THE SIMPLE EXCLUSION PROCESS ON THE CIRCLE HAS A DIFFUSIVE CUTOFF WINDOW

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Abstract. In this paper, we investigate the mixing time of the simple exclusion process on the circle with $N$ sites, with a number of particle $k(N)$ tending to infinity. For $k \leq N/2$ we show that the mixing time is asymptotically equivalent to $(8\pi^2)^{-1}N^2 \log k$, while the cutoff window which the time need for the distance to equilibrium to drop from $1 - \varepsilon$ to $\varepsilon$ for $\varepsilon > 0$ is identified to be $N^2$. We also prove that for most initial conditions, a time $O(N^2)$ is sufficient for mixing. Some portion of the proof remains valid in higher dimension.

Keywords: Markov chains, Mixing time, Particle systems, Cutoff Window

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1. Introduction

The Symmetric Simple Exclusion Process (which we will sometimes refer to simply as the Simple Exclusion) is one the simplest particle system with local interaction. It can be considered as a toy model for the relaxation of a gas of particles and was introduced by Spitzer in [15]. Since then, it has been the object of a large number of studies by mathematician and theoretical physicists, who investigated the evolution rules for the particle density, tried to derive Fick’s law from microscopic dynamics, studied to motion of an individual tagged particle (see [12] or [4] for reviews on the subject and references). More recently in the literature, an interest has been developed for the convergence to equilibrium of the process on a finite graph in terms of mixing time, which is the object of our study.

1.1. The Process. We consider $\mathbb{Z}_N := \mathbb{Z}/N\mathbb{Z}$, the discrete circle with $N$ sites and place $k \in \{1, \ldots, N-1\}$ particles on it, with at most one particle per site. With a slight abuse of notation, we will sometimes use elements of $\{1, \ldots, N\} \subset \mathbb{Z}$ to refer to elements of $\mathbb{Z}_N$.

The exclusion process on $\mathbb{Z}_N$ is a dynamical evolution of the particle system which can be described informally as follows: each particle tries to jump independently on its neighbors with transition rates $p(x, x+1) = p(x, x-1) = 1$, but the jumps are cancelled
if a particle tries to jump on a site which is already occupied (see Figure 1 in Section 5 for a graphical representation).

More formally, let our state-space be defined by
\[ \Omega = \Omega_{N,k} = \left\{ \eta \in \{0,1\}^{\mathbb{Z}_N} \mid \sum_{x=1}^{N} \eta(x) = k \right\}. \] (1.1)

Given \( \eta \in \Omega \) define \( \eta^x \) the configuration obtained by inverting the content of site \( x \) and \( x+1 \)
\[ \begin{align*}
\eta^x(x) &= \eta(x+1), \\
\eta^x(x+1) &= \eta(x), \\
\eta^x(y) &= \eta(y), \quad \forall y \notin \{x,x+1\}.
\end{align*} \] (1.2)

The exclusion process on \( \mathbb{Z}_N \) with \( k \) particles is the continuous time Markov process on \( \Omega_{N,k} \) whose generator is given by
\[ (Lf)(\eta) := \sum_{x \in \mathbb{Z}_N} f(\eta^x) - f(\eta). \] (1.3)

The only probability measure left invariant by \( L \) is the uniform probability measure on \( \Omega \) which we denote by \( \mu \). Given \( \chi \in \Omega \) we let \((\eta^\chi_t)_t \geq 0\) denote the trajectory of the Markov chain starting from \( \chi \).

We want to know how long we must wait to reach the equilibrium state of the particle system, for which all configurations are equally likely. We measure the distance to equilibrium is measured in terms of total variation distance. If \( \alpha \) and \( \beta \) are two probability measures on \( \Omega \), the total variation distance between \( \alpha \) and \( \beta \) is defined to be
\[ \|\alpha - \beta\|_{TV} := \frac{1}{2} \sum_{\omega \in \Omega} |\alpha(\omega) - \beta(\omega)| = \sum_{\omega \in \Omega} (\alpha(\omega) - \beta(\omega))_+. \] (1.4)

where \( x_+ = \max(x,0) \) is the positive part of \( x \). It measures how well one can couple two variables with law \( \alpha \) and \( \beta \). We define the distance to equilibrium of the Markov chain
\[ d(t) = d^{N,k}_t(t) := \max_{\chi \in \Omega_{N,k}} \|P^\chi_t - \mu\|_{TV}. \] (1.5)

For a given \( \varepsilon > 0 \) we define the \( \varepsilon \)-mixing-time to be the time needed for the system to be at distance \( \varepsilon \) from equilibrium
\[ T^{N,k}_{\text{mix}}(\varepsilon) := \inf\{t \geq 0 \mid d^{N,k}(t) \leq \varepsilon\}. \] (1.6)

Let us mention that the convergence to equilibrium has also been studied in terms of asymptotic rates: it has been known for a long time that for any reversible Markov chain
\[ t^{-1} \log d(t) = -\lambda_1 \] (1.7)
exists and that \( \lambda_1 > 0 \) is the smallest nonzero eigenvalue of \(-L\), usually referred to as the spectral gap (see [8, Chapter 12]).

The exclusion process can in fact be defined on an arbitrary graph and its mixing property have been the object of a large number of works. Let us mention a few of them here. In [3], the study of the exclusion on the complete graph with \( N/2 \) is reduced to the study of the birth and death chain and a sharp asymptotic for the mixing time is given using a purely algebraic approach (see also [5] for a probabilist approach of the problem for arbitrary \( k \)). In [13] the mixing time for the exclusion on \((\mathbb{Z}_N)^d\) with \( k \) particle is proved
to be of order $O(N^2 \log k)$ (see also [14] for a generalization on an arbitrary graph). In [1], it is shown that the spectral gap of the simple-exclusion on any graph is equal to that of the underlying simple random walk (e.g. in our case $\lambda_1 = 2(1 - \cos(2\pi/N))$). In [16], the mixing time of the exclusion process on the segment is proved to be larger than $(2\pi^2)^{-1}N^2 \log N$ and smaller than $(\pi^2)^{-1}N^2 \log N$, with the conjecture that the lower bound is sharp. This conjecture was brought on a rigorous ground in [6].

1.2. The main result. The main result of this paper is a sharp asymptotic for the exclusion on $\mathbb{Z}_N$. For a fixed $\varepsilon \in (0, 1)$, when $N$ and $k$ goes to infinity we are able to identify the asymptotic behavior of $T_{\text{mix}}(\varepsilon)$. We obtain that when $k \leq N/2$ (which by symmetry is not a restriction)

$$T_{\text{mix}}(\varepsilon) = \frac{N^2}{8\pi^2}(\log k)(1 + o(1)).$$

Note that here the dependence in $\varepsilon$ is not present in the asymptotic equivalent. This means that on a time window which is $o(N^2 \log k)$ the distance to equilibrium drops abruptly from 1 to 0. This sudden collapse to equilibrium was first observe by Diaconis and Shahshahani [2] in the case of the (mean-field) transposition shuffle. The term cutoff itself was coined in ?. It is believed that cutoff holds with some generality for reversible Markov chains as soon as the mixing time is much larger than the inverse of the spectral gap, but this remains a very challenging conjecture (see [8, Chapter 18] for more about cutoff and a counterexample in the non-reversible case).

A natural question is of course, then on what time scale does $d(t)$ decreases from, say, 3/4 to 1/4. This is what is called the cutoff window. We are able to show it is equal to $N^2$.

Let us mention that, in spite of numerous studies on the subject, this is result is, to our knowledge, the first sharp derivation of a cutoff window for a non mean-field interacting particle system.

Theorem 1.1. For any sequence $k(N)$ satisfying $k(N) \leq N/2$ and tending to infinity. We have for every $\varepsilon \in (0, 1)$

$$\lim_{N \to \infty} \frac{8\pi^2 T_{\text{mix}}^{N,k}(\varepsilon)}{N^2 \log k} = 1. \quad (1.8)$$

More precisely we have

$$\lim_{u \to -\infty} \limsup_{N \to \infty} d^{N,k} ((8\pi^2)^{-1}N^2 \log k + uN^2) = 0,$$

$$\lim_{u \to \infty} \liminf_{N \to \infty} d^{N,k} ((8\pi^2)^{-1}N^2 \log k + uN^2) = 1 \quad (1.9)$$

and the window is optimal in the sense that for any $u \in \mathbb{R}$

$$\limsup_{N \to \infty} d^{N,k} ((8\pi^2)^{-1}N^2 \log k + uN^2) < 1, \quad (1.10)$$

$$\liminf_{N \to \infty} d^{N,k} ((8\pi^2)^{-1}N^2 \log k + uN^2) > 0.$$  

Remark 1.2. Our result does not treat the case of a bounded number of particle. In this case there is no cutoff and the mixing time is of order $N^2$ for every $\varepsilon$ with a pre-factor which depends on $\varepsilon$ (a behavior very similar to the random-walk: case $k = 1$).
In the course of the proof, we are going to see that in fact for most initial conditions \( \chi \) the time needed for the distance to equilibrium \( \| P^X_t \chi - \mu \| \) to be small is much smaller than the mixing time, of order \( N^2 \). Let us illustrate this fact here by a statement which is a by-product of some of the intermediate results presented throughout the paper. Here and in what follows \( \mu(f(\xi)) \) denote the expectation of \( f(\xi) \) when \( \xi \) has law \( \mu \).

**Theorem 1.3.** For \( \chi \in \Omega_{N,k} \), set \( d(\chi, t) := \| P^X_t \chi - \mu \| \). For any sequence \( k(N) \) satisfying \( k(N) \leq N/2 \) and tending to infinity, we have for all \( s > 0 \)

\[
0 < \liminf_{N \to \infty} \mu(d(\xi, N^2 s)) \leq \limsup_{N \to \infty} \mu(d(\xi, N^2 s)) < 1 \tag{1.11}
\]

and

\[
\lim_{s \to \infty} \mu(d(\chi, N^2 s)) = 0. \tag{1.12}
\]

1.3. **Differences between the segment and the circle.** In [6], our proof to derive the mixing time for the exclusion on the segment heavily relied on monotonicity arguments: using the height-function representation of the particle system, we equipped the set of particle configuration with an order which was preserved by the dynamics (see [6, Section 3]). The monotonicity was then used in almost every step of the proof.

The drawback of this approach is that it is not very robust, and cannot be used for either higher dimension graphs (for instance \( \{1, \ldots, N\}^d \) with either free or periodic boundary condition) nor on the circle.

With this in mind, our idea when studying the exclusion on the circle is also to develop an approach to the problem which is more flexible, and could provide a step towards the rigorous identification of the cutoff threshold in higher dimension (see Section 2.4 for conjectures and rigorous lower-bounds). This goal is only partially achieved as even if we do not require monotonicity, a part of our proof relies on the interface representation of the process (see Section 5) which is a purely one-dimensional feature.

Apart from being more robust the method we develop give more precise result then the one in [6] as we identify exactly the width of the cutoff window (and it also extends to the segment). However, we could not extract from it the asymptotic mixing time for the adjacent transposition shuffle, we hope to be able to do it in a future work.

1.4. **Organization of the paper.** In Section 2 we prove the part of the results which corresponds to lower-bounds for the distance to equilibrium, that is to say, the first lines of (1.9) and (1.9). The proof of this statement is very similar to the one proposed by Wilson in [16], the only difference is that we work directly with the particle configuration instead of the height-function. Doing things in this manner underlines that the proof in fact does not rely much on the dimension. We also state the result for the higher dimensional torus as allows to improve the best existing bound in the literature [13] for \( d \geq 2 \) (see Section 2.4).

The main novelty in the paper is the strategy to prove upper-bound results (second lines of (1.9) and (1.9)). In Section 3 we explain how the proof is decomposed. In Section 4, we use a comparison inequality of Liggett [10] to control the density of fluctuation of particle after a time \( \frac{N^2}{8\pi^2} \log k \). Finally owe conclude by showing that configuration which have reasonable density fluctuation couples with equilibrium with time \( O(N^2) \), using interface representation for the particle system, and a coupling based on the graphical construction. The construction is detailed in Section 5, and the proof is performed using a multi-scale analysis in Section 6.
2. LOWER BOUND ON THE MIXING TIME

2.1. The statement. The aim of this Section is to prove the some lower bounds on the distance to equilibrium.

**Proposition 2.1.** For any sequence $k(N)$ satisfying $k(N) \leq N/2$ and tending to infinity, we have

$$\lim_{u \to -\infty} \lim_{N \to \infty} d^{N,k} ((8\pi^2)^{-1}N^2 \log k + uN^2) = 1,$$

and for any $u \in \mathbb{R}$

$$\liminf_{N \to \infty} d^{N,k} ((8\pi^2)^{-1}N^2 \log k + uN^2) > 0.$$  

2.2. Relaxation of the “first” Fourier coefficient. The main idea is to look at “the first” Fourier coefficient (a coefficient corresponding to one of the smallest eigenvalues of the discrete Laplacian on $\mathbb{Z}_N$), of $\eta_t$. For $\eta \in \Omega_{N,k}$, one defines

$$a_1(\eta) := \sum_{x \in \mathbb{Z}_N} \eta(x) \cos \left( \frac{2\pi x}{N} \right).$$

It is an eigenfunction of the generator $L$, (the reason for this being that each particle performs a diffusion for which $\cos(\frac{2\pi x}{N})$ is an eigenfunction), associated to the eigenvalue $-\lambda_1$ where

$$\lambda_1 := 2 \left(1 - \cos \left(\frac{2\pi}{N}\right)\right).$$

**Lemma 2.2.** The function $a_1$ is an eigenfunction of the generator $L$ with eigenvalue $-\lambda_1$, and as a consequence, for any initial condition $\chi \in \Omega$

$$M_t := e^{-t\lambda_1} a_1(\eta^\chi_t)$$

is a martingale for the filtration $\mathcal{F}$ defined by

$$\mathcal{F}_t := \sigma(\{\eta_s\}_{s \leq t}).$$

In particular we have

$$\mathbb{E}[a_1(\eta^\chi_t)] = e^{-t\lambda_1} a_1(\chi).$$

Furthermore one can find a constant such that for all $t \geq 0$

$$\text{Var}[a_1(\eta^\chi_t)] \leq 2k.$$  

**Proof.** We have

$$L a_1(\eta) := \sum_{x \in \mathbb{Z}_N} (a_1(\eta^x) - a_1(\eta))$$

$$= \sum_{x \in \mathbb{Z}_N} (\eta(x+1) - \eta(x)) \left( \cos \left(\frac{2\pi x}{N}\right) - \cos \left(\frac{2\pi (x+1)}{N}\right) \right)$$

$$= \sum_{x \in \mathbb{Z}_N} \eta(x) \left( 2 \cos \left(\frac{2\pi x}{N}\right) - \cos \left(\frac{2\pi (x-1)}{N}\right) - \cos \left(\frac{2\pi (x+1)}{N}\right) \right) = -\lambda_1 a_1(\eta)$$

where the second equality comes from a re-indexation of the sum and the last one from the identity

$$2 \cos \left(\frac{2\pi x}{N}\right) - \cos \left(\frac{2\pi (x-1)}{N}\right) - \cos \left(\frac{2\pi (x+1)}{N}\right) = -\lambda_1 \cos \left(\frac{2\pi x}{N}\right)$$
From the Markov property and the definition of the generator we have for every positive \(t\),
\[
\partial_s E[M_{t+s} \mid \mathcal{F}_t]_{s=0} = \lambda_1 M_t + e^{t\lambda_1} (\mathcal{L}a_1)(\eta^x_t) = 0. \tag{2.10}
\]
which implies that it is a martingale.

Now let us try to estimate the variance of \(M_t\): for the process with \(k\) particles, the maximal transition rate is \(2k\) (each of the \(k\) particles can jump in at most 2 directions independently with rate one). If a transition occurs at time \(s\), the value of \(M_s\) varies at most by an amount
\[
e^{\lambda_1 s} \max_{x \in \mathbb{Z}_N} \left| \cos \left( \frac{2\pi x}{N} \right) - \cos \left( \frac{2\pi (x+1)}{N} \right) \right| \leq e^{\lambda_1 s} \frac{2\pi}{N}.
\]

With this in mind we can obtain a bound on the bracket of \(M\)
\[
\langle M \rangle_t \leq 2k \int_0^t e^{2\lambda_1 s} \left( \frac{2\pi}{N} \right)^2 \, ds \tag{2.11}
\]
Then using the fact that \(\text{Var} M_t = E [\langle M \rangle_t]\), we have, for \(N\) sufficiently large, for any \(\chi \in \Omega_{N,k}\) and any \(t \geq 0\)
\[
\text{Var} [a_1 (\eta_{t}^x)] = e^{-2\lambda_1 t} E [\langle M \rangle_t] \leq 2k \int_0^t e^{2\lambda_1 (s-t)} \left( \frac{2\pi}{N} \right)^2 \, ds \leq \frac{4\pi^2 k}{N^2 \lambda_1} \leq 2k \tag{2.12}
\]
where the last inequality comes from the fact that \(\lambda_1 \sim 4\pi^2 N^{-2}\). \(\Box\)

At equilibrium (under the distribution \(\mu\)) \(a_1 (\eta)\) has mean zero and typical fluctuations of order \(\sqrt{k}\). The equilibrium variance can either be computed directly or one can use (2.7) for \(t \to \infty\) to obtain
\[
\text{Var}_\mu (a_1(\eta)) \leq 2k.
\]

From (2.7), if \(E [a_1 (\eta_{t}^x)]\) is much larger than \(\sqrt{k}\) then \(a_1 (\eta_{t}^x)\) is much larger than \(\sqrt{k}\) with large probability which implies \(\|P_t^x - \mu\|\) has to be large. We need to use this reasoning for a \(\chi\) which maximizes \(a_1\).

2.3. Proof of Proposition 2.1. Consider \(\chi = \chi_0\) being the configuration which minimizes \(a_1\).

\[
\chi_0 (x) := \begin{cases} 1_{x \in \{-p, \ldots, p\}} & \text{if } k = 2p + 1, \\ 1_{x \in \{-p+1, \ldots, p\}} & \text{if } k = 2p, \end{cases} \tag{2.13}
\]
It is rather straight-forward to check that for any \(N \geq 2, k \leq N/2\)
\[
a_1 (\chi_0) \geq k/2. \tag{2.14}
\]
Hence we have for all \(t > 0\)
\[
E [a_1 (\eta_{t}^{\chi_0})] \geq ke^{-\lambda_1 t}/2. \tag{2.15}
\]

Using [8, Proposition 7.8] (which is just a clever use of the Cauchy Schwartz inequality) we have
\[
\|P_t^{\chi_0} - \mu\|_{TV} \geq \frac{(E [a_1 (\eta_{t}^{\chi_0})])^2}{(E [a_1 (\eta_{t}^{\chi_0})])^2 + 2 \left[ \text{Var} (a_1 (\eta_{t}^{\chi_0})) + \text{Var}_\mu (a_1 (\eta)) \right]} \geq \frac{1}{1 + 32k^{-1} \exp(-2\lambda_1 t)}. \tag{2.16}
\]
Given \( u \in \mathbb{R} \), using the above inequality for \( t = t_N := uN^2 + \frac{N^2}{8\pi^2} \log k \) the reader can check that

\[
\lim_{N \to \infty} \inf \| P_{t_N}^{\chi_0} - \mu \|_{TV} \geq \lim_{N \to \infty} \frac{1}{1 + 32k^{-1}\exp(-2\lambda_1 t_N)} = \frac{1}{1 + 32e^{-8\pi^2u}}, \tag{2.17}
\]

which implies both (2.1) and (2.2).

\[\square\]

2.4. The exclusion in higher dimensions. Let us shortly present in this Section a generalization of Proposition 2.1 for the exclusion process in higher dimension \( d \geq 2 \).

For \( N \in \mathbb{N} \), and \( k \leq N^d/2 \) we define the state of particle configuration as

\[
\Omega_{N,k}^d := \{ \eta \in \{0, 1\}^{Z_N^d} \mid \sum_{x \in Z_N^d} \eta(x) = k \}. \tag{2.18}
\]

Given \( x \sim y \) a pair of neighbor on the torus \( Z_N^d \), we set

\[
\eta^{x,y} := \begin{cases}
\eta^{x,y}(x) = \eta(y), \\
\eta^{x,y}(y) = \eta(x), \\
\eta^{x,y}(z) = \eta(z), \text{ for } z \notin \{x, y\}.
\end{cases} \tag{2.19}
\]

and define the generator by

\[
\mathcal{L}f(\eta) := \sum_{x, y \in Z_N^d} f(\eta^{x,y}) - f(\eta). \tag{2.20}
\]

We set \( d_{N,k,d} \) to be the distance to equilibrium of the chain with generator \( \mathcal{L} \) at time \( t \) (see (1.5)). The we can adapt the proof of Proposition 2.1 and show that

**Proposition 2.3.** For any sequence \( k(N) \) satisfying \( k(N) \leq N^d/2 \) and tending to infinity, we have

\[
\lim_{u \to \infty} \lim_{N \to \infty} d_{N,k,d} ^{\mathcal{L}} \left((8\pi^2)^{-1}N^2 \log k - uN^2\right) = 1, \tag{2.21}
\]

and for any \( u \in \mathbb{R} \)

\[
\liminf_{N \to \infty} d_{N,k,d} ^{\mathcal{L}} \left((8\pi^2)^{-1}N^2 \log k + uN^2\right) > 0. \tag{2.22}
\]

**Remark 2.4.** Note that the result remains valid if the torus is replaced by the grid (i.e. if we drop the periodic boundary condition) in which case \((8\pi^2)^{-1}\) has to be replaced by \((2\pi^2)^{-1}\). In view of this result, and of the content of the next section, it is natural to conjecture that \((8\pi^2)^{-1}N^2 \log k\) is the mixing time of the exclusion process on the torus.

**Proof.** The proof is almost exactly the same. The eigenfunction which one has to consider is

\[
a_1(\eta) := \sum_{x \in Z_N^d} \eta(x) \cos\left(\frac{x_1 \pi}{N}\right). \tag{2.23}
\]

where \( x_1 \in \mathbb{Z}_N \) is the first coordinate of \( \mathbb{Z}_N \). It is not difficult to check that if \( \chi_0 \) is a maximizer of \( Z_N^d \) (there might be many of them) \( a_1(\chi_0) \) is larger than \( k/2 \). \[\square\]
3. Upper bound on the mixing time

3.1. Decomposition of the proof. To complete the proof of the main result, we have to prove

**Proposition 3.1.** For any sequence $k(N)$ satisfying $k(N) \leq N/2$ and tending to infinity, we have

$$\lim_{u \to \infty} \limsup_{N \to \infty} d^{N,k} ((8\pi^2)^{-1} N^2 \log k + uN^2) = 0,$$

(3.1)

and for any $u \in \mathbb{R}$

$$\limsup_{N \to \infty} d^{N,k} ((8\pi^2)^{-1} N^2 \log k + uN^2) < 1.$$

(3.2)

The proof of this statement is slightly more involved than that of Proposition (2.1) and relies on an explicit coupling of $P^\chi_t$ and the equilibrium measure $\mu$ for an arbitrary $\chi \in \Omega$, which requires two step.

In a first step we want to show that after a time $t_0 = (8\pi^2)^{-1} N^2 \log k$ the density of particle is close to $k/N$ everywhere on the torus and that the deviation from it are not larger than equilibrium fluctuation (which are of order $\sqrt{k}$). This part of the proof relies on comparison inequalities developed by Liggett [10], which allow to replace the exclusion process with $k$ independent random walks.

In a second step, we construct a dynamical coupling of the process starting $\chi$ which has fluctuations of order $\sqrt{k}$, with one starting from equilibrium, using the height-function representation. We show that the two height functions couple within a time $O(L^2)$ which is what we need to conclude. The construction of the coupling and heuristic explanations are given in Section 5, while the proof is performed in Section 6.

3.2. Control of the fluctuation of the particle density. To present the main proposition of the first step we need to introduce some notation. Given $x \neq y$ in $\mathbb{Z}_N$, we define the interval $[x,y]$ to be the smallest (for the inclusion) subset $I$ of $\mathbb{Z}_N$ which contains $x$ and which satisfies

$$\forall z \in I \setminus \{y\}, \ z + 1 \in I.$$

(3.3)

Let $f$ be a function defined on $\mathbb{Z}_N$ we use the notation

$$\sum_{z=x}^y f(z) := \sum_{z \in [x,y]} f(z).$$

(3.4)

We define the length of the interval (which we write $\# [x,y]$) to be the number of points in it (e.g. it is equal to $y - x + 1$ if $1 \leq x \leq y \leq N$). We will prove the following result: given $A \geq 0$ we set

$$t_A = \frac{N^2}{8\pi^2} \log N - AN^2.$$

(3.5)

**Proposition 3.2.** There exists a constant $c$ such that, for all $A \in \mathbb{R}$, for all $N$ sufficiently large (depending on $A$) for all initial condition $\chi \in \Omega_{N,k}$,

$$\mathbb{P} \left[ \exists x,y \in \mathbb{Z}_N, \left| \sum_{z=x+1}^y (\eta^\chi_A(z) - \frac{k}{N}) \right| \geq \left( s + 4e^{4\pi^2 A} \right) \sqrt{k} \right] \leq 2 \exp \left( -cs^2 \right).$$

(3.6)
3.3. **Coupling with small fluctuations.** In the second step of our proof, we show that starting from a configuration with small density fluctuation we can relax to equilibrium within time $O(N^2)$. Set
\[ \mathcal{G}_s := \left\{ \eta \in \Omega \mid \forall x, y \in \mathbb{Z}_N, \left| \sum_{z=x+1}^{y} (\eta_z^X(z) - \frac{k}{N}) \right| \leq s \sqrt{k} \right\} \quad (3.7) \]

The following Proposition establishes this diffusive relaxation to equilibrium in two ways: first it shows that one gets $\varepsilon$ close to equilibrium within a time $C(s, \varepsilon)$, but also that one the scale $N^2$ the distance becomes immediately bounded away from one for positive times.

**Proposition 3.3.** For any $s \geq 1$, given $\varepsilon > 0$ there exists a constant $C(s, \varepsilon)$ such that
\[ \forall \chi \in \mathcal{G}_s, \|P^X_{C(s, \varepsilon)N^2} - \mu\|_{TV} \leq \varepsilon. \quad (3.8) \]

For any $s > 0$, there exists $c(s) > 0$ which is such that
\[ \forall \chi \in \mathcal{G}_s, \|P^X_{N^2} - \mu\|_{TV} \leq 1 - c(s). \quad (3.9) \]

Now we show that Propositions 3.2 and 3.7 are sufficient to prove (3.1).

**Proof of Proposition 3.1.** For $A \geq 0$. We use the semi-group property at time $t_A$. We have for any $\chi \in \Omega$
\[ P^X_{t_A + CN^2} = \sum_{\chi' \in \Omega} P^X_{t_A}(\chi')P^X_{CN^2} (\cdot). \quad (3.10) \]

Hence, using the triangular inequality for any event $\mathcal{G}$ we have
\[ \|P^X_{t_A + CN^2} - \mu\| \leq \sum_{\chi' \in \Omega} P^X_{t_A}(\chi')\|P^X_{CN^2} - \mu\| \leq P^X_{t_A}(\mathcal{G}^c) + P^X_{t_A}(\mathcal{G}) \max_{\chi' \in \mathcal{G}} \|P^X_{CN^2} - \mu\| \quad (3.11) \]

We can now start the proof of (3.1). According to Proposition 3.2, if $s$ is sufficiently large, we have
\[ P^X_{t_0}(\mathcal{G}_s^c) \leq \varepsilon / 2. \quad (3.12) \]

Fixing such an $s$ (which we denote by $s(\varepsilon)$), according to Proposition 3.3 we can find a constant $C(\varepsilon)$ which is such that
\[ \max_{\chi' \in \mathcal{G}_s(\varepsilon)} \|P^X_{C(\varepsilon)N^2} - \mu\| \leq \varepsilon / 2, \quad (3.13) \]

which is enough to conclude, using (3.11) with $A = 0$, $\mathcal{G} = \mathcal{G}_s(\varepsilon)$, and $C = C(\varepsilon)$.

We now prove (3.2). For a fixed $u < -1$, for $A = 1 - u$, we can find $s(u)$ sufficiently large such that
\[ P^X_{t_A}(\mathcal{G}_{s(u)}^c) \leq \frac{1}{2}. \quad (3.14) \]

Using (3.9) for and $s = s(u)$, we find that there exists a constant $c(u)$ which is such that
\[ \max_{\chi' \in \mathcal{G}_{s(u)}} \|P^X_{N^2} - \mu\|_{TV} \leq 1 - 2c(u). \quad (3.15) \]

We can then conclude by using (3.11) for $A = s - 1$ and $\mathcal{G} = \mathcal{G}_{s(u)}$, that for large $N$
\[ \limsup_{N \to \infty} d^{N,k} \left( (8\pi^2)^{-1}N^2 \log k + uN^2 \right) < 1 - c(u). \quad (3.16) \]
\[ \square \]
4. Proof of Proposition 3.2

Proposition 4.1. There exists a constant c such that for all N sufficiently large, for all \( t \geq 3N^2 \) for all \( \chi \in \Omega_{N,k} \),

\[
P \left[ \exists x, y \in \mathbb{Z}_N, \left| \sum_{z=x+1}^{y} (\eta^\chi_t(z) - \mathbb{E}[\eta^\chi_t]) \right| \geq s\sqrt{k} \right] \leq 2 \exp\left(-cs^2\right). \tag{4.1}
\]

Remark 4.2. Note that by taking \( t = \infty \) with \( A = 0 \) in (4.1), we also have a result concerning density fluctuation for the equilibrium measure \( \mu \) which we will use during our proof.

\[
\mu \left( \exists x, y \in \mathbb{Z}_N, \left| \sum_{z=x+1}^{y} (\eta(z) - \frac{k}{N}) \right| \geq s\sqrt{k} \right) \leq 2 \exp\left(-cs^2\right). \tag{4.2}
\]

Proposition 3.2 can be deduced from Proposition 4.1 using the following Lemma which relies on standard heat-kernel estimates proved in the appendix.

Lemma 4.3. The following statement hold

(i) For \( N \) large enough, for all \( \chi \), all \( x \in \mathbb{Z}_N \) and \( t \geq N^2 \)

\[
\left| \mathbb{E}[\eta^\chi_t(x)] - \frac{k}{N} \right| \leq 4kN^{-1}e^{-\lambda t}. \tag{4.3}
\]

In particular

\[
\mathbb{E}[\eta^\chi_t(x)] \leq \frac{2k}{N}. \tag{4.4}
\]

(ii) If \( X_t \) is a nearest neighbor random walk starting from \( x_0 \in \mathbb{Z}_N \) one has for all \( t \geq N^2 \), all \( x, y \in \mathbb{Z}_N \)

\[
P[X_t \in [x, y]] \leq \frac{2\#[x, y]}{N} \tag{4.5}
\]

The idea of the proof is to control the Laplace transform of the number of particle in each interval (and then we roughly have to sum over all intervals to conclude). To control this Laplace transform, with use a comparison inequality due to Liggett [10], which allows to compare the simple exclusion with a particle system without exclusion, that is: \( k \) independent random walks on the torus. With this comparison at hand, the Laplace transform can be controlled simply by controlling the heat equation.

4.1. Estimate on the Laplace transform. From now on the initial condition \( \chi \) is fixed, and for convenience, does not always appear in the notation. For \( x \in \mathbb{Z}_N \) (\( x \in \{1, \ldots, N-1\} \)) and we set

\[
S_{x,y}(t) := \sum_{z=x+1}^{y} (\eta^\chi_t(z) - \mathbb{E}[\eta^\chi_t]), \tag{4.6}
\]

\[
S_{x}(t) := S_{0,x}(t),
\]

Lemma 4.4. For all \( x \in \mathbb{Z}_N \), for all \( t \geq N^2 \)

\[
\mathbb{E} \left[ e^{\alpha S_x(\eta)} \right] \leq \exp\left(2\frac{kx}{N} \alpha^2 \right). \tag{4.7}
\]
Remark 4.5. Of course the formula (4.7) remains valid for any interval of length $x$ by translation invariance.

Proof of Proposition 4.1. We fix $A$. Note that we can always consider in the proof that $s$ is sufficiently large (as the result is obvious for $s \leq ((\log 2)/c)^{1/2}$). By the triangular inequality we have for all $x, y \in \mathbb{Z}$.

$$|S_{x,y}(t)| \leq |S_x(t)| + |S_y(t)|$$  \hspace{1cm} (4.8)

For convenience we decide to replace $s$ by $16s$ in (4.1) (this only corresponds to changing the value of $c$ by a factor 256). Hence it is sufficient to prove that for any $t \geq N^2$ we have

$$P \left[ \exists x \in \mathbb{Z}_N, |S_x(t)| \geq 8s\sqrt{k} \right] \leq 2 \exp \left(-cs^2\right).$$  \hspace{1cm} (4.9)

Now let us show that we can replace $x \in \mathbb{Z}_N$ with a smaller subset. Let $q_0$ be such that

$$Ns/\sqrt{k} < 2^{q_0} \leq 2Ns/\sqrt{k}$$  \hspace{1cm} (4.10)

For $x \in \{0, \ldots, N-1\}$ (which we consider as an element of $\mathbb{Z}_N$, one can find $y \in 2^{q_0}\{0, \ldots, [(N-1)2^{-q_0}] \}$ (a multiple of $2^{q_0}$) such that $y \leq x \leq y^+ := \max(y + 2^{q_0}, N)$.

We have, from the definition of (4.6)

$$S_x(t) \geq S_y(t) - \sum_{z=y+1}^x P[\eta(t) = 1],$$  \hspace{1cm} (4.11)

$$S_x(t) \leq S_{y^+}(t) + \sum_{z=x+1}^{y^+} P[\eta(t) = 1]$$

From (4.4) and

$$(y^+ - y) \leq 2^q \leq 2Ns/(\sqrt{k}),$$

the second term of both equations in (4.11) is smaller than $4Ns/(\sqrt{k})$ and hence

$$|S_x(t)| \leq \max(|S_y(t)|, |S_{y^+}(t)|) + 4s\sqrt{k}. \hspace{1cm} (4.12)$$

Hence we can reduce (4.13) to proving that

$$P \left[ \exists y \in 2^{q_0}\{0, \ldots, [N2^{-q_0}] \}, |S_y(t)| \geq 4s\sqrt{k} \right] \leq 2 \exp \left(-cs^2\right). \hspace{1cm} (4.13)$$

The next step is a multi-scale analysis. Let $p$ be such that $N \in (2^p, 2^{p+1}]$. Given $y \in 2^{q_0}\{0, \ldots, [(N-1)2^{-q_0}] \}$, we can decompose in base 2 as follows

$$2^{q_0}y := \sum_{q=0}^{p-q_0} \varepsilon_q 2^{p-q}, \hspace{1cm} (4.14)$$

where $\varepsilon_q \in \{0, 1\}$. We set $y_{-1} := 0$ and for $r \in \{0, \ldots, p-q_0\}$

$$y_r := \sum_{q=0}^r \varepsilon_q 2^{p-q}. \hspace{1cm}$$

By the triangular inequality again we have
and hence
\[ \{ |S_y(t)| \geq 4s\sqrt{k} \} \Rightarrow \{ \exists r \in \{0, \ldots, p - q_0\} |S_{y_r-1,y_r}(t)| \geq \left( \frac{3}{4} \right)^r s\sqrt{k} \}. \tag{4.16} \]

Thus, the proof of Proposition 3.2 can be reduced to show the following

**Lemma 4.6.** Let us define
\[ \mathcal{H}(s, t) := \{ \exists q \in \{q_0, \ldots, p\}, \exists y \in \{1, \ldots, \lfloor N2^{-q} \rfloor\}, |S_{2q(y-1),2qy}(t)| \geq \left( \frac{3}{4} \right)^{p-q} s\sqrt{k} \} \tag{4.17} \]

For every \( t \geq N^2 \) we have
\[ \mathbb{P}[\mathcal{H}(s, t)] \leq 2e^{-cs^2} \tag{4.18} \]

Indeed from (4.16) and the reasoning taking place before, one has
\[ \{ \exists x \in \mathbb{Z}_N, |S_x(t)| \geq 8s\sqrt{N} \} \subset \mathcal{H}(s, t). \]

**Proof of Lemma 4.6.** We have by union bound
\[
\mathbb{P}[\mathcal{H}(s, t)] \leq \sum_{q=q_0}^{p} \sum_{y=1}^{\lfloor N2^{-q} \rfloor} \mathbb{P} \left[ |S_{2q(y-1),2qy}(t)| \geq \left( \frac{3}{4} \right)^{p-q} s\sqrt{k} \right] \\
\leq \sum_{q=0}^{p} 2^{p+1-q} \max_{y} \mathbb{P} \left[ |S_{2q(y-1),2qy}(t)| \geq \left( \frac{3}{4} \right)^{p-q} s\sqrt{k} \right] \tag{4.19} 
\]

Hence what we have to do is to find a bound on
\[ \mathbb{E} \left[ |S_{2q(y-1),2qy}(t)| \leq \left( \frac{3}{4} \right)^{p-q} s\sqrt{k} \right] \]

which is uniform in \( y \) and is such that the sum in the second line of (4.19) is smaller than \( 2e^{-cs^2} \). For what follows we can, without loss of generality consider only the case \( y = 1 \), as all the estimates we use are invariant by translation on \( \mathbb{Z}_N \).

Using Lemma 4.4 and the Markov inequality we have for any positive \( \alpha \leq \log 2 \)
\[ \mathbb{E} \left[ |S_{2q}(t)| \geq \left( \frac{3}{4} \right)^{p-q} s\sqrt{k} \right] \leq \exp \left( 2^{q+1} \alpha^2 \frac{k}{N} - \alpha \sqrt{k} \left( \frac{3}{4} \right)^{p-q} s \right). \tag{4.20} \]

We can check that the right-hand side is minimized for
\[ \alpha = \alpha_0 := 2^{-(q+2)} \left( \frac{3}{4} \right)^{p-q} s \frac{N}{\sqrt{k}}, \]

Note that for all \( q \geq q_0 \), (recall (4.10)) one has
\[ \alpha_0 \leq 2^{-(q_0+2)} \left( \frac{3}{4} \right)^{p-q_0} \frac{N}{s \sqrt{k}} \leq \frac{1}{4} \left( \frac{3}{4} \right)^{p-q_0} \leq \log 2. \]  
(4.21)

which ascertains the validity of (4.20), and hence

\[ \mathbb{E} \left[ |S_{2q}(t)| \right] \geq \left( \frac{3}{4} \right)^{p-q} s \sqrt{k} \leq \frac{1}{4} \left( \frac{3}{4} \right)^{p-q} \leq \log 2. \]  
(4.22)

Using the fact that \( N \geq 2^p \) we have

\[ \mathbb{P} \left[ |S_{2q}(t)| \geq \left( \frac{3}{4} \right)^{p-q} s \sqrt{k} \right] \leq e^{-\frac{s^2}{8}(\frac{3}{4})^{p-q}}. \]  
(4.23)

Using this in (4.19) allows us to conclude (choosing \( c \) appropriately).

\[ \square \]

4.2. Proof of Lemma 4.4. We use a result of Liggett [10] which provides a way to compare the simple exclusion with a simpler process composed of independent random walkers.

If \( f \) is a symmetric function on \( \mathbb{Z}_N^k \) and \( \eta \in \Omega_{N,k} \) we set

\[ f(\eta) := f(y_1, y_2, \ldots, y_k) \]  
(4.24)

where

\[ \{y_1, \ldots, y_k\} := \{x \mid \eta(x) = 1\}. \]

The above equation defines \( (y_1, \ldots, y_k) \) modulo permutation which is sufficient for the definition (4.24). We say that a function \( f \) defined on \( (\mathbb{Z}_N)^2 \) is positive definite if and only if for all \( \beta \) such that \( \sum_{x \in \mathbb{Z}_N} \beta(x) = 0 \), we have

\[ \sum_{x,y \in \mathbb{Z}_N^k} \beta(x)\beta(y)f(x,y) \geq 0. \]

We say that a function defined on \( (\mathbb{Z}_N)^k \) is positive definite if all its two marginals are.

Given \( \chi \in \Omega_{N,k} \), let \( X_t := (X_1(t), \ldots, X_k(t)) \) denote a set of independent random walk on \( \mathbb{Z}_N \), starting from initial condition \( (x^0_1, \ldots, x^0_k) \) with

\[ \{x^0_1, \ldots, x^0_k\} := \{x \mid \chi(x) = 1\}. \]  
(4.25)

Of course (4.25) defines \( (x^0_1, \ldots, x^0_k) \) only modulo permutation but this has no importance for what we are doing (e.g. we can fix \( (x^0_1, \ldots, x^0_k) \) to be minimal for the lexicographical order).

**Proposition 4.7.** If \( f \) is a symmetric definite positive then we have for all \( t \geq 0 \)

\[ \mathbb{E} [f(\eta(t))] \leq \mathbb{E} [f(X_t)] \]  
(4.26)

**Proof.** The proof in the case \( k = 2 \) is detailed in [9, Proof of Lemma 2.7] and it perfectly adapts to the case of general \( k \). \[ \square \]

Then, we want to apply this inequality to the function

\[ f(x_1, \ldots, x_k) := e^{\alpha \sum_{i=1}^k 1_{\{x_i \in [1,y]\}} - \mathbb{P}[X_i(t) \in [1,y]]} \]  
(4.27)

for \( y \in \{1, \ldots, N\} \). It is not difficult to check that \( f \) is indeed definite positive.
Lemma 4.8. For all \(N\) sufficiently large for all \(\alpha \in \mathbb{R}\), \(|\alpha| \leq \log 2\) we have, for all \(t \geq N^2\)
\[
\mathbb{E}\left[e^{\alpha \sum_{k=1}^{N} (1_{x_i(t) \in [1,y])} - \mathbb{P}[x_i(t) \in [0,y)]}\right] \leq \exp\left(\frac{2ky}{N} \alpha^2\right). \tag{4.28}
\]

To deduce (4.7) from (4.28) we just have to remark that
\[
\sum_{i=1}^{k} \mathbb{P}[X_i(t) \in [1,y)] = \sum_{z=1}^{y} \sum_{i=1}^{k} p_i(x_i^0, z) = \sum_{z=1}^{y} \mathbb{P}[\eta_t = 1].
\]

**Proof of Lemma 4.8.** We using the inequality
\[
\forall |x| \leq |\log 2|, \quad e^x \leq 1 + x + x^2
\]
for the variable \(Z = \alpha (1_{X_1(t) \in [1,y]} - \mathbb{P}[X_1(t) \in [1,y]])\). As \(\mathbb{E}[Z] = 0\) and from Lemma 4.3 (ii) we have for all \(t \geq N^2\),
\[
\mathbb{E}[Z^2] \leq \mathbb{P}[X_1(t) \in [1,y]) \leq 2y/N,
\]
the integrated inequality gives
\[
\mathbb{E}\left[e^{\alpha (1_{X_1(t) \in [1,y]} - \mathbb{P}[X_1(t) \in [1,y])}]\right] \leq 1 + \frac{2\alpha^2 y}{N} \leq \exp(2\alpha^2(y/N)). \tag{4.29}
\]

The independence of the \(X^i\) is then sufficient to conclude. \(\square\)

## 5. Coupling \(P^X_t\) with the equilibrium using the corner-flip dynamics

In this section we present the main tool which we use to prove Proposition 3.3: the corner-flip dynamics. The idea is to associate to each \(\eta \in \Omega\) an height function, and consider the dynamics associated with this rate function instead of the original one and use monotonicity properties of this latter dynamics.

### 5.1. The \(\xi\) dynamics

Let us consider the set of height functions of the circle.
\[
\Omega'_{N,k} := \left\{ \xi : \mathbb{Z}_N \to \mathbb{R} \mid \xi(x_0) \in \mathbb{Z}, \forall x \in \mathbb{Z}_N, \xi(x) - \xi(x+1) \in \left\{-\frac{k}{N}, 1 - \frac{k}{N}\right\} \right\}. \tag{5.1}
\]

Given \(\xi\) in \(\Omega'_{k,N}\), we define \(\xi^x\) as
\[
\begin{align*}
\xi^x(y) &= \xi(y), \quad \forall y \neq x, \\
\xi^x(x) &= \xi(x+1) + \xi(x-1) - 2\xi(x). 
\end{align*} \tag{5.2}
\]

and we let \(\xi_t\) be the irreducible Markov chain on \(\Omega'_{m,N}\) whose transition rates \(p\) are given by
\[
\begin{align*}
p(\xi, \xi^x) &= 1, \quad \forall x \in \mathbb{Z}_N, \\
p(\xi, \xi^x) &= 0, \quad \text{if } \xi \notin \{\xi^x \mid x \in \mathbb{Z}_N\}. \tag{5.3}
\end{align*}
\]

We call this dynamics the corner-flip dynamics, as the transition \(\xi \to \xi^x\) corresponds to flipping either a local maximum of \(\xi\) (a "corner" for the graph of \(\xi\)) to a local minimum \(\) vice versa. It is of course not positive recurrent, as the state space is infinite and translation invariant for the dynamics, however it is irreducible and recurrent.

The reader can check (see also Figure 1) that \(\Omega'_{N,k}\) is mapped onto \(\Omega_{N,k}\), by the transformation \(\xi \mapsto \nabla \xi\) defined by
\[
\nabla \xi(x) := \xi(x) - \xi(x+1) + \frac{k}{N}, \tag{5.4}
\]
and that the image $\nabla \xi_t$ of the corner-flip dynamics $\xi_t$ under this transformation is the simple exclusion, down and up-flips corresponding to jump $x \to x+1$, $x \to x-1$ of the particles respectively.

Figure 1. The correspondence between the exclusion process and the corner-flip dynamics. In red, a particle jump and its corner-flip counterpart are defined. Note that this is not a one to one mapping as a particle configuration gives the height function only modulo translation (the height function is drawn on a cylinder whose base is the circle on which the particles are moving).

There is a natural order on the set $\Omega'_{N,k}$ defined by

$$\xi \geq \xi' \iff \forall x \in \mathbb{Z}_N, \xi(x) \geq \xi'(x),$$

which has the property of being preserved by the dynamics in certain sense (see Section 5.3 for more details).

Given $\chi \in \Omega_{m,k}$, we define $(\xi^0_t)$ to be a process with transitions (5.3) starting from initial condition

$$\xi^0_0(x) := \sum_{z=0}^{x} \chi(z) - \frac{kx}{N},$$

It follows from the above remark that for all $t \geq 0$ we have

$$\mathbb{P} \left[ \nabla \xi^0_t \in \cdot \right] = P^\chi_t.$$  

Our idea is to construct another dynamic $\xi^1_t$ which starts from a stationary condition (the gradient is distributed according to $\mu$) and to try to couple it with $\xi^0_t$ within time $O(L^2)$. The difficulty here lies in finding the right coupling.

5.2. Construction the initial condition $\xi^1_0$ and $\xi^2_0$. In fact we define not one but two stationary dynamics $\xi^1_t$ and $\xi^2_t$, satisfying

$$\mathbb{P} \left[ \nabla \xi^1_t \in \cdot \right] = \mathbb{P} \left[ \nabla \xi^2_t \in \cdot \right] = \mu.$$  

$$\mathbb{E} \left[ \nabla \xi^0_t \in \cdot \right] = \mathbb{E} \left[ \nabla \xi^1_t \in \cdot \right] = \mu.$$
As $\mu$ is invariant for the dynamics $\nabla \xi$, (5.8) is satisfied for all $t$ as soon as it is satisfied for $t = 0$. As we wish to use monotonicity as a tool, we want to have

$$\forall t \geq 0, \quad \xi^1_t \leq \xi^0_t \leq \xi^2_t,$$  \hspace{1cm} (5.9)

Then our strategy to couple $\xi^0$ with equilibrium is in fact to couple $\xi^1$ with $\xi^2$ and remark that if (5.9) holds then

$$\forall t \geq 0, \quad \xi^1_t = \xi^2_t \Rightarrow \xi^1_t = \xi^0_t = \xi^2_t$$ \hspace{1cm} (5.10)

We first have to construct the initial condition $\xi^1_0$ and $\xi^2_0$ which satisfies (5.8)

$$\xi^1_0 \leq \xi^0_0 \leq \xi^2_0.$$ \hspace{1cm} (5.11)

Let us start with variable $\eta_0$ which has law $\mu$. We want to construct $\xi^1_0$ and $\xi^2_0$ which satisfies

$$\nabla \xi^i_0 = \eta_0.$$ \hspace{1cm} (5.12)

Somehow, we also want the vertical distance between $\xi^1_0$ and $\xi^2_0$ to be as small as possible. We set for arbitrary $\eta \in \Omega_{N,k}$, or $\xi \in \Omega'_{N,k}$

$$H(\eta) := \max_{x,y \in \mathbb{Z}_N} \left| \sum_{z=x+1}^{y} \left( \eta(z) - \frac{kN}{N} \right) \right|, \hspace{1cm} (5.13)$$

Finally set we set

$$H_0 := \left\lceil H(\eta_0) + s\sqrt{k} \right\rceil.$$ \hspace{1cm} (5.14)

and

$$\xi^1_0(x) := \sum_{z=1}^{x} \eta_0(x) - \frac{kx}{N} - H_0.$$ \hspace{1cm} (5.15)

$$\xi^2_0(x) := \sum_{z=1}^{x} \eta_0(x) - \frac{kx}{N} + H_0.$$ \hspace{1cm} (5.16)

Note that with this choice, (5.11) is satisfied for $\chi \in G_s$ (see Figure 2).

5.3. The graphical construction. Now we present a coupling which is such that (5.9) is satisfied. Note that in this case $\xi^1_t = \xi^0_t = \xi^2_t$ if and only if the area between the two paths, defined by

$$A(t) := \sum_{x \in \mathbb{Z}_N} \xi^2_t(x) - \xi^1_t(x),$$ \hspace{1cm} (5.16)

equals zero.

The idea is then to find among coupling which are order-preserving one for which the fluctuations of $A$ are the largest possible, so that it reaches zero faster. We want to make the corner-flips of $\xi^1$ and $\xi^2$ as independent as possible (of course making them completely independent is not an option since (5.9) would not hold)

We introduce now our coupling of the $\xi^i$, which is also a grand-coupling on $\Omega'_{N,k}$, in the sense that it allows to construct $\xi_t$ starting from all initial condition on the same probability space. The evolution of the $(\xi_t)_{t \geq 0}$ is completely determined by auxiliary Poisson processes which we call clock processes.
Figure 2. Representation of the three initial condition for the corner-flip dynamics. \( \xi_1^0 \) and \( \xi_2^0 \) are translated version of the same profile. The height \( H_0 \) is designed so that initially \( \xi_0 \) (whose variation are smaller than \( s\sqrt{k} \) if \( \chi \in G \)) is framed by \( \xi_1^0 \) and \( \xi_2^0 \). As the order is conserved by the graphical construction. \( \xi_0 \) couples with equilibrium when \( \xi_1^t = \xi_2^t \).

Set

\[
\Theta := \left\{ (x, z) \mid x \in \mathbb{Z}_N \text{ and } z \in \mathbb{Z} + \frac{kx}{N} \right\}
\]

And set \( T^\dagger \) and \( T^\urcorner \) to be two independent rate-one clock processes indexed by \( \Theta \) (\( T^\dagger_\theta \) and \( T^\urcorner_\theta \) are two independent Poisson processes of intensity one of each \( \theta \in \Theta \)). The trajectory of \( \xi_t \) given \( (T^\dagger, T^\urcorner) \) is given by the following construction

- \( \xi_t \) is a càdlàg, and does not jump until one of the clocks indexed by \( (x, \xi_t(x)) \), \( x \in \mathbb{Z}_N \).
- If \( T^\urcorner_{(x, \xi_t(x))} \) rings at time \( t \) and \( x \) is a local maximum for \( \xi_t \), then \( \xi_t = \xi_t^x \).
- If \( T^\dagger_{(x, \xi_t(x))} \) rings at time \( t \) and \( x \) is a local minimum for \( \xi_t \), then \( \xi_t = \xi_t^x \).

The coupling of \( \xi_t^0 \), \( \xi_t^1 \) and \( \xi_t^2 \) is obtained by using the same clock process for all of them. The reader can check that with this coupling (5.9) is a consequence of (5.11).

Remark 5.1. In fact, the corner flip dynamics which is considered here, is in one to one correspondence with the zero-temperature stochastic Ising model on an infinite cylinder. With this in mind the coupling we have constructed just corresponds to the graphical construction of this spin flip dynamics. See e.g. [4, Section 2.3 and Figure 3] for
more details in the case of the stochastic Ising model on a rectangle with mixed boundary condition.

To prove Proposition 3.3 it is sufficient to prove that $\xi_1^t$ and $\xi_2^t$ typically merge within a time $O(L^2)$. More precisely,

**Proposition 5.2.** For all $s > 0$, given $\varepsilon > 0$ there exists $C(\varepsilon, s)$ which is such that for all sufficiently large $N$,

$$P[\xi_{CN^2}^1 \neq \xi_{CN^2}^2] \leq \varepsilon. \quad (5.17)$$

Similarly there exists $c(s) > 0$ which is such that

$$P[\xi_{N^2}^1 \neq \xi_{N^2}^2] \leq 1 - c(s). \quad (5.18)$$

**Proof of Proposition 3.3.** Given $s > 0$ consider $(\xi_1^t)_{t \leq 0}$, $(\xi_2^t)_{t \geq 0}$, constructed as above. Then given $\chi \in G_s$, we construct $\xi^0_t$ the dynamics starting from the initial condition (5.6) and using the same clock process as $\xi_1^t$ and $\xi_2^t$. By definition of $G_s$, (5.11) is satisfied and thus so is (5.9) from the graphical construction. Recalling (5.7) and (5.8) we have

$$||P_\chi^t - \mu|| \leq P[\nabla \xi^0_t \neq \nabla \xi_1^t] \leq P[\xi^0_t \neq \xi_1^t] \leq P[\xi_1^t \neq \xi_2^t]. \quad (5.19)$$

for any $t > 0$, where the last inequality is a consequence of (5.10). According to the above inequality, Proposition 3.3 obviously is a consequence of Proposition 5.2.

□

6. The proof of Proposition 5.2

In order to facilitate the exposition of the proof, we choose to present it in the case $k = N/2$ first. We also chose to detail the proof of (5.17) first. The way to prove (5.18) and how to adapt the proofs for general $k$ is explained at the end of the section.

6.1. The randomly walking area. We are interested in bounding (recall (5.16))

$$\tau := \inf \{ t \geq 0 \mid A(t) = 0 \} = \inf \{ t \geq 0 \mid \xi_1^t = \xi_2^t \} \quad (6.1)$$

With our construction, $A(t)$, the area between the two curves is a $\mathbb{Z}_+$ valued martingale which only makes nearest neighbor jumps (corners flip one at a time).

Hence $A(t)$ is just a time changed symmetric nearest neighbor walk on $\mathbb{Z}_+$ which is absorbed at zero. In order to get a bound for the time at which $A$ hits zero, we need to have a reasonable control on the jump rate which depends on the particular configuration $(\xi_1^t, \xi_2^t)$ the system sits on. The jump rate is given by the number of places where corners can flip without flipping together for $\xi_1^t$ and $\xi_2^t$. More precisely, set

$$U_i(t) := \{ x \in \mathbb{Z}_N \mid \xi_i^t \text{ has a local extremum at } x \}$$

$$\exists y \in \{ x - 1, x, x + 1 \}, \xi_i^2(y) > \xi_i^1(y). \quad (6.2)$$

The jump rate of $A(t)$ is given by

$$u(t) := \#U_1(t) + \#U_2(t). \quad (6.3)$$

For $t \leq \int_0^\tau u(t) \, dt$ let us define

$$J(t) := \inf \left\{ s \mid \int_0^s u(v) \, dv \geq t \right\}. \quad (6.4)$$
By construction, the process \((X_t)_{t \geq 0}\) defined by
\[ X_t := A(J(t)) \] (6.5)
is a continuous time random walk on \(\mathbb{Z}_+\) which jumps up and down with rate \(1/2\). We have from the definition of the \(\xi_i^0\)
\[ X_0 = A(0) = 2H_0N \]
which is of order \(N^{3/2}\) and hence \(X_t\) needs a time of order \(N^3\) to reach 0. What we used to estimate \(A(0)\) is the following estimate which can be derived from (4.2) and the definition of \(H_0\) (5.14),
\[
\mathbb{P} \left[ A(0) \geq 2(s + r)N^{1/2} \right] = \mathbb{P} \left[ H_0 \geq (s + r)N^{1/2} \right] \leq \mathbb{P} \left[ H(\eta_0) \geq rN^{1/2} \right] \leq 2e^{-cr^2}. \tag{6.6}
\]
If \(u\) were of order \(N\) this would be sufficient to conclude that \(A(t)\) reaches zero within a time \(O(L^2)\). This is however not the case: the closer \(\xi_1^0\) and \(\xi_2^0\) get close to each other, the smaller \(u(t)\) becomes. A way out of this is to introduce a multi-scale analysis where the bound we require on \(u\) depends on how small \(A(t)\) already is.

6.2. Multi-scale analysis. We construct a sequence of intermediate stopping time \((\tau_i)_{i \geq 0}\) as follows.
\[ \tau_i := \inf \{ t \geq 0 \mid A(t) \leq N^{3/2}2^{-i} \} \tag{6.7} \]
We are in \(\tau_i\) for \(i \in \{0, \ldots, K\}\) with
\[ K_N := \left\lfloor \frac{1}{2} \log_2 N \right\rfloor. \tag{6.8} \]
Note that a number of \(\tau_i\) can be equal to zero if \(A(0) \leq N^{3/2}\). We set \(\tau_{-1} := 0\) for convenience.

To control the value of \(\tau\), our aim is to control the value of each \(\Delta \tau_i = \tau_i - \tau_{i-1}\) for \(i \leq K\) and that of \(\tau - \tau_K\). The way to do this is to first to control the equivalent of the \(\Delta \tau\) for the time rescaled process \(X_t\) (recall 6.5). We set for \(i \in \{0, \ldots, K\}\)
\[
\tau_i := \int_{\tau_{i-1}}^{\tau_i} u(t) \, dt, \quad \tau_\infty := \int_{\tau_K}^{\tau} u(t) \, dt. \tag{6.9}
\]
As \(X\) is diffusive, \(\tau_i\) is typically of order \((N^{3/2}2^{-i})^2 = N^{3/4}2^{-i}\), and \(\tau_\infty\) is of order \(N^2\). With this in mind it is not too hard to believe that

**Lemma 6.1.** Given \(\varepsilon\) there exists a constant \(C(\varepsilon, s)\) such that
\[
\mathbb{P} \left[ \exists i \in \{0, \ldots, K\}, \tau_i \geq C N^{3/2} 2^{-i} \right] \cup \{\tau_\infty \geq C N^2\} \leq \varepsilon. \tag{6.10}
\]

**Proof.** Let \(Z_t\) denote a nearest neighbor walk on \(\mathbb{Z}\) starting from 0 and \(T_a\) the first time \(Z\) reaches \(a\). It is rather standard that there exists a constant \(C_1\) such that for every \(a \geq 1\) and every \(u \geq 0\)
\[
\mathbb{P} \left[ T_a \geq u a^2 \right] \leq C_1 u^{-1/2}. \tag{6.11}
\]
Note that for \( i \geq 1 \), ignoring the effect of integer rounding, \( \tau_i \) has the same law as \( T_a \) with \( a = N^{3/2}i^{-1} \) and thus applying (6.11) we obtain that
\[
P[T_i \geq u N^{3-i}] \leq C_1 u^{-1/2} (3/4)^{i/2}. \tag{6.12}
\]
In the same manner we have
\[
P[T_\infty \geq u N^2] \leq C_1 u^{-1/2}. \tag{6.13}
\]
We can then choose \( u_0(\varepsilon) \) large enough in a way that
\[
C_1 u^{-1/2} \left( \sum_{i=1}^K (3/4)^{i/2} + 1 \right) \leq \varepsilon/2. \tag{6.14}
\]
Concerning \( T_0 \), from (6.6), one can find \( C_2(\varepsilon, s) \) such that
\[
P[A(0) \geq C_2 N^{3/2}] \leq \varepsilon/4. \tag{6.15}
\]
Conditionally on the event \( A(0) \leq C_2 N^{3/2} \), \( T_0 \) is stochastically dominated by \( T_a \) with \( a = C_2 N^{3/2} \) and hence using (6.11) and fixing \( u_1(\varepsilon, s) \) large enough (depending on \( C_1, C_2 \) and \( \varepsilon \)) we obtain
\[
P[T_0 \geq u_1(C_2)^2 N^3] \leq P[A(0) \geq C_2 N^{3/2}] + P[T_0 \geq u A(0)^2] \geq \varepsilon/2. \tag{6.16}
\]
Then we conclude by taking \( C(\varepsilon, s) := \max(u_0, u_1(C_2)^2) \).

Now, what we have to check then is that the value of \( u(t) \) is not too small in the time interval \([\tau_{i-1}, \tau_i]\) for all \( i \in \{0, \ldots, K\} \). What we want to use is that for any \( t \geq 0 \), \( \nabla \xi^1_t \) is at equilibrium so that \( \xi^1_t \) has to present a “density of flippable corners”.

We introduce an event \( \mathcal{A} \) which is aimed to materialize this fact. Given \( x \) and \( y \) in \( \mathbb{Z}_N \) we set
\[
j(x, y, \xi) := \# \{ z \in [x, y] \mid \xi(z) \text{ is a local extremum} \} \tag{6.17}
\]
We have
\[
\mu(j(x, y, \xi)) = \frac{N - 2}{2(N - 1)} \#[x, y]. \tag{6.18}
\]
We define
\[
\mathcal{A} := \left\{ \forall t \leq N^3, \forall (x, y) \in \mathbb{Z}_N^2, \#[x, y] \geq N^{1/4} \Rightarrow j(x, y, \xi^1_t) \leq \frac{1}{3} \#[x, y] \right\}, \tag{6.19}
\]
the event that a “large” interval with an anomalously low density of corner does not appear before \( N^3 \). We prove in the appendix that

**Lemma 6.2.** For all \( N \) sufficiently large
\[
P[\mathcal{A}^c] \leq \frac{1}{N}. \tag{6.20}
\]

Then when \( \mathcal{A} \) holds, we can derive an efficient lower bound on \( u \) which just depend on \( A(t) \). Recall (5.13)

**Lemma 6.3.** When \( \mathcal{A} \) holds we have for all \( t \) \( \leq N^3 \)
\[
u(t) \geq \frac{1}{3} \min \left( N, \frac{A(t)}{\max(H(\xi^1_t) + H(\xi^2_t), N^{1/2})} \right) \tag{6.21}
\]
Proof. If $\xi^1_t$ and $\xi^2_t$ have no contact with each other, then $u(t)$ is equal to the total number of flippable corner in $\xi^2_t$ and $\xi^1_t$. If $A$ holds, this number is larger than $N/3$, which, by definition of $A$, is a lower bound for the number of corners on $\xi^1_t$ alone.

If there exists $x$ such that $\xi^1_t(x) = \xi^2_t(x)$. Let us consider the set of active coordinates

$$C(t) := \{\exists y \in \{x-1, x, x+1\}, \xi^1_t(y) < \xi^2_t(y)\}.$$  (6.22)

Note that when one of the $\xi^i$ (or both) have a local maximum at $x \in C(t)$ then when the corresponding corner flips it changes the value of $A(t)$. Our idea is to find a way to bound from below the number of $x$ in $C(t)$ for which $\xi^1_t(x)$ has a flippable corner, using the assumption that $A$ holds.

Let use decompose $C(t)$ into connected components (for the graph $\mathbb{Z}_N$) which are intervals as defined in (3.3). Assume that $[a, b]$ is a connected component of $C(t)$, it corresponds to a “bubble” between $\xi^1_t$ and $\xi^2_t$ (see Figure 3). For each bubble, we want to have a bound on the number of flippable corners and compare it to the area of the bubble. Set

$$u_{[a,b]}(t) := j(a, b, \xi^1_t).$$  (6.23)

and

$$A_{[a,b]}(t) := \sum_{x=a}^{b} \xi^2_t(x) - \xi^1_t.$$  (6.24)

Note that

$$u(t) \geq \sum_{\text{all bubbles}} u_{[a,b]}(t),$$

$$A(t) = \sum_{\text{all bubbles}} A_{[a,b]}(t)$$  (6.25)

For small bubbles (of length smaller than $N^{1/4}$), $A$ does not give any information on the number of flippable corners. However we can simply observe that in any bubble, there is at least one flippable corners (e.g. where the minimum of $\xi^1_t|_{[a,b]}$ is attained).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{bubble_decomposition.pdf}
\caption{The bubble decomposition: The interval $[a,b]$ displayed here corresponds to a bubble. The large circle dots corresponds to corner which do not flip simultaneously for $\xi^1_t$ and $\xi^2_t$, the total number of them is $u(t)$. Among those, the white circles are the one which are counted in one of the $u_{[a,b]}(t)$ (note that some of the corners on $\xi^1_t$ are not counted). The smaller circles corresponds to corner which flip together for $\xi^1_t$ and $\xi^2_t$ and thus do not contribute to $u(t)$.}
\end{figure}
If $|[a, b]| \leq N^{1/4}$, the area of the bubble satisfies
\[ A_{[a, b]}(t) \leq N^{1/2}. \] (6.26)
This is because $\xi_t^2 - \xi_t^1$ is a Lipschitz function and cancels at both ends. Hence necessarily
\[ u_{[a, b]}(t) \geq \frac{A_{[a, b]}(t)}{N^{1/2}}. \] (6.27)

For large bubbles $|[a, b]| \geq N^{1/4}$ we can use the fact that $A$ holds. First, let us control the area: as $\xi_1^t$ and $\xi_2^t$ are in contact we have
\[ \max_{x \in \mathbb{Z}_N} \xi_2^t(x) - \xi_1^t(x) \leq H(\xi_1^t) + H(\xi_2^t). \] (6.28)
and hence
\[ A_{[a, b]}(t) \leq |[a, b]|(H(\xi_1^t) + H(\xi_2^t)). \] (6.29)
By the definition of $A$ there are at least $|a, b|/3$ flippable corners on the path $\xi_1^t$ restricted to $[a, b]$. And hence
\[ u_{[a, b]}(t) \geq \frac{A_{a, b}(t)}{3(H(\xi_1^t) + H(\xi_2^t))}. \] (6.30)
Hence for any value $|a, b|$ we have
\[ u_{a, b}(t) \geq \frac{A_{a, b}(t)}{3 \max(H(\xi_1^t) + H(\xi_2^t), \sqrt{N})}. \] (6.31)
and we conclude by summing (6.31) over all bubbles and using (6.25).

The previous lemma gives us some control over $u(t)$ (if $A$ holds) provided we can control $A(T)$ and
\[ H(t) := H(\xi_1^t) + H(\xi_2^t). \] (6.32)
To control the area, we use our multi-scale construction: for $t \in [\tau_{i-1}, \tau_i)$, we have
\[ A(t) \geq N^{3/2}2^{-i}. \]

To obtain a good control on $H(t)$ we use

**Lemma 6.4.** There exists a constant $c$ such that for any $t \geq 0$ and $r \geq 0$,
\[ \mathbb{P}[H(t) \geq r\sqrt{N}] \leq 2 \exp(-cr^2). \] (6.33)

**Proof.** This just comes from the fact that for any $t > 0$, $i = 1, 2$, $\nabla \xi_i^t$ are distributed according to $\mu$, and then we use (4.2). \qed

We use it to show that most of the time $H(t)$ is of order $\sqrt{N}$. In fact we need a slightly more twisted statement that fits the multi-scale analysis. For the remainder of the proof set
\[ \alpha := \left( \sum_{i \geq 0} (i + 1)^2 \right)^{-1} \] (6.34)

**Lemma 6.5.** For any $\varepsilon > 0$, there exists a constant $C(\varepsilon)$ such that for any $T \geq 0$
\[ \mathbb{P}\left[ \exists i \in \{0, \ldots, K\}, \int_0^T 1_{\{H(t) \geq C(i+1)^2\sqrt{N}\}} \, dt \geq (\alpha/2)(i + 1)^{-2}T \right] \leq \varepsilon. \] (6.35)
**Proof.** For a fixed $i$ from Lemma 6.4, we have
\[
E \left[ \int_0^T 1_{\{H(t) \geq r(i+1)^2 \sqrt{N}\}} dt \right] \leq 2Te^{-cr^2(i+1)^2}.
\] (6.36)

Hence from the Markov inequality
\[
E \left[ \int_0^T 1_{\{H(t) \geq r(i+1)^2 \sqrt{N}\}} dt \leq \left( \frac{\alpha}{2} \right)(i+1)^{-2}T \right] \leq 4\alpha^{-1}(i+1)^2 \exp(-cr^2(i+1)^2). \] (6.37)

Hence we obtain the result by choosing $C = r_0$ sufficiently large so that
\[
4 \sum_{i \geq 0} \alpha^{-1}(i+1)^2 \exp(-cr^2(i+1)^2) \leq \varepsilon. \] (6.38)

Combining Lemma 6.1, 6.2, 6.3, and 6.5 we can now conclude.

**Proof of (5.17).** Let $\varepsilon$ be fixed. We fix a constant $C_1(\varepsilon)$ such that Lemma 6.5 holds for $\varepsilon/3$ instead of $\varepsilon$. $C_2(\varepsilon, s)$ is chosen so that Lemma 6.1 holds for $\varepsilon/3$. We define the events
\[
B := \left\{ \forall i \in \{0, \ldots, K\}, \int_0^T 1_{\{H(t) \geq C_1(i+1)^2 \sqrt{N}\}} dt \leq \left( \frac{\alpha}{2} \right)(i+1)^{-2}T \right\},
\] (6.39)
\[
C := \{ \forall i \in \{0, \ldots, K\}, T_i \leq C_2N^{33-i} \} \cap \{ T_\infty \leq C_3N^2 \},
\]
where
\[
T := 6N^2C_1C_2\alpha^{-1} \max_{i \geq 0} [(i+1)^4(2/3)^i],
\] (6.40)
\[
T' := T + C_2N^2.
\]

We assume also that $N$ is large enough so that $A$ holds with probability larger than $1 - \varepsilon/3$ (cf. Lemma 6.3). Hence we have
\[
P[A \cap B \cap C] \geq 1 - \varepsilon
\]

Now what remains to prove is that
\[
\{ A \cap B \cap C \} \subset \{ \tau \leq T' \}. \] (6.41)

This implies (5.17), with
\[
C(\varepsilon, s) = T'/N^2 = C_2 \left( 6C_1\alpha^{-1} \max_{i \geq 0} [(i+1)^4(2/3)^i] + 1 \right). \] (6.42)

We split the proof of (6.41) in two statements. We want to show first that on the event $A \cap B \cap C$
\[
\tau - \tau_K \leq C_2N^2. \] (6.43)

and then that
\[
\forall i \in \{0, \ldots, K\}, \quad (\tau_i - \tau_{i-1}) \leq C_3(i+1)^{-2}N^2, \] (6.44)

where
\[
C_3 := 6C_2C_1 \max_{i \geq 0} [(i+1)^4(2/3)^i]
\]

These inequality combined give
\[
\tau \leq C_2N^2 + \sum_{i=0}^{K} C_3(i+1)^{-2}N^2 \leq T'. \] (6.45)
Note that the (6.43) is an immediate consequence of $\mathcal{C}$ as
\[
\mathcal{T}_\infty = \int_{\tau_K}^\tau u(t) \, dt \geq \tau - \tau_K. \tag{6.46}
\]

Let us turn to (6.44). Let us assume that the statement is false and let $i_0$ denote the smallest $i$ such that
\[
(\tau_i - \tau_{i-1}) > C_3(i + 1)^{-2}N^2.
\]
The definition of $i_0$ (the fact that it is the smallest) implies that
\[
\tau_{i_0-1} + C_3(i_0 + 1)^{-2}N^2 \leq T. \tag{6.47}
\]
From $\mathcal{B}$ we have (using (6.47) inequality for the second inequality)
\[
\int_{\tau_{i_0-1}}^{\tau_{i_0}} 1\{H(t) \leq C_1(1+i_0)^2\sqrt{N}\}
\geq
\int_{\tau_{i_0}}^{\tau_{i_0}+C_3(i_0+1)^{-2}N^2} 1\{H(t) \leq C_1(1+i_0)^2\sqrt{N}\} \, dt
\geq
C_3(i_0 + 1)^{-2}N^2 - \int_{0}^{T} 1\{H(t) \geq C_1(1+i_0)^2\sqrt{N}\} \, dt
\geq
C_3(i_0 + 1)^{-2}N^2 - (\alpha/2)T(i_0 + 1)^{-2} \geq \frac{1}{2} C_3(i_0 + 1)^{-2}N^2. \tag{6.48}
\]
Now from Lemma 6.3 and the assumption that $\mathcal{A}$ holds. For all $t \leq \tau_{i_0}$ we have $A(t) \geq N^{3/2}2^{-i_0}$ and hence
\[
u(t) \geq \frac{1}{3} \min\left(N, \frac{A(t)}{\max(H(t), N^{1/2})}\right) \geq \frac{N^{3/2}2^{-i_0}}{3C_1(i_0 + 1)^2\sqrt{N}} 1\{H(t) \leq C_1(i_0+1)^2\sqrt{N}\}. \tag{6.49}
\]
Hence from (6.64)
\[
\mathcal{T}_{i_0} \geq \frac{N^{3/2}2^{-i_0}}{C_1(i_0 + 1)^2\sqrt{N}} \int_{\tau_{i_0-1}}^{\tau_{i_0}} 1\{H(t) \leq C_1(i_0+1)^2\sqrt{N}\} \, dt
\geq \frac{1}{6C_1} C_3N^32^{i_0}(i_0 + 1)^{-4} \geq 3^{i_0}C_2N^2. \tag{6.50}
\]
where the last inequality comes from the definition of $C_1$. This brings a contradiction to the fact that $\mathcal{C}$ holds and ends the proof of (6.43).

\[\square\

6.3. Proof of (5.18). We want to prove now that starting from $\chi \in \mathcal{G}_s$, we get significantly closer to equilibrium after a time $2N^2$. The elements of the proof are essentially the same that for (5.17) but we have to be more careful. We can consider throughout the proof that $s$ is sufficiently large, by mononicity. Instead of (6.1) we have to prove the following statement

Lemma 6.6. There exists a constant $C$ such that for all $s$ sufficiently large
\[
P\left[\{\forall i \in \{0, \ldots, K\}, \mathcal{T}_i \leq s^{-6}N^{3}3^{-i}\} \cap \{\mathcal{T}_\infty \leq N^2\}\right] \geq e^{-Cs^s} \tag{6.51}
\]
Proof. A first observation is that the $\mathcal{T}_i$ are independent by the Markov property for the random walk $X_t = A(J(t))$ (recall (6.5)). To evaluate $\mathbb{P}[\mathcal{T}_i \leq s^{-6}N^33^{-i}]$, we are going to use a classical estimate of first hitting time of a given level for a simple symmetric random walk: the there exists a constant $C_1$ such that for all $a \geq 1$, for all $u \leq a$ one has (using the notation of (6.1))

$$
\mathbb{P}[\mathcal{T}_a \leq a^2 / u] \geq e^{-C_1 \max(u^{1/2},u)},
$$

(6.52)

(it is sufficient to check the estimate when $u$ is large as for $u$ close to zero it is just equivalent to (6.11)). The time $\mathcal{T}_\infty$ is stochastically dominated by $T_N$ and thus

$$
\mathbb{P}[\mathcal{T}_\infty \leq N^2] \geq e^{-C_1}.
$$

(6.53)

Neglecting the effect of integer rounding, for $i \geq 0$, $\mathcal{T}_i$ is equal in law to $T_{2^{-i}N\sqrt{2}}$. Hence from (6.11) we have

$$
\mathbb{P}[\mathcal{T}_i \geq s^{-6}N^33^{-i}] \geq e^{-C_1 \max((4/3)^{-i/2}s^3,(4/3)^{-i}6))}.
$$

(6.54)

Then note that there exists a constant $C_2$ such that for all $s \geq 1$

$$
C_1 \sum_{i=1}^{K} \max((4/3)^{-i/2}s^3,(4/3)^{-i}6) \geq C_2 s^6.
$$

(6.55)

For $i = 0$, $\mathcal{T}_0$ depends on initial value of the area. Now let us note that from (6.6) we have for $s$ sufficiently large

$$
\mathbb{P}[A(0) \leq 4sN] \geq 1 - 2e^{-cs^2} \geq 1/2.
$$

(6.56)

Then conditioned on the event $\{A(0) \leq 4sN\}$, $\mathcal{T}_0$ is dominated by $T_{4sN}$ and hence

$$
\mathbb{P}[\mathcal{T}_0 \geq s^{-6}N^3 | A(0) \leq 4sN] \geq \exp(-16C_1 s^8).
$$

(6.57)

Combined with (6.56), this gives us

$$
\mathbb{P}[\mathcal{T}_0 \geq s^{-4}N^3] \geq e^{-16C_1 s^8} / 2.
$$

(6.58)

Hence using the independence and multiplying the inequalities (6.58), (6.54) and (6.53) (and using (6.55) we obtain the result for some appropriate $C$. \hfill \Box

We also need an estimate on the probability that $H(t)$ (recall (6.32)) is too large which slightly differs from Lemma 6.5

**Lemma 6.7.** Recall $\alpha := (\sum_{i \geq 0} (i+1)^{-2})^{-1}$. There exists a constant $C$ such that for all $s$ sufficiently large

$$
\mathbb{P}\left[ \exists i \in \{0, \ldots, K\}, \int_0^{N^2} 1_{\{H(t) \geq s^8(i+1)^2/3\}} dt \geq (\alpha / 2)(i+1)^{-2}N^2 \right] \leq e^{-Cs^{10}}.
$$

(6.59)

**Proof.** As for Lemma 6.5 we just use Lemma 6.4 and the Markov inequality. \hfill \Box

**Proof of (5.18).** Set

$$
\mathcal{B}' := \left\{ \forall i \in \{0, \ldots, K\}, \int_0^T 1_{\{H(t) \geq s^8(i+1)^2/3\}} dt \leq (\alpha / 2)(i+1)^{-2}T \right\},
$$

$$
\mathcal{C}' := \left\{ \forall i \in \{0, \ldots, K\}, \mathcal{T}_i \leq s^{-6}N^33^{-1} \cap \mathcal{T}_\infty \leq N^2 \right\}.
$$

(6.60)
Then from Lemma 6.1, 6.2, and 6.5 we have for $s$ sufficiently large, for $N$ large enough (depending on $s$)

$$\mathbb{P} \left[ A \cap B' \cap C' \right] \geq \exp(-C_1 s^8) / 2. \quad (6.61)$$

What remains to prove is that $\tau \leq 2N^2$ on the event $A \cap B' \cap C'$. First, we notice that from the definition of $T^\infty (6.9)$, $C'$ readily implies that

$$\tau - \tau_K \leq T^\infty \leq N^2.$$  

Hence to conclude we need to show that

$$\forall i \in \{0, \ldots, K\}, \quad \tau_i - \tau_{i-1} \leq \alpha (i+1)^{-2} N^2. \quad (6.62)$$

Assume the statement is false and let $i_0$ be the smallest index which is such that it is not satisfied. Using Lemma 6.3 we have

$$T_{i_0} = \int_{\tau_{i_0} - 1}^{\tau_{i_0}} u(t) \, dt \geq \int_{\tau_{i_0} - 1}^{\tau_{i_0}} \frac{1}{3} \min \left( N, \frac{A(t)}{\max (H(t), N^{1/2})} \right) \, dt$$

$$\geq \frac{2^{-i_0} N}{3s^5(i+1)^2} \int_{\tau_{i_0} - 1}^{\tau_{i_0}} 1_{\{H(t) \leq s^5(i+1)^2N\}} \, dt. \quad (6.63)$$

Then one has from the definition of $i_0$

$$\tau_{i_0 - 1} + \alpha (i+0)^{-2} \leq N^2.$$  

Hence from $C'$

$$\int_{\tau_{i_0} - 1}^{\tau_{i_0}} 1_{\{H(t) \leq s^5(i+1)^2\sqrt{N}\}} \, dt$$

$$\geq \int_{\tau_{i_0} - 1}^{\tau_{i_0} + \alpha (i_0 + 1)^{-2} N^2} 1_{\{H(t) \leq s^5(1+i_0)^2\sqrt{N}\}} \, dt$$

$$\geq \alpha (i_0 + 1)^{-2} N^2 - \int_{0}^{N^2} 1_{\{H(t) \geq s^5(1+i_0)^2\sqrt{N}\}} \, dt$$

$$\geq (\alpha/2)(i_0 + 1)^{-2} N^2, \quad (6.64)$$

and thus

$$T_{i_0} \geq (\alpha/2)2^{-i_0}(i_0 + 1)^2N^3s^{-5} \geq C_2 3^{-i_0} N^3 s^{-5}. \quad (6.65)$$

for an explicit $C_2$. This brings a contradiction to $B'$ if $s$ is chosen sufficiently large.  

\[ \square \]

6.4. Proposition 5.2 for arbitrary $k$. The overall strategy is roughly the same, except that we start with an area which is of order $k^{1/2}N$. Hence most of the modifications in the proof can be performed just writing $\sqrt{k}$ instead of $N^{1/2}$. However Lemma 6.2 does not hold in has it is for small $k$ and one needs a deeper change there.

We define the $(\tau_i)_{i \geq 0}$ as follows $(\tau_{-1} := 0)$

$$\tau_i := \inf \{ t \geq 0 \mid A(t) \leq kN^{1/2}2^{-i} \}. \quad (6.66)$$

and we set

$$K_N := \left\lceil \frac{1}{2} \log_2 k \right\rceil. \quad (6.67)$$

Note that a number of $\tau_i$ can be equal to zero if $A(0) \leq N^{3/2}$. 

The time changed version $\mathcal{T}_i$, $i \in \{0, \ldots, K\} \cup \{\infty\}$ of $\Delta \tau_i$ are defined as in (6.9). Lemma (6.1) becomes

**Lemma 6.8.** Given $\varepsilon > 0$ there exists a constant $C_1(\varepsilon, s)$ such that
\[
\mathbb{P}[\{\exists i \in \{0, \ldots, K\}, \mathcal{T}_i \geq C_1 N k^3 s^{-1}\} \cup \{T_\infty > C_1 N^2\}] \leq \varepsilon. \tag{6.68}
\]
and a constant $C_2$ independent of the parameters
\[
\mathbb{P}[\{\exists i \in \{0, \ldots, K\}, \mathcal{T}_i \geq N^2 s^{-6} \} \cup \{T_\infty > N^2\}] \geq \exp(-C_2 s^8). \tag{6.69}
\]

We also have

**Lemma 6.9.** Recall $\alpha := (\sum_i 0(i_0 + 1)^{-2})^{-1}$. For any $\varepsilon > 0$, exists a constant $C_1(\varepsilon)$ such that for any $T \geq 0$
\[
\mathbb{P}\left[\exists i \in \{0, \ldots, K\}, \int_0^T 1_{\{H(t) > C_1(4/3)^{1/\sqrt{K}}\}} \, dt \geq (\alpha/4)(i + 1)^{-2} T \right] \leq \varepsilon. \tag{6.70}
\]
Moreover there exist a constant $C_2$ which is such that
\[
\mathbb{P}\left[\exists i \in \{0, \ldots, K\}, \int_0^{N^2} 1_{\{H(t) > s^5(4/3)^{1/\sqrt{K}}\}} \, dt \geq (\alpha/4)(i + 1)^{-2} N^2 \right] \, dt \leq e^{-C_2 s^{10}}. \tag{6.71}
\]

The only place where at major modification is needed is (6.2) as for small $k$, we cannot define an event similar to (6.19) which holds with high probability.

Set (recall (6.17))
\[
\mathfrak{A} := \left\{\xi \in \Omega_{k,N} \mid \#[x, y] \geq N/k \log k)^2 \Rightarrow j(x, y, \xi) \geq \frac{k}{10N} \#[x, y]\right\}
\]
\[
\cup \{\xi \mid \#[x, y] \leq N/k \log k)^2 \Rightarrow |\xi(x) - \xi(y)| \leq (\log k)^4\} =: \mathfrak{A}_1 \cup \mathfrak{A}_2. \tag{6.72}
\]

Note that $\mathfrak{A}$ is invariant by vertical translation of $\xi$ and thus only depends on $\nabla \xi$. Hence we can (improperly) consider it as a subset of $\Omega_{N,k}$. The following result is proved in the Appendix

**Lemma 6.10.** We have
\[
\mu(\mathfrak{A}) \leq \frac{1}{k^2}, \tag{6.73}
\]
as a consequence one has for every $T \geq 0$
\[
\mathbb{P}\left[\int_0^T 1_{\{\xi_1(t) \in \mathfrak{A}\}} \, dt \geq \frac{T}{k}\right] \leq 1/k. \tag{6.74}
\]

Moreover we have (recall (6.32))

**Lemma 6.11.** When $\xi_1(t) \in \mathfrak{A}$ we have
\[
u(t) \geq \frac{1}{10} \min\left(k, \frac{A(t)k}{N \max(H(t), (\log k)^6)}\right). \tag{6.75}
\]

**Proof.** If $\xi_1^1 \in \mathfrak{A}$ and $\xi_1^2(x) > \xi_1^1(x)$, for all $x \in \mathbb{Z}_N$, then all corners of $\xi_1^1$ give a contribution to $u(t)$ and from the assumption $\xi_1(t) \in \mathfrak{A}$ there are at least $k/10$ of them.

If there are some contact between $\xi_1^1$ and $\xi_2^1$ the idea of the proof is to control the contribution to the area and to $u(t)$ of each bubble. Recall (6.22) and (6.24). Assume
that the interval \([a, b]\) with \(#[a, b] \leq N/k(\log k)^2\) is a bubble. It has at least one flippable corner. For any \(x \in [a, b], i = 1, 2\) we have

\[
\xi_t^1(a) - \frac{k#[a + 1, x]}{N} \leq \xi_t^1(x) \leq \xi_t^1(b) + \frac{k#[x + 1, b]}{N},
\]

and hence

\[
\max_{x \in [a, b]} (\xi_t^2 - \xi_t^1)(x) \leq (\xi_t^1(b) - \xi_t^1(a)) + \frac{k#[a, b]}{N}.
\]  

(6.77)

From the definition of \(A\), the right-hand side is smaller than \(2(\log k)^4\) and hence

\[
A_{[a,b]}(t) \leq 2(\log k)^4#[a, b] \leq N/k(\log k)^6.
\]  

(6.78)

And hence (recall (6.23), (6.24))

\[
u_{[a,b]}(t) \geq 1 \geq \frac{A_{[a,b]}(t)k}{N(\log k)^6}.
\]  

(6.79)

For large bubbles with \(#[a, b] \geq \frac{2N}{k}(\log k)^2\)

\[
A_{[a,b]}(t) \leq H(t)#[a, b]
\]  

(6.80)

and thus from the definition of \(A\)

\[
u_{[a,b]}(t) \geq \frac{k}{10N}#[a, b] \geq \frac{A_{[a,b]}(t)k}{10N\alpha(t)}.
\]  

(6.81)

The lemma is proved by summing (6.79) and (6.81) over all bubbles. \(\Box\)

**Proof of Proposition 5.2 for arbitrary \(k\).** We fix a constant \(C_1(\varepsilon)\) such that (6.68) holds for \(\varepsilon/3\) instead of \(\varepsilon\). \(C_2(\varepsilon, s)\) is chosen so that (6.70) holds for \(\varepsilon/3\).

\[
A := \left\{ \int_0^T 1_{\{\xi_t(t) \notin A\}} \, dt \leq \left\{ \begin{array}{l}
1
\end{array} \right\} \right\},
\]

\[
B := \left\{ \forall i \in \{0, \ldots, K\}, \int_0^T 1_{\{H(t) \geq C_1(i+1)^2 \sqrt{k}\}} \leq (\alpha/4)(i + 1)^{-2}T \right\},
\]

\[
C := \left\{ \forall i \in \{0, \ldots, K\}, T_i \leq C_2N^2k^{-\frac{3}{2}} \cap \{T_\infty < C_2N^2\} \right\},
\]

where

\[
T := 40N^2C_1C_2\alpha^{-1} \max_{i \geq 0} [(i + 1)^4(2/3)^i],
\]

\[
T' := T + C_2N^2.
\]  

(6.83)

From our definitions one has

\[
P[A \cap B \cap C] \geq 1 - \varepsilon.
\]  

(6.84)

Then we conclude by doing the same reasoning as in the case \(k = N/2\) that

\(A \cap B \cap C \subset \{\tau \leq T'\}\)

using that for \(k\) sufficiently large, on the event \(A \cap B\) we have

\[
\forall i \in \{0, \ldots, K\}, \int_0^T 1_{\{H(t) \geq C_1(i+1)^2 \sqrt{k} \text{ or } \xi_t(t) \notin A\}} \leq (\alpha/2)(i + 1)^{-2}T
\]  

(6.85)
Concerning (5.18) we set

\[ A' := \left\{ \int_0^{N^2} 1_{\{\xi_1(t) \notin A\}} \, dt \leq (N^2/k) \right\}, \]

\[ B' := \left\{ \forall i \in \{0, \ldots, K\}, \int_0^{N^2} 1_{\{H(t) \geq s^{(i+1)^2}\}} \, dt \leq (\alpha/4)(i + 1)^{-2}N^2 \right\}, \]

\[ C' := \{ \forall i \in \{0, \ldots, K\}; T_i \leq s^{-6}N^2k^{-i} \} \cap \{ T_\infty < N^2 \}, \]

And use (6.69), (6.71) and (6.11) to have for all sufficiently large \( s \),

\[ \mathbb{P}[A \cap B \cap C] \geq e^{-C_s^{-6}} \quad (6.87) \]

and conclude as in the proof for \( k = N/2 \) that

\[ \{ \tau \leq 2N^2 \} \subset A' \cap B' \cap C', \]

using that if \( N \) (and thus \( k \)) is sufficiently large we have on the event \( A' \cap B' \)

\[ \forall i \in \{0, \ldots, K\}, \int_0^{N^2} 1_{\{H(t) \geq s^{(i+1)^2}\} \lor \xi_1(t) \notin A} \, dt \leq (\alpha/4)(i + 1)^{-2}N^2. \quad (6.88) \]

\[ \square \]

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**Appendix A. Technical computations**

**A.1. Estimate for the discrete heat equation.** Let \( p_t(x, y) \) denote the heat kernel of on the discrete circle (\( p_t(x, \cdot) \) corresponds to the probability distribution of a simple random walk starting from \( x \) at time \( t \)). For fixed \( x \in \mathbb{Z}_N \), the function

\[ u(y, t) := p_t(x, y) \]

is the solution of the discrete heat equation on \( \mathbb{Z}_N \)

\[ \partial_t u(y, t) = u(y + 1, t) + u(y - 1, t) - 2u(y, t), \]

\[ u(y, 0) := 1_{\{y = x\}}. \quad (A.1) \]

Without loss of generality one restrict to the case \( x = 0 \). The solution of the above equation can be found by a decomposition on a base of eigenvalues the discrete Laplacian. We set for all \( i \in \{1, \ldots, \lfloor N/2 \rfloor\} \)

\[ \lambda_i := 2 \left( 1 - \cos \left( \frac{2\pi i}{N} \right) \right), \]

to be the eigenvalue associated to the normalized eigenfunction,

\[ y \mapsto \sqrt{2/N} \cos(2\pi y/N) \]

, the factor in front being \( \sqrt{1/N} \) instead of \( \sqrt{2/N} \) for \( i = N/2 \). We have to add also sine eigenfunctions to have a base but the projection of \( 1_{\{y = 0\}} \) on these eigenfunction is equal
to zero. By Fourier decomposition, we have

$$\left| u(y, t) - \frac{1}{N} \right| = \left| \frac{1}{N} \sum_{i=1}^{[N/2]} (2 - 1_{i=N/2}) e^{-\lambda_i t} \cos(2i\pi y/N) \right|$$

$$\leq \frac{2}{N} \sum_{i=1}^{[N/2]} e^{-\lambda_i t} \leq \frac{2}{N} e^{\lambda_1 t} - 1. \quad (A.2)$$

where in the last inequality we just used $e^{-\lambda_i t} \leq e^{-i\lambda_1 t}$. When $t \geq N^2$, we have $e^{\lambda_1 t} \geq 3$ and hence we have

$$u(y, t) \leq \frac{2}{N},$$

$$\left| u(y, t) - \frac{1}{N} \right| \leq \frac{4}{N} e^{-\lambda_1 t}. \quad (A.3)$$

Both statements in Lemma 4.3 are derived from these inequalities. For the first one we need to use that $E[\eta_t(x)]$ is a solution of the discrete heat equation.

\[\Box\]

A.2. Proof of Lemma 6.2 and Lemma 6.10.

Proof of Lemma 6.2. Note that for any given time $t$, $\nabla \xi_t^1$ is distributed according to $\mu$ (because this is the case for $t = 0$ and $\mu$ is the equilibrium measure for the $\nabla \xi$ dynamics). Now let us estimate the probability of

$$E := \{ \eta \mid \forall x, y \in \mathbb{Z}_N, \#[x, y] \geq N^{1/4} \Rightarrow j(x, y, \eta) \leq \frac{1}{3} \#[x, y]\},$$

under the measure $\mu$ where $j(x, y, \eta)$ is defined like its counterpart for the height function (6.17) replacing ”$x$ is a local extremum” by ”$\eta(x) \neq \eta(x + 1)$”.

We consider $\tilde{\mu}$ an alternative measure on $\{0, 1\}^{2N}$, under which the $\eta(x)$ are i.i.d. Bernoulli random variables with parameter $1/2$. By the local central limit Theorem for the random walk we have

$$\mu(E) := \tilde{\mu}(E) \mid \sum_{x \in \mathbb{Z}_N} \eta(x) = N/2 \leq C_1 \sqrt{N} \tilde{\mu}(E). \quad (A.4)$$

Let us now estimate $\tilde{\mu}(E)$. First we remark that we can replace $\#[x, y] \geq N^{1/4}$ in the definition of $E$ by $\#[x, y] \in [N^{1/4}, 2N^{1/4}]$. Indeed, if the proportion of local maxima is smaller than $1/3$ on a long interval, it has to be smaller than $1/3$ on a subinterval whose length belongs to $[N^{1/4}, 2N^{1/4}]$.

Set $x \in \{[N^{1/4}], N - 1\}$, note that $1_{\eta(z) \neq \eta(z+1)}$, $z \in \{1, \ldots, x\}$ are IID Bernoulli variables of parameter $1/2$, and hence by standard large deviation results there exists a constant $C_2 > 0$ such that

$$\tilde{\mu} \left( \sum_{z=1}^{x} 1_{\eta(z) \neq \eta(z+1)} \right) \leq N/3 \leq e^{-C_2 x} \quad (A.5)$$

By translation invariance we can deduce similar bounds for any translation of the interval $[1, x]$. Then, summing over all intervals and using (A.4) we deduce that there exists $C_3$ such that for all $N$ sufficiently large
\[ \mu(\mathcal{E}) \leq e^{-cN^{1/4}}. \]

Now be set \((T_i)_{i \geq 0}\) be the times where the chain \(\xi^2\) makes a transition. The chain \((\xi^1_i)_{i \geq 0}\) is a discrete time Markov chain with equilibrium probability \(\mu\) and hence by union bound

\[ \mathbb{P}\left[ \exists t \leq T_i \ \nabla \xi^1_i(t) \notin \mathcal{E} \right] \leq ie^{-C_3N^{1/4}}. \]  \hspace{1cm} (A.6)

Hence

\[ \mathbb{P}[A] = \mathbb{P}\left[ \exists t \leq N^3 \nabla \xi^1_i(t) \notin \mathcal{E} \right] \leq ie^{-C_3N^{1/4}} + \mathbb{P}[T_i \leq N^3]. \]  \hspace{1cm} (A.7)

As the transitions occur with a rate which is at most \(N\), the second term is exponentially small e.g. for \(i = N^5\) and one can conclude. \(\square\)

**Proof of Lemma 6.10.** In this proof it is somehow easier to work with the particle system, hence we let \(\mu\) be the uniform measure on \(\Omega_{N,k}\). We consider \(\tilde{\mu}\) an alternative measure on \(\{0,1\}^{\mathbb{Z}_N}\), under which the \(\eta(x)\) are i.i.d. Bernoulli random variable with parameter \(k/N\).

From the local central limit Theorem (which in this case can be proved simply using the Stirling Formula), there exists a constant \(C_1\) such that for all choices of \(k\) and \(N\)

\[ \tilde{\mu}\left( \sum_{x \in \mathbb{Z}_N} \eta(x) = k \right) \geq \frac{1}{C_1 \sqrt{k}}. \]  \hspace{1cm} (A.8)

Hence for any event \(A \subset \{0,1\}^{\mathbb{Z}_N}\), we have

\[ \mu(A) = \tilde{\mu}\left( A \mid \sum_{x \in \mathbb{Z}_N} \eta(x) = k \right) \leq C_1 \sqrt{k} \tilde{\mu}(A). \]  \hspace{1cm} (A.9)

Hence to prove the result, we just have to prove a slightly stronger upper-bound for the probability \(\tilde{\mu}(\mathcal{A})\).

We start proving that

\[ \tilde{\mu}(\mathcal{A}_2) \leq \frac{1}{k^3}. \]  \hspace{1cm} (A.10)

In terms of particle, \(\mathcal{A}_2\) holds any interval of length smaller than \(N/k(\log k)^2\) contains at most \((\log k)^4/2\) particles. Set

\[ m_{k,N} = m := \lceil N/k(\log k)^2 \rceil. \]

It standard large (computing the Laplace transform and using the Markov inequality) to show that there exists a constant \(c\) which is such that

\[ \tilde{\mu}\left( \sum_{x=1}^{m} \eta(x) \geq \frac{(\log k)^4}{2} \right) \leq \exp(-c(\log k)^6). \]  \hspace{1cm} (A.11)

Hence by translation invariance, the probability that there exists an interval of the form \([(i-1)m+1, im], \ i \in \{1, \ldots, \lceil N/m \rceil + 1\} which contains at most \((\log k)^4/2\) particles is smaller than

\[ 2k(\log k)^2 \exp(-c(\log k)^6) \leq k^{-2}. \]

As every interval of length smaller than \(m\) is included in in the union of at most two intervals of the type \([(i-1)m+1, im]\) we have
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\[ \tilde{\mu}(A_2) \leq \tilde{\mu} \left( \exists i \in \{1, \ldots, \lceil N/m \rceil + 1\}, \sum_{x=(m-1)i+1}^{mi} \eta(x) \geq \frac{(\log k)^4}{2} \right) \leq k^{-2}. \]

We now prove

\[ \tilde{\mu}(A_1) \leq \frac{1}{k^3}. \quad (A.12) \]

In terms of particle, having a local extremum at \( x \) just corresponds to \( \eta(x) \neq \eta(x+1) \). Note that if \( A \) occurs then by dichotomy there exists necessarily an interval whose length belongs to \([m, 2m]\) in which the density of extrema is smaller than \( k/10N \) and hence the total number of extrema in it is smaller than \( kN/m \). Setting \( m' = \lfloor m/2 \rfloor \) this interval must include an interval of the type \([i-1)m'+1, im'] \), with \( i \in \{1, \ldots, \lceil N/m' \rceil \} \), in which there are at most \( kN/m \) extrema.

Hence, noting that \( \frac{k}{5N}m \) and \( \lceil N/m' \rceil \leq 3k(\log k)^2 \)

we have

\[ \tilde{\mu}(A) \leq \tilde{\mu} \left( \exists i \in \{1, \ldots, \lceil N/m' \rceil \}, \sum_{x=(i-1)m'+1}^{im'} \mathbf{1}_{\{\eta(x) \neq \eta(x+1)\}} \geq \frac{(\log k)^2}{5} \right) \leq 2k(\log k)^2 \tilde{\mu} \left( \sum_{i=1}^{m'} \mathbf{1}_{\{\eta(x) \neq \eta(x+1)\}} \geq \frac{(\log k)^2}{5} \right). \quad (A.13) \]

Now we remark that

\[ \tilde{\mu} (\eta(x) \neq \eta(x+1)) = \frac{2k(N-k)}{N^2} \geq \frac{k}{N}. \]

As the variables \( \mathbf{1}_{\eta(2x-1) \neq \eta(2x)} \) are i.i.d for \( x \in \{1, \lceil m'/2 \rceil \} \) we can use standard large deviation techniques for sums of i.i.d. variables and obtain that there exists a constant \( c \) for which

\[ \tilde{\mu} \left( \sum_{i=1}^{m'} \mathbf{1}_{\{\eta(x) \neq \eta(x+1)\}} \geq \frac{(\log k)^2}{5} \right) \leq \tilde{\mu} \left( \sum_{x=1}^{\lceil m'/2 \rceil} \mathbf{1}_{\{\eta(2x-1) \neq \eta(2x)\}} \geq \frac{(\log k)^2}{5} \right) \leq e^{-c(\log k)^2}. \quad (A.14) \]

This combined with (A.13) allows us to conclude.

\[ \square \]

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