Generalized associative algebras

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Abstract

We study diverse parametrized versions of the operad of associative algebra, where the parameter are taken in an associative semigroup $\Omega$ (generalization of matching or family associative algebras) or in its cartesian square (two-parameters associative algebras). We give a description of the free algebras on these operads, study their formal series and prove that they are Koszul when the set of parameters is finite. We also study operadic morphisms between the operad of classical associative algebras and these objects, and links with other types of algebras (diassociative, dendriform, post-Lie...).

Keywords. Family associative algebras, matching associative algebras, two-parameters associative algebras, associative semigroups.

AMS classification. 16S10 18M60 20M75 16W99

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Introduction

Recently, numerous parametrization of well-known operads were introduced. Choosing a set of parameters \( \Omega \), any product defining the considered operad is replaced by a bunch of products indexed by \( \Omega \) and various relations are defined on them, mimicking the relations defining the initial operads. One can for example require that any linear span of the parametrized products also satisfy the relations of the initial operads: this is the matching parametrization. For example, matching Rota-Baxter algebras, associative, dendriform, prelie algebras are introduced in \[11\] \[12\].

Another way is to use one or more semigroup structures on \( \Omega \): this it the family parametrization. In this spirit, family Rota-Baxter algebras, dendriform, prelie algebras are introduced and studied in \[11\] \[34\] \[35\] \[24\]. A way to obtain both these parametrizations for dendriform algebras is introduced in \[14\], with the help of a generalization of diassociative semigroups, called extended diassociative semigroups (briefly, EDS). Finally, a two-parameters version is given for dendriform algebras and prelie algebras is described in \[17\].

Our aim in this paper is the study of these parametrizations for the operad of associative algebras, which surprisingly did not receive a lot of attention for now. We start with two-parameters associative algebras \[17\]. If \( (\Omega, \rightarrow) \) is a semigroup, an \( \Omega \)-two-parameters associative algebra is given products \( *_{\alpha,\beta} \), with \( \alpha, \beta \in \Omega \), satisfying the following axiom:

\[
(x *_{\alpha,\beta} y) *_{\alpha\rightarrow\beta,\gamma} z = x *_{\alpha,\beta\rightarrow\gamma} (y *_{\beta,\gamma} z).
\]

When \( (\Omega, *) = (\mathbb{Z}/2\mathbb{Z}, \times) \), the two-parameters \( \Omega \)-associative algebras were described in \[1\], as an operad on bicolored trees. When \( \Omega \) is finite, the associated operad \( \mathcal{A}_{\Omega}^2 \) is finitely generated and quadratic. We prove that it is Koszul (Proposition \[13\]), and describe its Poincaré-Hilbert formal series \( P(X) \): if \( |\Omega| = \omega > 1 \), then

\[
P(X) = \frac{1 - \omega X - \sqrt{1 + 2\omega(1 - 2\omega)X + \omega^2 X^2}}{2\omega(\omega - 1)}
\]

\[= X + \omega^2 X^2 + (2\omega - 1)\omega^3 X^3 + (5\omega^2 - 5\omega + 1)\omega^4 X^4 + (2\omega - 1)(7\omega^2 - 7\omega + 1)\omega^5 X^5 + (42\omega^4 - 84\omega^3 + 56\omega^2 - 14\omega + 1)\omega^6 X^6 + \ldots\]

We deduce a formula for the dimension \( p_n(\omega) \) of \( \mathcal{A}_{\Omega}^2(n) \) with the help of Narayana numbers, (Corollary \[13\]), as well as properties of \( p_n(\omega) \), seen as a polynomial in \( \omega \) (Corollary \[14\]). We also give a combinatorial description of Koszul dual of \( \mathcal{A}_{\Omega}^2 \) in terms of words (Proposition \[17\]) and a description of free \( \mathcal{A}_{\Omega}^{1,1} \)-algebras.

In the second and third parts of this paper, we introduce and study \( \Omega \)-associative algebras. Here, the set of parameters \( \Omega \) is given two operations \( \rightarrow \) and \( \triangleright \). An \( \Omega \)-associative algebra is given bilinear products \( *_{\alpha} \), with \( \alpha \in \Omega \), with the following axioms:

\[
x *_{\alpha} (y *_{\beta} z) = (x *_{\alpha\triangleright\beta} y) *_{\alpha\rightarrow\beta} z.
\]

In order to have a suitable parametrised operad, we impose that free \( \Omega \)-associative algebras are of the form

\[
T_A(V) = \bigoplus_{n=1}^{\infty} \mathbb{K}\Omega^\otimes(n-1) \otimes V^\otimes n.
\]

Tensors of \( \mathbb{K}\Omega^\otimes(n-1) \otimes V^\otimes n \) will be called \( A \)-typed words of length \( n \) and will be denoted \( \alpha_1 \ldots \alpha_{n-1}v_1 \ldots v_n \). We impose that the products \( *_{\alpha} \) satisfy, among other conditions, that for any \( v_1, v_2 \in V \),

\[
v_1 *_{\alpha} v_2 = \alpha v_1 v_2.
\]
We prove in Theorem 3.5 that this holds if, and only if, the triple $(\Omega, \to, \Vdash)$ satisfies the following axioms:

\[
\begin{align*}
\alpha \to (\beta \to \gamma) &= (\alpha \to \beta) \to \gamma, \\
(\alpha \Vdash (\beta \to \gamma)) \to (\beta \Vdash \gamma) &= (\alpha \to \beta) \Vdash \gamma, \\
(\alpha \Vdash (\beta \to \gamma)) \Vdash (\beta \Vdash \gamma) &= \alpha \Vdash \beta.
\end{align*}
\]

Such a triple $(\Omega, \to, \Vdash)$ will be called an extended associative semigroup (briefly, EAS). For example:

- If $\Omega$ is a set, its trivial EAS structure is given, for any $\alpha, \beta \in \Omega$,
  \[
  \alpha \to \beta = \beta \Vdash \alpha = \beta.
  \]
  In this case, the $\Omega$-family algebras are the matching associative algebras \([33]\); the particular case when $\Omega$ contains two elements appears also in \([29]\). The underlying operads are also used in \([6]\).

- If $(\Omega, \to)$ is a semigroup, one can make it an EAS with, for any $\alpha, \beta \in \Omega$,
  \[
  \alpha \Vdash \beta = \alpha.
  \]
  In this case, the $\Omega$-family algebras are the family associative algebras of \([34]\).

- If $(\Omega, \star)$ is a group, one can make it an EAS with, for any $\alpha, \beta \in \Omega$,
  \[
  \alpha \to \beta = \beta, \quad \alpha \Vdash \beta = \alpha \star \beta^{-1}.
  \]

We give more examples of EAS, including a classification of EAS of cardinality two, in the second section. We in fact generalize these results in a linear setting: we first observe that if $(\Omega, \to, \Vdash)$ is a set with two operations, we consider the map:

\[
\phi : \Omega^2 \to \Omega^2 \quad \text{by} \quad (\alpha, \beta) \mapsto (\alpha \to \beta, \alpha \Vdash \beta),
\]

Then $(\Omega, \to, \Vdash)$ is an EAS if, and only if:

\[
(\text{Id} \times \phi) \circ (\phi \times \text{Id}) \circ (\text{Id} \times \phi) = (\phi \times \text{Id}) \circ (\text{Id} \times \tau) \circ (\phi \times \text{Id}),
\]

where $\tau : \Omega^2 \to \Omega^2$ is the usual flip:

\[
\tau : \Omega^2 \to \Omega^2 \quad \text{by} \quad (\alpha, \beta) \mapsto (\beta, \alpha).
\]

This can easily be generalized in the category of vector spaces: a linear extended associative semigroup (briefly, $\ell$EAS) is a pair $(A, \Phi)$, where $\Phi : A \otimes A \to A \otimes A$ is a linear map such that:

\[
(\text{Id} \otimes \Phi) \circ (\Phi \otimes \text{Id}) \circ (\text{Id} \otimes \Phi) = (\Phi \otimes \text{Id}) \circ (\text{Id} \otimes \tau) \circ (\Phi \otimes \text{Id}),
\]

where $\tau : A \otimes A \to A \otimes A$ is the usual flip. In particular, if $(\Omega, \to, \Vdash)$ is an EAS, then its algebra $K\Omega$ is an $\ell$EAS. We then introduce the notion of $\Phi$-associative algebra (Definition 3.4) and we describe free $\Phi$-associative algebras $T_\Phi(V)$ in term of tensor algebras in Theorem 3.5. In particular, as a vector space,

\[
T_\Phi(V) = \bigoplus_{n=1}^{\infty} A^{\otimes (n-1)} \otimes V^\otimes n.
\]
We prove in Proposition 3.6 that if $V$ is a $\Phi$-associative algebra, then $V \otimes A$ is naturally an associative algebra; if $\Phi$ is invertible, we prove conversely that any convenient associative product on $V \otimes A$ gives rise to a $\Phi$-associative algebra structure on $V$. Following these results, we study the algebra structure of $T_\Phi(V) \otimes A$ and, if $\Phi$ is invertible, we prove that it is freely generated by $V \otimes A$ (Proposition 3.7).

The description of free $\Phi$-algebras induce a combinatorial description of the operad $\text{As}_\Phi$ of $\Phi$-associative algebras (Proposition 3.8). We prove that, when $A$ is finite-dimensional, that the operad $\text{As}_\Phi$ is Koszul, and that its Koszul dual is the operad of $\text{As}_\Phi^\ast$-algebras, generalizing a well-known result for the operad $\text{As}$ of "classical" associative algebras (Proposition 3.9 and Theorem 3.10). We study operad morphisms between the operad of associative algebras and $\text{As}_\Phi$, which is related to eigenvectors of $\Phi$ (Proposition 3.12). We then give results on operadic maps between the operads $\text{As}$ and $\text{As}_\Phi$, and between the operads $\text{As}^2_{\Omega}$ and $\text{As}_\Phi$ (Propositions 3.14 and 3.15). The paper ends with various links with other types of algebras, such that diassociative, post-Lie, dendriform, tridendriform or duplicial algebras, and their Koszul duals.

Acknowledgements. The author acknowledges support from the grant ANR-20-CE40-0007 Combinatoire Algébrique, Résurgence, Probabilités Libres et Opérades.

Notations 0.1. Let $K$ be a commutative field. Any vector space in this text will be taken over $K$.

1 Two-parameters $\Omega$-associative algebras

Notations 1.1. In all this section, $(\Omega, \to)$ is an associative semigroup.

1.1 Definition

In the spirit of the notion of two-parameters dendriform or duplicial algebras of [17], we now introduce the notion of two-parameters associative algebras, which can be found in [1]:

Definition 1.1. A two-parameters $\Omega$-associative algebra is a family $(V, (*_{\alpha,\beta})_{\alpha,\beta \in \Omega})$, where $V$ is a vector space and, for any $(\alpha, \beta) \in \Omega^2$, $*_{\alpha,\beta} : V \otimes V \to V$ is a linear map such that:

$$\forall \alpha, \beta, \gamma \in \Omega, \forall x, y, z \in V, \quad (x *_{\alpha,\beta} y) *_{\beta,\gamma} z = x *_{\alpha,\beta \to \beta,\gamma} (y *_{\beta,\gamma} z).$$

Remark 1.1. If $|\Omega| = 1$, $\Omega$-associative algebras are associative algebras.

Such a structure is related to $(\Omega, \to)$-graded associative products on $V \otimes K\Omega$. For the sake of simplicity, we shall denote the tensor product $x \otimes \alpha$, with $x \in V$ and $\alpha \in \Omega$, by $x\alpha$.

Proposition 1.2. Let $V$ be a vector space, endowed with bilinear products $*_{\alpha,\beta}$ for any $(\alpha, \beta) \in \Omega^2$. We define a product $*$ on $V \otimes K\Omega$ by:

$$\forall x, y \in V, \forall \alpha, \beta \in \Omega, \quad x\alpha \ast y\beta = x \ast_{\alpha,\beta} y(\alpha \to \beta).$$

Then $*$ is associative if, and only if, $(V, (*_{\alpha,\beta})_{\alpha,\beta \in \Omega})$ is a two-parameters $\Omega$-associative algebra.

Proof. For any $x, y, z \in V$, any $\alpha, \beta, \gamma \in \Omega$:

$$(x\alpha \ast y\beta) \ast z\gamma = (x *_{\alpha,\beta} y) *_{\alpha \to \beta,\gamma} z(\alpha \to \beta \to \gamma),$$

$$x\alpha \ast (y\beta \ast z\gamma) = x *_{\alpha,\beta \to \beta,\gamma} (y *_{\beta,\gamma} z)(\alpha \to \beta \to \gamma).$$

The result is then immediate.
1.2 The operad of two-parameters $\Omega$-associative algebras

We refer to [23, 25, 26, 27, 32] for notations and usual results on operads.

**Notations 1.2.** We denote by $\text{As}_\Omega^2$ the nonsymmetric operad of two-parameters $\Omega$-associative algebras. It is generated by $\ast_{\alpha, \beta} \in \text{As}_\Omega^2(2)$, with $\alpha, \beta \in \Omega$, and the relations

\[
\forall \alpha, \beta, \gamma \in \Omega, \quad \ast_{\alpha \to \beta, \gamma} \circ_1 \ast_{\alpha, \beta} = \ast_{\alpha, \beta \to \gamma} \circ_2 \ast_{\beta, \gamma}.
\]

We assume in this section that $\Omega$ is finite, of cardinality denoted by $\omega$. Then the components of $\text{As}_\Omega^2$ are finite-dimensional, and the following Proposition allows to inductively compute their dimension:

**Proposition 1.3.** The operad $\text{As}_\Omega^2$ is Koszul. For any $n \geq 1$, let us put $p_n = \dim_{K}(\text{As}_\Omega^2(n))$ and

\[
P(X) = \sum_{n=1}^{\infty} p_n X^n \in \mathbb{Q}[X].
\]

Then:

\[
p_n = \omega(\omega - 1) \sum_{k=1}^{n-1} p_k p_{n-k} + \omega p_{n-1},
\]

or equivalently, if $|\omega| \geq 2$:

\[
P(X) = \frac{1 - \omega X - \sqrt{1 + 2\omega(1-2\omega)X + \omega^2X^2}}{2\omega(\omega - 1)}.
\]

**Proof.** We shall use the rewriting method of [2, 23]. We shall write elements of the free nonsymmetric operad generated by $\text{As}_\Omega^2(2)$ as planar trees which vertices are decorated by elements of $\Omega^2$. We will write indices on the vertices on the trees and put the corresponding decorations between parentheses, and we delete the symbols $\ast$ in order to enlighten the notations. For example, the operadic tree

![Operadic Tree](image)

will be shortly written $\bigotimes\left(((\alpha, \beta), (\gamma, \delta))\right)$. The rewriting rules are:

\[
\bigotimes\left(((\alpha \to \beta, \gamma), (\alpha, \beta))\right) \longrightarrow \bigotimes\left(((\alpha, \beta \to \gamma), (\beta, \gamma))\right)
\]

for any $\alpha, \beta, \gamma \in \Omega$. There is only one family of critical monomials, namely the monomials

\[
\bigotimes\left(((\alpha \to \beta) \to \gamma, \delta), (\alpha \to \beta, \gamma), (\alpha, \beta)\right),
\]

or equivalently, if $|\omega| \geq 2$:

\[
P(X) = \frac{1 - \omega X - \sqrt{1 + 2\omega(1-2\omega)X + \omega^2X^2}}{2\omega(\omega - 1)}.
\]
where $\alpha, \beta, \gamma \in \Omega$. Koszularity of $A_{\Omega}^2$ comes from the confluence of the following diagram:

\[
\begin{array}{c}
T_1 \\
\downarrow \\
T_2 \\
\downarrow \\
T_3 \\
\downarrow \\
T_4 \\
\downarrow \\
T_5
\end{array}
\]

with:

\[
\begin{align*}
T_1 &= \prod_{\omega} \left( (\alpha \to \beta) \to (\gamma, \delta), (\alpha \to \beta, \gamma), (\alpha, \beta) \right), \\
T_2 &= \prod_{\omega} \left( (\alpha \to \beta, \gamma) \to (\delta), (\alpha, \beta), (\gamma, \delta) \right), \\
T_3 &= \prod_{\omega} \left( (\alpha \to (\beta \to \gamma)) \to (\delta), (\alpha, \beta \to \gamma), (\beta, \gamma) \right), \\
T_4 &= \prod_{\omega} \left( (\alpha, (\beta \to \gamma)) \to (\delta), (\beta \to \gamma, \delta), (\gamma, \delta) \right), \\
T_5 &= \prod_{\omega} \left( (\alpha, (\beta \to \gamma)) \to (\delta), (\beta, \gamma \to \delta), (\gamma, \delta) \right).
\end{align*}
\]

Hence, the operad $A_{\Omega}^2$ is Koszul. Moreover, $A_{\Omega}^2(n)$ has for basis the set of rooted planar binary trees with $n - 1$ internal vertices decorated by $\Omega^2$, avoiding subtrees

\[
\prod_{\omega} \left( (\alpha \to \beta, \gamma), (\alpha, \beta) \right)
\]

for any $\alpha, \beta, \gamma \in \Omega$. For any planar binary tree $T$, let us denote by $p_T$ the number of decorations of the vertices of trees by elements of $\Omega^2$, avoiding these subtrees. If $T$ is a planar binary tree different from the tree $|$ (which is the unit of the operad $A_{\Omega}^2$), we denote by $T_l$ the left subtree born from the root of $T$, by $T_r$ the right subtree born from the root of $T$, and we write $T = T_l \vee T_r$. Then, looking at the possible decorations of the root:

\[
p_T = p_{T_l} p_{T_r} \omega \times \begin{cases} 
\omega & \text{if } T_2 = |, \\
\omega - 1 & \text{otherwise.}
\end{cases}
\]

Hence, if $n \geq 2$, denoting by $T_n$ the set of planar binary rooted trees with $n - 1$ internal vertices:

\[
p_n = \sum_{T \in T_n} p_T \\
= \omega^2 \sum_{T \in T_{n-1}} p_T + \omega(\omega - 1) \sum_{k=2}^{n-1} \sum_{T_l \in T_k} \sum_{T_r \in T_{n-k}} p_{T_l} p_{T_r} \\
= \omega^2 p_{n-1} + \omega(\omega - 1) \sum_{k=2}^{n-1} p_k p_{n-k} \\
= \omega(\omega - 1) \sum_{k=1}^{n-1} p_k p_{n-k} + \omega p_{n-1},
\]

\[
\]
which gives (1). Summing over \( n \), with \( p_1 = 1 \):

\[
P(X) = \omega(\omega - 1)P(X)^2 + \omega XP(X) + X.
\]  

(3)

if \( \omega = 1 \), we obtain that

\[
P(X) = XP(X) + X,
\]

so

\[
P(X) = \frac{X}{1 + X} = \sum_{n=1}^{\infty} X^n,
\]

recovering the formal series of the nonsymmetric operad of associative algebras. If \( \omega \geq 2 \), solving (3), with the initial condition \( P(0) = 0 \), we obtain (2).

\[\square\]

**Example 1.1.** We obtain:

\[
\begin{align*}
p_2(\omega) &= \omega^2, \\
p_3(\omega) &= (2\omega - 1)\omega^3, \\
p_4(\omega) &= (5\omega^2 - 5\omega + 1)\omega^4, \\
p_5(\omega) &= (2\omega - 1)(7\omega^2 - 7\omega + 1)\omega^5, \\
p_6(\omega) &= (42\omega^3 - 84\omega^3 + 56\omega^2 - 14\omega + 1)\omega^6, \\
p_7(\omega) &= (2\omega - 1)(66\omega^4 - 132\omega^3 + 84\omega^2 - 18\omega + 1)\omega^7, \\
p_8(\omega) &= (429\omega^6 - 1287\omega^5 + 1485\omega^4 - 825\omega^3 + 225\omega^2 - 27\omega + 1)\omega^8, \\
p_9(\omega) &= (2\omega - 1)(715\omega^6 - 2145\omega^5 + 2431\omega^4 - 1287\omega^3 + 319\omega^2 - 33\omega + 1)\omega^9.
\end{align*}
\]

This gives:

\[
\begin{array}{cccccccc}
\omega\backslash n & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 1 & 4 & 24 & 176 & 1440 & 12908 & 115584 \\
3 & 1 & 9 & 135 & 2511 & 52245 & 1164213 & 27173475 \\
4 & 1 & 16 & 448 & 15616 & 609280 & 25464832 & 1114882048 \\
5 & 1 & 25 & 1125 & 63125 & 3965625 & 266890625 & 18816328125 \\
6 & 1 & 36 & 2376 & 195696 & 18048096 & 1783238976 & 18457081536 \\
7 & 1 & 49 & 4459 & 506611 & 64454845 & 8785674373 & 125454699679 \\
8 & 1 & 64 & 7680 & 1150976 & 193167360 & 34733293568 & 6542642380800 \\
9 & 1 & 81 & 12393 & 2368521 & 506935665 & 116245810017 & 27925350157593 \\
\end{array}
\]

**Remark 1.2.** 1. If \( \omega = 2 \), the sequence \((p_n)_{n \geq 2} \) is referenced as \([A156017]\) in the OEIS. This is the sequence of dimensions of an operad given in [4], generated by four products \( \circ \), \( \triangleright \), \( \omega \) and \( \varnothing \), with eight relations, see (21) in [4]. This is a special example of a type of two-parameters \( \Omega \)-associative algebras, with \( \Omega = (\mathbb{Z}/2\mathbb{Z}, \times) \) and:

\[
\begin{align*}
\ast_{\omega\omega} &= \circ, \\
\ast_{\omega\triangleright} &= \omega, \\
\ast_{\triangleright\omega} &= \triangleright, \\
\ast_{\varnothing\omega} &= \varnothing.
\end{align*}
\]

Moreover:

\[
P(X)|_{\omega=2} = \frac{1 - 2X - \sqrt{1 - 6(2X) + (2X)^2}}{4}
\]

so for any \( n \geq 2 \), \( p_n = 2^{n-1}\text{schr}_n \), where \( \text{schr}_n \) is the \( n \)-th large Schröder number (sequence \([A006318]\) in the OEIS):

\[
\begin{array}{cccccccccccc}
n & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\text{schr}_n & 1 & 2 & 6 & 22 & 90 & 394 & 1806 & 8558 & 41586 & 206098 \\
\end{array}
\]

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Corollary 1.4. Let \( n \geq 1 \).

1. \( p_n \) is a polynomial in \( \mathbb{Z} [\omega] \), of degree \( 2n - 2 \). Its leading term is the \( n \)-th Catalan number \( \text{cat}_n \) (Sequence A000108 in the OEIS):

\[
\begin{array}{c|cccccccccc}
  n & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
  \text{cat}_n & 1 & 1 & 2 & 5 & 14 & 42 & 132 & 429 & 1430 & 4862 \\
\end{array}
\]

2. If \( n \geq 2 \), there exists a polynomial \( q_n \in \mathbb{Z} [\omega] \), such that \( p_n = \omega^n q_n \). Moreover, \( q_n(0) = (-1)^n \).

3. If \( n \) is odd and \( \geq 3 \), then \( p_n \left( \frac{1}{2} \right) = 0 \).

Proof. 1. and 2. We proceed by induction on \( n \). As \( p_1 = 1 \), this is obvious. Let us assume the result at all ranks \( < n \), with \( n \geq 2 \). The induction hypothesis gives that the following is a polynomial in \( \mathbb{Z} [\omega] \), of degree \( 2n - 2 \):

\[
\omega(\omega - 1) \sum_{k=1}^{n-1} p_k p_{n-k}.
\]

Its leading term is

\[
\sum_{k=1}^{n-1} \text{cat}_k \text{cat}_{n-k} = \text{cat}_n.
\]

We also obtain that \( \omega p_{n-1} \) is a polynomial in \( \mathbb{Z} [\omega] \), of degree \( 2n - 3 \). Summing in (1), we obtain the first point for \( p_n \). Still by (1):

\[
p_n = \omega(\omega - 1) \sum_{k=2}^{n-2} p_k p_{n-k} + 2\omega(\omega - 1) p_{n-1} + \omega p_{n-1}
\]

\[
= \omega(\omega - 1) \sum_{k=2}^{n-2} p_k p_{n-k} + \omega(2\omega - 1) p_{n-1}
\]

\[
= \omega^{n+1}(\omega - 1) \sum_{k=2}^{n-2} q_k q_{n-k} + \omega^n(2\omega - 1) q_{n-1}
\]

\[
= \omega^n \left( \omega(\omega - 1) \sum_{k=2}^{n-2} q_k q_{n-k} + (2\omega - 1) q_{n-1} \right) q_n.
\]

Moreover, \( q_n(0) = 0 - q_{n-1}(0) = (-1)^n \), which proves the second point.

3. For \( \omega = \frac{1}{2} \), we obtain:

\[
P(X)\big|_{\omega=\frac{1}{2}} = X - 2 + \sqrt{4 + X^2} = X + 2 \sum_{k=0}^{\infty} \frac{(-1)^k(2k - 2)!}{2^{2k-1}k!(k-1)!} X^{2k}.
\]

Corollary 1.5. For any \( n \geq 2 \),

\[
p_n = \frac{\omega^n}{n - 1} \left( \sum_{k=1}^{n-1} \binom{n-1}{k} \binom{n-1}{k-1} \omega^{n-1-k} (\omega - 1)^{k-1} \right).
\]
Proposition 1.6. Koszul dual

More precisely, with the notations of [5, Definition (1.13)]:

As 1.3 Koszul dual of In all this paragraph, Rem

Proof. Let us consider the Narayana numbers [31]:

Then:

These numbers appear in [5], where they are interpreted in terms of Catalan paths. More precisely, with the notations of [5 Definition (1.13)]:

Remark 1.3. These numbers appear in [3], where they are interpreted in terms of Catalan paths. More precisely, with the notations of [5 Definition (1.13)]:

Proof. Let us consider the Narayana numbers [31]:

and their formal series

\[ N(z, t) = \sum_{k,n \geq 1} N(n, k) z^n t^{k-1} = \frac{1 - z(t + 1) - \sqrt{1 - 2z(t + 1) + z^2(t - 1)^2}}{2tz}. \]

Then:

\[ P(X) = X + XN \left( \omega^2 X, \frac{\omega - 1}{\omega} \right) \]

\[ = X + \sum_{k,n \geq 1} N(n, k) \omega^{2n} X^{n+1} \left( \frac{\omega - 1}{\omega} \right)^{k-1} \]

\[ = X + \sum_{n=2}^{\infty} \left( \sum_{k=1}^{n-1} N(n - 1, k) \omega^{2n-2} \left( \frac{\omega - 1}{\omega} \right)^{k-1} \right) X^n \]

\[ = X + \sum_{n=2}^{\infty} \left( \sum_{k=1}^{n-1} N(n - 1, k) \omega^{2n-1-k} (\omega - 1)^{k-1} \right) X^n. \]

\[ \square \]

1.3 Koszul dual of As^2_{\Omega}

In this paragraph, \((\Omega, \rightarrow)\) is a finite semigroup.

Proposition 1.6. Koszul dual \(\text{As}_{\Omega}^{2!}\) of the operad \(\text{As}_{\Omega}^2\) is the quotient of \(\text{As}_{\Omega}^2\) by the trees

\[ \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\alpha, \beta, \gamma, \delta
\end{array}
\end{array}
\end{array} \]

\[ \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\alpha, \beta, \gamma, \delta
\end{array}
\end{array}
\end{array} \]

\[ \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\alpha, \beta, \gamma, \delta
\end{array}
\end{array}
\end{array} \]

In other terms, a \(\text{As}_{\Omega}^{2!}\)-algebra is a family \((V, *_{\alpha, \beta})_{\alpha, \beta \in \Omega}\), where \(V\) is a vector space and, for any \((\alpha, \beta) \in \Omega^2\), \(*_{\alpha, \beta} : V \otimes V \rightarrow V\) is a linear map such that:

\[ \forall \alpha, \beta, \gamma \in \Omega, \forall x, y, z \in V, \quad (x *_{\alpha, \beta} y) *_{\alpha \rightarrow \beta, \gamma} z = x *_{\alpha, \beta \rightarrow \gamma} (y *_{\beta, \gamma} z), \]

\[ \forall \alpha, \beta, \gamma, \delta \in \Omega, \forall x, y, z \in V, \quad (x *_{\alpha, \beta} y) *_{\gamma, \delta} z = 0 \text{ if } \alpha \rightarrow \beta \neq \gamma, \]

\[ x *_{\alpha, \beta} (y *_{\gamma, \delta} z) = 0 \text{ if } \beta \neq \gamma \rightarrow \delta. \]

For any \(n \geq 2\), \(\dim_{\mathbb{K}}(\text{As}_{\Omega}^{2!}(n)) = \omega^n\).

Proof. The presentation of \(\text{As}_{\Omega}^{2!}\) comes from a direct computation. Let us denote by \(Q(X)\) the Poincaré-Hilbert formal series of \(\text{As}_{\Omega}^{2!}\). As \(\text{As}_{\Omega}^2\) is Koszul, \(Q(X)\) is the inverse for the composition of \(-P(-X)\). From [3]:

\[ X = \frac{P(X) - \omega(\omega - 1)P(X)^2}{\omega P(X) + 1} = \frac{-P(-X) + \omega(\omega - 1)(-P(-X))^2}{1 - \omega(-P(-X))}, \]

9
Let us prove that $I$

A direct induction then proves that $w$

Let us prove the associativity: for any word $P$

So

3ě

For any word $P$

Proof.

Let us give a combinatorial presentation of $As_{\Omega}^{2}$:

Proposition 1.7. For any $n \geq 1$, let us put $P(n) = (\mathbb{K}\Omega)^{\otimes n}$. Elements of $P(n)$ are linear spans of words $\alpha_{1} \ldots \alpha_{n}$ in $\Omega$. Then $P = (P(n))_{n \geq 1}$ is given a structure of nonsymmetric operad with the following composition: for any $\alpha_{1}, \ldots, \alpha_{n} \in \Omega$, for any $\beta_{i,j} \in \Omega$,

$\alpha_{1} \ldots \alpha_{n} \circ (\beta_{1,1} \ldots \beta_{1,k_{1}}, \ldots, \beta_{n,1} \ldots \beta_{n,k_{n}}) = \left( \prod_{i=1}^{n} \delta_{\alpha_{i},\beta_{i,1} \ldots \beta_{i,k_{i}}} \right) \beta_{1,1} \ldots \beta_{1,k_{1}} \ldots \beta_{n,1} \ldots \beta_{n,k_{n}}.$

The unit is:

$I = \sum_{\alpha \in \Omega} \alpha$.

We define a suboperad $P_{0}$ isomorphic to $As_{\Omega}^{2}$ by:

$P_{0}(n) = \begin{cases} \mathbb{K}I & \text{if } n = 1, \\ P(n) & \text{if } n \geq 2. \end{cases}$

Proof. For any word $w = \alpha_{1} \ldots \alpha_{n}$ in $\alpha$, we put $|w| = \alpha_{1} \rightarrow \ldots \rightarrow \alpha_{n}$. The composition $\circ$ can be rewritten in the following way: for any words $w, w_{1}, \ldots, w_{n}$ with letters in $\Omega$, $w$ being of length $n$,

$w \circ (w_{1}, \ldots, w_{n}) = \delta_{w,|w_{1}| \ldots |w_{n}|} w_{1} \ldots w_{n}$.

Let us prove the associativity: for any word $w$ of length $n$, $w_{i}$, $1 \leq i \leq n$ of respective lengths $k_{i}$, $w_{i,j}$ with $1 \leq i \leq n$ and $1 \leq j \leq k_{i}$, all with letters in $\Omega$:

$w \circ (w_{1} \circ (w_{1}, \ldots, w_{1,k_{1}}), \ldots, w_{n} \circ (w_{n}, \ldots, w_{n,k_{n}}))$

$= \left( \prod_{i=1}^{n} \delta_{w_{i},|w_{i}| \ldots |w_{i,k_{i}}|} \delta_{w_{i,k_{i}}|w_{i+1}| \ldots |w_{n,k_{n}}|} \right) w_{1,1} \ldots w_{n,k_{n}}$

$= (\delta_{w_{1},w_{n},|w_{1}| \ldots |w_{n,k_{n}}|} \delta_{w_{1}|w_{1}| \ldots |w_{n}|} w_{1,1} \ldots w_{n,k_{n}})$

$= (w \circ (w_{1}, \ldots, w_{n})) \circ (w_{1,1}, \ldots, w_{n,k_{n}}).$

Let us prove that $I$ is a unit. Let $w = \alpha_{1} \ldots \alpha_{n}$ be a word with letters in $\Omega$.

$I \circ \alpha_{1} \ldots \alpha_{n} = \sum_{\alpha \in \Omega} \delta_{\alpha,|\alpha_{1} \ldots \alpha_{n}|} \alpha_{1} \ldots \alpha_{n} = \alpha_{1} \ldots \alpha_{n},$

$\alpha_{1} \ldots \alpha_{n} \circ (I, \ldots, I) = \sum_{\beta_{1}, \ldots, \beta_{n} \in \Omega} \prod_{i=1}^{n} \delta_{\alpha_{i},\beta_{i}} \beta_{1} \ldots \beta_{n}$

$= \alpha_{1} \ldots \alpha_{n}.$

So $P$ is an operad. Obviously, $P_{0}$ is a suboperad. Moreover, for any word $\alpha_{1} \ldots \alpha_{n}$ of length $\geq 3$:

$\alpha_{1} \ldots \alpha_{n} = (\alpha_{1} \rightarrow \alpha_{2}) \alpha_{3} \ldots \alpha_{n} \circ (\alpha_{1} \alpha_{2}, I, \ldots, I).$

A direct induction then proves that $P_{0}$ is generated by $P(2)$. Moreover, for any $\alpha, \beta, \gamma, \delta \in \Omega$:

$(\alpha \rightarrow \beta) \gamma \circ (\alpha \beta, I) = \alpha (\beta \rightarrow \gamma) \circ (I, \beta \gamma) = \alpha \beta \gamma,$

$\gamma \delta \circ (\alpha \beta, I) = 0$ if $\alpha \rightarrow \beta \neq \gamma,$

$\alpha \beta \circ (I, \gamma \delta) = 0$ if $\gamma \rightarrow \delta \neq \beta.$

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Therefore, the relations defining $\mathcal{A}s_{\Omega}^{2,1}$ are satisfied in $P_0$. Hence, there exists a surjective operad morphism:

$$\begin{cases} \mathcal{A}s_{\Omega}^{2,1} \longrightarrow P_0 \\ *_{\alpha,\beta} \longrightarrow \alpha \beta. \end{cases}$$

Comparing the formal series of $\mathcal{A}s_{\Omega}^{2,1}$ and $P_0$, we deduce that this is an isomorphism. □

**Corollary 1.8.** Let $V$ be a vector space. The free $\mathcal{A}s_{\Omega}^{2,1}$-algebra generated by $V$ is:

$$T_{\Omega}^2(V) = V \oplus \bigoplus_{n=2}^{\infty} (K\Omega \otimes V)^{\otimes n}.$$  

The product $*_{\alpha,\beta}$ is given in the following way: for any $u, v \in V$, for any $\alpha_1 u_1, \ldots, \alpha_k u_k$, $\beta_1 v_1, \ldots, \beta_l v_l \in K\Omega \otimes V$, with $k, l \geq 1$,

$$u *_{\alpha,\beta} v = \alpha u \beta v,$$

$$\alpha_1 u_1 \ldots \alpha_k u_k *_{\alpha,\beta} v = (\delta_{\alpha,\alpha_1 \rightarrow \ldots \rightarrow \alpha_k}) \alpha_1 u_1 \ldots \alpha_k u_k \beta v,$$

$$u *_{\alpha,\beta} \beta_1 v_1 \ldots \beta_l v_l = (\delta_{\beta,\beta_1 \rightarrow \ldots \rightarrow \beta_l}) \alpha u \beta_1 v_1 \ldots \beta_l v_l,$$

$$\alpha_1 u_1 \ldots \alpha_k u_k *_{\alpha,\beta} \beta_1 v_1 \ldots \beta_l v_l = (\delta_{\alpha,\alpha_1 \rightarrow \ldots \rightarrow \alpha_k} \delta_{\beta,\beta_1 \rightarrow \ldots \rightarrow \beta_l}) \alpha_1 u_1 \ldots \alpha_k u_k \beta_1 v_1 \ldots \beta_l v_l.$$

## 2 Extended associative semigroups

We here give the definition and few examples of extended associative semigroups. More results can be found in [16].

### 2.1 Definition and examples

**Definition 2.1.** An associative extended semigroup (briefly, EAS) is a triple $(\Omega, \rightarrow, \triangleright)$, where $\Omega$ is a nonempty set and $\rightarrow, \triangleright : \Omega^2 \longrightarrow \Omega$ are maps such that, for any $\alpha, \beta, \gamma \in \Omega$:

$$\alpha \rightarrow (\beta \rightarrow \gamma) = (\alpha \rightarrow \beta) \rightarrow \gamma,$$

$$\alpha \triangleright (\beta \rightarrow \gamma) \rightarrow (\beta \triangleright \gamma) = (\alpha \rightarrow \beta) \triangleright \gamma,$$

$$\alpha \triangleright (\beta \rightarrow \gamma) \triangleright (\beta \triangleright \gamma) = \alpha \triangleright \beta.$$  

**Example 2.1.**

1. Let $\Omega$ be a set. We put:

$$\forall \alpha, \beta \in \Omega, \begin{cases} \alpha \rightarrow \beta = \beta, \\
\alpha \triangleright \beta = \alpha. \end{cases}$$

Then $(\Omega, \rightarrow, \triangleright)$ is an EAS, denoted by $\text{EAS}(\Omega)$.

2. Let $(\Omega, \ast)$ be an associative semigroup. We put:

$$\forall \alpha, \beta \in \Omega, \quad \alpha \triangleright \beta = \alpha.$$ 

Then $(\Omega, \ast, \triangleright)$ is an EAS, which we denote by $\text{EAS}(\Omega, \ast)$.

3. Let $(\Omega, \ast)$ be a group. We put, for any $\alpha, \beta \in \Omega$:

$$\alpha \rightarrow \beta = \beta, \quad \alpha \triangleright \beta = \alpha \ast \beta^{-1}.$$ 

Then $(\Omega, \rightarrow, \triangleright)$ is an EAS, denoted by $\text{EAS}'(\Omega, \ast)$.  

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Definition 2.2. Let \((\Omega, \to, \rhd)\) be an EAS. We shall say that it is nondegenerate if the following map is bijective:

\[
\phi : \begin{cases}
\Omega^2 & \longrightarrow & \Omega^2 \\
(\alpha, \beta) & \longrightarrow & (\alpha \to \beta, \alpha \rhd \beta).
\end{cases}
\]

Example 2.2. 1. Let \(\Omega\) be a set. In \(\text{EAS}(\Omega)\), for any \(\alpha, \beta \in \Omega\), \(\phi(\alpha, \beta) = (\beta, \alpha)\), so \(\text{EAS}(\Omega)\) is nondegenerate, and \(\phi^{-1} = \phi\).

2. Let \((\Omega, *)\) be a group. Then \(\text{EAS}(\Omega, *)\) is nondegenerate. Indeed, for any \(\alpha, \beta \in \Omega\), \(\phi(\alpha, \beta) = (\alpha * \beta, \alpha)\), so \(\phi\) is a bijection, of inverse given by \(\phi^{-1}(\alpha, \beta) = (\beta, \beta^{-1} * \alpha)\).

3. Let \((\Omega, *)\) be an associative semigroup with the right inverse condition. Then \(\text{EAS}'(\Omega, *)\) is nondegenerate. Indeed, for any \(\alpha, \beta \in \Omega\), \(\phi(\alpha, \beta) = (\beta, \alpha \rhd \beta)\), so \(\phi\) is a bijection, of inverse given by \(\phi^{-1}(\alpha, \beta) = (\beta * \alpha, \alpha)\).

2.2 EAS of cardinality two

Here is a classification of EAS of cardinality two. The underlying set is \(\Omega = \{a, b\}\) and the products will be given by two matrices

\[
\begin{pmatrix}
a \to a & a \to b \\
b \to a & b \to b
\end{pmatrix}, \quad \begin{pmatrix}
a \rhd a & a \rhd b \\
b \rhd a & b \rhd b
\end{pmatrix}.
\]

We shall use the two maps:

\[
\pi_a : \begin{cases}
\Omega & \longrightarrow & \Omega \\
\alpha & \longrightarrow & a,
\end{cases} \quad \pi_b : \begin{cases}
\Omega & \longrightarrow & \Omega \\
\alpha & \longrightarrow & b.
\end{cases}
\]

We respect the indexation of EDS of [14].
\[
\begin{array}{|c|c|c|c|}
\hline
\text{Case} & \rightarrow & \leftarrow & \text{Description} \\
\hline
A1 & (a \ a) & (a \ a) & \text{EAS}(\Omega, \leftarrow) \\
A2 & (a \ a) & (a \ a) & \text{EAS}(\Omega, \rightarrow) \\
C1 & (a \ a) & (a \ a) & \text{EAS}(\Omega, \pi_a) \\
C3 & (a \ a) & (a \ a) & \text{EAS}(Z/2Z, \times) \\
C5 & (a \ a) & (a \ a) & \text{EAS}(Z/2Z, \pi_a) \\
C6 & (a \ a) & (a \ a) & \text{EAS}(\pi_a) \\
E1' – E2' & (a \ a) & (a \ a) & \text{EAS}(\rightarrow) \\
E3' & (a \ a) & (a \ a) & \text{EAS}(\rightarrow) \\
F1 & (a \ a) & (a \ a) & \text{EAS}(\rightarrow) \\
F3 & (a \ a) & (a \ a) & \text{EAS}(\rightarrow) \\
F4 & (a \ a) & (a \ a) & \text{EAS}(\rightarrow) \\
H1 & (a \ a) & (a \ a) & \text{EAS}(\rightarrow) \\
H2 & (a \ a) & (a \ a) & \text{EAS}(\rightarrow) \\
\hline
\end{array}
\]

The nondegenerate EAS are \( F_3, F_4 \) and \( H_2 \).

3 Generalized associative algebras

3.1 Discrete and linear versions

**Definition 3.1.** Let \((\Omega, \rightarrow, \leftarrow)\) be a set with two binary operations. Let \((V, \ast_\alpha)_{\alpha \in \Omega}\) be a family such that \(V\) is a vector space and, for any \(\alpha \in \Omega\), \(\ast_\alpha : V \otimes V \to V\) is a linear map. We shall say that it is an \(\Omega\)-associative algebra if:

\[
\forall x, y, z \in V, \forall \alpha, \beta \in \Omega, \quad x \ast_\alpha (y \ast_\beta z) = (x \ast_\alpha y) \ast_\beta z. \quad (7)
\]

**Example 3.1.**

1. If \(\Omega\) is a set, for \(\text{EAS}(\Omega)\), (7) becomes:

\[
\forall x, y, z \in V, \forall \alpha, \beta \in \Omega, \quad x \ast_\alpha (y \ast_\beta z) = (x \ast_\alpha y) \ast_\beta z.
\]

As a consequence, any linear span of \(\ast_\alpha\) is associative. We recover the notion of matching associative algebra [33].

2. If \((\Omega, \ast)\) is a semigroup, for \(\text{EAS}(\Omega, \ast)\), (7) becomes:

\[
\forall x, y, z \in V, \forall \alpha, \beta \in \Omega, \quad x \ast_\alpha (y \ast_\beta z) = (x \ast_\alpha y) \ast_{\alpha \beta} z.
\]

These are \((\Omega, \ast)\)-family associative algebras.
Remark 3.1. This does not include the multiassociative and the dual multiassociative algebras introduced by Giraudo in [18]. We shall see that the dimension of the \(n\)-th components of the operad of \(\Omega\)-associative algebras is \(|\Omega|^{n-1}\) for any \(n \geq 2\), whereas it is constant for dual multiassociative algebras and described by Narayana numbers for dual multiassociative algebras.

In order to linearize these axioms, let us first consider the following lemma, proved in [15]:

**Lemma 3.2.** Let \((\Omega, \rightarrow, \Rightarrow)\) be a set with two binary operations. We consider the maps

\[
\phi : \begin{cases} 
\Omega^2 & \rightarrow & \Omega^2 \\
(\alpha, \beta) & \rightarrow & (\alpha \rightarrow \beta, \alpha \Rightarrow \beta),
\end{cases} \quad \tau : \begin{cases} 
\Omega^2 & \rightarrow & \Omega^2 \\
(\alpha, \beta) & \rightarrow & (\beta, \alpha).
\end{cases}
\]

Then \((\Omega, \rightarrow, \Rightarrow)\) is an EAS if, and only if:

\[(\text{Id} \times \phi) \circ (\phi \times \text{Id}) \circ (\text{Id} \times \phi) = (\phi \times \text{Id}) \circ (\text{Id} \times \tau) \circ (\phi \times \text{Id}).\]  

(8)

This naturally leads to the following definition:

**Definition 3.3.** Let \(A\) be a vector space and let \(\Phi : A \otimes A \rightarrow A \otimes A\) be a linear map. We shall say that \((A, \Phi)\) is a linear extended associative semigroup (briefly, \(\ell\)EAS) if:

\[(\text{Id} \otimes \Phi) \circ (\Phi \otimes \text{Id}) \circ (\text{Id} \otimes \Phi) = (\Phi \otimes \text{Id}) \circ (\text{Id} \otimes \tau) \circ (\Phi \otimes \text{Id}),\]  

(9)

where \(\tau : A \otimes A \rightarrow A \otimes A\) is the usual flip:

\[
\tau : \begin{cases} 
A \otimes A & \rightarrow & A \otimes A \\
a \otimes b & \rightarrow & b \otimes a.
\end{cases}
\]

We shall say that \((A, \Phi)\) is nondegenerate if \(\Phi\) is invertible.

**Example 3.2.** 1. Let \((\Omega, \rightarrow, \Rightarrow)\) be an EAS and let \(A = \mathbb{K}\Omega\) be its algebra, that is to say the vector space generated by \(\Omega\). We define:

\[
\Phi : \begin{cases} 
\mathbb{K}\Omega \otimes \mathbb{K}\Omega & \rightarrow & \mathbb{K}\Omega \otimes \mathbb{K}\Omega \\
\alpha \otimes \beta & \rightarrow & (\alpha \rightarrow \beta) \otimes (\alpha \Rightarrow \beta),
\end{cases}
\]

where \(\alpha, \beta \in \Omega\). Lemma 3.2 implies that \((\mathbb{K}\Omega, \Phi)\) is an \(\ell\)EAS, which we call the linearization of \((\Omega, \rightarrow, \Rightarrow)\).

2. Here are examples of \(\ell\)EAS of dimension 2, which are not linearization of an EAS. In these examples, \(A\) is a two-dimensional space with basis \((x, y)\), and the maps \(\Phi\) are given by
their matrices in the basis \((x \otimes x, x \otimes y, y \otimes x, y \otimes y)\).

\[
\begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & \lambda & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & 0 & -1 & 0 \\
0 & 1 & -1 & 0 \\
0 & 0 & 2 & 1
\end{bmatrix},
\]

where \(\lambda\) is a scalar. More details on these examples can be found in \[16].

**Notations 3.1.** Let \((A, \Phi)\) be a pair, such that \(A\) is a vector space and \(\Phi : A \otimes A \to A \otimes A\) is a linear map. We use the Sweedler notation:

\[\Phi(a \otimes b) = \sum a' \to b' \otimes a'' \rhd b''.\]

Note that the operations \(\to\) and \(\rhd\) may not exist, nor the coproducts \(a' \otimes a''\) or \(b' \otimes b''\). With this notation, (9) can be rewritten as:

\[
\sum \sum \sum a' \to (b' \to c')' \otimes (a'' \rhd (b' \to c'))'' \to (b'' \rhd (b' \to c'))' \otimes (a'' \rhd (b' \to c'))'' \rhd (b'' \rhd (b' \to c'))'' = \sum (a' \to b')' \to c' \otimes (a' \to b'')' \rhd c'' \otimes a'' \rhd b''.
\]

**Definition 3.4.** Let \(V\) be a vector space and \(*\) a linear map:

\[
* : \begin{cases}
A & \to \text{hom}(V \otimes V, V) \\
a & \to \ast_a : \begin{cases}
V \otimes V & \to v \\
x \otimes y & \to x \ast_a y.
\end{cases}
\end{cases}
\]

We shall say that \((V, \ast)\) is a \(\Phi\)-associative algebra if:

\[
\forall x, y, z \in V, \forall a, b \in A, \quad x \ast_a (y \ast_b z) = \sum (x \ast_{a' \to b'} y) \ast_{a' \to b'} z. \tag{10}
\]

We shall say that \((V, \ast)\) is an opposite \(\Phi\)-associative algebra if:

\[
\forall x, y, z \in V, \forall a, b \in A, \quad \sum x \ast_{a' \to b'} (y \ast_{a' \to b'} z) = (x \ast_y y) \ast_a z. \tag{11}
\]

**Remark 3.2.**
1. We define \(\ast^{op}\) by \(x \ast^{op}_a y = y \ast_a x\). Then \((V, \ast)\) is a \(\Phi\)-associative algebra if, and only if, \((V, \ast^{op})\) is an opposite \(\Phi\)-associative algebra.

2. If \(\Phi\) is invertible, opposite \(\Phi\)-associative algebras and \(\Phi^{-1}\)-associative algebras are the same.
Remark 3.3. If \((Ω, →, ⇝)\) is an EAS and if \((KΩ, Φ)\) is its linearization, then the categories of \(Ω\)-associative algebras and of \(Φ\)-associative algebras are isomorphic: if \((V_\alpha)_{α ∈ Ω}\) is an \(Ω\)-algebra, then we obtain a \(Φ\)-associative algebra with the map:

\[
\ast : \begin{cases}
KΩ & → \text{hom}(V ⊗ V, V) \\
α ∈ Ω & → *α.
\end{cases}
\]

In this way, \(Ω\)-associative algebras can be seen as particular examples of \(Φ\)-associative algebras.

### 3.2 Free objects

**Notations 3.2.** Let \(V\) be a vector space. We put:

\[
T_A(V) = \bigoplus_{n=1}^∞ A^{⊗(n-1)} ⊗ V^{⊗n}.
\]

If \(a_1, \ldots, a_{n-1} ∈ A, x_1, \ldots, x_n ∈ V\), we shall denote their tensor product in \(A^{⊗(n-1)} ⊗ V^{⊗n}\) by \(a_1 \ldots a_{n-1}x_1 \ldots x_n\). Such a tensor will be called an \(A\)-typed word of length \(n\). We shall use the following map:

\[
\ast : \begin{cases}
T_A(V) ⊗ A ⊗ V & → T_A(V) \\
a_1 \ldots a_{n-1}x_1 \ldots x_n ⊙ a ⊗ x & → a_1 \ldots a_{n-1}x_1 \ldots x_n ⊙ ax = aa_1 \ldots a_{n-1}ax_1 \ldots x_n.
\end{cases}
\]

**Theorem 3.5.** For any vector space \(V\), we define bilinear products \(\ast_a\) on \(T_A(V)\) in the following way, by induction on the length of \(A\)-typed words:

\[
w \ast_a z = w \cdot az,
\]

\[
u \ast_a (v \cdot bz) = \sum(u \ast_a (v \cdot b)z) \cdot (a' → b')z,
\]

where \(u, v, w ∈ T_A(V), z ∈ V\) and \(a, b ∈ A\). The following conditions are equivalent:

1. \((A, Φ)\) is an \(ℓEAS\).
2. For any vector space \(V\), \((T_A(V), \ast)\) is a \(Φ\)-associative algebra.
3. There exists a nonzero vector space \(V\) such that \((T_A(V), \ast)\) is a \(Φ\)-associative algebra.

Moreover, if these conditions hold, then \((T_A(V), \ast)\) is the free \(Φ\)-associative algebra generated by \(V\).

**Proof.** Obviously, 2. \(→\) 3.

3. \(→\) 1. Let \(x, y, z, t\) be four nonzero elements of any nonzero vector space \(V\). For any \(a, b, c ∈ A\):

\[
x \ast_a (y \ast_b czt) = \sum ζ \sum \sum (a'' ⊡ (b' → c'))'' ⊡ (b'' ⊡ c'')''
\]

\[
(a'' ⊡ (b' → c'')') → (b'' ⊡ c''')' \quad a' → (b' → c')xyzt,
\]

\[
\sum(x \ast_{a'' \rightarrow b''} y) *_{a'' \rightarrow b''} czt = \sum ζ \sum (a'' ⊡ b'')(a' → b'')'' ⊡ c''((a' → b'')' → c')xyzt.
\]

This immediately gives \((9)\).

1. \(→\) 2. Let us prove that for any \(A\)-typed words \(u, v, w\), for any \(a, b ∈ A\),

\[
u \ast_a (v \ast_b w) = \sum(u \ast_{a'' \rightarrow b''} v) *_{a'' \rightarrow b''} w.
\]
We proceed by induction on the length $n$ of $w$. If $n = 1$, we put $w = z \in V$.

\[
u \ast_a (v \ast_b w) = u \ast_a (v \cdot bz) = \sum (u \ast_{a'} \ast_b' v) \cdot (a'' \rightarrow b'')z = \sum (u \ast_{a''} \ast_{b''} v \ast_{a'''} \ast_{b'''})z.
\]

Let us assume the result at rank $n - 1$. We put $w = w' \cdot cz$.

\[
u \ast_a (v \ast_b w) = \sum u \ast_a ((v \ast_{b' \rightarrow c'} w') \cdot (b' \rightarrow c''))z = \sum \sum (u \ast_{a''} \ast_{b''} \ast_{c''} v) \cdot (a'' \rightarrow b'')w' \cdot (a' \rightarrow b')' \rightarrow c' \cdot cz = \sum (u \ast_{a''} \ast_{b''} \ast_{c''} v \ast_{a'''} \ast_{b'''} \ast_{c'''})w'.
\]

We use the induction hypothesis for the third equality and $\Theta$ for the fourth one.

**Freeness.** Let us assume that conditions 1 and 2 hold. Let $W$ be an $A$-associative algebra, and $\Theta : V \rightarrow W$ be any linear map. Let us prove that it can be uniquely extended as a $\Phi$-associative algebra morphism $\Theta$ from $T_A(V)$ to $W$.

**Existence of $\Theta$.** We inductively define $\Theta(w)$ for any $A$-typed word $w$ by induction on its length $n$. If $n = 1$, then $w \in V$ and we put $\Theta(w) = \theta(w)$. Otherwise, let us write $w = w' \cdot az$. We put:

\[
\Theta(w) = \Theta(w') \ast_a \theta(z).
\]

Let $u, v$ be two $A$-typed words, and let us prove that for any $a \in A$, $\Theta(u \ast_a v) = \Theta(u) \ast_a \Theta(v)$. We proceed by induction on the length $n$ of $v$. If $n = 1$, then by definition of $\Theta$, this is true. Otherwise, let us put $v = v' \cdot bz$. Then:

\[
\Theta(u \ast_a v) = \sum \Theta((u \ast_{a''} \ast_{b''} v') \cdot (a' \rightarrow b')z) = \sum \Theta(u \ast_{a''} \ast_{b''} v') \ast_{a'''} \ast_{b'''} \theta(z) = \sum (\Theta(u) \ast_{a''} \ast_{b''2} (\Theta(v'))) \ast_{a'''} \ast_{b''} \theta(z) = \Theta(u) \ast_a (\Theta(v') \ast_b \theta(z)) = \Theta(u) \ast_a \Theta(v).
\]

So $\Theta$ is a $\Phi$-associative algebra morphism.

**Uniqueness of $\Theta$.** If $\Theta'$ is another morphism extending $\theta$, for any $A$-typed word $u$, for any $a \in A$, for any $z \in V$:

\[
\Theta'(u \cdot az) = \Theta'(u \ast_a z) = \Theta'(u) \ast_a \theta(z).
\]

An easy induction on the length proves that for any $A$-typed word $u$, $\Theta'(u) = \Theta(u)$. 

### 3.3 Links with associative algebras

**Proposition 3.6.** Let $(A, \Phi)$ such that $A$ is a vector space and $\Phi : A \otimes A \rightarrow A \otimes A$ is a linear map. We assume that $V$ is a vector space and $\ast : A \rightarrow \text{hom}(V \otimes V, V)$ is a linear map. We define a product on $A \otimes V$ by:

\[
\forall x, y \in V, \forall a, b \in A, \quad xa \ast yb = \sum x \ast_{a''} \ast_{b''} ya' \rightarrow b'.
\]

Then:
1. If \((A, \Phi)\) is an \(\ell\)EAS and \((V, \ast)\) is a \(\Phi\)-associative algebra, then \((A \otimes V, \ast)\) is an associative algebra.

2. If \((A, \Phi)\) is a nondegenerate \(\ell\)EAS and \((A \otimes V, \ast)\) is an associative algebra, then \((V, \ast)\) is a \(\Phi\)-associative algebra.

3. Let \(V\) be a nonzero vector space. If \((A \otimes T_{\ell}(V), \ast)\) is an associative algebra, then \((A, \Phi)\) is an \(\ell\)EAS.

Proof. Let \(a, b, c \in A\) and \(x, y, z \in V\). In \(A \otimes V:\)

\[
xa \ast (yb \ast zc) = \sum \sum \sum x \ast_{a'' \ast (b' \to c'')} (y \ast_{b'' \to c''} z)a' \to (b' \to c'),
\]

(12)

\[
(xa \ast yb) \ast zc = \sum \sum \sum (x \ast_{a'' \ast b''} y) \ast_{(a' \to b')'' \to c''} z(a' \to b') \to c'.
\]

1. We put

\[
x = \sum \sum a' \to (b' \to c')' \otimes a'' \to (b' \to c'') \otimes b'' \to c'' = (\Phi \otimes \text{Id})(a \otimes b \otimes c).
\]

By (9):

\[
(\text{Id} \otimes \Phi)(x) = (\Phi \otimes \text{Id}_A) \circ (\text{Id}_A \otimes \tau) \circ (\Phi \otimes \text{Id}_A)(a \otimes b \otimes c)
\]

\[
= \sum \sum (a' \to b')' \to c' \otimes (a' \to b')'' \otimes c'' \otimes a'' \otimes b''.
\]

As \(V\) is \(\Phi\)-associative, we obtain that \(\ast\) is associative.

2. By composition, the following map is bijective:

\[
\Psi = (\Phi \otimes \text{Id}_A) \circ (\text{Id}_A \times \Phi) : \begin{array}{cccc}
V^{\otimes 3} & \to & V^{\otimes 3} \\
\otimes \otimes \otimes & & \otimes \otimes \otimes \\
(a \otimes b \otimes c) & \to & (a' \to b')' \otimes a'' \to (b' \to c'') \otimes b'' \to c''.
\end{array}
\]

Let \(a \otimes b \otimes c \in A^{\otimes 3}\) and \(a_1 \otimes b_1 \otimes c_1 = \Psi^{-1}(a \otimes b \otimes c)\). For any \(x, y, z \in A:\)

\[
xa_1 \ast (yb_1 \ast zc_1) = (x \ast_{a} (y \ast_{c} z))a,
\]

\[
(xa_1 \ast yb_1) \ast zc_1 = \sum ((x \ast_{b'' \to c''} y) \ast_{b' \to c'} z)a.
\]

The associativity of \(\ast\) induces the axiom of \(\Phi\)-associative algebra for \(V\).

3. Let \(x, y, z \in V\), nonzero (not necessarily distinct). From the associativity of \(\ast\), we immediately deduce from (12) that:

\[
\sum \sum \sum a' \to (b' \to c')' \otimes (a'' \to (b' \to c''))' \to (b'' \to c'')' \otimes (a'' \to (b' \to c''))'' \to (b'' \to c'') = \sum \sum \sum (a' \to b')' \to c' \otimes (a' \to b')'' \otimes c'' \otimes a'' \otimes b''.
\]

So \((A, \Phi)\) is an \(\ell\)EAS.

**Remark 3.4.** As a corollary, if \((\Omega, \to, \leadsto)\) is an EAS, then \(\Omega\)-associative algebras are 2-parameters associative algebras with \(*_{\alpha, \beta} = *_{(\alpha \to \beta)}\). This will be formalized in Proposition 3.14 by an operad morphism.

**Proposition 3.7.** Let \((A, \Phi)\) be an \(\ell\)EAS and let \(V\) be a nonzero vector space.

1. The following conditions are equivalent:

   (a) The associative algebra \(T_{\ell}(V) \otimes A\) is generated by \(V \otimes A\).

   (b) \(\Phi\) is surjective.
2. The following conditions are equivalent:

(a) The subalgebra $T_A(V) \otimes A$ generated by $V \otimes A$ is free.

(b) $\Phi$ is injective.

Proof. We denote by $W$ the subalgebra of $T_A(V) \otimes A$ generated by $V \otimes A$. Note that it is graded by the length of words.

1. $(a) \implies (b)$. Let $a \otimes b \in A^{\otimes 2}$. Let us choose a nonzero element $x$ of $V$. Then $x x a b \in A$. Because of the graduation, we can write this element under the form:

$$x x a b = \sum_{i=1}^{n} x_i a_i \star y_i b_i = \sum_{i=1}^{n} x_i y_i (a''_i \Rightarrow b''_i)(a'_i \rightarrow b'_i).$$

Applying an element $f$ of $V^*$ such that $f(x) = 1$, we obtain

$$\Phi \left( \sum_{i=1}^{n} f(x_i) f(y_i) a_i \otimes b_i \right) = a \otimes b,$$

so $\Phi$ is surjective.

2. $(b) \implies (a)$. Let $x_1 \ldots x_n a_1 \ldots a_n$ be a word of length $n$, and let us prove that it belongs to $W$ by induction on $n$. This is obvious if $n = 1$. Otherwise, there exists $x = \sum b_{n-1} \otimes b_n \in A^{\otimes 2}$, such that

$$\Phi \left( \sum b_{n-1} \otimes b_n \right) = a_n \otimes a_{n-1}.$$ 

By the induction hypothesis, $x_1 \ldots x_{n-1} a_1 \ldots a_{n-2} b_{n-1} \in W$, so:

$$\sum x_1 \ldots x_{n-1} a_1 \ldots a_{n-2} b_{n-1} \star x_n b_n = \sum \sum x_1 \ldots x_n a_1 \ldots a_{n-2} (b''_{n-1} \Rightarrow b''_n)(b'_{n-1} \rightarrow b'_n) = x_1 \ldots x_n a_1 \ldots a_n \in W.$$

2. $(a) \implies (b)$. Because of the graduation, $W$ is freely generated by $V \otimes A$. Let $x$ be a nonzero element of $V$. If $\sum a_n \otimes b_n \neq 0$, by freeness, $\sum x a_n \star x b_n \neq 0$ and:

$$\sum x a_n \star x b_n = \sum \sum x x (a''_n \Rightarrow b''_n)(a'_n \rightarrow b'_n) \neq 0,$$

So $\Phi(\sum a_n \otimes b_n) \neq 0$.

2. $(b) \implies (a)$. We shall use the following map:

$$\Phi' = \tau \circ \Phi : \left\{ \begin{array}{l} A \otimes A \\ a \otimes b \end{array} \longrightarrow \left\{ \begin{array}{l} A \otimes A \\ \sum a'' \Rightarrow b'' \otimes a' \rightarrow b'. \end{array} \right. \right.$$

As $\Phi$ is injective, $\Phi'$ is injective. Let $x_1, \ldots, x_n \in V$ and let $a_1, \ldots, a_n \in A$. An easy induction on $n$ proves that:

$$x_1 a_1 \ldots \star x_n a_n = x_1 \ldots x_n \left( \text{Id}_A^{\otimes (n-2)} \otimes \Phi' \right) \circ \left( \text{Id}_A^{\otimes (n-3)} \otimes \Phi' \otimes \text{Id}_A \right) \circ \ldots \circ \left( \Phi' \otimes \text{Id}_A^{\otimes (n-2)} \right) (a_1 \otimes \ldots \otimes a_n).$$

As a consequence, the following algebra map is injective:

$$\left\{ \begin{array}{l} T(V \otimes A) \\ x_1 a_1 \ldots x_n a_n \end{array} \longrightarrow \left\{ \begin{array}{l} T_A(V) \otimes A \\ x_1 a_1 \ldots x_n a_n \end{array}. \right. \right.$$ 

So the image of this morphism, which is $W$, is freely generated by $V \otimes A$. □

Remark 3.5. Consequently, for any vector space $V$, $(T_A(V), \star)$ is freely generated by $V \otimes A$ if, and only if, $(A, \Phi)$ is nondegenerate.
3.4 Operadic aspects and Koszul duality

In this section, \((A, \Phi)\) is an \(\iota\)EAS.

**Notations 3.3.** We denote the nonsymmetric operad of \(\Phi\)-associative algebras by \(\text{As}_{\Phi}\), and the nonsymmetric operad of opposite \(\Phi\)-associative algebras by \(\text{As}'_{\Phi}\). In other words, \(\text{As}_{\Phi}\) is the nonsymmetric operad generated by \(A = \text{As}_{\Phi}(2)\), with the relations

\[
\alpha \circ_2 \beta = \sum a' \to b' \circ_1 a'' \rhd b'' ,
\]

whereas \(\text{As}'_{\Phi}\) is the nonsymmetric operad generated by \(A = \text{As}'_{\Phi}(2)\), with the relations

\[
\alpha \circ_1 \beta = \sum a' \to b' \circ_2 a'' \rhd b'' .
\]

We denote by \(\text{SymAs}_{\Phi}\), respectively by \(\text{SymAs}'_{\Phi}\), the operad of \(\Phi\)-associative algebras, respectively of opposite \(\Phi\)-associative algebras.

**Remark 3.6.**
1. \(\text{SymAs}_{\Phi}\), respectively \(\text{SymAs}'_{\Phi}\), is the symmetrisation of the nonsymmetric operad \(\text{As}_{\Phi}\), respectively \(\text{As}'_{\Phi}\).
2. If \(\Phi\) is nondegenerate, then \(\text{As}'_{\Phi} = \text{As}_{\Phi^{-1}}\).
3. \(\text{SymAs}_{\Phi}\) and \(\text{SymAs}'_{\Phi}\) are isomorphic operads, through the morphism

\[
\begin{cases}
\text{SymAs}_{\Phi} & \to \text{SymAs}'_{\Phi} \\
a \in A & \mapsto a^{op} = a^{(12)}. 
\end{cases}
\]

From the description of free \(\Phi\)-associative algebras, we obtain a combinatorial description of \(\text{As}_{\Phi}\):

**Proposition 3.8.** For any \(n \geq 1\), \(\text{As}_{\Phi}(n)\) is the vector space \(A^{\otimes (n-1)}\). For any \(a_k \ldots a_1 \in A^{\otimes k} = \text{As}_{\Phi}(k + 1)\), for any \(b_l \ldots b_1 \in A^{\otimes l} = \text{As}_{\Phi}(l + 1)\), for any \(i \in [k + 1]\

\[
a_k \ldots a_1 \circ_i b_l \ldots b_1 = \begin{cases} 
b_l \ldots b_1 a_k \ldots a_1 & \text{if } i = 1, \\
a_k \ldots a_i (\Phi \otimes \text{Id}^{\otimes (l-2)}) \circ \ldots \circ (\text{Id} \otimes \Phi \otimes \text{Id}^{\otimes (l-3)}) \circ (\text{Id}^{\otimes (l-2)} \otimes \Phi)(a_{i-1} b_l \ldots b_1) a_{i-2} \ldots a_1 & \text{if } i \geq 2. 
\end{cases}
\]

**Example 3.3.** Let us consider linearizations of EAS.

1. For \(\text{EAS}(A, \ast)\), this simplifies as:

\[
\alpha_1 \ldots \alpha_k \circ_i \beta_1 \ldots \beta_l = \alpha_1 \ldots \alpha_{i-1} (\alpha_{i-1} \ast \beta_1) \ldots (\alpha_{i-1} \ast \beta_l) \alpha_{i+1} \ldots \alpha_k.
\]

2. For \(\text{EAS}(\Omega)\), this simplifies as:

\[
\alpha_1 \ldots \alpha_k \circ_i \beta_1 \ldots \beta_l = \alpha_1 \ldots \alpha_{i-1} \beta_1 \ldots \beta_l \alpha_{i+1} \ldots \alpha_k.
\]

This operad is used in \([9]\). When \(\Omega\) has two elements, this gives the operad of duplexes of vertices of cubes defined in \([29]\) Section 6.3].

3. If \((A, \ast)\) is a group, we obtain for \(\text{EAS}'(A, \ast)\):

\[
\alpha_1 \ldots \alpha_k \circ_i \beta_1 \ldots \beta_l = \alpha_1 \ldots \alpha_{i-1} (\alpha_{i-1} \ast \beta_1^{-1} \ast \ldots \ast \beta_l^{-1}) \beta_1 \ldots \beta_l \alpha_{i+1} \ldots \alpha_k.
\]

**Proposition 3.9.** Let us assume that \(A\) is finite-dimensional.
1. Koszul dual of the nonsymmetric operad $\mathbf{As}_\Phi$ is isomorphic to $\mathbf{As}_\Phi^{* \ast}$.

2. Koszul dual of the nonsymmetric operad $\mathbf{As}_\Phi'$ is isomorphic to $\mathbf{As}_\Phi' \ast \ast$.

3. Koszul dual of the operad $\text{SymAs}_\Phi$ is isomorphic to $\text{SymAs}_\Phi^{* \ast}$.

**Proof.** 1. We identify $\mathbf{As}_\Phi(2)^* = A$ and $A^\ast$. This identification induces a pairing between the free nonsymmetric operad $F_A$ generated by $A$ and the free nonsymmetric operad $F_A^\ast$ generated by $A^\ast$. In particular, if $a, b \in A$, $f, g \in A^\ast$,

$$
\langle f \circ_1 g, a \circ_1 b \rangle = f(a)g(b),
\langle f \circ_2 g, a \circ_2 b \rangle = -f(a)g(b),
\langle f \circ_1 g, a \circ_2 b \rangle = 0,
\langle f \circ_2 g, a \circ_1 b \rangle = 0.
$$

We denote by $I$ the space of relations of $\mathbf{As}_\Phi(3)$: this is the subspace of $F_A$ generated by the elements

$$
\sum a' \rightarrow b' \circ_1 a'' \rhd b'' - a \circ_2 b,
$$

with $a, b \in A$. Note that $\mathbf{As}_\Phi^1$ is the quotient of $F_A^\ast$ by the operadic ideal generated by $I^\perp$. We also denote by $I'$ the space of relations of $\mathbf{As}_\Phi^*(3)$: this is the subspace of $F_A^\ast$ generated by the elements

$$
\sum f' \rightarrow g' \circ_2 f'' \rhd g'' - f \circ_1 g,
$$

with $f, g \in A^\ast$. Let $a, b \in A$ and $f, g \in A^\ast$.

$$
\langle \sum f' \rightarrow g' \circ_2 f'' \rhd g'' - f \circ_1 g, \sum a' \rightarrow b' \circ_1 a'' \rhd b'' - a \circ_2 b \rangle = -\Phi^\ast(f \otimes g)(a \otimes b) - (f \otimes g)(\Phi(a \otimes b)) = 0,
$$

so $I' \subseteq I^\perp$. Moreover:

$$
dim(F_A(3)) = 2 \dim(A)^2, \quad \dim(I) = \dim(I') = \dim(A)^2,
$$

so $\dim(I^\perp) = 2 \dim(A)^2 - \dim(A)^2 = \dim(A)^2 = \dim(I')$ and finally $I^\perp = I'$.

2. By duality.

3. By symmetrisation, $(\text{SymAs}_\Phi)^1 = \text{SymAs}_\Phi' \ast \ast$, which is isomorphic to $\text{SymAs}_\Phi^{* \ast}$, see Remark 3.6 $\square$

**Example 3.4.** Let $\Omega$ be a finite EAS. Koszul dual of the operad $\mathbf{As}_A$ of $\Omega$-associative algebra is generated by the products $\ast_{\alpha}$, with $\alpha \in \Omega$, and the relations

$$
\forall \alpha, \beta \in \Omega, \quad \left( \sum_{(\alpha', \beta') \in \Omega^2, \phi(\alpha', \beta') = (\alpha, \beta)} \ast_{\alpha'} \circ (I, \ast_{\beta'}) \right) = \ast_{\alpha}(\ast_{\beta}, I).
$$

**Theorem 3.10.** If $A$ is finite-dimensional, the nonsymmetric operads $\mathbf{As}_\Phi$ and $\mathbf{As}_\Phi'$ as well as the operad $\text{SymAs}_\Phi$ are Koszul.

**Proof.** We shall use the rewriting method of [2] [23]. We shall write elements of the free nonsymmetric operad generated by $\mathbf{As}_A(2)$ as planar trees which vertices are decorated by elements of $A$. The rewriting rules are:

$$
\Upsilon^2(a, b) \rightarrow \sum \Upsilon^2(a'' \rightarrow b'', a'' \rhd b'')
$$
for any $a, b \in A$. There is only one family of critical monomials, which are the trees

$$T_1 = \frac{\partial^3}{\partial a \partial b \partial c} (a, b, c)$$

with $a, b, c \in A$. Koszularity of $\mathbb{A}_S^A$ comes from the confluence of the following diagram:

(13)

with:

$$T_1 = \frac{\partial^3}{\partial a \partial b \partial c} (a, b, c),$$

$$T_2 = \sum \frac{\partial^3}{\partial a \partial b \partial c} (a' \rightarrow b', a'' \Rightarrow b'', c),$$

$$T_3 = \sum \frac{\partial^3}{\partial a \partial b \partial c} (a, b' \rightarrow c', b'' \Rightarrow c''),$$

$$T_4 = \sum \sum \frac{\partial^3}{\partial a \partial b \partial c} (a' \rightarrow (b' \rightarrow c)'), a'' \Rightarrow (b' \rightarrow c)'', b'' \Rightarrow c''$$

$$T_5 = \sum \sum \sum \frac{\partial^3}{\partial a \partial b \partial c} (((a' \rightarrow b')' \rightarrow c', (a' \rightarrow b'')' \Rightarrow c'', a'' \Rightarrow b'')$$

$$= \sum \sum \sum \frac{\partial^3}{\partial a \partial b \partial c} ((a' \rightarrow (b' \rightarrow c)'), (a'' \Rightarrow (b' \rightarrow c)'')' \rightarrow (b'' \Rightarrow c'')', (a'' \Rightarrow (b' \rightarrow c)'')'' \Rightarrow (b'' \Rightarrow c'')'').$$

The equality between the two expressions of $T_5$ is equivalent to (9).

Here is another application of Diagram (13):

**Proposition 3.11.** Let $\mathbf{P}$ be a nonsymmetric set operad such that for any $n \geq 1$, the following map is a linear isomorphism:

$$\iota_n : \left\{ \begin{array}{ccc} \mathbf{P}(2)^{\otimes (n-1)} & \rightarrow & \mathbf{P}(n) \\ p_1 \otimes \ldots \otimes p_{n-1} & \rightarrow & p_1 \circ_1 (p_2 \circ_1 \ldots \circ_1 (p_{n-2} \circ_1 p_{n-1}) \ldots) \end{array} \right\}. $$

Then there exists an $\ell$EAS $(A, \Phi)$ such that $\mathbf{P}$ is isomorphic to $\mathbb{A}_S^A$.

**Proof.** We put $A = \mathbf{P}(2)$ as a vector space. As $\iota_3$ is bijective, for any $a \otimes b \in A \otimes A$, there exists a unique $\Phi(a \otimes b) = \sum a' \rightarrow b' \otimes a'' \Rightarrow b'' \in A \otimes A$ such that

$$a \circ_2 b = \sum (a' \rightarrow b') \circ_1 (a'' \Rightarrow b''),$$

or, equivalently:

$$\mathbf{P}(2)(a, b) = \sum \mathbf{P}(2)(a', b', a'' \Rightarrow b'').$$
For any $a, b, c \in A$, let us compute $a \circ_2 (b \circ_2 c)$ into two different ways. This element is the tree $T_1$ of $\mathbf{P}(3)$, and, following the two paths of this diagram, we obtain that in $\mathbf{P}(3)$:

$$\sum (a' \rightarrow b')' \rightarrow c' \circ_1 ((a' \rightarrow b')'' \rhd c'' \circ_1 (a'' \rhd b''))$$

$$= \sum \sum a' \rightarrow (b' \rightarrow c')' \circ_1 (a'' \rhd (b' \rightarrow c')')' \rightarrow (b'' \rhd c'')' \circ ((a'' \rhd (b' \rightarrow c')'')' \rhd (b'' \rhd c'')').$$

As $\iota_4$ is an isomorphism, we obtain the axioms of $\ell\text{EAS}$ for $(A, \Phi)$. Hence, we obtain an operad isomorphism from $\text{As}_\Phi$ to $\mathbf{P}$, sending $*_a$ to $a$ for any $a \in A$.

\[\square\]

### 3.5 Associative products

We now look for operad morphisms from the operad of associative algebras to the operad $\text{SymAs}_\Phi$, where $(A, \Phi)$ is an $\ell\text{EAS}$, or equivalently to products $m \in \text{SymAs}_\Phi(2)$ which are associative, that is to say such that $m \circ_1 m = m \circ_2 m$.

**Proposition 3.12.** Let $(A, \Phi)$ be an $\ell\text{EAS}$. The associative products in $\text{SymAs}_\Phi(2)$ are the elements of the form

$$m = *_a \quad \text{or} \quad m = *^\text{op}_a,$$

where $a \in A$ is such that $\Phi(a \otimes a) = a \otimes a$. The products $m \in \text{SymAs}_\Phi(2)$ such that $m \circ_2 m = 0$ are the elements of the form

$$m = *_a,$$

where $a \in A$ is such that $\Phi(a \otimes a) = 0$.

**Proof.** Let $m = *_a + *^\text{op}_b \in \text{SymAs}_\Phi(2)$. Let $V = T_A(\text{Vect}(x, y, z))$ be the free $\Phi$-associative algebra generated by three elements $x, y, z$. In $V$:

$$m \circ (\text{Id} \otimes m)(x \otimes y \otimes z) = m(x \otimes (ayz + bzy))$$

$$= \tau \circ \Phi(a \otimes a)xyz + abyxz + \tau \circ \Phi(a \otimes b)xzy + bzyx,$$

$$m \circ (m \otimes \text{Id})(x \otimes y \otimes z) = m((axy + byx) \otimes z)$$

$$= aaxy +abayx + \tau \circ \Phi(b \otimes b)xyz + \tau \circ \Phi(b \otimes b)zxy.$$

1. If $m$ is associative, identifying the terms in $yzx$, we find $a \otimes b = 0$, so $a = 0$ or $b = 0$. Identifying the terms in $xyz$, we find that $\tau \circ \Phi(a \times a) = a \otimes a$. Similarly, the identification of the terms in $zyx$ gives that $\Phi(b \otimes b) = b \otimes b$. Conversely, if $\Phi(a) = a \otimes a$, then

$$a \circ_2 a = a,$$

$$a \circ_1 a = a,$$

so $*_a$ is associative, and its opposite $*^\text{op}_a$ is associative too.

2. If $m \circ_2 m = 0$, identifying the term in $zym$, we find that $b = 0$. Identifying the term in $xyz$, we find that $\tau \circ \Phi(a \times a) = 0$. Conversely, if $\Phi(a \otimes a) = 0$, then

$$a \circ_2 a = \Phi(a \otimes a) = 0.$$

**Remark 3.7.** If $(A, \Phi)$ is the linearization of an EAS $(\Omega, \rightarrow, \rhd)$, we obtain that:

- The associative elements $m \in \text{SymAs}_\Phi(2)$ are the elements of the form

$$m = \sum_{\alpha \in \Omega} \lambda_\alpha *_\alpha \quad \text{or} \quad m = \sum_{\alpha \in \Omega} \lambda_\alpha *^\text{op}_\alpha,$$

such that

$$\forall (\alpha, \beta) \in \Omega^2, \quad \lambda_\alpha \lambda_\beta = \sum_{(\gamma, \delta) \in \Omega^2, \phi(\gamma, \delta) = (\alpha, \beta)} \lambda_\gamma \lambda_\delta.$$ (14)

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• The elements \( m \in \text{SymAs}_\Phi(2) \) such that \( m \circ_2 m = 0 \) are the elements of the form
\[
m = \sum_{\alpha \in \Omega} \lambda_\alpha \ast_\alpha,
\]
such that
\[
\forall (\alpha, \beta) \in \Omega^2, \quad \sum_{(\gamma, \delta) \in \Omega^2, \phi(\gamma, \delta) = (\alpha, \beta)} \lambda_\gamma \lambda_\delta = 0.
\] (15)

**Example 3.5.** Working with \( \text{EAS}(\Omega) \), then \( \phi(\alpha, \beta) = (\beta, \alpha) \) and Condition (14) is empty: any linear combination of \( *_\alpha \) is associative, as well as their opposite.

**Example 3.6.** Let us give the associative products for EAS of cardinality two. We only mention the spans of \( *_\alpha \), their opposite should be added. Here, \( \lambda, \mu \) are scalars.

| Cases | Associative products | \( m \circ_2 m = 0 \) |
|-------|----------------------|----------------------|
| A1    | \( \lambda *_a \)    | \( \lambda(*_a *_{-b}) \) |
| A2    | \( \lambda *_a \)    | \( \lambda(*_a *_{-b}) \) |
| C1    | \( \lambda *_a \)    | 0                    |
| C2    | \( \lambda *_{a_1} \) | 0                    |
| C3    | \( \lambda *_{a_1} \) | 0                    |
| C4    | \( \lambda *_{b_1} \) | 0                    |
| E1' − E2' | \( \lambda *_{a_1} \) | \( \lambda(*_{a_1} *_{-b_1}) \) |
| E3'    | \( \lambda *_{a_1} \) | \( \lambda(*_{a_1} *_{-b_1}) \) |
| F1    | \( \lambda *_{a_1} \) | \( \lambda(*_{a_1} *_{-b_1}) \) |
| F3    | \( \lambda *_{a_1} + \mu *_{b_1} \) | 0                    |
| F4    | \( \lambda(*_{a_1} *_{-b_1}) \), \( \lambda *_{a_1} \) | 0                    |
| H1    | \( \lambda *_{a_1} \) | 0                    |
| H2    | \( \lambda(*_{a_1} *_{-b_1}) \), \( \lambda *_{a_1} \) | 0                    |

**Corollary 3.13.** Let \((\Omega, \ast)\) be a group. The nonzero associative products in \( \text{SymAs}_{\text{EAS}}(\Omega, \ast) \) or in \( \text{SymAs}_{\text{EAS'}}(\Omega, \ast) \) are the elements of one of the form
\[
\lambda \sum_{\alpha \in H} *_{\alpha} \quad \text{or} \quad \lambda \sum_{\alpha \in H} *_{\alpha}^{op},
\]
where \( \lambda \) is a nonzero scalar and \( H \) is a subgroup of \( \Omega \).

**Proof.** Case of \( \text{EAS}(\Omega, \ast) \). Then (14) becomes:
\[
\forall (\alpha, \beta) \in \Omega^2, \quad \lambda_{\alpha \ast \beta} \lambda_\alpha = \lambda_\alpha \lambda_\beta.
\]
Let \( H = \{ \alpha \in \Omega, \lambda_\alpha \neq 0 \} \). We assume that \( H \) is nonempty. If \( \alpha \in H \), for any \( \beta \in H \), \( \lambda_{\alpha \ast \beta} = \lambda_\beta \).

In particular:

• If \( \beta \in H \), then \( \alpha \ast \beta \in H \).

• If \( \beta = e_\Omega \), then \( \lambda_\alpha = \lambda_{e_\Omega} \neq 0 \). \( e_\Omega \in H \).

• If \( \beta = \alpha^{-1} \), then \( \lambda_{e_\Omega} = \lambda_{\alpha^{-1}} \neq 0 \). \( \alpha^{-1} \in H \).

Therefore, \( H \) is a subgroup of \( \Omega \). Let \( \alpha, \beta \in I \), then \( \alpha' = \alpha \ast \beta^{-1} \in H \). From (14), we deduce that \( \lambda_{\alpha' \ast \beta} = \lambda_\alpha = \lambda_\beta \). Let \( \lambda \) be the common value of \( \lambda_\alpha \) for any \( \alpha \in H \); the result is the immediate.

**Case of \( \text{EAS'}(\Omega, \ast) \).** Then (14) becomes:
\[
\forall (\alpha, \beta) \in \Omega^2, \quad \lambda_{\alpha \ast \beta^{-1}} \lambda_\alpha = \lambda_\alpha \lambda_\beta.
\]
The proof is similar to the case of \( \text{EAS}(\Omega, \ast) \).
3.6 Operadic morphisms between $\text{As}_{2}^{\Omega}$ and $\text{As}_{\Omega}$

Proposition 3.14. Let $(\Omega, \rightarrow, \rhd)$ and let $(A, \Phi)$ be its linearization, that is to say $A = \mathbb{K} \Omega$ and:

$$\Phi : \begin{cases} A \otimes A & \rightarrow A \otimes A \\ \alpha \otimes \beta & \rightarrow \alpha \rhd \beta \end{cases},$$

where $\alpha, \beta \in \Omega$. The following defines an operad morphism:

$$\Theta_{\Omega} : \begin{cases} \text{As}_{2}^{\Omega} & \rightarrow \text{As}_{\Phi} \\ *_{\alpha, \beta} & \rightarrow *_{(\alpha, \beta)} \end{cases}$$

Proof. Let us consider an $\Omega$-associative algebra $(A, (*_{\alpha})_{\alpha \in \Omega})$. For any $(\alpha, \beta) \in \Omega^{2}$, we put $*_{\alpha, \beta} = *_{\alpha \rhd \beta}$. Then, for any $x, y, z \in A$:

$$x *_{\alpha, \beta \rhd \gamma} (y *_{\beta, \gamma} z) = x *_{\alpha \rhd (\beta \rhd \gamma)} (y *_{\beta \rhd \gamma} z)$$

$$= (x *_{(\alpha \rhd (\beta \rhd \gamma)) \rhd (\beta \rhd \gamma)} y) *_{(\alpha \rhd (\beta \rhd \gamma)) \rhd (\beta \rhd \gamma)} z$$

$$= (x *_{(\alpha \rhd \beta) \rhd \gamma} y) *_{\alpha \rhd \beta} z$$

$$= (x *_{\alpha \rhd \beta, \gamma} y) *_{\alpha, \beta} z.$$

Hence, $(A, (*_{\alpha, \beta})_{\alpha, \beta \in \Omega})$ is a 2-parameter $\Omega$-associative algebra, which implies the existence of the operadic morphism $\Theta_{\Omega}$.  

Proposition 3.15. Let $(\Omega, \rightarrow)$ be an associative semigroup with the right inverse condition. We consider the EAS $\Omega' = \text{EAS}(\Omega, \rightarrow) \times \text{EAS}'(\Omega, \rightarrow)$ and denote by $(A', \Phi')$ its linearization. The following defines a surjective operad morphism:

$$\Theta'_{\Omega} : \begin{cases} \text{As}_{2}^{\Omega} & \rightarrow \text{As}_{\Phi'} \\ *_{\alpha, \beta} & \rightarrow *_{(\alpha, \beta)} \end{cases}$$

Proof. The EAS structure of $\Omega'$ is given by:

$$\forall (\alpha, \beta, \gamma, \delta) \in \Omega^{4}, \quad (\alpha, \beta) \rhd (\gamma, \delta) = (\alpha \rhd \gamma, \delta),$$

$$(\alpha, \beta) \rhd (\gamma, \delta) = (\alpha, \beta \rhd \delta).$$

Let $(A, (*_{(\alpha, \beta)})_{(\alpha, \beta) \in \Omega})$ be an $\Omega'$-associative algebra. For any $(\alpha, \beta) \in \Omega'$, we put $*_{\alpha, \beta} = *_{(\alpha, \beta)}$. Then, for any $x, y, z \in A$, using the right inverse property for the second equality:

$$(x *_{(\alpha, \beta)} y) *_{(\alpha \rhd \beta, \gamma)} z = (x *_{(\alpha, \beta)} y) *_{(\alpha \rhd \beta, \gamma)} z$$

$$= (x *_{(\alpha, (\beta \rhd \gamma)) \rhd \gamma} y) *_{(\alpha \rhd (\beta \rhd \gamma)) \rhd (\beta \rhd \gamma)} z$$

$$= (x *_{(\alpha, \beta \rhd \gamma)} y) *_{(\alpha, \beta \rhd \gamma) \rhd (\beta, \gamma)} z$$

$$= x *_{(\alpha, \beta \rhd \gamma)} (y *_{(\beta, \gamma)} z)$$

$$= x *_{\alpha, \beta \rhd \gamma} (y *_{\beta, \gamma} z).$$

Hence, $(A, (*_{\alpha, \beta})_{\alpha, \beta \in \Omega})$ is a 2-parameter $\Omega$-associative algebra. This implies the existence of the morphism $\Theta'_{\Omega}$.  

Remark 3.8. Except if $\omega = |\Omega| = 1$, this morphism is not bijective: the dimension of $\text{As}_{2}^{\Omega}(3)$ is $(2\omega - 1)\omega^{3}$, whereas the dimension of $\text{As}_{\Phi'}(3)$ is $\omega^{4}$.  

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4 Links with other operads

4.1 Post-Lie and ComTriAs algebras

Let us consider post-Lie algebras [30], see also [8, 19, 9, 7, 13, 11, 10] for applications and developments. Recall that a post-Lie algebra is a family $\mathcal{A}, \mathcal{t}, \mathcal{u}$, where $\mathcal{A}$ is a vector space and $\mathcal{t}$ and $\mathcal{u}$ are bilinear products on $\mathcal{A}$ such that $(\mathcal{A}, \mathcal{t}, \mathcal{u})$ is a Lie algebra and, for any $x, y, z \in \mathcal{A}$:

$$x \ast \{y, z\} = (x \ast y) \ast z - x \ast (y \ast z) - (x \ast z) \ast y + x \ast (z \ast y),$$

$$\{x, y\} \ast z = \{x \ast z, y\} + \{x, y \ast z\}.$$

Let us start with the Koszul dual of the operad of post-Lie algebras, namely the operad of ComTriAs algebras [36]:

**Definition 4.1.** A ComTriAs algebra is a family $(\mathcal{A}, \mathcal{t}, \mathcal{u})$, where $\mathcal{A}$ is a vector space and $\mathcal{t}$ and $\mathcal{u}$ are bilinear products on $\mathcal{A}$ such that for any $x, y, z \in \mathcal{A}$:

$$x \cdot y = y \cdot x,$$

$$(x \cdot y) \cdot z = x \cdot (y \cdot z),$$

$$(x \ast y) \ast z = x \ast (y \ast z),$$

$$(x \ast y) \ast z = x \ast (y \ast z),$$

$$(x \cdot y) \ast z = x(y \ast z).$$

Note that the products $\cdot$ and $\ast$ are respectively denoted by $\perp$ and $\downarrow$ in [36].

Apart from the first one, these axioms are the ones of a particular example of generalized associative algebra:

**Proposition 4.2.** Let $\Omega$ be the EAS $\mathbb{C}^3$ (that is to say the EAS associated to the semigroup $(\mathbb{Z}/2\mathbb{Z}, \times)$). Then any ComTriAs algebra $(\mathcal{A}, \cdot, \ast)$ is an $\Omega$-associative algebra, with $\mathcal{t} = \cdot$ and $\mathcal{u} = \downarrow$.

Consequently, we obtain an operad morphism from the operad of $\Omega$-associative algebra to the operad of ComTriAs algebras. Using Koszul duality:

**Corollary 4.3.** Let $(V, \Phi)$ be the EAS dual to $\mathbb{C}^3$: $V$ is two-dimensional, with basis $(e_1, e_2)$, and the basis of $\Phi$ is the basis $(e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2)$ is

$$\begin{pmatrix}
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.$$

Then any opposite $\Phi$-associative algebra is a post-Lie algebra, with for any $x, y \in \mathcal{A}$:

$$\{x, y\} = x \ast_2 y - y \ast_2 x,$$

$$x \ast y = x \ast_1 y.$$

We conjecture that the associated operad morphism from the operad of post-Lie algebras into the operad of opposite $\Phi$-associative algebra is injective.
4.2 Diassociative and dendriform algebras

Definition 4.4. \[27\] A diassociative algebra is a family \((A, -, \lhd)\) where \(A\) is a vector space and \(-\) and \(-\lhd\) are bilinear products on \(A\) such that for any \(x, y, z \in A:\)

\[
\begin{align*}
(x \lhd y) \lhd z &= x \lhd (y \lhd z), \quad (16) \\
(x \lhd y) \lhd z &= x \lhd (y \lhd z), \\
(x \lhd y) \lhd z &= x \lhd (y \lhd z), \\
(x \lhd y) \lhd z &= x \lhd (y \lhd z), \\
(x \lhd y) \lhd z &= x \lhd (y \lhd z). \\
\end{align*}
\]

Proposition 4.5. Let \((A, -, \lhd)\) be a diassociative algebra.

1. \((A, -, \lhd)\) is an opposite \(\Omega\)-associative algebra, with the EAS laws:

\[
\begin{align*}
\rightarrow \lhd \lhd & \quad \lhd \rightarrow \lhd \\
\lhd \rightarrow \lhd & \quad \lhd \lhd \rightarrow \\
\rightarrow \lhd \lhd & \quad \lhd \rightarrow \lhd \\
\lhd \rightarrow \lhd & \quad \lhd \lhd \rightarrow
\end{align*}
\]

These EAS are isomorphic to \(C_3\) and \(C_6\).

2. \((A, -, \lhd)\) is an \(\Omega\)-associative algebra, with the EAS laws:

\[
\begin{align*}
\rightarrow \lhd \lhd & \quad \lhd \rightarrow \lhd \\
\lhd \rightarrow \lhd & \quad \lhd \lhd \rightarrow \\
\rightarrow \lhd \lhd & \quad \lhd \rightarrow \lhd \\
\lhd \rightarrow \lhd & \quad \lhd \lhd \rightarrow
\end{align*}
\]

These EAS are isomorphic to \(C_6\) and \(C_3\).

Proof. 1. This is a reformulation of axioms (16), (18), (19) and (20), and of axioms (17), (18), (19) and (20).

2. This is a reformulation of axioms (16), (17), (18) and (19), and of axioms (16), (17), (18) and (20).

Using Koszul duality, we obtain dendriform algebras: recall that a dendriform algebra is a family \((A, <, >)\) where \(A\) is a vector space and \(<\) and \(>\) are bilinear products on \(A\) such that for any \(x, y, z \in A:\)

\[
\begin{align*}
(x < y) < z &= x < (y < z + y > z), \\
(x > y) < z &= x > (y < z), \\
x > (y > z) &= (x < y + x > y) > z.
\end{align*}
\]

Corollary 4.6. 1. Let \((V, \Phi)\) be one of the two following 2-dimensional \(\ell\)-EAS, where the matrix of \(\Phi\) is expressed in the basis \((e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2)\):

\[
\begin{align*}
&1 \ 0 \ 0 \ 0 \\
&0 \ 0 \ 0 \ 0 \\
&0 \ 1 \ 0 \ 0 \\
&0 \ 0 \ 1 \ 1 \\
\end{align*}
\]

Then any \(\Phi\)-associative algebra is a dendriform algebra, with \(<= *_1\) and \(>= *_2\).

2. Let \((V, \Phi)\) is one of the two following 2-dimensional \(\ell\)-EAS, where the matrix of \(\Phi\) is expressed in the basis \((e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2)\):

\[
\begin{align*}
&1 \ 0 \ 0 \ 0 \\
&1 \ 0 \ 0 \ 0 \\
&0 \ 1 \ 0 \ 0 \\
&0 \ 0 \ 1 \ 0 \\
\end{align*}
\]

Then any opposite \(\Phi\)-associative algebra is a dendriform algebra, with \(<= *_1\) and \(>= *_2\).
4.3 Triassociative and tridendriform algebras

Definition 4.7. \[ A \] A triassociative algebra is a family \( (A, \cdot, \triangleright, \triangleleft) \) where \( A \) is a vector space and \( \cdot, \triangleright \) and \( \triangleleft \) are bilinear products on \( A \) such that for any \( x, y, z \in A \):

\[
\begin{align*}
(x \triangleright y) \triangleleft z &= x \triangleleft (y \triangleright z), \\
(x \triangleright y) \triangleleft z &= x \triangleleft (y \cdot z), \\
(x \triangleright y) \triangleleft z &= x \triangleleft (y \perp z), \\
(x \perp y) \triangleleft z &= x \perp (y \cdot z), \\
(x \perp y) \perp z &= x \perp (y \cdot z), \\
(x \perp y) \perp z &= x \perp (y \cdot z).
\end{align*}
\]

Proposition 4.8. Let \((A, \triangleright, \triangleleft, \perp)\) be a triassociative algebra.

1. \((A, \triangleright, \triangleleft, \perp)\) is an opposite \(\Omega\)-associative algebra, with the EAS laws:

\[
\begin{array}{|c|c|c|c|c|}
\hline
\rightarrow & \rightarrow & \rightarrow & \perp \\
\hline
\rightarrow & \rightarrow & \rightarrow & \rightarrow \\
\hline
\rightarrow & \rightarrow & \rightarrow & \rightarrow \\
\hline
\end{array}
\quad
\begin{array}{|c|c|c|c|c|}
\hline
\triangleright & \triangleright & \triangleright & \perp \\
\hline
\triangleright & \triangleright & \triangleright & \rightarrow \\
\hline
\triangleright & \triangleright & \triangleright & \rightarrow \\
\hline
\end{array}
\quad
\begin{array}{|c|c|c|c|c|}
\hline
\triangleleft & \triangleleft & \triangleleft & \perp \\
\hline
\triangleleft & \triangleleft & \triangleleft & \rightarrow \\
\hline
\triangleleft & \triangleleft & \triangleleft & \rightarrow \\
\hline
\end{array}
\quad
\begin{array}{|c|c|c|c|c|}
\hline
\perp & \perp & \perp & \rightarrow \\
\hline
\perp & \perp & \perp & \rightarrow \\
\hline
\perp & \perp & \perp & \rightarrow \\
\hline
\end{array}
\]

2. \((A, \rightarrow, \rightarrow, \perp)\) is an \(\Omega\)-associative algebra, with the EAS laws:

\[
\begin{array}{|c|c|c|c|c|}
\hline
\rightarrow & \rightarrow & \perp & \rightarrow \\
\hline
\rightarrow & \rightarrow & \rightarrow & \rightarrow \\
\hline
\rightarrow & \rightarrow & \rightarrow & \rightarrow \\
\hline
\end{array}
\quad
\begin{array}{|c|c|c|c|c|}
\hline
\triangleright & \triangleright & \perp & \rightarrow \\
\hline
\triangleright & \triangleright & \rightarrow & \rightarrow \\
\hline
\triangleright & \triangleright & \rightarrow & \rightarrow \\
\hline
\end{array}
\quad
\begin{array}{|c|c|c|c|c|}
\hline
\triangleleft & \triangleleft & \perp & \rightarrow \\
\hline
\triangleleft & \triangleleft & \rightarrow & \rightarrow \\
\hline
\triangleleft & \triangleleft & \rightarrow & \rightarrow \\
\hline
\end{array}
\quad
\begin{array}{|c|c|c|c|c|}
\hline
\rightarrow & \rightarrow & \perp & \rightarrow \\
\hline
\rightarrow & \rightarrow & \rightarrow & \rightarrow \\
\hline
\rightarrow & \rightarrow & \rightarrow & \rightarrow \\
\hline
\end{array}
\]

Proof. 1. This is a reformulation of axioms((21) or (22) or (23)) and (24) – (30).

2. This is a reformulation of axioms (21) – (27), and ((28) or (29) or (30)).

Using Koszul duality, we obtain tridendriform algebras \[20\] \[3\] \[28\], that is to say families \((A, <, >, \cdot)\) where \(A\) is a vector space and \(<, >\) and \(\cdot\) are bilinear products on \(A\) such that for any \(x, y, z \in A\):

\[
\begin{align*}
(x < y) < z &= x < (y < z + y > z + y \cdot z), \\
(x > y) < z &= x > (y < z), \\
x > (y > z) &= (x < y + x > y + x \cdot y) > z, \\
(x > y) \cdot z &= x > (y \cdot z), \\
(x < y) \cdot z &= x \cdot (y > z), \\
(x \cdot y) < z &= x \cdot (y < z), \\
(x \cdot y) \cdot z &= x \cdot (y \cdot z).
\end{align*}
\]
Corollary 4.9. Let \((V, \Phi)\) is one of the three following 3-dimensional \(\ell\)-EAS, where the matrix of \(\Phi\) is expressed in the basis \(e_1 \otimes e_1, e_1 \otimes e_2, e_1 \otimes e_3, e_2 \otimes e_1, e_2 \otimes e_2, e_2 \otimes e_3, e_3 \otimes e_1, e_3 \otimes e_2, e_3 \otimes e_3\):

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
\]

Then any \(\Phi\)-associative algebra is a tridendriform algebra, with \(\preceq \ast_1, \succeq \ast_2 \) and \(\perp \ast_3\). Any opposite \(\Phi\)-associative algebra is a tridendriform algebra, with \(\preceq \ast_2, \succeq \ast_1 \) and \(\perp \ast_3\).

4.4 Dual duplicial and duplicial algebras

Definition 4.10. A dual duplicial algebra is a family \((A, <, >)\) where \(A\) is a vector space and \(<\) and \(>\) are bilinear products on \(A\) such that for any \(x, y, z \in A\):

\[
(x < y) < z = x < (y < z),
\]

\[
(x < x) > z = 0, 
\]

\[
(x > y) < z = x > (y < z),
\]

\[
0 = x < (y > z),
\]

\[
(x > y) > z = x > (y > z).
\]

Proposition 4.11. Let \((V, \Phi)\) be the following 2-dimensional \(\ell\)-EAS, where the matrix of \(\Phi\) is expressed in the basis \(e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2\):

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}
\]

Then any dual duplicial algebra \((A, <, >)\) is an opposite \(\Phi\)-associative algebra, with \(\ast_1 = <\) and \(\ast_2 = >\), and a \(\Phi\)-associative algebra, with \(\ast_1 = >\) and \(\ast_2 = <\).

Proof. This is a reformulation of axioms (31) – (33) and (35), and of axioms (31) and (33) – (35). \(\square\)

Using Koszul duality, we recover duplicial algebra, that is to say families \((A, <, >)\) where \(A\) is a vector space and \(<\) and \(>\) are bilinear products on \(A\) such that for any \(x, y, z \in A\):

\[
(x < y) < z = x < (y < z),
\]

\[
(x > y) < z = x > (y < z),
\]

\[
x > (y > z) = (x > y) > z.
\]

Corollary 4.12. Let \((V, \Phi)\) be the following 2-dimensional \(\ell\)-EAS, where the matrix of \(\Phi\) is expressed in the basis \(e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2\):

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}
\]

Then any \(\Phi\)-associative algebra is a duplicial algebra, with \(\ast_1 = <\) and \(\ast_2 = >\), and any opposite \(\Phi\)-associative algebra is a duplicial algebra, with \(\ast_1 = >\) and \(\ast_2 = <\).
References

[1] Marcelo Aguiar, Dendriform algebras relative to a semigroup, SIGMA Symmetry Integrability Geom. Methods Appl. 16 (2020), Paper No. 066, 15.

[2] Murray R. Bremner and Vladimir Dotsenko, Algebraic operads, CRC Press, Boca Raton, FL, 2016, An algorithmic companion.

[3] Frédéric Chapoton, Algèbres de Hopf des permutahèdres, associahèdres et hypercubes., Adv. Math. 150 (2000), no. 2, 264–275 (French).

[4] Frédéric Chapoton, Florent Hivert, and Jean-Christophe Novelli, A set-operad of formal fractions and dendriform-like sub-operads, J. Algebra 465 (2016), 322–355.

[5] Zhi Chen and Hao Pan, Identities involving weighted Catalan, Schröder and Motzkin paths, Adv. in Appl. Math. 86 (2017), 81–98.

[6] Camille Combe and Samuele Giraudo, Cliff operads: a hierarchy of operads on words, 2020.

[7] Charles Curry, Kurusch Ebrahimi-Fard, and Hans Munthe-Kaas, What is a post-Lie algebra and why is it useful in geometric integration, Numerical mathematics and advanced applications. ENUMATH 2017. Selected papers based on the presentations at the European conference, Bergen, Norway, September 25–29, 2017, Cham: Springer, 2018, pp. 429–437 (English).

[8] Charles Curry, Kurusch Ebrahimi-Fard, and Brynjulf Owren, The Magnus expansion and post-Lie algebras, Math. Comput. 89 (2020), no. 326, 2785–2799 (English).

[9] Vladimir Dotsenko, Functorial PBW theorems for post-Lie algebras, Commun. Algebra 48 (2020), no. 5, 2072–2080 (English).

[10] Kurusch Ebrahimi-Fard, Alexander Lundervold, and Hans Z. Munthe-Kaas, On the Lie enveloping algebra of a post-Lie algebra, J. Lie Theory 25 (2015), no. 4, 1139–1165 (English).

[11] Kurusch Ebrahimi-Fard, Igor Mencattini, and Hans Munthe-Kaas, Post-Lie algebras and factorization theorems, J. Geom. Phys. 119 (2017), 19–33 (English).

[12] Loïc Foissy, Algebraic structures on typed decorated rooted trees, arXiv:1811.07572, 2018.

[13] Loïc Foissy, Extension of the product of a post-Lie algebra and application to the SISO feedback transformation group, Computation and combinatorics in dynamics, stochastics and control. The Abel symposium, Rosendal, Norway, August 16–19, 2016. Selected papers, Cham: Springer, 2018, pp. 369–399 (English).

[14] Loïc Foissy, Generalized dendriform algebras and typed binary trees, arXiv:2002.12120, 2020.

[15] Loïc Foissy, Generalized prelie and permutative algebras, in preparation, 2021.

[16] Loïc Foissy, On extended associative semigroups, in preparation, 2021.

[17] Loïc Foissy, Dominique Manchon, and Yuanyuan Zhang, Families of algebraic structures, arXiv:2005.05116, 2020.

[18] Samuele Giraudo, Pluriassociative algebras II: The polydendriform operad and related operads, Adv. in Appl. Math. 77 (2016), 43–85.

[19] Vsevolod Gubarev, Poincaré-Birkhoff-Witt theorem for pre-Lie and post-Lie algebras, J. Lie Theory 30 (2020), no. 1, 223–238 (English).
[20] Jean-Louis Loday, *Dialgebras*, Dialgebras and related operads, Berlin: Springer, 2001, pp. 7–66 (English).

[21] _, Generalized bialgebras and triples of operads*, vol. 320, Paris: Société Mathématique de France, 2008 (English).

[22] Jean-Louis Loday and María Ronco, *Trialgebras and families of polytopes*, Homotopy theory: relations with algebraic geometry, group cohomology, and algebraic K-theory. Papers from the international conference on algebraic topology, Northwestern University, Evanston, IL, USA, March 24–28, 2002, Providence, RI: American Mathematical Society (AMS), 2004, pp. 369–398 (English).

[23] Jean-Louis Loday and Bruno Vallette, *Algebraic operads*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 346, Springer, Heidelberg, 2012.

[24] Dominique Manchon and Yuanyuan Zhang, *Free pre-Lie family algebras*, arXiv:2003.00917, 2020.

[25] Martin Markl, *Operads and PROPs*, Handbook of algebra. Vol. 5, Handb. Algebr., vol. 5, Elsevier/North-Holland, Amsterdam, 2008, pp. 87–140.

[26] Martin Markl, Steve Shnider, and Jim Stasheff, *Operads in algebra, topology and physics*, Mathematical Surveys and Monographs, vol. 96, American Mathematical Society, Providence, RI, 2002.

[27] Miguel A. Méndez, *Set operads in combinatorics and computer science*, SpringerBriefs in Mathematics, Springer, Cham, 2015.

[28] Jean-Christophe Novelli and Jean-Yves Thibon, *Construction de trigèbres dendriformes.*, C. R., Math., Acad. Sci. Paris 342 (2006), no. 6, 365–369 (French).

[29] Teimuraz Pirashvili, *Sets with two associative operations*, Cent. Eur. J. Math. 1 (2003), no. 2, 169–183.

[30] Bruno Vallette, *Homology of generalized partition posets*, J. Pure Appl. Algebra 208 (2007), no. 2, 699–725 (English).

[31] Tadepalli Venkata Narayana, *Sur les treillis formés par les partitions d’un entier et leurs applications à la théorie des probabilités*, C. R. Acad. Sci. Paris 240 (1955), 1188–1189.

[32] Donald Yau, *Colored operads*, Graduate Studies in Mathematics, vol. 170, American Mathematical Society, Providence, RI, 2016.

[33] Yi Zhang, Xing Gao, and Li Guo, *Matching Rota-Baxter algebras, matching dendriform algebras and matching pre-Lie algebras*, J. Algebra 552 (2020), 134–170.

[34] Yuanyuan Zhang and Xing Gao, *Free Rota-Baxter family algebras and (tri)dendriform family algebras*, Pacific J. Math. 301 (2019), no. 2, 741–766.

[35] Yuanyuan Zhang, Xing Gao, and Dominique Manchon, *Free (tri)dendriform family algebras*, J. Algebra 547 (2020), 456–493.

[36] G.W ZINBIEL, *Encyclopedia of types of algebras*, https://irma.math.unistra.fr/~loday/PAPERS/2010Zinbiel.pdf, 2010.