NON-ASSOCIATIVE FROBENIUS ALGEBRAS FOR SIMPLY LACED CHEVALLEY GROUPS

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Abstract. We provide an explicit construction for a class of commutative, non-associative algebras for each of the simple Chevalley groups of simply laced type. Moreover, we equip these algebras with an associating bilinear form, which turns them into Frobenius algebras. This class includes a 3876-dimensional algebra on which the Chevalley group of type $E_8$ acts by automorphisms. We also prove that these algebras admit the structure of (axial) decomposition algebras.

Introduction

In 1982, Robert R. Griess constructed the largest sporadic simple group as the automorphism group of a commutative non-associative algebra, the Griess algebra [Gri82]. In 1985, John Conway gave a different construction of the Griess algebra and observed that it has a peculiar feature [Con85]. It is generated by idempotents whose action by multiplication gives rise to a decomposition of the algebra that obeys a certain fusion law with respect to the multiplication. Alexander A. Ivanov axiomatized this property in his definition of Majorana algebras [Iva09]. Only recently, the definition has been further generalized to the definition of an axial algebra [HRS15] by Jonathan I. Hall, Felix Rehren and Sergey Shpectorov. A further generalization, called (axial) decomposition algebras, was given by Tom De Medts, Simon F. Peacock, S. Shpectorov and Michiel Van Couwenberghe [DMPSVC20].

The subject has received a lot of attention recently and there is an ongoing search for axial decomposition algebras admitting a given (simple) group as automorphism group.

Completely unrelated to this, Skip Garibaldi and Robert M. Guralnick showed that over any field of characteristic not 2 or 3, there exists a 3875-dimensional, commutative, non-associative, Frobenius algebra, the automorphism group of which is a simple algebraic group of type $E_8$ (as group schemes) [GG15, p. 15]. To the best of our knowledge, however, no explicit construction of this algebra was known.

Our paper aims to connect both worlds and shed light on the structure of this 3875-dimensional algebra. On the one hand, we will give an explicit construction...
of this algebra. On the other hand, we will be able to give it the structure of an axial decomposition algebra. Furthermore, this algebra fits into a larger class of algebras: the construction can be applied to any simple group of Lie type of type $ADE$ and each of these algebras will have the structure of a decomposition algebra.

**Organization of the paper.** We start in Section 1 by defining a commutative product on the symmetric square of a simple Lie algebra of simply laced type. This definition arises very naturally from the definition of the Lie bracket and will be the starting point for the construction of our algebra.

In Section 2, we will give an explicit, albeit impractical, construction of our algebra. The underlying module of the algebra will be a subrepresentation of the symmetric square of the Lie algebra. We will define a product on it by embedding it into the symmetric square, using the product from Section 1 and projecting it back onto our subrepresentation. We have postponed a technical character computation to Appendix A.

In Sections 3 and 4, we provide an explicit multiplication rule for our algebra. We start in Section 3 by constructing a very small subalgebra which we use in Section 4 to define the full algebra.

The algebra turns out to be unital, a fact that we prove in Section 5.

A very brief introduction into the realm of (axial) decomposition algebras is given in Section 6. Sections 7 and 8 explain how to make our algebra into a decomposition algebra. Once again, we do this for the small subalgebra first and then extend our results to the full algebra.

Since the algebra for $E_8$ has been our main algebra of interest, we give it some more attention in Section 9. We prove that the algebra is, in fact, an axial decomposition algebra. In particular, we prove the following, cf. Theorem 9.7.

**Theorem.** There exists a one-parameter family of non-associative, commutative, unital 3876-dimensional algebras $(A, \odot)$ on which the complex Chevalley group of type $E_8$ acts by automorphisms. Each of these contains a set $\Omega$ of idempotents. For each idempotent $e \in \Omega$, there exists a decomposition $A = \bigoplus_{1 \leq i \leq 6} A_i^e$ of $A$ as a vector space. Moreover, $a \odot e = \lambda_i a$ when $a \in A_i^e$ for $\lambda_1 = 1$, $\lambda_2 = 0$, $\lambda_3 = \frac{3}{4} c_1 - \frac{1}{6}$, $\lambda_4 = \lambda_5 = \frac{1}{2}$ and $\lambda_6 = c_1$ where $c_1 \in \mathbb{C}$ depends on the parameter. The linear map defined by

$$
\tau_e(a) := \begin{cases} 
a & \text{if } a \in \bigoplus_{1 \leq i \leq 4} A_i^e, \\
-a & \text{if } a \in A_5^e \oplus A_6^e,
\end{cases}
$$

defines an automorphism of $(A, \odot)$. These automorphisms $\tau_e$ for $e \in \Omega$ generate the complex Chevalley group of type $E_8$.

**Note.** Exactly one week after this paper was finished and posted on the arXiv, Maurice Chayet and Skip Garibaldi posted another paper on the arXiv about the exact same topic and also relying on the symmetric square of the Lie algebra. Their paper has now appeared; see [CG21]. Their construction is more general: it works for (almost) arbitrary base fields and is not restricted to the simply laced case. Their results focus on different aspects than ours, however, so our work is mostly complementary.
1. A PRODUCT ON THE SYMMETRIC SQUARE

Throughout, we will use the word “algebra” to mean a vector space equipped with a bilinear product. We do not assume this product to be associative nor our algebra to be unital. However, most of our algebras come equipped with a special bilinear form which turns them into Frobenius algebras.

Definition 1.1.

(i) A Frobenius algebra is a triple \((V, \theta, \eta)\) where \(V\) is a vector space over a field \(k\),

\[\theta: V \times V \to V: (v, w) \mapsto v \ast w\]

is a bilinear product and

\[\eta: V \times V \to k: (v, w) \mapsto \langle v, w \rangle\]

is a non-degenerate symmetric bilinear form such that

\[(1.1) \quad v_1 \ast v_2 = v_2 \ast v_1 \quad \text{and} \quad \langle v_1 \ast v_2, v_3 \rangle = \langle v_1, v_2 \ast v_3 \rangle\]

for all \(v_1, v_2, v_3 \in V\). The bilinear form \(\eta\) is called the Frobenius form for the algebra.

(ii) Let \(G\) be a group or a Lie algebra. We say that a Frobenius algebra \((V, \theta, \eta)\) is a Frobenius algebra for \(G\) if \(V\) is a linear \(G\)-representation and both \(\theta\) and \(\eta\) are \(G\)-equivariant, i.e. morphisms of \(G\)-representations.

(iii) Let \((V_i, \theta_i, \eta_i)\) be Frobenius algebras for \(i \in \{1, 2\}\). We say that a linear map \(\varphi: V_1 \to V_2\) is a morphism of Frobenius algebras if

\[\theta_2(\varphi(v), \varphi(w)) = \varphi(\theta_1(v, w)) \quad \text{and} \quad \eta_2(\varphi(v), \varphi(w)) = \varphi(\eta_1(v, w)) \quad \text{for all} \quad v, w \in V_1.\]

If both Frobenius algebras are Frobenius algebras for a group or a Lie algebra \(G\), then we say that \(\varphi\) is a morphism of Frobenius algebras for \(G\) if, in addition, \(\varphi\) is \(G\)-equivariant.

Example 1.2.

(i) Let \(V\) be a finite-dimensional vector space over a field \(k\) with characteristic \(\text{char}(k) \neq 2\). Then we can equip the vector space \(\text{End}(V)\) also with the Jordan product defined by

\[f \bullet g := \frac{1}{2}(fg + gf)\]

for all \(f, g \in \text{End}(V)\). This defines a Jordan algebra \((\text{End}(V), \bullet)\). Let \(\text{tr}: \text{End}(V) \to k\) denote the trace map. Then the bilinear form

\[B: \text{End}(V) \times \text{End}(V): fg \mapsto \text{tr}(fg),\]

is a Frobenius form for this algebra, i.e. \((\text{End}(V), \bullet, B)\) is a Frobenius algebra. This follows from the well-known identity \(\text{tr}(fg) = \text{tr}(gf)\) for all \(f, g, h \in \text{End}(V)\). The proof of the non-degeneracy of this form is an easy exercise, see for example [Lam99, Example 16.57, p. 443].

(ii) Suppose that, in addition, \(V\) itself is equipped with a non-degenerate bilinear form \(\kappa\) and that \(\text{char}(k) \neq 2\). Then we call an operator \(f \in \text{End}(V)\) symmetric if \(\kappa(f(a), b) = \kappa(a, f(b))\) for all \(a, b \in V\) and antisymmetric if \(\kappa(f(a), b) = -\kappa(a, f(b))\) for all \(a, b \in V\). Let \(S\) (resp. \(A\)) be the subspace of \(\text{End}(V)\) consisting of all symmetric (resp. antisymmetric) operators; then \(\text{End}(V) = S \oplus A\) as vector spaces. Then \(S\) is a subalgebra of the Jordan algebra \((\text{End}(V), \bullet)\). Moreover \(B(S, A) = 0\). Hence the restriction of
$B$ to $S$ is non-degenerate. Therefore $(S, \cdot, B)$ is a Frobenius subalgebra of $(\text{End}(V), \cdot, B)$.

We introduce some terminology and notation about Lie algebras that we will use throughout this paper. The relevant definitions can be found in [Hum78].

**Definition 1.3.**

(i) Let $\mathcal{L}$ be a complex simple Lie algebra of *simply laced type*, i.e., of type $A_n$, $D_n$ or $E_n$. To avoid some technicalities that appear when working with low rank, we assume that $n \geq 3$, $n \geq 4$ or $n \in \{6, 7, 8\}$ when $\mathcal{L}$ is of type $A_n$, $D_n$ or $E_n$ respectively. Consider a Cartan subalgebra $\mathcal{H}$ of $\mathcal{L}$ and the set of roots $\Phi \subseteq \mathcal{H}^*$ relative to $\mathcal{H}$. For each root $\alpha \in \Phi$, denote its coroot by $h_\alpha \in \mathcal{H}$. Denote the weight lattice by $\Lambda$. Let $\Delta$ be a base for $\Phi$ and denote the set of positive roots with respect to $\Delta$ by $\Phi^+$. To each root $\alpha \in \Phi$ we associate the reflection

$$s_\alpha : \mathcal{H} \to \mathcal{H}: h \mapsto h - \alpha(h)h_\alpha$$

across the root $\alpha$. The group $W$ generated by these reflections is called the Weyl group of $\Phi$.

(ii) Let $\{h_\alpha \mid \alpha \in \Delta\} \cup \{e_\alpha \mid \alpha \in \Phi\}$ be a Chevalley basis for $\mathcal{L}$ with respect to $\mathcal{H}$ and $\Delta$ [Hum78, §25]. For $\alpha, \beta \in \Phi$ for which $\alpha + \beta \in \Phi$, define $c_{\alpha, \beta} \in \mathbb{C}$ such that $[e_\alpha, e_\beta] = c_{\alpha, \beta}e_{\alpha + \beta}$. Then

$$[h_\alpha, h_\beta] = 0,$$

$$[h_\alpha, e_\beta] = \beta(h_\alpha)e_\beta,$$

$$[e_\alpha, e_{-\alpha}] = h_\alpha,$$

$$[e_\alpha, e_\beta] = c_{\alpha, \beta}e_{\alpha + \beta} \quad \text{if} \quad \alpha + \beta \in \Phi,$$

$$[e_\alpha, e_\beta] = 0 \quad \text{if} \quad \alpha + \beta \notin \Phi \quad \text{and} \quad \beta \neq -\alpha,$$

for all $\alpha, \beta \in \Phi$.

(iii) For $\ell \in \mathcal{L}$ let

$$\text{ad}_\ell : \mathcal{L} \to \mathcal{L} : l \mapsto [\ell, l].$$

The requirement that $\mathcal{L}$ is simple and of simply laced type is equivalent to the fact that $W$ acts transitively on $\Phi$. Therefore $t := \text{tr}(\text{ad}_{h_\alpha} \cdot \text{ad}_{h_\alpha})$ does not depend on the choice of $\alpha \in \Phi$. Define

$$\kappa(\ell_1, \ell_2) = 2t^{-1} \text{tr}(\text{ad}_{\ell_1} \cdot \text{ad}_{\ell_2})$$

for all $\ell_1, \ell_2 \in \mathcal{L}$. Then $\kappa$ is a rescaling of the Killing form of $\mathcal{L}$ such that $\kappa(h_\alpha, h_\alpha) = 2$ for all $\alpha \in \Phi$; in particular, $\kappa$ is non-degenerate. We will simply refer to $\kappa$ as the Killing form. This allows us to identify $\mathcal{H}^*$ with $\mathcal{H}$. Then $\alpha \in \mathcal{H}^*$ corresponds to $h_\alpha$ under this identification. In particular, $\alpha(h_\beta) = \kappa(\alpha, h_\beta) = \kappa(\alpha, \beta)$ and thus, since $\Phi$ is simply laced,

$$\kappa(\alpha, \beta) = \begin{cases} -2 & \text{if} \ \alpha = -\beta, \\ -1 & \text{if} \ \alpha + \beta \in \Phi, \\ 1 & \text{if} \ \alpha - \beta \in \Phi, \\ 2 & \text{if} \ \alpha = \beta, \\ 0 & \text{otherwise.} \end{cases}$$

(1.2)
Also note that
\[ \kappa(e_\alpha, e_\alpha) = -\frac{1}{2} \kappa(e_\alpha, [e_\alpha, h_\alpha]) = -\frac{1}{2} \kappa([e_\alpha, e_\alpha], h_\alpha) = \frac{1}{2} \kappa(h_\alpha, h_\alpha) = 1, \]
that
\[ \kappa(e_\alpha, h_\beta) = \frac{1}{2} \kappa([e_\alpha, h_\alpha], h_\beta) = \frac{1}{2} \kappa([h_\alpha, e_\alpha], h_\beta) = 0 \]
for all \( \alpha, \beta \), and hence that
\[ \kappa(e_\alpha, e_\beta) = \frac{1}{2} \kappa([h_\alpha, e_\alpha], e_\beta) = \frac{1}{2} \kappa([e_\alpha, e_\alpha], e_\beta) = 0 \]
for all \( \beta \neq -\alpha \).

(iv) The structure constants \( c_{\alpha, \beta} \) for \( \alpha, \beta \in \Phi \) with \( \alpha + \beta \in \Phi \) satisfy the following identities (see [Car72, Theorem 4.1.2]):

(i) \( c_{\alpha, \beta} = -c_{\beta, \alpha} \),
(ii) \( c_{\alpha, \beta} = -c_{-\alpha, -\beta} \),
(iii) \( c_{\alpha, \beta} = c_{\gamma, \alpha} c_{\alpha, \gamma} = c_{\alpha, \beta} \) for all \( \alpha, \beta, \gamma \in \Phi \) such that \( \alpha + \beta + \gamma = 0 \).

Since we assume that \( \Phi \) is simply laced, we also have \( c_{\alpha, \beta} = \pm 1 \).

(v) Let \( S^2(\mathcal{L}) \) be the symmetric square of \( \mathcal{L} \) considered as a representation for \( \mathcal{L} \). This means that \( S^2(\mathcal{L}) \) is the quotient of the \( \mathcal{L} \)-representation \( \mathcal{L} \otimes \mathcal{L} \) by the subrepresentation \( (\ell_1 \otimes \ell_2 - \ell_2 \otimes \ell_1) \mid \ell_1, \ell_2 \in \mathcal{L} \). We denote the image of \( \ell_1 \otimes \ell_2 \) under the natural projection onto \( S^2(\mathcal{L}) \) by \( \ell_1 \ell_2 \). We can also view \( S^2(\mathcal{L}) \) as a subrepresentation of \( \mathcal{L} \otimes \mathcal{L} \) by considering the section defined by

\[ \sigma: S^2(\mathcal{L}) \rightarrow \mathcal{L} \otimes \mathcal{L}: \ell_1 \ell_2 \mapsto \frac{1}{2}(\ell_1 \otimes \ell_2 + \ell_2 \otimes \ell_1). \]

Denote the action of \( \mathcal{L} \) on \( S^2(\mathcal{L}) \) by \( \cdot : \ell \cdot \ell_1 \ell_2 = [\ell, \ell_1] \ell_2 + \ell_1 [\ell, \ell_2] \)
for all \( \ell, \ell_1, \ell_2 \in \mathcal{L} \).

We define a product and bilinear form on the symmetric square \( S^2(\mathcal{L}) \) starting from the Lie bracket defined on \( \mathcal{L} \) and the Killing form \( \kappa \). Recall that a product \( \cdot \) on an \( \mathcal{L} \)-module \( A \) is called \( \mathcal{L} \)-equivariant if

\[ \ell \cdot (a \cdot b) = (\ell \cdot a) \cdot b + a \cdot (\ell \cdot b) \]

for all \( a, b \in A \) and all \( \ell \in \mathcal{L} \), and that a bilinear form \( B: A \times A \rightarrow \mathbb{C} \) is called \( \mathcal{L} \)-equivariant if

\[ B(\ell \cdot a, b) + B(a, \ell \cdot b) = 0 \]

for all \( a, b \in A \) and all \( \ell \in \mathcal{L} \). Notice that the Killing form \( \kappa: \mathcal{L} \times \mathcal{L} \rightarrow \mathbb{C} \) is \( \mathcal{L} \)-equivariant (with respect to the adjoint action).

**Definition 1.4.** The non-degeneracy of the Killing form \( \kappa \) allows us to identify \( \mathcal{L} \) with its dual \( \mathcal{L}^* \). Hence we can identify \( \mathcal{L} \otimes \mathcal{L} \) with \( \mathcal{L} \otimes \mathcal{L}^* \cong \text{Hom}(\mathcal{L}, \mathcal{L}) \) via the isomorphism defined by

\[ \zeta': \mathcal{L} \otimes \mathcal{L} \rightarrow \text{Hom}(\mathcal{L}, \mathcal{L}): \ell_1 \otimes \ell_2 \mapsto [\ell' \mapsto \kappa(\ell_2, \ell') \ell_1]. \]

Consider the \( \mathcal{L} \)-equivariant section from Definition 1.3 (vi):

\[ \sigma: S^2(\mathcal{L}) \rightarrow \mathcal{L} \otimes \mathcal{L}: \ell_1 \ell_2 \mapsto \frac{1}{2}(\ell_1 \otimes \ell_2 + \ell_2 \otimes \ell_1). \]

Let \( \zeta := \zeta' \circ \sigma \). Then \( \zeta \) is injective and its image consists of the symmetric operators. These are the operators \( f \in \text{Hom}(\mathcal{L}, \mathcal{L}) \) for which \( \kappa(f(a), b) = \kappa(a, f(b)) \) for all \( a, b \in \mathcal{L} \). We use the construction from Example 1.2 (ii) to turn \( S^2(\mathcal{L}) \) into a
Proof. Under the above correspondence, the maps \( \bullet \) and \( B \) are defined by

\[
\bullet : \quad S^2(\mathcal{L}) \times S^2(\mathcal{L}) \to S^2(\mathcal{L})
\]
\[
(\ell_1, \ell_2, \ell_3, \ell_4) \mapsto \frac{1}{4} \left( \kappa(\ell_1, \ell_3)\ell_2 \ell_4 + \kappa(\ell_1, \ell_4)\ell_2 \ell_3 + \kappa(\ell_2, \ell_3)\ell_1 \ell_4 + \kappa(\ell_2, \ell_4)\ell_1 \ell_3 \right),
\]

\[
B : \quad S^2(\mathcal{L}) \times S^2(\mathcal{L}) \to \mathbb{C}
\]
\[
(\ell_1, \ell_2, \ell_3, \ell_4) \mapsto \frac{1}{2} \left( \kappa(\ell_1, \ell_3)\kappa(\ell_2, \ell_4) + \kappa(\ell_1, \ell_4)\kappa(\ell_2, \ell_3) \right).
\]

**Proposition 1.5.** Consider the bilinear maps \( \bullet \) and \( B \) from Definition 1.4. Then \((S^2(\mathcal{L}), \bullet, B)\) is a Frobenius algebra for \( \mathcal{L} \).

**Proof.** The \( \mathcal{L} \)-equivariance follows immediately from the \( \mathcal{L} \)-equivariance of \( \kappa \). Now, this follows immediately from the construction and Example 1.2 (ii) \( \square \)

The following fact will be used throughout the following sections. Recall that, by definition, an element \( a \in S^2(\mathcal{L}) \) is a weight vector with weight \( w \in \mathcal{H}^* \) if \( h \cdot a = w(h)a \) for all \( h \in \mathcal{H} \).

**Lemma 1.6.** Let \( a, b \in S^2(\mathcal{L}) \) be weight vectors (with respect to the Cartan subalgebra \( \mathcal{H} \) of \( \mathcal{L} \)) with respective weights \( w_a, w_b \in \mathcal{H}^* \). Then

(i) \( a \bullet b \) is a weight vector with weight \( w_a + w_b \),

(ii) \( B(a, b) = 0 \) unless \( w_a + w_b = 0 \).

**Proof.**

(i) For all \( h \in \mathcal{H} \) we have

\[
h \cdot (a \bullet b) = (h \cdot a) \bullet b + a \bullet (h \cdot b) = (w_a + w_b)(h)(a \bullet b)
\]

because the product \( \bullet \) is \( \mathcal{L} \)-equivariant.

(ii) Let \( h \in \mathcal{H} \) such that \( (w_a + w_b)(h) \neq 0 \). Because \( B \) is \( \mathcal{L} \)-equivariant, we have \( w_a(h)B(a, b) = B(h \cdot a, b) = -B(a, h \cdot b) = -w_b(h)B(a, b) \). This implies that \( B(a, b) = 0 \). \( \square \)

2. Constructing the algebra

We will use the algebra from Proposition 1.5 to build a Frobenius algebra of smaller dimension for \( \mathcal{L} \). The highest occurring weight in \( S^2(\mathcal{L}) \), as an \( \mathcal{L} \)-representation, is the double of a root. Its weight space is one-dimensional. We will explicitly determine a generating set of the subrepresentation \( \mathcal{V} \) generated by this weight space in Proposition 2.7 below. Next, we will define an algebra product on the complement \( \mathcal{A} \) of \( \mathcal{V} \) in \( S^2(\mathcal{L}) \) with respect to \( B \). The algebra product on \( \mathcal{A} \) will be the composition of the algebra product from Proposition 1.5 and the projection onto \( \mathcal{A} \). We are grateful to Sergey Shpectorov for providing the central idea of this construction.

**Definition 2.1.** Let \( \mathcal{V} \) denote the subrepresentation of \( S^2(\mathcal{L}) \) generated by \( e_\omega e_\omega \), where \( \omega \) is the highest root with respect to the base \( \Delta \).

It will be fairly straightforward to find elements that lie in \( \mathcal{V} \). However, in order to determine whether they span \( \mathcal{V} \) as a vector space, we will first have to determine the multiplicity of each weight in \( \mathcal{V} \), a task requiring some work. We will use
the terminology of (formal) characters to describe these multiplicities; see \cite{Hum78} §22.5).

**Definition 2.2.** Let Λ be the weight lattice of Φ and consider the group ring \( \mathbb{Z}[\Lambda] \).
To avoid confusion, we denote the basis element of \( \mathbb{Z}[\Lambda] \) corresponding to a weight \( \lambda \in \Lambda \) by \( e^{\lambda} \) (so in particular, \( e^{0} = 1 \) and \( e^{\lambda}e^{\mu} = e^{\lambda+\mu} \) for all \( \lambda, \mu \in \Lambda \)). Let \( V \) be a representation for \( \mathcal{L} \). For each weight \( \lambda \in \Lambda \), we denote its weight-\( \lambda \)-space by \( V_{\lambda} \):

\[
V_{\lambda} := \{v \in V \mid h \cdot v = \lambda(h)v \text{ for all } h \in \mathcal{H}\}.
\]

Write \( m_{\lambda} := \dim(V_{\lambda}) \). Let \( \Pi \) be the set of weights \( \lambda \in \Lambda \) for which \( m_{\lambda} \neq 0 \). Then we define the formal character \( \text{ch}_V \) of \( V \) as

\[
\text{ch}_V = \sum_{\lambda \in \Pi} m_{\lambda}e^{\lambda} \in \mathbb{Z}[\Lambda].
\]

We introduce some notation to describe the weights and multiplicities of \( S^2(\mathcal{L}) \) and \( V \).

**Definition 2.3.**
(i) For \(-2 < i < 2\), let \( \Lambda_i := \{\alpha + \beta \mid \alpha, \beta \in \Phi, \kappa(\alpha, \beta) = i\} \). This means that \( \Lambda_i \) contains those weight vectors that can be represented as the sum of two roots \( \alpha, \beta \in \Phi \) such that \( \kappa(\alpha, \beta) = i \). Recall that \(-2 \leq \kappa(\alpha, \beta) \leq 2 \) for all \( \alpha, \beta \) as \( \Phi \) is simply laced.
(ii) For \( \lambda \in \bigcup_{i=-2}^{2} \Lambda_i \), let \( N_{\lambda} := \{\alpha, \beta \mid \alpha, \beta \in \Phi, \alpha + \beta = \lambda\} \) and \( n_{\lambda} = |N_{\lambda}| \).

Simply put, \( n_{\lambda} \) is the number of ways to write \( \lambda \) as the sum of two roots.

The elements of \( \Lambda_i \) for \(-2 \leq i \leq 2\) are the weights of \( S^2(\mathcal{L}) \) as an \( \mathcal{L} \)-representation, cf. Proposition 2.5. We prove a few easy statements about these weights.

**Lemma 2.4.**
(i) For each \( \lambda \in \Lambda_i \), we have \( \kappa(\lambda, \lambda) = 4 + 2i \). In particular, the sets \( \Lambda_i \) are disjoint.
(ii) \( \Lambda_{-2} = \{0\} \) and \( \Lambda_{-1} = \Phi \).
(iii) If \( \alpha + \beta \in \Lambda_i \) for \( \alpha, \beta \in \Phi \), then \( \kappa(\alpha, \beta) = i \).
(iv) Let \( \alpha \in \Phi \) and \( \lambda \in \Lambda_i \). Then \( \alpha + \beta = \lambda \) for some \( \beta \in \Phi \) if and only if \( \kappa(\alpha, \lambda) = 2 + i \).

**Proof.**
(i) For \( \lambda \in \Lambda_i \), we can write \( \lambda = \alpha + \beta \) where \( \alpha, \beta \in \Phi \) and \( \kappa(\alpha, \beta) = i \). Thus \( \kappa(\lambda, \lambda) = \kappa(\alpha + \beta, \alpha + \beta) = 4 + 2i \).
(ii) We have \( \kappa(\alpha, \beta) = -2 \) for \( \alpha, \beta \in \Phi \) if and only if \( \alpha = -\beta \). Therefore \( \Lambda_{-2} = \{0\} \). Also \( \Lambda_{-1} = \Phi \) because \( \alpha + \beta \in \Phi \) for \( \alpha, \beta \in \Phi \) if and only if \( \kappa(\alpha, \beta) = -1 \).
(iii) By [ii] we know that \( 4 + 2i = \kappa(\alpha + \beta, \alpha + \beta) = 4 + 2\kappa(\alpha, \beta) \) from which the assertion follows.
(iv) This is obvious by [ii] for \( i \in \{-2, -1\} \). Suppose that \( i \in \{0, 1, 2\} \). If \( \lambda = \alpha + \beta \) for some \( \beta \in \Phi \), then \( \kappa(\alpha, \beta) = i \) by [iii] and hence \( \kappa(\lambda, \lambda) = 2 + i \).

Conversely, suppose that \( \kappa(\alpha, \lambda) = 2 + i \). Write \( \lambda = \alpha' + \beta' \) for some \( \alpha', \beta' \in \Phi \), so \( \kappa(\alpha', \beta') = i \) and \( \kappa(\alpha, \alpha') + \kappa(\alpha', \beta') = 2 + i \).

If \( \alpha \in \{\alpha', \beta'\} \), then the conclusion is obvious, so we may assume \( \alpha \not\in \{\alpha', \beta'\} \). By [1.2], this
implies that \( i = 0 \) and \( \kappa(\alpha, \alpha') = \kappa(\alpha, \beta') = 1 \). Then \( \alpha' - \alpha \) is a root and \( \kappa(\alpha' - \alpha, \beta') = -1 \). So \( \lambda - \alpha = (\alpha' - \alpha) + \beta' \) is a root. \( \square \)

**Proposition 2.5.** The character of \( S^2(\mathcal{L}) \) is given by

\[
\text{ch}_{S^2(\mathcal{L})} = \left( \frac{n(n+1)}{2} + n_0 \right) + \sum_{\lambda \in \Lambda_{-1}} (n_\lambda + n)e^\lambda + \sum_{\lambda \in \Lambda_0} n_\lambda e^\lambda + \sum_{\lambda \in \Lambda_1 \cup \Lambda_2} e^\lambda.
\]

**Proof.** If \( \sum \lambda m_\lambda e^\lambda \) is the character of a representation, then its symmetric square has character \( \frac{1}{2} \sum_{\lambda, \mu} \lambda \mu e^{\lambda+\mu} + \frac{1}{2} \sum \lambda e^{2\lambda} \); see [FH91, Exercise 23.39]. Since the character of \( \mathcal{L} \) as \( \mathcal{L} \)-representation is given by

\[
n + \sum_{\alpha \in \Phi} e^\alpha,
\]

the statement follows from Definition 2.3. It is also possible to verify this more explicitly. The Chevalley basis of \( \mathcal{L} \) is a basis of weight vectors of \( \mathcal{L} \) with respect to \( \mathcal{H} \). Now, if \( b_1, \ldots, b_n \) is a basis of weight vectors of \( \mathcal{L} \) as \( \mathcal{L} \)-representation, then \( b_i b_j \) for \( i \leq j \) is a basis of weight vectors for \( S^2(\mathcal{L}) \) from which the character can be computed. \( \square \)

We are now ready to specify the formal character of \( \mathcal{V} \).

**Proposition 2.6.** The character of \( \mathcal{V} \) is given by

\[
\text{ch}_{\mathcal{V}} = n_0 + \sum_{\lambda \in \Lambda_{-1}} (n_\lambda + 1)e^\lambda + \sum_{\lambda \in \Lambda_0} (n_\lambda - 1)e^\lambda + \sum_{\lambda \in \Lambda_1 \cup \Lambda_2} e^\lambda.
\]

**Proof.** The character can be computed using Freudenthal’s formula [Hum78, §22.3]. We refer to Proposition A.2 for the details. \( \square \)

Next, we compute certain elements of \( \mathcal{V} \) and we use the character of \( \mathcal{V} \) to verify that these elements, in fact, span \( \mathcal{V} \) as a vector space.

**Proposition 2.7.** Let

\[
\Gamma_0 := \{2e_{\alpha}e_{-\alpha} - h_\alpha h_\alpha \mid \alpha \in \Phi\},
\]

\[
\Gamma_1 := \{e_{\alpha}h_\alpha \mid \alpha \in \Phi\},
\]

\[
\Gamma_2 := \{2e_{\alpha}e_{\beta} + c_{\alpha,\beta}e_{\alpha+\beta}(h_\beta - h_\alpha) \mid \alpha, \beta \in \Phi, \kappa(\alpha, \beta) = -1\},
\]

\[
\Gamma_3 := \{e_{\alpha}e_{\beta} + \frac{c_{\alpha,\gamma}}{c_{\beta,-\delta}}e_{\gamma}e_\delta \mid \alpha, \beta, \gamma, \delta \in \Phi, \kappa(\alpha, \beta) = 0, \{\gamma, \delta\} \in N_{\alpha+\beta} \setminus \{\{\alpha, \beta\}\}, \}
\]

\[
\Gamma_4 := \{e_{\alpha}e_{\beta} \mid \alpha, \beta \in \Phi, \kappa(\alpha, \beta) = 1\},
\]

\[
\Gamma_5 := \{e_{\alpha}e_{\alpha} \mid \alpha \in \Phi\}.
\]

Then \( \Gamma_0 \cup \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 \cup \Gamma_5 \) spans \( \mathcal{V} \) as a vector space.

**Proof.** The Weyl group \( W \) acts transitively on \( \Phi \) and \( e_\omega e_\omega \in \mathcal{V} \). Therefore \( \Gamma_5 \subseteq \mathcal{V} \). For any \( \alpha \in \Phi \), we have \( e_{-\alpha} \cdot e_{\alpha} e_{-\alpha} = -2e_\alpha h_\alpha \), thus \( \Gamma_1 \subseteq \mathcal{V} \). Hence \( e_{\alpha} \cdot e_{-\alpha} h_{\alpha} = 2e_{\alpha} e_{-\alpha} - h_{\alpha} h_{\alpha} \in \mathcal{V} \) for any \( \alpha \in \Phi \), which shows that \( \Gamma_0 \subseteq \mathcal{V} \). Now let \( \alpha, \beta \in \Phi \). Suppose that \( \kappa(\alpha, \beta) = 1 \). Then \( \alpha - \beta \in \Phi \) and \( e_{\alpha} e_{\beta} = 2e_{\alpha} e_{\beta} - e_{\alpha} e_{\beta} \in \mathcal{V} \). Therefore \( \Gamma_2 \subseteq \mathcal{V} \). Suppose next that \( \kappa(\alpha, \beta) = -1 \) such that \( \alpha + \beta \) is a root. Then \( e_{\beta} e_{\alpha} h_{\alpha} + e_{\alpha} e_{\beta} h_{\beta} = 2e_{\alpha} e_{\beta} + e_{\alpha} e_{\beta} (h_{\beta} - h_{\alpha}) \in \mathcal{V} \). Hence \( \Gamma_2 \subseteq \mathcal{V} \).
Finally, let $\alpha, \beta, \gamma, \delta \in \Phi$ such that $\kappa(\alpha, \beta) = 0$ and $\{\gamma, \delta\} \in N_{\alpha+\beta} \setminus \{\{\alpha, \beta\}\}$. Then $\alpha, \beta, \gamma, \delta$ generate a root subsystem of type $A_3$, $\kappa(\gamma, \delta) = 0$ and $\kappa(\alpha, \gamma) = \kappa(\alpha, \delta) = \kappa(\beta, \gamma) = \kappa(\beta, \delta) = 1$. Now $e_{\delta} \cdot (2e_{\gamma-\alpha}e_{\alpha} + c_{\gamma-\alpha,\alpha}e_{\gamma}(h_{\alpha} - h_{\gamma-\alpha})) = 2c_{\delta,\gamma-\alpha}e_{\alpha}e_{\beta} - 2c_{\delta,\gamma,\alpha}e_{\gamma}e_{\delta}$. Using the identities from Definition 2.3(iv) we see that $c_{\delta,\gamma-\alpha} = -\delta_{\beta,\gamma}$ and $c_{\gamma-\alpha,\alpha} = c_{\alpha,\gamma}$. This amounts to $\Gamma_3 \subseteq \mathcal{V}$.

In order to prove that $\Gamma := \bigcap_0 \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 \cup \Gamma_5$ spans $\mathcal{V}$, it suffices to check that the elements of weight $\lambda$ in $\Gamma$ span $\mathcal{V}_\lambda$, the weight-$\lambda$-space of $\mathcal{V}$. The dimension $\dim(\mathcal{V}_\lambda)$ of $\mathcal{V}_\lambda$ can be derived from Proposition 2.6.

The elements of $\Gamma$ of weight 0 are precisely those contained in $\Gamma_0$. Obviously $\dim(\langle \Gamma_0 \rangle) = \frac{|\Phi|}{2} = n_0$ and therefore $\langle \Gamma_0 \rangle = \mathcal{V}_0$.

Let $\alpha \in \Phi$. The elements of $\Gamma$ of weight $\alpha$ are $e_\alpha h_\alpha$ and the elements $2e_\beta e_\gamma + c_{\beta,\gamma}(h_{\gamma} - h_{\beta})$ where $\{\beta, \gamma\} \in N_\alpha$. Since these elements are linearly independent, they span a subspace of dimension $n_\alpha + 1$, which is the dimension of $\mathcal{V}_\alpha$ by Proposition 2.6.

For $\lambda \in \Lambda_0$, the elements of $\Gamma$ of weight $\lambda$ are those of the form $e_\alpha h_\alpha + e_\alpha h_\alpha$ where $\{\alpha, \beta\}$ and $\{\gamma, \delta\}$ are two different elements of $N_\Lambda$. Hence they span a subspace of dimension at least $n_\lambda - 1$ and at most $n_\lambda$. Since they are all contained in $\mathcal{V}$, they span $\mathcal{V}_\lambda$, a subspace of dimension $n_\lambda - 1$ by Proposition 2.6.

Finally, let $\lambda \in \Lambda_1 \cup \Lambda_2$. Then all elements of $S^2(\mathcal{L})$ of weight $\lambda$ are contained in $\langle \Gamma_4 \cup \Gamma_5 \rangle$. Therefore $\mathcal{V}_\lambda \leq \langle \Gamma \rangle$.

As we observed in the previous proof, the elements of $\Gamma_3$ of weight $\lambda \in \Lambda_0$ are linearly dependent. We introduce some notation to describe this linear dependence; this will be useful later.

**Definition 2.8.**

(i) Recall from Definition 2.3 that

$$\Lambda_0 = \{\alpha + \beta \mid \alpha, \beta \in \Phi, \kappa(\alpha, \beta) = 0\}.$$ 

For each $\lambda \in \Lambda_0$, fix elements $\alpha_\lambda, \beta_\lambda \in \Phi$ such that $\alpha_\lambda + \beta_\lambda = \lambda$. Note that it immediately follows that $\kappa(\alpha_\lambda, \beta_\lambda) = 0$ from Lemma 2.4(iii)

(ii) For all $\alpha, \beta \in \Phi$ such that $\kappa(\alpha, \beta) = 0$ (and therefore $\alpha + \beta \in \Lambda_0$) we write

$$f_{\alpha, \beta} := \begin{cases} 1 & \text{if } \{\alpha, \beta\} = \{\alpha_\lambda, \beta_\lambda\}, \\ -\frac{c_{\alpha,\beta}}{e_{\beta,\alpha}} & \text{otherwise.} \end{cases}$$

Notice that $f_{\alpha, \beta} \in \{\pm 1\}$.

**Proposition 2.9.** Let $\alpha, \beta \in \Phi$ such that $\kappa(\alpha, \beta) = 0$ and let $\lambda := \alpha + \beta \in \Lambda_0$. Then

(i) $f_{\alpha, \beta} = f_{\beta, \alpha}$ and,

(ii) $f_{-\alpha, -\beta} = f_{\alpha, \beta} f_{-\alpha, -\beta}$.

(iii) Let $\Gamma_\lambda$ be as in Proposition 2.7. The subspace of $\mathcal{V}$ spanned by $\Gamma_3$ is equal to the subspace of $\mathcal{V}$ spanned by

$$\{e_{\alpha_\lambda} e_{\beta_\lambda} - f_{\alpha, \beta} e_{\alpha} e_{\beta} \mid \lambda \in \Lambda_0; \alpha, \beta \in \Phi; \alpha + \beta = \lambda\}.$$
$e_\alpha e_\beta + \frac{c_{\alpha,-\beta}}{c_{\beta,-\beta}} e_{\gamma} e_\delta$ for $\alpha, \beta, \gamma, \delta \in \Phi$ with $\alpha + \beta = \gamma + \delta = \lambda$ and $\{\alpha, \beta\} \neq \{\gamma, \delta\}$ must be linearly dependent. In particular $e_\alpha e_\beta \notin \mathcal{V}$ for all $\alpha, \beta \in \Phi$ where $\kappa(\alpha, \beta) = 0$.

(i) This is obvious if $\{\alpha, \beta\} = \{\alpha_\lambda, \beta_\lambda\}$. Assume that $\{\alpha, \beta\} \neq \{\alpha_\lambda, \beta_\lambda\}$. Since $e_\alpha e_\beta = e_\beta e_\alpha$ it follows from the argument above that the elements $e_\alpha e_\beta + \frac{c_{\alpha,-\beta_\lambda}}{c_{\beta,-\beta_\lambda}} e_{\alpha_\lambda} e_{\beta_\lambda}$ and $e_\beta e_\alpha + \frac{c_{\beta,-\alpha_\lambda}}{c_{\alpha,-\beta_\lambda}} e_{\alpha_\lambda} e_{\beta_\lambda}$ must be linearly dependent. Therefore $e_\alpha e_\beta = \frac{c_{\alpha,-\alpha_\lambda}}{c_{\beta,-\beta_\lambda}}$ and thus $f_{\alpha,\beta} = f_{\beta,\alpha}$.

(ii) Since

$e_{\alpha} e_{-\beta} + \frac{c_{-\alpha,\alpha_\lambda}}{c_{-\beta,\beta_\lambda}} e_{-\alpha} e_{-\beta_\lambda} \in \Gamma_3,$

$e_{-\alpha} e_{-\beta} = -f_{-\alpha,-\beta} = -f_{-\alpha,-\beta} e_{\alpha_\lambda} e_{\beta_\lambda} \in \Gamma_3,$

these elements must be linearly dependent. Thus

$f_{-\alpha,-\beta} = \frac{-c_{-\alpha,\alpha_\lambda}}{c_{-\beta,\beta_\lambda}} f_{-\alpha,-\beta_\lambda}.$

The assertion follows because $c_{-\alpha,\alpha_\lambda} = -c_{\alpha,-\alpha_\lambda}$ and $c_{-\beta,\beta_\lambda} = -c_{\beta,-\beta_\lambda}$ (see Definition 1.3(iv)) and therefore

$\frac{-c_{-\alpha,\alpha_\lambda}}{c_{-\beta,\beta_\lambda}} = \frac{c_{\alpha,-\alpha_\lambda}}{c_{\beta,-\beta_\lambda}} = f_{\alpha,\beta}.$

(iii) This follows immediately because the elements of $\Gamma_3$ of weight $\lambda$ span a subspace of dimension $n_\lambda - 1$. $\square$

Next, we want to take a complement of $\mathcal{V}$ in $S^2(\mathcal{L})$ with respect to the bilinear form $B$. In order for this complement to be well-defined, we need $B$ to be non-degenerate on $\mathcal{V}$.

**Proposition 2.10.** The restriction of $B$ to $\mathcal{V} \times \mathcal{V}$ is non-degenerate.

*Proof.* Since $B$ is $\mathcal{L}$-equivariant, the radical $\{v \in \mathcal{V} \mid B(v, w) \text{ for all } w \in \mathcal{V}\}$ of $B|_{\mathcal{V} \times \mathcal{V}}$ is a subrepresentation of $\mathcal{V}$. However, $\mathcal{V}$ is irreducible as it is a highest weight representation. Since $B|_{\mathcal{V} \times \mathcal{V}}$ is non-zero (e.g., $B(e_\alpha e_\alpha, e_{-\alpha} e_{-\alpha}) = 1$), we conclude that the radical of $B|_{\mathcal{V} \times \mathcal{V}}$ is trivial. $\square$

The previous proposition allows us to define an orthogonal complement of $\mathcal{V}$ with respect to the bilinear form $B$. This will be the underlying representation of our algebra.

**Definition 2.11.**

(i) Let $\mathcal{A}$ be the orthogonal complement of $\mathcal{V}$ in $S^2(\mathcal{L})$ with respect to the $\mathcal{L}$-equivariant bilinear form $B$:

$\mathcal{A} := \{v \in S^2(\mathcal{L}) \mid B(v, w) = 0 \text{ for all } w \in \mathcal{V}\}.$

(ii) Denote the orthogonal projection of $S^2(\mathcal{L})$ onto $\mathcal{A}$ by $\text{pr}$. For each $v \in S^2(\mathcal{L})$, we will also denote $\text{pr}(v)$ by $\overline{v}$.

The character of $\mathcal{A}$ follows easily from the characters of $S^2(\mathcal{L})$ and $\mathcal{V}$.
Proposition 2.12. The character of $\mathcal{A}$ as a representation for $\mathcal{L}$ is given by

$$\text{ch}_\mathcal{A} = \frac{n(n+1)}{2} + \sum_{\alpha \in \Lambda_{-1}} (n-1)e^\alpha + \sum_{\lambda \in \Lambda_0} e^\lambda,$$

where $n$ is the rank of $\Phi$.

Proof. Since $S^2(\mathcal{L}) = \mathcal{A} \oplus \mathcal{V}$, we have $\text{ch}_\mathcal{A} = \text{ch}_{S^2(\mathcal{L})} - \text{ch}_\mathcal{V}$. The characters of $S^2(\mathcal{L})$ and $\mathcal{V}$ follow from Propositions 2.5 and 2.6.

Using Proposition 2.7 we can explicitly describe the weight spaces of $\mathcal{A}$.

Proposition 2.13. The weights of $\mathcal{A}$ are 0, the roots $\alpha \in \Phi = \Lambda_{-1}$ and the sums of orthogonal roots $\lambda \in \Lambda_0$. Any weight vector can be uniquely written as

(i) $\bar{\pi}$ for $a \in S^2(\mathcal{H}) \leq S^2(\mathcal{L})$ if the weight vector has weight 0;

(ii) $e^\alpha h$ for $h \in \alpha^1 := \{ h \in \mathcal{H} \mid \kappa(\alpha, h) = 0 \}$ if the weight vector has weight $\alpha \in \Phi$ (also note that $e^\alpha h_{\alpha} = 0$);

(iii) $ce^\alpha e^\beta_\lambda$ for $c \in \mathbb{C}$ when the weight vector has weight $\lambda \in \Lambda_0$.

Proof. Recall that $\mathcal{V}$ is the orthogonal complement of $\mathcal{A}$ in $S^2(\mathcal{L})$ with respect to the bilinear form $B$. The statement follows from the description of the generating set of $\mathcal{V}$ from Propositions 2.7 and 2.9.

The projection from Definition 2.11 can be computed explicitly. In fact, in what follows, we will only need formula 2.1, but for completeness, we also provide formulas 2.2 and 2.3.

Lemma 2.14. Let $\lambda \in \Lambda_0$. Then

$$e^{\alpha_\lambda}e^{\beta_\lambda} = \frac{1}{n_\lambda} \left( \sum_{\alpha + \beta = \lambda} f_{\alpha,\beta}e^\alpha e^\beta \right),$$

where the sum runs over all sets $\{\alpha, \beta\}$ where $\alpha, \beta \in \Phi$ such that $\alpha + \beta = \lambda$, or equivalently, over all elements $\{\alpha, \beta\} \in N_\lambda$. Also

$$e^{\alpha_\lambda}h_{\beta_\lambda} = \frac{1}{n_\lambda} \left( e^{\alpha_\lambda}h_{\beta_\lambda} + \sum e_\beta e^{\alpha_\lambda}e^{\beta_\lambda} e^\beta + e_\alpha e^{\alpha_\lambda}e^{\beta_\lambda} e^\alpha \right),$$

$$h^{\alpha_\lambda}h_{\beta_\lambda} = \frac{1}{n_\lambda} \left( h^{\alpha_\lambda}h_{\beta_\lambda} + \sum e_\alpha e^\alpha e^{\alpha_\lambda}e^{\beta_\lambda} e^\alpha - e_\alpha e^{\alpha_\lambda}e^{\beta_\lambda} e^\alpha - e_\alpha e^{\alpha_\lambda}e^{\beta_\lambda} e^\alpha \right),$$

where each sum runs over all $\{\alpha, \beta\} \in N_\lambda$ with $\{\alpha, \beta\} \neq \{\alpha_\lambda, \beta_\lambda\}$.

Proof. It is immediately verified that

$$B \left( v, \frac{1}{n_\lambda} \left( \sum_{\alpha + \beta = \lambda} f_{\alpha,\beta}e^\alpha e^\beta \right) \right) = 0$$

for all $v \in \Gamma_0 \cup \cdots \cup \Gamma_5$. Since

$$\frac{1}{n_\lambda} \left( \sum_{\alpha + \beta = \lambda} f_{\alpha,\beta}e^\alpha e^\beta \right) - e^{\alpha_\lambda}e^{\beta_\lambda} = \frac{1}{n_\lambda} \left( \sum_{\alpha + \beta = \lambda} \left( e^{\alpha_\lambda}e^{\beta_\lambda} - f_{\alpha,\beta}e^\alpha e^\beta \right) \right) \in \mathcal{V},$$

we have 2.1.
Recall the definition of $f_{\alpha,\beta}$ from Definition \[2.8\] and remember that $f_{\alpha,\beta} = f_{\beta,\alpha}$ by Proposition \[2.9\] Using these, we have, since the projection $pr$ is $\mathcal{L}$-equivariant,

$$e_{\alpha,\lambda} h_{\beta,\lambda} = e_{-\beta,\lambda} \cdot (-e_{\alpha,\lambda} e_{\beta,\lambda})$$

$$= \frac{1}{n_{\lambda}} \left( e_{\alpha,\lambda} h_{\beta,\lambda} - \sum (f_{\beta,\alpha} c_{-\beta,\alpha} e_{\alpha,\beta} - f_{\beta,\alpha} c_{-\beta,\alpha} e_{\beta,\lambda} e_{\alpha}) \right)$$

$$= \frac{1}{n_{\lambda}} \left( e_{\alpha,\lambda} h_{\beta,\lambda} + \sum (c_{\beta,\alpha} e_{\alpha,\alpha} e_{\beta,\lambda} + c_{\alpha,\alpha} e_{\alpha} e_{\beta,\lambda} e_{\alpha}) \right),$$

where each sum runs over the sets $\{\alpha, \beta\}$ with $\alpha, \beta \in \Phi$, $\alpha + \beta = \lambda$ and $\{\alpha, \beta\} \neq \{\alpha, \beta\}$. Similarly, we have

$$h_{\alpha,\lambda} h_{\beta,\lambda} = e_{-\alpha,\lambda} \cdot (-e_{\alpha,\lambda} e_{\beta,\lambda})$$

$$= \frac{1}{n_{\lambda}} \left( h_{\alpha,\lambda} h_{\beta,\lambda} - \sum (e_{\alpha} e_{-\alpha} + e_{\beta} e_{-\beta} - e_{\alpha} - e_{\alpha} e_{\alpha} - e_{\alpha} - e_{\beta} e_{\beta} - e_{\alpha} e_{\beta} e_{\alpha}) \right),$$

where we have used that

$$c_{\beta,\alpha} c_{-\alpha,\beta} = c_{\alpha,\beta} c_{-\alpha,\beta} = -1,$$

and

$$c_{\beta,\alpha} c_{-\alpha,\beta} = c_{\alpha,\beta} c_{-\alpha,\beta} = 1,$$

from Definition \[1.3(iv)\].

We finish this section by defining a suitable product $\ast$ and bilinear form $B$ for $A$ such that $(A, \ast, B)$ is a Frobenius algebra for $\mathcal{L}$.

**Proposition 2.15.** Consider the linear maps

$$\ast : A \times A \rightarrow A : (v, w) \mapsto v \bullet w,$$

$$B : A \times A \rightarrow \mathbb{C} : (v, w) \mapsto B(v, w).$$

Then $(A, \ast, B)$ is a Frobenius algebra for $\mathcal{L}$.

**Proof.** The maps $\ast$ and $B$ are $\mathcal{L}$-equivariant as a composition of $\mathcal{L}$-equivariant maps. The Frobenius property \[1.1\] follows from Proposition \[1.5\] and because $B(v, w) = B(v, w)$ if $w \in A$. \[2.15\]

### 3. The zero weight subalgebra

Consider the zero weight space $A_0$ of $A$ with respect to a fixed Cartan subalgebra $\mathcal{H}$ of $\mathcal{L}$. Since the product $\ast$ and bilinear form $B$ are $\mathcal{L}$-equivariant, $A_0$ is a Frobenius subalgebra of $A$. In this section, we describe this subalgebra explicitly. In order to keep a clear distinction with the construction of the previous section, we will denote vector spaces occurring in this new construction by gothic letters. First, we will use the isomorphism $\zeta : S^2(\mathcal{L}) \rightarrow \text{Hom}(\mathcal{L}, \mathcal{L})$ from Definition \[1.4\] to describe the zero weight space of $S^2(\mathcal{L})$ as a space of homomorphisms.

**Definition 3.1.** Recall the notation $e_{\alpha}$ and $h_{\alpha}$ for $\alpha \in \Phi$ from Definition \[1.3\] Then the zero weight subspace of $S^2(\mathcal{L})$ is spanned by the elements $h_{\alpha} h_{\beta}$ and $e_{\alpha} e_{-\alpha}$ for $\alpha, \beta \in \Phi$.

(i) Let $J$ be the subspace of $\text{Hom}(\mathcal{L}, \mathcal{L})$ spanned by the endomorphisms $j_{\alpha} := \zeta(h_{\alpha} h_{\alpha})$ for $\alpha \in \Phi$. Explicitly, the endomorphism $j_{\alpha}$ is defined by

$$j_{\alpha} : \mathcal{L} \rightarrow \mathcal{L} : \ell \mapsto \kappa(\ell, h_{\alpha}) h_{\alpha}. $$

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Since $h_{-\alpha} = -h_\alpha$, we have $j_\alpha = j_{-\alpha}$. Note that $j_\alpha(e_\beta) = 0$ and $j_\alpha(L) \subseteq H$ for all $\alpha, \beta \in \Phi$. Therefore we can, and will, view $J$ as a subspace of $\text{Hom}(H, H)$.

(ii) Let $J$ be the subspace of $\text{Hom}(L, L)$ spanned by the endomorphisms $z_\alpha := \zeta(e_\alpha e_{-\alpha})$ for $\alpha \in \Phi$. We have

$z_\alpha : L \to L : \ell \mapsto \frac{1}{2} (\kappa(\ell, e_\alpha) e_{-\alpha} + \kappa(\ell, e_{-\alpha}) e_\alpha).

Also $z_\alpha = z_{-\alpha}$.

(iii) Define $S_0 := J + Z$. Then $S_0 = J \oplus Z$ as vector spaces.

(iv) Consider the Jordan product $\bullet$ and bilinear form $B$ on $\text{Hom}(L, L)$ and on $\text{Hom}(H, H)$ as defined in Example 1.2 (i). This turns these vector spaces into Frobenius algebras.

We will prove that $S_0$ is a Frobenius subalgebra of $\text{Hom}(L, L)$. In fact, we have $S_0 = J \oplus Z$ as Frobenius algebras. In particular, $J$ is a subalgebra of $\text{Hom}(H, H)$. This subalgebra has already been studied by T. De Medts and F. Rehren in [DMR17] in a different context.

**Proposition 3.2.** The subspace $J$ is a Frobenius subalgebra of $\text{Hom}(H, H)$. More precisely,

$$j_\alpha \cdot j_\beta = \begin{cases} 2j_\alpha & \text{if } \alpha = \pm \beta, \\ 0 & \text{if } \kappa(\alpha, \beta) = 0, \\ \frac{1}{2}(j_\alpha + j_\beta - j_{s_\beta(\alpha)}) & \text{if } \kappa(\alpha, \beta) = \pm 1, \end{cases}$$

and

$$B(j_\alpha, j_\beta) = \kappa(\alpha, \beta)^2,$$

for all $\alpha, \beta \in \Phi$. It has dimension $\frac{n(n+1)}{2}$ and hence consists of all endomorphisms $f : H \to H$ for which $\kappa(f(a), b) = \kappa(a, f(b))$ for all $a, b \in H$. The Frobenius algebra $J$ is isomorphic to the Frobenius algebra from Example 1.2 (ii) for $V = H$.

**Proof.** The multiplication follows from [DMR17, Lemma 3.2]. However, this can also be calculated using the explicit description of these homomorphisms from Definition 3.1. The dimension follows from [DMR17, Lemma 3.3]. The endomorphisms $j_\alpha$ satisfy the condition that $\kappa(j_\alpha(a), b) = \kappa(a, j_\alpha(b))$ for all $a, b \in H$. Since the subspace of all such homomorphisms has dimension $n(n+1)/2$, this subspace must be equal to $J$. So, in fact, $J$ is precisely the Frobenius algebra from Example 1.2 (ii) for $V = H$.

Also $J$ and $S_0$ are Frobenius subalgebras of $\text{Hom}(L, L)$.

**Proposition 3.3.** The subspace $Z$ is a Frobenius subalgebra of $\text{Hom}(L, L)$. We have

$$z_\alpha \cdot z_\beta = \begin{cases} \frac{1}{2}z_\alpha & \text{if } z_\alpha = z_\beta, \\ 0 & \text{otherwise}, \end{cases}$$

and

$$B(z_\alpha, z_\beta) = \begin{cases} \frac{1}{2} & \text{if } z_\alpha = z_\beta, \\ 0 & \text{otherwise}, \end{cases}$$

for all $\alpha, \beta \in \Phi$. The subspace $Z$ has dimension $\frac{|\Phi|}{2}$. 

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Proof. Notice that \( \kappa(e_\alpha, e_{-\alpha}) = 1 \) and \( \kappa(e_\alpha, e_\beta) = 0 \) for all \( \beta \neq -\alpha \). The assertion now follows from an explicit calculation. \( \square \)

Proposition 3.4. The subspace \( \mathcal{G}_0 \) is a Frobenius subalgebra of \( \text{Hom}(\mathcal{L}, \mathcal{L}) \). In fact \( \mathcal{G}_0 = \mathcal{J} \oplus \mathcal{J} \) as Frobenius algebras. This means that \( \mathcal{G}_0 = \mathcal{J} \oplus \mathcal{J} \) as vector spaces and

\[
a \bullet b = 0 \quad \text{and} \quad B(a, b) = 0
\]

for all \( a \in \mathcal{J} \) and all \( b \in \mathcal{J} \).

Proof. Since the composition \( ab \) of the \( \mathcal{L} \)-endomorphisms \( a \) and \( b \) is zero, we also have \( a \bullet b = \frac{1}{2}(ab + ba) = 0 \) and \( B(a, b) = \text{tr}(ab) = 0 \). \( \square \)

Note that the zero weight space \( S^2(\mathcal{L})_0 \) of \( S^2(\mathcal{L}) \) is a Frobenius subalgebra of \( S^2(\mathcal{L}) \) because \( \bullet \) and \( B \) are \( \mathcal{L} \)-equivariant. It is isomorphic to \( \mathcal{G}_0 \) as a Frobenius algebra.

Proposition 3.5. Let \( \zeta \) be as in Definition 1.4 and let \( S^2(\mathcal{L})_0 \) be the zero weight subspace of \( S^2(\mathcal{L}) \) with respect to \( \mathcal{H} \). Then \( \zeta \) induces an isomorphism

\[
\zeta_S : S^2(\mathcal{L})_0 \rightarrow \mathcal{G}_0 : a \mapsto \zeta(a)
\]

of Frobenius algebras.

Proof. Of course, \( \mathcal{G}_0 \) must be contained in the image of \( S^2(\mathcal{L})_0 \) under \( \zeta \). However, since both have the same dimension, \( \mathcal{G}_0 \) must actually be equal to this image. Notice that this implies that \( \zeta(h_\alpha h_\beta) \in \mathcal{G}_0 \) for all \( \alpha, \beta \in \Phi \). Now it follows immediately from the construction of the product and bilinear form on \( S^2(\mathcal{L}) \) (see Definition 1.4) that this is an isomorphism of Frobenius algebras. \( \square \)

Next, we describe the zero weight space \( \mathcal{A}_0 \) of \( \mathcal{A} \). Recall that \( \mathcal{A} \) is defined as the orthogonal complement of the \( \mathcal{L} \)-invariant subspace \( \mathcal{V} \) with respect to \( B \). Since \( B \) is \( \mathcal{L} \)-equivariant, the zero weight space \( \mathcal{A}_0 \) of \( \mathcal{A} \) is the orthogonal complement of the zero weight space \( \mathcal{V}_0 \) of \( \mathcal{V} \) in \( S^2(\mathcal{L})_0 \). By Proposition 2.7, we know that the space \( \mathcal{V}_0 \) is spanned by the elements \( 2e_\alpha e_{-\alpha} - h_\alpha h_\alpha \). Therefore, we introduce the following definition.

Definition 3.6.

(i) For each \( \alpha \in \Phi \), let \( v_\alpha := \zeta(2e_\alpha e_{-\alpha} - h_\alpha h_\alpha) = 2z_\alpha - j_\alpha \) and let \( \mathcal{V} \) be the subspace of \( \mathcal{G}_0 \) spanned by these \( v_\alpha \). Then the restriction of \( B \) to \( \mathcal{V} \times \mathcal{V} \) is non-degenerate since \( B(v_\alpha, v_\alpha) = 6 \). Let \( \mathcal{A}_0 \) be the orthogonal complement of \( \mathcal{V} \) in \( \mathcal{G}_0 \) with respect to \( B \) and let \( \pi : \mathcal{G}_0 \rightarrow \mathcal{A}_0 \) be the orthogonal projection.

(ii) Define the following product and bilinear form on \( \mathcal{A}_0 \):

\[
a \circ b := \pi(a \bullet b) \quad \text{and} \quad B_A(a, b) := B(a, b)
\]

for all \( a, b \in \mathcal{A}_0 \).

Proposition 3.7.

(i) The triple \( (\mathcal{A}_0, \circ, B_A) \) is a Frobenius algebra.

(ii) The isomorphism \( \zeta \) from Definition 1.4 induces an isomorphism

\[
\mathcal{A}_0 \rightarrow \mathcal{A}_0 : a \mapsto \zeta(a)
\]

of Frobenius algebras.
Proof. This is obvious from Definition 3.6 and Proposition 3.5 since the subspace \( \mathcal{V}_0 \) corresponds to \( \langle v_\alpha \mid \alpha \in \Phi \rangle \) under \( \zeta \).

Remark 3.8. Before we continue, let us give a summary of the notation and the obtained results on the connection between the different algebras. The goal of this section is to get a better understanding of the zero weight space \( \mathcal{A}_0 \). With respect to \( \mathcal{H} \) of the \( \mathcal{L} \)-module from Definition 2.11. Recall from Definition 2.1 that \( \mathcal{V} \) is the subrepresentation of the symmetric square \( S^2(\mathcal{L}) \) of the adjoint module of \( \mathcal{L} \) generated by \( e_\omega e_\omega \), where \( \omega \) is the highest root, and that \( \mathcal{A} \) is its complement (with respect to \( \mathcal{B} \)). By Lemma 1.6, the zero weight space \( \mathcal{A}_0 \) is the complement of the zero weight space \( \mathcal{V}_0 \) of \( \mathcal{V} \) inside the zero weight space \( S^2(\mathcal{L})_0 \) of \( S^2(\mathcal{L}) \). So our first step is to obtain a better understanding of \( S^2(\mathcal{L})_0 \). We can decompose \( S^2(\mathcal{L})_0 \) as

\[ \langle h_\alpha h_\beta \mid \alpha, \beta \in \Phi \rangle \oplus \langle e_\alpha e_{-\alpha} \mid \alpha \in \Phi \rangle. \]

The first component of this decomposition can be identified with the symmetric square \( S^2(\mathcal{H}) \) of the Cartan subalgebra \( \mathcal{H} \). Now we consider the monomorphism \( \zeta : S^2(\mathcal{L}) \to \text{Hom}(\mathcal{L}, \mathcal{L}) \). Under this monomorphism the product \( \bullet \) and bilinear form \( B \) of \( S^2(\mathcal{L}) \) correspond, by definition, to the Jordan product and trace form on \( \text{Hom}(\mathcal{L}, \mathcal{L}) \). From the results above, we have that \( \zeta \) induces an isomorphism between the following structures:

\[
\begin{align*}
\zeta : (S^2(\mathcal{H}), \bullet, B) & \to (\mathfrak{J}, \bullet, B), \\
\zeta : (\langle e_\alpha e_{-\alpha} \mid \alpha \in \Phi \rangle, \bullet, B) & \to (\mathfrak{J}, \bullet, B), \\
\zeta : (S^2(\mathcal{L})_0, \bullet, B) & \to (\mathfrak{S}_0, \bullet, B), \\
\zeta : \mathcal{V}_0 & \to \mathfrak{S}, \\
\zeta : (\mathcal{A}_0, \circ, B) & \to (\mathfrak{A}_0, \circ, B_A).
\end{align*}
\]

It turns out that \( \mathfrak{J} \) and \( \mathfrak{A}_0 \) are isomorphic as vector spaces. It will be convenient in the next section to identify both, as the elements of \( \mathfrak{J} \) can be viewed as endomorphisms of \( \mathcal{H} \).

Proposition 3.9. The restriction \( \pi_J \) of \( \pi \) to \( \mathfrak{J} \) is an isomorphism of vector spaces.

Proof. Note that for each \( \alpha \in \Phi \), we have \( \pi(j_\alpha) = 2\pi(z_\alpha) \). Thus, since \( \pi \) is surjective, its restriction to \( \mathfrak{J} \) is surjective as well. Since \( \mathfrak{J} \) and \( \mathfrak{A}_0 \) have the same dimension, the restriction of \( \pi \) to \( \mathfrak{J} \) must be an isomorphism onto \( \mathfrak{A}_0 \).

Definition 3.10. In the next section we will identify \( \mathfrak{A}_0 \) with \( \mathfrak{J} \) using the isomorphism \( \pi_J \) from Proposition 3.9. In particular, we can transfer the product \( \circ \) and the bilinear form \( B_A \) to \( \mathfrak{J} \):

\[
a \circ b := \pi_J^{-1}(\pi_J(a) \circ \pi_J(b)), \\
B_A(a, b) := B_A(\pi_J(a), \pi_J(b)),
\]

for all \( a, b \in \mathfrak{J} \). From Proposition 3.7 it follows that \( (\mathfrak{J}, \circ, B_A) \) is isomorphic to \( (\mathfrak{A}_0, \circ, B) \) as Frobenius algebras. This will be the starting point of the next section.

Remark 3.11. In the spirit of Remark 3.8, we now have that \( \zeta^{-1} \circ \pi_J \) induces an isomorphism between \( (\mathfrak{J}, \circ, B_A) \) and \( (\mathfrak{A}_0, \circ, B) \).

The Weyl group of \( \mathcal{L} \) acts naturally on the zero weight space of \( S^2(\mathcal{L}) \). Since this zero weight space is isomorphic to \( \mathfrak{S}_0 \) by Proposition 3.5, also \( \mathfrak{S}_0 \) carries the structure of a representation of the Weyl group of \( \mathcal{L} \).
Definition 3.12. Consider the natural action of the Weyl group of \( \mathcal{L} \) on the zero weight space \( S^2(\mathcal{L})_0 \) of \( S^2(\mathcal{L}) \). Due to Proposition 3.5, we can transfer this action to \( S_0 \):
\[
w \cdot s := \zeta^{-1}_s(w \cdot \zeta(s))
\]
for all \( s \in S_0 \) and \( w \in W \). Notice that the product \( \cdot \) and bilinear form \( B \) are \( W \)-equivariant. Therefore \((S_0, \cdot, B)\) is a Frobenius algebra for \( W \). It is readily verified that \( w \cdot j_\alpha = j_{w \cdot \alpha} \), \( w \cdot z_\alpha = z_{w \cdot \alpha} \) and \( w \cdot v_\alpha = v_{w \cdot \alpha} \) for all \( \alpha \in \Phi \) and \( w \in W \).

Next, we will observe that the projection \( \pi \) is \( W \)-equivariant, and therefore also its image, \( \mathfrak{A}_0 \), is \( W \)-invariant. This fact will be used in Section 7 to give \( \mathfrak{A}_0 \) the structure of an axial decomposition algebra. We will prove that this also allows us to compute the projection \( \pi \) efficiently. (This is, however, not essential for the rest of our results.)

Definition 3.13.

(i) Consider the transitive action of \( W \) on the set \( X = \{ j_\alpha \mid \alpha \in \Phi^+ \} \). Let \( O_0, O_1, \ldots, O_d \) be the orbits of \( W \) on \( X \times X \), where \( O_0 \) is the diagonal \( O_0 := \{ (x, x) \mid x \in X \} \). Define the following intersection parameters for \( 0 \leq i, j, k \leq d \):
\[
p^k_{ij} := |\{ y \in X \mid (x, y) \in O_i \text{ and } (y, z) \in O_j \}|,
\]
where \((x, z)\) is any element of \( O_k \). Note that this does not depend on the choice of \((x, z)\). (In fact, \((X, \{O_i\}_{0 \leq i \leq d})\) is an association scheme; see, e.g., [BIS4, §2.2].)

(ii) For each \( \alpha \in \Phi \), we can write \( \pi(j_\alpha) \) uniquely as
\[
j_\alpha = \sum_{\beta \in \Phi^+} \mu_{(j_\alpha, j_\beta)} v_\beta
\]
for certain constants \( \mu_{(j_\alpha, j_\beta)} \in \mathbb{C} \).

Proposition 3.14. The projection \( \pi : S_0 \rightarrow \mathfrak{A}_0 \) is \( W \)-equivariant. Moreover, \( \mu_x = \mu_y \) for all \( x, y \in O_i \), \( 0 \leq i \leq d \).

Proof. Because \( w \cdot v_\alpha = v_{w \cdot \alpha} \) for all \( \alpha \in \Phi \) and all \( w \in W \), the subspace \( \mathfrak{V} \) of \( S_0 \) is \( W \)-invariant. Since the bilinear form \( B \) is \( W \)-equivariant, the orthogonal complement \( \mathfrak{A}_0 \) of \( \mathfrak{V} \) as well as the orthogonal projection \( \pi : S_0 \rightarrow \mathfrak{A}_0 \) with respect to \( B \) is \( W \)-equivariant. Thus on the one hand we have
\[
\pi(w \cdot j_\alpha) = j_{w \cdot \alpha} + \sum_{\beta \in \Phi^+} \mu_{(j_{w \cdot \alpha}, j_\beta)} v_\beta,
\]
while on the other hand
\[
\pi(w \cdot j_\alpha) = w \cdot \pi(j_\alpha) = j_{w \cdot \alpha} + \sum_{\beta \in \Phi^+} \mu_{(j_\alpha, j_\beta)} v_{w \cdot \beta}.
\]
Since the elements of \( \{j_{w \cdot \alpha}\} \cup \{v_\beta \mid \beta \in \Phi^+\} \) are linearly independent, we have \( \mu_{(j_\alpha, j_\beta)} = \mu_{(j_{w \cdot \alpha}, j_{w \cdot \beta})} \) for all \( w \in \Phi \). \( \square \)

Definition 3.15. Let \( 1 \leq i \leq d \). By Proposition 3.14 we can define \( \mu_i := \mu_x \) for any \( x \in O_i \). Since the bilinear form \( B \) is \( W \)-equivariant, we can also write \( b_i := B(j_\alpha, j_\beta) \) for any \( (j_\alpha, j_\beta) \in O_i \).
The following proposition allows us to compute the constants \( \mu_i \) and hence the projection \( \pi \) by solving a system of \( d + 1 \) linear equations.

**Proposition 3.16.** For all \( 0 \leq k \leq d \) we have

\[
\sum_{0 \leq i, j \leq d} p_{ij}^k b_j \mu_i = b_k - 2\mu_k.
\]

Moreover, these equations uniquely determine the constants \( \mu_i \).

**Proof.** We have that

\[
\pi(j_\alpha) = j_\alpha + \sum_{0 \leq i \leq d} \sum_{\beta \in \Phi^+} \mu_i v_\beta
\]

if and only if

\[
B(\pi(j_\alpha), v_\gamma) = B(j_\alpha, v_\gamma) + \sum_{0 \leq i \leq d} \sum_{\beta \in \Phi^+} \mu_i B(v_\beta, v_\gamma) = 0
\]

for all \( \gamma \in \Phi^+ \). If \( (j_\alpha, j_\gamma) \in O_k \) then we have, by Propositions 3.4 and 3.5

\[
B(\pi(j_\alpha), v_\gamma) = -b_k + \sum_{0 \leq i, j \leq d} \sum_{(j_\beta, j_\gamma) \in O_i} \mu_i B(j_\beta, j_\gamma) + 4\mu_k B(z_\gamma, z_\gamma)
\]

\[
= -b_k + \sum_{0 \leq i, j \leq d} p_{ij}^k \mu_i b_j + 2\mu_k.
\]

This proves the statement. \( \square \)

## 4. Extending the Product

The goal of this section is to explicitly describe the algebra \( \mathcal{A} \) from Section 2. More precisely, we will write the product of any two elements of \( \mathcal{A} \) in terms of the product on the zero weight subalgebra \( \mathcal{A}_0 \) studied in Section 3. It suffices to describe the product of any two weight vectors of \( \mathcal{A} \). These weight vectors are described in Proposition 2.13.

We will use the action of the Lie algebra \( \mathcal{L} \) on \( \mathcal{A} \) to accomplish this goal. Therefore it will be essential to get a good description of this action. Since \( \mathcal{L} \) is generated by the elements \( e_\alpha \) for \( \alpha \in \Phi \), it suffices to describe the action of \( e_\alpha \) on each of the weight-\( \lambda \)-spaces. This action will of course depend on the \( W \)-orbit of \( (\alpha, \lambda) \). Inevitably, we need to distinguish between each of those orbits which makes the following proposition look daunting at first sight. However, in each of the cases, the action is very natural.
**Proposition 4.1.** Let $\alpha \in \Phi$. Recall Definition 1.3 and the linear homomorphism $\zeta$ from Definition 1.4. The linear action of $e_\alpha$ on $A$ is uniquely determined as follows.

$$e_\alpha \cdot h_1 h_2 = -e_\alpha(\kappa(\alpha, h_1)h_2 + \kappa(\alpha, h_2)h_1) = -2e_\alpha(\zeta(h_1 h_2)(\alpha)),$$

$$e_\alpha \cdot e_\beta h = h_\alpha h$$  if $\beta = -\alpha$,

$$e_\alpha \cdot e_\beta h = e_{\alpha, \beta} e_{\alpha + \beta}(h + \kappa(h, \alpha)h_\beta)$$  if $\kappa(\alpha, \beta) = -1$,

$$e_\alpha \cdot e_\beta h = -\kappa(\alpha, h)\overline{e}_\alpha e_\beta$$  if $\kappa(\alpha, \beta) = 0$,

$$e_\alpha \cdot e_\beta h = 0$$  if $\kappa(\alpha, \beta) \geq 1$,

$$e_\alpha \cdot e_{\alpha, \lambda} e_{\beta, \lambda} = \int_{\lambda + \alpha}^{\alpha} \overline{e}_{\lambda + \alpha} h_\alpha$$  if $\kappa(\alpha, \lambda) = -2$, i.e. $\lambda + \alpha \in \Phi$,

$$e_\alpha \cdot e_{\alpha, \lambda} e_{\beta, \lambda} = e_{\alpha, \alpha} e_{\alpha + \alpha} e_{\beta, \lambda}$$  if $\kappa(\alpha, \alpha) = -1$ and $\kappa(\alpha, \beta_\lambda) = 0$,

$$e_\alpha \cdot e_{\alpha, \lambda} e_{\beta, \lambda} = e_{\alpha, \beta} e_{\alpha + \alpha} e_{\beta, \lambda}$$  if $\kappa(\alpha, \alpha) = 0$ and $\kappa(\alpha, \beta_\lambda) = -1$,

$$e_\alpha \cdot e_{\alpha, \lambda} e_{\beta, \lambda} = 0$$  if $\kappa(\alpha, \lambda) \geq 0$,

where $h_1, h_2 \in H$, $h \in \beta^\perp$, $\alpha, \beta \in \Phi$ and $\lambda \in \Lambda_0$.

**Proof.** First of all, note that this enumeration exhausts all possible weight vectors of $A$. Indeed, because the root system $\Phi$ is simply laced, we have for any root $\beta \in \Phi$ that $\kappa(\alpha, \beta) \in \{-2, -1, 0, 1, 2\}$, and $\kappa(\alpha, \beta) = -2$ if and only if $\alpha = -\beta$. Similarly, if $\lambda \in \Lambda_0$, then it is not hard to check (by writing $\lambda = \beta + \gamma$ with $\kappa(\beta, \gamma) = 0$ that $\kappa(\alpha, \lambda) \in \{-2, -1, 0, 1, 2\}$, and if $\kappa(\alpha, \lambda) = -2$, then $\lambda + \alpha \in \Phi$ by Lemma 2.13. The form of these weight vectors follows from Proposition 2.13.

The statements follow from explicit calculations using the rules from Definition 1.3 and the description of the generating set for $V$ from Proposition 2.7. We will do these calculations for the case when $\alpha, \beta \in \Phi$ such that $\kappa(\alpha, \beta) = -1$. The other cases are proven analogously. Let $h \in H$. By Proposition 2.7 it follows that $\overline{e_\beta h_\beta} = 0$. Thus $e_\beta h = e_\beta(h + \kappa(h, \alpha)h_\beta)$. Now we have

$$e_\alpha \cdot e_\beta h = e_\alpha \cdot e_\beta(h + \kappa(h, \alpha)h_\beta) = [e_\alpha, e_\beta](h + \kappa(h, \alpha)h_\beta) + e_\beta[e_\alpha, h + \kappa(h, \alpha)h_\beta] = c_{\alpha, \beta} e_{\alpha + \beta}(h + \kappa(h, \alpha)h_\beta) + e_\beta 0,$$

because the projection $\text{pr} : S^2(L) \to A : v \mapsto \overline{v}$ is $L$-equivariant. □

The next step is to write the product of any zero weight vector and an arbitrary weight vector in terms of products between zero weight vectors. The following lemma will be crucial.

**Lemma 4.2.** Let $\tau$ be an automorphism of the Lie algebra $L$. Then $\tau$ induces an automorphism of the Frobenius algebra $(A, *, B)$ via

$$\tau(\ell_1 \ell_2) := \overline{\tau(\ell_1) \tau(\ell_2)},$$

for all $\ell_1, \ell_2 \in L$.

**Proof.** Since $*$ and $B$ are $L$-equivariant, this is of course true if $\tau$ is an inner automorphism of the Lie algebra $L$. By [Hum78] §16.5 we can assume that $\tau$ leaves the Cartan subalgebra $H$ and a fixed Borel subalgebra containing $H$ invariant, in other
words, that it is a graph automorphism. Thus \( \tau \) acts on the set of highest weights of the irreducible subrepresentations of \( S^2(\mathcal{L}) \). Since \( \mathcal{V} \) is the only subrepresentation of \( S^2(\mathcal{L}) \) having the double of a root as its highest weight, \( \tau \) stabilizes the subrepresentation \( \mathcal{V} \) globally. Thus \( \tau \) commutes with the projection \( \pi: S^2(\mathcal{L}) \to \mathcal{A} \) from Definition 2.11. Therefore, by the definition of \( \ast \) and \( \mathcal{B} \) (see Proposition 2.15), \( \tau \) must preserve \( \ast \) and \( \mathcal{B} \).

**Remark 4.3.** Note that any automorphism of the root system \( \Phi \), i.e., any isometry \( \tau \) of \( \mathcal{H} \) such that \( \tau(\Phi) = \Phi \), extends to an automorphism of the Lie algebra \( \mathcal{L} \) via the isomorphism theorem [Hum78, §14.2].

The \( \mathcal{L} \)-module contains three different types of weights: the zero weight, the roots \( \alpha \in \Phi \) and the sums of two orthogonal roots \( \lambda \in \Lambda_0 \). In Section 3 we described the product of two vectors of weight zero. We determine the product of a zero weight vector and a vector of weight \( \alpha \in \Phi \) in Proposition 4.4. The computation of the product of a zero weight vector and a vector of weight \( \lambda \in \Lambda_0 \) is the subject of Proposition 4.5.

**Proposition 4.4.** Let \( v, w \in \mathcal{A} \) be weight vectors of respective weights \( 0 \) and \( \alpha \in \Phi \). Then

1. \( e_{\alpha} \cdot (e_{-\alpha} \cdot w) = 2w \),
2. \( (e_{\alpha} \cdot v) \ast (e_{-\alpha} \cdot w) = 0 \) and therefore
   \[ 2v \ast w = e_{\alpha} \cdot (v \ast (e_{-\alpha} \cdot w)) \],
3. \( \mathcal{B}(v, w) = 0 \).

**Proof.**

(i) By Proposition 2.13 we can write \( w = \overline{e_{\alpha} h} \) for some \( h \in \mathcal{H} \) with \( \kappa(\alpha, h) = 0 \). By Proposition 4.1 we have
   \[ e_{\alpha} \cdot (e_{-\alpha} \cdot w) = e_{\alpha} \cdot (e_{-\alpha} \cdot \overline{e_{\alpha} h}) = e_{\alpha} \cdot (\overline{h_{\alpha} h}) = 2e_{\alpha} \cdot h = 2w. \]

(ii) Recall that \( s_{\alpha} \) is the reflection about the hyperplane orthogonal to \( \alpha \). Now \( -s_{\alpha}: \mathcal{H} \to \mathcal{H} : h \mapsto -h^{s_{\alpha}} \) is an automorphism of the root system \( \Phi \). By the isomorphism theorem [Hum78, §14.2] there exists an extension \( \tau: \mathcal{L} \to \mathcal{L} \) of \( -s_{\alpha} \) which is an automorphism of the Lie algebra \( \mathcal{L} \) and such that \( \tau(e_{\alpha}) = e_{\alpha} \). By Lemma 4.2 the automorphism \( \tau \) induces an automorphism of the Frobenius algebra \( \mathcal{A}, \ast, \mathcal{B} \).

Due to Proposition 2.13 we can write any weight vector \( x \in \mathcal{A} \) of weight \( \alpha \) as \( x = e_{\alpha} \overline{h'} \) for some \( h' \in \mathcal{H} \) with \( \kappa(\alpha, h') = 0 \). Thus we have \( \tau(x) = \tau(e_{\alpha}) \tau(h') = -e_{\alpha} \overline{h'} = -x \) for any weight vector \( x \in \mathcal{A} \) of weight \( \alpha \). Now \( e_{\alpha} \cdot v \) is a weight vector with weight \( \alpha \) and thus \( \tau(e_{\alpha} \cdot v) = -e_{\alpha} \cdot v \). As we illustrated in part (i) we can write \( e_{-\alpha} \cdot w \) as \( \overline{h_{\alpha} h} \) for some \( h \in \mathcal{H} \) with \( \kappa(\alpha, h) = 0 \). Thus
   \[ \tau(e_{-\alpha} \cdot w) = \tau(\overline{h_{\alpha} h}) = \overline{h_{\alpha} h} = -e_{-\alpha} \cdot w. \]

Because \( \ast \) is \( \mathcal{L} \)-equivariant, the product \( (e_{\alpha} \cdot v) \ast (e_{-\alpha} \cdot w) \) is a weight vector of weight \( \alpha \). As a result
   \[ \tau((e_{\alpha} \cdot v) \ast (e_{-\alpha} \cdot w)) = -(e_{\alpha} \cdot v) \ast (e_{-\alpha} \cdot w). \]
On the other hand, because $\tau$ is an automorphism of $(A, *, B)$, we have
\[
\tau((e_\alpha \cdot v) \ast (e_{-\alpha} \cdot w)) = \tau(e_\alpha \cdot v) \ast \tau(e_{-\alpha} \cdot w) = (e_\alpha \cdot v) \ast (e_{-\alpha} \cdot w).
\]

We conclude that $(e_\alpha \cdot v) \ast (e_{-\alpha} \cdot w) = 0$. It follows that
\[
2v \ast w = v \ast (e_\alpha \cdot (e_{-\alpha} \cdot w)) = e_\alpha \cdot (v \ast (e_{-\alpha} \cdot w)) - (e_\alpha \cdot v) \ast (e_{-\alpha} \cdot w) = e_\alpha \cdot (v \ast (e_{-\alpha} \cdot w)),
\]
by (i) and the $L$-equivariance of $\ast$.

(iii) This follows from the fact that $B$ is $L$-equivariant and Lemma 1.6.\[\square\]

In a similar fashion, we will now express the product of a zero weight vector and a vector of weight $\lambda \in \Lambda_0$ in terms of products of zero weight vectors.

**Proposition 4.5.** Let $v, w \in A$ be weight vectors of respective weights 0 and $\lambda \in \Lambda_0$. Recall from Definition 2.3 that $n_\lambda$ is the number of ways to write $\lambda$ as the sum of two (orthogonal) roots. Write
\[
\epsilon_\lambda := \frac{1}{2n_\lambda} \sum_{\alpha \in \Phi, \kappa(\alpha, \lambda) = 2} \bar{c}_\alpha e_{-\alpha}.
\]

Then
(i) $e_{\alpha_\lambda} \cdot (e_{-\alpha_\lambda} \cdot w) = 2w$,
(ii) $v \ast w = B(v, \epsilon_\lambda)w$,
(iii) $B(v, w) = 0$.

**Proof.**

(i) By Proposition 2.13 we can write $w = ce_{\alpha_\lambda} e_{\beta_\lambda}$ for some $c \in \mathbb{C}$. As a result of Proposition 4.1 we have
\[
e_{\alpha_\lambda} \cdot (e_{-\alpha_\lambda} \cdot w) = ce_{\alpha_\lambda} \cdot (e_{-\alpha_\lambda} \cdot \bar{c}_{\alpha_\lambda} e_{\beta_\lambda}) = -ce_{\alpha_\lambda} \cdot h_{\alpha_\lambda} e_{\beta_\lambda} = 2ce_{\alpha_\lambda} e_{\beta_\lambda} = 2w.
\]

(ii) Note that by Proposition 2.13 it suffices to prove this for $w = \bar{c}_{\alpha_\lambda} e_{\beta_\lambda}$. Since $\ast$ is $L$-equivariant, the product $v \ast w$ is a weight vector of weight $\lambda$. Because the weight-$\lambda$-space of $A$ is only 1-dimensional, $v \ast w$ must be a scalar multiple of $w$. If $a \in A$ such that $B(w, a) \neq 0$, then this scalar multiple must be
\[
\frac{B(v \ast w, a)}{B(w, a)}.
\]

We claim that we can take $a = \bar{e}_{\alpha_{-\lambda}} e_{\beta_{-\lambda}}$. Recall the definition of $B$ from Proposition 2.15 and Definition 1.4, and notice that, as in Lemma 2.14, all sums of the form $\sum_{\alpha + \beta = \lambda}$ have to be interpreted as summations over sets
{α, β}. We then have

$$B(w, a) = B(e_{α,β} e_{α-λ}, e_{α-λ})$$

$$= \frac{1}{n^2 λ} \left( \sum_{α+β=λ} f_{α,β} e_α e_β, \sum_{γ+δ=−λ} f_{γ,δ} e_γ e_δ \right)$$

$$= \frac{1}{n^2 λ} \sum_{α+β=λ} f_{α,β} f_{−α,−β} B(e_α e_β, e_{−α} e_{−β})$$

$$= \frac{f_{−α,−β}}{2n^2 λ} \sum_{α+β=λ} 1$$

$$= \frac{f_{−α,−β}}{2n^2 λ}$$

by Lemma 2.14 Proposition 2.9 (ii) and because $n_λ = n_{−λ}$.

The triple $(A, *, B)$ is a Frobenius algebra, so we have $B(v * w, a) = B(v, w * a)$. We compute $w * a$ explicitly using the definition of $*$ and • from Proposition 2.15 and Definition 1.4. We have

$$w * a = e_{α,β} e_{α-λ,β-λ}$$

$$= \frac{1}{n^2 λ} \left( \sum_{α+β=λ} f_{α,β} e_α e_β, \sum_{γ+δ=−λ} f_{γ,δ} e_γ e_δ \right)$$

$$= \frac{1}{n^2 λ} \sum_{α+β=λ} f_{α,β} f_{−α,−β} (e_α e_β • e_{−α} e_{−β})$$

$$= \frac{f_{−α,−β}}{4n^2 λ} \sum_{α+β=λ} (e_α e_{−α} + e_β e_{−β})$$

once again by Lemma 2.14 Proposition 2.9 (ii) and because $n_λ = n_{−λ}$. By Lemma 2.4 (iv) we find $w * a = \frac{f_{−α,−β}}{2n^2 λ} e_λ$. As a result, we have

$$v * w = \frac{B(v, w * a)}{B(v, a)} w$$

$$= B(v, e_λ) w.$$  

(iii) This follows from Lemma 1.6

We are now ready to “build” the Frobenius algebra $(A, *, B)$. As a first step, we describe its underlying vector space.

**Definition 4.6.** Let $L$, $H$ and Φ be as in Definition 1.3 Let $A$ be the direct sum of the following spaces:

- the space $J$ from Definition 3.1
- for each $α ∈ Φ$, a copy $H_α$ of the subspace $α^⊥ := \{ h ∈ H \mid κ(h, α) = 0 \}$ of $H$;
- a vector space with basis $\{ x_λ \mid λ ∈ Λ_0 \}$ indexed by the set $Λ_0$.

For each $h ∈ H$, we will denote its orthogonal projection onto $H_α$ by $[h]_α$. 

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Definition 4.8. Let $\mathcal{A}$ be as in Definition 2.11. For each $\lambda \in \Lambda_0$, choose roots $\alpha, \beta \in \Phi$ such that $\alpha + \beta = \lambda$. Let $\theta$ be the linear map defined by

$$\theta : \mathfrak{A} \rightarrow \mathcal{A} : \begin{cases} f_{\alpha} \mapsto h_{\alpha}h_{\alpha} \\ [h]_{\alpha} \mapsto e_{\alpha}h \\ x_{\lambda} \mapsto e_{\alpha \lambda} \\
\end{cases}$$

for all $\alpha \in \Phi$, $h \in H$ and $\lambda \in \Lambda_0$. Then $\theta$ is an isomorphism of vector spaces.

Proof. Recall from Definition 3.1 that $f_{\alpha} = \zeta(h_{\alpha}h_{\alpha})$, hence the restriction of $\theta$ to $\mathfrak{J}$ is precisely the isomorphism $\zeta^{-1} \circ \pi_{J}$ from Remark 3.11. Since also $e_{\alpha}h_{\alpha} = 0$ by Proposition 2.13, the linear map $\theta$ is well-defined. By Proposition 2.13 it follows that $\theta$ is an isomorphism.

Now we translate the action of $\mathcal{L}$ on $\mathcal{A}$ to $\mathfrak{A}$ using this isomorphism. We also define bilinear maps $\mathfrak{J} \times \mathfrak{A} \rightarrow \mathfrak{A}$ based on Propositions 4.4 and 4.5.

Definition 4.8.

(i) Transfer the action of $\mathcal{L}$ on $\mathcal{A}$ to $\mathfrak{A}$ via the isomorphism $\theta$ from Proposition 4.7.

$$\ell \cdot v := \theta^{-1}(\ell \cdot \theta(v)).$$

Note that it is possible to write this action down explicitly using Proposition 4.1.

(ii) For $\lambda \in \Lambda_0$, let $e_{\lambda} := \theta^{-1}(e_{\lambda})$, i.e.

$$e_{\lambda} := \frac{1}{4n_{\lambda}} \sum_{\alpha \in \Phi, \kappa(\alpha, \lambda) = 2} j_{\alpha}.$$

(iii) Consider the Frobenius algebra $(\mathfrak{J}, \circ, B_{\mathcal{A}})$ from Definition 3.10. Note that for $h \in \alpha^\perp$ we have $e_{-\alpha} \cdot [h]_{\alpha} = -\theta^{-1}(h_{\alpha}h_{\alpha}) = e_{\lambda} \in \mathfrak{J}$. Now define bilinear maps

$$\ast : \mathfrak{J} \times \mathfrak{A} \rightarrow \mathfrak{A},$$

$$\mathcal{B} : \mathfrak{J} \times \mathfrak{A} \rightarrow \mathbb{C},$$

such that $\theta(v \ast w) = \theta(v) \ast \theta(w)$ for all $v \in \mathfrak{J}$ and $w \in \mathfrak{A}$. More precisely, we can use Definition 3.10 and Propositions 4.4 and 4.5 to define

$$v \ast w := v \circ w \quad \text{if } w \in \mathfrak{J},$$

$$v \ast w := e_{\alpha} \cdot (v \circ (e_{-\alpha} \cdot w)) \quad \text{if } w = [h]_{\alpha} \text{ for some } h \in \alpha^\perp \text{ and } \alpha \in \Phi,$$

$$v \ast w := B_{\mathcal{A}}(v, e_{\lambda})w \quad \text{if } w = cx_{\lambda} \text{ for some } c \in \mathbb{C} \text{ and } \lambda \in \Lambda_0.$$

We prove that we can uniquely extend the maps $\ast$ and $\mathcal{B}$ to $\mathfrak{A} \times \mathfrak{A}$.

Theorem 4.9. Let $\mathcal{L}$ be a simple complex Lie algebra with root system $\Phi$ of type $A_n$ ($n \geq 3$), $D_n$ ($n \geq 4$) or $E_n$ ($n \in \{6, 7, 8\}$). Let $\mathfrak{A}$ be as in Definition 4.6 equipped with the $\mathcal{L}$-action from Definition 4.8. The maps $\ast$ and $\mathcal{B}$ from Definition 4.8 uniquely extend to $\mathfrak{A} \times \mathfrak{A}$ such that $(\mathfrak{A}, \ast, \mathcal{B})$ is a Frobenius algebra for $\mathcal{L}$. Moreover, the isomorphism $\theta$ from Proposition 4.7 induces an isomorphism of Frobenius algebras for $\mathcal{L}$ with the Frobenius algebra $(\mathfrak{A}, \ast, \mathcal{B})$ from Proposition 2.13.
Proof. The extensions of * and \( \mathcal{B} \) must be \( \mathcal{L} \)-equivariant. Let \( \alpha \in \Phi \), \( h \in \alpha^\perp \) and \( v \in \mathfrak{A} \). By definition of the action of \( \mathcal{L} \) on \( \mathfrak{A} \) and Proposition 4.3(i) we have \( e_\alpha \cdot e_{-\alpha} \cdot [h]_\alpha = [2h]_\alpha \). Because * and \( \mathcal{B} \) must be \( \mathcal{L} \)-equivariant, we have
\[
[h]_\alpha \ast w = \frac{1}{2} e_\alpha \cdot ((e_{-\alpha} \cdot [h]_\alpha) \ast w) - \frac{1}{2} (e_{-\alpha} \cdot [h]_\alpha) \ast (e_\alpha \cdot w)
\]
and
\[
\mathcal{B}([h]_\alpha, w) = \frac{1}{2} \mathcal{B}(e_{-\alpha} \cdot [h]_\alpha, e_\alpha \cdot w).
\]
Since \( e_{-\alpha} \cdot [h]_\alpha \in \mathfrak{J} \), this uniquely extends * and \( \mathcal{B} \) to \( (\mathfrak{J} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{J}_\alpha) \times \mathfrak{A} \) with \( \mathfrak{J}_\alpha \) as in Definition 4.6. Analogously, we can extend * and \( \mathcal{B} \) to \( \mathfrak{A} \times \mathfrak{A} \) by using Proposition 4.3(i) which implies that \( e_{\alpha_1} \cdot e_{-\alpha_2} \cdot x_\lambda = 2x_\lambda \).

Now consider the Frobenius algebra \( (\mathfrak{A}, *, \mathcal{B}) \) from Proposition 2.15 and the isomorphism \( \theta : \mathfrak{A} \to \mathcal{A} \) from Proposition 4.7. Then also the bilinear maps
\[
\mathfrak{A} \times \mathfrak{A} \to \mathfrak{A} : (v, w) \mapsto \theta^{-1}(\theta(v) \ast \theta(w))
\]
and
\[
\mathfrak{A} \times \mathfrak{A} \to \mathbb{C} : (v, w) \mapsto \mathcal{B}(\theta(v), \theta(w))
\]
are \( \mathcal{L} \)-equivariant extensions of * and \( \mathcal{B} \). Therefore \( (\mathfrak{A}, *, \mathcal{B}) \) must be a Frobenius algebra isomorphic via \( \theta \) with \( (\mathcal{A}, *, \mathcal{B}) \).

Remark 4.10.

(i) The proof of Theorem 4.9 is constructive. It allows us to define the product recursively starting from the Frobenius algebra \( (\mathfrak{J}, \circ, B_A) \) and some structure constants, namely the constants \( c_{\alpha, \beta} \), of the Lie algebra. This is much more efficient than the construction from Section 2 where we start with the symmetric square of the Lie algebra.

(ii) From this explicit construction, it follows that we can pick a basis for \( \mathfrak{A} \) in such a way that the structure constants for the algebra \( (\mathfrak{A}, *, \mathcal{B}) \) are integers. This allows us to define this algebra over an arbitrary field by extension of scalars.

5. The story of the unit

Let \( (\mathcal{A}, *, \mathcal{B}) \) be the Frobenius algebra from Proposition 2.15. We prove that this algebra is unital, which means that there exists an element \( 1 \in \mathcal{A} \), called a unit, such that \( 1 \ast a = a \ast 1 = a \) for all \( a \in A(\Phi) \).

From Definition 1.4 we know that \( S^2(\mathcal{L}) \) corresponds to the symmetric operators \( \mathcal{L} \to \mathcal{L} \) via \( \zeta \). From the definition of the product \( \circ \) it will be immediately obvious that the identity operator \( \text{id} : \mathcal{L} \to \mathcal{L} : \ell \mapsto \ell \) corresponds to a unit for the algebra \( (S^2(\mathcal{L}), \circ) \).

Definition 5.1. Let \( \zeta \) be as in Definition 1.4. Since the identity operator \( \text{id} \) is a symmetric operator, it is contained in the image of \( \zeta \) and we can define
\[
C_\mathcal{L} := \zeta^{-1}(\text{id}).
\]

More explicitly, let \( \{b_1, b_2, \ldots, b_m\} \) be a basis of \( \mathcal{L} \) and let \( \{b_1^*, b_2^*, \ldots, b_m^*\} \) be the basis of \( \mathcal{L} \) dual to this basis with respect to the Killing form \( \kappa \). Then
\[
C_\mathcal{L} := \sum_{1 \leq i \leq m} b_i b_i^*.
\]
Note that $C_L$ is the Casimir element of $L$ [Hum78 §6.2]. Observe that
\[
B(C_L, \ell_1 \ell_2) = \frac{1}{2} \sum_{1 \leq i \leq m} \kappa(b_i, \ell_1) \kappa(b_i^*, \ell_2) + \frac{1}{2} \sum_{1 \leq i \leq m} \kappa(b_i, \ell_2) \kappa(b_i^*, \ell_1)
= \kappa(\ell_1, \ell_2).
\]
for all $\ell_1, \ell_2 \in L$ and $B$ as in Definition 1.4. Since $B$ is non-degenerate by Proposition 2.10, this also uniquely defines $C_L$.

**Proposition 5.2.** For all $a \in S^2(L)$ we have $C_L \cdot a = a$.

**Proof.** For any $\ell_1, \ell_2 \in L$, we have
\[
C_L \cdot \ell_1 \ell_2 = \frac{1}{4} \left( \sum_i \kappa(b_i, \ell_1) b_i^* \ell_2 + \sum_i \kappa(b_i^*, \ell_1) b_i \ell_2 
+ \sum_i \kappa(b_i, \ell_2) b_i^* \ell_1 + \sum_i \kappa(b_i^*, \ell_2) b_i \ell_1 \right) 
= \ell_1 \ell_2. \quad \square
\]

Next, we prove that $C_L \in A$ and that $C_L$ is also a unit for $(A, *)$.

**Proposition 5.3.**

(i) We have $\ell \cdot C_L = 0$ for all $\ell \in L$.

(ii) We have $C_L \in A$ and $C_L \ast a = a$ for all $a \in A$.

**Proof.**

(i) For any $\ell, \ell_1, \ell_2 \in L$ we have
\[
B(\ell \cdot C_L, \ell_1 \ell_2) = B(C_L, \ell \cdot \ell_1 \ell_2) 
= B(C_L, [\ell, \ell_1] \ell_2 + \ell_1 [\ell, \ell_2]) 
= \kappa([\ell, \ell_1], \ell_2) + \kappa(\ell_1, [\ell, \ell_2]) 
= 0
\]
because $B$ and $\kappa$ are $L$-equivariant. By Proposition 2.10, the bilinear form $B$ is non-degenerate and thus $\ell \cdot C_L = 0$ for all $\ell \in L$.

(ii) For any $\alpha \in \Phi$, we have $B(C_L, e_\alpha e_\alpha) = \kappa(e_\alpha, e_\alpha) = 0$. Recall that $V$ is the $L$-representation generated by the elements $e_\alpha e_\alpha$. By (i) we have $B(C_L, v) = 0$ for all $v \in V$ and thus $C_L \in A$.

It follows from the definition of $\ast$ (see Proposition 2.15) and Proposition 5.2 that $C_L \ast a = a$ for all $a \in A$. \quad \square

We transfer the unit $C_L$ from the algebra $(A, \ast)$ to the algebra $(\mathfrak{A}, \ast)$ via the isomorphism $\theta$ from Theorem 4.9.

**Definition 5.4.** Recall $\mathfrak{A}$ from Definition 4.6 and the isomorphism $\theta$ from Theorem 4.9. Write
\[
1 = \theta^{-1}(C_L).
\]

**Theorem 5.5.** Let $(\mathfrak{A}, \ast, B)$ be as in Theorem 4.9 and $1$ as in Definition 5.4. Then $1$ is a unit for the algebra $(\mathfrak{A}, \ast)$.

**Proof.** This follows immediately from Definition 5.4, Proposition 5.3 and Theorem 4.9. \quad \square
Note that by Proposition 5.3 (i) the element $1$ is contained in the zero weight space of $\mathfrak{g}$, this is $\mathfrak{z}$. We can write down $1$ explicitly as a linear combination of the generating set $\{j_\alpha \mid \alpha \in \Phi^+\}$ for $\mathfrak{z}$. First, we prove the following lemma.

**Lemma 5.6.** Let $\beta \in \Phi$ and let $r$ be the number of positive roots $\alpha \in \Phi^+$ such that $\kappa(\beta, \alpha) = \pm 1$. Then
\[
\sum_{\alpha \in \Phi^+} j_\alpha = \frac{4 + r}{2} \text{id}_\mathcal{H},
\]
where $\text{id}_\mathcal{H} : \mathcal{H} \to \mathcal{H} : h \mapsto h$.

*Proof.* Since $\Phi$ is irreducible and simply laced, the value of $r$ is independent of the choice of $\beta$. Let $\beta \in \Phi^+$ be arbitrary. Then
\[
B(\text{id}_\mathcal{H}, j_\beta) = \text{tr}(j_\beta) = \kappa(h_\beta, h_\beta) = 2.
\]
It now follows from Proposition 3.2 that
\[
B \left( \sum_{\alpha \in \Phi^+} j_\alpha, j_\beta \right) = 4 + r = B \left( \frac{4 + r}{2} \text{id}_\mathcal{H}, j_\beta \right).
\]
Because $B$ is non-degenerate on $\mathfrak{z}$ by Proposition 3.2, we have indeed $\sum_{\alpha \in \Phi^+} j_\alpha = \frac{4 + r}{2} \text{id}_\mathcal{H}$. □

**Remark 5.7.** Note that $r = 2n_\alpha$ for all $\alpha \in \Phi$ where $n_\alpha$ is as in Definition 2.3 because $\Phi$ is simply laced.

**Proposition 5.8.** We have
\[
1 = \frac{6 + r}{2} \text{id}_\mathcal{H}.
\]

*Proof.* Let $v_1, \ldots, v_n$ be an orthonormal basis for the Cartan subalgebra $\mathcal{H}$ of $\mathcal{L}$ with respect to the Killing form $\kappa$. Then $\{v_1, \ldots, v_n\} \cup \{e_\alpha \mid \alpha \in \Phi\}$ is a basis for $\mathcal{L}$. Note that for the dual basis, we have $v_i^* = v_i$ for all $i$ and $e_\alpha^* = e_{-\alpha}$ for all $\alpha \in \Phi$. Therefore we can write $C_L$ as
\[
C_L = \sum_i v_i v_i^* + \sum_{\alpha \in \Phi} e_\alpha e_{-\alpha}.
\]
By Lemma 5.6
\[
\sum_i v_i v_i = \zeta^{-1}(\text{id}_\mathcal{H}) = \frac{2}{4 + r} \sum_{\alpha \in \Phi^+} \zeta^{-1}(j_\alpha) = \frac{2}{4 + r} \sum_{\alpha \in \Phi^+} h_\alpha h_\alpha.
\]
Recall the projection $\text{pr} : S^2(L) \to A : v \mapsto \overline{v}$ from Definition 2.11. By Proposition 2.7 we have $2e_\alpha e_{-\alpha} = h_\alpha h_\alpha$. As a result
\[
\sum_{\alpha \in \Phi} e_\alpha e_{-\alpha} = \sum_{\alpha \in \Phi^+} h_\alpha h_\alpha.
\]
We have $C_L \in A$ by Proposition 5.3 (ii) so $C_L = \overline{C_L}$. Thus
\[
C_L = \left( \frac{2}{4 + r} + 1 \right) \sum_{\alpha \in \Phi^+} h_\alpha h_\alpha = \frac{6 + r}{4 + r} \theta \left( \sum_{\alpha \in \Phi^+} j_\alpha \right).
\]
The statement now follows from Lemma 5.6. □
6. Decomposition algebras

In the following sections, we will give our algebra \((A, \ast)\) the structure of a decomposition algebra. Decomposition algebras have only recently been introduced in [DMPSVC20] as a generalization of axial algebras. We repeat the definition here. The main idea is that the algebra decomposes (as a vector space) into many decompositions \(A = \bigoplus_{x \in X} A_x\), each of which is indexed by the same set \(X\). The parts \(A_x\) of each of these decompositions multiply according to a fixed fusion law. More precisely, the fusion law will tell us when \(A_x A_y\) has a non-zero component in \(A_z\) for \(x, y, z \in X\).

**Definition 6.1** ([DMPSVC20 §2]).

(i) A fusion law is a pair \((X, \ast)\) where \(X\) is a set and \(\ast\) is a map \(X \times X \to 2^X\), where \(2^X\) denotes the power set of \(X\).

(ii) An element \(1 \in X\) is called a unit for the fusion law \((X, \ast)\) if \(1 \ast x \subseteq \{x\}\) and \(x \ast 1 \subseteq \{x\}\) for all \(x \in X\).

(iii) Let \((X, \ast)\) and \((Y, \ast)\) be fusion laws. The product of these fusions laws is the fusion law \((X \times Y, \ast)\) given by the rule
\[
(x_1, y_1) \ast (x_2, y_2) = \{(x, y) \mid x \in x_1 \ast x_2, y \in y_1 \ast y_2\}
\]
for all \(x_1, x_2 \in X\) and \(y_1, y_2 \in Y\).

An important class of fusion laws comes from abelian groups.

**Definition 6.2** ([DMPSVC20 Definitions 2.10 and 3.1]). Let \(G\) be an abelian group.

(i) Let
\[
\ast : G \times G \to 2^G : (g, h) \mapsto \{gh\}
\]
Then we call \((G, \ast)\) the group fusion law of \(G\).

(ii) Suppose that \((X, \ast)\) is an arbitrary fusion law. A \(G\)-grading of \((X, \ast)\) is a map \(\xi : X \mapsto G\) such that
\[
\xi(x \ast y) \subseteq \xi(x) \ast \xi(y),
\]
for all \(x, y \in X\).

We are ready to formulate the definition of a decomposition algebra.

**Definition 6.3** ([DMPSVC20 §4]). Let \(k\) be a field and \(\mathcal{F} = (X, \ast)\) be a fusion law.

(i) An \(\mathcal{F}\)-decomposition of a \(k\)-algebra \(A\) (not assumed to be commutative, associative or unital) is a direct sum decomposition \(A = \bigoplus_{x \in X} A_x\) (as a vector space) indexed by the set \(X\) such that \(A_x A_y \subseteq \bigoplus_{z \in x \ast y} A_z\). For the sake of readability, we will denote \(\bigoplus_{z \in Z} A_z\) by \(A_Z\) for any \(Z \subseteq X\).

(ii) An \(\mathcal{F}\)-decomposition algebra is a triple \((A, \mathcal{I}, \Omega)\) where \(A\) is a \(k\)-algebra, \(\mathcal{I}\) is an index set and \(\Omega\) is a tuple of decompositions \(A = \bigoplus A^i_z\) indexed by the set \(\mathcal{I}\).

There is a close connection between decomposition algebras with a graded fusion law and groups. We refer to [DMPSVC20 §6] for more details.
Definition 6.4. Let $G$ be an abelian group and let $(X, \oplus)$ be a $G$-graded fusion law with grading map $\xi$. Let $(A, \mathcal{I}, \Omega)$ be an $(X, \oplus)$-decomposition algebra over a field $k$. For each linear character $\chi \in \text{Hom}(G, k^\times)$ and each $i \in \mathcal{I}$, we define an automorphism of $A$ as follows:

$$\tau_{i, \chi} : A \rightarrow A : a \mapsto \chi(\xi(x))a \quad \text{for all } x \in X \text{ and all } a \in A_x^i.$$ 

Since $A = \bigoplus_{x \in X} A_x^i$, this determines $\tau_{i, \chi}$, and the fact that this map is an automorphism of $A$ follows from the fusion law and its $G$-grading. The subgroup of $\text{Aut}(A)$ generated by all $\tau_{i, \chi}$ for $i \in \mathcal{I}$ and all $\chi \in \text{Hom}(G, k^\times)$ is called the Miyamoto group of the decomposition algebra (with respect to the grading $\xi$).

The definition of decomposition algebras originates from the theory of axial algebras. These are (usually commutative) decomposition algebras where the decompositions are given by eigenspace decompositions of operators of the form

$$\text{ad}_a : A \rightarrow A : b \mapsto ab$$

for certain elements $a \in A$, called axes. Axial decomposition algebras fulfill the role of axial algebras within the framework of decomposition algebras. They are more general than axial algebras since we only demand that these operators $\text{ad}_a$ act as a scalar on each part of the decomposition, allowing the possibility that some of these scalars coincide.

Definition 6.5 (\cite[§5]{DMPSVC20}). Let $\mathcal{F} = (X, \oplus)$ be a fusion law with a distinguished unit $1 \in X$.

(i) Let $\bigoplus_{x \in X} A_x$ be an $\mathcal{F}$-decomposition of a $k$-algebra $A$. We call a nonzero element $a \in A_1$ an axis for this decomposition if there exist scalars $\nu_x$ for each $x \in X$ such that

$$ab = \nu_x b$$

for all $b \in A_x$. The map $\nu : X \mapsto k : x \mapsto \nu_x$ is called the evaluation map of the axis.

(ii) An axial decomposition algebra with evaluation map $\nu : X \mapsto k : x \mapsto \nu_x$ is a quadruple $(A, \mathcal{I}, \Omega, e)$ such that $(A, \mathcal{I}, \Omega)$ is a decomposition algebra and $e : \mathcal{I} \rightarrow A$ is a map such that for each $i \in \mathcal{I}$, $e_i := e(i)$ is an axis for the decomposition $A = \bigoplus_{x \in X} A_x^i$ with evaluation map $\nu$.

We end this section by providing a procedure to obtain a decomposition algebra out of an algebra on which a group or Lie algebra acts by automorphisms.

Lemma 6.6. Let $G$ be a finite group or complex semisimple Lie algebra. Let $A$ be an algebra for $G$, i.e., a complex $G$-representation equipped with a $G$-equivariant bilinear product. Let $A = \bigoplus_{x \in X} A_x$ be the decomposition of $A$ into $G$-isotypic components. This means that for each $x \in X$ we have $A_x = nW_x \neq 0$ for some $n \in \mathbb{N} \setminus \{0\}$ and some irreducible $G$-representation $W_x$ and $W_x \not\cong W_y$ for $x \neq y$. Define

$$\otimes : X \times X \rightarrow 2^X : (x, y) \mapsto \{ z \in X \mid \text{Hom}_G(W_x \otimes W_y, W_z) \neq 0 \},$$

where $\text{Hom}_G$ stands for the space of homomorphisms of $G$-representations. Then $\bigoplus_{x \in X} A_x$ is an $(X, \otimes)$-decomposition of $A$.

Proof. The case where $G$ is a group follows from \cite[Theorem 7.4]{DMPSVC20}. The case where $G$ is a complex semi-simple Lie algebra is proven analogously. \qed


Remark 6.7. Determining whether $\text{Hom}_G(W_x \otimes W_y, W_z) \neq 0$ can be done using character theory. If $G$ is a finite group and $\chi_x, \chi_y$ and $\chi_z$ are the respective characters of $W_x, W_y$ and $W_z$ then $\text{Hom}_G(W_x \otimes W_y, W_z) \neq 0$ if and only if $\langle \chi_x, \chi_y, \chi_z \rangle \neq 0$ where $\langle , , \rangle$ denotes the inner product on the space of class functions of $G$. A similar argument works if $G$ is a semisimple complex Lie algebra where we have to use formal characters [Hum78, §22.5]. For a group or semisimple Lie algebra $G$, we write $\text{Irr}(G)$ for its set of irreducible characters.

We will use this lemma to obtain a “global decomposition” and a class of “local decompositions” for our algebra.

Definition 6.8. Let $G$ be a finite group or a complex semisimple Lie algebra. Let $\mathcal{I}$ be an index set and $(H_i \mid i \in \mathcal{I})$ an $\mathcal{I}$-tuple of conjugate subgroups or conjugate semisimple subalgebras respectively. Let $A$ be an algebra for $G$.

(i) Apply Lemma 6.6 to $G$ and $A$ to obtain a decomposition $A = \bigoplus_{x \in X_y} A_x$ of the algebra $A$. Its components are the $G$-isotypic components of $A$ as $G$-representation. We call this decomposition the global decomposition of $A$ with respect to $G$. Denote the fusion law by $(X_y, \otimes)$. (The subscript “g” stands for “global”. In Sections 7 and 8 we will denote elements of $X_y$ by letters.)

(ii) Apply Lemma 6.6 to each $H_i$ to obtain a decomposition $A = \bigoplus_{x \in X_i} A_x$ for each $i \in \mathcal{I}$. Note that these decompositions are all conjugate since we assume the $H_i$’s to be conjugate. Therefore, we can index each of those decompositions by the same index set $X_i$. We call these decompositions the local decompositions of $A$ with respect to $(H_i \mid i \in \mathcal{I})$. Also the corresponding fusion law $(X_i, \otimes)$ does not depend on $i \in \mathcal{I}$. (The subscript “l” stands for “local”. In Sections 7 and 8 we will denote elements of $X_i$ by numbers.)

We combine the global decomposition with each of the local decompositions to obtain a new decomposition which is a “refinement” of both.

Lemma 6.9. Consider the situation of Definition 6.8. For each $x \in X_y, y \in X_i$ and $i \in \mathcal{I}$ let $A_{x, y}^i = A_x \cap A_y^i$. Let $\mathcal{F}$ be the direct product of the fusion laws $(X_y, \otimes)$ and $(X_i, \otimes)$. Then $A = \bigoplus_{x, y} A_{x, y}^i$ is an $\mathcal{F}$-decomposition of $A$ for each $i \in \mathcal{I}$.

Proof. Because each $H_i$ is a subgroup or semisimple subalgebra of $G$, we have that $\bigoplus_{y} A_{x, y}^i$ must be the decomposition of $A_x$ into $H_i$-isotypic components. This proves that $A = \bigoplus_{x, y} A_{x, y}^i$ is a decomposition of $A$. Since $A = \bigoplus_{x \in X_y} A_x$ is an $(X_y, \otimes)$-decomposition of $A$ and $A = \bigoplus_{x \in X_i} A_x$ is an $(X_i, \otimes)$-decomposition of $A$, this decomposition is an $\mathcal{F}$-decomposition.

7. Decompositions of the zero weight subalgebra

The goal of this section is to give the algebra $(\mathfrak{g}, \diamond)$ the structure of a decomposition algebra. We will describe the general procedure and give the explicit decomposition for each of the possible types ($A_n, D_n$ or $E_n$) afterwards. From now on we will only consider the product $\diamond$ on the space $\mathfrak{g}$ so will simply omit it from our notation.

7.1. The general procedure. Note that $\mathfrak{g}$ is the zero weight space of $\mathfrak{g}$ as an $\mathcal{L}$-representation. Therefore, the Weyl group $W$ of $\mathcal{L}$ acts by automorphisms on
the algebra \( \mathfrak{J} \); see Definition 3.12. We can use the ideas and terminology from Definition 6.8 to obtain a decomposition algebra.

**Definition 7.1.**

(i) For each \( \alpha \in \Phi^+ \) let \( C_W(s_\alpha) \) be the centralizer in \( W \) of the reflection \( s_\alpha \). Since \( \Phi \) is irreducible and simply laced, these subgroups are conjugate inside \( W \).

(ii) Let \( \bigoplus_{x \in X^0} J_x \) be the global decomposition of \( \mathfrak{J} \) with respect to \( W \), cf. Definition 6.8. Denote its global fusion law by \( (X^0, \star) \). Let \( \bigoplus_{x \in X^0} J^\alpha_x \) be the local decompositions of \( \mathfrak{J} \) with respect to \( (C_W(s_\alpha) \mid \alpha \in \Phi^+) \). Write \( (X^0, \star) \) for the corresponding fusion law.

(iii) As in Lemma 6.9 let \( J^\alpha_{x,y} := J_x \cap J^\alpha_y \) for \( x \in X^0 \) and \( y \in X^0 \). Let \( F_0 \) be the direct product of the fusion laws \( (X^0, \star) \) and \( (X^0, \star) \). Write \( \Omega_0 \) for the \( \Phi^+ \)-tuple of \( F_0 \)-decompositions \( \bigoplus_{x,y} J^\alpha_{x,y} \). Then \( (\mathfrak{J}, \Phi^+, \Omega_0) \) is an \( F_0 \)-decomposition algebra.

We prove that \( C_W(s_\alpha) \) is a reflection subgroup of \( W \), this is, a subgroup generated by reflections. This makes it easier to determine the local decompositions and their fusion law.

**Proposition 7.2.** For each \( \alpha \in \Phi^+ \), the centralizer \( C_W(s_\alpha) \) is a reflection subgroup of \( W \). It is generated by the reflections \( s_\beta \) for which \( \kappa(\alpha, \beta) = 0 \) or \( \beta = \pm \alpha \). Its Dynkin diagram can be obtained by removing the neighbors of the extending node from the extended Dynkin diagram of \( \Phi \).

**Proof.** Recall that the extending node of the Dynkin diagram corresponds to the negative of the highest root [Bou02, Chapter VI, §3]. Since \( W \) acts transitively on \( \Phi \), it suffices to prove this when \( \alpha \) is the highest root of \( \Phi \). For \( w \in W \), we have \( w s_\alpha = s_\alpha \) if and only if \( w \) fixes the hyperplane orthogonal to \( \alpha \). This means that \( w \) must map \( \alpha \) to \( \pm \alpha \). From [Hum78, §10.3, Lemma B] it follows that those \( w \) that fix \( \alpha \) must be a product of reflections \( s_\beta \) with \( \kappa(\alpha, \beta) = 0 \). Since \( s_\alpha(\alpha) = -\alpha \), this proves the statement. \( \square \)

**Remark 7.3.** The following table gives the type of the subsystem

\[
\{ \pm \alpha \} \cup \{ \beta \in \Phi \mid \kappa(\alpha, \beta) = 0 \}
\]

for each of the possible types of \( \Phi \).

| Type | \( W \) | \( C_W(s_\alpha) \) |
|------|------|-----------------|
| \( A_n \) (\( n \geq 3 \)) | \( A_1 \times A_{n-2} \) | \( A_1 \times A_{n-2} \) |
| \( D_n \) (\( n \geq 4 \)) | \( D_2 \times D_{n-2} \) | \( D_2 \times D_{n-2} \) |
| \( E_6 \) | \( A_1 \times A_5 \) | \( A_1 \times A_5 \) |
| \( E_7 \) | \( A_1 \times D_6 \) | \( A_1 \times D_6 \) |
| \( E_8 \) | \( A_1 \times E_7 \) | \( A_1 \times E_7 \) |

Here we use the convention that \( D_2 \cong A_1 \times A_1 \) and \( D_3 \cong A_3 \).

We will show that the fusion law \( F_0 \) is \( \mathbb{Z}/2\mathbb{Z} \)-graded and that the corresponding Miyamoto group of the \( F_0 \)-decomposition algebra from Definition 7.1 is isomorphic to \( W \). First we prove that the fusion law \( (X^0, \star) \) is \( \mathbb{Z}/2\mathbb{Z} \)-graded.
Lemma 7.4. The fusion law \((X_0^0, \oplus)\) has a non-trivial \(\mathbb{Z}/2\mathbb{Z}\)-grading \(\xi: X_0^0 \to \mathbb{Z}/2\mathbb{Z}\). Let \(\chi\) be the non-trivial linear \(\mathbb{C}\)-character of \(\mathbb{Z}/2\mathbb{Z}\). The Miyamoto map \(\tau_{\alpha, \chi}\) of \((\mathfrak{J}, \Phi^+, \Omega_0)\) is precisely the element \(s_\alpha\) in its action on \(\mathfrak{J}\).

Proof. This follows from \cite[Example 7.3]{DMPSVC20}.

This \(\mathbb{Z}/2\mathbb{Z}\)-grading induces a \(\mathbb{Z}/2\mathbb{Z}\)-grading of the fusion law \(\mathcal{F}_0\).

Definition 7.5. The \(\mathbb{Z}/2\mathbb{Z}\)-grading \(\xi\) of \((X_0^0, \oplus)\) from Lemma 7.4 induces a \(\mathbb{Z}/2\mathbb{Z}\)-grading of \(\mathcal{F}_0\):

\[\xi: \mathcal{F}_0 \to \mathbb{Z}/2\mathbb{Z}: (x, y) \mapsto \xi(y)\]

Proposition 7.6. Let \((\mathfrak{J}, \Phi^+, \Omega_0)\) be the \(\mathcal{F}_0\)-decomposition algebra from Definition 7.1. The Miyamoto group with respect to the grading of \(\mathcal{F}_0\) from Definition 7.5 is the Weyl group \(W\) in its action on \(\mathfrak{J}\).

Proof. This follows from the definitions and Lemma 7.4.

Remark 7.7. There are two ways to refine the decomposition and fusion law.

(i) Of course, many of the intersections \(J_{x,y}^a = J_x \cap J_y^a\) are trivial. Therefore, we can omit them from our fusion law.

(ii) Instead of considering the global decomposition with respect to \(W\), we can consider the global decomposition with respect to the automorphism group \(\text{Aut}(\Phi)\) of the root system \(\Phi\). If the Dynkin diagram of \(\Phi\) admits a non-trivial graph automorphism, then this group is possibly larger than \(W\) but still acts by automorphisms on \(\mathfrak{J}\). This leads to another global decomposition \(\bigoplus_{x \in X_{\alpha}} J_x\) with fusion law \((X_0^0, \oplus)\). Let \(J_{x,y,z}^a := J_x \cap J_y^a \cap J_z^a\) for \(x \in X_0^0, y \in X_g^0\) and \(z \in X_l^0\). Then \(\bigoplus_{x,y,z} J_{x,y,z}^a\) will be a decomposition of \(\mathfrak{J}\) whose fusion law is the direct product of \((X_0^0, \oplus)\), \((X_g^0, \oplus)\) and \((X_l^0, \oplus)\).

It would be cumbersome to include all the computations that were needed to obtain the explicit decompositions. Instead, we will only present the results, hoping that the reader can fill in the details if necessary. First, we give a construction for the simply laced root systems.

Example 7.8.

(i) Consider a Euclidean space \(E\) of dimension \(n + 1\) and pick an orthonormal basis \(b_0, b_1, \ldots, b_n\) for \(E\). Then

\[\Phi = \{b_i - b_j \mid 0 \leq i, j \leq n, i \neq j\}\]

is a root system in the subspace consisting of the vectors \(\sum_{i=0}^{n} \lambda_i b_i\) for which \(\sum_{i=0}^{n} \lambda_i = 0\). The following vectors form a base for \(\Phi\).

\[b_0 - b_1 \quad b_1 - b_2 \quad b_{n-2} - b_{n-1} \quad b_{n-1} - b_n\]

With respect to this base, we have \(\Phi^+ = \{b_i - b_j \mid 0 \leq i < j \leq n\}\). This root system is of type \(A_n\). The action of its Weyl group can be extended to \(E\) such that it permutes the basis elements \(\{b_0, \ldots, b_n\}\). This defines an isomorphism with the symmetric group \(S_{n+1}\) on \(n + 1\) elements.

(ii) Let \(b_1, \ldots, b_n\) be an orthonormal basis for a Euclidean space of dimension \(n\). Then

\[\Phi = \{\pm b_i \pm b_j \mid 1 \leq i, j \leq n, i \neq j\}\]
forms a root system of type $D_n$. A base for $\Phi$ is given by the following vectors.

We have $\Phi^+ = \{b_i \pm b_j \mid 1 \leq i < j \leq n\}$.

Consider the subgroup $W_n$ of $\text{GL}(E)$ generated by the elements $\theta$ for which there exists a permutation $\pi$ of $\{1, \ldots, n\}$ such that $\theta(b_i) = \pm b_{\pi(i)}$ for all $1 \leq i \leq n$. Then $W_n$ is a split extension of $S_n$ by an elementary abelian 2-group of order $2^n$. The Weyl group of $\Phi$ is an index 2 subgroup of $W_n$. It consists of those elements of $W_n$ that have determinant one. We denote this group by $W'_n$. It is an extension of $S_n$ by an elementary abelian 2-group of order $2^{n-1}$.

(iii) Let $E$ be a Euclidean space of dimension 8 and $\{b_1, \ldots, b_n\}$ an orthonormal basis for $E$. Then

$$\Phi = \{\pm b_i \pm b_j \mid 1 \leq i, j \leq 8\} \cap \left\{ \frac{1}{2} \sum_{i=1}^{8} \epsilon_i b_i \mid \epsilon_i = \pm 1, \prod_{i=1}^{8} \epsilon_i = -1 \right\}$$

is a root system of type $E_8$. Consider the roots:

$$\alpha_1 := \frac{1}{2}(-b_1 - b_2 - b_3 - b_4 - b_5 + b_6 + b_7 + b_8),$$
$$\alpha_2 := \frac{1}{2}(-b_1 - b_2 - b_3 + b_4 + b_5 + b_6 + b_7 + b_8).$$

Then following roots form a base for $\Phi$.

The roots contained in the subspace of vectors $\sum_{i=1}^{8} \lambda_i b_i$ satisfying $\lambda_7 = \lambda_8$ form a root system of type $E_7$. We have the following base for this root system.

Next, we consider the roots contained in the subspace of vectors $\sum_{i=1}^{8} \lambda_i b_i$ satisfying $\lambda_7 = \lambda_8$ and $\sum_{i=1}^{6} \lambda_i = 0$. They form a root system of type $E_6$. A
The necessary information about the character theory of Weyl groups can be found in [GP00]. The irreducible characters of Weyl groups of type $A_n$ and $D_n$ can be described using compositions and partitions.

**Definition 7.9.** A composition of a positive integer $n$ is an ordered sequence $\lambda = (\lambda_1, \ldots, \lambda_r)$ such that $|\lambda| := \sum_{i=1}^{r} \lambda_i = n$. A partition of a positive integer $n$ is an unordered sequence $\lambda = [\lambda_1, \ldots, \lambda_r]$ such that $|\lambda| := \sum_{i=1}^{r} \lambda_i = n$. For a composition $\lambda$ we write $[\lambda]$ for its corresponding partition. For each partition we define a corresponding integer $a(\lambda)$, called the $a$-invariant of $\lambda$, by the formula

$$a(\lambda) := \sum_{1 \leq i < j \leq r} \min\{\lambda_i, \lambda_j\}.$$ 

If we order the sequence $\lambda$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$ then $a(\lambda) = \sum_{i=1}^{r}(i-1)\lambda_i$.

Let us now describe the characters of the Weyl group of type $A_n$, which is isomorphic to the symmetric group on $n+1$ elements; see Example 7.8(iii). Note that we use the notation $\text{Ind}^{G}_{H}(\chi)$ and $\text{Res}^{G}_{H}(\chi)$ for induced, respectively restricted, characters for groups $H \leq G$.

**Definition 7.10.** Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be a composition of $n+1$. We denote by $S_{\lambda}$ the subgroup of $S_{n+1}$ that permutes amongst themselves the first $\lambda_1$ numbers, the next $\lambda_2$ numbers and so on. Denote by $1_{\lambda}$ the trivial character of $S_{\lambda}$. Then we can index the irreducible characters of $S_{n+1}$ by the partitions of $n+1$. We write $\chi_\mu$ for the character corresponding to the partition $\mu$. This can be done in such a way that

$$\text{Ind}^{S_{n+1}}_{S_{\lambda}}(1_{\lambda}) = \chi_{[\lambda]} + \text{a linear combination of } \chi_\mu \text{ with } a(\mu) < a([\lambda]).$$

See [GP00] Theorem 5.4.7, p. 158 for a proof of this fact.

The characters of the Weyl group of type $D_n$ can be described in a similar manner. Recall the definitions of the groups $W_n$ and $W'_n$ from Example 7.8(ii).

**Definition 7.11.** The irreducible characters of $W_n$ can be indexed by pairs $(\lambda, \mu)$ of partitions such that $|\lambda| + |\mu| = n$. We allow that $|\lambda| = n$ or $|\mu| = n$ and in that case we write $\emptyset$ for the other partition. We write $\chi_{(\lambda,\mu)}$ for the character of $W_n$ corresponding to the partition $(\lambda, \mu)$. If $\lambda \neq \mu$ then the restriction $\chi'_{(\lambda,\mu)} := \text{Res}^{W'_n}_{W_n}\chi_{(\lambda,\mu)}$ is an irreducible character of $W'_n$. We have $\chi'_{(\lambda,\mu)} = \chi'_{(\mu,\lambda)}$. If $\lambda = \mu$, then $\text{Res}^{W'_n}_{W_n}\chi_{(\lambda,\lambda)}$ is the sum of two distinct irreducible characters for $W'_n$. We will denote these by $\chi_{(\lambda,+)}$ and $\chi_{(\lambda,-)}$. These characters exhibit all irreducible characters of $W'_n$.

Next, we illustrate how we can compute the representation fusion law for these groups. We will do this for the Weyl group of type $A_n$ or, equivalently, the symmetric group on $n+1$ elements. In order to determine the representation fusion
law of a group $G$, it suffices, by Remark 6.7, to decide whether $\langle \chi_1 \chi_2, \chi_3 \rangle = 0$ for $\chi_1, \chi_2, \chi_3 \in \text{Irr}(G)$. To this end, we can use the following lemma.

**Lemma 7.12.** Let $\lambda$ be a composition of $n+1$ and $\psi, \psi' \in \text{Irr}(S_{n+1})$. Then

$$\langle \chi_{\lambda} \psi, \psi' \rangle = \langle \text{Ind}_{S_{n+1}}^{S_{n+1}} \text{Res}_{S_{n+1}}^{S_{n+1}} \psi, \psi' \rangle - \langle (\text{Ind}_{S_{n+1}}^{S_{n+1}} 1_{\lambda} - \chi_{\lambda}) \psi, \psi' \rangle.$$

**Proof.** We have

$$\chi_{\lambda} \psi = (\text{Ind}_{S_{n+1}}^{S_{n+1}} 1_{\lambda}) \psi - (\text{Ind}_{S_{n+1}}^{S_{n+1}} 1_{\lambda} - \chi_{\lambda}) \psi$$

$$= \text{Ind}_{S_{n+1}}^{S_{n+1}} \text{Res}_{S_{n+1}}^{S_{n+1}} \psi - (\text{Ind}_{S_{n+1}}^{S_{n+1}} 1_{\lambda} - \chi_{\lambda}) \psi.$$

This proves the statement. \qed

We can compute the irreducible constituents of $\text{Ind}_{S_{n+1}}^{S_{n+1}} \text{Res}_{S_{n+1}}^{S_{n+1}} \psi$ and $\text{Ind}_{S_{n+1}}^{S_{n+1}} 1_{\lambda}$ using the Littlewood–Richardson rule; see [GP00, §6.1]. Recall that the partitions corresponding to the irreducible constituents of $\text{Ind}_{S_{n+1}}^{S_{n+1}} 1_{\lambda} - \chi_{\lambda}$ have $a$-invariant less than $a(\lambda)$. Therefore we can use induction on the $a$-invariant of $\lambda$ to compute $\langle \chi_{\lambda} \psi, \psi' \rangle$.

Determining the representation fusion law for $W_n$ is a bit more cumbersome but the same technique applies. The necessary background can be found in [GP00] §5.5, §5.6 and §6.1.

We are now ready to give local and global decompositions of $(\mathcal{J}, \circ)$ and their fusion law. Note that this does only depend on the structure of $\mathcal{J}$ as a representation for the Weyl group $W$.

For each of the possible types, we will give the following information:

(i) some more notation about the root system that allows a description of the decompositions;

(ii) the global decomposition $\bigoplus_{x \in X_0} J_x$ (elements of $X_0$ will be denoted by letters);

(iii) the characters and dimensions of the $W$-representations $J_x$ for $x \in X_0$;

(iv) the global fusion law $(X_0, \otimes)$;

(v) the elements of the full decomposition $\bigoplus_{x \in \mathcal{F}} J_x$ with respect to a root $\alpha$; from this the local decomposition $\bigoplus_{i \in \mathcal{F}} J_i$ can be derived (elements of $X_i$ will be denoted by numbers);

(vi) the characters and dimensions of the $C(W)(s_{\alpha})$-representations $J_i$ for $i \in X_0$;

(vii) the local fusion law $(X_i, \otimes)$.

### 7.2. Type $A_n$.

(i) We use the description of the root system of type $A_n$ from Example 7.8(1).

Denote the orthogonal projection onto $\langle \Phi \rangle = \langle b_i - b_j \mid 0 \leq i, j \leq n \rangle$ of a basis vector $b_i$ by $b'_i$. We identify $\mathcal{J}$ with $S^2(\mathcal{H}) = S^2(\langle \Phi \rangle)$ using Proposition 3.5.

We will give the full decomposition $\bigoplus_{x \in \mathcal{F}} J_x$ with respect to the root $\alpha := b_0 - b_n$. This is the highest root with respect to the base from Example 7.8(1).

We use the isomorphisms $W \cong S_{n+1}$ and $C_W(s_{\alpha}) \cong S_2 \times S_{n-1}$ to describe the characters.
(ii) 
\[ J_a := \sum_{i=0}^{n} b_i^t b_i^s, \]
\[ J_b := \langle b_i^t b_j^t - b_i^s b_j^s | 0 \leq i, j \leq n \rangle, \]
\[ J_c := \langle (b_i - b_j)(b_k - b_l) | 0 \leq i, j, k, l \leq n, \{i, j\} \cap \{k, l\} = \emptyset \rangle. \]

(iii) 
Table 1.

| Component | Character | Dimension |
|-----------|-----------|-----------|
| \( J_a \) | \( \chi_{[n+1]} \) | 1         |
| \( J_b \) | \( \chi_{[n,1]} \) | \( n \)    |
| \( J_c \) | \( \chi_{[n-1,2]} \) | \( \frac{n^2-n-2}{2} \) |

(iv) 
Table 2.

The fusion law \( (X^0_y, \circledast) \) for type \( A_n \). Entries marked with \( \dagger \) should be left out for \( n = 3 \).

| \( \circledast \) | \( a \) | \( b \) | \( c \) |
|------------------|---|---|---|
| \( a \)          | \( a \) | \( b \) | \( c \) |
| \( b \)          | \( b \) | \( a,b,c \) | \( b,c^\dagger \) |
| \( c \)          | \( c \) | \( b,c^\dagger \) | \( a,b^\dagger,c \) |

(v) 
\[ J_{a,1}^\alpha := \langle \sum_{i=0}^{n} b_i^t b_i^s \rangle, \]
\[ J_{b,1}^\alpha := \langle (n+1)(b_0^t b_0^s + b_n^t b_n^s) - 2 \sum_{k=0}^{n} b_k^t b_k^s \rangle, \]
\[ J_{b,2}^\alpha := \langle b_k^t b_k^s - b_l^t b_l^s | 1 \leq k, l \leq n-1 \rangle, \]
\[ J_{b,4}^\alpha := \langle b_0^t b_0^s - b_n^t b_n^s \rangle, \]
\[ J_{c,1}^\alpha := \langle n b_0^t b_0^s + n b_n^t b_n^s + n(n-1)b_0^t b_n^s - \sum_{k=0}^{n} b_k^t b_k^s \rangle, \]
\[ J_{c,2}^\alpha := \langle ((n-1)(b_0^t + b_n^t) + 2(b_k^t + b_l^t))(b_k^s - b_l^s) | 1 \leq k, l \leq n-1 \rangle, \]
\[ J_{c,3}^\alpha := \langle (b_{k_1}^t - b_{l_1}^t)(b_{k_2}^t - b_{l_2}^t) | 1 \leq k_1, k_2, l_1, l_2 \leq n-1, \{k_1, l_1\} \cap \{k_2, l_2\} = \emptyset \rangle, \]
\[ J_{c,5}^\alpha := \langle (b_0^t - b_n^t)(b_k^t - b_l^t) | 1 \leq k, l \leq n-1 \rangle. \]
Table 3.

Characters and dimensions for the local decomposition of $\mathfrak{J}$ for type $A_n$

| Component | Character | Dimension |
|-----------|-----------|-----------|
| $J_1^\alpha$ | $3 \cdot \chi[2] \times \chi[n-1]$ | $3 \cdot 1$ |
| $J_2^\alpha$ | $2 \cdot \chi[2] \times \chi[n-2,1]$ | $2 \cdot (n-2)$ |
| $J_3^\alpha$ | $\chi[2] \times \chi[n-3,2]$ | $\frac{n^2-5n+4}{2}$ |
| $J_4^\alpha$ | $\chi[1,1] \times \chi[n-1]$ | $1$ |
| $J_5^\alpha$ | $\chi[1,1] \times \chi[n-2,1]$ | $n-2$ |

Table 4.

The fusion law $(X_0, \otimes)$ for type $A_n$. Entries marked with † should be left out for $n = 3$ and those marked with ‡ should be left out for $n \in \{3, 4\}$.

| ⊗ | 1 | 2 | 3‡ | 4 | 5 |
|---|---|---|----|---|---|
| 1 | 1 | 2 | 3‡ | 4 | 5 |
| 2 | 2 | 1, 2‡, 3‡ | 2‡, 3‡ | 5 | 4, 5‡ |
| 3‡ | 3‡ | 2‡, 3‡ | 1‡, 2‡, 3‡ | 0 | 5‡ |
| 4 | 4 | 5 | 0 | 1 | 2 |
| 5 | 5 | 4, 5‡ | 5‡ | 2 | 1, 2‡, 3‡ |

7.3. Type $D_n$.

(i) The root system of type $D_n$ is described in Example 7.8(ii). Once again, we identify $\mathfrak{J}$ with $S^2(\mathcal{H}) = S^2((\Phi))$ using the isomorphism from Proposition 3.5. Local decompositions will be given with respect to the root $\alpha := b_1 + b_2$, the highest root with respect to the base from Example 7.8(ii). We use the isomorphisms $W \cong W'_n$ and $C_W(s_\alpha) \cong W'_2 \times W'_{n-2}$ to describe the characters of $W$ and $C_W(s_\alpha)$. For the global decomposition, we make a distinction between $n = 4$ and $n > 4$. For the local decomposition, we restrict to $n > 6$. The given decomposition remains a decomposition for $n \in \{4, 5, 6\}$ but the components are not isotypic.

(ii) $n = 4$

\[ J_a := \left\langle \sum_{i=1}^{4} b_i b_i \right\rangle, \]
\[ J_b := \left\langle b_i b_i - b_j b_j \mid 1 \leq i, j \leq 4 \right\rangle, \]
\[ J_c := \left\langle b_i b_j + b_k b_l \mid \{i, j, k, l\} = \{1, 2, 3, 4\} \right\rangle, \]
\[ J_d := \left\langle b_i b_j - b_k b_l \mid \{i, j, k, l\} = \{1, 2, 3, 4\} \right\rangle. \]

$n > 4$

\[ J_a := \left\langle \sum_{i=1}^{n+1} b_i b_i \right\rangle, \]
\[ J_b := \left\langle b_i b_i - b_j b_j \mid 1 \leq i < j \leq n \right\rangle, \]
\[ J_c := \left\langle b_i b_j \mid 1 \leq i < j \leq n \right\rangle. \]
(iii) \( n = 4 \)

**Table 5.**
Characters and dimensions for the global decomposition of \( J \) for type \( D_4 \)

| Component | Character | Dimension |
|-----------|-----------|-----------|
| \( J_a \) | \( \chi([4],\emptyset) \) | 1         |
| \( J_b \) | \( \chi([3,1],\emptyset) \) | 3         |
| \( J_c \) | \( \chi([2],+) \) | 3         |
| \( J_d \) | \( \chi([2],-) \) | 3         |

\( n > 4 \)

**Table 6.**
Characters and dimensions for the global decomposition of \( J \) for type \( D_n \) (\( n > 4 \))

| Component | Character | Dimension |
|-----------|-----------|-----------|
| \( J_a \) | \( \chi([n],\emptyset) \) | 1         |
| \( J_b \) | \( \chi([n-1,1],\emptyset) \) | \( n-1 \) |
| \( J_c \) | \( \chi([n-2],[2]) \) | \( \frac{n(n-1)}{2} \) |

(iv) \( n = 4 \)

**Table 7.**
The fusion law \( (X^0_g, \circledast) \) for type \( D_4 \)

\[
\begin{array}{cccccc}
\circledast & a & b & c & d \\
\hline
a & a & b & c & d \\
b & b & a, b & d & c \\
c & c & d & a, c & b \\
d & d & c & b & a, d \\
\end{array}
\]

\( n > 4 \)

**Table 8.**
The fusion law \( (X^0_g, \circledast) \) for type \( D_n \) (\( n > 4 \))

\[
\begin{array}{ccc}
\circledast & a & b & c \\
\hline
a & a & b & c \\
b & b & a, b & c \\
c & c & c & a, b, c \\
\end{array}
\]
(v) \( n > 6 \)

\[
J_{\alpha,1}^\alpha := \left( \sum_{i=1}^{n+1} b_ib_i \right),
\]

\[
J_{b,1}^\alpha := \langle nb_1b_2 - 2 \sum_{k=1}^{n} b_kb_k \rangle,
\]

\[
J_{b,2}^\alpha := \langle b_kb_l - b_lb_k \mid 3 \leq k, l \leq n \rangle,
\]

\[
J_{b,6}^\alpha := \langle b_1b_1 - b_2b_2 \rangle,
\]

\[
J_{c,1}^\alpha := \langle b_1b_2 \rangle,
\]

\[
J_{c,3}^\alpha := \langle b_kb_l \mid 3 \leq k < l \leq n \rangle,
\]

\[
J_{c,4}^\alpha := \langle (b_1 + b_2)b_k \mid 3 \leq k \leq n \rangle,
\]

\[
J_{c,5}^\alpha := \langle (b_1 - b_2)b_k \mid 3 \leq k \leq n \rangle.
\]

(vi) \( n > 6 \)

Table 9. Characters and dimensions for the local decomposition of \( \mathfrak{J} \) for type \( D_n (n > 6) \)

| Component | Character | Dimension |
|-----------|-----------|-----------|
| \( J_1 \) | \( 3 \cdot \chi'_2(\emptyset) \times \chi'_{(n-2},\emptyset) \) | \( 3 \cdot 1 \) |
| \( J_2 \) | \( \chi'_2(\emptyset) \times \chi'_3(\emptyset) \) | \( n - 3 \) |
| \( J_3 \) | \( \chi'_2(\emptyset) \times \chi'_{(n-2},2) \) | \( n^2 - 5n + 6 \) |
| \( J_4 \) | \( \chi_{(1,1)} \times \chi'_{(n-2},2) \) | \( n - 2 \) |
| \( J_5 \) | \( \chi_{(1,1),+} \times \chi'_{(n-2},2) \) | \( n - 2 \) |
| \( J_6 \) | \( \chi'_2(\emptyset) \times \chi'_{(n-2},\emptyset) \) | \( 1 \) |

(vii) \( n > 6 \)

Table 10. The fusion law \((\otimes, \oplus)\) for type \( D_n (n > 6) \)

| \( \otimes \) | 1 | 2 | 3 | 4 | 5 | 6 |
|-----------|---|---|---|---|---|---|
| 1         | 1 | 2 | 3 | 4 | 5 | 6 |
| 2         | 2 | 1,2 | 3 | 4 | 5 | . |
| 3         | 3 | 3 | 1,2,3 | 4 | 5 | . |
| 4         | 4 | 4 | 4 | 1,2,3 | 6 | 5 |
| 5         | 5 | 5 | 5 | 6 | 1,2,3 | 4 |
| 6         | 6 | 5 | 4 | 1 | . | . |

Remark 7.13. Observe that the local decomposition with respect to \( \alpha = b_1 + b_2 \) is the same as the one with respect to \( b_1 - b_2 \) up to the order of the terms. This is due to the fact that the centralizers of their corresponding reflections are equal. As a result, the local fusion law is \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \)-graded (as can be observed in Table 10).

7.4. Type \( E_n \).

(i) We use the description and notation for the root systems of type \( E_n \) from Example 7.8 (iii). As usual, we identify \( \mathfrak{J} \) with \( S^2(\mathcal{H}) \). Local decompositions will be given with respect to the highest root corresponding to the base
from Example 7.8 [iii]. Those are the roots $b_7 + b_8$, $b_7 + b_8$ and $b_7 - b_8$ for $n = 6, 7, 8$ respectively. The characters of the Weyl groups are given in the notation from [GP00, Table C.4 to C.6]. Also recall the Frobenius form $B_A$ for $\mathcal{J}$ from Definition 8.10.

(ii) \[
\begin{align*}
J_a &:= \langle \text{id} \in \mathcal{J} \rangle, \\
J_b &:= \langle v \in \mathcal{J} \mid B_A(v, \text{id}) = 0 \rangle.
\end{align*}
\]

(iii) $n = 6$\hspace{2cm} Table 11.

Characters and dimensions for the global decomposition of $\mathcal{J}$ for type $E_6$

| Component | Character | Dimension |
|-----------|-----------|-----------|
| $J_a$     | $1_p$     | 1         |
| $J_b$     | $20_p$    | 20        |

$n = 7$\hspace{2cm} Table 12.

Characters and dimensions for the global decomposition of $\mathcal{J}$ for type $E_7$

| Component | Character | Dimension |
|-----------|-----------|-----------|
| $J_a$     | $1_a$     | 1         |
| $J_b$     | $27_a$    | 27        |

$n = 8$\hspace{2cm} Table 13.

Characters and dimensions for the global decomposition of $\mathcal{J}$ for type $E_8$

| Component | Character | Dimension |
|-----------|-----------|-----------|
| $J_a$     | $1_x$     | 1         |
| $J_b$     | $35_x$    | 35        |

(iv)\hspace{2cm} Table 14.

The fusion law $(\mathcal{X}_g^0, \oplus)$ for type $E_n$

\[
\begin{array}{c|cc}
\oplus & a & b \\
\hline
a & a & b \\
b & b & a, b
\end{array}
\]
(v) \( n = 6 \) 

\[
J_{a,1}^\alpha := \left( \sum_{i=1}^{8} b_i b_i \right), \\
J_{b,1}^\alpha := \left( (b_7 + b_8)(b_7 + b_8) - \frac{1}{3} \sum_{i=1}^{8} b_i b_i \right), \\
J_{b,2}^\alpha := \left( b'_i b'_i - b'_j b'_j \mid 1 \leq i < j \leq 6 \right), \\
J_{b,3}^\alpha := \left( (b_i - b_j)(b_k - b_l) \mid 1 \leq i, j, k, l \leq 6, \{i, j\} \cap \{k, l\} = \emptyset \right), \\
J_{b,5}^\alpha := \left( (b_7 + b_8)(b_i - b_j) \mid 1 \leq i < j \leq 6 \right).
\]

\( n = 7 \) 

\[
J_{a,1}^\alpha := \langle \text{id} \rangle, \\
J_{b,1}^\alpha := \langle (b_7 + b_8)(b_7 + b_8) - \frac{2}{7} \sum_{i=1}^{8} b_i b_i \rangle, \\
J_{b,2}^\alpha := \langle b_i b_i - b_j b_j \mid 1 \leq i < j \leq 6 \rangle, \\
J_{b,3}^\alpha := \langle b_i b_j \mid 1 \leq i < j \leq 6 \rangle, \\
J_{b,5}^\alpha := \langle (b_7 + b_8)b_i \mid 1 \leq i \leq 6 \rangle.
\]

\( n = 8 \) 

\[
J_{a,1}^\alpha := \langle \text{id} \rangle, \\
J_{b,1}^\alpha := \langle \alpha \alpha - \frac{1}{4} \sum_{i=1}^{8} b_i b_i \rangle, \\
J_{b,3}^\alpha := \langle vw - \frac{\kappa(v, w)}{7} \sum_{i=1}^{8} b_i b_i + \frac{\kappa(v, w)}{14} \alpha \alpha \mid v, w \in \alpha^\perp \rangle, \\
J_{b,5}^\alpha := \langle \alpha v \mid v \in \alpha^\perp \rangle.
\]

(vi) \( n = 6 \) 

Table 15. 

| Component | Character | Dimension |
|-----------|-----------|-----------|
| \( J_1^\alpha \) | \( 2 \cdot \chi[2] \times \chi[6] \) | \( 2 \cdot 1 \) |
| \( J_2^\alpha \) | \( \chi[2] \times \chi[5,1] \) | 5 |
| \( J_3^\alpha \) | \( \chi[2] \times \chi[4,2] \) | 9 |
| \( J_5^\alpha \) | \( \chi[1,1] \times \chi[5,1] \) | 5 |
\section*{Table 16.}
Characters and dimensions for the local decomposition of $\widehat{\mathfrak{g}}$ for type $E_7$

| Component | Character | Dimension |
|-----------|-----------|-----------|
| $J_1^\alpha$ | $2 \cdot \chi[2] \times \chi((6),\emptyset)$ | $2 \cdot 1$ |
| $J_2^\alpha$ | $\chi[2] \times \chi((5,1),\emptyset)$ | $5$ |
| $J_3^\alpha$ | $\chi[2] \times \chi((4,1),\emptyset)$ | $15$ |
| $J_5^\alpha$ | $\chi_{[1,1]} \times \chi((5),(1))$ | $6$ |

\section*{Table 17.}
Characters and dimensions for the local decomposition of $\widehat{\mathfrak{g}}$ for type $E_8$

| Component | Character | Dimension |
|-----------|-----------|-----------|
| $J_1^\alpha$ | $2 \cdot \chi[2] \times 1_a$ | $2 \cdot 1$ |
| $J_3^\alpha$ | $\chi[2] \times 27_a$ | $27$ |
| $J_5^\alpha$ | $\chi_{[1,1]} \times 7_a$ | $7$ |

\section*{(vii) Table 18.}
The fusion law ($X_0^l$, $\oplus$) for type $E_6$

| $\oplus$ | 1 | 2 | 3 | 5 |
|-----------|---|---|---|---|
| 1 | 1 | 2 | 3 | 5 |
| 2 | 2 | 1,2,3 | 2,3 | 5 |
| 3 | 3 | 2,3 | 1,2,3 | 5 |
| 5 | 5 | 5 | 5 | 1,2,3 |

\section*{Table 19.}
The fusion law ($X_0^l$, $\oplus$) for type $E_7$

| $\oplus$ | 1 | 2 | 3 | 5 |
|-----------|---|---|---|---|
| 1 | 1 | 2 | 3 | 5 |
| 2 | 2 | 1,2 | 3 | 5 |
| 3 | 3 | 3 | 1,2,3 | 5 |
| 5 | 5 | 5 | 5 | 1,2,3 |

\section*{Table 20.}
The fusion law ($X_0^l$, $\oplus$) for type $E_8$

| $\oplus$ | 1 | 3 | 5 |
|-----------|---|---|---|
| 1 | 1 | 3 | 5 |
| 5 | 3 | 1,3 | 5 |
| 5 | 5 | 5 | 1,3 |
8. A decomposition of the algebra

In this section we will give the whole algebra \((\mathfrak{A}, \ast)\) the structure of a decomposition algebra. Once again, we apply the techniques from Definition 6.8 and Lemma 6.9. This time, we will use the fact that \((\mathfrak{A}, \ast)\) is an algebra for \(\mathcal{L}\). As for the decompositions of the zero weight subalgebra, we will first illustrate the general procedure and give the results for each of the possible types afterwards.

8.1. The general procedure. In order to use Definition 6.8 we look for a class of conjugate subalgebras of \(\mathcal{L}\) to obtain local decompositions. Recall the notation for a Chevalley basis from Definition 1.3. Since we used the reflection subgroups \(C_W(s_\alpha) = N_W(\langle s_\alpha \rangle)\) to obtain local decompositions of \(\mathfrak{J}\) in Section 7 a natural candidate is the subalgebras of the form \(N_{\mathcal{L}}(\langle h_\alpha, e_\alpha, e_{-\alpha} \rangle)\).

Definition 8.1. Let \(\mathfrak{J}\) be the class of subalgebras of \(\mathcal{L}\) conjugate to the subalgebra \(\langle h_\alpha, e_\alpha, e_{-\alpha} \rangle\) for some \(\alpha \in \Phi\). Note that \(i \cong \mathfrak{sl}_2(\mathbb{C})\) for each \(i \in \mathfrak{J}\).

Proposition 8.2. Let \(\alpha \in \Phi\). Consider the subalgebra \(i = \langle e_\alpha, e_{-\alpha} \rangle\) of \(\mathcal{L}\). Then
\[
N_{\mathcal{L}}(i) = \mathcal{H} \oplus \langle e_\beta | \beta \in \{\pm \alpha\} \cup \{\beta \in \Phi | \kappa(\alpha, \beta) = 0\} \rangle.
\]
In particular, \(N_{\mathcal{L}}(i)\) is reductive for each \(i \in \mathfrak{J}\). Moreover, all subalgebras \(N_{\mathcal{L}}(i)\) are conjugate.

Proof. Let \(\mathcal{H}\) be the Cartan subalgebra from Definition 1.3. Clearly we have \(\mathcal{H} \leq N_{\mathcal{L}}(i)\). Thus \(\mathcal{H}\) normalizes \(N_{\mathcal{L}}(i)\). As a result, the subalgebra \(N_{\mathcal{L}}(i)\) must be a direct sum of common eigenspaces of the adjoint action of \(\mathcal{H}\). This means that \(N_{\mathcal{L}}(i)\) is of the form
\[
\mathcal{H} \oplus \langle e_\beta | \beta \in S \rangle
\]
for some \(S \subseteq \Phi\). The first assertion follows because \(e_\beta \in N_{\mathcal{L}}(i)\) for \(\beta \in \Phi\) if and only if \(\beta \in \{\pm \alpha\} \cup \{\beta \in \Phi | \kappa(\alpha, \beta) = 0\}\). Now \(N_{\mathcal{L}}(i)\) is reductive by [Bou05, §VIII.3 Proposition 2]. Since all elements \(i \in \mathfrak{J}\) are conjugate, the same is true for the subalgebras \(N_{\mathcal{L}}(i)\).

Definition 8.3. Let \(\mathfrak{J}\) be as in Definition 8.1. For each \(i \in \mathfrak{J}\), let
\[
\mathcal{L}_i := [N_{\mathcal{L}}(i), N_{\mathcal{L}}(i)].
\]
By Proposition 8.2 and [Bou99, Chapter I, §6.3, Proposition 5] the subalgebra \(\mathcal{L}_i\) is semisimple. For example, we have
\[
\mathcal{L}_{\langle e_\alpha, e_{-\alpha} \rangle} = \langle e_\beta | \beta \in \{\pm \alpha\} \cup \{\beta \in \Phi | \kappa(\alpha, \beta) = 0\} \rangle.
\]
Since all elements \(i \in \mathfrak{J}\) are conjugate, so are all \(\mathcal{L}_i\) for \(i \in \mathfrak{J}\).

Remark 8.4. The type of \(\mathcal{L}_{\langle e_\alpha, e_{-\alpha} \rangle}\) is given by Remark 7.3. Note that the Weyl group of \(\mathcal{L}_{\langle e_\alpha, e_{-\alpha} \rangle}\) is precisely \(C_W(s_\alpha)\) by Proposition 7.2.

Definition 8.5.
(i) Let \(\bigoplus_{x \in X} A_x\) be the global decomposition of \(\mathfrak{A}\) with respect to \(\mathcal{L}\). Denote its fusion law by \((X_g, \otimes)\). Let \(\bigoplus_{x \in X} A_x^i\) be the local decomposition of \(\mathfrak{A}\) with respect to \((\mathcal{L}_i | i \in \mathfrak{J})\). Let \((X_l, \otimes)\) be the corresponding fusion law.

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(ii) For each \( x \in X_g \) and \( y \in X_I \), let \( A^1_{x,y} := A_x \cap A_y \) as in Lemma 8.6. Let \( \mathcal{F} \) be the direct product of the fusion laws \((X_g, \cdot)\) and \((X_I, \cdot)\). Then we know by Lemma 8.6 that \( \bigoplus_{x,y} A^1_{x,y} \) is an \( \mathcal{F} \)-decomposition of \( \mathfrak{A} \). Let \( \Omega \) be the \( I \)-tuple of these decompositions. Then \((\mathfrak{A}, I, \Omega)\) is an \( \mathcal{F} \)-decomposition algebra.

As in Section 7, we will define a \( \mathbb{Z}/2\mathbb{Z} \)-grading of the fusion law \( \mathcal{F} \) and determine the corresponding Miyamoto group of the \( \mathcal{F} \)-decomposition algebra \((\mathfrak{A}, I, \Omega)\). In Section 7 we obtained the \( \mathbb{Z}/2\mathbb{Z} \)-grading from Lemma 7.4 implicitly by restriction to the central subgroup \( \langle s_a \rangle \leq C_W(\langle s_a \rangle) \). Similarly, we will obtain a \( \mathbb{Z}/2\mathbb{Z} \)-grading by restricting to \( i \leq L_i \).

**Definition 8.6.**

(i) Let \( i \in I \). Recall that \( i \cong \mathfrak{sl}_2(\mathbb{C}) \). Let \( h \in i \) be one of the two coroots with respect to some Cartan subalgebra of \( i \). Write \( V \) for the standard representation of \( i \). Then the eigenvalues for the action of the element \( h \) on \( V \) are 1 and \(-1\). Therefore the eigenvalues of the adjoint action of \( h \) on \( V^{\otimes n} \) are odd (respectively even) integers if \( n \) is odd (respectively even). Since any irreducible representation of \( i \) is some subrepresentation of \( V^{\otimes n} \) for some \( n \in \mathbb{N} \), this divides the irreducible representations into two parts. The irreducible representations of \( i \) for which the eigenvalues of the action of \( h \) are odd (respectively even), are called odd (respectively even) representations. Also note that the tensor product of two odd (or two even) representations is a direct sum of even representations and that the tensor product of an odd and an even representation is a direct sum of odd representations. See also [FH91] §11.1, p. 150.

(ii) Obviously, \( i \) is an ideal of \( L_i \). Therefore, the \( L_i \)-representations \( A^1_x \) for \( x \in X_I \), restricted to \( i \), are isomorphic to \( nW_x \) for some irreducible representation \( W_x \) of \( i \) and some \( n \in \mathbb{N} \). Now define

\[
\xi : X_I \to \mathbb{Z}/2\mathbb{Z} : x \mapsto \begin{cases} 
0 & \text{if } W_x \text{ is even,} \\
1 & \text{if } W_x \text{ is odd.}
\end{cases}
\]

**Lemma 8.7.** The map \( \xi \) induces a non-trivial \( \mathbb{Z}/2\mathbb{Z} \)-grading of the fusion law \((X_I, \cdot)\).

**Proof.** The tensor product of two odd (or two even) representations is a direct sum of even representations and the tensor product of an odd and an even representation is a direct sum of odd representations. Since \((X_I, \cdot)\) is the representation fusion law on \( X_I \), it follows that \( \xi \) defines a grading of \((X_I, \cdot)\). To prove that this grading is non-trivial it suffices to show that the \( \mathfrak{A} \) has an odd irreducible \( \langle e_\alpha, e_{-\alpha} \rangle \)-subrepresentation. Equivalently, we need to show that one of the co-roots of \( \langle e_\alpha, e_{-\alpha} \rangle \), e.g. \( h_\alpha \), has an eigenvector in \( \mathfrak{A} \) with an odd eigenvalue. Since e.g. \( h_\alpha \cdot [h]_\beta = [h]_\beta \) for any roots \( \alpha, \beta \in \Phi \) with \( \kappa(\alpha, \beta) = 1 \) and any \( h \in \mathcal{H} \), it follows that the grading is non-trivial. \( \square \)

This grading induces a non-trivial grading of \( \mathcal{F} \).

**Definition 8.8.** The \( \mathbb{Z}/2\mathbb{Z} \)-grading \( \xi \) of \((X_I, \cdot)\) induces a \( \mathbb{Z}/2\mathbb{Z} \)-grading of \( \mathcal{F} \):

\[
\xi : X_g \times X_I \to \mathbb{Z}/2\mathbb{Z} : (x, y) \mapsto \xi(y).
\]
Now we determine the corresponding Miyamoto group of the $\mathcal{F}$-decomposition algebra $(A(\Phi), \mathfrak{I}, \Omega)$. It turns out that this Miyamoto group is isomorphic to the group of inner automorphisms (this is, the adjoint Chevalley group) of $L$. We repeat some terminology about inner automorphisms. We refer to [Ste68] for more details.

**Definition 8.9.** Let $L$ be an arbitrary complex, semisimple Lie algebra. Let $\rho : L \to \mathfrak{gl}(V)$ be a representation of $L$. Suppose that $\ell \in L$ such that $\rho(\ell)$ is nilpotent which means that $\rho(\ell)^k = 0$ for some $k \in \mathbb{N} \setminus \{0\}$. Let

$$\exp(\rho(\ell)) := \sum_{i=0}^{k-1} \frac{\rho(\ell)^i}{i!}.$$ 

Then $\exp(\rho(\ell))$ acts as an automorphism by conjugation on the Lie algebra $\rho(L)$. We call the subgroup of $\text{GL}(V)$ generated by these automorphisms for all possible choices of $\ell$ the Chevalley group of $(L, \rho)$ and denote it by $\text{Int}(L, \rho)$. The isomorphism class of $\text{Int}(L, \rho)$ only depends on the lattice $\Lambda(\rho)$ spanned by the weights of $\rho$. If $\Lambda(\rho)$ is equal to the weight lattice of $L$, then we call $\text{Int}(L, \rho)$ the fundamental Chevalley group of $L$ and denote it by $\hat{\text{Int}} L$. For any representation $\rho$ of $L$ there exists an epimorphism $\hat{\text{Int}} L \to \text{Int}(L, \rho)$ such that the kernel is contained in the center of $\hat{\text{Int}} L$. Any representation $\rho$ of $L$ can therefore be viewed as a representation for $\hat{\text{Int}}(L)$. On the other hand, if $\Lambda(\rho)$ is equal to the root lattice of $L$, then $\text{Int} L := \text{Int}(L, \rho)$ is called the adjoint Chevalley group of $L$. For any representation $\rho$ of $L$, there exist an epimorphism $\text{Int}(L, \rho) \to \text{Int} L$ with kernel contained in the center of $\text{Int}(L, \rho)$. If $L$ is simple, then so is $\text{Int} L$.

The following example explains how the grading coming from odd and even representations of $\mathfrak{sl}_2(\mathbb{C})$ gives rise to involutions.

**Example 8.10.** Consider a Lie algebra $i \cong \mathfrak{sl}_2(\mathbb{C})$ together with its standard 2-dimensional representation $\rho$. Then $\hat{\text{Int}} i = \text{Int}(i, \rho) \cong \text{SL}_2(\mathbb{C})$ while $\text{Int} i \cong \text{PSL}_2(\mathbb{C})$. Denote the unique non-trivial element in the center of $\hat{\text{Int}} i$ by $\sigma_1$. Then $\sigma_1$ acts trivially on the representation $\rho$ of $i$ (viewed as a representation for $\hat{\text{Int}} i$) if and only if the weight lattice $\Lambda(\rho)$ is equal to the root lattice of $i$. More precisely, $\sigma_1$ acts as 1 (respectively $-1$) on the even (respectively odd) representations of $i$.

Now we are ready to determine the Miyamoto group of $(\mathfrak{A}, \mathfrak{I}, \Omega)$.

**Theorem 8.11.** Let $\Phi$ be an irreducible simply laced root system. Consider the $\mathcal{F}$-decomposition algebra $(\mathfrak{A}, \mathfrak{I}, \Omega)$ from Definition 8.7. The Miyamoto group of this algebra corresponding to the $\mathbb{Z}/2\mathbb{Z}$-grading of $\mathcal{F}$ from Definition 8.8 is $\text{Int}(L, \mathfrak{A})$, the adjoint Chevalley group of type $\Phi$.

**Proof.** From the definition of the $\mathbb{Z}/2\mathbb{Z}$-grading of $\mathcal{F}$ (Definitions 8.9 and 8.8) and Example 8.10 it follows that the action of $\tau_{i, \chi}$ (with $\chi$ the non-trivial character of $\mathbb{Z}/2\mathbb{Z}$) corresponds to the action of $\sigma_1$. This action is non-trivial by Lemma 8.7. Since the index set $\mathfrak{I}$ is closed under the action of $\text{Int} L$, the elements $\{\sigma_1 | i \in \mathfrak{I}\}$ form a conjugacy class of involutions of $\text{Int}(L, \mathfrak{A})$. Since the weights of $\mathfrak{A}$ are...
contained in the root lattice $\mathcal{L}$, we have that $\text{Int}(\mathcal{L}, \mathfrak{A})$ is isomorphic to the adjoint Chevalley group of type $\Phi$ and therefore simple. So the group generated by the Miyamoto maps must be isomorphic to it.

\begin{remark}
Note that we have never used any information about the algebra product $\ast$ on $\mathfrak{A}$. Indeed, the technique that we used here is applicable to any algebra on which the Lie algebra $\mathcal{L}$ acts (non-trivially) by derivations, for example the Lie algebra itself. It will be possible to give the algebra the structure of a decomposition algebra with a $\mathbb{Z}/2\mathbb{Z}$-graded fusion law. If this grading is non-trivial then the corresponding Miyamoto group will be a Chevalley group of type $\Phi$ (but not necessarily adjoint).
\end{remark}

Let us now give an overview of some of the techniques that we used to explicitly obtain the local and global decompositions.

In Section 7, we described the decompositions of the zero weight space of $\mathfrak{A}$. We can use the results from [Bro95, Corollary 1] and [Ree98] to extend these decompositions of $\mathfrak{J}$ to decompositions in $\mathfrak{A}$. They introduce the terminology of small modules which means that the double of a root is not a weight of the module. More precisely, they prove that, if $V$ is a small module for a semisimple Lie algebra $\mathcal{L}$, then its zero weight space is (almost always) irreducible as a representation for the Weyl group of $\mathcal{L}$. Note that $\mathfrak{A}$ is small as a module for $\mathcal{L}$ or $\mathcal{L}_1$. If $V$ is now an irreducible subrepresentation of $\mathfrak{J}$ for $W$ (resp. $C_W(s_\alpha)$), then it follows from these results that the $\mathcal{L}$-module (resp. $\mathcal{L}_{\langle e_\alpha, e_{-\alpha} \rangle}$-module) generated by $V$ is also irreducible. Moreover, we can determine the highest weight of this module from the character of $V$. This already helps to get a lot of components of the global and local decompositions of $\mathfrak{A}$.

The representation fusion laws for $\mathcal{L}$ and $\mathcal{L}_1$ can be determined using the results from [FH91, §25.3].

For each of the types $A_n$, $D_n$ and $E_n$, we will continue to use the notation introduced in the corresponding subsection of Section 7. In particular, we recall the index sets $X^0_g$ and $X^0_l$ for the local and global decomposition. The global decomposition can then be given as follows. For each $x \in X^0_g$ we let $A_x$ be the $\mathcal{L}$-submodule generated by $J_x$. From the discussion above, it follows that each $A_x$ is an isotypic component of the $\mathcal{L}$-modules. In fact, these are all the isotypic components of $\mathfrak{A}$ as $\mathcal{L}$-module. Therefore we can take $X^0_g = X_g$. We will give the following additional information about the decompositions of $\mathfrak{A}$.

(i) For each isotypic component $A_x$ for $x \in X^0_g$, we will give its highest weight and dimension. We say that $A_x$ has highest weight $n \cdot w$ if $A_x$ is the isotypic component corresponding to the dominant weight $w$ and the weight $w$ has multiplicity $n$ in $A_x$. The weight $w$ is given with respect to the basis of fundamental weights. We have ordered this basis with respect to the numbering of the nodes of the Dynkin diagram from Fig. 1.

(ii) We give the global fusion law $(X_g, \otimes)$.

(iii) The full decomposition $\bigoplus_{(x,i) \in \mathcal{F}} A^1_{x,i}$ with respect to $i := \langle e_\alpha, e_{-\alpha} \rangle$ is given, where $\alpha$ is the highest root as in Section 7. For $(x,i) \in X^0_g \times X^0_l$ we let $A^1_{(x,i)}$
Figure 1. Dynkin diagrams of the irreducible simply laced root systems

be the $\mathcal{L}_i$-submodule generated by $J_{(x,i)}^\alpha$. We extend $X_i^0$ to $X_i$ and give $A_{(x,i)}^1$ for each $x \in X_i$ and $i \in X_i \setminus X_i^0$ for which $A_{(x,i)}^1 \neq 0$.

(iv) We give the highest weight and dimension of each of the components of the local decomposition $\bigoplus_{i \in X_l} A_i^1$.

(v) Lastly, the local fusion law $(X_l, \odot)$ is given.

8.2. Type $A_n$. We restrict to the case where $n > 3$ for the global decomposition and fusion law and to $n > 5$ for the local decomposition and fusion law.

(i)

| Component | Highest weight | Dimension |
|-----------|----------------|-----------|
| $A_a$     | $(0,0,\ldots,0)$ | 1         |
| $A_b$     | $(1,0,\ldots,0,1)$ | $n(n+2)$ |
| $A_c$     | $(0,1,0,\ldots,0,1,0)$ | $\frac{(n+2)(n+1)^2(n-2)}{4}$ |

(ii)

| $\oplus$ | $a$ | $b$ | $c$ |
|-----------|-----|-----|-----|
| $a$       | $a$ | $b$ | $c$ |
| $b$       | $b$ | $a,b,c$ | $b,c$ |
| $c$       | $c$ | $b,c$ | $a,b,c$ |
(iii)  
\[ A^i_{b,8} := \langle [b'_i + b'_i][b_{b_i - b_i}], [b'_{n+1} + b'_i][b_{b_{n+1} - b_i}] \mid 1 \leq i \leq n-1 \rangle, \]
\[ A^i_{b,9} := \langle [b'_i + b'_i][b_{b_i - b_i}], [b'_{n+1} + b'_i][b_{b_{n+1} - b_i}] \mid 1 \leq i \leq n-1 \rangle, \]
\[ A^i_{c,6} := \langle x_{b_i + b_{n+1} - b_i} \mid 1 \leq i < j \leq n-1 \rangle, \]
\[ A^i_{c,7} := \langle x_{b_i + b_j - b_i + b_{n+1}} \mid 1 \leq i < j \leq n-1 \rangle, \]
\[ A^i_{c,8} := \langle [b'_i][b_{n+1} - b_i], [b'_{n+1}][b_{b_i - b_i}] \mid 1 \leq i \leq n-1 \rangle, \]
\[ A^i_{c,9} := \langle [b'_i][b_{n+1}], [b'_{n+1}][b_{b_i - b_i}] \mid 1 \leq i \leq n-1 \rangle, \]
\[ A^i_{c,10} := \langle [b'_i][b_{b_i - b_j}], [b'_{n+1}][b_{b_{n+1} - b_j}] \mid x_{b_i + b_j - b_i - b_j} \rangle \]
\[ \leq i, j, k \leq n-1, \{i, j, k\} = 3 \rangle, \]
\[ A^i_{c,11} := \langle [b'_i][b_{b_i - b_j}], [b'_{n+1}][b_{b_{n+1} - b_j}] \mid x_{b_i + b_j - b_i - b_j} \rangle \]
\[ \leq i, j, k \leq n-1, \{i, j, k\} = 3 \rangle. \]

(iv)

Table 23. Highest weights and dimensions for the local decomposition of \( \mathfrak{A} \) for type \( A_n \) \( (n > 5) \)

| Component | Highest weight | Dimension |
|-----------|----------------|-----------|
| \( A^1_i \) | \( 3 \cdot (0; 0, \ldots, 0) \) | \( 3 \cdot 1 \) |
| \( A^2_i \) | \( 2 \cdot (0; 1, 0, \ldots, 0, 1) \) | \( 2 \cdot n(n-2) \) |
| \( A^3_i \) | \( (0; 0, 1, 0, \ldots, 0, 1, 0) \) | \( \frac{n(n-1)^2(n-4)}{4} \) |
| \( A^4_i \) | \( (2; 0, \ldots, 0) \) | \( 3 \) |
| \( A^5_i \) | \( (2; 1, 0, \ldots, 0, 1) \) | \( 3n(n-2) \) |
| \( A^6_i \) | \( (0; 0, \ldots, 0, 1, 0) \) | \( \frac{(n-1)(n-2)}{2} \) |
| \( A^7_i \) | \( (0; 0, 1, 0, \ldots, 0) \) | \( \frac{(n-1)(n-2)}{2} \) |
| \( A^8_i \) | \( (2; 1; 0, \ldots, 0, 1) \) | \( 2 \cdot 2(n-1) \) |
| \( A^9_i \) | \( 2 \cdot (1; 1; 0, \ldots, 0, 1) \) | \( 2 \cdot 2(n-1) \) |
| \( A^{10}_i \) | \( (1; 1, 0, \ldots, 0, 1, 0) \) | \( n(n-1)(n-3) \) |
| \( A^{11}_i \) | \( (1; 0, 1, 0, \ldots, 0, 1) \) | \( n(n-1)(n-3) \) |
8.3. Type $D_n$. We restrict to the case where $n > 5$ for the global decomposition and fusion law and to $n > 7$ for the local decomposition and fusion law.

(i)

Table 25.
Highest weights and dimensions for the global decomposition of $\mathfrak{a}$ for type $D_n$ ($n > 5$)

| Component | Highest weight | Dimension |
|-----------|----------------|-----------|
| $A_a$     | $(0,0,\ldots,0)$ | $1$       |
| $A_b$     | $(2,0,\ldots,0)$ | $(2n-1)(n+1)$ |
| $A_c$     | $(0,0,0,1,0,\ldots,0)$ | $(2n-3)(2n-1)(n-1)n$ |

(ii)

Table 26.
The fusion law $(X_g, \ast)$ for type $D_n$ ($n > 5$)

\[
\begin{array}{c|ccc}
\oplus & a & b & c \\
\hline
a & a & b & c \\
b & b & a & c \\
c & c & c & a, b, c \\
\end{array}
\]

(iii)

\[
\begin{align*}
A_{b,7}^i & := \langle [b_1]_{\pm b_1 \pm b_i}, [b_2]_{\pm b_2 \pm b_i} \mid 3 \leq i \leq n \rangle, \\
A_{c,7}^i & := \langle [b_2]_{\pm b_1 \pm b_i}, [b_1]_{\pm b_2 \pm b_i} \mid 3 \leq i \leq n \rangle, \\
A_{c,8}^i & := \langle [b_k]_{\pm b_1 \pm b_i}, [b_k]_{\pm b_2 \pm b_i}, x_{\pm b_1 \pm b_i \pm b_k \pm b_i}, x_{\pm b_2 \pm b_i \pm b_k \pm b_i} \mid 3 \leq i, j, k \leq n, |\{i, j, k\}| = 3 \rangle.
\end{align*}
\]
(iv)

Table 27.
Highest weights and dimensions for the local decomposition of \( \mathfrak{A} \) for type \( D_n (n > 7) \)

| Component | Highest weight | Dimension |
|-----------|----------------|-----------|
| \( A_1 \) | \( 3 \cdot (0;0;0,\ldots,0) \) | \( 3 \cdot 1 \) |
| \( A_2 \) | \( (0;0;2,0,\ldots,0) \) | \( (2n-5)(n-1) \) |
| \( A_3 \) | \( (0;0;0,0,0,1,0,\ldots,0) \) | \( (2n-7)(2n-5)(n-3)(n-2) \) |
| \( A_4 \) | \( (2;0;0,1,0,\ldots,0) \) | \( 3(2n-5)(n-2) \) |
| \( A_5 \) | \( (0;2;0,1,0,\ldots,0) \) | \( 3(2n-5)(n-2) \) |
| \( A_6 \) | \( (2;2;0,\ldots,0) \) | \( 9 \) |
| \( A_7 \) | \( 2 \cdot (1;1;1,0,\ldots,0) \) | \( 2 \cdot 8(n-2) \) |
| \( A_8 \) | \( (1;1;0,0,1,0,\ldots,0) \) | \( \frac{8(2n-5)(n-3)(n-2)}{3} \) |

(v)

Table 28.
The fusion law \( (X_1, \circledast) \) for type \( D_n (n > 7) \)

| \( \oplus \) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---------|---|---|---|---|---|---|---|---|
| 1       | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 2       | 2 | 1,2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 3       | 3 | 3 | 1,2,3 | 4 | 5 | 6 | 7 | 8 |
| 4       | 4 | 4 | 1,2,3,4 | 6 | 5 | 7,8 | 7,8 |
| 5       | 5 | 5 | 5 | 6 | 1,2,3,5 | 4 | 7,8 | 7,8 |
| 6       | 6 | 6 | 5 | 4 | 1,6 | 7 | 8 | |
| 7       | 7 | 7 | 8 | 7,8 | 7,8 | 7 | 1,2,4,5,6 | 3,4,5 |
| 8       | 8 | 8 | 7,8 | 7,8 | 7,8 | 8 | 3,4,5 | 1,2,3,4,5,6 |

8.4. Type \( E_n \).

(i) \( n = 6 \)

Table 29.
Highest weights and dimensions for the global decomposition of \( \mathfrak{A} \) for type \( E_6 \)

| Component | Highest weight | Dimension |
|-----------|----------------|-----------|
| \( A_a \) | \( (0,0,0,0,0,0) \) | 1 |
| \( A_b \) | \( (1,0,0,0,0,1) \) | 650 |

\( n = 7 \)

Table 30.
Highest weights and dimensions for the global decomposition of \( \mathfrak{A} \) for type \( E_7 \)

| Component | Highest weight | Dimension |
|-----------|----------------|-----------|
| \( A_a \) | \( (0,0,0,0,0,0,0) \) | 1 |
| \( A_b \) | \( (0,0,0,0,0,1,0) \) | 1539 |
\( n = 8 \)

Table 31.

Highest weights and dimensions for the global decomposition of \( \mathfrak{A} \) for type \( E_8 \)

| Component | Highest weight | Dimension |
|-----------|----------------|-----------|
| \( A_a \) | \((0,0,0,0,0,0,0,0)\) | 1         |
| \( A_b \) | \((1,0,0,0,0,0,0,0)\) | 3875      |

\[ (ii) \]

Table 32.

The fusion law \((X_g \odot \) for type \( E_n \)

\[ \odot \] 

\[ a \] \[ a \] \[ b \] \[ b \] 

\[ a \] \[ a \] \[ b \] \[ b \] 

\[ b \] \[ b \] \[ a \] \[ b \] 

\( n = 6 \)

Let \( S = \{ \beta \in \Phi \mid \kappa(\alpha, \beta) = \pm 1 \} \).

\( A_{b,12}^i := \langle [b_7 + b_8]_\beta \mid \beta \in S \rangle, \)

\( A_{b,13}^i := \langle [b_i - b_j]_\beta, x_{\beta+b_i-b_j} \mid 1 \leq i < j \leq 6, \beta \in S, \) \)

\( \kappa(b_i + b_j, \beta) = 1, \kappa(b_i - b_j, \beta) = 0 \rangle, \)

\( A_{b,14}^i := \langle [b_i - b_j]_\beta, x_{\beta+b_i-b_j} \mid 1 \leq i < j \leq 6, \beta \in S, \) \)

\( \kappa(b_i + b_j, \beta) = -1, \kappa(b_i - b_j, \beta) = 0 \rangle. \)

\( n = 7 \)

Also here we let \( S = \{ \beta \in \Phi \mid \kappa(\alpha, \beta) = \pm 1 \} \).

\( A_{b,9}^i := \langle [b_7 + b_8]_\beta \mid \beta \in S \rangle, \)

\( A_{b,10}^i := \langle [\gamma]_\beta, x_{\beta+\gamma} \rangle \gamma = \pm b_i \pm b_j \text{ for } 1 \leq i < j \leq 6, \kappa(\gamma, \beta) = 0 \rangle. \)

\( n = 8 \)

\( A_{b,6}^i := \langle [\alpha]_\beta \mid \beta \in \Phi, \kappa(\alpha, \beta) = \pm 1 \rangle, \)

\( A_{b,7}^i := \langle [\gamma]_\beta, x_{\beta+\gamma} \mid \gamma, \beta \in \Phi, \kappa(\alpha, \beta) = \pm 1, \kappa(\gamma, \alpha) = \kappa(\gamma, \beta) = 0 \rangle. \)

\( n = 6 \)

Table 33.

Highest weights and dimensions for the local decomposition of \( \mathfrak{A} \) for type \( E_6 \)

| Component | Highest weight | Dimension |
|-----------|----------------|-----------|
| \( A_1^1 \) | \(2 \cdot (0;0,0,0,0,0)\) | \(2 \cdot 1\)         |
| \( A_2^1 \) | \((0;1,0,0,0,1)\) | 35         |
| \( A_3^1 \) | \((0;0,1,0,1,0)\) | 189        |
| \( A_4^1 \) | \((2;1,0,0,0,1)\) | 105        |
| \( A_5^1 \) | \((1;0,0,1,0,0)\) | 40         |
| \( A_6^1 \) | \((1;1,1,0,0,0)\) | 140        |
| \( A_7^1 \) | \((1;0,0,0,1,1)\) | 140        |
Table 34.
Highest weights and dimensions for the local decomposition of $\mathfrak{A}$ for type $E_7$

| Component | Highest weight | Dimension |
|-----------|----------------|-----------|
| $A_1^1$   | $2 \cdot (0; 0, 0, 0, 0, 0, 0)$ | $2 \cdot 1$ |
| $A_2^1$   | $(0; 2, 0, 0, 0, 0, 0)$ | $77$ |
| $A_3^1$   | $(0; 0, 0, 0, 1, 0, 0)$ | $495$ |
| $A_5^1$   | $(2; 0, 1, 0, 0, 0, 0)$ | $198$ |
| $A_9^1$   | $(1; 0, 0, 0, 0, 1, 0)$ | $64$ |
| $A_{10}^1$| $(1; 1, 0, 0, 0, 0, 1)$ | $704$ |

Table 35.
Highest weights and dimensions for the local decomposition of $\mathfrak{A}$ for type $E_8$

| Component | Highest weight | Dimension |
|-----------|----------------|-----------|
| $A_1^1$   | $2 \cdot (0; 0, 0, 0, 0, 0, 0, 0)$ | $2 \cdot 1$ |
| $A_3^1$   | $(0; 0, 0, 0, 0, 1, 0, 0)$ | $1539$ |
| $A_5^1$   | $(2; 1, 0, 0, 0, 0, 0, 0)$ | $399$ |
| $A_6^1$   | $(1; 0, 0, 0, 0, 0, 1, 0)$ | $112$ |
| $A_{10}^1$| $(1; 0, 1, 0, 0, 0, 0, 0)$ | $1824$ |

Table 36.
The fusion law $(X_l \otimes)$ for type $E_6$

| $\otimes$ | 1    | 2    | 3    | 5    | 12   | 13   | 14   |
|-----------|------|------|------|------|------|------|------|
| 1         | 1    | 2    | 3    | 5    | 12   | 13   | 14   |
| 2         | 2    | 1, 2 | 3    | 2, 3 | 5    | 12, 13, 14 | 12, 13 | 12, 14 |
| 3         | 3    | 2, 3 | 1, 2 | 3    | 5    | 12, 13, 14 | 12, 13 | 12, 14 |
| 5         | 5    | 5    | 5    | 1, 2 | 3, 5 | 12, 13, 14 | 12, 13 | 12, 14 |
| 12        | 12   | 12, 13, 14 | 12, 13, 14 | 12, 13, 14 | 1, 2, 3, 5 | 2, 3, 5 | 2, 3, 5 |
| 13        | 13   | 12, 13 | 12, 13, 14 | 12, 13, 14 | 1, 2, 3, 5 | 3    | 1, 2, 3, 5 |
| 14        | 14   | 12, 14 | 12, 13, 14 | 12, 14, 2, 3, 5 | 1, 2, 3, 5 | 3    |

Table 37.
The fusion law $(X_l \otimes)$ for type $E_7$

| $\otimes$ | 1    | 2    | 3    | 5    | 9    | 10   |
|-----------|------|------|------|------|------|------|
| 1         | 1    | 2    | 3    | 5    | 9    | 10   |
| 2         | 2    | 1, 2 | 3    | 5    | 10   | 9, 10|
| 3         | 3    | 3    | 1, 2 | 3    | 5    | 9, 10| 9, 10|
| 5         | 5    | 5    | 5    | 1, 2 | 3, 5 | 9, 10| 9, 10|
| 9         | 9    | 10   | 9, 10| 9, 10| 1, 3, 5| 2, 3, 5|
| 10        | 10   | 9, 10| 9, 10| 9, 10| 2, 3, 5| 1, 2, 3, 5|
Proposition 9.2. The triple $(\mathfrak{A}, \circ, B_p)$ is a Frobenius algebra for $\mathcal{L}$ for any choice of the parameter $p \in \mathbb{C}$ with $1 + p\dim(\mathcal{L}) \neq 0$.

Proof. The $\mathcal{L}$-equivariance of $\circ$ follows from the definition of $\circ$ and Proposition 2.15. Next, notice that for all $a_1, a_2 \in A'$, we have $B(a_1 * a_2, 1) = B(a_1, a_2)$ by Proposition 2.15. So if we decompose $a_1 * a_2$ as $\alpha 1 + b$ with $\alpha \in \mathbb{C}$ and $b \in A'$, then $B(a_1, a_2) = \alpha B(1, 1)$. It now follows that

$$B_p((c_11 + a_1) \circ (c_21 + a_2), (c_31 + a_3))$$

$$= c_1c_2c_3B(1, 1)$$

$$+ (1 + pB(1, 1))(c_1B(a_2, a_3) + c_2B(a_1, a_3) + c_3B(a_1, a_2) + B(a_1 * a_2, a_3))$$

$$= B_p((c_{\pi(1)}1 + a_{\pi(1)}) \circ (c_{\pi(2)}1 + a_{\pi(2)}), (c_{\pi(3)}1 + a_{\pi(3)}))$$

for all $c_1, c_2, c_3 \in \mathbb{C}$, $a_1, a_2, a_3 \in A'$ and any permutation $\pi$ of $\{1, 2, 3\}$. The non-degeneracy of $B_p$ follows from the non-degeneracy of $B$ if $1 + pB(1, 1) \neq 0$. From the construction of the unit in Section 5, it follows that $B(1, 1) = \text{tr}(\text{id}_{\mathcal{L}}) = \dim(\mathcal{L})$. \qed

Next, we show that also the structure as a decomposition algebra from Definition 5.5 transfers to this new algebra product.

Proposition 9.3. Let $\mathfrak{I}$, $\Omega$ and $\mathcal{F}$ be as in Definition 8.2. Then $(\mathfrak{A}, \mathfrak{I}, \Omega)$ is an $\mathcal{F}$-decomposition algebra for the algebra product $\circ$ on $\mathfrak{A}$.

In this section, we direct some more attention to the case where $\Phi$ is of type $E_8$ since this was the original algebra of interest. We prove that $\mathfrak{A}$ belongs to a one-parameter family of algebras. Each of these can be given the structure of an axial decomposition algebra.

Definition 9.1. Let $\mathfrak{A}$, equipped with the $\mathcal{L}$-equivariant bilinear product $*$ and bilinear form $B$, be as in Section 4. Let $1$ be the unit for $(\mathfrak{A}, *)$ constructed in Section 5 and $A'$ the orthogonal complement of $(1)$ with respect to $B$. Consider a parameter $p \in \mathbb{C}$. Define the following product and bilinear form on $\mathfrak{A}$ that depends on the parameter $p$:

$$(c_11 + a_1) \circ (c_21 + a_2) := (c_1c_2 + pB(a_1, a_2))1 + c_1a_2 + c_2a_1 + a_1 * a_2,$$

$$B_p(c_11 + a_1, c_21 + a_2) := c_1c_2B(1, 1) + B(a_1, a_2)(1 + pB(1, 1)),$$

for all $c_1, c_2 \in \mathbb{C}$ and $a_1, a_2 \in A'$. Note that we retrieve the original product $*$ and bilinear form $B$ if we put $p = 0$.

The important properties of $*$ and $B$ from Proposition 2.15 still hold for this new product and bilinear form.

Table 38.

| $\circ$ | 1 | 3 | 5 | 6 | 7 |
|---|---|---|---|---|---|
| 1 | 1 | 3 | 5 | 6 | 7 |
| 3 | 3 | 1,3 | 5 | 6,7 | 6,7 |
| 5 | 5 | 5 | 1,3,5 | 6,7 | 6,7 |
| 6 | 6 | 6,7 | 6,7 | 1,3,5 | 3,5 |
| 7 | 7 | 6,7 | 6,7 | 3,5 | 1,3,5 |

9. An algebra for $E_8$
Proof. Let $X_g, X_l, L_i, A_x, A^i_y$ and $A^i_{x,y}$ for $x \in X_g$, $y \in X_l$ and $i \in I$ be as in Definition 8.5. Let $e_g \in X_g$ (resp. $e_l \in X_l$) be the element corresponding to the trivial $L$-module (resp. $L_i$-module). Then $A_{e_g} = A^i_{e_g,e_l} = (1)$ (as can be seen from Section 8.4). Note that $(e_g, e_l)$ is a unit for the fusion law $\mathcal{F}$ and since 1 is still a unit for the algebra $(\mathfrak{A}, \odot)$ we have indeed $A^i_{e_g,e_l} \odot A_{x,y} \subseteq A_{x,y}$ for all $(x, y) \in X_g \times X_l$. Also, since $\mathcal{B}$ is $L$-equivariant, we have $\mathcal{B}(A^i_{e_g,e_l}, A^i_{x,y}) = 0$ for all $(x, y) \in X_g \times X_l \setminus \{(e_g, e_l)\}$. Thus, for all $x_1, x_2 \in X_g$ and $y_1, y_2 \in X_l$ such that $(x_1, y_1) \neq (e_g, e_l) \neq (x_2, y_2)$:

$$A^i_{x_1,y_1} \odot A^i_{x_2,y_2} \subseteq A^i_{x_1,y_1} \ast A^i_{x_2,y_2} + \mathcal{B}(A^i_{x_1,y_1}, A^i_{x_2,y_2})(1) \subseteq A^i_{(x_1,y_1) \odot (x_2,y_2)} + \mathcal{B}(A^i_{x_1,y_1}, A^i_{x_2,y_2})(1).$$

Suppose that $\mathcal{B}(A^i_{x_1,y_1}, A^i_{x_2,y_2}) \neq 0$, then there exists an $L$-equivariant map $A_{x_1} \odot A_{x_2} \rightarrow (1)$ and an $L_i$-equivariant map $A_{y_1} \otimes A_{y_2} \rightarrow (1)$. By definition of $\mathcal{F}$ this means that $(e_g, e_l) \in (x_1, y_1) \odot (x_2, y_2)$. Therefore

$$A^i_{(x_1,y_1) \odot (x_2,y_2)} + \mathcal{B}(A^i_{x_1,y_1}, A^i_{x_2,y_2})(1) \subseteq A^i_{(x_1,y_1) \odot (x_2,y_2)}. \quad \square$$

In the remainder of this section we restrict to the case where $\Phi$ is of type $E_8$. The decomposition $\bigoplus_{x,i} A^i_{x,i}$ is given in Section 8.4. Note that there are only six non-zero components in this decomposition, namely $A^i_{1,1}, A^i_{1,2}, A^i_{1,3}, A^i_{1,4}, A^i_{1,5}$ and $A^i_{1,6}$ of respective dimensions 1, 1, 1539, 399, 112 and 1824. Each of these is irreducible as an $L_i$-representation. The corresponding sublaw of $\mathcal{F}$ on these components is given in Table 39. (To preserve space we have denoted $(x, i)$ by $x_i$.)

Table 39. The fusion law for type $E_8$

| $\oplus$ | $a_1$ | $b_1$ | $b_3$ | $b_5$ | $b_6$ | $b_7$ |
|--------|--------|--------|--------|--------|--------|--------|
| $a_1$  | $a_1$  | $b_1$  | $b_3$  | $b_5$  | $b_6$  | $b_7$  |
| $b_1$  | $b_1$  | $a_1$  | $b_3$  | $b_5$  | $b_6$  | $b_7$  |
| $b_3$  | $b_3$  | $b_3$  | $a_1, b_1, b_3$ | $b_5$ | $b_6, b_7$ | $b_6, b_7$ |
| $b_5$  | $b_5$  | $b_5$  | $b_5$  | $a_1, b_1, b_3, b_5$ | $b_6, b_7$ | $b_6, b_7$ |
| $b_6$  | $b_6$  | $b_6$  | $b_6, b_7$ | $b_6, b_7$ | $a_1, b_1, b_3, b_5$ | $b_3, b_5$ |
| $b_7$  | $b_7$  | $b_7$  | $b_6, b_7$ | $b_6, b_7$ | $b_3, b_5$ | $a_1, b_1, b_3, b_5$ |

We want to look for an axis for this decomposition on which $L_i$ acts trivially. Such an axis must be contained in $A^i_1 = A^i_{1,1} \oplus A^i_{1,2}$. Therefore, we need to know the action of $A^i_1$ on $\mathfrak{A}$ by multiplication. Consider a Chevalley basis of $L$ as in Definition 13.3 and take $i := \langle e_\alpha, e_\alpha \rangle$ for some root $\alpha \in \Phi^+$ as before. Let $a_\alpha$ be the projection with respect to $\mathcal{B}$ of $j_\alpha$ onto $A_{b_1}$. Then $A^i_1 = \langle 1, a_\alpha \rangle$ and $\mathcal{B}(1, a_\alpha) = 0$. Since 1 is a unit for $(\mathfrak{A}, \odot)$, it suffices to describe

$$\text{ad}_{a_\alpha} : \mathfrak{A} \rightarrow \mathfrak{A} : a \mapsto a_\alpha \odot a.$$

Note that $L_i$ fixes $a_\alpha$ and hence $\text{ad}_{a_\alpha}$ is an isomorphism of $L_i$-representations. Since each $L_i$-isotypic component $A^i_1$ is irreducible if $i \neq 1$, the operator $\text{ad}_{a_\alpha}$ must act as a scalar on each $A^i_1$ for $i \neq 1$, by Schur’s lemma.
Proposition 9.4. The linear map $\text{ad}_{a_\alpha}$ is defined by

\[
\begin{align*}
1 & \mapsto a_\alpha, \\
a_\alpha & \mapsto \left( \frac{1}{196} + \frac{1}{2}p \right) 1 + \frac{9}{98} a_\alpha, \\
a & \mapsto -\frac{3}{196} a \\
a & \mapsto \frac{9}{196} a \\
a & \mapsto \frac{9}{196} a \\
a & \mapsto 0 \\
\end{align*}
\]

if $a \in A_{b,3}^1$, if $a \in A_{b,5}^1$, if $a \in A_{b,6}^1$, if $a \in A_{b,7}^1$.

Proof. Since $\mathcal{L}_i$ acts trivially on $\langle a_\alpha \rangle$, it follows by Schur’s lemma that $\text{ad}_{a_\alpha}$ must act as a scalar on each $\mathcal{L}_i$-isotypic component that is not irreducible. More precisely, if $A_{x,y}^i$ is irreducible (this is true for $(x,y) \in \{(b,3),(b,5),(b,6),(b,7)\}$), then for $a \in A_{x,y}^i$ we have $a_\alpha \circ a = \lambda a$ for some $\lambda \in \mathbb{C}$ that does not depend on the choice of $a \in A_{x,y}^i$. So it suffices to compute $a_\alpha \circ a$ for any $a \in A_{x,y}^i \setminus \{0\}$ to determine $\lambda$. We can get such an element $a$ from the explicit description of the decomposition from Section 8.4. Thus we only have to compute a few products together with the product $a_\alpha \circ a_\alpha$. We have computed these products using a computer but the computation (although lengthy) can be done by hand. □

If $e_i$ is an axis, then we must have $e_i \circ e_i \in \langle e_i \rangle$. Therefore, we search for idempotents or nilpotents in $A_1^1$.

Proposition 9.5.

(i) If $p \neq -\frac{614}{74431}$, then the subalgebra $(A_1^1, \circ)$ of $(\mathfrak{g}, \circ)$ is generated by two primitive, orthogonal idempotents.

(ii) If $p = -\frac{614}{74431}$, then the subalgebra $(A_1^1, \circ)$ of $(\mathfrak{g}, \circ)$ is generated by 1 and a nilpotent element.

Proof. An arbitrary element of $A_1^1$ is of the form $c_1 1 + c_2 a_\alpha$ for $c_1, c_2 \in \mathbb{C}$. From Proposition 9.3 it follows that

\[(c_1 1 + c_2 a_\alpha)^2 = \left( c_1^2 + \left( \frac{1}{196} + \frac{1}{2}p \right) c_2^2 \right) 1 + \left( 2c_1 c_2 + \frac{9}{98} c_2^2 \right) a_\alpha.\]

Expressing that this element is an idempotent amounts to solving a system of two non-linear equations. A small calculation shows that we have 4 solutions (including the trivial solutions $c_1 = c_2 = 0$ and $c_1 = 1$, $c_2 = 0$) if $p \neq -\frac{614}{74431}$. If not, then we only have the two trivial solutions but then $1 - \frac{196}{9} a_\alpha$ is nilpotent. □

From now on we will always assume that $p \neq -\frac{614}{74431}$. (The case where $p = -\frac{614}{74431}$ is similar but more subtle because a nilpotent element does not have an “orthogonal nilpotent” as in the case of idempotents, which slightly disturbs the resulting fusion law.)

Definition 9.6. Let $e_i$ be one of the idempotents from Proposition 9.5 (i). Then $1 - e_i$ is the other idempotent. Write $e_i = c_1 1 + c_2 a_\alpha$ where $a_\alpha \in A'$. Explicitly we
have

\[ c_1 = \frac{1}{2} \pm \frac{9\sqrt{62}}{874431p + 614}, \]

\[ c_2 = \pm \frac{49\sqrt{62}}{2\sqrt{74431p + 614}}. \]

Notice that \( c_1 \neq \frac{1}{2} \). Therefore, we can distinguish between the two idempotents by computing \( B(e_i, 1) = c_i B(1, 1) \). Now we pick \( e_i \) for each \( i \in I \) such that \( B(e_i, 1) \) is constant for all \( i \in I \). Also let \( A^i_c := \langle e_i \rangle \) and \( A^i_{c'} := \langle 1 - e_i \rangle \).

**Theorem 9.7.** Let \( \Phi \) be an irreducible root system of type \( E_8 \), \( (\mathfrak{A}, \circ) \) the algebra parametrized by \( p \) from Definition [9.7] and \( c_1 \) as in Definition [9.6]. Let \( I \) be as in Definition [8.3]. For each \( i \in I \) let \( e_i, A^i_c \) and \( A^i_{c'} \) be as in Definition [9.6] and \( A^5_1, A^1_6 \) and \( A^1_7 \) as in Section [8.4].

(i) The decomposition \( A^1_1 \oplus A^1_{c'} \oplus A^1_3 \oplus A^1_5 \oplus A^1_6 \oplus A^1_7 \) is an \( \mathcal{F}' \)-decomposition of \( (\mathfrak{A}, \circ) \) where \( \mathcal{F}' \) is the fusion law from Table [80]. Both \( e \) and \( e' \) are units for \( \mathcal{F}' \).

(ii) Let \( \Omega \) be the tuple of decompositions from (i) indexed by \( I \). Then the quadruple \( (\mathfrak{A}, I, \Omega, i \mapsto e_i) \) is an axial decomposition algebra with evaluation map

\( e \mapsto 1 \),

\( e' \mapsto 0 \),

\( 3 \mapsto \frac{4}{3}c_1 - \frac{1}{6} \),

\( 5 \mapsto \frac{1}{2} \),

\( 6 \mapsto \frac{1}{2} \),

\( 7 \mapsto c_1 \).

(iii) The algebra \( (\mathfrak{A}, \circ) \) is generated by the idempotents \( e_i \) for \( i \in I \) if \( c_1 \neq 0 \). If \( c_1 = 0 \), then these idempotents generate the subalgebra \( A^i \) with \( A^i \) as in Definition [9.7].

**Proof.** Parts (i) and (ii) follow immediately from the calculations in Section [8.4] and Propositions [9.4] and [9.3]. Since the elements \( i \in I \) are conjugate for the action of \( \mathcal{L} \), also the idempotents \( e_i \) must be conjugate. Hence they span an \( \mathcal{L} \)-invariant subspace of \( \mathfrak{A} \). Assume that \( c_1 \neq 0 \). From the global decomposition (Table [31]) we know that \( \mathfrak{A} \) only has two proper \( \mathcal{L} \)-invariant subspaces, namely \( \langle 1 \rangle \) and its orthogonal complement with respect to \( B \). Since \( \mathcal{L} \) acts non-trivially on the idempotents \( e_i \), \( B(1, e_i) = c_i B(1, 1) \neq 0 \) it follows that \( \mathfrak{A} \) is spanned by the elements \( e_i \). In particular the algebra \( (\mathfrak{A}, \circ) \) is generated by them. If \( c_i = 0 \), then \( p = -\frac{1}{235} \) and \( B(1, e_i) = 0 \). Moreover, if \( a, b \in A' \), then \( B_p(a \circ b, 1) = B_p(a, b) = 0 \) by definition of \( B_p \). Therefore \( A' \) is a subalgebra of \( (\mathfrak{A}, \circ) \). Since \( A' \) is irreducible as \( \mathcal{L} \)-module, it must be spanned, and therefore generated, by the idempotents \( e_i \) for \( i \in I \). \( \square \)

**Remark 9.8.**

(i) Because the global decomposition for types \( A_n \) and \( D_n \) contains three terms (see Section [8]), it is possible to write down an \( \mathcal{L} \)-equivariant product, as in Definition [9.11] with two degrees of freedom instead of one. If we write \( A_n = \langle 1 \rangle, A_b \) and \( A_c \) for the components of the global decomposition, then
these subspaces are orthogonal with respect to the Frobenius form $B$. We can define a new product on $A$ with two parameters $p_1$ and $p_2$ such that

$$a \circ b = a \ast b + p_1B(a_b, b_b)1 + p_2B(a_c, b_c)1,$$

where $a_x$ (resp. $b_x$) is the projection of $a$ (resp. $b$) onto $A_x$ for $x \in \{b, c\}$.

(ii) If $\Phi$ is of type $A_n$, $D_n$, $E_6$ or $E_7$, we can also try to find idempotents $e_i$ in the subalgebra $A_i^1$. However, in Proposition 9.4 we used Schur’s lemma to derive the adjoint action of the elements of $A_i^1$. Note that this is no longer possible for the terms of the local decomposition that are not irreducible $L_i$-representations. This would lead to further difficulties when trying to establish the diagonalizability of the adjoint action of such an idempotent $e_i$.

### Appendix A. The character computation of $V$

In this section we prove Proposition 2.6 which gives the character of $V$ as an $L$-representation. We use Freudenthal’s formula [Hum78, §22.3] to compute this character in a combinatorial way. Although this character is essential in Proposition 2.7 the computation is quite technical. We use the notation from Definition 1.3. Recall the definition of $\Lambda_i$ for $-2 \leq i \leq 2$ and $n_\lambda$ for $\lambda \in \Lambda_i$ from Definition 2.3. In addition to Lemma 2.4 we prove a few more combinatorial properties about the weights $\lambda \in \Lambda_i$.

**Lemma A.1.**

(i) $n_0 = \frac{|\Phi|}{2}$ and $n_\lambda = 1$ for $\lambda \in \Lambda_1 \cup \Lambda_2$.

(ii) Suppose $\omega \in \Phi$ is the highest root of $\Phi$ and $\lambda \in \bigcup_{0 \leq i \leq 2} \Lambda_i$ is dominant. Then $\lambda = \omega + \psi$ for some $\psi \in \Phi^+$.

(iii) If $\lambda \in \Lambda_k$ and $\lambda + i\alpha \in \Lambda_j$ for $i \geq 1$, $\alpha \in \Phi$ and $-2 \leq j, k \leq 2$, then $j = i^2 + ik(\lambda, \alpha) + k$.

(iv) Let $\lambda \in \Lambda$ be dominant and $\omega \in \Phi$ the highest root. Suppose $f \in W$ such that $f(\lambda) = \lambda$ and $f(2\omega - \lambda) = \lambda - 2\omega$. Then $\sum_{\alpha \in \Phi^+} \kappa(2\omega - \lambda, \alpha) = \sum_{\kappa(\lambda, \alpha) = 0} \kappa(2\omega, \alpha)$.

(v) Suppose $\alpha, \beta \in \Phi$ such that $\kappa(\alpha, \beta) = 0$. The number of roots $\gamma \in \Phi^+$ such that $\kappa(\alpha, \gamma) = -\kappa(\beta, \gamma) = \pm 1$ is $2(n_{\alpha + \beta} - 1)$.

(vi) For each $\lambda \in \Lambda_0$, we have $n_\lambda > 1$. 

### Table 40. The fusion law $F'$

| $\otimes$ | $e$ | $e'$ | 3 | 5 | 6 | 7 |
|-----------|-----|------|---|---|---|---|
| $e$ | $e$ | $e'$ | 3 | 5 | 6 | 7 |
| $e'$ | $e'$ | $e$, $e'$, 3 | 5 | 6 | 7 | 7 |
| 3 | 3 | 3 | $e$, $e'$, 3 | 5 | 6, 7 | 6, 7 |
| 5 | 5 | 5 | $e$, $e'$, 3, 5 | 6, 7 | 6, 7 |
| 6 | 6 | 6, 7 | 6, 7 | $e$, $e'$, 3, 5 | 3, 5 |
| 7 | 7 | 6, 7 | 6, 7 | 3, 5 | $e$, $e'$, 3, 5 |
Proof.

(i) Of course $n_0 = |\Phi|/2$ because $\alpha + \beta = 0$ for $\alpha, \beta \in \Phi$ if and only if $\alpha = -\beta$. Let $\lambda \in \Lambda_1 \cup \Lambda_2$. Suppose $\lambda = \alpha + \beta = \alpha' + \beta'$ for some $\alpha, \beta, \alpha', \beta' \in \Phi$ for which $\kappa(\alpha, \beta) > 0$. Then $\kappa(\alpha', \alpha + \beta) \geq 3$. Since $\Phi$ is simply laced, $\kappa(\alpha', \alpha) = 2$ or $\kappa(\alpha', \beta) = 2$ and thus $\alpha' = \alpha$ or $\beta' = \beta$. Therefore $n_\lambda = 1$.

(ii) Let $\lambda \in \bigcup_{0 \leq i \leq 2} \Lambda_i$ be dominant and write $\lambda = \alpha + \beta$ for some $\alpha, \beta \in \Phi$. Suppose that $\alpha$ is maximal with respect to the partial order $\preceq$ induced by the base $\Delta$. If $\alpha \neq \omega$, then there exists a root $\gamma \in \Phi^+$ for which $\kappa(\alpha, \gamma) = -1$. Since $\lambda$ is dominant and $\gamma \in \Phi^+$, we have $\kappa(\beta, \gamma) \geq 1$. If $\kappa(\beta, \gamma) = 2$, then $\beta = \gamma$, so $\kappa(\alpha, \beta) = -1$, implying $\lambda = \alpha + \beta \in \Phi = \Lambda_{-1}$, contradicting our assumption. So $\kappa(\beta, \gamma) = 1$. Therefore, both $\alpha + \gamma$ and $\beta - \gamma$ are roots and $\lambda = (\alpha + \gamma) + (\beta - \gamma)$. This contradicts the fact that $\alpha$ was maximal. Thus $\lambda = \omega + \psi$ for some $\psi \in \Phi$. Because $\kappa(\lambda, \psi) > 1$, the root $\psi$ must be positive.

(iii) Suppose $\lambda + i\alpha \in \Lambda_j$ for some $i \geq 1$, $\alpha \in \Phi^+$ and $-2 \leq j \leq 2$. Then, by Lemma 2.4.1 and $-2 \leq j \leq 2$. Then, by Lemma 2.4(1) $4 + 2j = \kappa(\lambda + i\alpha, \lambda + i\alpha) = 4 + 2k + 2i\kappa(\lambda, \alpha) + 2i^2$. Thus $j = i^2 + i\kappa(\lambda, \alpha) + k$.

(iv) If $\alpha \in \Phi^+$ such that $\kappa(\lambda, \alpha) > 0$ then also $\kappa(\lambda, f(\alpha)) > 0$. Because $\lambda$ is dominant, $f(\alpha)$ must be positive. Now $\kappa(2\omega - \lambda, \alpha + f(\alpha)) = 0$.

(v) If $\gamma \in \Phi^+$ such that $\kappa(\alpha, \gamma) = -\kappa(\beta, \gamma) = \pm 1$ then $\{s_\gamma(\alpha), s_\gamma(\beta)\} \in N_{\alpha + \beta} \setminus \{\{\alpha, \beta\}\}$. Conversely, if $\{\alpha', \beta'\} \in N_{\alpha + \beta} \setminus \{\{\alpha, \beta\}\}$, then precisely two of the four roots $\pm(\alpha' - \alpha)$ and $\pm(\alpha' - \beta)$ are positive and satisfy the necessary requirement on $\gamma$. Hence the number of such roots is equal to $2 \cdot |N_{\alpha + \beta} \setminus \{\{\alpha, \beta\}\}| = 2(n_{\alpha + \beta} - 1)$.

(vi) Suppose that $\lambda = \omega + \psi$ for some $\omega, \psi \in \Phi$. We claim that there exists a root $\beta \in \Phi$ such that $\kappa(\omega, \beta) = \pm 1$ and $\kappa(\psi, \beta) = \pm 1$. If not, then every root would be orthogonal to either $\omega$ or $\psi$, which contradicts the irreducibility of $\Phi$. Now $\omega, \psi$ and $\beta$ form a root subsystem of $\Phi$ of type $A_3$ and inside this subsystem, we can find another way to write $\lambda$ as the sum of two orthogonal roots. □

We are ready to prove Proposition 2.6.

Proposition A.2. The character of $\mathcal{V}$ is given by

$$
\text{ch}_\lambda = n_0 + \sum_{\lambda \in \Lambda_{-1}} (n_\lambda + 1)e^\lambda + \sum_{\lambda \in \Lambda_0} (n_\lambda - 1)e^\lambda + \sum_{\lambda \in \Lambda_1 \cup \Lambda_2} e^\lambda.
$$

Proof. Let $m_\lambda$ be the dimension of the weight-$\lambda$-space of $\mathcal{V}$. Write $\rho$ for the half-sum of all positive roots, this is, $\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$ and $\omega$ for the highest root of $\Phi$. According to Freudenthal’s formula [Hum78, §22.3]:

$$(A.1) \quad (\kappa(2\omega + \rho, 2\omega + \rho) - \kappa(\lambda + \rho, \lambda + \rho))m_\lambda = 2 \sum_{\alpha \in \Phi^+} \sum_{i \geq 1} \kappa(\lambda + i\alpha, \alpha)m_{\lambda+i\alpha}.
$$

We can rewrite the left hand side as

$$
\left(\kappa(2\omega, 2\omega) - \kappa(\lambda, \lambda) + \sum_{\alpha \in \Phi^+} \kappa(2\omega - \lambda, \alpha)\right)m_\lambda.
$$

We compute the values of $m_\lambda$ inductively.
Claim \((m_\lambda = 1\) for all \(\lambda \in \Lambda_2\)). Since \(V\) is isomorphic to the highest weight representation of \(L\) of highest weight \(2\omega\), we have \(m_{2\omega} = 1\). Because \(\Phi\) is irreducible and simply laced, \(W\) acts transitively on \(\Phi\). Therefore \(m_{2\alpha} = 1\) for all \(\alpha \in \Phi\), or \(m_\lambda = 1\) for all \(\lambda \in \Lambda_2\).

Claim \((m_\lambda = 1\) for all \(\lambda \in \Lambda_1\)). Now, let \(\lambda \in \Lambda_1\) be dominant. Suppose \(\lambda + i\alpha \in \Lambda_j\) for some \(\alpha \in \Phi^+\) and \(i \geq 1\). Because \(\lambda\) is dominant \(\kappa(\lambda, \alpha) \geq 0\) and, by Lemma A.1(iii), this is only possible if \(j = 2\), \(i = 1\) and \(\kappa(\lambda, \alpha) = 0\). Thus \(\lambda + \alpha = 2\beta\) for some \(\beta \in \Phi\) and \(\kappa(\beta, \alpha) = 1\). Thus \(\beta - \alpha\) is a root. Because \(n_\lambda = 1\), \(\beta + (\beta - \alpha)\) is the unique way to write \(\lambda\) as the sum of two roots. By Lemma A.1(ii) \(\beta = \omega\) and \(\alpha = 2\omega - \lambda \in \Phi^+\). Thus there exists precisely one \(\alpha \in \Phi^+\) for which \(\lambda + \alpha \in \Lambda_2\). From equation (A.1) we have

\[
\left(\kappa(2\omega, 2\omega) - \kappa(\lambda, \lambda) + \sum_{\alpha \in \Phi^+} \kappa(2\omega - \lambda, \alpha)\right) m_\lambda = 2\kappa(2\omega, 2\omega - \lambda) = 4\kappa(\beta, \alpha) = 4.
\]

In order to calculate the left hand side, we use Lemma A.1(ii) to write \(\lambda = \omega + \psi\) for some \(\psi \in \Phi^+\). Then \(2\omega - \lambda = \omega - \psi\) is a positive root. Apply Lemma A.1(iv) with \(f = s_\omega - \psi\). If \(\alpha \in \Phi^+\) and \(\kappa(\lambda, \alpha) = 0\) then \(\{s_\alpha(\omega), s_\alpha(\psi)\} \in N_\lambda\). Because \(n_\lambda = 1\) either \(\kappa(\omega, \alpha) = 1\) or \(\alpha = \omega - \psi\). Thus we have

\[
\left(\kappa(2\omega, 2\omega) - \kappa(\lambda, \lambda) + \sum_{\alpha \in \Phi^+, \kappa(\lambda, \alpha) = 0} \kappa(2\omega, \alpha)\right) m_\lambda = 4,
\]

\[
(8 - 6 + \kappa(2\omega, \omega - \psi)) m_\lambda = 4.
\]

Hence \(m_\lambda = 1\) for all \(\lambda \in \Lambda_1\).

Claim \((m_\lambda = n_\lambda - 1\) for all \(\lambda \in \Lambda_0\)). Consider a dominant weight \(\lambda \in \Lambda_0\). If \(\lambda + i\alpha \in \Lambda_j\) for some \(i \geq 1\), \(\alpha \in \Phi^+\) and \(-2 \leq j \leq 2\) then, by Lemma A.1(iii), we have \(i = 1\), \(j \geq 1\) and \(\kappa(\lambda, \alpha) = j - 1\). If \(j = 2\) then \(\kappa(\lambda, \alpha) = 1\) and \(\kappa(\lambda, \alpha) = 3\) which is impossible since \(\lambda + \alpha\) has to be the double of a root. The only remaining case is where \(j = 1\) and \(\kappa(\lambda, \alpha) = 0\). Since \(\lambda + \alpha \in \Lambda_1\), \(\lambda + \alpha\) can be written uniquely as the sum of two roots \(\beta\) and \(\gamma\). Of course \(\alpha \neq \beta\) and \(\alpha \neq \gamma\) because otherwise \(\lambda\) would be a root. Thus \(\kappa(\alpha, \beta) = \kappa(\alpha, \gamma) = 1\) and \((\beta - \alpha) + \gamma\) and \(\beta + (\gamma - \alpha)\) are two ways to write \(\lambda\) as the sum of two roots. Conversely, if \(\lambda = \delta + \varepsilon\) with \(\delta, \varepsilon \in \Phi\) and \(\alpha \in \Phi^+\) such that \(\kappa(\delta, \alpha) = -\kappa(\varepsilon, \alpha) = \pm 1\), then \(\lambda + \alpha \in \Lambda_1\). Lemma A.1(v) and a double counting argument gives us the number of \(\alpha \in \Phi^+\) for which \(\lambda + \alpha \in \Lambda_1\): \(n_\lambda(n_\lambda - 1)\). The right hand side of (A.1) becomes \(4n_\lambda(n_\lambda - 1)\). As far as the left hand side goes, we write \(\lambda = \omega + \psi\) for some \(\psi \in \Phi^+\) using Lemma A.1(ii). By Lemma A.1(iv) we can find \(\{\omega', \psi'\} \in N_{\omega + \psi} \setminus \{\{\omega, \psi\}\}\). Let \(\beta_1 := \omega - \omega'\) and \(\beta_2 := \omega - \psi'\). Then \(\beta_1, \beta_2 \in \Phi\) and \(\kappa(\omega, \beta_1) = \kappa(\omega, \beta_2) = 1\) and \(\kappa(\psi, \beta_1) = \kappa(\psi, \beta_2) = -1\). Apply Lemma A.1(iv) with \(f = s_{\beta_1} s_{\beta_2}\). The left hand side of (A.1) reduces to

\[
\left(4 + \sum_{\alpha \in \Phi^+, \kappa(\lambda, \alpha) = 0} \kappa(2\omega, \alpha)\right) m_\lambda.
\]
The number of $\alpha \in \Phi^+$ for which $\kappa(\omega, \alpha) = 1$ and $\kappa(\lambda, \alpha) = 0$ is, because $\omega$ is dominant, equal to $2(n_\lambda - 1)$ by Lemma [A.1][v]. Also $\kappa(\omega, \alpha) > 0$ since $\omega$ is dominant and $\kappa(\omega, \alpha) = 2$ if and only if $\alpha = \omega$ (but then $\kappa(\lambda, \alpha) = 2 \neq 0$). Thus

$$\sum_{\alpha \in \Phi^+} \kappa(2\omega, \alpha) = 4(n_\lambda - 1).$$

We conclude, by (A.1), that $4n_\lambda m_\lambda = 4n_\lambda(n_\lambda - 1)$ and thus $m_\lambda = n_\lambda - 1$ for all $\lambda \in \Lambda_0$.

Claim ($m_\lambda = n_\lambda + 1$ for all $\lambda \in \Lambda_{-1}$). The only dominant weight in $\Lambda_{-1} = \Phi$ is $\omega$. Now, by Lemma [A.1][iii] if $\omega + i\alpha \in \Lambda_1$ for $i \geq 1$ and $\alpha \in \Phi^+$ then $i = 1$ and $\kappa(\omega, \alpha) = j$. Obviously, the converse is also true. Hence, for the right hand side of (A.1):

$$2 \sum_{\alpha \in \Phi^+} \kappa(\omega + \alpha, \alpha)m_{\omega+\alpha} = 2 \sum_{\alpha \in \Phi^+} 2 \cdot (n_{\omega+\alpha} - 1) + 2 \sum_{\alpha \in \Phi^+} 3 \cdot 2 + \sum_{\alpha \in \Phi^+} 4 \cdot 1.$$

Let $\alpha \in \Phi^+$ such that $\kappa(\omega, \alpha) = 0$. Let $\beta \in \Phi^+$ such that $\kappa(\beta, \omega) = 1$ and $\kappa(\beta, \alpha) = -1$. Since $\omega$ is dominant, there are precisely $2(n_{\omega+\alpha} - 1)$ choices for $\beta$ by Lemma [A.1][v]. Then $\{\beta, \omega - \beta\}$ and $\{\alpha + \beta, \omega - (\alpha + \beta)\}$ are two different elements of $N_\omega$. Conversely, let $\{\gamma, \delta\}$ and $\{\epsilon, \zeta\}$ be two different elements of $N_\omega$. Then $\kappa(\gamma, \epsilon) = 1$ or $\kappa(\gamma, \zeta) = 1$. Without loss of generality, assume that $\kappa(\gamma, \epsilon) = 1$. Also assume that $\gamma - \epsilon$ is positive (otherwise take $\epsilon - \gamma$). Since $\kappa(\omega, \epsilon) = 1$ and $\omega$ is dominant, the root $\epsilon$ must be positive. Thus $\alpha := \gamma - \epsilon \in \Phi^+$ and $\beta := \epsilon \in \Phi^+$ are positive roots for which $\kappa(\omega, \alpha) = 0$, $\kappa(\omega, \beta) = 1$ and $\kappa(\alpha, \beta) = -1$. The same reasoning applies when $\gamma$ is replaced by $\delta$ but leads to the same $\alpha$ and $\beta$. This double counting argument gives us

$$\sum_{\alpha \in \Phi^+} \kappa(\omega, \alpha) = 0 \quad 2 \cdot (n_{\omega+\alpha} - 1) = n_\omega(n_\omega - 1).$$

Now consider $\alpha \in \Phi^+$ such that $\kappa(\omega, \alpha) = 1$. Then $\{\alpha, \omega - \alpha\} \in N_\omega$. Conversely, if $\{\beta, \gamma\} \in N_\omega$ then $\kappa(\omega, \beta) = \kappa(\omega, \gamma) = 1$. Thus

(A.2) $\sum_{\alpha \in \Phi^+} 1 = n_\omega \cdot 2.$

The only root $\alpha \in \Phi^+$ such that $\kappa(\omega, \alpha) = 2$, is $\omega$ itself.

So the right hand side of (A.1) equals

$$2n_\omega(n_\omega - 1) + 12n_\omega + 8 = 2n_\omega^2 + 10n_\omega + 8 = 2(n_\omega + 1)(n_\omega + 4).$$

As far as the left hand side goes, we have

$$\left(\kappa(2\omega, 2\omega) - \kappa(\omega, \omega) + \sum_{\alpha \in \Phi^+} \kappa(\omega, \alpha)\right)m_\omega = \left(8 - 2 + \sum_{\alpha \in \Phi^+} 1 + 2 \right) m_\omega.$$

Using (A.2), we conclude

$$(2n_\omega + 8)m_\omega = 2(n_\omega + 1)(n_\omega + 4).$$

Hence $m_\lambda = n_\lambda + 1$ for all $\lambda \in \Lambda_{-1}$. 

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Claim \( (m_0 = n_0) \). Finally, we compute \( m_0 \), once again using (A.1). By (A.2), the left hand side equals \( 4(n_\omega + 3)m_0 \). Obviously \( 0 + ia \in \Lambda_j \) for \( i \geq 1 \) and \( \alpha \in \Phi^+ \) if and only if \( i = 1 \) and \( j = -1 \) or \( i = 2 \) and \( j = 2 \). Because \( W \) acts transitively on \( \Phi \), we have \( n_\alpha = n_\omega \) for all \( \alpha \in \Phi \). The right hand side becomes

\[
2 \sum_{\alpha \in \Phi^+} 2 \cdot (n_\alpha + 1) + 2 \sum_{\alpha \in \Phi^+} 4 \cdot 1 = \sum_{\alpha \in \Phi^+} (4n_\alpha + 12) = \frac{1}{2} |\Phi| (4n_\omega + 12).
\]

Hence \( m_0 = \frac{|\Phi|}{2} = n_0 \). \qed

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