Rate of equilibration of a one-dimensional Wigner crystal

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Abstract

We consider a system of one-dimensional spinless particles interacting via long-range repulsion. In the limit of strong interactions the system is a Wigner crystal, with excitations analogous to phonons in solids. In a harmonic crystal the phonons do not interact, and the system never reaches thermal equilibrium. We account for the anharmonism of the Wigner crystal and find the rate at which it approaches equilibrium. The full equilibration of the system requires umklapp scattering of phonons, resulting in exponential suppression of the equilibration rate at low temperatures.

Key words: equilibration, one-dimensional systems, Wigner crystal
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The low-temperature physics of interacting electron systems is usually described in the framework of the so-called Luttinger liquid theory [1]. The phenomenological nature of this approach enables one to study the systems with any interaction strength, provided that the physics is controlled by the low-energy excitations. On the other hand, even at low temperature T some phenomena involve excitations with energies much higher than T. In such cases microscopic approaches are usually more effective.

Fig. 1(a), are needed. The most effective such process includes backscattering of a particle at the very bottom of the band [2]. The small probability of finding an available final state deep below the Fermi level $E_F$ results in exponential suppression of the equilibration rate $\tau^{-1} \propto e^{-E_F/T}$ [3].

The key event in the process of equilibration is the backscattering of a hole at the bottom of the band. It is therefore helpful to focus on the motion of such a hole in momentum space. For electrons with quadratic spectrum $E_k = \hbar^2 k^2/2m$ the energy of the hole excitation is $\epsilon(q) = E_{k_F} - E_{k_F-q} = \hbar v_F q (1 - q/2k_F)$, assuming that the hole is created by moving an electron
from state \( k_F - q \) to the right Fermi point \( k_F \). Here \( k_F \) is the Fermi wave vector and \( v_F = \hbar k_F / m \) is the Fermi velocity. Backscattering occurs when the wave vector of the hole \( q \) crosses \( k_F \), Fig. 1(b).

Because the equilibration of the system involves excitations with energies of the order of the bandwidth, this phenomenon is not captured by the conventional Luttinger liquid theory [1]. This makes the generalization of the above picture beyond the weakly-interacting limit rather challenging. On the other hand, it is possible to develop a microscopic theory of this phenomenon in the case of strong Coulomb interactions [4]. Here we revisit this approach and obtain the full expression for the equilibration rate for arbitrary strong long-range repulsion. The key idea is that as repulsion of electrons becomes stronger, the system minimizes its energy by forming a periodic structure known as the Wigner crystal. Although the long-range order in such a system is destroyed by quantum fluctuations [5], the presence of the strong short-range order enables us to treat the system as anharmonic chain described by the Hamiltonian

\[
H = \sum_{l} \frac{p_l^2}{2m} + \frac{1}{2} \sum_{l,l'} V(x_l - x_{l'}).
\]  

(1)

Here \( p_l \) and \( x_l \) are the momentum and coordinate of the \( l \)-th particle and \( V(x) \) is the interaction potential.

The excitations of the system are essentially the phonons in the electronic crystal. They are conveniently described in terms of the displacements \( u_l = x_l - l a \) of electrons from their equilibrium positions, where \( a \) is the mean interparticle distance. Strong repulsion means small displacements, \( |u_l - u_{l'}| \ll |l - l'| a \), and the Hamiltonian can be approximated by that of a harmonic chain

\[
H_0 = \sum_{l} \frac{p_l^2}{2m} + \frac{1}{4} \sum_{l,l'} V_{l-l'}^{(2)} (u_l - u_{l'})^2,
\]  

(2)

where the \( \tau \)-th derivative of \( V(x) \) is denoted as

\[
V_{l-l'}^{(\tau)} = \frac{d^\tau V(x)}{dx^{\tau}} \bigg|_{x=la}.
\]  

(3)

The phonon modes of this Hamiltonian are easily found,

\[
\omega_q^2 = \frac{2}{m} \sum_{l=1}^{\infty} V_{l-l'}^{(2)} [1 - \cos(ql)].
\]  

(4)

At small wave vector \( qa \to 0 \) the excitation spectrum is linear, \( \omega_q = s|q| \), where \( s = (\sum_i V_i^{(2)} l^2 / m)^{1/2} \) is the “sound velocity” in the Wigner crystal measured in units of lattice spacings per unit time. The spectrum is periodic in \( q \), with the Brillouin zone \(-\pi < q < \pi \), Fig. 2.

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![Excitation spectrum of a one-dimensional Wigner crystal.](image)

**Fig. 2.** Excitation spectrum of a one-dimensional Wigner crystal. Equilibrium processes involve a phonon \( q_1 \) being scattered to a state \( q'_1 \) outside the Brillouin zone as a result of a collision with a thermal phonon \( q_2 \). Conservation of energy and momentum requires the latter to be backscattered to \( q''_2 \). This umklapp process is analogous to a hole crossing the point \( q = k_F \) in the weakly interacting Fermi gas, Fig. 1(b). (Note that if the distances are measured in units of \( a \), the Fermi wave vector \( k_F = \pi / a \).)

Any eigenstate of the harmonic chain (2) can be described by the set of occupation numbers \( N_q \) of the phonons. In the absence of interaction of phonons, the lifetime of any such state is infinite, and the system never reaches thermal equilibrium. On the other hand, the harmonic Hamiltonian (2) is merely the leading term of the expansion of Eq. (1) in the small parameter \( K = \pi \hbar / ma^2 s \). The next term is proportional to the third power of the displacement \( u_l \) and generates scattering processes involving three phonons. At low temperature \( T \ll \hbar \omega_s \) the typical quasimomenta of phonons are small, \( q \ll T / \hbar s \), umklapp scattering is suppressed, and the phonons remain in the first Brillouin zone upon scattering. This means that apart from energy, collisions conserve the total quasimomentum of the phonons. This yields the equilibrium phonon distribution

\[
N_q = \frac{1}{e^{\hbar (\omega_q - u_0)/T} - 1}
\]  

(5)

characterized by two parameters, the temperature \( T \) and the velocity \( u \) of the phonon gas with respect to the crystal.

On the other hand, even at low temperatures there are rare umklapp collisions of phonons, such as the one shown in Fig. 2, which do not conserve their quasimomentum. As a result, one expects the velocity \( u \) to relax gradually as \( \dot{u} = -u / \tau \) with a small relaxation rate \( \tau^{-1} \). To find it we notice that the most efficient umklapp processes involve a phonon \( q_1 \) near the boundary \( q = \pi \) of the Brillouin zone colliding with a thermal phonon with \( q_2 \sim T / \hbar s \), Fig. 2. At low temperature the resulting change of quasimomentum \( |q_1' - q_1| \sim T / \hbar \omega_s' \) is small compared to the typical scale \( q_T = (T / \hbar \omega_s')^{1/2} \) of the distribution (5) near the edge of the Brillouin zone. (Here \( \omega''_s = \partial^2 \omega_{q''_2} \).) Thus the high-energy phonon performs a slow diffusive motion in the momentum
space, and its distribution function \( N_q(t) \) obeys the Fokker-Planck equation

\[
\partial_t N_q = \partial_q \left[ \frac{B(q)}{2} \left( \frac{\hbar \omega_q}{T} + \partial_q \right) \right] N_q.
\]

Here

\[
B(q) = \sum_{\delta q} (\delta q)^2 W_{q,q+\delta q}
\]

(7)

has the meaning of the diffusion constant in momentum space and \( W_{q,q+\delta q} \) is the rate at which a phonon \( q \) changes its wave vector by \( \delta q \) as a result of collisions with other phonons.

The Fokker-Planck equation should be solved with the boundary conditions

\[
N_q = e^{-\hbar \omega_q/T} e^{\pm \pi \hbar \omega_q/T}, \quad \pi \ll \mp q - \pi \ll \pi
\]

(8)

obtained by extending the distribution (5) beyond the first Brillouin zone. Such solution [4] gives the relaxation law \( \dot{u} = -u/T \) with the rate

\[
\tau^{-1} = 3B \left( \frac{\hbar s}{T} \right)^3 \left( \frac{\hbar |\omega_q|}{2\pi T} \right)^{1/2} e^{-\hbar \omega_q/T}, \]

(9)

where \( B = B(\pi) \). The temperature dependence of the relaxation rate is dominated by the exponentially small probability of the occupation of phonon states near the edge \( q = \pi \) of the Brillouin zone. Expression (9) is analogous to the result \( \tau^{-1} \propto e^{-E_F/T} \) for weakly-interacting electrons. The strong interactions between electrons renormalize the activation temperature from \( E_F \) to \( \hbar \omega_q \) in Eq. (9).

The temperature dependence of the prefactor in Eq. (9) is determined by that of the diffusion constant \( B \) and by \( T^{-7/2} \) explicitly present in (9). The former can be deduced phenomenologically [6] by treating the phonon near \( q = \pi \) as a mobile impurity in a Luttinger liquid, for which \( B \) is known [7] to scale as

\[
B \propto T^3, \quad T \to 0.
\]

(10)

We therefore conclude that the equilibration rate scales with temperature as \( \tau^{-1} \propto T^{3/2} e^{-\hbar \omega_q/T} \).

The constant \( \chi \) in Eq. (10) has to be determined by microscopic evaluation of the scattering rate \( W_{q,q+\delta q} \) in Eq. (7). The dominant scattering process, illustrated in Fig. 2, involves two phonons in both the initial and final states. Such scattering can be accomplished either in the first order in four-phonon scattering amplitude or in the second order in three-phonon scattering amplitude. The resulting expression for the scattering rate has the form [4]

\[
W_{q_1,q_1'} = \frac{2\pi \hbar^2}{m^2 \hbar^2 N^2} \sum_{q_2,q_2'} \frac{\Lambda^2 N_{q_2} N_{q_2'} + 1}{\omega_{q_1} \omega_{q_2} \omega_{q_1'}^2} \delta_{q_1+q_2,q_1'+q_2'} \times \delta(\omega_{q_1} + \omega_{q_2} - \omega_{q_1'} - \omega_{q_2'}). \]

(11)

Here \( N \) is the total number of particles in the system.

The temperature dependence of the prefactor in Eqs. (11) and (12) are valid for any \( q_2 \) and \( q_2' \). At low temperature their magnitudes are small, \(|q_2|, |q_2'| \ll T/\hbar s \). Thus to find \( \chi \) in Eq. (10) we need to expand \( \Lambda \) in powers of \( q_2 \) and \( q_2' \). Carrying out such expansion and taking into account conservation of momentum and energy we find

\[
\Lambda = \frac{\pi^3}{8 s^2} \sum_{l=1}^{\infty} V_l^{(2)} \sin[(q_1 + q_2)l] - \sin(q_1l) - \sin(q_2l),
\]

(13)

\[
f_4(q_1,q_2) = \frac{1}{2} \chi T \sum_{l=1}^{\infty} V_l^{(2)} \left[ \sin((q_1 + q_2)l) + \sin((q_1 + q_2)l) - \sin(q_1l) - \sin(q_2l) \right].
\]

(14)

Equations (11), (13), and (14) determine the scattering rate \( W_{q,q+\delta q} \) for any \( q \). It is easy to see that \( W_{q,q+\delta q} \propto (\delta q)^2 \), and Eq. (7) immediately gives the temperature dependence (10). To find the coefficient \( \chi \) we set \( q = \pi \) and obtain

\[
\chi = \frac{4\pi^3 \left( \partial_n \omega \right)^2 \delta^2}{15 \hbar^2 m^2 v^2}, \quad T \to 0.
\]

(15)

Remarkably, \( \chi \) is fully determined by the phonon spectrum and its dependence on the particle density \( n \).

Equations (9), (10), and (15) give the complete expression for the equilibration rate of a one-dimensional Wigner crystal. In the case of pure Coulomb repulsion \( V(x) = e^2/|x| \) the velocity \( s \) diverges, and our treatment is inapplicable. However, in the experimental realizations of one-dimensional Wigner crystal, there is
usually a metal gate screening the interactions at large distances. In this case we obtain

\[
\frac{1}{\tau} = \eta \frac{\Delta}{\hbar \ln^{3/2} (d/a)} a(h/\Delta)^{3/2} e^{-\Delta/\tau},
\]

(16)

where \( \eta = 63 \pi^3 \zeta(3) \sqrt{\ln 2}/80 \sqrt{\pi} \) is a numerical prefactor, \( a_B = \hbar^2/m_e^2 \) is the Bohr’s radius, \( d \) is the distance to the gate, and

\[
\Delta = \hbar \omega_e = \left( \frac{7 \zeta(3) \hbar^2 e^2}{ma^3} \right)^{1/2}.
\]

(17)

A non-trivial test of our result (15) can be performed by considering the interaction potential

\[
V(x) = \frac{\gamma}{\sinh^2 cx}.
\]

(18)

It is well known [8] that the model (18) is integrable, i.e., it has an infinite number of integrals of motion. As a result the excitations of the system have infinite lifetimes, and one expects the diffusion constant \( D \) to vanish. Our approach applies only to the limit of strong repulsion \( \gamma \to \infty \), when the Wigner crystal approximation is applicable. On the other hand, the parameter \( c \) can take any value. It is easy to obtain analytic expressions for the phonon spectrum in the limiting cases \( c \ll n \) and \( c \gg n \).

At \( c \ll n \) one can approximate (18) with \( V(x) = \gamma/(cx)^2 \), the so-called Calogero-Sutherland model [8]. Then from Eq. (4) one finds

\[
\omega_q = \left( \frac{\gamma}{c^2 m} \right)^{1/2} n^2 (\pi q - q^2/2) e^{-c/n} \sin \frac{q}{2},
\]

(19)

\[
v_q = \left( \frac{\gamma}{c^2 m} \right)^{1/2} n (\pi - q) e^{-c/n} \cos \frac{q}{2}.
\]

(20)

and \( v \) is given by \( v_q \) at \( q = 0 \). Substitution of Eqs. (19) and (20) into (14) gives \( \Upsilon_q = 0 \) and therefore \( B = 0 \).

At \( c \gg n \) the interactions fall off very rapidly with the distance, \( V(x) = 4\gamma e^{-2cx} \). In this case only the interaction of the nearest neighbor particles in the Wigner crystal needs to be taken into account (Toda lattice). The spectrum takes the form

\[
\omega_q = 8c \left( \frac{\gamma}{m} \right)^{1/2} e^{-c/n} \sin \frac{q}{2},
\]

(21)

\[
v_q = 4c \left( \frac{\gamma}{m} \right)^{1/2} n e^{-c/n} \cos \frac{q}{2}.
\]

(22)

As expected, substitution of Eqs. (21) and (22) into (14) gives \( \Upsilon_q = 0 \). Finally, we have checked numerically that the expression (15) vanishes for potential (18) for any \( c \).

To summarize, we have obtained the equilibration rate \( \tau^{-1} \) of one-dimensional system of particles with strong long-rage repulsion. At low temperatures the rate is exponentially suppressed with the activation energy given by the Debye frequency \( \omega_e \) of the Wigner crystal. The prefactor can be expressed in terms of the phonon spectrum using Eqs. (9), (10), and (15). In the case of Coulomb repulsion the result is given by Eqs. (16) and (17). Finally, we have checked that the equilibration rate vanishes for the integrable model of particles with interactions in the form (18).

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