Inner functions of matrix argument and conjugacy classes in unitary groups

Yury A. Neretin

Denote by $B_n$ the set of complex square matrices of order $n$, whose Euclidean operator norms are $< 1$. Its Shilov boundary is the set $U(n)$ of all unitary matrices. A holomorphic map $B_m \rightarrow B_n$ is inner if it sends $U(m)$ to $U(n)$. On the other hand we consider a group $U(n + mj)$ and its subgroup $U(j)$ embedded to $U(n + mj)$ in a block-diagonal way ($m$ blocks $U(j)$ and a unit block of size $n$). For any conjugacy class of $U(n + mj)$ with respect to $U(j)$ we assign a 'characteristic function', which is a rational inner map $B_m \rightarrow B_n$. We show that the class of inner functions, which can be obtained as 'characteristic functions', is closed with respect to natural operations as pointwise direct sums, pointwise products, compositions, substitutions to finite-dimensional representations of general linear groups, etc. We also describe explicitly the corresponding operations on conjugacy classes.

1 Formulation of results

1.1. Some notation. Below:
— $\text{Mat}(n)$ is the space of square complex matrices of order $n$;
— $1_n$ is the unit matrix of order $n$;
— $\text{GL}(n, \mathbb{C})$ the group of invertible complex matrices of order $n$;
— $U(n)$ is the group of unitary matrices of order $n$.
— Let $V$, $W$ be linear spaces with bases $e_1, \ldots, e_p$ and $f_1, \ldots, f_q$ respectively. We order basis elements of the tensor product $V \otimes W$ as

$$e_1 \otimes f_1, \ldots, e_p \otimes f_1, e_1 \otimes f_2, \ldots, e_p \otimes f_2, \ldots, e_1 \otimes f_q, \ldots, e_p \otimes f_q.$$ 

According this we write tensor products of matrices.

1.2. Matrix balls and inner functions. Denote by $\| \cdot \|$ the operator norm in Euclidean space, i.e., $\|z\|^2$ is the maximal eigenvalue of the matrix $z^*z$. Denote $B_n$ (a matrix ball) the set of complex matrices $z$ of order $n$ such that $\|z\| < 1$, by $\overline{B}_n$ its closure, i.e., the set of matrices satisfying $\|z\| \leq 1$. By $\partial B_n$ we denote the boundary of $B_n$, i.e. the set of matrices with norm 1. The unitary group $U(n)$ is contained in $\partial B_n$ and is the Shilov boundary of $B_n$.

Recall that the pseudounitary group $U(n, n)$ acts on $B_n$ by biholomorphic transformations, and the space $B_n$ is the symmetric space

$$B_n \simeq U(n, n)/(U(n) \times U(n)).$$

1Supported by the grant FWF P31591.
We say that a holomorphic map $F : B_m \rightarrow B_\alpha$ is inner if its limit values on $U(m)$ are defined a.s. and $F$ sends $U(m)$ to $U(\alpha)$. Below we discuss only rational maps, so a meaning of the term 'limit values' here is clear.

**Remark.** Recall that inner functions $B_1 \rightarrow B_1$ (i.e., holomorphic maps of the unit disk $|z| < 1$ to itself that also send the circle $|z| = 1$ to itself) are a classical topic of function theory of complex variable, see, e.g. [8]. Inner functions $B_1 \rightarrow B_\alpha$ arose in the context of works of M.S.Livshits 1946-1954 on spectral theory of operators closed to unitary operators, see [11], [12], see also [23]. V.P.Potapov [22] obtained a multiplicative representation of such functions, see also [3]. Inner functions $B_m \rightarrow B_\alpha$ arose in [16]–[17] in representation theory of infinite-dimensional classical groups.

**1.3. Colligations and characteristic functions.** Fix $\alpha$ and $m \in \mathbb{N}$. $j = 0, 1, 2, \ldots$. Consider a unitary group $U(\alpha + mj)$ and its subgroup $U(j)$ embedded as

$$T \mapsto \begin{pmatrix} 1_{\alpha} & 0 & 0 & \cdots & 0 \\ 0 & T & 0 & \cdots & 0 \\ 0 & 0 & T & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & T \end{pmatrix} \in U(\alpha + mj).$$

Consider conjugacy classes of the group $U(\alpha + mj)$ with respect to the subgroup $U(j)$, i.e., matrices defined up to the equivalence

$$g \sim hgh^{-1}, \quad \text{where} \quad g \in U(\alpha + mj), \ h \in U(j).$$

We call such conjugacy classes by **colligations**. Let $S \in B_m$, let $s_{\mu\nu}$ be its matrix elements. We write $g$ as a block matrix of size $\alpha + \underbrace{j + \cdots + j}_{m \text{ times}}$,

$$g = \begin{pmatrix} a & b_1 & \cdots \\ c_1 & d_{11} & \cdots \\ \vdots & \vdots & \ddots \\ c_k & d_{m1} & \cdots \\ d_{1m} & \cdots & \ddots \end{pmatrix} \in U(\alpha + mj), \quad (1.1)$$

and consider the following relation:

$$\begin{pmatrix} p \\ x_1 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} a & b_1 & \cdots & b_m \\ c_1 & d_{11} & \cdots & d_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ c_k & d_{m1} & \cdots & d_{mm} \end{pmatrix} \begin{pmatrix} q \\ s_{11}x_1 + \cdots + s_{1m}x_m \\ \vdots \\ s_{m1}x_1 + \cdots + s_{mm}x_m \end{pmatrix}, \quad (1.2)$$

where columns $p, q \in \mathbb{C}^\alpha$, and $x_1, \ldots, x_m \in \mathbb{C}^j$. We eliminate variables $x_1, \ldots, x_m$ and get a dependence

$$p = \Theta[g; S]q,$$
where $\Theta(g; S)$ is a rational matrix-valued function of the variable $S$ depending on a parameter $g$. By [10], Theorem 4.1, this function depends only on the conjugacy class containing $g$ and is an inner function of the matrix variable $S$. We call $\Theta(g; S)$ by a characteristic function of a colligation.

Let us repeat the definition in other terms. Denote

$$1_j \otimes S := \begin{pmatrix} s_{11} \cdot 1_j & \cdots & s_{1m} \cdot 1_j \\ \vdots & \ddots & \vdots \\ s_{m1} \cdot 1_j & \cdots & s_{mm} \cdot 1_j \end{pmatrix}.$$

Then

$$\Theta(g; S) = a + b (1_j \otimes S) \left(1_{mj} - d (1_j \otimes S)\right)^{-1} c.$$

(1.3)

Remark. Let $H = H_1 \oplus H_2$ be a Hilbert space, let $(a \ b \ c \ d)$ be a unitary operator in $H$. The Livshits characteristic function, which arose in spectral theory of non-normal operators, see [11], [12] (see also [23], [7], [6]) is given by

$$\Theta(\lambda) = a + b(1 - \lambda d)^{-1} c.$$

(1.4)

In our construction this corresponds to $m = 1$. We prefer to use the original term 'characteristic function', which emphasize analogy with characteristic numbers and characteristic polynomials. Our equation (1.2) can be regarded as an extension of the equation $Ax = sx$. But the term 'characteristic function' is overburdened (it has two other common meanings: indicator functions and Fourier transforms of measures in probability). There is another term transfer function (see, e.g., [5]) for (1.4), which came from system theory.

Remark. Colligation type structures arise in representation theory of infinite dimensional classical groups as 'semigroups of double cosets'. Firstly, this was observed by G. I. Olshanski in [20], see also [14], Sect.IX.3-4. Such semigroups act in spaces of unitary representations of corresponding classical groups by certain operators with Gaussian kernels. A Gaussian kernel is determined by a matrix, and Olshanski showed that such matrices are given by expressions similar to matrix-valued characteristic functions of one variable. In fact, the origin of inner functions of matrix variables in [16], [17] was similar, but an initial point was a more general class of unitary representations from N. I. Nessonov [19]. From this point of view basic objects are semigroups of 'colligations' and inner functions are a tool for understanding of colligations. In considerations of this paper we do not refer to representation theory and regard inner function of matrix variables and colligations as abstract topics.

1.4. A conjecture. Denote by

$$\operatorname{Inn}(m, \alpha) = \operatorname{Inn}[B_m, B_\alpha]$$

the space of all rational interior maps $F : \overline{B}_m \to \overline{B}_\alpha$. Denote by $\operatorname{Inn}_c(m, \alpha)$ its subset consisting of maps $F$ such that $F(B_m) \subset B_\alpha$. By the maximum
modulus principle the last condition is equivalent to: for some \(z_0 \in B_m\) we have \(F(z_0) \in B_\alpha\).

We also define the space

\[
\text{Char}(m, \alpha) = \text{Char}[B_m, B_\alpha]
\]

consisting of characteristic functions determined by all possible elements of \(U(\alpha + mj)\) for \(j = 0, 1, 2, \ldots\). By \(\text{Char}_\circ(m, \alpha)\) we denote its subset, consisting of function \(\Theta[g; \cdot]\) sending \(B_m \to B_\alpha\). In notation (1.1), a map \(\Theta[g; \cdot]\) is contained \(\text{Char}_\circ(m, \alpha)\) if and only if \(\|a\| < 1\).

**Conjecture 1.1** Any rational inner function is a characteristic function of some colligation, i.e.,

\[
\text{Inn}(m, \alpha) = \text{Char}(m, \alpha).
\]

**Remark.** The conjecture was formulated in [17]. It is not doubtless since similar statement for inner functions in polydisks is false (or is valid under additional conditions to rational inner functions, cf. the case of polysisk in [9], [4]).

**1.5. Operations in \(\text{Char}(m, \alpha)\).** In this paper we chow that the class \(\bigsqcup_{m, \alpha}\text{Char}(m, \alpha) \subset \bigsqcup_{m, \alpha}\text{Inn}(m, \alpha)\) is closed with respect to several natural operations.\(^2\) In all cases we describe explicitly operations over colligations corresponding to operations over inner functions, these formulas are presented in proofs.

**Theorem 1.2** a) Let \(F_1 \in \text{Char}(m, \alpha), F_2 \in \text{Char}(m, \beta)\). Then \(F_1 \oplus F_2 \in \text{Char}(m, \alpha + \beta)\).

b) Let \(F \in \text{Char}(m, \alpha + \beta)\) admits a decomposition into a direct sum \(F = F_1 \oplus F_2\), where \(F_1 \in \text{Inn}(m, \alpha), F_2 \in \text{Inn}(m, \beta)\). Then \(F_1 \in \text{Char}(m, \alpha), F_2 \in \text{Char}(m, \beta)\).

The first (trivial) part of the statement is proved in Subsect. 3.1, the second part in Subsect. 3.3.

The following statement was obtained in [16], Theorem 4.1.

**Theorem 1.3** Let \(F_1, F_2 \in \text{Char}(m, \alpha)\). Then the pointwise product \(F_1F_2\) of matrix-valued functions \(F_1, F_2\) is contained in \(\text{Char}(m, \alpha)\).

**Theorem 1.4** Let \(F_1 \in \text{Char}(m, \alpha), F_2 \in \text{Char}(m, \beta)\). Then the pointwise tensor product \(F_1 \otimes F_2\) of matrix-valued functions \(F_1, F_2\) is contained in \(\text{Char}(m, \alpha \beta)\).

This is proved in Subsect. 3.3.

\(^2\)These statements were announced in [17].
Theorem 1.5 Let \( G \in \text{Char}(\beta, \gamma) \) be defined by a matrix \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U(\gamma + \beta j) \), \( F \in \text{Char}(\alpha, \beta) \) by a matrix \( \begin{pmatrix} p & q \\ r & t \end{pmatrix} \in U(\beta + \alpha i) \). Let

\[
\det(1_{\beta j} - d (1_j \otimes p)) \neq 0. \tag{1.5}
\]

Then \( G \circ F \in \text{Char}(\alpha, \gamma) \). More generally, the conclusion holds if

\[
\det(1_{\beta j} - d (1_j \otimes F(S_0))) \neq 0 \quad \text{for some } S_0 \in B_{\alpha}. \tag{1.6}
\]

Remarks. a) In particular, the condition (1.5) holds if \( \|d\| < 1 \) or if \( \|p\| < 1 \).

b) It can happened that the image of \( F \) is contained in the set of discontinuity of \( G \). The condition (1.6) is sufficient (and not necessary) for avoiding this situation.

Theorem is proved in Subsect. 3.4.

Next, consider a unitary finite-dimensional representation \( \rho \) of a unitary group \( U(n) \). Then (see, e.g., [24], §42) it admits a unique holomorphic continuation to a representation of \( GL(n, \mathbb{C}) \). The representation \( \rho \) is called polynomial, if all matrix elements of \( \rho(g) \) are polynomials in matrix elements of \( g \in GL(n, \mathbb{C}) \). Consider the semigroup \( \text{Mat}^\times(n) \) of all matrices of order \( n \) with respect to the multiplication. The group \( GL(n, \mathbb{C}) \) is dense in \( \text{Mat}^\times(n) \) and all polynomial representations of \( GL(n, \mathbb{C}) \) have a continuous extensions to the semigroup \( \text{Mat}^\times(n) \) (matrices \( \rho(\cdot) \) are determined by the same polynomials).

Theorem 1.6 Let \( \rho \) be a polynomial unitary representation of \( U(\alpha) \). Let \( F \in \text{Inn}(m, \alpha) \). Then \( \rho \circ F \) is contained in \( \text{Inn}(m, \dim \rho) \).

The statement is proved in Subsect. 3.6

Corollary 1.7 Let \( F \in \text{Inn}(m, \alpha) \). Then \( \det(F) \in \text{Inn}(m, 1) \).

1.6. Some remarks on behavior of inner functions on strata of boundaries. Recall (see [21], Sect.6, see below Subsect. 2.3) that the boundary of the domain \( B_m \subset \text{Mat}(m) \) is a disjoint union of a continual family of complex (open) manifolds (boundary components), these components are maximal complex manfolds that are contained in the boundary. Each component \( C \) is biholomorphically equivalent to some matrix ball \( B_\nu \), where \( \nu = 0, 1, \ldots m-1 \), and a biholomorphic map \( B_\mu \rightarrow C \) extends continuously to a homeomorphism of closures \( B_\mu \rightarrow \overline{C} \).

The following statements are obvious.

Proposition 1.8 a) Let \( F \in \text{Inn}[B_m, B_{\alpha}] \setminus \text{Inn}_0[B_m, B_{\alpha}] \). Then \( F(B_m) \) is contained in a unique boundary component \( C \subset B_{\alpha} \). Moreover, \( F \) is contained in \( \text{Inn}_0[B_m, C] \).

b) Let \( F \in \text{Inn}[B_m, B_{\alpha}] \) be continuous at some point of a boundary component \( C \subset B_m \). Then \( F \in \text{Inn}[C, B_{\alpha}] \).
The next statement is proved in Subsect. 3.7.

**Theorem 1.9** a) Let $F \in \text{Char}[B_m, B_\alpha]$ send $B_m$ into a boundary component $C \subset \overline{B}_\alpha$. Then $F \in \text{Char}[B_m, C]$.

b) Let $F \in \text{Char}[B_m, B_\alpha]$ be determined by a matrix \[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\in U(\alpha + mj).
\]
Let $C \subset \overline{B}_m$ be a boundary component. Let $\det(1_{mj} - d(1_j \otimes S_0)) \neq 0$ for some $S_0 \in C$. Then the restriction of $F$ to $C$ is contained in $\text{Char}[C, B_\alpha]$.

**1.7. On some extensions of the construction.** As we mentioned above, $B_n$ is a symmetric space. Our construction of inner functions can be automatically extended to Hermitian symmetric spaces of series $U(p, q)/U(p) \times U(q)$, $\text{Sp}(2n, \mathbb{R})/U(n)$, $\text{SO}^*(2n, \mathbb{R})$ (see, e.g., [21], they are called classical complex domains of I, II, and III types) and for direct products of such spaces (see [16], [18]).

However, initial spaces and target spaces in such constructions are not independent. For instance, this approach does not produce inner functions from the usual unit ball $U(n, 1)/(U(n) \times U(1))$ to the unit disk (they exist according A.B.Alexandrov [1] and E.Løw [13]), and, more generally, for functions from $U(n, 1)/(U(n) \times U(1))$ to the unit disk for $p \neq q$ (they also exist according Alexandrov [2]).

However it is possible to take $m = \infty$. There arises the following question.

**Question 1.10** Let $g$ be a unitary operator

\[
\mathbb{C} \oplus \ell^2 \oplus \ell^2 \to \mathbb{C} \oplus \ell^2,
\]

i.e., $gg^* = 1$, $g^*g = 1$. Represent $g$ in the block form

\[
g = \begin{pmatrix}
a & b_1 & b_2 \\
c & d_1 & d_2
\end{pmatrix}.
\]

Let $(s_1, s_2)$ be a point of the open unit ball in $\mathbb{C}^2$. Set

\[
\Theta[g; (s_1, s_2)] :=
\]

\[
:= a + (b_1 & b_2) \begin{pmatrix} s_1 \cdot 1_\infty \\
 & s_2 \cdot 1_\infty
\end{pmatrix} + \begin{pmatrix} 1_\infty & -d_1 & d_2 \\
 & s_1 \cdot 1_\infty
\end{pmatrix}^{-1}
\]

\[
c = a + (s_1 b_1 + s_2 b_2) [1 - s_1 d_1 - s_2 d_2]^{-1} c. \quad (1.7)
\]

Is it possible to find inner function of such a type? Is it possible to find conditions for $g$ under which $\Theta[g]$ is an inner function in the unit ball in $\mathbb{C}^2$?

An argument for this supposal is very simple. We write the following relation

\[
\begin{pmatrix} p \\
x
\end{pmatrix} = \begin{pmatrix}
a & b_1 & b_2 \\
c & d_1 & d_2
\end{pmatrix} \begin{pmatrix} q \\
s_1 x \\
s_2 x
\end{pmatrix}.
\]
Eliminating variables \(x\), we come to \(p = \Theta[g, (s_1, s_2)]q\). On the other hand, the matrix \(g\) is unitary, therefore \(|p|^2 + \|x\|^2 = |q|^2 + |s_1|^2\|x\|^2 + |s_2|^2\|x\|^2\), i.e.,
\[
|p|^2 = |q|^2 - (1 - |s_1|^2 - |s_2|^2)\|x\|^2.
\]
If \(|s_1|^2 + |s_2|^2 < 1\), then \(|p| \leq |q|\) and \(|\Theta[g, (s_1, s_2)]| \leq 1\). At first glance it seems that \(|s_1|^2 + |s_2|^2 = 1\) immediately implies \(|p| = |q|\). But it is not that simple, since the matrix in square brackets in (1.7) can be non-invertible in this case.

It is easy to present examples, when function \(\Theta[g, \cdot]\) is not inner, however a Livshits characteristic function \([1.3]\) also not always is inner, see [23], §VI.1.

2 Preliminaries

2.1. Linear fractional maps. We realize the pseudo-unitary group \(U(n, n)\) as the group of all complex block matrices \(g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\) of size \((n + n)\) satisfying the condition
\[
g \begin{pmatrix} -1_n & 0 \\ 0 & 1_n \end{pmatrix} g^* = \begin{pmatrix} -1_n & 0 \\ 0 & 1_n \end{pmatrix}.
\]
For each \(g \in U(n, n)\) we consider the following linear fractional transformation of the space \(\text{Mat}(n)\):
\[
\gamma[g; z] = (A + zC)^{-1}(B + zD). \tag{2.1}
\]
Such transformations send the matrix ball \(B_n\) to itself (see, e.g., [21], §6, [15], Sect. 2.3). This action of \(U(n, n)\) is transitive, the stabilizer of the point \(z = 0\) consists of matrices \(\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\), where \(a \in U(n), d \in U(n)\). So \(B_n\) is a homogeneous space
\[
B_n = U(n, n)/(U(n) \times U(n)).
\]

2.2. Realization of \(B_n\) as a domain in a Grassmannian. Consider the pseudo-Euclidean space
\[
V^{2n} = V^n_+ \oplus V^n_+ := \mathbb{C}^n \oplus \mathbb{C}^n
\]
equipped with the Hermitian form \(M = M_n\) determined by the matrix \(\begin{pmatrix} -1_n & 0 \\ 0 & 1_n \end{pmatrix}\).
Our group \(U(n, n)\) preserves this form.

Denote \(\text{Gr}(n)\) the Grassmanian of all \(n\)-dimensional subspaces in \(V^{2n}\). We say that a subspace \(L \in \text{Gr}(n)\) is negative (semi-negative) if the form \(M\) is negative (resp., semi-negative) defined on \(L\). A subspace \(L \in \text{Gr}(n)\) is isotropic if the form \(M\) is zero on \(L\). We denote the corresponding subsets in the Grassmannian by
\[
\text{Gr}^{<0}(n), \quad \text{Gr}^{\leq 0}(n), \quad \text{Gr}^{0}(n)
\]
respectively.
For any linear map $z : V^n \to V^+_n$ we consider the $n$-dimensional space $L[z]$ consisting of vectors of the form $(v_-, v_- z)$. If $z \in B_n$, then the form $\mathcal{M}$ is negative on $L(z)$. Vice versa, any $n$-dimensional negative subspace in $V$ has the form $L(z)$ for some $z \in B_n$. Formula (2.1) corresponds to the natural action of $U(n,n)$ on the set of negative subspaces. Also

$$\overline{B}_n \simeq \text{Gr}^{<0(n)} , \quad U(n) \simeq \text{Gr}^0(n).$$

2.3. The structure of the boundary of the matrix ball $\overline{B}_n$. The boundary of $B_n$ consists of $n$ orbits $O_j$, where $j = 1, 2, \ldots, n-1$, of the group $U(n,n)$, representatives of orbits are matrices of the form

$$\begin{pmatrix} 1_j & 0 \\ 0 & 0_{n-j} \end{pmatrix}.$$  

The Shilov boundary $U(n)$ corresponds to $j = n$.

On the language of the Grassmannian $\text{Gr}(n)$ the orbit $O_j$ corresponds to semi-negative subspaces $L$ such that the rank the form $\mathcal{M}$ on $L$ is $(n-j)$.

Any component of the boundary $\partial B_n$ can be reduced by a linear fractional transformation to the form

$$\begin{pmatrix} u & 0 \\ 0 & 1_j \end{pmatrix}, \quad \text{where } u \text{ ranges in } B_{n-j}.$$  

On the language of the Grassmannian, boundary components $C$ are enumerated by a number $j$ and a $j$-dimensional isotropic subspace $W \subset C^n \oplus C^n$. The corresponding component consists of all $n$-dimensional $\mathcal{M}$-semi-negative subspaces $L \supset W$ such that the kernel of $\mathcal{M}$ on $L$ coincides with $W$.

2.4. Linear relations. Let $V$, $W$ be linear spaces. A linear relation $Y : V \rightrightarrows W$ is a linear subspace in $V \oplus W$. For linear relations $Y : V \rightrightarrows W$ and $Z : W \rightrightarrows U$ we define their product $ZY : V \rightrightarrows U$ as the set of all $v \oplus u \in V \oplus U$, for which there exists $w \in W$ satisfying $v \oplus w \in Y$, $w \oplus u \in Z$.

Let $H \subset V$ be a linear subspace. The subspace $YH \subset W$ consists of $w \in W$, for which there exists $v \in H$ satisfying $v \oplus w \in Y$. We can consider $H$ as linear relation $0 \rightrightarrows V$, therefore we can understand $PH$ as a product of linear relations.

For a linear relation $Y : V \rightrightarrows W$ we define

- kernel $\ker Y \subset V$ as the intersection $Y \cap V$;
- domain $\text{dom } Y \subset V$ is the image of the projection $Y$ to $V$ along $W$;
- image $\text{im } Y \subset W$ is the image of projection of $Y$ to $W$;
- indefiniteness $\text{indef } Y$ is $Y \cap W$.

A product of linear relations $Y : V \rightrightarrows W$, $Z : W \rightrightarrows Y$ is a continuous operation $(Y,Z) \to ZY$ outside sets

$$\ker Z \cap \text{indef } Y \neq 0 \quad \text{and} \quad \text{im } Y + \text{dom } Z \neq W.$$  

2.5. Isotropic category. Objects of the isotropic category, see [15], Sect. 2.10, are spaces

$$V^{2n} = V^+_n \oplus V^-_n \simeq C^n \oplus C^n.$$  

where $n = 0, 1, 2, \ldots$. A morphism $V^{2n} \to V^{2m}$ is a linear relation $Y : V^{2n} \rightrightarrows V^{2m}$ satisfying conditions:

1) if $v \oplus v' \in Y$, then $M_n(v, v) = M_m(v', v')$;
2) dim $P$ is maximal possible, i.e., dim $P = m + n$.

A product of morphism is the product of linear relations. The group of automorphisms of $V^{2n}$ is $U(n, n)$.

Emphasize that the product has points of discontinuity.

Equip $V^{2n} \oplus V^{2m}$ with the difference of Hermitian forms in this space,

$$M_{m,n}(v \oplus w, v' \oplus w') := M_n(v, v') - M_m(w, w').$$

Then the subspace $V^n \oplus V^m \subset V^{2n} \oplus V^{2m}$ is negative with respect to the form $M_{m,n}$ and the subspace $V^m \oplus V^n \subset V^{2n} \oplus V^{2m}$ is positive. So we can apply above reasoning and get:

A relation $P : V^{2n} \rightrightarrows V^{2m}$ is isotropic if and only if $P$ is a graph of a unitary operator $V^n \oplus V^m \to V^n \oplus V^m$.

Thus the set of morphisms from $V^{2n}$ to $V^{2m}$ is in one-to-one correspondence with the unitary group $U(n + m)$ and the product of morphisms $Y : V^{2n} \rightrightarrows V^{2m}$, $Z : V^{2m} \rightrightarrows V^{2k}$ induces an operation

$$U(n + m) \times U(m + k) \to U(n + k).$$

For the following statement, see, e.g., [15], Theorem 2.8.4.

**Proposition 2.1** Let $Y : V^{2k} \rightrightarrows V^{2m}$ corresponds to a unitary matrix $\nu = \begin{pmatrix} p & q \\ r & t \end{pmatrix} \in U(k + m)$ and $Z : V^{2m} \rightrightarrows V^{2n}$ correspond to a unitary matrix $\zeta = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U(n + m)$. Let

$$\det(1 - pd)^{-1} \neq 0.$$

Then $ZY$ corresponds to the matrix

$$\zeta \ast \nu := \begin{pmatrix} a + b(1 - pd)^{-1}pc & b(1 - pd)^{-1}q \\ r(1 - dp)^{-1}c & t + rd(1 - pd)^{-1}q \end{pmatrix}.$$  \hspace{1cm} (2.2)

**2.6. Krein–Shmul’yan maps.** Let $L \in \text{Gr}^{<0}(m)$. Applying a morphism $Z : V^{2m} \rightrightarrows V^{2n}$ of the isotropic category to $L$ we again get an element of $\text{Gr}^{<0}(n)$, so we get a map $\overline{B}_m \to \overline{B}_n$ (see [15], Theorem 2.9.1). Let $\zeta = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be the unitary matrix corresponding to $Z$. Then the corresponding map $\sigma[\zeta]$ is given by the formula

$$\sigma[\zeta; u] : u \mapsto a + bu(1_m - ud)^{-1}c,$$ where $u \in \overline{B}_m$. \hspace{1cm} (2.3)

This holds if $u$ satisfies the condition $\det(1 - ud) \neq 0$. 

9
Notice, that

1) for any \( \zeta \in U(n + m) \) our map is continuous as a map \( B_m \to B_n \);

2) if \( \|d\| < 1 \), our formula determines a continuous map \( B_m \to B_n \);

3) if \( \|a\| < 1 \), then our formula determines a continuous map \( B_m \to B_n \).

Remark. A map \( \sigma(\zeta; z) \) is a special case of Krein–Shmul’yan maps, see \([10]\), see also \([15]\), Sect. 2.9.

Remark. A map \((2.3)\) \( B_m \to B_m \) is inner and moreover its characteristic function is determined by an element of \( U(n + m \cdot 1) \).

Lemma 2.2 Let \( \zeta \) and \( \upsilon \) be the same as in Proposition \((2.1)\). Then for any \( u \in B_k \) we have

\[
\sigma(\zeta; \sigma(\upsilon; u)) = \sigma(\zeta \circ \upsilon; u).
\] (2.4)

Remark. Cf. \([15]\), Theorem 2.9.4, but conditions of this theorem are not satisfied. Basically, Lemma 2.2 claims the associativity of product of linear relations \( 0 \Rightarrow V^{2k} \Rightarrow V^{2m} \Rightarrow V^{2n} \). However, formula (2.2) is not valid on the surface \( \det(1 - pd) = 0 \) and this requires some care. To avoid references to proofs or repetitions of proofs, we present a formal calculation.

Proof. We must transform the following expression to the Krein–Shmul’yan form:

\[
a + b \left( u(1 - tu)^{-1} \bigg|_{u=p+qz(1-tz)^{-1}r} \right) c.
\] (2.5)

Step 1. It is sufficient to examine the expression in big brackets, it is a sum \( I + J \) of two summands

\[
I := p \left[ 1 - dq - dqz(1 - tz)^{-1}r \right]^{-1};
\]

\[
J := qz(1 - tz)^{-1} \left[ 1 - dq - dqz(1 - tz)^{-1}r \right]^{-1}.
\]

First, we must show that the inverse matrix \([\ldots]^{-1}\) exists. Since \((1 - dp)^{-1}\) is invertible, we can transform \([\ldots]^{-1}\) as

\[
[\ldots]^{-1} = (1 - dq)^{-1} \left( 1 - dqz(1 - tz)^{-1} \cdot r(1 - dq)^{-1} \right)^{-1}.
\] (2.6)

Next, we notice that matrices \((1 - AB)\) and \((1 - BA)\) are invertible or non-invertible simultaneously. Therefore it is sufficient to verify existence of the matrix

\[
(1 - r(1 - dq)^{-1} \cdot dqz(1 - tz)^{-1})^{-1} = (1 - tz) \left( 1 - \{ t + r(1 - dq)^{-1} \cdot dq \} \cdot z \right)^{-1}.
\]

Since \( \|t\| \leqslant 1 \), \( \|z\| < 1 \) the matrix \((1 - tz)\) is invertible. Next, the expression in curly brackets coincides with lower right block of the matrix \((2.2)\). Therefore \( \|\{\ldots\}\| \leqslant 1 \) and the second factor is well-defined.
STEP 2. Transforming the summand $J$ with (2.6) we get

$$J = qz(1 - tz)^{-1} \cdot r(1 - dq)^{-1} \left(1 - dqz(1 - tz)^{-1} \cdot r(1 - dq)^{-1}\right)^{-1}.$$  

Keeping in mind the matrix identity

$$A(1 - BA)^{-1} = (1 - AB)^{-1}A,$$

we come to

$$J = qz(1 - tz)^{-1} \cdot \left(1 - r(1 - dq)^{-1} \cdot dqz(1 - tz)^{-1}\right)^{-1} r(1 - dq)^{-1} = qz \left(1 - \{t + r(1 - dq)^{-1} \cdot dq\} \cdot z\right)^{-1} r(1 - dq)^{-1}.$$  

Next, we transform the summand $I$ with (2.6) and apply the identity

$$(1 - C)^{-1} = 1 + C(1 - C)^{-1}$$

(2.7) to the second factor in (2.6). We come to

$$I = p(1 - dp)^{-1} + p(1 - dp)^{-1}d \cdot qz(1 - tz)^{-1}r(1 - dq)^{-1} \left(1 - dqz(1 - tz)^{-1}r(1 - dq)^{-1}\right)^{-1} = p(1 - dp)^{-1} + p(1 - dp)^{-1}d \cdot J$$

Thus

$$I + J = p(1 - dp)^{-1} + \left\{p(1 - pd)^{-1}d + 1\right\} \cdot J = (1 - pd)^{-1}p + \left\{(1 - pd)^{-1}\right\} \cdot qz \left(1 - \{t + r(1 - dq)^{-1} \cdot dq\} \cdot z\right)^{-1} r(1 - dq)^{-1}$$

(we applied (2.7) to the expression in curly brackets). We substitute the result to (2.5) instead of the expression in big brackets and get the desired formula. □

2.7. Characteristic functions and Krein–Shmul’yan maps. The formula (1.3) can be written as

$$\Theta \left[ \begin{pmatrix} a & b \\ c & d \end{pmatrix} ; S \right] = \sigma \left[ \begin{pmatrix} a & b \\ c & d \end{pmatrix} ; 1_j \otimes S \right].$$  

(2.8)

2.8. Polynomial representations of GL$(n, \mathbb{C})$. Irreducible holomorphic representations $\rho_\mathbf{m}(g)$ of GL$(n, \mathbb{C})$ are enumerated by ‘signatures’

$$\mathbf{m} := (m_1, \ldots, m_n), \quad \text{where} \ m_j \in \mathbb{Z} \ \text{and} \ m_1 \geq m_2 \geq \ldots \geq m_n,$$

see, e.g., [24], §49-50. Recall a semi-explicit construction of $\rho_\mathbf{m}$. Consider the space $\mathbb{C}^n$ with the standard basis $e_1, \ldots, e_n$. The representation $\lambda_k(g)$ with signature $(\underbrace{1, \ldots, 1}_k, 0, \ldots, 0)$ is the representation in the $k$-th
exterior power $\bigwedge^k \mathbb{C}^n$, its highest weight vector is $v_k = e_1 \wedge \cdots \wedge e_k$. The matrix element $\langle \rho(g)v_k, v_k \rangle$ is the $k$-th principal minor $\Delta_k(g)$ of $g$. The representation $\lambda_n$ is simply $\det(g)$ (in particular, we can consider its negative tensor powers).

The representation $\rho_m$ is a subrepresentation of

$$\bigotimes_{k=1}^{n-1} \lambda_k^{\otimes (m_k-m_{k+1})} (g) \otimes \lambda_k(g)^{m_n}. \quad (2.9)$$

More precisely, $\rho_m$ is the cyclic span of the highest weight vector

$$\xi_m := \bigotimes_{k=1}^{n-1} (e_1 \wedge \cdots \wedge e_k)^{\otimes (m_k-m_{k+1})} \otimes (e_1 \wedge \cdots \wedge e_n)^{\otimes m_n}. \quad (3.1)$$

The matrix element

$$\langle \rho_m(g)\xi_m, \xi_m \rangle = \prod_{k=1}^{n-1} \Delta_k^{m_k-m_{k+1}} \cdot \det(g)^{m_n}. \quad (3.2)$$

is polynomial if and only if $m_n \geq 0$. On the other hand, for $m_n \geq 0$ the representation $\rho_m$ is a polynomial representation by the construction.

3 Proofs

3.1. Direct sums. Proof of Theorem 1.2.a. Take an element $g \in U(\alpha + mi)$ written as

$$g = \begin{pmatrix} a & b_1 & b_2 & \cdots \\ c_1 & d_{11} & d_{12} & \cdots \\ c_2 & d_{21} & d_{22} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad (3.1)$$

and an element $\bar{g} \in U(\beta + mj)$ written as

$$\bar{g} = \begin{pmatrix} \bar{a} & \bar{b}_1 & \bar{b}_2 & \cdots \\ \bar{c}_1 & d_{11} & d_{12} & \cdots \\ \bar{c}_2 & d_{21} & d_{22} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad (3.2)$$

Consider the block matrix of order

$$\alpha + \beta + i + j + \cdots + i + j = (\alpha + \beta) + (i + j) + \cdots + (i + j) \quad \text{m times}$$
given by
\[
g(\oplus)\tilde{g} := \begin{pmatrix}
a & 0 & b_1 & 0 & b_2 & 0 & \ldots \\
0 & \tilde{a} & 0 & \tilde{b}_1 & 0 & \tilde{b}_2 & \ldots \\
c_1 & 0 & d_{11} & 0 & d_{12} & 0 & \ldots \\
0 & \tilde{c}_1 & 0 & \tilde{d}_{11} & 0 & \tilde{d}_{12} & \ldots \\
c_2 & 0 & d_{21} & 0 & d_{22} & 0 & \ldots \\
0 & \tilde{c}_2 & 0 & \tilde{d}_{21} & 0 & \tilde{d}_{22} & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
\]

By formula (2.8),
\[
\Theta[g(\oplus)\tilde{g}; S] = \Theta[g; S] \oplus \Theta[\tilde{g}; S]
\]
(we applied the Krein–Shmul’yan map determined by the matrix \(g(\oplus)\tilde{g}\) to a matrix \(S \otimes 1_{i+j}\)).

3.2. Pointwise products. Theorem 1.3 was observed in [16]. To be complete we present here the corresponding operation on colligations. Let \(g \in U(\alpha + mi)\) be represented as (3.1) and \(\tilde{g} \in U(\alpha + mj)\) as (3.2). We define the matrix \(g \odot \tilde{g}\) by
\[
g \odot \tilde{g} := \\
\begin{pmatrix}
a & b_1 & 0 & b_2 & 0 & \ldots \\
c_1 & d_{11} & 0 & d_{12} & 0 & \ldots \\
0 & 0 & 1_j & 0 & 0 & \ldots \\
c_2 & d_{21} & 0 & d_{22} & 0 & \ldots \\
0 & 0 & 0 & 0 & 1_j & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\begin{pmatrix}
\tilde{a} & 0 & \tilde{b}_1 & 0 & \tilde{b}_2 & 0 & \ldots \\
0 & \tilde{c}_1 & 0 & \tilde{d}_{11} & 0 & \tilde{d}_{12} & \ldots \\
0 & 0 & 0 & 1_i & 0 & \ldots \\
\tilde{c}_2 & 0 & \tilde{d}_{21} & 0 & \tilde{d}_{22} & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]
(3.3)

Then
\[
\Theta[g \odot \tilde{g}; S] = \Theta[g; S] \Theta[\tilde{g}; S].
\]

3.3. Pointwise tensor products. Proof of Theorem 1.4. This is a corollary of the previous statement. Since
\[
A \otimes B = (1 \otimes B) \cdot (A \otimes 1),
\]
it is sufficient to verify the claim for \(F_1 \otimes 1\) and \(1 \otimes F_2\).

The function \(F_1 \otimes 1\) is a direct sum of several copies of the function \(F_1\). By Theorem 1.2 it is a characteristic function. More precisely, if \(F_1\) is generated by
a matrix \( \begin{pmatrix} p & q \\ r & t \end{pmatrix} \), then

\[
\begin{pmatrix}
p & q_1 & \ldots & q_m \\
r_1 & t_{11} & \ldots & t_{1m} \\
\vdots & \vdots & \ddots & \vdots \\
r_m & t_{m1} & \ldots & t_{mm}
\end{pmatrix}; S \otimes 1_\beta =
\begin{pmatrix}
p \otimes 1_\beta & q_1 \otimes 1_\beta & \ldots & q_m \otimes 1_\beta \\
r_1 \otimes 1_\beta & t_{11} \otimes 1_\beta & \ldots & t_{1m} \otimes 1_\beta \\
\vdots & \vdots & \ddots & \vdots \\
r_m \otimes 1_\beta & t_{m1} \otimes 1_\beta & \ldots & t_{mm} \otimes 1_\beta
\end{pmatrix} ; S
\]

(3.4)

The identity

\[
1_\alpha \otimes \Theta \left[ \begin{pmatrix} a & b \\ c & d \end{pmatrix} ; S \right] = \Theta \left[ \begin{pmatrix} 1_\alpha \otimes a & 1_\alpha \otimes b \\ 1_\alpha \otimes c & 1_\alpha \otimes d \end{pmatrix} ; S \right]
\]

immediately follows from (2.8). Therefore

\[
\Theta \left[ \begin{pmatrix} p & q \\ r & t \end{pmatrix} ; S \right] \otimes \Theta \left[ \begin{pmatrix} a & b \\ c & d \end{pmatrix} ; S \right]
\]

is a characteristic function of the colligation

\[
\begin{pmatrix}
p \otimes a & q_1 \otimes 1_\beta & p \otimes b_1 & q_2 \otimes 1_\beta & \ldots & p \otimes b_m \\
r_1 \otimes a & t_{11} \otimes 1_\beta & r_1 \otimes b_1 & t_{12} \otimes 1_\beta & \ldots & r_1 \otimes b_m \\
1_\alpha \otimes c_1 & 0 & 1_\alpha \otimes d_{11} & 0 & \ldots & 1_\alpha \otimes d_{1m} \\
r_2 \otimes a & t_{21} \otimes 1_\beta & r_2 \otimes b_1 & t_{22} \otimes 1_\beta & \ldots & r_2 \otimes b_m \\
1_\alpha \otimes c_2 & 0 & 1_\alpha \otimes d_{21} & 0 & \ldots & 1_\alpha \otimes d_{2m} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots
\end{pmatrix}
\]

(3.6)

3.4. Compositions. Proof of Theorem 1.5. By (2.8) and (3.5) we have

\[
G \circ F(S) = \sigma \left[ \begin{pmatrix} a & b \\ c & d \end{pmatrix}; 1_j \otimes \sigma \left[ \begin{pmatrix} p & q \\ r & t \end{pmatrix}; 1_i \otimes S \right] \right] =
\]

\[
= G \circ F(S) = \sigma \left[ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \left[ \begin{pmatrix} 1_j \otimes p & 1_j \otimes q \\ 1_j \otimes r & 1_j \otimes t \end{pmatrix}; 1_i \otimes S \right] \right].
\]

If \( \det(1_\beta - d(1_j \otimes p)) \neq 0 \) (condition (1.5)), then we can apply formula (2.2) and Lemma 2.2. In this case we come to

\[
\sigma \left[ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \left[ \begin{pmatrix} 1_j \otimes p & 1_j \otimes q \\ 1_j \otimes r & 1_j \otimes t \end{pmatrix}; 1_{ij} \otimes S \right] \right],
\]

where \( \otimes \)-product of matrices is defined by (2.2). Since the both 'factors' are unitary matrices, we get an inner map \( B_\alpha \rightarrow B_\gamma \).
Corollary 3.1 Let $F \in \text{Char}[m, \alpha]$.

a) For $h \in U(\alpha, \alpha)$ we have $\gamma[h] \circ F \in \text{Char}[m, \alpha]$.

b) For $h' \in U(m, m)$ we have $F \circ \gamma[h'] \in \text{Char}[m, \alpha]$.

Indeed, in these cases our condition holds.

Let us finish the proof of Theorem 1.5. Let condition (1.6) holds, i.e.,

$$\det\left(1_{\beta_2} - d(1_{\beta_1} \otimes F(S_0))\right) \neq 0.$$ We take $h \in U(\alpha, \alpha)$ sending $0$ to $S_0$, then $F(\gamma[h; 0]) = F(S_0)$, we apply Corollary 3.1 to $G \circ (F \circ \gamma[h])$ and refer to the already proved part of Theorem 1.5. Thus $G \circ (F \circ \gamma[h]) \circ \gamma[h^{-1}]$ and get the desired statement.

3.5. Splitting off summands. Proof of Theorem 1.2.b. Let a characteristic function $F \in \text{Char}[m, \alpha + \beta]$ have a block form

$$F(z) := \begin{pmatrix} F_1(z) & 0 \\ 0 & F_2(z) \end{pmatrix}.$$ Let us show that $F_1 \in \text{Char}[m, \alpha]$.

First, assume that $F_2 \in \text{Inn} \circ [m, \beta]$. We consider a Krein–Shmul’yan map $G : B_{\alpha + \beta} \rightarrow B_\alpha$ determined by the matrix

$$
\begin{pmatrix}
0 & 1_\alpha & 0 \\
1_\alpha & 0 & 0 \\
0 & 0 & 1_\beta
\end{pmatrix}
$$

and take the composition $G \circ F$,

$$G \circ F(z) =
(1_\alpha \ 0) \begin{pmatrix} F_1(z) & 0 \\ 0 & F_2(z) \end{pmatrix} \left\{ \begin{pmatrix} 1_\alpha & 0 \\ 0 & 1_\beta \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1_\beta \end{pmatrix} \begin{pmatrix} F_1(z) & 0 \\ 0 & F_2(z) \end{pmatrix} \right\}^{-1} \begin{pmatrix} 1_\alpha \\ 0 \end{pmatrix}.
$$

The matrix in curly brackets is

$$
\begin{pmatrix} 1_\alpha & 0 \\ 0 & 1_\beta - F_2(z) \end{pmatrix},
$$

it is invertible, and by Theorem 1.5 the map $G \circ F$ is contained in $\text{Char}[m, \alpha]$. But

$$G \circ F = F_1(z)$$

and this implies our statement.

Second, let $F_2 \not\in \text{Inn}_\circ(m, \alpha)$. Then $F_2(B_m)$ is contained in some component $C$ of the boundary of $B_\beta$. By Corollary 3.1 we can assume that $C$ is in a canonical form, i.e., $C$ consists of matrices

$$
\begin{pmatrix} u & 0 \\ 0 & 1_k \end{pmatrix},
$$

where $u \in B_{\beta - k}$. 

15
and therefore $F_2(z)$ has the form

$$F_2(z) = \begin{pmatrix} R(z) & 0 \\ 0 & 1_k \end{pmatrix}, \text{where } \|R(z)\| < 1 \text{ for } z \in B_m. \quad (3.8)$$

Now we choose $\lambda \in \mathbb{C}$ such that $|\lambda| = 1$ and $\lambda \neq 1$. Instead of (3.7) we take the matrix

$$\begin{pmatrix} 0 & 1_\alpha & 0 \\ 1_\alpha & 0 & 0 \\ 0 & 0 & \lambda \cdot 1_\beta \end{pmatrix}$$

and the corresponding Krein–Shmul’yan map $G$. By (3.8) the matrix

$$\left\{ \begin{pmatrix} 1_\alpha & 0 \\ 0 & 1_\beta \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & \lambda 1_\beta \end{pmatrix} \begin{pmatrix} F_1(z) \\ 0 \end{pmatrix} \right\} = \begin{pmatrix} 1_\alpha & 0 \\ 0 & 1_\beta - \lambda F_2(z) \end{pmatrix}$$

is invertible, and by Theorem 1.5 we have $G \circ F = F_1(z) \in \text{Char}(m, \alpha)$.

### 3.6. Compositions with polynomial representations. Proof of Theorem 1.6
By Theorem 1.2a, it is sufficient to consider irreducible representations. By construction given in Subsect. 2.3 any irreducible polynomial representation $\rho_m = \rho_{m_1, \ldots, m_n}$ is contained in tensors

$$\bigotimes_{k=1}^n \left( \bigwedge^k \mathbb{C}^n \right)^{\otimes (m_k - m_{k-1})} \otimes \left( \bigwedge^n \mathbb{C}^n \right)^{m_n} \subset \left( \mathbb{C}^n \right)^{\otimes \left( \sum_{k=1}^n m_k \right)}.$$

By Theorem 1.4 for any characteristic function $F(z)$ a function $F(z)^\otimes \mathcal{L}$ is a characteristic function. By Theorem 1.2b we can split off a direct summand.

### 3.7. Boundary components. Proof of Theorem 1.9a
Without loss of generality we can assume that $C$ has the canonical form $\begin{pmatrix} u & 0 \\ 0 & 1_l \end{pmatrix}$. Therefore our function $F$ splits into a direct sum. By Theorem 1.2b we can split off a summand.

Proof of Theorem 1.9b Again, we can assume that $C \subset B_m$ consists of matrices $\begin{pmatrix} u & 0 \\ 0 & 1_l \end{pmatrix}$. As in Subsect. 3.4 we can assume that $S_0 := \begin{pmatrix} 0 & 0 \\ 0 & 1_l \end{pmatrix}$.

The identical embedding $u \mapsto \begin{pmatrix} u & 0 \\ 0 & 1_l \end{pmatrix}$ is a Krein–Shmul’yan map determined by the matrix

$$\begin{pmatrix} 0 & 0 & 1_{m-l} \\ 0 & 1_l & 0 \\ 1_{m-l} & 0 & 0 \end{pmatrix}.$$

Now we can apply Theorem 1.5.

### References

[1] Aleksandrov A. B. *The existence of inner functions in a ball*. Math. USSR-Sb., 46:2 (1983), 143-159.
[2] Aleksandrov, A. B. Inner functions on compact spaces. Funct. Anal. Appl., 1984, 18:2, 87-98.

[3] Alpay D. An advanced complex analysis problem book. Topological vector spaces, functional analysis, and Hilbert spaces of analytic functions. Birkhäuser, Springer, Cham, 2015.

[4] Ball J. A., Bolotnikov V. Canonical transfer-function realization for Schur-Agler-class functions of the polydisk. In A panorama of modern operator theory and related topics, 75-122, Birkhäuser/Springer, Basel, 2012.

[5] Bart H. Transfer functions and operator theory. Linear Algebra Appl. 84 (1986), 33-61.

[6] Brodskiǐ M. S. Unitary operator colligations and their characteristic functions. 33 (1978), no. 4, 159-191.

[7] Brodskiǐ V. M. On operator nodes and their characteristic functions. Sov. Math. Dokl., 12 (1971), 696-700.

[8] Garnett J. B. Bounded analytic functions. Academic Press, New York-London, 1981.

[9] Knese G. Rational inner functions in the Schur-Agler class of the polydisk. Publ. Mat. 55 (2011), no. 2, 343-357.

[10] Krein M. G., Shmul’yan Ju. L. Fractional linear transformations with operator coefficients. (Russian) Mat. Issled. 2 1967, vyp. 3, 64-96.

[11] Livshits M.S. On a certain class of linear operators in Hilbert space. Mat. Sb., N. Ser. 19(61), 239-262 (1946); English Transl. in Am. Math. Soc. Transl. (Ser 2), 13, 61-83 (1960).

[12] Livshits M.S. On spectral decomposition of linear non self-adjoint operators. Mat. Sbornik N.S. 34(76), 145-199 (1954). English transl. in Am. Math. Soc. Transl. (Ser 2), 5, 67-114 (1957).

[13] Løw E. A construction of inner functions on the unit ball in \( \mathbb{C}^p \). Invent. Math. 67 (1982), no. 2, 223-229.

[14] Neretin, Yu. A. Categories of symmetries and infinite-dimensional groups. The Clarendon Press, Oxford University Press, New York, 1996.

[15] Neretin Yu. A. Lectures on Gaussian integral operators and classical groups. European Mathematical Society (EMS), Zürich, 2011.

[16] Neretin Yu. A. Multi-operator colligations and multivariate characteristic functions. Anal. Math. Phys. 1 (2011), no. 2-3, 121-138.

[17] Neretin Yu. A. Sphericity and multiplication of double cosets for infinite-dimensional classical groups. Funct. Anal. Appl. 45 (2011), no. 3, 225-239.
[18] Neretin Yu. A. *Multiplication of the conjugacy classes, operator colligations and characteristic functions of a matrix argument*. Funct. Anal. Appl. 51 (2017), no. 2, 98-111.

[19] Nessonov N. I. *Factor representations of the group GL(∞) and admissible representations of GL(∞)^X*. (Russian) Mat. Fiz. Anal. Geom. 10 (2003), no. 4, 524-556.

[20] Ol’shanskiĭ G. I. *Unitary representations of infinite-dimensional pairs (G,K) and the formalism of R. Howe*. in Representation of Lie groups and related topics, 269-463, Gordon and Breach, New York, 1990.

[21] Pjateckii-Shapiro I. I. *Geometry of classical domains and theory of automorphic functions* (Russian) Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow 1961; English version *Automorphic functions and the geometry of classical domains*. Gordon and Breach, New York-London-Paris, 1969.

[22] Potapov V.P. *The multiplicative structure of J-contractive matrix functions*. Trudy Moskov. Mat. Obshch., 4 (1955), 125-236; English transl.: Amer. Math. Soc. Transl. (2), 15 (1960), 131-243.

[23] Sz.-Nagy B., Foiaş, C. *Analyse harmonique des opérateurs de l'espace de Hilbert*. (French) Akadémiai Kiadó, Budapest, 1967.

[24] Zhelobenko D.P. *Compact Lie groups and their representations*. American Mathematical Society, Providence, R.I., 1973.

Math. Dept., University of Vienna
&Institute for Theoretical and Experimental Physics (Moscow);
&MechMath Dept., Moscow State University;
&Institute for Information Transmission Problems;
URL: [http://mat.univie.ac.at/~neretin/](http://mat.univie.ac.at/~neretin/)