POLYNOMIAL BEHAVIOR OF THE HONDA FORMAL GROUP LAW

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Abstract. This note provides the calculation of the formal group law \( F(x, y) \) in modulo \( p \) Morava \( K \)-theory at prime \( p \) and \( s > 1 \) as an element in \( K(s)^*[[x]][[y]] \) and some applications to relevant examples.

1. Introduction

Let \( K(s)^*(-) \), be the \( s \)-th Morava \( K \)-theory at prime \( p \). The coefficient ring \( K(s)^*(pt) \) is the Laurent polynomial ring in one variable \( \mathbb{F}_p[v_s, v_s^{-1}] \), where \( \mathbb{F}_p \) is the field of \( p \) elements and \( \deg(v_s) = -2(p^s - 1) \).

Let \( F(x, y) \) be the formal group law in \( K(s)^*(-) \) theory. The purpose of this note is to prove that if \( s > 1 \), the formal group law \( F(x, y) \) is a polynomial modulo \( y^q \) (or equivalently modulo \( x^q \)) for any \( i \geq 1 \), see Theorem 2.1. This fact was never mentioned before in the literature even though the proof is quite simple. We also want to have a method for explicit calculation. The idea is to apply the Ravenel formula involving Witt’s symmetric polynomials. The proof does not work for \( s = 1 \). The particular case (2.3) of Theorem 2.1 was applied in several papers by the author, also [10, 11, 12], and [6]. Some other motivation is given in Section 3.

2. The statement

Recall the recursive formula from Ravenel’s green book, (see [8], 4.3.8) for the formal group law. In \( K(s)^*(-) \) theory it reads (we set \( v_s = 1 \) and \( q = p^s - 1 \) )

\[
F(x, y) = F(x + y, w_1(x, y)^q, w_2(x, y)^{q^2}, w_3(x, y)^{q^3}, \cdots)
\]

where \( F(x, y, z, \cdots) = x \oplus_F y \oplus_F z \oplus_F \cdots \) is the iterated \( \oplus_F y = F(x, y) \) and \( w_j \) are \( \mod(p) \) Witt’s integral symmetric homogeneous polynomials of degree \( p^j \):

\[
x^{p^j} + y^{p^j} = \sum_j p^j w_j(x, y)^{p^j - j}.
\]

In particular

\[
w_0 = x + y,
\]

\[
w_1 = - \sum_{0 < j < p} \left( \frac{p}{j} \right) x^j y^{p-j}.
\]

We will need that \( \deg(w_j) = p^j \) and that \( w_j(x, 0) = w_j(0, y) = 0 \), for \( j > 0 \).

Clearly we have

\[
F(x, y) = x + y \mod y^q.
\]

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One has for $s > 1$ (see [1])

$$F(x, y) = x + y + w_1(x, y)^q \text{ modulo } y^{q^2}.$$  

We now want to prove that for $s > 1$, $F(x, y)$ is again a polynomial modulo $y^{q^n}$ for any $n$. The idea is to apply (2.1).

**Theorem 2.1.** One has $F(x, y) \in K(s)^*[[x]][[y]]$ for the formal group law $F(x, y)$ in mod $p$ Morava $K(s)^*(-)$ theory at $p$ and $s > 1$.

A method for calculation of $F(x, y)$ modulo $y^{q^n}$ is given by the Ravenel formula (2.1) and induction on $n$.

**Proof.** By induction hypothesis, we have that $F(x, y)$ modulo $y^{q^k}$, $k \leq n$ is a polynomial, say $P_k(x, y)$. By (2.2) and (2.3) we have

$$P_1(x, y) = x + y, \quad P_2(x, y) = x + y + w_1(x, y)^q.$$  

Induction step: Let us work modulo $y^{q^{n+1}}$. Then (2.1) implies

$$F(x, y) \equiv F(x + y, w_1(x, y)^q, w_2(x, y)^{q^2}, \cdots, w_n(x, y)^{q^n}).$$  

By induction hypothesis we have

$$F(x, y) \equiv z_1 + w_n^{q^n}, \quad \text{where} \quad z_1 = F(x + y, w_1^q, \cdots, w_{n-1}^{q^{n-1}});$$

$$z_1 \equiv P_2(z_2, w_{n-1}^{q^{n-1}}), \quad z_2 = F(x + y, w_1^q, \cdots, w_{n-2}^{q^{n-2}});$$

$$z_2 \equiv P_3(z_3, w_{n-2}^{q^{n-2}}), \quad z_3 = F(x + y, w_1^q, \cdots, w_{n-3}^{q^{n-3}});$$

$$\cdots$$

$$z_{n-2} \equiv P_{n-1}(z_{n-1}, w_2^{q^2}), \quad z_{n-1} = F(x + y, w_1^q);$$

$$z_{n-1} \equiv P_n(x + y, w_1^q).$$

Accordingly $F(x, y)$ is again a polynomial modulo $y^{q^{n+1}}$ for any natural $n$. Therefore one can collect the coefficients at $y^j$, $j < q^{n+1}$ for any $n$ and write

$$F(x, y) = \sum A_i(x)y^i \in K(s)^*[[x]][[y]].$$

□

The proof above gives more, namely one can evaluate the degree of the polynomial $A_i(x)$.

**Proposition 2.2.** In $F(x, y) = \sum \alpha_{ij}x^iy^j$ we have $\alpha_{ij} = 0$ for $i > (pq)^n$ whenever $j < q^n$.

**Proof.** Base case is obvious. Induction step: By Theorem (2.1) we have modulo $y^{q^{n+1}}$

$$F(x, y) \equiv P_n(x + y, z) + w_n(x, y)^{q^n}, \quad z = F(w_1(x, y)^q, \cdots, w_{n-1}(x, y)^{q^{n-1}}).$$

The term $w_n^{q^n}$ is of degree $(pq)^n$ hence is irrelevant.

Let $\beta(z^j + (x+y)^j z^j)$ be any term of the polynomial $P_n(x + y, z)$. By induction hypothesis we have

$$i \leq (pq)^n \text{ whenever } j < q^n.$$  

Then $z^j$ is a polynomial in $w_1^q, \cdots, w_{n-1}^{q^{n-1}}$. Therefore it has the terms

$$(w_1^q)^{j_1} \cdots (w_{n-1}^{q^{n-1}})^{j_{n-1}} = (j_1 < q^n, \cdots, j_{n-1} < q^n),$$

as we work modulo $y^{q^{n+1}}$.}

Therefore $z^j$, as a polynomial in $x$ and $y$, has the terms of total degree

$$pqj_1 + (pq)^2j_2 + \cdots + (pq)^n-1j_{n-1} < pq^{n+1} + p^2q^{n+1} + \cdots + p^{n-1}q^{n+1}.$$ 

Thus for any term of $P_n(x+y,z)$

the total degree $< (pq)^n + q^{n+1} \sum_{1 \leq i \leq n-1} p^i < q^{n+1} \sum_{1 \leq l \leq n} p^l < (pq)^{n+1}$.

This completes the proof.

\[ \square \]

### 3. Some simple applications

The particular case $n = 2$ of Theorem 2.1 was already applied in several papers.

Consider an extension of $C_{p^k}$ by an elementary abelian $p$-group. That is $G$ fits into an extension

$$1 \to (C_p)^l \to G \to C_{p^k} \to 1.$$

It is known \[13, 17\] that $G$ is good, i.e., $K(s)^*(BG)$ is generated by Chern classes. However the explicit account of the ring structure was never done for $k > 1$.

The examples for the case $k = 1$ was considered in \[2, 3\]. Namely let $\xi$ be a complex $m$-plane bundle over the total space of a cyclic covering $\tau : X \to X/C_p$ of prime index $p$. Let $c$ be the Chern class of $X \times_{C_p} \mathbb{C} \to X/C_p$, the complex line bundle associated to covering $\tau$. In \[1\] we showed that modulo image of the transfer homomorphism the $i$-th Chern class $c_i$ of the transferred bundle $\xi$ can be written as a polynomial $A_i$ in Chern classes $c_p, c_{2p}, \cdots, c_{mp}$ and $c^{p-1}$. Using the polynomials $A_i$ in \[2, 3\], we showed for various examples of finite groups that $K(s)^*(BG)$ is the quotient of a polynomial ring by an ideal for which we listed explicit generators.

We recall that Morava $K$-theory for a cyclic group is the truncated polynomials \[9\]. In particular

$$K(s)^*(BC_{p^k}) = F_p[v_s, v_s^{-1}][u]/u^{p^k}.$$ 

Also

$$K(s)^*(BU(m)) = F_p[v_s, v_s^{-1}][c_1, \ldots, c_m]$$

and because of the Künneth isomorphisms

$$K(s)^*(BU(m) \times BC_{p^k}) = K(s)^*(BU(m)) \otimes K(s)^*(BC_{p^k}).$$

Theorem 2.1 enables to write explicitly the relations derived by formal group law and splitting principle as relations in Chern classes of complex representations.

In particular, let $\theta$ be the line complex bundle over $BG$, associated to covering $\tau : BH \to BG$, $H = (C_p)^l$, $\eta$ is the pullback by projection on the first factor $H \to C_p$ of the canonical bundle over $BC_{p^k}$ and $\pi\eta$ is the transferred $\eta$. Then we have the bundle relation over $BG$

$$\pi\eta \otimes \theta = \pi\eta.$$

The relation \[3.1\] holds because of Frobenius reciprocity of the transfer homomorphism of covering $\tau$ in complex $K$-theory:

$$\pi(\eta \otimes \theta) = \pi(\eta \otimes \pi^*(\theta)) = \pi(\eta \otimes 1) = \pi\eta.$$

This implies the relations

$$c_i(\pi\eta \otimes \theta) = c_i(\pi\eta)$$

in $K(s)^*(BG)$. If we want to write everything in the explicit form, we have to apply the splitting principle to \[3.1\] and write formally $\pi\eta$ as the sum of line bundles $\pi\eta = \eta_1 + \cdots + \eta_{p^k}$. Thus we have

$$\eta_1 + \cdots + \eta_{p^k} = \eta_1 \otimes \theta + \cdots + \eta_{p^k} \otimes \theta.$$

Using the elementary symmetric polynomials $\sigma_i$, $i = 1, \cdots, p^k$ we can write

$$c_i(\pi\eta) = \sigma_i(c_1(\eta_1), \cdots, c_1(\eta_{p^k})).$$

In fact we have the following equations

\[3.2\] $\sigma_i(c_1(\eta_1), \cdots, c_1(\eta_{p^k})) = \sigma_i(F(c_1(\eta_1), c_1(\theta)), \cdots, F(c_1(\eta_{p^k}), c_1(\theta))).$
To rewrite (3.2) explicitly, we apply Theorem 2.1 for each term \( F(c_1(\eta_j), c_1(\theta)) \) and write it as a polynomial in \( c_1(\eta_j) \) and \( u = c_1(\theta) \) as \( \theta^p = 1 \) implies \( u^{p^k} = 0 \). This is because \( c_1(\theta^p) = [p^k](c_1(\theta)) \) and \([p](x) = x^p\) for the Honda formal group law.

Finally we turn to the Chern classes. In this way one can try to compute \( K(s)^*(BG) \) as a quotient of a polynomial ring (as \( G \) is finite) by relations ideal. For this we have to establish two facts: the classes we define generate, and the list of relations is complete. To check the latter is easier if the relations are given as explicit polynomials.

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