Influence of modal loss on the quantum state generation via cross-Kerr nonlinearity

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In this work we investigate an influence of decoherence effects on quantum states generated as a result of the cross-Kerr nonlinear interaction between two modes. For Markovian losses (both photon loss and dephasing), a region of parameters when losses still do not lead to destruction of non-classicality is identified. We emphasize the difference in impact of losses in the process of state generation as opposed to those occurring in propagation channel. We show moreover, that correlated losses in modern realizations of schemes of large cross-Kerr nonlinearity might lead to enhancement of non-classicality.

I. INTRODUCTION

Nowadays cross-Kerr nonlinearity is considered as a promising tool for quantum computation and non-classical state generation [1]. An entanglement arising between modes participating in the cross-Kerr interaction can be used for establishing an interface between matter qubits and ‘flying’ light photonic [2], for generation of non-Gaussian states and Schrödinger-cat states [3,4], and for performing quantum gate operations [5]. An interest to cross-Kerr nonlinear interactions is heated up by both developing ways to implement effectively even very weak non-linearities (which one commonly expects to have in practice) [3,4,5], and by discovery of methods to produce sufficiently large self-Kerr and cross-Kerr nonlinearities (such as implementation of the electromagnetically induced transparency (EIT) [6,7] and photonic crystals [8]).

Decoherence is a main practical obstacle to implementations of schemes using Kerr and cross-Kerr nonlinearities. A genuine example of quantum state degradation due to losses is a decoherence of a quantum superposition state, the effect drastically enhanced if speaking of a macroscopic superposition state of the Schrödinger cat type. Already more than 20 years ago the self-Kerr nonlinearity was proposed as a tool for generating such a Schrödinger-cat state (more precisely, a superposition of two coherent states with the same amplitudes but opposite phases) [11]. However, photon losses turn this superposition into statistical mixture of two coherent states with the rate proportional to the square modulus of the amplitude of these states. Modifications of the scheme for the cross-Kerr nonlinearities or four-wave mixing brought no advantage with respect to photon loss [3]. This unfortunate circumstance made one look for the ways to circumvent the problem of decoherence that inevitably accompanies Kerr nonlinearity. Recent suggestions in this direction are based on the conditional preparation of desired states (which brings into consideration an additional problem of the finite detection efficiency), and are aimed to exploit weak nonlinearities [4,5,7]. Recently, even a way to produce cat-states ‘on demand’ was suggested using a source of single-photons [12].

In our work we want to discuss an aspect of the decoherence which has been seldom discussed when considering an influence of losses on states generated via Kerr nonlinearity. Namely, we address losses arising in the process of generation and not due to propagation of the generated state via lossy channels. We concentrate our attention on a feature that might be quite significantly pronounced in modern schemes of generating large Kerr nonlinearity: the modal loss can be strongly correlated. Indeed, the modes occupy the same volume and interact with the same physical systems which form the reservoirs. Also, if the Kerr-nonlinearity scheme implies a sufficiently strong dispersive coupling of light modes to emitters, then coupling of these emitters to dissipative reservoirs might also appear to be quite strong. As a result, this would mean strongly correlated modal losses. For example, in photonic crystals high density of states in the vicinity of a modal frequencies and emitter’s transition frequency can cause the strong emitter-field coupling; but it would also imply higher population loss of emitters due to coupling to radiative reservoirs. Dephasing losses of emitters would as well invoke a correlated modal dephasing.

Coupling to correlated reservoirs can drastically change state dynamics in comparison with loss to uncorrelated reservoirs. For example, it was demonstrated that coupling to the common reservoir preserves entanglement of a two-mode state [14]. Moreover, coupling to the common reservoir is capable of creating an entanglement between states of initially unentangled modes even in absence of any direct interaction between them [15,16].

In our work we demonstrate both how the correlated loss arises via Kerr nonlinear process, and how it affects the generated states. For this purpose we derive analytic solution generalizing a powerful and illustrative method of Chaturvedi and Srinivasan [20]. On a number of examples we show how the correlated loss enhances and creates intermodal correlations and even entanglement, and might lead to generation of entangled states quite different from...
those generated in the same scheme without loss. Correlated loss can result in the significantly enhanced robustness of the generation scheme.

The outline of the paper is as follows. In Sec. II and in the related Appendices we describe how cross-correlation terms emerge via correlations of Markovian reservoirs; we consider an example of the emitter-field interaction schemes producing correlated modal losses in the Section III. Then in Sec. IV we describe the method for obtaining exact solutions of the cross-Kerr nonlinear interaction between modes in presence of losses to uncorrelated reservoirs and give generalization of the method for some cases of correlated losses. In the Sec. V we analyze influence of losses in the nonclassical state generation process for the case of uncorrelated loss. Some examples of correlated losses are considered in Sec. VI.

II. MASTER EQUATION FOR CORRELATED AND UNCORRELATED LOSS

To illustrate clearly an influence of correlated and uncorrelated losses, we restrict ourselves to the Markovian loss accounting for photon losses and dephasing of interacting modes. We start from the general effective Hamiltonian $H(t)$ describing both self- and cross-interaction (for the moment we refrain from detailing it) and interaction of modes with reservoirs responsible for losses $V_{\text{loss}}(t)$:

$$
V(t) = H(t) + V_{\text{loss}}(t),
$$

$$
V_{\text{loss}}(t) = a_1^d \Gamma_1(t) + \Gamma_1^\dagger(t) a_1 + a_2^d \Gamma_2(t) + \Gamma_2^\dagger(t) a_2 + a_1^d a_1 D_1(t) + a_2^d a_2 D_2(t).
$$

Here we use the interaction picture with respect to the free Hamiltonians of reservoirs and the modes participating in the interaction process. These modes are described by usual bosonic creation and annihilation operators satisfying

$$
[a_1, a_1^\dagger] = [a_2, a_2^\dagger] = 1, \quad [a_1, a_2^\dagger] = [a_1, a_2] = 0.
$$

The operators $\Gamma_{1,2}(t)$ and $D_{1,2}(t)$ describe reservoirs responsible, correspondingly, to the photon losses in modes $a_1$ and $a_2$, and to dephasing of these modes. They may include also stochastic variables describing different realizations of reservoirs.

It should be emphasized that reservoir operators $\Gamma_{1,2}(t)$ and $D_{1,2}(t)$ (together with the initial state of the reservoir) completely describe the reservoir properties with respect to the interaction with the modes. These operators are built on the basis of underlying microscopic model and account for all relevant physical parameters. For example, if the photon loss reservoir of the first mode is composed of electromagnetic field modes with frequencies $w_j$, described by the creation and annihilation operators $b_j, b_j^\dagger$, then

$$
\Gamma_1(t) = \sum_j g_j b_j \exp\{-iw_j t\},
$$

where each $g_j$ is the constant of interaction of the mode $a_1$ with the $j$th mode of the reservoir. Sets of frequencies $w_j$ and interaction constants $g_j$ describe completely physical properties of the reservoir. In particular, if the reservoir is the set of electromagnetic modes of a non-absorbing dielectric structure, they are found from the eigensolutions of Maxwell’s equations for this structure [24].

We make a number of standard assumptions in considering decoherence: coupling with the reservoirs is weak, the initial state of the modes $a_j$ are uncorrelated with initial state of reservoirs, and correlation times of reservoirs are small enough to enable an implementation of the Born-Markov approximation. Then, using, for example, a time-convolutionless projection operator technique [18], one can obtain the master equation, the Liouville equation of the following form:

$$
\frac{d}{dt} \rho(t) = -i[H(t), \rho(t)] + \frac{1}{2} [\{\gamma_1 - \gamma_{12}\} L(a_1) + (\gamma_2 - \gamma_{12}) L(a_2) + (d_1 - d_{12}) L(a_1^\dagger a_1) + (d_2 - d_{12}) L(a_2^\dagger a_2)] \rho(t) + \frac{1}{2} \{\gamma_{12} L(a_1 + a_2) + d_{12} L(a_1^\dagger a_1 + a_2^\dagger a_2)\} \rho(t),
$$

where the superoperator $L(b)$ acts on the density matrix as

$$
L(b)\rho(t) = 2b\rho(t)b^\dagger - b^\dagger b\rho(t) - \rho(t)b b^\dagger.
$$

Here we set $\hbar = 1$ for simplicity. For more details of the derivation see Appendix A. By construction, the master equation [41] provides for non-negative definite $\rho(t)$ for arbitrary $t \geq 0$. 
Note that we call reservoirs "correlated", if their integrated cross-correlation function is non-zero, for example,

\[ \int d\tau \langle \Gamma_1^\dagger(t) \Gamma_2(\tau) \rangle_r \neq 0 \]

and, correspondingly, the coefficient \( g_{12} \) defined in Eq. (A2) is non-zero. Of course, one can always transform Eq. (4) to the diagonal form. However, in this case Lindblad operators (i.e. operators like \( b \) in the diagonal form (5)) will be the linear superpositions of the former Lindblad operators, and the transformed equation will be still describing a coupling between physical objects represented by these original Lindblad operators (modes \( a_{1,2} \) in our case). For example, the possibility that both modes are coupled to the same reservoir (either the photon loss reservoir or the dephasing one) corresponds to an equality in relations (A3) in Appendix A. In this case the operators describing the reservoir are proportional to each other, say, \( \Gamma_1(t) = x \Gamma_2(t) \) and \( D_1(t) = y D_2(t) \). Equation (4) then reduces to

\[ \frac{d}{dt} \rho(t) = -i[H(t), \rho(t)] + \frac{1}{2} [\gamma_1 \mathcal{L}(a_1 + xa_2) + d_1 \mathcal{L}(a_1^\dagger a_1 + ya_2^\dagger a_2)] \rho(t). \]

The most important point here is that in this equation both modes behave like a single object with respect to relaxation. Such a ‘decoherence’ is able to induce enduring entanglement between modes \( a_1 \) and \( a_2 \) (an example is shown in the Appendix B).

FIG. 1: Examples of level structures for generating cross-Kerr nonlinearities. Coupling to the quantized modes is shown by thin arrows and coupling to classical driving fields is shown by thick arrows.

Furthermore, interacting light modes might experience correlated losses due to fact that they both interact with the same atoms. One can illustrate the mechanism of such a loss by the following qualitative consideration. Let us consider a general Hamiltonian describing the light-atom interaction \( H_0 \) plus terms describing interaction of atoms...
with the dissipative reservoirs

\[ H_{\text{total}} = H_0(t; a_1, a_2, S_k) + \sum_j S_j \Gamma_j, \]  

(6)

where \( S_j \) and \( \Gamma_j \) are operators describing the atoms and dissipative reservoirs, respectively. The Hamiltonian of this form describes a general interaction between field modes and emitters of some type as the correlated loss can occur in different physical systems. Provided atom-reservoir interactions in (6) is sufficiently weak as not to perturb much interaction between light modes and atoms, the following approximation can be used for the atomic operators:

\[ S_j(t) \approx U(t) S_j(0) U(t)^\dagger \approx F_j(t; a_1, a_2, S_k(0)), \]

where \( U(t) = T \exp \left\{ -i \int_0^t \! \! d\tau H_0(t; a_1, a_2, S_k) \right\} \), and \( T \) denotes the time-ordering operator. After averaging out the atomic variable, the terms that describe reservoir-mode coupling in the effective interaction Hamiltonian will take the form of \( \sum_j f_j(t; a_1, a_2) \Gamma_j \). Here \( f_j(t; a_1, a_2) = \langle F_j(t; a_1, a_2, S_k(0)) \rangle_{\text{am}} \) and \( \langle \ldots \rangle_{\text{am}} \) denotes the averaging over atomic states (in general, over emitter’s states). Thus, one can see that coupling of light modes to the same atom (emitter) interacting with the dissipative reservoir under the condition of adiabatic elimination of emitter’s variables leads directly to the mode-reservoir interaction terms in the resulting effective Hamiltonian. Note, that these terms remain linear in reservoir operators \( \Gamma_j \). Hence one can derive a master equation averaging over the reservoir in a standard way. In the Appendix B we give examples of the derivation of effective Hamiltonians and the corresponding master equations discussing the simplest two-mode Jaynes-Cummings system (Fig. I(a)). The described scenario of how correlated loss emerge is quite general, and can take place for a wide range of schemes involving light-shift-induced photonic nonlinearities.

A three-level Λ-system interacting with classical driving and quantum fields represent a more realistic example of the scheme with the correlated loss. Consider the large cross-Kerr nonlinearity generation suggested in Ref. [10] and depicted in Fig. I(b). There two modes are coupled to the transition between 1 and 3 levels of the Λ-system in presence of two classical driving fields on transitions 1-2 and 1-3. In the rotating-wave approximation and in the interaction picture with respect to free Hamiltonian of the reservoir, the problem is described by the following interaction Hamiltonian:

\[ H_{\text{eff}} = [g_1 a_1^\dagger \exp(i(\Delta_1 - \delta) t) + g_2 a_2^\dagger \exp(i(\Delta_1 + \delta) t) + \Omega \exp(i\Delta_2 t) + \Gamma_{13}(t)] \sigma_{13} + \Omega_1 \exp(i\Delta_2 t) \sigma_{23} + \text{H. c.}, \]  

(7)

where \( \Omega \) is the Rabi frequency of the driving fields; \( \sigma_{kl} = |k\rangle \langle l|, k, l = 1, 2, 3; \) and \( g_{1,2} \) are the interaction constants for the coupling of a light mode to an emitter (atom). Here for simplicity we have taken into account only losses on the transition 1-3. The setup depicted in Fig. I(b) can be realized in toroidal microcavities, where \( a_1 \) and \( a_2 \) correspond to the clockwise and counter-clockwise propagating modes [27]. As usually, we assume the Markovian reservoir and the following conditions hold:

\[ \langle \Gamma_{13}(t) \rangle_\tau = 0, \quad \langle \Gamma_{13}(t) \Gamma_{13}(\tau) \rangle_\tau = \gamma \delta(t - \tau). \]

From the Hamiltonian (7), the dynamics is governed by

\[ \frac{d}{dt} \sigma_{31} \approx i[g_1 a_1^\dagger \exp(i(\Delta_1 - \delta) t) + g_2 a_2^\dagger \exp(i(\Delta_1 + \delta) t) + \Omega \exp(i\Delta_2 t)](\sigma_{33} - \sigma_{11}) + i\Omega \exp(i\Delta_2 t) \sigma_{21}. \]  

(8)

Using the approach described in Ref. [10], we assume that level 3 remains practically unpopulated, \( g_k / (\Delta_1 \pm \delta) \ll 1 \) and \( \Omega / \Delta_2 \ll 1 \), as well as \( g_k, \Omega \ll |\Delta_1 - \Delta_2| \). Thus the Λ-system is prepared in the superposition of the metastable levels 1 and 2 (namely, in the state \( (|1\rangle - |2\rangle) / \sqrt{2} \)). Then, neglecting small and rapidly oscillating terms, one obtains from Eqs. (7, 8) the following master equation

\[ \frac{d}{dt} \rho(t) \approx -i \frac{\Delta_2}{2\Omega^2} \left[ \left( \frac{g_1^2}{\Delta_1 - \delta} a_1^\dagger a_1 + \frac{g_2^2}{\Delta_1 + \delta} a_2^\dagger a_2 \right)^2, \rho(t) \right] + \frac{1}{2} \left[ (\gamma_1 - \gamma_{12}) \mathcal{L}(a_1) + (\gamma_2 - \gamma_{12}) \mathcal{L}(a_2) \right] \rho(t) + \frac{\gamma_{12}}{2} \mathcal{L}(a_1 + a_2) \rho(t), \]  

(9)

where

\[ \gamma_{1,2} = \gamma \frac{g_{1,2}^2}{(\Delta_1 \mp \delta)^2}, \quad \gamma_{12} = \gamma \frac{g_1 g_2}{(\Delta_1^2 - \delta^2)}. \]
The master equation (1) describes the cross-Kerr and self-Kerr interactions of two light modes plus their coupling to the correlated reservoirs. As follows from the model depicted in Fig. (1b), it is actually the same reservoir: one can see that $\gamma_1\gamma_2 = \gamma_1^2$. Naturally, loss rates $\gamma_{1,2}$ are much less than the loss rate of the emitter. However, if one deals with input modes in a coherent state of rather large amplitude for generating large cross-Kerr nonlinearities, then even comparatively small losses can strongly influence mode dynamics. Below we consider examples of such an influence, e.g., an example of a single-mode Schrödinger-cat state.

The occurrence of a correlated modal loss due to presence of the emitter (atom in our case) has been noticed in Ref. [10]. They have also pointed out that the loss rate should be proportional to the square of the ratio of the influence, e.g., an example of a single-mode Schrödinger-cat state. Then even comparatively small losses can strongly influence mode dynamics. Below we consider examples of such an influence, as it can be seen from Eq. (5).

III. CROSS-KERR INTERACTION MODEL

Now let us turn to the specific nonlinear interaction described by the same type of the Hamiltonian as in the examples above and in the Appendix B. We will consider the effective Hamiltonian in the master equation (1) $H(t) \equiv H_0$ in the following general form

$$H_0 = \sum_{k,l=1}^{2} \chi_{kl}a_k^\dagger a_k a_l^\dagger a_l.$$  

(10)

It describes the cross-Kerr and self-Kerr interaction with nonlinear coefficients $\chi_{kl}$ of two modes (or mode superpositions) $a_1$ and $a_2$. To solve the problem described by the master equation (1) with the Hamiltonian (10), we adopt a simple and illustrative ‘thermofield’ notation [10, 21]. Essentially, instead of a density matrix acting on some space $\mathcal{H}$, say, $\rho = \sum_{k,l} \rho_{kl} |k\rangle \langle l|$, where $|k\rangle$ is the Fock state with $k$ photons in $\mathcal{H}$, we consider a state vector $|\rho\rangle = \sum_{k,l} \rho_{kl} |k\rangle |\tilde{l}\rangle$ in an extended space $\mathcal{H} \otimes \mathcal{H}^*$, where $|\tilde{l}\rangle$ is the Fock state with $l$ photons in $\mathcal{H}^*$. So when the mode operators $a$ and $a^\dagger$ act on the density matrix from the left, one introduces operators $\tilde{a}$ and $\tilde{a}^\dagger$ is such a manner that

$$|k\rangle \langle l| a \longrightarrow \tilde{a}^\dagger |k\rangle \langle \tilde{l}|, \quad |k\rangle \langle \tilde{l}| a^\dagger \longrightarrow \tilde{a} |k\rangle \langle \tilde{l}|.$$ 

Obviously, operators $a$ and $a^\dagger$ commute with $\tilde{a}$ and $\tilde{a}^\dagger$. An action of the superoperator $\mathcal{L}(a)$ on the density matrix can be represented in the thermofield notation as

$$\mathcal{L}(a)|\rho\rangle \longrightarrow \mathcal{L}(a)|\rho\rangle = (2a\tilde{a} - a^\dagger \tilde{a} - \tilde{a}^\dagger \tilde{a})|\rho\rangle.$$ 

Also, the commutator of any function of the operators $a$ and $a^\dagger$, e.g., the Hamiltonian $H_0(t; a, a^\dagger)$ is re-written as:

$$[H_0(t; a, a^\dagger), \rho] \longrightarrow H_0(t)|\rho\rangle = (H_0(t; a, a^\dagger) - H_0(t; \tilde{a}, \tilde{a}^\dagger)) |\rho\rangle.$$ 

With help of these notations Eq. (11) can be represented in the ‘Hamiltonian’ form as

$$\frac{d}{dt} |\rho(t)\rangle = H_{total} |\rho(t)\rangle \equiv$$

$$\frac{1}{2}(-2iH_0(t) + (\gamma_{a1} - \gamma_{12}) L(a_1) + (\gamma_{12} - \gamma_{a2}) L(a_2) + (d_1 - d_{12}) L(a_1^\dagger a_1) + (d_2 - d_{12}) L(a_2^\dagger a_2)) |\rho(t)\rangle +$$

$$\frac{1}{2} (\gamma_{12} L(a_1 + a_2) + d_{12} L(a_1^\dagger a_1 + a_2^\dagger a_2)) |\rho(t)\rangle,$$  

(11)

Its solution is then of the form

$$|\rho(t)\rangle = \exp\{H_{total}t\} |\rho(0)\rangle.$$  

(12)

The advantage of using thermofield notation over more traditional algebraic manipulation with superoperators is that in many situations (and, particularly, ones of our interest) it enables to simplify, make more illustrative and less cumbersome finding the solution [12] and estimation of time-dependent matrix elements. In particular, it allows to represent in a simple form a factorization of the superoperator $\exp\{H_{total}t\}$ into multipliers with easily estimated actions on the number states [20].
To illustrate this, let us consider a simple problem of modal loss in a single mode described by the equation
\[
\frac{d}{dt}\rho(t) = \frac{1}{2}\gamma_0 L(a)\rho(t).
\]
The key to solving this equation lies in the observation that the operators
\[
A_+ \equiv a \hat{a}^\dagger, \quad A_- \equiv a \hat{a}, \quad A_3 \equiv (a^\dagger a + \hat{a}^\dagger \hat{a} + 1)/2
\]
generate the SU(1,1) algebra with the Casimir invariant \( A_0 \equiv a^\dagger a - \hat{a}^\dagger \hat{a} \). Using the disentangling theorem for this group, we arrive at the simple result
\[
|\rho(t)\rangle = \exp\left\{\frac{2a}{\gamma_0} t\right\} \exp[-\gamma_0 t A_3] \exp[(1 - e^{-\gamma_0 t}) A_-] |\rho(0)\rangle.
\]
Generally, the dynamics described by the solution leads to transforming initially pure states into mixtures. However, for a coherent initial state with the amplitude \( \alpha \),
\[
|\rho(0)\rangle = |\alpha\rangle|\alpha^*\rangle = \sum_{m,n=0}^\infty \exp\{-|\alpha|^2\} \frac{a^m \alpha^* n}{\sqrt{m! n!}} |m\rangle|n\rangle,
\]
equation gives the following result:
\[
|\rho(t)\rangle = |\alpha \exp(-\gamma_0 t/2)|\alpha^* \exp(-\gamma_0 t/2)\rangle.
\]
Returning now to the effective Hamiltonian that describes cross-Kerr and self-Kerr interaction of two modes (or modal superpositions) \( a_1 \) and \( a_2 \), we write it in the thermofield notation as follows:
\[
H_0 \longrightarrow H_0 = \sum_{k,l=1}^2 \chi_{kl} A_0^{(k)} (2A_3^{(l)} - 1).
\]
Operators in Eq. are \( A_0^{(k)} = a_k^\dagger a_k - a_k^\dagger a_k, \quad A_3^{(k)} = (a_k^\dagger a_k + \hat{a}_k^\dagger \hat{a}_k + 1)/2 \).

For completely uncorrelated reservoirs of different modes (i.e. \( \gamma_{12} = d_{12} = 0 \)) the master equation with the Hamiltonian \( H_0 \) given by Eq. can be solved exactly using the following factorization:
\[
\exp[H_{\text{total}} t] = \exp\left\{-\frac{d_1}{2} (A_0^{(1)})^2 - \frac{d_2}{2} (A_0^{(2)})^2 + ip_1 A_0^{(1)} + ip_2 A_0^{(2)} + \frac{\gamma_1 + \gamma_2}{2}\right\} \times \exp[(iP_1 - \gamma_1)A_3^{(1)}] \exp[(iP_2 - \gamma_2)A_3^{(2)}] \exp[\Gamma_{1-} A_1^{(1)}] \exp[\Gamma_{2-} A_1^{(2)}]
\]
where superoperators \( A_{1,2}^{(1,2)} \) are defined similarly as in Eq. and
\[
p_k = \sum_{l=1}^2 \chi_{kl} A_l^{(l)}, \quad \Gamma_{k-} = \frac{\gamma_k\{e^{ip_k} - \gamma_0\}}{ip_k - \gamma_k}, \quad P_k = \sum_{l=1}^2 \chi_{kl} A_l^{(l)}.
\]
It is useful to note that in Eq. all multipliers apart from two last ones are diagonal in the number-state basis. Also, it is easy to see that \( \Gamma_{k-}, A_0^{(k)} \) are diagonal in the number-state basis, and operators \( A_0^{(k)} \) are simply products of annihilation operators. Thus, Eq. provides for simple analytic solutions both for coherent initial states of interacting modes.

The solution for uncorrelated reservoirs can be straightforwardly generalized for some special cases of correlated reservoirs and the Hamiltonian. For example, let us consider the problem without dephasing, \( d_k = d_{12} = 0 \), and introduce the rotated mode operators \( b_k \) as
\[
a_1 = b_1 \cos(\phi) + b_2 \sin(\phi), \quad a_2 = b_2 \cos(\phi) - b_1 \sin(\phi),
\]
where \( \tan(2\phi) = 2\gamma_{12}/(\gamma_2 - \gamma_1) \). Then for the non-unitary part of the master equation one has
\[
(\gamma_1 - \gamma_{12})\mathcal{L}(a_1) + (\gamma_2 - \gamma_{12})\mathcal{L}(a_2) + \gamma_{12}\mathcal{L}(a_1 + a_2) \longrightarrow \tilde{\gamma}_1\mathcal{L}(b_1) + \tilde{\gamma}\mathcal{L}(b_2),
\]
where \( \tilde{\gamma}_1 = \gamma_1 \cos^2(\phi) + \gamma_2 \sin^2(\phi) - \gamma_{12} \sin(2\phi), \quad \tilde{\gamma} = \gamma_2 \cos^2(\phi) + \gamma_1 \sin^2(\phi) + \gamma_{12} \sin(2\phi) \). Obviously, if the transformation leaves the form of the Hamiltonian invariant, one can derive an exact solution in the way described in this Section.
IV. UNCORRELATED RESERVOIRS

A. General solution for the uncorrelated reservoirs

In this Section we consider specific effects of the uncorrelated losses in the process of the cross-Kerr nonlinear interaction. Notably, the losses in such a nonlinear process can lead to loss-mediated correlations between the modes, as seen from Eq. (10). These intermodal correlations modify the effect of losses on the quantum state generated in the cross-Kerr interaction with respect to the result, which one would intuitively expect treating the generation and loss separately, i.e., subjecting to loss a state that has been produced without losses.

Consider a particular problem of generating an entangled two-mode state from initially uncorrelated coherent states. We assume that the nonlinearity is given purely by the cross-Kerr interaction, i.e. we put \( \chi_{kl} = (1 - \delta_{kl}) \chi / 2 \) in the Hamiltonian (10). Producing entangled states this way is important in a number of schemes of quantum computation and communication using continuous variables [3, 4, 5]. We assume modes \( a_1 \) and \( a_2 \) to be initially in coherent states with amplitudes \( \alpha_1 \) and \( \alpha_2 \), respectively. As was pointed in the previous Section, for this choice of initial states, the solution given by Eq. (16) has a simple form:

\[
|\rho(t)\rangle = \exp \left[ -\frac{|d_1 A_0(1)|^2 + d_2 A_0(2)|^2 t}{2} \right] \exp \left\{ i \chi t(\alpha_1^* a_1 \alpha_2 a_2 - \alpha_1^* a_1 \alpha_2 a_2) \right\} \exp \left[ \frac{\gamma_1 t}{2} (a_1^* a_1 + \alpha_1^* \alpha_1) - \frac{\gamma_2 t}{2} (a_2^* a_2 + \alpha_2^* \alpha_2) \right]
\]

Then expressions (20) can be expanded as

\[
f^{(2)}_{mn}(t) = f^{(2)}_{mn}(t) = \frac{\gamma_1 (e^{i \chi t (m-n)} - \gamma_1 t - 1)}{i \chi (m-n) - \gamma_1} |\alpha_1|^2, \quad f^{(1)}_{kl}(t) = \frac{\gamma_2 (e^{i \chi t (k-l)} - \gamma_2 t - 1)}{i \chi (k-l) - \gamma_2} |\alpha_2|^2.
\]

B. Analysis: When is the purity of the state not broken by damping?

Now let us analyze the solution (18) in more detail and consider for the moment the case of no dephasing \( (d_1 = d_2 = 0) \). In this case the expression for the purity of the state given by the solution (18) is quite similar in structure to this solution itself:

\[
\text{Tr} \{ (\rho(t))^2 \} = e^{-2|\alpha_1|^2 - 2|\alpha_2|^2} \sum_{k,l,m,n=0}^{\infty} \frac{|\alpha_1|^{2(k+l)} |\alpha_2|^{2(m+n)}}{k!l!m!n!} \exp \left\{ -\frac{1}{2} (d_1 (k-l)^2 + d_2 (m-n)^2) t \right\}
\]

\[
\times \exp \left\{ i \chi t (km - ln) - \frac{\gamma_1 t}{2} (k+l) - \frac{\gamma_2 t}{2} (m+n) + f^{(2)}_{mn}(t) + f^{(1)}_{kl}(t) \right\} |k||l||m||n|, \quad (19)
\]

where

\[
f^{(2)}_{mn}(t) = \frac{\gamma_1 (e^{i \chi t (m-n)} - \gamma_1 t - 1)}{i \chi (m-n) - \gamma_1} |\alpha_1|^2, \quad f^{(1)}_{kl}(t) = \frac{\gamma_2 (e^{i \chi t (k-l)} - \gamma_2 t - 1)}{i \chi (k-l) - \gamma_2} |\alpha_2|^2. \quad (20)
\]
The third terms of these expansions already describe quite well typical effects produced by simultaneous damping and cross-Kerr nonlinearity. For example, for \( f_{mn}^{(2)}(t) \) in the limit of small times one has

\[
f_{mn}^{(2)}(t) \approx \gamma_1 |\alpha_1|^2 \left( 1 - \frac{\gamma_1 t}{2} + i \frac{\gamma_2 t}{2} \right)^2 - \frac{1}{6} |\gamma_2|^2 t^2 (m-n)^2 - \frac{i}{6} \gamma_1 \chi t^2 (m-n) \right).
\]

Obviously, first three terms in the round brackets of this expressions do not lead to breaking of the purity of the state. Retaining only them in expansion renders unity value for the right-hand side of Eq. (21). Indeed, assuming

\[
1 - \frac{\gamma_1 t}{2} \gg \frac{1}{6} |\gamma_1|^2 t^2 - \chi^2 t^2 (m-n)^2, \quad 1 - \frac{\gamma_2 t}{2} \gg \frac{1}{6} |\gamma_2|^2 t^2 - \chi^2 t^2 (k-l)^2,
\]

one obtains the time-dependent density matrix formally coinciding with the result for no photon loss [3]:

\[
|\rho(t)\rangle \approx \exp\{-|\alpha_1|^2/2\} \sum_{k,l=0}^\infty \left[ \frac{[\alpha_1(t)]^k [\alpha_1^*(t)]^l}{\sqrt{k!l!}} |k\rangle \langle l| \alpha_2(t)e^{i\chi kl}) |\alpha_2^*(t)e^{i\chi tl}. \right.
\]

Here time-dependent amplitudes do not depend on the numbers \( k, l \):

\[
\alpha_1(t) = \alpha_1 \exp \left\{ -\frac{\gamma_1 t}{2} - \frac{i}{2} \gamma_2 \chi \alpha_2^2 t^2 \right\}, \quad \alpha_2(t) = \alpha_2 \exp \left\{ -\frac{\gamma_2 t}{2} - \frac{i}{2} \gamma_1 \chi |\alpha_1|^2 t^2 \right\}. \tag{26}
\]

The state given by Eqs. (20) remains negligibly affected by losses if \( \gamma_{1,2} \ll 1 \). Thus, the considered scheme of non-classical state generation is quite robust with respect to photon loss (in drastic difference with propagation losses of already generated cat-state where the off-diagonal terms \(|k\rangle \langle l|, k \neq l \) will decay with the rates proportional to \( \gamma |\alpha|^2 \)). In addition, it is interesting to note, that one might be able to satisfy conditions \( \gamma_{1,2} \ll 1 \) in schemes involving dispersive atom-field interactions in QED where it is possible to restrict losses to the photon loss of cavity modes [10]. Moreover, further in this work we consider ways to circumvent an influence of losses by making them correlated.

Remarkably, a purity of the generated bimodal state can be preserved not only in the case of small losses, but also for large loss. Indeed, in the limits of large losses one can consider \( \chi(k-l) \) and \( \chi(m-n) \) as small quantities and expand functions [20] in the following manner:

\[
|\alpha_2|^{-2} f_{kl}^{(1)}(t) \approx (1 - e^{-\gamma_2 t}) - i \chi(k-l) \left[ \left( t + \frac{1}{\gamma_2} \right) e^{-\chi t} - \frac{1}{\gamma_2} \right] + \chi^2 (k-l)^2 \left[ \left( \frac{t^2}{2} + \frac{t}{\gamma_2} + \frac{1}{\gamma_2^2} \right) e^{-\gamma t} - \frac{1}{\gamma_2^2} \right], \tag{27}
\]

\[
|\alpha_1|^{-2} f_{mn}^{(2)}(t) \approx (1 - e^{-\gamma_1 t}) - i \chi(m-n) \left[ \left( t + \frac{1}{\gamma_1} \right) e^{-\chi t} - \frac{1}{\gamma_1} \right] + \chi^2 (m-n)^2 \left[ \left( \frac{t^2}{2} + \frac{t}{\gamma_1} + \frac{1}{\gamma_1^2} \right) e^{-\gamma t} - \frac{1}{\gamma_1^2} \right].
\]

Note that this approximation holds for arbitrary interaction times. If the interaction time is sufficiently large to fulfill the conditions

\[
1 - e^{-\gamma_2 t} \gg \chi^2 (k-l)^2 \left[ \left( \frac{t^2}{2} + \frac{t}{\gamma_2} + \frac{1}{\gamma_2^2} \right) e^{-\gamma_2 t} - \frac{1}{\gamma_2^2} \right], \tag{28}
\]

\[
1 - e^{-\gamma_1 t} \gg \chi^2 (m-n)^2 \left[ \left( \frac{t^2}{2} + \frac{t}{\gamma_1} + \frac{1}{\gamma_1^2} \right) e^{-\gamma_1 t} - \frac{1}{\gamma_1^2} \right], \tag{29}
\]

then the state given by Eqs. (19) is practically pure. Under the conditions (29) this state is of the form described by Eq. (25) with the time-dependent amplitudes given by

\[
\alpha_1(t) = \alpha_1 \exp \left\{ -\frac{\gamma_1 t}{2} - i \chi |\alpha_2|^2 \left[ t e^{-\gamma_2 t} - \chi^{-1} (1 - e^{-\gamma_2 t}) \right] \right\},
\]

\[
\alpha_2(t) = \alpha_2 \exp \left\{ -\frac{\gamma_2 t}{2} - i \chi |\alpha_1|^2 \left[ t e^{-\gamma_1 t} - \chi^{-1} (1 - e^{-\gamma_1 t}) \right] \right\}. \tag{30}
\]

Clearly, purity of the resulting state in the long time-limit is precisely a consequence of a strong photon loss. In this way strong photon loss paradoxically suppresses state mixing predicted by the general solution (19).
C. Survival of non-classicality for large losses

There is another interesting feature that distinguishes the losses occurring in the process of Kerr interaction from the losses that take place after the interaction. In particular, the non-Gaussian state generated by the cross-Kerr interaction in the scheme discussed in [3] can retain its non-classical features even for the loss level, which would completely eliminate any such features in case of free propagation of the state. To be specific, a typical signature of non-classicality of a quantum state is the fact that its Wigner function is negative in some regions of the phase space [22]. We will show that for a 50% photon loss occurring in the scheme [3] during the cross-Kerr interaction, the Wigner function of the output state retains its negativity while the same photon loss occurring after the interaction would make the Wigner function necessarily positive.

![FIG. 2: (Color online) Wigner function $W(u)$ of the non-classical non-Gaussian state of Ref. [3] exhibits strong negativity. Wigner function is viewed along the imaginary axis of the $u$ plane.](image1.png)

![FIG. 3: (Color online) Wigner function $W(u)$ for about 50% photon loss in mode $a$ corresponding to $\gamma_a t = 0.7$. The negative region of the Wigner function is now negligible, but still present, and the amplitude of the state is clearly damped.](image2.png)

In the scheme [3] for generating non-classical states, two coherent states in modes $a_1, a_2$ ($a, b$ in notations of [3]) interact via cross-Kerr effect in a non-linear medium and subsequently the $x$-quadrature of mode $a_2$ is measured. The resulting state of mode $a_1$ exhibits Wigner function with negative regions (see Fig. [2]) and a characteristic
Suppose mode $a_1$ is subject to losses during the cross-Kerr interaction, i.e., $\gamma_1 > 0$ in Eq. (26) while we assume $\gamma_2 = d_1 = d_2 = 0$. The photon loss in mode $a_1$ is given by the reduction of the coherent amplitude described by Eq. (20). Consider the situation when the mean photon loss is 50%. This corresponds to $e^{-\gamma_1 t} = 1/2 \Rightarrow \gamma_1 t = \ln 2 \approx 0.69$. The plot of the corresponding Wigner function is shown in Fig. 3 which shows clearly that although the negative region of the Wigner function is suppressed, it is still present.

Now compare this with the situation when losses are introduced to mode $a_1$ after the lossless cross-Kerr interaction has taken place. Such losses are equivalent to mixing mode $a_1$ with the vacuum state on a beam splitter (BS) and discarding one BS output. A 50% loss corresponds to a 50/50 BS. It is known $^{23}$ that for such a balanced beam splitter the Wigner function of one BS output can be expressed as a scaled Husimi Q-function of the input state:

$$W_{\text{out}}(\alpha) = 2Q_{\text{in}}(\sqrt{2} \alpha).$$  \hspace{1cm} (31)

The Q-function of a state $\rho$ is defined as $Q(\alpha) = \langle \alpha | \rho | \alpha \rangle / \pi$ and is clearly non-negative for all coherent state amplitudes $\alpha$, i.e., in the whole phase space of the mode. Therefore negative regions of the Wigner function cannot survive losses larger than 50% if these occur during propagation of the generated state. Hence losses that take place in the process of state generation via the cross-Kerr interaction are less harmful to the non-classicality of the output state than the same level of loss after the interaction.

D. Dephasing into independent reservoirs

As can be seen from the solution $^{11}$, an influence of dephasing into independent reservoirs can be profoundly destructive. Dephasing leads to diminishing of the off-diagonal elements in the number-state basis with rates proportional to the difference of these numbers. So for the coherent state with the amplitude $\alpha$, a condition $\delta_k |\alpha_k|^2 t \ll 1$ should be fulfilled for the interaction time $t$ to consider the influence of dephasing negligible. In the recently discussed QED schemes for generating non-linearity (including EIT-like ones), dephasing is usually disregarded without being estimated (see, for example, $^{11}$). However, since the rate of losses increases with increasing intensity of the coherent states used in the generation process, more caution is required with respect to the dephasing. For schemes involving large cross-Kerr nonlinearity in the solid-state structures (such as, for example, photonic crystals) emitter-mediated dephasing could be a major source of state decoherence and a reason for failure of schemes involving initial coherent states with large number of photons. However, remarkably, if the cross-Kerr nonlinear interaction scheme is designed in such a way that losses due to dephasing are correlated, then it may be possible to avoid their destructive effects. This is the subject of the next section.

V. CORRELATED RESERVOIRS

It is well established, that the states of quantum systems can be correlated and even entangled through interaction with the common reservoir $^{13, 16}$. This phenomenon can occur even in absence of any direct interaction between systems. In in Appendix B3 we give an example of such a phenomenon for a scheme of generating the cross-Kerr nonlinearity via dispersive interaction of modes with emitters. There a beam-splitting action of the common reservoir is considered, and it is shown how such a reservoir can produce a stationary entangled state of two modes.

Here we focus our attention on another important possibility: namely, on a way to neutralize a destructive influence of losses in the process of generation by rendering these losses correlated and exploit a correlating effect of the reservoir.

Let us consider an example of the realistic scheme to produce the large cross-Kerr nonlinearity described in Section II (see Eq. (7) and the text thereafter). We consider the case of

$$\frac{g_1^2}{\Delta_1 - \delta} = \frac{g_2^2}{\Delta_2 + \delta} = -\chi \frac{2\Omega^2}{\Delta_2},$$

so the master equation (9) now transforms as

$$\frac{d}{dt} \rho(t) \approx i\chi \left[ \left( a_1^\dagger a_1 + a_2^\dagger a_2 \right)^2, \rho(t) \right] + \frac{1}{2} \left[ (\gamma_1 - \gamma_{12}) \mathcal{L}(a_1) + (\gamma_2 - \gamma_{12}) \mathcal{L}(a_2) \right] \rho(t) + \frac{\gamma_{12}}{2} \mathcal{L}(a_1 + a_2) \rho(t).$$  \hspace{1cm} (32)

In absence of decoherence (i.e., $\gamma_k = \gamma_{12} = 0$), the scheme described by Eq. (9) is able to generate entangled superpositions of Schrödinger-cat states from initial coherent states of modes $a_1$ and $a_2$ (for similar schemes see, for
FIG. 4: (Color online) Examples of the Wigner function for the conditioned cat state of the rotated mode $b_1$. Figure (a) corresponds to absence of loss. Figure (b) corresponds to the perfectly correlated loss, $\gamma_1 = \gamma_2 = \gamma_{12} = 10\chi$; for figure (d) $\gamma_1 = \gamma_2 = \gamma_{12} = 3\chi$. Figure (c) corresponds to the completely uncorrelated loss, $\gamma_1 = \gamma_2 = \gamma_{12} = 3\chi$; figure (e) corresponds to the completely uncorrelated loss $\gamma_1 = \gamma_2 = 0.5\chi$, $\gamma_{12} = 0$. For all figures $\chi t = \pi/2$.

example, [11, 30]). It is easy to see that for $\chi t = \pi/2$ the scheme produces an entangled superposition of coherent states from a pair of initially uncorrelated coherent states, which reads as

$$\exp \left\{ \frac{i\pi}{2} \left( a_1^\dagger a_1 + a_2^\dagger a_2 \right)^2 \right\} |\alpha_1\rangle|\alpha_2\rangle = \frac{1}{\sqrt{2}} (i|\alpha_1\rangle|\alpha_2\rangle + | -\alpha_1\rangle| -\alpha_2\rangle).$$

(33)

In the presence of correlated reservoirs one can find a solution of Eq. (32) noticing that under the rotation (17) the Hamiltonian part of Eq. (32) remains invariant. Thus, performing the rotation one obtains

$$\frac{d}{dt} \rho(t) \approx i\chi \left[ \left( b_1^\dagger b_1 + b_2^\dagger b_2 \right)^2 , \rho(t) \right] + \frac{1}{2} \left[ \bar{\gamma}_1 \mathcal{L}(b_1) + \bar{\gamma}_2 \mathcal{L}(b_2) \right] \rho(t)$$

(34)

with the exact solution given by Eq. (16). Also, one immediately sees that for the completely correlated reservoirs (i.e. $\gamma_1 \gamma_2 = \gamma_{12}^2$), the mode $b_1$ is not affected by the loss, as $\bar{\gamma}_1 = 0$ then. Naturally, a cat state can be generated in this mode. Thus, in the limit of large loss, $\bar{\gamma}_2 t \gg 1$, and for $\chi t = \pi/2$ it follows from Eq. (34) that

$$|\rho(\pi/2\chi)\rangle \approx |\Psi\rangle|\overline{\Psi}\rangle; \quad |\Psi\rangle = \frac{1}{\sqrt{2}} (i|\tilde{\alpha}_1\rangle|\tilde{\alpha}_2\rangle + | -\tilde{\alpha}_1\rangle| -\tilde{\alpha}_2\rangle),$$

(35)
where \( \alpha_1 = \alpha_1 \cos^2(\phi) - \alpha_2 \cos(\phi) \sin(\phi) \), \( \alpha_2 = \alpha_2 \sin^2(\phi) - \alpha_1 \cos(\phi) \sin(\phi) \). We also assume that \( 2\chi |\alpha_1 \cos(\phi) - \alpha_2 \sin(\phi)|^2 < \tilde{\gamma}_2 \). From Eq. (35), we derive a conclusion that the only effect of completely correlated loss is a reduction of amplitudes of the coherent states forming the superposition (33).

Thus, we have seen that by making losses completely correlated one can completely avoid decoherence caused by these losses. Of course, in practice one can hardly have completely correlated reservoirs due to presence of additional uncorrelated loss (such as modal losses due to coupling to additional reservoirs etc.). Nevertheless, designing the scheme such as to have predominantly correlated losses might greatly enhance its robustness in production of nonclassical states. We illustrate this with the simple example of the conditioned cat-state generation from the solution of Eq. (12). If the rotated mode \( b_2 \) impinges on the detector, in the case of no signal on the detector the rotated mode \( b_1 \) is (up to the normalization factor) in the state:

\[
|\rho_1(t)\rangle \sim \sum_{k,l=0}^{\infty} \frac{\tilde{\alpha}_1^k \tilde{\alpha}_2^l}{\sqrt{k!l!}} \exp \left\{ \frac{i\chi t (k^2 - l^2)}{2} + \frac{\tilde{\gamma}_1 t}{2} (k + l) + f_{kl}(t) \right\} |k\rangle |l\rangle,
\]

where \( \tilde{\alpha}_1 = \alpha_1 \cos(\phi) - \alpha_2 \sin(\phi) \), \( \tilde{\alpha}_2 = \alpha_2 \cos(\phi) + \alpha_1 \sin(\phi) \), and

\[
f_{kl}(t) = \frac{\tilde{\gamma}_1 (e^{i\chi t (k-l)} - 1)}{i\chi (k - l - \tilde{\gamma}_1)} |\tilde{\alpha}_1|^2 + \frac{\tilde{\gamma}_2 (e^{i\chi t (k-l)} - 1)}{i\chi (k - l - \tilde{\gamma}_2)} |\tilde{\alpha}_2|^2.
\]

In Fig. 4 one can see examples of the Wigner function of the state (36). For no loss (Fig. 4(a)) and \( \chi t = \pi/2 \) the state (36) is a usual Schrödinger cat state with the pronounced oscillations near the origin. Large correlated loss (\( \gamma_{1,2} \gg \chi \)) changes the size of the cat and rotates it (Fig. 4(b)), but otherwise leaves it intact. Lower correlated loss distorts the cat (Fig. 4(d)) due to influence of additional mixing between modes in the interaction process (as it follows from Eq. (36)). Nevertheless, the state is strongly non-classical. Uncorrelated loss with the same rate eliminates the non-classicality outright (Fig. 4(c)). Correlated loss allows for the non-classicality to survive (Fig. 4(f)) even if the uncorrelated loss with the rate equal to difference between individual rates and the correlation rate (e.g., \( \gamma = \gamma_1 - \gamma_{12} \)) destroys the non-classicality completely (Fig. 4(e)).

VI. CONCLUSIONS

In what presented here, we followed the quest to find ways to impair the decoherence processes in quantum state generation and manipulation. For the particular class of nonlinear interaction processes, we have found two striking examples of loss dynamics, for which the losses themselves counteract decoherence: loss-mediated correlations between the interacting modes and losses to the correlated reservoirs. The latter result in strongly correlated modal loss.

These correlated losses influence the dynamics of the modes undergoing the cross-Kerr nonlinear interaction in a completely different way than losses into independent reservoirs, the aspect of quantum nonlinear dynamics to large extent unexplored so far. Thus, remarkably, designing the schemes for the generation of the Kerr nonlinearity in such a way that losses in this process are correlated, one can greatly diminish their destructive impact and even exploit them for entanglement generation (see also Appendix B). As to the origin of this effect, if both modes, for example, interact with the same emitter transition, emitter losses are likely to lead to the correlated loss of both of these modes. Note, that in this case a significant modal loss might occur even in the case when the emitter subject to losses stays in superposition of metastable levels with negligibly small probability to occupy higher, decaying levels. For more details and examples on the origin of the correlated losses and their entangling effect see Appendix B.

Turning to the other aforementioned unexpected aspect of the quantum dynamics, we have demonstrated that losses in the nonlinear process of state generation affect the quantum state in quite a different way to the propagation losses of already generated nonclassical state. This is mainly due to the loss-mediated correlations between the modes participating in the cross-Kerr interaction. In addition, losses through coupling to the correlated reservoirs (correlated loss) further enhance the difference between the decoherence processes during and after the state generation. In particular, non-classicality seems to be more robust with respect to the generation loss than to the propagation loss. We discussed an example of generation loss exceeding 50% with negative values of the Wigner function preserved, whereas the propagation loss exceeding 50% renders the Wigner function completely positive.

Acknowledgments

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APPENDIX A: DERIVATION OF THE MASTER EQUATION

We start from the general effective Hamiltonian $H(t)$ of Eq. (12) describing both self- and cross-interaction and interaction of modes with reservoirs responsible for losses (see Section III):

$$V(t) = H(t) + V_{\text{loss}}(t), \quad V_{\text{loss}}(t) = a_1^\dagger \Gamma_1(t) a_1 + a_2^\dagger \Gamma_2(t) a_2 + a_1^\dagger a_1 D_1(t) + a_2^\dagger a_2 D_2(t).$$

Setting $\hbar = 1$ for sake of simplicity, and using a time-convolutionless projection operator technique, one obtains the following master equation [18]:

$$\frac{d}{dt} \rho(t) = -i[H(t), \rho(t)] - \int_0^t dt \{ (V_{\text{loss}}(t)V_{\text{loss}}(\tau)), \rho(t) + \rho(t)(V_{\text{loss}}(\tau)V_{\text{loss}}(t)) - (V_{\text{loss}}(\tau)\rho(t)V_{\text{loss}}(t)) - (V_{\text{loss}}(t)\rho(t)V_{\text{loss}}(\tau)) \} \tag{A1}$$

where $\rho(t)$ denotes the density matrix of the system averaged over states of all reservoirs (and over possible stochastic variables, too), and $\langle \ldots \rangle_t$ denotes averaging over all reservoirs. Reservoir modes are assumed to be initially in the vacuum states. Further, we consider reservoirs of different types to be independent; correlation functions $\langle \Gamma_j(t)D_k(\tau) \rangle_t$, $j, k = 1, 2$ are taken to be zero. We consider the reservoirs of the same kind to be mutually correlated, i.e., for Markovian dephasing reservoirs we assume

$$\langle D_i(t)D_i(\tau) \rangle_t = \frac{1}{2} d_i \delta(t - \tau), \quad \langle D_1(t)D_2(\tau) \rangle_t = \frac{1}{2} d_{12} \delta(t - \tau).$$

Here $\delta(t - \tau)$ is the delta-function and the rates $d_1, d_2$, are real and non-negative. The cross-correlation parameter $d_{12}$ is taken to be real. For the photon loss reservoir, we assume that the term $a_1^\dagger \Gamma_1(t) a_1 + a_2^\dagger \Gamma_2(t) a_2$ in the Hamiltonian (1) preserves the total number of photons, i.e. only non-zero correlation functions are

$$\langle \Gamma_i(t)\Gamma_j^\dagger(\tau) \rangle_t = \frac{1}{2} \gamma_i \delta(t - \tau), \quad \langle \Gamma_1(t)\Gamma_2^\dagger(\tau) \rangle_t = \frac{1}{2} \gamma_{12} \delta(t - \tau). \tag{A2}$$

Also here, for simplicity, the rates $\gamma_1, \gamma_2$ are assumed to be real and non-negative and the cross-correlation parameter $\gamma_{12}$ to be real. For self-loss rates, $\gamma_i, d_i$, and ‘cross’-loss rates, $\gamma_{12}, d_{12}$, the following relations hold

$$d_a d_b \geq d_{ab}^2, \quad \gamma_a \gamma_b \geq \gamma_{ab}^2. \tag{A3}$$

Under the assumptions made above, the master equation of Eq. (4) is obtained from Eq. (A1).

APPENDIX B: ORIGIN OF CORRELATED LOSS

1. An example: dispersive two-mode Jaynes-Cummings model with damping

We will illustrate the process of appearance of a correlated modal photon loss in a process of off-resonant interaction between a mode and an emitter with a simple example. Consider two modes of the same frequency interacting off-resonantly with just a single two-level system (TLS), see Fig. (1a). In the rotating-wave approximation and in the interaction picture with respect to the free Hamiltonian of the reservoir, in the frame rotating with the TLS transition frequency $\omega_0$, one has the following Hamiltonian describing the problem:

$$H_{JK} = \Delta (a_1^\dagger a_1 + a_2^\dagger a_2) + \left( \sigma^+ (g_1 a_1 + g_2 a_2) + (g_1 a_1^\dagger + g_2 a_2^\dagger) \sigma^- \right) + (\sigma^+ \Gamma(t) + \Gamma^+(t) \sigma^-). \tag{B1}$$

Here $\Delta = \omega_0 - w$ and $w$ is the mode frequency; $g_{1,2}$ are interaction constants for the corresponding modes; $\sigma^\pm$ and $\sigma_z$ are Pauli operators for the TLS, $\sigma^+ = |2\rangle\langle 1|$, $\sigma^- = |1\rangle\langle 2|$, $\sigma_z = \sigma^+ \sigma^- - \sigma^- \sigma^+$; vectors $|k \rangle$, $k = 1, 2$ denote the lower and the upper TLS levels, correspondingly. The reservoir operator $\Gamma(t)$ describes the TLS energy loss. We assume this reservoir to be Markovian and the following relations hold:

$$\langle \Gamma(t) \rangle_t = 0, \quad \langle \Gamma(t) \Gamma^\dagger(\tau) \rangle_t = \gamma \delta(t - \tau)$$
where brackets $\langle \ldots \rangle$, denote an averaging over the reservoir. Here we are assuming that losses are weak (i.e., $\gamma \ll |g_{1.2}|$).

We adopt the usual conditions for an adiabatic elimination of the emitter, i.e. the TLS starts at the lower level, and the detuning $\Delta$ between the mode frequency and the TLS transition frequency is much larger than $g_{1.2}$. Thus, the TLS upper level remains practically unpopulated. Changing to the interaction picture with respect to the part of the Hamiltonian $\text{(B1)}$ corresponding to the absence of the TLS-field interaction, we get the following interaction Hamiltonian

$$V_{JK}(t) = G \{\sigma^+ C \exp\{i\Delta t\} + h. c.\} + (\sigma^+ \Gamma(t) + H. c.) \tag{B2}$$

with the bosonic annihilation operator for the collective mode

$$C = \frac{1}{\sqrt{g_1^2 + g_2^2}} (g_1 a_1 + g_2 a_2)$$

and $G = \sqrt{g_1^2 + g_2^2}$. A formal solution for $\sigma^+(t)$ without losses up to the third-order terms can be approximated as

$$\sigma^+(t) \approx \sigma^+(0) + i \int_0^t dt_1 [2F(t_1)F(t_1) - 1] X(t_1), \tag{B3}$$

where $X(t) = GC(t) \exp\{-i\Delta t\}$ and $F(t_1) \approx \sigma^+(0) \sigma_2(0) C(t_1) \frac{G}{\Delta} (1 - \exp\{-i\Delta t\})$, if one takes into account the fact that the modal dynamics is very slow on the scale of the TLS dynamics and $\sigma_2(t)$ can be considered as practically constant. From Eq. $\text{(B3)}$, neglecting small and rapidly oscillating terms, after averaging over the atomic variables one arrives to the following effective interaction Hamiltonian

$$V_{JK}(t) \approx 2G^2 C(t) C(t) + i \frac{G^4}{\Delta^3} C(t) C(t) C(t) C(t) C(t) + \frac{G}{\Delta} \left[C(t) \Gamma(t)(1 - \exp\{-i\Delta t\}) + H. c.\right]. \tag{B4}$$

Deriving the master equation in the standard manner, one obtains an equation describing the correlated photon losses

$$\frac{d}{dt} \rho(t) \approx -i[\delta w C(t) C(t) + \chi C(t) C(t) C(t) C(t) C(t)] + \gamma_L(C)\rho(t), \tag{B5}$$

where $\delta w = 2G^2/\Delta$, $\chi = 4G^4/\Delta^3$ and $\gamma_L = 2\gamma G^2/\Delta^2$.

Effectively, Eq. $\text{(B5)}$ describe both coupling between modes and their interaction with the same reservoir. Both these interaction might lead to the entanglement between modes. As seen in Section V, even in the absence of direct intermodal coupling (i.e. for $\delta w = 0$, $\chi = 0$) an interaction of modes with the reservoir entangles these modes.

The analysis made above can be readily generalized to different schemes of cross-Kerr nonlinearity generation through interaction of two modes with the same ensemble of emitters [9, 10]. In Subsection III we devise the procedure for the scheme of the giant cross-Kerr nonlinearity generation suggested in Ref. [10].

2. An example: dispersive two-mode Jaynes-Cummings model with dephasing

Here we illustrate an appearance of correlated modal dephasing with the example of the Jaynes-Cummings model (Fig. 1) considered in the previous Subsection. We model an influence of the dephasing reservoir as a stochastic fluctuation of the TLS transition frequency. In the rotating-wave approximation the problem is described by the following Hamiltonian:

$$H_{JK} = w(a_1^{\dagger} a_1 + a_2^{\dagger} a_2) + \frac{1}{2} [\omega_0 + \zeta(t)] \sigma_z + \left[\sigma^+(g_1 a_1 + g_2 a_2) + (g_1 a_1^{\dagger} + g_2 a_2^{\dagger}) \sigma^-ight], \tag{B6}$$

where $\zeta(t)$ is random process describing a small rapid stochastic modulation due to non-radiative interaction with surroundings. For simplicity we take $\zeta(t)$ to be just a white noise satisfying the following relations

$$\langle \zeta(t) \rangle = 0, \quad \langle \zeta(t) \zeta(\tau) \rangle = d \delta(t - \tau),$$

where $\langle \ldots \rangle$ denotes classical averaging. We consider the case of the weak loss, $d \ll |g_{1.2}|$. 

As before, we assume that the conditions for adiabatic elimination of the emitter hold. In the interaction picture with respect to the part of the Hamiltonian \((B6)\) corresponding to the absence of the TLS-field interaction, we have the following interaction Hamiltonian

\[
V_{JK}(t) = G \left( \sigma^+ C f(t) + \text{H. c.} \right), \quad f(t) = \exp \left\{ i\Delta t + i \int_0^t d\tau \zeta(\tau) \right\}. \tag{B7}
\]

A formal solution for \(\sigma^+(t)\) in this interaction picture can be approximated as

\[
\sigma^+(t) \approx \sigma^+(0) + iG \int_0^t dt_1 [2F^+(t_1)F(t_1) - 1]C^+(t_1)f^*(t_1), \tag{B8}
\]

where

\[
F^+(t) \approx \sigma^+(0) + iG\sigma_z(0)C^+(t) \int_0^t d\tau f^*(\tau). \tag{B9}
\]

After averaging over TLS states, the following effective interaction Hamiltonian can be obtained from Eq. \((B9)\):

\[
V_{JK}(t) \approx 2G^2C^+(t)C(t)p^{(2)}(t) + \frac{4G^4}{\Delta^3}[C^+(t)C(t)C^+(t)C(t)], \tag{B10}
\]

where

\[
p^{(2)}(t) = \text{Re} \left[ i f(t) \int_0^t d\tau f^*(\tau) \right].
\]

Averaging over dephasing noise and neglecting small terms, we get the following master equation using the standard technique implemented to derive Eq. \((A1)\):

\[
\frac{d}{dt} \rho(t) \approx -i[\delta w C^+ C + \chi C^+ C C^+ C, \rho(t)] + \bar{d} \mathcal{L}(C^+ C)\rho(t), \tag{B11}
\]

where

\[
\bar{d} = 4G^4 \left\langle \int_0^\infty d\tau p(0)p(\tau) \right\rangle_s \sim d\frac{4G^4}{\Delta^3}. \tag{B12}
\]

The calculations of the coefficients in \((B12)\) are carried out using the following property \([25, 26]\):

\[
\left\langle \exp \left\{ i \int_0^t d\tau \zeta(\tau) \right\} \right\rangle_s = \exp \left\{ - \int_0^t d\tau \int_0^{\tau} dx \langle \zeta(\tau)\zeta(x) \rangle_s \right\} = \exp \{-dt\}
\]

and the fact that the detuning \(\Delta\) is assumed to be large, \(\Delta \gg d\).

So, one can see that dephasing of the atom leads to appearance of the correlated modal dephasing practically in the same manner as atomic population losses lead to the correlated modal loss considered in the previous Subsection. Also, modal dephasing occurs notwithstanding the fact that the upper atomic level remains practically unpopulated.

3. Beam-splitting by decoherence

Finally, we demonstrate that the scheme described by the master equation \((B5)\) can effectively produce entanglement between the modes. In fact, this scheme can act as a kind of lossy beam-splitter even in the absence of intermodal interaction in unitary part of Eq. \((B5)\). Indeed, let us consider a completely uncorrelated single-photon initial state \(|\Psi(0)\rangle = |1\rangle_1|0\rangle_2\), and the initial density matrix \(|\rho(0)\rangle = |\Psi(0)\rangle \langle \Psi(0)|\). In the zero- and single-photon subspaces one can assume the following orthonormal basis:

\[
|\psi_+\rangle = \frac{1}{G} (g_1|1\rangle_1|0\rangle_2 + g_2|0\rangle_1|1\rangle_2), \quad |\psi_-\rangle = \frac{1}{G} (g_2|1\rangle_1|0\rangle_2 - g_1|0\rangle_1|1\rangle_2), \quad |v\rangle = |0\rangle_1|0\rangle_2.
\]
FIG. 5: Examples of the negativity, $N(\rho)$, dynamics given by the solution (B13) for the initially disentangled state of modes 1 and 2 (namely, single photon in the mode 1 and vacuum in the mode 2). The time, $T$, is given in units of $g_2$; solid, dotted and dashed lines correspond to $\bar{\gamma} = g_2, 0.25g_2, 2g_2$ and $g_1 = g_2$. Dash-dotted line corresponds to $\bar{\gamma} = g_2$ and $g_1 = 2g_2$. Here $\delta\omega + \chi = 0$ for all graphs.

One can easily see that the state $|\psi_-\rangle$ is not affected by the losses described by Eq. (B5) because $C|\psi_-\rangle = 0$. Also, the following relations are satisfied:

$$ C^\dagger C|\psi_+\rangle = |\psi_+\rangle, \quad C|v\rangle = 0. $$

Thus the system of equations for the density matrix elements can be obtained from Eq. (B5):

$$ \rho_{--}(t) = \rho_{--}(0), \quad \frac{d}{dt}\rho_{++}(t) = -2\bar{\gamma}\rho_{++}(t), \quad \frac{d}{dt}\rho_{+-}(t) = -[\bar{\gamma} + i(\delta\omega + \chi)]\rho_{+-}(t), $$

$$ \rho_{-v}(t) = \rho_{-v}(0), \quad \frac{d}{dt}\rho_{++}(t) = -[\bar{\gamma} + i(\delta\omega + \chi)]\rho_{++}(t). $$

The solution (B13) describes an emergence of entanglement form the initially uncorrelated state of both modes (single photon in the mode 1 and vacuum of the mode 2). Figure 5 depicts a measure of entanglement, a negativity as given in Ref. [28]

$$ N(\rho) = \frac{1}{2} \text{Tr} \sqrt{\rho \sigma^\dagger - 1}, $$

where $\sigma$ is the density matrix $\rho$ partially transposed with respect to the first mode. Non-zero value of the negativity means that the state is the entangled one. It can be seen that the decay rate into the common reservoir does not affect
the finally reached entanglement. This rate affect only time during which a stationary state is reached. It follows from this system of equations that the initial uncorrelated state $|1\rangle_1|0\rangle_2$ under the action of the correlated modal loss asymptotically turns into

$$|\rho(\infty)\rangle = \frac{g_2^2}{G^2}|\psi_-\rangle|\bar{\psi}_-\rangle + \left(1 - \frac{g_2^2}{G^2}\right)|\psi\rangle|\bar{\psi}\rangle.$$  (B14)

The state (B14) is entangled for an arbitrary $g_{1,2} > 0$. However, the maximal degree of asymptotic entanglement is reached when $g_1 = g_2$, and with increasing of difference between $g_1$ and $g_2$ the asymptotic entanglement decreases (Figure 5).

Note that the state (B14) is influenced neither by the cross-Kerr interaction of the modes nor by the linear excitation exchange, and one can set both $\delta w = \chi = 0$. The same type of state is produced by a correlated dephasing described by Eq. (B11). Effectively, the correlation of reservoirs allows for existence of decoherence-free subspaces to which the two-mode state eventually evolves [15]. Entangling through the common reservoir with appearance of the long-living state similar to the one described by Eq. (B14) might occur for emitters and collective reservoir modes near the band-edge in photonic crystals [29].

[1] P. Kok, W. J. Munro, K. Nemoto, T. C. Ralph, J. P. Dowling, and G. J. Milburn, Rev. Mod. Phys. 79, 135 (2007).
[2] J. Lee, M. Paternostro, C. Ogden, Y. W. Cheong, S. Bose and M. S. Kim, New J. Phys. 8, 23 (2006).
[3] T. Tyc and N. Korolkova, New J. Phys. 10, 023014 (2008).
[4] S. Glancy and H. M. de Vasconcelos, J. Opt. Soc. Am. B 25, 712 (2008).
[5] W. J. Munro, K. Nemoto and T. P. Spiller, New J. Phys. 7, 137 (2005).
[6] P. P. Rohde, W. J. Munro, T. C. Ralph, P. van Loock, and K. Nemoto, Quant. Inform. and Computing 8, 53 (2008).
[7] H. Jeong, Phys. Rev. A 72, 034305 (2005); ibid. Phys. Rev. A 73, 052320 (2006).
[8] M. Fleischhauer, A. Imamoglu, and J. P. Marangos, Rev. Mod. Phys. 77, 633 (2005).
[9] G. F. Sinclair and N. Korolkova, Phys. Rev. A 76, 033803 (2007).
[10] F. G. S. L. Brandao, M. J. Hartmann, and M. B. Plenio, New J. Phys. 10, 043010 (2008).
[11] B. Yurke and D. Stoler, Phys. Rev. Lett. 57, 13 (1986); Phys. Rev. A 35, 4846 (1987).
[12] M. S. Kim, M. Paternostro, J. Mod. Opt. 54, 1999 (2007).
[13] I. Fushman and J. Vuckovic, Opt. Express 15, 5559 (2007).
[14] J. S. Prauzner-Bechcicki, J. Phys. A: Math. Gen. 37 L173 (2004).
[15] D. Braun, Phys. Rev. Lett. 89, 277901 (2002).
[16] C. Horhammer and H. Buttner, Phys. Rev. A 77, 042305 (2008).
[17] G. J. Milburn, Phys. Rev. A 33, 674 (1986).
[18] H.P. Breuer, F. Petruccione, The Theory of Open Quantum Systems, Oxford University Press, Oxford (2002).
[19] H. Umezawa, H. Matsumoto, and M. Tachiki, Thermofield dynamics and condensed states, North Holland, Amsterdam, (1982).
[20] S. Chaturvedi and V. Srinivasan, Phys. Rev. A 43, 4054 (1991).
[21] K. Wódkiewicz and J. H. Eberly, J. Opt. Soc. Am. B 2, 458 (1985).
[22] D. F. Walls and G. J. Milburn, Quantum Optics, Springer, Berlin (2006).
[23] M. S. Kim and N. Imoto, Phys. Rev. A 52, 2401 (1995).
[24] R. Loudon, The Quantum Theory of Light (Clarendon, Oxford) 1973.
[25] P. A. Apanashevich, S. Ya. Kilin, and A. P. Nizovtsev, N. S. Onishchenko, J. Opt. Soc. Am. B 33, 587 (1986).
[26] C. W. Gardiner and P. Zoller, Quantum noise: A Handbook of Markovian and non-Markovian Quantum Stochastic Methods with Applications to Quantum Optics. Berlin: Springer-Verlag (1999).
[27] D. K. Armani, T. J. Kippenberg, S. M. Spillane and K. J. Vahala, Nature 421, 925 (2003); T. J. Kippenberg, S.M. Spillane, and K. J. Vahala, Phys. Rev. Lett. 93 083904 (2004); S. M. Spillane, T. J. Kippenberg, and K. J. Vahala, K. W. Goh, E. Wilecut, and H. J. Kimble, Phys. Rev. A71 013817 (2005).
[28] G. Vidal, R. F. Werner, Phys. Rev. A 65, 032314 (2002).
[29] D. Mogilevtsev, S. Kilin, S. B. Cavalcanti and J. M. Hickmann, Phys. Rev. A72, 043817 (2005).
[30] A. Mecozzi and P. Tombesi, Phys. Rev. Lett. 58, 1055 (1987).