I. INTRODUCTION

There is a strong interest in slow light structures and devices [1–14]. This is driven by the need for optical buffers and optical memory in optical integrated circuits and other applications. In order to have small group velocity, various methods have been exploited to engineer material and structural dispersions, such as electromagnetic induced transparency [15], coupled resonant structures [5, 16], and optical nonlinearity [3].

Tsakmakidis et al [6] proposed a novel scheme to realize trapped rainbow by using negative-index metamaterials. This slow light waveguide requires its core layer to be made of double-negative metamaterial (DNM) whose permittivity and permeability are both negative. It is quite a challenge to realize DNMs with very low loss [17] since even moderate loss will destroy the zero group velocity mode [18, 19]. Strictly speaking, when loss is present as is inevitable in passive systems, storing light indefinitely is impossible. However, gain may be introduced to compensate loss and makes zero group velocity possible [12]. Slow light waveguide with single-negative materials has also been proposed [12, 13].

In this paper we consider planar waveguide made of the so-called indefinite medium [20–23] whose permittivity and/or the permeability tensors are indefinite matrices. For an indefinite medium, the dispersion is hyperbolic for one polarization and elliptical for the other. Negative refraction, superlens imaging, and hyperlens focusing [24, 25] can be realized by using indefinite permittivity (ε) materials (IEMs). Nanowire waveguide made of IEM has been considered by Huang et al [9]. These IEM waveguides can support both forward- and backward-wave modes. High phase index can be obtained for these guided modes. Here we further exploit the unique properties of IEM waveguide by considering a more manageable planar waveguide geometry. We will reveal that these waveguides can also support degenerate modes which can be used to slow down and trap light, form open cavities, and generate strong longitudinal electric field. These unique properties can lead to a broad range of applications.

The paper is organized as follows. In Sec. II, we present the solutions for the TM modes supported by an anisotropic planar waveguide. The realization of IEMs will be proposed and light coupling into IEM waveguides will be discussed in Sec. III. In Sec. IV, various unique features such as slow light, open cavity formation, and large longitudinal electric field will be revealed. We conclude in Sec. V.

II. WAVE PROPAGATION IN AN ANISOTROPIC PLANAR WAVEGUIDE

A. Eigen modes equation

We consider a slab of planar waveguide made of anisotropic metamaterial in air. The wave propagation is along the z-direction with phase $e^{i(\beta z - \omega t)}$ and the transverse direction is in the x-direction. We consider the case of an IEM with

$$\varepsilon_z > 0, \quad \varepsilon_x < 0. \quad (1)$$

For the transverse magnetic (TM) modes, the magnetic field is in the y-direction. The transverse component $k_x$
of the wave vector inside the metamaterial is
\[ k_x = \sqrt{\varepsilon_x} \sqrt{\mu_y k_0^2 - \beta^2 / \varepsilon_x}. \] (2)

Here \( k_0 \) is the wave number in the vacuum. Since we do not consider magnetic materials, we set \( \mu_y = 1 \). Depending on the value of \( \varepsilon_y \), this extremely anisotropic medium can be uniaxial or biaxial [26–28]. However, since \( \varepsilon_y \) is not involved with the TM modes, its value is not of our interest in this paper. Due to the negativity of \( \varepsilon_x \), the dispersion Eq. (2) is hyperbolic instead of elliptic, which is shown in Fig. 1.

In order to facilitate the analysis of the eigen modes and subsequently the calculation of the group index \( n_g \), we introduce a parameter \( \xi \) with
\[ k_x = k_0 \sqrt{\varepsilon_x (1 - 1/\varepsilon_x) / \sqrt{1 - \xi^2}}, \]
\[ \kappa_0 = k_0 \sqrt{1 + \varepsilon_x / \sqrt{1 - \xi^2}}. \] (6)

For real \( \varepsilon_x \) and \( \varepsilon_z \), one has \( 0 < \xi < 1 \) for the guided modes. For complex permittivity, \( \xi \) takes complex values. We further introduce two parameters
\[ \rho = 1/\sqrt{-\varepsilon_x \varepsilon_z}, \quad \sigma = -1/\varepsilon_x. \] (7)

Note that one has \( \varepsilon_x = -1/\sigma \) and \( \varepsilon_z = \sigma / \rho^2 \). Thus \( \rho \) and \( \sigma \) can also be used to characterize the IEM.

In term of \( \xi, \sigma, \rho \), the eigen equation (5) becomes
\[ \sqrt{\varepsilon_\xi k_0 d} = F(\xi, \rho, \sigma) \] (8)
with the function \( F \) defined as
\[ F = \frac{\sqrt{1 - \xi^2}}{\sqrt{1 + \sigma}} (m \pi + 2 \arctan \frac{\xi}{\rho}). \] (9)

The phase index is
\[ n_p = \frac{1 + \xi^2 / \sigma}{\sqrt{1 - \xi^2}}. \] (10)

### B. Asymptotic solutions

The phase index \( n_p \) of the guided modes is given implicitly in the eigen equation (5) or (8). At a fixed wavelength or frequency, \( \varepsilon_x \) and \( \varepsilon_z \) are constant. For real values of \( \varepsilon_x \) and \( \varepsilon_z \), one can treat the thickness \( d \) as a function of \( n_p \) which is chosen as a free parameter. The band structure for TM modes on a slab waveguide with \( \varepsilon_x = -3 \) and \( \varepsilon_z = 2 \) is plotted in Fig. 2.

For complex values of \( \varepsilon_x \) and \( \varepsilon_z \), the phase index \( n_p \) will also be complex, which is not so easy to solve. One must resort to numerical methods. Nevertheless, asymptotic

![FIG. 1: (Color online) Hyperbolic dispersion relation for the TM waves in an IEM. The dashed line (blue) is for a perfect electronic conductor (PEC) grating whose transverse component of the wave vector is independent of the longitudinal component \( \beta \).](image1)

![FIG. 2: The phase index \( n_p \) of the guided \( \text{TM}_m \) modes on a free-standing planar waveguide of thickness \( d \) with \( \varepsilon_x = -3 \) and \( \varepsilon_z = 2 \).](image2)
solutions can be obtained analytically, which will reveal some of the unique properties of the guided modes on IEM waveguide.

For the modes near the light line, $\xi \to 0$, one has the following expansion

$$F \simeq \frac{1}{\sqrt{1+\sigma}} \left\{ m\pi + \frac{2\xi}{\partial} \left[ \frac{1}{2} - \frac{m\pi \vartheta}{4} \xi - \frac{1}{2} + \frac{1}{3\vartheta^2} \xi^2 \right] \right\}.$$ \hspace{1cm} (11)

Thus one has

$$\xi \simeq \frac{1}{2} \vartheta (\eta k_0 d - m\pi) + \frac{m\pi}{16} \vartheta^3 (\eta k_0 d - m\pi)^2 + \frac{1}{16} \left( 1 + \frac{2}{3\vartheta^2} + \frac{1}{4} m^2 \pi^2 \vartheta^2 \right) \vartheta^3 (\eta k_0 d - m\pi)^3.$$ \hspace{1cm} (12)

with $\eta = \sqrt{1+\sigma}\sqrt{\varepsilon_x} = \sqrt{\varepsilon_z^2 - 1}/\varepsilon_x$. The phase index for the modes near the light line is obtained as

$$n_p \simeq 1 + \frac{1}{2} (1 + \sigma^{-1}) \xi^2 - \frac{3}{4} (3 - \sigma^{-1} + \sigma^{-2}) \xi^4$$

$$\simeq 1 + \frac{1}{8} \vartheta^2 (1 + \sigma^{-1}) (\eta k_0 d - m\pi)^2 + \frac{1}{32} (1 + \sigma^{-1}) m\pi \vartheta^4 (\eta k_0 d - m\pi)^3 + \frac{1}{64} b \vartheta^4 (\eta k_0 d - m\pi)^4.$$ \hspace{1cm} (13)

with the coefficient $b$ given by

$$b = -1 + 3\sigma^{-1} - \sigma^{-2} + (1 + \sigma^{-1}) \left( \frac{4}{3} \vartheta - \frac{5}{8} m^2 \pi^2 \vartheta^2 \right).$$ \hspace{1cm} (14)

So the modes near the light line is not much different from that on a dielectric waveguide [29]. Thus these modes are forward-wave modes.

Due to the hyperbolic nature of the dispersion Eq. (2), IEM waveguide supports modes with $n_0 \to \infty$, thus $n_p \to \infty$. There is no upper bound for $n_p$, unlike the conventional dielectric waveguide. For these modes, $\xi \to 1$, one has the expansion

$$F \simeq \frac{\sqrt{2(1-\xi)}}{\sqrt{1+\sigma}} \left( m\pi + 2\vartheta \tan^{-1}\frac{1}{\vartheta} \right)$$ \hspace{1cm} (15)

and subsequently

$$\xi \simeq 1 - \frac{(1 + \sigma) \varepsilon_x (k_0 d)^2}{2 (m\pi + 2\vartheta \tan^{-1}\vartheta)^2}.$$ \hspace{1cm} (16)

It is evident that validity of $\xi \to 1$ requires $d \to 0$. Thus one obtains

$$n_p \simeq \frac{\vartheta}{\sigma k_0 d} \left[ m\pi + 2\vartheta \tan^{-1}\frac{1}{\vartheta} + \vartheta^{-1} \sigma (1 + \sigma) (k_0 d)^2 (m\pi + 2\vartheta \tan^{-1}\vartheta)^{-1} \right].$$ \hspace{1cm} (17)

From this expression of $n_p$, it is evident that the waveguide supports infinite number of TM modes at any thickness since all the modes with different parity index $m$ will approach the limit $n_p \to \infty$ as $d \to 0$. This leads to the divergence of the longitudinal electric field $E_z$ (see Eq. (4)), similar to that on a metallic nanowire [30]. Large longitudinal electric field will be discussed more in Sec. IV.

### C. Energy flow and group velocity

For waves inside the IEM, one has

$$\beta = \sqrt{\varepsilon_x k_0^2 - k_z^2}.$$ \hspace{1cm} (18)

If $k_z > \sqrt{\varepsilon_x k_0}$, $\beta$ will be real and negative if the imaginary part of the permittivity is ignored. So the waves inside the IEM will be left-handed, $\beta S_z < 0$ with the $z$-component of the Poynting vector [29]

$$S_z = \frac{1}{2} \text{Re}(E_x H_y^*).$$ \hspace{1cm} (19)

However for waves confined in the transverse direction, $S_z$ is no longer uniform. The energy flow of the guided modes is better characterized by a quantity $P_z$ which is defined as [29]

$$\frac{\beta}{|\beta|} P_z = \frac{1}{2} \int_{-\infty}^{\infty} E_z H_y^* dx.$$ \hspace{1cm} (20)

Here $P_z$ is actually the power per unit length (unit length in the $y$-direction). For the TM modes on the planar waveguide, $P_z$ consists of two parts, one inside and one outside the waveguide,

$$P_z = P_z^{\text{in}} + P_z^{\text{out}}$$ \hspace{1cm} (21)

with

$$P_z^{\text{in}} = \frac{|n_p|}{4} \frac{d}{\varepsilon_x} \left[ 1 + (-1)^m \text{sinc} k_x d \right]$$

$$= -\frac{|n_p|}{4} \frac{d}{F} \left( F + \frac{2\vartheta \xi}{\sqrt{1+\sigma} \vartheta^2 + \xi^2} \right),$$

$$P_z^{\text{out}} = \frac{|n_p|}{4} \frac{1 + (-1)^m \cos k_x d}{\kappa_0}$$

$$= \frac{|n_p|}{4} \frac{d}{F} \frac{2\vartheta}{\sqrt{1+\sigma} \xi (\vartheta^2 + \xi^2)}.$$ \hspace{1cm} (22)

The energy flow in the air $P_z^{\text{out}}$ is positive as expected. However, due to the negative sign of $\varepsilon_x$, the energy flow inside the waveguide is negative and thus contra-directional to that in the air. A guided wave is forward (backward) wave only if $P_z$ is positive (negative).

The group velocity for the TM modes on the IEM waveguide is given by

$$v_g = \frac{dk_0}{d\beta} = \frac{c}{n_g}$$ \hspace{1cm} (23)

with $c$ the speed of light in the vacuum and $n_g$ the group index defined as

$$n_g = n_p + k_0 \frac{dn_p}{dk_0}.$$ \hspace{1cm} (24)

From the above expression of $n_p$, one has

$$\frac{dn_p}{dk_0} = \frac{\partial n_p}{\partial \xi} \frac{d\xi}{dk_0} + \frac{\partial n_p}{\partial \vartheta} \frac{d\vartheta}{dk_0}.$$ \hspace{1cm} (25)
with
\[
\frac{\partial n_p}{\partial \xi} = \frac{\xi}{1 - \xi^2} \left( n_p + \frac{1}{\sigma n_p} \right),
\]
\[
\frac{\partial n_p}{\partial \sigma} = \frac{\xi^2}{2n_p\sigma^2(1 - \xi^2)}.
\]  
(26)

Taking the derivative on both sides of the eigen equation (8) with respect to \( k_0 \), one has
\[
\frac{F}{k_0} \left( 1 + \frac{k_0}{2\varepsilon_z} \frac{d\varepsilon_z}{dk_0} \right) = \partial_\xi F \frac{d\xi}{dk_0} + \partial_\sigma F \frac{d\sigma}{dk_0} + \partial_\varepsilon F \frac{d\varepsilon}{dk_0}.
\]  
(27)

Thus one obtains
\[
\frac{d\xi}{dk_0} = \frac{1}{\partial_\xi F} \left[ \frac{F}{k_0} \left( 1 + \frac{k_0}{2\varepsilon_z} \frac{d\varepsilon_z}{dk_0} \right) - \partial_\varepsilon F \frac{d\varepsilon}{dk_0} \right].
\]  
(28)

with
\[
\partial_\xi F = \frac{2\sigma}{\sqrt{1 + \sigma^2 + \xi^2}} \frac{\sqrt{1 - \xi^2}}{1 - \xi^2},
\]
\[
\partial_\varepsilon F = \frac{2}{\sqrt{1 + \sigma^2 + \xi^2}},
\]
\[
\partial_\sigma F = -\frac{F}{2(1 + \sigma)}.
\]  
(29)

After inserting the expressions (25)-(29) into Eq. (24), the final expression for the group index \( n_g \) can be obtained.

The expression for \( v_g \) is very involved. However, the group velocity of the modes near light line or with large phase index can be easily evaluated. From the asymptotic solutions we have derived in the previous subsection, one can see that the guided modes near the light line are fast modes with group velocity close to that in the vacuum. For the modes with very large phase index \( n_p \), the group index is also large, leading to the vanishing of the group velocity. Due to the finiteness of the material dispersion, the divergence of \( n_g \) is related to the vanishing of \( \partial_\xi F \) which is determined by the geometry of the waveguide. One can also verify that \( \partial_\xi F = 0 \) will lead to \( P_x = 0 \) by using the expressions in Eq. (22).

Using the above expression of \( P_x \), one can determine the nature of the modes. For the modes near the light line, \( n_p \rightarrow 1 \) and \( \xi \rightarrow 0 \), one has \( P_z^\text{in} \approx -\sigma d/4 \) and \( P_z^\text{out} \rightarrow +\infty \). Thus \( P_z > 0 \) and the modes are forward waves. For the modes with \( n_p \rightarrow \infty \) and \( \xi \rightarrow 1 \) which requires \( d \rightarrow 0 \), one has \( P_z^\text{in} \approx -\infty \) and \( P_z^\text{out} \rightarrow \sigma^2/2k_0(\sigma^2 + 1) \). Thus \( P_z < 0 \), the modes are backward waves. For modes with finite \( n_p \) and \( n_p > 1 \), \( P_z \) is finite. From the band structure shown in Fig. 2, one would expect that \( P_z \) will decrease with increasing \( n_p \). At certain critical point \( d_c \) or \( n_p,c \), \( P_z = 0 \). Further increasing \( n_p \), \( P_z \) will become negative, indicating the mode will be backward wave. The critical values of the phase index \( n_p,c \) and thickness \( d_c \) are determined by a critical value \( \xi_c \) which satisfies the condition
\[
\frac{2\sigma(1 - \xi^2)}{\xi_c(\sigma^2 + \xi^2)} = m\pi + 2\arctan \frac{\xi_c}{\sigma}.
\]  
(30)

The condition of zero group velocity is only determined by the parameter \( \varrho = 1/\sqrt{-\varepsilon_x\varepsilon_z} \). The above condition can be recast as
\[
\xi_c^2 = \left[ 1 + \frac{1}{2} \left( \varrho - \frac{\xi_c}{\varrho} \right) (m\pi + 2\arctan \frac{\xi_c}{\varrho}) \right]^{-1}.
\]  
(31)

The dependence of the critical values \( \xi_c \) on \( \varrho \) is plotted in Fig. 3. Once \( \xi_c \) is determined, the critical values of \( d_c \) and \( n_{p,c} \) can be calculated for a given value of \( \varrho \).

![FIG. 3: The dependence of the critical value \( \xi_c \) for the zero group velocity mode on the IEM parameter \( \varrho = 1/\sqrt{-\varepsilon_x\varepsilon_z} \) for the TM modes with different parity \( m \).](image-url)
III. REALIZATION AND LIGHT COUPLING TO INDEFINITE PERMITTIVITY MATERIAL WAVEGUIDE

Nature does allow the existence of single-negative materials with homogeneous permittivity or permeability being negative, such as noble metals, polaritonic materials, and ferrites. If the underlying crystal is anisotropic, the plasmon frequency can be anisotropic, leading to indefinite permittivity. One example is bismuth in the THz range [32], though bismuth must be placed in liquid helium temperatures in order to have very small loss.

Extremely anisotropic materials can be realized easily in a broad range of frequencies by proper design of metamaterials. One method is through the engineering of artificial electric resonances. Another method is through the homogenization [27, 33]. For example for a multilayered structure of dielectric $\varepsilon_a$ and metal $\varepsilon_m$, the effective permittivities can be obtained by using the effective medium theory [22, 33],

\[
\begin{align*}
\varepsilon_x &= f \varepsilon_m + (1 - f) \varepsilon_a, \\
\varepsilon_z &= \frac{\varepsilon_a \varepsilon_m}{f \varepsilon_a + (1 - f) \varepsilon_m}.
\end{align*}
\]

Here $f$ is the filling ratio of the metal. For $f > f_{\text{min}} = \frac{\varepsilon_a}{(\varepsilon_a - \text{Re} \varepsilon_m)}$, one has $\text{Re} \varepsilon_z < 0$. For noble metals such as gold or silver, $\text{Re} \varepsilon_m < 0$ in the near infrared and visible frequency range. For polar materials [34], such as LiTaO$_3$ crystal, between the transverse and longitudinal optical-mode frequencies, 26.7 THz and 46.9 THz, the real part of permittivity is also negative. Thus if the periodicity of the multilayered structure is much less than the operating wavelength, these negative permittivity materials can be used to design IEMs in their respective frequency ranges.

We design an IEM at $\lambda = 1.55 \mu$m. The metamaterial is formed by using alternative layers of silver and MgF$_2$. At this wavelength, one has $\varepsilon_m = -129 + 3.3i \ [35, 36]$ and $\varepsilon_a = 1.9$ [37]. Using Eq. (32) with filling ratio $f = 3.8\%$, we have $\varepsilon_z = -3.0742 + 0.1254i$ and $\varepsilon_x = 1.9762 + 0.00003i$. This IEM can be used to make waveguides of subwavelength thickness. When loss is present, the permittivity components in Eq. (1) are replaced by the real parts of the permittivity. All of our theoretical formulas in the previous section are still valid if complex $\varepsilon_x$ and $\varepsilon_z$ are used, which lead to guided modes with complex wave number $\beta$ and phase index $n_p$. Solutions must be sought numerically. If loss is ignored at this point, the guided TM modes can be easily calculated and are plotted in Fig. 4.

Due to its subwavelength thickness, coupling light into an IEM waveguide is a technical challenge. Currently there are 4 ways to couple light to waveguides [38]: butt coupling, end-fire coupling, prism coupling, and grating coupling. The coupling efficiency depends on the coupling method and the optical properties of the waveguide. The end-fire coupling is a butt coupling with a focal lens. Multiple modes will be excited in the IEM waveguide by butt coupling. However for a waveguide made of realistic metamaterial, loss is unavoidable, thus only one or two modes will survive over certain distance and eventually only one mode will survive after a certain distance.

Though an IEM waveguide supports infinite number of modes, selective excitation of a single mode is possible. Thus one can take the full advantage of the rich band structure provided by the IEM waveguide. In order to excite a single mode, the prism coupling or grating coupling should be used. However the phase-match condition must be satisfied for maximum energy transfer from the light source to the waveguide.

For a simple illustration, we use the prism coupling to excite the TM modes in a slab waveguide made of IEM at $\lambda = 1.55\mu$m. For simplicity, we first ignore the imaginary part of the permittivity. Dissipation will be considered later in this section. In the range $1.4 < n_p < 2.2$ and the slab thickness $d$ between 225 nm and 240 nm, only the TM$_0$ modes will be excited as shown in Fig. 4 (see Fig. 2 for band structure of similar parameters). The critical thickness, such that the forward TM$_0$ and backward TM$_0$ will be merged into a single mode of zero group velocity, is $d_c = 235.5$ nm with $n_p = 1.734$. At the thickness $d = 235$ nm, two TM$_0$ modes are allowed, with $n_p = 1.541, 1.981$. The first one is a forward-wave mode while the second is a backward-wave mode.

We place a silicon prism of refractive index $n = 3.518$ next to the IEM waveguide. The air gap between them is 600 nm. We then shine a Gaussian beam into the silicon prism. At an incident angle 25.97$^\circ$ inside the prism, the forward-wave mode will be excited while at an incident angle 34.27$^\circ$, the backward-wave mode will be excited, which are shown in Fig. 5.

In the above illustration, loss in IEM is ignored. When the loss is present, the eigen equation (8) will only allow complex solutions. Every mode acquires an imaginary
FIG. 5: (Color online) Gaussian beam excitation through prism coupling of the forward-wave (a) and backward-wave mode (b) on an IEM waveguide at incident angle 25.97° and 34.27°, respectively. The IEM has lossless permittivity $\varepsilon_x = -3.0742$ and $\varepsilon_z = 1.9762$ at $\lambda = 1.55\mu m$. The air gap between the prism and the waveguide ($d = 235$ nm) is $600$ nm. Plotted is the absolute value of the magnetic field $H_y$.

part. There is no clear cut distinction between propagating modes and decay modes. Due to the presence of loss in the IEM, the forward-wave TM$_0^F$ modes will not be degenerate with TM$_0^B$ at any thickness. This is shown in Fig. 6 in the complex-$\xi$ plane.

FIG. 6: (Color online) The splitting of the forward-wave (red) and backward-wave (blue) TM$_0$ modes in the complex-$\xi$ plane in the presence of loss in the IEM with $\varepsilon_x = -3.0742 + 0.1254i$ and $\varepsilon_z = 1.9762 + 0.00003i$ at $\lambda = 1.55\mu m$. The dot-dashed line (green) is the solution for TM$_0$ modes on the IEM waveguide without loss.

The splitting of solutions in the complex-$\xi$ plane will lead to the split of the phase index $n_p$ which is plotted in Fig. 7. Consequently, there is no well-defined critical thickness.

At a thickness $d = 235$ nm, the two TM$_0$ modes have $n_p = 1.5190 + 0.0672i$, $2.0026 - 0.1437i$ for the forward- and backward-wave modes, respectively. Thus the forward- and backward-wave modes will propagate a finite distance before lose most of their power. Since the forward-wave mode is closer to the light line, thus the loss will have a smaller effect while the backward-wave modes corresponding deeper penetration into the ohmic metal, thus will have shorter decay length. The excitation of these two modes is illustrated in Fig. 8.

IV. APPLICATIONS OF WAVEGUIDE MADE OF INDEFINITE PERMITTIVITY MATERIALS

A. Phase shifter and longitudinal electric field enhancement

One salient feature of the modes on the IEM waveguide is the large phase index. The high phase index is due to the hyperbolic dispersion in the metamaterial. Nanowires waveguide based on these slab waveguides can be used for phase shifter with small footprint in optical integrated circuits. The presence of loss in the metamaterials will restrict the use of the waveguide. However for many applications other than the long-haul transportation, short waveguides have the advantage of small sizes.

As examples, we consider a slab waveguide made of an IEM with $\varepsilon_x = -3.0742 + 0.1254i$ and $\varepsilon_z = 1.9762 + 0.00003i$ at $\lambda = 1.55\mu m$. The realization of this IEM was
discussed in the previous section. If no loss is present, the waveguide made of this IEM will support forward-wave and backward-wave modes as shown in Fig. 4.

At a thickness $d = 160$ nm, the waveguide supports two TM modes, one is a forward-wave mode with $n_p = 1.1211 + 0.0035i$ and the other is a backward-wave mode with $n_p = 4.0298 - 0.1200i$. The TM modes with phase index $Re n_p \sim 4$ can be used to create large phase shift over a short distance. In order to have a phase shift of $\pi$ by this slab waveguide, the propagating length can be about 192.3 nm. Over this distance, the damping is about 1.26 dB. For excitation of such high phase index modes, grating coupling should be used.

Besides the large phase index, modes with large longitudinal electric field can be excited on the IEM waveguide. Recently there is a strong interest in large longitudinal electric fields [39]. Large longitudinal electric field can have a lot of applications, such as superfocusing to beat the diffraction limit [40, 41] and trapping metallic nanoparticles in optical tweezer [42].

Due to the confinement in the transverse direction, there is a $\pi/2$ phase difference between $E_z$ and $E_x$ in air. The transverse electric field is

$$E_z(x) = \frac{\beta}{k_0} e^{\kappa_0(x+d/2)}, \quad x \leq -d/2,$$

$$= \frac{\beta}{\varepsilon_\perp k_0} (A e^{ik_\perp x} + B e^{-ik_\perp x}), \quad -d/2 \leq x \leq d/2,$$

$$= \frac{\beta}{k_0} e^{-\kappa_0(x-d/2)}, \quad x \geq d/2. \quad (33)$$

For the waveguide thickness $d = 160$ nm, the electric field of the two TM modes are plotted in Fig. 9.

Here we consider the ratio

$$s = \frac{|E_z|_{\text{max}}}{|E_x|_{\text{max}}}. \quad (34)$$

One can see that $s \sim 50\%$ for the TM$_0^F$ mode and $s \sim 100\%$ for the TM$_0^B$ mode. The ratio is comparable and even stronger than that on a silicon nanowire waveguide [39, 43]. If one shrinks the thickness, even larger ratio would be expected. The fact that the IEM nanowire waveguide will have large longitudinal electric field is due to the hyperbolic dispersion Eq. (2), which allows much stronger confinement of light in the transverse direction, thus stronger longitudinal electric field.

We further derive analytical expression for the ratio $s$. One has

$$|E_z|_{\text{max}} = |n_p| \text{max}(1, |\sigma\sqrt{1 + \xi^2/\sigma^2}|). \quad (35)$$

and

$$|E_z|_{\text{max}} = \left| \frac{\xi \sqrt{1 + 1/\sigma}}{\sqrt{1 - \xi^2}} \right|, \quad m = 0,$$

$$= \left| \frac{\xi + \sigma \sqrt{1 + \xi^2/\sigma^2}}{\sqrt{\varepsilon_z} \sqrt{1 - \xi^2}} \right|, \quad m > 0. \quad (36)$$

By using the above expressions and also the eigen equation (8), the ratio $s$ can be calculated for modes with different waveguide thickness $d$ as shown in Fig. 10. Depending on the value of $\varepsilon_\perp$ and $\varepsilon_z$, the ratio $s$ can be over 100%.

B. Light wheel and open cavity formation

Resonances are ubiquitous. They have many applications such as storing and confining energy, enhance the field concentration, and improve the detection accuracy. To have a resonance, a compact space is required, such as cavities. Examples are the quantum dots, microwave cavities, photonic crystal microcavities. Recently, negative-index metamaterials are also used to form open cavities [44–47], such as the checkerboard open resonators [48].
For optical integrated circuits, one of the most difficult tasks of nanofabrication is the alignment of different parts and devices. It would be more desirable to have an open cavity at any location.

Recently, a new concept, a light wheel, has been developed [49]. This is formed in a composite waveguide, which is made of an ordinary slab waveguide coupled with a properly designed slab waveguide made of DNM. If these two waveguides are separated infinitely away, the ordinary waveguide support a single forward-wave mode. The DNM waveguide supports a single backward-wave mode with the same phase index. Once these two waveguides are placed in the vicinity of each other, the composite waveguide no longer supports propagating modes at the same wavelength. Instead it will support complex-conjugate decay modes. These two decay modes will form at the same wavelength. Instead it will support complex-conjugate decay modes. For example at $d = 239$ nm, one has $n_p = 1.7286 \pm 0.0373i$. When an incident beam with $\beta = 1.7286k_0$ hits the waveguide, the two decay modes will be excited, one decays along $\beta$ and the other in the opposite direction of $\beta$, thus an open cavity will be formed. This cavity can be formed at any location along the IEM waveguide, which is shown in Fig. 11(a).

When dissipation is present, the decay modes no longer form a complex-conjugate pair. However the imaginary part of their complex phase indices will still have opposite signs, thus light wheel formation is still allowed. In our example as shown in Fig. 7, for $d = 239$ nm, the two modes has the complex phase index $n_p = 1.5877 + 0.1282i, 1.8730 - 0.2042$. The formation of light wheel is shown in Fig. 11(b).

C. Slow light waveguide

For the anisotropic waveguide we have considered, $\varepsilon_x < 0$, one has $P^\text{in} < 0$ and $P^\text{out} > 0$ if one sets $\beta > 0$ or $n_p > 0$. If $P_x = P^\text{in} + P^\text{out} < 0$, the mode is a backward-wave mode since the total energy flow is opposite to the phase velocity. Otherwise, the mode is a forward-wave mode. Unlike the dielectric waveguide or the PEC grating, the IEM waveguides support both forward and backward waves. At the critical thickness $d_c$, the backward and forward modes become degenerate, the energy flow

![Graph](image-url)
inside the waveguide cancels out that in the air. One can prove that at $d_e$, where $P_z=0$, the group velocity is indeed zero. One does not need to know the material dispersion to locate the zero group velocity point. This is due to the fact that for these waveguides, the dispersion due to geometric confinement dominates the material dispersion at and around the critical thickness.

This unique property of the modes on an IEM waveguide can be used to slow down and even trap light. Even though the waveguide supports infinite number of TM modes at any fixed thickness and frequency, with appropriate laser coupling, the excitation of the $T_{m}$ modes with $m \geq 1$ in the waveguide can be suppressed or even eliminated. Due to the material dissipation, the first TM mode will propagate the longest distance. The rest of the TM modes will all decay out at about half the decay length of the first TM mode. It is the $T_{0}$ band which can be used for slow light application, though other TM modes can also be used. Unlike the double negative waveguide [6], the IEM waveguide will slow down and trap light if one increases the thickness to the critical thickness $d_c$. These waveguides can thus be used as delay line in optical buffers [5].

For the butt-coupling, multiple modes are excited. To excite a single mode, prism coupling or grating coupling can be used. By tapering the waveguide thickness, any branches of the guided modes can be accessed in principle.

For an asymmetric waveguide where an IEM waveguide is placed on a dielectric substrate, the excitation of some unwanted TM modes can be avoided. A sketch of a tapered IEM waveguide on a glass substrate is illustrated in Fig. 12. Let the IEM have $\varepsilon_x = -3.0742$, $\varepsilon_z = 1.9762$ and the glass substrate have refractive index $n = 1.5$ at $\lambda = 1.55$ $\mu$m. The phase index $n_p$ as the function of thickness $d$ is plotted in Fig. 13. Due to the presence of substrate, the phase index of the guided $T_{0}$ modes starts at $n_p = 1.5$ with $d_0 = 116.5$ nm. The critical thickness of the $T_{0}$ modes is $d_c = 174.4$ nm. Thus if the initial thickness of a tapered IEM waveguide is less than 115.8 nm, the excitation of the forward-wave $T_{0}$ mode can be avoided. If one further ignores the excitation of $T_{m}$ with $m \geq 1$ due to the loss or the coupling strength, the tapered waveguide will allow only a single mode in practice. If the thickness is gradually increased to $d_e = 175.6$ nm, the wave will be stopped there.

However, loss is ubiquitous. The presence of loss in the IEM will prevent forward-wave modes and the backward-wave modes from joining together as shown in Fig. 7, leading to finite group velocity. For the IEM we have designed in the previous section, one has the material dispersion for the IEM

$$\frac{d\varepsilon_{x}}{d\kappa_{0}} = \frac{f\varepsilon_{z}^{2}}{\varepsilon_{m}} \frac{d\varepsilon_{m}}{d\kappa_{0}},$$

$$\frac{d\varphi}{d\kappa_{0}} = -\frac{1}{2}f\left(\frac{1}{\varepsilon_{x}} + \frac{1}{\varepsilon_{m}}\right) \frac{d\varepsilon_{m}}{d\kappa_{0}},$$

and

$$\frac{d\sigma}{d\kappa_{0}} = \frac{f}{\varepsilon_{z}^{2}} \frac{d\varepsilon_{m}}{d\kappa_{0}}.$$ 

Here we have ignored the dispersion of $\varepsilon_{x}$ in Eq. (32).

We further assume a Drude model dispersion for silver. Thus one has

$$\frac{d\varepsilon_{m}}{d\kappa_{0}} = \frac{1}{\kappa_{0}} \left[2(1 - \varepsilon_{m}) - i\varepsilon_{m} \frac{1 - \varepsilon_{m}}{\varepsilon_{s}^{*}} - \sigma \varepsilon_{m} \frac{1 - \varepsilon_{m}}{\varepsilon_{s}^{*}} \right].$$

For the exemplary IEM waveguide we have designed in the previous section, the group index of the modes is calculated and plotted in Fig. 14 by using the above dispersion of the IEM. One can clearly see that due to the presence of loss in the permittivity, especially in $\varepsilon_{x}$, the group index is greatly reduced. In the example we considered, we get $n_g \sim 10$. The fact that the group index $n_g$ for the backward-wave $T_{m}$ modes is negative is due to our convention of adopting positive phase index $n_p$ throughout the paper.

However gain can be introduced so that at a critical gain, the two branches of modes will recombine at a crit-
We point out that the effective medium theory for the metal and dielectric multilayered structure is still valid when one goes down in frequency from the visible, near infrared to THz waves and microwaves. The multilayered structure will still be able to support zero group velocity modes when one decreases the frequency. But the phase index when the group velocity is zero will be very high that in practice, a metallic grating structure will not be able to stop light. Due to the high phase index of the guided TM modes, strong reflection is expected along a tapered metallic grating. Only when the frequency is above the collision frequency, which is about 10 THz for silver [37], the dispersion of a metallic grating will be hyperbolic and the grating structure may be used to stop light.

We point out that the effective medium theory for the metal and dielectric multilayered structure is still valid when one goes down in frequency from the visible, near infrared to THz waves and microwaves. The multilayered structure will still be able to support zero group velocity modes when one decreases the frequency. But the phase index when the group velocity is zero will increase. In the limit of perfect electronic conductor, $\varepsilon_m \rightarrow \infty$, one has

$$
\varepsilon_x = \infty, \quad \varepsilon_z = \varepsilon_a/(1 - f).
$$

In order to give correct dispersion for waves in the metamaterial, $k_z = \sqrt{\varepsilon_a}k_0$ for the TM modes (see Fig. 1), one should have

$$
\mu_y = 1 - f.
$$

A free standing slab with the above permittivity and permeability will support guided TM modes. Waveguide can also be formed by sandwiching this metamaterial with dielectric on both sides, or dielectric on one side and PEC on the other. In the case of a PEC grating with finite depth, the above effective indices will give rise to the so-call spoof surface plasmons [52]. Only when the phase index goes to infinity, zero group velocity can be reached. This leads to the fact that the supported modes are all forward-wave modes since the energy flow inside the PEC grating is zero and that in the air are along the phase velocity. Though this type of waveguides can support TM modes with very small group velocity [11, 53], they can not be used to stop light. Even in the THz range, the phase index when the group velocity is zero will be very high that in practice, a metallic grating structure will not be able to stop light. Due to the high phase index of the guided TM modes, strong reflection is expected along a tapered metallic grating. Only when the frequency is above the collision frequency, which is about 10 THz for silver [37], the dispersion of a metallic grating will be hyperbolic and the grating structure may be used to stop light.

### V. CONCLUSIONS

In this paper, we consider wave propagation in a slab waveguide with anisotropic optical constant. For extremely anisotropic cylinder where the transverse component of the permittivity is negative and the longitudinal is positive, the waveguide supports infinite number of TM modes. Among the supported TM modes, at most only one mode can be forward wave. The rest of them are backward waves.

Possible realization of these IEM waveguides are proposed by utilizing alternative layers of metal and dielectric. Light couplings to IEM waveguides have also been discussed. To take full advantage of the rich band structure provided by the IEM waveguide, prism coupling and guiding coupling can be used to selectively excite the guided modes.

Four unique properties have been revealed for the modes on slab waveguides made of IEMs. The first is that the backward-wave modes can have very large phase index. These waveguide can be used as phase shifter in optics and telecommunications. The second is the large longitudinal electric field of the modes due to the hyperbolic dispersion of the metamaterial. The third is the formation of open cavity along in the waveguide due to its support of complex-conjugate decay modes above the critical thickness. The forth is that the waveguide supports modes of zero group velocity. This is due to the fact that the waveguide can support both forward- and backward-wave modes at a fixed thickness. If the waveguide is tapered, at certain critical thickness, the two modes will be degenerate and carry zero net energy flow. At other thickness, these waveguides support modes with small group velocity. The presence of loss in IEMs will destroy the zero group velocity modes but gain can be introduced to compensate loss. These waveguides can thus be used as ultra-compact delay line in optical buffers [5]. Many more interesting features many further emerge. The above features can lead to potential applications of...
these waveguides in optical integrated circuits.

In this paper, only indefinite ε metamaterials are considered for waveguide applications. Similar results can also be obtained for waveguide made of indefinite permeability (μ) metamaterials (IMMs). As the case for any application of metamaterials with negative parameters, loss will severely deteriorate the performance. The proposed design of IEM is by no means optimal. Similar to the IEM waveguide, IMM waveguide will have limited application due to large loss. A recent design of the split-ring resonators used in Ref. [54] for cloaking has very small loss of the permeability and can be used to realize the applications we have revealed in this work.

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