Stückelberg interferometry using spin-orbit-coupled cold atoms in an optical lattice

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Time evolution of spin-orbit-coupled cold atoms in an optical lattice is studied, with a two-band energy spectrum having two avoided crossings. A force is applied such that the atoms experience two consecutive Landau-Zener tunnelings while transversing the avoided crossings. Stückelberg interference arises from the phase accumulated during the adiabatic evolution between the two tunnelings. This phase is gauge field-dependent and thus provides new opportunities to measure the synthetic gauge field, which is verified via calculation of spin transition probabilities after a double passage process. Time-dependent and time-averaged spin probabilities are derived, in which resonances are found. We also demonstrate chiral Bloch oscillation and rich spin-momentum locking behavior in this system.

I. INTRODUCTION

Atom interferometry has been proven to be a powerful tool for precision metrology [1]. Besides that atom interferometry can be used to test fundamental physical theory such as general relativity [2–6]. Recent years have also witnessed growing interest in synthetic dimensions [43] and the tunneling problem realized using a single optical cavity with two independent synthetic SO coupling to create a pair of avoided crossings in the energy dispersion, the resulting interference fringes become frequency dependent. Besides scenarios in free space, synthetic SO coupling have also been implemented in ultracold atomic gas trapped in optical lattice via either optical clock transition [34–36], double-well optical lattice [37, 38], Raman-dressing [39, 40] and using a two-dimensional manifold of momentum states [41]. By considering spin as a synthetic dimension, these experiments can resemble the model of a two-leg ladder subject to a magnetic flux [42]. Two-leg ladder has been realized using a single optical cavity with two independent synthetic dimensions [43] and the tunneling problem of a ladder system has also been addressed with electronic setup [44, 45]. Many interesting phenomena such as chiral Bloch oscillation, bandgap closing, edge state and unconventional phases have been predicted in such a system by incorporating diagonal couplings, additional legs, strong and long-range interactions [46, 47].

Motivated by these developments, in this work we consider a Stückelberg interferometry using SO-coupled cold atoms in an optical lattice. We found the conditions to realize two avoided crossings in the energy dispersion of this two-band system, which can be achieved in ex-

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periment and thus the system can be used to perform St"uckelberg interference. The St"uckelberg interferometry represents an atom interferometry with synthetic gauge fields and provides new opportunity to study novel SO-coupled band structures. We demonstrate that the interference pattern reveals a phase which depends on the synthetic magnetic flux, from which the information on the synthetic gauge field can be derived. Although the synthetic magnetic flux can be directly probed via measuring the atomic SO momentum transfer, here St"uckelberg interference provides an alternative way to demonstrate the effect of synthetic gauge field without inquiring the atomic momentum information. The time evolution is also explicitly studied, where resonances and chiral Bloch oscillation are predicted.

The article is organized as follows: In Sec. II we present our model and the effective Hamiltonian is derived. The principle of St"uckelberg interference is illustrated in Sec. III. We use the adiabatic-impulse model to analyze a double passage process in which two avoided crossings are transversed. A more general discussion on the dynamics is also performed with Floquet-Bloch theory. Sec. IV is devoted to the discussion of chiral Bloch oscillation in the case that no LZ transitions take place and finally we conclude in Sec. V.

II. MODEL

In this work we consider the following model typically representing a two-leg bosonic ladder pierced by a magnetic flux, which can be described by the Hamiltonian ($\hbar = 1$)

$$\hat{H} = -\frac{\Delta}{2} \sum_l \left( \hat{c}_l^\dagger e^{i\phi_2} \hat{c}_{l+1} + \text{H.c.} \right) + \sum_l \hat{c}_l^\dagger \left( \frac{\Omega}{2} \hat{\sigma}_x - \frac{\delta}{2} \hat{\sigma}_z \right) \hat{c}_l. \quad (1)$$

Hamiltonian (1) can be implemented in the system of Raman-dressed $^{87}$Rb cold atoms trapped in a one-dimensional optical lattice [12, 39, 71] along the $z$-direction under lowest energy band truncation and tight-binding approximation, as shown in Fig. 1(a). Here the atomic hyperfine states $|\downarrow\rangle$ and $|\uparrow\rangle$ are coupled via Raman lasers, thus generating effective SO interaction. By considering the atomic pseudospin as a synthetic dimension, the system can be exactly mapped to a two-leg bosonic ladder, as shown in Fig. 1(b). The spin-momentum locking in Raman transition is equivalent to spin-dependent (leg-dependent) tunneling between neighboring sites, thus a particle hopping around an elementary plaquette picks up an Aharonov-Bohm phase $2\phi$, which is equivalent to the presence of an effective magnetic flux $2\phi$ per plaquette piercing the system. $\phi = \pi k_R / k_l$ with $k_R(l)$ the wavevector of Raman beams projected onto the $z$-axis and the laser forming the lattice, respectively. Here two-component annihilation operator $\hat{c}_l = (\hat{c}_{ql}, \hat{c}_{q\bar{l}})^T$ and $\Delta$ is half-bandwidth on the scale of kHz in typical experiments with $^{87}$Rb atoms [63]. The inter-leg coupling is characterized by the hopping amplitude $\Omega/2$ and detuning $\delta$.

![FIG. 1: Schematic diagram showing the system under consideration. (a) Setup: $^{87}$Rb cold atoms are confined in a one-dimensional optical lattice along the $z$-direction, inside which the effective SO interaction is induced via coupling the $|\downarrow\rangle$ and $|\uparrow\rangle$ hyperfine states with Raman lasers. A bias magnetic field $B$ causing quadratic Zeeman splitting is applied along the $z$-direction. (b) The hyperfine states can be treated as an effective synthetic dimension made by two sites connected with a coherent tunneling, resulting in a two-leg ladder pierced by a synthetic magnetic flux $2\phi$ per plaquette.](image)

Compared with the previously studied two-leg ladder [12], here the inter-leg detuning $\delta$ is additionally taken into account. We note that the two-photon detuning $\delta$ is usually available via bias magnetic field in typical experiments with Raman-dressed BECs [11, 37, 39]. Making the Fourier transform $\hat{c}_q = \sqrt{2\pi} \sum_l \hat{c}_l e^{-iqld} l$ with $\hat{c}_q = (\hat{c}_{ql}, \hat{c}_{q\bar{l}})^T$ and $d = \pi / k_l$ the lattice constant, which is equivalent to the transform of the system from the Wannier basis to the Bloch basis, Hamiltonian (1) can be rewritten in the quasi-momentum basis as $\hat{H} = \sum_q \hat{c}_q \hat{H}_q \hat{c}_q$ with

$$\hat{H}_q = -\Delta \cos \phi \cos (qd) \hat{1} + \frac{\Omega}{2} \hat{\sigma}_x + \left[ -\frac{\delta}{2} + \Delta \sin \phi \sin (qd) \right] \hat{\sigma}_z. \quad (2)$$

Hamiltonian $\hat{H}_q$ indicates a two-band structure with $\varepsilon_{\pm} (q) = -\Delta \cos \phi \cos (qd) \pm \sqrt{[-\delta/2 + \Delta \sin \phi \sin (qd)]^2 + (\Omega/2)^2}$. When $|\delta/2\Delta \sin \phi| < 1$, there are two avoided crossings (at which spacing between $\varepsilon_{\pm}$ takes minimal value $\Omega$) located in the first Brillouin zone, thus making the system ideal for implementing Landau-Zener-St"uckelberg interferometry.
III. STÜCKELBERG INTERFEROMETRY

The principle of Stückelberg interferometry is illustrated in Fig. 2, where we assume $\delta/2\Delta \sin \phi > 0$ without loss of generality. The two avoided crossings then locate at $A$ with $q_A d = \arcsin(\delta/2\Delta \sin \phi)$ and $B$ with $q_B d = \pi - \arcsin(\delta/2\Delta \sin \phi)$. Suppose that the system is initially prepared in the state $|q_0, \uparrow\rangle$ with $q_0 d \in [-\pi, \arcsin(\delta/2\Delta \sin \phi)]$ (for example $q_0$ could be the quasimomentum of the system ground state), which is marked as point $O$ in Fig. 2. Furthermore, a constant external force $F$ is exerted on the atoms via tilting the optical lattice, which drives Bloch oscillation of the atoms. Bloch oscillation depicts the traveling of atoms along the energy bands. Upon reaching the avoided crossing $A$, the atoms will coherently split into two components via LZ transition. The two components then separately travel along $\varepsilon_\pm(q)$ and thus acquire a different phase. Finally the two components recombine and interfere with each other after another LZ transition at the avoided crossing $B$, from which the information on the phase difference $\phi_S$ accumulated during traversing $\varepsilon_\pm(q)$ between $A$ and $B$ can be derived.

In the following we first study this double passage process (the avoided crossing is passed twice) using adiabatic-impulse model in Sec. III A, where a simple relation between spin population and Stückelberg phase is found. This model can help us gain physical insight into the Stückelberg interferometry. Then the spin dynamics is studied in Sec. III B from which one can calculate the time-averaged spin population. This is relevant to experiment and the corresponding interference patterns will be identified.

A. Double passage: Adiabatic-impulse model

Under the assumption that $F$ is weak enough not to induce interband transitions, the adiabatic approximation can be applied, under which the atoms move adiabatically along the energy band with the quasimomentum $q(t) = q_0 + F t$, except in the vicinity of the avoided crossings. The non-adiabatic evolution of atoms while traversing the avoided crossing region is considered to take place instantaneously, which validates at $\Omega^2 + (2\Delta \sin \phi)^2 >> (F d)^2$ [65]. The double passage process from $t_i = t_A^i$ to $t_f = t_B^f$ with $t_{A(B)} = (q_{A(B)} - q_0)/F$ can be described by a transfer matrix $T_D$ in the diabatic basis (bare spin basis) as

$$c(t_f) = T_D c(t_i),$$

where $c(t) = (\hat{c}_q) = (c_{\uparrow}, c_{\downarrow})^T$ represents atomic population amplitude in the diabatic basis (bare spin basis) and governed by the equations of motion $i\hbar \frac{dc(t)}{dt} = \hat{H}_g c(t)$. In the adiabatic-impulse approximation [66], $T_D$ can be devided into 3 parts:

i. The LZ transition at the avoided crossing $A$. To symmetrize the two diabatic energy levels, we treat $c(t) \exp \{i\Delta \cos \phi \int_0^t dt' \cos (q(t')d)\}$ as the wavevector, submit it into the equations of motion and can have

$$i\frac{dc_{\uparrow}(t)}{dt} = \frac{\Omega}{2} c_{\uparrow}(t) \pm \left[-\frac{\delta}{2} + \Delta \sin \phi \sin (qd)\right] c_{\downarrow}(t).$$

In the vicinity of $A$ with $|F d(t - t_A)| << 1$, Eq. (4) can be linearized as

$$i\frac{dc_{\uparrow}(t)}{dt'} = \frac{\Omega}{2} c_{\uparrow}(t') \pm \frac{v}{2} c_{\downarrow}(t')$$

with $v = 2\Delta F d \sin \phi \cos (q d A) = 2\Delta F d \sin \phi \sqrt{1 - (\delta/2\Delta \sin \phi)^2}$ and $t' = t - t_A$. Eq. [5] defines the standard LZ problem for which the exact solution can be expressed in terms of parabolic cylinder functions $\text{F}_\nu$ [19]. Then the nonadiabatic transition is described by $[c_{\uparrow} (+0), c_{\downarrow} (+0)]^T = T_A [c_{\uparrow} (-0), c_{\downarrow} (-0)]^T$ with the time-independent matrix

$$T_A = \left( \begin{array}{cc} \sqrt{P_{LL}} e^{i\varphi_{st}} & \sqrt{1 - P_{LL}} e^{-i\varphi_{st}} \\ \sqrt{1 - P_{LL}} e^{i\varphi_{st}} & \sqrt{P_{LL}} \\ \end{array} \right),$$

where the LZ transition probability $P_{LL} = \exp (-2\pi \xi)$ and the Stokes phase $\varphi_{st} = \pi/4 + \xi (\ln \xi - 1) + \arg \Gamma(1 - i\xi)$, with $\xi = \Omega^2/4v$ and the gamma function $\Gamma$.

ii. The adiabatic evolution in the region between the two avoided crossings $A$ and $B$. One can notice that in this region far from the avoided crossings the diabatic energy for spin-$\uparrow$ components coincides with the energy dispersion $\varepsilon_\pm(q)$ respectively, i.e., the excited eigenstate of $\hat{H}_g$ coincide with spin-$\uparrow$ while the ground eigenstate coincide with spin-$\downarrow$, as shown in Fig. 2. This enables us to write the evolution matrix as

$$U = \left( \begin{array}{cc} e^{-i\varphi_{st}} & 0 \\ 0 & e^{i\varphi_{st}} \\ \end{array} \right).$$
with \( \phi_s = \int_{q_0}^{q_{\beta}} dq [\varepsilon_+ (q) - \varepsilon_- (q)] / F. \)

iii. The LZ transition at the avoided crossing \( B. \) This process is identical to i except that \( v \rightarrow -v \) in Eq. (5). Then the LZ transition matrix \( T_B = T_A^T. \)

Combine the process i-iii, the transfer matrix \( T_D \) has the form

\[
T_D = T_B T_A = \left( \begin{array}{cc} \alpha & \beta \\ \beta^{*} & \alpha^{*} \end{array} \right),
\]

\[
\alpha = P_L Z e^{-i \frac{\phi_S}{2} + (1 - P_L Z) e^{i \frac{\phi_s}{2}}},
\]

\[
\beta = -2i \sqrt{P_L Z (1 - P_L Z)} \sin \left( \frac{\phi_S}{2} + \varphi_{st} \right). \quad (8)
\]

Eqs. (3) and (8) indicate that the transition probability from spin-up to down is \( |\beta|^2, \) which oscillates with the phase \( \phi_S / 2 + \varphi_{st}. \) Since \( \phi_S \propto 1 / F, \) then under adiabatic approximation with small force \( F \) the St"uckelberg phase \( \phi_S \) is dominant and the contribution from \( \varphi_{st} \) can be neglected [44], i.e., one can neglect the quantum phase and focus on the semiclassical lattice effect.

![FIG. 3: (a) Contour plot of the transition probability \(|\beta|^2\) in the \((\phi, \Omega)\) parameter plane with \( \delta = 0.23 \Delta \) and \( Fd = 0.05 \Delta. \) The dashed lines mark \( \Omega = 0.2 \Delta \) and 0.9 \( \Delta \) respectively with the corresponding oscillations shown in (b) and (c).](image)

We calculate the transition probability \( |\beta|^2 \) as a function of the synthetic magnetic flux \( \phi \) and the Raman coupling strength \( \Omega \) with the results shown in Fig. 3. This figure is symmetric with respect to \( \phi = \pi/2 \); thus \( |\beta|^2 \) is a function of \( \sin \phi. \) On both sides of \( \phi = \pi/2 \) it displays approximately periodic oscillation which can be recognized as St"uckelberg oscillation. The appearance of St"uckelberg oscillation versus \( \phi \) reflects the nearly monotonic dependence of the St"uckelberg phase \( \phi_S \) on \( \phi \) since the value of \( P_L Z \) is insensitive to the variation of \( \phi. \) The interval between two neighboring maximal/minimal transition probabilities is then given by \( \delta \phi_S \approx 2 \pi. \) The calculation is performed with \( Fd = 0.05 \Delta \) and the oscillation will become stronger for a smaller force \( F. \) For the parameters considered here, at \( \Omega = 0.2 \Delta \) we have \( P_L Z \approx 0.5 \) and hence the transition probability can reach the maxima 1, as shown in Fig. 3(b). Since \( \Omega \) determines the energy spacing at the avoided crossings and the LZ transition probability \( P_L Z \) is exponentially dependent on it, we can then generally expect that with increasing deviation from \( \Omega = 0.2 \Delta \) the transition probability \( |\beta|^2 \) will gradually decrease. This is the case except for the region around \( \phi = \pi/2. \) As \( \phi_S \) is a complex function of \( \Omega, \) another local maximum of the transition probability appears around \( \Omega = 0.39 \Delta, \) as shown in Fig. 3(c).

By measuring St"uckelberg interference fringes one can map out the novel band structure with SO coupling [29 32]. As compared with the experiment [33], here the Raman coupling is not periodically modulated. However in the scenario with optical lattice, it can take the effect as to engineer the dispersion and realize two avoided crossings, thus making the St"uckelberg interference feasible. The interference fringes in [33] rely on the frequency and amplitude of periodical modulation applied on the Raman beams, showing the effect of Floquet engineering on SO-coupled bands. Here the physics underlying St"uckelberg interference is the presence of synthetic magnetic flux. In the meanwhile considering the fact that the centre of interference (or zeroth order fringe corresponding to \( \phi = \pi/2 \)) can be easily identified due to the symmetric nature of the interference pattern shown in Fig. 3, it enables one to recognize the order of interference fringe which can be used to indicate the synthetic magnetic flux \( \phi. \) In this sense the St"uckelberg interferometer demonstrated here also provides a new opportunity to measure the synthetic gauge field.

**B. Dynamics: Floquet-Bloch theory**

The dynamics considered here is mathematically similar to Bloch oscillation in a two-band system with interband coupling, however many previous works have dealt with this kind of problem [67 68] without SO coupling. Here we perform the analysis in terms of the Floquet-Bloch operator [32 99]. We can start from Hamiltonian [1] with an additional term \( Fd \sum l \hat{c}_l ^\dagger \hat{c}_l \) describing that the lattice is tilted with on-site energies \( -i Fd. \) This term can be removed via making a unitary transform with the operator \( \exp \left( i Fd t \sum l \hat{c}_l ^\dagger \hat{c}_l \right) \). Then following the same process as that has been performed in Sec. [11] we obtain the resulting periodic Hamiltonian in the Bloch basis as

\[
\hat{H}_q'(t) = -\Delta \cos \phi \cos [(q + Ft) \hat{d}] \hat{1} + \Omega \hat{\sigma}_x
\]

\[
+ \left\{ -\frac{\delta}{2} + \Delta \sin \phi \sin [(q + Ft) \hat{d}] \right\} \hat{\sigma}_z. \quad (9)
\]

Introducing

\[
c_{\uparrow} (q, t) = \hat{c}_{\uparrow} (q, t) \exp \left\{ \pm i \delta t / 2 + i \Delta \int_0^t dt' \cos [(q + Ft') \hat{d} \rho \phi] \right\}
\]

to remove the diagonal terms in the equations of motion, we have

\[
\frac{\partial}{\partial t} \left( \hat{c}_{\uparrow} (q, t) \right) = \frac{\Omega}{2} \cos (\phi_D) \hat{\sigma}_x + \sin (\phi_D) \hat{\sigma}_y \left( \hat{c}_{\downarrow} (q, t) \right), \quad (10)
\]

where \( \phi_D (q, t) = \phi (q, t) + \delta t \).
interleg coupling $\Omega$ (which is indeed the case we are studying), one can have
\[
\left(\hat{c}_\uparrow (q, t), \hat{c}_\downarrow (q, t)\right) = \exp \left\{ \frac{-i}{2} \int_0^t dt' \{\cos[\phi_D (q, t')]\hat{\sigma}_x + \sin[\phi_D (q, t')]\hat{\sigma}_y\} \right\} \left(\hat{c}_\uparrow (q, 0), \hat{c}_\downarrow (q, 0)\right),
\]
which can be recognized as Magnus expansion to the first order [62]. For the atoms initially prepared in spin-$\uparrow$ leg, the spin transition probability is given by
\[
P_\uparrow (t) = \sin^2 \left[ \Omega \sum_n J_n (z) \frac{\delta_n^{\uparrow \downarrow} t + \sin (\delta_n t/2)}{\delta_n} \right],
\]
At some specific values of $n = m$ with $\delta_m \sim 0$ an interleg coupling is on resonance, then Eq. (13) becomes
\[
P_\uparrow (t) = \sin^2 \left[ \frac{\Omega}{2} \sum_n J_n (z) \frac{\sin (\delta_n t/2)}{\delta_n} \right],
\]
which typically represents a large period (determined by $2\pi/\Omega J_m (z)$) oscillation with small-amplitude high-frequency oscillations superimposed on it. Eq. (14) indicates that in long run the first term will dominate over other interleg couplings, thus the averaged probability $\bar{P}_\uparrow$ = 1/2. In the case that no resonance takes place and in the meanwhile if $z \leq 1$, the $n = 0$ term will dominate and then one can have
\[
P_\uparrow (t) \approx \sin^2 \left[ \frac{\Omega J_0 (z)}{\delta} \sin (\delta t/2) \right] = \left\{2 \sum_{n=1}^{\infty} J_{2n-1} (V_0) \sin \left[ \left( n - \frac{1}{2} \right) \delta t \right] \right\}^2,
\]
which can be approximately reduced to $V_0^2 \sin^2 (\delta t/2)$ if $|V_0| = |\Omega J_0 (z)/\delta| << 1$. This will lead to an averaged probability of $\bar{P}_\uparrow = V_0^2/2$. Combining the discussions above, one can understand that the averaged transition probability $\bar{P}_\uparrow$ will vanish at large values of $\delta$ unless resonances take place at $\delta = nFd$.

![Contour plot of the averaged probability $\bar{P}_\uparrow$ in the $(\delta, \phi)$ parameter plane with $\Omega = 0.2F_d$ and (a) $\Delta = 0.5F_d$; (b) $\Delta = 2F_d$. The numerical simulations are performed with Eq. (13).](image)

We plot the averaged probability $\bar{P}_\uparrow$ as a function of $\delta$ and $\phi$ in Fig. 4 based on Eq. (13). Fig. 4(a) is for $\phi < 1$ while Fig. 4(b) for $\phi > 1$. In both cases the resonances around $\delta = nFd$ can be clearly observed, indicating the system can be used for force detection. The precision of force detection is related to the uncertainty of two-photon detuning $\delta$, which can be restricted, for example, via performing Pound-Drever-Hall laser frequency stabilization on the Raman beams. Fig. 4(a) shows a weak dependence of $\bar{P}_\uparrow$ on $\delta$, however at $\phi > 1$ one can still expect differences while varying $\phi$, as displayed in Fig. 4(b). In this figure one can even observe that on the $\delta = 0$ resonance $\bar{P}_\uparrow$ vanishes at some specific values of $\phi$, which correspond to the points with $J_0 (z) = 0$.

**IV. CHIRAL BLOCH OSCILLATION**

In the case of weak force $F$ and large interleg coupling $\Omega$, i.e., no interband transitions, the traditional Bloch oscillation takes place in the present system with a period $T_B = 2\pi/F_d$ and an amplitude proportional to the bandwidth [20] [71]. For the Hamiltonian (3), at zero detuning $\delta = 0$ the eigenstate of lowest (highest) energy band has negative (positive) chirality. The chirality can be defined as $C = q \left( \langle \hat{c}_\uparrow \hat{c}_\downarrow \rangle - \langle \hat{c}_\downarrow \hat{c}_\uparrow \rangle \right) = q \langle \hat{\sigma}_z \rangle$. Then in the approximation under which the atoms move adiabatically along the energy band, the Bloch oscillation will exhibit chiral characteristics [49]. For an atomic wavepacket moving along the lowest band with negative chirality, the positive (negative) momentum will tend to associate with spin-$\downarrow$($\uparrow$) atoms, signaling spin-momentum locking. In addition to that, at $\delta = 0$ and small interleg coupling $\Omega$, the lowest energy band can have two energy degenerate minima, characteristic of spin-orbit coupled systems. Increasing $\Omega$ beyond a critical value, the two minima will merge into one minimum, signaling a quantum phase transition from the vortex phase to the Meissner phase, which was experimentally observed in [35]. This band curvature change upon the phase transition can also be captured via Bloch oscillation [49].

At any finite $\delta$, the energy band is asymmetric with respect to inversion in Brillouin zone ($q \rightarrow -q$), which breaks time reversal symmetry. As a result of time-re
universal symmetry breaking, Kramers degeneracy does not hold here. However the lowest energy band can still exhibit two nondegenerate local minima, typical of SO coupled systems as those have been demonstrated in [39]. In Fig. 5(a), where $|\langle \sigma_z \rangle|$ of the ground state is plotted as a function of $\Omega$, the first-order derivative discontinuity disappears at any finite $\delta$, indicating that there exists no quantum phase transition. In the meanwhile, chirality is not always negative for all the eigenstates in the lowest energy band, as shown in Fig. 5(b). For the parameters considered here, at finite detuning one can have negative chirality in the quasi-momentum region $[-\pi, 0]$ and $|\langle \sigma_z \rangle|$ are the same as those have been defined in Sec. IV while in the rest region of the 1st Brillouin zone the chirality becomes positive for ground eigenstates. This chirality change will be reflected in the Bloch oscillation dynamics.

We simulate the Bloch oscillation dynamics using the method of eigenstate expansion developed in [71]. Initially the atoms are assumed to be prepared in a state $|\psi\rangle = \sum_{l,\sigma} \psi_{l,\sigma} \tau_{l,\sigma} |0\rangle$ with

$$\psi_l (t = 0) = \frac{1}{(2\pi \sqrt{w})^{1/2}} e^{-(l-l_0)^2/2w^2+iq_{ld}(1)}, \quad (16)$$

representing a spin-balanced Gaussian wave packet with width $w$, center $l_0$ and initial quasi-momentum $q_0$. The results are shown in Fig. 6. The left column is for $\delta = 0$. The Bloch oscillation in the top row clearly reflects the two degenerate minima nature of the lowest band in the regime of the vortex phase. In the meanwhile the middle and bottom rows also show clear evidence of chiral Bloch oscillation and spin-momentum locking. The case of nonzero $\delta$ is shown in the right column. Despite that the initial state (16) is not exactly the eigenstate, the atoms can still adiabatically follow the lowest band and exhibit its curvature with two local minima. This process is associated with chirality change, as shown in the middle and bottom rows. Spin-momentum locking still exists, however it becomes nonmonotonic: The spin-$\uparrow (\downarrow)$ atoms don’t simply associate with negative (positive) momentum, but now they are associated with certain momentum range, which can be tuned via varying the detuning $\delta$.

V. SUMMARY AND OUTLOOK

We note that physically the Raman lasers inducing SO interaction also inevitably couple atoms to high-lying bands, which will affect the single-particle physics [72, 73] such as dynamical instability [39]. This heating problem can be circumvented with an optical clock transition [74] and deformation of interference patterns [75], which will be left for further investigation. The St"uckelberg interferometer can still be implemented at a small interaction energy per site ($<<k\text{Hz}$), which can be tuned by means of Feshbach resonance [76].

Besides the Stokes phase accumulated at the LZ transitions and the dynamical phase accumulated during adiabatical evolution between the LZ transitions, St"uckelberg interference also depends on a noncyclic geometric phase. This noncyclic geometric phase is nonvanishing in special energy spectra configuration such as those with Dirac cones [77]. The geometric phase can also be made gauge-dependent, for example as proposed in a periodically driven two-level system [78, 79]. A recent experiment have also tested theory for the noncyclic geometric phase [80]. It would be interesting to consider geometric St"uckelberg interferometer in future work and extend the discussion into the non-Hermitian case [81].

In summary, we have shown that SO-coupled cold atoms trapped in an optical lattice can be used to implement St"uckelberg interference. It represents an atom interferometry with synthetic gauge fields and provides new opportunities to measure the synthetic gauge field. Time-dependent and time-averaged spin probability is derived using Floquet-Bloch theory. Based on that the interference patterns are computed in the parameter space.
directly accessible in experiments and resonances are found. Finally we studied chiral Bloch oscillation and found that the system can display a rich spin-momentum locking via varying the detuning. The phenomena predicted in this work can be readily observed in current available experiments on atomic flux lattices.

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