Uniqueness of Solution of Cylindricity Objective Function

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Abstract. Based on the radial deviation measurement, the unconstrained optimization model is established for assessing cylindricity errors by the minimum zone method. The properties of the objective function concerned are thoroughly researched. On the basis of the modern theory on convex functions, it is strictly proved that the established cylindricity objective function is a continuous, non-differentiable and convex function defined on the four-dimensional Euclidean space $\mathbb{R}^4$. Therefore, the global minimal value of the objective function is unique and any of its minimal point must be its global minimal point. Thus, any existing optimization algorithm, as long as it is convergent, can be applied to solve the objective function to get the reliable minimum zone cylindricity errors. An example is given to verify the theoretical results presented.

1. Introduction
Optimization algorithms are generally used to seek the minimal values of form error objective functions by iteration when a microcomputer is applied to evaluate form errors by the minimum zone assessment methods. Nevertheless, the essential prerequisite to convergence of any optimization algorithm is that the objective function to be solved has only one minimal value on its domain of definition, i.e., it is a single valley function. If it is not determined whether a form error objective function is a single valley one, it can not be determined whether the minimal value sought by an algorithm is its global minimal value which is the wanted minimum zone form error. In addition, it is also important to determine such properties of an objective function as continuity and differentiability in order to select an appropriate algorithm for solving the objective function quickly and accurately. Researchers have paid close attention to these problems and have done much research by using various methods. Some researchers tried to improve algorithms theoretically in order to devise a type of algorithm which can seek the global minimal values of objective functions[1], but have made little progress for the present. Others put forward a method to analyze the single valley property of form error objective functions by isograms drawn by a computer[2, 3]. Another type of approach has been to work out some geometrical transformation ways which can seek the minimum zone form errors fast and accurately in terms of the geometrical criteria for the form error minimum zones[4–13]. However, these algorithms have their own defects so that they have limitation in application.

In this article, based on the radial deviation measurement, the unconstrained optimization model is established for assessing cylindricity errors by the minimum zone assessment method and the properties of the objective function concerned are thoroughly researched. On the basis of the modern theory on convex functions, it is strictly proved that the established cylindricity objective function is a continuous, non-differentiable, and convex one defined on the four-dimensional Euclidean space $\mathbb{R}^4$. © 2006 IOP Publishing Ltd
Therefore, the global minimal value of the objective function is unique and any of its minimal point must be its global minimal point. Thus, any existing optimization algorithm, so long as it is convergent, can be applied to solve the objective function to get the reliable minimum zone cylindricity errors. Finally, an example is given to verify the theoretical results presented.

2. Mathematical model

An instrument coordinate system $OXYZ$ is set on a cylindricity measurement apparatus of rotary table or spindle type and let the $Z$-axis be coincident with the rotation axis of the rotary table or spindle, as shown in figure 1(a). For the actual surface to be measured, a number of equi-spaced sampled cross-sections perpendicular to the $Z$-axis are taken. Then the discrete sampled data are successively measured at the equi-spaced sampled points in each of the sampled cross-sections, assuming that the sampled points are denoted by $P_{ij} (r_{ij}, \theta_{ij}, z_{ij})$ ($i=1,2,\ldots,m; j=1,2,\ldots,n$), where $r_{ij}$, $\theta_{ij}$ and $z_{ij}$ are, respectively, the polar distances, polar angles and $Z$ coordinates of $P_{ij}$.

![Figure 1. Schematic diagram for establishing mathematical model.](image)

Suppose that the axis $L$ of an assessment reference cylinder passes through the point $O_1(a,b,0)$ in the coordinate plane $XOY$ and a group of direction numbers of $L$ are $s$, $t$ and $u$. Then the equation of $L$ is

$$\frac{x-a}{s} = \frac{y-b}{t} = \frac{z}{u}. \tag{1}$$

Let $s/u=p$, $t/u=q$. Then (1) can be changed as

$$\begin{cases}
x = a + pz \\
y = b + qz
\end{cases} \tag{2}$$

Now the direction numbers of $L$ become $p$, $q$ and 1. Assume that the radius of the reference cylinder is $R$. Then there are five unknown parameters—$a$, $b$, $p$, $q$ and $R$, to be determined. In the $j$-th sampled cross-section, let the intersection point between $L$ and the cross-section be $O_j(a_j, b_j, z_j)$, as shown in figure 1(b). Then, by (2), we have

$$\begin{cases}
a_j = a + p z_j \\
b_j = b + q z_j
\end{cases} \tag{3}$$

Let $OO_j = e_j$. Then $e_j = \sqrt{a_j^2 + b_j^2}$ and $\tan \alpha_j = b_j/a_j$. Next let $e_{ij} = P_{ij}D_{ij}$ being the radial deviation of $P_{ij}$ to the reference cylinder. Then, it holds that

$$r_{ij} = e_j \cos(\theta_{ij} - \alpha_j) + [(R + e_{ij})^2 - e_j^2 \sin^2(\theta_{ij} - \alpha_j)]^{1/2}.$$

Since normally $e_j << R + e_{ij}$ for radial deviation method, hence we have

$$r_{ij} \approx e_j \cos(\theta_{ij} - \alpha_j) + R + e_{ij} = R + e_j \cos \theta_{ij} \cos \alpha_j + e_j \sin \theta_{ij} \sin \alpha_j + e_{ij}. \tag{4}$$
As \( e_j \cos \alpha_j = a_j = a + pz_j \) and \( e_j \sin \alpha_j = b_j = b + qz_j \), we get from (4)
\[
e_{ijj} = r_j - R - a \cos \theta_{ij} - b \sin \theta_{ij} - pz_j \cos \theta_{ij} - qz_j \sin \theta_{ij}.
\] (5)

Let \( r_j = r_0 + \Delta r_j \), where \( r_0 \) is a certain constant and \( \Delta r_j \) is the radial deviation measured at \( P_j \). Then the difference of the maximum and minimum deviations of all the sampled points on the actual surface to be measured to the reference cylinder can be expressed as
\[
F = \max_{i,j} \{e_{ijj}\} - \min_{i,j} \{e_{ijj}\} = \max_{i,j} \{\Delta r_j - a \cos \theta_{ij} - b \sin \theta_{ij} - pz_j \cos \theta_{ij} - qz_j \sin \theta_{ij}\} - \min_{i,j} \{\Delta r_j - a \cos \theta_{ij} - b \sin \theta_{ij} - pz_j \cos \theta_{ij} - qz_j \sin \theta_{ij}\}.
\] (6)

When \( a, b, p \) and \( q \) vary, \( F \) varies correspondingly. So \( F \) is a function of \( a, b, p, q \). According to the minimum zone evaluation method for cylindricity errors, \( F \) should reach its minimal value. Hence the following unconstrained optimization model is established
\[
\min F(a, b, p, q).
\] (7)

In the above optimization model, the optimal point (global minimal point) \([a^*, b^*, p^*, q^*]^{T}\) of the objective function \( F \) is the parameter vector of the axis of the reference cylinder satisfying the minimum condition of cylindricity and the optimal value (global minimal value) \( F(a^*, b^*, p^*, q^*) \) is the cylindricity error obtained by the minimum zone assessment method.

3. Properties of objective function

Next, the properties of the objective function \( F \) will be analyzed. Let
\[
f_{ij}(a, b, p, q) = \Delta r_j - a \cos \theta_{ij} - b \sin \theta_{ij} - pz_j \cos \theta_{ij} - qz_j \sin \theta_{ij},
\]
where \( i \in \{1, 2, ..., n\} \) and \( j \in \{1, 2, ..., m\} \). Then we have
\[
F(a, b, p, q) = \max_{i \in I, j \in J} \{f_{ij}(a, b, p, q)\} - \min_{i \in I, j \in J} \{f_{ij}(a, b, p, q)\}.
\] (8)

For the differentiability of \( F \), there is the following theorem.

**Theorem 1.** The function \( F \) is a piecewise smooth function so that it is non-differentiable at some points on the four-dimensional Euclidean space \( R^4 \).

The proof of the above theorem is quite troublesome and so is omitted here.

Next, the convexity of \( F \) will be analyzed. The following two theorems are the theoretical bases for the analysis [14].

**Theorem 2.** Let \( \{f(x)\}, i \in I \) be a finite or infinite collection of convex functions defined on the \( n \)-dimensional Euclidean space \( R^n \). For every \( x \in R^n \) define the pointwise supremum of this collection as
\[
f(x) = \sup_{i \in I} f_i(x).
\]
Then \( f(x) \) is a convex function on \( R^n \).

**Theorem 3.** Let \( \{f(x)\}, i \in I \) be a finite or infinite collection of concave functions defined on the \( n \)-dimensional Euclidean space \( R^n \). For every \( x \in R^n \) define the pointwise infimum of this collection as
\[
f(x) = \inf_{i \in I} f_i(x).
\]
Then \( f(x) \) is a concave function on \( R^n \).

The following theorem can be proven concerning the convexity of \( F \).

**Theorem 4.** The function \( F \) is a convex one defined on the four-dimensional Euclidean space \( R^4 \).

Proof. By definition
\[
F(a, b, p, q) = \max_{i \in I, j \in J} \{f_{ij}(a, b, p, q)\} - \min_{i \in I, j \in J} \{f_{ij}(a, b, p, q)\} = \sup_{i \in I, j \in J} f_{ij}(a, b, p, q) - \inf_{i \in I, j \in J} f_{ij}(a, b, p, q).
\]
Since \( f_{ij}(a,b,p,q) \) are all linear functions for \( i \in I \) and \( j \in J \), they are both convex and concave. By theorem 2 and theorem 3, it follows that \( \max_{i \in I, j \in J} \{f_{ij}(a,b,p,q)\} \) is a convex function and \( \min_{i \in I, j \in J} \{f_{ij}(a,b,p,q)\} \) is a concave one. Thus \(- \min_{i \in I, j \in J} \{f_{ij}(a,b,p,q)\}\) is a convex function. As the sum of two convex functions is also a convex one, \( F \) is a convex function on \( \mathbb{R}^4 \).

According to the property of the extrema of convex functions, any local minimal point of a convex function \( f \) on \( \mathbb{R}^n \) must be a global minimal point of \( f \). Hence the following corollary is obtained.

**Corollary 1.** The minimal value of \( F \) is unique. Any local minimal point of \( F \) must be its global minimal point.

In terms of the property of convex functions, the following corollary can be immediately got.

**Corollary 2.** The function \( F \) is continuous everywhere on \( \mathbb{R}^4 \).

From the above conclusions, it can be clearly known that when any existing optimization algorithm is used to solve the function \( F \), as long as the algorithm is convergent, the minimal point sought by the algorithm must be the global minimal point of \( F \) and the minimal value attained at the minimal point must be the global minimal value of \( F \).

### 4. Example

The author of this article has done many experiments to verify the theoretical results presented above. Due to limited space of the article, only one example is given here. A shaft with a diameter of 60 mm, machined by turning, was measured by means of a universal tool microscope. In measurement, five equi-spaced sampled cross-sections were taken with a distance of 10 mm between every two adjacent cross-sections. In each cross-section, the number of the equi-spaced sampled points was taken to be 72. Based on the optimization model established above, the pattern search algorithm, proposed by Hooke and Jeeves, was used to seek the minimal point and value of the objective function \( F \). The components of the minimal point sought are \( a^*=-0.2333 \) µm, \( b^*=0.3135 \) m, \( p^*=-0.3271 \times 10^{-3} \) and \( q^*=0.5757 \times 10^{-3} \), respectively, and the corresponding minimal value (i.e., the cylindricity error) is \( F(a^*,b^*,p^*,q^*)=11.06 \) µm. In order to show the properties of \( F \) visually, two groups of graphs and isograms of \( F \) were drawn by Matlab. Figure 2 shows the graph and isogram of \( F \) when let \( a \) and \( b \) be constants, while figure 3 shows those when let \( p \) and \( q \) be constants. The constants taken by \( a \), \( b \), \( p \) and \( q \), respectively, were obtained by the least squares assessment method for cylindricity. It can be clearly seen from figure 2 and figure 3 that \( F \) for the example is a continuous, non-differentiable and convex function which has the unique minimal point and minimal value.

![Figure 2](image-url)
5. Conclusion
The unconstrained optimization model established in the paper for assessing cylindricity errors by the minimum zone method is quite simple, so it is convenient for application. Since the objective function in the model is continuous, non-differentiable and convex, its minimal value is unique and any of its minimal point must be its global minimal point. Hence, any of existing algorithms, especially those without using derivatives, can be directly used to solve the objective function to get the reliable minimum zone cylindricity errors, which, undoubtedly, increase the practical value of the model.

Acknowledgments
This work is supported by National Science and Technology Project of China (No.2001BA206-3-4).

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