Hidden Supersymmetry of a

$P,T$–Invariant 3D Fermion System

Mikhail Plyushchay\textsuperscript{a}\footnote{On leave from Institute for High Energy Physics, Protvino, Moscow Region, Russia; e-mail: mikhail@cc.unizar.es} and Pasquale Sodano\textsuperscript{b}\footnote{E-mail: sodano@perugia.infn.it}

\textsuperscript{a}Departamento de Física Teórica, Facultad de Ciencias
Universidad de Zaragoza, 50009 Zaragoza, Spain
\textsuperscript{b}Dipartimento di Fisica and Sezione I.N.F.N.
Università di Perugia, via A. Pascoli, I-06100 Perugia, Italy

Abstract

We show that a (2+1)-dimensional $P,T$–invariant free fermion system, relevant to $P,T$–conserving models of high-$T_c$ superconductivity, has a U(1,1) dynamical symmetry as well as an $N=3$ supersymmetry with the even generator being a quadratic function of the spin operator and of the generator of chiral $U_c(1)$ transformations. We demonstrate that the hidden supersymmetry leads to a non-standard superextension of the (2+1)-dimensional Poincaré group. As a result, the one particle states of the $P,T$–invariant fermion system realize an irreducible representation of the Poincaré supergroup labelled by the zero eigenvalue of the superspin operator.
1. Planar gauge field theories have many interesting theoretical features such as gauge invariant local topological mass [1, 2], fractional spin [1, 2, 3, 4] and statistics [5]. The exotic spin and statistics of planar theories — together with the appearance of $P$– and $T$–breaking mass terms in the Lagrangians of the massive Dirac spinor field and topologically massive vector U(1) gauge field [1, 2] — are a consequence of the simple fact that in 2+1 dimensions the spin is a pseudoscalar quantity.

Especially when constructing gauge models of high-$T_c$ superconductors [6], one is interested in having a $P$– and $T$–invariant topologically massive vector gauge field and a $P$– and $T$–invariant massive Dirac spinor field. For this purpose one usually introduces doublets of these fields with mass terms having opposite signs [6, 7].

In this paper we shall investigate the hidden symmetries of the simplest $P$– and $T$–invariant fermion theory, namely, we shall investigate the model described by the Lagrangian

$$L = \bar{\psi}_u (p\gamma + m)\psi_u + \bar{\psi}_d (p\gamma - m)\psi_d. \quad (1)$$

This model is invariant under a global $U_c(1)$ symmetry describing chiral rotations, to which is associated the conserved chiral current $I_\mu = \frac{i}{2}(\bar{\psi}_u \gamma_\mu \psi_u - \bar{\psi}_d \gamma_\mu \psi_d)$. This global symmetry is promoted to a local gauge symmetry in all the models of refs. [6, 7].

We shall demonstrate that the free fermion system described by (1) has not only the global $U_c(1)$ symmetry, but a broader $U(1,1)=SU(1,1)\times U(1)$ symmetry as well as a hidden $N=3$ supersymmetry. As we shall see, these symmetries form a dynamical group symmetry of the one particle states, i.e. the quantum mechanical states, of the free fermion theory described by (1). Furthermore, the one particle states realize an irreducible representation of a nonstandard Poincaré supergroup, whose generators we shall construct explicitly. Therefore, the dynamical symmetries of the $P,T$–invariant fermion model described by (1) differ substantially from those of its $P,T$–non-invariant counterpart.

2. To begin, let us introduce the spinor function $\Psi$, $\Psi^f = (\psi^f_u, \psi^d_d)$, in terms of which the Lagrangian (1) is represented as $L = \bar{\Psi} D \Psi$, with

$$D = p\gamma \otimes 1 + m \cdot 1 \otimes \sigma_3. \quad (2)$$

In (2) we use the (2+1)-dimensional $\gamma$-matrices realized via the Pauli matrices, $\gamma^0 = \sigma_3$, $\gamma^i = i\sigma_i$, $i = 1,2$, which satisfy the relation $\gamma_\mu \gamma_\nu = -\eta_{\mu\nu} + i\epsilon_{\mu\nu\lambda}\gamma^\lambda$ with metric $\eta_{\mu\nu} = \text{diag}(-,+,+)$ and completely antisymmetric tensor $\epsilon^{\mu\nu\lambda}$ normalized as $\epsilon^{012} = 1$. To avoid a cumbersome notation we do not write explicitly the spinor indices on which the matrices $\gamma_\mu$ act. The second factor in the direct product in (3), being 1 or $\sigma_3$ (and $\sigma_{1,2}$ in other operators below), acts in the two-dimensional space with indices $u = 1$ and $d = 2$. With this notation the equations of motion are

$$D \Psi = 0. \quad (3)$$

The operator $D$ satisfies the relation

$$D^2 = -K + 4mN D, \quad (4)$$

where $K = p^2 + m^2$ is the Klein-Gordon operator and

$$N = \frac{1}{2} \cdot 1 \otimes \sigma_3. \quad (5)$$
The operators commuting with \((2)\) are called physical operators or integrals of motion since they are symmetry generators of the quantum mechanical system \([8]\). The operator \(N\) is then the integral of motion generating the global chiral rotations \(\Psi \rightarrow \Psi' = \exp(igN)\Psi\), where \(g\) is a constant transformation parameter. Another integral of motion is the spin operator

\[ S = -\frac{1}{2}\gamma^{(0)} \otimes 1. \]  

\(5\)

In the following we shall work in the momentum representation. In eq. \((6)\) we have introduced the notation \(\gamma^{(0)} = \gamma^\mu e^{(0)}_\mu\), \(e^{(0)}_\mu = p_\mu/\sqrt{-p^2}\), taking into account that, due to \((4)\), \(-p^2 \neq 0\) on the physical subspace \([3]\). We add to the time-like unit vector \(e^{(0)}_\mu\) the space-like quantities \(e^{(i)}_\mu = e^{(i)}_\mu(p), i = 1, 2\), in order to have a complete oriented triad \(e^{(\alpha)}_\mu, \alpha = 0, 1, 2\),

\[ e^{(\alpha)}_\mu \eta_{\alpha\beta} e^{(\beta)}_\nu = \eta_{\mu\nu}, \quad \epsilon_{\mu\nu\lambda} e^{(0)\mu} e^{(1)\nu} e^{(2)\lambda} = 1. \]  

\(7\)

Note that \(e^{(i)}_\mu, i = 1, 2\), are not Lorentz vectors; however, their explicit form is not needed for our derivation (see, e.g., ref. \([9]\)). We shall use also the notation \(\gamma^{(\alpha)} = \gamma^\mu e^{(\alpha)}_\mu\).

Let us consider the following mutually conjugate operators \(Q^\pm\),

\[ Q^\pm = (Q^\mp)^\dagger = \pm \frac{i}{8} \left( \gamma^{(1)} \mp i\gamma^{(2)} \right) \otimes (\sigma_1 \pm i\sigma_2). \]  

\(8\)

In \((8)\) the conjugation is with respect to the internal scalar product \((\Psi_1, \Psi_2) = \bar{\Psi}_1 \Psi_2\). This is an indefinite scalar product due to the presence of the \(\gamma^0 \otimes 1\) factor. The operators defined in \((8)\) satisfy the weak condition \([8]\)

\[ [Q^\pm, \mathcal{D}] = \pm 2Q^\pm \cdot \left( \sqrt{-p^2} - m \right) \approx 0 \]

on the physical subspace defined by \((3)\), implying that — in addition to \(N\) and \(S\) — also \(Q^\pm\) must be regarded as integrals of motion. One easily verifies that \(N, S\) and \(Q^\pm\) satisfy the algebra

\[ [N, Q^\pm] = \pm Q^\pm, \quad [S, Q^\pm] = \pm Q^\pm, \quad [Q^+, Q^-] = -\frac{1}{4}(N + S), \quad [N, S] = 0. \]

\(10\)

Upon considering the following linear combinations,

\[ Q_0 = -\frac{1}{2}(N + S), \quad Q_1 = Q^+ + Q^-, \quad Q_2 = i(Q^+ - Q^-), \]  

\(9\)

we find that the operators \(Q_\alpha, \alpha = 0, 1, 2\), form an \(su(1, 1)\) algebra:

\[ [Q_\alpha, Q_\beta] = -i\epsilon_{\alpha\beta\gamma} Q^\gamma. \]  

\(10\)

In \((10)\) \(Q^\alpha = \eta^{\alpha\beta} Q_\beta\). The operators \(Q_\alpha\), together with the operator

\[ \mathcal{U} = \frac{1}{2}(S - N), \]  

\(11\)

In \((10)\) \(Q^\alpha = \eta^{\alpha\beta} Q_\beta\). The operators \(Q_\alpha\), together with the operator
which coincides (up to a factor) with the operator $D$ on the physical subspace (3), are the generators of a U(1,1) = SU(1,1) × U(1) symmetry with the algebra given by (10) and $[U, Q_\alpha] = 0$. The Casimir operator of the $su(1,1)$ algebra is $C = Q_\alpha Q^\alpha$. Here it has the form

$$C = -\frac{3}{8}(4NS + 1),$$

(12)

and takes the value $C = -3/4$ on the physical subspace (3). Therefore, the equations of motion (3) imply that the physical subspace is formed by the states with $-p^2 = m^2$, which are annihilated by the U(1) generator $U$. On this subspace the hidden SU(1,1) symmetry acts irreducibly.

On the other hand, since $Q_1^2 = Q_2^2 = -Q_0^2$, the integrals (9) together with the Casimir operator (12) form also the following $s(3)$ superalgebra:

$$\{Q_\alpha, Q_\beta\} = \eta_{\alpha\beta} \cdot \frac{2}{3}C, \quad [Q_\alpha, C] = 0.$$  

(13)

Thus, the system described by (1) has a hidden $N = 3$ dynamical supersymmetry. In our construction, we treat the operator $D$ as the ‘hamiltonian’ since we consider the operators commuting with it as integrals of motion (8). Since the even generator of the superalgebra (13) differs from the ‘hamiltonian’ $D$, the $N = 3$ hidden supersymmetry turns out to be analogous to the hidden supersymmetry of a 3-dimensional monopole (11), where one of the even generators, being the square of one of the two odd supercharges, is also different from the corresponding hamiltonian. The latter system admits SU(1,1) as a subgroup of its dynamical symmetry group (11). Other systems showing similar hidden supersymmetries have been analyzed in ref. (12).

The integrals $S$ and $N$ as well as their linear combinations $Q_0$ and $U$ and the quadratic operator (12) have been written as covariant operators, whereas the integrals $Q_i$ have not since they are defined in terms of the noncovariant quantities $\gamma^{(i)} = \gamma^\mu e^{(i)}_\mu$. Nevertheless, one may introduce a vector operator

$$\Gamma_\mu = -\frac{1}{4}(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2})\gamma^\nu \otimes \sigma_2 - \frac{1}{4\sqrt{-p^2}}\epsilon_{\mu
u\lambda}p^\nu\gamma^\lambda \otimes \sigma_1,$$

such that $\Gamma_\mu e^{(i)\mu} = Q_i$. $\Gamma_\mu$ has only two independent components since it is transverse to $p_\mu$, $\Gamma_\mu p^\mu = 0$. In terms of $\Gamma_\mu$ one may construct the Lorentz vector operator

$$\hat{Q}_\mu = \Gamma_\mu + e^{(0)}_\mu Q_0$$

(14)

so that $\hat{Q}^{(a)} = Q^a$, $\hat{Q}_\mu \hat{Q}^\mu = C$. As a consequence, the operators defined in (14) and (12) satisfy the covariant (anti)commutation relations generalizing (10), (13). With this notation, eqs. (14) and (13) may be written as

$$\hat{Q}_\mu \hat{Q}_\nu = \eta_{\mu\nu} \cdot \frac{1}{3}C - \frac{i}{2}\epsilon_{\mu\nu\lambda} \hat{Q}^\lambda, \quad [\hat{Q}_\mu, C] = 0.$$  

(15)

Eqs. (15) together with the commutation relations

$$[\hat{Q}_\mu, U] = [U, C] = 0$$  

(16)
covariantly manifests the dynamical (super)symmetry of the system.

3. It is interesting to observe that the hidden dynamical symmetry yields a nonstandard super-extension of the Poincaré group. The generators of the superextended Poincaré group are $J_\mu$, $p_\mu$ and $\tilde{Q}_\mu$, where $J_\mu$ is the total angular momentum operator,

$$J_\mu = x_\mu p_\nu - x_\nu p_\mu - \frac{1}{2} \gamma_\mu \otimes 1,$$

and the vector generator $\tilde{Q}_\mu$,

$$[J_\mu, \tilde{Q}_\nu] = -i \epsilon_{\mu\nu\lambda} \tilde{Q}_\lambda,$$

is the odd generator satisfying the anticommutation relations given by eq. (13). Since it commutes with $p_\mu$, $[\tilde{Q}_\mu, p_\nu] = 0$, one of the Casimir operators of the Poincaré supergroup is $p^2$. In order to find the second Casimir operator and clarify its meaning, we consider the vector operator

$$J_\mu = J_\mu - \tilde{Q}_\mu.$$

It satisfies the same (2+1)-dimensional Lorentz algebra (or $su(1, 1)$ algebra) as the generators $J_\mu$ and $\tilde{Q}_\mu$ themselves,

$$[J_\mu, J_\nu] = -i \epsilon_{\mu\nu\lambda} J_\lambda,$$

and, moreover, commutes with $\tilde{Q}_\mu$. Thus, the second Casimir operator is

$$\tilde{S} = J_\mu e^{(0)}_\mu.$$

The operator $\tilde{S}$ is the superspin. Taking into account the explicit form of the operator $\tilde{Q}^{(0)}_\mu = -Q_0$ given by eq. (13), and the definition of spin, $J^{(0)}_\mu = S$, one gets:

$$\tilde{S} = U = \frac{1}{2} (S - N).$$

Therefore, the operator $U$ is the second Casimir operator of the Poincaré supergroup, and it has the meaning of a superspin. One may easily verify that the operator $C$ is a quadratic function of the superspin,

$$C = 3\tilde{S}^2 - \frac{3}{4}. \quad (17)$$

With the help of the explicit form of the spin operator $S$ and of the generator of the chiral $U_c(1)$ transformations, $N$, one easily finds that the spectrum of eigenvalues of the superspin $\tilde{S}$ is given by the series of numbers $(-1/2, 0, 0, 1/2)$. Thus, the equations of motion (3) single out the physical states as the subspace of states with zero superspin. These states satisfy also the Klein-Gordon equation.

4. We have displayed as a dynamical symmetry group the hidden $U(1,1) = SU(1,1) \times U(1)$ symmetry as well as the hidden $N=3$ supersymmetry of the planar $P,T$–invariant free fermion model described by the Lagrangian (14). The $N=3$ supersymmetry turns out to be analogous to the hidden supersymmetry of a 3-dimensional monopole since the fermionic generators of the hidden supersymmetry ($\tilde{Q}_\mu$) are the ‘square root’ of a bosonic integral of
motion \( (\frac{1}{2} \mathcal{C}) \) other than the ‘hamiltonian’ \( (\mathcal{D}) \). One may regard the superspin \( \tilde{S} \) as the Hamiltonian \( \mathcal{H} \) for all the matrix operator variables \( \gamma^\mu \otimes 1 \) (or \( \gamma^{(\alpha)} \otimes 1 \)) and \( 1 \otimes \sigma_a, a = 1, 2, 3 \). This can be easily seen \([11]\) from the pseudoclassical model \([14]\). Due to eq. \((17)\), the even generator of the \( N = 3 \) supersymmetry, \( \tilde{\mathcal{H}} \equiv \frac{2}{3} \mathcal{C} \), has the form

\[
\tilde{\mathcal{H}} = \mathcal{H}^2 + b, \tag{18}
\]

with \( \mathcal{H} = \sqrt{2} \tilde{\mathcal{S}} \) and \( b = -1/2 \). As it is known, a supersymmetry with an even generator of the form \([18]\) (with \( b = 0 \)) takes place also in the description of a relativistic electron in a uniform magnetic field \([13]\) and in the analysis of supersymmetric quantum mechanical systems for which the anticommutator of odd generators is a polynomial function of the hamiltonian \([16]\).

A \( P,T \)-non-invariant system of two free fermion fields with mass terms of one sign has a \( U(2) = SU(2) \times U(1) \) symmetry. The generators of the \( SU(2) \) symmetry are \( \frac{1}{2} \cdot 1 \otimes \sigma_a, a = 1, 2, 3 \), whereas the \( U(1) \) symmetry is generated by the operator \( \gamma^{(0)} \otimes 1 + 1 \otimes 1 \). At first sight, one may easily conjecture that — in order to properly describe the dynamical symmetries of a \( P,T \)-invariant planar free fermion system — one should transform the generators of \( SU(2) \) into the corresponding generators of \( SU(1,1) \) with the simple trick of changing one of them through multiplication by \( i \) (see, e.g., ref. \([4]\)). This is not a correct procedure for the model investigated in this letter since the \( SU(1,1) \) generators constructed by means of this trick are not integrals of motion of the free fermion model described by \((4)\). Furthermore, the \( SU(2) \) generators of a \( P,T \)-non-invariant planar free fermion model are Poincaré-invariant generators of a merely internal symmetry, while the \( SU(1,1) \) generators obtained with our construction form a translationally-invariant Lorentz vector. Our analysis clarifies an important property of the integrals of motion associated with the Lagrangian \((1)\): namely, it shows that the generators of \( SU(1,1) \), being simultaneously odd generators of an \( N = 3 \) supersymmetry, can be combined with the Poincaré generators. This results in the non-standard superextension of the Poincaré group with a vector supercharge explicitly exhibited in this letter. The one particle states of the system described by \((4)\) realize an irreducible representation of the Poincaré supergroup, labelled by the zero eigenvalue of the superspin operator.

It would be interesting to look also for hidden symmetries in the \( P,T \)-invariant system of two free topologically massive vector \( U(1) \) gauge fields, as well as for those arising in the \( P,T \)-invariant system of two massive Dirac fields interacting with a doublet of topologically massive gauge fields \([4, 2]\). Furthermore, it is intriguing to ascertain what happens to the hidden supersymmetry of the free fermion model described by \((4)\) if one switches on the interaction with an external \( U(1) \) gauge field \( \mathcal{A}_\mu \) coupled to the conserved chiral current \( I_\mu = \bar{\Psi} \gamma^\mu \Psi \) \([3]\). We hope to address these problems — as well as the possibility of having a similar non-standard Poincaré supergroup in (3+1)-dimensional space-time — in future publications.

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