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Study on the Algorithm of Judgment Matrix in Analytic Hierarchy Process

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Abstract. A new algorithm is proposed for the non-consistent judgment matrix in AHP. A primary judgment matrix is generated firstly through pre-ordering the targeted factor set, and a compared matrix is built through the top integral function. Then a relative error matrix is created by comparing the compared matrix with the primary judgment matrix which is regulated under the control of the relative error matrix and the dissimilar degree of the matrix step by step. Lastly, the targeted judgment matrix is generated to satisfy the requirement of consistence and the least dissimilar degree. The feasibility and validity of the proposed method are verified by simulation results.

1. Introduction

Analytic Hierarchy Process (AHP) is a systematic analysis method which proposed by Professor Thomas L. Saaty who was in the University of Pittsburgh in the mid-1970s. Its basic principle is to make decision by qualitative and quantitative analysis on the basis of goals, guidelines, programs and other levels which were decomposed by complex system. It has been widely used in auxiliary decision-making, pattern recognition, effectiveness evaluation and many other aspects in many areas such as social, economic, military, management, etc.

The judgment matrices are needed in AHP analysis, but the judgment matrices given by experts are often inconsistent and need to be adjusted. Research on the consistency adjustment of judgment matrix has been the difficulty and hotspot of AHP method. The traditional methods\[1\]\[2\] of consistency adjustment of judgment matrix is that the inconsistent matrix can be obtained by properly perturbing a completely consistent judgment matrix. By constructing such a perturbation matrix, we can find the elements that have the greatest disturbance to the original judgment matrix. Through the adjustment of the element we can adjust the consistency of the judgment matrix. However, this method does not take into account the expert judgment information provided by the judgment matrix, nor does it guarantee that the elements of the resulting judgment matrix must be in the range of 1 to 9 and its reciprocal, which is contrary to the principle of constructing judgment matrix proposed by Professor Saaty. In this paper, an AHP judgment matrix generation algorithm based on pre-ordering and round-up function was proposed, and it can solve this problem.

2. Traditional Method of Constructing AHP Judgment Matrix

Professor Saaty described the method of constructing AHP judgment matrices as follows:

2.1 Establish Target Hierarchical Hierarchy Model
Suppose that the problem can be divided into a hierarchical model as shown in Figure.1.
Aim

Criterion

Plan P

Figure 1 A Hierarchical Hierarchy Model

2.2 Construct The Judgment Matrix

The values of the elements in the judgment matrix are generally determined by $1 \sim 9$ scale method[3][4], mainly through expert assessments or from historical (empirical) data. Table 1 lists the meaning of the scale $1 \sim 9$.

| Scale | Meaning                                                      |
|-------|--------------------------------------------------------------|
| 1     | Indicating that the two factors are of equal importance      |
| 3     | Indicating that the former factor is slightly more important than the latter |
| 5     | Indicating that the former factor is significantly more important than the latter |
| 7     | Indicating that the former factor is strongly more important than the latter |
| 9     | Indicating that the former factor is extremely more important than the latter |
| 2, 4, 6, 8 | Indicating an intermediate value of the above-described adjacency determination reciprocal |

2.3 Consistency Check

In order to avoid the interference of other factors to the judgment matrix, it is required that the judgment matrix satisfy the general consistency in practice and the consistency check should be carried out. Only through the check, it can be determined that the judgment matrix is logically reasonable, and can continue to analyze the results. The consistency check of the judgment matrix according to the formula (1).

$$CR = CI / RI$$

In the formula, $CR$ is the consistency ratio. When $CR < 0.10$, it can be considered that the consistency of the judgment matrix is acceptable, otherwise the judgment matrix should be modified appropriately. $CI$ is the consistency index, which can be calculated by the formula (2)[5].

$$CI = (\lambda_{max} - n) / (n - 1)$$

In the formula, $\lambda_{max}$ is the maximum eigenvalue of judgment matrix, $n$ is the number of factors, $RI$ is the random consistency index, which can be found in table 2[4].

| Order | RI  |
|-------|-----|
| 3     | 0.58|
| 4     | 0.90|
| 5     | 1.12|
| 6     | 1.24|
| 7     | 1.32|
| 8     | 1.41|
| 9     | 1.45|
| 10    | 1.49|
| 11    | 1.52|
The maximum eigenvalue \( \lambda_{\text{max}} \) is the relative weight of each factor of the judgment matrix against its criterion. For the calculation of \( \lambda_{\text{max}} \), first we can use sum vector method[4][6] to solve the eigenvectors \( W \), then do as follows:

First, each column of the judgment matrix is normalized.

\[
\bar{a}_{ij} = \frac{a_{ij}}{\sum_{k=1}^{n} a_{kj}}, \quad i, j = 1, 2, \ldots, n
\]  

(3)

Next, each column of the normalized judgment matrix is added by rows

\[
\bar{W}_j = \sum_{i=1}^{n} \bar{a}_{ij}, \quad j = 1, 2, \ldots, n
\]  

(4)

Then, the vector \( \bar{w} = [\bar{w}_1, \bar{w}_2, \ldots, \bar{w}_n]^T \) is normalized

\[
W = \frac{\bar{W}_i}{\sum_{i=1}^{n} \bar{W}_i}, \quad i = 1, 2, \ldots, n
\]  

(5)

The result \( w = [w_1, w_2, \ldots, w_n]^T \) is the desired eigenvector.

Finally, the maximum eigenvalue of the matrix is as follows.

\[
\lambda_{\text{max}} = \sum_{i=1}^{n} \left( \frac{A\bar{W}}{nw_i} \right)
\]  

(6)

3. AHP Judgment Matrix Generation Algorithm Based on Pre-sort and Top-rounding Function

3.1 Definitions and Propositions

Definition 1: Initial Judgment Matrix: The matrix \( A = (a_{ij})_{n\times n} \) generated by the following rules is called the initial judgment matrix.

Step 1 Compare the importance of the factors belonging to the same goal, pre-sort the factor set, the factor set in descending order is \( f_1, f_2, \ldots, f_s \), in which \( s \geq 3 \).

Step 2 According to expert experience, construct a matrix \( A = (a_{ij})_{n\times n} \). Because the factor set has been pre-sorted, the first row of elements are all integers, and \( 1, 1, 1, 2, 1 \leq a_{i1} \leq a_{i2} \leq a_{in} \).

Definition 2: Comparison Matrix: The matrix \( b = (b_{ij})_{n\times n} \) generated by the following rules is called the comparison matrix.

Step 1 The first row of the matrix \( b \) has a relationship \( b_{1j} = \frac{1}{b_{j1}} \), \( b_{1j} \leq b_{2j} \leq \ldots \leq b_{nj} \), while the first column element’s value of \( b \) is \( b_{ij} = \frac{1}{b_{i1}} \), in which \( j = 1, 2, \ldots, n \).

Step 2 The kth row element value of the matrix \( b \) is \( b_{kj} = \left[ \frac{b_{j1}}{b_{k1}} \right] \), correspondingly, the kth column element value is \( b_{jk} = \left[ \frac{b_{j1}}{b_{jk}} \right] \), in which \( j = k, k+1, \ldots, n \), \( 2 \leq k \leq n \).

Step 3 Until the element value \( b_{kj} \) is calculated, the algorithm ends, the comparison matrix is generated.

Definition 3: Relative Error Matrix: The matrix \( e = (e_{ij})_{n\times n} \) generated by the following rules is called the relative error matrix.

The matrix \( e \) is obtained by subtracting the comparison matrix from the initial judgment matrix and comparing it with the comparison matrix for the corresponding position, and has a relationship \( e_{ij} = k_{ij} - b_{ij} \), \( i = 1, 2, \ldots, n \), \( j = 1, 2, \ldots, n \).
Definition 4: Transition Judgment Matrix: The generated matrix transformed from the initial judgment matrix is called the transition judgment matrix \( G = (g_{ij})_{n \times n} \).

Definition 5: Matrix Dissimilarity: The Euclidean distance between the principal eigenvectors of two matrices is called the matrix dissimilarity. The dissimilarity of matrices \( A = (a_{ij})_{n \times n} \) and \( B = (b_{ij})_{n \times n} \) is counted as \( d(A, B) \). There is

\[
d(A, B) = \sqrt{\sum_{i,j} (w_{A,i} - w_{B,j})^2}
\]

where \( w_{A,i} \) and \( w_{B,j} \) are the main eigenvectors of matrices \( A \) and \( B \). The smaller value indicates that the matrices \( A \) and \( B \) are more similar.

The smaller the value of \( d(A, B) \) indicates that the matrices \( A \) and \( B \) are more similar.

Definition 6: Target Judgment Matrix: Traverse the comparison matrix that has the same order with the initial judgment matrix, if the generated transition judgment matrix not only satisfies the consistency requirement but also has the smallest matrix dissimilarity with the initial judgment matrix, the transition judgment matrix is the target judgment matrix \( D = (d_{ij})_{n \times n} \).

Proposition 1: All comparison matrices meet the consistency requirements. The correctness of the proposition is verified by the simulation in this paper 3.2.

3.2 The Simulation Verification of Proposition 1
Whether the comparison matrix can meet the requirement of consistency, it’s need to check. Computer simulation is used to verify the consistency of the results. The following is the verification matrix consistency verification traversal nesting algorithm:

Step1 Initialize.
   I. Enter the order of the matrix \( n \), \( n \geq 3 \);
   II. Use variable \( m = n - 1 \) controle the nesting level of process.

Step2 For \( n_1 = 1 \) to 9;
   Step3 if( \( m - 1 \geq 1 \) ) For \( n_2 = 1 \) to 9;
   Step4 if( \( m - 1 \geq 1 \) ) For \( n_3 = 1 \) to 9;
   | Stepn-1 if\( (m - 2 \geq 1) \) For \( n_{v-2} = n_{v-3} \) to 9;
   Stepn if\( (m - 2 \geq 1) \) For \( n_{v-1} = n_{v-2} \) to 9;
   I. suppose \( b_{11} = 1, b_{12} = n_1, \ldots, b_{1, n_1} = n_{n_1 - 1} \), vector \( (b_{11}, b_{12}, \ldots, b_{1, n_1}) \) as the first row of the comparison matrix, call the pre-sort and the rounding function algorithm to construct the comparison matrix \( A \);
   II. Normalize the comparison matrix according to equation (3)
   III. The comparison matrix added by rows according to equation (4), get the vector \( \vec{w} \).
   IV. According to equation (5) get the normalized \( w \), which is the desired feature vector.
   V. Calculate the largest eigenvalue \( \lambda_{max} \) of the comparison matrix according to equation (6).
   VI. If \( \frac{CR - (R - n) / (n - 1) / RI \geq 0.10} {\lambda_{max} \geq 0.10} \), it does not satisfy the consistency requirement.

Stepn+2 The algorithm end.
Consistency checking of all possible AHP judgment matrices of the 3~30th order comparison matrix by the consistency verification traversal nesting algorithm, simulation results show that all meet the requirements. Table 3 lists the number of all possible AHP judgment matrices of 3~29th order comparison matrices.

If first row vector of the matrix given by experts is \( (1, 1, 2, 4, 6, 7, 8, 9, 9) \), the resulting comparison matrix is \( B \), which is as follows.
\[
B = \begin{bmatrix}
1 & 1 & 2 & 4 & 6 & 6 & 7 & 8 & 9 & 9 \\
1 & 1 & 2 & 4 & 6 & 6 & 7 & 8 & 9 & 9 \\
1/2 & 1/2 & 1 & 2 & 3 & 3 & 4 & 4 & 5 & 5 \\
1/4 & 1/4 & 1/2 & 1 & 2 & 2 & 2 & 2 & 3 & 3 \\
1/6 & 1/6 & 1/3 & 1/2 & 1 & 1 & 1 & 1 & 2 & 2 \\
1/6 & 1/6 & 1/3 & 1/2 & 1 & 1 & 1 & 1 & 2 & 2 \\
1/7 & 1/7 & 1/4 & 1/2 & 1 & 1 & 1 & 1 & 2 & 2 \\
1/8 & 1/8 & 1/4 & 1/2 & 1 & 1 & 1 & 1 & 2 & 2 \\
1/9 & 1/9 & 1/5 & 1/3 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/1 \\
1/9 & 1/9 & 1/5 & 1/3 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/1 \\
\end{bmatrix}
\]

After the consistency check, \( \lambda_{\infty} = 10.0629 \), \( CR = 0.0047 < 0.1 \), which is satisfied the consistency requirements.

### Table 3 The number of all possible AHP judgment matrices of 3~29th order comparison matrices

| Order | Number of matrices | Order | Number of matrices | Order | Number of matrices |
|-------|--------------------|-------|--------------------|-------|--------------------|
| 3     | 45                 | 12    | 75582              | 21    | 3108105           |
| 4     | 165                | 13    | 125970             | 22    | 4292145           |
| 5     | 495                | 14    | 203490             | 23    | 5852925           |
| 6     | 1287               | 15    | 319770             | 24    | 7888725           |
| 7     | 3003               | 16    | 490314             | 25    | 10518300          |
| 8     | 6435               | 17    | 735471             | 26    | 13884156          |
| 9     | 12870              | 18    | 1081575            | 27    | 18156204          |
| 10    | 24310              | 19    | 1562275            | 28    | 23535820          |
| 11    | 43758              | 20    | 2220075            | 29    | 30260340          |

### 3.3 Algorithm Flow

First, the initial judgment matrix is generated and the transition judgment matrix is equivalent to the initial judgment matrix. Then all the comparison matrices of the same order as the initial judgment matrix are traversed. According to the descending order of the elements of the error matrix, which is generated by the initial judgment matrix and the comparison judgment matrix, replacing the value of the corresponding place of the transition judgment matrix one by one with the value of the corresponding place of the comparison matrix, and replacing the element values of the corresponding symmetric place until the consistency requirement is satisfied. Finally compare and find out all the comparison matrices and the corresponding transition judgment matrix which satisfies the consistency requirement and the matrix dissimilarity is the smallest, the transition judgment matrix is the target judgment matrix. The specific algorithm flow is as follows.

**Step 1** Initialize

1. Input the order of the matrix \( n, n \geq 3 \).
2. Use variable \( m = n - 1 \) control the nesting level of process.
3. The smallest matrix dissimilarity is \( d = 2000000.0 \).

**Step 2** Generating the \( n \) order initial judgment matrix \( A = (a_{ij})_{n \times n} \), the transition judgment matrix \( G = (g_{ij})_{n \times n} = A \), the target judgment matrix \( D = (d_{ij})_{n \times n} = A \).

**Step 3** For \( n = 1 \) to 9

**Step 4** If \( n = 1 \) For \( n = n \) to 9

**Step 5** If \( n = 2 \) For \( n = n \) to 9

**Step n** If \( n = 2 \) For \( n = n \) to 9

**Step n+1** If \( n = 2 \) For \( n = n \) to 9
I. Make \( b_1 = \frac{1}{1}, b_2 = \frac{1}{2}, \ldots, b_n = \frac{1}{n} \), vector \( (b_1, b_2, \ldots, b_n) \) as the first row of the comparison matrix, generating the comparison matrix \( B \).

II. Generating the relative error matrix \( E = (e_{ij})_{n \times n} \) from matrices \( A \) and \( B \).

III. According to the descending order of the elements of the error matrix \( E \), replacing the value of the corresponding place of the transition judgment matrix one by one with the value of the corresponding place of the comparison matrix, and replacing the element values of the corresponding symmetric place until the consistency requirement is satisfied. If the matrix dissimilarity is smaller than the previous one, correct the dissimilarity, and the target judgment matrix is the transition judgment matrix. (The verification consistency algorithm is the same as the judgment matrix consistency verification traversal nesting algorithm).

Stepn+2 In this case, the target judgment matrix corresponding to the matrix dissimilarity value is the solution.

4. Example Analysis

4.1 Example

The above algorithm can be used to achieve by matlab programming, the following is an example of matrix adjustment.

Initial judgment matrix

\[
A = \begin{pmatrix}
1 & 2 & 5 & 9 \\
1/2 & 1 & 2 & 5 \\
1/5 & 1/2 & 1 & 1/3 \\
1/9 & 1/5 & 3 & 1
\end{pmatrix}
\]

The dominant eigenvector of matrix \( A \) is \( w_A = [0.5335, 0.2636, 0.0896, 0.1133] \), \( CR = 0.1874 > 0.10 \), so it does not satisfy the consistency requirements, it needs to be adjusted.

According to the generation algorithm in this paper, until the success of the adjustment, then

\[
B = \begin{pmatrix}
1 & 2 & 5 & 9 \\
1/2 & 1 & 3 & 5 \\
1/5 & 1/3 & 1 & 2 \\
1/9 & 1/5 & 1/2 & 1
\end{pmatrix},
E = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 1/3 & 0 \\
0 & 1/2 & 0 & 5/6 \\
0 & 0 & 5 & 0
\end{pmatrix},
D = \begin{pmatrix}
1 & 2 & 5 & 9 \\
1/2 & 1 & 2 & 5 \\
1/5 & 1/2 & 1 & 2 \\
1/9 & 1/5 & 1/2 & 1
\end{pmatrix}
\]

The dominant eigenvector of matrix \( D \) is \( w_D = [0.5526, 0.2689, 0.1202, 0.0583] \), \( CR = 0.0028 < 0.10 \), it satisfies the consistency requirements.

According to the adjustment method which is described by Saaty, The final judgment matrix is

\[
D_f = \begin{pmatrix}
1 & 2 & 5 & 9 \\
1/2 & 1 & 2 & 5 \\
1/5 & 1/2 & 1 & 2.1264 \\
1/9 & 1/5 & 0.4703 & 1
\end{pmatrix}
\]

The dominant eigenvector of matrix \( D_f \) is \( w_{D_f} = [0.5521, 0.2686, 0.1219, 0.0573] \), \( CR = 0.0027 < 0.10 \), it satisfies the consistency requirements.

4.2 Validity Analysis

The validity of the generating algorithm can be measured in terms of the reachability of the algorithm, matrix dissimilarity, and the maximum adjustment of the matrix elements.

The AHP judgment matrix generation algorithm based on pre-sort and round-up function adjusts the initial judgment matrix based on the comparison matrix which satisfies the consistency requirement, for the n-order initial judgment matrix which does not satisfy the consistency, the target judgment matrix can be solved by adjustment in at most n(n-1)/2 times. Therefore, the algorithm is convergent.

In the part 4.1, the dissimilarity between the target judgment matrix generated by the algorithm proposed in this paper and the initial judgment matrix is

\[
d(A, D) = \sqrt{\sum_{i=1}^{n} \left| w_A^i - w_D^i \right|^2} = 0.0660 .
\]

However, the
dissimilarity of the matrix generated by the method described by Saaty is
\[ d(A, D) = \sqrt{\sum_{k=1}^{n} \left( w_k^i - w_k^j \right)^2} = 0.0674 \]

thereby \( d(A, D) < d(A, D_s) \).

In the part 4.1, the algorithm proposed in this paper adjusts the element of the initial judgment matrix \( a_{i1} = 3 \) to \( d_{i1} = 0.5 \), the adjustment rate is 83.33\%. But the method described by Saaty adjusts the element of the initial judgment matrix \( a_{43} = 3 \) to \( d_{43} = 0.4703 \), the adjustment rate is 84.32\%.

5. Conclusion
Compared with the traditional Adjustment Algorithm for AHP Judgment Matrix, On the basis of making full use of the information of the initial judgment matrix, the AHP judgment matrix generation algorithm based on pre-sort and round-up function finds out a target judgment matrix which satisfies the consistency requirement, and the matrix dissimilarity and the adjusted ratio is smaller than the traditional AHP judgment matrix adjustment method, and can ensure the elements of the generated target adjustment matrix are in the range 1 to 9 and their reciprocal. Therefore, the AHP judgment matrix generation algorithm based on pre-sort and round-up function is reasonable and effective, and has strong operability and practicability.

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