Idea engines: A unified theory of innovation and obsolescence from markets and genetic evolution to science

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Innovation and obsolescence describes dynamics of ever-churning and adapting social and biological systems from the development of economic markets and scientific progress to biological evolution. The shared aspect of this picture is that agents destroy and extend the “idea lattice” in which they live, finding new possibilities and rendering old solutions irrelevant. We focus on this aspect with a simple model to study the central relationship between the rates at which replicating agents discover new ideas and at which old ideas are rendered obsolete. When these rates are equal, the space of the possible (e.g. ideas, markets, technologies, mutations) remains finite. A positive or negative difference distinguishes flourishing, ever-expanding idea lattices from Schumpeterian dystopias in which obsolescence causes the system to collapse. We map the phase space in terms of the rates at which new agents enter, replicate, and die. When we extend our model to higher dimensional graphs, cooperative agents, or inverted, obsolescence-driven innovation, we find that the essential features of the model are preserved. In all cases, we predict variation in the density profile of agents along the spectrum of new to old such as a drop in density close to both frontiers. When comparing our model to data, we discover that the density reveals a follow-the-leader dynamic in firm cost efficiency and biological evolution, whereas scientific progress reflects consensus that waits on old ideas to go obsolete. We show how the fundamental forces of innovation and obsolescence provide a unifying perspective on complex systems that may help us understand, harness, and shape their collective outcomes.

Understanding the dynamics and structure of innovation and obsolescence has been a sub-
ject of considerable interest across many domains ranging from business, economics, and technology to evolutionary biology, medicine, and the physical sciences. The forces of innovation and obsolescence define classical capitalist markets, summarized in Schumpeter’s iconic term “creative destruction,” based on the idea that new methods of production survive by eliminating existing ones (1). This is echoed in Spencer’s description of evolution as the “survival of the fittest,” and in Spielrein’s “destruction as the cause of coming into being” for psychological development (2). In the natural sciences, we have the more charitable adage from Newton that we build “on the shoulders of giants.” Each of these aphorisms implies that the new destroys or eclipses the old. A key point is that innovation of one thing often causes the obsolescence of another. A second key point is that agents, such as firms, organisms, or scientists, are themselves creating technologies, behaviors, or capacities that they could adopt from the set of the possible, while disregarding the irrelevant. In the substantial literature, each area has developed its own particular formalization of the problem that has masked their fundamental similarities. Furthermore, there have been few attempts to formulate the problem of innovation and obsolescence in an analytic framework that is quantitative, predictive, and testable. This paper fills that gap by developing a general theory that captures essential features of shared dynamics, leads to quantitative predictions supported by data, and thus connects diverse systems within a unified theory.

A primary distinction between fields is the relationship assumed for the dynamics between innovation and obsolescence. In Schumpeter’s creative destruction, the relationship is one of conservation, where productive innovation comes at the expense of an existing mode of production. A realization of this is the study of economic competition as new methods of production are innovated (3, 4). It follows that firms live on a line of productivity margins (5, 6), and obsolescence occurs endogenously either from desuetude or unsustainable profit margins (7). In the study of scientific progress (8–10), social change (11), and biological evolution (12) obsolescence is the complement of innovation: reduced citation rates imply that articles are forgotten, norms switch in a binary way from one to another (13), and extinction is a natural result of being outcompeted (2). Other examples, however, reveal more complicated relationships between innovation and obsolescence. In markets, obsolescence may be driven by external research developments funded by government programs (e.g. GPS, Internet, mRNA vaccines) or by products from technologically advanced neighbors. Some innovations open many more possibilities than they close. In biological evolution, obsolescence may depend on environmental shifts (14). Thus, a general theory must encompass different relationships between innovation and obsolescence of which “creative destruction” or complementarity are special cases.

A secondary distinction between fields is the choice of how to map a real system onto an “idea lattice” and agent. For example, a lattice site may represent a product offered or a manufacturing method used by a firm. Alternatively, the graph could be new mutations in a population or a time-ordered list of topics that have emerged in the scientific literature. The definition of the unit of innovation has been an especially vexing problem in biology, where innovations can be genotypic, phenotypic, behavioral, or environmental (15). Such distinctions
Agents are sets of single or multiple ideas on a dynamic idea graph with length $L$ that grows along the innovation front at $x = 0$ into the adjacent possible to the right and is obliterated along the obsolescence front to the left at $x = L - 1$. 

Figure 1: Model diagram on (a) a random graph structure, (b) linear lattice projection, (c) tree.
can be important. For example, resource constraints shape organism metabolism (16, 17), or physics constraints the distribution of new mutations (18, 19). Despite these differences, a shared, elementary process of competition and tension between innovation and obsolescence takes place in all these systems.

We picture this process to take place in a space of possibilities, an “idea lattice,” in which agents live and that is itself constantly churning as shown in Fig. 1a. Here, each vertex $x$ represents an “idea” in which an agent is invested. Ideas share an edge with related ideas, denoting either material similarity, shared inputs and skills, or common ancestry. Clustering of similar ideas allows us to compress related items into a single site, which leads to the linear lattice approximation in Fig. 1b. This simplification is equivalent to choosing a scale of analysis, where agents can be neatly assigned to sets of ideas, such as by clustering them with industry sector codes or distinct mutations (20, 21). These assumptions lead us to the linear idea lattice as shown in Fig. 1b, where we define an obsolescence front where ideas go defunct (vertices removed) and an innovation front, where agents drive ahead into the “adjacent possible” (vertices added) (22).

We consider simple agent dynamics that represent their mean tendencies to grow, die, and replicate in a way that summarizes more detailed processes. New agents such as firms, mutants, or publications enter the system with a rate $G$ distributed uniformly amongst the number of lattice sites $L(t)$ at time $t$. Agents may replicate into the adjacent innovative site with rate $r$ — for example, by copying more innovative ideas — and leave the system with death rate $r_d$. Our treatment considers these as two basic independent parameters whose variation and interrelationship maps to several types of dynamics and allows for a wide range of scenarios.

At the innovation front, agents encounter the additional difficulty of inventing what is possible before occupying it. In the case where agents innovate independently, the rate of successful innovation is proportional to how often they seek to expand, $r$, their innovativeness, $I$, and the number of agents at the innovation front, $n(x = 0, t)$, thereby summarizing the complex process of discovery in terms of a mean rate (23, 24) (see Appendix A).

Finally, we incorporate obsolescence by assuming that the oldest idea goes obsolete with a rate $r_o$. Consequently, the rate of change of the lattice size $\dot{L}(t)$ is the difference between the rates of innovation and obsolescence,

$$\dot{L}(t) = rIn(0, t) - r_o. \quad (1)$$

A minimum length $L = 1$ corresponds to when the two fronts coalesce. When the lattice size is stable $\dot{L}(t) = 0$, the system looks like an “innovation train” moving into the innovation frontier with innovativeness directly related to obsolescence with

$$n(0, t) = r_o(rI)^{-1}. \quad (2)$$

To simplify the mathematical treatment, we imagine “sitting” on the train and fixing the coordinate system such that the innovation front is at the origin, with its movement represented by the
Figure 2: Flourishing, static and dystopian dynamics in automaton simulation measured by (a) lattice width over time, (b) occupancy number at innovation front, and (c) shape of “pseudogap” averaged over time and lattice widths. (b) Dashed lines indicate steady-state conditions, which only \( r_o = 1/2 \) satisfies after initial transience. (c) Error bars show standard deviation over last \( 10^3 \) time steps. Shared colors indicate the same three conditions across all panels with blue for flourishing, orange for static, and green for dystopian. Parameters \( G = 4, r = 0.395, r_d = 0.4, I = 0.9 \).
train tracks moving past us. Putting these together, the rate of increase in the number of agents \( \dot{n}(x, t) \) at lattice site \( x \) at time \( t \) is

\[
\dot{n}(x, t) = \frac{G}{L(t)} + rn(x + 1, t) - rdn(x, t) - rIn(0, t)[n(x, t) - n(x - 1, t)].
\] (3)

The first term is the rate at which new firms enter the system, the second the rate at which they expand by mimetic innovation (25), the third the rate at which they leave the system, and the last term the effective shift from the motion of the innovation front. We solve Eqs 1 and 3 using both analytic approximation and numerical calculation including simulations that are further detailed in Appendix B. These results summarize the mean-field dynamics of a simple system of agents growing, dying, and innovating on a one-dimensional space.

Eq 1 predicts three regimes of idea graph dynamics: i) obsolescence outpaces innovation and the system collapses to only a few, transient ideas; ii) innovation outpaces obsolescence and the system grows indefinitely, providing an unbounded “marketplace” for exploitation; and iii) innovation and obsolescence are roughly balanced, leading to a typical size of the idea space. In order to demonstrate the three regimes, we provide three examples of a stochastic automaton simulation following the dynamics specified in Eqs 1 and 3, and illustrated in Fig. 2 (Appendix B). When the rate of obsolescence \( r_o \) is sufficiently large (green line), we find a small lattice that repeatedly collapses to its minimum size \( L = 1 \). As we decrease \( r_o \), we pass through a regime of fixed lattice size (orange line) to a regime of infinite growth (blue line). In analyzing these regimes, we start with a stationary configuration in which each innovation extinguishes one old idea.

Eq 2 states that obsolescence and innovation rates must match, which is a fundamentally collective property. Global stability also means that the number of agents on the leading edge is proportional to the rate at which ideas go obsolete. This is a direct consequence of the fact that the solution must be locally stable. In other words, when the number \( n(0, t) \) is above steady state, the innovation front temporarly drives ahead quickly, reducing the number of innovative agents. This reduced number then slows the innovation front down and allows for agents to flow in from the existing lattice, resulting in oscillations around steady state occupancy (Fig. 2b). Thus, the age of an idea is naturally related to the number of agents, which itself depends on the obsolescence rate. This result agrees with the intuition, for example, that firms expand into “adjacent markets” because there is less competition (26).

Following the logic that older ideas will have accumulated more agents, we expect the number of agents to be minimal near the innovation front and to increase as we move towards older ideas. This describes a “pseudogap”, or a drop in density (of excited states in physics), around \( x = 0 \). If this is assumed to be approximately linear at stationarity, i.e. \( \Delta n(x = 0) \approx \text{constant} \), then \( n(1) - n(0) \approx n(0) - n(-1) \). Since, by definition, the site at \( x = -1 \) is unoccupied, i.e. \( n(-1) = 0 \), this gives \( n(1) \approx 2n(0) \). Setting \( x = 0 \) in Eq 3 we can then solve for lattice size at stationarity:

\[
L = GIrr_o^{-1}[r_o + r_d - 2r]^{-1}.
\] (4)
While Eq 4 is only a crude approximation to a nontrivial solution with surprisingly complex variation (see Appendix A and Fig. A2), it provides a rather revealing starting point for capturing the essential characteristics of the resulting idea space.

Importantly, two critical points are indicated by Eq 4: (i) where obsolescence is perpetually outpaced by innovation and $L \to \infty$; and (ii) where obsolescence outpaces innovation and $L \to 1$ signaling the collapse of the system. Formally, the first can occur when $G$, $I$, or $r$ become infinite, or when $r_o = 0$, none of which is realistic. On the other hand, the singularity at $r_o + r_d - 2r = 0$ requires only balancing $r_o/r$ and $r_d/r$ such that

$$r_o/r \leq 2 - r_d/r. \quad (5)$$

We delineate this region in red in Fig. 3. Eq 5 indicates that the typical number of times an agent reproduces before it dies is a crucial order parameter, and that agent-level properties can drive unbounded growth of the idea lattice.

The second critical point is reached when the innovation front number falls below a self-sustaining threshold, $n(0,t) < (r/r_o)I$. From Eq 1, growth becomes negative, $\dot{L}(t) < 0$, driving the system to collapse to its minimum length $L \sim 1$. From Eq 4, we can solve for the corresponding cutoff obsolescence rate as

$$r_o/r \sim 1 - r_d/2r + \sqrt{(1 - r_d/2r)^2 + GI/r}. \quad (6)$$

In the limit $GI/r \to 0$, the boundary between collapse and growth shrinks to a line and eliminates the region of stability (white region in Fig. 3B). Consequently, we expect to find a sharp transition between collapse and growth, and the system can display long time scales and large fluctuations (Appendix D). Thus, Eqs 5 and 6 elucidate the three regimes of lattice dynamics: a collapsed and narrow regime, a growing divergent regime, and a balanced regime. These three regimes characterize how innovation and obsolescence determine idea diversity.

Furthermore, this picture holds for several key generalizations such as cooperative innovation, more complicated graph topologies, and inverted obsolescence-driven innovation, a reversed picture where obsolescence furthers system progress as we clarify below:

(i) Cooperative innovation implies that the front velocity scales nonlinearly with the number of agents as, for example, $rIn(0)^\alpha$. Here superlinearity, $\alpha > 1$, signals cooperation, whereas sublinearity, $\alpha < 1$, implies anti-cooperation. From Eq 2, these differences can be mapped back to Eq 3 by the simple transformation $I^\alpha \to I$ and $(r_o/r)^\alpha \to r_o/r$.

(ii) For tree graphs, shown in Fig. 1C, each sequential site branches into $Q - 1$ additional branches, one of which must be chosen by a new agent. If branches are equally likely, the replication term in Eq 3 acquires an additional factor $(Q - 1)^{-1}$ (see Eq S27), such that the number of agents systematically decreases towards the innovation front. This argument makes clear the importance of the relative dimensions of agent replication and the idea space. When next-generation agents do not fill all of the available space, then agents necessarily occupy a small fraction of the idea space. Nevertheless, the dimensional depletion effect does not
Figure 3: (a) Regimes in which lattice size collapses (blue), stabilizes (white), or grows indefinitely (red) from linear approximation in Eqs 5 and 6 along rescaled rates. Green star indicates fit to firm cost efficiency from Fig. 4. Parameters $G/r = 1$, $I = 1$. (b) Anti-cooperative innovation, $\alpha = 1/2$. (c) Cooperative innovation, $\alpha = 3/2$. (d) Growing tree with branching ratio $Q = 3$. 
fundamentally alter the dynamics. If we rescale \( r \rightarrow (Q - 1)r \) and \( I \rightarrow I/(Q - 1) \), then we again recover Eq 3. As argued in the Appendix, higher Euclidean dimensions can also be closely approximated by the linear model. Thus, important classes of dynamical or structural generalizations do not appreciably alter the basic model and can be captured by parameter rescaling.

(iii) Obsolescence-driven innovation is the antithesis of forward-looking innovative dynamics. By reversing the direction of the \( x \)-axis in Fig. 1 and setting the innovation front at the origin, agents now replicate towards obsolescence. For stable pseudogaps, the introduction of a new idea propels every agent towards innovation, making the entire system more innovative. As a result, innovation is driven at a rate proportional to the number of ideas on the verge of extinction. Newer ideas tend to beget agents with older ideas, which then drive themselves to extinction by eventually increasing occupancy at the obsolescence front. This is similar to scientific progress, where invalidation of old ideas permits new ideas, new ideas stimulate revision of existing topics, and the system as a whole progresses (32).

**Innovation distributions form a taxonomy**

The fact that Eq 3 incorporates these different extensions of the model implies that its resulting pseudogap form predicts a characteristic taxonomy: either an exponential rise to a skewed peak then with a rapid decay, or a sublinear rise to a flat occupancy function characterized by agents piling up near the innovation front (see Appendix Fig. D8). These constitute types of the innovative adjacent possible, which is complemented at the other end by types of the obsolescent adjacent possible. Between the sublinear and superlinear pseudogaps, we find that the pseudogap disappears as lattice length diverges. We illustrate several examples in Fig. 4 of innovation distributions in markets (Schumpeterian), genetic evolution (Darwinian), and science (Socratic) in terms of occupancy number vs. their distance from the innovation front. Do these predicted functional forms align with empirical examples?

Economics provides the classic example of Schumpeterian innovation measured in terms of cost efficiency, or the ability to extract profit from the same amount of investment (5–7, 33). This imposes a natural one-dimensional axis for ordering firms, where more innovative firms progressively decrease cost per unit output. Fig. 4b shows an example of the distribution of labor costs per value added for the US metal stamp industry in 1958 and 1963 from Iwai’s classic work (5, 6). The fewest number of firms are defined by being the most or least cost-efficient, and the distribution is skewed towards lower efficiency because many firms with higher costs still survive. This characteristic form also appears in recent distributions of cost efficiencies in Indian firms (27). In these cases, we find skewed peaks with flattened density near the innovation front, as in Figs. 4d and g, alongside which we sketch similar examples of Schumpeterian innovation number predicted by our model, which agrees well with the data.

As an example of biological, or Darwinian, innovation we consider the tree of SARS-CoV-2
Figure 4: Taxonomy of the innovative adjacent possible as innovation distributions in (a–c) markets, genetic evolution, and science. Occupancy number from (d, g) firms, (e) genetic mutations, and (f, i) scientific citations. Occupancy number for (d) U.S. metal stamping industry for two years (5) and (g) non-financial Indian firms grouped by size (27). Aligned model (black) aims to compare qualitative shapes for finite time points $t$ in simulation when labeled, otherwise at stationarity $t \to \infty$ in remaining panels. (e) SARS-CoV-2 diversity by number of base pair mutations from first detected strain hCoV-19/Wuhan/Hu-1/2019 normalized by branching rate (28, 29). We cannot match both the peak near zero mutations and the rise and decay of the peak because of functional constraints. Here, we fit to the rise and decay around the peaks, implying missing strains near the beginning of the pandemic (30). (f) Citation rates per year normalized by typical number of citations per patent annually for patents filed in 1994 and 1998. (i) Citation rate for *Physical Review B* articles published in 1980 normalized by the typical number of citations per paper annually and binned by total number of citations within the first five years. Sharp Einsteinian innovation events change the very parameters of innovation such as with (h) the emergence of the Delta clade in and (i) fast mutator strain in long-term *E. Coli* experiments (31) where starred. Details on alignment in Methods.
Table 1: Shortest distance of fit to systems to divergent regime. Starred example corresponds to green star in Fig. 3.

| system                                      | distance to divergence |
|---------------------------------------------|------------------------|
| SARS-CoV-2 (North Am.)                      | 0.33                   |
| SARS-CoV-2 (Europe)                         | 0.33                   |
| US patents                                  | 0.01                   |
| non-financial Indian firms (small)          | 0.02                   |
| non-financial Indian firms* (large)         | 0.02                   |
| Physical Review B (little cited)            | 0.16                   |
| Physical Review B (highly cited)            | 0.07                   |

clades measured from the GISAID repository of sequences (28, 29). The set of possible innovations, measured by base pair mutations, is an exponentially expanding phylogenetic tree with a branching ratio $Q \approx 2.3$ per unit phylogenetic branch length (see Methods). This estimate depends slightly on whether we consider the set of strains in North America or Europe. Since sequencing coverage was not uniform throughout time, we cannot use the date at which SARS-CoV-2 sequences were entered into the database. Instead, we take the number of mutations from the first known strain hCoV-19/Wuhan/Hu-1/2019 to order mutants in exact innovative order. As shown in Fig. 4e, the resulting occupancy plot again shows the characteristic form of replication-dominated systems that are driven by the innovative frontier determined by an evolutionary tree (34).

Notably, our model does not capture the sharp, temporary increase when the Delta variant emerged, indicating an important deviation from model predictions: these punctuated, unpredictable, and singular (i.e. Einsteinian) innovations reflect a natural violation of our model assumptions — though Einsteinian innovations could be incorporated into our model through the discontinuous shift in parameters. We highlight an example in Fig. 4h, where only the number of Delta clades are shown. A similar Einsteinian innovation is revealed in long-term evolution experiments, where the sudden emergence of a mutator strain increases the mutation rate in an *E. Coli* population, indicated by the discontinuity in accumulated mutations in Fig. 4i (31). These punctuated changes in innovation rates reveal a meta-dynamic that would change the very parameters of our model.

Science and technology, in contrast, is more systematic. New technological and scientific ideas must be tethered to the past and are often judged by their consistency with established knowledge, theory, and pedagogy (35). It is only when existing frameworks have been proved insufficient that a new idea can flourish. This suggests a reversed dynamics, where innovation is driven by the obsolescence of old ideas.

Remarkably, patent and scientific citation rates support this picture. Citation rates peak to maximal prominence quickly then slowly fall out of favor with age, a horizontally mirrored
version of the previous examples. In our formulation, the citation profile is proportional to the occupancy function in that agents move across a graph of sequential papers that mark the progression to new ideas. Since papers contribute different levels of innovation attributed to some intrinsic fitness \( (8, 32, 36, 37) \), we bin them based on their cumulative citation number in their first five years (see Methods). Fig. 4 shows examples of citation rates normalized to account for yearly variation of patents (panel f), and similarly of scientific papers (panel j). Again, our model captures the characteristic features of citation counts including the sharply rising peak and the slower decaying tail. For patents the tail is exponential \( (37) \), but is longer for scientific citations, reflecting both preferential attachment and proposed but debated “runaway” effects \( (37–39) \), neither of which we include. Importantly, however, innovation and obsolescence dynamics lead to a first-principles explanation for citation curves that complements microscopic models of how authors choose citations \( (40) \) despite limited attention \( (39) \).

These examples demonstrate how the dynamics of innovation and obsolescence predicts the distribution of social and biological agents from the innovative to the obsolete. We find across all these systems evidence for an exponentially growing pseudogap, which suggests a dominant agent replication rate, \( r \), and, intriguingly, that they all operate close to a divergence threshold. Indeed, a heuristic fit to each of the systems plotted alongside the data in Fig. 4 reveals parameters remarkably close to the respective divergent region, as shown in Table I. Since sizeable lattice lengths are all located near this boundary, our model suggests that only systems close to the edge of divergence manifest interesting innovation dynamics. Whether or not this dynamic is representative of the ensemble of systems in Nature, or whether evolutionary pressure drives systems towards this boundary, are questions that our framework raises by unifying diverse systems within a common space of innovative vitality.

**Discussion**

System constituents of social and biological systems constantly undergo turnover, expanding, exploiting, or reducing the space of realized capabilities in a sequence of innovations that eventually become obsolete. In any particular system, the exact details of this process may be determined by competition \( (7) \), innovative risks \( (41) \), resource constraints \( (42) \), and strategy \( (43) \), amongst other factors. Furthermore, we know that the process of innovation displays combinatorial dynamics \( (23, 24, 44) \), depends on the topology and dimension of the adjacent possible \( (32) \), and is influenced by agent interaction \( (24, 45) \). While important, such diversity obscures the fact that the same effective dynamics couple agents with the lattice on which they live. The power of our generalization originates from incorporating these processes into mean rates that highlight three fundamental innovative regimes common across dimensional, structural, and dynamical extensions of the basic model.

As a point of departure, we analyze variation in agents close and far from the innovative frontier for a fixed idea lattice size. Surprisingly, we discover strong commonality with the
predictions of innovative dynamics and firm productivity, genetic evolution, and scientific citation. We find a skewed density of agents with a peak opposite a much slower decaying tail indicating asymmetry in the relative rates of agent death and obsolescence. This is true even for contexts that reverse the orientation of the dynamics for innovation-driven vs. obsolescence-driven systems. Importantly, our theory fails to capture sudden changes in mutation rates or heavier densities near the innovative front for firms. Such effects could reflect unpredictable “Einsteinian” revolutions that modify the parameters of innovation, obsolescence, competitive effects, or atypical scientific papers with unusual citation trajectories. These deviations present additional, exciting questions to investigate as extensions of our basic model.

A natural question that arises from this work is whether our framework could be used to promote innovative economies, inhibit viral evolution, or shape scientific progress. Our starting principles lead to several relevant insights. First, some system parameters may be more opportune than others in forcing a transition in idea lattice dynamics. As we show in Eq 5 balancing relative death and obsolescence rate is realistic, but not growth or innovativeness. A second intriguing prediction of our model is that transitions from dystopian and flourishing regimes can be sudden (Appendix D). The fluctuations near the critical point highlight an opening for dynamical classification of systems through rate parameters or signaling when systems are on the verge of collapse (46), and it may reflect endogenous dynamics that drive large-fluctuations in biodiversity (14, 47), economic growth (48), or scientific decline (42). This puts forth the possibility of a comparative meta-dynamics, where we envision tracking systems in the innovative-obsolescence space with finer-scale dynamical data. Our model organizes these hypotheses and opens up a new framework for thinking about the forces of innovation and obsolescence. After all, an engine may explode from having too much fuel or putter out from having too little, so likewise, an idea engine also need be fine-tuned.
Methods

Data analysis

We test our model with several data sets. Here, we describe the sources of the data sets and how we calculated the values that we show in Fig. 4.

The distributions of firm cost efficiency in Fig. 4a are digitized from Fig. 1 of reference (5) by Iwai, and the densities in Fig. 4b are digitized from the third panel of Fig. 3 from reference (27) by Jangili, which derives an estimate of the relative cost efficiency of firms using a technique called stochastic frontier analysis (49). In short, this technique involves estimating from a given set of measures about firms (e.g. size, age, liquidity, leverage, capital labor ratio) that convey information about the costs firms incur, the maximal cost efficiency that firms could hope to obtain. When cost efficiency is unity, firms have reached such an ideal. As both Iwai’s and Jangili’s work shows, the shape of firm distribution is relatively consistent over long periods of time, and the distribution does not drift towards perfect cost efficiency of one. Iwai also notes that this observation seemed to generalize across industries at the time of his investigation, citing Sato’s 1975 publication. In agreement with this note, Jangili’s Figs. A.1 and A.2 are of particular interest, which show nearly the same inferred distribution of firm cost efficiency over 20 years. Another example of this characteristic shape is for European financial firms (50).

Since older, larger firms tend to become more cost-efficient over time, we anticipate that newer, smaller firms are closer to the innovative frontier. This is different from the standard orientation of the cost efficiency axis that takes more innovative firms to be the ones that are more cost efficient. That the distribution is preserved over time seems to imply, in our framework, that the underlying lattice is drifting to the right such that older firms (and thus larger ones (51)) become increasingly cost-efficient but also obsolete.

As is detailed in reference (5), Iwai obtains the data from Sato (1975) that were originally obtained from the U.S. Department of Commerce. As is detailed in reference (27), Jangili samples 11,410 non-financial Indian firms between the years 1995-2014 listed in the PROWESS database maintained by the Centre for Monitoring Indian Economy, but the data is not publicly available. Small firms and large firms are distinguished by either belonging to the first or last quartile, respectively, of the distribution of total assets.

We obtain the SARS-CoV-2 clade data from the Nextstrain project downloaded on January 5, 2022. We use the inferred phylogenetic trees based in the GISAID sequence repository that contains millions of global samples of SARS-CoV-2 strains. We focus on the European and North American subset since we expect these to be particularly well-sampled though generally it is impossible to track all circulating strains. After mapping the imputed phylogenetic tree from Nextstrain into distance marked by base pair mutations, we calculate the average number of branches into which any unit length of the tree branches. In other words, a non-branching unit

1. https://prowessiq.cmie.com
2. https://nextstrain.org/ncov/gisaid/global
contributes branching ratio of $Q = 2$, one that leads to two clades $Q = 3$, etc. We find for the downloaded phylogenetic history the averaged branching ratio $Q = 2.317$ for North America and $Q = 2.344$ for Europe. It is substantially larger for the Delta branch, $Q = 2.375$, and Omicron, $Q = 2.717$. Using the calculated value, we normalize the number of unique individual clades as by $(Q - 1)^y$, where $y$ is the number of mutations from the original detected strain, in order to calculate the typical number of strains per branch. This will lead to an exponential decay only if the number of detected strains remains constant as is the case in the data far from hCoV-19/Wuhan/Hu-1/2019.

The patent and article citation data in panels f and i come from the US Patent and Trademark Office and American Physical Society’s repository for Physical Review B (PRB), respectively. For PRB citations, we consider papers of the same cohort, published in 1980, and only citations to and from the universe of PRB papers. We first bin these papers by total cumulative citations to the papers within the first five years of publication as a measure of fitness. To obtain a citation rate that accounts for the approximately exponential growth in the number of scientific papers, we normalize the citation counts for each paper published in 1980 by the typical number of citations made by every paper published in each following year, i.e. we count effective citations relative to the typical number of citations made per paper per year. Patent citations show a similar pattern, where the number of patents increases over the years as well as the number of citations made by each patent (37). We again normalize citation count by the typical number of citations made by each patent for each year to obtain the trajectories in two different starting years. We do not consider highly cited patents separately from less cited patents in order to present a comparison of temporal dynamics with firm occupancy number in panel d.

These empirically derived functions are consistent with the hypothesis that scientific ideas are driven by the extinction of obsolete ones. In the case of patent citations, our model agrees with the exponential tail but not for scientific citations, though the exact shape is debated for scientific citations (37). Nevertheless, the overall shapes align with our obsolescence-driven formulation.

Our analysis implies that the majority of agents in the scientific system are closer to the innovative edge. While this means that the system as a whole is relatively innovative, this is not because agents are quickly catching up but because the edge is slow enough for agents to accumulate at it. This is partly a result of the fact that driving an idea obsolete in the stationary case is like making every individual more innovative and driving out the most innovative individuals from the system. The exit of the most innovative, remarkably, aligns with the evidence that scientific experts are not particularly good at valuing novel ideas (52). Firms and viral evolution, on the other hand, are forward-looking and that means that fewer agents are near the innovative edge. We emphasize that such a result implicitly depends on factors like financial viability, competitive advantage, or economic regulation, but it can also be stated as in terms of the age of an idea: when new agents are more innovative than previous

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3https://patentsview.org/download/data-download-tables
agents and are viable above a certain rate, a wide open space in occupancy appears. While the data is necessarily limited to the successful innovations, our results show how stripping away all but the essential details clarifies how the dynamics in different systems reflect a shared underlying process.

**Fitting the model**

Building an automated fitting routine to match the model with data is difficult because of non-linearities that lead to discrete jumps in \( L \) (see Figure A2) and degeneracies between parameter combinations. As a result, we find that numerical optimizers based on minimizing the distance between two curves struggle to find optimal parameters and fail even the base case of recovering the parameters of a curve obtained from the model. In order to handle such difficulties with data comparison, we rely on a heuristic approach that leverages intuition for each of the parameters to capture the main features of the curve.

First, we note that the super or sublinearity of the innovative pseudogap depends on the ratio \( 1 - r_d/r \) over \( 1 - r_o/r \) as revealed in the first-order ordinary differential equation presented in Appendix A.2. It tends to be the case that when this ratio is negative, we have a superlinear shape that curves upwards. When positive, we tend to have a sublinear shape that flattens out far from the innovation front. In all of the data sets we consider, we find indications of the superlinear regime.

Second, the size of the ratio determines the slope of the exponential. For example, the ratio \( r_d/r \) that deviates further from unity leads to faster decay, which effectively controls the slope of the exponential tail.

Third, lattice length grows with \( G \) and \( I \) in a way that does not affect strongly the previously mentioned features. If either of these are sufficiently small, we obtain a short hump like that fit to Iwai’s data. Increasing any of these parameters will lengthen the exponential tail.

Finally, we can in some cases rescale our model along the lattice coordinate and the occupancy number to convert the units to that of data. Clearly, rescaling of the lattice coordinate will also affect the rate at which the exponential tail decays, so this rescaling is not independent of the other rates. Rescaling the height by a factor of \( c \), on the other hand, is equivalent to rescaling \( G \rightarrow cG \) and \( I \rightarrow I/c \) when \( \alpha = 1 \), so it does not alter the parameterization of the model. We can leverage it to focus on parameter values that do not pose computational difficulties but otherwise retain the same shape.

From modifying these parameters in various combinations, it quickly becomes clear that the shape of the model is strongly constrained by the curvature of the peak, the decay length of the exponential, and the length of the exponential tail, which can be treated roughly independently of one another in the above-mentioned manipulations. Thus, our algorithm consists of first

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4One of the exceptions is the case of base pair mutations, in which the mapping of the lattice coordinate is clearly one-to-one.
matching the peak of the occupancy function. This will constrain the horizontal and vertical scales. Then, we iteratively make small modifications to the decay rate of the exponential tail with $r$ and $r_d$ while compensating for the shifting peak with small changes to the horizontal and vertical scales. When such changes consistently lead divergent or collapsed $L$, we rescale the entry rate $G$ to find a more computationally amenable parameter regime. As we show in Fig. 4, the limited flexibility of the occupancy number function means that heuristic implementation of this algorithm quickly leads to fits that allow us to assess the qualitative alignment of our model with the data.

Another aspect of our alignment process is the choice of the features of the data curves to which to fit. While one approach would be to minimize the area between the curves, this presents a practical difficulty (as we mention above), and a naïve fit neglects the fact that strongly constrained form of our model is not meant to match all the features of the empirical curves we consider. We aim to assess whether or not our model can reproduce the most prominent features of the empirical curves: the shape of the peak and the tail. We go through each of the panels presented in Fig. 4 and detail our logic for how we align the model with data.

d. Metal stamping firm data: Given that the data covers two different years, we present two different time points of a dynamical mean-field flow calculation. We choose parameters such that the difference between the innovation front number and the obsolescence front roughly aligns with the least cost efficient firms in the data. The time points of the simulation are then chosen such that the heights of the peaks roughly align with data maxima for each year in turn. Then, the location of the rightmost point is chosen to align the curvature of the model to pass through the points around the peak. As in the other example of firms in panel g, our model cannot capture the density of the most cost-efficient, or most “productive,” firms.

e. SARS-CoV-2 clades: We estimate the peak of the histogram of clades near the origin by taking a three-point moving average. Then, we adjust the tail to fit the decay in the data. Note that our model’s tail is not exactly exponential and shows more downwards curvature, which happens to match the data well. It is evident from observing the results of parameter variation that our model cannot match the location of the peak. We cannot do this by rescaling the lattice coordinate since a single mutation is exactly one lattice site away. By rescaling the height of the curve, however, we recognize that our model aligns well with the exponential increase on the left of the peak and the sudden decay on the right. We show this alignment to the curvature bounding the peak in normalized SARS-CoV-2 clades, which suggests that a substantial number of clades with few mutations are missing from the database relative to later mutations.

f. Patent citations: As with the SARS-CoV-2 clades, an attempt to match the location of the peaks while matching the tail is impossible. Given this limitation, we match roughly the height of the 1998 peak in the data and align the roughly exponential tail. Our model
fails to capture the slight deviation away from exponential decay in the data, which comes from some patents have many more citations than others.

g. Inferred cost efficiency for Indian firms: We match the height and curvature of the peak estimated by Jangili, allowing for the location of the obsolescence front to change. Then, we tune the decay of the tail to pass through the inferred density of firms. The sharp decay on the right side of the peak from our model does not agree with the densities obtained from stochastic frontier analysis, which itself relies on assumptions about the form of the curve.

j. Article citations: We align the location and heights of the curves, then align the exponential tails with the decays in the data. Since we know that citations rates tend to deviate from exponential for longer periods of time, we align to the time periods shortly following the peak from about 10-20 years.

To approximate the model occupancy densities, we rely on the second-order calculation of lattice length described in Appendix A along with the flow mean-field calculation in Appendix B.

**Code availability**

The code for processing the data, calculating the model, and producing all the plots as described below is available at https://github.com/eltrompetero/innovation and will be put on an archive repository to be determined upon publication.

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**Author contributions**

E.D.L. designed the research and wrote the code. All authors contributed to model and data analysis and edited the manuscript.
Competing interests

The authors declare no competing interests.

Materials & correspondence

Correspondence and questions about code should be sent to E.D.L.
A Model motivation, derivation, and solution

In Eq 3, we propose a general model for innovation and obsolescence dynamics of an idea lattice with replicating agents. The parameters in the model correspond to the rates at which various events occur, which could encapsulate multiple different processes at the level of our analysis. The proper mapping from each system of interest to model parameters must be operationalized, which essentially accounts for the proper units of the variables.

1. The entry rate $G$ at which new agents enter the system requires defining a threshold at which we observe new agents entering. In the context of firms, we might think this to be straightforwardly the entry of a new incorporated firm in an industry. As is discussed in more detail in the study of firm demographics, however, the “birth” of a firm is a matter of measurement and definition. It depends on the universe of study such as listed public firms vs. hard-to-measure private firms (53). The study of the former at the exclusion of the latter effectively establishes a minimum capital size; in other words, the effective rate at which agents enter can also depend on the level of data resolution. For viral evolution, entry captures the emergence of new strains that do not derive from existing ones, which must be from the fact that we can only observe a small fraction of circulating strains at any given time and must necessarily impute the missing pieces of the phylogenetic tree. For scientific citations, entry rate combines the rates at which new authors enter as well as that of existing authors moving into a new area.

2. The rate of agent replication $r$ refers to the typical timescale at which an agent spawns a more innovative copy of itself, and the death rate $r_d$ refers to the typical timescale on which agents leave the system. Though these can be treated separately, they in combination describe various situations. As one example, $r = r_d$ corresponds to the scenario where agents hop from one site to the adjacent one because death is commensurate with a single replication event. When $r > r_d$, agents tend to stimulate more innovative copies, whereas when $r < r_d$, agents tend to die before they can replicate, akin to the scenario where agents only progress a few steps towards the innovative frontier before dying. The parameters individually can encapsulate several different types of events. For example, $r$ may be the sum of two different process like mimetic innovation and replication, but events that we would sum together into a single rate because they effectively increase the number of agents on the adjacent innovative site. Likewise, death could consist of the disappearance of an agent from a site because of mimetic innovation and death. The key assumption is that the combination of these types of events can be separately considered as mean rates that can be then summed together.

3. The rate of obsolescence $r_o$ details when ideas fall into desuetude. While we explicitly write down a rate that acts on the idea lattice, usually it is considered in terms of an endogenous variable like when a production method is no longer used by firms. As we point out in the main text, not all obsolescent events can be traced back to an endogenous cause, and so this parameter is a way of representing the results of either kind of dynamics.
4. Innovativeness $I$ sets a scale on which attempts at innovating into the adjacent possible are successful, distinguishing mimetic innovation from serendipitous innovation at the frontier. As with the other rates, this parameterization assumes that there is a typical rate at which innovative events occur as when new technologies are created, unseen mutations emerge, or ideas are introduced into the literature.

When an Einsteinian innovation occurs, the parameters themselves must be changed, which violates the assumption of fixed mean rates.

Accounting for these rates means that we only need describe the number of agents $n(x, t)$ at some lattice site $x$ at time $t$. As a start, we take the innovation front to be the rightmost point in the linear lattice as pictured in Fig. 1B. First, we account for the rates entry $G$ over $L(t)$ lattice sites and the rate at which any single agent dies $r_d$. Next, the probability that at least one agent successfully advances the innovation front is $1 - (1 - rIdt)^{n(L,t)} = In(L, t)dt + \mathcal{O}(dt^2)$, where the latter term refers to all terms of order $dt^2$ and smaller. On the other hand, with probability $(1 - rIdt)^{n(L,t)} \approx 1 - rIn(L, t)dt$, the front does not move and $rIn(x - 1, t)dt$ agents move in from the left. To arrive at Eq 3, we move into a fixed reference frame that does not change as the lattice moves. We reverse the coordinate system such that the innovation front is fixed at $x = 0$. Flow comes in from the right and vice versa, and innovation events shift the occupancy number to the right by one. This mathematical simplifications focuses our attention on the innovation front and the shape of the pseudogap around it.

At stationarity, we obtain a formal solution to the time-independent occupancy number

$$n(x) = In(0)[n(x - 1) - n(x - 2)] + r_d n(x - 1) - \frac{G}{rL}. \quad (S1)$$

By rescaling parameters in units of $r$ as is indicated by bars, we obtain a form that reveals that replication rate sets a shared relative timescale for all the remaining parameters,

$$n(x) = \bar{r}_o[n(x - 1) - n(x - 2)] + \bar{r}_d n(x - 1) - \bar{G}. \quad (S2)$$

It is the relative differences between entry, obsolescence, and death with respect to replication that distinguish the regimes of the model.

To solve for the innovation front density, we use the fact that $n(-1) = 0$. We also take as an assumption $n(1) - n(0) = n(0)$, which is to say that the shape of the occupancy number about the innovation front is linear, or that the pseudogap is rather wide.

$$\dot{n}(0, t) = \frac{\bar{G}}{L} + (2 - \bar{r}_d)n(0, t) - In(0, t)^2$$

(S3)
At stationarity, we find the innovation front number to be

\[ n(0) = \frac{1}{2I} \left[ 2 - \bar{r}_d + \sqrt{(2 - \bar{r}_d)^2 + \frac{4GI}{L}} \right], \quad (S4) \]

having put aside the unphysical negative solution to the quadratic equation. Eq (S4) reveals that the density at the innovation front is inversely proportional to the difficulty of advancing the front to the adjacent possible \( I \). Increasing the replication rate \( r \) or \( G \) also leads to higher innovation front density as agents pile up faster. Thus, we find simple dependence of the innovation front density on the local dynamics of agent innovation, death, and replication.

**A.1 Improving the linearized estimate for lattice width**

While the linearity assumption \( n(1) \approx 2n(0) \) was revealing, it was meant as starting approximation. We refine the calculation of \( L \), essentially solving the first-order differential equation, by considering corrections to the linear approximation. We start with the first-order ordinary differential equation (ODE)

\[ 0 = \frac{G}{L} + (r - r_d) n(x) + r (1 - In(0)) n'(x), \quad (S5) \]

but we add on top of our local Taylor expansion to first order at \( x = 0 \) a correction

\[ n(1) = 2[n(0) + \epsilon]. \quad (S6) \]

This correction really comes from the fact that derivatives of higher order matter in the full Taylor series expansion, or

\[ n(-1) = n(0) - n'(0) + n''(0)/2! - \cdots. \quad (S7) \]

In this context, the linearization in Eq (S5) determines the form of these higher derivatives in a compact way. Since \( n(-1) = 0 \),

\[ n'(0) = n(0) + n''(0)/2! - n^{(3)}(0)/3! + \cdots \quad (S8) \]

This is a restatement of our assertion that if \( n''(0) \) and all higher order derivatives are sufficiently small, then we could use a linear approximation to specify \( n(x = 1) \).

Returning to Eq (S5) we solve for the higher order derivatives. We can do this by taking the \( k \)th derivative with \( k \geq 1 \),

\[ 0 = (r - r_d) n^{(k)}(x) + r [1 - In(0)] n^{(k+1)}(x) \quad (S9) \]
and the recursion relation
\[ n^{(k)}(0) = z^{k-1} n'(0) \]  
(S10)

having defined
\[ z \equiv \frac{r - r_d}{r[In(0) - 1]} = -\frac{1 - \bar{r}_d}{1 - \bar{r}_o}. \]  
(S11)

This means that our approximation for linearity at the front is also the condition that \( z \ll 1 \), an approximation that fails as \( 1 - \bar{r}_o \to 0 \) for finite \( 1 - \bar{r}_d \). We show the effects of this factor in terms of the expected stationary value at \( n(0) \) in Fig. B5.

Putting these calculations together, the corrected pseudogap slope from Eq S8 is
\[ n'(0) = n(0) + n'(0)z^{-1}\left[ \frac{z^2}{2} - \frac{z^3}{3!} + \cdots \right] \]
\[ = n(0) + n'(0)z^{-1}\left[ e^{-z} - 1 + z \right] \]  
(S12)

Going back to the stationary condition in Eq S5 we can solve for \( L \) but now accounting for this correction
\[ L = -\bar{G}\left( n(0)\left[ 1 - \bar{r}_d + \frac{1}{1-C} (1-In(0)) \right] \right)^{-1} \]  
(S13)

Replacing \( n(0) \) with the stationary condition for innovation front occupancy
\[ = \bar{G}I\left( \bar{r}_o \left[ \bar{r}_d + \frac{\bar{r}_o}{1-C} - \left( 1 + \frac{1}{1-C} \right) \right] \right)^{-1} \]  
(S14)

In order to check what happens to the singularity where \( L \) diverges with \( \bar{r}_o \), we check when the denominator of Eq S14 goes to 0,
\[ \frac{\bar{r}_o}{1-C} = \left( 1 + \frac{1}{1-C} \right) - \bar{r}_d, \]  
(S15)

where \( C \) depends implicitly on both \( \bar{r}_d \) and \( \bar{r}_o \). Eq S14 modifies the boundaries of the phase space as graphed in Fig. A1.
A.2 First-order solution

We can also take a direct approach to solving the first-order ODE by simply integrating Eq S5. Then,

$$n(x) = \left( \frac{\bar{r}_o}{I} + \frac{\bar{G}}{L(1 - \bar{r}_d)} \right) \exp \left( -\frac{1 - \bar{r}_d}{1 - \bar{r}_o} x \right) - \frac{\bar{G}}{L(1 - \bar{r}_d)},$$  \hspace{1cm} (S16)

having used the starting assumption \(n(0) = \frac{\bar{r}_o}{I}\). This approximation shows clearly that the slope of the exponential rise at \(x = 0\) and its concavity are determined by the competition between death \(\bar{r}_d\) and competition \(\bar{r}_o\).

We obtain an a simple form for lattice length by using the boundary condition \(n(x = -1) = 0\),

$$L = \frac{\bar{G}}{1 - \bar{r}_d \bar{r}_o} \left[ \exp \left( -\frac{1 - \bar{r}_d}{1 - \bar{r}_o} \right) - 1 \right].$$  \hspace{1cm} (S17)

Eq [S17] is equal to Eq [S14], but the latter gives a more transparent form that aligns with the derivation presented in the main text.

We note that lattice length is not a physical quantity in the first-order ODE because the occupancy function never intersects with the \(x\)-axis beyond \(x = -1\). This observation suggests that we must at least go to a second derivative of \(n(x)\) before we can expect to approximate well the exact occupancy function.
A.3 Second-order solution

A solvable, continuum formulation that captures the rise and drop required for a finite lattice would be helpful for solving for lattice length $L$. Here, we present a second-order ordinary differential equation approximation for an interpolation through the iterative solution in Eq S1.

Expanding the discrete formulation about $x$, we obtain

$$\dot{n}(x,t) = \frac{G}{L(t)} + (r - r_d)n(x) + r[1 - I n(0)]n'(x) + \frac{r}{2}[1 + I n(0)]n''(x),$$  \hspace{1cm} (S18)

where we have stopped at second-order to leverage the two conditions, $n(0) = r_o (r I)^{-1}$ and $n(-1) = 0$. Replace $n(0)$ with the stationary condition and taking rescaled rates, we obtain

$$0 = \frac{\bar{G}}{L(t)} + (1 - \bar{r}_d)n(x) + [1 - \bar{r}_o]n'(x) + \frac{1}{2}[1 + \bar{r}_o]n''(x).$$  \hspace{1cm} (S19)

Eq [S19] is an inhomogeneous second-order differential equation that we can solve using the standard method of variation of coefficients after having solved the homogenous equation. Since the inhomogenous term is a constant, the particular solution is likewise a constant. Then, the particular solution with characteristic eigenvalues $\lambda_+$ and $\lambda_-$ from the homogenous equation is

$$n(x) = Ae^{\lambda_+ x} + Be^{\lambda_- x} + C$$

$$\lambda_{\pm} = \frac{1}{1 + \bar{r}_o} \left( \bar{r}_o - 1 \pm \sqrt{(1 - \bar{r}_o)^2 - 2(1 - \bar{r}_d)(1 + \bar{r}_o)} \right).$$  \hspace{1cm} (S20)

By using Eqs [S19] and [S20] we find

$$C = -\frac{\bar{G}}{L(1 - \bar{r}_d)}. \hspace{1cm} (S21)$$

We then solve for $A$ and $B$ using the boundary conditions.

$$A = \frac{Ge^{\frac{\bar{r}_o - 1 - \sqrt{z}}{1 + \bar{r}_o}}}{L(1 - \bar{r}_d)} - \frac{2G(1 + \bar{r}_o)}{L[(1 - \bar{r}_o)^2 - z]} + \frac{\bar{r}_o}{T}$$

$$B = \frac{Ge^{\frac{\bar{r}_o + 1 + \sqrt{z}}{1 + \bar{r}_o}}}{L(1 - \bar{r}_d)} - \frac{2G(1 + \bar{r}_o)}{L[(1 - \bar{r}_o)^2 - z]} + \frac{\bar{r}_o}{T} \hspace{1cm} (S22)$$

$$z = (1 - \bar{r}_o)^2 - 2(1 - \bar{r}_d)(1 + \bar{r}_o).$$

Eq [S20] provides us two potential consistency equations for solving for $L$. Since we know
Figure A2: Numerical calculation of phase space from refined, second-order differential equation approximation in Appendix A.3. Solved by setting the initial condition to the linear approximation in Eq 4 and the self-consistency condition in Eq S24. Static region is shaded such that darker is a longer lattice as indicated by the color map. Critical point at $r_o = r_d = r$ shows different behavior depending on angle of approach. For example, we anticipate from the first-order solution that concavity of the curve at the pseudogap depends on the ratio $(1 - \bar{r}_d)/(1 - \bar{r}_o)$, which can diverge, be negative, or be positive depending on the limit taken towards the critical point.

that $n(-1) = 0$, we have the following transcendental equation that must be satisfied:

$$L = \frac{\bar{G}}{1 - \bar{r}_d} \left( A e^{-\lambda_+} + B e^{-\lambda_-} \right)^{-1}.$$  \hspace{1cm} (S23)

Alternatively, we have the condition $n(L) = 0$,

$$L = \frac{\bar{G}}{1 - \bar{r}_d} \left( A e^{-\lambda_+ L} + B e^{-\lambda_- L} \right)^{-1}.$$  \hspace{1cm} (S24)

Eqs [S23] and [S24] do not in general return the same result for $L$, but we find that the latter agrees better with automaton simulations. For some parameter regimes, this condition is numerically degenerate for $L$. In these cases, we rely on either the heuristic solution for the iterative equation as discussed in Appendix B.1 or the linear approximation discussed in the main text.

**B Numerical calculations**

We calculate the occupancy number using the iterative form of Eq [S1], the analytic approximation presented in Appendix A.3, an automaton simulation of agents, and a mean-field flow
implementation of the dynamics in Eq. Each of these calculations have respective weaknesses and strengths that we discuss briefly in the following bullet points. More details about the implementation follow, and the code for each solution is located in the aforementioned repository.

- While it is exact, the iterative calculation can be subject to numerical precision errors that grow with the lattice coordinate. It also requires a precise estimate of $L$, without which it may diverge or return negative values. Because of such divergence, the iterative calculation does not always permit an easy estimate of $L$ using self-consistency of the equation.

- The analytic solution, on the other hand, provides a ready way to estimate $L$ by using the consistency condition in Eq. It is an excellent approximation to the iterative solution for small densities and lattice widths, but its accuracy decreases as higher derivatives become increasingly important far from the pseudogap and for larger values of $n$.

- The automaton model allows us to access non-stationary dynamics and stochastic fluctuations, but it is relatively slow to calculate. It only gives a stochastic estimate of the lattice width $L$, which is necessary to condition upon to compare with the mean-field models of occupancy number.

- Finally, the dynamical mean-field flow calculation couples flow of agent density between adjacent lattice sites. This approach obtains temporal trajectories like with the automaton model but is much faster. It is, however, subject to discrete lattice corrections and requires knowledge of $L$ beforehand.

We leverage the set of alternative calculations to shed light on corrections and details that arise with each respective calculation. Some of the limitations may be important to consider when applying our results to practice. We describe the calculations in more detail below.

**B.1 Iterative calculation**

Assuming that we have an estimate of lattice length $L$, we start with the stipulation $n(-1) = 0$ and $n(0) = r_o(rI)^{-1}$ to calculate the occupancy number up to $x = [L - 1]$. Note that we set the innovation front to be at $x = L - 1$ and $n(x = L) = 0$ such that the lattice consists of $L$ sites.

A heuristic approach to estimating the lattice length for a given set of parameter relies on the observation that agent number must be finite and positive when stationarity. We run the iterative solution starting with the initial estimate for $L$ from linearized mean-field theory. When $L$ is slightly to large, the function diverges quickly to infinity and when it is slightly too small it quickly diverges to negative infinity. Using the wiggling of the tail, we can in many cases narrow our estimate of $L$ precisely. We do not expect $L$ to always depend so sensitively on errors this heuristic technique is not guaranteed to work.
B.2 Analytic solution

The solution to the second-order approximation of the difference equation presents a self-consistency condition for $L$. We use a standard Broyden–Fletcher–Goldfarb–Shanno (BFGS) minimization algorithm implemented in NumPy to solve for $L$ starting with the initial condition given by the linearized mean-field approximation.

B.3 Automaton model

We simulate the dynamics with an automaton model that consists of individual agents that follow the rules stipulated in Eq 3 but with a moving coordinate system (instead of with an innovation front fixed at $x = 0$). At a given step in time, each individual agent dies with probability $r_d dt$, replicates towards the innovation front with probability $r d t$, and the agents on the front each extend with probability $r I dt$. The obsolescence front progresses with probability $r_o dt$. In the limit $dt \to 0$, each of these steps commutes with the others (the non-commuting corrections are of lowest order $dt^2$). Furthermore, we no longer need to consider each agent separately but just the sum of the rates of all the agents on any site $n(x)$. Thus, the small time step limit allows us simplify the calculation as is specified in the code repository and summarized below.

Assuming that $x = -1$ is the obsolescence front and $x = L - 1$ is the innovation front, one algorithm is

1. Instantiate lattice of size 1.
2. Extend right side of lattice at $x = L(t) - 1$ by unit length with probability $r I n(L - 1) dt$.
3. Remove a lattice site from the left hand side with probability $r_o dt$ unless lattice is already of size 1.
4. For each lattice site $x$ in the order of the rightmost to the leftmost do the following:
   a. If $x < L(t) - 1$, add a new agent to site $x + 1$ with probability $r n(x) dt$.
   b. Remove one agent with probability $r d n(x) dt$.
   c. Add one agent with probability $G dt / L(t)$.
5. Return to Step 2.

Note that variations of this algorithm give exactly the same results in the limit $dt \to 0$ though some may converge to the limit slower than others with $dt$. The important point is that terms of order $dt$ be preserved in the calculation.

The mean-field model that we discuss will not generally agree with the time or ensemble averaged occupancy number of automaton simulations. This is because the mean-field is based on the assumption that the average lattice width is the same as the average of its inverse. From the Cauchy inequality, however, we know that the typical width of the lattice in the automaton
simulation will always be smaller, effectively mapping the automaton model to another mean-field equation. In order to align the mean-field theory with automaton results, we recognize that it is necessary to rescale $L \to cL$ and $x \to cx$ in Eq S1. This transformation corresponds to the rescaling $G \to cG$ and $I \to cI$. A second factor that we must account for is the variation about $L$. To obtain an averaged occupancy number function for comparison, we restrict the average to the snapshots that are close to the stable value of $L$. In this way, we simply average over the occupancy number in the lattice relative to the innovation front up to the length of the shortest considered lattice (unless specified to the contrary). We show just one example of such a comparison that accounts for these two corrections in Fig. B3, which shows close alignment between automaton and the corresponding mean-field calculation.

**B.4 Mean-field flow**

Having fixed the lattice size to be $L$ using the algorithm specified for the analytic calculation, we couple adjacent site densities from Eq S3 and evolve the densities in time increments of $dt$. We stop the iteration when a convergence criterion has been met for the maximum absolute change $\dot{n}(x,t)$. This allows us to track the evolution of the occupancy number function from any initial condition as in Fig. B4.
Figure B4: Simulated dynamics from mean-field flow. Numerical solution is close to expected density at the innovation front when (a) the stationarity condition is fulfilled for \( r < r_d \) and (b) \( r > r_d \) but not when (c) stationarity is violated in the growing regime.
C Model extensions

C.1 Cooperative innovation

Cooperative innovation implies that front velocity scales nonlinearly with the number of agents, or that \( r In(0)^\alpha \) for \( \alpha \in [0, \infty] \). The stationary condition is now \( n(0)^\alpha = r_o/r I \). When \( \alpha > 1 \), we have cooperative behavior, and \( \alpha < 1 \) implies anti-cooperativity. The nonlinearity means that the innovation fronts coincide under the rescaling \( n(x)^{1/\beta} \) as in Fig. C6. Modifying Eq 3 accordingly, we find that \( L = G[n(0)(r_d + r In(0)^\alpha - 2r)]^{-1} \), corresponding to the phase boundaries shown in Fig. 3b and c. The boundary separating stasis from growth is

\[
\bar{r}_o \leq (2 - \bar{r}_d)^{1/\alpha} I^{(\alpha-1)/\alpha}
\]  

(S25)

whereas the boundary separating stasis from collapse requires solving for the zeros of the fractional polynomial

\[
G = (\bar{r}_o/I)^{1/\alpha} (\bar{r}_d + \bar{r}_o - 2).
\]  

(S26)

With the transformation \( I^\alpha \to I \) and \( (r_o/r)^\alpha \to r_o/r \), the stationary cooperative equation maps back to Eq 3. Thus, cooperativity does not alter the qualitative aspects of the model.

C.2 Bethe lattices and higher dimensions

Bethe lattices and Euclidean graphs of higher dimension imply that the number of agents per site decreases every sequential step from the origin. In the Bethe lattice picture from Fig. 1c, each sequential site branches into \( Q - 1 \) additional branches. A new agent must choose a branch and
thus the replication term from Eq \[3\] acquires a factor of \((Q - 1)^{-1}\), and agent number decreases towards the innovation front faster than in a linear graph. As we show in Fig. 3b, collapsed and stable regimes grow larger at the expense of the growing regime. This argument makes clear the importance of the relative dimensions of agent replication and the idea space: when next-generation agents do not fill all of the available space, then agents necessarily occupy a small fraction of the idea space.

To make this explicit, we show the transformed equations for the Bethe lattice. Assuming that every step further out into the adjacent possible opens up \(Q\) possibilities, where \(Q = 2\) corresponds to the linear case where each lattice site has a parent and a child, then the density per site should decrease per branching point as agents decide on a branch to take. The site specific dynamics — including death, startup, and shift — remain the same. Then, the analogous equation to Eq \[3\] is

\[
\dot{n}(x,t) = \frac{\bar{G}}{L(t)} - \bar{r}_d n(x,t) + \frac{1}{Q-1} n(x+1,t) - In(0,t)[n(x,t) - n(x-1,t)].
\]

Eq \[S27\] is the dynamical equation for a Bethe lattice, where \(L(t)\) refers to the number of branching steps (not sites) between the innovation and obsolescent fronts at time \(t\).
The stationary solution is

\[ n(x) = (Q - 1) \left\{ I n(0)[n(x - 1) - n(x - 2)] + \bar{r}_d n(x - 1) - \frac{G}{L} \right\}. \] (S28)

Then, the collapse condition is \(0 \leq \bar{r}_o \leq 2/(Q - 1) - \bar{r}_d\), and the divergent growth condition from Eq 6 is \(\bar{r}_o \sim (Q - 1)^{-1} - \bar{r}_d/2 + \sqrt{((Q - 1)^{-1} - \bar{r}_d/2)^2 + GI/L}\). The dimensional depletion effect does not fundamentally alter the dynamics. If we rescale \(r \rightarrow (Q - 1)r\) and \(I \rightarrow I/(Q - 1)\), then we recover Eq 3.

In a similar sense, we consider how the local density of agents living in a \(d\)-dimensional space must divide themselves as they move outwards from the origin at distance \(x\). The local density decreases as

\[
\frac{(x - 1)^{d-1}}{x^{d-1}} = (1 - 1/x)^{d-1} \\
\approx 1 - (d - 1)1/x
\] (S29)

(S30)

Clearly, the local curvature of the surface determines how thinly agents have to spread themselves out as they move further out. In the limit of a long-running lattice, \(x \rightarrow \infty\), and finite dimension, this case reduces to the one-dimensional model. Thus, considering a tree-like lattice or higher dimensions does not appreciably alter the basic model.

### C.3 Obsolescence-driven innovation

Obsolescence-driven innovation is the antithesis of forward-looking innovative systems. By reversing the direction of the \(x\)-axis and fixing the new innovation front at the origin, agents now replicate towards obsolescence. For stable pseudogaps, the introduction of a new idea drives every agent to left, making the whole system more innovative. As a result, innovation is driven at a rate proportional to the number of ideas on the verge of being extinguished. Newer ideas tend to beget agents on older ideas with rate \(r\), which then drive themselves to extinction by eventually increasing occupancy at the obsolescence front.

We have a mirrored version of Eq 3:

\[
\dot{n}(x, t) = \frac{\bar{G}}{L(t)} - \bar{r}_d n(x, t) - n(x - 1, t) + I n(0, t)[n(x, t) - n(x + 1, t)].
\] (S31)

Let us consider the stationary case. Since sites are now added at the obsolescent front, the addition of a new site shifts all the agents one lattice site towards the innovation front. In other words, an innovation in this system means that all agents move to the left, introducing a new site adjacent to the obsolescence front and removing all agents previously at the innovation front. The removal of agents at the innovation front effectively imposes an threshold below which we are unable to detect innovative agents at stationarity. These agents beyond the known innovative
frontier do not contribute any longer to lattice dynamics. We might interpret this as the fact that innovative ideas are around in some form before they are measured, so this establishes a threshold above which we recognize an innovative idea.

This threshold is linked to the obsolescence rate $r_o$, which determines how quickly the most innovative agents leave the system. This process, the disappearance of the most innovative agents, either is slow enough for lattice growth or sufficiently rapid for collapse. In this sense, it is more appropriate to call the obsolescence rate an indulgence of unconventional ideas, which, in the stationary case, is equal to the innovation rate.

D Critical lines

Eq. S14 suggests that the dynamics of lattice width are determined by a competition between forces that drive the system towards 0 such as via small innovativeness $I \to 0$ and those that drive the system towards infinite growth. In a randomly growing system, we expect such gyrations to be most prominent when slight changes to the innovation front number $n(0,t)$ push us quickly from one extreme to another. Indeed, we noted exactly this in the collapsed limit in Eq. 6, where for $Glr \to 0$, the width of the stable regime shrank to 0. Recalling that we had substituted in stationary values for $n(0)$, we recognize the essential role of the innovation front number $n(0)$ in determining the balance of the innovation and obsolescence fronts. This suggests as an order parameter, the typical fluctuation, or the noise-to-signal ratio, of the innovation front number,

$$\sigma^2/\mu = \left(\left\langle n(0,t)^2\right\rangle - \left\langle n(0,t)\right\rangle^2\right) / \left\langle n(0,t)\right\rangle.$$

We use Eq. S32 to characterize the fluctuations in the system when $I \ll 1$, and we approach the critical point $r^*$ in Fig. D7. The appearance of a critical point is clear in the normalized variance of the order parameter $n_0$, which takes finite values beyond the critical $r$. As we approach the critical point, we see increasingly large fluctuations in the width of the lattice as shown in the example trajectory in Fig. D7A. The maximum durations for which the lattice breaks away from the collapsed state $\Delta t$ become longer as we approach the critical point, extending the cutoff of a heavy-tailed distribution as in Fig. D7C. Short of the critical point, the system eventually returns to the collapsed configuration.

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5 Even in the original formulation, an assumption of a continuous density function that smoothly goes to 0 would imply that some agents exist in the innovative adjacent possible and in the obsolescent adjacent possible.
Figure D7: Large fluctuations at the critical point generate flourishing periods that eventually collapse. (a,b) Time series from automaton simulation near critical point \( r = 0.4 \) for lattice width and innovation front number, respectively. (c) Normalized variance at innovation front around critical replication rate \( r \). Critical point seems to be continuous because of slow, sublinear scaling of lattice size with simulation time beyond the critical point, but simulations show that lattice length will diverge for sufficiently long times. (d) Distribution of time periods of flourishing ideas defined as periods during which lattice width was greater than the mean.
Figure D8: Examples of pseudogap types. Sublinear (blue), none (orange), and superlinear (green). These have been rescaled along both axes to facilitate comparison.
Table S2: Model dynamics and possible attributions. The provided set extends beyond the data sets considered in the main text.

| agent          | growth $G$            | expansion $r_e$ | death $r_d$           | obsolescence $r_o$                                 | innovativeness $I$          |
|----------------|-----------------------|-----------------|-----------------------|---------------------------------------------------|-----------------------------|
| **firms**      | start-up              | spin-off        | bankruptcy            | no longer profitable tech.                       | availability of public research |
|                | above min.            | improved        | improved              | no demand                                         | ease of tech. challenge     |
|                | capitalization        | productivity    | productivity          |                                                  |                             |
|                | IPO                   | market          | market                | prohibited by legal means or cultural norms      | robustness of new markets  |
|                |                       | reposition      | reposition            |                                                  |                             |
|                |                       | merger          | merger                | natural resource input                            |                             |
|                |                       | below min.      | below min.            |                                                  |                             |
|                |                       | capitalization  | capitalization        |                                                  |                             |
| **genetic evolution** | crossover from unobserved lineage | base pair mutation | extinction from neutral drift or selection | non-viable adaptation in changing environment | rate of viable mutation |
|                |                       |                 |                       |                                                  | environmental diversity    |
| **citation**   | new author            | follow-up work  | retirement            | disproved theory                                  | rate of “jumps”             |
|                | existing author in    |                 |                       |                                                  |                             |
|                | new field             |                 |                       |                                                  |                             |
|                |                       |                 |                       |                                                  |                             |

\cite{fosterTraditionInnovation2015}
Table S3: Model structure and possible attributions. The provided set extends beyond the data sets considered in the main text.

| context          | agent                        | lattice site                      | edge                             | order                   |
|------------------|------------------------------|-----------------------------------|----------------------------------|-------------------------|
| markets          | firm                         | technology used                   | shared patent citation           | time                    |
|                  | unit of invested capital     | cost efficiency                   | small distance                   |                         |
|                  |                              | material input composition       |                                  |                         |
|                  |                              | labor skill composition          |                                  |                         |
| genetic evolution| clade                        | base pair mutations               | small distance                   | time                    |
|                  |                              |                                   |                                  | no. of mutations        |
| science          | publication                  | publication                       | temporally similar               | time                    |
|                  | keystone publication         | patent                            | similar impact                   | citation hierarchy      |
|                  | author                       | terminology                       | shared topic                     | from less to more refined |
|                  | author combination           | model                             | shared authors                   | models                  |
|                  | patent                       | technology                        | shared technology                |                         |
Figure D9: Example SARS-CoV-2 phylogeny tree from Nextstrain for North America.
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