ERGODICITY AND TOPOLOGICAL ENTROPY OF GEODESIC FLOWS ON SURFACES

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Abstract. We consider reversible Finsler metrics on the 2-sphere and the 2-torus, whose geodesic flow has vanishing topological entropy. Following a construction of A. Katok, we discuss examples of Finsler metrics on both surfaces with large ergodic components for the geodesic flow in the unit tangent bundle. On the other hand, using results of J. Franks and M. Handel, we prove that ergodicity and dense orbits cannot occur in the full unit tangent bundle of the 2-sphere, if the Finsler metric has conjugate points along every closed geodesic. In the case of the 2-torus, we show that ergodicity is restricted to strict subsets of tubes between flow-invariant tori in the unit tangent bundle. The analogous result applies to monotone twist maps.

1. Introduction and Main Results

Let $M$ be a closed, orientable surface, endowed with a reversible Finsler metric $F: TM \to \mathbb{R}_+$, that is a strongly convex norm in each tangent space $T_x M$. We refer to [6] for more information on Finsler metrics. In this paper we study how the vanishing of the topological entropy of the geodesic flow of $F$ relates to questions of integrability on the one hand and dense orbits and ergodicity on the other.

Let us refer to [25] for the definition and more information on the topological entropy $h_{\text{top}}(\phi^t) = h_{\text{top}}(\phi^1) \in [0, +\infty]$ of a dynamical system $\phi^t: X \to X$ ($t \in \mathbb{Z}$ or $\mathbb{R}$) in a compact metrizable space $X$. It measures the growth of distinguishable orbits on an exponential scale. By a result of A. Katok (Corollary 4.3 in [24]), $h_{\text{top}}(\phi^t) > 0$ is in low dimensional systems equivalent to the existence of non-trivial, hyperbolic invariant sets (horseshoes).

We write $SM = \{F = 1\} \subset TM$ for the $F$-unit tangent bundle of $M$ and $\phi^t_F : SM \to SM$ for the geodesic flow of $F$. By a result of E. I. Dinaburg (Corollary 4.2 in [12]) we find

$$h_{\text{top}}(\phi^t_F) > 0,$$

if the genus of $M$ is at least two. Hence, in this paper we shall assume that $M$ is either the 2-sphere $M = \mathbb{S}^2$ or the 2-torus $M = \mathbb{T}^2$. Conversely, both surfaces admit whole families of completely integrable geodesic flows [9]. Due to a result...
of G. P. Paternain [26], these completely integrable Hamiltonian systems have vanishing topological entropy. Hence, on \( M = \mathbb{S}^2, \mathbb{T}^2 \) there exists a variety of examples of Finsler metrics \( F \) with
\[
h_{\text{top}}(\phi_F^t) = 0.
\]

The flow \( \phi_F^t \) leaves the Liouville measure \( \mu_L \) in \( SM \) invariant, which is the smooth measure induced by the pullback of the canonical contact form on the unit cotangent bundle of \( M \) under the Legendre transform associated to \( \frac{1}{2} F^2 \). If a closed set \( A \subseteq SM \) with \( \mu_L(A) > 0 \) is \( \phi_F^t \)-invariant, we call \( \phi_F^t|_A \) ergodic if this is the case with respect to the Liouville measure \( \mu_L \). It is well-known that in this case \( \phi_F^t|_A \) is transitive, i.e. there exists an orbit of \( \phi_F^t \), which is dense in \( A \) (cf. Proposition 4.1.18 in [25]). For integrable geodesic flows we find
\[
\phi_F^t|_A \text{ transitive} \implies \mu_L(A) = 0.
\]

By the result of G. P. Paternain [26], integrable geodesic flows on surfaces are a subclass of geodesic flows with vanishing entropy (provided that the critical points of the additional integral form submanifolds of \( SM \)). Hence, it is a natural question, whether the implication (1) is true more generally under the assumption of \( h_{\text{top}}(\phi_F^t) = 0 \). As we will see, the answer is negative. In particular, vanishing entropy does not imply integrability. On the other hand, we will study how large closed invariant sets \( A \) with transitive \( \phi_F^t|_A \) can be, assuming \( h_{\text{top}}(\phi_F^t) = 0 \). We will have to distinguish between the case \( M = \mathbb{S}^2 \) and \( M = \mathbb{T}^2 \).

1.1. The 2-sphere. In [23], A. Katok gave the following examples of Finsler metrics on the 2-sphere \( \mathbb{S}^2 \), showing that \( h_{\text{top}}(\phi_F^t) = 0 \) does not force ergodic components to be small.

**Theorem 1.1 (Katok).** There exist reversible Finsler metrics \( F \) on \( \mathbb{S}^2 \), arbitrarily \( C^k \)-close to the standard round metric for fixed \( 2 \leq k < \infty \), whose geodesic flow has vanishing topological entropy \( h_{\text{top}}(\phi_F^t) = 0 \) and which admits two closed, \( \phi_F^t \)-invariant sets \( A_0, A_1 \subseteq \mathbb{S}^2 \), such that \( A_1 = \{ v : v \in A_0 \} \), \( \phi_F^t|_{A_1} \) is ergodic and such that the measure \( \mu_L(\mathbb{S}^2 - (A_0 \cup A_1)) \) is positive but arbitrarily small. In particular, \( \phi_F^t|_{A_1} \) is transitive.

Dropping the reversibility assumption \( F(v) = F(-v) \) on \( F \), Katok [23] was able to construct similar examples, where \( \phi_F^t \) is ergodic in the whole unit tangent bundle \( \mathbb{S} \mathbb{S}^2 \). The question is, whether Theorem 1.1 is optimal in the sense that transitivity or ergodicity can be obtained in the whole unit tangent bundle in the reversible case.

To answer such a question we recall the notion of conjugate points along an arc-length \( F \)-geodesic \( c : \mathbb{R} \rightarrow \mathbb{S}^2 \). We fix preliminarily the notation \( \pi : \mathbb{S} \mathbb{S}^2 \rightarrow \mathbb{S}^2 \) for the bundle projection and \( V_v = \ker d\pi(v) \subset T_v \mathbb{S} \mathbb{S}^2 \) for the vertical line bundle. Then two points \( x_0, x_1 \in c(\mathbb{R}) \) are conjugate along \( c \), if there exist times \( t_0 < t_1 \) with \( x_i = c(t_i) \), such that for the geodesic flow
\[
d\phi_F^{t_1-t_0}(c(t_0)) V_{c(t_0)} = V_{c(t_1)},
\]
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Figure 1. An orbit \( \dot{c} \) of the geodesic flow \( \phi^t_F \) in the unit tangent bundle, together with the vertical line bundles \( V_\dot{c} \) under the action of \( d\phi^t_F \). The planes are the transverse sections of \( \phi^t_F \) given by the kernel of the Liouville 1-form (i.e. the contact structure). Here we see the situation \( d\phi^{t_1-t_0}_F(\dot{c}(t_0))V_{\dot{c}(t_0)} = V_{\dot{c}(t_1)} \), i.e. two conjugate points along \( c \). Projecting to \( M = S^2 \) via \( \pi \) yields a geodesic variation of \( c \), which intuitively focuses at the conjugate endpoints \( c(t_0) \) and \( c(t_1) \). The arrows at \( V_{\dot{c}(t_i)} \) indicate how the geodesic flow twists \( V_{\dot{c}(t_i)} \).

cf. Figure 1. The closeness of the Finsler metrics in Theorem 1.1 to the round metric implies that any geodesic in the examples in Theorem 1.1 possesses conjugate points. Hence, our following result shows that Katok’s example in Theorem 1.1 is indeed optimal.

Theorem 1.2. Let \( F \) be a reversible Finsler metric on \( S^2 \), such that every closed \( F \)-geodesic has a pair of conjugate points along itself. If the geodesic flow \( \phi^t_F : S^2 \to S^2 \) is transitive, then

\[ h_{\text{top}}(\phi^t_F) > 0. \]

Hence, under the condition on conjugate points (which is fulfilled, e.g., if \( F \) has positive flag curvatures), dense geodesics and in particular ergodicity of the geodesic flow in \( S\mathbb{S}^2 \) imply the existence of a non-trivial, hyperbolic invariant set for \( \phi^t_F \), cf. Corollary 4.3 in [24]. Assuming conversely, that the topological entropy vanishes there will be a certain amount of structure in the dynamics of \( \phi^t_F \) due to the results of J. Franks and M. Handel [16], which we apply in order to prove Theorem 1.2, cf. Sections 3 and 4.

We can ask the following questions.

(i) Does transitivity of the geodesic flow \( \phi^t_F \) in \( S\mathbb{S}^2 \) of a general reversible Finsler (or Riemannian) metric \( F \) on \( S^2 \) imply \( h_{\text{top}}(\phi^t_F) > 0 \)?

(ii) Do there exist Riemannian metrics \( g \) on \( S^2 \) with everywhere strictly positive curvature, whose geodesic flow \( \phi^t_g \) is ergodic in \( S\mathbb{S}^2 \)?
Question (i) was posed by G. Knieper. Our result answers the question affirmatively in the presence of conjugate points, in particular in the case of positive curvature. The general case is a topic for future research. Question (ii) is a long-standing open problem. For instance, by the results of V. J. Donnay [13] there exist Riemannian metrics on $S^2$, whose geodesic flow is ergodic in $S\mathbb{S}^2$; however, for these metrics there exist large regions of negative curvature and it is the negative curvature that creates ergodicity by means of hyperbolicity. Our result shows that ergodicity and positive curvature would necessarily entail hyperbolicity in the geodesic flow by means of topological entropy.

1.2. The 2-torus. In this subsection we discuss the 2-torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. Here, the assumption $h_{\text{top}}(\phi^t_F) = 0$ leads to strong integrable behavior on a large scale. The following theorem on geodesic flows on $\mathbb{T}^2$ with vanishing topological entropy is due to E. Glasmachers and G. Knieper [17], [18] with an earlier version for monotone twist maps given by S. Angenent in [3]. As before, $\pi : ST^2 \to \mathbb{T}^2$ stands for the canonical projection.

**Theorem 1.3** (Glasmachers, Knieper). Let $F$ be a reversible Finsler metric on $\mathbb{T}^2$ with $h_{\text{top}}(\phi^t_F) = 0$. Then there exists a function $\rho : ST^2 \to S^1$ with the following properties.

(i) $\rho$ is continuous, surjective, fiberwise monotone of degree one and invariant under the geodesic flow $\phi^t_F$.

(ii) If $\tilde{c} : \mathbb{R} \to \mathbb{R}^2$ is a lift of an arc-length $F$-geodesic $c : \mathbb{R} \to \mathbb{T}^2$, then

$$\lim_{t \to \pm\infty} |\tilde{c}(t)| = \infty, \quad \lim_{t \to \pm\infty} \frac{\tilde{c}(t)}{|\tilde{c}(t)|} = \pm \rho(\tilde{c}(0)),$$

where $|.|$ denotes the euclidean norm in $\mathbb{R}^2$.

(iii) If $\zeta \in S^1$ has irrational slope, then $\rho^{-1}(\zeta) \subset ST^2$ is a Lipschitz graph over $\mathbb{T}^2$ (meaning that $\pi|_{\rho^{-1}(\zeta)} : \rho^{-1}(\zeta) \to \mathbb{T}^2$ is a bi-Lipschitz homeomorphism), whose Lipschitz constant depends only on $F$.

(iv) If $\zeta \in S^1$ has rational or infinite slope, then the boundary of $\rho^{-1}(\zeta) \subset ST^2$ consists of two Lipschitz graphs $\Gamma_{\pm}$ over $\mathbb{T}^2$, such that the non-empty intersection $\cap \Gamma_{\pm}$ consists of closed geodesics in the prime homotopy class in $\mathbb{R}_{>0}\zeta \cap \mathbb{Z}$. The set $\cup \Gamma_{\pm} \cap \cap \Gamma_{\pm}$ consists of heteroclinic connections between the closed geodesics in $\cap \Gamma_{\pm}$.

For intuition about the invariant sets in Theorem 1.3, cf. Figure 2. Of course, in item (iv) of Theorem 1.3 it could happen that $\Gamma_- = \Gamma_+$, in which case the interior of $\rho^{-1}(\zeta)$ is empty and $\rho^{-1}(\zeta)$ is a Lipschitz graph filled by periodic orbits of $\phi^t_F$. A well-understood example is given by the rotational torus: The short inner closed geodesic $c_0$ has direction $\rho(c_0) = e_1$, $\rho_0(\mathbb{R}) = \Gamma_- \cap \Gamma_+ \subset \rho^{-1}(e_1)$ and there are geodesics homoclinic to $c_0$, winding once around the torus forming the graphs $\Gamma_{\pm}$; the long outer closed geodesic lies in the interior of $\rho^{-1}(e_1)$; for all $\zeta \in S^1 - \{e_1, e_1\}$ the sets $\rho^{-1}(\zeta)$ are (smooth) invariant graphs in $ST^2$. Indeed, one finds that the dynamical behavior of the geodesic flow of the rotational
Figure 2. The invariant graphs and flow lines of $\phi^t_\mathcal{F}$ in the unit tangent bundle $ST^2 \cong T^3$ occurring in Theorem 1.3 in the case of $h_{\text{top}}(\phi^t_F) = 0$; here $\zeta$ has rational, $\zeta'$ irrational slope. The horizontal plane can be thought of as the base $T^2$, $\pi$ being the usual vertical projection. One can see an elliptic tube $E$ of direction $\zeta = (1,0)$ enclosed by two graphs $\Gamma_\pm$, containing a $\phi^t_F$-invariant sub-tube $A$, where $\phi^t_F|_A$ might be ergodic due to Theorem 1.5.

What remains open in Theorem 1.3 is a description of the dynamics of $\phi^t_F$ in the interiors of $\rho^{-1}(\zeta)$ for $\zeta \in S^1$ with rational slope. The following terminology is motivated by the often found presence of elliptic closed geodesics in the interiors of $\rho^{-1}(\zeta)$.

**Definition 1.4.** Let $F$ be a Finsler metric on $T^2$ with $h_{\text{top}}(\phi^t_F) = 0$ and $\zeta \in S^1$ have rational or infinite slope. An **elliptic tube (of direction $\zeta$)** is a connected component $E \subset ST^2$ of the interior $\text{Int}(\rho^{-1}(\zeta))$ of $\rho^{-1}(\zeta)$.

Note that, due to Theorem 1.3 (iv), the projections $\pi(E) \subset T^2$ of different elliptic tubes of a common direction are disjoint and each $\pi(E)$ is bounded by a pair of closed geodesics. Moreover, all lifts to $\mathbb{R}^2$ of geodesics in an elliptic tube $E$ of direction $\zeta$ move to infinity along the direction $\zeta$.

Our next theorem shows that, even if $h_{\text{top}}(\phi^t_F) = 0$, elliptic tubes can contain complicated dynamical behavior of $\phi^t_F$. For the proof we apply Katok’s construction, which also led to the examples in Theorem 1.1, to a particular rotational metric on $T^2$. 
Theorem 1.5. There exist reversible Finsler metrics $F$ on $\mathbb{T}^2$ with vanishing topological entropy $h_{\text{top}}(\phi^t_F) = 0$ and an elliptic tube $E \subset S\mathbb{T}^2$ containing a $\phi^t_F$-invariant, closed sub-tube $A \subset E$ with non-empty interior, such that $\phi^t_F|_A$ is ergodic, hence transitive. The measure $\mu_1(E - A)$ is positive, but can be made arbitrarily small.

In the spirit of Theorem 1.2, our next theorem shows that complicated behavior in all of $E$ is excluded.

Theorem 1.6. Let $F$ be a reversible Finsler metric on $\mathbb{T}^2$ with vanishing topological entropy $h_{\text{top}}(\phi^t_F) = 0$ and suppose that $E \subset S\mathbb{T}^2$ is an elliptic tube for $\phi^t_F$. Then the set

$$\{ v \in E : \inf_{t \in \mathbb{R}} d(\partial E, \phi^t_F v) \geq 1/n \}$$

is nowhere dense in $E$. In particular, $\phi^t_F|_E$ cannot be transitive, hence not ergodic.

Recall that a nowhere dense set is a set whose closure has empty interior. In particular, the complement of a nowhere dense set contains an open and dense set. Hence, Theorem 1.6 shows that topologically almost every $\phi^t_F$-orbit in $E$ is bounded away from the boundary $\partial E$ of the elliptic tube. This indicates that there are large, closed, $\phi^t_F$-invariant sets in the interior of $E$, not touching the boundary $\partial E$. A possible picture would be a sequence $\{A_n\}$ of nested, invariant, closed tubes inside the elliptic tube $E$, such that $E = \bigcup A_n$ (observe the smoothly bounded invariant tubes, which we will find in the example in Theorem 1.5). However, the sets

$$A_n : = \{ v \in E : \inf_{i \in \mathbb{R}} d(\partial E, \phi^t_F v) \geq 1/n \}$$

might a priori be quite exotic and Theorem 1.6 guarantees that $\bigcup A_n$ equals $E$ only up to a topologically small set.

We close the introduction with two remarks.

Remark 1.7. The above results about geodesic flows on $\mathbb{T}^2$ remain true for monotone twist maps of the compact annulus. One can show that the example in Theorem 1.5 can be translated into a twist map. Moreover, for the proof of Theorem 1.6 we work with first-return maps to annuli, so we could have done the same with monotone twist maps.

Remark 1.8. Our results about geodesic flows in $\mathbb{T}^2$ are stated in terms of reversible Finsler metrics. It seems quite plausible that Theorem 1.6 holds also in the non-reversible case, opposed to the case of the 2-sphere, as we remarked after Theorem 1.1. The main ingredient depending on the topology of $\mathbb{T}^2$ is Theorem 1.3, which is used to construct Poincaré sections in Section 5. Apart from the fact that orbits in elliptic tubes tend to infinity, Theorem 1.3 has been generalized to non-reversible Finsler metrics and Tonelli Lagrangian systems by the author in [27].

In the case of the 2-sphere, one could possibly admit non-reversible Finsler metrics, assuming a pinching condition on the curvature and the existence of at
least three closed geodesics (opposed to Katok’s example in the non-reversible
case, which has only two closed geodesics), cf. Remark 4.4 below.

**Structure of this paper.** In Section 2, we study rotational metrics on $S^2$ and
$T^2$ and apply a result due to A. Katok from [23] to prove Theorems 1.1 and 1.5.
The main ingredient to prove Theorems 1.2 and 1.6 are the results of J. Franks
and M. Handel from [16], which we recall and extend in Section 3. In Sections
4 and 5 we prove Theorems 1.2 and 1.6, respectively.

2. Examples with large ergodic components

The examples in Theorems 1.1 and 1.5 depend on a perturbation of degen-
erate integrable systems. The starting point is the geodesic flow of the round
2-sphere. We will denote the infinite cylinder by $C := (\mathbb{R}/2\pi \mathbb{Z}) \times \mathbb{R} \ni x = (x_1, x_2)$
and by $\langle ., . \rangle, |.|$ the euclidean inner product and its norm.

**Lemma 2.1.** There exists a diffeomorphism $\varphi: C \to S^2 - \{-e_3, e_3\}$, such that for
the pullback $\varphi^* g$ of the round metric $g$ on $S^2$ we have

$$(\varphi^* g)_x = f_0^2(x_2) \cdot \langle ., . \rangle, \quad f_0(t) = \frac{2e^t}{1 + e^{2t}}.$$  

**Proof.** The function

$h: \mathbb{R} \to (-\pi/2, \pi/2), \quad h(t) := 2 \arctan(e^t) - \pi/2$  
solves $h' = \cos h$ and $h(0) = 0$. Consider

$\varphi(x) := (\cos x_1 \cos h(x_2), \sin x_1 \cos h(x_2), \sin h(x_2))$.  

The map $\varphi$ is obtained from the parametrization of $S^2$ as a surface of revolution,
composed with the diffeomorphism $C \to (\mathbb{R}/2\pi \mathbb{Z}) \times (-\pi/2, \pi/2)$ given by $x \mapsto
(x_1, h(x_2))$. Using $h' = \cos h$, one easily calculates

$$(\varphi^* g)_x = (h'(x_2))^2 \langle ., . \rangle,$$  

while $h' = f_0$.  

It will be convenient to work in the Hamiltonian setting, as we shall deal with
several Finsler metrics. We identify $T^*C \equiv C \times \mathbb{R}^2 \ni (x, v), \quad T^*C \equiv C \times \mathbb{R}^2 \ni (x, p)$
and endow $T^*C$ with the canonical symplectic structure $\omega$, such that the Hamilton-
ian vector field of $H: T^*C \to \mathbb{R}$ is given by

$$X_H = \left( \frac{\partial}{\partial p} H, -\frac{\partial}{\partial x} H \right).$$  

The Hamiltonian flow of $X_H$ is denoted by $\phi_H^t$.

Consider the Legendre transform associated to the Riemannian metric $f_0^2 \langle ., . \rangle$
on $C$, given by

$$\mathcal{L}: T^*C \to T^*C, \quad \mathcal{L}(x, v) = f_0^2(x_2) \cdot v.$$
\( \mathcal{L} \) conjugates the geodesic flow of \( f_0^2(.,.) \) to the Hamiltonian flow of \( H_0^2/2 \), where the dual norm \( H_0 : T^*\mathcal{C} \to \mathbb{R} \) is given by

\[
H_0(x, p) := \frac{|p|}{f_0(x_2)}.
\]

One easily shows \( H \cdot X_H = X_{H^2/2} \), so the Hamiltonian flows of \( H_0 \) and \( H_0^2/2 \) are reparametrizations of each other in \( \{ p \neq 0 \} \). If \( \{.,.\} \) denotes the Poisson bracket of \( \omega \), then using that \( H_0 \) is independent of \( x_1 \) we find

\[
\{H_0, H_1\} = 0 \quad \text{in} \quad \{ p \neq 0 \},
\]

where

\[
H_1(x, p) := p_1.
\]

We say that \( U \subset T^*\mathcal{C} \) is a fiberwise cone, if \( (x, r \cdot p) \in U \) for \( (x, p) \in U \) and \( r > 0 \). Consider the fiberwise cones

\[
A_c := \{ H_0 > 0, H_1 / H_0 \geq f_0(c) \} \subset T^*\mathcal{C}, \quad c \in \mathbb{R}.
\]

**Lemma 2.2.** Let \( c \in \mathbb{R} \). Then \( A_c \subset \{ p_1 > 0, |x_2| \leq |c| \} \) and for \( i = 0, 1 \) the Hamiltonian flows \( \phi^{t_i}_{H_i} \) leave \( A_c \) invariant and are 2\( \pi \)-periodic in \( A_c \).

**Proof.** Note that for \((x, p) \in A_c\),

\[
0 < f_0(c) \leq \frac{H_1(x, p)}{H_0(x, p)} = \frac{f_0(x_2)}{f_0(x_2)} \frac{p_1}{\sqrt{p_1^2 + p_2^2}} \leq f_0(x_2).
\]

Hence, \( A_c \) lies in the set \( \{ p_1 > 0 \} \). Moreover, since \( f_0 \) is even and increasing in \( (-\infty, 0) \) we find \( |t| \leq |c| \) if and only if \( f_0(t) \geq f_0(c) \), so the first claim follows. The invariance of \( A_c \) follows, as \( A_c \) is defined in terms of the commuting integrals \( H_0, H_1 \). Due to the homogeneity of \( H_0 \) we have

\[
\phi^{t}_{H_0} \circ \psi^{r} = \psi^{r} \circ \phi^{t}_{H_0} \quad \forall r > 0,
\]

where \( \psi^{r}(x, p) = (x, r \cdot p) \). Hence, the 2\( \pi \)-periodicity of \( \phi^{t}_{H_0} \) in the energy level \( H_0^{-1}(1) \cap \{ p_1 > 0 \} \) extends to all of \( \{ p_1 > 0 \} \supset A_c \). For \( H_1 \) observe \( \phi^{t}_{H_1}(x, p) = (x + t e_1, p) \), so the periodicity is trivial. \( \square \)

Let us recall Theorem A from [23], adapted to our situation, where we are given two commuting Hamiltonians \( H_0, H_1 \) with periodic flows (i.e. defining a \( \mathbb{T}^2 \)-action) in the symplectic manifold with boundary given by the fiberwise cone \( A_c \). Note that the Hamiltonian vector fields \( X_{H_0}, X_{H_1} \) are linearly dependent precisely in the 2-dimensional cylinder

\[
A_0 = \{(x, p) : x_2 = p_2 = 0, \ p_1 > 0\},
\]

which corresponds to the equator in various velocities.

**Theorem 2.3** (Katok). Let \( H_0, H_1 : T^*\mathcal{C} \to \mathbb{R}, c > 0 \) and \( A_c \subset T^*\mathcal{C} \) be defined as above. Then for any \( \varepsilon > 0, k \geq 2 \) and any compact subset \( K \subset A_c \) there exists a \( C^\infty \)-function \( H : A_c \to \mathbb{R} \) with the following properties:

(i) \( H(x, r \cdot p) = r \cdot H(x, p) \) for all \( (x, p) \in A_c, \ r > 0 \),

(ii) \( \| H - H_0 \|_{C^k(K)} \leq \varepsilon \).
(iii) in \( A_0 \cup \partial A_\varepsilon \), the function \( H \) coincides together with all its derivatives with a function of the form \( H_0 + \alpha \cdot H_1 \) with \( |\alpha| \leq \varepsilon \).

(iv) the Hamiltonian flow \( \phi^t_H \) of \( H \) is ergodic in each level set \( H^{-1}(r) \) for \( r > 0 \) with respect to the volume induced by the canonical contact form, \( \phi^t_H \) has no closed orbits in \( A_\varepsilon - A_0 \).

We can now construct the examples mentioned in the introduction.

**Proof of Theorems 1.1 and 1.5.** Choose \( 0 < a < b \) and let \( f : \mathbb{R} \to (0, \infty) \) with \( f = f_0 \) in \([-b, b] \). Let \( \eta : \mathbb{R} \to [0, 1] \) be a smooth function with \( \eta(t) = 0 \) for \( t \leq f_0(b) \) and \( \eta(t) = 1 \) for \( t \geq f_0(a) \). Theorem 2.3 yields a function \( H : A_a \to \mathbb{R} \), which coincides with a function \( H_a := H_0 + \alpha H_1 \) in \( \partial A_a \) together with all its derivatives. Writing \( \chi_S \) for the characteristic function of a set \( S \), let

\[
H' := \frac{|p|}{f(x_2)} + \alpha \cdot \eta(H_1/H_0) \cdot \chi_{\{|x_2| \leq b\}} \cdot H_1.
\]

By \( H_1/H_0 \leq f_0(x_2) \) we find that \( H' \) is smooth in \( T^*\mathcal{C} - \{p = 0\} \). Moreover, \( H' = H_a \) in \( A_a \) and \( H' = \frac{|p|}{f(x_2)} \) in \( T^*\mathcal{C} - A_b \). We replace \( H' \) by a Hamiltonian \( H'' \) by changing \( H_a \) into \( H \) in \( A_a \) and consider the reversible Hamiltonian

\[
F^*(x, p) := \begin{cases} H''(x, p) & \text{if } p_1 \geq 0 \\ H''(x, -p) & \text{if } p_1 < 0 \end{cases}
\]

which is smooth in \( T^*\mathcal{C} - \{p = 0\} \), since \( H'' = \frac{|p|}{f(x_2)} \) in the neighborhood \( T^*\mathcal{C} - A_b \) of \( \{p_1 = 0\} \). Writing \( A_\pm = \{(x, p) : (x, \pm p) \in A_a\} \), the Hamiltonian flow of \( F^* \) is ergodic in the sets \( \{F^* = 1\} \cap A_\pm \) by Theorem 2.3 (iv). Moreover, \( \phi^t_{F^*} \) has the equator \( A_0 \) as its only closed orbit in \( A_\pm \) by Theorem 2.3 (v), so the topological entropy \( h_{\text{top}}(\phi^t_{F^*} |_{A_\pm \cap \{F^* = 1\}}) = 0 \), as there is only subexponential growth of closed orbits, cf. Corollary 4.4 in [24]. In the set \( T^*\mathcal{C} - \cup A_\pm^\varepsilon \) the Hamiltonian flow of \( F^* \) is completely integrable, as \( F^* \) is defined in terms of the commuting Hamiltonians \( \frac{|p|}{f(x_2)}, H_1 \). Moreover, we can make the constant \( \alpha \) in Theorem 2.3 (iii) sufficiently small, such that (having fixed \( a, b \) and \( \eta \)), the derivative of \( H' \) stays linearly independent of the derivative of \( H_1 \) in \( (\cup A_\pm^\varepsilon - \cup A_\pm^\varepsilon) \cap \{F^* = 1\} \). In \( \{F^* = 1\} - \cup A_\pm^\varepsilon \), the derivatives of \( H' \) and \( H_1 \) become linearly dependent precisely in the set \( \{(x, p) : f'(x_2) = 0, p_2 = 0\} \), which is a closed union of straight lines, hence of submanifolds. This means that we can apply Theorem 1 in [26] to obtain \( h_{\text{top}}(\phi^t_{F^*} |_{\{F^* = 1\} - A_\pm^\varepsilon}) = 0 \). Putting things together we find

\[
h_{\text{top}}(\phi^t_{F^*} |_{\{F^* = 1\} - A_\pm^\varepsilon}) = 0.
\]

By Theorem 2.3 (ii), we can choose \( H \) close to \( H_0 \), such that w.l.o.g. the Hessian \( \frac{\partial^2}{\partial p^2} (F^*)^2 \) is positive definite everywhere in \( T^*\mathcal{C} - \{p = 0\} \). This together with the homogeneity of \( H \) in Theorem 2.3 (i) shows that \( F^* \) defines a dual Finsler metric on \( \mathcal{C} \).

We move \( F^* \) to \( T^*\mathcal{C} \) by using its Legendre transformation

\[
\mathcal{L}_*: T^*\mathcal{C} - \{p = 0\} \to T\mathcal{C} - \{\nu = 0\}, \quad \mathcal{L}_*(x, p) = (x, \frac{1}{2} \frac{\partial}{\partial p}(F^*)^2(x, p)),
\]

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which yields a reversible Finsler metric $F = F^* \circ \mathcal{L}^{-1}_s : T^* \mathbb{C} \to \mathbb{R}$ with $h_{\text{top}}(\phi_T^t) = 0$ and ergodic components $\mathcal{L}_s(A^*_T)$. To find a metric on $\mathbb{S}^2$, consider the case $f = f_0$ on all of $\mathbb{R}$, so we can use the diffeomorphism $\varphi$ in Lemma 2.1. This yields the desired Finsler metric on $\mathbb{S}^2$, which equals the round metric over a small neighborhood of $\{ -e_3, e_3 \} \subset \mathbb{S}^2$. Letting $a \to \infty$, the size of $\mathbb{S}^2 - u \mathcal{L}_s(A^*_T)$ becomes arbitrarily small. In the case of the torus, we choose a $2\pi$-periodic function $f$, such that $f = f_0$ on $[-\pi + \varepsilon, \pi - \varepsilon]$. If $f$ has global minima precisely in $\pi + 2\pi \mathbb{Z}$, then the set $E := \mathcal{L}_s(\{H_0 > 0, H_1/H_0 > f(\pi)\})$ projected to $T(\mathbb{R}^2/2\pi \mathbb{Z}^2)$ is an elliptic tube of direction $e_1$ and choosing $f$ and $\varepsilon > 0$ properly, we can obtain arbitrarily small complements $E - \mathcal{L}_s(A_{-2\varepsilon})$.

\[ \square \]

3. The results of J. Franks and M. Handel

In this section we recall results of J. Franks and M. Handel from [16]. We start by reviewing the setting of that paper.

Let $\mu$ be a measure on the 2-sphere $\mathbb{S}^2$ topologically conjugate to the Lebesgue measure (i.e. there exists a homeomorphism of $\mathbb{S}^2$ conjugating $\mu$ to the Lebesgue measure). Let $N$ be a surface diffeomorphic to $\mathbb{S}^2$ with $n$ disjoint, smoothly bounded, open discs removed. Collapsing each boundary circle $\partial_i N$ of $N$ into a point $p_i \in \mathbb{S}^2$ defines a $C^0$-quotient map $\pi_N : N \to \mathbb{S}^2$, whose restriction $\text{Int} N \to \mathbb{S}^2 - P$ is a $C^\infty$-homeomorphism, where $P = \{ p_1, \ldots, p_n \}$. If $\phi : N \to N$ is an orientation-preserving $C^\infty$-diffeomorphism, leaving each boundary component $\partial_i N$ invariant, we can define a homeomorphism $\psi : \mathbb{S}^2 \to \mathbb{S}^2$ by $\psi \circ \pi_N = \pi_N \circ \phi$, such that $P \subset \text{Fix}(\psi)$. We denote by $\text{Diff}(\mathbb{S}^2, P, \mu)$ the set of all so obtained homeomorphisms $\psi : \mathbb{S}^2 \to \mathbb{S}^2$, that in addition preserve the measure $\mu$.

We reformulate parts of Theorem 1.2 from [16].

**Theorem 3.1** (Franks, Handel). Let $\psi \in \text{Diff}(\mathbb{S}^2, P, \mu)$ have infinite order and $h_{\text{top}}(\psi) = 0$. Consider the set $\mathcal{A}$ of maximal, $\psi$-invariant, open annuli $U \subset \mathbb{S}^2 - \text{Fix}(\psi)$. Then the elements of $\mathcal{A}$ are pairwise disjoint and the union $\bigcup_{U \in \mathcal{A}} U$ is (open and) dense in $\mathbb{S}^2 - \text{Fix}(\psi)$.

Franks and Handel prove more on the dynamics of $\psi$ in the annuli $U \in \mathcal{A}$. For instance, each restriction $\psi|_U$ has a continuous integral of motion given by the rotation number. In order to apply the results of Franks and Handel to questions on transitivity, we prove the following theorem, combining the results in [16] with Theorem (3.4) in [15] and Brouwer theory. We write $\overline{\mathcal{O}}_\psi(x) = \{ \psi^n x : n \in \mathbb{Z} \}$ for the orbit of $x$ under $\psi$.

**Theorem 3.2.** Let $\psi \in \text{Diff}(\mathbb{S}^2, P, \mu)$ have infinite order and $h_{\text{top}}(\psi) = 0$, let $\mathcal{A}$ be as in Theorem 3.1 and $U \in \mathcal{A}$. Assume that one component $\partial_0 U$, say, of the boundary $\overline{U} - U = \partial_0 U \cup \partial_1 U$ consists of more than one point (closures taken in $\mathbb{S}^2$). Then the set

$$\{ x \in U : \overline{\mathcal{O}}_\psi(x) \cap \partial_0 U \neq \emptyset \}$$

has empty interior and is disjoint from the other boundary component $\partial_1 U$. 


**Proof.** Given \( U \in \mathcal{A} \), there exists an annulus homeomorphism \( h : \mathcal{A} \to \mathcal{A} \) of the compact annulus \( \mathcal{A} = \mathbb{R}/\mathbb{Z} \times [0, 1] \), such that \( h|_{\text{Int}\mathcal{A}} \) is smoothly conjugated to \( \psi|_{U} \) (\( h \) is called the annular compactification of \( \psi|_{U} \), cf. Notation 2.7 in [16]). We shall in the following identify \( U \) with the interior \( \text{Int}\mathcal{A} \). The set \( \partial_{0}U \subset \mathbb{S}^{2} - \cup_{V \in \mathcal{A}} V \) is closed and by Theorem 1.2 (3) in [16] contains a fixed point of \( \psi \). By the properties of the prime-end compactification used to define \( h \) in the case where \( \partial_{0}U \) consists of more than one point (see [16]), we obtain a fixed point of \( h \) in the boundary component \( \partial_{0}\mathcal{A} \) corresponding to \( \partial_{0}U \). Moreover, by Theorem 1.4 in [16] we obtain a continuous, \( h \)-invariant function \( \rho : \mathcal{A} \to \mathbb{R}/\mathbb{Z} \) given by the rotation number (see Definition 2.1 in [16]) with the following property:

1. If \( C \) is a connected component of a level set \( \rho^{-1}(c) \), then \( C \) is \( h \)-invariant; if \( C \) is disjoint from the boundary \( \partial\mathcal{A} \), then \( C \) is essential in \( \mathcal{A} \), i.e. \( \mathcal{A} - C \) has two components, each containing a component of \( \partial\mathcal{A} \).

Using the fixed point of \( h \) in \( \partial_{0}\mathcal{A} \), we find \( \rho|_{\partial_{0}\mathcal{A}} = 0 \). Denoting by \( C_{0} \subset \mathcal{A} \) the connected component of \( \partial_{0}\mathcal{A} \) in \( \rho^{-1}(0) \) and by \( \Omega_{0} \) the set in the statement of the theorem, we thus find by (1)

\[
\Omega_{0} \subset C_{0}.
\]

Moreover, by (1), \( \mathcal{A} - C_{0} \) contains \( \partial_{1}\mathcal{A} \), provided \( \rho \) is non-constant. Hence the second part of the theorem follows if \( \rho \) is non-constant.

Let \( A \) be a connected component of the interior \( \text{Int}C_{0} \subset \text{Int}\mathcal{A} \) (assuming \( \text{Int}C_{0} \neq \emptyset \)). We claim that

\[
\overline{A} \cap \partial_{0}\mathcal{A} = \emptyset \quad \implies \quad A \cap \Omega_{0} = \emptyset.
\]

Indeed, by \( \mu(A) > 0 \), Poincaré recurrence and \( h \)-invariance of \( \text{Int}C_{0} \) we find some \( q \in \mathbb{N} \) with \( h^{q}(A) = A \). Hence, for any \( x \in A \) we have

\[
\overline{\mathcal{O}_{\psi}(x)} \subset \cup_{i=0}^{q-1} h^{i}(A) = \cup_{i=0}^{q-1} h^{i}(\overline{A}),
\]

while the last set is by assumption disjoint from \( \partial_{0}\mathcal{A} \).

Considering the set \( \Omega_{0}^{*} \) given by \( C_{0} \) minus the components \( A \) of \( \text{Int}C_{0} \) with \( \overline{A} \cap \partial_{0}A = \emptyset \) as above, we find

\[
\Omega_{0} \subset \Omega_{0}^{*},
\]

while \( \Omega_{0}^{*} \) is closed. We claim that \( \Omega_{0}^{*} \) has empty interior, then the first part of the theorem follows. For this we assume that \( A \) is a component of \( \text{Int}C_{0} \subset \text{Int}\mathcal{A} \) with \( \overline{A} \cap \partial_{0}\mathcal{A} \neq \emptyset \). We will show that the existence of such components \( A \) lead to a fixed point of \( h \) in \( \text{Int}\mathcal{A} \), which contradicts \( U \subset \mathbb{S}^{2} - \text{Fix}(\psi) \). In particular, the rotation number \( \rho \) cannot be constant, which we used to prove the second part of the theorem.

Let \( A \subset \text{Int}C_{0} \subset \text{Int}\mathcal{A} \) be a component with \( \overline{A} \cap \partial_{0}\mathcal{A} \neq \emptyset \) as above. If \( A \subset \text{Int}\mathcal{A} \) is not simply connected, then using property (1) above, \( A \) has to be a neighborhood of \( \partial_{0}\mathcal{A} \) and \( A \) becomes simply connected, when we contract \( \partial_{0}\mathcal{A} \) to a point and add it to \( A \). In any case, using the Riemann mapping theorem, \( A \) is either an open disc or an open annulus. In the latter case, \( A \) is essentially
embedded in $\mathbb{A}$ and by construction, the rotation number of the restriction $h|_{\mathbb{A}}$ vanishes identically. In this case, the $\mu$-preserving homeomorphism $h|_{\mathbb{A}}$ has a fixed point by Theorem (3.4) in [15] (note that $h|_{\mathbb{A}}$ is isotopic to the identity by the fixed point in $\partial \mathbb{A} \supset \partial_0 \mathbb{A}$). Let us now assume that $A$ is an open disc. We consider the lift $\tilde{h} : \tilde{A} \to \tilde{A}$ of $h$ to the strip $\tilde{A} = \mathbb{R} \times [0,1]$, such that the fixed point of $h$ in $\partial_0 \mathbb{A}$ lifts to a fixed point of $\tilde{h}$ in $\partial_0 \mathbb{A}$, denoted by $x_0 \in \partial_0 \mathbb{A}$. We find an open disc $\tilde{A} \subset \tilde{A}$ projecting onto the open disc $A$. By the same argument as above, we find some $q \in \mathbb{N}$ with $\tilde{h}^q(A) = A$ and denoting by $\tau(x) = x + (1,0)$ the shift in $\tilde{A}$, we find some $p \in \mathbb{Z}$ with
\[ \tilde{h}^q(A) = \tau^p(A). \]
We claim that $p = 0$. For this choose $k \in \mathbb{Z}$, such that for the fixed point $x_0 \in \partial_0 \mathbb{A}$ of $\tilde{h}$, the closure of $\tilde{A}$ intersects the half-open interval $I := [\tau^k x_0, \tau^{k+1} x_0] \times [0) \subset \partial_0 \mathbb{A}$. As $\tilde{h}^q$ leaves $I$ invariant by $x_0$ being a fixed point, $p = 0$ follows and hence, $\tilde{h}^q(A) = \tilde{A}$.

In this case, classical Brouwer theory (see [11] or Theorem 1.5 in [10]) shows that $\tilde{h}|_{\text{Int} \tilde{A}}$ has a fixed point, which projects to an interior fixed point of $h$. $\square$

In our applications, $\psi \in \text{Diff}(\mathbb{S}^2, P, \mu)$ is obtained from a first-return map $\phi : N \to N$ of a Poincaré section $N$, and in this situation we will have an invariant measure $\nu$ defined by a smooth volume form only in $\text{Int} N \equiv \mathbb{S}^2 - P$. Hence, the following observation will be useful, which is a special case of Corollary A2.6 in [1].

**Lemma 3.3.** If $\nu$ is a measure in $\mathbb{S}^2 - P$ induced by a smooth area form defined in $\mathbb{S}^2 - P$, such that $\nu(\mathbb{S}^2 - P) < \infty$, then the measure $\mu$ in $\mathbb{S}^2$ defined by
\[ \mu(A) = \nu(A - P) \]
is topologically conjugate to a Lebesgue measure (i.e. the properly rescaled standard Lebesgue measure).

In order to apply Theorems 3.1 and 3.2 to Poincaré sections in $\mathbb{ST}^2$ for Finsler metrics on the 2-torus, we need the following observation, which under certain conditions allows the boundary circles of the surface $N$ above to be only continuous, instead of $C^\infty$.

**Lemma 3.4.** Let $\gamma_1, \ldots, \gamma_n \subset \mathbb{S}^2$ be disjoint, continuous, simple, closed curves and let $N \subset \mathbb{S}^2$ be the compact surface obtained from $\mathbb{S}^2$ by cutting out interiors of the $\gamma_i$. Then Theorems 3.1 and 3.2 continue to hold for $\mu$-preserving homeomorphisms $\psi : \mathbb{S}^2 \to \mathbb{S}^2$ obtained from diffeomorphisms $\phi : N \to N$ by collapsing each $\gamma_i$ into a fixed point $p_i \in P$ as above with the additional condition that $\phi : N \to \mathbb{S}^2$ extends to a $C^\infty$-embedding of an open neighborhood $U \subset \mathbb{S}^2$ of $N$ into $\mathbb{S}^2$.

**Proof.** The only two places, where Franks and Handel use the smoothness of $\partial N$ in [16] is to prove the following two statements:
(i) Let \( \sigma : [0,1] \to N \) be a smooth curve segment and \( \ell \) denote the length with respect to some Riemannian metric in \( U \). Then
\[
\limsup_{n \to \infty} \frac{1}{n} \log \ell(\phi^n(\sigma)) \leq h_{\text{top}}(\phi).
\]

(ii) There exists a finite family \( \mathcal{R} \) of essential, non-peripheral, non-parallel, simple, closed curves in \( \mathbb{S}^2 - \text{Fix}(\psi) \), such that the homeomorphism \( \psi \in \text{Diff}(\mathbb{S}^2, P, \mu) \) is isotopic relative to \( \text{Fix}(\psi) \) to a composition of non-trivial Dehn twists in the elements of \( \mathcal{R} \).

Considering the embedding \( \phi : U \to \mathbb{S}^2 \), we show how to find a \( C^\infty \)-diffeomorphism \( F : \mathbb{S}^2 \to \mathbb{S}^2 \), such that \( F|_N = \phi \). Item (i) then follows from applying Theorem 1.4 in [29] to \( F \) and smooth curves \( \sigma \) in the \( F \)-invariant set \( N \subset \mathbb{S}^2 \). For item (ii) one proceeds as in Section 4 of [16].

Let us construct \( F \). Choose within \( U \) \( n \) disjoint, smoothly bounded, compact annuli \( A_1, \ldots, A_n \), each \( A_i \) containing \( \gamma_i \) in its interior. There exist paths of smooth embeddings \( \{f_t^i : A_i \to \mathbb{S}^2\}_{t \in [0,1]} \) with \( f_0^i \) the inclusion \( A_i \to \mathbb{S}^2 \) and \( f_1^i = \phi|_{A_i} \). We show how to find \( \{f_t^i\} \). Consider a smooth isotopy \( \{h_t^i\}_{t \in [0,1]} \) of \( \mathbb{S}^2 \) with \( h_0^i = \text{id}_{\mathbb{S}^2} \) and \( h_1^i(A_i) = \phi(A_i) \). Note that \( \pi_1(\mathbb{S}^2) \) is trivial, so \( h_t^i \) can be obtained from isotopying the boundary circles of \( A_i \) into those of \( \phi(A_i) \). Choosing a suitable isotopy \( h_t^i \), the map \( g_t := (h_t^i)^{-1} \circ \phi : A_i \to A_i \) is a diffeomorphism of the closed annulus \( A_i \), which leaves the boundary circles invariant individually due to the invariance of \( N \) and \( \gamma_i \) under \( \phi \). First isotopying \( g_t \) near the boundary circles to obtain \( g_t|_{\partial A_i} = \text{id}_{A_i} \) and then using the fact that the mapping class group of \( A_i \), \( \mathfrak{A} \), is generated by the Dehn twist, which itself is isotopic to \( \text{id}_{A_i} \) when not fixing the boundary, we find an isotopy \( \{g_t^i\} \) of \( A_i \) with \( g_0^i = \text{id}_{A_i} \) and \( g_1^i = g_i \). Now \( f_t^i = h_t^i \circ g_t^i \) is the desired path of embeddings. The complement \( \mathbb{S}^2 - (N \cup \cup A_i) \) consists of \( n \) disjoint discs \( D_i \). In each disc \( A_i \cup D_i \), we use the isotopy extension theorem (Theorem 1.4 on p. 180 of [22]) to obtain a path of \( \tilde{f}_t^i \) of embeddings \( A_i \cup D_i \to \mathbb{S}^2 \) with \( \tilde{f}_t^i|_{A_i} = f_t^i \). The time-1 maps \( \tilde{f}_1^i \) yield the extension of \( F \) in the discs \( D_i \).

**Remark 3.5.** We will apply the results in this Section 3 to first-return maps to Poincaré sections. In both applications, the return-times to the Poincaré section are uniformly bounded. This is enough to deduce that the topological entropy of the occurring geodesic flows bounds the topological entropy of the occurring first-return maps from above, cf. the arguments in the proof of Proposition 2.1 in [7].

4. The case of the 2-sphere

In this section we let \( (\mathbb{S}^2, F) \) be the 2-sphere endowed with a reversible Finsler metric \( F \). In the following discussion we will assume that every closed geodesic of \( F \) has conjugate points. Under this assumption we claim that the existence of a dense orbit of the geodesic flow \( \varphi_F \) in \( \mathbb{S}^2 \) implies \( h_{\text{top}}(\varphi_F) > 0 \) (Theorem 1.2). In order to prove this, we want to apply the results of J. Franks and M. Handel from Section 3, i.e. we need Poincaré sections in \( \mathbb{S}^2 \).
construction is due to G. D. Birkhoff (cf. Section VI.10 of [8]), which we will recall now.

It is a classical fact, that in a Riemannian 2-sphere \((S^2, g)\) there exists a simple, closed geodesic \(c: \mathbb{R} / T \to S^2\), \(T > 0\) being the minimal period of \(c\). The proof is delicate and depends on the existence of a curve-shortening method with certain properties, cf. also the discussion in [5], pp. 8f for an overview of the literature. A sufficient method of curve-shortening is given by the curve-shortening flow discussed in [19], which has been extended to the reversible Finsler case in [4]. Hence, we have the following result.

**Theorem 4.1.** If \(F\) is a reversible Finsler metric on \(S^2\), then there exists a simple, closed \(F\)-geodesic \(c: \mathbb{R} / T \to S^2\).

Letting \(N: \mathbb{R} / T \to T S^2\) be a non-vanishing vector field along the simple closed geodesic \(c\) found in Theorem 4.1, orthogonal to \(\dot{c}\) with respect to the standard round metric denoted by \(\langle \ldots \rangle\) and letting \(\pi: SS^2 \to S^2\) be the bundle projection, we define a smoothly bounded, compact annulus

\[A := \{ v \in SS^2 \mid \exists t \in \mathbb{R} / T : \pi v = c(t), \langle v, N(t) \rangle \geq 0 \} \subset SS^2.\]

We call \(A\) the Birkhoff annulus with base geodesic \(c\) in direction \(N\).

We give a proof of the following lemma along the lines of V. Bangert’s arguments, cf. Section 4 of [5].

**Lemma 4.2** (Birkhoff, Bangert). If \(F\) is a reversible Finsler metric on \(S^2\), then the Birkhoff annulus \(A \subset SS^2\) of any base geodesic \(c: \mathbb{R} / T \to S^2\) is everywhere transverse to the geodesic flow \(\phi^t F\): \(SS^2 \to SS^2\) in the interior \(\text{Int}\ A\). Moreover, if every closed geodesic of \(F\) possesses conjugate points, then every \(\phi^t F\)-orbit in \(SS^2 - \{\dot{c}(t), -\dot{c}(t) : t \in \mathbb{R} / T\}\) hits \(\text{Int}\ A\) in uniformly bounded positive and negative times.

*Proof.* By \(d \frac{d}{dt} |_{t=0} \pi \circ \phi^t F v = v\) and \(\pi(A) = c(\mathbb{R} / T)\) we have transversality of the projected flow line \(\pi \circ \phi^t F v\) to \(\pi(A)\) for \(v \in \text{Int}\ A\). Hence, transversality of \(\phi^t F\) to \(\text{Int}\ A\) follows.

Let \(L\) be the supremum of times that unit speed geodesics other than \(c(t)\) and \(c(-t)\) take to hit \(\text{Int}\ A\) and assume \(L = \infty\). Then there exists a sequence of arc-length geodesic segments \(c_n: [0, n] \to S^2\) with \(n \to \infty\), disjoint from the base geodesic \(c: \mathbb{R} / T \to S^2\) of \(A\). Letting \(v_n := c_n(n/2) \in SS^2\), take a convergent subsequence \(v_n \to v_0\), then the geodesic \(c_0: \mathbb{R} \to S^2\) with \(c_0(0) = v_0\) is entirely disjoint from the base geodesic \(c(\mathbb{R} / T)\). By the assumption that the closed geodesic \(c\) possesses conjugate points, it is not possible for \(c_0\) to come arbitrarily close to \(c\) without intersecting it, as then the geodesic flow would take the orbit \(c_0(t)\) across \(c\). Remember, indeed, that the existence of conjugate points forces \(\phi^t F\) to rotate along the closed orbit \(\dot{c}(\mathbb{R} / T)\), cf. Figure 1. Hence \(\inf_{t \in \mathbb{R}} d(c(\mathbb{R} / T), c_0(t)) > 0\).

Let us write \(D \subset S^2\) for the closed disc bounded by \(c\), containing \(\overline{c_0(\mathbb{R})}\) in its interior. Considering the \(\omega\)-limit set of \(c_0\), we can assume that \(c_0\) is recurrent. If \(c_0\) has no self-intersections it is a simple closed geodesic in \(\text{Int}\ D\) by a
Poincaré-Bendixon argument, using recurrence. Here, let $U \subset \text{Int} D$ be the open annulus bounded by the two simple closed geodesics $c, c_0$. If $c$ does have a self-intersection at $c_0(t_1) = c_0(t_2)$ with $t_1 < t_2$, then using recurrence we can find another self-intersection at $c_0(t_3) = c_0(t_4)$ with $t_2 < t_3 < t_4$. In this case, consider the connected component $U$ of $\text{Int} D - c_0[t_1, t_4]$ touching $c$, which is bounded by geodesic arcs forming angles at most $\pi$. Again, $U$ is topologically an open annulus.

We show how to find a closed geodesic in $U$ without conjugate points, thus obtaining a contradiction. Using the assumption that closed geodesics have conjugate points, we choose $k \geq 1$, such that a pair of conjugate points can be found along $c$ in the interval $[0, kT]$ and another pair along $c_0$ in $[0, kT_0]$ in the case where $c_0$ is a simple closed geodesic ($T, T_0$ being the prime periods of $c, c_0$, respectively). We move to the $k$-fold cover $\hat{U}^k$ of the open annulus $U$. Consider a sequence $\{\hat{\gamma}^k_n\}_{n \in \mathbb{N}}$ of smooth, simple, closed curves in the prime homotopy class of $\hat{U}^k$, such that the $F$-lengths $l_F(\hat{\gamma}^k_n)$ decrease with $n \to \infty$ to the infimum of lengths of such curves in $\hat{U}^k$. For each $n$ deform $\hat{\gamma}^k_n$ into a simple closed geodesic $\gamma^k_n$ in $\overline{U}^k$ by means of the curve-shortening flow for reversible Finsler metrics from [4]. Note that under the curve-shortening flow all curves stay in $\overline{U}^k$ due to $U$ having piecewise geodesic boundary with interior angles at most $\pi$ (cf. [2]). Passing to a limit geodesic of $\{\hat{\gamma}^k_n\}$, we obtain a shortest, simple, closed geodesic $\gamma^k$ in the closed annulus $\overline{U}^k$. Moreover, this closed geodesic has to be contained in the interior $\hat{U}^k$: if $\gamma^k$ touches a boundary component tangentially, it has to be equal to this boundary component by being a geodesic; moreover, $\gamma^k$ cannot be a closed geodesic in $\partial \hat{U}^k$, as such a component can be approximated within $\hat{U}^k$ by shorter curves, using the conjugate points (cf. Chapter 7.4 of [6]). It now follows from classical arguments of G. A. Hedlund (cf. Section 5 in [21] or Lemma 3.1 and Theorem 3.2 in [28]), that $\gamma^k$ is prime periodic, when projected to $U$ and locally minimizing on arbitrarily long subsegments in the universal cover of $U$. Hence, it has to be free of conjugate points.

As a corollary, we obtain a smooth first-return map

$$\phi : \text{Int} A \to \text{Int} A,$$

which (as a map coming from a Hamiltonian flow) is well-known to preserve a smooth area form, also defined in the interior $\text{Int} A$. We call the map $\phi$ the Birkhoff annulus map.

In order to apply the results from Section 3, we need a smooth continuation of the Birkhoff annulus map $\phi$ to all of $A$.

**Lemma 4.3.** If $F$ is a reversible Finsler metric on $\mathbb{S}^2$ with conjugate points along every closed geodesic, then the Birkhoff annulus map of any simple, closed geodesic extends to a $C^\infty$-diffeomorphism $\phi : A \to A$.

**Proof.** Let $c : \mathbb{R}/TZ \to \mathbb{S}^2$ be the base geodesic of the Birkhoff annulus $A$. We show that the first-return time $\tau : \text{Int} A \to \mathbb{R}$ given by Lemma 4.2 can be extended to a $C^\infty$-function $\hat{\tau} : A \to \mathbb{R}$. If $\phi_F^t$ is the geodesic flow, then $\phi(v) = \phi_F^{\hat{\tau}(v)} v$ is the...
smooth continuation of \( \phi \) to \( A \). We shall discuss the continuation of \( \tau \) near the boundary component \( \partial_0 A := \hat{c}(\mathbb{R}/TZ) \) of \( A \), the other boundary component \( \partial_1 A := -\hat{c}(\mathbb{R}/TZ) \) being treated analogously. In the following we assume that \( \mathbb{S}^2 \) has been embedded into \( \mathbb{R}^3 \) as the round sphere in such a way that \( c(\mathbb{R}/TZ) \) is the equator \( \mathbb{S}^2 \cap \{ x_3 = 0 \} \). For \( t \in \mathbb{R}/TZ \) and \( \theta \in \mathbb{R}/2\pi \mathbb{Z} \) we let \( \varphi(t, \theta) \) be the vector forming an angle \( \theta \) with \( \hat{c}(t) \) (using the euclidean metric in \( \mathbb{R}^3 \supset \mathbb{S}^2 \) and the oriented frame \((\hat{c}, N)\), where \( N \) is the vector field along \( c \) giving the direction of the Birkhoff annulus \( A \)). Writing \( \pi : T\mathbb{S}^2 \to \mathbb{S}^2 \) for the projection and \( V_\varphi := \ker d\pi(\nu) \) for the vertical bundle as in Subsection 1.1, we then have

\[
V_{\varphi(t, \theta)} = \mathbb{R} \cdot \left. \frac{\partial}{\partial \theta} \varphi(t, \theta) \right\} \quad \forall \theta \in \mathbb{R}/2\pi \mathbb{Z}.
\]

First we extend \( \tau \) continuously near \( \partial_0 A \). Consider the geodesic variation

\[
h : \mathbb{R} \times (\mathbb{R}/TZ) \times (\mathbb{R}/2\pi \mathbb{Z}) \to \mathbb{R}^3, \quad h(s, t, \theta) := \pi \circ \varphi_{\hat{c}} \circ \varphi(t, \theta).
\]

The derivative

\[
J_\tau(s) := \left. \frac{\partial}{\partial \theta} \right\} h(s, t, \theta) = d\pi(\hat{c}(t+s)) \cdot d\varphi_{\hat{c}}(\hat{c}(t)) \cdot \left. \frac{\partial}{\partial \theta} \right\} \varphi(t, \theta)
\]

is a non-constant Jacobi field along \( c \) with \( J_\tau(0) = 0 \). By the assumed existence of conjugate points along the periodic geodesic \( c \), there exists a discrete set of times (given by the points conjugate along \( c \) to \( c(t) \))

\[
s_0(t) = 0 < s_1(t) < s_2(t) < s_3(t) < \ldots,
\]

where \( J_\tau(s) \) vanishes. We consider the function

\[
\sigma : (\mathbb{R}/TZ) \times [0, \pi) \to \mathbb{R}, \quad \sigma(t, \theta) := \begin{cases} 
\tau \circ \varphi(t, \theta) & \text{if } \theta \neq 0 \\
\sigma_2(t) & \text{if } \theta = 0.
\end{cases}
\]

Observe that in first approximation near \( \theta = 0 \) we have (in \( \mathbb{R}^3 \))

\[
h(s, t, \theta) \approx c(t+s) + \theta \cdot J_\tau(s).
\]

Using that the Jacobi fields \( J_\tau \) are solutions to a second order differential equation, one can easily verify the continuity of \( \sigma \).

Next, we show that \( \sigma \) is indeed a \( C^\infty \)-function and hence \( \hat{\tau} = \sigma \circ \varphi^{-1} \) is the desired smooth extension of \( \tau \). Writing \( h = (h_1, h_2, h_3) \in \mathbb{R}^3 \) for the above geodesic variation and observing \( h_3(s, t, 0) = 0 \) for all \( s, t \), the function

\[
f : \mathbb{R} \times (\mathbb{R}/TZ) \times \mathbb{R} \to \mathbb{R}, \quad f(s, t, \theta) = \begin{cases} 
\frac{1}{0} h_3(s, t, \theta) & \text{if } \theta \neq 0 \\
\frac{\partial}{\partial \theta} \big|_{\theta=0} h_3(s, t, \theta) & \text{if } \theta = 0
\end{cases}
\]

is continuous. Using Taylor expansions, one shows that \( f \) is even \( C^\infty \). Furthermore, by definition of \( f \) and \( \sigma \) we have

\[
f(\sigma(t, \theta), t, \theta) = 0 \quad \forall (t, \theta) \in (\mathbb{R}/TZ) \times [0, \pi).
\]

In order to show that \( \sigma \) is smooth near \( \theta = 0 \), we wish to apply the implicit function theorem and find \( \sigma \) as the locally unique solution of \( f(\sigma(t, \theta), t, \theta) = 0 \).
Using the linearity of the projection $x \mapsto x_3$ in $\mathbb{R}^3$, we compute

$$\frac{\partial}{\partial s} f(s, t, 0) = \left( \frac{d}{ds} J_t(s) \right)_3.$$ 

Evaluating at $s = \sigma(t, 0) = s_2(t)$, this derivative is non-zero. Indeed, otherwise the non-constant Jacobi field $J_t(s)$ would be tangent to $c$ and hence never vanish except at $s = 0$. The lemma follows.

We can thus prove Theorem 1.2 from the introduction.

**Proof of Theorem 1.2.** By Lemmata 4.2 and 4.3, the geodesic flow of $F$ can be reduced to the Birkhoff annulus map $\phi$, which is smooth in the closed annulus $A$. Assuming $h_{\text{top}}(\phi^j_F) = 0$, the topological entropy of $\phi$ vanishes by Remark 3.5 (note that the first-return time is uniformly bounded). Hence, we are in the setting of Section 3 and obtain a homeomorphism $\psi \in \text{Diff}(\mathbb{S}^2, P, \mu)$ with $\text{card} P = 2$ and $h_{\text{top}}(\psi) = 0$. Note that Theorem 1.2 is void, if $\psi$ has finite order, hence we assume that $\psi$ has infinite order, such that Theorems 3.1 and 3.2 can be applied. By the existence of infinitely many closed geodesics on $(\mathbb{S}^2, F)$ due to J. Franks and V. Bangert [15], [5] and its generalization to the reversible Finsler case due to H. Duan and Y. Long [14], we obtain an interior periodic point of the Birkhoff annulus map of period $q \geq 1$, say, and hence $\psi^q$ has a third fixed point in $\mathbb{S}^2 - P$. We apply Theorem 3.1 to $\psi^q$ and let $U \subset \mathbb{S}^2 - \text{Fix}(\psi^q)$ be a $\psi^q$-invariant, open annulus. Observe that one end $\partial_0 U$, say, of $U$ consists of more than one point in $\mathbb{S}^2$; otherwise we have $\mathbb{S}^2 - P = U \subset \mathbb{S}^2 - \text{Fix}(\psi^q)$, contradicting the fact that $\psi^q$ has a third fixed point. Assuming that $\phi^j_F$ is transitive we find some $x \in U$ (using that $U$ is open), such that the orbit $\overline{\phi^q(x)}$ of $x$ under $\psi$ is dense in $U$. We find

$$\overline{U} = \overline{\phi^q(x)} = \bigcup_{i=0}^{q-1} \phi^i \overline{\phi^q(x)} = \bigcup_{i=0}^{q-1} \overline{\phi^q(x)}(\psi^i x).$$

Hence, $\overline{\phi^q(x_0)} \cap \partial_0 U \neq \emptyset$ and moreover the interior $\text{Int} \overline{\phi^q(x_0)} \neq \emptyset$, where $x_0 = \psi^{i_0} x$ for some $i_0 \in \{0, ..., q - 1\}$. By the second condition, the point $x_0$ enters under $\psi^q$ the set

$$\{ y \in U : \overline{\phi^q(y)} \cap \partial_0 U = \emptyset \},$$

as this set contains an open and dense subset of $U$ by Theorem 3.2. This is a contradiction.

**Remark 4.4.** By the results of A. Harris and G. P. Paternain [20], also for $(1 - \frac{1}{r+1})^2$-pinched non-reversible Finsler metrics ($r$ being the reversibility of $F$) and, more generally, for dynamically convex Reeb flows on $\mathbb{S}^3$ there exist well-behaved, disc-like, global Poincaré sections. Note that dynamical convexity replaces the condition of having conjugate points. One might be able to prove smoothness of the arising first-return maps on the closure of the disc-like Poincaré section and then apply the results of J. Franks and M. Handel; here one can use that the surfaces of section arise from pseudo-holomorphic curves. Hence, it is quite possible that Theorem 1.2 generalizes to $(1 - \frac{1}{r+1})^2$-pinched..
non-reversible Finsler metrics and dynamically convex Reeb flows on $S^3$, assuming the existence of at least three closed orbits in $S^2$, $S^3$, respectively.

5. Non-ergodicity in elliptic tubes for the 2-torus

We fix a reversible Finsler metric $F$ on $T^2$ with geodesic flow $\phi^t_F: ST^2 \to ST^2$ and assume $h_{\text{top}}(\phi^t_F) = 0$. Moreover, we fix some integer point $z \in \mathbb{Z}^2 - \{0\}$ and assume in this section that for the asymptotic direction $\rho: ST^2 \to S^1$ in Theorem 1.3 the preimage $\rho^{-1}(z/|z|)$ has non-empty interior. In order to prove Theorem 1.6, we construct a Poincaré section for the geodesic flow $\rho_1.3$, the preimage $\rho_1.3$, the definition of $\rho_1.3$.

Proof. Let $z = (z_1, z_2) \in \mathbb{Z}^2 - \{0\}$ as above and let $z^\perp := (-z_2, z_1)$. Choose a minimal axis $c^\perp: \mathbb{R} \to \mathbb{R}^2$ of the translation $\mathbb{R}^2 \to \mathbb{R}^2$ associated to $z^\perp$ and consider the torus $T^2 = \mathbb{R}^2 / (z \mathbb{Z} \oplus z^\perp \mathbb{Z})$.

We consider a non-vanishing vector field $N: \mathbb{R} \to TT^2_z$ along $c^\perp$, orthogonal to $c^\perp$ with respect to the euclidean metric, such that $(N, c^\perp)$ has the same orientation as $(z, z^\perp)$. Writing $(..)$ for the euclidean inner product on $\mathbb{R}^2 \cong T_z^1 T^2_z$ and $\pi: TT^2_z \to T^2_z$ for the canonical projection, we consider the open annulus $A_z := \{v \in ST^2_z | \exists t \in \mathbb{R}: \pi v = c^\perp(t), \langle v, N(t) \rangle > 0 \} \subset ST^2_z$.

Recall the $\phi^t_F$-invariant function $\rho(v) = \lim_{t \to -\infty} \frac{\tilde{c}_v(t)}{c_v(t)} \in S^1$ for $v \in ST^2_z$ and a lift $\tilde{c}_v: \mathbb{R} \to \mathbb{R}^2$ of the geodesic $c_v: \mathbb{R} \to T^2$, $c_v(t) := \pi \circ \phi^t_F v$, as discussed in Theorem 1.3.

**Lemma 5.1.** $A_z$ is transverse to the geodesic flow $\phi^t_F: ST^2_z \to ST^2_z$. If $h_{\text{top}}(\phi^t_F) = 0$, then every orbit $\phi^t_F v$ with $v \in ST^2_z$ and $\rho(v)$ lying in the connected component of $z/|z|$ in $S^1 - \mathbb{R} z^\perp$ hits $A_z$ after finite positive and negative times.

Note, however, that the return time becomes infinite for $\rho(v)$ close to $\pm z^\perp/|z^\perp|$. Indeed, if $v_n \to v$ in $A_z$ with e.g. $\rho(v_n) \to z^\perp/|z^\perp|$, then $c_v = c^\perp$ using Theorem 1.3. Moreover, $\rho(v)$ for $v \in A_z$ lies in the connected component of $z/|z|$ in $S^1 - \mathbb{R} z^\perp$.

**Proof.** Let $v \in A_z$, then the geodesic $c_v(t) = \pi(\phi^t_F v)$ is transverse in $t = 0$ to $\pi(A_z) = c^\perp(\mathbb{R})$, showing transversality. The second claim follows directly from the definition of $\rho$, $|\tilde{c}_v(t)| \to \infty$ for $v \in ST^2_z$ and $|t| \to \infty$ and the symmetry $\rho(-v) = -\rho(v)$.

The first-return map to the Poincaré section $A_z$ is a $C^\infty$-diffeomorphism

$$\phi: A_z \to A_z, \quad A_z \cong \mathbb{R} / \mathbb{Z} \times \mathbb{R},$$
preserving a smooth area $\nu$ (by being a first-return map of a flow conjugated to a Hamiltonian flow in $T^*T^2_\mathbb{Z}$). The dynamics of $\phi$ on $A_z$ and the corresponding invariant graphs can be seen by taking a transverse section of Figure 2. Recall the notation $\Gamma_\pm \subset \partial \rho^{-1}(z/|z|)$ in Theorem 1.3 (iv) for the two $\phi_{F}^t$-invariant Lipschitz graphs in $ST^2_\mathbb{Z}$ with asymptotic direction $z/|z| \in \mathbb{S}^1$. Moreover, remember that $\rho^{-1}(\zeta)$ is itself an invariant Lipschitz graph for any $\zeta \in \mathbb{S}^1$ with irrational slope. We choose two irrational directions $\zeta_-, \zeta_+ \in \mathbb{S}^1$, such that in the counterclockwise order of $\mathbb{S}^1$

$$-\frac{\zeta_-}{|z|} < \zeta_+ < \frac{\zeta_+}{|z|} < \frac{\zeta_+}{|z|}$$

and set

$$\gamma_\pm := \rho^{-1}(\zeta_\pm).$$

We use the same notation for the intersections of these graphs with $A_z$: $\gamma_\pm$ and $\Gamma_\pm$ then appear as $\phi$-invariant, (Lipschitz) continuous, simple, closed and non-contractible curves in $A_z$.

We want to study the behavior of $\phi : A_z \to A_z$ in the space between the $\phi$-invariant Lipschitz curves $\Gamma_\pm \subset A_z$. We restrict ourselves to the compact subset $N$ of $A_z$ between $\gamma_-$ and $\gamma_+$, restricting the area $\nu$ to Int $N$. By Remark 3.5, we find

$$h_{\text{top}}(\phi|_N) \leq h_{\text{top}}(\phi_{F}^t) = 0.$$ Collapsing $\gamma_\pm$ into two points $p_\pm \in \mathbb{S}^2$, we are precisely in the situation of Lemmata 3.3, 3.4 and hence can apply Theorems 3.1 and 3.2 in Section 3. The components $E$ of $N - \cup \Gamma_\pm$ between $\Gamma_-$ and $\Gamma_+$ are the (intersections with $A_z$ of) elliptic tubes of direction $z/|z|$ from Definition 1.4. Since the non-empty intersection $\cap \Gamma_\pm$ consists of fixed points for $\phi_t$ elliptic tubes are $\phi$-invariant, instead of merely being permuted. Cf. Figure 3 for the notation.

![Figure 3](image-url)  

**Figure 3.** The annulus $A_z$ with the curves $\gamma_\pm, \Gamma_\pm$ and fixed points $p_0, p_1$ in the boundary of an elliptic tube $E$. The arrows indicate the principle direction of the dynamics of $\phi$.

We can now prove Theorem 1.6. In the following proof, we fix an elliptic tube $E \subset N$, which has two fixed points $p_0, p_1 \in \partial E$, cf. Figure 3. We have to prove
the set \( \{ v \in E : \overline{O_\phi(v) \cap \partial E} \neq \emptyset \} \) is nowhere dense in \( E \), where \( O_\phi(v) \) is the orbit of \( v \) under \( \phi \).

(ii) \( \phi \) has no dense orbit in \( E \).

Proof of Theorem 1.6. First note that item (ii) follows from (i). Indeed, if some \( v \in E \) had a dense orbit in \( E \) under \( \phi \), then by item (i) there exists some \( n \in \mathbb{Z} \) with \( \phi^n v \) lying in the complement of the set in item (i). But such points cannot have a dense orbit in \( E \) by definition.

Let us prove item (i). By Theorem 3.1, the union of all maximal, \( \phi \)-invariant annuli \( \bigcup_{U \in \mathcal{A}} U \) is open and dense in \( E - Fix(\phi) \). Let us fix some \( U \in \mathcal{A} \) with \( U \cap E \neq \emptyset \) and let \( p_0, p_1 \in \partial E \) be the fixed points.

Let us first observe that no boundary component of \( U \) can consist of only one point \( p \in \{ p_0, p_1 \} \). Indeed, suppose e.g. that \( \partial_1 U = \{ p_1 \} \) and write \( G \subset \Gamma_+ \) for the open subarc of \( \Gamma_+ \) connecting \( p_0, p_1 \). By invariance of \( U \) we have \( G \subset U \), connecting the two boundary components \( \partial_0 U, \partial_1 U \) (note that \( p_0 \notin U \)). Of course, by the heteroclinic dynamics in \( \Gamma_+ \) we have \( G \subset \{ v \in U : \overline{O_\phi(v) \cap \partial E} \neq \emptyset \} \).

Let now \( v \in U \) with \( \overline{O_\phi(v) \cap \partial E} \neq \emptyset \), then by the heteroclinic dynamics in \( \partial E - \{ p_0, p_1 \} \), we obtain \( \overline{O_\phi(v) \cap \partial E} \neq \emptyset \). Hence, one point \( p \in \{ p_0, p_1 \} \) lies in one end \( \partial_0 U \), say, of \( U \) in \( N \) and \( \overline{O_\phi(v) \cap \partial E} \neq \emptyset \). We proved before, that \( \partial_0 U \neq \{ p \} \) and hence Theorem 3.2 shows that

\[
\{ v \in U : \overline{O_\phi(v) \cap \partial E} \neq \emptyset \}
\]

is nowhere dense in \( U \). To finish the proof, observe that

\[
\text{Int} \, Fix(\phi) \bigcup_{U \in \mathcal{A}} U
\]

is open and dense in \( E \). \( \blacksquare \)

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