LOCALLY COMPACT SUBGROUP ACTIONS ON TOPOLOGICAL GROUPS

SERGEY A. ANTONYAN

ABSTRACT. Let $X$ be a Hausdorff topological group and $G$ a locally compact subgroup of $X$. We show that $X$ admits a locally finite $\sigma$-discrete $G$-functionally open cover each member of which is $G$-homeomorphic to a twisted product $G \times H S_i$, where $H$ is a compact large subgroup of $G$ (i.e., the quotient $G/H$ is a manifold). If, in addition, the space of connected components of $G$ is compact and $X$ is normal, then $X$ itself is $G$-homeomorphic to a twisted product $G \times K S$, where $K$ is a maximal compact subgroup of $G$. This implies that $X$ is $K$-homeomorphic to the product $G/K \times S$, and in particular, $X$ is homeomorphic to the product $\mathbb{R}^n \times S$, where $n = \dim G/K$. Using these results we prove the inequality $\dim X \leq \dim X/G + \dim G$ for every Hausdorff topological group $X$ and a locally compact subgroup $G$ of $X$.

1. Introduction

By a $G$-space we mean a completely regular Hausdorff space together with a fixed continuous action of a given Hausdorff topological group $G$ on it.

The notion of a proper $G$-space was introduced in 1961 by R. Palais \cite{22} with the purpose to extend a substantial portion of the theory of compact Lie group actions to the case of noncompact ones.

Recall that a $G$-space $X$ is called proper (in the sense of Palais \cite{22} Definition 1.2.2), if each point of $X$ has a, so called, small neighborhood, i.e., a neighborhood $V$ such that for every point of $X$ there is a neighborhood $U$ with the property that the set $\langle U, V \rangle = \{g \in G \mid gU \cap V \neq \emptyset\}$ has compact closure in $G$.

Clearly, if $G$ is compact, then every $G$-space is proper. Important examples of proper $G$-spaces are the coset spaces $G/H$ where $H$ is a compact subgroup of a locally compact group $G$.

In \cite{2} we have shown that if $G$ is a locally compact subgroup of a Hausdorff topological group $X$, then $X$ is a proper $G$-space with respect to the natural action $g \ast x = xg^{-1}$ (or equivalently, $g \ast x = gx$) of $G$ on $X$. Based on the theory of proper actions, we proved in \cite{22} that many topological properties (among which are normality and paracompactness) are transferred from $X$ to its quotient space $X/G$.

This paper is continuation of \cite{2}; here we further apply the results and methods of the theory of proper actions in the study of topological groups with respect to the natural translation action of a locally compact subgroup.

2010 Mathematics Subject Classification. 22A05, 22F05, 54H11, 54H15, 54F45.

Key words and phrases. Proper $G$-space; orbit space; locally compact group; paracompact space; dimension.
Below all topological groups are assumed to satisfy at least the Hausdorff separation axiom.

All the notions involved in the formulations of the following main results are defined in the next section:

**Theorem 1.1.** Let $X$ be a topological group and $G$ a locally compact subgroup of $X$. Then there exists a compact subgroup $H \subset G$ such that:

1. the quotient $G/H$ is a manifold, and
2. there exists a locally finite $\sigma$-discrete cover of $X$ consisting of $G$-functionally open sets $W_i$ such that the closure $\overline{W}_i$ is a $G$-tubular set with the slicing subgroup $H$ (that is to say, there exists a $G$-equivariant map $\overline{W}_i \to G/H$).
3. each $W_i$ is $G$-homeomorphic to a twisted product $G \times_H S_i$.

We recall that a locally compact group is called *almost connected* if its space of connected components is compact.

**Theorem 1.2.** Let $X$ be a topological group, $G$ an almost connected subgroup of $X$, and $K$ a maximal compact subgroup of $G$. Then there exists a locally finite $\sigma$-discrete cover of $X$ consisting of $G$-functionally open sets $W_i$ such that the closure $\overline{W}_i$ is a $G$-tubular set with the slicing subgroup $K$ (that is to say, there exists a $G$-equivariant map $\overline{W}_i \to G/K$). In particular, each $\overline{W}_i$ is homeomorphic to a product $G/K \times S_i$.

**Theorem 1.3.** Let $X$ be a normal topological group, $G$ an almost connected subgroup of $X$, and $K$ a maximal compact subgroup of $G$. Then there exists a global $K$-slice $S$ in $X$ (equivalently, there exists a $G$-equivariant map $X \to G/K$). In particular, $X$ is homeomorphic to a product $G/K \times S$.

Combining Theorem 1.3 with a result of Abels [1, Theorem 2.1], we obtain the following:

**Corollary 1.4.** Let $X$ be a normal group, $G$ an almost connected subgroup of $X$, and $K$ a maximal compact subgroup of $G$. Then there exists a $K$-invariant subset $S \subset X$ such that $X$ is $K$-homeomorphic to the product $G/K \times S$ endowed with the diagonal action of $K$ defined as follows: $k \ast (gK,s) = (kgK,ks)$. In particular, $X$ is homeomorphic to a product $G/K \times S$, where $n = \dim G/K$.

**Corollary 1.5.** Let $X$ be a normal topological group, $G$ an almost connected subgroup of $X$, and $X/G$ the quotient space of all right cosets $xG = \{xg \mid g \in G\}$, $x \in X$. Then there exists a closed subset $S \subset X$ such that the restriction $p|_S : S \to X/G$ is a perfect, open, surjective map.

This fact has the following two immediate corollaries about transfer of properties from $X$ to $X/G$ and vice versa.

**Corollary 1.6.** Let $P$ be a topological property stable under open perfect maps and also inherited by closed subsets. Assume that $X$ is a normal topological group with the property $P$ and let $G$ be an almost connected subgroup of $X$. Then the quotient space $X/G$ also has the property $P$.

Among properties stable under open perfect maps and also inherited by closed subsets we highlight strong paracompactness and realcompactness (see [12, Exercises 5.3.C(c), 5.3H(d), and Theorem 3.11.4 and Exercises 3.11.G]).
Corollary 1.7. Let $\mathcal{P}$ be a topological property that is both invariant and inverse invariant under open perfect maps, and also stable under multiplication by a locally compact group. Assume that $X$ is a normal topological group and let $G$ be an almost connected subgroup of $X$ such that the quotient space $X/G$ has the property $\mathcal{P}$. Then the group $X$ also has the property $\mathcal{P}$.

Among such properties we highlight realcompactness (see [12, Theorem 3.11.14 and Exercise 3.11.G, and also take into account that every locally compact group is realcompact]).

Note that A. V. Arhangel’skii [8] has studied properties which are transferred from $X/G$ to $X$ for an arbitrary topological group $X$ and its locally compact subgroup $G$; see also [6, Corollary 1.10].

Theorems 1.1 and 1.3 are further applied to prove the following Hurewicz type formula in dimension theory which, for a paracompact group $X$ and an almost connected subgroup $G \subset X$, was proved in [6, Theorem 1.12]:

**Theorem 1.8.** Let $G$ be a locally compact subgroup of a topological group $X$. Then

$$\dim X \leq \dim X/G + \dim G.$$  

**Remark 1.9** ([23], [24]). If in this theorem $X$ is a locally compact group then, in fact, equality holds:

$$\dim X = \dim X/G + \dim G.$$  

All the proofs are given in section 3.

2. Preliminaries

Unless otherwise stated, by a group we shall mean a topological group $G$ satisfying the Hausdorff separation axiom; by $e$ we shall denote the unity of $G$.

All topological spaces are assumed to be Tychonoff (= completely regular and Hausdorff). The basic ideas and facts of the theory of $G$-spaces or topological transformation groups can be found in G. Bredon [10] and in R. Palais [21]. Our basic reference on proper group actions is Palais’ article [22] (see also [1], [2], [11]).

For the convenience of the reader we recall, however, some more special definitions and facts below.

By a $G$-space we mean a topological space $X$ together with a fixed continuous action $G \times X \to X$ of a topological group $G$ on $X$. By $gx$ we shall denote the image of the pair $(g, x) \in G \times X$ under the action.

If $Y$ is another $G$-space, a continuous map $f : X \to Y$ is called a $G$-map or an equivariant map, if $f(gx) = gf(x)$ for every $x \in X$ and $g \in G$.

If $X$ is a $G$-space, then for a subset $S \subset X$ and for a subgroup $H \subset G$, the $H$-hull (or $H$-saturation) of $S$ is defined as follows: $H(S) = \{hs \mid h \in H, s \in S\}$. If $S$ is the one point set $\{x\}$, then the $G$-hull $G(\{x\})$ usually is denoted by $G(x)$ and called the orbit of $x$. The orbit space $X/G$ is always considered in its quotient topology.

A subset $S \subset X$ is called $H$-invariant if it coincides with its $H$-hull, i.e., $S = H(S)$. By an invariant set we shall mean a $G$-invariant set.

For any $x \in X$, the subgroup $G_x = \{g \in G \mid gx = x\}$ is called the stabilizer (or stationary subgroup) at $x$. 
A compatible metric \( \rho \) on a metrizable \( G \)-space \( X \) is called invariant or \( G \)-invariant, if \( \rho(gx,gy) = \rho(x,y) \) for all \( g \in G \) and \( x,y \in X \). If \( \rho \) is a \( G \)-invariant metric on any \( G \)-space \( X \), then it is easy to verify that the formula

\[
\tilde{\rho}(G(x), G(y)) = \inf\{ \rho(x',y') \mid x' \in G(x), y' \in G(y) \}
\]

defines a pseudometric \( \tilde{\rho} \), compatible with the quotient topology of \( X/G \). If, in addition, \( X \) is a proper \( G \)-space then \( \tilde{\rho} \) is, in fact, a metric on \( X/G \) (Theorem 4.3.4).

For a closed subgroup \( H \subset G \), by \( G/H \) we will denote the \( G \)-space of cosets \( \{gH \mid g \in G\} \) under the action induced by left translations.

A locally compact group \( G \) is called almost connected, if the space of connected components of \( G \) is compact. Such a group has a maximal compact subgroup \( K \), i.e., every compact subgroup of \( G \) is conjugate to a subgroup of \( K \) (see [24] Ch. H, Theorem 32.5]).

In what follows we shall need also the definition of a twisted product \( G \times_K S \), where \( K \) is a closed subgroup of \( G \), and \( S \) a \( K \)-space. \( G \times_K S \) is the orbit space of the \( K \)-space \( G \times S \) on which \( K \) acts by the rule: \( k(g,s) = (gk^{-1}, ks) \). Furthermore, there is a natural action of \( G \) on \( G \times_K S \) given by \( g' [g,s] = [g'g,s] \), where \( g' \in G \) and \( [g,s] \) denotes the \( K \)-orbit of the point \( (g,s) \in G \times S \). We shall identify \( S \), by means of the \( K \)-equivariant embedding \( s \mapsto [e,s] \), \( s \in S \), with the \( K \)-invariant subset \( \{[e,s] \mid s \in S\} \) of \( G \times_K S \). This \( K \)-equivariant embedding \( S \hookrightarrow G \times_K S \) induces a homeomorphism of the \( K \)-orbit space \( S/K \) onto the \( G \)-orbit space \( (G \times_K S)/G \) (see [10] Ch. II, Proposition 3.3)).

The twisted products are of a particular interest in the theory of transformation groups (see [10] Ch. II, § 2). It turns out that every proper \( G \)-space locally is a twisted product. For a more precise formulation we need to recall the following well-known notion of a slice (see [22] p. 305):

**Definition 2.1.** Let \( X \) be a \( G \)-space and \( K \) a closed subgroup of \( G \). A \( K \)-invariant subset \( S \subset X \) is called a \( K \)-kernel if there is a \( G \)-equivariant map \( f : G(S) \to G/K \), called the slicing map, such that \( S = f^{-1}(eK) \). The saturation \( G(S) \) is called a \( G \)-tubular or just a tubular set, and the subgroup \( K \) will be referred to as the slicing subgroup.

If in addition \( G(S) \) is open in \( X \) then we shall call \( S \) a \( K \)-slice in \( X \).

If \( G(S) = X \) then \( S \) is called a global \( K \)-slice of \( X \).

It turns out that each tubular set \( G(S) \) with a compact slicing subgroup \( K \) is \( G \)-homeomorphic to the twisted product \( G \times_K S \); namely the map \( \xi : G \times_K S \to G(S) \) defined by \( \xi([g,s]) = gs \) is a \( G \)-homeomorphism (see [10] Ch. II, Theorem 4.2)). In what follows we will use this fact without a specific reference.

One of the fundamental results in the theory of topological transformation groups states (see [22] Proposition 2.3.1]) that, if \( X \) is a proper \( G \)-space with \( G \) a Lie group, then for any point \( x \in X \), there exists a \( G_x \)-slice \( S \in X \) such that \( x \in S \). In general, when \( G \) is not a Lie group, it is no longer true that a \( G_x \)-slice exists at each point of \( X \) (see [11] for a discussion). However, the following approximate version of Palais’ slice theorem for non-Lie group actions holds true, which plays a key role in the proof of Theorem 1.8.
Theorem 2.2 (Approximate slice theorem [7]). Let $G$ be a locally compact group, $X$ a proper $G$-space and $x \in X$. Then for any neighborhood $O$ of $x$ in $X$, there exist a large subgroup $K \subset G$ with $G_x \subset K$, and a $K$-slice $S$ such that $x \in S \subset O$.

Recall that by a large subgroup here we mean a compact subgroup $H \subset G$ such that the quotient space $G/H$ is a manifold.

Thus, every proper $G$-space is covered by invariant open subsets each of which is a twisted product.

A version of this theorem, without requiring $K$ to be a large subgroup was obtained earlier in [2] (see also [1] for the case of compact non-Lie group actions, and [3] for the case of almost connected groups). We emphasize that namely the property “$K$ is a large subgroup” is responsible for the applications of Theorem 2.2 in this paper. We refer the reader to [7] for discussion of further properties of large subgroups.

On any group $G$ one can define two natural (but equivalent) actions of $G$ given by the formulas:

$$g \cdot x = gx, \quad \text{and} \quad g \ast x = xg^{-1},$$

respectively, where in the right parts the group operations are used with $g, x \in G$.

Throughout we shall use the second action only.

By $U(G)$ we shall denote the Banach space of all right uniformly continuous bounded functions $f : G \to \mathbb{R}$ endowed with the supremum norm. Recall that $f$ is called right uniformly continuous, if for every $\varepsilon > 0$ there exists a neighborhood $O$ of the unity in $G$ such that $|f(y) - f(x)| < \varepsilon$ whenever $yx^{-1} \in O$.

We shall consider the induced action of $G$ on $U(G)$, i.e.,

$$(gf)(x) = f(xg), \quad \text{for all} \quad g, x \in G.$$ 

It is easy to check that this action is continuous, linear and isometric (see e.g., [3, Proposition 7]).

Proposition 2.3. Let $G$ be a topological group. Then for every $f \in U(G)$, the map $f_* : G \to U(G)$ defined by $f_*(x)(g) = f(xg^{-1})$, $x, g \in G$, is a right uniformly continuous $G$-map.

Proof. A simple verification. \hfill \Box

The following lemma of H. Abels [1, Lemma 1.5] is used in the proof of Theorem 2.5.

Lemma 2.4. Suppose $G$ is an almost connected group and $K$ a maximal compact subgroup of $G$. Let $X$ be any $G$-space, $A_1$ and $A_2$ two $G$-subsets of $X$ such that $X = A_1 \cup A_2$ and the intersection $A_1 \cap A_2$ is closed in $X$, and let $f_i : A_i \to G/K$, $i = 1, 2$, be $G$-maps. If the inverse image $f_2^{-1}(eK)$ is a normal space then there exists a $G$-map $f : X \to G/K$ such that $f|_{A_1} = f_1$.

In the proof of Theorem 2.8 the following result due to K. Morita [19] plays a key role:

Theorem 2.5. Let $X$ be a Tychonoff space while $Y$ is a locally compact paracompact space. Then

$$(2.1) \quad \dim X \times Y \leq \dim X + \dim Y,$$

where $\dim$ stands for the covering dimension.
One should mention that the covering dimension of an arbitrary Tychonoff space is obtained by a slight modification of the usual definition of the dimension of a normal space; namely, one should just replace “open covering” by “functionally open covering”. It coincides with the usual definition of covering dimension for normal spaces. This modification is due to Katětov [18] (see also Smirnov [25]).

An invariant subset \( A \) of a \( G \)-space \( X \) is called \( G \)-functionally open if there exists a continuous invariant function \( f : X \to [0, 1] \) such that \( A = f^{-1}((0, 1]) \). Putting here \( G = \{e\} \), the trivial group, we obtain the corresponding notion of a functionally open set. Clearly, “\( G \)-functionally open” implies “functionally open”.

Recall that in the literature instead of “functionally open” also the term “cozero set” is used.

It is clear that every open invariant subset \( A \) of a \( G \)-space \( X \) which is metrizable by a \( G \)-invariant metric is necessarily \( G \)-functionally open. In fact, if \( \rho \) is such a metric on \( X \), then the continuous function \( f : X \to [0, 1] \) defined by

\[
f(x) = \min\{1, \rho(x, X \setminus A)\}, \quad x \in X,
\]

is invariant and \( A = f^{-1}((0, 1]) \).

This observation will be used in the proof of Proposition 3.1 below.

In the proof of Theorem 1.8 we shall also use the following simple assertion:

**Proposition 2.6.** Let \( G \) be a topological group, \( X \) a \( G \)-space and \( A \) a \( G \)-functionally open subset of \( X \). Then \( A/G \) is a functionally open subset of \( X/G \).

**Proof.** A simple verification. \( \square \)

Finally we recall that by a \( \sigma \)-discrete cover of a space \( X \) we mean a cover which is the countable union \( \bigcup_{n=1}^{\infty} U_n \), where every family \( U_n \) is discrete, i.e., for each point \( x \in X \) there is a neighborhood \( V \) of \( x \) such that \( V \cap A \neq \emptyset \) for at most one element \( A \) of \( U_n \).

3. **Proofs**

Recall that a cover \( \omega \) of a \( G \)-space is called invariant if each member \( U \in \omega \) is an invariant set, i.e., \( G(U) = U \).

A cover \( \{U_i \mid i \in I\} \) of a \( G \)-space \( X \) is called \( G \)-functionally open whenever each \( U_i \) is a \( G \)-functionally open subset of \( X \). Putting here \( G = \{e\} \), the trivial group, we obtain the corresponding notion of a functionally open cover.

**Proposition 3.1.** Let \( G \) be a locally compact group, \( X \) a proper \( G \)-space and \( \{U_s\}_{s \in S} \) an invariant open cover of \( X \). Suppose that there exists a \( G \)-map \( f : X \to Z \) to a (not necessarily proper) \( G \)-space \( Z \) which is metrizable by a \( G \)-invariant metric and has an invariant open cover \( \{O_t\}_{t \in T} \) such that the cover \( \{f^{-1}(O_t)\}_{t \in T} \) refines \( \{U_s\}_{s \in S} \). Then \( \{U_s\}_{s \in S} \) admits a \( G \)-functionally open refinement \( \{V_t\}_{t \in R} \) which is both locally finite and \( \sigma \)-discrete.

**Proof.** Consider the following composition:

\[
X \xrightarrow{f} Z \xrightarrow{q} Z/G
\]

where \( q \) is the \( G \)-orbit map.

Since \( Z \) is metrizable by a \( G \)-invariant metric, the orbit space \( Z/G \) is pseudometrizable (see Preliminaries). Hence the open cover \( \{q(O_t) \mid t \in T\} \) of \( Z/G \)
admits a refinement, say \( \{ W_i \mid i \in \mathcal{I} \} \) which is both locally finite and \( \sigma \)-discrete (see [12] Theorem 4.4.1 and Remark 4.4.2).

Then, clearly, \( \{ q^{-1}(W_i) \mid i \in \mathcal{I} \} \) is an invariant open refinement of \( \{ G(O_t) \mid t \in T \} \), and hence, \( \{ f^{-1}(q^{-1}(W_i)) \mid i \in \mathcal{I} \} \) is an invariant open refinement of \( \{ f^{-1}(G(O_t)) \mid t \in T \} \). But \( f^{-1}(G(O_t)) = G(f^{-1}(O_t)) \) since \( f \) is a \( G \)-map. Since each \( f^{-1}(O_t) \) is contained in some \( U_s \) and \( U_s \) is invariant, we infer that \( G(f^{-1}(O_t)) \) is contained in \( U_s \). This yields that the cover \( \{ f^{-1}(q^{-1}(W_i)) \mid i \in \mathcal{I} \} \) is a refinement of \( \{ U_s \mid s \in S \} \).

Further, since \( \{ W_i \mid i \in \mathcal{I} \} \) is locally finite and \( \sigma \)-discrete and \( qf \) is continuous we infer that \( \{ f^{-1}(q^{-1}(W_i)) \mid i \in \mathcal{I} \} \) is also locally finite and \( \sigma \)-discrete.

Besides, each \( q^{-1}(W_i) \) is \( G \)-functionally open because it is an invariant open subset of \( Z \) which is metrizable by a \( G \)-invariant metric (see the observation before Proposition 2.6). Next, since \( f \) is a \( G \)-map, the inverse image \( f^{-1}(q^{-1}(W_i)) \) is also \( G \)-functionally open. Thus, \( \{ f^{-1}(q^{-1}(W_i)) \mid i \in \mathcal{I} \} \) is the desired refinement of \( \{ U_s \mid s \in S \} \).

\[ \square \]

**Proof of Theorem 7.7** By [9] Theorem 1.1, \( X \) is a proper \( G \)-space. According to Theorem 2.2, there exist a \( G \)-invariant neighborhood \( V \) of the unity in \( X \), a large subgroup \( H \subset G \) and a \( G \)-equivariant map \( \varphi : V \to G/H \) such that \( \varphi(e) = eH \). Due to regularity of the quotient space \( X/G \), one can choose a \( G \)-invariant neighborhood \( P \) of the unity such that \( T \subset V \).

Choose a right uniformly continuous function \( f : X \to [0, 1] \) such that
\[
(3.1) \quad f(e) = 0 \quad \text{and} \quad f^{-1}([0, 1]) \subset P.
\]
Such a function exists because, as is well known, uniformly continuous bounded functions separate points from closed sets in any uniform space (see, e.g., [17] p. 7, I.13; for topological groups see [9] Theorem 3.3.11).

Then, by Proposition 2.3, \( f \) induces an \( X \)-equivariant map \( f_* : X \to U(X) \) defined by the rule:
\[
f_*(x)(g) = f(xg^{-1}), \quad x, g \in X.
\]

Denote by \( Z \) the image \( f_*(X) \). Clearly, \( Z \) is the \( X \)-orbit of the point \( f_*(e) \) in the \( X \)-space \( U(X) \), and the metric of \( U(X) \) induces an \( X \)-invariant metric on \( Z \).

It follows from (3.1) and the \( X \)-equivariance of \( f_* \) that
\[
(3.2) \quad f_*^{-1}(\Gamma_{x,Q}) \subset x^{-1}P, \quad \text{for every} \quad x \in X,
\]
where \( Q = [0, 1] \) and \( \Gamma_{x,Q} = \{ \varphi \in Z \mid \varphi(x) \in Q \} \), which is an open subset of \( Z \).

Besides, since \( f_*(x) \in \Gamma_{x,Q} \) for every \( x \in X \), we see that the sets \( \Gamma_{x,Q}, x \in X \), constitute a cover of \( Z \).

In what follows we restrict ourselves only to the induced actions of the subgroup \( G \subset X \), i.e., we will consider \( X \) and \( Z \) just as \( G \)-spaces. Note that \( Z \) may not be a proper \( G \)-space and we do not need this fact in the sequel.

It follows from (3.2) and from the \( G \)-equivariance of \( f_* \) that
\[
(3.3) \quad f_*^{-1}(G(\Gamma_{x,Q})) \subset x^{-1}P, \quad \text{for every} \quad x \in X.
\]

Thus, the hypotheses of Proposition 3.1 are fulfilled for the \( G \)-map \( f_* : X \to Z \) and the \( G \)-invariant open covers \( \{ xP \mid x \in X \} \) and \( \{ G(\Gamma_{x,Q}) \mid x \in X \} \) of \( X \) and \( Z \), respectively. Then by Proposition 3.1 \( \{ xP \mid x \in X \} \) admits a \( G \)-functionally open
refinement $\{W_i\}_{i \in I}$ which is both locally finite and $\sigma$-discrete. Hence, each $W_i$ is contained in some $xP$ which implies that $\overline{W_i} \subset xP \subset xV$. Now observe that each set $xV$ is $G$-tubular; the corresponding slicing $G$-map $\psi: xV \to G/H$ is just the composition

$$xV \xrightarrow{\eta} V \xrightarrow{\varphi} G/H$$

where $\eta(xv) = v$ for all $v \in V$. Perhaps it is in order to emphasize here that $V$ and $xV$ are $G$-invariant subsets of $X$ which is equipped with the action $g \cdot t = tg^{-1}$, where $g \in G$ and $t \in X$. Then, clearly $\eta$ is a $G$-map, and hence, the composition $\psi = \varphi \eta$ is also a $G$-map.

Consequently, each $\overline{W_i}$ being a $G$-invariant subset of $xV$, is itself a $G$-tubular set with the slicing subgroup $H$, as required.

In conclusion, it remains to observe that statement (2) follows immediately from statement (2) (see Section 2, the first paragraph after Definition 2.1).

$\square$

**Proof of Theorem 1.2.** By Theorem 1.1, $X$ is covered by a locally finite $\sigma$-discrete family $\{W_i\}_{i \in I}$ of $G$-functionally open sets $W_i$ such that the closure $\overline{W_i}$ is a $G$-tubular set associated with a large slicing subgroup $H \subset G$.

Let $\psi_i: \overline{W_i} \to G/H$ be the corresponding slicing map. Since, by the maximality of $K$, there exists an evident $G$-map $\xi: G/H \to G/K$, the composition $\varphi_i = \xi \psi_i$ is a $G$-map $\varphi_i: \overline{W_i} \to G/K$. In this case $\overline{W_i}$ is $G$-homeomorphic to the twisted product $G \times_K S_i$, where $S_i = \varphi_i^{-1}(eK)$ (see Section 2). Furthermore, by a result of H. Abels [1, Theorem 2.1], $\overline{W_i}$ is homeomorphic (in fact, $K$-equivariantly homeomorphic) to the cartesian product $G/K \times S_i$, as required.

$\square$

**Proof of Theorem 1.3.** By Theorem 1.1, cover $X$ by a locally finite collection $\{C_i\}_{i \in I}$ of closed $G$-tubular sets. Let $\psi_i: C_i \to G/H$, $i \in I$, be the corresponding slicing $G$-map, where $H$ is a large subgroup of $G$. Since, by the maximality of $K$, there exists an evident $G$-map $\xi: G/H \to G/K$, the composition $\varphi_i = \xi \psi_i$ is a $G$-map $\varphi_i: C_i \to G/K$.

Well-order the index set $I$, and for every $i \in I$ put

$$A_i = \bigcup_{j < i} C_j$$

which is closed in $X$ by the local finiteness of the cover $\{C_i\}_{i \in I}$.

We aim at constructing inductively a $G$-map $\phi: X \to G/K$ as follows.

Assume that $i \in I$ and the $G$-maps $\phi_j: A_j \to G/K$ are defined for all $j < i$ in such a way that $\phi_j|_{A_k} = \phi_k$ whenever $k < j$.

Define $\phi_i: A_i \to G/K$ as follows. If $i$ is a limit ordinal, then

$$A_i = \bigcup_{j < i} A_j.$$  

Setting $\phi_i|_{A_j} = \phi_j$ for every $j < i$ we get a well-defined $G$-map $\phi_i: A_i \to G/K$. The continuity of $\phi_i$ follows from the local finiteness of the cover $\{C_i\}_{i \in I}$ and from continuity of all maps $\phi_j$, $j < i$.

If $i$ is the successor to $j$, then

$$A_i = A_j \cup C_j.$$
Observe that each $C_k$, $k \in \mathcal{I}$, is closed in $X$ and hence it is a normal space. Since $A_j \cap C_j$ is a closed $G$-invariant subset of $C_j$, the restriction $\phi_j|_{A_j \cap C_j}$ extends to $C_j$ by Lemma 2.4 and hence, we get a $G$-map $\phi_i : A_i \to G/K$. This completes the inductive step and the proof. □

Proof of Corollary 1.5 Since $G$ is almost connected, it has a maximal compact subgroup $K$ (see Preliminaries). By Theorem 1.3, $X$ admits a global $K$-slice $S$, and hence, it is $G$-homeomorphic to the twisted product $G \times_K S$ (see Preliminaries). Since the group $K$ is compact, it then follows that the $K$-orbit map

$$\xi : G \times S \to G \times_K S \cong_G X$$

is open and perfect. This yields immediately that the restriction $f = p|_S : S \to X/G$ of the $G$-orbit map $p : X \to X/G$ is an open and perfect surjection. Indeed, it suffices to observe that for every set $A \subset S$, one has $f^{-1}(f(A)) = KA$, which is open (respectively, closed or compact) whenever $A$ is so. This completes the proof. □

Proof of Theorem 1.8 Consider two cases.

Case 1. Assume that $G$ is connected. In this case $G$ has a maximal compact subgroup, say, $K$ (see Preliminaries). Due to Theorem 1.1, we can cover $X$ by a locally finite collection $\{\Phi_i\}_{i \in \mathcal{I}}$ of $G$-functionally open tubular sets. Let $\psi_i : \Phi_i \to G/H$ be the corresponding slicing $G$-map, where the slicing subgroup $H$ is a (compact) large subgroup of $G$. Since, by the maximality of $K$, there exists an evident $G$-map $\xi : G/H \to G/K$, the composition $\phi_i = \xi \psi_i$ is a $G$-map $\phi_i : \Phi_i \to G/K$.

Since $\{\Phi_i\}_{i \in \mathcal{I}}$ is a locally finite functionally open cover of $X$, by virtue of the locally finite sum theorem of Nagami [20, Theorem 2.5], it suffices to show that $\dim \Phi \leq \dim X/G + \dim G$, for every member $\Phi$ of the cover $\{\Phi_i\}_{i \in \mathcal{I}}$.

Let $\varphi : \Phi \to G/K$ be the slicing map corresponding to the tubular set $\Phi \in \{\Phi_i\}_{i \in \mathcal{I}}$. In this case $\Phi$ is $G$-homeomorphic to the twisted product $G \times_K S$, where $S = \varphi^{-1}(eK)$ (see Preliminaries). Furthermore, by a result of H. Abels [1, Theorem 2.1], $\Phi$ is homeomorphic (in fact, $K$-equivariantly homeomorphic) to the cartesian product $G/K \times S$.

Since the group $G$ is locally compact, and hence, paracompact, we infer that the quotient space $G/K$ is also locally compact and paracompact. Then, according to Theorem 2.5, we have:

(3.4) \[ \dim \Phi = \dim (G/K \times S) \leq \dim G/K + \dim S. \]

Since $K$ is compact, according to a result of V.V. Filippov [14], one has the inequality:

(3.5) \[ \dim S \leq \dim S/K + \dim q \]

where $q : S \to S/K$ is the $K$-orbit projection and $\dim q = \sup \{\dim q^{-1}(a) \mid a \in S/K\}$.

Besides, since $K$ acts freely on $S$, we see that each $K$-orbit $q^{-1}(a)$ is homeomorphic to $K$, and hence, $\dim q = \dim K$.

Consequently, combining (3.4) and (3.5), one obtains:

(3.6) \[ \dim \Phi = \dim (G/K \times S) \leq \dim G/K + \dim K + \dim S/K. \]
Next, since $\Phi/G \cong (G \times_K S)/G \cong S/K$ (see Preliminaries) and $\dim G/K + \dim K = \dim G$ (see Remark 1.9), it then follows from (3.6) that

$$\dim \Phi \leq \dim \Phi/G + \dim G.$$ 

Observe that the quotient $X/G$ is a Tychonoff space. This follows from a standard fact about proper actions [22, Proposition 1.2.8] if we remember that $X$ is a proper $G$-space [6, Theorem 1.1]. It is worth noting that a quotient space $X/M$ is Tychonoff also for any closed (not necessarily locally compact) subgroup $M \subset X$ (see [16, Ch. II, § 8, (8.14)]).

Further, since $\Phi/G$ is a functionally open subset of $X/G$ (see Proposition 2.6), due to monotonicity of the dimension by functionally open subsets (see [20, Theorem 1.1]), one has $\dim \Phi/G \leq \dim X/G$. This, together with the previous inequality, yields that

$$\dim \Phi \leq \dim X/G + \dim G,$$

as required.

Case 2. Let $G$ be arbitrary locally compact. By case 1, we have:

$$\dim X \leq \dim X/G_0 + \dim G_0,$$

where $G_0$ is the unity component of $G$.

Now, since $\dim G_0 \leq \dim G$, it remains to show that

$$\dim X/G_0 \leq \dim X/G.$$

Due to Theorem 1.1, we can cover $X$ by a locally finite collection $\{\Phi_i\}_{i \in I}$ of $G$-functionally open tubular sets. Then $\{\Phi_i/G_0\}_{i \in I}$ constitutes a functionally open locally finite cover of $X/G_0$ (see Proposition 2.6).

Consequently, by virtue of the locally finite sum theorem of Nagami [20, Theorem 2.5], it suffices to show that $\dim \Phi/G_0 \leq \dim X/G$ for every member $\Phi/G_0$ of the cover $\{\Phi_i/G_0\}_{i \in I}$.

Let $\psi : \Phi \to G/H$ be the slicing $G$-map corresponding to the tubular set $\Phi \in \{\Phi_i\}_{i \in I}$, where $H$ is a large subgroup of $G$. Then the quotient $G/H$ is locally connected (in fact, it is a manifold).

Since the natural map $G/H \to G/G_0H$ is open and the local connectedness is invariant under open maps, we infer that $G/G_0H$ is locally connected. On the other hand, the following natural homeomorphism holds:

$$G/G_0H \cong \frac{G/G_0}{G_0H/G_0}.$$

Consequently, $G/G_0H$, being the quotient space of the totally disconnected group $G/G_0$, is itself totally disconnected (this follows from Remark 1.9 and from the fact that in the realm of locally compact spaces total disconnectedness is equivalent to zero-dimensionality [13, Theorem 1.4.5]). Hence, $G/G_0H$ should be discrete, implying that $G_0H$ is an open subgroup of $G$.

Further, the $H$-slicing $G$-map $\psi : \Phi \to G/H$ induces a $G$-map

$$f : \Phi/G_0 \to \frac{G/H}{G_0} = G/G_0H.$$

Since $G/G_0H$ is a discrete group, we infer that if we put $S = f^{-1}(eG_0H)$, then each $gS$ is closed and open in $\Phi/G_0$, and $\Phi/G_0$ is the disjoint union of the sets $gS$. 


one \( g \) out of every coset in \( G/G_0H \). In other words, \( \Phi/G_0 \) is just homeomorphic to the product \((G/G_0H) \times S\).

Thus,

\[
\Phi/G_0 \cong (G/G_0H) \times S.
\]

Consider the natural continuous open homomorphism \( \pi : G \to G/G_0 \) and denote \( L = G_0H/G_0 \). Since \( L = \pi(G_0H) = \pi(H) \) while \( G_0H \) is open and \( H \) is compact in \( G \), we infer that \( L \) is an open compact subgroup of \( G/G_0 \).

Next, observe that the composition of \( f \) with the isomorphism from (3.8) is just the following \( G \)-map:

\[
\Psi : \Phi/G_0 \to \frac{G/G_0}{G_0H/G_0} = \frac{G/G_0}{L}.
\]

Since the quotient group \( G/G_0 \) acts on the spaces \( \Phi/G_0 \) and \( \frac{G/G_0}{L} \) by the induced actions and since \( S = \Psi^{-1}(L) \), we conclude that \( S \) is a global \( L \)-slice of the \( G/G_0 \)-space \( \Phi/G_0 \).

This yields (see Preliminaries) that the \( G/G_0 \)-orbit space of \( \Phi/G_0 \) is just homeomorphic to the \( L \)-orbit space \( S/L \), i.e.,

\[
S/L \cong \frac{\Phi/G_0}{G/G_0}.
\]

In its turn,

\[
\frac{\Phi/G_0}{G/G_0} \cong \Phi/G.
\]

So, in sum we get that

\[
S/L \cong \Phi/G.
\]

Next, since \( \Phi/G_0 = (G/G_0H) \times S \) and \( G/G_0H \) is discrete, we infer that

\[
\dim \Phi/G_0 = \dim S.
\]

(3.9) \( \dim \Phi/G_0 = \dim S \).

Further, since \( L \) is a compact group, due to the above quoted result of V.V. Filippov [13], we have

\[
(3.10) \quad \dim S \leq \dim S/L + \dim q,
\]

where \( q : S \to S/L \) is the \( L \)-orbit projection and \( \dim q = \sup \{\dim q^{-1}(a) \mid a \in S/L\} \).

Since \( G \) acts freely on \( \Phi \) it follows that \( G/G_0 \) acts freely on \( \Phi/G_0 \). But \( L \) is a subgroup of \( G/G_0 \), and hence, its action on \( S \) is also free. This yields that each \( L \)-orbit \( q^{-1}(a) \) is homeomorphic to \( L \), and hence, \( \dim q = \dim L \).

Further, since \( G/G_0 \) is totally disconnected, it is zero-dimensional (see [13, Theorem 1.4.5]). It then follows from Remark [1.9] that \( \dim L = 0 \). This, together with (3.10), implies that

\[
(3.11) \quad \dim S \leq \dim S/L.
\]

Now, taking into account that \( \dim S/L = \dim \Phi/G \), from (3.9) and (3.11), we get:

\[
\dim \Phi/G_0 = \dim \Phi/G = \dim S \leq \dim S/L = \dim \Phi/G.
\]

But, as we have mentioned above, \( \dim \Phi/G \leq \dim X/G \), and we finally get the desired inequality \( \dim \Phi/G_0 \leq \dim X/G \).

\( \square \)
Acknowledgement. The author would like to thank the referee for useful comments. This research was supported in part by grants IN102608 from PAPIIT (UNAM) and 79536 from CONACYT (Mexico).

References

1. H. Abels, Parallelizability of proper actions, global K-slices and maximal compact subgroups, Math. Ann. 212 (1974), 1–19.
2. H. Abels, A universal proper G-space, Math. Z. 159 (1978), 143-158.
3. S. A. Antonyan, Equivariant embeddings into G-AR’s, Glasnik Matematicki 22 (42) (1987), 503–533.
4. S. A. Antonyan, Existence of a slice for arbitrary compact transformation groups, Matematicheskie Zametki 56:5 (1994) 3-9; English transl. in: Math. Notes 56 (5-6) (1994), 101-1104.
5. S. A. Antonyan, Orbit spaces and unions of equivariant absolute neighborhood extensors, Topology Appl. 146-147 (2005), 289-315.
6. S. A. Antonyan, Proper actions on topological groups: Applications to quotient spaces, Proc. AMS, 138, no. 10 (2010), 3707-3716.
7. S. A. Antonyan, Equivariant extension properties of coset spaces of locally compact groups and approximate slices, ArXiv preprint no. arXiv:1103.0804v1 [math. GN].
8. A. V. Arhangel’skii, Quotients with respect to locally compact subgroups, Houston J. Math. 31, no. 1 (2005), 215-226.
9. A. Arhangel’skii and M. Tkachenko, Topological Groups and Related Structures, Atlantis Press/World Scientific, Amsterdam-Paris, 2008.
10. G. Bredon, Introduction to compact transformation groups, Academic Press, 1972.
11. E. Elfving, The G-homotopy type of proper locally linear G-manifolds. II, Manuscripta Math. 105 (2001), 235-251.
12. R. Engelking, General Topology, PWN-Pol. Sci. Publ., Warsaw, 1977.
13. R. Engelking, Dimension Theory, PWN-Pol. Sci. Publ., Warsaw, 1978.
14. V. V. Filippov, Dimensionality of spaces with the action of a bicompact group, Math. Notes, 25, no. 3 (1979), 171-174.
15. G. Hochshild, The structure of Lie groups, Holden-Day Inc., San Francisco, 1965.
16. E. Hewitt and K. Ross, Abstract Harmonic Analysis, V. I, Springer-Verlag, 1963.
17. J. R. Isbell, Uniform spaces, Amer. Math. Soc., Providence, R.I., 1964.
18. M. Katetov, A theorem on the Lebesgue dimension, Časopis Pěst. Mat. Fys. 75 (1950), 79-87.
19. K. Morita, On the dimension of the product of Tychonoff spaces, General Topol. Appl. 3 (1973), 125-133.
20. K. Nagami, Dimension of non-normal spaces, Fund. Math. 109, no. 2 (1980), 113-121.
21. R. Palais, The classification of G-spaces, Memoirs of the AMS, 36 (1960).
22. R. Palais, On the existence of slices for actions of non-compact Lie groups, Ann. of Math. 73, (1961), 295-323.
23. B. A. Pasynkov, On coincidence of different definitions of dimension for quotient spaces of locally bicompact groups, Uspekhi Mat. Nauk, 17, 5(107) (1962), 129-135.
24. E. G. Skljarenko, On the topological structure of locally bicompact groups and their quotient spaces, Amer. Math. Soc. Transl., Ser. 2, 39 (1964), 57-82.
25. Yu. M. Smirnov, On the dimension of proximity spaces (in Russian), Mat. Sb. 38 (1956) 283–302.
26. M. Stroppel, Locally compact groups, European Math. Soc., 2006

Departamento de Matemáticas, Facultad de Ciencias, Universidad Nacional Autónoma de México, 04510 México Distrito Federal, México.

E-mail address: antonyan@unam.mx