FREDHOLM DETERMINANTS IN THE MULTI-PARTICLE HOPPING ASYMMETRIC DIFFUSION MODEL

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Abstract. In this paper we treat the multi-particle hopping asymmetric diffusion model with the initial configuration \((0,0, \cdots)\). For the multi-particle hopping asymmetric diffusion model with two free parameters, we find a Fredholm determinant representation for the probability distribution of the \(m\)th left-most particle’s position at time \(t\). For the one-parameter MADM with partial asymmetry we show that the Tracy-Widom distribution, \(F_2(s)\), arises in the regime where \(m, t \to \infty\) with \(m/t\) fixed in \((0,1)\).

1. Introduction and Statement of results

The coordinate Bethe Ansatz has been a quite useful tool to study a certain class of interacting particle systems on the one-dimensional integer lattice \(\mathbb{Z}\). It provides a direct way to find transition probabilities of the finite systems \([8,10,19,21]\). For the coordinate Bethe Ansatz to be applicable to an interacting particle system on \(\mathbb{Z}\), the model should meet the \textit{two-particle reducibility} \([7]\). For some interacting particle systems generally defined on \(\mathbb{Z}\), it may be possible to impose a certain condition on the model so that the model meets the \textit{two-particle reducibility}. These conditions are usually for the parameters of exponential clocks in the system. The multi-particle hopping asymmetric diffusion model (MADM) \([18]\), the asymmetric exclusion process with pushing dynamics in both directions \([1]\), the \(q\)-totally asymmetric zero range process \((q\text{-TAZRP})\) \([13,17]\), the asymmetric avalanche process \([14]\), the \((q,\mu,\nu)\)-Boson process \([15]\), the \((q,\mu,\nu)\)-TASEP \([5]\) and the \(q\)-PushASEP \([6]\) are some examples of the models to which the coordinate Bethe Ansatz is applicable.

For the asymmetric simple exclusion process (ASEP) with the step initial condition, Tracy and Widom obtained the probability distribution of the random variable \(x_m(t)\) representing the \(m\)th left-most particle’s position at time \(t\) \([21]\). This probability distribution is described by a contour integral of which integrand includes a Fredholm determinant \([22]\). Also, it was shown that the Tracy-Widom distribution, \(F_2\) \([20]\), arises in the regime where \(m\) and \(t\) go to infinity with \(m/t\) fixed in \((0,1)\) \([23]\). A Fredholm determinant representation in the wider range of models was systematically studied by Borodin and Corwin \([3]\). Indeed, through
Borodin and Corwin’s approaches, Tracy and Widom’s Fredholm determinant formula in the ASEP with the step initial condition was rediscovered and Fredholm determinant formulas for some other models were obtained [3–6].

The dynamics of the MADM considered in this paper is two-sided in the sense that particles jump to the right and to the left. If we restrict the model to be one-sided, that is, particles jump only in one direction, then the model becomes a special case of \((q, \mu, \nu)\)-TASEP [5,15]. In the \((q, \mu, \nu)\)-TASEP that also generalizes the \(q\)-TASEP, particles jump only in one direction with special rules. So, the MADM considered in this paper is not a special case of the \((q, \mu, \nu)\)-TASEP. To the best of the author’s knowledge, whether or not it is possible to extend the \((q, \mu, \nu)\)-TASEP to a model with two-sided dynamics is not known.

The main result of this paper is that the Tracy-Widom distribution arises in the MADM when initially infinitely many particles occupy a single site and all other sites are empty. (See [25] for the asymptotics in the \((q, \mu, \nu)\)-TASEP with the step initial condition.) In this paper, we mostly rely on Tracy and Widom’s program established for the ASEP [22–24]. Although the rules of the dynamics of the ASEP and the MADM are different, interestingly, their formulas are very similar. It could be alluded from the author’s earlier work in [10] that the required calculations for the asymptotics of the MADM would be similar to those of the ASEP. Indeed, what we want to present in this paper is that Tracy and Widom’s fine calculations well suited for the ASEP can be used in the MADM by slightly modifying their methodology. For this reason, we will follow the notations used in Tracy and Widom’s papers as much as possible.

1. Definition of the MADM.

1.1. Finite systems. The finite MADM is an interacting particle system on the one-dimensional integer lattice introduced by Sasamoto and Wadati [18]. The rules for the MADM with \(N\) particles are as follows.

\( (i) \) A site \( x \in \mathbb{Z} \) can be occupied up to \( N \) particles, that is, the state of \( x \) is \( n \in \{0, 1, 2, \cdots, N\} \) and each site is equipped with \( 2N \) exponential clocks with rates \( R_1, \cdots, R_N \) and \( L_1, \cdots, L_N \).

\( (ii) \) For \( n \leq n \), if the exponential clock with the rate \( R_n \) alarms at \( x \) in the state \( n \), then \( n \) particles among \( n \) particles at \( x \) jump to \( x + 1 \) all together, and if the exponential clock with the rate \( L_n \) alarms at \( x \) in the state \( n \), then \( n \) particles among \( n \) particles at \( x \) jump to \( x - 1 \) all together.
(iii) If \( n > n \) and \( x \) is in the state \( n \), then the particles at \( x \) do not respond to the exponential clocks with the rates \( R_n \) and \( L_n \).

The rates \( R_n \) and \( L_n \) are given by

\[
R_n = u \frac{1 - \frac{v}{u}}{1 - \left( \frac{v}{u} \right)^n} = \frac{u}{\lfloor n \rfloor u} = \frac{u}{\lfloor n \rfloor \tau}
\]  

(1.1)

and

\[
L_n = v \frac{1 - \frac{u}{v}}{1 - \left( \frac{u}{v} \right)^n} = \frac{v}{\lfloor n \rfloor v} = \frac{v}{\lfloor n \rfloor \tau^{-1}}
\]  

(1.2)

where \( u + v = 1 \), \( u, v \neq 0 \) and \( \tau = v/u \). These conditions are essential for the integrability of the model. In this paper we consider the system with partial asymmetry \( u > v > 0 \).

1.2. Infinite systems. Ultimately, we will be interested in an infinite system of countably many particles with a special initial configuration. The existence of the Markov dynamics of the infinite MADM is not obvious at first glance and we will not discuss the existence of the infinite system in this paper. (See [11] for the existence of the infinite system of the models with exclusion property.) Instead, for the purpose of this paper, we formally define the infinite system by extending (1.1), (1.2), and the state space of each site. Additionally, we impose a condition on the order of particles because we will denote a configuration of the system by \( X = (x_1, x_2, \cdots) \) with \( x_1 \leq x_2 \leq \cdots \). The rules for the infinite system are as follows.

\((i')\) The state of a site \( x \) is \( n \in \{0, 1, 2, \cdots \} \cup \{\infty\} \), which implies that \( n \) particles are stacked upward at \( x \), and each site is equipped with countably many exponential clocks with rates \( R_n \) and \( L_n \) for each \( n \in \{1, 2, \cdots \} \cup \{\infty\} \). The rates \( R_n \) and \( L_n \) are given by (1.1) and (1.2), respectively, and \( R_\infty = \lim_{n \to \infty} R_n \) and \( L_\infty = \lim_{n \to \infty} L_n = 0 \). Here, the state \( \infty \) implies that infinitely many particles are stacked upward.

\((ii')\) At most one site can have the state \( \infty \). If \( x \) is in the state \( \infty \), then no site on the right of \( x \) is allowed to be occupied by a particle.

\((iii')\) (Jump to the right) For \( n \leq n \), if the exponential clock with the rate \( R_n \) alarms at \( x \) in the finite state \( n \), then a stack of \( n \) particles from the top jumps to \( x + 1 \) and they occupy the bottom portion at \( x + 1 \) by pushing upward the particles that already existed at \( x + 1 \). If \( n = \infty \), then particles do not respond to any exponential clock of the rate \( R_n \) with finite \( n \), but when the exponential clock of the rate \( R_\infty \) alarms, all the particles at \( x \) jump to \( x + 1 \).

\((iv')\) (Jump to the left) For \( n \leq n \), if the exponential clock with the rate \( L_n \) alarms at \( x \), then a stack of \( n \) particles from the bottom jumps to \( x - 1 \) and they are stacked on the particles that already existed at \( x - 1 \).
(v') If \( n > n \) and \( x \) is in the state \( n \), then the particles at \( x \) do not respond to the exponential clocks with the rates \( R_n \) and \( L_n \). When a stack of particles jumps to the right or to the left, there is no change in the order of particles in the stack.

2. Statement of Results. The first result is the Fredholm determinant representation in the MADM with two free parameters. Although the original definition of the MADM by Sasamoto and Wadati has one free parameter \( u \) for the system, defined by (1.1) and (1.2), it is possible to extend Sasamoto and Wadati’s model to the model with two free parameters [1, 10]. In this case (1.1) and (1.2) are generalized to

\[
R_n = p \frac{1 - \frac{v}{n}}{1 - \left( \frac{v}{n} \right)^n} = \frac{p}{[n]_{\tau}} = \frac{p}{[n]_{\tau}}
\]

and

\[
L_n = q \frac{1 - \frac{u}{n}}{1 - \left( \frac{u}{n} \right)^n} = \frac{q}{[n]_{\tau}} = \frac{q}{[n]_{\tau-1}}
\]

where \( p + q = 1 \). Here, \( p \) describes the asymmetry for noninteracting particles and \( u \) describes the asymmetry for interacting particles. We will call the MADM with (1.3) and (1.4) the two-parameter MADM and the MADM with (1.1) and (1.2) the one-parameter MADM. In the two-parameter MADM, if \( p = 1 \), then the model becomes a special case of the \((q, \mu, \nu)\)-TASEP [5]. Recall that \( u > v > 0 \), \( u + v = 1 \), and \( \tau = v/u \). Let \( \gamma = u - v \) and

\[
K_{x,t}(\xi, \xi') = v \frac{\xi^v e^{\varepsilon(\xi)/\gamma}}{u + v \xi - \xi} \frac{\tau^{\xi'} - 1}{1 - \tau}
\]

where

\[
\varepsilon(\xi') = \frac{p}{\xi'} + q \xi' - 1.
\]

It can be shown that \( K_{x,t}(\xi, \xi') \) is analytic on \( C_R \times C_R \) where \( 1 < R < \tau^{-1} \), so we define a trace-class operator \( K_{x,t} \) on \( L^2(C_R) \) by

\[
K_{x,t} f(\xi) = \int_{C_R} K_{x,t}(\xi, \xi') f(\xi') d\xi'.
\]

We consider an initial configuration such that infinitely many particles are at the origin and all other sites are empty at time \( t = 0 \). We denote this initial configuration by \((0, 0, \cdots)\). Due to the rules for the infinite system, we may label particles from the left-most one. Let \( x_m(t) \) be the random variable for the position of the \( m \)th left-most particle at time \( t \). Let \( \mathbb{P} \) be the probability measure of the system with the initial configuration \((0, 0, \cdots)\).

**Proposition 1.1.** In the two-parameter MADM with the initial configuration \((0, 0, \cdots)\),

\[
\mathbb{P}(x_m(t/\gamma) \leq x) = \int_C \frac{\det(I - \lambda K_{x,t}) d\lambda}{\prod_{i=1}^{m} (1 - \lambda \tau^i) \cdot \lambda}
\]

where the contour \( C \) is a circle centered at 0 of radius larger than \( \tau^{-m} \).
Proposition 1.1 is derived from the formula $P_Y(x_m(t) = x)$ for the finite two-parameter MADM with an initial configuration $Y = (y_1, \cdots, y_N)$, where $P_Y$ is the probability measure of the system with the initial configuration $Y$. This formula is found in Theorem 3 in [10]. In the ASEP, Tracy and Widom reformulated the formula corresponding to (1.7) into a formula suitable for a saddle point analysis from which they derived the Tracy-Widom distribution [23]. Later, the weakly asymmetric limit of Tracy and Widom’s formula in the ASEP was studied [2, 16]. The corresponding formula in the one-parameter MADM for a saddle point analysis is given in the following theorem.

**Theorem 1.2.** Let
\[
f(\mu, z) = \sum_{k=-\infty}^{\infty} \frac{\tau^k}{1 - \tau^k \mu} z^k,
\]
\[
\Lambda_{t,m,x}(\zeta) = x \log(1 - \zeta) + \frac{\zeta}{1 - \zeta} t + m \log \zeta,
\]
and $J_{t,m,x,\mu}$ be an operator with kernel $J(\eta, \eta')$ defined by
\[
J(\eta, \eta') = \int_{C_\zeta} \exp \left( \Lambda_{t,m,x}(\zeta) - \Lambda_{t,m,x}(\eta') \right) \prod_{n=0}^{\infty} \frac{\tau^{-1} - \tau^n \eta \bar{f}(\mu, \zeta/\eta')}{\tau^{-1} - \tau^n \zeta \eta' (\zeta - \eta')} \, d\zeta
\]
acting on a circle centered at 0 of radius $r \in (\tau, 1)$ where $C_\zeta$ is a circle centered at 0 with radius $r' \in (1, r/\tau)$. In the one-parameter MADM with the initial configuration $(0, 0, \cdots)$,
\[
\mathbb{P}(x_m(t/\gamma) \leq x) = \int_{C_2} \prod_{k=1}^{\infty} (1 - \mu \tau^k) \cdot \det(I + \mu J_{t,m,x,\mu}) \frac{d\mu}{\mu}
\]
where $C_2$ is a circle centered at 0 with radius larger than 1 but not equal to any $\tau^{-k}, k \geq 0$.

We note that (1.8) and (1.9) are very similar to the ASEP’s counterparts. There is an extra factor of infinite products in (1.9), which is not found in the ASEP’s formula. However, a good thing about these infinite products is that all the singularities coming from these infinite products are outside the contours so they do not affect the contour deformation if the deformation is made inside the circle centered at 0 of radius $\tau^{-1}$. The following theorem states that if the random variable $x_m(t/\gamma)$ for fixed $\gamma$ is properly centered and scaled by $t^{1/3}$, then it is governed by $F_2(s)$ in the regime of $m, t \to \infty$ with $m/t$ fixed in $(0, 1)$.

**Theorem 1.3.** Let $\sigma = m/t$ be fixed in a compact subset of $(0, 1)$. In the one-parameter MADM with the initial configuration $(0, 0, \cdots)$, we have
\[
\lim_{t \to \infty} \mathbb{P} \left( \frac{x_m(t/\gamma) + c_1 t}{c_2 t^{1/3}} \leq -s \right) = 1 - F_2(s)
\]
uniformly for $\sigma$, where $c_1 = -1 + 2\sqrt{\sigma}$ and $c_2 = \sigma^{-1/6}(1 - \sqrt{\sigma})^{2/3}$.
2. Fredholm Determinants and the Tracy-Widom distribution

1. Notations. Let us denote a configuration of the two-parameter MADM with \(N\) particles by a partition with reversed indices, that is, \(X = (x_1, \ldots, x_N)\) with \(x_1 \leq \cdots \leq x_N\). Here, each \(x_i\) represents the position of the \(i\)th left-most particle. Let \(S = \{z_1, \cdots, z_k\} \subset \{1, \cdots, N\}\) with \(z_1 < \cdots < z_k\) and \(k \geq 1\). Let \(\sigma(S) = z_1 + \cdots + z_k, \ |S|\) be the number of elements in \(S\),

\[
[N] = \frac{u^N - v^N}{u - v},
\]

and

\[
[N]! = [N][N - 1] \cdots [1], \quad \left[ \frac{N}{m} \right] = \frac{[N]!}{[m]![N - m]!}
\]

with \([0]! = 1\). We agree that all contour integrals over closed curves in this paper include \(\frac{1}{2\pi i}\).

2. \(\mathbb{P}(x_m(t) \leq x)\) for the infinite system. Let \(\mathbb{P}^Y\) be the probability measure of the finite system with the initial configuration \(Y = (y_1, \cdots, y_N)\). We start with the probability \(\mathbb{P}^Y(x_m(t) = x)\) that the \(m\)th left-most particle’s position is \(x\) at time \(t\) in the two-parameter MADM. This was obtained by the author’s earlier work, Theorem 3 in [10]. We state the formula for \(\mathbb{P}^Y(x_m(t) = x)\) with the notations in this paper after summing \(\mathbb{P}^Y(x_m(t) = x)\) over \(x\) from \(-\infty\) to \(x\).

**Proposition 2.1.** Let 0 < \(v < u\), \(u + v = 1\) and \(\tau = v/u\). For the two-parameter MADM with \(N\) particles, we have

\[
\mathbb{P}^Y(x_m(t) \leq x) = \sum_{S \text{ with } |S| \geq m} (-1)^m (uv)^{m(m-1)/2} \left[ \frac{|S| - 1}{|S| - m} \right] \frac{v^{\sigma(S) - m|S|}}{v^{\sigma(S) - |S|(|S|+1)/2}} \prod_{i<j} \xi_{z_i} - \xi_{z_j} \prod_{i=1}^k \xi_{z_i}^{x - y_i} e^{\varepsilon(\xi_i)t} \prod_{i=1}^k \frac{1}{(1 - \xi_{z_i})} \ d\xi_{z_1} \cdots d\xi_{z_k}
\]

where \(C_{R_i}\) is a circle centered at 0 with counterclockwise orientation of radius \(R_i\) such that \(1 < R_1 < \cdots < R_N < \tau^{-1}\) and \(\varepsilon(\xi_i)\) is given by (1.6).

**Remark 2.1.** The formula (2.12) is very similar to Tracy and Widom’s formula in the ASEP but there are a few differences. Note that the contours are all different in (2.12). Setting aside the issue of the contours, it is not clear whether or not the symmetrization of the integrand for the initial configuration \((1, 2, 3, \cdots)\) and \((2, 4, 6, \cdots)\) yields a desirable result to the author’s knowledge. See the relevant works in the ASEP [9, 21].
The infinite system with the initial configuration \((0, 0, \cdots)\) can be regarded as the limiting system of the finite system with the initial configuration \((0, \cdots, 0)\), and therefore we substitute \(y_{z_i} = 0\) for each \(i\) in (2.12) and then take the limit as \(N \to \infty\). One motivation of considering the initial configuration \((0, 0, \cdots)\) is that it is possible to obtain a simple formula after symmetrizing the integrand if we substitute \(y_{z_i} = 0\) and set aside the issue of contours for a moment. Indeed, if we substitute \(y_{z_i} = 0\), we see that the integral is independent of \(S\) and hence (2.12) is written as

\[
\sum_{k \geq m} \sum_{|S| = k} \left( \frac{v}{u} \right)^{z_1 + \cdots + z_k} (-1)^m (uv)^{m(m-1)/2} \left[ k - 1 \atop k - m \right] \frac{v^{-mk}}{u^{-k(k+1)/2}} \quad (2.13)
\]

\[
\times \int_{\mathcal{C}_R_k} \cdots \int_{\mathcal{C}_R_1} \prod_{i<j} \frac{\xi_i - \xi_j}{u + v\xi_i\xi_j - \xi_j} \prod_{i=1}^k \frac{\xi_i^\tau e^{\varepsilon(\xi_i)t}}{(1 - \xi_i)} \, d\xi_1 \cdots d\xi_k.
\]

The inner sum over \(S\) is written as the sum over all strict partitions \(1 \leq z_1 < z_2 < \cdots < z_k\) in the limit \(N \to \infty\), so we have

\[
\sum_{|S| = k} \left( \frac{v}{u} \right)^{z_1 + \cdots + z_k} = \sum_{1 \leq z_1 < z_2 < \cdots < z_k} \tau^{z_1} \cdots \tau^{z_k} = \frac{\tau^{k(k+1)/2}}{(1 - \tau)(1 - \tau^2) \cdots (1 - \tau^k)}.
\]

Now, we want to symmetrize the integrand, but the contours in (2.13) are all different. However, it is possible to deform all the contours to a single contour without encountering any singularities while deforming those contours. The reason is as follows. First, the pole of the variable \(\xi_i\) from \(u + v\xi_i\xi_j - \xi_j\) is outside \(R_j\) by Lemma 1 in [10] and it can be also shown that the pole of the variable \(\xi_j\) from \(u + v\xi_i\xi_j - \xi_j\) is inside \(R_i\) by a simple calculation. Hence, we can deform all the contours sequentially in the order of \(\mathcal{C}_{R_k}, \cdots, \mathcal{C}_{R_1}\) to any circle \(\mathcal{C}_R\) centered at 0 with radius \(R\) such that \(R_k \leq R < \tau^{-1}\). If we set \(\xi = \frac{1 + z}{1 + z\tau}\) and use (1.4) of Chapter III in [12], we have

\[
\sum \prod_{\sigma \in \mathcal{S}_k} \frac{u + v\xi_{\sigma(i)}\xi_{\sigma(j)} - \xi_{\sigma(i)}}{\xi_{\sigma(j)} - \xi_{\sigma(i)}} = u^{k(k-1)/2} \prod_{i=1}^k \frac{1 - \tau^i}{1 - \tau}.
\]

Therefore, (2.13) is written, after taking the limit \(N \to \infty\) and symmetrizing the integrand, as

\[
\sum_{k \geq m} (-1)^m (uv)^{m(m-1)/2} \left[ k - 1 \atop k - m \right] \frac{v^{-mk}u^{k-m(k-m-1)/2}}{k!u^{-k(k+1)/2}} \tau^{k(k+1)/2} \prod_{i=1}^k d\xi_i.
\]
where \([ \big] \) is the Gaussian binomial coefficient defined by

\[
{m \choose r}_\tau = \frac{(1 - \tau^m)(1 - \tau^{m-1}) \cdots (1 - \tau^{m-r+1})}{(1 - \tau)(1 - \tau^2) \cdots (1 - \tau^r)}.
\]

Recalling that

\[
\det(I - \lambda K_{x,t}) = \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{k!} \int_{C_R} \cdots \int_{C_R} \det(K_{x,t}(\xi_i, \xi_j))_{1 \leq i, j \leq k} \, d\xi_1 \cdots d\xi_k
\]

and using (3) and (8) in [23], we obtain \((1.7)\) in Proposition 1.1. Comparing \((1.7)\) with the ASEP’s formula (2) in [23], we note that there is an extra factor \(\tau \xi' - 1 \quad 1 - \tau\) in the kernel \((1.5)\), the denominator of the integrand is slightly different, and the kernel \((1.5)\) has two independent parameters \(p\) and \(u\).

3. Fredholm determinant for the one-parameter MADM. In this section, we show that if \(p = u\) and \(q = v\), then \((1.7)\) can be rewritten as a formula suitable for a saddle point analysis. In [23], Tracy and Widom made Möbius transformations

\[
\xi = \frac{1 - \tau \eta}{1 - \eta}, \quad \xi' = \frac{1 - \tau \eta'}{1 - \eta'} \quad \text{where} \quad \tau = \frac{p}{q},
\]

so that they could express the Fredholm determinant of the auxiliary operator as an infinite product. (See Proposition 4 in [23] for details.) However, these transformations do not give desirable results in the one-parameter MADM because of the extra factor \(\frac{\tau \xi' - 1}{1 - \tau}\) in the kernel \((1.5)\). We find that if we use, instead of \((2.15)\),

\[
\xi = \frac{1 - \eta}{1 - \tau \eta}, \quad \xi' = \frac{1 - \eta'}{1 - \tau \eta'} \quad \text{where} \quad \tau = \frac{v}{u},
\]

then an asymptotic result for the one-parameter MADM can be obtained in parallel to Tracy and Widom’s result in the ASEP. We explain why \((2.16)\) will work if we use Tracy and Widom’s techniques in [23]. Recall the kernel

\[
K^{ASEP}_{2}(\eta, \eta') = \frac{\varphi^{ASEP}(\eta')}{\eta' - \tau \eta}
\]

where

\[
\varphi^{ASEP}(\eta) = \left(\frac{1 - \tau \eta}{1 - \eta}\right)^x e^{\left[\frac{1}{1 - \tau \eta} - \frac{1}{1 - \tau \eta}\right]t}
\]

in the ASEP, which is obtained by \((2.15)\). The features to be noted in \((2.17)\) are that the singularities of \(\varphi^{ASEP}(\tau^k \eta)\) are only at \(\eta = \tau^{-k}, \tau^{-k-1}\) for each \(k \in \mathbb{Z}\) and that more importantly \(\varphi^{ASEP}(0) = 1\). (See Proposition 3,4,5 in [23] to understand how these features
are used in the proof.) In the one-parameter MADM, by the transformations (2.16), we obtain a new kernel $K_2(\eta, \eta')$ of the operator $K_{x,t}$

$$K_2(\eta, \eta') = \frac{\varphi(\eta')}{\eta' - \tau \eta}$$  \quad (2.18)

acting on $C \times C$ where $C$ is the image of $C_R$ under (2.16) and

$$\varphi(\eta) = \left( \frac{1 - \eta}{1 - \tau \eta} \right)^{x} e^{\left[ \frac{1}{1 - \eta} - \frac{1}{1 - \tau \eta} \right] t} \cdot \frac{1}{\tau - 1 - \eta}.$$  \quad (2.19)

We note that the singularities of $\varphi(\tau^k \eta)$ are also only at $\eta = \tau^{-k}, \tau^{-k-1}$ for each $k \in \mathbb{Z}$ but $\varphi(0) = \tau$, which will not make a big trouble.

**Lemma 2.2.** Under the transformations (2.16) the circle $C_R$ in (2.14) is mapped on the circle centered $\eta_0$ with $1 < \eta_0 < \tau^{-1}$ describing clockwise, and having $\tau^{-1}$ on the outside and $1$ on the inside.

**Proof.** By (2.16) we see that the center of the image circle is at $\frac{1 - \tau R^2}{1 - \tau R^2}$ and the radius of the image circle is $R(1 - \tau)$. The condition $1 < R < \tau^{-1}$ immediately gives the proof. \qed

We define the auxiliary operator $K_1$ by its kernel

$$K_1(\eta, \eta') = \frac{\varphi(\tau \eta)}{\eta' - \tau \eta}$$  \quad (2.20)

With this setting and Lemma 2.2, we may have the preliminary results corresponding to Proposition 3.4.5 in [23]. First, the statement of Proposition 3 in [23] is true if we take it in our context. The proof is almost the same because of the analyticity of $\frac{1}{\tau - 1 - \eta}$.

**Proposition 2.3.** [23] Let $\Gamma$ be any closed curve going around $\eta = 1$ once counterclockwise with $\eta = \tau^{-1}$ on the outside. Then the Fredholm determinant of $K_{x,t}(\xi, \xi')$ acting on $C_R$ has the same Fredholm determinant as $K_2(\eta, \eta') - K_1(\eta, \eta')$ acting on $\Gamma$.

Proposition 4 in [23] should be modified because $\varphi(0) = \tau$ in our case. Our version of Proposition 4 in [23] is as follows.

**Proposition 2.4.** Suppose the contour $\Gamma$ of Proposition 2.3 is star-shaped with respect to $\eta = 0$. Then the Fredholm determinant of $K_1$ acting on $\Gamma$ is equal to

$$\prod_{k=1}^{\infty} (1 - \lambda \tau^k).$$
Proof. We replace $K_0(\eta, \eta') = \frac{1}{\eta' - \tau \eta}$ and $\varphi(\eta)$ in Proposition 4 in [23] by $\frac{\tau}{\eta' - \tau \eta}$ and our $\varphi(\eta)$, \((2.19)\), respectively, and follow the steps in [23]. □

The statement of Proposition 5 in [23] is also true in our context because of the analyticity of the extra term $\frac{1}{\tau - \eta}$ in \((2.19)\).

**Proposition 2.5.** [23] Assume that $\Gamma$ is star-shaped with respect to 0 with 1 inside and $\tau^{-1}$ outside. Then for sufficiently small $\lambda$,

$$R(\eta, \eta'; \lambda) = \sum_{n=1}^{\infty} \lambda^n \frac{\varphi_n(\tau \eta)}{\eta' - \tau^n \eta}$$

where $R(\eta, \eta'; \lambda)$ is the kernel of $R = \lambda(I - \lambda K_1)^{-1}K_1$.

The proof is the same as Proposition 5 in [23]. Now, we may rewrite \((1.7)\) by a Fredholm determinant of which kernels are of variables $\eta, \eta'$. We obtain the following formula corresponding to (6) in [23] by the same manner as the ASEP case. See Section 2 in [23].

**Proposition 2.6.** Let $K_2$ and $K_1$ be the operators with the kernels \((2.18)\) and \((2.20)\) acting on a circle $\Gamma$ centered at 0 of radius $r \in (1, \tau^{-1})$. Then we have

$$\mathbb{P}(x_m(t/\gamma) \leq x) = \int_{C_1} \prod_{k=m+1}^{\infty} (1 - \lambda \tau^k) \det(I + \lambda K_2(I + R)) \frac{d\lambda}{\lambda} \quad (2.22)$$

where the contour $C_1$ is a circle of radius larger than $\tau^{-m}$ but not equal to $\tau^{-k}$ for any $k \geq 0$ and $R = \lambda(I - \lambda K_1)^{-1}K_1$.

4. The Tracy-Widom distribution. In Tracy and Widom’s works on the ASEP, the formula (6) in [23] was transformed to a formula so that it becomes suitable for a saddle point analysis. This type of reformulation is also possible in the one-parameter MADM. If we set $\lambda = \tau^{-m} \mu$ in \((2.22)\), then \((2.22)\) becomes

$$\mathbb{P}(x_m(t/\gamma) \leq x) = \int_{C_2} \prod_{k=1}^{\infty} (1 - \mu \tau^k) \det(I + \tau^{-m} \mu K_2(I + R)) \frac{d\mu}{\mu} \quad (2.23)$$

where the contour $C_2$ is now a circle centered at 0 of radius larger than 1 but not equal to $\tau^{-k}$ for any $k \geq 0$. We define

$$\varphi_{\infty}(\eta) := \prod_{n=0}^{\infty} \tau^{-1} \varphi(\tau^n \eta) = (1 - \eta)^x e^{\frac{\eta}{\tau^{-1}}} \cdot \prod_{n=0}^{\infty} \frac{\tau^{-1}}{\tau^{-1} - \tau^n \eta}$$

so that

$$\varphi_n(\eta) = \tau^{-n} \frac{\varphi_{\infty}(\eta)}{\varphi_{\infty}(\tau^n \eta)}.$$
Then
\[ K_2 R(\eta, \eta') = \sum_{n=1}^{\infty} \tau^{-1}(\lambda \tau^{-1})^n \int \frac{\varphi_\infty(\zeta)}{\varphi_\infty(\tau^n \zeta)(\zeta - \tau \eta)(\eta' - \tau^n \zeta)} d\zeta \]
by (2.21), and if we follow the rest part of the proof of Lemma 4 in [23] with the same arguments, then we may obtain
\[ \det( I + \tau^{-m} \mu K_2 (I + R) ) = \det( I + \mu J_{t,m,x,\mu} ). \]

Now, to prove the convergence of the Fredholm determinant in (1.10) to the Tracy-Widom distribution in the regime we consider, we sketch Tracy and Widom's arguments in our context because the proof is essentially the same as Tracy and Widom's. (See the proof of Theorem 3 in [23] for more details.) Looking at the exponential part (1.8) for a saddle point analysis, the only difference from the ASEP is the sign of \( x \). Hence, if we choose \( x = -c_1 t - c_2 st^{1/3} \) where \( c_1 \) and \( c_2 \) are defined as in (1.11), then the exponential part (1.8) is exactly the same as the corresponding exponential part in the ASEP's formula, so we have the same saddle points and the same steepest descent curves. Tracy and Widom deformed the original contours in their formula to the contours described in Lemma 5 in [23] because one might encounter singularities if the deformation were made to the steepest descent curves. That might happen in our case, too, if we deform the contours for the variables \( \eta \) and \( \zeta \) in the Fredholm determinant to the steepest descent curves. Recalling that the contour for the variable \( \eta \) is inside of the unit circle and the contour for the variable \( \zeta \) is between the unit circle and the circle centered at 0 of radius \( \tau^{-1} \), and noting that the singularities \( \tau^{-k} \), \( k = 1, 2, \ldots \), coming from the infinite product in (1.9) are to the right of the contour for \( \zeta \), we do not encounter any singularities if we deform our original contours to the contours Tracy and Widom used for their analysis. The deformation of contours does not change the Fredholm determinant by Proposition 1 in [23], and we do not encounter singularities from \( f(\mu, \zeta/\eta') \) while deforming contours as in Tracy and Widom's.

As in [23], the deformed contours may be replaced by their portions, which are rays near the point \( \xi \) where the two saddle points coincide because the contribution to the Fredholm determinant on the part other than the rays is exponentially small. Here, the rays for the variable \( \zeta \) starts from \( \xi - t^{-1/3} \) in the direction of \( \pm 2\pi i/3 \) and the rays for the variable \( \eta \) starts from \( \xi \) in the direction of \( \pm \pi i/3 \). These rays were further restricted to \( t^{-a} \) neighborhood of the saddle point \( \xi \) where \( a < 1/3 \) and, then we make substitutions for \( \eta, \eta' \), and \( \zeta \)
\[ \eta \to \xi - c_3^{-1} t^{-1/3} \eta, \quad \eta' \to \xi - c_3^{-1} t^{-1/3} \eta', \quad \zeta \to \xi - c_3^{-1} t^{-1/3} \zeta \]
on the rays where \( c_3 = c_3(1 - \sqrt{\sigma}) \). The new contour for the variable \( \eta \) are the rays from 0 to \( c_3 t^{1/3-a} e^{\pm 2\pi i/3} \) where the directions of the rays have changed, and for the variable \( \zeta \), we do not change the phase and instead we will multiply by \(-1\) the integral, so the contours for the variable \( \zeta \) are the rays from \(-c_3 \) to \(-c_3 + c_3 t^{1/3-a} e^{\pm 2\pi i/3} \). So, for each \( t \), we have
Re(ζ − η′) < 0 on these new contours. The factor of the infinite products in $J(η, η')$ becomes, after rescaling,

$$\prod_{n=0}^{\infty} \frac{\tau^{-1} - \tau^n η}{\tau^{-1} - \tau^n ζ} = 1 + o(1).$$

and the remaining part in $J(η, η')$ is the same as the ASEP’s. Hence, the rescaled kernel of $μJ(η, η')$ in the limit $t \to \infty$ after the substitutions is

$$- \int_{Γ_ζ} e^{ζ^3/3 - s(η')^3/3 + x(η' - ζ)} \frac{dζ}{(η - ζ)(ζ - η')}, \quad (2.24)$$

where $Γ_ζ$ is the straight line from $-c_3$ to $-c_3 + \infty e^{±2\pi i/3}$. Since Re(ζ − η′) < 0, we have

$$\int_{-∞}^{s} e^{x(η' - ζ)} dx = \frac{e^{x(η' - ζ)}}{η' - ζ},$$

so that (2.24) is equal to

$$\int_{-∞}^{s} \int_{Γ_ζ} e^{ζ^3/3 - (η')^3/3 + x(η' - ζ)} \frac{dζ}{η - ζ} dx. \quad (2.25)$$

The Fredholm determinant of the operator with kernel (2.25) is equal to the Fredholm determinant of the operator, acting on $L^2(−∞, s)$, with kernel

$$\int_{Γ_ζ} \int_{Γ_η} e^{-η^3/3 + yη + ζ^3/3 - xζ} \frac{dη}{η - ζ} dζ = -K_{Airy}(x, y).$$

Hence, we have shown that for $x = -c_1 t - c_2 s t^{1/3}$

$$\det (I + μJ_{t,m,x,µ}) \to \det (I - K_{Airy}(−∞,s)) = 1 - F_2(s)$$

uniformly for $μ$ on $C_2$ in (1.10) and uniformly for $σ = m/t$ in a compact subset of (0, 1) as $t \to \infty$. Finally, Theorem 1.3 is immediately obtained.

Remark 2.2. The one-parameter MADM with the initial configuration $(0, 0, \cdots)$ can be interpreted as the ASEP with the two-sided pushing dynamics with the step initial condition $(1, 2, \cdots)$ [11,10]. Theorem 1.3 implies that when there is a drift to the right in the ASEP with the two-sided pushing dynamics with the step initial condition $(1, 2, \cdots)$,

$$\lim_{t \to \infty} \mathbb{P} \left( \frac{x_m(t/γ) + c_1 t}{c_2 t^{1/3}} \leq s \right) = 1 - F_2(-s).$$

On the other hand, Tracy and Widom’s result states that when there is a drift to the left in the ASEP with the step initial condition $(1, 2, \cdots)$,

$$\lim_{t \to \infty} \mathbb{P} \left( \frac{x_m(t/γ) - c_1 t}{c_2 t^{1/3}} \leq s \right) = F_2(s).$$
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