Calculation of QCD jet cross sections at next-to-leading order

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Abstract
A general method for calculating next-to-leading order cross sections in perturbative QCD is presented. The algorithm is worked out for calculating $N$-jet cross sections in hadron-hadron collisions. The generalization of the scheme to performing calculations for other QCD processes, such as electron-positron annihilation or in deep inelastic scattering is also straightforward. As an illustration several three-jet cross section distributions in electron-positron annihilation, calculated using the algorithm, are presented.

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1 Introduction

It is well known that owing to large scale dependence any result of a leading order calculation in perturbative QCD can be regarded only as a one parameter fit to the data, but not a real theoretical prediction. The scale dependence is expected to decrease substantially with the inclusion of next-to-leading order corrections, which in a sense fixes the scale. Consequently, the calculation of cross sections at next-to-leading order in perturbative QCD is highly desirable. During the last sixteen years a good number of such calculations have been performed.

A common feature of these calculations is that they use the specific, usually simple kinematics of the problem in order to achieve the analytic cancelation of infrared singularities. This specialization brings an element of art into next-to-leading order calculations, which for more complex problems like the calculation of four-jet production in $e^+e^-$ annihilation or three-jet production in hadron-hadron collisions becomes an obstacle to overcome. One would rather like to apply a sort of “standard technique” in order to discriminate the problems of book-keeping from those of the theory. The first steps into the direction of drawing a standard picture were done by Ellis, Kunszt and Soper in refs. [1, 2] and by Giele, Glover and Kosower in refs. [3, 4]. The common characteristics of these works is the recognition that the factorization properties of the QCD amplitudes or squared matrix elements [5] as well as that of the phase space in the limits when one particle becomes soft, or an external pair becomes collinear can be utilized for devising a universal scheme for the calculation of any infrared safe physical cross section at next-to-leading order in perturbative QCD.

The algorithm developed in refs. [1, 2] — called the subtraction method — was applied for calculating inclusive one-jet [7] as well as for two-jet [8] production at next-to-leading order. The same algorithm however, cannot be directly applied to more complex cases — like the ones mentioned above —, because the evaluation of certain integrals used the specific $2 \rightarrow 2$ kinematics of the problem considered and also because the algorithm relies on single singular decomposition of the squared matrix element. The latter is not a problem in principle, however, the last decade has proved that one has to use helicity amplitudes both at tree (see eg. [9]) as well as at loop level [10, 11, 12] in order to obtain higher order results and it is rather cumbersome to square these amplitudes analytically and perform the single singular factoring of the squared matrix elements. The subtraction method has been generalized to the calculation of three-jet cross sections in hadron collisions in a recent paper by Frixione, Kunszt and Signer [13]. In this paper however, the physical quantity — the “measurement function” — was used for coping with the problem of single singular factoring, which makes the generalization to different types of cross sections than the one discussed in the paper non-trivial.

The algorithm of refs. [3, 4] — called the slicing method — avoid the above obstacles offering a general scheme, but at the price of introducing an unphysical parameter $s_{\text{min}}$ and calculating the result to $O(s_{\text{min}})$ accuracy. In principle $s_{\text{min}}$ can be chosen infinitesimal, thus an exact result can be recovered. However, in practice the choice of a very small $s_{\text{min}}$ adversely affects the numerical convergence of the Monte Carlo integrals and one has to carry out a balancing procedure between the error of the Monte Carlo integration and the one introduced by the choice of finite $s_{\text{min}}$ in order to minimize the theoretical error.

\footnote{More recently an elegant scheme has been outlined by Catani and Seymour in ref. [6].}
This balancing procedure can be inconvenient in those cases when the matrix elements are complicated and their numerical evaluation is time consuming.

The aim of the present paper is to provide a simple generalization of the subtraction method that can be used for calculating any infrared safe physical quantity at next-to-leading order in perturbative QCD if the required tree and one-loop level helicity amplitudes are known. In order to minimize the theoretical error of the Monte Carlo integrals, we apply important sampling. There are two ways of achieving efficient important sampling. One is when the integrand is decomposed into single singular factors and the important sampling is performed in the variable controlling the singularity. As stated above, however, single singular factoring is better avoided. The second possibility, that we apply in this paper, is a decomposition of the phase space into regions, where the integrand can become singular due to the vanishing of only one Lorentz invariant of the external momenta. This decomposition can be done quite generally, without any reference to the squared matrix element, or to the physical quantity being calculated. We describe the algorithm in detail for the case of hadron collisions, which is the most general case one can encounter. Algorithms for other processes can be obtained by leaving out certain terms as it will be explained later.

In section 2, we discuss how infrared safe cross sections can be calculated in perturbative QCD. In the following sections we describe the cancelation scheme in detail. The scheme is based upon the soft and collinear factorization properties of the squared matrix elements of QCD and that of the phase space. The singularity structure of the one-loop amplitudes for QCD processes involving arbitrary number of external partons has been discussed in ref. [14]. Using those results it is not difficult to find the universal structure of the singularities in the next-to-leading order matrix element of the virtual corrections. That universal structure has already been given in refs. [2, 15]. For the sake of completeness as well as for setting some of the notation, we recall the necessary formulas in section 3.

Section 4 contains the essence of our algorithm. Here we discuss the decomposition of the phase space, the singularity structure of the real corrections. We describe how the phase space is generated to achieve the necessary important sampling. We define local soft and collinear subtraction terms that make the integral of the real corrections over the $N+1$ particle phase space finite. The explicit expression for this finite integral is also given. We integrate out the variables of the soft or collinear particle analytically in sections 5 and 6. We show that the remaining expression has the form of the $2 \to N$ integrals (like the Born and virtual corrections), so they can be combined and the analytic cancelation of the infrared divergences is demonstrated. The remaining finite $2 \to N$ integral is explicitly given. Section 7 contains some sample results for three-jet cross section calculation in $e^+e^-$ annihilation. We conclude in section 8. The appendix is a collection of the analytic integrals that were used in the main text for the demonstration of the cancelation of the infrared divergences.

2 Infrared safe cross sections at next-to-leading order

At order $\alpha_s^{(N+1)}$, one calculates cross sections, with infrared divergences controlled using dimensional regularization in $d = 4 - 2\varepsilon$ dimensions, for the two incoming hadrons to collide and produce either $N$ or $N+1$ final state partons. According to the factorization theorem,
the next-to-leading order infrared safe physical cross section is a sum of two integrals,

$$\sigma = I[2 \rightarrow N] + I[2 \rightarrow N + 1],$$  \hspace{1cm} (2.1)

where

$$I[2 \rightarrow N] = \sum_{a_A,a_B,a_1,\ldots,a_N} \int dx_A \hat{f}_A(a_A,x_A) \int dx_B \hat{f}_B(a_B,x_B) \times \frac{1}{2s} \int d\Gamma^{(N)}(p_1^\mu, \ldots, p_N^\mu) (|\mathcal{M}(2 \rightarrow N)|^2) S_N(p_1^\mu, \ldots, p_N^\mu)$$  \hspace{1cm} (2.2)

and

$$I[2 \rightarrow N + 1] = \sum_{a_A,a_B,a_1,\ldots,a_{N+1}} \int dx_A f_A(a_A,x_A) \int dx_B f_B(a_B,x_B) \times \frac{1}{2s} \int d\Gamma^{(N+1)}(p_1^\mu, \ldots, p_{N+1}^\mu) (|\mathcal{M}(2 \rightarrow N + 1)|^2) S_{N+1}(p_1^\mu, \ldots, p_{N+1}^\mu).$$  \hspace{1cm} (2.3)

In these equations $\hat{s} = x_A x_B s$. In the phase space measures,

$$d\Gamma^{(n)}(p_1^\mu, \ldots, p_n^\mu) = \frac{1}{n!} \prod_{i=1}^{n} \left( \frac{\mu^{2\varepsilon}d^{d-1}P_i}{(2\pi)^{d-2}E_i} \right) (2\pi)^d \mu^{-2\varepsilon} \delta^d \left( p_A^\mu + p_B^\mu - \sum_{i=1}^{n} p_i^\mu \right)$$  \hspace{1cm} (2.4)

($n = N, N + 1$) we included the identical particle factor $1/n!$ which is present if we treat all final state partons identical and sum over all possible parton flavors $a_i = u, \bar{u}, d, \bar{d}, \ldots, g$.

The parton distribution functions for incoming partons $A$ and $B$ defined in the $\overline{\text{MS}}$ renormalization scheme are denoted by $f_A(a_A, x_A)$ and $f_B(a_B, x_B)$.

In order to factor the dependence of the cross section on the physics of low transverse momenta out of the partonic cross section and into these $\overline{\text{MS}}$ parton distributions, in the $2 \rightarrow N$ cross section one uses the modified parton distribution $\hat{f}(a, x)$ that satisfies

$$\hat{f}(a, x) = \sum_b \int \frac{dz}{z} f(b, x/z)$$  \hspace{1cm} (2.5)

$$\times \left[ \delta_{ab}\delta(1 - z) + \frac{(4\pi)^\varepsilon}{\varepsilon \Gamma(1 - \varepsilon)} \frac{\alpha_s}{2\pi} P_{a/b}(z) + \mathcal{O}(\alpha_s^2) \right],$$

with $P_{a/b}(z)$ being the full Altarelli-Parisi kernel, for the $b \rightarrow a$ splitting:

$$P_{a/b}(z) = \hat{P}_{a/b}(z) - \delta_{ab} \frac{2C(a)}{1 - z} + \delta_{ab} \frac{2C(a)}{(1 - z)^+} + \delta_{ab} \gamma(a) \delta(1 - z),$$  \hspace{1cm} (2.6)

where, for instance, in the case of $g \rightarrow gg$ splitting

$$\hat{P}_{g/g}(z) = 2C(g) \left( \frac{1 - z}{z} + \frac{z}{1 - z} + z(1 - z) \right).$$  \hspace{1cm} (2.7)

$C(g) = N_c$ is the color charge of the gluon, while $C(q) = V/(2N_c)$ is that of the quark ($V = (N_c^2 - 1)$), while the $\gamma$ constants represent the contribution of the virtual graphs to the Altarelli-Parisi kernel,

$$\gamma(g) = \frac{11N_c - 2N_f}{6}, \hspace{1cm} \gamma(q) = \frac{3V}{4N_c}. \hspace{1cm} (2.8)$$
The notation $1/(1 - z)_+$ is the usual “+” prescription,

$$\int_0^1 \frac{f(z)}{(1 - z)_+} = \int_0^1 \frac{f(z) - f(1)}{1 - z}.$$  \hspace{1cm} (2.9)

This factorization recipe is discussed in ref. \[16\]. The $\langle |M(2 \to n)|^2 \rangle$ functions are the $2 \to n$ squared matrix elements averaged over initial state and summed over final state spins and colors:

$$\langle |M(a + b \to n)|^2 \rangle = \frac{1}{\omega(a)\omega(b)} \sum_{(\text{spin color})} |M(a + b \to n)|^2.$$  \hspace{1cm} (2.10)

In the conventional $\overline{\text{MS}}$ scheme, we need $\langle |M|^2 \rangle$ in $d = 4 - 2\varepsilon$ dimensions. However, it was shown in ref. \[15\] that simple rules exist which tell us how to obtain the finite $2 \to N$ hard scattering cross section of the conventional $\overline{\text{MS}}$ scheme at next-to-leading order using the expressions for $\langle |M(2 \to n)|^2 \rangle$ obtained in the dimensional reduction scheme. Therefore, we use $\omega(g) = 2V$ and $\omega(q) = 2N_c$, which are valid in $d = 4$ dimensions and the four-dimensional expressions for the squared matrix elements. Finally, the functions $S_n$ define the physical quantity to be calculated.

In equation (2.1) both terms are singular when the regularization is removed, $\varepsilon \to 0$. When $\varepsilon \neq 0$ the singularities are represented as $1/\varepsilon^2$ and $1/\varepsilon$ poles. These poles cancel between the $I[2 \to N]$ and $I[2 \to N + 1]$ terms, provided the physical measurement, represented by the functions $S_n$, is infrared safe. This means that the emission of a soft or a collinear parton must not influence the result of the measurement. Therefore, the measurement functions $S_n$ must possess the following properties:

$$\lim_{p_i^\mu \to 0} S_{N+1}(p_1^\mu, \ldots, p_{N+1}^\mu) = S_N(p_1^\mu, \ldots, \nu^\mu, \ldots, p_{N+1}^\mu), \quad i \in [1, N + 1];$$  \hspace{1cm} (2.11)

$$\lim_{p_i^\mu \to z p_j^\mu} S_{N+1}(p_1^\mu, \ldots, p_{N+1}^\mu) = S_N(p_1^\mu, \ldots, \nu^\mu, p_j^\mu, \ldots, \nu^\mu, \ldots, p_{N+1}^\mu),$$  \hspace{1cm} (2.12)

$$\lim_{p_i^\mu \to (1-z)p_{N+1}^\mu} S_{N+1}(p_1^\mu, \ldots, p_{N+1}^\mu) = S_N(p_1^\mu, \ldots, \nu^\mu, \ldots, \nu^\mu, \ldots, p_{N+1}^\mu),$$  \hspace{1cm} (2.12)

$$m = A, B, \quad j \in [1, N + 1], \quad z \in [0, 1].$$

3 The $2 \to N$ integral

In this section, our aim is to write the $I[2 \to N]$ integral in such a form that will make the cancelation of divergent pieces against corresponding divergent terms in the $I[2 \to N + 1]$ integral as simple as possible. The discussion is a generalization of the corresponding discussion in ref. \[2\] given for the $I[2 \to 2]$ integral to the $2 \to N$ case. There are some differences however. Firstly, we do not specify the integration variables, but leave it for the reader to use a preferred choice. Secondly, in order we could use the results for the five-parton
one-loop QCD helicity amplitudes \[10, 11, 12\] for a next-to-leading order calculation of three-jet production in hadron collisions, we perform the analysis using dimensional reduction scheme and add the necessary transition terms at the end to obtain the correct formula in conventional $\overline{\text{MS}}$ scheme as they are given in ref. \[15\].

In order to simplify the book-keeping of the various factors, we introduce the integration measure
\[
D_N(\varepsilon) = \frac{1}{2s} \frac{\alpha_s^N}{N!} \frac{1}{(2\pi)^{2N-4}} \, dx_A \, dx_B \, \prod_{i=1}^{N} \left[ (2\mu)^{2\varepsilon} \, d^{4-2\varepsilon} p_i \, 2\delta(p_i^2) \right],
\]
(3.1)

where the subscript on $D$ reminds us that this measure is related to the phase space integration measure of $N$ particles according to the relation
\[
\frac{(4\pi\alpha_s)^N}{2s} \, dx_A \, dx_B \, d\Gamma^{(N)}(p_1^{\mu}, \ldots, p_N^{\mu}) = D_N(\varepsilon)(2\mu)^{-2\varepsilon} \delta^d \left( p_A^{\mu} + p_B^{\mu} - \sum_{i=1}^{N} p_i^{\mu} \right). \quad (3.2)
\]

It is convenient to write the perturbative expansion of the squared matrix element summed over final spins and colors and averaged over initial spins and colors and with ultraviolet renormalization in the $\overline{\text{MS}}$ renormalization scheme included in terms of functions $\Psi^{(2N)}_{\text{DR}}(\vec{p})$ and $\Psi^{(2N+2)}_{\text{DR}}(\vec{p})$,
\[
\langle |M(2 \rightarrow N)|^2 \rangle = \frac{(4\pi\alpha_s)^N}{\omega(a_A)\omega(a_B)} \left\{ \Psi^{(2N)}_{\text{DR}}(\vec{a}, \vec{p}) + \frac{\alpha_s}{2\pi} c_T \left( \frac{\mu^2}{Q_{\text{ES}}^2} \right)^\varepsilon \Psi^{(2N+2)}_{\text{DR}}(\vec{a}, \vec{p}) + O \left( \left( \frac{\alpha_s}{2\pi} \right)^2 \right) \right\},
\]
(3.3)

where
\[
c_T = (4\pi)^\varepsilon \frac{\Gamma^2(1-\varepsilon)\Gamma(1+\varepsilon)}{\Gamma(1-2\varepsilon)}, \quad (3.4)
\]

and the subscript DR refers to expressions obtained using dimensional reduction. We use the notation $\vec{a} = (a_A, a_B, a_1, \ldots, a_N)$ for the collection of external parton flavors and $\vec{p} = (p_A^{\mu}, p_B^{\mu}, p_1^{\mu}, \ldots, p_N^{\mu})$ for the collection of external four-momenta. The variable $Q_{\text{ES}}$ is an arbitrary parameter of mass dimensions introduced to facilitate writing the result \[17\]. The dependence of the function $\Psi^{(2N+2)}_{\text{DR}}$ on $Q_{\text{ES}}$ is such that the squared matrix element does not actually depend on $Q_{\text{ES}}$. The first term in the curly braces is the Born $2 \rightarrow N$ matrix element squared, without the $g^2N$ coupling factor, while the second term is the next-to-leading order contribution, which in the dimensional reduction scheme can be expressed in terms of the helicity amplitudes \[17\],
\[
(4\pi\alpha_s)^N \frac{\alpha_s}{2\pi} c_T \left( \frac{\mu^2}{Q_{\text{ES}}^2} \right)^\varepsilon \Psi^{(2N+2)}_{\text{DR}}(\vec{a}, \vec{p}) = \sum_{\text{hel}} \sum_{\text{col}} \left( A^{(1)} A^{(0)*} + A^{(1)*} A^{(0)} \right).
\]
(3.5)

On the right hand side the superscript (0) and (1) refers to the tree and one-loop helicity amplitudes respectively. The helicity amplitudes can be decomposed in color space in terms of gauge invariant color subamplitudes. For instance, in the case of pure gluon processes,
\[
A^{(i)}(g_1, \ldots, g_N) = g^N \left( \frac{g}{4\pi} \right)^{(2i)} \sum_n C_n^{g_1 \ldots g_N} a_n^{(i)}(1, \ldots, N), \quad (3.6)
\]
where $C^{g_1\ldots g_N}$ is a general invariant matrix in color space with upper indices in the adjoint representation and summation on $n$ runs over a linearly independent set of such matrices. Explicit examples of this decomposition can be found in reference [2]. The one-loop color subamplitudes and thus the helicity amplitudes can naturally be decomposed into singular terms containing at most double poles in $\varepsilon$ and into terms that are finite when $\varepsilon \to 0$,

$$\mathcal{A}^{(1)} = \mathcal{A}^{(1)}_{S} + \mathcal{A}^{(1)}_{NS}. \quad (3.7)$$

Looking at the explicit form of the singular terms of one-loop five-parton color subamplitudes [3], we see that the imaginary parts of the factors $-1/\varepsilon^2 (-s_{ij}/Q^2_{ES})^{-\varepsilon}$ do not contribute to the function $\Psi^{(8)}_{DR}$. In such cases the $\Psi^{(N+2)}_{DR}$ functions have the following structure:

$$\Psi^{(N+2)}_{DR}(\vec{a}, \vec{p}) = \Psi^{(N)}_{DR}(\vec{a}, \vec{p}) \left\{ -\frac{1}{\varepsilon^2} \sum_{n=A,B,1,\ldots,N} C(a_n) - \frac{1}{\varepsilon} \sum_{n=A,B,1,\ldots,N} \gamma(a_n) \right\} + \frac{1}{2\varepsilon} \sum_{(m,n,A,B,1,\ldots,N)} \ell(s_{mn}) \Psi^{(2N,c)}_{mn,DR}(\vec{a}, \vec{p})$$

$$+ \Psi^{(N)}_{DR}(\vec{a}, \vec{p}) \ell(\mu^2) \sum_{n=A,B,1,\ldots,N} \gamma(a_n) - \frac{1}{4} \sum_{(m,n,A,B,1,\ldots,N)} \ell_2(s_{mn}) \Psi^{(2N,c)}_{mn,DR}(\vec{a}, \vec{p})$$

$$+ 2 \left[ (4\pi\alpha_s)^N \frac{\rho}{\alpha_s} \sum_{n=1}^{N} (2\pi\mu^2)^{1/2} \sum_{\lambda=1}^{N} \left( \mathcal{A}^{(1)}_{NS} \mathcal{A}^{(0)} + \mathcal{A}^{(1)}_{NS} \mathcal{A}^{(0)} \right) + \mathcal{O}(\varepsilon). \right]$$

Here the $\Psi^{(2N,c)}_{mn}$ functions are the color correlated Born squared matrix elements defined in ref. [2]. The factor $2 \left[ (4\pi\alpha_s)^N (\alpha_s/2\pi) \rho (\mu^2/Q^2_{ES})^\varepsilon \right]^{-1} = \left[ g^{2N} (g/4\pi)^2 \right]^{-1}$ cancels against the coupling factors in $\mathcal{A}^{(1)}_{NS} \mathcal{A}^{(0)}$. The functions $\ell(x)$ and $\ell_2(x)$ are defined as

$$\ell(x) = \ln \left| \frac{x}{Q^2_{ES}} \right|, \quad \ell_2(x) = \ell_2(x) - \pi^2 \Theta(x). \quad (3.9)$$

Substituting the integration measure of eq. (3.2), the perturbative expression for the squared matrix element, eq. (3.3) and the expression for the modified effective parton distribution functions as defined in eq. (2.5), we can write the $2 \to N$ cross section as

$$\mathcal{I}_{2 \to N}^{\text{DR}} =$$

$$\sum_{a_A, a_B, a_1, \ldots, a_N} \int D_N(\varepsilon) S_N(p^\mu_A, \ldots, p^\mu_N)(2\pi\mu)^{-2\varepsilon} \delta^d \left( p^\mu_A + p^\mu_B - \sum_{i=1}^{N} p^\mu_i \right)$$

$$\times \left\{ L(a_A, a_B, x_A, x_B) [\Psi^{(N)}_{DR}(\vec{a}, \vec{p}) + \frac{\alpha_s}{2\pi} \rho (\mu^2/Q^2_{ES})^\varepsilon \Psi^{(N+2)}_{DR}(\vec{a}, \vec{p})] \right.$$ 

$$+ \sum_{a'_A} \omega(a'_A) \int_{x_A}^1 \frac{dz}{z^2} L(a'_A, a_B, x'_A, x_B) \frac{(4\pi)^\varepsilon}{\varepsilon \Gamma(1-\varepsilon)} \frac{\alpha_s}{2\pi} P_{a_A/a'_A}(z) \Psi^{(N)}_{DR}(\vec{a}, \vec{p})$$

$$+ \sum_{a'_B} \omega(a'_B) \int_{x_B}^1 \frac{dz}{z^2} L(a_A, a'_B, x_A, x'_B) \frac{(4\pi)^\varepsilon}{\varepsilon \Gamma(1-\varepsilon)} \frac{\alpha_s}{2\pi} P_{a_B/a'_B}(z) \Psi^{(N)}_{DR}(\vec{a}, \vec{p}) \right\}. \quad (3.10)$$
The function $L$ used here and elsewhere describes the parton luminosity:

$$L(a_A, a_B, x_A, x_B) = \frac{f(a_A, x_A) f(a_B, x_B)}{\omega(a_A) x_A \omega(a_B) x_B}. \quad (3.11)$$

According to the factorization and Kinoshita-Lee-Nauenberg theorems, and we shall see it at next-to-leading order explicitly in the following sections, the pole terms in eq. (3.10) cancel against poles emerging in the phase space integral of the bremsstrahlung contributions. Therefore, it is only the Born function $\Psi^{(2N)}$ and the finite part of the $\Psi^{(2N+2)}$ function — the last two lines of eq. (3.8) — that is really integrated in eq. (3.10). However, we need the corresponding finite expressions valid in conventional dimensional regularization. In ref. [15] it was shown that simple terms are to be added in order to obtain the correct formula we need for a next-to-leading order calculation in the conventional $\overline{\text{MS}}$ scheme. Thus the function resulting from the loop corrections that we need for a next-to-leading order Monte Carlo program is the non-singular function

$$\Psi^{(2N+2)}_{NS}(\vec{a}, \vec{p}) = \Psi^{(2N)}_{DR}(\vec{a}, \vec{p}) \left[ \sum_{n=A,B,1,...,N} \left[ \ell(\mu^2)\gamma(a_n) - \tilde{\gamma}(a_n) \right] + N \frac{N_c}{6} \right] + \frac{1}{4} \sum_{m,n=A,B,1,...,N} \ell_2(s_{mn}) \Psi^{(2N;c)}_{mn, DR}(\vec{a}, \vec{p})$$

$$+ \left[ g^{2N} \left( \frac{g}{4\pi} \right)^2 \right]^{-1} \frac{1}{2} \sum_{\text{hel}} \sum_{\text{col}} (A_{NS}^{(1)} A^{(0)*} + A_{NS}^{(1)*} A^{(0)}) ,$$

where the transition terms $\tilde{\gamma}(a_n)$ are given by

$$\tilde{\gamma}(g) = \frac{1}{6} C(g), \quad \tilde{\gamma}(q) = \frac{1}{2} C(q). \quad (3.13)$$

4 The $2 \to N+1$ integral

In this section, we separate the $I[2 \to N+1]$ integral into terms containing $1/\varepsilon^2$ and $1/\varepsilon$ poles, which cancel against the corresponding poles of the $I[2 \to N]$ integral, and terms that are finite when $\varepsilon \to 0$ and, therefore, can be integrated numerically. The $1/\varepsilon^p$ singularities arise from integrating the square of the matrix element over the $(N+1)$-particle phase space when a gluon becomes soft, or two partons become collinear. Firstly, we organize the integration domain so as to reduce the complexity of the problem.

4.1 The domain of integration

We must integrate over the momenta of the $N+1$ final state particles treating them identical. If we do not fix a definite label to each particle, then we integrate over each event topology $(N+1)!$ times. We can however, simplify the calculation by

1. first splitting the phase space in the parton-parton c.m. system into two parts: in the first one, the smallest angular distance $r_{ij} = s_{ij}/(E_i E_j)$ is between final state particles, while in the second region the smallest angular distance is between an initial state particle and a final state particle;
2. secondly cutting into the first region by fixing that label of the smallest Lorentz invariant of final state particle pairs to which the smaller energy in the parton-parton c.m. system belongs to be $j = N + 1$, and cutting into the second region by fixing the final state label of the smallest Lorentz invariant of pairs involving an initial state and a final state particle to be $j = N + 1$.

With this distinction of parton $(N + 1)$, we have to integrate over each event topology only $N!$ times and there is a corresponding symmetry factor $1/N!$ associated with the integration:

$$d\Gamma^{(N+1)}(p_1^\mu, \ldots, p_{N+1}^\mu) = \frac{1}{N!} \prod_{i=1}^{N+1} \left( \frac{\mu^2 \delta - d - 1}{(2\pi)^d-1} \right) (2\pi)^d \mu^{-2\varepsilon} \delta \left( p_A^\mu + p_B^\mu - \sum_{i=1}^{N+1} p_i^\mu \right) (\text{4.1})$$

$$\times \left[ \Theta(r_{\text{min}}^{(N+1;i)} > r_{\text{min}}^{(N+1;i)}) \Theta(s_{\text{min}}^{(N;i)} > s_{k,N+1}^{\text{min}}) \Theta(E_k > E_{N+1}) \right]$$

$$+ \left[ \Theta(r_{\text{min}}^{(N+1;i)} > r_{\text{min}}^{(N+1;i)}) \Theta(s_{\text{min}}^{(N;i)} > s_{X,N+1}^{\text{min}}) \right],$$

where

$$d_{\text{min}}^{(n,f)} = \text{min}(d_{ij} : i, j = 1, \ldots, n, i \neq j), \quad (4.2)$$
$$d_{\text{min}}^{(n,i)} = \text{min}(d_{Xj} : X = A, B, j = 1, \ldots, n), \quad (4.3)$$
$$d_{k,N+1}^{\text{min}} = \text{min}(d_{j,N+1} : j = 1, \ldots, N), \quad (4.4)$$
$$d_{X,N+1}^{\text{min}} = \text{min}(d_{A,N+1}, d_{B,N+1}), \quad (4.5)$$

with $d$ meaning either Lorentz invariant $s$ or angular distance $r$. In eq. (4.4) the index $k$ denotes that $j$ for which the minimum value is assumed.

In the cut phase space the only singularities that can occur when a single Lorentz invariant vanishes are

- parton $N + 1$ is soft (in both regions);
- parton $N + 1$ is collinear to a final state parton $i \in [1, N]$ (in the first region);
- parton $N + 1$ is collinear to an initial state parton A or B (in the second region).

We are not interested in configurations when two Lorentz invariants involving four different labels vanish simultaneously, because those emerge only in $(N-1)$-jet configurations that are not considered here.

Next, we study the singularity structure of the squared matrix element.

### 4.2 Singularity structure of the squared matrix element

In order to find the singularity structure of $\mathcal{I}[2 \to N + 1]$ over the cut phase space of eq. (4.1) explicitly, it is useful to strip off the spin and color averaging and the coupling of the squared matrix element. We define the function

$$\psi^{(2N+2)}(A, B, 1, \ldots, N + 1) = \frac{\omega(a_A)\omega(a_B)}{(4\pi\alpha_s)^{N+1}} \langle |M(2 \to N + 1)|^2 \rangle,$$  

(4.6)
where the argument denotes:

\[(A, B, 1, \ldots, N + 1) \equiv (a_A, a_B, a_1, \ldots, a_{N+1}; p_A^\mu, p_B^\mu, p_1^\mu, \ldots, p_{N+1}^\mu). \tag{4.7}\]

The singularity structure of the function $\Psi^{(2N+2)}$ in four dimensions can most easily be obtained from the factorization properties of helicity amplitudes \[9\]. Citing only the results, we find for soft gluon labeled $j = N + 1$

\[
\lim_{p_j^\mu \to 0} \Psi^{(2N+2)}(A, B, 1, \ldots, N + 1) = \sum_{(m, n = A, B, 1, \ldots, N)} \delta_{a_j g} \frac{2s_{mn}}{s_{mj}s_{jn}} \Psi^{(2N;c)}_{mn}(\vec{a}, \vec{p}) + O\left(\frac{1}{\sqrt{s_{mj}}}, \frac{1}{\sqrt{s_{nj}}}\right), \tag{4.8}\]

which is called “soft identity” in ref. [2]. In order to make the cancelation of the infrared singularities as transparent as possible, it is useful to perform single singular decomposition of the eikonal factor in eq. (4.8):

\[
\frac{2s_{mn}}{s_{mj}s_{jn}} = \frac{2s_{mn}}{s_{mj}(s_{mj} + s_{nj})} + \frac{2s_{mn}}{s_{nj}(s_{mj} + s_{nj})}. \tag{4.9}\]

With this decomposition and using the symmetry of the $\Psi^{(2N;c)}_{mn}$ functions in the $m, n$ indices we can write eq. (4.8) in the form

\[
\lim_{p_j^\mu \to 0} \Psi^{(2N+2)}(A, B, 1, \ldots, N + 1) = \sum_{(m, n = A, B, 1, \ldots, N)} \delta_{a_j g} \Psi^{(2N+2)}_{S, mn}(\vec{a}, \vec{p}, p_j^\mu), \tag{4.10}\]

where

\[
\Psi^{(2N+2)}_{S, mn}(\vec{a}, \vec{p}, p_j^\mu) = \frac{2s_{mn}}{s_{mj}(s_{mj} + s_{nj})} \Psi^{(2N;c)}_{mn}(\vec{a}, \vec{p}). \tag{4.11}\]

In the collinear limit of two final state partons $i$ and $j = N + 1$, we introduce a pseudo particle $P$ with $a_P$ flavor that splits into gluons $i$ and $j$: $p_i^\mu = p_i^\mu + p_j^\mu$. The flavor $a_P = a_i$ if $a_j = g$ and $a_P = g$ if $a_i = q$, $a_j = \bar{q}$. The momentum fraction $z$ is defined by $p_i^\mu = zp_P^\mu$. Then for the collinear limit of $\Psi^{(2N+2)}$ one finds:

\[
\lim_{\substack{p_i^\mu \to zp_P^\mu \\
(p_j^\mu \to (1-z)p_P^\mu)}} \Psi^{(2N+2)}(A, B, 1, \ldots, N + 1) \tag{4.12}
\]

\[
= \frac{2}{s_{ij}} \Psi^{(2N+2)}_{C, ij}(z, P; A, B, 1, \ldots, N + 1) + O\left(1/\sqrt{s_{ij}}\right),
\]

where

\[
\Psi^{(2N+2)}_{C, ij}(z, P; A, B, 1, \ldots, N + 1) \tag{4.13}
\]

\[
= \overline{P}_{a_i/a_P}(z)\Psi^{(2N)}(A, B, \ldots, \mp P, \ldots, \mp \ldots) + 2\Re \left(Q_{P \to ij}(z)\Phi^{(2N)}(P; A, B, \ldots, \mp \ldots, \mp \ldots)\right).
\]

In this equation $\Psi^{(2N)}(A, B, \ldots, \mp P, \ldots, \mp \ldots)$ is the $\Psi^{(2N)}$ function of $2 + N$ partons obtained from $\Psi^{(2N+2)}(2 \to N + 1)$ by deleting labels $i$ and $j$ and adding the pseudo particle
label $P$, \( \tilde{P}_{a/p}(z) \) is the Altarelli-Parisi splitting function for the process $P \to ij$ in four dimensions without $z = 1$ regulation (eq. (2.7) in the case of gluon splitting). The $Q_{P \to ij}(z)$ functions are calculated from the tree-level splitting amplitudes, \( \text{Split}^\text{tree}_{ij}(i^h, j^h) \) of ref. [19] according to the formula

\[
\frac{2}{s_{ij}} Q_{P \to ij}(z) = \sum_{i,j} \sum_{h_i, h_j = \pm} c(i, j, P)c(i, j, P)^* \text{Split}^\text{tree}_{ij}(i^h, j^h) \text{Split}^\text{tree}_{ij}(i^h, j^h)^* \tag{4.14}
\]

where $c(i, j, P)$ is the color matrix of the $P \to ij$ vertex. In the case of gluon splitting

\[
Q_{g \to gq}(z) = -2C(g)z(1 - z)^{\frac{\langle ij \rangle}{[ij]}}, \quad Q_{g \to q\bar{q}}(z) = z(1 - z)^{\frac{\langle ij \rangle}{[ij]}}, \tag{4.15}
\]

while in the case of quark splitting $Q_{q \to ij}(z) = 0$, which is also understood from helicity conservation along fermion lines. The function $\Phi^{(2N)}$ does not depend on the momenta $p^\mu_i$ and $p^\mu_j$ only on their sum, $p^\mu_p$. The $\Phi^{(2N)}$ functions are calculated from the tree-level helicity amplitudes as the Born function $\Psi^{(2N)}$, except that the summation over the helicity of parton $P$ is not carried out:

\[
\Phi^{(2N)}(P; A, B, \ldots, \bar{X}, \ldots, \bar{X} \ldots) = \sum_{\text{color}} \sum_{h_A, h_B, \ldots} \mathcal{A}^{(0)}(A^{h_A}, B^{h_B}, \ldots, \bar{X}P^+, \ldots, \bar{X} \ldots) \times \mathcal{A}^{(0)}(A^{h_A}, B^{h_B}, \ldots, \bar{X}P^-, \ldots, \bar{X} \ldots)^* \tag{4.16}
\]

In the collinear limit of a final state parton $j = N + 1$ with an initial state parton $A$, we let $A$ split into partons $P$ and $j$: $p^\mu_A = p^\mu_P + p^\mu_j$, with momentum fraction $z$ defined as $p^\mu_P = zp^\mu_A$, followed by an $P + B \to 1, \ldots, N$ hard-scattering process. From the crossing of $A$ and $P$ in relation (1.13), for the collinear limit of $\Psi^{(2N+2)}$ one obtains:

\[
\lim_{p^\mu_j \to (1 - z)p^\mu_A} \Psi^{(2N+2)}(A, B, 1, \ldots, N + 1) = \frac{-2}{s_{A,j}} (-1)^{f(a_A) + f(a_P)} \Psi^{(2N+2)}_{C, A,j}(1/z; P; A, B, 1, \ldots, N + 1) + O\left(1/\sqrt{s_{A,j}}\right), \tag{4.17}
\]

where

\[
f(g) = 0, \quad f(q) = 1. \tag{4.18}
\]

We can write the right hand side of eq. (4.17) in a more explicit form using the crossing relation of the Altarelli-Parisi splitting functions,

\[
\tilde{P}_{b/a}(z) = -(-1)^{f(a_A) + f(b)} \frac{\omega(b)}{\omega(a)} z \tilde{P}_{a/b}(1/z), \tag{4.19}
\]

where $\bar{a}$ is the antiparticle of particle $a$. Thus we find

\[
-(-1)^{f(a_A) + f(a_P)} \Psi^{(2N+2)}_{C, A,j}(1/z; P; A, B, 1, \ldots, N + 1) = \frac{\omega(a_A)}{\omega(a_P)} \frac{1}{z} \tilde{P}_{a_P/a_A}(z) \Psi^{(2N)}(P, B, \ldots, \bar{X} \ldots) - \frac{-2}{s_{A,j}} (-1)^{f(a_A) + f(a_P)} 2 \text{Re} \left( Q_{P \to A,j}(1/z) \Phi^{(2N)}(P; B, \ldots, \bar{X} \ldots) \right). \tag{4.20}
\]
We close the analysis of the singularity structure of $\Psi^{(2N+2)}$ with considering the limit when the soft gluon is also collinear with parton $m$. Thus we take $p_j^\mu = (1-z)p_m^\mu$ with $z \to 1$. Using the “soft-collinear” identity of ref. [2],

$$\sum_{(n=A,B,1,...,N) \atop n \neq m} \Psi^{(2N+2)}_{mn}(\vec{p}) = 2C(a_{m})\Psi^{(2N)}(A, B, 1, \ldots, N), \quad (4.21)$$

we obtain from eq. (4.11)

$$\lim_{p_j^\mu \to (1-z)p_m^\mu} \sum_{(n=A,B,1,...,N) \atop n \neq m} \delta_{a_j g} \Psi^{(2N+2)}_{S;mn}(\vec{p}) = 2C(a_{m})\delta_{a_j g} \frac{2}{(1-z)s_{mj}} \Psi^{(2N)}(\vec{p}). \quad (4.22)$$

In the same limit, eqs. (4.13) and (4.20) yield

$$\lim_{z \to 1} \frac{2}{s_{mj}} \Psi^{(2N+2)}_{C;mn}(z, P; A, B, 1, \ldots, N + 1) = 2C(a_{m})\delta_{a_j g} \frac{2}{(1-z)s_{mj}} \Psi^{(2N)}(\vec{p}) \quad (4.23)$$

and

$$\lim_{z \to 1} \frac{2}{s_{Aj}} \Psi^{(2N+2)}_{C;mn}(1/z, P; A, B, 1, \ldots, N + 1) = 2C(a_{A})\delta_{a_j g} \frac{2}{(1-z)s_{Aj}} \Psi^{(2N)}(\vec{p}), \quad (4.24)$$

in agreement with eq. (4.22).

The function $\Psi^{(2N+2)}$ does not posses any other poles when parton $N + 1$ is soft or collinear to another parton. Knowing the singularity structure of the function $\Psi^{(2N+2)}$, we can decompose the integral $I[2 \to N + 1]$ into three terms:

$$I[2 \to N + 1] = I[soft] + I[coll] + I[fin]. \quad (4.25)$$

The first two of these integrals possess divergences in $\varepsilon$, but they are sufficiently simple to calculate the pole structure of the Laurent expansion in $\varepsilon$ around zero analytically. The third one is complicated, but contains at most square-root singularities over the modified $(N + 1)$-particle phase space of eq. (4.1), therefore, can be integrated numerically yielding a finite contribution as $\varepsilon \to 0$. We further decompose the “soft” and “collinear” contributions into sums of $N + 2$ terms,

$$I[soft] = \sum_{m=A,B,1,...,N} I[soft]_m, \quad (4.26)$$

$$I[coll] = \sum_{m=A,B,1,...,N} I[coll]_m, \quad (4.27)$$

where $I[soft]_m$ is associated with the integral of the soft terms $\Psi^{(2N+2)}_{S;mn}$, while $I[coll]_m$ is associated with the integral of the collinear term $\Psi^{(2N+2)}_{C;mn,N+1}$. We shall call these integrals subtraction terms for the obvious reason that subtracting them from the $I[2 \to N + 1]$ integral the finite term remains. In order to define the soft and collinear subtraction terms precisely, we first give a decomposition of the phase space into such regions that in any one of them only one pair of external particles can become collinear.
4.3 Decomposition of the phase space integral

In this subsection, our goal is to write the phase space integral in those variables that allow the most efficient Monte Carlo integration. We write the integration measure of eq. (1.1) as

$$\frac{(4\pi\alpha_s)^{(N+1)}}{2s} dx_A \, dx_B \, d\Gamma^{(N+1)}(p_1^\mu, \ldots, p_{N+1}^\mu) = \frac{\alpha_s}{(2\pi)^2} \left[ \sum_{m_f=1}^{N} D_{N+1}^{n,m_f}(\varepsilon) + \sum_{m_i=A,B} D_{N+1}^{ini;m_i}(\varepsilon) \right],$$

(4.28)

where

$$D_{N+1}^{n;m}(\varepsilon) = D_N(\varepsilon)(2\pi\mu)^{2\varepsilon}d^{4-2\varepsilon}p \, 2 \, \delta(p^\mu p_\mu) \, (2\pi\mu)^{-2\varepsilon} \delta^d \left( p_A^\mu + p_B^\mu - \sum_{i=1}^{N+1} p_i^\mu \right)$$

(4.29)

$$\times \Theta(r_{\min}^{(N+1;i)} > r_{\min}^{(N+1;f)}) \Theta(s_{\min}^{(N;f)} > s_{k,N+1}^{\min}) \Theta(E_k > E_{N+1})$$

$$\times \Theta(\min(r_i,N_+ : i = 1, \ldots, N, i \neq m) > r,m,N+1),$$

(4.30)

and

$$D_{N+1}^{ini;A}(\varepsilon) = D_N(\varepsilon)(2\pi\mu)^{2\varepsilon}d^{4-2\varepsilon}p \, 2 \, \delta(p^\mu p_\mu) \, (2\pi\mu)^{-2\varepsilon} \delta^d \left( p_A^\mu + p_B^\mu - \sum_{i=1}^{N+1} p_i^\mu \right)$$

(4.31)

$$\times \Theta(r_{\min}^{(N+1;f)} > r_{\min}^{(N+1;i)}) \Theta(s_{\min}^{(N;i)} > s_{k,N+1}^{\min}) \Theta(r_{B,N+1} > r_{A,N+1}),$$

with $p^\mu \equiv p_{N+1}^\mu$ and $D_N(\varepsilon)$ is defined in equation (3.1). The measure $D_{N+1}^{ini;B}$ is defined as $D_{N+1}^{ini;A}$ with $A$ and $B$ interchanged. The advantage of this decomposition of the phase space should be clear: in each region there is only one pair of particles that can become collinear. As a result, the single singular factor decomposition of the integrand is substituted by a (much simpler) decomposition of the phase space.

In order to write the integration measure $D_{N+1}^{fin;m}$ in the required form, we utilize a vector $p_Q^\mu$ of invariant mass $Q^2$ that splits into the vectors $p_m^\mu$ and $p^\mu \equiv p_{N+1}^\mu$ and use the mathematical identity

$$D_{N+1}^{fin;m}(\varepsilon) = \int_0^s \frac{dQ^2}{2\pi} \left[ D_N(\varepsilon)(2\pi\mu)^{-2\varepsilon} \delta^d \left( p_A^\mu + p_B^\mu - \sum_{i=1}^{N} p_i^\mu \right) \right]_{m \to Q}$$

(4.32)

$$\times \frac{\mu^2 d^{d-1} P_m}{(2\pi)^{d-1} 2E_m} (2\pi\mu)^{2\varepsilon}d^{d-1} \frac{p}{E} (2\pi\mu)^{-2\varepsilon} \delta^d \left( p_Q^\mu - p_m^\mu - p^\mu \right)$$

$$\times \Theta(r_{\min}^{(N+1;f)} > r_{\min}^{(N+1;i)}) \Theta(s_{\min}^{(N;i)} > s_{k,N+1}^{\min}) \Theta(E_k > E)$$

$$\times \Theta(\min(r_i,N_+ : i = 1, \ldots, N, i \neq m) > r,m,N+1).$$

We can use the second $\delta$ function for integrating over the $d - 1$ momenta of particle $m$ and over $Q^2$. We obtain

$$\int_0^s \frac{dQ^2}{2\pi} \frac{\mu^2 d^{d-1} P_m}{(2\pi)^{d-1} 2E_m} \frac{(2\pi\mu)^{2\varepsilon}d^{d-1} P}{E} \left( p_Q^\mu - p_m^\mu - p^\mu \right) = \frac{E_m + E}{E_m} \left( 2\pi\mu \right)^{2\varepsilon} \frac{d^{d-1} P}{E}.$$ 

Next we choose a coordinate system in the parton-parton c.m. frame which has $z$ axis showing into the direction of the three-momentum $p_Q$. We denote the polar and azimuthal coordinates of parton $N + 1$ by $\vartheta$ and $\varphi$, so the four-vector $p^\mu$ in $d$ dimensions is

$$p^\mu = E(1, \cos \vartheta, \sin \vartheta \cos \varphi, \sin \vartheta \sin \varphi, \ldots), \quad 0 < \vartheta, \varphi < \pi.$$ 

(4.33)
The first four components are energy, $z$, $x$ and $y$ components of the three-momentum and the dots mean $d-4$ unspecified components. In this coordinate system

$$
(2\pi\mu)^{2\varepsilon}\frac{d^{d-1}p}{E} = E \left(\frac{2\pi\mu}{E}\right)^{2\varepsilon} dE (\sin \vartheta)^{(1-2\varepsilon)} d\vartheta d^1 \varphi
$$

(4.34)

$$
\equiv E \left(\frac{2\pi\mu}{E}\right)^{2\varepsilon} dE (\sin \vartheta)^{(1-2\varepsilon)} d\vartheta (\sin \varphi)^{-2\varepsilon} d\varphi d^2\Omega
$$

with $\Omega$ being the solid angle in $d-4$ dimensions. We change integration variables from $(E, \cos \vartheta)$ to $(z, \cos \omega)$, where $z = E_m/(E_m + E)$, so

$$
E = \frac{1 - z}{z} E_m
$$

(4.35)

and $\omega$ is the angle between the three-momenta $p_m$ and $p$ in this coordinate system,

$$
\cos \vartheta = \frac{1 - z + z \cos \omega}{\rho},
$$

(4.36)

with $\rho$ being the ratio $p_Q/(E_m + E)$ that can be expressed in terms of $z$ and $\cos \omega$ as

$$
\rho = \sqrt{1 - 2z(1 - z)(1 - \cos \omega)}.
$$

(4.37)

In the collinear limit of particles $m$ and $N+1$, i.e. $\omega \to 0$, the definition of $z$ given here is identical to the one given in the previous subsection and therefore, extends this variable to non-collinear configurations naturally. We remark that this definition of $z$ is not boost invariant.

Finally, we change variable from $\omega$ to $Q^2 = s_{m,N+1}$. The necessary relation is

$$
\cos \omega = 1 - \frac{Q^2}{2z(1 - z)E_Q^2}
$$

(4.38)

with $E_Q^2$ being dependent on $Q^2$. We obtain

$$
\mathcal{D}_{N+1}^{m;N}(\varepsilon) = \left[ \mathcal{D}_N(\varepsilon) \delta^{(4)}(p_A^\mu + p_B^\mu - \sum_{i=1}^{N} p_i^\mu) \right]_{m \to Q}
$$

(4.39)

$$
\times \frac{1}{2\rho^{2-4\varepsilon}} \left( \frac{E_Q^2}{p_Q^2} \right)^{2\varepsilon} \left[ 2 - \frac{Q^2}{2z(1 - z)E_Q^2} \right]^{-\varepsilon} dz
$$

$$
\times (\sin \varphi)^{-2\varepsilon} d\varphi d^2\Omega \Theta(4z(1 - z)E_Q^2 > Q^2)
$$

$$
\times \Theta(r_{m;N+1}^{(N+1;i)} > r_{m;N+1}^{(N+1;f)}) \Theta(s_{m;N+1}^{(N;i)} > s_{m;N+1}^{(N;i)}) \Theta(E_k > E_{N+1})
$$

$$
\times \Theta(\min(r_{i,N+1} : i = 1, \ldots, N, i \neq m) > r_{m,N+1}), \quad \varphi \in [0, \pi].
$$
In this equation the \( p^\mu_m = p^\mu_Q - p^\mu \) momentum conservation constraint is implicitly understood. We shall use the four dimensional limit of this integration measure:

\[
D_{N+1}^{\text{fin};m}(\varepsilon = 0) = \left[ D_N(\varepsilon = 0) \delta^{(4)} \left( p^\mu_A + p^\mu_B - \sum_{i=1}^{N} p^\mu_i \right) \right]_{m \to Q} \tag{4.40}
\]

\[
\times \frac{1}{2p^\mu} \left( \frac{p^2_Q}{E_Q} \right)^2 \text{d}z \text{d}Q^2 \text{d}\varphi \Theta(4z(1-z)E_Q^2 > Q^2) \times \Theta(r_{\text{min}}^{(N+1);i} > r_{\text{min}}^{(N+1);f}) \Theta(s_{\text{min}}^{(N);i} > s_{\text{min}}^{k,N+1}) \Theta(E_k > E_{N+1}) \\
\times \Theta(\min(r_{i,N+1} : i = 1, \ldots, N, i \neq m) > r_{m,N+1}), \quad \varphi \in [-\pi, \pi].
\]

We now turn to the discussion of the integration measure \( D_{N+1}^{\text{ini};A} \). In order to write it in the required form, we imagine a \( 2 \to 2 \) scattering, \( A + B \to Q + (N+1) \), followed by the decay of particle \( Q \) into \( N \) particles. We write the integration measure \( D_{N+1}^{\text{ini};A} \) in the form

\[
D_{N+1}^{\text{ini};A}(\varepsilon) = D_N(\varepsilon) \delta^d \left( p^\mu_Q - \sum_{i=1}^{N} p^\mu_i \right) \bigg|_{p^\mu_Q = p^\mu_A + p^\mu_B - p^\mu} (2\pi\mu)^{2}\text{d}^4\varepsilon \text{d}^2p \delta(p^\mu p_\mu) \tag{4.41}
\]

\[
\times \Theta(r_{\text{min}}^{(N+1);f} > r_{\text{min}}^{(N+1);i}) \Theta(s_{\text{min}}^{(N);i} > s_{\text{min}}^{X,N+1}) \Theta(r_{B,N+1} > r_{A,N+1}).
\]

For the invariant measure of particle \( N + 1 \) we use the the variables \( \xi \) and \( W \) (introduced in ref. [1]) that are defined in the hadron-hadron c.m. frame such that the four-momentum of particle \( N + 1 \) in light cone coordinates \( (p^\mu = (p^+, p^-, p^z, p^+)\bigg/\sqrt{2}) \) can be written as

\[
\xi = \left( \xi \sqrt{\frac{s}{2}} \right), \quad W = \left( \frac{\xi W^2}{\sqrt{2}} \right),
\]

In these variables, the soft limit is controlled by \( \xi \to 0 \), while the limit when particle \( N + 1 \) becomes collinear to particle \( A \) is controlled by \( W \to 0 \) and the invariant measure becomes

\[
(2\pi\mu)^{2}\text{d}^4\varepsilon \text{d}^2p \delta(p^\mu p_\mu) = \xi^{1-2\varepsilon} \text{d}\xi (2\pi\mu)^{2}\text{d}^2\varepsilon W
\]

\[
\equiv \xi W \left( \frac{2\pi\mu}{\xi W} \right)^{2\varepsilon} \text{d}\xi \text{d}W (\sin \phi)^{-2\varepsilon} \text{d}\phi \text{d}^{-2\varepsilon} \Omega,
\]

where \( W = |W| \) and \( \phi \) is the azimuthal angle of particle \( N + 1 \). In light cone coordinates the incoming particles have four-momenta

\[
p^\mu_A = \left( x_A \sqrt{\frac{s}{2}}, 0, 0, 0 \right), \quad p^\mu_B = \left( 0, x_B \sqrt{\frac{s}{2}}, 0, 0 \right),
\]

hence, \( \xi < x_A \). The invariant mass squared of particle \( Q \) has to be greater then zero, which constraints the upper value of \( \xi W^2, \xi W^2 < x_Q x_B s / x_A \) with \( x_Q = x_A - \xi \). For later use we record the four-dimensional limit of the integration measure \( D_{N+1}^{\text{ini};A} \):

\[
D_{N+1}^{\text{ini};A}(\varepsilon = 0) = D_N(\varepsilon = 0) \delta^{(4)} \left( p^\mu_Q - \sum_{i=1}^{N} p^\mu_i \right) \bigg|_{p^\mu_Q = p^\mu_A + p^\mu_B - p^\mu} \frac{1}{2} \xi \text{d}\xi \text{d}W^2 \text{d}\phi \tag{4.45}
\]

\[
\times \Theta(r_{\text{min}}^{(N+1);f} > r_{\text{min}}^{(N+1);i}) \Theta(s_{\text{min}}^{(N);i} > s_{\text{min}}^{X,N+1}) \Theta(r_{B,N+1} > r_{A,N+1}).
\]
The decomposition of the phase space into initial and final pieces (see eq. (4.28)) naturally decomposes the subtraction terms and the finite contribution as well,

\[ \mathcal{I}_{[\text{soft}]} = \sum_{m_f=1}^{N} \mathcal{I}_{[\text{soft}]}^{m_f} + \sum_{m_i=A,B} \mathcal{I}_{[\text{soft}]}^{m_i}, \]  
(4.46)

\[ \mathcal{I}_{[\text{coll}]} = \sum_{m_f=1}^{N} \mathcal{I}_{[\text{coll}]}^{m_f} + \sum_{m_i=A,B} \mathcal{I}_{[\text{coll}]}^{m_i}, \]  
(4.47)

\[ \mathcal{I}_{[\text{fin}]} = \sum_{m_f=1}^{N} \mathcal{I}_{[\text{fin}]}^{m_f} + \sum_{m_i=A,B} \mathcal{I}_{[\text{fin}]}^{m_i}. \]  
(4.48)

In the following subsections, we define these terms precisely.

### 4.4 Soft subtractions

In this subsection we define the \( \mathcal{I}_{[\text{soft}]}^{x} \) integrals for the cases \( m = A, B, 1, \ldots, N \) and \( x = A, B, 1, \ldots, N \). We start with the integrals involving the measure \( D_{N+1}^{\text{fin};m} \).

In the soft limit, the second and third \( \Theta \) functions in eq. (4.31) take the value one and so does the factor \( (E_m + E)/E_m \) in eq. (4.32). We find

\[
\lim_{\mu \to 0} D_{N+1}^{\text{fin};m}(\varepsilon) = \left[ D_N(\varepsilon) (2\pi\mu)^{-2}\delta^d \left( p_A^\mu + p_B^\mu - \sum_{i=1}^{N} p_i^\mu \right) \right]_{m \to P} \]  
(4.49)

\[
\times (2\pi\mu)^2 d^{1-2\varepsilon} \frac{p^2}{2} \delta(p^\mu p_\mu) \left[ \Theta(r_{m,N+1}^{(N+1):i} > r_{m,N}^{(N+1);f}) \right] \]  
\[
\times \Theta(\min(r_i,N+1 : i = 1, \ldots, N, i \neq m) > r_{m,N+1}) \right]_{m \to P}. \]

We shall also use the four-dimensional limit of this measure expressed in terms of \( z \) and \( Q^2 \):

\[
\lim_{\mu \to 0} D_{N+1}^{\text{fin};m}(\varepsilon = 0) = \left[ D_N(\varepsilon = 0) \delta^{(4)} \left( p_A^\mu + p_B^\mu - \sum_{i=1}^{N} p_i^\mu \right) \right]_{m \to P} \]  
(4.50)

\[
\times \frac{1}{2} d z \, d Q^2 \, d \varphi \Theta(4E_P^2(1 - z) \geq Q^2) \Theta(\min(r_i,N+1 : i = 1, \ldots, N, i \neq m) > r_{m,N+1}). \]

This expression is consistent with the soft limit of eq. (4.40). In the soft limit, \( z \) and \( Q^2 \) are related to the energy and polar angle according to the relations

\[
z = 1 - \frac{E}{E_P}, \quad Q^2 = 2(1 - z)E_P^2(1 - \cos \vartheta). \]  
(4.51)

Using the soft limit of \( D_{N+1}^{\text{fin};m}(\varepsilon) \), we define the soft terms \( \mathcal{I}_{[\text{soft}]}^{m_f} \) (\( m = A, B, 1, \ldots, N \)) as
\[ I^{[\text{soft}]}_{m_f} = \sum_{a_A, a_B, a_1, \ldots, a_N} \int \frac{\alpha_s}{(2\pi)^2} \left[ D_N(\varepsilon)(2\pi\mu)^{-2\varepsilon} \delta^d \left( p_A^\mu + p_B^\mu - \sum_{i=1}^N p_i^\mu \right) \right]_{m_f \to P} \]

\[ \times (2\pi\mu)^{2\varepsilon} d^{4-2\varepsilon} \sum_{j=1}^N \sum_{i=1}^N \Psi_{S_{mN}}^{(2N+2)}(\vec{a}, \vec{p}, p_{N+1}^\mu) \Theta(\alpha_{mn} > s_{m,N+1} + s_{n,N+1}) \]

\[ \times \left. L(a_A, a_B, x_A, x_B) S_{N+1}(p_1^\mu, \ldots, p_N^\mu) \right|_{p_{N+1}^\mu = 0}, \quad 0 < \alpha < \alpha_{\text{max}} \leq 1/2. \]

The indices \( m_f \) and \( P \) label the same momentum, therefore, the \( m_f \to P \) substitution is formal in this equation and we shall omit it in the rest of the paper. In eq. (4.52), we introduced an upper bound — \( \alpha \) times the energy of parton \( m \) — for the energy of parton \( N + 1 \) in the c.m. frame of partons \( m \) and \( n \), expressed in terms of invariants. Physical quantities will not depend on this bound. The \( z = 1 - E/E_P > 0 \) relation introduces an upper bound on \( E \) which is not present in eq. (4.49). Thus the use of the soft approximation is justified only if

\[ \Theta(\alpha_{mn} > s_{m,N+1} + s_{n,N+1}) \Theta(E_P > E) S_{N+1}(d_{\text{cut}}) \bigg|_{p_{N+1}^\mu = 0} = 0 < \alpha < \alpha_{\text{max}} \leq 1/2. \]

The second line contains the invariant measure of particle \( N + 1 \), so we define the soft terms \( I^{[\text{soft}]}_{m} \) \((m = A, B, 1, \ldots, N)\) as

\[ \lim_{\xi \to 0} D_{N+1}^{\text{ini}; A}(\varepsilon) = D_N(\varepsilon)(2\pi\mu)^{-2\varepsilon} \delta^d \left( p_A^\mu + p_B^\mu - \sum_{i=1}^N p_i^\mu \right) \]

\[ \times \xi W \left( \frac{2\pi\mu}{\xi W} \right)^{2\varepsilon} \frac{\partial W}{\partial \phi} (\sin \phi)^{-2\varepsilon} d\phi d^{-2\varepsilon} \Omega \]

\[ \times \Theta(r_{\text{min}}^{(N+1;i)} > r_{\text{min}}^{(N+1;f)}) \Theta(r_{B,N+1} > r_{A,N+1}). \]

The second line contains the invariant measure of particle \( N + 1 \), so we define the soft terms \( I^{[\text{soft}]}_{m} \) \((m = A, B, 1, \ldots, N)\) as
\[ I[\text{soft}]^A_m = \sum_{a_A, a_B, a_1, \ldots, a_N} \int \frac{\alpha_s}{(2\pi)^2} \mathcal{D}_N(\varepsilon) (2\pi \mu)^{-2\varepsilon} \delta^d \left( p_A^\mu + p_B^\mu - \sum_{i=1}^N p_i^\mu \right) \times (2\pi \mu)^{2\varepsilon} d^{4-2\varepsilon} p \delta(p^\mu p_\mu) \times \Theta(r_{(N+1);f}) \Theta(r_{(N+1);i}) \Theta(r_{B,N+1} > r_{A,N+1}) \times \sum_{n=A,B;1 \ldots N, n \neq m} \Psi_{S;mn}^{(2N+2)}(\vec{a}, \vec{p}, p_{N+1}^\mu) \Theta(\alpha s_{mn} > s_{m,N+1} + s_{n,N+1}) \times L(a_A, a_B, x_A, x_B) \mathcal{S}_N+1(p_1^\mu, \ldots, p^\mu) \bigg|_{p^\mu \to 0}, \quad 0 < \alpha < \alpha_{\text{max}}. \]

Here, it is useful to choose \( \alpha_{\text{max}} \) such that

\[ \Theta(x_A > \xi) \Theta((x_A - \xi) x_B s > x_A \xi W^2) \times \Theta(\alpha s_{mn} > s_{m,N+1} + s_{n,N+1}) \mathcal{S}_N+1(d_{\text{cut}}) \bigg|_{p_{N+1}^\mu = 0} = \Theta(\alpha s_{mn} > s_{m,N+1} + s_{n,N+1}) \mathcal{S}_N+1(d_{\text{cut}}) \bigg|_{p_{N+1}^\mu = 0} \]

for any \( m \) and \( n \). In this case the generation of the soft phase space in (4.69) becomes simpler.

The definition of the soft term \( I[\text{soft}]^B_m \) is analogous. One simply interchanges labels \( A \) and \( B \) in the second \( \Theta \) function of the third line of eq. (4.55). We remark that the sum of the \( N + 2 \) soft terms \( I[\text{soft}]^m_A \) and \( I[\text{soft}]^m_B \) is independent of the phase space decomposition:

\[ I[\text{soft}]_m = \sum_{a_A, a_B, a_1, \ldots, a_N} \int \frac{\alpha_s}{(2\pi)^2} \mathcal{D}_N(\varepsilon) (2\pi \mu)^{-2\varepsilon} \delta^d \left( p_A^\mu + p_B^\mu - \sum_{i=1}^N p_i^\mu \right) \times (2\pi \mu)^{2\varepsilon} d^{4-2\varepsilon} p \delta(p^\mu p_\mu) \times \sum_{n=A,B;1 \ldots N, n \neq m} \Psi_{S;mn}^{(2N+2)}(\vec{a}, \vec{p}, p_{N+1}^\mu) \Theta(\alpha s_{mn} > s_{m,N+1} + s_{n,N+1}) \times L(a_A, a_B, x_A, x_B) \mathcal{S}_N+1(p_1^\mu, \ldots, p^\mu) \bigg|_{p^\mu \to 0}. \]

This relation will be the starting point for the evaluation of the integrals over the invariant measure of particle \( N + 1 \) in the soft subtraction terms \( I[\text{soft}]_m \).

We close the definition of the soft subtraction terms with spelling out the four-dimensional limit of the measure (4.54),

\[ \lim_{\xi \to 0} \mathcal{D}_{N+1}^{\text{int};A}(\varepsilon = 0) = \mathcal{D}_N(\varepsilon = 0) \delta^{(4)} \left( p_A^\mu + p_B^\mu - \sum_{i=1}^N p_i^\mu \right) \frac{1}{2\pi^2} d^2 x dW^2 d\phi \]

\[ \times \Theta(r_{(N+1);f}) \Theta(r_{(N+1);i}) \Theta(r_{B,N+1} > r_{A,N+1}) \]

This form is consistent with the soft limit of eq. (4.43). It will be used for giving an explicit expression for the finite integral \( I[\text{fin}] \) in four dimensions.
4.5 Collinear subtractions

In this subsection we define the $\mathcal{I}\text{[coll]}_A^{\varepsilon}$ integrals for the cases $m = A, B, 1, \ldots, N$ and $x = A, B, 1, \ldots, N$. We start with the integrals involving the measure $\mathcal{D}_{N+1}^{\text{ini},X}$ and we discuss in detail the case $X = A$. The treatment of case $X = B$ is analogous.

In the collinear limit $p^\varepsilon_\mu = z p^\varepsilon_A$, $p^\mu = (1-z)p^\mu_A$, the first $\Theta$ function in the integration measure (4.39) is zero, therefore, the terms $\mathcal{I}\text{[coll]}_A^{\varepsilon}$ for $x = 1, \ldots, N$ are defined to be zero. Also in this limit, the $\Theta$ function $\Theta(t_{A,N+1} > t_{B,N+1})$ appearing in the measure $\mathcal{D}_{N+1}^{\text{ini},B}$ becomes zero, therefore, the term $\mathcal{I}\text{[coll]}_B^{\varepsilon}$ is defined to be zero. In the same limit the measure $\mathcal{D}_{N+1}^{\text{ini},A}(\varepsilon)$ becomes

$$
\lim_{W \to 0} \mathcal{D}_{N+1}^{\text{ini},A}(\varepsilon) = \mathcal{D}_{N}(\varepsilon)(2\pi\mu)^{-2\varepsilon} \delta^d \left( z p^\mu_A + p_B^\mu - \sum_{i=1}^{N} p_i^\mu \right) \xi \left( \frac{2\pi\mu}{\xi} \right)^{2\varepsilon} d\xi d^{2-2\varepsilon} W.
$$

where the momentum fraction $z$ can be expressed in terms of actual integration variables $x_A$ and $\xi$ using the momentum conservation relation for the “+” component of the momenta,

$$
z = \frac{x_A - \xi}{x_A}. \tag{4.60}
$$

The collinear pole $2/s_{A,N+1}$ equals $2/(x_A \xi W^2)$, therefore, the collinear term $\mathcal{I}\text{[coll]}_A^{\varepsilon}$ is defined as

$$
\mathcal{I}\text{[coll]}_A^{\varepsilon} = \sum_{a_A,a_B,a_1,\ldots,a_N} \frac{\alpha_s}{2\pi^2} \mathcal{D}_{N}(\varepsilon) L(a_A, a_B, x_A, x_B) \left( \frac{2\pi\mu}{\xi} \right)^{2\varepsilon} d\xi d^{2-2\varepsilon} W \frac{1}{x_A W^2}
\times \left[ (2\pi\mu)^{-2\varepsilon} \delta^d \left( z p^\mu_A + p_B^\mu - \sum_{i=1}^{N} p_i^\mu \right) \Theta(z \xi s_A > W^2) \Theta(x_A > \xi) \right.
\times \sum_{a_{N+1}} (-1)^{1+f(a_A)+f(a_B)} \Psi(2N+2)_{C,A,N+1}(1/z, P; A, B, 1, \ldots, N+1)
\left. \times S_{N+1}(p^\mu_1, \ldots, p^\mu_{N+1}) \Big|_{p^\mu=(1-z)p^\mu_A} - (2\pi\mu)^{-2\varepsilon} \delta^d \left( p^\mu_A + p_B^\mu - \sum_{i=1}^{N} p_i^\mu \right) \Theta(s \xi s_A > W^2) \Theta(z > 1 - \alpha) \right]
\times \frac{2C(a_A)}{(1-z)} \Psi(2N)_{(\vec{a}, \vec{p})} S_{N+1}(p^\mu_1, \ldots, p^\mu_{N+1}) \Big|_{p^\mu=0}, \quad 0 < \delta < \delta_{\text{max}} < 1,
$$

where we introduced a convenient upper bound for the $W$ integral. We have subtracted a term in the soft-collinear limit ($z \to 1$) in order to keep $\mathcal{I}\text{[coll]}_A^{\varepsilon}$ from having a soft divergence when $\varepsilon \to 0$. According to eq. (4.60), in this subtracted term the $z > 1 - \alpha$ constraint is equivalent to $\alpha x_A > \xi$ and $1/(1-z) = x_A/\xi$. It is useful to choose $\delta_{\text{max}}$ such that

$$
\Theta((x_A - \xi)x_B s > z \xi s_{\text{max}}) S_{N+1}(d_{\text{cut}}) \Big|_{p^\mu_{N+1}=(1-z)p^\mu_A} = S_{N+1}(d_{\text{cut}}) \Big|_{p^\mu_{N+1}=(1-z)p^\mu_A}, \tag{4.62}
$$

in which case the generation of the collinear phase space in (4.69) becomes simpler.
We shall also use the four-dimensional limit of measure \( \Theta \):

\[
\lim_{W \to 0} D_{N+1}^{\text{init}}(\varepsilon = 0) = D_N(\varepsilon = 0)\delta^{(4)}(z p_A^\mu + p_B^\mu - \sum_{i=1}^{N} p_i^\mu) \frac{1}{2} \xi d\xi dW^2 d\phi. \quad (4.63)
\]

This form is consistent with the collinear limit of eq. \( \Theta \). It will be used for giving an explicit expression for the finite integral \( \mathcal{I}[\text{fin}] \) in four dimensions.

Next we consider the collinear subtraction terms involving the measure \( D_{N+1}^{\text{fin}} \). The collinear limit of particles \( m \) and \( N+1 \) implies taking \( \omega_{m,N+1} = \omega \to 0 \), \( Q^2 \to 0 \). In this limit, in the integration measure \( \Theta \), the first two \( \Theta \) functions become one, the third one becomes \( \Theta(z > 1/2) \) and the last one becomes \( \delta_{mnf} \). We find

\[
\lim_{\omega, Q^2 \to 0} D_{N+1}^{\text{fin};m,f}(\varepsilon) = \left[D_N(\varepsilon) \frac{(2\pi \mu)^{-2\varepsilon}}{E_P} \delta^d(p_A^\mu + p_B^\mu - \sum_{i=1}^{N} p_i^\mu) \right]_{m,f \to P} \delta_{mnf}
\]

\[
\times \frac{1}{2} \left(\frac{2\pi \mu}{E_P}\right)^{2\varepsilon} |z(1-z)|^{-\varepsilon} dz \left(\frac{Q^2}{E_P^2}\right)^{-\varepsilon} dQ^2 (\sin \varphi)^{-2\varepsilon} d\varphi d^{1-2\varepsilon} \Omega
\]

\[
\times \Theta(4z(1-z)E_P^2 > Q^2) \Theta(z > 1/2).
\]

The collinear pole \( 2/s_{mf,N+1} = 2/Q^2 \), so we define the collinear terms \( \mathcal{I}[\text{coll}]_{m,f} \) as

\[
\mathcal{I}[\text{coll}]_{m,f} = \sum_{a,A,a_B,a_1,\ldots,a_N} \int \frac{\alpha_s}{(2\pi)^2} \left[D_N(\varepsilon) \frac{(2\pi \mu)^{-2\varepsilon}}{E_P} \delta^d(p_A^\mu + p_B^\mu - \sum_{i=1}^{N} p_i^\mu) \right]_{m,f \to P} \delta_{mnf}
\]

\[
\times L(a_A, x_A, x_B) \left(\frac{2\pi \mu}{E_P}\right)^{2\varepsilon} (1-z)^{-\varepsilon} dz \left(\frac{Q^2}{E_P^2}\right)^{-\varepsilon} dQ^2 Q^2 d^{1-2\varepsilon} \varphi
\]

\[
\times \left[z^{-\varepsilon} \Theta(4z(1-z)E_P^2 \delta > Q^2)
\right.
\]

\[
\times \sum_{a_{N+1}} \Psi_{C,m,f,N+1}^{(2N+2)}(z, P; A, B, 1, \ldots, N + 1)
\]

\[
\times \Theta(z > 1/2) S_{N+1}(p_1^\mu, \ldots, p_N^\mu, p_{m,f}^\mu) |p_m^\mu = z p_P^\mu,
\]

\[
- \Theta(4(1-z)E_P^2 \delta > Q^2) \left[\frac{2C(a_m)}{1-z} \Psi_{(2N)}(\vec{p}) \right]_{m,f \to P}, 0 < \delta < \delta_{\text{max}}.
\]

Here we have subtracted the integrand at \( z = 1 \) in order to keep \( \mathcal{I}[\text{coll}]_{m,f} \) from having a soft divergence when \( \varepsilon \to 0 \). After making the indicated substitutions, the right hand side does not contain the indices \( m \) and \( m_f \), but the index \( P \). In equation \( \Theta \), for the case \( \delta = 1 \) the upper limit on the \( Q^2 \) integration derives from the \( \cos \omega \geq -1 \) constraint with the use of the relations

\[
Q^2 = 2z(1-z)E_P^2(1-\cos \omega), \quad Q^2 = 2(1-z)E_P^2(1-\cos \omega)
\]

\( (4.66) \)
in the collinear and soft-collinear limits respectively. We remind the reader that particle $P$ is the massless limit of particle $Q$. In the collinear limit $p_{\mu}^m = z p_{\mu}^P$, $p_{\mu} \to (1 - z) p_{\mu}^P$ the first $\Theta$ function in the measure (4.41) becomes zero, therefore, the collinear subtraction terms $I_{\text{coll}}[m]$ for $x = A, B$ are defined to be zero, therefore,

$$I_{\text{coll}}[m] = \sum_{m_f = 1}^{N} I_{\text{coll}}[m_f].$$

(4.67)

Finally we spell out the four-dimensional limit of the measure (4.64), which is consistent with the collinear limit of the measure (4.40):

$$\lim_{\omega \to 0} D_{N+1}^\text{fin:m} (\varepsilon = 0) = \left[ D_N(\varepsilon = 0) \delta^{(4)} \left( p_A^\mu + p_B^\mu - \sum_{i=1}^{N} p_i^\mu \right) \right]_{m \to P} \times \frac{1}{2} dz dQ^2 d\varphi (4z(1 - z)E_P^2 \geq Q^2) \Theta(z > 1/2).$$

(4.68)

4.6 The finite contribution

The precise definition of the soft and collinear terms fixes the finite term by eqs. (2.3) and (4.25). For the sake of completeness, in this subsection we write it in terms of the actual integration variables. The integrand contains at most integrable square-root singularities over the integration domain, therefore, it suffices to write the integral in $d = 4$ dimensions.

We give the integrals $I_{\text{fin}}[m_i]$ explicitly for the case $m_i = A$ only.

$$I_{\text{fin}}[A] = \sum_{a_A, a_B, a_1, \ldots, a_N} \int \frac{\alpha_s}{(2\pi)^2} D_N(0) \frac{1}{2} z d\xi dW^2 d\phi \mathcal{L}(a_A, a_B, x_A, x_B) \times$$

$$\left\{ \delta^{(4)} \left( p_A^\mu + p_B^\mu - \sum_{i=1}^{N} p_i^\mu \right) \Theta(r_{\text{min}}^{(N+1:f)} > r_{\text{min}}^{(N+1;i)}) \times \Theta(x_A > \xi) \Theta((x_A - \xi)x_B > x_A \xi W^2) \times \Theta(s_{\text{min}}^{(N;i)} > s_{N,N+1}^{(N;i)}) \Theta(r_{B,N+1} > r_{A,N+1}) \times \sum_{a_{N+1}} \Psi^{(2N+2)}(A, B, 1, \ldots, N + 1) S_{N+1}(p_{1}^\mu, \ldots, p_{N+1}^\mu) \right.$$
\[ \times \Psi_{C;A,N+1}(1/z, P; A, B, 1, \ldots, N + 1)S_{N+1}(P_1^\mu, \ldots, P_N^\mu) \bigg|_{p^\mu = (1-z)p_A^\mu} \]

\[ + \delta^{(4)} \left( p_A^\mu + p_B^\mu - \sum_1^N p_i^\mu \right) \Theta(s\delta/x_A > W^2) \Theta(\alpha x_A > \xi) \]

\[ \times 2C(a_A) \frac{2}{\xi^2 W^2} \Psi^{(2N)}(\vec{a}, \vec{p}) \bigg|_{p^\mu = 0} \]

\[ = \int \frac{\alpha_s}{(2\pi)^2} \frac{1}{2} \frac{d\varphi}{dQ^2} \frac{dQ^2}{d\varphi} L(a_A, a_B, x_A, x_B) \]

\[ \Theta(\hat{r}_{(N+1; i)} > r_{(N+1; f)}^\text{min}) \Theta(s_{(N; f)}^\text{min} > s_{(m,N+1)}^\text{min}) \Theta(E_k > E_{N+1}) \]

\[ \times \sum_{m,n = 1}^{N+1} \frac{2s_{mn}}{s_{m,N+1}s_{n,N+1}} \Psi^{(2N;c)}_{m,n}(\vec{a}, \vec{p}) \Psi^{(2N+2)}_{C;m_j,N+1}(z, P; A, B, 1, \ldots, N + 1) \]

\[ \times S_{N+1}(P_1^\mu, \ldots, P_N^\mu) \bigg|_{p^\mu = 0} \]

\[ \times 2C(a_{m_j}) \frac{2}{(1-z)Q^2} \Psi^{(2N)}(\vec{a}, \vec{p}) \bigg|_{p^\mu = 0} \}

with \( \phi \in [-\pi, \pi] \) and \( z \) given in eq. (4.60). The integral \( \mathcal{I}[\text{fin}]^B \) is analogous with the labels \( A \) and \( B \) and the momentum components \( p_{N+1}^\mu \) and \( p_{N+1}^\mu \) interchanged. The \( \Theta \) functions assure that the integrand can become singular only when \( \xi W^2 \to 0 \). For the reconstruction of the four-momentum \( p_{N+1}^\mu \) see eq. (4.43).

The explicit form of the integral \( \mathcal{I}[\text{fin}]^{mf} \) is

\[ \mathcal{I}[\text{fin}]^{mf} = \sum_{a_A,a_B,a_1,\ldots,a_N} \int \frac{\alpha_s}{(2\pi)^2} \frac{1}{2} \frac{d\varphi}{dQ^2} \frac{dQ^2}{d\varphi} L(a_A, a_B, x_A, x_B) \]

\[ \times \Theta(\hat{r}_{(N+1; i)} > r_{(N+1; f)}^\text{min}) \Theta(s_{(N; f)}^\text{min} > s_{(m,N+1)}^\text{min}) \Theta(E_k > E_{N+1}) \]

\[ \times \sum_{m,n = 1}^{N+1} \frac{2s_{mn}}{s_{m,N+1}s_{n,N+1}} \Psi^{(2N;c)}_{m,n}(\vec{a}, \vec{p}) \Psi^{(2N+2)}_{C;m_j,N+1}(z, P; A, B, 1, \ldots, N + 1) \]

\[ \times S_{N+1}(P_1^\mu, \ldots, P_N^\mu) \bigg|_{p^\mu = 0} \]

\[ \times 2C(a_{m_j}) \frac{2}{(1-z)Q^2} \Psi^{(2N)}(\vec{a}, \vec{p}) \bigg|_{p^\mu = 0} \}

with \( z \in [0, 1] \) unless otherwise indicated and \( \varphi \in [-\pi, \pi] \). In these equations we used the
\[
\sum_{(m,n=A,B,1,\ldots,N)}^{\sum_n} \frac{2s_{mn}}{s_{m,N+1}(s_{m,N+1} + s_{n,N+1})} \Psi^{(2N;c)}_{mn}(\vec{a}, \vec{p}) \Theta(\alpha s_{mn} > s_{m,N+1} + s_{n,N+1}) \quad (4.71)
\]

in order to simplify the sum of the soft subtraction terms. The \(D_N(0)\) factor with the accompanying momentum conservation \(\delta\) function contains the \(N\)-body phase space, the \(x_A, x_B\) integration (and other trivial factors). This \(N\)-body phase space can be generated according to the reader’s preference. For instance, one can use the well-known phase space generating routine RAMBO [20]. The reconstruction of the momentum of particle \((N+1)\) from \(z, Q^2\) and \(\varphi\) was given in previous subsections (see formulas (4.35), (4.38) and (4.51)).

We have given the precise definition of all terms in eq. (4.25). The integrals \(I_{[soft]}\) and \(I_{[coll]}\) contain poles in the Laurent expansion in \(\varepsilon\) around zero. According to the factorization and Kinoshita-Lee-Nauenberg theorems, these poles cancel against similar poles in the \(I_{[2 \to N]}\) contribution. In order to see the cancelation of infrared divergences explicitly, we have to analyze the integrals over the invariant measure of gluon \(N+1\) in the soft and collinear contributions, which is the subject of the next two sections.

5 Soft integrals

In this section, we evaluate the integrals in \(I_{[soft]}\) \((m = A, B, 1, \ldots, N)\) over the invariant measure of particle \(N+1\).

At the soft point, \(p^\mu = 0\) the measurement function \(S_{N+1}(p_1^\mu, \ldots, p_N^\mu) = S_N(p_1^\mu, \ldots, p_N^\mu)\), which is the manifestation of the requirement of infrared safe measurement. The only dependence in eq. (4.57) on the variables of gluon \(N+1\) is in the eikonal factor, therefore, we can write \(I_{[soft]}\) as

\[
I_{[soft]} = \sum_{a_A, a_B, \ldots} \int \frac{\alpha_s}{(2\pi)^2} D_N(\varepsilon) (2\pi\mu)^{-2\varepsilon} \delta^d(p_A^\mu + p_B^\mu - \sum_{i=1}^N p_i^\mu) L(a_A, a_B, x_A, x_B) S_N(p_1^\mu, \ldots, p_N^\mu) \sum_{(n=A,B,1,\ldots,N)} J_{mn}(\vec{p}) \Psi^{(2N;c)}_{mn}(\vec{a}, \vec{p}) \quad (5.1)
\]

where

\[
J_{mn}(\vec{p}) = \int (2\pi\mu)^{2\varepsilon} d^4 q_{N+1} 2 \delta(p_{N+1}^2) \times \frac{2s_{mn}}{s_{m,N+1}(s_{m,N+1} + s_{n,N+1})} \Theta(\alpha s_{mn} > s_{m,N+1} + s_{n,N+1}) \quad (5.2)
\]

This integral is evaluated in the appendix. The result can be obtained exactly, however, for our purposes the Laurent expansion in the form,

\[
J_{mn}(\vec{p}) = 2\pi c_T \left(\frac{\mu^2}{Q_{ES}^2}\right)^\varepsilon \left[ \frac{1}{2\varepsilon^2} - \frac{1}{\varepsilon} \ln \alpha + \frac{1}{2\varepsilon} \ln \frac{s_{mn}}{Q_{ES}^2} + \tilde{J}_{mn}(\vec{p}) + O(\varepsilon) \right] \quad (5.3)
\]
is better suited. The function $\tilde{J}_{mn}$ is independent of $\epsilon$ and is very simple:

$$\tilde{J}_{mn} = \frac{1}{4} \ln^2 \left( \alpha^2 \frac{s_{mn}}{Q_{ES}} \right) - \frac{\pi^2}{12}. \quad (5.4)$$

Substituting this result for the $J_{mn}$ soft integral into eq. (5.1) and using the soft-collinear identity, formula (5.21), we see that $I_{[\text{soft}]}$ assumes very similar form to the $I_{[2 \to N]}$ integral, eq. (3.10):

$$I_{[\text{soft}]} = \sum_{a_A,a_B,a_1,\ldots,a_N} \int D_N(\epsilon) S_N(p_1^\mu, \ldots, p_N^\mu) (2\pi\mu)^{-2\epsilon} \delta^d \left( p_A^\mu + p_B^\mu - \sum_{i=1}^N p_i^\mu \right) \times L(a_A, a_B, x_A, x_B) \frac{\alpha_s}{2\pi} c_T \left( \frac{\mu^2}{Q_{ES}} \right)^\epsilon \psi_{m}^\text{soft} (\vec{a}, \vec{p}),$$

where

$$\psi_{m}^\text{soft} (\vec{a}, \vec{p}) = \psi^{(2N)}(\vec{a}, \vec{p}) \left[ \frac{1}{\epsilon} C(a_m) - \frac{1}{\epsilon} 2C(a_m) \ln \alpha \right]$$

$$+ \sum_{n=A,B,1,\ldots,N} \psi_{mn}^{(2N+2)}(\vec{a}, \vec{p}) \left[ -\frac{1}{2\epsilon} \ln \left( \frac{s_{mn}}{Q_{ES}} \right) + \tilde{J}_{mn}(\vec{p}) \right]. \quad (5.6)$$

## 6 Collinear integrals

In this section, we evaluate the integrals in $I_{[\text{coll}]}^m (m = A, B, 1, \ldots, N)$ over the invariant measure of parton $N + 1$. Before going into the details, we have to make a remark. The collinear subtraction terms were defined using the four-dimensional expressions for the collinear limit of the squared matrix element. That was sufficient for the evaluation of the $I_{[\text{fin}]}$ integral. Strictly speaking however, the subtraction scheme applied in this paper is defined in $d$ dimensions. It was shown in ref. [15] that with making use of process independent transition terms, one can use four-dimensional expressions for the helicity independent part for the collinear limit of the squared matrix element except for the Altarelli-Parisi splitting functions, $\tilde{P}_{a/b}$ that have to be calculated in $d$ dimensions. As for the helicity dependent part, the analysis of its general structure in $d$ dimensions shows it has vanishing azimuthal integral in $d$ dimensions [2, 13]. Therefore, we drop the helicity dependent part of $\psi^{(2N+2)}$ in the following considerations. This causes some inconsistency in our notation, but the physical cross section remains unchanged.

We start with the evaluation of the integrals in $I_{[\text{coll}]}^m (m = A, B)$ over the invariant measure of parton $N + 1$. At the collinear point $W \to 0$ with $\xi$ fixed, the measurement function $S_{N+1}(p_1^\mu, \ldots, p_N^\mu) = S_N(p_1^\mu, \ldots, p_N^\mu)$. We change integration variables in the first term of eq. (4.61) form $x_A$ (which is hidden in $D_N$ and in the luminosity factor) and $\xi$ to $x_P = x_A - \xi$ and $z$ with $z$ defined in eq. (4.60). The Jacobian for this transformation is $x_A/z$. The lower limit for the $z$ integral is defined by the $x_A \leq 1$ relation, hence $z \geq x_P$, while the upper limit is obviously one. The limits on $x_P$ are zero and one, just as was on $x_A$. After this change of variables we can rename the index $P$ to $A$ (and simultaneously...
the flavor index $a_A$ to $b$). In the second term of eq. (I.61), we change variable from $\xi$ to $z$.

Keeping the helicity independent term, we can now write $I[\text{coll}]_A^A$ as

$$I[\text{coll}]_A^A =$$

$$\sum_{a_A,a_B,a_1,\ldots,a_N} \int \frac{\alpha_s}{2\pi^2} D_N(\varepsilon)(2\pi\mu)^{-2\varepsilon} \delta^d \left( p_A^\mu + p_B^\mu - \sum_{i=1}^N p_i^\mu \right) \Psi^{(2N)}(\vec{p}) S_N(p_1^\mu, \ldots, p_N^\mu)$$

$$\times \int dz \left[ \left( \frac{1-z}{z} x_A \right)^{-2\varepsilon} \frac{1}{z^2} \Theta(z > x_A) \right]$$

$$\times \int \sum_b \frac{\omega(b)}{\omega(a_A)} \tilde{P}_{a_A/b}(z, \varepsilon) L \left( b, a_B, \frac{x_A}{z}, x_B \right)$$

$$\times (2\pi\mu)^{2\varepsilon} \int \frac{d^2 z}{W^2} \Theta \left( s\delta z^2 / x_A > W^2 \right)$$

$$- (1-z) x_A^{2\varepsilon} \Theta(z > 1-\alpha) L(a_A, a_B, x_A, x_B) \frac{2C(a_A)}{1-z}$$

$$\times (2\pi\mu)^{2\varepsilon} \int \frac{d^2 z}{W^2} \Theta \left( s\delta / x_A > W^2 \right).$$

Evaluation of the integral over $W$ (see eq. (A.12)) results in

$$I[\text{coll}]_A^A =$$

$$- \sum_{a_A,a_B,a_1,\ldots,a_N} \int D_N(\varepsilon) S_N(p_1^\mu, \ldots, p_N^\mu)(2\pi\mu)^{-2\varepsilon} \delta^d \left( p_A^\mu + p_B^\mu - \sum_{i=1}^N p_i^\mu \right)$$

$$\times \Psi^{(2N)}(\vec{p}) \frac{\alpha_s}{2\pi \varepsilon \Gamma(1-\varepsilon)} \left( \frac{\mu^2}{x_A s\delta} \right)^\varepsilon$$

$$\times \int dz (1-z)^{-2\varepsilon}$$

$$\times \left[ \frac{1}{z^2} \Theta(z > x_A) \sum_b \frac{\omega(b)}{\omega(a_A)} \tilde{P}_{a_A/b}(z, \varepsilon) L \left( b, a_B, \frac{x_A}{z}, x_B \right) \right.$$ 

$$\left. - \Theta(z > 1-\alpha) L(a_A, a_B, x_A, x_B) \frac{2C(a_A)}{1-z} \right].$$

In order that we could combine this contribution with the collinear factorization counter term for hadron $A$, we use the relations

$$\frac{(4\pi)^\varepsilon}{\varepsilon \Gamma(1-\varepsilon)} = c_T \left( \frac{\mu^2}{Q_{ES}^2} \right)^\varepsilon \left[ \frac{1}{\varepsilon} + \ln \left( \frac{Q_{ES}^2}{\mu^2} \right) + O(\varepsilon) \right]$$

and

$$X^\varepsilon \int_0^1 dz (1-z)^{-2\varepsilon} \left[ f(z) P(z, \varepsilon) \Theta(z > x) - f(1) \frac{2C}{1-z} \Theta(z > 1-\alpha) \right]$$

$$= \int_x^1 dz f(z) \left[ P(z, 0) - \frac{2C}{1-z} + \frac{2C}{1-z_+} + \gamma \delta(1-z) \right] - f(1) \left[ \gamma + 2C \ln \alpha \right]$$

$$+ \varepsilon \left\{ \int_x^1 dz \left[ \ln \frac{X}{(1-z)^2} \left( f(z) P(z, \varepsilon) - f(1) \frac{2C}{1-z} + f(z) P'(z) \right) \right. \right.$$

$$\left. + 2C f(1) \left[ \ln X (\ln(1-x) - \ln \alpha) + \ln^2 \alpha - \ln^2(1-x) \right] \right\} + O(\varepsilon)$$
The latter equation holds if \((1 - z) P(z, 0) \to 2C\) as \(z \to 1\) and for any function \(f(z)\) that is not singular in \(z = 1\). The function \(\tilde{P}'(z)\) is defined by the relation \(\tilde{P}(z, \varepsilon) = \tilde{P}(z) + \varepsilon \tilde{P}'(z)\). For the complete collinear integral \(I[\text{coll}]^A\) we obtain

\[
I[\text{coll}]^A = \sum_{a_A, a_B, a_1, \ldots, a_N} \int D_N(\varepsilon) S_N(p_{1}^{\mu}, \ldots, p_{N}^{\mu})(2\pi\mu)^{-2\varepsilon} \delta^{d}(p_{A}^{\mu} + p_{B}^{\mu} - \sum_{i=1}^{N} p_{i}^{\mu}) \times \left\{ \frac{\alpha_s}{2\pi} \frac{\mu^2}{Q_{ES}^2} \right\}^{\varepsilon} \Psi_A^{coll}(\vec{a}, \vec{p}) L(a_A, a_B, x_A, x_B) \\
- \int_{x_A}^{1} \frac{dz}{z^2} \sum_{a} \frac{\omega(b)}{\omega(a_A)} L(a_A, a_B, x_A/z, x_B) \frac{(4\pi)^{\varepsilon}}{\varepsilon \Gamma(1 - \varepsilon)} \frac{\alpha_s}{2\pi} P_{a_A/b}(z) \Psi(2N)(\vec{a}, \vec{p}) \\
- \frac{\alpha_s}{2\pi} \frac{\mu^2}{Q_{ES}^2} \left\{ \Psi(2N)(\vec{a}, \vec{p}) L(a_A, a_B, x_A, x_B) \times \left[ \int_{x_A}^{1} \frac{dz}{z} \ln X_A(z) \frac{1}{z^2} \sum_{a} \frac{\omega(b)}{\omega(a_A)} L(b, a_B, x_A/z, x_B) \tilde{P}_{a_A/b}(z) - \frac{2C(a_A)}{1 - z} \right] \\
+ \frac{1}{z^2} \sum_{a} \frac{\omega(b)}{\omega(a_A)} L(a_A, a_B, x_A, x_B) \tilde{P}_{a_A/b}(z) \right\} + O(\varepsilon),
\]

where \(X_A(z) = \mu^2/((1 - z)^2 x_A \delta)\) and

\[
\Psi_A^{coll}(\vec{a}, \vec{p}) = \Psi(2N)(\vec{a}, \vec{p}) \left\{ \frac{1}{\varepsilon} [\gamma(a_A) + 2C(a_A) \ln \alpha] + \ln \left( \frac{Q_{ES}^2}{\mu^2} \right) [\gamma(a_A) + 2C(a_A) \ln \alpha] \right\}.
\]

When two final state particles become collinear, \(p_{m}^{\mu} = z p_{p}^{\mu}\), \(p^{\mu} = (1 - z) p_{p}^{\mu}\), the measurement function \(S_{N+1}(p_{1}^{\mu}, \ldots, p_{N}^{\mu}) = S_N(p_{1}^{\mu}, \ldots, p_{N}^{\mu})\) holds. Keeping the helicity independent term in the collinear integral, we have

\[
I[\text{coll}]^m = \sum_{a_A, a_B, a_1, \ldots, a_m} \int \left[ D_N(\varepsilon) S_N(p_{1}^{\mu}, \ldots, p_{N}^{\mu})(2\pi\mu)^{-2\varepsilon} \delta^{d}(p_{A}^{\mu} + p_{B}^{\mu} - \sum_{i=1}^{N} p_{i}^{\mu}) \right]_{m \to P} \times \frac{\alpha_s}{(2\pi)^2} L(a_A, a_B, x_A, x_B) \left( \frac{2\pi\mu}{E_P} \right)^{2\varepsilon} \Psi(2N)(\vec{a}, \vec{p})_{m \to P} \\
	imes \int (1 - z)^{-\varepsilon} dz \left( \frac{Q^2}{E_P^2} \right)^{-\varepsilon} dQ^2 d1^{2-2\varepsilon} \phi \\
	imes \left[ z^{-\varepsilon} \Theta(4z(1 - z) E_P^2 \delta > Q^2) \Theta(z > 1/2) \sum_{a_m} \tilde{P}_{a_m/a_P}(z, \varepsilon) \\
- \Theta(4(1 - z) E_P^2 \delta > Q^2) \Theta(z > 1 - \alpha) \frac{2C(a_m)}{1 - z} \right].
\]

25
The $Q^2$, $\phi$ integrations can be calculated immediately. The required integral is given in the appendix, eq. (A.11). The remaining integral over $z$ is also straightforward. After performing these steps and leaving out the formal $m \to P$ substitutions, we see that the term $I[\text{coll}]_m$ has a form very similar to that of $I[2 \to N]$:

$$
I[\text{coll}]_m = \sum_{a_A,a_B,a_1,\ldots,a_N} \int D_N(\varepsilon) S_N(p^\mu_A, \ldots, p^\mu_N)(2\pi \mu)^{-2\varepsilon} \delta^d \left( p^\mu_A + p^\mu_B - \sum_{i=1}^N p^\mu_i \right) \times L(a_A, a_B, x_A, x_B) \frac{\alpha_s}{2\pi} c_T \left( \frac{\mu^2}{Q^2_{\text{ES}}} \right) \varepsilon \Psi_{m,\text{coll}}(\vec{a}, \vec{p}),
$$

where

$$
\Psi_{m,\text{coll}}(\vec{a}, \vec{p}) = -\frac{1}{\varepsilon} \left( \frac{Q^2_{\text{ES}}}{4E_m^2 \delta} \right) \mathcal{Z}_{a_m}(\alpha) \Psi^{(2N)}(\vec{a}, \vec{p}) + O(\varepsilon)
$$

with $\mathcal{Z}_{a_m}(\alpha)$ given in eq. (A.10).

At this point we see that the sum of the integrals $I[2 \to N]$, $I[\text{soft}]$ and $I[\text{coll}]$ is free of any poles of $\varepsilon$, therefore, it can be calculated in $d = 4$ dimensions:

$$
(I[2 \to N] + I[\text{soft}] + I[\text{coll}]) \bigg|_{\varepsilon=0} = \sum_{a_A,a_B,a_1,\ldots,a_N} \int D_N(0) S_N(p^\mu_A, \ldots, p^\mu_N)\delta^d \left( p^\mu_A + p^\mu_B - \sum_{i=1}^N p^\mu_i \right) L(a_A, a_B, x_A, x_B) \Psi^{(2N)}(\vec{a}, \vec{p})
$$

$$
\times \left\{ 1 + \frac{\alpha_s}{2\pi} \left( \frac{\Psi^{(2N+2)}_{\text{NS}}(\vec{a}, \vec{p})}{\Psi^{(2N)}(\vec{a}, \vec{p})} + \sum_{m,n=A,B,1,\ldots,N} \mathcal{J}_m \mathcal{J}_n \right) \frac{\Psi^{(2N,c)}_{m,n}(\vec{a}, \vec{p})}{\Psi^{(2N)}(\vec{a}, \vec{p})} \right\}
$$

$$
+ \sum_{m=1}^N \left[ \gamma(a_m) + 2C(a_m) \ln \alpha \right] \ln \frac{Q^2_{\text{ES}}}{4E_m^2 \delta}
$$

$$
+ \sum_{m=A,B} \left[ \gamma(a_m) + 2C(a_m) \ln \alpha \right] \ln \frac{Q^2_{\text{ES}}}{\mu^2}
$$

$$
- \sum_{m=1}^N \left[ 2C(a_m) \left( \frac{\ln^2 \alpha + \pi^2}{3} \right) + \gamma'(a_m) \right]
$$

$$
+ \sum_{m=A,B} 2C(a_m) \left[ \ln X_m(0) \ln \frac{\alpha}{1-x_m} + \ln^2 (1-x_m) - \ln^2 \alpha \right]
$$

$$
- \int_{x_A}^1 dz \ln X_A(z) \left[ \frac{1}{z^2} \sum_b L(b, a_B, x_A/z, x_B) \tilde{P}_{a_A/b}(z) - \frac{2C(a_B)}{1-z} \right]
$$

$$
- \int_{x_B}^1 dz \ln X_B(z) \left[ \frac{1}{z^2} \sum_b L(b, a_A, x_B/z, x_B) \tilde{P}_{a_B/b}(z) - \frac{2C(a_B)}{1-z} \right]
$$

$$
- \int_{x_A}^1 dz \frac{1}{z^2} \sum_b L(b, a_B, x_A/z, x_B) \tilde{P}'_{a_A/b}(z)
$$

$$
- \int_{x_B}^1 dz \frac{1}{z^2} \sum_b L(a_B, b, x_A, x_B) \tilde{P}'_{b_A/b}(z) \right\},
$$

26
The logarithmic dependence on the unphysical parameters $\alpha$ and $\delta$ in this equation gets canceled when this contribution is combined with the integral $\mathcal{I}[^{\text{fin}}]$ (eq. (6.11)) in order to obtain the infrared safe physical cross section at next-to-leading order:

$$\sigma = \mathcal{I}[^{\text{fin}}] |_{\varepsilon=0} + (\mathcal{I}[2 \to N] + \mathcal{I}[\text{soft}] + \mathcal{I}[\text{coll}]) |_{\varepsilon=0}. \quad (6.11)$$

We can also demonstrate the independence of the $N$-body integral of the auxiliary parameter $Q^{\text{ES}}_N$ explicitly by making use of eqs. (3.12), (4.21), (5.4) as well as the definition of the $\ell_2$ function, formula (3.9):

$$\mathcal{I}[2 \to N] + \mathcal{I}[\text{soft}] + \mathcal{I}[\text{coll}] |_{\varepsilon=0} = \sum_{a_A,a_B,a_1,\ldots,a_N} \mathcal{D}_N(0) \mathcal{S}_N(p^\mu_1, \ldots, p^\mu_N) \delta^{(4)} \left( p_\mu_A + p_\mu_B - \sum_{i=1}^N p_\mu_i \right) \left. L(a_A, a_B, x_A, x_B) \Psi^{(2N)}(\vec{a}, \vec{p}) \right|_{\varepsilon=0}$$

$$\times \left\{ 1 + \frac{\alpha_s}{2\pi} \left[ g^2 \left( \frac{g}{4\pi} \right)^2 \right] - \frac{1}{2} \sum_{\text{hel}} \sum_{\text{col}} \left( A_{NS}^{(1)} A_{NS}^{(0)} + A_{NS}^{(1)} A_{NS}^{(0)} \right) / \Psi^{(2N)}(\vec{a}, \vec{p}) \right\} \quad (6.12)$$

$$+ \sum_{m=1}^N \left[ \frac{\pi^2}{4} \Theta(s_{mn}) + \ln \left( \frac{s_{mn}}{4E_{\mu}^2 \delta} \right) \right] \frac{\Psi^{(2N,c)}(\vec{a}, \vec{p})}{\Psi^{(2N)}(\vec{a}, \vec{p})}$$

$$+ \sum_{m=1}^N \left[ \pi^2 s_{mn} / x_m s_{n} \right] \Gamma(a_m) \ln \left( 1 - x_m \right) \ln \left( \frac{x_m}{x_m - 1} \right) s_{\delta} - \frac{\pi^2}{12} - \gamma(a_m) \right]$$

$$+ \sum_{m=1}^N \left[ 2C(a_m) \left( \ln \left( 1 - x_m \right) \ln \left( \frac{x_m}{x_m - 1} \right) s_{\delta} - \frac{\pi^2}{12} \right) - \gamma(a_m) \right]$$

$$- \int_{x_A}^1 dz \ln x_A(z) \left[ \frac{1}{z^2} \sum_b L(b, a_B, x_A/z, x_B) \tilde{P}_{a_A/b}(z) \right] \frac{2C(a_A)}{1 - z}$$

$$- \int_{x_B}^1 dz \ln x_B(z) \left[ \frac{1}{z^2} \sum_b L(b, a_A, x_A/x_B) \tilde{P}_{a_B/b}(z) \right] \frac{2C(a_B)}{1 - z}$$

$$- \int_{x_A}^1 dz \left[ \frac{1}{z^2} \sum_b L(b, a_B, x_A/z, x_B) \tilde{P}_{a_A/b}(z) \right]$$

$$- \int_{x_B}^1 dz \left[ \frac{1}{z^2} \sum_b L(b, a_A, x_A/x_B) \tilde{P}_{a_B/b}(z) \right].$$

This equation together with eqs. (4.69) and (4.70) define explicitly those integrals that are needed for the calculation of a jet cross section at next-to-leading order in perturbative QCD.

### 7 Numerical results

In this section we present some numerical results of the first non-trivial application of our algorithm, namely the calculation of three-jet cross sections in $e^+e^-$ annihilation. Thus our results can be compared with those of ref. [21].
We use the matrix elements of ref. [22] for the construction of the various $\Psi$ functions. The algorithm can easily be altered for performing jet cross section calculations in the case of $e^+e^-$ annihilation. One simply drops all terms in the integrals (4.70) and (6.12) that carry $A$ or $B$ indices, leaves out the $x_A$, $x_B$ integrations from $D_N(0)$ and then the sum of integrals (4.70) and (6.12) immediately gives the physical cross section. We implemented such an algorithm in a Monte Carlo program. The results are in good agreement with those of ref. [21]. As an example we show the next-to-leading order coefficients for the thrust, C-parameter distributions multiplied by $(1 - t)$ and $C$ respectively and distributions for the Jade E and $k_\perp$ jet clustering algorithms multiplied by the jet resolution parameter in fig. 1.

We find that the numerical convergence is similar to the program of ref. [21].
8 Conclusion

In this paper we have presented a simple generalization of the subtraction method of ref. [3] for the calculation of any infrared safe physical cross section in perturbative QCD. The apparent conflict between the need for important sampling in order to achieve sufficient numerical precision and the increasing difficulty of performing partial fractioning in the tree-level next-to-leading order matrix element was overcome by a decomposition of the phase space such that in one region only one Lorentz invariant of the external momenta can become singular in the calculation of an $N$-jet observable. We wish to emphasize the simplicity of our algorithm: the necessary analytic integrals are rather trivial, while the numerical implementation is only a little more complicated than a tree-level Monte Carlo program.

We have given all the necessary integrals that define any next-to-leading order QCD jet cross section explicitly (see eqs. (4.69), (4.70) and (6.12)). The phase space integrations in these integrals can be programmed for any number of jets. Once having such a master program the only ingredients that have to be changed in a modular fashion are the

- the Born-level and next-to-leading order tree matrix elements in four dimensions ($\Psi^{(2n)}$ ($n = N, N + 1$) functions);
- the color linked Born matrix elements ($\Psi^{(2N;c)}$ functions);
- the non-singular part of the one-loop helicity amplitudes ($A_{NS}$ functions);
- the $S_n$ ($n = N, N + 1$) measurement functions.

The algorithm can be trivially altered for calculating QCD jet cross sections in other processes, like in $e^+e^-$ annihilation or deep-inelastic scattering. As an example, we have shown results of the next-to-leading order thrust, C-parameter and jet distributions in the case of $e^+e^-$ annihilation. The numerical convergence was found to be similar to that of the program of ref. [21] in the case of $e^+e^-$ annihilation into three jets. In the case of three-jet production in hadron collisions such a benchmark calculation does not exist yet. In order to demonstrate the applicability of the algorithm in such calculations, in a companion paper [23], we give results of a next-to-leading order calculation of three-jet cross section in hadron collisions in the simplified case of pure Yang-Mills theory. The structure of the algorithm is essentially the same when quarks are included therefore, the conclusions are expected to be similar in the full QCD case.

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Appendix

This appendix is added for the readers’ convenience. It contains a collection of the integrals that were used in the main text. The explicit evaluation of these integrals is rather trivial, therefore, minimal details are given.

First we calculate the soft integral

\[ J_{mn}(\vec{p}) = \int (2\pi \mu)^{2\epsilon} d^4p_{N+1} \frac{2 \delta(p_{N+1})}{s_{m,N+1}(s_{m,N+1} + s_{n,N+1})} \Theta(\alpha s_{mn} > s_{m,N+1} + s_{n,N+1}). \]  

(A.1)

for \( m, n = 1, \ldots, N \), \( m \neq n \). The integrand depends only on Lorentz invariants therefore, the integral can be calculated in any frame. We choose the “\( m-n \)” system, where the four-momenta of gluon \( m, n \) and \( N + 1 \) take the form (first four components are energy, \( z, x \) and \( y \) components of the three-momentum)

\[ p_m^\mu = \frac{\sqrt{s_{mn}}}{2}(1, 1, 0, 0, \ldots), \]  

(A.2)

\[ p_n^\mu = \frac{\sqrt{s_{mn}}}{2}(1, -1, 0, 0, \ldots), \]  

(A.3)

\[ p_{N+1}^\mu = E(1, \cos \vartheta, \sin \vartheta \cos \varphi, \sin \vartheta \sin \varphi, \ldots), \]  

(A.4)

where the dots mean \( d-4 \) zeros. With this choice the invariants are

\[ s_{m,N+1} = \sqrt{s_{mn}}E(1 - \cos \vartheta), \]  

(A.5)

\[ s_{n,N+1} = \sqrt{s_{mn}}E(1 + \cos \vartheta). \]  

(A.6)

Consequently, in the “\( m-n \)” system the integral takes the following simple form:

\[ J_{mn} = \int \left( \frac{2\pi \mu}{E} \right)^{2\epsilon} \frac{dE}{E} (\sin \vartheta)^{-2\epsilon} d\vartheta (\sin \varphi)^{-2\epsilon} d\varphi d^{-2\epsilon} \Omega \frac{\sin \vartheta}{1 - \cos \vartheta} \Theta(\alpha \sqrt{s_{mn}/2} > E). \]  

(A.7)

This integral is easily evaluated and one obtains the exact result:

\[ J_{mn} = \frac{\pi}{\epsilon^2} \left( \frac{4\pi \mu^2}{s_{mn}} \right)^\epsilon \alpha^{-2\epsilon} \frac{\Gamma(1 - \epsilon)}{\Gamma(1 - 2\epsilon)}. \]  

(A.8)

In the main text, we have also used the following integrals:

\[ \int \left( \frac{Q^2}{E_F^2} \right)^{-\epsilon} \frac{dQ^2}{Q^2} \Theta(4Q_{\text{max}}^2 > Q^2) (\sin \varphi)^{-2\epsilon} d\varphi d^{-2\epsilon} \Omega = -\frac{2\pi}{\epsilon} Q^{-2\epsilon} \frac{(4\pi)^{-\epsilon}}{\Gamma(1 - \epsilon)}, \]  

(A.9)

\[ Z_a(\alpha) = \int dz (1 - z)^{-2\epsilon} \left[ z^{-2\epsilon} \sum_b \tilde{P}_{a/b}(z) \Theta(z > 1/2) - \frac{2C(a)}{1 - z} \Theta(z > 1 - \alpha) \right] \]  

(A.10)

\[ = -\gamma(a) - 2C(a) \ln \alpha + \epsilon \gamma'(a) + 2\epsilon C(a) \left( \ln^2 \alpha + \frac{\pi^2}{3} \right), \]  

where \( \epsilon \) is the regularization parameter.
where

\[ \gamma'(g) = -\frac{67}{9} N_c + \frac{23}{18} N_f, \quad \gamma'(q) = -\frac{13}{4} \frac{V}{N_c}, \tag{A.11} \]

and

\[ (2\pi\mu)^{2\varepsilon} \int \frac{d^2W}{W^2} \Theta(W_{\text{max}} > W) = -\frac{1}{\varepsilon} \frac{\pi}{\Gamma(1 - \varepsilon)} \left( \frac{4\pi\mu^2}{W_{\text{max}}^2} \right)^\varepsilon, \tag{A.12} \]

The derivation of these results is sufficiently simple so that we can omit the details.

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