A$_1$-REGULARITY AND BOUNDEDNESS OF CALDERON-ZYGMUND OPERATORS

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Abstract. The Coifman-Fefferman inequality implies quite easily that a Calderon-Zygmund operator $T$ acts boundedly in a Banach lattice $X$ on $\mathbb{R}^n$ if the Hardy-Littlewood maximal operator $M$ is bounded in both $X$ and $X'$. We discuss this phenomenon in some detail and establish a converse result under the assumption that $X$ satisfies the Fatou property and $X$ is $p$-convex and $q$-concave with some $1 < p, q < \infty$: if a linear operator $T$ is bounded in $X$ and $T$ is nondegenerate in a certain sense (for example, if $T$ is a Riesz transform) then $M$ is bounded in both $X$ and $X'$.

0. Introduction

The problem of characterizing the spaces in which (and between which) the operators of harmonic analysis act boundedly lies in the core of the modern harmonic analysis, and it definitely has far-reaching consequences in terms of applications. These operators in a vast number of cases can be represented by (or the corresponding questions reduced to the study of) a general Calderon-Zygmund operator. The study of such operators has received a lot of attention over the past several decades and significant advancements have been made. To mention a few highlights: the quest for practical conditions that guarantee boundedness of a Calderon-Zygmund operator in $L_2$ led to useful $T1$ theorems, new approaches to the classical proofs have made it possible to significantly relax the doubling condition on the underlying measurable space, the action of such operators was studied in detail in a wide variety of spaces beyond the classical Lebesgue spaces $L_p$, and a number of representations for such operators were developed together with highly refined techniques that recently yielded answers to several long-standing problems such as the $A_2$-hypothesis (positive) and the $A_1$ conjecture of Muckenhoupt and Wheeden (negative). Although it seems that the focus has always been on particular classes of spaces, weighted Lebesgue spaces $L_p(w)$ being of a particular interest (not least because of their rather general nature which has long been noted), results extending

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various useful relationships to fairly general classes of spaces, and indeed sometimes demonstrating exhaustively the true scope of what has been known for many years, recently began to emerge.

The purpose of the present work is to establish the following theorem showing that the boundedness of Calderon-Zygmund singular integral operators $T$ and the boundedness of the Hardy-Littlewood maximal operator $M$ in both the lattice and its dual is actually the same property in a fairly general class of Banach lattices. It constitutes a substantial improvement over the respective results of [27].

The (standard) definitions and basic facts concerning Banach lattices and Calderon-Zygmund operators can be found in Section 1. The notion of an $A_2$-nondegenerate lattice is introduced in Definition 11 below; for now we say that $R$ can be any of the Riesz transforms $\{R_j\}_{j=1}^n$. We fix a $\sigma$-finite measurable space $(\Omega, \mu)$ which we understand as a space for the second variable $\omega$ in $(x, \omega) \in \mathbb{R}^n \times \Omega$ (unless indicated otherwise, all operators are assumed to act in the first variable $x$ only); this allows us to naturally include lattices with mixed norm such as $X(l^1)$ in this setting.

**Theorem 1.** Suppose that $X$ is a Banach lattice of measurable functions on $\mathbb{R}^n \times \Omega$ that satisfies the Fatou property and $X$ is $p$-convex and $q$-concave with some $1 < p, q < \infty$. Let $R$ be a Calderon-Zygmund operator in $L_2(\mathbb{R}^n)$ such that both $R$ and $R^*$ are $A_2$-nondegenerate. The following conditions are equivalent.

1. The Hardy-Littlewood maximal operator $M$ acts boundedly in $X$ and in the order dual $X'$ of $X$.
2. All Calderon-Zygmund operators act boundedly in $X$.
3. $R$ acts boundedly in $X$.

We will explore several proofs of implication $1 \Rightarrow 2$ in Section 3 below. Although it is hard to come by this sufficient condition for boundedness of Calderon-Zygmund operators in the literature, it is certainly not new; see, e. g., [13, Remark 4.3]. Implication $2 \Rightarrow 3$ is trivial, and implication $3 \Rightarrow 1$, which is in a sense the main point of the present work, is established in Section 6, although the argument itself is technically simple, it relies heavily on the theory of $A_{p'}$-regular Banach lattices, a part of which we develop further in Section 5 and the proof taken as a whole involves overall two distinct applications of the Ky-Fan–Kakutani fixed point theorem and a variant of the Maurey–Krivine factorization theorem which is based on the Grothendieck theorem.

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1 A note of caution concerning how this paper is laid out seems to be necessary: since the author tried to properly introduce and discuss at length all elements (more or less well-known with possibly a few exceptions) leading to this result in order to explore possible connections and extensions, it was convenient to postpone the main argument until the very end. Thus the impatient reader who wishes to study the proof of the implication $3 \Rightarrow 1$ is advised to skip right away down to Section 5.
As it will be seen, the sufficiency of Condition 3 of Theorem 1 for the other conditions actually extends to a wide class of singular operators that are nondegenerate in a certain sense. The proof, which is covered by Proposition 6 in Section 3 and by Theorem 25 in Section 6 below can easily be generalized to the case of a general space of homogeneous type instead of just $\mathbb{R}^n$ if there exists a suitable nondegenerate operator $R$; it is not clear whether every space of homogeneous type has at least one such operator. It is easy to see that the proof of Theorem 1 also works in the vector-valued case, i.e., for lattices of measurable functions like $X(l^r)$, where $X$ is a lattice on $\mathbb{R}^n$. The $p$-convexity and $q$-concavity assumptions are probably not necessary (they are not used in the implication 1 $\Rightarrow$ 2) and I conjecture that they in themselves are a consequence of any of the conditions of Theorem 1 that Condition 1 implies $p$-convexity and $q$-concavity with some $1 < p, q < \infty$ is known to hold true at least in the case of the variable exponent Lebesgue spaces (see, e.g., [6, Theorem 4.7.1]), and it seems that it is possible to adapt the same argument to cover suitable nondegenerate singular integral operators as well. Recently in [4, Theorem 5.42] it was established that if all Riesz transforms $R_j$ are bounded in $L^p(\cdot)$ then the exponent $p(\cdot)$ is bounded away from 1 and $\infty$ (and thus lattice $L^p(\cdot)$ satisfies the $p$-convexity and $q$-concavity assumptions in this case). It is also interesting to note that implication 1 $\Rightarrow$ 2 easily extends in a certain natural way to the case of operators acting between different Banach lattices; see Theorem 9 in Section 3 below.

Let us briefly outline some of the contributions that led to this result. In the standard part of the theory describing the properties of the Calderon-Zygmund operators (see, e.g., [29]) in the Lebesgue space setting the maximal operator plays an essential part. In the case of weighted Lebesgue spaces $L_p(w)$ Theorem 1 of course, follows from the theory of the Muckenhoupt weights (see, e.g., [29, Chapter 5]) that individually links the conditions of Theorem 1 to the Muckenhoupt condition of the weight $w$. Of a particular interest in this regard is the Coifman-Fefferman inequality [3]

$$\int |T f|^p \omega \leq C \int (M f)^p \omega, \quad 0 < p < \infty,$$

with $C$ independent of $f$, that holds true for Calderon-Zygmund operators $T$ and any weight $\omega \in A_\infty$ for all locally summable functions $f$ such that the right-hand part of (1) is finite. Thus $T$ is estimated in terms of $M$ for a relatively wide class of Muckenhoupt weights even though $M$ may not act boundedly in the corresponding weighted Lebesgue space. There is a large number of various extensions of (1); see, e.g.,

and refer to the rest of the paper and to [27] as necessary. An abridged version of this paper made for submission to a journal is also available upon request (or by configuring the \LaTeX sources in a certain way).
On the other hand, making use of the duality and the famous construction due to Rubio de Francia allowed a large number of very useful extrapolation results that essentially exploit a very simple idea: if $M$ is bounded in $X$ then any $f \in X$ can be pointwise dominated with a controlled increase of norm by some weight $w \in A_1$, and the converse is also trivially true. It is natural to call such lattices $A_1$-regular by analogy with BMO-regularity (see [27]). For example, this idea works very well in the case of variable exponent Lebesgue spaces $L_{p(\cdot)}$ where the behavior of boundedness of $M$ under duality and certain scaling operations is nice and well understood; see, e. g., [6, §7.2]. The Coifman-Fefferman inequality (1) with $p = 1$ gives a very easy proof of the implication $1 \Rightarrow 2$ of Theorem 1; see Proposition 6 in Section 3 below. And this is far from the only way to establish this implication; we will also discuss in Section 3 below how some of the recent results by A. Lerner [19], [20] and [21] also give the necessary tools to effortlessly establish this implication.

The study of the duality of BMO-regularity, initially motivated by certain problems in the theory of interpolation of Hardy-type spaces, eventually led in [27] to a refinement and generalization to the general spaces of homogeneous type of certain properties and results concerning the interplay of various majorization and boundedness properties that were previously known only in the case of the unit circle $\mathbb{T}$. In particular, the main result of [27] is similar to Theorem 1 because it links boundedness of $T$ and $M$ in lattices of the form $X^\alpha L_1^{1-\alpha}$ for $0 < \beta < 1$ and sufficiently small $0 < \alpha < 1$ to another property, namely to BMO-regularity of $X$. This, admittedly, still left much more to be desired in terms of refinements, since unlike $A_1$-regularity the BMO-regularity property, which proved to be very useful in certain questions pertaining to spaces on the unit circle $\mathbb{T}$, so far does not seem to be as useful in the case of the spaces on $\mathbb{R}^n$ in the same capacity. The results of the present work can be regarded as an extension and an application of the techniques described in [27].

The paper is organized as follows. In Section 1 we introduce some basic notions pertaining to Banach lattices and spaces of homogeneous type. In Section 2 certain known facts about Muckenhoupt weights and $A_{p^*}$-regular spaces are outlined. In Section 3 we briefly describe Calderon-Zygmund operators and show several ways to obtain the implication $1 \Rightarrow 2$ of Theorem 1. Then in Section 4 we discuss some results having to do with operators that are nondegenerate in a certain sense. Section 5 contains a new result that gives a sufficient condition for a lattice $X$ to be $A_1$-regular in terms of $A_1$-regularity of lattice $X^\delta$ and $A_p$-regularity of lattice $X$. Finally, in Section 6 we prove the converse implication $3 \Rightarrow 1$ of Theorem 1.
1. Preliminaries

In this section we briefly go over the basic definitions and facts used by the rest of the paper. For the generalities on the Banach lattices and their properties see, e. g., [12, Chapter 10], [23]. A space of homogeneous type \((S, \nu)\) is a quasimetric space equipped with a Borel measure \(\nu\) that has the doubling property, i. e. \(\nu(B(x, 2r)) \leq c \nu(B(x, r))\) for all \(x \in S\) and \(0 < r < \infty\) with some constant \(c\), where \(B(x, r)\) is the ball of radius \(r\) centered at \(x\). The main example here is \(S = \mathbb{R}^n\) equipped with the Lebesgue measure.

A quasi-normed lattice of measurable functions \(X\) is a quasi-normed space of measurable functions \(X\) in which the norm is compatible with the natural order; that is, if \(|f| \leq g\ a. e.\) for some function \(g \in X\) then \(f \in X\) and \(\|f\|_X \leq \|g\|_X\). For simplicity we only work with lattices \(X\) such that \(\text{supp } X = S \times \Omega\). For a Banach lattice of measurable functions \(X\), any order continuous functional \(f\) on \(X\) (order continuity is understood in the sense that for any sequence \(x_n \in X\) such that \(\sup n |x_n| \in X\) and \(x_n \to 0\ a. e.\) one also has \(f(x_n) \to 0\)) has an integral representation \(f(x) = \int xy\) for some measurable function \(y\) which can be identified with \(f\). The set of all such functionals \(X'\) is a Banach lattice with the norm defined by \(\|f\|_{X'} = \sup_{g \in X, \|g\|_X = 1} \int |fg|\).

The lattice \(X'\) is called the order dual of the lattice \(X\). The norm of a lattice \(X\) is said to be order continuous if for any nonincreasing sequence \(x_n \in X\) converging to \(0\ a. e.\) one also has \(\|x_n\|_X \to 0\).

Order continuity of the norm of a Banach lattice \(X\) is equivalent to \(X^{**} = X'\), and it is also equivalent to density of the simple functions in \(X\). A lattice \(X\) has the Fatou property if for any \(f_n, f \in X\) such that \(\|f_n\|_X \leq 1\) and the sequence \(f_n\) converges to \(f\ a. e.\) it is also true that \(f \in X\) and \(\|f\|_X \leq 1\). The Fatou property of a lattice \(X\) is equivalent to \((\nu \times \mu)\)-closedness of the unit ball \(B_X\) of the lattice \(X\) (here and elsewhere \((\nu \times \mu)\)-convergence denotes the convergence in measure in any measurable set \(E\) such that \((\nu \times \mu)(E) < \infty\)). If the lattice \(X\) is Banach then the Fatou property is equivalent to order reflexivity of \(X\), i. e. to the relation \(X^{**} = X\). For a lattice \(X\) either one of the Fatou property or the order continuity of norm property is sufficient to guarantee that the lattice \(X'\) is norming for \(X\), i. e. that \(\|f\|_X = \sup g \in X', \|g\|_X = 1 \int fg\) for all \(f \in X\).

For any two quasi-normed lattices \(X\) and \(Y\) on the same measurable space the set of pointwise products of their functions \(XY = \{fg \mid f \in X, g \in Y\}\) is a quasi-normed lattice with the norm defined by \(\|h\|_{XY} = \inf_{h=fg} \|f\|_X \|g\|_Y\).

If both lattices \(X\) and \(Y\) satisfy the Fatou property then the lattice \(XY\) also has the Fatou property. If either of the lattices \(X\) and \(Y\) has
order continuous quasi-norm then the quasi-norm of $XY$ is also order continuous.

For any $\delta > 0$ and a quasi-normed lattice $X$ the lattice $X^\delta$ consists of all measurable functions $f$ such that $|f|^{1/\delta} \in X$ with a quasi-norm $\|f\|_{X^\delta} = \|f^{1/\delta}\|_X^\delta$. For example, $L^\delta_p = L^\delta_{\infty}$. It is easy to see that $(XY)^\delta = X^\delta Y^\delta$ for any $X$, $Y$ and $\delta$, and $X^\delta$ naturally inherits many properties from $X$. For any $0 < \delta \leq 1$, if $X$ is a Banach lattice then $X^\delta$ is also a Banach lattice. If $X$ and $Y$ are Banach lattices then for any $0 < \delta < 1$ lattice $X^{1-\delta}Y^\delta$, sometimes called the Calderon-Lozanovsky product of $X$ and $Y$, is also Banach; moreover, one has a very useful relation $(X^{1-\delta}Y^\delta)' = (X')^{1-\delta}(Y')^\delta$ (see [2], [24]). If $Z = X^{1-\delta}Y^\delta$ has the Fatou property or order continuous norm then $Z$ is an exact interpolation space of exponent $\delta$ between $X$ and $Y$; see, e. g., [25], [2], [10].

Let $1 \leq p, q < \infty$. A Banach lattice $X$ is said to be $p$-convex with constant $C$ if $\left\| \left( \sum_{j=1}^N |f_j|^p \right)^{\frac{1}{p}} \right\|_X \leq C \left( \sum_{j=1}^n \|f_j\|_X^p \right)^{\frac{1}{p}}$ for any $\{f_j\}_{j=1}^N \subset X$; lattice $X$ is said to be $q$-concave with constant $c$ if $\left( \sum_{j=1}^N \|f_j\|_X^q \right)^{\frac{1}{q}} \leq c \left\| \left( \sum_{j=1}^n |f_j|^q \right)^{\frac{1}{q}} \right\|_X$ for any $\{f_j\}_{j=1}^N \subset X$. If $X$ is $p$-convex then $X'$ is $p'$-concave, and if $X$ is $q$-concave then $X'$ is $q'$-convex. It is well known (see, e. g., [23, Book II, Proposition 1.d.8]) that a Banach lattice that is $p$-convex and $q$-concave can be renormed to make both its $p$-convexity and $q$-concavity constants $C = c = 1$. The assumption of $p$-convexity imposed on a lattice $X$ enables us to raise $X$ to a power $1 < p < \infty$ without it becoming quasi-Banach since $p$-convexity of $X$ is equivalent to $1$-convexity of $Y = X^p$. This in turn implies that $X = Y^{\frac{1}{p}}$ and $X' = (Y')^{\frac{1}{p}}L^{1-\frac{1}{p}}_1$ provided that $X$ has the Fatou property, so in this case $X'$ has order continuous norm and therefore $X = X'' = X^*$. By the same argument, if a lattice $X$ has the Fatou property and $X$ is $q$-concave for some $1 < q < \infty$ then $X$ has order continuous norm and $X' = X^*$. Thus a lattice $X$ which is both $p$-convex and $q$-concave with some $1 < p, q < \infty$ is reflexive, and also both $X$ and $X'$ have order continuous norm and enjoy many other nice properties.

For a quasi-normed lattice $X$ and weights $w$ such that $0 \leq w \leq \infty$ almost everywhere the weighted lattice $X(w)$ is defined by

$$X(w) = \{ g \mid g_w \in X \}$$

with the quasi-seminorm defined by $\|f\|_{X(w)} = \|fw^{-1}\|_X$. This somewhat cumbersome definition is needed because the more natural definition $X(w) = \{ wh \mid h \in X \}$ is meaningless if the weight $w$ takes value $+\infty$ on a set of positive measure and it seems to be easier to allow this
in the definition and work with weighted lattices that may be quasi-normed rather than negotiate finiteness of $w$ every time. Thus in this setting one has $g = 0$ on the set where $w = 0$, $g$ restricted on the set $\{w = +\infty\}$ is an arbitrary measurable function, and $\|\cdot\|_{X(w)}$ is a norm for weights $w$ such that $(\nu \times \mu)(\{w = +\infty\}) = 0$. If $w = 0$ on a set of positive measure, we regard $X(w)$ as merely a set of functions with a seminorm under our conventions, since then $\text{supp } X(w) \neq \text{supp } X$. In majorization arguments it is usually possible to avoid dealing with “bad” weights with the help of the following proposition.

**Proposition 2** ([27, Proposition 3.2]). Suppose that $X$ is a Banach lattice on $(\Sigma, \mu)$. Then for every $f \in X$ such that $f \neq 0$ identically and $\varepsilon > 0$ there exists $g \in X$ such that $g > |f|$ a. e. and $\|g\|_X \leq (1 + \varepsilon)\|f\|_X$.

The construction of a weighted lattice yields $L_\infty(w) = \{f \mid |f| \leq Cw \text{ a. e.} \}$. It is easy to see that $[X(w)]' = X'(w^{-1})$. Notice that this definition of the weighted Lebesgue space $L_p(w)$ differs from the “classical” one with the norm defined by $\|f\|_{L_p(w)} = \int |f|^p w$, which is often used in the literature; the latter norm corresponds to the norm of the lattice $L_p\left(w^{-\frac{1}{p}}\right)$ in our notation. Thus all weighted lattices are defined in the same way everywhere in this paper; however, one has to pay attention to this difference. We adopt the natural conventions $0^{-1} = \infty$ and $\infty^{-1} = 0$ in all expressions involving weights.

2. **Muckenhoupt weights and $A_p$-majorants**

In this section we introduce some useful notions having to do with the Muckenhoupt weights; for more detail see, e. g., [29, Chapter 5]. The (centered) Hardy-Littlewood maximal operator

$$Mf(x, t) = \sup_{r > 0} \frac{1}{\nu(B(x, r))} \int_{B(x, r)} |f(z, t)|d\nu(z), \quad x \in S, \quad t \in \Omega,$$

is well-defined for a. e. $x \in S, t \in \Omega$, and the measurable functions $f$ on $(S \times \Omega, \nu \times \mu)$ that are locally summable in the first variable. We say that a non-negative measurable function $w$ on $(S \times \Omega, \nu \times \mu)$ belongs to the Muckenhoupt class $A_p$ for some $1 \leq p < \infty$ with a constant $C$ if

$$\text{ess sup}_{t \in \Omega} \|M\|_{L_p(w^{-1/p}(\cdot, t)) \rightarrow L_p(w^{-1/p}(\cdot, t))} \leq C.$$

In the case $p > 1$ this condition is equivalent to

$$\text{ess sup}_{t \in \Omega} \|M\|_{L_p(w^{-1/p}(\cdot, t))} \leq C'$$

with a constant $C'$ estimated in terms of $C$ and $p$. The class $A_1$ is characterized by the estimate $Mw \leq C'w$ almost everywhere, while
classes $A_p$ for $p > 1$ are characterized by the well-known Muckenhoupt condition

$$
(2) \quad \text{ess sup}_{x \in S, t \in \Omega} \sup_{r > 0} \left[ \frac{1}{\nu(B(x, r))} \int_{B(x, r)} w(u, t) d\nu(u) \right] \times \left[ \frac{1}{\nu(B(x, r))} \int_{B(x, r)} w(u, t) \frac{1}{r^p} d\nu(u) \right]^{p-1} < \infty.
$$

The class $A_{\infty}$ is defined as the class of weights $w$ satisfying the reverse Hölder inequality

$$
(3) \quad \text{ess sup}_{x \in S, t \in \Omega} \sup_{r > 0} \left[ \frac{1}{\nu(B(x, r))} \int_{B(x, r)} [w(u, t)]^q d\nu(u) \right]^{\frac{1}{q}} \times \left[ \frac{1}{\nu(B(x, r))} \int_{B(x, r)} w(u, t) d\nu(u) \right]^{-1} < \infty
$$

with some $q > 1$, and for certainty we will take for the value of the supremum in (3) for the $A_{\infty}$ constant of the weight $w$. It is well known that $w \in A_{\infty}$ if and only if $w \in A_p$ with some $1 < p < \infty$ and the $A_p$ constant of the weight $w$ depending only on the $A_{\infty}$ constant of the weight $w$ and vice versa.

The following notion is a natural refinement of the BMO-regularity property which was apparently first introduced by N. Kalton in [11].

**Definition 3.** A quasi-normed lattice $X$ on $(S \times \Omega, \nu \times \mu)$ is $A_p$-regular with constants $(C, m)$ if for any $f \in X$ there exists a majorant $g \in X$, $g \geq |f|$ such that $\|g\|_X \leq m\|f\|_X$ and $g \in A_p$ with constant $C$.

This property was formally introduced and studied to some extent in [27]; we will reference here only the results used in the present work.

**Proposition 4** ([27, Proposition 1.2]). A quasi-normed lattice $X$ on $(S \times \Omega, \nu \times \mu)$ is $A_1$-regular if and only if the maximal operator $M$ is bounded in $X$.

Sufficiency is trivial, and necessity quickly follows from an application of the famous Rubio de Francia construction.

As a consequence of the reverse Hölder inequality we see that the $A_1$-regularity property is self-improving, which is the subject of the following proposition. (It is not difficult to see that the general $A_p$-regularity property is also self-improving in this manner, but we will not need it in the present work). There is also a fairly general approach that makes it possible to establish this property using certain methods originating in the geometry of Banach spaces; see [22].
Proposition 5. Suppose that $X$ is an $A_1$-regular Banach lattice on $(S \times \Omega, \nu \times \mu)$ with constants $(C, m)$. Then $X^r$ is also an $A_1$-regular lattice for some $r > 1$ depending only on $C$.

Indeed, let $r > 1$ be a constant of the reverse Hölder inequality that is satisfied for all $A_1$ weights with constant $C$, and let $g$ be an $A_1$-majorant for $|f|^\frac{1}{r}$ in $X$ with constants $(C, m)$. Then $g^r$ is an $A_1$-majorant for $f$ with constants independent of $f$, because by the reverse Hölder inequality we have an estimate

$$\frac{1}{\nu(B(x, \rho))} \int_{B(x, \rho)} g^r(u, \omega) d\nu(u) \leq c \left( \frac{1}{\nu(B(x, \rho))} \int_{B(x, \rho)} g(u, \omega) d\nu(u) \right)^r \leq c C^r [g(x, \omega)]^r$$

for almost all $x \in S$, $\omega \in \Omega$ and $\rho > 0$ with a constant $c$ independent of $f$, $x$, $\omega$ and $\rho$.

3. Calderon-Zygmund operators

In this section we will show how certain conditions on the lattices are sufficient for boundedness of the Calderon-Zygmund operators in the general setting. Namely, we give 4 somewhat independent proofs of the implication $1 \Rightarrow 2$ of Theorem 1 that use different properties of a Calderon-Zygmund operator. For generalities on the real harmonic analysis see, e. g., [7], [29]. Although it seems possible to extend all of the results used here to the general setting of spaces of homogeneous type and beyond, for simplicity we will only discuss the standard setting of $\mathbb{R}^n$ with the usual Lebesgue measure $d\nu = dm$. We will also use, in contrast to the definition introduced in Section 2, the (uncentered) Hardy-Littlewood maximal operator defined by

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| dy, \quad x \in \mathbb{R}^n,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ with sides parallel to the coordinate axes. However, it is well known that this definition is pointwise equivalent to the one given before.

We say that $T$ is a Calderon-Zygmund operator if $T$ is a singular integral operator that is bounded in $L^2(\mathbb{R}^n)$ and its kernel $K(x, y)$ satisfies

$$|K(x, s) - K(x, t)| \leq C_K \frac{|s - t|^{\gamma}}{|x - s|^{n+\gamma}}, \quad x, s, t \in \mathbb{R}^n, \quad |x - s| > 2|s - t|$$

with some $\gamma > 0$, and the kernel $K(y, x)$ of the adjoint operator $T^*$ satisfies the same estimates. It is well known that $T$ is bounded in $L^p$ for all $1 < p < \infty$. We begin with the more classical approach, which
is also very simple. The following proposition contains the implication $1 \Rightarrow 2$ of Theorem 1 (see also Corollary 22 in Section 5 below).

**Proposition 6.** Suppose that $X$ is a Banach lattice on $\mathbb{R}^n$ having either the Fatou property or order continuous norm, $X$ is $A_1$-regular and $X'$ is $A_\infty$-regular. Then any Calderon-Zygmund operator $T$ is bounded in $X$.

Indeed, let $f \in X$ and $g \in X'$, and let $h$ be an $A_\infty$-majorant of $g$ in $X'$. Then

$$
\int (Mf)h \leq \|Mf\|_X \|h\|_{X'} \leq c_1 \|f\|_X \|g\|_{X'} < \infty,
$$

and the Coifman-Fefferman inequality (1) with $p = 1$ implies that

$$
\int (Tf)g \leq \int |Tf|h \leq c \int (Mf)h \leq c_1 \|f\|_X \|g\|_{X'},
$$

with certain constants $c$ and $c_1$ independent of $f$ and $g$, which implies that $T$ acts boundedly in $X$. Compared to the other approaches that follow (that, at least in their presently available form, significantly rely in their details on the structure of the dyadic cubes in $\mathbb{R}^n$ which makes it harder to carry the arguments over to a general space of homogeneous type), it is easy to see that the Coifman-Fefferman inequality and other parts of the proof remain valid in the general case of Calderon-Zygmund operators on $\sigma$-finite spaces of measurable functions on $S \times \Omega$ where $S$ is a space of homogeneous type.

Now let us briefly describe another approach to establishing a slightly weaker version of Proposition 6 that uses the classical Fefferman-Stein inequality that was recently generalized to general Banach lattices. The Fefferman-Stein maximal function $f^\#$ on $\mathbb{R}^n$ is defined for a locally integrable function $f$ by

$$
f^\#(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy, \quad x \in \mathbb{R}^n,
$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ containing $x$ with sides parallel to the coordinate axes and $f_Q = \frac{1}{|Q|} \int_Q f(z)dz$ is the average of $f$ over $Q$ with respect to the Lebesgue measure. This maximal function is very useful in the estimates of the Calderon-Zygmund operators $T$. On the one hand, we have the well-known (and rather simple) pointwise estimate

$$
(Tf)^\# \leq c_r (M|f|_r)^{\frac{1}{r}}
$$

almost everywhere for any $1 < r < \infty$; see, e.g., [29, Chapter 4, §4.2]. On the other hand, there is the classical Fefferman-Stein inequality

$$
\|f\|_{L_p} \leq c \|f^\#\|_{L_p}
$$

for $1 < p < \infty$. The latter was recently generalized to general Banach lattices as follows (see [20] for more information). In the rather convenient notation $S_0$ denotes the set of all measurable functions $f$ on $\mathbb{R}^n$.
such that their nonincreasing rearrangement \( f^* \) satisfies \( f^*(+\infty) = 0 \), and the main tools of [20] that will appear shortly in this section work for this class of functions rather than just the locally summable ones. Surely \( S_0 \) contains all measurable functions with compact support, and thus it is easy to see that \( S_0 \) is dense in a Banach lattice \( X \) if, for example, \( X \) has order continuous norm. The converse, however, is not true: take, for example, \( X = L_\infty(w) \) with a weight \( w \) satisfying \( w^*(+\infty) = 0 \).

**Theorem 7** ([20, Corollary 4.3]). Suppose that \( X \) is an \( A_1 \)-regular real Banach lattice of measurable functions on \( \mathbb{R}^n \) having the Fatou property. Then the following conditions are equivalent.

1. \( X' \) is \( A_1 \)-regular.
2. There exists some \( c > 0 \) such that \( \|f\|_X \leq c\|f\|_X \) for all \( f \in S_0 \cap X \).

These two ingredients allow us to easily establish Proposition 6 under the additional assumption that \( S_0 \) is dense in \( X \). Indeed, by the assumed density property it is sufficient to estimate \( \|Tf\|_X \) for all \( f \in S_0 \cap X \). An application of Theorem 7, (5) and Proposition 5 yields

\[
\|Tf\|_X \leq c c_r \left\| M^\frac{1}{r} |f|^r \right\|_X \leq c c_1 \|f\|_X
\]

with some \( r > 1 \) and constants \( c, c_1 \) and \( c_r \) independent of \( f \). The assumption in Theorem 7 that \( X \) is a real Banach lattice is easy to lift; see, e.g., [28, Proposition 6].

We need some more preliminaries before further discussion. The Strömbärg local sharp maximal function is defined for \( f \in S_0 \) by

\[
M^\lambda f(x) = \sup_{Q \ni x} \inf_{c \in \mathbb{R}} \left( (f - c) \chi_Q \right)^*(\lambda|Q|), \quad x \in \mathbb{R}^n,
\]

where the supremum is taken over all cubes \( Q \subset \mathbb{R}^n \) containing \( x \). Functions \( M^\lambda f \) and \( f^\lambda \) are closely related via the following estimate which holds true with all sufficiently small \( 0 < \lambda < 1 \) and some \( c_0, c_1 > 1 \) for all locally summable \( f \):

\[
c_0 M^\lambda f(x) \leq f^\lambda(x) \leq c_1 M^\lambda f(x), \quad x \in \mathbb{R}^n;
\]

see, e.g., [10], [18]. On the other hand, \( M^\lambda \) in many cases provides estimates that are significantly finer than those obtained by the means of the Fefferman-Stein sharp maximal function. For example (see [1], [10]),

\[
M^\lambda(Tf) \leq cMf
\]

almost everywhere for all locally summable functions \( f \) with \( c \) independent of \( f \). Estimate (8) is sharper than (5), as it corresponds to the missing limiting case \( r = 1 \) in (5), and we can easily obtain (5) from
using (7) and [29, Chapter 5, §5.2]. Finally, there is the following result similar to the well-known duality relation between $H_1$ and BMO.

**Theorem 8** ([19, Theorem 1]).

$$\int |fg| \leq c \int M^t_\lambda f Mg$$

for any $f \in S_0$ and locally summable function $g$ with some $c$ and $\lambda$ independent of $f$ and $g$.

Examining the details of the previous argument contained in Theorem 7 quickly leads to the following observation.

**Theorem 9.** Suppose that $X$, $Y$ and $Z$ are Banach lattices on $\mathbb{R}^n$ having the Fatou property, $S_0$ is dense in $X$, and suppose that the Hardy-Littlewood maximal operator $M$ acts boundedly from $X$ to $Z$ and from $Y'$ to $Z'$. Then any operator $T$ that satisfies estimate (8) acts boundedly from $X$ to $Y$.

Theorem 9 follows at once from Theorem 8, since for any $f \in X \cap S_0$ and $g \in Y'$ we have an estimate

$$\int |(Tf)g| \leq c \int \left[ M^t_\lambda (Tf) \right] Mg \leq c_1 \int (Mf)(Mg) \leq c_1 \|Mf\|_Z \|Mg\|_{Z'} \leq c_2 \|f\|_X \|g\|_{Y'},$$

with some $c$, $c_1$ and $c_2$ independent of $f$ and $g$.

Since $M$ is a positive operator and $Mg \geq g$ almost everywhere for any locally summable $g$, conditions of Theorem 9 imply that $X \subset Z$ and $Y' \subset Z'$ in the sense of continuous inclusions, which in turn implies that $X \subset Z \subset Y$. Unlike the case $X = Y = Z$ it is presently unclear whether Theorem 9 admits a converse similar to Theorem 25 below. In other words, if a suitable Calderon-Zygmund operator $T$ acts boundedly from $X$ to $Y$, does it follow that there exists a lattice $Z$ satisfying the conditions of Theorem 9? There are still other approaches to establishing estimate (9). Let us now describe a recent result [21]. First, we need some preliminaries. If $T$ is a Calderon-Zygmund operator with kernel $K$ then there is a maximal truncation operator

$$T^*_\varepsilon f(x) = \sup_{0 < \varepsilon < A} \int_{|x-y| < \varepsilon} K(x,y) f(y) \, dy, \quad x \in \mathbb{R}^n,$$

associated with $T$ defined for all locally summable functions $f$. It is well known (see, e. g., [22], Chapter 1, §7]) that this operator is bounded in $L_p$ for all $1 < p < \infty$. The maximal truncated operator $T^*_\varepsilon$ dominates the family of truncations

$$T_{\varepsilon,A} f(x) = \int_{|x-y| < \varepsilon} K(x,y) f(y) \, dy, \quad x \in \mathbb{R}^n,$$
of \( T \), so this family of operators has a weak limit \( T_* \) in \( L_2 \) as \( \varepsilon \to 0 \) and \( A \to \infty \), and there exists some \( a \in L_\infty \) such that
\[
T f(x) = T_* f(x) + a(x) f(x)
\]
for all \( f \in L_2 \) and almost all \( x \in \mathbb{R}^n \). Since multiplication by a bounded function \( a \) is bounded in any lattice, boundedness of \( T \) in a given lattice \( X \) is thus implied by boundedness of the maximal truncation operator \( T^\# \) and vice versa.

A dyadic grid \( \mathcal{D} \) is a collection of cubes \( Q \) in \( \mathbb{R}^n \) with sides parallel to the coordinate axes such that their lengths \( \ell(Q) \) only take values \( 2^k \), \( k \in \mathbb{Z} \), for any \( Q, R \in \mathcal{D} \) we have \( Q \cap R \in \{Q, R, \emptyset\} \), and the cubes \( \{Q \in \mathcal{D} \mid \ell(Q) = 2^k\} \) form a partition of \( \mathbb{R}^n \) for any \( k \in \mathbb{Z} \). A collection \( \mathcal{S} = \{Q^k_j\} \subset \mathcal{D} \) is called a sparse family of dyadic cubes if it satisfies the following properties.

1. Cubes \( Q^k_j \) are pairwise disjoint in \( j \) with \( k \) fixed.
2. If \( \Omega_k = \bigcup_j Q^k_j \) then \( \Omega_{k+1} \subset \Omega_k \).
3. \( |\Omega_{k+1} \cap Q^k_j| \leq \frac{1}{2} |Q^k_j| \) for any \( j \) and \( k \).

For any family of cubes \( \mathcal{S} \) we define an operator
\[
A_{\mathcal{D},\mathcal{S}} f(x) = A_{\mathcal{S}} f(x) = \sum_{Q \in \mathcal{S}} f_Q \chi_Q(x)
\]
acting on locally summable functions \( f \), where as usual \( f_Q = \frac{1}{|Q|} \int_Q f \).

It turns out that these operators with sparse families can be used to estimate Calderon-Zygmund operators in the general setting.

**Theorem 10** ([21, Theorem 1.1]). Suppose that \( X \) is a Banach lattice on \( \mathbb{R}^n \) having the Fatou property. Then
\[
\|T_\sharp f\|_X \leq c_{T,n} \sup_{\mathcal{D},\mathcal{S}} \|A_{\mathcal{D},\mathcal{S}} f\|_X
\]
for any Calderon-Zygmund operator \( T \) and locally summable function \( f \) with compact support, where the supremum is taken over arbitrary dyadic grids \( \mathcal{D} \) and sparse families \( \mathcal{S} \subset \mathcal{D} \).

Theorem 10 is based on a number of results that only recently were developed to a sufficient extent, including a representation of Calderon-Zygmund operators as an average of dyadic shifts and the local mean oscillation decomposition that represents every function \( f \in S_0 \) as \( A_{\mathcal{S}} f \) for some sparse family \( \mathcal{S} \) in a given dyadic grid \( \mathcal{D} \) with good pointwise control on \( f - A_{\mathcal{S}} f \); see [21] for a brief history of the techniques.

As it was shown in [21], Theorem 10 has many interesting corollaries, including the so-called \( A_2 \) conjecture and certain two-weight estimates. Let us verify a replacement for the first line of the estimate (9) for a suitable function \( f \in X \) with operator \( T_\sharp \) in place of \( T \). By Theorem 10 there exists a dyadic grid \( \mathcal{D} \) and a sparse family \( \mathcal{S} \subset \mathcal{D} \) such
that $\|T♯f\|_Y \leq c\|A_{D,S}|f|\|_Y$ with some constant $c$ independent of $f$. Therefore there exist some $g \in Y'$, $\|g_0\|_{Y'} \leq 1$, such that
\begin{equation}
\|T♯f\|_Y \leq 2c \int (A_{D,S}|f|) g.
\end{equation}
We may assume that $g \geq 0$. The integral on the right-hand side of (10) can now be estimated using a kind of a stopping time argument [21, (2.2)] which we are going to reproduce here. Let $\{Q^k_j\} = S$ and $\Omega_k$ be the cubes and sets in the definition of the sparse family $S$, and let $E^k_j = Q^k_j \setminus \Omega_{k+1}$, so that $|E^k_j| \geq \frac{1}{2}|Q^k_j|$ and $\{E^k_j\}$ is a collection of pairwise disjoint sets. Then
\[
\int (A_{D,S}|f|) g = \sum_{j,k} \frac{1}{|Q^k_j|} \int_{Q^k_j} |f| \int_{Q^k_j} g = \\
\sum_{j,k} |Q^k_j| \left( \frac{1}{|Q^k_j|} \int_{Q^k_j} |f| \right) \left( \frac{1}{|Q^k_j|} \int_{Q^k_j} g \right) \leq \\
2 \sum_{j,k} |E^k_j| \left( \frac{1}{|Q^k_j|} \int_{Q^k_j} |f| \right) \left( \frac{1}{|Q^k_j|} \int_{Q^k_j} g \right) = \\
2 \sum_{j,k} \int_{E^k_j} |f| \left( \frac{1}{|Q^k_j|} \int_{Q^k_j} g \right) \leq \\
2 \sum_{j,k} \int_{E^k_j} (Mf)(Mg) \leq 2 \int (Mf)(Mg),
\]
which together with (10) provides a suitable replacement for the first line in estimate (9). Another estimate for Calderon-Zygmund and certain other operators that can also be used to establish (11) can be found in [9].

4. Nondegenerate singular operators

In this section we will try to give a more or less precise meaning to the nondegeneracy conditions that a singular operator in Condition 2 of Theorem 1 must satisfy in order to have a converse implication $2 \Rightarrow 1$ as well as discuss certain restrictions on the spaces that are implied by boundedness of certain classes of operators. Recall that the Muckenhoupt weights $w \in A_2$ are precisely those for which the Hardy-Littlewood maximal operator is bounded in the corresponding weighted space $L^2_w \left( w^{-\frac{1}{2}} \right)$. There is, however, a large class of operators that also characterize Muckenhoupt weights in this sense.

**Definition 11.** A mapping $T : L^2 \to L^2$ is called $A_2$-nondegenerate with a constant $C$ if boundedness of $T$ in a lattice $L^2 \left( w^{-\frac{1}{2}} \right)$ implies $w \in A_2$ with constant $C$. 

This definition is stated for the general setting of a homogeneous space $S$ and measurable functions on $S \times \Omega$. It is worth mentioning that for linear maps $Tf(x, y) = [T_0f(\cdot, y)](x)$ that act in the first variable $x \in S$ only, i.e. uniformly in $y \in \Omega$, nondegeneracy of $T_0$ on $S$ implies nondegeneracy of $T$ on $S \times \Omega$; see [27, Proposition 3.7]. For simplicity we will only work with a single variable from $S = \mathbb{R}^n$ in this section.

Although it is not clear yet how nondegeneracy in the sense of Definition 11 can be characterized in terms of the kernel of a singular integral operator $T$, there are some useful sufficient conditions that illustrate this phenomenon.

**Definition 12.** We say that a mapping $T : L^2 \to L^2$ is nondegenerate along a direction $x_0 \in \mathbb{R}^n \setminus \{0\}$ if there exists a constant $c > 0$ such that for any ball $B \subset \mathbb{R}^n$ of radius $r > 0$ and any locally summable nonnegative function $f$ supported on $B$ we have

$$|Tf(x)| \geq cf_B$$

for all $x \in B \pm rx_0$.

It is well known that singularity of a mapping $T$ in the sense of Definition 12 implies $A_2$-nondegeneracy of $T$; in Proposition 13 below we will establish a somewhat more general result. In terms of the kernel $K$ of $T$ condition (11) roughly means that $K(x, y)$ as a function of $|x - y|$ has to increase at 0 as quickly and decay at infinity as slowly as $|x - y|^{-n}$ along a certain direction; this statement is made more precise in Proposition 16 below. It is not clear whether the class of mappings described by Definition 11 is actually wider than that described by Definition 12.

Let $S = \{Q_l\}$ be a collection of cubes or balls. In addition to $A_S$ we introduce the following “square” averaging operator

$$A^\square_S f(x) = \left( \sum_{Q \in S} (f_Q)^2 \chi_Q(x) \right)^{\frac{1}{2}}$$

for all locally summable functions $f$. It is easy to see that if the cubes or balls from $S$ are pairwise disjoint then $A^\square_S f = A_S f$ almost everywhere for nonnegative functions $f$.

**Proposition 13.** Suppose that a linear operator $T$ that is nondegenerate along a direction $x_0$ is bounded with norm $C$ in a Banach lattice $X$ having the Fatou property. Then for any collection of cubes or balls $S = \{Q_l\}$ we have

$$\|A^\square_S f\|_X \leq c_a \left\| f \left( \sum_l \chi_{Q_l} \right)^{\frac{1}{2}} \right\|_X$$

for all $f$ such that the right-hand part of (12) is well-defined with a constant $c_a$ independent of $f$ and $S$. 
To prove Proposition \[13\] let \( S' = \{ Q'_l \} \) with \( Q'_l = Q_l + x_0 \) being the cubes or balls \( Q_l \) shifted by \( x_0 \) and set \( f_l = f \chi_{Q_l} \). We may assume that \( f \) is nonnegative and that the right-hand part of (12) is finite. It follows that the sequence valued function \( F = \{ f_l \} \) belongs to \( X(l^2) \) with \( \| F \|_{X(l^2)} = \| f (\sum l \chi_{Q_l})^{1/2} \|_X \). Using the nondegeneracy assumption and the Grothendieck theorem (see, e. g., [17]) we can easily obtain the estimate

\[
(13) \quad c^{-1} \left\| \left( \sum_l \chi_{Q'_l}(f_{Q_l})^2 \right)^{1/2} \right\|_X \leq \left\| \left( \sum_l \chi_{Q'_l}|T f_l|^2 \right)^{1/2} \right\|_X \leq \| T F \|_{X(l^2)} \leq C K_G \| F \|_{X(l^2)} = C K_G \left\| f \left( \sum l \chi_{Q_l} \right)^{1/2} \right\|_X,
\]

\( K_G \) being the Grothendieck constant. Repeating this estimate for function \( G = \{ g_l \} \), \( g_l = \chi_{Q'_l} f_{Q_l} \), in place of \( F \) and with the order of \( Q_l \) and \( Q'_l \) reversed yields

\[
(14) \quad c^{-1} \| A_S^G f \|_X = c^{-1} \left\| \left( \sum l \chi_{Q_l}(f_{Q_l})^2 \right)^{1/2} \right\|_X \leq \left\| \left( \sum l \chi_{Q_l}|T g_l|^2 \right)^{1/2} \right\|_X \leq \| T G \|_{X(l^2)} \leq C K_G \| G \|_{X(l^2)} = C K_G \left\| \left( \sum l \chi_{Q'_l}(f_{Q_l})^2 \right)^{1/2} \right\|_X.
\]

Combining (13) and (14) together yields (12) with \( c_a = (C c K_G)^2 \).

The following corollary is essentially well known; see, e. g., remarks after [6, Lemma 5.2.2].

**Corollary 14.** Suppose that a linear operator \( T \) is nondegenerate along a direction \( e \). Then \( T \) is \( A_2 \)-nondegenerate.

Indeed, suppose that \( T \) is bounded in \( L_2 \left( w^{-1/2} \right) \) as in Definition 11.

Taking in (12) a family \( S = \{ B \} \) consisting of a single ball \( B \subset \mathbb{R}^n \), \( X = L_2 \left( w^{-1/2} \right) \) and a nonnegative locally summable function \( f \) supported in \( B \) yields

\[
(15) \quad f_B \left( \int_B w \right)^{1/2} = \| A_S f \|_X \leq c \| f \|_X = c \left( \int_B f^2 w \right)^{1/2}.
\]
By rearranging the terms of \((15)\) we arrive at
\[(f_B)^2 \leq c^2 \frac{1}{|B|} \int_B f^2 w,\]
which is a well-known characterization of the Muckenhoupt weights from [29, Chapter 5, §1.4]; setting \(f = (w + \varepsilon)^{-1}\) and passing to the limit \(\varepsilon \to 0\) quickly leads to \((2)\).

Observe that Proposition 13 also implies that if a suitably nondegenerate operator \(T\) acts boundedly in \(X\) then all operators \(A_S\) with disjoint collections \(S\) of cubes or balls are uniformly bounded in \(X\). It is not clear in general how this property is related to other properties. One, of course, immediately notices that such operators \(A_S\) are bounded in \(L^p\) for both \(p = 1\) and \(p = \infty\) so their uniform boundedness in a lattice \(X\) does not imply per se that \(X\) is \(A_1\)-regular. However, and somewhat surprisingly, this implication holds true at least in the case of variable exponent Lebesgue spaces if we also assume that \(X\) is \(p\)-convex and \(q\)-concave for some \(1 < p, q < \infty\); see [6, Theorem 5.7.2].

This rather involved result together with Proposition 13 provides at once the converse implication \(3 \Rightarrow 1\) of Theorem 1 in the case of variable exponent Lebesgue spaces.

**Corollary 15.** Suppose that \(p(\cdot)\) is a measurable function on \(\mathbb{R}^n\) such that \(1 < \text{ess inf}_{x \in \mathbb{R}^n} p(x) \leq \text{ess sup}_{x \in \mathbb{R}^n} p(x) < \infty\) and a linear operator \(T\) is nondegenerate along a direction and bounded in \(L^{p(\cdot)}\). Then both \(L^{p(\cdot)}\) and \(L^{p'(\cdot)}\) are \(A_1\)-regular.

Thus not only is the converse to [11, Theorem 5.39] true for nondegenerate operators, which answers positively [11, Problem A.17], there is also no need to involve the somewhat complicated machinery of the main results of this paper.

Now we give a standard condition sufficient for a Calderon-Zygmund operators to be nondegenerate along a direction.

**Proposition 16** ([29, Chapter 5, §4.6]). Suppose that \(T\) is a Calderon-Zygmund operator with kernel \(K\) and there exist some \(u \in \mathbb{R}^n\) and a constant \(c\) such that for any \(x \in \mathbb{R}^n\) and \(t \neq 0\) we have
\[(16) \quad |K(x, x + tu)| \geq ct^{-n}.
Then \(T\) is nondegenerate along the direction \(x_0 = su\) with some \(s > 0\) and hence \(T\) is \(A_2\)-nondegenerate.

The two typical examples are the Hilbert transform \(H\) on \(\mathbb{R}\) with kernel \(K(x, y) = \frac{\text{sgn} x}{x - y}\) and Riesz transforms \(R_j, 1 \leq j \leq n\) on \(\mathbb{R}^n\) with kernels \(K_j(x, y) = \frac{c_n(y_j - x_j)}{|y - x|^{n+1}}\), where \(c_n \neq 0\) are some constants. It is evident that these kernels satisfy condition \((16)\) for \(u = e_j, e_j\) being the \(j\)-th coordinate basis vector of \(\mathbb{R}^n\).
For completeness, let us prove Proposition 16. Indeed, condition (1) on the kernel $K$ implies that
\begin{equation}
|K(x, x + r[-x_0 + v]) - K(x, x - r x_0)| \leq C_K \frac{|r v|^\gamma}{|r x_0|^{n + \gamma}} \leq c' s^{-n-\gamma} r^{-n}
\end{equation}
for all $v \in \mathbb{R}^n$, $|v| < \frac{1}{2}|x_0| = \frac{1}{2} s |u|$, and any $r \neq 0$ with some constant $c'$ independent of $s$. By taking $s$ sufficiently large we may assume that (17) holds true for all $|v| < 1$. Therefore (16) implies that
\[
|T f(x)| = \left| K(x, x - r x_0) \int_B f(y) dy + \int_B [K(x, y) - K(x, x - r x_0)] f(y) dy \right|
\geq |K(x, x - r x_0)| \int_B f(y) dy - \left| \int_B [K(x, y) - K(x, x - r x_0)] f(y) dy \right|
\geq |K(x, x - r x_0)| \int_B f(y) dy - \int_B |K(x, y) - K(x, x - r x_0)| |f(y)| dy
\geq \int_B f(y) dy \cdot (c(rs)^{-n} - c's^{-n-\gamma} r^{-n}) = (cs^{-n} - c's^{-n-\gamma}) f_B
\]
for any $f$ supported on the ball $B \subset \mathbb{R}^n$ having radius $r$ and centered at the origin and for any $x \in B \pm r x_0$. Choosing $s$ sufficiently large yields (11), so $T$ is indeed nondegenerate along the direction $x_0$ and is therefore $A_2$-nondegenerate by Corollary 14. The proof of Proposition 16 is complete.

5. A LEMMA ABOUT $A_p$-REGULARITY

In this section we establish the following auxiliary result that we will need in Section 6 below.

Theorem 17. Suppose that $X$ is a Banach lattice of measurable functions on $(S \times \Omega, \mu \times \nu)$ such that $X$ satisfies the Fatou property, and

1. $X$ is $A_p$-regular with constants $(c_1, m_1)$ for some $1 < p < \infty$,
2. $X^\delta$ is $A_1$-regular with constants $(c_2, m_2)$ for some $\delta > 0$.

Then lattice $X$ is $A_1$-regular with an estimate for the constants depending only on the corresponding $A_p$-regularity constants of $X$, $A_1$-regularity constants of $X^\delta$ and the value of $\delta$.

This theorem is easily derived from the corresponding result for $A_p$ weights with the help of a fixed point argument.

Lemma 18. Suppose that a weight $w$ on $(S \times \Omega, \mu \times \nu)$ satisfies $w \in A_p$ and $w^\delta \in A_1$ with some $1 < p < \infty$ and $\delta > 0$. Then $w \in A_1$ with
an estimate for the constants depending only on $\delta$, the corresponding constants of the $A_p$ condition for $w$ and the $A_1$ condition for $w^\delta$.

Lemma \[18\] is essentially a particular case $X = L_\infty(w)$ of Theorem \[17\]. This result is suggested by a very simple observation: by the factorization of $A_p$ weights (see, e.g., \[29\], Chapter 5, §5.3) we have $w = \omega_0 \omega_1^{1-p}$ with $\omega_j \in A_1$, and since we also have $w^\delta \in A_1$, $w$ is bounded away from 0 on every ball, which indicates that the singularities of the denominator factor $\omega_1$ have to be dominated by the singularities of the nominator factor $\omega_0$ in some sense and $\omega_1$ should essentially cancel out in this factorization. To prove Lemma \[18\], fix some $\omega \in \Omega$ such that $w(\cdot, \omega) \in A_p$ and $w^\delta(\cdot, \omega) \in A_1$, and let $B(x, r) \subset S$, $x \in S$, $r > 0$, be an arbitrary ball of $S$. Then sequential application of the $A_p$ condition satisfied by weight $w$, the Jensen inequality with convex function $t \mapsto t^{-(p-1)}$, $t > 0$, and the $A_1$ condition satisfied by the weight $w^\delta$ yields

\begin{equation}
\frac{1}{\nu(B(x, r))} \int_{B(x, r)} w(u, \omega) d\nu(u) \leq c \left[ \frac{1}{\nu(B(x, r))} \int_{B(x, r)} [w(u, \omega)]^{1-p\delta} d\nu(u) \right]^{-\delta(p-1)} \leq c \left[ \frac{1}{\nu(B(x, r))} \int_{B(x, r)} [w(u, \omega)]^{\delta} d\nu(u) \right]^{\frac{1}{\delta}} \leq c' w(x, \omega)
\end{equation}

for almost all $x \in S$ with some constants $c$ and $c'$ depending only on the corresponding constants of the $A_p$ condition for $w$, the $A_1$ condition for $w^\delta$ and the value of $\delta$. Since $\omega$, $x$ and $B$ are arbitrary, \[18\] implies that $w \in A_1$ with the necessary estimates of the constants, which concludes the proof of Lemma \[18\].

In order to reduce Theorem \[17\] to Lemma \[18\] we need to show that under the conditions of Theorem \[17\] an arbitrary function $f \in X$ has a majorant $w$ such that with the appropriate estimates on the constants $w$ is an $A_p$-majorant of $f$ in $X$ and simultaneously $w^\delta$ is an $A_1$-majorant of $|f|^\delta$ in $X^\delta$. At a first glance it may seem that there is little reason to suspect existence of a common majorant in sets that look vastly different (for example, a majorant $w$ such that $w^\delta \in A_1$ may not even be locally summable in the first variable, while on the other hand a majorant $w \in A_p$ may vanish near some points); however, careful application of the celebrated Ky-Fan–Kakutani fixed point theorem allows us to establish the existence of a common majorant in this setting with relative ease.
Theorem (8). Suppose that $K$ is a compact set in a locally convex linear topological space. Let $\Phi$ be a mapping from $K$ to the set of nonempty convex compact subsets of $K$. If the graph
$$
\Gamma(\Phi) = \{(x, y) \in K \times K \mid y \in \Phi(x)\}
$$
of $\Phi$ is closed in $K \times K$ then $\Phi$ has a fixed point, i.e. $x \in \Phi(x)$ for some $x \in K$.

We will also need the following sets of nonnegative a.e. measurable functions $w$ on $(S \times \Omega, \nu \times \mu)$ (see also [27, Section 3]):

$$
\text{BA}_p(C) = \left\{w \mid \text{ess sup}_{\omega \in \Omega} \|M\|_{L_p(w^{-\frac{1}{p}}, \nu)} \leq C \right\};
$$
$$
\text{BA}_1(C) = \left\{w \mid \text{ess sup} \frac{Mw}{w} \leq C \right\}.
$$

These are the sets of Muckenhoupt weights with fixed bounds on the constants (“the Ball of $A_p$”).

Proposition 19 ([27, Proposition 3.4]; see also [11, Lemma 4.2]). Suppose that $1 \leq p < \infty$ a.e. and $C \geq 0$. The set $\text{BA}_p(C)$ is a nonempty convex cone which is also logarithmically convex and closed in measure.

The proof of convexity and closedness is quite routine; the logarithmic convexity is a bit harder but we will not need it under the assumptions of Theorem 17.

We are now ready to prove Theorem 17. The technical details of this proof as well as the general pattern are similar to the main result of [27]. By using [27, Proposition 3.6] it is sufficient to establish the existence of a suitable majorant for every function $f \in X$, $\|f\|_X \leq 1$, such that $E = \text{supp} f$ has positive finite measure and $f \geq \beta$ on $E$ with some $\beta > 0$, since the set of such functions is dense in measure in the nonnegative part of the closed unit ball $B$ of $X$. We fix such a function $f$.

By Proposition 2 there exists some function $a \in X'$, $\|a\|_{X'} = 1$, such that $a > 0$ almost everywhere. This implies that for any $u \in B$ we have $\int |u| a \leq \|u\|_X \|a\|_{X'} \leq 1$, i.e. $\|u\|_{L_1(\alpha^{-1})} \leq 1$. Let $0 < \alpha \leq \beta \leq 1$ be a sufficiently small number to be determined later, and let
$$
D = \{\chi_E \log g \mid g \in B, g \geq \chi_E \alpha\}.
$$

It is easy to see that $D$ is a bounded set in $Y = L_2(a^{-\frac{1}{2}})$ for any given $E$ and $\alpha$ because
$$
\int_{E \cap \{g < 1\}} |\log g|^2 a \leq |\log \alpha|^2 \|\chi_E\|_X \|a\|_{X'} \leq \frac{1}{\beta} |\log \alpha|^2
$$
and
$$
\int_{E \cap \{g \geq 1\}} |\log g|^2 a = \int_{E \cap \{g \geq 1\}} 4 |\log \left(g^{\frac{1}{2}}\right)|^2 a \leq 4 \int |g| a \leq 4
$$
for any \( x \in E \), \( \log g \in D \); \( D \) is convex because \( B \) is logarithmically convex and \( D \) is closed in measure, so \( D \) is compact in the weak topology of \( Y \).

Observe that since \( A_1 \)-regularity of \( X \) implies \( A_1 \)-regularity of \( X^\circ \) for all \( 0 < \gamma < 1 \) we may assume that \( 0 < \delta < 1 \), otherwise the conclusion of Theorem 17 is immediate. We define a set-valued map \( \Phi \) in \( D \times D \) by

\[
\Phi((\log u, \log v)) = \{(\log u_1, \log v_1) \mid u_1, v_1 \in X, \quad u_1 \in B \cap BA_p(c_1), \quad v_1^\delta \in B \cap BA_1(c_2), \quad f \vee (u \vee v) \leq A(u_1 \wedge v_1)\}
\]

Since for any \( (\log u, \log v) \in D \times D \) we have \( w = f \vee u \vee v \in X \) with \( \|w\|_X \leq 3 \) and by the assumptions there exist some \( a, b \in X \) such that \( a \in BA_p(c_1), b^\delta \in BA_1(c_2), \alpha \geq w, b \geq w \) and \( \|a\|_X \leq 3m_1, \quad \|b\|_X \leq (3m_2)^\delta \). Thus choosing \( A = (3m_1) \vee (3m_2)^\frac{1}{\delta} \) and \( \alpha = \beta \wedge A^{-1} \) yields \( (\log u_1, \log v_1) \in \Phi((\log u, \log v)) \) with \( u_1 = \frac{1}{\delta}a \) and \( v_1 = \frac{1}{\delta}b \), so \( \Phi \) takes nonempty values. The condition \( f \vee (u \vee v) \leq A(u_1 \wedge v_1) \) is of course equivalent to (and a shorthand for) the six inequalities \( f \leq Au_1, f \leq Av_1, u \leq Au_1, v \leq Av_1, u \leq Av_1 \) and \( v \leq Av_1 \). It is easy to see using Proposition 19 that the graph \( \Gamma \) of \( \Phi \) is a convex set and \( \Gamma \) is closed with respect to the convergence in measure. Let us verify that \( \Gamma \) is closed in \( Y \times Y \). Indeed, the weak topology of \( Y \times Y \) is metrizable on a bounded set \( D \times D \). If \( x_j \in \Gamma \) and \( x_j \to x \in Y \times Y \) then there exists some sequence \( y_j \) of convex combinations of \( x_j \) such that \( y_j \to x \) in the strong topology of \( Y \times Y \), and \( y_j \in \Gamma \) by the convexity of \( \Gamma \). Strong convergence in \( Y \) implies convergence in measure, so \( y_j \to x \) in measure. Since \( \Gamma \) is closed in measure, it follows that \( x \in \Gamma \) and thus \( \Gamma \) is indeed closed in \( Y \times Y \). From this we also infer that the values of \( \Phi \) are convex and closed in the compact set \( D \times D \) and thus they are compact in \( Y \times Y \).

By the Ky Fan–Kakutani fixed point theorem there exists some \( (\log u, \log v) \in D \times D \) such that \( (\log u, \log v) \in \Phi((\log u, \log v)) \). This implies that \( u \) and \( v \) are pointwise equivalent to one another with the constant of equivalence depending only on \( A \) (which, in turn, only depends on the values of \( m_1, m_2 \) and \( \delta \)), and so \( w = Au \) is a majorant of \( f \) such that \( w \in A_p \) and \( w^\delta \in A_1 \) with the appropriate estimates on the constants. By Lemma 18 it follows that \( w \in A_1 \) with suitable estimates on the constants, which concludes the proof of Theorem 17.

We will need the following proposition, which is a simple consequence of duality and the properties of \( A_p \) weights.

**Proposition 20** ([27, Proposition 2.3]). Suppose that \( X \) is a Banach lattice on \((S \times \Omega, \nu \times \mu)\) such that \( X^\prime \) is a norming space for \( X \). If \( X^\prime \) is \( A_1 \)-regular then \( X^{\prime \prime} \) is \( A_1 \)-regular for all \( q \geq 1 \). If \( X^\prime \) is \( A_p \)-regular with some \( p > 1 \) then \( X^{\prime \prime} \) is \( A_1 \)-regular.

Theorem 17 has an interesting immediate application.
Proposition 21. Let $X$ be a Banach lattice on $(S \times \Omega, \nu \times \mu)$ having the Fatou property. Suppose that both $X$ and $X'$ are $A_\infty$-regular. Then both $X$ and $X'$ are $A_1$-regular.

Indeed, since $X$ and $X'$ are $A_\infty$-regular, they are also $A_p$-regular with some $p > 1$, which by Proposition 20 means that both $X'^{1/\delta}$ and $X^\delta$ are $A_1$-regular, and it remains to apply Theorem 17 to $X$ and $X'$ with $\delta = \frac{1}{p}$.

Corollary 22. Suppose that $X$ is a Banach lattice on $\mathbb{R}^n$ having the Fatou property, and both $X$ and $X'$ are $A_\infty$-regular. Then any Calderon-Zygmund operator $T$ is bounded in $X$.

This corollary, which strengthens Proposition 6, immediately follows from Proposition 21 and Proposition 6.

6. Necessity of $A_1$-regularity

In this section we establish the converse implication $3 \Rightarrow 1$ of Theorem 1. We will need the following fairly well known result, the proof of which in the present setting can be found in [27, Theorem 2.6].

Theorem 23. Suppose that $Y$ is a Banach lattice on $(S \times \Omega, \nu \times \mu)$ with an order continuous norm. If a linear operator $T$ is bounded in $Y^{1/2}$ then for every $f \in Y'$, $m > 1$ and $a > K_G^2\|T\|_{Y^{1/2}}$, $K_G$ being the Grothendieck constant, there exists a majorant $w \geq |f|$, $\|w\|_{Y'} \leq \frac{m}{m-1}\|f\|_{Y'}$, such that $\|T\|_{L^2(w^{-1/2})} \leq \frac{a}{\sqrt{m}}$.

This theorem essentially says that for suitably nondegenerate operators $T$ boundedness of $T$ in a lattice $Y^{1/2}$ implies that $Y'$ is $A_2$-regular, which binds the boundedness property of certain operators in a lattice back to a regularity property for some related lattices. The proof of Theorem 23 given in [27, §6] is merely a slight refinement of the proof of [14, Theorem 3.5], which is in turn a variant of the well-known Maurey-Krivine factorization theorem (see [23]). For the first time these ideas were exploited in a similar context in [26].

Theorem 24 ([27, Theorem 1.6]). Suppose that $X$ is a Banach lattice on $(S \times \Omega, \nu \times \mu)$ having the Fatou property. Suppose also that $XL_q$ for some $1 < q < \infty$ is a Banach lattice and $XL_q$ is $A_p$-regular for some $1 \leq p < \infty$. Then $X$ is $A_{p+1}$-regular.

Theorem 24, which is rather involved, is a direct precursor to a very deep and nontrivial fact that the so-called BMO-regularity property is self-dual at least for Banach lattices having the Fatou property; see [11], [15], [27].

Theorem 25. Suppose that $X$ is a Banach lattice of measurable functions on $(S \times \Omega, \nu \times \mu)$ such that $X$ is $p$-convex and $q$-concave for some
1 < p, q < ∞ and X satisfies the Fatou property. Let T be a linear operator on $L_2(S \times \Omega)$ such that both $T$ and $T^*$ are $A_2$-nondegenerate and $T$ acts boundedly in $X$ and in all $L_s$ for $1 < s < \infty$. Then lattices $X$ and $X'$ are $A_1$-regular.

Let us now prove Theorem 25. By the $p$-convexity condition $X^p$ is also a Banach lattice with the Fatou property, and so $X^{p(1-\theta)}L^\theta_t$ is also a Banach lattice for all $1\leq t \leq \infty$ and $0 < \theta < 1$. Choosing $\theta = 1 - \frac{1}{p}$ shows that $Y_s = XL_s$ is a Banach lattice for all sufficiently large s. Lattice $Y_s$ satisfies the Fatou property and has order continuous norm (because $L_s$ has order continuous norm for $s < \infty$). Since $T$ is bounded in $X$ and in $L_s$ for all $1 < s < \infty$, by the interpolation theorem mentioned in Section 1 operator $T$ is also bounded in $X^{\frac{1}{2}}L^{\frac{1}{2}}_s = Y^{\frac{1}{2}}_s$ for all $1 < s < \infty$. Theorem 23 and $A_2$-nondegeneracy of $T$ then imply that lattice $Y'_s = X'L'_s$ is $A_2$-regular for all sufficiently large $s$. By Theorem 24 it follows that lattice $X'$ is $A_3$-regular, and furthermore by Proposition 20 lattice $X^{\frac{1}{2}}$ is $A_1$-regular. Since the convexity assumptions of Theorem 25 imply that lattices $X$ and $X'$ have order continuous norm, we have $X' = X^*$ and $X = (X')^*$, and moreover $X \cap L_2$ is dense in $X$ and $X' \cap L_2$ is dense in $X'$, so the duality relation $\int (Tf)g = \int f(T^*g)$ for $f \in X \cap L_2$ and $g \in X' \cap L_2$ shows that boundedness of $T$ in $X$ implies boundedness of the conjugate operator $T^*$ in $X'$ and vice versa. Repeating the argument above with lattice $X'$ in place of $X$ (which is a $q'$-convex lattice since $X$ is $q$-concave) and operator $T^*$ in place of $T$ shows that lattice $X$ is $A_3$-regular and lattice $(X')^{\frac{1}{2}}$ is $A_1$-regular. Finally, we apply Theorem 17 to $X$ and to $X'$ with $p = 3$ and $\delta = \frac{1}{3}$, which establishes that lattices $X$ and $X'$ are both $A_1$-regular. The proof of Theorem 25 is complete.

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