THE STRUCTURE, CAPABILITY AND THE SCHUR MULTIPLIER OF GENERALIZED HEISENBERG LIE ALGEBRAS

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ABSTRACT. From [Problem 1729, Groups of prime power order, Vol. 3], Berkovich et al. asked to obtain the Schur multiplier and the representation of a group $G$, when $G$ is a special $p$-group minimally generated by $d$ elements and $|G'| = p^{\frac{1}{2}d(d-1)}$. Since there are analogies between groups and Lie algebras, we intend to give an answer to this question similarly for nilpotent Lie algebras. Furthermore, we give some results about the tensor square and the Schur multiplier of some nilpotent Lie algebras of class two.

1. Motivation and Preliminaries

The Schur multiplier of groups is appeared in the works of Schur in 1904. There are many results on the Schur multiplier of finite $p$-groups and the reader can see for instance [3, 6, 7]. According to [1, 2, 9, 16, 17], we may define the Schur multiplier, $M(L)$, for a Lie algebra $L$. Many papers in the literature make an attempt to generalize the results on finite $p$-groups to the theory of Lie algebras. Although there are some sporadic results for the Lie algebra that does not coincide with the results for groups. However, there are analogies between groups and Lie algebras, but the analogies are not completely identical and most of them should be checked carefully. Recently in [18], Rai tried to give an answer to [3, Problem 1729]. The motivation of this paper is not only to answer this question for Lie algebra but we also determine which one of these Lie algebras are capable (a Lie algebra $L$ is called capable if and only if $L \cong E/Z(E)$ for a Lie algebra $E$). By $d(L)$ we denote the minimal number of elements required to generate a Lie algebra $L$. Roughly speaking, for a generalized Heisenberg Lie algebra $H$, $H^2 = Z(H)$, such that $d(H) = d$ and $\dim H^2 = \frac{1}{2}d(d-1)$, we intend to describe the structure of $H$ and its Schur multiplier and we also specify when $H$ is capable. In this paper, we are going to consider the generalized Heisenberg Lie algebra $H$ with the maximum dimension of the derived subalgebra, that means, if $d(H) = d$, then $\dim H^2 = \frac{1}{2}d(d-1)$. Our technique depends on the results that recently obtained in [14] and completely is different from [18]. Furthermore, we give the structure of Schur multiplier and tensor square of Lie algebras of class two with the maximum dimension of the derived subalgebra as an application and we show such Lie algebras are capable. Throughout the paper, $H(m)$ and $A(n)$ are used to denote the Heisenberg and abelian Lie algebra of dimension $2m + 1$ and $n$, respectively. For the convenience of the reader, we give some results and definitions in the following.

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Lemma 1.1. [15] Lemma 2.6] We have

(i) $\dim \mathcal{M}(A(n)) = \frac{1}{2}n(n - 1)$.
(ii) $\dim \mathcal{M}(H(1)) = 2$.
(iii) $\dim \mathcal{M}(H(m)) = 2m^2 - m - 1$ for all $m \geq 2$.

For a Lie algebra $L$, we use notation $L^{(ab)}$ instead of $L/L^2$.

Theorem 1.2. [2] Proposition 3] Let $A$ and $B$ be two Lie algebras. Then

$$\mathcal{M}(A \oplus B) \cong \mathcal{M}(A) \oplus \mathcal{M}(B) \oplus (A^{(ab)} \otimes_{\text{mod}} B^{(ab)}),$$

in where $A^{(ab)} \otimes_{\text{mod}} B^{(ab)}$ is the standard tensor product $A$ and $B$.

From [5], let $L \wedge L$ and $L \otimes L$ denote the exterior square and the tensor square of a Lie algebra $L$, respectively. The authors assume that the reader is some what familiar with the exterior square and the tensor square of a Lie algebra. See for instance $[5, 13]$. Recall from [1] that the concept exterior centre of a Lie algebra $L$ is denoted by $Z^\wedge(L)$ and it is equal to the set $\{x \in L \mid x \wedge g = 1_{L \wedge L} \text{ for all } g \in L\}$. It is well-known that $L$ is capable if and only if $Z^\wedge(L) = 0$.

The next lemma shows the kernel of the commutator map $\kappa'$ is exactly the Schur multiplier.

Lemma 1.3. [5] Theorem 35 (iii)] Let $L$ be a Lie algebra. Then $0 \to \mathcal{M}(L) \to L \wedge L \xrightarrow{\kappa'} L^2 \to 0$ is exact, in where $\kappa' : L \wedge L \to L^2$ is given by $l \wedge l_1 \mapsto [l, l_1]$.

In the following, we provide a criterion for detecting the capability of a Lie algebra $L$.

Lemma 1.4. [20] Corollary 4.6] A Lie algebra $L$ is capable if and only if the natural map $\mathcal{M}(L) \to \mathcal{M}(L/\langle x \rangle)$ has a non-trivial kernel for all non-zero elements $x \in Z(L)$.

Recall that a Lie algebra $L$ is called Heisenberg provided that $L^2 = Z(L)$ and $\dim L^2 = 1$. Such algebras are odd dimension and have the following presentation $H(m) \cong \langle a_1, b_1, \ldots, a_m, b_m, z \mid [a_l, b_l] = z, 1 \leq l \leq m \rangle$.

Definition 1.5. A Lie algebra $H$ is called generalized Heisenberg of rank $n$ if $H^2 = Z(H)$ and $\dim H^2 = n$.

The following result allows us to work only on the generalized Heisenberg Lie algebras when working on the capability of Lie algebras of class 2.

Proposition 1.6. [14] Proposition 2.2] and [10] Proposition 3.1] Let $L$ be a finite dimensional nilpotent Lie algebra of nilpotency class 2. Then $L = H \oplus A$ and $Z^\wedge(L) = Z^\wedge(H)$, where $A$ is abelian and $H$ is a generalized Heisenberg Lie algebra.

Let $L$ be a free Lie algebra on the set $X = \{x_1, x_2, \ldots\}$ From [21], the basic commutator on the set $X$ defined inductively.

(i) The generators $x_1, x_2, \ldots, x_n$ are basic commutators of length one and ordered by setting $x_i < x_j$ if $i < j$.

(ii) If all the basic commutators $d_t$ of length less than $t$ have been defined and ordered, then we may define the basic commutators of length $t$ to be all commutators of the form $[d_i, d_j]$ such that the sum of lengths of $d_i$ and $d_j$
is $t$, $d_i > d_j$, and if $d_i = [d_s, d_t]$, then $d_j \geq d_t$. The basic commutators of length $t$ follow those of lengths less than $t$. The basic commutators of the same length can be ordered in any way, but usually the lexicographical order is used.

The number of all basic commutators on a set $X = \{x_1, x_2, \ldots, x_d\}$ of length $n$ is denoted by $l_d(n)$. Thanks to [8], we have

$$l_d(n) = \frac{1}{n} \sum_{m|n} \mu(m)d^m,$$

where $\mu(m)$ is the Möbius function, defined by $\mu(1) = 1, \mu(k) = 0$ if $k$ is divisible by a square, and $\mu(p_1 \ldots p_s) = (-1)^s$ if $p_1, \ldots, p_s$ are distinct prime numbers.

Using the topside statement and looking [19, Lemma 1.1] and [21], we have the following.

**Theorem 1.7.** Let $F$ be a free Lie algebra on set $X$, then $F^c/F^{c+i}$ is an abelian Lie algebra with the basis of all basic commutators on $X$ of lengths $c, c+1, \ldots, c-i+1$ for all $0 \leq i < c$. In particular, $F^c/F^{c+1}$ is an abelian Lie algebra of dimension $l_d(c)$, where $F^c$ is the $c$-th term of the lower central series of $F$.

### 2. Main results

In this section, we intend to obtain the structure of a generalized Heisenberg Lie algebra $H$ of rank $\frac{1}{2}d(d-1)$, in where $d = d(H)$. Furthermore, we describe the Schur multiplier of such Lie algebras and then we detect which ones of them are capable. It gives an affirmative answer to [3, Problem 1729] and generalized and enriched the result of [18] with a quite different way.

The following proposition determines the minimal number of elements required to generate a finite dimensional nilpotent Lie algebra.

**Proposition 2.1.** Let $L$ be a $d$-generator nilpotent Lie algebra of dimension $n$ with the derived subalgebra of dimension $m$. Then $d = \dim L/L^2 = n-m$.

**Proof.** We can choose a basis set $\{x_1 + L^2, \ldots, x_{n-m} + L^2\}$ for $L/L^2$. Thus $L = \langle x_1, \ldots, x_{n-m} \rangle + L^2$. By [11 Corollary 2], $L^2$ is equal to the intersection of all maximal subalgebras of $L$. Now [22 Lemma 2.1] implies $L = \langle x_1, \ldots, x_{n-m} \rangle$ and so $d \leq n-m$. Let now $L = \langle y_1, \ldots, y_d \rangle$. Therefore $L/L^2 = \langle y_1 + L^2, \ldots, y_d + L^2 \rangle$. Thus we have $\dim L/L^2 = n - m \leq d$. Hence $d = \dim L/L^2 = n - m$, as required. \qed

We need the following easy lemma.

**Lemma 2.2.** [12 Lemma 14] Let $L$ be a Lie algebra such that $\dim(L/Z(L)) = n$. Then $\dim L^2 \leq \frac{1}{2}n(n-1)$.

**Corollary 2.3.** Let $H$ be an $n$-dimensional generalized Heisenberg Lie algebra of rank $m$. Then $d(H) = \dim(H/Z(H)) = n-m$ and $\dim H^2 \leq \frac{1}{2}(n-m)(n-m-1)$.

**Proof.** It is a conclusion of Proposition [21] and Lemma [22] \qed

We are a position to determine the structure a $d$-generator generalized Heisenberg Lie algebra $H$ of rank $\frac{1}{2}d(d-1)$. 

The proof is completed.

Proposition 2.4. Let $H$ be a $d$-generator generalized Heisenberg Lie algebra of rank $\frac{1}{2}d(d-1)$. Then $\dim H = \frac{1}{2}d(d+1)$ and $H$ has the presentation $(x_1, \ldots, x_d, y_{ij} | [x_i, x_j] = y_{ij}, 1 \leq i < j \leq d)$. 

Proof. By Corollary 2.3, we have $d(H) = \dim(H/Z(H)) = d$. We can choose a basis set $\{x_1 + Z(H), \ldots, x_d + Z(H)\}$ for the $H/Z(H)$ such that $[x_i, x_r]$ is non-trivial for all $1 \leq r < d$ and $t \neq r$. It is clear to see that the set $\{[x_i, x_j] | 1 \leq i < j \leq d\}$ is a basis of $H^2$. Since $\dim H^2 = \frac{1}{2}d(d-1)$, we have $H = \langle x_1, \ldots, x_d, y_{ij} | [x_i, x_j] = y_{ij}, 1 \leq i < j \leq d \rangle$ and $\dim H = \frac{1}{2}d(d+1)$. The result follows.

In the following, we give the structure of the Schur multiplier of a $d$-generator generalized Heisenberg Lie algebra $H$ of rank $\frac{1}{2}d(d-1)$.

Proposition 2.5. Let $H$ be a $d$-generator generalized Heisenberg Lie algebra of rank $\frac{1}{2}d(d-1)$. Then $\mathcal{M}(H) \cong A(l_d(3))$.

Proof. By Proposition 2.4, we have $H \cong \langle x_1, \ldots, x_d, y_{ij} | [x_i, x_j] = y_{ij}, 1 \leq i < j \leq d \rangle$. Using the method of Hardy and Stitzinger in [9], since $\mathcal{M}(H)$ is abelian, we just need to compute $\dim \mathcal{M}(H)$. Start with

\[
[x_i, x_j] = y_{ij} + s_{ij}, 1 \leq i < j \leq d, \\
[y_{ij}, x_i] = z_{ijt}, 1 \leq i < j \leq d \text{ and } 1 \leq t \leq d, \\
[y_{ij}, y_{ik}] = a_{ijk}, 1 \leq i < j \leq d \text{ and } 1 \leq k < t \leq d,
\]

where $s_{ij}, z_{ijt}$, and $a_{ijk}$ generate $\mathcal{M}(H)$. Putting $y'_{ij} = y_{ij} + s_{ij}, 1 \leq i < j \leq d$. A change of variables allows that $s_{ij} = 0$, for all $1 \leq i < j \leq d$. Using of the Jacobi identity, we have $a_{ijk} = [y_{ij}, y_{ik}] = [x_i, x_j, y_{ik}] = [z_{ijt}, y_{ik}] = 0$, for all $1 \leq i < j \leq d$ and $1 \leq k < t \leq d$. We know that $z_{ijt} = y_{ij}t$ for all $1 \leq i < j \leq d$, and $1 \leq t \leq d$ is a simple commutator of the length three. Put $S = \langle z_{ijt} | 1 \leq i < j \leq d \rangle$. Clearly, the set of all simple basic commutators of length three is a basis of $S$. Thus $\mathcal{M}(H) = S$ and $\dim S = l_d(3)$, by using Theorem 1.7. The proof is completed.

We are ready to decide about the capability of a $d$-generator generalized Heisenberg Lie algebra $H$ of rank $\frac{1}{2}d(d-1)$. At first, we need the following proposition.

Proposition 2.6. Let $H$ be a $d$-generator generalized Heisenberg Lie algebra of rank $\frac{1}{2}d(d-1)$ such that $K$ be a one-dimensional subalgebra containing in $H^2$. Then $\mathcal{M}(H/K) \cong A(l_d(3) - d + 1)$.

Proof. By Proposition 2.4, we have $H \cong \langle x_1, \ldots, x_d, y_{ij} | [x_i, x_j] = y_{ij}, 1 \leq i < j \leq d \rangle$. Clearly $K = \langle [x_i, x_j] \rangle$ for all $1 \leq i < j \leq d$. Therefore $H/K \cong \langle x_1 + K, \ldots, x_d + K, y_{lm} + K | [x_i, x_m] + K = y_{lm} + K, 1 \leq l < m \leq d, \text{ and } l \neq i, m \neq j \rangle$. Using the method of Hardy and Stitzinger in [9], we compute $\dim \mathcal{M}(H/K)$. Start with

\[
[x_i, x_m] = y_{lm} + s_{lm}, 1 \leq l < m \leq d \text{ for } l \neq i \text{ and } m \neq j, \\
[x_i, x_j] = s_{ij}, \\
[y_{lm}, x_i] = z_{lm}, 1 \leq l < m \leq d \text{ and } 1 \leq t \leq d \text{ for } l \neq i \text{ and } m \neq j, \\
[y_{lm}, y_{ks}] = a_{lmks}, 1 \leq l < m \leq d \text{ (l \neq i and m \neq j) and } 1 \leq k < t \leq d \\
(k \neq i \text{ and } t \neq j)
\]

where $s_{lm}, s_{ij}, z_{lm},$ and $a_{lmks}$ generate $\mathcal{M}(H/K)$. Putting $y'_{lm} = y_{lm} + s_{lm}, 1 \leq l < m \leq d$, for $l \neq i$ and $m \neq j$. A change of variables allows that $s_{lm} = 0$, for all
1 \leq l < m \leq d(l \neq i \text{ and } m \neq j). \text{ Use of the Jacobi identity, } a_{tmk} = [y_m, y_k] = [x_t, x_m, y_k] = [z_{tm}, y_k] = 0 \text{ for all } 1 \leq l < m \leq d(l \neq i \text{ and } m \neq j) \text{ and } 1 \leq k < t \leq d(k \neq i \text{ and } t \neq j). \text{ Thus the set } \{z_{tm}, s_i | 1 \leq l < m \leq d \text{ and } 1 \leq t \leq d \text{ for } l \neq i \text{ and } m \neq j \} \text{ generates } \mathcal{M}(H/K). \text{ We know that } z_{pq} = [y_p, x_l] \text{ is a simple commutator of length three for all } 1 \leq p < q \leq d \text{ and } 1 \leq t \leq d \text{. Put } S = \langle z_{pq} | 1 \leq p < q \leq d \text{ and } 1 \leq t \leq d \rangle. \text{ By the definition of basic commutators, the generators } x_1, x_2, \ldots, x_d \text{ are basic commutators of length one and ordered by setting } x_i < x_j \text{ if } i < j. \text{ Clearly, the set of all simple basic commutators of the length three is a basis of } S. \text{ Thus } S = \langle [x_j, x_i, x_l] | 1 \leq i < j \leq d \text{ and } 1 \leq t \leq d \rangle \text{ and } \dim S = l_d(3), \text{ by using Theorem } 1.7. \text{ Since } [s_{ij}, x_k] = [x_i, x_j, x_k] = 0, \text{ we have } [x_j, x_i, x_k] \notin \mathcal{M}(H/K) \text{ for all } i \leq k \leq d. \text{ By using Jacobian identity, we have } [x_j, x_r, x_i] = [x_t, x_r, x_j] \text{ for all } 1 \leq r \leq i - 1. \text{ Thus the set } \{[x_j, x_i, x_k] | 1 \leq i < j \leq d \text{ and } 1 \leq t \leq d \} = \{[x_j, x_i, x_k], [x_j, x_r, x_i] | i \leq k \leq d \text{ and } 1 \leq r \leq i - 1\}. \text{ The set } \{s_{ij}\} \text{ is a basis of } \mathcal{M}(H/K) \text{ and so } \dim \mathcal{M}(H/K) = l_d(3) - d + 1, \text{ as required.} \hfill \Box

The following theorem shows all such Lie algebras are capable.

**Theorem 2.7.** Let } H \text{ be a } d \text{-generator generalized Heisenberg Lie algebra of rank } \frac{1}{2}d(d - 1). \text{ Then } H \text{ is capable.}

**Proof.** Propositions 2.5 and 2.6 imply } \dim \mathcal{M}(H) = l_d(3) \text{ and } \dim \mathcal{M}(H/K) = l_d(3) - d + 1 \text{ for every one-dimensional central ideal } K \text{ of } H. \text{ Since } \dim \mathcal{M}(H) > \dim \mathcal{M}(H/K), \text{ the homomorphism } \mathcal{M}(H) \to \mathcal{M}(H/K) \text{ is not monomorphism. Thus the result follows from Lemma 1.4} \hfill \Box

Now we show that the converse of Proposition 2.5 is also true.

**Theorem 2.8.** Let } H \text{ be a } d \text{-generator generalized Heisenberg Lie algebra. Then } \dim H^2 = \frac{1}{2}d(d - 1) \text{ if and only if } \mathcal{M}(H) \cong A(l_d(3)).

**Proof.** Let } \mathcal{M}(H) \cong A(l_d(3)). \text{ By a similar to the proof of Proposition 2.6 we can see that if } \dim H^2 < \frac{1}{2}d(d - 1), \text{ then } \dim \mathcal{M}(H) < l_d(3). \text{ Thus } \dim H^2 = \frac{1}{2}d(d - 1). \text{ The converse holds by Proposition 2.5} \hfill \Box

Recall that a pair of Lie algebras } (K, M) \text{ is said to be a defining pair for } L \text{ if } M \subseteq Z(K) \cap K^2 \text{ and } K/M \cong L. \text{ When } L \text{ is finite-dimensional then the dimension of } K \text{ is bounded. If } (K, M) \text{ is a defining pair for } L, \text{ then a } K \text{ of maximal dimension is called a cover for } L. \text{ Moreover, from } [11][12], \text{ in this case } M \cong \mathcal{M}(L).

**Theorem 2.9.** Let } H \text{ be a } d \text{-generator generalized Heisenberg Lie algebra of rank } \frac{1}{2}d(d - 1). \text{ Then } H^* \text{ is covering of } H \text{ if and only if } H^* \text{ is nilpotent of class } 3, \dim (H^*)^3 = \dim \mathcal{M}(H), Z(H^*) \subseteq (H^*)^2 \text{ and } H^*/(H^*)^3 \cong H.

**Proof.** Let } H^* \text{ be a covering Lie algebra of } H. \text{ Then there exists an ideal } B \text{ of } H^\ast \text{ such that } B \cong \mathcal{M}(H), B \subseteq Z(H^*) \cap (H^*)^2 \text{ and } H^*/B \cong H. \text{ Since } Z(H^*/B) = (H^*/B)^2 = (H^*)^2/B, \text{ we have } Z(H^*) \subseteq (H^*)^2. \text{ We claim that } H^* \text{ is nilpotent of class } 3. \text{ Clearly } H^* \text{ is not abelian since } H \text{ is non-abelian. By contrary, let } H^* \text{ be nilpotent of class } 2. \text{ Since } B \subseteq (H^*)^2 = Z(H^*), \text{ we have } d = \dim H/Z(H) = \dim H/H^2 = \dim(H^*/B)/((H^*)^2) = \dim(H^*/(H^*)^2) = \dim(H^*/Z(H^*)). \text{ By Lemma 2.22, we have } \dim(H^*)^2 \leq \frac{1}{2}d(d - 1). \text{ Now Proposition 2.25 shows } \dim B = \dim \mathcal{M}(H) = l_d(3) \leq \dim(H^*)^2 \leq \frac{1}{2}d(d - 1) = l_d(2). \text{ It is a contradiction. Thus } H^*
is nilpotent of class 3 and so \((H^*)^3 \subseteq B\) and \(\dim(H^*)^2 - \dim B = \dim H^2\). Since \(H^*/(H^*)^3\) is of class 2 and \(\dim((H^*)^2/(H^*)^3)^{ab} = d\), we have \(\dim((H^*)^2/(H^*)^3) \leq \frac{1}{2}d(d-1)\), by Lemma \[2.2\]

Therefore

\[\frac{1}{2}d(d-1) + \dim B - \dim(H^*)^3 = \dim H^2 + \dim B - \dim(H^*)^3 = \dim(H^*)^2 - \dim(H^*)^3 \leq \frac{1}{2}d(d-1).\]

Hence \(\dim B \leq \dim(H^*)^3\) and \((H^*)^3 \subseteq B\). Thus \(B = (H^*)^3\). The converse is clear. \(\square\)

We are going to characterize the structure of the Schur multiplier and tensor square of a Lie algebra \(L\) of class two such that \(\dim(L/Z(L)) = d\) and \(\dim L^2 = \frac{1}{2}d(d-1)\).

**Theorem 2.10.** Let \(L\) be an \(n\)-dimensional Lie algebra of class two such that \(\dim(L/Z(L)) = d\) and \(\dim L^2 = \frac{1}{2}d(d-1)\). Then \(L = H \oplus A(t)\) for some \(t \geq 0\) such that \(H \cong \langle x_1, \ldots, x_d, y_i \mid [x_i, x_j] = y_{ij}, 1 \leq i < j \leq d\rangle\), \(Z(H) = H^2 = L^2 \cong A(\frac{1}{2}d(d-1))\), \(n = \frac{1}{2}d(d+1) + t\), and \(\mathcal{M}(L) \cong A(l_d(3) + l_t(2) + dt)\).

**Proof.** Propositions \[1.6\] and \[2.3\] imply \(L = H \oplus A(t)\), where \(H \cong \langle x_1, \ldots, x_d, y_i \mid [x_i, x_j] = y_{ij}, 1 \leq i < j \leq d\rangle\) and \(Z(H) = H^2 = L^2 \cong A(\frac{1}{2}d(d-1))\) for some \(t \geq 0\).

By using Lemma \[1.1\] (i), Theorem \[1.2\] and Proposition \[2.6\] we have

\[
\dim \mathcal{M}(L) = \dim \mathcal{M}(H) + \dim \mathcal{M}(A(t)) + \dim((H)^{ab} \otimes_{mod} A(t)) = l_d(3) + \frac{1}{2}t(t-1) + dt = l_d(3) + l_t(2) + dt.
\]

Hence the result follows. \(\square\)

**Corollary 2.11.** Let \(L\) be an \(n\)-dimensional Lie algebra of class two such that \(\dim(L/Z(L)) = d\) and \(\dim L^2 = \frac{1}{2}d(d-1)\). Then \(L \wedge L \cong \mathcal{M}(L) \oplus L^2 \cong A(l_d(3) + l_t(2) + dt + l_d(2))\) such that \(n = \frac{1}{2}d(d+1) + t\) for some \(t \geq 0\).

**Proof.** Theorem \[2.10\] implies that \(L \cong \langle x_1, \ldots, x_d, y_i \mid [x_i, x_j] = y_{ij}, 1 \leq i < j \leq d\rangle \oplus A(t)\) for some \(t \geq 0\). Since

\([l_1 \wedge l_2, l_3 \wedge l_4] = [l_1, l_2] \wedge [l_3, l_4] = l_1 \wedge [l_2, [l_3, l_4]] - l_2 \wedge [l_1, [l_4, l_3]],\]

for all \(l_1, l_2, l_3, l_4 \in L\), we have \((L \wedge L)^2 = 0\). Thus \(L \wedge L \cong \mathcal{M}(L) \oplus L^2 \cong A(l_d(3) + l_t(2) + dt + l_d(2))\) such that \(n = \frac{1}{2}d(d+1) + t\) for some \(t \geq 0\), by using Lemma \[1.3\] and Theorem \[2.10\]. \(\square\)

Recall that \(L \boxtimes L \cong \langle l \otimes l \mid l \in L \rangle\). Using \[13\] Theorem \[2.5\], we have

**Corollary 2.12.** Let \(L\) be an \(n\)-dimensional Lie algebra of class two such that \(\dim(L/Z(L)) = d\) and \(\dim L^2 = \frac{1}{2}d(d-1)\). Then

\[
L \otimes L \cong \mathcal{M}(L) \oplus L^2 \oplus L \boxtimes L \cong \mathcal{M}(L) \oplus L^2 \oplus (L/L^2 \boxtimes L/L^2) \cong A(l_d(3) + l_t(2) + dt + l_d(2) + \frac{1}{2}(d+t)(d+t+1))
\]

for some \(t \geq 0\).
Proof. By Corollary 2.11 [13, Lemmas 2.2 and 2.3], we have $L/L^2 \cong A(\frac{1}{2}(d+t)(d+t+1))$ and so

$$L \otimes L \cong A(l_d(3) + l_t(2) + dt) \oplus A(\frac{1}{2}d(d-1)) \oplus A(\frac{1}{2}(d+t)(d+t+1))$$

$$\cong A(l_d(3) + l_t(2) + dt + l_d(2) + \frac{1}{2}(d+t)(d+t+1)),$$

as required. □

Corollary 2.13. Let $L$ be an $n$-dimensional Lie algebra of class two such that $\dim(L/Z(L)) = d$ and $\dim L^2 = \frac{1}{2}d(d - 1)$. Then $L$ is capable.

Proof. The result follows from Proposition 1.6, Theorems 2.7 and 2.10. □

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