A UNIFORMLY BOUNDED COMPLETE
EUCLIDEAN SYSTEM

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Abstract. A uniformly bounded complete orthonormal system of functions \( \Theta = \{ \theta_n \}_{n=1}^\infty \) with \( \| \theta_n \|_{L^\infty[0,1]} \leq M \) is constructed such that
\[
\sum_{n=1}^\infty a_n \theta_n \quad \text{converges almost everywhere on } [0,1]
\]
if \( \{ a_n \}_{n=1}^\infty \in l^2 \) and \( \sum_{n=1}^\infty a_n \theta_n \) diverges a.e. for any \( \{ a_n \}_{n=1}^\infty \not\in l^2 \).

Thus Menshov’s theorem on the representation of measurable, almost everywhere finite, functions by almost everywhere convergent trigonometric series cannot be extended to the class of uniformly bounded complete orthonormal systems.

1. Introduction

The history of the study of pointwise convergence of the expansions by general orthogonal complete systems goes back to the beginning of the twentieth century. Among others one can recall the example constructed by H. Steinhaus \([12],[1]\) of a complete orthonormal system such that the expansion by the system of an integrable function diverges almost everywhere. An orthonormal system (ONS) \( \{ \varphi_n \}_{n=1}^\infty \) of functions defined on a closed interval \([a,b]\) is called a convergence system if
\[
\sum_{n=1}^\infty a_n \varphi_n \quad \text{converges almost everywhere (a.e.) for any } \{ a_n \}_{n=1}^\infty \in l^2 .
\]

The history of studies in convergence and divergence of orthogonal series has a long story (see \([4],[11],[13]\)). P.L. Ul’yanov (see \([13]\)) posed various problems in this area which stimulated research in this area. Particularly, B.S. Kashin \([6]\) responding to a problem posed in \([13]\) prove that there exists a complete ONS \( \{ \varphi_n \}_{n=1}^\infty \) of functions defined on \([0,1]\) which is a convergence system and for any \( \{ a_n \}_{n=1}^\infty \not\in l^2 \) the series
\[
\sum_{n=1}^\infty a_n \varphi_n \quad \text{diverges on some set of positive measure.}
\]

An ONS \( \{ \varphi_n \}_{n=1}^\infty \) of functions defined on a closed interval \([a,b]\) is called a divergence system if the series \( \sum_{n=1}^\infty a_n \varphi_n \) diverges a.e. on \([a,b]\) for any \( \{ a_n \}_{n=1}^\infty \not\in l^2 \).

Another problem posed in \([13],p.695\) asks if there exists a complete ONS which is simultaneously a convergence and a divergence system.

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B.S. Kashin indicated in [6] that this problem remains open. An affirmative answer was given by the author [8].

An ONS \( \{ \varphi_n \}_{n=1}^\infty \) is called a Euclidean system if it is both a convergence and a divergence system.

A system of functions \( \{ \phi_n \}_{n=1}^\infty \) defined on \([a, b]\) is called uniformly bounded if there exists \( M > 0 \) such that

\[
\| \phi_n \|_{L^\infty([a, b])} \leq M \quad \text{for all} \quad n \in \mathbb{N}.
\]

In the present paper we construct a uniformly bounded complete Euclidean system. We prove the following

**Theorem 1.** For any \( M > 1 + \sqrt{2} \) there exists a complete Euclidean system \( \Theta = \{ \theta_n \}_{n=1}^\infty \) in \( L^2_{[0,1]} \) such that

\[
\| \theta_n \|_{L^\infty_{[0,1]}} \leq M \quad \text{for all} \quad n \in \mathbb{N}.
\]

As an immediate corollary we obtain that Menshov’s theorem [10] on the representation of measurable, almost everywhere finite, functions by almost everywhere convergent trigonometric series cannot be extended to the class of uniformly bounded complete orthonormal systems. Moreover, the system \( \Theta \) is not a representation system for the classes \( L^r_{[0,1]}, 0 \leq r < 2 \) if we want to represent the functions from those classes by a series which converges pointwise on sets of positive measure, even if those sets depend on the function. Other corollaries of Theorem 1 can be find in [7].

In the theory of general orthogonal series uniformly bounded ONS are one of the main objects that have been studied systematically. In the survey article [13] Ul’yanov posed the problem of the existence of complete uniformly bounded convergence system. It was motivated by the known open problem about the a.e. convergence of the Fourier series. Giving an answer to Ul’yanov’s problem Olevskii [11] constructed such a convergence system. The idea of the construction can be described as follows. At first step construct a complete ONS of bounded functions which can be divided into two convergence systems such that the second one is uniformly bounded. In the construction it is the Rademacher system. Afterwards any element of the first convergence system is “dissolved” by the Rademacher functions in such a way that the resulting functions are uniformly bounded. This process is performed by special orthogonal matrices. Those special matrices afterwards were used for various constructions. It created among some experts an impression that those matrices are remarkable by themselves. Probably this believe do not permit to some group of experts to admit that those matrices were known in applied mathematics much earlier as Haar matrices. We will return to the Haar matrices later on. The important novelty in Olevskii’s construction was the idea of dissolution by orthogonal transformations of “bad” elements of a complete ONS by “good” ones. Of course, at first one should be able to obtain
a CONS for which such a construction can be applied. It should be mentioned that the idea of sticking together orthogonal functions by some orthogonal transformations was applied earlier by V. Kostitzin [9]. We should also mention L. Carleson’s [2] famous article where it was proved that the trigonometric system is a convergence system.

Let us explain what we understand by saying that some function is “dissolved” by the Rademacher functions. Moreover, it is done in such a way that the resulting functions are uniformly bounded. Let $M > 1 + \sqrt{2}$ and suppose we have two functions $\phi_0, \phi_1$ such that

$$\|\phi_0\|_{\infty} = \lambda \geq M \quad \text{and} \quad \|\phi_1\|_{\infty} \leq 1,$$

then the orthogonal transformation of those functions by the matrix

$$A_0 = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

will give two functions $\psi_i = \frac{1}{\sqrt{2}}(\phi_0 + (-1)^{i-1}\phi_1)$ ($i = 1, 2$) such that

$$\|\psi_i\|_{\infty} \leq 2^{-1/2}(1 + \lambda) < \lambda \quad i = 1, 2.$$

Repeating the process for the pairs $(\psi_1, \phi_3), (\psi_2, \phi_4)$, where

$$\|\phi_i\|_{\infty} \leq 1 (i = 3, 4)$$

and, if necessary, for the obtained new functions one can easily check that on some step the obtained functions will have $L^\infty_{[0,1]}$-norm less than $M$. If we have an infinite subsystem of functions uniformly bounded by $C > 0$ then the same process will give functions with $L^\infty_{[0,1]}$-norm less than $C \cdot M$. It seems that the solution of the following conjecture needs some new ideas.

**Conjecture 2.** There is no complete Euclidean system $\{\varphi_n\}_{n=1}^{\infty}$ in $L^2_{[0,1]}$ such that

$$|\varphi_n(t)| = 1 \quad \text{a.e. on } [0, 1] \quad \text{for any } n \in \mathbb{N}.$$

The present paper consists of 5 sections. In Section 2 are given definitions and auxiliary results many of which can be consulted in the previous papers [7] and [8] of the author. In Section 3 we repeat the construction of two auxiliary complete orthonormal systems from [8], where it was proved that those systems are convergence systems. At the end of Section 3 a Euclidean system $\{\Upsilon_k(x)\}_{k=1}^{\infty}$ is constructed such that by adding a subsystem of the Rademacher functions to $\{\Upsilon_k(x)\}_{k=1}^{\infty}$ we will obtain a complete orthonormal system. In Section 4 one can find the proof that $\{\Upsilon_k(x)\}_{k=1}^{\infty}$ is a system of divergence. Moreover, we prove an essentially stronger result (see Theorem 13) which is fundamental for the proof of Theorem 1. That the system $\Theta$ is a convergence system follows immediately from Proposition 6 and from the construction of the auxiliary system $\{\chi_k(x)\}_{k=1}^{\infty}$. 
2. Definitions and auxiliary results

We repeat some notations from [7]. For \([a, b] \subset \mathbb{R}\) and \(k \in \mathbb{N}\) let
\[
\mathcal{E}^k_{[a, b]} = \left\{ f : f(x) = a_i \text{ if } x \in \left(a + (i - 1)\frac{b - a}{2^k}, a + i\frac{b - a}{2^k}\right) \right\},
\]
where \(1 \leq i \leq 2^k\) and the inner product is defined in the same way as in \(L^2_{[a, b]}\). Further we ignore the values at the points of discontinuity of functions from \(\mathcal{E}^k_{[a, b]}\). In the paper we will use also the following notation:
\[
\mathcal{E}^j_{[a, b]} = \mathcal{E}^k_{[a, b]} \oplus \mathcal{E}^{k,j}_{[a, b]},
\]
where \(j > k\). In what follows we will denote by \(I_E(x)\) the characteristic function of a measurable set \(E\).

One of the main tools in our construction will be the Menshov functions \(M_k, k \geq 3\) which are odd 2–periodic functions defined on the real line \(\mathbb{R}\) and \(M_k \in \mathcal{E}^k_{[-1, 1]}\). For any natural \(k \geq 3\) we define \(M_k \in \mathcal{E}^k_{[-1, 1]}\) to be an odd 2–periodic function on the real line \(\mathbb{R}\) satisfying to the following equations:
\[
M_k(x) = \begin{cases}
\frac{1}{8i} \cdot 2^k, & \text{if } x \in (\frac{i-1}{2^k}, \frac{i}{2^k}) \quad 1 \leq |i| \leq 2^k - 1, \\
0, & \text{if } x \in (-\frac{1}{2^k}, 0) \cup (1 - \frac{1}{2^k}, 1).
\end{cases}
\]
Denote \(M_{k,i}(x) = M_k(x - i \cdot 2^{-k})\) where \(i \in \mathbb{N}\).

The following lemma was proved in [7].

**Lemma 3.** For any \(k \in \mathbb{N}\) there exist an orthonormal system \(\{f^i_k\}_{i=0}^{2^k-1}\) in \(L^2_{[-2,2]}\) such that
\[
f^i_k(x) = \begin{cases}
M_{k,i}(x), & \text{if } x \in [-1, 1]; \\
0, & \text{if } x \in [-2, -1);
\end{cases}
\]
\[
\int_{-2}^{2} f^i_k(x) dx = 0 \quad \text{for all } 0 \leq i \leq 2^k - 1,
\]
and \(f^i_k|_{[1,2]} \in \mathcal{E}^{k+1}_{[1,2]}\).

The Haar functions are defined in the following way: for all \(t \in [0, 1]\) we will take \(h_1(t) = 1\) and for \(k = 0, 1, 2, \ldots; j = 1, 2, \ldots, 2^k\), let
\[
h^{(k)}_j(t) = \begin{cases}
2^{\frac{k}{2}}, & \text{if } \frac{2^j-2}{2^{k+1}} < t < \frac{2^j-1}{2^{k+1}}; \\
-2^{\frac{k}{2}}, & \text{if } \frac{2^{j-1}}{2^{k+1}} < t < \frac{2^j}{2^{k+1}}; \\
0, & \text{otherwise}.
\end{cases}
\]
If \(n = 2^k + j\) we denote \(h_n = h^{(k)}_j\). The closure of the support of a the Haar function \(h_n\) will be denoted by \(\Delta_n\) or by \(\Delta^{(k)}_j\). It can be easily checked that for any \(k \in \mathbb{N}\) the Haar functions \(\{h_n\}_{n=1}^{2^k}\) constitute an orthonormal basis in the space \(\mathcal{E}^k_{[0,1]}\).
Recall orthogonal Haar matrices \( H_k, k \in \mathbb{N} \), that arise from the Haar system. For any \( k \in \mathbb{N} \) we take the midpoints \( x_j^{(k)} = (j - \frac{1}{2})2^{-k} \) of the intervals \( \Delta_j^{(k)} = \left( \frac{j-1}{2^k}, \frac{j}{2^k} \right) \) and set

\[
H_k = \left( a_{ij}^{(k)} \right)_{1 \leq i, j \leq 2^k, k \in \mathbb{N}},
\]

where

\[
a_{ij}^{(k)} = 2^{-\frac{k}{2}} h_i(x_j^{(k)}).
\]

The Rademacher system \( \{ r_n(t) \}_{n=0}^{\infty} \) is an orthonormal system of functions defined on the closed interval \([0, 1]\). It is convenient for us to consider the Rademacher functions defined on the real line:

\[
r_n(t) = \text{sgn} \left( \sin 2^{n+1} \pi t \right) \quad t \in \mathbb{R}, n = 0, 1, \ldots
\]

For our construction it is useful to note that

\[
r_n(\cdot) \in E_{[0,1]}^{n,n+1}, \quad h_i(\cdot) \in E_{[0,1]}^{n,n+1} \quad \text{for all} \quad n \in \mathbb{N}.
\]

We also need the following two lemmas from [8] (see Lemma 2.9 and Lemma 2.7).

**Lemma 4.** Let \( \{ f_k \}_{k=1}^{N+1} \) be a collection of functions on \([0, 1]\) such that for any \( \{ a_k \}_{k=1}^{N+1} \subset \mathbb{R} \)

\[
\int_{[0,1]} \sup_{m} \left| \sum_{k=1}^{m} a_k f_k(x) \right|^2 dx \leq C \sum_{k=1}^{N+1} a_k^2
\]

for some \( C > 0 \). Then for any \( \{ b_k \}_{k=1}^{N+1} \subset \mathbb{R} \)

\[
\int_{[0,1]} \max_{1 \leq m \leq N} \left| \sum_{k=1}^{m} b_k \tilde{f}_k(x) \right|^2 dx \leq 14C \sum_{k=1}^{N} b_k^2,
\]

where

\[
\tilde{f}_j(x) = -\delta_N \sum_{i=1}^{N} f_{i+1}(x) + f_j(x) + \frac{1}{\sqrt{N+1}} f_1(x) \quad (1 \leq j \leq N),
\]

and \( \delta_N = \frac{1}{\sqrt{N+1}} \).

We give the proof of the following lemma because the value of the constant \( C_p \) is adjusted. Of course the exact value of the constant is not important for the proof of the main result but the lemma may be interesting by itself.

**Lemma 5.** Let \( f \in L_p^{[0,1]}, 1 \leq p < \infty \) and

\[
\Lambda_f(t) := |\{ x \in [0, 1] : |f(x)| > t \}|.
\]

Define

\[
f_p^*(x) = 2^{k/p} f \left( 2^k (x - 2^{-k}) \right) \quad \text{when} \quad x \in (2^{-k}, 2^{-k+1}], \quad k \in \mathbb{N}.
\]
Then
\[ \Lambda f^p_p(t) \leq C_p t^{-p} \| f \|_p^p, \quad \text{where} \quad C_p = \frac{2}{p(2/p - 1)}. \]

**Proof.** The proof is straightforward. We have that
\[ \Lambda f^p_p(t) = \sum_{k=1}^{\infty} \left| \{ x \in (2^{-k}, 2^{-k+1}] : |f^p(x)| > t \} \right| \]
\[ = \sum_{k=1}^{\infty} 2^{-k} \left| \{ x \in [0, 1] : |f(x)| > 2^{-k/p}t \} \right| = \sum_{k=1}^{\infty} 2^{-k} \Lambda f(2^{-k/p}t). \]

Afterwards, we write
\[ 2^{-k} \Lambda f(2^{-k/p}t) = t^{-p} (2^{-k/p}t)^{p-1} \Lambda f(2^{-k/p}t) \left( 2^{-k/p} - 2^{-k/p-1} \right) t \cdot \frac{1}{1 - 2^{-1/p}}. \]

Observe that
\[ (2^{-(k+1)/p}t)^{p-1} \Lambda f(2^{-k/p}t) \leq t^{p-1} \Lambda f(x) \quad \text{if} \quad x \in (2^{-(k+1)/p}t, 2^{-k/p}t) \]
Hence,
\[ (2^{-(k+1)/p}t)^{p-1} \Lambda f(2^{-k/p}t) (2^{-k/p} - 2^{-k/p-1}) t \leq \int_{2^{-(k+1)/p}t}^{2^{-k/p}t} x^{p-1} \Lambda f(x) \, dx \]
and the proof is easily finished recalling the formula
\[ \| f \|_p^p = p \int_0^{\infty} t^{p-1} \Lambda f(x) \, dx. \]

Recall that a system of functions \( \{ f_k \}_{k=1}^{\infty} \) defined on \([0, 1] \) is called an \( S_p \)-system \( (2 < p < \infty) \) if for any \( \{a_k\}_{k=1}^{N} \subset \mathbb{R} \)
\[ \left\| \sum_{k=1}^{m} a_k f_k(\cdot) \right\|_{p} \leq C_p \left( \sum_{k=1}^{N} a_k^2 \right)^{1/2} \]
for some \( C_p > 0 \). The following result is well known (see [3]).

**Proposition 6.** Let \( \{ f_k \}_{k=1}^{\infty} \) be an \( S_p \)-system \( (2 < p < \infty) \). Then for any \( \{a_k\}_{k=1}^{\infty} \in l^2 \)
\[ \left\| \sup_m \sum_{k=1}^{m} a_k f_k(\cdot) \right\|_{p} \leq C'_p \left( \sum_{k=1}^{\infty} a_k^2 \right)^{1/2} \]
for some \( C'_p > 0 \).

The Khintchine inequalities (see [5]) show that the Rademacher system is an \( S_p \)-system \( (2 < p < \infty) \). The definition of a set/system of independent functions can be consulted in [5], [3] and others.
3. Construction of a CONS of bounded functions

For the completeness of the exposition we repeat the construction of two auxiliary complete orthonormal systems from [8]. For the convenience of the reader we will maintain some notations of the cited paper.

3.1. Construction of the first auxiliary CONS. We suppose that the orthonormal set of functions \( \{f_k^i\}_{i=0}^{2^k - 1}, k \in \mathbb{N} \) defined in Lemma 3 are extended periodically with period 4 on the whole line and define

\[
g_1^i(x) = \begin{cases} 
2f_2^3(8x - 2), & \text{if } x \in [0, \frac{1}{2}); \\
r_9 + i(x)I(\frac{1}{2}, 1], & \text{if } x \in [\frac{1}{2}, 1],
\end{cases}
\]

for \( 0 \leq i \leq 2^2 - 1 \). It is easy to check that the functions \( \{g_1^i(x)\}_{i=0}^{3} \) are orthonormal in the space \( L^2_{[0,1]} \). Let \( k_1 \) be the smallest natural number such that

\[ g_1^i \in \mathcal{E}_{[0,1]}^{k_1} \quad 0 \leq i \leq 2^2 - 1. \]

We take a set of orthonormal functions

\[ \psi_{\nu} \in \mathcal{E}_{[0,1]}^{k_1+2}, \quad 1 \leq \nu \leq 2^{k_1+2} - 6 = m_1, \]

\[ \psi_1(t) = 1 \quad \text{if} \quad t \in [0, 1], \quad \text{that are orthogonal in} \quad L^2_{[0,1]}, \quad \text{to the functions} \]

\[ g_1^i(0 \leq i \leq 2^2 - 1), r_{k_1} \quad \text{and} \quad h_1^{(k_1+1)}. \]

By (5) it is obvious that the functions (6) constitute an orthonormal set of functions. According to our construction the set of the functions

\[ \{g_1^i(x)\}_{i=0}^{3} \bigcup \{r_{k_1}\} \bigcup \{h_1^{(k_1+1)}\} \bigcup \{\psi_{\nu}(x)\}_{\nu=1}^{m_1} \]

is an orthonormal basis in \( \mathcal{E}_{[0,1]}^{k_1+2} \).

At the \( n \)-th step, \( n > 1 \), of our construction we define

\[
\hat{g}_n^i(x) = \begin{cases} 
-12n^2f_2^n(2^{2n^2+2}(x - 2^{-2n^2})) & \text{if } x \in [0, 2^{-2n^2}); \\
-n^{-1}2^{\frac{k}{2}}f_2^{i}(2^{2^{k}+2}(x - 2^{-k})) & \text{if } x \in (2^{-k}, 2^{-k+1}], \\
r_{k_{n-1}+i+1}(x)I(\frac{1}{2}, 1], & \text{if } x \in (\frac{1}{2}, 1],
\end{cases}
\]

and \( 0 \leq i \leq 2^{2n} - 1 \). Then as above we extend the functions \( \hat{g}_n^i, 0 \leq i \leq 2^{2n} - 1 \) periodically with period 1 to the whole line and denote

\[ g_n^i(x) = \hat{g}_n^i(2^{k_{n-1}+2^{2n^2+2}}x) \quad \text{for all} \quad 0 \leq i \leq 2^{2n} - 1. \]

It is easy to check that the functions \( \{g_n^i(x) : 0 \leq i \leq 2^{2n} - 1\} \) are orthonormal in the space \( L^2_{[0,1]} \) and if \( k_n \) is the smallest natural number such that

\[ g_n^i \in \mathcal{E}_{[0,1]}^{k_n} \quad \text{for all} \quad 0 \leq i \leq 2^{2n} - 1, \]
then by the definition of the set of functions \( \{ \hat{g}^i_n(x) : 0 \leq i \leq 2^{2n} - 1 \} \) and Lemma 3
\[
\hat{g}^i_n \in \mathcal{E}^{k_{n-1}+1,k_n}_{[0,1]} \quad 0 \leq i \leq 2^{2n} - 1.
\]
We take a set of orthonormal functions
\[
\psi_\nu \in \mathcal{E}^{k_{n-1}+2,k_n+2}_{[0,1]}, m_{n-1} + 1 \leq \nu \leq 2^{k_n+2} - 2^{2n} - 2 = m_n,
\]
that are orthogonal in \( L^2_{[0,1]} \) to the functions
\[
r_{k_n}, h_{1}^{(k_n+1)} \quad \text{and} \quad g^i_n \quad \text{for all} \quad (0 \leq i \leq 2^{2n} - 1).
\]
As above we conclude that the set of functions
\[
\{ g^i_n(x) \}_{i=0}^{2^{2n}-1} \bigcup \{ r_{k_n}(x) \} \bigcup \{ h_1^{(k_n+1)}(x) \} \bigcup \{ \psi_\nu(x) \}_{\nu=m_{n-1}+1}^{m_n}
\]
is an orthonormal basis in \( \mathcal{E}^{k_{n-1}+2,k_n+2}_{[0,1]} \). Hence,
\[
\bigcup_{n=1}^{\infty} \bigcup_{i=0}^{2^{2n}-1} \{ g^i_n(x) \} \bigcup_{n=1}^{\infty} \{ r_{k_n}(x) \} \bigcup_{n=1}^{\infty} \{ h_1^{(k_n+1)}(x) \} \bigcup_{\nu=1}^{\infty} \{ \psi_\nu(x) \}
\]
is a CONS in \( L^2_{[0,1]} \). From our construction it follows immediately that
\[
\int_{\Delta^{(k)}_{\nu}} \left( \sum_{i=0}^{2^{2n}-1} a_i g_i^j(x) \right)^2 dx = |\Delta^{(k)}_{\nu}| \sum_{i=0}^{2^{2n}-1} a_i^2
\]
for any \( \Delta^{(k)}_{\nu} = \left( \frac{\nu-1}{2^k}, \frac{\nu}{2^k} \right) \), where \( 1 \leq \nu \leq 2^k, 1 \leq k \leq k_{n-1} + 2n^2 + 2 \).
Moreover, by \( \mathcal{F} \) we obtain that for any \( \omega \in \mathbb{R} \)
\[
\int_{\Delta^{(k)}_{\nu}} \left( \omega + \sum_{i=0}^{2^{2n}-1} a_i g_i^j(x) \right)^2 dx = |\Delta^{(k)}_{\nu}| \left( \omega^2 + \sum_{i=0}^{2^{2n}-1} a_i^2 \right).
\]
We also have that for any \( n \in \mathbb{N} \)
\[
\| \psi_\nu \|_\infty \leq \sqrt{2^{k_n+2}} \quad \text{for all} \quad m_{n-1} + 1 \leq \nu \leq m_n, m_0 = 0
\]
just because they belong to the space \( \mathcal{E}^{k_{n-1}+2}_{[0,1]} \). Note also that
\[
m_n < 2^{k_n+2} \quad \text{for any} \quad n \in \mathbb{N}.
\]
According to our construction and Lemma 3 of \( \mathcal{F} \) the following assertion holds.

**Proposition 7.** For all \( n \in \mathbb{N} \) and for any collection of nontrivial functions
\[
F_j(x) = \sum_{i=1}^{2^{j+2}-1} a_i^{(j)} g_i^j(x) \quad 1 \leq j \leq n
\]
the functions \( \{ F_j(x), r_{k_j}(x) \}_{j=1}^n \) constitute a set of independent functions.

The following propositions were proved in \( \mathcal{F} \).
Proposition 8. For any sequence \( \{c_\nu\}_{\nu=1}^\infty \in L^2 \)
\[
\int_{[0,1]} \sup_n \left| \sum_{\nu=1}^{m_n} c_\nu \psi_\nu(x) \right|^2 dx \leq C \sum_{\nu=1}^\infty c_\nu^2,
\]
for some \( C > 0 \) independent of the coefficients.

Proposition 9. The system \( \{g^j_i(x) : 0 \leq i \leq 2^{2n} - 1\}_{n=1}^\infty \) is an orthonormal system of convergence.

3.2. Construction of the second auxiliary CONS. In this section our aim is to transform the set of orthogonal functions
\[
\{h_1^{(k_n+1)}(x)\}_{n=1}^\infty \bigcup \{\psi_\nu(x)\}_{\nu=1}^\infty
\]
into an orthonormal system of convergence \( \{\xi_l(x)\}_{l=1}^\infty \). We will do that by the help of the orthogonal matrices (see [4], Proposition 1)
\[
K_N = (\kappa^{(N+1)}_{ij}) = \begin{pmatrix}
\sqrt{N+1} & 1 - \delta_N & 1 & \cdots & 1 \\
-1 & -\delta_N & \sqrt{N+1} & \cdots & -\delta_N \\
1 - \delta_N & 1 & -\delta_N & \cdots & 1 - \delta_N \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
-\delta_N & -\delta_N & \cdots & 1 - \delta_N & \sqrt{N+1}
\end{pmatrix},
\]
where \( \delta_N = \frac{1}{2^N} (1 + \frac{1}{\sqrt{N+1}}) \). Moreover, we will obtain some estimates on \( \|\xi_l\|_{L_1^n} \), where \( \eta_l \to 0 \) as \( l \to \infty \). Let
\[
q_0 = 0, \quad q_n = \left(2^{2(k_n+1)} - 1\right) (m_n - m_{n-1}) \quad \text{for any } n \in \mathbb{N}
\]
and put
\[
(11) \quad p(n) = 2^{2(k_n+1)} \quad \text{and} \quad q_\nu(n) = \sum_{i=1}^{n-1} q_i + (\nu - m_{n-1} - 1) (2^{2(k_n+1)} - 1)
\]
for all \( n \in \mathbb{N} \) and \( \nu(m_{n-1} + 1 \leq \nu \leq m_n) \).
Afterwards for any \( \nu(m_{n-1} + 1 \leq \nu \leq m_n) \) and \( 1 \leq j \leq p(n) \) we define
\[
(12) \quad \phi_j^\nu(x) = \kappa_{ij}^{(p(n))} \psi_\nu(x) + \sum_{i=2}^{p(n)} \kappa_{ij}^{(p(n))} h_1^{k_\nu(n)+i-1+1} (x).
\]
By (9) and (11),(12) we have that for any \( \nu \in [m_{n-1} + 1, m_n] \cap \mathbb{N} \)
\[
(13) \quad |\phi_j^\nu(x)| \leq \sqrt{2 - k_n} \quad \forall x \in [2^{-k_\nu(n)} - 1, 1] \quad \text{and} \quad \forall j \in [1, p(n)] \cap \mathbb{N}.
\]
From (2),(10) and (12) follows that for any \( \nu \in [m_{n-1} + 1, m_n] \cap \mathbb{N} \) and any \( n \in \mathbb{N} \)
\[
(14) \quad \phi_j^\nu \cdot I_{[2^{-k_n-2},1]} \in \mathcal{E}_{[0,1]}^{k_n+2} \quad \text{for} \quad 1 \leq j \leq p(n).
\]
In order to enumerate the obtained functions we put

$$\rho_0 = 0, \rho_n = \sum_{j=1}^{n} 2^{2(k_j+1)}(m_j - m_{j-1})$$

and denote

$$\rho_{\nu}(n) = \rho_{n-1} + (\nu - m_{n-1} - 1)2^{2(k_{n}+1)}$$

for all $n \in \mathbb{N}$ and $\nu(m_{n-1} + 1 \leq \nu \leq m_n)$. Afterwards we denote

$$\xi_l(x) = \phi^j(x) \text{ if } l = \rho_{\nu}(n) + j, 1 \leq j \leq p(n)$$

and $m_{n-1} + 1 \leq \nu \leq m_n$. Hence, from (14) it follows that for any $l \in \mathbb{N}$ such that

$$\rho_n - 1 < l \leq \rho_n$$

and any dyadic interval $\Delta^{(k_{n}+2)}_\nu = (\nu - \frac{1}{2^{k_n+2}}, \nu + \frac{1}{2^{k_n+2}})$, where $2 \leq \nu \leq 2^{k_n+2}$ the function $\xi_l$ is constant on the interval $\Delta^{(k_{n}+2)}_\nu$:

$$\xi_l(x) = \omega^j_\nu \text{ for } x \in \Delta^{(k_{n}+2)}_\nu (2 \leq \nu \leq 2^{k_n+2}).$$

Thus, according to our construction, the set of functions

$$\bigcup_{n=1}^{\infty} \{g_n^j(x)\}_{i=0}^{2^{2n}-1} \bigcup \{r_n(x)\}_{n=1}^{\infty} \bigcup \{\xi_l(x)\}_{l=1}^{\infty}$$

is a CONS in $L^2_{[0,1]}$.

We also have

**Proposition 10.** For any sequence $\{c_l\}_{l=1}^{\infty} \in \ell^2$

$$\int_{[0,1]} \sup_n \sum_{l=1}^{m_n} c_l \xi_l(x)^2 dx \leq C \sum_{l=1}^{\infty} c_l^2,$$

for some $C > 0$ independent of coefficients.

**Proof.** The Proposition 10 follows immediately from Proposition 8 and Lemma 4. We should check that the conditions of Lemma 4 are satisfied for the system $\{h_1^{(k_n+1)}(x)\}_{n=1}^{\infty}$.

If we denote $h(x) = \sum_{n=1}^{\infty} a_{k_n} h_1^{(k_n+1)}(x)$ then its orthogonal projection $P(f)$ onto the subspace $E^{k_{n}+2}_{[0,1]}$ equals

$$|\Delta^{(k_{m}+2)}_{\nu}|^{-1} \int_{\Delta^{(k_{m}+2)}_{\nu}} h(t) dt \text{ on the interval } \Delta^{(k_{m}+2)}_{\nu}, 1 \leq \nu \leq 2^{k_{m}+2}.$$
3.3. Construction of a Euclidean system of bounded functions.
On this step we construct a Euclidean system of bounded functions by transformation of finite collections of functions from the ONS
\[ \bigcup_{n=1}^{\infty} \{g_n^i(x)\}_{i=0}^{2^{2n}-1} \bigcup_{i=1}^{\infty} \{\xi_i(x)\}_{i=1}^{\infty} \]
applying the orthogonal matrices \( K_N \). For any \( n \in \mathbb{N} \) we put \( l(n) = 2^{2n}+1 \) and define
\[ \Upsilon^n_j(x) = \kappa_{ij}^{(l(n))} \xi_n(x) + \sum_{i=2}^{2^{2n}+1} \kappa_{ij}^{(l(n))} g_n^{i-2}(x) \quad 1 \leq j \leq 2^{2n}+1. \]
Evidently, the obtained system of functions
\[ \left\{ \{\Upsilon^n_j(x)\}_{j=1}^{2^{2n}+1} \right\}_{n=1}^{\infty} \]
is again an ONS. We enumerate them in the natural order: for
\[ \mu_0 = 0, \quad \mu_m = \mu_{m-1} + 2^{2m} + 1 \]
we put
\[ \Upsilon_k(x) = \Upsilon^n_j(x) \quad \text{if} \quad k = \mu_{m-1} + j, \quad \text{where} \quad 1 \leq j \leq 2^{2m}+1. \]

**Theorem 11.** The ONS \( \{\Upsilon_k(x)\}_{k=1}^{\infty} \) is a system of convergence.

**Proof.** The theorem is an immediate consequence of Lemma 4 and Propositions 9 and 10. \( \square \)

**Theorem 12.** The ONS \( \{\Upsilon_k(x)\}_{k=1}^{\infty} \) is a system of divergence.

The proof of Theorem 12 is a particular case of the proof of Theorem 13 which will be given in the next section. However, while explaining the idea of the construction of the system \( \Theta \) we will refer the system \( \{\Upsilon_k(x)\}_{k=1}^{\infty} \) as a Euclidean system.

4. A uniformly bounded complete Euclidean system

We have constructed a Euclidean system \( \{\Upsilon_k(x)\}_{k=1}^{\infty} \) such that
\[ \{\Upsilon_k(x)\}_{k=1}^{\infty} \bigcup \{\tau_{kn}(x)\}_{n=1}^{\infty} \]
is a complete ONS. If we enumerate the system (23) in some order then a priori it is not clear that the obtained system will be a complete Euclidean system. Evidently it will be a convergence system. Whether the obtained system is a divergence system or not is far from being clear. In our particular case this problem is solved mainly with the help of Proposition 7. Moreover, for the proof of Theorem 1 we need a stronger property which we explain after enumerating the system
{∀_{k=1}^{\infty} \cup \{r_{k_n}(x)\}_{n=1}^{\infty}$ in a special way. Let $M > \sqrt{2} + 1$, be the constant from Theorem 1 and let $l_0 \in \mathbb{N}$ be such that 
\[\sqrt{2} - l_0 < M - \sqrt{2} - 1.\]

Afterwards, we put
\[\chi_0^{(i)}(x) = r_{k_i}(x), \quad 1 \leq i \leq \nu_0,\]
where $\nu_0 \in \mathbb{N}$ is such that
\[\Upsilon_1 \in E^{k_{\nu_0}}_{[0,1]}.
\]

Then we define
\[\chi^{(1)}_1(x) = \Upsilon_1(x) \quad \text{and} \quad \chi^{(i)}_1(x) = r_{k_i}^{(i)}(x), \quad 1 < i \leq 2^{n_1},\]
where $n_1 \in \mathbb{N}$ is such that $n_1 \geq k_{\nu_0} + l_0$
\[\Upsilon_2 \in E^{k_{\nu_0}}_{[0,1]} \quad \text{and} \quad \nu_1 = 2^{n_1} + \nu_0.
\]

In the same way for any $j \geq 2$ we define
\[\chi^{(j)}_j(x) = \Upsilon_j(x) \quad \text{and} \quad \chi^{(i)}_j(x) = r_{k_{\nu_j}}^{(i)}(x), \quad 1 < i \leq 2^{n_j},\]
where $n_j \in \mathbb{N}$ is such that $n_j \geq k_{\nu_j} + l_0$
\[\Upsilon_{j+1} \in E^{k_{\nu_j}}_{[0,1]} \quad \text{and} \quad \nu_j = 2^{n_j} + \nu_{j-1}.
\]

If we transform the functions
\[\Upsilon_j(x), r_{k_{\nu_j}}^{(i)}(x), \quad 1 < i \leq 2^{n_j}\]
by the Haar matrix $H_{n_j}$ then it is easy to check that the obtained orthonormal functions are bounded by the constant $M$.

By (21)–(22) we can consider that the numbers $k_{\nu_j}$ are chosen so that for any $m \in \mathbb{N}$
\[\Upsilon_j \in E^{k_{\nu_m}}_{[0,1]} \quad \text{for all} \quad \mu_{m-1} + 1 \leq j \leq \mu_m.
\]

Denote
\[\chi_k(x) = \chi_k^{(k)}(x), \quad \text{where} \quad 1 \leq k \leq \nu_0
\]
\[\chi_k(x) = \chi_k^{(i)}(x), \quad \text{where} \quad k = \nu_{j-1} + j + i - 1; \quad 1 \leq i \leq \nu_{j-1} + 2^{n_j} \quad \text{and} \quad j \geq 1.
\]

Then the system $\{\chi_k(x)\}_{k=1}^{\infty}$ will be a complete ONS. Moreover, the following assertion is true.

**Theorem 13.** For any $\{a_k\}_{k=1}^{\infty} \notin l^2$ the partial sums
\[S_j(x) = \sum_{k=1}^{\nu_{j-1}} a_k \chi_k(x), \quad j \in \mathbb{N},
\]
diverge a.e. on $[0,1]$, when $j \to +\infty$. 
4.1. Proof of Theorem 13. Let us rewrite the series \( \sum_{k=1}^{\infty} a_k \chi_k(x) \) as

\[
\sum_{i=1}^{\nu_0} a^{(i)}_0 \chi_0^{(i)}(x) + \sum_{j=1}^{\infty} 2^{n_j} \sum_{i=1}^{\nu_0} a^{(i)}_j \chi_j^{(i)}(x)
\]

and observe that the partial sums

\[
\sum_{i=1}^{\nu_0} a^{(i)}_0 \chi_0^{(i)}(x) + \sum_{j=1}^{N} 2^{n_j} \sum_{i=1}^{\nu_0} a^{(i)}_j \chi_j^{(i)}(x)
\]

coincide with the corresponding partial sums (30).

If \( \sum_{j=1}^{\infty} |a_j^{(1)}|^2 < +\infty \) then by Theorem 11 and well known properties of the Rademacher system (see [14]) we immediately obtain that the sequence (30) diverges a.e. when \( j \to +\infty \). Thus we have to consider only the case when

\[
\sum_{k=1}^{\infty} |a_k^{(1)}|^2 = +\infty.
\]

For any \( m \in \mathbb{N} \) we put

\[
\alpha_j^m = a_k^{(1)} \text{ if } k = \mu_{m-1} + j, \text{ where } 1 \leq j \leq 2^{2m} + 1.
\]

According to the construction we have

\[
\sum_{k=\mu_{m-1}+1}^{\mu_m} \sum_{i=1}^{2^{n_k}} a_k^{(i)} \chi_k^{(i)}(x) = \sum_{k=\mu_{m-1}+1}^{\mu_m} a_k^{(1)} \chi_k(x) + \Psi_m(x)
\]

\[
= \sum_{j=1}^{2^{2m}+1} \alpha_j^m \chi_j^m(x) + \Psi_m(x)
\]

\[
= \beta_m \xi_m(x) + \sum_{i=0}^{2^{2m}-1} \gamma_i \gamma_i^j g_m(x) + \Psi_m(x),
\]

where

\[
\Psi_m(x) = \sum_{k=\mu_{m-1}+1}^{\mu_m} \sum_{i=1}^{2^{n_k}} a_k^{(i)} \chi_k^{(i)}(x) = \sum_{k=\mu_{m-1}+1}^{\mu_m} \sum_{i=2}^{2^{n_k}} a_k^{(i)} r_{k+i+j-1}(x)
\]

and

\[
\beta_m = \frac{1}{\sqrt{2^{2m} + 1}} \sum_{j=1}^{2^{2m}+1} \alpha_j^m = \int_0^1 \left[ \sum_{j=1}^{2^{2m}+1} \alpha_j^m \chi_j^m(x) \right] \xi_m(x) dx.
\]

Let

\[
\mathcal{M}_m^2 = \sum_{k=\mu_{m-1}+1}^{\mu_m} |a_k^{(1)}|^2 = (\beta_m)^2 + \sum_{i=0}^{2^{2m}-1} (\gamma_i^j)^2.
\]
The Cauchy inequality yields

$$|\beta_m| \leq M_m$$

for all \( m \in \mathbb{N} \).

For any \( \epsilon \in (0, 1] \) we denote

$$\Omega_\epsilon = \left\{ m \in \mathbb{N} : \frac{\epsilon}{4} M_m \leq |\beta_m| \right\}$$

and

$$\Omega_\epsilon^c = \mathbb{N} \setminus \Omega_\epsilon = \left\{ m \in \mathbb{N} : \frac{\epsilon}{4} M_m > |\beta_m| \right\}.$$  

Evidently \( \Omega_\epsilon \supseteq \Omega_\delta \) for any \( 0 < \epsilon < \delta \leq 1 \). The proof will be divided into two main parts.

4.1.1. The case \( \sum_{s=1}^{\infty} M_m^2(s) = +\infty \), for some \( \{m(s)\}_{s=1}^{\infty} \subset \Omega_\epsilon \). Suppose \( \{m(s)\}_{s=1}^{\infty} \subset \Omega_\epsilon \) is any subsequence of natural numbers in \( \Omega_\epsilon \) such that

$$\sum_{s=1}^{\infty} M_m^2(s) = +\infty.$$  

Without loss in generality we can suppose that

$$2^{-s} \leq M_m(s)$$

because \( \text{(38)} \) remains true after deleting the terms that does not satisfy the condition \( \text{(39)} \). Let

$$S_m^v(x) = \sum_{k=\mu_m+1}^{v} \sum_{i=1}^{2^n k} a_k^{(i)} \chi_k^{(i)}(x) = \sum_{k=\mu_m+1}^{v} a_k^{(1)} \chi_k(x) + \Psi_m^v(x),$$

where \( \mu_m + 1 \leq v \leq \mu_m \) and

$$\Psi_m^v(x) = \sum_{k=\mu_m+1}^{v} \sum_{i=2}^{2^n k} a_k^{(i)} \chi_k^{(i)}(x) = \sum_{k=\mu_m+1}^{v} \sum_{i=2}^{2^n k} a_k^{(i)} r_{k+i\nu-1}(x).$$

We put

$$\sigma_m^v(x) = \sup_{1 \leq l \leq 2^{2m}} \left| \sum_{j=1}^{l} \alpha_j^m \chi_j^m(x) \right|$$

$$= \sup_{1 \leq l \leq 2^{2m}} \left| \sum_{j=1}^{l} \alpha_j^m \left( \frac{1}{\sqrt{2^{2m} + 1}} \xi_m + \sum_{i=2}^{2^{2m} + 1} k_{ij}^{(2^{2m} + 1)} g_{m-2}(x) \right) \right|.$$  

By \( \text{(18)}, \text{(17)} \) we will have that for

$$\rho_{n-1} < m \leq \rho_n \quad \text{and} \quad x \in \Delta_{\nu}^{(k_{\nu}+2)}, 2 \leq \nu \leq 2^{k_{\nu}+2}$$

$$\sigma_m^v(x) = \sup_{1 \leq l \leq 2^{2m}} \left| \sum_{j=1}^{l} \alpha_j^m \left( \frac{1}{\sqrt{2^{2m} + 1}} \omega_m + \sum_{i=2}^{2^{2m} + 1} k_{ij}^{(2^{2m} + 1)} g_{m-2}(x) \right) \right|.$$
If we denote by \( n_s \) the number that corresponds to \( n \) in the condition (32) when \( m \) is replaced by \( m(s) \) it is easy to observe (see (15), (42)) that
\[
m(s) > n_s + 2 \quad \text{for all} \quad s \geq 4.
\]
Thus if \( x \in \Delta^{(k_m-1+2)}_\nu \), where \( \nu \) is such that \( 2^{k_m-1-n_s+1} \leq \nu \leq 2^{k_m-1+2} \) then
\[
\sigma^*_m(x) = \sup_{1 \leq v \leq 2^{2m}} \left| \sum_{j=1}^v \alpha^{m}_{j} \left( \frac{1}{\sqrt{2^{2m} + 1}} \nu^* + \sum_{i=2}^{2^{2m}+1} \kappa_{ij}^{(2^{2m}+1)} g_{m}^{(i-2)}(x) \right) \right|
\]
where \( \omega^*_\nu = \omega^{m}_{\nu} \) for \( \Delta^{(k_m-1+2)}_\nu \subset \Delta^{(k_m+2)}_{\nu_1} \).

For any \( s \in \mathbb{N} \) we define \( \tau_s \in \mathbb{N} \) so that
\[
\sqrt{2^{\tau_s-1} M_{m(s)}} < 1 \leq \sqrt{2^{\tau_s} M_{m(s)}},
\]
otherwise we put \( \tau_s = 2 \).

Hence by (39) we will have that
\[
2 \leq \tau_s \leq 2s + 1 \quad \text{for all} \quad s \in \mathbb{N}.
\]
Define
\[
E_{s,\nu} = \left\{ x \in \Delta^{(k_m(s)-1+2)}_\nu : I_{[2^{-\tau_s}, 2^{-\tau_s}+1]}(2^{k_m(s)-1+2m(s)^2+2}x) = 1 \right\},
\]
where we suppose that the characteristic function \( I_E \) is extended with the period 1 to the whole line. Evidently,
\[
|E_{s,\nu}| = 2^{-\tau_s} |\Delta^{(k_m(s)-1+2)}_\nu|.
\]
Then we write
\[
\sigma^*_m(x) \geq \sup_{1 \leq v \leq 2^{2m}} \left| \sum_{j=1}^v \alpha^{m}_{j} g_{m(s)}^{j-1}(x) \right|
- \delta_{2^{2m(s)}+1} \left| \frac{\delta^{(-1)}_{2^{2m(s)}+1} \nu^*}{\sqrt{2^{2m(s)} + 1}} - \sum_{j=1}^{2^{2m(s)}} \alpha^{m}_{j} g_{m(s)}^{j-1}(x) \right|
= \tilde{\sigma}^*_m(x) - \tilde{R}^*_\nu(x) \quad \text{if} \quad x \in E_{s,\nu}
\]
\[
\left( 2^{k_m(s)-1-k_n+1} \leq \nu \leq 2^{k_m(s)-1+2} \right).
\]
Applying the equality (8) we obtain

\[
\int_{E_{s,\nu}} \tilde{R}_{\nu}^s(x)dx \leq \sqrt{|E_{s,\nu}|} \left( \int_{E_{s,\nu}} (\tilde{R}_{\nu}^s(x))^2 dx \right)^{\frac{1}{2}} \\
\leq 4M_{m(s)}\sqrt{|E_{s,\nu}|} \frac{1}{\sqrt{2m(s) + 1}} \\
\times \left( \int_{E_{s,\nu}} \left( (2^{2m(s)} + 1)(\omega_{\nu}^s)^2 + \sum_{j=1}^{2^{m(s) + 1}} (g_{j,m(s)}(x))^2 \right) dx \right)^{\frac{1}{2}} \\
\leq 4|E_{s,\nu}|M_{m(s)}\sqrt{(\omega_{\nu}^s)^2 + m(s)^{-2}} \\
\leq 4\sqrt{2^{-k_{s,\nu} + m(s)^{-2}}} |E_{s,\nu}| M_{m(s)}
\]

for all \(\nu(2^{m(s) - k_{s,\nu} + 1} \leq \nu \leq 2^{m(s) + 2})\).

The last inequality follows from the conditions (13), (16) and (18).

On the other hand by the definition of the functions \(\hat{g}_n^i(x)\) and (7) we deduce that for all \(\nu(2^{m(s) - k_{s,\nu} + 1} \leq \nu \leq 2^{m(s) + 2})\)

\[
\int_{E_{s,\nu}} \tilde{g}_{m(s)}^s(x)dx \\
= \int_{E_{s,\nu}} \sup_{1 \leq v \leq 2^{m(s)}} \left| \sum_{j=1}^{v} \alpha_{j}^{m(s)} \hat{g}_{m(s)}^{j-1}(2^{m(s) + 1 + 2m(s)^2 + 2} x) \right|dx \\
= m(s)^{-1}2^{-\frac{7}{2}} |\Delta_{\nu}^{(k_{m(s)} - 1)}| \\
\times \int_{2^{-\tau_s} + 1}^{2^{-\tau_s}} \sup_{1 \leq v \leq 2^{m(s)}} \left| \sum_{j=1}^{v} \alpha_{j}^{m(s)} \hat{g}_{m(s)}^{j-1}(2^{m(s)} (2^{\tau_s + 2} x - 4) \right|dx \\
= m(s)^{-1}2^{-\frac{7}{2}} |\Delta_{\nu}^{(k_{m(s)} - 1)}| \int_{0}^{4} \sup_{1 \leq v \leq 2^{m(s)}} \left| \sum_{j=1}^{v} \alpha_{j}^{m(s)} \hat{g}_{2m(s)}^{j-1}(y) \right|dy \\
\geq \sqrt{\frac{|E_{s,\nu}|}{6}} 2^{-m(s)} \left| \sum_{j=1}^{2^{m(s)}} \alpha_{j}^{m(s)} \right|.
\]

In order to obtain the last two inequalities we have applied consecutively Lemma 2 and Lemma 1 of [7]. In the last inequality we applied
also the equality \(14\). We have that
\[
\left| \sum_{j=1}^{2m(s)} \alpha_j^m \right| \geq \left| \sum_{j=1}^{2m(s)+1} \alpha_j^m \right| - \left| \alpha_{2m(s)+1}^m \right|
\geq \left| \sum_{j=1}^{2m(s)+1} \alpha_j^m \right| - \mathcal{M}_{m(s)} \geq \frac{1}{2} \left| \sum_{j=1}^{2m(s)+1} \alpha_j^m \right|
\]
for any \(m(s) \geq N_\epsilon\), where \(N_\epsilon = 2 \log_2 \left( \frac{\epsilon}{2} \right)\). Hence, for any \(m(s) \geq N_\epsilon\), we have
\[
\int_{E_{s,\nu}} \hat{\sigma}_{m(s)}^*(x) dx \geq 2 \frac{\epsilon}{96} \left| E_{s,\nu} - 2m(s) \right| \left| \sum_{j=1}^{2m(s)+1} \alpha_j^m \right|.
\]
Thus if we take \(N_\epsilon \geq N_\epsilon\) so that
\[
4\sqrt{2^{-k_{\nu}} + m(s)^{-2}} < \frac{\epsilon}{96} \quad \text{when} \quad m(s) > N_\epsilon
\]
then we will obtain that for all \(\nu \left( 2^{k_{m(s)}+1+k_{\nu}+1} \leq \nu \leq 2^{k_{m(s)}+2} \right)\)
\[
\int_{E_{s,\nu}} \sigma_{m(s)}^*(x) dx \geq \int_{E_{s,\nu}} \hat{\sigma}_{m(s)}^*(x) dx - \int_{E_{s,\nu}} \hat{R}_{m(s)}^*(x) dx
\geq 2 \frac{\epsilon}{96} \left| E_{s,\nu} - 2m(s) \right| \left| \sum_{j=1}^{2m(s)+1} \alpha_j^m \right|
- 4 \sqrt{2^{-k_{\nu}} + m(s)^{-2}} |E_{s,\nu}| \mathcal{M}_{m(s)}
\geq 2 \frac{\epsilon}{96} |E_{s,\nu}| \mathcal{M}_{m(s)}.
\]
Applying the Menshov-Rademacher theorem by the definition of the functions \(\hat{\sigma}_{n_i}^*(x)\) and \(14\) we have that
\[
\int_{E_{s,\nu}} \left| \hat{\sigma}_{m(s)}^*(x) \right|^2 dx
= \int_{E_{s,\nu}} \sup_{1 \leq v \leq 2^{2m(s)}} \left| \sum_{j=1}^{v} \alpha_j^{m(s)} g_m^{j-1} \left( 2^{k_{m(s)}+1+2m(s)^2+2} x \right) \right|^2 dx
= |\Delta_{m(s)}^{(k_{m(s)}+2)}| \int_{2^{-r_{\tau}}}^{2^{-r_{\tau}+1}} \sup_{1 \leq v \leq 2^{2m(s)+1}} \left| \sum_{j=1}^{v} \alpha_j^{m(s)} g_m^{j-1} \left( x \right) \right|^2 dx
= m(s)^{-2} 2^{-r_{\tau}} |\Delta_{m(s)}^{(k_{m(s)}+1+2)}|
\times \int_{2^{-r_{\tau}}}^{2^{-r_{\tau}+1}} \sup_{1 \leq v \leq 2^{2m(s)}} \left| \sum_{j=1}^{v} \alpha_j^{m(s)} f_{2m(s)}^{j-1} \left( 2^{-r_{\tau}+2} x - 4 \right) \right|^2 dx
\leq C^2 |\Delta_{m(s)}^{(k_{m(s)}+1+2)}| \left( \frac{m(s) + 2}{m(s)} \right)^2 \mathcal{M}_{m(s)}^2 \leq 4C^2 |\Delta_{m(s)}^{(k_{m(s)}+1+2)}| \mathcal{M}_{m(s)}^2,
\]
where $C > 0$ is an absolute constant. Hence, if $m(s) > \hat{N}_\epsilon$

\[
\left( \int_{E_{s,\nu}} (\sigma^*_{m(s)}(x))^2 dx \right)^{\frac{1}{2}} \leq \left( \int_{E_{s,\nu}} (\hat{\sigma}^*_{m(s)}(x))^2 dx \right)^{\frac{1}{2}} + \left( \int_{E_{s,\nu}} (\hat{R}^*_\nu(x))^2 dx \right)^{\frac{1}{2}} \leq 2C|\Delta_{\nu}^{(k_m(s)-1+2)}|^\frac{1}{2}M_m(s)
\]

\[+|\Delta_{\nu}^{(k_m(s)-1+2)}|^\frac{1}{2}M_m(s) \leq (2C + 1)|\Delta_{\nu}^{(k_m(s)-1+2)}|^\frac{1}{2}M_m(s)\]

for any $\nu(2^{k_m(s)-1-k_{ns+1}} \leq \nu \leq 2^{k_m(s)-1+2})$.

For the function $\sigma^*_{m(s)}$ we apply Lemma C of [7] (see also [5], p.8) to estimate the Lebesgue measure of the set

\[E^*_s = \{ x \in E_{s,\nu} : \sigma^*_{m(s)}(x) \geq \frac{2\xi}{200}M_m(s) \} .\]

After normalizing the Lebesgue measure on the set $E_{s,\nu}$ and easily computing the increase of the norms $L^1$ and $L^2$ we will obtain that for any $\nu(2^{k_m(s)-1-k_{ns+1}} \leq \nu \leq 2^{k_m(s)-1+2})$, $m(s) \in \Omega_\epsilon$, $m_s \geq \hat{N}_\epsilon$

\[|E_{s,\nu}|^{-1}|E^*_s| \geq \left( \frac{2\xi |E_{s,\nu}|^{\frac{1}{2}} \epsilon}{400(2C + 1)|\Delta_{\nu}^{(k_m(s)-1+2)}|^\frac{1}{2}} \right)^2
\]

\[= \left( \frac{\epsilon}{400(2C + 1)} \right)^2 = C_\epsilon > 0.\]

By (26) and (27) we observe (see (40), (41)) that for any $t \in \mathbb{R}$

\[\left\{ x \in E_{s,\nu} : S^\nu_m(x) > t \right\} = \left\{ x \in E_{s,\nu} : S^\nu_m(x) - 2\Psi^\nu_m(x) > t \right\} .\]

Hence if we define

\[S^*_m(x) = \sup_{\mu_{m-1+1} \leq \nu \leq \mu_m} |S^\nu_m(x)|\]

then we obtain that the measure of the set

\[\hat{E}_{s,\nu} = \left\{ x \in E_{s,\nu} : S^*_m(x) \geq \frac{2\xi}{200}M_m(s) \right\}
\]

is greater than or equal $\frac{1}{2}|E^*_s|$.

The sequence of partial sums (30) diverges a.e. on the set

\[E = \limsup_{s \to \infty} \bigcup_{s \nu=2^{k_m(s)-1-k_{ns+1}}} \hat{E}_{s,\nu}
\]

when $j \to +\infty$. By (44) and (45) it follows that for any dyadic interval

\[\Delta_{\nu}^{(k_m(s)-1+2)}(2^{k_m(s)-1-k_{ns+1}} \leq \nu \leq 2^{k_m(s)-1+2})\]
\[
\left\{ x \in \Delta_{\nu}^{(k_m(s) - 1 + 2)} : S_m^s(x) \geq \frac{\epsilon}{200} \right\} \geq C_\epsilon 2^{-\tau_s - 1}|\Delta_{\nu}^{(k_m(s) - 1 + 2)}| \\
\geq \frac{1}{4} C_\epsilon M_m^s|\Delta_{\nu}^{(k_m(s) - 1 + 2)}|. 
\]

Hence one easily derives that \(|E| = 1\).

4.1.2. The case \(\sum_{m \in \Omega} M_m^2 < +\infty\) for all \(\epsilon \in (0, 1]\). The proof of this part is similar to the proof given in \([7]\). In this case we have that
\[
(45) \quad \sum_{m \in \Omega^c} M_m^2 = +\infty \quad \forall \epsilon \in (0, 1].
\]

Let \(\{m_j(\epsilon)\}_{j=1}^{\infty} = \Omega^c\), where
\[
m_1(\epsilon) < m_2(\epsilon) < \cdots < m_j(\epsilon) < m_{j+1}(\epsilon) < \cdots
\]
and study two subcases.

a) When \(\limsup_j M_{m_j(\epsilon)} > 0\) for some \(\epsilon \in (0, 1]\);

b) When \(\lim_{j \to \infty} M_{m_j(\epsilon)} = 0\) for any \(\epsilon \in (0, 1]\).

In the case a) the reader must take into account the conditions \((24)\) and \((27)\) to assure that the Rademacher functions that appear between the functions \(\Psi_j(\mu_m - 1 \leq j \leq \mu_m)\) do not affect on the proof.

The proof of Theorem 13 in the case b) is also similar to the proof given in \([7]\) for the corresponding case. Here one should use the conditions \((24)\), \((27)\) and Proposition 4 to guarantee that the same arguments work. Thus the proof of Theorem 13 is finished.

5. Construction of the system \(\Theta\)

As we have explained in Section 4 we will obtain the system \(\Theta\) with the help of corresponding orthogonal transformations. It is easy to check that dissolution process will work if instead of the matrix \((2)\) one takes any \(2 \times 2\) orthogonal matrix with elements by modulus strictly less than one. The advantage of the matrix \((2)\) resides, particularly, on the fact that the elements of the first row of the resulting matrix are equal. In fact if we apply the process \(2^n - 1\) times then we will obtain the Haar matrix. But for our purposes we do not need such details.

We define
\[
(46) \quad \theta_i(x) = \chi^{(i)}_0(x), \quad 1 \leq i \leq \nu_0.
\]

Afterwards any block of functions \(\chi^{(i)}_j(x)(1 < i \leq 2^{n_j})\) will be transformed by the corresponding orthogonal matrix \(H_{n_j}\).

We define
\[
(47) \quad \theta^{(i)}_j(x) = \sum_{k=1}^{2^{n_j}} a^{(j)}_{ki} \chi^{(k)}_j(x) \quad 1 \leq i \leq 2^{n_j}; \quad j \in \mathbb{N}
\]
(see (11)) and denote

\[ \theta_k(x) = \theta_j^{(i)}(x), \quad \text{where} \quad k = \varpi_{j-1} + i; \]

\[ 1 \leq i \leq 2^{n_j}, \quad j \in \mathbb{N} \quad \text{and} \quad \varpi_{j} = \nu_0 + \sum_{i=1}^{j} 2^{n_i}. \]

By (46)-(48) and the definition of the orthogonal matrices \( H_j \) we obtain that \( \Theta = \{\theta_k(x)\}_{k=1}^{\infty} \) is a complete ONS and (1) holds. Moreover, we also have that for any collection of coefficients \( \{c_i\}_{i=1}^{2^{n_j}} \) and any \( j \in \mathbb{N} \)

\[ \sum_{i=1}^{2^{n_j}} c_i \theta_j^{(i)}(x) = \sum_{i=1}^{2^{n_j}} b_i \chi_j^{(i)}(x), \]

where

\[ \sum_{i=1}^{2^{n_j}} |c_i|^2 = \sum_{i=1}^{2^{n_j}} |b_i|^2. \]

Hence from Theorem 13 follows that \( \Theta \) is a divergence system. To show that the system \( \Theta \) is a convergence system we observe that the system

\[ \theta_j^{(i)}(x) - 2^{-\varpi_j} \chi_j^{(1)}(x) \quad 1 \leq i \leq 2^{n_j}; \quad j \in \mathbb{N} \]

is an \( S_p \) system for any \( p > 2 \). From (24) we decompose any function \( \theta_k(x), k > \nu_0 \) in the following form:

\[ \theta_j^{(i)}(x) = 2^{-\varpi_j} \Upsilon_j(x) + \left[ \theta_j^{(i)}(x) - 2^{-\varpi_j} \chi_j^{(1)}(x) \right] \quad 1 \leq i \leq 2^{n_j}; \quad j \in \mathbb{N} \]

Hence by Theorem 14 and Proposition 6 we easily obtain that \( \Theta \) is a convergence system. Proof of Theorem 1 is finished.
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