Maker-Breaker Percolation Games II: Escaping to Infinity

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Abstract

Let Λ be an infinite connected graph, and let $v_0$ be a vertex of Λ. We consider the following positional game. Two players, Maker and Breaker, play in alternating turns. Initially all edges of Λ are marked as unsafe. On each of her turns, Maker marks $p$ unsafe edges as safe, while on each of his turns Breaker takes $q$ unsafe edges and deletes them from the graph. Breaker wins if at any time in the game the component containing $v_0$ becomes finite. Otherwise if Maker is able to ensure that $v_0$ remains in an infinite component indefinitely, then we say she has a winning strategy. This game can be thought of as a variant of the celebrated Shannon switching game. Given $(p, q)$ and $(Λ, v_0)$, we would like to know: which of the two players has a winning strategy?

Our main result in this paper establishes that when Λ = $\mathbb{Z}^2$ and $v_0$ is any vertex, Maker has a winning strategy whenever $p \geq 2q$, while Breaker has a winning strategy whenever $2p \leq q$. In addition, we completely determine which of the two players has a winning strategy for every pair $(p, q)$ when Λ is an infinite $d$-regular tree. Finally, we give some results for general graphs and lattices and pose some open problems.

1 Introduction

Let Λ be an infinite connected (multi)graph, and let $v_0$ be a vertex of Λ. We consider the following positional game, which we call the $(p,q)$-percolation game on $(Λ, v_0)$.

Definition 1.1 ($(p,q)$-percolation game). Two players, Maker and Breaker, play in alternating turns, with Maker playing first. Initially all edges of Λ are marked as unsafe. On each of her turns, Maker marks $p$ unsafe edges as safe, while on each of his turns Breaker takes $q$ unsafe edges and deletes them from the graph. Breaker wins if at any time in the game the component containing $v_0$ becomes finite. Otherwise if Maker is able to ensure that $v_0$ remains in an infinite component indefinitely, then we say she has a winning strategy.

If Λ is a vertex-transitive graph, then the choice of $v_0$ does not matter, and we simply speak of the $(p,q)$-percolation game on Λ. Our main concern in this paper is to determine for given $(p,q)$ and $(Λ, v_0)$ which of the two players has a winning strategy in the corresponding $(p,q)$-percolation game.

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1.1 Main results

The *d*-dimensional integer lattice is the graph with vertex set $\mathbb{Z}^d$ whose edges consist of pairs of vertices $v, w \in \mathbb{Z}^d$ lying at Euclidean distance $\|v - w\| = 1$ from each other. In a standard abuse of notation, we write $\mathbb{Z}^d$ to denote this graph. Our main results in this paper are the following theorems.

**Theorem 1.2.** Maker has a winning strategy for the $(1,1)$-percolation game on $\mathbb{Z}^2$. More generally, Maker has a winning strategy for the $(p,p)$-percolation game on $\mathbb{Z}^d$ for every integer $1 \leq p < d$.

**Theorem 1.3.** Let $p, q \in \mathbb{N}$ If $p \geq 2q$, then Maker has a winning strategy for the $(p,q)$-percolation game on $\mathbb{Z}^2$.

**Theorem 1.4.** Let $p, q \in \mathbb{N}$. If $q \geq 2p$, then Breaker has a winning strategy for the $(p,q)$-percolation game on $\mathbb{Z}^2$.

**Theorem 1.5.** Let $p, q, d \in \mathbb{N}$, and let $T_d$ denote the infinite $d$-regular tree. Maker has a winning strategy for the $(p,q)$-percolation game on $T_d$ if and only if $p(d - 2) \geq q$.

**Definition 1.6.** For integers $2 \leq a \leq b$, let $T_{a,b}$ denote the infinite bi-regular tree with vertices of two types, Type I and Type II, defined as follows: each vertex of Type I is adjacent to exactly $a$ vertices of Type II, while each vertex of Type II is adjacent to exactly $b$ vertices of Type I.

**Theorem 1.7.** Let $p, q, a, b \in \mathbb{N}$. Maker has a winning strategy in the $(p,q)$-percolation game on $T_{a,b}$ if and only if

$$p(b - 2) - \left\lceil \frac{p}{a} \right\rceil (b - a) \geq q,$$

irrespective of the choice of the root.

1.2 Background and discussion

The $(p,q)$-percolation game can be thought of as a variant of the celebrated Shannon switching game, where the identity of the winner under optimal play was determined by by Lehman [11]. The Shannon switching game is played on a finite connected graph $G$ with two pre-specified vertices $u, v$. Two players, Short and Cut, play in alternating turns. Initially all edges of $G$ are marked as unsafe. In each of her turns Short selects an unsafe edge and marks it as safe, while in each of his turns Cut selects an unsafe edge and deletes it. Short wins if she managed to create a path of safe edges from $u$ to $v$, otherwise Cut wins.

The possibility of selecting multiple edges radically changes the nature of the game, and in particular Lehman’s arguments for the Shannon switching game do not appear to carry over to the $(p,q)$ setting. Mention should also be made here of the game of Gale, or Bridg-it, a specific class of Shannon switching games where $G$ is a rectangular grid and Short seeks to construct a path of safe edges from the left-hand side to the right-hand side. Bridg-it was popularised by Martin Gardner [7] and made into a commercially available game sold by Hasbro. Several winning strategies for Bridg-it are known, but again they do not seem to generalise to the $(p,q)$ case.

As the names given to the two players indicate, $(p,q)$-percolation games are an instance of the more general class of Maker–Breaker games. Such games are played on a board $X$ (i.e. a set — in our case, the edge-set of $\Lambda$), and a collection $\mathcal{W}$ of subsets of $X$ called winning sets (in our case, paths from $v_0$ to some target set of vertices). Two players, Maker and Breaker, take turns to claim elements of $X$. Maker (typically) plays first, and claims $a$ elements in each of her turns, while
Breaker claims $b$ elements on each of his. Maker wins if she manages to claim all the elements from some winning set $W \in W$, while Breaker wins if he thwarts her by claiming at least one element from each winning set. Since the board is finite, no draws are allowed, and the main question is to determine who has a winning strategy.

Strictly speaking, in the $(p, q)$-percolation game one could argue that Breaker rather than Maker is trying to claim all elements from some winning set — namely he seeks to claim all edges from a cut-set in $\Lambda$ that disconnects $\Lambda$ in such a way that the origin lies in a finite component. However, there are two reasons to name the players in our percolation games as we do. The first is that, informally speaking, we think of Maker as trying to build a ‘path to infinity’ from the origin, and would like to view these infinite paths as her winning sets. The second reason is that in Theorem 2.2 in Section 2 we show that Maker has a winning strategy for the $(p, q)$-percolation game on $\Lambda$ if and only if she has winning Maker strategies for certain collections of Maker–Breaker games played on finite subsets of $\Lambda$. It thus seems apt to call her Maker as we do.

Maker–Breaker games on graphs have been extensively investigated since the foundational work of Chvátal and Erdős [4] in the late 1970s. Important examples of such games include the connectivity game, the $k$-clique game and the Hamiltonicity game, where the board $X$ consists of the edges of a complete graph on $n$ vertices and the winning sets are spanning trees, $k$-cliques and Hamiltonian cycles respectively.

Chvátal and Erdős proved that, for a variety of such games, if $n$ is sufficiently large, then Maker has a winning strategy in the case where $a = b = 1$. In each case they then asked how large a bias $b = b(n)$ was required for the $(1, b)$ versions of these games to turn into Breaker’s win and provided a surprising and influential random graph heuristic for determining the value of these threshold biases. Namely, according to this heuristic the threshold bias $b_\star$ at which Breaker has a winning strategy should lie close to the threshold $b$ for a set of $\binom{n}{2}$ edges chosen uniformly at random to fail, with high probability, to contain any winning set. Almost twenty years after this heuristic was formulated, Bednarska and Łuczak [1] were able to rigorously establish its correctness in the case of $H$-games, where the winning sets are copies of some fixed finite graph $H$ containing at least three non-isolated vertices. In a different direction, Stojaković and Szabó [13] considered playing these Maker–Breaker games on random boards, by having $X$ consist of the edges of an Erdős–Rényi random graph $G_{n, \theta}$. As having fewer edges cannot help Maker, the natural question in this setting is: what is the threshold $\theta_\star$ such that if $\theta \gg \theta_\star$, then with probability $1 - o(1)$ Maker has a winning strategy for the $(1, 1)$-game on $G_{n, \theta}$, while if $\theta \ll \theta_\star$, then with probability $1 - o(1)$ Breaker has a winning strategy? Stojaković and Szabó showed that for some games, such as the connectivity games, $1/b_\star$ and $\theta_\star$ are of the same order, but that for others, such as the triangle game, no such relationship holds.

These subtle and intriguing connections between Maker–Breaker games and resilience phenomena in discrete random structures (in addition to their obvious combinatorial appeal) have led to an abiding interest in Maker–Breaker games. Our main motivation in this paper is to investigate whether any connections similar to the Erdős–Chvátal random graph heuristic exist when we move over from the world of finite graphs to the world of percolation on infinite graphs. Percolation theory is the branch of probability theory concerned, broadly speaking, with the study of random subgraphs of infinite graphs, and in particular with the almost-sure emergence of infinite connected components. Since its inception in Oxford in the late 1950s, it has blossomed into a rich and active area of mathematical research. One of the cornerstones of the theory is the celebrated Harris–Kesten Theorem [9, 10] which we state below. The $\theta$-random measure $\mu_\theta$ on an infinite graph $\Lambda$ is, informally, the probability measure on subsets of $E(\Lambda)$ that includes each edge with probability $\theta$, independently of all the others. (We eschew measure-theoretic subtleties here; for a rigorous
Theorem (Harris–Kesten Theorem). Let $G_\theta$ denote a $\mu_\theta$-random subgraph of $\mathbb{Z}^2$. Then

- if $\theta \leq \frac{1}{2}$, then almost surely $G_\theta$ does not contain an infinite component;
- if $\theta > \frac{1}{2}$, then almost surely $G_\theta$ contains an infinite component.

The motivation for the present paper is the following question, inspired by the Harris–Kesten theorem and the work of Chvátal and Erdős.

**Question 1.8.** Consider the $(p,q)$-percolation game on $\mathbb{Z}^2$. Does there exist a critical bias $b_\star > 0$ such that for any $\varepsilon > 0$ and all $p$ sufficiently large, $(b_\star + \varepsilon)p < q$ implies Breaker has a winning strategy, while $(b_\star - \varepsilon)p > q$ implies Maker has a winning strategy?

Our Theorems 1.3 and 1.4 show that there exist constants $b^- = \inf\{b : \forall (p,q) \text{ with } bp \geq q, \text{Maker has a winning strategy for the } (p,q)\text{-percolation game on } \mathbb{Z}^2\}$, $b^+ = \sup\{b : \forall (p,q) \text{ with } bp \leq q, \text{Breaker has a winning strategy for the } (p,q)\text{-percolation game on } \mathbb{Z}^2\}$ with

$$\frac{1}{2} \leq b^- \leq b^+ \leq 2.$$  

Furthermore, Theorem 1.2 shows that $b^+ \geq 1$. Based on the Harris–Kesten theorem and a random graph heuristic, it would be tempting to guess that $b^- = b^+ = b_\star = 1$ — however, we currently have far too little evidence in favour of this guess to make it a formal conjecture.

On the other hand, we are able to establish some correspondence between percolation theory and our Maker-Breaker games in the case of infinite $d$-regular trees, also known as Bethe lattices in the context of percolation theory. It is an easy exercise to determine that the critical probability for percolation on an infinite $d$-regular tree is $\theta_c = 1/(d-1)$. Our Theorem 1.5 establishes that if $(d-2)p \geq q$ then Maker has a winning strategy for the $(p,q)$-percolation game on the $d$-regular tree, while if $(d-2)p < q$ Breaker has a winning strategy. Thus, in this case we do have a critical bias $b_\star = d-2$, and it matches what one might expect from a random graph heuristic, that is, $\theta_c = 1/(b_\star + 1)$.

We also show the existence of a critical bias $b_\star = b_\star - \frac{b-a}{a} = \frac{ab-a-b}{a}$ for $(p,q)$-percolation games on the bi-regular trees $T_{a,b}$. In this case our result does not quite match what one might expect from a random graph heuristic, since $1/(b_\star + 1) = a/(b(a-1))$, whereas the critical probability for percolation is (via standard application of the theory of branching processes) is $\theta_c = 1/\sqrt{(a-1)(b-1)}$.

Finally, we should note that many of the modern proofs of the Harris–Kesten theorem rely on Russo–Seymour–Welsh (RSW) lemmas on the probability of having a crossing path in a $\theta$-random subset of a rectangular grid. Before considering Question 1.8 and the $(p,q)$-percolation, it thus makes sense to first investigate certain crossing games — in effect, generalisations of Bridg-it. This was done by the authors in a prequel [5] to this work. In the present paper we apply some results on crossing games from [5] together with some other ideas to obtain Theorems 1.3 and 1.4.

### 1.3 Organisation of the paper

In Section 2 we prove a compactness result, Theorem 2.2, which reduces a percolation game on an infinite connected graph to certain families of Maker–Breaker games on finite connected (multi)graphs. In Section 3 we prove a result for general graphs as well as Theorems 1.2, 1.5 and 1.7 on percolation games on integer lattices and infinite trees. In Section 4 we prove our results for percolation games on $\mathbb{Z}^2$, that is, Theorems 1.3 and 1.4. We end the paper in Section 5 with some concluding remarks and many questions and open problems.
1.4 Notation

A (multi)graph is a pair $G = (V,E)$, where $V = V(G)$ is a set of vertices and $E = E(G)$ is a collection of pairs from $V$ (some of which may be included multiple times), which form the edges of $G$. Given $S \subseteq V(G)$, the subgraph of $G$ induced by $S$ is $(S,\{e \in E(G) : e \subseteq S\})$. In this paper, we often omit the prefix (multi) and identify a graph with its edge-set when the underlying vertex-set is clear from context. A graph $G$ is connected if for any pair of vertices $v, v' \in V(G)$ there is a finite path of edges of $G$ from $v$ to $v'$. We study percolation games on rooted infinite connected (multi)graphs, or RIC-graphs, which is to say pairs $(\Lambda,v_0)$ where $\Lambda$ is an infinite, connected graph and the root $v_0$ is a vertex of $\Lambda$. We often refer to $v_0$ as the origin of our RIC-graph. In the case where $\Lambda$ is vertex-transitive, we may in a slight abuse of notation write $\Lambda$ for the RIC-graph $(\Lambda,v_0)$ and take it that $v_0$ is arbitrarily specified.

The $(p,q)$-percolation game on the RIC-graph $(\Lambda,v_0)$ was formally defined above in Definition 1.1. When analysing it, it is convenient to consider that both players choose the edges they will select sequentially, so e.g. on each of her turns Maker selects an ordered sequence of $p$ edges, $e_1,e_2,\ldots,e_p$, she marks as safe one by one (rather than a $p$-set of edges which she marks safe all at once) and similarly Breaker selects an ordered sequence of $q$ edges to destroy one by one. This clearly makes no difference to the game, and allows us to define a notion of time: we say a game is at time $t$ if a combined total of $t$ edges have been played. For example, at time $t = 9$ in a $(2,2)$-percolation game, it is Maker’s third turn and she has already claimed the first (but not the second) of the two edges she will mark as safe in that turn.

2 Compactness

In this section we prove a compactness theorem on percolation games, which says that they are essentially equivalent to certain generalised Shannon switching games. Let $(\Lambda,v_0)$ be an RIC-graph, and let $X$ be a finite connected induced subgraph of $\Lambda$ such that $v_0 \in V(X)$. The boundary of $X$ in $\Lambda$ is the collection $\partial_\Lambda(X)$ of vertices in $X$ that send an edge to a vertex in $V(\Lambda) \setminus V(X)$.

Definition 2.1 (Escape game). In the $(p,q)$-escape game on $(X,\Lambda,v_0)$, two players, Maker and Breaker, claim edges of the finite graph $X$ in alternating turns, with Maker playing first. On each of her turns, Maker claims $p$ as-yet-unclaimed edges for herself, while Breaker on each of his turns claims $q$ such edges for himself. Maker wins the game if she manages to claim all of the edges of a path from the $v_0$ to $\partial_\Lambda(X)$, while Breaker wins if he claims at least one edge from every such path.

When there is no ambiguity regarding $p,q,\Lambda,v_0$ we just refer to the escape game on $X$. Observe that the escape game cannot end in a draw, and terminates within $2|E(X)|/(p+q)$ turns. By contracting all vertices in $\partial_\Lambda(X)$ to a single vertex $u_0$, we obtain a (multi)graph $G$. The escape game on $X$ is thus seen to be equivalent to a generalised Shannon switching game on $G$ where Short seeks to create a path of safe edges from $u_0$ to $v_0$ and where Short and Cut mark respectively $p$ and $q$ edges on each of their turns.

Theorem 2.2. Let $\Lambda$ be an RIC-graph with origin $v_0$. Maker has a winning strategy for the $(p,q)$-percolation game on $\Lambda$ if and only if Maker has winning strategies for every $(p,q)$-escape game on $(X,\Lambda,v_0)$, for all finite connected induced subgraphs $X$ of $\Lambda$ with $v_0 \in V(X)$.

Proof. One direction of the statement is straightforward. Suppose there exists some finite connected induced subgraph $X$ of $\Lambda$ with $v_0 \in V(X)$ such that Breaker has a winning strategy for the $(p,q)$-escape game on $X$. Then the following is a winning Breaker strategy for the $(p,q)$-percolation game
on \( \Lambda \): ignore the rest of the board and focus exclusively on \( X \). Whenever Maker, on one of her turns, claims \( p' \leq p \) edges of \( X \), Breaker selects \( p - p' \) as-yet-unclaimed edges of \( X \) arbitrarily and pretends that Maker claimed these edges too. Breaker then responds using the winning strategy for the \((p, q)\)-escape game on \( X \). Within at most \( 2|E(X)|/(p + q) \) turns, Breaker will have claimed (and destroyed) at least one edge from every path from \( v_0 \) to \( \partial_\Lambda(X) \). In particular by this time, \( v_0 \) is contained in a finite component which is a strict subset of \( V(X) \) and Breaker has won the percolation game on \( \Lambda \).

In the other direction, suppose that Maker has a winning strategy for the \((p, q)\)-escape game on \( X \) for all finite connected induced subgraphs \( X \) of \( \Lambda \) with \( v_0 \in V(X) \). Let \((X_i)_{i \geq 0}\) be an arbitrary sequence of finite connected induced subgraphs of \( \Lambda \) such that \( v_0 \in V(X_0) \), \( X_i \subseteq X_{i+1} \) for all \( i \geq 0 \) and \( \bigcup_{i \geq 0} X_i = \Lambda \). (For example, one could take each \( X_i \) to be the subgraph induced by the vertices of \( \Lambda \) that lie at graph distance at most \( i \) from \( v_0 \) in \( \Lambda \).)

By our assumption, at the beginning of Maker’s first turn each of the boards \( X_i \) is in a winning position for her in the \((p, q)\)-escape game on that \( X_i \). We shall construct a Maker strategy that ensures this remains true for all subsequent turns, i.e. that at the beginning of her every turn Maker has a winning strategy for the \((p, q)\)-escape game on \( (X'_i, \Lambda', v_0) \) for every \( i \geq 0 \), where \( X'_i \) and \( \Lambda' \) are the (multi)graphs obtained from \( X_i \) and \( \Lambda \) by deleting all edges claimed by Breaker and contracting all edges claimed by Maker up to that point.

We first show that if we are able to achieve the above, then this is a winning strategy for Maker in the \((p, q)\)-percolation game on \( \Lambda \). Suppose this is not the case, and that at some time \( T \) in the game the component \( C \) containing \( v_0 \) becomes finite. Let \( D \) be the set of vertices in in \( V(\Lambda) \setminus C \) that are adjacent (by any edge of \( \Lambda \) regardless of who may have claimed it) to at least one vertex in \( C \). Then every edge between \( C \) and \( D \) must have been claimed by Breaker. Moreover, as Breaker has certainly claimed no more than \( T \) edges we have that \( D \) is finite. Thus there exists some \( i \) such that \( C \cup D \subseteq X_i \). We must have that \( C \cap \partial(X_i) = \emptyset \), as otherwise there would be some vertex in \( D \cap (V(\Lambda) \setminus X_i) \), which is not possible. Therefore every path from \( v_0 \) to \( \partial(X_i) \) meets at least one Breaker edge, which contradicts the fact that \( X_i \) was in a winning position for Maker in the escape game on \( X_i \).

We now specify our strategy. Assume that at the beginning of her turn, Maker is in a winning position in the \((p, q)\)-escape game on \( X_i \) for every \( i \geq 0 \) (taking into account the edges of \( X_i \) claimed by Maker and Breaker during past turns). Then for each \( i \geq 0 \) there is a winning Maker strategy for the \((p, q)\)-game on \( X_i \) from the current position specifying some set \( P_i \) of \( p \) as-yet-unclaimed edges of \( X_i \) as Maker’s next move in the escape game. We use the sequence of sets \((P_i)_{i \geq 0}\) to produce a set \( A \) of \( |A| \leq p \) edges, using the following algorithm. Set \( A_0 = \emptyset \). Given a set of edges \( A_j \), let

\[
S_{i,j} = \begin{cases} 
    P_i \setminus A_j & \text{if } A_j \subseteq P_i, \\
    \emptyset & \text{if } A_j \not\subseteq P_i.
\end{cases}
\]

If any edge appears in infinitely many members of the sequence \((S_{i,j})_{i \geq 0}\), pick\(^1\) one such edge \( e \) and set \( A_{j+1} = A_j \cup \{e\} \). Otherwise, set \( A = A_j \) and terminate the process. As \( |A_j| = j \) and \( |S_{i,j}| \leq p - j \), the algorithm will terminate after at most \( p \) iterations, outputting a set \( A \) of at most \( p \) edges. Maker’s move for the percolation game is then to claim all edges in \( A \) together with a set \( A' \) of \( p - |A| \) arbitrary edges. Breaker then answers by claiming some set \( B \) of \( q \) edges. We claim that irrespective of the choices of \( A' \) and \( B \), Maker is still in a winning position in the \((p, q)\)-escape game on \( X_i \) for every \( i \geq 0 \) at the beginning of her next turn.

\(^1\)If one assumes \( V(\Lambda) = \mathbb{N} \), one can e.g. pick the least such edge in the lexicographic order and thereby avoid using the axiom of choice.
Indeed, suppose for contradiction that Maker is in a losing position on \( X_i_0 \), for some \( i_0 \in \mathbb{N} \), i.e. that Breaker has a winning strategy for the \((p,q)\)-escape game in that position with Maker playing first. By construction of \( A \), there exists some \( i_1 > i_0 \) such that \( P_{i_1} \cap X_{i_0} = A \cap X_{i_0} \). Now, as Maker is in a losing position on \( X_i_0 \), we have that Breaker has some strategy that can guarantee that Maker never builds a path from \( v_0 \) to \( \partial(X_{i_0}) \). Thus, when playing the \((p,q)\)-escape game on \( X_{i_1} \), Breaker can use this same strategy to ensure that Maker never builds a path from \( v_0 \) to \( \partial(X_{i_1}) \). However, this contradicts the fact that \( P_{i_1} \) was a winning Maker move for the \((p,q)\)-escape game on \( X_{i_1} \). Therefore no such \( X_{i_0} \) can exist, and so every \( X_i \) is in a winning position for Maker in the corresponding \((p,q)\)-escape game.

\( \square \)

**Remark 2.3.** Lehman \([11]\) showed that Short has a winning strategy in the Shannon switching game on a multigraph \( G \) with distinguished vertices \( u,v \) if and only there is an induced subgraph \( H \) of \( G \) containing two edge-disjoint spanning trees connecting \( u \) to \( v \). Together with Theorem 2.2, this in principle gives a criterion for determining which of Maker and Breaker has a winning strategy in the \((1,1)\)-percolation game on \( \Lambda \) for any RIC-graph \( \Lambda \): for each \( i \in \mathbb{N} \) consider the subgraph \( X_i \) of \( \Lambda \) induced by vertices at graph distance at most \( i \) from the root \( v_0 \), contract the boundary \( \partial(\Lambda(X_i)) \) to a single point \( u_0 \) to obtain a graph \( G_i \) and then use Lehman’s result to determine which of Short and Cut has a winning strategy for the Shannon switching game played on \( G_i \) with distinguished vertices \( u_0 \) and \( v_0 \).

### 3 General graphs, integer lattices and regular trees

#### 3.1 Path colourability and a proof of Theorem 1.2

We say that a \( k \)-colouring of the edges of an RIC-graph \( \Lambda \) is a \( k \)-path-colouring if, for every vertex \( v \) of \( \Lambda \) and every colour \( i \), there exists an infinite path through \( v \) consisting only of edges of \( \Lambda \) in colour \( i \). If such a colouring exists, we say that \( \Lambda \) is \( k \)-path-colourable; \( k \)-path colourability can be viewed as a generalisation of the necessary and sufficient condition for Short to have a winning strategy in the Shannon switching game, see \([12]\).

**Theorem 3.1.** Let \( \Lambda \) be a \((k+1)\)-path-colourable RIC-graph. Then Maker has a winning strategy for the \((p,p)\)-percolation game on \( \Lambda \) for every \( p \leq k \) (irrespective of the choice of the root).

What is more, Maker can forgo her first turn and in addition ensure that every vertex in \( \Lambda \) remains in an infinite component at all times in the game.

**Proof.** Let \( c \) be a \((k+1)\)-path-colouring of \( \Lambda \) with colours from \([k+1] = \{1,2,\ldots,k+1\}\). Given finite, pairwise-disjoint edge-sets \( X,Y \subseteq \Lambda \), let \( (\Lambda - X)/Y \) denote the graph obtained from \( \Lambda \) by first deleting every edge in \( X \) and then contracting every edge in \( Y \).

**Lemma 3.2.** Let \( c \) be a \((k+1)\)-path-colouring of an RIC-graph \( \Lambda' \) with colours from \([k+1] \). Then for every edge \( e \in \Lambda' \) and every colour \( j \in [k+1] \setminus \{c(e)\} \), there exists an edge \( f \in \Lambda' \) with \( c(f) = j \) such that \((\Lambda - \{e\})/\{f\} \) is \((k+1)\)-path colourable.

**Proof.** Let \( \Gamma = \{e' \in E(\Lambda) : c(e') = c(e)\} \). As \( c \) is a \((k+1)\)-path-colouring of \( \Lambda \), every component of \( \Gamma \) is infinite. In particular, \( \Gamma - \{e\} \) contains at most one finite component. If \( \Gamma - \{e\} \) contains no finite component, then let \( f \) be an arbitrary edge of \( \Lambda \) with \( c(f) = j \). Otherwise, \( \Gamma - \{e\} \) has a unique finite component \( D \). Let \( v \) be any vertex in \( D \). As \( c \) is a \((k+1)\)-path-colouring of \( \Lambda \), there is an infinite path \( P \) in \( \Lambda \) through \( v \) all of whose edges are assigned colour \( j \neq c(e) \) by \( c \). As
\[ P \text{ is infinite and } D \text{ is finite, there exists an edge } f = \{v_1, v_2\} \in P \text{ such that } v_1 \in D, v_2 \notin D. \text{ In particular there exists a finite path } P_1 \text{ from } v \text{ to } v_1 \text{ and an infinite path } P_2 \text{ through } v_2 \text{ such that both } P_1 \text{ and } P_2 \text{ are monochromatic with colour } c(e) \text{ (with respect to the colouring } c). \text{ Thus, when we delete } e \text{ from our } (k + 1)-\text{coloured graph } \Lambda \text{ and contract the edge } f \text{ to obtain } (\Lambda - \{e\})/\{f\}, \text{ we have that every vertex } v \in D \text{ is part of an infinite component in colour } c(e) \text{ — and hence the same holds for every vertex } v \in V(\Lambda). \text{ This immediately implies that our } (k + 1)-\text{coloured graph } (\Lambda - \{e\})/\{f\} \text{ is } (k + 1)\text{-path-colourable (the property that every vertex is part of an infinite monochromatic component in colour } i \text{ for every } i \in [k + 1]\{c(e)\} \text{ is inherited from } \Lambda. \]

For each } s \in \mathbb{N}, \text{ let } M_s \text{ (respectively } B_s) \text{ be the set of edges claimed by Maker (respectively Breaker) on her (respectively his) } s\text{-th turn of the game. Set } \Lambda^0 = \Lambda, \text{ and for each } s \in \mathbb{N} \text{ let } \Lambda^s = (\Lambda^{s-1} - B_s)/M_{s+1}. \text{ We construct below a strategy for Maker that ensures } \Lambda^s \text{ is } (k + 1)\text{-path-colourable for every } s \geq 0. \text{ This is a winning strategy in the strongest sense possible, as it guarantees that every vertex } v \in V(\Lambda) \text{ (not just } v_0) \text{ remains in an infinite component at all times throughout the game.}

By assumption, } \Lambda^0 \text{ is } (k + 1)\text{-path-colourable. On her first turn, Maker claims } p \text{ arbitrary edges, or forgoes her turn entirely (this makes no difference to our analysis). On subsequent turns, she responds to Breaker’s moves as follows. Suppose } \Lambda^{s-1} \text{ has a } (k + 1)\text{-path-colouring } c \text{ with colours from } [k + 1], \text{ and that in his } s\text{-th turn Breaker claims the edges } B_s = \{b_1, \ldots, b_p\}. \text{ As } |B_s| = p \leq k, \text{ there is some colour } j \in [k + 1] \text{ which does not appear in } \{c(b_1), c(b_2), \ldots, c(b_p)\}. \text{ Set } c_0 = c, \Lambda_0^s = \Lambda^{s-1}, \text{ and define sequences of } (k + 1)\text{-path-colourings and graphs } c_i \text{ and } \Lambda_i^s \text{ as follows. For } 1 \leq i \leq p, \text{ assume } c_{i-1} \text{ is a } (k + 1)\text{-path-colouring of the graph } \Lambda_{i-1}^s \text{ with colours from } [k + 1]. \text{ By Lemma 3.2, given an edge } b_i \text{ with } c_{i-1}(b_i) \neq j \text{ there exists an edge } f_i \text{ in colour } c_{i-1}(f_i) = j \text{ such that the graph } \Lambda_i^s = (\Lambda_{i-1}^s - \{b_i\})/\{f_i\} \text{ is } (k + 1)\text{-path-colourable by the colouring } c_i \text{ it inherits from } \Lambda_{i-1}^s. \text{ It follows that the graph } \Lambda_p^s = (\Lambda^{s-1} - B_s)/\{f_1, f_2, \ldots, f_p\} \text{ is } (k + 1)\text{-path-colourable. Setting } M_{s+1} = \{f_1, f_2, \ldots, f_p\} \text{ to be the set of edges claimed by Maker on her } (s + 1)\text{-th turn, we have that } \Lambda^s \text{ is } (k + 1)\text{-path-colourable. By induction on } s, \text{ this gives our desired winning strategy for Maker.}

\[ \text{Theorem 1.2 now follows as an easy corollary:} \]

\[ \text{Proof of Theorem 1.2. Label the axes of } \mathbb{Z}^d \text{ as } x_1, x_2, \ldots, x_d. \text{ For each } i \in [d], \text{ colour all edges parallel to the } x_i\text{-axis with colour } i. \text{ This is a } d\text{-path-colouring of } \mathbb{Z}^d. \text{ By Theorem 3.1, it follows that Maker has a winning strategy for the } (p, p)\text{-percolation game on } \mathbb{Z}^d \text{ for all integers } p: 1 \leq p < d. \]

### 3.2 Infinite trees

We begin this section by proving a general lemma on percolation games played on trees. Let } T \text{ be any RIC-tree with root } v_0. \text{ Given two edges } e_1, e_2 \text{ in } T, \text{ we say that } e_1 \text{ is an ancestor of } e_2 \text{ if the unique path between } e_2 \text{ and } v_0 \text{ contains the edge } e_1. \text{ Conversely if } e_1 \text{ is an ancestor of } e_2, \text{ then we say that } e_2 \text{ is a descendant of } e_1. \text{ Note that every edge is both an ancestor and a descendant of itself.}

Let } C_t \text{ be the component of Maker edges that, at time } t, \text{ includes } v_0. \text{ Throughout this subsection, whenever we refer to } C_t \text{ we will always take it to mean the component containing } v_0 \text{ at the relevant point in time, and forgo writing “at time } t”.

\[ \text{Lemma 3.3. We may assume that under optimal play in the } (p, q)\text{-percolation game on } T, \text{ Maker and Breaker only claim edges adjacent to } C_t. \]
Proof. We begin by showing this holds for Breaker. Suppose Breaker’s strategy requires him to claim an edge \( e \) that is not adjacent to \( C_t \). Let \( e' \) be the ancestor of \( e \) that is closest to \( v_0 \) but is not in \( C_t \); this edge \( e' \) has not yet been claimed by Maker as it does not lie in \( C_t \). Furthermore, as far as the outcome of the percolation game is concerned, Breaker claiming \( e' \) is equivalent to Breaker simultaneously claiming both \( e' \) and all its descendants (a set of edges that includes \( e \)). Thus claiming \( e' \) is at least as good for Breaker as claiming \( e \), and we may thus assume without loss of generality that Breaker does claim \( e' \) rather than \( e \) when following an optimal strategy.

We next show that we may assume that at time \( t \) Maker only claims edges adjacent to \( C_t \). Suppose that \( S \) is a winning strategy for Maker in the \((p,q)\)-percolation game on \( T \), and at some point in the game \( S \) requires Maker to claim an edge \( e \) that is not adjacent to \( C_t \). Once again, let \( e' \) be the ancestor of \( e \) that is closest to \( v_0 \) but is not in \( C_t \). If \( e' \) has already been claimed by Breaker, than as far as the outcome of the game is concerned this is equivalent to Breaker having claimed every descendant of \( e' \). In particular Maker playing \( e \) has no bearing on the outcome of the game, and we may thus assume without loss of generality that Maker claims some other edge adjacent to \( C_t \) rather than \( e \) when following a winning strategy.

On the other hand, suppose that Breaker has not claimed \( e' \). Then we construct a new Maker strategy \( S' \), in which Maker claims \( e' \) instead of \( e \) at time \( t \), but then follows what the strategy \( S \) would dictate if she had claimed \( e \) (i.e. she plays \( e' \) but pretends she played \( e \) and keeps following \( S \) accordingly). Three things can happen.

(i) If \( S \) at some future time \( t' > t \) requires Maker to claim \( e' \), then our new strategy \( S' \) will require her to pick \( e \) at that time instead. From then on \( S \) and \( S' \) are identical winning strategies for Maker.

(ii) If at some future time \( t' > t \) Breaker claims the edge \( e \), then Maker is in no worse a position than if she had been following the strategy \( S \) and Breaker had claimed the edge \( e' \) at time \( t' \) instead. This is because, once again, Breaker claiming the edge \( e' \) is equivalent to them claiming \( e' \) and all its descendants (a set of edges that includes \( e \)). Thus Maker may pretend that Breaker picked \( e' \) at time \( t' \) (and continue to pretend that she picked \( e \) at time \( t \)). From then on, \( S \) and \( S' \) are identical winning strategies for Maker.

(iii) If neither of the above ever occurs, then \( S \) and \( S' \) are indistinguishable winning strategies for Maker from time \( t + 1 \) onwards.

We have thus constructed from \( S \) a new winning strategy \( S' \) for Maker in which she only picks edges adjacent to \( C_t \) at time \( t \). This concludes the proof of the Lemma.

Recall that \( T_d \) denotes the rooted \( d \)-regular infinite tree. We are now ready to prove Theorem 1.5, which stated that Maker wins the \((p,q)\)-percolation game on \( T_d \) if and only if \( p(d-2) \geq q \).

Proof of Theorem 1.5. As before, let \( C_t \) be the component of Maker edges that contains the root at time \( t \). By Lemma 3.3 we may assume that both Maker and Breaker always play their edges adjacent to \( C_t \). For each \( t \in \mathbb{N} \), let \( B_t \) be the set of edges that Breaker has claimed at time \( t \), and let \( \delta_t \) be the number of unclaimed edges adjacent to \( C_t \) at time \( t \). Breaker wins the percolation game if and only if there exists some \( t \in \mathbb{N} \) such that \( \delta_t = 0 \). Note that \( \delta_0 = d \). Suppose \( \delta_t > 0 \) at some time \( t \); if it is Maker’s turn to claim an edge, then \( \delta_{t+1} = \delta_t + d - 2 \), while if it is Breaker’s turn then \( \delta_{t+1} = \delta_t - 1 \). In particular if Maker has not lost before the end of Breaker’s \( N \)-th turn, for \( N \geq 0 \), then

\[
\delta_{N(p+q)} = \delta_0 + N (p(d-2) - q) = d + N (p(d-2) - q)
\]
and it is Maker’s turn to play. Thus, if \( p(d - 2) - q \geq 0 \), then \( \delta_t \geq d > 0 \) for all \( t \in \mathbb{N} \), and the game goes on indefinitely, which is a win for Maker. On the other hand, if \( p(d - 2) - q < 0 \) then the game lasts at most \( \left\lceil \frac{d}{q - p(d-2)} \right\rceil \) rounds before \( \delta_t \) hits 0 and Breaker wins the game.

With a little more work, we can use Lemma 3.3 to prove Theorem 1.7 for percolation games on bi-regular trees. Recall the definition of \( T_{a,b} \) from the introduction. For \( \epsilon \in \{I, II\} \) we denote by \( T^\epsilon_{a,b} \) the rooted tree obtained from \( T_{a,b} \) by selecting an arbitrary vertex of Type \( \epsilon \) to be the root.

**Proof of Theorem 1.7.** Let \( T = T^\epsilon_{a,b} \). As before, let \( C_t \) denote the component of Maker edges containing the origin at time \( t \). By Lemma 3.3, we may assume without loss of generality that at each time \( t \), both players only claim edges adjacent to \( C_t \). Let \( X_t \) and \( Y_t \) denote the collections of vertices in \( V(T) \setminus C_t \) of Type I and II respectively that are adjacent, by an unclaimed edge, to a vertex of \( C_t \) at time \( t \).

The effects of Maker’s and Breaker’s moves on the vector \((|X_t|, |Y_t|)\) are easy to describe. If Maker claims an edge between \( C_t \) and \( X_t \), then \((|X_{t+1}|, |Y_{t+1}|) = (|X_t| - 1, |Y_t| + a - 1)\), while if she claims an edge between \( C_t \) and \( Y_t \), then \((|X_{t+1}|, |Y_{t+1}|) = (|X_t| + b - 1, |Y_t| - 1)\). Similarly, if Breaker claims an edge between \( C_t \) and \( X_t \), then \((|X_{t+1}|, |Y_{t+1}|) = (|X_t| - 1, |Y_t|)\), while if he claims an edge between \( C_t \) and \( Y_t \), then \((|X_{t+1}|, |Y_{t+1}|) = (|X_t|, |Y_t| - 1)\).

Write \( d_N \) for the \( \ell_1 \)-distance from \((|X_t|, |Y_t|)\) to \((0, 0)\) at the beginning of Maker’s \((N+1)\)-th turn, i.e. \( d_N := |X_{N(p+q)}| + |Y_{N(p+q)}| \). Note that Breaker wins the game if and only if he can ensure \( d_N = 0 \) for some \( N \in \mathbb{Z}_{\geq 0} \). Finally, set \( r_* = \left\lceil \frac{p}{a} \right\rceil \), and write \( \Delta = p(b-2) - r_*(b-a) - q \).

Our claim is that Maker has a winning strategy if and only if \( \Delta \geq 0 \). Before we dive into the proof, let us give a heuristic for this. Both players in the \((p,q)\)-percolation game on \( T \) have an obvious greedy strategy, namely: claim any edge whose addition to \( C_t \) would result in the largest increase in the edge-boundary (in other words, pick an edge from \( C_t \) to \( Y_t \) if \( Y_t \neq \emptyset \), otherwise pick an edge from \( C_t \) to \( X_t \)). As we show in Lemma 3.4 below, in the long run the quantity \( \Delta \) essentially bounds the change in the \( \ell_1 \)-distance \( d_N \) between successive turns under these strategies.

In particular, if \( \Delta \) is nonnegative then Maker can ensure that \( d_N > 0 \) for all \( N \geq 0 \), while if \( \Delta < 0 \) then Breaker can over time reduce \( d_N \) to 0 and win the game. For Maker, this is essentially the whole story. For Breaker’s winning strategy, there are some subtleties: Breaker must first reduce \( |Y_t| \) to zero before he can reduce the distance \( d_N \). Further, if \( b = a + 1 \), then he cannot ensure that \( |Y_t| \) remain zero at the start of every Maker turn, and even showing that he can ensure \( |Y_t| = 0 \) at the end of some turn requires a little care.

Having outlined the proof, we now give the details. We begin by dealing with a trivial case. By definition, we know \( a \leq b \). If \( a = b \) then in fact we can merge the two types into just one: \( T \) is then just a copy of \( T_a \), the rooted \( a \)-regular infinite tree. By Theorem 1.5, we thus have that Maker has a winning strategy in the \((p,q)\)-percolation game if and only if \( q \leq p(b-2) = p(b-2) - r_*(b-a) \), as desired. So from now on, let us assume \( a < b \).

**Lemma 3.4.** Suppose \( d_N > 0 \). Then the following hold:

(i) by using her greedy strategy, Maker can ensure that \( d_{N+1} \geq d_N + \Delta \);

(ii) if \( Y_{N(p+q)} = \emptyset \), then by using his greedy strategy, Breaker can ensure that \( d_{N+1} \leq d_N + \Delta \).

**Proof.**

**Part (i):** proving the first part of this lemma is simple accounting. If \( Y_{N(p+q)} = \emptyset \), then Maker’s strategy dictates that she first claims an edge from \( C_t \) to a Type I vertex (such an edge exists since \( d_N > 0 \)) and then edges from \( C_t \) to Type II vertices whenever possible. In particular, she claims exactly \( r_* \) edges to Type I vertices and \( (p-r_*) \) edges to Type II vertices. Otherwise if \( Y_{N(p+q)} \neq \emptyset \)
Maker can get away with claiming only \( r \) edges to Type I vertices, for some \( r \leq r_* \). Either way, Maker’s moves increase the \( \ell_1 \)-distance from \( (|X_t|, |Y_t|) \) to \( (0,0) \) by at least

\[
r_*(a - 2) + (p - r_*)(b - 2) = \Delta + q.
\]

Breaker’s move then reduce the distance again by \( q \), whence at the start of Maker’s next turn \( d_{N+1} \geq d_N + \Delta \).

**Part (ii):** as noted above, if \( Y_{N(p+q)} = \emptyset \) then during her \((N+1)\)-th turn Maker must claim \( r \) edges from \( C_t \) to \( X_t \) for some \( r: r_* \leq r \leq p \). Thus \(|X_t| + |Y_t|\) increases by at most \( \Delta + q \) as a result of Maker’s moves. By following his greedy strategy, Breaker then ensures that \( d_{N+1} \leq d_N + \Delta \).

Lemma 3.4 immediately implies that Maker has a winning strategy for the \((p,q)\)-percolation game on \( T \) whenever \( \Delta \geq 0 \). Thus all that remains to show is that Breaker has a winning strategy if \( \Delta < 0 \). Since \( p, q, a, b \) are integers and since increasing Breaker’s power can only help him, it suffices to show that Breaker has a winning strategy if \( \Delta = -1 \), i.e. if \( q = p(b - 2) - r_*(b - a) + 1 \). We have two cases to consider.

**Case 1:** \( b \geq a + 2 \). In this case, observe that in any turn, Maker’s moves can increase \( Y_t \) by at most \( p(a - 1) \). Since we have \( a \geq 2, p \geq 1 \) and

\[
q - p(a - 1) = (p - r_*)(b - a) - p + 1 \geq 2 \left( p - \frac{p + a - 1}{a} \right) - p + 1
\]

\[
= \frac{(a - 2)(p - 1)}{a} \geq 0,
\]

(3.1)

Breaker’s greedy strategy ensures that the number of edges to Type II vertices available to Maker at the start of her turn is non-increasing. Moreover in any turn where the number of such edges has not strictly decreased, the number of edges to Type I vertices available to Maker must have gone down by at least \( p \). Thus within at most \(|X_o|/p\) turns, \(|Y_{N(p+q)}|\) must have decreased by at least 1. What is more, at the beginning of the first turn \( N \) on which \(|Y_{N(p+q)}| < |Y_0|\) we have \(|X_{N(p+q)}| \leq |X_0| + p(b - 1)\) (since \(|X_{N-1}(p+q)}| \leq |X_0|\) and since Maker’s moves during her \(N\)-th turn can increase \(|X_t|\) by at most \(p(b - 1)\)). This readily implies that within at most

\[
\frac{1}{p} \left( |X_0| + (|X_0| + p(b - 1)) + (|X_0| + 2p(b - 1)) + \cdots + (|X_0| + (|Y_0| - 1)p(b - 1)) \right) \leq (b - 1)(d_0)^2
\]

turns, \( Y_t \) is empty at the beginning of Maker’s turn. Suppose this occurs at the start of turn \( N_0 \), i.e. \( Y_{N_0(p+q)} = \emptyset \). Then Breaker’s greedy strategy further ensures that \( Y_{(N_0+1)(p+q)} = \emptyset \) and, by Lemma 3.4 part (ii), that \( d_{N_0+1} \leq d_{N_0} + \Delta = d_{N_0} - 1 \). Within a further \( N_1 = d_{N_0} \) turns, Breaker will therefore win the game. Breaker’s greedy strategy thus ensures his victory in the \((p,q)\)-percolation game within at most a total of \( N_0 + N_1 \leq N_0 + (p(b - 2) - q)N_0 \leq (p(b - 2) - q + 1)(b - 1)(d_0)^2 \) turns, where the bound on \( N_1 \) is from the trivial upper bound on the change in the \( \ell_1 \)-distance \( d_N \) between consecutive Maker turns. So even if Maker were given a head start in the game (by being allowed to claim some edges before the game starts, thus making \( d_0 \) large), Breaker would still win.

**Case 2:** \( b = a + 1 \). In this case, \( q - p(a - 1) = -(r_* - 1) \). In particular, Breaker cannot prevent Maker from increasing the number of edges to Type II vertices between turns, and more care is need to show his greedy strategy is successful.

Suppose that the game starts from some position \((X_0, Y_0)\), and that during the first \( N \) turns, Breaker is able to claim only edges to Type II vertices. Let \( \theta N p \) denote the number of edges to
Type I vertices claimed by Maker during her first \( N \) turns. Then the following hold:

\[
|X_{N(p+q)}| = |X_0| + Np\left(-\theta + a(1-\theta)\right),
\]
\[
|Y_{N(p+q)}| = |Y_0| + Np\left((a-1)\theta - (1-\theta) - \frac{q}{p}\right).
\]

Since \( |X_{N(p+q)}|, |Y_{N(p+q)}| \geq 0 \), inequalities (3.2) and (3.3) together imply

\[
-\frac{|Y_0|}{aNp} + \frac{p + q}{ap} \leq \theta \leq \frac{a}{a + 1} + \frac{|X_0|}{(a + 1)Np}.
\]

Since

\[
\frac{(p + q)}{pa} - \frac{a}{a + 1} = \left(\frac{pa - r_* + 1}{pa}\right) - 1 + \frac{1}{a + 1} + \frac{1}{pa(a + 1)} \left(\frac{pa - p(a + 1)}{a}\right) = \frac{1}{a(a + 1)} \left(a - 1 - \frac{1}{a}\right) > 0,
\]

it follows from (3.4) that there exists some finite constant \( N_0 \) depending only on \( a, p \) and \( d_0 = |X_0| + |Y_0| \) such that on his \( N_0 \)-th turn Breaker is forced to claim at least one edge of Type I; explicitly, we can take

\[
N_0 = \left\lceil \frac{d_0(2a + 1)}{p(a^2 - a - 1)} \right\rceil.
\]

We now divide the game into phases. At the beginning of each phase it is Maker’s turn to play, and \( Y_t \) is empty. Throughout a phase, Breaker follows his greedy strategy and claims only edges to Type II vertices; a phase then ends with the first turn in which Breaker claims an edge of Type I. For completeness, we say Phase 0 begins with Maker’s first move and ends at the end of the first turn in which Breaker claims an edge to a Type I vertex.

We have just shown above that Breaker’s greedy strategy ensures every phase is finite. We now claim that for all \( k \geq 1 \), the \( \ell_1 \)-distance \( d_N \) at the start of Phase \( (k + 1) \) is strictly smaller than it was at the start of Phase \( k \). Observe that this will immediately imply that Breaker’s greedy strategy wins the game for Breaker within some finite number of turns, thereby completing the proof of Theorem 1.7.

For convenience, let us shift the game-time so that Phase \( k \) begins at time 0, with thus \( d_0 = |X_0| \) and \( |Y_0| = 0 \), and let us assume the Phase \( k \) ends at the \( N \)-th turn. Let \( \theta Np \) denote the number of edges to Type I vertices claimed by Maker during Phase \( k \). Then we have

\[
d_N = |X_{N(p+q)}| = d_0 + Np\left(-\theta + a(1-\theta)\right) - q',
\]
\[
0 = |Y_{N(p+q)}| = Np\left((a-1)\theta - (1-\theta) - \frac{q}{p}\right) + q',
\]

where \( q' \in [q] \) is the number of edges to Type I vertices Breaker claims in his last turn of the phase. As in (3.4), we can derive a lower bound

\[
\theta = \frac{q'}{aNp} + \frac{p + q}{pa} \geq \frac{q}{ap} + \frac{p + q}{ap} = \frac{1}{a}.
\]
On the other hand, combining (3.6) and (3.7), we obtain
\[
d_N = d_0 + aNp - Np\theta - Nq \leq d_0 + N\left(p(a-1) - \frac{p}{a} - q\right) < d_0 + N\left(p(a-1) - \frac{p}{a} - p(a-1) + \frac{p}{a}\right) = d_0.
\]

Thus \(d_N < d_0\), as claimed, and Breaker’s greedy strategy is a winning one. Appealing to (3.3), we can moreover give an upper bound of
\[
N_0 + N_0(p(a-1) - q) \left\lceil \frac{N_0(p(a-1) - q)(2a + 1)}{p(a^2 - a - 1)} \right\rceil < 108pa^3(d_0)^2.
\]
on the number of turns before Breaker’s victory.

**Remark 3.5.** Our proof of Theorem 1.7 establishes that if \(p(b-2) - \lceil \frac{p}{a} \rceil(b-a) < q\), then Breaker has a winning strategy in a very strong sense: he wins even if Maker is allowed to pre-emptively claim some arbitrary finite number of edges before the start of the \((p, q)\)-percolation game on \(T\). This is somewhat reminiscent of the phenomenon in percolation theory whereby the existence of an infinite connected component is independent of what happens in any finite region (this is a special case of the celebrated Kolmogorov 0-1 law). It is natural to ask whether a similar phenomenon might also occur for percolation games.

**Question 3.6.** Let \(\Lambda\) be a connected unrooted infinite graph. Allow Maker to select a vertex \(v_0\) of \(\Lambda\) to be the root, and then play the \((p, q)\)-percolation game on \(\Lambda\) as normal. Suppose Breaker has a winning strategy. Is it true that Breaker still has a winning strategy even if Maker is allowed to pre-emptively claim some finite number of edges before the game starts?

The proof of Theorem 1.7 essentially boils down to the analysis of a vector game: with each of the edges she picks, Maker changes \((X_t, Y_t)\) by \((-1, a-1)\) or \((b-1, -1)\), while each edges claimed by Breaker \((X_t, Y_t)\) by \((-1, 0)\) or \((0, 1)\). This is a instance of the broader class of energy games and vector addition systems with states (VASS), which are widely studied within computer science, albeit primarily from a complexity perspective (see e.g. [3, 4]). Such games are motivated by the problem of a manufacturer who uses raw materials to produce various items. The manufacturer can buy batches of raw materials, while customers buy items from the manufacturer. Customer demand is fickle, perhaps even adversarial, and the manufacturer’s aim is to maintain sufficient levels of raw materials and cash to run his operation indefinitely. This is clearly reminiscent of Maker’s aim to avoid \((X_t, Y_t) = (0, 0)\) in the percolation game on the tree \(T\) above, and points to a possible application of our Maker-Breaker games and especially of our winning Maker strategies.

## 4 The square integer lattice

Our goal in this section is to prove Theorems 1.3 and 1.4 on percolation games on the square integer lattice \(\mathbb{Z}^2\). Since \(\mathbb{Z}^2\) is a planar graph (with its natural embedding in \(\mathbb{R}^2\)), we can define its planar dual \((\mathbb{Z}^2)^*\), which is the graph with a vertex for each face of \(\mathbb{Z}^2\) and an edge \(e^*\) for each edge \(e\) of \(\mathbb{Z}^2\), where \(e^*\) joins the two vertices of \((\mathbb{Z}^2)^*\) corresponding to the two faces of \(\mathbb{Z}^2\) in whose boundary \(e\) lies; we say that such edges \(e, e^*\) form a dual pair.

A key property of the square integer lattice is that it is self-dual: \((\mathbb{Z}^2)^*\) is isomorphic to \(\mathbb{Z}^2\). It is customary to represent \((\mathbb{Z}^2)^*\) in the plane as a copy of \(\mathbb{Z}^2\) shifted by \((1/2, 1/2)\), so that its
vertex set is $\mathbb{Z}^2 + (1/2, 1/2)$ and the straight line segments corresponding to a dual pair of edges $(e, e^*)$ intersect in their midpoint. If $e$ is a horizontal edge in $\mathbb{Z}^2$, that is $e = \{(x, y), (x + 1, y)\}$ for some $x, y \in \mathbb{Z}$, then we denote both $e$ and its dual $e^*$ by their midpoint, so that $e = (x + 0.5, y)$ and $e^* = (x + 0.5, y)^*$. Similarly, for a vertical edge $e = \{(x, y), (x, y + 1)\}$ for some $x, y \in \mathbb{Z}$, we denote both $e$ and its dual $e^*$ by their midpoint, so that $e = (x, y + 0.5)$ and $e^* = (x, y + 0.5)^*$. This identification of edges/dual edges with their midpoints will allows us to view our Maker–Breaker games as being played on two boards: the two players compete to claim midpoints, with one player claiming them to obtain edges from the board $\mathbb{Z}^2$, while the other does so to obtain edges from the dual board $(\mathbb{Z}^2)^*$.

A key tool used in this section will be a result from [5] on the following auxiliary game.

**Definition 4.1** ($q$-double-response-game). Let $\mathbb{Z} \times P_n$ be the subgraph of $\mathbb{Z}^2$ induced by the vertex set $\{(x, y) : x \in \mathbb{Z}, y \in [n]\}$ and let $(\mathbb{Z} \times P_n)^*$ be the graph of edges dual to the edges in $\mathbb{Z} \times P_n$. The $q$-double-response-game is a positional game played by a horizontal player $\mathcal{H}$ and a vertical player $\mathcal{V}$. The game begins with $\mathcal{V}$ playing first. On each of his turns, $\mathcal{V}$ first picks an integer $r \in [q]$ and then claims $r$ as-yet unclaimed edges in $\mathbb{Z} \times P_n$ for himself; $\mathcal{H}$ then responds by claiming $2r$ as-yet unclaimed edges for herself in response to $\mathcal{V}$’s move.

The vertical player $\mathcal{V}$ wins if he is able to claim a set of edges $D$ whose dual $D^*$ is a top-bottom crossing path in $(\mathbb{Z} \times P_n)^*$ (equivalently, the edges in $D$ are a vertical cut through $\mathbb{Z} \times P_n$). The horizontal player $\mathcal{H}$ wins if she is able to indefinitely prevent $\mathcal{V}$ from building such a dual path.

**Proposition 4.2.** ([5 Theorem 4.1]) If $n \geq q + 1$, then $\mathcal{H}$ has a winning strategy for the $q$-double-response-game on $\mathbb{Z} \times P_n$.

**Remark 4.3.** If $\mathcal{H}$ has a winning strategy for the $q$-double-response game on $\mathbb{Z} \times P_n$, then for any $m \geq 0$ she also has a winning strategy for the $q$-double-response game restricted to the subgraph $P_m \times P_n$ induced by the vertices $(x, y): x \in [m], y \in [n]$. By [5 Lemma 2.1], this implies she has a strategy for building a left-right crossing path through the rectangle $P_m \times P_n$.

Theorem 1.3 follows as an easy consequence of Proposition 4.2.

**Proof of Theorem 1.3** first turn she claims all $p$ edges of the vertical path $P$ from the origin $(0, 0)$ to $(0, p)$. From then on, Breaker can win the game only if he manages to claim a set of edges $D$ such that the corresponding set of dual edges $D^*$ contains a cycle in $(\mathbb{Z}^2)^*$ with $P$ in its interior; in particular, $D \cap \mathbb{Z} \times P_p$ would have to contain a top-bottom dual crossing path of $\mathbb{Z} \times P_p$. After completing her first turn, Maker follows the horizontal player $\mathcal{H}$’s winning strategy for the $q$-double-response-game on $\mathbb{Z} \times P_p$ (a strategy whose existence is given by Proposition 4.2 and our assumption that $p \geq 2q \geq q + 1$). This prevents Breaker from ever creating a top-bottom dual crossing path of $\mathbb{Z} \times P_p$, and, as such, prevents Breaker from ever winning the $(p, q)$-percolation game on $\mathbb{Z}^2$. Maker thus has a winning strategy in this case.

To prove Theorem 1.4, we shall need a result about a variant of a specific instance of the box-game. The box-game was introduced by Chvátal and Erdős in [4], and solved in full generality by Hamidoune and Las Vergnas [8]. Our variant of the box-game is played as follows.

**Definition 4.4** ($(q, M, N)$-double-response-box-game). In the $(q, M, N)$-double-response-box-game, two players, BoxBreaker and BoxMaker, play in alternating turns on a board consisting of $N$ boxes, each containing $M$ items. BoxBreaker plays first, and on each of his turns he picks an integer $r \in [q]$ and removes $r$ boxes from the board. On each of her turns, BoxMaker claims up to a total of $2r$ items from the boxes remaining on the board (where $r$ was the number of boxes removed by
BoxMaker wins the game if she manages to claim all $M$ items from some box before BoxBreaker removes it.

**Lemma 4.5.** If $N \geq 4(q + 2)^M$, then BoxMaker has a winning strategy for the $(q, M, N)$-double-response-box-game.

**Proof.** BoxMaker’s winning strategy proceed in $M + 1$ phases. For every integer $k \in [0, M]$, her strategy (described below) will guarantee that when Phase $k$ begins the following hold:

(i) it is BoxBreaker’s turn to play

(ii) there are at least $4(q + 2)^{M-k}$ boxes on the board from which BoxMaker has claimed $k$ items.

In particular, the above entails that at the beginning of Phase $M$ there is at least one box on the board from which BoxMaker has claimed all $M$ items, so this is a winning strategy for BoxMaker.

We now show by induction on $k$ that BoxMaker can guarantee (i) and (ii) for every $k \in \{0, 1, \ldots, M\}$. The assumption on $N$ and the definition of the $(q, M, N)$-double-response-box-game give us that both (i) and (ii) hold at the beginning of Phase 0. Suppose now that we have reached the beginning of Phase $k$ for some $k < M$, and that (i) and (ii) are both satisfied. On each of his turns throughout Phase $k$, BoxBreaker chooses some $r \in [q]$ and removes $r$ boxes from the board; BoxMaker’s response will then be to claim one item each from $2r$ different boxes from which she has previously claimed $k$ items, unless there are strictly fewer than $2r$ such boxes left, in which case she claims $2r$ arbitrary items and declares Phase $k$ over.

Let $A$ and $B$ denote the number of boxes remaining on the board from which BoxMaker has claimed $k + 1$ and $k$ items respectively. Set $S = 2A + B$, and consider how $S$ changes throughout phase $k$. Initially $A = 0$ while $B \geq 4(q + 2)^{M-k}$, and so $S \geq 4(q + 2)^{M-k}$. On each of his turns, BoxBreaker removes $r$ boxes for some $r \in [q]$, thereby reducing $S$ by at most $2r$. If $B \geq 2r$, then BoxMaker’s response ensures that $S$ increases back by $2r$. If on the other hand $B < 2r$, then BoxMaker first chooses her items arbitrarily, in which case $S$ potentially does not change, and then declares Phase $k$ to be over. This implies that at the end of BoxMaker’s last turn of Phase $k$ we have $S \geq 4(q + 2)^{M-k} - 2q$ and $B < 2q$. In particular we have

$$A = \frac{S - B}{2} \geq \frac{4(q + 2)^{M-k} - 4q}{2} \geq 4(q + 2)^{M-k-1},$$

and it is BoxBreaker’s turn to play, so that (i) and (ii) are both satisfied at the beginning of Phase $k + 1$, as required.

**Proof of Theorem 1.4.** Before we get into the technical details, let us give an outline of Breaker’s winning strategy. We may assume without loss of generality that $q = 2p$ — any edges Breaker gets to claim above $2p$ can be played arbitrarily without prejudice to Breaker. Recall that the $\ell_\infty$-norm of a point $u = (u_1, u_2)$ in $\mathbb{Z}^2$ is $\|u\|_\infty = \max(|u_1|, |u_2|)$.

Let us denote the origin of $\mathbb{Z}^2$ by $0$. Breaker will restrict his attention to the finite subset of the square lattice induced by vertices at $\ell_\infty$ distance at most $N(p + 1)$ from the $0$, for some suitably chosen $N$. He partitions this subset into square annuli, where the $k$-th annulus $A(k)$ corresponds to vertices with $\ell_\infty$ norm between $k(p + 1)$ and $(k + 1)(p + 1)$. Note that for such annuli, any path in $\mathbb{Z}^2$ from the inner square to the outer one must contain at least $p + 1$ edges.

Each of these annuli can be further subdivided into a set of four rectangular strips $R(k)$ and a set of four ‘corners’ $C(k)$. Breaker will play double-response games with Maker on each of the $R(k)$ and on the union of the $C(k)$: for every edge that Maker claims from $R(k)$ in her turn, Breaker will
claim two suitably chosen edges from \( R(k) \) in his response; and for every edge that Maker claims from a corner in her turn, Breaker will claim two edges from suitably chosen corners in his response.

By Proposition 4.2 Breaker has a strategy in (a variant of) the \( p \)-double-response game on \( R(k) \) for preventing Maker from claiming the edges of a path crossing a strip in \( R(k) \) from the inner square of the annulus \( A(k) \) to its outer square. Furthermore, our choice of \( N \) will ensure that by Lemma 4.5 Breaker has a strategy for (a variant of) the \( p \)-double-response box-game played on the collection of corners \( C(0), C(1), \ldots C(N−1) \) that guarantees he is able to claim all the edges of some \( C(k₀) \). (The crucial property of the \( C(k) \) for this argument to work is that, unlike the \( R(k) \), the sizes of the \( C(k) \) is constant.) By following these double-response strategies, Breaker will be able to ensure the component of \( 0 \) is eventually wholly contained inside some finite set \( \bigcup_{k≤k₀} A(k₀) \), \( k₀ ≤ N−1 \), and hence that he wins the percolation game.

We now fill in the details. Set \( N = 4(p + 2)^{8p} \). As discussed above, for \( k ∈ \{0, 1, \ldots N−1\} \), let \( A(k) \) denote the annulus \( A(k) = \{ u : \|u\|₁ ≤ \{k(p + 1), (k + 1)(p + 1)\}\} \). Inside \( A(k) \), let \( R_i(k) \) denote the subgraph of \( \mathbb{Z}^2 \) induced by the vertices \( \{(x, y) : |x| ≤ k(p + 1), k(p + 1) ≤ y ≤ (k + 1)(p + 1)\} \).

We then let \( R_1(k) \) denote the subgraph of \( R_i(k) \) obtained by removing any edge of \( R_i(k) \) that lies inside \( A(k + 1) \) or \( A(k − 1) \). We further let \( R_2(k), R_3(k) \) and \( R_4(k) \) be the subgraphs obtained by rotating \( R_1(k) \) around the origin by angles of \( π/2, π \) and \( 3π/2 \) respectively. Thus \( R_1(k), R_2(k), R_3(k) \) and \( R_4(k) \) are respectively part of the topmost, leftmost, bottommost and rightmost rectangular strips in the square annulus \( A(k) \). Let \( R(k) \) denote the union of the \( R_i(k) \), \( i ∈ \{4\} \). Denote by \( L(k) \) the collection of all edges which are from a vertex in \( R(k) \) to a vertex in \( A(k) \) \( \setminus V(R(k)) \). We then let \( L(k) \) denote the subset of \( L(k) \) obtained by removing any edge that lies in \( A(k + 1) \) or \( A(k − 1) \). The set of dual edges corresponding to \( L(k) \) consist of four L-shapes, each containing \( 2p \) edges. Let us write \( \overline{R}_i^⋆(k) \) (respectively \( \overline{L}^⋆(k) \)) for the set of edges dual to \( R_i^⋆(k) \) (respectively \( L(k) \)). These four strips \( \overline{R}_i^⋆(k), i ∈ \{4\} \), may be viewed as (rotated, translated) subgraphs of \( \mathbb{Z} × P_{p + 1} \). See Figure 1 for an example of these sets.

We can now formally describe Breaker’s winning strategy. Set \( L = \bigcup_{k≤N−1} L(k) \). In any given turn, Maker claims \( p_L \) edges from \( L \) and \( p_{i,k} \) edges from \( R_i(k) \) for each \( i ∈ \{4\} \) and each \( k ∈ \{0, 1, \ldots N−1\} \), where \( p_L + \sum_{i,k} p_{i,k} ≤ p \). Breaker treats \( L \) as an instance of the box game, where the boxes are the \( L(k) \), each of which contains exactly \( 8p \) elements. He responds to Maker’s moves on \( L \) by claiming \( 2p_L \) edges from \( L \) according to the winning strategy for BoxMaker in the \( (p, 8p, N) \)-double response box-game given in Lemma 4.5 (which is possible since we chose \( N ≥ 4(p + 2)^{8p} \)). Furthermore, for each \( i, k \), Breaker treats \( R_i^⋆(k) \) as a (rotated, translated and truncated) instance of the \( p \)-double response game on \( \mathbb{Z} × P_{p + 1} \); he responds to Maker’s moves on \( R_i(k) \) by claiming \( 2p_{i,k} \) edges from \( R_i^⋆(k) \) according to the winning strategy for the horizontal player \( \mathcal{H} \) given by Proposition 4.2. This uses up \( 2p_L + \sum_{i,k} 2p_{i,k} ≤ 2p \) of the edges Breaker is allowed to claim. If he has any edges left over (for example if Maker played fewer than \( p \) edges inside \( \bigcup_{k≤N−1} (A(k) \setminus \mathcal{A}(k)) \) in her preceding turn), then Breaker claims some arbitrary edges inside \( S_p := \bigcup_{k≤N−1} A(k) \). This ensures that between successive turns for Breaker, at least \( 2p \) edges of \( S_p \) are claimed by the players. Thus within \( T ≤ 4(2N + 1)^2/2p \) turns, all edges inside \( S_p \) have been claimed by one of the players.

Since Breaker’s strategy on \( L \) is a winning one we have that by turn \( T \) of the game, there exists some \( k₀ ∈ \{0, 1, \ldots N−1\} \) such that Breaker has claimed all of the edges from \( L(k₀) \). Since Breaker’s strategy on each of the \( R_i^⋆(k₀) \) is a winning one, by turn \( T \) Maker has failed for every \( i ∈ \{4\} \) to claim the edges of a path in \( R_i(k₀) \) from the inner square of \( A(k₀) \) to its outer square. By a standard result on planar duality (see e.g. [2] Chapter 3, Lemma 1]), this implies Breaker has claimed edges corresponding to dual paths \( π_i, i ∈ \{4\} \), where \( π₁ \) and \( π₂ \) are left-right crossing
Figure 1: This figure shows some of the sets involved in our winning Breaker strategy in the special case \( p = 2 \). The blue vertices above represent the vertices of \( \mathbb{Z}^2 \), while the red vertices represent the dual vertices from \((\mathbb{Z}^2)^* \approx \mathbb{Z}^2 + (0.5, 0.5)\). We see in the figure the boundaries of the annulus \( A(1) \) (in light blue), the edge-set \( L^*(1) \) (in green), and the sets \( R^*_1(1), R^*_2(1), R^*_3(1) \) and \( R^*_4(1) \) (in pink), as well as crossing paths \( \pi_i \) (in red) for each of the \( R^*_i(1) \). The circled vertex is the origin of \( \mathbb{Z}^2 \).

Note that the union of the red dual edges with the green dual edges contains a dual cycle around the origin; claiming the corresponding edges constitutes a win for Breaker in the percolation game.

paths of \( R^*_1(k_0) \) and \( R^*_3(k_0) \) and \( \pi_2 \) and \( \pi_4 \) are top-bottom crossing paths of \( R^*_2(k_0) \) and \( R^*_4(k_0) \) respectively. Now, the union of the \( \pi_i \) together with \( L^*(k_0) \) contains a dual cycle \( \sigma \) inside \( A(k_0) \) surrounding the origin (see Figure 1 for an example in the case \( p = 2, k = 1 \)). By turn \( T \) Breaker has thus claimed edges whose dual is such a cycle \( \sigma \). This implies that by that turn, the origin is contained in a component strictly contained inside the finite square \( \bigcup_{k \leq k_0} A(k) \subseteq S_p \). In particular, Breaker has won the \((p, 2p)\)-percolation game.

Remark 4.6. The proof above shows that in the \((p, 2p)\)-percolation game on \( \mathbb{Z}^2 \), Breaker can ensure the origin is contained in a connected component of order \( p^{16p+O(1)} \) by the end of the game. It is natural to ask whether in fact Breaker can do a lot better: can he ensure, for instance, that the origin is contained in a component of order subexponential in \( p \)?

Question 4.7. Set \( M_p \) to be the least integer such that Breaker has a winning strategy in the \((p, 2p)\)-game on \( \mathbb{Z}^2 \) ensuring the origin is contained in a component of order at most \( M_p \) by the time the game ends. What is the asymptotic behaviour of \( M_p \) as \( p \to \infty \)?

Our analysis in the proof of Theorem 1.4 is certainly wasteful, so we cannot hazard a guess as to what the right answer to this question might be.

5 Concluding remarks and open questions

There are many questions arising from our work. Given our motivation for looking at Maker-Breaker games in the context of percolation, the one we would most like answered is Question 1.8, which asks about a Maker-Breaker analogue of the Harris–Kesten theorem: does there exist a critical bias
for the \((p, q)\)-percolation game on the square integer lattice \(\mathbb{Z}^2\) such that \(b_p p < q\) implies Breaker has a winning strategy while \(b_p p > q\) implies Maker has a winning strategy?

More generally, for fixed \(p\) it is natural to ask for the exact threshold value \(q\) at which the \((p, q)\)-percolation game goes from being a Maker win to a Breaker win under optimal play.

**Question 5.1.** Let \(p \geq 2\) be a fixed integer. What is the least integer \(q\) such that Breaker has a winning strategy for the \((p, q)\)-percolation game on the square integer lattice \(\mathbb{Z}^2\)?

Theorem 1.4 shows this threshold is at most \(2p\), while Theorem 1.3 shows it is at least \(p/2\). One natural way of investigating Question 5.1 would be to settle who has a winning strategy in the \((p, q)\)-percolation game for small values of \(p\) and \(q\). The smallest open case is when \(p = q = 2\). We believe this \((2, 2)\)-case already contains many of the difficulties inherent in the general case.

**Question 5.2.** Which of Maker and Breaker has a winning strategy for the \((2, 2)\)-percolation game on \(\mathbb{Z}^2\)?

In a more theoretical direction, one could ask about the existence of a critical bias in a more general setting (possibly ignoring some finite number of ‘bad’ pairs \((p, q)\)).

**Question 5.3.** Let \(\Lambda\) be a vertex-transitive infinite, locally finite, connected graph. Does there exist a critical bias \(b_\star = b_\star(\Lambda) \geq 0\) for the \((p, q)\)-percolation game on \(\Lambda\), in the sense that for all \(\varepsilon > 0\) there exists \(p_0 \in \mathbb{N}\) such that for all \(p \geq p_0\), \((b_\star + \varepsilon)p < q\) implies Breaker has a winning strategy while \((b_\star - \varepsilon)p > q\) implies Maker has a winning strategy?

To answer such a question, it would be good to understand what, if anything, the existence of a winning strategy for one of the players in the \((p, q)\)-percolation game says about the \((mp, mq)\)-percolation game, and vice-versa, where \(m \in \mathbb{N}\). We observe that if any relationship between the two games does exist, it is not entirely trivial—note for instance that, as we showed, Maker wins the \((1, 1)\)-percolation game on \(\mathbb{Z}^2\) even if she forfeits her first turn, while clearly Breaker will win the \((4, 4)\)-percolation game on the same graph if he is allowed to play first.

In this paper, we focussed on Bethe lattices and integer lattices, though we gave a definition of our percolation game which is valid in a much more general context. It is natural to ask what happens on other lattices, such as the triangular lattice. We also focussed on games where the players take turns claiming edges in the graph (which corresponds to bond percolation in percolation theory); it would be interesting to study the variant where they take turns claiming vertices in the graph instead (which would be the analogue of site percolation). Observe this is a more general class of games, since claiming edges in a graph \(\Lambda\) is equivalent to claiming vertices in the line graph of \(\Lambda\). Of particular interest would be percolation game analogues of the celebrated game of Hex, which corresponds to site percolation on the triangular lattice.

Another open problem is to extend our results on trees. We determined which of Maker and Breaker has a winning strategy for the \((p, q)\)-percolation game on regular and bi-regular trees. Is there a general criterion for determining who has a winning strategy on arbitrary infinite trees? One difficulty here may be that global invariants such as the branching number may ignore some local bottlenecks, which don’t affect the asymptotic growth of the tree but do allow Breaker to overwhelm Maker and cut off the root. One way to address this might be to play the percolation game on non-rooted trees and allow Maker in her first turn to choose the location of the root she has to protect.

Finally, given our motivation for looking into percolation games, it would be natural to ask what happens when they are played on random boards: given a rooted, connected infinite graph...
Λ, let Λθ be a θ-random subgraph obtained by including each edge of Λ in Λθ with probability θ, independently of all other edges. For suitable values of θ ∈ [0, 1], the subgraph Λθ will almost surely (a.s.) contain an infinite connected component. One can then play a percolation game on Λθ by allowing Maker in her first turn to choose the location of a root, and then playing the (p, q)-percolation game as normal. In particular, given Maker wins the (1, 1)-percolation on Z², one could ask how much one can ‘thin’ the board (i.e. how small a θ we can take) while ensuring she a.s. retains her advantage in the (1, 1)-game over Maker.

**Question 5.4.** Set

\[ \theta_* = \inf \left\{ \theta \in [0, \frac{1}{2}] : \text{Maker a.s. has a winning strategy for the (1, 1)-percolation game on } (\mathbb{Z}^2)_\theta \right\}. \]

What is the value of \( \theta_* \)?

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