LATTICE POINTS PROBLEM, EQUIDISTRIBUTION AND ERGODIC THEOREMS FOR CERTAIN ARITHMETIC SPHERES

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ABSTRACT. We establish an asymptotic formula for the number of lattice points in the sets

\[ S_{h_1,h_2,h_3}(\lambda) := \{ x \in \mathbb{Z}^3 : [h_1(x_1)] + [h_2(x_2)] + [h_3(x_3)] = \lambda \} \quad \text{with} \quad \lambda \in \mathbb{Z}_+; \]

where functions \( h_1, h_2, h_3 \) are constant multiples of regularly varying functions of the form

\[ h(x) := x^c \ell_h(x), \]

where the exponent \( c > 1 \) (but close to 1) and a function \( \ell_h(x) \) is taken

from a certain wide class of slowly varying functions. Taking \( h_1(x) = h_2(x) = h_3(x) = x^c \)

we will also derive an asymptotic formula for the number of lattice points in the sets

\[ S^3_h(\lambda) := \{ x \in \mathbb{Z}^3 : |x_1|^c + |x_2|^c + |x_3|^c = \lambda \} \quad \text{with} \quad \lambda \in \mathbb{Z}_+; \]

which can be thought of as a perturbation of the classical Waring problem in three variables.

We will use the latter asymptotic formula to study, the main results of this paper, norm and pointwise convergence of the ergodic averages

\[ \frac{1}{\#S^3_h(\lambda)} \sum_{n \in S^3_h(\lambda)} f(T_1^n T_2^n T_3^n x) \quad \text{as} \quad \lambda \to \infty; \]

where \( T_1, T_2, T_3 : X \to X \) are commuting invertible and measure-preserving transformations of a \( \sigma \)-finite measure space \((X, \nu)\) for any function \( f \in L^p(X) \) with \( p > \frac{11}{11-\frac{1}{c}} \). Finally, we will study the equidistribution problem corresponding to the spheres \( S^3_h(\lambda) \).

1. INTRODUCTION

Let \( \lambda \in \mathbb{Z}_+ \) be a positive integer and define the set \( S^3_h(\lambda) \) of all lattice points on a two-dimensional sphere of radius \( \lambda^{1/2} \) by

\[ S^3_h(\lambda) := \{ x \in \mathbb{Z}^3 : x_1^2 + x_2^2 + x_3^2 = \lambda \}. \]

The study of the behavior of \( S^3_h(\lambda) \) as \( \lambda \to \infty \) is a central problem in number theory, which has gone through a period of considerable change and development in the past three decades. One of many interesting features of this problem is that the sets \( S^3_h(\lambda) \) might be empty for some choices of \( \lambda \in \mathbb{Z}_+ \). A celebrated result of Legendre, whose complete proof was given by Gauss [31], states that \( S^3_h(\lambda) \neq \emptyset \) if and only if \( \lambda \neq 4^m(8n + 7) \) for \( m, n \in \mathbb{N} \) (see also to [33]). Hence, for \( \lambda \neq 4^m(8n + 7) \) it makes sense to study the counting function

\[ r_2(\lambda) := \#S^3_h(\lambda). \]

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The behavior of \( r_2(\lambda) \) as \( \lambda \to \infty \) is very delicate. On the one hand, \( r_2(4^m) = 6 \) for \( m \in \mathbb{Z}_+ \). It is well known, (see e.g. the paper of Bateman \[10\]), that

\[
r_2(\lambda) = \frac{\pi^{3/2}}{\Gamma(3/2)} \lambda^{1/2} \mathcal{G}_3(\lambda) = \frac{2^3 \Gamma(1 + 1/2)^3}{\Gamma(3/2)} \lambda^{3/2 - 1} \mathcal{G}_3(\lambda),
\]

(1.1)

where \( \Gamma \) is the standard Gamma function and the factor \( \mathcal{G}_3(\lambda) \) is called the singular series, see \[10\] for more details. Formula (1.1) can be also obtained using the spectral theory of automorphic forms \[28, 46\]. We also refer to \[47\] Theorem 20.15, p. 478] for a more extensive treatment of the formula for \( r_2(\lambda) \) and its relations with the Kloosterman circle method. The formula (1.1) is a genuine asymptotic only if the singular series \( \mathcal{G}_3(\lambda) \) does not vanish. However, this is a very subtle question. The upper bound \( r_2(\lambda) \lesssim \lambda^{2+o(1)} \) can be obtained by analyzing the singular series \( \mathcal{G}_3(\lambda) \) as \( \lambda \to \infty \). Moreover, if \( \lambda \neq 0, 4, 7 \mod 8 \), then there is also a lower bound \( r_2(\lambda) \gtrsim \lambda^{2-o(1)} \), which also follows from (1.1) and Siegel’s bounds \[67\], which ensure that \( |\mathcal{G}_3(\lambda)| \gtrsim \lambda^{-o(1)} \); see also \[47\] Remark below formula (20.130), p. 479], where more details are given.

Having many lattice points in \( S^2_2(\lambda) \) as \( \lambda \to \infty \) and \( \lambda \neq 0, 4, 7 \mod 8 \), it is natural to understand distribution of their projections

\[
P^2_3(\lambda) := \{ \lambda^{-1/2}x : x \in S^2_2(\lambda) \} \subset S^2
\]

on the unit sphere \( S^2 \subset \mathbb{R}^3 \). This line of investigations were initiated by Linnik \[54\], who proved under the Generalized Riemann Hypothesis, that the projected lattice points \( P^2_3(\lambda) \) become equidistributed on the unit sphere \( S^2 \) as \( \lambda \to \infty \) and \( \lambda \neq 0, 4, 7 \mod 8 \). More precisely, if \( \Omega \subset S^2 \) is a “nice” set then

\[
\frac{\#(P^2_3(\lambda) \cap \Omega)}{r_2(\lambda)} \sim \nu_2(\Omega)
\]

as \( \lambda \to \infty \) and \( \lambda \neq 0, 4, 7 \mod 8 \), where \( \nu_2 \) is a normalized area measure on \( S^2 \). Linnik’s result was proved unconditionally (without (GRH)) by Duke \[28\] and Golubeva and Fomenko \[32\] following a breakthrough paper by Iwaniec \[46\]. Linnik’s ergodic method and related topics were recently carefully discussed by Ellenberg, Michel and Venkantesh in \[29\]. We also refer to the recent paper of Bourgain, Rudnick and Sarnak \[19\], where the spatial distribution of point sets on the sphere \( S^2 \) obtained from the representation of a large integer as a sum of three squares were investigated. The authors gave a strong evidence to the thesis that the solutions behave randomly, which stands in sharp contrast to what happens with sums of two or four or more squares.

In this paper we consider perturbations of the discrete spheres \( S^2_3(\lambda) \) and we will be mainly concerned with three-dimensional variants of the following sets

\[
S^d_c(\lambda) := \{ x \in \mathbb{Z}^d : ||x_1||^c + \ldots + ||x_d||^c = \lambda \}
\]

(1.2)

for any \( \lambda \in \mathbb{Z}_+ \), where \( d \in \mathbb{Z}_+ \) and \( c > 1 \). The sets \( S^d_c(\lambda) \) will be called the arithmetic spheres or arithmetic \( c \)-spheres. We see that \( S^d_c(\lambda) \) with \( c = 2 \) coincides with the discrete \( d \)-dimensional Euclidean spheres \( S^2_3(\lambda) \). We will be mainly focused on the case \( d = 3 \), and our main aim is to show that the sets \( S^3_c(\lambda) \) can be used as much simpler models than \( S^2_3(\lambda) \) or even toy models (especially for \( c > 1 \), which is close to 1) to study fundamental problems in number theory and ergodic theory as discussed above. The arithmetic spheres also provide a good source of examples exhibiting some seemingly counterintuitive phenomena. These aspects will be discussed in detail later in the paper.
We now briefly highlight the main results of this article.

1. **Lattice point problems in \(Z^3\).** Using a variant of the circle method we establish a precise asymptotic formula (in the spirit of the classical Waring problem) for the number of lattice points in \(S^3_c(\lambda)\) for \(c \in (1,9/8)\), i.e.,

\[
 r_c(\lambda) = \#S^3_c(\lambda) \sim \frac{2^{3}\Gamma(1+1/c)^3}{\Gamma(3/c)} \lambda^{3/c-1} \quad \text{as} \quad \lambda \to \infty,
\]

see Corollary 1.3 and compare with the formula for \(r_2(\lambda)\) from (1.1). This contrasts sharply with the situation for \(S^2_2(\lambda)\), where the asymptotic formula for the number of lattice points is still unknown. The argument that we present will also cover a more general situation; generalized arithmetic spheres (1.9) which are induced by certain regularly varying functions, see Definition 1.1 and Theorem 1.2, which is the main result of this subsection.

2. **Ergodic theorems and corresponding maximal estimates in \(Z^3\) and \(R^3\).** The asymptotic formula for the number of lattice points in \(S^3_c(\lambda)\) allows us to study the main results of this paper; norm and pointwise convergence for ergodic averages (1.15) over the arithmetic spheres \(S^3_c(\lambda)\). Let \((X,\nu,T)\) be a measure-preserving system equipped with a family \(T = (T_1, T_2, T_3)\) of commuting invertible and measure-preserving transformations \(T_1, T_2, T_3 : X \to X\). In Theorem 1.5 we show that for every \(p > \frac{11-4c}{11-7c}\) and every \(f \in L^p(X)\) the ergodic averages

\[
 \frac{1}{\#S^3_c(\lambda)} \sum_{n \in S^3_c(\lambda)} f(T_1^{m_1}T_2^{m_2}T_3^{m_3}x)
\]

converge in \(L^p(X)\) norm and almost-everywhere on \(X\) as \(\lambda \to \infty\). Similar problems where considered for the discrete spheres induced by the Euclidean norm but only in dimensions \(d \geq 5\). Here we show that \(S^3_c(\lambda)\) may be used to illustrate these kind of phenomena in dimension \(d = 3\). We will obtain these results by establishing maximal ergodic theorems for the ergodic averages \(A_{\lambda}^c\) from (1.15), see Theorem 1.5. Pointwise convergence will be established by studying \(r\)-variational estimates. We will also prove sharp lacunary maximal estimates for averaging operators over \(S^3_c(\lambda)\), which are in marked contrast to the behavior of averaging operators over the discrete Euclidean spheres in \(Z^d\), see Theorem 1.6. Corresponding maximal and variational estimates for continuous averaging operators (1.20) over the spheres \(S^2_c \subset \mathbb{R}^3\) (see Section 2 for a definition of \(S^2_c\)) will be also discussed, see Theorem 1.7.

3. **Equidistribution problems.** Finally we will discuss a variant of equidistribution problem for the arithmetic spheres \(S^3_c(\lambda)\). More precisely, we will study the following projections

\[
P^3_c(\lambda) := \{\lambda^{-1/c} x : x \in S^3_c(\lambda)\}\]

on a neighborhood of the unit sphere \(S^2_c \subset \mathbb{R}^3\), where

\[
 S^2_c := \{x \in \mathbb{R}^3 : |x|_c = 1\}
\]

(see Section 2 for a definition of the norm \(|\cdot|_c\)). Even though \(P^3_c(\lambda) \not\subseteq S^2_c\) we will show that the points from \(P^3_c(\lambda)\) can be interpreted as equidistributed on the unit
sphere $S^2_c \subset \mathbb{R}^3$ as $\lambda \to \infty$. Namely for some nice functions $\phi$ one has
\[
\frac{1}{r_c(\lambda)} \sum_{x \in \mathbb{P}^3(\lambda)} \phi(x) \xrightarrow{\lambda \to \infty} \int_{S^2_c} \phi(x) d\nu_c(x),
\]
where $\nu_c$ is a probability measure on $S^2_c$ obtained by normalization of the measure $\mu_c$, see Theorem 1.8 and (1.23). We will achieve this by upgrading the circle method that led us to the asymptotic formula for the number of lattice points in $S^2_c(\lambda)$. We will also study the discrepancy function corresponding to the sets $\mathbb{P}^3_c(\lambda)$.

In the next three subsections, which will correspond to the topics described above, we will formulate the main results of this paper and present a broader context of undertaken problems and give a brief overview of the proofs.

1.1. Lattice point problems in $\mathbb{Z}^3$. As in [60, 61, 62] we begin with introducing a class of regularly varying or $c$-regularly varying functions $F_c$, which will be of our interest.

**Definition 1.1.** Let $c \in [1, 2)$ and $F_c$ be the family of all functions $h : [x_0, \infty) \to [1, \infty)$ for some $x_0 \geq 1$, satisfying the following properties

(i) $h \in C^3([x_0, \infty))$ and

\[
h'(x) > 0, \quad h''(x) > 0, \quad \text{for every } x \geq x_0.
\]

(ii) There exists a real valued function $\vartheta \in C^2([x_0, \infty))$ and a constant $C_h > 0$ such that

\[
h(x) = C_h x^c \ell_h(x), \quad \text{where } \ell_h(x) = \exp \left( \int_{x_0}^x \frac{\vartheta(t)}{t} dt \right), \quad \text{for every } x \geq x_0,
\]

and if $c > 1$, then

\[
\lim_{x \to \infty} \vartheta(x) = 0, \quad \lim_{x \to \infty} x\vartheta'(x) = 0, \quad \lim_{x \to \infty} x^2\vartheta''(x) = 0.
\]

(iii) If $c = 1$, then $\vartheta(x)$ is positive, decreasing and for every $\varepsilon > 0$ we have

\[
\frac{1}{\vartheta(x)} \lesssim x^\varepsilon, \quad \text{and} \quad \lim_{x \to \infty} \frac{x}{\vartheta(h(x))} = 0.
\]

**Furthermore,**

\[
\lim_{x \to \infty} \vartheta(x) = 0, \quad \lim_{x \to \infty} \frac{x\vartheta'(x)}{\vartheta(x)} = 0, \quad \lim_{x \to \infty} \frac{x^2\vartheta''(x)}{\vartheta(x)} = 0.
\]

Definition 1.1 will be essential to formulate the main result of this subsection for the arithmetic spheres as well as generalized arithmetic spheres.

For a fixed $c \in [1, 2)$ and a fixed function $h \in F_c$ we define the sets

\[
\mathbb{N}_h := \{ [h(m)] : m \geq N_0 \},
\]

where $N_0 \in \mathbb{Z}_+$ is a sufficiently large absolute constant depending only on the function $h$.

For each $k \in [3]$ we fix $c_k \in [1, 2)$ and a function $h_k \in F_{c_k}$. Since $h_k'(x) \geq 1$ for each $k \in [3]$, see 3.1, we may assume without loss of generality that there exists an absolute constant $N_0 \in \mathbb{Z}_+$ such that for $x \geq N_0$ the functions $x \mapsto [h_k(x)]$ are well defined and injective. From now on we assume that for each $k \in [3]$ the set $\mathbb{N}_{h_k}$ is defined with this choice of $N_0$. We now define generalized arithmetic spheres with radius $\lambda \in \mathbb{Z}_+$ by

\[
S_{h_1, h_2, h_3}(\lambda) := \{ x \in (\mathbb{Z}_+ \setminus [N_0 - 1])^3 : [h_1(x_1)] + [h_2(x_2)] + [h_3(x_3)] = \lambda \}.
\]
Let us define the corresponding counting function by
\[
rt_{h_1,h_2,h_3}(\lambda) := \#S_{h_1,h_2,h_3}(\lambda).
\]  
(1.10)

Then it is not difficult to see that
\[
rt_{h_1,h_2,h_3}(\lambda) = \#\{(n_1, n_2, n_3) \in \mathbb{N}_{h_1} \times \mathbb{N}_{h_2} \times \mathbb{N}_{h_3} : n_1 + n_2 + n_3 = \lambda\}.
\]

We now formulate the first result of this paper establishing the asymptotic formula for (1.10).

**Theorem 1.2.** For \(k \in [3]\) let \(c_k \in [1, 4/3]\) and \(\gamma_k := 1/c_k\) be such that
\[
4(1 - \gamma_1) + 5(1 - \gamma_2)/2 + 5(1 - \gamma_3)/2 < 1,
\]
\[
5(1 - \gamma_1)/2 + 4(1 - \gamma_2) + 5(1 - \gamma_3)/2 < 1,
\]
\[
5(1 - \gamma_1)/2 + 5(1 - \gamma_2)/2 + 4(1 - \gamma_3) < 1.
\]

Assume that \(h_k \in \mathcal{F}_{c_k}\) for each \(k \in [3]\), and let \(\varphi_k\) be its inverse. Then
\[
rt_{h_1,h_2,h_3}(\lambda) = \frac{\Gamma(\gamma_1)\Gamma(\gamma_2)\Gamma(\gamma_3)}{\Gamma(\gamma_1 + \gamma_2 + \gamma_3)} \lambda^{\gamma_1\gamma_2'(\lambda)\varphi_2'\varphi_3'(\lambda)} + o(\lambda^{\gamma_1\gamma_2'(\lambda)\varphi_2'(\lambda)\varphi_3'(\lambda)}),
\]
where \(\Gamma\) denotes the standard Gamma function.

In other words Theorem 1.2 says that every sufficiently large integer \(\lambda \in \mathbb{Z}_+\) can be represented as a sum of three integers \(n_1 \in \mathbb{N}_{h_1}, n_2 \in \mathbb{N}_{h_2}\) and \(n_3 \in \mathbb{N}_{h_3}\) for some fixed regularly varying functions \(h_1, h_2\) and \(h_3\). Theorem 1.2 can be thought of as a low-dimensional counterpart of the classical Waring problem asserting that every sufficiently large positive integer is the sum of a bounded number of \(k\)-th powers of positive integers for a fixed integer \(k \geq 2\). We refer to [17] and [65] for more details about the classical Waring problem. Our interest of the behavior of the counting function \(rt_{h_1,h_2,h_3}(\lambda)\) was rekindled by the fact that \(S^3_{h_1,h_2,h_3}(\lambda) = \emptyset\) if and only if \(\lambda = 4^m(8n + 7)\) for any \(m, n \in \mathbb{N}\). Then we started to investigate whether it is possible to perturb the squares in the spheres \(S^3_{h_1,h_2,h_3}(\lambda)\) by regularly varying functions \(h_1 \in \mathcal{F}_{c_1}, h_2 \in \mathcal{F}_{c_2}\) and \(h_3 \in \mathcal{F}_{c_3}\), which are close (in terms of the order of magnitude of their exponents \(c_1, c_2\) and \(c_3\)) to the identity \(\text{id}(x) := x\) function, to make sure that the set \(S_{h_1,h_2,h_3}(\lambda) \neq \emptyset\) for all sufficiently large integers \(\lambda \in \mathbb{Z}_+\).

The analogy of Theorem 1.2 to the Waring problem for the \(k\)-th powers will be more apparent if we assume that \(h_1(x) = h_2(x) = h_3(x) = x^c\), then the asymptotic formula for the number of lattice points
\[
r_c(\lambda) := \#S^3_c(\lambda)
\]
in the arithmetic spheres \(S^3_c(\lambda)\), see (1.2), is very much in the spirit of the Hardy and Littlewood asymptotic formula from the solution of the Waring problem for the \(k\)-th powers. We refer to [17] Theorem 20.2, p. 456 and [65] Theorem 5.7, p. 146 for detailed expositions of the Hardy and Littlewood theorem as well as to the Kloosterman circle method [17] Theorem 20.9, p. 472], which handles positive definite quadratic forms with integer coefficients in \(d \geq 4\) variables. We now see that the arithmetic spheres \(S^3_c(\lambda)\) can be thought of as a perturbation of the classical spheres \(S^3(\lambda)\) by a noninteger exponent \(c > 1\), which is close to 1, in place of the squares. Theorem 1.2 can be easily applied to deduce the asymptotic formula for \(r_c(\lambda)\).

**Corollary 1.3.** If \(c \in (1, 9/8)\), then for every \(\varepsilon > 0\) we have
\[
r_c(\lambda) = \frac{2^3\Gamma(1 + 1/c)^3}{\Gamma(3/c)} \lambda^{3/c-1} + O_\varepsilon(\lambda^{3/c-1-(9-8c)/(5c)+\varepsilon}).
\]
We note that the formula for $r_c(\lambda)$ gives us a genuine information about the asymptotic behavior of the number of lattice points in $S^3_c(\lambda)$. Comparing the formula for $r_c(\lambda)$ with the formula $r_2(\lambda)$ from (1.1) we see that the leading terms are of the same kinds. The only difference is the lack of the singular series in the formula for $r_c(\lambda)$. This contrasts sharply with the Euclidean situation for $S^2_c(\lambda)$, where the asymptotic formula for $r_2(\lambda)$ greatly depends on the behavior of the underlying singular sequence, which is very irregular, see [47, Theorem 20.15, p. 478] as well as [47, Remark below formula (20.130), p. 479]. The lack of the singular series in our situation can be easily explained. We work with non-polynomial functions, which is much simpler than the polynomial situation, and the only major arc that is expected to be significant in analysis of $r_c(\lambda)$ is the one centered at the origin. Therefore the singular series does not arise in the case of arithmetic $c$-spheres or can be thought of as a sum containing the only one term. This is the reason why the arithmetic spheres (1.2) may be used as toy models to examine various phenomena in number theory and ergodic theory as we shall see later in the paper.

Finally, let us mention that Deshouillers [27] and later Arkhipov and Zhitkov [7] studied variants of Waring’s problem with noninteger exponents. Specifically, it is known from [7] that every sufficiently large integer $\lambda \in \mathbb{Z}_+$ can be written in the form

$$\lfloor x_1^c \rfloor + \ldots + \lfloor x_d^c \rfloor = \lambda \quad (1.13)$$

for some positive integers $x_1, \ldots, x_d \in \mathbb{Z}_+$, whenever $c > 12$ and $d > 22c^2(\log c + 4)$. Moreover, the number of all $d$-tuples $(x_1, \ldots, x_d) \in \mathbb{Z}_+^d$ that solve (1.13) behaves like

$$\frac{\Gamma(1 + 1/c)^d}{\Gamma(d/c)} \lambda^{d/c - 1} (1 + o(1)) \quad \text{as} \quad \lambda \to \infty. \quad (1.14)$$

If we were allowed to take $c \in (1, 9/8)$ and $d = 3$ in (1.14) we would see that the asymptotic formula for $r_c(\lambda)$ coincides with (1.14) up to the factor $2^3$ that arises from considering all integer triples in $S^3_c(\lambda)$ instead of only positive triples from $S^3_c(\lambda) \cap \mathbb{Z}_+^3$. Neither the method from [27] nor from [7] does seem to work in the case of small exponents $c \in (1, 9/8)$. Here we propose a different approach to prove Theorem 1.2, our strategy will be briefly described later in the paper. The proofs of Theorem 1.2 and Corollary 1.3 are given in Section 3.

The asymptotic formula from Corollary 1.3 was the starting point to study the main results of this article concerning norm and pointwise convergence for the ergodic averages over the arithmetic spheres.

1.2. Ergodic theorems and corresponding maximal estimates in $\mathbb{Z}^3$ and $\mathbb{R}^3$. Let $(X, \nu, T)$ be a measure-preserving system, where $(X, \nu)$ is a $\sigma$-finite measure space equipped with a family $T = (T_1, T_2, T_3)$ of commuting invertible and measure-preserving transformations $T_1, T_2, T_3 : X \to X$. We will also write $T^n := T_{1}^{n_1}T_{2}^{n_2}T_{3}^{n_3}$ for any $n \in \mathbb{Z}^3$; and for each $k \in [3]$ we define $T^0_k := Id$ to be the identity map on $X$.

We now fix $c \in (1, 11/10)$, and let $\lambda(c) \in \mathbb{Z}_+$ be the smallest integer such that $S^3_c(\lambda) \neq \emptyset$ for all $\lambda \geq \lambda(c)$. Such an integer exists in view of the asymptotic formula for $r_c(\lambda)$ from Corollary 1.3. Now for every $f \in L^0(X)$ (see Section 2 for a definition of this space) and every integer $\lambda \geq \lambda(c)$ we define the ergodic averages over the arithmetic spheres $S^3_c(\lambda)$ by

$$A^3_c f(x) := A^3_\lambda f(x) := \frac{1}{\#S^3_c(\lambda)} \sum_{n \in S^3_c(\lambda)} f(T^n x), \quad x \in X. \quad (1.15)$$

We see that the average (1.15) is well defined for all $\lambda \geq \lambda(c)$. This was also our motivation to study the asymptotic formula for $r_c(\lambda) = \#S^3_c(\lambda)$ in Corollary 1.3. Without this asymptotic
it is even not clear whether \( S_3^3(\lambda) \neq \emptyset \). From now on when we consider averages (1.15) we always assume that \( \lambda \geq \lambda(c) \). For the further reference, let \( A_{c,d}^\lambda \) denote the ergodic average, which is an analogue of average (1.15), over the \( d \)-dimensional \( c \)-sphere \( S_3^d(\lambda) \) as in (1.16).

**Example 1.4** (Integer shift system on \( \mathbb{Z}^3 \)). The integer shift system \( (\mathbb{Z}^3, \nu_{\mathbb{Z}^3}, T_{\mathbb{Z}^3}) \) is the three-dimensional integer grid \( \mathbb{Z}^3 \) equipped with counting measure \( \nu_{\mathbb{Z}^3} \) and the family of shifts \( T_{\mathbb{Z}^3} = (T_{\mathbb{Z}^3,1}, T_{\mathbb{Z}^3,2}, T_{\mathbb{Z}^3,3}) \), with \( T_{\mathbb{Z}^3,k}(x) := x - e_k \) for \( k \in [3] \), where \( e_k \) is the \( k \)-th basis vector from the standard basis on \( \mathbb{R}^3 \). In view of the Calderón transference principle, see [16], the integer shift system is thought of as a “universal” system for all other measure-preserving systems. It will be convenient as we shall see in a moment to work with this system due to the extensive Fourier-analytic structure available on \( \mathbb{Z}^3 \). The averages \( A_{c}^\lambda \) on \( (\mathbb{Z}^3, \nu_{\mathbb{Z}^3}, T_{\mathbb{Z}^3}) \) will be denoted by \( M_{\lambda}^c \) and we have for any function \( f : \mathbb{Z}^3 \to \mathbb{C} \) that

\[
M_{\lambda}^c f(x) := M_{\lambda}^{c,3} f(x) := \frac{1}{\# S_3^3(\lambda)} \sum_{n \in S_3^3(\lambda)} f(x - n), \quad x \in \mathbb{Z}^3, \quad \lambda \geq \lambda(c). \tag{1.16}
\]

Taking \( \sigma_{\lambda}(x) := \mathbb{1}_{S_3^3(\lambda)}(x) \) for \( x \in \mathbb{Z}^3 \) we immediately see that the discrete average \( M_{\lambda}^c \) is a convolution operator and we have

\[
M_{\lambda}^c f(x) = \frac{1}{\tau_{\lambda}(\lambda)} \sigma_{\lambda} \ast f(x), \quad x \in \mathbb{Z}^3, \quad \lambda \geq \lambda(c).
\]

For the further reference, let \( M_{c,d}^\lambda \) denote the discrete average, which is an analogue of average (1.16), over the \( d \)-dimensional \( c \)-sphere \( S_3^d(\lambda) \) as in (1.16).

We say that a set \( \mathbb{D} = \{ \lambda_n : n \in \mathbb{Z}^+ \} \subset (0, \infty) \) is \( \lambda_0 \)-lacunary if

\[
\lambda_0 := \inf_{n \in \mathbb{Z}^+} \frac{\lambda_{n+1}}{\lambda_n} > 1.
\]

We shall be concerned with norm and pointwise convergence for the averages \( A_{c}^\lambda \) defined for \( \lambda \in \mathbb{Z}^+ \) as well as for \( \lambda \in \mathbb{D} \), where \( \mathbb{D} = \{ \lambda_n : n \in \mathbb{Z}^+ \} \subset \mathbb{Z}^+ \) is a lacunary set. The lacunary cases will exhibit different phenomena than the averages \( A_{c}^\lambda \) defined for \( \lambda \in \mathbb{Z}^+ \).

The main result of the paper is the following ergodic theorem for averages (1.15).

**Theorem 1.5.** Let \( (X, \nu, T) \) be a measure-preserving system. Fix \( c \in (1, 11/10) \) and let \( A_{c}^\lambda \) be the average from (1.15) defined for all integers \( \lambda \geq \lambda(c) \). Then for every \( f \in L^p(X) \), where \( (11 - 4c)/(11 - 7c) < p < \infty \) we have that:

(i) (Mean ergodic theorem). The averages \( A_{c}^\lambda f \) converge in \( L^p(X) \) norm as \( \lambda \to \infty \).

(ii) (Pointwise ergodic theorem). The averages \( A_{c}^\lambda f \) converge pointwise almost everywhere as \( \lambda \to \infty \).

(iii) (Maximal ergodic theorem). The maximal inequality holds

\[
\left\| \sup_{\lambda \in \mathbb{Z}^+} |A_{c}^\lambda f| \right\|_{L^p(X)} \lesssim_{c,p} \|f\|_{L^p(X)}. \tag{1.17}
\]

Inequality (1.17) also holds for \( p = \infty \). Moreover, if \( \lambda \geq \lambda(c) \) in the average \( A_{c}^\lambda \) is restricted to a \( \lambda_0 \)-lacunary set \( \mathbb{D} \subset \mathbb{Z}^+ \), then (i), (ii) and (iii) remain valid for any \( f \in L^p(X) \) and any \( p \in (1, \infty) \). However, the implicit constant in (1.17) additionally depends on \( \lambda_0 > 1 \).

We immediately see that (i) is a simple consequence of (ii), (iii) and the dominated convergence theorem. Thus it suffices to establish pointwise ergodic theorem (ii) and maximal ergodic theorem (iii). A detailed proof of Theorem 1.5 will be given in Section 6. We first prove maximal inequality (1.17) and then use it to deduce pointwise convergence for \( A_{c}^\lambda \).
The maximal ergodic theorem (iii), in fact, will follow from Theorem 1.6 below by appealing to the Calderón transference principle as in [16]. We prove Theorem 1.6 in Section 4.

The main maximal result of this article is the boundedness for the discrete averages $M^c_{\lambda}$.

**Theorem 1.6.** Fix $c \in (1, 11/10)$ and let $M^c_{\lambda}$ be the discrete average from (1.16) defined for all integers $\lambda \geq \lambda(c)$. Then $M^c_{\lambda}$ can be extended.

As we mentioned above, inequality (1.18) implies (1.17) upon appealing to the Calderón transference principle as in [16]. By the same argument we conclude that (1.19) implies a corresponding lacunary variant of inequality (1.17). Therefore, the conclusion of Theorem 1.6 (iii) is entirely reduced to the conclusion of Theorem 1.6 which will be proved using Fourier methods in Section 4. An essential part of our approach is the fact that the averages $M^c_{\lambda}$ from (1.16) may be thought of as discrete analogues of spherical averaging operators in $\mathbb{R}^3$, which are dilates of the sphere $S^2_c$. Let $\mu_c$ be a measure on $S^2_c$ arising in a natural way from the polar decomposition with respect to the norm $|\cdot|_c$, see (4.13) for a precise description of $\mu_c$. Then continuous spherical averaging operators are defined by

$$A^c_c f(x) := \int_{S^2_c} f(x - t\theta) \, d\mu_c(\theta), \quad x \in \mathbb{R}^3, \quad t > 0, \quad f \in C_c^\infty(\mathbb{R}^3), \quad (1.20)$$

As before, for the further reference, let $A^c_{f,d}$ denote the average, which is an analogue of average (1.20), over the sphere $S^{d-1}_c \subset \mathbb{R}^d$ (see Section 2 for a definition of this set). Our next result subsumes maximal and $r$-variational estimates for the operators $A^c_f$, which will be proved in Section 5 (see Section 2 for a definition of $r$-variational seminorm $V^r$ and its simple properties).

**Theorem 1.7.** Let $c \in (1, 2)$ be fixed, and let $A^c_f$ for $t > 0$ be the spherical averaging operator defined in (1.20). Then for every $p \in (3/2, 4)$ and $r \in (2, \infty)$ we have

$$\|V^r(A^c_f f : t > 0)\|_{L^p(\mathbb{R}^3)} \lesssim_{c,p,r} \|f\|_{L^p(\mathbb{R}^3)}, \quad f \in L^p(\mathbb{R}^3), \quad (1.21)$$

as well as, for all $p \in (3/2, \infty]$, the maximal estimate

$$\|\sup_{t > 0} |A^c_f f(t)|\|_{L^p(\mathbb{R}^3)} \lesssim_{c,p} \|f\|_{L^p(\mathbb{R}^3)}, \quad f \in L^p(\mathbb{R}^3), \quad (1.22)$$

If we consider only lacunary times the range of $p$ in (1.21) and (1.22) can be extended. Namely, for every $p \in (1, \infty)$ and $r \in (2, \infty)$ we have

$$\|V^r(A^c_f f : t \in \mathbb{D})\|_{L^p(\mathbb{R}^3)} \lesssim_{c,p,r,\lambda_0} \|f\|_{L^p(\mathbb{R}^3)}, \quad f \in L^p(\mathbb{R}^3), \quad (1.23)$$

and for all $p \in (1, \infty]$, the maximal estimate holds

$$\|\sup_{t \in \mathbb{D}} |A^c_f f(t)|\|_{L^p(\mathbb{R}^3)} \lesssim_{c,p,\lambda_0} \|f\|_{L^p(\mathbb{R}^3)}, \quad f \in L^p(\mathbb{R}^3), \quad (1.24)$$
whenever $D \subset (0, \infty)$ is $\lambda_0$-lacunary set. Moreover, the range of $p$ in (1.21) and in (1.22) is sharp from below in the sense that for every $p \in [1, 3/2]$ the estimates (1.21) and (1.22) do not hold.

Averaging operators over the Euclidean spheres ($S^{d-1}_c$ with $c = 2$) were extensively studied over the years. It is very well known from the results of Stein [65] for all $d \geq 3$, and Bourgain [13] for $d = 2$ that the maximal spherical function $\sup_{t > 0} |A_{1,t}^{2,d} f|$ is bounded on $L^p(\mathbb{R}^d)$ if and only if $\frac{d}{d-1} < p \leq \infty$. If $t$ is restricted to a lacunary set $\mathbb{D} \subset (0, \infty)$ then the lacunary spherical maximal function $\sup_{t \in \mathbb{D}} |A_{1,t}^{2,d} f|$ is bounded on $L^p(\mathbb{R}^d)$ for all $1 < p \leq \infty$ as it was shown by Calderón [20] and independently by Coifman and Weiss [21].

Consider the $c$-sphere $S^{d-1}_c$ as defined in (2.1). If $c$ is an even integer, $> 2$, it follows from (13, see also [15]) that the corresponding maximal operator is bounded for $p > \max \left\{ \frac{d}{d-1}, \frac{d}{c} \right\}$ (sharp). Similar methods yield the same result for any $c \geq 2$. The case $1 < c < 2$ is a bit different due to the lack of smoothness and the fact just as in the case $c = 2$, the Gaussian curvature does not vanish. We are able to handle this case in Theorem 1.7 above using a suitable Fourier decay estimate (see inequality (4.25)) that allows to follow the general outline of Stein’s original argument for the sphere in dimensions three and higher. While Theorem 1.7 is only stated and proved in three dimensions, the maximal estimate easily extends to all $d \geq 3$ with $p > \frac{3}{2}$ replaced by $p > \frac{d}{d-1}$.

Variational estimates for the spherical averages $A_{1,t}^{2,d}$ were studied by Jones, Seeger and Wright [48] who showed that $V^r(A_{1,t}^{2,d} f : t > 0)$ is bounded on $L^p(\mathbb{R}^d)$ for all $r > 2$ if $\frac{d}{d-1} < p \leq 2d$, where the conditions for $r$ and $p$ are both sharp. If $p > 2d$ then $V^r(A_{1,t}^{2,d} f : t > 0)$ is bounded on $L^p(\mathbb{R}^d)$ as long as $r > p/d$ and this is not true when $r < p/d$. Recently, also Beltran, Oberlin, Roncal, Seeger and Stovall [11] showed the endpoint estimate, which states that if $d \geq 3$ and $p > 2d$ then $V^{p/d}(A_{1,t}^{2,d} f : t > 0)$ is of restricted weak type $(p,p)$.

Theorem 1.7 gives sharp estimates in (1.22), (1.23) and (1.24). However, the range of $p$ and $r$ in (1.21) is not sharp. The exponent $c \in (1,2)$ does not affect the maximal estimates (1.22) and (1.24), which coincide with their Euclidean counterparts (with $c = 2$ and $d = 3$). This phenomenon (as mentioned above) can be easily explained by examining Fourier transform estimates of the spherical measure $\mu_c$ on $S^{d-1}_c$, which for all $c \in (1,2)$ does have precisely the same bounds as the Fourier transform of the spherical measure on $S^2_2$, see (4.25). We have not found this result in the existing literature, and since it may be of independent interest we included the proof in Section 5. Moreover, the Fourier transform estimates (4.25) of the measure $\mu_c$ will be essential in the proofs of Theorem 1.5 and Theorem 1.6.

Discrete analogues $M^{2,d}_A$ of the Euclidean spherical averages $A_{1,t}^{2,d}$ were investigated by Magyar [53], and Magyar, Stein and Wainger [59]. In the latter work a complete result was proved, which asserts that the discrete maximal spherical function $\sup_{\lambda \in \mathbb{Z}^d} |M^{2,d}_A f|$ is bounded on $\ell^p(\mathbb{Z}^d)$ if and only if $d \geq 5$ and $\frac{d}{d-2} < p \leq \infty$. It is a remarkable result that illustrates a difference between discrete and continuous operators. The restricted weak-type endpoint result was also proved for $\sup_{\lambda \in \mathbb{Z}^d} |M^{2,d}_A f|$ by Ionescu [48].

Magyar, Stein and Wainger paper [59] was the starting point of three different interesting lines of investigations in the discrete harmonic analysis and related areas.

1. The first line, initiated by Magyar [56], extended the result of Magyar–Stein–Wainger [59] to discrete averages defined over a certain wide class of positive definite hypersurfaces in the spirit of Birch [12] and Davenport [26]. Magyar [56] also initiated investigations of norm and
pointwise ergodic theorems for these kind of averages as well as equidistribution problems \[57\] and \[58\] in the context of discrepancy function, see also the next subsection for more details. Magyar’s results were extended by Hughes \[11\] to \(k\)-spheres \(S_k^d(\lambda)\) for any integer \(k \geq 2\) and sufficiently large dimensions \(d \in \mathbb{Z}_+\). Hughes’ results were recently improved by Anderson, Cook, Hughes and Kumchev in \[3\]. In the latter paper the authors also continued their investigations of the so-called ergodic Waring–Goldbach problems \[2\], where averages are taken over \(k\)-spheres in primes \(S_k^d(\lambda) \cap \mathbb{P}^d\) for any integer \(k \geq 2\) and sufficiently large dimensions \(d \in \mathbb{Z}_+\). Discrete maximal averages over Birch forms were recently studied by Cook \[22\].

2. The second line of investigation, initiated by Hughes \[39\], on the one hand, asks about \(\ell^p(\mathbb{Z}^d)\) bounds for discrete spherical maximal operators obtained from restricting the supremum to lacunary sets. On the other hand, Hughes observed (even though the Magyar–Stein–Wainger theorem is sharp) that it also makes sense to study \(\sup_{\lambda \in 2\mathbb{N}^*+1} |M^{2,d}_\lambda f|\) for \(d = 4\) upon restricting the radii \(\lambda\) to odd integers. Specifically, the author \[39\] constructed a sophisticated lacunary set of radii \(\mathbb{D}\) such that \(\sup_{\lambda \in \mathbb{D}} |M^{2,d}_\lambda f|\) is bounded on \(\ell^p(\mathbb{Z}^d)\) for every \(d/2 \leq p \leq \infty\) and \(d \geq 4\). Hughes’ result was recently extended by Kesler, Lacey and Mena \[52\], where it was proved that \(\sup_{\lambda \in \mathbb{D}} |M^{2,d}_\lambda f|\) for any lacunary sequence \(\mathbb{D} \subset \mathbb{Z}_+\) is bounded on \(\ell^p(\mathbb{Z}^d)\) for every \(d/2 < p \leq \infty\) and \(d \geq 4\). The case \(d = 4\) was recently established by Anderson and Madrid \[4\] for \(d+1/2 < p \leq \infty\) and all lacunary sequences \(\mathbb{D} \subset \mathbb{Z}_+ \setminus 4\mathbb{Z}_+\). Cook and Hughes \[25\] considered similar questions in the context of Birch forms, and specifically recovered the main result from \[52\]. They also showed in \[25\] that in contrast to the continuous case, no such inequalities can hold close to \(\ell^1(\mathbb{Z}^d)\). More precisely, for any \(1 < p < \frac{d}{d-1}\) there exists a set of lacunary radii \(\mathbb{D} \subset \mathbb{Z}_+\) such that \(\sup_{\lambda \in \mathbb{D}} |M^{2,d}_\lambda f(x)|\) is unbounded on \(\ell^p(\mathbb{Z}^d)\). This negative result is also true \[25\] for averages over more general forms in the spirit of Birch. This is another remarkable phenomenon that exhibit some peculiar features in the discrete world. Finally, let us emphasize that the negative result from \[25\] does not exclude positive results for \(1 < p < \frac{d}{d-1}\). Namely, Cook showed \(\ell^p(\mathbb{Z}^d)\) maximal bounds for all \(1 < p \leq \infty\) by considering \(M^{2,d}_\lambda f\) for \(d \geq 5\) with a very sparse sequence of radii \[21\] as well as by studying averages associated to a certain class of homogeneous algebraic hypersurfaces in \[22\].

3. The third line, initiated by Kesler, Lacey and Mena \[52, 53\] who proposed to study sparse estimates in the context of discrete spherical averages. It is a very successful line of research, which significantly enhanced the field of discrete harmonic analysis. In \[53\] Kesler, Lacey and Mena proved conjecturally sharp sparse bounds for \(\sup_{\lambda \in \mathbb{Z}_+} |M^{2,d}_\lambda f|\). Using these bounds the authors recovered in a fairly unified way the main results from Magyar–Stein–Wainger paper \[59\] as well as the endpoint result of Ionescu \[43\]. Sparse estimates \[49, 50, 51\] turned out to be very efficient in \(\ell^p(\mathbb{Z}^d)\)-improving estimates for the Magyar–Stein–Wainger averages \(M^{2,d}_\lambda\), which were initially studied by Hughes \[42\]. We also refer to Anderson’s paper \[1\] which obtained \(\ell^p(\mathbb{Z}^d)\)-improving bounds for spherical averages along the primes.

Theorem \[1.5\] and Theorem \[1.6\] contribute to the first and the second line of research. Firstly, it is the first paper that investigates arithmetic spheres \(S^3_c(\lambda)\) from \[1.2\] with non-integer exponents \(c \notin \mathbb{Z}_+\) in the context of norm and pointwise ergodic theorems, see Theorem \[1.5\]. Secondly, Theorem \[1.6\] establishes full (all radii \(\lambda \in \mathbb{Z}_+\)) maximal estimates \[1.18\] as
well as sharp (for all exponents $p \in (1, \infty]$) lacunary maximal estimates \([11, 19]\) for the discrete averages $M^c_\lambda$ over the arithmetic spheres $S_\lambda^c(\lambda)$ in dimension $d = 3$. To the best of our knowledge it is the first result in the discrete setup that handles three-dimensional case. Thirdly, since $S_\lambda^c(\lambda)$ is induced by non-polynomial functions the circle method that lies behind of the maximal estimates in Theorem \([15]\) and Theorem \([16]\) is much simpler. The only major arc that is expected to be significant in analysis of Fourier multipliers corresponding to the averages $M^c_\lambda$ is the one centered at the origin and the arithmeticity that was apparent in all papers mentioned above does not enter into our proofs at all. Therefore, the proofs are conceptually closer to the situation that arises in the continuous setup, this allows us to think that averages $A^c_\lambda$ may be thought of as toy models to study the questions of this type in ergodic theory and discrete harmonic analysis. This gives strong motivation to understand the situation more thoroughly.

1.3. Equidistribution problems. We will discuss equidistribution problems corresponding to the arithmetic spheres $S_\lambda^c(\lambda)$. The last goal of the article is to study the following projections $P_\lambda^c(\lambda)$, see \([11, 13]\). In other words this is the set of all projections of lattice points from $S_\lambda^c(\lambda)$ via the dilatation $x \mapsto \lambda^{-1/c}x$ on a neighborhood of the unit sphere $S_\lambda^c \subset \mathbb{R}^3$. Even though $P_\lambda^c(\lambda) \nsubseteq S_\lambda^c$ we can show that the points from $P_\lambda^c(\lambda)$ are asymptotically close to $S_\lambda^c$ since

$$\sup_{x \in P_\lambda^c(\lambda)} |x|_c^c - 1 \leq 3/\lambda \xrightarrow[\lambda \to \infty]{} 0,$$

and in fact the points from $P_\lambda^c(\lambda)$ can be interpreted as equidistributed on the unit sphere $S_\lambda^c \subset \mathbb{R}^3$ as $\lambda \to \infty$, in the sense of the following result.

**Theorem 1.8.** Let $c \in (1,9/8)$ be fixed. Then for every $\phi \in C^\infty(\mathbb{R}^3)$ we have

$$\frac{1}{r_c(\lambda)} \sum_{x \in P_\lambda^c(\lambda)} \phi(x) \xrightarrow[\lambda \to \infty]{} \int_{S_\lambda^c} \phi(x) d\nu_c(x),$$

where $\nu_c$ is a probability measure on $S_\lambda^c$ obtained by normalization of the measure $\mu_c$, i.e.

$$\nu_c := \frac{\mu_c}{\mu_c(S_\lambda^c)} = \frac{8\Gamma(1/c)^3}{c^3\Gamma(3/c)}.$$

Theorem \([11, 18]\) can be thought of as a variant of Linnik’s problem \([51]\) for the arithmetic spheres $S_\lambda^c(\lambda)$, see also \([28]\) and \([32]\) for unconditional variants of Linnik’s result.

In fact, being motivated by Magyar \([57]\) and \([58]\), we obtain a much stronger result by investigating the rate of equidistribution of the sets $P_\lambda^c(\lambda)$ on the neighborhood of the sphere $S_\lambda^c$ as $\lambda \to \infty$. For this purpose we shall investigate the discrepancy of the sets $P_\lambda^c(\lambda)$ with respect to the following spherical caps

$$C_{a,\xi} := \{x \in S_\lambda^c : x \cdot \xi \geq a\}, \quad \xi \in S^2, \quad a > 0.$$  

Here and later on on $S^2 = S_2^3$, see \([21]\). Due to the technical reason caused by the fact that $P_\lambda^c(\lambda) \nsubseteq S_\lambda^c$, we also need to consider

$$C_{a,\xi} := \{x \in \mathbb{R}^3 : 100 \geq x \cdot \xi \geq a\}, \quad \xi \in S^2, \quad a > 0.$$  

Observe that $C_{a,\xi} \subseteq C_{a,\xi}$ because $|x \cdot \xi| \leq 1$ for $x \in S_\lambda^c$ and $\xi \in S^2$. Then the discrepancy function of the set $P_\lambda^c(\lambda)$ with respect to the caps $C_{a,\xi}$ and $C_{a,\xi}$ associated with a given
direction $\xi \in S^2$ is defined by
\[ D_c(\lambda, \xi) := \sup_{a > 0} |D_c(\lambda, \xi, a)|, \quad \lambda \in \mathbb{Z}^+, \ \xi \in S^2, \]
where
\[ D_c(\lambda, \xi, a) := \#(P^3_c(\lambda) \cap C_{a, \xi}) - r_c(\lambda)\nu_c(C_{a, \xi}), \quad \lambda \in \mathbb{Z}^+, \ \xi \in S^2, \ a > 0. \]
Using the ideas from [58] we are able to prove the following estimate.

**Theorem 1.9.** Let $c \in (1, 9/8)$ be fixed. Then for every $\epsilon > 0$ we have
\[ D_c(\lambda, \xi) \lesssim \epsilon^{3c-1-(9-8c)/(5c)+\epsilon}, \quad \lambda \in \mathbb{Z}^+, \ \xi \in S^2. \]
The implicit constant is uniform in $\lambda$ and $\xi$.

Theorem 1.9 is a three-dimensional analogue of Magyar’s result [57] for the Euclidean spheres (as well as some hypersurfaces corresponding to homogeneous polynomials with integer coefficients) for all dimensions $d \geq 4$, which establishes bounds for the discrepancy function with respect to spherical caps along diophantine directions $\xi \in S^{d-1}$, (see [57] and [58] for a definition of diophantine directions). Our result controls the discrepancy function $D_c(\lambda, \xi)$ uniformly in $\xi \in S^2$ in contrast to Magyar’s results [57, 58], which exclude some sets of directions that is of Lebesgue measure zero in $\mathbb{R}^{d-1}$. The reason is exactly the same as in the previous two problems undertaken in this paper. The only major arc that is expected to be significant in analysis of the discrepancy function $D_c(\lambda, \xi)$ is the one centered at the origin and we do not need to exclude non-diophantine directions from the picture in our case.

We show a detailed proof of Theorem 1.9 in Section 7. Then we easily deduce, proceeding in a similar way as in the proof of Theorem 1.9, the convergence in Theorem 1.8.

### 1.4. Structure of the paper.

The paper is organized as follows. Theorem 1.2 is proved in Section 3. In our proof we combine the ideas of Heath–Brown [37] with Vinogradov’s ideas from the ternary Goldbach problem (see [65], Section 8, p. 211 and also [47], Chapter 13, p. 336) as it was done in the Balog and Friedlander paper [9], where the ternary Goldbach problem is solved in the Piatetski–Shapiro primes $N_h \cap P$ with $h(x) = x^c$ for some $c > 1$. The important part of the argument relies on the estimates contained in Proposition 3.8 (an asymptotic formula giving the leading term for $r_{h_1,h_2,h_3}$ in Theorem 1.2) and Proposition 3.12 (exponential sum estimates giving the error term in Theorem 1.2).

In Section 4 we establish maximal inequalities (1.18) and (1.19) from Theorem 1.6 which is the first main result of this article. Maximal inequalities (1.18) and (1.19) corresponding to the averages $M^c_\lambda$ and the Magyar–Stein–Wainger maximal inequality [59] corresponding to the averages $M^{2,d}_\lambda$ have the same starting point — the Hardy–Littlewood circle method — however completely different proofs. In our situation, there is only one major arc (centered at the origin) and respectively the only one minor arc. This phenomenon is caused by the non-integer nature of functions $x^c$, $c \in (1, 2)$, which is in marked contrast with the classical polynomial situation arising in [59], where major arcs consist of a collection of small neighborhoods of rationals with sufficiently small denominators. From this point of view our model is much simpler, but we cannot directly follow the ideas from [59] or even the subsequent papers discussed above. Maximal estimates corresponding to the averages $M^c_\lambda$ on the minor arc (see Theorem 4.11) are reduced upon analyzing various Fourier expansions to certain exponential sum estimates. The latter exponential sums, following the ideas of Heath–Brown [37], can be estimated using the van der Corput second derivative test for exponential sums in place of the usual Weyl’s inequality like in the polynomial situation.
Maximal estimates corresponding to the averages $M^c_\lambda$ on the major arc (see Theorem 4.7) can be controlled by maximal functions associated to continuous convolution operators with the kernels $K^\lambda$ as in (4.20) upon applying a comparison principle from Lemma 4.9. Boundedness of the latter operators are obtained by standard arguments as in [63], which in turn are reduced to the Fourier transform estimates of the surface measure $\mu_c$ on the hypersurface $S_c^2$, see (4.25). Since we work with noninteger $c \in (1, 2)$ it is easy to see that $S_c^2$ is locally nonsmooth and the Fourier transform estimates of $\mu_c$ in (4.25) are much more delicate than the Euclidean one. We have not found these kind of estimates in the existing literature therefore the details (which are interesting in their own right) are provided in Section 5, see Lemma 5.2. The proof of Theorem 1.7 is also provided in Section 5 as a consequence of the Fourier transform estimates (4.25) and the techniques from [63].

Theorem 1.5, the second main result of this paper, is proved in Section 6. In fact, as mentioned above, we only need to establish pointwise convergence for the averages $A^c_\lambda$. This is achieved by splitting the average $A^c_\lambda$ into two parts corresponding respectively to major and minor arc. The minor arc part of the operator $A^c_\lambda$ converges to zero almost-everywhere. To understand the major arc part we study various $r$-variational estimates, see (6.5) and (6.6). The proof is also to a large extent based on the Fourier transform estimates (4.25) for $\mu_c$ and the techniques from [63]. Our approach allows us to handle also $\sigma$-finite measure-preserving systems in contrast to probability systems usually studied in this context.

Finally, in Section 7 we prove Theorem 1.9 by adapting Magyar’s approach [57, 58] to our setup. The key ingredients in this process are tools that we developed to prove Theorem 1.2 as well as Theorem 1.6 see especially the proof of Lemma 7.3. The technique from the proof of Lemma 7.3 is also used to establish Theorem 1.8.

1.5. Open problems. We close this section with a discussion of some open problems that arise naturally out of this project.

1. In this paper we are only concerned with dimension 3, however it makes sense to consider all these problems in higher dimensional setting. Therefore, do Theorem 1.2, Theorem 1.5, Theorem 1.6, Theorem 1.7, Theorem 1.8 and Theorem 1.9 remain valid in higher dimensional setup? We expect that these theorems should have higher dimensional analogues. However, then the relations between the exponent $c$ and the dimension $d$ must be understood and should enter somehow into play.

2. Does Theorem 1.2 remain valid for all $c \in (1, 2)$ or even $c > 1$? In our proof it was important that $c > 1$ but very close to 1. Our method can be refined to obtain a larger range of $c$ in Theorem 1.2 though still very far from $c \in (1, 2)$.

3. In all theorems except Theorem 1.2 we were concerned with the arithmetic spheres $S^3_c(\lambda)$, however it is interesting to know whether they remain true with the generalized spheres $S_{h_1, h_2, h_3}(\lambda)$ in place of $S^3_c(\lambda)$. An important ingredient in the proofs is the polar decomposition (4.23) for $S^3_c$, which is rather not available for generalized spheres from (1.9). This was the obstacle that we could not overcome.

4. In Theorem 1.5 and Theorem 1.6 we obtained sharp results it terms of the range $p \in (1, \infty)$ for the averages taken over $A_0$-lacunary sets. It is interesting to know what is the sharp range of exponents in Theorem 1.5 and Theorem 1.6 when averages are taken over the full set of integers $\mathbb{Z}_+$. So far we have $(11 - 4c)/(11 - 7c) < p < \infty$ and $c \in (1, 11/10)$. We expect that these results should hold for all $p \in (3/2, \infty)$ at least.
for any \( c \in (1,11/10) \). The question about all \( c > 1 \) may be very hard, especially taking into account the case of \( c \in \mathbb{Z}_+ \), which as we have seen \([4, 23, 39, 52]\) is very different. This question is also very interesting in higher dimensions when averages \( A^{c,d}_\lambda \) are taken over \( S^d(\lambda) \) for any \( d \geq 3 \). Then the range of \( p \) should depend on \( d \) and also on \( c \) especially when \( c > 2 \). It is most likely that the sharp range should be for the exponents \( p > \frac{d}{p-c} \) if \( c > 2 \) and \( d \geq 3 \). This would correspond to the conjecture formulated by Cook and Hughes \([23, \text{Conjecture 1}]\).

5. In Theorem \(1.5\) we established pointwise convergence by considering \( r \)-variational estimates corresponding to various pieces of the underlying averages. However, it would be nice to have \( r \)-variational estimates for the averages \( A^{c,d}_\lambda \) themselves.

6. A similar question concerns the continuous averages \( A^{c,d}_\lambda \). In view of Jones, Seeger and Wright \( r \)-variational result \([48]\) for the classical spherical averages and the recent endpoint result from \([11]\) we ask about sharp \( L^p(\mathbb{R}^d) \) estimates of \( r \)-variations for \( A^{c,d}_\lambda \) in terms of parameters \( r, p, c \) and \( d \). It is a very intriguing question due to unclear relations of the underlying parameters \( r, p, c \) and \( d \). These relations are expected to be different for \( c \in (1, 2) \) and for \( c \geq 2 \). Here one will have to extend the Fourier transform estimates from \([1, 25]\) to higher dimensions and understand their interactions with the exponent \( c \) and the dimension \( d \).

2. Notation

We now set up notation that will be used throughout the paper.

2.1. Basic notation. The sets \( \mathbb{Z}, \mathbb{R}, \mathbb{C} \) and \( \mathbb{T} := \mathbb{R}/\mathbb{Z} \) have standard meaning. The set of positive integers and nonnegative integers will be denoted respectively by \( \mathbb{Z}_+ := \{1, 2, \ldots\} \) and \( \mathbb{N} := \{0, 1, 2, \ldots\} \). The set \( \mathbb{P} \subset \mathbb{Z}_+ \) denotes the set of the prime numbers. For any real number \( N > 0 \) we define

\[
[N] := \mathbb{Z}_+ \cap (0, N].
\]

For any \( x \in \mathbb{R} \) we will use the floor function \( \lfloor x \rfloor := \max \{n \in \mathbb{Z} : n \leq x\} \), the fractional part \( \{x\} := x - \lfloor x \rfloor \) and the distance to the nearest integer \( \|x\| := \text{dist}(x, \mathbb{Z}) \). For \( x, y \in \mathbb{R} \) we shall also write \( x \lor y := \max \{x, y\} \) and \( x \land y := \min \{x, y\} \).

For two nonnegative quantities \( A, B \) we write \( A \lesssim B \) \((A \gtrsim B)\) if there is an absolute constant \( C_\delta > 0 \) (which possibly depends on \( \delta > 0 \)) such that \( A \leq C_\delta B \) \((A \geq C_\delta B)\). We will write \( A \asymp B \) when \( A \lesssim B \) and \( A \gtrsim B \) hold simultaneously. We will omit the subscript \( \delta \) if irrelevant. For a function \( f : X \to \mathbb{C} \) and positive-valued function \( g : X \to (0, \infty) \), write \( f = O(g) \) if there exists a constant \( C > 0 \) such that \( |f(x)| \leq Cg(x) \) for all \( x \in X \). We will also write \( f = O_\delta(g) \) if the implicit constant depends on \( \delta \). For two functions \( f, g : X \to \mathbb{C} \) such that \( g(x) \neq 0 \) for all \( x \in X \) we write \( f = o(g) \) if \( \lim_{x \to \infty} f(x)/g(x) = 0 \).

We use \( \mathbb{1}_A \) to denote the indicator function of a set \( A \). If \( S \) is a statement we write \( \mathbb{1}_S \) to denote its indicator, equal to 1 if \( S \) is true and 0 if \( S \) is false. For instance \( \mathbb{1}_A(x) = \mathbb{1}_{x \in A} \).

2.2. Euclidean spaces. For every \( d \in \mathbb{Z}_+ \) the set \( \{e_i \in \mathbb{R}^d : i \in [d]\} \) will always denote the standard basis in \( \mathbb{R}^d \). The standard inner product on \( \mathbb{R}^d \) is denoted by

\[
x \cdot \xi := \sum_{k \in [d]} x_k \xi_k
\]
for every \( x = (x_1, \ldots, x_d) \) and \( \xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d \). The inner product induces the Euclidean norm \( |x|_2 := \sqrt{x \cdot x} \), which will be abbreviated to \( |x| \). We will also consider \( \mathbb{R}^d \) with the following norms

\[
|x|_p := \begin{cases} 
(\sum_{i=1}^d |x_i|^p)^{1/p} & \text{if } p \in [1, \infty), \\
\max_{k=1}^d |x_k| & \text{if } p = \infty.
\end{cases}
\]

The \( d \)-dimensional torus \( \mathbb{T}^d := \mathbb{R}^d / \mathbb{Z}^d \) is a priori endowed with the periodic norm

\[
\|\xi\| := \left( \sum_{k=1}^d \|\xi_k\|^2 \right)^{1/2} \quad \text{for} \quad \xi = (\xi_1, \ldots, \xi_d) \in \mathbb{T}^d,
\]

where \( \|\xi_k\| = \text{dist}(\xi_k, \mathbb{T}) \) for all \( \xi_k \in \mathbb{T} \) and \( k \in [d] \). However, identifying \( \mathbb{T}^d \) with \([-1/2, 1/2]^d\), we see that the norm \( \| \cdot \| \) coincides with the Euclidean norm \( | \cdot | \) restricted to \([-1/2, 1/2]^d\).

The unit sphere in \( \mathbb{R}^d \) induced by the norm \( | \cdot |_p \) is defined by

\[
S_{d-1}^p := \{ x \in \mathbb{R}^d : |x|_p = 1 \}.
\]

If \( p = 2 \) we shall abbreviate \( S_{d-1}^p \) to \( S^{d-1} \), which is the standard Euclidean sphere.

### 2.3. The \( L^p \) spaces.

The triple \((X, \mathcal{B}(X), \mu)\) is a measure space \( X \) with \( \sigma \)-algebra \( \mathcal{B}(X) \) and \( \sigma \)-finite measure \( \mu \). The space of all \( \mu \)-measurable complex-valued functions defined on \( X \) will be denoted by \( L^0(X) \). The space of all functions in \( L^0(X) \) whose modulus is integrable with \( p \)-th power is denoted by \( L^p(X) \) for \( p \in (0, \infty) \), whereas \( L^\infty(X) \) denotes the space of all essentially bounded functions in \( L^0(X) \). In our case we will usually have \( X = \mathbb{R}^d \) or \( X = \mathbb{T}^d \) equipped with the Lebesgue measure, and \( X = \mathbb{Z}^d \) endowed with the counting measure. If \( X \) is endowed with counting measure we will abbreviate \( L^p(X) \) to \( \ell^p(X) \).

Let \( X \) be a locally compact Hausdorff space. Then \( C(X) \) denotes the space of all continuous functions on \( X \). \( C_c(X) \) denotes the space of all continuous and compactly supported functions on \( X \). \( C_0(X) \) denotes the space of all continuous functions on \( X \) that vanish at infinity. Finally, let \( U \subseteq \mathbb{R}^d \) be open, for any \( n \in \mathbb{Z}_+ \) let \( C^n(U) \) denote the space of all functions \( f \) on \( U \) whose partial derivatives of order \( \leq n \) all exist and are continuous. We also set \( C^\infty(U) := \bigcap_{n \in \mathbb{Z}_+} C^n(U) \) and \( C^\infty_c(\mathbb{R}^d) \) denotes the set of all compactly supported smooth functions on \( \mathbb{R}^d \). If \( U \subseteq \mathbb{R}^d \) it also make sense to consider the spaces \( C^n(U) \). In this case we say that \( f \in C^n(U) \) if \( f \in C^n(V) \) for some open \( V \supseteq U \).

The partial derivative of a function \( f : \mathbb{R}^d \to \mathbb{C} \) with respect to the \( j \)-th variable \( x_j \) will be denoted by \( \partial_{x_j} f = \partial_j f \), while the \( m \)-th partial derivative with respect to the \( j \)-th variable will be denoted by \( \partial_j^m f = \partial_j^{m_j} f \). The gradient of a function \( f : \mathbb{R}^d \to \mathbb{C} \) is the following vector \( \nabla f := (\partial_{x_1} f, \ldots, \partial_{x_d} f) := (\partial_1 f, \ldots, \partial_d f) \). If \( \alpha \in \mathbb{N}^d \) is a multi-index \( |\alpha| := \alpha_1 + \ldots + \alpha_d \) denotes its size, and it will be always clear from the context that \( |\alpha| \) is not the Euclidean norm of \( \alpha \in \mathbb{N}^d \). We shall also write \( \alpha! := \alpha_1! \ldots \alpha_d! \) and \( (\alpha)^{\beta} := \prod_{j=1}^d (\alpha_j - \beta_j)! \) for all multi-indices \( \alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d \) and \( \beta = (\beta_1, \ldots, \beta_d) \in \mathbb{N}^d \) such that \( \alpha \geq \beta \), where the last relation means that \( \alpha_j \geq \beta_j \) for all \( j \in [d] \). For any \( \alpha \in \mathbb{N}^d \) let \( \partial^\alpha f \) denote the derivative \( \partial_{x_1}^{\alpha_1} \ldots \partial_{x_d}^{\alpha_d} f = \partial_1^{\alpha_1} \ldots \partial_d^{\alpha_d} f \) operator of total order \( |\alpha| \). If \( f : \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \to \mathbb{C} \) for some \( d_1, d_2 \in \mathbb{Z}_+ \), and let \((x, y) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \) then \( \nabla f(x, y) \) denotes the gradient of \( f \) with respect to the variable \( x \in \mathbb{R}^{d_1} \) and we write \( \nabla_x f(x, y) := (\partial_{x_1} f(x, y), \ldots, \partial_{x_{d_1}} f(x, y)) \) and \( \partial_x^2 f(x, y) \) denote the partial differential operator of order \( \alpha \in \mathbb{N}^{d_1} \) with respect to the
variable \( x \in \mathbb{R}^d \) and we write \( \partial_t^n f(x,y) := \partial_{x_1}^{a_1} \ldots \partial_{x_d}^{a_d} f(x,y) \). We have analogous definitions for the variable \( y \in \mathbb{R}^2 \).

2.4. \textit{r-variations.} For any family \((a_t : t \in \mathbb{I})\) of elements of \( \mathbb{C} \) indexed by a totally ordered set \( \mathbb{I} \), and any exponent \( 1 \leq r < \infty \), the \( r \)-variation seminorm is defined to be

\[
V^r(a_t : t \in \mathbb{I}) := \sup_{J \subseteq \mathbb{N}} \sup_{t_1, t_2 \in \mathbb{I}} \left( \sum_{j=0}^{J-1} |a(t_{j+1}) - a(t_j)|^r \right)^{1/r},
\]

where the supremum is taken over all finite increasing sequences in \( \mathbb{I} \).

It is easy to see that for every \( t_0 \in \mathbb{I} \) one has

\[
\sup_{t \in \mathbb{I}} |a_t| \leq |a_{t_0}| + V^r(a_t : t \in \mathbb{I}).
\]

We will usually use \( r \)-variations with \( \mathbb{I} = \mathbb{Z}_+ \) or \( \mathbb{I} = (0, \infty) \) or \( \mathbb{I} = \mathbb{D} \), where \( \mathbb{D} \) is \( \lambda_0 \)-lacunary subset of \((0, \infty)\). Then if \( V^r(a_t : t \in \mathbb{I}) < \infty \) we see that the underlying sequence \((a_t : t \in \mathbb{I})\) is a Cauchy sequence and consequently in each case the limit \( \lim_{t \to \infty} a_t \) exists, and in the second case when \( \mathbb{I} = (0, \infty) \) the limit \( \lim_{t \to 0} a_t \) also exists.

2.5. \textit{Fourier transform.} We will write \( e(z) := e^{2\pi iz} \) for every \( z \in \mathbb{C} \). The Fourier transform and the inverse Fourier transform of \( f \in L^1(\mathbb{R}^d) \) will be denoted respectively by

\[
\mathcal{F}_{\mathbb{R}^d} f(\xi) := \int_{\mathbb{R}^d} f(x) e(-x \cdot \xi) \, dx, \quad \xi \in \mathbb{R}^d,
\]

\[
\mathcal{F}_{\mathbb{R}^d}^{-1} f(x) := \int_{\mathbb{R}^d} f(\xi) e(x \cdot \xi) \, d\xi, \quad x \in \mathbb{R}^d.
\]

The Fourier coefficient of \( f \in L^1(\mathbb{T}^d) \), and the Fourier series of \( g \in \ell^1(\mathbb{Z}^d) \) will be denoted respectively by

\[
\mathcal{F}_{\mathbb{T}^d} f(n) := \int_{\mathbb{T}^d} f(x) e(-n \cdot \xi) \, d\xi, \quad n \in \mathbb{Z}^d,
\]

\[
\mathcal{F}_{\mathbb{Z}^d}^{-1} g(\xi) := \sum_{n \in \mathbb{Z}^d} g(n) e(n \cdot \xi), \quad \xi \in \mathbb{T}^d.
\]

3. \textbf{Proof of Theorem 1.2} \textbf{Asymptotic formula for} \( r_{h_1,h_2,h_3}(\lambda) \)

We begin with the observation that the asymptotic behaviour of \( r_{h_1,h_2,h_3}(\lambda) \) is not affected by removing the restriction that all coordinates of a given triple are larger than \( N_0 \).

\textbf{Remark 3.1.} We note that the values of functions \( h_1, h_2, h_3 \) for \( n \in [N_0 - 1] \) do not change the asymptotic formula of \( r_{h_1,h_2,h_3}(\lambda) \) obtained in Theorem 1.2. More precisely, assume for any \( k \in [3] \) that \( h_k \in \mathcal{F}_{c_k} \), and they take arbitrary values for \( n \in [N_0 - 1] \). We define

\[
\tilde{r}_{h_1,h_2,h_3}(\lambda) := \# \{(m_1, m_2, m_3) \in \mathbb{Z}_+^3 : [h_1(m_1)] + [h_2(m_2)] + [h_3(m_3)] = \lambda \}.
\]

Then one can easily show that

\[
0 \leq \tilde{r}_{h_1,h_2,h_3}(\lambda) - r_{h_1,h_2,h_3}(\lambda) = o \left( \lambda^2 \varphi_1'(\lambda) \varphi_2(\lambda) \varphi_3(\lambda) \right).
\]
3.1. Properties of $h$ and $\varphi$. In this subsection we gather some properties of functions $h \in F_c$ and their inverses, which will be used throughout the paper. We pick a large number $N_0 \in \mathbb{Z}_+$ such that for every $x \geq N_0$ we have

$$h'(x) \geq 1.$$  \hfill (3.1)

Then for every $n \geq h(N_0)$, as in [60] Lemma 2.12, p. 624, one has

$$[-\varphi(n)] - [-\varphi(n + 1)] = \mathbb{1}_{N_h}(n),$$  \hfill (3.2)

where $\varphi$ is the inverse of $h$. The next lemma shows that the function $\varphi$ is $1/c$-regular.

**Lemma 3.2.** Assume that $c \in [1, 2)$, $h \in F_c$, $\gamma = 1/c$ and let $\varphi : [h(x_0), \infty) \to [x_0, \infty)$ be the inverse of $h$. Then

$$\varphi(x) := x^\gamma \ell_\varphi(x), \quad \text{where} \quad \ell_\varphi(x) := \exp\left(\int_{h(x_0)}^x \frac{\theta(t)}{t} dt + D\right),$$  \hfill (3.3)

for every $x \geq h(x_0)$, where $D = \log (x_0/h(x_0)^\gamma)$ and

$$\theta(x) := \frac{1}{(c + \vartheta(\varphi(x)))} - \gamma = -\frac{\vartheta(\varphi(x))}{c(1 + \vartheta(\varphi(x)))},$$

with $\vartheta$ as in Definition [L.1(ii)], and

$$\lim_{x \to \infty} \theta(x) = 0.$$  \hfill (3.4)

If $L(x) = \ell_h(x)$ or $L(x) = \ell_\varphi(x)$, then for every $\varepsilon > 0$ we have

$$\lim_{x \to \infty} x^{-\varepsilon} L(x) = 0, \quad \text{and} \quad \lim_{x \to \infty} x^\varepsilon L(x) = \infty,$$  \hfill (3.5)

and consequently, for every $\varepsilon > 0$

$$x^{1-\varepsilon} \lesssim_\varepsilon \varphi(x) \quad \text{and} \quad \lim_{x \to \infty} \frac{\varphi(x)}{x} = 0.$$  \hfill (3.6)

Furthermore, $x \mapsto x \varphi(x)^{-\delta}$ is increasing for every $\delta < c$, (if $c = 1$, even $\delta \leq 1$ is allowed) and for every $x \geq h(x_0)$ we have

$$\varphi(x) \simeq \varphi(2x) \quad \text{and} \quad \varphi'(x) \simeq \varphi'(2x).$$  \hfill (3.7)

*Proof.* The proof of Lemma 3.2 can be found in [60] Lemma 2.6, p. 623, we also refer to [61] as well as to [62]. □

The properties of the derivatives of function $\varphi$ are gathered in the lemma below.

**Lemma 3.3.** Assume that $c \in [1, 2)$, $h \in F_c$, $\gamma = 1/c$ and let $\varphi : [h(x_0), \infty) \to [x_0, \infty)$ be its inverse. Then for every $n \in [3]$ there exists a function $\theta_n : [h(x_0), \infty) \to \mathbb{R}$ such that

$$\lim_{x \to \infty} \theta_n(x) = 0$$

and

$$x \varphi^{(n)}(x) = \varphi^{(n-1)}(x)(\beta_n + \theta_n(x)), \quad \text{for every} \quad x \geq h(x_0),$$  \hfill (3.8)

where $\beta_n = \gamma - n + 1$, $\theta_1(x) = \theta(x)$ and

$$\theta_i(x) = \theta_{i-1}(x) + \frac{x \varphi'(x)}{\beta_{i-1} + \theta_{i-1}(x)}, \quad i = 2, 3.$$

If $c = 1$, then there exist a positive function $\sigma : [h(x_0), \infty) \to (0, \infty)$ and a function $\tau : [h(x_0), \infty) \to \mathbb{R}$ such that [3.8] with $n = 2$ reduces to

$$x \varphi''(x) = \varphi'(x)\sigma(x)\tau(x), \quad \text{for every} \quad x \geq h(x_0).$$  \hfill (3.9)
Moreover, \( \sigma(x) \) is decreasing, \( \lim_{x \to \infty} \sigma(x) = 0 \), \( \sigma(2x) \simeq \sigma(x) \), and \( \sigma(x)^{-1} \lesssim x^\varepsilon \), for every \( \varepsilon > 0 \). Finally, there are constants \( 0 < c_3 \leq c_4 \) such that and \( c_3 \leq -\tau(x) \leq c_4 \) for every \( x \geq h(x_0) \).

**Proof.** The proof of Lemma 3.3 can be found in [60] Lemma 2.14, p. 625, we also refer to [61] as well as to [62]. \( \square \)

We will also need the following result.

**Lemma 3.4.** For \( k \in \mathbb{Z} \), let \( c_k \in [1, 2) \), \( \gamma_k = 1/c_k \), \( h_k \in \mathcal{F}_{c_k} \), and let \( \varphi_k \) be their inverses, respectively. Further, let \( L_\varphi(x) := \varphi'(x)/x^{\gamma-1} \). Then

\[
\lim_{\lambda \to \infty} \int_{N_0/\lambda}^{1-N_0/\lambda} x^{\gamma_1-1}(1-x)^{\gamma_2-1} \frac{L_{\varphi_1}(\lambda x)}{L_{\varphi_1}(\lambda)} \frac{L_{\varphi_2}(\lambda(1-x))}{L_{\varphi_2}(\lambda)} \, dx = \frac{\Gamma(\gamma_1)\Gamma(\gamma_2)}{\Gamma(\gamma_1 + \gamma_2)}.
\]

**Proof.** By the definition of the Beta function it suffices to show

\[
\lim_{\lambda \to \infty} \int_{N_0/\lambda}^{1-N_0/\lambda} x^{\gamma_1-1}(1-x)^{\gamma_2-1} \frac{L_{\varphi_1}(\lambda x)}{L_{\varphi_1}(\lambda)} \frac{L_{\varphi_2}(\lambda(1-x))}{L_{\varphi_2}(\lambda)} \, dx = \int_0^1 x^{\gamma_1-1}(1-x)^{\gamma_2-1} \, dx.
\]

For this purpose we will use the dominated convergence theorem, which reduces the matter to proving, with \( \varphi = \varphi_1 \) or \( \varphi = \varphi_2 \), that

\[
\lim_{\lambda \to \infty} \frac{L_{\varphi}(\lambda x)}{L_{\varphi}(\lambda)} = 1, \quad x \in (0, 1),
\]

and

\[
\frac{L_{\varphi}(\lambda x)}{L_{\varphi}(\lambda)} \lesssim x^{-\gamma/100}, \quad x \geq N_0/\lambda.
\]

We first deal with (3.10). Using (3.8) and (3.3) we conclude

\[
\varphi'(x) = \frac{\varphi(x)}{x} (\gamma \theta_1(x)) = x^{-\gamma_1} \ell_\varphi(x)(\gamma + \theta(x)),
\]

which yields \( L_\varphi(x) = \ell_\varphi(x)(\gamma + \theta(x)) \). Applying (3.3) once again we infer that

\[
\frac{L_{\varphi}(\lambda x)}{L_{\varphi}(\lambda)} = \exp \left( -\int_{\lambda x}^{\lambda} \frac{\theta(t)}{t} \right) \frac{\gamma + \theta(\lambda x)}{\gamma + \theta(\lambda)},
\]

for sufficiently large \( \lambda \in \mathbb{Z}_+ \). By (3.4) the last fraction converges to 1 as \( \lambda \to \infty \). Therefore, to verify (3.10) we observe

\[
\left| \int_{\lambda x}^{\lambda} \frac{\theta(t)}{t} \right| dt \leq \int_{\lambda x}^{\lambda} \frac{|\theta(t)|}{t} dt \leq \max_{t \in [\lambda x, \lambda]} \frac{|\theta(t)| \log(1/x)}{\lambda} \xrightarrow[\lambda \to \infty]{x} 0.
\]

We now prove (3.11). Using again (3.4) (notice that \( \lambda \geq \lambda x \geq N_0 \)) we see

\[
\frac{\gamma + \theta(\lambda x)}{\gamma + \theta(\lambda)} < \frac{101 \gamma/100}{99 \gamma/100} = 101/99, \quad x \geq N_0/\lambda.
\]

Applying (3.13) and (3.4) we obtain

\[
\exp \left( -\int_{\lambda x}^{\lambda} \frac{\theta(t)}{t} dt \right) \leq x^{-\gamma/100}, \quad x \geq N_0/\lambda.
\]

These estimates together with (3.12) lead to (3.11). The proof of Lemma 3.4 is finished. \( \square \)
3.2. Some asymptotic formulas. For \( k \in [3] \), let \( c_k \in [1, 2) \), \( h_k \in F_{c_k} \) and \( \varphi_k \) be its inverse. We will describe an asymptotic behavior of the following function

\[
J_{\varphi'_1, \varphi'_2, \varphi'_3}(\lambda) := \sum_{n_1 + n_2 + n_3 = \lambda \atop n_1, n_2, n_3 \geq N_0} \varphi'_1(n_1) \varphi'_2(n_2) \varphi'_3(n_3), \quad \lambda \in \mathbb{Z}_+,
\]

We begin with a simpler object

\[
J_{\varphi'_1, \varphi'_2}(\lambda) := \sum_{m = N_0}^{\lambda - N_0} \varphi'_1(m) \varphi'_2(\lambda - m), \quad \lambda \in \mathbb{Z}_+,
\]

and we prove the following result.

**Lemma 3.5.** For \( k \in [2] \), let \( c_k \in [1, 2) \), \( h_k \in F_{c_k} \) and \( \varphi_k \) be its inverse. Then

\[
J_{\varphi'_1, \varphi'_2}(\lambda) = \frac{\Gamma(\gamma_1) \Gamma(\gamma_2)}{\Gamma(\gamma_1 + \gamma_2)} \lambda \varphi'_1(\lambda) \varphi'_2(\lambda) + o(\lambda \varphi'_1(\lambda) \varphi'_2(\lambda)).
\]

**Proof.** We first show for sufficiently large \( \lambda \in \mathbb{Z}_+ \) that

\[
\left| \sum_{m = N_0}^{\lambda - N_0} \varphi'_1(m) \varphi'_2(\lambda - m) - \int_{N_0}^{\lambda - N_0} \varphi'_1(x) \varphi'_2(\lambda - x) \, dx \right| \lesssim \lambda^{3/4} \varphi'_1(\lambda) \varphi'_2(\lambda). \tag{3.14}
\]

Using (3.8) and (3.6) we may write \( \varphi'_1(\lambda - N_0) \varphi'_2(N_0) \lesssim \varphi'_1(\lambda) \lesssim \lambda^{3/4} \varphi'_1(\lambda) \varphi'_2(\lambda) \). Thus (3.14) will follow if we show that

\[
\sum_{m = N_0}^{\lambda - N_0 - 1} \int_{m}^{m+1} \left| \varphi'_1(m) \varphi'_2(\lambda - m) - \varphi'_1(x) \varphi'_2(\lambda - x) \right| \, dx \lesssim \lambda^{3/4} \varphi'_1(\lambda) \varphi'_2(\lambda). \tag{3.15}
\]

Let \( \psi_\lambda(x) = \varphi'_1(x) \varphi'_2(\lambda - x), \; N_0 \leq x \leq \lambda - N_0 \). Using (3.8) with \( n = 2 \) we have

\[
\psi'_\lambda(x) = \varphi'_1(x) \varphi'_2(\lambda - x) \left( \frac{\gamma_1 - 1 + \theta_{1,2}(x)}{x} - \frac{\gamma_2 - 1 + \theta_{2,2}(\lambda - x)}{\lambda - x} \right),
\]

where \( \theta_{k,2} \) is a \( \theta \) function from Lemma 3.3 corresponding to \( \varphi_k \) for \( k \in [2] \). This together with (3.7) shows that

\[
|\psi_\lambda(m) - \psi_\lambda(x)| \lesssim \varphi'_1(x) \varphi'_2(\lambda - x) \left( \frac{1}{x} + \frac{1}{\lambda - x} \right), \quad x \in (m, m+1).
\]

Using (3.7) again, we see that the left-hand side of (3.15) is controlled by

\[
\int_{N_0}^{\lambda - N_0} \varphi'_1(x) \varphi'_2(\lambda - x) \left( \frac{1}{x} + \frac{1}{\lambda - x} \right) \, dx
\]

\[
\lesssim \int_{N_0}^{\lambda/2} \frac{\varphi'_1(N_0) \varphi'_2(\lambda)}{x} \, dx + \int_{\lambda/2}^{\lambda - N_0} \frac{\varphi'_1(\lambda) \varphi'_2(N_0)}{\lambda - x} \, dx
\]

\[
\lesssim \varphi'_2(\lambda) \log \lambda + \varphi'_1(\lambda) \log \lambda
\]

\[
\lesssim \lambda^{3/4} \varphi'_1(\lambda) \varphi'_2(\lambda),
\]

where in the last estimate we have used Lemma 3.3 and (3.6). We are reduced to estimate the integral instead of the sum. Changing the variables, using the identity \( \varphi'_k(x) = x^{\gamma_k - 1} L_{\varphi_k}(x) \) for \( k \in [2] \), and applying Lemma 3.4 we conclude

\[
\int_{N_0}^{\lambda - N_0} \varphi'_1(x) \varphi'_2(\lambda - x) \, dx
\]
Corollary 3.7.

For \( \log \) replaced in the statement of this result by \( \gamma \)
the details.

Remark 3.6.

Lemma 3.5 is optimal in a sense that the error term \( \square \)
The proof of Lemma 3.5 is finished.

Proposition 3.8.

Proof. Repeating the arguments used in the proof of Lemma 3.5 the cla
aim follows. We omit

The proof will be in some sense inductive. Observe that

More precisely, this would imply that there exist \( s \)
3.4 below. More precisely, this implies that \( \sqrt{1-c} \)

Using Lemma 3.5 we will describe an asymptotic behavior of

\[ J_{\varphi'_{1}, \varphi'_{2}}(\lambda) = \frac{\Gamma(\gamma_{1})\Gamma(\gamma_{2})}{\Gamma(\gamma_{1} + \gamma_{2})} \lambda^{\gamma_{1} + \gamma_{2} - 1} + O(\lambda^{\gamma_{1} + \gamma_{2} - 1 - \gamma_{1} \wedge \gamma_{2})]. \]

Proof. Repeating the arguments used in the proof of Lemma 3.5 the claim follows. We omit

Using Lemma 3.5 we will describe an asymptotic behavior of \(J_{\varphi'_{1}, \varphi'_{2}}(\lambda)\).

Proposition 3.8. For \( k \in [3] \), let \( c_{k} \in [1, 2] \), \( \gamma_{k} = 1/c_{k} \) and \( \varphi_{k}(x) = x^{\gamma_{k}} \), then we have

\[ J_{\varphi'_{1}, \varphi'_{2}}(\lambda) = \frac{\Gamma(\gamma_{1})\Gamma(\gamma_{2})}{\Gamma(\gamma_{1} + \gamma_{2})} \gamma_{1}\gamma_{2}\lambda^{\gamma_{1} + \gamma_{2} - 1} + O(\lambda^{\gamma_{1} + \gamma_{2} - 1 - \gamma_{1} \wedge \gamma_{2})]. \]

Proof. The proof will be in some sense inductive. Observe that

\[ J_{\varphi'_{1}, \varphi'_{2}}(\lambda) = \sum_{n_{1}=N_{0}}^{\lambda-N_{0}} \varphi'_{1}(n_{1}) \sum_{n_{2}+n_{3}=\lambda-n_{1}}^{\lambda-N_{0}} \varphi'_{2}(n_{2})\varphi'_{3}(n_{3}) = \sum_{n_{1}=N_{0}}^{\lambda-N_{0}} \varphi'_{1}(n_{1})J_{\varphi'_{2}, \varphi'_{3}}(\lambda - n_{1}). \]

Let \( \psi(x) := x\varphi'_{2}(x)\varphi'_{3}(x) \). Using Lemma 3.5 we see that

\[ J_{\varphi'_{1}, \varphi'_{2}}(\lambda) = \frac{\Gamma(\gamma_{2})\Gamma(\gamma_{3})}{\Gamma(\gamma_{2} + \gamma_{3})} \sum_{n_{1}=N_{0}}^{\lambda-N_{0}} \varphi'_{1}(n)\psi(\lambda-n) + \sum_{n=N_{0}}^{\lambda-N_{0}} \varphi'_{1}(n)h(\lambda-n), \quad (3.17) \]

where \( h \) is a function satisfying \( h(\lambda) = o(\psi(\lambda)) \).
We claim that
\[
\sum_{n=N_0}^{\lambda-N_0} \varphi_1'(n) \psi(\lambda - n) = \frac{\Gamma(\gamma_1)\Gamma(\gamma_2 + \gamma_3)}{\Gamma(\gamma_1 + \gamma_2 + \gamma_3)} \lambda \varphi_1'(\lambda) \psi(\lambda) + o(\lambda \varphi_1'(\lambda) \psi(\lambda)).
\] (3.18)

Assume for a moment that we can verify (3.18). Then applying this to the first term on the right-hand side of (3.17) it suffices to show that
\[
\sum_{n=N_0}^{\lambda-N_0} \varphi_1'(\lambda - n) h(n) = o(\lambda \varphi_1'(\lambda) \psi(\lambda)).
\] (3.19)

To prove (3.19) we split the summation in (3.19) into two parts \(\sum_{n=N_0}^{N_1} \) and \(\sum_{n=N_1+1}^{\lambda-N_0} \), where \(N_1 \in \mathbb{Z}_+\) is a number such that for all \(n > N_1\) the asymptotic \(h(n) = o(\psi(n))\) will get efficient. For \(n \in [N_0, N_1]\) we apply (3.8) (for the first derivative) together with (3.3) and (3.5) to conclude \(\varphi_1'(\lambda - n) h(n) = o(\lambda \varphi_1'(\lambda) \psi(\lambda))\), since \(h(n) = O(1)\) for \(n \in [N_0, N_1]\). Further, for the remaining \(n > N_1\) we use the fact that \(h(\lambda) = o(\psi(\lambda))\) and once again (3.18) to obtain (3.19). The proof of Proposition 3.8 will be completed if we verify (3.18).

To prove (3.18) we shall proceed much the same way as in the proof of Lemma 3.5. For large \(\lambda \in \mathbb{Z}_+\) we show that
\[
\left| \sum_{n=N_0}^{\lambda-N_0} \varphi_1'(n) \psi(\lambda - n) - \int_{N_0}^{\lambda-N_0} \varphi_1'(x) \psi(\lambda - x) \, dx \right| \lesssim \lambda^{3/4} \varphi_1'(\lambda) \psi(\lambda).
\]

Observe that \(\varphi_1'(\lambda - N_0) \psi(N_0) \lesssim \varphi_1'(\lambda) \lesssim \lambda^{3/4} \varphi_1'(\lambda) \psi(\lambda)\). It suffices to show that
\[
\sum_{n=N_0}^{\lambda-N_0-1} \int_n^{n+1} |g_\lambda(n) - g_\lambda(x)| \, dx \lesssim \lambda^{3/4} \varphi_1'(\lambda) \psi(\lambda),
\] (3.20)

where \(g_\lambda(x) := \varphi_1(x) \psi(\lambda - x)\), \(N_0 \leq x \leq \lambda - N_0\). Taking Lemma 3.3 into account we have
\[
g_\lambda'(x) = g_\lambda(x) \left( \frac{\gamma_1 - 1 + \theta_{1,2}(x)}{x} - \frac{\gamma_2 + \gamma_3 - 1 + \theta_{2,2}(\lambda - x) + \theta_{3,2}(\lambda - x)}{\lambda - x} \right),
\]

where \(\theta_{k,2}\) is a \(\theta_2\) function from Lemma 3.3 corresponding to \(\varphi_k\) for \(k \in [3]\). This shows that
\[
|g_\lambda'(x)| \lesssim \frac{g_\lambda(x)}{x \wedge (\lambda - x)}, \quad N_0 \leq x \leq \lambda - N_0.
\]

Therefore combining this with (3.7) we see that the left-hand side of (3.20) is dominated by
\[
\int_{N_0}^{\lambda-N_0} \frac{g_\lambda(x)}{x \wedge (\lambda - x)} \, dx \lesssim \int_{N_0}^{\lambda/2} \varphi_1'(N_0) \psi(\lambda) \, dx \int_{\lambda/2}^{\lambda-N_0} \varphi_1'(\lambda) \psi(N_0) \, dx \quad \int_{\lambda/2}^{\lambda-N_0} \varphi_1'(\lambda) \psi(N_0) \, dx \quad \int_{\lambda/2}^{\lambda-N_0} \varphi_1'(\lambda) \psi(N_0) \, dx
\]
\[
\lesssim \frac{\psi(\lambda) \log \lambda + \varphi_1'(\lambda) \log \lambda}{\lambda - x} \lesssim \lambda^{3/4} \varphi_1'(\lambda) \psi(\lambda),
\]

where in the last estimate we have used Lemma 3.3 and (3.6). To justify (3.18), it suffices to verify (3.18) with \(\int_{N_0}^{\lambda-N_0} g_\lambda(x) \, dx\) in place of \(\sum_{n=N_0}^{\lambda-N_0} \varphi_1'(n) \psi(\lambda - n)\).

Changing the variable \(x \mapsto \lambda x\) and using the identity \(\varphi_k'(x) = x^{\gamma_k-1} L_{\varphi_k}(x)\) for \(k \in [3]\), we infer that
\[
\int_{N_0}^{\lambda-N_0} g_\lambda(x) \, dx = \lambda^2 \varphi_1'(\lambda) \varphi_2'(\lambda) \varphi_3'(\lambda) \int_{N_0/\lambda}^{1-N_0/\lambda} x^{\gamma_1-1} (1 - x)^{\gamma_2+\gamma_3-1} L_{\varphi_1,\varphi_2,\varphi_3}(\lambda, x) \, dx,
\]
Proposition 3.12. Let $U$ be a subinterval of $[a,b]$ such that $\sum$ and (3.11) as in Lemma 3.4. The proof of Proposition 3.8 is complete. □

Proof. It suffices to proceed as in the proof of Proposition 3.8 and use Corollary 3.7 instead of Proposition 3.8 and use Corollary 3.7 instead of Lemma 3.5. The details are left to the reader. □

3.3. Exponential sum estimates. To prove Theorem 1.2, we need some auxiliary results. We will need the discrete version of the Van der Corput lemma.

Lemma 3.10 (Van der Corput, [47, Corollary 8.13]). Assume that $a, b \in \mathbb{R}$ are such that $b - a \geq 1$. Let $F \in C^2([a,b])$ be a real valued function and let $I$ be a subinterval of $[a,b]$ such that $|I| \geq 1$. If there exist $\eta > 0$ and $r \geq 1$ such that

$$
\eta \lesssim |F''(x)| \lesssim r\eta, \quad x \in I,
$$

then

$$
\left| \sum_{k \in I} e(F(k)) \right| \lesssim r|I|\eta^{1/2} + \eta^{-1/2}.
$$

We shall also need the following variant of the summation by parts formula.

Lemma 3.11 (Summation by parts, [65, Theorem A.4, p. 304]). Let $u(n)$ and $g(n)$ be arithmetic functions and let $x, y \in \mathbb{R}$ be such that $0 \leq y < x$. Then, for $g \in C^1([y,x])$ we have

$$
\sum_{y < n \leq x} u(n)g(n) = U(x)g(x) - \int_y^x U(z)g'(z) \, dz,
$$

where $U(z) = \sum_{\lfloor y \rfloor + 1 \leq n \leq \lfloor z \rfloor} u(n)$.

Let $c \in [1, 2)$, $h \in \mathcal{F}_c$ and let $\varphi$ be the inverse of $h$. Then, we define

$$
F_\lambda(t) := \sum_{N_0 \leq n \leq \lambda} \varphi'(n)e(nt), \quad t \in \mathbb{T}, \quad \lambda \geq N_0, \quad (3.21)
$$

$$
G_\lambda(t) := \sum_{n \in \mathbb{N}_c \cap [\lambda]} e(nt), \quad t \in \mathbb{T}, \quad \lambda \geq N_0. \quad (3.22)
$$

Proposition 3.12 will be essential in the proof of Theorem 1.2.

Proposition 3.12. Let $c \in [1, 4/3)$, $\gamma = 1/c$, $h \in \mathcal{F}_c$ and $\varphi$ be its inverse. Further, assume that $\chi > 0$ is such that $4(1 - \gamma) + 5\chi < 1$. Then,

$$
||F_\lambda - G_\lambda||_{L^\infty(\mathbb{T})} \lesssim \varphi(\lambda)\lambda^{-\chi}, \quad \lambda \geq N_0.
$$
Noting that $\chi < \gamma$ and using (3.5) we have
\[ 1 \lesssim \varphi(\lambda) \lambda^{-\chi}, \quad \lambda \geq N_0, \tag{3.23} \]
hence the summation in $F_N$ can be extended to all $n \in [\lambda]$ no matter how $\varphi$ is defined for $n \in [N_0 - 1]$ and Proposition 3.12 remains valid.

For $x \in \mathbb{R}$ define a sawtooth function by $\Phi(x) := \{x\} - 1/2$. Expanding $\Phi$ in a Fourier series, see [37, Section 2], we obtain
\[ \Phi(x) = \sum_{0 < |m| \leq M} \frac{1}{2\pi i m} e(mx) + O\left( \min\left\{ 1, \frac{1}{M \|x\|} \right\} \right), \tag{3.24} \]
for every $M \in \mathbb{Z}_+$, recall that $\|x\| = \min\{ |x - n| : n \in \mathbb{Z} \}$ is the distance of $x \in \mathbb{R}$ to the nearest integer. Moreover, for any $M \in \mathbb{Z}_+$ we have
\[ \min\left\{ 1, \frac{1}{M \|x\|} \right\} = \sum_{m \in \mathbb{Z}} b_m e(mx), \quad x \in \mathbb{R}, \tag{3.25} \]
where
\[ |b_m| \lesssim \min\left\{ 1 + \log M \frac{M}{M |m|^2} \right\}, \quad M \in \mathbb{Z}_+, \quad m \in \mathbb{Z}. \tag{3.26} \]
Note that $b_m$ depends also on $M$, but we are not going to emphasize this fact in the sequel.

**Proof of Proposition 3.12.** Using (3.2) and the identity $\lfloor x \rfloor = x - \{x\}$ we may write
\[ G_\lambda(t) = O(1) + \sum_{N_0 \leq n \leq \lambda} e(nt)(\lfloor -\varphi(n) \rfloor - \lfloor -\varphi(n + 1) \rfloor) \]
\[ = O(1) + \sum_{N_0 \leq n \leq \lambda} e(nt)(\varphi(n + 1) - \varphi(n)) \]
\[ + \sum_{N_0 \leq n \leq \lambda} e(nt)(\Phi(-\varphi(n + 1)) - \Phi(-\varphi(n))). \]
Further, we have
\[ G_\lambda(t) = O(1) + F_\lambda(t) + I_1 + I_2, \]
where
\[ I_1 := \sum_{N_0 \leq n \leq \lambda} e(nt)(\varphi(n + 1) - \varphi(n) - \varphi'(n)), \]
\[ I_2 := \sum_{N_0 \leq n \leq \lambda} e(nt)(\Phi(-\varphi(n + 1)) - \Phi(-\varphi(n))). \]
Now, it suffices to show that
\[ |I_1| + |I_2| \lesssim \varphi(\lambda) \lambda^{-\chi}, \quad \lambda \geq N_0, \quad t \in \mathbb{T}. \]
We first deal with $I_1$. Using the fact that
\[ -\varphi''(2x) \simeq -\varphi''(x), \quad x \geq N_0, \tag{3.27} \]
which easily follows from (3.7) and Lemma 3.3 we obtain
\[ |\varphi(n + 1) - \varphi(n) - \varphi'(n)| = \left| \int_n^{n+1} \int_n^x \varphi''(y) dy \ dx \right|. \]
where

\[ \sigma(t_m) = 1, \quad \epsilon > 0 \text{ is fixed and such that } 4(1 - \gamma) + 5\chi' + 7\epsilon < 1. \]

Now applying (3.24) with \( M := \frac{p^{1+ \chi' + \epsilon}}{\varphi(P)\sigma(P)} \) where \( \epsilon > 0 \) is fixed and such that \( 4(1 - \gamma) + 5\chi' + 7\epsilon < 1 \). Now applying (3.24) with \( M \) as above we infer that

\[ S_{P, P'}(t) \lesssim S_1 + S_2, \quad N_0 \leq P < P' \leq 2P, \quad t \in \mathbb{T}. \]

where

\[ S_1 := \left| \sum_{P \leq n < P'} e(nt) \sum_{0 < |m| \leq M} \frac{\varphi(n) + 1}{2\pi im} \right|, \]

\[ S_2 := \sum_{P \leq n < P'} \min \left\{ 1, \frac{1}{M\|\varphi(n)\|} \right\}. \]

In order to show \( 3.28 \) it is enough to prove that

\[ S_1 + S_2 \lesssim \varphi(P)P^{-\chi'}, \quad N_0 \leq P < P' \leq 2P, \quad t \in \mathbb{T}. \]

We first analyze \( S_2 \). By \( 3.25 \) we may write

\[ S_2 = \sum_{P \leq n \leq P', m \in \mathbb{Z}} b_m e(m\varphi(n)) \leq \sum_{m \in \mathbb{Z}} |b_m| \sum_{P \leq n \leq P'} e(m\varphi(n)). \]  

Let us denote

\[ T_{P, P'}(t, m) := \left| \sum_{P \leq n \leq P'} e(nt + m\varphi(n)) \right|, \quad N_0 \leq P \leq P' \leq 2P, \quad t \in \mathbb{T}, \quad m \in \mathbb{Z}, \]

and observe that \( T_{P, P'}(0, m) \) is exactly the inner exponential sum in \( 3.29 \). We shall prove

\[ T_{P, P'}(t, m) \lesssim \begin{cases} \frac{|m|^{1/2} P}{(\varphi(P)\sigma(P))^{1/2}}, & m \neq 0, \\ P, & m = 0, \end{cases} \quad N_0 \leq P \leq P' \leq 2P, \quad t \in \mathbb{T}. \]

Using this estimate we conclude

\[ |I_1| \lesssim \sum_{N_0 \leq n \leq N \leq \lambda} (\varphi'(n) - \varphi'(n + 1)) = \varphi'(N_0) - \varphi'(\lambda + 1) \lesssim 1, \]

which in view of \( 3.23 \) gives us the desired estimate for \( I_1 \).

Next, we focus on \( I_2 \). Decomposing the sum defining \( I_2 \) into dyadic pieces we see that

\[ I_2 = \sum_{0 \leq l \leq \log_2 \lambda} \sum_{2^l \leq n \leq 2^{l+1}} e(nt) \left( \Phi(-\varphi(n + 1)) - \Phi(-\varphi(n)) \right). \]

For \( 1 \leq P < P' \leq 2P \) and \( t \in \mathbb{T} \), define the auxiliary functions

\[ S_{P, P'}(t) := \left| \sum_{N_0 \vee P \leq n < P'} e(nt) \left( \Phi(-\varphi(n + 1)) - \Phi(-\varphi(n)) \right) \right|. \]

Let \( \sigma \) be the function from \( 3.3 \) if \( c = 1 \), or \( \sigma \equiv 1 \) if \( c > 1 \). Further, let

\[ M := \frac{p^{1+ \chi' + \epsilon}}{\varphi(P)\sigma(P)} \]

where \( \epsilon > 0 \) is fixed and such that \( 4(1 - \gamma) + 5\chi' + 7\epsilon < 1 \). Now applying \( 3.24 \) with \( M \) as above we infer that

\[ S_{P, P'}(t) \lesssim S_1 + S_2, \quad N_0 \leq P < P' \leq 2P, \quad t \in \mathbb{T}. \]
The case \( m = 0 \) is trivial so we focus on \( m \neq 0 \). In order to prove (3.30) we use Lemma 3.10. For \( x \in [P, 2P] \) let \( F(x) := tx + m\varphi(x) \). Using (3.27) and Lemma 3.3 we see that
\[
|F''(x)| = -|m|\varphi''(x) \simeq |m| \frac{\varphi(P)\sigma(P)}{P^2}, \quad x \in [P, 2P].
\]
Therefore an application of Lemma 3.10 with \( I = [P, P'] \subseteq [P, 2P] \), \( \eta = |m| \frac{\varphi(P)\sigma(P)}{P^2} \) and \( r = 1 \) leads us to
\[
T_{P,P'}(t, m) \lesssim P|m|^{1/2} \frac{(\varphi(P)\sigma(P))^{1/2}}{P} + |m|^{-1/2} \frac{P}{(\varphi(P)\sigma(P))^{1/2}} \lesssim |m|^{1/2} \frac{P}{(\varphi(P)\sigma(P))^{1/2}}.
\]
The last estimate follows from inequality \( \varphi(x)\sigma(x) \lesssim x \) for \( x \geq N_0 \), which is a simple consequence of (3.5) if \( c > 1 \) and (3.6) if \( c = 1 \). Therefore we have justified (3.30). Now combining this estimate with (3.20) we obtain
\[
S_2 \lesssim \frac{1 + \log M}{M} P + \sum_{0 < |m| \leq M} \frac{1 + \log M}{M} \frac{|m|^{1/2} P}{(\varphi(P)\sigma(P))^{1/2}} + \sum_{|m| \geq M+1} \frac{M}{|m|^2} \frac{|m|^{1/2} P}{(\varphi(P)\sigma(P))^{1/2}}.
\]
Since \( \sigma(x)^{-1} \lesssim x^\delta \) for every \( \delta > 0 \) (see Lemma 3.3), we have \( \log P \lesssim \log P \) and consequently
\[
S_2 \lesssim \frac{\log P + 1}{P^{x^\delta + \varepsilon}} \varphi(P)\sigma(P) + (\log P + 1) \frac{P^{(3+\chi'+\varepsilon)/2}}{\varphi(P)\sigma(P)}
\]
\[
= \varphi(P)P^{-\chi'} \left( \frac{\log P + 1}{P^{\varepsilon}} \varphi(P)\sigma(P) + (\log P + 1) \frac{P^{(3+3\chi'+\varepsilon)/2}}{\varphi(P)^2\sigma(P)} \right).
\]
The required bound for \( S_2 \) follows because \( \log P + 1 \lesssim P^\varepsilon \), \( \sigma(P) \lesssim 1 \) and \( (3+3\chi'+\varepsilon)/2 < 2\gamma \).

Now we analyze \( S_1 \). We see that
\[
S_1 \lesssim \sum_{0 < |m| \leq M} \frac{1}{|m|} \left| \sum_{P \leq n \leq P'} e(nt + m\varphi(n))(e(m\varphi(n+1) - \varphi(n)) - 1) \right|.
\]
Let \( u(n) = e(nt + m\varphi(n)) \) and \( g(z) = e(m\varphi(z+1) - \varphi(z)) - 1 \). Clearly \( g \in C^1([P - 1, 2P]) \) and using (3.7), Lemma 3.3 and (3.27) we obtain
\[
|g(z)| \lesssim |m|\varphi'(P) \simeq |m| \frac{\varphi(P)}{P}, \quad z \in [P - 1, 2P],
\]
\[
|g'(z)| \lesssim |m| |\varphi'(z+1) - \varphi'(z)| \simeq -|m|\varphi''(P) \simeq |m| \frac{\varphi(P)\sigma(P)}{P^2}, \quad z \in [P - 1, 2P].
\]
Further, using (3.30) we have
\[
|U(z)| = \left| \sum_{P \leq n \leq z} u(n) \right| = T_{P,\lfloor z \rfloor}(t, m) \lesssim |m|^{1/2} \frac{P}{(\varphi(P)\sigma(P))^{1/2}}, \quad z \in [P - 1, 2P].
\]
Therefore applying Lemma 3.11 to the inner summation in (3.31) and then using the above estimates we arrive at
\[
S_1 \lesssim \sum_{0 < |m| \leq M} \frac{1}{|m|} \left( |m|^{1/2} \frac{P|m|}{(\varphi(P)\sigma(P))^{1/2}} \frac{\varphi(P)}{P} + |m|^{1/2} \frac{P|m|}{(\varphi(P)\sigma(P))^{1/2}} \frac{\varphi(P)\sigma(P)}{P^2} \right)
\]
\[
= \sum_{0 < |m| \leq M} \frac{1}{|m|} \left( \frac{|m|^{1/2} \varphi(P)}{P} + \frac{|m|^{1/2} \varphi(P)\sigma(P)}{P^2} \right)
\]
\[
= \frac{1}{P} \sum_{0 < |m| \leq M} \frac{|m|^{1/2} \varphi(P)}{P} + \frac{1}{P^2} \sum_{0 < |m| \leq M} \frac{|m|^{1/2} \varphi(P)\sigma(P)}{P^2}.
\]
\[
\lesssim \sum_{0 < |m| \leq M} |m|^{1/2} \left( \frac{\varphi(P)}{\sigma(P)} \right)^{1/2} \lesssim M^{3/2} \left( \frac{\varphi(P)}{\sigma(P)} \right)^{1/2}.
\]

By \( \sigma(P)^{-1} \lesssim P^\epsilon \), we further obtain
\[
S_1 \lesssim \frac{P^{3+3\chi'+3\epsilon}/2}{\varphi(P)\sigma(P)^2} \lesssim \varphi(P)P^{-\chi'}P^{3+5\chi'+7\epsilon}/2 \varphi(P)^2
\]
and the desired estimate for \( S_1 \) follows since \((3+5\chi'+7\epsilon)/2 < 2\gamma\).

The proof of (3.28) is completed, we will show that |\( I_2 \)| \( \lesssim \varphi(\lambda)\lambda^{-\chi} \). Let \( \delta < (\chi' - \chi)/2 \). Using (3.28) (with \( P = 2^l, P' = 2^{l+1} \wedge (\lambda + 1) \)), (3.5) and (3.6) we conclude
\[
|I_2| \lesssim 1 + \sum_{\log_2 N_0 \leq l \leq \log_2 \lambda} \varphi(2^l)2^{-l\chi'} \lesssim \sum_{0 \leq l \leq \log_2 \lambda} 2^l(\gamma - \chi' + \delta) \lesssim \lambda^{\gamma - \chi' + \delta} \lesssim \varphi(\lambda)\lambda^{-\chi' + 2\delta} \lesssim \varphi(\lambda)\lambda^{-\chi}.
\]
This yields the required estimate for \( I_2 \) and completes the proof of Proposition 3.12. \( \square \)

### 3.4. Proof of Theorem 1.2

Here we will proceed as in the proof of [9] Theorem 1, p. 46] or [62, Theorem 1.6, p. 30]. For \( k \in [3] \), let \( F^k_\lambda \) and \( G^k_\lambda \) be the functions defined in (3.21) and (3.22) respectively, that correspond to \( h_k \). We first observe that
\[
r_{h_1,h_2,h_3}(\lambda) = \sum_{\substack{n_1,n_2,n_3 \leq \lambda \\text{odd}, \\lambda}} \int_0^1 e(t(n_1 + n_2 + n_3 - \lambda)) \, dt = \int_0^1 G^1_\lambda(t)G^2_\lambda(t)G^3_\lambda(t)e(-t\lambda) \, dt.
\]

Proceeding in a similar way we obtain
\[
\int_0^1 F^1_\lambda(t)F^2_\lambda(t)F^3_\lambda(t)e(-t\lambda) \, dt = \sum_{\substack{n_1 + n_2 + n_3 = \lambda \\text{odd}, \\lambda}} \varphi_1(n_1)\varphi_2(n_2)\varphi_3(n_3) = J_{\varphi_1,\varphi_2,\varphi_3}(\lambda).
\]

Therefore, using Proposition 3.8 and (3.8) we reduce our problem to checking that
\[
\int_0^1 |G^1_\lambda(t)G^2_\lambda(t)G^3_\lambda(t) - F^1_\lambda(t)F^2_\lambda(t)F^3_\lambda(t)| \, dt = o \left( \frac{\varphi_1(\lambda)\varphi_2(\lambda)\varphi_3(\lambda)}{\lambda} \right).
\]  

Using Hölder’s inequality we see that the left-hand side of (3.32) is controlled by
\[
||F^1_\lambda - G^1_\lambda||_{L^\infty(T)}||F^2_\lambda||_{L^2(T)}||F^3_\lambda||_{L^2(T)} + ||G^1_\lambda||_{L^2(T)}||F^2_\lambda - G^2_\lambda||_{L^\infty(T)}||F^3_\lambda||_{L^2(T)} + ||G^1_\lambda||_{L^2(T)}||G^2_\lambda||_{L^2(T)}||F^3_\lambda - G^3_\lambda||_{L^\infty(T)}.
\]

By the Plancherel theorem and the fact that \( \varphi_k(n) \lesssim 1 \) and \( \varphi'_k(n) \simeq \varphi_k(n+1) - \varphi_k(n) \) for \( n \geq N_0 \) and \( k \in [3] \) (see (3.7), we obtain
\[
||F^k_\lambda||^2_{L^2(T)} = \sum_{N_0 \leq n \leq \lambda} \varphi_k(n)^2 \lesssim \varphi_k(\lambda + 1) - \varphi_k(N_0) \lesssim \varphi_k(\lambda), \quad \lambda \geq N_0,
\]
and
\[
||G^k_\lambda||^2_{L^2(T)} = \sum_{n \in \mathbb{N}_0 \cap [\lambda]} 1 \lesssim \varphi_k(\lambda + 1) \simeq \varphi_k(\lambda), \quad \lambda \geq N_0.
\]
Let
\[ \chi_k := \sum_{j=1}^{3} (1 - \gamma_j) / 2 + \varepsilon, \quad k \in [3]. \]

Using the assumption \([1.11]\) we can choose \(\varepsilon > 0\) such that \(4(1 - \gamma_k) + 5\chi_k < 1\) for \(k \in [3]\).

By Proposition \(3.12\) we have
\[ \|F_\chi^k - G_\chi^k\|_{L^\infty(T)} \lesssim \varphi_k(\lambda)\lambda^{-\chi_k}, \quad \lambda \geq N_0. \]

Combining this with \((3.34)\) and \((3.35)\) we see that \((3.33)\) is bounded by
\[ \frac{\varphi_1(\lambda)\varphi_2(\lambda)\varphi_3(\lambda)}{\lambda^{1+\varepsilon/2}} \lesssim \frac{\varphi_1(\lambda)\varphi_2(\lambda)\varphi_3(\lambda)}{\lambda^{1+\varepsilon/2}} \]

The last estimate above is a straightforward consequence of \((3.6)\) and the fact that
\[ 1 - \chi_k + \varepsilon/2 < \sum_{j=1}^{3} \gamma_j/2, \quad k \in [3]. \]

The estimate \((3.32)\) follows and the proof of Theorem \(1.2\) is completed. \(\Box\)

We finally establish the asymptotic formula for the number of lattice points in the arithmetic spheres \(S_\lambda(x)\) from \((1.2)\).

**Proof of Corollary \(1.3\)** Let \(r_{c,Z_+}(\lambda) := \# \{ x \in \mathbb{Z}_+^3 : [x_1] + [x_2] + [x_3] = \lambda \} \) for any \(\lambda \in \mathbb{Z}_+\), and let \(\gamma = 1/c\). Since \(r_c(\lambda) = 2^3 r_{c,Z_+}(\lambda) + O(\lambda^\gamma)\), it is enough to prove that for every \(\varepsilon > 0\) one has
\[ r_{c,Z_+}(\lambda) = \frac{\gamma^3 \Gamma(\gamma)^3}{\Gamma(3\gamma)} \lambda^{3\gamma-1} + O_\varepsilon (\lambda^{3\gamma-1-(9\gamma-8c)/(5c)+\varepsilon}). \]

This, however, follows by a careful repetition of the arguments from the proof of Theorem \(1.2\).

We provide some details. Let \(F_N\) and \(G_N\) be the functions defined in \((3.21)\) and \((3.22)\) respectively, that correspond to \(h(x) = x^c\) and \(N_0 = 1\). Let \(\varphi(x) = x^\gamma\) and observe that
\[ r_{c,Z_+}(\lambda) = \int_0^1 G_\lambda(t)^3 e^{-2\pi i t \lambda} dt, \quad J_{\varphi,\varphi',\varphi'}(\lambda) = \int_0^1 F_\lambda(t)^3 e^{-2\pi i t \lambda} dt, \quad \lambda \in \mathbb{Z}_+. \]

We take sufficiently small \(\varepsilon > 0\) and apply Proposition \(3.12\) with \(\chi = 1-\gamma - \varepsilon > 0\) (notice that \(4(1 - \gamma) + 5\chi < 1\)) together with the estimate \(\|F_\lambda\|_{L^2(T)} + \|G_\lambda\|_{L^2(T)} \lesssim \lambda^\gamma\) for \(\lambda \in \mathbb{Z}_+\), and conclude that
\[ |r_{c,Z_+}(\lambda) - J_{\varphi,\varphi',\varphi'}(\lambda)| \lesssim \varepsilon \lambda^{3\gamma-1-(9\gamma-8c)/(5c)+\varepsilon}, \quad \lambda \in \mathbb{Z}_+. \]

Now the claim immediately follows from Corollary \(3.9\) \(\Box\)
4. Proof of Theorem 4.6 \( \ell^p(\mathbb{Z}^3) \) bounds for maximal functions

4.1. Preliminary reductions. Throughout this section \( c \in (1,11/10) \) is fixed and we define

\[
\kappa := \kappa_c := \frac{3 - 4c}{4c} = \frac{3}{4c} - 1. \tag{4.1}
\]

We also fix a \( \lambda_0 \)-lacunary set \( \mathbb{D} := \{ \lambda_n : n \in \mathbb{Z}_+ \} \subset \mathbb{Z}_+ \) with \( \lambda_0 = \inf_{n \in \mathbb{Z}_+} \frac{\lambda_{n+1}}{\lambda_n} > 1 \).

Let \( \eta \in C_c^\infty(\mathbb{R}^3) \) be such that \( 0 \leq \eta(x) \leq 1 \) for all \( x \in \mathbb{R}^3 \), \( \text{supp} \eta \subseteq [-10,10]^3 \) and \( \eta(x) = 1 \) whenever \( |x|^3_c \leq 4 \). We also assume that \( \eta \) is of product type, i.e. \( \eta(x) := \prod_{j=1}^3 \eta_j(x_j) \) for some even functions \( \eta_j \in C_c^\infty(\mathbb{R}) \) with \( j \in [3] \). We employ the ideas from the circle method and for every \( \lambda \in \mathbb{Z}_+ \) we split the unit interval \([-1/2,1/2]\) into major \( \mathcal{M}_\lambda \) and minor \( m_\lambda \) arcs, which are defined respectively by

\[
\mathcal{M}_\lambda := \left( -\frac{\lambda^\kappa}{2c} \frac{\lambda^\kappa}{2c}, \right), \\
m_\lambda := [-1/2,1/2] \setminus \mathcal{M}_\lambda.
\]

Take a smooth partition of unity \( \psi(t) + \tilde{\psi}(t) = 1 \) for any \( t \in \mathbb{R} \), where \( \psi \) is supported in \((-1/2,1/2)\) and \( \psi(t) = 1 \) for \( t \in (-1/(2c),1/(2c)) \). We consider

\[
\psi_\lambda(t) = \psi\left( \frac{t}{\lambda^\kappa} \right) , \quad \tilde{\psi}_\lambda(t) = \tilde{\psi}\left( \frac{t}{\lambda^\kappa} \right) , \quad t \in \mathbb{R} , \quad \lambda \geq 1.
\]

It is easy to see that \( \tilde{\psi}_\lambda(t) \leq 1_{m_\lambda}(t) \) for any \( t \in [-1/2,1/2] \). Simple computations show that

\[
\sigma_\lambda(x) = \eta\left( \frac{x}{\lambda^{1/c}} \right) \int_{-1/2}^{1/2} e((Q(x) - \lambda)t) (\psi_\lambda(t) + \tilde{\psi}_\lambda(t)) dt = \sigma^m_\lambda(x) + \sigma^n_\lambda(x), \tag{4.2}
\]

where

\[
\sigma^m_\lambda(x) := \lambda^\kappa \eta\left( \frac{x}{\lambda^{1/c}} \right) \mathcal{F}^{-1}_{\mathbb{R}} \psi\left( \lambda^\kappa (Q(x) - \lambda) \right) , \tag{4.3}
\]

\[
\sigma^n_\lambda(x) := \eta\left( \frac{x}{\lambda^{1/c}} \right) \int_{-1/2}^{1/2} e((Q(x) - \lambda)t) \tilde{\psi}_\lambda(t) dt \tag{4.4}
\]

and

\[
Q(x) = [\|x_1\|^3_c] + [\|x_2\|^3_c] + [\|x_3\|^3_c] , \quad x \in \mathbb{R}^3.
\]

Consequently we obtain

\[
\sup_{\lambda \in \mathbb{Z}_+} \frac{1}{r_\lambda(\lambda)} |\sigma_\lambda * f(x)| \leq \sup_{\lambda \in \mathbb{Z}_+} \frac{1}{\lambda^{3/c-1}} |\sigma^m_\lambda * f(x)| + \sum_{n=0}^{\infty} \sup_{\lambda \leq 2^{n+1}} \frac{1}{\lambda^{3/c-1}} |\sigma^n_\lambda * f(x)| , \tag{4.5}
\]

and

\[
\sup_{n \in \mathbb{Z}_+} \frac{1}{r_\lambda(\lambda_n)} |\sigma_{\lambda_n} * f(x)| \leq \sup_{n \in \mathbb{Z}_+} \frac{1}{\lambda_n^{3/c-1}} |\sigma^m_{\lambda_n} * f(x)| + \sum_{n=0}^{\infty} \frac{1}{\lambda_n^{3/c-1}} |\sigma^n_{\lambda_n} * f(x)| . \tag{4.6}
\]

4.2. Minor arc estimate. In this section we show the following result.

**Theorem 4.1.** Let \( c \in (1,11/10) \). Then for every \( N \geq 1 \) and \( f \in \ell^2(\mathbb{Z}^3) \) we have

\[
\| \sup_{N \leq \lambda \leq 2N} \lambda^{-3/c+1} |\sigma^m_\lambda * f| \|_{\ell^2(\mathbb{Z}^3)} \lesssim N^{-(11-10c)/(6c)} \log^2(N+1) \|f\|_{\ell^2(\mathbb{Z}^3)}.
\]
To prove Theorem 4.1, we need some preparatory results. We note that
\[ e(-x\{y\}) = \sum_{|m| \leq M} c_m(x)e(my) + O\left( \min\left\{ 1, \frac{1}{M||y||} \right\} \right). \quad (4.7) \]
This expansion is uniform in \(0 < |x| \leq 1/2, M \in \mathbb{Z}_+\) and \(y \in \mathbb{R}\). Moreover, one has
\[ c_m(x) = \begin{cases} \frac{1-e(-x)}{2\pi i(x+m)}, & x + m \neq 0, \\ 1, & x + m = 0. \end{cases} \]

**Remark 4.2.** Note that (4.7) is not true uniformly for all \(x \in \mathbb{R}\). To see this, take \(y = 0, x = M + 1/2\) and let \(M \to \infty\).

For a fixed \(c \in (1, 2)\) let us define
\[ U_{P, P'}(t, \xi) := \left| \sum_{P \leq n \leq P'} e(n^c t + n \xi) \right|, \quad 1 \leq P \leq P' \leq 2P, \quad t \in \mathbb{R}, \quad \xi \in \mathbb{T}, \quad (4.8) \]
and
\[ V_{P, P'}(M) := \sum_{P \leq n \leq P'} \min\left\{ 1, \frac{1}{M||n^c||} \right\}, \quad M \in \mathbb{Z}_+, \quad 1 \leq P \leq P' \leq 2P. \quad (4.9) \]

**Lemma 4.3.** Let \(c \in (1, 2)\) be fixed. Then for every \(1 \leq P \leq P' \leq 2P, t \neq 0\) and \(\xi \in \mathbb{T}\) one has
\[ U_{P, P'}(t, \xi) \lesssim P^{c/2}|t|^{1/2} + P^{1-c/2}|t|^{-1/2}. \]

**Proof.** It is enough to apply Lemma 3.10 (with \(F(x) = x^c t + x \xi, \eta = P^c-2|t|, r = 1\)). \(\square\)

**Lemma 4.4.** Let \(c \in (1, 2)\) be fixed. Then for every \(M \in \mathbb{Z}_+\) and \(1 \leq P \leq P' \leq 2P\) one has
\[ V_{P, P'}(M) \lesssim (1 + \log M)(M^{-1}P + P^{c/2}M^{1/2}). \]

**Proof.** Using (3.25) we see that
\[ V_{P, P'}(M) = \sum_{P \leq n \leq P'} \sum_{m \in \mathbb{Z}} b_m e(mn^c) \leq \sum_{m \in \mathbb{Z}} |b_m| U_{P, P'}(m, 0). \]
By Lemma 3.10 (with \(F(x) = mx^c, \eta = |m|P^{c-2}, r = 1\)) we deduce
\[ U_{P, P'}(m, 0) \lesssim \begin{cases} 1, & m = 0, \\ |m|^{1/2}P^{c/2}, & m \neq 0. \end{cases} \]
Consequently, combining this with (3.26) we obtain
\[ V_{P, P'}(M) \lesssim \frac{1 + \log M}{M} \left( P + \sum_{|m| \leq M} |m|^{1/2}P^{c/2} \right) + \sum_{|m| \geq M+1} M|m|^{-3/2}P^{c/2} \lesssim \frac{1 + \log M}{M} P + (1 + \log M)P^{c/2}M^{1/2}. \]
This finishes the proof of Lemma 4.3. \(\square\)

For \(c \in (1, 2)\) and \(g \in C^\infty_c(\mathbb{R})\) we introduce
\[ \Pi^c_{s, \eta}(\xi) := \sum_{n \in \mathbb{Z}} e(|n^c| t + n \xi) g\left( \frac{n}{s^{1/c}} \right), \quad s \geq 1, \quad t \in \mathbb{R}, \quad \xi \in \mathbb{T}, \quad (4.10) \]
and the quantity
\[ N_c := (2e)^{-1}(2N)^c. \]

**Lemma 4.5.** Let \( c \in (1, 2) \) and \( g \in C_c^\infty(\mathbb{R}) \) be fixed. Then for all \( N \geq 1 \) one has
\[ \sup_{1 \leq s \leq 2N} \sup_{\xi \in T} \sup_{N_c \leq |t| \leq 1/2} |\Pi_{t,s}^g(\xi)| \lesssim N^{1/3+1/(3c)} \log(N + 1). \]

**Proof.** Observe that there exists \( K_0 > 0 \) such that \( \text{supp} \, g \subset [-K_0, K_0] \) and consequently \( \text{supp} \, g(\frac{\cdot}{s}) \subset [-K_0s^{1/c}, K_0s^{1/c}] \). Thus the summation in (4.11) can be restricted to the set \([-K_0s^{1/c}, K_0s^{1/c}] \cap \mathbb{Z} \). Then summation by parts, see Lemma 3.11 reduces the proof of Lemma 4.5 to establishing that for any fixed \( K > 0 \) and for every \( N \geq 1 \) we have
\[ \sup_{1 \leq s \leq KN} \sup_{1 \leq Z \leq K_0s^{1/c}} \sup_{\xi \in T} \sup_{N_c \leq |t| \leq 1/2} \sum_{n \in [Z]} e(|n^c t + n\xi|) \lesssim N^{1/3+1/(3c)} \log(N + 1). \] (4.11)

Appealing to (4.7) with \( M = \lfloor N^{2(1/c-1/2)} \rfloor \geq 1 \) we may write
\[ \sum_{n \in [Z]} e(|n^c t + n\xi|) = \sum_{n \in [Z]} e(n^c t + n\xi)e(-\{n^c\}t) = I_1(t, \xi, Z) + I_2(t, \xi, Z), \]
where
\[ I_1(t, \xi, Z) := \sum_{|m| \leq M} c_m(t) \sum_{n \in [Z]} e(n^c(t + m) + n\xi), \]
\[ I_2(t, \xi, Z) := O\left( \sum_{n \in [Z]} \min \left\{ 1, \frac{1}{M\|n^c\|} \right\} \right). \]

We first deal with \( I_2(t, \xi, Z) \). Let \( V_{p, p'}(M) \) be defined as in (4.8). By Lemma 4.3 we conclude
\[ I_2(t, \xi, Z) \lesssim \sum_{0 \leq l \leq \log_2 Z} \sum_{2^l \leq n < 2^{l+1}} \min \left\{ 1, \frac{1}{M\|n^c\|} \right\} = \sum_{0 \leq l \leq \log_2 Z} V_{2^l, 2^{l+1}-1}(M) \lesssim (1 + \log M) \left( 2^l M^{-1} + 2^{l+1} / M^{1/2} \right) \lesssim (1 + \log M) \left( N^{1/c} M^{-1} + N^{1/2} M^{1/2} \right) \lesssim N^{1/3+1/(3c)} \log(N + 1), \]
as desired.

Now we focus on \( I_1(t, \xi, Z) \). Notice that
\[ \sup_{N_c \leq |t| \leq 1/2} |I_1(t, \xi, Z)| \leq \sup_{N_c \leq |t| \leq 1/2} \sum_{|m| \leq M} |c_m(t)| \left| \sum_{n \in [Z]} e(n^c(t + m) + n\xi) \right| \leq \sup_{N_c \leq |t| \leq 1/M+1/2} \left| \sum_{n \in [Z]} e(n^c t + n\xi) \right| \sup_{|t| \leq 1/2} \sum_{|m| \leq M} |c_m(t)|. \]

Since we have
\[ \sum_{|m| \leq M} |c_m(t)| \lesssim 1 + \sum_{0 < |m| \leq M} \frac{1}{|m|} \lesssim \log M, \quad |t| \leq 1/2, \]

an application of Lemma 4.3 gives us

\[
\sup_{N \leq |t| \leq 1/2} |I_1(t, \xi, Z)| \lesssim \log M \sup_{N \leq |t| \leq M+1/2} \sum_{0 \leq i \leq \log_2 Z} \sum_{2^i \leq n \leq (2^{i+1} - 1) \wedge Z} e(n^c t + n \xi) \\
= \log M \sup_{N \leq |t| \leq M+1/2} \sum_{0 \leq i \leq \log_2 Z} U_{2^i, (2^{i+1} - 1) \wedge Z}(t, \xi) \\
\lesssim \log M \sum_{0 \leq i \leq \log_2 Z} \left(2^{i+1/2} |t|^{1/2} + 2^{(i+1/2)|t|} \right) \\
\lesssim \log M \left(N^{1/2} M^{1/2} + N^{1/c - 1/2} N^{-1/2} \right) \\
\lesssim N^{1/3 + 1/(3c)} \log(N + 1),
\]

as claimed. This finishes the proof of Lemma 4.5.

Now we are ready to prove Theorem 4.7.

Proof of Theorem 4.7. Let \( \Upsilon(x) = x \cdot \nabla \eta(x), \ x \in \mathbb{R}^3, \) and observe that

\[
\eta\left(\frac{x}{\lambda^{1/c}}\right) = \eta\left(\frac{x}{N^{1/c}}\right) + \int_{N}^{\lambda} \frac{d}{ds} \left(\eta\left(\frac{x}{s^{1/c}}\right)\right) ds = \eta\left(\frac{x}{N^{1/c}}\right) - c^{-1} \int_{N}^{\lambda} \Upsilon\left(\frac{x}{s^{1/c}}\right) \frac{ds}{s}.
\]

Notice that \( \Upsilon \in C^\infty_c(\mathbb{R}^3) \) and that \( \Upsilon \) is a sum of three functions of product type. Now for a fixed \( g \in C^\infty_c(\mathbb{R}^3) \) let us define the auxiliary function

\[
\Delta_{t,s}^g(x) = c(Q(x) t)g\left(\frac{x}{s^{1/c}}\right), \quad t \in \mathbb{R}, \ s \geq 1, \ x \in \mathbb{R}^3.
\]

Elementary computations show

\[
\sigma_{\lambda}^m * f(x) = \sum_{y \in \mathbb{Z}^3} \eta\left(\frac{y}{\lambda^{1/c}}\right) \int_{-1/2}^{1/2} e((Q(y) - \lambda)t) \tilde{\psi}_{\lambda}(t) dt f(x - y) \\
= \int_{-1/2}^{1/2} \sum_{y \in \mathbb{Z}^3} e(Q(y)t) \eta\left(\frac{y}{N^{1/c}}\right) f(x - y) e(-\lambda t) \tilde{\psi}_{\lambda}(t) dt \\
- c^{-1} \int_{-1/2}^{1/2} \sum_{y \in \mathbb{Z}^3} e(Q(y)t) \int_{N}^{\lambda} \Upsilon\left(\frac{y}{s^{1/c}}\right) \frac{ds}{s} f(x - y) e(-\lambda t) \tilde{\psi}_{\lambda}(t) dt \\
= \int_{-1/2}^{1/2} \Delta_{t,N}^g * f(x) e(-\lambda t) \tilde{\psi}_{\lambda}(t) dt - c^{-1} \int_{-1/2}^{1/2} \int_{N}^{\lambda} \Delta_{t,s}^\Upsilon * f(x) \frac{ds}{s} e(-\lambda t) \tilde{\psi}_{\lambda}(t) dt.
\]

Since \( N \leq \lambda \leq 2N, \) we have \((2c)^{-1} \lambda^\kappa \geq N_c\) and \( \tilde{\psi}_{\lambda}(t) \leq 1_{m_\lambda}(t) \leq 1_{N_c \leq |t| \leq 1/2} \) for \( t \in [-1/2, 1/2], \) Consequently, we get

\[
\sup_{N \leq \lambda \leq 2N} |\sigma_{\lambda}^m * f(x)| \leq \int_{N_c \leq |t| \leq 1/2} |\Delta_{t,N}^g * f(x)| \, dt + \int_{N_c \leq |t| \leq 1/2} \int_{N}^{2N} |\Delta_{t,s}^\Upsilon * f(x)| \frac{ds}{s} \, dt.
\]
Now using Minkowski’s and Hölder’s inequalities we obtain
\[
\left\| \sup_{N \leq \lambda \leq 2N} |\sigma^m_\lambda * f|_{\ell^2(\mathbb{Z}^3)} \right\| \leq \int_{Nc \leq |t| \leq 1/2} \|\Delta_{t,N}^0 * f\|_{\ell^2(\mathbb{Z}^3)} \, dt \\
+ \int_{Nc \leq |t| \leq 1/2} \int_0^{2N} \|\Delta_{t,s}^Y * f\|_{\ell^2(\mathbb{Z}^3)} \, ds \, dt \\
\lesssim \left( \int_{Nc \leq |t| \leq 1/2} \|\Delta_{t,N}^0 * f\|_{\ell^2(\mathbb{Z}^3)} \right)^{1/2} \\
+ \left( \int_{Nc \leq |t| \leq 1/2} \int_0^{2N} \|\Delta_{t,s}^Y * f\|_{\ell^2(\mathbb{Z}^3)}^2 \, ds \, dt \right)^{1/2}.
\] (4.12)

Let \( g \in C^\infty_c(\mathbb{R}^3) \) be a fixed function of product type, i.e. \( g(x) := \prod_{j=1}^3 g_j(x_j) \). Then
\[
F_{\mathbb{Z}^3}^{-1} \Delta_{t,s}^g(\xi) = \prod_{j=1}^3 \Pi_{t,s}^{\partial_j}(\xi_j), \quad s \geq 1, \quad t \in \mathbb{R}, \quad \xi \in \mathbb{T}^3,
\]
where \( \Pi_{t,s}^{\partial_j}(\xi_j) \) is given as in (4.10) for any \( j \in [3] \). By Plancherel’s theorem applied twice and Lemma 4.5 we obtain
\[
\int_{Nc \leq |t| \leq 1/2} \|\Delta_{t,s}^g * f\|^2_{\ell^2(\mathbb{Z}^3)} \, dt = \int_{Nc \leq |t| \leq 1/2} \int_{\mathbb{T}^3} |F_{\mathbb{Z}^3}^{-1} \Delta_{t,s}^g(\xi)|^2 |F_{\mathbb{Z}^3}^{-1} f(\xi)|^2 \, d\xi \, dt \\
\lesssim \sup_{N \leq \lambda \leq 2N} \sup_{\xi, t, s \in \mathbb{T}} \sup_{Nc \leq |t| \leq 1/2} |\Pi_{t,s}^{\partial_j}(\xi_j)|^2 \int_{\mathbb{T}^3} |\Pi_{t,s}^{\partial_j}(\xi_j)|^2 \, dt \int_{|t| \leq 1/2} |\Pi_{t,s}^{\partial_j}(\xi_j)|^2 \, d\xi \right|^2 \, d\xi \\
\lesssim N^{4(1/3+1/(3c))} \log^4(N + 1) N^{1/c} \|f\|^2_{\ell^2(\mathbb{Z}^3)}.
\]
Consequently, taking \( g = \eta \) or \( g = \Upsilon \) and plugging this estimate into (4.12) we conclude
\[
\| \sup_{N \leq \lambda \leq 2N} \|\sigma^m_\lambda * f\|_{\ell^2(\mathbb{Z}^3)} \lesssim N^{2/3 + 7/(6c)} \log^2(N + 1) \|f\|_{\ell^2(\mathbb{Z}^3)}, \quad N \geq 1.
\]
The proof of Theorem 4.1 is finished. \( \square \)

**Corollary 4.6.** Let \( c \in (1, 11/10) \) be fixed. Then for every \( 1 < p \leq 2 \) there exists \( \delta > 0 \) such that
\[
\| \lambda^{-3/c+1} \sigma^m_\lambda * f \|_{\ell^p(\mathbb{Z}^3)} \lesssim_N \lambda^{-\delta} \|f\|_{\ell^p(\mathbb{Z}^3)}, \quad \lambda \geq 1, \quad f \in \ell^p(\mathbb{Z}^3).
\]
Moreover, for every \( (11 - 4c)/(11 - 7c) < p \leq 2 \) there exists \( \delta > 0 \) such that
\[
\| \sup_{N \leq \lambda \leq 2N} \lambda^{-3/c+1} \|\sigma^m_\lambda * f\|_{\ell^p(\mathbb{Z}^3)} \lesssim_N \|f\|_{\ell^p(\mathbb{Z}^3)}, \quad N \geq 1, \quad f \in \ell^p(\mathbb{Z}^3).
\]

**Proof.** To prove Corollary 4.6 it is enough to interpolate \( \ell^2(\mathbb{Z}^3) \) estimate from Theorem 4.1 with the trivial bounds
\[
\| \lambda^{-3/c+1} \sigma^m_\lambda * f \|_{\ell^1(\mathbb{Z}^3)} \lesssim \|f\|_{\ell^1(\mathbb{Z}^3)}, \quad \lambda \geq 1, \quad f \in \ell^1(\mathbb{Z}^3),
\]
\[
\| \sup_{N \leq \lambda \leq 2N} \lambda^{-3/c+1} \|\sigma^m_\lambda * f\|_{\ell^1(\mathbb{Z}^3)} \lesssim \|f\|_{\ell^1(\mathbb{Z}^3)}, \quad N \geq 1, \quad f \in \ell^1(\mathbb{Z}^3),
\]
which follow from the fact that \( \frac{1}{c(\lambda)} \sigma_\lambda * f \) is an averaging operator and
\[
\| \lambda^{-3/c+1} \sigma^m_\lambda * f \|_{\ell^1(\mathbb{Z}^3)} \lesssim \|f\|_{\ell^1(\mathbb{Z}^3)}, \quad \lambda \geq 1. \quad (4.13)
\]
In order to prove (4.13) we invoke (4.21), Lemma 4.9 and (4.28), which are proved in the subsection below.

4.3. Major arc estimate. We now estimate the maximal functions corresponding to the major arc. Our main result of this subsection is the following maximal theorem.

Theorem 4.7. Let \( c \in (1, 2) \) be fixed. Then the following two maximal inequalities hold

\[
\left\| \sup_{\lambda \in \mathbb{Z}^+} \lambda^{-3/c+1} |\sigma^\omega_{\lambda} * f| \right\|_{\ell^p(\mathbb{Z}^3)} \lesssim \|f\|_{\ell^p(\mathbb{Z}^3)}, \quad f \in \ell^p(\mathbb{Z}^3), \quad 3/2 < p \leq \infty, \quad (4.14)
\]

\[
\left\| \sup_{n \in \mathbb{Z}^+} \lambda^{-3/c+1} |\sigma^\omega_{\lambda_n} * f| \right\|_{\ell^p(\mathbb{Z}^3)} \lesssim \|f\|_{\ell^p(\mathbb{Z}^3)}, \quad f \in \ell^p(\mathbb{Z}^3), \quad 1 < p \leq \infty. \quad (4.15)
\]

Since \( \psi \in C_c^\infty(\mathbb{R}) \) thus for any \( x \in \mathbb{R}^3 \) and \( \lambda \geq 1 \) we may write, see (4.3),

\[
|\sigma^\omega_{\lambda}(x)| \lesssim \lambda^\epsilon \frac{1}{1 + (\lambda^\epsilon|Q(x) - \lambda|)} \lesssim \lambda^\epsilon \omega_\lambda(x),
\]

where

\[
\omega_\lambda(x) := \frac{1}{1 + (\lambda^\epsilon|x| - \lambda)}^{10}, \quad x \in \mathbb{R}^3, \quad \lambda \geq 1. \quad (4.16)
\]

Lemma 4.8. Let \( c \geq 1 \) and \( a > 0 \) be fixed. Then,

\[
\omega_\lambda(t) \simeq \omega_\lambda(t + \gamma),
\]

uniformly in \( \lambda \geq 1 \) and \( t, \gamma \in \mathbb{R}^3 \) satisfying \( |\gamma|_\infty \leq a \).

Proof. By symmetry it suffices to justify only the estimate \( \lesssim \). Equivalently we show that

\[
\left( \lambda^\epsilon|t + \gamma|_c^c - \lambda \right)^{10} \lesssim 1 + \left( \lambda^\epsilon|t|_c^c - \lambda \right)^{10}, \quad (4.18)
\]

uniformly in \( \lambda \geq 1 \) and \( t, \gamma \in \mathbb{R}^3 \) satisfying \( |\gamma|_\infty \leq a \). Since the case \( |t|_\infty \leq 2a \) is obvious, from now on we may assume that \( |t|_\infty > 2a \). This implies that \( |t + \gamma|_\infty \simeq |t|_\infty \) and consequently there exists a constant \( C_a \geq 1 \) such that

\[
C_a^{-1}|t|_c^c \leq |t + \gamma|_c^c \leq C_a|t|_c^c, \quad |t|_\infty > 2a, \quad |\gamma|_\infty \leq a.
\]

Moreover, for every \( j \in [3] \), we have \( |t_j + \gamma_j|_c^c - |t_j|_c^c| \lesssim |t_j|_c^{c-1} + 1 \), which implies

\[
|t + \gamma|_c^c - |t|_c^c| \leq \sum_{j=1}^3 |t_j + \gamma_j|_c^c - |t_j|_c^c| \lesssim |t|_c^{c-1} + 1 \leq |t|_c^{c-1} + 1. \quad (4.19)
\]

We now distinguish two cases.

Case 1. Assume that \( |t|_c^c > 2C_a\lambda \). Then we have \( |t|_c^c \geq 1 \) and \( |t|_c^c - \lambda \simeq |t|_c^c \). Using (4.18) we see that (4.13) follows, since

\[
|t + \gamma|_c^c - \lambda \leq (|t|_c^c - \lambda) + |t + \gamma|_c^c - |t|_c^c| \lesssim (|t|_c^c - \lambda) + |t|_c^{c-1} \lesssim |t|_c^c - \lambda.
\]
Indeed, using Lemma 4.8 (applied with $t$), as in the previous case, we may write
\[ |t + \gamma|^c - \lambda| \leq |t|^c - \lambda| + |t|^{c-1} + 1 \leq |t|^c - \lambda| + \lambda^{1-1/c} + 1. \]
Thus the left-hand side of (4.18) is controlled by
\[ ( \lambda^\kappa |t|^c - \lambda|)^{10} + \left( \lambda^\kappa (\lambda^{1-1/c} + 1) \right)^{10} \leq ( \lambda^\kappa |t|^c - \lambda|)^{10} + 1, \]
as required. □

We define
\[ K_\lambda(x) := \lambda^{-9/(4c)} \frac{1}{1 + (\lambda^\kappa |x|^c - \lambda|)^{10}}, \quad x \in \mathbb{R}^3, \lambda \geq 1, \quad (4.20) \]
and observe that
\[ \lambda^{-3/c+1} |\sigma_\lambda^{3n}(x)| \lesssim K_\lambda(x), \quad x \in \mathbb{Z}^3, \lambda \geq 1. \quad (4.21) \]

For any function $f : \mathbb{Z}^3 \to \mathbb{C}$ let us consider its extension $F_f : \mathbb{R}^3 \to \mathbb{C}$ by
\[ F_f(x) := \sum_{y \in \mathbb{Z}^3} f(y) \mathbb{1}_{[-1/2,1/2]^3}(x-y), \quad x \in \mathbb{R}^3. \]

If $f \in \ell^p(\mathbb{Z}^3)$ with $p \in [1, \infty]$ it is easy to see that $F_f \in L^p(\mathbb{R}^3)$ and $\|f\|_{\ell^p(\mathbb{Z}^3)} = \|F_f\|_{L^p(\mathbb{R}^3)}$.

The next lemma is a variant of a comparison principle that was recently used in [17, Theorem 1, p. 859], see also [18]. Lemma 4.9 will transfer the problem from the set of integers $\mathbb{Z}^3$ to the continuous setting $\mathbb{R}^3$, where the properties of the maximal functions corresponding to the kernels $K_\lambda$ will be investigated.

**Lemma 4.9.** Let $c \geq 1$ and $p \in [1, \infty]$ be fixed. Then for any $\Pi \subseteq [1, \infty)$ one has
\[ \left\| \sup_{\lambda \in \Pi} |K_\lambda \ast f| \right\|_{\ell^p(\mathbb{Z}^3)} \lesssim \left\| \sup_{\lambda \in \Pi} |F_f| \right\|_{L^p(\mathbb{R}^3)}. \]

**Proof.** We assume that $p \in [1, \infty)$; the case of $p = \infty$ can be handled in a similar way and thus is omitted. Since $K_\lambda$ is nonnegative, it is clear that it suffices to prove that
\[ \left( \sup_{\lambda \in \Pi} |K_\lambda \ast f(x)| \right)^p \leq \int_{x+[1/2,1/2]^3} \left( \sup_{\lambda \in \Pi} |F_f(y)| \right)^p \, dy, \quad x \in \mathbb{Z}^3. \]
This, in turn, will follow once we show the estimate
\[ K_\lambda \ast f(x) \lesssim K_\lambda \ast |F_f|(y), \quad x \in \mathbb{Z}^3, \quad y \in x + [1/2,1/2]^3, \lambda \geq 1. \quad (4.22) \]
Indeed, using Lemma 4.8 (applied with $t = x - z, \gamma = y - x + z - t$ and $a = 1$) we obtain for all $x, z \in \mathbb{Z}^3$ and $y \in x + [1/2,1/2]^3$ and $t \in z + [1/2,1/2]^3$ that
\[ K_\lambda(x - z) \lesssim K_\lambda(y - t), \quad \lambda \geq 1. \]
Consequently, uniformly in $y \in x + [1/2,1/2]^3$ we obtain
\[ K_\lambda \ast |f|(x) = \sum_{z \in \mathbb{Z}^3} |f(z)|K_\lambda(x - z) \lesssim \sum_{z \in \mathbb{Z}^3} |f(z)| \int_{z+[1/2,1/2]^3} K_\lambda(y - t) \, dt = K_\lambda \ast |F_f|(y). \]
This finishes the proof of Lemma 4.9. □
In order to study $L^p(\mathbb{R}^3)$ bounds of the maximal functions corresponding to the kernels $K_\lambda$ efficiently, we will need a polar decomposition with respect to the norm $|\cdot|_c$. Proceeding as in [30, Theorem 2.49, p. 78] we can easily show that there exists the unique Borel measure on $\mathbb{S}^2_c$ denoted by $\mu_c$ such that for every $f \geq 0$ or $f \in L^1(\mathbb{R}^3)$ we have the identity

$$\int_{\mathbb{R}^3} f(x) \, dx = \int_0^\infty r^2 \int_{\mathbb{S}^2_c} f(ry) \, d\mu_c(y) \, dr.$$  \tag{4.23}

Further, using this formula one can easily show that for $f \geq 0$ or $f \in L^1(\mathbb{S}^2_c, d\mu_c)$ we have

$$\int_{\mathbb{S}^2_c} f(x) \, d\mu_c(x) = \int_{y \in \mathbb{R}^2 \, \mid |y| \leq 1} \left( f(y, (1 - |y_c|^1/c) + f(y, (1 - |y_c|^1/c)) \right)(1 - |y_c|^1/c-1) \, dy.$$  \tag{4.24}

**Remark 4.10.** Using the identity \(4.24\) with $f = 1$ and the well-known relation between the Beta and Gamma functions we can easily compute that $\mu_c(\mathbb{S}^2_c) = \frac{8\Gamma(1/c)^3}{c\Gamma(3/c)^3}$.

Now is not difficult to see that inequality \(4.21\) and Lemma \(4.9\) reduce the proof of Theorem \(4.7\) to Theorem \(4.11\) below. The key ingredient in the proof of Theorem \(4.11\) will be the Fourier transform estimates for the measures $\mu_c$ associated with the spheres $\mathbb{S}^2_c$. Namely, for every fixed $c \in (1, 2)$ one has

$$|\mathcal{F}_{\mathbb{R}^3}\mu_c(\xi)| + |\nabla \mathcal{F}_{\mathbb{R}^3}\mu_c(\xi)| \lesssim (1 + |\xi|)^{-1}, \quad \xi \in \mathbb{R}^3.$$  \tag{4.25}

The proof of \(4.25\) is the most technical part of this paper and hence has been postponed to Section \(5\). We assume momentarily that inequality \(4.25\) has been proven and we use it to establish Theorem \(4.11\).

**Theorem 4.11.** Let $c \in (1, 2)$ be fixed, and $K_\lambda$ be the kernel defined in \(4.20\). Then the following two inequalities hold

$$\left\| \sup_{1 \leq \lambda < \infty} |K_\lambda * f| \right\|_{L^p(\mathbb{R}^3)} \lesssim_p \|f\|_{L^p(\mathbb{R}^3)}, \quad f \in L^p(\mathbb{R}^3), \quad p > 3/2,$$  \tag{4.26}

$$\left\| \sup_{n \in \mathbb{Z}^+} |K_{\lambda n} * f| \right\|_{L^p(\mathbb{R}^3)} \lesssim_p \|f\|_{L^p(\mathbb{R}^3)}, \quad f \in L^p(\mathbb{R}^3), \quad p > 1.$$  \tag{4.27}

**Proof.** Our aim is to prove that for every $\lambda \geq 1$ the following estimates hold

$$\|K_\lambda\|_{L^1(\mathbb{R}^3)} \simeq 1,$$  \tag{4.28}

$$|\mathcal{F}_{\mathbb{R}^3}K_\lambda(\xi) - \mathcal{F}_{\mathbb{R}^3}K_\lambda(0)| \lesssim |\lambda^{1/c}\xi|,$$  \tag{4.29}

$$|\mathcal{F}_{\mathbb{R}^3}K_\lambda(\xi)| \lesssim |\lambda^{1/c}\xi|^{-1},$$  \tag{4.30}

$$\left| \frac{d}{dt} \mathcal{F}_{\mathbb{R}^3}K_\lambda(\xi) \right|_{t=\lambda} \lesssim \lambda^{-1},$$  \tag{4.31}

with the implicit constants independent of $\lambda \geq 1$ and $\xi \in \mathbb{R}^3$.

Once \(4.28\), \(4.29\), \(4.30\) and \(4.31\) are established we may proceed much the same way as in [63] to deduce \(4.26\) and \(4.27\), we refer also to [18, (4.7) and (4.8), pp. 16–19] or [17]. More precisely, to prove \(4.27\) it suffices to use \(4.28\), \(4.29\), \(4.30\) with the standard Littlewood–Paley theory and appeal to [63, Theorem 2.14, p. 537]. To prove \(4.26\) we use estimates \(4.28\), \(4.29\), \(4.30\), \(4.31\) with the standard Littlewood–Paley theory and...
Lemma 5.1. Let \( j \) Moreover, for each \( g \) useful construction which will allow us to estimate the oscillatory integrals from (4.25). We claim. To prove (4.30) we use (4.33) and the estimate
\[
|F| \leq \int_0^\infty \frac{\mu_c(S_3^2)\,dt}{1 + (\lambda^{k+1}|t^c - 1|)^{10}}.
\]

For every \( c \geq 1 \) and \( t \geq 0 \) we have \(|t^c - 1| \geq |t - 1|\), thus (4.28) follows, since
\[
\int_0^\infty \frac{\lambda^{k+1}\,dt}{1 + (\lambda^{k+1}|t^c - 1|)^{10}} \lesssim \int_0^\infty \frac{\lambda^{k+1}\,dt}{1 + (\lambda^{k+1}|t - 1|)^8} \lesssim \int_0^\infty \frac{dt}{1 + t^8} \lesssim 1. \tag{4.32}
\]

Invoking (4.23) and changing variable \( r = \lambda^{1/c}t \), we obtain
\[
F_{R^3}K_\lambda(\xi) = \lambda^{-9/(4c)} \int_0^\infty \frac{1}{1 + (\lambda^{k}|t^c - \lambda|)^{10}} \int_{S_3^2} e(-rx \cdot \xi) \,d\mu_c(x) r^2 \,dr
\]
\[
= \lambda^{k+1} \int_0^\infty \frac{t^2 dt}{1 + (\lambda^{k+1}|t^c - 1|)^{10}} F_{R^3} \mu_c(\lambda^{1/c} t) dt, \tag{4.33}
\]

since \( F_{R^3} \mu_c(\xi) = \int_{S_3^2} e(-x \cdot \xi) \,d\mu_c(x) \).

We now prove (4.29). By (4.33) we may conclude
\[
|F_{R^3}K_\lambda(\xi) - F_{R^3}K_\lambda(0)| \lesssim |\lambda^{1/c}| \int_0^\infty \frac{\lambda^{k+1}\,dt}{1 + (\lambda^{k+1}|t^c - 1|)^{10}},
\]

since \( |F_{R^3} \mu_c(\xi) - F_{R^3} \mu_c(0)| \lesssim |\xi| \), hence arguing in a similar way as in (4.32) we obtain the claim. To prove (4.30) we use (4.33) and the estimate \(|F_{R^3} \mu_c(\xi)| \lesssim |\xi|^{-1}\) from (4.25) and we are done. Finally, to prove (4.31) we differentiate (4.33) (this produces a factor \( \lambda^{-1}\)) and then we use \(|\nabla F_{R^3} \mu_c(\xi)| \lesssim |\xi|^{-1}\) from (4.25) and we are also done. \( \square \)

5. Proof of inequality (4.25). Fourier transform estimates

We start by introducing a (non-smooth) decomposition of the unity on \( S_3^2 \). It is a very useful construction which will allow us to estimate the oscillatory integrals from (4.25). We define for any \( j \in \{3\} \) the homeomorphisms
\[
\varphi_j^\pm : \{\xi \in S_3^2 : \pm \xi_j > 0\} \to \{(x_1, x_2) \in \mathbb{R}^2 : |x_1|^c + |x_2|^c < 1\},
\]
where
\[
\varphi_j^j(\xi) := (\xi_1, \ldots, \xi_{j-1}, \xi_{j+1}, \ldots, \xi_3).
\]

Lemma 5.1. Let \( c \in (1, 2) \) be fixed. Then, for each \( j \in \{3\} \) there exist continuous functions \( g_j^\pm : S_3^2 \to [0, 1] \) such that \( \text{supp} \ g_j^\pm = \{\xi \in S_3^2 : \pm \xi_j \geq (14/48)^{1/c}\} \) and satisfy
\[
\sum_{j=1}^3 (g_j^+ + g_j^-)(\xi) = 1, \quad \xi \in S_3^2. \tag{5.1}
\]
Moreover, for each \( j \in \{3\} \) there exist smooth functions \( M_j^\pm \) on
\[
A := \{(x_1, x_2, y) \in \mathbb{R}^3 : |y| < 1 - \frac{14}{48} \text{ or } \frac{14}{48} < |x_1|^c < 2 - \frac{14}{48} \text{ or } \frac{14}{48} < |x_2|^c < 2 - \frac{14}{48}\},
\]
\[
\text{supp} \ M_j^\pm \subset A.
\]
such that the functions
\[\{(x_1, x_2) \in \mathbb{R}^2 : |x_1|^c + |x_2|^c < 1\} \ni (x_1, x_2) \mapsto M_j^\pm(x_1, x_2, |x_1|^c + |x_2|^c)\]
are supported in \(\{(x_1, x_2) \in \mathbb{R}^2 : |x_1|^c + |x_2|^c \leq 1 - 14/48\}\) and we have
\[g_j^\pm \circ (\varphi_j^\pm)^{-1}(x_1, x_2) = M_j^\pm(x_1, x_2, |x_1|^c + |x_2|^c), \quad \text{for } |x_1|^c + |x_2|^c < 1. \quad (5.2)\]

**Proof.** Let \(H : \mathbb{R} \to [0, 1]\) be a compactly supported smooth function such that \(H(x) = 1\) for \(x \in [-1 + 15/48, 1 - 15/48]\), \(\supp H \subset [-1 + 14/48, 1 - 14/48]\) and \(H(x) > 0\) for \(x \in (-1 + 14/48, 1 - 14/48)\). Further, we let
\[h(x_1, x_2) := H(|x_1|^c + |x_2|^c), \quad x_1, x_2 \in \mathbb{R}.\]
Observe that \(h\) is smooth everywhere except possibly at the points \((x_1, x_2)\) such that \(x_1 = 0\) or \(x_2 = 0\). Furthermore, we consider
\[h_j^\pm(\xi) := \begin{cases} h \circ \varphi_j^\pm(\xi), & \pm \xi_j > (13/48)^{1/c}, \\
0, & \pm \xi_j < (14/48)^{1/c}. \end{cases} \quad \xi \in S_c^2.\]
Notice that \(h_j^\pm\) are well-defined and continuous because \(\supp H \subset [-1 + 14/48, 1 - 14/48]\).
Moreover, we have
\[\supp h_j^\pm = \{\xi \in S_c^2 : \pm \xi_j \geq (14/48)^{1/c}\}, \quad j \in [3].\]
Let us consider
\[g(\xi) := \sum_{j=1}^{3} (h_j^+(\xi) + h_j^-(\xi)), \quad \xi \in S_c^2,\]
and observe that \(g(\xi) > 0\) on \(S_c^2\). Finally, we define
\[g_j^\pm(\xi) := \frac{h_j^\pm(\xi)}{g(\xi)}, \quad \xi \in S_c^2, \quad j \in [3].\]
Notice that \(g(\xi) > 0\) holds, thus it remains to prove the existence of the functions \(M_j^\pm\). By symmetry it is enough to focus on \(M_3^+\). Observe that for \(|x_1|^c + |x_2|^c < 1\) we obtain
\[h_3^+ \circ (\varphi_3^+)^{-1}(x_1, x_2) = H(|x_1|^c + |x_2|^c), \quad h_3^- \circ (\varphi_3^-)^{-1}(x_1, x_2) = 0,\]
\[(h_3^+ + h_3^-) \circ (\varphi_3^+)^{-1}(x_1, x_2) = H(1 - |x_1|^c), \quad (h_3^+ + h_3^-) \circ (\varphi_3^-)^{-1}(x_1, x_2) = H(1 - |x_2|^c),\]
and consequently we can take
\[M_3^+(x_1, x_2, y) := \frac{H(y)}{H(y) + H(1 - |x_1|^c) + H(1 - |x_2|^c)}, \quad (x_1, x_2, y) \in A.\]
Now the mapping properties of \(M_3^+\) and \(g(\xi) > 0\) hold easily from the mapping properties of \(H\) and the inclusion
\[\{(x_1, x_2, |x_1|^c + |x_2|^c) \in \mathbb{R}^3 : |x_1|^c + |x_2|^c < 1\} \subseteq A.\]
This finishes the proof of Lemma \(5.1\) \(\square\)

In order to prove \((4.25)\) we establish a slightly more general result, which will be used in the proof of Theorem \(1.3\).
Lemma 5.2. Assume that $c \in (1, 2)$ is fixed and let $f \in C(\mathbb{R}^3) \cap C^\infty((\mathbb{R} \setminus \{0\})^3)$ be such that for every fixed $\beta \in \mathbb{N}^3$ we have the estimate
\[
|\partial^\beta f(x)| \lesssim \prod_{j=1}^3 |x_j|^{-\beta_j + (c-1)\alpha_j \geq 1}, \quad x \in (\mathbb{R} \setminus \{0\})^3. 
\] (5.3)

Then the following estimate holds
\[
\left| \int_{\mathbb{R}^2} f(w)e(-w \cdot \xi) d\mu_c(w) \right| \lesssim (1 + |\xi|)^{-1}, \quad \xi \in \mathbb{R}^3. 
\] (5.4)

It is clear that only the values of $f$ on $S_\xi^2$ are essential. Therefore we may think that $f$ has compact support and in practice it suffices to verify the assumption (5.3) only for $|x| \leq C$, where $C > 0$ is a large absolute constant.

Proof. By symmetry, taking into account Lemma 5.1 and (1.24), we see that our task is reduced to showing that
\[
\text{it is reduced to proving}
\]
\[
|\partial^\alpha x \partial^\beta \cdot \partial^\gamma f(x, \phi(x)) M(x, |x|_c^\xi) dx| \lesssim (1 + |\xi|)^{-1}, \quad \xi \in \mathbb{R}^3, 
\] (5.5)

where $x = (x_1, x_2)$, $|x|_c^\xi = |x_1|^c + |x_2|^c$, $\phi(x) := (1 - |x_1|^c - |x_2|^c)^{1/c}$ and $M$ is a fixed smooth function on $\mathbb{R}$ such that $\supp M(\cdot, | \cdot |_c^\xi) \subseteq \{x \in \mathbb{R}^2 : |x|_c^\xi \leq 1 - 14/48\}$. Let us define
\[
F(x) := f(x, \phi(x)) M(x, |x|_c^\xi), \quad x \in \mathbb{R}^2.
\]

It is clear that $F \in C_c(\mathbb{R}^2) \cap C^\infty((\mathbb{R} \setminus \{0\})^2)$ and $\supp F \subseteq \{x \in \mathbb{R}^2 : |x|_c^\xi \leq 1 - 14/48\}$. Further, one can show that for every fixed $\alpha \in \mathbb{N}^2$ we have
\[
|\partial^\alpha x \partial^\beta \cdot \partial^\gamma f(x, \phi(x)) M(x, |x|_c^\xi)| \lesssim \prod_{j=1}^2 |x_j|^{-\alpha_j + (c-1)\alpha_j \geq 1}, \quad x \in (\mathbb{R} \setminus \{0\})^2. 
\] (5.6)

Indeed, taking into account the fact that the assumptions imposed on $f$ are symmetric it suffices to check (5.6) for $x_1, x_2 > 0$ satisfying $|x|_c^\xi \leq 1 - 14/48$. In view of the Leibniz rule it is reduced to proving
\[
|\partial^\alpha \partial^\beta \cdot \partial^\gamma f(x, \phi(x))(x)| \lesssim \prod_{j=1}^2 |x_j|^{-\alpha_j + (c-1)\alpha_j \geq 1}, \quad x_1, x_2 > 0, \quad |x|_c^\xi \leq 1 - 14/48, 
\] (5.7)

\[
|\partial^\alpha \partial^\beta \cdot \partial^\gamma \partial^\gamma f(x, \phi(x))(x)| \lesssim \prod_{j=1}^2 |x_j|^{-\alpha_j + (c-1)\alpha_j \geq 1}, \quad x_1, x_2 > 0, \quad |x|_c^\xi \leq 1 - 14/48. 
\]

By a simple induction we obtain the formula
\[
\partial^\alpha \partial^\beta \cdot \partial^\gamma f(x, \phi(x))(x) = \sum_{\beta_1 + \gamma_1 + \delta_1 \leq \alpha_1} \sum_{\beta_2 + \gamma_2 + \delta_2 \leq \alpha_2} \sum_{\theta \leq \alpha_1 + \alpha_2} \partial^{(\beta_1 + \beta_2 + \gamma_1 + \gamma_2)} f(x, \phi(x)) P_{\alpha, \beta, \gamma, \delta, \theta}(x_1, x_1^\xi, x_2, x_2^\xi, \phi(x)) \times \left( \prod_{j=1}^2 |x_j|^{-\gamma_j - \delta_j + (c-1)\alpha_j \geq 1} \right) (1 - |x|_c^\xi)^{-\theta},
\]

for some polynomials $P_{\alpha, \beta, \gamma, \delta, \theta}$. This immediately implies (5.7), the second bound can be obtained similarly. Now we come back to the proof of (5.5). Observe that this estimate is trivial for $|\xi| \leq 1$, therefore we may assume that $|\xi| \geq 1$. We shall distinguish two cases.
Case 1. Assume that $|\xi_3| \leq \tau|\xi'|$, where $\xi' = (\xi_1, \xi_2)$ and $\tau > 0$ is a constant such that $2\tau|\nabla \phi(x)| \leq 1$ for all $|x'| < 3/4$. Let $\rho : \mathbb{R} \to [0, 1]$ be a smooth function such that $\rho(x) = 1$ for $x \in [-1, 1]$ and $\text{supp} \rho \subset [-2, 2]$. To prove (5.5) we take $\delta := |\xi'|^{-1} \leq 1$ and show that

$$\int_{\mathbb{R}^2} e\left( - (x' + \phi(x)\xi_3) \right) (1 - \rho(x_1/\delta)) (1 - \rho(x_2/\delta)) F(x) \, dx \leq |\xi'|^{-1}. \quad (5.8)$$

Let us define $\tilde{\phi}(x, \xi) := \frac{\xi' \cdot \xi}{|\xi'|} \phi(x)$ and observe that $\nabla_x \tilde{\phi}(x, \xi) = \frac{\xi' \cdot \xi}{|\xi'|^2} \nabla \phi(x)$. Consequently, this gives us $|\nabla_x \tilde{\phi}(x, \xi)| \geq 1/2$. Now integrating by parts with respect to the first order differential operator

$$D_x := \sum_{j=1}^2 (\partial_x j, \tilde{\phi}(x, \xi)) \partial_{x_j}$$

and using (5.6) we see that the left-hand side of (5.8) is controlled by

$$|\xi'|^{-1} \int_{\mathbb{R}^2} \sup_{1 \leq k, l \leq 2} \frac{|\partial_x k \partial_x l \tilde{\phi}(x, \xi)|}{|\nabla_x \tilde{\phi}(x, \xi)|^2} (1 - \rho(x_1/\delta)) (1 - \rho(x_2/\delta)) |F(x)|$$

$$\leq \frac{1}{|\nabla_x \phi(x, \xi)|} \nabla_x \left( (1 - \rho(x_1/\delta)) (1 - \rho(x_2/\delta)) F(x) \right) \right) \, dx$$

$$\leq |\xi'|^{-1} \int_{|x'| < 3/4} \left( |x_1| c^{-2} + |x_2| c^{-2} + \delta^{-1}(1_{\delta \leq |x_1| \leq 2\delta} + 1_{\delta \leq |x_2| \leq 2\delta}) \right) \, dx$$

$$\leq |\xi'|^{-1}. \quad (5.9)$$

This proves (5.5).

Case 2. We now assume that $|\xi_3| > \tau|\xi'|$. To prove (5.5) we need to show the estimate

$$\int_{\mathbb{R}^2} e\left( - (x' + \phi(x)\xi_3) \right) F(x) \, dx \leq |\xi_3|^{-1}. \quad (5.9)$$

Using $\psi_0 \in C^\infty([-4, -1/2] \cup [1/2, 4])^2$ we introduce a dyadic partition of unity such that for $0 < |x_1|, |x_2| \leq 1$ we have

$$\sum_{j_1, j_2 \geq 0} \psi_0(2^j \circ x) = 1,$$

where $2^j \circ x := (2^{j_1} x_1, 2^{j_2} x_2)$ for all $j \in \mathbb{Z}^2$. Now the left-hand side of (5.9) is dominated by

$$\sum_{j_1, j_2 \geq 0} \int_{\mathbb{R}^2} e\left( - (x' + \phi(x)\xi_3) \right) F(x) \psi_0(2^j \circ x) \, dx$$

$$\leq \sum_{j_1, j_2 \geq 0} 2^{-j_1 - j_2} \int_{\mathbb{R}^2} e\left( - (x(2^{-j_2} \circ x') + \phi(2^{-j_2} \circ x)\xi_3) \right) \psi_0(2^j \circ x) \, dx.$$

Since $|2^{-j_2} \circ \xi'| \leq |\xi'|$, it is enough to prove for all $j = (j_1, j_2) \in \mathbb{N}^2$ that

$$\int_{\mathbb{R}^2} e\left( - (x' + \phi(2^{-j_2} \circ x)\xi_3) \right) F(2^{-j_2} \circ x) \psi_0(x) \, dx \leq |\xi_3|^{-1} 2^{j_1 c/2 + j_2 c/2}. \quad (5.10)$$

It is enough to prove (5.10) only for $j \in \mathbb{N}^2$ satisfying $j_1 \vee j_2 \geq C_c$, where $C_c > 0$ is a large absolute constant depending only on $c \in (1, 2)$. A more precise value of $C_c$ will be specified later. In the opposite case we have $j_1, j_2 \leq C_c$, this means that we only need to handle finitely many terms in which both the phase and the amplitude functions are smooth and
the determinant of the Hessian of the phase is non-zero on the support of the amplitude. Therefore, for \( j_1, j_2 \leq C_c \) (in fact with an arbitrary \( C_c > 0 \)) the estimate (5.10) follows from the general theory, see [64] Lemma 4.15, p. 87. So from now on we assume that \( j_1 \vee j_2 \geq C_c \).

By the symmetry we may assume that \( \text{supp} \psi_0 \subset [1/2, 4]^2 \). Let us define

\[
I_j(\xi) := \int_{\mathbb{R}^2} e(-\xi_3 \Phi_j(x, \xi)) F(2^{-j} \circ x) \psi_0(x) \, dx,
\]

where

\[
\Phi_j(x, \xi) := \frac{x_\xi}{\xi_3} + \phi_j(x), \quad \text{with} \quad \phi_j(x) := \phi(2^{-j} \circ x).
\]

Observe that

\[
|I_j(\xi)|^2 = \int_{\mathbb{R}^2} J_j(u, \xi) \, du,
\]

where

\[
J_j(u, \xi) := \int_{\mathbb{R}^2} e(\xi_3 (\Phi_j(x + u, \xi) - \Phi_j(x, \xi))) \Psi_j(x, u) \, dx,
\]

\[
\Psi_j(x, u) := \psi_0(x + u) F(2^{-j} \circ (x + u)) \psi_0(x) F(2^{-j} \circ x).
\]

Our aim is to show that

\[
|J_j(u, \xi)| \lesssim |\xi_3|^{-3} |2^{-j} \circ u|^{-3}, \quad j_1 \vee j_2 \geq C_c. \tag{5.11}
\]

Notice that combining (5.11) with the trivial bound \( |J_j(u, \xi)| \lesssim 1 \), we see that the square of the left-hand side in (5.10) is controlled by

\[
|I_j(\xi)|^2 \lesssim \int_{\mathbb{R}^2} \left( |\xi_3|^{-3} |2^{-j} \circ u|^{-3} \right) \wedge 1 \, du
\]

\[
= |\xi_3|^{-2} 2^{j_1c+j_2c} \int_{\mathbb{R}^2} |u|^{-3} \wedge 1 \, du
\]

\[
\lesssim |\xi_3|^{-2} 2^{j_1c+j_2c}.
\]

Therefore we have reduced the proof of Lemma 5.2 to showing (5.11). Let us define

\[
a_j(x, u) := \frac{\nabla_x \Phi_j(x + u, \xi) - \nabla_x \Phi_j(x, \xi)}{|\nabla_x \Phi_j(x + u, \xi) - \nabla_x \Phi_j(x, \xi)|^2} = \frac{b_j(x, u)}{|b_j(x, u)|^2},
\]

where

\[
\langle b_j(x, u) := \nabla_x \phi_j(x + u) - \nabla_x \phi_j(x) \rangle.
\]

We shall write \( a_j(x, u) = (a_{j,1}(x, u), a_{j,2}(x, u)) \), \( b_j(x, u) = (b_{j,1}(x, u), b_{j,2}(x, u)) \). Further, we consider the following differential operators

\[
L_j f(x) := \frac{1}{2\pi i \xi_3} (a_j(x, u) \nabla_x f(x) = \frac{1}{2\pi i \xi_3} \sum_{k=1}^2 a_{j,k}(x, u) \partial_{x_k} f(x),
\]

\[
L^*_j f(x) = -\frac{1}{2\pi i \xi_3} \sum_{k=1}^2 \partial_{x_k} (a_{j,k}(x, u) f(x)).
\]

We shall write \( L_{j,t} \) and \( L^*_{j,t} \) to indicate that \( L_j \) and \( L^*_j \), respectively, act on a variable \( t \). Notice that \( L^*_j \) is an adjoint operator to \( L_j \) and we have the identity

\[
L_{j,x} (e(\xi_3 (\Phi_j(x + u, \xi) - \Phi_j(x, \xi))) = e(\xi_3 (\Phi_j(x + u, \xi) - \Phi_j(x, \xi))).
\]
Therefore

$$|J_j(u, \xi)| \leq \int_{\mathbb{R}^2} \left| (L_{j,x}^*)^3 \Psi_j(x,u) \right| \, dx.$$  \hfill (5.12)

We now show that

$$|b_j(x, u)| \simeq \left| 2^{-j_c} \circ u \right|, \quad j_1 \vee j_2 \geq C_c, \quad (x, u) \in \text{supp } \Psi_j.$$  \hfill (5.13)

Letting \( \gamma(t) := 2^{-j} \circ x + t 2^{-j} \circ u \) for \( t \in [0,1] \), we see that for \( k \in [2] \) we have

\[
 b_{j,k}(x, u) = 2^{-j_k} \left( (\partial_k \phi)(2^{-j} \circ (x + u)) - (\partial_k \phi)(2^{-j} \circ x) \right)
\]

\[= 2^{-j_k} \int_0^1 \left( (\partial_k \partial_1 \phi)(\gamma(t)) \gamma'_1(t) + (\partial_k \partial_2 \phi)(\gamma(t)) \gamma'_2(t) \right) \, dt. \]

Therefore simple computations produce

\[
|b_{j,1}(x, u)| \simeq 2^{-j_1} \left| \int_0^1 \left( 1 - |\gamma(t)| \right)^{1/c-2} \left( (\gamma_1(t)^{c-2}(1 - \gamma_2(t)^c) 2^{-j_1} u_1 + (\gamma_1(t) \gamma_2(t))^{c-1} 2^{-j_2} u_2 \right) \, dt, \right|
\]

\[
|b_{j,2}(x, u)| \simeq 2^{-j_2} \left| \int_0^1 \left( 1 - |\gamma(t)| \right)^{1/c-2} \left( (\gamma_1(t) \gamma_2(t))^{c-1} 2^{-j_2} u_1 + (\gamma_2(t)^c(1 - \gamma_1(t)^c) 2^{-j_2} u_2 \right) \, dt. \right|
\]

Since \( (x, u) \in \text{supp } \Psi_j \) we see that \( \gamma(t) \in 2^{-j} \circ [1/2, 4]^2 \) and \( |\gamma(t)| \leq 3/4, \ t \in [0,1] \). Using this we can easily obtain the estimate \( (\bar{\gamma}) \) in (5.13). Indeed, we have

\[
|b_j(x, u)| \lesssim \int_0^1 \left( 2^{-j_1} u_1 + 2^{-j_1} u_2 \right) \, dt \simeq \left| 2^{-j_c} \circ u \right|.
\]

Therefore, it suffices to prove \((\bar{\gamma})\) in (5.13). Here is the place where we adjust the value of the constant \( C_c > 0 \). Let \( \kappa > 0 \) be a number which will be chosen in a moment. If \( 2^{-j_1} u_1 \leq \kappa 2^{-j_2} u_2 \), then using the fact that \( 1 - \gamma_1(t)^c \geq 1/4 \) one can check that

\[
(\gamma_1(t) \gamma_2(t))^{c-1} 2^{-j_1} u_1 \leq (\kappa C_c^{1/2} - c^{j_2}) \gamma_2(t)^c - (1 - \gamma_1(t)^c) 2^{-j_2} u_2//2
\]

for some absolute constant \( C_c^{1} > 0 \). In the opposite case, if \( 2^{-j_1} u_1 > \kappa 2^{-j_2} u_2 \) we have a similar bound

\[
(\gamma_1(t) \gamma_2(t))^{c-1} 2^{-j_2} u_2 \leq (\kappa^{-1} C_c^{1/2} - c^{j_1}) \gamma_1(t)^c - (1 - \gamma_2(t)^c) 2^{-j_1} u_1/\|2
\]

for some absolute constant \( C_c^{2} > 0 \). We now require \( \kappa^{-1} C_c^{1/2} - c^{j_1} \leq 1 \) and \( \kappa C_c^{1/2} - c^{j_2} \leq 1 \). It suffices to take \( C_c > 0 \) such that \( \max\{C_c^1, C_c^2\} \leq 2^C \), and set \( \kappa := \max\{C_c^1, C_c^2\} \) if \( j_1 \geq C_c \) or \( \kappa := \max\{C_c^1, C_c^2\}^{-1} \) if \( j_1 \geq C_c \). With this choice of \( \kappa \) and \( C_c \) we see that (5.13) is proved, since by (5.13), we obtain

\[
|b_{j,2}(x, u)| \simeq 2^{-j_2} u_2 \int_0^1 \gamma_2(t)^c - (1 - \gamma_1(t)^c) 2^{-j_2} dt \simeq 2^{-j_c} u_2 \simeq |2^{-j_c} \circ u|,
\]

whereas by (5.15) we conclude

\[
|b_{j,1}(x, u)| \simeq 2^{-j_1} u_1 \int_0^1 \gamma_1(t)^c - (1 - \gamma_2(t)^c) 2^{-j_1} dt \simeq 2^{-j_c} u_1 \simeq |2^{-j_c} \circ u|.
\]

Observe that for any \( \alpha \in \mathbb{N}^2 \) and \( k \in [2] \) we also have the identity

\[
\partial^\alpha_k b_{j,k}(x, u) = 2^{-j_1 \alpha_1 - j_2 \alpha_2} \left( (\partial^\alpha_1 + \epsilon \phi)(2^{-j} \circ (x + u)) - (\partial^\alpha_1 + \epsilon \phi)(2^{-j} \circ x) \right).
\]

Proceeding analogously and using the estimate

\[
|\partial^\alpha_k \partial^\beta_l \phi(x)| \lesssim_{k,l} x_1^{-k+\epsilon} x_2^{-l+\epsilon} x_1 x_2 > 0, \quad x_1 + x_2 \leq 3/4,
\]

...
we can easily show that for every $\alpha \in \mathbb{N}^2$ we have
\[
|\partial^\alpha_x b_j(x,u)| \lesssim_\alpha |2^{-j} \circ u|, \quad j_1, j_2 \geq 0, \quad (x,u) \in \text{supp } \Psi_j.
\] (5.16)

We now show that for any $\alpha \in \mathbb{N}^2$ the following estimate holds
\[
|\partial^\alpha_x a_j(x,u)| \lesssim_\alpha |2^{-j} \circ u|^{-1}, \quad j_1 \vee j_2 \geq C_c, \quad (x,u) \in \text{supp } \Psi_j.
\] (5.17)

Using the identity
\[
\partial^\alpha_x a_j(x,u) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \partial^{\alpha-\beta}_x b_j(x,u) \partial^\beta_x \left( \frac{1}{|b_j(x,u)|^2} \right)
\]
and estimate (5.16) the task is reduced to showing that for a fixed $\alpha \in \mathbb{N}^2$ we have
\[
|\partial^\alpha_x \left( \frac{1}{|b_j(x,u)|^2} \right)| \lesssim_\alpha |2^{-j} \circ u|^{-2}, \quad j_1 \vee j_2 \geq C_c, \quad (x,u) \in \text{supp } \Psi_j.
\]

This, however, follows from estimates (5.13), (5.16) and the fact that $\partial^\alpha_x (|b_j(x,u)|^{-2})$ is a linear combination of the terms of the form
\[
|b_j(x,u)|^{-2(1+\sum_{\beta \leq \alpha} p_\beta) \prod_{\beta \leq \alpha} (\partial^\beta_x |b_j(x,u)|^2)^{p_\beta},
\]
where the indices run over the set of all non-negative integers $p_\beta$, $\beta \in \mathbb{N}^2$, $\beta \leq \alpha$, satisfying $\sum_{\beta \leq \alpha} p_\beta \leq |\alpha|$. The latter can be easily justified by using the induction argument. Now by (5.12) and (5.17) we see that (5.11) will be proven if we show that for every fixed $\alpha \in \mathbb{N}^2$ one has
\[
|\partial^\alpha_x \Psi_j(x,u)| \lesssim_\alpha 1, \quad j_1, j_2 \geq 0,
\]
which is a straightforward consequence of the Leibniz rule and the estimates (5.6). The proof of (5.11) and consequently Lemma 5.2 is completed.

As a simple consequence of the previous lemma we obtain Corollary 5.3, which will be used in the proof of Theorem 1.5.

**Corollary 5.3.** Assume that $c \in (1, 2)$ is fixed and let $\Upsilon_c : \mathbb{R} \to \mathbb{R}$ be defined as follows
\[
\Upsilon_c(x) := (|x|^c)' := c \text{sgn } (x)|x|^{c-1}, \quad x \in \mathbb{R}.
\] (5.18)

Further, for $\alpha \in \{0, 1\}^3$ let us define $h_{c,\alpha} : \mathbb{R}^3 \to \mathbb{R}$ by setting
\[
h_{c,\alpha}(x) := \prod_{j=1}^3 \Upsilon_c(x_j)^{\alpha_j}, \quad x \in \mathbb{R}^3.
\] (5.19)

Then for every $h \in C^\infty(\mathbb{R}^3)$ the function $\mathbb{R}^3 \ni x \mapsto h(x)h_{c,\alpha}(x)$ satisfies the assumptions of Lemma 5.2 and consequently we have
\[
\left| \int_{\mathbb{R}^3} h(w)h_{c,\alpha}(w)e(-w \cdot \xi) \, d\mu_c(w) \right| \lesssim (1 + |\xi|)^{-1}, \quad \xi \in \mathbb{R}^3.
\] (5.20)

We now close this section by giving a proof of Theorem 1.7, where the $L^p(\mathbb{R}^3)$ bounds for $r$-variations corresponding to the spherical averages $\mathcal{A}_r^c$ are studied.
Proof of Theorem 1.7. If \( p = \infty \) there is nothing to prove and the estimates (1.22) and (1.24) follow in this case. We now prove (1.21) for \( p \in (3/2, 4) \), and (1.23) for \( p \in (1, \infty) \) with any \( r > 2 \). These in turn will imply maximal estimates (1.22) and (1.24) in the same ranges of \( p \), since \( r \)-variations dominate the maximal functions. In order, to obtain (1.22) for all \( p \in (3/2, \infty) \) we will simply interpolate maximal estimate (1.22) for \( p \in (3/2, 4) \) with (1.22) at \( p = \infty \). To prove (1.21) and (1.23) we observe that for all \( \xi \in \mathbb{R}^3 \) we have

\[
|\mathcal{F}_{\mathbb{R}^3} \mu_c(\xi) - \mathcal{F}_{\mathbb{R}^3} \mu_c(0)| \lesssim |\xi|,
\]

(5.21)

\[
|\mathcal{F}_{\mathbb{R}^3} \mu_c(\xi)| \lesssim (1 + |\xi|)^{-1},
\]

(5.22)

\[
|\langle \xi, \nabla \mathcal{F}_{\mathbb{R}^3} \mu_c(\xi) \rangle| \lesssim 1,
\]

(5.23)

where (5.22) and (5.23) immediately follow from Lemma 5.2, see also (4.25). Now using positivity of the operators \( \mathcal{A}_c^r \) and combining (5.21), (5.22) and (5.23) with the standard Littlewood–Paley theory and appealing to [63, Theorem 2.14, p. 537] and [63, Theorem 2.39, p. 543] we obtain (1.21) for \( p \in (3/2, 4) \) and \( r > 2 \). To prove (1.23) for \( p \in (1, \infty) \) and \( r > 2 \) we proceed much the same way as before, but we only use (5.21) and (5.22) with the standard Littlewood–Paley theory and invoke [63, Theorem 1.14, p. 537].

We now assume that \( p \in [1, 3/2] \) and we show that estimate (1.22) does not hold in this range. Observe that

\[
\sup_{t > 0} \mathcal{A}_c^t f(x) = \mu_c(S_c^2) A_c f(x), \quad x \in \mathbb{R}^3, \quad f \in C_c^\infty(\mathbb{R}^3), \quad f \geq 0;
\]

(5.24)

where \( A_c \) is a maximal function over annuli defined as

\[
A_c f(x) := \sup_{0 < a < b} \frac{1}{|A_{a,b}|} \left| \int_{A_{a,b}} f(x-y) dy \right|,
\]

with \( A_{a,b} := \{y \in \mathbb{R}^3 : a < |y|_c < b \} \). To prove (5.24) note first that if \( f \in C_c^\infty(\mathbb{R}^3) \) and \( f \geq 0 \), then for any \( r > 0 \) and \( x \in \mathbb{R}^3 \) we have

\[
\frac{1}{\mu_c(S_c^2)} \int_{S_c^2} f(x-ry) d\mu_c(y) = \lim_{h \to 0^+} \frac{1}{|A_{r-h,r+h}|} \int_{A_{r-h,r+h}} f(x-y) dy \leq A_c f(x),
\]

therefore taking supremum over \( r > 0 \) we conclude \( \frac{1}{\mu_c(S_c^2)} \sup_{t > 0} \mathcal{A}_c^t f(x) \leq A_c f(x) \).

On the other hand, using (4.23) we see for any \( 0 < a < b \) and \( x \in \mathbb{R}^3 \) that

\[
\frac{1}{|A_{a,b}|} \int_{A_{a,b}} f(x-y) dy = \left( \int_a^b \frac{1}{r^2} \int_{S_c^2} f(x-ry) d\mu_c(y) dr \right) \leq \frac{1}{|A_{a,b}|} \int_a^b r^2 dr \sup_{t > 0} \mathcal{A}_c^t f(x) = \frac{1}{\mu_c(S_c^2)} \sup_{t > 0} \mathcal{A}_c^t f(x),
\]

and taking supremum over all annuli proves (5.24).

Consider \( f \in C_c^\infty(\mathbb{R}^3) \) with the property \( 1_{\{y \in \mathbb{R}^3 : |y|_c < 1/2 \}}(x) \leq f(x) \leq 1_{\{y \in \mathbb{R}^3 : |y|_c < 1 \}}(x) \). Then for \( x \in \mathbb{R}^3 \) such that \( |x|_c > 100 \) we have

\[
A_c f(x) \geq \frac{1}{|A_{|x|_c-1, |x|_c+1}|} \int_{A_{|x|_c-1, |x|_c+1}} f(x-y) dy \simeq \frac{|\{y \in \mathbb{R}^3 : |y|_c < 1/2 \}|}{|A_{|x|_c-1, |x|_c+1}|} \simeq |x|_c^{-2}.
\]

Therefore using (1.23) we get an estimate

\[
\left\| A_c f \right\|_{L^p(\mathbb{R}^3)} \simeq \int_{\{y \in \mathbb{R}^3 : |y|_c > 100 \}} |x|_c^{-2p} dx \simeq \int_{100}^\infty r^{-2p+2} dr
\]
and the last integral converges if and only if \( p > 3/2 \).

6. Proof of Theorem 1.5. Ergodic theorems

We now establish pointwise convergence (ii) from Theorem 1.5. We fix \( c \in (1, 11/10) \) and we shall abbreviate \( A_\lambda^\chi \) to \( A_\lambda \). We may assume that \( f \in L^1(X) \cap L^\infty(X) \); since from Theorem 1.6 and the Calderón transference principle [16] we deduce that for every \( f \in L^p(X) \) we have

\[
\| \sup_{\lambda \in \mathbb{Z}_+} |A_\lambda f| \|_{L^p(X)} \lesssim \| f \|_{L^p(X)} \quad \text{if} \quad (11 - 4c)/(11 - 7c) < p \leq \infty;
\]

and

\[
\| \sup_{n \in \mathbb{N}} |A_\lambda f| \|_{L^p(X)} \lesssim \| f \|_{L^p(X)} \quad \text{if} \quad 1 < p \leq \infty.
\]

Using \( \sigma_\lambda^m \) from (4.3) and \( \sigma_\lambda^m \) from (4.4) we may write \( A_\lambda f(x) := A_\lambda^\chi f(x) + A_\lambda^m f(x) \), where

\[
A_\lambda^\chi f(x) := \frac{1}{r_c(\lambda)} \sum_{n \in \mathbb{Z}^3} \sigma_\lambda^\chi(n) f(T^n x),
\]

\[
A_\lambda^m f(x) := \frac{1}{r_c(\lambda)} \sum_{n \in \mathbb{Z}^3} \sigma_\lambda^m(n) f(T^n x).
\]

We now show that both limits \( \lim_{\lambda \to \infty} A_\lambda^{\chi} f(x) \) and \( \lim_{\lambda \to \infty} A_\lambda^{m} f(x) \) exist almost everywhere for any \( f \in L^1(X) \cap L^\infty(X) \), which is dense in \( L^p(X) \) for any \( p \in [1, \infty) \). This in turn, in view of (6.1) and (6.2), by a standard density argument establishes the desired claim in (ii) of Theorem 1.5. The proof will be split into five steps.

Step 1. We first show that \( \lim_{\lambda \to \infty} A_\lambda^m f(x) = 0 \) almost everywhere on \( X \). Indeed, suppose for a contradiction that \( \limsup_{\lambda \to \infty} |A_\lambda^m f(x)| > 0 \) on a set of positive measure. Then there is \( \varepsilon > 0 \) such that

\[
\nu( \{ x \in X : \limsup_{\lambda \to \infty} |A_\lambda^m f(x)| > \varepsilon \} ) > \varepsilon.
\]

Since \( \nu \) is \( \sigma \)-finite, by a standard argument, we can find a rapidly increasing sequence \( (\lambda_k) \in \mathbb{Z}_+ \), satisfying \( \lambda_{k+1} \geq 2\lambda_k \) and such that

\[
\nu( \{ x \in X : \sup_{\lambda_k \leq \lambda \leq \lambda_{k+1}} |A_\lambda^m f(x)| > \varepsilon \} ) > \varepsilon, \quad k \in \mathbb{Z}_+.
\]

This implies that

\[
K^{-1} \sum_{k=1}^{K} \| \sup_{\lambda_k \leq \lambda \leq \lambda_{k+1}} |A_\lambda^m f| \|_{L^2(X)}^2 > \varepsilon^2, \quad K \in \mathbb{Z}_+.
\]

We now claim that there is \( \delta > 0 \) such that

\[
\| \sup_{n \geq N} |A_\lambda^m f| \|_{L^2(X)} \lesssim N^{-\delta} \| f \|_{L^2(X)}, \quad N \in \mathbb{Z}_+,
\]

which contradicts (6.3), since \( \lim_{K \to \infty} K^{-1} \sum_{k=1}^{K} \lambda_k^{-\delta} = 0 \). In order to justify (6.4) it suffices to prove that

\[
\| \sup_{N \leq \lambda \leq 2N} |A_\lambda^m f| \|_{L^2(X)} \lesssim N^{-\delta} \| f \|_{L^2(X)}, \quad N \in \mathbb{Z}_+.
\]

This, however, is a consequence of Theorem 4.1, the fact that \( r_c(\lambda) \simeq \lambda^{3/c-1} \), upon invoking the Calderón transference principle [16].
\textbf{Step 2.} Now we split the operator $A^2_\kappa f$. Let $n_c := \left(\frac{\kappa}{2}\right)^3 \Gamma(3/c)$, and observe that

$$A^2_\kappa f(x) := B^1_\kappa f(x) + n_c B^2_\kappa f(x),$$

where

$$B^2_\kappa f(x) := \sum_{n \in \mathbb{Z}^3} K^2_\kappa(n) f(T^n x), \quad \lambda \in \mathbb{Z}_+, \quad j \in [2],$$

and for $\lambda \in \mathbb{Z}_+$, $n \in \mathbb{Z}^3$, we put

$$K^1_\kappa(n) := \lambda^{\kappa} \eta \left(\frac{n}{\lambda^{1/c}}\right) \left[ \frac{1}{f_c(\lambda)} \mathcal{F}_\mathbb{R}^{-1} \psi (\lambda^\kappa (Q(n) - \lambda)) \right],$$

$$K^2_\kappa(n) := \lambda^{3/c + 1} \lambda^\kappa \eta \left(\frac{n}{\lambda^{1/c}}\right) \mathcal{F}_\mathbb{R}^{-1} \psi (\lambda^\kappa (|n|_c^c - \lambda)).$$

We show that for each $j \in [2]$ the limit $\lim_{\lambda \to \infty} B^j_\kappa f(x)$ exists almost everywhere on $X$. For this purpose, we prove that for sufficiently large $r > 2$ we have

$$\|V^r(B^1_\kappa f : \lambda \in \mathbb{Z}_+)\|_{L^r(X)} \lesssim \|f\|_{L^r(X)}, \quad (6.5)$$

and we also prove that for every $r > 2$ we have

$$\|V^r(B^2_\kappa f : \lambda \in \mathbb{Z}_+)\|_{L^2(X)} \lesssim \|f\|_{L^2(X)}. \quad (6.6)$$

Once (6.5) and (6.6) are established the proof of (ii) in Theorem 1.5 follows, since variational seminorms imply pointwise almost everywhere convergence for the underlying sequences. These estimates for (6.5) and (6.6) will be proved in the next two steps.

\textbf{Step 3.} To prove (6.5), by the Calderón transference principle \cite{16}, it suffices to show that for sufficiently large $r$ we have

$$\|V^r(K^1_\kappa * g : \lambda \in \mathbb{Z}_+)\|_{\ell^r(\mathbb{Z}^3)} \lesssim \|g\|_{\ell^r(\mathbb{Z}^3)}, \quad g \in \ell^r(\mathbb{Z}^3).$$

Since we have $V^r(K^1_\kappa * g(m) : \lambda \in \mathbb{Z}_+) \lesssim \|K^1_\kappa * g(m)\|_{\ell^r(\lambda)}$, $m \in \mathbb{Z}^3$, we conclude

$$\|V^r(K^1_\kappa * g : \lambda \in \mathbb{Z}_+)\|_{\ell^r(\mathbb{Z}^3)} \lesssim \left( \sum_{\lambda \in \mathbb{Z}_+} \|K^1_\kappa * g\|_{\ell^r(\mathbb{Z}^3)}^r \right)^{1/r} \lesssim \left( \sum_{\lambda \in \mathbb{Z}_+} \|K^1_\kappa\|_{\ell^r(\mathbb{Z}^3)}^r \right)^{1/r} \|g\|_{\ell^r(\mathbb{Z}^3)}, \quad g \in \ell^r(\mathbb{Z}^3).$$

Therefore the task is reduced to showing that there is $\delta > 0$ such that

$$\|K^1_\kappa\|_{\ell^1(\mathbb{Z}^3)} \lesssim \lambda^{-\delta}, \quad \lambda \in \mathbb{Z}_+. \quad (6.7)$$

Using Corollary \cite{13} and the mean value theorem we may write

$$|K^1_\kappa(n)| \leq \varepsilon \lambda^\kappa \lambda^{-3/c + 1 - (9 - 8c)/5c + \omega_\lambda(n)} + \lambda^2 \lambda^{-3/c + 1} \omega_\lambda(n), \quad (6.8)$$

for every $\varepsilon > 0$, where $\omega_\lambda$ is defined in (4.16). In the above estimate we use the fact that $Q(n) - |n|_c^c \leq 3$ and the estimate

$$1 + \lambda^\kappa |\theta - \lambda| \simeq 1 + \lambda^\kappa |n|_c^c - \lambda|,$$
uniformly in $\lambda \geq 1$, $n \in \mathbb{Z}^3$ for $\theta$ being a convex combination of $Q(n)$ and $|n|_c^c$. Further, using Lemma 4.8 polar decomposition (4.23) and then changing the variable $r \mapsto \lambda^{1/c} t$ we obtain uniformly in $\lambda \geq 1$ that

$$\sum_{n \in \mathbb{Z}^3} \omega_\lambda(n) \lesssim \int_{\mathbb{R}^3} \omega_\lambda(x) \, dx \lesssim \int_0^\infty \frac{r^2 \, dr}{1 + (\lambda^{c+1}(t^c - 1))^{10}}$$

$$= \lambda^{3/c} \int_0^\infty \frac{t^2 \, dt}{1 + (\lambda^{c+1}(t^c - 1))^{10}}$$

$$\lesssim \lambda^{3/c} \int_0^\infty \frac{dt}{1 + (\lambda^{c+1}(t^c - 1))^{3}}$$

$$\lesssim \lambda^{3/c-c-1}, \quad (6.9)$$

Combining this with (6.8) we obtain (6.7) and the pointwise convergence for $B_\lambda^1 f$ is justified. **Step 4.** As in the previous step, to prove (6.6), by the Calderón transference principle [16], it suffices to show that for every $r > 0$

$$\|V^r(K^3_\lambda * g : \lambda \in \mathbb{Z}+)\|_{\ell^2(\mathbb{Z}^3)} \lesssim \|g\|_{\ell^2(\mathbb{Z}^3)}, \quad g \in \ell^2(\mathbb{Z}^3). \quad (6.10)$$

Let us define

$$K^3_\lambda(x) := \lambda^{-3/c+1} \lambda^\kappa \mathcal{F}_\mathbb{R}^{-1} \psi(\lambda^\kappa (|x|_c^c - \lambda)), \quad x \in \mathbb{R}^3, \quad \lambda \geq 1,$$

and observe that to prove (6.10) it suffices to show that

$$\sum_{\lambda \in \mathbb{Z}_+} \|K^3_\lambda - K^3_\lambda\|_{\ell^1(\mathbb{Z}^3)} < \infty, \quad (6.11)$$

$$\|V^r(K^3_\lambda * g : \lambda \in \mathbb{Z}+)\|_{\ell^2(\mathbb{Z}^3)} \lesssim \|g\|_{\ell^2(\mathbb{Z}^3)}, \quad g \in \ell^2(\mathbb{Z}^3). \quad (6.12)$$

For (6.11) notice that for every fixed $N \in \mathbb{Z}_+$ we have uniformly in $\lambda \geq 2$ that

$$\|K^3_\lambda - K^3_\lambda\|_{\ell^1(\mathbb{Z}^3)} \lesssim N \sum_{|n|_c \geq 4^{1/c}\lambda^{1/c}} \frac{\lambda^{-3/c+1} \lambda^\kappa}{1 + (\lambda^\kappa (|n|_c^c - \lambda))^{N}}$$

$$\lesssim N \sum_{|n|_c \geq 4^{1/c}\lambda^{1/c}} \frac{\lambda^{-3/c+1} \lambda^\kappa}{1 + (\lambda^\kappa (|n|_c^c - \lambda))^{N}}$$

$$\lesssim N \lambda^{-3/c+1} \lambda^\kappa \lambda^{-3N/(8c)} \lambda^{-3c/c}. \quad (6.11)$$

Taking sufficiently large $N$ we obtain (6.11). To prove (6.12) we will proceed in a similar way as in [63]. For the reader convenience we will give the details. For this purpose, in this step we show that the following estimates hold

$$\|K^3_\lambda\|_{\ell^1(\mathbb{Z}^3)} \lesssim 1, \quad \lambda \in \mathbb{Z}_+, \quad (6.13)$$

$$|\mathcal{F}^{-1}_{\mathbb{Z}^3} K^3_\lambda(\xi)| \lesssim (\lambda^{1/c}||\xi||)^{-1}, \quad \lambda \in \mathbb{Z}_+, \quad \xi \in \mathbb{T}^3, \quad (6.14)$$

$$|\mathcal{F}^{-1}_{\mathbb{Z}^3} K^3_\lambda(\xi) - \mathcal{F}^{-1}_{\mathbb{Z}^3} K^3_\lambda(0)| \lesssim \lambda^{1/c}||\xi||, \quad \lambda \in \mathbb{Z}_+, \quad \xi \in \mathbb{T}^3, \quad (6.15)$$

$$|\mathcal{F}^{-1}_{\mathbb{Z}^3} K^3_\lambda(\xi) - \mathcal{F}^{-1}_{\mathbb{Z}^3} K^3_\lambda(0)| \lesssim (s/\lambda)^{3/4}, \quad 0 \leq s \leq \lambda, \quad \lambda \in \mathbb{Z}_+, \quad \xi \in \mathbb{T}^3, \quad (6.16)$$

$$V^2(\mathcal{F}^{-1}_{\mathbb{Z}^3} K^3_\lambda(0) : \lambda \in \mathbb{Z}_+) < \infty. \quad (6.17)$$
In fact the exponent $3/4$ in (6.10) can be replaced with any $\delta \in (1/2, 1)$. The estimate (6.13) is a direct consequence of (6.9), since

$$|\mathcal{F}^{-1}_\mathbb{R} \psi(\lambda^n(n|\mathcal{C} - \lambda))| \lesssim \omega_\lambda(n).$$

In order to prove the remaining estimates we may assume, without any loss of generality, that $\mathbb{T}^3 = [-1/2, 1/2]^3$. Further, we will use the Poisson summation formula (its application is possible thanks to the rapid decay of $K_\lambda^3(x)$ in $x \in \mathbb{R}^3$ and the estimate (6.19) below), which in our case says that

$$\mathcal{F}^{-1}_{\mathbb{Z}^3} K_\lambda^3(\xi) = \sum_{n \in \mathbb{Z}^3} I_\lambda(n, \xi), \quad \xi \in \mathbb{T}^3, \quad \lambda \geq 1,$$

where

$$I_\lambda(n, \xi) := \int_{\mathbb{R}^3} K_\lambda^3(x)e(x \cdot (\xi - n)) \, dx, \quad n \in \mathbb{Z}^3, \quad \xi \in \mathbb{T}^3, \quad \lambda \geq 1.$$

**Estimate (6.14).** By (6.18) inequality (6.14) is a simple consequence of the following bound

$$|I_\lambda(n, \xi)| \lesssim \left( \prod_{j=1}^3 \frac{1}{1 + |n_j|} \right) \frac{1}{1 + \lambda^{1/3} |\xi - n|}, \quad \lambda \geq 1, \quad \xi \in \mathbb{T}^3, \quad n \in \mathbb{Z}^3. \quad (6.19)$$

To justify (6.19) we need to introduce the functions $h_{c,\alpha}$ with $\alpha \in \{0, 1\}^3$ as in (5.19). Notice that $h_{c,\alpha}$ are homogeneous of order $(c - 1)|\alpha|$, where $|\alpha| := \alpha_1 + \alpha_2 + \alpha_3$. More precisely, we have

$$h_{c,\alpha}(rw) = r^{(c-1)|\alpha|}h_{c,\alpha}(w), \quad r > 0, \quad w \in \mathbb{R}^3.$$ 

We define $\alpha = \alpha(n) \in \{0, 1\}^3$ in such a way that for every $j \in [3]$ we put $\alpha_j = 0$ if $n_j = 0$ and $\alpha_j = 1$ if $n_j \neq 0$. Further, for $\kappa$ defined in (4.1) we notice that $-1 < \kappa < 0$ and $\kappa + (c-1)/c < 0$, which will be frequently used below.

We now prove (6.19). Integrating by parts we obtain

$$I_\lambda(n, \xi) = \frac{C_\alpha}{(\xi - n)^\alpha} \int_{\mathbb{R}^3} \partial_x^\alpha K_\lambda^3(x)e(x \cdot (\xi - n)) \, dx$$

for some constants $C_\alpha \in \mathbb{C}$. Further, taking $\psi_\alpha := (\mathcal{F}^{-1}_\mathbb{R} \psi(|\alpha|)$ observe that

$$\partial_x^\alpha K_\lambda^3(x) = \lambda^{-3(c+1+\kappa+|\alpha|)}\psi_\alpha(\lambda^n(|x|^c - \lambda))h_{c,\alpha}(x). \quad (6.20)$$

Therefore using (1.23) and changing the variable $r \mapsto \lambda^{1/c}t$ we obtain

$$I_\lambda(n, \xi) = \frac{C_\alpha}{(\xi - n)^\alpha} \lambda^{\kappa+1+|\alpha|(1-1)/c+\kappa+|\alpha|} \int_0^\infty t^{2(c-1)|\alpha|} \psi_\alpha(\lambda^{\kappa+1}(t^c - 1))$$

$$\times \int_{\mathbb{R}^3} h_{c,\alpha}(w)e(\lambda^{1/c}tw \cdot (\xi - n)) \, d\mu_c(w) \, dt. \quad (6.21)$$

Therefore using (5.20) and the estimate $|t - 1| \lesssim |t^c - 1|$ for $t > 0$, we see that for every fixed $N \in \mathbb{N}$ we have

$$|I_\lambda(n, \xi)| \lesssim_N \left( \prod_{j=1}^3 \frac{1}{1 + |n_j|} \right) \lambda^{\kappa+1} \int_0^\infty \frac{t^{2(c-1)|\alpha|}}{1 + (\lambda^{\kappa+1}(t^c - 1))^{N}} \frac{dt}{1 + \lambda^{1/c}|\xi - n|}$$

$$\lesssim_N \left( \prod_{j=1}^3 \frac{1}{1 + |n_j|} \right) \lambda^{\kappa+1} \int_0^\infty \frac{1}{1 + (\lambda^{\kappa+1}(t^c - 1))^{N-1-(c-1)|\alpha|}} \frac{t \, dt}{1 + \lambda^{1/c}|\xi - n|}.$$
Now splitting the above integral into the intervals \((0, 1/2)\) and \((1/2, \infty)\) and using the relation
\[
\int_0^{1/2} \frac{t \, dt}{1 + At} \approx \frac{1}{1 + A}, \quad A > 0,
\]
we obtain \((6.19)\) and consequently \((6.14)\).

**Estimate \((6.15)\).** Let \(J_\lambda(n, \xi) := I_\lambda(n, \xi) - I_\lambda(n, 0)\) and note that \((6.15)\) is reduced to proving
\[
|J_\lambda(n, \xi)| \lesssim \left( \prod_{j=1}^3 \frac{1}{1 + |n_j|} \right)^{\lambda^{1/c} \|\xi\|}, \quad \lambda \geq 1, \quad \xi \in \mathbb{T}^3, \quad n \in \mathbb{Z}^3. \quad (6.22)
\]
Integrating by parts, using the Leibniz rule, \((6.20)\), \((4.23)\) and finally changing the variable \(r \mapsto \lambda^{1/c} t\) we obtain for some constants \(C_\alpha, C_{\alpha, \beta} \in \mathbb{C}\) that
\[
J_\lambda(n, \xi) = \frac{C_\alpha}{n^\alpha} \int_{\mathbb{R}^3} e(-x \cdot n) \partial_x^\alpha [K_\lambda(x)(e(x \cdot \xi) - 1)] \, dx
\]
\[
= \sum_{0 \leq \beta \leq \alpha} \frac{C_{\alpha, \beta}}{n^\alpha} \int_{\mathbb{R}^3} e(-x \cdot n) \partial_x^\beta K_\lambda(x) \partial_x^{\alpha-\beta} (e(x \cdot \xi) - 1) \, dx
\]
\[
= \sum_{0 \leq \beta \leq \alpha} \frac{C_{\alpha, \beta}}{n^\alpha} \lambda^{3/c + 1 + \kappa + \kappa |\beta|} \int_{\mathbb{R}^3} e(-x \cdot n) \psi_\beta(\lambda^c (|x|^c - \lambda)) \, dx
\]
\[
\quad \times (e(x \cdot \xi) - 1_{\alpha=\beta}) \, dx
\]
\[
= \sum_{0 \leq \beta \leq \alpha} \frac{C_{\alpha, \beta}}{n^\alpha} \lambda^{1 + \kappa + \kappa |\beta| + |\beta(c-1)/c|} \int_0^\infty \int_{\mathbb{R}^3} 2^{|\beta|} \lambda^{c-1} \psi_\beta(\lambda^{c+1} (t^{c-1} - 1)) J_{\lambda, \alpha, \beta}(n, \xi, t) \, dt.
\]
where
\[
J_{\lambda, \alpha, \beta}(n, \xi, t) = \xi^{\alpha-\beta} \int_{S^2_n} h_{\epsilon, \beta}(w) e(-\lambda^{1/c} t w \cdot n) (e(\lambda^{1/c} t w \cdot \xi) - 1_{\alpha=\beta}) \, d\mu_c(w).
\]
Next, we show that
\[
|J_{\lambda, \alpha, \beta}(n, \xi, t)| \lesssim \frac{(t + 1)^2}{t(|n| + 1)} \lambda^{1/c} \|\xi\|, \quad t > 0, \quad \lambda \geq 1, \quad n \in \mathbb{Z}^3, \quad \xi \in \mathbb{T}^3. \quad (6.23)
\]
Indeed, if \(\alpha \neq \beta\) then using \((5.20)\) we obtain
\[
|J_{\lambda, \alpha, \beta}(n, \xi, t)| \lesssim \frac{\|\xi\|}{1 + \lambda^{1/c} |\xi - n|} \lesssim \frac{(t + 1) \|\xi\|}{t(|n| + 1)}
\]
and \((6.23)\) follows. In the opposite case when \(\alpha = \beta\) we combine the identity
\[
eq (w \cdot \xi) - 1 = 2\pi i \int_0^1 (w \cdot \xi) e(wu \cdot \xi) \, du, \quad w, \xi \in \mathbb{R}^3,
\]
with \((5.20)\) and we get
\[
|J_{\lambda, \alpha, \beta}(n, \xi, t)| \lesssim \int_0^1 \int_{S^2_n} \left( \sum_{j=1}^3 \lambda^{1/c} t w_j \xi_j \right) h_{\epsilon, \beta}(w) e(\lambda^{1/c} t w \cdot (u \xi - n)) \, d\mu_c(w) \, du
\]
\[
\lesssim \int_0^1 \frac{\lambda^{1/c} |\xi|}{1 + \lambda^{1/c} |u \xi - n|} \, du \lesssim \frac{t + 1}{|n| + 1} \lambda^{1/c} \|\xi\|.
\]
This directly leads to (6.23). Next, applying (6.23) for every fixed $N \in \mathbb{N}$ we arrive at

$$|J_\lambda(n, \xi)| \lesssim \left( \prod_{j=1}^{3} \frac{1}{1 + |n_j|} \right) \sum_{0 \leq \beta \leq \alpha} \lambda^{1+\kappa} \int_{0}^{\infty} \frac{t^{2+(c-1)|\beta|}}{1 + (\lambda^{\kappa+1}|t| - 1)^N} \frac{(t + 1)^2}{t(|n| + 1)} \lambda^{1/c} \xi \| dt$$

$$\lesssim \left( \prod_{j=1}^{3} \frac{1}{1 + |n_j|} \right) \frac{1}{|n| + 1} \lambda^{1/c} \xi \| \sum_{0 \leq \beta \leq \alpha} \lambda^{1+\kappa} \int_{0}^{\infty} \frac{dt}{1 + (\lambda^{\kappa+1}|t - 1|)^{N-4-(c-1)|\beta|}},$$

which gives (6.22) and consequently (6.15) is justified.

Estimate (6.16). We begin with showing that

$$|\partial_\lambda I_\lambda(n, \xi)| \lesssim \lambda^{-1} \left( \prod_{j=1}^{3} \frac{1}{1 + |n_j|} \right), \quad \lambda \geq 1, \quad \xi \in \mathbb{T}^3, \quad n \in \mathbb{Z}^3. \quad (6.24)$$

Taking into account (6.21) we obtain

$$\partial_\lambda I_\lambda(n, \xi) = \frac{1}{(\xi - n)^\alpha} \lambda^{\kappa+1+|\alpha|(c-1)/c+\kappa|\alpha|} \left( \int_{0}^{\infty} t^{2+(c-1)|\alpha|} \int_{\mathbb{R}^2} h_c,\alpha(w)e^{\frac{1}{c}tw \cdot (\xi - n)} \right)$$

$$\times \left[ C_{\alpha,1} \lambda^{-1} \psi_\alpha(\lambda^{\kappa+1}(t^c - 1)) + C_{\alpha,2} \psi_\alpha'(\lambda^{\kappa+1}(t^c - 1)) \lambda^\kappa(t^c - 1) \right.$$

$$+ C_{\alpha,3} \psi_\alpha(\lambda^{\kappa+1}(t^c - 1)) \lambda^{1/c-1}tw \cdot (\xi - n) \bigg] \, d\mu_c(w) \, dt,$$

for some constants $C_{\alpha,j} \in \mathbb{C}$ for $j \in [3]$. Using (5.20) for every fixed $N \in \mathbb{N}$ we obtain

$$|\partial_\lambda I_\lambda(n, \xi)| \lesssim \left( \prod_{j=1}^{3} \frac{1}{1 + |n_j|} \right) \lambda^{\kappa+1} \int_{0}^{\infty} t^{2+(c-1)|\alpha|}$$

$$\times \left( \frac{\lambda^{-1}}{1 + (\lambda^{\kappa+1}|t| - 1)^N} + \frac{\lambda^\kappa|t^c - 1|}{1 + (\lambda^{\kappa+1}|t| - 1)^N} \right) \, dt$$

$$\lesssim \left( \prod_{j=1}^{3} \frac{1}{1 + |n_j|} \right) \lambda^\kappa \int_{0}^{\infty} \frac{t^{2+(c-1)|\alpha|}}{1 + (\lambda^{\kappa+1}|t - 1|)^{N-4-(c-1)|\alpha|}} \, dt$$

$$\lesssim \left( \prod_{j=1}^{3} \frac{1}{1 + |n_j|} \right) \lambda^\kappa \int_{0}^{\infty} \frac{dt}{1 + (\lambda^{\kappa+1}|t - 1|)^{N-3-(c-1)|\alpha|}}.$$

Therefore we obtain (6.24) as claimed. Consequently, using (6.24) we infer that

$$|I_{\lambda+s}(n, \xi) - I_\lambda(n, \xi)| = \left| \int_{\lambda}^{\lambda+s} \partial_\lambda I_\lambda(n, \xi) \, dt \right|$$

$$\lesssim \left( \prod_{j=1}^{3} \frac{1}{1 + |n_j|} \right) \frac{s}{\lambda}, \quad 0 \leq s \leq \lambda, \quad \lambda \geq 1, \quad \xi \in \mathbb{T}^3, \quad n \in \mathbb{Z}^3.$$

On the other hand, from (6.19) we conclude

$$|I_{\lambda+s}(n, \xi) - I_\lambda(n, \xi)| \lesssim \left( \prod_{j=1}^{3} \frac{1}{1 + |n_j|} \right) \frac{1}{1 + |n|}, \quad 0 \leq s \leq \lambda, \quad \lambda \geq 1, \quad \xi \in \mathbb{T}^3, \quad n \in \mathbb{Z}^3.$$
Therefore fixing $\delta \in [0, 1]$ and combining these estimates we have
\[
|I_{\lambda+s}(n, \xi) - I_\lambda(n, \xi)| \lesssim_\delta \left( \prod_{j=1}^3 \frac{1}{1 + |n_j|^{\lambda}} \right) \left( \frac{1}{1 + |n|} \right)^{1-\delta} (s/\lambda)^\delta,
\]
uniformly in $0 \leq s \leq \lambda$, $\lambda \geq 1$, $\xi \in \mathbb{T}^3$ and $n \in \mathbb{Z}^3$. This, in view of (6.18), proves (6.16).

**Estimate (6.17).** Since $\sum_{\lambda \in \mathbb{Z}_+} \lambda^{-2(\kappa+1)} < \infty$, it is enough to prove that
\[
\mathcal{F}_{\mathbb{Z}^3}^{-1} K_\lambda^3(0) = \mu_c(S^3_c)/c + O(\lambda^{-\kappa-1}), \quad \lambda \geq 1.
\]
Taking into account (6.18) and (6.19) our aim reduces to proving that
\[
I_\lambda(0, 0) = \mu_c(S^3_c)/c + O(\lambda^{-\kappa-1}), \quad \lambda \geq 1.
\]
Using (6.21) and changing the variable $t^\kappa - 1 \mapsto s$ we get
\[
I_\lambda(0, 0) = \mu_c(S^3_c)c^{-1}\lambda^{\kappa+1} \int_{-1}^\infty (s + 1)^{3/c-1} \mathcal{F}_\mathbb{R}^{-1} \psi(\lambda^{\kappa+1}s) \, ds
\]
\[
= \mu_c(S^3_c)c^{-1}\lambda^{\kappa+1} \int_{-1/2}^{1/2} (s + 1)^{3/c-1} \mathcal{F}_\mathbb{R}^{-1} \psi(\lambda^{\kappa+1}s) \, ds + O(\lambda^{-100})
\]
\[
= \mu_c(S^3_c)c^{-1}\lambda^{\kappa+1} \int_{-1/2}^{1/2} \mathcal{F}_\mathbb{R}^{-1} \psi(\lambda^{\kappa+1}s) \, ds + O(\lambda^{-\kappa-1}).
\]
Since $\int_{\mathbb{R}} \mathcal{F}_\mathbb{R}^{-1} \psi(s) \, ds = \psi(0) = 1$, we conclude (6.26). This finishes the Step 4.

**Step 5.** Here using (6.13), (6.14), (6.15), (6.16) and (6.17) we finally justify (6.12). To proceed, let $(P_t)_{t>0}$ be the discrete Poisson semigroup defined by
\[
\mathcal{F}_{\mathbb{Z}^3}^{-1} (P_t f)(\xi) := p_t(\xi) \mathcal{F}_{\mathbb{Z}^3}^{-1} f(\xi), \quad t > 0, \quad \xi \in \mathbb{T}^3,
\]
where
\[
p_t(\xi) := e^{-2\pi \|\xi\|_\infty}, \quad |\xi|_\infty := \left( \sum_{j=1}^3 (\sin(\pi \xi_j))^2 \right)^{1/2}, \quad t > 0, \quad \xi \in \mathbb{T}^3.
\]
Notice that $|\xi|_\infty \approx \|\xi\|$. It was recently proved in [17, Section 5.1, p. 893] that for every $1 < p < \infty$ and $r > 2$ we have the variational estimate
\[
\|V^r(P_t f : t > 0)\|_{L^p(\mathbb{Z}^3)} \lesssim \|f\|_{L^p(\mathbb{Z}^3)}, \quad f \in L^p(\mathbb{Z}^3).
\]
Using this, (6.17) and the fact that
\[
V^r(a_N b_N : N \in \mathbb{Z}_+) \leq (\sup_{N \in \mathbb{Z}_+} |a_N|) V^r(b_N : N \in \mathbb{Z}_+) + (\sup_{N \in \mathbb{Z}_+} |b_N|) V^r(a_N : N \in \mathbb{Z}_+)
\]
we see that our task reduces to showing that
\[
\|V^r(T_\lambda f : \lambda \in \mathbb{Z}_+)\|_{L^2(\mathbb{Z}^3)} \lesssim_r \|f\|_{L^2(\mathbb{Z}^3)}, \quad f \in \ell^2(\mathbb{Z}^3), \quad r > 2,
\]
where
\[
T_\lambda f := K_\lambda^3 * f - \mathcal{F}_{\mathbb{Z}^3}^{-1} K_\lambda^3(0) P_{\lambda^{1/c}} f, \quad \lambda \in \mathbb{Z}_+.
\]
Splitting into long and short variations we see that it suffices to prove that
\[
\left( \sum_{k \in \mathbb{Z}_+} |T_2^k f|^2 \right)^{1/2} \|_{L^2(\mathbb{Z}^3)} \lesssim \|f\|_{L^2(\mathbb{Z}^3)}, \quad f \in \ell^2(\mathbb{Z}^3), \quad (6.27)
\]
\[ \left\| \left( \sum_{k \in \mathbb{Z}_+} V^2(T_\lambda f : \lambda \in [2^k, 2^{k+1}])^2 \right)^{1/2} \right\|_{L^2(\mathbb{Z}^3)} \lesssim \|f\|_{L^2(\mathbb{Z}^3)}, \quad f \in \ell^2(\mathbb{Z}^3). \quad (6.28) \]

Let \( \mathcal{F}_{\mathbb{Z}^3}^{-1} T_\lambda \) denote the multiplier of the operator \( T_\lambda \). Using (6.14) and (6.15) we see that
\[ |\mathcal{F}_{\mathbb{Z}^3}^{-1} T_\lambda(\xi)| \lesssim (\lambda^{1/c}\|\xi\|)^{-1} \wedge (\lambda^{1/c}\|\xi\|), \quad \xi \in \mathbb{T}^3, \quad \lambda \in \mathbb{Z}_+, \quad (6.29) \]
and applying the Plancherel theorem we see that (6.27) follows. In order to justify (6.28) we use the following version of the Rademacher–Menshov inequality, see [63, Lemma 2.5],
\[ V^2(T_\lambda f : \lambda \in [2^k, 2^{k+1}]) \lesssim \sum_{l=0}^k \left( \sum_{m=0}^{2^l-1} |T_{2^k+2^{k-l}(m+1)}f - T_{2^k+2^{k-l}m}f|^2 \right)^{1/2}, \]
which together with the triangle inequality shows that
\[ \text{LHS of (6.28)} \lesssim \sum_{l \geq 0} \left( \sum_{k \geq l \vee 1} \sum_{m=0}^{2^l-1} \|T_{2^k+2^{k-l}(m+1)}f - T_{2^k+2^{k-l}m}f\|_{L^2(\mathbb{Z}^3)}^2 \right)^{1/2}. \]
Combining (6.16) with (6.29) we see that uniformly in \( 0 \leq l \leq k, 0 \leq m \leq 2^l - 1, \xi \in \mathbb{T}^3 \), we have
\[ |\mathcal{F}_{\mathbb{Z}^3}^{-1} T_{2^k+2^{k-l}(m+1)}(\xi) - \mathcal{F}_{\mathbb{Z}^3}^{-1} T_{2^k+2^{k-l}m}(\xi)| \lesssim (2^{k/c}\|\xi\|)^{-1} \wedge (2^{k/c}\|\xi\|) \wedge 2^{-3l/4}, \]
and consequently using the Plancherel theorem we get (6.28). This concludes the proof of (6.12). This finishes the Step 5 and the proof of Theorem 1.5 is completed.

7. Proof of Theorem 1.9 The upper bound for the discrepancy

To prove Theorem 1.9 we need some preparatory results, which will be gathered below. We first observe that
\[ D_c(\lambda, \xi, a) = D_c(\lambda, \xi; 1_{[a,100]}), \quad \lambda \in \mathbb{Z}_+, \quad \xi \in \mathbb{S}^2, \quad a > 0, \]
where for any function \( f : \mathbb{R} \to \mathbb{R} \) we have
\[ D_c(\lambda, \xi; f) := \sum_{m \in \mathbb{S}^2(\lambda)} \left( f\left( \frac{m \cdot \xi}{\lambda^{1/c}} \right) - r_c(\lambda) \int_{\mathbb{S}^2} f(x : \xi) \, dv(x) \right), \]
and \( v_c \) is a normalized measure \( \mu_c \) to have mass one.

Proceeding as in [58, Lemma 9] we show that the function \( 1_{[a,100]} \) in \( D_c(\lambda, \xi; 1_{[a,100]}) \) can be replaced by smooth functions \( \phi_{a,\delta}^\pm \). Indeed, let \( f \in C^\infty(\mathbb{R}) \) be such that \( 0 \leq f(t) \leq 1 \), supp \( \phi \subset [-1,1] \) and \( \int f(x) \, dx = 1 \). Further, let \( \phi_{a,\delta}^\pm := 1_{[a \pm \delta,100]} \ast \phi_{\delta} \), where \( \phi_{\delta} \) is an \( L^1 \) dilatation of \( \phi \), i.e. \( \phi_{\delta}(t) := \delta^{-1}\phi(\delta^{-1}t) \). We now prove an analogue of [58, Lemma 9].

**Lemma 7.1.** Let \( c \in (1,2) \) be fixed. Then the following bound holds
\[ D_c(\lambda, \xi) \leq \sup_{0 < a \leq 100} |D_c(\lambda, \xi; \phi_{a,\delta}^+)| \vee \sup_{0 < a \leq 100} |D_c(\lambda, \xi; \phi_{a,\delta}^-)| + O(\delta^{1/2}r_c(\lambda)), \]
uniformly in \( \lambda \in \mathbb{Z}_+, \xi \in \mathbb{S}^2 \) and \( 0 < \delta \leq 1 \).

**Proof.** Since \( D_c(\lambda, \xi, a) = 0 \) for \( a > 100, \) we may assume that \( 0 < a \leq 100 \). Observe that
\[ \phi_{a,\delta}^\pm(t) \leq 1_{[a,100]}(t) \leq \phi_{a,\delta}^\pm(t), \quad |t| \leq 99, \quad 0 < a \leq 100, \quad 0 < \delta \leq 1. \]
Consequently, we have
\[
\sum_{m \in S_2^c(\lambda)} \phi^+_{a,\delta,\lambda} \left( \frac{m \cdot \lambda}{\lambda_1^c} \right) \leq \sum_{m \in S_2^c(\lambda)} \mathbb{1}_{[a,100]} \left( \frac{m \cdot \lambda}{\lambda_1^c} \right) \leq \sum_{m \in S_2^c(\lambda)} \phi^-_{a,\delta,\lambda} \left( \frac{m \cdot \lambda}{\lambda_1^c} \right),
\]
\[
r_c(\lambda) \int_{S_2^c} \phi^-_{a,\delta,\lambda} (x \cdot \lambda) \mathrm{d}\nu_c(x) \geq r_c(\lambda) \int_{S_2^c} \mathbb{1}_{[a,100]} (x \cdot \lambda) \mathrm{d}\nu_c(x) \geq r_c(\lambda) \int_{S_2^c} \phi^+_{a,\delta,\lambda} (x \cdot \lambda) \mathrm{d}\nu_c(x),
\]
for \( \lambda \in \mathbb{Z}_+, \xi \in \mathbb{S}^2, 0 < a \leq 100 \) and \( 0 < \delta \leq 1 \). Subtracting the above inequalities we see that
\[
|D_c(\lambda, \xi, a)| \leq |D_c(\lambda, \xi, \phi^+_{a,\delta})| \lor |D_c(\lambda, \xi, \phi^-_{a,\delta})| + r_c(\lambda) \int_{S_2^c} |\phi^-_{a,\delta,\lambda} (x \cdot \lambda) - \phi^+_{a,\delta,\lambda} (x \cdot \lambda)| \mathrm{d}\nu_c(x).
\]
Since we have
\[
\int_{S_2^c} |\phi^-_{a,\delta,\lambda} (x \cdot \lambda) - \phi^+_{a,\delta,\lambda} (x \cdot \lambda)| \mathrm{d}\nu_c(x) \lesssim \int \phi_\delta(y) \left( \int_{S_2^c} \mathbb{1}_{[a-\delta+y,a+\delta+y]} (x \cdot \lambda) \mathrm{d}\mu_c(x) \right) \mathrm{d}y,
\]
the proof of Lemma 7.4 will be completed if we show
\[
\int_{S_2^c} \mathbb{1}_{[a,a+\delta]} (x \cdot \lambda) \mathrm{d}\mu_c(x) \lesssim \delta^{1/2}, \quad \xi \in \mathbb{S}^2, \quad \alpha \in \mathbb{R}, \quad \delta > 0.
\]

Using the decomposition of the unity on \( \mathbb{S}_2^c \) and (4.24) we see that our problem reduces to showing that
\[
\int_{|x_1|+|x_2|<3/4} \mathbb{1}_{[a,a+\delta]} (x_1 \xi_1 + x_2 \xi_2 + (1 - |x_1|^c - |x_2|^c)^{1/c} \xi_3) \mathrm{d}x \lesssim \delta^{1/2}, \quad (7.1)
\]
uniformly in \( \xi \in \mathbb{S}^2, \alpha \in \mathbb{R}, 0 < \delta \leq 1 \). By symmetry we may assume that \( x_1, x_2 > 0 \) and \( \xi_3 \geq 0 \). In what follows we distinguish two cases.

**Case 1.** \( \xi_3 \leq 1/100 \). Here we have \( |\xi_1| \geq 1/4 \) or \( |\xi_2| \geq 1/4 \) and without any loss of generality we may assume that \( |\xi_2| \geq 1/4 \). Then, we see that
\[
\partial_{x_2} (x_1 \xi_1 + x_2 \xi_2 + (1 - x_1^c - x_2^c)^{1/c} \xi_3) = \xi_2 - \xi_3 (g_{x_1}(x_2))^{-1}, \quad x_1^c + x_2^c < 3/4,
\]
where \( g_b : (0, (3/4 - b^c)^{1/c}) \to (0, \infty) \), and \( 0 < b < (3/4)^{1/c} \), are defined by
\[
g_b(y) := \frac{y}{(1 - b^c - y^c)^{1/c}}, \quad b^c + y^c < 3/4, \quad b, y > 0.
\]
Since \( g_{x_1}(x_2) \leq 4 \) for \( x_1^c + x_2^c < 3/4 \) we infer that
\[
|\partial_{x_2} (x_1 \xi_1 + x_2 \xi_2 + (1 - x_1^c - x_2^c)^{1/c} \xi_3)| \asymp 1.
\]
This forces that uniformly in \( 0 < x_1 < (3/4)^{1/c} \) we have
\[
|\{0 < x_2 < (3/4 - x_1^c)^{1/c} : x_1 \xi_1 + x_2 \xi_2 + (1 - x_1^c - x_2^c)^{1/c} \xi_3 \in [\alpha, \alpha + \delta]\}| \lesssim \delta
\]
and the conclusion follows.
Case 2. $\xi_3 \geq 1/100$. We see that the left-hand side of (7.3) is equal to

$$\int_{0<x_1<(3/4)^{1/c}} \int_{x_1^2<3/4-x_1^2} \mathbb{1}_{[\alpha-x_1,\alpha+\delta-x_1]}(x_2 \xi_2 + (1-x_1-c-x_2)^{1/c} \xi_3) \, dx_2 \, dx_1.$$ 

It suffices to verify that

$$\int_{0<y<(3/4-b^c)^{1/c}} \mathbb{1}_{[\alpha,\alpha+\delta]}(y \xi_2 + (1-b^c-y^c)^{1/c} \xi_3) \, dy \lesssim \delta^{1/2},$$

(7.2)

uniformly in $0 < b < (3/4)^{1/c}$, $\alpha \in \mathbb{R}$, $0 < \delta \leq 1$, $\xi_3 \geq 1/100$ such that $|\xi_2|^2 + |\xi_3|^2 \leq 1$. Let

$$f(y) := y \xi_2 + (1-b^c-y^c)^{1/c} \xi_3,$$

and observe that

$$f'(y) = \xi_2 - \xi_3 (g_b(y))^{c-1}, \quad 0 < y < (3/4-b^c)^{1/c}.$$ 

We consider two sets

$$I_b := \{0 < y < (3/4-b^c)^{1/c} : |f'(y)| \geq \delta^{1/2}\},$$

$$J_b := \{0 < y < (3/4-b^c)^{1/c} : |f'(y)| \leq \delta^{1/2}\}.$$ 

Note that $J_b$ is an interval and $I_b$ is a sum of at most two intervals, which follow from the fact that the function $y \mapsto g_b(y)$ is increasing. More precisely, we have

$$g_b'(y) = \frac{1-b^c}{(1-b^c-y^c)^{1/c+1}} \approx 1, \quad 0 < y < (3/4-b^c)^{1/c}, \quad 0 < b < (3/4)^{1/c}. \quad (7.3)$$

By the mean value theorem we conclude

$$|\{y \in I_b : f(y) \in [\alpha,\alpha+\delta]\}| \lesssim \delta^{1/2}, \quad \alpha \in \mathbb{R}, \quad 0 < b < (3/4)^{1/c}, \quad 0 < \delta \leq 1.$$ 

Thus to prove (7.2) it is enough to check that

$$|J_b| \lesssim \delta^{1/2}, \quad 0 < b < (3/4)^{1/c}, \quad 0 < \delta \leq 1. \quad (7.4)$$

Observe that

$$J_b = \{0 < y < (3/4-b^c)^{1/c} : 0 \vee \left(\frac{\xi_2 - \delta^{1/2}}{\xi_3}\right)^{1/(c-1)} \leq g_b(y) \leq \left(\frac{\xi_2 + \delta^{1/2}}{\xi_3}\right)^{1/(c-1)}\}.$$

Using (7.3) we see that for all $0 \leq A \leq B < \infty$ and $0 < b < (3/4)^{1/c}$ one has

$$|\{0 < y < (3/4-b^c)^{1/c} : g_b(y) \in [A,B]\}| \lesssim B - A,$$

which shows that

$$|J_b| \lesssim (\xi_2 + \delta^{1/2})^{1/(c-1)} - (\xi_2 - \delta^{1/2})^{1/(c-1)} \vee 0 \lesssim \delta^{1/2}.$$ 

This gives (7.2) and consequently leads to (7.4). The proof of Lemma 7.1 is completed. \(\square\)

Lemma 7.2. Let $c \geq 1$, $C > 0$ and $M > 1$ be fixed. Then

$$\int_{\mathbb{R}^3} \mathbb{1}_{|x| \leq C \lambda^{1/c}} \frac{dx}{(1 + A||x||_c - \lambda)^M} \lesssim \lambda^{3/c-1} A^{-1}, \quad A, \lambda > 0.$$
Proof. Let $C' > 0$ be a constant such that $\{ x \in \mathbb{R}^3 : |x| \leq C \lambda^{1/c} \} \subseteq \{ x \in \mathbb{R}^3 : |x|_c \leq C' \lambda^{1/c} \}$. Then, using polar coordinates, see (4.23), and changing the variable $r \mapsto \lambda^{1/c} t$ we see that the left-hand side in question is controlled by

$$
\int_0^{C \lambda^{1/c}} \frac{r^2 \, dr}{(1 + A|t^c - \lambda|)^M} \lesssim \lambda^{3/c} \int_0^{C'} \frac{dt}{(1 + A|t^c - 1|)^M} \lesssim \lambda^{3/c} \int_{\mathbb{R}} \frac{dt}{(1 + A|t|)^M} \lesssim \lambda^{3/c - 1} A^{-1},
$$

uniformly in $A, \lambda > 0$, and we are done. \qed

**Lemma 7.3.** Let $c \in (1, 5/4)$ be fixed. Then for every $\varepsilon > 0$ we have

$$
\mathcal{F}_{\mathbb{Z}^3}^{-1} \sigma_\lambda (\xi) = \lambda^\varepsilon \int_{\mathbb{R}^3} e(x \cdot \xi) \eta \left( \frac{x}{\lambda^{1/c}} \right) \mathcal{F}_{\mathbb{R}}^{-1} \psi (\lambda^\varepsilon (|x|_c^\varepsilon - \lambda)) \, dx
$$

$$
+ O_\varepsilon (\lambda^{3/c - 1 - (5-4c)/(3c)+\varepsilon}) + \lambda^{3/c - 1} O (|\xi| + \lambda^{-1/(4c)}),
$$

uniformly in $\lambda \geq 1$ and $\xi \in \mathbb{R}^3$.

**Proof.** Using (4.12) we see that

$$
\mathcal{F}_{\mathbb{Z}^3}^{-1} \sigma_\lambda (\xi) = \sum_{m \in \mathbb{Z}^3} e(m \cdot \xi) \sigma_\lambda (m)
$$

$$
= \lambda^\varepsilon \sum_{m \in \mathbb{Z}^3} e(m \cdot \xi) \eta \left( \frac{m}{\lambda^{1/c}} \right) \mathcal{F}_{\mathbb{R}}^{-1} \psi (\lambda^\varepsilon (Q(m) - \lambda))
$$

$$
+ \sum_{m \in \mathbb{Z}^3} e(m \cdot \xi) \eta \left( \frac{m}{\lambda^{1/c}} \right) \int_{-1/2}^{1/2} e((Q(m) - \lambda)t) \tilde{\psi}_\lambda (t) \, dt.
$$

We first show that

$$
\left| \sum_{m \in \mathbb{Z}^3} e(m \cdot \xi) \eta \left( \frac{m}{\lambda^{1/c}} \right) \int_{-1/2}^{1/2} e((Q(m) - \lambda)t) \tilde{\psi}_\lambda (t) \, dt \right| \lesssim \lambda^{3/c - 1 - (5-4c)/(3c)} \log (\lambda + 1), \quad \lambda \in \mathbb{Z}_+, \quad \xi \in \mathbb{R}^3,
$$

(7.6)

and

$$
\sum_{m \in \mathbb{Z}^3} \mathbb{1}_{|m| \leq \lambda^{1/c}} \left| \mathcal{F}_{\mathbb{R}}^{-1} \psi (\lambda^\varepsilon (Q(m) - \lambda)) - \mathcal{F}_{\mathbb{R}}^{-1} \psi (\lambda^\varepsilon (|m|_c^\varepsilon - \lambda)) \right| \lesssim \lambda^{3/c - 1}, \quad \lambda \in \mathbb{Z}_+.
$$

(7.7)

We first justify (7.6). Setting $\lambda_c := (2c)^{-1} (2\lambda)^c$ and using respectively Lemma 4.3 (with $g = \eta_1$), the Cauchy–Schwarz inequality and the Plancherel theorem we see that the left-hand side of (7.6) is controlled by

$$
\int_{\lambda_c \leq |t| \leq 1/2} \left| \prod_{j=1}^{3} \Pi_{t, \lambda}^{\eta_j} (\xi_j) \right| \, dt \lesssim \lambda^{1/3 + 1/(3c)} \log (\lambda + 1) \int_{|t| \leq 1/2} \left| \prod_{j=2}^{3} \Pi_{t, \lambda}^{\eta_j} (\xi_j) \right| \, dt
$$

$$
\lesssim \lambda^{1/3 + 1/(3c)} \log (\lambda + 1) \prod_{j=2}^{3} \left( \int_{|t| \leq 1/2} \left| \Pi_{t, \lambda}^{\eta_j} (\xi_j) \right|^2 \, dt \right)^{1/2}
$$
Lemma 7.2 (with \(A\) uniformly in \(|m|_c^\xi\) dominated by \(F(m)\)), we see that the left-hand side of (7.7) is dominated by

\[
\sum_{m \in \mathbb{Z}^3} \frac{1}{1 + (\lambda^\xi|\xi|_c^\xi - \lambda)^{10}},
\]

which, in turn, by Lemma 4.8 and Lemma 7.2 (with \(A = \lambda^\xi\)) is controlled by

\[
\int_{\mathbb{R}^3} \frac{1}{1 + (\lambda^\xi|\xi|_c^\xi - \lambda)^{10}} \lesssim \lambda^{3/c-1},
\]

giving the claim in (7.7).

Combining (7.5) with (7.6) and (7.7) we see that the proof of Lemma 7.3 will be completed if we show, uniformly in \(\lambda \in \mathbb{Z}_+\) and \(\xi \in \mathbb{R}^3\), that

\[
\lambda^\xi \sum_{m \in \mathbb{Z}^3} e(m \cdot \xi) \eta\left(\frac{m}{\lambda^{1/c}}\right) F_{\mathbb{R}}^{-1} \psi\left(\lambda^\xi(|m|_c^\xi - \lambda)\right) - \int_{\mathbb{R}^3} e(x \cdot \xi) \eta\left(\frac{x}{\lambda^{1/c}}\right) F_{\mathbb{R}}^{-1} \psi\left(\lambda^\xi(|x|_c^\xi - \lambda)\right) dx \lesssim \lambda^{3/c-1} (|\xi| + \lambda^{-1/(4c)}).
\]  

(7.8)

For \(x \in m + [0, 1]^3\), \(\lambda \in \mathbb{Z}_+\), and \(\xi \in \mathbb{R}^3\), we observe that

\[
\left| e(m \cdot \xi) \eta\left(\frac{m}{\lambda^{1/c}}\right) - e(x \cdot \xi) \eta\left(\frac{x}{\lambda^{1/c}}\right) \right| \lesssim (|\xi| + \lambda^{-1/c}) 1_{|x| \leq \lambda^{1/c}}.
\]

Applying the mean value theorem we obtain

\[
\eta\left(\frac{x}{\lambda^{1/c}}\right) \left| F_{\mathbb{R}}^{-1} \psi\left(\lambda^\xi(|m|_c^\xi - \lambda)\right) - F_{\mathbb{R}}^{-1} \psi\left(\lambda^\xi(|x|_c^\xi - \lambda)\right) \right| \lesssim \frac{1}{1 + (\lambda^\xi|\theta_{m,x} - \lambda|)^{10}},
\]

where \(\theta_{m,x}\) is a convex combination of \(|m|_c^\xi\) and \(|x|_c^\xi\); here we have used the fact that

\[
|m|_c^\xi - |m|_c^\xi \lesssim \lambda^{(c-1)/c} \leq \lambda^{-\xi}
\]

uniformly in \(|m| \lesssim \lambda^{1/c}, x \in m + [0, 1]^3, \lambda \in \mathbb{Z}_+.\) Applying these estimates together with Lemma 7.2 (with \(A = \lambda^\xi\)) we see that the left-hand side of (7.8) is controlled by

\[
\lambda^\xi \int_{\mathbb{R}^3} \frac{1}{1 + (\lambda^\xi|\xi|_c^\xi - \lambda)^{10}} \lesssim \lambda^{3/c-1} (|\xi| + \lambda^{-1/(4c)}),
\]

as desired. This finishes the proof of Lemma 7.3. □
Lemma 7.4. We have the following estimates
\[
\int_{\mathbb{R}} |F \phi_{a, \delta}^\pm(t)| dt \lesssim \log(1 + \delta^{-1}), \quad 0 < a \leq 100, \quad 0 < \delta \leq 1/2, \quad (7.9)
\]
and
\[
\int_{\mathbb{R}} (1 + |t|) |F \phi_{a, \delta}^\pm(t)| dt \lesssim \delta^{-1}, \quad 0 < a \leq 100, \quad 0 < \delta \leq 1/2. \quad (7.10)
\]
Proof. Since \( \phi_{a, \delta}^\pm = 1_{[a \pm \delta, 100]} \ast \phi_{\delta} \), we have
\[
F \phi_{a, \delta}^\pm(t) = F(1_{[a \pm \delta, 100]} \ast \phi_{\delta})(t) = F(1_{[a \pm \delta, 100]})(t)F \phi_{\delta}(t),
\]
Taking into account the estimate
\[
|F(1_{[a, \delta]})(t)| \lesssim (1 + |t|)^{-1}, \quad t \in \mathbb{R}, \quad -\infty < a < b \leq 100,
\]
we see that for every \( M \in \mathbb{Z}_+ \) we have
\[
|F \phi_{a, \delta}^\pm(t)| \lesssim_M (1 + |t|)^{-1}(1 + \delta|t|)^{-M}, \quad t \in \mathbb{R}, \quad 0 < a \leq 100, \quad \delta > 0.
\]
To prove the estimate in (7.9) it suffices to use the above estimate and split the integral in question into two pieces \( |t| \leq \delta^{-1} \) and \( |t| > \delta^{-1} \). Then, the conclusion easily follows. The proof of (7.10) is even simpler. It is enough to make the change of variable \( \delta t \mapsto t \).

Now we are ready to prove Theorem 1.9.

Proof of Theorem 1.9. We set \( \delta := \lambda^{-1/(2c)} \). Using the inverse Fourier transform formula and applying Lemma 7.3 and Lemma 7.4, we infer that
\[
\sum_{m \in S^1(\lambda)} \phi_{a, \delta}^\pm(m \cdot \xi) = \int_{\mathbb{R}} F \phi_{a, \delta}^\pm(t)F^{-1}_{\mathbb{Z}^2} \sigma_{\lambda} \left( \frac{t \xi}{\lambda^{1/c}} \right) dt \quad (7.11)
\]
uniformly in \( \lambda \geq 1, \xi \in \mathbb{S}^2, 0 < a \leq 100 \), where
\[
I_\lambda(\xi) := \lambda^{\kappa} \int_{\mathbb{R}} F \phi_{a, \delta}^\pm(t) \int_{\mathbb{R}^3} e(\lambda^{-1/c}x \cdot t\xi) \eta \left( x \lambda^{1/c} \right) F^{-1}_{\mathbb{R}} \psi(\lambda^{\kappa}(|x|^{c} - \lambda)) dx dt.
\]
Changing the variable \( x \mapsto \lambda^{1/c}y \) and using the polar decomposition, see (4.23), we obtain
\[
I_\lambda(\xi) = \lambda^{3/c + \kappa} \int_{\mathbb{R}} F \phi_{a, \delta}^\pm(t) \int_{\mathbb{R}^3} e(y \cdot t\xi) \eta(y) F^{-1}_{\mathbb{R}} \psi(\lambda^{\kappa}(|y|^{c} - 1)) dy dt
\]
which, using the polar decomposition, see (4.23), we obtain
\[
I_\lambda(\xi) = \lambda^{3/c + \kappa} \int_{\mathbb{R}} F \phi_{a, \delta}^\pm(t) \int_{\mathbb{R}^3} e(rw \cdot t\xi) \eta(rw) F^{-1}_{\mathbb{R}} \psi(\lambda^{\kappa}(r^{c} - 1)) d\mu_c(w) dr dt.
\]
We now show that
\[
\int_{\mathbb{R}} |F \phi_{a, \delta}^\pm(t)| \left| \int_{0}^{\infty} r^2 e(rw \cdot t\xi) \eta(rw) F^{-1}_{\mathbb{R}} \psi(\lambda^{\kappa}(r^{c} - 1)) dr \right. - \left. \int_{0}^{\infty} e(w \cdot t\xi) F^{-1}_{\mathbb{R}} \psi(\lambda^{\kappa}(r^{c} - 1)) dr \right| dt \lesssim \lambda^{-1/c}, \quad (7.12)
\]
uniformly in \( \lambda \in \mathbb{Z}_+, \xi \in \mathbb{S}^2, 0 < a \leq 100 \) and \( w \in \mathbb{S}^2_c \). Indeed, by definition \( \eta(w) = 1 \) for \( w \in \mathbb{S}^2_c \) and we see
\[
|r^2 e(rw \cdot t\xi) \eta(rw) - e(w \cdot t\xi)| \lesssim |r - 1(1 + |t|), \quad t \in \mathbb{R}, \quad r > 0, \quad \xi \in \mathbb{S}^2, \quad w \in \mathbb{S}^2_c.
\]
This together with (7.10) shows that the left-hand side of (7.12) is bounded by
\[ \lambda^{1/(2c)} \int_0^\infty \frac{|r - 1| dr}{1 + |\lambda^{\kappa+1}(r^c - 1)|^3} \lesssim \lambda^{1/(2c)} \int_{\mathbb{R}} \frac{|r| dr}{1 + |\lambda^{\kappa+1}r|^3} \lesssim \lambda^{-1/c}, \]
which is the asserted estimate. Let
\[ J_{\lambda}(\xi) := \lambda^{3/c+\kappa} \int_{\mathbb{R}} F_{\mathbb{R}} \phi_{a,\delta}^\pm(t) \int_0^\infty \int_{S^2} e(w \cdot t\xi) F_{\mathbb{R}}^{-1} \psi(\lambda^{\kappa+1}(r^c - 1)) d\mu_c(w) dr dt. \]
Then estimate (7.12) yields
\[ |I_{\lambda}(\xi) - J_{\lambda}(\xi)| \lesssim \lambda^{3/c-1-1/(4c)}, \quad \lambda \in \mathbb{Z}_+, \quad \xi \in S^2, \quad 0 < a \leq 100. \quad (7.13) \]
Changing the variable \( r^c \mapsto s + 1 \) we infer that for every \( M \in \mathbb{Z}_+ \) we have
\[ J_{\lambda}(\xi) = c^{-1} \lambda^{3/c+\kappa} \int_{\mathbb{R}} F_{\mathbb{R}} \phi_{a,\delta}^\pm(t) \int_{-1/2}^{1/2} \int_{S^2} e(w \cdot t\xi) F_{\mathbb{R}}^{-1} \psi(\lambda^{\kappa+1}s) d\mu_c(w)(s + 1)^{1/c-1} ds dt + O_M(\lambda^{-M}) \]
\[ = c^{-1} \lambda^{3/c+\kappa} \int_{\mathbb{R}} F_{\mathbb{R}} \phi_{a,\delta}^\pm(t) \int_{-1/2}^{1/2} \int_{S^2} e(w \cdot t\xi) F_{\mathbb{R}}^{-1} \psi(\lambda^{\kappa+1}s) d\mu_c(w) ds dt + O(\lambda^{3/c-1-1/(4c)}), \]
uniformly in \( \lambda \in \mathbb{Z}_+, \xi \in S^2 \) and \( 0 < a \leq 100 \).
On the other hand, using the formula
\[ \int_{S^2} \phi_{a,\delta}(x \cdot \xi) d\nu_c(x) = \int_{\mathbb{R}} F_{\mathbb{R}} \phi_{a,\delta}^\pm(t) F_{\mathbb{R}^3}^{-1}(\nu_c)(t\xi) dt, \]
and Corollary 1.3 we see that for every \( \varepsilon > 0 \) we have
\[ r_c(\lambda) \int_{S^2} \phi_{a,\delta}^\pm(x \cdot \xi) d\nu_c(x) = c^{-1} \lambda^{3/c-1} \int_{\mathbb{R}} F_{\mathbb{R}} \phi_{a,\delta}^\pm(t) F_{\mathbb{R}^3}^{-1}(\nu_c)(t\xi) dt + O_{\varepsilon}(\lambda^{3/c-1-9/8c/(5c) + \varepsilon}), \]
uniformly in \( \lambda \in \mathbb{Z}_+, \xi \in S^2 \) and \( 0 < a \leq 100 \). Combining this with (7.11), (7.13) and (7.14), and taking into account Lemma 7.1 it suffices to verify that
\[ |\lambda^{\kappa+1} \int_{\mathbb{R}} F_{\mathbb{R}} \phi_{a,\delta}^\pm(t) F_{\mathbb{R}^3}^{-1}(\mu_c)(t\xi) F_{\mathbb{R}}^{-1} \psi(\lambda^{\kappa+1}s) ds dt - \int_{\mathbb{R}} F_{\mathbb{R}} \phi_{a,\delta}^\pm(t) F_{\mathbb{R}^3}^{-1}(\mu_c)(t\xi) dt| \lesssim \lambda^{-1}, \]
uniformly in \( \lambda \in \mathbb{Z}_+, \xi \in S^2 \) and \( 0 < a \leq 100 \). Changing the variable \( \lambda^{\kappa+1}s \mapsto u \) in the first integral above, using the estimate \( |F_{\mathbb{R}}^{-1}(\mu_c)(\zeta)| \lesssim 1 \) for \( \zeta \in \mathbb{R}^3 \), and Lemma 7.4 (a), we see that the above estimate will follow once we show that
\[ \left| \int_{-\lambda^{\kappa+1}/2}^{\lambda^{\kappa+1}/2} F_{\mathbb{R}}^{-1} \psi(u) du - 1 \right| \lesssim \lambda^{-2}, \quad \lambda \geq 1. \]
This, however, is a direct consequence of the fact that \( \int_{\mathbb{R}} F_{\mathbb{R}}^{-1} \psi(t) dt = \psi(0) = 1 \). This completes the proof of Theorem 1.9. \( \square \)

Finally we prove our equidistribution theorem.

**Proof of Theorem 1.8.** Here we will proceed in a similar way as in the proof of Theorem 1.9. For the convenience of the reader we give a sketch of the proof. Since only the values of \( \phi \) on a neighborhood of \( S^2_c \) play a role, without any loss of generality we may assume that
\( \phi \in C_c(\mathbb{R}^3) \). By Corollary \([3]\), one can replace \( r_c(\lambda) \) by \( \left( \frac{2}{\pi} \right)^{3} \frac{\Gamma(1/3)}{\Gamma(3/3)} \lambda^{3/3} \) on the left-hand side of \([1,25]\). Now using the inverse Fourier transform formula we obtain

\[
\sum_{x \in \mathbb{P}^3_3(\lambda)} \phi(x) = \int_{\mathbb{R}^3} \mathcal{F}_{\mathbb{R}^3} \phi(\xi) \mathcal{F}_{\mathbb{Z}^d}^{-1} \sigma_{\lambda} \left( \frac{\xi}{\lambda^{1/3}} \right) d\xi,
\]

and changing the variable \( x \rightarrow \mathbf{r} \).

By Remark \([1,10]\) we have \( \nu_c = \frac{c^2 \Gamma(3/2)}{\Gamma(1/3)} \mu_c \), we see that our task is reduced to showing that

\[
\lambda^{-3/3+1} \int_{\mathbb{R}^3} \mathcal{F}_{\mathbb{R}^3} \phi(\xi) \mathcal{F}_{\mathbb{Z}^d}^{-1} \sigma_{\lambda} \left( \frac{\xi}{\lambda^{1/3}} \right) d\xi \xrightarrow{\lambda \to \infty} \int_{\mathbb{R}^3} \mathcal{F}_{\mathbb{R}^3} \phi(\xi) \mathcal{F}_{\mathbb{R}^3}^{-1}(\mu_c)(\xi) d\xi.
\]

Now applying Lemma \([7,3]\) we see that we can further reduce our problem to proving that

\[
I_{\lambda} \xrightarrow{\lambda \to \infty} \int_{\mathbb{R}^3} \mathcal{F}_{\mathbb{R}^3} \phi(\xi) \mathcal{F}_{\mathbb{R}^3}^{-1}(\mu_c)(\xi) d\xi,
\]

where

\[
I_{\lambda} := \lambda^{-3/3+1+\kappa} \int_{\mathbb{R}^3} \mathcal{F}_{\mathbb{R}^3} \phi(\xi) \int_{\mathbb{R}^3} e \left( \frac{x \cdot \xi}{\lambda^{1/3}} \right) e^{\left( \frac{x}{\lambda^{1/3}} \right)} \mathcal{F}_{\mathbb{R}^3}^{-1} \psi(\lambda^{\kappa}(|\mathbf{r}|^{3} - \lambda)) d\xi.
\]

Changing the variable \( x \to \lambda^{1/3} y \) and using the polar decomposition \([1,23]\), we obtain

\[
I_{\lambda} = \lambda^{1+\kappa} \int_{\mathbb{R}^3} \mathcal{F}_{\mathbb{R}^3} \phi(\xi) \int_{\mathbb{R}^3} e \left( y \cdot \xi \right) e^{\left( \frac{y}{\lambda^{1/3}} \right)} \mathcal{F}_{\mathbb{R}^3}^{-1} \psi\left( \lambda^{\kappa+1}\left( |y|^{3} - 1 \right) \right) d\xi.
\]

Applying the bound

\[ |r^2 e(rw \cdot \xi) \eta(rw) - e(w \cdot \xi)| \leq |r - 1|(|\xi| + 1), \quad r > 0, \quad \xi \in \mathbb{R}^3, \quad w \in \mathbb{S}^2, \]

and changing the variable \( r^c \to s + 1 \) we see that

\[
I_{\lambda} = \lambda^{1+\kappa} \int_{\mathbb{R}^3} \mathcal{F}_{\mathbb{R}^3} \phi(\xi) \mathcal{F}_{\mathbb{R}^3}^{-1}(\mu_c)(\xi) \int_{0}^{\infty} \mathcal{F}_{\mathbb{R}^3}^{-1} \psi(\lambda^{\kappa+1}(r^c - 1)) dr d\xi + O(\lambda^{-(1+\kappa)})
\]

\[ = \lambda^{1+\kappa} \int_{\mathbb{R}^3} \mathcal{F}_{\mathbb{R}^3} \phi(\xi) \mathcal{F}_{\mathbb{R}^3}^{-1}(\mu_c)(\xi) \int_{0}^{\infty} \mathcal{F}_{\mathbb{R}^3}^{-1} \psi(\lambda^{\kappa+1} s)(s + 1) \frac{1}{s} ds d\xi + O(\lambda^{-(1+\kappa)})
\]

\[ = \lambda^{1+\kappa} \int_{\mathbb{R}^3} \mathcal{F}_{\mathbb{R}^3} \phi(\xi) \mathcal{F}_{\mathbb{R}^3}^{-1}(\mu_c)(\xi) \int_{1/2}^{1} \mathcal{F}_{\mathbb{R}^3}^{-1} \psi(\lambda^{\kappa+1} s) ds d\xi + O(\lambda^{-(1+\kappa)}).
\]

Since \( \psi(0) = 1 \) we have

\[
\lim_{\lambda \to \infty} \lambda^{1+\kappa} \int_{1/2}^{1/2} \mathcal{F}_{\mathbb{R}^3}^{-1} \psi(\lambda^{\kappa+1} s) ds = \lim_{\lambda \to \infty} \int_{-\lambda^{\kappa+1/2}}^{\lambda^{\kappa+1/2}} \mathcal{F}_{\mathbb{R}^3}^{-1} \psi(u) du
\]

\[ = \int_{\mathbb{R}} \mathcal{F}_{\mathbb{R}^3}^{-1} \psi(u) du = \psi(0),
\]

using the dominated convergence theorem we see that the identity \([7,15]\) follows. \( \square \)
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