Symbol-Level Precoding Made Practical for Multi-Level Modulations via Block-Level Rescaling

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Abstract—In this letter, we propose an interference exploitation symbol-level precoding (SLP) method for multi-level modulations via an in-block power allocation scheme to greatly reduce the signaling overhead. Existing SLP approaches require the symbol-level broadcast of the rescaling factor to the users for correct demodulation, which hinders the practical implementation of SLP. The proposed approach allows a block-level broadcast of the rescaling factor as done in traditional block-level precoding, greatly reducing the signaling overhead for SLP without sacrificing the performance. Our derivations further show that the proposed in-block power allocation enjoys an exact closed-form solution and thus does not increase the complexity at the base station (BS). In addition to the significant alleviation of the signaling overhead validated by the effective throughput results, numerical results demonstrate that the proposed power allocation approach also improves the error-rate performance of the existing SLP. Accordingly, the proposed approach enables the practical use of SLP in multi-level modulations.

Index Terms—MIMO, symbol-level precoding, constructive interference, multi-level modulations, signaling overhead.

I. INTRODUCTION

Precoding design has been extensively studied for multi-user transmission in the field of multi-antenna communication systems [1]. By exploiting the channel state information (CSI), both linear and nonlinear precoding methods have been studied in the literature [2], [3], all of which attempt to suppress, minimize or limit interference. However, the above solutions have ignored the fact that the information of the data symbols is also available at the base stations (BSs) before transmission, which is not fully exploited. Recently, interference exploitation symbol-level precoding (SLP) that employs the information of both the data symbols to be transmitted and the channel knowledge has emerged as a new precoding method [4], [5]. Compared to common block-level precoding where the precoding matrix is applied to a block of data symbols, SLP applies different precoding matrix to different data symbols or directly designs the precoded signals to be transmitted at the antenna port, which allows to observe interference that is inherent in multi-user transmission on a symbol-by-symbol basis and further exploit it instead of cancelling it, greatly enhancing the performance of multi-antenna communication systems. Due to such benefits, SLP has gained increasing research attention in recent years.

The study of interference exploitation precoding originates from the adaptation of traditional zero-forcing (ZF) and regularized ZF (RZF) precoding to closed-form SLP, where the concept of constructive interference (CI) and destructive interference (DI) is characterized [6], [7]. More specifically, [6] retains the CI with a dynamic linear precoding scheme while fully removing the DI via ZF, while a more superior scheme is proposed in [7] that further rotates the phases of DI such that all the interfering signals are constructive to the users. As a step further, optimization-based CI precoding is studied in [9] for PSK signaling under the context of vector perturbation precoding, where CI in the form of symbol scaling is proposed. In [10]-[12], CI precoding for PSK signaling is studied based on phase rotation, where the superiority of non-strict phase rotation is presented over the strict phase rotation considered in [6], [7], and [13] has further extended the CI precoding to QAM modulations. Importantly, [14] and [15] have further derived the optimal precoding structure for CI precoding with PSK signaling and QAM signaling respectively, which greatly reduces the computational costs of CI precoding. It should be noted that the above SLP solutions for QAM modulations have assumed that the rescaling factor is known to the users, which would however result in an excessive signaling overhead and hinder the practical implementation of CI precoding, since the rescaling factor needs to be broadcast to the users on a symbol level for SLP.

Therefore in this paper, we aim to reduce such excessive signaling overhead for SLP and propose a practical SLP scheme for multi-level modulations via an in-block power allocation. Specifically, as opposed to traditional SLP schemes that consider a uniform power allocation for each symbol duration within a transmission block, we propose to allocate the available transmit power of SLP dynamically for each symbol duration within a transmission block, based on which a joint optimization problem on the precoding matrix and the power allocation strategy is formulated. By decoupling the joint optimization problem, we prove via the Karush-Kuhn-Tucker (KKT) conditions that the proposed dynamic power allocation will lead to an identical rescaling factor within a transmission block when optimality is achieved, which then allows the block-level broadcast of the rescaling factor to the users as usually done in traditional block-level precoding and greatly reduces the signaling overhead for SLP. Our derivations further reveal an exact closed-form solution for the proposed in-block power allocation strategy, leading to only
negligible complexity increase at the BS. Numerical results have validated the significant signaling overhead reduction brought by the proposed in-block power allocation scheme without sacrificing the error-rate performance of SLP, which further boosts the practical deployment of SLP for multi-level modulations.

Notations: \(a, \mathbf{a}, \) and \(\mathbf{A}\) denote scalar, column vector and matrix, respectively. \((\cdot)^T, (\cdot)^H, (\cdot)^{-1}\) denote transposition, conjugate transposition and inverse, respectively. \(\mathbb{C}^{n \times n}\) and \(\mathbb{R}^{n \times n}\) represent the sets of \(n \times n\) complex- and real-valued matrices, respectively. \(\Re \{\cdot\}\) and \(\Im \{\cdot\}\) extract the real and imaginary part. \(\odot\) is the Hadamard product, \(\|\cdot\|_2\) represents the \(\ell_2\)-norm and \(j\) is the imaginary unit.

II. SYSTEM MODEL AND CONSTRUCTIVE INTERFERENCE

A. System Model

We focus on a generic multi-user multiple-input single-output (MU-MISO) communication system in the downlink, where the BS with \(N_T\) transmit antennas communicates with a total number of \(K\) single-antenna users simultaneously in the same time-frequency resource, where \(K \leq N_T\). Assume that a transmission block consists of \(M\) symbol durations, within which the wireless channel stays constant, and we can express the data symbol matrix as

\[
\mathbf{S} = \begin{bmatrix} \mathbf{s}^{(1)} & \mathbf{s}^{(2)} & \cdots & \mathbf{s}^{(M)} \end{bmatrix} \in \mathbb{C}^{K \times M},
\]

where \(\mathbf{s}^{(m)} = \begin{bmatrix} s_1^{(m)} & s_2^{(m)} & \cdots & s_K^{(m)} \end{bmatrix}^T \in \mathbb{C}^{1 \times K}\) is the data symbol vector in the \(n\)-th symbol duration drawn from a nominal QAM constellation, which is a representative example for multi-level modulations. Accordingly, the received signal for user \(k\) in the \(m\)-th symbol duration can be expressed as

\[
y_k^{(m)} = \sqrt{p^{(m)}} \cdot \mathbf{h}_k^T \mathbf{W}^{(m)} \mathbf{s}^{(m)} + n_k^{(m)},
\]

where \(y_k^{(m)}\) is the received signal for user \(k\) in the \(m\)-th symbol duration, \(p^{(m)}\) represents the allocated transmit power for the \(m\)-th symbol duration, \(\mathbf{h}_k \in \mathbb{C}^{1 \times N_T}\) is the flat-fading Rayleigh channel between the BS and the users that is constant within the entire transmission block, \(\mathbf{W}^{(m)} \in \mathbb{C}^{N_T \times K}\) denotes the precoding matrix, and \(n_k^{(m)}\) is the corresponding additive Gaussian noise at the users with zero mean and variance \(\sigma^2\). Since we focus on SLP and have included the allocated transmit power \(p^{(m)}\) in \(\mathbf{W}^{(m)}\), the symbol-level power constraint is enforced as \(\|\mathbf{W}^{(m)} \mathbf{s}^{(m)}\|_2^2 \leq 1\) and we have \(\sum_{m=1}^M p^{(m)} \leq P_T\), where \(P_T\) represents the total available transmit power for the transmission block.

At the receiver side, \(\tilde{y}_k^{(m)}\) needs to be scaled for correct demodulation when multi-level modulations are employed, and the signals ready for demodulation can be expressed as

\[
r_k^{(m)} = f^{(m)} \tilde{y}_k^{(m)} = f^{(m)} \sqrt{p^{(m)}} \cdot \mathbf{h}_k^T \mathbf{W}^{(m)} \mathbf{s}^{(m)} + f^{(m)} n_k^{(m)},
\]

where \(f^{(m)}\) is the rescaling factor for the \(m\)-th symbol duration, also known as the noise amplification factor, which needs to be broadcast to the users on a symbol level for SLP.

III. PROPOSED IN-BLOCK POWER ALLOCATION

A. Problem Formulation

In traditional SLP approach, uniform transmit power is assumed for each symbol duration, i.e., \(p^{(m)} = \frac{P_T}{M}, \forall m \in \mathcal{M}\), where \(\mathcal{M} = \{1, 2, \cdots, M\}\). In this letter, we aim to jointly optimize the transmit power and the precoding matrix for each symbol duration such that the minimum CI effect within the considered transmission block is maximized, and such a joint optimization will return an identical rescaling factor within the transmission block, as will be mathematically shown in Section III-B. Following \[15\], we divide the constellation points for a QAM modulation into 4 types, as shown in Fig. 1. Type A corresponds to the inner constellation points that cannot exploit CI, type B corresponds to the outer constellation points whose real part can exploit CI, type C corresponds to the outer constellation points whose imaginary part can exploit CI, and type D corresponds to the outer constellation points at the corner, whose real and imaginary part can both exploit CI. Accordingly, we follow the symbol-scaling CI metric in \[15\] and decompose each constellation point into

\[
s_k^{(m)} = s_k^A + s_k^B,
\]
where $s_k^m = \Re \{ s_k^{(m)} \}$ and $s_k^m = j \cdot \Im \{ s_k^{(m)} \}$. Following a similar principle, we decompose the noiseless received signal $h_k^T W^{(m)} s^{(m)}$ into

$$h_k^T W^{(m)} s^{(m)} = \alpha_k^A s_k^A + \alpha_k^B s_k^B,$$

where $\alpha_k^A$ and $\alpha_k^B$ are two introduced real scalars that jointly determine the effect of interference on $s_k^{(m)}$, and for simplicity of notation we have removed the index of symbol duration for $\alpha_k^A, \alpha_k^B, s_k^A$ and $s_k^B$. The optimization problem on $W^{(m)}$ to maximize the CI effect for the $m$-th symbol duration can then be formulated as [15]

$$\mathcal{P}_1 : \max_{W^{(m)}} t^{(m)}$$

s.t. $\mathbf{C}_1 : h_k^T W^{(m)} s^{(m)} = \alpha_k^A s_k^A + \alpha_k^B s_k^B, \forall k \in \mathcal{K}$; $\mathbf{C}_2 : t^{(m)} = \alpha_k^T, \forall \alpha_k^T \in \mathcal{T}$; $\mathbf{C}_3 : \| W^{(m)} s^{(m)} \|_2^2 \leq 1$;

where $\mathcal{O}$ and $\mathcal{T}$ consist of the real scalars corresponding to the constellation points that can exploit CI and cannot exploit CI, respectively, and $\mathcal{K} = \{1, 2, \cdots, K\}$. As can be observed, the constraint $\mathbf{C}_5$ indicates that the nominal constellation is scaled by $t^{(m)}$ due to the effect of the wireless channel, and together with $\mathbf{C}_2$, the rescaling factor for the received signals in the $m$-th symbol duration can be obtained as:

$$f^{(m)} = \frac{1}{t^{(m)} \sqrt{p^{(m)}}}$$

Accordingly, the joint optimization on $p^{(m)}$ and $W^{(m)}$ to maximize the minimum CI effect within a transmission block, which is equivalent to minimizing the maximum noise amplification effect, can be constructed as

$$\mathcal{P}_2 : \max_{W^{(m)}, p^{(m)}} \min_m t^{(m)} \sqrt{p^{(m)}}$$

s.t. $\mathbf{C}_1 : h_k^T W^{(m)} s^{(m)} = \alpha_k^A s_k^A + \alpha_k^B s_k^B, \forall k \in \mathcal{K}$; $\mathbf{C}_2 : t^{(m)} \leq \alpha_k^T, \forall \alpha_k^T \in \mathcal{O}$; $\mathbf{C}_3 : t^{(m)} = \alpha_k^T, \forall \alpha_k^T \in \mathcal{T}$; $\mathbf{C}_4 : \| W^{(m)} s^{(m)} \|_2^2 \leq 1$; $\mathbf{C}_5 : \sum_{m=1}^M p^{(m)} \leq P_I$.

### B. Closed-form Solution

A closer look at $\mathcal{P}_2$ reveals that the optimization on $W^{(m)}$ is independent of $p^{(m)}$, and $\mathcal{P}_2$ can thus be decoupled into two sub-problems. Given that the optimization on $W^{(m)}$ has been well studied in [15], in the following we focus on the optimization on $p^{(m)}$ for given $t^{(m)}$, which degenerates to the following optimization problem:

$$\mathcal{P}_3 : \max_{p^{(m)}} \min_m t^{(m)} \sqrt{p^{(m)}}$$

s.t. $\mathbf{C}_1 : \sum_{m=1}^M p^{(m)} \leq P_I$.  

By introducing an auxiliary variable $u_m = \sqrt{p^{(m)}}$, the above problem can be further expressed in a standard convex form:

$$\mathcal{P}_4 : \begin{array}{ll}
\min & g \\
\text{s.t.} & C_1 : g - t^{(m)} u_m \leq 0, \forall m \in \mathcal{M}; \\
& C_2 : \sum_{m=1}^M u_m^2 - P_I \leq 0,
\end{array}$$

based on which the following proposition is obtained.

**Proposition 1**: When the optimality of $\mathcal{P}_4$ is achieved, we arrive at an identical rescaling factor for each symbol duration in the transmission block, i.e., $f^{(1)} = f^{(2)} = \cdots = f^{(M)}$.

**Proof**: We prove this proposition via the KKT conditions, where the Lagrangian of $\mathcal{P}_4$ is formulated as

$$\mathcal{L}(u_m, g, \delta_m, \vartheta) = -g + \sum_{m=1}^M \delta_m \left( g - t^{(m)} u_m \right) + \vartheta \left( \sum_{m=1}^M u_m^2 - P_I \right).$$

In (11), $\delta = [\delta_1, \delta_2, \cdots, \delta_M]^T$, $u$ and $t$ are similarly defined, and $1 = [1, 1, \cdots, 1]^T$.

Based on (11), the corresponding KKT conditions for $\mathcal{P}_4$ are given by

$$\frac{\partial \mathcal{L}}{\partial g} = (1^T \delta - 1) = 0 \quad (12a)$$

$$\frac{\partial \mathcal{L}}{\partial u} = \vartheta \cdot u - (\delta \circ t)^T u - \vartheta \cdot P_I = 0 \quad (12b)$$

$$\delta_m \left( g - t^{(m)} u_m \right) = 0, \forall m \in \mathcal{M} \quad (12c)$$

$$\vartheta \left( \sum_{m=1}^M u_m^2 - P_I \right) = 0 \quad (12d)$$

By observing (12), firstly we obtain that $\vartheta \neq 0$, since $\delta = 0$ results in $\delta_m = 0, \forall m \in \mathcal{M}$ based on (12b), which contradicts with (12a). This means that the transmit power constraint is active when optimality is achieved, i.e., $\sum_{m=1}^M u_m^2 = P_I$. Subsequently, according to (12c) and given that $\delta_m \neq 0, \forall m \in \mathcal{M}$, we further obtain $g - t^{(m)} u_m = 0, \forall m \in \mathcal{M}$, which is equivalent to

$$t^{(1)} u_1 = t^{(2)} u_2 = \cdots = t^{(M)} u_M,$$

and with $u_m = \sqrt{p^{(m)}}$, we arrive at

$$t^{(1)} \sqrt{p^{(1)}} = t^{(2)} \sqrt{p^{(2)}} = \cdots = t^{(M)} \sqrt{p^{(M)}},$$

which based on (7) completes the proof. ■

Moreover, we can further derive the closed-form solution of the optimal transmit power value $p^{(m)}$, given by the following proposition.

**Proposition 2**: The value of the optimal transmit power $p^{(m)}$ can be obtained in a closed form as

$$p^{(m)} = \frac{1}{\sum_{m=1}^M t^{(m)} \sqrt{p^{(m)}}} \cdot P_I.$$  

(15)
Proof: The closed-form solution can readily be obtained based on the results in Proposition 1 where \( \sum_{m=1}^{M} p^{(m)} = P_T \) and \( t(1) \sqrt{p[1]} = t(2) \sqrt{p[2]} = \ldots = t(M) \sqrt{p[M]} \).

IV. NUMERICAL RESULTS

In this section, we evaluate the performance of SLP with the proposed in-block power allocation via Monte Carlo simulations. We consider a practical communication system where the BS employs \( B \) bits to broadcast the rescaling factor \( f \) to the users, which is subject to quantization errors due to the limited feed-forwarding, given by

\[
\hat{f} = f + \epsilon,
\]

where \( \hat{f} \) is the rescaling factor received at the users, and \( f \) is the ideal rescaling factor known at the BS as in (3). \( \epsilon \sim \mathcal{CN} (0, \nu) \) is the quantization error, whose variance is modeled according to [9], [16], [17] as

\[
\nu = \frac{f_{max}}{2^B},
\]

where \( f_{max} \) is the maximum value of the rescaling factor with non-zero probability. Accordingly, we evaluate the performance of different precoding techniques via the effective throughput for \( M \)-QAM, defined as throughput minus the total number bits required for feed forwarding, given by

\[
T_{eff} = \max \left\{ \left( 1 - BLER \right) \cdot \log_2 (M) \cdot N_T - N_{overhead}, 0 \right\} = \max \left\{ \left( 1 - P_B \right) M \cdot \log_2 (M) \cdot N_T - N_{overhead}, 0 \right\},
\]

where \( BLER \) is the block error rate, and \( P_B \) is the bit error rate (BER). \( N_{overhead} \) is the total number of bits for signaling overhead, and \( N_{overhead} = B \) for ZF, RZF, and SLP with in-block power allocation, while \( N_{overhead} = M \cdot B \) for traditional SLP with uniform power allocation. Without loss of generality, we assume that the length of a transmission block is \( M = 10 \) symbols, the number of bits for broadcasting the rescaling factor to the users is \( B = 4 \) bits. The maximum available transmit power in a transmission block is \( P_T = 1 \), and the transmit SNR in each symbol duration is accordingly defined as \( \rho = \frac{1}{1+B} \). We compare our proposed scheme with traditional block-level ZF and RZF precoding, and both 16QAM and 64QAM modulations are considered in the simulations.

Fig. 2 validates the effectiveness of the proposed in-block power allocation in reducing the signaling overhead for both 16QAM and 64QAM. Compared to traditional SLP approach where the value of the rescaling factor varies for each symbol duration in the transmission block, which requires a symbol-level broadcast of these values to the users for correct demodulation, SLP with the proposed in-block power allocation scheme returns an identical rescaling factor within a transmission block, which then enables the block-level broadcast of the rescaling factors as done in traditional block-level precoding, validating the effectiveness of the proposed scheme.

Fig. 3 compares the BER of SLP with the proposed in-block power allocation with ZF precoding, RZF precoding and SLP with uniform power allocation, for both 16QAM and 64QAM. Since we have considered quantization errors in broadcasting the rescaling factors to the users as in practical communication systems, the traditional SLP technique is observed to exhibit BER losses when the transmit SNR increases and becomes inferior to traditional ZF/RZF precoding. On the contrary, the SLP method with the proposed in-block power allocation has alleviated such performance degradation, and meanwhile still offers transmit SNR gains over traditional schemes.

Fig. 4 depicts the effective throughput for different pre-
coding schemes to further highlight the significance of the proposed in-block power allocation strategy for SLP. As can be observed, traditional SLP methods have shown the worst effective throughput performance due to the requirement of symbol-level broadcast of the rescaling factor, which results in an excessive signaling overhead. Meanwhile, we observe that the proposed scheme can greatly improve the effective throughput for SLP and achieves the highest throughput performance over ZF and RZF precoding. Both of the BER and throughput results above have exhibited the superiority and significance of the proposed scheme for existing SLP with multi-level modulations.

V. CONCLUSIONS

In this letter, we have designed an in-block power allocation scheme for SLP with multi-level modulations, which returns an identical rescaling factor for the entire transmission block. The proposed scheme can thus greatly reduce the signaling overhead of existing SLP schemes by reducing the frequency for the broadcast of the rescaling factor from symbol level to block level, achieving a similar signaling overhead to traditional block-level precoding methods without sacrificing the performance. By decoupling the precoding design and the power allocation process, our derivations have shown that the in-block power allocation enjoys a closed-form solution, which is thus efficient to deploy. In addition, numerical results demonstrate that the proposed power allocation scheme exhibits significant performance improvements for SLP in terms of the effective throughput over existing SLP techniques without in-block power allocation.

REFERENCES

[1] L. Zheng and D. N. C. Tse, “Diversity and Multiplexing: A Fundamental Tradeoff in Multiple-Antenna Channels,” *IEEE Trans. Inf. Theory*, vol. 49, no. 5, pp. 1073–1096, May 2003.

[2] C. B. Peel, B. M. Hochwald, and A. L. Swindlehurst, “A Vector-Perturbation Technique for Near-Capacity Multiantenna Multiuser Communication-part I: Channel Inversion and Regularization,” *IEEE Trans. Commun.*, vol. 53, no. 1, pp. 195–202, Jan. 2005.

[3] A. Li and C. Masouros, “A Two-Stage Vector Perturbation Scheme for Adaptive Modulation in Downlink MU-MIMO,” *IEEE Trans. Veh. Tech.*, vol. 65, no. 9, pp. 7785–7791, Sept. 2016.

[4] A. Li, D. Spano, J. Krvovicha, E. Domouchoitsidis, C. G. Tsinos, C. Masouros, S. Chatzinotas, Y. Li, B. Vucetic, and B. Ottersten, “A Tutorial on Interference Exploitation via Symbol-Level Precoding: Overview, State-of-the-Art and Future Directions,” *IEEE Commun. Surveys & Tut.*, vol. 22, no. 2, pp. 796–839, Secondquarter 2020.

[5] M. Alodeh, D. Spano, A. Kalantari, C. G. Tsinos, D. Christopoulos, S. Chatzinotas, and B. Ottersten, “Symbol-Level and Multicast Precoding for Multiuser Multiantenna Downlink: A State-of-the-Art, Classification, and Challenges,” *IEEE Commun. Surveys & Tut.*, vol. 20, no. 3, pp. 1733–1757, Thirdquarter 2018.

[6] C. Masouros and B. Alsusa, “Dynamic Linear Precoding for the Exploitation of Known Interference in MIMO Broadcast Systems,” *IEEE Trans. Wireless Commun.*, vol. 8, no. 3, pp. 1396–1404, Mar. 2009.

[7] C. Masouros, “Correlation Rotation Linear Precoding for MIMO Broadcast Communications,” *IEEE Trans. Sig. Process.*, vol. 59, no. 1, pp. 252–262, Jan. 2011.

[8] C. Masouros, T. Ratnarajah, M. Sellathurai, C. B. Papadias, and A. K. Shukla, “Known Interference in the Cellular Downlink: A Performance Limiting Factor or a Source of Green Signal Power?” *IEEE Commun. Mag.*, vol. 51, no. 10, pp. 162–171, Oct. 2013.

[9] C. Masouros, M. Sellathurai, and T. Ratnarajah, “Vector Perturbation based on Symbol Scaling for Limited Feedback MISO Downlinks,” *IEEE Trans. Sig. Process.*, vol. 62, no. 3, pp. 562–571, Feb. 2014.

[10] C. Masouros and G. Zheng, “Exploiting Known Interference as Green Signal Power for Downlink Beamforming Optimization,” *IEEE Trans. Sig. Process.*, vol. 63, no. 14, pp. 3628–3640, July 2015.

[11] M. Alodeh, S. Chatzinotas, and B. Ottersten, “Constructive Multiuser Interference in Symbol Level Precoding for the MISO Downlink Channel,” *IEEE Trans. Sig. Process.*, vol. 63, no. 9, pp. 2239–2252, May 2015.

[12] ——, “Energy-Efficient Symbol-Level Precoding in Multiuser MISO based on Relaxed Detection Region,” *IEEE Trans. Wireless Commun.*, vol. 15, no. 5, pp. 3755–3767, May 2016.

[13] ——, “Symbol-Level Multiuser MISO Precoding for Multi-Level Adaptive Modulation,” *IEEE Trans. Wireless Commun.*, vol. 17, no. 8, pp. 5511–5524, Aug. 2017.

[14] A. Li and C. Masouros, “Interference Exploitation Precoding Made Practical: Optimal Closed-Form Solutions for PSK Modulations,” *IEEE Trans. Wireless Commun.*, vol. 17, no. 11, pp. 7661–7676, Nov. 2018.

[15] A. Li, C. Masouros, Y. Li, B. Vucetic, and A. L. Swindlehurst, “Interference Exploitation Precoding for Multi-Level Modulations: Closed-Form Solutions,” *arXiv preprint*, available online: https://arxiv.org/abs/1811.03289, 2018.

[16] D. J. Ryan, I. B. Collings, I. V. L. Clarkson, and R. W. Heath, “Performance of Vector Perturbation Multiuser MIMO Systems with Limited Feedback,” *IEEE Trans. Commun.*, vol. 57, no. 9, pp. 2635–2644, Sept. 2009.

[17] R. G. Gallager, *Information Theory and Reliable Communication*, 1st ed. Wiley, Jan. 1968.