STABILITY OF THE CHARI-PRESSLEY-LOKTEV BASES FOR LOCAL WEYL MODULES OF $\mathfrak{sl}_2[t]$

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Abstract. We prove stability of the Chari-Pressley-Loktev bases for natural inclusions of local Weyl modules of the current algebra $\mathfrak{sl}_2[t]$. These modules being known to be Demazure submodules in the level 1 representations of the affine Lie algebra $\hat{\mathfrak{sl}}_2$, we obtain, by passage to the direct limit, bases for the level 1 representations themselves.

1. Introduction

Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra and $\mathfrak{g}[t]$ the corresponding current algebra: recall that the Lie bracket on $\mathfrak{g}[t]$ is obtained from that on $\mathfrak{g}$ merely by the extension of scalars to the polynomial ring $\mathbb{C}[t]$ in one variable. Local Weyl modules, introduced in Chari-Pressley [3], are interesting finite dimensional representations. Corresponding to every dominant integral weight $\lambda$ of $\mathfrak{g}$, there is one local Weyl module for $\mathfrak{g}[t]$ denoted by $W(\lambda)$. The $W(\lambda)$ is universal among finite dimensional $\mathfrak{g}[t]$-modules generated by a highest weight vector of weight $\lambda$, in the sense that any such module is uniquely (up to scaling) a quotient of $W(\lambda)$ [1,3].

In [3], Chari-Pressley also produced nice monomial bases of local Weyl modules in the case $\mathfrak{g} = \mathfrak{sl}_2$. Chari-Loktev [1] clarified and extended the construction of these bases to the case $\mathfrak{g} = \mathfrak{sl}_m$. Their work was motivated by the following conjecture [2,3] about the dimension of the local Weyl modules for $\mathfrak{g}$ simply laced:

$$\dim W(\lambda) = \prod_{i=1}^\ell (\dim W(\varpi_i))^{a_i} \quad \text{for} \quad \lambda = a_1\varpi_1 + \cdots + a_\ell\varpi_\ell \quad (1.1)$$

where $\ell$ is the rank and $\varpi_1, \ldots, \varpi_\ell$ the fundamental weights of $\mathfrak{g}$. Using their bases, Chari-Loktev were able to obtain, for $\mathfrak{g} = \mathfrak{sl}_m$:

- the formula (1.1) above,
- an identification of $W(\lambda)$ as a Demazure module of a level 1 representation of the untwisted affine Lie algebra $\hat{\mathfrak{g}}$ corresponding to $\mathfrak{g}$,
- a fermionic formula for the graded character of $W(\lambda)$.

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Later, by very different means, Fourier-Littelmann [4] obtained, for \( \mathfrak{g} \) simply laced, the identification of local Weyl modules as level 1 Demazure modules, and then deduced formula (1.1). Their methods do not however give explicit bases or graded character formulas.

In this paper, we investigate further the Chari-Pressley bases for local Weyl modules for \( \mathfrak{g} = \mathfrak{sl}_2 \), taking into account the perspective gained from [1]. The dominant integral weights for \( \mathfrak{g} = \mathfrak{sl}_2 \) being parametrized by the non-negative integers, there is one local Weyl module \( W(n) \) for every integer \( n \geq 0 \). Let us restrict ourselves in this introduction, for the sake of simplicity, to the case when \( n \) is even. The local Weyl modules then get identified with Demazure modules of the basic representation \( L(\Lambda_0) \) of \( \mathfrak{sl}_2 \). As such, they are related by a chain of inclusions: \( W(0) \hookrightarrow W(2) \hookrightarrow W(4) \hookrightarrow \cdots \). It is natural to ask if the Chari-Pressley bases for the individual \( W(n) \) respect these inclusions. This question is the main focus of this paper.

As a first step, we define—see Equation (3.4)—a normalized version of the Chari-Pressley bases by replacing the powers in the monomials by divided powers and introducing a sign factor. These normalized bases, which we refer to throughout as the CPL (short for Chari-Pressley-Loktev) bases, have better properties with respect to inclusions of local Weyl modules. Indeed in our main result (Theorem 8) we show that the CPL bases respect inclusions “in the limit”. Note that it is too much to expect any result of this nature without passage to the limit (Example 7).

To state a little more precisely what we do, let \( \mathcal{P}(n) \) denote the parametrizing set of the CPL basis of \( W(n) \): the elements of \( \mathcal{P}(n) \) are pairs \((P, \pi)\) where \( P \) is a Gelfand-Tsetlin pattern for \( \mathfrak{sl}_2 \) with bounds \( n, 0 \) and \( \pi \) is a partition whose Young diagram fits into an \((n - p) \times p\) box, where \( n - 2p \) is the weight of the pattern \( P \). We first define a weight preserving embedding of \( \mathcal{P}(n) \) into \( \mathcal{P}(n + 2) \) for each \( n \), thereby obtaining a chain \( \mathcal{P}(0) \hookrightarrow \mathcal{P}(2) \hookrightarrow \mathcal{P}(4) \hookrightarrow \cdots \). Given an element \( \xi \) of \( \mathcal{P}(n) \), let \( \xi_k \) be its image in \( \mathcal{P}(n + 2k) \) (where \( k \) is a non-negative integer), and let \( c(\xi_k) \) be the corresponding CPL basis element. Consider the sequence \( c(\xi_k), k = 0, 1, 2, \ldots \), of elements in \( L(\Lambda_0) \). Our main result (Theorem 8) implies that this sequence stabilises for large \( k \). In fact, it says that \( c(\xi_k) \) equals the stable value as soon as \( k \) is such that the weight space of \( W(n + 2k) \) corresponding to the weight of \( \xi \) equals that of \( L(\Lambda_0) \). Passing to the direct limit, we obtain a basis of \( L(\Lambda_0) \) consisting of the stable CPL basis elements (see (3.3)). Moreover, we obtain an explicit description of the stable CPL basis in terms of elements of the Fock space of the homogeneous Heisenberg subalgebra of \( \mathfrak{sl}_2 \) (Equations (4.17), (5.3)).

As to the generalization of our results to the case \( \mathfrak{g} = \mathfrak{sl}_m \), we now briefly describe the issues that crop up. The parametrizing set \( \mathcal{P}(\lambda) \) of the Chari-Loktev basis of a local Weyl module \( W(\lambda) \) has a neat combinatorial description (see [7] for details): namely, it is the set of partition overlaid Gelfand-Tsetlin patterns with boundary row \( \lambda \). Further a natural normalization of the basis (analogue of Equation (3.1)) suggests itself. Denoting by \( \theta \) the highest root of \( \mathfrak{g} \), the identification of local Weyl modules as Demazure modules (of some fundamental representation of \( \widehat{\mathfrak{sl}_m} \)) gives us a natural chain of inclusions \( W(\lambda) \hookrightarrow W(\lambda + \theta) \hookrightarrow W(\lambda + 2\theta) \hookrightarrow \cdots \). Mirroring this, we have on the other hand a chain of weight preserving embeddings \( \mathcal{P}(\lambda) \hookrightarrow \mathcal{P}(\lambda + \theta) \hookrightarrow \mathcal{P}(\lambda + 2\theta) \hookrightarrow \cdots \). Given an element \( \xi \) of \( \mathcal{P}(\lambda) \), let \( \xi_k \) denote its image in \( \mathcal{P}(\lambda + k\theta) \), and \( c(\xi_k) \) the corresponding normalized basis element. It is tempting to conjecture, based on the evidence of the present paper, that the sequence \( c(\xi_k) \) stabilises, and further that \( c(\xi_k) \) equals the stable value once the weight space of \( W(\lambda + k\theta) \)
corresponding to the weight of $\xi$ stabilises. However, generalizing our methods to $g = sl_n$ presents formidable technical difficulties, and we hope to address these in future work.

This paper is organised as follows: in §2 we set up the notation and recall some fundamental facts concerning local Weyl modules; in §3 we give the definition of the CPL bases, the statement of our main theorem (Theorem 8), as well as an application to constructing bases of level 1 representations of $sl_2$. Also described in §3 is another variant of the Chari-Pressley basis, which is more natural when one thinks of Weyl modules as Demazure modules. The remaining sections are devoted to the proof of Theorem 8 in stages; the main case is when $n$ is even and the Gelfand-Tsetlin pattern of weight zero, and the proof of this occurs in §4. We show how the other cases can be reduced to this one using the translation operators of Frenkel-Kac (§5) and automorphisms of $sl_2$ (§6).

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2. Notation and Preliminaries

2.1. The affine Lie algebra $\widehat{sl}_2$. Let $sl_2$ be the Lie algebra of $2 \times 2$ trace zero matrices over the field $\mathbb{C}$ of complex numbers with standard basis

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Let $\mathfrak{h} = \mathbb{C} h$ be the standard Cartan subalgebra and $(A, B) \mapsto \text{trace}(AB)$ the normalized invariant bilinear form on $sl_2$.

Let $\mathbb{C}[t, t^{-1}]$ be the ring of Laurent polynomials in an indeterminate $t$. Let $\widehat{sl}_2$ be the affine Lie algebra defined by

$$\widehat{sl}_2 = sl_2 \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C} c \oplus \mathbb{C} d,$$

where $c$ is central and the other Lie brackets are given by

$$[At^m, Bt^n] = [A, B] t^{m+n} + m \delta_{m,-n} (A, B) c$$

$$[d, At^m] = m (At^m)$$

for all $A, B \in sl_2$ and integers $m, n$: here, as throughout the paper, $At^s$ is shorthand for $A \otimes t^s$. We let $\mathfrak{h} = \mathbb{C} h \oplus \mathbb{C} c \oplus \mathbb{C} d$, and regard $\mathfrak{h}^*$ as a subspace of $\mathfrak{h}^*$ by setting $\langle \lambda, c \rangle = \langle \lambda, d \rangle = 0$ for $\lambda \in \mathfrak{h}^*$.

Let $\alpha_0, \alpha_1$ denote the simple roots of $\widehat{sl}_2$ and let $\alpha_0^\vee = c - h, \quad \alpha_1^\vee = h$ be the corresponding coroots. Let $e_i, f_i (i = 0, 1)$ denote the Chevalley generators of $\widehat{sl}_2$; these are given by

$$e_1 = x, \quad f_1 = y, \quad e_0 = yt, \quad f_0 = xt^{-1}$$

We have

$$\langle \alpha_1, h \rangle = 2, \quad \langle \alpha_1, c \rangle = 0 \quad \langle \alpha_1, d \rangle = 0 \quad \text{and} \quad \langle \alpha_0, h \rangle = -2 \quad \langle \alpha_0, c \rangle = 0 \quad \langle \alpha_0, d \rangle = 1$$

Let $\delta = \alpha_0 + \alpha_1$ denote the null root, $\widehat{Q} = \mathbb{Z} \alpha_0 + \mathbb{Z} \alpha_1$ the root lattice, and $\widehat{Q}^+$ the non-negative integer span of $\alpha_0, \alpha_1$. The weight lattice (resp. the set of dominant weights) is defined by

$$\widehat{P} \ (\text{resp.} \ \widehat{P}^+) = \{ \lambda \in \mathfrak{h}^*: \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z} \ (\text{resp.} \ \mathbb{Z}_{\geq 0}), \ i = 0, 1 \}.$$

We define $\Lambda_0 \in \widehat{P}^+$ by $\langle \Lambda_0, h \rangle = 0, \langle \Lambda_0, c \rangle = 1, \langle \Lambda_0, d \rangle = 0$. 

The Weyl group $\hat{W}$ of $\hat{\mathfrak{sl}}_2$ is the subgroup of $GL(\mathfrak{h}^*)$ generated by the simple reflections $s_0, s_1$. These are defined by $s_i(\lambda) = \lambda - (\lambda, \alpha_i^\vee) \alpha_i$ for $\lambda \in \mathfrak{h}^*$, and $i = 0, 1$. There is a non-degenerate, symmetric, bilinear $\hat{W}$-invariant form $\langle \cdot , \cdot \rangle$ on $\mathfrak{h}^*$, given by requiring that $C\alpha_1$ be orthogonal to $\mathbb{C}\delta + \mathbb{C}\Lambda_0$, together with the relations $(\alpha_1|\alpha_1) = 2, (\delta|\delta) = (\Lambda_0|\Lambda_0) = 0, (\delta|\Lambda_0) = 1$.

Given $\alpha \in \mathfrak{h}^*$, we define $t_\alpha \in GL(\hat{\mathfrak{h}}^*)$ by

$$t_\alpha(\lambda) = \lambda + (\lambda|\delta) \alpha - (\lambda|\alpha) \delta - \frac{1}{2} (\lambda|\delta)(\alpha|\alpha) \delta \quad \text{for} \ \lambda \in \hat{\mathfrak{h}}^*.$$ (2.3)

The translation subgroup $T$ of $\hat{W}$ is defined by $T = \{t_{j\varpi_1} : j \in \mathbb{Z}\}$. We have $\hat{W} = W \ltimes T$, where $W = \{1, s_1\}$ is the underlying finite Weyl group. Now let $\varpi_1 = \alpha_1/2$; then $Q = \mathbb{Z}\alpha_1$ and $P = \mathbb{Z}\varpi_1$ are the root and weight lattices of the underlying $\mathfrak{sl}_2$. We also let $P^+ = \mathbb{Z}_{\geq 0} \varpi_1$ be the set of dominant weights of the underlying finite type diagram.

The extended affine Weyl group $\hat{W}_{ex}$ is the semi-direct product

$$\hat{W}_{ex} = W \ltimes T_{ex}$$

where $T_{ex} = \{t_{j\varpi_1} : j \in \mathbb{Z}\}$. Now consider the element $\sigma = s_1 t_{-\varpi_1} \in \hat{W}_{ex}$. This induces the diagram automorphism of the Dynkin diagram of $\hat{\mathfrak{sl}}_2$; we have

$$\sigma \alpha_0 = \alpha_1, \sigma \alpha_1 = \alpha_0, \sigma \rho = \rho.$$ Here, $\rho \in \mathfrak{h}^*$ is the Weyl vector, defined by $\langle \rho, \alpha_i^\vee \rangle = 1$ for $i = 0, 1$ and $\langle \rho, d \rangle = 0$. We also have $\hat{W}_{ex} = \hat{W} \ltimes \Sigma$, where $\Sigma = \{1, \sigma\}$ is the subgroup generated by $\sigma$.

2.2. The basic representation $L(\Lambda_0)$ of $\hat{\mathfrak{sl}}_2$. Given $\Lambda \in \hat{P}^+$, let $L(\Lambda)$ be the irreducible $\hat{\mathfrak{sl}}_2$-module with highest weight $\Lambda$. It is the cyclic $\mathfrak{sl}_2$-module generated by $v_\Lambda$, with defining relations

$$hv_\Lambda = \langle \Lambda, h \rangle v_\Lambda \quad \forall h \in \hat{\mathfrak{h}}$$ (2.4)

$$e_iv_\Lambda = 0 \quad (i = 0, 1)$$ (2.5)

$$f_i^{(\Lambda, \alpha_i^\vee) + 1} v_\Lambda = 0 \quad (i = 0, 1)$$ (2.6)

It has weight space decomposition $L(\Lambda) = \bigoplus_{\mu \in \hat{\mathfrak{h}}^*} L(\Lambda)_\mu$. The $\mu$ for which $L(\Lambda)_\mu \neq 0$ are the weights of $L(\Lambda)$. The module $L(\Lambda_0)$ is particularly well-understood; the following well-known proposition describes the weight spaces of $L(\Lambda_0)$ [9].

**Proposition 1.**

1. The set of weights of $L(\Lambda_0)$ is $\{t_{j\alpha_1}(\Lambda_0) - d\delta \mid j \in \mathbb{Z}, d \in \mathbb{Z}_{\geq 0}\}$.

2. $\dim \left( L(\Lambda_0)_{t_{j\alpha_1}(\Lambda_0) - d\delta} \right) = p(d)$, the number of partitions of $d$.

We let $\Lambda_1 = \sigma \Lambda_0$. Then, $\Lambda_0, \Lambda_1$ are (a choice of) fundamental weights corresponding to the coroots $\alpha_0^\vee, \alpha_1^\vee$, i.e., $\langle \Lambda_i, \alpha_j^\vee \rangle = \delta_{ij}$ for $i, j \in \{0, 1\}$. We let $v_{\Lambda_i}$ denote a highest weight vector of $L(\Lambda_i)$ for $i = 0, 1$. 


2.3. The current algebra and its Weyl modules. Let $\mathbb{C}[t]$ be the polynomial ring in an indeterminate $t$. The current algebra $\mathfrak{sl}_2[t] = \mathfrak{sl}_2 \otimes \mathbb{C}[t]$ is a Lie algebra with Lie bracket obtained from that of $\mathfrak{sl}_2$ by extension of scalars to $\mathbb{C}[t]$: $[At^m, Bt^n] = [A, B]t^{m+n}$ for all $A, B$ in $\mathfrak{sl}_2$ and non-negative integers $m, n$. As such, it is a subalgebra of $\mathfrak{sl}_2$.

**Definition 2.** (see [1 §1.2.1]) Given $n \in \mathbb{Z}_{\geq 0}$, the local Weyl module $W(n)$ is the cyclic $\mathfrak{sl}_2[t]$-module with generator $w_n$ and relations:

$$(xt^s)w_n = 0, \quad (ht^{s+1})w_n = 0, \quad hw_n = nw_n, \quad y^{n+1}w_n = 0 \quad \text{for all } s \geq 0 \quad (2.7)$$

2.4. Weyl modules as Demazure modules. We recall that the standard Borel subalgebra of $\mathfrak{sl}_2$ is

$$\hat{\mathfrak{b}} = \mathfrak{sl}_2 \otimes t\mathbb{C}[t] \oplus \mathbb{C}x \oplus \hat{\mathfrak{h}}.$$ 

Let $w$ be in $\hat{\mathcal{W}}$ and $\Lambda$ in $\hat{\mathcal{P}}^+$. The weight space $L(\Lambda)_{w\Lambda}$ of $L(\Lambda)$ has dimension one (since two weights that are Weyl group conjugates have the same multiplicities).

Define $V_w(\Lambda) := \mathfrak{U}\hat{\mathfrak{b}} (L(\Lambda)_{w\Lambda})$ where $\mathfrak{U}\hat{\mathfrak{b}}$ denotes the universal enveloping algebra of $\hat{\mathfrak{b}}$. Then, $V_w(\Lambda)$ is a $\mathfrak{U}\hat{\mathfrak{b}}$-submodule of $L(\Lambda)$, called the Demazure module of $L(\Lambda)$ associated to $w$. More generally, given an element $w$ of the extended affine Weyl group $\hat{\mathcal{W}}_{\mathrm{ex}}$, we write $w = u\tau$ with $u \in \hat{\mathcal{W}}, \tau \in \Sigma$ and define, following [4], the associated Demazure module by $V_w(\Lambda) := V_u(\tau(\Lambda))$.

We will consider the modules $V_{\lambda}(\Lambda_0)$ for $\lambda \in P$. It is convenient to use the notation of [4] and set

$$D(1, \lambda) := V_{\lambda}(\Lambda_0).$$

Since $\Sigma = \{1, \sigma\}$, the $D(1, \lambda)$ are Demazure modules for $L(\Lambda_0)$ (when $\lambda \in Q$) or $L(\Lambda_1)$ (when $\lambda$ is a nontrivial element of $P/Q$). Further, $D(1, \lambda)$ is $\mathfrak{sl}_2[t]$-stable (not just $\hat{\mathfrak{b}}$-stable) if and only if $\lambda \in P^+$ [4].

The following theorem identifies the $\mathfrak{sl}_2[t]$-stable Demazure modules with the Weyl modules of the current algebra:

**Theorem 3.** (Chari-Loktev [1], Fourier-Littelmann [4]) The local Weyl module $W(n)$ is isomorphic to the Demazure module $D(1, n\varpi_1)$, as modules for the current algebra $\mathfrak{sl}_2[t]$.

This isomorphism maps the generator $w_n$ of $W(n)$ to a vector of $L(\Lambda_\varpi)$, which we will also denote $w_n$. Here $\varpi$ is 0 if $n$ is even and 1 if $n$ is odd. By [1 Corollary 1.5.1] (see also [4 corollary 1]), the weight $\gamma$ of the vector $w_n \in L(\Lambda_\varpi)$ is a Weyl conjugate of $\Lambda_\varpi$. Further, we must have $\langle \gamma, h \rangle = n$. It follows from (2.3) that $\gamma = t_{n\alpha_1/2}(\Lambda_0)$ (respectively $t_{(n-1)\alpha_1/2}(\Lambda_1)$) if $n$ is even (respectively, if $n$ is odd).

Since the $\gamma$-weight space of $L(\Lambda_\varpi)$ is one-dimensional, this isomorphism identifying the Weyl module as a Demazure module is unique up to scaling. We will fix the following choice of $w_n$ for the rest of the paper:

$$w_n := \begin{cases} 
(t^{-\frac{n}{2}})^{(\#)} v_{\Lambda_0} & \text{if } n \text{ even} \\
(t^{-\frac{n+1}{2}})^{(\#)} v_{\Lambda_1} & \text{if } n \text{ odd}
\end{cases} \quad (2.8)$$
where we have used the “divided power notation”: $X^{(p)} := X^p/p!$. It is clear that $w_n$ has weight $\gamma$; the fact that $w_n \neq 0$ will follow from Proposition 16(1) for $n$ even, and from the arguments of §6.2 for $n$ odd. We will henceforth identify $W(n)$ with $D(1, n\varpi_1)$ by the isomorphism defined by this choice of $w_n$, and think of $W(n)$ as a subspace of $L(\Lambda_0)$. 

2.5. Inclusions of Weyl modules. Let $\Lambda \in \hat{P}^+$ and $\hat{W}_\Lambda := \{ w \in \hat{W} | w\Lambda = \Lambda \}$. For elements $w_1 \leq w_2$ of $\hat{W}/\hat{W}_\Lambda$, where $\leq$ denotes the Bruhat order on $\hat{W}/\hat{W}_\Lambda$, the Demazure module $V_{w_1}(\Lambda)$ is included in $V_{w_2}(\Lambda)$ (as submodules of $L(\Lambda)$). Specializing to our case, we have, for $n$ even, 

\[
W(n) = V_{t-n\varpi_1}(\Lambda_0) \subseteq V_{t-(n+2)\varpi_1}(\Lambda_0) = W(n+2)
\] 

(2.9) since $t-n\varpi_1 \leq t-(n+2)\varpi_1 = s_1s_0t-n\varpi_1$. For $n$ odd, we have $W(n) = V_{t-(n-1)\varpi_1}(\Lambda_1)$, since $t-\varpi_1 = s_1\sigma$. A similar argument to the above establishes $W(n) \subset W(n+2)$ in this case as well. We thus have the following chains of embeddings:

\[
W(0) \hookrightarrow W(2) \hookrightarrow \ldots \hookrightarrow W(2n) \hookrightarrow W(2n+2) \hookrightarrow \ldots \hookrightarrow L(\Lambda_0).
\] 

(2.10) 

\[
W(1) \hookrightarrow W(3) \hookrightarrow \ldots \hookrightarrow W(2n+1) \hookrightarrow W(2n+3) \hookrightarrow \ldots \hookrightarrow L(\Lambda_1).
\] 

(2.11) 

3. The main results

3.1. Bases for Weyl modules. We first recall some results of Chari-Pressley [3] (see also Chari-Loktev [1]) which give a basis for the local Weyl module $W(n)$. We begin by introducing some notation. Let $\mathcal{Y}$ denote the set of all partitions. Elements of $\mathcal{Y}$ are infinite sequences $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$ of non-negative integers such that (i) $\lambda_i \geq \lambda_{i+1}$ for all $i \geq 1$ and (ii) $\lambda_j = 0$ for all sufficiently large $j$. We let $|\lambda| = \sum_i \lambda_i$, and write $\lambda \vdash r$ to mean $\lambda \in \mathcal{Y}$ with $|\lambda| = r$. Let $\text{supp} \lambda = \min \{ j \geq 0 : \lambda_{j+1} = 0 \}$. Given non-negative integers $a, b$, let

\[
\mathcal{Y}(a, b) := \{ \lambda \in \mathcal{Y} : \lambda_1 \leq b \text{ and } \text{supp} \lambda \leq a \}
\] 

(3.1) 

We identify partitions with Young diagrams in the standard way: the Young diagram corresponding to a partition $\lambda$ is also denoted $\lambda$ and consists of an arrangement of square boxes, all of the same size (the sides are of unit length), numbering $|\lambda|$ in all, arranged left-and top-justified, $\lambda_1$ on the first row, $\lambda_2$ on the second row (which is below the first row), and so on:

```
\lambda_1 
\lambda_2 
\lambda_3 
\vdots 
\lambda_a 
```

where $s = \text{supp} \lambda$. In this language, $\mathcal{Y}(a, b)$ is the set of partitions whose Young diagrams fit into a rectangular $a \times b$ box:
Next, we define the set which will parametrize bases of local Weyl modules:

$$\mathfrak{P} := \{(m,k,\lambda) : m, k \in \mathbb{Z} \text{ with } m \geq k \geq 0, \text{ and } \lambda \in \mathcal{Y}(m-k,k)\}. \quad (3.2)$$

In light of \([1]\) a triple \((m,k,\lambda) \in \mathfrak{P}\) should be thought of as the pair \((\text{GT}_{m,k}, \lambda)\) where

$$\text{GT}_{m,k} = \begin{pmatrix} k & 0 \\ m & k \end{pmatrix}$$

is a Gelfand-Tsetlin pattern for \(\mathfrak{sl}_2\). Associated to this pattern is a box of size \((m-k) \times (k-0)\), and the condition in (3.2) says that the Young diagram of \(\lambda\) should fit into this box.

For each non-negative integer \(n\), we also define

$$\mathfrak{P}(n) := \{(m,k,\lambda) \in \mathfrak{P} : m = n\}.$$ 

Given \(\xi = (n,k,\lambda) \in \mathfrak{P}(n)\) with \(\lambda = (\lambda_1, \lambda_2, \lambda_3, \cdots)\), define the following element of \(W(n)\):

$$B(\xi) := \left(\prod_{i=1}^{n-k} y^{t_{k-\lambda_i}}\right) w_n. \quad (3.3)$$

We note that since \([y^j, y^k] = 0\) for all \(j, k \geq 0\), the order of terms in the product in equation (3.3) is immaterial. We now have the following important theorem due to Chari and Pressley (see also \([1]\)):

**Theorem 4.** (Chari-Pressley \([3]\)) Let \(n \geq 0\). Then \(\{B(\xi) | \xi \in \mathfrak{P}(n)\}\) is a basis of the local Weyl module \(W(n)\).

### 3.2. The CPL basis elements \(c(\xi)\).

Our primary goal in this paper is to study the compatibility of the Chari-Pressley bases of \(W(n)\) with the chain of embeddings in equations (2.10) and (2.11). As a first step, we slightly modify the definition of these bases, introducing normalization factors and parametrizing them by the complements of partitions in \(\mathcal{Y}(n-k,k)\), rather than by the partitions themselves. More precisely, given \(\xi = (n,k,\lambda) \in \mathfrak{P}(n)\), define

$$c(\xi) := z(\xi) \left(\prod_{i=1}^{n-k} y^{t_{k-\lambda_i}}\right) w_n. \quad (3.4)$$

where \(z(\xi)\) is a normalization factor. To specify \(z(\xi)\), we first let \(m_j := \#\{i : \lambda_i = j\}\) denote the multiplicity of the part \(j\) in \(\lambda\) for each \(j \geq 1\), and let \(m_0 := n-k-\text{supp} \lambda\). Then, we have

$$\prod_{i=1}^{n-k} y^{t_{k-\lambda_i}} = \prod_{j=0}^{k} (y^{t_{k-j}})^{m_j}.$$ 

The normalization factor is given by

$$z(\xi) := \frac{(-1)^{\frac{1}{2}(\frac{n-k}{2})} \prod_{j=0}^{k} (m_j)!}{\prod_{j=0}^{k} m_j!}.$$
Here, \([x]\) denotes the greatest integer less than or equal to \(x\). We may also rewrite (3.4) in terms of divided powers; we have
\[
c(\xi) = \epsilon(\xi) y^{(m_k)} (y^{t_1})^{(m_{k-1})} \cdots (y^{t_k})^{(m_0)} w_n
\]
where \(\epsilon(\xi) = (-1)^{[\frac{n}{2}] - [\frac{n-k}{2}]}\).

Given \(\xi = (n, k, \lambda) \in \mathcal{P}(n)\), with \(s = \text{supp} \lambda\), define \(\lambda^c \in \mathcal{Y}(n-k, k)\) by
\[
\lambda^c := (k, k, \cdots, k, k - \lambda_s, k - \lambda_{s-1}, \cdots, k - \lambda_1, 0, 0, \cdots),
\]
where the initial string of \(k\)’s is of length \(n-k-s\). The Young diagrams of \(\lambda^c\) and \(\lambda\), the latter rotated by \(180^\circ\) and appropriately translated, are complements of each other in the \((n-k) \times k\) box:

\[
\begin{array}{c}
\begin{array}{c}
\lambda^c \\
\end{array}
\end{array}
\]

Letting \(\xi^c = (n, k, \lambda^c)\), it is clear that \(\xi^c \in \mathcal{P}(n)\) and \(c(\xi) = z(\xi) B(\xi^c)\). This of course implies that the set
\[
\mathcal{C}(n) := \{c(\xi) : \xi \in \mathcal{P}(n)\}
\]
is also a basis of \(W(n)\). We call this the \textit{CPL basis} of \(W(n)\).

We now view \(W(n)\) as a subspace of \(L(\Lambda_n)\) as in equations (2.10) and (2.11). The weight of \(c(\xi)\) in \(L(\Lambda_n)\) is given by the following lemma.

\textbf{Lemma 5.} Let \(\xi = (n, k, \lambda) \in \mathcal{P}\). Then

(1) Weight of \(c(\xi) = t_{(k-n) \alpha_1} (\text{weight of } w_n) - |\lambda| \delta\).

(2) If \(n\) is even, the weight of \(c(\xi)\) in \(L(\Lambda_0)\) is \(t_{(k-\frac{n}{2}) \alpha_1} (\Lambda_0) - |\lambda| \delta\).

(3) If \(n\) is odd, the weight of \(c(\xi)\) in \(L(\Lambda_1)\) is \(t_{(k-\frac{n+1}{2}) \alpha_1} (\Lambda_1) - |\lambda| \delta\).

\textbf{Proof.} From (3.4), we have
\[
\text{wt}(c(\xi)) = \text{wt}(w_n) - (n-k) \alpha_1 + \delta \sum_{i=1}^{n-k} (k-\lambda_i)
\]
\[
= \text{wt}(w_n) + (k-n) \alpha_1 + k(n-k) \delta - |\lambda| \delta.
\]
Let \(\beta = \frac{n}{2} \alpha_1\) if \(n\) is even, and \(\frac{n-1}{2} \alpha_1\) if \(n\) is odd. Then \(\text{wt}(w_n) = t_\beta(\Lambda_n)\). Since \(t_{(k-n) \alpha_1}\) and \(t_\beta\) commute, the first part of the lemma is implied by the following identity, which can be verified directly using (2.3):
\[
t_{(k-n) \alpha_1} (\Lambda_n) = \Lambda_n + (k-n) t_{-\beta}(\alpha_1) + k(n-k) \delta.
\]
Assertions (2) and (3) are obvious from (1). \(\square\)
3.3. The main theorem: stability of the CPL bases. We wish to study the compatibility of the bases \( C(n) \) and \( C(n + 2) \) with respect to the embedding \( W(n) \hookrightarrow W(n + 2) \). As a first step, we define a weight preserving embedding at the level of the parametrizing sets of these bases. Define the map \( \psi : \mathfrak{P} \rightarrow \mathfrak{P} \) by

\[
\psi(n, k, \lambda) = (n + 2, k + 1, \lambda).
\]

This is well defined, since \( \mathcal{Y}(n - k, k) \) is a subset of \( \mathcal{Y}(n - k + 1, k + 1) \). Further, \( \psi \) is injective, and maps \( \mathfrak{P}(n) \) to \( \mathfrak{P}(n + 2) \) for all \( n \). Now, the following is immediate from Lemma 5.

**Lemma 6.** Let \( \xi \in \mathfrak{P}(n) \). Then the basis vectors \( c(\xi) \in W(n) \) and \( c(\psi(\xi)) \in W(n + 2) \) lie in the same weight space of \( L(\Lambda) \).

However, it is not true in general that \( c(\xi) \) and \( c(\psi(\xi)) \) are equal as elements of \( L(\Lambda) \), as the following example shows.

**Example 7.** Let \( \lambda \) be the partition \( 2 + 1 \), i.e., \( \lambda = (2, 1, 0, 0, \cdots) \). Let \( \xi = (4, 2, \lambda) \). Then \( \xi \in \mathfrak{P}(4) \), and \( \psi(\xi) = (6, 3, \lambda) \). Using (3.4), (2.8) and the commutation relations in \( \mathfrak{sl}_2 \), it is easy to compute:

\[
c(\xi) = \frac{1}{3} \left( h^{-3} - (h^{-1})^3 \right) v_{\Lambda_0}
\]

\[
c(\psi(\xi)) = \left( h^{-3} + h^{-2}h^{-1} \right) v_{\Lambda_0}.
\]

Both these vectors have weight \( \Lambda_0 - 3\delta \). It is well known that the vectors \( h^{-3}v_{\Lambda_0} \), \( h^{-2}h^{-1}v_{\Lambda_0} \), \( (ht^{-1})^3 v_{\Lambda_0} \) form a basis of the weight space \( L(\Lambda_0)_{\Lambda_0 - 3\delta} \). Thus, we conclude \( c(\xi) \neq c(\psi(\xi)) \). \( \square \)

We will however see below that \( c(\xi) = c(\psi(\xi)) \) for all stable \( \xi \). More precisely, let

\[
\mathfrak{P}^{\text{stab}}(n) := \begin{cases} 
\{ (n, k, \lambda) \in \mathfrak{P}(n) : |\lambda| \leq \min(n - k, k) \} & \text{if } n \text{ is even} \\
\{ (n, k, \lambda) \in \mathfrak{P}(n) : |\lambda| \leq \min(n - k, k - 1) \} & \text{if } n \text{ is odd}
\end{cases}
\]

and \( \mathfrak{P}^{\text{stab}} = \bigsqcup_{n \geq 0} \mathfrak{P}^{\text{stab}}(n) \).

We note that \( \xi \in \mathfrak{P}^{\text{stab}}(n) \) implies \( \psi(\xi) \in \mathfrak{P}^{\text{stab}}(n + 2) \). The following is the main result of this paper.

**Theorem 8.** Let \( n \) be a non-negative integer and \( \xi = (n, k, \lambda) \in \mathfrak{P}^{\text{stab}} \). Then

\[
c(\xi) = c(\psi(\xi)),
\]

i.e., they are equal as elements of \( L(\Lambda) \).

This theorem is proved in §4.

3.4. Passage to the direct limit: a basis for \( L(\Lambda_0) \). Theorem 8 allows us to construct a basis of \( L(\Lambda_p) \) \((p = 0, 1)\) by taking the direct limit of the \( C(n) \) (for \( n \equiv p \) (mod 2)). We explain this below for \( p = 0 \), the case \( p = 1 \) being similar. Consider \( L(\Lambda_0) \), and let \( \mu = t_{j\alpha_1}(\Lambda_0) - d\delta \) \((j \in \mathbb{Z}, d \in \mathbb{Z}_{\geq 0})\) be a weight of this module. Define

\[
\mathfrak{P}_\mu := \{ (n, k, \lambda) \in \mathfrak{P} : k - \frac{n}{2} = j \text{ and } |\lambda| = d \}.
\]

(3.6)
We note that \( \xi = (n, k, \lambda) \in \Psi_\mu \) forces \( n \) to be even; further, it is clear from Lemma 3 that \( \z(\xi) \) has weight \( \mu \) iff \( \xi \in \Psi_\mu \).

Now, let \( \Psi_\mu(n) = \Psi_\mu \cap \Psi(n) \). This set parametrizes the basis elements of \( W(n) \) of weight \( \mu \). By (3.6), the cardinality of \( \Psi_\mu(n) \) is the number of partitions of \( d \) which fit into a \( (\frac{d}{2} - j) \times (\frac{d}{2} + j) \) box. Thus, for large enough \( n \), \( \Psi_\mu(n) \) contains exactly \( p(d) \) (the number of partitions of \( d \)) elements; in particular this implies that \( \psi \) induces a bijection of the sets \( \Psi_\mu(n) \) and \( \Psi_\mu(n + 2) \). Further, it is also clear that for large \( n \), every \( \xi \in \Psi_\mu(n) \) is stable. More precisely, we have

\[
|\Psi_\mu(n)| = p(d) \quad \text{and} \quad \Psi_\mu(n) \subset \Psi^{\text{stab}} \quad \text{for all even } n \geq 2(d + |j|).
\]

Choosing any such \( n \), say \( n = 2(d + |j|) \), we define the following (linearly independent) subset of \( L(\Lambda_0)_\mu \):

\[
B_\mu := \{ \z(\xi) : \xi \in \Psi_\mu(n) \}.
\]

By Theorem 8 and the remarks above, this is independent of the choice of \( n \). Since by Proposition 1 the dimension of \( L(\Lambda_0)_\mu \) is also \( p(d) \), we conclude that \( B_\mu \) is a basis of the weight space \( L(\Lambda_0)_\mu \).

Finally, to obtain a basis of \( L(\Lambda_0) \), we take the disjoint union over the weights of \( L(\Lambda_0) \):

\[
B := \bigcup_\mu B_\mu.
\]

We may view \( B \) as a direct limit of the CPL bases \( C(n) \) (\( n \) even) of the Demazure modules (=local Weyl modules) of \( L(\Lambda_0) \).

3.5. A variation on the theme. We note that the generator \( w_n \) of \( W(n) = D(1, n\varpi_1) \) is not a lowest weight vector of the Demazure module \( D(1, n\varpi_1) \); while the lowest weight in \( D(1, n\varpi_1) \) is \( t_{-n\varpi_1}(\Lambda_0) \), the weight of \( w_n \) is in fact \( t_{n\varpi_1}(\Lambda_0) \). From the basis \( B(\xi) \) of equation (3.3), it is easy to construct a basis consisting of monomials in the raising operators of the current algebra acting on a lowest weight vector \( v_n \) of the Demazure module. Given \( \xi = (n, k, \lambda) \in \Psi(n) \), with \( \lambda = (\lambda_1, \lambda_2, \lambda_3, \cdots) \), define the following element of \( W(n) \):

\[
\overline{B}(\xi) := \left( \prod_{i=1}^{n-k} x^\lambda_i \right) v_n.
\]

We now have:

**Proposition 9.** The set \( \{ \overline{B}(\xi) | \xi \in \Psi(n) \} \) is a basis of the local Weyl module \( W(n) \).

The proof appears in [8]. This basis also admits a normalised version which exhibits similar stabilization behaviour as the CPL basis.

4. Proof of the main case of the main theorem

4.1. In this section we prove Theorem 1 in the special case that \( \xi = (n, k, \lambda) \in \Psi^{\text{stab}} \) with \( n \) even and \( k = n/2 \). In this case, the weight of \( \z(\xi) \) in \( L(\Lambda_0) \) is \( \Lambda_0 - |\lambda| \delta \). From equations (3.4) and (2.8), we have

\[
\z(\xi) := z(\xi) \left( \prod_{i=1}^{k} y^{k_i - \lambda_i} \right) (xt^{-k})^{(k)} v_{\Lambda_0}.
\]
Now, let \( t = \oplus_{n \in \mathbb{Z}} L(h) \oplus \mathbb{C} c \) denote the homogeneous Heisenberg subalgebra of \( \hat{sl}_2 \). Recall that the subspace \( \oplus_{n \geq 0} L(h) \Lambda_n \) is invariant under \( t \) and is isomorphic to the canonical commutation relations representation (Fock space) of \( t \). Thus, each element of this subspace can be uniquely expressed as a polynomial in (the infinitely many variables) \( h, t^{-1}, t^{-2}, \ldots \), acting on \( v_{\Lambda_0} \). In particular, there is a unique polynomial \( f_\xi(h, t^{-1}, t^{-2}, \ldots) \) such that

\[
c(\xi) = f_\xi(h, t^{-1}, t^{-2}, \ldots) v_{\Lambda_0}.
\]

\( \ominus \)

Our first goal is to determine \( f_\xi \) explicitly by applying the straightening rules in \( \mathfrak{H} \hat{sl}_2 \) to equation (4.1). We will then show that \( f_\xi = f_{\psi(\xi)} \) for \( \xi \in \mathfrak{P}^{\text{stab}} \), thereby establishing Theorem 8 in this case. The details appear in the next subsections.

4.2. For \( r \geq 1 \), we let \( [r] := \{1, 2, \ldots , r\} \). Let \( \pi \in \mathcal{Y} \) be a partition such that \( \mid \pi \mid = r \) and \( \text{supp} \pi = s \). A set partition of \( [r] \) of type \( \pi \) is a collection \( B = \{B_1, B_2, \ldots , B_s\} \) of pairwise disjoint subsets of \( [r] \) such that \( \bigcup_{i=1}^s B_i = [r] \) and \( |B_i| = \pi_i \) for all \( i \in [s] \). We let \( \mathcal{Y}(\pi) \) denote the set of all set partitions of \( [r] \) of type \( \pi \).

Now, let \( B = \{B_1, B_2, \ldots , B_s\} \in \mathcal{Y}(\pi) \); given \( \sigma \in S_r \) (the symmetric group on \( r \) letters), \( p = (p_1, p_2, \ldots , p_r) \in \mathbb{N}^r \) and \( q = (q_1, q_2, \ldots , q_r) \in \mathbb{N}^r \), define the following element of \( \mathfrak{H} t \):

\[
W(B, \sigma; p, q) := \prod_{j=1}^s h^{\sum_{i \in B_j}(p_i - q_{\sigma(i)})}.
\]

We also define

\[
\mathcal{H}(\pi; p, q) := \frac{1}{\pi_1! \ldots \pi_s!} \sum_{B \in \mathcal{Y}(\pi)} W(B, \sigma; p, q).
\]

With these notations we can state the following theorem.

**Theorem 10.** Let \( r \geq 1 \). For every triple \((p, q, v)\) with \( p = (p_1, p_2, \ldots , p_r) \in \mathbb{N}^r \), \( q = (q_1, q_2, \ldots , q_r) \in \mathbb{N}^r \) and \( v \in L(h) \), satisfying

1. \( p_i < q_j \) for all \( i, j \in [r] \),
2. \( \sum_{i \in A} p_i \geq \sum_{j \in B} q_j \) for all subsets \( A, B \) of \([r]\) such that \( |A| = |B| + 1 \),
3. \( yt^{\sum_{i \in A} p_i - \sum_{j \in B} q_j} v = 0 \) for all subsets \( A, B \) of \([r]\) such that \( |A| = |B| + 1 \),

we have

\[
\left( \prod_{i=1}^r yt^{p_i} \right) \left( \prod_{j=1}^r xt^{-q_j} \right) v = (-1)^r \sum_{\pi \in \mathcal{Y}} C(\pi) \mathcal{H}(\pi; p, q) v,
\]

where for \( \pi = (\pi_1, \pi_2, \ldots) \), \( C(\pi) = \prod_{i=1}^{\text{supp} \pi} \pi_i! (\pi_i - 1)! \).

**Proof.** We proceed by induction on \( r \). First, for \( r = 1 \), consider \( yt^{p_1} xt^{-q_1} v \). Since \( yt^{p_1} v = 0 \) and \( p_1 \neq q_1 \), we have

\[
yt^{p_1} xt^{-q_1} v = [yt^{p_1}, xt^{-q_1}] v = -ht^{p_1 - q_1} v.
\]
as required. Now let \( r \geq 2 \), and assume the result for \( r - 1 \). Consider \((\prod_{i=1}^{r} y^{p_{i}})(\prod_{j=1}^{r} x^{t-q_{j}})v\). Since \( y^{p_{r}}v = 0 \) and \( p_{r} \neq q_{j} \) for all \( j \), we may replace \( y^{p_{r}}(\prod_{j=1}^{r} x^{t-q_{j}})v \) by

\[
[y^{p_{r}}, \prod_{j=1}^{r} x^{t-q_{j}}]v = (-1) \sum_{l=1}^{r} \left( \prod_{j=l+1}^{r} x^{t-q_{j}} \right) h^{p_{r}-q_{l}} \left( \prod_{j=1}^{l-1} x^{t-q_{j}} \right) v.
\]

Next, using \([ht^{p_{r}-q_{l}}, x^{t-q_{j}}] = 2x^{t-q_{l}+p_{r}}\), we can commute the \( ht^{p_{r}-q_{l}} \) term past the \((\prod_{j=1}^{r-1} x^{t-q_{j}})\). This yields

\[
(−1)^{r} \prod_{i=1}^{r} y^{p_{i}} \prod_{j=1}^{r} x^{t-q_{j}} v = \sum_{l=1}^{r} \prod_{j=1}^{r} y^{p_{j}} \prod_{j \neq l}^{r} x^{t-q_{j}} (ht^{p_{r}-q_{l}} v) + 2 \sum_{m < l, m < l}^{r} \prod_{j=1}^{r} y^{p_{j}} \prod_{j \neq l, m}^{r} x^{t-q_{j}} (xt^{-q_{m}-q_{l}+p_{r}} v). \tag{4.5}
\]

We now consider the first sum in equation \((4.5)\). Fix \( l \in [r] \) and let \( p' \) and \( q' \) denote the \( r-1 \) tuples obtained by deleting \( p_{r} \) from \( p \) and \( q_{r} \) from \( q \) respectively. We also let \( v' = ht^{p_{r}-q_{l}} v \). Then, we claim that the triple \((p', q', v')\) satisfies the hypotheses (1)-(3) of the theorem. The first two hypotheses are clear; now given \( A \subset [r-1] \) and \( B \subset [r]\{l\} \) with \(|A| = |B| + 1\), we have

\[
yt(\sum_{i \in A} p_{i} - \sum_{j \in B} q_{j})v' = \left[ yt(\sum_{i \in A} p_{i} - \sum_{j \in B} q_{j}), ht^{p_{r}-q_{l}} \right] v = 2yt(\sum_{i \in A \cup \{r\}} p_{i} - \sum_{j \in B \cup \{l\}} q_{j}) v = 0,
\]

thereby verifying hypothesis (3). By the induction hypothesis, we obtain

\[
\prod_{i=1}^{r-1} \prod_{j=1}^{r} x^{t-q_{j}} (ht^{p_{r}-q_{l}} v) = (-1)^{r-1} \sum_{\pi' \vdash r-1} C(\pi') \mathcal{H}(\pi'; p', q') ht^{p_{r}-q_{l}} v. \tag{4.7}
\]

The second sum in equation \((4.5)\) is treated analogously. Fix \( l, m \in [r] \) with \( m < l \) and let \( q'' \) denote the \( r-1 \) tuple obtained from \( q \) by deleting \( q_{l} \) and appending \( q_{m} - p_{r} \). We also let \( p'' = (p_{1}, p_{2}, \ldots, p_{r-1}) \) and \( v'' = v \). The triple \((p'', q'', v'')\) evidently satisfies the hypotheses of the theorem. Again, the induction hypothesis implies

\[
\prod_{i=1}^{r-1} \prod_{j=1}^{r} x^{t-q_{j}} (xt^{-q_{m}-q_{l}+p_{r}} v) = (-1)^{r-1} \sum_{\pi'' \vdash r-1} C(\pi'') \mathcal{H}(\pi''; p'', q'') v. \tag{4.8}
\]

Fix a partition \( \pi \vdash r \), with \( \pi = (\pi_{1}, \pi_{2}, \ldots) \) and \( s = \text{supp} \pi \). We can now find the coefficient \( C(\pi) \) that occurs in equation \((4.3)\). Since \( y^{p_{i}} \) commute pairwise and likewise the \( x^{t-q_{j}} \), it is clear that the expression for \((\prod_{i=1}^{r} y^{p_{i}})(\prod_{j=1}^{r} x^{t-q_{j}}) v \) is invariant under the \( S_{r} \times S_{s} \) action that
permutes the $p_i$ and $-q_j$ amongst themselves. Thus, to find $C(\pi)$ it is enough to find the coefficient of the canonical word,

$$ht^{\sum_{i=1}^s (p_i - q_i)} \prod_{i=1}^s (p_i - q_i) \ldots ht^{\sum_{i=1}^{s-1} (p_i - q_i)} \ldots ht^{\sum_{i=1}^{s-1} (p_i - q_i)}$$

in the RHS of (4.5).

We consider two cases (a) $\pi_s = 1$, and (b) $\pi_s \geq 2$. In case (a), it is clear from equations (4.5), (4.7) and (4.8) that the canonical word above occurs only in $\prod_{j=1}^{r-1} yt^{p_j} \prod_{j=1}^{r-1} xt^{-q_j} (ht^{p_r - q_r} v)$, and with coefficient $C(\pi')$ where $\pi' = (\pi_1, \pi_2, \ldots, \pi_{s-1}) \vdash r - 1$. Thus,

$$C(\pi) = C(\pi') = \prod_{i=1}^{s-1} \pi_i!(\pi_i - 1)! = \prod_{i=1}^{s} \pi_i!(\pi_i - 1)!$$

since $\pi_s = 1$.

In case (b), we have $\pi_s \geq 2$. Again, examining equations (4.5), (4.7) and (4.8), it follows that the canonical word in this case occurs only in

$$\prod_{i=1}^{r-1} yt^{p_i} \prod_{j=1}^{r} xt^{-q_j} (xt^{-q_m - q_r + p_r}) v$$

for all $l, m$ such that

$$\pi_1 + \cdots + \pi_{s-1} + 1 \leq m < l \leq r.$$

Each such pair $(l, m)$ contributes a coefficient $C(\pi'')$ where $\pi'' = (\pi_1, \pi_2, \ldots, \pi_{s-1}, \pi_s - 1) \vdash r - 1$. Since $r - \sum_{i=1}^{s-1} \pi_i = \pi_s$, we get

$$C(\pi) = \binom{\pi_s}{2} 2C(\pi'') = \pi_s(\pi_s - 1) \prod_{i=1}^{s-1} \pi_i!(\pi_i - 1)! (\pi_s - 1)! (\pi_s - 2)! = \prod_{i=1}^{s} \pi_i!(\pi_i - 1)!$$

as required. This proves Theorem 10.

4.3. Let $\lambda = (\lambda_1, \lambda_2, \cdots)$ be a partition with $\supp \lambda = r \geq 1$. Let $\pi \vdash r$ with $\supp \pi = s$, and let $B = \{B_1, B_2, \cdots, B_s\}$ be an element of $\mathcal{Y}(\pi)$. Define the following elements of $\mathcal{H}(h \otimes t^{-1} C[t^{-1}])$:

$$W(B, \lambda) := \prod_{p=1}^{s} ht^{-\sum_{j \in B_p} \lambda_j}, \quad \text{and}$$

$$\mathcal{H}(\pi, \lambda) := \sum_{B \in \mathcal{Y}(\pi)} W(B, \lambda).$$

**Example 11.** $\mathcal{H}(\pi = (3), \lambda = (\lambda_1, \lambda_2, \lambda_3)) = ht^{-(\lambda_1 + \lambda_2 + \lambda_3)}$

$\mathcal{H}(\pi = (2, 1), \lambda = (\lambda_1, \lambda_2, \lambda_3)) = ht^{-(\lambda_1 + \lambda_2) + ht^{-\lambda_3} + ht^{-(\lambda_1 + \lambda_2)} + ht^{-(\lambda_1 + \lambda_3)} + ht^{-(\lambda_2 + \lambda_3)} + ht^{-(\lambda_1 + \lambda_2 + \lambda_3)}}$

$\mathcal{H}(\pi = (1, 1, 1), \lambda = (\lambda_1, \lambda_2, \lambda_3)) = ht^{-\lambda_1} ht^{-\lambda_2} ht^{-\lambda_3}.$

We now have the following important corollary to Theorem 10.
Corollary 12. Let \( r \geq 1 \). Fix a partition \( \lambda = (\lambda_1, \lambda_2, \cdots) \) with \( \text{supp} \lambda = r \). Then, for all \( k \geq |\lambda| \), we have
\[
\left( \prod_{i=1}^{r} y^{k_i - \lambda_i} \right) \left( xt^{-k} \right)^{(r)} v_{\Lambda_0} = (-1)^r \sum_{\pi \vdash r} C'(\pi) \mathcal{H}(\pi, \lambda) v_{\Lambda_0}. \tag{4.13}
\]

Here, for \( \pi = (\pi_1, \pi_2, \cdots) \), \( C'(\pi) \) is given by
\[
C'(\pi) = \prod_{i=1}^{\text{supp} \pi} (\pi_i - 1)!. \tag{4.14}
\]

Proof. Consider \( p = (p_1, p_2, \cdots, p_r) \) and \( q = (q_1, q_2, \cdots, q_r) \) with \( p_i = k - \lambda_i \) and \( q_i = k \) for all \( i \in [r] \). We claim that the triple \( (p, q, v_{\Lambda_0}) \) satisfies the hypotheses of Theorem 10. To see this, observe first that \( p_i < q_j \) for all \( i, j \in [r] \). Further, if \( A \) is a non-empty subset of \( [r] \), we have
\[
\sum_{i \in A} p_i = \sum_{i \in A} (k - \lambda_i) \geq |A| k - |\lambda| \geq (|A| - 1) k. \tag{4.15}
\]

Finally, the highest weight vector \( v_{\Lambda_0} \in L(\Lambda_0) \) clearly satisfies \( y^{p} v_{\Lambda_0} = 0 \) \( \forall \) \( p \geq 0 \). Thus, by Theorem 10 we obtain
\[
\left( \prod_{i=1}^{r} y^{k_i - \lambda_i} \right) \left( xt^{-k} \right)^{r} v_{\Lambda_0} = (-1)^r \sum_{\pi \vdash r} C(\pi) \mathcal{H}(\pi; p, q). \tag{4.16}
\]

with \( C(\pi) = \prod_{i=1}^{\text{supp} \pi} \pi_i ! (\pi_i - 1)! \). Now since \( q_j = k \) for all \( j \), it is clear from equations (4.3) and (4.12) that \( \mathcal{H}(\pi; p, q) = \frac{r!}{\pi_1! \pi_2 ! \cdots \pi_s !} \mathcal{H}(\pi, \lambda) \). Equations (4.14) and (4.15) complete the proof.

We observe that while the expression on the left hand side of equation (4.13) depends on \( k \), the one on the right hand side is independent of it. The fact that these two expressions are equal for \( k \geq |\lambda| \) is precisely what leads to the stability properties of interest.

4.4. We can now deduce the key special case of Theorem 8 that we are after, namely for \( \xi \) of the form \( (n, n/2, \lambda) \) with \( n \) even. Firstly, given a partition \( \lambda \in \mathcal{Y} \), let \( r = \text{supp} \lambda \) and \( m_j(\lambda) = \#\{i : \lambda_i = j\} \) denote the multiplicity of the part \( j \) in \( \lambda \) for each \( j \geq 1 \). If \( r \geq 1 \), define the following element of \( \mathfrak{H}(h \otimes t^{-1} C[t^{-1}]) \):
\[
f_{\lambda}(ht^{-1}, ht^{-2}, \cdots) := \frac{(-1)^r}{\prod_{j \geq 1} m_j(\lambda)!} \sum_{\pi \vdash r} C'(\pi) \mathcal{H}(\pi, \lambda), \tag{4.16}
\]

where \( C'(\pi) = \prod_{i=1}^{\text{supp} \pi} (\pi_i - 1)! \) as in Corollary 12. If \( r = 0 \), i.e., \( \lambda \) is the empty partition, we let \( f_{\lambda} := 1 \).

Now, let \( \xi = (n, n/2, \lambda) \in \mathcal{P}^{\text{stab}} \) with \( n \) even. As mentioned before, the weight of \( c(\xi) \) in this case is \( \Lambda_0 - |\lambda| \delta \). The expression of \( c(\xi) \) as a polynomial in \( ht^{-1}, ht^{-2}, \cdots \) acting on \( v_{\Lambda_0} \) is given by the following theorem.
Theorem 13. Let \( n \) be even and let \( \xi = (n,n/2,\lambda) \in \mathcal{P}^{\text{stab}} \). Then
\[
\mathcal{C}(\xi) = f_\lambda(ht^{-1},ht^{-2},\ldots)v_{\Lambda_0}.
\] (4.17)

Proof. Let \( r = \supp \lambda \) and \( k = n/2 \). If \( r = 0 \), then \( \mathcal{C}(\xi) = (yt^k)^{(n-k)}w_n = (yt^k)^{(k)}(xt^{-k})^{(k)}v_{\Lambda_0} = v_{\Lambda_0} \), by Lemma 19(1). Now for \( r \geq 1 \),
\[
\prod_{i=1}^{r} yt^{k-\lambda_i}(yt^k)^{(n-k-r)}w_n = (\prod_{i=1}^{r} yt^{k-\lambda_i})(yt^k)^{(k-r)}(xt^{-k})^{(k)}v_{\Lambda_0} = (\prod_{i=1}^{r} yt^{k-\lambda_i}) (xt^{-k})^{(r)}v_{\Lambda_0},
\]
again by Lemma 19(1). The theorem now follows from this and equations (4.13), (4.4) and (4.16). \( \square \)

We now observe that \( f_\lambda \) depends only on \( \lambda \) and not on \( n \), thereby proving Theorem 8 when \( \xi \) is of the form \((n,n/2,\lambda)\):

Corollary 14. Let \( n \) be even and let \( \xi = (n,n/2,\lambda) \in \mathcal{P}^{\text{stab}} \). Then \( \mathcal{C}(\xi) = \mathcal{C}(\psi(\xi)) \).

5. The General Case When \( n \) is Even

We now turn to the remaining cases of Theorem 8 for even \( n \), i.e., \( \xi = (n,k,\lambda) \in \mathcal{P}^{\text{stab}} \) with \( n \) even and \( k \neq n/2 \). We will now show how to reduce these to the case \( k = n/2 \) using the translation operators of Frenkel and Kac. We recall the necessary facts from [5], stated for our context.

Let \( \Delta := \{\alpha_1, -\alpha_1\} \) be the set of all roots of \( \mathfrak{sl}_2 \), and set \( E_{\alpha_1} := x \) and \( E_{-\alpha_1} := y \). Let \((V,\pi)\) be an integrable representation of \( \mathfrak{sl}_2 \) with weight space decomposition \( V = \bigoplus_{\mu \in \hat{\mathfrak{h}}} V_\mu \). For a real root \( \alpha = \gamma + k\delta \ (\gamma \in \Delta, k \in \mathbb{Z}) \) of \( \mathfrak{sl}_2 \) we define
\[
r_\alpha^\pi := e^{-\pi(E_\alpha)} e^{\pi(E_{-\alpha})} e^{-\pi(E_\alpha)}.
\] (5.1)
where \( E_\alpha := E_\gamma t^k \). The operator \( r_\alpha^\pi \) is a linear automorphism of \( V \) such that \( r_\alpha^\pi(V_\mu) = V_{s_\alpha(\mu)} \), where \( s_\alpha \in \hat{\mathcal{W}} \) is the reflection defined by \( \alpha \).

Next, we introduce the translation operators \( T_\beta^\pi \) on \( V \) for each \( \beta \in Q = \mathbb{Z}\Delta \). For \( \gamma \in \Delta \), define
\[
T_\gamma^\pi := r_{\beta-\gamma}^\pi r_\gamma^\pi.
\] (5.2)
and let \( T_\mu^\pi := (T_\mu^\pi)^p \forall p \in \mathbb{Z}_{\geq 0} \). These operators satisfy \( T_\mu^\pi(V_\mu) = V_{t^\mu(\mu)} \) for all \( \mu \in \hat{\mathfrak{h}}^*, \beta \in Q \).

We will only need these operators in two cases, namely when \((V,\pi)\) is either the adjoint representation or the basic representation of \( \mathfrak{sl}_2 \). We note that \( T_\beta^\text{ad} \) is in fact a Lie algebra automorphism of \( \mathfrak{sl}_2 \). For ease of notation, we will denote the translation operators corresponding to the basic representation simply by \( T_\beta \), suppressing the \( \pi \) in the superscript.

The key properties of the translation operators are given by the following proposition:

**Proposition 15.** (Frenkel-Kac [5])
\begin{enumerate}
\item \( T_{p\alpha_1}^\text{ad}(xt^k) = xt^{k-2p} \forall p, k \in \mathbb{Z} \).
\item \( T_{p\alpha_1}^\text{ad}(yt^k) = yt^{k+2p} \forall p, k \in \mathbb{Z} \).
\item \( T_{p\alpha_1} T_{q\alpha_1} = T_{(p+q)\alpha_1} \forall p, q \in \mathbb{Z} \).
\end{enumerate}
Proposition 16. Let \( n \) be even. Then, we have:

1. \( w_n = (-1)^{\frac{n}{2}} T_{n \alpha / 2} (v_{A_0}) \).
2. Given \( 0 \leq k \leq n \), let \( \gamma = (k - n/2) \alpha_1 \). Then
   \[
   (-1)^{\frac{n}{2}} w_n = (-1)^{\frac{n-k}{2}} T_{\gamma}(w_{2(n-k)}).
   \]
3. Given \( \xi = (n, k, \lambda) \in \mathcal{P}^\text{stab} \), let \( \xi^\dagger = (2(n-k), n-k, \lambda) \) and \( \gamma(\xi) = (k-n/2) \alpha_1 \). Then
   \[
   \psi(\xi) = T_{\gamma(\xi)} \left( \psi(\xi^\dagger) \right) .
   \]

Proof. The proof of (1) will be given in the appendix (see Lemma 19(8)). Equation (5.3) follows easily from (1) and Proposition 15(3). To prove (3), we start with equation (3.4) and use Proposition 15 again to obtain
\[
T_{-\gamma(\xi)} (\psi(\xi)) = z(\xi) \left( \prod_{i=1}^{n-k} T_{-\gamma(\xi)} (yt^{k-\lambda_i}) \right) (T_{-\gamma(\xi)} w_n) .
\]

Now, \( T_{-\gamma(\xi)} (yt^{k-\lambda_i}) = yt^{n-k-\lambda_i} \). Further, it is clear from definition that \( z(\xi) = (-1)^{\frac{n}{2} - \frac{n-k}{2}} \psi(\xi^\dagger) \). Plugging these and (5.3) into (5.5), we obtain (5.4). □

We can now complete the proof of Theorem 8 for \( n \) even. Given \( \xi = (n, k, \lambda) \in \mathcal{P}^\text{stab} \), recall that \( \psi(\xi) = (n+2, k+1, \lambda) \). It is now immediate from the definitions that
\[
\gamma(\xi) = \gamma(\psi(\xi)) \text{ and } \psi(\xi^\dagger) = \psi(\xi)^\dagger.
\]
Proposition 16 and Corollary 14 now imply Theorem 8 for the case that \( n \) is even. □

6. \( n \) odd

6.1. In this section, we show how to reduce the case of \( n \) odd to that of \( n \) even, using automorphisms of \( \widehat{\mathfrak{sl}_2} \). Let \( \tau \) be an automorphism of \( \widehat{\mathfrak{sl}_2} \) such that \( \tau h = \hat{h} \). We have the induced action of \( \tau \) on \( \hat{h}^* \) by \( \langle \tau \lambda, h \rangle = \langle \lambda, \tau^{-1} h \rangle \). Given an \( \mathfrak{sl}_2 \)-module \( V \), let \( V^\tau \) denote the module with the twisted action
\[
g \circ v = \tau^{-1}(g) v \text{ for } g \in \mathfrak{sl}_2, v \in V.
\]
Observe that for automorphisms \( \tau_1, \tau_2 \), we have \( V^{\tau_1 \tau_2} \simeq (V^{\tau_2})^{\tau_1} \).

We now study the twisted actions on \( L(\Lambda_0) \) by two specific automorphisms \( \hat{\sigma}, \hat{\phi} \) of \( \widehat{\mathfrak{sl}_2} \). First, recall from 12 that \( \sigma = s_t - s_{-1} \in \widehat{W}_{ex} \) is an automorphism of the Dynkin diagram of \( \widehat{\mathfrak{sl}_2} \); it swaps \( \alpha_0, \alpha_1 \) and fixes \( \rho \). Consider the Lie algebra automorphism \( \hat{\sigma} \) of \( \widehat{\mathfrak{sl}_2} \) given by the relations
\[
\hat{\sigma}(e_i) = e_{1-i}, \hat{\sigma}(f_i) = f_{1-i}, \hat{\sigma}(\alpha_i^\vee) = \alpha_{1-i}^\vee (i = 0, 1) \text{ and } \hat{\sigma}(\rho^\vee) = \rho^\vee.
\]
Here $\rho' \in \hat{\mathfrak{g}}$ is the unique element for which $\langle \alpha_0, \rho' \rangle = 1, \langle \alpha_1, \rho' \rangle = 1$ and $\langle \Lambda_0, \rho' \rangle = 0$. Clearly $\tilde{\sigma}$ is an involution, and

$$\tilde{\sigma}(yt^m) = xt^{m-1}, \tilde{\sigma}(xt^m) = yt^{m+1}, \tilde{\sigma}(ht^m) = -ht^m + \delta_{m0} c \quad \forall m \in \mathbb{Z}.$$  

Further, $\tilde{\sigma}$ leaves $\hat{\mathfrak{g}}$ invariant, and its induced action on $\hat{\mathfrak{g}}^*$ coincides with $\sigma$.

To define the second automorphism $\tilde{\phi}$, we employ the following simple lemma, which follows directly from the Lie bracket relations (2.1), (2.2).

**Lemma 17.** Let $\phi$ be an automorphism of $\mathfrak{sl}_2$, which preserves the Killing form. Then $\phi$ can be extended to an automorphism $\tilde{\phi}$ of $\hat{\mathfrak{sl}}_2$ by defining $\tilde{\phi}(c) = c, \tilde{\phi}(d) = d$ and $\tilde{\phi}(A t^m) = \phi(A) t^m \forall A \in \mathfrak{sl}_2, m \in \mathbb{Z}$.

Now, consider the involution $\phi$ of $\mathfrak{sl}_2$ defined by

$$\phi(x) = y, \phi(y) = x, \phi(h) = -h.$$  

(6.1)

This preserves the Killing form, so by Lemma 17, it extends to an automorphism (in fact, an involution) $\tilde{\phi}$ of $\hat{\mathfrak{sl}}_2$. It is again clear that (i) $\tilde{\phi}$ preserves $\hat{\mathfrak{g}}$, and (ii) the induced action of $\tilde{\phi}$ on $\hat{\mathfrak{g}}^*$ coincides with the simple reflection $s_1$.

**Proposition 18.** With notation as above, we have (i) $L(\Lambda_0)^{\tilde{\sigma}} \simeq L(\Lambda_1)$, and (ii) $L(\Lambda_0)^{\tilde{\phi}} \simeq L(\Lambda_0)$.

**Proof.** To prove (i), consider the $\mathfrak{sl}_2$-linear map $L(\Lambda_1) \to L(\Lambda_0)^{\tilde{\sigma}}$ which sends $v_{\Lambda_1}$ to $v_{\Lambda_0}$. To show this is well defined, we only need to check that $v_{\Lambda_0} \in L(\Lambda_0)^{\tilde{\sigma}}$ satisfies the relations (2.4)-(2.6) for $\Lambda = \Lambda_1$. Since $\tilde{\sigma}$ interchanges each pair $(e_0, e_1), (f_0, f_1)$ and acts as $\sigma$ on $\mathfrak{g}^*$, all three relations follow. Now, this map is a surjection, since $v_{\Lambda_0}$ generates $L(\Lambda_0)^{\tilde{\sigma}}$. Since $L(\Lambda_1)$ is irreducible, it must be an isomorphism.

A similar argument establishes (ii). We map $L(\Lambda_0) \to L(\Lambda_0)^{\tilde{\phi}}$ by sending $v_{\Lambda_0}$ to $v_{\Lambda_0}$. To show that this extends to a well-defined $\mathfrak{sl}_2$-linear map on all of $L(\Lambda_0)$, we verify that $v_{\Lambda_0} \in L(\Lambda_0)^{\tilde{\phi}}$ satisfies (2.4)-(2.6) for $\Lambda = \Lambda_0$. As above, (2.4) holds since the action of $\tilde{\phi}$ on $\hat{\mathfrak{g}}^*$ coincides with $s_1$, and $s_1\Lambda_0 = \Lambda_0$. Further, in $L(\Lambda_0)$, we have $\tilde{\phi}^{-1}(e_0)v_{\Lambda_0} = xt v_{\Lambda_0} = 0$ and $\tilde{\phi}^{-1}(e_1)v_{\Lambda_0} = yv_{\Lambda_0} = 0$. This establishes (2.5). Finally, for (2.6), we compute in $L(\Lambda_0)$: $\tilde{\phi}^{-1}(f_1)v_{\Lambda_0} = xv_{\Lambda_0} = 0$, and $\tilde{\phi}^{-1}(f_0)^2 v_{\Lambda_0} = (yt^{-1})^2 v_{\Lambda_0}$. Since $yt^{-1}$ is in a real root space of $\mathfrak{sl}_2$, it is easy to see that this last term is also zero by a standard $\mathfrak{sl}_2$ argument (using the $\mathfrak{sl}_2$ spanned by $xt, yt^{-1}$ and $h + c$). The fact that it is an isomorphism follows as in (i). \qed

Let $\tau = \tilde{\sigma} \tilde{\phi}$. Then Proposition 18 implies

$$L(\Lambda_1) \simeq L(\Lambda_0)^{\tilde{\sigma}} \simeq (L(\Lambda_0)^{\tilde{\phi}})^{\tilde{\sigma}} \simeq L(\Lambda_0)^{\tau}.$$  

The isomorphism $F: L(\Lambda_1) \to L(\Lambda_0)^{\tau}$ maps $v_{\Lambda_1} \mapsto v_{\Lambda_0}$. It is then determined on all of $L(\Lambda_1)$ by $\mathfrak{sl}_2$-linearity, i.e., by the relation

$$F(Xv) = \tau^{-1}(X)F(v) \quad \forall X \in \mathfrak{sl}_2, \ v \in L(\Lambda_1).$$
6.2. We now prove Theorem 8 for \( \xi = (n, k, \lambda) \in \mathcal{P}^{\text{stab}} \) with \( n \) odd. From (3.4) and (2.3), we have

\[
\mathcal{c}(\xi) = z(\xi) \left( \prod_{i=1}^{n-k} y t^{k-\lambda_i} \right) (x t^{-\frac{n-1}{2}})^{(\frac{n-1}{2})} v_{\Lambda_1}.
\]

Applying the isomorphism \( F \), we obtain

\[
F(\mathcal{c}(\xi)) = z(\xi) \left( \prod_{i=1}^{n-k} y t^{k-\lambda_i-1} \right) (x t^{-\frac{n-1}{2}})^{(\frac{n-1}{2})} v_{\Lambda_0} = \mathcal{c}(n-1, k-1, \lambda),
\]

since \((-1)^{\lfloor \frac{n}{2} \rfloor} = (-1)^{\lfloor \frac{n-1}{2} \rfloor} \) for \( n \) odd. Observe by (3.5) that \((n, k, \lambda) \in \mathcal{P}^{\text{stab}} \) for \( n \) odd, implies that \((n-1, k-1, \lambda) \) is also in \( \mathcal{P}^{\text{stab}} \). Theorem 8 now follows for \( \xi \) since we have already proved it for all even \( n \). This completes the proof of that theorem in all cases.

6.3. Finally, we observe that the above ideas also give us a proof of Proposition 9. With notation as in that proposition, first let \( n \) be even. If \( G : L(\Lambda_0) \to L(\Lambda_0)^{\bar{\phi}} \) is the isomorphism constructed in the proof of Proposition 15, observe that \( G(w_n) = (y t^{-\frac{n}{2}} (\frac{r}{2})) v_{\Lambda_0} = v_n \), say, is a lowest weight vector of \( D(1, n \pi_1) \). Further, for \( \xi \in \mathcal{P}(n) \), we have \( G(B(\xi)) = B(\bar{\xi}) \), thereby proving Proposition 9 in this case. The stable basis elements in this set up are simply the images of the \( \mathcal{c}(\xi), \xi \in \mathcal{P}^{\text{stab}} \), under the appropriate isomorphism \( G \). The case of odd \( n \) is analogous, via the isomorphism \( G' : L(\Lambda_1) \to L(\Lambda_1)^{\bar{\phi} \bar{d} \bar{\delta}^{-1}} \).

7. Appendix

The following lemma collects together the straightening rules in \( L(\Lambda_0) \) that are used in the course of proving our main theorem. In principle, these can all be proved directly by working in the vertex operator realization of \( L(\Lambda_0) \). The proofs below are simpler, and are included here for the sake of completeness.

**Lemma 19.** Let \( v_{\Lambda_0} \) denote a highest weight vector of \( L(\Lambda_0) \). Then

1. \( (y t^m(l)(x t^{-m}(m)) v_{\Lambda_0} = (x t^{-m}(m-l)) v_{\Lambda_0} \) for all \( 1 \leq l \leq m \).
2. \( \prod_{i=1}^r x t^{2i-1} \quad \prod_{i=1}^r y t^{-2i-1} v_{\Lambda_0} = v_{\Lambda_0} \forall r \in \mathbb{N} \).
3. \( \prod_{i=1}^r y t^{2i-1} \quad \prod_{i=1}^r x t^{-2i-1} v_{\Lambda_0} = v_{\Lambda_0} \forall r \in \mathbb{N} \).
4. Let \( p > q \geq 0 \) and let \( v \in L(\Lambda_0) \) satisfy \( y t^p v = h t^{p-q} v = 0 \). Then
   \[
y t^p (x t^{-q}) v = -(x t^{-q})^{s-2} x t^{p-2q} v \quad \forall s \geq 2.
   \]
5. For \( r \in 2\mathbb{N} \) and \( 0 \leq j \leq \frac{r}{2} \), we have
   \[
   \left( \prod_{i=1}^{\frac{r}{2}+j} x t^{-r}(2j) \prod_{i=1}^{\frac{r}{2}-j} x t^{-2i-1} \right) v_{\Lambda_0} = (-1)^j v_{\Lambda_0}.
   \]
(6) For $r \in 2\mathbb{N} - 1$ and $0 \leq j \leq \frac{r-1}{2}$, we have

$$
\left( \prod_{i=1}^{\frac{r+1}{2}+j} y^{t^{2i-1}} \right) (x^{t^{-r}})^{(2j+1)} \left( \prod_{i=1}^{\frac{r-1}{2}+j} x^{-t^{-2i-1}} \right) v_{\Lambda_0} = (-1)^j v_{\Lambda_0}.
$$

(7) $(\prod_{i=1}^{r} y^{t^{2i-1}}) (x^{t^{-r}})^{(r)} v_{\Lambda_0} = (-1)^{\frac{r(r-1)}{2}} v_{\Lambda_0}$ $\forall r \in \mathbb{N}$.

(8) $(x^{t^{-r}})^{(r)} v_{\Lambda_0} = (-1)^{\frac{r(r-1)}{2}} T_{r\alpha_1} (v_{\Lambda_0}) = 0$ $\forall r \in \mathbb{N}$.

Proof. (1) Consider the Lie subalgebra of $\tilde{\mathfrak{sl}}_2$ spanned by $E := y^{t^{m}}, F := x^{t^{-m}}$ and $H := -h + mc$. This is isomorphic to $\mathfrak{sl}_2$. Further, $E, F$ act locally nilpotently on $L(\Lambda_0)$, and we have $H v_{\Lambda_0} = m v_{\Lambda_0}$, $E v_{\Lambda_0} = 0$. The standard $\mathfrak{sl}_2$ calculation now shows $E^{(l)} F^{(m)} v_{\Lambda_0} = F^{(m-l)} v_{\Lambda_0}$.

(2) Using Proposition 15, it is easy to see that this is just a restatement of the identity $T_{r\alpha_1} T_{r\alpha_1} v_{\Lambda_0} = v_{\Lambda_0}$.

(3) As in (2), this is now the identity $T_{r\alpha_1}, T_{r\alpha_1}, v_{\Lambda_0} = v_{\Lambda_0}$.

(4) With the given hypotheses, we compute

$$
y^{t_l} (x^{t^{-q}})^{s} v = [y^{t_l}, (x^{t^{-q}})^{s}] v = -\sum_{i=0}^{s-1} (x^{t^{-q}})^{i} h^{t_l-q} (x^{t^{-q}})^{s-1-i} v.
$$

(7.1)

We also have $[h^{t_l-q}, (x^{t^{-q}})^{u}] = 2u (x^{t^{-q}})^{u-1} x^{t_l-2q}$ for all $u \geq 1$. Applying this to (7.1) completes the proof.

(5) For $j = 0$, this is just the statement of (3). For $1 \leq j \leq \frac{r}{2}$, define $v_j := \prod_{i=1}^{\frac{r-1}{2}+j} x^{-t^{-(2i-1)}} v_{\Lambda_0}$. From weight considerations, it can be easily seen that $v_j$ satisfies $y^{t^{r+2j-1}} v_j = 0 = h t^{j-1} v_j$. Thus, by (4), we obtain

$$
y^{t^{r+2j-1}} (x^{t^{-r}})^{(2j)} v_j = -(x^{t^{-r}})^{(2j-2)} x^{-t^{-(r-2j-1)}} v_j = -(x^{t^{-r}})^{(2j-2)} v_{j-1}.
$$

The result now follows by induction on $j$.

(6) This is analogous to (5).

(7) For $r$ even, put $j = \frac{r}{2}$ in (5) to obtain

$$
\prod_{i=1}^{r} y^{t^{2i-1}} (x^{t^{-r}})^{(r)} v_{\Lambda_0} = (-1)^{\frac{r}{2}} v_{\Lambda_0}.
$$

(7.2)

Similarly, for $r$ odd, put $j = \frac{r-1}{2}$ in (6):

$$
\prod_{i=1}^{r} y^{t^{2i-1}} (x^{t^{-r}})^{(r)} v_{\Lambda_0} = (-1)^{\frac{r-1}{2}} v_{\Lambda_0}.
$$

(7.3)

Equations (7.2) and (7.3) together give us the desired result for all $r \in \mathbb{N}$. 
(8) Let \( r \in \mathbb{N} \). Then \((x^t-r)^r v_{\Lambda_0}\) and \(T_{\alpha_1}(v_{\Lambda_0}) = \prod_{i=1}^{r} x t^{-(2i-1)} v_{\Lambda_0}\) belong to the 1-dimensional space \( L(\Lambda_0)_{\Lambda_0 + r\alpha_1 - r\delta} \), and so we must have

\[
(x^t-r)^r v_{\Lambda_0} = a \prod_{i=1}^{r} x t^{-(2i-1)} v_{\Lambda_0}
\]

for some \( a \in \mathbb{C} \). But by (3) and (7), it follows that \( a = (-1)^{|\delta|/2} \) and that these vectors are non-zero. \( \square \)

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