Approximation on hexagonal domains by
Taylor-Abel-Poisson means

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Abstract. Approximative properties of the Taylor-Abel-Poisson linear summation method of Fourier series are considered for functions of several variables, periodic with respect to the hexagonal domain, in the integral metric. In particular, direct and inverse theorems are proved in terms of approximations of functions by the Taylor-Abel-Poisson means and K-functional generated by radial derivatives. Bernstein type inequalities for $L_1$-norm of high-order radial derivatives of the Poisson kernel are also obtained.

Keywords: direct approximation theorem, inverse approximation theorem, K-functional, Taylor-Abel-Poisson means, Poisson kernel, hexagon.

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1 Introduction

In this paper, the approximative properties of the Taylor-Abel-Poisson linear summation method of the Fourier series are considered for functions of several variables, periodic with respect to the hexagonal domain. This type of periodicity is defined by a lattice which is a discrete subgroup defined by $AZ^d$, where $A$ is a nonsingular matrix and the periodic function satisfies the relation $f(x + Ak) = f(x)$ for all $k \in Z^d$ and $x \in R^d$. With such a periodicity, one works with exponentials or trigonometric functions of the form $e^{2\pi i \alpha \cdot x}$, where $\alpha$ and $x$ are in certain sets of $R^d$, not necessarily the usual trigonometric polynomials. Lattices appeared prominently in various problems of analysis and combinatorics (see, for example, [7], [10]).

In the Euclidean plane $R^2$, besides the standard lattice $Z^2$ and the rectangular domain $[-\frac{1}{2}, \frac{1}{2})^2$, one of the lattices is the hexagon lattice and the corresponding spectral set is the regular hexagon. The study of problems connected with Fourier analysis for functions given on the hexagonal domains goes back to the papers [1], [24]. Orthogonal polynomials with respect to the area measure on the regular hexagon were studied in [9], where an algorithm was developed for generating an orthogonal polynomial basis. Let us note the papers [16], [25] and [26] in which important problems of trigonometric approximation and Fourier analysis on a hexagon were considered. In [16], in particular, the discrete Fourier analysis on the regular hexagon is developed in detail. [25] deals with the problems of the Abel and Cesàro summation of Fourier series, the degree of approximation, and best approximation.
by trigonometric polynomials were considered. In [25], especially, the properties of the Poisson and Cesàro kernels were studied and direct and inverse theorems were proved in the terms of best approximations of functions by trigonometric polynomials and their moduli of smoothness.

Let us also note the papers of Guven (see, for example, [11], [12] and the references cited therein), in which the author investigated the problems of approximation of functions defined on the hexagonal domains by various linear summation methods of Fourier series.

In this paper, the approximative properties of the Taylor-Abel-Poisson linear summation method are considered in the hexagonal domain. This method defines the operators that possess the main properties of the Abel–Poisson and Taylor operators. However, they can also be adapted to smoothness properties of functions of arbitrarily large order. The Taylor-Abel-Poisson operators $A_{g,r}$ were first studied in [21] where, in terms of these operators, the author gave the constructive characteristic of Hardy-Lipschitz classes of functions of one variable, holomorphic on the unit disc of the complex plane. In [22], in terms of approximation estimates by such operators in the spaces $S^p$ of Sobolev type, the authors give a constructive description of classes of functions of several variables whose generalised derivatives belong to the classes $S^pH_w$. Approximations of functions of one variable by similar operators of polynomial type were studied in [15], [4], [6], [14], [18], [5] etc.

In the integral metrics, for $2\pi$-periodic functions and for functions of several variables $2\pi$-periodic in each variable, direct and inverse theorems of approximation by the operators $A_{g,r}$ were given in the terms of $K$-functionals of functions generated by their radial derivatives in [19] and [20] respectively.

In this paper, we prove direct and inverse theorems of approximation by the Taylor-Abel-Poisson operators for functions periodic with respect to hexagonal domains. We also obtain Bernstein-type inequalities for $L_1$-norm of high-order radial derivatives of the Poisson kernel.

2 Preliminaries

2.1 Fourier series on the regular hexagon.

Let us give basic definitions of the hexagon lattice and the hexagonal Fourier series. More detailed information can be found in [16], [25], [24]. The hexagonal lattice is given by $\mathcal{H}\mathbb{Z}^2$, where the matrix $\mathcal{H}$ and the spectral set $\Omega_\mathcal{H}$ are given by

$$
\mathcal{H} = \begin{pmatrix} \sqrt{3} & 0 \\ -1 & 2 \end{pmatrix}, \quad \Omega_\mathcal{H} = \left\{ (x_1, x_2) : -1 \leq x_2, \frac{\sqrt{3}}{2} x_1 \pm \frac{1}{2} x_2 \leq 1 \right\},
$$

respectively. The reason why $\Omega_\mathcal{H}$ contains only half of its boundary is described in [24]. We use the homogeneous coordinates $(t_1, t_2, t_3)$ which satisfy the equality $t_1 + t_2 + t_3 = 0$. If we set

$$
t_1 := -\frac{x_2}{2} + \frac{\sqrt{3}x_1}{2}, \quad t_2 := x_2, \quad t_3 := -\frac{x_2}{2} - \frac{\sqrt{3}x_1}{2}, \quad (2.1)
$$

then the hexagonal domain $\Omega_\mathcal{H}$ becomes

$$
\Omega = \left\{ (t_1, t_2, t_3) : \ t_1 + t_2 + t_3 = 0, \ -1 \leq t_1, t_2, t_3 \leq 1 \right\}
$$
which is the intersection of the plane \( t_1 + t_2 + t_3 = 0 \) with the cube \([-1,1]^3\) in \( \mathbb{R}^3 \).

Further, for convenience we denote by \( \mathbb{R}^3_H \) and \( \mathbb{Z}^3_H \) the sets of all triples \( t = (t_1, t_2, t_3) \) from the corresponding sets \( \mathbb{R}^3 \) and \( \mathbb{Z}^3 \) such that \( t_1 + t_2 + t_3 = 0 \).

If we treat \( x \in \mathbb{R}^2 \) and \( t \in \mathbb{R}^3_H \) as column vectors, then it follows from (2.1) that
\[
x = \frac{1}{3} \mathcal{H}(t_1 - t_3, t_2 - t_3)^{\text{tr}} = \frac{1}{3} \mathcal{H}(2t_1 + t_2, t_1 + 2t_2)^{\text{tr}}
\] (2.2)
upon using the fact that \( t_1 + t_2 + t_3 = 0 \). Computing the Jacobian of the change of variables shows that \( dx = \frac{2\sqrt{3}}{3} dt_1 dt_2 \).

A function \( f \) is called periodic with respect to the hexagonal lattice \( \mathcal{H} \) (or \( \mathcal{H} \)-periodic) if
\[
f(x) = f(x + \mathcal{H}k), \quad k \in \mathbb{Z}^2, \quad x \in \mathbb{R}^2.
\]
In homogeneous coordinates, \( x \equiv y \) (mod \( \mathcal{H} \)) becomes, as easily seen using (2.2), \( t \equiv s \) (mod 3) defined as \( t_1 - s_1 \equiv t_2 - s_2 \equiv t_3 - s_3 \) (mod 3). Thus, a function \( f(t) \) is \( \mathcal{H} \)-periodic, i.e., \( f(t) = f(t + j) \) whenever \( j \equiv 0 \) (mod 3).

Let \( L_p = L_p(\Omega) \), \( 1 \leq p < \infty \), be the space of all measurable functions \( f \), given on the hexagonal domain \( \Omega \), with finite norm
\[
\|f\|_p := \left( \frac{1}{|\Omega_H|} \int_{\Omega_H} |f(x_1, x_2)|^p dx_1 dx_2 \right)^{1/p}
= \left( \frac{1}{|\Omega|} \int_{\Omega} |f(t)|^p dt \right)^{1/p},
\]
where \( |\Omega_H| \) and \( |\Omega| \) denote the areas of \( \Omega_H \) and \( \Omega \) respectively. As usual, by \( L_\infty = L_\infty(\Omega) \) we denote the space of all measurable functions bounded almost everywhere on \( \Omega \) with norm
\[
\|f\|_\infty := \text{ess sup}_{(x_1, x_2) \in \Omega_H} |f(x_1, x_2)|
= \text{ess sup}_{t \in \Omega} |f(t)|.
\]

The inner product on the hexagonal domain is defined by
\[
\langle f, g \rangle_{\mathcal{H}} = \frac{1}{|\Omega_H|} \int_{\Omega_H} f(x_1, x_2)\overline{g(x_1, x_2)} dx_1 dx_2
= \frac{1}{|\Omega|} \int_{\Omega} f(t)\overline{g(t)} dt.
\] (2.3)

For any integer triple \( k = (k_1, k_2, k_3) \in \mathbb{Z}^3_H \) and every point \( t = (t_1, t_2, t_3) \in \mathbb{R}^3_H \) of the plane, we consider the following trigonometric monomials:
\[
\phi_k(t) := e^{\frac{2\pi i}{3} k \cdot t}, \quad \text{where} \quad k \cdot t = k_1 t_1 + k_2 t_2 + k_3 t_3.
\]
It is known that these functions are \( \mathcal{H} \)-periodic and orthogonal with respect to the inner product (2.3) and for \( k, j \in \mathbb{Z}^3_H \), \( \langle \phi_k, \phi_j \rangle_{\mathcal{H}} = \delta_{k, j} \). Moreover, the set \( \{ \phi_j : j \in \mathbb{Z}^3_H \} \) is an orthonormal basis of \( L_2(\Omega) \) [24], and for any function \( f \in L_1(\Omega) \), the Fourier series with respect to the system \( \phi \) has the form
\[
S[f](t) = \sum_{k \in \mathbb{Z}^3_H} \hat{f}(k) \phi_k(t), \quad \text{where} \quad \hat{f}(k) := \langle f, \phi_k \rangle.
\] (2.4)
2.2 Poisson integral and Taylor-Abel-Poisson means

Let $f \in L^1(\Omega)$. Set $I_\nu := \{ k \in \mathbb{Z}^3_{\nu} : |k| := \max_j \{|k_j|\} = \nu \}$, $\nu = 0, 1, \ldots$, and for an arbitrary number $\varrho \in [0, 1)$, denote by $P(f) (\varrho, t)$ the Poisson integral of $f$, i.e.,

$$P(f) (\varrho, t) := \frac{1}{|\Omega|} \int_{\Omega} f(t + s) P(\varrho, s) ds, \quad (2.5)$$

where $P(\varrho, t)$ is the Poisson kernel, i.e.,

$$P(\varrho, s) := \sum_{\nu=0}^{\infty} \varrho^\nu \sum_{k \in I_\nu} \phi_k(s). \quad (2.6)$$

For any $\varrho \in [0, 1)$ and $r \in \mathbb{N}$, consider the transformation

$$A_{\varrho,r}(f)(t) := \sum_{\nu=0}^{\infty} \lambda_{\nu,r}(\varrho) \sum_{k \in I_\nu} \hat{f}(k) \phi_k(t), \quad (2.7)$$

where for $\nu = 0, 1, \ldots, r-1$, the coefficients are defined by $\lambda_{\nu,r}(\varrho) \equiv 1$ and for $\nu = r, r+1, \ldots$,

$$\lambda_{\nu,r}(\varrho) := \sum_{j=0}^{r-1} \binom{\nu}{j} (1 - \varrho)^j \varrho^{\nu-j}$$

$$= \sum_{j=0}^{r-1} \frac{(1 - \varrho)^j}{j!} \frac{\partial^j}{\partial \varrho^j} \varrho^\nu, \quad \varrho \in [0, 1). \quad (2.8)$$

The transformation $A_{\varrho,r}$ can be considered as a linear operator on $L^1(\Omega)$ into itself. Indeed, $\lambda_{k,r}(0)=0$ and for all $k = r, r + 1, \ldots$ and $\varrho \in (0, 1)$, we have

$$\sum_{j=0}^{r-1} \binom{\nu}{j} (1 - \varrho)^j \varrho^{\nu-j} \leq rq^{\nu}r^{-1}, \quad \text{where } 0 < q := \max\{1 - \varrho, \varrho\} < 1.$$  

Therefore, for any function $f \in L^1(\Omega)$ and for any $0 < \varrho < 1$, the series on the right-hand side of (2.7) is majorized by the convergent series $2r\|f\|_1 \sum_{\nu=r}^{\infty} q^{\nu}r^{-1}$.

**Lemma 2.1.** Assume that $f \in L^1(\Omega)$. Then, for any numbers $r \in \mathbb{N}$, $\varrho \in [0, 1)$ and $t \in \Omega$,

$$A_{\varrho,r}(f)(t) = \sum_{k=0}^{r-1} \frac{(1 - \varrho)^k}{k!} \frac{\partial^k}{\partial \varrho^k} P(f) (\varrho, t). \quad (2.9)$$

**Proof.** By virtue of (2.5) and (2.6), we have the following decomposition of the Poisson integral into a uniformly convergent series:

$$P(f) (\varrho, t) = \sum_{\nu=0}^{\infty} \varrho^\nu \sum_{k \in I_\nu} \hat{f}(k) \phi_k(t) \quad \text{for all } \varrho \in [0, 1), \ t \in \Omega. \quad (2.10)$$
Differentiating (2.10) with respect to \( \varrho \), we realize that for any \( k = 0, 1, \ldots \)

\[
\sum_{k=0}^{r-1} \frac{(1 - \varrho)^k}{k!} \frac{\partial^k}{\partial \varrho^k} P(f)(\varrho, t) = \sum_{k=0}^{r-1} \sum_{\nu=k}^{\infty} \binom{\nu}{k} (1 - \varrho)^k \varrho^{-k} \sum_{k \in J_\nu} \hat{f}(k) \phi_k(t).
\]  

(2.11)

By changing the summation order on the right-hand side of (2.11) and using the identity

\[
\sum_{k=0}^{\nu} \binom{\nu}{k} (1 - \varrho)^k \varrho^{-k} = 1, \ \nu = 0, 1, \ldots,
\]  

(2.12)

we obtain

\[
\sum_{k=0}^{r-1} \frac{(1 - \varrho)^k}{k!} \frac{\partial^k}{\partial \varrho^k} P(f)(\varrho, t) = \sum_{\nu=0}^{r-1} \sum_{k=0}^{\nu} \binom{\nu}{k} (1 - \varrho)^k \varrho^{-k} \sum_{k \in J_\nu} \hat{f}(k) \phi_k(t)
\]

\[
+ \sum_{\nu=r}^{\infty} \sum_{j=0}^{r-1} \binom{\nu}{j} (1 - \varrho)^j \varrho^{-j} \sum_{k \in J_\nu} \hat{f}(k) \phi_k(t)
\]

\[
= A_{\varrho, r}(f)(t).
\]

2.3 Radial derivatives and \( K \)-functionals

If for a function \( f \in L_1(\Omega) \) and for a positive integer \( n \), there exists a function \( g \in L_1(\Omega) \) such that

\[
\hat{g}_k = \begin{cases} 
0, & \text{if } |k| < n, \\
\frac{|k|!}{(|k| - n)!} \hat{f}_k, & \text{if } |k| \geq n, 
\end{cases} \quad k \in \mathbb{Z}_n^3,
\]

then we say that for the function \( f \), there exists the radial derivative \( g \) of order \( n \), for which we use the notation \( f^{[n]} \).

Let us note that if the function \( f^{[n]} \in L_1(\Omega) \), then its Poisson integral can be represented as

\[
P(f^{[n]})(\varrho, t) = P(f)[n](\varrho, t) = \varrho^n \frac{\partial^n}{\partial \varrho^n} P(f)(\varrho, t), \quad \varrho \in [0, 1), \ t \in \Omega.
\]  

(2.13)

In the space \( L_p(\Omega) \), the \( K \)-functional of a function \( f \) (see, for example, [8, Chap. 6]) generated by the radial derivative of order \( n \), is the following quantity:

\[
K_n(\delta, f)_p := \inf \left\{ \|f - h\|_p + \delta^n \left\| h^{[n]} \right\|_p : h^{[n]} \in L_p(\Omega) \right\}, \quad \delta > 0.
\]

3 Main results

3.1 Bernstein-type inequality

In the following assertion, we give Bernstein-type inequalities for the \( L_1 \)-norm of radial derivatives of arbitrary order of the Poisson kernel.
Theorem 3.1. For any $r = 0, 1, \ldots$, and for any $0 \leq \varrho < 1$, the following inequality holds:

$$I(\varrho) := \int_{\Omega} \left| \frac{\partial^r}{\partial \varrho^r} P(\varrho, t) \right| \frac{dt}{|\Omega|} \leq \frac{C_r}{(1 - \varrho)^r},$$  \hspace{1cm} (3.1)$$

where the constant $C_r \geq 1$ depends only on $r$.

Before proving Theorem 3.1, let us give a few auxiliary results.

Let $P_\varrho(t)$ be the usual Poisson kernel, i.e.,

$$P_\varrho(t) = \frac{1 - \varrho^2}{|1 - \varrho e^{it}|^2},$$

where $\varrho \in [0, 1)$, $t \in \mathbb{R}$ and $q_\varrho(t) = 1 - 2\varrho \cos t + \varrho^2$.

Lemma 3.1. For any $r = 0, 1, 2, \ldots$, and for any $0 \leq \varrho < 1$, the following relation holds:

$$\left| \frac{\partial^r}{\partial \varrho^r} P_\varrho(t) \right| \leq \frac{2r!}{(1 - \varrho)^{r+1}}.$$  \hspace{1cm} (3.2)$$

Proof. Since

$$P_\varrho(t) = \frac{1 - \varrho^2}{|1 - \varrho e^{it}|^2} = \frac{1}{1 - \varrho e^{it}} + \frac{1}{1 - \varrho e^{-it}} - 1,$$  \hspace{1cm} (3.3)$$

we have for any $r = 0, 1, 2, \ldots$

$$\frac{\partial^r}{\partial \varrho^r} P_\varrho(t) = \frac{r! e^{i rt}}{(1 - \varrho e^{it})^{r+1}} + \frac{r! e^{-i rt}}{(1 - \varrho e^{-it})^{r+1}},$$

and the inequality (3.2) follows. \hfill \Box

Lemma 3.2. Assume that $z_1 = \frac{2\varrho}{3}(t_2 - t_3)$, $z_2 = \frac{2\varrho}{3}(t_3 - t_1)$, $z_3 = \frac{2\varrho}{3}(t_1 - t_2)$, $r = 0, 1, \ldots$, and $0 \leq \varrho < 1$. The following assertions holds:

i) for any $j = 1, 2, 3$,

$$I_{1,r}(\varrho) := \int_{\Omega} \left| \frac{\partial^r}{\partial \varrho^r} P_\varrho(z_j) \right| \frac{dt}{|\Omega|} \leq \frac{2r!}{(1 - \varrho)^r}.$$  \hspace{1cm} (3.4)$$

ii) for any numbers $j, k = 1, 2, 3$ and $r_1, r_2 = 0, 1, \ldots$ such that $j \neq k$ and $r_1 + r_2 = r$,

$$I_{2,r}(\varrho) := \int_{\Omega} \left| \frac{\partial^{r_1}}{\partial \varrho^{r_1}} P_\varrho(z_j) \frac{\partial^{r_2}}{\partial \varrho^{r_2}} P_\varrho(z_k) \right| \frac{dt}{|\Omega|} \leq \frac{4r_1! r_2!}{(1 - \varrho)^r}. $$  \hspace{1cm} (3.5)$$

iii) for any $r_j = 0, 1, \ldots$, $j = 1, 2, 3$, such that $r_1 + r_2 + r_3 = r$,

$$I_{3,r}(\varrho) := \int_{\Omega} \left| \frac{\partial^{r_1}}{\partial \varrho^{r_1}} P_\varrho(z_1) \frac{\partial^{r_2}}{\partial \varrho^{r_2}} P_\varrho(z_2) \frac{\partial^{r_3}}{\partial \varrho^{r_3}} P_\varrho(z_3) \right| \frac{dt}{|\Omega|} \leq \frac{8r_1! r_2! r_3!}{(1 - \varrho)^{r+1}}. $$  \hspace{1cm} (3.6)$$
Proof. First, consider the case \( r = 0 \). In this case, the integrals in the inequalities (3.4)–(3.6) can be found exactly. Let us determine the integral \( I_{1,0}(\varrho) \). Consider, for example, the case \( j = 1 \) (the proof in the other cases is similar). Then, \( z_j = z_1 = \frac{2\pi}{3}(t_2 - t_3) \) and by virtue of (3.3), we have

\[
I_{1,0}(\varrho) = \int_\Omega P_\varrho(z_1) \frac{dt}{|\Omega|} = (1 - \varrho^2) \int_\Omega \frac{1}{|1 - \varrho e^{iz_1}|^2} \frac{dt}{|\Omega|} = (1 - \varrho^2) \int_\Omega \left| \sum_{l=0}^{\infty} \varrho^l e^{iz_1} \right|^2 \frac{dt}{|\Omega|} = (1 - \varrho^2) \int_\Omega \left| \sum_{l=0}^{\infty} \varrho^l e^{2\pi i (t_2 - t_3)} \right|^2 \frac{dt}{|\Omega|}.
\]

(3.7)

In (3.7), the last sum does not contain two terms with identical harmonics. Moreover, each harmonic in it is equal to \( \phi_{k_l}(t) \), where \( k_l = (0, l, -l) \in \mathbb{Z}_H^3 \). Therefore, using the orthonormality of \( \{\phi_k\} \), we get

\[
I_{1,0}(\varrho) = (1 - \varrho^2) \sum_{l=0}^{\infty} \varrho^{2l} = (1 - \varrho^2) \frac{1}{1 - \varrho^2} = 1.
\]

Let us find the integral \( I_{2,0}(\varrho) \). Similarly, consider, for example the case \( j = 1 \) and \( k = 2 \). Then, \( z_j = z_1 = \frac{2\pi}{3}(t_2 - t_3), \) \( z_k = z_2 = \frac{2\pi}{3}(t_3 - t_1) \), and

\[
\frac{I_{2,0}(\varrho)}{(1 - \varrho^2)^2} = \frac{1}{(1 - \varrho^2)^2} \int_\Omega P_\varrho(z_1)P_\varrho(z_2) \frac{dt}{|\Omega|} = \int_\Omega \frac{1}{|(1 - \varrho e^{iz_1})(1 - \varrho e^{iz_2})|^2} \frac{dt}{|\Omega|} = \int_\Omega \left| \sum_{l,n=0}^{\infty} \varrho^{l+n} e^{i(lz_1 + nz_2)} \right|^2 \frac{dt}{|\Omega|} = \int_\Omega \left| \sum_{l,n=0}^{\infty} \varrho^{l+n} e^{\frac{2\pi i}{3}(-nt_1 + t_2 + (n-l)t_3)} \right|^2 \frac{dt}{|\Omega|}.
\]

(3.8)

It is easy to show that the last sum in (3.8) does not contain two terms with identical harmonics. Moreover, each harmonic in it is equal to \( \phi_{k_{l,n}}(t) \), where \( k_{l,n} = (-n, l, n-l) \in \mathbb{Z}_H^3 \).
Therefore, using the orthonormality of \( \{ \phi_k \} \), we get

\[
I_{2,0}(\varrho) = (1 - \varrho^2)^2 \sum_{l,n=0}^{\infty} \varrho^{2(l+n)}
\]

\[
= (1 - \varrho^2)^2 \frac{1}{(1 - \varrho^2)^2}
\]

\[
= 1. \quad (3.9)
\]

In [25, Proposition 3.1], it was shown that for all \( t \in \Omega \), the Poisson kernel \( P(\varrho, t) \) satisfies the following relations:

\[
P(\varrho, t) = \frac{(1 - \varrho)^3(1 - \varrho^3)}{q_\varrho \left( \frac{2\pi}{3}(t_1 - t_2) \right) q_\varrho \left( \frac{2\pi}{3}(t_2 - t_3) \right) q_\varrho \left( \frac{2\pi}{3}(t_3 - t_1) \right)}
\]

\[
+ \frac{\varrho(1 - \varrho)^2}{q_\varrho \left( \frac{2\pi}{3}(t_1 - t_2) \right) q_\varrho \left( \frac{2\pi}{3}(t_2 - t_3) \right)}
\]

\[
+ \frac{\varrho(1 - \varrho)^2}{q_\varrho \left( \frac{2\pi}{3}(t_1 - t_2) \right) q_\varrho \left( \frac{2\pi}{3}(t_3 - t_1) \right)}
\]

\[
+ \frac{\varrho(1 - \varrho)^2}{q_\varrho \left( \frac{2\pi}{3}(t_2 - t_3) \right) q_\varrho \left( \frac{2\pi}{3}(t_3 - t_1) \right)}.
\]

(3.10)

and

\[
\int_{\Omega} P(\varrho, t) \frac{dt}{|\Omega|} = 1. \quad (3.11)
\]

Taking into account the notations above, relation (3.10) can be represented as

\[
P(\varrho, t) = \frac{1 - \varrho^3}{(1 + \varrho)^3} P(\varrho(z_1)) P(\varrho(z_2)) P(\varrho(z_3))
\]

\[
+ \frac{\varrho}{(1 + \varrho)^2} (P(\varrho(z_1)) P(\varrho(z_2)) + P(\varrho(z_1)) P(\varrho(z_3)) + P(\varrho(z_2)) P(\varrho(z_3))).
\]

(3.12)

Combining (3.12), (3.11) and (3.9), we get

\[
I_{3,0}(\varrho) = \int_{\Omega} P(\varrho(z_1)) P(\varrho(z_2)) P(\varrho(z_3)) \frac{dt}{|\Omega|}
\]

\[
= \left( 1 - \frac{3\varrho}{(1 + \varrho)^2} \right) \frac{(1 + \varrho)^3}{1 - \varrho^3}
\]

\[
= \frac{1 + \varrho^3}{1 - \varrho^3}. \quad (3.13)
\]

Now let \( r = 1, 2, \ldots \). By virtue of (3.2) and the Parseval’s identity, for any \( j = 1, 2, 3, \)
we have

\[ I_{1,r}(\varrho) \leq \frac{2r!}{(1 - \varrho)^{r-1}} \int_{\Omega} \frac{1}{|1 - \varrho e^{iz_j}|^2} \, dt \leq \frac{2r!}{(1 - \varrho)^{r-1}} \frac{1}{1 - \varrho^2} \leq \frac{2r!}{(1 - \varrho)^r}. \]  

(3.14)

If one of the numbers \( r_i, i = 1, 2, \) is zero, for example, \( r_1 = 0 \), then \( r_2 = r \) and taking into account (3.3) and (3.2), we have

\[ I_{2,r}(\varrho) = \frac{2r!}{(1 - \varrho)^{r-1}} \int_{\Omega} \frac{1}{|1 - \varrho e^{iz_j}(1 - \varrho e^{iz_k})|^2} \, dt \leq \frac{2r!}{(1 - \varrho)^{r-1}} \frac{1}{(1 - \varrho^2)^2} \leq \frac{2r!}{(1 - \varrho)^r}. \]

(3.15)

If \( r_1, r_2 \neq 0 \), then by virtue of (3.2), we similarly get the estimate (3.5):

\[ I_{2,r}(\varrho) \leq \frac{4r_1!r_2!}{(1 - \varrho)^{r_1+r_2-2}} \int_{\Omega} \frac{1}{|1 - \varrho e^{iz_j}(1 - \varrho e^{iz_k})|^2} \, dt \leq \frac{4r_1!r_2!}{(1 - \varrho)^r}. \]

In the case when one or two of the \( r_j \) are equal to zero, the estimates of the integral \( I_{3,r}(\varrho) \) are obtained similarly to (3.15). If all \( r_j \neq 0 \), then the inequality (3.6) is easily obtained by applying estimates (3.2) and (3.5):

\[ I_{3,r}(\varrho) \leq \frac{2r_1!}{(1 - \varrho)^{r_1+1}} \int_{\Omega} \left| \frac{\partial^{r_2} P_e(z_2)}{\partial \varrho^{r_2}} \frac{\partial^{r_3} P_e(z_3)}{\partial \varrho^{r_3}} \right| \, dt \leq \frac{8r_1!r_2!r_3!}{(1 - \varrho)^{r_1+1}(1 - \varrho)^{r_2+r_3}} = \frac{8r_1!r_2!r_3!}{(1 - \varrho)^{r+1}}. \]

Proof of Theorem 3.1. According to (3.11), we see that the inequality (3.1) is satisfied for
Let us verify that it is also satisfied in the case \( r = 1 \). By virtue of (3.12), we have
\[
\frac{\partial}{\partial \varrho} P(\varrho, t) = \frac{-3(1 + \varrho^2)}{(1 + \varrho)^4} \varrho P(\varrho(z_1), \varrho(z_2), \varrho(z_3)) + \frac{1 - \varrho^3}{(1 + \varrho)^3} \left( \frac{\partial P(\varrho(z_1))}{\partial \varrho} + \frac{\partial P(\varrho(z_2))}{\partial \varrho} + \frac{\partial P(\varrho(z_3))}{\partial \varrho} \right)
\]
\[
+ \frac{3(1 - \varrho)}{(1 + \varrho)^2} \left( \frac{\partial P(\varrho(z_1))}{\partial \varrho} + \frac{\partial P(\varrho(z_2))}{\partial \varrho} + \frac{\partial P(\varrho(z_3))}{\partial \varrho} \right)
\]
\[
+ \frac{1 - \varrho}{(1 + \varrho)^3} \left( \varrho P(\varrho(z_1)) + \varrho P(\varrho(z_2)) + \varrho P(\varrho(z_3)) \right)
\]
\[
+ \frac{\varrho}{(1 + \varrho)^4} \left( \frac{\partial P(\varrho(z_1))}{\partial \varrho} + \frac{\partial P(\varrho(z_2))}{\partial \varrho} + \frac{\partial P(\varrho(z_3))}{\partial \varrho} \right)
\]
\[
+ \frac{\varrho}{(1 + \varrho)^4} \left( \frac{\partial P(\varrho(z_1))}{\partial \varrho} + \frac{\partial P(\varrho(z_2))}{\partial \varrho} + \frac{\partial P(\varrho(z_3))}{\partial \varrho} \right)
\]

Then, using the estimates (3.13), (3.6), (3.9), (3.15), we get
\[
\int_\Omega \left| \frac{\partial}{\partial \varrho} P(\varrho, t) \right| \frac{dt}{|\Omega|} \leq \frac{3(1 + \varrho^2)}{(1 + \varrho)^4} \frac{1 + \varrho^3}{1 - \varrho^3} + \frac{1 - \varrho^3}{(1 + \varrho)^3} \left( \frac{24}{(1 + \varrho)^4} \right)
\]
\[
+ \frac{3(1 - \varrho)}{(1 + \varrho)^2} + \frac{\varrho}{(1 + \varrho)^2} \frac{12}{1 - \varrho}
\]
\[
\leq \frac{C_1}{1 - \varrho}.
\]

Therefore, in the case \( r = 1 \), the inequality (3.1) is indeed satisfied. For \( r > 1 \), the validity of (3.1) is verified in a similar way using the Leibniz rule of differentiation and Lemma 3.2. □

**Corollary 3.1.** Assume that \( f \in L_p(\Omega) \), \( 1 \leq p \leq \infty \) and \( \varrho \in (0, 1) \). Then, for any \( r = 0, 1, \ldots \),
\[
\left\| \frac{\partial^r}{\partial \varrho^r} P(f(\varrho, \cdot)) \right\|_p \leq C_r \frac{\|f\|_p}{(1 - \varrho)^r},
\]
where the constant \( C_r \) depends only on \( r \).

**Proof.** For any \( f \in L_p(\Omega) \), by virtue of the integral Minkowski inequality, we have
\[
\left\| \frac{\partial^r}{\partial \varrho^r} P(f(\varrho, \cdot)) \right\|_p \leq \|f\|_p \int_\Omega \left| \frac{\partial^r}{\partial \varrho^r} P(\varrho, t) \right| \frac{dt}{|\Omega|}.
\]
Therefore, to prove Corollary 3.1, it is sufficient to apply relation (3.1) to estimate the integral on the right-hand side of the last inequality. □
3.2 Direct and inverse approximation theorems

Let $\mathcal{Z}$ and $\mathcal{Z}_n$, $n \in \mathbb{N}$, denote the sets of all continuous strictly increasing functions $\omega(t)$, $t \in [0, 1]$, with $\omega(0) = 0$ satisfying the following conditions (3.17) and (3.18), respectively:

$$\int_0^\delta \frac{\omega(t)}{t} \, dt = O(\omega(\delta)), \quad \delta \to 0+$$

(3.17)

and

$$\int_\delta^1 \frac{\omega(t)}{t^\alpha} \, dt = O\left(\frac{\omega(\delta)}{\delta^\alpha}\right), \quad \delta \to 0+.$$  

(3.18)

Conditions (3.17) and (3.18) are well-known (see, for example, [2]).

**Theorem 3.2.** Assume that $f \in L_p(\Omega)$, $1 \leq p \leq \infty$, $n, r \in \mathbb{N}$, $n \leq r$ and $\omega \in \mathcal{Z}$. If there exists the derivative $f^{[r-n]} \in L_p(\Omega)$ and

$$K_n \left(\delta, f^{[r-n]}\right)_p = O(\omega(\delta)), \quad \delta \to 0+,$$

then

$$\|f - A_{\varrho, r}(f)\|_p = O \left( (1 - \varrho)^{r-n} \omega(1 - \varrho) \right), \quad \varrho \to 1-. \quad (3.20)$$

**Theorem 3.3.** Assume that $f \in L_p(\Omega)$, $1 \leq p \leq \infty$, $n, r \in \mathbb{N}$, $n \leq r$ and $\omega \in \mathcal{Z} \cap \mathcal{Z}_n$. If the relation (3.20) holds, then $f^{[r-n]} \in L_p(\Omega)$ and (3.19) also holds.

The proof of the Theorems 3.2 and 3.3 will be given in the Subsection 3.6. Here, we give some comments.

**Remark 3.1.** For a given $n \in \mathbb{N}$, from condition (3.18) it follows that

$$\lim_{\delta \to 0+} \inf (\delta^{-n} \omega(\delta)) > 0$$

or, equivalently, that $(1 - \varrho)^{r-n} \omega(1 - \varrho) \leq C(1 - \varrho)^r$ as $\varrho \to 1-$. Therefore, if condition (3.18) is satisfied, then the quantity on the right-hand side of (3.20) decreases to zero as $\varrho \to 1-$ not faster than the function $(1 - \varrho)^r$.

Here and below, $C, C_{r,n}, c_r, c_{r,n}, ...$ are positive numbers that do not depend on $\varrho$.

Consider the case $\omega(t) = t^\alpha$, $\alpha > 0$.

**Corollary 3.2.** Assume that $f \in L_p(\Omega)$, $1 \leq p \leq \infty$, $n, r \in \mathbb{N}$, $0 < \alpha \leq n \leq r$. The following two relations are equivalent:

$$\|f - A_{\varrho, r}(f)\|_p = O \left( (1 - \varrho)^{r-n+\alpha} \right), \quad \varrho \to 1-, \quad (3.21)$$

and

$$f^{[r-n]} \in L_p(\Omega) \quad \text{and} \quad K_n \left(\delta, f^{[r-n]}\right)_p = O(\delta^\alpha), \quad \delta \to 0+.$$  

(3.22)

**Remark 3.2.** If $\alpha = n$, then the relation (3.21) does not depend on $n$ and $\alpha$. Therefore, the sets of functions $f \in L_p(\Omega)$ such that $f^{[r-n]} \in L_p(\Omega)$ and $K_n \left(\delta, f^{[r-n]}\right)_p = O(\delta^\alpha)$, $\delta \to 0+$, coincide for any positive integer $n \leq r$. 

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Recall (see, e.g., [3, Ch. 5], [23, Ch. 2]) that a summation method generated by the operator $A_{\varrho,r}(f)$ is saturated in the space $L_p(\Omega)$ if there exists a positive function $\varphi$ (which is called the saturation order) defined on the interval $[0,1)$, monotonically decreasing to zero as $\varrho \to 1-$, and such that each function $f \in L_p(\Omega)$ satisfying the relation

$$
\|f - A_{\varrho,r}(f)\|_p = o(\varphi(\varrho)), \quad \varrho \to 1-,
$$

(3.23)
is an invariant element of the operator $A_{\varrho,r}(f)$ (i.e., $A_{\varrho,r}(f) = f$) and, furthermore, the set

$$
\Phi(A_{\varrho,r}) = \left\{ f \in L_p(\Omega) : \|f - A_{\varrho,r}(f)\|_p = O(\varphi(\varrho)), \quad \varrho \to 1- \right\}
$$

(3.24)
which is called the saturation class) contains at least one noninvariant element.

Setting $n = \alpha = 1$ in Corollary 3.2, we see that for the function $f \in L_p(\Omega)$, the relation

$$
\|f - A_{\varrho,r}(f)\|_p = O((1 - \varrho)^r), \quad \varrho \to 1-,
$$

holds, if and only if

$$
K_1(\delta, f^{(r-1)}) = O(\delta), \quad \delta \to 0+.
$$

(3.25)
Furthermore, by virtue of (2.7), for $k \in \mathbb{J}_\nu$, $\nu = 0, 1, \ldots$, we have

$$
\frac{1}{|\Omega|} \int_{\Omega} (f(t) - A_{\varrho,r}(f)(t))\hat{\phi}_k(t)dt = (1 - \lambda_{\nu,r}(\varrho))\hat{f}(k),
$$

where the coefficients $\lambda_{\nu,r}(\varrho) = 1$ when $\nu = 0, 1, \ldots, r-1$, and for $\nu \geq r$, they are given by (2.8). This yields that for any $\varrho \in [0,1)$, $r \in \mathbb{N}$, $k \in \mathbb{J}_\nu$ and $\nu \geq r$,

$$
\|f - A_{\varrho,r}(f)\|_p \geq |1 - \lambda_{\nu,r}(\varrho)| \left| \hat{f}(k) \right| = \left( 1 - \sum_{j=0}^{r-1} \binom{\nu}{j} (1 - \varrho)^j \varrho^{r-j} \right) \left| \hat{f}(k) \right|.
$$

By virtue of (2.12), for $k \in \mathbb{J}_\nu$ and $\nu \geq r$, we have

$$
\left| \hat{f}(k) \right| \binom{\nu}{r} = \lim_{\varrho \to 1} \frac{|\hat{f}(k)|}{(1 - \varrho)^r} \sum_{j=0}^{r-1} \binom{\nu}{j} (1 - \varrho)^j \varrho^{r-j} \left| \hat{f}(k) \right|
$$

$$
= \left| \hat{f}(k) \right| \lim_{\varrho \to 1} \frac{|1 - \lambda_{\nu,r}(\varrho)|}{(1 - \varrho)^r}
$$

$$
\geq \lim_{\varrho \to 1} \frac{\|f - A_{\varrho,r}(f)\|_p}{(1 - \varrho)^r}.
$$

Therefore, the relation $\|f - A_{\varrho,r}(f)\|_p = o((1 - \varrho)^r)$, $\varrho \to 1-$, only holds in the case when $f(t) = \sum_{\nu=0}^{r-1} \sum_{k \in \mathbb{J}_\nu} f(k) \hat{\phi}_k(t)$ is a polynomial of order not exceeding $r-1$, that is $A_{\varrho,r}(f) = f$.

Thus, we conclude that the linear summation method generated by the operator $A_{\varrho,r}(f)$ is saturated in the space $L_p(\Omega)$. The saturation order is the function $\varphi(\varrho) = (1 - \varrho)^r$ and the saturation class $\Phi(A_{\varrho,r})$ is the set of all functions $f \in L_p(\Omega)$ satisfying (3.25).

In [21], a similar fact was proved in the Hardy spaces $H_p$ of functions of one variable, holomorphic on the unit disc of the complex plane.
3.3 Example

Let \( k_0 = (k_{0,1}, k_{0,2}, k_{0,3}) \in \mathbb{Z}^3 \) be any triple such that \( |k_0| := \max_j \{|k_{0,j}|\} = r, r = 1, 2, \ldots \). Consider the function \( f_0 = \phi_{k_0}(t) = e^{2\pi i k_0 \cdot t} \). Then

\[
\|f_0 - A_{\varrho,r}(f_0)\|_p = \|(1 - \lambda_{r,r}(\varrho))\phi_{k_0}(t)\|_p = \left\| \left(1 - \sum_{j=0}^{r-1} \binom{r}{j} (1 - \varrho)^j \varrho^{r-j}\right) \phi_{k_0}(t) \right\|_p = (1 - \varrho)^r.
\]

Let us also show that for \( 0 < \delta \leq \frac{1}{r} \)

\[
K_1 \left( \delta, f_0^{[r-1]} \right)_p = r \cdot r! \cdot \delta.
\]  

(3.26)

For this, we first prove the following assertion.

Lemma 3.3. Let \( 1 \leq p \leq \infty, n \in \mathbb{N}, k \in \mathbb{J}_n \) and \( f = a + \phi_k, a \in \mathbb{C} \). Then

\[
K_1(\delta, f)_p = \begin{cases} 
\delta n, & \text{if } 0 \leq \delta \leq \frac{1}{n}, \\
1, & \text{if } \delta \geq \frac{1}{n}.
\end{cases}
\]  

(3.27)

**Proof.** Let \( 0 \leq \delta \leq \frac{1}{n} \). Since \( f^{[1]} = n\phi_k \), we have the upper estimate \( K_1(\delta, f)_p \leq \delta n \|\phi_k\|_p = \delta n \).

To prove the lower estimate for a fixed arbitrary small \( \varepsilon > 0 \), consider a function \( h \) such that \( h^{[1]} \in L_p(\Omega) \) and

\[
\|f - h\|_p + \delta \|h^{[1]}\|_p \leq K_1(\delta, f)_p + \varepsilon.
\]  

(3.28)

From the formulas

\[
\left(1 - \hat{h}(k)\right) \phi_k(t) = \frac{1}{|\Omega|} \int_\Omega (f - h)(s) \phi_k(t - s)ds
\]

and

\[
\hat{h}(k)\phi_k(t) = \frac{1}{n|\Omega|} \int_\Omega h^{[1]}(s) \phi_k(t - s)ds,
\]

we get, respectively,

\[
\left|1 - \hat{h}(k)\right| \leq \|f - h\|_p \quad (3.29)
\]

and

\[
\left|\hat{h}(k)\right| \leq \frac{1}{n} \|h^{[1]}\|_p. \quad (3.30)
\]

In view of (3.28) and the obvious estimate \( K_1(\delta, f)_p \leq 1 \), the last inequality gives us

\[
\left|\hat{h}(k)\right| \leq \|f - h\|_p + \frac{1}{n} \|h^{[1]}\|_p \leq K_1 \left( \frac{1}{n}, f \right)_p + \varepsilon \leq 1 + \varepsilon.
\]
Therefore, for all $\delta \in [0, \frac{1}{n}]$ we get

$$\left| 1 - \hat{h}(k) \right| \geq 1 - \left| \hat{h}(k) \right| \geq \delta n \left( 1 - \left| \hat{h}(k) \right| + \varepsilon \right) - \varepsilon.$$  

Combining this inequality with (3.28)–(3.30), we obtain

$$\delta n \leq \delta n + \left( \left| 1 - \hat{h}(k) \right| - \delta n \left( 1 - \left| \hat{h}(k) \right| + \varepsilon \right) + \varepsilon \right)$$

$$= \left| 1 - \hat{h}(k) \right| + \delta n \left| \hat{h}(k) \right| + \varepsilon (1 - \delta n) \leq K_1(\delta, f)_p + 2\varepsilon,$$

which proves (3.27) for $0 \leq \delta \leq \frac{1}{n}$, since $\varepsilon$ is arbitrary small.

For $\delta \geq \frac{1}{n}$ the result (3.27) follows on observing that the function $\delta \mapsto K_1(\delta, f)$ increases on $\mathbb{R}_+$ while staying bounded by 1. \qed

To prove relation (3.26) it is sufficient to apply Lemma 3.3 for the function $f_0^{[p-1]} = r! \phi_{kn}(t)$.

### 3.4 Discussion

Note that the results of Subsection 3.2 can be applied to characterize classes of functions given by $K$-functionals and moduli of smoothness. In particular, they can be used to study the properties of classical moduli of smoothness, their connection with $K$-functionals generated by radial and usual derivatives. The methods and results of Subsection 3.1, in particular, Theorem 3.1, Lemma 3.2, and Corollary 3.1, are crucial for studying properties of the Poisson kernel in the spaces $L_p(\Omega)$. In addition, interpolation methods for approximation over hexagonal grids are discussed in [16], [25]. The results now available can be used to examine the approximation orders whether they are best possible. The same applies to the papers [26] and [11], in which certain summation methods are used similar to the Taylor-Abel-Poisson means studied here.

### 3.5 Auxiliary statements

In the proof of the Theorems 3.2 and 3.3, we mainly use the scheme from [21], [19] and [20] modifying it in due consideration of the peculiarities of the spaces $L_p(\Omega)$. First, let us give some auxiliary results. The proofs are similar to those of Lemmas 3.6–3.8 in [20].

For any function $f \in L_p(\Omega)$, $1 \leq p \leq \infty$, $0 \leq \varrho < 1$ and $r = 0, 1, 2, \ldots$, we define

$$M_p(\varrho, f, r) := \varrho^r \left\| \frac{\partial^r}{\partial \varrho^r} P(f)(\varrho, \cdot) \right\|_p = \left\| P(f)^r(\varrho, \cdot) \right\|_p.$$  

(3.31)

**Lemma 3.4.** Assume that $f \in L_p(\Omega)$, $1 \leq p \leq \infty$, and $\varrho \in (0, 1)$. Then, for any nonnegative integers $r$ and $n$, the following inequality holds:

$$\left\| A^{[r]}_{\varrho,n}(f) \right\|_p \leq C_{r,n} \frac{\| f \|_p}{(1 - \varrho)^r}$$  

(3.32)
where the constant \(C_{r,n}\) depends only on \(r\) and \(n\).

**Proof.** According to (2.9) and (2.13), for nonnegative integers \(r \in \mathbb{N}\) we have

\[
A_{\rho,n}^{[r]}(f)(t) = \left( \sum_{k=0}^{n-1} \frac{P(f)^{[k]}(\rho, \cdot)}{\rho^k k!} (1 - \rho)^k \right)^{[r]}(t)
\]

\[
= \sum_{k=0}^{n-1} \frac{(P(f)^{[k]}(\rho, \cdot))^{|r|}(t)}{\rho^k k!} (1 - \rho)^k.
\]

Since for any nonnegative integers \(k\) and \(r\)

\[
(P(f)^{|r|}(\rho, \cdot))^{[k]}(t) = (P(f)^{[k]}(\rho, \cdot))^{|r|}(t),
\]

we obtain

\[
A_{\rho,n}^{[r]}(f)(t) = \sum_{k=0}^{n-1} \frac{(P(f)^{[k]}(\rho, \cdot))^{|r|}(t)}{\rho^k k!} (1 - \rho)^k.
\]

This yields

\[
\|A_{\rho,n}^{[r]}(f)\|_p \leq \sum_{k=0}^{n-1} \frac{\|P(f)^{[k]}(\rho, \cdot)^{|r|}\|_p}{\rho^k k!} (1 - \rho)^k \leq \sum_{k=0}^{n-1} \left\| \frac{\partial^k}{\partial \rho^k} P(f)^{|r|}(\rho, \cdot) \right\|_p \frac{(1 - \rho)^k}{k!},
\]

where by virtue of Corollary 3.1 and (3.31)

\[
\left\| \frac{\partial^k}{\partial \rho^k} P(f)^{|r|}(\rho, \cdot) \right\|_p \leq C_k \frac{\|P(f)^{|r|}(\rho, \cdot)\|_p}{(1 - \rho)^k} = \frac{C_k}{(1 - \rho)^k} M_p(\rho, f, r).
\]

Therefore,

\[
\|A_{\rho,n}^{[r]}(f)\|_p \leq \sum_{k=0}^{n-1} \frac{C_k}{k!} M_p(\rho, f, r)
\]

and applying Corollary 3.1, we get inequality (3.32).

\[\Box\]

**Lemma 3.5.** Assume that \(f \in L_p(\Omega), 1 \leq p \leq \infty\). Then, for any numbers \(n \in \mathbb{N}\) and \(\rho \in [1/2, 1)\),

\[
c_{1,n}(1 - \rho)^n M_p(\rho, f, n) \leq K_n (1 - \rho, f)_p \leq c_{2,n} \left( \|f - A_{\rho,n}(f)\|_p + (1 - \rho)^n M_p(\rho, f, n) \right),
\]

where the constants \(c_{1,n}\) and \(c_{2,n}\) depend only on \(n\).

**Proof.** First, let us note that the statement of Lemma 3.5 is trivial in the case when \(f\) is a polynomial of order not exceeding \(n - 1\) as well as in the case \(\rho = 0\). Therefore, from now on we exclude these two cases.
Let $g$ be a function such that $g^{[n]} \in L_p(\Omega)$. Using Corollary 3.1 and relation (3.31), we get
\[
\left\| \frac{\partial^n}{\partial \varrho^n} P(f)(\varrho, \cdot) \right\|_p = \left\| \frac{\partial^n}{\partial \varrho^n} P(f - g)(\varrho, \cdot) + \frac{\partial^n}{\partial \varrho^n} P(g)(\varrho, \cdot) \right\|_p \\
\leq C_n \| f - g \|_p \| \frac{\partial^n}{\partial \varrho^n} P(g)(\varrho, \cdot) \|_p.
\]
Taking into account (2.13) and Corollary 3.1, we conclude that for any $\varrho \in (0, 1),$
\[
(1 - \varrho)^n M_p(\varrho, f, n) \leq C_n \varrho^n \| f - g \|_p + (1 - \varrho)^n \| P(g^{[n]})(\varrho, \cdot) \|_p \\
\leq C_n \| f - g \|_p + C_0 (1 - \varrho)^n \| g^{[n]} \|_p.
\]
Considering the infimum over all functions $g$ such that $g^{[n]} \in L_p(\Omega)$, we imply
\[
c_{1,n} (1 - \varrho)^n M_p(\varrho, f, n) \leq K_n (1 - \varrho, f)_p.
\]
On the other hand, from the definition of the $K$–functional, it follows that
\[
K_n (1 - \varrho, f)_p \leq \| f - A_{\varrho,n}(f) \|_p + (1 - \varrho)^n \| (A_{\varrho,n}(f))^{[n]} \|_p.
\]
Using estimate (3.35) with $r = n$, we obtain the right-hand inequality in (3.36).

\begin{lemma}
Assume that $f \in L_p(\Omega), 1 \leq p \leq \infty, 0 \leq \varrho < 1$ and $r = 2, 3, \ldots$ such that
\[
\int_{\varrho}^{1} \left\| \frac{\partial^r}{\partial \zeta^r} P(f)(\zeta, \cdot) \right\|_p (1 - \zeta)^{r-1} d\zeta < \infty.
\]
Then for almost all $t \in \Omega,$
\[
f(t) - A_{\varrho,r}(f)(t) = \frac{1}{r!} \int_{\varrho}^{1} (1 - \zeta)^{r-1} \frac{\partial^r}{\partial \zeta^r} P(f)(\zeta, t) d\zeta.
\]
\end{lemma}

\begin{proof}
For fixed $r = 2, 3, \ldots$ and $0 \leq \varrho < 1$, the integral on the right-hand side of (3.39) defines a certain function $F(t)$. By virtue of (3.38) and the integral Minkowski inequality, we conclude that the function $F$ belongs to the space $L_p(\Omega)$. Let us find the Fourier coefficients of $F$ and compare them with the Fourier coefficients of the function $G := f - A_{\varrho,r}(f)$.

Since for any $\nu = r, r + 1, \ldots,$
\[
\frac{1}{(r - 1)!}(\nu - r)! \int_{\varrho}^{\varrho_1} (1 - \zeta)^{r-1} d\zeta = \sum_{j=0}^{r-1} \frac{\varrho_1^{\nu-j}(1 - \varrho_1)^j - \varrho^{\nu-j}(1 - \varrho)^j}{j!(\nu - j)!},
\]
we have in view of (2.11) for a fixed $\varrho_1 \in (\varrho, 1)$
\[
\frac{1}{(r - 1)!}(\nu - r)! \int_{\varrho}^{\varrho_1} (1 - \zeta)^{r-1} \frac{\partial^r}{\partial \zeta^r} P(f)(\zeta, t) d\zeta
= \sum_{\nu=r}^{\infty} \sum_{k \in \mathbb{Z}^r} \frac{\nu!}{(r - 1)!}(\nu - r)! \int_{\varrho}^{\varrho_1} (1 - \zeta)^{r-1} d\zeta
= \sum_{\nu=r}^{\infty} \sum_{k \in \mathbb{Z}^r} \hat{f}_k \phi_k(t) \sum_{j=0}^{r-1} \left(\frac{\nu}{j}\right) \left(\varrho_1^{\nu-j}(1 - \varrho_1)^j - \varrho^{\nu-j}(1 - \varrho)^j\right).
\]
\end{proof}
Now, if in the equality (3.40), the value $\varrho_1$ tends to $1^-$, then we observe that the Fourier coefficients $\hat{F}_k$ of the function $F$ are equal to zero when $k \in J_\nu$, $\nu < r$ and for $k \in J_\nu$, $\nu \geq r$,

$$\hat{F}_k = \hat{f}_k \left( 1 - \sum_{j=0}^{r-1} \binom{\nu}{j} (1 - \varrho)^j \varrho^{r-j} \right) = (1 - \lambda_{\nu,r}(\varrho)) \hat{f}_k. \quad (3.41)$$

Therefore, for all $k \in \mathbb{Z}^3_H$ we have $\hat{F}_k = (1 - \lambda_{\nu,r}(\varrho)) \hat{f}_k$. Hence, for almost all $t \in \Omega$, the representation (3.39) holds. \hfill \Box

### 3.6 Proof of the Theorems 3.2 and 3.3.

**Proof of Theorem 3.2.** Assume that the function $f$ is such that $f^{[r-n]} \in L_p(\Omega)$ and relation (3.19) is satisfied. Let us apply the left-hand side inequality of Lemma 3.5 to the function $f^{[r-n]}$. In view of (2.13) and (3.31), we obtain

$$c_1, n (1 - \varrho)^n M_p(\varrho, f, r) \leq K_n (1 - \varrho) f^{[r-n]}_n. \quad (3.42)$$

This yields

$$M_p(\varrho, f, r) = \mathcal{O}(1)(1 - \varrho)^{-n} \omega(1 - \varrho), \quad \varrho \to 1 -. \quad (3.43)$$

Using relations (3.17), (3.31), (3.42) and the integral Minkowski inequality, we obtain the following estimate

$$\int_{\varrho}^1 \left\| \frac{\partial^r}{\partial \zeta^r} P(f)(\zeta, \cdot) \right\|_p (1 - \zeta)^{r-1} d\zeta \leq \int_{\varrho}^1 M_p(\zeta, f, r) \frac{(1 - \zeta)^{r-1}}{\zeta^r} d\zeta \leq c_1 (1 - \varrho)^{r-n} \int_{\varrho}^1 \omega(1 - \zeta) d\zeta = \mathcal{O}((1 - \varrho)^{r-n} \omega(1 - \varrho)) \quad (3.43)$$

for all $0 \leq \varrho < 1$.

Therefore, for almost all $t \in \Omega$, relation (3.39) holds. Hence, by virtue of (3.39), using the integral Minkowski inequality and (3.43), we finally get (3.20):

$$\|f - A_{\varrho,r}(f)\|_p \leq \frac{1}{(r-1)!} \int_{\varrho}^1 M_p(\zeta, f, r) \frac{(1 - \zeta)^{r-1}}{\zeta^r} d\zeta = \mathcal{O}((1 - \varrho)^{r-n} \omega(1 - \varrho)), \quad \varrho \to 1 -. \quad \Box$$

**Proof of Theorem 3.3.** First, let us note that for any function $f \in L_p(\Omega)$ and all fixed numbers $r, s \in \mathbb{N}$ and $\varrho \in (0, 1)$, we have
\[ \|A^{[s]}_{\varrho,r}(f)\|_p = \left\| \sum_{\nu=s}^{\infty} \frac{\nu!}{(\nu - s)!} A^{[\nu]}_{\varrho,r}(\varrho) \sum_{k \in I_\nu} \hat{f}_k \phi_k(t) \right\|_p \]
\[ \leq 2r\|f\|_p \left( \sum_{\nu=s}^{\max\{r,s\}-1} \frac{\nu!}{(\nu - s)!} + \sum_{\nu \geq \max\{r,s\}} q^{\nu+\nu^{-1}} \right) \]
\[ < \infty, \quad (3.44) \]

where \(0 < q = \max\{1 - \varrho, \varrho\} < 1\). In the case where \(s \geq r\), the sum \(\sum_{\nu=s}^{s-1}\) is set equal to zero.

Put \(\varrho_k := 1 - 2^{-k}\), \(k \in \mathbb{N}\), and \(A_k := A_k(f) := A^{[\varrho_k,r]}_{\varrho_k,r}(f)\). For any \(t \in \Omega\) and \(s \in \mathbb{N}\), consider the series
\[ A^{[s]}_0(f)(t) + \sum_{k=1}^{\infty} (A^{[s]}_k(f)(t) - A^{[s]}_{k-1}(f)(t)). \quad (3.45) \]

According to the definition of the operator \(A_{\varrho,r}\), we conclude that for any \(\varrho_1, \varrho_2 \in [0,1)\) and \(r \in \mathbb{N}\),
\[ A^{[\varrho_1,r]}(A^{[\varrho_2,r]}(f)) = A^{[\varrho_1,r]}(A^{[\varrho_1,r]}(f)). \]

By virtue of Lemma 3.4 and relation (3.20), for any \(k \in \mathbb{N}\) and \(s \in \mathbb{N}\), we have
\[ \left\| A^{[s]}_k - A^{[s]}_{k-1} \right\|_p = \left\| A^{[s]}_k f - A^{[s]}_{k-1} f - A^{[s]}_{k-1} f + A^{[s]}_{k-1} (f - A^{[s]}_k f) \right\|_p \]
\[ \leq \left\| A^{[s]}_k f - A^{[s]}_{k-1} f \right\|_p + \left\| A^{[s]}_{k-1} f - A^{[s]}_k f \right\|_p \]
\[ \leq C_{s,r} \frac{\|f - A^{[s]}_{k-1} f\|_p}{(1 - \varrho_k)^s} + C_{s,r} \frac{\|f - A^{[s]}_k f\|_p}{(1 - \varrho_{k-1})^s} \]
\[ = \mathcal{O}\left( \frac{\omega(1 - \varrho_{k-1})}{(1 - \varrho_k)^s} \right) + \mathcal{O}\left( \frac{\omega(1 - \varrho_k)}{(1 - \varrho_{k-1})^s} \right). \quad (3.46) \]

Therefore, if \(s \leq r - n\), then
\[ \left\| A^{[s]}_k - A^{[s]}_{k-1} \right\|_p = \mathcal{O}\left( \omega(1 - \varrho_{k-1}) \right) = \mathcal{O}\left( \omega(2^{1-k}) \right), \quad k \to \infty. \quad (3.47) \]

Consider the sum \(\sum_{k=1}^{N} \omega(2^{1-k})\), \(N \in \mathbb{N}\). Taking into account the monotonicity of the function \(\omega\) and (3.17), we observe that for all \(N \in \mathbb{N}\),
\[ \sum_{k=1}^{N} \omega(2^{1-k}) \leq \omega(1) + \int_{1}^{N} \omega(2^{1-t})dt \]
\[ = \omega(1) + \int_{2^{1-N}}^{1} \frac{\omega(\tau)d\tau}{\tau \ln 2} < \infty. \]

Combining the last relation and (3.47), we conclude that the series in (3.45) converges in the norm of the space \(L^p(\Omega)\), \(1 \leq p \leq \infty\). Hence, by virtue of the Banach–Alaoglu theorem,
there exists the following subsequence if $0 ≤ s ≤ r - n$:

$$S_{N_j}^{[s]}(t) = A_0^{[s]}(f)(t) + \sum_{k=1}^{N_j} (A_k^{[s]}(f)(t) - A_{k-1}^{[s]}(f)(t)), \quad j = 1, 2, \ldots$$

(3.48)

of partial sums of this series converging to a certain function $g \in L_p(\Omega)$ almost everywhere on $\Omega$ as $j \to \infty$.

Let us show that $g = f^{[s]}$. For this, let us find the Fourier coefficients of the function $g$. For any fixed $k \in \mathbb{Z}^3_H$ and all $j = 1, 2, \ldots$, we have

$$\hat{g}_k := \frac{1}{|\Omega|} \int_{\Omega} S_{N_j}^{[s]}(t) \overline{\phi_k(t)} \, dt + \frac{1}{|\Omega|} \int_{\Omega} (g(t) - S_{N_j}^{[s]}(t)) \overline{\phi_k(t)} \, dt.$$

Since the sequence $\{S_{N_j}^{[s]}\}_{j=1}^{\infty}$ converges almost everywhere on $\Omega$ to the function $g$, the second integral on the right-hand side of the last equality tends to zero as $j \to \infty$. By virtue of (3.48) and the definition of the radial derivative, for $|k| = \nu < s$ the first integral is equal to zero, and for all $|k| = \nu \geq s$,

$$\frac{1}{|\Omega|} \int_{\Omega} S_{N_j}^{[s]}(t) \overline{\phi_k(t)} \, dt = \lambda_{\nu r} (1 - 2^{-N_j}) \frac{\nu!}{(\nu - s)!} \hat{f}_k \to \nu! \hat{f}_k.$$

Therefore, the equality $g = f^{[s]}$ holds true. Hence, for the function $f$ and all $0 ≤ s ≤ r - n$, there exists the derivative $f^{[s]}$, and $f^{[s]} \in L_p(\Omega)$.

Now, let us prove the estimate (3.42). By virtue of (3.31), (3.46), for any $k \in \mathbb{N}$ and $\rho \in (0, 1)$, we have

$$M_p (\rho, A_k - A_{k-1}, r) \leq \left\| A_k^{[r]} - A_{k-1}^{[r]} \right\|_p$$

$$= \mathcal{O} \left( \frac{\omega(1 - \theta_{k-1})}{(1 - \theta_k)^n} \right) + \mathcal{O} \left( \frac{\omega(1 - \theta_k)}{(1 - \theta_{k-1})^n} \right)$$

$$= \mathcal{O} \left( 2^{kn} \omega(2^{k+1}) + 2^{(k-1)n} \omega(2^{-k}) \right)$$

$$= \mathcal{O} \left( 2^{(k-1)n} \omega(2^{-(k-1)}) \right), \quad k \to \infty.$$  

(3.49)

By virtue of (3.31), (3.16) and (3.20), for any $r \in \mathbb{N}$ and $\rho \in (0, 1)$, we obtain

$$M_p (\rho, f - A_{\rho r}(f), r) = \mathcal{O}(1) \frac{\| f - A_{\rho r}(f) \|_p}{(1 - \rho)^r}$$

$$= \mathcal{O} \left( \frac{\omega(1 - \rho)}{(1 - \rho)^n} \right), \quad \rho \to 1 - .$$

Therefore, for $N \to \infty$,

$$M_p (\rho_N, f - A_N(f), r) = \mathcal{O} \left( \frac{\omega(1 - \theta_N)}{(1 - \theta_N)^n} \right)$$

$$= \mathcal{O} \left( 2^{Nn} \omega(2^{-N}) \right).$$  

(3.50)

(3.51)
Consider the sum \( \sum_{k=1}^{N} 2^{(k-1)n} \omega(2^{-(k-1)}) \), \( N \in \mathbb{N} \). For any \( \omega \in \mathbb{Z}_n \), the function \( \omega(t)/t^n \) almost decreases on \((0, 1] \), i.e., there exists a number \( c_2 > 0 \) such that \( \omega(t_1)/t_1^n \geq c_2 \omega(t_2)/t_2^n \) for any \( 0 < t_1 < t_2 \leq 1 \) (see, for example [2]). Therefore,

\[
\sum_{k=1}^{N} 2^{(k-1)n} \omega(2^{-(k-1)}) \\
\leq c_2 \left( 2^{(N-1)n} \omega(2^{-(N-1)}) + \int_{1}^{N} 2^{(t-1)n} \omega(2^{-(t-1)}) \text{d}t \right) \\
\leq c_2 \left( 2^{(N-1)n} \omega(2^{-(N-1)}) + \int_{2^{-N+1}}^{1} \omega(\tau) \frac{\text{d}\tau}{\tau^{n+1} \ln 2} \right) \\
= \mathcal{O} \left( 2^{(N-1)n} \omega(2^{-(N-1)}) \right) \\
= \mathcal{O} \left( 2^N n \omega(2^{-N}) \right) \quad N \to \infty. \tag{3.52}
\]

Putting \( \varrho = \varrho_N \) and taking into account the relations (3.49), (3.50), (3.52) and

\[
A_0(t) = S_{r-1}(f)(t) = \sum_{|k| \leq r-1} \hat{f}_k \phi_k(t),
\]
we obtain

\[
M_p \left( \varrho_N, f, r \right) = M_p \left( \varrho_N, f - S_{r-1}(f), r \right) \\
= M_p \left( \varrho_N, f - A_{\varrho_N} + \sum_{k=1}^{N} (A_k - A_{k-1}), r \right) \\
= \mathcal{O} \left( \sum_{k=1}^{N} \left( 2^{(k-1)n} \omega(2^{-(k-1)}) \right) \right) \tag{3.53} \\
= \mathcal{O} \left( 2^N n \omega(2^{-N}) \right) \\
= \mathcal{O} \left( (1 - \varrho_N)^{-n} \omega(1 - \varrho_N) \right) \quad N \to \infty. \tag{3.54}
\]

If the function \( \omega \) satisfies the condition \((\mathbb{Z}_n)\), then \( \sup_{t \in [0,1]} (\omega(2t)/\omega(t)) < \infty \) (see, for example [2]). Furthermore, for all \( \varrho \in [\varrho_{N-1}, \varrho_N] \), we have \( 1 - \varrho_N \leq 1 - \varrho \leq 2(1 - \varrho_N) \). Hence, relation (3.54) yields the estimate (3.42).

Now, applying the right-hand side inequality in Lemma 3.5 to the function \( f^{[r-n]} \), we get

\[
K_n \left( 1 - \varrho, f^{[r-n]} \right)_p \leq c_{2n} \left( \|f^{[r-n]} - A_{\varrho,n}(f^{[r-n]})\|_p + (1 - \varrho)^n M_p(\varrho, f, r) \right). \tag{3.55}
\]

By virtue of (3.31) and (3.42), we conclude that for \( \varrho \in [1/2, 1) \),

\[
\int_{\varrho}^{1} \left\| \frac{\partial^n}{\partial \zeta^n} P(f^{[r-n]})(\zeta, \cdot) \right\|_p (1 - \zeta)^{n-1} \text{d}\zeta = \int_{\varrho}^{1} \left\| P(f)^{[r]}(\zeta, \cdot) \right\|_p \frac{(1 - \zeta)^{n-1}}{\zeta^n} \text{d}\zeta \\
\leq c_1 \int_{\varrho}^{1} \frac{1 - \zeta}{1 - \xi} \text{d}\zeta \\
= \mathcal{O} \left( \omega(1 - \varrho) \right). \tag{3.56}
\]

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Therefore, we can apply Lemma 3.6 to the function $f^{[r-n]}$. Taking into account (3.31), we obtain

$$f^{[r-n]}(t) - A_{\varrho,n}(f^{[r-n]})(t) = \frac{1}{(n-1)!} \int_\varrho^1 P(f)^{[r]}(\zeta, t) \frac{(1-\zeta)^{n-1}}{\zeta^n} d\zeta.$$

Using the integral Minkowski inequality and (3.56), we conclude

$$\|f^{[r-n]} - A_{\varrho,n}(f^{[r-n]})\|_p \leq \frac{1}{(n-1)!} \int_\varrho^1 M_p(\zeta, f, r) \frac{(1-\zeta)^{n-1}}{\zeta^n} d\zeta \leq \mathcal{O}(\omega(1-\varrho)), \quad \varrho \to 1-. \quad (3.57)$$

Combining the relations (3.55), (3.42) and (3.57), we finally get (3.19).

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References

[1] Arestov, V. V., Berdysheva, E. E. *Turán’s problem for positive definite functions with supports in a hexagon*. In: Proc. Steklov Inst. Math. 2001, Approximation Theory. Asymptotical Expansions, Suppl. 1, 20–29.

[2] Bari N. K., Stechkin S. B. *Best approximations and differential properties of two conjugate functions*, Tr. Mosk. Mat. Obshch. 1956, no. 5, 483–522.

[3] Butzer P. L., Nessel R. *Fourier Analysis and Approximation*. One–Dimensional Theory, Basel–New York, 1971.

[4] Butzer P. L., Sunouchi G. *Approximation theorems for the solution of Fourier’s problem and Dirichlet’s problem*, Math. Ann. 1964, no. 155, 316–330.

[5] Chandra P., Mohapatra R. N. *Approximation of functions by $(J, q_n)$ means of Fourier series*, Approx. Theory Appl. 4 (1988), no. 2, 49–54.

[6] Chui C. K., Holland A. S. B. *On the order of approximation by Euler and Taylor means*, J. Approx. Theory, 39 (1983), no. 1, 24–38.

[7] Conway J. H., Sloane N. J. A. *Sphere Packings, Lattices and Groups*, 3rd ed., Springer, New York, 1999.

[8] DeVore R. A., Lorentz G. G. *Constructive Approximation*, Berlin: Springer–Verlag, 1993.
[9] Dunkl C. F. *Orthogonal polynomials on the hexagon.* SIAM J. Appl. Math. **47** (1987), 343–351.

[10] Ebeling W., Lattices and Codes, Vieweg, Braunschweig/Wiesbaden, 1994.

[11] Guven A. *Approximation by (C,1) and Abel-Poisson means of Fourier series on hexagonal domains.* Math. Inequal. Appl. **16** (2013), no. 1, 175-191.

[12] Guven A. *Degree of approximation by means of hexagonal Fourier series.* Turkish Journal of Mathematics. **44** (2020), no. 3, Paper no. 25, 17 p.

[13] Higgins J. R., Sampling Theory in Fourier and Signal Analysis, Foundations, Oxford Science Publications, New York, 1996.

[14] Holland A. S. B., Sahney B. N., Mohapatra R. N. *L_p approximation of functions by Euler means,* Rend. Mat. **3** (7) (1983), no. 2, 341–355.

[15] Leis R. *Approximationssätze für stetige Operatoren,* Arch. Math. 1963, no. 14, 120–129.

[16] Li, H., Sun, J., Xu, Y. *Discrete Fourier analysis, cubature and interpolation on a hexagon and a triangle.* SIAM J. Numer. Anal. **46** (2008), 1653–1681.

[17] Marks II R. J., Introduction to Shannon Sampling and Interpolation Theory, Springer, New York, 1991.

[18] Mohapatra R. N., Holland A. S. B., Sahney B. N. *Functions of class Lip(α,p) and their Taylor mean,* J. Approx. Theory, **45** (1985), no. 4, 363–374.

[19] Prestin J., Savchuk V. V., Shidlich A. L. *Direct and inverse approximation theorems of 2π-periodic functions by Taylor-Abel-Poisson means,* Ukr. Mat. Journ. **69** (2017), no. 5, 766-781.

[20] Prestin J., Savchuk V. V., Shidlich A. L. *Approximation theorems for multivariate Taylor-Abel-Poisson means,* Stud. Univ. Babeş-Bolyai Math. **64** (2019), no. 3, 313-329.

[21] Savchuk V. V. *Approximation of holomorphic functions by Taylor-Abel-Poisson means,* Ukr. Mat. Journ. **59** (2007), no. 9, 1397-1407.

[22] Savchuk V. V., Shidlich A. L. *Approximation of functions of several variables by linear methods in the space S^p,* Acta Sci. Math. **80** (2014), no. 3–4, 477–489.

[23] Stepanets A. I. Methods of approximation theory. VSP, Leiden (2005), 919 pp.

[24] Sun J., Multivariate Fourier series over a class of non tensor-product partition domains, J. Comput. Math. **21** (2003), no. 1, 53-62.

[25] Xu Y. Fourier Series and Approximation on Hexagonal and Triangular Domains, Constr. Approx. **31** (2010), 115–138.

[26] Xu Y. Positivity and Fourier integrals over regular hexagon, J. Approx. Theory **2007** (2016), 193–206.