Infinite-dimensional stochastic differential equations related to Bessel random point fields

Ryuich Honda, Hiroyumi Osada
Faculty of Mathematics, Kyushu University,
Fukuoka 819-0395, Japan
osada@math.kyushu-u.ac.jp

Abstract

We solve the infinite-dimensional stochastic differential equations (ISDEs) describing an infinite number of Brownian particles in \( \mathbb{R}^+ \) interacting through the two-dimensional Coulomb potential. The equilibrium states of the associated unlabeled stochastic dynamics are Bessel random point fields. To solve these ISDEs, we calculate the logarithmic derivatives, and we prove that the random point fields are quasi-Gibbsian.

Keywords: Interacting Brownian particles Bessel random point fields random matrices infinite-dimensional stochastic differential equations Coulomb potentials hard edge scaling limit

2000 MSC: 82C22 15A52 60J60 60K35 82B21

Abstract

We solve the infinite-dimensional stochastic differential equations (ISDEs) describing an infinite number of Brownian particles in \( \mathbb{R}^+ \) interacting through the two-dimensional Coulomb potential. The equilibrium states of the associated unlabeled stochastic dynamics are Bessel random point fields. To solve these ISDEs, we calculate the logarithmic derivatives, and we prove that the random point fields are quasi-Gibbsian.

1 Introduction

Bessel random point fields \( \mu_\alpha \) \((-1 < \alpha < \infty)\) are probability measures on the configuration space \( S \) over \( S = [0, \infty) \), whose \( n \)-point correlation functions \( \rho^n_\alpha \) (see (2.3)) with respect to the Lebesgue measure are given by

\[
\rho^n_\alpha(x_1, \ldots, x_n) = \det[K_\alpha(x_i, x_j)]_{1 \leq i, j \leq n}.
\]
Here $K_\alpha(x, y)$ is the Bessel kernel defined with the Bessel function $J_\alpha$ of order $\alpha$ such that

$$K_\alpha(x, y) = \frac{J_\alpha(\sqrt{x}) \sqrt{y} J'_\alpha(\sqrt{y}) - \sqrt{x} J'_\alpha(\sqrt{x}) \sqrt{y} J_\alpha(\sqrt{y})}{2(x - y)}. \quad (1.2)$$

We note that $0 \leq K_\alpha \leq \text{Id}$ as an operator on $L^2(S, dx)$. By definition $\mu_\alpha$ are determinantal random point fields with Bessel kernels $K_\alpha$ (see [21]).

It is known that these random point fields arise as a scaling limit at the hard left edge of the distributions $\mu^n_\alpha$ of the spectrum of the Laguerre ensemble. The random point fields $\mu_\alpha$ represent the thermodynamic limit of the $n$-particle systems $\mu^n_\alpha$, whose labeled densities $\sigma^n_\alpha(x)dx$ are given by

$$\sigma^n_\alpha(x) = 1 \frac{\prod_{j=1}^n x^\alpha_j \prod_{k<l} |x_k - x_l|^2. \quad (1.3)}{Z^n_\alpha \sum_{i=1}^n x^{1/4n} \prod_{j=1}^n x^\alpha_j \prod_{k<l} |x_k - x_l|^2. \quad (1.3)}$$

Very loosely, by taking $n$ to infinity, we obtain an informal expression for the $\mu_\alpha$ as follows:

$$\mu_\alpha(dx) = 1 \frac{\prod_{j=1}^\infty x^\alpha_j \prod_{k<l} |x_k - x_l|^2 \prod_{m=1}^\infty dx_m. \quad (1.4)}{Z^\infty_\alpha \prod_{j=1}^\infty x^\alpha_j \prod_{k<l} |x_k - x_l|^2 \prod_{m=1}^\infty dx_m. \quad (1.4)}$$

Hence we regard the $\mu_\alpha$ as random point fields with free potentials $\Phi_\alpha(x) = -\alpha \log x$ and interaction potential $\Psi(x) = -2 \log |x|$. Unlike Ruelle’s class of interaction potentials, one can not justify this by use of the Dobrushin-Lanford-Ruelle (DLR) equations. Instead, we will proceed in terms of logarithmic derivatives in Theorem 2.3.

We next turn to the stochastic dynamics associated with the $\mu^n_\alpha$. To prevent the particles from hitting the origin, we suppose that $1 \leq \alpha$. Then, from Eq. (1.3), it can be seen that the natural $n$-particle stochastic dynamics $X^n = (X^{n,1}_t, \ldots, X_t^n)$ are given by the stochastic differential equations (SDEs)

$$dX^{n,i}_t = dB^i_t + \left\{-\frac{1}{8n} + \frac{\alpha}{2X^{n,i}_t} + \sum_{j \neq i}^n \frac{1}{X^{n,i}_t - X^{n,j}_t}\right\}dt \quad (1 \leq i \leq n). \quad (1.5)$$

Hence, taking $n$ to infinity, we come to the infinite-dimensional stochastic differential equations

$$dX^i_t = dB^i_t + \left\{\frac{\alpha}{2X^i_t} + \sum_{j \neq i}^\infty \frac{1}{X^i_t - X^j_t}\right\}dt \quad (i \in \mathbb{N}). \quad (1.6)$$

2
The purpose of this paper is to solve these ISDEs in such a way that the equilibrium states of the associated unlabeled dynamics \( X_t = \sum_{i=1}^{\infty} \delta_{X_i^t} \) are Bessel random point fields \( \mu_\alpha \).

For a given free potential \( \Phi \) and interaction potential \( \Psi \), the interacting Brownian motions in infinite dimensions are the stochastic dynamics given by ISDEs of the form

\[
\frac{dX_i^t}{dt} = \frac{\beta}{2} \nabla \Phi(X_i^t) dt + \frac{\beta}{2} \sum_{j \neq i} \nabla \Psi(X_i^t, X_j^t) dt \quad (i \in \mathbb{N}).
\]

(1.7)

Here \( \{B_i^t\}_{i \in \mathbb{N}} \) is a sequence of independent copies of \( d \)-dimensional Brownian motions. The study of interacting Brownian motions in infinite dimensions was initiated by Lang [8], [9], followed by Shiga [19], Fritz [5], Tanemura [24], and others. In these works, \( \Psi \) is assumed to be a Ruelle type potential: that is, \( \Psi \) is super-stable and integrable at infinity. In addition, \( \Psi \) is assumed to be of class \( C_0^3 \) ([8, 9, 19, 5]) or to decay exponentially at infinity with a hard core ([24, 25]). Hence, polynomial decay potentials are excluded even for Ruelle’s category.

Recently, an interesting class of random point fields has appeared from random matrix theory such as the sine, Airy, and Bessel random point fields in one-dimensional space and the Ginibre random point field in two dimensions. These represent the thermodynamic limits of the distributions of Gaussian random matrices. There are many other such random point fields that emerge from random matrix theory, but these examples are of particular note. The sine, Airy, and Bessel random point fields describe the universality classes called bulk, soft-edge, and hard-edge scaling limits, respectively. The Ginibre random point field is rotation and translation invariant, and thus is the typical example in two dimensions.

In these random point fields, the interactions always have logarithmic potentials and so represent the outer side of the classical theory of interacting Brownian motions in infinite dimensions. In [17, 18, 16] the second author (H.O.) developed the theory applicable to these examples except for Bessel random point fields. Here we solve the ISDEs for these random point fields, which describe the remaining universality class in one dimension.

This paper is organized as follows: In Section 2 we establish the mathematical framework and state the main results (Theorems 2.1–2.4). In Section 3, we prove Theorems 2.1 and 2.2 by using Theorems 2.3 and 2.4 in
combination with the general theory developed in [17, 18, 16]. In Section 4, we set forth Theorem 4.1 in preparation for Section 5, where we calculate the logarithmic derivatives of Bessel random point fields and prove Theorem 2.3. In Section 6, we prove that they are quasi-Gibbsian (Theorem 2.4). In Section 7, we prove Lemma 5.2.

2 Set up and main results

Let $S = [0, \infty)$ and $S_r = \{x \in S; \ x < r\}$. Let

$$S = \{s = \sum_{i} \delta_{a_i}; \ s_i \in S, \ s(S_r) < \infty \ \text{for all} \ r \in \mathbb{N}\},$$

where $\delta_a$ stands for the delta measure at $a$. We endow $S$ with the vague topology, under which $S$ is a Polish space. $S$ is called the configuration space over $S$. We write $s(x) = s(\{x\})$. Let

$$S_{s.i.} = \{s \in S; \ s(x) \leq 1 \ \text{for all} \ x \in S, \ s(S) = \infty\}. \quad (2.1)$$

By definition, $S_{s.i.}$ is the set of the configurations consisting of an infinite number of single point measures.

For an infinite or finite product $S^k$ of $S$ we define the map $u$ from $S^k$ to the set of measures on $S$ by $u((s_j)) = \sum_{j=1}^{k} \delta_{s_j}$. We omit $k$ from the notation of $u$. Let $u_{\text{path}}$ be the map defined by

$$u_{\text{path}}(X) = \{\sum_{j=1}^{k} \delta_{X_t^j}\}_{0 \leq t < \infty}, \quad (2.2)$$

where $X = \{(X_t^j)_{j=1}^{k}\}$. We set $X = u_{\text{path}}(X)$. We call $X$ (resp. $X$) the (fully) labeled process (unlabeled process). Moreover, for $k \in \{0\} \cup \mathbb{N}$, the process $(X_t^1, \ldots, X_t^k; \sum_{j>k} \delta_{X_t^j})$ is said to be the $k$-labeled process. When $k = 0$, $k$-labeled process equals the unlabeled process $u_{\text{path}}(X)$.

A symmetric locally integrable function $\rho^n : S^n \to [0, \infty)$ is called the $n$-point correlation function of a probability measure $\mu$ on $S$ w.r.t. the Lebesgue measure if $\rho^n$ satisfies

$$\int_{A_1 \times \cdots \times A_n} \rho^n(x_1, \ldots, x_n)dx_1 \cdots dx_n = \int_{S} \prod_{i=1}^{n} \frac{s(A_i)!}{(s(A_i) - k_i)!} d\mu \quad (2.3)$$

for any sequence of disjoint bounded measurable subsets \( A_1, \ldots, A_m \subset S \) and a sequence of natural numbers \( k_1, \ldots, k_m \) satisfying \( k_1 + \cdots + k_m = n \). When \( s(A_i) - k_i < 0 \), according to our interpretation, \( s(A_i)!/(s(A_i) - k_i)! = 0 \) by convention. It is known that under a mild condition \( \{\rho^n\}_{n \in \mathbb{N}} \) determines the measure \( \mu \) [21].

Let \( \mu_\alpha \) be Bessel random point fields. By definition \( \mu_\alpha \) are probability measures on \( S \) whose \( n \)-point correlation functions \( \rho^n_\alpha \) are given by (1.1).

**Theorem 2.1.** Assume that \( 1 \leq \alpha < \infty \). Then for each \( \alpha \) there exists a set \( S_{be} \) such that

\[
\mu_\alpha(S_{be}) = 1, \quad S_{be} \subset S_{s.i}, \tag{2.4}
\]

and that, for all \( s \in u^{-1}(S_{be}) \), there exists a \([0, \infty)^{\mathbb{N}}\)-valued continuous process \( X = (X^i)_{i \in \mathbb{N}} \), and \( \mathbb{R}^{\mathbb{N}} \)-valued Brownian motion \( B = (B^i)_{i \in \mathbb{N}} \) satisfying

\[
dX^i_t = dB^i_t + \left\{ \frac{\alpha}{2X^i_t} + \sum_{j \neq i} \frac{1}{X^i_t - X^j_t} \right\} dt \quad (i \in \mathbb{N}), \tag{2.5}
\]

\[
X_0 = s. \tag{2.6}
\]

Moreover, \( X \) satisfies

\[
P(u(X_t) \in S_{be}, \ 0 \leq \forall t < \infty) = 1. \tag{2.7}
\]

**Remark 2.1.** When \(-1 < \alpha < 1\), the left most particle hits the origin. Hence a coefficient coming from the boundary condition will appear in the ISDEs. Since we suppose \( 1 \leq \alpha \), particles never hit the origin. It would be an interesting problem to study the case \(-1 < \alpha < 1\) where the boundary condition would appear.

A diffusion with state space \( S_0 \) is a family of continuous stochastic processes with the strong Markov property starting at each point of the state space \( S_0 \). In general, the notion of the Markov property depends on the filtering. We always consider the natural filtering in the present paper [4].

**Theorem 2.2.** Assume that \( 1 \leq \alpha < \infty \). Let \( S_{be} = u^{-1}(S_{be}) \). Let \( P_s \) be the distribution of \( X \) given by Theorem 2.1. Then \( \{P_s\}_{s \in S_{be}} \) is a diffusion with state space \( S_{be} \).

\(^1\)\( \alpha \) boundary term, \((0, \infty)\)
We deduce Theorem 2.1 and Theorem 2.2 from a general theory developed in [17, 18, 16]. The key point for this is to calculate the logarithmic derivative of the measure \( \mu_\alpha \) and to prove the quasi-Gibbs property of \( \mu_\alpha \). These two notions play an important role in the proof of Theorem 2.1 and Theorem 2.2.

The logarithmic derivative of \( \mu_\alpha \) will be calculated in Theorem 2.1. We will use Theorem 4.1 to prove Theorem 2.1 in Section 5. The quasi-Gibbs property of \( \mu_\alpha \) will be proved in Theorem 2.4. Theorem 2.4 will be proved in Section 6.

To introduce the notion of the logarithmic derivative of random point fields we recall the definitions of reduced Palm measures and Campbell measures.

Let \( \mu \) be a probability measure on \((S, \mathcal{B}(S))\). A probability measure \( \mu_x \) is called the reduced Palm measure conditioned at \( x = (x_1, \ldots, x_k) \in S^k \) if \( \mu_x \) is the regular conditional probability defined by

\[
\mu_x = \mu(\cdot - \sum_{i=1}^k \delta_{x_i} \mid s(x_i) \geq 1 \text{ for } i = 1, \ldots, k).
\]

(2.8)

Let \( \rho^k \) be the \( k \)-point correlation function of \( \mu \) with respect to the Lebesgue measure. Let \( \mu^k \) be the measure on \( S^k \times S \) defined by

\[
\mu^k(A \times B) = \int_A \mu_x(B) \rho^k(x) dx.
\]

(2.9)

Here we set \( dx = dx_1 \cdots dx_k \) for \( x = (x_1, \ldots, x_k) \in S^k \). The measure \( \mu^k \) is called the \( k \)-Campbell measure.

**Definition 2.1.** We call \( d^\mu \in L^1_{\text{loc}}(\mu^1) \) the logarithmic derivative of \( \mu \) if \( d^\mu \) satisfies

\[
\int_{S \times S} d^\mu f d\mu^1 = -\int_{S \times S} \frac{\partial f(x,s)}{\partial x} d\mu^1 \quad \text{for all } f \in C^\infty_0((0, \infty)) \otimes C_b(S).
\]

(2.10)

Very loosely, (2.10) can be written as \( d^\mu = \partial \log \mu^1(x,s)/\partial x \). This intuitive expression is the reason why we call \( d^\mu \) the logarithmic derivative of \( \mu \).
Theorem 2.3. Assume that $1 \leq \alpha < \infty$. Then $\mu_\alpha$ has the logarithmic derivative $d^{\mu_\alpha} \in L^2_{\text{loc}}(\mu_1^1)$ defined by

$$d^{\mu_\alpha}(x, y) = \frac{\alpha}{x} + \sum_{i \in \mathbb{N}} \frac{2}{x - y_i}. \quad (2.11)$$

Here $y = \sum_{i \in \mathbb{N}} \delta_{y_i}$.

Remark 2.2. If $-1 < \alpha < 1$, then the left most particle hits the origin. Hence in the definition of logarithmic derivative, it would be more natural to take $C^\infty_0((0, \infty)) \otimes C_b(S)$ as a space of test functions. In this case, the logarithmic derivative contains a term arising from the boundary condition. Although it would be interesting to study this case, we do not pursue this here.

We next introduce the notion of the quasi-Gibbs property.

For two measures $\nu_1, \nu_2$ on a measurable space $(\Omega, \mathcal{B})$ we write $\nu_1 \leq \nu_2$ if $\nu_1(A) \leq \nu_2(A)$ for all $A \in \mathcal{B}$. We say a sequence of finite Radon measures $\{\nu^n\}$ on a Polish space $\Omega$ converge weakly to a finite Radon measure $\nu$ if $\lim_{n \to \infty} \int f \, d\nu^n = \int f \, d\nu$ for all $f \in C_b(\Omega)$.

Let $S^m_r = \{x \in S : x(S_r) = m\}$ and $\Lambda^m_r = \Lambda(\cdot \cap S^m_r)$, where $\Lambda$ is the Poisson random point field whose intensity is the Lebesgue measure.

Definition 2.2. A probability measure $\mu$ is said to be a $(\Phi, \Psi)$-quasi Gibbs measure if, for all $r, m \in \mathbb{N}$ and for $\mu^m_{r,k}$-a.e. $s \in S$,

$$c_1 e^{-\mathcal{H}_r(x)} \Lambda^m_r(dx) \leq \mu^m_{r,k,s}(dx) \leq c_1 e^{-\mathcal{H}_r(x)} \Lambda^m_r(dx). \quad (2.12)$$

Here $c_1 = \prod_{r, m, k} \pi^c_r(s)$ is a positive constant and $\mu^m_{r,k,s}$ is the regular conditional probability measure of $\mu^m_{r,k}$ defined by

$$\mu^m_{r,k,s}(dx) = \mu^m_{r,k}(\pi_r \in dx \mid \pi^c_r(s)) \quad (2.13)$$

and $\mathcal{H}_r(x)$ is the Hamiltonian on $S_r$ defined by

$$\mathcal{H}_r(x) = \sum_{x_i \in S_r} \Phi(x_i) + \frac{1}{2} \sum_{x_i, x_j \in S_r, i \neq j} \Psi(x_i, x_j). \quad (2.14)$$

Here we set $x = \sum_{i} \delta_{x_i}$.
The notion of quasi-Gibbsian is first introduced in [17] by the second author (H.O.). The original definition of the quasi-Gibbs measures is slightly general than the present version, and is essentially the same. We adopt here a restrictive version for the sake of simplicity.

We remark that we do not assume the symmetry of the interaction potential $\Psi$. Hence we take the unordered summation of $\Psi(x_i, x_j)$, and put $1/2$ in the sum of (2.14).

**Theorem 2.4.** Let $-1 < \alpha < \infty$. Then $\mu_\alpha$ is a $(\alpha \log x, 2 \log|x-y|)$-quasi Gibbs measure.

Combining Theorem 2.4 with a general theory [17, Corollary 2.1], we obtain a natural unlabeled $\mu_\alpha$-reversible diffusion $(X, P)$.

**Theorem 2.5.** Let $-1 < \alpha < \infty$. Let $\mathcal{E}^{\mu_\alpha}$ and $\mathcal{D}^{\mu_\alpha}$ be as in (3.4) and (3.5) with $k = 0$ and $\mu = \mu_\alpha$. Then $(\mathcal{E}^{\mu_\alpha}, \mathcal{D}^{\mu_\alpha})$ is closable on $L^2(S, \mu_\alpha)$. There exists a diffusion $(X, P)$ associated with the closure of $(\mathcal{E}^{\mu_\alpha}, \mathcal{D}^{\mu_\alpha})$ on $L^2(S, \mu_\alpha)$.

**Remark 2.3.** (1) If $-1 < \alpha < 1$, then the left most particle hits the origin.

(2) We write the diffusion $X$ in Theorem 2.5 as $X_t = \sum_{i \in \mathbb{N}} \delta_{X^i_t}$. Since the particles never collide each other, the infinite-dimensional labeled paths $(X^i_t)_{i \in \mathbb{N}}$ is well defined. Then with suitable labeling of the unlabeled particles $X$ at time $t = 0$, the solution of the ISDE $(X^i_t - X^0_0)_{i \in \mathbb{N}}$ becomes an infinite-dimensional additive functional of the unlabeled diffusion $(X, P)$. We remark that this additive functional is not Dirichlet process because no coordinate functions $x_i$ $(i \in \mathbb{N})$ belong to the domain of the Dirichlet form even if locally.

(3) There are other approaches for this kind of unlabeled stochastic dynamics related to random matrix theory. See [22], [6], [7], [1], [2], and [12]. These approaches are more algebraic, and restricted to one dimensional system with inverse temperature $\beta = 2$.

### 3 Proof of Theorems 2.1 and 2.2

The purpose of this section is to prove Theorems 2.1 and 2.2. For this we will use Theorem 2.3 and Theorem 2.4, and a result from [10] in a reduced form being sufficient for the present problem.

Let $k \in \{0\} \cup \mathbb{N}$. We introduce Dirichlet forms describing the $k$-labeled process introduced at the beginning of Section 2. For a subset $A \subset S$ we
define the map $\pi_A : S \to S$ by $\pi_A(s) = s(A \cap \cdot)$. We say a function $f : S \to \mathbb{R}$ is local if $f$ is $\sigma[\pi_A]$-measurable for some compact set $A \subset S$. We say $f$ is smooth if $f$ is smooth, where $\tilde{f}((s_i))$ is the permutation invariant function in $(s_i)$ such that $f(s) = \tilde{f}((s_i))$ for $s = \sum_i \delta_{s_i}$.

Let $\mathcal{D}_o$ be the set of all local, smooth functions on $S$. For $f, g \in \mathcal{D}_o$ we set $\mathbb{D}[f, g] : S \to \mathbb{R}$ by

$$
\mathbb{D}[f, g](s) = \frac{1}{2} \sum_i \frac{\partial \tilde{f}(s)}{\partial s_i} \frac{\partial \tilde{g}(s)}{\partial s_i}.
$$

(3.1)

Here $s = \sum_i \delta_{s_i}$ and $s = (s_i)$. For given $f$ and $g$ in $\mathcal{D}_o$, it is easy to see that the right-hand side of (3.1) depends only on $s$. So $\mathbb{D}[f, g]$ is well defined. For $f, g \in C_0^\infty(S^k) \otimes \mathcal{D}_o$ let $\nabla^k[f, g]$ be the function on $S^k \times S$ defined by

$$
\nabla^k[f, g](x, s) = \frac{1}{2} \sum_{j=1}^k \frac{\partial f(x, s)}{\partial x_j} \frac{\partial g(x, s)}{\partial x_j}.
$$

(3.2)

where $x = (x_j) \in S^k$. We set $\mathbb{D}^k$ for $k \geq 1$ by

$$
\mathbb{D}^k[f, g](x, s) = \nabla^k[f, g](x, s) + \mathbb{D}[f(x, \cdot), g(x, \cdot)](s).
$$

(3.3)

Let $(\mathcal{E}^{\mu^k}, \mathcal{D}^{\mu^k})$ be the bilinear form defined by

$$
\mathcal{E}^{\mu^k}(f, g) = \int_{S^k \times S} \mathbb{D}^k[f, g]d\mu^k,
$$

(3.4)

$$
\mathcal{D}^{\mu^k} = \{ f \in C_0^\infty(S^k) \otimes \mathcal{D}_o \cap L^2(S^k \times S, \mu^k) : \mathcal{E}^{\mu^k}(f, f) < \infty \}.
$$

(3.5)

When $k = 0$, we take $\mathbb{D}^0 = \mathbb{D}$, $\mu^0 = \mu$, and $\mathcal{E}^\mu = \mathcal{E}$. We set $L^2(\mu) = L^2(S, \mu)$ and $L^2(\mu^k) = L^2(S^k \times S, \mu^k)$ and so on.

We assume that there exists a probability measure $\mu$ on $S$ with correlation functions $\{\rho^k\}_{k \in \mathbb{N}}$ satisfying (A.1)–(A.5):

(A.1) $\rho^k$ is locally bounded for each $k \in \mathbb{N}$.

(A.2) There exists a logarithmic derivative $d^\mu$ in the sense of (2.10).

(A.3) $(\mathcal{E}^{\mu^k}, \mathcal{D}^{\mu^k})$ is closable on $L^2(\mu^k)$ for each $k \in \{0\} \cup \mathbb{N}$.

(A.4) $\text{Cap}^\mu(\{S_{s, i} \}^c) = 0$.

(A.5) There exists a $T > 0$ such that for each $R > 0$

$$
\liminf_{r \to \infty} \left\{ \int_{|x| \leq r + R} \rho^1(x)dx \right\} \left\{ \int_{\sqrt{\delta^2 + R^2}}^\infty e^{-u^2/2}du \right\} = 0.
$$

(3.6)
Let \((E^\mu, D^\mu)\) be the closure of \((E^\mu, D^\mu_0)\) on \(L^2(\mu^k)\). It is known \([15, \text{Lemma } 2.3]\) that \((E^\mu, D^\mu)\) is quasi-regular and that the associated diffusion \((P^k, X^k)\) exists. We refer to \([10]\) for the definition and necessary background of quasi-regular Dirichlet forms. We remark that \(\text{Cap}^\mu\) in \((A.4)\) is the capacity of the Dirichlet space \((E^\mu, D^\mu, L^2(\mu))\).

The assumptions \((A.4)\) and \((A.5)\) have clear dynamical interpretations. Indeed, \((A.4)\) means that particles never collide with each other. Moreover, \((A.5)\) means that no labeled particle ever explodes \([15]\).

We quote two theorems from \([16]\).

**Theorem 3.1** \([16, \text{Theorem } 26]\). Assume \((A.1)\)–\((A.5)\). Then there exists an \(S_0\) such that

\[
\mu(S_0) = 1, \quad S_0 \subset S_\text{a.i.},
\]

and that, for all \(s \in u^{-1}(S_0)\), there exists an \(S^\alpha\)-valued continuous process \(X = (X^i)_{i \in \mathbb{N}},\) and \((\mathbb{R})^N\)-valued Brownian motion \(B = (B^i)_{i \in \mathbb{N}}\) satisfying

\[
dX^i_t = dB^i_t + \frac{1}{2}d\mu(X^i_t, X^*_t)dt \quad (i \in \mathbb{N}),
\]

\[
X_0 = s.
\]

Moreover, \(X\) satisfies

\[
P(u(X_t) \in S_0, \ 0 \leq \forall t < \infty) = 1.
\]

**Theorem 3.2** \([16, \text{Theorem } 27]\). Let \(S_0\) be the subset of \(S^\alpha\) defined by \(S_0 = u^{-1}(S_0)\). Let \(P_s\) be the distribution of \(X\) given by Lemma \(2.1\). Then \(\{P_s\}_{s \in S_0}\) is a diffusion with state space \(S_0\).

We take \(\mu = \mu_\alpha\). Then the assumptions \((A.1)\), \((A.4)\), and \((A.5)\) are easily checked as we see in the next lemma.

**Lemma 3.3.** \(\mu_\alpha\) satisfy \((A.1), (A.4),\) and \((A.5)\).

**Proof.** \((A.1)\) and \((A.5)\) are clear because the correlation functions \(\{\rho^n_\alpha\}\) of \(\mu_\alpha\) are given by the equation \((1.1)\) and the kernels \(K^n_\alpha\) are locally bounded in \((0, \infty)\) and bounded in \([1, \infty)\). \((A.4)\) follows from \([14, \text{Theorem } 2.1]\) because the kernel \(K^n_\alpha\) is locally Lipschitz continuous.

We next deduce Theorems \(2.1\) and \(2.2\) from Theorems \(2.3\) and \(2.4\).
Proof of Theorems 2.1 and 2.2. We will use Theorems 3.1 and 3.2 to prove Theorems 2.1 and 2.2. For this we check the assumptions (A.1)–(A.5) with a help of Theorems 2.3 and 2.4.

The assumption (A.2) follows from Theorem 2.3. From Lemma 3.3 we have already known that $\mu_\alpha$ satisfy (A.1), (A.4), and (A.5). From Theorem 2.4 we see that $\mu_\alpha$ are quasi-Gibbsian with continuous potentials. In [17, Lemma 3.6], it was proved that, when potentials are upper semi-continuous, the closability in (A.3) for $k = 0$ follows from the quasi-Gibbs property. Then we have (A.3) for $k = 0$. The closability for general $k \geq 1$ also follows from the quasi-Gibbs property of $\mu_\alpha$ in a similar fashion. Hence we obtain (A.3) for $\mu_\alpha$.

We have thus seen that the assumptions (A.1)–(A.5) are fulfilled. Hence Theorems 2.1 and 2.2 follows from Theorems 3.1 and 3.2, respectively. □

In the rest of the paper we devote to the proof of Theorems 2.3 and 2.4.

4 Logarithmic derivative of random point fields.

Let $\mu$ be a probability measure on $S$ with locally bounded $n$-point correlation function $\rho^n$ for each $n \in \mathbb{N}$. Let $\mu^1$ be the measure defined by (2.9) with $k = 1$. In this section we present a sufficient condition for the existence of the logarithmic derivative $d\mu$ in $L^p_{\text{loc}}(\mu^1)$.

Let $S_r = \{x \in S; |x| < r\}$ and $S_r^n$ denote the $n$-product of $S_r$. Here and after, $\cdot^n$ denotes the $n$-product of the set $\cdot$. Let $\{\mu^n\}$ be a sequence of probability measures on $S$. We assume that their $n$-point correlation functions $\{\rho^{n,n}\}$ satisfy for each $r \in \mathbb{N}$

\[
\lim_{n \to \infty} \rho^{n,n}(x) = \rho^n(x) \quad \text{uniformly on } S_r^n, \quad (4.1)
\]

\[
\sup_{n \in \mathbb{N}} \sup_{x \in S_r^n} \rho^{n,n}(x) \leq c(r)^n n^3, \quad (4.2)
\]

where $0 < c_2(r) < \infty$ and $0 < c_3(r) < 1$ are constants independent of $n \in \mathbb{N}$.

Let $g : S^2 \to \mathbb{R}$ be measurable functions. For $(x, y) \in S \times S$ we set

\[
g_s(x, y) = \sum_{|x-y_j| < s} g(x, y_j), \quad w_s(x, y) = \sum_{s \leq |x-y_j|} g(x, y_j), \quad (4.3)
\]

where $y = \sum_i \delta_{y_i}$. As for $w_s$, we define only for $y$ such that $y(S) < \infty$ in order to make the sum $w_s(x, y) = \sum_{s \leq |x-y_j|} g(x, y_j)$ finite. We note that $g_s + w_s$ are independent of $s$. 

11
Let \( u, u^n : S \to \mathbb{R} \) and \( 1 < \hat{p} < \infty \). Assume that \( \mu^n \) has a logarithmic derivative \( d^n \) for each \( n \) satisfying the following.

\[
d^n(x, y) = u^n(x) + g_s(x, y) + w_s(x, y), \tag{4.4}
\]

\[
\lim_{n \to \infty} u^n = u \text{ in } L^p_{\text{loc}}(S, dx), \tag{4.5}
\]

\[
\lim_{s \to \infty} \limsup_{n \to \infty} \int_{S \times S} |w_s(x, y)|^{\hat{p} \mu^n} dy = 0. \tag{4.6}
\]

We quote:

**Theorem 4.1** ([16, Theorem 45]). Let \( 1 < p < \hat{p} \). Assume (4.1)–(4.6). Then the logarithmic derivative \( d^\mu \) exists in \( L^p_{\text{loc}}(\mu^1) \) and is given by

\[
d^\mu(x, y) = u(x) + \lim_{s \to \infty} g_s(x, y). \tag{4.7}
\]

The convergence \( \lim g_s \) takes place in \( L^p_{\text{loc}}(\mu^1) \).

**Remark 4.1.** Theorem 4.1 is a special case of [16, Theorem 45]. In [16, Theorem 45] extra terms such as \( v^n \) and \( w(x) \) appeared. These terms are vanished here. However, this is not the case for the Ginibre random point field and the Airy random point fields.

In practice, to check the condition (4.6) is the most hard part of the proof. So we quote a sufficient condition for this in terms of correlation functions.

**Lemma 4.2.** We set \( S_{s \infty}^x = \{ y \in S; s \leq |x-y| < \infty \} \). Let \( \rho^n_{x, n} \) be the n-point correlation function of the reduced Palm measure \( \mu^n_x \). Then (4.6) with \( \hat{p} = 2 \) follows from the following:

\[
\lim_{s \to \infty} \limsup_{n \to \infty} \sup_{x \in S_r} |\int_{S_{s \infty}^x} g(x, y) \rho^{n,1}(y) dy| = 0, \tag{4.8}
\]

\[
\lim_{s \to \infty} \limsup_{n \to \infty} \sup_{x \in S_r} |\int_{S_{s \infty}^x} g(x, y) \{ \rho^{n,1}(y) - \rho^{n,1}(y) \} dy| = 0, \tag{4.9}
\]

\[
\lim_{s \to \infty} \limsup_{n \to \infty} \sup_{x \in S_r} |\int_{S_{s \infty}^x} |g(x, y)|^2 \rho^{n,1}(y) dy - \int_{(S_{s \infty}^x)^2} g(x, y) \cdot g(x, z) \rho^{n,2}(y, z) dydz| = 0, \tag{4.10}
\]

\[
\lim_{s \to \infty} \limsup_{n \to \infty} \sup_{x \in S_r} |\int_{S_{s \infty}^x} |g(x, y)|^2 \{ \rho^{n,1}(y) - \rho^{n,1}(y) \} dy - \int_{(S_{s \infty}^x)^2} g(x, y) \cdot g(x, z) \{ \rho^{n,2}(y, z) - \rho^{n,2}(y, z) \} dydz| = 0. \tag{4.11}
\]
Proof. Lemma 4.2 is a special case of [16, Lemma 52].

5 Finite particle approximations and proof of Theorem 2.3

In this section, we prove Theorem 2.3. For this we use Theorem 4.1. We will check the assumptions (4.1)–(4.6) posed in Theorem 4.1.

We begin by giving finite particle approximations for Bessel random point fields. Let \{L_\alpha^n\} denote the Laguerre polynomials. Then by definition

\[ L_\alpha^n(x) = \sum_{m=0}^{n} (-1)^m \frac{(n + \alpha)^m x^m}{m!}. \]  

(5.1)

The associated monic polynomials \{p_\alpha^n\} are given by

\[ p_\alpha^n(x) = (-1)^n n! L_\alpha^n(x) = (-1)^n \Gamma(n + 1) L_\alpha^n(x). \]  

(5.2)

Let \( w_\alpha(x) = x^\alpha e^{-x} \). Then it is known [27, 301p, 302p] that for \( m, n \in \{0\} \cup \mathbb{N} \)

\[ \int_0^\infty L_\alpha^m(x)L_\alpha^n(x)w_\alpha(x)dx = \delta_{m,n} \frac{\Gamma(n + \alpha + 1)}{\Gamma(n + 1)}. \]  

(5.3)

From (5.2) and (5.3) we immediately deduce that

\[ \int_0^\infty p_\alpha^{n-1}(x)^2 w_\alpha(x)dx = \Gamma(n + \alpha)\Gamma(n). \]  

(5.4)

Let \( k^n_\alpha : (0, \infty)^2 \to \mathbb{R} \) be such that

\[ k^n_\alpha(x, y) = \sqrt{w_\alpha(x)w_\alpha(y)} \sum_{m=0}^{n-1} \frac{p_\alpha^m(x)p_\alpha^m(y)}{\int_0^\infty p_\alpha^m(z)^2 w_\alpha(z)dz}. \]  

(5.5)

Then we deduce from the Christoffel-Darboux formula [3 Proposition 5.1.3.] that for \( x \neq y \)

\[ k^n_\alpha(x, y) = \frac{\sqrt{w_\alpha(x)w_\alpha(y)} \left( p_\alpha^n(x)p_\alpha^n(y) - p_\alpha^{n-1}(x)p_\alpha^{n-1}(y) \right)}{\int_0^\infty p_\alpha^{n-1}(z)^2 w_\alpha(z)dz} \frac{x - y}{x^\alpha e^{-x} - y^\alpha e^{-y}}. \]  

(5.6)
From (5.2)–(5.4) combined with a straightforward calculation, we obtain that
\[
 k_n^\alpha(x, y) = \sqrt{w_\alpha(x)w_\alpha(y)} \frac{\Gamma(n + 1) L_{n-1}^{[\alpha]}(x)L_n^{[\alpha]}(y) - L_n^{[\alpha]}(x)L_{n-1}^{[\alpha]}(y)}{\Gamma(n + \alpha)} x - y. \tag{5.7}
\]

We now introduce the rescaled kernels $K_n^\alpha$ as follows.
\[
 K_n^\alpha(x, y) = \frac{1}{4n} k_n^\alpha\left(\frac{x}{4n}, \frac{y}{4n}\right). \tag{5.8}
\]

Let $\mu_n^\alpha$ be the determinantal random point field over $(0, \infty)$ generated by $(K_n^\alpha, dx)$. Then by construction the $n$-point correlation functions of $\mu_n^\alpha$ are given by
\[
 \rho_n^n(x_1, \ldots, x_n) = \det[K_n^\alpha(x_i, x_j)]_{i,j=1,\ldots,n}. \tag{5.9}
\]

It is known that $\mu_n^\alpha(s(S) = n) = 1$, and so the labeled density function of $\mu_n^\alpha$ is $\rho_n^n$ up to the normalizing constant. Moreover, by the standard theory of the random matrix, we obtain, because of the calculation of Vandermond determinant,
\[
 \rho_n^n(x_1, \ldots, x_n) = c_n e^{-\sum_{i=1}^n x_i/4n} \prod_{j=1}^n x_j^\alpha \prod_{k<l} |x_k - x_l|^2 \tag{5.10}
\]
with the normalizing constant $c_n$. We easily deduce from (5.10) that the logarithmic derivative $d\mu_n^\alpha$ of $\mu_n^\alpha$ are given by
\[
 d\mu_n^\alpha(x, y) = -\frac{1}{4n} + \frac{\alpha}{x} + \sum_{i=1}^{n-1} \frac{2}{x - y_i}. \tag{5.11}
\]

**Lemma 5.1.** Let $1 \leq \alpha < \infty$. Then $\{\mu_n^\alpha\}$ satisfy (4.1)–(4.5) with
\[
 u_n^\alpha(x) = -\frac{1}{4n} + \frac{\alpha}{x}, \quad g(x, y) = \frac{2}{x - y}. \tag{5.12}
\]

**Proof.** It is known ([3, 290p]) that, for each $(x, y) \in (0, \infty)^2$,
\[
 \lim_{n \to \infty} K_n^\alpha(x, y) = K_\alpha(x, y). \tag{5.13}
\]

Furthermore, one can easily see that the convergence takes place compact uniformly in $(x, y) \in [0, \infty)^2$. We deduce (4.1) from (5.9) and (5.13) immediately.
The condition (4.2) follows from (5.9) and (5.13). In fact, from (5.13) and the definition (5.8) of the kernel $K_n^{\alpha}$, we deduce that the norm $k_{n,i}^{\alpha,n}(x_1,\ldots,x_n)$ of the $i$th row vector of the matrix $[K_n^{\alpha}(x_i,x_j)]_{i,j=1,\ldots,n}$ satisfies the following inequality.

$$\sup_{(x_1,\ldots,x_n)\in S_r^n} k_{n,i}^{\alpha,n}(x_1,\ldots,x_n) \leq c_{4,n}^{1/2}. \quad (5.14)$$

Here $c_4 = c_{4}(r)$ is a positive constant independent of $n$ and $n$. Hence from Hadamard’s inequality we deduce that

$$\sup_{(x_1,\ldots,x_n)\in S_r^n} |\det[K_n^{\alpha}(x_i,x_j)]_{i,j=1,\ldots,n}| \leq c_{4,n}^{n/2}. \quad (5.15)$$

The conditions (4.3)–(4.5) are obvious from construction and (5.11). □

By Lemma 5.1, it only remains to prove (4.6). Taking Lemma 4.2 into account, we will deduce (4.6) from (4.8)–(4.11). The key point of this is the estimate (5.17) in Lemma 5.3 which control the 1-point correlation functions $\rho_{n,1}^{\alpha}$ of $\mu_{\alpha}$. To prove (5.17) we prepare a bound of $\rho_{n,1}^{\alpha}$.

**Lemma 5.2.** Let $\alpha > -1$ and $\omega > 1$. Then for all $n \in \mathbb{N}$

$$\rho_{n,1}^{\alpha}(x) \leq \frac{c_5}{\sqrt{x}} \quad \text{for } 1 \leq x \leq 4n\omega. \quad (5.16)$$

Here $c_5 = c_{5}(\alpha,\omega)$ is a positive constant independent of $x$ and $n$.

This lemma follows from an asymptotic formula of Hilb’s type from [26, 8.22.5], 199 p.]. Since the proof is long, although straightforward, we postpone it in Appendix (Section 7).

The next result is the most significant step of the proof.

**Lemma 5.3.** The condition (4.8) is satisfied. Furthermore, it holds that

$$\lim_{s\to\infty} \limsup_{n\to\infty} \sup_{x\in S_r^s} \int_{S_{s\infty}^x} \frac{1}{|x-y|}\rho_{n,1}^{\alpha}(y)dy = 0. \quad (5.17)$$

**Proof.** Since (4.8) follows from (5.17), we only prove (5.17). We divide $S_{s\infty}^x$ into two parts $S_{s\infty}^x \cap [s,\omega n]$ and $S_{s\infty}^x \cap [\omega n,\infty)$, where $\omega$ is a positive constant.
We begin by the first case $S_{s}^{x} \cap [s, \omega n]$. From (5.16) we deduce that
\[
\sup_{n \in \mathbb{N}} \sup_{x \in S_{r}} \int_{S_{s}^{x} \cap [s, \omega n]} \frac{\rho_{n}^{1}(y)}{|x - y|} dy \leq \sup_{x \in S_{r}} \int_{S_{s}^{x}} \frac{1}{|x|} dy \rightarrow 0
\] (5.18)
as $s \rightarrow \infty$. As for the second case $S_{s}^{x} \cap [\omega n, \infty)$, we see that for $r < \omega n$
\[
\sup_{x \in S_{r}} \int_{S_{s}^{x} \cap [\omega n, \infty)} \frac{\rho_{n}^{1}(y)}{|x - y|} dy \leq \frac{1}{n \omega - r} \int_{0}^{\infty} \rho_{n}^{1}(y) dy = \frac{n}{n \omega - r}.
\] (5.19)
Then we deduce from (5.19) that
\[
\lim_{s \rightarrow \infty} \limsup_{n \rightarrow \infty} \sup_{x \in S_{r}} \int_{S_{s}^{x} \cap [\omega n, \infty)} \frac{\rho_{n}^{1}(y)}{|x - y|} dy \leq \frac{1}{\omega}.
\] (5.20)
Combining (5.18) and (5.20), we deduce that
\[
\lim_{s \rightarrow \infty} \limsup_{n \rightarrow \infty} \sup_{x \in S_{r}} \int_{S_{s}^{x}} \frac{\rho_{n}^{1}(y)}{|x - y|} dy \leq \frac{1}{\omega}.
\] (5.21)
Taking $\omega > 0$ to be arbitrary large in (5.21) yields (5.17). \(\square\)

We next prepare two properties (5.22) and (5.23) of determinantal kernels. We will repeatedly use these in the sequel. Let $\mu_{\alpha,x}$ be the reduced Palm measure of $\mu_{\alpha}$ conditioned at $x$ and let $\rho_{\alpha,x}^{n}$ be its $n$-point correlation function as before. Then $\mu_{\alpha,x}$ has a determinantal structure with kernel
\[
K_{\alpha,x}(y, z) = K_{\alpha}(y, z) - \frac{K_{\alpha}(y, x) K_{\alpha}(x, z)}{K_{\alpha}(x, x)}.
\] (5.22)
This relation follows from a general theorem on determinantal random point fields [20, Theorem 1.7]. Applying the Schwarz inequality to (5.5), we deduce from (5.8) that
\[
|K_{\alpha}(x, y)| \leq \sqrt{K_{\alpha}(x, x)} \sqrt{K_{\alpha}(y, y)} = \sqrt{\rho_{\alpha}^{1}(x)} \sqrt{\rho_{\alpha}^{1}(y)}.
\] (5.23)
Here the equality in (5.23) follows from (5.9) with $n = 1$.

**Lemma 5.4.** The condition (4.9) is satisfied. Furthermore, it holds that
\[
\lim_{s \rightarrow \infty} \limsup_{n \rightarrow \infty} \sup_{x \in S_{r}} \int_{S_{s}^{x}} \frac{|\rho_{\alpha,x}^{1}(y) - \rho_{\alpha}^{1}(y)|}{|x - y|} dy = 0.
\] (5.24)
Proof. From (1.1), (5.22), and (5.23), we deduce that

\[ |\rho_{\alpha,x}^n(y) - \rho_{\alpha}^n(y)| = \frac{|K_{\alpha}^n(y,x)K_{\alpha}^n(x,y)|}{K_{\alpha}^n(x,x)} \leq K_{\alpha}^n(y,y) = \rho_{\alpha}^n(y). \] (5.25)

Hence (5.24) is immediate from (5.17). The condition (4.9) follows from (5.24) immediately. \( \square \)

Lemma 5.5. The condition (4.10) is satisfied. Furthermore, it holds that

\[ \lim_{s \to \infty} \limsup_{n \to \infty} \sup_{x \in S} \int_{S \times S} \frac{\rho_{\alpha,x}^n(y)}{|x-y|^2} dy + \int_{(S \times S)^2} \frac{\rho_{\alpha}^n(y,z)}{|x-y||x-z|} dydz = 0. \] (5.26)

Proof. From Lemma 5.3 we easily deduce that

\[ \lim_{s \to \infty} \limsup_{n \to \infty} \sup_{x \in S} \int_{S \times S} \frac{\rho_{\alpha,x}^n(y)}{|x-y|^2} dy = 0. \] (5.27)

From (5.9) and (5.23) we see that

\[ \rho_{\alpha}^n(y,z) = \rho_{\alpha,x}^n(y)\rho_{\alpha}^n(z) - K_{\alpha}^n(y,z)K_{\alpha}^n(z,y) \leq 2\rho_{\alpha,x}^n(y)\rho_{\alpha}^n(z). \] (5.28)

Hence from (5.28) and Fubini’s theorem, we deduce that

\[ \int_{(S \times S)^2} \frac{\rho_{\alpha}^n(y,z)}{|x-y||x-z|} dydz \leq 2 \int_{(S \times S)^2} \frac{2\rho_{\alpha,x}^n(y)\rho_{\alpha}^n(z)}{|x-y||x-z|} dydz \] (5.29)

\[ \leq 2 \left( \int_{S \times S} \frac{\rho_{\alpha,x}^n(y)}{|x-y|} dy \right)^2. \]

Then from (5.29) and Lemma 5.3 we deduce that

\[ \lim_{s \to \infty} \limsup_{n \to \infty} \sup_{x \in S} \int_{(S \times S)^2} \frac{\rho_{\alpha}^n(y,z)}{|x-y||x-z|} dydz = 0. \] (5.30)

From (5.27) and (5.30), we conclude (5.26). This implies (4.10). \( \square \)

Lemma 5.6. The condition (4.11) is satisfied. Furthermore, it holds that

\[ \lim_{s \to \infty} \limsup_{n \to \infty} \sup_{x \in S} \int_{S \times S} \frac{|\rho_{\alpha,x}^n(y) - \rho_{\alpha}^n(y)|}{|x-y|^2} dy \] (5.31)

\[ + \int_{(S \times S)^2} \frac{|\rho_{\alpha,x}^n(y,z) - \rho_{\alpha}^n(y,z)|}{|x-y||x-z|} dydz = 0. \]
Proof. We deduce from Lemma 5.4 that
\[
\lim_{s \to \infty} \limsup_{n \to \infty} \sup_{x \in S_s} \int_{S_{s,x}} \frac{|\rho_{n,1}(y) - \rho_{n,1}(y)|}{|x - y|^2} dy = 0. \tag{5.32}
\]
To estimate the second term of (5.31) we observe that
\[
\rho_{n,2}(y, z) - \rho_{n,2}(y, z) = -K_n(y, x)K_n(x, y)K_n(z, z) - K_n(z, x)K_n(x, z)K_n(y, y) + K_n(y, x)K_n(x, z)K_n(z, y)K_n(x, x) + K_n(y, z)K_n(z, x)K_n(x, y)K_n(x, x).
\tag{5.33}
\]
Then applying (5.23) to each term of the right-hand side, we obtain
\[
|\rho_{n,2}(y, z) - \rho_{n,2}(y, z)| \leq 4\rho_{n,1}(y)\rho_{n,1}(z). \tag{5.34}
\]
Hence from (5.34) and Fubini’s theorem, we deduce that
\[
\int_{(S_{s,x})^2} \frac{|\rho_{n,2}(y, z) - \rho_{n,2}(y, z)|}{|x - y||x - z|} dydz \leq \int_{(S_{s,x})^2} \frac{4\rho_{n,1}(y)\rho_{n,1}(z)}{|x - y||x - z|} dydz \tag{5.35}
= 4 \left\{ \int_{S_{s,x}} \frac{\rho_{n,1}(y)}{|x - y|} dy \right\}^2.
\]
Then from (5.35) and Lemma 5.3 we see that
\[
\lim_{s \to \infty} \limsup_{n \to \infty} \sup_{x \in S_s} \int_{(S_{s,x})^2} \frac{|\rho_{n,2}(y, z) - \rho_{n,2}(y, z)|}{|x - y||x - z|} dydz = 0. \tag{5.36}
\]
From (5.32) and (5.36), we obtain (5.31). This implies (4.11). \qed

Proof of Theorems 2.3. From Lemma 5.1 we see that \(\mu_\alpha\) satisfy (4.1)–(4.5). From Lemma 5.3–Lemma 5.6, we deduce that \(\mu_\alpha\) satisfy (4.8)–(4.11). This combined with Lemma 4.2 yields (4.6). We thus see that all the conditions (4.1)–(4.6) of Theorem 4.1 are fulfilled. Hence from Theorem 4.1 we obtain Theorem 2.3 with the logarithmic derivative d given by (4.7). \qed
6 Proof of Theorem 2.4

In this section we prove Theorem 2.4. For this we use [18, Theorem 2.2]. We prepare a result from [18], which is a special case of [18, Theorem 2.2].

In the next theorem, we take $S = [0, \infty)$ or $S = \mathbb{R}$. Let $\mu$ be a random point field on $S$. We assume three conditions.

\begin{enumerate}
    \item [(B.1)] The random point field $\mu$ has a locally bounded, $n$-point correlation function $\rho^n$ for each $n \in \mathbb{N}$.
    \item [(B.2)] There exists a sequence of random point fields $\{\mu^n\}_{n \in \mathbb{N}}$ over $S$ satisfying the following.
        \begin{enumerate}
            \item [\(1\)] The $n$-point correlation functions $\rho^{n,n}$ of $\mu^n$ satisfy
                \begin{equation}
                    \lim_{n \to \infty} \rho^{n,n}(x_n) = \rho^n(x_n) \quad \text{a.e. for all } n \in \mathbb{N},
                \end{equation}
                \begin{equation}
                    \sup\{\rho^{n,n}(x_n); n \in \mathbb{N}, x_n \in S^n\} \leq \{c_6^n, c_7^n\} \quad \text{for all } n, r \in \mathbb{N},
                \end{equation}
                where $x_n = (x_1, \ldots, x_n) \in S^n$, $c_6 = (0)(r) > 0$, and $c_7 = (7)(r) < 1$ are constants depending on $r \in \mathbb{N}$.
            \item [\(2\)] $\mu^n(s(S) = N_n) = 1$ for each $n$, where $N_n \in \mathbb{N}$ are strictly increasing.
            \item [\(3\)] $\mu^n$ is a $(\Phi^n, -\beta \log |x - y|)$-canonical Gibbs measure for each $n$.
            \item [\(4\)] $\Phi^n$ satisfy the following.
                \begin{equation}
                    \lim_{n \to \infty} \Phi^n(x) = \Phi(x) \quad \text{for a.e. } x, \inf \inf_{n \in \mathbb{N}, x \in S} \Phi^n(x) > -\infty.
                \end{equation}
        \end{enumerate}
    \item [(B.3)] There exists an $\ell_0$ such that $\ell_0 \in \mathbb{N}$ and that
        \begin{equation}
            \sup_{n \in \mathbb{N}} \int_{1 \leq |x| < \infty} \frac{1}{|x|^{\ell_0}} \rho^{n,1}(x)dx < \infty
        \end{equation}
        and that, for each $1 \leq \ell < \ell_0$,
        \begin{equation}
            \lim \sup_{s \to \infty} \sup_{n \in \mathbb{N}} \|\nu_{\ell,s,n}\|_{L^1(S,\mu^n)} = 0.
        \end{equation}
\end{enumerate}

Let $x = \sum_i \delta_{x_i}$. For $1 \leq r < s \leq \infty$ let $\nu_{\ell,rs}: S \to \mathbb{R}$ such that
\begin{equation}
    \nu_{\ell,rs}(x) = \beta \left\{ \sum_{x_i \in S_r \setminus S_s} \frac{1}{x_i^\ell} \right\} (\ell \geq 1).
\end{equation}
Note that the sum in (6.4) makes sense for $\mu^n\text{-a.s. } x$ even if $s = \infty$. Indeed, by (2) of (B.2), the total number of particles is $N_n$ under $\mu^n$. Hence, $\nu_{\ell,rs}(x)$ is well defined and finite for $\mu^n\text{-a.s. } x$, for all $n \in \mathbb{N}$.

(B.3) There exists an $\ell_0$ such that $\ell_0 \in \mathbb{N}$ and that
\begin{equation}
    \sup_{n \in \mathbb{N}} \int_{1 \leq |x| < \infty} \frac{1}{|x|^{\ell_0}} \rho^{n,1}(x)dx < \infty
\end{equation}
and that, for each $1 \leq \ell < \ell_0$,
\begin{equation}
    \lim \sup_{s \to \infty} \sup_{n \in \mathbb{N}} \|\nu_{\ell,s,n}\|_{L^1(S,\mu^n)} = 0.
\end{equation}
When $\ell_0 = 1$, we interpret that \((6.6)\) always holds. The following is a special case of [18, Theorem 2.2]. We remark that the assumptions (B.1), (B.2) and (B.3) correspond to (H.1), (H.2) and (H.4) in [18, Theorem 2.2], respectively.

**Theorem 6.1** ([18, Theorem 2.2]). Assume (B.1), (B.2) and (B.3). Then $\mu$ is a $(\Phi, \Psi)$-quasi-Gibbs measure.

**Proof of Theorem 6.1.** We check the assumptions (B.1), and (B.2), and (B.3) in Theorem 6.1. We take $\mu = \mu_\alpha$ and $\mu^n = \mu^n_\alpha$. Then (B.1) and (B.2) are satisfied. Furthermore, we take $\ell_0 = 1$. Then from (1.2) and Lemma 5.3, we deduce (6.5) easily. Hence we conclude Theorem 6.1.

### Appendix: Proof of Lemma 5.2.

In this section we prove Lemma 5.2. Let $-1 < \alpha < \infty$. Let $L^{[\alpha]}_n$ denote the Laguerre polynomial and $w_\alpha(x) = e^{-x}x^\alpha$ as before. Let

$$M^n_\alpha(x) = \{w^{\frac{1}{2}}_{\alpha+1}L^{[\alpha+1]}_{n-1}w^{\frac{1}{2}}_\alpha L^{[\alpha]}_n - w^{\frac{1}{2}}_\alpha L^{[\alpha]}_n w^{\frac{1}{2}}_{\alpha+1}L^{[\alpha+1]}_{n-1}\}(x). \tag{7.1}$$

Then from a straightforward calculation we obtain the following.

**Lemma 7.1.** There exists a positive constant $c_8$ such that

$$\rho^{n,1}_\alpha(y) \leq \frac{c_8}{\sqrt{y}} \frac{1}{n^{\alpha - \frac{1}{2}}} M^n_\alpha\left(\frac{y}{4n}\right) \tag{7.2}$$

for all $n \in \mathbb{N}$ and $y \in (0, \infty)$.

**Proof.** From (5.8) and (5.9), we see that

$$\rho^{n,1}_\alpha(y) = K^n_\alpha(y, y) = \frac{1}{4n} k^n_\alpha\left(\frac{y}{4n}, \frac{y}{4n}\right). \tag{7.3}$$
Hence we will estimate \( k_n^\alpha(x, x) \). Taking \( y \to x \) in (5.7), we deduce that

\[
k_n^\alpha(x, x) = w_\alpha(x) \frac{\Gamma(n+1)}{\Gamma(n+\alpha)} \left\{ - \frac{dL_n^{[\alpha]}}{dx} L_n^{[\alpha]} - L_n^{[\alpha]} \frac{dL_n^{[\alpha]}}{dx} \right\}(x)
\]

(7.4)

\[
= w_\alpha(x) \frac{\Gamma(n+1)}{\Gamma(n+\alpha)} \left\{ L_n^{[\alpha+1]} L_n^{[\alpha]} - L_n^{[\alpha]} L_n^{[\alpha+1]} \right\}(x)
\]

\[
= \frac{w_\alpha(x)^{\frac{1}{2} \alpha}}{w_{\alpha+1}(x)^{\frac{1}{2} \alpha}} \frac{\Gamma(n+1)}{\Gamma(n+\alpha)} \left\{ w_{\alpha+1}^{\alpha} L_n^{[\alpha+1]} - w_\alpha^{\alpha} L_n^{[\alpha]} w_{\alpha+1}^{\alpha} L_n^{[\alpha+1]} \right\}(x)
\]

\[
= \frac{1}{\sqrt{x}} \frac{\Gamma(n+1)}{\sqrt{n+\alpha}} M_n^\alpha(x).
\]

Here we used the formula \( \frac{dl_n^{[\alpha]}}{dx} = -L_n^{[\alpha+1]} \) (see [26, 102 p]) for the second line, and (7.1) for the last line. Taking \( x = \frac{y}{4n} \) in (7.4), we obtain that

\[
\frac{1}{4n} k_n^\alpha \left( \frac{y}{4n}, \frac{y}{4n} \right) = \frac{1}{\sqrt{4ny}} \frac{\Gamma(n+1)}{\Gamma(n+\alpha)} M_n^\alpha \left( \frac{y}{4n} \right).
\]

(7.5)

Clearly, there exists a positive constant \( c_8 \) such that

\[
\frac{1}{\sqrt{4n}} \frac{\Gamma(n+1)}{\Gamma(n+\alpha)} \leq c_8 n^{-\alpha+\frac{1}{2}} \quad \text{for all } n \in \mathbb{N}.
\]

(7.6)

From (7.5) and (7.6) we obtain (7.2).

From Lemma 7.1, our next task is to prove

\[
M_n^\alpha \left( \frac{y}{4n} \right) = O(n^{\alpha-\frac{1}{2}}).
\]

(7.7)

Here the bound \( O(n^{\alpha-\frac{1}{2}}) \) is taken to be uniform in \( c_9 n^{-1} \leq x \leq \omega \). For this we quote an asymptotic formula of Hilb’s type from [26].

Lemma 7.2 ([26, (8.22.5), 199 p.]). Let \(-1 < \alpha < \infty\). Let \( c_9, \omega > 0 \) be fixed. Then each Laguerre polynomial \( L_n^{[\alpha]} \) satisfies, for all \( c_9 n^{-1} \leq x \leq \omega \),

\[
w_\alpha(x)^{\frac{1}{2} \alpha} L_n^{[\alpha]}(x) = A_{n, \alpha} J_\alpha(\sqrt{4Nx}) + x^{\frac{1}{2}} O(n^{\alpha/2-\frac{3}{4}}).
\]

(7.8)

Here \( N = n + (\alpha + 1)/2 \) and the bound \( O(n^{\alpha/2-\frac{3}{4}}) \) holds uniformly in \( c_9 n^{-1} \leq x \leq \omega \). Furthermore, \( A_{n, \alpha} \) is defined by

\[
A_{n, \alpha} = \frac{\Gamma(n+1+\alpha)}{N^{\alpha/2} \Gamma(n+1)}.
\]

(7.9)
From Lemma 7.2 we see the following.

**Lemma 7.3.** For all \( n \in \mathbb{N} \) and \( cn^{-1} \leq x \leq \omega \)

\[
M^\alpha_n\left(\frac{y}{4n}\right) = O(n^{\alpha+\frac{1}{2}}) \tag{7.10}
\]

\[
\cdot \left[ J_\alpha\left(\left(\frac{N-1}{n}y\right)^{\frac{1}{2}}\right) + \left(\frac{y}{4n}\right)^{\frac{3}{2}}O(n^{-\frac{3}{4}}) \right] \left\{ J_{\alpha+1}\left(\left(\frac{N-1}{n}y\right)^{\frac{1}{2}}\right) + \left(\frac{y}{4n}\right)^{\frac{3}{2}}O(n^{-\frac{3}{4}}) \right\} 
- \left\{ J_{\alpha+1}\left(\left(\frac{N-2}{n}y\right)^{\frac{1}{2}}\right) + \left(\frac{y}{4n}\right)^{\frac{3}{2}}O(n^{-\frac{3}{4}}) \right\} \left\{ J_\alpha\left(\left(\frac{N}{n}y\right)^{\frac{1}{2}}\right) + \left(\frac{y}{4n}\right)^{\frac{3}{2}}O(n^{-\frac{3}{4}}) \right\} \right].
\]

**Proof.** From (7.8) we easily deduce that, for all \( cn^{-1} \leq x \leq \omega \),

\[
w_\alpha(x)^{\frac{1}{2}}L^\alpha_n(x) = A_{n,\alpha} \left\{ J_\alpha(\sqrt{4Nx}) + x^\frac{5}{4}O(n^{-\frac{3}{4}}) \right\} . \tag{7.11}
\]

Then taking \( x = y/4n \) in (7.11) we deduce that

\[
w_\alpha(\frac{y}{4n})^{\frac{1}{2}}L^\alpha_n(y/4n) = A_{n,\alpha} \left\{ J_\alpha\left(\frac{n}{N}y\right)^{\frac{1}{2}} + \left(\frac{y}{4n}\right)^{\frac{3}{2}}O(n^{-\frac{3}{4}}) \right\} . \tag{7.12}
\]

A simple calculation shows that

\[
A_{n,\alpha} = O(n^{\frac{4}{3}\alpha}) \tag{7.13}
\]

Hence we deduce (7.10) from (7.1), (7.12) and (7.13) immediately. \( \Box \)

**Proof of Lemma 5.2.** We will calculate the right-hand side of (7.10).

Recall that \( N = n + \frac{1}{2}(\alpha + 1) \). Then \( N^2 - \frac{1}{n}N = \frac{1}{n} \). Hence from Taylor expansion of \( J_\alpha(\sqrt{y}) \) and \( J_{\alpha+1}(\sqrt{y}) \), we deduce that

\[
J_\alpha\left(\frac{N}{n}y\right)^{\frac{1}{2}} - J_{\alpha+1}\left(\frac{N}{n}y\right)^{\frac{1}{2}} \tag{7.14}
\]

\[
= \frac{1}{n} \left\{ J_{\alpha+1}\left(\frac{N}{n}y\right)^{\frac{1}{2}} - J_\alpha\left(\frac{N}{n}y\right)^{\frac{1}{2}} \right\} J_\alpha\left(\frac{N}{n}y\right)^{\frac{1}{2}}
+ J_\alpha\left(\frac{N}{n}y\right)^{\frac{1}{2}} n \left\{ J_{\alpha+1}\left(\frac{N}{n}y\right)^{\frac{1}{2}} - J_\alpha\left(\frac{N}{n}y\right)^{\frac{1}{2}} \right\}
= O\left(\frac{1}{n}\right) \left\{ \frac{d}{dy} J_\alpha(\sqrt{y}) \cdot J_{\alpha+1}(\sqrt{y}) - J_\alpha(\sqrt{y}) \cdot \frac{d}{dy} J_{\alpha+1}(\sqrt{y}) \right\}
= O\left(\frac{1}{n}\right).
\]
We also see that
\[ J_\alpha\left(\left[\frac{N}{n}\right]y^{\frac{4}{n}}\left[\frac{y}{4n}\right]^\frac{4}{n}\right)O(n^{-\frac{4}{n}}) = O(1)y^{-\frac{4}{n}}\left[\frac{y}{4n}\right]^\frac{4}{n}n^{-\frac{4}{n}} = O\left(\frac{1}{n}\right). \tag{7.15} \]

Here we used \(|J_\alpha(t)| \leq O(1)t^{-1/2}\) and \(\frac{y}{4n} = O(1)\). Substituting (7.14) and (7.15) together with similar relations into (7.10), we deduce that

\[ M_\alpha^n\left(\frac{y}{4n}\right) = O(n^{\alpha+\frac{1}{2}})O\left(\frac{1}{n}\right) = O(n^{\alpha-\frac{1}{2}}). \tag{7.16} \]

This together with Lemma 7.4 completes the proof of Lemma 5.2.

References

[1] Borodin A., Olshanski G., Markov processes on the path space of the Gelfand-Tsetlin graph and on its boundary, J. Functional Analysis 263 (2012) 248-303.

[2] Borodin A., Gorin V., Markov processes of infinitely many nonintersecting random walks, Probab. Theory Relat. Fields 155 (2013) 935-997.

[3] Forrester, Peter J., Log-gases and Random Matrices, London Mathematical Society Monographs, Princeton University Press (2010).

[4] Fukushima, M., et al. Dirichlet forms and symmetric Markov processes, 2nd ed., Walter de Gruyter (2011).

[5] Fritz, J. Gradient dynamics of infinite point systems, Ann. Probab. 15 (1987) 478-514.

[6] Katori, M., Tanemura, H., Non-equilibrium dynamics of Dyson’s model with an infinite number of particles, Commun. Math. Phys. 293, 469?497 (2010).

[7] Katori, M., Tanemura, H.: Markov property of determinantal processes with extended sine, Airy, and Bessel kernels. Markov processes and related fields 17, 541-580 (2011)

[8] Lang, R., Unendlich-dimensionale Wienerprozesse mit Wechselwirkung I, Z. Wahrscheinverw. Gebiete 38 (1977) 55-72.

[9] Lang, R., Unendlich-dimensionale Wienerprozesse mit Wechselwirkung II, Z. Wahrscheinverw. Gebiete 39 (1978) 277-299.
[10] Ma, Z.-M., Röckner, M., *Introduction to the theory of (non-symmetric) Dirichlet forms*, Berlin: Springer-Verlag 1992.

[11] Mehta, M., *Random matrices*, (Third Edition) Elsevier 2004.

[12] Olshanski G. *Laguerre and Meixner symmetric functions, and infinite-dimensional diffusion processes*, J. Mathematical Sciences, 174, (2011) 41-57.

[13] Osada, H., *Dirichlet form approach to infinitely dimensional Wiener processes with singular interactions*, Commun. Math. Physic. (1996), 117-131.

[14] Osada, H., *Non-collision and collision properties of Dyson’s model in infinite dimensions and other stochastic dynamics whose equilibrium states are determinantal random point fields*, in Stochastic Analysis on Large Scale Interacting Systems, eds. T. Funaki and H. Osada, Advanced Studies in Pure Mathematics 39, 2004, 325-343.

[15] Osada, H., *Tagged particle processes and their non-explosion criteria*, J. Math. Soc. Japan, 62, No. 3 (2010)867-894.

[16] Osada, H., *Infinite-dimensional stochastic differential equations related to random matrices*, Probability Theory and Related Fields, 153, 471-509 (2012)

[17] Osada, H., *Interacting Brownian motions in infinite dimensions with logarithmic interaction potentials*, Ann. of Probab. 41(2013)1-49.

[18] Osada, H., *Interacting Brownian motions in infinite dimensions with logarithmic interaction potentials II: Airy random point field*, Stochastic Processes and their applications 123(2013) 813-838.

[19] Shiga, T. *A remark on infinite-dimensional Wiener processes with interactions*, Z. Wahrscheinverw. Gebiete 47 (1979) 299-304.

[20] Shirai, T., Takahashi, Y. *Random point associated with certain Fredholm determinants I: Fermion, Poisson and Boson processes*, J. Funct. Anal. 205: (2003), 414-463.

[21] Soshnikov, A., *Determinantal random point fields*, Russian Math. Surveys 55:5 (2000) 923-975.

[22] Spohn, H., *Interacting Brownian particles: a study of Dyson’s model*, In: Hydrodynamic Behavior and Interacting Particle Systems, ed. by G.C. Papanicolaou, IMA Volumes in Mathematics 9, Springer-Verlag (1987) 151-179.
[23] Tanemura, H., *Uniqueness of Dirichlet forms associated with systems of infinitely many Brownian balls in $\mathbb{R}^d$*, Probab. Theory Relat. Fields **109** (1997) 275-299.

[24] Tanemura, H., *A system of infinitely many mutually reflecting Brownian balls in $\mathbb{R}^d$*, Probab. Theory Relat. Fields **104** (1996) 399-426.

[25] Fradon, M., Roelly S., Tanemura, H., *An infinite system of Brownian balls with infinite range interaction*, Stochastic Process. Appl. **90** (2000) 43-66.

[26] G.Szego, *Orthogonal Polynomials*, American Mathematical Society Providence, (1939).

[27] G.Sansone, *Orthogonal Functions*, Interscience Publishers, (1959).