THE THIRD HOMOLOGY OF STEM-EXTENSIONS AND
WHITEHEAD’S QUADRATIC FUNCTOR

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Abstract. Let $A \rightarrow G \rightarrow Q$ be a stem-extension and let $\rho : A \times G \rightarrow G$ be the multiplication map. We show that there is a natural map

$$\varphi : H_1(\Sigma^2_\ast \text{Tor}_1^Z(A, A)) \rightarrow H_3(G, \mathbb{Z})/\rho_* (A \otimes_\mathbb{Z} H_2(G, \mathbb{Z}))$$

such that, the image of $\varphi$ coincides with the image of the natural map $H_3(A, \mathbb{Z}) \rightarrow H_3(G, \mathbb{Z})/\rho_* (A \otimes_\mathbb{Z} H_2(G, \mathbb{Z}))$. An important tool used here is Whitehead’s quadratic functor $\Gamma$. As part of our proof of the main result, we give a precise homological description of the kernel of the natural map $\Gamma(A) \rightarrow A \otimes_\mathbb{Z} A, \gamma(a) \mapsto a \otimes a$.

Introduction

It is usually difficult to calculate the (integral) homologies of a given group $G$. One way to deal with this problem is to take a normal subgroup $N$ of $G$ and study the Lyndon-Hochschild-Serre spectral sequence associated to the extension $N \rightarrow G \rightarrow G/N$, which ties up the homologies of these groups:

$$E^{2}_{p,q} = H_p(G/N, H_q(N, \mathbb{Z})) \Rightarrow H_{p+q}(G, \mathbb{Z}).$$

This is a powerful tool. There are many interesting cases that the homologies of $N$ and $G/N$ are more accessible, and the above spectral sequence can be used to study the homologies of $G$.

The natural homology maps $H_n(N, \mathbb{Z}) \rightarrow H_n(G, \mathbb{Z})$ and $H_n(G, \mathbb{Z}) \rightarrow H_n(G/N, \mathbb{Z})$ are hidden, in a way, in the above spectral sequence. For example $H_n(N, \mathbb{Z}) \rightarrow H_n(G, \mathbb{Z})$ is the composite $H_n(N, \mathbb{Z}) \rightarrow E^{\infty}_{0,n} \leftarrow H_n(G, \mathbb{Z})$.

In this article we will study the image of the map $H_3(A, \mathbb{Z}) \rightarrow H_3(G, \mathbb{Z})$, when $A \rightarrow G \rightarrow Q$ is a quasi stem-extension. We say that a central extension is a quasi stem-extension if the maps $\text{Tor}_i^Z(A, A) \rightarrow \text{Tor}_i^Z(A, \text{H}_1(G, \mathbb{Z}))$, induced by $A \rightarrow G$, are trivial for $i = 0, 1$. For example perfect central extensions are quasi stem-extensions.
As our main theorem (Theorem 4.1) we show that when \( A \hookrightarrow G \to Q \) is a quasi stem-extension, then there is a natural map

\[
\varphi : H_1(\Sigma_2^\infty, \text{Tor}_1^\mathbb{Z}(2\infty A, 2\infty A)) \to H_3(G, \mathbb{Z})/\rho_*(A \otimes \mathbb{Z} H_2(G, \mathbb{Z}))
\]
such that the image of \( \varphi \) coincides with the natural image of \( H_3(A, \mathbb{Z}) \) in \( H_3(G, \mathbb{Z})/\rho_*(A \otimes \mathbb{Z} H_2(G, \mathbb{Z})) \). In particular if the extension is a universal central extension, then the image of \( \varphi \), which is 2-torsion. In the above map \( 2\infty A \) is the 2-power torsion subgroup of \( A \), \( \Sigma_2 := \{\text{id}, \sigma^e\} \) is the symmetric group with two elements and \( \sigma^e \) being the involution on \( \text{Tor}_1^\mathbb{Z}(2\infty A, 2\infty A) \) induced by the involution \( A \times A \to A \times A, (a, b) \mapsto (b, a) \).

An important tool to investigate our main problem is Whitehead’s quadratic functor. This functor, which first appeared in the context of algebraic topology, is a functor from the category of abelian groups to itself and usually is denoted by \( \Gamma \). Most of important aspects of this functor is known and it has been generalised in various ways.

For an abelian group \( A \), we give a precise homological description of the kernel of the natural map

\[
\Gamma(A) \to A \otimes \mathbb{Z} A, \quad \gamma(a) \mapsto a \otimes a,
\]
which it is known to be 2-torsion. In fact we prove that (Theorem 3.1) we have the exact sequence

\[
0 \to H_1(\Sigma_2^\infty, \text{Tor}_1^\mathbb{Z}(2\infty A, 2\infty A)) \to \Gamma(A) \to A \otimes \mathbb{Z} A \to H_2(A, \mathbb{Z}) \to 0.
\]

1. **Whitehead’s quadratic functor**

A function \( \psi : A \to B \) of (additive) abelian groups is called a quadratic map if

1. for any \( a \in A \), \( \psi(a) = \psi(-a) \),
2. the function \( A \times A \to B \) with \( (a, b) \mapsto \psi(a+b) - \psi(a) - \psi(b) \) is bilinear.

For any abelian group \( A \), there is a universal quadratic map

\[
\gamma : A \to \Gamma(A)
\]
such that for any quadratic map \( \psi : A \to B \), there is a unique group homomorphism \( \Psi : \Gamma(A) \to B \) such that \( \Psi \circ \gamma = \psi \). It is easy to see that \( \Gamma \) is a functor from the category of abelian groups to itself.

The functions \( \phi : A \to A/2 \) and \( \psi : A \to A \otimes \mathbb{Z} A \), given by \( \phi(a) = a \bar{a} \) and \( \psi(a) = a \otimes a \) respectively, are quadratic maps. Thus we get the canonical homomorphisms

\[
\Phi : \Gamma(A) \to A/2, \quad \gamma(a) \mapsto \bar{a} \quad \text{and} \quad \Psi : \Gamma(A) \to A \otimes \mathbb{Z} A, \quad \gamma(a) \mapsto a \otimes a.
\]

Clearly \( \Phi \) is surjective and \( \text{coker}(\Psi) = A \wedge A \simeq H_2(A, \mathbb{Z}) \). Furthermore we have the bilinear pairing

\[
[,] : A \otimes \mathbb{Z} A \to \Gamma(A), \quad [a, b] := \gamma(a + b) - \gamma(a) - \gamma(b).
\]
It is easy to see that for any \(a, b, c \in A\), \([a, b] = [b, a]\), \(\Phi[a, b] = 0\), \(\Psi[a, b] = a \otimes b + b \otimes a\) and \([a + b, c] = [a, c] + [b, c]\). Using (1) and this last equation, for any \(a, b, c \in A\), we obtain

(a) \(\gamma(a) = \gamma(-a)\),

(b) \(\gamma(a + b + c) - \gamma(a + b) - \gamma(a + c) - \gamma(b + c) + \gamma(a) + \gamma(b) + \gamma(c) = 0\).

Using these properties we can construct \(\Gamma(A)\). Let \(A\) be the free abelian group generated by the symbols \(w(a), a \in A\). Set \(\Gamma(A) := A/\mathcal{R}\), where \(\mathcal{R}\) denotes the relations (a) and (b) with \(w\) replaced by \(\gamma\). Now \(\gamma : A \to \Gamma(A)\) is given by \(a \mapsto w(a)\).

Using this properties one can show that for any nonnegative integer \(n\), we have

\[\gamma(na) = n^2 \gamma(a).\]

It is known that the sequence

\[A \otimes Z A \xrightarrow{[\cdot, \cdot]} \Gamma(A) \xrightarrow{\Phi} A/2 \to 0\]

is exact and the kernel of \([\cdot, \cdot]\) is generated by the elements of the form \(a \otimes b - b \otimes a\), \(a, b \in A\). Therefore we have the exact sequence

\[0 \to H_0(\Omega_2, A \otimes Z A) \xrightarrow{[\cdot, \cdot]} \Gamma(A) \xrightarrow{\Phi} A/2 \to 0,\]

where \(\Omega_2 := \{\text{id}, \omega\}\) and \(\omega\) is the involution \(\omega(a \otimes b) = b \otimes a\) on \(A \otimes Z A\).

It is easy to see that the composition

\[A \otimes Z A \xrightarrow{[\cdot, \cdot]} \Gamma(A) \xrightarrow{\Psi} A \otimes Z A\]

takes \(a \otimes b\) to \(a \otimes b + b \otimes a\). Moreover the composition

\[\Gamma(A) \xrightarrow{\Psi} A \otimes Z A \xrightarrow{[\cdot, \cdot]} \Gamma(A)\]

coincide with multiplication by 2. Thus \(\ker(\Psi)\) is 2-torsion.

To give a homological description of the kernel of \(\psi\), we will need the following fact.

**Proposition 1.1.** For any abelian group \(A\), \(\Gamma(A) \simeq H_3(K(A, 2), \mathbb{Z})\), where \(K(A, 2)\) is the Eilenberg-Maclane space of type \((A, 2)\).

**Proof.** See [4, Theorem 21.1] \(\square\)

2. **Tor-functor and third homology of abelian groups**

Let \(A\) and \(B\) be abelian groups. For any positive integer \(n\) there is a natural homomorphism

\[\tau_n : nA \otimes Z nB \to n\text{Tor}_1^Z(A, B).\]

We denote the image of \(a \otimes b\), under \(\tau_n\) by \(\tau_n(a, b)\).
For any pair of integers \( s \) and \( n \) such that \( n = sm \), the maps \( \tau_n \) are related by the commutative diagrams

\[
\begin{array}{c}
\xymatrix{
 nA \otimes Z nB 
\ar[rr]^{\text{id} \otimes i_m} 
\ar[dd]_{\tau_n} 
\ar[dr]_{p_m \otimes \text{id}} & & nA \otimes Z nB, \\
\tau_s & & \tau_s \\
_s A \otimes Z sB 
\ar[rr]_{i_m \otimes \text{id}} 
\ar[dd]_{\tau_s} & & sA \otimes Z sB, \\
\tau_n & & \tau_n \\
\end{array}
\]

\[
\begin{array}{c}
\xymatrix{
 nA \otimes Z nB 
\ar[rr]^{\text{id} \otimes p_m} 
\ar[dd]_{\tau_n} 
\ar[dr]_{\text{id} \otimes \text{id}} & & nA \otimes Z nB, \\
\tau_s & & \tau_s \\
_s A \otimes Z sB 
\ar[rr]_{i_m \otimes \text{id}} 
\ar[dd]_{\tau_s} & & sA \otimes Z sB, \\
\tau_n & & \tau_n \\
\end{array}
\]

in which \( i_m : sA \to nA \) and \( p_m : nA \to sA \) are the inclusion and the map induced by multiplication by \( m \) respectively. The commutativity of these diagrams expresses the relations

\[
\tau_n(a, b) = \tau_s(ma, b), \quad \text{for } a \in nA \text{ and } b \in sB,
\]

and

\[
\tau_n(a', b') = \tau_s(a', mb'), \quad \text{for } a' \in sA \text{ and } b' \in nB.
\]

The following proposition is well-known \([2, \text{Proposition 3.5}]\).

**Proposition 2.1.** The induced map \( \tau : \lim_I(nA \otimes nB) \to \text{Tor}_1^n(A, B), \) where \( I \) is the inductive system of objects \( nA \otimes Z nB \) determined by the above diagrams for varying \( n \), is an isomorphism.

Let \( \sigma_0 : A \otimes B \to B \otimes A \) and \( \sigma_1 : \text{Tor}_1^n(A, B) \to \text{Tor}_1^n(B, A) \) be induced by interchanging the groups \( A \) and \( B \). It is well known that the diagram

\[
\begin{array}{c}
\xymatrix{
 nA \otimes Z nB 
\ar[rr]^{\sigma_0} 
\ar[d]_{\tau_n} & & nB \otimes Z nA \\
\tau_n & & \tau_n \\
\end{array}
\]

commutes. By passing to the inductive limit, the same is true for the diagram

\[
\begin{array}{c}
\xymatrix{
\lim_I(nA \otimes Z nB) 
\ar[rr]^{\sigma_0} 
\ar[d]_{\tau} & & \lim_I(nB \otimes Z nA) \\
\tau' & & \tau'
\end{array}
\]

\[
\begin{array}{c}
\xymatrix{
\text{Tor}_1^n(A, B) 
\ar[rr]^{\sigma_1} & & \text{Tor}_1^n(B, A).}
\end{array}
\]
It is useful to observe that the map $\sigma_1 : \text{Tor}_1^\mathbb{Z}(A, B) \to \text{Tor}_1^\mathbb{Z}(B, A)$ is indeed induced by the involution $A \otimes \mathbb{Z} B \to B \otimes \mathbb{Z} A$ given by $a \otimes b \mapsto -b \otimes a$ and therefore $-\sigma_1$ is induced by the involution $a \otimes b \mapsto b \otimes a$.

Let $\Sigma_2$ be the symmetric group of order 2. For an abelian group $A$, $\Sigma_2$ acts on $A \otimes \mathbb{Z} A$ and $\text{Tor}_1^\mathbb{Z}(A, A)$, through $\sigma_0$ and $\sigma_1$. Let us denote the symmetric group by $\Sigma^\varepsilon_2$, rather than simply by $\Sigma_2$, when it acts on $\text{Tor}_1^\mathbb{Z}(A, A)$ as $(\sigma^\varepsilon, x) \mapsto -\sigma_1(x)$.

We need the following well-known lemma on the third homology of abelian groups [8, Lemma 5.5], [2, Section 6].

**Proposition 2.2.** For any abelian group $A$ we have the exact sequence

$$0 \to \bigwedge^3 \mathbb{Z} A \to H_3(A, \mathbb{Z}) \to \text{Tor}_1^\mathbb{Z}(A, A)^{\Sigma^\varepsilon_2} \to 0,$$

where the right side homomorphism is obtained from the composition

$$H_3(A, \mathbb{Z}) \xrightarrow{\Delta_A} H_3(A \times A, \mathbb{Z}) \to \text{Tor}_1^\mathbb{Z}(A, A),$$

$\Delta_A$ being the diagonal map $A \to A \times A$, $a \mapsto (a, a)$.

3. The kernel of $\Psi : \Gamma(A) \to A \otimes A$

We study the kernel of $\Psi : \Gamma(A) \to A \otimes \mathbb{Z} A$. If $\Theta = [\ , \ ] : A \otimes \mathbb{Z} A \to \Gamma(A)$, then from the commutative diagram

$$
\begin{array}{ccc}
0 & \longrightarrow & \ker(\Theta) \\
\downarrow & & \downarrow \Theta \\
0 & \longrightarrow & A \otimes \mathbb{Z} A \\
\ker(\Psi) & \longrightarrow & \Gamma(A) \\
\gamma & \longrightarrow & A \otimes \mathbb{Z} A
\end{array}
$$

and exact sequence (1.1) we obtain the exact sequence

$$\ker(\Psi) \to A/2 \xrightarrow{\delta} (A \otimes \mathbb{Z} A)_{\Omega_2} \to H_2(A, \mathbb{Z}) \to 0,$$

where $(A \otimes \mathbb{Z} A)_{\Omega_2} = (A \otimes \mathbb{Z} A)/\langle a \otimes b + b \otimes a | a, b \in A \rangle$ and $\delta(\overline{a}) = a \otimes a$. But the sequence

$$0 \to A/2 \to (A \otimes \mathbb{Z} A)_{\Omega_2} \to H_2(A, \mathbb{Z}) \to 0$$

is always exact. Thus the map $\ker(\Psi) \to A/2$ is trivial, which shows that

$$\ker(\Gamma(A) \xrightarrow{\Psi} A \otimes \mathbb{Z} A) \subseteq \text{im}(A \otimes \mathbb{Z} A \xrightarrow{\gamma} \Gamma(A)).$$

We give a precise description of the kernel of $\Psi$.

**Theorem 3.1.** For any abelian group $A$, we have the exact sequence

$$0 \to H_1(\Sigma^\varepsilon_2, \text{Tor}_1^\mathbb{Z}(2 \times A, 2 \times A)) \to \Gamma(A) \xrightarrow{\Psi} A \otimes \mathbb{Z} A \to H_2(A, \mathbb{Z}) \to 0.$$
Proof. If \( A \rightarrow B \rightarrow C \) is an extension of abelian groups, then standard classifying space theory gives a (homotopy theoretic) fibration of Eilenberg-MacLane spaces \( K(A,1) \rightarrow K(B,1) \rightarrow K(C,1) \). From this we obtain the fibration [5, Lemma 3.4.2]

\[
K(B,1) \rightarrow K(C,1) \rightarrow K(A,2).
\]

For the group \( A \), the morphism of extensions

\[
\begin{array}{ccc}
A & \xrightarrow{i_1} & A \times A \\
\downarrow & & \downarrow p_2 \\
A & \xrightarrow{=} & A
\end{array}
\]

where \( i_1(a) = (a, 1) \), \( p_2(a,b) = b \) and \( \mu(a,b) = ab \), induces the morphism of fibrations

\[
\begin{array}{ccc}
K(A \times A, 1) & \xrightarrow{} & K(A, 1) \\
\downarrow & & \downarrow \\
K(A, 1) & \xrightarrow{} & K(\{1\}, 1)
\end{array}
\]

By analysing the Serre spectral sequences associated to this morphism of fibrations, we obtain the exact sequence

\[
0 \rightarrow \ker(\Psi) \rightarrow H_4(K(A,2)) \xrightarrow{\Psi} A \otimes \mathbb{Z} A \rightarrow H_2(A) \rightarrow 0,
\]

where \( \ker(\Psi) \simeq H_3(A, \mathbb{Z})/\mu_*(A \otimes \mathbb{Z} H_2(A, \mathbb{Z}) \oplus \text{Tor}_1^2(A, A)) \).

By Proposition 2.2 we have the exact sequence

\[
0 \rightarrow \bigwedge^3 \mathbb{Z} A \rightarrow H_3(A, \mathbb{Z}) \rightarrow \text{Tor}_1^2(A, A) \rightarrow 0.
\]

Clearly \( \mu_*(A \otimes \mathbb{Z} H_2(A, \mathbb{Z})) \subseteq \bigwedge^3 \mathbb{Z} A \). Therefore

\[
\ker(\Psi) \simeq \text{Tor}_1^2(A, A)^{\Sigma_2}/(\Delta_1 \circ \mu)_*(\text{Tor}_1^2(A, A)).
\]

We prove that the map \( \Delta \circ \mu : A \times A \rightarrow A \times A \), which is given by \((a, b) \mapsto (ab, ab)\), induces the map

\[
id + \sigma^c : \text{Tor}_1^2(A, A) \rightarrow \text{Tor}_1^2(A, A).
\]

By studying the map \( (\Delta \circ \mu)_* : H_2(A \times A, \mathbb{Z}) \rightarrow H_2(A \times A, \mathbb{Z}) \) using the fact that \( A \otimes A \simeq H_2(A \times A, \mathbb{Z})/(H_2(A, \mathbb{Z}) \oplus H_2(A, \mathbb{Z})) \) (the Künneth Formula), one sees that \( \Delta \circ \mu \) induces the map

\[
A \otimes A \rightarrow A \otimes A, \quad a \otimes b \mapsto a \otimes b - b \otimes a,
\]

Thus to study the induced map on \( \text{Tor}_1^2(A, A) \) by \( \Delta \circ \mu \) we should study the map induced on \( \text{Tor}_1^2(A, A) \) by the map

\[
A \otimes A \rightarrow A \otimes A, \quad a \otimes b \mapsto a \otimes b + b \otimes a = (\text{id} + \iota)(a \otimes b),
\]

where \( \iota : A \otimes A \rightarrow A \otimes A \) is given by \( a \otimes b \mapsto b \otimes a \). Let

\[
0 \rightarrow F_1 \xrightarrow{\partial} F_0 \xrightarrow{\epsilon} A \rightarrow 0
\]
be a free resolution of \( A \). Then the sequence

\[
0 \to F_1 \otimes F_1 \xrightarrow{\partial_2} F_0 \otimes F_1 \oplus F_1 \otimes F_0 \xrightarrow{\partial_1} F_0 \otimes F_0 \to 0
\]

can be used to calculate \( \text{Tor}^F_1(A, A) \), where \( \partial_2 = (\partial \otimes \text{id}_{F_1}, -\text{id}_{F_1} \otimes \partial) \), \( \partial_1 = \text{id}_{F_0} \otimes \partial + \partial \otimes \text{id}_{F_0} \). The map \( \text{id} + \iota : A \otimes A \to A \otimes A \) can be extended to the morphism of complexes

\[
0 \to F_1 \otimes F_1 \xrightarrow{\partial_2} F_0 \otimes F_1 \oplus F_1 \otimes F_0 \xrightarrow{\partial_1} F_0 \otimes F_0 \to 0
\]

where

\[
f_0(x \otimes y) := x \otimes y + y \otimes x,
\]

\[
f_1(x \otimes y, y' \otimes x') := (x \otimes y + x' \otimes y', y \otimes x + y' \otimes x'),
\]

\[
f_2(x \otimes y) := x \otimes y - y \otimes x.
\]

Since

\[
f_1(x \otimes y, y' \otimes x') = (x \otimes y, y' \otimes x') + (x' \otimes y', y \otimes x),
\]

\( \Delta \circ \mu \) induces the map \( \text{id} + \sigma \circ: \text{Tor}^F_1(A, A) \to \text{Tor}^F_1(A, A) \). Therefore

\[
\ker(\Psi) \simeq \text{Tor}^F_1(A, A)^{\Sigma_2}/(\text{id} + \sigma \circ(\text{Tor}^F_1(A, A))) = H_1(\Sigma_2, \text{Tor}^F_1(A, A)).
\]

Finally since \( \text{Tor}^F_1(A, A) = \text{Tor}^F_1(A_T, A_T) \), \( A_T \) being the subgroup of torsion elements of \( A \), and since for any torsion abelian group \( B \), \( B \simeq \bigoplus_p \text{prime} \ p \sim B \), we have the isomorphism

\[
H_1(\Sigma_2, \text{Tor}^F_1(A, A)) \simeq H_1(\Sigma_2, \text{Tor}^F_1(2^\sim A, 2^\sim A)).
\]

This completes the proof of the theorem. \( \square \)

**Corollary 3.2.** For any abelian group \( A \), we have the exact sequence

\[
0 \to \lim_{\to} H_1(\Sigma_2, 2^n A \otimes Z 2^n A) \to \Gamma(A) \xrightarrow{\Psi} A \otimes Z A \to H_2(A, Z) \to 0.
\]

In particular if \( 2^\sim A \) is finite then we have the exact sequence

\[
0 \to H_1(\Sigma_2, 2^\sim A \otimes Z 2^\sim A) \to \Gamma(A) \xrightarrow{\Psi} A \otimes Z A \to H_2(A, Z) \to 0.
\]

**Proof.** This follows from Theorem 3.1 and Proposition 2.1. \( \square \)

### 4. The Third Homology of Stem-extensions

A central extension \( A \hookrightarrow G \twoheadrightarrow Q \) is called a **stem-extension** if the natural map \( A = H_1(A, Z) \to H_1(G, Z) \) is trivial and it is called a **weak stem-extension** if the natural map \( A \otimes A \to A \otimes Z H_1(G, Z) \) is trivial [7].
Let \( A \rightarrow G \rightarrow Q \) be a weak stem-extension and let \( n \geq 2 \). From the commutative diagram

\[
\begin{array}{ccc}
A \times A & \xrightarrow{\mu} & A \\
\downarrow & & \downarrow \\
A \times G & \xrightarrow{\rho} & G,
\end{array}
\]

where \( \mu \) and \( \rho \) are the usual multiplications, we obtain the commutative diagram

\[
H_{n-2}(A, \mathbb{Z}) \otimes_{\mathbb{Z}} H_1(A, \mathbb{Z}) \otimes_{\mathbb{Z}} H_1(A, \mathbb{Z}) \rightarrow H_n(A, \mathbb{Z})
\]

It follows that the composite \( \bigwedge^n_{\mathbb{Z}} A \rightarrow H_n(A, \mathbb{Z}) \rightarrow H_n(G, \mathbb{Z}) \) is trivial. In particular, the natural image of \( H_n(A, \mathbb{Z}) \) in \( H_n(G, \mathbb{Z}) \) is torsion, because the map \( \bigwedge^n_{\mathbb{Z}} A \rightarrow H_n(A, \mathbb{Z}) \) has torsion cokernel [3, Theorem 6.4, Chap. V].

We say that a central extension \( A \rightarrow G \rightarrow Q \) is a quasi stem-extension if the natural maps \( \operatorname{Tor}_i^\mathbb{Z}(A, A) \rightarrow \operatorname{Tor}_i^\mathbb{Z}(A, H_1(G, \mathbb{Z})) \) are trivial for \( i = 0, 1 \). In particular quasi stem-extensions are weak stem extensions. The following theorem generalizes [6, Theorem 2].

**Theorem 4.1.** Let \( A \rightarrow G \rightarrow Q \) be a quasi stem-extension. Then there is a natural map

\[
\varphi : H_1(\Sigma_2, \operatorname{Tor}_1^\mathbb{Z}(2\Sigma, 2\Sigma A)) \rightarrow H_3(G, \mathbb{Z})/\rho_*(A \otimes H_2(G, \mathbb{Z}))
\]

such that the image of \( \varphi \) coincides with the natural image of \( H_3(A, \mathbb{Z}) \) in \( H_3(G, \mathbb{Z})/\rho_*(A \otimes H_2(G, \mathbb{Z})) \). In particular the image of \( H_3(A, \mathbb{Z}) \) in \( H_3(G, \mathbb{Z})/\rho_*(A \otimes H_2(G, \mathbb{Z})) \) is 2-torsion.

**Proof.** We argue as the proof of [6, Theorem 2]. By Proposition 2.2 we have the exact sequence

\[
0 \rightarrow \bigwedge^3_{\mathbb{Z}} A \rightarrow H_3(A, \mathbb{Z}) \rightarrow \operatorname{Tor}_1^\mathbb{Z}(A, A)^{\Sigma_2} \rightarrow 0,
\]

where the homomorphism on the right side of the exact sequence is obtained from the composition \( H_3(A, \mathbb{Z}) \xrightarrow{\Delta} H_3(A \times A, \mathbb{Z}) \rightarrow \operatorname{Tor}_1^\mathbb{Z}(A, A) \), \( \Delta \) being the diagonal map \( A \rightarrow A \times A \), \( a \mapsto (a, a) \). From (4.1), we obtain the commutative diagram

\[
\begin{array}{ccc}
\tilde{H}_3(A \times A) & \xrightarrow{\mu^*} & \tilde{H}_3(A, \mathbb{Z}) \\
\downarrow & & \downarrow \\
\tilde{H}_3(A \times G) & \xrightarrow{\rho^*} & \tilde{H}_3(G, \mathbb{Z}),
\end{array}
\]

where

\[
\tilde{H}_3(A \times A) := \ker(H_3(A \times A, \mathbb{Z}) \xrightarrow{(p_1, p_2, \ast)} H_3(A, \mathbb{Z}) \oplus H_3(A, \mathbb{Z})),
\]
\( \tilde{H}_3(A \times G) := \ker(H_3(A \times G, \mathbb{Z}) / p_1^{*}p_2^{*}) \to H_3(A, \mathbb{Z}) \oplus H_3(G, \mathbb{Z}) \).

The previous lemma implies that the composite
\[ \bigwedge_{\mathbb{Z}}^3 A \to H_3(A, \mathbb{Z}) \to H_3(G, \mathbb{Z}) \]
is trivial. These facts together with the Künneth formula give us the commutative diagram
\[
\begin{array}{c}
\text{Tor}^\mathbb{Z}_1(A, A) & \xrightarrow{\mu_*} & \text{Tor}^\mathbb{Z}_1(A, A) / \bigwedge_{\mathbb{Z}}^3 A \\
\tilde{H}_3(A \times A) / \bigoplus_{i=1}^2 H_1(A, \mathbb{Z}) \otimes \bigoplus_{i=1}^2 H_3-i(A, \mathbb{Z}) & \xrightarrow{\mu_*} & H_3(A, \mathbb{Z}) / \bigwedge_{\mathbb{Z}}^3 A \\
\tilde{H}_3(A \times G) / \bigoplus_{i=1}^2 H_1(A, \mathbb{Z}) \otimes \bigoplus_{i=1}^2 H_3-i(G, \mathbb{Z}) & \xrightarrow{\rho_*} & H_3(G, \mathbb{Z}) / M \\
\text{Tor}^\mathbb{Z}_1(A, H_1(G, \mathbb{Z})) & \simeq & \\
\end{array}
\]
where \( M = \rho_* (A \otimes \mathbb{Z} H_2(G, \mathbb{Z})) \). Note that \( \text{im}(H_2(A, \mathbb{Z}) \otimes \mathbb{Z} H_1(G, \mathbb{Z}) \to H_3(G, \mathbb{Z})) \subseteq M \) (see [7, Proposition 4.4, Chap. V]). Since the map
\[ \text{Tor}^\mathbb{Z}_1(A, A) \to \text{Tor}^\mathbb{Z}_1(A, H_1(G, \mathbb{Z})) \]
is trivial, \( \rho_* \circ \text{inc}_* \circ \alpha^{-1} = 0 \). This shows that the composite map \( \text{inc}_* \circ \beta^{-1} \circ \tilde{\mu}_* \) is trivial. Therefore the image of \( H_3(A, \mathbb{Z}) \) in \( H_3(G, \mathbb{Z}) / M \) is equal to the image of
\[ \text{Tor}^\mathbb{Z}_1(A, A) / \bigwedge_{\mathbb{Z}}^3 A \to \text{Tor}^\mathbb{Z}_1(A, A) / \bigwedge_{\mathbb{Z}}^3 A / (\Delta_A \circ \mu)_*(\text{Tor}^\mathbb{Z}_1(A, A)) \]
Now as in the proof of Theorem 3.1, one can show that
\[ \text{Tor}^\mathbb{Z}_1(A, A) / \bigwedge_{\mathbb{Z}}^3 A / (\Delta_A \circ \mu)_*(\text{Tor}^\mathbb{Z}_1(A, A)) \simeq \ker(\Psi) \]
which by Theorem 3.1 is isomorphic to \( H_1(\Sigma^6, \text{Tor}^\mathbb{Z}_1(2 \otimes A, 2 \otimes A)) \). This complete the proof of the theorem. \( \Box \)

Remark 4.2. In general \( \varphi \) does not necessarily factors through \( \Gamma(A) \). In fact we have the following commutative diagram
\[
\begin{array}{c}
H_1(\Sigma^6, \text{Tor}^\mathbb{Z}_1(2 \otimes A, 2 \otimes A)) \xrightarrow{\varphi} H_3(G, \mathbb{Z}) / \rho_* (A \otimes H_2(G, \mathbb{Z})) \\
\Gamma(A) \xrightarrow{} H_3(G, \mathbb{Z}) / \rho_* (\tilde{H}_3(A \times G)).
\end{array}
\]
Thus \( \varphi \) factors through \( \Gamma(A) \) if the extension is a stem-extension.

Corollary 4.3. Let \( A \to G \to Q \) be a universal central extension. Then the image of \( H_3(A, \mathbb{Z}) \) in \( H_3(G, \mathbb{Z}) \) coincides with the image of
\[ \varphi : H_1(\Sigma^6, \text{Tor}^\mathbb{Z}_1(2 \otimes A, 2 \otimes A)) \to H_3(G, \mathbb{Z}). \]
In particular the image of $H_3(A, \mathbb{Z})$ in $H_3(G, \mathbb{Z})$ is 2-torsion.

**Remark 4.4.** Let $A \to G \to Q$ be a perfect central extension, i.e. $G$ is a perfect group and the extension is central. By an easy analysis of the Lyndon-Hochschild-Serre spectral sequence of the extension, we obtain the exact sequence

$$H_4(Q, \mathbb{Z}) \to T \to H_3(G, \mathbb{Z})/N \to H_3(Q, \mathbb{Z}) \to 0,$$

where $T = \ker(A \otimes A \to H_2(A, \mathbb{Z})) = \langle a \otimes a : a \in A \rangle$ and $N := \rho_*(H_3(A, \mathbb{Z}) \otimes A \otimes H_2(G, \mathbb{Z})).$

Moreover from the fibration of Eilenberg-MacLane spaces $K(A, 1) \to K(G, 1) \to K(Q, 1),$ we obtain the fibration [5, Lemma 3.4.2]

$$K(G, 1) \to K(Q, 1) \to K(A, 2).$$

By studying the Serre spectral sequence associated to this fibration we obtain the exact sequence

$$H_4(Q, \mathbb{Z}) \to \Gamma(A) \to H_3(G, \mathbb{Z})/M \to H_3(Q, \mathbb{Z}) \to 0,$$

where $M := \rho_*(A \otimes \mathbb{Z} H_2(G, \mathbb{Z})).$ Now we have the following commutative diagram with exact rows and columns:

$$
\begin{array}{ccccccccc}
H_4(Q, \mathbb{Z}) & \longrightarrow & H_4(Q, \mathbb{Z}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & H_1(\Sigma_2, \text{Tor}^\mathbb{Z}_1(2\infty A, 2\infty A)) & \longrightarrow & \Gamma(A) & \longrightarrow & T & \longrightarrow & 0 \\
\uparrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
H_3(A, \mathbb{Z})/\bigwedge_2^3 A & \longrightarrow & H_3(G, \mathbb{Z})/M & \longrightarrow & H_3(G, \mathbb{Z})/N & \longrightarrow & 0. \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
H_3(Q, \mathbb{Z}) & \longrightarrow & H_3(Q, \mathbb{Z}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0 & & 0 & & 0 & & 0
\end{array}
$$

**Example 4.5.** For any sequence of abelian groups $A_n,$ $n \geq 2,$ Berrick and Miller constructed a perfect group $Q$ such that $H_n(Q, \mathbb{Z}) \simeq A_n$ [1, Theorem 1].

Let $A$ be an abelian group. Choose a perfect group $Q$ such that $H_2(Q, \mathbb{Z}) \simeq A$ and $H_4(Q, \mathbb{Z}) = 0.$ Then if $A \to G \to Q$ is the universal central extension of $Q,$ we have the exact sequence

$$0 \to \Gamma(A) \to H_3(G, \mathbb{Z}) \to H_3(Q, \mathbb{Z}) \to 0.$$

Thus the map $\varphi : H_1(\Sigma_2^3, \text{Tor}^\mathbb{Z}_1(2\infty A, 2\infty A)) \to H_3(G, \mathbb{Z})$ is injective here. This example also show that the kernel of the natural map $H_3(G, \mathbb{Z}) \to H_3(Q, \mathbb{Z})$ usually is larger than the image of $\varphi.$
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