Computing multiple zeros by using a parameter in Newton–Secant method

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Abstract In this paper, we modify the Newton–Secant method with third order of convergence for finding multiple roots of nonlinear equations. This method requires two evaluations of the function and one evaluation of its first derivative per iteration. This method has the efficiency index equal to $3^{1/3} \approx 1.44225$. We describe the analysis of the proposed method along with numerical experiments including comparison with existing methods. Moreover, the attraction basins of the proposed method are shown and compared with other existing methods.

Keywords Multi-point iterative methods · Newton–Secant method · Multiple roots · Basin of attraction

Mathematics Subject Classification 65H05

1 Introduction

Solving nonlinear equations based on iterative methods is a basic and extremely valuable tool in all fields of science as well as in economics and engineering. The important aspects related...
to these methods are order of convergence and number of function evaluations. Therefore, it is favorable to attain the highest possible convergence order with a fixed number of function evaluations per iteration. The aim of this paper is to modify the third order Newton–Secant method to solve nonlinear equations for multiple zeros with the same order of convergence and efficiency index. The efficiency index of an iterative method of order \( p \) requiring \( k \) function evaluations per iteration is defined by \( E(k, p) = \sqrt[\frac{k}{p}]{p} \), see [19].

Let \( \alpha \) be multi roots of \( f(x) = 0 \) with multiplicity \( m \) i.e., \( f^{(i)}(\alpha) = 0, i = 0, 1, \ldots, m - 1 \) and \( f^{(m)}(\alpha) \neq 0 \). If functions \( f^{(m-1)} \) and \( f^{1/m} \) have only a simple zero at \( \alpha \), any of the iterative methods for a simple zero may be used [10,25]. The Newton method for finding a simple zero \( \alpha \) has been modified by Scheroder to find multiple zeros of a non-linear equation which is of the form \( x_{n+1} = x_n - m \frac{f(x_n)}{f'(x_n)} \) with convergence of two [21].

In recent years, there have been several attempts to construct iterative methods for finding simple roots as well as comparison of various iterative methods, see [1,4–8,13,17,20,23,24] and for finding multiple roots such as the work of Chun et al. [10], Dong [12], Hansen and Patrick [15], Osada [18], Victory and Neta [27] which propose various iterative methods for finding multiple zeros \( \alpha \) of a nonlinear equation \( f(x) = 0 \) where multiplicity \( m \) is known.

This paper is organized as follow: Sect. 2 is devoted to the construction and convergence analysis of a new method with convergence order three. Computational aspects, graphical comparison with other methods are illustrated in Sect. 3. Finally, a conclusion is provided in Sect. 4.

2 Description of the method and convergence analysis

In this section, we propose a new modification of Newton–Secant method to find multiple zeros of nonlinear equations. Newton–Secant’s method is

\[
\begin{align*}
y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\
x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \frac{f(x_n)}{f'(y_n)}, \quad (n = 0, 1, \ldots),
\end{align*}
\]

which the order of convergence is three for simple roots [25]. We aim to extend the method (1) for multiple roots and build a method according to (1) without any additional evaluations of the function or its derivatives using an additional parameter. In other words, the convergence order of Newton–Secant method for approximating simple zeros of nonlinear equations is three, whereas the convergence order of this method is linear for finding multiple zeros. In order to do so, we add a parameter \( \theta \) in the second term of (1).

We have

\[
\begin{align*}
y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\
x_{n+1} &= x_n - \frac{\theta f(x_n)}{\theta f(x_n) - f(y_n)} \frac{f(x_n)}{f'(x_n)}, \quad (n = 0, 1, \ldots),
\end{align*}
\]

The order of convergence of the preceding method will be analyzed and the method will be adjusted accordingly to prove the following expected theorem.

**Theorem 1** Let \( \alpha \in D \) be a multiple zero of a sufficiently differentiable function \( f : D \subset \mathbb{R} \to \mathbb{R} \) for an open interval \( D \) with the multiplicity \( m > 1 \), which includes \( x_0 \) as an initial approximation of \( \alpha \). Then, the method (2) has order three and \( \theta = \left( \frac{-1+m}{m} \right)^{-1+m} \).
Proof Let $e_n := x_n - \alpha$, $e_{n,y} := y_n - \alpha$, $c_i := m! c^{(m+i)}(\alpha)/f^{(m)}(\alpha)$. Using the fact that $f(\alpha) = 0$, Taylor expansion of $f$ at $\alpha$ yields

$$f(x_n) = e_n^m (c_0 + c_1 e_n + c_2 e_n^2 + c_3 e_n^3) + O(e_n^4),$$

and

$$f'(x_n) = e_n^{m-1} (m + (m + 1) c_1 e_n + (m + 2) c_2 e_n^2 + (m + 3) c_3 e_n^3 + O(e_n^4)).$$

Therefore

$$\frac{f(x_n)}{f'(x_n)} = \frac{1}{m} e_n - \frac{c_1}{m^2 c_0} e_n^2 + \frac{-(1 + m) c_1^2 + 2 mc_0 c_2}{m^3 c_0^2} e_n^3 + O(e_n^4),$$

and hence

$$e_{n,y} = y_n - \alpha = \frac{-1 + m}{m} e_n - \frac{c_1}{m^2 c_0} e_n^2 + \frac{-(1 + m) c_1^2 + 2 mc_0 c_2}{m^3 c_0^2} e_n^3 + O(e_n^4).$$

For $f(y_n)$ we also have

$$f(y_n) = e_{n,y}^m (c_0 + c_1 e_{n,y} + c_2 e_{n,y}^2 + c_3 e_{n,y}^3) + O(e_{n,y}^4).$$

Substituting (3)–(7) in (2), we obtain

$$e_{n+1} = D_1 e_n + D_2 e_n^2 + D_3 e_n^3 + O(e_n^4),$$

where

$$D_1 = 1 + \frac{\theta}{m \left(-\theta + \left(\frac{-1 + m}{m}\right)^m\right)},$$

and

$$D_2 = \frac{\theta m^{-2+m} (-m(-1 + m)^m + \theta m^m (-1 + m)) c_1}{(-1 + m)((-1 + m)^m - \theta m^m)^2 c_0},$$

and

$$D_3 = \frac{\theta m^{-3+m} A}{2((-1 + m)^2((-1 + m)^m - \theta m^m)^3 c_0^2},$$

where

$$A = (-1 + m)^2 (-1 + m + 2 m^2)(c_1^2 - 2(-1 + m)c_0 c_2) + 2 \theta^2((-1 + m)^2 m^2 ((1 + m)c_1^2 - 2 mc_0 c_2) - \theta(-1 + m)^{1+m}) (m(3 + 4m) c_1^2 + 2(1 + m - 4m^2)c_0 c_2).$$

Therefore, to provide the three order of convergence, it is necessary to choose $D_i = 0$ ($i = 1, 2$), thus we have

$$\theta = \left(\frac{-1 + m}{m}\right)^{-1+m},$$

and the error equation becomes

$$e_{n+1} = \left(\frac{mc_1^2 - 2(-1 + m)c_0 c_2}{2m^2 c_0^2}\right) e_n^3 + O(e_n^4),$$

and method (2) has convergence order of three, which proves the theorem. \qed
3 Numerical performance and graphical comparison

3.1 Numerical results

In this section we apply the new method (2) to several benchmark examples and compare the results with existing methods that have the same order of convergence.

The new method is given by

\[ y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \]

\[ x_{n+1} = x_n - \frac{(\frac{1}{1+m}) - \frac{m f(x_n)}{f'(x_n)}}{1 + \frac{m f(x_n)}{f'(x_n)}}, \] (8)

The Osada’s method [18], is given by

\[ x_{n+1} = x_n - \frac{1}{2} \frac{m (m+1) f'(x_n)}{f'(x_n)} + \frac{1}{2} (m-1)^2 \frac{f''(x_n)}{f''(x_n)}. \] (9)

The Dong’s method [12], is given by

\[ y_n = x_n + \sqrt{m} \frac{f(x_n)}{f'(x_n)}, \]

\[ x_{n+1} = y_n - m \left( 1 - \frac{1}{\sqrt{m}} \right)^{1-m} \frac{f(y_n)}{f'(x_n)}. \] (10)

The Chun’s method [10], is given by

\[ x_{n+1} = x_n - \frac{m ((2 \gamma - 1)m + 3 - 2 \gamma)}{2} \frac{f(x_n)}{f'(x_n)} + \frac{\gamma (m - 1)^2}{2} \frac{f'(x_n)}{f''(x_n)} - \frac{(1 - \gamma) m^2 f(x_n)^2 f''(x_n)}{2 f'(x_n)^3}. \] (11)

In the numerical experiments of this paper we use \( \gamma = -1 \).

We have tested the method (8) on a number of nonlinear equations. To obtain a high accuracy and avoid the loss of significant digits, we employ multi-precision arithmetic with 100 significant decimal digits in the programming package of Mathematica 8 [16].

In order to test our proposed method (8) and compare it with the methods (9), (10) and (11), we compute the error and the computational order of convergence (COC) by the approximate formula [28]

\[ \text{COC} \approx \frac{\ln \left| \frac{x_{n+1} - x_n}{x_n - x_0} \right|}{\ln \left| \frac{x_n - x_{n-1}}{x_{n-1} - x_{n-2}} \right|}. \] (12)

The approximated computational order of convergence, (ACOC) is calculated by using the formula [11]

\[ \text{ACOC} \approx \frac{\ln \left| \frac{x_{n+1} - x_n}{x_n - x_{n-1}} \right|}{\ln \left| \frac{x_n - x_{n-1}}{x_{n-1} - x_{n-2}} \right|}. \]

For four nonlinear equations presented in Table 1, our new method (8) is compared with the methods (9), (10) and (11) in Table 2.
Table 1 Test functions \( f_1, \ldots, f_4 \) and root \( \alpha \)

| \( f_1, x_0 = 0.1 \) | \( f_2, x_0 = 8 \) | \( f_3, x_0 = 3.1 \) | \( f_4, x_0 = 9 \) |
|-----------------|-----------------|-----------------|-----------------|
| \( |x_1 - x^*| \) | \( |x_2 - x^*| \) | \( |x_3 - x^*| \) | \( |x_4 - x^*| \) |
| 0.739e–3 | 0.162e–2 | 0.955e–3 | 0.112e–2 |
| 0.364e–9 | 0.106e–7 | 0.111e–8 | 0.215e–8 |
| 0.434e–28 | 0.304e–23 | 0.175e–26 | 0.149e–25 |
| COC | COC | COC | COC |
| 3.0000 | 3.0000 | 3.0000 | 3.0000 |
| ACOC | ACOC | ACOC | ACOC |
| 2.9999 | 2.9994 | 2.9998 | 2.9998 |

Table 2 Errors, COC and ACOC for methods (8), (9), (10) and (11)

| Test function \( f_N \) | Root \( \alpha \) |
|-----------------|-----------------|
| \( f_1(x) = (\sin^2 x + x)^5 \) | 0 |
| \( f_2(x) = (\ln x + \sqrt{x - 5})^3 \) | 8.3094326942315717953469556827 |
| \( f_3(x) = (e^{x^2+7x-30} - 1)^6 \) | 3 |
| \( f_4(x) = (\sqrt{x - \frac{1}{5} - 3})^7 \) | 9.6335955628326951924063127092 |

3.2 Graphical comparison by means of attraction basins

We observe that all methods converge if the initial guess is chosen properly. We now investigate the stability region. In other words, we numerically approximate the domain of attraction of the zeros as a qualitative measure of stability. To answer the important question on the dynamical behavior of the algorithms, we investigate the attraction basins of the new method and compare them with common and well-performing methods from the literature. For more details one can consult [2, 3, 14, 17, 22, 24, 26].

Let \( G : \mathbb{C} \to \mathbb{C} \) be a rational map on the complex plane. For \( z \in \mathbb{C} \), we define its orbit as the set \( \text{orb}(z) = \{ z, G(z), G^2(z), \ldots \} \). A point \( z_0 \in \mathbb{C} \) is called periodic point with minimal
period $m$ if $G^m(z_0) = z_0$, where $m$ is the smallest integer with this property. A periodic point with minimal period 1 is called fixed point. Moreover, a point $z_0$ is called attracting if $|G'(z_0)| < 1$, repelling if $|G'(z_0)| > 1$, and neutral otherwise. The Julia set of a nonlinear map $G(z)$, denoted by $J(G)$, is the closure of the set of its repelling periodic points. The complement of $J(G)$ is the Fatou set $F(G)$, where the basin of attraction of the different roots lie [9].

For the dynamical point of view, in fact, we take a $512 \times 512$ grid of the square $[-3, 3] \times [-3, 3] \in \mathbb{C}$ and assign a color to each point $z_0 \in D$ according to the root to which the corresponding orbit of the iterative method starting from $z_0$ converges. We mark the point as black if the orbit does not converge to a root, in the sense that after at most 100 iterations it has a distance to any of the roots, which is larger than $10^{-3}$. In this way, we distinguish the attraction basins by their color for different methods.

We have tested several different examples, and the results on the performance of the tested methods were similar. Therefore, we report the general observation here for following test problems which are presented in Table 3.

| Test problem | Roots |
|--------------|-------|
| $p_1(z) = (z^3 - 1)^{10}$ | $1, -0.5 \pm 0.866025i$ |
| $p_2(z) = (z^5 - z^2 + 1)^{15}$ | $-0.808731, -0.464912 \pm 1.07147i, 0.869278 \pm 0.388269i$ |
| $p_3(z) = (2z^4 - z)^8$ | $0, -0.39685 \pm 0.687365i, 0.793701$ |

Fig. 1 Comparison of basin of attraction of method (1), for test problems $p_1(z) = z^3 - 1$, $p_2(z) = z^5 - z^2 + 1$ and $p_3(z) = 2z^4 - z$ respectively, without any multiplicity

Fig. 2 Comparison of basin of attraction of method (8), for test problems $p_1(z) = (z^3 - 1)^{10}$, $p_2(z) = (z^5 - z^2 + 1)^{15}$ and $p_3(z) = (2z^4 - z)^8$ respectively
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In the following figures different colors are used for different roots. In the basins of attraction, the number of iterations needed to achieve the root is shown by the brightness. A brighter color means less iteration steps. Note that black color denotes lack of convergence to any of the roots or convergence to infinity.

In each of Figs. 1, 2, 3, 4 and 5, basins of attractions of methods (1), (8), (9), (10) and (11) are illustrated for three test problems $p_1(z) = (z^3 - 1)^{10}$, $p_2(z) = (z^5 - z^2 + 1)^{15}$ and $p_3(z) = (2z^4 - z)^8$ respectively.

Fig. 3 Comparison of basin of attraction of method (9), for test problems $p_1(z) = (z^3 - 1)^{10}$, $p_2(z) = (z^5 - z^2 + 1)^{15}$ and $p_3(z) = (2z^4 - z)^8$ respectively.

Fig. 4 Comparison of basin of attraction of method (10), for test problems $p_1(z) = (z^3 - 1)^{10}$, $p_2(z) = (z^5 - z^2 + 1)^{15}$ and $p_3(z) = (2z^4 - z)^8$ respectively.

Fig. 5 Comparison of basin of attraction of method (11), for test problems $p_1(z) = (z^3 - 1)^{10}$, $p_2(z) = (z^5 - z^2 + 1)^{15}$ and $p_3(z) = (2z^4 - z)^8$ respectively.

In the following figures different colors are used for different roots. In the basins of attraction, the number of iterations needed to achieve the root is shown by the brightness. A brighter color means less iteration steps. Note that black color denotes lack of convergence to any of the roots or convergence to infinity.

In each of Figs. 1, 2, 3, 4 and 5, basins of attractions of methods (1), (8), (9), (10) and (11) are illustrated for three test problems $p_1(z) = (z^3 - 1)^{10}$, $p_2(z) = (z^5 - z^2 + 1)^{15}$ and $p_3(z) = (2z^4 - z)^8$ respectively. From the pictures, we can easily judge the behavior and suitability of any method depending on the circumstances. As a result, in Fig. 2 method (8) seems to produce larger basins of attraction than other methods. In addition, modified Newton–Secant method (8) produces larger basins of attraction than original Newton–Secant method (1) as well.
4 Conclusion

In this paper, Newton–Secant’s method for simple zeros was modified for finding multiple zeros of non-linear equations with the same order of convergence and without any additional evaluations of the function or its derivatives. A numerical comparison with other methods shows that our new method is a valuable alternative to the existing methods. In addition, a numerical investigation of the basins of attraction of the solutions illustrated that the stability region of our method it typically larger than that of other methods.

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