DUALITY AND SELF-DUALITY
FOR DYNAMICAL QUANTUM GROUPS

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Abstract. We define a natural concept of duality for the \( h \)-Hopf algebroids introduced by Etingof and Varchenko. We prove that the special case of the trigonometric SL(2) dynamical quantum group is self-dual, and may therefore be viewed as a deformation both of the function algebra \( \mathcal{F}(SL(2)) \) and of the enveloping algebra \( U(sl(2)) \). Matrix elements of the self-duality in the Peter–Weyl basis are \( 6j \)-symbols; this leads to a new algebraic interpretation of the hexagon identity or quantum dynamical Yang–Baxter equation for quantum and classical \( 6j \)-symbols.

1. Introduction

Quantum groups may be viewed as an algebraic framework for studying solutions to the Yang–Baxter equation, which exists in several versions. Although the quantum dynamical Yang–Baxter (QDYB) equation is much older (cf. §2.3 for some historical comments) than the perhaps more well-known equation with spectral parameters, the corresponding quantum groups have only been studied in recent years. The main reason is perhaps that these objects are not Hopf algebras, but more involved structures such as Hopf algebroids \([EV, EV2, X]\), weak Hopf algebras \([EN]\) and quasi-Hopf algebras \([BBB, JKOS]\).

The dynamical quantum groups of this paper are so called “\( h \)-Hopf algebroids”, a notion introduced by Etingof and Varchenko \([EV]\), motivated by earlier work of Felder and Varchenko \([F, FV]\). They are constructed from solutions to the QDYB equation in a manner analogous to the Faddeev–Reshetikhin–Sklyanin–Takhtajan (FRST) construction.

In \([EV2]\) it was suggested that a large class of dynamical quantum groups are self-dual. This should mean that they are analogues both of function algebras and of envelopping algebras, a rather intriguing fact. However, to quote \([EN]\): “It is not very convenient to formulate such a statement precisely, because of difficulties with the notion of a dual Hopf algebroid.” It is the purpose of this paper to resolve these difficulties. We show how to formulate a duality theory for \( h \)-Hopf algebroids and prove self-duality in the SL(2) case. The authors of \([EN]\) chose a different approach and obtained a general self-duality theorem for dynamical quantum groups within the framework of weak Hopf algebras. Though it may be possible to transfer the results of \([EN]\) to \( h \)-Hopf algebroids, the approach of this paper is completely different and should be of independent interest.
It is interesting to compare dynamical quantum groups with another type of self-dual quantum groups: the braided groups of Majid. For braided groups, the self-duality degenerates as the deformation parameter \( q \to 1 \). By contrast, for dynamical quantum groups there is nothing special (from an algebraic viewpoint) about the case \( q = 1 \); in particular self-duality still holds.

Let us summarize the contents of the paper. In §2 we recall the necessary algebraic background. References for this material are \([EV]\) and \([KR]\). In §3 we show that there is a working duality theory for \( \h \)-Hopf algebroids. This leads naturally to the definition of a cobraiding on an \( \h \)-Hopf algebroid. The main result in this section is Corollary 3.20, which shows that applying the generalized FRST construction of \([EV]\) to a dynamical \( R \)-matrix automatically gives a cobraided \( \h \)-bialgebroid. A corresponding statement is true for Hopf algebras, but in interesting examples the cobraiding will be degenerate. In §4 we prove that for the trigonometric \( SL(2) \) dynamical quantum group, the cobraiding is “almost” non-degenerate. The radical of the cobraiding serves to eliminate those representations which correspond to some covering group rather than \( SL(2) \). Finally we show that the matrix elements of the cobraiding in the Peter–Weyl basis are quantum (classical if \( q = 1 \)) \( 6j \)-symbols. This allows us to recover the first known instance of the QDYB equation: the hexagon identity for \( 6j \)-symbols found by Wigner in 1940.

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2. Preliminaries

2.1. \( \h \)-algebra. Throughout the paper, \( \h^* \) will be a finite-dimensional complex vector space. In the context of dynamical quantum groups, it appears as the dual of a Cartan subalgebra of the corresponding Lie algebra. We denote by \( M_{\h^*} \), the field of meromorphic functions on \( \h^* \).

An \( \h \)-prealgebra \( A \) is a complex vector space, equipped with a decomposition \( A = \bigoplus_{\alpha, \beta \in \h^*} A_{\alpha \beta} \) and two left actions \( \mu_l, \mu_r : M_{\h^*} \to \text{End}_\C(A) \) (the left and right moment maps) which preserve the bigrading, such that the images of \( \mu_l \) and \( \mu_r \) commute. A homomorphism of \( \h \)-prealgebras is a linear map which preserves the moment maps and the bigrading. It is convenient to introduce two right actions by

\[
\mu_l(f)a = a\mu_l(T_a f), \quad \mu_r(f)a = a\mu_r(T_\beta f), \quad a \in A_{\alpha \beta}, \quad f \in M_{\h^*},
\]

where \( T_\alpha \) denotes the automorphism \( T_\alpha f(\lambda) = f(\lambda + \alpha) \) of \( M_{\h^*} \).

We need two different tensor products on \( \h \)-prealgebras. The first one, denoted \( A \otimes B \), equals \( A \otimes \C B \) modulo the relations

\[
a\mu_l^A(f) \otimes b = a \otimes \mu_l^B(f)b, \quad a\mu_r^A(f) \otimes b = a \otimes \mu_r^B(f)b.
\]

The bigrading \( A_{\alpha \beta} \otimes A_{\gamma \delta} \subseteq (A \otimes B)_{\alpha + \gamma, \beta + \delta} \) and the moment maps

\[
\mu_l^A \otimes B(f)(a \otimes b) = \mu_l^A(f)a \otimes b, \quad \mu_r^A \otimes B(f)(a \otimes b) = \mu_r^A(f)a \otimes b
\]

make \( A \otimes B \) an \( \h \)-prealgebra.

Another kind of tensor product, denoted \( A \bar{\otimes} B \), equals \( \bigoplus_{\alpha \beta \gamma} A_{\alpha \gamma} \otimes \C B_{\gamma \beta} \) modulo the relations

\[
\mu_r^A(f)a \otimes b = a \otimes \mu_l^B(f)b, \quad a \in A, \quad b \in B, \quad f \in M_{\h^*}.
\]
The bigrading $A_{\alpha \beta} \otimes B_{\beta \gamma} \subseteq (A \otimes B)_{\alpha \gamma}$ and the moment maps

$$
\mu_t^{A \otimes B}(f)(a \otimes b) = \mu_t^A(f)a \otimes b, \quad \mu_r^{A \otimes B}(f)(a \otimes b) = a \otimes \mu_r^B(f)b
$$

make $A \otimes B$ an $\mathfrak{h}$-prealgebra.

An $\mathfrak{h}$-algebra is an $\mathfrak{h}$-prealgebra which is also an associative algebra with 1. It is required that the decomposition is a bigrading: $A_{\alpha \beta}A_{\gamma \delta} \subseteq A_{\alpha + \gamma, \beta + \delta}$. Considering $\mu_t$ and $\mu_r$ as algebra embeddings $M_{\mathfrak{h}^r} \to A_{00}$ through $\mu_t(f) = \mu_t(f)1$, it is moreover required that (2.2) hold as relations in the algebra. A homomorphism of $\mathfrak{h}$-algebras is an $\h$-coalgebroid homomorphism which is also an algebra homomorphism. If $A$ and $B$ are $\mathfrak{h}$-algebras, then so is $A \otimes B$, with the multiplication $(a \otimes b)(c \otimes d) = ac \otimes bd$.

We denote by $D_\mathfrak{h}$ the algebra of difference operators on $M_{\mathfrak{h}^r}$, consisting of finite sums $\sum_i f_i T_{\beta_i}$, $f_i \in M_{\mathfrak{h}^r}$, $\beta_i \in \mathfrak{h}^*$. This is an $\mathfrak{h}$-algebra with the bigrading defined by $f T_{-\beta} \in (D_\mathfrak{h})_{\beta}$ and both moment maps equal to the natural embedding. For any $\mathfrak{h}$-algebra $A$, there are canonical $\mathfrak{h}$-algebra isomorphisms $A \approx A \otimes D_\mathfrak{h} \approx D_\mathfrak{h} \otimes A$, defined by

$$
(2.2) \quad x \simeq x \otimes T_{-\beta} \simeq T_{-\alpha} \otimes x, \quad x \in A_{\alpha \beta}.
$$

An $\mathfrak{h}$-coalgebroid is an $\mathfrak{h}$-prealgebra equipped with two $\mathfrak{h}$-prealgebra homomorphisms, $\Delta : A \to A \otimes A$ (the coproduct) and $\varepsilon : A \to D_\mathfrak{h}$ (the counit), such that $(\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta$ and, under the identifications (2.2), $(\varepsilon \otimes \text{id}) \circ \Delta = (\text{id} \otimes \varepsilon) \circ \Delta = \text{id}$. An $\mathfrak{h}$-coalgebroid homomorphism $\phi : A \to B$ is an $\mathfrak{h}$-coalgebroid homomorphism with $(\phi \otimes \phi) \circ \Delta^A = \Delta^B \circ \phi$, $\varepsilon^B \circ \phi = \varepsilon^A$.

If $A$ and $B$ are $\mathfrak{h}$-coalgebroids, then $\Delta^{A \otimes B} = \sigma_{23} \circ (\Delta^A \otimes \Delta^B)$ and $\varepsilon^{A \otimes B}(a \otimes b) = \varepsilon^A(a) \circ \varepsilon^B(b)$ define an $\mathfrak{h}$-coalgebroid structure on $A \otimes B$. Here, $\sigma_{23}$ is a quotient of the map $a \otimes b \otimes c \otimes d \mapsto a \otimes c \otimes b \otimes d$.

For completeness we note here (this is not in the references) that the product $\otimes$ has a unit element. Namely, let $I_\mathfrak{h} = M_\mathfrak{h}^r \otimes_{\mathbb{C}} M_\mathfrak{h}^r$ with the bigrading $I_\mathfrak{h} = (I_\mathfrak{h})_{00}$, the moment maps $\mu_t(f) = f \otimes 1$, $\mu_r(f) = 1 \otimes f$, the coproduct $\Delta(f \otimes g) = f \otimes 1 \otimes 1 \otimes g$ and the counit $\varepsilon(f \otimes g) = fg$. Then $I_\mathfrak{h}$ is an $\mathfrak{h}$-coalgebroid. Moreover, for any $\mathfrak{h}$-coalgebroid $A$ there are canonical $\mathfrak{h}$-coalgebroid isomorphisms $A \approx A \otimes I_\mathfrak{h} \approx I_\mathfrak{h} \otimes A$, defined by

$$
(2.3) \quad x \simeq x \otimes 1 \otimes 1 \simeq 1 \otimes 1 \otimes x, \quad x \in A.
$$

An $\mathfrak{h}$-bialgebroid is an $\mathfrak{h}$-algebra which is also an $\mathfrak{h}$-coalgebroid, such that $\Delta$ and $\varepsilon$ are $\mathfrak{h}$-algebra homomorphisms. It follows that the multiplication in $A$ factors to an $\mathfrak{h}$-coalgebroid homomorphism $A \otimes A \to A$ and that $f \otimes g \mapsto \mu_t(f) \mu_r(g)$ defines an $\mathfrak{h}$-coalgebroid homomorphism $I_\mathfrak{h} \to A$.

In particular, if $A$ and $B$ are $\mathfrak{h}$-bialgebroids, then $A \otimes B$ is an $\mathfrak{h}$-coalgebroid and $A \otimes B$ an $\mathfrak{h}$-algebra. We stress that none of these spaces carry a natural $\mathfrak{h}$-bialgebroid structure.
An $\mathfrak{h}$-Hopf algebroid is an $\mathfrak{h}$-bialgebroid $A$ equipped with a map $S \in \text{End}_C(A)$ (the antipode), such that

\begin{equation}
S(\mu_r(f)a) = S(a)\mu(r(f)), \quad S(a\mu_l(f)) = \mu_r(f)S(a), \quad a \in A, \quad f \in M_{h^*},
\end{equation}

\begin{equation}
m \circ (\text{id} \otimes S) \circ \Delta(a) = \mu_l(\varepsilon(a)1), \quad a \in A,
\end{equation}

\begin{equation}
m \circ (S \otimes \text{id}) \circ \Delta(a) = \mu_r(T_\alpha(\varepsilon(a)1)), \quad a \in A_{\alpha\beta},
\end{equation}

where $m$ denotes multiplication and where $\varepsilon(a)$ is the action of the difference operator $\varepsilon(a)$ on the function $1 \in M_{h^*}$. In [KR] it was proved that $S$ is unique and is an antihomomorphism (in a natural sense) of $\mathfrak{h}$-bialgebroids.

The $\mathfrak{h}$-algebra $D_{\mathfrak{h}}$ is an $\mathfrak{h}$-Hopf algebroid with $\Delta_{D_{\mathfrak{h}}}$ the canonical isomorphism as in (2.2), $\varepsilon_{D_{\mathfrak{h}}}$ the identity map and the antipode defined by $S_{D_{\mathfrak{h}}}(fT_\alpha) = (T_{-\alpha}f)T_{-\alpha}$, or equivalently $S_{D_{\mathfrak{h}}}(x) = T_\alpha \circ x \circ T_\alpha$, $x \in (D_{\mathfrak{h}})_{\alpha\alpha}$.

The $\mathfrak{h}$-coalgebroid $I_{\mathfrak{h}}$ is an $\mathfrak{h}$-Hopf algebroid with the product $(f \otimes g)(h \otimes k) = fh \otimes gk$ and the antipode $S(f \otimes g) = g \otimes f$.

### 2.2. Representation theory.

When considering representations and corepresentations of $\mathfrak{h}$-algebroids, the representation spaces must also carry an $\mathfrak{h}$-structure. More precisely, by an $\mathfrak{h}$-space we mean an $\mathfrak{h}$-graded vector space over $M_{h^*}$, $V = \bigoplus_{\alpha \in h^*} V_{\alpha}$. A morphism of $\mathfrak{h}$-spaces is a grade-preserving $M_{h^*}$-linear map. In analogy with (2.1), we will write

$$vf = T_{-\alpha}(f)v, \quad f \in M_{h^*}, \quad v \in V_{\alpha}.$$  

If $A$ is an $\mathfrak{h}$-prealgebra and $V$ an $\mathfrak{h}$-space, we define $A \tilde{\otimes} V = \bigoplus_{\alpha\beta} A_{\alpha\beta} \otimes_C V_{\beta}$ modulo the relations

$$\mu_r^A(f)a \otimes v = a \otimes fv.$$  

The grading $A_{\alpha\beta} \tilde{\otimes} V_{\beta} \subseteq (A \tilde{\otimes} V)_\alpha$ and the $M_{h^*}$-linear structure $f(a \otimes v) = \mu_r^A(f)a \otimes v$ make $A \tilde{\otimes} V$ into an $\mathfrak{h}$-space. A corepresentation of an $\mathfrak{h}$-coalgebroid $A$ on an $\mathfrak{h}$-space $V$ is an $\mathfrak{h}$-space morphism $\pi : V \rightarrow A \tilde{\otimes} V$ such that $(\Delta \otimes \text{id}) \circ \pi = (\text{id} \otimes \pi) \circ \pi$, $(\varepsilon \otimes \text{id}) \circ \pi = \text{id}$. The second equality is in the sense of the isomorphism $D_{\mathfrak{h}} \tilde{\otimes} V \simeq V$ defined by $f T_{-\alpha} \otimes v \simeq fv$, $f \in M_{h^*}$, $v \in V_{\alpha}$. If $\{v_k\}_k$ is a homogeneous basis (over $M_{h^*}$) of $V$, $v_k \in V_{\omega(k)}$, the matrix elements $t_{kj} \in A_{\omega(k),\omega(j)}$ of $\pi$ with respect to this basis are given by

\begin{equation}
\pi(v_k) = \sum_j t_{kj} \otimes v_j.
\end{equation}

They satisfy

\begin{equation}
\Delta(t_{kj}) = \sum_l t_{kl} \otimes t_{lj}, \quad \varepsilon(t_{kj}) = \delta_{kj} T_{-\omega(k)}.
\end{equation}

An intertwiner $\Phi : V \rightarrow W$ of corepresentations is an $\mathfrak{h}$-space morphism which satisfies $\pi_W \circ \Phi = (\text{id} \otimes \Phi) \circ \pi_V$. Given two bases $\{v_k\}_k$, $\{w_k\}_k$ of $V$ and $W$, respectively, and writing $\Phi(v_k) = \sum_j \Phi_{kj}w_j$, the intertwining property may be written

\begin{equation}
\sum_j \mu_r(\Phi_{ji})t_{kj}^V = \sum_j \mu_l(\Phi_{kj})t_{jl}^W \quad \text{for all } k \text{ and } l.
\end{equation}
Given two $\mathfrak{h}$-spaces $V$ and $W$, their tensor product $V \hat{\otimes} W$ is defined as $V \otimes W$ modulo the relations
\[ vf \otimes w = v \otimes fw \]
and equipped with the $\mathfrak{h}$-space structure $V_\alpha \hat{\otimes} W_\beta \subseteq (V \hat{\otimes} W)_{\alpha+\beta}$, $f(v \otimes w) = fv \otimes w$. If $V$ and $W$ are corepresentations of an $\mathfrak{h}$-bialgebroid, then so is $V \hat{\otimes} W$. Explicitly, if $t^V_{kj}$ and $t^W_{kj}$ are matrix elements as in (2.8), then
\[(2.9) \quad \pi_{V \hat{\otimes} W}(v_j \otimes w_k) = \sum_{lm} t^V_{jl} t^W_{km} \otimes v_l \otimes w_m.\]

Next we turn to representations. For $V$ an $\mathfrak{h}$-space, let $(D_V)_{\alpha\beta}$ be the space of $\mathbb{C}$-linear operators $U$ on $V$ such that $U(gv) = T_{-\beta}(g)U(v)$ and $U(V_\gamma) \subseteq V_{\gamma+\beta-\alpha}$ for all $g \in M_{\mathfrak{h}^r}$, $v \in V$ and $\gamma \in \mathfrak{h}^*$. Then the space $D_V = \bigoplus_{\alpha\beta \in \mathfrak{h}^r} (D_V)_{\alpha\beta}$ is an $\mathfrak{h}$-algebra with moment maps
\[(2.10) \quad \mu_l (f)(v) = vf, \quad \mu_r (f)(v) = fv.\]

An $\mathfrak{h}$-representation of an $\mathfrak{h}$-algebra $A$ on $V$ is an $\mathfrak{h}$-algebra homomorphism $A \to D_V$. (In [EV] this was called a dynamical representation. The attribute $\mathfrak{h}$- or dynamical should remind us that we are not merely representing the underlying associative algebra, but also the $\mathfrak{h}$-structure.) An intertwiner $\Phi : V \to W$ of $\mathfrak{h}$-representations is an $\mathfrak{h}$-space morphism which satisfies $\Phi \circ \pi_V(a) = \pi_W(a) \circ \Phi$ for all $a \in A$.

Given two $\mathfrak{h}$-representations $V$ and $W$ of an $\mathfrak{h}$-bialgebroid $A$, $V \hat{\otimes} W$ is an $\mathfrak{h}$-representation with
\[ \pi_{V \hat{\otimes} W}(a)(v \otimes w) = \sum_i \pi_V(a_i^\prime)v \otimes \pi_W(a_i^\prime)w, \quad \Delta(a) = \sum_i a_i^\prime \otimes a_i^\prime. \]

Since this is an analogue of the representation $W \otimes V$ for bialgebras, it is perhaps more natural to work with the tensor product opposite to $\hat{\otimes}$ (denoted $\hat{\otimes}$ in [EV]), but we avoid that in the present paper.

**2.3. Generalized FRST construction.** Let $X$ be a finite index set, $\omega : X \to \mathfrak{h}^*$ an arbitrary function and $R = (R^a_{xy})_{x,y,a,b \in X}$ a matrix with entries in $M_{\mathfrak{h}^r}$ such that $R^a_{xy} = 0$ if $\omega(x) + \omega(y) \neq \omega(a) + \omega(b)$. To this data is associated an $\mathfrak{h}$-bialgebroid $A_R$ generated by $\{L_{xy}\}_{x,y \in X}$ together with two copies of $M_{\mathfrak{h}^r}$, embedded as subalgebras. We write the elements of these copies as $f(\lambda)$, $f(\mu)$, respectively. The defining relations of $A_R$ are
\[(2.11) \quad f(\lambda)L_{xy} = L_{xy}f(\lambda + \omega(x)), \quad f(\mu)L_{xy} = L_{xy}f(\mu + \omega(y)), \quad f(\lambda)g(\mu) = g(\mu)f(\lambda)\]
for $f, g \in M_{\mathfrak{h}^r}$, together with
\[(2.12) \quad \sum_{xy} R^a_{xy}(\lambda)L_{xb}L_{yd} = \sum_{xy} R^{bd}_{xy}(\mu)L_{cy}L_{ax}, \quad a, b, c, d \in X.\]

The bigrading on $A_R$ is defined by $L_{xy} \in A_{\omega(x), \omega(y)}$, $f(\lambda), f(\mu) \in A_{00}$, and the moment maps by $\mu_l (f) = f(\lambda), \mu_r (f) = f(\mu)$. The coproduct and counit are defined
by

\[
\Delta(L_{ab}) = \sum_{x \in X} L_{ax} \otimes L_{xb}, \quad \Delta(f(\lambda)) = f(\lambda) \otimes 1, \quad \Delta(f(\mu)) = 1 \otimes f(\mu),
\]

\[
\varepsilon(L_{ab}) = \delta_{ab} T_{-\omega(a)}, \quad \varepsilon(f(\lambda)) = \varepsilon(f(\mu)) = f.
\]

Let \( V \) be a complex vector space with basis \( \{ v_x \}_{x \in X} \), viewed as an \( \mathfrak{g} \)-module through \( v_x \in V_{\omega(x)} \). We identify \( R \) with the meromorphic function \( \mathfrak{g}^* \to \text{End}_\mathbb{C}(V \otimes V) \) defined by

\[
(2.13) \quad R(\lambda)(v_a \otimes v_b) = \sum_{xy} R^{ab}_{xy}(\lambda) v_x \otimes v_y.
\]

Then \( R \) is called a dynamical \( R \)-matrix if it satisfies the quantum dynamical Yang–Baxter (QDYB) equation

\[
(2.14) \quad R^{12}(\lambda - h^{(3)}) R^{13}(\lambda) R^{23}(\lambda - h^{(1)}) = R^{23}(\lambda) R^{13}(\lambda - h^{(2)}) R^{12}(\lambda).
\]

This is an identity in the algebra of meromorphic functions \( \mathfrak{g}^* \to \text{End}(V \otimes V \otimes V) \). Here, \( h \) indicates the action of \( \mathfrak{g} \), and the upper indices refer to the factors in the tensor product. For instance, \( R^{12}(\lambda - h^{(3)}) \) denotes the operator

\[
R^{12}(\lambda - h^{(3)})(v \otimes v \otimes w) = (R(\lambda - \mu)(v \otimes v)) \otimes w, \quad w \in V_{\mu}.
\]

Evaluating both sides of (2.14) on a tensor product \( v_a \otimes v_b \otimes v_c \) and identifying the coefficient of \( v_d \otimes v_e \otimes v_f \) gives the expression for the QDYB equation in terms of the matrix elements of \( R \):

\[
(2.15) \quad \sum_{xyz} R^{xy}_{de}(\lambda - \omega(f)) R^{az}_{xf}(\lambda) R^{bc}_{yz}(\lambda - \omega(a)) = \sum_{xyz} R^{yz}_{ef}(\lambda) R^{sc}_{dz}(\lambda - \omega(y)) R^{ab}_{xy}(\lambda).
\]

Although the FRST construction works for general \( R \), the case when \( R \) is a dynamical \( R \)-matrix is the most interesting (cf. [EV] and Corollary 3.20 below).

The dynamical Yang–Baxter equation first occurred in the work of Wigner [W]. The matrix elements \( R^{ab}_{xy} \) are then 6\( j \)-symbols and each side of (2.14) a 9\( j \)-symbol, so the QDYB equation expresses a symmetry of the 9\( j \)-symbol. Although it is common knowledge that Wigner’s identity is a kind of Yang–Baxter equation, it seems that it was first written down in the form (2.14) by Gervais and Neveu [GN] (where it arose independently and in a different context). Therefore, the QDYB equation has also been called the Gervais–Neveu equation.

2.4. The \( \text{SL}(2) \) dynamical quantum group. Our main example is the trigonometric \( \text{SL}(2) \) dynamical quantum group. In this example \( \mathfrak{g} = \mathfrak{g}^* = \mathbb{C} \), \( X = \{ +, - \} \), \( \omega(\pm) = \pm 1 \) and \( R \) is the dynamical \( R \)-matrix

\[
(2.16) \quad \begin{pmatrix}
R_{++} & R_{+-} & R_{-+} & R_{--} \\
R_{-+} & R_{++} & R_{-+} & R_{--} \\
R_{+-} & R_{-+} & R_{++} & R_{-+} \\
R_{--} & R_{-+} & R_{--} & R_{++}
\end{pmatrix} = \begin{pmatrix}
q & 0 & 0 & 0 \\
0 & 1 & 0 & q^{-1} \\
0 & \frac{q^{-1} - q}{q^{2(\lambda + 1)} - 1} & \frac{q^{2(\lambda + 1)} - q^{-1}}{q^{2(\lambda + 1)} - 1} & 0 \\
0 & 0 & 0 & q
\end{pmatrix},
\]

with \( q \) a fixed parameter, \( 0 < q < 1 \). This is the \( R \)-matrix arising from 6\( j \)-symbols of the standard quantum group \( \mathcal{U}_q(\text{sl}(2)) \) evaluated in a two-dimensional representation.
It can also be obtained by a twisting construction from the $R$-matrix of $U_q(\text{sl}(2))$, using certain “dynamical boundaries” discovered by Babelon [B, BBB]; cf. also [R].

We write the generators of the corresponding $\mathfrak{g}$-bialgebroid as

$$\alpha = L_{++}, \quad \beta = L_{+-}, \quad \gamma = L_{-+}, \quad \delta = L_{--}. \quad (2.17)$$

In terms of the auxiliary functions

$$F(\lambda) = \frac{q^{2(\lambda+1)} - q^{-2}}{q^{2(\lambda+1)} - 1}, \quad G(\lambda) = \frac{(q^{2(\lambda+1)} - q^2)(q^{2(\lambda+1)} - q^{-2})}{(q^{2(\lambda+1)} - 1)^2},$$

$$H(\lambda, \mu) = \frac{(q - q^{-1})(q^{2(\lambda+\mu+2)} - 1)}{(q^{2(\lambda+1)} - 1)(q^{2(\mu+1)} - 1)}, \quad I(\lambda, \mu) = \frac{(q - q^{-1})(q^{2(\mu+1)} - q^{2(\lambda+1)})}{(q^{2(\lambda+1)} - 1)(q^{2(\mu+1)} - 1)},$$

the sixteen relations (2.12) reduce to the six independent equations

$$\alpha \beta = qF(\mu - 1)\beta \alpha, \quad \alpha \gamma = qF(\lambda)\gamma \alpha, \quad \beta \delta = qF(\lambda)\delta \beta, \quad \gamma \delta = qF(\mu - 1)\delta \gamma,$$

$$\alpha \delta - \delta \alpha = H(\lambda, \mu)\gamma \beta, \quad \beta \gamma - G(\lambda)\gamma \beta = I(\lambda, \mu)\alpha \delta. \quad (2.18)$$

It is possible to obtain an $\mathfrak{g}$-Hopf-algebroid, which we denote $\mathcal{F}_R = \mathcal{F}_R(\text{SL}(2))$, from this example by adjoining the relation

$$\alpha \delta - qF(\lambda)\gamma \beta = 1$$

and defining the antipode by

$$S(\alpha) = \frac{F(\lambda)}{F(\mu)} \delta, \quad S(\beta) = -\frac{q^{-1}}{F(\mu)} \beta, \quad S(\gamma) = -qF(\lambda)\gamma, \quad S(\delta) = \alpha.$$

The Hopf algebra $\mathcal{F}_q(\text{SL}(2))$ is recovered as the formal limit of $\mathcal{F}_R$ when the dynamical variables $\lambda, \mu \to -\infty$. We will refer to this as the non-dynamical limit. Another interesting limit is the rational limit $q \to 1$. We stress that all our results survive the rational limit. That is, the rational $\text{SL}(2)$ dynamical quantum group is an essentially self-dual Hopf algebroid, which is constructed from the classical $6j$-symbols of Racah and Wigner (very natural objects in the representation theory of $\text{SL}(2)$), and in fact provides an alternative algebraic framework for deriving their main properties.

An important difference between the dynamical and non-dynamical case is that $\mathcal{F}_R$ has a non-trivial center. In fact, the element

$$\Xi = q^{\lambda-\mu-1} + q^{\mu-\lambda+1} - q^{-(\lambda+\mu+2)}(1 - q^{2(\lambda+1)})(1 - q^{2(\mu+1)})\beta \gamma \quad (2.19)$$

is central. In [KR] it was observed that $\Xi$ plays the role of Casimir element of $\mathcal{F}_R$. We will obtain a precise version of this statement in Proposition 4.2 below.

### 3. Duality for $\mathfrak{g}$-bialgebroids

#### 3.1. Algebraic duals.

In this section we show that there is a working duality theory for $\mathfrak{g}$-bialgebroids. As in the case of Hopf algebras, it is more convenient in practice to work with pairings between two objects than to work directly with the algebraic dual. Consequently, the reader may wish to skip this part of the paper and pass directly to the definition of a pairing in §3.3.
Let us first try to motivate our construction. If \( A \) is an \( \mathfrak{h} \)-coalgebroid, we expect the counit \( \varepsilon^A \) to be the unit in the dual \( \mathfrak{h} \)-algebra \( A' \). Thus, we want \( A' \) to be a subspace of \( \text{Hom}_C(A, D_\mathfrak{h}) \). It will be convenient to write
\[
\langle a, \phi \rangle = \phi(a), \quad a \in A, \ \phi \in \text{Hom}_C(A, D_\mathfrak{h}).
\]
It follows from the counit axioms that if \( \Delta(a) = \sum_i a_i' \otimes a_i'' \), \( a_i' \in A_{\alpha\beta}, \ a_i'' \in A_{\beta\gamma} \), then
\[
(3.1) \quad \langle x, \phi \rangle = \sum_i \langle a_i', \varepsilon \rangle T_{\beta_i} \langle a''_i, \phi \rangle = \sum_i \langle a_i', \phi \rangle T_{\beta_i} \langle a''_i, \varepsilon \rangle
\]
for suitable \( \phi \in \text{Hom}_C(A, D_\mathfrak{h}) \). By suitable, we mean that the above expression should be independent of the choice of representative in \( A \otimes A \) for \( \Delta(a) \in A \otimes A \), that is, that
\[
\sum_i \langle \mu_r(f)a_i', \varepsilon \rangle T_{\beta_i} \langle a''_i, \phi \rangle = \sum_i \langle a_i', \phi \rangle T_{\beta_i} \langle \mu_l(f)a''_i, \varepsilon \rangle
\]
for all \( f \in M_{\mathfrak{h}'} \). A sufficient condition for this to hold is that
\[
(3.2) \quad \langle \mu_l(f)a, \phi \rangle = f \circ \langle a, \phi \rangle, \quad \langle a\mu_r(f), \phi \rangle = \langle a, \phi \rangle \circ f, \quad a \in A, \ f \in M_{\mathfrak{h}'}.
\]
Here and below we write \( f \) for the operator \( fT_0 \in D_\mathfrak{h} \).

Let us write \( A'_{\alpha\beta} \) for the subspace of \( \text{Hom}_C(A, D_\mathfrak{h}) \) consisting of elements \( \phi \) satisfying (3.2) such that
\[
\langle A_{\gamma\delta}, A'_{\alpha\beta} \rangle \subseteq (D_\mathfrak{h})_{\gamma+\beta,\delta+\alpha}
\]
(in particular, \( \langle A_{\gamma\delta}, A'_{\alpha\beta} \rangle = 0 \) if \( \gamma + \beta \neq \delta + \alpha \)), and define \( A' = \bigoplus_{\alpha\beta \in \mathfrak{h}'} A'_{\alpha\beta} \). It is easy to check that \( A' \) is closed under the multiplication
\[
(3.3) \quad \langle a, \phi\psi \rangle = \sum_i \langle a_i', \phi \rangle T_{\beta_i} \langle a''_i, \psi \rangle.
\]
It follows from the coproduct axiom that this product is associative, and from (3.1) that \( \varepsilon \) is a unit element. Finally, we introduce the moment maps
\[
\langle a, \mu_l(f) \rangle = f \circ \langle a, \varepsilon \rangle, \quad \langle a, \mu_r(f) \rangle = \langle a, \varepsilon \rangle \circ f,
\]
so that
\[
\langle a, \mu_l(f)\phi \rangle = f \circ \langle a, \phi \rangle = \langle \mu_l(f)a, \phi \rangle, \quad \langle a, \phi\mu_r(f) \rangle = \langle a, \phi \rangle \circ f = \langle a\mu_r(f), \phi \rangle.
\]
It is then easy to check that \( A' \) is an \( \mathfrak{h} \)-algebra. We call it the dual \( \mathfrak{h} \)-algebra of the \( \mathfrak{h} \)-coalgebroid \( A \).

**Remark 3.1.** A product related to (3.3),
\[
\langle a, \phi \ast \psi \rangle = \sum_i \langle a_i', \phi \rangle \langle a''_i, \psi \rangle,
\]
which makes sense on a certain subspace of \( \text{Hom}_C(A, B) \) for \( B \) an arbitrary \( \mathfrak{h} \)-algebra, was used and studied in [KR]. This product has a left unit coming from \( \mu^B_r \) and a
right unit coming from $\mu^B$. It is the fact that $\mu^{D_b}_i = \mu^{D_b}_r$ that allows us to modify it into a product with a two-sided unit.

Next we want to define the dual $\mathfrak{h}$-bialgebroid of an $\mathfrak{h}$-bialgebroid. We will need the following lemmas. We omit the straight-forward proof of the first one.

**Lemma 3.2.** Let $A$ and $B$ be $\mathfrak{h}$-coalgebroids and $\chi : A \to B$ an $\mathfrak{h}$-coalgebroid homomorphism. Then $\chi'(\phi) = \phi \circ \chi$ defines an $\mathfrak{h}$-algebra homomorphism $\chi' : B' \to A'$.

**Lemma 3.3.** Let $A$ be an $\mathfrak{h}$-coalgebroid and let $\psi_1, \ldots, \psi_n \in A'$ be linearly independent over $\mu^{-1}_i(M_{\mathfrak{h}})$. Then there exists $b \in A$ with $\langle b, \psi_i \rangle = \delta_{i1}$.

**Proof.** We view $A$ and $A'$ as vector spaces over $M_{\mathfrak{h}}$ through $\mu^A_i$, $\mu^A_i$. Let $X = \bigoplus_{i=1}^n M_{\mathfrak{h}} \psi_i$. Then $\rho(a)(\psi) = \langle a, \psi \rangle 1$ defines an $M_{\mathfrak{h}}$-linear map $\rho : A \to X^*$, where $X^*$ is the $M_{\mathfrak{h}}$-dual of $X$. We want to prove that $\rho$ is surjective. Since $X$ is finite-dimensional, it suffices to prove that $\rho(A)$ separates points on $X$. Let $0 \neq \psi \in X$, and choose $c \in A$ with $\langle c, \psi \rangle \neq 0$. After decomposing with respect to the bigrading of $D_{\mathfrak{h}}$, we may assume that $\langle c, \psi \rangle \in (D_{\mathfrak{h}})_{\delta \delta}$ for some $\delta$, which implies $\rho(c)(\psi) = \langle c, \psi \rangle T_\delta \neq 0$. This proves that $\rho$ is surjective. In particular, we can find $b \in A$ with $\rho(b)(\psi_i) = \delta_{i1}$.

**Lemma 3.4.** Let $A$ be an $\mathfrak{h}$-coalgebroid. Then the map
\[
\langle a \otimes b, \iota(\phi \otimes \psi) \rangle = \langle a, \phi \rangle T_\beta \langle b, \psi \rangle, \quad \phi \in A'_\alpha \beta, \ \psi \in A'_{\beta \gamma},
\]
defines an $\mathfrak{h}$-algebra embedding $\iota : A' \otimes A' \hookrightarrow (\hat{A} \hat{\otimes} A)'$.

**Proof.** First we must check that $\iota$ is well-defined, that is, that
\[
\langle a \otimes b, \iota(\mu_r(f) \phi \otimes \psi) \rangle = \langle a \otimes b, \iota(\phi \otimes \mu_l(f)) \rangle,
\]
\[
\langle a \mu_l(f) \otimes b, \iota(\phi \otimes \psi) \rangle = \langle a \otimes \mu_l(f) b, \iota(\phi \otimes \psi) \rangle,
\]
\[
\langle a \mu_l(f) \otimes b, \iota(\phi \otimes \psi) \rangle = \langle a \otimes \mu_l(f) b, \iota(\phi \otimes \psi) \rangle
\]
for $f \in M_{\mathfrak{h}}$. This is straight-forward. For instance, to prove the second identity, we assume $\phi \in A'_\alpha \beta$, $a \in A_{\gamma \delta}$. We must prove that
\[
\langle a \mu_l(f), \phi \rangle T_\beta \langle b, \psi \rangle = \langle a, \phi \rangle T_\beta \langle \mu_l(f) b, \psi \rangle,
\]
or equivalently, by (2.1) and (3.2), that
\[
T_{-\gamma} f \circ \langle a, \phi \rangle \circ T_\beta \circ \langle b, \psi \rangle = \langle a, \phi \rangle \circ T_\beta \circ f \circ \langle b, \psi \rangle.
\]
This is clear from the fact that $\langle a, \phi \rangle \in (D_{\mathfrak{h}})_{\gamma + \beta, \delta + \alpha} = M_{\mathfrak{h}} T_{-\gamma - \beta}$.

It is easy to check that $\iota$ maps into the subspace $(\hat{A} \hat{\otimes} A)'$ of Hom$_C(\hat{A} \hat{\otimes} A, D_{\mathfrak{h}})$ and that $\iota$ is an $\mathfrak{h}$-prealgebra homomorphism. Let us write out the proof that $\iota$ is multiplicative, that is, that
\[
\langle a \otimes b, \iota((\phi_1 \otimes \phi_2)(\psi_1 \otimes \psi_2)) \rangle = \langle a \otimes b, \iota(\phi_1 \otimes \phi_2)(\psi_1 \otimes \psi_2) \rangle.
\]
Assume that $\phi_1 \in A'_{\alpha \gamma}$, $\psi_1 \in A'_{\beta \gamma}$, and write $\Delta(a) = \sum_i a'_i \otimes a''_i$, $\Delta(b) = \sum_j b'_j \otimes b''_j$, where $a'_i \in A_{i \gamma}, b'_j \in A_{j \delta}$. The left-hand side of (3.4) is then
\[
\langle a, \phi_1 \psi_1 \rangle T_{\alpha + \beta} \langle b, \phi_2 \psi_2 \rangle = \sum_{i j} (a'_i, \phi_1) T_{\gamma i} (a''_i, \psi_1) T_{\alpha + \beta} (b'_j, \phi_2) T_{\delta j} (b''_j, \psi_2),
\]
while the right-hand side is
\[ \sum_{ij} \langle a'_i \otimes b'_j, \iota(\phi_1 \otimes \phi_2) \rangle T_{\gamma_i + \delta_j} \langle a''_j \otimes b''_j, \iota(\psi_1 \otimes \psi_2) \rangle \]
\[ = \sum_{ij} \langle a'_i, \phi_1 \rangle T_{\alpha} \langle b'_j, \phi_2 \rangle T_{\gamma_i + \delta_j} \langle a''_j, \psi_1 \rangle T_{\beta} \langle b''_j, \psi_2 \rangle. \]

These expressions are equal since \( \langle a''_j, \psi_1 \rangle \in M_{b'} T_{-\gamma_i - \beta}, \langle b''_j, \psi_2 \rangle \in M_{b'} T_{-\delta_j - \alpha}. \)

To prove that \( \iota \) is injective, let \( 0 \neq x \in A' \otimes A'. \) We choose a representative \( \sum_{i=1}^n \phi_i \otimes \psi_i \) of \( x \) such that \( \langle \psi_i \rangle_{i=1}^n \) are independent over \( \mu_i(M_{b'}) \) and \( \phi_i \neq 0 \) for all \( i. \) Choosing \( b \) as in Lemma 3.3, we have \( \langle a \otimes b, \iota(x) \rangle 1 = \langle a, \phi_1 \rangle 1. \) Again by Lemma 3.3, there exists \( a \in A \) with \( \langle a, \phi_1 \rangle 1 \neq 0. \) Thus \( \iota(x) \neq 0. \) This completes the proof. \( \square \)

We now let \( A \) be an \( \mathfrak{h} \)-bialgebroid, and apply Lemma 3.2 to the multiplication, or rather to its quotient \( m : A \otimes A \rightarrow A. \) This gives a map \( m' : A' \rightarrow (A \otimes A)' \) defined by \( (x \otimes y, m'(\phi)) = (xy, \phi). \) Now let
\[ A^* = \{ \phi \in A' ; m'(\phi) \in \text{Im}(\iota) \}. \]
It follows from the fact that \( m' \) and \( \iota \) are \( \mathfrak{h} \)-algebra homomorphisms that \( A^* \) is an \( \mathfrak{h} \)-subalgebra of \( A'. \)

**Lemma 3.5.** The map \( \Delta = \iota^{-1} \circ m'|_{A^*} \) is an \( \mathfrak{h} \)-algebra homomorphism \( A^* \rightarrow A^* \otimes A^*. \)

**Proof.** It remains to prove that \( \Delta \) takes values in \( A^* \otimes A^*. \) Since \( \Delta \) preserves the bigrading, it is enough to consider \( \Delta(\chi) \) for \( \chi \in A^*_{\alpha \beta}. \) We choose a representative \( \sum_{i=1}^n \phi_i \otimes \psi_i \) of \( \Delta(\chi) \) such that \( \phi_i \in A'_{\alpha \gamma_i}, \psi_i \in A'_\gamma, (\phi_i)_{i=1}^n \) are linearly independent over \( \mu_{\gamma} A'(M_{b'}) \) and \( (\psi_i)_{i=1}^n \) are independent over \( \mu_{\gamma} M_{b'} \). Then
\[ \langle ab, \chi \rangle = \sum_{i=1}^n \langle a, \phi_i \rangle T_{\gamma_i} \langle b, \psi_i \rangle, \quad a, b \in A. \]
Choosing \( b \) as in Lemma 3.3 and applying the above identity to \( (ac)b = a(cb) \) gives
\[ \langle ac, \phi_1 \rangle 1 = \sum_{i=1}^n \langle a, \phi_i \rangle T_{\gamma_i} \langle cb, \psi_i \rangle 1, \quad a, c \in A. \]
We may assume that \( b = \sum \epsilon b_\epsilon \) with \( b_\epsilon \in A_{\epsilon + \gamma - \beta, \epsilon}. \) Define \( \tilde{\psi}_i : A \rightarrow D_\hbar \) through \( \langle c, \tilde{\psi}_i \rangle = \sum \epsilon \langle cb_\epsilon, \psi_i \rangle T_\epsilon. \) It is then easily checked that \( \tilde{\psi}_i \in A'_{\gamma i}, \) which gives
\[ m'(\phi_1) = \iota \left( \sum_{i=1}^n \phi_i \otimes \tilde{\psi}_i \right) \]
so that \( \phi_1 \in A^*. \) By symmetry, \( \phi_i \in A^* \) for all \( i, \) and by a similar argument we may conclude that also \( \psi_i \in A^*. \) This completes the proof. \( \square \)

It is now clear how to define the dual of an \( \mathfrak{h} \)-bialgebroid.

**Definition 3.6.** For \( A \) an \( \mathfrak{h} \)-bialgebroid, we define the dual \( \mathfrak{h} \)-bialgebroid \( A^* \) to be the \( \mathfrak{h} \)-subalgebra \( A^* \subseteq A' \), equipped with the coproduct \( \Delta = \iota^{-1} \circ m' \) and the counit \( \varepsilon : A^* \rightarrow D_\hbar \) defined by \( \varepsilon(\psi) = (1, \psi). \)
It is easy to check that the coproduct and counit axioms are satisfied.

**Proposition 3.7.** If $A$ is an $h$-Hopf algebra, then $A^*$ is an $h$-Hopf algebra with the antipode $S^* = D_h \circ \phi \circ S^A$.

**Proof.** It is easy to check that $S(A') \subseteq A'$. To see that $S$ preserves $A^*$, choose $\phi \in A^*$ with $\Delta(\phi) = \sum_i \phi_i' \otimes \phi_i''$, and check that $m'(S(\phi)) = \iota(\sum_i S(\phi_i'') \otimes S(\phi_i'))$. Next, check that $S(A_{\alpha \beta}^*) \subseteq A_{-\beta, -\alpha}^*$. The condition (2.4) is then equivalent to $S(\mu_t(f)\phi) = S(\phi)\mu_t(f)$, $S(\phi\mu_t(f)) = \mu_t(f)S(\phi)$, which is easily verified.

To check the first identity in (2.5), we let $\phi \in A_{\alpha \beta}^*$ and $x \in A_{\gamma \delta}$ with $\Delta(\phi) = \sum_i \phi_i' \otimes \phi_i''$, $\phi_{11}'' \in A_{\xi \eta}^*$, $\Delta(x) = \sum_j x_j' \otimes x_j''$, $x_j'' \in A_{\eta \delta}$. Then

$$\langle x, m(\text{id} \otimes S)\Delta(\phi) \rangle = \sum_i \langle x, \phi_i' S(\phi_i'') \rangle = \sum_{ij} \langle x_j', \phi_i' \rangle T_{\eta j} \langle x_j'' S(\phi_i'') \rangle = \sum_{ij} \langle x_j', \phi_i' \rangle T_{\xi j} \langle x_j'' S(\phi_i'') \rangle T_{\beta - \delta},$$

where we used that $\langle S(x_j''), \phi_i'' \rangle \in (D_h)_{\beta - \delta, \xi - \eta}$. By the definition of $\Delta^\ast$ and by (2.3) for $S^A$, this equals

$$\sum_j \langle x_j'' S(x_j''), \phi \rangle T_{\beta - \delta} = \langle \mu_t(\langle x, \varepsilon \rangle 1), \phi \rangle T_{\beta - \delta} = \langle x, \varepsilon \rangle 1 \circ \langle 1, \phi \rangle \circ T_{\beta - \delta}.$$  

On the other hand,

$$\langle x, \mu_t(\varepsilon(\phi) 1) \rangle = \langle x, \mu_t(\langle 1, \phi \rangle 1) \rangle = \langle 1, \phi \rangle 1 \circ \langle x, \varepsilon \rangle.$$  

These two expressions are equal since $\langle 1, \phi \rangle \in M_{h^*} T_{-\beta}$, $\langle x, \varepsilon \rangle \in M_{h^*} T_{-\delta}$. This proves the first identity in (2.3) and the second one may be proved similarly. $\square$

**Proposition 3.8.** For $A$ an $h$-Hopf algebra, there exists a homomorphism of $h$-Hopf algebroids $\kappa : A \rightarrow A^{**}$, defined by $\langle \phi, \kappa(x) \rangle = \langle x, \phi \rangle$.

The proof is straightforward.

**Example 3.9.** Since $D_h$ is the unit object for $h$-algebras (as a tensor category) and $I_h$ the unit object for $h$-coalgebroids, it is natural to expect that $I_h$ and $D_h$ are mutually dual. Indeed, it is easy to check that $I_h^* \simeq D_h$ through the isomorphism

$$\langle f \otimes g, x \rangle = f \circ x \circ g, \quad f, g \in M_{h^*}, \ x \in D_h.$$

Moreover, the embedding $\iota : D_h \otimes D_h \hookrightarrow (I_h \otimes I_h)'$ of Lemma 3.7 is an isomorphism. In fact, it can be identified with the identity map on $D_h$ through the canonical isomorphisms (2.2), (2.3). Thus we have also an equivalence of $h$-Hopf algebroids $I_h^* \simeq D_h$.

Next, we let $\phi \in D_h'$ and introduce the function $f_\phi : h^* \oplus h^* \rightarrow \mathbb{C} \cup \{\infty\}$ through

$$f_\phi(\lambda, \mu) = \langle T_{\mu - \lambda}, \phi \rangle 1(\lambda).$$

Then $\phi \mapsto f_\phi$ gives an equivalence

$$D_h' \simeq \{ f : h^* \oplus h^* \rightarrow \mathbb{C} \cup \{\infty\} ; \lambda \mapsto f(\lambda, \lambda + \mu) \text{ is meromorphic for all } \mu \in h^* \}.$$
The space $D_h^*$ is identified with the subspace of tensors:

$$D_h^* \simeq \left\{ f \in D_h^* ; \ f(\lambda, \mu) = \sum_{\text{finite}} g_i(\lambda) h_i(\mu) \right\}.$$ 

We have set this up so that the map $\kappa : I_h \to I_h^{**} = D_h^*$ of Proposition 3.8 is the natural embedding $\kappa(g \otimes h)(\lambda, \mu) = g(\lambda) h(\mu)$. Strictly speaking, $D_h^*$ is bigger than $I_h$, as can be seen by considering $f(\lambda, \mu) = g(\lambda) / g(\mu)$ with $g$ a discontinuous character, $g(\lambda + \mu) = g(\lambda) g(\mu)$.

**Example 3.10.** This is a pathological example which shows that the map $\kappa$ of Proposition 3.8 is not always an embedding. Let $f \in M_{h^*}$ and let $K_f \subseteq I_h$ be the principal ideal generated by $f \otimes 1 - 1 \otimes f$. One may check that $I_h / K_f$ is an $h$-Hopf algebroid (cf. Lemma 4.6 below). As in the previous example, any $\phi \in (I_h / K_f)^*$ is given by an element $x \in D_h$ through (3.3). However, $x \in D_h$ defines an element of $(I_h / K_f)^*$ if and only if $f \circ x = x \circ f$, which for generic $f$ implies $x \in (D_h)_{0,0}$. But then $\phi$ vanishes on every element of the form $a = g \otimes 1 - 1 \otimes g$, $g \in M_{h^*}$, so that $\kappa(a) = 0$. Since not necessarily $a \in K_f$, we have proved that $\kappa$ is not injective.

### 3.2. Dual representations.

In this section we will establish a correspondence between corepresentations of an $h$-bialgebroid $A$ and $h$-representations of the dual algebroid $A^*$.

When $V$ is an $h$-space we define $V^+$ to be the $h$-space $V^+ = \bigoplus_{\alpha} V^+_\alpha$, where $V^\alpha_\alpha = \text{Hom}_{M_{h^*}} (V_{-\alpha}, M_{h^*})$ (note the minus sign!), with the $M_{h^*}$-linear structure

$$\langle v, \xi f \rangle = \langle v, \xi \rangle f, \quad v \in V, \ \xi \in V^+, \ f \in M_{h^*},$$

where we write $\langle v, \xi \rangle = \xi(v) \in M_{h^*}$. Note that, by definition, $\langle f v, \xi \rangle = f \langle v, \xi \rangle$.

**Proposition 3.11.** Let $\pi : V \to A \tilde{\otimes} V$ be a corepresentation of an $h$-bialgebroid $A$ on an $h$-space $V$. Then there exists an $h$-representation $\pi^+$ of $A^*$ on $V^+$, defined by

$$\langle v, \pi^+(\phi) \xi \rangle = \langle \pi(v), \phi \otimes \xi \rangle, \quad v \in V, \ \phi \in A^*, \ \xi \in V^+, $$

where the pairing on the right-hand side is defined by

$$\langle a \otimes v, \phi \otimes \xi \rangle = \langle a, \phi \rangle T_\beta \langle v, \xi \rangle 1, \quad a \in A_{\alpha\beta}.$$

We omit the straight-forward proof, which is largely parallel to the first part of the proof of Lemma 3.4.

Let $\{v_k\}_k$ be a basis of $V$, $t_{kj} \in A$ the corresponding matrix elements as in (2.6) and $v^+_k \in V^+$ the dual basis elements defined by $\langle f v_j, v^+_k \rangle = f \delta_{jk}$. Then the representation $\pi^+$ is given by

$$\pi^+(\phi) v^+_k = \sum_j v^+_j (t_{jk}; \phi) 1. \quad (3.6)$$

The coproduct $\Delta$ defines a corepresentation of any $h$-coalgebroid on itself (the regular corepresentation). It is less easy to define the regular $h$-representation of an $h$-algebra on itself. The naive definition $\pi(a) b = ab$ does not work since then (2.10) would imply a connection between the left and right moment map. However, we can define it using duality.
Definition 3.12. For $A$ an $h$-algebra, we define the regular representation of $A$ to be the $h$-representation $\rho \circ \kappa : A \to D_{A^+}$, where $\rho : A^{**} \to A_{A^+}$ is the dual of the regular corepresentation of $A^*$ and $\kappa : A \to A^{**}$ is as in Proposition 3.8.

Example 3.13. Consider the regular representation of the $h$-algebra $I_h$. According to Example 3.9, it is realized on the $h$-space $D_h^+$. We identify elements of $I_h$ with functions on $h^* \oplus h^*$ through $(f \otimes g)(\lambda, \mu) = f(\lambda)g(\mu)$ and elements $\xi \in D_h^+$ with functions through

$$\xi(\lambda, \mu) = \langle T_{\mu-\lambda}, \xi \rangle(\lambda).$$

Then $D_h^+$ is identified with the space of finite sums

$$f(\lambda, \mu) = \sum_{\alpha \in h^*} g_\alpha(\lambda) \delta_{\mu-\lambda, \alpha}, \quad g_\alpha \in M_h,$$

equipped with the $h$-space structure $g(\lambda)\delta_{\mu-\lambda, \alpha} \in (D_h^+)_{\alpha\alpha},$

$$(gf)(\lambda, \mu) = f(\lambda)g(\lambda, \mu), \quad g \in D_h^+, \ f \in M_h^*.$$

Note that it follows that $(fg)(\lambda, \mu) = f(\mu)g(\lambda, \mu)$; this gives the connection between the left and right moment maps needed to define the regular representation. It is the $h$-representation $\pi$ of $I_h$ on $D_h^+$ given by pointwise multiplication of functions:

$$(\pi(g)h)(\lambda, \mu) = g(\lambda, \mu)h(\lambda, \mu), \quad g \in I_h, \ h \in D_h^+.$$

For completeness, we state the following fact, which shows that the the duality of Proposition 3.12 extends to the level of tensor categories. Again we omit the straight-forward proof.

Proposition 3.14. Let $V$ and $W$ be corepresentations of an $h$-bialgebroid $A$. If $\Phi : V \to W$ is an intertwiner of corepresentations, then $\langle v, \Phi^*(\xi) \rangle = \langle \Phi(v), \xi \rangle$ defines an intertwiner of $h$-representations $\Phi^+ : W^+ \to V^+$.

Moreover, there exists an intertwiner (an equivalence in the finite-dimensional case) of $h$-representations $\iota : W^+ \otimes V^+ \to (V \otimes W)^+$ defined by

$$\langle x \otimes y, \iota(\eta \otimes \xi) \rangle = \langle x, \xi \rangle T_{-\alpha}(\langle y, \eta \rangle), \quad x \in V_al.$$

3.3. Pairings and cobraidings. In view of the results of §3.3, the following definition of a pairing of $h$-bialgebroids is natural.

Definition 3.15. For $A$ and $B$ $h$-bialgebroids, we define a pairing to be a $\mathbb{C}$-bilinear map $\langle \cdot, \cdot \rangle : A \times B \to D_h$, with

$$\langle A_{\alpha\beta}, B_{\gamma\delta} \rangle \subseteq (D_h)_{\alpha+\delta, \beta+\gamma},$$

$$\langle \mu_t(f)a, b \rangle = \langle a, \mu_t(f)b \rangle = f \circ \langle a, b \rangle,$$

$$\langle a, \mu_t(f)b \rangle = \langle a, b\mu_t(f) \rangle = \langle a, b \rangle \circ f,$$

$$\langle ab, c \rangle = \sum_i \langle a_i, c_i \rangle T_{\beta_i} \langle b, c''_i \rangle, \quad \Delta(c) = \sum_i c'_i \otimes c''_i, \quad c''_i \in B_{\beta_i, \gamma_i},$$

$$\langle a, bc \rangle = \sum_i \langle a'_i, b \rangle T_{\beta_i} \langle a''_i, c \rangle, \quad \Delta(a) = \sum_i a'_i \otimes a''_i, \quad a''_i \in A_{\beta_i, \gamma_i},$$

$$\langle a, 1 \rangle = \varepsilon(a), \quad \langle 1, b \rangle = \varepsilon(b).$$
A pairing is said to be non-degenerate if \( \langle a, B \rangle = 0 \Rightarrow a = 0 \) and \( \langle A, b \rangle = 0 \Rightarrow b = 0 \).

It is clear that a pairing defines homomorphisms (embeddings in the non-degenerate case) of \( h \)-bialgebroids \( A \rightarrow B^* \) and \( B \rightarrow A^* \).

A pairing between \( h \)-Hopf algebroids should in addition satisfy

\[
\langle Sa, b \rangle = S_{DBh}(\langle a, Sb \rangle).
\]

For \( A \) an \( h \)-bialgebroid, we denote by \( A^{\text{cop}} \) the \( h \)-bialgebroid which equals \( A \) as an associative algebra but has the opposite \( h \)-coalgebroid structure, that is,

\[
A^{\text{cop}}_{\alpha\beta} = A_{\beta\alpha}, \quad \mu^{A^{\text{cop}}}_I = \mu^A, \quad \mu^{A^{\text{cop}}}_r = \mu^A, \quad \Delta^{A^{\text{cop}}} = \sigma \circ \Delta^A, \quad \varepsilon^{A^{\text{cop}}} = \varepsilon^A,
\]

where \( \sigma(x \otimes y) = y \otimes x \). If \( A \) is an \( h \)-Hopf-algebroid with invertible antipode \( S \), then \( A^{\text{cop}} \) is an \( h \)-Hopf-algebroid with antipode \( S^{-1} \).

For Hopf algebras, pairings on \( A^{\text{cop}} \times A \) are of special interest since they are related to quasitriangular or braided structures on \( A^* \). The following definition will turn out to be appropriate in the dynamical case.

**Definition 3.16.** A cobraiding on an \( h \)-bialgebroid \( A \) is a pairing \( \langle \cdot, \cdot \rangle \) on \( A^{\text{cop}} \times A \) which satisfies

\[
\sum_{ij} \mu_I(\langle a'_i, b'_j \rangle) 1 \ a''_i b'_j = \sum_{ij} \mu_r(\langle a''_i, b'_j \rangle) 1 \ b'_j a'_i,
\]

where \( \Delta^A(a) = \sum_i a'_i \otimes a''_i \) and similarly for \( b \).

Note that, in terms of the equivalence \( (2.2) \), we may write \( (3.10) \) as

\[
\sum_{ij} a''_i b''_j \otimes \langle a'_i, b'_j \rangle = \sum_{ij} \langle a''_i, b''_j \rangle \otimes b'_j a'_i.
\]

To prove that something is a cobraiding one needs the following lemma, which shows that it is enough to verify \( (3.10) \) for a set of generators. We omit the straightforward proof; cf. [K] for the case of Hopf algebras.

**Lemma 3.17.** If the condition \( (3.10) \) holds with \( (a, b) \) replaced by \( (x, y) \), \( (x, z) \) and \( (y, z) \), then it also holds for \( (xy, z) \) and \( (x, yz) \).

A cobraiding on an \( h \)-bialgebroid yields a braiding (in the dynamical sense) on its corepresentations.

**Proposition 3.18.** Let \( A \) be an \( h \)-bialgebroid equipped with a cobraiding, and let \( V \) and \( W \) be two corepresentations of \( A \). Choosing bases and introducing matrix elements as in \( (2.6) \), the equation

\[
\Phi(v_j \otimes w_k) = \sum_{lm} \langle t^W_{kl}, t^V_{jm} \rangle 1 \ w_l \otimes v_m
\]

defines an intertwiner \( \Phi : V \otimes W \rightarrow W \otimes V \).

Let \( V_0 \subseteq V \), \( W_0 \subseteq W \) be the complex subspaces spanned by \( \{v_k\}_k \), \( \{w_k\}_k \), and define \( \Phi_0 : h^* \rightarrow \text{Hom}_h(V_0 \otimes W_0, W_0 \otimes V_0) \) through

\[
\Phi_0(\lambda)(v_j \otimes w_k) = \sum_{lm} ((t^W_{kl}, t^V_{jm}) \langle \lambda \rangle)(\lambda) \ w_l \otimes v_m.
\]
Define $R = R_{V_0 W_0} : \mathfrak{h}^* \to \text{End}_0(V_0 \otimes W_0)$ by $\Phi_0 = \sigma \circ R$ where $\sigma(x \otimes y) = y \otimes x$. Then, given also a third corepresentation $U$ with distinguished complex subspace $U_0$, $R$ satisfies the QDYB equation of the form

\[
R_{U_0 V_0}^{12}(\lambda - h^{(3)}) R_{U_0 W_0}^{13}(\lambda) R_{V_0 W_0}^{23}(\lambda - h^{(1)}) = R_{V_0 W_0}^{23}(\lambda) R_{U_0 W_0}^{13}(\lambda - h^{(2)}) R_{U_0 V_0}^{12}(\lambda).
\]

(3.11)

Note that it is not possible to factor $\Phi$ as $\sigma \circ \hat{R}$ for some $\hat{R} \in \text{End}_b(V \otimes W)$, since $\sigma$ does not make sense as an operator on $V \otimes W$. As was pointed out in [EV], it is this non-naturality of the flip map that gives rise to the QDYB equation instead of the elementary braid relation $R_{12}^{12}R_{13}^{13}R_{23}^{23} = R_{23}^{23}R_{13}^{13}R_{12}^{12}$.

**Proof.** It follows from (2.8) and (2.9) that the equation

\[
\Phi(v_j \otimes w_k) = \sum_{lm} \Phi^{lm}_{jk} w_l \otimes v_m
\]

defines an intertwiner if and only if

\[
\sum_{ij} \mu_i(\Phi^{ij}_{kl}) t^{W}_{im} t^{V}_{jn} = \sum_{ij} \mu_r(\Phi^{mn}_{ji}) t^{V}_{ji} t^{W}_{kn}.
\]

Choosing $a = t^{W}_{im}$ and $b = t^{V}_{kn}$ in (3.10) and using (2.7), we see that this indeed holds for $\Phi^{lm}_{jk} = (t^{W}_{ik}, t^{V}_{jm})_1$.

To prove the second statement we apply $x \mapsto \langle x, t^{ij}_{pq} \rangle$ to both sides of (3.12). Writing $R_{lm}^{ij} = \Phi^{ml}_{jk} = (t^{W}_{km}, t^{V}_{jl})_1$, the left-hand side gives

\[
\sum_{ij} \langle t^{W}_{im} t^{V}_{jn} \mu_r A^{\text{cop}}(T_{\omega(i)+\omega(j)} R^{kl}_{ji}), t^{W}_{pq} \rangle
\]

\[
= \sum_{hi} \langle t^{W}_{im}, t^{V}_{ln} \rangle \circ T_{\omega(h)} \circ \langle t^{V}_{jn}, t^{V}_{hi} \rangle \circ T_{\omega(i)+\omega(j)} \circ R^{kl}_{ji} \circ T_{-\omega(i)-\omega(j)}
\]

\[
= \sum_{hi} R^{pi}_{hm} \circ T_{-\omega(m)} \circ R^{hj}_{jn} \circ T_{\omega(i)-\omega(h)} \circ R^{kl}_{ji} \circ T_{-\omega(i)-\omega(j)}
\]

\[
= \sum_{hi} R^{pi}_{hm}(\lambda) R^{hj}_{jn}(\lambda - \omega(m)) R^{kl}_{ji}(\lambda - \omega(p)) T_{-\omega(m)-\omega(n)-\omega(q)}.
\]

Computing the right-hand side similarly gives the identity

\[
\sum_{hi} R^{pi}_{hm}(\lambda) R^{hj}_{jn}(\lambda - \omega(m)) R^{kl}_{ji}(\lambda - \omega(p)) = \sum_{hi} R^{ji}_{in}(\lambda) R^{nk}_{ij}(\lambda) R^{hl}_{qi}(\lambda - \omega(j)) - \sum_{hi} R_{in}(\lambda) R^{nk}_{ij}(\lambda) R_{qi}(\lambda - \omega(j)).
\]

Replacing $(h, i, j, k, l, m, n, p, q) \mapsto (x, z, y, b, c, f, e, a, d)$ and comparing with (2.15), we see that this is indeed the QDYB equation.

3.4. Cobraidings on dynamical quantum groups. We now turn to the case of an $\mathfrak{h}$-bialgebroid $A_R$ constructed via the generalized FRST construction from a matrix $R$. We will show that the QDYB equation for $R$ gives a cobraidings on $A_R$.
Proposition 3.19. In the setting of (2.3), let \( L : \mathfrak{h}^* \to \text{End}_k(V \otimes V) \) be a meromorphic function. We introduce its matrix elements \( L_{xy}^{ab} \in M_{h^*} \) as in (2.13); note that \( L_{xy}^{ab} = 0 \) if \( \omega(x) + \omega(y) \neq \omega(a) + \omega(b) \). Then the following three statements are equivalent:

(i) There exists a pairing \( A^\text{cop}_R \times A_R \to D_h \) defined by

\[
\langle L_{ij}, L_{kl} \rangle = L_{ik}^{jl} T_{-\omega(i) - \omega(k)}.
\]

(ii) \( L \) satisfies the relations

\[
\begin{align*}
R^{12}(\lambda - h^{(3)})L^{13}(\lambda)L^{23}(\lambda - h^{(1)}) &= L^{23}(\lambda)L^{13}(\lambda - h^{(2)})R^{12}(\lambda), \\
L^{12}(\lambda - h^{(3)})L^{13}(\lambda)L^{23}(\lambda - h^{(1)}) &= R^{23}(\lambda)L^{13}(\lambda - h^{(2)})L^{12}(\lambda).
\end{align*}
\]

(iii) There exist two \( \mathfrak{h} \)-representations \( \pi, \rho \) on \( M_{h^*} \otimes V \) of \( A_R, A^\text{cop}_R \), respectively, defined by

\[
\begin{align*}
\pi(L_{ij})(g \otimes v_k) &= \sum_i L_{ij}^{ik} T_{-\omega(j)} g \otimes v_l, \\
\rho(L_{ij})(g \otimes v_k) &= \sum_i L_{ij}^{ik} T_{-\omega(i)} g \otimes v_l.
\end{align*}
\]

If these conditions are satisfied, the pairing is a cobrading if and only if

\[
\sum_{xy} \langle L_{ax}, L_{cy} \rangle(\lambda) L_{xb}L_{yd} = \sum_{xy} \langle L_{xb}, L_{yd} \rangle(\mu) L_{cy}L_{ax} \quad \text{for all } a, b, c, d.
\]

Choosing \( L = R \) in Proposition 3.19 and recalling (2.12) leads to the following algebraic interpretation of the QDYB equation.

Corollary 3.20. The following two conditions are equivalent:

(i) There exists a pairing \( A^\text{cop}_R \times A_R \to D_h \) defined by

\[
\langle L_{ij}, L_{kl} \rangle = R_{ik}^{jl} T_{-\omega(i) - \omega(k)}.
\]

(ii) \( R \) satisfies the QDYB equation (2.14).

Moreover, this pairing is automatically a cobrading.

Proof of Proposition 3.19. Let \( X \) and \( Y \) denote the left- and right-hand sides of (2.12), respectively. Assuming the existence of a pairing, we have

\[
\langle L_{ef}, X \rangle = \sum_{xyz} R_{ac}^{xy}(\lambda) \langle L_{zf}, L_{xb} \rangle T_{\omega(z)} \langle L_{ez}, L_{yd} \rangle
\]

\[
= \sum_{xyz} R_{ac}^{xy}(\lambda) L_{\lambda x}^{fy}(\lambda - \omega(x)) T_{-\omega(a) - \omega(c) - \omega(e)}.
\]

Computing \( \langle L_{ef}, Y \rangle \) similarly one obtains the identity

\[
\sum_{xyz} R_{ac}^{xy}(\lambda) L_{\lambda x}^{fy}(\lambda - \omega(x)) = \sum_{xyz} L_{\lambda x}^{fy}(\lambda) L_{\lambda y}^{ze}(\lambda - \omega(c)) R_{xy}^{zd}(\lambda - \omega(f)).
\]

Replacing \((a, b, c, d, e, f, x, y, z) \mapsto (e, b, f, c, d, a, y, z, x)\) and comparing with (2.13) we see that this is equivalent to (3.14b). Similarly, the identity \( \langle X, L_{ef} \rangle = \langle Y, L_{ef} \rangle \) implies (3.14a). Thus, condition (i) implies condition (ii). The converse is proved.
sarily: using \( (\ref{eq:17}) \) and \( (\ref{eq:3.8}) \) we may extend \( (\ref{eq:3.13}) \) to the algebra generated by \( f(\lambda) \), \( g(\mu) \), \( L_{x\lambda} \) subject to relations \( (\ref{eq:2.11}) \). By the above argument, \( (\ref{eq:3.14}) \) guarantees that the resulting form factors through the relations \( (\ref{eq:2.12}) \).

The equivalence of (ii) and (iii) is contained in \( \text{EV} \), Proposition 4.5.

The “only if”-part of the final statement follows by choosing \( a = L_{ab} \), \( b = L_{cd} \) in \( (\ref{eq:3.10}) \). The “if”-part then follows using Lemma 3.17. \( \square \)

4. Self-duality of the \( \text{SL}(2) \) Dynamical Quantum Group

4.1. A pairing on \( \mathcal{F}_R(\text{SL}(2)) \). We now turn to the case of Proposition 3.19 when \( R \) is given by \( (\ref{eq:2.10}) \). As in the non-dynamical case, the solution \( L = R \) has the disadvantage that the corresponding pairing does not factor through the determinant relation \( (\ref{eq:2.18}) \). Instead, one must work with \( L = q^{-1/2}R \), which clearly satisfies \( (\ref{eq:3.14}) \) and \( (\ref{eq:3.13}) \). We will denote the corresponding pairing by \( \{ \cdot, \cdot \} \). To see that it factors through \( (\ref{eq:2.18}) \), it is enough to check that \( \{ \cdot, \cdot \} = 0 \) for all other combinations of generators.

Proposition 4.1. There exists a cobrading \( \{ \cdot, \cdot \} \) on \( \mathcal{F}_R \) determined by
\[
\{ L_{ij}, L_{kl} \} = q^{-i/2} R_{ijkl}^j T_{-\omega(i) - \omega(k)}.
\]

Explicitly, the cobrading is defined by
\[
\{ \alpha, \alpha \} = q^{1/2} T_{-2}, \quad \{ \alpha, \delta \} = q^{-1/2}, \quad \{ \delta, \alpha \} = q^{-1/2} G(\lambda), \quad \{ \delta, \delta \} = q^{1/2} T_2,
\]
\[
\{ \beta, \gamma \} = q^{-1/2} \frac{q^{-1} - q}{q^{2(\lambda+1)} - 1}, \quad \{ \gamma, \beta \} = q^{-1/2} \frac{q^{-1} - q}{q^{-2(\lambda+1)} - 1};
\]

all other combinations of generators give zero. Although we will not need it, we note that the antipode axiom \( (\ref{eq:3.9}) \) is satisfied.

For our purposes (especially to simplify the proof of Lemma 4.3 below), it will be more convenient to work with a related pairing. It is straightforward to check that
\[
\Psi(\alpha) = q^{1/2(\lambda-\mu)} \alpha, \quad \Psi(\beta) = q^{-1/2(\lambda+\mu)} F(\lambda) \gamma,
\]
\[
(4.1) \quad \Psi(\gamma) = q^{1/2(\lambda+\mu)} \frac{1}{F(\mu)} \beta, \quad \Psi(\delta) = q^{1/2(\mu-\lambda)} \frac{F(\lambda)}{F(\mu)} \delta
\]
defines an isomorphism of \( \mathfrak{g}\)-Hopf algebroids \( \Psi : \mathcal{F}_R \to \mathcal{F}_R^\text{cop} \). Therefore, we may construct a pairing on \( \mathcal{F}_R \times \mathcal{F}_R \) as \( \langle x, y \rangle = \{ \Psi(x), y \} \). Explicitly,
\[
\langle \alpha, \alpha \rangle = T_{-2}, \quad \langle \alpha, \delta \rangle = \langle \delta, \alpha \rangle = 1, \quad \langle \delta, \delta \rangle = G(\lambda + 1) T_2,
\]
\[
(4.2) \quad \langle \beta, \beta \rangle = \frac{q - q^{-1}}{q^\lambda - q^{-\lambda}}, \quad \langle \gamma, \gamma \rangle = \frac{q^{-1} - q}{q^{\lambda+2} - q^{-\lambda-2}}.
\]

It is not hard to obtain the above pairing from a general Ansatz, without using Proposition 3.19. Note also that it is symmetric. In fact, one can prove that this is the case for all pairings on \( \mathcal{F}_R \times \mathcal{F}_R \).

In the formal limit \( f(\lambda) T_{-\alpha} \to f(-\infty) \) we obtain the highly degenerate pairing
\[
\langle \alpha, \alpha \rangle = \langle \delta, \delta \rangle = \langle \alpha, \delta \rangle = \langle \delta, \alpha \rangle = 1, \quad \langle \beta, \beta \rangle = \langle \gamma, \gamma \rangle = 0.
\]
on $\mathcal{F}_q(\text{SL}(2))$. We will see in Theorem 4.7 that, although the pairing (4.2) is also degenerate, its kernel is so small that $\mathcal{F}_R$ is “almost” self-dual. In particular, $\mathcal{F}_R(\text{SL}(2))$ can be viewed as a deformation both of the function algebra $\mathcal{F}(\text{SL}(2))$ and of the universal enveloping algebra $\mathcal{U}(\text{sl}(2))$.

4.2. $\mathcal{F}_R(\text{SL}(2))$ as a deformed enveloping algebra. In this section we work out the formal limit from $\mathcal{F}_R(\text{SL}(2))$ to $\mathcal{U}_q(\text{sl}(2))$ explicitly. To this end we introduce the elements

$$X_+ = q^{-1} \frac{q^{\lambda+1} - q^{-\lambda-1}}{q - q^{-1}} \beta, \quad X_- = -q \frac{q^\mu - q^{-\mu}}{q - q^{-1}} \gamma, \quad K = q^{\frac{1}{2}(\lambda-\mu)} (4.3)$$

of $\mathcal{F}_R$. Note that these elements degenerate in the non-dynamical limit $\lambda, \mu \to -\infty$, but not in the rational limit $q \to 1$.

**Proposition 4.2.** The elements $X_+, X_-, K, K^{-1}$ generate a subalgebra of $\mathcal{F}_R$ isomorphic to $\mathcal{U}_q(\text{sl}(2))$, that is,

$$X_+X_- - X_-X_+ = \frac{K^2 - K^{-2}}{q - q^{-1}}, \quad KX_\pm = q^{\pm 1} X_\pm K, \quad KK^{-1} = K^{-1} K = 1. (4.4)$$

Moreover, the pairing (4.2) satisfies

$$\langle X_+, \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \rangle = \begin{pmatrix} 0 & F(\lambda - 1) \\ 0 & 0 \end{pmatrix}, \quad \langle X_-, \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \rangle = \begin{pmatrix} 0 & 0 \\ 1/F(\lambda) & 0 \end{pmatrix},$$

$$\langle K^\pm, \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \rangle = \begin{pmatrix} q^{\pm \frac{1}{2} T_{-1}} & 0 \\ 0 & q^{\mp \frac{1}{2} T_1} \end{pmatrix},$$

and the Casimir element

$$C = \frac{q^{-1} K^2 + qK^{-2} - 2}{(q^{-1} - q)^2} + X_+X_-$$

is related to the element $\Xi$ defined in (2.19) by

$$C = \frac{1}{(q^{-1} - q)^2} (\Xi - 2).$$

The first relation in (4.4) corresponds to

$$\beta \gamma - \frac{F(\lambda)}{F(\mu - 1)} \gamma \beta = I(\lambda, \mu),$$

which follows from the defining relations of $\mathcal{F}_R$, and the remaining statements are straightforward to check.

Note that $\mathcal{U}_q(\text{sl}(2))$ is not embedded into $\mathcal{F}_R$ as a Hopf subalgebra. For instance, the coproduct rule $\Delta(\beta) = \alpha \otimes \beta + \beta \otimes \delta$ can be written as

$$\Delta(X_+) = \left( \frac{q^{\lambda+1} - q^{-\lambda-1}}{q^\mu+1 - q^{-\mu-1}} \alpha \right) \otimes X_+ + X_+ \otimes \delta. (4.5)$$
To see how the Hopf structure on $\mathcal{U}_q(\text{sl}(2))$ arises in the non-dynamical limit, rewrite the relations of $\mathcal{F}_R$ using the generators $X_\pm, K^\pm$ instead of $\beta, \gamma$, and then formally let $\lambda, \mu \to -\infty$. For instance (1.5) becomes

$$\Delta(X_+) \sim \left( \frac{q^{\lambda+1}}{q^{\mu+1}} \alpha \right) \otimes X_+ + X_+ \otimes \delta = K^2 \alpha \otimes X_+ + X_+ \otimes \delta.$$ 

Thus we are effectively rescaling $\beta$ and $\gamma$ in the limit, and also considering $\lambda - \mu$ as fixed.

This results in a Hopf algebra $H$ with generators $X_+, X_-, K, K^{-1}, \alpha, \delta$ satisfying relations (1.4). The remaining relations say that $\alpha$ and $\delta$ are central elements with $\alpha \delta = \delta \alpha = 1$. The Hopf structure on $H$ is given by

$$\Delta(\alpha) = \alpha \otimes \alpha, \quad \Delta(\delta) = \delta \otimes \delta, \quad \Delta(K^\pm) = K^\pm \otimes K^\pm,$$

$$\Delta(X_+) = X_+ \otimes \delta + K^2 \alpha \otimes X_+, \quad \Delta(X_-) = X_- \otimes K^{-2} \alpha + \delta \otimes X_-,$$

$$\varepsilon(\alpha) = \varepsilon(\delta) = \varepsilon(K^\pm) = 1, \quad \varepsilon(X_+) = 0,$$

$$S(\alpha) = \delta, \quad S(\delta) = \alpha, \quad S(K^\pm) = K^\mp,$$

$$S(X_+) = -K^{-2} X_+, \quad S(X_-) = -X_- K^2.$$

It is clear from these relations that $\alpha - 1$ and $\delta - 1$ generate a Hopf ideal $I$ of $H$ and that $H/I \simeq \mathcal{U}_q(\text{sl}(2))$ with the standard Hopf structure.

It is curious that $\mathcal{U}_q(\text{sl}(2))$ is contained in $\mathcal{F}_R$ as an associative algebra. It would be interesting to know whether the corresponding statement is true for the dynamical quantum groups constructed in [EV], [EV2].

4.3. Finite-dimensional representations of $\mathcal{F}_R(\text{SL}(2))$. Let $\pi$ be a corepresentation of $\mathcal{F}_R$ on an $\hbar$-space $V$. In Proposition 3.11 we defined the dual $\hbar$-representation $\pi^+$ of $\mathcal{F}_R^*$ on the dual space $V^+$. Composing it with the morphism $\mathcal{F}_R \to \mathcal{F}_R^*$ induced from the pairing (4.3), we obtain an $\hbar$-representation of $\mathcal{F}_R$ on $V^+$ which we also denote $\pi^+$.

We are interested in the dual of the finite-dimensional corepresentations studied in [KR]. Namely, let $V_N$ be the subspace of $\mathcal{F}_R$ spanned by $\{\mu_i(M_C)\gamma^{N-k} \alpha^k\}^N_{k=0}$, viewed as an $\hbar$-space through $\gamma^{N-k} \alpha^k \in (V_N)_{2k-N}, f v = \mu_i(f) v$. Then $\pi_N = \Delta|_{V_N}$ defines a corepresentation of $\mathcal{F}_R$ on $V_N$. Its matrix elements $t^N_{kj}$ in $\mathcal{F}_R$ are defined by

$$\Delta(\gamma^{N-k} \alpha^k) = \sum_{j=0}^N t^N_{kj} \otimes \gamma^{N-j} \alpha^j.$$ 

In [KR], the expression

$$\begin{align*}
t^N_{kj} &= \sum_{l=\max(0,j-k-N)}^{\min(j,k)} \binom{N-k}{j-l} \binom{k}{l} q^2 \frac{q^{2j+2k-N+l(3l-3k-3j+N)}}{q^2} \\
&\quad \times \frac{(q^{2j-N-\mu-1}; q^2)_{j-l}}{(q^{2j+k-l-N-\mu-1}; q^2)_{j-l}} \gamma^{j-l} \delta^{N-k-j+l} \alpha^l \beta^{k-l}
\end{align*}$$

(4.6)
Lemma 4.3. The pairing (4.2) satisfies
\[ \langle t_{kj}^N, \alpha \rangle = \delta_{kj} T_{N-2k-1}, \]
\[ \langle t_{kj}^N, \beta \rangle = \delta_{k,j+1} \frac{[k]}{[\lambda + N - 2k + 1]} T_{N-2k-1}, \]
\[ \langle t_{kj}^N, \gamma \rangle = \delta_{k+1,j} \frac{[k-N]}{[\lambda + 2]} T_{N-2k-1}, \]
\[ \langle t_{kj}^N, \delta \rangle = \delta_{kj} \frac{[\lambda - k + 1][\lambda + N - k + 2]}{[\lambda + 2][\lambda + N - 2k + 1]} T_{N-2k+1}, \]
\[ \langle t_{kj}^N, \Xi \rangle = \delta_{kj} (q^{N+1} + q^{-N-1}) T_{N-2k}, \]
where we temporarily write \([x] = (q^x - q^{-x})/(q - q^{-1}).\]

Proof. Iterating (3.8) and recalling the notation (2.17) we have in general
\[ \langle x_1 x_2 \cdots x_N, L_{ij} \rangle = \sum_{k_1, \ldots, k_{n-1} \in \{+,-\}} \langle x_1, L_{ik_1} \rangle T_{\omega(k_1)} \langle x_2, L_{k_1 k_2} \rangle T_{\omega(k_2)} \cdots \langle x_N, L_{k_{n-1}} \rangle \]
(recall that \(\omega(\pm) = \pm 1).\) Using this rule and (4.6) to compute \(\langle t_{kj}^N, \delta \rangle\) we get two non-zero terms, corresponding to \(j = k = l + 1, k_1 = \cdots = k_{N-1} = +\) and \(j = k = l, k_1 = \cdots = k_{N-1} = -\). Thus,

\[ \langle t_{kj}^N, \delta \rangle = \delta_{kj} \left( \langle [N-k][k] \gamma \delta^{N-k-1} \alpha^{k-1} \beta \frac{[\mu+k+1]}{[\mu+k]}, \delta \rangle + \langle \delta^{N-k} \alpha^{k}, \delta \rangle \right) \]
\[ = \delta_{kj} \left( \langle [N-k][k] \gamma T_{1} \left( \langle \delta, \alpha \rangle T_{1} \right)^{N-k-1} \langle \alpha, \alpha \rangle T_{1} \right)^{k-1} \langle \beta, \beta \rangle \frac{[\lambda+k+1]}{[\lambda+k]} \]
\[ + \left( \langle \delta, \delta \rangle T_{-1} \right)^{N-k} \langle \alpha, \delta \rangle T_{-1} \right)^{k} \times T_{1} \right) \]
\[ = \delta_{kj} \left( \langle [N-k][k] \frac{-1}{[\lambda+2]} T_{N-2k+1} \frac{1}{[\lambda+2]} \frac{[\lambda+k+1]}{[\lambda+k]} \right) + \left( G(\lambda + 1) T_{1} \right)^{N-k} T_{-k} \right). \]

Using that \(G(\lambda) = F(\lambda)/F(\lambda - 1)\) and \(F(\lambda) = q^{-1}[\lambda+2]/[\lambda+1],\) we have
\[ \left( G(\lambda + 1) T_{1} \right)^{N-k} = \frac{F(\lambda + N - k)}{F(\lambda)} T_{N-k} = \frac{[\lambda+N-k+2]}{[\lambda+N-k+1]} \frac{[\lambda+1]}{[\lambda+2]} T_{N-k}. \]
Plugging this into the previous expression gives
\[
\{ t_{k,j}^N, \delta \} = \delta_{kj} \frac{[\lambda + N - k + 2]}{[\lambda + N - k + 1] \cdot [\lambda + 2] \cdot [\lambda + N - 2k + 1]} 
\times \left\{ [\lambda + 1] \cdot [\lambda + N - 2k + 1] - [k] \cdot [N - k] \right\} T_{N-2k+1}.
\]

The expression in brackets equals \([\lambda - k + 1][\lambda + N - k + 1]\), which proves the statement for \(\delta\). For \(\alpha, \beta, \gamma\) the proof is similar but simpler, since we only get one non-zero term. Finally we prove the statement for \(\Xi\) using (2.7).

\[\square\]

Remark 4.4. The \(h\)-representations obtained above may be compared with those studied in [KR]. Let, for \(\omega \in \mathbb{C}\), \(H^\omega\) be the \(h\)-space with basis \(\{e_k\}_{k=0}^\infty\), where \(e_k \in H^\omega_{N+2k}\). In [KR] it was shown that the following equations define an \(h\)-representation \(\pi^\omega\) of \(F_R\) on \(H^\omega\):

\[
\pi^\omega(\alpha)e_k = q^{-k}\frac{1 - q^{2(\lambda - \omega - k + 1)}}{1 - q^{2(\lambda - \omega - 2k + 1)}} e_k,
\]

\[
\pi^\omega(\beta)e_k = \frac{(1 - q^{2k})(1 - q^{2(\omega + k - 1)})}{(1 - q^{2(\lambda + 1)})(1 - q^{2(\omega + 2k - \lambda - 3)})} e_{k-1},
\]

\[
\pi^\omega(\gamma)e_k = -q^{-1}e_{k+1},
\]

\[
\pi^\omega(\delta)e_k = q^k\frac{1 - q^{2(\lambda - 1 - k)}}{1 - q^{2(\lambda + 1)}} e_k.
\]

We will consider the case \(\omega = -N \in \mathbb{Z}_{\leq 0}\), for which \(H_N^\omega = \bigoplus_{k=N+1}^\infty M_{\mathbb{C}} e_k\) is an \(h\)-submodule. We write \(W_N = H^\omega / H_N^\omega\); this is a finite-dimensional \(h\)-representation of \(F_R\). It is natural to expect that it is equivalent to the representation \(\pi_N^\omega\) defined in (4.7). Indeed, one may check that this is true, the equivalence being given by \(e_k = v_{N-k}^N N_k\), where \(N_k(\lambda) = q^{k(\lambda + 2)}(q^2; q^2)_k/(q^2; q^2)_k\). This explains the fact (cf. [KR]) that the Clebsch–Gordan coefficients for \(\pi_N\) can be formally obtained from those of \(\pi^\omega\) by substituting \(\omega = -N\).

4.4. The radical of the pairing. In this section we will compute the radical of the pairing (1.2). We need the basic facts about ideals of \(h\)-Hopf algebroids; these have not previously appeared in the literature.

Definition 4.5. A subspace \(I\) of an \(h\)-Hopf algebroid \(A\) is called an \(h\)-Hopf ideal if the following conditions are satisfied:

(i) \(I\) is a two-sided ideal of \(A\) as an associative algebra.

(ii) \((A_{\alpha}\beta + I) \cap (A_{\gamma}\delta + I) = I\) if \((\alpha, \beta) \neq (\gamma, \delta)\).

(iii) \(\Delta(I) \subseteq A \otimes I + I \otimes A\).

(iv) \(\varepsilon(I) = 0\) and \(S(I) = I\).

Note that (ii) guarantees that \(A/I\) is bigraded.

Lemma 4.6. For \(\Phi : A \rightarrow B\) a homomorphism of \(h\)-Hopf algebroids, \(\text{Ker } \Phi = \{ x \in A; \Phi(x) = 0 \}\) is an \(h\)-Hopf ideal. Moreover, for any \(h\)-Hopf ideal \(I\) in \(A\), \(A/I\) has a structure of an \(h\)-Hopf algebroid such that the projection \(A \rightarrow A/I\) is an \(h\)-Hopf algebroid homomorphism.
The least trivial point is to prove that \( I = \ker \Phi \) satisfies condition \((iii)\) of Definition \([4.3]\). Let us write \( \mu_c \) for the “central” moment map \( \mu_c(f)(a \otimes b) = \mu_r(f)a \otimes b = a \otimes \mu(f)b \) and consider each copy of \( A \) in \( A \otimes A \) as a vector space over \( \mu_c(M_{\theta}) \). By elementary linear algebra, we can then for each \( x \in A \) write \( \Delta(x) = \sum_{i,j \in \Lambda_1 \cup \Lambda_2} \mu_c(f_{ij})x'_i \otimes x''_j \), where \( x'_i, x''_j \in \ker(\Phi) \) for \( i \in \Lambda_1 \) and where the families \( (\Phi(x'_i))_{i \in \Lambda_2}, (\Phi(x''_i))_{i \in \Lambda_2} \) are linearly independent over \( \mu_c(M_{\theta}) \). If \( x \in \ker(\Phi) \) we have
\[
0 = \Delta(\Phi(x)) = (\Phi \otimes \Phi)\Delta(x) = \sum_{i,j \in \Lambda_2} \mu_c(f_{ij}) \Phi(x'_i) \otimes \Phi(x''_j).
\]
Since \( \tilde{\otimes} \) means tensor product over \( \mu_c(M_{\theta}) \), the linear independence gives \( f_{ij} = 0 \) for \( i, j \in \Lambda_2 \), which implies \( \Delta(x) \in \ker(\Phi) \otimes A + A \otimes \ker(\Phi) \).

We want to compute the radical \( \mathcal{F}^1_R \) of the pairing \((4.2)\), which is the space of \( x \in \mathcal{F}_R \) such that \( \langle x, y \rangle = 0 \) for all \( y \in \mathcal{F}_R \), or equivalently the kernel of the \( \mathfrak{h} \)-Hopf algebroid homomorphism \( \Phi : \mathcal{F}_R \to \mathcal{F}^*_R \) induced by the pairing.

**Theorem 4.7.** Let \( I \subseteq \mathcal{F}_R \) be the left ideal generated by all elements \( f(\lambda, \mu) \), where \( f \in M_{\mathbb{C}} \otimes M_{\mathbb{C}} \) satisfies \( f(\lambda, \lambda + k) = 0 \) for all \( k \in \mathbb{Z} \). Then \( \mathcal{F}^1_R = I \). In particular, \( I \) is an \( \mathfrak{h} \)-Hopf ideal and \( \mathcal{F}_R/I \) is an \( \mathfrak{h} \)-Hopf algebroid with a non-degenerate pairing on \( \mathcal{F}_R/I \times \mathcal{F}_R/I \).

To get a better understanding of \( \mathcal{F}_R/I \) we consider its representations.

**Proposition 4.8.** Let \( \pi \) be an \( \mathfrak{h} \)-representation of \( \mathcal{F}_R \) on an \( \mathfrak{h} \)-space \( V \). Then the following are equivalent:

(i) \( \pi \) factors to an \( \mathfrak{h} \)-representation of \( \mathcal{F}_R/I \).

(ii) \( V \) has the grading \( V = \bigoplus_{k \in \mathbb{Z}} V_k \) (where some \( V_k \) may be zero).

(iii) Writing \( K = q^{\frac{1}{2}(\lambda - \mu)} \) as in \([4.3]\), the spectrum of \( K \) is contained in \( q^{\frac{1}{2}\mathbb{Z}} \).

This follows immediately from the definitions. Note that condition \((iii)\) is natural both in mathematics and physics: it corresponds to particles with half-integer spin and to representations of SL(2) rather than a covering group. We conclude that \( \mathcal{F}_R/I \) is itself a good analogue of the SL(2) Lie group. In particular, we can interpret Theorem \([4.7]\) as saying that “the” dynamical SL(2) quantum group is self-dual.

Our main tool to prove Theorem \([4.7]\) is the Peter–Weyl theorem for \( \mathcal{F}_R \), proved in \([KR]\), which says that the matrix elements \( t_{k_j}^N \) form a basis for \( \mathcal{F}_R \) over \( \mu(M_{\mathbb{C}})\mu_r(M_{\mathbb{C}}) \).

Roughly speaking, we will prove Theorem \([4.7]\) by constructing the dual basis with respect to our pairing. For this we also need Lemma \([4.3]\).

First we construct the (truncated) projectors onto the isotypic components of the Peter–Weyl decomposition.

**Lemma 4.9.** Let, for \( L, M \in \mathbb{Z}_{\geq 0} \), \( P_{ML} \in \mathcal{F}_R \) be the central element
\[
P_{ML} = \prod_{0 \leq i \leq L \atop i \neq M} (1 - q^{l+1}\Xi + q^{2l+2})
\]
and let \( x, y \in \mathcal{F}_R \). Assume that \( x = \sum_{N \leq L} x_N \) with \( x_N = \sum_{k_j} f_{kj}(\lambda, \mu)t_{k_j}^N \). Then
\[
\langle x, y P_{ML} \rangle = C(x_M, y), \quad 0 \neq C \in \mathbb{C}.
\]
In particular, if \( x \in \mathcal{F}_R^\perp \), then each \( x_N \in \mathcal{F}_R^\perp \).

**Proof.** Iterating (2.7) gives in general

\[
\langle t^N_{kj}, x_1 x_2 \cdots x_m \rangle = \sum_{l_1, \ldots, l_m=0}^N \langle t^N_{kl_1}, x_1 \rangle T_{2l_1-N} \langle t^N_{l_1l_2}, x_2 \rangle T_{2l_2-N} \cdots \langle t^N_{l_{m-1}l_m}, x_m \rangle.
\]

Using this and Lemma 4.3 we see that

\[
\langle t^N_{kj}, p(\Xi) \rangle = \delta_{kj} p(q^{N+1} + q^{-N-1}) T_{N-2k}
\]

for any complex polynomial \( p \). Thus, for \( N \leq L \),

\[
\langle t^N_{kj}, P_{ML} \rangle = \delta_{kj} \prod_{0 \leq j < L \atop l \neq M} (1 - q^{l+1} (q^{N+1} + q^{-N-1}) + q^{2l+2}) T_{N-2k}
\]

\[
= \delta_{kj} \prod_{0 \leq j < L \atop l \neq M} (q^{N+1} - q^{l+1})(q^{-N-1} - q^{l+1}) T_{N-2k} = \delta_{kj} \delta_{MN} C T_{N-2k}
\]

with \( C \neq 0 \). This gives

\[
\langle t^N_{kj}, y P_{ML} \rangle = \sum_t \langle t^N_{kl}, y \rangle T_{2l-N} \langle t^N_{l_j}, P_{ML} \rangle = \delta_{MN} C \langle t^N_{kj}, y \rangle,
\]

from which the statement readily follows. \( \square \)

Next we consider the projection from an isotypic component onto the span of a single matrix element.

**Lemma 4.10.** For \( y \in (\mathcal{F}_R)_{st} \) and \( f \in M_C \otimes M_C \),

\[
\langle f(\lambda, \mu) t^N_{kj}, \gamma^{N-\lambda} \beta^{N-\mu} \rangle = \delta_{kl} \delta_{jm} f(\lambda, \lambda + j - k - s) C(\lambda) T_{-k} \langle t^N_{00}, y \rangle T_{-j}
\]

with \( 0 \neq C \in M_C \).

To prove this we again use the expression (4.8) together with Lemma 4.3. To get a non-zero contribution we must choose \( l_1 = k + 1, l_2 = k + 2 \) and so on up to \( l_{N-l} = k + N - l \), which gives \( k \leq l \). The next \( N \) indices decrease down to \( l_{2N-l} = k - l \), which gives \( k \geq l \); this accounts for the factor \( \delta_{kl} \). Similarly, starting from the right accounts for the factor \( \delta_{jm} \). Keeping track of the bigrading completes the proof.

We are now ready to prove Theorem 4.7. Assume that \( x \in \mathcal{F}_R^\perp \). Using first Lemma 4.3 and then Lemma 4.10 we may assume that \( x = f(\lambda, \mu) t^N_{kj} \), where

\[
f(\lambda, \lambda + j - k - s) \langle t^N_{00}, y \rangle = 0, \quad y \in (\mathcal{F}_R)_{st}.
\]

Choosing \( y = \alpha^s \) for \( s \geq 0 \) and \( y = \delta^{-s} \) for \( s < 0 \), it follows from (4.8) and Lemma 4.3 that \( \langle t^N_{00}, y \rangle \neq 0 \). Thus, \( f(\lambda, \lambda + k) = 0 \) for all \( k \in \mathbb{Z} \). This shows that \( \mathcal{F}_R^\perp \subseteq I \). The reverse inclusion is easy to prove.

**Remark 4.11.** The parameter \( L \) in Lemma 4.9 is only needed to ensure that the projector belongs to \( \mathcal{F}_R \). In particular, we may consider the infinite product \( P_{0\infty} \) as the projector from the regular to the trivial corepresentation, that is, as the Haar functional. More precisely, let \( I_C/I \) be the space \( M_C \otimes M_C \) modulo the ideal generated...
by elements satisfying \( f(\lambda, \lambda + k) = 0 \) for all \( k \in \mathbb{Z} \). We write \( D_Z = (I_{C}/I)^* \); it is easy to check that \( D_Z \simeq \bigoplus_{k \in \mathbb{Z}} (D_C)_{kk} \). We define the Haar functional on \( \mathcal{F}_R/I \) to be the map \( h : \mathcal{F}_R/I \to I_{C}/I \) given by

\[
h(f(\lambda, \mu), t_{kJ}^N) = \delta_{N0} f(\lambda, \mu),
\]
cf. also [KR], and its dual \( h^* : D_Z \to (\mathcal{F}_R/I)^* \) by \( \langle h(a), x \rangle = \langle a, h^*(x) \rangle \). It follows from Lemma 4.9 that

\[
h^*(fT_k) = \begin{cases} f(\lambda) \delta^k P, & k \geq 0, \\ f(\lambda) \alpha^{-k} P, & k < 0, \end{cases}
\]
where

\[
P = \prod_{l=1}^{\infty} \frac{1 - q^{l+1} \Xi + q^{2l+2}}{1 - q^{l+1}(q + q^{-1}) + q^{2l+2}} = \frac{(q^2 \xi, q^2 \xi^{-1}; q)_{\infty}}{(q, q^3; q)_{\infty}};
\]

here \( \xi \) is a formal quantity satisfying \( \xi + \xi^{-1} = \Xi \) and we use the standard notation

\[
(a, b; q)_{\infty} = \prod_{j=0}^{\infty} (1 - aq^j)(1 - bq^j).
\]

The pairing gives a meaning to \( P \) as an element of \((\mathcal{F}_R/I)^*\).

4.5. 6j-symbols. Next we turn to the problem of computing \( \langle t_{kJ}^M, t_{mk}^N \rangle \). This gives the expression for our pairing in the Peter–Weyl basis, or equivalently the action of the basis elements in the representation \( \pi^+_{NL} \). Using [18] together with [10] and Lemma 4.3 to compute \( \langle t_{kJ}^M, t_{mk}^N \rangle \), it is clear that we get a single sum, which turns out to be a terminating \( sW_T \) [GR], or equivalently a quantum 6j-symbol [KR] or \( q \)-Racah polynomial [AW]. In view of Remark 4.4, we may alternatively deduce this from [KR], where the corresponding result for the representations \( \pi^\omega \) was derived.

Due to the large symmetry group of quantum 6j-symbols, there is in fact a large number of single sum expressions. Omitting the details of the derivation, we write down one such expression which exhibits the symmetry of the pairing. We have checked that both methods indicated above yield the same result.

**Theorem 4.12.** One has the identity \( \langle t_{kJ}^M, t_{mk}^N \rangle = \delta_{k+m, j+l = L} fT_{M+N-2L} \), where

\[
f(\lambda) = (-1)^{j+l+L} q^{j+l-L}(\lambda+1+j+l-M-N)-2km \frac{(q^2, q^2, q^2, q^2, q^2)_{M-N}; \cdots}{(q^2, q^2, q^2, q^2)_{M-N}; \cdots} \times \frac{1}{1 - q^{2(\lambda+1-d)}} \frac{q^{2(\lambda+1-d)}; q^2}{q^{2(\lambda+1-d)}; q^2} \frac{q^{2(\lambda+1-d)}; q^2}{q^{2(\lambda+1-d)}; q^2} \times 4\phi_3 \left[ q^{2L}; q^{2L}; q^{2L}; q^{2L}; q^{2L}; q^{2L}; \right]
\]

The \( 4\phi_3 \)-sum appearing here is defined by [GR]

\[
4\phi_3 \left[ q^{-k}, q^{-m}, a, b; c, d, e; q, z \right] = \sum_{n=0}^{\min(k,m)} \frac{(q^{-k}; q)_n (q^{-m}; q)_n (a; q)_n (b; q)_n}{(q; q)_n (c; q)_n (d; q)_n (e; q)_n} z^n.
\]
Lemma 4.13. One has
\[ \Psi(t_{kj}^N) = C_{kj}^N(\lambda, \mu) t_{jk}^N, \]
where
\[ C_{kj}^N(\lambda, \mu) = q^{\frac{1}{2} \lambda(2j-N)+\frac{1}{2} \mu(N-2k)+(j-k)(j+k+2-N)} \frac{\binom{N}{j}}{\binom{N}{k}} q^2 (q^{2(\lambda+2)}; q^2)_{N-j} \frac{(q^{2(\mu+2)}; q^2)_{N-k}}{(q^{2(\mu+2)}; q^2)_{N-j} (q^{2(\lambda+1-j)}; q^2)_{N-j}}. \]

This can be proved similarly as Proposition 3.12 of [KR].

Combining Proposition 3.18 with Theorem 1.12 and Lemma 4.13 gives that the matrix
\[ R_{lm}^{jk}(\lambda; M, N) = \{ t_{km}^N, t_{jl}^M \} 1(\lambda) = \frac{1}{C_{mk}^M(\lambda + M - 2j, \lambda)} \langle t_{mk}^N, t_{jl}^M \rangle 1(\lambda) \]
\[ = \delta_{j+k,l+m=L} (-1)^{j+l} q^{(l-j)(2j+2l-M-L-1)+\frac{1}{2} MN-Mk-Nj} \]
\[ \times \frac{\binom{q^2; q^2}{(q^2; q^2)_j} (q^2)_{M-j} (q^2; q^2)_{M-l} (q^2; q^2)_{M-j} (q^2; q^2)_{M-l}}{(q^2; q^2)_j (q^2; q^2)_{M-j} (q^2; q^2)_{M-l} (q^2; q^2)_{M-j}} \]
\[ \times 4\varphi_3 \left[ \frac{q^{-2j}, q^{-2m}, q^{2(j-M-\lambda-1)}, q^{2(m-N-\lambda-1)}, q^{-2\lambda}}{q^{-2L}, q^{2(L-M-N-\lambda-1)}, q^{2(L-M-N-\lambda-1)}, q^{-2\lambda}} \right] \]
satisfies the dynamical Yang–Baxter equation of the form (here \((l_1, m_1, n_1, l_2, m_2, n_2)\) corresponds to \((a, b, c, d, e, f)\) in (2.13))
\[ \min(M, m_1+n_1, l_2+m_2) \sum_{y=\max(0, m_1+n_1-N, l_2+m_2-L)} R_{l_2,m_2}^{l_1+m_2-y, y} (\lambda - 2n_2 + N; L, M) R_{l_2+m_2-y, n_2}^{l_1, m_1+n_1-y} (\lambda, L, N) \]
\[ \times R_{y,m_1+n_1-y}^{m_1+n_1} (\lambda - 2l_1 + L; M, N) = \min(M, l_1+m_1+m_2+n_2) \sum_{y=\max(0, l_1+m_1-L, m_2+n_2-N)} R_{m_2+n_2-y}^{y,m_2+n_2} (\lambda; M, N) \]
\[ \times R_{l_2,m_2+n_2-y}^{l_1+m_1+n_1-y} (\lambda - 2y + M; L, N) R_{l_1+m_1-y, y}^{l_1+m_1} (\lambda; L, M). \]

This is (for \(\lambda\) discrete) the hexagon identity for \(6j\)-symbols, first proved by Wigner [W] for \(q = 1\) and by Kirillov and Reshetikhin [KR] in general. Thus, dynamical quantum groups provide an alternative algebraic framework for studying \(6j\)-symbols (in [KR] we derived the pentagon or Biedenharn–Elliott relation using this framework).

REFERENCES

[AW] R. Askey and J. A. Wilson, A set of orthogonal polynomials that generalize the Racah coefficients or 6-j symbols, SIAM J. Math. Anal. 10 (1979), 1008–1016.
[BBB] O. Babelon, D. Bernard and E. Billey, A quasi-Hopf algebra interpretation of quantum 3-j and 6-j symbols and difference equations, Phys. Lett. B 375 (1996), 89–97.
[EN] P. Etingof and D. Nikshych, Dynamical quantum groups at roots of 1, Duke Math. J. 108 (2001), 135-168.
[EV] P. Etingof and A. Varchenko, Solutions of the quantum dynamical Yang–Baxter equation and dynamical quantum groups, Comm. Math. Phys. 196 (1998), 591-640.
[EV2] P. Etingof and A. Varchenko, Exchange dynamical quantum groups, Comm. Math. Phys. 205 (1999), 19–52.
[F] G. Felder, *Elliptic quantum groups*, XIth International Congress of Mathematical Physics (Paris, 1994), 211–218, Internat. Press, Cambridge, MA, 1995.

[FV] G. Felder and A. Varchenko, *On representations of the elliptic quantum group \( E_{\tau,\eta}(\mathfrak{sl}_2) \)*, Comm. Math. Phys. 181 (1996), 741–761.

[GR] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, Cambridge University Press, Cambridge, 1990.

[GN] J.-L. Gervais and A. Neveu, *Novel triangle relation and absence of tachyons in Liouville string field theory*, Nucl. Phys. B 238 (1984), 125–141.

[JKOS] M. Jimbo, S. Odake, H. Konno and J. Shiraishi, *Quasi-Hopf twistors for elliptic quantum groups*, Transform. Groups 4 (1999), 303–327.

[K] C. Kassel, *Quantum Groups*, Springer-Verlag, New York, 1995.

[KIR] A. N. Kirillov and N. Yu. Reshetikhin, *Representations of the algebra \( U_q(\mathfrak{sl}(2)) \), \( q \)-orthogonal polynomials and invariants of links*, Infinite-dimensional Lie Algebras and Groups, 285–339, World Sci. Publishing, Teaneck, NJ, 1989.

[KR] E. Koelink and H. Rosengren, *Harmonic analysis on the \( SU(2) \) dynamical quantum group*, Acta Appl. Math., to appear.

[M] S. Majid, *Braided matrix structure of the Sklyanin algebra and of the quantum Lorentz group*, Comm. Math. Phys. 156 (1993), 607–638.

[R] H. Rosengren, *A new quantum algebraic interpretation of the Askey–Wilson polynomials*, Contemp. Math. 254 (2000), 371–394.

[W] E. P. Wigner, *On the matrices which reduce the Kronecker products of representations of S. R. groups* (1940), in L. C. Biedenharn and H. Van Dam (eds.), *Quantum Theory of Angular Momentum*, 87–133, Academic Press, New York, 1965.

[X] P. Xu, *Quantum groupoids*, Comm. Math. Phys. 216 (2001), 539–581.