Poisson-Lie T-Duality and Bianchi Type Algebras

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Abstract

All Bianchi bialgebras have been obtained. By introducing a non-degenerate adjoint invariant inner product over these bialgebras the associated Drinfeld doubles have been constructed, then by calculating the coupling matrices for these bialgebras several $\sigma$-models with Poisson-Lie symmetry have been obtained. Two simple examples as prototypes of Poisson-Lie dual models have been given.

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1 Introduction

It is a well known fact that one of the most important symmetries of string theories, or in general $\sigma$-models, is target-space, T-duality. There was a common belief among the experts in this field that the existence of T-duality mainly should depend on the isometry symmetry of the original target manifold such that, depending on the kind of isometry it was called abelian or non-abelian T-duality (for a review see [1]). Of course there were some difficulties in the non-abelian case, for example it was not possible to obtain the original model from the dual one, because the latter might not have isometry [2]. Indeed Klimcik and Severra, by introducing the Poisson-Lie T-duality in their pioneering work [3], could remove the requirement of isometry in obtaining the dual model. Hence they could remove the above mentioned difficulties in T-duality investigation. Indeed, Poisson-Lie duality does not require the isometry in the original target manifold; the integrability of the Noether’s current associated with the action of group $G$ on the target manifold is good enough to have this symmetry. In other words, the components of the Noether’s current play the role of flat connection, that is, they satisfy Maurer-Cartan equations with group structure of $\tilde{G}$ (with the same dimension as $G$) [3]; such that $G$ and $\tilde{G}$ have Poisson-Lie structure and their Lie algebras form a bialgebra [4, 5].

Classically the dual models are equivalent, since one can obtain one from the other through a canonical transformation in their respective phase space. This canonical equivalence has already been shown, both in Poisson-Lie T-duality [3, 6, 7] and abelian or non-abelian T-duality [8]. Actually, the quantum equivalence of both abelian and non-abelian dual models have already been investigated. This equivalence can be shown by obtaining a general relation between the Weyl-anomaly coefficients (and $\beta$-functions) of the original model and those of its dual [9, 10]. But the quantum equivalence of the Poisson-Lie dual models is still a challenging problem; the quantum equivalence has been shown only in some special examples [10, 11]. Therefore, in order to understand the quantum features of the Poisson-Lie T-duality we have a long way ahead. Hence, we need to investigate too many $\sigma$-models with Poisson-Lie symmetry, since so far few examples with Poisson-Lie symmetry have been obtained [3, 10, 11, 12]. This suggests us trying to find further examples with Poisson-Lie symmetry. In this paper, we find explicitly all possible dual algebras of all three dimensional real Lie algebras (Bianchi algebras). Then by introducing a non-degenerate adjoint invariant inner product over the Lie algebras of Drinfeld doubles, we find many bialgebras, where we have associated a pair of Poisson-Lie dual $\sigma$-models to each of these bialgebras which contain all examples mentioned above and many other examples. This paper is organized as follows. In order the paper to be self-contained and also to fix the notations, we give a brief review of the Poisson-Lie T-duality in section 2. In section 3 we obtain all possible dual algebras of each of Bianchi algebras, where the bialgebras thus obtained have been listed in Table 2. Then, we introduce a non-degenerate adjoint invariant inner product over (six dimensional) Lie algebras of Drinfeld doubles, such that each of bialgebras are isotropic with respect to this inner product. In section 4 some of the general formulas related to the calculation of coupling matrices of $\sigma$-models, associated with the bialgebras of Table 2, have been constructed. Actually these informations help us to write a pair of Poisson-Lie dual $\sigma$-models associated with each bialgebra of Table 2. At the end of this section we give two simple examples. Finally, the paper ends with a brief conclusion and two appendices.
2 Poisson-Lie T-duality

In this section we review briefly the Poisson-Lie T-duality. According to \[3\] the Poisson-Lie duality based on the concepts of Drinfeld double and Manin triple. Drinfeld double \(D\) is a Lie group where its Lie algebra \(\mathcal{D}\) can be decomposed (as a vector space) into direct sum of two Lie subalgebras \(\mathfrak{g}\) and \(\tilde{\mathfrak{g}}\), such that this is maximal isotropic with respect to a non-degenerate invariant bilinear form \(<, >\) over \(\mathcal{D}\). The doublet \((\mathfrak{g}, \tilde{\mathfrak{g}})\) and triplet \((\mathcal{D}, \mathfrak{g}, \tilde{\mathfrak{g}})\) are called bialgebra and Manin triple respectively \[4, 5\]. Actually taking the sets \(\{X_i\}\) and \(\{\tilde{X}^i\}\) as the bases of the Lie algebras \(\mathfrak{g}\) and \(\tilde{\mathfrak{g}}\), respectively, we have:

\[
[X_i, X_j] = f_{ij}^k X_k, \quad [\tilde{X}^i, \tilde{X}^j] = \tilde{f}^{ij}{}_k \tilde{X}^k, \quad [X_i, \tilde{X}^j] = \tilde{f}^{jk}{}_i \tilde{X}^k.
\]

(1)

The isotropy of the subalgebras with respect to bilinear form means that:

\[
<X_i, X_j> = <\tilde{X}^i, \tilde{X}^j> = 0, \quad <X_i, \tilde{X}^j> = \delta_i^j.
\]

(2)

As we will see in the next section the above relations put rather strong constraint on the structure constants of subalgebras. Hence, obtaining of the bialgebra and Manin triple becomes rather a tough job. In order to define \(\sigma\) models with Poisson-Lie duality symmetry, we need to consider the following relations \[3\]:

\[
g^{-1}X_ig = a(g)^i_j X_j, \quad g^{-1}\tilde{X}^ig = b(g)^{ij} \tilde{X}^j, \quad \Pi(g) = b(g)a^{-1}(g),
\]

(4)

with \(g\) and \(\tilde{g}\) as elements of Lie groups \(G\) and \(\tilde{G}\) associated with Lie algebras \(\mathfrak{g}\) and \(\tilde{\mathfrak{g}}\), respectively. The invariance of inner product with respect to adjoint action of group together with \(\mathfrak{g}\) and \(\mathfrak{h}\) requires the above matrices to possess the following properties:

\[
d(g) = a^{-1}(g), \quad a^{-1}(g) = a(g^{-1}), \quad b'(g) = b(g^{-1}), \quad \Pi'(g) = -\Pi(g).
\]

(5)

The matrices \(a(g), \tilde{b}(\tilde{g})\) and \(\tilde{\Pi}(\tilde{g})\) associated with group \(\tilde{G}\) can be defined in a similar way. Now, we can define below the \(\sigma\)-model with d-dimensional target manifold \(M\), where the group \(G\) acts freely on it \[3, 4, 6\]:

\[
S = -\frac{1}{2} \int d\xi^+ d\xi^- [E_{ij}(\partial_+ gg^{-1})^i (\partial_- gg^{-1})^j + \Phi^{(1)}_{\alpha i} (\partial_+ gg^{-1})^i \partial_+ y^\alpha + \Phi^{(2)}_{\alpha i} \partial_+ y^\alpha (\partial_- gg^{-1})^i + \Phi_{\alpha \beta} \partial_+ y^\alpha \partial_+ y^\beta],
\]

(6)

where the coupling matrices are:

\[
E = (E_0^{-1} + \Pi)^{-1}, \quad \Phi^{(1)} = EE_0^{-1} F^{(1)}, \quad \Phi^{(2)} = F^{(2)} E_0^{-1} E,
\]

\[3\]
\[ \Phi = F - F^{(2)} \Pi E_0^{-1} F^{(1)}. \]  

The \( y^\alpha \) with \( \alpha = 1, \ldots, d - \text{dim} G \) are coordinates of \( M/G \) manifold. The matrices \( E_0^{-1}, F^{(1)}, F^{(2)} \) and \( F \) are arbitrary functions of \( y^\alpha \) only. Similarly, the coupling matrices of the dual \( \sigma \)-model can be written as [3, 4]:

\[ \tilde{E} = (E_0 + \tilde{\Pi})^{-1}, \quad \tilde{\Phi}^{(1)} = \tilde{E} F^{(1)}, \quad \tilde{\Phi}^{(2)} = -F^{(2)} \tilde{E}, \]

\[ \Phi = F - F^{(2)} \tilde{E} F^{(1)}. \]  

The target space of dual model is \( d \)-dimensional manifold \( \tilde{M} \) with the group \( \tilde{G} \) acting freely on it. The corresponding dual action can be written as:

\[ \tilde{S} = -\frac{1}{2} \int d\xi^+ d\xi^- [\tilde{E}^{ij}(\partial_+ \tilde{g}^{-1})_i (\partial_- \tilde{g}^{-1})_j + \tilde{\Phi}^{(1)}_{ij} \alpha (\partial_+ \tilde{g}^{-1})_i \partial_- \tilde{g}^{-1} \partial_- \tilde{g}^{-1} \partial_- \tilde{g}^{-1} ] + \tilde{\Phi}_{\alpha \beta} \partial_+ y^\alpha \partial_- y^\beta. \]  

The actions (6) and (9) correspond to Poisson-Lie dual \( \sigma \) models [3, 4]. Notice that if the group \( G(\tilde{G}) \) besides having free action on \( M(\tilde{M}) \), acts transitively over it, then the corresponding manifolds \( M(\tilde{M}) \) will be the same as the groups \( G(\tilde{G}) \). In this case only the first term appears in the actions (6) and (9). Also if the group \( G \) becomes the isometry group of the manifold \( M \) with dual abelian group \( \tilde{G} \), then we get the standard nonabelian duality [2].

### 3 Bianchi Bialgebras

In this section, we use [3] to obtian the dual Lie algebras \( \tilde{\mathfrak{g}} \) of a given 3-dimensional real Lie algebras \( \mathfrak{g} \). Then, by introducing a nondegenerate adjoint invariant bilinear form we get the corresponding Manin triples. According to Behr’s classification [13, 14] of 3-dimensional Bianchi Lie algebras [15], the commutation relation of these algebras can be generally written as:

\[ [X_1, X_2] = -a X_2 + n_3 X_3, \quad [X_2, X_3] = n_1 X_1, \]

\[ [X_3, X_1] = n_2 X_2 + a X_3, \]  

where the structure constants are given in Table 1 [3].

| Table 1 : Bianchi classification of three dimensional Lie algebras. |
|---------------------------------------------------------------|
| 1 Actually the Noether’s current associated with the action of \( G(\tilde{G}) \) on \( M(\tilde{M}) \) satisfy the Maurer-Cartan equation with structure constants of dual Lie group \( G(\tilde{G}) \). |
| 2 For three algebras III, VIa and VIIa, with nonzero parameters \( a, n_2 \) and \( n_3 \), rescaling of the basis yields the ratio \( \frac{n_2^2}{n_3} \) constant. For other algebras one can choose a basis with structure constants \( \pm 1 \), as in Table 1. |

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4
For all Lie algebras of Table 1 we have \( an_1 = 0 \). The algebras with \( a = 0 \) are called class A while those with nonvanishing \( a \) are called class B. Now we can determine the dual Lie algebra \( \tilde{g} \), by using the structure constant of Lie algebra \( g \) given in (10). Due to the antisymmetricity of (3) with respect to the set of indices \((i, j)\) and \((k, l)\), these relations, after some algebraic calculations and with the help of (10), lead to the following nine equations

\[
\begin{align*}
&n_1 \tilde{f}^{23} - n_2 \tilde{f}^{13} = n_3 \tilde{f}^{12}, \\
n_2 \tilde{f}^{13} = n_2 \tilde{f}^{12}, &\quad n_3 \tilde{f}^{13} = n_3 \tilde{f}^{12} - n_1 \tilde{f}^{12}, &\quad n_1 \tilde{f}^{23} = n_1 \tilde{f}^{13}, \\
-n_1 \tilde{f}^{32} + n_3 \tilde{f}^{12} = -n_2 \tilde{f}^{13}, &\quad a(\tilde{f}^{32} + \tilde{f}^{31}) = n_3(\tilde{f}^{12} + \tilde{f}^{23}), \\
a(\tilde{f}^{32} + \tilde{f}^{31}) = n_2(\tilde{f}^{13} + \tilde{f}^{23}), &\quad a(\tilde{f}^{32} + \tilde{f}^{31}) = n_3(\tilde{f}^{12} + \tilde{f}^{23}), \\
-a(\tilde{f}^{31} + n_2 \tilde{f}^{31} - n_3 \tilde{f}^{21} - a \tilde{f}^{21} = n_1 \tilde{f}^{23}).
\end{align*}
\]

In the above equations some coefficients can be eliminated, but the reason for their presence is due to their vanishing for some algebras. Now, in order to determine the structure constants of dual Lie algebras \( \tilde{g} \) associated with those listed in Table 1, we could write (11) for each of them. Then, by using the Jacobi identity for \( \tilde{g} \), we can determine \( \tilde{f}^{ij} \). The algebras \( \tilde{g} \) reduces to Lie algebras of Bianchi type if (a) we use the following general form of structure constant of a given 3-dimensional Lie algebra:

\[
\tilde{f}^{ij} = \epsilon^{ijk} \tilde{n}_{lk} + \delta^i_k \tilde{a}^k - \delta^j_k \tilde{a}^i,
\]

with \( \tilde{a}^i \) as components of a 3-vector and \( \tilde{n}_{ij} \) as elements of symmetric matrix with the constraint \( \tilde{n}_{ij} \tilde{a}^j = 0 \); and (b) if we work in a basis that diagonalizes the matrix \( \tilde{n}_{ij} \) with diagonal elements \( (\tilde{n}_1, \tilde{n}_2, \tilde{n}_3) \) and vector \( \tilde{a}^k \) with components \( (\tilde{a}, 0, 0) \). But, instead of performing the above steps, we can choose another method with rather less computations and obtain the same results, as follows. Similar to \( f^{ij} \) we assume that the \( \tilde{f}^{ij} \) have the following form:

\[
[\tilde{X}_1, \tilde{X}_2] = -\tilde{a}\tilde{X}_2 + \tilde{n}_3\tilde{X}_3, \quad [\tilde{X}_2, \tilde{X}_3] = \tilde{n}_1\tilde{X}_1.
\]

After using (11) for each Lie algebra, the number of equations becomes less than nine.
\[ [\tilde{X}_3, \tilde{X}_1] = \tilde{n}_2 \tilde{X}_2 + \tilde{a} \tilde{X}_3. \]  

(12)

Then (11) reduces to

\[ n_1 \tilde{n}_1 = 0, \quad a\tilde{a} = n_2 \tilde{n}_2 = n_3 \tilde{n}_3. \]  

(13)

Hence the structure constants \((\tilde{a}, \tilde{n}_1, \tilde{n}_2, \tilde{n}_3)\) of the dual Lie algebra \(\tilde{g}\) can be determined in terms of the structure constants \((a, n_1, n_2, n_3)\) of the Lie algebra \(g\).

As an example, for \(g = IX\), from (13) and Table 1 we have: \(\tilde{n}_1 = \tilde{n}_2 = \tilde{n}_3 = 0\) and \(\tilde{a}\) is an arbitrary constant. Now, for \(\tilde{a} = 0\) we obtain Bianchi of type I and for \(\tilde{a} \neq 0\) we get Bianchi of type V if we rescale the basis such that \(\tilde{a} = 1\). Therefore, the Lie algebra \(IX\) has two different dual Lie algebras I, V. These bialgebras have already been used in references [10, 13].

As another example we consider \(g = VII_0\), then we have \(\tilde{n}_1 = \tilde{n}_2 = 0\) and \(\tilde{a}\) together with \(\tilde{n}_3\) become arbitrary. In this case the dual Lie algebra \(\tilde{g}\) can be one of the following four different Bianchi Lie algebras:

1) For \(\tilde{n}_3 = \tilde{a} = 0\), we get Bianchi of type I.

2) For \(\tilde{n}_3 \neq 0\) and \(\tilde{a} = 0\) via rescaling and change of basis, we get Bianchi of type II.

3) For \(\tilde{n}_3 = 0\) and \(\tilde{a} \neq 0\) via rescaling of basis, we get Bianchi of type V.

4) Finally, for \(\tilde{n}_3 \neq 0\) and \(\tilde{a} \neq 0\) via rescaling of basis, we get Bianchi of type IV.

Therefore the Lie algebra \(VII_0\) has four different dual Lie algebras I, II, V, IV. We have also determined the possible dual Lie algebras of other Bianchi Lie algebras, where the results are given in Table 2.

### Table 2: Bianchi Bialgebras.

| \(g\) (class A) | \(\tilde{g}\) |
|-----------------|----------------|
| I               | all Bianchi algebras |
| II              | all Bianchi algebras except types of IX, VIII |
| VII₀           | \(I, II, V, IV\) |
| VI₀           | \(I, II, V, IV\) |
| IX            | \(I, V\) |
| VIII          | \(I, V\) |

| \(g\) (class B) | \(\tilde{g}\) |
|-----------------|----------------|
| V               | class A |
| IV             | \(I, II, VII₀, VI₀\) (class A except types of IX, VIII) |
| VII₀          | \(I, II, VII₀ \tilde{a} (\tilde{a} = \frac{1}{a})\) |
| III           | \(I, II, III\) |
| VI₀           | \(I, II, VI₀ \tilde{a} (\tilde{a} = \frac{1}{a})\) |

As in Table 2, all Bianchi Lie algebras, except types of IX and VIII, have more than three different dual Lie algebras. Also Bianchi algebras II, III, VI₀ and VII₀ can be self dual. The
number of different bialgebras listed in Table 2, is 28\footnote{It is clear that if the pair \((g, \tilde{g})\) form a bialgebra, then, also, the pair \((\tilde{g}, g)\) will lead to another bialgebra. Therefore the total number of bialgebras of Table 2 is 56.}. So far we have been able to obtain all possible bialgebras associated with Bianchi algebras. As it is mentioned in the previous section, in order to have a Manin triple associated with these bialgebras, we need to have a nondegenerate ad-invariant inner product over the algebra \(D = g \oplus \tilde{g}\), such that the algebras \(g\) and \(\tilde{g}\) become isotropic with respect to it. In order to obtain this inner product we choose the set \(\{T_A\} = \{X_1, X_2, X_3, \tilde{X}_1, \tilde{X}_2, \tilde{X}_3\}\) as the basis of \(D\). Notice that we have more than one \(D\) Lie algebra, but considering the algebras \(g\) and \(\tilde{g}\) with commutation relation (10) and (12), respectively, we can give the general form of \(D\).

Now, writing the commutation relation of the algebras \(D\) as:

\[
[T_A, T_B] = C_{AB}^C T_C,
\]

with \(A, B = 1, ..., 6\), the structure constants \(C_{AB}^C\) can be obtained by using (1), (10) and (12). Then the basis \(T_A\) will have the following matrix form in the adjoint representation of \(D\):

\[
T_i = \begin{pmatrix} X_i & 0 \\ \tilde{Y}_i & -(X_i)^t \end{pmatrix}, \quad T_{i+3} = \begin{pmatrix} -(\tilde{X}^i)^t & Y_i \\ 0 & \tilde{X}^i \end{pmatrix},
\]

where \(X_i\), the adjoint representation of basis of Bianchi Lie algebra \(g\), and the antisymmetric matrices \(Y_i\) are:

\[
X_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a & -n_3 \\ 0 & n_2 & a \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & -a & n_3 \\ 0 & 0 & 0 \\ -n_1 & 0 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & -n_2 & -a \\ n_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]

\[
Y_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -n_1 \\ 0 & n_1 & 0 \end{pmatrix}, \quad Y_2 = \begin{pmatrix} 0 & a & n_2 \\ -a & 0 & 0 \\ -n_2 & 0 & 0 \end{pmatrix}, \quad Y_3 = \begin{pmatrix} 0 & -n_3 & a \\ n_3 & 0 & 0 \\ -a & 0 & 0 \end{pmatrix}.
\]

Similarly, \(\tilde{X}^i\), adjoint representation of the basis of dual Bianchi Lie algebras \(\tilde{g}\), and \(\tilde{Y}^i\) have the same form as \(X_i\) and \(Y_i\) with the difference that we must replace the set \((a, n_1, n_2, n_3)\) with \((\tilde{a}, \tilde{n}_1, \tilde{n}_2, \tilde{n}_3)\).

Since, in general, Lie algebra \(D\) is non-semisimple, the nondegenerate ad-invariant inner product on it can not be obtained via the trace of bilinear product of those algebras in adjoint representation. In other words Killing form is degenerate for these algebras. Therefore, in order to obtain the nondegenerate ad-invariant metric \(\Omega_{AB}\) in the above basis, we have to solve the following equation\footnote{Equation (17) must be solved to find the metric \(\Omega\).}:

\[
C_{AB}^D \Omega_{CD} + C_{AC}^D \Omega_{BD} = 0.
\]

Equation (16) means that the matrices \(T_A \Omega\) must be antisymmetric matrices. Hence we must find symmetric matrix \(\Omega\) such that the above relations holds. Now, in order to determine \(\Omega\) for
each bialgebra of Table 2, we must write $T_A$ explicitly and then, using the antisymmetricity of $T_A, \Omega$, we find the explicit form of $\Omega$. As an example, for the pair of $(V, IX)$, the form of $\Omega$ is:

$$
\Omega = \begin{pmatrix}
-b & 0 & 0 & k & 0 & 0 \\
0 & 0 & 0 & 0 & k & -b \\
0 & 0 & 0 & b & k & 0 \\
k & 0 & 0 & b & 0 & 0 \\
0 & k & b & 0 & b & 0 \\
0 & -b & k & 0 & 0 & b \\
\end{pmatrix}, \quad det\Omega = -(k^2 + b^2)^3.
$$

Now by choosing $k = 1$ and $b = 0$ we get:

$$
\Omega = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (17)
$$

Actually, by using (14) and (15) one can easily show that the symmetric metric (17) can be ad-invariant metric of all 28 different bialgebras of Table 2, that is

$$
T_A, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
$$

is antisymmetric for all bialgebras of Table 2. Hence, we choose $\Omega$ as the required inner product. It is also interesting to see that the Lie subalgebras of all bialgebras of Table 2 are isotropic with respect to this inner product, that is we have:

$$
<T_A, T_B> = \Omega_{AB},
$$

$$
<X, \tilde{X}^j> = <T_i, T_{j+3}> = \Omega_{i,j+3} = \delta_{ij},
$$

$$
<X, X_j> = <T_{i+3}, T_j> = \Omega_{i+3,j} = \delta_{ij}. \quad (18)
$$

Therefore, all bialgebras of Table 2, together with the ad-invariant metric (17), form Manin triples. Now we can construct $\sigma$-models with Poisson-Lie symmetries associated with the Manin triples obtained above.

## 4 $\sigma$-models and Bianchi Lie Groups

As mentioned in section 2, in order to construct $\sigma$-models with Poisson-Lie symmetry associated with bialgebra $(g, \tilde{g})$, we need to know $a(g)$ and $b(g)$. Indeed for every bialgebras of Table 2, we can evaluate $a(g)$ and $b(g)$. Hence we will have a $\sigma$-model as well as its dual, which possess the Poisson-Lie symmetry. In this section we give general formulas for $a(g)$ and $b(g)$ in terms of the sets of parameters $(a, n_1, n_2, n_3)$ and $(\tilde{a}, \tilde{n}_1, \tilde{n}_2, \tilde{n}_3)$. Therefore, by appropriate choices of these parameters, $a(g)$ and $b(g)$ of all bialgebras of Table 2 can be evaluated. Hence we can construct all $\sigma$-models associated with them. To achieve this goal we parametrize the group $G$ in the following form:

$$
g = e^{\chi_1 X_1} e^{\chi_2 X_2} e^{\chi_3 X_3}, \quad (19)
$$
where $\chi_i$ are coordinates of the group manifold $G$. In order to calculate $a(g)$ and $b(g)$, we need to calculate expressions such as $e^{-\chi_i X_i} e^{\chi_i X_i}$ and $e^{-\chi_i X_i} \tilde{X}_j e^{\chi_i X_i}$. To do this, we must use the algebraic relations (1), (10) and (12). To make the calculation simple, we need to work with basis of $D$, where we have:

$$e^{-\chi_i X_i} T_A e^{\chi_i X_i} = e^{-\chi_i T_i} T_A e^{\chi_i T_i} = (e^{\chi_i T_i})^B_A T_B. \tag{20}$$

Therefore, we must calculate matrices $e^{\chi_i T_i}$, where due to the particular block diagonal form of $T_i$ (14), in general, they have the following form:

$$e^{\chi_i T_i} = \begin{pmatrix} e^{\chi_i X_i} & 0 \\ B_i & e^{-\chi_i (X_i)^t} \end{pmatrix}. \tag{21}$$

To calculate (21) we need to know the block diagonal form $D_i$ of matrices $T_i$ together with matrices $S_i$, then we have:

$$S_i T_i S_i^{-1} = D_i, \quad e^{\chi_i T_i} = S_i^{-1} e^{\chi_i D_i} S_i. \tag{22}$$

Now we put each $T_i$ into its block diagonal form separately. For $T_1$ we have:

$$S_1 = \tilde{n}_1 \begin{pmatrix} N \frac{2a}{n_1} \frac{1}{0} \\ 0 0 1 \end{pmatrix}, \quad D_1 = \begin{pmatrix} - (X_1)^t & 0 \\ 0 & X_1 \end{pmatrix}, \tag{23}$$

where

$$N = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$ 

On the other hand, by using the following matrices:

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{\frac{n_3}{n_2+n_3}} & \sqrt{\frac{n_1}{n_2+n_3}} \\ 0 & -i \sqrt{\frac{n_3}{n_2+n_3}} & i \sqrt{\frac{n_1}{n_2+n_3}} \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a + i \sqrt{n_2 n_3} & 0 \\ 0 & 0 & a - i \sqrt{n_2 n_3} \end{pmatrix},$$

we write the matrix $X_1$ in digonal form, then using (22) for these matrices we calculate $e^{\chi_1 X_1}$:

$$e^{\chi_1 X_1} = e^{a \chi_1} \begin{pmatrix} e^{-a \chi_1} & 0 & 0 \\ 0 & \cos \eta_1 & -\sqrt{\frac{n_3}{n_2}} \sin \eta_1 \\ 0 & \sqrt{\frac{n_3}{n_2}} \sin \eta_1 & \cos \eta_1 \end{pmatrix}, \tag{24}$$

where $\eta_1 = \sqrt{n_2 n_3} \chi_1$. With the help of (22), (23) and (24) and after some algebraic calculations we have:

$$B_1 = - \frac{\tilde{n}_1}{a} \sinh a \chi_1 \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sqrt{\frac{n_3}{n_2}} \sin \eta_1 & \cos \eta_1 \\ 0 & - \cos \eta_1 & \sqrt{\frac{n_3}{n_2}} \sin \eta_1 \end{pmatrix}. \tag{25}$$

Here, repeated indices do not imply summation.
Similarly, for the matrices $\mathcal{T}_2$ and $\mathcal{T}_3$, we have calculated the matrices $S_2$ and $S_3$, which transform them into their Jordan forms $D_2$ and $D_3$ (these matrices are given in appendix A). Finally, using these Jordan forms together with (22) and considering (21), we obtain the following expressions for matrices $e^{\chi_2 \mathcal{T}_2}$ and $e^{\chi_3 \mathcal{T}_3}$:

$$
e^{\chi_2 \mathcal{T}_2} = \begin{pmatrix}
\cos \eta_2 & -\frac{a}{\sqrt{n_1 n_3}} \sin \eta_2 & \sqrt{\frac{n_2}{n_1}} \sin \eta_2 \\
0 & 1 & 0 \\
-\sqrt{\frac{n_1}{n_3}} \sin \eta_2 & -\frac{a}{n_3} (\cos \eta_2 - 1) & \cos \eta_2
\end{pmatrix}, \quad (26)
$$

where $\eta_2 = \sqrt{n_1 n_3} \chi_2$, and

$$
B_2 = \begin{pmatrix}
-\frac{\sqrt{n_1}}{n_1} \chi_2 \tilde{n}_2 \sin \eta_2 & \frac{\xi}{n_1} \sin \eta_2 + \frac{\tilde{n}_2 \chi_2}{n_3} \cos \eta_2 & -\frac{2a_n}{n_3} \sin \eta_2 - \frac{\tilde{n}_2 \chi_2}{n_3} \cos \eta_2 \\
-\tilde{n}_2 \chi_2 \cos \eta_2 & \frac{\xi}{n_1} \sin \eta_2 + \frac{\tilde{n}_2 \chi_2}{n_3} \cos \eta_2 & \frac{\xi}{n_1} (\cos \eta_2 - 1) - \frac{\tilde{n}_2 \chi_2}{n_3} \sin \eta_2 - \sqrt{\frac{n_2}{n_1}} \tilde{n}_2 \sin \eta_2 \\
\frac{\xi}{n_1} \sin \eta_2 + \frac{\tilde{n}_2 \chi_2}{n_3} \cos \eta_2 & \frac{\xi}{n_1} \sin \eta_2 + \frac{\tilde{n}_2 \chi_2}{n_3} \cos \eta_2 & \frac{\xi}{n_1} (\cos \eta_2 - 1) + \frac{\tilde{n}_2 \chi_2}{n_3} \sin \eta_2
\end{pmatrix}, \quad (27)
$$

where $\xi = \tilde{a}n_3 + \tilde{n}_2 a$.

Also

$$
e^{\chi_3 \mathcal{T}_3} = \begin{pmatrix}
\frac{\xi}{n_2} \sin \eta_3 & \frac{\xi}{n_2} \sin \eta_3 & \frac{\xi}{n_2} \cos \eta_3 \\
\frac{\xi}{n_2} \cos \eta_3 & \frac{\xi}{n_2} \cos \eta_3 & \frac{\xi}{n_2} \sin \eta_3 \\
\frac{\xi}{n_2} \sin \eta_3 & \frac{\xi}{n_2} \cos \eta_3 & \frac{\xi}{n_2} \sin \eta_3
\end{pmatrix}, \quad (28)
$$

where $\eta_3 = \sqrt{n_1 n_2} \chi_3$, and

$$
B_3 = \begin{pmatrix}
\frac{\sqrt{n_2}}{n_2} \chi_3 \tilde{n}_3 \sin \eta_3 & -\tilde{n}_3 \chi_3 \cos \eta_3 & \frac{\xi}{n_2} \sin \eta_3 - \frac{\tilde{n}_3 \chi_3}{n_2} \cos \eta_3 \\
\tilde{n}_3 \chi_3 \cos \eta_3 & -\frac{\sqrt{n_2}}{n_2} \chi_3 \tilde{n}_3 \sin \eta_3 & \frac{\xi}{n_2} \cos \eta_3 - \frac{\tilde{n}_3 \chi_3}{n_2} \sin \eta_3 \\
-\frac{\xi}{n_2} \sin \eta_3 + \frac{\tilde{n}_3 \chi_3}{n_2} \cos \eta_3 & -\frac{\xi}{n_2} \cos \eta_3 + \frac{\tilde{n}_3 \chi_3}{n_2} \sin \eta_3 & \frac{\xi}{n_2} (\cos \eta_3 - 1) - \frac{\tilde{n}_3 \chi_3}{n_2} \cos \eta_3
\end{pmatrix}, \quad (29)
$$

where $\tilde{\xi} = \tilde{a}n_2 + \tilde{n}_3 a$.

Now, considering the definitions of $b(g)$ and $a(g)$ given in (4), in general we have [3]:

$$
g^{-1} \begin{pmatrix}
T_1 \\
\vdots \\
T_6
\end{pmatrix} g = \begin{pmatrix}
a(g) & 0 \\
\frac{b(g)}{d(g)} & d(g)
\end{pmatrix} \begin{pmatrix}
T_1 \\
\vdots \\
T_6
\end{pmatrix}. \quad (30)
$$

Hence due to (19) and (20) we have:

$$
e^{\chi_1 \mathcal{T}_1} e^{\chi_2 \mathcal{T}_2} e^{\chi_3 \mathcal{T}_3} = \begin{pmatrix}
a(g) & 0 \\
\frac{b(g)}{d(g)} & d(g)
\end{pmatrix}. \quad (31)
$$

Finally, by using (21) we get:

$$
a(g) = e^{\chi_1 \mathcal{T}_1} e^{\chi_2 \mathcal{T}_2} e^{\chi_3 \mathcal{T}_3}, \\
b(g) = B_1 e^{\chi_2 \mathcal{T}_2} e^{\chi_3 \mathcal{T}_3} + e^{-\chi_1 \mathcal{T}_1} B_2 e^{\chi_3 \mathcal{T}_3} + e^{-\chi_1 \mathcal{T}_1} e^{-\chi_2 \mathcal{T}_2} B_3. \quad (32)
$$
Indeed by the above mentioned prescription, we can evaluate $\tilde{a}(\tilde{g})$ and $\tilde{b}(\tilde{g})$. Since the set of $a(g)$ and $b(g)$ associated with bialgebra $(\tilde{g}, g)$ is the same as the set $\tilde{a}(\tilde{g})$ and $\tilde{b}(\tilde{g})$ of bialgebra $(g, \tilde{g})$, therefore in order to write $\tilde{a}(\tilde{g})$ and $\tilde{b}(\tilde{g})$, it is enough to replace the parameters $\chi_i$ and $(a, n_1, n_2, n_3)$ with $\tilde{\chi}_i$ and $(\tilde{a}, \tilde{n}_1, \tilde{n}_2, \tilde{n}_3)$ in (31) and (32), respectively. As an example, in order to determine $\tilde{a}(\tilde{g})$ and $\tilde{b}(\tilde{g})$ of the pair $(IX, V)$, we just need to calculate $a(g)$ and $b(g)$ of the pair $(V, IX)$, provided that we replace the coordinates $\chi_i$ with $\tilde{\chi}_i$.

So far all calculations have been done on the basis of the parametrization (19) for Bianchi Lie groups $G$. In the case of usual parametrization of each of Bianchi groups \footnote{ indeed by the above mentioned prescription, we can evaluate $\tilde{a}(\tilde{g})$ and $\tilde{b}(\tilde{g})$. Since the set of $a(g)$ and $b(g)$ associated with bialgebra $(\tilde{g}, g)$ is the same as the set $\tilde{a}(\tilde{g})$ and $\tilde{b}(\tilde{g})$ of bialgebra $(g, \tilde{g})$, therefore in order to write $\tilde{a}(\tilde{g})$ and $\tilde{b}(\tilde{g})$, it is enough to replace the parameters $\chi_i$ and $(a, n_1, n_2, n_3)$ with $\tilde{\chi}_i$ and $(\tilde{a}, \tilde{n}_1, \tilde{n}_2, \tilde{n}_3)$ in (31) and (32), respectively. As an example, in order to determine $\tilde{a}(\tilde{g})$ and $\tilde{b}(\tilde{g})$ of the pair $(IX, V)$, we just need to calculate $a(g)$ and $b(g)$ of the pair $(V, IX)$, provided that we replace the coordinates $\chi_i$ with $\tilde{\chi}_i$.}

In caculating these matrices we must remove a series of ambiguities.

\footnote{Indeed by the above mentioned prescription, we can evaluate $\tilde{a}(\tilde{g})$ and $\tilde{b}(\tilde{g})$. Since the set of $a(g)$ and $b(g)$ associated with bialgebra $(\tilde{g}, g)$ is the same as the set $\tilde{a}(\tilde{g})$ and $\tilde{b}(\tilde{g})$ of bialgebra $(g, \tilde{g})$, therefore in order to write $\tilde{a}(\tilde{g})$ and $\tilde{b}(\tilde{g})$, it is enough to replace the parameters $\chi_i$ and $(a, n_1, n_2, n_3)$ with $\tilde{\chi}_i$ and $(\tilde{a}, \tilde{n}_1, \tilde{n}_2, \tilde{n}_3)$ in (31) and (32), respectively. As an example, in order to determine $\tilde{a}(\tilde{g})$ and $\tilde{b}(\tilde{g})$ of the pair $(IX, V)$, we just need to calculate $a(g)$ and $b(g)$ of the pair $(V, IX)$, provided that we replace the coordinates $\chi_i$ with $\tilde{\chi}_i$.}

In caculating these matrices we must remove a series of ambiguities.
Actually this is a self-dual model with respect to Poisson-Lie dual transformation. Indeed if we choose $E_0$ as an arbitrary invertible constant matrix then the dual $\sigma$-model again will be the same as the original one but we must replace elements of $E_0$ with those of $E_0^{-1}$. Actually this particular example is similar to two dimensional Borelian dual models of reference [3].

Another example is related to the bialgebra $(II, V)$. Again, after calculating the matrices (24-29) given in appendix B, $\Pi(g)$ can be written as:

$$
\Pi(g) = \begin{pmatrix}
0 & \chi_2 & \chi_3 \\
-\chi_2 & 0 & 0 \\
-\chi_3 & 0 & 0 \\
\end{pmatrix}.
$$

(38)

Here $dg.g^{-1}$ has the same form as (36). Hence for $E_0 = I$, from (33) the action of the original model is:

$$
S = -\frac{1}{2} \int d\sigma d\tau \left\{ \frac{1}{1 + \chi_2^2 + \chi_3^2} \left[ \partial_\mu \chi_1 \partial^\mu \chi_1 + 2\chi_2 \partial_\mu \chi_1 \partial^\mu \chi_3 + (1 + \chi_3^2) \partial_\mu \chi_2 \partial^\mu \chi_2 \\
+ (1 + 2\chi_2^2) \partial_\mu \chi_3 \partial^\mu \chi_3 \right] + \partial_\mu t \partial^\mu t \right\}.
$$

(39)

Finally, by calculating matrices (24-29), as given in appendix B, we determine $\tilde{\Pi}(\tilde{g})$:

$$
\tilde{\Pi}(\tilde{g}) = e^{-\tilde{x}_1} \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & -\sinh \tilde{x}_1 \\
0 & \sinh \tilde{x}_1 & 0 \\
\end{pmatrix}.
$$

(40)

On the other hand, we have:

$$
d\tilde{g}.\tilde{g}^{-1} = d\tilde{x}_1 \tilde{X}^1 + e^{-\tilde{x}_1} d\tilde{x}_2 \tilde{X}^2 + e^{-\tilde{x}_1} d\tilde{x}_3 \tilde{X}^3.
$$

(41)

Then from (34) the dual model can be written as:

$$
\tilde{S} = -\frac{1}{2} \int d\sigma d\tau [\partial_\mu \tilde{x}_1 \partial^\mu \tilde{x}_1 + \frac{e^{-2\tilde{x}_1}}{1 + e^{-2\tilde{x}_1}} \left( \partial_\mu \tilde{x}_2 \partial^\mu \tilde{x}_2 + \partial_\mu \tilde{x}_3 \partial^\mu \tilde{x}_3 \right) + \partial_\mu t \partial^\mu t] 
$$

(42)
Concluding Remarks

We have obtained dual algebras of all Bianchi type algebras. Then by introducing a non-degenerate adjoint invariant inner product over (six dimensional) Lie algebra of Drinfeld double, we have obtained many real bialgebras listed in Table 2. As it has been shown above, we can associate a pair of Poisson-Lie dual $\sigma$-models to every pair of bialgebras of Table 2. This point is explained by two examples; the rest will appear in future works. Besides introducing associated $\sigma$-models with Poisson-Lie dualities, there are other important problems to be studied, and we mention some of them in the following.

Actually determination of the Drinfeld doubles of Table 2 can be very important for the following reasons. Due to the existence of non-degenerate adjoint invariant metric on these groups, we can construct some important physical models such as WZNW models or gauge theories over them (mostly non-compact). Also one can determine the modular space of each of these Drinfeld doubles. Below we give the known doubles. Two of these doubles have already been known, namely the double $(IX,V)$ is $SO(3,1)$ and $(I,I)$ is $(U(1))^6$. Our investigations show that the bialgebras of Table 2 are the following types. The Drinfeld doubles of $(III,I), (III,II), (VII_0,I), (VII_0,I)$ and $(V,I)$ are real forms of the complexification of $H_4 \otimes U(1) \otimes U(1)$. Also $(III,III)$ is $U(2) \otimes U(1) \otimes U(1)$. On the other hand $(V,II)$ and $(VII_a,I)$ are real forms of the complexification of six dimensional Heisenberg algebra. Also the doubles $(VII_0,V), (VII_0,V), (IX,I)$ and $(VIII,I)$ correspond to the real forms of $ISO(3,c)$. The doubles $(II,II)$ and $(VIII,V)$ are real forms of $SO(4,c)$. Finally doubles $(IV,I), (IV,II), (VII_a,II)$ and $(VII_a,II)$ are solvable. The exact and thorough characterization of these doubles and those mentioned in Table 2 are under further investigation.

On the other hand, by using the information of Table 2 together with the prescription of references [20] and [21] one can study the Poisson-Lie T-duality in $N=2$ superconformal WZNW models related to the bialgebras of Table 2, for both classical and quantum cases, similar to [20] and [21]. Furthermore for some bialgebras of Table 2 one can extend these studies to $N=4$ superconformal WZNW models.

Also, because of canonical transformation nature of Poisson-Lie T-duality, it would be interesting to find the generating function of these canonical transformations for each bialgebra of Table 2 [7]; similar to the work done in [11] for $(IX,V)$. Notice that here the algebra $\mathcal{D}$ can have more than one decomposition such as $\mathcal{D} = (g \oplus \tilde{g})$ and $\mathcal{D} = (k \oplus \tilde{k})$. Hence it is possible to find canonical transformation which relates the $\sigma$-model with group $G$ to the one with group $K$ [3]. On the other hand, one can research the commutativity of the renormalisation flow with Poisson-Lie T-duality, by explicit computing of the beta functions and Weyl anomaly coefficients at the 1-loop level for these $\sigma$-models and their duals. Of course this work has already been done in [10] for $(IX,V)$. Finally, there is a possibility that some of these models have application in string cosmology [19]; then the Poisson-Lie T-duality will play a key role in interrelating different cosmological models and their solutions.

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Appendix A

Here in this appendix we give the required similarity transformations for obtaining Jordan forms of the matrices $T_2$ and $T_3$. The similarity transformation related to the matrix $T_2$ and its Jordan form are:

\[
S_2 = \begin{pmatrix}
0 & 1 & -\xi n_1 & 0 & n_3 & a \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & -i\xi n_3 & 0 & \sqrt{n_1 n_3} & 0 & -i n_1 \\
\frac{1}{\sqrt{n_1 n_3}} & ia & -in_3 & 0 & 0 & 0 \\
0 & i\xi n_3 & 0 & \sqrt{n_1 n_3} & 0 & in_1 \\
-in_2 n_1 & -a\tilde{n}_2 \sqrt{n_1 n_3} & \tilde{n}_2 \sqrt{n_1 n_3} & 0 & 0 & 0
\end{pmatrix},
D_2 = \begin{pmatrix}
0 & 0 & 0 \\
0 & A_1 & 0 \\
0 & 0 & A_2
\end{pmatrix},
\]

where:

\[
A_1 = \begin{pmatrix}
i\sqrt{n_1 n_3} & i\tilde{n}_2 \sqrt{n_1 n_3} \\
0 & i\sqrt{n_1 n_3}
\end{pmatrix},
A_2 = \begin{pmatrix}
-i\sqrt{n_1 n_3} & 1 \\
0 & -i\sqrt{n_1 n_3}
\end{pmatrix}.
\]

Similarly, the similarity transformation related to the matrix $T_3$ and its Jordan form are:

\[
S_3 = \begin{pmatrix}
0 & -\xi n_1 & 0 & 0 & a & -n_2 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & -i\xi n_2 & \sqrt{n_1 n_2} & in_1 & 0 \\
\frac{1}{\sqrt{n_1 n_2}} & in_2 & ia & 0 & 0 & 0 \\
0 & 0 & i\xi n_2 & \sqrt{n_1 n_2} & -in_1 & 0 \\
-in_3 n_1 & -\tilde{n}_3 \sqrt{n_1 n_2} & \tilde{n}_3 \sqrt{n_1 n_2} & 0 & 0 & 0
\end{pmatrix},
D_3 = \begin{pmatrix}
0 & 0 & 0 \\
0 & A_3 & 0 \\
0 & 0 & A_4
\end{pmatrix},
\]

where $A_3$ and $A_4$ are similar to $A_1$ and $A_2$ except that we must replace $(n_3, \tilde{n}_2)$ by $(n_2, \tilde{n}_3)$, respectively.

Appendix B

In this appendix we give coupling matrices of bialgebras of the examples quoted at the end of section 4. For bialgebra $(II, II)$ we have:

\[
e^{\chi_1 x_1} = I,\quad B_1 = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & -\chi_1 \\
0 & \chi_1 & 0
\end{pmatrix},\quad B_2 = B_3 = 0,
\]

\[
e^{\chi_2 x_2} = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
-\chi_2 & 0 & 1
\end{pmatrix},\quad e^{\chi_3 x_3} = \begin{pmatrix}
1 & 0 & 0 \\
\chi_3 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

Similarly, for bialgebra $(II, V)$ we have:

\[
e^{\chi_1 x_1} = I,\quad B_1 = 0,\quad B_2 = \begin{pmatrix}
0 & \chi_2 & 0 \\
-\chi_2 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},\quad B_3 = \begin{pmatrix}
0 & 0 & \chi_3 \\
0 & 0 & 0 \\
-\chi_3 & 0 & 0
\end{pmatrix},
\]
where the matrices $e^{\chi_2 X_2}$ and $e^{\chi_3 X_3}$ have the same form as the matrices for bialgebra ($II, II$), as mentioned above. Finally the matrices related to the dual model of bialgebra ($II, V$) are:

$$e^{\tilde{\chi}_1 \tilde{X}_1} = e^{\tilde{\chi}_1 I}, \quad \tilde{B}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\sinh \tilde{\chi}_1 \\ 0 & \sinh \tilde{\chi}_1 & 0 \end{pmatrix}, \quad \tilde{B}_2 = \tilde{B}_3 = 0,$$

$$e^{\tilde{\chi}_2 \tilde{X}_2} = \begin{pmatrix} 1 & -\tilde{\chi}_2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad e^{\tilde{\chi}_3 \tilde{X}_3} = \begin{pmatrix} 1 & 0 & -\tilde{\chi}_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$}

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