Approximate analytical solution of the Graetz problem

A V Eremin

Samara State Technical University, Molodogvardeiskaya street 244, Samara, Russia, 443100

Abstract. The study on velocity and temperature distribution in fluid flows is important both for the theory and practical applications. The design of efficient heat-exchange equipment, the development of heat and thermomechanical modes of product treatment, the determination of heat losses in the pipeline systems include the need to determine the velocity and temperature fields in fluid flows. The key aspects of the method have been considered with help of using the example of solving the Graetz problem for parallel and cylindrical channels.

It is shown that finding the solution to a partial differential equation with respect to the temperature function can be reduced to integrating an ordinary differential equation with respect to the new unknown function \( q(\eta) \) which is the law of temperature change in the center of the channel. The combined use of the heat balance integral method and additional boundary characteristics made it possible to obtain a simple in form analytical solution to the problem under consideration. It is noted that the accuracy of the solutions obtained depends on the number of approximations used, i.e. the number of terms of the approximating series.

When using only one term, i.e. already in the first approximation, the relative error of the method is not more than 8 percent in the range of the longitudinal coordinate change \( 0.1 \leq \eta < \infty \) and decreases to 4 percent in the second approximation. The analytical form of the solutions obtained provides analyzing the isotherms fields, calculating the average temperature, the Nusselt number, etc.

1. Introduction

Currently, heat exchangers using the incompressible fluid as a heat carrier have been widely spread. Such heat exchangers can be used both for heating and for cooling of fluid flows. It is widely used in heating, cooling, air conditioning, at petrochemical enterprises, in solar energy, etc. The exchangers have various design features depending on the purpose. However, in most cases, the channels through which the fluid moves have simple shapes that allows applying the analytical methods to describe the heat transfer in them (parallel and cylindrical channels, cylindrical annular, tubes). Such heat exchangers include, for example, double-pass heat exchangers [1], plate heat exchangers, vacuum tube solar collector [2], microchannel heat sinks [3] etc.

The mathematical description of heat and mass transfer processes in fluid flows includes classical laws of fluid mechanics such as the Navier – Stokes equations, the continuity equation, and the energy equation [4], [5]. Supplemented by the dependences of fluid physical properties on temperature and pressure, these equations form a closed system of equations describing convective heat exchange and fluid dynamics. The use of accurate analytical methods for solving
these problems is possible only in some simple cases. For example, when solving linear boundary value problems for plane bodies and bodies with central axial symmetry, the variable separation method (Fourier), Green’s functions method [6, 7] and integral transform methods with finite and infinite limits of integration (Hankel, Laplace, Legendre transformations, etc.) [8, 9] are used. Such solutions, as a rule, are expressed by complex analytical dependencies, infinite series containing special functions that significantly limit their practical use.

Currently, numerical methods for studying the processes of heat and mass transfer in fluid and gas flows are widely used [10, 11, 12, 13]. Modern software products make it possible to build meshes, solve systems of linear equations and provide a wide range of tools for analyzing the results obtained automatically. However, analytical solutions have some significant advantages over numerical ones. For example, the solutions obtained in an analytical form provide performing parametric analysis of the system under study, parametric identification, setup and programming of measuring devices, planning of control actions in production processes, etc [14]. Thereby, approximate analytical methods for mathematical modeling of transfer processes in fluid flows such as various modifications of the heat balance integral method [15, 16, 17, 18], Ritz method [19, 20, 21], Kantorovich method [22], Galerkin method [23, 24, 25, 26] etc. have rapid development.

This article presents the development results of an approximate analytical method for solving one-dimensional heat and mass transfer problems which provide obtaining simple in form solutions with accuracy sufficient for engineering applications. Using the example of solving the Graetz problem [27, 28, 29, 30], it was shown that already in the second approximation the solution error does not exceed 4 percent.

2. Formulation of the problem

As a specific example of the use of this method, consider the heat transfer problem in a stabilized incompressible fluid flow moving in a parallel channel of width \(2h\) (see Fig. 1). To derive the differential equation describing this process, assume the following: 1) the flow is steady-state and stabilized; 2) the fluid is incompressible; 3) the thermophysical properties are constant; 4) the fluid flow model is laminar; 5) internal heat sources, as well as energy dissipation, are not taken into account. Under these assumptions, the process of non-stationary heat transfer is described by the Navier - Stokes equations in combination with the energy equation and the continuity equation

\[
\frac{\partial T}{\partial \tau} + \mathbf{\omega} \cdot \nabla T = \alpha \nabla^2 T, \tag{1}
\]

\[
\rho \frac{\partial \mathbf{\omega}}{\partial t} + \rho \mathbf{\omega} \cdot \nabla \mathbf{\omega} = -\nabla P + \mu \nabla^2 \mathbf{\omega}, \tag{2}
\]

\[
\text{div} \mathbf{\omega} = 0. \tag{3}
\]

It is known that the heat exchange process with constant physical properties of a fluid does not effect on the fluid flow. In this case, the fluid moves as if the flow were isothermal.

The coordinate system was introduced as it shown in 1. In this case, equation (2) with regard to (3) is as follows

\[
\frac{d^2 \omega_x(y)}{dy^2} = -\frac{\Delta p}{\mu l},
\]

where \(x, y\) – longitudinal and transverse spatial coordinates; \(\omega_x\) – velocity vector projection on the \(Ox\) axis; \(\Delta p/l = \text{const}\) – pressure drop in the channel section of length \(l\); \(\mu\) – dynamic viscosity.
The well-known law of velocity distribution in a parallel channel was obtained with help of using the solution of the motion equation and considering the fact that the flow velocity on the channel surface is zero \( \omega_x(h) = \omega_x(-h) = 0 \) \[4, 5\].

\[
\omega_x(y) = \frac{\Delta p h^2}{2\mu l} \left[1 - \left(\frac{y}{h}\right)^2\right] = \frac{3}{2} \omega_{av} \left[1 - \left(\frac{y}{h}\right)^2\right],
\]

where \( \omega_{av} = \frac{1}{3} \frac{\Delta p h^2}{\mu l} \) - average flow rate.

The energy equation (1) with regard to the found velocity profile is as follows

\[
\frac{3}{2} \omega_{av} \left[1 - \left(\frac{y}{h}\right)^2\right] \frac{\partial T(x,y)}{\partial x} = \frac{\partial^2 T(x,y)}{\partial y^2}.
\]

The problem (4), (5) can be presented in a non-dimensional form. To do this, introduce the following non-dimensional parameters:

\[
\Theta = \frac{T - T_{in}}{T_{in}}; \xi = \frac{y}{h}; \eta = \frac{2ax}{3h^2 \omega_{av}}; A = \frac{3Bh^2 \omega_{av}}{2T_{in}a},
\]
where $\Theta$ – non-dimensional temperature; $\xi, \eta$ – non-dimensional transverse and longitudinal coordinates. Problem (4), (5) with respect to the introduced notation is as follows

$$(1 - \xi^2) \frac{\partial \Theta(\eta, \xi)}{\partial \eta} = \frac{\partial^2 \Theta(\eta, \xi)}{\partial \xi^2};$$

$$\Theta(0, \xi) = 0;$$

$$\Theta(\eta, 1) = A\eta;$$

$$\frac{\partial \Theta(\eta, 0)}{\partial \xi} = 0.$$

3. Method of solution

3.1. First approximation

According to the method developed a new unknown function - the law of temperature change in the center of a parallel channel along its length was introduced

$$q(\eta) = \Theta(\eta, 0).$$

The problem solution should be found in the form of an algebraic or trigonometric series. To solve the problem (6) - (9), use a trigonometric series

$$\Theta(\eta, \xi) = A\eta + \sum_{k=1}^{N} \cos \left[ \frac{r\pi}{2} \xi \right] b_k(\eta),$$

where $r = 2k - 1$; $b_k(\eta)$ – unknown coefficients.

Due to the chosen system of coordinate functions, the boundary conditions (8) and (9) are satisfied at any values of the longitudinal coordinate $\eta$.

To obtain a solution to problem (6) - (9) in a first approximation, we require that relation (11) satisfies the additional boundary condition (10). Note that an approximation number is the number of terms under the sum sign in relation (11).

The condition (10) introduction doesn’t change the initial problem statement because the temperature $\Theta(\eta, 0)$ on the axis of the channel is the new value. It is also worth noting that the function $q(x)$ is not equal to 0 at any value $x > 0$. This is due to the fact that the derivation of the energy equation is based on the Fourier hypothesis, according to which a change in the temperature gradient leads to the instantaneous heat flux. It was shown in [30, 31] that this assumption results in the fact that the rate of heat transfer is infinite.

To obtain a solution in the first approximation ($N = 1$), substitute (11) into (10)

$$q(\eta) = A\eta + b_1(\eta).$$

Expressing $b_1(\eta)$ using (12) and substituting into (11), we obtain

$$\Theta(\eta, \xi) = A\eta + \cos \frac{\pi}{2} \xi (q(\eta) - A\eta).$$

To determine the unknown function $q(\eta)$, we require that relation (13) satisfies not the initial differential equation but the integral equation – the heat balance integral [15, 16, 17, 18, 31]

$$1 \int_0^1 (1 - \xi^2) \frac{\partial \Theta(\eta, \xi)}{\partial \eta} d\xi = \int_0^1 \frac{\partial^2 \Theta(\eta, \xi)}{\partial \xi^2} d\xi.$$
Calculating the integral (14), we obtain the following differential equation

\[
\frac{\partial q(\eta)}{\partial \eta} + \frac{\pi^4}{32} q(\eta) - \frac{\pi^4 A}{32} \eta + \frac{\pi^3 A}{24} - A = 0,
\]

so, the general solution is written as

\[
q(\eta) = C_1 e^{-\frac{\pi^4 A}{32} \eta} + A \left( \eta - \frac{4}{3} \pi \right).
\] (15)

Substituting (15) into (13), we obtain

\[
\Theta(\eta, \xi) = \cos\left(\frac{\pi \xi}{2}\right) \left( C_1 e^{-\frac{\pi^4 A}{32} \eta} - \frac{4A}{3\pi} \right) + A \eta.
\] (16)

Relation (16) exactly satisfies the boundary conditions (8), (9), additional condition (10), and also the heat balance integral (averaged over the range of the transverse coordinate to equation (6)). To fulfill the boundary condition (7), we find its residual and require the orthogonality of the residual to the coordinate function, i.e.

\[
\int_0^1 \Theta(0, \xi) \cos\left(\frac{\pi \xi}{2}\right) d\xi = 0.
\] (17)

Using solution (17) we can find \( C_1 = \frac{4A}{3\pi} \). Taking into account the found integration constant, the solution of problem (6) – (9) in the first approximation can be written as

\[
\Theta(\eta, \xi) = A \eta + \frac{4A}{3\pi} \left( e^{-\frac{\pi^4 A}{32} \eta} - 1 \right) \cos\left(\frac{\pi \xi}{2}\right).
\] (18)

The results of temperature calculations using formula (18) in comparison with a numerical solution are shown in Fig. 2. Their analysis shows that already in the first approximation the relative error of the method is not more than 8 percent. The true value of the temperature in determining the resulting error was the value obtained by a numerical calculation. An algorithm for the solution of the problem (6) - (9) is presented in the appendix.

3.2. The second approximation

To increase the accuracy of the solutions obtained, it is necessary to increase the number of terms of the series (11), here the number of unknown coefficients \( b_k(\eta) \) will increase. To determine them (in addition to condition (10)) we use additional boundary conditions (characteristics). To obtain them, the initial differential equation is repeatedly differentiated with respect to the points \( \xi = 0 \) and \( \xi = 1 \). It was shown in [32] that the satisfaction of the additional boundary conditions obtained in this way is equivalent to the fulfillment of the initial differential equation at the boundary points. In addition, it was noted in [32, 33] that the use of additional boundary characteristics with an increase in the number of approximations leads to the fulfillment of the equation inside the area as well.

To obtain the second additional boundary condition (the first one is relation (10)), write equation (6) with regard to the point \( \xi = 0 \)

\[
\frac{\partial \Theta(\eta, 0)}{\partial \eta} = \frac{\partial^2 \Theta(\eta, 0)}{\partial \xi^2}.
\] (19)
Relation (19) with respect to (10) can be written as
\[ \frac{dq(\eta)}{d\eta} = \frac{\partial^2 \Theta(\eta, 0)}{\partial \xi^2}. \]  
(20)

We obtain one more additional condition by writing equation (6) at the point \( \xi = 1 \)
\[ \frac{\partial^2 \Theta(\eta, 1)}{\partial \xi^2} = 0. \]

To obtain the following additional boundary characteristics, we differentiate the initial differential equation (6) with respect to the variable \( \xi \), as a result we have
\[ -2\xi \frac{\partial \Theta(\eta, \xi)}{\partial \eta} + (1 - \xi^2) \frac{\partial^2 \Theta(\eta, \xi)}{\partial \xi \partial \eta} = \frac{\partial^3 \Theta(\eta, \xi)}{\partial \xi^3}. \]  
(21)

We can now write relation (21) with regard to the point \( \xi = 0 \)
\[ \frac{\partial^2 \Theta(\eta, 0)}{\partial \xi \partial \eta} = \frac{\partial^3 \Theta(\eta, 0)}{\partial \xi^3}. \]  
(22)

Relation (22) with respect to (9) is the third additional boundary condition
\[ \frac{\partial^3 \Theta(\eta, 0)}{\partial \xi^3} = 0. \]

Writing relation (21) at a point \( \xi = 1 \), we can obtain the fourth additional boundary condition
\[ -2A = \frac{\partial^3 \Theta(\eta, 1)}{\partial \xi^3}. \]
Additional boundary characteristics can also be obtained for subsequent approximations by this method. To obtain a solution to problem (6) - (9) in the second approximation ($N = 2$), we use two additional conditions. To do this, substitute (11) into (10) and (20)

$$\{ b_1(\eta) + b_2(\eta) - q(\eta) + A\eta = 0, \frac{\partial q(\eta)}{\partial (\eta)} + \frac{\pi^2}{4}[b_1(\eta) + 9b_2(\eta)] = 0. \tag{23}$$

The solution of the system of algebraic equations is used to determine the unknown coefficients $b_1(\eta)$ and $b_2(\eta)$

$$b_1(\eta) = -\frac{9}{8}[A\eta - q(\eta)] + \frac{1}{2\pi^2} \frac{dq(\eta)}{d\eta}, b_2(\eta) = \frac{1}{8}[A\eta - q(\eta)] + \frac{1}{2\pi^2} \frac{dq(\eta)}{d\eta}. \tag{24}$$

To determine the unknown function $q(\eta)$, as in the first approximation, we require the unknown solution (11) to satisfy the heat balance integral (14). Calculating (14), we obtain an ordinary second-order differential equation with respect to the unknown function $q(\eta)$

$$0.02711 \frac{d^2 q(\eta)}{d\eta^2} + 0.90123 \frac{dq(\eta)}{d\eta} + 2.3562(q(\eta) - A\eta) + 0.08375 = 0. \tag{25}$$

Its solution is as follows

$$q(\eta) = C_1 e^{-30.383\eta} + C_2 e^{-2.8606\eta} + (\eta - 0.41804)A. \tag{26}$$

The integration constants can be obtained by finding the residual of the boundary condition (7) and the requirement of the orthogonality of the residual to all coordinate functions, i.e.

$$\int_0^1 \Theta(0, \xi) \cos \left( \frac{r\pi\xi}{2} \right) d\xi = 0 (r = 2k - 1; k = \overline{1, N}). \tag{27}$$

For the second approximation ($N = 2$), relation (27) is a system of two algebraic equations which are used to find

$$C_1 = -0.0071153A, C_2 = 0.42515A. \tag{28}$$

Substituting (24), (26) and (28) into (11), we obtain the unknown temperature distribution function in the parallel channel in the second approximation.

The calculation results of the non-dimensional temperature are presented in Fig. 2. A significant increase in accuracy is observed in comparison with the first approximation. So, the error in calculating the temperature in the given range of the variable $\eta$ changes is not more than 4 percent. To further increase the accuracy, it is necessary to use more terms of the series (11), and additional boundary characteristics are used as well.

### 3.3. Determination of heat transfer coefficient

The average mass temperature can be found by formula

$$\overline{T}(x) = \frac{\int_0^h \omega_\parallel(y)T(x, y)dy}{\int_0^h \omega_\parallel(y)dy} = \frac{1}{h\omega_\parallel} \int_0^h \omega_\parallel(y)T(x, y)dy. \tag{29}$$
Relation (29) can be reduced to a non-dimensional formula

$$\Theta(\eta) = \frac{\bar{T}(x) - T_m}{T_m} = \frac{3}{2} \int_0^1 \Theta(\eta, \xi)(1 - \xi^2)d\xi.$$  \hspace{1cm} (30)

Nusselt criterion is determined according to the relation

$$Nu = \frac{2ah}{\lambda} = -2(\Theta(\eta) - A\eta)^{-1}\frac{\partial \Theta(\eta, \xi)}{\partial \xi}.$$  

The limit value of the Nusselt number $x \to \infty$ in the second approximation was $Nu_\infty = 4.104$. Graphs of changes in the Nusselt criterion and average mass temperature along the channel length are shown in Fig. 3.

3.4. Analysis of isotherms
The temperature distribution functions obtained according to the above method have a simple analytical form that makes it possible to study the heat transfer process in the fields of isotherms. The law of distribution of isotherms in the first approximation can be obtained from relation (18) by expressing the variable $\xi$ as a function of temperature and longitudinal coordinate

$$\xi(\Theta, \eta) = 2(1 - \pi^{-1} arccos(0.75A^{-1}\pi[A\eta - \Theta][exp(-3.044\eta) - 1]^{-1}) \hspace{1cm} (31)$$

The results of building isotherms using formula (29) are presented in Fig. 4. The analysis shows that isotherms occur on the surface of the body (at a point $\xi = 0$) and propagate along the longitudinal coordinate $\eta$. Note, that for each isotherm it is possible to determine two characteristic coordinates - the coordinates in which it appears and disappears.
4. Results and Discussion

The main advantage of this method is the possibility of solving the equations with non-separable variables [34] with the temperature dependence of the thermophysical properties of substances, under boundary conditions of the second and third kind, for channels of various shapes. For example, for a cylindrical channel the heat transfer process is described by the following differential equation

\[ (1 - \xi^2) \frac{\partial \Theta(\eta, \xi)}{\partial \eta} = \frac{1}{\xi} \frac{\partial \Theta(\eta, \xi)}{\partial \xi} + \frac{\partial^2 \Theta(\eta, \xi)}{\partial \xi^2}. \]

This problem is called the Graetz problem. Its exact solution is expressed in an infinite series in [35]. Using the method considered in the article, a simple in form approximate solution of the Graetz problem can be found. Under the boundary conditions (7) - (9) in the first approximation, it is as follows

\[ \Theta(\eta, \xi) = A[\eta - 0.125(1 - e^{-12\eta})(1 - \xi^2)]. \]

Thus, the proposed method is recommended for obtaining the analytical solutions that are simple in form, in cases where an error (3 percent) can be considered as satisfactory. It is worth noting that this method can be used to obtain high-precision solutions, however, in this case, the formula of the resulting solution is rather complicated, and the amount of computational work increases significantly.

5. Conclusion

Based on the applied additional boundary conditions and an additional unknown function, the approximate solution to the heat transfer problem in a parallel channel has been obtained. It is shown that the problem of finding a solution to a partial differential equation with respect to the temperature function can be reduced to integrating an ordinary differential equation with respect to the new unknown function. The error in determining the temperature, in this case, is found by the number of performed approximations and is no more than 3 percent already in the second approximation. The obtained results can be useful to designers and operators of heat exchange equipment.
Appendix A. Supplementary data
Supplementary data associated with this article can be found, in the online version, at https://data.mendeley.com/datasets/7mj2d3zw6n/1.

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