Gelfand-Tsetlin modules over $\mathfrak{gl}(n)$ with arbitrary characters

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Abstract

A Gelfand-Tsetlin tableau $T(v)$ induces a character $\chi_v$ of the Gelfand-Tsetlin subalgebra $\Gamma$ of $\mathcal{U}(\mathfrak{gl}(n, \mathbb{C}))$. By a theorem due to Ovsienko, for each tableau $T(v)$ there exists a finite number of nonisomorphic irreducible Gelfand-Tsetlin modules with $\chi_v$ in its support, though explicit examples of such modules are only known for special families of characters. In this article we build a family of Gelfand-Tsetlin modules parametrized by characters, such that each character appears in its corresponding module. We also find the support of these modules, with multiplicities.

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1 Introduction

The notion of a Gelfand-Tsetlin module (see Definition 3.1) has its origin in the classical article [GC50], where I. Gelfand and M. Tsetlin gave an explicit presentation of all finite dimensional irreducible representations of $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$ in terms of certain combinatorial objects, which have come to be known as Gelfand-Tsetlin tableaux, or GT tableaux for short. A GT tableau is a triangular array of $\frac{n(n+1)}{2}$ complex numbers, with $k$ entries in the $k$-th row; given a point $v \in \mathbb{C}^{\frac{n(n+1)}{2}}$ we denote the corresponding array by $T(v)$. The group $G = S_1 \times S_2 \times \cdots \times S_n$ acts on the set of all tableaux, with $S_k$ permuting the elements in the $k$-th row. Gelfand and Tsetlin’s theorem establishes that any finite dimensional irreducible representation of $\mathfrak{g}$ has a basis parameterized by GT tableaux with integer entries satisfying certain betweenness relations. Identifying the elements of the basis with the corresponding GT tableaux, the action of an element of $\mathfrak{g}$ on a tableau is given by rational functions in its entries. These rational functions are known as the Gelfand-Tsetlin formulas; their poles form an infinite hyperplane array in $\mathbb{C}^{\frac{n(n+1)}{2}}$.

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The enveloping algebra $U = U(\mathfrak{g})$ contains a large (indeed, maximal) commutative subalgebra $\Gamma$ called the Gelfand-Tsetlin subalgebra of $U$. A Gelfand-Tsetlin module is a $U$-module that can be decomposed as the direct sum of generalized eigenspaces for $\Gamma$. The characters of $\Gamma$ are in one-to-one correspondence with GT tableaux modulo the action of $G$ (see [Zhe73]), and in the original construction of Gelfand and Tsetlin each tableau $T(v)$ is an eigenvector of $\Gamma$ whose eigenvalue is precisely the character $\chi_v : \Gamma \to \mathbb{C}$ corresponding to $v$. Since no two tableaux in this construction are in the same $G$-orbit, the multiplicity of this character (i.e. the number of eigenvectors of eigenvalue $\chi_v$) is one.

Ovsienko proved in [Ovs02, Ovs03] that for each character $\chi : \Gamma \to \mathbb{C}$ there exists a nonzero finite number of Gelfand-Tsetlin $U$-modules with $\chi$ in its character support. Many such modules have been constructed for different classes of characters, such as standard [GC50], generic [DFO94], 1-singular [FGR16a], index 2 [FGR16b], etc. However, no explicit construction of such modules is known for arbitrary characters.

The first work in this direction is due to Y. Drozd, S. Ovsienko and V. Futorny, who introduced a large family of infinite dimensional $\mathfrak{g}$-modules in [DFO94]. These GT modules have a basis parameterized by Gelfand-Tsetlin tableaux with complex coefficients such that no pattern is a pole for the rational functions appearing in the GT formulae (such tableaux are called generic, hence the name “generic Gelfand-Tsetlin module”). While each character in the decomposition of a generic Gelfand-Tsetlin module appears with multiplicity one, there are examples of non-generic GT modules with higher multiplicities. These examples were first encountered in [Fut86, Fut91] for $\text{sl}(3)$.

In [FGR16a] V. Futorny, D. Grantcharov and the first named author constructed a GT module with 1-singular characters, i.e. characters associated to tableaux over which the Gelfand-Tsetlin formulas may have singularities of order at most 1. These modules have a basis in terms of so-called derived tableaux, new objects which, according to the authors “are not new combinatorial objects” but rather formal objects in a large vector-space that contains classical GT tableaux. This construction was expanded and refined in the articles [FGR16b, Zad17, Vis17] for characters with more general singularities. The aim of this article is to extend this construction to arbitrary characters and calculate the multiplicity of the corresponding characters. This is achieved in section 5 in the process we disprove the statement above and give a combinatorial interpretation of derived tableaux.

The general idea of our construction is the following. Let $K$ be the field of rational functions over the space of GT tableaux, and denote by $V$ the $K$ vector-space of arbitrary integral GT tableaux. This is a $U$-module, which we call the big GT module, with the action of $\mathfrak{g}$ given by the Gelfand-Tsetlin formulas. These rational functions lie in the algebra $A$ of regular functions over generic tableaux, and hence the $A$-lattice $L_A$ whose $A$-basis is the set of all integral tableaux is a $U$-submodule of $V$; now given a generic tableau $T(v)$, we can recover the corresponding generic module by specializing $V_A$ at $v$. This idea breaks down if $T(v)$ is a singular tableau, and in that case we replace $A$ with an algebra $B \subset K$ such that (1) evaluation at $v$ makes sense and (2)
there exists a $B$-lattice $L_B \subset V$ which is also a $U$-submodule. Once this is done, the rest of the construction follows as in the generic case.

Denote by $\mu$ the composition of $\frac{n(n+1)}{2}$ given by $(1,2,\ldots,n)$. Each point $v \in \mathbb{C}^{\frac{n(n+1)}{2}}$, or rather its class modulo $G$, defines a refinement $\eta(v)$ of $\mu$, and it turns out that the structure of the associated GT module $V(T(v))$ depends heavily on $\eta(v)$. While it is possible in principle to choose an algebra $B$ and a lattice $L_B$ that works for all $v$ simultaneously, we get more information by fixing a refinement $\eta$ and focusing on characters with $\eta(v) = \eta$, thus obtaining an algebra $B_\eta$ and a $B_\eta$-lattice $L_\eta$. Each $L_\eta$ has a basis of derived tableaux, and changing $\eta$ changes this basis in an essential way.

As shown in [FGR15], the generic GT modules from [DFO04] are universal, in the sense that any irreducible GT module with a generic character in its support is isomorphic to a subquotient of the corresponding generic GT module. The 1-singular GT modules built in [FGR16a] are also universal with respect to 1-singular characters, see [FGR17]. Since these modules are special cases of our construction, we expect the singular GT modules built in this article to be universal with respect to the characters in their support.

While finishing this paper the article [Vis17] by E. Vishnyakova was uploaded to the ArXiv, containing a similar construction of $p$-singular GT modules, where $p \in \mathbb{N}_{\geq 2}$. This is a special class of singular modules, associated to classes $v \in \mathbb{C}^{\frac{n(n+1)}{2}}/G$ where the composition $\eta(v)$ has only one nontrivial part, which is equal to $p$.

The article is organized as follows. In section 2 we set the notation used for the combinatorial invariants associated to tableaux. In section 3 we review some basic facts on GT modules, including the construction of generic and 1-singular GT modules in terms of the big GT module. In section 4 we study certain operators related to divided differences which play a central role in the construction of the lattices $L_\eta$. Finally in section 5 we present the lattices $L_\eta$, build the GT modules associated to a character $\chi_v$ and find its character support with the corresponding multiplicites.

## 2 Preliminaries

2.1. Let $n,m \in \mathbb{N}$. We write $[n,m] = \{k \in \mathbb{N} \mid n \leq k \leq m\}$ and $[n] = [1,n]$. We denote by $S_n$ the symmetric group on $n$ elements. Recall that for each $\sigma \in S_n$ the length of $\sigma$, denoted by $\ell(\sigma)$, is the number of inversions of $\sigma$, i.e. the number of pairs $(i,j) \in [n]^2$ such that $i < j$ but $\sigma(i) > \sigma(j)$. There exists a unique longest word $w_0 \in S_n$ such that $\ell(w_0) = \frac{n(n-1)}{2}$. Also $\ell(\sigma) = \ell(\sigma^{-1})$, and $\ell(\sigma^{-1}w_0) = \ell(w_0) - \ell(\sigma)$. For each $i \in [n-1]$ the $i$-th simple transposition is $s_i = (i,i+1) \in S_n$. Simple transpositions generate $S_n$, and the length of $\sigma \in S_n$ is the minimal $l$ such that $\sigma$ can be written as $s_{i_l}s_{i_{l-1}}\cdots s_{i_1}$. Any such writing is called a reduced decomposition of $\sigma$.

2.2. Compositions. Recall that a composition of $n$ is a sequence $\mu = (\mu_1,\ldots,\mu_r)$ of positive integers such that $\sum_{i=1}^{r} \mu_i = n$. The $\mu_k$ are called the parts of $\mu$. Now let $\mu = (\mu_1,\ldots,\mu_r)$ be a composition of $n$. For each $k \in [r]$ set $a_k = a_k(\mu) = \sum_{j=1}^{k-1} \mu_j + 1$
and \( \beta_k = \beta_k(\mu) = \sum_{j=1}^{k} \mu_j \), so the interval \([\alpha_k, \beta_k]\) has \(\mu_k\) elements; we refer to this interval as the \(k\)-th block of \(\mu\).

Denote by \(S_\mu \subset S_\mu\) the subgroup of bijections \(\sigma\) such that \(\sigma([\alpha_k, \beta_k]) = [\alpha_k, \beta_k]\) for each \(k \in [r]\). This is a parabolic subgroup of \(S_n\) in the sense of [BB05] section 2.4], and we review some of the properties discussed there. By definition each \(\sigma \in S_\mu\) is a product of the form \(\sigma = \sigma(1)\sigma(2) \cdots \sigma(r)\) with each \(\sigma(i)\) the identity on each \(\mu\)-block except the \(j\)-th. The length of an element on \(S_\mu\) is \(\ell(\sigma) = \ell(\sigma(1)) + \cdots + \ell(\sigma(r))\), and \(S_\mu\) has a unique longest word \(w_\mu\) with \(w_\mu^{(k)}\) the longest word of the permutation group of \([\alpha_k, \beta_k]\). We set \(\mu! = \#S_\mu = \mu_1! \mu_2! \cdots \mu_r!\).

Put \(\Sigma(\mu) = \{ (k, i) \mid k \in [r], i \in [\mu_k] \}\) and let \(\gamma_\mu : \Sigma(\mu) \rightarrow [n]\) given by \(\gamma_\mu(k, i) = i + \sum_{j=1}^{k-1} \mu_j\). This map is a bijection, and through it \(S_\mu\) acts on \(\Sigma(\mu)\). We will often identify a permutation \(\sigma \in S_\mu\) by its action on \(\Sigma(\mu)\); for example, we denote by \(s_i^{(k)}\) the simple transposition in \(S_\mu\) which acts on \(\Sigma(\mu)\) by interchanging \((k, i)\) and \((k, i+1)\), leaving all other elements fixed. With this notation, \(\sigma^{(k)}\) leaves all elements of the form \((j, l)\) with \(j \neq k\) fixed.

Set

\[
\text{sym}_\mu = \frac{1}{\mu!} \sum_{\sigma \in S_\mu} \sigma; \quad \text{asym}_\mu = \frac{1}{\mu!} \sum_{\sigma \in S_\mu} \sg(\sigma)\sigma.
\]

These are idempotent elements of the group algebra \(\mathbb{C}[S_\mu]\) and given a \(\mathbb{C}[S_\mu]\)-module \(V\), multiplication by \(\text{sym}_\mu\), resp. \(\text{asym}_\mu\), is the projection onto the symmetric, resp. antisymmetric, component of \(V\).

2.3. Refinements. A refinement of \(\mu\) is a collection of compositions \(\eta = (\eta^{(1)}, \ldots, \eta^{(r)})\) with each \(\eta^{(k)}\) a composition of \(\mu_k\). If \(\eta\) is a refinement of \(\mu\) then the concatenation of the \(\eta^{(k)}\)'s is also a composition of \(n\), which by abuse of notation we will also denote by \(\eta\).

If \(\eta\) refines \(\mu\) then \(S_\eta \subset S_\mu\). We say that \(\sigma \in S_\mu\) is a \(\eta\)-shuffle if it is increasing in each \(\eta\)-block. Among the elements of a coclass \(\sigma S_\mu \subset S_\mu/S_\eta\) there is exactly one \(\eta\)-shuffle, and this is the unique element of minimal length in the coclass. We denote the set of all \(\eta\)-shuffles in \(S_\mu\) by \(\text{Shuffle}_n^{\eta}\). The group \(S_\eta\) acts on \(\Sigma(\mu)\) by restriction, and the orbits of this action are also called the \(\eta\)-blocks of \(\mu\).

2.4. We write \(\mathbb{C}_\mu = \mathbb{C}^{\mu_1} \oplus \mathbb{C}^{\mu_2} \oplus \cdots \oplus \mathbb{C}^{\mu_r};\) thus \(v \in \mathbb{C}_\mu\) is an \(r\)-uple of vectors \((v_1, \ldots, v_r)\) with \(v_k \in \mathbb{C}^{\mu_k}\). We refer to the elements of \(\mathbb{C}_\mu\) as \(\mu\)-points, or simply points if the composition \(\mu\) is fixed. For each \((k, i) \in \Sigma(\mu)\) we write \(v_{k,i}\) for the \(i\)-th coordinate of \(v_k\). We refer to the \(v_k\)'s as the \(\mu\)-blocks of \(v\), and to the \(v_{k,i}\)'s as the entries of \(v\). We say that a \(\mu\)-point \(v\) is integral if all its entries lie in \(\mathbb{Z}\). Clearly \(\mathbb{C}_\mu\) is an affine variety, and we denote by \(\mathbb{C}[X_\mu]\) the polynomial algebra generated by \(x_{k,i}\) with \((k, i) \in \Sigma(\mu)\), which is the coordinate ring of \(\mathbb{C}_\mu\). We also denote by \(\mathbb{C}(X_\mu)\) the field of fractions of \(\mathbb{C}[X_\mu]\). The group \(S_\mu\) acts on \(\mathbb{C}_\mu\) in an obvious way, and this induces actions on \(\mathbb{C}[X_\mu]\) and \(\mathbb{C}(X_\mu)\).

If \(\eta\) is a refinement of \(\mu\) then there exists an isomorphism \(\mathbb{C}_\eta \cong \mathbb{C}_\mu\), so we can talk
of the \( \eta \)-blocks of \( v \in \mathbb{C}^n \). Since \( \eta \) refines \( \mu \) we get an inclusion \( S_\eta \subset S_\mu \), and so \( S_\eta \) acts on \( \mathbb{C}^n \) by restriction. The isomorphism \( \mathbb{C}^n \cong \mathbb{C}^n \) is \( S_\eta \)-equivariant.

3 Gelfand-Tsetlin modules

For the rest of this article we fix \( n \in \mathbb{N} \) and set \( N = \frac{n(n+1)}{2} \).

3.1. For each \( k \in [n] \) we denote by \( U_k \) the enveloping algebra of \( \mathfrak{gl}(k, \mathbb{C}) \), and set \( U = U_n \). Inclusion of matrices in the top left corner induces a chain
\[
\mathfrak{gl}(1, \mathbb{C}) \subset \mathfrak{gl}(2, \mathbb{C}) \subset \cdots \subset \mathfrak{gl}(n, \mathbb{C}),
\]
which in turn induces a chain \( U_1 \subset U_2 \subset \cdots \subset U_n \). Denote by \( Z_k \) the center of \( U_k \) and by \( \Gamma \) the subalgebra of \( U \) generated by \( \bigcup_{k=1}^n Z_k \). This algebra is the Gelfand-Tsetlin subalgebra of \( U \), and it is generated by the elements
\[
c_{k,i} = \sum_{(r_1, \ldots, r_i) \in [k]^i} E_{r_1} \cdots E_{r_i}, \quad (k, i) \in \Sigma(\mu).
\]

By work of Zhelobenko there exists an isomorphism \( i : \Gamma \rightarrow \mathbb{C}[X_\mu]^{S_\mu} \), given by \( i(c_{k,i}) = \gamma_{k,i} \) where
\[
\gamma_{k,i} = \sum_{j=1}^{k} (x_{k,j} + k - 1)^j \prod_{m \neq j} \left( 1 - \frac{1}{x_{k,j} - x_{k,m}} \right),
\]
see [FGR16a, subsection 3.1] for details. It follows that \( \text{Specm} \Gamma \cong \mathbb{C}^n / S_\mu \), and so every \( \mu \)-point \( v \) induces a character \( \chi_v : \Gamma \rightarrow \mathbb{C} \) by setting \( c \in \Gamma \mapsto i(c)(v) \), and two \( \mu \)-points induce the same character if and only if they lie in the same \( S_\mu \)-orbit.

**Definition.** A finitely generated \( U \)-module \( M \) is called a Gelfand-Tsetlin module if
\[
M = \bigoplus_{m \in \text{Specm} \Gamma} M[m],
\]
where \( M[m] = \{ x \in M \mid m^k v = 0 \text{ for some } k \geq 0 \} \).

Let \( M \) be a Gelfand-Tsetlin module. Identifying \( m \) with the character \( \chi : \Gamma \rightarrow \Gamma / m \cong \mathbb{C} \), we also write \( M[\chi] \) for \( M[m] \). We say that \( \chi \) is a Gelfand-Tsetlin character of \( M \) if \( M[\chi] \neq 0 \) and define the multiplicity of \( \chi \) in \( M \) as \( \dim_{\mathbb{C}} M[\chi] \). The Gelfand-Tsetlin support of \( M \) is the set of all of its Gelfand-Tsetlin characters. We will often abbreviate Gelfand-Tsetlin for GT, or omit it completely when it is clear from the context.

3.2. Gelfand-Tsetlin tableaux. For the rest of this document we denote by \( \mu \) the composition \( (1, 2, \ldots, n) \) of \( N \). To each \( \mu \)-point we assign a triangular array
\[
T(v) = \begin{array}{ccccc}
v_{n,1} & v_{n,2} & \cdots & v_{n,n-1} & v_{n,n} \\
v_{n-1,1} & \cdots & v_{n-1,n-1} \\
\vdots & \ddots & \ddots & \ddots \\
v_{2,1} & v_{2,2} \\
v_{1,1}
\end{array}
\]
Such an array is known as a *Gelfand-Tsetlin tableau*. With this identification the group $S_n$ acts on the space of all tableaux. The main difference between the space of $\mu$-points and the space of all tableaux is that the first is obviously a $\mathbb{C}$-vector-space, while the second one is not. In particular $T(v + w) \neq T(v) + T(w)$, since the second expression does not make sense (though we will eventually consider a vector-space generated by tableaux).

A *standard* $\mu$-point is one in which $v_{k,i} - v_{k-1,i} \in \mathbb{Z}_{\geq 0}$ and $v_{k-1,i} - v_{k,i+1} \in \mathbb{Z}_{\geq 0}$ for all $1 \leq i < k \leq n$. We denote by $\mathbb{C}_\mu^{\text{std}}$ the set of all standard $\mu$-points. Notice that given $\lambda = (\lambda_1, \ldots, \lambda_n)$, there exists finitely many standard tableaux with top row equal to $\lambda - (0, 1, 2, \ldots, n - 1)$, and that this number is nonzero if and only if $\lambda_i - \lambda_{i+1} \in \mathbb{Z}_{\geq 0}$, i.e. $\lambda$ must be a dominant integral weight of $\mathfrak{gl}(n, \mathbb{C})$. This definition was introduced by Gelfand and Tsetlin in their article [GC50] in order to give an explicit presentation of irreducible $\mathfrak{gl}(n, \mathbb{C})$-modules.

**Theorem (GC50, [Zhe73])**. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a dominant integral weight of $\mathfrak{gl}(n, \mathbb{C})$, and let $V(\lambda)$ be the complex vector-space freely generated by the set

$$\{ T(v) \mid v \in \mathbb{C}_\mu^{\text{std}} \text{ and } v_{n,1} = \lambda_1, v_{n,2} = \lambda_2 - 1, \ldots, v_{n,n} = \lambda_n - n + 1 \}$$

(by convention, if $v$ is non-standard then $T(v) = 0$ in $V(\lambda)$). The vector-space $V(\lambda)$ can be endowed with a $\mathfrak{gl}(n, \mathbb{C})$-module structure, with the action of the canonical generators given by

$$E_{k,k+1}T(v) = - \sum_{i=1}^{k} \frac{\prod_{j=1}^{k-1} (v_{k,i} - v_{k+1,j})}{\prod_{j \neq i} (v_{k,i} - v_{k,j})} T(v + \delta_{k,i}),$$

$$E_{k+1,k}T(v) = \sum_{i=1}^{k} \frac{\prod_{j=1}^{k-1} (v_{k,i} - v_{k-1,j})}{\prod_{j \neq i} (v_{k,i} - v_{k,j})} T(v - \delta_{k,i}),$$

$$E_{k,k}T(v) = \left( \sum_{j=1}^{k} v_{k,j} - \sum_{j=1}^{k-1} v_{k-1,j} + k - 1 \right) T(v),$$

where $\delta_{k,i}$ is the element of $\mathbb{Z}^\mu$ with a 1 in position $(k, i)$ and 0’s elsewhere. Furthermore, for each $c \in \Gamma$ we have $c_{k,i}T(v) = \gamma_{k,i}(v)T(v)$.

The $U$-module $V(\lambda)$ is an irreducible finite dimensional representation of maximal weight $\lambda$, so this theorem provides an explicit presentation of all finite dimensional simple $\mathfrak{gl}(n, \mathbb{C})$-modules. The last statement of the theorem, giving the action of the generators of $\Gamma$ on $V(\lambda)$, is due to Zhelobenko.

**3.3.** Given $1 \leq i < k < n$ we set

$$e_{k,i}^+ = \frac{\prod_{j=1}^{k-1} (x_{k,j} - x_{k+1,j})}{\prod_{j \neq i} (x_{k,i} - x_{k,j})}, \quad e_{k,i}^- = \frac{\prod_{j=1}^{k-1} (x_{k,i} - x_{k-1,j})}{\prod_{j \neq i} (x_{k,i} - x_{k,j})},$$

which are elements of $\mathbb{C}(X_\mu)$. These are the rational functions that appear in Gelfand and Tsetlin’s presentation of finite dimensional irreducible $U$-modules, and we refer to them as the Gelfand-Tsetlin functions.
A \( \mu \)-point \( v \), or equivalently the corresponding GT tableaux, is called generic if 
\( v_{ik} - v_{ij} \notin \mathbb{Z} \) for all \( 1 \leq i < j \leq k < n \), otherwise it is called singular. Notice that a 
generic tableau may have entries in the top row whose difference is an integer.

Let \( \mathcal{Z}_0^\mu \) be the set of all integral \( \mu \)-points with \( z_{n,i} = 0 \) for all \( 1 \leq i \leq n \) (i.e., the 
corresponding GT tableau has its top row filled with zeros). If \( v \) is generic then the 
set \( \{ v + z \mid z \in \mathcal{Z}_0^\mu \} \) contains no poles of the Gelfand-Tsetlin functions. This idea was 
used by Drozd, Futorny and Ovsienko in [DFO94] to give a \( U \)-module structure to the 
vector-space

\[
V(T(v)) = \langle T(v + z) \mid z \in \mathcal{Z}_0^\mu \rangle_{\mathbb{C}}.
\]

Since the action of \( U \) is given by the Gelfand-Tsetlin functions, each tableau \( T(v) \) is an 
eigenvector of \( \Gamma \) and hence \( V(T(v)) \) is a GT module with support \( \{ \chi_{v+z} \mid z \in \mathcal{Z}_0^\mu \} \) and 
all multiplicities equal to 1.

3.4. The main goal of this article is to give a nontrivial \( U \)-module structure to the 
space \( V(T(v)) \) for an arbitrary \( \mu \)-point \( v \). The approach from [DFO94] breaks down if 
\( v \) is not generic, but Futorny, Grantcharov and the first named author extended this 
construction to a subset of singular characters in [FGR16b, FGR16a], and in this article 
we extend their work to arbitrary \( \mu \)-points. In order to do this, we introduce a large 
\( U \)-module which serves as a formal model of a GT module.

Set \( K = \mathbb{C}(X_\mu) \) to be the field of rational functions over \( \mu \)-points. As mentioned 
above \( e_{ki}^\pm \in K \) for all \( 1 \leq i \leq k < n \). We set \( V_C \) to be the \( \mathbb{C} \)-vector-space with basis 
\( \{ T(z) \mid z \in \mathcal{Z}_0^\mu \} \), and \( V_K = K \otimes_{\mathbb{C}} V_C \). Since \( S_\mu \) acts naturally on both \( K \) and \( V_C \), it acts 
on \( V_K \) by the diagonal action, i.e. given \( \sigma \in S_\mu, f \in K \) and \( z \in \mathcal{Z}_0^\mu \) the action is the 
linear extension of \( \sigma \cdot f \otimes T(z) = \sigma(f) \otimes T(\sigma(z)) \). The group \( \mathcal{Z}_0^\mu \) also acts on \( \mathbb{C}[X_\mu] \), 
with \( \delta^{k,i} \cdot x_{ij} = x_{ij} + \delta_{k,i} \delta_{ij} \). This action extends to \( K \), and given \( f \in K, z \in \mathcal{Z}_0^\mu \) we 
sometimes write \( f(x + z) \) instead of \( z \cdot f \). The actions of \( S_\mu \) and \( \mathcal{Z}_0^\mu \) do not commute, 
since \( \sigma(z \cdot f) = \sigma(z) \cdot \sigma(f) \).

**Proposition.** The vector-space \( V_K \) has an \( S_\mu \)-equivariant \( U \)-module structure, with the action of 
the generators given by

\[
E_{k,k+1} T(z) = - \sum_{i=1}^{k} e_{k,i}^z T(z + \delta_{i,j});
\]

\[
E_{k+1,k} T(z) = \sum_{i=1}^{k} e_{k,i}^z T(z - \delta_{i,j});
\]

\[
E_{k,k} T(z) = \left( \sum_{j=1}^{k} (x_{k,j} + z_{k,j}) - \sum_{j=1}^{k-1} (x_{k-1,j} + z_{k-1,j}) + k - 1 \right) T(z).
\]

Furthermore, for each \( c \in \Gamma \) we have \( cT(z) = \iota(c)(x + z)T(z) \).

**Proof.** The proof that \( V_K \) is a \( U \)-module is identical to the proof of [Zad17, Proposition 1]. It follows from the definitions that \( \sigma \cdot e_{ki}^\pm = e_{ki}^\pm \sigma(i)^{(i)} \), and using this it is easy to 
check that the canonical generators of \( U \) act by \( S_\mu \)-equivariant operators. \( \square \)
We will refer to the $U$-module $V_k$ as the “big GT module”. The big GT module was introduced in [Zad17], where it was proved that certain sublattices can serve as universal models for modules with generic or 1-singular characters (1-singular here means that there is at most one pair of entries in the same $\mu$-block differing by an integer). In the following sections of this article we will extend this argument to arbitrary characters without any restriction.

4 Symmetrized divided difference operators

In this section we recall some results on divided difference operators, and introduce a symmetrized version of them. These operators will play a central role in the construction of the sublattices of the big GT module.

4.1. Throughout this section we fix $m \in \mathbb{N}$ and $\eta = (\eta_1, \ldots, \eta_r)$ a composition of $m$. Recall that $\mathbb{C}[X_\eta] = \mathbb{C}[x_{k,i} \mid (k, i) \in \Sigma(\eta)]$, on which $S_\eta$ acts by $\sigma(x_{k,i}) = x_{\sigma(k), \sigma(i)}$. We set $F = \mathbb{C}(X_\eta)$, the fraction field of $\mathbb{C}[X_\eta]$, with its obvious $S_\eta$-action.

Set
\[
\Delta_k = \prod_{1 \leq i < j \leq \eta_k} (x_{k,i} - x_{k,j}), \quad \Delta_\eta = \prod_{k=1}^r \Delta_k.
\]
If $f \in \mathbb{C}[X_\eta]$ is a polynomial such that $\sigma(f) = \text{sgn}(\sigma)f$ for all $\sigma \in S_\eta$ then $f = g\Delta_\eta$, with $g$ an $S_\eta$-invariant polynomial.

4.2. Divided difference operators. Since the action of $S_\eta$ extends to $F$ we can form the smash product $F\#S_\eta$, which is the $\mathbb{C}$-algebra whose underlying vector-space is $F \otimes \mathbb{C}[S_\eta]$ and product given by $(f \otimes \sigma) \cdot (g \otimes \tau) = f\sigma(g) \otimes \sigma\tau$ for all $f, g \in F$ and all $\sigma, \tau \in S_\eta$. We will usually omit the tensor product symbol when writing elements in $F\#S_\eta$, so $f\sigma$ stands for $f \otimes \sigma$. We must be careful to distinguish the action of $\sigma \in S_\eta$ on a rational function $f \in F$, denoted by $\sigma(f)$, from their product in $F\#S_\eta$, which is $\sigma \cdot f = \sigma(f)\sigma$.

Recall that for each $(k, i) \in \Sigma(\eta)$ we denote by $s_i^{(k)}$ the unique simple transposition in $S_\eta$ which acts on $\Sigma(\eta)$ by interchanging $(k, i)$ and $(k, i + 1)$, while leaving the other elements fixed. The divided difference associated to $s_i^{(k)}$ is
\[
\partial_i^{(k)} = \frac{1}{s_{k,i} - s_{k,i+1}} (id - s_i^{(k)}) \in F\#S_\eta.
\]
These elements satisfy the relations
\[
\begin{align*}
\partial_i^{(k)} = 0, & \quad \text{if } l \neq k \text{ or } |i - j| > 1; \\
\partial_i^{(k)}\partial_j^{(l)} = \partial_j^{(l)}\partial_i^{(k)} & \quad \text{for } 1 \leq i \leq k - 2.
\end{align*}
\]

Let $\sigma = \sigma^{(1)} \cdots \sigma^{(n)} \in S_n$. For each $k \in [n]$ we can write $\sigma^{(k)}$ as a reduced composition $s_{i_1}^{(k)} s_{i_2}^{(k)} \cdots s_{i_\ell}^{(k)}$, with $\ell(\sigma^{(k)}) = \ell$. We set $\partial_\sigma^{(k)} = \partial_{i_1}^{(k)} \cdot \partial_{i_2}^{(k)} \cdots \partial_{i_\ell}^{(k)}$, and $\partial_\sigma = \partial_{\sigma^{(1)}} \cdot \partial_{\sigma^{(2)}} \cdots \partial_{\sigma^{(n)}}$; the relations among the divided difference operators imply that $\partial_\sigma^{(k)}$ and hence $\partial_\sigma$, is independent of the reduced composition we choose for $\sigma^{(k)}$. 

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The equality $\ell(\sigma) + \ell(\tau) = \ell(\sigma\tau)$ holds if and only if the concatenation of a reduced composition of $\sigma$ with a reduced composition of $\tau$ is a reduced composition of $\sigma\tau$, and this implies that $\partial_\sigma \cdot \partial_\tau = \partial_{\sigma\tau}$. If equality does not hold then the concatenation is not reduced, which implies that in the product $\partial_\sigma \cdot \partial_\tau$ there must be a term of the form $\partial_i^{(k)} \cdot \partial_j^{(k)} = 0$. Thus for all $\sigma, \tau \in S_\eta$ we get

$$\partial_\sigma \cdot \partial_\tau = \begin{cases} \partial_{\sigma\tau} & \text{if } \ell(\sigma) + \ell(\tau) = \ell(\sigma\tau); \\ 0 & \text{otherwise.} \end{cases}$$

4.3. Divided differences are usually defined as operators on the polynomial algebra $\mathbb{C}[X_\eta]$, although they make sense over any $F$-vector space with an equivariant $S_\eta$-action. Since these operators play an important role in the sequel, we gather some of their basic properties in the following lemma.

**Lemma.** Let $w_\eta$ be the longest word in $S_\eta$. The following equalities hold in $F\#S_\eta$.

1. $\partial_i^{(k)} \cdot f = s_i^{(k)}(f)\partial_i^{(k)} + \partial_i^{(k)}(f)$ for all $(k, i) \in \Sigma(\eta), f \in F$.
2. $\partial_{w_\eta} \cdot f \partial_\sigma = \partial_{w_\eta} \cdot \partial_{e^{-1}}(f)$ for all $\sigma \in S_\eta, f \in F$.
3. $\frac{\eta!}{\Delta_\eta} \partial_{w_\eta} = \text{asym}_\eta = \text{sym}_\eta \cdot \frac{1}{\Delta_\eta}$

**Proof.** Item [1] is an easy computation following from the definition. Now set $s = s_i^{(k)}$ and $\partial_s = \partial_i^{(k)}$. By definition $\partial_s \cdot s = -\partial_s$ and $s \cdot \partial_s = \partial_s$, and since $w_\eta = (w_\eta s)s$ with $\ell(w_\eta s) = \ell(w_\eta) - 1$ we get

$$\partial_{w_\eta} \cdot s = \partial_{w_\eta s} \cdot \partial_s \cdot s = -\partial_{w_\eta s} \cdot \partial_s = -\partial_{w_\eta}.$$ 

This along with the previous item gives

$$0 = \partial_{w_\eta} \cdot (\partial_s \cdot f) = \partial_{w_\eta} \cdot (s(f)\partial_s + \partial_{w_\eta} \cdot \partial_s(f)$$

$$= \partial_{w_\eta} \cdot (s \cdot fs) \cdot \partial_s + \partial_{w_\eta} \cdot \partial_s(f) = -\partial_{w_\eta} \cdot f \partial_s + \partial_{w_\eta} \cdot \partial_s(f)$$

which proves that item [2] holds for $\sigma = s$. Since $k$ and $i$ are arbitrary, the general case follows by induction on the length of $\sigma$.

A similar argument as above shows that $\sigma \cdot \partial_{w_\eta} = \partial_{w_\eta}$ for all $\sigma \in S_\eta$. Induction on the length of $\sigma$ shows that $\partial_\sigma = \sum_{\tau \in S_\eta} \frac{1}{f_{r,\tau}} \tau$ with $f_{r,\sigma} \in \mathbb{C}[X_\eta]$ homogeneous of degree $\ell(\sigma)$. If we put $f_\sigma = f_{r,\sigma}$, the equalities $\sigma \cdot \partial_{w_\eta} = \partial_{w_\eta}$ and $\partial_{w_\eta} \cdot \sigma = \text{sg}(\sigma)\partial_{w_\eta}$ imply that $f_\sigma = \sigma(f_\sigma) = \text{sg}(\sigma)f_\sigma$, so $f_\sigma$ is an $S_\eta$-antisymmetric polynomial of degree $\ell(w_\eta)$, i.e. a scalar multiple of $\Delta_\eta$. Now $\partial_{w_\eta}(\Delta_\eta) = \eta!$, so $f_\sigma = \frac{\eta!}{\Delta_\eta}$ and

$$\partial_{w_\eta} = \frac{\eta!}{\Delta_\eta} \sum_{\sigma \in S_\eta} \text{sg}(\sigma)\sigma = \eta! \sum_{\sigma \in S_\eta} \sigma \cdot \frac{1}{\Delta_\eta}.$$ 

This completes the proof of item [3].
Many subalgebras of $F$ are stable by the action of divided differences. It is a well-known fact that this is the case for $\mathbb{C}[X_\eta]$. Assume $\mathbb{C}[X_\eta] \subset A \subset F$ is closed under the action of $S_\eta$. Then any rational function in $A$ can be written as a quotient $p/q$ with $p, q \in \mathbb{C}[X_\eta]$ and $q$ $S_\eta$-invariant, so $\partial_\alpha(f) = \partial_\alpha(p/q) = \partial_\alpha(p)/q \in A$ for each $\alpha \in S_\eta$ and $A$ is also closed under divided differences.

4.4. We focus now on the action of divided differences over the polynomial ring $\mathbb{C}[X_\eta]$. First, for every $\sigma \in S_\eta$ the operator $\partial_\sigma : \mathbb{C}[X_\eta] \rightarrow \mathbb{C}[X_\eta]$ is homogeneous of degree $-\ell(\sigma)$. Now let $p_\eta$ be the ideal generated by $\{x_{k,j} - x_{i,j} | (k,i),(j,k) \in \Sigma(\eta)\}$, and let $\mathbb{C}[p_\eta] \subset \mathbb{C}[X_\eta]$ be the algebra generated by $p_\eta$; clearly $\Delta_\eta \in p_\eta$. Since $\partial_{\eta}^{(k)}(x_{i,j} - x_{i,k}) \in \mathbb{Z}$, the algebra $\mathbb{C}[p_\eta]$ is stable by the action of divided differences. It follows that $\partial_\alpha \Delta_\eta \in p_\eta$ if $\ell(\alpha) \leq \deg \Delta_\eta = \eta!$, while $\partial_{w_\eta} \Delta_\eta = \eta!$.

Let $\sigma,\tau \in S_\eta$. By item 2 of Lemma 4.3 we have

$$\frac{1}{\eta!} \partial_{w_\eta}((\partial_\tau \Delta_\eta)(\partial_\sigma \Delta_\eta)) = \frac{1}{\eta!} \partial_{w_\eta}(\Delta(\partial_{\tau-1} \partial_\sigma \Delta_\eta)) = \text{sym}_\eta(\partial_{\tau-1} \partial_\sigma \Delta_\eta).$$

Now if $\ell(\tau^{-1}) + \ell(\sigma) \geq \ell(w_\eta)$ this is 0 unless $\tau^{-1} \sigma = w_\eta$, in which case it equals $\eta!$. If $\ell(\tau^{-1}) + \ell(\sigma) < \ell(w_\eta)$ then the best we can say is that this element lies in $p_\eta^{\delta_\eta}$. Consider a total order $<$ on $S_\eta$ such that $\ell(\sigma) < \ell(\tau)$ implies $\sigma < \tau$, and use this to index the rows and columns of matrices in $M_{n!(F)}$ by elements in $S_\eta$. For each $X \in M_{n!(F)}$ write $X^\sigma_\tau$ for the entry in the $\sigma$-th row and the $\tau$-th column, so if $Y$ is another matrix then $(XY)^\sigma_\tau = \sum_{\rho \in S_\eta} X^\sigma_\rho Y^\rho_\tau$. Consider the matrices $X,Y \in M_{n!(F)}$ defined by $X^\sigma_\tau = \tau \left(\frac{\partial_\sigma \Delta_\eta}{\eta! \Delta_\eta}\right)$ and $Y^\sigma_\tau = \frac{1}{\eta!} \rho(\partial_{w_\eta} \Delta_\eta)$. Then

$$(XY)^\sigma_\tau = \frac{1}{\eta!} \sum_{\tau \in S_\eta} \tau \left(\frac{\partial_\tau \Delta_\eta}{\partial_{w_\eta} \Delta_\eta} \right) = \frac{1}{\eta!} \partial_{w_\eta}((\partial_\tau \Delta_\eta)(\partial_{w_\eta} \Delta_\eta)).$$

By the previous discussion $XY$ is an upper triangular matrix with ones in the diagonal, and the nonzero elements in the upper-triangular part lie in $p_\eta^{\delta_\eta}$. From this we deduce that $X$ is invertible, and its inverse is of the form $Y + C$ with $C$ a matrix with entries in the ideal generated by $p_\eta^{\delta_\eta}$.

The rational function $X^\sigma_\tau$ is homogeneous of degree $-\ell(\sigma)$, hence its determinant is also homogeneous of degree $-\sum_{\alpha \in S_\eta} \ell(\alpha) = -\eta!$. This implies that $\deg(Y + C)^\sigma_\tau = \eta! + \ell(\tau) - \ell(\sigma) = \deg Y^\sigma_\tau$, so the entries of $C$ are also homogeneous polynomials. Also since $X^\sigma_\tau = \tau(X^\sigma_\tau)$ and $Y^\sigma_\tau = \rho(Y^\sigma_\tau)$, we must have $C^\sigma_\tau = \rho(C^\sigma_\tau)$. We summarize these results in the following lemma.

**Lemma.** For each $\sigma \in S_\eta$ set $(\partial_\sigma \Delta)^* = (Y + C)^{\delta_\eta}_{n-1}$, and let $I$ be the ideal generated by $p_\eta^{\delta_\eta}$ in $\mathbb{C}[X_\eta]$. Then $(\partial_\sigma \Delta)^*$ is a homogeneous polynomial of degree $\ell(\sigma)$, and $(\partial_\sigma \Delta)^* \equiv \frac{1}{\eta} \partial_{\sigma^{-1} \tau} \Delta_\eta$ mod $I$. Furthermore $(Y + C)^{\delta_\eta}_{n-1} = \tau((\partial_\tau \Delta)^*)$ for each $\tau \in S_\eta$.

4.5. There is a second family of elements in $F\#S_\eta$ that we need to distinguish before going on. For each $\sigma \in S_\eta$ we define the symmetrized divided difference operator $D^\sigma_\tau = \frac{1}{\eta!}$.
sym. It follows from the discussion at the end of 4.3 that if $A \subset F$ is a $\mathbb{C}[X_\eta]$-algebra stable by the action of $S_\eta$ then $D^\eta(A) \subset A$.

**Proposition.** For each $\sigma \in S_\eta$ we have

$$D^\eta = D^\eta_w \cdot \partial_{\eta-1} \Delta_\eta = \frac{1}{\eta!} \sum_{\tau \in S_\eta} \tau \left( \frac{\partial_{\tau-1} \Delta_\eta}{\Delta_\eta} \right),$$

$$\sigma = \sum_{\tau \in S_\eta} \sigma((\partial_\tau \Delta_\eta)^*) D^\eta_\tau.$$ 

**Proof.** By item 3 of Lemma 4.3, $D^\eta_w = \frac{1}{\eta!} \partial_{w_\eta} \cdot \Delta_\eta$, so by item 2 of the same lemma

$$D^\eta = \frac{1}{\eta!} \partial_{w_\eta} \cdot \Delta_\eta \partial_\sigma = \frac{1}{\eta!} \partial_{w_\eta} \cdot \partial_{\eta-1} \Delta_\eta = \text{sym} \eta \cdot \frac{\partial_\sigma \Delta_\eta}{\Delta_\eta}.$$ 

Using the fact that $\tau \cdot f = \tau(f) \tau$ the first equality follows. In terms of the matrices $X, Y$ from the previous paragraph, this says that $D^\eta = \sum_{\tau \in S_\eta} X^{-1}_{\tau} \tau$. Since $X$ is invertible with inverse $Y + C$, we get that

$$\sigma = \sum_{\tau \in S_\eta} (Y + C)^\sigma_{\tau-1} D^\eta_\tau$$

which is the second equality. \[\square\]

As a nice application of this proposition notice that for each $f \in F$

$$f = \sum_{\sigma \in S_\eta} D^\eta(f)(\partial_\sigma \Delta_\eta)^*.$$ 

In particular, if $A$ is a $\mathbb{C}[X_\mu]$-subalgebra of $F$ and it is stable by the action of $S_\eta$ then the set $\{(\partial_\sigma \Delta_\eta)^* \mid \sigma \in S_\eta\}$ is a basis of $A$ over $A^S_\eta$, and symmetrized divided differences give the coefficients of each element in this basis.

## 5 Singular GT modules

Recall that we have fixed $n \in \mathbb{N}$ and $U = U(\mathfrak{gl}(n, \mathbb{C}))$. Also, we set $N = \frac{n(n+1)}{2}$ and we put $\mu = (1,2,\ldots,n)$ and we denote by $\mathbb{Z}^\mu_0$ the set of all integral $\mu$-points with $z_{n,i} = 0$ for all $1 \leq i \leq n$. Recall that if $\eta$ is a refinement of $\mu$ then every $\mu$-point can be seen as an $\eta$-point, and we may speak freely of the $\eta$-blocks of a $\mu$-point. Finally, recall $K = \mathbb{C}(X_\mu)$

**5.1.** To each $\mu$-point $v$, or to its corresponding tableau $T(v)$, we associate a refinement of $\mu$ which we denote by $\eta(v)$ which will act as a measure of how far is a tableau from being generic. For any $k \in \llbracket n-1 \rrbracket$ form a graph with vertices $\llbracket k \rrbracket$, and put an edge between $i$ and $j$ if and only if $v_{k,i} - v_{k,j} \in \mathbb{Z}$; the resulting graph is the disjoint union of complete graphs, and we set $\eta^{(k)}$ to be the cardinalities of each connected
component arranged in descending order. Finally we set \( \eta(v) = (\eta^{(1)}, \ldots, \eta^{(n-1)}, 1^n) \), where \( 1^n \) denotes the composition of \( n \) consisting of \( n \) ones. Thus if \( v \) is generic then \( \eta^{(k)}(v) = 1^k \), and if it is a singular tableau then \( \eta(v) \) will have at least one part larger than 1.

**Definition.** Given \( v \in \mathbb{C}^\mu \) the composition \( \eta(v) \) is called the singularity of \( v \). Given \( \theta \) a refinement of \( \mu \), we will say that \( v \) is \( \theta \)-singular if \( \eta(v) = \theta \).

Recall that characters of the GT subalgebra \( \Gamma \) are in one to one correspondence with \( S_\mu \)-orbits of \( \mathbb{C}^\mu \). Now if \( v \in \mathbb{C}^\mu \) and \( \sigma \in S_\mu \) then \( \eta(v) = \eta(\sigma(v)) \), so the singularity of a character is well-defined. We say that \( v \) is in normal form if \( v_{k,i} - v_{k,j} \in \mathbb{Z}_{\geq 0} \) implies that \( v_{k,i} \) and \( v_{k,j} \) lie in the same \( \eta(v) \)-block of \( v \) and that \( i > j \).

We denote by \( \text{st}(v) \) the stabilizer of \( v \) in \( S_\eta \). We say that \( v \) is fully critical if it is in normal form and \( \text{st}(v) = S_\eta \), or in other words if \( v_{k,i} - v_{k,j} \in \mathbb{Z} \) implies \( v_{k,i} = v_{k,j} \). We will say that \( v \) is fully \( \eta \)-critical if it is both \( \eta \)-singular and fully critical. By definition every character has at least one representative in normal form (though it may have many), and if \( v \in \mathbb{C}^\mu \) is in normal form then there exists \( z \in \mathbb{Z}^\mu_0 \) such that \( v + z \) is fully critical.

**Example.** Suppose \( n = 5 \) and let \( v \in \mathbb{C}^\mu \) be the point whose corresponding tableau is

\[
T(v) = \begin{array}{ccccc}
* & * & * & * & * \\
 a & b - 1 & b & a + 1 \\
 c & c + 1 & d \\
e & e & & & f \\
\end{array}
\]

where \( a, b, c, d, e, f \in \mathbb{C} \) are \( \mathbb{Z} \)-linearly independent. Its singularity is given by the refinement \( \eta(v) = ( (1), (2), (2, 1), (2, 2), (1, 1, 1, 1, 1) ) \). It is not in normal form, since for example in the third row \( c \) is to the left of \( c + 1 \), and in the fourth row the entries differing by integers are not organized in \( \eta(v) \)-blocks. The \( \mu \)-points \( v', v'' \) whose tableaux are

\[
\begin{array}{ccccc}
* & * & * & * & * \\
 a + 1 & a & b & b - 1 \\
 c + 1 & c & d \\
e & e & & & f \\
\end{array}
\]

\[
\begin{array}{ccccc}
* & * & * & * & * \\
 b & b - 1 & a + 1 & a \\
 c + 1 & c & d \\
e & e & & & f \\
\end{array}
\]

are in normal form, and since both are in the \( S_\mu \)-orbit of \( v \) they define the same character of \( \Gamma \). None of these \( \mu \)-points is fully critical, but it is clear that we may obtain a fully critical \( \mu \)-point by adding a suitable \( z \in \mathbb{Z}^\mu_0 \) to either \( v' \) or \( v'' \). Notice that in all these cases the entries in the top row are irrelevant.

5.2. Let \( C = \{ x_{k,i} - x_{k,j} - z \mid 1 \leq i < j \leq k < n, z \in \mathbb{Z} \setminus \{0\} \} \subset \mathbb{C}[X_\mu] \). We put \( B = C^{-1}\mathbb{C}[X_\mu] \). Let \( v \in \mathbb{C}^\mu \) be fully critical, and fix \( \eta = \eta(v) \). We denote by \( B_\eta \) the localization of \( B \) at the set of all \( x_{k,i} - x_{k,j} \) such that \( (k,i) \) and \( (k,j) \) lie in different orbits of \( S_\eta \). The algebra \( B_\eta \) is closed under the action of \( S_\eta \), and hence by the discussion at
the end of paragraph 4.3 if \( f \in B_\eta \) then \( D_\eta^0(f) \in B_\eta \).

**Definition.** We set \( L_\eta \subseteq V_K \) to be the \( B_\eta \)-span of \( \{D_\eta^0 T(z) \mid \sigma \in S_\eta, z \in \mathbb{Z}_0^\mu \} \).

We will now prove that \( L_\eta \) is a \( U \)-submodule of \( V_K \), but in order to do this we first need to show that it is a free \( B_\eta \)-module.

Notice that if \( z \in \mathbb{Z}_0^\mu \) then \( \mu \)-point \( v + z \) is in normal form if and only if each \( \eta \)-block of \( z \) is a descending sequence, so the set \( \mathcal{N}_\eta = \{ z \in \mathbb{Z}_0^\mu \mid v + z \text{ is in normal form} \} \) depends only of \( \eta \) and not of \( v \). Also, there is exactly one element in the orbit of \( z \) which lies in \( \mathcal{N}_\eta \), so this set is a family of representatives of \( \mathbb{Z}_0^\mu / \mathcal{N}_\eta \). We denote by \( \text{st}(z) \) the stabilizer of \( z \) in \( \mathcal{N}_\eta \). If \( z \in \mathcal{N}_\eta \) then \( \text{st}(z) = S_\epsilon(z) \) with \( \epsilon(z) \) a refinement of \( \eta \).

**Lemma.** The set \( B = \{ D_\eta^0 T(z) \mid z \in \mathcal{N}_\eta, \sigma \in \text{Shuffle}_\epsilon^\eta(z) \} \) is a basis of \( L_\eta \) as \( B_\eta \)-module.

**Proof.** For each \( z \in \mathbb{Z}_0^\mu \) we denote by \( O(z) \) the \( K \)-vector-space generated by \( \{ T(\sigma(z)) \mid \sigma \in S_\eta \} \), and by \( D(z) \) the \( B_\eta \)-module generated by \( \{ D_\eta^0 T(z) \mid \sigma \in S_\eta \} \). Clearly \( L_\eta = \sum_{z \in \mathbb{Z}_0^\mu} D(z) \). By Proposition 4.3

\[
D_\eta^0 T(z) = \frac{1}{\eta!} \sum_{\tau \in S_\eta} \tau \left( \frac{\partial_{\tau^{-1}}\Delta_\eta}{\Delta_\eta} \right) T(\tau(z))
\]

\[
T(\sigma(z)) = \sum_{\tau \in S_\eta} \sigma((\partial_{\tau}\Delta_\eta)^*) D_\eta^0 T(z),
\]

so \( D(z) \subseteq O(z) \) and \( O(z) \subseteq K \otimes_{B_\epsilon} D(z) \).

Let \( v \in V_K \) be \( S_\eta \)-invariant and let \( s \in S_\eta \) be a simple transposition. Then for each \( f \in K \) we have \( \partial_s(fv) = \partial_s(f)s(v) + f\partial_s(v) = \partial_s(f)v \). A simple induction shows that \( \partial_v(fv) = \partial_v(f)v \), and hence \( D_\eta^0(fv) = D_\eta^0(f)v \) for all \( \sigma \in S_\eta \). Thus for each \( v \in S_\eta \)

\[
D_\eta^0 T(\sigma(z)) = D_\eta^0 \left( \sum_{\tau \in S_\eta} \sigma((\partial_{\tau}\Delta_\eta)^*) D_\eta^0 T(z) \right) = \sum_{\tau \in S_\eta} D_\eta^0(\sigma((\partial_{\tau}\Delta_\eta)^*) D_\eta^0 T(z),
\]

which shows that \( D(z) = \mathcal{D}(\tau(z)) \), and hence \( L_\eta = \sum_{z \in \mathcal{N}_\eta} D(z) \). Since \( D(z) \subseteq O(z) \), the sum is direct and \( L_\eta = \bigoplus_{z \in \mathcal{N}_\eta} D(z) \). Hence to prove the statement it is enough to show that for each \( z \in \mathcal{N}_\eta \) the \( B_\eta \)-module \( D(z) \) is free with basis \( B(z) = B \cap D(z) \).

Let \( z \in \mathcal{N}_\eta \) and put \( \epsilon = \epsilon(z) \). If \( \sigma \in S_\eta \) is not an \( \epsilon \)-shuffle then \( \sigma \) can be written as \( \delta s \) with \( s \in S_\epsilon \) a simple transposition. Since \( s T(z) = T(z) \) we get

\[
D_\eta^0 T(z) = \text{sym}_\eta(\partial_v(\partial_s T(z))) = 0.
\]

This implies that \( B(z) \) generates \( D(z) \) over \( B_\eta \), and since \( O(z) = K \otimes_{B_\eta} D(z) \) it also generates \( O(z) \) over \( K \). The set \( \text{Shuffle}_\epsilon^\eta \) is a complete set of representatives of \( S_\eta / S_\epsilon \), so \( \#B(z) = \#\text{Shuffle}_\epsilon^\eta = \eta! / \epsilon! \). On the other hand \( \dim_K O(z) = \#(S_\eta / S_\epsilon) = \eta! / \epsilon! \), so \( B(z) \) is a basis of \( O(z) \), and in particular it is linearly independent over \( B_\eta \).

We refer to the elements of \( B \) as derived tableaux, and the elements in \( B(z) \) as the derived tableaux of \( T(z) \).
5.3 Theorem. The $B_\eta$-lattice $L_\eta$ is a $U$-submodule of $V_K$.

Proof. Let $z \in \mathcal{N}_\eta$ and let $\epsilon$ be the unique refinement of $\eta$ such that $S_{\epsilon} \subset S_\eta$ is the stabilizer of $z$ in $S_\eta$. For each $k \in [n]$ the $K$-vector-space $O(z)$ consists of eigenvectors of $E_{k,k}$ with the same eigenvalue; since every derived tableaux of $T(z)$ lies in $O(z)$, we get that $E_{k,k}L_\eta \subset L_\eta$. Hence we only need to show that $EL_\eta \subset L_\eta$ for $E \in \{E_{k,k+1}, E_{k+1,k} \mid 1 \leq k \leq n-1\}$.

It follows from the definitions that $\Delta_\epsilon e^\epsilon_{k,n}(x+z) \in B_\eta$ for each $k \in [n-1]$, so $\Delta_\epsilon ET(z)$ is a linear combination of tableaux $T(w)$. Now since $T(w) \in L_\eta$ for all $w \in \mathbb{Z}_0^n$ and $B$ is a basis of $L_\eta$, we can write

$$ET(z) = \sum_{D^\eta_{\Delta_\epsilon}(w) \in B} \frac{f_{\epsilon,w}}{\Delta_\epsilon} D^\eta_{\Delta_\epsilon} T(w)$$

with $f_{\epsilon,w} \in B_\eta$ unique.

Since the action of $U$ is $S_\eta$-equivariant and $T(z)$ is stable by $S_\epsilon$, the same is true for $ET(z)$, and hence the right hand side of the equation is also $S_\epsilon$-invariant. Since derived tableaux are $S_\eta$-invariant we see that $\frac{f_{\epsilon,w}}{\Delta_\epsilon}$ is an $S_\epsilon$-invariant element of $K$, and hence $\tau(f_{\epsilon,w}) = sg(\tau)f_{\sigma,\epsilon \tau}$ for each $\tau \in S_\epsilon$. This implies that $f_{\epsilon,w} = g_{\epsilon,w} \Delta_\epsilon$, with $g_{\epsilon,w} \in B_\eta\epsilon$, and hence $ET(z) \in L_\eta$. Finally, since the action of $U$ is both $K$-linear and $S_\eta$-equivariant we obtain

$$ED^\eta_{\Delta_\epsilon}(z) = D^\eta_{\Delta_\epsilon}(ET(z)) = \sum_{D^\eta_{\Delta_\epsilon}(w) \in B} D^\eta_{\Delta_\epsilon}(g_{\epsilon,w}) D^\eta_{\Delta_\epsilon} T(w).$$

Now $B_\eta$ is stable by the action of $S_\eta$ and closed under divided differences, so $D^\eta_{\Delta_\epsilon}(g_{\epsilon,w}) \in B_\eta$ and $ED^\eta_{\Delta_\epsilon}(z) \in L_\eta$. 

\[\square\]

5.4. Let $v \in \mathbb{C}^n$ be an $\eta$-critical point. Then there is a well-defined map $\pi_\eta : B_\eta \rightarrow \mathbb{C}$ given by $\pi_\eta(f) = f(v)$, and it is clear by definition that $\pi_\eta \subset \ker \pi_\eta$. From this we obtain a one-dimensional representation of $B_\eta$, which we denote by $\mathbb{C}_v$. We fix a nonzero element $1_\eta \in \mathbb{C}_v$.

Definition. Let $v \in \mathbb{C}^n$ be an $\eta$-critical point. We define $V(T(v))$ to be the complex vector-space $\mathbb{C}_v \otimes_{B_\eta} L_\eta$, with the $U$-module structure given by the action of $U$ on $L_\eta$.

Given $z \in N$ and $\sigma \in \text{Shuffle}_{\epsilon(z)}^\eta$ we write $\overline{D}_{\epsilon(z)}^\eta(v+z) = 1_\eta \otimes D_{\epsilon(z)}^\eta(v)$. It follows from Lemma that the set $\{\overline{D}_{\epsilon(z)}^\eta(v+z) \mid z \in N_\eta, \sigma \in \text{Shuffle}_{\epsilon(z)}^\eta\}$ is a basis of $V(T(v))$ as a complex vector-space.

Example. Fix $n = 4$, let $\eta = (1,1^2,3,1^4)$ and let $v$ be any $\eta$-singular $\mu$-point. The following table shows the nonzero derived tableaux of $T(v+z)$, classified according to the composition $\epsilon(z); we always assume $a > b > c$. 

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5.5. We finish with the proof that \( V(T(v)) \) is always a Gelfand-Tsetlin module, and we find its support along with the multiplicity of each character. We need the following preliminary result.

**Lemma.** Let \( \sigma, \nu, \tau \in S_\eta \). Then \( d_{v, \tau}^\nu = D_\nu^0 ((\partial_\sigma \Delta_\eta)^* (\partial_\tau \Delta_\eta)^*) \equiv 0 \mod p_\eta \) unless \( \ell(\sigma) + \ell(\tau) = \ell(v) \). Furthermore \( d_{v, id}^\nu = d_{id, v}^\nu = 1 \).

**Proof.** Recall that \((\partial_\sigma \Delta_\eta)^* = \frac{1}{\eta!} \partial_{\sigma^{-1} 1 \nu} \Delta_\eta + c_{\sigma^{-1} 1 \nu} \) with \( c_{\sigma^{-1} 1 \nu} \) lying in the ideal generated by \( p_\eta \), and that this is a homogeneous polynomial of degree \( \ell(\sigma) \). Recall also that the algebra \( \mathbb{C}[p_\eta] \) generated by \( p_\eta \) is closed by the action of the symmetrized divided difference operators, and that a polynomial in \( \mathbb{C}[p_\eta] \) lies in \( p_\eta \) if and only if it is of strictly positive degree.

By definition \( \deg(\partial_\sigma \Delta_\eta)^* (\partial_\tau \Delta_\eta)^* = \ell(\sigma) + \ell(\tau) \). This implies that if \( \ell(v) > \ell(\sigma) + \ell(\tau) \) then \( d_{v, \tau}^\nu = 0 \), so we may assume from now on that \( \ell(v) \leq \ell(\sigma) + \ell(\tau) \). Now if \( f \in p_\eta \), then \( D_\nu^0 (fg) = f D_\nu^0 (g) \) for any \( g \in \mathbb{C}[X_{\mu}] \), so \( d_{v, \tau}^\nu \equiv \frac{1}{\eta!^2} D_\nu^0 ((\partial_{\nu^{-1} 1 \nu} \Delta_\eta)(\partial_{\tau^{-1} 1 \nu} \Delta_\eta)) \mod p_\eta \).

The polynomial in the right hand side of this congruence lies in \( \mathbb{C}[p_\eta] \), and if \( \ell(v) < \ell(\sigma) + \ell(\tau) \) then its degree is positive and hence it lies in \( p_\eta \). On the other hand \( d_{v, id}^\nu = d_{id, v}^\nu = \frac{1}{\eta!^2} \text{sym}_\nu \partial_\nu ((\partial_{\nu^{-1} 1 \nu} \Delta_\eta)(\partial_{\nu \nu} \Delta_\eta)) = \frac{1}{\eta!^2} \text{sym}_\nu (\partial_{\nu \nu} \Delta_\eta)^2 = 1 \).

\[ \square \]

5.6 **Theorem.** Let \( v \in \mathbb{C}^\mu \) be a critical \( \mu \)-point, and set \( \eta = \eta(v) \). The \( U \)-module \( V(T(v)) \) is a Gelfand-Tsetlin module whose support is the set \( \{ \chi_{v+z} \mid z \in \mathcal{N}_\eta \} \). Furthermore the multiplicity of \( \chi_{v+z} \) in \( V(T(v)) \) is \( \eta! / \varepsilon(z) \).

**Proof.** Fix \( c \in \Gamma \) and let \( \gamma = i(c) \). Since \( cT(z) = \gamma(x+z)T(z) \) for each \( z \in \mathbb{Z}_0^\mu \), we see that \( cD_\nu^0 T(z) = D_\nu^0 (\gamma(x+z)T(z)) \).

Now using Proposition 4.5

\[ \gamma(x+z) = \sum_{\sigma \in S_\eta} (\partial_\sigma \Delta_\eta)^* D_\sigma (\gamma(x+z)), \]

\[ T(z) = \sum_{\tau \in S_\eta} (\partial_\tau \Delta_\eta)^* D_\tau T(z), \]

| \(z_3\) | \(e(z)^{(3)}\) | Nonzero derived tableaux |
|---|---|---|
| (a, b, c) | (1, 1, 1) | \( D_{id}^0(v+z), D_{(12)}^0(v+z), D_{(23)}^0(v+z), D_{(13)}^0(v+z) \) |
| (a, a, b) | (2, 1) | \( D_{id}^0(v+z), D_{(12)}^0(v+z), D_{(23)}^0(v+z) \) |
| (a, a, b) | (1, 2) | \( D_{id}^0(v+z), D_{(12)}^0(v+z), D_{(13)}^0(v+z) \) |
| (a, a, a) | (3) | \( D_{id}^0(v+z) \) |
and plugging this in \( D^\eta_\theta (\gamma(x+z)T(z)) \) we get
\[
cD^\eta_\theta T(z) = \sum_{\sigma, \tau \in S_\eta} D^\eta_\theta ((\partial_{\tau} \Delta_\eta)^* (\partial_{\tau} \Delta_\eta)^*) D^\eta_\theta (\gamma(x+z)) D^\eta_\theta T(z).
\]

By Lemma 5.4 we get
\[
cD^\eta_\theta T(z) \equiv \text{sym}_\eta (\gamma(x+z)) D^\eta_\theta T(z) \\
+ \sum_{\ell(\sigma) + \ell(\tau) = \ell(\nu)} d^\nu_{\sigma, \tau} D^\eta_\theta (\gamma(x+z)) D^\eta_\theta T(z) \mod p_\eta L_\eta.
\]

and using the fact that \( p_\eta \subset \text{ker} \pi_v \) and that \( \pi_v (\text{sym}_\eta f) = f(\nu) \) for each \( f \in \mathbb{C}[X_\eta] \),
\[
cD^\eta_\theta (v + z) = \gamma(v + z) D^\eta_\theta (v + z) \\
+ \sum_{\ell(\sigma) + \ell(\tau) = \ell(\nu)} d^\nu_{\sigma, \tau} \pi_v(D^\eta_\theta (\gamma(x+z))) D^\eta_\theta T(v + z).
\]

This shows that \((c - \gamma(v+z))^{\ell(\nu)} D^\eta_\theta (v + z) = 0 \) and hence \( D^\eta_\theta (v + z) \in V(T(\nu))|_{\chi_{v+z}} \). If it follows that
\[
V(T(\nu))|_{\chi_{v+z}} = (D^\eta_\theta (v + z) \mid v \in \text{Shuffle}^\eta_{(z)}) \subset \mathbb{C}
\]
so the multiplicity of \( \chi_{v+z} \) in \( V(T(\nu)) \) is \( \eta! / e(z)! \).

5.7. Remark. Let \( \theta \) be a refinement of \((1,2,\ldots, n-1,1^\eta)\). Let \( v \in \mathbb{C}^\mu \) be fully critical and suppose \( \eta = \eta(v) \) is a refinement of \( \theta \). Then \( B_\theta \subset B_\eta \) and \( C_\theta \) is a \( B_\eta \)-module by restriction, so we get a \( U \)-module by setting \( W(T(\nu)) = C_\theta \otimes_{B_\eta} L_\theta \). It is natural to ask whether \( W(T(\nu)) \) is equal to \( V(T(\nu)) \).

The answer to this question is yes. Since \( S_\eta \subset S_\theta \) the elements \( D^\theta_\eta (T(z)) \) are \( S_\eta \)-invariant for each \( \tau \in S_\theta \), and hence
\[
D^\eta_\theta T(z) = \sum_{\tau \in S_\eta} D^\eta_\theta ((\partial_{\tau} \Delta_\theta)^*) D^\eta_\theta T(z) \in L_\eta.
\]

This implies that \( L_\eta \subset B_\eta \otimes_{B_\theta} L_\theta \).

Now if \( f \in B_\eta \) then there exist \( f_{\sigma} \in B_\eta^S_\eta \) such that \( f = \sum_{\sigma \in S_\eta} f_{\sigma} \partial_{\tau-1} \Delta_\eta \), and so
\[
\text{sym}_\eta \left( \frac{f}{\Delta_\eta} T(z) \right) = \sum_{\sigma \in S_\eta} f_{\sigma} \text{sym}_\eta \left( \frac{\partial_{\tau-1} \Delta_\eta}{\Delta_\eta} T(z) \right) = \sum_{\sigma \in S_\eta} f_{\sigma} D^\theta_\eta T(z) \in L_\eta.
\]

for each \( z \in \mathbb{Z}_\eta^\mu \). Now let \( R \subset S_\theta \) be a set of representatives of left \( S_\eta \)-coclasses of \( S_\theta \), so \( S_\theta = \bigsqcup_{\sigma \in R} S_\eta \sigma \). Then
\[
D^\eta_\theta T(z) = \sum_{\sigma \in R} \text{sym}_\eta \left( \frac{\text{sg}(\sigma)}{\sigma(\partial_{\tau-1} \Delta_\theta)(\Delta_\eta / \Delta_\theta)} T(\sigma(z)) \right).
\]

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Since $\sigma(\sigma((\partial_{\tau} - 1) \Delta_\theta)) \in \mathbb{C}[X_\eta]$ and $\Delta_\eta / \Delta_\theta \in B_\eta$ we get that $D_\theta^\eta T(z) \in L_\eta$ and hence $L_\eta = B_\eta \otimes_{B_\theta} L_\theta$, so

$$V(T(v)) = \mathbb{C}_v \otimes_{B_\eta} L_\eta = \mathbb{C}_v \otimes_{B_\eta} (B_\eta \otimes_{B_\theta} L_\theta) = \mathbb{C}_v \otimes_{B_\theta} L_\theta = \mathcal{W}(T(v)).$$

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