Exact regularity of pseudo-splines

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Abstract

In this paper we review and refine a technique of Rioul to determine the Hölder regularity of a large class of symmetric subdivision schemes from the spectral radius of a single matrix. These schemes include those of Dubuc and Deslauriers, their dual versions, and more generally all the pseudo-spline and dual pseudo-spline schemes. We also derive various comparisons between their regularities using the Fourier transform. In particular we show that the regularity of the Dubuc-Deslauriers family increases with the size of the mask.

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1 Introduction

Subdivision is a recursive method for generating curves, surfaces and other geometric objects. Rather than having a complete description of the object of interest at hand, subdivision generates the object by repeatedly refining its description starting from a coarse set of control points. Since subdivision schemes are often easy to implement and very flexible, they provide a powerful tool for modelling geometry. However, analyzing their smoothness, or regularity, can be difficult. The purpose of this paper is to review and

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refine a method proposed by Rioul [12] to determine the Hölder regularity of a surprisingly large class of subdivision schemes from the spectral radius of a single matrix. A joint spectral radius analysis is not required.

Consider the scheme

\[ f_{j+1,k} = \sum_\ell a_{k-2\ell} f_{j,\ell}, \]

with finitely supported mask \( a = (a_k)_{k \in \mathbb{Z}} \), and coefficients \( a_k \in \mathbb{R} \), acting on the initial data \( f_{0,k} \in \mathbb{R}, k \in \mathbb{Z} \). At each subdivision level \( j \geq 0 \), let \( f_j \) be the piecewise linear function with value \( f_{j,k} \) at the point \( 2^{-j}k \). The scheme is \textit{convergent} if it has a pointwise limit \( f := \lim_{j \to \infty} f_j \). We assume that only a finite number of the initial data \( f_{0,k} \) are non-zero, in which case \( f \) has compact support. In the special case of the cardinal data \( f_{0,k} = \delta_{k,0} \), the support of the limit \( f \) is the interval \([K, L]\) if \( a_K, a_L \neq 0 \) and \( a_k = 0 \) for all \( k < K \) and \( k > L \). Note that shifting the \( a_k \) merely shifts \( f \).

The Laurent polynomial

\[ a(z) = \sum_k a_k z^k \]

is the \textit{symbol} of the scheme. It is well known [10], that a necessary condition for convergence of (1) is that

\[ \sum_k a_{2k} = \sum_k a_{2k+1} = 1, \]

and so we will make this assumption. This condition can be expressed in terms of the symbol as

\[ a(-1) = 0 \quad \text{and} \quad a(1) = 2. \] (2)

Let us now suppose, after shifting the coefficients \( a_k \) as necessary, that \( a(z) \) can be factorized as

\[ a(z) = 2^{-r}(1 + z)^{r+1} b(z) \] (3)

for some \( r \geq 0 \), and that \( b = (b_k)_{k \in \mathbb{Z}} \), the mask corresponding to \( b(z) \), is symmetric about \( b_0 \), i.e., \( b_k = b_{-k} \). Then the Fourier transform of \( b \),

\[ B(\xi) := b(e^{-i\xi}) = \sum_k b_k e^{-ik\xi}, \quad \xi \in \mathbb{R}, \]
is both periodic with period $2\pi$ and real.

Rioul showed in [12] that a lower bound on the Hölder regularity of (1), defined below, can be determined from the spectral radius of a single matrix if $B \geq 0$, i.e., if $B(\xi) \geq 0$ for all $\xi \in [-\pi, \pi]$. A surprisingly large class of schemes are of this type. For example, one can easily check that they include all the pseudo-spline schemes, both primal and dual [2, 6, 7, 5, 4], from explicit formulas for $a(z)$.

Rioul further showed that in the special case that the scheme (1) is interpolatory, the lower bound is optimal. We will show more: that the lower bound is optimal whenever the cardinal function of the scheme has $\ell^\infty$-stable integer translates. Using a characterization of such stability due to Jia and Micchelli [11], this leads us to conclude that the lower bound is optimal under the slightly stricter condition that $B > 0$. Such schemes include again all the pseudo-spline and dual pseudo-spline schemes.

We apply these results to compute and tabulate the regularity of the pseudo-spline schemes, primal and dual, for low orders. We then obtain new information about these regularities: by making pointwise comparisons between the Fourier transforms of two schemes, we derive inequalities on their regularities. As an example, we show that the regularity of the Dubuc-Deslauriers scheme [8, 3] increases with the size of the mask.

## 2 Regularity

The limit function $f$ has Hölder regularity $\alpha$, $0 < \alpha < 1$, written $f \in C^\alpha$, if

$$|f(x) - f(y)| \leq C|x - y|^{\alpha}$$

for all $x, y \in \mathbb{R}$, and we write $f \in C^{q+\alpha}$ for $q \in \mathbb{N}_0$, $0 < \alpha < 1$, if $f \in C^q$, i.e., $f$ is $q$ times continuously differentiable, and $f^{(q)} \in C^\alpha$. Correspondingly, we shall say that the scheme (1) has Hölder regularity $\gamma$ for some real $\gamma \geq 0$, if, for $\beta < \gamma$, $f \in C^\beta$ for all initial data, and, for $\beta > \gamma$, $f \not\in C^\beta$ for some initial data.

The regularity of $f$ is related to the behaviour of the divided differences of the scheme. For each integer $s \geq 0$, let $f_{\delta}^{[s]}$ denote the divided difference of the values $f_{j,k-s}, \ldots, f_{j,k}$ at the corresponding dyadic points $2^{-j}(k-s), \ldots, 2^{-j}k$. Then $f_{\delta}^{[0]} = f_{j,k}$ and for $s \geq 1$,

$$f_{\delta}^{[s]} = \frac{2^j}{s}(f_{\delta}^{[s-1]} - f_{\delta}^{[s-1]}).$$  \hfill (4)
Under conditions (2) and (3), there is a scheme for the $f_{s,j,k}^{[s]}$ for $s = 0, 1, \ldots, r+1$. For such $s$, if we define the associated Laurent polynomial as

$$f_{j}^{[s]}(z) = \sum_{k} f_{j,k}^{[s]} z^{k},$$

then

$$f_{j+1}^{[s]}(z) = a^{[s]}(z) f_{j}^{[s]}(z^{2})$$

where

$$a^{[s]}(z) = \frac{2^{s}}{(1+z)^{s}} a(z),$$

from which we obtain the derived scheme

$$f_{j+1,k}^{[s]} = \sum_{\ell} a_{k-2\ell}^{[s]} f_{j,\ell}^{[s]}.$$  \hfill (5)

Then, with

$$g_{j,k}^{[r]} := f_{j,k}^{[r]} - f_{j,k-1}^{[r]},$$

it can be shown \cite{10} that if

$$|g_{j,k}^{[r]}| \leq C\lambda^{j},$$  \hfill (6)

for some constants $C$ and $\lambda < 1$, for large enough $j$, then $f \in C^{r}$. Moreover, if $1/2 < \lambda < 1$, then $f \in C^{r-\log_{2}(\lambda)}$.

## 3 Reduction procedure

How can we use (6) in the case that it holds with $\lambda \geq 1$? Then we do not know whether $f \in C^{r}$, but if $r \geq 1$ we can use the ‘reduction procedure’ of Daubechies, Guskov, and Sweldens \cite{1} to obtain information about lower order derivatives. Although the procedure was shown to work for interpolatory schemes in \cite{1}, it also applies to the more general scheme \cite{1}.

**Lemma 1.** Suppose (3) holds for some $r \geq 1$. If (6) holds with $\lambda > 1$ then

$$|g_{j,k}^{[r-1]}| \leq D_{1} 2^{-j} \lambda^{j},$$

while if it holds with $\lambda = 1$,

$$|g_{j,k}^{[r-1]}| \leq (D_{2} + D_{3}j) 2^{-j},$$

for constants $D_{1}, D_{2}, D_{3}$. 

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Proof. By the divisibility assumption, $a^{[r]}(-1) = 0$, and by the assumption that $a(1) = 2$ in (2), it also follows that $a^{[r]}(1) = 2$. Therefore,

$$\sum_k a^{[r]}_{2k} = \sum_k a^{[r]}_{2k+1} = 1,$$

and from (5) using summation by parts there is a constant $C_1$ such that

$$|f^{[r]}_{j+1,2k+s} - f^{[r]}_{j,k}| \leq C_1 \max_k |g^{[r]}_{j,k}|, \quad s = 0, 1.$$

So, for any $j \geq 1$, if we represent any $k \in \mathbb{Z}$ in binary form as $k = k_j$, where

$$k_\ell = 2k_{\ell-1} + s_\ell, \quad \ell = j, j-1, \ldots, 1,$$

for some $k_0 \in \mathbb{Z}$ and $s_1, \ldots, s_j \in \{0, 1\}$, then

$$|f^{[r]}_{j,k} - f^{[r]}_{0,k_0}| \leq \sum_{\ell=1}^j |f^{[r]}_{\ell,k_\ell} - f^{[r]}_{\ell-1,k_{\ell-1}}| \leq C_1 C (1 + \lambda + \cdots + \lambda^{j-1}).$$

Hence,

$$|f^{[r]}_{j,k}| \leq C_2 + C_1 C (1 + \lambda + \cdots + \lambda^{j-1}),$$

and since

$$g^{[r-1]}_{j,k} = 2^{-j} f^{[r]}_{j,k},$$

this gives the result in the two cases $\lambda > 1$ and $\lambda = 1$. \hfill \square

By applying this procedure recursively, it follows that if (6) holds for any $\lambda$ with $1/2 < \lambda < 2^r$ then $f \in C^{\lambda - \log_2 \lambda}$ if $\log_2 \lambda$ is not an integer, and $f \in C^{\lambda - \log_2 \lambda - \epsilon}$ for any small $\epsilon > 0$ if $\log_2 \lambda$ is an integer.

4 Rioul’s method

With $r$ in (3) now fixed, let $g_{j,k} = g^{[r]}_{j,k}$ and $g_j(z) = \sum_k g_{j,k} z^k$. Then

$$g_{j+1}(z) = b(z) g_j(z^2), \quad (7)$$

with $b(z)$ as in (3), or equivalently,

$$g_{j+1,k} = \sum_\ell b_{k-2\ell} g_{j,\ell}. \quad (8)$$
Suppose now that b is symmetric, and therefore, after shifting the coefficients as necessary, it has the form

$$b = (b_p, \ldots, b_1, b_0, b_1, \ldots, b_p), \quad b_p \neq 0,$$

for some $$p \geq 0$$, in which case

$$B(\xi) = b_0 + 2 \sum_{k=1}^{p} b_k \cos(k\xi).$$

If $$p = 0$$ we must have $$b_0 = 1$$ and so we can take $$\lambda = 1$$. In this case the scheme (11) is the B-spline scheme of degree $$r$$ and this merely confirms the well-known fact that the limit $$f$$, being a spline of degree $$r$$, belongs to $$C^\beta$$ for any $$\beta < r$$. Thus, we assume from now on that $$p \geq 1$$.

Iterating (7) gives

$$g_j(z) = b_j(z)g_0(z^{2^j}),$$

where

$$b_j(z) := b(z)b(z^2) \cdots b(z^{2^{j-1}}).$$

But then

$$b_{j+1}(z) = b(z)b_j(z^2),$$

and so $$b_j(z)$$ is the Laurent polynomial of the data $$b_{j,k}$$, where $$b_{0,k} = \delta_{k,0}$$ and

$$b_{j+1,k} = \sum_{\ell} b_{k-2\ell} b_{j,\ell}.$$

In particular, $$b_{1,k} = b_k$$. Since (10) can be written as

$$g_{j,k} = \sum_{\ell} b_{j,k-2\ell} g_0,\ell,$$

it follows that

$$|g_{j,k}| \leq \max_m |b_{j,m}| \sum_{\ell} |g_0,\ell|,$$

and so (6) holds if there is some constant $$C'$$ such that

$$\max_k |b_{j,k}| \leq C'\lambda^j.$$

By induction on $$j$$, the values $$b_{j,k}$$ are zero whenever $$k < -p_j$$ or $$k > p_j$$, where $$p_j := (2^j - 1)p.$$
Lemma 2 (Riou). If \( b(z) \) in (3) has the form (4) and \( B \geq 0 \) then
\[
\max_k |b_{j,k}| = b_{j,0} \quad \text{for all } j \geq 0.
\]

Proof. Since
\[
B_j(\xi) := \sum_{k=-p_j}^{p_j} b_{j,k} e^{-ik\xi}
\]
is a Fourier series, we have
\[
b_{j,k} = \frac{1}{2\pi} \int_{-\pi}^{\pi} B_j(\xi) e^{ik\xi} d\xi,
\]
and therefore
\[
|b_{j,k}| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |B_j(\xi)| d\xi.
\]
By the symmetry of the \( b_k \), it follows from (13) by induction on \( j \) that the \( b_{j,k} \) are symmetric for all \( j \), i.e., \( b_{j,-k} = b_{j,k} \). Therefore, \( B_j \) is real, and by induction on \( j \) from (12), \( B_j(\xi) \geq 0 \) for all \( \xi \), and it follows that
\[
|b_{j,k}| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} B_j(\xi) d\xi = b_{j,0}. \quad \Box
\]

5 Spectral radius

Under the assumption of Lemma 2 it follows that (6) holds if
\[
b_{j,0} \leq C\lambda^j
\]
for large enough \( j \). One way to determine such \( \lambda \) is to study the vector of coefficients
\[
b_j = (b_{j,-p+1}, \ldots, b_{j,p-1})^T,
\]
since it includes the central coefficient \( b_{j,0} \) and is self-generating in the sense that \( b_{j+1} = Mb_j \), where, from (13), \( M \) is the matrix of dimension \( 2p-1 \) defined by
\[
M = (b_{k-2\ell})_{k,\ell=-p+1,\ldots,p-1}.
\]
The first few examples of $M$, with $p = 1, 2, 3$, are

$$
\begin{bmatrix}
 b_0 \\
 b_1 & b_{-1} & 0 \\
 b_2 & b_0 & b_{-2} \\
 0 & b_1 & b_{-1}
\end{bmatrix},
\begin{bmatrix}
 b_2 & b_0 & b_{-2} & 0 & 0 \\
 b_3 & b_1 & b_{-1} & b_{-3} & 0 \\
 0 & b_2 & b_0 & b_{-2} & 0 \\
 0 & b_3 & b_1 & b_{-1} & b_{-3} \\
 0 & 0 & b_2 & b_0 & b_{-2}
\end{bmatrix}.
$$

**Theorem 1.** If $B \geq 0$ then

$$
\lim_{j \to \infty} b_{j,0}^{1/j} = \rho,
$$

where $\rho$ is the spectral radius of $M$, and if $\rho \geq 1/2$, a lower bound for the regularity of the scheme (1) is $r - \log_2(\rho)$.

**Proof.** Since $b_j = M^j b_0$,

$$b_{j,0} = \|b_j\|_\infty \leq \|M^j\|_\infty \|b_0\|_\infty = \|M^j\|_\infty.$$

On the other hand, from equation (14),

$$M^j = (b_{j,k-2}\ell)_{k,\ell=-p+1,...,p-1},$$

and so

$$\|M^j\|_\infty \leq (2p - 1) \max_k |b_{j,k}| = (2p - 1)b_{j,0}.$$

Therefore,

$$(2p - 1)^{-1/j} \|M^j\|_\infty^{1/j} \leq b_{j,0}^{1/j} \leq \|M^j\|_\infty^{1/j},$$

and letting $j \to \infty$ proves (16). It follows from (16) that (6) holds with $C = 1$ for any $\lambda > \rho$, and this proves the lower bound on the regularity of the scheme.

### 6 Alternative matrices

Due to the assumption that $b$ is symmetric we can compute $\rho$ in (16) as the spectral radius of a matrix of roughly half the size of $M$, namely of dimension $p$ instead of $2p - 1$. Since $b_{j,-k} = b_{j,k}$, the vector of coefficients

$$b_j = (b_{j,0}, b_{j,1}, \ldots, b_{j,p-1})^T,$$

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also includes \( b_{j,0} \) and is self-generating. Indeed, from (13),

\[
b_{j+1,k} = b_k b_{j,0} + \sum_{\ell \geq 1} (b_{k-2\ell} + b_{k+2\ell})b_{j,\ell},
\]

and using the fact that \( b_{-k} = b_k \) implies

\[
b_{j+1,k} = b_k b_{j,0} + \sum_{\ell \geq 1} (b_{k-2\ell} + b_{k+2\ell})b_{j,\ell},
\]

and it follows that \( b_{j+1} = M b_j \), where \( M \) is the matrix of dimension \( p \),

\[
M = (m_{k,\ell})_{k,\ell=0,...,p-1}, \quad m_{k,\ell} = \begin{cases} b_k, & \ell = 0; \\
(b_{k-2\ell} + b_{k+2\ell}), & \ell \geq 1,
\end{cases} \quad (17)
\]

For \( p = 1, 2, 3, 4 \), this ‘folded’ matrix is

\[
\begin{bmatrix}
    b_0 \\
    b_1 \\ b_0 2b_2 \\
\end{bmatrix}, \quad 
\begin{bmatrix}
    b_0 & 2b_2 & 0 \\
    b_1 & b_1 + b_3 & b_3 \\
    b_2 & b_0 & b_2 \\
\end{bmatrix}, \quad 
\begin{bmatrix}
    b_0 & 2b_2 & 2b_4 & 0 \\
    b_1 & b_1 + b_3 & b_3 & 0 \\
    b_2 & b_0 + b_4 & b_2 & b_4 \\
    b_3 & b_1 & b_1 & b_3
\end{bmatrix}.
\]

Rioul obtained \( \rho \) in (16) in two alternative, but equivalent ways, working from an alternative to (12). From (11) there is another recursion:

\[
b_{j+1}(z) = b(z^{2^j})b_j(z),
\]

which gives

\[
b_{j+1,k} = \sum_{\ell} b_{\ell} b_{j,2\ell}. 
\]

This means that the subset of coefficients \( b_{j,k} \) whose indices increase in steps of \( 2^j \), rather than 1, i.e., \( c_{j,k} := b_{j,2^j} \), satisfy the recursion

\[
c_{j+1,k} = \sum_{\ell} b_{\ell} c_{j,2\ell - \ell} = \sum_{\ell} b_{2\ell - \ell} c_{j,\ell}. \quad (18)
\]

It follows that the vector

\[
c_j = (c_{j,-p+1}, \ldots, c_{j,p-1})^T
\]

is self-generating and includes \( b_{j,0} \) because \( b_{j,0} = c_{j,0} \), and so we deduce that \( \rho \) in (16) is also the spectral radius of the matrix \( N \) of dimension \( 2p - 1 \).
where \( c_{j+1} = Nc_j \). However, by comparing (18) with (13) we see that \( N \) is simply the transpose of \( M \) in (15) and so these two approaches to computing \( \rho \) are equivalent. Rioul computed \( \rho \) in (16) from a folded version of \( N \) of dimension \( p \), analogous to \( M \) in (17), using the reduced vector

\[
c_j = (c_{j,0}, \ldots, c_{j,p-1})^T.
\]

Theorem 1 holds with \( M \) replaced by each of these alternative matrices, the proof being similar.

7 Optimality

In this section we show that under a slightly stricter condition, the lower bound on the regularity of Theorem 1 is optimal.

**Theorem 2.** If \( B > 0 \) the lower bound of Theorem 1 is optimal.

To prove this we first establish a lemma that shows that the bound is optimal whenever the cardinal function of the scheme has \( \ell^\infty \)-stable integer translates. The main point in proving this lemma is that the stability allows us to bound divided differences of the scheme by corresponding divided differences of the limit function.

Let \( \phi \) denote the cardinal function of the scheme (I), i.e., its limit when the initial data is the cardinal data \( f_{0,k} = \delta_{k,0} \). Then the limit function for general data can be expressed as the linear combination

\[
f(x) = \sum_\ell f_{0,\ell} \phi(x - \ell).
\]

Following Jia and Micchelli [11], we shall say that \( \phi \) has \( \ell^\infty \)-stable integer translates if there is some constant \( K > 0 \) such that for any sequence \( c = (c_\ell)_\ell \) in \( \ell^\infty (\mathbb{Z}) \),

\[
\| \sum_\ell c_\ell \phi(\cdot - \ell) \|_{L^\infty (\mathbb{R})} \geq K\| c \|_{\ell^\infty (\mathbb{Z})}.
\]

(19)

**Lemma 3.** Suppose \( \phi \) has \( \ell^\infty \)-stable integer translates and \( f \) has regularity \( q + \alpha \) for some \( q \in \mathbb{N}_0 \) and \( 0 < \alpha < 1 \). Then for any integer \( r \geq q \), there is a constant \( C \) such that

\[
|g_{j,k}^{[r]}| \leq C 2^j (r-q-\alpha).
\]

(20)
Proof. As is well known, see e.g. the review by Dyn and Levin [10], $\phi$ satisfies the two-scale difference equation

$$
\phi(x) = \sum_k a_k \phi(2x - k),
$$

(21)

and therefore, for any $j \geq 0$,

$$
f(x) = \sum_{\ell} f_{j,\ell} \phi(2^j x - \ell).
$$

(22)

We can use this equation to relate any divided difference of $f$ of the form

$$
\tilde{f}_{j,y}^q := [2^{-j}(y - q), 2^{-j}(y - q + 1), \ldots, 2^{-j}(y)] f,
$$

for $y \in \mathbb{R}$, to the divided differences of the scheme. Putting $x = 2^{-j}(y - k)$ in (22) gives

$$
f\left(2^{-j}(y - k)\right) = \sum_{\ell} f_{j,\ell-k} \phi(y - \ell),
$$

and, using the cases $k = 0, 1, \ldots, q$, and the linearity of divided differences,

$$
\tilde{f}_{j,y}^q = \sum_{\ell} f_{j,\ell}^q \phi(y - \ell).
$$

Similarly, if

$$
\tilde{g}_{j,y}^q := \tilde{f}_{j,y}^q - \tilde{f}_{j,y-1}^q,
$$

then

$$
\tilde{g}_{j,y}^q = \sum_{\ell} g_{j,\ell}^q \phi(y - \ell).
$$

Recalling that $f$ has compact support, if $f$ has regularity $q + \alpha$, there is some $C > 0$ such that for any $\xi_0, \xi_1 \in \mathbb{R}$,

$$
|f^{(q)}(\xi_1) - f^{(q)}(\xi_0)| \leq C|\xi_1 - \xi_0|^{\alpha},
$$

and, by a standard property of divided differences, for each $j$ and $y$,

$$
|\tilde{g}_{j,y}^q| = |f^{(q)}(\xi_1) - f^{(q)}(\xi_0)|/q!,
$$

for $\xi_0, \xi_1 \in \left(2^{-j}(y - q - 1), 2^{-j}(y)\right)$. Therefore, for any $y$,

$$
|\tilde{g}_{j,y}^q| \leq C'2^{-ja},
$$

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where

\[ C' = C(q + 1)^\alpha /q!. \]

Therefore,

\[ \left\| \sum_{\ell} g_{j, \ell}^{[q]} \phi(\cdot - \ell) \right\|_{L^\infty(\mathbb{R})} \leq C'^{2^{-j\alpha}}, \]

and by (19) it follows that for any \( \ell \in \mathbb{Z} \),

\[ |g_{j, \ell}^{[q]}| \leq K^{-1} C'^{2^{-j\alpha}}. \]

Finally, by applying the divided difference definitions (4) recursively, \( r - q \) times, we obtain (20).

**Lemma 4.** If \( \phi \) has \( \ell^\infty \)-stable integer translates then the lower bound, \( r - \log_2(\rho) \), of Theorem 1 is optimal.

**Proof.** Let \( f \) be the limit of the scheme with any initial data for which \( g_{0,k}^{[r]} = \delta_{k,0}, -p + 1 \leq k \leq p - 1 \), and with only a finite number of initial data \( f_{0,k} \) non-zero. Then \( f \) has compact support. Suppose that \( f \in C^{r-\log_2(\rho)+\varepsilon} \) for some small \( \varepsilon > 0 \) and write the exponent as

\[ r - \log_2(\rho) + \varepsilon = q + \alpha, \]

for \( q \in \mathbb{N}_0 \) and \( 0 < \alpha < 1 \). If \( \rho > 1/2 \), we have \( r \geq q \), and so Lemma 3 can be applied, implying

\[ |g_{j,k}^{[q]}| \leq C'2^{j(\log_2(\rho) - \varepsilon)} = C'\rho^j 2^{-j\varepsilon}. \]

Thus,

\[ \limsup_{j \to \infty} |g_{j,0}^{[r]}|^{1/j} \leq \rho 2^{-\varepsilon}. \]

But \( g_{j,0}^{[r]} = b_{j,0} \) of equation (16) and so this contradicts (16).

Using this lemma we can now prove Theorem 2 by comparing the cardinal function \( \phi \) with B-splines, which are known to be stable. A similar idea was used by Dong and Shen [6 Lemma 2.2] to show that pseudo-splines are stable.
Proof of Theorem 2. By Lemma 4 it is sufficient to show that \( \phi \) has \( \ell^\infty \)-stable integer translates if \( B > 0 \). If the scheme (1) is interpolatory, in the sense that \( a_{2k} = \delta_{k,0} \), then \( \phi(k) = \delta_{k,0} \) and so the stability condition (19) holds with \( K = 1 \). To show stability in the general case, we apply some results by Jia and Micchelli [11]. We denote the (continuous) Fourier transform of \( \phi \) by

\[
\hat{\phi}(\xi) = \int_{\mathbb{R}} \phi(x) e^{-i\xi x} \, dx, \quad \xi \in \mathbb{R}.
\]

Since the scheme (1) has constant precision,

\[
\sum_{\ell} \phi(x - \ell) = 1, \quad x \in \mathbb{R},
\]

and, as shown by Jia and Micchelli [11, Theorem 2.4], \( \hat{\phi}(0) = 1 \). Since the Fourier transform of (21) is

\[
\hat{\phi}(\xi) = 2^{-1} A(\xi/2) \hat{\phi}(\xi/2),
\]

it follows that

\[
\hat{\phi}(\xi) = \prod_{j=1}^{\infty} \left( 2^{-1} A(\xi/2^j) \right).
\]

A sufficient condition [11, Theorem 3.5] for \( \phi \) to have \( \ell^\infty \)-stable integer translates is that

\[
\sup_{\ell \in \mathbb{Z}} |\hat{\phi}(\xi + 2\pi\ell)| > 0, \quad \forall \xi \in \mathbb{R}. \tag{23}
\]

Consider then again the case that the derived scheme (8) holds. Then

\[
A(\xi) = 2 \cos^{r+1}(\xi/2) B(\xi),
\]

where, since \( A(0) = 2 \) under the assumption of convergence, \( B(0) = 1 \). In the B-spline scheme of degree \( r \) we have \( b(z) = 1 \), in which case we can write the symbol as \( a_r(z) = (1 + z)^{r+1}/2^r \). The cardinal function \( \phi_r \) is the B-spline of degree \( r \) centred at 0, and we have

\[
\hat{\phi}_r(\xi) = \prod_{j=1}^{\infty} \cos^{r+1} \left( \frac{\xi}{2^{j+1}} \right) = \left( \frac{\sin(\xi/2)}{\xi/2} \right)^{r+1}.
\]

It then follows that

\[
\hat{\phi}(\xi) = \hat{\phi}_r(\xi) \prod_{j=1}^{\infty} B(\xi/2^j).
\]
Since the condition (23) holds for the B-spline \( \phi_r \), we deduce that \( \phi \) has \( \ell_\infty \)-stable integer translates if \( B(\xi) > 0 \) for all \( \xi \in [-\pi, \pi] \). \( \square \)

**Example 1.** Consider the quintic Dubuc-Deslauriers scheme \([8, 3]\) with mask

\[
a = \frac{1}{256}(3, 0, -25, 0, 150, 256, 150, 0, -25, 0, 3).
\]

There is a factorization (3) up to \( r = 5 \), in which case one finds

\[
b = (b_{-2}, b_{-1}, b_0, b_1, b_2) = \frac{1}{8}(3, -18, 38, -18, 3),
\]

with \( p = 2 \), and

\[
B(\xi) = \frac{1}{8}(38 - 36 \cos \xi + 6 \cos 2\xi).
\]

Making the substitution \( s = \sin^2(\xi/2) \) yields

\[
B(\xi) = 1 + 3s + 6s^2 > 0
\]

for any \( \xi \in [-\pi, \pi] \). Thus Theorems 1 and 2 both apply. We find \( \rho \) either as the spectral radius of the matrix \( (15) \),

\[
\frac{1}{8} \begin{bmatrix} -18 & -18 & 0 \\ 3 & 38 & 3 \\ 0 & -18 & -18 \end{bmatrix},
\]

or of the smaller, folded matrix matrix \( (17) \),

\[
\frac{1}{8} \begin{bmatrix} 38 & 6 \\ -18 & -18 \end{bmatrix}.
\]

In both cases we find \( \rho = 9/2 \) and therefore the scheme has regularity

\[
5 - \log_2(9/2) \approx 2.8301.
\]

### 8 Pseudo-splines

In the remainder of the paper we focus on the pseudo-spline schemes and their dual versions, all of which satisfy the conditions of Theorem 1 and Theorem 2. We first compute numerically their regularities from the spectral radius of \( M \) in (17) and tabulate them. Then, by making pointwise comparisons among their Fourier transforms, we derive various comparisons among their regularities. For example, we show that the regularity of the Dubuc-Deslauriers family of schemes increases with the polynomial degree used to define them.
8.1 Computing regularities

For integers \( m \geq 1 \) and \( \ell = 1, \ldots, m - 1 \), the (primal) pseudo-spline scheme can be defined in terms of its symbol as

\[
a_{m,\ell}(z) = 2\sigma^m(z)b_{m,\ell}(z), \quad b_{m,\ell}(z) = \sum_{k=0}^\ell \binom{m - 1 + k}{k} \delta^k(z),
\]

where

\[
\sigma(z) = \frac{(1 + z)^2}{4z}, \quad \delta(z) = -\frac{(1 - z)^2}{4z}.
\]

The Fourier transform of \( a_{m,\ell} \) is then

\[
A_{m,\ell}(\xi) = 2 \cos^{2m}(\xi/2) B_{m,\ell}(\xi),
\]

\[
B_{m,\ell}(\xi) = \sum_{k=0}^\ell \binom{m - 1/2 + k}{k} \sin^{2k}(\xi/2).
\]

These schemes can be viewed as a blend between the B-spline and Dubuc-Deslauriers schemes: when \( \ell = 0 \) the scheme is B-spline subdivision of degree \( 2m - 1 \) and when \( \ell = m - 1 \) the scheme is \((2m)\)-point Dubuc-Deslauriers subdivision. These schemes were introduced by Daubechies, Han, Ron, and Shen [2], and further studied by Dong and Shen in [6] and [7].

A family of ‘dual’ Dubuc-Deslauriers schemes was studied by Dyn, Floater, and Hormann [9] and generalized by Dyn, Hormann, Sabin, and Shen [5] to a family of dual pseudo-spline schemes defined by the symbol

\[
\tilde{a}_{m,\ell}(z) = \frac{1 + z}{z} \sigma^m(z)\tilde{b}_{m,\ell}(z), \quad \tilde{b}_{m,\ell}(z) = \sum_{k=0}^\ell \binom{m - 1/2 + k}{k} \delta^k(z),
\]

for integers \( m \geq 1 \) and \( \ell = 1, \ldots, m - 1 \). The Fourier transform of \( \tilde{a}_{m,\ell} \) is

\[
\tilde{A}_{m,\ell}(\xi) = 2e^{i\xi/2} \cos^{2m+1}(\xi/2) \tilde{B}_{m,\ell}(\xi),
\]

\[
\tilde{B}_{m,\ell}(\xi) = \sum_{k=0}^\ell \binom{m - 1/2 + k}{k} \sin^{2k}(\xi/2).
\]

Since both \( B \geq 1 > 0 \) and \( \tilde{B} \geq 1 > 0 \), Theorems [1] and [2] apply to both kinds of scheme.
Table 1: Regularities for $a_{m,l}$.

| $m$ | $l = 1$   | $l = 2$   | $l = 3$   | $l = 4$   | $l = 5$   | $l = 6$   | $l = 7$   |
|-----|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| 2   | 2         |           |           |           |           |           |           |
| 3   | 3.67807   | 2.83007   |           |           |           |           |           |
| 4   | 5.41504   | 4.34379   | 3.55113   |           |           |           |           |
| 5   | 7.19265   | 5.92502   | 4.19357   |           |           |           |           |
| 6   | 9         | 7.55781   | 6.43997   | 5.53250   | 4.77675   |           |           |
| 7   | 10.83007  | 9.23111   | 7.97187   | 6.93577   | 5.31732   |           |           |
| 8   | 12.67807  | 10.93702  | 9.54804   | 8.39272   | 7.41006   | 6.56398   | 5.82944   |

Example 2. The primal scheme $a_{4,3}(z)$ is the eight-point Dubuc-Deslauriers scheme. We can take $r = 7$ in the factorization (3) and we have $b(z) = b_{4,3}(z) = b_{-3}z^{-3} + \cdots + b_2z^3$, with

$$(b_{-3}, \ldots, b_3) = \frac{1}{16}(-5, 40, -131, 208, -131, 40, -5).$$

So $p = 3$ and the folded matrix $M$ in (17) has dimension 3. Thus $\rho$ is the largest root in absolute value of the cubic polynomial

$$\det(M - \lambda I) = \det \begin{bmatrix} b_0 - \lambda & 2b_2 & 0 \\ b_1 & b_1 + b_3 - \lambda & b_3 \\ b_2 & b_0 & b_2 - \lambda \end{bmatrix} = -\lambda^3 + 7\lambda^2 + \frac{217}{4} \lambda - 125,$$

which is $\rho \approx 10.91976$ and so the scheme has regularity $7 - \log_2(\rho) \approx 3.55113$.

Similarly, one can compute the regularities of the schemes $a_{m,l}(z)$ and $\tilde{a}_{m,l}(z)$ as the log$_2$ of algebraic numbers of degree at most $\ell$. These are shown, to five decimal places, in Tables 1 and 2 respectively for $1 \leq \ell < m \leq 8$. These numbers agree to four decimal places with those computed from a joint spectral radius in Dong, Dyn, and Hormann [4].

8.2 Comparisons

In order to make comparisons between the regularities of the various primal and dual pseudo-spline schemes, we will show that it is sufficient to make pointwise comparisons between their corresponding Fourier transforms. Consider two subdivision schemes defined by their Fourier transforms $A$ and $\tilde{A}$,
and suppose that for some integers \( r, \tilde{r} \geq 0 \),

\[
A(\xi) = 2 \cos^{r+1}(\xi/2)B(\xi),
\]

\[
\tilde{A}(\xi) = 2 \cos^{\tilde{r}+1}(\xi/2)\tilde{B}(\xi),
\]

where \( B \) and \( \tilde{B} \) are real and symmetric in \( \xi \) and \( B > 0 \) and \( \tilde{B} > 0 \). Let \( \gamma \) and \( \tilde{\gamma} \) be the respective regularities of the two schemes.

**Lemma 5.** If there is a constant \( C \geq 1 \) such that

\[
\tilde{B}(\xi) \leq CB(\xi), \quad \xi \in [-\pi, \pi],
\]

then

\[
\tilde{\gamma} \geq \gamma + \tilde{r} - r - \log_2 C.
\]

**Proof.** Applying (11) twice,

\[
\tilde{B}_j(\xi) = \tilde{B}(\xi)\tilde{B}(2\xi)\cdots\tilde{B}(2^j\xi) \leq C^j B_j(\xi),
\]

for all \( \xi \in [-\pi, \pi] \) and \( j \geq 0 \). Therefore,

\[
\tilde{b}_{j,0} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{B}_j(\xi) d\xi \leq C^j \frac{1}{2\pi} \int_{-\pi}^{\pi} B_j(\xi) d\xi = C^j b_{j,0}, \quad j \geq 0,
\]

and so

\[
\tilde{\rho} = \lim_{j \to \infty} (\tilde{b}_{j,0})^{1/j} \leq C \lim_{j \to \infty} (b_{j,0})^{1/j} = C \rho,
\]

from which the result follows since

\[
\gamma = r + 1 - \log_2 \rho, \quad \tilde{\gamma} = \tilde{r} + 1 - \log_2 \tilde{\rho}.
\]
As an example of the use of this lemma, suppose that \( A \) and \( \tilde{A} \) are the \( 2m \)- and \( 2(m + 1) \)-point Dubuc-Deslauriers schemes respectively, and that their regularities are \( \gamma \) and \( \tilde{\gamma} \) respectively. The lemma implies that \( \tilde{\gamma} \geq \gamma \) if \( \tilde{B}(\xi) \leq 4B(\xi) \) for all \( \xi \in [-\pi, \pi] \). In turns out that this latter inequality holds. This is part of the proof of the following more general result.

**Theorem 3.** Let \( \gamma_{m,\ell} \) be the regularity of the pseudo-spline scheme defined by \( a_{m,\ell} \), and let \( \gamma_m = \gamma_{m,m-1} \). Then

(i) \( \gamma_{m,\ell} \) is decreasing in \( \ell \), and moreover,

\[
\gamma_{m,\ell-1} - \log_2 \left( \frac{m + \ell}{\ell} \right) \leq \gamma_{m,\ell} \leq \gamma_{m,\ell-1},
\]

(ii) \( \gamma_{m,\ell} \) is increasing in \( m \), and moreover,

\[
\gamma_{m,\ell} + \log_2 \left( \frac{4m}{m + \ell} \right) \leq \gamma_{m+1,\ell} \leq \gamma_{m,\ell} + 2,
\]

(iii) \( \gamma_m \) is increasing in \( m \), and moreover,

\[
\gamma_m + \log_2 \left( \frac{2m + 2}{2m + 1} \right) \leq \gamma_{m+1} \leq \gamma_m + 2.
\]

**Proof.** Part (i) follows from applying Lemma 5 with \( r = \tilde{r} = 2m \). Since \( B_{m,\ell-1}(\xi) \leq B_{m,\ell}(\xi) \) for \( \xi \in [-\pi, \pi] \), the lemma implies the second inequality in (i). To prove the first inequality in (i) we look for a constant \( C \geq 1 \) such that

\[
B_{m,\ell}(\xi) \leq CB_{m,\ell-1}(\xi), \quad \xi \in [-\pi, \pi],
\]

or equivalently, such that

\[
p(s) := C \sum_{k=0}^{\ell-1} \binom{m - 1 + k}{k} s^k - \sum_{k=0}^{\ell} \binom{m - 1 + k}{k} s^k \geq 0, \quad 0 \leq s \leq 1.
\]

Letting

\[
c_k := (C - 1) \binom{m - 1 + k}{k}, \quad 0 \leq k \leq \ell - 1,
\]

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we can express \( p \) as
\[
p(s) = \sum_{k=0}^{\ell-1} c_k s^k - (m-1+\ell) s^\ell = \sum_{k=0}^{\ell-1} c_k (s^k - s^\ell) + c_\ell s^\ell,
\]
where
\[
c_\ell := \sum_{k=0}^{\ell-1} c_k - \binom{m-1+\ell}{\ell} = (C-1) \binom{m+\ell-1}{\ell-1} - \binom{m-1+\ell}{\ell}.
\]
For \( s \in [0,1] \), \( p(s) \geq 0 \) if \( c_k \geq 0 \), \( 0 \leq k \leq \ell \). Clearly, if \( C \geq 1 \), \( c_k \geq 0 \) for \( 0 \leq k \leq \ell-1 \), and since
\[
c_\ell = \frac{(m+\ell-1)!}{\ell! m!} (C-1) \ell - m,
\]
c\( \ell \geq 0 \) if \( C \geq (m+\ell)/\ell \). Thus (26) holds with \( C = (m+\ell)/\ell \), and Lemma 5 with this value of \( C \) gives the first inequality of (i).

To prove Part (ii), we apply Lemma 5 with \( r = 2m \) and \( \tilde{r} = 2m+2 \), in which case \( \tilde{r} - r = 2 \). Since \( B_{m+1,\ell}(\xi) \geq B_{m,\ell}(\xi) \) for \( \xi \in [\pi, \pi] \), the lemma implies the second inequality in (ii). To prove the first inequality in (ii) we look for a constant \( C \geq 1 \) such that
\[
B_{m+1,\ell}(\xi) \leq C B_{m,\ell}(\xi), \quad \xi \in [-\pi, \pi],
\]
or equivalently, such that
\[
p(s) := C \sum_{k=0}^{\ell} \binom{m-1+k}{k} s^k - \sum_{k=0}^{\ell} \binom{m+k}{k} s^k \geq 0, \quad 0 \leq s \leq 1.
\]
Since
\[
p(s) = \sum_{k=0}^{\ell} \frac{(m-1+k)!}{k! m!} (Cm - (m+k)) s^k,
\]
(26) holds with \( C = (m+\ell)/m \), and with this \( C \), Lemma 5 implies the first inequality in (ii).

To prove Part (iii), we again apply Lemma 5 with \( r = 2m \) and \( \tilde{r} = 2m+2 \), in which case \( \tilde{r} - r = 2 \). Since \( B_{m,m-1}(\xi) \leq B_{m+1,m}(\xi) \) for \( \xi \in [-\pi, \pi] \), the
lemma then implies the second inequality of (iii). To prove the first inequality we look for a constant $C \geq 1$ such that

$$B_{m+1,m}(\xi) \leq CB_{m,m-1}(\xi), \quad \xi \in [-\pi, \pi],$$

(27)

or equivalently, such that

$$p(s) := C \sum_{k=0}^{m-1} \left( \frac{m - 1 + k}{k} \right) s^k - \sum_{k=0}^{m} \left( \frac{m + k}{k} \right) s^k \geq 0, \quad 0 \leq s \leq 1.$$

Letting

$$c_k := C \left( \frac{m - 1 + k}{k} \right) - \left( \frac{m + k}{k} \right), \quad 0 \leq k \leq m - 1,$$

we can express $p$ as

$$p(s) = \sum_{k=0}^{m-1} c_k s^k - \left( \frac{2m}{m} \right) s^m = \sum_{k=0}^{m-1} c_k (s^k - s^m) + c_m s^m,$$

where

$$c_m := \sum_{k=0}^{m-1} c_k - \left( \frac{2m}{m} \right) = C \left( \frac{2m - 1}{m - 1} \right) - \left( \frac{2m + 1}{m} \right).$$

Similar to part (ii), we have $c_k \geq 0, 0 \leq k \leq m - 1$, if $C \geq (2m - 1)/m$. On the other hand,

$$c_m = \frac{(2m - 1)!}{(m - 1)!(m + 1)!} \left( C(m + 1) - 2(2m + 1) \right),$$

and so $c_m \geq 0$ if $C \geq 2(2m + 1)/(m + 1)$. Thus, (27) holds with $C = 2(2m + 1)/(m + 1)$, and Lemma 5 then yields the first inequality of (iii).

Similar comparisons can be made for the dual schemes, and also between the primal and dual ones. To see this observe that Lemma 5 also holds if $A$ in (24) is replaced by

$$A(\xi) = 2e^{i\xi/2} \cos^{r+1}(\xi/2) B(\xi),$$

with $B$ having the same properties as before, and the lemma also holds with a similar replacement of $\tilde{A}$ in (25).

Consider then the dual schemes.
Theorem 4. Let $\tilde{\gamma}_{m,\ell}$ be the regularity of the dual pseudo-spline scheme $\tilde{a}_{m,\ell}$, and let $\tilde{\gamma}_m = \tilde{\gamma}_{m,m-1}$. Then

(i) $\tilde{\gamma}_{m,\ell}$ is decreasing in $\ell$, and moreover,

$$\tilde{\gamma}_{m, \ell - 1} - \log_2 \left( \frac{m + \ell + 1/2}{\ell} \right) \leq \tilde{\gamma}_{m,\ell} \leq \tilde{\gamma}_{m, \ell - 1},$$

(ii) $\tilde{\gamma}_{m,\ell}$ is increasing in $m$, and moreover,

$$\tilde{\gamma}_{m+1, \ell} \leq \tilde{\gamma}_{m,\ell} + 2,$$

(iii) $\tilde{\gamma}_m$ is increasing in $m$, and moreover,

$$\tilde{\gamma}_m + \log_2 \left( \frac{4m(m+1/2)}{m + \ell + 1/2} \right) \leq \tilde{\gamma}_{m+1} \leq \tilde{\gamma}_m + 2.$$

Proof. The proof of Part (i) is similar to that of Theorem 3 but with $C$ replaced by $(m + \ell + 1/2)/\ell$. Part (ii) is also similar to that of Theorem 3 but with $C$ replaced by $(m + \ell + 1/2)/(m + 1/2)$. Part (iii) is again similar, but we now have $c_k \geq 0$ for $0 \leq k \leq m - 1$ if $C \geq (2m - 1/2)/(m + 1/2)$, and

$$c_m = C \left( \frac{2m - 1/2}{m - 1} \right) - \left( \frac{2m + 3/2}{m} \right) \geq 0$$

if $C \geq (2m + 1/2)(2m + 3/2)/(m(m + 3/2))$. \qed

Finally, we compare the regularities of the primal and dual pseudo-splines.

Theorem 5. For $m \geq 1$ and $0 \leq \ell \leq m - 1$,

$$\gamma_{m, \ell} + \log_2 \left( 2 \prod_{n=0}^{\ell-1} \frac{m + n}{m + 1/2 + n} \right) \leq \tilde{\gamma}_{m,\ell} \leq \gamma_{m,\ell} + 1,$$  \hspace{1cm} (28)

$$\tilde{\gamma}_{m+1, \ell} \leq \gamma_{m,\ell} + 1,$$  \hspace{1cm} (29)

where an empty product is understood to mean 1.
Proof. We only prove (28), since the proof of (29) is similar. We apply Lemma 5 with \( r = 2m - 1 \) and \( \tilde{r} = 2m \), in which case \( \tilde{r} - r = 1 \). Since \( \tilde{B}_{m, \ell}(\xi) \geq B_{m, \ell}(\xi) \) for \( \xi \in [-\pi, \pi] \), the lemma implies the second inequality in (28). To prove the first inequality in (28) we look for a constant \( C \geq 1 \) such that
\[
\tilde{B}_{m, \ell}(\xi) \leq CB_{m, \ell}(\xi), \quad \xi \in [-\pi, \pi],
\]
or equivalently, such that
\[
p(s) := \sum_{k=0}^{\ell} c_k s^k \geq 0, \quad 0 \leq s \leq 1,
\]
where
\[
c_k = C \binom{m - 1 + k}{k} - \binom{m - 1/2 + k}{k}.
\]
For any \( k = 0, \ldots, \ell \), one has \( c_k \geq 0 \) if and only if
\[
C \geq \binom{m - 1/2 + k}{k} / \binom{m - 1 + k}{k} = \prod_{n=0}^{k-1} \frac{m + 1/2 + n}{m + n}.
\]
So (30) holds if we take
\[
C = \prod_{n=0}^{l-1} \frac{m + 1/2 + n}{m + n},
\]
and applying Lemma 5 gives the first inequality in (28).

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