“Exotic” black holes with torsion

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In the context of three-dimensional gravity with torsion, the concepts of standard and “exotic” Bañados–Teitelboim–Zanelli black holes are generalized by going over to black holes with torsion. This approach provides a unified insight into thermodynamics of black holes, with or without torsion.

I. INTRODUCTION

Recently, Townsend and Zhang examined thermodynamics of “exotic” Bañados–Teitelboim-Zanelli (BTZ) black holes—the solutions of a class of 3D gravity models for which the metric coincides with the standard BTZ metric, but the conserved charges, energy and angular momentum, are, in a sense, reversed (as explained in chapter III). Their analysis was focused on a simple model of this type, described by the parity-odd gravitational Lagrangian that Witten named “exotic”. In this framework, the authors discussed basic thermodynamic properties of the “exotic” BTZ black holes (that is, the standard BTZ black holes viewed as solutions of the “exotic” model).

In Ref. 1, general relativity with a cosmological constant (GRΛ) and the “exotic” gravity are treated as independent models, based on Riemannian geometry of spacetime. In the present note, we show that these two models can be naturally interpreted as different sectors of a single model—the Mielke–Baekler (MB) model of 3D gravity with torsion 4. This approach offers a unified view at GRΛ and the “exotic” gravity, revealing a new, “interpolating” role of torsion with respect to Riemannian theories of gravity. In this, more general setting, standard BTZ black hole solutions can be generalized to BTZ-like black holes with torsion 5–7, see also 8. At the same time, their thermodynamic properties 3,10 allow us not only to simplify the considerations presented in 1, but also to generalize them.

II. 3D GRAVITY WITH TORSION

In the framework of Poincaré gauge theory 11,12,13, where the triad e\textsuperscript{i} and the Lorentz connection ω\textsuperscript{i} are basic dynamical variables (1-forms), and their field strengths are the torsion T\textsuperscript{i} = de\textsuperscript{i} + ε\textsuperscript{ijk}e\textsuperscript{j}ek and the curvature R\textsuperscript{i} = dω\textsuperscript{i} + 1/2ε\textsuperscript{ijk}ω\textsuperscript{j}ω\textsuperscript{k} (2-forms), the MB model is defined by the Lagrangian (3-form)

\[ L_{MB} = 2ae^iR_i - \frac{A}{3}ε_{ijk}e^ie^je^k + α_3L_{CS}(ω) + α_4e^iT_i. \]  

Here, \( L_{CS}(ω) := ω^iω_i + \frac{1}{3}ε_{ijk}ω^iω^jω^k \) is the Chern–Simons Lagrangian for ω\textsuperscript{i}, the exterior product is omitted for simplicity, and (\( α_3, α_4 \)) are free parameters. In the non-degenerate case \( α_3α_4 - α^2 \neq 0 \), the variation of \( L_{MB} \) with respect to \( e^i \) and \( ω^i \) leads to the gravitational field equations in vacuum:

\[ 2T^i = pe^i_jk e^j ∧ e^k, \quad 2R^i = qε^i_jk e^j ∧ e^k, \]  

where

\[ p = \frac{α_3A + α_4a}{α_3α_4 - α^2}, \quad q = -\frac{(α_4)^2 + aA}{α_3α_4 - α^2}. \]  

Using Eqs. (2) and the formula \( ω^i = ˜ω^i + K^i \), where \( ˜ω^i \) is the Riemannian piece of the curvature, \( R = ˜R( ˜ω) \), reads:

\[ 2 ˜R^i = ˜A_{eff} ε^i_jkek, \quad ˜A_{eff} := q - \frac{1}{4}p^2, \]  

where \( ˜A_{eff} \) is the effective cosmological constant.

In the anti-de Sitter (AdS) sector with \( ˜A_{eff} = -1/ℓ^2 \), the MB model admits a new type of black hole solutions, known as the BTZ-like black holes with torsion 5,6,7, see also 8. These solutions can be determined in two steps. First, by combining the form the BTZ black hole metric,

\[ ds^2 = N^2dt^2 - N^{-2}dr^2 - r^2( dφ + N_φ dt)^2, \]

\[ N^2 = \left( -8Gm + \frac{r^2}{ℓ^2} + \frac{16Gj^2}{r^2} \right) \quad N_φ = \frac{4Gj}{r^2}, \]

with the relation \( ds^2 = η_{ij}e^i.e^j \), one concludes that the triad field can be chosen in the simple, diagonal form:

\[ e^0 = N dt, \quad e^1 = N^{-1} dr, \quad e^2 = r \left( dφ + N_φ dt \right). \]  

Then, the connection is determined by the first field equation in 2:

\[ ω^i = ˜ω^i + \frac{p}{2}e^i. \]  

The pair \((e^i, ω^i)\) determined in this way represents the BTZ-like black hole with torsion 5,6,7. The thermodynamic aspects of the new black holes are given as follows.

Energy and angular momentum of the black hole with torsion, defined as the on-shell values of the asymptotic...
generators for time translations and spatial rotations, have the following form [5,14]:

\[
E = 16\pi G \left[ \left( a + \frac{\alpha_3 p}{2} \right) m - \frac{\alpha_3}{\ell^2} j \right],
\]

\[
J = 16\pi G \left[ \left( a + \frac{\alpha_3 p}{2} \right) j - \alpha_3 m \right].
\]

In contrast to GR, where \( E = m \) and \( J = j \), the presence of the Chern–Simons term \( (\alpha_3 \neq 0) \) modifies \( E \) and \( J \) into linear combinations of \( m \) and \( j \).

After choosing the AdS asymptotic conditions, the Poisson bracket algebra of the asymptotic symmetry is given by two independent Virasoro algebras with different central charges [5,14]:

\[
e^\mp = 24\pi \left[ \left( a + \frac{\alpha_3 p}{2} \right) \ell \mp \alpha_3 \right].
\]

The partition function of the MB model, calculated in the semiclassical approximation around the black hole with torsion, yields the following expression for the black hole entropy [3]:

\[
S = 8\pi^2 \left[ \left( a + \frac{\alpha_3 p}{2} \right) r_+ - \alpha_3 \frac{r_-}{\ell} \right],
\]

where \( r_\pm \) are the outer and inner horizons of the black hole, defined as the zeros of \( N^2 \). The gravitational entropy [5] coincides with the corresponding statistical entropy [8], obtained by combining Cardy’s formula with the central charges [7]. The existence of torsion is shown to be in complete agreement with the first law of black hole thermodynamics.

### III. A SPECIAL CASE: THE RESULTS OF TOWNSEND AND ZHANG

After clarifying basic thermodynamic aspects of black holes with torsion, the two types of black holes discussed in [1] can be given a unified treatment by considering the related limiting cases of the MB model.

For \( \alpha_3 = \alpha_4 = 0 \) and \( 16\pi G a = 1 \), the MB model reduces to GR, the spacetime geometry is Riemannian \( (p = 0) \), and formulas (6), (7) and (8) produce the standard expressions for the conserved charges, central charges and entropy:

\[
E = m, \quad J = j, \quad c^\mp = \frac{3\ell}{2G}, \quad S = \frac{2\pi r_+}{4G}.
\]

Similarly, for \( a = \Lambda = 0 \), the MB model reduces to Witten’s “exotic” gravity with Riemannian geometry of spacetime. By choosing \( 16\pi G a_3 = -\ell \), one arrives at the “exotic” conserved charges, central charges and entropy,

\[
E = \frac{2}{\ell}, \quad J = \ell m, \quad c^\mp = \pm \frac{3\ell}{2G}, \quad S = \frac{2\pi r_+}{4G},
\]

which coincide with those in [3]. Since \( \Lambda_{\text{eff}} = -1/\ell^2 \) implies \( 16\pi G a_4 = -1/\ell \), the corresponding “exotic” Lagrangian is also the same as in [3].

These considerations, based on our earlier studies of black holes with torsion, provide a simple way to understand somewhat enigmatic relation between the standard and “exotic” black hole thermodynamics.

### IV. GENERALIZATION: STANDARD AND “EXOTIC” BLACK HOLES WITH TORSION

In the previous section, the concepts of standard and “exotic” black holes are used in the context of simple gravitational models with Riemannian geometry of spacetime. Here, we wish to generalize these concepts by going over to black holes with torsion.

The form of the general results (6), (7) and (8) suggests to introduce standard black holes with torsion by imposing the following requirements:

\[
\alpha_3 = 0, \quad 16\pi G a = 1.
\]

In this case, the general formulas reduce to the standard form (9), and the corresponding 2-parameter Lagrangian is given by:

\[
L_S = \frac{1}{8\pi G} e^i R_i - \frac{A}{3} \varepsilon_{ijk} e^i e^j e^k + \alpha_4 e^i T_i.
\]

The AdS condition,

\[
\Lambda_{\text{eff}} = \frac{3}{4} \left( \frac{\alpha_4}{a} \right)^2 + \frac{A}{a} - \frac{1}{\ell^2},
\]

implies \( \Lambda < 0 \).

Similar considerations lead to the following definition of “exotic” black holes with torsion:

\[
a + \frac{\alpha_3 p}{2} = 0, \quad 16\pi G a_3 = -\ell,
\]

which implies that the conserved charges, central charges and entropy take the “exotic” form (10). The corresponding 2-parameter Lagrangian can be written in the form

\[
L_E = \frac{1}{16\pi G} \left[ 2\beta e^i R_i + \frac{\beta (\beta^2 + 3)}{3\ell^2} \varepsilon_{ijk} e^i e^j e^k - \ell L_{CS} - \frac{\beta^2 + 1}{\ell} e^i T_i \right],
\]

where \( \beta := 16\pi G a \) and \( \ell \) are free parameters.

In the limit \( p = 0 \), \( L_S \) and \( L_E \) describe torsionless theories discussed by Townsend and Zhang [1]; thermodynamic aspects of the corresponding black holes are given in Eqs. (9) and (10). All the other limits define the standard and “exotic” gravities with torsion. In particular, for the choice \( q = 0 \) (that is, by taking \( (\alpha_4)^2 + A/16\pi G = 0 \) in \( L_S \) and \( \beta = 1 \) in \( L_E \)), the geometry of these models becomes teleparallel \( (R^i = 0) \).

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