Quantitative stability estimates for a two-phase Serrin-type overdetermined problem

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Abstract

In this paper, we deal with an overdetermined problem of Serrin-type with respect to a two-phase elliptic operator in divergence form with piecewise constant coefficients. In particular, we consider the case where the two-phase overdetermined problem is close to the one-phase setting. First, we show quantitative stability estimates for the two-phase problem via a one-phase stability result. Furthermore, we prove non-existence for the corresponding inner problem by the aforementioned two-phase stability result.

Key words. two-phase, overdetermined problem, Serrin’s problem, transmission condition, stability.

AMS subject classifications. 35B35, 35J15, 35N25, 35Q93.

1 Introduction and main results

Let $\Omega$ be a bounded domain of $\mathbb{R}^N$ $(N \geq 2)$ and let $D$ be an open set such that $\overline{D} \subset \Omega$. In this paper, we consider the following two-phase Dirichlet boundary value problem:

\[
\begin{cases}
-\text{div} (\sigma \nabla u) = 1 \text{ in } \Omega, \\
u = 0 \text{ on } \partial \Omega,
\end{cases}
\]  

(1.1)

where $\sigma = \sigma(x)$ is the piecewise constant function defined by $\sigma(x) = 1 + (\sigma_c - 1)\chi_D$ for some $\sigma_c > 0$. More precisely, we consider the problem given by adding an overdetermined condition of Serrin-type to (1.1). That is, we focus on the following overdetermined problem:

\[
\begin{cases}
-\text{div} (\sigma \nabla u) = 1 \text{ in } \Omega, \\
u = 0 \text{ on } \partial \Omega, \\
\partial_n u = c \text{ on } \partial \Omega,
\end{cases}
\]  

(1.2)
where $n$ denotes the outward unit normal vector of $\partial \Omega$ and $\partial_n$ is the corresponding normal derivative. By integration by parts, it is easy to see that, if the overdetermined problem (1.2) is solvable, then the parameter $c$ must be given by

$$c = -\frac{|\Omega|}{|\partial \Omega|}.$$  

(1.3)

There are two different approaches for studying the solutions $(D, \Omega)$ of the overdetermined problem above. Indeed, the overdetermined problem (1.2) can be either regarded as an “inner problem” or as an “outer problem”. Roughly speaking, the outer problem consists in determining the domain $\Omega$ given $D$, while the inner problem consists in determining the inclusion $D$ given $\Omega$ (for a precise definition of the inner problem and outer problem, see [CY2020i]).

![Figure 1: Problem setting](image)

When $\sigma_c = 1$ (or, equivalently, $D = \emptyset$), it is known from Serrin’s paper [Se1971] that the overdetermined problem (1.2) is solvable if and only if the domain $\Omega$ is a ball. In this paper, we will refer to the original Serrin’s overdetermined problem as the “one-phase problem”.

The two-phase setting, that is, when $\sigma_c \neq 1$ and $D \neq \emptyset$, is more complicated since solutions of the overdetermined problem (1.2) are affected by the geometry of the inclusion $D$ or the domain $\Omega$. The first author and the third author, in [CY2020i], proved local existence and uniqueness for the outer problem near concentric balls under some non-criticality condition on the coefficients and then gave a numerical algorithm for finding the solutions to the outer problem based on the Kohn–Vogelius functional and the augmented Lagrangian method. Furthermore, in [CY2020ii], they proved that there exist symmetry-breaking solutions of (1.2) for certain critical values of $\sigma_c$. Similar problems involving
two-phase conductors have been studied in several situations. We refer to [MT1997i, MT1997ii, CMS2009, CLM2012, L2014, CSU2019, Ca2020, CMS2021, Ca2021].

Let \((D, \Omega)\) denote a solution of the overdetermined problem (1.2). One would expect that, if either \(\sigma_c \simeq 1\) or \(D\) is small enough in some sense, then \(\Omega\) must be close to a ball (the solution of the one-phase problem). This was conjectured in the paper [CY2020i] from the numerical results. The purpose of this paper is to give quantitative stability estimates that show how close the solution \(\Omega\) is to a ball when either \(\sigma_c \simeq 1\) or \(|D|\) is small.

![Figure 2: Numerical result when \(\sigma_c \simeq 1\)](image1)

![Figure 3: Numerical result when \(|D|\) is small](image2)

We begin by setting some relevant notations. The diameter of \(\Omega\) is indicated by \(d_\Omega\). For a point \(z \in \Omega\), \(\rho_i\) and \(\rho_e\) will denote the radius of the largest ball contained in \(\Omega\) and that of the smallest ball that contains \(\Omega\), both centered at \(z\) (see Figure 4); in formulas, \[
\rho_i = \min_{x \in \partial \Omega} |x - z| \quad \text{and} \quad \rho_e = \max_{x \in \partial \Omega} |x - z|.
\] (1.4)

In what follows, the point \(z\) will be always taken as later specified in Theorem 4.

![Figure 4: \(\rho_i\) and \(\rho_e\).](image3)
If $\partial\Omega$ is of class $C^{1,\alpha}$ (see [GT1983, p.94] for a definition), then from the compactness of $\partial\Omega$, there exist two positive constants $K$ and $\rho_0$ such that for all $x \in \partial\Omega$ and $0 < \rho \leq \rho_0$ there exists $x_0 \in \partial\Omega$ and a one-to-one mapping $\Psi$ of $B_\rho(x_0)$ onto $\omega \subset \mathbb{R}^N$ such that $x \in B_\rho(x_0)$ and

$$\Psi \left( B_\rho(x_0) \cap \Omega \right) \subset \{ x_N > 0 \}, \quad \Psi \left( B_\rho(x_0) \cap \partial\Omega \right) \subset \{ x_N = 0 \},$$

$$\|\Psi\|_{C^{1,\alpha}(B_\rho(x_0))} \leq K, \quad \left\|\Psi^{-1}\right\|_{C^{1,\alpha}(\omega)} \leq K.$$  

We will refer to the pair $(K, \rho_0)$ as the $C^{1,\alpha}$ modulus of $\partial\Omega$ (see also [ABR1999, BNST2008] for a similar definition in the case of $C^{2,\alpha}$ domains and [LV2000] for another definition of the $C^{1,\alpha}$ modulus).

In what follows, we state the main theorems of this paper. The following stability result for the one-phase problem will be crucial to establish quantitative stability estimates of the two-phase overdetermined problem (1.2).

**Theorem I** (Stability for the one-phase problem with $L^2$ deviation in terms of the $C^{1,\alpha}$ modulus of $\partial\Omega$). Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with boundary $\partial\Omega$ of class $C^{1,\alpha}$ and let $c$ be the constant defined in (1.3). Let $v$ be the solution of (1.1) with $\sigma_c = 1$ and let $z \in \Omega$ be a point such that $v(z) = \max_{\Omega} v$. Then, there exists a positive constant $C_1$ such that

$$\rho_e - \rho_i \leq C_1 \|\partial_nv - c\|_{L^2(\partial\Omega)}^2,$$  

with the following specifications:

(i) $\tau_2 = 1$;

(ii) $\tau_3$ is arbitrarily close to one, in the sense that for any $\theta > 0$, there exists a positive constant $C_1$ such that (1.5) holds with $\tau_3 = 1 - \theta$;

(iii) $\tau_N = 2/(N - 1)$ for $N \geq 4$.

The constant $C_1$ depends on $N$, $d_\Omega$, the $C^{1,\alpha}$ modulus of $\partial\Omega$, and $\theta$ (only in the case $N = 3$).

**Remark 1.1.** The proof of Theorem I relies on (and is hugely an adaptation of) the techniques developed by Magnanini and the second author in [Po2019ii, MP2020i, MP2020ii]. When the $C^{1,\alpha}$ modulus of $\partial\Omega$ is replaced by the uniform interior and exterior touching ball condition, Theorem I is contained in [Po2019ii, MP2020ii]. We point out that Theorem I provides a new extension of [MP2020ii, Theorem 3.1] in which the constant $C_1$ appearing
in (1.5) depends on the $C^{1,\alpha}$ modulus of $\partial \Omega$ instead of the radii of the uniform interior and exterior touching ball condition (as it happened in [MP2020]). We stress that the uniform interior and exterior touching ball condition is equivalent to the $C^{1,1}$ regularity of $\partial \Omega$ (see, for instance, [Ba2009, Theorem 1.0.9] or [ABMMZ2011, Corollary 3.14]). The weaker $C^{1,\alpha}$ (with $0 < \alpha < 1$) regularity that we are considering here, is equivalent to a uniform interior and exterior touching pseudoball condition (see [ABMMZ2011, Theorem 1.3 and Corollary 3.14]).

Thanks to Theorem I, we can obtain quantitative stability estimates for the two-phase overdetermined problem (1.2) when $\sigma_c \simeq 1$ and $|D|$ is small.

**Theorem II** (Stability for $\sigma_c \simeq 1$). Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and let $D$ be an open set satisfying $\overline{D} \subset \Omega$. Moreover, suppose that the pair $(D, \Omega)$ is a solution to the overdetermined problem (1.2). Then, we have that

$$\rho_e - \rho_i \leq C_2 |\sigma_c - 1|^\tau_N,$$

where $\tau_N$ is defined as in Theorem I and the constant $C_2 > 0$ depends on $N$, $d_\Omega$, the $C^{1,\alpha}$ modulus of the boundary $\partial \Omega$, and $\theta$ (only in the case $N = 3$).

**Theorem III** (Stability for $|D|$ small). Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and let $D$ be an open set satisfying $\overline{D} \subset \Omega$. Moreover, suppose that the pair $(D, \Omega)$ is a solution to the overdetermined problem (1.2). Then, we have that

$$\rho_e - \rho_i \leq C_3 |D|^\tau_N,$$

where $\tau_N$ is defined as in Theorem I and the constant $C_3 > 0$ depends on $N$, $d_\Omega$, $\sigma_c$, the $C^{1,\alpha}$ modulus of the boundary $\partial \Omega$, the distance between $\overline{D}$ and $\partial \Omega$, and $\theta$ (only in the case $N = 3$).

**Remark 1.2** (On the regularity). Even without imposing any regularity assumptions on $\partial \Omega$ (in Theorems I and III), [Vo1992, Theorem 1] guarantees that if $u$ satisfies (1.2) (where the boundary conditions are interpreted in the appropriate weak sense), then $\partial \Omega$ is of class $C^{2,\gamma}$, with $0 < \gamma < 1$. In particular, the $C^{1,\alpha}$ modulus of $\partial \Omega$ is well defined, and the notation $\partial_n u = c$ on $\partial \Omega$ is well posed in the classical sense. Furthermore, the regularity of $\partial \Omega$ can be bootstrapped even more. Indeed, once one knows that $(D, \Omega)$ is a classical solution of (1.2), then the local result [KN1977, Theorem 2] implies that $\partial \Omega$ must be an analytic surface.
Remark 1.3. Theorem III should be compared with the results obtained (with a different approach) by Dipierro, Valdinoci, and the second author in [DPV2021]. Although the results in [DPV2021] apply to the more general setting in which the equation is not known (and could be arbitrary) in $D$, in the case of the two-phase problem (1.2) considered here, Theorem III provides substantial improvements. First, in [DPV2021] the closeness of $\Omega$ to a ball is controlled by $|\partial D|$, while Theorem III provides a stronger control in terms of $|D|$. Also, the constant $C$ appearing in the estimates in [DPV2021] also depends on the $C^2$ norm of $u$ on $\partial D$, and that dependence does not appear in Theorem III. We mention that, in the present setting, such regularity of $u$ up to $\partial D$ would be available at the cost of assuming some regularity of $\partial D$ (see [XB2013]), which is not assumed in Theorem III.

From Theorem II and III we can show the non-existence for the inner problem of the two-phase overdetermined problem (1.2) when $\sigma_c \simeq 1$ and $|D|$ is small.

**Corollary I** (Non-existence for $\sigma_c \simeq 1$). Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and suppose that $\Omega$ is not a ball (that is, $\rho_e - \rho_i > 0$). Then, the overdetermined problem (1.2) does not admit a solution of the form $(D, \Omega)$ if

$$|\sigma_c - 1| < C_4 (\rho_e - \rho_i)^{1 \over \tau_N},$$

where $\tau_N$ is defined as in Theorem I and the constant $C_4$ can be explicitly written as

$$C_4 = (C_2)^{-1/\tau_N},$$

where $C_2$ is the constant that appears in the statement of Theorem II.

**Corollary II** (Non-existence for $|D|$ small). Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and suppose that $\Omega$ is not a ball (that is, $\rho_e - \rho_i > 0$). Then, the overdetermined problem (1.2) does not admit a solution of the form $(D, \Omega)$ if

$$|D| < C_5 (\rho_e - \rho_i)^{2 \over \tau_N},$$

where $\tau_N$ is defined as in Theorem I and the constant $C_5$ can be explicitly written as

$$C_5 = (C_3)^{-2/\tau_N},$$

where $C_3$ is the constant that appears in the statement of Theorem III.

This paper is organized as follows. In Section 2, we provide stability results for the one-phase problem and prove Theorem I. Section 3 is devoted to the proof of Theorem
II by the implicit function theorem for Banach spaces and a corollary of Theorem I. In Section 4, we prove Theorem III by a perturbation argument using Green’s function of the Dirichlet boundary value problem for the Laplace operator and a corollary of Theorem I. In Section 5, we show the non-existence for the inner problem of the two-phase overdetermined problem (1.2) from Theorems II and III.

2 Proof of Theorem I

In this section, we consider $v$ solution of (1.1) with $\sigma_c = 1$, that is,

$$-\Delta v = 1 \text{ in } \Omega, \quad v = 0 \quad \text{on } \partial \Omega. \quad (2.6)$$

The stability issue for the classical Serrin’s problem has been deeply studied by several authors in [ABR1999, BNST2008, CMV2016, Fe2018, MP2019, Po2019i, Po2019ii, MP2020i, MP2020ii, GO2021, MP2021]. A more detailed overview and comparison of those results can be found in [Ma2017, Po2019ii, MP2020i, MP2020ii].

We now give the proof of Theorem I.

Proof of Theorem I. As already mentioned, the result with the $C^{1,\alpha}$ modulus of $\partial \Omega$ replaced by the uniform interior and exterior touching ball condition has been obtained in [Po2019ii, MP2020ii]. Here, we hugely exploit tools and techniques developed in [Po2019ii, MP2020ii], adapting them to our (more general) setting. More precisely, we are going to point out how to modify the proof of [MP2020ii, Theorem 3.1] in the present setting, referring the reader to [Po2019ii, MP2020ii] for the remaining details.

In this proof, we use the letter $C$ to denote a positive constant whose value could change by line to line; the parameters on which $C$ depends will be specified each time.

Step 1 (Fundamental identity). By following [MP2020ii] and taking into account that here a different normalization of (2.6) is adopted, we introduce the function $q(x) = -\frac{|x-z|^2}{2N}$ (where $z$ is a global maximum point of $v$ in $\Omega$) and the harmonic function $h = v - q$. In the present setting, Identity (3.1) in [MP2020ii] reads

$$\int_{\Omega} v |\nabla^2 h|^2 \, dx = \frac{1}{2} \int_{\partial \Omega} \left( c^2 - (\partial_n v)^2 \right) \partial_n h \, dS_x, \quad (2.7)$$

where $c$ is the constant given by (1.3).

Notice that, by definition $h$ is harmonic and, being $h = -q$ on $\partial \Omega$, we have that

$$\text{osc}_{\partial \Omega} h := \max_{\partial \Omega} h - \min_{\partial \Omega} h = \frac{\rho_2^2 - \rho_1^2}{2N}.$$
This last relation and the inequality $\rho_e + \rho_i \geq \rho_e \geq d_\Omega/2$ immediately lead to

$$\rho_e - \rho_i \leq \frac{4N}{d_\Omega} \text{osc } h. \tag{2.8}$$

**Step 2** (Optimal growth of $v$ from the boundary). We prove that

$$v(x) \geq C \delta_{\partial \Omega}(x) \quad \text{for any } x \in \overline{\Omega}, \tag{2.9}$$

where $\delta_{\partial \Omega}(x) := \text{dist}(x, \partial \Omega)$ denotes the distance function to $\partial \Omega$, and $C$ is a constant only depending on $N$ and the $C^{1,\alpha}$ modulus of $\partial \Omega$.

By the Hopf-Olenik lemma for $C^{1,\alpha}$ domains\(^1\) (see, for instance, the more general version contained in \[ABMMZ2011\] Theorem 4.4), for any $x_0 \in \partial \Omega$ we have that

$$v(x_0 - tn) \geq kt \quad \text{for any } 0 < t < \delta, \tag{2.10}$$

where $k$ and $\delta$ are two constants only depending on $N$ and the $C^{1,\alpha}$ modulus of $\partial \Omega$. This, together with the rough estimate

$$v(x) \geq \delta_{\partial \Omega}(x)^2 \quad \text{for any } x \in \overline{\Omega}, \tag{2.11}$$

easily leads to the global inequality (2.9) with $C = \max\{k, \delta^2/2N\}$, where $k$ and $\delta$ are those in (2.10). For a proof of (2.11) see, for instance, the first claim in \[MP2020i\] Lemma 3.1.

**Step 3** (Key inequality). Here, we prove that

$$\rho_e - \rho_i \leq C \|\delta_{\partial \Omega}^{1/2} \nabla^2 h\|_{L^2(\Omega)}^{\tau_N}, \tag{2.12}$$

where $\tau_N$ is as in the statement of Theorem \[2\] and $C$ only depends on $N$, $d_\Omega$, the $C^{1,\alpha}$ modulus of $\partial \Omega$, and $\theta$ (only in the case $N = 3$).

The reference result here is \[MP2020ii\] Theorem 2.8. To extend \[MP2020ii\] Theorem 2.8 in the present setting, we need an appropriate extension of \[MP2020ii\] Lemma 2.7, which is provided in \[MP2020iii\]. Here, it is enough to apply \[MP2020iii\] Theorem 3.1 with $L = \Delta$, $v = h$, $\alpha = 1$ to get that

$$\text{osc }_{\partial \Omega} h \leq C \|\nabla h\|_{L^\infty(\Omega)}^{N/(N+p)} \|h - h_\Omega\|_{L^p(\Omega)}^{p/(N+p)}, \tag{2.13}$$

where $h_\Omega$ denotes the mean value of $h$ on $\Omega$ and $C$ only depends on $N$, $p$, $d_\Omega$, and the $C^{1,\alpha}$ modulus\(^2\) of $\partial \Omega$. Notice that, $\|\nabla h\|_{L^\infty(\Omega)}$ can be estimated in terms of $N$, $d_\Omega$, and $\theta$ only in the case $N = 3$.

\(^1\)Hopf-Olenik Lemma for $C^{1,\alpha}$ domains is due to Giraud \[Gi1933\]. We refer to \[ABMMZ2011\] Section 4.1 for a historical perspective on this subject.

\(^2\)[MP2020ii] Theorem 3.1] has been proved for domains satisfying a uniform interior cone condition. This class of domains contains that of Lipschitz domains, which in turn contains $C^{1,\alpha}$ domains. Of course, the parameters of the uniform interior cone condition appearing in the estimate in \[MP2020ii\] Theorem 3.1 can be bounded in terms of the $C^{1,\alpha}$ modulus of $\partial \Omega$.\[8\]
the $C^{1,\alpha}$ modulus of $\partial \Omega$, by putting together
\[
\|\nabla h\|_{L^\infty(\Omega)} = \max_{\Omega} |\nabla h| \leq \max_{\Omega} |\nabla v| + \frac{d\Omega}{N}
\]
and the classical Schauder estimate for $\max_{\Omega} |\nabla v|$. Thus, (2.8) and (2.13) ensure that
\[
\rho_e - \rho_i \leq C \|h - h_{\Omega}\|_{L^p(\Omega)}^{p/(N+p)}
\] (2.14)
holds true with a constant $C$ only depending on $N, p, d\Omega$, and the $C^{1,\alpha}$ modulus of $\partial \Omega$.

With this at hand, one can directly check that replacing [MP2020ii, Equation (1.13)] and [MP2020ii, Lemma 2.7] with (2.8) and (2.14) in the proof of [MP2020ii, Theorem 2.8] leads to (2.12).

**Step 4** (Final estimate for the left-hand side of (2.7)). Putting together (2.12) and (2.9) immediately gives that
\[
\rho_e - \rho_i \leq C \left( \int_{\Omega} v |\nabla^2 h|^2 \, dx \right)^{\tau_N/2},
\] (2.15)
where $\tau_N$ is as in the statement of Theorem II and $C$ only depends on $N, d\Omega$, the $C^{1,\alpha}$ modulus of $\partial \Omega$, and $\theta$ (only in the case $N = 3$).

**Step 5** (Estimate for the right-hand side of (2.7)). We start by estimating from above the right-hand side of (2.7) by using Hölder’s inequality as follows:
\[
\int_{\partial \Omega} \left( c^2 - (\partial_n v)^2 \right) \partial_n h \, dS_x \leq \left( c + \max_{\Omega} |\nabla v| \right) \|\partial_n v - c\|_{L^2(\partial \Omega)} \|\partial_n h\|_{L^2(\partial \Omega)}.
\] (2.16)
Notice that $c + \max_{\Omega} |\nabla v|$ can be bounded above by a constant depending on $N$, $d_{\Omega}$, and the $C^{1,\alpha}$ modulus of $\partial \Omega$; this easily follows in light of the classical Schauder estimates for $\max_{\Omega} |\nabla v|$, and estimating $|c|$ by putting together [1,3], the isoperimetric inequality

$$|\partial \Omega| \geq N |B_1|^{1/N} |\Omega|^{(N-1)/N},$$

and the trivial bound

$$|\Omega| \leq |B_1|(d_{\Omega}/2)^N,$$

where $B_1$ denotes a unit ball in $\mathbb{R}^N$.

Now, reasoning as in [MP2020ii, Lemma 2.5] we can prove that

$$\|\partial_n h\|_{L^2(\partial \Omega)}^2 \leq C \int_{\Omega} v |\nabla^2 h|^2 \, dx,$$ \hfill (2.17)

where $C$ is a constant only depending on $N$, $d_{\Omega}$, and the $C^{1,\alpha}$ modulus of $\partial \Omega$.

As in the proof of [MP2020ii, Theorem 3.1], putting together (2.7), (2.16) and (2.17) gives that

$$\|\partial_n h\|_{L^2(\partial \Omega)} \leq C \|\partial_n v - c\|_{L^2(\partial \Omega)},$$ \hfill (2.18)

now with a constant $C$ only depending on $N$, $d_{\Omega}$, and the $C^{1,\alpha}$ modulus of $\partial \Omega$. Thus, combining (2.7), (2.16) and (2.18) gives

$$\int_{\Omega} v |\nabla^2 h|^2 \, dx \leq C \|\partial_n v - c\|_{L^2(\partial \Omega)}^2,$$ \hfill (2.19)

where $C$ is a constant only depending on $N$, $d_{\Omega}$, and the $C^{1,\alpha}$ modulus of $\partial \Omega$. The conclusion of Theorem II immediately follows by combining (2.19) and (2.15). \hfill $\square$

We remark that, in the proofs of Theorems II and III we are able to obtain an upper bound on the uniform norm of the deviation of $\partial_n v$ from $c$. We are therefore interested in an estimate similar to (1.5) but where the $L^2$ norm is replaced by the uniform norm. In other words, what we really need in the proofs of Theorems II and III is the following Corollary of Theorem I. Nevertheless, since Theorem I is of independent interest, we decided to state it in its full generality in the introduction of this paper.

\footnote{We can repeat the proof of [MP2020ii, (i) of Lemma 2.5] (with $u = v$ and $v = h$) just by replacing [MP2020ii, (1.15)] with (2.9) and [MP2019, Theorem 3.10] with

$$-\partial_n v \geq k,$$

where $k$ is the constant appearing in (2.10),

which easily follows from (2.10). Also, we took into account that a different normalization of the problem (2.6) was adopted in [MP2020ii].}
Corollary 2.1 (Stability for the one-phase problem with uniform deviation in terms of the $C^{1,\alpha}$ modulus of $\partial \Omega$). Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with boundary $\partial \Omega$ of class $C^{1,\alpha}$ and $c$ be the constant defined in (1.3). Let $v$ be the solution of (2.6) and let $z \in \Omega$ be a point such that $v(z) = \max_{\Omega} v$. Then there exists a positive constant $C_6$ such that
\[
\rho_e - \rho_i \leq C_6 \|\partial_n v - c\|_{L^\infty(\partial \Omega)},
\]
where $\tau_N$ is defined as in Theorem I and the constant $C_6 > 0$ only depends on $N$, $d_\Omega$, the $C^{1,\alpha}$ modulus of the boundary $\partial \Omega$, and $\theta$ (only in the case $N = 3$).

Proof. Since
\[
\|\partial_n v - c\|_{L^2(\partial \Omega)} \leq |\partial \Omega|^{1/2} \|\partial_n v - c\|_{L^\infty(\partial \Omega)}
\]
trivially holds true, the desired result can be easily deduced by (1.5). It only remains to notice that we can get rid of the dependence on $|\partial \Omega|$ appearing in (2.21), thanks to the bound
\[
|\partial \Omega| \leq \frac{|\Omega|}{k}, \quad \text{where } k \text{ is the constant appearing in (2.10)}.
\]
The last bound follows by putting together the identity
\[
|\Omega| = \int_{\Omega} (-\Delta v) \, dx = \int_{\partial \Omega} (-\partial_n v) \, dS_x
\]
and the inequality $-\partial_n v \geq k$, which easily follows from (2.10). $\square$

3 Proof of Theorem II

In this section, we prove Theorem II. First, we will show the Fréchet differentiability of the solution of (1.1) with respect to the parameter $\sigma_c$.

Lemma 3.1. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain of class $C^{1,\alpha}$ and $D$ be an open set such that $\overline{D} \subset \Omega$. Moreover, let $U \subset \overline{U} \subset \Omega$ be an open neighborhood of $\overline{D}$ of class $C^{1,\alpha}$. For $t \in (-1, \infty)$, let $u(t) \in H^1_0(\Omega)$ denote the solution of (1.1) with respect to $\sigma(t) = 1 + t \chi_D$ (that is, $\sigma_c = 1 + t$). Then, $u(\cdot)$ defines a Fréchet differentiable map
\[
t \mapsto u(t) \in H^1_0(\Omega) \cap C^{1,\alpha}(\overline{\Omega} \setminus U).
\]
Moreover, for every $t_0 \in (-1, \infty)$, the Fréchet derivative $u'(t_0)$ is given by the solution of the following boundary value problem.
\[
\begin{cases}
- \text{div} \left( \sigma(t) \nabla u'(t_0) \right) = - \text{div} \left( \chi_D \nabla u(t_0) \right) & \text{in } \Omega, \\
u'(t_0) = 0 & \text{on } \partial \Omega.
\end{cases}
\tag{3.22}
\]
The proof of Lemma 3.1 relies on a standard method (see [HP2018, proof of Theorem 5.3.2, pp.206–207] for an application to shape-differentiability) based on the following implicit function theorem for Banach spaces (see [AP1983, Theorem 2.3, p.38] for a proof).

**Theorem 3.2** (Implicit function theorem). Let $\Psi \in C^k(\Lambda \times W, Y)$, $k \geq 1$, where $Y$ is a Banach space and $\Lambda$ (resp. $U$) is an open set of a Banach space $T$ (resp. $X$). Suppose that $\Psi(\lambda^*, w^*) = 0$ and that the partial derivative $\partial_w \Psi(\lambda^*, w^*)$ is a bounded invertible linear transformation from $X$ to $Y$.

Then there exist neighborhoods $\Theta$ of $\lambda^*$ in $T$ and $W^*$ of $w^*$ in $X$, and a map $g \in C^k(\Theta, X)$ such that the following hold:

(i) $\Psi(\lambda, g(\lambda)) = 0$ for all $\lambda \in \Theta$,

(ii) If $\Psi(\lambda, u) = 0$ for some $(\lambda, u) \in \Theta \times U^*$, then $u = g(\lambda)$,

(iii) $g'(\lambda) = -(\partial_u \Psi(p))^{-1} \circ \partial_\lambda \Psi(p)$, where $p = (\lambda, g(\lambda))$ and $\lambda \in \Theta$.

**Proof of Lemma 3.1**. For arbitrary $t \in (-1, \infty)$ and $u \in H^1_0(\Omega)$, let $V(t, u)$ denote the solution to the following boundary value problem:

\[
\begin{aligned}
-\Delta V &= -\operatorname{div}(\sigma(t)\nabla u) - 1 \quad \text{in } \Omega, \\
V &= 0 \quad \text{on } \partial\Omega.
\end{aligned}
\]  

(3.23)

A functional analytical interpretation of this mapping is the following: we are identifying $V \in H^1_0(\Omega)$ with the element $-\operatorname{div}(\sigma(t)\nabla u) - 1 \in H^{-1}(\Omega)$ whose action on $H^1_0(\Omega)$ is defined via integration by parts, that is, for $\varphi \in H^1_0(\Omega)$,

\[
(V, \varphi)_{H^1_0} = \int_\Omega \nabla V \cdot \nabla \varphi = \int_\Omega \sigma(t)\nabla u \cdot \nabla \varphi - \int_\Omega \varphi = \langle -\operatorname{div}(\sigma(t)\nabla u) - 1, \varphi \rangle.
\]

By the classical Schauder estimates for the Dirichlet problem near the boundary (see, for instance [GT1983, Theorem 8.33] and the subsequent remarks) and the $L^\infty$ estimates [GT1983, Theorem 8.16], we notice that $V(\cdot, \cdot)$ defines a mapping $(-1, \infty) \times X \to X$, where $X$ is the Banach space

\[X := H^1_0(\Omega) \cap C^{1,\alpha}(\overline{\Omega} \setminus U).\]

By the defining properties of $V(t, u)$, it is clear that $u$ solves (1.1) with $\sigma = \sigma(t)$ if and only if $V(t, u) \equiv 0$. In particular, for all $t_0 \in (-1, \infty)$, the pair $(t_0, u(t_0))$ is a zero of $V$ by definition.
We will now show that the map $V$ is (totally) Fréchet differentiable jointly in the variables $t$ and $u$. By the definition of $\sigma(t)$ we can expand the left-hand side of (3.23) as $-\Delta u - t \text{div} (\chi_D \nabla u) - 1$. By the linearity of problem (3.23), this implies that the map $V(t, u)$ can be decomposed as the sum of three parts:

$$V(t, u) = V_1(u) + V_2(t, u) + V_3,$$

where $V_i (i = 1, 2, 3)$ are the solution of $-\Delta V = f_i$ with with Dirichlet zero boundary condition corresponding to

$$f_1 = -\Delta u, \quad f_2 = -t \text{div} (\chi_D \nabla u), \quad f_3 = -1.$$

Now, notice that, by construction, $V_1(u)$ is linear and continuous in $u$, $V_2(t, u)$ is bilinear and continuous in $(t, u)$ and $V_3$ does not depend on either $t$ or $u$. In particular, we get that $V_1$, $V_2$ and $V_3$ are all Fréchet differentiable. As a consequence, we get the Fréchet differentiability of the map $(t, u) \mapsto V(t, u)$ in the appropriate Banach spaces. Now, a simple computation yields that, for fixed $t_0 \in (-1, \infty)$, the partial Fréchet differential $\partial_u V(t_0, u(t_0))$ is given by the mapping from the Banach space $X$ into itself defined as:

$$X \ni \phi \mapsto \partial_u V(t_0, u(t_0))[\phi] = W(t_0, \phi),$$

where $W(t_0, \phi) \in X$ is the unique solution to the following boundary value problem:

$$\begin{cases} 
-\Delta W = - \text{div} (\sigma(t_0) \nabla \phi) & \text{in } \Omega, \\
W = 0 & \text{on } \partial \Omega.
\end{cases} \quad (3.24)$$

By “inverting the roles” of the right and left-hand side in the above and applying once again the classical Schauder estimates for the Dirichlet problem near the boundary and the $L^\infty$ estimates as before, we can conclude that the map $\phi \mapsto \partial_u V(t_0, u(t_0))[\phi]$ is invertible (that is, problem (3.24) is well posed in the appropriate Banach spaces), as required. We can, therefore, apply the implicit function theorem to the map $(t, u) \mapsto V(t, u)$ at its zero $(t_0, u(t_0))$. This yields the existence of a Fréchet differentiable branch

$$(t_0 - \varepsilon, t_0 + \varepsilon) \ni t \mapsto \tilde{u}(t) \in X \quad \text{such that } V(t, \tilde{u}(t)) = 0.$$ 

In other words, $\tilde{u}(t)$ also solves (1.1). Now, by the unique solvability of (1.1), $\tilde{u}(t) = u(t)$, and therefore, the map $t \mapsto u(t) \in X$ is Fréchet differentiable, as claimed. Finally, (3.22) is derived by simple differentiation with respect to $t$ of the weak form

$$\int_\Omega \sigma(t) \nabla u(t) \cdot \nabla \varphi = \int_\Omega \varphi \quad \text{for all } \varphi \in H_0^1(\Omega).$$

The proof is completed. \qed
Proof of Theorem II. As above, let \( u(t) \) denote the solution to (1.1) with \( \sigma = \sigma(t) \). Moreover, suppose that, for some small \( t_0 \in (-1, 1) \), the function \( u(t_0) \) satisfies the overdetermined condition

\[
\partial_n u(t_0) = c \quad \text{on } \partial \Omega.
\]

Consider the map

\[
(-1, \infty) \ni t \mapsto \partial_n u(t) \big|_{\partial \Omega} \in C^\alpha(\partial \Omega).
\]

Lemma 3.1 tells us that the map defined by (3.25) is Fréchet differentiable. In particular, for all \( x \in \partial \Omega \), the map \( t \mapsto \partial_n u(t)(x) \in \mathbb{R} \) is differentiable. By the fundamental theorem of calculus we have

\[
\partial_n u(t_0)(x) - \partial_n u(0)(x) = \int_0^{t_0} \partial_n u'(\tau)(x) \, d\tau.
\]

Therefore,

\[
\left\| \partial_n u(t_0) - \partial_n u(0) \right\|_{C^\alpha(\partial \Omega)} \leq |t_0| \max_I \left\| \partial_n u'(\tau) \right\|_{C^\alpha(\partial \Omega)},
\]

where \( I = [\min(0, t_0), \max(0, t_0)] \). Again, by two applications of the classical Schauder estimates for the Dirichlet problem near the boundary [GT1983, Theorem 8.33] and the \( L^\infty \) estimate [GT1983, Theorem 8.16], we can estimate the right-hand side in the inequality above to get

\[
\left\| \partial_n v - c \right\|_{L^\infty(\partial \Omega)} \leq \left\| \partial_n v - c \right\|_{C^\alpha(\partial \Omega)} = \left\| \partial_n u(t_0) - \partial_n u(0) \right\|_{C^\alpha(\partial \Omega)} \leq C_7 |t_0|,
\]

where the constant \( C_7 > 0 \) depends only on \( |\Omega|, N \) and the \( C^{1,\alpha} \) modulus of the boundary \( \partial \Omega \). By applying Corollary 2.1, we get the following estimate:

\[
\rho_e - \rho_i \leq C_2 |\sigma - 1|^{\tau_N},
\]

where \( \tau_N \) is defined as in Theorem I and the constant \( C_2 \) depends on \( N, d_\Omega \), the \( C^{1,\alpha} \) modulus of \( \partial \Omega \), and \( \theta \) (only in the case \( N = 3 \)). This is the desired estimate.

Remark 3.3. The result of Theorem II can be immediately extended to the case where \( \partial \Omega \) is of class \( C^{1,\alpha} \) and the overdetermined condition in (1.2) reads

\[
\partial_n u(x) = c + \eta(x) \quad \text{for } x \in \partial \Omega,
\]

where the function \( \eta \in L^\infty(\partial \Omega) \) has vanishing mean over \( \partial \Omega \). Instead of (3.27) we get

\[
\left\| \partial_n v - c \right\|_{L^\infty(\partial \Omega)} \leq \left\| \partial_n v - \partial_n u \right\|_{L^\infty(\partial \Omega)} + \left\| \partial_n u - c \right\|_{L^\infty(\partial \Omega)} \leq C_7 |\sigma - 1| + ||\eta||_{L^\infty(\partial \Omega)}.
\]

Now, by applying Corollary 2.1 we get the following estimates:

\[
\rho_e - \rho_i \leq C_6 \left( C_7 |\sigma - 1| + ||\eta||_{L^\infty(\partial \Omega)} \right)^{\tau_N}.
\]
4 Proof of Theorem III

In this section, we prove Theorem III. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain of class $C^{1,\alpha}$ and $D$ be an open set such that $\overline{D} \subset \Omega$. We also assume that $D$ satisfies

$$\text{dist}(D, \partial \Omega) \geq \frac{1}{M}, \quad (4.31)$$

where $M$ is a positive constant that for simplicity will be taken to be greater than 1. Let us put $w = u - v$, where $u, v$ are the solutions of (1.1) and (2.6), respectively. The function $w$ satisfies the following boundary value problem:

$$\begin{cases}
-\Delta w = \text{div} \left( (\sigma_c - 1) \chi_D \nabla u \right) & \text{in } \Omega, \\
w = 0 & \text{on } \partial \Omega.
\end{cases} \quad (4.32)$$

We consider a perturbation argument by using Green’s function. Let $G(x, y)$ be the Green’s function of the Dirichlet boundary value problem for the Laplace operator in $\Omega$. By [GT1983, pp.17–19], the Green’s function $G$ is represented by

$$G(x, y) = \Gamma(x - y) - h(x, y),$$

where $\Gamma$, defined for $x \in \mathbb{R}^N \setminus \{0\}$, is the fundamental solution of Laplace’s equation:

$$\Gamma(x) = \begin{cases}
-\frac{1}{2\pi} \log |x| & (N = 2), \\
\frac{1}{N(N-2)\omega_N} \frac{1}{|x|^{N-2}} & (N \geq 3),
\end{cases} \quad (4.33)$$

(here $\omega_N$ denotes the volume of the unit ball in $\mathbb{R}^N$) and for $y \in \Omega$, $h(\cdot, y)$ is the solution to the following Dirichlet boundary value problem:

$$\begin{cases}
-\Delta_x h(x, y) = 0 & x \in \Omega, \\
h(x, y) = \Gamma(x - y) & x \in \partial \Omega.
\end{cases} \quad (4.34)$$

The following gradient estimate for Green’s function $G$ will be useful in the proof of Theorem III

**Lemma 4.1.** Let $U := \left\{ x \in \Omega \middle| \text{dist}(x, D) > \frac{1}{2M} \right\}$. Then, there exists a positive constant $C^*$ depending on $N$, $|\Omega|$ and the $C^{1,\alpha}$ modulus of $\partial \Omega$ such that

$$\sup_{(x,y) \in U \times D} |\nabla_x \nabla_y G(x, y)| \leq C^* M^{N+1}. \quad (4.35)$$
Proof. Fix \((x, y) \in U \times D\) and let \(\beta\) be a multi-index with \(|\beta| \geq 1\). From the definition of the fundamental solution (4.33), by direct calculation we obtain the estimate
\[
|D^\beta \Gamma(x - y)| \leq \frac{C(N)}{|x - y|^{N - 2 + |\beta|}} \leq C(N)M^{N - 2 + |\beta|}, \quad (4.35)
\]
where \(C(N) > 0\) is a constant depending only on \(N\).

In what follows, let us show the gradient estimate for \(h\). First, notice that the function \(\partial_y h(\cdot, y)\) is harmonic on \(\Omega\) and verifies
\[
\partial_y h(z, y) = \partial_y \Gamma(z - y) \quad \text{for } z \in \partial \Omega.
\]
Now, by the classical Schauder estimates for the Dirichlet problem near the boundary \([GT1983\text{, Theorem 8.33}]\) and the \(L^\infty\) estimate \([GT1983\text{, Theorem 8.16}]\), we can estimate
\[
|\partial_x \partial_y h(x, y)| \leq \|\partial_y h(\cdot, y)\|_{C^{1, \alpha}(\Omega)} \leq C\|\partial_y \Gamma(\cdot - y)\|_{C^{1, \alpha}(\partial \Omega)},
\]
where the constant \(C > 0\) only depends on \(|\Omega|, N\) and the \(C^{1, \alpha}\) modulus of \(\partial \Omega\). Now, up to redefining \(C\), one can estimate the right-hand side in the above as follows
\[
C\|\partial_y \Gamma(\cdot - y)\|_{C^{1, \alpha}(\partial \Omega)} \leq C\|\Gamma(\cdot - \cdot)\|_{C^{3}(\partial \Omega \times D)} \leq CM^{N + 1},
\]
where we made use of (4.35) in the last inequality. \(\square\)

Proof of Theorem III. Let us consider the Dirichlet boundary value problem (4.32). For any \(x \in U \cap \Omega\), Green’s representation formula gives us
\[
w(x) = \int_{\Omega} G(x, y) \text{div}_y ((\sigma c - 1) \chi_D \nabla u(y)) \, dy
\]
\[
= - (\sigma c - 1) \int_{D} \nabla_y G(x, y) \cdot \nabla y u(y) \, dy.
\]
Then, we have
\[
\partial_x w(x) = - (\sigma c - 1) \int_{D} \partial_{x_i} \nabla y G(x, y) \cdot \nabla y u(y) \, dy.
\]
Now, by Lemma 4.1 and the Cauchy–Schwarz inequality, we may write (as usual, up to redefining \(C\))
\[
|\nabla_x w| \leq \sqrt{N}|\sigma c - 1| \int_{D} |\nabla_x \nabla y G(x, y)||\nabla y u(y)| \, dy
\]
\[
\leq |\sigma c - 1|C^* M^{N + 1}|D|^{1/2} \left( \int_{\Omega} |\nabla y u(y)|^2 \, dy \right)^{1/2}. \quad (4.36)
\]
Consider the weak form of (1.1). For any \( \varphi \in H^1_0(\Omega) \),
\[
\int_{\Omega} \sigma \nabla y u \cdot \nabla y \varphi \, dy = \int_{\Omega} \varphi \, dy.
\]
Taking \( \varphi = u \), then
\[
\min\{\sigma_c, 1\} \int_{\Omega} |\nabla y u|^2 \, dy \leq \int_{\Omega} |\nabla y u|^2 \, dy = \int_{\Omega} u \, dy \leq |\Omega|^{1/2} \|u\|_{L^2(\Omega)}. \tag{4.37}
\]
Let now \( \lambda_1(\Omega) \) denote the first eigenvalue of the Laplace operator with Dirichlet zero boundary condition, that is
\[
\lambda_1(\Omega) = \inf_{f \in H^1_0(\Omega), f \neq 0} \frac{\|\nabla f\|_{L^2(\Omega)}}{\|f\|_{L^2(\Omega)}}.
\]
Since \( u \in H^1_0(\Omega) \) and \( u \neq 0 \), we have \( \lambda_1(\Omega) \leq \frac{\|\nabla y u\|_{L^2(\Omega)}}{\|u\|_{L^2(\Omega)}} \). Thus, we obtain
\[
\|u\|_{L^2(\Omega)} \leq \frac{\|\nabla y u\|_{L^2(\Omega)}}{\lambda_1^{1/2}(\Omega)}. \tag{4.38}
\]
Combining (4.37) with (4.38),
\[
\|\nabla y u\|_{L^2(\Omega)} \leq \frac{|\Omega|^{1/2}}{\lambda_1^{1/2}(\Omega) \min\{\sigma_c, 1\}}. \tag{4.39}
\]
By (4.36) and (4.39), we have
\[
|\nabla x w| \leq |\sigma_c - 1| |C^* M^{N+1} D|^{1/2} \frac{|\Omega|^{1/2}}{\lambda_1^{1/2}(B^*) \min\{\sigma_c, 1\}}.
\]
By the Faber–Krahn inequality, there exists a ball \( B^* \) such that
\[
\lambda_1(B^*) \leq \lambda_1(\Omega), \quad |B^*| = |\Omega|.
\]
Therefore, we obtain
\[
|\nabla x w| \leq |\sigma_c - 1| |C^* M^{N+1} D|^{1/2} \frac{|\Omega|^{1/2}}{\lambda_1^{1/2}(B^*) \min\{\sigma_c, 1\}}. \tag{4.40}
\]
By the classical Schauder estimates for the Dirichlet problem near the boundary (see, for instance [GT1983 Theorem 8.33] and the subsequent remarks), \( \nabla x w \) is continuous up to the boundary \( \partial \Omega \). If we let \( x \) tend to \( \partial \Omega \), then we realize that (4.40) also holds true for \( x \in \partial \Omega \). Let us recall that the solution \( u \) of (1.1) satisfies the overdetermined condition \( \partial_n u = c \) on \( \partial \Omega \). Therefore, for any \( x \in \partial \Omega \), we obtain
\[
|\partial_n v - c| \leq |\nabla x w| \leq |\sigma_c - 1| |D|^{1/2} \frac{|C^* M^{N+1} |\Omega|^{1/2}}{\lambda_1^{1/2}(B^*) \min\{\sigma_c, 1\}}.
\]
By applying Corollary 2.1 we get the following estimate:
\[ \rho_e - \rho_i \leq C_3 |D|^{\frac{1}{2N}}, \]
where \( \tau_N \) is defined as in Theorem I. The constant \( C_3 \) depends on \( N, d_\Omega, \sigma_c, M, \) the \( C^{1,\alpha} \) modulus of \( \partial \Omega, \) and \( \theta \) (only in the case \( N = 3 \)).

Remark 4.2. It is clear that the proof of Theorem III can also be used to obtain a stability estimate in the spirit of Theorem II. However, notice that such a proof would lead to a (weaker) version of Theorem II in which the constant \( C_2 \) also depends on the distance between \( D \) and \( \partial \Omega, \) and \( |D| \).

Remark 4.3. Whenever an apriori bound for \( \|\nabla u\|_{L^\infty(D)} \) is available, the stability exponent of Theorem III can be improved (that is \( \tau_N / 2 \) can be replaced by \( \tau_N \)), at the cost of allowing the constant \( C_3 \) to depend also on the above mentioned bound for \( \|\nabla u\|_{L^\infty(D)} \).

This can be obtained by replacing in the proof of Theorem III (4.36) with
\[ |\nabla_x w| \leq |\sigma_c - 1| C^* M^{N+1} |D| \|\nabla_y u\|_{L^\infty(D)}. \]

5 Non-existence for the inner problem when \( \sigma_c \simeq 1 \) or \( |D| \) is small

In this section we show how one can employ the results of Theorems II and III to prove non-existence for the inner problem corresponding to (1.2) when \( \sigma_c \simeq 1 \) or \( |D| \) is small.

Proof of Corollary 1. Let \( \Omega \) be a bounded domain of \( \mathbb{R}^N \) different from a ball and set \( c := -|\Omega|/|\partial \Omega| \). Since, by hypothesis, \( \Omega \) is not a ball, we have
\[ \rho_e - \rho_i > 0. \]

Now, let \( C_4 \) and \( \tau_N \) be the same constants as in the statement of Corollary I and suppose by contradiction that there is an open set \( D \subset \overline{D} \subset \Omega \) such that the overdetermined problem (1.2) admits a solution \( u \) for some \( \sigma_c \) satisfying
\[ |\sigma_c - 1| < C_4 (\rho_e - \rho_i)^{\frac{1}{2N}}, \tag{5.41} \]
(notice that there exist infinitely many such values of \( \sigma_c \) because \( \rho_e - \rho_i > 0 \) by construction). Finally, Theorem II yields
\[ \rho_e - \rho_i \leq C_2 |\sigma_c - 1|^{\tau_N} < C_2 C_4^{\tau_N} (\rho_e - \rho_i) = \rho_e - \rho_i, \]
which is a contradiction. \( \square \)
Remark 5.1. By applying the result of Remark \[3.3\], we can extend Corollary \[4\] to the case where the overdetermined condition in \[1.2\] is replaced by \[3.28\] for some \(\eta \in L^\infty(\partial \Omega)\) with vanishing mean over \(\partial \Omega\). In this case, given a bounded domain \(\Omega\) of class \(C^{1,\alpha}\) that is not a ball, the overdetermined problem given by \[1.1\] and \[3.28\] does not admit a solution of the form \((D, \Omega)\) if

\[|\sigma_c - 1| < \frac{1}{C_7} \left\{ \left( \frac{\rho_e - \rho_i}{C_6} \right)^{\frac{1}{\tau_N}} - \|\eta\|_{L^\infty(\partial \Omega)} \right\},\]

where \(C_6\) and \(\tau_N\) are as in Corollary \[2.1\], while \(C_7\) is the constant in \[3.27\]. Notice that the set of values \(\sigma_c\) satisfying the inequality above is not empty if the norm \(\|\eta\|_{L^\infty(\partial \Omega)}\) is small enough.

Proof of Corollary \[7\]. It follows from Theorem \[3\] by arguing by contradiction. The proof will be omitted because it is completely analogous to that of Corollary \[4\].

Remark 5.2. We remark that the constant \(C_5\) of Corollary \[7\] depends on the distance between \(D\) and \(\partial \Omega\). Indeed, given \(\Omega\), \(\sigma_c > 0\) and \(M > 1\), Corollary \[7\] tells us that there does not exist a solution of \[1.2\] of the form \((D, \Omega)\), where \(D\) is an open set belonging to the class

\[D_M := \left\{ D \subset \Omega : \text{dist}(D, \partial \Omega) \geq \frac{1}{M} \right\},\]

and the volume \(|D|\) is small enough (namely, smaller than \(C_5(\rho_e - \rho_i)^{\frac{2}{\tau_N}}\)).
Indeed, Corollary II does not preclude the existence of a family of “wild solutions” \( \{(D_k, \Omega)\}_{k \geq 1} \) with \( D_k \in \mathcal{D}_{M_k} \) such that

\[
\lim_{k \to \infty} M_k = \infty.
\]

Geometrically speaking, this suggests the possibility of “wild solutions” \((D_k, \Omega)\) where the inclusion \(D_k\) becomes closer and closer to the boundary \(\partial \Omega\) as \(k \to \infty\). We conjecture that this could happen in many ways. For example, when \(D\) takes the form of a thin layer increasingly close to \(\partial \Omega\) or when one allows the formation of increasingly many connected components that give rise to a microstructure. Indeed, both such configurations seem likely to affect the global behavior of the solution of (1.1) near the boundary (see Figure 3). Such behaviors are linked to the so-called homogenization phenomena (see [BCF1980, Fr1980, MT1997a, MT1997b, Ya2019, ACMOY2019] and the references therein).

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