ON A FAMILY OF HOPF ALGEBRAS OF DIMENSION 72

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Abstract. We investigate a family of Hopf algebras of dimension 72 whose coradical is isomorphic to the algebra of functions on $S_3$. We determine the lattice of submodules of the so-called Verma modules and as a consequence we classify all simple modules. We show that these Hopf algebras are unimodular (as well as their duals) but not quasitriangular; also, they are cocycle deformations of each other.

Introduction

The study of finite dimensional Hopf algebras over an algebraically closed field $k$ of characteristic 0 is split into two different classes: the class of semisimple Hopf algebras and the rest. The Lifting Method from [AS] is designed to deal with non-semisimple Hopf algebras whose coradical is a Hopf subalgebra. Pointed Hopf algebras, that is Hopf algebras whose coradical is a group algebra, were intensively studied by this Method. It is natural to consider next the class of Hopf algebras whose coradical is the algebra $kG$ of functions on a non-abelian group $G$. This class seems to be interesting at least by the following reasons:

- The categories of Yetter-Drinfeld modules over the group algebra $kG$ and $kG$, $G$ a finite group, are equivalent. Thence, a lot sensible information needed for the Lifting Method (description of Yetter-Drinfeld modules, determination of finite dimensional Nichols algebras) can be translated from the pointed case to this case—or vice versa.

- The representation theory of Hopf algebras whose coradical is the algebra of functions on a non-abelian group looks easier than the representation theory of pointed Hopf algebras with non-abelian group, because the representation theory of $kG$ is easier than that of $G$. Indeed, $kG$ is a semisimple abelian algebra and we may try to imitate the rich methods in representation theory of Lie algebras, with $kG$ playing the role of the Cartan subalgebra. We believe that the representation theory of Hopf algebras with coradical $kG$ might be helpful to study Nichols algebras and deformations.

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1An adaptation to general non-semisimple Hopf algebras was recently proposed in [AC].
We have started the consideration of this class in [AV], where finite dimensional Hopf algebras whose coradical is $\mathbb{k}S_3$ were classified and, in particular, a new family of Hopf algebras of dimension 72 was defined. The purpose of the present paper is to study these Hopf algebras. We first discuss in Section 1 some general ideas about modules induced from simple $\mathbb{k}G$-modules, that we call Verma modules. We introduce in Section 2 a new family of Hopf algebras, as a generalization of the construction in [AV], attached to the class of transpositions in $S_n$ and depending on a parameter $a$. Our main contributions are in Section 3: we determine the lattice of submodules of the various Verma modules and as a consequence we classify all simple modules over the Hopf algebras of dimension 72 introduced in [AV]. Some further information on these Hopf algebras is given in Section 4 and Section 5. We assume that the reader has some familiarity with Yetter-Drinfeld modules and Nichols algebras $B(V)$; we refer to [AS] for these matters.

Conventions.

If $V$ is a vector space, $T(V)$ is the tensor algebra of $V$. If $S$ is a subset of $V$, then we denote by $\langle S \rangle$ the vector subspace generated by $S$. If $A$ is an algebra and $S$ is a subset of $A$, then we denote by $(S)$ the two-sided ideal generated by $S$ and by $\mathbb{k}(S)$ the subalgebra generated by $S$. If $H$ is a Hopf algebra, then $\Delta$, $\epsilon$, $S$ denote respectively the comultiplication, the counit and the antipode. We denote by $\hat{R}$ the set of isomorphism classes of a simple $R$-modules, $R$ an algebra; we identify a class in $\hat{R}$ with a representative without further notice. If $S$, $T$ and $M$ are $R$-modules, we say that $M$ is an extension of $T$ by $S$ when $M$ fits into an exact sequence $0 \to S \to M \to T \to 0$.

1. Preliminaries

1.1. The induced representation.

We collect well-known facts about the induced representation. Let $B$ be a subalgebra of an algebra $A$ and let $V$ be a left $B$-module. The induced module is $\text{Ind}_B^A V = A \otimes_B V$. The induction has the following properties:

- Universal property: if $W$ is an $A$-module and $\varphi : V \to W$ is morphism of $B$-modules, then it extends to a morphism of $A$-modules $\overline{\varphi} : \text{Ind}_B^A V \to W$. Hence, there is a natural isomorphism (called Frobenius reciprocity): $\text{Hom}_B(V, \text{Res}^A_B W) \simeq \text{Hom}_A(\text{Ind}_B^A V, W)$. In categorical terms, induction is left-adjoint to restriction.

- Any finite dimensional simple $A$-module is a quotient of the induced module of a simple $B$-module.

Indeed, let $S$ be a finite dimensional simple $A$-module and let $T$ be a simple $B$-submodule of $S$. Then the induced morphism $\text{Ind}_B^A T \to S$ is surjective.

- If $B$ is semisimple, then any induced module is projective.

The induction functor, being left adjoint to the restriction one, preserves projectives, and any module over a semisimple algebra is projective.
If \( A \) is a free right \( B \)-module, say \( A \cong B^{(I)} \), then \( \text{Ind}_B^A V = B^{(I)} \otimes_B V = V^{(I)} \) as \( B \)-modules, and a fortiori as vector spaces.

We summarize these basic properties in the setting of finite dimensional Hopf algebras, where freeness over Hopf subalgebras is known [NZ]. Also, finite dimensional Hopf algebras are Frobenius, so that injective modules are projective and vice versa.

**Proposition 1.** Let \( A \) be a finite dimensional Hopf algebra and let \( B \) be a semisimple Hopf subalgebra.

- If \( T \in \hat{B} \), then \( \dim \text{Ind}_B^A T = \frac{\dim T \dim A}{\dim B} \).
- Any finite dimensional simple \( A \)-module is a quotient of the induced module of a simple \( B \)-module.
- The induced module of a finite dimensional \( B \)-module is injective and projective.

\[ \square \]

1.2. Representation theory of Hopf algebras with coradical a dual group algebra.

An optimal situation to apply the Proposition 1 is when the coradical of the finite dimensional Hopf algebra \( A \) is a Hopf subalgebra; in this case \( B = \text{coradical of } A \) is the best choice. It is tempting to say that the induced module of a simple \( B \)-module is a Verma module of \( A \).

Assume now the coradical \( B \) of the finite dimensional Hopf algebra \( A \) is the algebra of functions \( k^G \) on a finite group \( G \). In this case, we have:

- Any simple \( B \)-module has dimension 1 and \( \hat{B} \cong G \); for \( g \in G \), the simple module \( k_g \) has the action \( f \cdot 1 = f(g)1 \), \( f \in k^G \). Thus any simple \( A \)-module is a quotient of a Verma module \( M_g := \text{Ind}_{k^G}^A k_g \), for some \( g \in G \).
  - The ideal \( A\delta_g \) is isomorphic to \( M_g \) and \( A \cong \bigoplus_{g \in G} M_g \); here \( \delta_g \) is the characteristic function of the subset \( \{g\} \).

- Let \( g \in G \) such that \( \delta_g \) is a primitive idempotent of \( A \). Since \( A \) is Frobenius, \( M_g \cong A\delta_g \) has a unique simple submodule \( S \) and a unique maximal submodule \( N \); \( M_g \) is the injective hull of \( S \) and the projective cover of \( M_g/N \). See [CR, (9.9)].

- In all known cases, \( \text{gr} A \cong \mathcal{B}(V) \# k^G \), where \( V \) belongs to a concrete and short list. Hence, \( \dim M_g = \dim \mathcal{B}(V) \) for any \( g \in G \). More than this, in all known cases we dispose of the following information:

  - There exists a rack \( X \) and a 2-cocycle \( q \in Z^2(X,k^X) \) such that \( V \cong (kX,c^q) \) as braided vector spaces, see [AG] for details.
  - There exists an epimorphism of Hopf algebras \( \phi : T(V) \# k^G \to A \), see [AV, Subsection 2.5] for details. Note that \( \phi(f \cdot x) = \text{ad} f(\phi(x)) \) for all \( f \in k^G \) and \( x \in T(V) \).
o Let $X$ be the set of words in $X$, identified with a basis of the tensor algebra $T(V)$. There exists $B \subset X$ such that the classes of the monomials in $B$ form a basis of $B(V)$. The corresponding classes in $A$ multiplied with the elements $\delta_g \in k^G, g \in G$, form a basis of $A$.

o If $x \in X$, then there exists $g_x \in G$ such that $\delta_h \cdot x = \delta_{hx} x$ for all $h \in G$. We extend this to have $g_x \in G$ for any $x \in X$.

o If $x \in X$, then $x^2 = 0$ in $B(V)$ and there exists $f_x \in k^G$ such that $x^2 = f_x$ in $A$.

Let $g \in G$. If $x \in B$, then we denote by $m_x$ the class of $x$ in $M_g$. Hence $(m_x)_{x \in B}$ is a basis of $M_g$. We may describe the action of $A$ on this basis of $M_g$, at least when we know explicitly the relations of $A$ and the monomials in $B$. To start with, let $f \in k^G$ and $x \in B$. Then

$$f \cdot m_x = f x \otimes 1 = f(1) \cdot x f(2) \otimes 1 = f(1) \cdot x \otimes f(2) \cdot 1$$\hspace{1cm}(1)$$

Let now $x = x_1 \ldots x_t$ be a monomial in $B$, with $x_1, \ldots, x_t \in X$. Set $y = x_2 \ldots x_t$; observe that $y$ need not be in $B$.

$$x_1 \cdot m_x = x_1^2 x_2 \ldots x_t \otimes 1 = f x_1 y \otimes 1 = f x_1 (g_y g) y \otimes 1.$$\hspace{1cm}(2)$$

Let now $M$ be a finite dimensional $A$-module. It is convenient to consider the decomposition of $M$ in isotypic components as $k^G$-module: $M = \bigoplus_{g \in G} M[g]$, where $M[g] = \delta_0 \cdot M$. Note that

$$x \cdot M[g] = M[ge] \quad \text{for all } x \in B, g \in G.$$\hspace{1cm}(3)$$

For instance, (1) says that the isotypic components of the Verma module $M_g$ are $M_g[H] = \langle m_x : x \in B, g_x g = h \rangle$.

2. Hopf algebras related to the class of transpositions in the symmetric group

2.1. Quadratic Nichols algebras.

Let $n \geq 3$; denote by $O_2^n$ the conjugacy class of $(12)$ in $S_n$ and by $\text{sgn} : C_{S_n}(12) \to k$ the restriction of the sign representation of $S_n$ to the centralizer of $(12)$. Let $V_n = M((12), \text{sgn}) \in k^{S_n} \mathcal{YD}$; $V_n$ has a basis $(x_{(ij)})_{(ij) \in \mathcal{D}_n}$ such that the action $\cdot$ and the coaction $\delta$ are given by

$$\delta_h \cdot x_{(ij)} = \delta_{h(ij)} x_{(ij)} \quad \forall h \in S_n \text{ and } \delta(x_{(ij)}) = \sum_{h \in S_n} \text{sgn}(h) \delta_h \otimes x_{h^{-1}(ij)h}.$$\hspace{1cm}(4)$$

Let $n = 3, 4, 5$. By [MS, G], we know that $B(V_n)$ is quadratic and finite dimensional; actually, the ideal $J_n$ of relations of $B(V_n)$ is generated by

$$x_{(ij)}^2,$$

$$R_{(ij)(kl)} := x_{(ij)} x_{(kl)} + x_{(kl)} x_{(ij)},$$

$$R_{(ij)(ik)} := x_{(ij)} x_{(ik)} + x_{(ik)} x_{(jk)} + x_{(jk)} x_{(ij)}$$\hspace{1cm}(5)$$

\hspace{1cm}(6)$$
for \((ij), (kl), (ik) \in \mathcal{O}^n_2\) with \(#\{i, j, k, l\} = 4\).

For \(n \geq 6\), we define the quadratic Nichols algebra \(\mathcal{B}_n\) in the same way, that is as the quotient of the tensor algebra \(T(V_n)\) by the ideal generated by the quadratic relations (4), (5) and (6) for \((ij), (kl), (ik) \in \mathcal{O}^n_2\) with \(#\{i, j, k, l\} = 4\). It is however open whether:

- \(\mathcal{B}(V_n)\) is quadratic, i.e. isomorphic to \(\mathcal{B}_n\);
- the dimension of \(\mathcal{B}(V_n)\) is finite;
- the dimension of \(\mathcal{B}_n\) is finite.

But we do know that the only possible finite dimensional Nichols algebras\(^2\) over \(\mathbb{S}_n\) are related to the orbit of transpositions and a pair of characters [AFGV, Th. 1.1]. Also, the Nichols algebras related to these two characters are twist-equivalent [Ve].

\[2.2. \text{The parameters.}\]

We consider the set of parameters

\[\mathfrak{A}_n := \left\{a = (a_{(ij)})_{(ij) \in \mathcal{O}^n_2} \in k^{\mathcal{O}^n_2} : \sum_{(ij) \in \mathcal{O}^n_2} a_{(ij)} = 0 \right\}.\]

The group \(\Gamma_n := k^\times \times \text{Aut}(\mathbb{S}_n)\) acts on \(\mathfrak{A}_n\) by

\[(7) \quad (\mu, \theta) \triangleright a = \mu(a_{\theta(ij)}), \quad \mu \in k^\times, \quad \theta \in \text{Aut}(\mathbb{S}_n), \quad a \in \mathfrak{A}_n.\]

Let \([a] \in \Gamma_n \backslash \mathfrak{A}_n\) be the class of \(a\) under this action. Let \(\triangleright\) denote also the conjugation action of \(\mathbb{S}_n\) on itself, so that\(^3\) \(\mathbb{S}_n < \{e\} \times \text{Aut}(\mathbb{S}_n) < \Gamma_n\). Let \(\mathbb{S}_n^a = \{g \in \mathbb{S}_n | g \triangleright a = a\}\) be the isotropy group of \(a\) under the action of \(\mathbb{S}_n\).

We fix \(a \in \mathfrak{A}_n\) and introduce

\[(8) \quad f_{ij} = \sum_{g \in \mathbb{S}_n} (a_{(ij)} - a_{g^{-1}(ij)}g) \delta_g \in k^{\mathbb{S}_n}, \quad (ij) \in \mathcal{O}^n_2.\]

Clearly,

\[(9) \quad f_{ij}(ts) = f_{ij}(s) \quad \forall t \in C_{\mathbb{S}_n}(ij), \quad s \in \mathbb{S}_n.\]

\[\text{Definition 2.}\] We say that \(g\) and \(h \in \mathbb{S}_n\) are \(a\)-linked, denoted \(g \sim_a h\), if either \(g = h\) or else there exist \((i_m j_m), \ldots, (i_1 j_1) \in \mathcal{O}^n_2\) such that

- \(g = (i_m j_m) \cdots (i_1 j_1) h,\)
- \(f_{i_s j_s}((i_s j_s) \cdots (i_1 j_1) h) \neq 0\) for all \(1 \leq s \leq m.\)

In particular, \(f_{i_1 j_1}(h) \neq 0\) by (9). We claim that \(\sim_a\) is an equivalence relation. For, if \(g\) and \(h \in \mathbb{S}_n\) are \(a\)-linked, then \(h = (i_1 j_1) \cdots (i_m j_m) g\) and

\(^2\)There is one exception when \(n = 4\) that is finite dimensional and two exceptions when \(n = 5\) and 6 that are not known.

\(^3\)It is well-known that \(\mathbb{S}_n\) identifies with the group of inner automorphisms and that this equals \(\text{Aut}(\mathbb{S}_n)\), except for \(n = 6\).
\[ f_{i_s} ((i_s j_s) (i_{s+1} j_{s+1}) \cdots (i_n j_n) g) = f_{i_s} ((i_{s-1} j_{s-1}) \cdots (i_1 j_1) h) \]
\[ = f_{i_s} ((i_s j_s) (i_{s-1} j_{s-1}) \cdots (i_1 j_1) h) \neq 0. \]

In the same way, we see that if \( g \sim_a h \) and also \( h \sim_a z \), then \( g \sim_a z \).

2.3. A family of Hopf algebras.

We fix \( a \in S_n \); recall the elements \( f_{ij} \) defined in (8). Let \( \mathcal{I}_a \) be the ideal of \( T(V_n) \# k^S_n \) generated by (5), (6) and
\[ (10) \quad x_{(ij)}^2 - f_{ij}, \]
for all \( (ij), (kl), (ik) \in \mathcal{O}_2^n \) such that \#\{\( i, j, k, l \)\} = 4. Then
\[ A[a] := T(V_n) \# k^S_n / \mathcal{I}_a \]
is a Hopf algebra, see Remark 3. Also, if \( gr A[a] \simeq B(V_n) \# k^S_n \simeq gr A[b] \), then \( A[a] \simeq A[b] \) if and only if \( [a] = [b] \), what justifies the notation. If \( n = 3 \), then \( gr A[a] \simeq B(V_3) \# k^S_3 \) and \( dim A[a] = 72 [AV] \); for \( n = 4, 5 \) the dimension is finite but we do not know if it is the "right" one; for \( n \geq 6 \), the dimension is unknown to be finite.

**Remark 3.** A straightforward computation shows that
\[ \Delta(x_{(ij)}^2) = x_{(ij)}^2 \otimes 1 + \sum_{h \in S_n} \delta_h \otimes x_{h^{-1}(ij)}^2 \] \[ \Delta(f_{ij}) = f_{ij} \otimes 1 + \sum_{h \in S_n} \delta_h \otimes f_{h^{-1}(i) h^{-1}(j)}. \]
Then \( J = \langle x_{(ij)}^2 - f_{ij} : (ij) \in \mathcal{O}_2^n \rangle \) is a coideal. Since \( f_{ij}(e) = 0 \), we have that \( J \subset ker \epsilon \) and \( S(J) \subset k^S_n J \). Thus \( \mathcal{I}_a = (J) \) is a Hopf ideal and \( A[a] \) is a Hopf algebra quotient of \( T(V_n) \# k^S_n \). We shall say that \( k^S_n \) is a subalgebra of \( A[a] \) to express that the restriction of the projection \( T(V_n) \# k^S_n \rightarrow A[a] \) to \( k^S_n \) is injective.

Let us collect a few general facts on the representation theory of \( A[a] \).

**Remark 4.** Assume that \( k^S_n \) is a subalgebra of \( A[a] \) and let \( M \) be an \( A[a] \)-module. Hence
(a) If \((ij) \in \mathcal{O}_2^n \) satisfies \( f_{ij}(h) \neq 0 \), then \( \rho(x_{(ij)}) : M[h] \rightarrow M[(ij)h] \) is an isomorphism.
(b) Let \( g \sim_a h \in S_n \). Then \( \rho(x_{(i_m j_m)} \cdots \rho(x_{(i_1 j_1)} : M[h] \rightarrow M[g] \) is an isomorphism.

**Proof.** \( \rho(x_{(ij)}) : M[h] \rightarrow M[(ij)h] \) is injective and \( \rho(x_{(ij)}) : M[(ij)h] \rightarrow M[h] \) is surjective, by (10). Interchanging the roles of \( h \) and \( (ij)h \), we get (a). Now (b) follows from (a). \( \square \)

This Remark is particularly useful to compare Verma modules.
Lemma 7. Assume that $h$ is simple, then the last assertion of the lemma follows. □

Definition 6. We say that the parameter $a$ is generic when any of the following equivalent conditions holds.

(a) $a_{(ij)} \neq a_{(kl)}$ for all $(ij) \neq (kl) \in \mathcal{O}_2^n$.
(b) $a_{(ij)} \neq a_{h^{ij}(ij)}$ for all $(ij) \in \mathcal{O}_2^n$ and all $h \in S_n - C_{S_n}(ij)$.
(c) $f_{ij}(h) \neq 0$ for all $(ij) \in \mathcal{O}_2^n$ and all $h \in S_n - C_{S_n}(ij)$.

Proof. (a) $\implies$ (b) is clear, since $(ij) \neq h \triangleright (ij)$ by the assumption on $h$.
(b) $\implies$ (a) follows since any $(kl) \neq (ij)$ is of the form $(kl) = h \triangleright (ij)$, for some $h \notin S_n^i$. (b) $\iff$ (c): given $(ij)$, we have
\[
\{h \in S_n : a_{(ij)} = a_{h^{ij}(ij)}\} = \{h \in S_n : f_{ij}(h) = 0\};
\]
hence, one of these sets equals $C_{S_n}(ij)$ iff the other does. □

Lemma 7. Assume that $a$ is generic, so that $g \sim_a h$ for all $g,h \in S_n - \{e\}$. If $kS_n$ is a subalgebra of $A[a]$, then

(a) If $A[a]$ is finite dimensional, then the Verma modules $M_g$ and $M_h$ are isomorphic, for all $g,h \in S_n - \{e\}$.
(b) If $M$ is an $A[a]$-module, then $\dim M[h] = \dim M[g]$ for all $g,h \in S_n - \{e\}$. Thus $\dim M = (n! - 1) \dim M[(ij)] + \dim M[e]$.
(c) If $M$ is simple and $n = 3$, then $\dim M[h] \leq 1$ for all $h \in S_3 - \{e\}$.

Proof. Let $(ij) \in S_n$ and $g \in S_n - \{e\}$.

- If $g = (ik)$, then $g \sim_a (ij)$, as $(ik) = (jk)(ij)(jk)$ and $a$ is generic.
- If $g = (kl)$ with $\#\{i,j,l,k\} = 4$, then $(ij) \sim_a (ik)$ and $(ik) \sim_a (kl)$, hence $(ij) \sim_a (kl)$.
- If $g = (i_1i_2\cdots i_r)$ is an $r$-cycle, then $g = (i_1i_r)(i_1i_2\cdots i_{r-1})$. Hence $g \sim_a (ij)$ by induction on $r$.
- Let $g = g_1 \cdots g_m$ be the product of the disjoint cycles $g_1, \ldots, g_m$, with $m \geq 2$; say $g_1 = (i_1 \cdots i_r)$, $g_2 = (i_{r+1} \cdots i_{r+s})$ and denote $y = g_3 \cdots g_m$. Then $g = (i_1i_{r+1})(i_1 \cdots i_{r+s})y$ and $y \in C_{S_n}(i_1i_{r+1})$. Hence $g$ and $(ij)$ are linked by induction on $m$.

Now (a) follows from Proposition 5 and (b) from Remark 4. If $n = 3$ and $M$ is simple, then $\dim A[a] = 72 > (\dim M)^2 \geq 25(\dim M[(12)])^2$ and the last assertion of the lemma follows. □

The characterization of all one dimensional $A[a]$-modules is not difficult. Let $\equiv$ be the equivalence relation in $\mathcal{O}_2^n$ given by $(ij) \equiv (kl)$ iff $a_{(ij)} = a_{(kl)}$.
Let $O^n_2 = \prod_{s \in \Upsilon} C_s$ be the associated partition. If $h \in S_n$, then
\[(11) \quad f_{ij}(h) = 0 \forall (ij) \in O^n_2 \iff h^{-1}C_s h = C_s \forall s \in \Upsilon \iff h \in S^n_n.
\]

**Lemma 8.** Assume that $kS^n$ is a subalgebra of $A_a$ and let $h \in S^n_n$. Then $k_h$ is a $A_a$-module with the action given by the algebra map $\zeta_h : A_a \to k$.

\[(12) \quad \zeta_h(x_{(ij)}) = 0, \quad (ij) \in O^n_2 \quad \text{and} \quad \zeta_h(f) = f(h), \quad f \in kS^n.
\]

The one-dimensional representations of $A_a$ are all of this form.

**Proof.** Clearly, $\zeta_h$ satisfies the relations of $T(V_3)\# kS^n$, (5) and (6); (10) holds because $h$ fulfills (11). Now, let $M$ be a module of dimension 1. Then $M = M[h]$ for some $h$; thus $f_{ij}(h) = 0$ for all $(ij) \in O^n_2$ by Remark 4. \[\square\]

3. **Simple and Verma modules over Hopf algebras with coradical $kS^n$**

3.1. **Verma modules.**

In this Section, we focus on the case $n = 3$. Let $a \in \mathfrak{A}_3$. Explicitly, $A_a$ is the algebra $(T(V_3)\# kS^n)/I_a$ where $I_a$ is the ideal generated by
\[(13) \quad R_{(13)(23)}, \quad R_{(23)(13)}, \quad x_{(ij)}^2 - f_{ij}, \quad (ij) \in O^n_2,
\]
where
\[(14) \quad f_{13} = (a_{13} - a_{23})(\delta_{12} + \delta_{23} + \delta_{32}) + (a_{13} - a_{12})(\delta_{23} + \delta_{12} + \delta_{32}),
\]
\[(15) \quad f_{23} = (a_{23} - a_{12})(\delta_{13} + \delta_{23} + \delta_{32}) + (a_{23} - a_{13})(\delta_{12} + \delta_{13} + \delta_{23}),
\]
\[(16) \quad f_{12} = (a_{12} - a_{13})(\delta_{23} + \delta_{13} + \delta_{32}) + (a_{12} - a_{23})(\delta_{12} + \delta_{13} + \delta_{23}).
\]

We know from [AV] that $A_a$ is a Hopf algebra of dimension 72 and coradical isomorphic to $kS^n$, for any $a \in \mathfrak{A}_3$. Furthermore, any finite dimensional non-semisimple Hopf algebra with coradical $kS^n$ is isomorphic to $A_a$ for some $a \in \mathfrak{A}_3$; $A_{[b]} \simeq A_a$ iff $[a] = [b]$. Let $\Omega = f_{13}(12) - f_{13}$, that is
\[(15) \quad \Omega = (a_{23} - a_{13})(\delta_{12} - \delta_e) + (a_{13} - a_{12})(\delta_{13} - \delta_{12} - \delta_{23}) + (a_{12} - a_{23})(\delta_{23} - \delta_{12}).
\]

The following formulae follow from the defining relations:
\[(16) \quad x_{(12)}x_{(13)}x_{(12)} = x_{(13)}x_{(12)}x_{(12)} + x_{(23)}(a_{13} - a_{12}),
\]
\[(17) \quad x_{(23)}x_{(12)}x_{(23)} = x_{(12)}x_{(23)}x_{(12)} - x_{(23)}(a_{23} - a_{12}) \quad \text{and}
\]
\[(18) \quad x_{(23)}x_{(12)}x_{(13)} = x_{(12)}x_{(12)}x_{(23)} + x_{(12)}\Omega.
\]

Let
\[
\mathbb{B} = \left\{ 1, \quad x_{(13)}, \quad x_{(13)}x_{(12)}, \quad x_{(13)}x_{(12)}x_{(13)}, \quad x_{(13)}x_{(12)}x_{(23)}x_{(12)}, \quad x_{(23)}, \quad x_{(12)}x_{(13)}, \quad x_{(12)}x_{(23)}x_{(12)}x_{(23)}, \quad x_{(12)}x_{(23)} \right\}.
\]

Then $\{x\delta_g | x \in \mathbb{B}, \quad g \in S_3\}$ is a basis of $A_{[a]} [AV]$. Fix $g \in G$. The classes of the monomials in $\mathbb{B}$ form a basis of the Verma module $M_g$. Denote by
the class of \(x_{ij}\ldots x_{rs}\); we simply set \(m_{\text{top}} = m_{(13)(12)(23)(12)}\).

The action of \(A_\theta\) on \(M_g\) is described in this basis by the following formulae:

\[
(19) \quad f \cdot m_1 = f(g)m_1, \quad f \in \mathbb{k}^3_1; \\
(20) \quad f \cdot m_{(ij)\ldots (rs)} = f((ij)\ldots (rs)g)m_{(ij)\ldots (rs)}, \quad f \in \mathbb{k}^3_1; \\
(21) \quad x_{(ij)} \cdot m_1 = m_{(ij)}, \quad (ij) \in \mathbb{O}^3_2; \\
(22) \quad x_{(ij)} \cdot m_{(ij)} = f_{ij}(g)m_1, \quad (ij) \in \mathbb{O}^3_2; \\
(23) \quad x_{(13)} \cdot m_{(23)} = -m_{(23)(12)} - m_{(12)(13)}, \\
(24) \quad x_{(13)} \cdot m_{(12)} = m_{(13)(12)}, \\
(25) \quad x_{(23)} \cdot m_{(13)} = -m_{(12)(23)} - m_{(13)(12)}, \\
(26) \quad x_{(23)} \cdot m_{(12)} = m_{(23)(12)}, \\
(27) \quad x_{(12)} \cdot m_{(13)} = m_{(12)(13)}, \\
(28) \quad x_{(12)} \cdot m_{(23)} = m_{(12)(23)}; \\
(29) \quad x_{(13)} \cdot m_{(13)(12)} = f_{13}(12)g)m_{(12)}, \\
(30) \quad x_{(13)} \cdot m_{(12)(13)} = m_{(13)(12)(13)}, \\
(31) \quad x_{(13)} \cdot m_{(23)(12)} = -m_{(13)(12)(13)} - f_{13}(12)g)m_{(23)} \\
(32) \quad x_{(13)} \cdot m_{(12)(23)} = m_{(13)(12)(23)}, \\
(33) \quad x_{(23)} \cdot m_{(13)(12)} = -m_{(12)(23)(12)} - f_{12}(13)m_{(13)}, \\
(34) \quad x_{(23)} \cdot m_{(12)(13)} = m_{(13)(12)(23)} + \Omega(g)m_{(12)}, \\
(35) \quad x_{(23)} \cdot m_{(12)(23)} = f_{23}(12)g)m_{(12)}, \\
(36) \quad x_{(23)} \cdot m_{(12)(23)} = m_{(13)(23)(12)} - m_{(13)}f_{23}(13), \\
(37) \quad x_{(12)} \cdot m_{(13)(12)} = m_{(13)(12)(13)} + m_{(23)}f_{13}(23)), \\
(38) \quad x_{(12)} \cdot m_{(12)(13)} = f_{12}(13)g)m_{(13)}, \\
(39) \quad x_{(12)} \cdot m_{(12)(23)} = m_{(12)(23)(12)}, \\
(40) \quad x_{(12)} \cdot m_{(12)(23)} = f_{12}(23)g)m_{(23)}; \\
(41) \quad x_{(13)} \cdot m_{(13)(12)(13)} = f_{13}(12)g)m_{(12)(13)}, \\
(42) \quad x_{(13)} \cdot m_{(12)(23)(12)} = m_{\text{top}}, \\
(43) \quad x_{(13)} \cdot m_{(13)(12)(23)} = f_{13}(12)(23)g)m_{(12)(23)}, \\
(44) \quad x_{(23)} \cdot m_{(13)(12)(13)} = m_{\text{top}} - (f_{12}\Omega + (a_{(13)} - a_{(12)})f_{23})g)m_1, \\
(45) \quad x_{(23)} \cdot m_{(12)(23)(12)} = f_{12}(12)g)m_{(12)(23)} + (a_{(12)} - a_{(23)})m_{(13)(12)}, \\
(46) \quad x_{(23)} \cdot m_{(13)(12)(23)} = f_{23}(12)g)m_{(12)(13)} - \Omega(g)m_{(23)(12)}, \\
(47) \quad x_{(12)} \cdot m_{(13)(12)(13)} = (f_{13}(g) + f_{12}(23))m_{(13)(12)} + f_{12}(23)m_{(12)(23)}, \\
(48) \quad x_{(12)} \cdot m_{(12)(23)(12)} = f_{12}(23)g)m_{(23)(12)}, \\
(49) \quad x_{(12)} \cdot m_{(13)(12)(23)} = -m_{\text{top}} + (f_{13}(23)f_{23} - f_{12}(13)f_{13})(g)m_1;
Lemma gives the following result.

Let
\[ g, h \]
Let
\[ a \]
are reduced to consider the Verma modules
\[ M \]
maximal submodules of the various Verma modules. By Lemma 7 (a), we
\[ \hat{\omega} \]
This weight space is
\[ A \]
considered of the algebras
\[ A_{[a]} \]
to determine the simple
\[ k \]
case. Up to isomorphism, cf. (7), we may assume
\[ a_{(12)} = a_{(13)} = a_{(23)} \]
For shortness, we shall say that
\[ a \]
is generic.

To proceed with the description of the simple modules, we split the con-
\[ S \]
 consider the Verma module
\[ M \]
as
\[ k \]
modules. The isotypic components of the Verma module
\[ M_e \]
M_{e}[12] = \langle m_{(12)}, m_{(13)(12)(23)} \rangle,
\[ M_e[(13)] = \langle m_{(13)}, m_{(12)(23)(12)} \rangle, \]
\[ M_e[(23)] = \langle m_{(23)}, m_{(13)(12)(13)} \rangle, \]
\[ M_e[(132)] = \langle m_{(13)(12)}, m_{(23)(12)} \rangle. \]

Let
\[ g, h \in S_3, \ (ij) \in S_3 \]
By (20) and (3), we have
\[ M_g[h] = M_g[hg^{-1}], \]
\[ x_{(ij)} \cdot M_g[h] \subseteq M_g[(ij)h]. \]

It is convenient to introduce the following elements:
\[ m_{soc} = f_{13}(23) f_{23}(13) m_1 - m_{top}, \]
\[ m_o = m_{(13)(12)(13)} + f_{13}(23) m_{(23)}. \]

3.2. Case \( a \in A_3 \) generic.

To determine the simple \( A_{[a]} \)-modules, we just need to determine the
maximal submodules of the various Verma modules. By Lemma 7 (a), we
are reduced to consider the Verma modules
\[ M_e \]
and
\[ M_g \]
for some fixed
\[ g \neq e. \]
We choose
\[ g = (13)(23); \]
for the sake of an easy exposition, we write the
\[ S_3 \]
elements as products of transpositions.

We start with the following observation. Let
\[ M \]
be a cyclic \( A_{[a]} \)-module, generated by
\[ v \in M[(13)(23)]. \]
By (55) and acting by the monomials in our
basis of
\[ A_{[a]}, \]
we see that
\[ M[(13)(23)] = (x_{(13)} x_{(23)} \cdot v, x_{(23)} x_{(12)} \cdot v, x_{(12)} x_{(13)} \cdot v). \]
This weight space is \( \neq 0 \) by Lemma 7 (b), and a further application of this
Lemma gives the following result.
Remark 9. Let $M$ be a cyclic $A_{[a]}$-module, generated by $v \in M[(13)(23)]$. If $\dim M[(23)(13)] = 1$, then

\begin{align}
M[(23)] &= \langle x_{(23)} \cdot v \rangle, & M[e] &= \langle x_{(12)} x_{(23)} \cdot v, x_{(13)} x_{(12)} \cdot v \rangle, \\
M[(12)] &= \langle x_{(23)} \cdot v \rangle, & M[(13)] &= \langle x_{(12)} \cdot v \rangle, \\
M[(13)(23)] &= \langle v \rangle, & M[(23)(13)] &= \langle x_{(13)} x_{(23)} \cdot v \rangle.
\end{align}

(58)

Thus, any cyclic module as in the Remark has either dimension 5, 6 or 7. Moreover, there is a simple module $L$ like this; $L$ has a basis $\{v_g | e \neq g \in S_3\}$ and the action is given by

\begin{equation}
(59) \quad v_g \in L[g], \quad x_{(ij)} \cdot v_g = \begin{cases} 
    v_{(ij)g} & \text{if } \sgn g = 1, \\
    f_{(ij)g}(v_{(ij)g}) & \text{if } \sgn g = -1.
\end{cases}
\end{equation}

Let $k_e$ be as in Lemma 8. We shall see that $L$ and $k_e$ are the only simple modules of $A_{[a]}$.

The Verma module $M_e$ projects onto the simple submodule $k_e$, hence the kernel of this projection is a maximal submodule; explicitly this is

$$N_e = A_{[a]} : M_e [(13)(23)] = \oplus_{g \sim a(13)(23)} M_e [g] \oplus \langle m_{\top} \rangle.$$  

We see that this is the unique maximal submodule, as consequence of the following description of all submodules of $M_e$.

Lemma 10. The submodules of $M_e$ are

$$\langle m_{\top} \rangle \subset A_{[a]} : v \subset N_e \subset M_e$$

for any $v \in M_e [(13)(23)] - 0$. The submodules $A_{[a]} : v$ and $A_{[a]} : u$ coincide iff $v \in \langle u \rangle$. The quotients $A_{[a]} : v/(m_{\top})$ and $N_e / A_{[a]} : v$ are isomorphic to $L$; and $M_e / N_e$ and $\langle m_{\top} \rangle$ are isomorphic to $k_e$.

Proof. By (51), (50) and (52), we have $x_{(ij)} \cdot m_{\top} = 0$ for all $(ij) \in O_2^2$. Let

$$v = \lambda m_{(23)(12)} + \mu m_{(12)(13)} \in M_e [(13)(23)] - 0,$$

$$w = \mu m_{(12)(23)} + (\mu - \lambda) m_{(13)(12)} \in M_e [(23)(13)].$$

Using the formulae (23) to (49), we see that $x_{(13)} x_{(23)} \cdot v$, $x_{(23)} x_{(12)} \cdot v$ and $x_{(12)} x_{(13)} \cdot v$ are non-zero multiples of $w$. That is, $\dim (A_{[a]} : v) [(23)(13)] = 1$.

Also, $x_{(12)} x_{(23)} \cdot v = -\mu m_{\top}$ and $x_{(13)} x_{(12)} \cdot v = \lambda m_{\top}$. Hence

$$\left\{ v, x_{(23)} \cdot v, x_{(12)} \cdot v, x_{(13)} \cdot v, w, m_{\top} \right\}$$

is a basis of $A_{[a]} : v$ by Remark 9.

Let now $N$ be a (proper, non-trivial) submodule of $M_e$. If $N \neq \langle m_{\top} \rangle$, then there exists $v \in N [(13)(23)] - 0$. Hence $A_{[a]} : v$ is a submodule of $N$ and $N[e] = \langle m_{\top} \rangle$ because $m_1 \in M[e]$ and $\dim M[e] = 2$. Therefore $N = A_{[a]} : N [(13)(23)]$. \quad \Box

It is convenient to introduce the following $A_{[a]}$-modules which we will use in the Section 4.
Definition 11. Let \( t \in \mathfrak{S}_3 \). We denote by \( W_t(L, k_e) \) the \( A_3 \)-module with basis \( \{ w_g : g \in \mathfrak{S}_3 \} \) and action given by

\[
w_g \in W_t(L, k_e)[g], \quad x_{(ij)} \cdot w_g = \begin{cases} 0 & \text{if } g = e, \\ w_{(ij)g} & \text{if } g \neq e \text{ and } \sgn g = 1, \\ f_{ij}(g)w_{(ij)g} & \text{if } g \neq (ij) \text{ and } \sgn g = -1, \\ t_{(ij)}w_e & \text{if } g = (ij). \end{cases}
\]

The well-definition of \( W_t \) follows from the next lemma.

Lemma 12. Let \( t, \tilde{t} \in \mathfrak{S}_3 \).

(a) If \( t = (0, 0, 0) \), then \( W_t(L, k_e) \simeq k_e \oplus L \).

(b) If \( t \neq (0, 0, 0) \), then there exists \( v \in M_e[(13)(23)] = 0 \) such that \( W_t(L, k_e) \simeq A_3 \cdot v \).

(c) If \( v \in M_e[(13)(23)] \neq 0 \), then there exists \( t \neq (0, 0, 0) \) such that \( W_t(L, k_e) \simeq A_3 \cdot v \).

(d) \( W_t(L, k_e) \) is an extension of \( L \) by \( k_e \).

(e) \( W_t(L, k_e) \simeq W_{\tilde{t}}(L, k_e) \) if and only if \( t = \mu \tilde{t} \) with \( \mu \in k^\times \).

Proof. (a) is immediate. If we prove (b), then (d) follows from Lemma 10. (b) We set \( w_{(13)(23)} = t_{(13)}m_{(23)(12)} - t_{(12)}m_{(12)(13)} \in M_e[(13)(23)] - 0 \),

\[
w_{(23)} = x_{(13)} \cdot w_{(13)(23)} , \quad w_{(13)} = x_{(12)} \cdot w_{(13)(23)} , \quad w_{(12)} = x_{(23)} \cdot w_{(13)(23)} .
\]

\[
w_{(23)(13)} = \frac{1}{f_{23}((13))} x_{(23)} x_{(12)} \cdot w_{(13)(23)} \quad \text{and} \quad w_e = m_{\text{top}}.
\]

By the proof of Lemma 10 and (17), we see that \( W_t(L, k_e) \simeq A_3 \cdot w_{(13)(23)} \).

(c) follows from the proof of Lemma 10. (e) Let \( \{ w_g : g \in \mathfrak{S}_3 \} \) be the basis of \( W_t(L, k_e) \) according to Definition 11. Let \( F : W_t(L, k_e) \rightarrow W_{\tilde{t}}(L, k_e) \) be an isomorphism of \( A_3 \)-modules. Since \( F \) is an isomorphism of \( k^3 \)-modules, there exists \( \mu_g \in k^\times \) for all \( g \in \mathfrak{S}_3 \) such that \( F(w_g) = \mu_g w_g \). In particular, \( F \) induces an automorphism of \( L \). Since \( L \) is simple (cf. Theorem 1), \( \mu_g = \mu_L \) for all \( g \neq e \). Since \( F(x_{(ij)} \cdot w_{(ij)}) = x_{(ij)} \cdot F(w_{(ij)}) \), we see that \( t = \frac{\mu_L}{\mu_e} \tilde{t} \).

Conversely, \( F \) is well defined for all \( \mu_e \) and \( \mu_L \) such that \( \mu = \frac{\mu_L}{\mu_e} \).

The Verna module \( M_{(13)(23)} \) projects onto the simple module \( L \), hence the kernel of this projection is a maximal submodule; explicitly this is

\[
N_{(13)(23)} = A_3 \cdot M_{(13)(23)}[e] = M_{(13)(23)}[e] \oplus A_3 \cdot m_{\text{soc}}.
\]

We see that this is the unique maximal submodule, as consequence of the following description of all submodules of \( M_{(13)(23)} \). Recall \( m_{\text{soc}} \) from (56).

Lemma 13. The submodules of \( M_{(13)(23)} \) are

\[
A_3 \cdot m_{\text{soc}} \subseteq A_3 \cdot v \subseteq N_{(13)(23)} \subseteq M_{(13)(23)}
\]

for all \( v \in M_{(13)(23)}[e] - 0 \). The submodules \( A_3 \cdot v \) and \( A_3 \cdot u \) coincide iff \( v \in \langle u \rangle \). The quotients \( A_3 \cdot v/A_3 \cdot m_{\text{soc}} \) and \( N_{(13)(23)}/A_3 \cdot v \) are isomorphic to \( k_e \); and \( M_{(13)(23)}/N_{(13)(23)} \) and \( A_3 \cdot m_{\text{soc}} \) are isomorphic to \( L \).
Proof. Let \( v = \lambda m_1 + \mu m_{\text{top}} \in M_{(13)(23)}[(13)(23)] - 0 \) and \( N = A_{[a]} \cdot v \). Using the formulae (23) to (49), we see that
\[
\begin{align*}
x_{(12)}x_{(13)} \cdot v &= \lambda m_{(12)(13)} - \mu f_{13}((23))^2 m_{(23)(12)} \\
x_{(23)}x_{(12)} \cdot v &= \mu f_{23}((13))^2 m_{(12)(13)} + (\lambda + 2\mu f_{13}((23)) f_{23}((13))) m_{(23)(12)}.
\end{align*}
\]
Thus, \( \dim N[(13)(23)] = 1 \) iff \( \lambda + \mu f_{13}((23)) f_{23}((13)) = 0 \), that is iff \( v \in \langle m_{\text{soc}} \rangle - 0 \). In this case,
\[
\left\{ v, x_{(23)} \cdot v, x_{(12)} \cdot v, x_{(13)} \cdot v, x_{(12)} x_{(13)} \cdot v \right\}
\]
is a basis of \( A_{[a]} \cdot m_{\text{soc}} \) by Remark 9.

Let now \( N \) be an arbitrary submodule of \( M_{(13)(23)} \). If \( \dim N[(13)(23)] = 2 \), then \( N = M_{(13)(23)} \). If \( \dim N[(13)(23)] = 0 \), then \( N \subset M_{(13)(23)}[e] \) by Lemma 7. But this is not possible since \( \ker x_{(13)} \cap \ker x_{(23)} \cap \ker x_{(12)} = 0 \), what is checked using the formulae (23) to (52). It remains the case \( \dim N[(13)(23)] = 1 \). By the argument at the beginning of the proof, the lemma follows.

It is convenient to introduce the following \( A_{[a]} \)-modules which we will use in the Section 4.

Definition 14. Let \( t \in \mathfrak{a}_3 \). We denote by \( W_t(k_e, L) \) the \( A_{[a]} \)-module with basis \( \{ w_g : g \in S_3 \} \) and action given by
\[
w_g \in W_t(k_e, L)[g], \quad x_{(ij)} \cdot w_g = \begin{cases} t_{(ij)} w_{(ij)} & \text{if } g = e, \\ f_{ij}(g) w_{(ij)} & \text{if } g \neq e \text{ and } \sgn g = 1, \\ w_{(ij)} & \text{if } \sgn g = -1. \end{cases}
\]

The well-definition of \( W_t(k_e, L) \) follows from the next lemma.

Lemma 15. Let \( t, \tilde{t} \in \mathfrak{a}_3 \).

(a) If \( t = (0, 0, 0) \), then \( W_t(k_e, L) \cong L \oplus k_e \).
(b) If \( t \neq (0, 0, 0) \), then there exists \( v \in M_{(13)(23)}[e] - 0 \) such that \( W_t(k_e, L) \cong A_{[a]} \cdot v \).
(c) If \( v \in M_{(13)(23)}[e] - 0 \), then there exists \( t \neq (0, 0, 0) \) such that \( W_t(k_e, L) \cong A_{[a]} \cdot v \).
(d) \( W_t(k_e, L) \) is an extension of \( k_e \) by \( L \).
(e) \( W_t(k_e, L) \cong W_{\tilde{t}}(k_e, L) \) if and only if \( t = \mu \tilde{t} \) with \( \mu \in k^{\times} \).

Proof. (a) is immediate. If we prove (b), then (d) follows from Lemma 13.

(b) We set \( w((13)(23)) = m_{\text{soc}} \in M_{(13)(23)}[(13)(23)] \),
\[
w_{(23)} = \frac{x_{(13)} \cdot w_{(13)(23)}}{f_{13}((13)(23))}, \quad w_{(13)} = \frac{x_{(12)} \cdot w_{(13)(23)}}{f_{12}((13)(23))}, \quad w_{(12)} = \frac{x_{(23)} \cdot w_{(13)(23)}}{f_{23}((13)(23))},
\]
\[
w_{(23)(13)} = x_{(23)} x_{(12)} \cdot w_{(13)(23)} \quad \text{and} \quad w_e = -t_{(12)} m_{(13)(12)} + t_{(13)} m_{(12)(13)} \neq 0.
\]
Using the formulae (23) to (49), it is not difficult to see that \( W_t(k_e, L) \cong A_{[a]} \cdot w_e \). (c) follows using the formulae (23) to (49). The proof of (e) is similar to the proof of Lemma 12 (e).
Theorem 1. Let $a \in \mathfrak{A}_3$ be generic. There are exactly 2 simple $\mathcal{A}_{[a]}$-modules up to isomorphism, namely $k_e$ and $L$. Moreover, $M_e$ is the projective cover, and the injective hull, of $k_e$; also, $M_{(13)(23)}$ is the projective cover, and the injective hull, of $L$.

Proof. We know that $k_e$ and $L$ are the only two simple $\mathcal{A}_{[a]}$-modules up to isomorphism by Proposition 1 and Lemmata 7 (a), 10 and 13. Hence, a set of primitive orthogonal idempotents has at most 6 elements [CR, (6.8)]. Since the $\delta_g$, $g \in S_3$, are orthogonal idempotents, they must be primitive. Therefore $M_e$ and $M_{(13)(23)}$ are the projective covers (and the injective hulls) of $k_e$ and $L$, respectively by [CR, (9.9)], see page 3.

3.3. Case $a \in \mathfrak{A}_3$ sub-generic.

Through this subsection, we suppose that $a_{(12)} \neq a_{(13)} = a_{(23)}$. Then the equivalence classes of $S_3$ by $\sim_a$ are

$$\{e\}, \quad \{(12)\} \quad \text{and} \quad \{(13), (23), (13)(23), (23)(13)\}.$$ 

In fact,

- $e$ and (12) belong to the isotropy group $S_3^a$.
- $(13) = (23)(12)(23)$ with $f_{12}((23)) = a_{(12)} - a_{(13)} \neq 0$ and $f_{23}((12)(23)) = a_{(23)} - a_{(12)} \neq 0$.
- $(123) = (13)(23)$ with $f_{13}((23)) = a_{(13)} - a_{(12)} \neq 0$.
- $(132) = (23)(13)$ with $f_{23}((13)) = a_{(23)} - a_{(12)} \neq 0$.

To determine the simple $\mathcal{A}_{[a]}$-modules, we proceed as in the subsection above; that is, we just need to determine the maximal submodules of the Verma modules $M_e$, $M_{(12)}$ and $M_{(13)(23)}$, see Proposition 5.

Let $M$ be a cyclic $\mathcal{A}_{[a]}$-module generated by $v \in M[[(13)(23)]$. Here again, we can describe the weight spaces of $M$. By (55) and acting by the monomials in our basis, we see that $M[[(13)(23)] = \langle x_{(13)}x_{(23)} \cdot v, x_{(12)}x_{(23)} \cdot v, x_{(12)}x_{(13)} \cdot v \rangle$. This weight space is $\neq 0$ by Remark 4 applied to $(13)(23) \sim_a (23)(13)$, and a further application of this Remark gives the following result.

Remark 16. Let $M$ be a cyclic $\mathcal{A}_{[a]}$-module generated by $v \in M[[(13)(23)]$. If $\dim M[[(13)(23)] = 1$, then

$$M[e] = \langle x_{(23)}x_{(13)} \cdot v, (x_{(12)}x_{(23)}) \cdot v, x_{(12)}x_{(13)} \cdot v \rangle,$$

$$M[[(12)] = \langle (x_{(23)} \cdot v), (x_{(13)}x_{(12)}x_{(13)}) \cdot v \rangle,$$

$$M[[(23)(13)] = \langle x_{(12)}x_{(13)} \cdot v \rangle.$$

There is a simple module $L$ like this; $\{v_{(13)}, v_{(23)}, v_{(13)(23)}, v_{(23)(13)}\}$ is a basis of $L$ and the action is given by

$$v_g \in L[g], \quad x_{(ij)} \cdot v_g = \begin{cases} 0 & \text{if } g = (ij) \\
m_{(ij)g} & \text{if } g \neq (ij), \ \text{sgn } g = -1, \\
f_{ij}(g)m_{(ij)g} & \text{if } \text{sgn } g = 1. \end{cases}$$
Let \( k_{(12)} \) and \( k_e \) be as in Lemma 8. We shall see that \( L, k_{(12)} \) and \( k_e \) are the only simple modules of \( A_{[a]} \).

The Verma module \( M_e \) projects onto the simple module \( k_e \), hence the kernel of this projection is a maximal submodule; explicitly this is

\[
N_e = A_{[a]} \cdot (M_e[13](23) \oplus M_e[12]) = \oplus_{g \sim a} (13)(23) M_e[g] \oplus M_e[12] \oplus \langle m_{\text{top}} \rangle.
\]

We see that this is the unique maximal submodule, as consequence of the following description of all submodules of \( M_e \).

**Lemma 17.** The lattice of (proper, non-trivial) submodules of \( M_e \) is displayed in (62), where \( v \) and \( w \) satisfy

\[
M_e[13](23) = \langle v, m_{(23)(12)} \rangle, \quad M_e[12] = \langle w, m_{(13)(12)(23)} \rangle.
\]

The submodules \( A_{[a]} \cdot v \) (resp. \( A_{[a]} \cdot w \)) and \( A_{[a]} \cdot v_1 \) (resp. \( A_{[a]} \cdot w_1 \)) coincide iff \( v \in \langle v_1 \rangle \) (resp. \( w \in \langle w_1 \rangle \)). The labels on the arrows indicate the quotient of the module on top by the module on the bottom.

![Diagram](62)

**Proof.** Let

\[
v = \lambda m_{(23)(12)} + \mu m_{(12)(13)} \quad \in M_e[13](23) - 0,
\]

\[
\tilde{v} = \mu m_{(12)(23)} + (\mu - \lambda)m_{(13)(12)} \quad \in M_e[23](13).
\]

Using the formulae (23) to (49), we see that \( x_{(23)} x_{(12)} \cdot v \) and \( x_{(12)} x_{(13)} \cdot v \) are non-zero multiples of \( \tilde{v} \). That is, \( \dim(A_{[a]} \cdot v)[[23](13)] = 1 \). Moreover, \( x_{(12)} x_{(23)} \cdot v = -\mu m_{\text{top}} \) and \( x_{(13)} x_{(12)} \cdot v = \lambda m_{\text{top}} \), and \( x_{(23)} \cdot v \) and \( (x_{(13)} x_{(12)} x_{(13)}) \cdot v \) are non-zero multiples of \( \mu m_{(13)(12)(23)} \). By Remark 16, we obtain a basis for \( A_{[a]} \cdot v \):

\[
\left\{ v, x_{(12)} \cdot v, x_{(13)} \cdot v, \tilde{v}, m_{\text{top}}, \mu m_{(13)(12)(23)} \right\};
\]

if \( \mu = 0 \), we obviate the last vector.
By (51), (50) and (52), \( x_{ij} \cdot m_{\text{top}} = 0 \) for all \( (ij) \in \mathcal{O}_2^3 \). Then
\[
\mathcal{A}_a \cdot m_{\text{top}} = \langle m_{\text{top}} \rangle
\]
and \( \mathcal{A}_a \cdot u = \mathcal{A}_a \cdot m_1 = M_e \) if \( u \in M_e[e] \) is linearly independent to \( m_{\text{top}} \).

By (43), (46) and (49), \( x_{ij} \cdot m_{(13)(12)(23)} = -\delta_{(12)}((ij))m_{\text{top}} \) for all \( (ij) \in \mathcal{O}_2^3 \). Then
\[
\mathcal{A}_a \cdot m_{(13)(12)(23)} = \langle m_{\text{top}}, m_{(13)(12)(23)} \rangle.
\]

By (22), (24) and (26), \( x_{ij} \cdot m_{(12)} = \delta_{(13)}((ij))m_{(13)(12)} + \delta_{(23)}((ij))m_{(23)(12)} \) for all \( (ij) \in \mathcal{O}_2^3 \). Then
\[
\mathcal{A}_a \cdot w = \mathcal{A}_a \cdot m_{(23)(12)} \oplus \langle w \rangle
\]
by (63) and Remark 4, if \( w \in M_e[(12)] \) is linearly independent to \( m_{(13)(12)(23)} \).

Let now \( N \) be a (proper, non-trivial) submodule of \( M_e \) which is not \( \langle m_{\text{top}} \rangle \). We set \( \tilde{N} = \mathcal{A}_a \cdot N[(12)] + \mathcal{A}_a \cdot N[(13)(23)] \). Then \( \tilde{N}[g] = N[g] \) for all \( g \neq e \) by Remark 4. By the argument at the beginning of the proof, \( \langle m_{\text{top}} \rangle \subset \tilde{N} \). Then \( \tilde{N}[e] = \langle m_{\text{top}} \rangle = N[e] \) because otherwise \( N = M_e \). Therefore \( N = \tilde{N} \). To finish, we have to calculate the submodules of \( M_e \) generated by homogeneous subspaces of \( M_e[(12)] \oplus M_e[(13)(23)] \); this follows from the argument at the beginning of the proof. \( \square \)

The Verma module \( M_{(13)(23)} \) projects onto the simple module \( L \), hence the kernel of this projection is a maximal submodule; explicitly this is
\[
N_{(13)(23)} = \mathcal{A}_a \cdot (M_{(13)(23)}[e] \oplus M_{(13)(23)}[(12)])
= M_{(13)(23)}[e] \oplus M_{(13)(23)}[(12)] \oplus \mathcal{A}_a \cdot m_{\text{soc}}.
\]

We see that this is the unique maximal submodule, as consequence of the following description of all submodules of \( M_{(13)(23)} \).

**Lemma 18.** The lattice of (proper, non-trivial) submodules of \( M_{(13)(23)} \) is

\[
\begin{array}{ccc}
\mathcal{A}_a \cdot M_{(13)(23)}[e] & \mathcal{A}_a \cdot M_{(13)(23)}[(12)] \\
\mathcal{A}_a \cdot v & \mathcal{A}_a \cdot \langle m_o, m_{(12)(23)} \rangle & \mathcal{A}_a \cdot w \\
\mathcal{A}_a \cdot m_o & \mathcal{A}_a \cdot m_{\text{soc}} \\
\end{array}
\]

\[
\begin{array}{ccc}
\mathcal{A}_a \cdot M_{(13)(23)}[e] & \mathcal{A}_a \cdot M_{(13)(23)}[(12)] \\
\mathcal{A}_a \cdot v & \mathcal{A}_a \cdot \langle m_o, m_{(12)(23)} \rangle & \mathcal{A}_a \cdot w \\
\mathcal{A}_a \cdot m_o & \mathcal{A}_a \cdot m_{\text{soc}} \\
\end{array}
\]
Here \( v \) and \( w \) satisfy \( M_{(13)(23)}[e] = \langle v, m_{(12)(23)} \rangle \), \( M_{(13)(23)}[(12)] = \langle w, m_o \rangle \). The submodules \( A_{[a]} \cdot v \) (resp. \( A_{[a]} \cdot w \)) and \( A_{[a]} \cdot v_1 \) (resp. \( A_{[a]} \cdot w_1 \)) coincide iff \( v \in \langle v_1 \rangle \) (resp. \( w \in \langle w_1 \rangle \)). The labels on the arrows indicate the quotient of the module on top by the module on the bottom.

**Proof.** Let \( u = \lambda m_1 + \mu m_{\text{top}} \in M_{(13)(23)}[(13)(23)] - 0 \). Using the formulae (23) to (49), we see that

\[
x_{(12)}x_{(13)} \cdot u = \lambda m_{(12)(13)} - \mu f_{13}(23)2m_{(23)(12)} \quad \text{and} \quad
x_{(23)}x_{(12)} \cdot u = \mu f_{23}(13)2m_{(12)(13)} + (\lambda + 2\mu f_{13}(23)f_{23}(13))m_{(23)(12)}.
\]

Thus, \( \dim N[(23)(13)] = 1 \) iff \( \lambda + \mu f_{13}(23)f_{23}(13) = 0 \), that is iff \( u \in \langle m_{\text{soc}} \rangle - 0 \). By Remark 16,

\[
A_{[a]} \cdot m_{\text{soc}} = \langle m_{\text{soc}}, x_{(12)} \cdot m_{\text{soc}}, x_{(13)} \cdot m_{\text{soc}}, x_{(12)}x_{(13)} \cdot m_{\text{soc}} \rangle
\]

and \( A_{[a]} \cdot u = A_{[a]} \cdot m_1 = M_{(13)(23)} \), if \( u \in M_{(13)(23)}[(13)(23)] \) is linearly independent to \( m_{\text{soc}} \).

By the formulae (23) to (52), if \( u \in (M_{(13)(23)}[e] \oplus M_{(13)(23)}[(12)]) - 0 \), then \( 0 \neq \langle x_{(13)} \cdot u, x_{(23)} \cdot u \rangle \subset A_{[a]} \cdot m_{\text{soc}} \). Therefore

\[
A_{[a]} \cdot m_{\text{soc}} \subset A_{[a]} \cdot u
\]

by Remark 4. Also, if \( v \) and \( w \) satisfy \( M_{(13)(23)}[e] = \langle v, m_{(12)(23)} \rangle \) and \( M_{(13)(23)}[(12)] = \langle w, m_o \rangle \), then

\[
\langle x_{(12)} \cdot v \rangle = \langle m_o \rangle \quad \text{and} \quad \langle x_{(12)} \cdot w \rangle = \langle m_{(12)(23)} \rangle.
\]

Let now \( N \) be a (proper, non-trivial) submodule of \( M_{(13)(23)} \) which is not \( A_{[a]} \cdot m_{\text{soc}} \). We set \( \tilde{N} = A_{[a]} \cdot N[e] + A_{[a]} \cdot N[(12)] \). Then \( \tilde{N}[g] = N[g] \) for \( g = e, (12) \) by Remark 4. By the argument at the beginning of the proof, \( A_{[a]} \cdot m_{\text{soc}} \subset \tilde{N} \). Then \( \oplus_{g \sim a(13)(23)} N[g] = A_{[a]} \cdot m_{\text{soc}} = \oplus_{g \sim a(13)(23)} \tilde{N}[g] \) because otherwise \( N = M_{(13)(23)} \). Therefore \( N = \tilde{N} \). To finish, we have to calculate the submodules of \( M_{(13)(23)} \) generated by homogeneous subspaces of \( M_{(13)(23)}[(12)] \oplus M_{(13)(23)}[e] \); this follows from the argument at the beginning of the proof. \( \square \)

The Verma module \( M_{(12)} \) projects onto the simple module \( \mathbb{k}_{(12)} \), hence the kernel of this projection is a maximal submodule; explicitly this is

\[
N_{(12)} = A_{[a]} \cdot (M_{(12)}[(13)(23)] \oplus M_{(12)}[e])
= \oplus_{g \sim a(13)(23)} M_{(12)}[g] \oplus M_{(12)}[e] \oplus \langle m_{\text{top}} \rangle.
\]

We see that this is the unique maximal submodule, as consequence of the following description of all submodules of \( M_{(12)} \).
Lemma 19. The lattice of (proper, non-trivial) submodules of $M_{(12)}$ is

![Diagram](image_url)

Here $v$ and $w$ satisfy $M_{(12)}[[13](23)] = \langle v, m_\circ \rangle$, $M_{(12)}[e] = \langle w, m_{(13)(12)(23)} \rangle$. The submodules $A[a] \cdot v$ (resp. $A[a] \cdot w$) and $A[a] \cdot v_1$ (resp. $A[a] \cdot w_1$) coincide iff $v \in \langle v_1 \rangle$ (resp. $w \in \langle w_1 \rangle$). The labels on the arrows indicate the quotient of the module on top by the module on the bottom.

Proof. Let $v = \lambda m_{(23)} + \mu m_{(13)(12)(13)} \in M_{(12)}[[13](23)]$ be a non-zero element. By Remark 16 and using the formulae (23) to (52), we see that

$$\langle A[a] \cdot v \rangle[[13](23)] = \langle v \rangle,$$

$$\langle A[a] \cdot v \rangle[[13]] = \langle (f_{13}(23)) \mu - \lambda m_{(12)(23)} - \mu f_{13}(23)m_{(13)(12)} \rangle,$$

$$\langle A[a] \cdot v \rangle[[23]] = \langle (f_{13}(23)) \mu - \lambda m_{(12)(13)} - \lambda m_{(23)(12)} \rangle,$$

$$\langle A[a] \cdot v \rangle[[23](13)] = \langle (f_{13}(23)) \mu - \lambda f_{23}(13)m_{(13)} + \lambda m_{(12)(23)}(12) \rangle,$$

$$\langle A[a] \cdot v \rangle[[12]] = \langle m_{0} \rangle$$

and $A[a] \cdot u = A[a] \cdot m_1 = M_e$, if $u \in M_{(12)[(12)]}$ is linearly independent to $m_{0}$. By (43), (46) and (49), $x_{(ij)} \cdot m_{(13)(12)(23)} = - \delta_{(12)}((ij)) m_{0}$ for all $(ij) \in \mathcal{O}_2^3$. Then

$$A[a] \cdot m_{0} = \langle m_{0} \rangle$$

and $A[a] \cdot u = A[a] \cdot m_1 = M_e$, if $u \in M_{(12)[(12)]}$ is linearly independent to $m_{0}$. By (43), (46) and (49), $x_{(ij)} \cdot m_{(13)(12)(23)} = - \delta_{(12)}((ij)) m_{0}$ for all $(ij) \in \mathcal{O}_2^3$. Then

$$A[a] \cdot m_{(13)(12)(23)} = \langle m_{0}, m_{(13)(12)(23)} \rangle.$$

By (22), (24) and (26), $x_{(ij)} \cdot m_{(12)} = \delta_{(13)}((ij)) m_{(13)(12)} + \delta_{(23)}((ij)) m_{(23)(12)}$ for all $(ij) \in \mathcal{O}_2^3$. Then

$$A[a] \cdot w = A[a] \cdot m_\circ \oplus \langle w \rangle$$

by (64) and Remark 4, if $w \in M_{(12)}[e]$ is linearly independent to $m_{(13)(12)(23)}$. 


Let now $N$ be a (proper, non-trivial) submodule of $M_{(12)}$ which is not $\langle m_{\text{top}} \rangle$. We set $\tilde{N} = A_{[a]} \cdot N[e] + A_{[a]} \cdot N[(13)(23)]$. Then $\tilde{N}[g] = N[g]$ for all $g \neq (12)$ by Remark 4. By the argument at the beginning of the proof, $\langle m_{\text{top}} \rangle \subset \tilde{N}$. Therefore $N = \tilde{N}$. To finish, we have to calculate the submodules of $M_{(12)}$ generated by homogeneous subspaces of $M_{(12)}[(13)(23)] \oplus M_{(12)}[e]$; this follows from the argument at the beginning of the proof. 

As a consequence, we obtain the simples modules in the sub-generic case. The proof of the next theorem runs in the same way as that of Theorem 1.

**Theorem 2.** Let $a \in A_3$ with $a_{(12)} \neq a_{(13)} = a_{(23)}$. There are exactly 3 simple $A_{[a]}$-modules up to isomorphism, namely $k_e$, $k_{(12)}$, and $L$. Moreover, $M_e$ is the projective cover, and the injective hull, of $k_e$; $M_{(12)}$ is the projective cover, and the injective hull, of $k_{(12)}$; and $M_{(13)(23)}$ is the projective cover, and the injective hull, of $L$.

**Proof.** We know that $k_e$, $k_{(12)}$ and $L$ are the only two simple $A_{[a]}$-modules up to isomorphism by Proposition 1 and Lemmata 17, 18 and 19. Hence, a set of primitive orthogonal idempotents has at most 6 elements [CR, (6.8)]. Since the $\delta_g$, $g \in S_3$ are orthogonal idempotents, they must be primitive. Therefore $M_e$, $M_{(12)}$ and $M_{(13)(23)}$ are respectively the projective covers (and the injective hulls) of $k_e$, $k_{(12)}$ and $L$ by [CR, (9.9)], see page 3. 

4. **Representation type of $A_{[a]}$**

In this section, we assume that $n = 3$ as in the preceding one. We will determine the $A_{[a]}$-modules which are extensions of simple $A_{[a]}$-modules. As a consequence, we will show that $A_{[a]}$ is not of finite representation type for all $a \in A_3$.

4.1. **Extensions of simple modules.** By the following lemma, we are reduced to consider only submodules of the Verma modules for to determine the extensions of simple $A_{[a]}$-modules. Then we shall split the consideration into three different cases like Section 3 and use the lemmata there.

**Lemma 20.** Let $a \in A_3$ be non-zero. Let $S$ and $T$ be simple $A_{[a]}$-modules and $M$ be an extension of $T$ by $S$. Hence either $M \simeq S \oplus T$ as $A_{[a]}$-modules or $M$ is an indecomposable submodule of the Verma module which is the injective hull of $S$.

**Proof.** If there exists a proper submodule $N$ of $M$ which is not $S$, then $M \simeq S \oplus T$ as $A_{[a]}$-modules. In fact, $N \cap S$ is either 0 or $S$ because $S$ is simple. Let $\pi$ be as in (65). Since $T$ is simple, $\pi_{|N} : N \to T$ results an epimorphism. Therefore $M \simeq S \oplus T$ since $\dim N = \dim(N \cap S) + \dim T$. 


Let $M_S$ be the Verma module which is the injective hull of $S$. Then we have the following commutative diagram

\[
\begin{array}{ccc}
0 & \rightarrow & S \\
\downarrow & & \downarrow \pi \\
& & T \\
M_S & \rightarrow & 0
\end{array}
\]

\[
S \leftarrow f \rightarrow M
\]

Therefore either $M \simeq S \oplus T$ as $A_{[a]}$-modules or $f$ is injective. If $f$ is injective, then $M$ results indecomposable by Lemmata 10 and 13 in the generic case, and by Lemmata 17, 18 and 19 in the sub-generic case. \hfill $\Box$

Recall the modules $W_t(L, k_e)$ and $W_t(k_e, L)$ from Definitions 11 and 14. The next results follow from Lemmata 10, 13, 17, 18 and 19 by Lemma 20.

**Lemma 21.** Let $a \in \mathfrak{A}_3$ be generic. Let $S$ and $T$ be simple $A_{[a]}$-modules and $M$ be an extension of $T$ by $S$.

(a) If $S \simeq T$, then $M \simeq S \oplus S$.
(b) If $S \simeq k_e$ and $T \simeq L$, then $M \simeq W_t(L, k_e)$ for some $t \in \mathfrak{A}_3$.
(c) If $S \simeq L$ and $T \simeq k_e$, then $M \simeq W_t(k_e, L)$ for some $t \in \mathfrak{A}_3$. \hfill $\Box$

**Lemma 22.** Let $a \in \mathfrak{A}_3$ with $a_{(12)} \neq a_{(13)} = a_{(23)}$. Let $S$ and $T$ be simple $A_{[a]}$-modules and $M$ be an extension of $T$ by $S$.

(a) If $S \simeq T$, then $M \simeq S \oplus S$.
(b) If $S \simeq k_e$ and $T \simeq k_{(12)}$, then $M \simeq A_{[a]} \cdot m_{(13)(12)(23)} \subset M_e$.
(c) If $S \simeq k_{(12)}$ and $T \simeq k_e$, then $M \simeq A_{[a]} \cdot m_{(13)(12)(23)} \subset M_{(12)}$.
(d) If $S \simeq k_e$ and $T \simeq L$, then $M \simeq A_{[a]} \cdot m_{(23)(12)} \subset M_e$.
(e) If $S \simeq L$ and $T \simeq k_e$, then $M \simeq A_{[a]} \cdot m_{(12)(23)} \subset M_{(13)(23)}$.
(f) If $S \simeq k_{(12)}$ and $T \simeq L$, then $M \simeq A_{[a]} \cdot m_{(12)} \subset M_{(12)}$.
(g) If $S \simeq L$ and $T \simeq k_{(12)}$, then $M \simeq A_{[a]} \cdot m_{(12)} \subset M_{(13)(23)}$. \hfill $\Box$

**Lemma 23.** Let $k_g$ and $k_h$ be one-dimensional simple $A_{[(0,0,0)]}$-modules and $M$ be an extension of $k_h$ by $k_g$. Hence

(a) If $\text{sgn} g = \text{sgn} h$, then $M \simeq k_g \oplus k_h$.
(b) If $\text{sgn} g \neq \text{sgn} h$ and $M$ is not isomorphic to $k_g \oplus k_h$, then $g = (st)h$ for a unique $(st) \in O_3^2$ and $M$ has a basis $\{w_g, w_h\}$ such that $\langle w_g \rangle \simeq k_g$ as $A_{[a]}$-modules, $w_h \in M[h]$ and $x_{(ij)}w_h = \delta_{(ij), (st)}w_g$.

**Proof.** $M = M[g] \oplus M[h]$ as $k^S_{a_3}$-modules and $M[g] \simeq k_g$ as $A_{[a]}$-modules. Since $x_{(ij)} \cdot M[h] \subset M[(ij)h]$, the lemma follows. \hfill $\Box$

### 4.2. Representation type.

We summarize some facts about the representation type of an algebra.

Let $R$ be an algebra and $\{S_1, \ldots, S_t\}$ be a complete list of non-isomorphic simple $R$-modules. The *separated quiver of $R$* is constructed as follows. The set of vertices is $\{S_1, \ldots, S_t, S'_1, \ldots, S'_t\}$ and we write $\dim \text{Ext}^1_R(S_i, S_j)$ arrows from $S_i$ to $S'_j$, cf. [ARS, p. 350]. Let us denote by $\Gamma_R$ the underlying graph of the separated quiver of $R$. 


A characterization of the hereditary algebras of finite and tame representation type is well-known, see for example [DR2]. As a consequence, the next well-known result is obtained. If $R$ is of finite representation type, then it is Theorem D of [DR1] or Theorem X.2.6 of [ARS]. The proof given in [ARS] adapts immediately to the case when $R$ is of tame representation type.

**Theorem 3.** Let $R$ be a finite dimensional algebra with radical square zero. Then $R$ is of finite (resp. tame) representation type if and only if $\Gamma_R$ is a finite (resp. affine) disjoint union of Dynkin diagrams.

In order to use the above theorem, we know that

**Remark 24.** If $r$ is the radical of $R$, then the separated quiver of $R$ is equal to the separated quiver of $R/r^2$, see for example [GI, Lemma 4.5].

We obtain the following result by combining Corollary VI.1.5 and Proposition VI.1.6 of [ARS].

**Proposition 25.** Let $R$ be an artin algebra, $\chi$ an infinite cardinal and assume there are $\chi$ non-isomorphic indecomposable modules of length $n$. Then $R$ is not of finite representation type.

Here is the announced result.

**Proposition 26.** $A_{((0,0,0))}$ is of wild representation type. If $a \in \mathfrak{A}_3$ is non-zero, then $A_{[a]}$ is not of finite representation type.

**Proof.** If $a \in \mathfrak{A}_3$ is generic, we can apply Proposition 25 by Lemma 12 and Lemma 14. Hence $A_{[a]}$ is not of finite representation type for all $a \in \mathfrak{A}_3$ generic.

Let $a \in \mathfrak{A}_3$ be sub-generic or zero. Then $\dim \text{Ext}^1_{A_{[a]}}(T, S) = 0$ if $S \simeq T$ by Lemma 22 and 23, and $\dim \text{Ext}^1_{A_{[a]}}(T, S) = 1$ in otherwise. In fact, suppose that $a_{(12)} \neq a_{(13)} = a_{(23)}$, $S \simeq k_e$ and $T \simeq L$. By Lemma 18 and Theorem 2, $L$ admits a projective resolution of the form

$$
\ldots \xrightarrow{} P^2 \xrightarrow{} M_e \oplus M_{(12)} \xrightarrow{F} M_{(13)(23)} \xrightarrow{} L \xrightarrow{} 0,
$$

where $F$ is defined by $F|_{M_e}(m_1) = v$ and $F|_{M_{(12)}}(m_1) = w$; here $v$ and $w$ satisfy $M_{(13)(23)}[e] = \langle v, m_{(12)(23)} \rangle$, $M_{(13)(23)}[\{12\}] = \langle w, m_\circ \rangle$. Then

$$
0 \xrightarrow{} \text{Hom}_{A_{[a]}}(M_{(13)(23)}, k_e) \xrightarrow{\partial_1} \text{Hom}_{A_{[a]}}(M_e \oplus M_{(12)}, k_e) \xrightarrow{\partial_2} \ldots
$$

and $\text{Ext}^1_{A_{[a]}}(L, k_e) = \ker \partial_2/\text{Im} \partial_1$. Since $M_h$ is generated by $m_1 \in M_h[h]$ for all $h \in \mathfrak{S}_3$, $\text{Hom}_{A_{[a]}}(M_{(13)(23)}, k_e) = 0$ and $\dim \text{Hom}_{A_{[a]}}(M_e \oplus M_{(12)}, k_e) = 1$. By Lemma 22, we know that there exists a non-trivial extension of $L$ by $k_e$ and therefore $\dim \text{Ext}^1_{A_{[a]}}(L, k_e) = 1$ because it is non-zero. For other $S$ and $T$ and for the case $a = (0, 0, 0)$, the proof is similar.
Hence if \( a \in A_{[a]} \) is sub-generic and \( a_{(12)} \neq a_{(13)} = a_{(23)} \), the separated quiver of \( A_{[a]} \) is

\[
\begin{array}{c}
\mathbb{k}_e \\
\downarrow \\
L' \\
\downarrow \\
L \\
\downarrow \\
\mathbb{k}'_{(12)} \\
\mathbb{k}'_{(13)} \\
\mathbb{k}'_{(23)} \\
\mathbb{k}_{(13)(23)} \\
\mathbb{k}_{(23)(13)} \\
\end{array}
\]

and the separated quiver of \( A_{[(0,0,0)]} \) is

\[
\begin{array}{c}
\mathbb{k}_e \\
\downarrow \\
L' \\
\downarrow \\
L \\
\downarrow \\
\mathbb{k}'_{(12)} \\
\mathbb{k}'_{(13)} \\
\mathbb{k}'_{(23)} \\
\mathbb{k}'_{(13)(23)} \\
\mathbb{k}'_{(23)(13)} \\
\mathbb{k}_{(23)} \\
\end{array}
\]

Therefore the lemma follows from Theorem 3 and Remark 24.

\[\square\]

**Remark 27.** Let \( a \in \mathfrak{A}_3 \) be generic. It is not difficult to prove that the separated quiver of \( A_{[a]} \) is

\[
\begin{array}{c}
\mathbb{k}_e \\
\downarrow \\
L' \\
\downarrow \\
L \\
\downarrow \\
\mathbb{k}'_{(12)} \\
\mathbb{k}'_{(13)} \\
\mathbb{k}'_{(23)} \\
\mathbb{k}'_{(13)(23)} \\
\mathbb{k}'_{(23)(13)} \\
\mathbb{k}_{(23)} \\
\end{array}
\]

5. **On the structure of \( A_{[a]} \)**

In this section, we assume that \( n = 3 \) as in the preceding one.

5.1. **Cocycle deformations.**

We show in this subsection that the algebras \( A_{[a]} \) are cocycle deformation of each other. For this, we first recall the following theorem due to Masuoka.

If \( K \) is a Hopf subalgebra of a Hopf algebra \( H \) and \( J \) is a Hopf ideal of \( K \), then the two-sided ideal \((J)\) of \( H \) is in fact a Hopf ideal of \( H \).

**Theorem 4.** [M, Thm. 2], [BDR, Thm. 3.4]. Suppose that \( K \) is Hopf subalgebra of a Hopf algebra \( H \). Let \( I, J \) be Hopf ideal of \( K \). If there is an algebra map \( \psi \) from \( K \) to \( \mathbb{k} \) such that

- \( J = \psi \rightarrow I \leftarrow \psi^{-1} \)
- \( H/(\psi \rightarrow I) \) is nonzero,

then \( H/(\psi \rightarrow I) \) is a \((H/(I), H/(J))\)-biGalois object and so the quotient Hopf algebras \( H/(I), H/(J) \) are monoidally Morita-Takeuchi equivalent. If \( H/(I) \) and \( H/(J) \) are finite dimensional, then \( H/(I) \) and \( H/(J) \) are cocycle deformations of each other. \[\square\]

We will need the following lemma to apply the Masuoka's theorem.

**Lemma 28.** If \( W \) is a vector space and \( U \) is a vector subspace of \( W^\otimes n \), then the subalgebra of \( T(W) \) generated by \( U \) is isomorphic to \( T(U) \).
Let \( W \) where \( \psi_T \) be a basis of \( W^{\otimes n} \). Since the \( X_i \)'s are all homogeneous elements of the same degree in \( T(W) \), we only have to prove that \( \{X_i \cdots X_m : i_1, \ldots, i_m \in I^{\times n}\} \) is linearly independent in \( T(W) \) for all \( m \geq 1 \) and this is true because \( B \) is a basis of monomials of the same degree.

Here is the announced result. Observe that this gives an alternative proof to the fact that \( \dim A_{[\alpha]} = 72 \), proved in [AV] using the Diamond Lemma.

**Proposition 29.** For all \( \alpha \in \mathfrak{A}_3 \), \( A_{[\alpha]} \) is a Hopf algebra monoidally Morita-Takeuchi equivalent to \( B(V_3) \# k^{S_3} \).

**Proof.** To start with, we consider the algebra \( K_\alpha := T(V_3) \# k^{S_3} / \mathcal{J}_\alpha \), \( \alpha \in \mathfrak{A}_3 \), where \( \mathcal{J}_\alpha \) is the ideal generated by

\[
R_{(13)(23)}, \ R_{(23)(13)} \quad \text{and} \quad x_{(ij)}^2 + \sum_{g \in S_3} a_{g^{-1}(ij)g} \delta_g, \quad (ij) \in \mathcal{O}_2^3.
\]

Let \( M_3 = k^{S_3} \) with the regular representation. For all \( \alpha \in \mathfrak{A}_3 \), \( M_3 \) is an \( K_\alpha \)-module with action given by

\[
x_{(ij)} \cdot m_g = \begin{cases} m_{(ij)g} & \text{if } \text{sgn } g = -1, \\ -a_{g^{-1}(ij)g} m_{(ij)g} & \text{if } \text{sgn } g = 1. \end{cases}
\]

We have to check that the relations defining \( K_\alpha \) hold in the action. Then

\[
\delta_h(x_{(ij)} \cdot m_g) = \delta_h(\lambda_g m_{(ij)g}) = \lambda_g \delta_h((ij)g)m_{(ij)g} = \lambda_g \delta_{g^{-1}(ij)g}(g)m_{(ij)g} = x_{(ij)} \cdot (\delta_{g^{-1}(ij)g} \cdot m_g)
\]

with \( \lambda_g \in k \) according to the definition of the action. Note that

\[
x_{(ij)} \cdot (x_{(ik)} \cdot m_g) = \begin{cases} -a_{g^{-1}(ik)(ij)g} m_{(ij)(ik)g} & \text{if } \text{sgn } g = -1, \\ -a_{g^{-1}(ik)g} m_{(ij)(ik)g} & \text{if } \text{sgn } g = 1. \end{cases}
\]

In any case, we have that \( x_{(ij)}^2 \cdot m_g = -a_{g^{-1}(ij)g} m_g \) and

\[
R_{(ij)(ik)} \cdot m_g = -(\sum_{(st) \in \mathcal{O}_2^3} a_{g^{-1}(st)g} m_{(ij)(ik)g}) = 0.
\]

Let \( W = \langle R_{(13)(23)}, R_{(23)(13)}, x_{(ij)}^2 \rangle : (ij) \in \mathcal{O}_2^3 \rangle \) and \( K \) be the subalgebra of \( T(V_3) \) generated by \( W \); \( K \) is a braided Hopf subalgebra because \( W \) is a Yetter-Drinfeld submodule contained in \( \mathcal{P}(T(V_3)) \) the primitive elements of \( T(V_3) \). Then \( K \# k^{S_3} \) is a Hopf subalgebra of \( T(V_3) \# k^{S_3} \). For each \( \alpha \in \mathfrak{A}_3 \), by Lemma 28 we can define the algebra morphism \( \psi = \psi_K \otimes \epsilon \) : \( K \# k^{S_3} \to k \) where

\[
\psi_K|_{W[g]} = 0 \text{ if } g \neq e \quad \text{and} \quad \psi_K(x_{(ij)}^2) = -a_{(ij)}^\forall (ij) \in \mathcal{O}_2^3.
\]

If \( J \) denotes the ideal of \( K \# k^{S_3} \) generated by the generator of \( K \), then \( \psi^{-1} \to J \leftarrow \psi \) is the ideal generated by the generators of \( \mathcal{I}_\alpha \). In fact, \( \psi^{-1} = \)
$\psi \circ \mathcal{S}$ is the inverse element of $\psi$ in the convolution group $\text{Alg}(K \# k^{S_3}, k)$, $\mathcal{S}(W)[g] \subset (K \# k^{S_3})[g^{-1}]$ and $\mathcal{S}(x^2_{(ij)}) = - \sum_{h \in S_3} \delta_h^{-1} x_{h^{-1}(ij)h}$. Then our claim follows if we apply $\psi \otimes \text{id} \otimes \psi^{-1}$ to $(\Delta \otimes \text{id})\Delta(x^2_{(ij)})$

$$= x^2_{(ij)} \otimes 1 \otimes 1 + \sum_{h \in S_3} \delta_h \otimes x^{-1}_{h^{-1}(ij)h} \otimes 1 + \sum_{h,g \in S_3} \delta_h \otimes \delta_g \otimes x^2_{g^{-1}h^{-1}(ij)hg}$$

and $(\Delta \otimes \text{id})\Delta(x) = x \otimes 1 \otimes 1 + x^{-1} \otimes x_0 \otimes 1 + x^{-2} \otimes x^{-1} \otimes x_0$ for $g \neq e$ and $x \in W[g]$; note that also $x_0 \in W[g]$.

The ideal $\psi^{-1} \to J$ is generated by

$$R_{(13)(23)}, R_{(23)(13)} \text{ and } x^2_{(ij)} + \sum_{g \in S_3} a_g^{-1}(ij)g \delta_g \forall (ij) \in \mathcal{O}_2^3.$$ 

Now $K_a = T(V_3) \# k^{S_3}/(\psi^{-1} \to J) \neq 0$ because it has a non-zero quotient in $\text{End}(M_3)$. Hence $A_{[a]}$ is monoidally Morita-Takeuchi equivalent to $B(V_3) \# k^{S_3}$, by Theorem 4.

5.2. **Hopf subalgebras and integrals of $A_{[a]}$.**

We collect some information about $A_{[a]}$. Let

$$\chi = \sum_{g \in S_3} \text{sgn}(g)\delta_g, \quad y = \sum_{(ij) \in \mathcal{O}_2^3} x_{(ij)}.$$ 

It is easy to see that $\chi$ is a group-like element and that $y \in \mathcal{P}_{1,\chi}(A_{[a]})$.

**Proposition 30.** Let $a \in \mathfrak{A}_3$. Then

(a) $G(A_{[a]}) = \{1, \chi\}$.

(b) $\mathcal{P}_{1,\chi}A_{[a]} = \langle 1 - \chi, y \rangle$.

(c) $k\langle \chi, y \rangle$ is isomorphic to the 4-dimensional Sweedler Hopf algebra.

(d) The Hopf subalgebras of $A_{[a]}$ are $k^{S_3}$, $k\langle \chi \rangle$ and $k\langle \chi, y \rangle$.

(e) $S^2(a) = \chi a \chi^{-1}$ for all $a \in A_{[a]}$.

(f) The space of left integrals is $\langle m_{\top}^\dagger \delta_e \rangle$; $A_{[a]}$ is unimodular.

(g) $(A_{[a]})^*$ is unimodular.

(h) $A_{[a]}$ is not a quasitriangular Hopf algebra.

**Proof.** We know that the coradical $(A_{[a]})_0$ of $A_{[a]}$ is isomorphic to $k^{S_3}$ by [AV]. Since $G(A_{[a]}) \subset (A_{[a]})_0$, (a) follows.

(b) Recall that $V_3 = M((12), \text{sgn}) \in k^{S_3}_{+} \mathcal{YD}$, see Subsection 2.1. Then $\mathcal{P}_{1,\chi}A_{[a]}/\langle 1 - \chi \rangle$ is isomorphic to the isotypic component of the comodule $V_3$ of type $\chi$. That is, if $z = \sum_{(ij) \in \mathcal{O}_2^3} \lambda_{(ij)} x_{(ij)} \in (V_3)_{\chi}$, then

$$\delta(z) = \sum_{h \in G,(ij) \in \mathcal{O}_2^3} \text{sgn}(h)\lambda_{(ij)} \delta_h \otimes x_{h^{-1}(ij)h} = \chi \otimes z.$$ 

Evaluating at $g \otimes \text{id}$ for any $g \in S_3$, we see that $\lambda_{(ij)} = \lambda_{(12)}$ for all $(ij) \in \mathcal{O}_2^3$. Then $z = \lambda_{(12)}y$. The proof of (c) is now evident.
(d) Let \( A \) be a Hopf subalgebra of \( A_3 \). Then \( A_0 = A \cap (A_3)_0 \subseteq k^{S_3} \) by [Mo, Lemma 5.2.12]. Hence \( A_0 \) is either \( k\langle \chi \rangle \) or else \( k^{S_3} \). If \( A_0 = k\langle \chi \rangle \), then \( A \) is a pointed Hopf algebra with group \( \mathbb{Z}/2 \). Hence \( A \) is either \( k\langle \chi \rangle \) or else \( k\langle \chi, y \rangle \) by (b) and [N] or [CD]. If \( A_0 = k^{S_3} \), then \( A \) is either \( k^{S_3} \) or else \( A = A_3 \) by [AV].

To prove (e), just note that \( \chi x_{ij} \chi^{-1} = -x_{ij} \).

(f) follows from Subsections 3.2 and 3.3. Let \( \Lambda \) be a non-zero left integral of \( A_3 \). By Lemma 8, the distinguished group-like element of \( (A_3)^* \) is \( \zeta_3 \) for some \( h \in S_3 \), hence \( A \delta_h = \zeta_3 (\delta_h)\Lambda = \Lambda \). Let us consider \( A_3 \) as a left \( k^{S_3} \)-module via the left adjoint action, see page 3. Let \( \Lambda_2 \in (A_3)[g] \) such that \( \Lambda = \sum_{g \in S_3} \Lambda_2 \). Then \( \Lambda = \delta_e \Lambda = \sum_{g \in S_3} \text{ad} \delta_s (\Lambda g) \delta_{g^{-1}} \delta_h = \Lambda h^{-1} \delta_h \).

Since \( M_h \cong A_3 \delta_h \), we can use the lemmata of the Section 3 to compute \( \Lambda \).

If \( a \) is generic, then \( h = e \) by Theorem 1. Since \( x_{ij} \Lambda = 0 \) for all \( (ij) \in S_3 \), \( \Lambda = m_{top} \delta_e \) by Lemma 10.

If \( a \) is sub-generic, we assume that \( a_{12} \neq a_{13} = a_{23} \), then either \( \Lambda = \Lambda e \delta_e \) or \( a_{12} \delta_{12} \) by Theorem 2. Since \( x_{ij} \Lambda = 0 \) for all \( (ij) \in S_3 \), \( \Lambda = m_{top} \delta_e \) by Lemma 17 and Lemma 19.

(g) By (e), \( S^4 = \text{id} \). By Radford’s formula for the antipode and (f), the distinguished group-like element of \( A_3 \) is central, hence trivial. Therefore, \( (A_3)^* \) is unimodular.

(h) If there exists \( R \in A_3 \otimes A_3 \) such that \( (A_3, R) \) is a quasitriangular Hopf algebra, then \( (A_3, R) \) has a unique minimal subquasitriangular Hopf algebra \( (A_R, R) \) by [R]. We shall show that such a Hopf subalgebra does not exist using (d) and therefore \( A_3 \) is not a quasitriangular Hopf algebra.

By [R, Prop. 2, Thm. 1] we know that there exist Hopf subalgebras \( H \) and \( B \) of \( A_3 \) such that \( A_R = HB \) and an isomorphism of Hopf algebras \( H^\text{cop} \to B \). Then \( A_R \neq A_3 \). In fact, let \( M(d, k) \) denote the matrix algebra over \( k \) of dimension \( d^2 \). Then the coradical of \( (A_3)^* \) is isomorphic to

- \( k^6 \) if \( a = (0, 0, 0) \).
- \( k \otimes M(5, k)^* \) if \( a \) is generic by Theorem 1.
- \( k^2 \otimes M(4, k)^* \) if \( a \) is sub-generic by Theorem 2.

Since \( (A_3)_0 \cong k^{S_3} \), \( A_3 \) is not isomorphic to \( (A_3)^{\text{cop}} \) for all \( a \in S_3 \). Clearly, \( A_R \) cannot be \( k^{S_3} \). Since \( A_3 \) is not cocommutative, \( R \) cannot be \( 1 \otimes 1 \). The quasitriangular structures on \( k\langle \chi \rangle \) and \( k\langle \chi, y \rangle \) are well known, see for example [R]. Then it remains the case \( A_R \subseteq k\langle \chi, y \rangle \) with \( R = R_0 + R_0 \) where \( R_0 = \frac{1}{2}(1 \otimes 1 + \chi \otimes 1 + 1 \otimes \chi - \chi \otimes \chi) \) and \( R_0 = \frac{1}{2}(y \otimes y + y \otimes y + y \otimes y) \) for some \( \alpha \in k \). Since \( \Delta(\delta_g)^{\text{cop}} R = R \Delta(\delta_g) \) for all \( g \in S_3 \), then

\[ \Delta(\delta_g)^{\text{cop}} R_0 = R_0 \Delta(\delta_g) = \Delta(\delta_g) R_0 \text{ in } k^{S_3}; \]

but this is not possible because \( R_0^2 = 1 \otimes 1 \) and \( k^{S_3} \) is not cocommutative. \( \Box \)

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The classification of all finite dimensional pointed Hopf algebras with group \( \mathbb{Z}/2 \) also follows easily performing the Lifting method [AS].
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