Error estimates for the Cahn–Hilliard equation with dynamic boundary conditions

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Abstract A proof of convergence is given for bulk–surface finite element semi-discretisation of the Cahn–Hilliard equation with Cahn–Hilliard-type dynamic boundary conditions in a smooth domain. The semi-discretisation is studied in the weak formulation as a second order system. Optimal-order uniform-in-time error estimates are shown in the $L^2$ and $H^1$ norms. The error estimates are based on a consistency and stability analysis. The proof of stability is performed in an abstract framework, based on energy estimates exploiting the anti-symmetric structure of the second order system. Numerical experiments illustrate the theoretical results. Cahn–Hilliard equation; dynamic boundary conditions; bulk–surface finite elements; error estimates; stability; energy estimates.
1 Introduction

In the present paper we analyse a bulk–surface finite element discretisation of the Cahn–Hilliard equation with Cahn–Hilliard-type dynamic boundary conditions, introduced in [GMS11], in a domain with curved boundary. The studied finite element discretisation is based on the (usual) reformulation of the problem into a system of second-order equations. We show optimal-order error estimates that are uniform-in-time for the $L^2$ and $H^1$ norms for both variables, over time intervals where the solutions are sufficiently regular. The chemical potentials in the equation are only required to satisfy only very mild local Lipschitz conditions.

The Cahn–Hilliard equation with Cahn–Hilliard-type dynamic boundary conditions is often used in models which account for interactions with non-permeable walls in a confined system, see, e.g., [KEM+01, RZ03, Gol06, Gal08, GM09, GMS11], which also contain analytic results for existence, uniqueness and regularity.

In this paper we first rewrite the weak formulation of the second order system corresponding to the Cahn–Hilliard equation with Cahn–Hilliard-type dynamic boundary conditions in a general abstract setting (very similar to that of [KL17]). The anti-symmetric structure of this abstract weak formulation will play a crucial role later on. The (non-conforming) bulk–surface finite element semi-discretisation is also rewritten in a semi-discrete version of the abstract formulation, which naturally preserves the anti-symmetry.

Our main theorem, the optimal-order time uniform error estimates, will be shown by separately studying the stability and consistency of the method.

Proving stability, i.e. a uniform-in-time bound of the errors in terms of the defects and their time derivatives, is the main issue in the paper. The main idea of the stability proof is to exploit the anti-symmetric structure of the semi-discrete error equations and combine it with multiple energy estimates, testing with the errors and also with the time derivative of the errors. The main idea of the stability proof was originally developed for Willmore flow [KLL20]. A key issue in the stability proof is to establish uniform-in-time $L^\infty$ norm bounds for the errors, which are then used to estimate the non-linear terms. The $L^\infty$ bounds are obtained from the time-uniform $H^1$ norm error estimates via an inverse estimate. The stability proof is completely independent of any geometric approximation errors. Since the stability proof is entirely performed in an abstract setting, it can be easily generalised to other problems which can be cast in the same setting, for instance the Cahn–Hilliard equation with standard boundary conditions, or more general semi-linear Cahn–Hilliard equations, etc., see Section 5.3. The versatility of the stability analysis is clear in view of the related stability proofs in [KLL20] and [BK20].

The consistency analysis, i.e. proving estimates for the defects (the error obtained upon inserting the Ritz map of the exact solutions into the method) and their time derivatives, uses geometric approximation error estimates and error estimates for the interpolation and the Ritz map for bulk–surface finite elements.
Let us briefly review the numerical analysis literature for problems with dynamic boundary conditions in general.

The paper [ER13] was the first to analyse isoparametric bulk–surface finite element approximations for an elliptic bulk–surface PDE in curved domains, and they have established many fundamental results which are due to the non-conformity of the method.

A conforming finite element discretisation of the Cahn–Hilliard equation with dynamic boundary conditions in a rectangle based slab — in contrast to our curved domain — was analysed in [CPP10] and [CP14]. In both papers the problem is endowed with dynamic boundary conditions except along the first axis where a periodic boundary condition is posed. Both papers use a system formulation of the Cahn–Hilliard equation. The case of Allen–Cahn-type dynamic boundary conditions was analysed in [CPP10], while the case of Cahn–Hilliard-type dynamic boundary conditions was analysed in [CP14]. In the paper [CP14] optimal-order error bounds for both variables are shown, but — in contrast to our results — these estimates are only time-uniform for the original variable and of $L^2$-in-time for the auxiliary variable. The analysis therein also takes advantage of continuous differentiability of the nonlinearities (which are assumed to be $C^2$), and also uses a positivity condition at infinity for their first derivatives. The analysis in the present paper does not use such assumptions.

General linear and semi-linear parabolic equations (of second order) with dynamic boundary conditions were analysed in [KL17], casting the problems in a general abstract setting and showing optimal-order error bounds for a wide range of problems. The Cahn–Hilliard equation was not analysed therein, although Section 2.3.3 of [KL17] contains a possible approach. The present paper uses a different one. Although, [Fai79] already gave error estimates for a conforming Galerkin method for a class of linear parabolic problems, it went unnoticed in the dynamic boundary conditions community, possibly due to the fact that the term dynamic has not appeared at all in his paper. The above mentioned four papers, [Fai79, CPP10, CP14, KL17], are the only publications on the finite element discretisations of parabolic problems with dynamic boundary conditions that we know of.

On the other hand, the numerical analysis of wave equations with dynamic boundary conditions has also developed rapidly over the past few years, see [Hip17, HHS18, HK20] for error estimates for linear problems in a bounded domain, [HL19] for semi-linear wave equations (and [BLN20] for wave equations in an unbounded domain).

The paper is structured as follows. In Section 2 we introduce the basic notation, then formulate the fourth order problem, and rewrite it as a system of second order equations. We formulate the weak formulation corresponding to the system, which is then rewritten in a general abstract setting. In the remainder of the paper we work in this abstract setting. Section 3.1 describes the spatial semi-discretisation using non-conforming bulk–surface finite elements and the abstract semi-discrete setting. Section 4 contains the main result of this paper, Theorem 4.1, which proves optimal-order uniform-in-time error
estimates for the semi-discretisation of the Cahn–Hilliard equation with Cahn–Hilliard-type dynamic boundary conditions. The proof of Theorem 4.1 is shown by separating the issues of stability and consistency. Section 5 contains the stability result based on the novel energy estimates using the anti-symmetric structure of the second order system. Section 5.3 extends the stability results to various problems. Consistency of the method is studied in Section 6 (which also collects various geometric errors bounds and approximation estimates). The combination of the results of these two sections, i.e. the proof of the main theorem, is performed in Section 7. In Section 8 we give a brief outlook on linearly implicit backward difference time discretisations, which preserved the mentioned anti-symmetric structure, and are also used in our numerical experiments. Section 9 is devoted to numerical experiments, which illustrate our theoretical results.

2 Cahn–Hilliard equation with dynamic boundary conditions

Let us briefly introduce some notations. Let the bulk $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) be a bounded domain, with an (at least) $C^2$ boundary $\Gamma = \partial \Omega$, which is referred to as the surface. Further, let $n$ denote the unit outward normal vector to $\Gamma$. Then the surface (or tangential) gradient on $\Gamma$, of a function $u : \Gamma \to \mathbb{R}$, is denoted by $\nabla_{\Gamma} u$, and is given by $\nabla_{\Gamma} u = \nabla \bar{u} - (\nabla \bar{u} \cdot n)n$, (where $\bar{u}$ is an arbitrary extension of $u$ in a neighbourhood of $\Gamma$), while the Laplace–Beltrami operator on $\Gamma$ is given by $\Delta_{\Gamma} u = \nabla_{\Gamma} \cdot \nabla_{\Gamma} u$. For more details see, e.g., [DE13a]. Moreover, $\gamma u$ denotes the trace of $u$ on $\Gamma$, and $\partial_n u$ denotes the normal derivative of $u$ on $\Gamma$. Finally, temporal derivatives are denoted by $\dot{\cdot} = \frac{d}{dt}$.

In this paper we consider the Cahn–Hilliard equation with dynamic boundary conditions of Cahn–Hilliard-type, or – as we will also refer to it – the Cahn–Hilliard/Cahn–Hilliard coupling, that is the fourth-order equation, for a function $u : \Omega \times [0, T] \to \mathbb{R}$,

$$
\dot{u} = \Delta(-\Delta u + W'^{\Omega}(u)) \quad \text{in } \Omega, \tag{2.1a}
$$

$$
\dot{u} = \Delta_{\Gamma}(-\Delta_{\Gamma} u + W'^{\Gamma}(u) + \partial_n u) - \partial_n(-\Delta_{\Gamma} u + W'^{\Gamma}(u) + \partial_n u) \quad \text{on } \Gamma, \tag{2.1b}
$$

with continuous (and sufficiently regular) initial condition $u(0) = u_0$. The scalar functions $W^\Omega$ and $W^\Gamma$ are chemical potentials and are only assumed to be locally Lipschitz, with locally Lipschitz derivatives. By $W'$ we simply denote the derivative of $W$.

In typical examples double well potentials are often used, i.e. $W(u) = (u^2 - 1)^2$. The solution $u \in [-1, 1]$ models the concentration of two fluids, with $u = \pm 1$ indicating the pure occurrences of each. To model the interactions within systems confined by a non-permeable wall, dynamic boundary conditions in this context were introduced in [GMS11].
2.1 Weak formulation as a second order system

By introducing an auxiliary function \( w : \Omega \times [0, T] \to \mathbb{R} \) we rewrite the Cahn–Hilliard equation (2.1) into a system of second order partial differential equations: For \( u, w : \Omega \times [0, T] \to \mathbb{R} \)

\[
\begin{align*}
\dot{u} &= \Delta w & \text{in } \Omega \quad (2.2a) \\
\dot{w} &= -\Delta u + W'_u(u) & \text{in } \Omega, \quad (2.2b)
\end{align*}
\]

with dynamic Cahn–Hilliard boundary conditions

\[
\begin{align*}
\dot{u} &= \Delta_{\Gamma} w - \partial_n w & \text{on } \Gamma \quad (2.2c) \\
\dot{w} &= -\Delta_{\Gamma} u + W'_{\Gamma}(u) + \partial_n u & \text{on } \Gamma. \quad (2.2d)
\end{align*}
\]

Before we turn to the weak formulation of the above problem, let us introduce some notations for function spaces. We will use standard Sobolev spaces \( H^k(\Omega) \) and \( H^k(\Gamma) \), for \( k \geq 0 \), in the bulk and on the surface, respectively, for more details see [DE13a] and [ER13]. The variational formulation will also use the Hilbert spaces

\[
V = \{ v \in H^1(\Omega) \mid \gamma u \in H^1(\Gamma) \} \quad \text{and} \quad H = L^2(\Omega) \times L^2(\Gamma),
\]

with a dense embedding from \( V \) into \( H \) by their first argument \( v \).

The weak formulation of the Cahn–Hilliard equation with dynamic boundary conditions of Cahn–Hilliard-type is derived by multiplying (2.2a) with the test function \( \Phi \in V \), and (2.2b) with \( \phi \in V \). Then both equations are integrated over the domain \( \Omega \), and by applying Green’s formula for both, we obtain

\[
\begin{align*}
\int_{\Omega} \dot{u} \Phi &= -\int_{\Omega} \nabla w \cdot \nabla \Phi + \int_{\Gamma} \partial_n w \gamma \Phi, \\
\int_{\Omega} w \Phi &= \int_{\Omega} \nabla u \cdot \nabla \Phi + \int_{\Omega} W'(u) \Phi - \int_{\Gamma} \partial_n u \gamma \Phi.
\end{align*}
\]

Plugging in the dynamic boundary conditions (2.2c) and (2.2d) into the boundary terms above, applying Green’s formula now on the boundary, and collecting the terms, yield

\[
\begin{align*}
\left( \int_{\Omega} \dot{u} \Phi + \int_{\Gamma} \gamma u \gamma \Phi \right) + \left( \int_{\Omega} \nabla w \cdot \nabla \Phi + \int_{\Gamma} \nabla_{\Gamma} w \cdot \nabla_{\Gamma} \Phi \right) &= 0, \quad (2.5a) \\
\left( \int_{\Omega} w \Phi + \int_{\Gamma} \gamma w \gamma \Phi \right) - \left( \int_{\Omega} \nabla u \cdot \nabla \Phi + \int_{\Gamma} \nabla_{\Gamma} u \cdot \nabla_{\Gamma} \Phi \right) &= \int_{\Omega} W'(u) \Phi + \int_{\Gamma} W'_{\Gamma}(u) \gamma \Phi, \quad (2.5b)
\end{align*}
\]
where for brevity we write $\nabla_R v$ instead of $\nabla_R (\gamma v)$, and have also suppressed the trace operator in $W'_R(u)$. We will employ these conventions throughout the paper.

### 2.2 Abstract formulation

It is insightful to formulate the variational formulation in an abstract setting. To this end we introduce the bilinear forms, on $V$ and $H$, respectively,

$$
\begin{align*}
a(u, v) &= \int_{\Omega} \nabla u \cdot \nabla v + \int_{\Gamma} \nabla_{R} u \cdot \nabla_{R} v \\
m(u, v) &= \int_{\Omega} uv + \int_{\Gamma} (\gamma u)(\gamma v),
\end{align*}
$$

and let

$$a^*(\cdot, \cdot) = a(\cdot, \cdot) + m(\cdot, \cdot).$$

We note here that the norms (2.4) on $V$ and $H$ are directly given by

$$\|u\|^2 = a^*(u, u) = a(u, u) + m(u, u) \quad \text{and} \quad |u|^2 = m(u, u).$$

Furthermore, on $V$ the bilinear form $a(\cdot, \cdot)$ generates the semi-norm

$$\|u\|_a^2 := a(u, u).$$

For the non-linear term, under the expression $m(W'(u), \varphi^w)$ we mean

$$m(W'(u), \varphi^w) = \int_{\Omega} W'(u)\varphi^w + \int_{\Gamma} W'_R(\gamma u)\gamma \varphi^w.$$

Using these bilinear forms, we rewrite the weak formulation (2.5). The Cahn–Hilliard equation with Cahn–Hilliard-type dynamic boundary conditions in the above abstract setting reads: Find $u \in C^1([0, T], H) \cap L^2([0, T], V)$ and $w \in L^2([0, T], V)$ such that, for time $0 < t \leq T$ and all $\varphi^u, \varphi^w \in V$,

$$
\begin{align*}
m(\dot{u}(t), \varphi^u) + a(u(t), \varphi^u) &= 0, \\
m(w(t), \varphi^w) - a(u(t), \varphi^w) &= m(W'(u(t)), \varphi^w),
\end{align*}
$$

for given initial data $u(0) = u^0 \in H$.

From now on we will use this (rather general) abstract formulation for the Cahn–Hilliard equation with Cahn–Hilliard-type dynamic boundary condition.

**Remark 2.1** Other types of dynamic boundary conditions also fit into this framework by using different Hilbert spaces, and changing the boundary integrals in the bilinear forms $m$ and $a$. For more details see Section 5.3.
3 Semi-discretisation of Cahn–Hilliard equations with dynamic boundary conditions

For the numerical solution of the above examples we consider a linear finite element method both in the bulk and on the surface. In the following, from [ER13], [KL17, Section 3.2.1], and [HK20], we will briefly recall the construction of the discrete domain, the finite element space and the lift operation, the discrete bilinear forms, which will be used to discretize the Cahn–Hilliard problem of Section 2.

3.1 The bulk–surface finite elements

The domain $\Omega$ is approximated by a triangulation $T_h$ with maximal mesh width $h$. The union of all elements of $T_h$ defines the polyhedral domain $\Omega_h$ whose boundary $\Gamma_h := \partial \Omega_h$ is an interpolation of $\Gamma$, i.e. the vertices of $\Gamma_h$ are on $\Gamma$. Analogously, we denote the outer unit normal vector of $\Gamma_h$ by $n_h$. We assume that $h$ is sufficiently small to ensure that for every point $x \in \Gamma_h$ there is a unique point $p \in \Gamma$ such that $x - p$ is orthogonal to the tangent space $T_p \Gamma$ at $p$. For convergence results, we consider a quasi-uniform family of such triangulations $T_h$ of $\Omega_h$.

The finite element space $V_h \subset H^1(\Omega)$ corresponding to $T_h$ is spanned by continuous, piecewise linear nodal basis functions on $\Omega_h$, satisfying for each node $(x_k)_{k=1}^N$
\[ \phi_j(x_k) = \delta_{jk}, \quad \text{for } j, k = 1, \ldots, N. \]
Then the finite element space is given as
\[ V_h = \text{span}\{\phi_1, \ldots, \phi_N\}. \]

We note here that the restrictions of the basis functions to the boundary $\Gamma_h$ again form a surface finite element basis over the approximate boundary elements.

Following [Dzi88], we define the lift operator $\ell : V_h \to V$ to compare functions in $V_h$ with functions in $V$. For functions $v_h : \Gamma_h \to \mathbb{R}$, we define the lift as
\[ v_h^\ell : \Gamma \to \mathbb{R} \quad \text{with} \quad v_h^\ell(p) = v_h(x), \quad \forall p \in \Gamma, \] (3.1)
eq lift definition

where $x \in \Gamma_h$ is the unique point on $\Gamma_h$ with $x - p$ orthogonal to the tangent space $T_p \Gamma$. We further consider the lift of functions $v_h : \Omega_h \to \mathbb{R}$ to $v_h^\ell : \Omega \to \mathbb{R}$ by setting $v_h^\ell(p) = v_h(x)$ if $x \in \Omega_h$ and $p \in \Omega$, where the two points are related as described in detail in [ER13, Section 4]. The mapping $G_h : \Omega_h \to \Omega$ is defined piecewise, for an element $E \in T_h$, by
\[ G_h|_E(x) = F_e((F_E)^{-1}(x)), \quad \text{for } x \in E, \] (3.2)
eq bulk mapping

where $F_e$ is a $C^1$ map (see [ER13, equation (4.2) & (4.4)]) from the reference element onto the smooth element $e \subset \Omega$, and $F_E$ is the standard affine linear map between the reference element and $E$, see, e.g. [ER13, equation (4.1)].
Finally, the lifted finite element space is denoted by $V_h^\ell$, and is given as $V_h^\ell = \{ v_h^\ell \mid v_h \in V_h \}$.

Note that both definitions of the lift coincide on $\Gamma$.

### 3.2 Discrete spaces and discrete bilinear forms

The discrete bilinear forms on $V_h$, i.e. the discrete counterparts of $a$ and $m$, are given, for $u_h, v_h \in V_h$, by

\[
a_h(u_h, v_h) = \int_{\Omega_h} \nabla u_h \cdot \nabla v_h + \int_{\Gamma_h} \nabla_\Gamma u_h \cdot \nabla_\Gamma v_h, \\
m_h(u_h, v_h) = \int_{\Omega_h} u_h v_h + \int_{\Gamma_h} (\gamma_h u_h)(\gamma_h v_h),
\]

and let $a_h^\ast(\cdot, \cdot) = a_h(\cdot, \cdot) + m_h(\cdot, \cdot)$.

The discrete norms on $V_h$, corresponding to $\| \cdot \|$ and $| \cdot |$, are given by

\[
\| u_h \|^2_h := \| u_h \|^2_{H^1(\Omega_h)} + \| \gamma_h u_h \|^2_{H^1(\Gamma_h)} = a_h(u_h, u_h) + m_h(u_h, u_h), \\
| u_h |^2_h := \| u_h \|^2_{L^2(\Omega_h)} + \| \gamma_h u_h \|^2_{L^2(\Gamma_h)} = m_h(u_h, u_h),
\]

and the discrete semi-norm induced by $a_h(\cdot, \cdot)$ (analogously to the continuous case)

\[
\| u_h \|^2_{a_h} := \| \nabla u_h \|^2_{L^2(\Omega_h)} + \| \nabla_\Gamma u_h \|^2_{L^2(\Gamma_h)} = a_h(u_h, u_h).
\]

Later on in Section 6.1.2, it will be shown that these discrete norms and their continuous counterparts are $h$-uniformly equivalent.

### 3.3 A Ritz map

We will now define a Ritz map, which will be used in the error analysis, and also for prescribing the initial data for the semi-discrete problem.

From [KL17, Section 3.4] we recall the Ritz map $\tilde{R}_h : V \to V_h$ which is defined, for $u \in V$, by

\[
a_h^\ast(\tilde{R}_h u, \varphi_h) = a(\cdot, \cdot)_{V_h}, \quad \text{for every } \varphi_h \in V_h.
\]

The above Ritz map is well-defined for all $u \in V$ due to the ellipticity of the bilinear form $a_h^\ast$, see [KL17, Section 3.4]. Note that the Ritz map $\tilde{R}_h u$ is the Riesz representation of $u$. Further, note that, the bilinear forms $a^\ast$ and $a_h^\ast$ contain boundary integrals which influence $\tilde{R}_h$. The lifted Ritz map will be denoted by $R_h u := (\tilde{R}_h u)^\ell \in V_h^\ell$.
3.4 Semi-discrete problem

The semi-discrete problem then reads: Find \( u_h \in C^1([0,T],V_h) \) and \( w_h \in L^2([0,T],V_h) \) such that, for time \( 0 < t \leq T \) and all \( \phi_h^u, \phi_h^w \in V \),

\[
\begin{align*}
    m_h(\dot{u}_h(t), \phi_h^u) + a_h(u_h(t), \phi_h^u) &= 0, \\
    m_h(w_h(t), \phi_h^u) - a_h(u_h(t), \phi_h^w) &= m_h(W'(u_h(t)), \phi_h^w),
\end{align*}
\]

(3.5a)  
(3.5b)

where the initial data \( u_h(0) = u_h^0 \in V_h \) is chosen to be the Ritz map of \( u^0 \), i.e. \( u_h(0) = u_h^0 = \tilde{R}_h u^0 \in V_h \), and the initial value for \( w_h \) is obtained by solving the elliptic equation (3.5b).

3.5 A modified semi-discrete problem

Since the initial value for \( w_h \) is not freely chosen, but is determined by the equations, it is to be expected that they will be involved in the error analysis. Our error analysis will provide an optimal order error bound in both norms \( \| \cdot \| \) and \( | \cdot | \), provided that this initial value is \( O(h^2) \) close to the Ritz map of the exact initial value in the stronger norm.

Since such an \( O(h^2) \) estimate can only be verified in a weaker norm, we modify the second equation (3.5b) with a time-independent term to obtain optimal-order error estimates. The additional term corrects the initial value \( \bar{w}_h(0) \in V_h \) (obtained from (3.5b)) at \( t = 0 \) to coincide with the Ritz map of the exact initial value \( w_h^*(0) \). Let \( \theta_h \in V_h \) be given by

\[
\theta_h = w_h^*(0) - \bar{w}_h(0),
\]

(3.6)

then the modified semi-discrete problem then reads: Find \( u_h \in C^1([0,T],V_h) \) and \( w_h \in L^2([0,T],V_h) \) such that, for time \( 0 < t \leq T \) and all \( \phi_h^u, \phi_h^w \in V \),

\[
\begin{align*}
    m_h(\dot{u}_h(t), \phi_h^u) + a_h(u_h(t), \phi_h^u) &= 0, \\
    m_h(w_h(t), \phi_h^u) - a_h(u_h(t), \phi_h^w) &= m_h(W'(u_h(t)), \phi_h^w) + m_h(\theta_h, \phi_h^w),
\end{align*}
\]

(3.7a)  
(3.7b)

The initial data for (3.7a) is still \( u_h(0) = u_h^0 = \tilde{R}_h u^0 \in V_h \), while the initial data for \( w_h \), obtained from (3.7b), is

\[
\begin{align*}
    m_h(u_h(0), \phi_h^u) &= a_h(u_h(0), \phi_h^u) + m_h(W'(u_h(0)), \phi_h^w) + m_h(\theta_h, \phi_h^w) \\
    &= m_h(u_h^*(0), \phi_h^u),
\end{align*}
\]

via (3.5b).

The advantage of the modified system, is that the errors in the initial data for \( w_h \) are included into the problem similarly to a defect, which allows for feasible weaker norm estimate of this error. Note that for the linear case, this is nothing else but shifting the solutions to a particular initial value using a constant inhomogeneity.
4 Main results: optimal-order semi-discrete error estimates

We are now able to state the main theorem of this paper.

**Theorem 4.1** Let \( u \) and \( w \) be sufficiently smooth solutions to the Cahn–Hilliard equation with Cahn–Hilliard-type dynamic boundary conditions (2.2), sufficient regularity assumptions are (4.1).

Then there exists an \( h_0 > 0 \) such that for all \( h \leq h_0 \) the error between the solutions \( u \) and \( w \) and the linear finite element semi-discretisations \( u_h \) and \( w_h \) of (3.7) satisfy the optimal-order uniform-in-time error estimates in both variables, for \( 0 \leq t \leq T \),

\[
\|u_h(t) - u(t, s)\|_{L^2(\Omega)} + \|\gamma(u_h(t) - u(t, s))\|_{L^2(\Gamma)} + h\left(\|u_h'(t) - u'(t, s)\|_{H^1(\Omega)} + \|\gamma(u_h'(t) - u'(t, s))\|_{H^1(\Gamma)}\right) \leq Ch^2,
\]

whereas for the time derivatives of the errors in \( u \) satisfy, for \( 0 \leq t \leq T \),

\[
\left(\int_0^t \left(\|\partial_t u_h(t, s) - u(t, s)\|_{L^2(\Omega)} + \|\partial_t \gamma(u_h(t, s) - u(t, s))\|_{L^2(\Gamma)}\right)^2 \, ds\right)^{1/2} \leq Ch^2.
\]

The constant \( C > 0 \) depends on Sobolev norms of the solutions, and on the final time \( T \), but it is independent of \( h \) and \( t \).

For Theorem 4.1 sufficient regularity conditions are

\( u \in C^4([0, T], H^1(\Omega)) \cap H^2([0, T], H^2(\Omega)) \cap W^{1, \infty}([0, T], L^\infty(\Omega)) \) with \( \gamma u \in C^4([0, T], H^1(\Gamma)) \cap H^2([0, T], H^2(\Gamma)) \cap W^{1, \infty}([0, T], L^\infty(\Gamma)) \),

and

\( w \in C([0, T], H^1(\Gamma)) \cap H^1([0, T], H^2(\Omega)) \) with \( \gamma w \in C([0, T], H^1(\Gamma)) \cap H^1([0, T], H^2(\Gamma)) \).

Our main theorem, Theorem 4.1, will be proved by separately studying the questions of stability and consistency in Section 5 and 6, respectively, and combining their results in Section 7.

5 Stability

5.1 Error equations

Let us consider the Ritz map of the exact solutions \( u \) and \( w \) of (2.2), which are denoted by

\[
u_h(t) = R_h u(t) \in V_h, \quad \text{and} \quad w_h(t) = R_h w(t) \in V_h.
\]
The Ritz maps of the exact solutions satisfy the system (3.5) only up to some defects $d_h^u$ and $d_h^w$ in $V_h$:

\begin{align}
  m_h(\bar{u}_h^u(t), \bar{\varphi}_h^u) + a_h(u_h^u(t), \varphi_h^u) &= m_h(d_h^u(t), \varphi_h^u), \quad (5.1a) \\
  m_h(w_h^u(t), \bar{\varphi}_h^w) - a_h(u_h^u(t), \varphi_h^w) &= m_h(W'(u_h(t)), \varphi_h^w) + m_h(d_h^w(t), \varphi_h^w), \quad (5.1b)
\end{align}

with initial data as the Ritz map of the exact initial data, i.e. $u_h^u(0) = \tilde{u}$. The errors between the semi-discrete solutions and the Ritz maps of the exact solutions are denoted by $e_h^u = u_h - u_h^u$ and $e_h^w = w_h - w_h^u$ in $V_h$. By subtracting (5.1) from (3.7) we obtain that the errors $e_h^u$ and $e_h^w$ satisfy the following error equations:

\begin{align}
  m_h(e_h^u(t), \varphi_h^u) + a_h(e_h^u(t), \varphi_h^u) &= m_h(d_h^u(t), \varphi_h^u), \quad (5.2a) \\
  m_h(e_h^w(t), \varphi_h^w) - a_h(e_h^w(t), \varphi_h^w) &= m_h(W'(u_h(t)) - W'(u_h^u(t)), \varphi_h^w) + m_h(d_h^w(t), \varphi_h^w), \quad (5.2b)
\end{align}

with vanishing initial values $e_h^u(0) = 0$ and $e_h^w(0) = 0$. Since, $e_h^u(0)$ and $e_h^w(0)$ satisfies the error equation (5.2b) at $t = 0$, we obtain that

\[ \partial_h = d_h^w(0). \]

The defects will be estimated using a discrete dual norm on the space $V_h$ defined by

\[ \|d_h\|_{\ast, h} = \sup_{0 \neq v_h \in V_h} \frac{m(d_h, v_h^f)}{\|v_h\|_h}. \]

It is easy to see that, as in the continuous case, there exist constants $c, C > 0$ (independent of $h$) such that

\[ c\|v_h\|_{\ast, h} \leq |v_h|_h \leq C\|v_h\|_h. \]

5.2 Stability bounds

**Proposition 5.1** Assume that the defects satisfy

\[ \|d(t)\|_{\ast, h} \leq ch^2 \quad \text{for} \quad d = d_h^u, d_h^w, d_h^\varphi, \text{and for } 0 \leq t \leq T. \]

Then there exists $h_0 > 0$ such that the following stability estimate holds for all $h \leq h_0$ and $0 \leq t \leq T$:

\begin{align}
  \|e_h^u(t)\|_h^2 + \|e_h^w(t)\|_h^2 + \int_0^t \|e_h^u(s)\|_h^2 ds + \int_0^t \|e_h^w(s)\|_h^2 ds \\
  \leq C \left( \|e_h^u(0)\|_h^2 + \|d_h^u(t)\|_h^2 + \int_0^t \|d_h^w(s)\|_h^2 ds \right) \quad \text{(5.6)}
\end{align}

where the constant $C > 0$ is independent of $h$ and $t$, but depends on the final time $T$. 

The main idea of the stability proof is to exploit the anti-symmetric structure of the semi-discrete error equations (5.2) and combine it with multiple energy estimates, and which was originally developed for Willmore flow [KLL20]. The key issue in the stability proof is to establish a uniform-in-time $L^\infty(\Omega)$ norm bound for $e^u_h$, which is then used to estimate the non-linear terms. Such time-uniform $L^\infty(\Omega)$ bounds are obtained from the time-uniform $H^1(\Omega)$ norm error estimates via an inverse estimate. The stability proof is completely independent of any geometric approximation errors, which only enter the consistency analysis. Since the stability proof is entirely performed in the semi-discrete abstract setting of Section 3, we strongly expect that it can be generalised to other problems which can be cast in the same setting, for further details see Section 5.3.

In Section 6 we will show that the assumed bounds (5.5) indeed hold. Hence, together with the stability bound (5.6), the consistency bounds imply the estimates for the errors $e^u_h$ and $e^w_h$, for $0 \leq t \leq T$,

$$\|e^u_h(t)\|_h \leq Ch^2 \quad \text{and} \quad \|e^w_h(t)\|_h \leq Ch^2.$$ 

Proof In order to achieve the uniform-in-time stability bound, two sets of energy estimates are needed. These energy estimates strongly exploit the anti-symmetric structure of (5.2). (i) In the first, a uniform-in-time energy estimate is proved for $e^u_h$, but which comes with a critical term involving $\dot{e}^u_h$. (ii) The second estimate uses the time derivative of (5.2b), and leads to a bound of this critical term and also to a uniform-in-time bound for $e^w_h$. The combination of these two energy estimates will give the stated stability bound. The structure and basic idea of the proof is sketched in Figure 5.1.

In order to handle the semi-linear term we first prove the stability bound on a time interval where the $L^\infty$ norm of $e^u_h$ is small enough, and then show that this time interval can be enlarged up to $T$.
In the following $c$ and $C$ are generic constants that may take different values on different occurrences. Whenever it is possible, without confusion, we omit the argument $t$. By $\rho > 0$ we will denote a small number, used in Young’s inequality, and hence we will often incorporate $h$ independent multiplicative constants into the those yet unchosen factors.

Let $t^* \in (0, T]$ be the maximal time such that the following estimate holds

$$
\|\gamma_h e_h^u(t)\|_{L^\infty(\Omega_h)} \leq \|e_h^u(t)\|_{L^\infty(\Omega_h)} \leq h^{1-d/4} \quad \text{for} \quad t \in [0, t^*]. \tag{5.7}
$$

Recall, that here we consider domains of dimension $d = 2$ or 3, and note that for finite element functions the first inequality holds in general. We note that such a positive $t^*$ exists, since the initial error in $u$ is identically zero $e_h^u(0) = 0$.

Energy estimate (i): We test (5.2a) with $e_h^w$ and (5.2b) with $e_h^w$ then sum up the two equations, and use the symmetry of $a_h(\cdot, \cdot)$, to obtain

$$
m_h(e_h^w, e_h^w) + m_h(e_h^w, e_h^w) = m_h(W'(u_h) - W'(u_h^*), e_h^w) - m_h(d_h^w, e_h^w) - m_h(d_h^w, e_h^w) + m_h(\vartheta_h, e_h^w). \tag{5.8}
$$

Similarly, we test (5.2a) with $e_h^w$ and (5.2b) with $e_h^w$ now subtracting the two equations and using the symmetry of $m_h(\cdot, \cdot)$, we obtain

$$
a_h(e_h^w, e_h^w) + a_h(e_h^w, e_h^w) = m_h(W'(u_h) - W'(u_h^*), e_h^w) - m_h(d_h^w, e_h^w) + m_h(d_h^w, e_h^w) - m_h(\vartheta_h, e_h^w). \tag{5.9}
$$

Taking the linear combination of the equations (5.8) and (5.9), we then use that for the bilinear form $a^* = a + m$ the following holds

$$
a_h^*(e_h^w, e_h^w) = \frac{1}{2} \frac{d}{dt} \|e_h^w\|_h^2, \tag{5.10}
$$

and hence we obtain

$$
\frac{1}{2} \frac{d}{dt} \|e_h^w\|_h^2 + \|e_h^w\|_h^2 = m_h(W'(u_h) - W'(u_h^*), e_h^w) + m_h(W'(u_h) - W'(u_h^*), e_h^w) + m_h(d_h^w, e_h^w) + m_h(d_h^w, e_h^w) - m_h(\vartheta_h, e_h^w).
$$

Now we estimate all the terms on the right-hand side separately.

For the non-linear terms, using the bound (5.7) together with the local Lipschitz continuity of the functions $W''_D$ and $W''_F$, we obtain

$$
m_h(W'(u_h) - W'(u_h^*), e_h^w) \leq c|e_h^w|_h |e_h^w|_h \leq c|e_h^w|_h^2 + \frac{1}{8}|e_h^w|_h^2. \tag{5.12}
$$

Analogously, for the other non-linear term we obtain

$$
m_h(W'(u_h) - W'(u_h^*), e_h^w) \leq c|e_h^w|_h |e_h^w|_h \leq c|e_h^w|_h^2 + \rho|e_h^w|_h^2. \tag{5.13}
$$

In both cases the for the last inequalities we have used Young’s inequality (in the second case with a small factor $\rho > 0$ chosen later on).
For the terms with defects, using the Cauchy–Schwarz and Young’s inequalities (often with a small factor $\rho > 0$ chosen later on), we obtain
\[
\begin{aligned}
& m_h(\delta_U^+, e_h^w) + m_h(\delta_U^-, e_h^w) + m_h(d_h^w, e_h^w) - m_h(d_h^w, e_h^w) \\
& \leq \|d_h^w\|_{-h} \|e_h^w\|_h + \|d_h^w\|_{-h} \|e_h^w\|_h + \|d_h^w\|_{-h} \|e_h^w\|_h + \|d_h^w\|_{-h} \|e_h^w\|_h \\
& \leq \epsilon\|e_h^w\|_h^2 + \rho\|e_h^w\|_h^2 + \frac{1}{2}\|e_h^w\|_h^2 + c\left(\|d_h^w\|_{-h}^2 + \|d_h^w\|_{-h}^2\right).
\end{aligned}
\] (5.14)

The terms involving the correction $\vartheta_h$ are estimated similarly, in combination with (5.3), as
\[
\begin{aligned}
& m_h(\vartheta_h, e_h^w) - m_h(\vartheta_h, e_h^w) \\
& \leq \|\vartheta_h\|_{-h} \|e_h^w\|_h + \|\vartheta_h\|_{-h} \|e_h^w\|_h \\
& \leq \rho\|e_h^w\|_h^2 + \frac{1}{2}\|e_h^w\|_h^2 + c\|d_h^w\|_{-h}^2.
\end{aligned}
\] (5.15)

Altogether, by plugging in the estimates (5.12)–(5.15) into (5.11), and integrating from 0 to $t$, we first differentiate the system (5.17), and then sum up the two equations separately, integrate in time (in (a) and (b), respectively), and then take their weighted combination (in (c)).

(a) We test (5.17a) with $\delta_U^w$ and (5.17b) with $\vartheta_h^w$, then sum up the two equations, and use the symmetry of $a_h(\cdot, \cdot)$, to obtain
\[
\begin{aligned}
& m_h(\delta_U^w, \vartheta_h^w) = m_h\left(\frac{d}{dt}(W'(u_h(t)) - W'(\dot{u}_h^w(t))), \vartheta_h^w\right) - m_h(\delta_U^w, \vartheta_h^w).
\end{aligned}
\] (5.16)

(b) Testing the error equation system (5.17) in two sets as before, and then taking directly the linear combination of the obtained equations would not lead to a feasible energy estimate, due to a critical term involving $\dot{e}_h^w$. Therefore, we first estimate the two equations separately, integrate in time (in (a) and (b), respectively), and then take their weighted combination (in (c)).

For the non-linear terms, using the bound (5.7) together with the local Lipschitz continuity of the functions $W'_I$, $W'_I$, and $W''_I$, $W''_I$, and using that
we rewrite it as follows:

\[ m_h \left( \frac{d}{dt} (W'(u_h(t)) - W'(u_h^*(t))), e_h^w \right) = m_h \left( \frac{d}{dt} W'(u_h), e_h^w \right) - m_h \left( \frac{d}{dt} W'(u_h^*), e_h^w \right) \]

\[ = \int_0^1 \frac{d}{d\theta} m_h \left( \frac{d}{dt} W'(u_h), e_h^w \right) d\theta \]

\[ = \int_0^1 \frac{d}{d\theta} m_h \left( W''(u_h) \dot{u}_h, e_h^w \right) d\theta \]

\[ = \int_0^1 m_h \left( W''(u_h \dot{u}_h^w, e_h^w) \right) d\theta \]

\[ \leq c (|e_h^w|_h + |\dot{e}_h^w|_h) |e_h^w|_h. \]  

Therefore, for the non-linear term in (5.18) we obtain

\[ m_h \left( \frac{d}{dt} (W'(u_h(t)) - W'(u_h^*(t))), e_h^w \right) \leq c (|e_h^w|_h + |\dot{e}_h^w|_h) |e_h^w|_h \]

\[ \leq \rho |e_h^w|_h^2 + c |\dot{e}_h^w|_h^2 + c |\ddot{e}_h^w|_h^2. \]  

(5.19)

For the terms with defects, using the Cauchy–Schwarz and Young’s inequalities (often with a small factor \( \rho > 0 \) chosen later on), we obtain

\[ -m_h (d_h^u, \dot{e}_h^w) - m_h (\dot{d}_h^w, e_h^w) \leq \|d_h^u\|_{*,h} \|\dot{e}_h^w\|_h + \|\dot{d}_h^w\|_{*,h} \|e_h^w\|_h \]

\[ \leq \rho \|e_h^w\|_h^2 + c |\dot{e}_h^w|_h^2 + c (\|d_h^u\|_{*,h}^2 + \|\dot{d}_h^w\|_{*,h}^2). \]  

(5.20)

Combining the above estimates for (5.18), using (5.10), integrating in time from 0 to \( t \leq t^* \), and multiplying by two, we altogether obtain

\[ \int_0^t |e_h^w(s)|_h^2 ds + |e_h^w(t)|_h^2 \leq \rho \int_0^t \|e_h^w(s)|_h^2 ds + c \int_0^t (\|e_h^w(s)|_h^2 + |\dot{e}_h^w(s)|_h^2) ds \]

\[ + c \int_0^t (\|d_h^u(s)|_h^2 + \|\dot{d}_h^w(s)|_h^2) ds. \]  

(5.21)

(b) We now test (5.17a) with \( \dot{e}_h^w \) and (5.17b) with \( \ddot{e}_h^w \) now subtracting the two equations and using the symmetry of \( m_h(\cdot, \cdot) \), we obtain

\[ a_h (e_h, \dot{e}_h^w) + a_h (e_h^w, \dot{e}_h^w) = -m_h \left( \frac{d}{dt} (W'(u_h(t)) - W'(u_h^*(t))), \dot{e}_h^w \right) \]

\[ -m_h (d_h^u, \dot{e}_h^w) + m_h (\dot{d}_h^w, e_h^w) \]

The term \( \dot{e}_h^w \) cannot be absorbed or controlled, therefore, by the product rule we rewrite it as follows:

\[ -m_h (d_h^u, \dot{e}_h^w) = -\frac{d}{dt} m_h (d_h^u, e_h^w) + m_h (\dot{d}_h^w, e_h^w). \]
The combination of the above two equations then gives

\[
a_h(e_h^w, \dot{e}_h^w) + a_h(\dot{e}_h^u, \dot{e}_h^u) = -m_h \left( \frac{d}{dt} (W'(u_h(t)) - W'(u_h^0(t))) \right) + \frac{d}{dt} m_h(d_h^w, e_h^w) + m_h(d_h^u, e_h^u) + m_h(d_h^w, \dot{e}_h^w). \tag{5.22}
\]

For the non-linear term in (5.22), analogously as before, we obtain

\[
m_h \left( \frac{d}{dt} (W'(u_h(t)) - W'(u_h^0(t))) \right), \dot{e}_h^k \leq c(\|e_h^w\|_h + |e_h^u|) |\dot{e}_h^k|_h, \tag{5.23}
\]

with a particular \((h\text{ independent})\) constant \(c_0 > 0\).

The defect terms without a time derivative are bounded, similarly as before, by

\[
m_h(d_h^w, e_h^w) + m_h(d_h^u, e_h^u) \leq \|d_h^w\|_{\ast, h} \|e_h^w\|_h + \|d_h^u\|_{\ast, h} \|e_h^u\|_h \leq \rho \|\dot{e}_h^w\|^2 + c\|e_h^w\|^2 + c(\|d_h^w\|^2_{\ast, h} + \|d_h^u\|^2_{\ast, h}). \tag{5.24}
\]

Combining the above estimates for (5.22), using (5.10) for the bilinear form \(a_h(\cdot, \cdot)\), integrating in time from \(0\) to \(t\), and multiplying by two, we altogether obtain

\[
\int_0^t \|\dot{e}_h^w(s)\|^2_{e_h^w} ds + \frac{1}{2}\|e_h^w(t)\|^2_{e_h^w} \leq \frac{1}{2}\|e_h^w(0)\|^2_{e_h^w} + c_0 \int_0^t |e_h^w(s)|^2_{e_h^w} ds + \rho \int_0^t \|e_h^w(s)\|^2_{e_h^w} ds + \rho \int_0^t \|e_h^u(s)\|^2_{e_h^u} ds
\]

\[
+ c \int_0^t (\|e_h^w(s)\|^2 + \|e_h^u(s)\|^2) ds
\]

\[
- m_h(d_h^w(t), e_h^w(t)) + c \int_0^t (\|d_h^w(s)\|^2_{\ast, h} + \|d_h^u(s)\|^2_{\ast, h}) ds.
\]

Finally, by estimating the pointwise defect terms similarly as, e.g., in (5.24), we obtain

\[
\int_0^t \|\dot{e}_h^w(s)\|^2_{e_h^w} ds + \frac{1}{2}\|e_h^w(t)\|^2_{e_h^w} \leq c_0 \int_0^t |e_h^w(s)|^2_{e_h^w} ds + \rho \int_0^t \|e_h^w(s)\|^2_{e_h^w} ds + \rho \|e_h^u(t)\|^2_{e_h^u}
\]

\[
+ c \int_0^t (\|e_h^w(s)\|^2 + \|e_h^u(s)\|^2) ds
\]

\[
+ \rho \|e_h^w(t)\|^2 + c\|d_h^w(t)\|^2_{\ast, h}
\]

\[
+ c \int_0^t (\|d_h^w(s)\|^2_{\ast, h} + \|d_h^u(s)\|^2_{\ast, h}) ds. \tag{5.25}
\]
(c) We now multiply (5.21) by $2c_0$ and (5.25) by 1, and take this weighted combination of the two inequalities. After collecting the terms, we obtain

$$2c_0 \int_0^t |\dot{e}_h^n(s)|^2_h ds + \int_0^t |\dot{e}_h^n(s)|^2_h ds + 2c_0 |e_h^n(t)|^2_h + \frac{1}{2} |e_h^n(t)|^2_h \leq c_0 \int_0^t |\dot{e}_h^n(s)|^2_h ds + \rho \int_0^t |\dot{e}_h^n(s)|^2_h ds + \rho |e_h^n(t)|^2_h + c \int_0^t (||e_h^n(s)||^2_h + ||e_h^n(s)||^2_h) ds + c \int_0^t (||d_h^n(t)||^2_{2,h} + ||d_h^n(0)||^2_{2,h}) + c \int_0^t (||d_h^n(s)||^2_{2,h} + ||d_h^n(s)||^2_{2,h} + ||\dot{d}_h^n(s)||^2_{2,h} + ||\dot{d}_h^n(s)||^2_{2,h}) ds. $$

\[ \text{eq:energy est - ii - post sum} \]

The first term with $\dot{e}_h^n$ on the right-hand side is directly absorbed into the first term on the left-hand side. After combining the norms on the left-hand side, the second term on the right-hand side is absorbed by choosing $\rho > 0$ small enough. We use a further absorption into the term $e_h^n(t)$, and then divide both sides by $\min\{1/4, c_0\}$, which yields the second energy estimate

$$\int_0^t |\dot{e}_h^n(s)|^2_h ds + |e_h^n(t)|^2_h \leq c_0 \int_0^t (||e_h^n(s)||^2_h + ||e_h^n(s)||^2_h) ds + c (||d_h^n(t)||^2_{2,h} + ||d_h^n(0)||^2_{2,h}) + c \int_0^t (||d_h^n(s)||^2_{2,h} + ||d_h^n(s)||^2_{2,h} + ||\dot{d}_h^n(s)||^2_{2,h} + ||\dot{d}_h^n(s)||^2_{2,h}) ds. $$

\[ \text{eq:energy estimate - ii} \]

Combining the energy estimates: The sum of the energy estimates (5.16) and (5.27) gives

$$||e_h^n(t)||^2_h + |e_h^n(t)|^2_h + \int_0^t ||\dot{e}_h^n(s)||^2_h ds + \int_0^t |e_h^n(s)|^2_h ds \leq \rho \int_0^t ||e_h^n(s)||^2_h ds + c \int_0^t (||e_h^n(s)||^2_h + ||e_h^n(s)||^2_h) ds + ct ||d_h^n(0)||^2_{2,h} + c \int_0^t (||d_h^n(s)||^2_{2,h} + ||d_h^n(s)||^2_{2,h} + ||\dot{d}_h^n(s)||^2_{2,h} + ||\dot{d}_h^n(s)||^2_{2,h}) ds. $$

\[ \text{eq:energy estimate - pre Gronwall} \]

A final absorption, and Gronwall’s inequality gives the stated stability estimate in $[0, t^*]$. It is left to show that in fact $t^*$ coincides with $T$ for sufficiently small $h \leq h_0$. We prove this by the following argument: By the proven stability bound and the assumed bounds (5.5) the error $e_h^n(t)$ in $[0, t^*]$ satisfies

$$|e_h^n(t)| \leq C h^2.$$
Then, by the inverse estimate, [BS08, Theorem 4.5.11], we have, for $t \in [0, t^*)$,
\[
\|e_u^h\|_{L^\infty(\Omega_h)} \leq c h^{-d/2} \|e_u^h\|_{L^2(\Omega_h)} \\
\leq c h^{-d/2} \|e_u^h\|_h \\
\leq c Ch^{-d/2} h^2 \leq \frac{1}{2} h^{1-d/4},
\]
for sufficiently small $h$. Therefore, we can extend the bounds (5.7) beyond $t^*$, which contradicts the maximality of $t^*$ unless $t^* = T$. Hence, we have the stability bound (5.6) for $t \in [0, T]$.

5.3 Generalisations

In this section we illustrate the versatility of the above stability analysis of Section 5.2.

First, let us consider the standard Cahn–Hilliard equation [CH58] in a smooth domain with homogeneous Neumann boundary conditions. Written as a second order system it reads: The functions $u, w : \Omega \times [0, T] \to \mathbb{R}$ satisfying the PDEs (2.2a)–(2.2b) in $\Omega$, which are endowed with the boundary conditions
\[
\partial_n u = \partial_n w = 0 \quad \text{on } \Gamma.
\]
This problem can be cast in the abstract setting of Section 2.2, by setting $V = \{v \in H^1(\Omega) \mid \int_\Omega u = 0\}$ and $H = L^2(\Omega)$, with the bilinear forms on these spaces:
\[
a(u,v) = \int_\Omega \nabla u \cdot \nabla v, \quad \text{and} \quad m(u,v) = \int_\Omega uv.
\]
Therefore, the weak formulation to the Cahn–Hilliard equation with boundary conditions (5.30) reads exactly as (2.7). Hence, with the analogous modifications, the bulk–surface finite element semi-discretisation can be written exactly as (3.7). The proof of Proposition 5.1 immediately holds.

A similar setting can be used for inhomogeneous Neumann, or Dirichlet, or mixed boundary conditions.

Let us now consider a generalised semi-linear Cahn–Hilliard equation with Cahn–Hilliard-type dynamic boundary conditions, written in the abstract setting of Section 2.2: Find $u \in C^1([0, T], H) \cap L^2([0, T], V)$ and $w \in L^2([0, T], V)$ such that, for time $0 < t \leq T$ and all $\varphi^u, \varphi^w \in V$,
\[
m(\dot{u}(t), \varphi^u) + a(u(t), \varphi^u) = m(f(u(t), w(t))), \varphi^u), \quad (5.31a) \\
m(w(t), \varphi^w) - a(u(t), \varphi^w) = m(W'(u(t)), \varphi^w), \quad (5.31b)
\]
which only differs from (2.7) in that the non-linear term also depends on $w$, i.e.
\[
m(f(u, w), \varphi) = \int_\Omega f_\Omega(u, w)\varphi + \int_\Gamma f_\Gamma(\gamma u, \gamma w)\gamma\varphi
\]
with \( f, f_T : \mathbb{R}^2 \to \mathbb{R} \). The corresponding modified system (with \( \vartheta_h \) as before) is analogous.

To this problem the proof of Proposition 5.1 holds only subject to the following modifications:

- Since the semi-linear term \( f \) now depends on \( w \) (and \( u \)) as well, the definition of \( t^* \) has to be extended by a condition on \( \| e^w_h(t) \|_{L^\infty(\Omega_h)} \) analogous to (5.7), using the exact same argument.
- The estimates of the semi-linear terms in the first error equation (corresponding to (5.31a)), are estimated analogously to (5.12) and (5.13), etc.
- In order to prove that \( t^* = T \), the argument (5.29) has to be repeated for \( e^w_h \) as well, using the exact same arguments.

The Cahn–Hilliard equation with Allen–Cahn-type dynamic boundary conditions [CPP10], i.e. the PDE system (2.2a)–(2.2b) in \( \Omega \) endowed with the dynamic boundary conditions

\[
\dot{u} = \Delta u - W'(u) - \partial_n u, \quad \partial_n w = 0, \quad \text{on } \Gamma, \tag{5.32}
\]

Unfortunately, does not fit into the abstract framework of this paper: compare the weak formulation (10)–(11) in [CPP10] with (2.5), or (2.7) above.

## 6 Consistency

The consistency analysis relies on error estimates of the nodal interpolations in the bulk and on the surface, error estimates for the Ritz map, geometric approximation errors in the bilinear forms, and a technical result for estimating norms on a boundary layer.

### 6.1 Geometric errors

Let us recall our assumptions on the bulk and the surface, and on their discrete counterparts: the bounded domain \( \Omega \subset \mathbb{R}^d \) (\( d = 2 \) or 3) has an (at least) \( C^2 \) boundary \( \Gamma \); the quasi-uniform triangulation \( \Omega_h \) (approximating \( \Omega \)) whose boundary \( \Gamma_h := \partial \Omega_h \) is an interpolation of \( \Gamma \).

#### 6.1.1 Interpolation and Ritz map error estimates

The piecewise linear finite element interpolation operator \( I_h v \in V_h \), with lift \( I_h v = (I_h L f)^T \in V_h^T \) satisfies the following bounds.

**Lemma 6.1** For \( v \in H^2(\Omega) \), such that \( \gamma v \in H^2(\Gamma) \). The piecewise linear finite element interpolation satisfies the following estimates:

(i) **Interpolation error in the bulk:** see [Ber89, ER13]:

\[
\| v - I_h v \|_{L^2(\Omega)} + h \| \nabla (v - I_h v) \|_{L^2(\Omega)} \leq C h^2 \| v \|_{H^2(\Omega)}.
\]
Lemma 6.2 For any \( v \in H^2(\Omega) \) with \( \gamma v \in H^2(\Gamma) \) the error of the Ritz map (3.4) satisfies the following bounds, for \( h \leq h_0 \),

\[
\| v - R_h v \|_{H^1(\Omega)} + \| \gamma (v - R_h v) \|_{H^1(\Gamma)} \leq C h \| v \|_{H^2(\Omega)} + \| \gamma v \|_{H^2(\Gamma)},
\]

\[
\| v - R_h v \|_{L^2(\Omega)} + \| \gamma (v - R_h v) \|_{L^2(\Gamma)} \leq C h^2 \| v \|_{H^2(\Omega)} + \| \gamma v \|_{H^2(\Gamma)},
\]

where the constant \( C \) is independent of \( h \) and \( v \).

6.1.2 Geometric approximation errors

The following technical result from \([ER13, Lemma 6.3]\) helps to estimate norms on a layer of triangles around the boundary.

Lemma 6.3 For all \( v \in H^1(\Omega) \) the following estimate holds:

\[
\| v \|_{L^2(B_h^2)} \leq Ch^2 \| v \|_{H^1(\Omega)},
\]

where \( B_h^2 \) collects the lifts of elements which have at least two nodes on the boundary.

The bilinear forms \( a_h \) and \( m_h \), and \( m \) and \( m_h \), from (2.6) and (3.3), satisfy the following geometric approximation estimate, proved in \([KL17, Lemma 3.9]\).

Lemma 6.4 The bilinear forms (2.6) and their discrete counterparts (3.3) satisfy the following estimates for \( h \leq h_0 \), for any \( v_h, w_h \in V_h \),

\[
|a(v_h^*, w_h^*) - a_h(v_h, w_h)| \leq C h \| \nabla v_h^* \|_{L^2(B_h^2)} \| \nabla w_h^* \|_{L^2(B_h^2)}
\]

\[
+ Ch^2 \left( \| \nabla v_h^* \|_{L^2(\Omega)} \| \nabla w_h^* \|_{L^2(\Omega)} + \| \nabla w_h^* \|_{L^2(\Gamma)} \| \nabla v_h^* \|_{L^2(\Gamma)} \right),
\]

\[
|m(v_h^*, w_h^*) - m_h(v_h, w_h)| \leq C h \| v_h^* \|_{L^2(B_h^2)} \| w_h^* \|_{L^2(B_h^2)}
\]

\[
+ Ch^2 \left( \| v_h^* \|_{L^2(\Omega)} \| w_h^* \|_{L^2(\Omega)} + \| \gamma v_h^* \|_{L^2(\Gamma)} \| \gamma w_h^* \|_{L^2(\Gamma)} \right).
\]

The combination of the two estimates of Lemma 6.4 yields a similar estimate between the bilinear forms \( a^* \) and \( a_h^* \).

As a consequence we also have the \( h \)-uniform equivalence of the norms \( \| \cdot \| \) and \( \| \cdot \|_h \) induced by the bilinear forms \( a^* \) and \( a_h^* \), respectively, of the norms \( \| \cdot \| \) and \( \| \cdot \|_h \) induced by the bilinear forms \( m \) and \( m_h \), respectively:

\[
\| v_h^* \| \sim \| v_h \|_h \quad \text{and} \quad |v_h^*| \sim |v_h|_h \quad \text{uniformly in} \ h.
\]
6.2 Defect bounds

In this section we prove bounds for the defects and for their time derivatives, i.e. we prove that condition (5.5) of Proposition 5.1 is satisfied.

**Proposition 6.1** Let \((u, w)\) be a solution of (2.2) that satisfies the regularity conditions (4.1). Then the defects \(d_h^u(\cdot, t)\) and \(d_h^w(\cdot, t)\) \(\in V_h\) from (5.1) and their time derivatives satisfy the bounds, for \(0 \leq t \leq T\),

\[
\|d_h^u(\cdot, t)\|_{*, h} \leq C h^2 \quad \text{and} \quad \|d_h^u(\cdot, t)\|_{*, h} \leq C h^2, \quad (6.3)
\]
\[
\|d_h^w(\cdot, t)\|_{*, h} \leq C h^2 \quad \text{and} \quad \|d_h^w(\cdot, t)\|_{*, h} \leq C h^2, \quad (6.4)
\]

where the constant \(C > 0\) depends on the final time \(T\), on the Sobolev norms of the solution, but it is independent from \(h\) and \(t\).

**Proof** The proof is in the standard spirit of consistency estimates comparing the exact solutions to their Ritz maps, and uses geometric approximation errors from above. For such proofs see, e.g., [DE13b] for linear evolving surface PDEs, and [KL17, Hip17, HK20] for problems with dynamic boundary conditions. Again, within the proof we omit the time dependencies. For more details to the present proof we refer to [Har19].

We will first prove the consistency bounds for the defect in \(u\) and its time derivative, and then, by using analogous techniques, we will show the same estimates for the defect in \(w\) and its time derivative.

**Bounds for \(d_h^u\) and \(d_h^w\):** To estimate the defects in \(u\), we start by subtracting (2.7a) from (5.1a) with \(\varphi_h^u \in V_h^e\) and \(\varphi_h \in V_h\), respectively, as test functions, then use the definition of the Ritz map (3.4) to obtain

\[
m_h(d_h^u, \varphi_h^u) = m_h(\tilde{R}_h \dot{u}, \varphi_h) - m(\dot{u}, \varphi_h^u) + a_h(\tilde{R}_h w, \varphi_h) - a(w, \varphi_h^u) = (m_h(\tilde{R}_h \dot{u}, \varphi_h) - m(R_h \dot{u}, \varphi_h^u)) + m(R_h \dot{u} - \dot{u}, \varphi_h^u) - (m(R_h w, \varphi_h) - m(R_h w, \varphi_h^u)) - m(R_h w - w, \varphi_h^u),
\]

where for the last equality we added and subtracted the appropriate intermediate terms. The terms in the last two lines are estimated separately: the first terms by the geometric approximation estimates of Lemma 6.4 together with Lemma 6.3, the second terms by a Cauchy–Schwarz inequality and then by the Ritz map error bounds from Lemma 6.2. Altogether we obtain

\[
m_h(d_h^u, \varphi_h) \leq ch^2 \|R_h \dot{u}\| \|\varphi_h^u\| + ch^2 (\|\dot{u}\|_{H^2(\Gamma)} + \|\gamma \dot{u}\|_{H^2(\Gamma)}) |\varphi_h^u|.
\]

By using the \(\| \cdot \|\) norm error estimates for the Ritz map within the first term here, we obtain:

\[
\|R_h \dot{u}\| \leq \|R_h \dot{u} - \dot{u}\| + \|\dot{u}\| \leq (1 + ch)(\|\dot{u}\|_{H^2(\Gamma)} + \|\gamma \dot{u}\|_{H^2(\Gamma)}).
\]
 Altogether, after recalling the definition of the discrete dual norm (5.4), we obtain that the defect in $u$ is bounded by

$$\|d_h^u\|_{*,h} \leq ch^2(\|\tilde{u}\|_{H^2(\Omega)} + \|\gamma u\|_{H^2(\Gamma)}).$$

The time derivative of the defect $d_h^u$ is bounded by the same techniques. We take the time derivative of the equation (6.5), use that $\varphi_h \in V_h^\ell$ and we again add terms to gain the structure as before:

$$\|\dot{d}_h^u\|_{*,h} \leq ch^2(\|\dot{u}\|_{H^2(\Omega)} + \|\gamma u\|_{H^2(\Gamma)}).$$

**Bounds for $d_h^w$ and $\dot{d}_h^w$**: To estimate the defects in $w$ we use the same approach as for $u$. We start by subtracting (2.7b) from (5.1b) with $\varphi_h^\ell \in V_h^\ell$ and $\varphi_h \in V_h$, respectively, as test functions. Using again the definition of the Ritz map (3.4) and we again add terms to gain the structure as above:

$$m_h(d_h^w, \varphi_h^\ell) = (m_h(\tilde{R}_h w, \varphi_h) - m(R_h w, \varphi_h^\ell)) + m(R_h w - w, \varphi_h^\ell)
+ (m_h(\tilde{R}_h u, \varphi_h) - m(R_h u, \varphi_h^\ell)) + m(R_h u - u, \varphi_h^\ell)
- m_h(W'(\tilde{R}_h u, \varphi_h) - m((W'(\tilde{R}_h u))^\ell, \varphi_h^\ell)) - m(W'(R_h u) - W'(u), \varphi_h^\ell),$$

where for the non-linear term we have used the fact that, for an arbitrary function $f$ and for any $v_h \in V_h$, there holds $f(v_h^\ell) = f(v_h \circ G_h^{-1}) = (f \circ v_h) \circ G_h^{-1} = (f(v_h))^\ell$, with $G_h$ defined in (3.2).

The terms in the first two lines on the right-hand side are estimated exactly as before for $d_h^u$, as

$$ch^2(\|u\|_{H^2(\Omega)} + \|\gamma u\|_{H^2(\Gamma)} + \|w\|_{H^2(\Omega)} + \|\gamma w\|_{H^2(\Gamma)}).$$

The remaining non-linear terms are bounded similarly as before. The first term by the geometric approximation estimates Lemma 6.4 together with Lemma 6.3 as

$$m_h(W'(R_h u), \varphi_h) - m((W'(\tilde{R}_h u))^\ell, \varphi_h^\ell) \leq ch^2\|W'(R_h u)\|\|\varphi_h^\ell\|.$$ 

The second term is estimated by a Cauchy–Schwarz inequality and then by the Ritz map error bounds from Lemma 6.2.

$$m(W'(R_h u) - W'(u), \varphi_h^\ell) \leq \|W'(R_h u) - W'(u)\|\|\varphi_h^\ell\|.$$ 

It is left to estimate the non-linear terms involving the Ritz map. We first establish a bound for $\|W'(R_h u)\|$, and start by decomposing this norm into its bulk and surface parts

$$\|W'(R_h u)\| \leq \|\nabla(W'_\Omega(R_h u))\|_{L^2(\Omega)} + \|W'_\Omega(R_h u)\|_{L^2(\Omega)}
+ \|\nabla_R(W'_\Gamma(\Gamma(R_h u)))\|_{L^2(\Omega)} + \|W'_\Gamma(\Gamma(R_h u))\|_{L^2(\Gamma)}.$$
The above terms on the right-hand side are estimated separately, but by analogous techniques. For the first term we have

$$\| \nabla (W'(R_h u)) \|_{L^2(\Omega)} \leq \| W''(R_h u) \|_{L^\infty(\Omega)} \| \nabla R_h u \|_{L^2(\Omega)}$$

$$\leq \| W''(R_h u) \|_{L^\infty(\Omega)} (\| \nabla R_h u - u \|_{L^2(\Omega)} + \| u \|_{L^2(\Omega)})$$

$$\leq \| W''(R_h u) \|_{L^\infty(\Omega)} (1 + ch^2) \| u \|_{H^2(\Omega)},$$

which is bounded by a constant, provided an $L^\infty(\Omega)$ norm bound on $R_h u$. Under the same condition, the other three terms are also bounded independently of $h$, by a similar argument.

To show a bound for $\| R_h u \|_{L^\infty(\Omega)}$, we use an inverse estimate [BS08, Theorem 4.5.11] (recall that $d = 2$ or 3) and the $L^\infty(\Omega)$-stability of the finite element interpolation, which yield

$$\| R_h u \|_{L^\infty(\Omega)} \leq \| R_h u - I_h u \|_{L^\infty(\Omega)} + \| I_h u \|_{L^\infty(\Omega)}$$

$$\leq ch^{-d/2} \| R_h u - I_h u \|_{L^2(\Omega)} + \| I_h u \|_{L^\infty(\Omega)}$$

$$\leq ch^{-d/2} \| R_h u - u \|_{L^2(\Omega)} + ch^{-d/2} \| u - I_h u \|_{L^2(\Omega)} + \| I_h u \|_{L^\infty(\Omega)}$$

$$\leq ch^{-d/2} \| u \|_{H^2(\Omega)} + ch^{-d/2} \| u \|_{H^2(\Omega)} + \| u \|_{L^\infty(\Omega)}.$$

Note that, there holds $\| \gamma(R_h u) \|_{L^\infty(\Gamma)} \leq \| R_h u \|_{L^\infty(\Omega)}$.

Using the $L^\infty$ norm bound on the Ritz map (6.7) and the local Lipschitz continuity of $W'$, via (5.4), we obtain that the defect in $w$ is bounded by

$$\| d_h^w \|_{L^\infty(\Omega)} \leq ch^2 (\| u \|_{L^\infty(\Omega)} + \| u \|_{H^2(\Omega)} + \| \gamma u \|_{H^2(\Omega)} + \| w \|_{H^2(\Omega)} + \| \gamma w \|_{H^2(\Omega)}).$$

The time derivative of the defect $d_h^w$ is bounded by the same techniques. We take the time derivative of the equation (6.6), use that $\varphi_h \in V_h$ is time independent and that $d/dt (R_h u) = R_h \dot{u}$, the $L^\infty$ bounds on the Ritz map of $\dot{u}$ (obtained analogously as for $\ddot{u}$), and then use the same estimates as above to obtain

$$\| d_h^{\dot{w}} \|_{L^\infty(\Omega)} \leq ch^2 (\| \dot{u} \|_{L^\infty(\Omega)} + \| \dot{u} \|_{H^2(\Omega)} + \| \gamma \dot{u} \|_{H^2(\Omega)} + \| \ddot{u} \|_{H^2(\Omega)} + \| \gamma \ddot{u} \|_{H^2(\Omega)}).$$

### 7 Proof of Theorem 4.1

Combining the results of the two previous sections we now prove the semidiscrete convergence theorem.

**Proof (Proof of Theorem 4.1)** The proof combines the results of the previous two sections on stability and consistency.

The error between the lifted numerical solutions $u_h^f(\cdot, t)$ and $w_h^f(\cdot, t)$ and the exact solutions $u(\cdot, t)$ and $w(\cdot, t)$ decomposes to (omitting the time $t$):

$$u_h^f - u = (u_h - \bar{R}_h u)^f + (R_h u - u) = (e_h^u)^f + (R_h u - u),$$

$$w_h^f - w = (w_h - \bar{R}_h w)^f + (R_h w - w) = (e_h^w)^f + (R_h w - w),$$
where the second equalities follow upon recalling the definitions of the errors from Section 5.1. A similar decomposition holds for the time derivatives as well.

The first terms are estimated by combining the stability, Proposition 5.1, with the bounds for the defects, Proposition 6.1, and the fact that \( e^u_h(0) = 0 \) and \( e^w_h(0) = 0 \), which altogether immediately implies the assumed bounds in (5.5). Altogether, using a norm equivalence (6.2), we obtain, for \( 0 \leq t \leq T \),

\[
\left( \left( |(e^u_h)^f(\cdot, t)|^2 + |(e^w_h)^f(\cdot, t)|^2 + \int_0^t |(e^u_h)^f(\cdot, s)|^2 \, ds \right) \right)^{1/2} \leq \left( \left( \left\| (e^u_h)^f(\cdot, t) \right\|_2^2 + \left\| (e^w_h)^f(\cdot, t) \right\|_2^2 + \int_0^t \left\| (e^u_h)^f(\cdot, s) \right\|_2^2 \, ds \right) \right)^{1/2} \leq Ch^2,
\]

where the first inequality holds by the natural estimate \( |\cdot| \leq \| \cdot \| \).

The second terms are estimated directly by the Ritz map error estimates of Lemma 6.2. By translating back from the abstract functional analytic setting, in the stronger \( V \) (i.e. \( H^1 \)) norm and in the weaker \( H \) (i.e. \( L^2 \)) norm, we respectively obtain

\[
\| R_h u - u \|_{H^1(\Omega)} + \| \gamma(R_h u - u) \|_{H^1(\Gamma)} \leq ch, \\
| R_h u - u |_{L^2(\Omega)} + \| \gamma(R_h u - u) \|_{L^2(\Gamma)} \leq ch^2.
\]

For \( w \) we have the same bounds.

Combining these estimates yields the stated convergence result.

8 Linearly implicit backward difference time discretisation

The semi-discrete problem (3.7) is first rewritten in the matrix–vector form, with \( u(t) \) and \( w(t) \) collecting the nodal values of the finite element functions \( u_h(\cdot, t) \) and \( w_h(\cdot, t) \) in \( V_h \), respectively,

\[
\begin{align*}
M \dot{u}(t) - A w(t) &= 0, \quad (8.1a) \\
M w(t) + A u(t) &= W'(\tilde{u}(t)). \quad (8.1b)
\end{align*}
\]

Here \( A \) and \( M \) denote the stiffness and mass matrix given through (3.3), while the vector \( W'(\tilde{u}(t)) \) is given via the right-hand side of (3.7b).

As a time discretisation of the system (8.1), we consider the linearly implicit \( q \)-step backward differentiation formulae (BDF). For a step size \( \tau > 0 \), and with \( t_n = n\tau \leq T \), we determine the approximations to the variables \( u^n \) to \( u(t_n) \) and \( w^n \) to \( w(t_n) \) by the fully discrete system of linear equations, for \( n \geq 0 \),

\[
\begin{align*}
M \dot{u}^n - A w^n &= 0, \quad (8.2a) \\
M w^n + A u^n &= W'(\tilde{u}^n), \quad (8.2b)
\end{align*}
\]
where the discretised time derivative is determined by

\[ \dot{u}^n = \frac{1}{\tau} \sum_{j=0}^{q} \delta_j u^{n-j}, \quad n \geq q, \quad (8.3) \]

eq: backward differences def

while the non-linear term uses an extrapolated value, and is given by:

\[ W'(\tilde{u}^n) := W'(\frac{s-1}{\sum_{j=0}^{q-1} \gamma_j u^{n-1-j}}), \quad n \geq s. \]

The starting values \( u^i \) \( (i = 0, \ldots, q-1) \) are assumed to be given. They can be precomputed using either a lower order method with smaller step sizes, or an implicit Runge–Kutta method. The initial values \( w^i \) \( (i = 0, \ldots, q-1) \) are computed from the already obtained \( u^i \).

The method is determined by its coefficients, given by \( \delta(\zeta) = \sum_{j=0}^{q} \delta_j \zeta^j = \sum_{j=0}^{q} \frac{1}{j!} (1 - \zeta)^j \) and \( \gamma(\zeta) = \sum_{j=0}^{q-1} \gamma_j \zeta^j = (1 - (1 - \zeta)^q)/\zeta \). The classical BDF method is known to be zero-stable for \( q \leq 6 \) and to have order \( q \); see [HW96, Chapter V]. This order is retained by the linearly implicit variant using the above coefficients \( \gamma_j \); cf. [AL15,ALL17].

The anti-symmetric structure of the system (3.7), is preserved by the above time discretisation, and can be observed in (8.2). Using the \( G \)-stability theory of [Dah78] and the multiplier technique of [NO81], the energy estimates used in the proof of Proposition 5.1 can be transferred to linearly implicit BDF full discretisations (up to order 5). Therefore, we strongly expect that Proposition 5.1 translates to the fully discrete case, and so does the convergence result Theorem 4.1, with classical convergence order in time. The successful application of these techniques to the analogous (linearly implicit) BDF discretisation applied to evolving surface PDEs, e.g. [LMV13,KPG18,KP16] showing optimal-order error bounds for various problems on evolving surfaces, strengthens this statement. Linearly implicit BDF methods were also analysed for various geometric surface flows: for \( H^1 \)-regularised surface flows [KL18], and for mean curvature flow [KLL19], both proving optimal-order error bounds for full discretisations, using the above mentioned techniques and energy estimates testing with the time derivative of the error.

9 Numerical experiments

In this section we present some numerical experiments to illustrate our theoretical results. We consider the Cahn–Hilliard equation with Cahn–Hilliard-type dynamic boundary conditions (2.2) in a disk \( \Omega \) with its boundary \( \Gamma \). We present two numerical experiments: a convergence test, and a numerical test reporting on the evolution of the \( u \) component when the equation is started from random initial data. In both cases for the spatial discretization we use linear bulk–surface finite elements, while for time discretization we use the BDF method of third order \( k = 3 \).
9.1 Convergence test

For the convergence experiment, additional inhomogeneities are added to each equation in (2.2), such that the exact solution is known to be \( u(x,t) = w(x,t) = e^{-t}x_1x_2 \). The experiment is performed with the double-well potential both in the bulk and on the surface, i.e. with the non-linearities \( W_\Omega(u) = W_\Gamma(u) = \frac{1}{4}(u^2 - 1)^2 \).

The domain \( \Omega \) is the unit disk, and the final time is \( T = 1 \). For this experiment we used a sequence of time step sizes \( \tau = (0.05, 0.025, 0.0125, 0.005, 0.0025, 0.00125) \) (having an approximate ratio of 2), and a sequence of initial meshes with degrees of freedom \( 2^k \cdot 10 \) for \( k = 1, \ldots, 8 \).

In Figure 9.1 we report on the bulk–surface \( L^\infty(L^2) \) norm errors (left) and \( L^\infty(H^1) \) norm errors (right) between the (linear bulk–surface FEM / BDF3) numerical approximation and the exact solution for both variables, i.e. both the bulk and the surface error for both variables \( u \) and \( w \). The logarithmic plots show the errors against the mesh width \( h \), the lines marked with different symbols correspond to different time step sizes.

In Figure 9.1 we can observe the spatial discretisation error dominates, and matches the order of convergence of our theoretical results (note the reference lines). Note that, however, in the \( H^1 \) norm we observe a better convergence rate than the linear convergence order proven in Theorem 4.1.
9.2 The Cahn–Hilliard equation with dynamic boundary conditions and a double-well potential

An illustration of the phenomena of phase separation described by the Cahn–Hilliard equation with dynamic Cahn–Hilliard boundary conditions is shown in Figure 9.2. We consider a non-linear the Cahn–Hilliard equation with potentials \( W_\Omega(u) = W_\Gamma(u) = 10(u^2 - 1)^2 \). We again used the linear finite element method and the linearly implicit BDF method of order 3. For our plots shown in the figure we generated random initial data \( u_0 \in \{-1, 1\} \), using the disk with radius 10 as a domain, a mesh with 640 nodes, and a time step size \( \tau = 0.00125 \). Each subplot in Figure 9.2 shows the solution \( u \) of the problem, i.e. the phase separation, at the displayed times. The shown colorbar is valid for every subplot.

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Fig. 9.2 Evolution of the $u_h$ component of the (linear bulk–surface FEM / BDF3) approximation to the non-linear Cahn–Hilliard equation with Cahn–Hilliard-type dynamic boundary conditions $W_{Ω}(u) = W_{Γ}(u) = 10(u^2 - 1)^2$. 

**figure:solution**
Error estimates for the Cahn–Hilliard equation with dynamic b.c. 29

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