Reduction formula of form factors for the integrable spin-$s$ XXZ chains and application to correlation functions

Tetsuo Deguchi

Department of Physics, Graduate School of Humanities and Sciences, Ochanomizu University, 2-1-1 Ohtsuka, Bunkyo-ku, Tokyo 112-8610, Japan
E-mail: deguchi@phys.ocha.ac.jp

Received 30 August 2011
Accepted 3 March 2012
Published 2 April 2012

Abstract. For the integrable spin-$s$ XXZ chain we express explicitly any given spin-$s$ form factor in terms of a sum over the scalar products of the spin-$1/2$ fundamental operators of the algebraic Bethe ansatz. Here, they are given by the operator-valued matrix elements of the monodromy matrix of the spin-$1/2$ XXZ spin chain. We derive the reduction formula of higher-spin form factors by the fusion method in detail. We call an arbitrary matrix element of a local operator between two Bethe eigenstates a form factor of the operator in this paper. We thus revise the derivation of the higher-spin XXZ form factors given in a previous paper. The revised method has several interesting applications in mathematical physics. For instance, we express the spin-$s$ XXZ correlation function of an arbitrary entry at zero temperature in terms of a sum of multiple integrals.

Keywords: correlation functions, form factors, integrable spin chains (vertex models), quantum integrability (Bethe ansatz)

ArXiv ePrint: 1105.4722
# Contents

1. Introduction 3

2. Affine quantum group and the monodromy matrix 6
   2.1. Spin-$\ell/2$ representations of the quantum group $U_q(sl_2)$ ......................... 6
   2.2. Operators acting on the tensor product space ............................................. 8
   2.3. $R$ matrix and the monodromy matrix of type $(1, 1^{\otimes L})$ ....................... 9
   2.4. Rapidities forming complete $n$-strings .................................................. 10
   2.5. Monodromy matrix of type $(1, (\ell)^{\otimes N_s})$ associated with homogeneous grading 10
   2.6. Gauge transformations for the spin-$\ell/2$ representation ............................ 12
   2.7. Spin-$\ell/2$ monodromy matrix associated with principal grading ......................... 12
   2.8. Higher-spin monodromy matrices of type $(\ell, (2s)^{\otimes N_s})$ ........................ 13

3. Reduction of higher-spin elementary operators 14
   3.1. Spin-$\ell/2$ elementary operators associated with homogeneous and principal gradings .......................................................... 14
   3.2. Two expressions of a product of spin-$1/2$ elementary operators ...................... 14
   3.3. Reduction into the spin-$1/2$ elementary operators ...................................... 16
   3.4. General spin-$\ell/2$ elementary operators ................................................ 18
   3.5. Quantum inverse-scattering problem for the spin-$\ell/2$ operators ..................... 20
   3.6. Non-regularity of the transfer matrix at special points .................................. 20

4. Reduction of the matrix elements of spin-$\ell/2$ operators 21
   4.1. Definition of the spin-$\ell/2$ matrix elements and form factors ......................... 21
   4.2. Commutation relations with projection operator ....................................... 22
   4.3. Reduction of the spin-$\ell/2$ matrix elements into the spin-$1/2$ ones .............. 23
   4.4. Consequence of the continuity assumption of the Bethe roots ........................ 25
   4.5. Spin-$\ell/2$ form factors reduced into the spin-$1/2$ ones ............................ 27

5. Spin-$\ell/2$ form factors via the spin-$1/2$ scalar products 28
   5.1. Fundamental commutation relations ...................................................... 28
   5.2. Form factors as a sum of the spin-$1/2$ scalar products ............................. 29

6. Spin-$s$ XXZ correlation functions in a massless region 32
   6.1. Conjecture of the spin-$s$ ground state solution ...................................... 33
   6.2. Multiple-integral representations for arbitrary matrix elements .................. 34
   6.3. The multiple-integral representation in the XXX limit ................................ 36

Acknowledgments 39

Appendix A. Derivation of proposition 3.7 39

Appendix B. Reduction of spin-$\ell/2$ Hermitian elementary operators 39

Appendix C. Non-regularity of the transfer matrix 41

Appendix D. Reducing spin-$\ell/2$ Bethe states with principal grading 41

doi:10.1088/1742-5468/2012/04/P04001
1. Introduction

The multiple-integral representations of correlation functions of the spin-1/2 XXZ spin chain have attracted much interest during the last two decades in the mathematical physics of integrable quantum spin chains [1]–[8]. They are also derived for the integrable higher-spin XXX spin chains through the algebraic Bethe ansatz method [9,10]. The multiple-integral representations of the finite-temperature correlation functions of the integrable isotropic spin-1 chain have been explicitly derived [11].

The Hamiltonian of the spin-1/2 XXZ spin chain under the periodic boundary conditions (PBC) is given by

$$\mathcal{H}_{XXZ} = \frac{1}{2} \sum_{j=1}^{L} (\sigma_j^X \sigma_{j+1}^X + \sigma_j^Y \sigma_{j+1}^Y + \Delta \sigma_j^Z \sigma_{j+1}^Z). \quad (1.1)$$

Here $\sigma_j^a$ ($a = X,Y,Z$) are the Pauli matrices defined on the $j$th site and $\Delta$ denotes the anisotropy of the exchange coupling. The PBC are given by $\sigma_{L+1}^a = \sigma_1^a$ for $a = X,Y,Z$. In terms of the $q$ parameter of the quantum group $U_q(sl_2)$, we express $\Delta$ by $\Delta = \frac{1}{2}(q + q^{-1})$. We define the parameter $\eta$ by $q = \exp \eta$. The transfer matrix of the XXZ spin chain has free parameters which we call the inhomogeneity parameters $w_j$ for $j = 1, 2, \ldots, L$.

Recently, a systematic method for evaluating the form factors and correlation functions of the integrable higher-spin XXZ spin chain has been proposed by applying the fusion method [12]–[14]. However, the proposed method was not completely correct [15]. In this paper, we revise the previous method for evaluating the spin-$\ell/2$ form factors and formulate systematic formulae by which we can express any given spin-$\ell/2$ form factor in terms of a sum over the scalar products of the spin-1/2 operators. We derive the formulae of the revised method, explicitly. In particular, we show how the spin-$\ell/2$ elementary operators depend on the grading of the evaluation representation for the affine quantum group $U_q(sl_2)$. In fact, the expression of the basis vectors depends on the grading. In [15] the revised expression of a given spin-$\ell/2$ form factor is presented only for the principal grading without any explicit derivation. Throughout this paper we call an arbitrary matrix element of a local spin-$s$ operator between two Bethe eigenvectors a form factor of the operator as in [2,6].

Applying the revised method we derive a concise multiple-integral representation of correlation functions of the integrable spin-$s$ XXZ spin chain in a gapless region with $\eta = i\zeta$ and $0 \leq \zeta < \pi/2s$. The correlation function of any given entry is expressed in terms of a sum of multiple integrals. In [13,14] it was expressed in terms of a single term of a multiple integral, which is not derived by the revised method. Sending $\zeta$ to 0 we obtain the multiple-integral representation of correlation functions for the integrable spin-$s$ XXX chain from that of the integrable spin-$s$ XXZ chain. The expression is similar to that of [9].

Let us review the fusion method for evaluating the higher-spin form factors briefly and point out where it was wrong. We consider the integrable spin-$\ell/2$ XXZ spin chain...
for an integer $\ell$ with $\ell > 1$. In the fusion method we construct the spin-$\ell/2$ XXZ transfer matrix in the following two steps: we first construct the spin-$1/2$ transfer matrix with $w_j = w_j^{(\ell)}$ from the product of the spin-$1/2$ $R$ matrices with their rapidities shifted by the inhomogeneity parameters $w_j$ which are given by the $N_s$ pieces of the complete $\ell$-strings $w_j^{(\ell)}$ such as $w_j = w_j^{(\ell)}$ for $j = 1, 2, \ldots, L$ (see section 2.4); we then multiply the product with the spin-$\ell/2$ projection operators. Here, the spin-$\ell/2$ chain with $N_s$ sites is defined on the spin-$1/2$ chain with $L$ sites, where $L = \ell N_s$.

When we evaluate the spin-$\ell/2$ form factors with the fusion method, we reduce each of the spin-$\ell/2$ operators into a sum of products of the local spin-$1/2$ operators, which we want to express in terms of the operator-valued matrix elements of the spin-$1/2$ monodromy matrix through the formula of the quantum inverse-scattering problem (QISP). We then want to calculate the expectation value or the matrix elements of the sums of products of the local spin-$1/2$ operators with respect to the Bethe states by making use of Slavnov’s formula of scalar products for the spin-$1/2$ Bethe ansatz operators. Here we recall that the inhomogeneity parameters $w_j$ of the spin-$1/2$ monodromy matrix are given by the $N_s$ pieces of the complete $\ell$-strings, $w_j = w_j^{(\ell)}$, for $j = 1, 2, \ldots, L$.

However, the QISP formula does not hold if one of the transfer matrices appearing in it is non-regular. Here we remark that it has the product of the inverse operators of the transfer matrices where the spectral parameters $\lambda$ are given by some of the inhomogeneity parameters $w_j$. In fact, we can show that the spin-$1/2$ transfer matrix with $w_j = w_j^{(\ell)}$ is non-regular at $\lambda = w_{(k-1)+1}^{(\ell)}$, the first rapidity of the $k$th complete $\ell$-string for an integer $1 \leq k \leq N_s$ (see section 3.6). Consequently, the QISP formula does not hold in the straightforward form for the fusion method. Thus, we want to avoid such special values of the spectral parameter when we evaluate the matrix elements or expectation values of the higher-spin local operators through the fusion method. In the revised method we avoid directly putting the complete $\ell$-strings $w_j^{(\ell)}$ into the inhomogeneity parameters $w_j$, as we shall see later.

The main result of the present paper has several interesting applications in the mathematical physics of exactly solvable models. For instance, we can evaluate the form factors of various solvable quantum spin chains associated with the affine quantum group [16]. For the spin-$s$ XXZ spin chain we can derive the multiple-integral representation of correlation functions through the revised method, as mentioned in the above. Moreover, we can calculate some form factors and matrix elements of the local spin operators for the superintegrable chiral Potts chains through the present method [17]. Furthermore, we can construct integrable quantum impurity models such as consisting of one spin-$s_1$ site with $N_s$ spin-$1/2$ sites through the fusion method. Some details should be discussed elsewhere.

This paper consists of the following. In section 2 we introduce finite-dimensional representations of the quantum group and construct the monodromy matrices of the integrable higher-spin XXZ spin chains through the fusion method. We introduce the complete $\ell$-strings, $w_j^{(\ell)}$, where the $k$th complete $\ell$-string $w_{(k-1)+1}^{(\ell)}$ for $1 \leq \alpha \leq \ell$ is given

\[ \text{doi:10.1088/1742-5468/2012/04/P04001} \]
by $\ell$ complex numbers shifted by $\eta$ successively, such as $\xi_k, \xi_k - \eta, \ldots, \xi_k - (\ell - 1)\eta$. We then construct the spin-$\ell/2$ monodromy matrix associated with homogeneous grading, and that of principal grading, systematically. In section 3, we define the higher-spin elementary operators $E^{\ell;i,j}$, which have only one nonzero matrix element of entry $(i, j)$ with respect to the basis vectors and their conjugate vectors. Then, we explicitly derive a formula (proposition 3.7), by which we can reduce any given product of the higher-spin elementary operators into a sum of products of the spin-1/2 elementary operators. In order to prove it we derive two expressions for a given product of the spin-1/2 elementary operators. The overall factor of the higher-spin form factor depends on whether the operator is associated with principal or homogeneous grading. In order to make the form factors independent of the grading $w$ we define the general spin-$\ell/2$ elementary operators $E^{\ell;i,j}(\ell w)$. We show in section 3.6 that the spin-1/2 transfer matrix with $w_j = w_j^{(\ell)}$ is non-regular at $\lambda = w_{\ell(k-1)+1} = \xi_k$. In section 4, we reduce the form factor of a product of the spin-$\ell/2$ elementary operators into those of the spin-1/2 elementary operators (proposition 4.2). We introduce the almost complete $\ell$-strings, $w_j^{(\ell \epsilon)} (1 \leq j \leq L)$, a set of inhomogeneity parameters that are slightly different from the complete $\ell$-strings $w_j^{(\ell)}$ by the order of a small parameter $\epsilon$. We reduce the spin-$\ell/2$ form factors into a sum of the spin-1/2 scalar products by making use of the QISP formula with inhomogeneity parameters given by the almost complete $\ell$-strings and by sending $\epsilon$ to 0.2 We can revise the expressions of the higher-spin form factors given in [12] making use of proposition 4.5. In section 5, we show an explicit formula by which we can calculate every spin-$\ell/2$ form factor in terms of the scalar products of the spin-1/2 operators, i.e. the reduction formula of spin-$\ell/2$ form factors (proposition 5.3). In section 6, we express the correlation function of an arbitrary entry for the integrable spin-$s$ XXZ spin chain in terms of a sum of multiple integrals. Moreover, the normalization factors are systematically shown for the general spin-$\ell/2$ elementary operators $E^{\ell;i,j}(\ell w)$. The expression of the correlation functions is different from that of [13] mainly with respect to the sum over the multiple integrals. Here we remark that the multiple-integral representations of the spin-$s$ XXZ correlation functions are based on the Bethe ansatz solutions for the ground state of the integrable spin-$s$ XXZ spin chain. It has been demonstrated by numerical and analytical research that in a massless region with $0 \leq \zeta < \pi/2s$ the ground state of the integrable spin-$s$ XXZ chain is given by a set of 2-string solutions [18]–[30]. Finally, in the XXX limit we derive

---

2 Here we assume that, for a given solution $\{\lambda_\gamma\}$ of the Bethe ansatz equations of the spin-$\ell/2$ chain, there is a solution of the Bethe ansatz equations, $\{\lambda_\gamma(\epsilon)\}$, for the spin-1/2 chain with inhomogeneity parameters given by the almost complete $\ell$-strings, i.e. $w_j = w_j^{(\ell \epsilon)} (1 \leq j \leq L)$, and also that the solution $\{\lambda_\gamma(\epsilon)\}$ approaches $\{\lambda_\gamma\}$ continuously at $\epsilon = 0$.<ref>

In section 4.3 of [12] the spin-$\ell/2$ form factors were discussed by making use of the invalid QISP formulae, and the following expressions are not valid: (4.40) and (4.42) for $X^{-(\ell \epsilon)}$, (4.44) and (4.48) for $K^{(\ell)}$, (4.50) for $S^Z_\ell$, (4.52) for $E^m,m(\ell)$ and (4.54) for $E^m,m-1,m(\ell)$. Here (a,b,c) denote equation (a,b,c) of [12]; see also erratum of [15].

4 In [13] equations (3.27) and (3.28) are not valid, which were derived from equation (3.26) under an invalid assumption that the spin-1/2 monodromy matrix with $w_j = w_j^{(\ell \epsilon)}$ should commute with projection $P_{12}^{(2s)}$ within the order of $\epsilon$. Thus, the derivation of equation (4.16) is not valid. However, equation (4.16) itself is valid, since we can show it by the revised method. Here, for the emptiness formation probability, we have only one set of $x_k (1 \leq k \leq \epsilon)$. Moreover, the multiple-integral representation of the correlation functions given in equation (6.14) should be replaced with equation (6.10) of the present paper for most of the cases. Here (a,b,c) denotes equation (a,b,c) of [13].

---

doi:10.1088/1742-5468/2012/04/P04001
the multiple-integral representation of the spin-s XXX correlation functions from that of the spin-s XXZ correlation functions. It illustrates the advantage of the symmetry (4.13) among the form factors.

2. Affine quantum group and the monodromy matrix

After introducing finite-dimensional representations of the quantum group $U_q(sl_2)$, we construct through the fusion method higher-spin monodromy matrices, in particular, those associated with principal grading systematically via higher-spin gauge transformations. The fusion construction of the higher-spin monodromy matrices associated with principal grading should be presented for the first time. Here we remark that the symmetric higher-spin $R$ matrices and the symmetric higher-spin monodromy matrices are associated with principal grading.

2.1. Spin-$\ell/2$ representations of the quantum group $U_q(sl_2)$

The quantum group $U_q(sl_2)$ is an associative algebra over $\mathbb{C}$ generated by $X^\pm, K^\pm$ with the following relations [31]–[33]:

$$K X^\pm K^{-1} = q^{\pm 2} X^\pm, \quad KK^{-1} = K^{-1} K = 1,$$

$$[X^+, X^-] = \frac{K - K^{-1}}{q - q^{-1}}.$$  \hfill (2.1)

The algebra $U_q(sl_2)$ is also a Hopf algebra over $\mathbb{C}$ with comultiplication

$$\Delta(X^+) = X^+ \otimes 1 + K \otimes X^+, \quad \Delta(X^-) = X^- \otimes K^{-1} + 1 \otimes X^-,$$

$$\Delta(K) = K \otimes K,$$  \hfill (2.2)

and antipode $S(K) = K^{-1}, S(X^+) = -K^{-1}X^+, S(X^-) = -X^-K$ and coproduct $\epsilon(X^\pm) = 0 = \epsilon(K) = 1$.

Let us introduce symbols of $q$-analogs. We define the $q$-integer of an integer $n$ by $[n]_q = (q^n - q^{-n})/(q - q^{-1})$ and the $q$-factorial of $n$ by $[n]_q! = [n]_q[n-1]_q \cdots [1]_q$.  \hfill (2.3)

For integers $m$ and $n$ satisfying $m \geq n \geq 0$ we define the $q$-binomial coefficients as follows:

$$\left[ \frac{m}{n} \right]_q = \frac{[m]_q!}{[m-n]_q ![n]_q!}.$$  \hfill (2.4)

We first formulate the spin-1/2 representation $V^{(1)}$. Let $|\alpha\rangle$ ($\alpha = 0, 1$) be the basis vectors. We have $X^-|0\rangle = |1\rangle, X^-|1\rangle = 0, X^+|0\rangle = 0, X^+|1\rangle = |0\rangle$ and $K|\alpha\rangle = q^{1-2\alpha}|\alpha\rangle$ for $\alpha = 0, 1$.

Let $\ell$ be a positive integer. We construct the spin-$\ell/2$ representation $V^{(\ell)}$ of $U_q(sl_2)$ in the tensor product space $(V^{(1)})^\otimes \ell$ of the spin-1/2 representations $V^{(1)}$. We now introduce the basis vectors of $V^{(\ell)}$, $|\ell, n\rangle$, for $n = 0, 1, \ldots, \ell$. First, we define the highest weight vector $|\ell, 0\rangle$ by

$$|\ell, 0\rangle = |0\rangle_1 \otimes |0\rangle_2 \otimes \cdots \otimes |0\rangle_\ell.$$  \hfill (2.5)
Here \(|\alpha\rangle_j\) for \(\alpha = 0, 1\) denote the basis vectors of the spin-1/2 representation defined on the \(j\)th component of the tensor product \((V^{(1)})^{\otimes \ell}\). We remark that 0 and 1 correspond to up-spin, \(\uparrow\), and down-spin, \(\downarrow\), respectively. We also remark that \(K\|\ell, 0\rangle = q^{\ell}\|\ell, 0\rangle\). We define \(\|\ell, n\rangle\) for \(n \geq 1\) by

\[
\|\ell, n\rangle = \left(\Delta^{(\ell-1)}(X^-)\right)^n \|\ell, 0\rangle \frac{1}{[n]_q!}.
\]  

(2.6)

We then have the following [12]:

\[
\|\ell, i\rangle = \sum_{1 \leq a(1) < \cdots < a(i) \leq \ell} \sigma^-_{a(1)} \cdots \sigma^-_{a(i)} \|\ell, 0\rangle q^{a(1)+a(2)+\cdots+a(i)-i\ell+i(i-1)/2} \quad \text{for } i = 0, 1, \ldots, \ell.
\]

(2.7)

Here the sum is taken over all such integers \(a(1), a(2), \ldots, a(i)\) that satisfy \(1 \leq a(1) < \cdots < a(i) \leq \ell\). We denote by \(\sigma_j^\pm\) the Pauli matrices acting on the \(j\)th site.

We denote by \(F(\ell, n)\) the square length of vectors \(\|\ell, n\rangle\) as follows:

\[
F(\ell, n) = (\|\ell, n\rangle)^T \cdot \|\ell, n\rangle.
\]

(2.8)

Here the superscript \(T\) denotes the matrix transposition. Setting \((\|\ell, 0\rangle)^T \cdot \|\ell, 0\rangle = 1\), we have

\[
F(\ell, n) = \left[ \frac{\ell}{n} \right]_q^{-n(\ell-n)}.
\]

(2.9)

We thus define conjugate vectors \(\langle \ell, j|\) by

\[
\langle \ell, j| = (\|\ell, j\rangle)^T / F(\ell, j) \quad \text{for } j = 0, 1, \ldots, \ell.
\]

(2.10)

Explicitly we have the following:

\[
\langle \ell, j| = \left[ \frac{\ell}{j} \right]_q^{-j(\ell-j)} \sum_{1 \leq b(1) < \cdots < b(j) \leq \ell} \langle \ell, 0| \sigma^+_{b(1)} \cdots \sigma^+_{b(j)} q^{b(1)+b(2)+\cdots+b(j)-j\ell+j(j-1)/2}.
\]

(2.11)

It is easy to derive the normalization factor (2.9) by making use of the following lemma:

**Lemma 2.1.** For an integer \(n\) with \(0 \leq n \leq \ell\) we have

\[
\sum_{1 \leq a(1) < \cdots < a(n) \leq \ell} q^{2a(1)+\cdots+2a(n)} = q^n \left[ \frac{\ell}{n} \right]_q.
\]

(2.12)

**Proof.** Let us consider the \(q\)-binomial theorem:

\[
\prod_{k=1}^\ell (1 - zq^{2k}) = \sum_{n=0}^{\ell} (-1)^n z^n q^{n(n+1)} \left[ \frac{\ell}{n} \right]_q.
\]

(2.13)

Expanding the left-hand side of (2.13) with respect to \(z\) we have

\[
\prod_{k=1}^\ell (1 - zq^{2k}) = \sum_{n=0}^{\ell} (-z)^n \sum_{1 \leq a(1) < \cdots < a(n) \leq \ell} q^{2a(1)+\cdots+2a(n)}.
\]

(2.14)

Hence we have (2.12). \qed

doi:10.1088/1742-5468/2012/04/P04001 7
We remark that, when \( q \) is complex and not real, the conjugate vector \( \langle \ell, j \| \) is different from the Hermitian conjugate of a given vector \( \| \ell, j \)\). Thus, the pairing of \( \langle \ell, j \| \) and \( \| \ell, k \) does not give a standard scalar product if \( q \) is complex and not real. However, the Hermitian conjugate of a vector \( \| \ell, j \)\) is not covariant with respect to the quantum group \( U_q(sl_2) \) if \( q \) is complex and not real, while the transposed vector is covariant. Therefore, we express the transposed vector as the conjugate vector \( \langle \ell, j \| \)\).

We now introduce the projection operator which maps the tensor product of the spin-1/2 representations \( (V^{(1)})^{\otimes \ell} \) to the spin-\( \ell/2 \) representation \( V^{(\ell)} \). In terms of the basis vectors and their conjugate vectors we define it by

\[
P^{(\ell)} = \sum_{n=0}^{\ell} \| \ell, n \rangle \langle \ell, n \|.
\]  

(2.15)

We shall denote it also by \( P^{(\ell)}_{1\ldots \ell} \), since it is defined on the tensor product \( (V^{(1)})^{\otimes \ell} \).

2.2. Operators acting on the tensor product space

Let us denote by \( V_{1}^{(\ell)} \) the \(( \ell + 1)\)-dimensional representation of \( U_q(sl_2) \) with complex parameter \( \lambda_j \). It corresponds to the evaluation representation of the affine quantum group \( U_q(sl_2) \) with evaluation parameter \( \lambda_j \) associated with homogeneous grading [4,31].

In order to construct the integrable spin-\( \ell/2 \) XXZ spin chain of \( N_s \) sites, we introduce the tensor product space \( V_{1}^{(\ell)} \otimes \cdots \otimes V_{N_s}^{(\ell)} \) of the \(( \ell + 1)\)-dimensional representations \( V_{j}^{(\ell)} \) for \( j = 1,2,\ldots, N_s \). We call it the quantum space of the monodromy matrix of the spin-\( \ell/2 \) XXZ spin chain. We introduce another representation, \( V_{0}^{(\ell)} \), which we call the auxiliary space. We consider the tensor product of the auxiliary space \( V_{0}^{(1)} \) and the quantum space \( V_{1}^{(\ell)} \otimes \cdots \otimes V_{N_s}^{(\ell)} \):

\[
V_{0}^{(1)} \otimes (V_{1}^{(\ell)} \otimes \cdots \otimes V_{N_s}^{(\ell)}).
\]  

(2.16)

On the tensor product space (2.16) we shall define the monodromy matrix of the integrable spin-\( \ell/2 \) XXZ spin chain in section 2.5. In the fusion method we construct it from the monodromy matrix of the spin-1/2 XXZ spin chain of \( L \) sites with \( L = \ell N_s \). In the quantum space \( V_{1}^{(\ell)} \otimes \cdots \otimes V_{N_s}^{(\ell)} \) we shall assign a free parameter \( \xi_b \) to the \( b \)th component \( V_{b}^{(\ell)} \) for each integer \( b \) of \( 1 \leq b \leq N_s \).

For the most general case, we consider the tensor product of the auxiliary space \( V_{0}^{(2s_0)} \) and the quantum space \( V_{1}^{(2s_1)} \otimes \cdots \otimes V_{r}^{(2s_r)} \) such as \( V_{0}^{(2s_0)} \otimes (V_{1}^{(2s_1)} \otimes \cdots \otimes V_{r}^{(2s_r)}) \), where \( V_{j}^{(2s_j)} \) have parameters \( \lambda_j = \xi_j \) for \( j = 1,2,\ldots, r \), respectively. Here \( s_j \) are given by integers or half-integers for \( j = 0,1,2,\ldots, r \). We set \( L \) by \( L = 2s_1 + \cdots + 2s_r \).

We now introduce operators on the tensor product space. We denote by \( \epsilon^{a,b} \) such 2 × 2 matrices that have only one nonzero element equal to 1 at the entry of \( (a,b) \) for \( a,b = 0,1 \). Let \( s \) be an integer or a half-integer. We define \( E_{a,b}^{(2s)} \) by \( (2s+1) \times (2s+1) \) matrices with unique nonzero element 1 at the entry of \( (a,b) \) for \( a,b = 0,1,\ldots,2s \). If it is defined on the \( j \)th component of the quantum space, we denote it by \( E_{j}^{a,b} \). For a given set of
matrix elements $A_{b,\beta}^{a,\alpha}$ with $a, b = 0, 1, \ldots, 2s_j$ and $\alpha, \beta = 0, 1, \ldots, 2s_k$, we define operators $A_{j,k}$ by

$$A_{j,k} = \sum_{a,b=1}^{2s_j} \sum_{\alpha,\beta=1}^{2s_k} A_{b,\beta}^{a,\alpha} I_{2s_0} \otimes I_{2s_1} \otimes \cdots \otimes I_{2s_{j-1}} \otimes E_{a,b}^{(2s_j)} \otimes I_{2s_{j+1}} \otimes \cdots \otimes I_{2s_{r-1}} \otimes E_{\alpha,\beta}^{(2s_k)} \otimes I_{2s_{r+1}} \otimes \cdots \otimes I_{2s_r}.$$  

(2.17)

Similarly, for a set of matrix elements $B_{b}^{a}$ for $a, b = 0, 1, \ldots, 2s_j$, we define operators $B_{j}$ by

$$B_{j} = \sum_{a,b=1}^{2s_j} B_{b}^{a} I_{2s_0} \otimes I_{2s_1} \otimes \cdots \otimes I_{2s_{j-1}} \otimes E_{a,b}^{(2s_j)} \otimes I_{2s_{j+1}} \otimes \cdots \otimes I_{2s_r}.$$  

(2.18)

2.3. $R$ matrix and the monodromy matrix of type $(1,1^L)$

Let us introduce the $R$ matrix of the XXZ spin chain [1, 5–7]. We recall that $V_1^{(1)}$ and $V_2^{(2)}$ denote two-dimensional representations with parameters $\lambda_1$ and $\lambda_2$, respectively. The $R$ matrix acting on $V_1 \otimes V_2$ associated with homogeneous grading of type $w = +$ is given by

$$R^{(1+)}_{12}(\lambda_1 - \lambda_2) = \sum_{a,b,c,d=0}^{1} R^{(1+)}_{ab} e_{c}^{a,d} \otimes e_{d}^{b,c} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b(u) & c^{-}(u) & 0 \\ 0 & c^{+}(u) & b(u) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{[1,2]},$$  

(2.19)

where $u = \lambda_1 - \lambda_2$, $b(u) = \sinh u / \sinh(u + \eta)$ and $c^{\pm}(u) = \exp(\pm u) \sinh \eta / \sinh(u + \eta)$.

Here, the suffix [1, 2] in equation (2.19) denotes that the matrix acts on the tensor product of $V_1^{(1)}$ and $V_2^{(2)}$.

We denote by $R^{(1p)}(u)$, or simply by $R(u)$, the symmetric $R$ matrix where $c^{\pm}(u)$ of (2.19) are replaced by $c(u) = \sinh \eta / \sinh(u + \eta)$ [12]. The symmetric $R$ matrix is compatible with principal grading of the affine quantum group $U_q(\hat{sl}_2)$ [12]. We remark that the $R$ matrix associated with homogeneous grading of type $w = -$, $R^{(1-)}_{12}(\lambda_1 - \lambda_2)$, is given by exchanging all the $\pm$ signs in (2.19) [12, 13].

In the massless regime, we set $\eta = i\zeta$ with a real number $\zeta$ and we have $\Delta = \cos \zeta$. We mainly consider the region $0 \leq \zeta < \pi/2s$ for the correlation functions of the integrable spin-$s$ XXZ spin chain in this paper. In the massive regime, we assign $\eta$ a real nonzero number and we have $\Delta = \cosh \eta > 1$.

We now define the spin-1/2 monodromy matrix. Let us consider the tensor product of the auxiliary space $V_0^{(1)}$ and the quantum space which is given by the $L$th tensor product of $V_j^{(1)}$ for $j = 1, 2, \ldots, L$, i.e. $V_0^{(1)} \otimes (V_1^{(1)} \otimes \cdots \otimes V_L^{(1)})$. We call it of type $(1,1^L)$. We introduce a set of free parameters $w_j$ for $j = 1, 2, \ldots, L$. In the quantum space we set $\lambda_j = w_j$ for $j = 1, 2, \ldots, L$. In the auxiliary space we denote $\lambda_0$ by $\lambda$. We define the monodromy matrix of the spin-1/2 XXZ spin chain associated with grading of type $w$ ($w = \pm, p$) by

$$T^{(1w)}_{0,12,\ldots,L}(\lambda; \{w_j\}_L) = P_{0L}^{(1w)}(\lambda - w_L) \cdots P_{01}^{(1w)}(\lambda - w_1).$$  

(2.20)
Here \( R_{jk}^{(1w)} \) denote the \( R \) matrices associated with grading of type \( w \) with parameters \( \lambda_j \) and \( \lambda_k \). We call parameters \( w_j (j = 1, 2, \ldots, L) \) inhomogeneity parameters. We denote the set of \( w_j \) by \( \{ w_j \}_L \). In equation (2.20) the superscript \((1,1w)\) shows that the monodromy matrix is of type \((1,1\otimes L)\) associated with grading of type \( w \). We denote the right-hand side of equation (2.20) by \( R^{(1w)}_{0,12,\ldots,L} \).

### 2.4. Rapidities forming complete \( n \)-strings

Let \( n \) be a positive integer. We define a complete \( n \)-string by a set of \( n \) rapidities as follows:

\[
\lambda_{\alpha} = \Lambda + (n - 2\alpha + 1)\eta/2, \quad \text{for } \alpha = 1, 2, \ldots, n. \tag{2.21}
\]

Here we call parameter \( \Lambda \) the center of the complete \( n \)-string.

We next introduce almost complete \( n \)-strings, which are sets of \( n \) rapidities close to complete \( n \)-strings. Let \( \epsilon \) be an infinitesimally small number, i.e. \( |\epsilon| \ll 1 \). We take generic parameters \( r_{\beta} \) for \( \beta = 1, 2, \ldots, n \). We consider a set of \( n \) rapidities of the following form:

\[
\lambda_{\beta} = \Lambda + (n - 2\beta + 1)\eta/2 + \epsilon r_{\beta}, \quad \text{for } \beta = 1, 2, \ldots, n. \tag{2.22}
\]

We call it an almost complete \( n \)-string. They are different from the complete \( n \)-string with center \( \Lambda \) by the small numbers \( \epsilon r_{\beta} \).

When we construct the integrable spin-\( \ell/2 \) XXZ spin chain of \( N_s \) sites by the fusion method, we consider the spin-1/2 chain of \( L \) sites with \( L = \ell N_s \). Here we recall that \( \ell \) is a positive integer. We introduce \( N_s \) sets of almost complete \( \ell \)-strings \( w_j^{(\ell)} \) for \( 1 \leq j \leq L \) as follows:

\[
w^{(\ell)}_{(b-1)+\beta} = \xi_b - (\beta - 1)\eta + \epsilon r_{\beta}^{(b)} \quad \text{for } \beta = 1, 2, \ldots, \ell; \quad b = 1, 2, \ldots, N_s. \tag{2.23}
\]

Here, arbitrary parameters \( \xi_b \) for \( b = 1, 2, \ldots, N_s \) are assigned to the quantum space \( V^{(1)}_1 \otimes V^{(2)}_2 \otimes \cdots \otimes V^{(1)}_{N_s} \). We remark that suffix \( j \) of \( w^{(\ell)}_j \) corresponds to \( \ell (b - 1) + \beta \) in (2.23).

We recall that parameters \( r_{\beta}^{(b)} \) are generic. In the fusion construction we shall put the almost complete \( \ell \)-strings \( w^{(\ell)}_{(b-1)+\beta} \) into inhomogeneous parameters \( w_j (j = 1, 2, \ldots, L) \) of the monodromy matrix of the spin-1/2 XXZ spin chain defined on the \( L \) sites and derive the spin-\( \ell/2 \) monodromy matrix by sending \( \epsilon \) to zero. At \( \epsilon = 0 \), the \( N_s \) pieces of almost complete \( \ell \)-strings \( w^{(\ell)}_j \) reduce to \( N_s \) pieces of complete \( \ell \)-strings. We denote them by \( w_j^{(\ell)} \), i.e. \( w_j^{(\ell)} = w_j^{(\ell)}_0 \) for \( j = 1, 2, \ldots, L \):

\[
w^{(\ell)}_{(b-1)+\beta} = \xi_b - (\beta - 1)\eta \quad \text{for } \beta = 1, 2, \ldots, \ell; \quad b = 1, 2, \ldots, N_s. \tag{2.24}
\]

### 2.5. Monodromy matrix of type \((1, (\ell)^{\otimes N_s})\) associated with homogeneous grading

We now derive the spin-\( \ell/2 \) monodromy matrix for the spin-\( \ell/2 \) XXZ spin chain of \( N_s \) sites associated with homogeneous grading of \( w = + \). We construct the quantum space \((V^{(1)})^{\otimes N_s}\) in \((V^{(1)})^{\otimes L}\), where we set \( L = \ell N_s \). Furthermore, we consider the tensor product of the auxiliary space \( V^{(1)} \) and the quantum space \((V^{(1)})^{\otimes N_s}\), i.e. \( V^{(1)} \otimes (V^{(1)})^{\otimes N_s} \).

We have abbreviated by the symbol \((V^{(1)})^{\otimes L}\) the \( L \)th tensor product \( V^{(1)}_1 \otimes \cdots \otimes V^{(1)}_L \) with inhomogeneity parameters \( w_j \) for \( j = 1, 2, \ldots, L \). Here, the \( j \)th component \( V^{(1)}_j \) has

doii:10.1088/1742-5468/2012/04/P04001

J. Stat. Mech. (2012) P04001
gives the fundamental operators of the algebraic Bethe ansatz. For instance, the \( \lambda \)-operator of the algebraic Bethe ansatz:

\[
V^{(\ell)}_b \otimes \cdots \otimes V^{(\ell)}_{N_s}, \quad \text{where we put } w^{(\ell;\epsilon)}_{j} \quad \text{for } j = 1, 2, \ldots, L \text{ in the spin-1/2 monodromy matrix } T^{(1,1w)}(\lambda; \{w_j\}_L). \]

We construct the spin-1/2 monodromy matrix with grading \( w \):

\[
T^{(1,\ell;\epsilon;w)}_{0,12-\cdots-L}(\lambda) = T^{(1,1w)}_{0,12-\cdots-L}(\lambda; \{w^{(\ell;\epsilon)}_{j}\}_L), \quad \text{for } w = \pm, p, \tag{2.25}
\]

where we put \( w^{(\ell;\epsilon)}_{j} = w^{(\ell;\epsilon)}_{j;\epsilon} \) for \( j = 1, 2, \ldots, L \) in the spin-1/2 monodromy matrix \( T^{(1,1w)}(\lambda; \{w_j\}_L) \). We also introduce the following:

\[
T^{(1,\ell;\epsilon;+;0)}_{0,12-\cdots-L}(\lambda) = \lim_{\epsilon \to 0} T^{(1,\ell;\epsilon;\cdot;\cdot;0)}_{0,12-\cdots-L}(\lambda). \tag{2.26}
\]

The matrix elements of the spin-1/2 monodromy matrix \( T^{(1,\ell;\epsilon;+;0)}(\lambda) \) correspond to the fundamental operators of the algebraic Bethe ansatz. For instance, the \( (0, 1) \) element gives the \( B \) operator:

\[
B^{(\ell;\epsilon;+;0)}(\lambda) = (T^{(1,\ell;\epsilon;+;0)}_{0,12-\cdots-L}(\lambda))_{0,1}. \tag{2.27}
\]

Let us denote by \( P^{(\ell)}_{(b-1)+1} \) the projection operator which maps the tensor product of the spin-1/2 representations \( V^{(1)}_{(b-1)+1} \otimes \cdots \otimes V^{(1)}_{(b-1)+\ell} \) to the \( b \)-th component of the \( N_s \)-th tensor product \( (V^{(\ell)}) \otimes N_s \). Here \( b \) is an integer satisfying \( 1 \leq b \leq N_s \), and the tensor product \( V^{(1)}_{(b-1)+1} \otimes \cdots \otimes V^{(1)}_{(b-1)+\ell} \) corresponds to the \( \ell(b-1)+1 \) th to the \( \ell \) th component of the \( L \) th tensor product \( (V^{(1)}) \otimes L \). We define \( P^{(\ell)}_{1-\cdots-L} \) by

\[
P^{(\ell)}_{1-\cdots-L} = \prod_{b=1}^{N_s} P^{(\ell)}_{(b-1)+1}. \tag{2.28}
\]

We construct the spin-\( \ell/2 \) monodromy matrix \( T^{(1,\ell;+)\cdot;0}_{0,12-\cdots-N_s}(\lambda) \) associated with homogeneous grading by applying the projection operator \( P^{(\ell)}_{1-\cdots-L} \) as follows [12]:

\[
T^{(1,\ell;+)}_{0,12-\cdots-N_s}(\lambda; \{\xi_b\}_{N_s}) = P^{(\ell)}_{1-\cdots-L} T^{(1,\ell;+;0)}_{0,12-\cdots-L}(\lambda) P^{(\ell)}_{1-\cdots-L}. \tag{2.29}
\]

The matrix elements of the spin-\( \ell/2 \) monodromy matrix \( T^{(1,\ell;+)}(\lambda) \) give the fundamental operators of the algebraic Bethe ansatz:

\[
T^{(1,\ell;+)}_{0,12-\cdots-N_s}(\lambda) = \begin{pmatrix}
A^{(\ell;+)}(\lambda; \{\xi_b\}_{N_s}) & B^{(\ell;+)}(\lambda; \{\xi_b\}_{N_s}) \\
C^{(\ell;+)}(\lambda; \{\xi_b\}_{N_s}) & D^{(\ell;+)}(\lambda; \{\xi_b\}_{N_s})
\end{pmatrix}. \tag{2.30}
\]

It follows from definition (2.29) that they are expressed in term of the matrix elements of the spin-1/2 monodromy matrix \( T^{(1,\ell;+;0)}_{0,12-\cdots-L}(\lambda) \) such as shown in (2.27) as follows:

\[
T^{(1,\ell;+;0)}_{0,12-\cdots-N_s}(\lambda) = \begin{pmatrix}
P^{(\ell)}_{1-\cdots-L} A^{(\ell;+;0)}(\lambda; \{\xi_b\}_{N_s}) P^{(\ell)}_{1-\cdots-L} & P^{(\ell)}_{1-\cdots-L} B^{(\ell;+;0)}(\lambda; \{\xi_b\}_{N_s}) P^{(\ell)}_{1-\cdots-L} \\
P^{(\ell)}_{1-\cdots-L} C^{(\ell;+;0)}(\lambda; \{\xi_b\}_{N_s}) P^{(\ell)}_{1-\cdots-L} & P^{(\ell)}_{1-\cdots-L} D^{(\ell;+;0)}(\lambda; \{\xi_b\}_{N_s}) P^{(\ell)}_{1-\cdots-L}
\end{pmatrix}. \tag{2.31}
\]
2.6. Gauge transformations for the spin-\(\ell/2\) representation

We shall now introduce higher-spin gauge transformations in order to formulate higher-spin monodromy matrices associated with principal grading.

In the spin-1/2 case we introduce a 2 × 2 diagonal matrix \(\Phi(w) = \text{diag}(1, \exp(w))\). In terms of tensor product notation (2.18) we define \(\Phi_j(w)\) for \(j = 1, 2, \ldots, L\), which act on the tensor product space \((V^{(1)})^\otimes L\). We define the spin-1/2 gauge transformation \(\chi_{12-\ldots-L}\) by

\[
\chi_{12-\ldots-L} = \Phi_1(w_1)\Phi_2(w_2)\cdots\Phi_L(w_L).
\]

Here \(w_j\) denote the inhomogeneity parameters of the spin-1/2 transfer matrix of the XXZ spin chain for \(j = 1, 2, \ldots, L\).

We now introduce the spin-\(\ell/2\) gauge transformation. Here we recall that the quantum space is given by the \(N_s\)th tensor product of the spin-\(\ell/2\) representations, \((V^{(\ell)})^\otimes N_s\), and also that \(w_{\ell(b-1)+\beta}^{(\ell)}\) form \(N_s\) sets of complete \(\ell\)-strings (2.24) for \(\beta = 1, 2, \ldots, \ell\) and \(b = 1, 2, \ldots, N_s\). We define an \((\ell + 1) \times (\ell + 1)\) diagonal matrix \(\Phi^{(\ell)}(w)\) by

\[
\Phi^{(\ell)}(w)\|\ell, n\rangle = \exp(nw)\|\ell, n\rangle \quad \text{for } n = 0, 1, \ldots, \ell.
\]

We then define the spin-\(\ell/2\) gauge transformation \(\chi_{12-\ldots-N_s}^{(\ell)}\) on the quantum space \((V^{(\ell)})^\otimes N_s\) by

\[
\chi_{12-\ldots-N_s}^{(\ell)} = \Phi_1^{(\ell)}(\Lambda_1)\Phi_2^{(\ell)}(\Lambda_2)\cdots\Phi_{N_s}^{(\ell)}(\Lambda_{N_s}).
\]

Here we have defined \(\Lambda_b\) by

\[
\Lambda_b = \xi_b - (\ell - 1)\eta/2, \quad \text{for } b = 1, 2, \ldots, N_s.
\]

For each integer \(b\), parameter \(\Lambda_b\) denotes the center of the \(b\)th complete \(\ell\)-string, \(w_{\ell(b-1)+\beta}^{(\ell)}\) \((\beta = 1, 2, \ldots, \ell)\).

Let us consider the tensor product of the auxiliary space \(V^{(1)}\) and the quantum space \((V^{(\ell)})^\otimes N_s\). We define the gauge transformation \(\chi_{0,12-\ldots-N_s}^{(1,\ell)}\) on \(V^{(1)} \otimes (V^{(\ell)})^\otimes N_s\) by

\[
\chi_{0,12-\ldots-N_s}^{(1,\ell)} = \Phi_0\Phi_1^{(\ell)}\cdots\Phi_{N_s}^{(\ell)}.
\]

Similarly, we define the spin-1/2 gauge transformation \(\chi_{0,12-\ldots-L}^{(1)}\) by

\[
\chi_{0,12-\ldots-L}^{(1)} = \Phi_0(\lambda)\Phi_1(w_1)\cdots\Phi_L(w_L),
\]

which acts on the tensor product \(V_0^{(1)} \otimes (V^{(1)})^\otimes L\).

2.7. Spin-\(\ell/2\) monodromy matrix associated with principal grading

We now construct the higher-spin monodromy matrix of type \((1, (\ell)^\otimes N_s)\) associated with principal grading. We denote it by \(T_{0,12-\ldots-N_s}^{(1,\ell)}(\lambda; \{\xi_b\}_{N_s})\), which acts on the tensor product of the auxiliary space \(V_0^{(1)}\) and the quantum space \(V_1^{(\ell)} \otimes \cdots \otimes V_{N_s}^{(\ell)}\). In the fusion construction we define it by

\[
T_{0,12-\ldots-N_s}^{(1,\ell)}(\lambda) = (\chi_{0,12-\ldots-N_s}^{(1,\ell)})^{-1}T_{0,12-\ldots-N_s}^{(1,\ell+\ldots)}(\lambda; \{\xi_b\}_{N_s})(\chi_{0,12-\ldots-N_s}^{(1,\ell)})
\]

\[
= (\chi_{0,12-\ldots-N_s}^{(1,\ell)})^{-1}(P_{12-\ldots-L}^{(\ell)}T_{0,12-\ldots-L}^{(1,\ell+\ldots)}(\lambda)P_{12-\ldots-L}^{(\ell)})(\chi_{0,12-\ldots-N_s}^{(1,\ell)}).
\]

\[\text{doi:10.1088/1742-5468/2012/04/P04001}\]
Thus, we construct the spin-\(\ell/2\) monodromy matrix associated with principal grading in the following three procedures: (i) we put complete \(\ell\)-strings \(w_j^{(\ell)}\) into inhomogeneity parameters \(w_j\) of the spin-1/2 monodromy matrix \(T_{0,12\cdots L}^{(1,1+)}(\lambda; \{w_j\}_L)\) of \(L\) sites associated with homogeneous grading \((w = +)\), where \(L = \ell N_s\); (ii) we multiply the spin-\(\ell/2\) projection operators \(P_{12\cdots L}^{(\ell)}\) to the spin-1/2 monodromy matrix \(T^{(1,\ell+0)}(\lambda)\); and (iii) we apply the spin-\(\ell/2\) gauge transformation \(\chi_{0,12\cdots N_s}^{(1,\ell)}\) to the spin-\(\ell/2\) monodromy matrix \(T^{(1,\ell+)}(\lambda; \{\xi_b\}_{N_s})\) with inhomogeneity parameters \(w_{\ell(b-1)+1}^{(\ell)} = \xi_b\) for \(b = 1, 2, \ldots, N_s\).

The matrix elements of the monodromy matrix of type \((1, (\ell)^{\otimes N_s})\) associated with principal grading give the fundamental operators of the algebraic Bethe ansatz for the integrable spin-1/2 XXZ spin chain:

\[
T_{0,12\cdots N_s}^{(1,\ell p)}(\lambda; \{\xi_b\}_{N_s}) = \begin{pmatrix}
A^{(\ell p)}(\lambda; \{\xi_b\}_{N_s}) & B^{(\ell p)}(\lambda; \{\xi_b\}_{N_s}) \\
C^{(\ell p)}(\lambda; \{\xi_b\}_{N_s}) & D^{(\ell p)}(\lambda; \{\xi_b\}_{N_s})
\end{pmatrix}
\]

(2.39)

Hereafter we shall often denote \(T^{(1,\ell w)}(\lambda)\) by \(T^{(\ell w)}(\lambda)\), briefly.

### 2.8. Higher-spin monodromy matrices of type \((\ell, (2s)^{\otimes N_s})\)

We now define the monodromy matrix of type \((\ell, (2s)^{\otimes N_s})\) associated with homogeneous grading. It acts on the tensor product of the auxiliary space \(V_{a_1\cdots a_\ell}\) and the quantum space \(V_1^{(2s)} \otimes \cdots \otimes V_{N_s}^{(2s)}\). Here we remark that for the quantum space we put \(2s\) into \(\ell\), i.e. \(\ell = 2s\), and set \(L = 2s N_s\). We construct \((V^{(2s)})^{\otimes N_s}\) in \((V^{(1)})^{\otimes L}\).

Let us express the tensor product \(V_0^{(\ell)} \otimes (V_1^{(2s)} \otimes \cdots \otimes V_{N_s}^{(2s)})\) by the following symbol:

\[
(\ell, (2s)^{\otimes N_s}) = (\ell, 2s, 2s, \ldots, 2s).
\]

(2.40)

Here we recall that \(V_0^{(\ell)}\) abbreviates \(V_{a_1\cdots a_\ell}\). For the auxiliary space \(V_0^{(\ell)}\) we define the monodromy matrix of type \((\ell, (2s)^{\otimes N_s})\) by

\[
T_{0,12\cdots N_s}^{(\ell,2s+)} = P_{a_1a_2\cdots a_\ell}^{(\ell)} T_{a_1,12\cdots N_s}^{(1,2s+)}(\lambda_{a_1}) T_{a_2,12\cdots N_s}^{(1,2s+)}(\lambda_{a_2} - \eta) \cdots
\]

\[
\times T_{a_\ell,12\cdots N_s}^{(1,2s+)}(\lambda_{a_\ell} - (\ell - 1)\eta) P^{(2s)}_{a_1a_2\cdots a_\ell}.
\]

(2.41)

Here we remark that it is associated with homogeneous grading.

Let us construct the higher-spin monodromy matrix of type \((\ell, (2s)^{\otimes N_s})\) associated with principal grading, which acts on the quantum space \(V_1^{(2s)} \otimes \cdots \otimes V_{N_s}^{(2s)}\). From the higher-spin monodromy matrices associated with homogeneous grading we derive them through the inverse of the gauge transformation as follows:

\[
T^{(\ell,2s+)}(\lambda) = (\chi_{a_1\cdots a_\ell,12\cdots N_s}^{(\ell,2s)})^{-1} T^{(\ell,2s+)}(\lambda)(\chi_{a_1\cdots a_\ell,12\cdots N_s}^{(\ell,2s)}).
\]

(2.42)

Here \(\chi_{a_1\cdots a_\ell,12\cdots N_s}^{(\ell,2s)}\) is given by

\[
\chi_{a_1\cdots a_\ell,12\cdots N_s}^{(\ell,2s)} = \Phi_{a_1\cdots a_\ell}^{(\ell)}(\Lambda_0) \Phi_{1}^{(2s)}(\Lambda_1) \cdots \Phi_{N_s}^{(2s)}(\Lambda_{N_s}),
\]

(2.43)

where \(\Lambda_0\) denotes the string center, \(\Lambda_0 = \lambda_{a_1} - (\ell - 1)\eta/2\).
3. Reduction of higher-spin elementary operators

3.1. Spin-ℓ/2 elementary operators associated with homogeneous and principal gradings

Let us consider the spin-ℓ/2 representation $V^{(\ell)}$ constructed in the $\ell$th tensor product space $(V^{(1)})^{\otimes \ell}$. We define the spin-ℓ/2 elementary operators associated with homogeneous grading, $E_{i,j}^{(\ell)}$, by

$$E_{i,j}^{(\ell)} = \|\ell, i\rangle\langle \ell, j\| \quad \text{for } i, j = 0, 1, \ldots, \ell. \quad (3.1)$$

We define the spin-ℓ/2 elementary matrices associated with principal grading, $E_{i,j}^{(\ell,p)}$, also by

$$E_{i,j}^{(\ell,p)} = \|\ell, i\rangle\langle \ell, j\| \quad \text{for } i, j = 0, 1, \ldots, \ell. \quad (3.2)$$

In this paper we define it by the same operator as that of homogeneous grading. We have

$$E_{i,j}^{(\ell)} = E_{i,j}^{(\ell,p)} = \|\ell, i\rangle\langle \ell, j\|. \quad (3.3)$$

Through (2.15), which expresses the projection operator in terms of the basis vectors and conjugate vectors, we have the following:

**Lemma 3.1.** In the $(\ell + 1)$-dimensional representation $V^{(\ell)}$ for the spin-ℓ/2 elementary operators with grading of $w = \pm, p$ and the spin-ℓ/2 projection operator we have

$$P^{(\ell)} E_{i,j}^{(\ell w)} = E_{i,j}^{(\ell w)} P^{(\ell)} = E_{i,j}^{(\ell w)}. \quad (3.4)$$

Let us recall that we have set $L = \ell N_s$ and the quantum space $(V^{(\ell)})^{\otimes N_s}$ is constructed in the $L$th tensor product space $(V^{(1)})^{\otimes L}$. We now introduce the spin-ℓ/2 elementary operators associated with grading of $w$, $E_k^{(\ell, tw)}$, acting on the $k$th component of the quantum space $(V^{(\ell)})^{\otimes N_s}$ as follows:

$$E_k^{(\ell, tw)} = (I^{(\ell)})^{\otimes (k-1)} \otimes E^{(\ell w)} \otimes (I^{(\ell)})^{\otimes (N_s-k)} \quad \text{for } k = 1, 2, \ldots, N_s. \quad (3.5)$$

3.2. Two expressions of a product of spin-1/2 elementary operators

For a given product of the spin-1/2 elementary operators we shall express it in another form. Let us first consider the simplest example. In terms of the highest weight vector $\|\ell, 0\rangle = |0\rangle_1 \otimes \cdots \otimes |0\rangle_\ell$ we have the following:

$$\|\ell, 0\rangle\langle \ell, 0\| = |0\rangle_1 \otimes \cdots \otimes |0\rangle_\ell \langle 0\rangle_1 \otimes \cdots \otimes \langle 0\rangle_\ell$$

$$= e_1^{0,0} \cdots e_\ell^{0,0}. \quad (3.6)$$

Thus, the product of the spin-1/2 elementary operators $e_1^{0,0} \cdots e_\ell^{0,0}$ is also expressed as $\|\ell, 0\rangle\langle \ell, 0\|$. Here we remark that $|0\rangle_1 = (1, 0)$ and $|0\rangle_1 = (1, 0)^T$, where the superscript $T$ denotes the matrix transposition. We thus have

$$|0\rangle_1 \langle 0\rangle_1 = (1, 0)^T (1, 0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = e_1^{0,0}. \quad (3.7)$$

We shall generalize relation (3.6) in the following.
Let us introduce symbols for expressing sequences. If a sequence of numbers, \(a_1, a_2, \ldots, a_N\), are given, we denote it by \((a_j)_N\), briefly, i.e. we have
\[
(a_j)_N = (a_1, a_2, \ldots, a_N).
\]
(3.8)

Here we recall that we denote by \(\{\mu_k\}_N\) a set of \(N\) parameters \(\mu_k\), i.e. \(\mu_1, \mu_2, \ldots, \mu_N\).

We now consider two sequences consisting of only two values 0 or 1, \((\varepsilon'_\alpha)_{\ell}\) and \((\varepsilon_\beta)_{\ell}\).

Here, the values of \(\varepsilon'_\alpha\) and \(\varepsilon_\beta\) are given by 0 or 1 for \(\alpha, \beta = 1, 2, \ldots, \ell\). Given such sequences \((\varepsilon'_\alpha)_{\ell}\) and \((\varepsilon_\beta)_{\ell}\) we consider the following product of the spin-1/2 elementary operators:
\[
\prod_{k=1}^{\ell} e^{\varepsilon'_\alpha} e^\varepsilon = e^{\varepsilon'_1} e^{\varepsilon_1} \cdots e^{\varepsilon'_\ell} e^{\varepsilon_\ell}.
\]
(3.9)

Here we recall that \(e^{\varepsilon'_\alpha} e^\varepsilon\) for \(\varepsilon', \varepsilon = 0, 1\) denote the \(2 \times 2\) matrices defined on the \(k\)th sites with unique nonzero element 1 at the entry \((\varepsilon', \varepsilon)\) for integers \(k\) satisfying \(1 \leq k \leq \ell\).

Let us give another expression of product (3.9). We define a set \(\bm{\alpha}^-\) by the set of integers \(k\) satisfying \(\varepsilon'_k = 1\) for \(1 \leq k \leq \ell\) and a set \(\bm{\alpha}^+\) by the set of integers \(k\) satisfying \(\varepsilon_k = 0\) for \(1 \leq k \leq \ell\), respectively:
\[
\bm{\alpha}^-(\{\varepsilon'_\alpha\}) = \{\alpha; \varepsilon'_\alpha = 1(1 \leq \alpha \leq \ell)\}, \quad \bm{\alpha}^+(\{\varepsilon_\beta\}) = \{\beta; \varepsilon_\beta = 0(1 \leq \beta \leq \ell)\}.
\]
(3.10)

Let us denote by \(\Sigma_\ell\) the set of integers 1, 2, \ldots, \(\ell\), i.e. \(\Sigma_\ell = \{1, 2, \ldots, \ell\}\). In terms of sets \(\bm{\alpha}^\pm\) we express the product of elementary operators given by (3.9) as
\[
\prod_{a \in \bm{\alpha}^-} \sigma^-_a \|\ell, 0\rangle \langle \ell, 0\| \prod_{b \in \Sigma_\ell \setminus \bm{\alpha}^+} \sigma^+_b.
\]
(3.11)

Now we derive the expression of (3.9) from that of (3.11) in detail. Let us denote by \(r\) and \(r'\) the number of elements of the set \(\bm{\alpha}^-\) and \(\bm{\alpha}^+\), respectively. We express the elements of \(\bm{\alpha}^-\) as \(a(k)\) for \(k = 1, 2, \ldots, r\) and those of \(\Sigma_\ell \setminus \bm{\alpha}^+\) as \(b(k)\) for \(k = 1, 2, \ldots, r'\), respectively. Expressing \(r\) and \(\ell - r'\) by \(i\) and \(j\), respectively, we have
\[
\bm{\alpha}^- = \{a(1), a(2), \ldots, a(i)\}, \quad \Sigma_\ell \setminus \bm{\alpha}^+ = \{b(1), b(2), \ldots, b(j)\}.
\]
(3.12)

Hereafter, if not specified, we shall put them in increasing order: \(1 \leq a(1) < \cdots < a(i) \leq \ell\) and \(1 \leq b(1) < \cdots < b(j) \leq \ell\), respectively. Here we recall \(i = r\) and \(j = \ell - r'\). We thus express the product of the elementary operators in terms of \(a(k)\) and \(b(k)\) as follows:
\[
\prod_{a \in \bm{\alpha}^-} \sigma^-_a \|\ell, 0\rangle \langle \ell, 0\| \prod_{b \in \Sigma_\ell \setminus \bm{\alpha}^+} \sigma^+_b = \sigma^-_{a(1)} \cdots \sigma^-_{a(i)} \|\ell, 0\rangle \langle \ell, 0\| \sigma^+_{b(1)} \cdots \sigma^+_{b(j)} = e^{1,0} \cdots e^{1,0} e^{0,1} \cdots e^{0,1}.
\]
(3.13)

Calculating products of \(2 \times 2\) matrices, from expression (3.11) we derive the expression in terms of products of the spin-1/2 elementary operators, \(e^{\varepsilon'_1} e^{\varepsilon_1} \cdots e^{\varepsilon'_\ell} e^{\varepsilon_\ell}\), such as given in (3.9). Here, we derive the sequence \((\varepsilon'_\alpha)_{\ell}\) by setting \(\varepsilon'_\alpha = 1\) for \(k = 1, 2, \ldots, i\) while \(\varepsilon'_\alpha = 0\) for \(\alpha \neq a(k)\) with \(k = 1, 2, \ldots, i\):\[
\varepsilon'_\alpha = \begin{cases} 1 & \text{if } \alpha = a(k)(1 \leq k \leq \ell), \\ 0 & \text{otherwise}. \end{cases}
\]
(3.14)
Similarly, we derive sequence \((\varepsilon'_\beta)_{\ell}\) by setting \(\varepsilon_{b(k)} = 1\) for \(k = 1, 2, \ldots, j\) while \(\varepsilon_\beta = 0\) for \(\beta \not= b(k)\) with \(k \leq 1 \leq k \leq j\).

Let us introduce useful notation. Suppose that we have a sequence \((\varepsilon'_{\alpha})_{\ell}\) such that \(\varepsilon'_{\alpha} = 0\) or 1 for all integers \(\alpha\) with \(1 \leq \alpha \leq \ell\) and the number of integers \(\alpha\) satisfying \(\varepsilon'_{\alpha} = 1\) (\(1 \leq \alpha \leq \ell\)) is given by \(i\). Then, we denote \(\varepsilon'_{\alpha}\) by \(\varepsilon'_{\alpha}(i)\) for each integer \(\alpha\) and the sequence \((\varepsilon'_{\alpha})_{\ell}\) by \((\varepsilon'_{\alpha}(i))_{\ell}\). In the same way, we denote by \((\varepsilon_{\beta}(j))_{\ell}\) a sequence of 0 or 1 such that the number of integers \(\beta\) satisfying \(\varepsilon_{\beta}(j) = 1\) for \(1 \leq \beta \leq \ell\) is given by \(j\).

The two expressions of a product of the spin-1/2 elementary matrices are summarized as follows.

**Lemma 3.2.** Sequences \((\varepsilon'_{\alpha(i)})_{\ell}\) and \((\varepsilon_{\beta(j)})_{\ell}\) are related to integers \(a(1) < a(2) < \cdots < a(i)\) and \(b(1) < b(2) < \cdots < b(j)\), respectively, by

\[
e_{1}^{(i)} \cdots e_{\ell}^{(j)} = e_{a(1)}^{1} \cdots e_{a(i)}^{0} \cdots e_{b(1)}^{0} \cdots e_{b(j)}^{1},
\]

\[
\prod_{k=1}^{\ell} e_{k}^{(i)} e_{k}^{(j)} = \prod_{a \in \alpha^-} \sigma_{a}^{-\|} \langle \ell, 0 \rangle \langle \ell, 0 \rangle \prod_{b \in \Sigma \alpha^+} \sigma_{b}^{+}.
\]

**3.3. Reduction into the spin-1/2 elementary operators**

We shall express the spin-\(\ell/2\) elementary operators \(E_{i,j}^{(\ell+)}\) for integers \(i\) and \(j\) satisfying \(1 \leq i, j \leq \ell\) in terms of sums of products of the spin-1/2 elementary matrices. It follows from (2.7) and (2.11) that we have

\[
\langle \ell, i \rangle \langle \ell, j \rangle = \sum_{(\varepsilon_{i}(i))_{\ell}} \sum_{(\varepsilon_{j}(j))_{\ell}} g_{ij}(\varepsilon'_{\alpha(i)}, \varepsilon_{\beta(j)}) e_{\ell}^{(i)} \cdots e_{\ell}^{(j)}.
\]

Here the sum is taken over all sequences \((\varepsilon'_{\alpha(i)})_{\ell}\) and \((\varepsilon_{\beta(j)})_{\ell}\). The coefficients \(g_{ij}(\varepsilon'_{\alpha(i)}, \varepsilon_{\beta(j)})\) are given explicitly as follows:

\[
g_{ij}(\varepsilon'_{\alpha(i)}, \varepsilon_{\beta(j)}) = \left[ \begin{array}{l} \ell \\ j \end{array} \right]^{-1} q^{a(1)+\cdots+a(i)+b(1)+\cdots+b(j)-(i+j)\ell+i(i-1)/2+j(j-1)/2}.
\]

The ket vectors \(\langle \ell, i \rangle\) satisfy the following symmetry, which plays a central role in the fusion method for evaluating the spin-\(\ell/2\) form factors.

**Lemma 3.3.** Let \(\alpha^-\) be a set of distinct integers \(\{a(1), \ldots, a(i)\}\) satisfying \(1 \leq a(1) < \cdots < a(i) \leq \ell\): we have the following:

\[
\langle \ell, i \| \sigma^{-a(1)} \cdots \sigma^{-a(i)} \| \ell, 0 \rangle q^{-(a(1)+\cdots+a(i))+i} = \left[ \begin{array}{l} \ell \\ i \end{array} \right]^{-1} q^{-i(i-1)/2},
\]

which is independent of the set \(\alpha^- = \{a(1), a(2), \ldots, a(i)\}\).

**Proof.** From the explicit expression (2.11) of the conjugate vector \(\langle \ell, i \|\) we have (3.19).

Expressing the matrix elements of the matrix \(\Phi(w)\) as \((\Phi(w))_{a,b} = \delta(a, b) \exp(aw)\) for \(a, b = 0, 1\), we show the gauge transformation \(\chi_{12-\ell}\) on the spin-1/2 elementary operators as follows.
Lemma 3.4. Recall that $\varepsilon_\alpha(i)$ and $\varepsilon_\beta(j)$ are related to $a(k)$ and $b(k)$ via (3.15). Every product of the spin-$1/2$ elementary operators is transformed with the gauge transformation as
\[
\chi_{12-\ell}^{\varepsilon_1'(i),\varepsilon_1(j)} \cdots \varepsilon_\ell'(i),\varepsilon_\ell(j) \chi_{12-\ell}^{-1} = e_1^{\varepsilon_1'(i)} \cdots e_\ell^{\varepsilon_\ell'(i)} q^{-(a(1)+\cdots+a(i)-i)+(b(1)+\cdots+b(j)-j)} e^{(i-j)\xi_1}.
\]
(3.20)

It is useful to express lemma 3.3 in the following form.

Corollary 3.5. For a pair of integers $i$ and $j$ with $1 \leq i, j \leq \ell$, let us consider sequences $(\varepsilon_\alpha'(i))_\ell$ and $(\varepsilon_\beta(j))_\ell$, which correspond to sets $\alpha^-$ and $\alpha^+$, respectively, through (3.15) and (3.16). The product of the spin-$1/2$ elementary operators multiplied by the projection operator from the left and multiplied also by $q^{-(a(1)+\cdots+a(i)+i)}$ does not depend on the set $\alpha^-$:
\[
P^{(l)}e_1^{\varepsilon_1'(i),\varepsilon_1(j)} \cdots e_\ell^{\varepsilon_\ell'(i),\varepsilon_\ell(j)} q^{-(a(1)+\cdots+a(i)+i)} = \left[ \begin{array}{c} \ell \\ i \end{array} \right]_q^{-1} q^{-(i-1)/2\parallel \ell, i)} \prod_{\beta \in \Sigma \alpha^+} \sigma_\beta^+.\]
(3.21)

In terms of the gauge transformation we express relation (3.21) as
\[
P^{(l)}\chi_{12-\ell}^{\varepsilon_1'(i),\varepsilon_1(j)} \cdots \varepsilon_\ell'(i),\varepsilon_\ell(j) \chi_{12-\ell}^{-1} = \left[ \begin{array}{c} \ell \\ i \end{array} \right]_q^{-1} q^{-(i-1)/2\parallel \ell, i)} \prod_{b \in \Sigma \alpha^+} \sigma_b^+ q^{b(1)+\cdots+b(j)-j}.
\]
(3.22)

Lemma 3.6. The sum of coefficients $g_{ij}(\varepsilon_\alpha'(i),\varepsilon_\beta(j))$ over all sequences $(\varepsilon_\alpha'(i))_\ell$ multiplied by $q^{a(1)+\cdots+a(i)-i}$ is given by the following:
\[
\sum_{(\varepsilon_\alpha'(i))_\ell} g_{ij}(\varepsilon_\alpha'(i),\varepsilon_\beta(j))q^{a(1)+\cdots+a(i)-i} = \left[ \begin{array}{c} \ell \\ i \end{array} \right]_q^{-1} q^{b(1)+\cdots+b(j)-j} q^{i(i-1)/2+j(j-1)/2}.
\]
(3.23)

Here we remark that we take the sum over all sequences of the form of $(\varepsilon_\alpha'(i))_\ell$.

Proof. Putting $n = i$ in (2.12) and observing that the sum over sequences $(\varepsilon_\alpha'(i))_\ell$ corresponds to the sum over integers $a(1),\ldots,a(i)$ satisfying $1 \leq a(1) < \cdots < a(i) \leq \ell$, we have (3.23) from (2.12).

We thus show the main formula for reducing the spin-$\ell/2$ operators into the spin-$1/2$ ones.

Proposition 3.7. For every pair of integers $i$ and $j$ with $1 \leq i, j \leq \ell$ the spin-$\ell/2$ elementary operator associated with grading $w$, $E_1^{i,j(\ell w)}$, is decomposed into a sum of products of the spin-$1/2$ elementary operators as follows:
\[
E_1^{i,j(\ell w)} = \left[ \begin{array}{c} \ell \\ i \end{array} \right]_q q^{-(i-1)/2-j(j-1)/2} e^{-(i-j)\xi_1} P^{(l)}_{12-\ell} \sum_{(\varepsilon_\beta(j))_\ell} \chi_{12-\ell}^{\varepsilon_1'(i),\varepsilon_1(j)} \cdots \varepsilon_\ell'(i),\varepsilon_\ell(j) \chi_{12-\ell}^{-1}.
\]
(3.24)

Here, we fix a sequence $(\varepsilon_\alpha'(i))_\ell$. Furthermore, the expression (3.24) does not depend on the order of $\varepsilon_\alpha'(i)$ with respect to $\alpha$.
We shall show the derivation of proposition 3.7 explicitly in appendix A.

In terms of the string center: $\Lambda_1 = \xi_1 - (\ell - 1)\eta/2$, the $q$ factors in equation (3.24) are expressed as follows:

\[
q^{i(i-1)/2-j(j-1)/2}e^{-(i-j)\xi_1} = \frac{q^{-i(\ell-i)/2+j(\ell-j)/2}e^{-(i-j)(\xi_1-(-1)\eta/2)}}{q^{-i(\ell-i)/2+j(\ell-j)/2}e^{-(i-j)\Lambda_1}}.
\]

(3.25)

Thus, introducing the symbol

\[
N_{i,j}^{\ell} = \frac{\ell}{i} \frac{\ell}{j} q^{-i(\ell-i)/2+j(\ell-j)/2},
\]

we express (3.24) compactly as follows:

\[
E_{i,j}^{\ell}(w) = N_{i,j}^{\ell} e^{-(i-j)\Lambda_1} P_{12}^{\ell} \sum_{(\epsilon_\ell(\ell)}) \chi_{12} e^{\frac{1}{2}(i,j)\Lambda_1} e^{(i-j)\eta/2} e^{(i-j)(\xi_1-(-1)\eta/2)}.
\]

(3.27)

In [13] the Hermitian elementary operators $E_{i,j}^{\ell}(w)$ are introduced. The expectation values of the Hermitian elementary operators are the same as those of the standard elementary operators, $E_{i,j}^{\ell}(w)$. We shall show the reduction formula for the Hermitian elementary operators in appendix B.

3.4. General spin-$\ell/2$ elementary operators

Let us consider a similarity transformation of the basis vectors as follows:

\[
\|\ell, m\| \rightarrow \|\ell, m\|/g(m), \quad \langle\ell, n\| \rightarrow g(n)\langle\ell, n\|, \quad \text{for } m, n = 0, 1, \ldots, \ell.
\]

(3.28)

In the spin-$\ell/2$ representation constructed in the $\ell$th tensor product space $(V^{(1)})^{\otimes \ell}$, we define the general spin-$\ell/2$ elementary operators associated with principal grading, $\hat{E}_{i,j}^{\ell}(w)$, by

\[
\hat{E}_{i,j}^{\ell}(w) = \|\ell, i\rangle \langle\ell, j\| \frac{g(j)}{g(i)}, \quad \text{for } i, j = 0, 1, \ldots, \ell.
\]

(3.29)

Then, through the spin-$\ell/2$ gauge transformation we define the general spin-$\ell/2$ elementary operators associated with homogeneous grading by

\[
\hat{E}_{i,j}^{\ell}(\pm) = \chi_{12}^{\ell} \hat{E}_{i,j}^{\ell}(w)(\chi_{12}^{\ell})^{-1}.
\]

(3.30)

We explicitly have

\[
\hat{E}_{i,j}^{\ell}(\pm) = \|\ell, i\rangle \langle\ell, j\| \frac{g(j)}{g(i)} e^{\ell-2(j-\ell)(\xi_1-(-1)\eta/2)}, \quad \text{for } i, j = 0, 1, \ldots, \ell.
\]

(3.31)

Here we recall that the quantity $\xi - (\ell - 1)\eta/2$ corresponds to the string center of the $\ell$-string: $\xi_1, \xi - \eta, \ldots, \xi - (\ell - 1)\eta$. They are originally the evaluation parameters of the $\ell$th tensor product of the spin-1/2 representations, $(V^{(1)})^{\otimes \ell}$.

We remark that the definition of the general elementary operators $\hat{E}_{i,j}^{\ell}(w)$ associated with grading $w$ are covariant under the gauge transformations. We also remark that if we put $g(j) = \exp(j(\xi - (\ell - 1)\eta/2))$, then expression (3.31) reduces to that of $E_{i,j}^{\ell}(\pm)$. 

doi:10.1088/1742-5468/2012/04/P04001
Corollary 3.9. Let us take integers $i$ and $j$ satisfying $0 \leq i, j \leq \ell$. The product of the general spin-$\ell/2$ elementary operators, $\hat{E}^{i_1,j_1}_{k_1}(\ell w) \ldots \hat{E}^{i_m,j_m}_{k_m}(\ell w)$, is expressed in terms of a sum of products of the spin-$1/2$ elementary operators as
\[
\prod_{k=1}^{m} \hat{E}^{i_k,j_k}_{k}(\ell w) = \prod_{k=1}^{m} \left( \hat{N}^{(\ell)}_{i, k} - (i_k - j_k) \Lambda_k \delta(w, p) \right) P^{(\ell)}_{1 \ldots \ell} \sum_{(\varepsilon^{(i)}_{1}(j_1))_\ell} \ldots \sum_{(\varepsilon^{(m)}_{j}(j_m))_\ell} \chi^{(\ell \ldots \ell)}_{12 \ldots \ell}.
\] (3.37)

Here we fix a sequence $(\varepsilon^{(i)}_{\alpha}(i))_\ell$ for each integer $k$ of $1 \leq k \leq m$.
The expression (3.37) is useful for deriving the multiple-integral representations of correlation functions for the integrable higher-spin XXZ spin chain, as we shall see in section 6.

### 3.5. Quantum inverse-scattering problem for the spin-$\ell/2$ operators

Let us recall the formula of the quantum inverse-scattering problem (QISP) for the spin-1/2 XXZ spin chain [6, 7]:

$$x_n = \prod_{k=1}^{n-1} (A^{(1w)} + D^{(1w)})(w_k) \cdot \text{tr}_0(x_0 T_{0,12...L}^{(1w)}(w_n)) \cdot \prod_{k=1}^{n} (A^{(1w)} + D^{(1w)})^{-1}(w_k).$$

(3.38)

Here we assume that inhomogeneity parameters $w_j$ are given by generic values so that the transfer matrices $(A^{(1w)} + D^{(1w)})(w_k)$ are regular for $k = 1, 2, \ldots, n$.

Making use of the QISP formula (3.38) we have the following expressions for $b = 1, 2, \ldots, N_s$:

$$e_{\ell(b-1)+1}^{\epsilon_1} \cdots e_{\ell(b-1)+\ell}^{\epsilon_{\ell}} = \prod_{k=1}^{\ell(b-1)} (A^{(1w)}(w_k) + D^{(1w)}(w_k))$$

$$\times T_{\epsilon_1,\epsilon_1}^{(1w)}(w_{\ell(b-1)+1}) \cdots T_{\epsilon_\ell,\epsilon_\ell}^{(1w)}(w_{\ell(b-1)+\ell}) \prod_{k=1}^{\ell b} (A^{(1w)}(w_k) + D^{(1w)}(w_k))^{-1}. \quad (3.39)$$

Here we have denoted by $T_{\alpha,\beta}(\lambda)$ the $(\alpha, \beta)$ element of the monodromy matrix $T(\lambda)$.

Applying (3.39) to reduction formula (3.24) (or (3.36)) we obtain the QISP formula for the spin-$\ell/2$ local operators. For an illustration, we show the case of $b = 1$ as follows:

$$\hat{E}_{1}^{ij(w)} = \hat{N}_{1,3}^{(1)} e^{-(i-j)A_{12}(w,p)} P_{1-\ell}^{(1)} \sum_{(\epsilon_{3}(j))_{\ell}} T_{\epsilon_1,\epsilon_1}^{(1w)}(w_1) \cdots T_{\epsilon_{\ell},\epsilon_{\ell}}^{(1w)}(w_{\ell})$$

$$\times \prod_{k=1}^{\ell} (A^{(1w)}(w_k) + D^{(1w)}(w_k))^{-1} \chi_{12-\ell}. \quad (3.40)$$

Here, we fix a sequence $(\epsilon_{\alpha}(i))_{\ell}$.

### 3.6. Non-regularity of the transfer matrix at special points

Let us consider the sector of $M$ down-spins on the spin-1/2 chain with $L$ sites.

**Proposition 3.10.** In the sector of $M$ down-spins with $1 \leq M \leq L - 1$, the spin-1/2 transfer matrix $A^{(\ell w; 0)}(\lambda) + D^{(\ell w; 0)}(\lambda)$ is non-regular at $\lambda = w_{\ell(k-1)+1}^{(\ell)} + n\pi \sqrt{-1}$ for $k = 1, 2, \ldots, N_s$ and $n \in \mathbb{Z}$. Here, $w_{\ell(k-1)+1}^{(\ell)}$ is the first rapidity of the $k$th complete $\ell$-string.

**Proof.** Calculating the matrix elements of the transfer matrix $A^{(\ell w; 0)}(\lambda) + D^{(\ell w; 0)}(\lambda)$ in the sector of $M$ down-spins we show that there exists a pair of column vectors that are parallel to each other if $\lambda = \xi_k$. 

\[ \text{doi:10.1088/1742-5468/2012/04/P04001} \]
Thus, the inverse matrix of the spin-1/2 transfer matrix $A^{(fw;0)}(\lambda) + D^{(fw;0)}(\lambda)$ does not exist at the special points. We remark that, in the sector of $M = 0$ (and $M = L$), it is regular at $\lambda = \xi_k + n\pi\sqrt{-1}$ for $k = 1, 2, \ldots, N_s$ and $n \in \mathbb{Z}$.

For an illustration, let us consider the case of $L = 2$ with $\ell = 1$ and $N_s = 1$. The operators $A$ and $D$ are explicitly given by

$$A_{12}^{(2;0)}(\lambda) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b_{02} & c_{01}^+c_{02}^- & 0 \\ 0 & 0 & b_{01} & 0 \\ 0 & 0 & 0 & b_{01}b_{02} \end{pmatrix}_{[1,2]},$$

$$D_{12}^{(2;0)}(\lambda) = \begin{pmatrix} b_{01}b_{02} & 0 & 0 & 0 \\ 0 & b_{01} & 0 & 0 \\ 0 & c_{01}^-c_{02}^+ & b_{02} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{[1,2]},$$

(3.41)

Here we have introduced $b_{0j}$ and $c_{0j}^\pm$ for $j = 1, 2$ by $b_{0j} = b(\lambda - w_j^{(2)})$ and $c_{0j}^\pm = \exp(\pm(\lambda - w_j^{(2)}))c(\lambda - w_j^{(2)})$ for $j = 1, 2$, respectively. Putting $\lambda = w_1^{(2)} = \xi_1$ we have

$$A_{12}^{(2;0)}(\xi_1) + D_{12}^{(2;0)}(\xi_1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2\eta} & q_1 & 0 \\ 0 & 0 & \frac{1}{2\eta} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{[1,2]},$$

(3.42)

We thus show that the transfer matrix is non-regular at $\lambda = w_1^{(2)} = \xi_1$:

$$\det(A_{12}^{(2;0)}(\xi_1) + D_{12}^{(2;0)}(\xi_1)) = 0.$$  

(3.43)

In the sector of $M = 1$ the determinant is given by

$$\det(A_{12}^{(2;0)}(\lambda) + D_{12}^{(2;0)}(\lambda)) \bigg|_{M=1} = \frac{4\sinh(\lambda - \xi_1)}{\sinh(\lambda - \xi_1 + 2\eta)}.$$  

(3.44)

For an illustration, we shall show in appendix C that there exists a pair of column vectors that are parallel to each other if we set $\lambda = \xi_k$, in the sector of $M = 1$, for the case of $L = 3$ with $w_1 = w_1^{(2)}$, $w_1 = w_2^{(2)}$ and $w_3 = \xi_2$.

Consequently, the QISP formula does not hold in the straightforward form for the operator-valued matrix elements of the monodromy matrix $T^{(fw;0)}(\lambda)$ for $w = \pm, p$ at $\lambda = w_\ell^{(k-1)+1}$ for $k = 1, 2, \ldots, N_s$. Here we recall that the monodromy matrix $T^{(fw;0)}(\lambda)$ is given by the spin-1/2 monodromy matrix $T^{(fw);\ell}(\lambda)$ by putting $\epsilon = 0$.

4. Reduction of the matrix elements of spin-$\ell/2$ operators

4.1. Definition of the spin-$\ell/2$ matrix elements and form factors

Let $|0\rangle$ be the vacuum vector of the spin-1/2 chain of $L$ sites, i.e. $|0\rangle = |\uparrow\rangle_1 \otimes \cdots \otimes |\uparrow\rangle_L$. Here we recall that the symbol $\{\lambda_\alpha\}_M$ denotes a set of $M$ parameters $\lambda_\alpha$ for $\alpha = 1, 2, \ldots, M$.  

doi:10.1088/1742-5468/2012/04/P04001
We consider two sets of arbitrary parameters, \( \{ \mu_\alpha \}_N \) and \( \{ \lambda_\beta \}_M \). We define the off-shell Bethe covectors and vectors, \( \langle \{ \mu_\alpha \}_N^{(w)} \rangle \) and \( \{ \lambda_\beta \}_M^{(w)} \), respectively, for \( w = \pm, p \) as follows:

\[
\langle \{ \mu_\alpha \}_N^{(w)} \rangle = \langle 0 \rangle \prod_{\alpha=1}^N C^{(w)}(\mu_\alpha), \quad \{ \lambda_\beta \}_M^{(w)} = \prod_{\beta=1}^M B^{(w)}(\lambda_\beta) |0\rangle. \tag{4.1}
\]

Furthermore, we define the spin-\( \ell/2 \) off-shell matrix elements of \( E^{i_k,j_k(\ell w)}_k \) for \( w = \pm, p \) by

\[
M^{i_k,j_k(\ell w)}_k(\{ \mu_\alpha \}_N, \{ \lambda_\beta \}_M) = \langle \{ \mu_\alpha \}_N^{(w)} \| E^{i_k,j_k(\ell w)}_k \| \{ \lambda_\beta \}_M^{(w)} \rangle, \tag{4.2}
\]

and those of the general elementary operators \( \hat{E}^{i_k,j_k(\ell w)}_k \) for \( w = \pm, p \) by

\[
\hat{M}^{i_k,j_k(\ell w)}_k(\{ \mu_\alpha \}_N, \{ \lambda_\beta \}_M) = \langle \{ \mu_\alpha \}_N^{(w)} \| \hat{E}^{i_k,j_k(\ell w)}_k \| \{ \lambda_\beta \}_M^{(w)} \rangle. \tag{4.3}
\]

If \( \{ \mu_\alpha \}_N \) and \( \{ \lambda_\beta \}_M \) satisfy the Bethe ansatz equations, we call the covectors and the vectors in (4.1) the Bethe covectors and vectors, respectively, and also call them \textit{on-shell}. Furthermore, if \( \{ \mu_\alpha \}_N \) and \( \{ \lambda_\beta \}_M \) satisfy the Bethe ansatz equations, we call the off-shell matrix elements in (4.2) and (4.3) \textit{on-shell}.

Let us assume that \( \{ \mu_\alpha \}_N \) and \( \{ \lambda_\beta \}_M \) satisfy the Bethe ansatz equations, i.e. they are two sets of Bethe roots. In terms of the Bethe covectors and vectors, \( \langle \{ \mu_\alpha \}_N^{(w)} \rangle \) and \( \{ \lambda_\beta \}_M^{(w)} \), we define the spin-\( \ell/2 \) form factors of \( E^{i_k,j_k(\ell w)}_k \) for \( w = \pm, p \) by

\[
F^{i_k,j_k(\ell w)}_k(\{ \mu_\alpha \}_N, \{ \lambda_\beta \}_M) = \langle \{ \mu_\alpha \}_N^{(w)} \| E^{i_k,j_k(\ell w)}_k \| \{ \lambda_\beta \}_M^{(w)} \rangle, \tag{4.4}
\]

and those of the general elementary operators \( \hat{E}^{i_k,j_k(\ell w)}_k \) for \( w = \pm, p \) by

\[
\hat{F}^{i_k,j_k(\ell w)}_k(\{ \mu_\alpha \}_N, \{ \lambda_\beta \}_M) = \langle \{ \mu_\alpha \}_N^{(w)} \| \hat{E}^{i_k,j_k(\ell w)}_k \| \{ \lambda_\beta \}_M^{(w)} \rangle. \tag{4.5}
\]

We have defined the form factors of a local operator by the \textit{on-shell} matrix elements of the operator, i.e. by the matrix elements between all pairs of the Bethe eigenvectors. However, it is often the case in many papers that only the matrix elements between the ground state and excited states are called form factors.

### 4.2. Commutation relations with projection operator

**Lemma 4.1.** If spectral parameter \( \lambda \) is distinct from discrete values such as \( w_j^{(f)} - \eta + n\pi \sqrt{1} \) for \( j = 1, 2, \ldots, L \) and \( n \in \mathbb{Z} \), the projection operator \( P^{(f)}_{12-..L} \) commutes with the matrix elements of the monodromy matrix \( T^{(f)}_{0,12-..L}(\lambda) = T^{(1+)}_{0,12-..L}(\lambda; \{ w_j^{(f)} \}_L) \) as follows:

\[
P^{(f)}_{12-..L} T^{(1+)}_{0,12-..L}(\lambda; \{ w_j^{(f)} \}_L) P^{(f)}_{12-..L} = P^{(f)}_{12-..L} T^{(1+)}_{0,12-..L}(\lambda; \{ w_j^{(f)} \}_L). \tag{4.6}
\]

Let us assume that all the parameters in \( \{ \mu_\alpha \}_N \) and \( \{ \lambda_\beta \}_M \) are different from the discrete values given by \( w_j^{(f)} - \eta + n\pi \sqrt{1} \) for \( j = 1, 2, \ldots, L \) and \( n \in \mathbb{Z} \). Here we recall that they correspond to \( \hat{N}_s \) pieces of complete \( \ell \)-strings minus \( \eta \) modulo \( \pi \sqrt{-1} \) and the
transfer matrix is singular at these points. Applying lemma 4.1 we have

\[ |\{\lambda \beta\}_M^{(\ell+)}\rangle = \prod_{\beta=1}^{M} (P_{12...L}^{(\ell)} B^{(\ell;0)}(\lambda \beta) P_{12...L}^{(\ell)}) |0\rangle = P_{12...L}^{(\ell)} \prod_{\beta=1}^{M} B^{(\ell;0)}(\lambda \beta) |0\rangle \]  

(4.7)

and

\[ \langle\{\mu \alpha\}_N^{(\ell+)}| = \langle 0| \prod_{\alpha=1}^{N} (P_{12...L}^{(\ell)} C^{(\ell;0)}(\mu \alpha) P_{12...L}^{(\ell)}) = \langle 0| \prod_{\alpha=1}^{N} C^{(\ell;0)}(\mu \alpha). \]  

(4.8)

Here we remark that for the off-shell Bethe covectors we can absorb the projection operator acting to the left as follows:

\[ \langle 0| \prod_{\alpha=1}^{N} C^{(\ell;0)}(\mu \alpha) \cdot P_{12...L}^{(\ell)} = \langle 0| \prod_{\alpha=1}^{N} C^{(\ell;0)}(\mu \alpha). \]  

(4.9)

However, in equation (4.7) we cannot remove the projection operator acting to the right.

It follows from (4.7) and (4.8) that we can evaluate the spin-\(\ell/2\) off-shell matrix elements by calculating the spin-1/2 ones. For instance, applying (4.7) and (4.8) we reduce every spin-\(\ell/2\) off-shell matrix element into an off-shell matrix element between the spin-1/2 off-shell Bethe covector and vector as follows:

\[ \hat{M}_k^{i,j(\ell+)}(\{\mu \alpha\}_N, \{\lambda \beta\}_M) = \langle 0| \prod_{\alpha=1}^{N} C^{(\ell;0)}(\mu \alpha) \hat{E}_k^{i,j(\ell+)} P_{12...L}^{(\ell)} \prod_{\beta=1}^{M} B^{(\ell;0)}(\lambda \beta) |0\rangle \]

\[ = \langle 0| \prod_{\alpha=1}^{N} C^{(\ell;0)}(\mu \alpha) \hat{E}_k^{i,j(\ell+)} \prod_{\beta=1}^{M} B^{(\ell;0)}(\lambda \beta) |0\rangle. \]  

(4.10)

Here we have made use of lemma 3.1 in order to delete the projection operator.

We remark that in [12]–[14] there was a nontrivial assumption that the projection operator should commute with the operator-valued matrix elements of the spin-1/2 monodromy matrix \(T^{(\ell;w;0)}(\lambda)\) at an arbitrary value of the spectral parameter \(\lambda\). In fact, the spin-1/2 monodromy matrix becomes singular if \(\lambda\) is equal to some discrete values such as \(w_j - \eta\). At \(\lambda = w_j - \eta\) the commutation relation of the monodromy matrix with the projection operator becomes nontrivial. If we multiply it with normalization factor \(\sinh(\lambda - w_j + \eta)\) and define the normalized monodromy matrix, then its commutation relation with the projection operator becomes valid at \(\lambda = w_j - \eta\).

4.3. Reduction of the spin-\(\ell/2\) matrix elements into the spin-1/2 ones

For homogeneous gradings with \(w = \pm\) and principal grading with \(w = p\), we define \(\sigma(w)\) by

\[ \sigma(w) = \begin{cases} 
+1 & \text{for } w = \pm, \\
0 & \text{for } w = p.
\end{cases} \]  

(4.11)

We denote by \(S_n\) the symmetric group of \(n\) elements.

doi:10.1088/1742-5468/2012/04/P04001
Reduction formula of form factors for the integrable spin-$\alpha$ XXZ chains and application to correlation functions

**Proposition 4.2.** Let $i_1$ and $j_1$ be integers satisfying $1 \leq i_1, j_1 \leq \ell$. For arbitrary parameters $\{\mu_\alpha\}_N$ and $\{\lambda_\beta\}_M$ with $i_1 - j_1 = N - M$ we have

\[
\hat{M}_{i_1,j_1}(\{\mu_k\}_N, \{\lambda_\beta\}_M) = \langle 0 | \prod_{k=1}^{N} C^{(\ell w)}(\mu_k) \cdot \hat{E}_{i_1,j_1}(\ell w) \cdot \prod_{\beta=1}^{M} B^{(\ell w)}(\lambda_\beta) | 0 \rangle
\]

\[
= \hat{N}_{i_1,j_1}^{(\ell)} e^{\sigma(w)(\sum_k \mu_k - \sum_\gamma \lambda_\gamma)} \sum_{(c_\beta(j))_\ell} \langle 0 | \prod_{\alpha=1}^{N} C^{(\ell p;0)}(\mu_\alpha) \cdot \epsilon_1^+(i_1),\epsilon_1^+(j_1) \cdots \\
\times \epsilon_\ell^+(i_1),\epsilon_\ell^+(j_1) \cdot \prod_{\beta=1}^{M} B^{(\ell p;0)}(\lambda_\beta) | 0 \rangle. \tag{4.12}
\]

Each summand is symmetric with respect to exchange of $\epsilon_\alpha^+(i_1)$, i.e. the following expression is independent of any permutation $\pi \in S_\ell$:

\[
\langle 0 | \prod_{\alpha=1}^{N} C^{(\ell p;0)}(\mu_\alpha) \cdot \epsilon_1^+(i_1),\epsilon_1^+(j_1) \cdots \epsilon_\ell^+(i_1),\epsilon_\ell^+(j_1) \cdot \prod_{\beta=1}^{M} B^{(\ell p;0)}(\lambda_\beta) | 0 \rangle. \tag{4.13}
\]

**Proof.** In the case of homogeneous grading with $w = +$, we put (3.36) into (4.10) and we have (4.12) through the gauge transformation

\[
\langle 0 | \prod_{\alpha=1}^{N} C^{(\ell+0)}(\mu_\alpha) = e^{\sigma^0} \langle 0 | \prod_{\alpha=1}^{N} C^{(\ell p;0)}(\mu_\alpha) \cdot \chi_{1\ldots L}^{-1}, \tag{4.14}
\]

\[
\prod_{\beta=1}^{M} B^{(\ell+0)}(\lambda_\beta) | 0 \rangle = \chi_{1\ldots L} \cdot \prod_{\beta=1}^{M} B^{(\ell p;0)}(\lambda_\beta) | 0 \rangle e^{-\sum_\beta \lambda_\beta}.
\]

In the case of principal grading with $w = p$ we shall show explicitly in appendix D the following:

\[
\langle 0 | \prod_{\alpha=1}^{N} C^{(\ell p)}(\mu_\alpha) = \langle 0 | \prod_{k=1}^{N} C^{(\ell p;0)}(\mu_k) \cdot \chi_{1\ldots L}^{-1} P_{1\ldots L}^{(\ell)} \chi_{1\ldots N_\alpha}, \tag{4.15}
\]

\[
\prod_{\alpha=1}^{N} B^{(\ell p)}(\lambda_\alpha) | 0 \rangle = \left(\chi_{1\ldots N_\alpha}^{(\ell)}\right)^{-1} P_{1\ldots L}^{(\ell)} \cdot \prod_{\alpha=1}^{M} B^{(\ell p;0)}(\lambda_\alpha) | 0 \rangle.
\]

We thus evaluate the form factor as follows:

\[
F_{i_1,j_1}(\{\mu_\alpha\}_N, \{\lambda_\beta\}_M) = \langle 0 | \prod_{\alpha=1}^{N} C^{(\ell p)}(\mu_\alpha) \cdot E_{i_1,j_1}^{(\ell p)} \cdot \prod_{\beta=1}^{M} B^{(\ell p)}(\lambda_\beta) | 0 \rangle
\]

\[
= \langle 0 | \prod_{\alpha=1}^{N} C^{(\ell p;0)}(\mu_\alpha) \cdot \chi_{1\ldots L}^{-1} P_{1\ldots L}^{(\ell)} \cdot E_{i_1,j_1}^{(\ell p)} \\
\cdot \left(\chi_{1\ldots N_\alpha}^{(\ell)}\right)^{-1} P_{1\ldots L}^{(\ell)} \cdot \prod_{\beta=1}^{M} B^{(\ell p;0)}(\lambda_\beta) | 0 \rangle
\]

doi:10.1088/1742-5468/2012/04/P04001
Reduction formula of form factors for the integrable spin-$\alpha$ XXZ chains and application to correlation functions

$$
\langle 0 | \prod_{\alpha=1}^{N} C^{(\ell p)(0)}(\mu_{\alpha}) \cdot \chi_{1-L}^{(\ell)} \cdot \prod_{\beta=1}^{M} B^{(\ell p)(0)}(\lambda_{\beta}) | 0 \rangle = \prod_{\alpha=1}^{N} \sum L = \prod_{\beta=1}^{M} B^{(\ell p)(0)}(\lambda_{\beta}) | 0 \rangle.
$$

(4.16)

Here we remark that

$$
\chi_{1-L}^{(\ell)} \cdot \sum_{i}^{(\ell)} = E_{1}^{(\ell)} \cdot (\chi_{1-L}^{(\ell)})^{-1} = E_{1}^{(\ell)} \cdot (\chi_{1-L}^{(\ell)})^{(\ell)} = E_{1}^{(\ell)} \cdot (\chi_{1-L}^{(\ell)})^{(\ell)}.
$$

(4.17)

Applying proposition 3.7 we obtain equation (4.12) for the case of $w = p$.

\[\square\]

**Corollary 4.3.** Let us take integers $i_{k}$ and $j_{k}$ satisfying $1 \leq i_{k}, j_{k} \leq L$ for $k = 1, 2, \ldots, m$. For arbitrary sets of parameters $\{\mu_{\alpha}\}_{N}$ and $\{\lambda_{\beta}\}_{M}$ with $\sum_{k=1}^{m} i_{k} - \sum_{k=1}^{m} j_{k} = N - M$, we have the matrix element for the $m$th product of the spin-$\ell/2$ elementary operators with entries $(i_{k}, j_{k})$ $(1 \leq k \leq m)$ associated with principal grading as follows:

$$
\langle 0 | \prod_{\alpha=1}^{N} C^{(\ell w)(\mu_{\alpha})} \cdot \prod_{k=1}^{m} \hat{E}_{k}^{(\ell w)}(\mu_{\alpha}) \cdot \prod_{\beta=1}^{M} B^{(\ell w)(\lambda_{\beta})} | 0 \rangle = \prod_{k=1}^{m} \hat{N}^{(\ell)}_{i_{k}, j_{k}} \cdot e^{\sigma(w)(\sum_{k} m_{k} - \sum_{\gamma} \gamma_{\gamma})} \times \sum_{(\ell_{\beta}(\mu_{\alpha}(k)))}^{(\ell_{\beta}(\mu_{\alpha}(k)))} \prod_{k=1}^{m} e^{e^{(\ell_{\beta}(\mu_{\alpha}(k)))}_{(\ell_{\beta}(\mu_{\alpha}(k)))}} \cdot \prod_{\beta=1}^{M} B^{(\ell p)(0)}(\lambda_{\beta}) | 0 \rangle.
$$

(4.18)

The summand is symmetric with respect to the exchange of $\epsilon_{\alpha}(i_{k})$s for each $k$ of $1 \leq k \leq m$, i.e. the following expression is independent of any permutation $\pi^{(k)} \in S_{k}$ for $k = 1, 2, \ldots, m$:

$$
\langle 0 | \prod_{\alpha=1}^{N} C^{(\ell p)(0)}(\mu_{\alpha}) \cdot \prod_{k=1}^{m} e^{e^{(\ell_{\beta}(\mu_{\alpha}(k)))}_{(\ell_{\beta}(\mu_{\alpha}(k)))}} \cdot \prod_{\beta=1}^{M} B^{(\ell p)(0)}(\lambda_{\beta}) | 0 \rangle.
$$

(4.19)

\[\square\]

**4.4. Consequence of the continuity assumption of the Bethe roots**

We now consider the Bethe ansatz equations for the integrable spin-$\ell/2$ XXZ spin chain with inhomogeneity parameters $\xi_{b}$ for $b = 1, 2, \ldots, N_{s}$:

$$
\frac{d^{(\ell)}(\lambda_{\alpha})}{d^{(\ell)}(\lambda_{\alpha}; \{\xi_{k}\}_{N_{s}})} = \prod_{\beta=1, \beta \neq \alpha}^{M} \sinh(\lambda_{\alpha} - \lambda_{\beta} + \eta) \sinh(\lambda_{\alpha} - \lambda_{\beta} - \eta) \quad (\alpha = 1, 2, \ldots, M).
$$

(4.20)

Here $M$ denotes the number of down-spins. We recall $L = \ell N_{s}$. For a positive integer $\ell$ we have set $a^{(\ell)}(\lambda_{\alpha}) = 1$ and defined $d^{(\ell)}(\mu; \{\xi_{k}\}_{N_{s}})$ by

$$
d^{(\ell)}(\mu; \{\xi_{k}\}_{N_{s}}) = \prod_{k=1}^{N_{s}} \frac{\sinh(\mu - \xi_{k})}{\sinh(\mu - \xi_{k} + \ell \eta)}.
$$

(4.21)

Let $\{\lambda_{\gamma}\}_{M}$ be a solution of the Bethe ansatz equations of the spin-$\ell/2$ chain with inhomogeneity parameters $\{\xi_{k}\}_{N_{s}}$. Suppose that $\{\lambda_{\beta}(\epsilon)\}_{M}$ denotes a solution of the spin-$1/2$ Bethe ansatz equations with inhomogeneity parameters $w_{j}$ being given by the $N_{s}$ sets

doi:10.1088/1742-5468/2012/04/P04001
Reduction formula of form factors for the integrable spin-$\alpha$ XXZ chains and application to correlation functions

of the almost complete $\ell$-strings, $w_j^{(\ell\epsilon)}$ for $j = 1, 2, \ldots, L$. They satisfy the Bethe ansatz equations for the spin-1/2 XXZ spin chain:

$$\frac{a(\lambda_\alpha(\epsilon))}{d(\lambda_\alpha(\epsilon); \{w_j^{(\ell\epsilon)}\}_L)} = \prod_{\beta=1; \beta \neq \alpha}^M \frac{\sinh(\lambda_\alpha(\epsilon) - \lambda_\beta(\epsilon) + \eta)}{\sinh(\lambda_\alpha(\epsilon) - \lambda_\beta(\epsilon) - \eta)} \quad (\alpha = 1, 2, \ldots, M). \tag{4.22}$$

Here we have set $a(\mu) = 1$ and defined $d(\mu; \{w_j\}_L)$ by

$$d(\mu; \{w_j\}_L) = \prod_{j=1}^L b(\mu - w_j) = \prod_{j=1}^L \frac{\sinh(\mu - w_j)}{\sinh(\mu - w_j + \eta)}. \tag{4.23}$$

Then, the Bethe ansatz equations (4.22) for the spin-1/2 XXZ chain with $w_j = w_j^{(\ell\epsilon)}$ (1 ≤ $j$ ≤ $L$) become those of the spin-$\ell$/2 XXZ chain by sending $\epsilon$ to zero. Here we remark the following:

$$\lim_{\epsilon \to 0} d(\mu; \{w_j^{(\ell\epsilon)}\}) = d^{(\ell)}(\mu; \{\xi_k\}_{N_\ell}), \quad \text{if } \mu \neq w_j^{(\ell)} \text{ for } j = 1, 2, \ldots, L. \tag{4.24}$$

Let us now assume that the Bethe roots $\{\lambda_\beta(\epsilon)\}_M$ approach the Bethe roots $\{\lambda_\beta\}_M$ continuously in the limit of sending $\epsilon$ to 0. It follows that each entry of the Bethe ansatz eigenstate of the Bethe roots $\{\lambda_\beta(\epsilon)\}_M$ is continuous with respect to $\epsilon$. For a set of arbitrary parameters $\{\kappa_k\}_N$ we therefore have

$$\langle 0 | \prod_{\alpha=1}^N C^{(\ell\epsilon\alpha)}(\mu_\alpha) \cdot e_{\ell_1}^{(\epsilon_1)} \cdots e_{\ell_\ell}^{(\epsilon_\ell)} \cdot \prod_{\beta=1}^M B^{(\ell\epsilon\beta)}(\lambda_\beta) | 0 \rangle = \lim_{\epsilon \to 0} \langle 0 | \prod_{\alpha=1}^N C^{(\ell\epsilon\alpha)}(\mu_\alpha) \cdot e_{\ell_1}^{(\epsilon_1)} \cdots e_{\ell_\ell}^{(\epsilon_\ell)} \cdot \prod_{\beta=1}^M B^{(\ell\epsilon\beta)}(\lambda_\beta) | 0 \rangle. \tag{4.25}$$

The inhomogeneity parameters $w_j = w_j^{(\ell\epsilon)}$ for $j = 1, 2, \ldots, L$ are generic since the small number $\epsilon$ takes generic values and parameters $r_k^{\beta}$ are also generic. Putting $w_j = w_j^{(\ell\epsilon)}$ in (3.38) we have the QISP formula with suffix $(1w)$ replaced by $(\ell w; \epsilon)$ for local operator $x_n$. We thus have the following expressions for $b = 1, 2, \ldots, N_\ell$:

$$e_{\ell_1}^{(\epsilon_1)} \cdots e_{\ell_\ell}^{(\epsilon_\ell)} = \prod_{k=1}^{(\ell-1)} (A^{(\ell w_\ell)}(w_k^{(\ell w_\ell)}) + D^{(\ell w_\ell)}(w_k^{(\ell w_\ell)})) \times T_{\epsilon_1}^{(\ell w_\ell)}(w_1^{(\ell w_\ell)} - 1) \cdots T_{\epsilon_\ell}^{(\ell w_\ell)}(w_\ell^{(\ell w_\ell)} - 1) \prod_{k=1}^\delta (A^{(\ell w_\ell)}(w_k^{(\ell w_\ell)}))^{-1}. \tag{4.26}$$

For instance, in the case of $b = 1$, applying formula (4.26) of $w = p$ we have

$$\langle 0 | \prod_{\alpha=1}^N C^{(\ell\epsilon\alpha)}(\mu_\alpha) \cdot e_{\ell_1}^{(\epsilon_1)} \cdots e_{\ell_\ell}^{(\epsilon_\ell)} \cdot \prod_{\beta=1}^M B^{(\ell\epsilon\beta)}(\lambda_\beta) | 0 \rangle = \phi_\ell(\{\lambda_\beta\}; \{w_j^{(\ell)}\}) \langle 0 | \prod_{\alpha=1}^N C^{(\ell\epsilon\alpha)}(\mu_\alpha) \cdot T_{\epsilon_1}^{(\ell\epsilon\alpha)}(w_1^{(\ell\epsilon\alpha)}) \cdots T_{\epsilon_\ell}^{(\ell\epsilon\alpha)}(w_\ell^{(\ell\epsilon\alpha)}) \cdot \prod_{\beta=1}^M B^{(\ell\epsilon\beta)}(\lambda_\beta) | 0 \rangle. \tag{4.27}$$

doi:10.1088/1742-5468/2012/04/P04001
Proposition 4.4. Let \( \{ \mu_k \}_N \) be a set of arbitrary parameters and \( \{ \lambda_\alpha \}_M \) a solution of the spin-\( \ell/2 \) Bethe ansatz equations. We denote by \( \{ \lambda_\alpha(\epsilon) \}_M \) a solution of the Bethe ansatz equations for the spin-1/2 XXZ chain whose inhomogeneity parameters \( w_j \) are given by the \( N_e \) pieces of the almost complete \( \ell \)-strings: \( w_j = w_j^{(\ell \epsilon)} \) for \( 1 \leq j \leq L \). We assume that the set \( \{ \lambda_\alpha(\epsilon) \}_M \) approaches \( \{ \lambda_\alpha \}_M \) continuously when we send \( \epsilon \) to zero. For the Bethe states \( \{ \mu_k \}_N \) and \( \{ \lambda_\alpha \}_M \), which are off-shell and on-shell, respectively, we evaluate the matrix elements of a given product of elementary operators \( \epsilon^e_{\ell \alpha} \cdots \epsilon^e_{\ell \alpha} \) as follows:

\[
\begin{align*}
\langle 0 | & \prod_{\alpha=1}^{N} C^{(\ell \epsilon \alpha)}(\mu_\alpha) \epsilon^e_{\ell \alpha} \cdots \epsilon^e_{\ell \alpha} \prod_{\beta=1}^{M} B^{(\ell \epsilon \beta)}(\lambda_\beta) | 0 \rangle = \phi_\ell(\{ \lambda_\alpha \}; \{ w_j^{(\ell \epsilon)} \}) \lim_{\epsilon \to 0} \langle 0 | \\
& \quad \times \prod_{\alpha=1}^{N} C^{(\ell \epsilon \alpha)}(\mu_\alpha) T^{(\ell \epsilon \alpha \beta)}_{\epsilon_{\ell \alpha} \epsilon_{\ell \alpha}}(w_1^{(\ell \epsilon)} \cdots w_\ell^{(\ell \epsilon)}) \prod_{\beta=1}^{M} B^{(\ell \epsilon \beta)}(\lambda_\beta(\epsilon)) | 0 \rangle,
\end{align*}
\]

where \( \phi_m(\{ \lambda_\alpha \}) \) has been defined by \( \phi_m(\{ \lambda_\alpha \}; \{ w_j \}) = \prod_{j=1}^{m} \prod_{\alpha=1}^{M} b(\lambda_\alpha - w_j) \) with \( b(u) = \sinh(u)/\sinh(u + \eta) \).

4.5. Spin-\( \ell/2 \) form factors reduced into the spin-1/2 ones

Combining propositions 4.2 and 4.4 we have the following:

Proposition 4.5. Let \( i_1 \) and \( j_1 \) be integers satisfying \( 1 \leq i_1, j_1 \leq \ell \). We set \( i_1 - j_1 = N - M \). Let \( \{ \mu_k \}_N \) be a set of arbitrary \( N \) parameters. For a set of Bethe roots \( \{ \lambda_\alpha(\epsilon) \}_M \) which approaches \( \{ \lambda_\alpha \}_M \) continuously at \( \epsilon = 0 \) we have the following:

\[
\begin{align*}
\langle 0 | & \prod_{\alpha=1}^{N} C^{(\ell \epsilon \mu_\alpha)}(\mu_\alpha) \cdot \hat{E}_{i_1,j_1}^{(\ell \epsilon \mu_\alpha)} \cdot \prod_{\beta=1}^{M} B^{(\ell \epsilon \lambda_\beta)}(\lambda_\beta) | 0 \rangle = \hat{N}_{i_1,j_1}^{(\ell \epsilon \mu_\alpha)} e^{\sigma(w)(\sum_k \mu_k - \sum_\alpha \lambda_\alpha)} \phi_\ell(\{ \lambda_\alpha \}; \{ w_j^{(\ell \epsilon)} \}) \\
& \quad \times \sum_{(\epsilon_\beta(j_1))} \lim_{\epsilon \to 0} \langle 0 | \prod_{\alpha=1}^{N} C^{(\ell \epsilon \alpha)}(\mu_\alpha) T^{(\ell \epsilon \alpha \beta)}_{\epsilon_{\ell \alpha} \epsilon_{\ell \alpha}}(w_1^{(\ell \epsilon)} \cdots w_\ell^{(\ell \epsilon)}) \prod_{\beta=1}^{M} B^{(\ell \epsilon \beta)}(\lambda_\beta(\epsilon)) | 0 \rangle,
\end{align*}
\]

Let us recall a product of the general spin-\( \ell/2 \) elementary operators, \( \hat{E}_{i_1,j_1}^{(\ell \epsilon \mu_\alpha)} \cdots \hat{E}_{m,n}^{(\ell \epsilon \mu_\alpha)} \), which we have introduced in corollary 3.9. We also recall variables \( \epsilon^{(k)}_{\alpha}(i_k) \) and \( \epsilon^{(k)}_{\beta}(j_k) \) which take only two values 0 or 1 for \( k = 1, 2, \ldots, m \) and \( \alpha, \beta = 0, 1, \ldots, \ell \). We have the following:

Corollary 4.6. Let us take integers \( i_k \) and \( j_k \) satisfying \( 1 \leq i_k, j_k \leq \ell \) for \( k = 1, 2, \ldots, m \). We set \( \sum_k i_k - \sum_k j_k = N - M \). Let \( \{ \mu_k \}_N \) be a set of arbitrary \( N \) parameters. If the set of the Bethe roots \( \{ \lambda_\alpha(\epsilon) \}_M \) approaches the set of the Bethe roots \( \{ \lambda_\alpha \}_M \) continuously

\[\text{doi:10.1088/1742-5468/2012/04/P04001}\]
Reduction formula of form factors for the integrable spin-$s$ XXZ chains and application to correlation functions

at $\epsilon = 0$, we have the following:

$$
\langle 0 | \prod_{\alpha=1}^{N} C^{(\ell_{w})}(\mu_{\alpha}) \cdot \prod_{k} \tilde{E}_{k}^{i_{k}j_{k}(\ell_{w})} \cdot \prod_{\beta=1}^{M} B^{(\ell_{w})}(\lambda_{\beta}) | 0 \rangle
$$

$$
= \left( \prod_{k=1}^{m} \tilde{N}_{i_{k}j_{k}}^{(\ell_{k})} \right) \cdot e^{\sigma(\omega)(\sum_{k=1}^{N} \mu_{k} - \sum_{\gamma=1}^{M} \lambda_{\gamma})} \phi_{0}(\{\lambda_{\beta}\}; \{w^{(\ell)}_{j}\})
$$

$$
\times \sum_{(\epsilon_{\alpha}^{(\ell)}(j))_{\epsilon}} \cdots \sum_{(\epsilon_{\beta}^{(\ell)}(j))_{\epsilon}} \lim_{\epsilon \to 0} (0) \prod_{\alpha=1}^{N} C^{(\ell \rho \epsilon)}(\mu_{\alpha})
$$

$$
\times \prod_{k=1}^{m} T^{(\ell \rho \epsilon)}_{\epsilon_{\alpha}^{(k)}(j_{n}), \epsilon_{b}^{(k')}_{\alpha} (j_{k})} (w^{(\epsilon \rho \epsilon)}_{1}) \cdots T^{(\ell \rho \epsilon)}_{\epsilon_{\alpha}^{(k)}(j_{n}), \epsilon_{b}^{(k')}_{\alpha} (j_{k})} (w^{(\epsilon \rho \epsilon)}_{\ell})
$$

$$
\times \prod_{\beta=1}^{M} B^{(\ell \rho \epsilon)}(\lambda_{\beta}(\epsilon)) | 0 \rangle.
$$

(4.30)

Here we have chosen sequences $\epsilon_{k}^{[k]'}(j_{k})$ for each integer $k$ of $1 \leq k \leq m$.

5. Spin-$\ell/2$ form factors via the spin-$1/2$ scalar products

5.1. Fundamental commutation relations

For given sequences $(\epsilon_{\alpha}^{(\ell)})_{m}$ and $(\epsilon_{\beta}^{(\ell)})_{m}$ we consider sets $\alpha^{\pm}$ defined by equations (3.10). We also denote the sets $\alpha^{\pm}$ by $\alpha^{\pm}(\{\epsilon_{\alpha}^{(\ell)}\})$ and $\alpha^{\pm}(\{\epsilon_{\beta}^{(\ell)}\})$, respectively, in order to show their dependence on the sequences $(\epsilon_{\alpha}^{(\ell)})_{m}$ and $(\epsilon_{\beta}^{(\ell)})_{m}$ explicitly. We take a set of distinct integers $a_{j}$ for $j \in \alpha^{-}$ and $a'_{k}$ for $k \in \alpha^{+}$ such that they satisfy $1 \leq a_{j} \leq N$ for $j \in \alpha^{-}$ and $1 \leq a'_{k} \leq N + k$ for $k \in \alpha^{+}$. For the given set of $a_{j}, a'_{j}$, we introduce $A_{j}$ and $A'_{j}$ by

$$
A_{j} = \{ b; 1 \leq b \leq N + m, b \neq a_{j}, a'_{j} \text{ for } k < j \},
$$

$$
A'_{j} = \{ b; 1 \leq b \leq N + m, b \neq a_{k}, a'_{k} \text{ for } k \leq j, b \neq a'_{k} \text{ for } k < j \}.
$$

(5.1)

Setting rapidities $\mu_{N+j}$ by

$$
\mu_{N+j} = w_{j}, \quad \text{for } j = 1, 2, \ldots, m,
$$

(5.2)

we can show the fundamental commutation relations as follows [7]:

$$
\langle 0 | \prod_{\alpha=1}^{N} C^{(1p)}(\mu_{\alpha}) T^{(1p)}_{\epsilon_{\alpha}^{(1)}, \epsilon_{\alpha}^{(1)}}(\mu_{N+1}) \cdots T^{(1p)}_{\epsilon_{\alpha}^{(N)}, \epsilon_{\alpha}^{(N)}}(\mu_{N+m})
$$

$$
= \left( \prod_{j \in \alpha^{-}(\{\epsilon_{\alpha}^{(1)}\})} \sum_{a_{j}=1}^{N} \prod_{j \in \alpha^{+}(\{\epsilon_{\beta}^{(1)}\})} \sum_{a'_{j}=1}^{N+j} C_{\{a_{j}, a'_{j}\}}^{(\epsilon_{\alpha}^{(1)}, \epsilon_{\beta}^{(1)})} ((\mu_{k})_{N+m}) | 0 \rangle
$$

$$
\times \prod_{k \in A_{m+1}(\{a_{j}, a'_{j}\})} C^{(1p)}(\mu_{k}),
$$

doi:10.1088/1742-5468/2012/04/P04001
where coefficients \( G^{(\varepsilon_{a_j}, \varepsilon_{a_j'})}_{(a_j, a_j')}(\{\mu_\alpha\}_{N+m}) \) are given by

\[
G^{(\varepsilon_{a_j}, \varepsilon_{a_j'})}_{(a_j, a_j')}(\{\mu_\alpha\}_{N+m}) = \prod_{j \in \alpha^+(\varepsilon_{a_j})} \left( \prod_{b=1}^{N+j-1} \sinh(\mu_b - \mu_{a_j} + \eta) \right) \prod_{b=1}^{N+j} \sinh(\mu_b - \mu_{a_j'})
\]

\[
\times \prod_{j \in \alpha^-(\varepsilon_{a_j'})} \left( d(\mu_{a_j}; \{w_k\}_L) \prod_{b=1}^{N+j-1} \sinh(\mu_{a_j} - \mu_b + \eta) \right). \tag{5.3}
\]

Here we recall

\[
d(\mu; \{w_k\}_L) = \prod_{k=1}^{L} b(\mu - w_k). \tag{5.4}
\]

We consider the sums over integers \( a_j \) and \( a_j' \) such that they satisfy \( 1 \leq a_j \leq N \) and 
\( 1 \leq a_j' \leq N + j \) for \( j \in \alpha^- \) and \( k \in \alpha^+ \), respectively. Hereafter, we express the products of the sums over \( a_j \) and \( a_j' \) by the symbol \( \sum_{\{\alpha}, \{\beta\}\} \), as follows:

\[
\sum_{\{\alpha}, \{\beta\}\} = \prod_{j \in \alpha^-} \left( \sum_{a_j=1}^{N} \right) \prod_{j \in \alpha^+} \left( \sum_{a_j'=1}^{N+j} \right). \tag{5.5}
\]

5.2. Form factors as a sum of the spin-1/2 scalar products

We shall evaluate the spin-\( \ell/2 \) form factor \( \tilde{F}_k^{(\ell, \alpha)}(\{\mu_\alpha\}_{N}, \{\lambda_\beta\}_{M}) \).

We first define the scalar product in the spin-1/2 case for two sets of \( M \) parameters \( \{\mu_\alpha\}_{M} \) and \( \{\lambda_\gamma\}_{M} \) by

\[
S_M^{(1)}(\{\mu_1, \ldots, \mu_M\}, \{\lambda_1, \ldots, \lambda_M\}; \{w_j\}_L) = \langle 0 | \prod_{k=1}^{M} C^{(1p)}(\mu_k) \prod_{\gamma=1}^{M} B^{(1p)}(\lambda_\gamma) | 0 \rangle. \tag{5.6}
\]

Here \( \{\mu_\alpha\}_{M} \) and \( \{\lambda_\gamma\}_{M} \) are not necessarily solutions of the Bethe ansatz equations.

We define the scalar product for the spin-\( \ell/2 \) operators \( B^{(\ell p)}(\mu_k) \) and \( C^{(\ell p)}(\lambda_k) \) for \( k = 1, 2, \ldots, M \) by

\[
S_M^{(\ell)}(\{\mu_\alpha\}, \{\lambda_\beta\}; \{\xi_k\}_N) = \langle 0 | \prod_{\alpha=1}^{M} C^{(\ell p)}(\mu_\alpha) \prod_{\beta=1}^{M} B^{(\ell p)}(\lambda_\beta) | 0 \rangle. \tag{5.7}
\]

Here we also recall that \( \{\mu_\alpha\}_{M} \) and \( \{\lambda_\gamma\}_{M} \) are not necessarily Bethe roots.

Let us first review Slavnov’s formula of scalar products in the spin-1/2 case [2]: if \( \lambda_1, \lambda_2, \ldots, \lambda_M \) satisfy the spin-1/2 Bethe ansatz equations for the spin-1/2 XXZ spin chain with inhomogeneity parameters \( w_j \), the scalar product is expressed in terms of the determinant:

\[
S_M^{(1)}(\{\mu_\alpha\}, \{\lambda_\beta\}; \{w_j\}_L) = \frac{\det \hat{H}(\{\lambda_\beta\}_{M}; \{\mu_k\}_{M}; \{w_j\}_L)}{\prod_{1 \leq j < k \leq M} \sinh(\mu_j - \mu_k) \prod_{1 \leq \alpha < \beta \leq M} \sinh(\lambda_\beta - \lambda_\alpha)}. \tag{5.8}
\]
Here we recall that \( d \) with respect to given bra and ket vectors, 
\[ \langle \{ \lambda \}_{a, b}, \{ w \}_{L} \rangle = \frac{\sinh \eta}{\sinh(\lambda_a - \mu_b)} \left( a(\mu_b) \prod_{k=1; k \neq a}^{M} \sinh(\lambda_k - \mu_b + \eta) \right. \]
\[ \left. - d(\mu_b; \{ w \}_{L}) \prod_{k=1; k \neq a}^{M} \sinh(\lambda_k - \mu_b - \eta) \right). \] (5.9)

Here we recall that \( d(\mu; \{ w \}_{L}) = \prod_{j=1}^{L} \sinh(\mu - w_j)/\sinh(\mu - w_j + \eta) \).

We remark that it is sometimes useful to make use of the following relations:

**Lemma 5.1.** For two sets of arbitrary parameters \( \{ \mu_k \}_{N+m} \) and \( \{ \lambda_j \}_M \) we have
\[
\langle 0 | \prod_{a=1}^{N} C^{(1p)}(\mu_a) \cdot T^{(1p)}_{\epsilon_1, \epsilon_2} (\mu_{N+1}) \cdots T^{(1p)}_{\epsilon_m, \epsilon_m} (\mu_{N+m}) \cdot \prod_{\gamma=1}^{M} B^{(1p)}(\lambda_\gamma) | 0 \rangle = \langle 0 | \prod_{\beta=1}^{M} C^{(1p)}(\lambda_\beta) \cdot T^{(1p)}_{\epsilon'_m, \epsilon_m} (\mu_{N+m}) \cdots T^{(1p)}_{\epsilon'_1, \epsilon'_1} (\mu_{N+1}) \cdot \prod_{a=1}^{N} B^{(1p)}(\mu_a) | 0 \rangle. \] (5.10)

We now consider the matrix element of an \( m \)th product of the spin-1/2 operators with respect to given bra and ket vectors, \( \{ \{ \mu_\alpha \}_N \}^{(\ell, p=0)} \) and \( \{ \{ \lambda_\gamma \}_N \}^{(\ell, p=0)} \), respectively. We define \( P \) by \( P = N - M \), where \( P \) can be negative. Setting \( \mu_{N+j} = w_j \) for \( j = 1, 2, \ldots, m \), we have
\[
\langle 0 | \prod_{a=1}^{N} C^{(1p)}(\mu_a) \cdot T^{(1p)}_{\epsilon_1, \epsilon_2} (\mu_{N+1}) \cdots T^{(1p)}_{\epsilon_m, \epsilon_m} (\mu_{N+m}) \cdot \prod_{\gamma=1}^{M} B^{(1p)}(\lambda_\gamma) | 0 \rangle = \sum_{\{ a_j, a'_k \}} G(\epsilon_m)_{\{ a_j, a'_k \}} ((\mu_k)_{N+m}) \langle 0 | \prod_{k \in A_{m+1}(\{ a_j, a'_k \})} C(\mu_k) \cdot \prod_{\gamma=1}^{M} B(\lambda_\gamma) | 0 \rangle \]
\[
= \sum_{\{ a_j, a'_k \}} G(\epsilon_m)_{\{ a_j, a'_k \}} ((\mu_k)_{N+m}) \times S_{\lambda}(\epsilon_1, \epsilon_2, \ldots, \epsilon_m) \{ a_j, \mu_a, w_1, \ldots, w_m \} \{ \mu_a, \mu_{a'_j} \}_{m+p} \{ \lambda_\gamma \}_M \{ w \}_{L}. \] (5.11)

Here we remark that the number of elements in the set \( \{ a_j, a'_k \} \) is given by \( m + P \).

Let us now consider the case of \( m = \ell \) for the form factor of the spin-\( \ell/2 \) operators. For a given sequence of parameters \( \mu_k \) for \( 1 \leq k \leq N \) we extend it into a sequence of length \( N + \ell \) by setting \( \mu_k = w_{k-N}^{(\ell)} \) for \( k = N+1, \ldots, N+\ell \). We define another sequence \( \mu_k(\epsilon) \) for \( k = 1, 2, \ldots, N + \ell \) by
\[
\mu_k(\epsilon) = \begin{cases} 
\mu_k & \text{for } 1 \leq k \leq N, \\
w_{k-N}^{(\ell)} & \text{for } N < k \leq N + \ell.
\end{cases} \] (5.12)
Substituting (5.11) into (4.28) we have

\[
\langle 0| \prod_{\alpha=1}^{N} C^{(\ell p; 0)}(\mu_a) e^{\epsilon_1 \epsilon_1} \cdots e^{\epsilon_\ell \epsilon_\ell} \prod_{\beta=1}^{M} B^{(\ell p; 0)}(\lambda_\beta) | 0 \rangle = \phi_\ell(\{\lambda_\beta\}; \{w_{j(\ell)}\}) \sum_{\{a_j, a'_k\}} G_{\{a_j, a'_k\}}^{(\ell p; 0; 0)}((\mu_k)_{N+\ell})
\]

\[
\times \lim_{\epsilon \to 0} \sum_{\{a_j, a'_k\}} G_{\{a_j, a'_k\}}^{(\ell p; 0; 0)}((\mu_k)_{N+\ell}) \times \lim_{\epsilon \to 0} \sum_{\{a_j, a'_k\}} G_{\{a_j, a'_k\}}^{(\ell p; 0; 0)}((\mu_k)_{N+\ell}) \times \lim_{\epsilon \to 0} \sum_{\{a_j, a'_k\}} G_{\{a_j, a'_k\}}^{(\ell p; 0; 0)}((\mu_k)_{N+\ell})
\]

\[
\times \lim_{\epsilon \to 0} \sum_{\{a_j, a'_k\}} G_{\{a_j, a'_k\}}^{(\ell p; 0; 0)}((\mu_k)_{N+\ell}) \times \lim_{\epsilon \to 0} \sum_{\{a_j, a'_k\}} G_{\{a_j, a'_k\}}^{(\ell p; 0; 0)}((\mu_k)_{N+\ell}) \times \lim_{\epsilon \to 0} \sum_{\{a_j, a'_k\}} G_{\{a_j, a'_k\}}^{(\ell p; 0; 0)}((\mu_k)_{N+\ell})
\]

\[
\langle 0| \prod_{\alpha=1}^{N} C^{(\ell p; 0)}(\mu_a) e^{\epsilon_1 \epsilon_1} \cdots e^{\epsilon_\ell \epsilon_\ell} \prod_{\beta=1}^{M} B^{(\ell p; 0)}(\lambda_\beta) | 0 \rangle = \phi_\ell(\{\lambda_\beta\}; \{w_{j(\ell)}\}) \sum_{\{a_j, a'_k\}} G_{\{a_j, a'_k\}}^{(\ell p; 0; 0)}((\mu_k)_{N+\ell})
\]

We shall explicitly express the limiting procedure in the last line of (5.13). Here we recall the Bethe ansatz equations (BAE) for the spin-\(\ell/2\) case (4.20) and the limiting procedure (4.24), where the spin-\(\ell/2\) BAE is derived from the spin-1/2 BAE with almost complete strings \(w_{j(\ell)}^{(\ell \epsilon)}\) by sending \(\epsilon\) to 0. Let us now consider the case where some of \(\mu_k\) are given by inhomogeneity parameters \(w_j\). We first define the following function:

\[
d^{(\ell \epsilon)}(\mu; \{\xi_k\}_{N_+}) = \begin{cases} 0, & \text{if } \mu = w_j^{(\ell \epsilon)} (j = 1, 2, \ldots, L), \\ \sum_{k=1}^{N_\alpha} \sinh(\mu - \xi_k) - \sinh(\mu - \xi_k + \ell \eta), & \text{otherwise.} \end{cases}
\]

We next introduce the matrix \(\hat{H}^{(\ell \epsilon)}\). We define the matrix elements of the matrix \(\hat{H}^{(\ell \epsilon)}\) with entry \((a, b)\) for \(a, b = 1, 2, \ldots, M\) by

\[
\hat{H}_{a, b}^{(\ell \epsilon)} (\{\lambda\}_M; \mu_b; \{\xi_k\}_{N_+}) = \frac{\sinh \eta}{\sinh(\lambda_a - \mu_b)}\left(a(\mu_b) \prod_{k=1; k\neq a}^{M} \sinh(\lambda_k - \mu_b + \eta) - d^{(\ell \epsilon)}(\mu_b; \{\xi_k\}_{N_+}) \prod_{k=1; k\neq a}^{M} \sinh(\lambda_k - \mu_b - \eta)\right).
\]

If \(\{\lambda_\beta\}_M\) satisfy the Bethe ansatz equations of the spin-\(\ell/2\) XXZ spin chain with inhomogeneity parameters \(\xi_k\), we define the expression \(S_M^{(\ell \epsilon)}\) by

\[
S_M^{(\ell \epsilon)} (\{\mu_k\}_M; \{\lambda\}_M; \{\xi_k\}_{N_+}) = \det \hat{H}^{(\ell \epsilon)}(\{\lambda(a)\}_M; \{\mu_k\}_M; \{\xi_k\}_{N_+}) \prod_{1 \leq j < k \leq M} \sinh(\mu_j - \mu_k) \prod_{1 \leq a < b \leq M} \sinh(\lambda_b - \lambda_a).
\]

In equation (5.13), sending \(\epsilon\) to zero, for a given set of integers \(\{a_j, a'_k\}\) we have

\[
\lim_{\epsilon \to 0} S_M^{(\ell \epsilon)} (\{\mu_k(\epsilon)\}_N+\ell \setminus \{\mu_a(\epsilon), \mu_a^{(\ell \epsilon)}(\epsilon)\}_{\ell+P}, \{\lambda_\beta(\epsilon)\}_M; \{w_j^{(\ell \epsilon)}\}_L) = S_M^{(\ell \epsilon)} (\{\mu_k\}_N+\ell \setminus \{\mu_a, \mu_a^{(\ell \epsilon)}\}_{\ell+P}, \{\lambda_\beta\}_M; \{\xi_k\}_{N_+}).
\]

We summarize the result as follows:

**Lemma 5.2.** Let \(\{\lambda_\gamma\}_M\) be a solution of the Bethe ansatz equations for the spin-\(\ell/2\) chain with inhomogeneity parameters \(\xi_k\) (\(1 \leq k \leq N_+\)), and \(\{\mu_k\}_N\) a set of arbitrary parameters. We assume that \(\{\lambda_\beta(\epsilon)\}\) is a solution of the Bethe ansatz equations for the spin-1/2 chain with \(w_j = w_j^{(\ell \epsilon)}\) (\(1 \leq j \leq L\)) and it approaches \(\{\lambda_\gamma\}_M\) continuously at \(\epsilon = 0\). We express
Here, we have fixed a sequence \( P \) as follows: 

\[
\text{elementary operators such as } \hat{\epsilon}_1 \text{ for the spin-1/2 regime: } 0 \leq \gamma < \pi / 2s. \text{ Here we remark that integer } 2s \text{ corresponds to integer } \ell \text{ of } V^{(\ell)}. \text{ We show only the main results. In fact, we derive them by following mainly the procedures of [13] except for the evaluation of the expectation values of products of the spin-s operators.}
\]

\[
\text{The matrix elements of a product of the spin-1/2 operators in the limit of sending } \epsilon \text{ to } 0 \text{ in terms of the modified scalar product } S^{(e)} \text{ as follows:}
\]

\[
\lim_{\epsilon \to 0} \left\{ \prod_{k=1}^{N} C^{(\ell \epsilon)}(\mu_k) T^{(\ell \epsilon)}(w^\epsilon_1) \cdots T^{(\ell \epsilon)}(w^\epsilon_\ell) \right\} \prod_{\beta=1}^{M} B^{(\ell)\beta}(\lambda\beta(\epsilon)) |0\rangle = \sum_{\{a_j, a'_{k}\}} \tilde{G}^{(e)}_{\{a_j, a'_{k}\}}((\mu_k)_{N+\ell}) \\
\times S^{(\ell)}_{\beta}(\{\mu_1, \ldots, \mu_N, w^\epsilon_1, \ldots, w^\epsilon_\ell\}) \setminus \{\mu_{a_j}, \mu_{a'_{k}}\}_{\ell+P}, \{\lambda\gamma\}_{M}; \{\xi\}_{N_s}). \tag{5.18}
\]

Here, \( P = N - M \) and we have set \( \mu_{N+j} = w^\epsilon_j \) for \( j = 1, 2, \ldots, \ell \).

Through lemma 5.2 we show the following: 

**Proposition 5.3.** Let us take integers \( i_1 \) and \( j_1 \) satisfying \( 1 \leq i_1, j_1 \leq \ell, \) and integers \( N \) and \( M \) with \( i_1 - j_1 = N - M \). Let \( \{\lambda\gamma\}_{M} \) be a solution of the Bethe ansatz equations for the spin-1/2 chain with inhomogeneity parameters \( \lambda_k \) (\( 1 \leq k \leq N_s \)) and \( \{\mu_k\}_{N} \) a set of arbitrary parameters. We assume that there exists such a solution of the Bethe ansatz equations for the spin-1/2 chain with \( w_j = w^{(\ell \epsilon)}_j \) \( (1 \leq j \leq L) \) that approaches the set of Bethe roots \( \{\lambda\gamma\}_{M} \) in the limit of \( \epsilon \to 0 \). Then, every spin-1/2 off-shell matrix element associated with Bethe roots \( \{\lambda\gamma\}_{M} \) is expressed as a sum of the spin-1/2 scalar products as follows: 

\[
\tilde{N}^{i_1,j_1(\ell \epsilon)}_{(s)}(\{\mu_1\}_{N}, \{\lambda\gamma\}_{M}) = (0| \prod_{\alpha=1}^{N} C^{(\ell \epsilon)}(\mu_\alpha) \cdot \hat{E}^{i_1,j_1(\ell \epsilon)}_{(s)} \cdot \prod_{\beta=1}^{M} B^{(\ell \epsilon)}(\lambda\beta(\epsilon)) |0\rangle = \tilde{N}^{(e)}_{i_1,j_1(\ell \epsilon)} e^{\sigma(|\sum_k \mu_k - \sum_k \lambda\gamma|) \phi(\{\lambda\gamma\}_{M}); \{\gamma\}_{N_s}} \\
\times \sum_{(\ell \epsilon)_{j_1}} \sum_{\{a_j, a'_{k}\}} \tilde{G}^{(e)}_{\{a_j, a'_{k}\}}((\mu_k)_{N+\ell}) \\
\times S^{(\ell)}_{\beta}(\{\lambda_1, \ldots, \mu_N, w^\epsilon_1, \ldots, w^\epsilon_\ell\}) \setminus \{\mu_{a_j}, \mu_{a'_{k}}\}_{\ell+P}, \{\lambda\gamma\}_{M}; \{\xi\}_{N_s}). \tag{5.19}
\]

Here, we have fixed a sequence \( (\epsilon^\ell_{\alpha}(i_1))_{\ell} \). We have set \( P = N - M \) and \( \mu_{N+j} = w^\epsilon_j \) for \( j = 1, 2, \ldots, \ell \).

Substituting a set of Bethe roots to \( \{\mu_k\} \) we can evaluate the form factor \( \tilde{F}^{i_1,j_1(\ell \epsilon)}_{k=1} \). It is easy to generalize the formula (5.19) to that of a given product of the spin-1/2 elementary operators such as \( E^{i_1,j_1(\ell \epsilon)}_{(s)} \hat{E}^{i_2,j_2(\ell \epsilon)} \cdots \). 

**6. Spin-s XXZ correlation functions in a massless region**

Applying the reduction formula we now derive the multiple-integral representations of the correlation functions of the integrable spin-s XXZ spin chain in a region of the massless regime: \( 0 \leq \zeta < \pi / 2s \). Here we remark that integer \( 2s \) corresponds to integer \( \ell \) of \( V^{(\ell)} \). We show only the main results. In fact, we derive them by following mainly the procedures of [13] except for the evaluation of the expectation values of products of the spin-s operators.
Let us review the main procedures for deriving the multiple-integral representation of the spin-s XXZ correlation functions, briefly. First, we introduce the spin-s elementary operators as the basic blocks for constructing the local operators of the integrable spin-s XXZ spin chain. Second, we reduce them into a sum of products of the spin-1/2 elementary operators, which we express in terms of the matrix elements of the spin-1/2 monodromy matrix through the spin-1/2 QISP formula. We then evaluate their scalar products with Slavnov’s formula of the Bethe ansatz scalar products we have shown in section 5. Here, the expectation value of a physical quantity is expressed as a sum of the ratios of the Bethe ansatz scalar products to the norm of the ground state Bethe ansatz eigenvector, and the ratios are expressed in terms of the determinants of some matrices. Third, by solving the integral equations for the matrices in the thermodynamic limit, we derive the multiple-integral representation of the correlation functions. Here, the integrals and their solutions for the spin-s case are given in [13].

6.1. Conjecture of the spin-s ground state solution

Let us now consider the ground state of the integrable spin-s XXZ spin chain in the massless regime. Here we remark that integer 2s corresponds to integer $\ell$ of $V^{(\ell)}$. In the massless regime we set $\eta = i\zeta$ with $0 \leq \zeta < \pi$. For the spin-s case, in the region $0 \leq \zeta < \pi/2s$ we assume that the spin-s ground state $|\psi^{(2s)}_g\rangle$ is given by $N_s/2$ sets of the 2s-strings [18]–[21]:

$$\lambda^{(a)}_a = \mu_a - (\alpha - 1/2)\eta + \delta^{(a)}_a, \quad \text{for } a = 1, 2, \ldots, N_s/2 \quad \text{and} \quad \alpha = 1, 2, \ldots, 2s. \quad (6.1)$$

We also assume that string deviations $\delta^{(s)}_a$ are small enough when $N_s$ is large enough [28]–[30]. In terms of $\lambda^{(a)}_a$, the spin-s ground state associated with grading $w$ is given by

$$|\psi^{(2sw)}_g\rangle = \prod_{a=1}^{N_s/2} \prod_{\alpha=1}^{2s} B^{(2sw)}(\lambda^{(a)}_a; \{\xi_\alpha\}_{N_s})|0\rangle. \quad (6.2)$$

Here we have $M$ Bethe roots with $M = 2sN_s/2 = sN_s$.

To the spin-s XXX and XXZ Bethe ansatz equations the 2s-string solutions (6.1) were evaluated numerically for the ground state of the integrable spin-s XXX and XXZ spin chains by several authors [22]–[25]. It was found that string deviations $\delta^{(s)}_a$ are approximately given by pure imaginary numbers, and the strings are slightly wider than complete strings which have zero deviations: For $a \leq s$, $\delta^{(s)}_a = i|\delta^{(s)}_a|$ and $|\delta^{(s)}_a| > |\delta^{(s)}_a|$; for $a > s$, $\delta^{(s)}_a = -i|\delta^{(s)}_a|$ and $|\delta^{(s)}_a| < |\delta^{(s)}_a|$. The behavior of string deviations is also shown by analytical methods [28]–[30].

The behavior of string deviations is analytically important to determine the contours of the multiple-integral representation of correlation functions. It gives the small numbers $\epsilon_{\ell,k}$ in the denominators of equation (6.11), as we shall see in section 6.2.

We give a remark here. We consider only the region $0 \leq \zeta < \pi/2s$ in the massless regime of $0 \leq \zeta < \pi$ throughout this paper. In fact, for $\zeta \geq \pi/2s$, the ground state consists of different types of solutions of the Bethe ansatz equations [21,26,27]. Furthermore, it seems that, for $\zeta \geq \pi/2s$, the ground state solutions have not been numerically studied, yet.
6.2. Multiple-integral representations for arbitrary matrix elements

Let us now formulate the multiple-integral representations of the spin-$s$ XXZ correlation functions in the general case for the massless region: $0 \leq \zeta < \pi/2s$. We define the zero-temperature correlation function for a given product of the general spin-$s$ elementary operators with grading $w$, $\hat{E}_{1}^{i_{1},j_{1}(2s w)} \cdots \hat{E}_{m}^{i_{m},j_{m}(2s w)}$, which are $(2s+1) \times (2s+1)$ matrices, by

$$
\hat{F}_{m}^{(2s p)}(\{i_{k}, j_{k}\}) = \langle \psi_{g}^{(2s w)} | \prod_{k=1}^{m} \hat{E}_{k}^{i_{k},j_{k}(2s w)} | \psi_{g}^{(2s w)} \rangle / \langle \psi_{g}^{(2s w)} | \psi_{g}^{(2s w)} \rangle. 
$$  \hfill (6.3)

For the $m$th product of elementary operators, we introduce the sets of variables $\varepsilon_{\alpha}^{[k]}$ and $\varepsilon_{\beta}^{[k]}$'s ($1 \leq k \leq m$) such that the number of $\varepsilon_{\alpha}^{[k]} = 1$ with $1 \leq a \leq 2s$ is given by $i_{k}$ and the number of $\varepsilon_{\beta}^{[k]} = 1$ with $1 \leq b \leq 2s$ by $j_{k}$, respectively. Here, the variables $\varepsilon_{\alpha}^{[k]}$ and $\varepsilon_{\beta}^{[k]}$ take only two values 0 or 1 (see also corollary 3.9). We then express them by integers $\varepsilon_{j}$'s and $\varepsilon_{j}$'s for $j = 1, 2, \ldots, 2sm$ as follows:

$$
\varepsilon_{2s(k-1)+\alpha}^{[k]} = \varepsilon_{\alpha}^{[k]} \quad \text{for } \alpha = 1, 2, \ldots, 2s; \quad k = 1, 2, \ldots, m, 
$$

$$
\varepsilon_{2s(k-1)+\beta}^{[k]} = \varepsilon_{\beta}^{[k]} \quad \text{for } \beta = 1, 2, \ldots, 2s; \quad k = 1, 2, \ldots, m. 
$$  \hfill (6.4)

For given sets of $\varepsilon_{j}$ and $\varepsilon_{j}^{'}$ for $j = 1, 2, \ldots, 2sm$ we define $\alpha^{-}$ by the set of integers $j$ satisfying $\varepsilon_{j}^{'} = 1$ and $\alpha^{+}$ by the set of integers $j$ satisfying $\varepsilon_{j} = 0$:

$$
\alpha^{-}(\{\varepsilon_{j}^{'}\}) = \{j; \varepsilon_{j}^{'} = 1\}, \quad \alpha^{+}(\{\varepsilon_{j}\}) = \{j; \varepsilon_{j} = 0\}. 
$$  \hfill (6.5)

We denote by $r$ and $r'$ the number of elements of the set $\alpha^{-}$ and $\alpha^{+}$, respectively. Due to charge conservation, we have $r + r' = 2sm$. Precisely, we have $r = \sum_{k=1}^{m} i_{k}$ and $r' = 2sm - \sum_{k=1}^{m} j_{k}$.

For sets $\alpha^{-}$ and $\alpha^{+}$, which correspond to $\{\varepsilon_{a}\}$ and $\{\varepsilon_{b}\}$, respectively, we define integral variables $\tilde{\lambda}_{j}$ for $j \in \alpha^{-}$ and $\tilde{\lambda}_{j}^{'}$ for $j \in \alpha^{+}$, respectively, by the following:

$$
(\tilde{\lambda}_{j_{\max}}^{'} \ldots, \tilde{\lambda}_{j_{\min}}^{'} \ldots, \tilde{\lambda}_{j_{\min}} \ldots, \tilde{\lambda}_{j_{\max}}) = (\lambda_{1}, \ldots, \lambda_{2sm}). 
$$  \hfill (6.6)

We now introduce a matrix $S = S((\lambda_{j})_{2sm}; (w_{j}^{(2s)})_{2sm})$. For each integer $j$ satisfying $1 \leq j \leq 2sm$, we define an integer $\gamma(j)$ by an integer $\gamma$ satisfying $1 \leq \gamma \leq 2s$ if $\lambda_{j}$ is related to an integral variable $\mu_{j}$ through $\lambda_{j} = \mu_{j} - (\gamma - 1/2)\eta$ or if $\lambda_{j}$ takes a value close to $w_{k}^{(2s)}$ with $\beta(k) = \gamma$. Thus, $\mu_{j}$ corresponds to the ‘string center’ of $\lambda_{j}$. Here we have defined $\beta(j)$ by

$$
\beta(j) = j - 2s[[(j - 1)/2s]] \quad (1 \leq j \leq M). 
$$  \hfill (6.7)

Here $[[x]]$ denotes the greatest integer less than or equal to $x$. We define the $(j, k)$ element of the matrix $S$ by

$$
S_{j,k} = \rho(\lambda_{j} - w_{k}^{(2s)} + \eta/2)\delta(\alpha(\lambda_{j}), \beta(k)), \quad \text{for } j, k = 1, 2, \ldots, 2sm. 
$$  \hfill (6.8)

Here $\delta(\alpha, \beta)$ denotes the Kronecker delta and $\rho(\lambda)$ the density of string centers [13]:

$$
\rho(\lambda) = \frac{1}{2\zeta \cosh(\pi\lambda/\zeta)}. 
$$  \hfill (6.9)

doi:10.1088/1742-5468/2012/04/P04001

34
We obtain the following multiple-integral representation:

\[
\hat{F}_m^{(2s\nu)}(\{i_k, j_k\}) = \hat{C}^{(2s)}(\{i_k, j_k\}) \\
\times \left( \int_{-\infty+i\epsilon}^{\infty+i\epsilon} + \int_{-\infty-i\epsilon+i\epsilon}^{\infty-i\epsilon+i\epsilon} + \cdots + \int_{-\infty-i(2s-1)i\epsilon+i\epsilon}^{\infty-i(2s-1)i\epsilon+i\epsilon} \right) d\lambda_1 \cdots \\
\times \left( \int_{-\infty+i\epsilon}^{\infty+i\epsilon} + \int_{-\infty-i\epsilon+i\epsilon}^{\infty-i\epsilon+i\epsilon} + \cdots + \int_{-\infty-i(2s-1)i\epsilon+i\epsilon}^{\infty-i(2s-1)i\epsilon+i\epsilon} \right) d\lambda_{\nu'} \\
\times \left( \int_{-\infty+i\epsilon}^{\infty+i\epsilon} + \int_{-\infty-i\epsilon+i\epsilon}^{\infty-i\epsilon+i\epsilon} + \cdots + \int_{-\infty-i(2s-1)i\epsilon+i\epsilon}^{\infty-i(2s-1)i\epsilon+i\epsilon} \right) d\lambda_{\nu'+1} \cdots \\
\times \sum_{\alpha^+ (\{\epsilon_j\})} Q(\{\epsilon_j, \epsilon'_j\}; \lambda_1, \ldots, \lambda_{2s_m}) \det S(\lambda_1, \ldots, \lambda_{2s_m}). \tag{6.10}
\]

Here the sum of \(\alpha^+ (\{\epsilon_j\})\) is taken over all \(\{\epsilon_j\}\) corresponding to \(\{\epsilon_{b}^{[k]}\}\) (1 ≤ \(k\) ≤ \(m\)) such that the number of \(\epsilon_{b}^{[k]} = 1\) with 1 ≤ \(b\) ≤ 2s is given by \(j_k\). The function \(Q(\{\epsilon_j, \epsilon'_j\}; \lambda_1, \ldots, \lambda_{2s_m})\) is given by

\[
Q(\{\epsilon_j, \epsilon'_j\}; \lambda_1, \ldots, \lambda_{2s_m}) = (-1)\nu \prod_{j \in \alpha^- (\{\epsilon'_j\})} \left( \prod_{k=1}^{j-1} \sinh(\tilde{\lambda}_j - w_k^{(2s)}) + \eta \prod_{k=j+1}^{2s_m} \sinh(\tilde{\lambda}_j - w_k^{(2s)}) \right) \\
\times \prod_{1 \leq k < \ell \leq 2s_m} \sinh(\lambda_\ell - \lambda_k + \eta + \epsilon_{k,\ell}) \\
\times \prod_{j \in \alpha^+ (\{\epsilon_j\})} \left( \prod_{k=1}^{j-1} \sinh(\tilde{\lambda}'_j - w_k^{(2s)}) - \eta \prod_{k=j+1}^{2s_m} \sinh(\tilde{\lambda}'_j - w_k^{(2s)}) \right) \prod_{1 \leq k < \ell \leq 2s_m} \sinh(w_k^{(2s)} - w_\ell^{(2s)}). \tag{6.11}
\]

In the denominator we set \(\epsilon_{k,\ell} = i\epsilon\) for \(\text{Im}(\lambda_k - \lambda_\ell) > 0\) and \(\epsilon_{k,\ell} = -i\epsilon\) for \(\text{Im}(\lambda_k - \lambda_\ell) < 0\), where \(\epsilon\) is an infinitesimally small positive number.

In (6.11) we may take any \(\alpha^- (\{\epsilon'_j\})\) corresponding to \(\epsilon_{b}^{[k]}\)'s for \(k = 1, 2, \ldots, m\), as far as the number of \(\epsilon_{b}^{[k]} = 1\) with 1 ≤ \(\alpha\) ≤ 2s is given by \(i_k\) for each \(k\).

The coefficient \(\hat{C}^{(2s)}(\{i_k, j_k\})\) in equation (6.10) is given by

\[
\hat{C}^{(2s)}(\{i_k, j_k\}) = \prod_{k=1}^{m} N_{i_k, j_k}^{(2s)} \prod_{k=1}^{m} \left( \frac{g(j_k) F(2s, i_k)}{g(i_k) F(2s, j_k)} \right)^{i_k(2s-i_k)/2 - j_k(2s-j_k)/2}. \tag{6.12}
\]

Here we have made use of (4.30) and (5.19). If we put \(g(2s, j) = \sqrt{F(2s, j)}\) for \(j = 0, 1, \ldots, 2s\) into (6.12), we have

\[
\hat{C}^{(2s)}(\{i_k, j_k\}) = \prod_{k=1}^{m} \sqrt{\left[ \frac{2s}{i_k} \right]_q \left[ \frac{2s}{j_k} \right]_q^{-1}}. \tag{6.13}
\]
Here we remark that the general elementary operators $\hat{U}_k^{(2s)}$ with $g(2s,j) = \sqrt{F(2s,j)}$ correspond to the basis vectors such that the spin-$s$ $R$ matrix becomes symmetric [16]. It is sometimes useful to express some factors in the denominator of equation (6.11) as follows:

\[
\prod_{1 \leq k < \ell \leq 2sm} \sinh(w_k^{(2s)} - w_\ell^{(2s)}) = \left( \prod_{\alpha=1}^{2s-1} \sinh^{\alpha} \eta \cdot [\alpha]_q \right)^m \prod_{1 \leq a < b \leq m} \prod_{\alpha=1}^{2s} \prod_{\beta=1}^{2s} \sinh(\xi_a - \xi_b - (\alpha - \beta)\eta). \tag{6.14}
\]

We can show the symmetric expression for the multiple-integral representation of the spin-$s$ correlation function $\hat{F}_m^{(2sw)}(\{i_k, j_k\})$ as follows:

\[
\hat{F}_m^{(2sw)}(\{i_k, j_k\}) = \hat{C}^{(2s)}(\{i_k, j_k\}) \prod_{1 \leq a < b \leq 2s} \sinh^{m}(\beta - \alpha)\eta \prod_{1 \leq k < \ell \leq m} \prod_{j=1}^{2s} \prod_{r=1}^{2s} \sinh(\xi_k - \xi_\ell + (r - j)\eta) \times \sum_{\sigma \in S_{2sm}/(S_m)^{2s}} \text{sgn} \sigma \prod_{j=1}^{r'} \int_{-\infty + i\epsilon}^{\infty + i\epsilon} d\mu_{\sigma j} \prod_{j=r'+1}^{2sm} \int_{-\infty - i\epsilon}^{\infty - i\epsilon} d\mu_{\sigma j} \times \sum_{\{\epsilon_\beta^{(k)}\}} \cdots \sum_{\{\epsilon_\lambda^{(2s_m)}\}} Q'(\{\epsilon_j, \epsilon'_j\}; \lambda_1, \ldots, \lambda_2(2sm)) \times \left( \prod_{j=1}^{2sm} \prod_{b=1}^{m} \prod_{\beta=1}^{2s-1} \sinh(\lambda_j - \xi_b + \beta\eta) \right) \times \frac{i^{2sm^2}}{(2\xi)^{2sm}} \prod_{\gamma=1}^{2s} \prod_{1 \leq k < \ell \leq 2sm} \sinh(\pi(\mu_{2s(a-1)+\gamma} - \mu_{2s(b-1)+\gamma})/\xi). \tag{6.15}
\]

Here $\lambda_j$ are given by $\lambda_j = \mu_j - (\beta(j) - 1/2)\eta$ for $j = 1, \ldots, 2sm$, and (sgn $\sigma$) denotes the sign of permutation $\sigma \in S_{2sm}/(S_m)^{2s}$. The coefficient $\hat{C}^{(2s)}(\{i_k, j_k\})$ is given by (6.12) and $Q'(\{\epsilon_j, \epsilon'_j\}; \lambda_1, \ldots, \lambda_{2(2sm)})$ is given by $Q'(\{\epsilon_j, \epsilon'_j\}; \lambda_1, \ldots, \lambda_{2(2sm)}) = Q(\{\epsilon_j, \epsilon'_j\}; \lambda_1, \ldots, \lambda_{2(2sm)}) \times \prod_{1 \leq k < \ell \leq 2sm} \sinh(w_k^{(2s)} - w_\ell^{(2s)})$. We recall that the sums over $\{\epsilon_\beta^{(k)}\}$ are taken over all $\epsilon_\beta^{(k)}$'s for $1 \leq k \leq m$ such that the number of integers $\beta$ with $\epsilon_\beta^{(k)} = 1$ and $1 \leq \beta \leq \ell$ is equal to $j_k$ for each $k$.

In appendix E we shall explain the derivation of the symmetric expression for the multiple-integral representation of the correlation functions.

6.3. The multiple-integral representation in the XXX limit

Sending $\zeta$ to zero, we shall derive the multiple-integral representation of correlation functions for the integrable spin-$s$ XXX chain from that of the integrable spin-$s$ XXZ chain.

We first define the basis vectors of the spin-$s$ representation of $U(sl(2))$ as follows:

\[
|2s, i\rangle = \sum_{1 \leq a(1) < \cdots < a(i) \leq \ell} \sigma_{-a(1)}^- \cdots \sigma_{-a(i)}^- |2s, 0\rangle \quad \text{for} \ i = 0, 1, \ldots, 2s. \tag{6.16}
\]

doi:10.1088/1742-5468/2012/04/P04001
Here the sum is taken over all such integers $a(1), a(2), \ldots, a(i)$ that satisfy $1 \leq a(1) < \cdots < a(i) \leq \ell$. We define conjugate vectors $\langle 2s, j \rangle$ by

$$\langle 2s, j \rangle = (|2s, j\rangle)^T / f(2s, j),$$

for $j = 0, 1, \ldots, 2s,$

(6.17)

where $f(2s, j)$ are given by the binomial coefficients

$$f(\ell, n) = \binom{\ell}{n}.$$  

(6.18)

We define the general spin-$\ell/2$ elementary operators, $\hat{X}^{i,j(\ell)}$, by

$$\hat{X}^{i,j(2s)} = |2s, i\rangle \langle 2s, j| g(j)/g(i),$$  

for $i, j = 0, 1, \ldots, 2s.$

(6.19)

Let us define the correlation function of the integrable spin-$s$ XXX spin chain for a given product of the spin-$s$ elementary operators by

$$\hat{f}_m^{(2s)}(\{i_k, j_k\}) = \langle \bar{\psi}_g^{(2s)} \rangle \prod_{k=1}^m \hat{X}_{k}^{i_k,j_k(2s)} \langle \bar{\psi}_g^{(2s)} \rangle / \langle \bar{\psi}_g^{(2s)} | \bar{\psi}_g^{(2s)} \rangle.$$  

(6.20)

Here we denote by $|\bar{\psi}_g^{(2s)}\rangle$ the ground state of the integrable spin-$s$ XXX spin chain.

We now take the XXX limit. Setting $\lambda = \epsilon \lambda$ we have

$$\rho(\lambda) \, d\lambda = \bar{\rho}(\bar{\lambda}) \, d\bar{\lambda}$$  

(6.21)

where $\bar{\rho}(\bar{\lambda}) = \frac{1}{2} \cosh(\pi \bar{\lambda})$. We send $\epsilon$ to zero. Then, we denote $\bar{\lambda}_j$ by $\lambda_j$ for simplicity. Thus, from the multiple-integral representation (6.10) we have

$$\hat{f}_m^{(2s)}(\{i_k, j_k\}) = \hat{C}_{XXX}^{(2s)}(\{i_k, j_k\})$$

$$\times \left( \int_{-\infty - i\epsilon}^{\infty + i\epsilon} + \int_{-\infty - i\epsilon}^{\infty + i\epsilon} + \cdots + \int_{-\infty - i(2s-1) + i\epsilon}^{\infty + i(2s-1) + i\epsilon} \right) d\lambda_1 \cdots$$

$$\times \left( \int_{-\infty - i\epsilon}^{\infty + i\epsilon} + \cdots + \int_{-\infty - i(2s-1) + i\epsilon}^{\infty + i(2s-1) + i\epsilon} \right) d\lambda_{\nu}$$

$$\times \left( \int_{-\infty - i\epsilon}^{\infty + i\epsilon} + \int_{-\infty - i\epsilon}^{\infty + i\epsilon} + \cdots + \int_{-\infty - i(2s-1) - i\epsilon}^{\infty + i(2s-1) - i\epsilon} \right) d\lambda_{\nu+1} \cdots$$

$$\times \left( \int_{-\infty - i\epsilon}^{\infty + i\epsilon} + \cdots + \int_{-\infty - i(2s-1) - i\epsilon}^{\infty + i(2s-1) - i\epsilon} \right) d\lambda_{2sm}$$

$$\times \sum_{\alpha^+(\{\epsilon_j\})} \tilde{Q}(\{\epsilon_j^\prime\}, \lambda_1, \ldots, \lambda_{2sm}) \det \tilde{S}(\lambda_1, \ldots, \lambda_{2sm}).$$

(6.22)

Here, the $(j, k)$ element of the matrix $\tilde{S}$ is given by

$$\tilde{S}_{j,k} = \bar{\rho}(\lambda_j - \omega_{k}^{(2s)} + i/2) \delta(\alpha(\lambda_j), \beta(k)), \quad \text{for } j, k = 1, 2, \ldots, 2sm.$$  

(6.23)

We recall that in (6.22) the sum of $\alpha^+(\{\epsilon_j\})$ is taken over all $\{\epsilon_j\}$ corresponding to $\{\epsilon_{b}^{[k]}\}$ ($1 \leq k \leq m$) such that the number of $\epsilon_{b}^{[k]} = 1$ with $1 \leq b \leq 2s$ is given by $j_k$. The doi:10.1088/1742-5468/2012/04/P04001
function is given by
\[ \tilde{Q}(\{\varepsilon_j, \varepsilon'_j\}; \lambda_1, \ldots, \lambda_{2s_m}) = (-1)^r \prod_{j \in \alpha^- \{\varepsilon_j\}} \left( \prod_{k=1}^{j-1} (\tilde{\lambda}_j - \tilde{w}_{2s,k}^{(2)}) + i \prod_{k=j+1}^{2s_m} (\tilde{\lambda}_j - \tilde{w}_{2s,k}^{(2)}) \right) \prod_{1 \leq k < \ell \leq 2s_m} (\lambda_{\ell} - \lambda_k + i + \varepsilon_{k,\ell}) \times \prod_{j \in \alpha^+ \{\varepsilon_j\}} \left( \prod_{k=1}^{j-1} (\tilde{\lambda}_j - \tilde{w}_{2s,k}^{(2)}) - i \prod_{k=j+1}^{2s_m} (\tilde{\lambda}_j - \tilde{w}_{2s,k}^{(2)}) \right) \right). \] (6.24)

Here \( \tilde{w}_{2s(a-1)+\alpha} = \tilde{\xi}_a - i(\alpha - 1) \) for \( \alpha = 1, 2, \ldots, 2s \) and \( a = 1, 2, \ldots, N_s \). The parameters \( \tilde{\xi}_a \) are the inhomogeneity parameters of the integrable spin-s XXX chain. In the denominator of (6.24) we have made use of equation (6.14). We recall that in the denominator we set \( \varepsilon_{k,\ell} = i\epsilon \) for \( \text{Im}(\lambda_k - \lambda_\ell) > 0 \) and \( \varepsilon_{k,\ell} = -i\epsilon \) for \( \text{Im}(\lambda_k - \lambda_\ell) < 0 \), where \( \epsilon \) is an infinitesimally small positive number. We also recall that in (6.24) we may take any \( \alpha^- \{\varepsilon_{j}^{(k)}\} \) corresponding to \( \varepsilon_{\alpha}^{(k)} \) for \( k = 1, 2, \ldots, m \), if the number of \( \varepsilon_{\alpha}^{(k)} = 1 \) with \( 1 \leq \alpha \leq 2s \) is given by \( i_k \) for each \( k \).

The coefficient \( \tilde{C}_{\text{XXX}}^{(2s)}(\{i_k, j_k\}) \) in equation (6.22) is given by
\[ \tilde{C}_{\text{XXX}}^{(2s)}(\{i_k, j_k\}) = \prod_{k=1}^{m} \left( \frac{g(i_k) f(2s, i_k)}{g(k) f(2s, j_k)} \right). \] (6.25)

In order to associate with the symmetric \( R \) matrix of the integrable spin-s XXX chain [16] we put \( g(2s, j) = \sqrt{f(2s, j)} \) for \( j = 0, 1, \ldots, 2s \) into (6.25). We thus have
\[ \tilde{C}_{\text{XXX}}^{(2s)}(\{i_k, j_k\}) = \prod_{k=1}^{m} \sqrt{\left( \frac{2s}{i_k} \right) \left( \frac{2s}{j_k} \right)^{-1}}. \] (6.26)

The multiple-integral representation of correlation functions for the integrable spin-s XXX spin chain (6.22) with equations (6.23), (6.24) and (6.26) corresponds to that of [9], in particular, equations (6.12)–(6.14) of [9] for the integrable spin-s XXX chain with some different points in the expression. It is the most significant point that in [9] the sum over \( \alpha^+(\{\varepsilon_j\}) \) in (6.22) is replaced by the factor
\[ \prod_{k=1}^{m} \left( \frac{2s}{i_k} \right) \] (6.27)
so that the normalization coefficient is given by
\[ \tilde{C}_{\text{XXX}}^{(2s)}(\{i_k, j_k\}) = \prod_{k=1}^{m} \sqrt{\left( \frac{2s}{i_k} \right) \left( \frac{2s}{j_k} \right)^{-1}}. \] (6.28)

It is an open problem how to derive the expression with the factor (6.27) instead of the sum over \( \alpha^+(\{\varepsilon_j\}) \) in (6.22), in particular, through the fusion method. In fact, if we apply the formulae of the quantum inverse-scattering problem in section 3 of [9] straightforwardly, we have the multiple-integral representation not only with the sum over \( \alpha^+(\{\varepsilon_j\}) \) but also with the sum over \( \alpha^-\{\varepsilon_j^{(k)}\} \), where the coefficient \( \tilde{C}_{\text{XXX}}^{(2s)}(\{i_k, j_k\}) \) of (6.22) is given by the inverse of (6.28). Applying the symmetry relations (4.13) we have the factor \( \prod_{k=1}^{m} f(2s, i_k) \) instead of the sum over \( \alpha^-\{\varepsilon_j^{(k)}\} \) and obtain the expression (6.22).
Appendix B. Reduction of spin-\(\ell/2\) Hermitian elementary operators

Let us introduce vectors \(\|\tilde{\ell}, n\rangle\) which are Hermitian conjugate to \(\langle \ell, n\|\) for positive integers \(\ell\) with \(n = 0, 1, \ldots, \ell\) [13]. Setting the norm of \(\|\tilde{\ell}, n\rangle\) such that...
Reduction formula of form factors for the integrable spin-$s$ XXZ chains and application to correlation functions

\[ \langle \ell, n | \widetilde{\ell}, n \rangle = 1, \] vectors \( \| \widetilde{\ell}, n \| \) are given by

\[ \| \widetilde{\ell}, n \| = \sum_{1 \leq i_1 < \cdots < i_n \leq \ell} \sigma_{i_1}^- \cdots \sigma_{i_n}^- \| \ell, 0 \| q^{-(i_1 + \cdots + i_n) + n \ell - n(n-1)/2} \left[ \begin{array}{c} \ell \\ n \end{array} \right] q^{-n(n-1)/2} \left( \frac{\ell}{n} \right)^{-1}. \] (B.1)

We define the spin-$\ell/2$ Hermitian elementary matrices associated with homogeneous grading, \( \widetilde{E}^{i,j(\ell,\pm)} \), by

\[ \widetilde{E}^{i,j(\ell,\pm)} = \| \widetilde{\ell}, i \| \langle \ell, j \|. \] (B.2)

Introducing \( \tilde{g}_{i,j} \) by

\[ \| \widetilde{\ell}, i \| \langle \ell, j \| = \sum_{(\varepsilon_\alpha')_\ell} \sum_{(\varepsilon_\beta)_\ell} \tilde{g}_{i,j}(\varepsilon_\alpha'(i), \varepsilon_\beta(j)) e^{\varepsilon_1'(i)} e^{\varepsilon_\ell'(i)} \cdots e^{\varepsilon_{\ell'}(i)} e^{\varepsilon_1(j)} \cdots e^{\varepsilon_j(j)}, \] (B.3)

we have

\[ \tilde{g}_{i,j}(\varepsilon_\alpha'(i), \varepsilon_\beta(j)) = \left[ \begin{array}{c} \ell \\ i \end{array} \right]_q q^{\ell(i-1)/2} \left[ \begin{array}{c} j \\ i \end{array} \right]_q q^{j(j-1)/2} q^{-(\alpha(1)+\cdots+\alpha(i)-) + (b(1)+\cdots+b(j)-)}. \] (B.4)

We derive the reduction formula for the Hermitian elementary operators \( \widetilde{E}^{i,j(\ell,\pm)} \) as follows:

\begin{align*}
\widetilde{E}^{i,j(\ell,\pm)} &= \mathcal{P}(\ell) \widetilde{E}^{i,j(\ell,\pm)} \\
&= \left[ \begin{array}{c} \ell \\ i \end{array} \right]_q q^{\ell(i-1)/2} \left[ \begin{array}{c} j \\ i \end{array} \right]_q q^{j(j-1)/2} \langle \ell, i \| \widetilde{\ell}, i \| \langle \ell, j \| \\
&\times \sum_{(\varepsilon_\alpha)(i)_{\ell}} \sum_{(\varepsilon_\beta)(i)_{\ell}} (\langle \ell, i \| \sigma_{\alpha(1)}^- \cdots \sigma_{\alpha(i)}^- \| \ell, 0 \| q^{-(\alpha(1)+\cdots+\alpha(i)-)}) \\
&\times \langle \ell, 0 \| \sigma_{b(1)}^+ \cdots \sigma_{b(j)}^+ q^{b(1)+\cdots+b(j)-1}. \] (B.5)
\end{align*}

Here, applying lemma 3.3 we show that the inside of the parentheses (or the round brackets) is independent of \( a(k) \)’s. Making use of the following:

\[ \sum_{(\varepsilon_\alpha'(i))_{\ell}} 1 = \left( \begin{array}{c} \ell \\ i \end{array} \right), \] (B.6)

we thus have

\begin{align*}
\widetilde{E}^{i,j(\ell,\pm)} &= \left[ \begin{array}{c} \ell \\ i \end{array} \right]_q q^{\ell(i-1)/2} \left[ \begin{array}{c} j \\ i \end{array} \right]_q q^{j(j-1)/2} \langle \ell, i \| \widetilde{\ell}, i \| \langle \ell, j \| \\
&\times \left( \begin{array}{c} \ell \\ i \end{array} \right) \sum_{(\varepsilon_\alpha)(i)_{\ell}} \sigma_{\alpha(1)}^- \cdots \sigma_{\alpha(i)}^- \langle \ell, 0 \| \sigma_{b(1)}^+ \cdots \\
&\times \sigma_{b(j)}^+ q^{-(\alpha(1)+\cdots+\alpha(i)-)} q^{b(1)+\cdots+b(j)-} \\
&= \left[ \begin{array}{c} \ell \\ i \end{array} \right]_q q^{\ell(i-1)/2} \left[ \begin{array}{c} j \\ i \end{array} \right]_q q^{j(j-1)/2} \langle \ell, i \| \langle \ell, j \| e^{-(i-j)\xi_1} \sum_{(\varepsilon_\beta)} 1 \chi_{1-\ell} e^{\varepsilon_1} \cdots e^{\varepsilon_{\ell'}} \chi_{1-\ell}^{-1}. \] (B.7)
\end{align*}

Here we have applied lemma 3.4 to derive the last line of equation (B.7).
Appendix C. Non-regularity of the transfer matrix

Let us consider the case of \( L = 3 \). We introduce \( b_{0j} \) and \( c_{0j}^\pm \) for \( j = 1, 2, 3 \) by

\[
b_{0j} = b(\lambda - w_j^{(2)}) \quad \text{and} \quad c_{0j}^\pm = \exp(\pm(\lambda - w_j^{(2)}))c(\lambda - w_j^{(2)}) \quad \text{for} \quad j = 1, 2, 3,
\]

respectively.

The matrix elements of the operator \( A_{123}^{(1+)}(\lambda) \) in the sector of \( M = 1 \) are given by

\[
A_{123}^{(1+)}(\lambda) \bigg|_{M=1} = \left( \begin{array}{ccc} b_{03} & c_{02}^+ c_{03} & c_{01}^+ b_{02} c_{03} \\ 0 & b_{02} & c_{01}^+ c_{02}^- \\ 0 & 0 & b_{01} \end{array} \right),
\]

and those of the operator \( A_{123}^{(1+)}(\lambda) \) in the sector of \( M = 1 \) are given by

\[
A_{123}^{(2+0)}(\lambda) \bigg|_{M=1} = \left( \begin{array}{ccc} b_{01} b_{02} & 0 & b_{01} b_{03} \\ b_{01}^- c_{02}^+ c_{03} & b_{01} b_{03} & 0 \\ c_{01}^+ c_{03} & c_{01}^+ c_{02}^- b_{03} & b_{02} b_{03} \end{array} \right).
\]

Let us set \( w_1 = w_1^{(2)} = \xi_1, w_2 = w_2^{(2)} = \xi_1 - \eta \) and \( w_3 = \xi_2 \). Setting \( \lambda = \xi_1 \) we have

\[
\left( A_{123}^{(2+0)}(\xi_1) + D_{123}^{(2+0)}(\xi_1) \right) \bigg|_{s^z=1/2} = \left( \begin{array}{ccc} b_{13} & q \xi_{13}^- & \xi_{13}^- \\ 0 & 1 & \xi_{13}^- \\ c_{13}^+ & \xi_{13}^- b_{13} & \xi_{13}^- b_{13} \end{array} \right).
\]

Here, the second and third columns are parallel. Thus, the determinant of the spin-1/2 transfer matrix in the sector of \( M = 1 \) is non-regular.

Appendix D. Reducing spin-\( \ell/2 \) Bethe states with principal grading

In order to evaluate the spin-\( s \) form factors for the spin-\( s \) elementary operators \( E_i^{(\ell)} \) associated with principal grading, we first transform them into those of homogeneous grading, and then apply to them the formula for expressing the spin-\( s \) elementary operators in terms of the sum of products of spin-1/2 elementary operators.

Let us recall the gauge transformation \( \chi_{12\ldots N_s}^{(1\ell)} \), which maps the higher-spin transfer matrix associated with principal grading of type \((1,\ell^{\otimes N_s})\) to that of homogeneous grading:

\[
T_{0,12\ldots N_s}^{(1,\ell)}(\lambda) = \chi_{0,12\ldots N_s}^{(1,\ell)} T_{0,12\ldots N_s}^{(1,\ell)}(\lambda) \chi_{0,12\ldots N_s}^{(1,\ell)}
\]

\[
= \left( \begin{array}{ccc} \chi_{12\ldots N_s}^{(\ell)} A^{(\ell)}(\lambda) \chi_{12\ldots N_s}^{(\ell)} & -e^{-\lambda} \chi_{12\ldots N_s}^{(\ell)} B^{(\ell)}(\lambda) \chi_{12\ldots N_s}^{(\ell)} \\ e^{-\lambda} \chi_{12\ldots N_s}^{(\ell)} C^{(\ell)}(\lambda) \chi_{12\ldots N_s}^{(\ell)} & \chi_{12\ldots N_s}^{(\ell)} D^{(\ell)}(\lambda) \chi_{12\ldots N_s}^{(\ell)} \end{array} \right).
\]

We also recall that the \( C \) operator acting on the tensor product of the spin-\( \ell/2 \) representations \((V^{(\ell)})^{\otimes N_s}\) is derived from the \( C \) operator acting on the tensor product of the spin-1/2 representations \((V^{(1)})^{\otimes N_s}\) multiplied by the projection operators:

\[
C^{(\ell)}(\mu) = P_{12\ldots L}^{(\ell)} C^{(\ell+0)}(\mu) P_{12\ldots L}^{(\ell)} e^\mu.
\]

Here, through the spin-1/2 gauge transformation we have

\[
C^{(\ell+0)}(\mu) = \chi_{12\ldots L} C^{(\ell+0)}(\mu) \chi_{12\ldots L}^{-1} e^\mu.
\]
For the spin-1 case, let us express the double sum 
\[ \sum_{\alpha=1}^{N} C^{(\ell \mu)}(\mu_{\alpha}) = \langle 0| \prod_{k=1}^{N} ( (\chi_{1 \ldots N_{s}}^{(\ell)} )^{-1} P_{1 \ldots L}^{(\ell)} C^{(\ell \mu)}(\mu_{\alpha}) \chi_{1 \ldots N_{s}}^{(\ell)} \rangle \]

Here we have made use of the commutation relation with the projection \( P_{1 \ldots L}^{(\ell)} \). Thus, we have

\[ \langle \{ \mu_{\alpha}\}_{N}^{(\ell \mu)} \rangle = \langle 0| \prod_{k=1}^{N} ( (\chi_{1 \ldots N_{s}}^{(\ell)} )^{-1} C^{(\ell \mu)}(\mu_{k}) \chi_{1 \ldots N_{s}}^{(\ell)} \rangle \]  

Similarly we have

\[ | \{ \lambda_{\alpha}\}_{M}^{(\ell \mu)} \rangle = \prod_{\alpha=1}^{M} B^{(\ell \mu)}(\lambda_{\alpha}) |0\rangle \]

\[ = \prod_{\alpha=1}^{M} \left( (\chi_{1 \ldots N_{s}}^{(\ell)} )^{-1} P_{1 \ldots L}^{(\ell)} B^{(\ell \mu)}(\lambda_{\alpha}) \chi_{1 \ldots N_{s}}^{(\ell)} \right) \langle 0| \]

\[ = \left( (\chi_{1 \ldots N_{s}}^{(\ell)} )^{-1} P_{1 \ldots L}^{(\ell)} \right) \prod_{\alpha=1}^{M} B^{(\ell \mu)}(\lambda_{\alpha}) |0\rangle. \]  

**Appendix E. Symmetric multiple-integral representations**

For the spin-1 case, let us express the double sum \( \sum_{c_{1}}^{M} \sum_{c_{2}}^{M} f(c_{1}, c_{2}) \) in the symmetric form which leads to the symmetric expression of the multiple-integral representations. Here \( c_{1} \) and \( c_{2} \) run through from 1 to \( M \) corresponding to all the 2-strings of the ground state.

doi:10.1088/1742-5468/2012/04/P04001
Recall that variable $c_j$ ($1 \leq j \leq 2sm$) takes integers from 1 to $M = N_s$ which correspond to $N_s/2$ sets of 2-strings. We express them in terms of integers $a(j, \beta)$ for $\beta = 1, 2$, where $a(j, \beta)$ take integral values from 1 to $N_s/2$. We first express the sum over $c_1$ in terms of $a(1, \beta)$ as follows:

$$
\sum_{c_1} = \sum_{a(1,1)=1}^{M/2} + \sum_{a(1,2)=1}^{M/2}.
$$

(E.1)

More precisely we have

$$
\sum { f(c_1) = \sum_{a(1,1)=1}^{M/2} f(2(a(1,1) - 1) + 1) + \sum_{a(1,2)=1}^{M/2} f(2(a(1,2) - 1) + 1).}
$$

(E.2)

For the spin-1 case with one-point function ($m = 1$) we have

$$
\sum_{c_1=1}^{M'} \sum_{c_2=1}^{M} f(c_1, c_2) = \left( \sum_{a(1,1)=1}^{M'/2} + \sum_{a(1,2)=1}^{M'/2} \right) \left( \sum_{a(2,1)=1}^{M/2} + \sum_{a(2,2)=1}^{M/2} \right) f(c_1, c_2)
$$

$$
= \left( 0 + \sum_{a(1,1)=1}^{M'/2} \sum_{a(2,1)=1}^{M'/2} + \sum_{a(1,2)=1}^{M'/2} \sum_{a(2,2)=1}^{M'/2} \right) f(c_1, c_2). \tag{E.3}
$$

Here we recall that $f(c_1, c_2)$ vanishes if the types of string rapidities $c_1$ and $c_2$ are the same.

Let us now introduce variables $a_j$ ($j = 1, 2$) which correspond to the centers of the 2-strings. We define an integer-valued variable $\hat{c}_j$ which is a function of $a_j$ as follows:

$$
\hat{c}_j = 2(a_j - 1) + \beta(j).
$$

(E.4)

Then, in terms of permutations $\pi$ in the symmetric group $S_2$ we express the sum as follows:

$$
\sum_{c_1=1}^{M'} \sum_{c_2=1}^{M} f(c_1, c_2) = \sum_{\pi \in S_2} \sum_{a_{\pi_1}=1}^{M'/2} \sum_{a_{\pi_2}=1}^{M/2} f(\hat{c}_{\pi_1}, \hat{c}_{\pi_2})
$$

$$
= \sum_{\pi \in S_2} \sum_{a_{\pi_1}=1}^{M'/2} \sum_{a_{\pi_2}=1}^{M/2} f(\hat{c}_{(12)1}, \hat{c}_{(12)2})
$$

$$
= \sum_{\pi \in S_2} \sum_{a_{\pi_1}=1}^{M'/2} \sum_{a_{\pi_2}=1}^{M/2} f(\hat{c}_{\pi_1}, \hat{c}_{\pi_2}). \tag{E.5}
$$

We thus have

$$
\sum_{c_1=1}^{M'} \sum_{c_2=1}^{M} f(c_1, c_2) = \sum_{\pi \in S_2} \sum_{a_{\pi_1}=1}^{M'/2} \sum_{a_{\pi_2}=1}^{M/2} f(\hat{c}_{\pi_1}, \hat{c}_{\pi_2}). \tag{E.6}
$$

The result leads to the symmetric expression of the multiple-integral representation.
References

[1] Korepin V E, Bogoliubov N M and Izergin A G, 1993 Quantum Inverse Scattering Method and Correlation Functions (Cambridge: Cambridge University Press)

[2] Slavnov N A, Calculation of scalar products of wavefunctions and form factors in the framework of the algebraic Bethe ansatz, 1989 Theor. Math. Phys. 79 502

[3] Jimbo M, Miki K, Miwa T and Nakayashiki A, Correlation functions of the XXZ model for $\Delta < -1$, 1992 Phys. Lett. A 168 256

[4] Jimbo M and Miwa T, 1995 Algebraic Analysis of Solvable Lattice Models (Providence, RI: American Mathematical Society)

[5] Maillet J M and Sanchez de Santos J, Drinfel'd twists and algebraic Bethe ansatz, 2000 L D Faddeev's Seminar on Mathematical Physics (Amer. Math. Soc. Transl. vol 201, Ser. 2) ed M Semenov-Tian-Shansky (Providence, RI: American Mathematical Society) pp 137–78

[6] Kitanine N, Maillet J M and Terras V, Form factors of the XXZ Heisenberg spin-1/2 finite chain, 1999 Nucl. Phys. B 554 647 [FS]

[7] Kitanine N, Maillet J M and Terras V, Correlation functions of the XXZ Heisenberg spin-1/2 chain in a magnetic field, 2000 Nucl. Phys. B 567 554 [FS]

[8] Göhmann F, Klümper A and Seel A, Integral representations for correlation functions of the XXZ chain at finite temperature, 2004 J. Phys. A: Math. Gen. 37 7625

[9] Kitanine N, Correlation functions of the higher spin XXX chains, 2001 J. Phys. A: Math. Gen. 34 8151

[10] Castro-Alvaredo O A and Maillet J M, Form factors of integrable Heisenberg (higher) spin chains, 2007 J. Phys. A: Math. Theor. 40 7451

[11] Göhmann F, Seel A and Suzuki J, Correlation functions of the integrable isotropic spin-1 chain at finite temperature, 2010 J. Stat. Mech. P11011

[12] Deguchi T and Matsui C, Form factors of integrable higher-spin XXZ chains and the affine quantum-group symmetry, 2009 Nucl. Phys. B 814 405 [FS]

[13] Deguchi T and Matsui C, Correlation functions of the integrable higher-spin XXX and XXZ spin chains through the fusion method, 2010 Nucl. Phys. B 831 359 [FS]

[14] Deguchi T and Matsui C, Algebraic aspects of the correlation functions of the integrable higher-spin XXZ spin chains with arbitrary entries, 2011 New trends in Quantum Integrable Systems ed B Feigin et al (Singapore: World Scientific) pp 11–33

[15] Deguchi T and Matsui C, On the evaluation of form factors and correlation functions for the integrable spin-$s$ XXZ chains via the fusion method, 2011 arXiv:1103.4206

For form factors, see also Deguchi T and Matsui C, Erratum to form factors of integrable higher-spin XXZ chains and the affine quantum-group symmetry, 2009 Nucl. Phys. B 814 405

Deguchi T and Matsui C, 2011 Nucl. Phys. B 851 238

[16] Deguchi T and Motegi K, Quantum spin Hamiltonians and form factors of solvable models associated with $U_q(sl_2)$, in preparation

[17] Nishino A, Dotsenko V and Deguchi T, An algebraic derivation of the eigenspaces associated with an Ising-like spectrum of the superintegrable chiral Potts model, 2008 J. Stat. Phys. 133 587

[18] Takhtajan L A, The picture of low-lying excitations in the isotropic Heisenberg chain of arbitrary spins, 1982 Phys. Lett. A 87 479

[19] Babujian H M, Exact solution of the isotropic Heisenberg chain with arbitrary spins: thermodynamics of the model, 1983 Nucl. Phys. B 215 317 [FS7]

[20] Sogo K, Ground state and low-lying excitations in the Heisenberg XXZ chain of arbitrary spin $s$, 1984 Phys. Lett. A 104 51

[21] Kirillov A N and Reshetikhin N Yu, Exact solution of the integrable XXZ Heisenberg model with arbitrary spin. I. The ground state and the excitation spectrum, 1987 J. Phys. A: Math. Gen. 20 1565

[22] Alcaraz F C and Martins M J, Conformal invariance and critical exponents of the Takhtajan-Babujian models, 1988 J. Phys. A: Math. Gen. 21 4397

[23] Dörfel B-D, Finite-size corrections for spin-$S$ Heisenberg chains and conformal properties, 1989 J. Phys. A: Math. Gen. 22 L657

[24] Avdeev L V, The lowest excitations in the spin-$s$ XXX magnet and conformal invariance, 1990 J. Phys. A: Math. Gen. 23 L485

[25] Alcaraz F C and Martins M J, Conformal invariance and the operator content of the XXZ model with arbitrary spin, 1989 J. Phys. A: Math. Gen. 22 1829

[26] Frahm H, Yu N-C and Fowler M, The integrable XXZ Heisenberg model with arbitrary spin: construction of the Hamiltonian, the ground-state configuration and conformal properties, 1990 Nucl. Phys. B 336 396
Reduction formula of form factors for the integrable spin-$s$ XXZ chains and application to correlation functions

[27] Frahm H and Yu N-C, \textit{Finite-size effects in the XXZ Heisenberg model with arbitrary spin}, 1990 J. Phys. A: Math. Gen. \textbf{23} 2115

[28] de Vega H J and Woynarovich F, \textit{Solution of the Bethe ansatz equations with complex roots for finite size: the spin $S \geq 1$ isotropic and anisotropic chains}, 1990 J. Phys. A: Math. Gen. \textbf{23} 1613

[29] Klümper A and Batchelor M T, \textit{An analytic treatment of finite-size corrections in the spin-1 antiferromagnetic XXZ chain}, 1990 J. Phys. A: Math. Gen. \textbf{23} L189

[30] Klümper A, Batchelor M T and Pearce P A, \textit{Central charge of the 6- and 19-vertex models with twisted boundary conditions}, 1991 J. Phys. A: Math. Gen. \textbf{24} 3111

[31] Jimbo M, \textit{A $q$-difference analogue of $U(g)$ and the Yang–Baxter equation}, 1985 Lett. Math. Phys. \textbf{10} 63

[32] Jimbo M, \textit{A $q$-analogue of $U(gl(N + 1))$, Hecke algebra and the Yang–Baxter equation}, 1986 Lett. Math. Phys. \textbf{11} 247

[33] Drinfel’d V G, \textit{Quantum groups}, 1986 Proc. ICM (Berkeley) pp 798–820