general: NP and CDP are equivalent, and attackers with more partial knowledge are stronger.

We mentioned previously that NP was not monotonous: an attacker with more knowledge can be considered to be less powerful. In Example 1 an attacker $A$ who did not know $\mu$ could learn $\mu$ by observing $M(D)$. This increases her knowledge about $D(i)$; she now knows that $D(i)$ is probably around $\mu$, a fact previously unknown. However, an attacker $B$, who already knew $\mu$, does not increase her knowledge as much when observing $M(D)$. Examples 2 and 3 show that DDP and CDP also suffer from this issue. How can we fix this problem? The informal goal is that an attacker with more background knowledge should gain more information. We must make sure that the privacy quantification $\varepsilon$ cannot be artificially inflated by non-sensitive information learned by the attacker.

In all previous examples, the attacker’s partial knowledge is strongly correlated with the sensitive information. Thus, $\varepsilon$ does not measure the privacy loss due to the mechanism, but also takes into account the prior knowledge from the attacker about the sensitive attribute. Modeling the attacker’s uncertainty is necessary to formalize her partial knowledge, but the only thing that should be captured by $\varepsilon$ is the mechanism’s privacy leakage. To ensure this is the case, we argue that the partial knowledge must be independent from the sensitive information, and we propose a formalization that enforces this distinction between sensitive information and partial knowledge.

To this end, we propose an alternative way to model the attacker’s uncertainty, and suggest to normalize the distribution $\theta$ to cleanly separate sensitive information and partial knowledge. We show that if such a normalization exists, then an attacker with more partial knowledge is more powerful. Thus, the existence of such a normalization is a desirable property for privacy definitions that model an attacker with partial knowledge, and we argue that it should serve as a criterion that must be satisfied before using such definitions, in order to get meaningful results.
How to formalize the intuition that the sensitive information should be separated from the partial knowledge? The core idea is to express θ as the output of a generative function, with independent random parameters. Each possible value of these parameters corresponds to a possible database.

**Definition 7** (Normalization of data-generating distributions). A normalization of a probability distribution θ is a family of mutually independent random variables (ϕ₀, . . . , ϕₖ), and an injective, deterministic function ̂θ, such that θ = ̂θ(ϕ₀, . . . , ϕₖ).

A normalization de-correlates the distribution: it splits its randomness into independent parts ϕᵢ. The ϕᵢ can then play distinct roles: one parameter can capture the sensitive property, while the others can model the attacker’s partial knowledge. We capture this additional requirement in the following definition.

**Definition 8** (Acceptable parameters). Given a distribution θ with values in D × B, an acceptable normalization of θ at index i is a normalization ̂θ(ϕ₀, . . . , ϕₖ) where:

1. ϕ₀ entirely determines the sensitive attribute D(i): there exists a function f such that for all possible values of D(i), Pθ[f(ϕ₀) = D(i)] = 1.
2. Some of the ϕ₁, . . . , ϕₖ entirely determine the partial knowledge: there exists I ⊆ {1, . . . , k} and an injective function g such that Pθ[g((ϕₗ)ₗ∈I)] = 1.

A family of distributions Θ is acceptable if each θ ∈ Θ has an acceptable normalization at all indices i.

In practice, we can simply consider a family (ϕᵢ)ᵢ∈I for some I to be the attacker’s partial knowledge, rather than using a bijection. When partial knowledge is defined using a subset of parameters, an attacker has more partial knowledge when she knows more parameters.

Informally, ϕ₀ must contain enough information to retrieve the sensitive attribute, and the partial knowledge must be independent from it. How can we normalize the θ in Example [1]? We cannot have one parameter for μ, and n parameters for the noise added to each record: the sensitive attribute D(i) would require two parameters to express. Rather, ϕ₀ could be a pair containing both μ and the noise added at i, D(i) − μ. The attacker’s partial knowledge can be D(j) − μ, for j ≠ i, without μ. In other words, μ itself is also sensitive. What if we do not want to consider μ as sensitive? Then, ϕ₀ can be the value of the noise added to the record i, D(i) − μ, and μ is another parameter that can (or not) be part of the attacker’s partial knowledge.

As this example shows, this separation between the partial knowledge and the sensitive value forces us to carefully choose the sensitive value, and it prevents us from comparing scenarios where the sensitive value varies. This formalism can now be used to precisely define the relative strength of two attackers, based on their partial knowledge, and show that an attacker with more knowledge is more powerful.

**Definition 9** (Relative strength of partial knowledge). Given probability distributions θ₁ and θ₂ with values in D × B, we say that θ₁ has more background knowledge than θ₂ if the three following conditions are satisfied.

1. Dθ₁ = Dθ₂: the only difference between the probability distributions is the partial knowledge.
2. For all i, there exists an acceptable normalization ̂θ(ϕ₀, . . . , ϕₖ) given i that is common to θ₁ and θ₂.
3. For all i, if we denote I₁ and I₂ the set of parameters that correspond to θ₁ and θ₂ respectively in this acceptable normalization, then I₁ ⊆ I₂.

Given two families of probability distributions Θ₁ and Θ₂, we say that Θ₁ has more background knowledge than Θ₂ if for all θ₂ ∈ Θ₂, there exists θ₁ ∈ Θ₁ such that θ₁ has more background knowledge than θ₂.

**Proposition 3.** Let Θ₁ and Θ₂ be two families of distributions such that Θ₁ has more background knowledge than Θ₂. If a mechanism ℳ satisfies (Θ₁, ε)-NP, it also satisfies (Θ₂, ε)-NP.

**Proof.** Suppose that ℳ is (Θ₁, ε)-NP. For a distribution θ₂ ∈ Θ₂ and an index i, let θ₁ ∈ Θ₁ be such that θ₁ is stronger than θ₂. By definition, there exists an acceptable normalization ̂θ(ϕ₀, . . . , ϕₖ) common to θ₁ and θ₂. Let f be the function extracting the sensitive value from ϕ₀ in this normalization. Denoting I₁ and I₂ the set of parameters corresponding respectively to θ₁ and θ₂, as a simplification, we assume
that $I_2 = \emptyset$ and $I_1 = \{1\}$; it is straightforward to adapt the proof to the more generic case. For any output $O$, and all values $a, b \in T$, we can decompose:

$$P[\mathcal{M}(D) = O | f(\phi_0) = a] = \sum_B P[\phi_1 = \hat{B} | f(\phi_0) = a] \cdot P[\mathcal{M}(D) = O | f(\phi_0) = a, \phi_1 = \hat{B}].$$

The $\phi_i$ are independent: $P[\phi_1 = \hat{B} | f(\phi_0) = a]$ is the same as $P[\phi_1 = \hat{B} | f(\phi_0) = b]$. Since $\mathcal{M}$ satisfies $(\Theta_1, \varepsilon)$-NP, we also have for all $\hat{B}$:

$$P[\mathcal{M}(D) = O | f(\phi_0) = a, \phi_1 = \hat{B}] \leq e^\varepsilon P[\mathcal{M}(D) = O | f(\phi_0) = b, \phi_1 = \hat{B}].$$

Thus:

$$P[\mathcal{M}(D) = O | f(\phi_0) = a] \leq e^\varepsilon \sum_B P[\phi_1 = \hat{B} | f(\phi_0) = b] \cdot P[\mathcal{M}(D) = O | f(\phi_0) = b \land \phi_1 = \hat{B}] \leq e^\varepsilon P[\mathcal{M}(D) = O | f(\phi_0) = b]$$

and thus, $\mathcal{M}$ satisfies $(\Theta_2, \varepsilon)$-NP. Adapting the proof to cases where $I_1 \subseteq I_2$ is straightforward.

This proposition states that, for acceptable distributions, partial background knowledge can be formalized in a reasonable and intuitive way, guaranteeing that attackers with more background knowledge are stronger. Another advantage is that when this criterion holds, NP and CDP are equivalent.

**Proposition 4.** If $\Theta$ is an acceptable distribution, then for any $\varepsilon$ and $\delta$, $(\Theta, \varepsilon, \delta)$-NP is equivalent to $(\Theta, \varepsilon, \delta)$-CDP.

The proof is the same as for Proposition 2. Figure 1 summarizes the relations between the definitions introduced in this section.

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**Figure 1:** Relations between definitions introduced in Section 2, assuming $\delta = 0$.

Acceptable normalizations force practitioners to define which information is considered private. If, like in DP, the sensitive information is the value of a given record, with an acceptable normalization, the attacker’s partial knowledge cannot contain records correlated with the target record. This might seem overly restrictive: what if the attacker does know some information correlated with the target record? In this case, one must change the sensitive information to only consider the decorrelated part as sensitive.

To illustrate this process, consider a medical database where the diagnostic records of different patients might be correlated. To find an acceptable normalization, we must choose between two options. The first is to consider attackers that do not have information correlated with the diagnosis of a given patient: in that case, we must protect the information of multiple patients at once, similarly to DP under correlations (a variant of DP defined in [CFY14]). Another option is for the sensitive property to be the diagnosis of someone given the diagnosis of those they are correlated with. We formally present a simpler example below, in Example 6.

**Example 6.** Consider a referendum, where people vote in pairs, with some amount of correlation between pairs. More precisely, $\theta$ is a distribution that generates a database of $2n$ records according to the following process:
• $D(2i) = 1$ with probability $p_i$, and 0 with probability $1 - p_i$;
• $D(2i + 1) = D(2i)$ with probability $p_c$, and $1 - D(2i)$ with probability $1 - p_c$.

Suppose the attacker is interested in $D(1)$. There are two ways of modeling this with an acceptable normalization, depending on whether we want to allow the attacker to know $D(0)$.

• We can pick $\phi_i$ to determine the value of both $D(2i)$ and $D(2i + 1)$. This way, $\phi_0$ is sufficient to know the value of the sensitive information $D(i)$, and is independent from all $\phi_i$ for $i > 0$.
• We can also pick $\phi_{2i}$ to be the event “$D(2i) = D(2i + 1)$”, and $\phi_{2i+1}$ to be the value of $D(2i)$. In that case, the attacker is allowed to know $D(0)$, but the sensitive value has changed: the sensitive value is whether $D(0) = D(1)$; which is independent from the actual value of $D(0)$.

3 Passive vs. active attackers

In this section and the rest of this paper, we assume that the criterion introduced in Definition 8 is satisfied: the data-generating distributions considered are always acceptable. For simplicity, we also assume that the sensitive property is the value of one record. Under these conditions, any value of the target dataset is independent from the actual value of $D(0)$.

Correlations in the data are not the only issue to account for when limiting the attacker’s background knowledge in DP. Another important question is whether the attacker simply receives partial knowledge, and we explore the relationship between the two corresponding definitions. To help understand this distinction, in this section, we consider the following example.

Example 7 (Thresholding). 1000 people take part in a Yes/No-referendum. Each person votes “Yes” with some probability, independently from the others. The mechanism $M$ counts the number of “Yes” votes, but only returns this if it is above 100; otherwise it returns ⊥. The partial knowledge $\hat{B}$ contains the votes of 100 participants, and the attacker wants to know the vote of another individual $D(i)$. We will see that in case the probability that each person votes “Yes” is very small (say, $10^{-7}$), the privacy of this scheme will depend on whether the attacker is passive or active.

3.1 Privacy loss random variable

Let us recall the privacy loss random variable (PLRV) of an output $O$. For simplicity, we only consider the case where the set of possible outputs of the mechanism, $\mathcal{O} = \bigcup_D \text{support}(M(D))$, is countable.

Definition 10 (Privacy Loss Random Variable). Given a mechanism $M$, and two datasets $D_1$ and $D_2$, the privacy loss random variable (PLRV) of an output $O \in \mathcal{O}$ is defined as

$$\mathcal{L}_{\mathcal{D}_1/\mathcal{D}_2}(O) = \ln \frac{\mathbb{P}[M(D_1) = O]}{\mathbb{P}[M(D_2) = O]}.$$  

Using the convention $x/0 = \infty$ for all $x$, the PLRV can be ±∞.

$(\varepsilon, \delta)$-DP can be reformulated using the PLRV.

Lemma 1 ([AMM18] Lemma 1). A mechanism $M$ is $(\varepsilon, \delta)$-DP iff for all neighboring databases $D_1, D_2 \in \mathcal{D}$ (differing in 1 record):

$$\mathbb{P}_{O \sim M(D_1)} \left[ \max(0, 1 - e^{\varepsilon \mathcal{L}_{D_1/D_2}(O)}) \right] \leq \delta.$$  

Suppose the attacker only has partial knowledge about the data: the data comes from a distribution $\theta$, and the attacker tries to distinguish between $D(i) = a$ and $D(i) = b$ by observing $M(D)$, given partial knowledge $\hat{B}$. Since $\hat{B}$ is given to the attacker prior to $M(D)$, we must condition the probabilities by $\hat{B}$.
Definition 11 (PLRV for partial knowledge). Given a mechanism $\mathcal{M}$, a distribution $\theta$ with values in $\mathcal{D} \times \mathcal{B}$, an index $i$, and values $a, b \in T$, the PLRV of an output $O \in \mathcal{O}$ given partial knowledge $\hat{B}$ is:

$$\mathcal{L}_{i,a,b}(O, \hat{B}) = \ln \frac{\mathbb{P}_\theta[\mathcal{M}(D) = O | D(i) = a, \hat{B}]}{\mathbb{P}_\theta[\mathcal{M}(D) = O | D(i) = b, \hat{B}]}$$

using the convention $x/0 = \infty$ for all $x$.

This PLRV captures the same idea as for classical DP: it quantifies the attacker’s information gain. If $\hat{B} = D_{-i}$, this definition is the same as the classical PLRV.

Now that we translated the concept of PLRV to account for partial knowledge, we can use it to adapt the privacy definition. The formula in Lemma 11 averages the PLRV over all possible outputs $O$, but the PLRV with partial knowledge has a second parameter, $\hat{B}$. How should this new parameter be handled? There are at least two reasonable possibilities.

3.2 Active partial knowledge

The first option is to quantify over all possibilities for the attacker’s partial knowledge. We assume the worst: we consider the case where the attacker’s partial knowledge causes the privacy to be the greatest. This models a scenario where the attacker can not only see, but also influence the data. If the attacker can, for example, add fake users to the database, then she can choose the values associated to these new records to maximize the chances of information gain. We therefore call this option active partial knowledge under active attacks”.

Definition 12 (APKDP). Given a family of distributions $\Theta$, a mechanism $\mathcal{M}$ is $(\Theta, \varepsilon, \delta)$-APKDP (Active Partial Knowledge Differential Privacy) if for all distributions $\theta \in \Theta$, all indices $i$, all $a, b \in T$, and all $\hat{B} \in \mathcal{B}$:

$$\max_{\theta, D(i) = a, \hat{B} \sim \mathcal{M}(D)} \mathbb{E}_{O \sim \mathcal{M}(D)} \left[ \max \left( 0, 1 - e^{-\varepsilon \mathcal{L}_{i,a,b}(O, \hat{B})} \right) \right] \leq \delta,$$

or, equivalently:

$$\mathcal{M}(D)_{\theta, D(i) = a, \hat{B}} \approx_{(\varepsilon, \delta)} \mathcal{M}(D)_{\theta, D(i) = b, \hat{B}}.$$

The proof of this equivalence is the same as in [MM18, Lemma 1], which makes it explicit that APKDP is the same as NP in its reformulation in [BGKS13]. As shown in [TSD17], APKDP and DP coincide whenever the attacker has full knowledge (when $\hat{B} = D_{-i}$).

With APKDP, a fixed part of the distribution can be arbitrarily determined. In Example 7 this corresponds to the attacker controlling some percentage of voters. Such an active attacker can simply add many fake “Yes” votes to the database to reach the threshold of 100, rendering the thresholding pointless. $\mathcal{M}$ then becomes a simple counting query without providing privacy. With high probability, everybody votes “No”, and the only uncertainty left is over the attacker’s target.

In addition to modeling an active attacker, APKDP can also be used in scenarios where $\theta$ is unknown, but can be approximated. The “partial knowledge” can represent the error between the true distribution and the approximation and if APKDP is satisfied, then the privacy property also holds for the true database.

Note that in this context, explicitly modeling the background knowledge $\hat{B}$ is technically not necessary. Instead, we could simply create a new family of probability distributions $\Theta'$ by conditioning each $\theta \in \Theta$ by the value of each possible $\hat{B}$. We make this background knowledge explicit instead, so APKDP is easier to compare with PPKDP, defined in the next section.

3.3 Passive partial knowledge

APKDP represents situations where the attacker can modify the data. An example is an online service that publishes statistics about its use, where the attacker can interact with the service before the usage statistics are published. Now, what if the attacker cannot interact with the data? Consider e.g. researchers publishing the results of a clinical study about patients having a medical condition. A typical attacker cannot influence the clinical data, but might have some partial knowledge about other participants to the survey.

How can we model such a passive attacker, which has access to some information about the data, but cannot influence it? It no longer makes sense to quantify over arbitrary partial knowledge. In the same
way that the reformulation of $(\varepsilon, \delta)$-DP using the PLRV averages the PLRV over all possible outputs, we must average the PLRV over all possible values of the partial knowledge.

**Definition 13 (PPKDP).** Given a family of distributions $\Theta$, a mechanism $M$ is $(\Theta, \varepsilon, \delta)$-PPKDP (Passive Partial Knowledge Differential Privacy) if for all distributions $\theta \in \Theta$, all indices $i$, and all $a, b \in \mathcal{T}$:

$$\mathbb{E}_{\theta(\cdot) \sim O \sim M(D)} \left[ \max \left( 0, 1 - e^{-\varepsilon} M^\delta_{\theta} (O, B) \right) \right] \leq \delta.$$ 

In this context, $\delta$ has a similar meaning as in $(\varepsilon, \delta)$-DP: it captures the probability that the attacker is lucky. In $(\varepsilon, \delta)$-DP, it means that $O$ allows the attacker to distinguish between $D_1$ and $D_2$ with high probability, or, equivalently, that the PLRV associated to output $O$ is large. In $(\varepsilon, \delta)$-PPKDP however, $\delta$ captures the probability of the attacker getting either a favorable output $O$, or a favorable partial knowledge $B$.

With PPKDP, the thresholding mechanism of Example 7 is private. Indeed, with high probability, the partial knowledge will have only “No” votes; and almost certainly, the mechanism will output $\perp$ and gives no information. We formalize this intuition in Section 4.2.

Note that as opposed to APKDP, PPKDP cannot be easily reformulated using $(\varepsilon, \delta)$-indistinguishability. Since the statement $B = \hat{B}$ **conditions** both probabilities, the $\delta$ in $(\varepsilon, \delta)$-indistinguishability only applies to the randomness of $M$. To use an indistinguishability-based formulation, we would need to use $\delta$ twice, and (for example) explicitly require that $(\varepsilon, \delta)$-indistinguishability holds with probability $1 - \delta$ over the choice of $\hat{B}$.

**Remark.** PPKDP shares some characteristics with inference-based distributional differential privacy (IBDDP), introduced in [BGKST13]. A mechanism $M$ satisfies IBDDP if there is a simulator $Sim$ such that for all probability distributions $\theta \in \Theta$, and indices $i$, the statement:

$$D(i)_{|\theta, M(D) = O, B} \approx_{(\varepsilon, \delta)} D(i)_{|\theta, Sim(D_{-i}) = O, B}$$

holds with probability $1 - \delta$ over the choice of $\hat{O}$ and $\hat{B}$.

Leaving aside the simulator, note that $\delta$ is used in two separate parts of the definition: both over the choice of $O$ and $B$, and in the indistinguishability. As such, it is difficult see intuitively what $\delta$ corresponds to, and the interpretation based on the “probability that the attacker gets more information than $e^\varepsilon$” is not correct. This is one of the reasons why a PLRV-based formulation is more convenient: $\delta$ can simply be interpreted in the same way as in $(\varepsilon, \delta)$-differential privacy.

Further, the strict implication between DDP and IBDDP proven in [BGKST13] can be explained by a similar distinction between an active and a passive attacker, even if it is not made explicit in the original paper. Indeed, when $\delta > 0$, this $\delta$ applies only to the indistinguishability property of DDP, and DDP quantifies over all possible values of the background knowledge: the attacker is assumed to be able to choose the most favorable value of the background knowledge. In contrast, with IBDDP, $\delta$ is applied both to the indistinguishability property and to the choice of the background knowledge; hence the attacker is implicitly assumed to get the background knowledge randomly.

### 3.4 Relation between definitions

In this section, we formalize the relation between PPKDP and APKDP and show basic results on those definitions. First, APKDP and PPKDP satisfy both privacy axioms proposed in [KL10]. These axioms express natural properties that we expect to be true for any reasonable definition of privacy.

**Proposition 5.** PPKDP satisfies the post-processing axiom: if a mechanism $M$ is $(\Theta, \varepsilon, \delta)$-PPKDP, then for any function $f$, the mechanism $M'$ defined by $M'(D) = f(M(D))$ is also $(\Theta, \varepsilon, \delta)$-PPKDP. It also satisfies the convexity axiom: if two mechanisms $M_1$ and $M_2$ are both $(\Theta, \varepsilon, \delta)$-PPKDP, then the mechanism $M$ that applies $M_1$ with some probability $p$ and $M_2$ with probability $1 - p$ is also $(\Theta, \varepsilon, \delta)$-PPKDP.

APKDP also satisfies these axioms.

**Proof.** For APKDP, the reformulation of the definition using classical $(\varepsilon, \delta)$-indistinguishability makes the result straightforward: the proof is the same as for $(\varepsilon, \delta)$-DP. For PPKDP, we first reformulate the definition using $f$-divergence. Let $f(x) = \max(0, 1 - e^\varepsilon x)$. Then a mechanism $M$ is $(\Theta, \varepsilon, \delta)$-PPKDP iff:

$$\sum_B \mathbb{P} \left[ \hat{B} \mid D(i) = a \right] D_f \left( O_{a, B} \| O_{b, B} \right) \leq \delta$$
where \( O_{\hat x,\hat B} = \mathcal{M}(D)_{\theta, D(i) = \hat x, \hat B} \). This view allows us to use the monotonicity and joint convexity properties of the \( f \)-divergence to immediately prove the result for PPKDP. \[\square\]

We saw that \((\varepsilon, \delta)\)-APKDP bounds the probability mass of the PLRV above \( \varepsilon \) by \( \delta \), for all possible partial knowledge \( \hat B \). By contrast, PPKDP bounds the same probability mass, averaged over all possible values of \( \hat B \), weighted by their likelihood. We formalize this interpretation and use it to show that APKDP is, as expected, stronger than PPKDP. More surprisingly, we also use it to show that when \( \delta = 0 \), both definitions are equivalent.

**Theorem 1.** Given a distribution \( \theta \), a mechanism \( \mathcal{M} \), an index \( i \), two values \( a, b \in T \), an output \( O \), a possible value of the partial knowledge \( \hat B \), and a fixed \( \varepsilon > 0 \), let us denote \( m(O, \hat B) = \max \left( 0, 1 - e^{\varepsilon - \mathcal{L}_{\hat x, \hat B}(O, \hat B)} \right) \). The respective quantities bounded by the requirements of APKDP and PPKDP are:

\[
APK_{i,a,b,\hat B} = \mathbb{E}_{\theta, D(i) = a, \hat B} m(O, \hat B)
\]

and:

\[
PPK_{i,a,b} = \mathbb{E}_{\theta, D(i) = a} m(O, \hat B)
\]

As an immediate consequence, if \( \mathcal{M} \) is \((\Theta, \varepsilon, \delta)\)-APKDP, then it is also \((\Theta, \varepsilon, \delta)\)-PPKDP. Further, \((\Theta, \varepsilon)\)-PPKDP and \((\Theta, \varepsilon)\)-APKDP are equivalent.

**Proof.** We decompose PPK\(_{i,a,b}\) depending on \( \hat B \):

\[
PPK_{i,a,b} = \sum_{\hat B} \mathbb{P}[\hat B | D(i) = a] \mathbb{E}_{\hat B, \mathcal{M}(\hat B)} m(O, \hat B)
\]

\[
= \sum_{\hat B} \mathbb{P}[\hat B | D(i) = a] \sum_{\hat D} \mathbb{P}[\hat D | D(i) = a, \hat B] \mathbb{E}_{\hat D, \mathcal{M}(\hat D)} m(O, \hat B)
\]

\[
= \sum_{\hat B} \mathbb{P}[\hat B | D(i) = a] \mathbb{E}_{\theta, D(i) = a, \hat B, \mathcal{M}(\hat B)} m(O, \hat B)
\]

\[
= \mathbb{E}_{\theta, D(i) = a} APK_{i,a,b,\hat B}
\]

For the second part of the statement, if the mechanism is \((\varepsilon, \delta)\)-APKDP, then for all \( i, a, b, \) and \( \hat B \), APK\(_{i,a,b,\hat B} \leq \delta \), so:

\[
PPK_{i,a,b} = \mathbb{E}_{\theta, D(i) = a} APK_{i,a,b,\hat B} \leq \mathbb{E}_{\theta, D(i) = a} [\delta] \leq \delta.
\]

For the last part of the statement, assume that \( \mathcal{M} \) is \((\Theta, \varepsilon)\)-PPKDP. Then:

\[
0 = PPK_{i,a,b} = \mathbb{E}_{\theta, D(i) = a} APK_{i,a,b,\hat B}.
\]

All summands are non-negative, so the sum can only be 0 if all summands are 0: for all \( \hat B \), APK\(_{i,a,b,\hat B} = 0 \), and \( \mathcal{M} \) is \((\Theta, \varepsilon)\)-APKDP. \[\square\]

When \( \delta > 0 \), the implication is strict: the PLRV can be arbitrarily higher for certain values \( \hat B \) of the background knowledge. Thus, quantifying over all possible values \( \hat B \) can lead to much larger values of \( \varepsilon \) and \( \delta \) than averaging over all possible values of the background knowledge: Example 7 illustrates this phenomenon. When \( \delta = 0 \) however, \( \varepsilon \)-APKDP and \( \varepsilon \)-PPKDP are both worst-case properties, like \( \varepsilon \)-DP: an attacker’s ability to choose the background knowledge does not matter, since even for the passive attacker, we need to consider the worst possible output \( O \) and background knowledge \( \hat B \).
3.5 Θ-reducible mechanisms

For some mechanisms and probability distributions, active attackers are, perhaps surprisingly, not more powerful than passive attackers, even when δ > 0. We introduce here a necessary condition for APKDP and PPKDP to be equivalent and we show that this condition appears in natural contexts.

Consider the example of a referendum where 2000 users take part in a vote with two options. Each user i votes “yes” with probability p_i, and “no” with probability 1 − p_i. The mechanism M returns the exact tally of the vote. We assume that the attacker knows half of the votes: their partial knowledge is the vote of 1000 users. They might know that e.g. 500 of these users voted “yes”, 500 voted “no”, and the remaining 1000 votes are unknown. The attacker aims to get information on the vote of her target.

Does it matter in this situation whether the attacker is passive or active? If the attacker can choose the votes of 1000 users, she can decide that each known user will vote “yes”. Yet, changing these votes will only modify the tally in a predictable way: the attacker can remove these votes from the total tally. So, it does not matter whether these known users all vote “yes”, “no”, or have any other behavior known to the attacker. Without dependency relationships between users, the attacker’s uncertainty solely resides in the unknown votes; so, a passive attacker is not weaker than an active attacker.

We generalize this intuition via the concept of Θ-reducibility, which characterizes that all possible values of background knowledge are equivalent from a privacy perspective. We show that under this condition APKDP is equivalent to PPKDP and formalize the above example to show that it satisfies this condition.

**Definition 14 (Θ-reducibility).** A mechanism M is Θ-reducible if for all indices i, and all B_1, B_2 ∈ B, there is a bijective mapping φ_i, B_1, B_2 : O → O such that for all a ∈ T and for all O ∈ O:

\[ \mathbb{P}_a [M(D) = O(D(i) = a, B = B_1)] = \mathbb{P}_a [M(D) = φ_i, B_1, B_2(O) | D(i) = a, B = B_2]. \]

This equivalence between possible outputs under B_1 and B_2 can be translated to an equivalence between the corresponding PLRVs: if M is Θ-reducible, then L_i, a, b, Θ(O, B_1) | O−M(D), D(i) = a has the same global behavior as L_i, a, b, Θ(O, B_2) | O−M(D), D(i) = a. This equivalence between PLRVs enables us to show that Θ-reducibility implies APKDP and PPKDP are the same, even when δ > 0. So there are mechanisms and background knowledge functions for which an active attacker is not stronger than a passive one.

**Theorem 2.** Let Θ be a family of probability distributions, and let M be a Θ-reducible mechanism. Then M is (Θ, ε, δ)-APKDP iff it is (Θ, ε, δ)-PPKDP.

**Proof.** First, we show that for all O ∈ O and all B_1, B_2 ∈ B:

\[ L_i, a, b, Θ(O, B_1) = L_i, a, b, Θ(φ_i, B_1, B_2(O), B_2). \]

This statement directly follows by unfolding the definition of L_i, a, b, Θ(O, B_1) (Definition 11) and Θ-reducibility (Definition 14).

We can now prove Theorem 2. Since APK_i, a, b, B is the expected value of PPK_i, a, b, B (Theorem 1), it is enough to prove that for any B_1 and B_2, APK_i, a, b, B_1 = APK_i, a, b, B_2. Recall that m(O, B) = max \( \{0, 1 - e^{−L_i, a, b, Θ(O, B)}\} \), and fix B_1 and B_2 in B. Abusing notations, we denote B_1 (resp. B_2) the event “B = B_1” (resp. B = B_2), Ō the event “D = D̂”, and a the event “D(i) = a”. We have:

\[ APK_i, a, b, B_1 = \sum_O \mathbb{P} \left[ M(D) = \hat{O} | a, B_1 \right] m(\hat{O}, B_1) \]

\[ = \sum_O \mathbb{P} \left[ M(D) = φ_i, B_1, B_2(\hat{O}) | a, B_2 \right] m(φ_i, B_1, B_2(\hat{O}), B_2) \]
Proof. Fix $B$ present in $M$ independently, and assume that the background knowledge of the attacker is $k$ records not present in $B$. Then for all $O \in \mathcal{O}$ and all $a, B \in \mathcal{B}$:

$$\mathbb{P}_\theta [M(D) = O | D(i) = a, B = B_1] = \mathbb{P}_{D' \sim \omega} [M(D' \cup B_1) = O | (D' \cup B_1)_i = a]$$

$$= \mathbb{P}_{D' \sim \omega} [M(D' \cup B_2) = \phi_{B_1, B_2,i}^{\text{count}}(O) | (D' \cup B_2)_i = a]$$

$$= \mathbb{P}_\theta [M(D) = \phi_{B_1, B_2,i}^{\text{count}}(O) | D(i) = a, B = B_2] .$$

For linear queries, we use a similar mapping to $\phi$, which also depends on the mapping. Let $I$ be the indices of records present in $B$; then a linear query is $\Theta$-reducible with:

$$\phi_{B_1, B_2,i}^{\text{linear}}(M(D)) = M(D) + \sum_{j \in I} \alpha_j (B_2)_j - (B_1)_j .$$

This also proves the result for the arithmetic mean.

Similarly, it is easy to verify that the following $\phi$ functions show $\Theta$-reducibility for the geometric, harmonic and quadratic mean.

$$\phi_{B_1, B_2,i}^{\text{geometric}}(M(D)) = \left( M(D)^n \cdot \prod_{j \in I} (B_2)_j \right)^{1/n}$$

$$\phi_{B_1, B_2,i}^{\text{harmonic}}(M(D)) = n \left( \frac{M(D)}{n} \right)^{-1} + \sum_j \left( \frac{1}{(B_2)_j} - \frac{1}{(B_1)_j} \right)^{-1}$$

$$\phi_{B_1, B_2,i}^{\text{quadratic}}(M(D)) = \sqrt{n \cdot M(D)^2 + \sum_j (B_2)_j^2 - (B_1)_j^2}$$

The injectivity of each of these functions is clear.

Note that in Proposition 6, it is crucial that the records in the attacker’s partial knowledge are fixed. In general, knowing the records of $k$ users is not equivalent to knowing the records of $k$ other users. Indeed, if these records are generated with different probabilities, the randomness from the unknown records differs between both scenarios, and it is in general impossible to convert one into the other.
Remark. Observation 1 in [GK17] claims that the partial knowledge of $k$ records in a database of size $n$ is the same as no partial knowledge in a database of size $n - k$. For counting queries, this holds for the same reason that counting queries are $\Theta$-reducible: one can “remove” the $k$ known records from the mechanism output and obtain a bijection between the cases with and without partial knowledge. Thus, the partial knowledge is irrelevant to the mechanism’s privacy and can be ignored.

This observation, however, does not hold in general: we later show in Example 7 that counting queries with thresholding are not $\Theta$-reducible, and in this case, the knowledge of $k$ records in a database of size $n$ has a very different effect than no partial knowledge in a database of size $n - k$.

4 Applications

In this section, we show that the foundations laid in the previous sections can be applied to practical problems. First, we propose improved bounds on the privacy of noiseless counting queries under partial background knowledge. Second, we investigate counting queries with thresholding and show that thresholding can be used to improve privacy for these queries.

These two results formalize intuitions that privacy practitioners frequently use when trying to assess the risk of releasing aggregated data. Theorem 3 provides a rigorous explanation of why, under reasonable assumptions on the background knowledge of a realistic attacker, counting queries over a large user population do not necessarily leak individual information. Theorem 4 shows that thresholding provides protection in those cases where a counting query only captures a small number of users; this provides a new interpretation of this technique as giving formal privacy guarantees when the attacker has partial background knowledge.

4.1 Counting queries

The initial motivation for limiting the attacker’s background knowledge was to show that, under this assumption, some noiseless mechanisms preserve the individuals’ privacy [BBG+11]. A typical example is a counting query, which answers the question “How many users satisfy $P$?” for some property $P$. We can model this by a data-generating distribution $\theta$ where each record $D(i)$ is either 0 or 1 with some probability $p_i$, and we want to measure the privacy of the mechanism $M(D) = \sum_i D(i)$. Records are assumed to be independent, and the adversary is assumed to know some portion of the records. As an immediate consequence of Theorem 2 and Proposition 6, it does not matter whether the attacker can modify, or only see, this portion of records: the values of $\varepsilon$ and $\delta$ are identical for APKDP and PPKDP.

Furthermore, the closer $p_i$ are to 0 or 1, the less randomness is present in the data. For extremely small or large values of $p_i$, the situation is very similar to one where the attacker exactly knows $D(i)$. As such, it is natural to assume that among the records that are unknown by the attacker, all $p_i$ are between $\lambda$ and $1 - \lambda$, for some $\lambda$ not too close to 0. This assumption can easily be communicated to non-specialists: “we assume that there are at least 1000 records that the attacker does not know, and that her level of uncertainty is at least 10% for these records.”

Initial asymptotic results in this context appeared in [BBG+11] and more precise bounds were derived in [GK17]. In the special case where all $p_i$ are equal to a fixed value $p$, Theorem 5 in [BBG+11] and Theorem 1 in [GK17] show that counting queries are APKDP with $\varepsilon = O\left(\sqrt{\frac{\ln(1/\delta)}{n}}\right)$ (for small $\delta$, and increasing $n$) and $\delta = e^{-\Omega(\varepsilon^2 n)}$ (for small $\varepsilon$, and increasing $n$). This provides tiny values of $\varepsilon$ and $\delta$ for moderate values of $n$ and $p$. However, the assumption that all $p_i$ are identical is unrealistic: in practice, there is no reason to assume that all users have an equal chance of satisfying $P$. Theorem 7 in [BBG+11] and Theorem 2 in [GK17] show that without this assumption, the $\varepsilon$ obtained is still small: $\varepsilon = O\left(\sqrt{\frac{\ln(n)}{n}}\right)$, but the upper bound obtained on $\delta$ is significantly larger: $\delta = O\left(\frac{1}{\sqrt{n}}\right)$. This is more than what is typically acceptable; a common recommendation is to choose $\delta = o\left(\frac{1}{n}\right)$.

In the following theorem, we show that the exponential decrease of $\delta$ with $n$ still holds in the general case where all $p_i$ are different. For simplicity, we assume that the attacker has no background knowledge: because all records are independent, adding some partial knowledge has a fixed, reversible effect on the output space, similarly to $\Theta$-reducibility. In this case, having the attacker know $m$ records out of $n$ is the same as having the attacker know no records among $n - m$.

Theorem 3. Let $\theta$ be a distribution that generates $n$ records, where $D(i)$ is the result of an independent Bernoulli trial of probability $p_i$. Let $\lambda$ be such that for all $i$, $\lambda < p_i < 1 - \lambda$. Let $M$ be defined by
\[ \mathcal{M}(D) = \sum_i D(i). \] Then \( \mathcal{M} \) is \((\Theta, \varepsilon, \delta)\)-APKDP, for any \( \varepsilon \) and \( \delta \) such that:

\[ \delta \geq \Pr \left[ \frac{X}{Y} \geq \varepsilon \right] \]

where \( X \) and \( Y \) are independent random variables sampled from a binomial distribution with \( n-1 \) trials and success probability \( 2\lambda \). For a fixed \( \varepsilon \leq 1 \), this condition is satisfied if:

\[ \varepsilon \geq \max \left( \sqrt{\frac{14\ln(1/\delta)}{\lambda(n-1)}}, \frac{27}{\lambda(n-1)} \right) \]

which gives \( \delta = e^{-\Omega(\varepsilon^2 \lambda n)} \).

**Proof.** The proof uses existing results on privacy amplification by shuffling: in \cite{EFM19, BBGN19}, the authors show that adding noise independently to each data point, and then shuffling the results (hiding from the attacker which record comes from which user), provide strong DP guarantees. Even though our problem looks different, the same reasoning can be applied. First, we show that \( \theta \) can be seen as applying randomized response on each record. Then, since a counting query is a symmetric boolean function, it can be composed with a shuffle of its input, which allows us to use amplification by shuffling.

Let us formalize this intuition. A Bernoulli trial of probability \( p_i \) (denoted Bernoulli \( (p_i) \)), with \( \lambda < p_i < 1 - \lambda \), can be decomposed into the following process:

- Generate \( b \sim \text{Bernoulli}(2\lambda) \).
- If \( b = 0 \), return \( b_{uv} \sim \text{Bernoulli} \left( \frac{p_i - \lambda}{1 - 2\lambda} \right) \).
- If \( b = 1 \), return \( b_{vr} \sim \text{Bernoulli}(0.5) \).

This can be seen as a *randomized response* process applied on some input \( b_{uv} \), itself random: \( \theta = \mathcal{R}_{2\lambda}(\nu) \), where \( \nu \) is a distribution generating \( n \) records, where the \( i \)-th record is generated by \( b_{uv} \sim \text{Bernoulli} \left( \frac{p_i - \lambda}{1 - 2\lambda} \right) \), and \( \mathcal{R}_{2\lambda} \) is a binary randomized response process with parameter \( 2\lambda \).

Note that \( \mathcal{M} \) can be seen as the composition between itself and a pre-shuffling phase: \( \mathcal{M} = \mathcal{M} \circ \mathcal{S} \), where \( \mathcal{S} : \mathcal{D} \to \mathcal{D} \) is a function that applies a random permutation to the input records. Thus, \( \mathcal{M}(D)_{i=D^{\circ} \theta} = \mathcal{M}(S(\mathcal{R}_{2\lambda}(D))_{i=D^{\circ} \theta}) \). We can now apply Theorem 3.1 in \cite{BBGN19} and its proof to show that \( S \circ \mathcal{R}_{2\lambda} \) is \((\varepsilon, \delta)\)-DP for \( \varepsilon \) within the constraints above (with \( k = 2 \) and \( \gamma = 2\lambda \)). By post-processing, \( \mathcal{M} \circ S \circ \mathcal{R}_{2\lambda} \) is also \((\varepsilon, \delta)\)-DP, which directly yields that \( \mathcal{M} \) is \((\varepsilon, \delta)\)-APKDP.

Note that we omitted a small technical detail: conditioning \( \theta \) on \( D(i) = a \) is not identical to conditioning \( \nu \) on \( D(i) = a \), since no noise is added to the record \( i \) in the former case. To fix this, we need to define \( \mathcal{R}_{\lambda} \) as randomizing all records except a fixed one \( i \). The proof of Theorem 3.1 in \cite{BBGN19} assumes that no noise is added to the target record, so the result still holds.

We compare this result with the previous state-of-the-art. First, we reformulate a previously known result from \cite{GK17} that applies to our setting.

**Proposition 7** (Theorem 3 in \cite{GK17}). Let \( \theta \) be a distribution that generates \( n \) records, where \( D(i) \) is the result of an independent Bernoulli trial of probability \( p_i \), and let \( \mathcal{M} \) be defined by \( \mathcal{M}(D) = \sum_i D(i) \). Let \( \mu_2 = \frac{1}{n} \sum_i p_i(1-p_i) \) and \( \mu_3 = \frac{1}{n} \sum_i p_i(1-p_i)(1-2p_i) \) be respectively the average second moment and average absolute third moment of the \( D(i) \). Then for any \( \delta_2 \geq 1.25e^{-n\mu_2/2} \), \( \mathcal{M} \) is \((\theta, \varepsilon, \delta)\)-APKDP, with

\[ \varepsilon = \sqrt{\frac{2 \ln(1.25/\delta_2)}{n\mu_2}} \quad \text{and} \quad \delta = \frac{1.12\mu_3}{\sqrt{n\mu_2}} (1 + e^\varepsilon) + \delta_2. \]

**Proof.** For \( \delta_2 = \frac{4}{\sqrt{n}} \), this is a direct application of Theorem 3 in \cite{GK17}. Changing the value \( \delta_2 \) in its proof (Appendix A.2) allows us to obtain the more general formula above. This requires Fact 1 to be true, which is the case when \( \varepsilon \leq 1 \) (the authors omit this detail), or equivalently, when \( \delta_2 \geq 1.25e^{-n\mu_2/2} \).

The comparison between this result and Theorem 3 is not completely straightforward. Aside from \( n \), Theorem 3 only depends on a global bound on the “amount of randomness” \( (p_i) \) of each user, while Proposition 7 depends on the average behavior of all users. As such, the global bound \( \lambda \) can be small because of one single user having a low \( p_i \), even if all other users have a lot of variance because their \( p_i \).
Figure 2: Comparison of \((\varepsilon, \delta)\) bounds given by Theorem 3 and Proposition 7 for \(\lambda = 0.05, n = 100\) (left) and \(n = 10,000\) (right).
Case 1: all but one \(p_i\) are 0.5. Case 2: the \(p_i\) are distributed uniformly over \([0.05, 0.95]\).

Figure 3: Comparison of \(\varepsilon\) bounds given by the numerical computation of Theorem 3 with varying \(\lambda\), for various values of \(n\) and \(\delta = 0.01/n\).

is close to 0.5. We therefore provide two experimental comparisons. In the first one, \(p_1 = \lambda = 0.05\) and \(p_i = 0.5\) for all \(i > 1\). This case is designed to have the parameters of Theorem 3 underperform (as we underestimate the total amount of randomness) and those of Proposition 7 perform well. In the second one, the \(p_i\) are uniformly distributed in \([\lambda, 1 - \lambda] = [0.05, 1 - 0.05]\). In both cases, we compare the \((\varepsilon, \delta)\) graphs obtained for \(n = 10^3\) and \(n = 10^5\), and present the results in Figure 2.

The graphs show that if we consider the smallest possible \(\varepsilon\) given by the definitions, our theorem leads to a large \(\delta\): with \(\varepsilon = \Theta(1/\sqrt{n})\), we obtain \(\delta = \Theta(1)\); in contrast, Proposition 7 leads to \(\delta = \Theta(1/\sqrt{n})\). However, increasing \(\varepsilon\) to slightly larger values quickly leads to tiny values of \(\delta\), which was impossible with the previous state-of-the-art results. They also show that the closed-form bound from [BBGN19] is far from tight, as numerically computing these bounds improves them by several orders of magnitude. This leads to a natural open question: is there a better asymptotic formulation of the bounds given by amplification by shuffling for randomized response?

What is the impact of \(\lambda\) on the privacy guarantees? In Figure 3, we plot the \(\varepsilon\) obtained for \(\delta = 0.01/n\) as a function of \(\lambda\), for various values of \(n\).

One natural application for this result is voting: in typical elections, the total tally is released without any noise. Adding noise to the election results, or not releasing them, would both be unacceptable. Thus, the results are not \((\varepsilon, \delta)\)-DP for any \((\varepsilon, \delta)\) parameters, even though publishing the tally is not perceived as a breach of privacy. The intuitive explanation for this is that attackers are assumed not to have complete background knowledge of the secret votes. Our results confirm this intuition and quantify it.
These results can easily be extended to votes between multiple candidates.

**Corollary 1.** Let \( \theta \) be a distribution that generates \( n \) records, where \( D(i) \in \{1, \ldots, K\} \) for \( K > 1 \), and where \( \mathbb{P}[D_i = k] = p_{i,k} \), and every record is independent from all others. Let \( \lambda \) be such that for all \( i \) and all \( k \), \( \lambda < p_{i,k} \). Let \( M \) return the histogram of all values: \( \mathcal{M}(D) = (N_1, \ldots, N_K) \), where \( N_k \) is the number of records \( i \) such that \( D(i) = k \). Then \( \mathcal{M} \) is \((\Theta, \varepsilon, \delta)\)-APKDP, for any \( \varepsilon \leq 1 \) and \( \delta > 0 \) such that:

\[
\varepsilon \geq \max \left( \frac{14 \ln(1/\delta)}{\lambda(n-1)}, \frac{27}{\lambda(n-1)} \right).
\]

**Proof.** The proof is the same as for Theorem 3. With multiple options, the parameter \( \gamma \) of the multicategory randomized response is \( \gamma = K\lambda \), which leads to the same \((\varepsilon, \delta)\) parameters.

The results in this section apply to *individual* counting queries. This applies to scenarios like votes, but in many practical use cases, multiple queries are released. Can the results of this section be generalized to these cases? In general, noiseless mechanism do not compose. For example, fixing an individual \( t \), queries like “How many people voted 1?” and “How many people who are not \( t \) voted 1?” can both be private on their own. However, publishing both results will reveal \( t \)'s vote: the composition of both queries cannot be private. Are there special cases where noiseless counting queries can be composed?

One such case happens when each counting query contains the data of a number of new users, independent from users in previous counting queries. This can happen in situations where statistics are collected on actions that each user can only do once, for example, registering with an online service. In this case, we can restrict the privacy analysis of each new query to the set of independent users in its input, and use the previous results from this section: this approach is formalized and proven in Theorem 1 in \([\text{BBG}+11]\).

What if this approach is impossible, for example if there are dependencies between each input record of each query? For example, a referendum could ask voters multiple questions, with correlations between the different possible answers. Another example could be app usage statistics published every day, where the data for day \( d + 1 \) for each user is correlated to the user’s data on the previous day \( d \). In this case, to compute the privacy loss of the first \( d \) binary queries, we can consider them as a single query with \( 2^d \) options. Afterwards, we can take into account the temporal correlations to compute the probabilities associated with each option, and use Corollary 1.

### 4.2 Thresholding

Theorem 3 gives good \( \varepsilon \) and \( \delta \) parameters when there are many people who vote with “enough randomness”: there is a \( \lambda \) such that \( \lambda < p_i < 1 - \lambda \). The parameters have a dependency on \( \lambda n \), which in practice translates to scenarios where both options have large counts with high probability. In many practical applications, however, it is hard to know in advance whether this will be the case. Consider, for example, a mobile app gathering usage metrics on possible sequences of actions carried by users within the app. Some sequences will be very probable, and have high counts. But if there are arbitrarily many such sequences, some will be very rare: like many practical distributions, there will be a long tail.

To protect the data from these outlier users, a typical protection employed is thresholding: only return the user count associated with a sequence if it is larger than a given threshold \( T \). What level of protection does such a technique provide? In this section, we formalize the intuition given in Example 7 and show that, assuming a passive attacker, thresholding provides protection when all voters vote with a very small or a very large probability. First, we formalize the notion of a thresholding mechanism in a simple context.

**Definition 15** (Simple thresholding). Given a database \( D = (D(1), \ldots, D(n)) \) with values in \( \{0, 1\} \) and a threshold \( T \), the \( T \)-thresholding mechanism \( \mathcal{M}_T \) evaluates \( \hat{k} = \sum_i D(i) \) and returns \( \perp \) if \( \hat{k} \leq T \), and \( k \) otherwise.

Note that \( \mathcal{M}_T \) only thresholds low counts. In many practical situations, however, thresholding is applied in both directions: it must also catch the case where, with high probability, almost all records are 1. This situation is symmetrical to thresholding low counts: without loss of generality, we can assume \( T < n/2 \), and the symmetric version of all results in this section hold.

Let us now show the main result of this section: if participants vote 1 with low probability, then thresholding protects against passive attackers, and in some cases also against certain active attackers. This privacy property only holds if the expected value of the count is lower than the threshold; and the
level of protection depends on the ratio between the threshold and the expected value (denoted by \( r \) below).

**Theorem 4.** Let \( \theta \) be a distribution that returns \( n \) independent records, each of which is 1 with low probability: \( D(i) \sim \text{Ber}(p_i) \), and \( p_i < p \) for all \( i \); moreover, let \( \Theta = \{ \theta \} \). Suppose that there is no partial knowledge, i.e., \( |B| = 0 \), and let us denote by \( f(s, n, p) \) the probability that a random variable following a binomial distribution with parameters \( n \) and \( p \) has value \( s \): \( f(s, n, p) = p^s(1-p)^{n-s} \).

Then, if \( r = \frac{p(n-1)}{(1-p)k} < 1 \), \( \mathcal{M}_T \) is \((\Theta, \varepsilon, \delta)\)-APKDP (and thus, \((\Theta, \varepsilon, \delta)\)-PPKDP), with:

\[
\varepsilon = -\ln \left( 1 - \frac{f(T, n-1, p)}{1 - r} \right) \quad \delta = \frac{f(T, n-1, p)}{1 - r}.
\]

For a large \( n \), assuming \( p_n \) is fixed, we can use the Poisson approximation and get \( \varepsilon \approx \frac{(p_n)^T e^{-n}}{(1-r)T} \). If this quantity is small enough, \( \varepsilon \approx \delta \).

If the background knowledge \( B \) is not empty, assume that the attacker knows a subset \( |B| \) of records.

Let \( b_{\text{max}} \) be such that \( r_b = \frac{\binom{|B|}{n} \cdot p^{\binom{n-1}{k}}}{(1-p)b_{\text{max}}} < 1 \) and \( r' = \frac{p(n-|B|-1)}{(1-p)(T-b_{\text{max}})} < 1 \). Then \( \mathcal{M}_T \) is \((\Theta, \varepsilon, \delta)\)-PPKDP, with:

\[
\varepsilon = -\ln \left( 1 - \frac{f(T - b_{\text{max}}, n - |B| - 1, p)}{1 - r'} \right) \quad \delta = \frac{f(b_{\text{max}}, |B|, p)}{1 - r} + \frac{f(T - b_{\text{max}}, n - |B| - 1, p)}{1 - r'}.
\]

**Proof.** The proof is presented in three stages.

1. First, we consider the simpler case where all \( p_i \) are equal and there is no background knowledge. This allows us to compute the PLRV exactly, and we can then split the output space into two parts. Most of its mass will be in the \( \perp \) event, and we can compute it there exactly. All other events are captured by \( \delta \).

2. Second, we extending this to non-empty partial knowledge in a similar fashion: for some \( b_{\text{max}} \), with high probability, the background knowledge will not have more than \( b_{\text{max}} \) records whose value is 1: the rest of the probability mass goes in the \( \delta \), and this allows us to use the previous idea with a new threshold \( T' = T - b_{\text{max}} \).

3. Finally, we use a coupling argument to extend this to the case where the \( p_i \) are not all the same.

First, let us compute the PLRV for \( \mathcal{M}_T \) depending on the output \( k \) and the value of the background knowledge \( \hat{B} \), assuming a simple distribution \( \theta \) where records are i.i.d. Denote by \( b \) the number of records in \( \hat{B} \) which are 1 and by \( b = |B| - b \) the number of records that are 0. The targeted record will be called \( D(t) \), and we assume it is never part of \( \hat{B} \). Let \( \theta \) be a distribution that returns \( n \) i.i.d records according to \( D(i) \sim \text{Ber}(p) \). Then we can directly compute:

\[
\mathcal{L}_{i,0,1}^{\mathcal{M}_T, \theta}(k, \hat{B}) = \ln \frac{P_{\mathcal{M}(D) = k \mid D(t) = 0, B = \hat{B}}}{P_{\mathcal{M}(D) = k \mid D(t) = 1, B = \hat{B}}} = \begin{cases} \ln \frac{\sum_{s=0}^{T-k} f(s, n-|B|-1, p)}{\sum_{s=0}^{T-k} f(s, n-|B|-1, p)} & \text{if } k = \perp \\ \ln \frac{\binom{n-|B|-k}{n-k} (1-p)^{|B|-b}}{(1-p)^k} & \text{otherwise.} \end{cases}
\]

Note that if \( k = \perp \), then \( \mathcal{L}_{i,0,1}^{\mathcal{M}_T, \theta}(k, \hat{B}) = \infty \). The case where \( k < b \) is impossible regardless of \( D(i) \): there cannot be more 1s in the background knowledge than the mechanism outputs. In the case where there is no background knowledge, this becomes:

\[
\mathcal{L}_{i,0,1}^{\mathcal{M}_T, \theta}(k) = \begin{cases} \ln \frac{\sum_{s=0}^{T-k} f(s, n-1, p)}{\sum_{s=0}^{T-k} f(s, n-1, p)} & \text{if } k = \perp \\ \ln \frac{\binom{n-k}{n-k}}{(1-p)^k} & \text{otherwise.} \end{cases}
\]

This calculation allows us to bound \( \varepsilon \) and \( \delta \) when there is no background knowledge. First, we need a technical lemma to bound the probability mass of the tail of the binomial distribution appearing above.
Lemma 2. For any \( n, p, \) and \( m \) such that \( m > \frac{pn}{1-p} \):
\[
\sum_{s=m}^{n} f(s, n, p) < \frac{f(m, n, p)}{1 - \frac{pn}{(1-p)m}}.
\]

Proof. Note that for all \( s \geq m \):
\[
\frac{f(s + 1, n, p)}{f(s, n, p)} = \frac{p (n - s)}{1 - p} < \frac{pm}{(1-p)m}.
\]
Since \( m > \frac{pn}{1-p} \), this is strictly lower than 1, so the sum converges at least as fast as a geometric series, which directly gives the desired result. \( \square \)

Now, let us prove the main theorem when there is no background knowledge and all \( p_i \) are equal. We need to consider both \( L_{i,1,0}^{M_T,\theta} \) and \( L_{i,0,1}^{M_T,\theta} \). First, we have:
\[
\mathbb{P}_\theta [ M_T(D) \neq \perp | D(i) = 0 ] < \mathbb{P}_\theta [ M_T(D) \neq \perp | D(i) = 1 ]
\]
\[
= 1 - \sum_{s=0}^{T-1} f(s, n-1, p)
\]
\[
= \sum_{s=T}^{n-1} f(s, n-1, p)
\]
\[
< \frac{f(T, n-1, p)}{1 - \frac{p(n-1)}{(1-p)T}}.
\]

since \( T > \frac{p(n-1)}{1-p} \), so we can use Lemma 2. Let us denote this quantity as \( \delta \). Now, we have
\[
L_{i,1,0}^{M_T,\theta} (\perp) = \ln \left( \frac{\sum_{s=0}^{T-1} f(s, n-1, p)}{\sum_{s=0}^{T} f(s, n-1, p)} \right) < 0.
\]
Furthermore:
\[
L_{i,0,1}^{M_T,\theta} (\perp) = \ln \left( \frac{\sum_{s=0}^{T} f(s, n-1, p)}{\sum_{s=0}^{T-1} f(s, n-1, p)} \right)
< \ln \left( \frac{1}{1 - \sum_{s=T}^{n-1} f(s, n-1, p)} \right)
< - \ln \left( \frac{f(T, n-1, p)}{1 - \frac{p(n-1)}{(1-p)T}} \right).
\]

Thus, when the output is thresholded, the PLRV is smaller than \( \varepsilon = - \ln \left( 1 - \frac{f(T, n-1, p)}{1-p} \right) \), and the event “the output is not thresholded” only happens with a probability smaller than \( \delta \), which proves the initial statement in the simpler case.

Now, in the case where the background knowledge is non-empty, we must not only split the output space, but also \( B \) as well. Denoting \( b \) the number of “1” entries in \( B \), there are three cases we must consider:
1. \( b \geq b_{\text{max}} \): if \( b_{\text{max}} \) is large enough, this happens with small probability, which we put in the \( \delta \) term;
2. \( b < b_{\text{max}} \) and \( M_T(D) \neq \perp \): if \( T' = T - b_{\text{max}} \) is large enough, this happens with small probability, which we put in the \( \delta \);
3. \( b < b_{\text{max}} \) and \( M_T(D) = \perp \): this is the event in which most of the probability mass is concentrated on, so we bound its privacy loss to obtain \( \varepsilon \).

The probability of the first event can be bounded by:
\[
\mathbb{P}_\theta [ b \geq b_{\text{max}} ] = \sum_{s=b_{\text{max}}}^{|B|} f(s, |B|, p)
< \frac{f(b_{\text{max}}, |B|, p)}{1 - \frac{p|B|}{(1-p)b_{\text{max}}}}.
\]
by Lemma 2. Similarly, the probability of the second event can be bounded by:

\[
\Pr_\theta \left[ \mathcal{M}_T(D) \neq \bot | b < b_{\text{max}}, D(i) = 1 \right] < \sum_{s = T'}^{n - |B| - 1} f(s, n - |B| - 1, p) < \frac{f(T', n - |B| - 1, p)}{1 - p(n - |B| - 1)}
\]

so we can bound \( \delta \) by the sum of those two terms. Now, let us compute the privacy loss for the third case. Assuming \( b < b_{\text{max}} \), we have:

\[
\mathcal{L}_{i,1,0}^{\mathcal{M}_T,\theta} (\bot, \hat{B}) = \ln \frac{\sum_{s=0}^{T-b-1} f(s, n - |B| - 1, p)}{\sum_{s=0}^{T-b-1} f(s, n - |B| - 1, p)} < 0
\]

and:

\[
\mathcal{L}_{i,0,1}^{\mathcal{M}_T,\theta} (\bot, \hat{B}) = \ln \frac{\sum_{s=0}^{T-b} f(s, n - |B| - 1, p)}{\sum_{s=0}^{T-b} f(s, n - |B| - 1, p)} < \ln \left( \frac{1}{1 - \frac{f(T', n - |B| - 1, p)}{1 - r'}} \right).
\]

Denoting this by \( \varepsilon \), this proves the theorem in the special case where all \( p_i \) are equal to a constant \( p \).

Now, we extend the first case (where the background knowledge is empty) to the case where all \( p_i \) are different, and \( p_i < p \) for all \( p \). Let us denote \( \theta_p \), the distribution where all users vote with the same probability \( p \). Let \( g(s, n, p) = \Pr_\theta \left[ \mathcal{M}_T(D) = s \right] \). By a simple coupling argument between \( \theta \) and \( \theta_p \), we have for all \( t \):

\[
\sum_{s=t+1}^{n} g(s, n, p) = \Pr_\theta \left[ \mathcal{M}_T(D) > t \right] \\
\leq \Pr_{\theta_p} \left[ \mathcal{M}_T(D) > t \right] \\
= \sum_{s=t+1}^{n} f(s, n, p).
\]

We can then use this fact throughout the previous proof. This is immediate for \( \delta \), and for \( \varepsilon \), we have:

\[
\mathcal{L}_{i,0,1}^{\mathcal{M}_T,\theta} (\bot, \hat{B}) = \ln \frac{\sum_{s=0}^{T} g(s, n - 1, p)}{\sum_{s=0}^{T} g(s, n - 1, p)}.
\]

Bound the numerator by 1 and expand the denominator:

\[
\sum_{s=0}^{T-1} g(s, n - 1, p) = 1 - \sum_{s=T}^{n-1} g(s, n - 1, p) \\
\geq 1 - \sum_{s=T}^{n-1} f(s, n - 1, p) \\
= \sum_{s=0}^{T-1} f(s, n - 1, p)
\]

so we can reuse the previous bound. The bounds translate to the case where the background knowledge is not empty by a similar argument.

As shown in Figure 4, when the threshold is above the expected value, the values \( \varepsilon \) and \( \delta \) given by Theorem 4 are very close. Moreover, for large \( n \), these values are extremely small. This shows that thresholding counts constitutes a good practice, which can be used to meaningfully improve user privacy.
Figure 4: $\epsilon$ and $\delta$ from Theorem 4 as a function of the threshold $T$, where $|B| = 0$, $p = 0.5\%$, and three different values of $n$: $n = 1000$ (top), $n = 10,000$ (middle), and $n = 100,000$ (bottom).

without having to know about the data distribution in advance, like in the usage statistics example at the beginning of this section.

A practitioner can apply the following reasoning: for each possible sequence of actions captured by the system collecting app usage statistics, either many users are likely to have a value of 1, in which case Theorem 3 applies and thresholding will likely not impact data utility; or the vast majority of users will have a value of 0, in which case Theorem 4 applies and thresholding will protect the rare users whose value is 1.

What if the attacker has non-zero partial knowledge, but is able to interact with the system? We saw in Example 7 that if this partial knowledge is larger than the threshold, the mechanism is not private. But if this partial knowledge is small enough, then privacy is still possible: it is equivalent to reducing the threshold for an attacker with no partial knowledge.

**Proposition 8.** Let $\theta$ be the same distribution as in Theorem 4 with background knowledge of size $|B| \leq T$. Let $\theta'$ be the equivalent distribution but with $n' = n - |B|$, and no partial knowledge. Then for any $\epsilon$ and $\delta$, $\mathcal{M}_T$ is (\{\theta\}, $\epsilon$, $\delta$)-APKDP iff $\mathcal{M}_{T-|B|}$ is (\{\theta'\}, $\epsilon$, $\delta$)-PPKDP.

**Proof.** In this case, APK$_{1,a,b,B}$ only depends on $T - b$ (b being the number of ones $b$ in $B$). The same applies for $n$ and $|B|$, which only appear as $n - |B|$. Thus, $\mathcal{M}_T$ is (\{\theta\}, $\epsilon$, $\delta$)-APKDP iff $\mathcal{M}_{T-|B|}$ is (\{\theta'\}, $\epsilon$, $\delta$)-APKDP. As APKDP and PPKDP are the same when there is no background knowledge, the statement follows.

### 4.3 Application to k-anonymity

Sections 4.1 and 4.2 formalize two intuitive phenomena under partial knowledge. First, if the attacker has a significant enough uncertain about enough people, counting queries do not leak too much information.
about individuals. Second, for counting queries that apply to rare enough behavior, thresholding provides meaningful protection against a passive attacker. This suggests a link to an older anonymization notion: $k$-anonymity. In this section, we formalize that link, and combine these two intuitions to provide a relation between $k$-anonymity and differential privacy under partial knowledge.

$k$-anonymity, introduced in \cite{Sam01, Swe02}, requires each record in a database to be indistinguishable from at least $k - 1$ other records. The intuition is that blending in a large enough crowd provides protection; this intuition is close to the results of Section 4.4: $k$-anonymity is generally obtained by generalizing the data to group similar records together, then dropping the groups with less than $k$ records. The link with the results of Section 1.2 is obvious.

To formalize it, we need to clarify the notion of a $k$-anonymity mechanism. For simplicity, we will simply assume that such a mechanism groups records by their value, and returns a truncated histogram, where all values with less than $k$ records have been removed.

**Definition 16** ($k$-anonymity mechanism). The $k$-anonymity mechanism $M_k$ takes a dataset in $D$ as input, and returns a histogram in $(N \cup \perp)^T$. For each $t \in T \cup \{\perp\}$, $M_k(D)$ is defined as:

- for all $t \in T$, $M_k(D)(t) = |\{i \mid D(i) = t\}|$ if this number is at least $k$ (if there are less than $k$ records with value $t$ in $D$);
- $M_k(D)(t) = \perp$ otherwise.

If an input record is not in $T$, it is ignored by $M_k$.

Note that we skipped the generalization step. The results below can be easily extended to any fixed generalization strategy, i.e. a fixed mapping between $T$ and an arbitrary space forming the support of the histogram. It is important that this strategy is fixed. If this function depends on the data, arbitrary correlations can be embedded in the output, which might leak additional information; minimality attacks \cite{WFWP09} provide an example of this phenomenon.

Now, under which condition is such a mechanism private? The distribution that captures the attacker’s uncertainty must be such that for all possible values $t \in T$, either this value is rare enough to be thresholded with high probability, either there is sufficient randomness in the input data that releasing the exact value does not leak too much information.

In addition, we assume that it is possible for a given record to have the value $\perp$, representing their absence in the dataset. The count corresponding to $\perp$ are never released. We discuss later the importance of such a special value, and its practical interpretation.

**Theorem 5.** Let $\theta$ be a distribution that generates $n$ independent records in $T \cup \{\perp\}$. Assume that there is a $\lambda$ such that for all $t \in T$:

- either for all indices $i$, $P[D(i) = t] \leq \lambda$,
- or for all indices $i$, $\lambda \leq P[D(i) = t] \leq 1 - \lambda$;

furthermore, assume that for all indices $i$, $\lambda \leq P[D(i) = \perp] \leq 1 - \lambda$, and that the attacker does not have any background knowledge.

Let $T$ be a threshold such that $r = \frac{\lambda(n-1)}{1-\lambda} < 1$. Then $M_T$ is $((\theta), \varepsilon, \delta)$-APKDP for all $\delta \geq \delta_0$, where:

$$\delta_0 = \frac{2 \cdot f(T, n-1, \lambda)}{1-r}$$

$$\varepsilon = 2 \cdot \max \left(-\ln \left(1 - \frac{f(T, n-1, \lambda)}{1-r}\right), \varepsilon_c\right)$$

and $\varepsilon_c$ is such that $\delta \geq P \left[\frac{X}{Y} \geq \varepsilon_c\right]$, where $X$ and $Y$ are two independent random variables sampled from a binomial distribution with $n - 1$ trials and success probability $2\lambda$.

**Proof.** For a given index $i$ and a possible record $t \in T$, we compare the events $D(i) = t$ and $D(i) = \perp$. If we find $\varepsilon$ and $\delta$ such that $M(D)|_{D(i)=t} \approx_{\varepsilon, \delta} M(D)|_{D(i)=\perp}$, then we have $M(D)|_{D(i)=t} \approx_{2\varepsilon, 2\delta} M(D)|_{D(i)=t'}$ for all $t', t' \in T$, which would conclude the proof immediately.

We must consider two options: either for all indices $i$, $P[D(i) = t] \leq \lambda$, or for all $i$, $\lambda \leq P[D(i) = t] \leq 1 - \lambda$. 

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In the first case, we can reuse the proof of Theorem 4 with probability \(1 - \delta_0 = 1 - \frac{\epsilon_0}{\ln \left( 1 - \frac{1}{1 - \gamma} \right)}\), the result is thresholded, and the privacy loss is bounded by \(\epsilon_0 = -\ln \left( 1 - \frac{1}{1 - \gamma} \right)\). Importantly, comparing \(D(i) = t\) and \(D(i) = \perp\) allows us to restrict our analysis to the value of \(\mathcal{M}_T(D)(t)\): for all \(t' \neq t\), \(\mathcal{M}_T(D)(t')\) is the same when \(\theta\) is conditioned on \(D(i) = t\) or \(D(i) = \perp\).

In the second case, we reuse the proof of Theorem 5; the distribution of \(\mathcal{M}_T(D)(t)\) can be seen as the sum of records, each of whom has been randomized using a binary randomized response process with parameter \(2\lambda\). As \(\mathcal{M}_T(D)(\perp)\) also follows this binary randomized response process, we can directly apply the proof with \(\delta_0\).

Combining both cases leads to the desired result, using the indistinguishability between \(\mathcal{M}_T(D)(t)\) and \(\mathcal{M}_T(D)(\perp)\) to get the result between arbitrary \(t\) and \(t'\).

Theorem 5 is relatively complex, and depends on a number of conditions. Let us discuss its limitations. Some of them are necessary for the result to be true, others could be overcome with a more careful analysis, at the cost of simplicity.

First, we assume that the attacker has no partial knowledge over the data. The result can easily be extended to the case where the attacker has non-zero passive partial knowledge of \(m\) records over the data: for the counting case, we can simply remove these \(m\) records and obtain the results with \(n - m\) instead of \(m\), and for the thresholding case, we can apply Theorem 4 directly. The discussion in Section 4 shows that cannot be easily extended to the case where the attacker has the ability to influence the data, unless a very small number of records can be influenced (as in Proposition 5). This captures the correct intuition that \(k\)-anonymity is vulnerable against active attackers.

Second, the choice distribution \(\theta\) might seem artificial, carefully chosen so the previous results can be applied. Why would there be a value \(\lambda\) such that all records have a probability lower than \(\lambda\) of being in a fixed category, or larger than \(\lambda\)? The first option is reasonable: many real-life distributions are long-tailed; some types of actions, or characteristics, are simply very rare. The second option is less natural: maybe a characteristic that is common for many people is extremely rare in others, so requiring all records to have a high enough probability for this record seems too restrictive. However, note that this high probability captures the attacker’s uncertainty: if the attacker knows that some records have a particularly low probability of having a certain record, it is possible to over-approximate this knowledge, and simply consider these records as known by the attacker. We can then use the previous point to still get an upper bound on the attacker’s information gain.

Third, what is the meaning of the \(\perp\) special case, and is it necessary for the proof of Theorem 5 to work? We use it to prove the desired indistinguishability property in the second case of the proof. Without it, it turns out that subtle problems can arise. Suppose, for example, that \(T = \{a, b, c\}\), and that for all \(i\), \(P[D(i) = a]\) is infinitesimally small, while \(P[D(i) = b]\) and \(P[D(i) = c]\) are both close to 0.5. If the total number of records is fixed (and implicitly assumed to be known by the attacker), note that thresholding the count for \(a\) is pointless: with high probability, we can retrieve it by computing the difference between \(n\) and the counts for \(b\) and \(c\). This phenomenon is a real vulnerability of \(k\)-anonymity when the total number of participants is known: any result showing that \(k\)-anonymity protects privacy under partial knowledge must find a way of guaranteeing that this does not happen.

Creating an artificial category \(\perp\) whose count is never released solves this problem, assuming that this category has sufficient uncertainty. This hides the total number of participants and mitigates this vulnerability. Another way would be to impose that the distribution \(\theta\) has multiple \(t \in T\) whose counts will likely be thresholded, and that these \(t\) together have enough uncertainty to hide the total count. This is also realistic in practice, given that most distributions are long-tailed, but would likely require a more complex analysis, as well as complicate the theorem statement.

Note that a link between \(k\)-anonymity and differential privacy was already introduced in [LQS11]. We use the same notion of a \(k\)-anonymity mechanism, however, we model the attacker’s partial knowledge differently. In [LQS11], the attacker is assumed to know the value of every single record from the original dataset, but not which records have been randomly sampled from it. Arguably, the only way to satisfy that assumption in practice is to have the mechanism actually sample the data before applying \(k\)-anonymity. In that case, the original differential privacy definition is satisfied. By contrast, our setting assumes an attacker that has some uncertainty about the value of the records themselves; we argue that this is a much more natural way of capturing the natural assumption that the attacker has partial knowledge over the data.
5 Composition

Composition theorems enable the modular analysis of complex systems and the continued usage of mechanisms over time. In this section, we study two kinds of composition. Sequential composition, where multiple mechanisms are applied to the same data, and nested composition, where post-processing noise is added to the result of the aggregation.

5.1 Sequential composition

We saw in the previous section that noiseless mechanisms could be private under partial knowledge. For such mechanisms, composition does not hold in general. We explain why dependencies between mechanisms are the root cause of composition failing, and we explain how bounding this dependencies allow us to derive usable composition results. First, we show that noiseless composition fails in general.

Example 8. Going back to the voting example, consider the queries “How many people voted 1?” and “How many people who are not t voted 1?” for some individual t. As shown in Section 4.2, each query can be private on its own. However, publishing both results reveals t’s vote: the composition of both queries is not private.

Are there special cases where noiseless counting queries can be composed? In this section, we propose a criterion, \((\mu, \nu)\)-boundedness, under which sequential composition does hold.

The core problem with Example 8 is that the two queries are heavily dependent on each other. In fact, knowing the result to the first query only leaves two options for the result of the second query: it drastically reduces an attacker’s uncertainty about the second query’s result. We show that this dependency between queries is the main obstacle towards a composition result and prove that mechanisms where the dependency is bounded \((\text{Definition } 18)\) can actually be composed \((\text{Theorem } 6)\).

How can we formalize the bounded dependency between mechanisms? A natural approach is to quantify how much the additional knowledge of the first mechanism impacts the privacy loss of the second mechanism.

Definition 17. Given two mechanisms \(\mathcal{M}_1\) and \(\mathcal{M}_2\), two \(a, b \in \mathcal{T}\), two outputs \(O_1, O_2\), a distribution \(\theta\), an index \(i\), a possible value of the background knowledge \(B\) compatible with \(D(i) = a\) and \(D(i) = b\), the dependency of \(\mathcal{M}_2\) on \(\mathcal{M}_1\) to distinguish \(D(i) = t\) and \(D(i) = t'\) is the function \(O^2 \rightarrow \mathbb{R} \cup \{-\infty, \infty\}\) defined by:

\[
\text{Dep}_{i, a, b}^{\mathcal{M}_1, \mathcal{M}_2, \theta} (O_1, O_2 | B) = \mathbb{L}_{i, a, b}^{\mathcal{M}_2, \theta} (O_2, (B, \mathcal{M}_1(D) = O_1)) - \mathbb{L}_{i, a, b}^{\mathcal{M}_2, \theta} (O_2, B)
\]

using the convention \(\pm \infty - x = \pm \infty\) for all \(x\).

Intuitively, this value quantifies the amount of additional information that \(\mathcal{M}_1\) gives the attacker when analyzing the privacy loss of \(\mathcal{M}_2\). In Example 8, the first term is \(\pm \infty\), as knowing both the results of \(\mathcal{M}_1\) and \(\mathcal{M}_2\) leaks the value of \(D(i)\), while the second term is typically finite. So \(\text{Dep}_{i, a, b}^{\mathcal{M}_1, \mathcal{M}_2, \theta} (O_1, O_2 | B)\) takes infinite values, which captures the fact that the two mechanisms together leak a lot of information.

Bounding this dependency can be done in the same way as using the PLRV to define differential privacy: we bound \(\text{Dep}\) by \(\mu\) almost everywhere, and use a small quantity \(\nu\) to capture rare events where \(\text{Dep} > \mu\).

Definition 18 \((\mu, \nu)\)-bounded dependency. Given a family of distributions \(\Theta\), two mechanism \(\mathcal{M}_1\) and \(\mathcal{M}_2\) are \((\mu, \nu)\)-bounded dependent for \(\Theta\) if for all \(\theta \in \Theta\), all indices \(i\) and records \(a, b \in \mathcal{T}\), and all \(\hat{B} \in \mathcal{B}\):

\[
\mathbb{E}_{\theta | D(i) = \hat{B}, O_1 \sim \mathcal{M}_1(D), O_2 \sim \mathcal{M}_2(D)} \left[ \max \left( 0, 1 - e^{-\mu \cdot \text{Dep}_{i, a, b}^{\mathcal{M}_1, \mathcal{M}_2, \theta} (O_1, O_2 | \hat{B})} \right) \right]
\]

is smaller or equal to \(\nu\).
This notion formalizes the intuition that the result of the first mechanism should not impact “too much” the result of the second mechanism. As we show in the following theorem, the dependency of $M_2$ on $M_1$ can be used to express the PLRV of the composed mechanism as a function of the PLRV of the two original mechanisms. As a direct consequence, we show that two $(\mu, \nu)$-bounded dependent mechanisms can be sequentially composed.

**Theorem 6.** Given a distribution $\Theta$, two mechanisms $M_1, M_2$, an indice $i$, records $a, b \in T$, and $\hat{B} \in B$, the PLRV of the composed mechanism $M(D) := (M_1(D), M_2(D))$ satisfies:

$$L_{i,a,b}^{M,\Theta}(O, \hat{B}) = 2 \cdot \text{Dep}_{i,a,b}^{M_1,\Theta}(O_1, O_2 \mid \hat{B}) + L_{i,a,b}^{M_1,\Theta}(O_1, \hat{B}) + L_{i,a,b}^{M_2,\Theta}(O_2, \hat{B}).$$

As a corollary, if $M_1, M_2$ are $(\mu, \nu)$-bounded dependent, if $M_1$ is $(\Theta, \varepsilon_1, \delta_1)$-APK, and if $M_2$ is $(\Theta, \varepsilon_2, \delta_2)$-APK for $\Theta$, then $M$ is $(\Theta, 2\mu + \varepsilon_1 + \varepsilon_2, \delta_1 + \delta_2 + \nu)$-APK.

**Proof.** To prove the main statement, we decompose:

$$L_{i,a,b}^{M,\Theta}(O, \hat{B}) = \ln \frac{P_\Theta[M(D) = (O_1, O_2) \mid D(i) = a, \hat{B}]}{P_\Theta[M(D) = (O_1, O_2) \mid D(i) = b, \hat{B}]} = \ln \frac{P_\Theta[M_2(D) = O_2 \mid M_1(D) = O_1, D(i) = a, \hat{B}]}{P_\Theta[M_2(D) = O_2 \mid M_1(D) = O_1, D(i) = b, \hat{B}]} + \ln \frac{P_\Theta[M_1(D) = O_1 \mid D(i) = a, \hat{B}]}{P_\Theta[M_1(D) = O_1 \mid D(i) = b, \hat{B}]}$$

and use the definition of $\text{Dep}_{i,a,b}^{M_1,\Theta}(O_1, O_2 \mid \hat{B})$.

To prove the composition theorem, we have to show that, if we denote $\varepsilon = \varepsilon_1 + \varepsilon_2 + \mu$ and $\delta = \delta_1 + \delta_2 + \nu$:

$$E_{\Theta, D(i) = a, b \sim M(D)} \left[ \max \left( 0, 1 - e^\varepsilon - L_{i,a,b}^{M,\Theta}(O, \hat{B}) \right) \right] \leq \delta.$$

The function $f(x) = \max(0, 1 - e^x)$ satisfies “for all $x$ and $y$, $f(x + y) \leq f(x) + f(y)$”: if $x > 0$, then $f(x) = 0$, and $1 - e^{x+y} \leq 1 - e^x$. The same holds when $y > 0$. If $x \leq 0$ and $y \leq 0$, then we must show that $1 - e^{x+y} \leq 2 - e^x - e^y$, which is equivalent to $(e^{x+y} - 1) (e^x - 1) \geq 0$.

We then define $a = \varepsilon_1 - L_{i,a,b}^{M_1,\Theta}(O_1, \hat{B}), b = \varepsilon_2 - L_{i,a,b}^{M_2,\Theta}(O_2, \hat{B})$, and $c = \mu - \text{Dep}_{i,a,b}^{M_1,\Theta}(O_1, O_2 \mid \hat{B})$, and we use subadditivity: $f(a + b + c) \leq f(a) + f(b) + f(c)$. As $a + b + c = e - L_{i,a,b}^{M,\Theta}(O, \hat{B})$, we can directly plug this into the expression above and use the theorem assumption to prove the $\delta_1 + \delta_2 + \nu = \delta$ bound.

Note that the characterization of $L_{i,a,b}^{M,\Theta}(O, \hat{B})$ enables the use of more sophisticated composition bounds for differential privacy, such as the advanced composition theorem [DR14], Rényi Differential Privacy [Mir17], or privacy buckets [MM18]. For simplicity, the proof uses the standard (non-tight) composition bound for DP.

A common special case directly leads to $(0, 0)$-bounded dependent mechanisms: two mechanisms that work on distinct parts of a database are $(0, 0)$-bounded dependent if these two parts are independent.

**Proposition 9.** Let $\Theta$ be a family of distributions, and let $M_1$ and $M_2$ be mechanisms. Assume that for any $\Theta$, there are functions $\pi_1$ and $\pi_2$ such that $\pi_1(D) \sim \Theta$ and $\pi_2(D) \sim \Theta$ are independent, and functions $M_1'$ and $M_2'$ such that $M_1(D) = M_1'(\pi_1(D))$ and $M_2(D) = M_2'(\pi_2(D))$. Then $M_1$ and $M_2$ are $(0, 0)$-bounded dependent.

We now present an natural example of a practical scenario where we can use these composition results.
Example 9. Consider a regularly updated database, like usage information about an online service. Statistics \( q \) are computed from this database: for example, among registered users, how many of them used a specific feature on any given day. This count is released daily, and we want to understand how the privacy of a particular user is impacted over time.

This can be represented by a database \( D \) where each record \( i \) is a series of binary values \((D(i))_j\), where \( j = 0, 1, 2, \ldots \), and we release a series of mechanisms \( M_j(D) = \sum q(D(i)_j) \). The results of Section 4 can be used to determine the privacy of each \( M_j \) depending on the data-generating distribution \( \theta \). The goal is to determine the privacy of multiple queries, assuming independence between \( D(i)_{i_1} \) and \( D(i)_{i_2} \) for all \( i_1 \neq i_2 \).

The analysis of the privacy guarantees offered by this setting over time depends on \( \theta \), and on the correlations between the different values of a record. If \( D(i)_j \) is independent from \( D(i)_{j_2} \) for all \( j_1 \neq j_2 \), then the result is direct. Otherwise, we must quantify the maximum amount of correlation between \( D(i)_j \) and \( D(i)_{j+1} \). Quantifying this can be done using indistinguishability: we can assume, for example, that there is a \( c \geq 0 \) such that for all \( a \in T \) and all indices \( i \) and \( j \):

\[
(D(i)_{j+1})_{|D(i)_j = a} \approx c D(i)_j.
\]

Under this assumption, it is easy to verify that mechanisms \( M_j \) and \( M_{j+1} \) are \((2c, 0)\)-bounded dependent, so we can use the composition result of Theorem 7 and derive bounds on the privacy leakage over time.

This approach can be extended to other scenarios, for example if only a subset of users participate to each update, or if a referendum contains multiple questions, whose answers are correlated. Another possible scenario is if only a subset of users participate to each update. We can represent this by having \( D(i)_j \) be either a categorical value (which encodes e.g. the type of interaction) or a special value \( \perp \) that encodes “user \( i \) did not participate to this update”. The probabilities and correlation relationships of different values associated with the same user can be set to capture different scenarios (e.g. the probability that \( D(i)_j = \perp \) can be large, to capture a scenario where few users participate every round).

5.2 Composing partial knowledge with post-processing noise

The results of Sections 4.2 to 4.3 show that noiseless mechanisms can be considered private, assuming some additional assumptions on the attacker’s background knowledge. With enough records, even pessimistic assumptions (considering an attacker who knows a large fraction of records) can lead to very small values of \( \varepsilon \) and \( \delta \). However, one could still consider these assumptions as too brittle, and decide to add a small amount of additional noise to the mechanism to have it satisfy differential privacy in its original form.

Such mechanisms have a double privacy guarantee: under realistic assumptions, their privacy level is very high thanks to the attacker’s uncertainty, and the additional noise provides a “worst-case” privacy level that the mechanism satisfies independently of the attacker capabilities. Without noise, we can use results like Theorem 5 to show that a given aggregation over \( n \) records is \((\Theta(\varepsilon(B)), \varepsilon(|B|), \delta(|B|))\)-APKDP (or PPKDP), where \( |B| \) is the number of records that the attacker knows. In situations like ones we have seen so far, \( \varepsilon(|B|) \) and \( \delta(|B|) \) can be very small when \( |B| \) is close to 0, but might become unacceptably high when \( |B| \) gets close to \( n \). Adding noise can be a way to guarantee that \( \varepsilon(|B|) \) and \( \delta(|B|) \) never get above a certain point: when there is not enough randomness coming from the data anymore, the guarantee from post-processing noise take over. Figure 5 illustrates this phenomenon.

It is also natural to wonder whether the two sources of uncertainty could be combined. The privacy guarantees from Theorem 5 come from the shape of the binomial distribution, just like the shape of Laplace noise is the reason why adding it to the result of an aggregation can provide \( \varepsilon \)-DP. It seems intuitively that combining two sources of noise would have a larger effect.

In some cases, this effect can be numerically estimated. Given a noise distribution \( X \) added to a mechanism of sensitivity \( s \), the PLRV can be obtained by comparing the distributions of \( X \) and \( X + s \). To estimate the PLRV coming from two noise sources summed together (for example, binomial and geometric noise), we can simply compute the convolution of the corresponding two distributions, and use the result to compute the PLRV, and thus, the \((\varepsilon, \delta)\) graph. We demonstrate this approach in Figure 6 where we add two-sided geometric noise to a noiseless counting query.

Definition 19 (Two-sided geometric distribution). The two-sided geometric distribution of mean 0 and of parameter \( p \in (0, 1) \) is the probability distribution such that a random variable \( X \) sampled from the
Figure 5: $\varepsilon$ from the closed-form formula of Theorem 3 for $\delta = 10^{-10}$, $\lambda = 0.05$, and $n = 100,000$, as a function of the number of records known by the attacker $|B|$. We compare two scenarios: either we do not add any post-processing noise, or we add Laplace noise of scale 2 to the output.

Figure 6: Numerical computation of the $(\varepsilon,\delta)$ bounds given by Theorem 3 with $n = 10,000$ and $\lambda = 0.05$ (in dashed blue), compared to the bounds obtained by adding two-sided geometric noise of parameter $\rho = 0.5$ or $\rho = 0.75$ (see Definition 19) and combining both distributions.

distribution follows, for all integers $k$:

$$P[X = k] = \frac{1 - \rho}{1 + \rho^{|k|}}.$$

It is natural to ask whether we could obtain generic results that quantify the combined effect of noise coming from the input data and noise added after the aggregation mechanism, without numerical evaluation. In [GK17], the authors propose such a result, based on the fact that Gaussian distributions are closed under convolution. The noise from the input data is approximated by a Gaussian using the central limit theorem, and their Theorem 6 shows that adding Gaussian noise leads to a smaller $\varepsilon$. However, since the $\delta$ term comes from the central limit theorem approximation, it cannot be improved beyond $\delta = O(1/\sqrt{n})$ in general.

We could solve this by simply making the assumption that the input data unknown from the attacker actually follows a Gaussian distribution. Sadly, the corresponding result would be very brittle: if an attacker does not conform exactly to this approximation, then the result no longer holds. This is a major criticism of privacy definitions which assume the input data has inherent randomness [SU20]. The results of this paper are not so brittle, as the privacy guarantees degrade gracefully with the assumptions we make on the attacker’s partial knowledge (e.g. the number of records known, or the value of $\rho$ in Theorems 3 or 4).
Another approach would be to choose the noise added as post-processing based on the natural noise distributions emerging from the partial knowledge assumption. For example, since the proof of Theorem 3 uses the fact that the attacker uncertainty corresponds to binomial noise, we could also add binomial noise as post-processing, since \( B(n, p) + B(m, p) = B(n + m, p) \). Yet, this property depends on the exact value of \( p \), which creates a brittleness we were trying to avoid.

The question of computing the privacy loss in situations where multiple sources of randomness are combined appears in other scenarios. Amplification by sampling or amplification by shuffling are examples of such results. These two classes of results are \textit{generic}: they do not depend on the exact mechanism used to obtain the initial \((\varepsilon, \delta)\)-DP guarantee. It is unlikely that such generic results exist when combining two arbitrary sources of noise, each of which satisfies \((\varepsilon, \delta)\)-DP.

Other results depend on additional assumptions on the noise distribution, like amplification by iteration \[\text{[FMTT18]}\] or amplification by mixing and diffusion mechanisms \[\text{[BBGG19]}\]. These do not seem to bring significant improvements in scenarios like \[\text{Theorem 3}\] with post-processing noise: amplification by iteration characterizes adding noise many times (not only once), while amplification by mixing and diffusion requires stronger assumptions on the original noise distribution.

An generic result on the privacy guarantee of chained \(\varepsilon\)-DP mechanisms appears in \[\text{[EFM+20]}\] (Appendix B). This tight result is only valid for pure \(\varepsilon\)-DP, but the main building block holds for \((\varepsilon, \delta)\)-DP mechanisms: proving a fully generic chained composition result is equivalent to solving the special case where the input and output of both mechanisms have values in \{0, 1\}. This result can likely be extended to the \((\varepsilon, \delta)\)-DP, although the analysis is surprisingly non-trivial, and fully generic optimality results do not necessarily mean optimality for the special case of additive noise mechanisms.

6 Conclusion

We identified issues that arise with existing definitions in the presence of correlations in the data. We proposed a criterion that resolves these issues and unifies different approaches, and we showed that an attacker with partial knowledge can be either passive or active. We delineated these cases in two definitions, and we proved fundamental results about these definitions. We then quantified the privacy guarantees of natural, practical mechanisms, under realistic assumptions. We improved known results on the noiseless privacy of counting queries, and showed that thresholding can protect the privacy of individuals even in cases where is little randomness in the original data. Finally, we showed a natural relationship linking how correlated two mechanisms are to how well their privacy guarantees compose, and proposed initial results on nested composition. We hope that this work will encourage the privacy analysis of different natural mechanisms under partial knowledge, extending this kind of analysis to systems more complex than individual queries.

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