Abstract. Over an algebraically closed field $k$ of characteristic zero, the Drinfeld double $D_n$ of the Taft algebra that is defined using a primitive $n$th root of unity $q \in k$ for $n \geq 2$ is a quasitriangular Hopf algebra. Kauffman and Radford have shown that $D_n$ has a ribbon element if and only if $n$ is odd, and the ribbon element is unique; however there has been no explicit description of this element. In this work, we determine the ribbon element of $D_n$ explicitly. For any $n \geq 2$, we use the R-matrix of $D_n$ to construct an action of the Temperley-Lieb algebra $TL_k(\xi)$ with $\xi = -(q^{\frac{1}{2}} + q^{-\frac{1}{2}})$ on the $k$-fold tensor power $V^\otimes k$ of any two-dimensional simple $D_n$-module $V$. This action is known to be faithful for arbitrary $k \geq 1$. We show that $TL_k(\xi)$ is isomorphic to the centralizer algebra $\text{End}_{D_n}(V^\otimes k)$ for $1 \leq k \leq 2n - 2$.

1. Introduction

Quasitriangular Hopf algebras are widely studied in representation theory, knot theory, tensor categories, and quantum physics [13, 21, 23]. Well-known examples include quantum groups associated to various Lie algebras and superalgebras. Motivated by the study of quantum groups, Drinfeld [12] introduced the notion of a quantum double of any Hopf algebra, and the resulting quasitriangular Hopf algebra is now referred to as the Drinfeld double.

The Drinfeld double $D_n$ of the Taft Hopf algebra $A_n$ at a primitive $n$th root of unity $q$ is an example of a finite-dimensional nonsemisimple Hopf algebra, and it admits more complex behavior than semisimple counterparts. Representations of $D_n$ in characteristic zero have been studied in depth in [6–10, 36]. In particular, $D_n$ has simple modules $V(\ell, r)$, for $1 \leq \ell \leq n$ and $r \in \mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$, and the fusion rules for decomposing tensor products of simple and projective modules are known. These fusion rules were used in [1] by the authors of this article to construct various types of McKay matrices for $D_n$, to determine their eigenvalues and eigenvectors, and to relate them to characters of $D_n$-modules. Chebyshev polynomials of the second, third and fourth kinds play an essential role in describing these eigenvectors and eigenvalues, and in expressing the characters of the simple modules $V(\ell, r)$ when they are evaluated on the grouplike elements of $D_n$.

The algebra $D_n$ is a quasitriangular Hopf algebra. This entails the existence of a distinguished element of $D_n \otimes D_n$, the so-called R-matrix, which induces a $D_n$-module endomorphism on tensor products. The R-matrix can be used to define an element $u$ with special properties (see (3.1)). In particular, $u$ is central and acts as a scalar on each simple module. Some quasitriangular Hopf algebras admit a ribbon element $\upsilon$, which can be regarded as a square root of $u$. The ribbon element satisfies axioms compatible with the quasitriangular structure (see Section 3.1). It has been used extensively to construct invariants of framed links in three-dimensional space and invariants of 3-manifolds (see for example, [16, 24, 26, 32, 34]).
Our focus is on the quasitriangular structure of \( D_n \) and on tensor modules of the form \( V(2, r)^{\otimes k} \), where \( V(2, r) \) is any of the two-dimensional simple \( D_n \)-modules. Kauffman and Radford [21, Prop. 7] proved that \( D_n \) has a ribbon element if and only if \( n \) is odd, and that the ribbon element of \( D_n \) is unique. However, there has been no explicit description of this element. In this work, we determine the unique ribbon element in \( D_n \) and show that the following theorem holds. In the statement, \( b \) and \( c \) are two of the four generators of \( D_n \), and the grouplike elements of \( D_n \) are exactly the elements \( b^ic^k \) for \( i, k \in \mathbb{Z}_n \). The modules \( V(\ell, r) \), \( 1 \leq \ell \leq n, r \in \mathbb{Z}_n \), are all the simple \( D_n \)-modules.

**Theorem 1.1.** (Theorem 3.6) For \( n \) odd, \( n \geq 3 \), and \( q \) a primitive \( n \)th root of unity, the unique ribbon element of the Drinfeld double \( D_n \) of the Taft algebra \( A_n \) is \( v = ub^{q^{-1}}c^{q^{-1}} = u(b^c)^{q^{-1}} \). Moreover, \( v \) acts by the scalar \( q^{(r+\ell-1)+\frac{1}{2}(n-1)(\ell-1)} \) on the simple \( D_n \)-module \( V(\ell, r) \) for all \( 1 \leq \ell \leq n \) and \( r \in \mathbb{Z}_n \).

For any quasitriangular Hopf algebra \( H \) and any finite-dimensional \( H \)-module \( V \), the R-matrix \( R \) can be used to construct elements of \( \text{End}_H(W^{\otimes k}) \) that satisfy axioms (QT1)-(QT3) in Section 3.1 where (QT3) is the well-known Yang-Baxter relation. As a result, tensor modules for \( H \) admit an action of the braid group, which sometimes factors through an action of the Iwahori-Hecke algebra. Leduc and Ram [23] describe a general recipe for constructing such an action on the \( H \)-module \( W^{\otimes k} \). Each generator \( s_i, 1 \leq i \leq k-1 \), of the Iwahori-Hecke algebra acts by applying a scalar multiple of \( R \) to positions \( i \) and \( i+1 \) of \( W^{\otimes k} \) followed by switching those two tensor factors. The action of the Iwahori-Hecke algebra on \( W^{\otimes k} \) is not faithful in general, and there is a kernel. In Section 4.1 we describe the action of the Iwahori-Hecke algebra \( H_k(q) \) on \( V^{\otimes k} \) for any two-dimensional simple \( D_n \)-module \( V = V(2, r) \), \( r \in \mathbb{Z}_n \).

The Temperley-Lieb algebra is a quotient of the Iwahori-Hecke algebra, and it has generators \( t_i, 1 \leq i \leq k-1 \), that satisfy relations (R1’)-(R4’) in Section 4.1. It was introduced in [37] to study the partition function of the Potts model of interacting spins in statistical mechanics and later shown to have applications in the study of von Neumann algebras and subfactors [18], tensor categories [13], canonical bases of the quantum group \( U_q(sl_2) \) for \( q \) generic [14], \( sl_2 \)-tensor invariants and webs [41], and countless other topics in mathematics and physics [33][19][20][22]. The Temperley-Lieb algebra \( TL_k(\xi) \) that arises in our work is a quotient of the Iwahori-Hecke algebra \( H_k(q) \) and depends on the parameter \( \xi = -(q^{1/2}+q^{-1/2}) \). The main result of Section 4 is the following.

**Theorem 1.2.** (Theorem 4.3) Assume \( q \) is a primitive \( n \)th root of unity for any \( n \geq 2 \). Let \( V = V(2, r) \), for \( r \in \mathbb{Z}_n \), be any two-dimensional simple \( D_n \)-module, and set \( \xi = -(q^{1/2}+q^{-1/2}) \). There is an injective algebra homomorphism \( TL_k(\xi) \to \text{End}_{D_n}(V^{\otimes k}) \) for \( k \geq 2 \) given by \( t_i \mapsto q^{\frac{1}{2}(\lambda_i^{-1}R_i - \text{id})} \), for \( 1 \leq i \leq k-1 \), where \( \lambda_r = q^{-r(r+1)} \), and \( R_i \) is the \( D_n \)-module homomorphism obtained by applying the R-matrix of \( D_n \) to tensor slots \( i \) and \( i+1 \) of \( V^{\otimes k} \) and then interchanging those two factors.

When \( 3 \leq \ell \leq n \) and \( V \) is the \( D_n \)-module \( V = V(\ell, r) \), the action of the braid group on \( V^{\otimes k} \) satisfies additional relations beyond the braid group relations, and this setting is considered in Section 4.2. When \( n \) is odd, \( n \geq 3 \), and \( \ell \) satisfies \( 2\ell \leq n+1 \), the ribbon element is used to compute the eigenvalues of \( R^2 \) on \( V(\ell, r)^{\otimes k} \) in Proposition 4.4. In Proposition 4.5 we provide further relations on the action of \( R_i \) when \( n \) is any integer \( \geq 5 \) and \( \ell = 3 \). For arbitrary \( n \geq 2 \) and \( 2\ell \leq n+1 \) we compute two eigenvalues of \( R_i \) on \( V(\ell, r)^{\otimes 2} \) in Proposition 4.7 and suggest a formula for all eigenvalues of \( R_i \) on \( V(\ell, r)^{\otimes k} \) in Conjecture 4.8.

In Section 5 we compare the Bratteli diagram for tensoring with \( n \) simple two-dimensional \( D_n \)-module \( V \) with the Bratteli diagram for tensoring with the simple module \( C^2 \) for the quantum
group $U_q(\mathfrak{sl}_2)$ at a generic value of $q \in \mathbb{C}$. In the $U_q(\mathfrak{sl}_2)$-case, nodes in the $k$th row of the Bratteli diagram can be labeled by partitions $\beta$ of $k$ with at most two parts. This corresponds to the fact that the finite-dimensional simple $U_q(\mathfrak{sl}_2)$-modules have highest weights that can be labeled by such partitions. The comparison enables us to prove the following result giving the dimension of the centralizer algebra $\text{End}_{D_n}(V^\otimes k)$ for any two-dimensional simple $D_n$-module $V$. In the statement, we suppose $S_1, S_2, \ldots, S_{n^2}$ is a listing of the simple $D_n$-modules; $P_1, P_2, \ldots, P_{n^2}$ are respectively their projective covers; $I_k$ is the set of all $i \in \{1, 2, \ldots, n^2\}$ such that $S_i$ or $P_i$ or both occur in $V^\otimes k$; $s_i$ (resp. $p_i$) is the multiplicity of $S_i$ (resp. $P_i$) in $V^\otimes k$ for $i \in I_k$; and $P_i, P'_i$ are two projective covers with the same composition factors but arranged differently.

**Theorem 1.3.** (Theorem 3.4) For any $n \geq 2$ and any two-dimensional simple $D_n$-module $V$, the dimension of the centralizer algebra $\text{End}_{D_n}(V^\otimes k)$ for $k \geq 1$ is

$$\dim_k \text{End}_{D_n}(V^\otimes k) = \sum_{i \in I_k} p_i^2 + \sum_{i \in I_k} (s_i + p_i)^2 + 2 \sum_{i \in I_k} p_ip_i'.$$

In this expression, $s_i, p_i, p'_i$ represent the numbers of paths of length $k$ from $V$ at level 1 in the Bratteli diagram to the summands $S_i, P_i, P'_i$, respectively, at level $k$ in the Bratteli diagram that is determined by $V$.

Using Theorem 1.3 we prove our final result.

**Theorem 1.4.** (Theorem 5.9) For any $n \geq 2$ and any $r \in \mathbb{Z}_n$, the algebra homomorphism $\pi : TL_k(\xi) \to \text{End}_{D_n}(V(2,r)^\otimes k)$ is an isomorphism when $1 \leq k \leq 2n - 2$.

The module $V^\otimes k$, $V = V(2,r)$, is completely reducible when $1 \leq k \leq n - 1$, resulting in a relatively simple argument to prove this theorem for such values of $k$ by comparing the dimensions of $TL_k(\xi)$ and $\text{End}_{D_n}(V^\otimes k)$. In particular, both dimensions can be regarded as sums of squares of the number of paths in a certain Bratteli diagram $\Gamma$, which is essentially a truncated Pascal's triangle (see Lemma 5.7).

When $n \leq k \leq 2n - 2$, $V^\otimes k$ is no longer completely reducible, and it decomposes into a direct sum of simple and indecomposable projective modules that yields a modified Bratteli graph $\Gamma_n$ (displayed in Section 5.11 for $n = 5$ (Figure 1) and compared to the Bratteli diagram $\Gamma$ of partitions with at most two parts (Figure 2)). Nevertheless, the path counts in $\Gamma_n$ can be identified with those in the graph $\Gamma$ (as in Proposition 5.8), although the dimension argument is more complex. Once the dimension match for $1 \leq 2n - 2$ is obtained, the isomorphism result follows immediately from the injectivity statement. When $k > 2n - 2$, examples in Table 1 (see Example 5.5) for $n = 5$ show that the dimension of $\text{End}_{D_n}(V^\otimes k)$ is larger than that of $TL_k(\xi)$, and therefore the map fails to be an isomorphism beyond $k = 2n - 2$.

This result is analogous to the well-known double-centralizer property [5][15] between the quantum group $U_q(\mathfrak{sl}_2)$ and the Temperley-Lieb algebras on tensor powers of the natural $U_q(\mathfrak{sl}_2)$-module $\mathbb{C}^2$ when $q \in \mathbb{C}$ is generic. The centralizer results for $D_n$ are similar to those for the small quantum group $u_q(\mathfrak{sl}_2)$ at $q$ a root of unity (which is a quotient of $D_n$) acting on its unique simple two-dimensional self-dual module.

2. Preliminaries

Throughout this work, $n$ is an integer $\geq 2$, $k$ is an algebraically closed field of characteristic zero, and $q$ is a primitive $n$th root of unity in $k$. All tensor products are over $k$, and we adopt Sweedler's
notation for the coproduct $\Delta$ applied to an element $x$ of a Hopf algebra,

$$\Delta(x) = \sum_x x_1 \otimes x_2.$$  

**Modules for the Drinfeld double of a Taft algebra.** The Drinfeld double $D_n$ has a presentation as the Hopf algebra over $\mathbb{k}$ with generators $a, b, c, d$ that satisfy the following relations:

$$ba = qab, \quad db = qbd, \quad bc = cb, \quad ca = qac, \quad dc = qcd, \quad da - qad = 1 - bc, \quad a^n = 0 = d^n, \quad b^n = 1 = c^n. \quad (2.1)$$

It is an algebra of dimension $n^4$, and the elements $a^i b^j c^k d^l$, $0 \leq i, j, k, l \leq n - 1$, determine a basis for $D_n$. The coproduct, counit, and antipode of $D_n$ are given by

$$\Delta(a) = a \otimes b + 1 \otimes a, \quad \Delta(d) = d \otimes c + 1 \otimes d,$$

$$\Delta(b) = b \otimes b, \quad \Delta(c) = c \otimes c,$$

$$\epsilon(a) = 0 = \epsilon(d), \quad \epsilon(b) = 1 = \epsilon(c),$$

$$S(a) = -ab^{-1}, \quad S(b) = b^{-1}, \quad S(c) = c^{-1}, \quad S(d) = -dc^{-1}. \quad (2.2)$$

The Taft algebra $A_n$ is the Hopf subalgebra generated by $a$ and $b$, and Drinfeld’s process of doubling $A_n$ to get $D_n$ was shown in [6] to yield the presentation above. It follows from (2.2) and the fact that $\Delta$ is an algebra homomorphism that the elements $g = b^i c^k$ for $0 \leq i, k \leq n - 1$ are grouplike, that is, $\Delta(g) = g \otimes g$, $\epsilon(g) = 1 \in \mathbb{k}$, and $S(g) = g^{-1} = g^{n-1}$.

The simple $D_n$-modules $V(\ell, s)$ are indexed by a pair $(\ell, s)$ where $\ell \in \{1, 2, \ldots, n\}$ and $s \in \mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ (the integers modulo $n$). Then $V(\ell, s)$ is a $\mathbb{k}$-vector space of dimension $\ell$ with basis $v_1, v_2, \ldots, v_\ell$ and with $D_n$-action given by

$$a.v_j = v_{j+1}, \quad 1 \leq j < \ell, \quad a.v_\ell = 0,$$

$$b.v_j = q^{s+j-1}v_j, \quad c.v_j = q^{j-1}\ell v_j, \quad 1 \leq j \leq \ell,\quad d.v_j = \alpha_{j-1}(\ell)v_{j-1}, \quad 1 < j \leq \ell, \quad d.v_1 = 0, \quad \text{where}$$

$$\alpha_i(\ell) = \frac{(q^i - 1)(1 - q^{i-\ell})}{q - 1} \quad \text{for} \quad 1 \leq i \leq n - 1. \quad (2.3)$$

From [7][10], we know the following:

1. $V(1, 0)$ is the trivial $D_n$-module with action given by the counit $\epsilon$.
2. $V(\ell, s) \otimes V(1, r) \cong V(\ell, r + s)$.
3. $V(\ell, s) \otimes V(\ell', r)$ is completely reducible if and only if $\ell + \ell' \leq n + 1$. In this case, if $m = \min(\ell, \ell')$, then

$$V(\ell, s) \otimes V(\ell', r) \cong \bigoplus_{j=1}^m V(\ell + \ell' + 1 - 2j, r + s + j - 1). \quad (2.5)$$

Let $P(\ell, s)$ be the projective cover of $V(\ell, s)$ for $1 \leq \ell < n$ and $s \in \mathbb{Z}_n$. Chen [8] has shown that any indecomposable projective left $D_n$-module is isomorphic to one of the modules $P(\ell, s)$ for $1 \leq \ell < n$ or to $V(n, s)$ for some $s \in \mathbb{Z}_n$, where the indecomposable module $P(\ell, s)$ has the following structure:

There is a chain of submodules $P(\ell, s) \supset \text{soc}^2(P(\ell, s) \supset \text{soc}(P(\ell, s)) \supset (0)$ such that

1. $\text{soc}(P(\ell, s))$ is the socle of $P(\ell, s)$ (the sum of all the simple submodules), and $\text{soc}(P(\ell, s)) \cong V(\ell, s)$;
2. $\text{soc}^2(P(\ell, s))/\text{soc}(P(\ell, s)) \cong V(n - \ell, s + \ell) \oplus V(n - \ell, s + \ell)$;
The element $u$ and $R$ are defined.

The projective indecomposable modules $P(\ell, s)$ and $P(n - \ell, s + \ell)$ have the same composition factors, but arranged differently. The dimension of $P(\ell, s)$ is $2n$ for $1 \leq \ell < n$. The modules $V(n, s)$ for $s \in \mathbb{Z}_n$ are the only $D_n$-modules that are both simple and projective.

It follows from [25] and [10] Prop. 3.1, Thms. 3.3 and 3.5 that for the two-dimensional simple module $V = V(2, r)$,

1. $V(1, s) \otimes V \cong V(2, r + s)$;
2. $V(\ell, s) \otimes V \cong V(\ell + 1, r + s) \oplus V(\ell - 1, r + 1 + s)$ for $2 \leq \ell < n$;
3. $V(n, s) \otimes V \cong P(n - 1, r + 1 + s)$;
4. $P(1, s) \otimes V \cong P(2, r + s) \oplus 2V(n, r + 1 + s)$;
5. $P(\ell, s) \otimes V \cong P(\ell + 1, r + s) \oplus P(\ell - 1, r + 1 + s)$ for $2 \leq \ell < n - 1$;
6. $P(n - 1, s) \otimes V \cong P(n - 2, r + 1 + s) \oplus 2V(n, r + s)$.

In Section [5] we will use these relations to determine the decomposition of $V^\otimes k$ into simple and projective summands and to relate the corresponding Bratteli diagram to that obtained from tensor powers of the natural two-dimensional module for the quantum group $U_q(\mathfrak{sl}_2)$, where $q$ is generic.

3. The ribbon structure on $D_n$

The algebra $D_n$ belongs to the class of quasi-triangular Hopf algebras. The distinguishing feature of a quasi-triangular Hopf algebra is the existence of an R-matrix, which induces a module homomorphism on tensor products. The R-matrix can be used to construct central elements in the algebra, and some quasi-triangular Hopf algebras admit a special type of central element termed a ribbon element, which plays an essential role in constructing knot and link invariants. Kauffman and Radford [21] Prop. 7] have shown that $D_n$ has a unique ribbon element whenever $n$ is odd. However, the exact expression for this ribbon element has not been known. In this section, we review needed results on quasi-triangular Hopf algebras and use them to determine an explicit expression for the unique ribbon element of $D_n$. When the field is algebraically closed, the ribbon element acts as a scalar on the simple $D_n$-modules because it is central, and we also determine that scalar for each simple $D_n$-module in Theorem 3.6.

3.1. Background on quasi-triangular Hopf algebras. Throughout Section 3.1 we assume $H$ is a finite-dimensional Hopf algebra over $k$ with antipode $S$, counit $\varepsilon$, and coproduct $\Delta(x) = \sum_x x_{(1)} \otimes x_{(2)}$ for $x \in H$, and $H^*$ is the $k$-dual Hopf algebra.

The quasi-triangular property. (See, for example, [29] Section 10.1 or [5] Section 4.2.) A Hopf algebra $H$ is quasi-triangular if there is an invertible element $R \in H \otimes H$ such that

1. $R \Delta(x) R^{-1} = \Delta^{op}(x)$ for all $x \in H$, where $\Delta^{op}(x)$ has the tensor factors in $\Delta(x)$ interchanged, and
2. $(\Delta \otimes id)(R) = R_{13} R_{23}$,
3. $(id \otimes \Delta)(R) = R_{13} R_{12}$,

where if $R = \sum_i x_i \otimes y_i$, then $R_{12} = \sum_i x_i \otimes y_i \otimes 1$, $R_{13} = \sum_i x_i \otimes 1 \otimes y_i$, and $R_{23} = \sum_i 1 \otimes x_i \otimes y_i$. Let $R^{op} = \sum_i y_i \otimes x_i$

The element $u$. We assume $R = \sum_i x_i \otimes y_i$ as above and use the antipode $S$ to define

\begin{equation}
(3.1) \quad u = \sum_i S(y_i) x_i \in H.
\end{equation}
Then the following hold
\[ u x u^{-1} = S^2(x) \text{ for all } x \in H \quad \text{and} \quad \Delta(u) = (R^\text{op} R)^{-1}(u \otimes u). \]

**A ribbon Hopf algebra.** A quasitriangular Hopf algebra \( H \) is a ribbon Hopf algebra if there is an invertible element \( v \) (the ribbon element) in the center of \( H \) such that
\[
\begin{align*}
&v^2 = u S(u), \\
&S(v) = v, \\
&\varepsilon(v) = 1, \\
&\Delta(v) = (R^\text{op} R)^{-1}(v \otimes v),
\end{align*}
\]
where \( u \) is as in (3.1). Then \( v^{-1}u \) is grouplike: \( \Delta(v^{-1}u) = v^{-1}u \otimes v^{-1}u \).

**The tensor power centralizer algebra.** Let \( V \) be a module over the quasitriangular Hopf algebra \( H \), and assume \( R \in \text{End}_H(V \otimes^2 V) \) gives the action of \( R \) on \( V \otimes^2 V \). Suppose \( \sigma: V \otimes^2 V \to V \otimes^2 V \) is the interchange map \( \sigma(w \otimes x) = x \otimes w \), and set \( \tilde{R} = \sigma R \in \text{End}_H(V \otimes^2 V) \). Assume in \( \text{End}_H(V \otimes^k) \) that
\[
\tilde{R}_i := \text{id}_V \otimes \cdots \otimes \text{id}_V \otimes \tilde{R} \otimes \text{id}_V \otimes \cdots \otimes \text{id}_V,
\]
for \( 1 \leq i \leq k - 1 \), where \( \tilde{R} \) occupies tensor slots \( i \) and \( i + 1 \). Then the following hold (see, for example, [23] Prop. 2.18):
\begin{itemize}
  \item (QT1) \( \tilde{R}_i \) belongs to the centralizer algebra \( \text{End}_H(V \otimes^k) \) of transformations on \( V \otimes^k \) commuting with the \( H \)-action.
  \item (QT2) \( \tilde{R}_i \tilde{R}_j = \tilde{R}_j \tilde{R}_i \), for \( |i - j| > 1 \).
  \item (QT3) \( \tilde{R}_i \tilde{R}_{i+1} \tilde{R}_i = \tilde{R}_{i+1} \tilde{R}_i \tilde{R}_{i+1} \), for \( 1 \leq i \leq k - 2 \).
\end{itemize}
This tells us that the subalgebra of \( \text{End}_H(V \otimes^k) \) generated by the \( \tilde{R}_i \) is a homomorphic image of the group algebra of the braid group on \( k - 1 \) strands.

**Action of the ribbon element.** Suppose \( \{ U_\omega \}_{\omega \in \Omega} \) are the simple \( H \)-modules for the ribbon Hopf algebra \( H \). Then since the field is algebraically closed, the ribbon element \( \upsilon \) acts as a scalar, \( \upsilon_\omega \) on \( U_\omega \) by Schur’s lemma.
\begin{itemize}
  \item When \( U_\mu \otimes U_\nu \) is a completely reducible \( H \)-module, then \( R^\text{op} R \) acts on a simple summand \( U_\omega \) of \( U_\mu \otimes U_\nu \) by the scalar
\[
\frac{\upsilon_\mu \upsilon_\nu}{\upsilon_\omega}.
\]
\end{itemize}
More details can be found in [23].

**Integrals and quasiribbon elements.** To determine the ribbon element of \( D_n \) in the next section, we will use several well-known facts about integrals and quasiribbon elements for a finite-dimensional Hopf algebra \( H \). For \( h \in H \) and \( \alpha \) in the dual space \( H^* \), we define
\[
\langle \alpha, h \rangle := \alpha(h) \in k.
\]
The following results on integrals can be found, e.g., in [27] Secs. 12.1.1, 12.1.2 or [29] Chap. 2:
\begin{itemize}
  \item The right integrals and left integrals of \( H \) are respectively
\[
\int_H^r = \{ \Lambda \in H \mid \Lambda h = \langle \varepsilon, h \rangle \Lambda \text{ for all } h \in H \} \quad \text{and} \quad \int_H^l = \{ \Lambda' \in H \mid h \Lambda' = \langle \varepsilon, h \rangle \Lambda' \text{ for all } h \in H \}.
\]
  \item These spaces are one-dimensional (see [27] Thm. 10.9(b)) and are related by the antipode:
\[
S(\int_H^r) = \int_H^l, \quad S(\int_H^l) = \int_H^r.
\]
When \( \int_H^l = \int_H^r \), then \( H \) is said to be unimodular.
• Fix \( \Lambda \neq 0 \) in \( J^+ \). For every \( h \in H \), \( h\Lambda \in J^+ \), hence \( h\Lambda \) is a scalar multiple of \( \Lambda \). This implies that there is an \( \tilde{\alpha} \in H^* \), such that \( h\Lambda = \tilde{\alpha}(h)\Lambda \), for all \( h \in H \). The element \( \tilde{\alpha} \) is a grouplike element of \( H^* \), referred to as the distinguished grouplike element of \( H^* \). The condition that \( H \) is unimodular is equivalent to \( \tilde{\alpha} = \varepsilon \).

Analogously, the dual algebra \( H^* \) has a right integral, which we denote \( \lambda \), and for any \( h^* \in H^* \),
\[
\lambda h^* = h^*(1_H)\lambda.
\]
Corresponding to a nonzero right integral \( \lambda \in H^* \), there is a distinguished grouplike element \( \tilde{g} \in H \) such that for any \( h^* \in H^* \),
\[
h^* \lambda = h^*(\tilde{g})\lambda.
\]
As above, assume the R-matrix is \( R = \sum_{i} x_i \otimes y_i \), and define
\[
g_{\tilde{\alpha}} = \sum_{i} x_i \tilde{\alpha}(y_i), \quad \text{and} \quad h_{\tilde{\alpha}} = g_{\tilde{\alpha}} \tilde{g}^{-1},
\]
where \( \tilde{\alpha} \) is the distinguished grouplike element of \( H^* \), and \( \tilde{g} \) is the distinguished grouplike element of \( H \).

A quasiribbon element of the Hopf algebra \( H \) is an element satisfying all the ribbon conditions in (3.2) except for the requirement that it be central. Our approach to finding an explicit formula for the ribbon element of \( D_n \) is to use the following results from [21] on quasiribbon elements.

**Theorem 3.1.** [21] Thm. 1] Suppose \( h_{\tilde{\alpha}}' \) is any element of \( H \) such that \( (h_{\tilde{\alpha}}')^2 = h_{\tilde{\alpha}} \), i.e. \( h_{\tilde{\alpha}}' \) is any square root of the element \( h_{\tilde{\alpha}} \) in (3.5). Then \( v = uh_{\tilde{\alpha}}' \) is a quasiribbon element, where \( u \) is as in (3.1).

**Corollary 3.2.** [21] Cor. 2] When \( H \) has odd dimension, its quasiribbon element is unique.

**Remark 3.3.** Since the Drinfeld double \( D_n \) of the Taft algebra \( A_n \) has a unique ribbon element when \( n \) is odd by [21] Prop. 7], it is the only possible candidate for the quasiribbon element. Moreover, by [31] Thm. 4] or [10] Prop. 3.4], \( D_n \) is unimodular, so that \( \tilde{\alpha} = \varepsilon \). Thus for \( D_n \), the ribbon element \( v = uh_{\tilde{\alpha}}' \), where \( h_{\tilde{\alpha}}' \) is a square root of \( h_{\tilde{\alpha}} = h_{\varepsilon} = g_{\varepsilon} \tilde{g}^{-1} \). We will use this fact in the next section to compute \( v \) explicitly and to determine its action on simple \( D_n \)-modules.

### 3.2. Computation of the ribbon element in \( D_n \)

Throughout this section \( n \) is an odd integer \( n \geq 3 \). We combine the description of the ribbon element of \( D_n \) in Remark 3.3] with the next result due to Radford to obtain an explicit expression for the ribbon element of \( D_n \). We identify \( D_n \) with \( A_n^* \otimes A_n \), where \( A_n \) is the Taft algebra, and \( A_n^* \) is its dual, and we assume that \( \alpha_0 \) and \( g_0 \) are the distinguished grouplike elements in \( A_n^* \) and \( A_n \), respectively. Under these identifications, we have

**Proposition 3.4.** [21] Cor. 7] The distinguished grouplike element in \( D_n \) is given by \( \alpha_0 \otimes g_0 \).

To understand this element and its relation to the ribbon element of \( D_n \) more precisely, we need a better grasp of the isomorphism identifying \( D_n \) and \( A_n^* \otimes A_n \), which is detailed in the following results of Chen. (In comparing the statements below with the results of [6], it is helpful to note that our \( A_n^* \) is \( A_n(q) \), and our \( A_n \) is \( H_n(q) \) in the notation of [5].) The expressions for the coproduct in (3.7) and (3.8) below require the quantum binomial coefficient
\[
\binom{m}{i} = \frac{[m]!}{[i]! [m-i]!},
\]
for \( m, i \in \mathbb{Z}_{\geq 0} \), with \( m \geq i \). Here \([m]!\) is the quantum factorial \([m]! = [m][m-1] \cdots [1]\), where \([m]\) is the quantum integer \([m] = 1 + q + \cdots + q^{m-1}\) for \( m \geq 1 \), and \([0] = [0]! = 1\).
Lemma 3.5. \( (a^{n-1}b)^* \) is a right integral in \( \mathbb{A}_n^* \), and the distinguished grouplike element of \( \mathbb{A}_n \) is \( b \).

(b) \( (cd^{m-1})^* \) is a right integral in \( (\mathbb{A}_n^*)^* \), and the distinguished grouplike element of \( \mathbb{A}_n^* \) is \( c \).

Proof. We use the following characterization of a right integral of \( H^* \) given in [27, (12.2)] with \( H = \mathbb{A}_n \):

\[(3.10)\quad \lambda \in \int_{\mathbb{A}_n^*}^r \iff \lambda(x)1_{\mathbb{A}_n} = \sum_x \lambda(x_{(1)})x_{(2)} \quad \text{for all} \quad x \in \mathbb{A}_n.\]

(a) For \( \lambda = (a^{n-1}b)^* \) and \( x = a^mb^w \), we have

\[
\langle (a^{n-1}b)^*, a^mb^w \rangle 1_{\mathbb{A}_n} = \begin{cases}
1_{\mathbb{A}_n} & \text{if } m = n-1 \text{ and } w = 1 \pmod{n}, \\
0 & \text{otherwise}.
\end{cases}
\]

By (3.7), the expression on the right-hand side of (3.10) is

\[
\sum_{i=0}^{m} \binom{m}{i} \langle (a^{n-1}b)^*, a^i b^w \rangle \otimes a^{m-i} b^{w+i}.
\]

This is zero unless \( i = n-1 = m \) and \( w = 1 \pmod{n} \), in which case \( a^{n-1}b^{w+i} = a^0 b^0 = 1_{\mathbb{A}_n} \). Thus, \( (a^{n-1}b)^* \) is a right integral of \( \mathbb{A}_n^* \) by (3.10).
To determine the distinguished grouplike element associated to \((a^{n-1}b)^*\), we use (3.7) and (3.9) to compute
\[
\langle (a^s b^t)^* (a^{n-1}b)^*, a^m b^w \rangle = \sum_{i=0}^{m} \left[ \sum_{i}^{m} \langle (a^s b^t)^*, a^i b^w \rangle \langle (a^{n-1}b)^*, a^{m-i} b^{w+i} \rangle \right]
\]
\[
= \sum_{s}^{m} \langle (a^s b^t)^*, a^s b^w \rangle \langle (a^{n-1}b)^*, a^{m-s} b^{w+s} \rangle
\]
\[
= \sum_{s}^{m} \delta_{t,w} \delta_{n-1,m-s} \delta'_1 \delta_{w+s},
\]
where in the Kronecker delta \(\delta'_{w+s}\) the term \(w+s\) should be interpreted mod \(n\). This expression is zero, unless \(w = 1 - s \mod n\) and \(m = n - 1 + s\). Therefore, \((a^s b^t)^* (a^{n-1}b)^* = 0\) if \(s \geq 1\), and when \(s = 0\),
\[
(b^t)^* (a^{n-1}b)^* = \delta_{t,1} (a^{n-1}b)^*.
\]
Consequently, for \(0 \leq s \leq n - 1\), and \(t \in \mathbb{Z}_n\),
\[
(a^s b^t)^* (a^{n-1}b)^* = \langle (a^s b^t)^*, b \rangle (a^{n-1}b)^* = \begin{cases} 0 & \text{if } (s,t) \neq (0,1), \\ 1 & \text{if } (s,t) = (0,1). \end{cases}
\]
This shows that \(b\) is the distinguished grouplike element in \(A_n\).

Part (b) follows by a similar argument using (3.8).

Let \(\alpha_0\) and \(g_0\) be the distinguished grouplike elements defined prior to Proposition 3.4. The previous lemma implies that \(\alpha_0 = C\) and \(g_0 = B\), under the identifications made in (i) of Section 3.2. Therefore \(C \otimes B\) is the distinguished group element in \(D_n\) by Proposition 3.4.

Recall that the R-matrix \(R = \sum_i x_i \otimes y_i\) gives rise to the element \(u = \sum_j S(y_j) x_i\) in Section 3.1. We will use the explicit expression for the R-matrix of the quasitriangular Hopf algebra \(D_n\) in [6, Thm. 3.6]:
\[
(3.11) \quad R = \frac{1}{n} \sum_{m,s,t=0}^{n-1} \frac{q^{-tm}}{[s]!} a^s b^t \otimes c^m d^s.
\]

**Theorem 3.6.** Assume that \(n\) is an odd integer \(n \geq 3\).

(a) The unique ribbon element in \(D_n\) is
\[
(3.12) \quad v = u (bc)^{\frac{n-1}{2}}, \quad \text{where}
\]
\[
u = \frac{1}{n} \sum_{m,s,t=0}^{n-1} \frac{q^{-(s+1)-t(m+s)}}{[s]!} (-1)^s d^s c^{-s-m} b^a s.
\]

(b) The ribbon element \(v\) of \(D_n\) acts on any simple \(D_n\)-module \(V(\ell, r)\) by the scalar
\[
q^{r(\ell+\ell-1)+\frac{1}{2}(n-1)(\ell-1)} \quad \text{for } 1 \leq \ell \leq n \text{ and } r \in \mathbb{Z}_n.
\]

**Proof.** (a) We adopt the previous conventions and set \(g_\alpha = g_\varepsilon\), (which holds as \(D_n\) is unimodular). Now combining (3.11) and (3.5), we have
\[
(3.13) \quad g_\varepsilon = \frac{1}{n} \sum_{m,s,t=0}^{n-1} \frac{q^{-tm}}{[s]!} a^s b^t \varepsilon (c^m d^s).
\]
Since \( \varepsilon(d) = 0 \) and \( \varepsilon(c) = 1 \), and \( \varepsilon \) is an algebra homomorphism, only the terms with \( s = 0 \) survive, and therefore
\[
g_{\varepsilon} = \frac{1}{n} \sum_{m, t = 0}^{n - 1} q^{-tm} b^t = \frac{1}{n} \sum_{t = 0}^{n - 1} \left( \sum_{m = 0}^{n - 1} q^{-tm} \right) b^t.
\]
Observe that
\[
\sum_{m = 0}^{n - 1} q^{-tm} = \frac{1 - (q^{-t})^n}{1 - q^{-t}} = 0
\]
unless \( t = 0 \), in which case \( \sum_{m = 0}^{n - 1} q^{-tm} = n \). Therefore \( g_{\varepsilon} = 1_{D_n} \).

By the discussion prior to this theorem, the distinguished grouplike element in \( D_n \simeq A_n^* \otimes A_n \) is \( \tilde{g} = C \otimes B \). We aim to show that this is in fact \( cb \). The expression for the product in \( D_n \simeq A_n^* \otimes A_n \) in (3.6) when applied to \( cb \) reduces to a single term:
\[
cb = (C \otimes 1)(1 \otimes B) = \tau(1, 1)C \otimes B r^{-1}(1, 1) = C \otimes B = \tilde{g}.
\]
This is mainly due to the fact that \( \Delta(1) = 1 \otimes 1 \), and the coproduct factors of \( \Delta^2(1) \) are \( 1_{(1)} = 1_{(2)} = 1_{(3)} = 1 \).

By (3.5), \( h_{\varepsilon} = g_{\varepsilon}\tilde{g}^{-1} = (bc)^{-1} = (bc)^{n-1} \). When \( n \) is odd, the square root \( h'_{\varepsilon} \) of \( h_{\varepsilon} \) is unique, because \( h_{\varepsilon} \), and therefore \( h'_{\varepsilon} \), has odd order. Thus,
\[
(3.14) \quad h'_{\varepsilon} = (bc)^{n-1} \quad \text{and} \quad v = uh_{\varepsilon} = u (bc)^{n-1}
\]
by Theorem 3.6 as asserted in part (a).

It remains to show that \( u \) has the expression in (3.12). Recall that \( u = \sum_i S(y_i)x_i \), where \( R = \sum_i x_i \otimes y_i = \frac{1}{n} \sum_{m, s, t = 0}^{n - 1} \frac{q^{-tm}}{[s]!} a^s b^t \otimes c^m d^s \). Thus,
\[
u = \frac{1}{n} \sum_{m, s, t = 0}^{n - 1} \frac{q^{-tm}}{[s]!} S(c^m d^s)a^s b^t = \frac{1}{n} \sum_{m, s, t = 0}^{n - 1} \frac{q^{-tm}}{[s]!} (-dc^{-1})^s c^{-m} a^s b^t
= \frac{1}{n} \sum_{m, s, t = 0}^{n - 1} \frac{q^{s(s-1)/2}}{[s]!} (-1)^s d^sc^{-m} a^s b^t = \frac{1}{n} \sum_{m, s, t = 0}^{n - 1} \frac{q^{s(s-1)/2 - t(m+s)}}{[s]!} (-1)^s d^sc^{-m} b^t a^s.
\]
As a result, (a) holds.

(b) Since \( v \) is central and the field is assumed to be algebraically closed, \( v \) acts on any simple module \( V(\ell, r) \) by a scalar. It suffices to compute the action of \( v \) on any vector of \( V(\ell, r) \), in particular, on the last basis element \( v_\ell \), and for this we have
\[
bc.v_\ell = q^{r+\ell-1}q^{r-\ell+\ell}v_\ell = q^{\ell-1}v_\ell.
\]
Note that in the expression (3.12) for \( u \), all the summands act on \( v_\ell \) as zero except for the terms with \( s = 0 \). The scalar action of \( u \) on \( v_\ell \) reduces to the action of
\[
\overline{u} = \frac{1}{n} \sum_{m, t = 0}^{n - 1} q^{-mt} c^{-m} b^t
\]
on \( v_\ell \), and the scalar determined by \( \overline{u} \) is
\[
\frac{1}{n} \sum_{m, t = 0}^{n - 1} q^{-mt} q^{-m(-r)} q^{(r+\ell-1)t} = \frac{1}{n} \sum_{t = 0}^{n - 1} q^{t(r+\ell-1)} \sum_{m = 0}^{n - 1} q^m(r-t) = q^{r(r+\ell-1)}.
\]
Thus, there is a linear action of the braid group on $k$-fold tensor powers. Part (b) follows from multiplying the scalars for the action of $u$ and $(bc)^{n-1}$ on $v_\ell$. □

4. Temperley-Lieb actions

We continue to assume that $k$ is an algebraically closed field of characteristic zero, and take $q \in k$ to be a primitive $n$th root of unity for any $n \geq 2$. For $k \geq 2$, we describe an action of the Iwahori-Hecke algebra $H_k(q)$ on $V(2,r)\otimes^k$ for $r \in \mathbb{Z}_n$, that comes from the R-matrix of $D_n$. We show that this action factors through a Temperley-Lieb algebra. Section 5 will continue our focus on tensor powers of $V(2,r)$.

4.1. Action of the Iwahori-Hecke algebra. Assume $k$ is an integer $\geq 2$. The group algebra of the braid group on $k-1$ strands has generators $s_1, s_2, \ldots, s_{k-1}$ over $k$ subject to relations (R1) and (R2) below. The Iwahori-Hecke algebra (of type A) is the associative algebra $H_k(q)$ over $k$ that is the quotient of the group algebra of the braid group by the additional relation (R3) below:

(R1) $s_is_j = s_js_i$ \quad $|i - j| > 1$,
(R2) $s_is_{i+1}s_i = s_{i+1}s_is_{i+1}$ \quad $1 \leq i \leq k - 2$,
(R3) $(s_i - 1)(s_i - q^{-1}) = 0$ \quad $1 \leq i \leq k - 1$.

Let $t_i = q^{\frac{i}{2}}(s_i - 1)$, and fix the parameter $\xi = -(q^{\frac{1}{2}} + q^{-\frac{1}{2}})$. It is straightforward to check that $H_k(q)$ has an alternative presentation with generators $t_1, t_2, \ldots, t_{k-1}$ subject to relations

(R1') $t_it_j = t_jt_i$ \quad $|i - j| > 1$,
(R2') $t_it_{i+1}t_i - t_i = t_{i+1}t_it_{i+1} - t_{i+1}$ \quad $1 \leq i \leq k - 2$,
(R3') $t_i^2 = \xi t_i$ \quad $1 \leq i \leq k - 1$.

The Temperley-Lieb algebra $\mathcal{TL}_k(\xi)$ with $\xi = -(q^{\frac{1}{2}} + q^{-\frac{1}{2}})$ is the quotient of $H_k(q)$ by the further relations (compare to [14,30]):

(R4') $t_it_{i+1}t_i = t_i$ \quad $1 \leq i \leq k - 2$.

The dimension of the Temperley-Lieb algebra $\mathcal{TL}_k(\xi)$ is the Catalan number $C_k$ for any value of $\xi$. Thus, $\dim \mathcal{TL}_1(\xi) = \dim \mathcal{TL}_0(\xi) = 1$, $\dim \mathcal{TL}_2(\xi) = 2$, $\dim \mathcal{TL}_3(\xi) = 5$, $\dim \mathcal{TL}_4(\xi) = 14$, and so forth.

The action of the Iwahori-Hecke algebra on tensor powers. Let $V$ be a module over a quasitriangular Hopf algebra $H$. Recall that the mappings $\tilde{R}_i := id \otimes \cdots \otimes id \otimes \tilde{R} \otimes \otimes id \otimes \cdots \otimes id$ in (8.3), where $\tilde{R}$ occupies tensor slots $i$ and $i+1$ for $1 \leq i \leq k-1$, satisfy the relations (QT1)-(QT3). Thus, there is a linear action of the braid group on the $k$-fold tensor power $V^\otimes k$ given by

$$s_i \mapsto \alpha \tilde{R}_i,$$

for some $\alpha \in k^\times$.

This is the standard action of the braid group with $k-1$ strands on the tensor power $V^\otimes k$, and it is well-studied in the literature, see e.g., [17,23].

When $V = V(2,r)$, $r \in \mathbb{Z}_n$, is any two-dimensional $D_n$-module, we use the R-matrix of $D_n$ to explicitly describe an action of the Iwahori-Hecke algebra $H_k(q)$ on $V^\otimes k$ that induces an action of the Temperley-Lieb algebra $\mathcal{TL}_k(\xi)$ on $V^\otimes k$. Note that this action does not depend on $r$ and the resulting representation coincides with the standard action on tensor powers described above.
Recall that the R-matrix for $D_n$ is given by

\begin{equation}
\mathcal{R} = \frac{1}{n} \sum_{m,s,t=0}^{n-1} \frac{q^{-tm}}{[s]!} a^s b^t \otimes c^m d^s = \frac{1}{n} \sum_{m,s,t=0}^{n-1} \frac{q^{-tm-ts}}{[s]!} b^t a^s \otimes c^m d^s.
\end{equation}

We assume $\mathcal{R} \in \text{End}_{D_n}(V^{\otimes 2})$ gives the action of $\mathcal{R}$ on $V^{\otimes 2}$, where the terms $a^s b^t$ act on the first tensor factor of $V^{\otimes 2}$ and the $c^m d^s$ on the second. Let $\sigma : V^{\otimes 2} \rightarrow V^{\otimes 2}$ be the map interchanging the factors, $\sigma(w \otimes x) = x \otimes w$, and set $\tilde{\mathcal{R}} = \sigma \mathcal{R} \in \text{End}_{D_n}(V^{\otimes 2})$.

For ease of notation, in what follows we will use id for the identity map; the space it is acting upon should be apparent from the context.

**Lemma 4.1.** Let $V = V(2, r)$ for any $r \in \mathbb{Z}_n$. Then the transformation $\tilde{\mathcal{R}}$ on $V^{\otimes 2}$ satisfies

\begin{equation}
(\tilde{\mathcal{R}} - \lambda_r \text{id})(\tilde{\mathcal{R}} + \lambda_r q^{-1} \text{id}) = 0, \quad \text{where} \quad \lambda_r = q^{-r(r+1)}.
\end{equation}

The eigenspace of $\tilde{\mathcal{R}}$ corresponding to the eigenvalue $\lambda_r$ is spanned by $v_1 \otimes v_1, v_1 \otimes v_2 + q^r v_2 \otimes v_1,$ and $v_2 \otimes v_2$. The eigenspace corresponding to $-\lambda_r q^{-1}$ is spanned by $v_1 \otimes v_2 - q^{r+1} v_2 \otimes v_1$.

**Proof.** By (2.3), the actions of the generators of $D_n$ relative to the basis elements $\{v_1, v_2\}$ of $V = V(2, r)$ are given by the following matrices:

\[
\begin{align*}
\begin{pmatrix} a \rightarrow 0 & 0 \\ 1 & 0 \end{pmatrix} & \quad \begin{pmatrix} b \rightarrow q^r & 0 \\ 0 & q^{r+1} \end{pmatrix} & \quad \begin{pmatrix} c \rightarrow q^{-(r+1)} & 0 \\ 0 & q^{-r} \end{pmatrix} & \quad \begin{pmatrix} d \rightarrow 0 & \alpha_1(2) \\ 0 & 0 \end{pmatrix},
\end{align*}
\]

where $\alpha_1(2) = 1 - q^{-1}$.

To compute the action of $\mathcal{R}$ on $V^{\otimes 2}$, note that $a^s$ and $d^s$ act as 0 on $V$ for $s \geq 2$. Then

\[
\mathcal{R}(v_1 \otimes v_1) = \frac{1}{n} \sum_{m,t=0}^{n-1} q^{-tm} b^t v_1 \otimes c^m v_1 = \frac{1}{n} \left( \sum_{m=0}^{n-1} q^{-m(r+1)} \left( \sum_{t=0}^{n-1} q^{t(r-m)} \right) \right) v_1 \otimes v_1 = q^{-r(r+1)} v_1 \otimes v_1,
\]

since the inner summation over $t$ is 0 unless $m = r$, in which case it is $n$. Therefore,

\begin{equation}
\tilde{\mathcal{R}}(v_1 \otimes v_1) = \sigma \mathcal{R}(v_1 \otimes v_1) = q^{-r(r+1)} v_1 \otimes v_1 = \lambda_r v_1 \otimes v_1.
\end{equation}

Similarly,

\[
\mathcal{R}(v_2 \otimes v_2) = \frac{1}{n} \sum_{m,t=0}^{n-1} q^{-tm} b^t v_2 \otimes c^m v_2 = \frac{1}{n} \left( \sum_{m=0}^{n-1} q^{-m(r+1) - mr} v_2 \otimes v_2 \right) = \frac{1}{n} \left( \sum_{m=0}^{n-1} q^{-mr} \left( \sum_{t=0}^{n-1} q^{t(r+1-m)} \right) \right) v_2 \otimes v_2 = q^{-(r+1)} v_2 \otimes v_2, \quad \text{so that}
\]

\begin{equation}
\tilde{\mathcal{R}}(v_2 \otimes v_2) = \lambda_r v_2 \otimes v_2.
\end{equation}

Now

\[
\mathcal{R}(v_1 \otimes v_2) = \frac{1}{n} \sum_{m,t=0}^{n-1} q^{-tm} b^t v_1 \otimes c^m v_2 + \frac{1}{n} \sum_{m,t=0}^{n-1} q^{-t(m+1)} b^t v_2 \otimes c^m \alpha_1(2) v_1
\]
\[
R(v_2 \otimes v_1) = \frac{1}{n} \sum_{m,t=0}^{n-1} q^{-tm} b^t v_2 \otimes c^m v_1 = \frac{1}{n} \sum_{m,t=0}^{n-1} q^{-tm+t(r+1)-m(r+1)} v_2 \otimes v_1
\]

Finally,

\[
R(v_2 \otimes v_1) = \frac{1}{n} \left( \sum_{m=0}^{n-1} q^{-m(r+1)} \left( \sum_{t=0}^{n-1} q^{t(r+1-m)} \right) \right) v_2 \otimes v_1 = q^{-(r+1)^2} v_2 \otimes v_1,
\]

and as a result,

\[
\hat{R}(v_2 \otimes v_1) = q^{-(r+1)^2} v_2 \otimes v_2.
\]

On the subspace of \(V^{\otimes 2}\) spanned by \(v_1 \otimes v_2\) and \(v_2 \otimes v_1\), the transformation \(\hat{R}\) has matrix

\[
\begin{pmatrix}
\lambda_r - \lambda_r q^{-1} & q^{-(r+1)^2} \\
q^{r-2} & 0
\end{pmatrix}
\]

It is easily seen that this matrix has eigenvalues \(\lambda_r = q^{-r(r+1)}\) and \(-\lambda_r q^{-1} = -q^{-r(r+1)-1}\) with corresponding eigenvectors \(v_1 \otimes v_2 + q^r v_2 \otimes v_1\), \(v_1 \otimes v_2 - q^{r+1} v_2 \otimes v_1\), respectively. Thus, \(\hat{R}\) has a three-dimensional eigenspace on \(V^{\otimes 2}\) with basis \(v_1 \otimes v_1\), \(v_1 \otimes v_2 + q^r v_2 \otimes v_1\), \(v_2 \otimes v_2\) corresponding to the eigenvalue \(\lambda_r\), and a one-dimensional eigenspace with basis element \(v_1 \otimes v_2 - q^{r+1} v_2 \otimes v_1\) corresponding to the eigenvalue \(-\lambda_r q^{-1}\).

Recall that the mappings \(\hat{R}_i := \text{id} \otimes \cdots \otimes \text{id} \otimes \hat{R} \otimes \text{id} \otimes \cdots \otimes \text{id}\) in (3.3), where \(\hat{R}\) occupies tensor slots \(i\) and \(i + 1\) for \(1 \leq i \leq k - 1\), satisfy the relations (QT1)-(QT3). Hence, they belong to \(\text{End}_D(V(2,r)^{\otimes k})\). As we noted earlier, the subalgebra of \(\text{End}_D(V(2,r)^{\otimes k})\) generated by the \(\hat{R}_i\) is a homomorphic image of the group algebra of the braid group on \(k - 1\) strands, but in fact, we can now prove that the centralizer algebra of the \(D_n\)-action on \(V^{\otimes k}\) affords a representation of the Iwahori-Hecke algebra.

**Proposition 4.2.** Let \(V = V(2,r)\), \(r \in \mathbb{Z}_n\), and assume \(k \geq 2\). There is an algebra homomorphism \(H_k(q) \to \text{End}_D(V^{\otimes k})\) given by \(s_i \mapsto \lambda_r^{-1} \hat{R}_i\) for \(1 \leq i \leq k - 1\), where \(\hat{R}_i\) is as in (3.3) and \(\lambda_r = q^{-r(r+1)}\).

**Proof.** We know from (4.2) that \(\hat{R}\) satisfies the relation \((\hat{R} - \lambda_r \text{id})(\hat{R} + \lambda_r q^{-1} \text{id}) = 0\) on \(V^{\otimes 2}\) when \(V = V(2,r)\). Because \(\hat{R}_i\) is just the operator \(\hat{R}\) applied to factors \(i\) and \(i + 1\) in \(V^{\otimes k}\), and \(\lambda_r \neq 0\), we have that the relation

\[
(\lambda_r^{-1} \hat{R}_i - \text{id})(\lambda_r^{-1} \hat{R}_i + q^{-1} \text{id}) = 0
\]

holds on \(V^{\otimes k}\). Therefore, the map \(s_i \mapsto \lambda_r^{-1} \hat{R}_i\) for \(1 \leq i \leq k - 1\) extends to an algebra homomorphism \(H_k(q) \to \text{End}_D(V^{\otimes k})\), since the \(s_i\) and \(\lambda_r^{-1} \hat{R}_i\) satisfy the same relations (see (QT1)-(QT3) above).

Relation (R3) of the definition of the Iwahori-Hecke algebra is just (4.7).
The standard action of the Iwahori-Hecke algebra $H_k(q)$ on a $k$-fold tensor power $V^\otimes k$ factors through an action of the Temperley-Lieb algebra when $\dim(V) = 2$, see e.g., [17, Remark 3]. Using the action of $H_k(q)$ on $V(2,r)^\otimes k$ given by Proposition 4.2, we obtain the following result.

**Theorem 4.3.** Assume $q$ is a primitive $n$th root of unity for $n \geq 2$. Let $V = V(2,r)$ for any $r \in \mathbb{Z}$, and set $\xi = -(q^{\frac{1}{2}} + q^{-\frac{1}{2}})$. Then for $k \geq 2$,

(a) There is an algebra homomorphism $\pi: TL_k(\xi) \to \text{End}_{\mathbb{D}_n}(V^\otimes k)$ given by $t_i \mapsto q^\frac{i}{2}(\lambda_i^{-1} \tilde{R}_i - \text{id})$ for $1 \leq i \leq k - 1$, where $\lambda_i = q^{-r(r+1)}$, and

(b) $\pi$ is an injection, that is, the action of $TL_k(\xi)$ on $V^\otimes k$ is faithful.

**Proof.** (a) Suppose that $\tilde{R}_i$ is the twist of the $R$-matrix of $D_n$ applied to the tensor factors $i$ and $i + 1$ of $V^\otimes k$ as before. Recall that by Proposition 4.2, the map $s_i \mapsto \lambda_i^{-1} \tilde{R}_i$, $1 \leq i \leq k - 1$, extends to an algebra homomorphism $H_k(q) \to \text{End}_{\mathbb{D}_n}(V^\otimes k)$. Therefore, setting $t_i = q^\frac{i}{2}(s_i - \text{id})$, we know that the relations (R1'), (R2') (R3') for the alternative presentation of $H_k(q)$ hold for $\xi = -(q^{\frac{1}{2}} + q^{-\frac{1}{2}})$. Consequently, there is an algebra homomorphism $H_k(q) \to \text{End}_{\mathbb{D}_n}(V^\otimes k)$ given by $t_i \mapsto q^\frac{i}{2}(\lambda_i^{-1} \tilde{R}_i - \text{id})$ for $1 \leq i \leq k - 1$. What remains to be shown is that $t_i t_{i+1} t_i - t_i$ is in the kernel of this homomorphism for all $1 \leq i \leq k - 1$, and there is an induced algebra homomorphism $\pi: TL_k(\xi) \to \text{End}_{\mathbb{D}_n}(V^\otimes k)$ by relation (R1').

It suffices to check that $t_1 t_2 t_1 = t_1$ on $V^\otimes 3$, as $t_i t_{i+1} t_i - t_i$ acts as the identity map on all factors of $V^\otimes k$, except for the factors in positions $i, i + 1, i + 2$, and the action is the same as for $t_1 t_2 t_1 - t_1$ on $V^\otimes 3$. By the proof of Lemma 4.11 on the basis $\{v_1 \otimes v_1, v_1 \otimes v_2, v_2 \otimes v_1, v_2 \otimes v_2\}$ for $V^\otimes 2$, the transformation $\tilde{R}$ has matrix

$$
\begin{pmatrix}
\lambda_r & 0 & 0 & 0 \\
0 & \lambda_r - \lambda_r q^{-1} & q^{-(r+1)^2} & 0 \\
0 & q^{-r^2} & 0 & 0 \\
0 & 0 & 0 & \lambda_r
\end{pmatrix}
$$

Since $t_1 = q^\frac{1}{2}(\lambda_1^{-1} \tilde{R}_1 - \text{id})$ it follows that $t_1$ is zero on the vectors $v_1 \otimes v_1 \otimes v_1, v_1 \otimes v_1 \otimes v_2, v_2 \otimes v_2 \otimes v_1$, and $v_2 \otimes v_2 \otimes v_2$ so that $t_1 t_2 t_1 = t_1$ holds on the subspace spanned by these four vectors. Using the fact that $t_1 t_2 t_1 - t_1$ is central in $H_3(q)$, it can be shown that $t_1 t_2 t_1 = t_1$ on all of $V^\otimes 3$. For example

$$v_2 \otimes v_1 \otimes v_2 = q^{(r+1)^2} \tilde{R}_2(v_2 \otimes v_2 \otimes v_1) = q^{(r+1)^2} \lambda_r(q^{-\frac{3}{2}} t_2 + \text{id})(v_2 \otimes v_2 \otimes v_1)$$

so that

$$(t_1 t_2 t_1 - t_1)(v_2 \otimes v_1 \otimes v_2) = q^{(r+1)^2} \lambda_r(t_1 t_2 t_1 - t_1)(q^{-\frac{1}{2}} t_2 + \text{id}))(v_2 \otimes v_2 \otimes v_1) = q^{(r+1)^2} \lambda_r(q^{-\frac{1}{2}} t_2 + \text{id}))(t_1 t_2 t_1 - t_1)(v_2 \otimes v_2 \otimes v_1) = 0.$$

Then similarly using

$$v_1 \otimes v_2 \otimes v_2 = q^{(r+1)^2} \tilde{R}_1(v_2 \otimes v_1 \otimes v_2),$$

$$v_1 \otimes v_2 \otimes v_1 = q^{r^2} (\tilde{R}_2 - \lambda_r(1 - q^{-1}) \text{id})(v_1 \otimes v_1 \otimes v_2),$$

$$v_2 \otimes v_1 \otimes v_1 = q^{r^2} (\tilde{R}_1 - \lambda_r(1 - q^{-1}) \text{id})(v_1 \otimes v_2 \otimes v_1),$$

we obtain that $t_1 t_2 t_1 = t_1$ on all of $V^\otimes 3$.

(b) The action that comes from the standard Iwahori-Hecke action and passing to the Temperley-Lieb action is known to be faithful, see e.g., [15, Theorem 2.4] and [28, Main Theorem].
4.2. Action for $V(3, r)$. In this section we give an application of the explicit formula in Theorem 3.6 for the ribbon element when $n$ is odd. We show that it can be used to compute the eigenvalues of $\hat{R}$ to obtain further relations on the action of the braid group on $V(\ell, r)^{\otimes k}$ for arbitrary integers $1 \leq \ell \leq n$, $r \in \mathbb{Z}_n$, and $n \geq 2$. We illustrate these results in the case when $\ell = 3$.

The condition $2\ell \leq n + 1$ implies that $V(\ell, r)^{\otimes 2}$ is completely reducible, see (2.5).

Our first result holds for $n$ odd and arbitrary $\ell$, with $2\ell \leq n + 1$, and uses the action of the ribbon element.

**Proposition 4.4.** Let $n \geq 3$ be odd, $2\ell \leq n + 1$, and $V = V(\ell, r)$, $r \in \mathbb{Z}_n$, be a simple $\mathbb{D}_n$-module of dimension $\ell$. The braid group action where the generators act via $\hat{R}_i$ on $V^{\otimes k}$ satisfies the further relations for $1 \leq i \leq k - 1$:

$$\prod U (\hat{R}_i^2 - c_U \text{id}) = 0,$$

where the product is over all simple $\mathbb{D}_n$-modules $U$ occurring in the tensor product $V^{\otimes 2}$. For $U = V(a, b)$ the scalar $c_U$ is computed as the follows:

$$c_U = q^{2r^2+(2r-1)(\ell-1)-b(a+b-1)-\frac{1}{2}(n-1)(a-1)}.$$

Furthermore, the eigenvalues of $\hat{R}_i$ on $V^{\otimes k}$ come from among the values $\{\pm \sqrt{c_U}\}$, where $U$ ranges over all irreducible summands of $V^{\otimes 2}$.

**Proof.** We use the eigenspace decomposition of $V^{\otimes 2} = V(\ell, r)^{\otimes 2}$ under the action of $\hat{R}$. Let $W \subseteq V^{\otimes 2}$ be an eigenspace corresponding to eigenvalue $\alpha$. It is straightforward to check that $W$ is a $\mathbb{D}_n$-module. Since $V^{\otimes 2}$ is completely reducible, $W$ has a direct sum decomposition $W = \bigoplus_j W_j$ into irreducible $\mathbb{D}_n$-summands $W_j$. For $1 \leq i \leq k - 1$, since $\hat{R}_i$ is just the operator $\hat{R}$ applied to factors $i$ and $i + 1$ in $V^{\otimes k}$, its eigenspace corresponding to eigenvalue $\alpha$ is $V^{\otimes i-1} \otimes (\bigoplus_j W_j) \otimes V^{\otimes k-i-1}$. That is, $\hat{R}$ and $\hat{R}_i$ share the same eigenvalues.

Property (1) of the quasitriangular properties in Section 3.1 is equivalent to $\hat{R} \Delta(x) = \Delta(x) \hat{R}$ for all $x \in \mathbb{D}_n$. In other words, the action of $\hat{R}$ on a simple summand $U_{\omega} \subset U_{\mu} \otimes U_{\nu}$ should be a scalar, because $\hat{R} \in \text{End}_{\mathbb{D}_n}(U_{\omega})$, and the latter space is one-dimensional by Schur’s Lemma. Since $\mathbb{R}^{op}\mathbb{R} = \mathbb{R}^2$, this scalar can be computed using (3.4) and Theorem 3.6 (b). This gives the desired expression for $c_U$. As $\alpha^2 = c_U$ is an eigenvalue of $\hat{R}_i^2$, we see from the above discussion that an eigenvalue of $\hat{R}_i$ is a square root of $c_U$, determined up to a sign.

The remaining computations give a refinement of the relation in Proposition 4.4 by finding eigenvalues for the action of $\hat{R}_i$ rather than their squares. In other words, we determine the sign of the square root of $c_U$.

The following result gives the full list of eigenvalues specifically when $\ell = 3$ and $n$ is any integer, $n \geq 5$, and hence we have an additional relation for the action of the braid group on $V(3, r)^{\otimes k}$.

**Proposition 4.5.** Let $n$ be an arbitrary integer $n \geq 5$. The action of the braid group on $V(3, r)^{\otimes k}$ factors through the further relations:

$$(\hat{R}_i - q^{-r^2-2r+2}\text{id})(\hat{R}_i + q^{-r^2-2r-2}\text{id})(\hat{R}_i - q^{-r^2-2r-3}\text{id}) = 0,$$

for $1 \leq i \leq k - 1$.

**Proof.** Since $\hat{R}_i$ is just the operator $\hat{R}$ acting on two tensor factors of $V(3, r)^{\otimes k}$, it suffices to prove the statement for the action of $\hat{R}$ on $V(3, r)^{\otimes 2}$.

For a $\mathbb{D}_n$-module $V(3, r)$ with basis $v_1, v_2$, and $v_3$,

$$V(3, r) \otimes V(3, r) \simeq V(5, 2r) \oplus V(3, 2r + 1) \oplus V(1, 2r + 2).$$
On the three summands, \( \hat{R} \) acts by \( \pm q^{r-2-r} \), \( \pm q^{r-2-r-2} \), \( \pm q^{r-2-r-3} \) respectively, where each of the eigenvalues of \( \hat{R} \) has either the plus or minus sign. By computing the action of \( \hat{R} \) on \( v_1 \otimes v_1 \) one obtains the first eigenvalue \( q^{r-2-r} \) (a more general computation will be given in the next proposition). We now show the second eigenvalue takes the minus sign. We note that for \( s \geq 3 \), \( a^s \) and \( d^s \) act as 0 on \( V(3, r) \). A straightforward computation shows that

\[
R(v_1 \otimes v_2) = q^{r-2}(v_1 \otimes v_2) + q^{r-2}(1 - q^{-2})v_2 \otimes v_1, \quad R(v_2 \otimes v_1) = q^{r-2}(r+1)v_2 \otimes v_1.
\]

Therefore, on the subspace spanned by \( v_1 \otimes v_2 \) and \( v_2 \otimes v_1 \), \( \hat{R} = \sigma R \) acts via the matrix

\[
\begin{pmatrix}
(1 - q^{-2})q^{(r-2)} & q^{(r-2)(r+1)} \\
q^{(r-1)r} & 0
\end{pmatrix},
\]

and therefore it has an eigenvalue \( -q^{r-2-r-2} \) with eigenvector \( v_1 \otimes v_2 - q^{r+2}v_2 \otimes v_1 \).

For the remaining one-dimensional module, we compute the action of \( \hat{R} \) on the subspace spanned by \( v_1 \otimes v_3 \), \( v_2 \otimes v_2 \), \( v_3 \otimes v_1 \). Here the summation is over all \( 0 \leq t, m \leq n - 1 \) unless otherwise specified. We have

\[
R(v_1 \otimes v_3) = \frac{1}{n} \sum_{m,t} q^{-mt-2t} \frac{\alpha_1(3)\alpha_2(3)}{2n} \sum_{m,t} q^{-mt-2t+(r+2)t+(-r-2)m} v_3 \otimes v_1 + \frac{\alpha_2(3)}{n} \sum_{m,t} q^{-mt-t+(r+1)t+(-r-1)m} v_2 \otimes v_2
\]

and

\[
R(v_3 \otimes v_1) = \frac{\alpha_1(3)\alpha_2(3)}{2n} \sum_{t} (q^{tr} \sum_{m} q^{m(t-r-2)}) v_3 \otimes v_1 + \frac{\alpha_2(3)}{n} \sum_{t} (q^{tr} \sum_{m} q^{m(-r-1-t)}) v_2 \otimes v_2
\]

By similar calculations, one computes the action on \( v_2 \otimes v_2 \) and \( v_3 \otimes v_1 \), and obtains that the matrix of the action of \( \hat{R} - q^{r-2-2r-3} \) id on the subspace spanned by \( v_1 \otimes v_3 \), \( v_2 \otimes v_2 \), \( v_3 \otimes v_1 \) is

\[
\begin{pmatrix}
\frac{1}{2} \alpha_1(3)\alpha_2(3) & q^{r-2-2r-3} - q^{r-2-2r-3} & 0 \\
q^{r(r+1)} \alpha_2(3) & q^{r-2-2r-3} - q^{r-2-2r-3} & 0 \\
q^{-r^2} & q^{-r^2} - q^{-r^2} - q^{-r^2-2r-3} & 0
\end{pmatrix}.
\]

This matrix has rank 2, since \( u = q^{-r-1}v + q^{-2r-1}w = 0 \), where the first, second, and third rows of the matrix (in that order) are \( u, v, \) and \( w \). Therefore \( q^{r-2-2r-3} \) is an eigenvalue. We have recovered all eigenvalues for the action of \( \hat{R} \) on \( V(3, r)^{\otimes 2} \).

**Remark 4.6.** One may argue that on the given summands, Proposition 4.4 is a stronger result than Proposition 4.3, as it gives us all eigenvalues of \( \hat{R}_0 \) explicitly for any \( n \geq 5 \), not just for \( n \) odd. However, in practice, when Proposition 4.3 applies, it gives the eigenvalues up to a sign, and it is much easier to verify that a scalar is an eigenvalue than to find it without prior knowledge.

Next we give a first attempt at computing eigenvalues of the action of \( \hat{R}_0 \) on \( V(\ell, r)^{\otimes k} \) for any integer \( \ell \) with \( 2\ell \leq n + 1 \) by computing two eigenvalues of \( \hat{R} \) on \( V(\ell, r)^{\otimes 2} \).
Remark 5.1. Since the simple and projective direct summands of $V$ on vertices in $\Gamma$ captures the decomposition of tensoring repeatedly with the simple comparing Bratteli diagrams.

Proposition 4.7. Assume that $n \geq 2$ and $2\ell \leq n + 1$. Then $q^{r(1-r-\ell)}$ and $-q^{-r^2+r-r\ell-\ell+1}$ are eigenvalues of $\hat{R}$ on $V(\ell, r)^{\otimes 2}$.

Proof. We show that $v_1 \otimes v_1$ is an eigenvector corresponding to eigenvalue $q^{r(1-r-\ell)}$:

$$\hat{R}(v_1 \otimes v_1) = \frac{1}{n} \sum_{m,t=0}^{n-1} q^{-mt} q^{r^1(1-r-\ell)} v_1 \otimes v_1$$

$$= \frac{1}{n} \sum_{m=0}^{n-1} q^{r^1(1-r-\ell)} \sum_{t=0}^{n-1} q^{r^1(r-m)t} v_1 \otimes v_1$$

$$= q^{r(1-r-\ell)} v_1 \otimes v_1.$$

Similarly, one can compute that on the subspace spanned by $v_1 \otimes v_2$ and $v_2 \otimes v_1$, $\hat{R} = \sigma \hat{R}$ acts via the following matrix

$$\begin{pmatrix}
q^{r^1(1-r-\ell)} (1-q^{r^1-\ell}) & q^{r^1(r+1)(1-r-\ell)} \\
q^{r^2(2-r-\ell)} & 0
\end{pmatrix},$$

which has eigenvalue $-q^{-r^2+r-r\ell-\ell+1}$. \hfill \Box

Conjecture 4.8. For any simple $D_n$-module $V(\ell, r)$ with $n \geq 2$ and $2\ell \leq n + 1$, the braid group action where the generators act via $\hat{R}_i$ on $V(\ell, r)^{\otimes k}$ satisfies the further relations for $1 \leq i \leq k - 1$:

$$\prod_{j=1}^{\ell} (\hat{R}_i - c_j \text{id}) = 0.$$

The product is over all simple $D_n$-modules $V(a_j, b_j)$ occurring in $V(\ell, r)^{\otimes 2}$, with $a_j = 2\ell + 1 - 2j$, $b_j = 2r + j - 1$, and $c_j = (-1)^{j+1} q^{r^2+\frac{r}{2}(2r-1)(\ell-1)-\frac{1}{2}q(a_j+b_j-1)-\frac{1}{2}(n-1)(a_j-1)}$.

5. Bratteli diagrams and centralizer algebras

In this section we focus on the $k$-fold tensor power of any two-dimensional simple $D_n$-module $V = V(2, r)$ for any integer $n \geq 2$. Our main result, Theorem\textsuperscript{5.3}, provides an isomorphism between the centralizer algebra $\text{End}_{D_n}(V^{\otimes k})$ and the Temperley-Lieb algebra $\text{TL}_k(\xi)$ for $1 \leq k \leq 2n - 2$.

5.1. Comparing Bratteli diagrams. We establish a correspondence between the Bratteli diagram $\Gamma$ of partitions of at most two parts as in Section\textsuperscript{5.1} and the Bratteli diagram $\Gamma_n$ which captures the decomposition of tensoring repeatedly with the simple $D_n$-module $V = V(2, 0)$. The vertices in $\Gamma_n$ lie in rows labeled by positive integers, where the vertices in Row $k$ correspond to the simple and projective direct summands of $V(2, 0)^{\otimes k}$. There are $s$ directed edges from $W$ to $W'$, if and only if $W'$ occurs as a summand of $W \otimes V(2, 0)$ with multiplicity $s$.

Remark 5.1. Since $V(2, r)^{\otimes k} \cong V(2, 0)^{\otimes k} \otimes V(1, kr)$ for any $r \in \mathbb{Z}_n$, it suffices to consider the case $r = 0$. The Bratteli diagram for repeated tensoring with $V(2, r)$ can be obtained from one for $V(2, 0)$ below simply by adding $kr$ to the second coordinate of each label in Row $k$ of $\Gamma_n$.

In Figure 1 we display Rows $k = 1, 2, \ldots, 11$ of the Bratteli graph $\Gamma_5$. We use $(\ell, r)$ to signify the simple $D_n$-module $V(\ell, r)$, and $(\ell, r)$ for its projective cover $P(\ell, r)$, where the second component $r$ should be read modulo $n = 5$. The directed edges come from the decomposition formulas in Section\textsuperscript{2} for tensoring with $V = V(2, 0)$. A dotted arrow means the absence of an edge (we include
it as a “fake” edge for visual completeness), and a double arrow from $W$ to $W'$ means that two copies of $W'$ occur in $W \otimes V(2,0)$. The vertical dashed lines will be explained later on. In Figure 2 we display the first 11 rows of the graph $\Gamma$ of partitions with at most two parts. Thus on row $k$ of $\Gamma$, a partition $\beta \vdash k$ is represented by a pair $\beta = \{\beta_1, \beta_2\}$ where $\beta_1 \geq \beta_2$, and $\beta_1 + \beta_2 = k$.

Row

\[
\begin{array}{cccccccc}
1 & & & \gamma = 2 & \gamma = 1 & \gamma = 1 & (2,0) \\
2 & & \gamma = 1 & (3,0) & (1,1) \\
3 & & (4,0) & (2,1) \\
4 & & (5,0) & (3,1) & (1,2) \\
5 & \vdash (4,1) & (2,2) \\
6 & \vdash (3,2) & (5,1) & (3,2) & (1,3) \\
7 & \vdash (2,3) & (4,2) & (4,2) & (2,3) \\
8 & \vdash (1,4) & (3,3) & (5,2) & (3,3) & (1,4) \\
9 & \vdash (2,4) & (4,3) & (4,3) & (2,4) \\
10 & \vdash (4,6) & (3,4) & (5,3) & (3,4) & (1,5) \\
11 & (3,7) & (5,6) & (2,5) & (4,4) & (4,4) & (2,5)
\end{array}
\]

Figure 1: First 11 rows of the Bratteli diagram $\Gamma_5$. 
depends on an $n$ and the centralizing action of a certain Temperley-Lieb algebra
defines a one-to-one correspondence between such partitions of $\gamma$
We will consider several cases that depend on the expression in (5.1).
\begin{equation}
\beta_1 - \beta_2 + 1 = \gamma n + \delta, \quad \text{where } 0 \leq \delta \leq n - 1.
\end{equation}
We will consider several cases that depend on the expression in (5.1).

**Case 1:** $\gamma = 0$ so that $1 \leq \beta_1 - \beta_2 + 1 = \delta \leq n - 1$

In this case,
\[ \{\beta_1, \beta_2\} \leftrightarrow (\beta_1 - \beta_2 + 1, \beta_2) \]
defines a one-to-one correspondence between such partitions of $k$ to the right of the first critical line (the line where $\gamma = 1$) in $\Gamma$, and the simple $D_n$-modules to the right of the $\gamma = 1$ line of the Bratteli diagram $\Gamma_n$.

For example, when $n = 5$ and $\beta = \{5, 2\} \uparrow 7$, then $\beta_1 - \beta_2 + 1 = 4 = 0 \cdot 5 + 4$, and we have $\{5, 2\} \leftrightarrow (4, 2) \leftrightarrow V(4, 2)$.

**Case 2:** $\gamma > 0$ and $\delta = 0$ so that $1 \leq \beta_1 - \beta_2 + 1 = \gamma n$

Here $\beta$ is on the $\gamma$th critical line of $\Gamma$, and $\beta = \{\beta_1, \beta_2\} \leftrightarrow (n, \beta_2)$.
For example, when \( n = 5 \), and \( \beta = \{6,2\} \vdash 8 \), then \( \beta_1 - \beta_2 + 1 = 5 \), so \( \gamma = 1 \) and \( \delta = 0 \), and \( \{6,2\} \leftrightarrow \{5,2\} \leftrightarrow V(5,2) \).

**Case 3:** \( \gamma > 0 \) and \( \delta \neq 0 \)

In this case, \( \delta \) measures how far \( \beta = \{\beta_1, \beta_2\} \) is to the left of the \( \gamma \)th critical line. Then \( \beta = \{\beta_1, \beta_2\} \underleftarrow{\gamma} \{\beta_1 - \delta, \beta_2 + \delta\} \) reflects the partition from the left of the \( \gamma \)th critical line to the right of it.

For example, when \( \beta = \{11\} \) we have \( 11 - 0 + 1 = 12 = 2 \cdot 5 + 2 \), so \( \gamma = 2 \), and \( \delta = 2 \). Then \( \{11\} \underleftarrow{2} \{9,2\} \) is a reflection about the second \( (\gamma = 2) \) critical line. Since \( 9 - 2 + 1 = 8 = 1 \cdot 5 + 3 \), we need to do another reflection, this time corresponding to the critical line \( \gamma = 1 \). So \( \{11\} \underleftarrow{2} \{9,2\} \underleftarrow{1} \{6,5\} \). Now with \( \{6,5\} \), we are to the right of the first critical line, and we know from Case 1, that \( \{6,5\} \leftrightarrow (6 - 5 + 1, 5) = (2,5) \leftrightarrow V(2,5) \).

Tensoring a projective module with a finite-dimensional module always gives a sum of projective indecomposable modules. The region to the left of the \( \gamma = 1 \) critical line in \( \Gamma_n \) has nodes labeled by projective modules. So subsequent tensoring of them with \( V(2,0) \) will yield only projective modules. Since \( \{11\} \) is to the left of the second critical line, it corresponds to a projective module, and since \( \{9,2\} \) is between the first and second critical lines, it also corresponds to a projective module. Which ones?

We have \( \{9,2\} \leftrightarrow \{2,5\} \leftrightarrow P(2,5) \) (recall labels in the second component are in \( \mathbb{Z}_n(= \mathbb{Z}_5) \) here). Now projective modules on opposite sides of critical lines have complementary labels. Thus, \( \{11\} \leftrightarrow (3,5 + 2) = (3,7) \leftrightarrow P(3,7) \).

Here is what the two 11th rows look like under the correspondence described above. The dashed lines indicate the critical lines. Some partitions and module labels fall on the critical lines.

\[
\begin{array}{cccc}
\Gamma : & \{11\} & \{10,1\} & \{9,2\} & \{8,3\} & \{7,4\} & \{6,5\} \\
\Gamma_5 : & (3,7) & (5,6) & (2,5) & (4,4) & (4,4) & (2,5)
\end{array}
\]

This example illustrates several phenomena worth noting. Complementary projective modules in \( \Gamma_n \) are ones with the same composition factors, i.e. \( P(\ell,s) \) and \( P(n-\ell,\ell+s) \). There can exist projective modules that occur in a row of \( \Gamma_n \), for which the corresponding simple module does not occur in the same row. The projective module \( P(3,7) \) is an example of that.

### 5.2. Computing the dimension of the centralizer algebra

Our objective here is to develop a formula for the dimension of the \( k \)th centralizer algebra \( \text{End}_{D_n}(V^{\otimes k}) \) for all \( k \geq 1 \), where \( V \) is any simple two-dimensional \( D_n \)-module \( V(2,r) \). The result is independent of the value of \( r \), and it will be a consequence of the following lemma, which also holds for any algebraically closed field of characteristic zero.

**Lemma 5.2.** For \( 1 \leq \ell, \ell' \leq n \) and \( s, s' \in \mathbb{Z}_n \), the following hold:

\[
\begin{align*}
\text{(5.2)} \quad \dim_k \text{Hom}_{D_n}(V(\ell,s), V(\ell', s')) &= \delta(\ell,s), (\ell',s') \\
\text{(5.3)} \quad \dim_k \text{Hom}_{D_n}(V(\ell,s), P(\ell', s')) &= \delta(\ell,s), (\ell',s') \\
\text{(5.4)} \quad \dim_k \text{Hom}_{D_n}(P(\ell,s), V(\ell', s')) &= \delta(\ell,s), (\ell',s') \\
\text{(5.5)} \quad \dim_k \text{Hom}_{D_n}(P(\ell,s), P(\ell', s')) &= \begin{cases} 2 & \text{if } (\ell',s') = (\ell,s) \text{ or } (n-\ell, s+\ell), \\ 0 & \text{otherwise}. \end{cases}
\end{align*}
\]
Proof. The fact that \( \dim_k \text{Hom}_{D_n}(V(\ell, s), V(\ell, s')) = \delta_{\ell,s},(\ell',s') \) follows from simplicity of the modules and Schur’s lemma. The assertion in (5.3) holds because \( \mathcal{P}(\ell', s') \), \( \ell' \neq n \), has a unique simple submodule, \( V(\ell', s') \), which is its socle. The module \( \mathcal{P}(\ell, s), \ell \neq n \), has a unique maximal submodule, and a unique simple quotient \( V(\ell, s) \), which implies (5.4). Finally, (5.5) is a direct consequence of the fact that \( \mathcal{P}(\ell, s) \) has only the simple modules \( V(\ell, s) \) and \( V(n-\ell, s+\ell) \) as composition factors, each with multiplicity 2, and then Proposition 5.3 below can be applied to derive the result. \( \square \)

**Proposition 5.3.** (See [13, Prop. 9.2.3].) For a finite-dimensional algebra \( H \), let \( S \) be a simple module whose cover is \( \mathcal{P} \). Then for any finite-dimensional \( H \)-module \( N \),

\[
\dim_k \text{Hom}_H(\mathcal{P}, N) = [N : S],
\]

the multiplicity of \( S \) in a Jordan-Hölder series of \( N \).

Suppose now that \( S_1, S_2, \ldots, S_{n^2} \) is a listing of the simple \( D_n \)-modules, and \( P_1, P_2, \ldots, P_{n^2} \) are respectively their projective covers. Let \( I_k \) be the set of all \( i \in \{1, 2, \ldots, n^2\} \) such that \( S_i \) or \( P_i \) or both occur in \( V^{\otimes k} \). For \( i \in I_k \), let \( s_i \) (resp. \( p_i \)) be the multiplicity of \( S_i \) (resp. \( P_i \)) in \( V^{\otimes k} \). Hence, not both \( s_i \) and \( p_i \) are zero when \( i \in I_k \). Because the module \( V(n, s) \) is both simple and projective, when \( V(n, s) \) occurs as a summand of \( V^{\otimes k} \) for some \( s \in \mathbb{Z}_n \), we will assume that \( p_i = 0 \) and \( s_i \) is the multiplicity of \( V(n, s) \) in \( V^{\otimes k} \). We adopt one further convention: When \( P_i = \mathcal{P}(\ell, s) \) for some \( \ell \neq n \), and \( p_i \) is its multiplicity in \( V^{\otimes k} \), then we let \( P'_i \) be \( P(n-\ell, s+\ell) \) and use \( p'_i \) to denote the multiplicity of \( P'_i \) in \( V^{\otimes k} \). Thus, \( P(n-\ell, s+\ell) \) is \( P_j \) for some \( j \), and it is also denoted as \( P_j' \) for \( P_i = \mathcal{P}(\ell, s) \). When \( k = 1 \) in the next result, then \( I_k = \{1\} \), there is a unique summand \( S_1 = V \), and \( s_1 = 1 \) is the only nonzero term in (5.6).

**Theorem 5.4.** For any two-dimensional simple \( D_n \)-module \( V \), the dimension of the centralizer algebra \( \text{End}_{D_n}(V^{\otimes k}) \) for \( k \geq 1 \) is given by

\[
\dim_k \text{End}_{D_n}(V^{\otimes k}) = \sum_{i \in I_k} p_i^2 + \sum_{i \in I_k} (s_i + p_i)^2 + 2 \sum_{i \in I_k} p_i p'_i.
\]

**Proof.** It follows from the decomposition of \( V^{\otimes k} \) into simple and indecomposable projective summands that

\[
\text{End}_{D_n}(V^{\otimes k}) \cong \bigoplus_{i \in I_k} \text{Hom}_{D_n}(S_i^{\oplus s_i}, S_i^{\oplus s_i}) \bigoplus_{i \in I_k} \text{Hom}_{D_n}(S_i^{\oplus p_i}, \mathcal{P}_i^{\oplus p_i}) \\
\bigoplus_{i \in I_k} \text{Hom}_{D_n}(\mathcal{P}_i^{\oplus p_i}, S_i^{\oplus s_i}) \bigoplus_{i \in I_k} \text{Hom}_{D_n}(\mathcal{P}_i^{\oplus p_i}, \mathcal{P}_i^{\oplus p_i}) \\
\bigoplus_{i \in I_k} \text{Hom}_{D_n}(\mathcal{P}_i^{\oplus p_i}, \mathcal{P}_i^{\oplus p_i}'),
\]

where if \( P_i = \mathcal{P}(\ell, s) \), then \( P'_i = \mathcal{P}(n-\ell, s+\ell) \). Counting dimensions, we have

\[
\dim_k \text{End}_{D_n}(V^{\otimes k}) = \sum_{i \in I_k} (s_i^2 + s_i p_i + p_i s_i + 2 p_i^2 + 2 p_i p'_i) \\
= \sum_{i \in I_k} p_i^2 + \sum_{i \in I_k} (s_i + p_i)^2 + 2 \sum_{i \in I_k} p_i p'_i.
\]

**Example 5.5.** Pictured below are Rows 1-11 of the Bratteli diagram \( \Gamma_5 \), without the directed edges, but with the multiplicities displayed as subscripts. The multiplicities are the number of
paths from (2,0) at the top of the diagram to a particular vertex. Recall that \((\ell,s)\) is shorthand for \(V(\ell,s)\) and \((\ell,s')\) is shorthand for \(P(\ell,s)\), where the second component should be interpreted modulo \(n=5\) in this example.

In Table 1 we compare the Catalan number \(C_k\) with \(\dim_k\text{End}_{D_n}(V^{\otimes k})\) for \(n=5\).

| \(k\) | \(C_k\) | \(\sum_{i \in I_k} p_i^2\) | \(\sum_{i \in I_k} (s_i + p_i)^2\) | \(2 \sum_{i \in I_k} p_i p'_i\) | \(\dim_k\text{End}_{D_n}(V^{\otimes k})\) |
|---|---|---|---|---|---|
| 1 | 1 | 0 | 1\(^2\) | 0 | 1 |
| 2 | 2 | 0 | 1\(^2\) + 1\(^2\) | 0 | 2 |
| 3 | 5 | 0 | 1\(^2\) + 2\(^2\) | 0 | 5 |
| 4 | 14 | 0 | 1\(^2\) + 3\(^2\) + 2\(^2\) | 0 | 14 |
| 5 | 42 | 1\(^2\) | 4\(^2\) + 5\(^2\) | 0 | 42 |
| 6 | 132 | 1\(^2\) | 9\(^2\) + 5\(^2\) + 5\(^2\) | 0 | 132 |
| 7 | 429 | 1\(^2\) + 6\(^2\) | (14\(^2\)) + (14\(^2\)) | 0 | 429 |
| 8 | 1430 | 1\(^2\) + 7\(^2\) | (20\(^2\)) + (28\(^2\)) + (14\(^2\)) | 0 | 1430 |
| 9 | 4862 | 8\(^2\) + (27\(^2\)) | 2\(^2\) + (42\(^2\)) + (48\(^2\)) | 0 | 4865 |
| 10 | 16796 | 2\(^2\) + 8\(^2\) + (35\(^2\)) | (75\(^2\)) + (90\(^2\)) + (42\(^2\)) | 2 \cdot 2 \cdot 8 + 2 \cdot 8 \cdot 2 | 16846 |
| 11 | 58786 | 2\(^2\) + (43\(^2\)) + (110\(^2\)) | (20\(^2\)) + (165\(^2\)) + (132\(^2\)) | 2 \cdot 2 \cdot 43 + 2 \cdot 43 \cdot 2 | 59346 |

**Table 1. Catalan number \(C_k\) and \(\dim_k\text{End}_{D_n}(V^{\otimes k})\) for \(n=5\) and \(1 \leq k \leq 11\).**

**Remark 5.6.** The last summand \(2 \sum_{i \in I_k} p_i p'_i\) does not occur for \(1 \leq k \leq 2n-1\). Row \(k = 2n-2\) is the first time a module of the form \(P(1,s)\) appears in \(\Gamma_n\) for some \(s \in \mathbb{Z}_n\), and when tensored with \(V\), it produces 2 copies of \(V(n,s+1)\) on Row \(2n-1\). Going down the left-hand diagonal of the Bratteli diagram \(\Gamma_n\), all multiplicities have been equal to 1 prior to this shift at Row \(2n-1\) to multiplicity 2. The difference \(2^2 - 1^2 = 3\) accounts for the fact that \(4865 = 4862 + 3\) when \(n = 5\) and \(k = 9\).

Some further observations are recorded in the next lemma.

**Lemma 5.7.** For the top \(2n-2\) rows of \(\Gamma\) and \(\Gamma_n\), the following are true:

(a) The first \(n-1\) rows of \(\Gamma\) and \(\Gamma_n\) are isomorphic as graphs, where vertices are identified in the order they appear. More precisely, vertex \(\beta = \{\beta_1, \beta_2\} \downarrow k\) in \(\Gamma\) corresponds to \((\ell,r)\) in Row \(k\) of \(\Gamma_n\) under the correspondence \(\ell = \beta_1 - \beta_2 + 1\) and \(r = \beta_2\).
(b) The first $2n - 2$ rows of $\Gamma$ and $\Gamma_n$ differ exactly at the diamonds lying on the first critical line. These diamonds have as their top vertices $\{n - 1\}$ in $\Gamma$ and $(n, 0)$ in $\Gamma_n$ and are referred to as irregular diamonds of $\Gamma_n$. An irregular diamond in $\Gamma_n$ has no edges in the northeast corner and double edges on the southwest corner.

(c) In the first $2n - 2$ rows, if both $(\ell, r)$ and $(\ell, r')$ occur in Row $k$, then they appear as reflections across the first critical line. Moreover, in such a row, if $(\ell, r)$ occurs, then $(\ell, r')$ must occur.

5.3. **Comparing multiplicities of summands in $\Gamma$ and $\Gamma_n$.** Since our arguments in this section will involve using multiplicities in various rows of the Bratteli graphs $\Gamma$ and $\Gamma_n$, we will adopt the following notational conventions:

Recall that the first (rightmost) critical line in $\Gamma_n$ (the one corresponding to $\gamma = 1$) separates the vertices corresponding to simple summands and the vertices corresponding to projective summands. In Row $k$, starting from the first critical line and moving to the left, the path counts for the vertices corresponding to the projective modules will be recorded here as $p_1^{(k)}, p_2^{(k)}, p_3^{(k)}, \ldots$, from right to left (see the picture below). From the first critical line moving to the right, the path counts will be recorded as $s_1^{(k)}, s_2^{(k)}, s_3^{(k)}, \ldots$, from left to right. If the first critical line cuts through a vertex in Row $k$, the path count for that vertex is recorded as $s_0^{(k)}$, and we set $p_0^{(k)} = 0$. Otherwise both $s_0^{(k)}$ and $p_0^{(k)}$ are assumed to be 0. When the same simple module occurs in a later row, say in row $\ell > k$ for some $\ell$, its path count will have the label $s_j^{(\ell)}$ (with the same subscript). Analogously, $p_j^{(k)}$ records the multiplicity of its projective cover, and $s_0^{(k)}$ records an occurrence of a module $V(n, s)$, which is both simple and projective among the summands. Furthermore, when a summand does not occur in a certain row, we assume the corresponding value $p_j^{(k)}$ or $s_j^{(k)}$ is 0. In particular, since there are no projective summands for $V^{\otimes k}$ when $1 \leq k \leq n - 1$, it follows that $p_j^{(k)} = 0$ for all $j$ and all $1 \leq k \leq n - 1$.

We set up the notation for path counts in $\Gamma$ in a similar fashion denoting ones on Row $k$ to the right of the first critical line by $\tilde{s}_i^{(k)}$ from left to right, and ones to the left of the first critical line starting with $\tilde{p}_1^{(k)}, \tilde{p}_2^{(k)}$, and proceeding right to left.

Recall that in Part (b) of Lemma 5.7 in Rows 1 through $2n - 2$, the vertices of $\Gamma$ and $\Gamma_n$ can be identified, so that the edges only differ at the irregular diamonds. This labeling of path counts depends on $n$, even though the underlying graph $\Gamma$ remains the same.

The following is key in our dimension argument.

**Proposition 5.8.** For $1 \leq k \leq 2n - 2$,

(a) $p_i^{(k)} = \tilde{p}_i^{(k)}$,

(b) $\tilde{p}_i^{(k)} + s_i^{(k)} = \tilde{s}_i^{(k)}$,

where the index $i$ runs over all subscripts that occur in Row $k$.

**Proof.** The first $n - 1$ rows of $\Gamma$ and $\Gamma_n$ are isomorphic as graphs. Therefore, the assertions in Proposition 5.8 are straightforward in light of the fact that $p_i^{(k)} = 0$ for $1 \leq k \leq n - 1$, and direct summands that are projective, but not simple, occur only after Row $n - 1$.
For \( n \leq k \leq 2n - 2 \), we proceed by induction on \( k \), where the base case \( k = n \) is as follows:

\[
\begin{align*}
\text{Row } n - 1 & \\
& \quad s_0^{(n-1)} \quad s_1^{(n-1)} \quad s_2^{(n-1)} \quad \ldots \\
\text{Row } n & \\
& \quad p_1^{(n)} \quad s_1^{(n)} \quad s_2^{(n)} \quad s_3^{(n)}
\end{align*}
\]

By applying Lemma 5.7 to both \( \Gamma \) and \( \Gamma_n \), we have \( p_1^{(n)} = s_0^{(n-1)} = s_0^{(n-1)} = p_1^{(n)} \), where the middle equality holds by the statement for \( k = n - 1 \), discussed in the semisimple case (i.e., when \( 1 \leq k \leq n - 1 \)).

By a similar argument, \( p_1^{(n)} + s_1^{(n)} = s_0^{(n-1)} + s_1^{(n-1)} = s_0^{(n-1)} + s_1^{(n-1)} = s_1^{(n)} \), where the last equality is applying Lemma 5.7 to \( \Gamma \) at \( s_1^{(n)} \) (notice the dashed arrow above is replaced by a solid arrow in \( \Gamma \)). To see that (b) is true for \( i > 1 \), observe that \( p_i^{(n)} = 0 \) for \( i > 1 \), and \( s_i^{(n)} \) has two solid incoming arrows. Therefore, \( s_i^{(n)} + p_i^{(n)} = s_i^{(n)} = s_{i-1}^{(n)} + s_i^{(n-1)} = s_{i-1}^{(n)} + s_1^{(n-1)} = s_1^{(n)} \).

We now proceed with the main induction step. Suppose statements (a) and (b) are true for \( k = j - 1 \). We claim the statements are true for \( k = j \) by a discussion of two cases.

**Case 1:** Row \( j \) has no vertices lying on the first critical line (i.e. no \( s_0^{(j)} \) occurs).

\[
\begin{align*}
\text{Row } j - 1 & \\
& \quad \ldots \quad p_i^{(j-1)} \quad s_i^{(j-1)} \quad s_{i+1}^{(j-1)} \quad \ldots \\
\text{Row } j & \\
& \quad p_2^{(j)} \quad s_2^{(j)} \quad p_3^{(j)} \quad s_3^{(j)}
\end{align*}
\]

By the induction hypothesis, \( p_0^{(j-1)} + s_0^{(j-1)} = s_0^{(j-1)} \), but \( p_0^{(j-1)} = 0 \) by definition, therefore \( s_0^{(j-1)} = s_0^{(j-1)} \). Also \( p_1^{(j-1)} = p_1^{(j-1)} \) by induction hypothesis. It follows that \( p_1^{(j)} = p_1^{(j-1)} + s_0^{(j-1)} = p_1^{(j-1)} + s_0^{(j)} = s_1^{(j)} \). For \( i > 1 \), it is straightforward that \( p_i^{(j)} = p_i^{(j-1)} + p_{i-1}^{(j-1)} = p_i^{(j-1)} + p_{i-1}^{(j-1)} = s_i^{(j)} \), because arrows to the left of the first critical line are all solid, between Rows \( j - 1 \) and \( j \). Now we turn to the path counts to the right of the critical line. Notice

\[
p_1^{(j)} + s_1^{(j)} = p_1^{(j-1)} + s_0^{(j-1)} + s_1^{(j-1)} = s_0^{(j)} + s_1^{(j)} = s_1^{(j)}
\]

where the equality marked by * is given by the induction hypothesis for (b) when \( k = j - 1 \) and \( i = 1 \). Also \( s_0^{(j-1)} = s_0^{(j-1)} \) for the next equality, as proven earlier. The argument is easier for \( i > 1 \) because the edges that are identified in \( \Gamma_n \) and \( \Gamma \) are away from the axis:

\[
p_i^{(j)} + s_i^{(j)} = p_i^{(j-1)} + p_{i-1}^{(j-1)} + s_i^{(j-1)} + s_{i-1}^{(j-1)} = s_i^{(j-1)} + s_{i-1}^{(j-1)} = s_i^{(j)}.
\]

**Case 2:** Row \( j \) has a vertex lying on the first critical line (i.e. \( s_0^{(j)} \) occurs):

\[
\begin{align*}
\text{Row } k - 1 & \\
& \quad \ldots \quad p_i^{(k-1)} \quad p_k^{(k-1)} \quad p_{i+1}^{(k-1)} \quad \ldots \\
\text{Row } k & \\
& \quad p_2^{(k)} \quad p_3^{(k)} \quad p_4^{(k)} \quad s_1^{(k)} \quad s_2^{(k)}
\end{align*}
\]

Here, (a) is true for all \( i \) by an argument similar to that in Case 1, due to the fact all \( p_i^{(k)} \) have two incoming arrows, which are identified with those edges in \( \Gamma \). To see (b) is true for \( i = 0 \), observe that

\[
p_0^{(k)} + s_0^{(k)} = 2p_1^{k-1} + s_1^{k-1} = p_1^{k-1} + s_1^{k-1} = p_1^{k-1} + s_1^{k-1} = s_0^{k}.
\]
The remaining multiplicities for (b) when $i > 0$ are true by an argument similar to that in Case 1. This concludes the proof of Proposition 5.8.

In Theorem 5.9 we will use the path count comparisons in Proposition 5.8 together with the result in Theorem 5.4 to show that the Temperley-Lieb algebra $\text{TL}_k(\xi)$ is indeed isomorphic to the centralizer algebra $\text{End}_{D_n}(V(2,r)^{\otimes k})$ for any $r \in \mathbb{Z}_n$ and $1 \leq k \leq 2n - 2$.

**Theorem 5.9.** Assume $n \geq 2$, and $V = V(2,r)$ for any $r \in \mathbb{Z}_n$. Then the algebra homomorphism $\pi : \text{TL}_k(\xi) \to \text{End}_{D_n}(V^{\otimes k})$ in Theorem 4.3 is an isomorphism for $1 \leq k \leq 2n - 2$.

**Proof.** It follows immediately from the dimension count (see Proposition 5.8) that

$$\dim C \text{TL}_k(\xi) = \sum_{i \in I_k} s_i^2 + \sum_{i \in I_k} p_i^2 = \sum_{i \in I_k} (p_i + s_i)^2 + \sum_{i \in I_k} p_i^2 = \dim C \text{End}_{D_n}(V^{\otimes k}),$$

where the last equality is a consequence of Theorem 5.4 and the fact that $p_ip_i' = 0$ for $i \in I_k$ when $1 \leq k \leq 2n - 2$. Since the map $\pi$ is injective for all $k$, it follows by the above dimension count that it is also surjective, hence an isomorphism, when $1 \leq k \leq 2n - 2$. □

### 5.4. Further questions.

In Theorem 4.3 of this paper, we have shown that for an arbitrary two-dimensional simple $D_n$-module $V = V(2,r)$, there exists an injective algebra homomorphism $\pi : \text{TL}_k(\xi) \to \text{End}_{D_n}(V^{\otimes k})$ for $\xi = -(q^\frac{1}{2} + q^{-\frac{1}{2}})$, and when $1 \leq k \leq 2n - 2$, $\pi$ is an isomorphism according to Theorem 5.9.

1. What generates the full centralizer algebra $\text{End}_{D_n}(V(2,r)^{\otimes k})$ when $k \geq 2n - 1$?
2. For arbitrary $r \in \mathbb{Z}_n$, can $\text{End}_{D_n}(V(2,r)^{\otimes k})$ be realized as a diagram algebra?
3. Is the category of $D_n$-modules a highest weight category, in the sense of [11]? If so, can we give an explicit description of the standard, costandard and tilting modules as in [35]?
4. If the above is true, do the Jones-Wenzl projectors (see e.g., [19]) provide projections onto each standard summand of $V(\ell, r)^{\otimes k}$?
5. Do the $p$-Jones-Wenzl projectors of [2] provide projections onto the tilting summands of $V(\ell, r)^{\otimes k}$?
6. For other simple $D_n$-modules $V(\ell, r)$ with $\ell \geq 3$, what is the decomposition of $V(\ell, r)^{\otimes k}$ into simple and projective summands, and what is the centralizer algebra $\text{End}_{D_n}(V(\ell, r)^{\otimes k})$?

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**References**

[1] G. Benkart, R. Biswal, E. Kirkman, V.C. Nguyen, J. Zhu, McKay matrices for finite-dimensional Hopf algebras. *Canad. J. Math.* published online by Cambridge University Press: 08 February 2021, pp. 1–46, arXiv:2007.05510.

[2] G. Burrull, N. Libedinsky, P. Sentinelli, $p$-Jones-Wenzl idempotents. *Adv. Math.* 352 (2019), 246–264.
[3] J.S. Carter, D.E. Flath, M. Saito, The classical and quantum 6j-symbols. Mathematical Notes 43. *Princeton University Press, Princeton, NJ*, 1995.

[4] S. Cautis, J. Kamnitzer, S. Morrison, Webs and quantum skew Howe duality. *Math. Ann. 360* (2014), no. 1–2, 351–390.

[5] V. Chari and A. Pressley, A guide to quantum groups. *Cambridge University Press*, Cambridge, 1994.

[6] H.-X. Chen, A class of noncommutative and noncocommutative Hopf algebras: the quantum version. *Comm. Algebra 27* (1999), no. 10, 5011–5032.

[7] H.-X. Chen, Irreducible representations of a class of quantum doubles. *J. Algebra 225* (2000), 391–409.

[8] H.-X. Chen, Finite-dimensional representations of a quantum double. *J. Algebra 251* (2002), 751–789.

[9] H.-X. Chen, Representations of a class of Drinfeld doubles. *Comm. Algebra 33* (2005), 2809–2825.

[10] H.-X. Chen, H.S.E. Mohammed, H. Sun, Indecomposable decomposition of tensor products of modules over Drinfeld doubles of Taft algebras. *J. Pure Appl. Algebra 221* (2017), no. 11, 2752–2790.

[11] E. Cline, B. Parshall, L. Scott, Finite-dimensional algebras and highest weight categories. *J. Reine Angew. Math. 391* (1988), 85–99.

[12] V. Drinfeld, Quantum groups. *Proceedings of the International Congress of Mathematicians, Vol. 1–2 (Berkeley, Calif., 1986)*, 798–820. *Amer. Math. Soc., Providence, RI*, 1987.

[13] P. Etingof, O. Golberg, S. Hensel, T. Liu, A. Schwendner, D. Vaintrob, E. Yudovina, Introduction to representation theory. *Student Mathematical Library, 59. American Mathematical Society, Providence, RI*, 2011.

[14] I.B. Frenkel and M.G. Khovanov, Canonical bases in tensor products and graphical calculus for $U_q(sl_2)$. *Duke Math. J. 87* (1997), no. 3, 409–480.

[15] F.M. Goodman, H. Wenzl, The Temperley-Lieb algebra at roots of unity. *Pacific J. Math. 161* (1993), 307–334.

[16] M.A. Hennings, Invariants of links and 3-manifolds obtained from Hopf algebras. *J. London Math. Soc. 54* (1996), 594–624.

[17] M. Jimbo, A $q$-analogue of $U_q(gl(N + 1))$, Hecke algebra, and the Yang-Baxter equation. *Lett. Math. Phys. 11* (1986), no. 3, 247–252.

[18] V. Jones and V.S. Sunder, Introduction to subfactors. *London Mathematical Society Lecture Note Series, 234*. Cambridge University Press, Cambridge, 1997.

[19] L. Kauffman, Knots and physics. Fourth edition. Series on Knots and Everything, 53. *World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ*, 2013. xviii+846 pp.

[20] L.H. Kauffman and S.L. Lins, Temperley-Lieb recoupling theory and invariants of 3-manifolds. Annals of Mathematics Studies 134. *Princeton University Press, Princeton, NJ*, 1994.

[21] L. Kauffman and D. Radford, A necessary and sufficient condition for a finite-dimensional Drinfeld double to be a ribbon Hopf algebra. *J. Algebra 159* (1993), no. 1, 98–114.

[22] A.N. Kirillov and N.Yu. Reshetikhin, Representations of the algebra $U_q(sl(2))$, $q$-orthogonal polynomials and invariants of links. Infinite-dimensional Lie algebras and groups (Luminy-Marseille, 1988). 285–339, Adv. Ser. Math. Phys., 7, *World Sci. Publ., Teaneck, NJ*, 1989.

[23] R. Leduc and A. Ram, A ribbon Hopf algebra approach to the irreducible representations of centralizer algebras: the Brauer, Birman-Wenzl, and type A Iwahori-Hecke algebras. *Adv. Math. 125* (1997), no. 1, 1–94.

[24] W.B.R. Lickorish, Three-manifolds and the Temperley-Lieb algebra. *Math. Ann. 290* (1991), no. 4, 657–670.

[25] W.B.R. Lickorish, Calculations with the Temperley-Lieb algebra. *Comment. Math. Helv. 67* (1992), no. 4, 571–591.

[26] W.B.R. Lickorish, Quantum invariants of 3-manifolds, *Handbook of Geometric Topology*, 707–734, North-Holland, Amsterdam, 2002.

[27] M. Lorenz, A tour of representation theory. *Graduate Studies in Mathematics 193 American Math. Soc., Providence, RI*, 2018.

[28] P.P. Martin, On Schur-Weyl duality, $A_n$ Hecke algebras and quantum $sl(N)$ on $\otimes^{n+1}C^N$. *Int. J. Mod. Phys. A 7* Suppl.1B (1992) 645–673.

[29] S. Montgomery, Hopf algebras and their actions on rings. CBMS Regional Conference Series in Mathematics, 82. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the *American Math. Soc., Providence, RI*, 1993.

[30] S. Morrison, A formula for the Jones-Wenzl projections. *Proceedings of the 2014 Maui and 2015 Qinhuangdao conferences in honour of Vaughan F. R. Jones’ 60th birthday*, 367–378, Proc. Centre Math. Appl. Austral. Nat. Univ., 46, Austral. Nat. Univ., Canberra, 2017.
[31] D. Radford, Minimal quasitriangular Hopf algebras. *J. Algebra* **157** (1993), no. 2, 285–315.

[32] D. Radford and S. Westreich, Trace-like functionals on the double of the Taft Hopf algebra. *J. Algebra* **301** (2006), no. 1, 1–34.

[33] N.Yu. Reshetikhin and V.G. Turaev, Ribbon graphs and their invariants derived from quantum groups. *Comm. Math. Phys.* **127** (1990) 1–26.

[34] N.Yu. Reshetikhin and V.G. Turaev, Invariants of 3-manifolds via link polynomials and quantum groups. *Invent. Math.* **103** (1991) 547–597.

[35] S. Riche and G. Williamson, Tilting modules and the $p$-canonical basis. *Astérisque* **397** (2018).

[36] H. Sun, H.S.E. Mohammed, W. Lin, H.-X. Chen, Green rings of Drinfeld doubles of Taft algebras. *Comm. Algebra* **48** (2020), no. 9, 3933–3947.

[37] H.N.V. Temperley and E.H. Lieb, Relations between the “percolation” and “colouring” problem and other graph-theoretical problems associated with regular planar lattices: some exact results for the ”percolation” problem. *Proc. Roy. Soc. London Ser. A* **322** (1971), no. 1549, 251–280.

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