Abstract

In the 1980’s, the work of Frenkel, Lepowsky and Meurman, along with that of Borcherds, culminated in the notion of vertex operator algebra, and an example whose full symmetry group is the largest sporadic simple group: the Monster. Thus it was shown that the vertex operators of mathematical physics play a role in finite group theory. In this article we describe an extension of this phenomenon by introducing the notion of enhanced vertex operator algebra, and constructing examples that realize other sporadic simple groups, including one that is not involved in the Monster.

1 Motivation

We begin not with the problem that motivates the article, but with motivation for the tools that will furnish the solution to this problem. The tools we have in mind are called vertex operator algebras (VOAs); here follows one way to motivate the notion.

In mathematics there are various kinds of finite dimensional algebras that have proven to be significant or interesting in some respect. For example,

1. semisimple Lie algebras (with invariant bilinear form)
2. simple Jordan algebras (of type A, B, or C)
3. the Chevalley algebra (see [Che54])
4. the Griess algebra (see [Gri82])

The items of this list are very different from each other in terms of their properties and structure theory. Perhaps the only thing they have in common (as algebras) is finite dimensionality.

Nonetheless, it turns out that there is such a process called affinization which associates a certain infinite dimensional algebra structure (let’s say affine algebra), to each finite dimensional example in this list. Not only this, but the corresponding affine algebra has a distinguished representation...
(infinite dimensional) for which the action of the affine algebra extends in a natural way to a new kind of algebra structure called vertex operator algebra structure.

Thus we obtain objects of a common category for each distinct example here, and the notion of vertex operator algebra (VOA) furnishes a framework within which these distinct finite dimensional algebra structures may be unified.

2 VOAs

Let us now present a definition of the notion of VOA. For our purposes it is more natural to consider the larger category of super vertex operator algebras (SVOAs).

An SVOA is a quadruple \((U, Y, 1, \omega)\) where

- \(U = U_0 \oplus U_1\) is a super vector space over a field \(F\) say. (We will take \(F\) to be \(\mathbb{R}\) or \(\mathbb{C}\).)
- \(Y\) is a map \(U \otimes U \to U((z))\), so that the image of the vector \(u \otimes v\) under \(Y\) is a Laurent series with coefficients in \(U\). This series is denoted \(Y(u, z)v = \sum_n u_n v z^{-n-1}\), and the operator \(Y(u, z)\) is called the vertex operator associated to \(u\).
- \(1 \in U_0\) is called the vacuum vector, and is a kind of identity for \(U\) in the sense that we should have \(Y(1, z)u = u\) and \(Y(u, z)1|_{z=0} u = u\) for all \(u \in U\).
- \(\omega \in U_0\) is called the conformal element, and is such that for \(Y(\omega, z) = \sum L(n) z^{-n-2}\) the operators \(L(n)\) should satisfy the relations

\[
[L(m), L(n)] = (m-n)L(m+n) + \frac{m^3-m}{12}c \delta_{m+n,0} \text{Id}
\]

for some scalar value \(c \in F\). In other words, the Fourier modes of \(Y(\omega, z)\) should generate a representation of the Virasoro algebra, with some central charge \(c\).

The main axiom for the vertex operators is the Jacobi identity, which states that

- for \(\mathbb{Z}/2\)-homogeneous \(u, v \in U\) and arbitrary \(a \in U\) we should have

\[
z_0^{-1} \delta \left( \frac{z_1 - z_2}{z_0} \right) Y(u, z_1) Y(v, z_2)
- (-1)^{|u||v|} z_0^{-1} \delta \left( \frac{z_2 - z_1}{-z_0} \right) Y(v, z_2) Y(u, z_1)
= z_2^{-1} \delta \left( \frac{z_1 - z_0}{z_2} \right) Y(Y(u, z_0)v, z_2)
\]

\(\text{Id}\)
where $|u|$ is 1 or 0 as $u$ is odd or even, and similarly for $|v|$, and the expression $z_0^{-1} \delta((z_1 - z_2)/z_0)$, for example, denotes the formal power series

$$z_0^{-1} \delta \left( \frac{z_1 - z_2}{z_0} \right) = \sum_{n,k \in \mathbb{Z}, k \geq 0} (-1)^k \binom{n}{k} z_0^{-n-1} z_1^{-n-k} z_2^k.$$  

(3)

From (2) and from the properties of $1$ we see the extent to which the triple $(U, Y, 1)$ behaves like a (super)commutative associative (super)algebra with identity, since the formal series (3) may be regarded as a “delta function supported at $z_1 - z_2 = z_0$ (and expanded in $|z_1| > |z_2|$)”. The Jacobi identity (2) thus encodes, among other things, some sense in which the compositions $Y(v, z_2)Y(u, z_1)$, and $Y(Y(u, z_1 - z_2)v, z_2)$ all coincide.

The structure in $(U, Y, 1, \omega)$ which has no analogue in the ordinary superalgebra case is that furnished by the conformal element $\omega$. This structure furnished by $\omega$ (we will call it conformal structure) manifests in two important axioms.

- The action of $L(0)$ on $U$ should be diagonalizable with eigenvalues in $\frac{1}{2} \mathbb{Z}$ and bounded from below. We write $U = \bigoplus_n U_n$ for the corresponding grading on $U$, and we call $U_n$ the subspace of degree $n$.
- The operator $L(-1)$ should satisfy $Y(L(-1)u, z) = D_z Y(u, z)$ for all $u \in U$, where $D_z$ denotes differentiation in $z$.

The conformal structure is essential for the construction of characters associated to an SVOA. Zhu has shown [Zhu90] that these characters (under certain finiteness conditions on the SVOA) span a representation of the Modular Group $PSL_2(\mathbb{Z})$.

We often write $U$ in place of $(U, Y, 1, \omega)$. The scalar $c$ of (1) is called the rank of $U$. Modules and module morphisms can be defined in a natural way. We say that an SVOA is self-dual in the case that it is irreducible as a module over itself, and has no other inequivalent irreducible modules.

## 3 Sporadic groups

The Classification of Finite Simple Groups (see [Gri98], [Sol00]) states that in addition to the

- cyclic groups of prime order
- alternating groups $A_n$ for $n \geq 5$
- finite groups of Lie type (like $PSL_n(q)$, $PSU_n(q)$, $G_2(q)$, &c.)

there are exactly 26 other groups that are finite and simple, and can be included in none of the infinite families listed. These 26 groups are called the sporadic groups.

The largest of the sporadic groups is called the Monster, and was first constructed by Robert L. Griess, Jr., who obtained this result by explicitly constructing a certain commutative non-associative algebra (with invariant bilinear form) of dimension 196883, with the Monster group $M$ as its full group of automorphisms. This algebra is named the Griess algebra.

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It is also called the Friendly Giant
The Monster group furnishes a setting in which the majority (but not all) of the other sporadic groups may be analyzed, since it involves 19 of the other sporadic simple groups. Here we say that a group $G$ is involved in the Monster if $G$ is the homomorphic image of some subgroup of the Monster; that is, if there is some $H$ in $\mathbb{M}$ with normal subgroup $N$ such that $H/N$ is isomorphic to $G$. The sporadic groups that are involved in the Monster are called the Monstrous sporadic groups.

The remaining 6 sporadic groups not involved in the Monster are called the non-Monstrous sporadic groups, or more colorfully, the Pariahs.

4 Vertex operators and the Monster

It is a surprising and fascinating fact that the Griess algebra can appear in our list of finite dimensional algebras admitting affinization (see §1).

The fact that it does is due to the work of Frenkel, Lepowsky and Meurmann, who extended the notion of affinization (known to exist for certain Lie algebras) to the Griess algebra using vertex operators, obtaining an infinite dimensional version: the affine Griess algebra. They constructed a distinguished infinite dimensional module over this affine Griess algebra called the Moonshine Module. They built upon the work of Borcherds [Bor86] so as to arrive at the notion of vertex operator algebra, and showed that the Moonshine Module admits such a structure. Finally, they showed that the full automorphism group of this structure is the Monster simple group.

In contrast to the situation with Lie algebras or Jordan algebras, the Griess algebra is an object which is hard to axiomatize. It is perhaps not clear that there is any reasonable category of algebras (in the orthodox sense) which includes the Griess algebra as an example (see [Con85]). The notion of VOA thus provides a remedy to this situation: a setting within which the Griess algebra can be axiomatized.

A related question is: “How might the Monster group be characterized?” Having found that such an extraordinary group is the symmetry group of some structure, we would like to be able recognize this structure as distinguished in its own right, so that our group might be defined to be just the group of automorphisms of this distinguished object. In particular, our structure should belong to some family of similar structures; a family equipped with invariants, sufficiently rich that they can distinguish our particularly interesting examples from all others, and sufficiently simple that we can communicate them easily.

This question of characterization can also be addressed (conjecturally, at least) within the theory of VOAs. Let us call to attention three invariants for VOAs:

- rank
- self-duality
- degree (vanishing conditions)

One may check that the Moonshine Module satisfies the following three properties.

- rank 24
- self-dual
• degree 1 subspace vanishes

It is a conjecture due to Frenkel, Lepowsky and Meurman that the Moonshine Module is uniquely determined by these properties. Modulo a proof of the conjecture, VOA theory thus provides a compelling definition of the Monster group: the automorphism group of the Moonshine Module, a beautifully characterized object in the category of VOAs.

5 Vertex operators and Monstrous groups

We have seen that vertex operators may be fruitfully applied to one of the sporadic groups, and we may wonder if there is anything they can do for the remaining 25. Let us formulate The Problem:

∗ Given a sporadic group $G$, find a VOA whose automorphism group is $G$, and characterize it.

The fact that 20 of the sporadic groups are involved in the Monster suggests that The Problem may have solutions, at least for $G$ a Monstrous group. After all, if $G$ is such a group, then a cover $\hat{G} = N.G$ say, of $G$, is a subgroup of the Monster, and in particular, acts on the Moonshine Module. This action is probably very reducible, but by choosing an appropriate irreducible subalgebra for example, we may well obtain an object – a VOA even – which serves as a reasonable analogue of the Moonshine Module for our new group $\hat{G}$. We may even find that the normal subgroup $N$ acts trivially, so that our analogue of the Moonshine Module actually realizes $G$ itself, and not a cover of $G$. This outline is extremely speculative, and certainly does not constitute an acceptable solution to The Problem, but it does at least give us somewhere to start, at least in the case that $G$ is a Monstrous group.

It is important to mention that something like this has been carried out rigourously and successfully in at least one case: that in which $G$ is the Baby Monster $BM$, and the group $\hat{G}$ is a double cover of $G$, the centralizer of a so-called $2A$ involution in the Monster. The precise method is due to Gerald Höhn [Höh96], and the result is a self-dual SVOA of rank $23\frac{1}{2}$ whose full automorphism group is a direct product $2 \times BM$ of the Baby Monster with a group of order 2.

6 Vertex operators and the Conway group

At this point we would like to describe a solution to The Problem for a specific Monstrous group $G$, which is nonetheless not along the lines just described in §5. The group we have in mind is the largest sporadic group of Conway, $Co_1$. The solution we have in mind is the object of the following Theorem.

Theorem 1 ([Dun07]). Among nice rational $N = 1$ SVOAs, there is a unique one satisfying

• rank 12
• self-dual
• degree $1/2$ subspace vanishes
Let us name this structure $A_{C_0}$. We can see that it admits a convenient characterization. That $A_{C_0}$ is a solution to one of our problems is shown by the next Theorem.

**Theorem 2 ([Dun07]).** The full automorphism group of $A_{C_0}$ is the sporadic group $C_{11}$.

We should go no further before addressing the new terminology that has arisen. The terms *nice* and *rational* refer to certain technical conditions on SVOAs, and it will be convenient to put aside their precise meaning, and refer the interested reader to the article [Dun07]. (One would expect such technical conditions to arise in a precise formulation of the uniqueness conjecture for the Moonshine Module.) Of more importance for our present purpose is the term $N = 1$ SVOA.

**Definition.** An $N = 1$ SVOA is a quadruple $(U, Y, 1, \{\omega, \tau\})$, such that $(U, Y, 1, \omega)$ is an SVOA, and $\tau$ is a distinguished vector of degree $3/2$ satisfying $\tau(0)\tau = 2\omega$, and such that the Fourier coefficients of $Y(\tau, z)$ generate a representation of the Neveu–Schwarz Lie superalgebra on $U$.

The Neveu–Schwarz superalgebra is a natural super-analogue of the Virasoro algebra. It is also known as the $N = 1$ Virasoro superalgebra. Thus an $N = 1$ SVOA is just like an ordinary SVOA except there is some extra structure: the role of the Virasoro algebra is now played by the $N = 1$ Virasoro superalgebra.

Let us consider for a moment longer, the difference between SVOA structure and $N = 1$ SVOA structure. Looking back at Theorem 1 experts will notice that the conclusion remains true if we drop the the “$N = 1$” from “$N = 1$ SVOA” in the hypothesis. That is to say, there is indeed a unique self-dual SVOA with rank 12 that has vanishing degree 1/2 subspace. In fact, it is a reasonably familiar object as SVOAs go: it is the lattice SVOA associated to the integral lattice $D_{12}^+$ (the unique self-dual integral lattice of rank 12 with no vectors of unit norm).

One may be surprised to see a sporadic group, or even a finite group here, since the SVOA underlying $A_{C_0}$ has infinite automorphism group. In fact, there is an action by Spin$_{24}$ (faithful up to some subgroup of order 2). The crux of the matter is that

1. for a suitably chosen vector in this Spin$_{24}$-module $A_{C_0}$ the fixing group is $C_{11}$,
2. the precise choice is made for us by the $N = 1$ Virasoro superalgebra.

Let us also emphasize that the uniqueness result for $A_{C_0}$ furnishes a compelling definition of the Conway group: as the full automorphism group of $A_{C_0}$, a well characterized object in the category of $N = 1$ SVOAs.

## 7 Enhanced SVOAs

With the example of $A_{C_0}$ in mind, and also with the suspicion that it may be interesting to consider other extensions of the Virasoro algebra, we formulate the notion of *enhanced SVOA*.

Roughly speaking, an enhanced SVOA is a quadruple $(U, Y, 1, \Omega)$ where $\Omega$ is a finite subset of $U$ (the set of *conformal generators*) containing a vector $\omega$ for which $(U, Y, 1, \omega)$ is an SVOA. (We

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3. The definition of $N = 1$ SVOA here is almost identical to that of $N = 1$ Neveu–Schwarz vertex operator superalgebra without odd formal variables ($N = 1$ NS-VOSA) which was introduced earlier by Barron [Bar00].
refer to [Dun06a, §2] for the precise definition.) The subSVOA of $U$ generated by the elements of $\Omega$ is called the conformal subSVOA.

The rank of an enhanced SVOA is just the rank of the underlying SVOA. We say that an enhanced SVOA is self-dual just when it is self-dual as an SVOA. The automorphism group of an enhanced SVOA is the subgroup of the automorphism group of the underlying SVOA that fixes every conformal generator.

We see then that an ordinary SVOA is an enhanced SVOA with $\Omega = \{\omega\}$, and an $N = 1$ SVOA is an enhanced SVOA with $\Omega = \{\omega, \tau\}$, and conformal subSVOA a copy of the SVOA associated to the vacuum representation of the $N = 1$ Virasoro superalgebra. In order to show that there are other interesting examples of enhanced SVOA structure, we present the following result.

**Theorem 3** ([Dun06b]). There exists a self-dual enhanced SVOA $A_{Suz}$ of rank 12

$$A_{Suz} = (A_{Suz}, Y, 1, \{\omega, j, \nu, \mu\})$$

with $\text{Aut}(A_{Suz}) \cong 3.\text{Suz}$.

It turns out that the conformal algebra in this example contains the direct product of a pair of $N = 1$ Virasoro superalgebras, at central charges 11 and 1, respectively. The SVOA underlying $A_{Suz}$ coincides with that underlying $A_{Co}$, and taking the diagonal $N = 1$ Virasoro superalgebra generated by $\tau = \nu + \mu$ (with central charge 12), we recover the enhanced SVOA structure with automorphism group $Co_{1}$.

### 8 Beyond the Monster

We have seen already that vertex operators have a role to play in the analysis of several Monstrous sporadic groups. In the cases of Conway’s group, Suzuki’s group, the Monster, and the Baby Monster, precise theorems have been formulated. A very significant question is whether or not there is any application to sporadic groups beyond the Monster; that is, to pariahs. We observed at the very beginning that the notion of VOA can unify such disparate notions as ‘semi-simple Lie algebra’ and ‘commutative non-associative algebra’.

The principal idea of this paper is that vertex operators do play a role in the representation theory of non-Monstrous sporadic groups.

At this point let us introduce the sporadic group of Rudvalis, $Ru$. This sporadic simple group is not involved in the Monster; it is one of the pariahs. It has order

$$145926144000 = 2^{14} \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 13 \cdot 29 \approx \frac{3}{2} \times 10^{11}. \quad (5)$$

The largest maximal subgroup is of the form $^{2}F_4(2)$ and has index 4060 in $Ru$. (The Tits group has index two in this group.) The next largest maximal subgroup is a non-split extension of the form $^{2}G_2(2)$, and has index 188500. The smallest non-trivial irreducible representations of $Ru$ have degree 378.
9 Vertex operators and Rudvalis’s group

Consider The Problem for $G = Ru$. The main theorem we wish to present is the following.

**Theorem 4 ([Dum06a], [Dum06b])**. There exists a self-dual enhanced SVOA $A_{Ru}$ of rank 28

$$A_{Ru} = (A_{Ru}, Y, 1, \{\omega, j, \nu, \rho\})$$

(6)

with $\text{Aut}(A_{Ru}) \cong 7 \times Ru$.

(Compare this with the statement of Theorem 3.) We will now provide a description of the enhanced SVOA $A_{Ru}$.

At the level of SVOAs, we have an isomorphism

$$A_{Ru} \cong V_{D_{28}^+}$$

(7)

where $D_{28}^+$ denotes a self-dual integral lattice of rank 28 with no vectors of unit length, and with $D_{28}$ as its even part. We have observed already that there are analogous statements for the SVOAs underlying the enhanced SVOAs $A_{Co}$ and $A_{Suz}$.

$$A_{Co} \cong A_{Suz} \cong V_{D_{12}^+}, \quad \text{as SVOAs.}$$

(8)

In the case of the enhanced SVOA $A_{Suz}$, the conformal vectors $j$ and $\omega$ have degree 1 and 2 respectively, and the two conformal vectors beyond these; viz. $\nu$ and $\mu$ are both found in the degree 3/2 subspace. In the case of $A_{Ru}$, the degree 3/2 subspace is trivial. In fact the degree 1/2, 3/2, and 5/2 subspaces are all trivial for $A_{Ru}$. The extra conformal vectors $\nu$ and $\rho$ for $A_{Ru}$ are found in the degree 7/2 subspace. This space is very large; the dimension is the number of vectors of square-length 7 in the lattice $D_{28}^+$.

$$\dim(A_{Ru})_{7/2} = 2^{28}/2 \approx 10^8$$

(9)

The most effort in the construction of $A_{Ru}$ goes into determining a precise description of the vectors $\nu$ and $\rho$. It is a remarkable fact that the finite group eventually obtained has almost no other point-wise invariants\(^4\) in its action on this $2^{27}$ dimensional space.

10 The 28 dimensional representation

It is important for our construction of $A_{Ru}$ that the Rudvalis group admits a perfect double cover $2.Ru$ which has irreducible representations of degree 28 (writable over $\mathbb{Z}[i]$). That the group $2.Ru$ preserves a lattice of rank 28 over $\mathbb{Z}[i]$ was observed independently by Meurman\(^5\) and by Conway and Wales [CW73] (see also [Con77] and [Wil84]), and this lattice is in fact self-dual when regarded as a lattice (of rank 56) over $\mathbb{Z}$. We choose to view this 28 dimensional representation in terms of the maximal of $G_2$-type: $2^6.G_2(2)$, which becomes $2^7.G_2(2)$ in the double cover $2.Ru$.

\(^4\)In fact, there is just one other invariant in addition to $\nu$ and $\rho$. It turns out to be $f(0)\nu$.

\(^5\)private communication
The action of the group $2^7G_2(2)$ in the 28 dimensional representation can be understood in the following way in terms of the $E_8$ lattice. Let $\Lambda$ denote a copy of the $E_8$ lattice, the unique self-dual even lattice of rank 8. (A lattice is a free $\mathbb{Z}$-module equipped with a bilinear form, and an even lattice is a lattice for which the square-norm of every vector is an even integer.) We may take

$$\Lambda = \left\{ \sum_{i \in \Pi} n_i h_i \mid \sum_{i \in \Pi} n_i \in 2\mathbb{Z}; \text{ all } n_i \in \mathbb{Z}, \text{ or all } n_i \in \mathbb{Z} + \frac{1}{2} \right\} \quad (10)$$

where the bilinear form is defined so that $\langle h_i, h_j \rangle = \delta_{ij}$. Then $\Lambda$ supports a structure of non-associative algebra over $\mathbb{Z}$ which makes it a copy of the integral Cayley algebra, or what is the same, a maximal integral order in the Octonions. To see such a structure explicitly, we assume that the index set $\Pi = \{\infty, 0, 1, 2, 3, 4, 5, 6\}$ is a copy of the projective line over $\mathbb{F}_7$, and then offer the following defining relations (taken from [CCN+85])

1. $1 = \frac{1}{2} \sum_{i \in \Pi} h_i \quad (11)$
2. $2h_i^2 = h_i - 1 \quad (12)$
3. $2h_{\infty}h_0 = 1 - h_3 - h_5 - h_6 \quad (13)$
4. $2h_0h_{\infty} = 1 - h_2 - h_1 - h_4 \quad (14)$

and also the images of these relations under the natural action of $L_2(7)$ on the indices. The sublattice of doubles $2\Lambda$ is an ideal in this algebra, and we may consider the quotient $\tilde{\Lambda} = \Lambda/2\Lambda$ which becomes a copy of the Cayley algebra over the finite field $\mathbb{F}_2$; what we call the binary Cayley algebra. We should note that the automorphism group of this algebra is the finite group $G_2(2)$ (which contains the simple group $U_3(3)$ to index 2).

There are just $2^8 = 256$ elements in the binary Cayley algebra. We can count them.

| Type                | Count |
|---------------------|-------|
| zeroes              | 1     |
| identities           | 1     |
| involutions          | 63    |
| square roots of 0    | 63    |
| idempotents          | 72    |
| cube roots of 1      | 56    |

There is a natural pairing on the elements of $\tilde{\Lambda}$ obtained by sending $x$ to the pair \{x, 1 + x\}. This association pairs the zero with the identity, the involutions with the idempotents, and partitions the idempotents and cube roots into 36 and 28 pairs, respectively. The group $G_2(2)$ acts transitively on each of these different sets of pairs.

The 28 cube root pairs are in a sense the basis upon which the enhanced SVOA will be constructed. Let us denote them by $\Delta$. A typical cube root of unity in $\tilde{\Lambda}$ is $(h_i - h_j)$ for $i \neq j \in \Pi$. A typical involution in $\tilde{\Lambda}$ is given by $(h_i + h_j)$ for $i \neq j \in \Pi$, and it is important that for any given involution there are exactly 24 cube roots (12 pairs of cube roots) that are not orthogonal to the chosen involution. The corresponding 12-subsets of $\Delta$ are called dozens.
We now introduce a complex vector space \( \mathfrak{r} \) of dimension 28, with Hermitian form \((\cdot, \cdot)\) and an orthonormal basis \( \{a_i\}_{i \in \Delta} \), indexed by our cube root pairs. The structures arising from the binary Cayley algebra described above allow us to define an action by the group \( 2.2^6.G_2(2) \) on this space. For example, the normal subgroup \( 2.2^6 \) is generated by the transformations that change sign on the coordinates of a given dozen. With a somewhat finer analysis of the geometry of \( \Lambda \) we can define the action of the rest of the group (cf. [Dun06a]); we call this group the monomial group and we denote it \( M \). (Warning: the action of \( G_2(2) \) cannot be realized as coordinate permutations. Instead we must write generators as coordinate permutations followed by multiplications by \( \pm 1 \) or \( \pm i \) on particular coordinates. The group \( M \) is a non-split extension of \( G_2(2) \).) The action of \( M \) on \( \mathfrak{r} \) preserves the Hermitian form.

### 11 The conformal elements

Assume now that we have a Hermitian space \( \mathfrak{r} \) and a unitary action on this space of the monomial group \( M \) of the shape \( 2.2^6.G_2(2) \). We assume further that \( M \) is regarded as a matrix group with respect to the basis \( \{a_i\} \), and consists of monomial matrices (having one non-zero entry in each row and column). We set \( \mathfrak{u} = \mathfrak{r} \oplus \mathfrak{r}^\ast \), where \( \mathfrak{r}^\ast \) denotes the dual space to \( \mathfrak{r} \), and is equipped with the induced Hermitian form. The space \( \mathfrak{u} \) then comes equipped with a Hermitian form (obtained by taking direct sum of those associated to the summands) and also a bilinear form \( \langle \cdot, \cdot \rangle \), induced by the canonical pairing \( \mathfrak{r} \times \mathfrak{r}^\ast \to \mathbb{C} \). We define \( \text{Cliff}(\mathfrak{u}) \) to be the Clifford algebra of \( \mathfrak{u} \) defined with respect to this bilinear form.

\[
\text{Cliff}(\mathfrak{u}) = T(\mathfrak{u})/\langle u \otimes u + \langle u, u \rangle \mid u \in \mathfrak{u} \rangle
\]  

We define \( \text{CM}_X \) to be the module over \( \text{Cliff}(\mathfrak{u}) \) spanned by a vector \( 1_X \) satisfying \( u1_X = 0 \) whenever \( u \in \mathfrak{r}^\ast \). We claim that the isomorphism

\[
\text{CM}_X \cong \bigoplus \wedge^n(\mathfrak{r})1_X
\]  

holds when these spaces are viewed as modules over \( \text{Cliff}(\mathfrak{r}) \) (the subalgebra of \( \text{Cliff}(\mathfrak{u}) \) generated by \( \mathfrak{r} \hookrightarrow \text{Cliff}(\mathfrak{u}) \)). Next we claim that the degree \( 7/2 \) subspace of \( A_{Ru} \) may be naturally identified with the even part of \( \text{CM}_X \).

\[
(A_{Ru})_{7/2} \longleftrightarrow \text{CM}_X^0 \cong \bigoplus \wedge^{2n}(\mathfrak{r})
\]  

The Clifford algebra \( \text{Cliff}(\mathfrak{u}) \) naturally contains a copy of the group \( \text{Spin}(\mathfrak{u}) \), and the space \( \text{CM}_X^0 \) is an irreducible module for this group \( \text{Spin}(\mathfrak{u}) \).

Recall that our goal is to define the elements \( \nu \) and \( \varrho \). We now define \( \nu \) by setting

\[
\nu = 1_X + a_\Delta 1_X
\]  

where \( a_\Delta = a_\infty a_1 \cdots a_{27} \in \text{Cliff}(\mathfrak{u}) \) say. Let us write \( \overline{M} \) for the copy of \( G_2(2) \) obtained by replacing each non-zero entry with a 1 in each matrix in \( M \). The vector \( \varrho \) will be expressed in terms of orbits of \( \overline{M} \) on monomials \( a_I 1_X \) for \( I \subset \Delta \) with \( |I| = 14 \).
It turns out that there are 80 such orbits, but only 68 give rise to invariants for the monomial group \( M \) in \( \wedge^{14}(r)1_X \subset CM_X^0 \). The vector \( \varrho \) is a linear sum of these invariants, \( t_i \) say, where the coefficients \( r_i \) can be taken to lie in \( \mathbb{Z}[i] \).

\[
\varrho = \sum_{i=1}^{68} r_i t_i \tag{20}
\]

The orbits are paired under complementation, and the coefficients of invariants corresponding to complementary orbits are conjugate (up to sign), so that ultimately, we require to specify 34 values. We refer to [Dun06a, §5.2] for the details.

Finally, we take \( A_{Ru} = (A_{Ru}, Y, 1, \{\omega, j, \nu, \varrho\}) \) as in the statement of Theorem 4, and this completes the construction.

It follows from the isomorphism [7], with the lattice SVOA for \( D_+^{28} \), that \( A_{Ru} \) is self-dual, and has rank 28. To prove that the automorphism group is of the stated form we prove first that it is finite, by showing that it is a reductive algebraic group with trivial Lie algebra (cf. [Dun06a, §5.4]). It follows that it has dimension 0, and hence, is finite. We can explicitly construct generators for 2\( .Ru \) acting on \( A_{Ru} \) and fixing the vectors \( \{\omega, j, \nu, \varrho\} \), and we may employ an argument from [NRS01], to show that the only other symmetries possible are scalar multiples of the identity. It is then easy to check that such multiples must be 14\( ^{th} \)-roots of unity (since they must preserve \( \varrho \in \wedge^{14}(r)1_X \)). Finally we obtain a central product 14\( \times \)2\( .Ru \), but the central \( \mathbb{Z}/2 \) here acts trivially on \( A_{Ru} \), and so the full automorphism group is just 7 \( \times \)Ru.

12 The character

We conclude with consideration of the character of the enhanced SVOA \( A_{Ru} \).

The action of the Rudvalis group \( Ru \) on \( A_{Ru} \) preserves a certain vector \( j \) of degree 1. The residue of the corresponding vertex operator \( Y(j, z) \) is denoted \( J(0) \), commutes with the Virasoro operator \( L(0) \), and has diagonalizable action on \( A_{Ru} \), thus giving rise to a grading by \textit{charge}. It is natural then to consider the two variable series

\[
\text{tr}
|_{A_{Ru}}(p^J(0))q^L(0)^{-c/24}
\]

which we call the \textit{(2 variable) character} of \( A_{Ru} \). Recall the Jacobi theta function given by

\[
\vartheta_3(z|\tau) = \sum_{m \in \mathbb{Z}} e^{2izm + \pi\tau m^2} \tag{22}
\]

and also the Dedekind eta function

\[
\eta(\tau) = q^{1/24} \prod_{m \geq 1} (1 - q^m) \tag{23}
\]

written here according to the convention \( q = e^{2\pi i\tau} \). Let us also convene to write \( p = e^{2\pi iz} \). Then we have
Table 1: The Character of $A_{Ru}$

| $m$ | 0    | 1/2  | 1    | 3/2  | 5/2  |
|-----|------|------|------|------|------|
| 0   | 1    | 1    | 1/2  | 1    | 1/2  |
| 1   | 784  | 378  | 784  | 378  | 784  |
| 2   | 14452| 92512| 20475|      |      |
| 3   | 11327232| 8128792| 2843568| 376740| 2843568|
| 4   | 490068257| 373673216| 161446572| 35904960| 35904960|
| 5   | 2096760960| 1649657520| 794670240| 226546320| 226546320|
| 6   | 13668945136| 10818453324| 528448352| 1513872360| 1513872360|
| 7   | 56547022140| 45624923820| 23757475560| 7766243940| 7766243940|

Proposition 5 ([Dun06b]). The character of $A_{Ru}$ is given by

$$
\begin{align*}
\text{tr}|_{A_{Ru}} p^{J(0)} q^{L(0)} - c/24 &= \frac{1}{2} \left( \frac{\vartheta_3(\pi z | \tau)^{28}}{\eta(\tau)^{28}} + \frac{\vartheta_3(\pi z + \pi/2 | \tau)^{28}}{\eta(\tau)^{28}} \right) \\
+ \frac{1}{2} p^{14} q^{7/2} \left( \frac{\vartheta_3(\pi z + \pi/2 | \tau)^{28}}{\eta(\tau)^{28}} + \frac{\vartheta_3(\pi z + \pi/2 + \pi/2 | \tau)^{28}}{\eta(\tau)^{28}} \right)
\end{align*}
$$

The terms of lowest charge and degree in the character of $A_{Ru}$ are recorded in Table 1. The column headed $m$ is the coefficient of $p^m$ (as a series in $q$), and the row headed $n$ is the coefficient of $q^{n-c/24}$ (as a series in $p$). The coefficients of $p^{-m}$ and $p^m$ coincide, and all subspaces of odd charge vanish.

Many irreducible representations of $Ru$ are visible in the entries of Table 1. For example, we have the following equalities, where the left hand sides are the dimensions of homogeneous subspaces of $A_{Ru}$, and the right hand sides indicate decompositions into irreducibles for the Rudvalis group.

$$
\begin{align*}
378 &= 378 \\
784 &= 1 + 783 \\
20475 &= 20475 \\
92512 &= (2)378 + 406 + 91350 \\
144452 &= (3)1 + (3)783 + 65975 + 76125 \\
376740 &= 27405 + 65975 + 75400 + 102400
\end{align*}
$$

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