Wiener criteria for existence of large solutions of nonlinear parabolic equations with absorption in a non-cylindrical domain.

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Abstract

We obtain a necessary condition and a sufficient condition, both expressed in terms of Wiener type tests involving the parabolic $W^{2,1}_q$-capacity, where $q' = \frac{q}{q-1}$, for the existence of large solutions to equation $\partial_t u - \Delta u + u^q = 0$ in non-cylindrical domain, where $q > 1$. Also, we provide a sufficient condition associated with equation $\partial_t u - \Delta u + e^{u} - 1 = 0$. Besides, we apply our results to equation: $\partial_t u - \Delta u + a|\nabla u|^p + bu^q = 0$ for $a, b > 0, 1 < p < 2$ and $q > 1$.

Keywords. Bessel capacities; Hausdorff capacities; parabolic boundary; Riesz potential; maximal solutions.

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1 Introduction

The aim of this paper is to study the problem of existence of large solutions to nonlinear parabolic equations with superlinear absorption in an arbitrary bounded open set $O \subset \mathbb{R}^{N+1}$, $N \geq 2$. These are solutions $u \in C^{2,1}(O)$ of equations

$$\partial_t u - \Delta u + |u|^{q-1}u = 0 \quad \text{in } O,$$

$$\lim_{\delta \to 0} \inf_{O \cap Q_{\delta}(x,t)} u = \infty \quad \text{for all } (x,t) \in \partial_p O,$$

and

$$\partial_t u - \Delta u + \text{sign}(u)(e^{|u|} - 1) = 0 \quad \text{in } O,$$

$$\lim_{\delta \to 0} \inf_{O \cap Q_{\delta}(x,t)} u = \infty \quad \text{for all } (x,t) \in \partial_p O,$$

where $q > 1$ and $\partial_p O$ is the parabolic boundary of $O$, i.e, the set all points $X = (x,t) \in \partial O$ such that the intersection of the cylinder $Q_{\delta}(x,t) := B_{\delta}(x) \times (t-\delta^2, t)$ with $O^c$ is not empty for any $\delta > 0$. By the maximal principle for parabolic equations we can assume that all solutions of (1.1) and (1.2) are positive. Henceforth we consider only positive solutions of the preceding equations.

In [22], we studied the existence and the uniqueness of solution of general equations in a cylindrical domain,

$$\partial_t u - \Delta u + f(u) = 0 \quad \text{in } \Omega \times (0,\infty),$$

$$u = \infty \quad \text{in } \partial_p (\Omega \times (0,\infty)),$$

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where $\Omega$ is a bounded open set in $\mathbb{R}^N$ and $f$ is a continuous real-valued function, nondecreasing on $\mathbb{R}$ such that $f(0) \geq 0$ and $f(a) > 0$ for some $a > 0$. In order to obtain the existence of a maximal solution of $\partial_t u - \Delta u + f(u) = 0$ in $\Omega \times (0, \infty)$ there is need to assume

\begin{align}
(i) & \quad \int_a^\infty \left( \int_0^s f(\tau) d\tau \right)^{-\frac{3}{2}} ds < \infty,
(ii) & \quad \int_a^\infty (f(s))^{-1} ds < \infty. \tag{1.4}
\end{align}

Condition (i), due to Keller and Osserman, is a necessary and sufficient for the existence of a maximal solution to $-\Delta u + f(u) = 0$ in $\Omega$. \tag{1.5}

Condition (ii) is a necessary and sufficient for the existence of a maximal solution of the differential equation

$$\varphi' + f(\varphi) = 0 \quad \text{in } (0, \infty),$$

and this solution tends to $\infty$ at $0$. In [22], it is shown that if for any $m \in \mathbb{R}$ there exists $L = L(m) > 0$ such that

for any $x, y \geq m \Rightarrow f(x + y) \geq f(x) + f(y) - L,$

and if (1.5) has a large solution, then (1.3) admits a solution.

It is not always true that the maximal solution to (1.5) is a large solution. However, if $f$ satisfies

$$\int_1^\infty s^{-2(N-1)/(N-2)} f(s) ds < \infty \quad \text{if } N \geq 3,$$

or

$$\inf \left\{ a \geq 0 : \int_0^\infty f(s) e^{-as} ds < \infty \right\} < \infty \quad \text{if } N = 2,$$

then (1.5) has a large solution for any bounded domain $\Omega$, see [16].

When $f(u) = u^q$, $q > 1$ and $N \geq 3$, the first above condition is satisfied if and only if $q < q_c := \frac{N}{N-2}$; this is called the sub-critical case. When $q \geq q_c$, a necessary and sufficient condition for the existence of a large solution to

$$-\Delta u + u^q = 0 \quad \text{in } \Omega; \tag{1.7}$$

is expressed in term of a Wiener-type test,

$$\int_0^1 \frac{\text{Cap}_{2,q'}(\Omega^c \cap B_r(x))}{r^{N-2}} dr = \infty \quad \text{for all } x \in \partial \Omega. \tag{1.8}$$

In the case $q = 2$ it is obtained by probabilistic methods involving the Brownian snake by Dhersin and Le Gall [5], also see [13, 14]: this method can be extended for $1 < q \leq 2$ by using ideas from [7, 8]. In the general case the result is proved by Labutin, using purely analytic methods [12]. Here, $q' = \frac{q}{q-1}$ and Cap$_{2,q'}$ is the capacity associated to the Sobolev space $W^{2,q'}(\mathbb{R}^N)$.

In [19] we obtain sufficient conditions when $f(u) = e^u - 1$, involving the Hausdorff $H^{N-2}_1$-capacity in $\mathbb{R}^N$, namely,

$$\int_0^1 \frac{H_{1}^{N-2}(\Omega^c \cap B_r(x))}{r^{N-2}} dr = \infty \quad \text{for all } x \in \partial \Omega. \tag{1.9}$$
We refer to [17] for investigation of the initial trace theory of (1.3). In [9], Evans and Gariepy establish a Wiener criterion for the regularity of a boundary point (in the sense of potential theory) for the heat operator \( L = \partial_t - \Delta \) in an arbitrary bounded set of \( \mathbb{R}^{N+1} \). We denote by \( \mathcal{M}(\mathbb{R}^{N+1}) \) the set of Radon measures in \( \mathbb{R}^{N+1} \) and, for any compact set \( K \subset \mathbb{R}^{N+1} \), by \( \mathcal{M}_K(\mathbb{R}^{N+1}) \) the subset of \( \mathcal{M}(\mathbb{R}^{N+1}) \) of measures with support in \( K \). Their positive cones are respectively denoted by \( \mathcal{M}_+^+(\mathbb{R}^{N+1}) \) and \( \mathcal{M}^+_K(\mathbb{R}^{N+1}) \). The capacity used in this criterion is the thermal capacity defined by

\[
\text{Cap}_{\partial\Omega}(K) = \sup \{ \mu(K) : \mu \in \mathcal{M}_K(\mathbb{R}^{N+1}), \mathbb{H} \ast \mu \leq 1 \},
\]

for any \( K \subset \mathbb{R}^{N+1} \) compact, where \( \mathbb{H} \) is the heat kernel in \( \mathbb{R}^{N+1} \). It coincides with the parabolic Bessel \( G_1 \)-capacity \( \text{Cap}_{G_{1,2}} \),

\[
\text{Cap}_{G_{1,2}}(K) = \sup \left\{ \int_{\mathbb{R}^{N+1}} |f|^2 \, dx \, dt : f \in L^2_x(\mathbb{R}^{N+1}), \, G_1 \ast f \geq \chi_K \right\},
\]

here \( G_1 \) is the parabolic Bessel kernel of first order, see [20, Remark 4.12]. Garofalo and Lanconelli [10] extend this result to the parabolic operator \( L = \partial_t - \text{div}(A(x,t)\nabla) \), where \( A(x,t) = (a_{i,j}(x,t)) \), \( i,j = 1,2,\ldots,N \) is a real, symmetric, matrix-valued function on \( \mathbb{R}^{N+1} \) with \( C^\infty \) entries for which there holds

\[
C^{-1} |\xi|^2 \leq \sum_{i,j=1}^N a_{i,j}(x,t) \xi_i \xi_j \leq C |\xi|^2 \quad \forall (x,t) \in \mathbb{R}^{N+1}, \forall \xi \in \mathbb{R}^N,
\]

for some constant \( C > 0 \).

Less is known concerning the equation

\[
\partial_t u - \Delta u + f(u) = 0 \tag{1.10}
\]

in a bounded open set \( O \subset \mathbb{R}^{N+1} \), where \( f \) is a continuous function in \( \mathbb{R} \), Gariepy and Ziemer [11, 23] prove that if there are \( (x_0,t_0) \in \partial O \), \( l \in \mathbb{R} \) and a weak solution \( u \in W^{1,2}(O) \cap L^\infty(O) \) of (1.10) such that \( \eta(-l - \varepsilon + u)^+ + \eta(l - \varepsilon - u)^+ \in W^{1,2}_0(O) \) for any \( \varepsilon > 0 \) and \( \eta \in C^\infty(B_r(x_0) \times (-r^2 + t_0, r^2 + t_0)) \) for some \( r > 0 \) and if

\[
\int_0^1 \text{Cap}_{\partial\Omega} \left( O^e \cap \{B_r(x_0) \times (t_0 - \frac{2}{\tau} \alpha \rho^2, t_0 - \frac{5}{\tau} \alpha \rho^2)\} \right) \, d\rho = \infty \text{ for some } \alpha > 0
\]

then

\[
\lim_{{(x,t) \to (x_0,0)}} u(x,t) = l.
\]

This result is not easy to use because it is not clear whether (1.10) has a weak solution \( u \in W^{1,2}(O) \). In this article we show that (1.10) admits a maximal solution \( u \in C^{2,1}(O) \) in an arbitrary bounded open set \( O \), by approximation by dyadic parabolic cubes from inside \( O \), provided that \( f \) is as in (1.3) and satisfies (1.4).

Our main purpose of this article is to extend the result of Labutin [12] to nonlinear parabolic equation (1.10). Namely, we give a necessary and a sufficient condition for the existence of solutions to (1.10) in a bounded non-cylindrical domain \( O \subset \mathbb{R}^{N+1} \), expressed in terms of a Wiener test based upon the parabolic \( W^{2,1}_q \)-capacity in \( \mathbb{R}^{N+1} \). We also give a sufficient condition associated (1.2) where the parabolic \( W^{2,1}_q \)-capacity is replaced the parabolic Hausdorff \( PH^q_\alpha \)-capacity. These capacities are defined as follows: if \( K \subset \mathbb{R}^{N+1} \) is compact, we set

\[
\text{Cap}_{2,1,q}^p(K) = \inf \|\|\varphi\|_{W^{2,1}_q(\mathbb{R}^{N+1})}^q : \varphi \in S(\mathbb{R}^{N+1}), \varphi \geq 1 \text{ in a neighborhood of } K \}
\]

where

\[
\|\varphi\|_{W^{2,1}_q(\mathbb{R}^{N+1})} = \|\partial_\varphi\|_{L^q(\mathbb{R}^{N+1})} + \|\nabla \varphi\|_{L^q(\mathbb{R}^{N+1})} + \sum_{i,j=1,2,\ldots,N} \|\partial_{x_i \partial x_j} \varphi\|_{L^q(\mathbb{R}^{N+1})},
\]

for some constant \( C > 0 \).
and for Suslin set $E \subset \mathbb{R}^{N+1}$,

$$\text{Cap}_{2,1,q'}(E) = \sup\left\{\text{Cap}_{2,1,q'}(D) : D \subset E, D \text{ compact}\right\}.$$ 

This capacity has been used in order to obtain potential theory estimates that are most helpful for studying quasilinear parabolic equations (see e.g. [3, 4, 20]). Thanks to a result due to Richard and Bagby [2], the capacities $\text{Cap}_{2,1,p}$ and $\text{Cap}_{G,2,q'}$ are equivalent in the sense that, for any Suslin set $K \subset \mathbb{R}^{N+1}$, there holds

$$C^{-1}\text{Cap}_{2,1,q'}(K) \leq \text{Cap}_{G,2,q'}(K) \leq C\text{Cap}_{2,1,p}(K),$$

for some $C = C(N, q)$, where $\text{Cap}_{G,2,q'}$ is the parabolic Bessel $G_2$–capacity, see [20]. For $E \subset \mathbb{R}^{N+1}$, we define $\mathcal{P}\mathcal{H}^N_\rho(E)$ by

$$\mathcal{P}\mathcal{H}^N_\rho(E) = \inf \left\{ \sum_j r_j^N : E \subset \bigcup B_{r_j}(x_j) \times (t_j - r_j^2, t_j + r_j^2), r_j \leq \rho \right\}.$$ 

It is easy to see that, for $0 < \sigma \leq \rho$ and $E \subset \mathbb{R}^{N+1}$, there holds

$$\mathcal{P}\mathcal{H}^N_\rho(E) \leq \mathcal{P}\mathcal{H}^N_{\sigma}(E) \leq C(N) \left( \frac{\rho}{\sigma} \right)^2 \mathcal{P}\mathcal{H}^N_\rho(E).$$ (1.11)

With these notations, we can state the two main results of this paper.

**Theorem 1.1** Let $N \geq 2$ and $q \geq q_* := \frac{N+2}{N}$. Then

(i) The equation

$$\partial_t u - \Delta u + u^q = 0 \text{ in } O$$

admits a large solution if

$$\sum_{k=1}^{\infty} \frac{\text{Cap}_{2,1,q'}(O^c \cap (B_{r_k}(x) \times (t - 1168r_k^2, t - 1136r_k^2)))}{r_k^N} = \infty,$$ (1.13)

for any $(x, t) \in \partial_p O$, where $r_k = 4^{-k}$, and $N \geq 3$ when $q = q_*$. 

(ii) If equation (1.12) admits a large solution, then

$$\int_0^1 \frac{\text{Cap}_{2,1,q'}(O^c \cap Q_{\rho}(x,t))}{\rho} \frac{d\rho}{\rho} = \infty,$$ (1.14)

for any $(x, t) \in \partial_p O$, where $Q_{\rho}(x,t) = B_{\rho}(x) \times (t - \rho^2, t)$.

**Theorem 1.2** Let $N \geq 2$. The equation

$$\partial_t u - \Delta u + e^u - 1 = 0 \text{ in } O$$

admits a large solution if

$$\sum_{k=1}^{\infty} \frac{\mathcal{P}\mathcal{H}^N_1(O^c \cap (B_{r_k}(x) \times (t - 1168r_k^2, t - 1136r_k^2)))}{r_k^N} = \infty,$$ (1.16)

for any $(x, t) \in \partial_p O$, with $r_k = 4^{-k}$. 

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From properties of the $W^{2,1}_{q'}$-capacity and the $\mathcal{PH}_1^N$-capacity, relation (1.13) holds if
\[ \sum_{k=1}^{\infty} r_k^{-N} |O^c \cap (B_{r_k}(x) \times (t - 1168r_k^2, t - 1136r_k^2))|^{1 - \frac{2q}{q'}} = \infty \text{ when } q > q_*, \]
and
\[ \sum_{k=1}^{\infty} r_k^{-N} \log \left( |O^c \cap (B_{r_k}(x) \times (t - 1168r_k^2, t - 1136r_k^2))|^{-1} \right) = \infty \text{ when } q = q_. \]

Similarly, identity (1.16) is verified if
\[ \sum_{k=1}^{\infty} r_k^{-N} |O^c \cap (B_{r_k}(x) \times (t - 1168r_k^2, t - 1136r_k^2))|^{\frac{N}{2 - \alpha}} = \infty. \]

Therefore, when $O = \{ (x, t) \in \mathbb{R}^{N+1} : |x|^2 + |t|^2 < 1 \}$ for some $\lambda > 0$, we see that $\partial O = \partial_p O$, (1.14) holds for any $(x, t) \in \partial_p O$, (1.13) and (1.16) hold for any $(x, t) \in \partial_p O \setminus \{(0, \sqrt{\lambda})\}$. However, (1.13) and (1.16) are also true at $(x, t) = (0, \sqrt{\lambda})$ if $\lambda > 2272^2$ and not true if $\lambda < 2272^2$.

As a consequence of Theorem 1.1 we derive a sufficient condition for the existence of large solution of some viscous Hamilton-Jacobi parabolic equations.

**Theorem 1.3** Let $q_1 > 1$. If there exists a large solution $v \in C^{2,1}(O)$ of
\[ \partial_t v - \Delta v + v^a = 0 \quad \text{in} \quad O, \]
then, for any $a, b > 0$, $1 < q < q_1$ and $1 < p < \frac{2q}{q_1 + 1}$, problem
\[ \partial_t u - \Delta u + a|\nabla u|^p + bu^q = 0 \quad \text{in} \quad O, \quad u = \infty \quad \text{on} \quad \partial_p O, \] (1.17)

admits a solution $u \in C^{2,1}(O)$ which satisfies
\[ u(x, t) \geq C \min \left\{ a - \frac{1}{q_1 - 1} R \frac{\alpha}{\alpha + 1}, b - \frac{1}{q_1 - 1} R \frac{\alpha}{\alpha + 1}, \left( v(x, t) \right)^{\frac{q_1}{q_1 - 1}} \right\}, \]
for all $(x, t) \in O$ where $R > 0$ is such that $O \subset \tilde{Q}_R(x_0, t_0)$, $C = C(N, p, q, q_1) > 0$ and $\alpha = \max \left\{ \frac{2(p - 1)}{(q_1 - 1)(2 - p)}, \frac{q}{q_1 - 1} \right\} \in (0, 1)$.

## 2 Preliminaries

Throughout the paper, we denote $Q_\rho(x, t) = B_\rho(x) \times (t - \rho^2, t)$ and $\tilde{Q}_\rho(x, t) = B_\rho(x) \times (t - \rho^2, t + \rho^2)$ for $(x, t) \in \mathbb{R}^{N+1}$, $\rho > 0$ and $r_k = 4^{-k}$ for all $k \in \mathbb{Z}$. We also denote $A \lesssim (\gtrsim) B$ if $A \lesssim (\gtrsim) CB$ for some $C$ depending on some structural constants, $A \asymp B$ if $A \lesssim B \lesssim A$.

**Definition 2.1** Let $R \in (0, \infty]$ and $\mu \in M^+(\mathbb{R}^{N+1})$. We define $R-$truncated Riesz parabolic potential $I_2^R$ of $\mu$ by
\[ I_2^R[\mu](x, t) = \int_0^R \frac{\mu(\tilde{Q}_\rho(x, t))}{\rho} \, d\rho \quad \text{for all} \quad (x, t) \in \mathbb{R}^{N+1}, \]
and the $R-$truncated fractional maximal parabolic potential $M_2^R$ of $\mu$ by
\[ M_2^R[\mu](x, t) = \sup_{0 < \rho < R} \frac{\mu(\tilde{Q}_\rho(x, t))}{\rho^N} \quad \text{for all} \quad (x, t) \in \mathbb{R}^{N+1}. \]
We recall two results in [20].

**Theorem 2.2** Let $q > 1$, $R > 0$ and $K$ be a compact set in $\mathbb{R}^{N+1}$. There exists $\mu := \mu_K \in \mathcal{M}^+(\mathbb{R}^{N+1})$ with compact support in $K$ such that

$$\mu(K) \leq \text{Cap}_{2,1,q}(K) := \int_{\mathbb{R}^{N+1}} (|I_2^R[\mu]|)^q \, dx \, dt$$

where the constants of equivalence depend on $N, q$ and $R$. The measure $\mu_K$ is called the capacitary measure of $K$.

**Theorem 2.3** For any $R > 0$, there exist positive constants $C_1, C_2$ such that for any $\mu \in \mathcal{M}^+(\mathbb{R}^{N+1})$ such that $||M_2^R[\mu]||_{L^\infty(\mathbb{R}^{N+1})} \leq 1$, there holds

$$\int_Q \exp(C_1 \|x_Q\|) \, dx \, dt \leq C_2,$$

for all $Q = \tilde{Q}, (y,s) \subset \mathbb{R}^{N+1}$, $R > 0$, where $\chi_Q$ is the indicator function of $Q$.

Frostman’s Lemma in [21, Th. 3.4.27] is at the basis of the dual definition of Hausdorff capacities with doubling weight. It is easy to see that it is valid for the parabolic Hausdorff $\mathcal{P}\mathcal{H}_\rho^N$ capacity version. As a consequence we have

**Theorem 2.4** There holds

$$\sup \{\mu(K) : \mu \in \mathcal{M}^+(\mathbb{R}^{N+1}), \text{supp}(\mu) \subset K, ||M_2^R[\mu]||_{L^\infty(\mathbb{R}^{N+1})} \leq 1\} \leq \mathcal{P}\mathcal{H}_\rho^N(K)$$

for any compact set $K \subset \mathbb{R}^{N+1}$ and $\rho > 0$, where equivalent constant depends on $N$.

For our purpose, we need the same results about the behavior of the capacity with respect to dilations.

**Proposition 2.5** Let $K \subset \tilde{Q}_{100}(0,0)$ be a compact set and $1 < p < \frac{N+2}{2}$. Then

$$\text{Cap}_{2,1,p}(K) \geq |K|^1 \cdot \frac{2^p}{\pi^p}, \quad \text{Cap}_{2,1,\frac{N+2}{2}}(K) \geq \left( \log \left( \frac{\tilde{Q}_{100}(0,0)}{|K|} \right) \right)^{-\frac{N}{2}}, \quad (2.1)$$

and

$$\text{Cap}_{2,1,p}(K_\rho) \geq \rho^{N+2-2p} \text{Cap}_{2,1,p}(K), \quad \frac{1}{\text{Cap}_{2,1,\frac{N+2}{2}}(K_\rho)} \leq \frac{1}{\text{Cap}_{2,1,\frac{N+2}{2}}(K)} + (\log(2/\rho))^{N/2} \quad (2.2)$$

for any $0 < \rho < 1$, where $K_\rho = \{(\rho x, \rho^2 t) : (x,t) \in K\}$.

**Proposition 2.6** Let $K \subset \tilde{Q}_1(0,0)$ be a compact set and $1 < p \leq \frac{N+2}{2}$. Then, there exists a function $\varphi \in C^{\infty}_c(\tilde{Q}_{3/2}(0,0))$, $0 \leq \varphi \leq 1$ and $\varphi|_D = 1$ for some open set $D \supset K$ such that

$$\int_{\mathbb{R}^{N+1}} \{|D^2 \varphi|^p + |\nabla \varphi|^p + |\varphi|^p + |\partial_t \varphi|^p\} \, dx \, dt \leq \text{Cap}_{2,1,p}(K). \quad (2.4)$$

We will give proofs of the above two propositions in the Appendix. It is well known that there exists a semigroup $e^{tA}$ corresponding to equation

$$\partial_t u - \Delta u = \mu \quad \text{in } \tilde{Q}_R(0,0), \quad u = 0 \quad \text{on } \partial_t \tilde{Q}_R(0,0), \quad (2.5)$$
Theorem 2.7
framework.

We have
\[ u(x,t) = \int_0^t \left( e^{(t-s)\Delta} \mu \right)(x,s)ds \quad \text{for all } (x,t) \in \tilde{Q}_R(0,0). \]

We denote by \( H \) the heat kernel:
\[ H(x,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}} \chi_{t>0}. \]

We have
\[ |u(x,t)| \leq (H * \mu)(x,t) \quad \text{for all } (x,t) \in \tilde{Q}_R(0,0). \]

In [20, Th. 2.5] (with \( \Delta \) replaced by a uniformly elliptic quasilinear operator) we show that
\[ |(H * \mu)(x,t)| \leq C_1(\mu)2^R ||\mu||(x,t) \quad \text{for all } (x,t) \in \tilde{Q}_R(0,0). \]

Here \( \mu \) is extended by 0 in \((\tilde{Q}_R(0,0))^c\). Thus,
\[ |\int_0^t \left( e^{(t-s)\Delta} \mu \right)(x,s)ds| \leq C_1(\mu)2^R ||\mu||(x,t) \quad \text{for all } (x,t) \in \tilde{Q}_R(0,0). \quad (2.6) \]

Moreover, we also prove in [20], that if \( \mu \geq 0 \) then for \( (x,t) \in \tilde{Q}_R(0,0) \) and \( B_\rho(x) \subset B_R(0) \),
\[ \int_0^t \left( e^{(t-s)\Delta} \mu \right)(x,s)ds \geq C_2(\mu) \sum_{k=0}^\infty \frac{\mu(Q_{2^k}(x,t-\frac{35}{128}\rho_k^2))}{\rho_k^2}. \quad (2.7) \]

with \( \rho_k = 4^{-k}\rho \).

It is easy to see that estimates (2.6) and (2.7) also holds for any bounded Radon measure \( \mu \) in \( \tilde{Q}_R(0,0) \). The following result is proved in [3] and [18], and also in [20] in a more general framework.

Theorem 2.7 Let \( q > 1 \), \( R > 0 \) and \( \mu \) be bounded Radon measure in \( \tilde{Q}_R(0,0) \).

(i) If \( \mu \) is absolutely continuous with respect to \( \text{Cap}_{2,1,q} \) in \( \tilde{Q}_R(0,0) \), then there exists a unique weak solution \( u \) to equation
\[ \partial_t u - \Delta u + |u|^{q-1} u = \mu \quad \text{in } \tilde{Q}_R(0,0), \]
\[ u = 0 \quad \text{on } \partial_t \tilde{Q}_R(0,0). \]

(ii) If \( \exp \left( C_1(\mu)2^R ||\mu|| \right) \in L^1(\tilde{Q}_R(0,0)) \) then there exists a unique weak solution \( v \) to equation
\[ \partial_t v - \Delta v + \text{sign}(v)(|v|^q - 1) = \mu \quad \text{in } \tilde{Q}_R(0,0), \]
\[ v = 0 \quad \text{on } \partial_t \tilde{Q}_R(0,0), \]

where the constant \( C_1(\mu) \) is the one of inequality (2.6).

From estimates (2.6) and (2.7) and using comparison principle we get the estimates from below of the solutions \( u \) and \( v \) obtained in Theorem 2.7.

Proposition 2.8 If \( \mu \geq 0 \) then the functions \( u \) and \( v \) of the previous theorem are nonnegative and satisfy
\[ u(x,t) \geq C_2(\mu) \sum_{k=0}^\infty \frac{\mu(Q_{2^k}(x,t-\frac{35}{128}\rho_k^2))}{\rho_k^2} - C_1(\mu)2^R \left[ (\|2^R||\mu||)^q \right] (x,t) \quad (2.8) \]
and
\[ v(x,t) \geq C_2(\mu) \sum_{k=0}^\infty \frac{\mu(Q_{2^k}(x,t-\frac{35}{128}\rho_k^2))}{\rho_k^2} - C_1(\mu)2^R \left[ \exp \left( C_1(\mu)2^R ||\mu|| \right) - 1 \right] (x,t). \quad (2.9) \]

for any \( (x,t) \in \tilde{Q}_R(0,0) \) and \( B_\rho(x) \subset B_R(0) \) and \( \rho_k = 4^{-k}\rho \).
3 Maximal solutions

In this section we assume that $O$ is an arbitrary non-cylindrical and bounded open set in $\mathbb{R}^{N+1}$ and $q > 1$. We will prove the existence of a maximal solution of

$$\partial_t u - \Delta u + u^q = 0$$  \hfill (3.1)

in $O$. We also get analogous result where $u^q$ is replaced by $e^u - 1$.

It is easy to see that if $u$ satisfies (3.1) in $\bar{Q}_r(0,0)$ then $u_a(x,t) = a^{-2/(q-1)}u(ax,a^2t)$ satisfies (3.1) in $\bar{Q}_{r/a}(0,0)$ for any $a > 0$.

If $X = (x,t) \in O$, the parabolic distance from $X$ to the parabolic boundary $\partial_P O$ of $O$ is defined by

$$d(X, \partial_P O) = \inf_{(y,s) \in \partial_P O} \max\{ |x - y|, |t - s| \}.$$  

It is easy to see that there exists $C = C(N,q) > 0$ such that the function $V$ defined by

$$V(x,t) = C \left( (\rho^2 + t)^{-\frac{q}{4}} + \left( \frac{2 - |x|^2}{\rho} \right)^{-\frac{q}{4}} \right) \text{ in } B_\rho(0) \times (-\rho^2,0)$$

satisfies

$$\partial_t V - \Delta V + V^q \geq 0 \text{ in } B_\rho(0) \times (-\rho^2,0).$$  \hfill (3.2)

**Proposition 3.1** There exists a maximal solution $u \in C^{2,1}(O)$ of (3.1) and it satisfies

$$u(x,t) \leq C(d((x,t), \partial_P O))^{-\frac{q}{4}} \text{ for all } (x,t) \in O$$  \hfill (3.3)

for some $C = C(N,q)$.

**Proof.** Let $\mathcal{D}_k$, $k \in \mathbb{Z}$ be the collection of all the dyadic parabolic cubes (abridged $p$-cubes) of the form

$$\{(x_1, \ldots, x_N, t) : m_j 2^{-k} \leq x_j \leq (m_j + 1)2^{-k}, j = 1, \ldots, N, m_{N+1}4^{-k} \leq t \leq (m_{N+1} + 1)4^{-k}\}$$

where $m_{j} \in \mathbb{Z}$. The following properties hold,

**a.** For each integer $k$, $\mathcal{D}_k$ is a partition of $\mathbb{R}^{N+1}$ and all $p$-cubes in $\mathcal{D}_k$ have the same sidelengths.

**b.** If the interiors of two $p$-cubes $Q$ in $\mathcal{D}_{k_1}$ and $P$ in $\mathcal{D}_{k_2}$, denoted $\tilde{Q}, \tilde{P}$, have nonempty intersection then either $Q$ is contained in $R$ or $Q$ contains $R$.

**c.** Each $Q$ in $\mathcal{D}_k$ is union of $2^{N+2}$ $p$-cubes in $\mathcal{D}_{k+1}$ with disjoint interiors.

Let $k_0 \in \mathbb{N}$ be such that $Q \subset O$ for some $Q \in \mathcal{D}_{k_0}$. Set $O_k = \bigcup_{Q \subset \bar{O}} Q$ \forall $k \geq k_0$, we have $O_k \subset O_{k+1}$ and $O = \bigcup_{k \geq k_0} O_k = \bigcup_{k \geq k_0} \tilde{O}_k$. More precisely, there exist real numbers $a_1, a_2, \ldots, a_{n(k)}$ and open sets $\Omega_1, \Omega_2, \ldots, \Omega_{n(k)}$ in $\mathbb{R}^{N}$ such that

$$a_i < a_i + 4^{-k} \leq a_{i+1} < a_{i+1} + 4^k \text{ for } i = 1, \ldots, n(k) - 1$$

and

$$\tilde{O}_k = \bigcup_{i=1}^{n(k)-1} (\Omega_i \times (a_i, a_i + 4^{-k})) \bigcup \left( \Omega_{n(k)} \times (a_{n(k)}, a_{n(k)} + 4^{-k}) \right).$$
For \( k \geq k_0 \), we claim that there exists a solution \( u_k \in C^{2,1}(\hat{O}_k) \) to problem

\[
\begin{align*}
\partial_t u_k - \Delta u_k + u_k^q &= 0 & \text{in } \hat{O}_k, \\
u_k(x,t) &\to \infty & \text{as } d((x,t),\partial_p\hat{O}_k) \to 0.
\end{align*}
\] (3.4)

Indeed, by \([6, 15]\) for \( m > 0 \) one can find nonnegative solutions \( v_i \in C^{2,1}(\Omega_i \times (a_i, a_i + 4^{-k})) \cap C(\Omega_i \times [a_i, a_i + 4^{-k}]) \) for \( i = 1, ..., n(k) \) to equations

\[
\begin{align*}
\partial_t v_i - \Delta v_i + v_i^q &= 0 & \text{in } \Omega_i \times (a_i, a_i + 4^{-k}), \\
v_i(x,t) &= m & \text{on } \partial\Omega_i \times (a_i, a_i + 4^{-k}), \\
v_i(x,t_i) &= m & \text{in } \Omega_i,
\end{align*}
\]

and

\[
\begin{align*}
\partial_t v_i - \Delta v_i + v_i^q &= 0 & \text{in } \Omega_i \times (a_i, a_i + 4^{-k}), \\
v_i(x,t) &= m & \text{on } \partial\Omega_i \times (a_i, a_i + 4^{-k}), \\
v_i(x,t_i) &= \begin{cases} m \text{ in } \Omega_i \\
m\chi_{\Omega_i \setminus \Omega_{i-1}}(x) + v_{i-1}(x, a_{i-1} + 4^{-k})\chi_{\Omega_{i-1}}(x) & \text{if } a_i > a_{i-1} + 4^{-k}, \\
\end{cases} \\
& \text{otherwise .}
\end{align*}
\]

Clearly,

\[
u_{k,m} = v_i \text{ in } \Omega_i \times (a_i, a_i + 4^{-k}) \text{ for } i = 1, ..., n(k)
\]
is a solution in \( C^{2,1}(\hat{O}_k) \cap C(\hat{O}_k) \) to equation

\[
\begin{align*}
\begin{cases}
\partial_t u_{k,m} - \Delta u_{k,m} + u_{k,m}^q &= 0 & \text{in } \hat{O}_k, \\
u_{k,m} &= m & \text{on } \partial_p\hat{O}_k.
\end{cases}
\end{align*}
\]

Moreover, for \( (x,t) \in \hat{O}_k \), we can see that \( B_{\frac{d}{4}}(x) \times (t - \frac{d_2^2}{4}, t) \subset \hat{O}_k \) where \( d = d((x,t),\partial_p\hat{O}_k) \). From \([6, 15]\), we verify that

\[
U(y,s) := V(y-x,s-t) = C \left( \left( \rho^2 + s - t \right)^{-\frac{d}{4}} + \left( \frac{\rho^2 - |x-y|^2}{\rho} \right)^{-\frac{d}{4}} \right)
\]

with \( \rho = d/2 \), satisfies

\[
\partial_t U - \Delta U + U^q \geq 0 \text{ in } B_{\frac{d}{4}}(x) \times (t - \frac{d_2^2}{4}, t). \] (3.5)

Applying the comparison principle we get

\[
u_{k,m}(y,s) \leq U(y,s) \text{ in } B_{\frac{d}{4}}(x) \times (t - \frac{d_2^2}{4}, t],
\]

which implies

\[
u_{k,m}(x,t) \leq C \left( d((x,t),\partial_p\hat{O}_k) \right)^{-\frac{d}{4}} \text{ for all } (x,t) \in \hat{O}_k.
\] (3.6)

From this, we also obtain uniform local bounds for \( \{u_{k,m}\}_m \). By standard regularity theory see \([6, 15]\), \( \{u_{k,m}\}_m \) is uniformly locally bounded in \( C^{2,1} \). Hence, up to a subsequence, \( u_{k,m} \to u_k \) \( C^{1,0}_{\text{loc}}(\hat{O}_k) \). Passing the limit, we derive that \( u_k \) is a weak solution of \((3.4)\) in \( \hat{O}_k \), which satisfies \( u_k(x,t) \to \infty \) as \( d((x,t),\partial_p\hat{O}_k) \to 0 \) and

\[
u_k(x,t) \leq C \left( d((x,t),\partial_p\hat{O}_k) \right)^{-\frac{d}{4}} \text{ for all } (x,t) \in \hat{O}_k.
\]
Let \( m > 0 \) and \( k \geq k_0 \). Since \( u_{k+1,m} \leq u_m \), it follows by the comparison principle applied to \( u_{k+1,m} \) and \( u_{k,m} \) in the sub-domains \( \Omega_1 \times (a_1,a_1+4^{-k}), \Omega_2 \times (a_2,a_2+4^{-k}), \ldots, \Omega_{n(k)} \times (a_{n(k)},a_{n(k)}+4^{-k}) \) of \( \hat{O}_k \) to obtain that \( u_{k+1,m} \leq u_{k,m} \), and thus \( u_{k+1} \leq u_k \). In particular, \( \{u_k\}_k \) is uniformly locally bounded in \( \mathbb{F}^\infty_{loc} \). We use the same compactness property as above to obtain that \( u_k \to u \) where \( u \) is a solution of (3.1) and satisfies (3.3). By construction \( u \) is the maximal solution.

**Remark 3.2** Let \( R \geq 2r \geq 2 \), \( K \) be a compact subset in \( \overline{Q_r}(0,0) \). Arguing as one can easily it is clear that there exists a maximal solution of

\[
\partial_t u - \Delta u + u^q = 0 \quad \text{in} \quad \hat{Q}_R(0,0) \backslash K,
\]

\[
u = 0 \quad \text{on} \quad \partial \hat{Q}_R(0,0),
\]

which satisfies

\[
u(x,t) \leq C(d((x,t),\partial \hat{Q}_R(0,0) \backslash K))^{-\frac{r}{2}} \quad \forall \ (x,t) \in \hat{Q}_R(0,0) \backslash K,
\]

for some \( C = C(N,q) \). Furthermore, assume \( K_1, K_2, \ldots, K_m \) are compact subsets in \( Q_r(0,0) \) and \( K = K_1 \cup K_2 \cup \ldots \cup K_m \). Let \( u, u_1, \ldots, u_m \) be the maximal solutions of (3.7) in \( Q_R(0,0) \backslash K, Q_R(0,0) \backslash K_1, Q_R(0,0) \backslash K_2, \ldots, Q_R(0,0) \backslash K_m \), respectively, then

\[
u \leq \sum_{j=1}^m u_j \quad \text{in} \quad \hat{Q}_R(0,0) \backslash K.
\]

**Remark 3.3** If the equation (3.1) admits a large solution for some \( q > 1 \), then for any \( 1 < q_1 < q \), equation

\[
\partial_t u - \Delta u + u^{q_1} = 0 \quad \text{in} \quad O
\]

admits also a large solution.

Indeed, assume that \( u \) is a large solution of (3.1) and \( v \) is the maximal solution of (3.10). Take \( R > 0 \) such that \( O \subset B_R(0) \times (-R^2,R^2) \), then the function \( V \) defined by

\[
V(x,t) = (q-1)^{-\frac{1}{q-1}}(2R^2 + t)^{-\frac{1}{q-1}}R^{-\frac{1}{q-1}}
\]

satisfies (3.1). It follows for all \( (x,t) \in O \)

\[
u(x,t) \geq \inf_{(y,s) \in O} V(x,t) \geq (q-1)^{-\frac{1}{q-1}}R^{-\frac{1}{q-1}} =: a_0.
\]

Thus, \( \tilde{u} \equiv \frac{a_0}{a_0^q}u \) is a subsolution of (3.10). Therefore \( v \geq \frac{a_0}{a_0^q}u \) in \( O \), thus \( v \) is a large solution.

**Remark 3.4 (Sub-critical case)** Assume that \( 1 < q < q_* \). One easily see that the function

\[
U(x,t) = \frac{C}{t^{\frac{n+q}{q}}} e^{-\frac{|x|^2}{2t^{\frac{n+q}{q}}}}
\]

is a subsolution of (3.1) in \( \mathbb{R}^{N+1} \backslash \{(0,0)\} \), where \( C = \left( \frac{2}{n+q} - \frac{n}{2} \right)^{\frac{1}{n+q}} \). Therefore, the maximal solutions \( u \) of (3.1) in \( O \) verify

\[
u(x,t) \geq C \frac{1}{(t-s)^{\frac{n+q}{q}}} e^{-\frac{|x-y|^2}{2(t-s)^{\frac{n+q}{q}}}} \chi_{t> s},
\]

(3.12)
for all \((x,t) \in O\) and \((y,s) \in O^c\).

If for any \((x,t) \in \partial_p O\) there exist \(\varepsilon \in (0,1)\) and a decreasing sequence \(\{\delta_n\} \subset (0,\varepsilon)\) converging to 0 as \(n \to \infty\) such that \((B_{\delta_n}(x) \times (-\delta_n^2 + t, -\delta_n^2 + t)) \cap O^c \neq \emptyset\) for any \(n \in \mathbb{N}\), then \(u\) is a large solution. For proving this, we need to show that \(\lim_{\rho \to 0} \inf_{\partial\cap(B_{\rho}(x) \times (-\rho^2 + t, \rho^2 + t))} u = \infty\).

Let \(0 < \rho < \delta_1\), and \(n \in \mathbb{N}\) such that \(\sqrt{n}\delta_{n+1} \leq \rho < \sqrt{n}\delta_n\). Since \((B_{\delta_n}(x) \times (-\delta_n^2 + t, -\delta_n^2 + t)) \cap O^c \neq \emptyset\), there is \((x_n,t_n) \in O^c\) such that \(|x_n| < \delta_n\) and \(-\delta_n^2 + t < t_n < -\delta_n^2 + t\). Hence, from (3.12) we have

\[
u(x,t) \geq C \frac{1}{(t-t_n)^2} e^{-\frac{|x-x_n|^2}{4(t-t_n)^2}} \quad \forall \,(x,t) \in O \cap (B_{\rho}(x) \times (-\rho^2 + t, \rho^2 + t)),\]

which implies

\[
\inf_{O \cap (B_{\rho}(x) \times (-\rho^2 + t, \rho^2 + t))} u \geq C \frac{1}{\delta_n^{-2}} e^{-\frac{(n+1)2}{\delta_n^2}} \to \infty \text{ as } \rho \to 0.
\]

**Remark 3.5** Note that if \(u \in C^{2,1}(O)\) is a solution of (3.1) for some \(q > 1\) then, for \(a,b > 0\) and \(1 < p \leq 2, v = b^\frac{1}{q-1} u\) is a super-solution of

\[
\partial_t v - \Delta v + a|\nabla v|^p + bv^q = 0 \quad \text{in } O. \quad (3.13)
\]

Thus, we can apply the argument of the previous proof, with equation (3.1) replaced by (3.13), and deduce that there exists a maximal solution \(v \in C^{2,1}(O)\) of (3.13) satisfying

\[
v(x,t) \leq Cb^{-\frac{1}{q-1}}(d((x,t), \partial_p O))^{-\frac{1}{q-1}} \quad \text{for all } (x,t) \in O.
\]

Furthermore, if \(1 < q < q_*\), \(q = \frac{2p}{p+1}\), \(a,b > 0\) then the function \(U\) in Remark 3.4 is a subsolution of (3.13) in \(\mathbb{R}^{N+1}\backslash \{(0,0)\}\), for some \(C = C(N,p,q,a,b)\). Therefore, we conclude that every maximal solution of \(v \in C^{2,1}(O)\) of (3.13) satisfy

\[
v(x,t) \geq C \frac{1}{(t-s)^{\frac{N+1}{2}}} e^{-\frac{|x-s|^2}{4(t-s)^2}} \chi_{t < s} \quad (3.14)
\]

for all \((x,t) \in O\) and \((y,s) \in \partial_p O\).

As in Remark 2.4, if for any \((x,t) \in \partial_p O\) there exist \(\varepsilon \in (0,1)\) and a decreasing sequence \(\{\delta_n\} \subset (0,\varepsilon)\) converging to 0 as \(n \to \infty\) such that \((B_{\delta_n}(x) \times (-\delta_n^2 + t, -\delta_n^2 + t)) \cap O^c \neq \emptyset\) for any \(n \in \mathbb{N}\), then \(v\) is a large solution.

Next, we consider the following equation

\[
\partial_t u - \Delta u + e^u - 1 = 0. \quad (3.15)
\]

It is easy to see that the two functions

\[
V_1(t) = -\log\left(\frac{t + \rho^2}{1 + \rho^2}\right) \quad \text{and} \quad V_2(x) = C - 2 \log\left(\frac{\rho^2 - |x|^2}{\rho}\right)
\]

satisfy

\[
V'_1 + e^{V_1} - 1 \geq 0 \quad \text{in } (-\rho^2,0]
\]

and

\[
-\Delta V_2 + e^{V_2} - 1 \geq 0 \quad \text{in } B_\rho(0)
\]

for some \(C = C(N)\). Using \(e^a + e^b \leq e^{a+b} - 1\) for \(a,b > 0\), we obtain that \(V_1 + V_2\) is a supersolution of equation (3.15) in \(B_\rho(0) \times (-\rho^2,0]\). By the same argument as in Proposition 3.1 and the estimate of the above supersolution, we obtain
Proposition 3.6 There exists a maximal solution \( u \in C^{2,1}(O) \) of
\[
\partial_t u - \Delta u + e^u - 1 = 0 \quad \text{in} \quad O
\] (3.16)
and it satisfies
\[
u(x, t) \leq C - \log \left( \frac{(d((x, t), \partial B_0))}{4 + (d((x, t), \partial B_0))^2} \right) \quad \text{for all} \quad (x, t) \in O, \tag{3.17}
\]
for some \( C = C(N) \).

The next three propositions will be useful to prove Theorem 1.1-(ii).

Proposition 3.7 Let \( K \subset \overline{Q}_1(0, 0) \) be a compact set and \( q > 1, R \geq 100 \). Let \( u \) be a solution of (3.7) in \( \overline{Q}_R(0, 0) \backslash K \) and \( \varphi \) as in Proposition 2.6 with \( p = q' \). Set \( \xi = (1 - \varphi)^{2q} \).

Then,
\[
\int_{\overline{Q}_n(0, 0)} u(|\Delta \xi| + |\nabla \xi| + |\partial \xi|) \, dx \, dt \lesssim \text{Cap}_{2, q'}(K) \tag{3.18}
\]
\[
u(x, t) \lesssim \text{Cap}_{2, q'}(K) + R^{-\frac{q'}{2}} \quad \text{for any} \quad (x, t) \in \overline{Q}_{R/5}(0, 0) \backslash \overline{Q}_2(0, 0), \tag{3.19}
\]
and
\[
\int_{\overline{Q}_2(0, 0)} u \xi \, dx \, dt \lesssim \text{Cap}_{2, q'}(K) + R^{-\frac{q'}{2}} \tag{3.20}
\]
where the constants in above inequalities depend only on \( N, q \).

Proof. Step 1. We claim that
\[
\int_{\overline{Q}_n(0, 0)} u^q \xi \, dx \, dt \lesssim \text{Cap}_{2, q'}(K). \tag{3.21}
\]
Actually, using by parts integration and the Green formula, one has
\[
\int_{\overline{Q}_n(0, 0)} u^q \xi \, dx \, dt = - \int_{\overline{Q}_n(0, 0)} \partial_t u \xi \, dx \, dt + \int_{\overline{Q}_n(0, 0)} (\Delta u) \xi \, dx \, dt
\]
\[
eq \int_{\overline{Q}_n(0, 0)} u \partial_t \xi \, dx \, dt + \int_{\overline{Q}_n(0, 0)} u \Delta \xi \, dx \, dt + \int_{\overline{R}^2} \int_{\partial B_R(0)} \left( \xi \frac{\partial u}{\partial \nu} - u \frac{\partial \xi}{\partial \nu} \right) \, dS \, dt
\]
where \( \nu \) is the outer normal unit vector on \( \partial B_R(0) \). Clearly,
\[
\frac{\partial u}{\partial \nu} \leq 0 \quad \text{and} \quad \frac{\partial \xi}{\partial \nu} = 0 \quad \text{on} \quad \partial B_R(0).
\]
Thus,
\[
\int_{\overline{Q}_n(0, 0)} u^q \xi \, dx \, dt \leq \int_{\overline{Q}_n(0, 0)} u |\partial_t \xi| \, dx \, dt + \int_{\overline{Q}_n(0, 0)} u |\Delta \xi| \, dx \, dt
\]
\[
\leq 2q' \int_{\overline{Q}_n(0, 0)} u (1 - \varphi)^{2q - 1} |\partial_t \varphi| \, dx \, dt + 2q'(2q' - 1) \int_{\overline{Q}_n(0, 0)} u (1 - \varphi)^{2q - 2} |\nabla \varphi|^2 \, dx \, dt
\]
\[
+ 2q' \int_{\overline{Q}_n(0, 0)} u (1 - \varphi)^{2q - 1} |\Delta \varphi| \, dx \, dt
\]
\[
\leq 2q' \int_{\overline{Q}_n(0, 0)} u^{1/q} |\partial_t \varphi| \, dx \, dt + 2q'(2q' - 1) \int_{\overline{Q}_n(0, 0)} u^{1/q} |\nabla \varphi|^2 \, dx \, dt
\]
\[
+ 2q' \int_{\overline{Q}_n(0, 0)} u^{1/q} |\Delta \varphi| \, dx \, dt. \tag{3.22}
\]
In the last inequality, we have used the fact that \((1 - \phi)^{2q' - 1} \leq (1 - \phi)^{2q' - 2} = \xi^{1/q}.

Hence, by Hölder’s inequality,

\[
\int_{Q\eta(0,0)} u^q |\xi|dxdt \leq \int_{Q\eta(0,0)} |\partial_t \varphi|^q \, dxdt + \int_{Q\eta(0,0)} |\nabla \varphi|^{2q'} \, dxdt \\
+ \int_{Q\eta(0,0)} |\Delta \varphi|^q \, dxdt.
\]

By the Gagliardo-Nirenberg inequality,

\[
\int_{Q\eta(0,0)} |\nabla \varphi|^{2q'} \, dxdt \leq \|\varphi\|_{L^q(Q\eta(0,0))}^q \int_{Q\eta(0,0)} |D^2 \varphi|^q \, dxdt \\
\leq \int_{Q\eta(0,0)} |D^2 \varphi|^q \, dxdt.
\]

Hence, we find

\[
\int_{Q\eta(0,0)} u^q |\xi|dxdt \leq \int_{Q\eta(0,0)} (|\partial_t \varphi|^q + |D^2 \varphi|^q) \, dxdt
\]

and derive (3.21) from (2.4). In view of (3.22), we also obtain

\[
\int_{Q\eta(0,0)} u(|\Delta \xi| + |\partial_t \xi|)dxdt \leq \text{Cap}_{2,1,q'}(K)
\]

and

\[
\int_{Q\eta(0,0)} u|\nabla \xi|dxdt \leq \text{Cap}_{2,1,q'}(K),
\]

since

\[
\int_{Q\eta(0,0)} u|\nabla \xi|dxdt = 2q' \int_{Q\eta(0,0)} u^{(2q' - 1)/2} |\nabla \varphi|dxdt \\
\leq 2q' \int_{Q\eta(0,0)} u^{1/q} |\nabla \varphi|dxdt \\
\leq \int_{Q\eta(0,0)} u^q |\xi|dxdt + \int_{Q\eta(0,0)} |\nabla \varphi|^q \, dxdt.
\]

It yields (3.18).

Step 2. Relation (3.19) holds. Let \(\eta\) be a cut off function on \(\hat{Q}_{R/4}(0,0)\) with respect to \(\hat{Q}_{R/3}(0,0)\) such that \(|\partial_t \eta| + |D^2 \eta| \lesssim R^{-2}\) and \(|\nabla \eta| \lesssim R^{-1}\). We have

\[
\partial_t (\eta \xi u) - \Delta (\eta \xi u) = F \in C_c(\hat{Q}_{R/3}(0,0)).
\]

Hence, we can write

\[
(\eta \xi u)(x,t) = \int_{\mathbb{R}^N} \int_{-\infty}^t \frac{1}{(4\pi(t-s))^{\frac{N}{2}}} e^{-\frac{|y-s|^2}{4(t-s)}} F(y,s) \, ds \, dy \quad \forall (x,t) \in \mathbb{R}^{N+1}.
\]

Now, we fix \((x,t) \in \hat{Q}_{R/5}(0,0) \setminus \hat{Q}_2(0,0)\). Since \(\text{supp}\{|\nabla \eta|\} \cap \text{supp}\{|\nabla \xi|\} = \emptyset\) and

\[
F = \eta \xi (\partial_t u - \Delta u) - 2 (\eta \nabla \xi + \xi \nabla \eta) \nabla u + (\xi \partial_t \eta + \eta \partial_t \xi - 2 \nabla \eta \nabla \xi - \Delta \eta \xi - \eta \Delta \xi) u \\
\leq -2 (\eta \nabla \xi + \xi \nabla \eta) \nabla u + (\xi \partial_t \eta + \eta \partial_t \xi - \xi \Delta \eta - \eta \Delta \xi) u,
\]

we get

\[
(\eta \xi u)(x,t) = \int_{\mathbb{R}^N} \int_{-\infty}^t \frac{1}{(4\pi(t-s))^{\frac{N}{2}}} e^{-\frac{|y-s|^2}{4(t-s)}} F(y,s) \, ds \, dy \quad \forall (x,t) \in \mathbb{R}^{N+1}.
\]
there holds
\[ u(x,t) = (\eta \xi u)(x,t) \leq -2 \int_{\mathbb{R}^N} \int_{-\infty}^t \frac{1}{(4\pi(t-s))^{N/2}} e^{-\frac{|x-y|^2}{4(t-s)}} (\eta \nabla \xi + \xi \nabla \eta) u \, dy ds \]
\[ + \int_{\mathbb{R}^N} \int_{-\infty}^t \frac{1}{(4\pi(t-s))^{N/2}} e^{-\frac{|x-y|^2}{4(t-s)}} (\eta \partial_t \xi - \eta \Delta \xi) u \, dy ds \]
\[ + \int_{\mathbb{R}^N} \int_{-\infty}^t \frac{1}{(4\pi(t-s))^{N/2}} e^{-\frac{|x-y|^2}{4(t-s)}} (\partial_t \eta \xi - \xi \Delta \eta) u \, dy ds \]
\[ = I_1 + I_2 + I_3. \]

By parts integration
\[ I_1 = 2(4\pi)^{-N/2} \int_{-\infty}^t \int_{\mathbb{R}^N} \frac{(x-y)}{(t-s)^{N+2}/2} e^{-\frac{|x-y|^2}{4(t-s)}} (\eta \nabla \xi + \xi \nabla \eta) u \, dy ds \]
\[ 2(4\pi)^{-N/2} \int_{-\infty}^t \int_{\mathbb{R}^N} \frac{1}{(t-s)^{N/2}} e^{-\frac{|x-y|^2}{4(t-s)}} (\xi \Delta \eta) u \, dy ds. \]

Note that
\[ \frac{1}{(t-s)^{N/2}} e^{-\frac{|x-y|^2}{4(t-s)}} \lesssim \left( \max\{|x-y|, |t-s|^{1/2}\} \right)^{-N}, \]
\[ \left| \frac{(x-y)}{(t-s)^{N+2}/2} e^{-\frac{|x-y|^2}{4(t-s)}} \right| \lesssim \left( \max\{|x-y|, |t-s|^{1/2}\} \right)^{-N-1}, \]
and
\[ \max\{|x-y|, |t-s|^{1/2}\} \geq 1 \quad \forall (y,s) \in \text{supp}\{|D^n \xi| \cup \text{supp}\{|\partial_t \xi|\}, \]
\[ \max\{|x-y|, |t-s|^{1/2}\} \geq R \quad \forall (y,s) \in \text{supp}\{|D^n \eta| \cup \text{supp}\{|\partial_t \eta|\} \quad \forall |\alpha| \geq 1. \]

We deduce
\[ I_1 \lesssim \int_{\mathbb{R}^{N+1}} \left( \max\{|x-y|, |t-s|^{1/2}\} \right)^{-N} (\eta \nabla \xi + \xi \nabla \eta) u \, dy ds \]
\[ + \int_{\mathbb{R}^{N+1}} \left( \max\{|x-y|, |t-s|^{1/2}\} \right)^{-N} (\xi \Delta \eta + \eta \Delta \xi) u \, dy ds \]
\[ \lesssim \int_{\mathbb{R}^{N+1}} (|\nabla \xi| + |\Delta \xi|) u \, dy ds + \int_{\hat{Q}_{R/2}(0,0) \setminus \hat{Q}_{R/4}(0,0)} (R^{-N-1} |\nabla \eta| + R^{-N} |\Delta \eta|) u \, dy ds \]
\[ \lesssim \int_{\mathbb{R}^{N+1}} (|\nabla \xi| + |\Delta \xi|) u \, dy ds + \sup_{\hat{Q}_{R/3}(0,0) \setminus \hat{Q}_{R/4}(0,0)} u, \]
\[ I_2 \lesssim \int_{\mathbb{R}^{N+1}} \left( \max\{|x-y|, |t-s|^{1/2}\} \right)^{-N} (|\partial_t \xi| + |\Delta \xi|) u \, dy ds \]
\[ \lesssim \int_{\mathbb{R}^{N+1}} (|\partial_t \xi| + |\Delta \xi|) u \, dy ds, \]
and
\[ I_3 \lesssim \int_{\mathbb{R}^{N+1}} \left( \max\{|x-y|, |t-s|^{1/2}\} \right)^{-N} (|\partial_t \eta| + |\Delta \eta|) u \, dy ds \]
\[ \lesssim \int_{\hat{Q}_{R/2}(0,0) \setminus \hat{Q}_{R/4}(0,0)} R^{-N} (|\partial_t \eta| + |\Delta \eta|) u \, dy ds \]
\[ \lesssim \sup_{\hat{Q}_{R/3}(0,0) \setminus \hat{Q}_{R/4}(0,0)} u. \]
Hence, 
\[ u(x, t) \leq I_1 + I_2 + I_3 \leq \int_{\mathbb{R}^{N+1}} \left( |\partial_2 \xi| + |\nabla \xi| + |\Delta \xi| \right) u \, dyds + \sup_{Q_{R(0,0)} \setminus Q(0,0)} u. \]

Combining this with (3.18) and (3.8), we obtain (3.19).

Step 3. End of the proof. Let \( \theta \) be a cut off function on \( \tilde{Q}_3(0, 0) \) with respect to \( \tilde{Q}_4(0, 0) \). As above, we have for any \((x, t) \in \mathbb{R}^{N+1}\)

\[ (\theta \xi u)(x, t) \leq \int_{\mathbb{R}^{N+1}} \left( \max\{|x - y|, |t - s|^{1/2}\}\right)^{-N-1}(\theta|\nabla \xi| + \xi|\nabla \theta|) u \, dyds \]

\[ + \int_{\mathbb{R}^{N+1}} \left( \max\{|x - y|, |t - s|^{1/2}\}\right)^{-N}(\theta|\partial_t \xi| + \theta|\Delta \xi|) u \, dyds \]

\[ + \int_{\mathbb{R}^{N+1}} \left( \max\{|x - y|, |t - s|^{1/2}\}\right)^{-N}(\xi|\partial_t \theta| + \xi|\Delta \theta|) u \, dyds. \]

Hence, by Fubini theorem,

\[ \int_{Q_2(0,0)} \eta u \, dx \, dt = \int_{Q_2(0,0)} \theta \eta u \, dx \, dt \]

\[ \leq A \int_{\mathbb{R}^{N+1}} \left( \max\{|x - y|, |t - s|^{1/2}\}\right)^{-N} + (\max\{|x - y|, |t - s|^{1/2}\}\right)^{-N-1}) \, dx \, dt. \]

Therefore we obtain (3.20) from (3.18) and (3.11).

**Proposition 3.8** Let \( K \subset \{(x, t) : \varepsilon < \max\{|x|, |t|^{1/2}\} < 1\} \) be a compact set, \( 0 < \varepsilon < 1 \) and \( u \) be the maximal solution of (3.7) in \( \tilde{Q}_R(0, 0) \setminus K \) with \( R \geq 100 \). Then

\[ \sup_{\tilde{Q}_{\varepsilon/4}(0,0)} u \leq \sum_{j=0}^{j_2} \frac{\text{Cap}_{2,1,q}(K \cap \tilde{Q}_{\rho_j}(0,0))}{\rho_j^q} + j_2 \varepsilon^{-\frac{q_\ast}{q}} \text{ if } q > q_\ast, \]

and

\[ \sup_{\tilde{Q}_{\varepsilon/4}(0,0)} u \leq \sum_{j=0}^{j_2} \frac{\text{Cap}_{2,1,q}(K_j)}{\rho_j^q} + j_2 \varepsilon^{-\frac{q_\ast}{q}} \text{ if } q = q_\ast, \]

where \( \rho_j = 2^{-j}, \) \( K_j = \{(x/\rho_{j+3}, t/\rho_{j+3}^2) : \langle x, t \rangle \in K \cap \tilde{Q}_{\rho_{j+3}}(0,0)\} \) and \( j \in \mathbb{N} \) is such that \( \rho_j \leq \varepsilon < \rho_{j-1} \).

**Proof.** For \( j \in \mathbb{N} \), we define \( S_j = \{x : \rho_j \leq \max\{|x|, |t|^{1/2}\} \leq \rho_{j-1}\} \).

Fix any \( 1 \leq j \leq j_2 \). We cover \( S_j \) by \( L = L(N) \in \mathbb{N}^* \) closed cylinders

\[ \tilde{Q}_{\rho_{j+3}}(x_k, t_k), \quad k = 1, \ldots, L(N)\]

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where \((x_{k,j}, t_{k,j}) \in S_j\).
For \(k = 1, \ldots, L(N)\), let \(u_j, u_{k,j}\) be the maximal solutions of \((3.4)\) where \(K\) is replaced by \(K \cap S_j\) and \(K \cap \hat{Q}_{\rho_j+3}(x_{k,j}, t_{k,j})\), respectively. Clearly the function \(\hat{u}_{k,j}\) defined by
\[
\hat{u}_{k,j}(x, t) = \rho_j^{3+q} u_{k,j}(\rho_j^{3+q} x + x_{k,j}, \rho_j^{3+q} t + t_{k,j})
\]
is the maximal solution of \((3.4)\) when \((K_{k,j}, \hat{Q}_{R/\rho_j+3}(-x_{k,j}/\rho_j+3, -t_{k,j}/\rho_j^{2+3}))\) is replacing \((K, \hat{Q}_R(0,0))\), with
\[
K_{k,j} = \{(y/\rho_j+3, s/\rho_j^{2+3}) : (y, s) \in -(x_{k,j}, t_{k,j}) + K \cap \hat{Q}_{\rho_j+3}(x_{k,j}, t_{k,j})\} \subset \hat{Q}_1(0,0).
\]
Let \(\tilde{u}_{k,j}\) be the maximal solution of \((3.4)\) with \((K, \hat{Q}_R(0,0))\) replaced by \((K_{k,j}, \hat{Q}_{2R/\rho_j+3}(0,0))\). Since \(\hat{Q}_{R/\rho_j+3}(-x_{k,j}/\rho_j+3, -t_{k,j}/\rho_j^{2+3}) \subset \hat{Q}_{2R/\rho_j+3}(0,0)\), then, by the comparison principle as in the proof of Proposition 3.1, we get \(\hat{u}_{k,j} \leq \tilde{u}_{k,j}\) in \(\hat{Q}_{R/\rho_j+3}(-x_{k,j}/\rho_j+3, -t_{k,j}/\rho_j^{2+3})\)\(\cap K_{k,j}\) and thus
\[
\hat{u}_{k,j}(x, t) \leq \text{Cap}_{2,1,q}(K_{k,j}) + (R/\rho_j+3)^{1/1+q},
\]
for any \((x, t) \in \hat{Q}_{2R/(5\rho_j+3)}(0,0) \cap \hat{Q}_{R/\rho_j+3}(-x_{k,j}/\rho_j+3, -t_{k,j}/\rho_j^{2+3})\)\(\setminus \hat{Q}_2(0,0) = D\).
Fix \((x_0, t_0) \in \hat{Q}_{4}(0,0)\). Clearly, \((x_0 - x_{k,j})/\rho_j+3, (t_0 - t_{k,j})/\rho_j^{2+3}) \in D\), hence
\[
u_{k,j}(x_0, t_0) = \rho_j^{3+q} \hat{u}_{k,j}(x_0 - x_{k,j})/\rho_j+3, (t_0 - t_{k,j})/\rho_j^{2+3}) \leq \frac{\text{Cap}_{2,1,q}(K_{k,j})}{\rho_j^{2+3}} + R^{-\frac{q}{2+3}}.
\]
Therefore, using \((3.9)\) in Remark 3.2 and the fact that
\[
\text{Cap}_{2,1,q}(K_{k,j}) = \text{Cap}_{2,1,q}(K_{k,j} + (x_{k,j}/\rho_j+3, t_{k,j}/\rho_j^{2+3})) \leq \text{Cap}_{2,1,q}(K_j),
\]
we derive
\[
u(x_0, t_0) \leq \sum_{j=1}^{j_x} \nu_j(x_0, t_0) \leq \sum_{j=1}^{j_x} \sum_{k=1}^{L(N)} u_{k,j}(x_0, t_0)
\leq \sum_{j=0}^{j_x} \frac{\text{Cap}_{2,1,q}(K_j)}{\rho_j^{2+3}} + \sum_{j=0}^{j_x} \frac{\text{Cap}_{2,1,q}(K_j)}{\rho_j^{2+3}} + j_x R^{-\frac{q}{2+3}},
\]
which yields \((3.24)\). If \(q > q_*\), then by \((2.2)\) in Proposition 2.5, we have
\[
\text{Cap}_{2,1,q}(K_j) \lesssim \rho_j^{-N+2+2q}\text{Cap}_{2,1,q}(K \cap \hat{Q}_{\rho_j(0,0)}),
\]
which implies \((3.25)\). \thinspace \blacksquare

**Proposition 3.9** Let \(K, u, \xi\) be as in Proposition \((3.4)\). For any compact set \(K_0\) in \(Q_1(0,0)\) with positive measure \(|K_0|\), there exists \(\varepsilon = \varepsilon(N, q, |K_0|) > 0\) such that
\[
\text{Cap}_{2,1,q}(K) \leq \varepsilon \Rightarrow \sup_{K_0} u \leq \int_{Q_2(0,0)} u \xi \, dx \, dt,
\]
where the constant in the inequality \(\lesssim\) depends on \(K_0\). In particular,
\[
\text{Cap}_{2,1,q}(K) \leq \varepsilon \Rightarrow \inf_{K_0} u \lesssim \text{Cap}_{2,1,q}(K) + R^{-\frac{q}{2+3}}. \tag{3.25}
\]
Proof. It is enough to prove that there exists $\varepsilon > 0$ such that
\[
\text{Cap}_{2,1,q'}(K) \leq \varepsilon \Rightarrow |K|_1 \geq 1/2|K_0|
\] (3.26)
where $K_1 = \{(x, t) \in K_0 : \xi(x, t) \geq 1/2\}$. By (2.11) in Proposition 2.5 we have the following estimates
\[
|K_0 \setminus K_1|^{1 - \frac{q'}{2q}} \lesssim \text{Cap}_{2,1,q'}(K_0 \setminus K_1)
\]
if $q > q_*$, and
\[
\left(\log \left(\frac{|Q_{200}(0,0)|}{|K_0 \setminus K_1|}\right)\right)^{-\frac{q}{q'}} \lesssim \text{Cap}_{2,1,q'}(K_0 \setminus K_1)
\]
if $q = q_*$. On the other hand,
\[
\text{Cap}_{2,1,q'}(K_0 \setminus K_1) = \text{Cap}_{2,1,q'}(\{K_0 : \varphi > 1 - (1/2)^{1/(2q')}\})
\]
\[
\leq (1 - (1/2)^{1/(2q')})^{-q'} \int_{\mathbb{R}^N} \left(|D^2\varphi|^{q'} + |\nabla\varphi|^{q'} + |\varphi|^{q'} + |\partial_i\varphi|^{q'}\right) dxdt
\]
\[
\lesssim \text{Cap}_{2,1,q'}(K)
\]
where $\varphi$ is in Proposition 3.7. Henceforth, one can find $\varepsilon = \varepsilon(N, q, |K_0|) > 0$ such that
\[
\text{Cap}_{2,1,q'}(K) \leq \varepsilon \Rightarrow |K_0 \setminus K_1| \leq 1/2 |K_0|.
\]
This implies (3.26).

4 Large solutions

In the first part of this section, we prove theorem 1.1(ii), then we prove theorems 1.1(i) and 1.2, at end we consider a parabolic viscous Hamilton-Jacobi equation.

4.1 Proof of Theorem 1.1(ii)

Let $R_0 \geq 4$ such that $O \subset \bar{Q}_{R_0}(0,0)$. Assume that the equation (1.12) has a large solution $u$. Take any $(x, t) \in \partial_q O$. We will to prove that (1.14) holds. We can assume $(x, t) = (0,0)$. Set $K = Q_{2R_0}(0,0)\setminus O$ and define
\[
T_j = \{x : \rho_{j+1} \leq \max\{|x|, |t|^{1/2}\} \leq \rho_j, t \leq 0\},
\]
\[
\tilde{T}_j = \{x : \rho_{j+3} \leq \max\{|x|, |t|^{1/2}\} \leq \rho_{j+2}, t \leq 0\}.
\]
Here $\rho_j = 2^{-j}$. For $j \geq 3$, let $u_1, u_2, u_3, u_4$ be the maximal solutions of (3.7) when $K$ is replaced by $K \cap \bar{Q}_{j+3}(0,0), K \cap \tilde{T}_j, (K \cap \bar{Q}_1(0,0)) \setminus Q_{\rho_{j+2}}(0,0)$ and $K \setminus Q_1(0,0)$ respectively and $R \geq 100R_0$. From (3.9) in Remark 3.2 we can assert that
\[
u \leq u_1 + u_2 + u_3 + u_4 \quad \text{in} \quad O \cap \{(x, t) \in \mathbb{R}^{N+1} : t \leq 0\}.
\]
Thus,
\[
\inf_{T_j} u \leq ||u_1||_{L^\infty(T_j)} + ||u_3||_{L^\infty(T_j)} + ||u_4||_{L^\infty(T_j)} + \inf_{\tilde{T}_j} u_2.
\] (4.1)

Case 1: $q > q_*$. By (3.8) in Remark 3.2
\[
||u_4||_{L^\infty(T_j)} \lesssim 1.
\] (4.2)
By (3.23) in Proposition 3.8
\[ \|u_3\|_{L^\infty(T_j)} \lesssim \sum_{i=-2}^{j-4} \frac{\text{Cap}_{2,1,q'}(K \cap Q_{\rho_i}(0,0))}{\rho_i^q} + jR^{-\frac{q}{q-1}}. \tag{4.3} \]

Since \((x, t) \mapsto \pi_1(x, t) = \rho_{j+3}^2/(q-1)u_1(\rho_{j+3}x, \rho_{j+3}^2t)\) is the maximal solution of (3.7) when \((K, \bar{Q}_R(0,0))\) is replaced by \(\{(y/\rho_{j+3}, s/\rho_{j+3}^2) : (y, s) \in K \cap \bar{Q}_{\rho_{j+3}}(0,0)\}, \bar{Q}_{R/\rho_{j+3}}(0,0)\), we derive, thanks to (3.19) in Proposition 3.7 and (2.2) in Proposition 2.5,
\[ \|\pi_1\|_{L^\infty(T_{-3})} \lesssim \frac{\text{Cap}_{2,1,q'}(K \cap Q_{\rho_{j+3}}(0,0))}{\rho_{j+3}^2} + (R/\rho_{j+3})^{-\frac{2}{q-1}}, \]
from which follows
\[ \|u_1\|_{L^\infty(T_j)} \lesssim \frac{\text{Cap}_{2,1,q'}(K \cap Q_{\rho_{j+3}}(0,0))}{\rho_{j+3}^q} + R^{-\frac{q}{q-1}}. \tag{4.4} \]

Since, \((x, t) \mapsto \pi_2(x, t) = \rho_{j-2}^2/(q-1)u_2(\rho_{j-2}x, \rho_{j-2}^2t)\) is the maximal solution of (3.7) when the couple \((K, \bar{Q}_R(0,0))\) is replaced by \(\{(y/\rho_{j-2}, s/\rho_{j-2}^2) : (y, s) \in K \cap \bar{T}_j\}, \bar{Q}_{R/\rho_{j-2}}(0,0)\), Proposition 3.9 and relation (2.2) in Proposition 2.5 yield
\[ \frac{\text{Cap}_{2,1,q'}(K \cap \bar{T}_j)}{\rho_{j-2}^q} \leq \varepsilon \Rightarrow \inf_{T_j} \pi_2 \lesssim \frac{\text{Cap}_{2,1,q'}(K \cap \bar{T}_j)}{\rho_{j-2}^q} + (R/\rho_{j-2})^{-\frac{2}{q-1}}, \]
which implies
\[ \frac{\text{Cap}_{2,1,q'}(K \cap Q_{\rho_{j-3}}(0,0))}{\rho_{j-3}^q} \leq \varepsilon \Rightarrow \inf_{T_j} u_2 \lesssim \frac{\text{Cap}_{2,1,q'}(K \cap Q_{\rho_{j-3}}(0,0))}{\rho_{j-3}^q} + R^{-\frac{q}{q-1}}, \tag{4.5} \]
for some \(\varepsilon = \varepsilon(N, q) > 0\).

First, we assume that there exists \(J \in \mathbb{N}, J \geq 10\) such that
\[ \frac{\text{Cap}_{2,1,q'}(K \cap Q_{\rho_{j-3}}(0,0))}{\rho_{j-3}^q} \leq \varepsilon \quad \forall \ j \geq J. \]

Then, from (4.1) and (4.2), (4.3), (4.4), (4.5), we have
\[ \inf_{T_j} u \lesssim \sum_{i=-2}^{j+2} \frac{\text{Cap}_{2,1,q'}(K \cap Q_{\rho_i}(0,0))}{\rho_i^q} + jR^{-\frac{q}{q-1}} + 1, \]
for any \(j \geq J\). Letting \(R \to \infty\),
\[ \inf_{T_j} u \lesssim \sum_{i=-2}^{j+2} \frac{\text{Cap}_{2,1,q'}(K \cap Q_{\rho_i}(0,0))}{\rho_i^q} + 1. \]

Since \(\inf_{T_j} u \to \infty\) as \(j \to \infty\), we get
\[ \sum_{i=0}^{\infty} \frac{\text{Cap}_{2,1,q'}(K \cap Q_{\rho_i}(0,0))}{\rho_i^q} = \infty, \]
which implies that (1.14) holds with \((x, t) = (0, 0)\).

Alternatively, assume that for infinitely many \(j\)
\[ \frac{\text{Cap}_{2,1,q'}(K \cap Q_{\rho_{j-3}}(0,0))}{\rho_{j-3}^q} > \varepsilon. \]
Then,
\[
\frac{\text{Cap}_{2,1,q'}(K \cap Q_{\rho_{j-3}}(0,0))}{\rho_j^{N-2}} > \rho_j^{2-2q'} \varepsilon \to \infty \quad \text{when} \; j \to \infty.
\]
We also derive that (1.14) holds with \((x, t) = (0, 0)\). This proves the case \(q > q_*\).

Case 2: \(q = q_*\). Similarly to Case 1, we have: for \(j \geq 6\)
\begin{align*}
||u_4||_{L^\infty(T_j)} & \lesssim 1, \\
||u_3||_{L^\infty(T_j)} & \lesssim \sum_{i=0}^{j-2} \frac{\text{Cap}_{2,1,q'}(K_j)}{\rho_i^N} + jR^{-\frac{2}{N-1}}, \\
||u_1||_{L^\infty(T_j)} & \lesssim \frac{\text{Cap}_{2,1,q'}(K_j)}{\rho_j^N} + R^{-\frac{2}{N-1}}, \\
\text{Cap}_{2,1,q'}(K_{j-5}) & \leq \varepsilon \Rightarrow \inf_{T_j} u_2 \lesssim \frac{\text{Cap}_{2,1,q'}(K_{j-5})}{\rho_j^N} + R^{-\frac{2}{N-1}},
\end{align*}
where \(K_j = \{(x/\rho_{j-3}, t/\rho_{j-3}^2) : (x, t) \in K \cap Q_{\rho_{j-3}}(0,0)\}\) and \(\varepsilon = \varepsilon(N) > 0\).

From (2.2) in Proposition 2.5 we have
\[
\frac{1}{\text{Cap}_{2,1,q'}(K \cap Q_{\rho_{j-3}}(0,0))} \leq \frac{c}{\text{Cap}_{2,1,q'}(K_j)} + cj^{N/2}
\]
for any \(j \geq 4\) where \(c = c(N)\). If there are infinitely many \(j \geq 4\) such that
\[
\text{Cap}_{2,1,q'}(K \cap Q_{\rho_{j-3}}(0,0)) \geq \frac{1}{2c j^{N/2}},
\]
then (1.14) holds with \((x, t) = (0, 0)\) since
\[
\frac{\text{Cap}_{2,1,q'}(K \cap Q_{\rho_{j-3}}(0,0))}{\rho_j^N} > \frac{2^{j-3}}{2c j^{N/2}} \to \infty \quad \text{when} \; j \to \infty.
\]
Now, we assume that there exists \(J \geq 6\) such that
\[
\text{Cap}_{2,1,q'}(K \cap Q_{\rho_{j-3}}(0,0)) \leq \frac{1}{2c j^{N/2}}.
\]
Then,
\[
\text{Cap}_{2,1,q'}(K_j) \leq 2\varepsilon \text{Cap}_{2,1,q'}(K \cap Q_{\rho_{j-3}}(0,0)) \quad \forall \; j \geq J.
\]
This leads to
\[
\text{Cap}_{2,1,q'}(K_j) \leq 2\varepsilon \text{Cap}_{2,1,q'}(K \cap Q_{\rho_{j-3}}(0,0)) \leq \varepsilon \quad \forall \; j \geq j' + J,
\]
for some \(J' = J'(N)\). Hence, from (1.6)-(1.9) we have, for any \(j \geq j' + J + 3\),
\begin{align*}
||u_4||_{L^\infty(T_j)} & \lesssim 1, \\
||u_3||_{L^\infty(T_j)} & \lesssim \sum_{i=j'+J+1}^{j-2} \frac{\text{Cap}_{2,1,q'}(K \cap Q_{\rho_{i-3}}(0,0))}{\rho_i^N} + C(j' + J) + jR^{-\frac{2}{N-1}}, \\
||u_1||_{L^\infty(T_j)} & \lesssim \frac{\text{Cap}_{2,1,q'}(K \cap Q_{\rho_{j-3}}(0,0))}{\rho_j^N} + R^{-\frac{2}{N-1}}, \\
\inf_{T_j} u_2 & \lesssim \frac{\text{Cap}_{2,1,q'}(K \cap Q_{\rho_{j-3}}(0,0))}{\rho_j^N} + R^{-\frac{2}{N-1}},
\end{align*}

where \( C(J' + J) = \sum_{i=0}^{j'} \frac{\text{Cap}_{p,1,q}(K_i)}{r_i^N} \).

Consequently we derive

\[
infty \sum_{i=0}^{j} \frac{\text{Cap}_{p,1,q}(K \cap Q_{p_i}(0,0))}{r_i^N} + C(J' + J) + 1 + jR^{-\frac{\varepsilon}{2}} \quad \forall j \geq J' + J + 3
\]

from (4.1). Letting \( R \to \infty \) and \( j \to \infty \) we obtain

\[
infty \sum_{i=0}^{ \infty } \frac{\text{Cap}_{p,1,q}(K \cap Q_{p_i}(0,0))}{r_i^N} = \infty,
\]

i.e \((1.14)\) holds with \((x,t) = (0,0)\). This completes the proof of Theorem 1.1-(ii).

### 4.2 Proof of Theorem 1.1-(i) and Theorem 1.2

Fix \((x_0, t_0) \in \partial \mathcal{Q}O\). We can assume that \((x_0, t_0) = 0\). Let \( \delta \in (0,1/100) \). For \((y_0, s_0) \in (B_{\delta}(0) \times (-\delta^2, \delta^2)) \cap O\), we set

\[
M_k = O^c \cap \left( B_{r_{k+1}}(y_0) \times [s_0 - (73 + \frac{1}{2})r_{k+2}^2, s_0 - (70 + \frac{1}{2})r_{k+2}^2]\right)
\]

and

\[
S_k = \{(x, t) : r_{k+1} \leq \max\{|x - y_0|, |t - s_0|^\frac{1}{2}\} < r_k\} \quad \text{for} \quad k = 1, 2, \ldots
\]

where \( r_k = 4^{-k} \). Note that \( M_k = \emptyset \) for \( k \) large enough and \( M_k \subset S_k \) for all \( k \). Let \( R_0 \geq 4 \) such that \( O \subset \subset \tilde{Q}_{R_0}(0,0) \). By Theorem 2.2 and 2.4 and estimate \((1.11)\) there exist two sequences \( \{\mu_k\}_k \) and \( \{\nu_k\}_k \) of nonnegative Radon measures such that

\[
\text{supp}(\mu_k) \subset M_k, \quad \text{supp}(\nu_k) \subset M_k, \quad (4.10)
\]

\[
\mu_k(M_k) \asymp \text{Cap}_{p,1,q}(M_k) \asymp \int_{R^{N+1}} \left( \frac{2R_0}{r_k} \right)^q \mu_k \, dxdt \quad (4.11)
\]

and

\[
\nu_k(M_k) \asymp \mathcal{PH}^N(M_k), \quad ||M_1^{2R_0}[\nu_k]||_{L^\infty(R^{N+1})} \leq 1 \quad \text{for} \quad k = 1, 2, \ldots, \quad (4.12)
\]

where the constants of equivalence depend on \( N, q, R_0 \).

Take \( \varepsilon > 0 \) such that \( \exp\left( C_1 \varepsilon \frac{2R_0}{r_k} \sum_{k=1}^{\infty} \nu_k \right) \in L^1(\tilde{Q}_{R_0}(0,0)) \) where the constant \( C_1 = C_1(N) \) is the one of inequality \((2.10)\). By Theorem 2.7 and Proposition 2.8 there exist two nonnegative solutions \( U_1, U_2 \) of problems

\[
\partial_t U_1 - \Delta U_1 + U_1^q = \varepsilon \sum_{k=1}^{\infty} \mu_k \quad \text{in} \quad \tilde{Q}_{R_0}(0,0),
\]

\[
U_1 = 0 \quad \text{on} \quad \partial p \tilde{Q}_{R_0}(0,0).
\]

and

\[
\partial_t U_2 - \Delta U_2 + eU_2^2 - 1 = \varepsilon \sum_{k=1}^{\infty} \nu_k \quad \text{in} \quad \tilde{Q}_{R_0}(0,0),
\]

\[
U_2 = 0 \quad \text{on} \quad \partial p \tilde{Q}_{R_0}(0,0),
\]

respectively which satisfy

\[
U_1(y_0, z_0) \geq \sum_{i=0}^{\infty} \sum_{k=1}^{\infty} \varepsilon \frac{\mu_k(B_{r_i^N}(y_0) \times (s_0 - \frac{37}{128}r_i^2, s_0 - \frac{35}{128}r_i^2))}{r_i^N}
\]

\[
- \frac{2R_0}{r_k} \left[ \left( \frac{2R_0}{r_k} \sum_{k=1}^{\infty} \mu_k \right)^q \right] (y_0, s_0) =: A \quad (4.13)
\]
and
\[ U_2(y_0, z_0) \gtrsim \sum_{i=0}^{\infty} \sum_{k=1}^{\infty} \nu_k (B_{r_i} (y_0) \times (s_0 - \frac{37}{128} r_i^2, s_0 - \frac{35}{128} r_i^2)) \]
\[ - \|2 R_0 \left[ \exp \left( C_1 R_0^2 \sum_{k=1}^{\infty} \nu_k \right) - 1 \right] (y_0, s_0) =: B \] (4.14)

and \( U_1, U_2 \in C^{2,1}(O) \).

Let \( u_1, u_2 \) be the maximal solutions of equations (3.1) and (3.16) respectively. We have \( u_1(y_0, s_0) \geq U_1(y_0, s_0) \) and \( u_2(y_0, s_0) \geq U_2(y_0, s_0) \). Now, we claim that
\[ A \gtrsim \sum_{k=1}^{\infty} \text{Cap}_{2,1,q}(M_k) \frac{r_i^N}{r_i^N} \] (4.15)
and
\[ B \gtrsim -c_1 (R_0) + \sum_{k=1}^{\infty} \mathcal{P} \mathcal{H}_{1,q}^N(M_k) \frac{r_i^N}{r_i^N} \] (4.16)

**Proof of assertion (4.15).** From (4.11) we have
\[ A \gtrsim \sum_{k=1}^{\infty} \text{Cap}_{2,1,q}(M_k) \frac{r_i^N}{r_i^N} - e^q A_0 \] (4.17)
with
\[ A_0 = \|2 R_0 \left[ \left( R_0^2 \sum_{k=1}^{\infty} \mu_k \right) \right] (y_0, s_0). \]

Take \( i_0 \in \mathbb{Z} \) such that \( r_{i_0+1} < \max \{2 R_0, 1\} \leq r_{i_0} \). Then
\[ A_0 \lesssim \sum_{i=i_0}^{\infty} r_i^{-N} \int_{S_i} \mathbb{Q}_{r_i} (y_0, s_0) \left( \|2 R_0 \left[ \sum_{k=1}^{\infty} \mu_k \right] \right) \] \[ \lesssim \sum_{j=0}^{\infty} \sum_{i=i_0}^{j} r_i^{-N} \int_{S_j} \left( \|2 R_0 \left[ \sum_{k=1}^{\infty} \mu_k \right] \right) \] \[ \lesssim \sum_{j=0}^{\infty} \sum_{i=0}^{j} r_i^{-N} \int_{S_j} \left( \|2 R_0 \left[ \sum_{k=1}^{\infty} \mu_k \right] \right) \] \[ \gtrsim \sum_{j=0}^{\infty} r_j^{-N} \int_{S_j} \left( \|2 R_0 \left[ \sum_{k=1}^{\infty} \mu_k \right] \right) \] \[ \gtrsim \frac{4}{3} r_j^{-N} \text{ for all } j. \]
Setting $\mu_k \equiv 0$ for all $i_0 - 1 \leq k \leq 0$, the previous inequality becomes

$$A_0 \lesssim \sum_{j=i_0}^{\infty} r_j^{-N} \int_{S_j} \left( \|2^{2R_0}[\mu_j + \sum_{k=i_0-1}^{j-1} \mu_k + \sum_{k=j+1}^{\infty} \mu_k]^q \right) dx dt$$

Next, using (4.10) we have for any $(x, t) \in S_j$ if $k \geq j + 1$,

$$\|2^{2R_0}[\mu_k](x, t) = \int_{r_{j+1}}^{2R_0} \frac{\mu_k(\tilde{Q}_\rho(x, t))}{\rho^N} \, d\rho \lesssim \frac{\mu_k(\mathbb{R}^{N+1})}{r_k^q}$$

and if $k \leq j - 1$

$$\|2^{2R_0}[\mu_k](x, t) = \int_{r_{k+1}}^{2R_0} \frac{\mu_k(\tilde{Q}_\rho(x, t))}{\rho^N} \, d\rho \lesssim \frac{\mu_k(\mathbb{R}^{N+1})}{r_k^q}$$

Thus,

$$A_2 \lesssim \sum_{j=i_0}^{\infty} r_j^{-N} \left( \sum_{k=i_0-1}^{j-1} \frac{\mu_k(\mathbb{R}^{N+1})}{r_k^N} \right)^q$$

and

$$A_3 \lesssim \sum_{j=i_0}^{\infty} q r_j^{-N} q \left( \sum_{k=j+1}^{\infty} \frac{\mu_k(\mathbb{R}^{N+1})}{r_k^N} \right)^q.$$

Noticing that $(a + b)^q - a^q \leq q(a + b)^{q-1}b$ for any $a, b \geq 0$, we get

$$(1 - 4^{-2}) \sum_{j=i_0}^{\infty} r_j^{-N} \left( \sum_{k=i_0-1}^{j-1} \frac{\mu_k(\mathbb{R}^{N+1})}{r_k^N} \right)^q$$

$$= \sum_{j=i_0}^{\infty} r_j^{-N} \left( \sum_{k=i_0-1}^{j-1} \frac{\mu_k(\mathbb{R}^{N+1})}{r_k^N} \right)^q - \sum_{j=i_0+1}^{j_{i_0+1}} r_j^{-N} \left( \sum_{k=i_0-1}^{j-1} \frac{\mu_k(\mathbb{R}^{N+1})}{r_k^N} \right)^q$$

$$\leq \sum_{j=i_0}^{\infty} q r_j^{-N} \left( \sum_{k=i_0-1}^{j-1} \frac{\mu_k(\mathbb{R}^{N+1})}{r_k^N} \right)^{q-1} \frac{\mu_{j-1}(\mathbb{R}^{N+1})}{r_{j-1}^{N(q-1)}}.$$
Similarly, we also have
\[
(1 - 4^{2-Nq}) \sum_{j=1}^{\infty} r_j^{2-Nq} \left( \sum_{k=j+1}^{\infty} \mu_k(\mathbb{R}^{N+1}) \right)^q \\
\leq \sum_{j=1}^{\infty} q r_j^{2-Nq} \left( \sum_{k=j+1}^{\infty} \mu_k(\mathbb{R}^{N+1}) \right)^{q-1} \mu_{j+1}(\mathbb{R}^{N+1}).
\]

Therefore,
\[
A_2 + A_3 \lesssim \sum_{j=1}^{\infty} r_j^{2-Nq} \left( \sum_{k=j+1}^{\infty} \frac{\mu_k(\mathbb{R}^{N+1})}{r_k^{N}} \right)^{q-1} \frac{\mu_{j+1}(\mathbb{R}^{N+1})}{r_{j+1}^{N}} \\
+ \sum_{j=1}^{\infty} r_j^{2-Nq} \left( \sum_{k=j+1}^{\infty} \mu_k(\mathbb{R}^{N+1}) \right)^{q-1} \mu_{j+1}(\mathbb{R}^{N+1}).
\]

Since \(\mu_k(\mathbb{R}^{N+1}) \lesssim r_k^{N+2-2q} \) if \(q > q_* \) and \(\mu_k(\mathbb{R}^{N+1}) \lesssim \min\{k^{-\frac{1}{q_*}}, 1\} \) if \(q = q_* \) for any \(k\), we infer that
\[
r_j^{2-Nq} \left( \sum_{k=j+1}^{\infty} \mu_k(\mathbb{R}^{N+1}) \right)^{q-1} \lesssim 1
\]

and
\[
r_j^{2-Nq} \left( \sum_{k=j+1}^{\infty} \mu_k(\mathbb{R}^{N+1}) \right)^{q-1} \lesssim r_{j+1}^{-N} \quad \text{for any } j.
\]

In the case \(q = q_* \) we assume \(N \geq 3\) in order to ensure that
\[
\sum_{j=1}^{\infty} \mu_k(\mathbb{R}^{N+1}) \lesssim \sum_{k=1}^{\infty} k^{-\frac{1}{q_*}} < \infty.
\]

This leads to
\[
A_2 + A_3 \lesssim \sum_{k=1}^{\infty} \frac{\mu_k(\mathbb{R}^{N+1})}{r_k^{N}}.
\]

Combining this with (4.19) and (4.18), we deduce
\[
A_0 \lesssim \sum_{k=1}^{\infty} \frac{\text{Cap}_{2,q}(M_k)}{r_k^{N}}.
\]

Consequently, we obtain (4.15) from (4.17), for \(\varepsilon\) small enough.

**Proof of assertion (4.16).** From (4.12) we get
\[
B \gtrsim \varepsilon \sum_{k=1}^{\infty} \frac{\mathcal{P}H_{1,N}^{N}(M_k)}{r_k^{N}} - B_0,
\]

where
\[
B_0 = \sum_{k=1}^{\infty} \exp \left( C_1 \|\mathbb{R}_{R_0}^{2} \varepsilon \sum_{k=1}^{\infty} \nu_k \right) - 1 \right) (y_0, s_0).
\]

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We show that
\[ B_0 \leq c(N, q, R_0) \quad \text{for } \varepsilon \text{ small enough.} \tag{4.22} \]

In fact, as above we have
\[ B_0 \lesssim \sum_{j=0}^{\infty} r_j^{-N} \int_{S_j} \exp \left( C_1 \varepsilon \|I^2_{2R_0}[\nu_k]\| \right) \, dx \, dt. \]

Consequently,
\[ B_0 \lesssim \sum_{j=0}^{\infty} r_j^{-N} \int_{S_j} \exp \left( 3C_1 \varepsilon \|I^2_{2R_0}[\nu_j]\| \right) \, dx \, dt 
+ \sum_{j=0}^{\infty} r_j^2 \exp \left( 3C_1 \varepsilon \sum_{k=0}^{j-1} \|\|I^2_{2R_0}[\nu_k]\|\|_{L^\infty(S_j)} \right) 
+ \sum_{j=0}^{\infty} r_j^2 \exp \left( 3C_1 \varepsilon \sum_{k=j+1}^{\infty} \|I^2_{2R_0}[\nu_k]\|_{L^\infty(S_j)} \right) 
= B_1 + B_2 + B_3. \tag{4.23} \]

Here we have used the inequality \( \exp(a + b + c) \leq \exp(3a) + \exp(3b) + \exp(3c) \) for all \( a, b, c \).

By Theorem 2.3 we have
\[ \int_{S_j} \exp \left( 3C_1 \varepsilon I^2_{2R_0}[\nu_j] \right) \, dx \, dt \lesssim r_j^{N+2} \quad \text{for all } j, \]

for \( \varepsilon > 0 \) small enough. Hence,
\[ B_1 \lesssim \sum_{j=0}^{\infty} r_j^2 \lesssim (\max\{2R_0, 1\})^2. \tag{4.24} \]

Note that estimates (4.20) and (4.21) are also true with \( \nu_k \); we deduce
\[ B_2 + B_3 \lesssim \sum_{j=0}^{\infty} r_j^2 \exp \left( c_2 \varepsilon \sum_{k=0}^{j-1} \frac{\mu_k(\mathbb{R}^{N+1})}{r_k^j} \right) 
+ \sum_{j=0}^{\infty} r_j^2 \exp \left( c_2 \varepsilon \sum_{k=j+1}^{\infty} \frac{\mu_k(\mathbb{R}^{N+1})}{r_k^j} \right) \]
\[ \lesssim \sum_{j=0}^{\infty} \exp \left( c_3 \varepsilon (j - i_0) - 4 \log(2) j \right) r_{j_0}^2 
\leq c_4(N, q, R_0) \quad \text{for } \varepsilon \text{ small enough.} \]

From (4.12) we have \( \mu_k(\mathbb{R}^{N+1}) \lesssim r_k^N \) for all \( k \), therefore
\[ B_2 + B_3 \lesssim \sum_{j=0}^{\infty} r_j^2 \exp (c_3 \varepsilon (j - i_0)) + \sum_{j=0}^{\infty} r_j^2 \exp (c_3 \varepsilon ) \]
\[ \lesssim \sum_{j=0}^{\infty} \exp (c_3 \varepsilon (j - i_0) - 4 \log(2) j) + r_{j_0}^2 
\leq c_4(N, q, R_0) \quad \text{for } \varepsilon \text{ small enough.} \]

Combining this with (4.24) and (4.23) we obtain (4.22).

This implies straightforwardly \( \exp \left( C_1 \varepsilon I^2_{2R_0}[\sum_{k=1}^{\infty} \nu_k] \right) \in L^1(\tilde{Q}_{R_0}(0, 0)). \)

We conclude that for any \( (y_0, s_0) \in (B_s(0) \times (-\delta^2, \delta^2)) \cap O, \)
\[ u_1(y_0, s_0) \gtrsim \sum_{k=1}^{\infty} \frac{\text{Cap}_{2,1,q} \left( M_k(y_0, s_0) \right)}{r_k^N} \]

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and 
\[ u_2(y_0, s_0) \geq -c_1(R_0) + \sum_{k=1}^{\infty} \frac{\mathcal{P}H_1^N(M_k(y_0, s_0))}{r_k^N}, \]
where \( r_k = 4^{-k} \) and 
\[ M_k(y_0, s_0) = O^c \cap \left( B_{r_{k+2}}(y_0) \times [s_0 - (73 + \frac{1}{2})r_{k+2}^2, s_0 - (70 + \frac{1}{2})r_{k+2}^2] \right). \]

Take \( r_{k_0+1} \leq \delta < r_{k_0+3} \), we have for \( 1 \leq k \leq k_0 \)
\[ M_k(y_0, s_0) \supset O^c \cap \left( B_{r_{k+2}}(0) \times \left( \delta^2 - (73 + \frac{1}{2})r_{k+2}^2, -\delta^2 - (70 + \frac{1}{2})r_{k+2}^2 \right) \right) \supset O^c \cap \left( B_{r_{k+3}}(0) \times (-71r_{k+2}^2, -1168r_{k+3}^2) \right), \]
Finally
\[
\inf_{(y_0, s_0) \in (B_{r_{k+2}}(0) \times (-\delta^2, \delta^2)) \cap O} u_1(y_0, s_0) \sim \sum_{k=1}^{k_0+3} \text{Cap}_{2,1,q} \left( O^c \cap \left( B_{r_{k+3}}(0) \times (-1168r_{k+3}^2, -1136r_{k+2}^2) \right) \right) \to \infty \quad \text{as} \quad \delta \to 0,
\]
and
\[
\inf_{(y_0, s_0) \in (B_{r_{k+2}}(0) \times (-\delta^2, \delta^2)) \cap O} u_2(y_0, s_0) \geq -c_1(R_0) + \sum_{k=4}^{k_0+3} \frac{\mathcal{P}H_1^N(M_k(y_0, s_0))}{r_k^N} \to \infty \quad \text{as} \quad \delta \to 0.
\]
This completes the proof of Theorem 1.1-(i) and Theorem 1.2.

### 4.3 The viscous Hamilton-Jacobi parabolic equations

In this section we apply our previous result to the question of existence of a large solution of the following type of parabolic viscous Hamilton-Jacobi equation
\[
\partial_t u - \Delta u + a|\nabla u|^p + bu^q = 0 \quad \text{in} \quad O,
\]
\[
u = \infty \quad \text{on} \quad \partial_p O, \tag{4.25}
\]
where \( a > 0, b > 0 \) and \( 1 < p \leq 2, q \geq 1 \). First, we show that such a large solution to \[(4.25)\]
does not exist when \( q = 1 \). Equivalently namely, for \( a > 0, b > 0 \) and \( p > 1 \) there exists no function \( u \in C^{2,1}(O) \) satisfying
\[
\partial_t u - \Delta u + a|\nabla u|^p \geq -bu \quad \text{in} \quad O,
\]
\[
u = \infty \quad \text{on} \quad \partial_p O. \tag{4.26}
\]
Indeed, assuming that such a function \( u \in C^{2,1}(O) \), exists, we define
\[ U(x, t) = u(x, t)e^{bt} - \frac{\varepsilon}{2} |x|^2, \]
for \( \varepsilon > 0 \) and denote by \((x_0, t_0) \in O \setminus \partial_p O\) the point where \( U \) achieves it minimum in \( O \), i.e. 
\[ U(x_0, t_0) = \inf \{ U(x, t) : (x, t) \in O \}. \]
Clearly, we have
\[ \partial_t U(x_0, t_0) \leq 0, \quad \Delta U(x_0, t_0) \geq 0 \quad \text{and} \quad \nabla U(x_0, t_0) = 0. \]
Thus,
\[ \partial_t u(x_0, t_0) \leq -bu(x_0, t_0), \quad -\Delta u(x_0, t_0) \leq -\varepsilon Ne^{-bt_0} \quad \text{and} \quad a|\nabla u(x_0, t_0)|^p = a\varepsilon^p|x_0|e^{-pbt_0}, \]
from which follows
\[ \partial_t u(x_0, t_0) - \Delta u(x_0, t_0) + a|\nabla u(x_0, t_0)|^p \leq -bu(x_0, t_0) + \varepsilon e^{-bt_0} \left( -N + a\varepsilon^{p-1}|x_0|^p e^{-(p-1)bt_0} \right) \]
\[ \leq -bu(x_0, t_0) \]
for \( \varepsilon \) small enough, which is a contradiction.

**Proof of Theorem 1.3.** By Remark 3.3 we have
\[ \inf \{ v(x, t); (x, t) \in O \} \geq (q_1 - 1)^{-\frac{1}{q_1-1}} R^{-\frac{2}{q_1-1}}. \]
Take \( V = \lambda v^\frac{1}{p} \in C^{2,1}(O) \) for \( \lambda > 0 \). Thus \( v = \lambda^{-\alpha} V^\alpha \),
\[ \inf \{ V(x, t); (x, t) \in O \} \geq (q_1 - 1)^{-\frac{1}{q_1-1}} R^{-\frac{2}{q_1-1}}, \]
and
\[ \partial_t v - \Delta v + \varepsilon v^q = \alpha \lambda^{-\alpha} V^{\alpha-1} \partial_t V - \alpha \lambda^{-\alpha} V^{\alpha-1} \Delta V + a(1 - \alpha) \lambda^{-\alpha} V^{\alpha-1} \frac{|\nabla V|^2}{V} + \lambda^{-\alpha q_1} V^{\alpha q_1}. \]
This leads to
\[ \partial_t V - \Delta V + (1 - \alpha) \frac{|\nabla V|^2}{V} + \alpha^{-1} \lambda^{-\alpha(q_1-1)} V^{\alpha q_1 - \alpha + 1} = 0 \quad \text{in} \quad O. \]
Using Hölder’s inequality,
\[ (1 - \alpha) \frac{|\nabla V|^2}{V} + (2\alpha)^{-1} \lambda^{-\alpha(q_1-1)} V^{\alpha q_1 - \alpha + 1} \geq c_1 |\nabla V|^p \lambda^{-\alpha(q_1-1)(2-p)} V^{\alpha(q_1-1)(2-q) - (p-1)} \]
\[ \geq c_2 |\nabla V|^p \lambda^{-p - 2 + \frac{2(p-1)}{q_1-1}} V^q, \]
and
\[ (2\alpha)^{-1} \lambda^{-\alpha(q_1-1)} V^{\alpha q_1 - \alpha + 1} \geq c_3 \lambda^{-q_1} R^{-2 + \frac{2(q_1-1)}{q_1-1}} V^q. \]
If we choose
\[ \lambda = \min \left\{ c_2 \frac{1}{q_1-1}, c_3 \frac{1}{q_1-1} \right\} \min \left\{ a^{-\frac{1}{p-1}} R^{-\frac{2(p-1)}{q_1-1} - \frac{2}{q_1-1}}, b^{-\frac{1}{q_1-1}} R^{-\frac{2}{q_1-1} - \frac{2}{q_1-1}} \right\} \]
then
\[ c_2 \lambda^{-p - 2 + \frac{2(p-1)}{q_1-1}} \geq a, \]
\[ c_3 \lambda^{-q_1} R^{-2 + \frac{2(q_1-1)}{q_1-1}} \geq b, \]
from what follows
\[ \partial_t V - \Delta V + a|\nabla V|^p + bV^q \leq 0 \quad \text{in} \quad O. \]
By Remark 3.3 there exists a maximal solution \( u \in C^{2,1}(O) \) of
\[ \partial_t u - \Delta u + a|\nabla u|^p + bu^q = 0 \quad \text{in} \quad O. \]

Therefore, \( u \geq V = \lambda v^\frac{1}{p} \) and \( u \) is a large solution of (4.25). This completes the proof of Theorem 1.3. 

\[ \]
5 Appendix

Proof of Proposition 2.5.

Step 1. We claim that the following relation holds:

\[ \int_{\mathbb{R}^{N+1}} (\underline{\mu}^1 \mu)(x,t)^{(N+2)/N} \, dx \, dt \leq \int_{\mathbb{R}^{N+1}} (\mu(\tilde{Q}_r(x,t)))^{2/N} \, dr \, d\mu(x,t). \]  \tag{5.1}

In fact, we have for \( \rho_j = 2^{-j}, j \in \mathbb{Z} \),

\[ \sum_{j=1}^{\infty} \int_{\mathbb{R}^{N+1}} (\mu(\tilde{Q}_{\rho_j}(x,t)))^{2/N} \, dx \, dt \lesssim \int_{\mathbb{R}^{N+1}} (\mu(\tilde{Q}_r(x,t)))^{2/N} \, dr \, d\mu(x,t). \]

Note that for any \( j \in \mathbb{Z} \)

\[ \rho_j^{-N/2} \int_{\mathbb{R}^{N+1}} (\mu(\tilde{Q}_{\rho_j}(x,t)))^{(N+2)/N} \, dx \, dt \lesssim \int_{\mathbb{R}^{N+1}} (\mu(\tilde{Q}_{\rho_j}(x,t)))^{2/N} \, dr \, d\mu(x,t) \]

Hence,

\[ \sum_{j=2}^{\infty} \rho_j^{-N} \int_{\mathbb{R}^{N+1}} (\mu(\tilde{Q}_{\rho_j}(x,t)))^{(N+2)/N} \, dx \, dt \lesssim \int_{\mathbb{R}^{N+1}} (\mu(\tilde{Q}_r(x,t)))^{2/N} \, dr \, d\mu(x,t) \]

Thus,

\[ \sum_{j=-1}^{\infty} \rho_j^{-N} \int_{\mathbb{R}^{N+1}} (\mu(\tilde{Q}_{\rho_j}(x,t)))^{(N+2)/N} \, dx \, dt \lesssim \int_{\mathbb{R}^{N+1}} (\mu(\tilde{Q}_r(x,t)))^{(N+2)/N} \, dx \, dt. \]

This yields

\[ \int_{\mathbb{R}^{N+1}} (\underline{\mu}^2 \mu)(x,t)^{(N+2)/N} \, dx \, dt \lesssim \int_{\mathbb{R}^{N+1}} (\mu(\tilde{Q}_r(x,t)))^{2/N} \, dr \, d\mu(x,t) \lesssim \int_{\mathbb{R}^{N+1}} (\underline{\mu}^2 \mu)(x,t)^{(N+2)/N} \, dx \, dt. \]

By [20, Theorem 4.2],

\[ \int_{\mathbb{R}^{N+1}} (\underline{\mu}^2 \mu)(x,t)^{(N+2)/N} \, dx \, dt \lesssim \int_{\mathbb{R}^{N+1}} (\mu(\tilde{Q}_r(x,t)))^{(N+2)/N} \, dx \, dt, \]

thus we obtain (5.1).

Step 2. End of the proof. The first inequality in (2.1) is proved in [20]. We now prove the second inequality. By Theorem 2.4, there is \( \mu \in \mathcal{M}^1(\mathbb{R}^{N+1}), \text{supp}(\mu) \subset K \) such that

\[ ||\underline{\mu}^2 \mu||_{L^\infty(\mathbb{R}^{N+1})} \leq 1 \text{ and } \mu(K) = \mathcal{P} \mathcal{H}_2^N(K) \gtrsim |K|^{N/(N+2)}. \]  \tag{5.2}
Thanks to \[(5.4)\], we have for \(\delta = \min\{1, (\mu(K))^{1/N}\} \]
\[
\|\sqrt{2\mu}\|_{L^{(N+2)/N}(\mathbb{R}^{N+1})} \leq \int_{\mathbb{R}^{N+1}} \int_{0}^{1} (\mu(\bar{Q}_r(x,t)))^{2/N} \frac{dr}{r} \mu(x,t) \]
\[
\leq \int_{\mathbb{R}^{N+1}} \left( \int_{0}^{\delta} + \int_{\delta}^{1} \right) (\mu(\bar{Q}_r(x,t)))^{2/N} \frac{dr}{r} \mu(x,t) \]
\[
\leq \int_{0}^{\delta} \frac{r^2}{r} \int_{\mathbb{R}^{N+1}} d\mu(x,t) + \int_{\delta}^{1} \frac{1}{r} \left( \int_{\mathbb{R}^{N+1}} d\mu(x,t) \right) \]
\[
\leq (\mu(K))^{(N+2)/N} \left( 1 + \log_+ \left( (\mu(K))^{-1} \right) \right) \]
\[
\leq (\mu(K))^{(N+2)/N} \log \left( \frac{|\hat{Q}_{200}(0,0)|}{|K|} \right). \]

Set \(\tilde{\mu} = \left( \log \left( \frac{Q_{200}(0,0)}{|K|} \right) \right)^{-N/(N+2)} \mu/\mu(K)\), then \(\|\sqrt{2\tilde{\mu}}\|_{L^{(N+2)/N}(\mathbb{R}^{N+1})} \lesssim 1\).

It is well known that
\[
\text{Cap}_{2,1,\frac{N+2}{N}}(K) \simeq \sup \{ (\omega(K))^{(N+2)/2} : \omega \in \mathcal{M}^+(K), \|\sqrt{2\omega}\|_{L^{(N+2)/N}(\mathbb{R}^{N+1})} \lesssim 1 \} \quad (5.3) \]
see [20] Section 4. This gives the second inequality in \[(2.1)\]. It is easy to prove \[(2.2)\] from its definition. Moreover, \[(5.3)\] implies that
\[
\frac{1}{\text{Cap}_{2,1,\frac{N+2}{N}}(K)^{2/N}} \simeq \inf \left\{ \int_{\mathbb{R}^{N+1}} \int_{0}^{1} (\omega(\bar{Q}_r(x,t)))^{2/N} \frac{dr}{r} d\mu(x,t) : \omega \in \mathcal{M}^+(K), \omega(K) = 1 \right\}. \quad (5.4) \]

We deduce from \[(3.1)\] that
\[
\frac{1}{\text{Cap}_{2,1,\frac{N+2}{N}}(K)^{2/N}} \simeq \inf \left\{ \int_{\mathbb{R}^{N+1}} \int_{0}^{1} (\omega(\bar{Q}_r(x,t)))^{2/N} \frac{dr}{r} d\mu(x,t) : \omega \in \mathcal{M}^+(K), \omega(K) = 1 \right\}. \quad (5.4) \]

As in [12] proof of Lemma 2.2, it is easy to derive \[(2.3)\] from \[(5.4)\].

**Proof of Proposition \[(2.4)\]**. Thanks to the Poincaré inequality, it is enough to show that there exists \(\varphi \in C_c^\infty(\tilde{Q}_{3/2}(0,0))\) such that \(0 \leq \varphi \leq 1\), with \(\varphi = 1\) in an open neighborhood of \(K\) and
\[
\int_{\mathbb{R}^{N+1}} (|D^2 \varphi|^p + |\partial_t \varphi|^p) dx dt \leq \text{Cap}_{2,1,p}(K). \quad (5.5) \]

By definition, one can find \(0 \leq \phi \in S(\mathbb{R}^{N+1}), \phi \geq 1\) in a neighborhood of \(K\) such that
\[
\int_{\mathbb{R}^{N+1}} (|D^2 \phi|^p + |\nabla \phi|^p + |\phi|^p + |\partial_t \phi|^p) dx dt \leq 2\text{Cap}_{2,1,p}(K). \]

Let \(\eta\) be a cut off function on \(\tilde{Q}_1(0,0)\) with respect to \(\tilde{Q}_{3/2}(0,0)\) and \(H \in C^\infty(\mathbb{R})\) such that
\(0 \leq H(t) \leq t^+, \ t||H''(t)|| \leq 1\) for all \(t \in \mathbb{R}\), \(H(t) = 0\) for \(t \leq 1/4\) and \(H(t) = 1\) for \(t \geq 3/4\).

We claim that
\[
\int_{\mathbb{R}^{N+1}} (|D^2 \varphi|^p + |\partial_t \varphi|^p) dx dt \leq \int_{\mathbb{R}^{N+1}} (|D^2 \phi|^p + |\nabla \phi|^p + |\phi|^p + |\partial_t \phi|^p) dx dt, \quad (5.6) \]
where \(\varphi = \eta H(\phi)\). Indeed, we have
\[
|D^2 \varphi| \lesssim |D^2 \eta|H(\phi) + |\nabla \eta||H''(\phi)||\nabla \phi| + \eta|H''(\phi)||\nabla \phi|^2 + \eta|H'(\phi)||D^2 \phi| \]
and 

\[ |\partial_t \varphi| \lesssim |\partial_t \eta| H(\phi) + \eta|H'(\phi)|\partial_t \eta|, \quad H(\phi) \leq \phi, \quad \phi|H''(\phi)| \lesssim 1. \]

Thus,

\[
\int_{\mathbb{R}^{N+1}} (|D^2 \varphi|^p + |\partial_t \varphi|^p) dx dt \lesssim \int_{\mathbb{R}^{N+1}} (|D^2 \phi|^p + |\nabla \phi|^p + |\phi|^p + |\partial_t \phi|^p) dx dt \\
+ \int_{\mathbb{R}^{N+1}} \frac{|\nabla \phi|^{2p}}{\phi^p} dx dt.
\]

This implies (5.6) since, according to [1], one has

\[
\int_{\mathbb{R}^N} \frac{|\nabla \phi(t)|^{2p}}{\phi(t)^p} dx \lesssim \int_{\mathbb{R}^N} |D^2 \phi(t)|^p dx \quad \forall t \in \mathbb{R}.
\]

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