Inherent stochasticity of superconductive-resistive switching in nanowires

Nayana Shah, David Pekker, Paul M. Goldbart
Department of Physics, University of Illinois at Urbana-Champaign,
1110 West Green Street, Urbana, Illinois 61801-3080, USA

Hysteresis in the current-voltage characteristic in a superconducting nanowire reflects an underlying bistability. As the current is ramped up repeatedly, the state switches from a superconductive to a resistive one, doing so at random current values below the equilibrium critical current. Can a single phase-slip event somewhere along the wire—during which the order-parameter fluctuates to zero—induce such switching, via the local heating it causes? We address this and related issues by constructing a stochastic model for the time-evolution of the temperature in a nanowire whose ends are maintained at a fixed temperature. The model indicates that although, in general, several phase-slip events are necessary to induce switching, there is indeed a temperature and current-range for which a single event is sufficient. It also indicates that the statistical distribution of switching currents initially broadens, as the temperature is reduced. Only at lower temperatures does this distribution show the narrowing with cooling naively expected for resistive fluctuations consisting of phase slips that are thermally activated.

The essential qualitative characteristics of quasi-one-dimensional superconducting nanowires are controlled by fluctuations of the superconducting order parameter, these fluctuations being predominantly thermal or quantal, depending on the temperature regime. Bulk superconductors undergo a sharp transition from an electrically resistanceless (i.e. superconducting) to a resistive (i.e. normal) state, e.g., with increasing temperature. In contrast, as explained by Little and Langer and Ambegaokar, in quasi-one-dimensional superconductors the resistanceless (and truly long-range-ordered) state is destabilized by a certain class of accessible order-parameter fluctuations that connect topologically distinct sectors of current-carrying states. In so doing, these fluctuations, which are known as phase-slip events, can dissipate supercurrent, and because of them such systems undergo a broad evolution between the (nominally) superconductive and normal states, e.g., with increasing temperature.

Recent advances in the fabrication of ultra-narrow superconducting wires—using carbon nanotube or DNA-templating—have spurred renewed interest in quasi-one-dimensional superconductivity and opened up new avenues for investigating the impact of order-parameter fluctuations. One setting in which order-parameter fluctuations in superconducting nanowires have been widely investigated, both theoretically and experimentally, is that of transport properties in the vicinity of the normal-to-superconducting quantum phase transition. In this setting, the primary mechanism underlying destruction of (nominal) superconducting order is depairing associated with magnetic fields or magnetic impurities. As is well known, applied currents also cause depairing and, if larger than a certain value (known as the thermodynamic critical or depairing current), would render the superconducting state locally unstable (regardless of the role of phase-slip fluctuations).

However, phase-slip fluctuations, which are responsible for the broad resistive transition in quasi-one-dimensional superconductors, also allow for premature switching to the resistive state, i.e. a nonequilibrium transition from the (nominally) superconducting, low-resistivity
state to the (nominally) normal, high-resistivity one. If damping of the order-parameter dynamics were low, a single phase-slip event would induce such switching, in analogy with what happens in underdamped Josephson junctions. By contrast, nanowires are generally overdamped, and so, whilst causing resistance, phase slippage does not, by itself, induce switching. As discussed in Ref. [14], this resistance causes Joule heating which, if not overcome sufficiently rapidly by conductive cooling, effectively reduces the depairing current, ultimately to below the applied current, thus causing switching to the highly resistive state. Naturally, this switching is not deterministic, owing to the underlying stochasticity of the phase-slip events that are responsible for the resistance. Rather, for a given subcritical applied current there is a statistical distribution of times at which switching occurs, characterized by a mean switching time (i.e. a superconducting state “lifetime”).

Our focus here is on stochastic aspects of the superconducting-to-resistive switching dynamics, an area that has not received much attention, to date. Inter alia, by obtaining the current-dependent mean switching time and convolving it with the sweep rate of the applied current that describes the experimental protocol, we shall determine the statistical distribution of currents at which switching occurs. Besides its fundamental significance, the characterization of switching dynamics in nanowires seems likely to have technological implications, such as for the integration of superconducting wires into electronic circuitry as controllable (current-liming) switching elements, the implementation of nanowire-based devices, and the exploration of the use of nanowires in quantum computers.

Having in mind the configuration in recent and ongoing experiments on superconducting nanowires, we consider a free-standing wire of length $L$ and cross-sectional area $A$, the ends of which are held at a fixed temperature $T_b$, as shown in Fig. [1] The fact that the wire is free-standing (i.e. lacks any substrate) is conducive to a clear interpretation of the measurements. On the other hand, the absence of an overall thermal bath means that any heat generated locally in the wire by a source term $Q$ can be taken away only through the ends; the corresponding heat conduction equation for the temperature $\Theta(x, t)$ at position $x$ along the wire at time $t$ reads

$$C_v(\Theta) \partial_t \Theta(x, t) = \partial_x [K_s(\Theta) \partial_x \Theta(x, t)] + Q(x, t), \quad (1)$$

and is characterized by the specific heat $C_v(\Theta)$ and thermal conductivity $K_s(\Theta)$ of the wire, together with the boundary condition $\Theta(\pm L/2, t) = T_b$ at its ends. Note that although our analysis rests on the premise that there are no additional heat-removing channels, it can readily be extended to account for such possibilities.

Before addressing dynamical issues, let us dwell briefly on the steady-state solutions of the heat conduction equation, obtained by setting $\partial_t \Theta(x, t) = 0$ and assuming that the wire is subjected to temporally continuous Joule heating at a rate given by $ALQ(x) = I^2 R(\Theta(x), I)$. Here, the function $R(\Theta, I)$ is to be understood as the resistance of an entire wire held at a uniform temperature $\Theta(x)$ = $\Theta$. The system-wide I-V characteristic at a given boundary value $T_b$ can be traced by obtaining the temperature profile $\Theta(x)$ for every value of current $I$ in both up and down (parametric) sweeps of $I$. On determining $R(\Theta, I)$ via the current-biased version of LAMH theory\textsuperscript{15,19} for phase-slips, the I-V curves are indeed found to become progressively more hysteretic in $I$ as $T_b$ is lowered (see Fig. [2]). This steady-state problem was previously studied by Tinkham et al\textsuperscript{20}, who used for $R(\Theta, I)$ the experimental linear-response resistance measured at $T_b = \Theta$, leading to qualitative agreement with the hysteresis observed in MoGe nanowires\textsuperscript{14}.

Our aim here is to study the inherent stochasticity in the switching process, and therefore it is necessary for us to explicitly take into consideration the fact that the resistive fluctuations of the superconducting nanowire consist of discrete phase-slip events (labelled by $i$) that take place at random moments of time $t_i$ and are centered at random spatial locations $x_i$. The work done on the wire by a phase slip may be obtained from the time integral of $IV(t)$, in which the Josephson relation $d\phi/dt = 2eV/\hbar$ may be used to relate the voltage pulse to the rate of change of the phase difference\textsuperscript{12}, via fundamental constants $\hbar$ and $e$. Hence, a single phase slip (or anti-phase slip), which corresponds to a decrease (or increase) of $\phi$ by $2\pi$, will heat (or cool) the wire by a “quantum” of energy $\hbar I/2e$. Thus we arrive at the central thrust of our paper: the dynamics of switching from the superconducting to the resistive state in the nanowires is controlled by a heat conduction equation that is stochastic by virtue of its source term:

$$Q(x, t) = \frac{\hbar I}{2eA} \sum_i \sigma_i F(x - x_i) \delta(t - t_i), \quad (2)$$

where $F(x - x_i)$ is a normalized (to unity) form factor representing the relative spatial distribution of heat produced by the $i^{th}$ phase-slip event, and $\sigma_i = \pm 1$ for phase
ducting (at low-)

FIG. 3: Effective potential \( U(T, T_b, I) \) (dashed line) and the mean first-passage time \( \tau \) as functions of the temperature \( T \) of the central segment for various bias currents \( I \) and for \( T_b = 1.2 \text{K} \). The marks on the temperature axis indicate the temperatures that the central segment would have after 1, 2, ..., 10 phase slips in the absence of cooling (as given by Eq. (5) for \( T_0 = T_b \)).

\[
\frac{dT}{dt} = -\alpha(T, T_b)(T - T_b) + \eta(T, I) \sum_i \delta(t - t_i), \quad (3)
\]

where the second term on the RHS corresponds to heating by phase slips, and the first term to cooling as a result of conduction of heat from the central segment to the external bath via the two end-segments, each of length \((L - l)/2\). The temperature-dependent cooling rate \( \alpha \) is given by

\[
\alpha(T, T_b) = \frac{4}{l(L - l)C_v(T)T - T_b} \int_{T_b}^{T} dT' K_v(T'). \quad (4)
\]

If \( T_i \) and \( T_l \) are temperatures before and after a phase slip then, using

\[
A l \int_{T_l}^{T_i} C_v(T') dT' = \frac{hI}{2e}, \quad (5)
\]

we can express the temperature ‘impulse’ due to a phase slip, i.e., \( T_l - T_i \equiv \eta(T_l, I) = \bar{\eta}(T_l, I) \), as function of either \( T_l \) or \( T_i \), depending on the context.

Let us now elucidate the physical and mathematical structure of Eq. (3). To begin with, we shall consider the continuous-heating limit, \( \eta(T, I) \bar{\Gamma}(T, I) \), for the source term, and express Eq. (3) as \( dT/dt = -\partial U/\partial T \). In Fig. 3, we illustrate the form of the ‘potential’ \( U(T, T_b, I) \) for fixed \( T_b \); there is a range of currents \( I \) for which \( U \) has two local minima, corresponding to the superconducting (at low-\( T \)) and the resistive (at high-\( T \)) states, separated by a local maximum. The resulting bistability is central to the underlying physics. On the one hand, it explains the origin of the hysteretic behavior; on the other hand, it provides a basis for phrasing the question of stochastic switching dynamics in superconducting nanowires in terms of an existing general framework for stochastic bistable systems. In what follows, we focus on the stochastic variable \( T(t) \) to ease the notation we do not display the dependences on \( I \) and \( T_b \) unless essential. To continue the analysis of the stochastic equation, imagine turning off the cooling term. If we now start with an initial temperature \( T_0 \) then

\[
T_0, T_0 + \eta(T_0), T_0 + \eta(T_0) + \eta(T_0 + \eta(T_0)), ...
\]

defines the discrete sequence of values that \( T \) jumps to, as marked on the horizontal axes in Fig. 3 for \( T_0 = T_b \). The probability per unit time, \( \Gamma(T) \), to make a jump changes at each step, and so does the size \( \eta(T) \) of the jump, owing to their explicit dependence on temperature. On the other hand, if we turn off the heating term then we have a deterministic problem in which \( T \) would decay at a rate \( \alpha(T) \), from its initial value \( T_0 > T_b \) to the bath temperature \( T_b \), which is the lowest value \( T \) can have. It is the competition between the discrete heating and the continuous cooling that makes for a rather rich stochastic problem. We hope that our solution will also furnish insight into other physical problems that possess a similar mathematical structure.

The master equation for \( P(T, t) \), the probability for the temperature of the central segment of the nanowire to be \( T \) at a time \( t \) (given that it had some initial value
\( T_0 \) at time \( t_0 \), reads

\[
\partial_t P(T, t) = \partial_T \left[ (T - T_0) \alpha(T) P(T, t) \right] - \Gamma(T) P(T, t) \\
+ \Gamma(T - \eta(T)) P(T - \eta(T), t) \left( 1 - \partial_T \eta(T) \right),
\]

(7)

where the first (i.e. the transport) term corresponds to the effect of cooling, and the last two terms correspond to the effects of heating. Note that the term \((1 - \partial_T \eta(T))\) appears because of the dependence of the jump size on \( T \), as given by \( \eta(T) \). The fundamental quantity of interest is the mean switching time \( \tau_s(T_b, I) \), i.e. the mean time required for the wire to switch from being superconducting to resistive, assuming that the wire has temperature \( T = T_b \) when the current \( I \) is turned on at time \( t = 0 \).

The master equation, Eq. (7), provides the starting point for generalizing the standard procedure for computing \( \tau_s \) via the evaluation of the mean first-passage time\(^\text{22}\).

The mean first-passage time \( \tau(T \to T^*) \), to go past a point \( T = T^* \) for the first time having started from some \( T < T^* \), can be shown to satisfy the equation

\[
-(T_b - T) \alpha(T) \partial_T \tau(T) + \Gamma(T) \left[ \tau(T) - \tau(T + \eta(T)) \right] = 1,
\]

(8)

together with the conditions \( \tau(T) = 0 \) for \( T > T^* \) and \( d\tau(T)/dT = 0 \) at \( T = T_b \), which are appropriate for our problem. Some illustrative plots for \( \tau(T \to T^*) \), obtained by numerically solving Eq. (8) are shown in Fig. 3 with the choice of \( T^* \) being somewhat larger than the location of the local maximum of \( U \). From these plots we see that the mean first-passage time has a plateau at low values of \( T \) and then rapidly decreases in the vicinity of the potential barrier. At very high currents, as can be seen from the last panel of Fig. 3, the local stability of the superconducting state disappears, and so does the plateau in the mean first-passage time. In these plots, the tick marks on the \( T \) axes correspond to the temperatures given by the sequence \( T_b \) for \( T_0 = T_b \).

As long as the high-\( T \) minimum is lower than the low-\( T \) one, and \( T^* \) is chosen to be appreciably past the intervening potential maximum (in order to eliminate the possibility of reversion to the superconducting state), we can make the identification \( \tau_s(T_b, I) \equiv \tau(T_b \to T^*, T_b, I) \). The number of tick marks (see sequence \( T_b \)) between \( T_b \) and \( T^* \) is nothing but the number \( N(T_b, I) \) of phase-slip events required to raise the temperature of the central segment from \( T_b \) to \( T^* \) in the absence of cooling. Accordingly, \( N(T_b, I) \) also provides an estimate of the number of phase-slip events needed to overcome the potential barrier if the timespan of these events is insufficient to allow significant cooling to occur. ‘Thermal runaway’—heating by rare sequences of closely-spaced phase slips that overcome the potential barrier—constitutes the mechanism of superconductiveto-resistive switching within our model. As the \( N(T_b, I) \) becomes large, the total number of phase-slip events taking place before switching can happen, and correspondingly the value of \( \tau_s(T_b, I) \), may indeed be quite large.

Our key findings are summarized in Fig. 4. There is a region of \( I \) and \( T_b \) for which the occurrence of just one phase slip is sufficient to cause the nanowire to switch from the superconducting to the resistive state\(^\text{22}\); in this case \( \tau_s^{-1} = \alpha \). A switching measurement in this range can thus provide a way of detecting and probing a single phase-slip fluctuation. As, outside this range, several phase-slip events are required for switching, \( \tau_s^{-1} \) deviates from \( \alpha \) (see panel 3c). A graphical representation of the contour lines for a few values of \( \tau_s^{-1} \) and \( \alpha \), chosen in an
experimentally accessible range, is provided in panel I. Whilst the spacing between the \( \Gamma \) contour lines decreases monotonically on lowering \( T_b \), the spacing between \( \tau_s^{-1} \) lines can be seen to behave non-monotonically.

The mean switching time \( \tau_s \) in bistable current-biased systems can be either directly measured or extracted from the switching-current statistics, \( \tau_s \), generated via the repeated tracing of the \( I-V \) characteristic by ramping the current up and down at some sweep rate \( r \). For this reason, in Fig. IIb we have illustrated the behavior of this distribution of switching currents in superconducting nanowires based on the theory presented here. Upon raising \( T_b \), one would naively expect the distribution to become broader for a model involving thermally activated phase slips. Such an broadening in the distribution-width becomes broader for a model involving thermally activated raising nanowires based on the theory presented here. Upon this distribution of switching currents in superconducting, switching is induced by single phase slips. However, on continuing to raise \( T_b \), but now through temperatures above \( T_b^{cr}(r) \), the distribution-width shows a seemingly anomalous decrease. This is a manifestation of the now-decreasing spacing between the \( \tau_s \) contour lines. This striking behavior above \( T_b^{cr} \) may be understood by the following reasoning: the larger the number of phase-slips in the sequence inducing the superconductive-to-resistive thermal runaway, the smaller the stochasticity in the switching process and, hence, the sharper the distribution of switching currents. This non-monotonicity in the temperature dependence of the width of switching-current distribution, along with the existence of a regime in which a single phase-slip event can be probed, are the two key predictions of our theory.

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