On constant solutions of SU(2) Yang–Mills–Dirac equations

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For the first time, a complete classification of all constant solutions of the Yang–Mills–Dirac equations with SU(2) gauge symmetry in Minkowski space $\mathbb{R}^{1,3}$ is given. The explicit form of all solutions is presented. We use the method of hyperbolic singular value decomposition of real and complex matrices and the two-sheeted covering of the group $SO(3)$ by the group SU(2). In the degenerate case of zero potential, we use the pseudo-unitary symmetry of the Dirac equation. Nonconstant solutions can be considered in the form of series of perturbation theory using constant solutions as a zeroth approximation; the equations for the first approximation in the expansion are written.

I. INTRODUCTION

At present, the laws of particle physics are described by gauge theories. The Yang–Mills theory with the gauge non-Abelian Lie group $U(1) \times SU(2)$ describes electroweak interactions, with the gauge non-Abelian Lie group SU(3) it describes strong interactions. Exact solutions of the classical Yang–Mills equations are important for the development of gauge theory, in particular, for describing the vacuum structure of the theory and a more complete understanding of the gauge theory. Also the possible way is to quantize the gauge theory around a classical solution of Yang–Mills equations. For other reasons to consider the classical Yang–Mills theory and its solutions, see the review\textsuperscript{9}. Note that secondary quantization issues are not considered in this paper. For the quantum meaning of classical field theory, see the paper\textsuperscript{10}.

The complexity of studying the Yang–Mills equations is explained by the nonlinearity of these equations, they are cubic with respect to the potential $A$. In the mode of a small coupling constant, these equations are solved approximately in the form of perturbation theory series. There are many works on particular classes of solutions of the system of the Yang–Mills equations with zero current (note the classical paper\textsuperscript{11} and review\textsuperscript{12} and others). Constant solutions of the Yang–Mills equations and their significance for describing the physical vacuum are discussed in\textsuperscript{13}--\textsuperscript{18}. We discuss some particular classes of solutions of the Yang–Mills equations and Yang–Mills–Proca equations in\textsuperscript{14}--\textsuperscript{22}.

The Dirac equation in an external field is linear with respect to the spinor $\Psi$. There are well-known solutions of this equation, for example, the plane wave solutions. The Dirac equation with respect to the variables $(\Psi, A)$ is nonlinear because it contains the products of the spinor $\Psi$ and the components of the potential $A$. The right-hand side of the Yang–Mills–Dirac equations contains the current, which is also nonlinear (quadratic) with respect to the spinor $\Psi$. The system of the Yang–Mills–Dirac equations is a highly nontrivial nonlinear system of partial differential equations. This system is too complex to obtain exact solutions. Some particular solutions of this system are known; note the paper\textsuperscript{23} and the review\textsuperscript{24}.

In this paper, we present a compete classification and explicit form of all constant solutions of the Yang–Mills–Dirac equations with SU(2) gauge symmetry in Minkowski space $\mathbb{R}^{1,3}$. Note that the constant solutions of the Yang–Mills–Dirac equations are essentially nonlinear solutions and, from this point of view, are especially important for applications. Theorems\textsuperscript{2} and \textsuperscript{3} are new. Nonconstant solutions can be considered in the form of series of perturbation theory using all constant solutions as a zeroth approximation. For the first approximation, the problem reduces to solving the systems of linear partial differential equations with constant coefficients, which can be studied using different methods of the theory of linear partial differential equations and numerical analysis. For the second and subsequent approximations, the problem reduces to solving the systems of linear partial differential equations with variable coefficients. These facts can be used in the future for local classification of all (nonconstant) solutions of the classical Yang–Mills–Dirac equations.

In this paper, we use the method of hyperbolic singular value decomposition (HSVD) of real and complex matrices. The method of ordinary singular value decomposition (SVD)\textsuperscript{25} is rather standard and is widely used in different applications. The polar decomposition is related to the SVD and is used in the Yang–Mills context in\textsuperscript{26}. The method of hyperbolic singular value decomposition was proposed in 1999\textsuperscript{27} and is used in signal and image processing, engineering, and computer science. The new version of the HSVD without the use of hypersurface matrices is presented in the paper\textsuperscript{29}. This new version of HSVD naturally includes the ordinary SVD. We use the HSVD in this paper two times (for the real matrix $A \in \mathbb{R}^{4 \times 3}$ of potential of the Yang–Mills field and for the complex matrix $\Psi \in \mathbb{C}^{4 \times 2}$) to obtain all constant solutions of the Yang–Mills–Dirac equations.

In Section\textsuperscript{30} we discuss the Yang–Mills–Dirac equations. In Section\textsuperscript{31} we present classification and explicit form of all constant solutions of the Yang–Mills–Dirac equations with SU(2) gauge symmetry in Minkowski space $\mathbb{R}^{1,3}$. We present explicit formulas for the spinor $\Psi$, the potential $A$, the current $J$, and the invariant $F^2$, which is used in the Lagrangian of the Yang–Mills field. In Section\textsuperscript{32} we discuss the possibility of studying nonconstant solutions of the Yang–Mills–Dirac equations in the form of series of perturbation theory using constant solutions as a zeroth approximation; the equations for the first approximation in
the system (4), (5), (6) is gauge invariant with respect to the following transformation

The potential of the Yang–Mills field and the current can be represented in the form

II. THE YANG–MILLS–DIRAC EQUATIONS

Let us consider the Minkowski space \( \mathbb{R}^{1,3} \) with Cartesian coordinates \( x^\mu, \mu = 0, 1, 2, 3 \). We denote partial derivatives by \( \partial_\mu = \frac{\partial}{\partial x^\mu} \). The metric tensor of \( \mathbb{R}^{1,3} \) is given by the diagonal matrix

\[
\eta = (\eta_{\mu \nu}) = (\eta^{\mu \nu}) = \text{diag}(1, -1, -1, -1).
\] (1)

We can raise and lower indices of components of tensor fields with the aid of the metric tensor. For example,

\[
F^{\mu \nu} = \eta^{\mu \alpha} \eta^{\nu \beta} F_{\alpha \beta}.
\] (4)

We can substitute (4) into (5) and obtain

\[
\partial_\mu (\partial_\mu A^\nu - \partial_\nu A_\mu - [A_\mu, A_\nu]) - [A_\mu, \partial_\mu A^\nu - \partial_\nu A_\mu - [A_\mu, A_\nu]] = J^\nu.
\] (7)

The system (4), (5), (6) is gauge invariant with respect to the following transformation

\[
A^\mu \to S^{-1} A^\mu S - S^{-1} \partial_\mu S, \quad F_{\mu \nu} \to S^{-1} F_{\mu \nu} S, \quad \Psi \to \Psi S, \quad J^\mu \to S^{-1} J^\mu S, \quad S : \mathbb{R}^{1,3} \to \text{SU}(2).
\] (12)

The potential of the Yang–Mills field and the current can be represented in the form

\[
A^\mu = A_\mu^a \tau^a, \quad J^\mu = J_\mu^a \tau^a, \quad A_\mu^a, J_\mu^a : \mathbb{R}^{1,3} \to \mathbb{R}.
\] (13)
III. CLASSIFICATION AND explicit form of all constant solutions of the Yang-Mills-Dirac equations

Let us consider the system for constant solutions (the matrices $\Psi \in \mathbb{C}^{4 \times 2}, A \in \mathbb{R}^{4 \times 3}$ do not depend on $x \in \mathbb{R}^{1,3}$) of the system (4), (5), (6):

$$[A_\mu, [A^\mu, A^\nu]] = J^\nu := i \Psi^\dagger \gamma^\nu \gamma^\rho \gamma^\mu \Psi - \frac{1}{2} \text{tr}(\Psi^\dagger \gamma^\rho \gamma^\nu \Psi) J_2,$$

$$i \gamma^\mu \Psi A_\mu - m \Psi = 0, \quad m \geq 0.$$  \hspace{1cm} (14)

For the strength, we get $F^\mu_\nu = -[A^\mu, A^\nu]$. In this paper, we present explicit form of all solutions $(\Psi, A)$ of the system of equations (14), (15). Also we give the explicit formulas for the corresponding current $J = (J^\mu_\nu)$ and the invariant $F^2 = F^\mu_\nu F^\nu_\mu$.

The system of equations (14), (15) is invariant under the transformation

$$A^\mu \rightarrow S^{-1} A^\mu S, \quad \Psi \rightarrow \Psi S, \quad J^\mu \rightarrow S^{-1} J^\mu S, \quad S \in \text{SU}(2),$$  \hspace{1cm} (16)

because of the gauge invariance (12). The element $S \in \text{SU}(2)$ does not depend on $x$ now. Using the two-sheeted cover of the orthogonal group $\text{SO}(3)$ by the spin group $\text{Spin}(3) \cong \text{SU}(2)$

$$S^{-1} \tau^a S = \rho^a_b \tau^b, \quad P = (\rho^a_b) \in \text{SO}(3), \quad \pm S \in \text{SU}(2),$$  \hspace{1cm} (17)

we conclude that the system (14), (15) is invariant under the transformation

$$A \rightarrow AP, \quad \Psi \rightarrow \Psi S, \quad J \rightarrow JP, \quad S \in \text{SU}(2), \quad P \in \text{SO}(3),$$  \hspace{1cm} (18)

where $P$ and $S$ are related as (17).

The systems of equations (4), (5), (6) and (14), (15) are invariant under the Lorentz transformation of coordinates. Namely, let us consider the transformation $x^\mu \rightarrow q^\mu_\nu x^\nu, Q = (q^\mu_\nu) \in \text{O}(1,3)$. The system (14), (15) is invariant under the transformation

$$A \rightarrow QA, \quad J \rightarrow QJ, \quad \Psi \rightarrow T \Psi, \quad Q \in \text{O}(1,3), \quad T \in \text{Pin}(1,3)$$  \hspace{1cm} (19)

where $T$ and $Q$ are related as the two-sheeted cover

$$T^{-1} \gamma^\mu T = q^\mu_\nu \gamma^\nu, \quad Q = (q^\mu_\nu) \in \text{O}(1,3), \quad \pm T \in \text{Pin}(1,3),$$  \hspace{1cm} (20)

and $\gamma^\mu, \mu = 0, 1, 2, 3$, are the Dirac gamma-matrices (11).

Combining transformations (18) and (19), we conclude that the system (14), (15) is invariant under the transformation

$$\Psi \rightarrow T \Psi S, \quad A \rightarrow QAP, \quad J \rightarrow QJP, \quad S \in \text{SU}(2), \quad Q \in \text{O}(1,3), \quad P \in \text{SO}(3), \quad T \in \text{Pin}(1,3)$$  \hspace{1cm} (21)

where $P$ and $S$ are related as (17) and $Q$ and $T$ are related as (20).

The degenerate cases of zero potential $A = 0$ or zero $\Psi = 0$ are considered below. First, let us consider the nondegenerate case $(\Psi, A)$ with $\Psi \neq 0$ and $A \neq 0$, which are more interesting from the physical point of view.

Suppose we have the solution $(\Psi, A)$ of the system (14), (15). We can always choose the matrices $P \in \text{SO}(3)$ and $Q \in \text{O}(1,3)$ such that $QAP$ is in the canonical form (22). We use the following theorem on the hyperbolic singular value decomposition (see (22)).

Theorem 1. For an arbitrary matrix $A \in \mathbb{R}^{n \times N}$, there exist matrices $R \in \text{O}(N)$ and $L \in \text{O}(p, q), p + q = n$, such that

$$L^T A R = \Sigma^A, \quad \Sigma^A = \begin{pmatrix} I_d & 0 & 0 & 0 \\ 0 & X_3 & 0 & 0 \\ 0 & 0 & O & 0 \\ I_d & 0 & 0 & 0 \\ 0 & O & Y_3 & 0 \\ 0 & 0 & 0 & O \end{pmatrix}$$  \hspace{1cm} (22)

where the first block of the matrix $\Sigma^A$ has $p$ rows and the second block has $q$ rows, $X_3$ and $Y_3$ are diagonal matrices of corresponding dimensions $x$ and $y$ with all positive uniquely determined diagonal elements (up to a permutation), $I_d$ is the identity matrix of dimension $d$. We denote zero blocks by $O$.

Moreover, choosing $R$, one can swap columns of the matrix $\Sigma^A$. Choosing $L$, one can swap rows in individual blocks but not across blocks. Thus we can always arrange diagonal elements of the matrices $X_3$ and $Y_3$ in decreasing order.

Here we have

$$d = \text{rank}(A) - \text{rank}(A^T \eta A), \quad x + y = \text{rank}(A^T \eta A),$$

where $x$ is the number of positive eigenvalues of the matrix $A^T \eta A$, $y$ is the number of negative eigenvalues of the matrix $A^T \eta A$. 

$$x \geq d, \quad y \geq N-d.$$
Let us use Theorem 1 in the case \( p = 1, q = 3, n = p + q = 4, N = 3 \). Note that the matrix \( \Sigma^A \) has 4 rows and 3 columns in this case. This means that some blocks of the matrix \( \Sigma^A \) have size zero.

Note that we can always find the matrix \( R = P \in SO(3) \) from the special orthogonal group in the HSVD. If it has the determinant \(-1\), then we can change the sign of all elements of the matrices \( P \) and \( Q \).

The canonical form of the matrix \( A \neq 0 \) is characterized by the three parameters \( d_A, x_A, \) and \( y_A \). The values of these parameters in all the nine possible cases are presented in Table I.

The canonical form of the matrix \( R \) can always be chosen in one of the three types of solutions:

\[ \Sigma(21) \rightarrow \text{the following three types of solutions:} \]

1. the solutions with nonzero mass and nonzero current

\[ \Psi = \begin{pmatrix} G & iG \\ iD & D \end{pmatrix}, \quad A = (A^G) = \begin{pmatrix} 2m & 0 & 0 \\ 0 & 0 & 2m \\ 0 & 0 & 0 \end{pmatrix}, \quad J = (J^G) = \begin{pmatrix} 16m^3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad F^2 = 8m^4I_2, \quad m > 0, \]

where \( L \in \mathbb{C} \) satisfies \(|L| = 4m^2\).

2. the solutions with zero mass and nonzero current

\[ m = 0, \quad \Psi = \begin{pmatrix} G & iG \\ iD & D \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a_1 & 0 \\ 0 & 0 & a_3 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a_1^2 & 0 \\ 0 & a_3^2 & 0 \end{pmatrix}, \quad F^2 = -a_1^4/2I_2, \quad a_1 > 0, \]

Finally, we get Theorem 2.

**Theorem 2.** All solutions \((\Psi, A)\) with \( \Psi \neq 0, A \neq 0 \) of the system of equations (14), (15) can be reduced using transformations (21) to the following three types of solutions:

1. the solutions with nonzero mass and nonzero current

\[ \Psi = \begin{pmatrix} G & iG \\ iD & D \end{pmatrix}, \quad A = (A^G) = \begin{pmatrix} 2m & 0 & 0 \\ 0 & 0 & 2m \\ 0 & 0 & 0 \end{pmatrix}, \quad J = (J^G) = \begin{pmatrix} 16m^3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad F^2 = 8m^4I_2, \quad m > 0, \]

where \( L \in \mathbb{C} \) satisfies \(|L| = 4m^2\).

2. the solutions with zero mass and nonzero current

\[ m = 0, \quad \Psi = \begin{pmatrix} G & iG \\ iD & D \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a_1 & 0 \\ 0 & 0 & a_3 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a_1^2 & 0 \\ 0 & a_3^2 & 0 \end{pmatrix}, \quad F^2 = -a_1^4/2I_2, \quad a_1 > 0, \]
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where

\[ |G|^2 + |K|^2 = |D|^2 + |N|^2, \quad \text{Re}(\bar{K}G + \bar{N}D) = 0, \quad \text{Re}(\bar{N}G + \bar{K}D) = 0, \quad \text{Im}(\bar{N}G - \bar{K}D) = a_1^3, \quad G, D, K, N \in \mathbb{C}. \] (28)

For example, we have solutions with

\[ K = N, \quad G = -D, \quad \text{Im}(\bar{K}G) = \frac{a_1^3}{2}. \] (29)

3. the solutions with zero mass and zero current

\[ m = 0, \quad \Psi = \begin{pmatrix} G & B \\ C & D \\ G & B \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad J = 0, \quad F^2 = 0; \] (30)

where

\[ |G|^2 + |C|^2 = |B|^2 + |D|^2, \quad \bar{G}B + \bar{C}D = 0, \quad G, B, C, D \in \mathbb{C}. \] (31)

For example, we have solutions with

\[ G = D, \quad B = -C. \] (32)

Proof. The detailed proof is presented in Appendix A. □

Now let us consider the degenerate solutions \((\Psi, A)\) with \(A = 0\) or \(\Psi = 0\).

If \(\Psi = 0\), then (15) holds for arbitrary \(A\), the current is zero \(J = 0\), and (14) is reduced to

\[ [A_\mu, [A^\mu, A^\nu]] = 0. \] (33)

Using the results of (23), we get Theorem 3. The parameters \(d_A, x_A, y_A\) for these solutions are presented in Table [II].

**Table II. The degenerate solutions with \(\Psi = 0\)**

| case | \(d_A\) | \(x_A\) | \(y_A\) | \(d_J\) | \(x_J\) | \(y_J\) |
|------|--------|--------|--------|--------|--------|--------|
| 1    | 0      | 0      | 0      | 0      | 0      | 0      |
| 2    | 1      | 0      | 0      | 0      | 0      | 0      |
| 3    | 0      | 1      | 0      | 0      | 0      | 0      |
| 4    | 0      | 0      | 1      | 0      | 0      | 0      |

**Theorem 3.** All solutions \((\Psi, A)\) with \(\Psi = 0\) of the system of equations (14), (15) can be reduced using transformations (27) to the following four types of solutions:

1.

\[ \Psi = 0, \quad A = 0, \quad J = 0, \quad F^2 = 0; \] (34)

2.

\[ \Psi = 0, \quad A = \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad a \in \mathbb{R} \setminus \{0\}, \quad J = 0, \quad F^2 = 0; \] (35)

3.

\[ \Psi = 0, \quad A = \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad a \in \mathbb{R} \setminus \{0\}, \quad J = 0, \quad F^2 = 0; \] (36)
where $S$ is arbitrary.

Using $\Psi = \bar{\Psi}$, we conclude that the system (14), (15) is invariant under the transformation

$$\Psi \rightarrow e^{-i\epsilon \Gamma^\mu W}, \quad W \in SU(2,2).$$

(42)

Combining the transformations (18) and (42), we conclude that the system (14), (15) is invariant under the transformation

$$\Psi \rightarrow e^{-i\epsilon \Gamma^\mu W}, \quad W \in SU(2,2), \quad S \in SU(2), \quad P \in SO(3).$$

(43)

where $S$ and $P$ are related as (17).

We can choose the matrices $W^{-1} \in SU(2,2)$ and $S \in SU(2)$ such that $\Psi$ is in canonical form. We use the following theorem on the hyperbolic singular decomposition for the complex matrix $\Psi \in C^{4 \times 2}$ (see [12]). Note that the matrix $\Sigma^\Psi$ has 4 rows and 2 columns; this means that some blocks of the matrix $\Sigma^\Psi$ have size zero.

**Theorem 4.** Assume $\omega = \text{diag}(1, 1, -1, -1)$. For an arbitrary matrix $\Psi \in C^{4 \times 2}$, there exist $R \in U(2)$ and $L \in U(2,2)$ such that

$$L^\dagger \Psi R = \Sigma^\Psi,$$

(44)

where

$$\Sigma^\Psi = \begin{pmatrix}
I_d & O & O & O \\
O & X & O & O \\
O & O & Y & O \\
O & O & O & O
\end{pmatrix} \quad \in \mathbb{R}^{4 \times 2},$$

(45)
where the first block has 2 rows and the second block has 2 rows, $X_1$ and $Y_1$ are diagonal matrices of corresponding dimensions $x$ and $y$ with all positive uniquely determined diagonal elements (up to a permutation). We denote zero blocks by $O$.

Moreover, choosing $R$, one can swap columns of the matrix $\Sigma^\Psi$. Choosing $L$, one can swap rows in individual blocks but not across blocks. Thus we can always arrange diagonal elements of the matrices $X_1$ and $Y_1$ in decreasing (or ascending) order. Here we have

$$d = \text{rank}(\Psi) - \text{rank}(\Psi^\dagger \omega \Psi), \quad x + y = \text{rank}(\Psi^\dagger \omega \Psi),$$

$x$ is the number of positive eigenvalues of the matrix $\Psi^\dagger \omega \Psi$, and $y$ is the number of negative eigenvalues of the matrix $\Psi^\dagger \omega \Psi$.

We can always find the matrices $W^{-1} \in \text{SU}(2,2)$ and $S \in \text{SU}(2)$ from the special (pseudo-)unitary groups, neglecting the requirement that the elements of the matrix $\Sigma^\Psi$ are positive. If the matrix $S$ has the determinant $-1$, then we can change the sign of one of the columns of the matrix $S$ (which is the eigenvector of the matrix $\Psi^\dagger \omega \Psi$, see the details in (23) and at the same time we should change the sign of the corresponding column of the matrix $\Sigma^\Psi$. If the matrix $W^{-1}$ has the determinant $-1$, then we can change the sign of one of the rows of the matrix $W^{-1}$ (which is the eigenvector or the generalized eigenvector in the case $d \neq 0$ of the matrix $\omega \Psi \Psi^\dagger$, see the details in (23) and at the same time we should change the sign of the corresponding row of the matrix $\Sigma^\Psi$.

The canonical form of $\Psi$ is characterized by three parameters $d\nu$, $x\xi$, and $y\eta$. The possible values of these parameters are presented in Table [III]

| case | $d\nu$ | $x\xi$ | $y\eta$ | $d\nu$ | $x\xi$ | $y\eta$ | $d\nu$ | $x\xi$ | $y\eta$ |
|------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| 1    | -      | -      | -      | -      | -      | -      | 1      | 0      | 0      |
| 2    | 0      | 0      | 0      | 0      | 0      | 0      | 2      | 0      | 0      |
| 3    | -      | -      | -      | -      | -      | -      | 1      | 1      | 0      |
| 4    | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 1      | 0      |
| 5    | -      | -      | -      | -      | -      | -      | 0      | 1      | 0      |
| 6    | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 2      |
| 7    | -      | -      | -      | -      | -      | -      | 0      | 0      | 1      |
| 8    | -      | -      | -      | -      | -      | -      | 0      | 1      | 1      |

The matrix $\Psi$ has the following explicit form in these nine cases:

$$
\begin{bmatrix}
1 & 0 \\
0 & 0 \\
0 & 1 \\
0 & 0
\end{bmatrix}
, \quad
\begin{bmatrix}
1 & 0 \\
0 & \pm 1 \\
0 & 1 \\
0 & 0
\end{bmatrix}
, \quad
\begin{bmatrix}
\psi_1 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 1
\end{bmatrix}
, \quad
\begin{bmatrix}
0 & 1 \\
0 & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix}
, \quad
\begin{bmatrix}
\psi_1 & 0 \\
0 & \psi_2 \\
0 & 0 \\
0 & 0
\end{bmatrix}
, \quad
\begin{bmatrix}
\psi_1 & 0 \\
\psi_2 & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix}
, \quad
\begin{bmatrix}
0 & 0 \\
0 & 0 \\
\psi_1 & 0 \\
0 & \psi_2
\end{bmatrix}
, \quad
\begin{bmatrix}
0 & 0 \\
0 & 0 \\
\psi_1 & 0 \\
0 & \psi_2
\end{bmatrix}
, \quad
\begin{bmatrix}
\psi_1 & 0 \\
\psi_2 & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix}
\in \mathbb{R}^{4 \times 2},
$$

where the hyperbolic singular values $\psi_1$ and $\psi_2$ are positive real numbers except in the following cases (because of the reasons discussed above on the determinants of the matrices $S$ and $W$): in the third and fourth cases $\psi_1$ can be negative; in the fifth, seventh, and ninth cases $\psi_2$ can be negative; also note that in the second case all four possible pairs of signs can be realized.

The matrices $\Psi$ do not depend on the point $x \in \mathbb{R}^{1,3}$, that’s why we can use standard Dirac gamma-matrices for $W^{-1} \Psi^\dagger W$.

Finally, we get Theorem [5].

**Theorem 5.** All solutions $(\Psi, A)$ with $A = 0$ of the system of equations (44), (45) with $m = 0$ can be reduced using transformations (43) to the following three types of solutions:

1. $\Psi = \begin{bmatrix}
\psi & 0 \\
0 & \pm \psi \\
0 & 0 \\
0 & 0
\end{bmatrix}$, $\psi > 0$, $A = 0$, $J = 0$, $F^2 = 0$;
2. \[ \Psi = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \psi & 0 \\ 0 & \pm \psi \end{pmatrix}, \quad \psi > 0, \quad A = 0, \quad J = 0, \quad F^2 = 0; \] (49)

3. \[ \Psi = \begin{pmatrix} 1 & 0 \\ 0 & \pm 1 \\ 1 & 0 \\ 0 & \mp 1 \end{pmatrix}, \quad A = 0, \quad J = 0, \quad F^2 = 0. \] (50)

Note that \( \psi \) is a real positive number and \( \Psi \in \mathbb{R}^{4 \times 2} \) in all three cases.

Proof. The detailed proof is given in Appendix B.

Note that in the general case, we can not choose the matrices \( Q \in O(1, 3), P \in SO(3), W \in SU(2, 2), S \in SU(2), T \in \text{Pin}(1, 4) \) such that the solutions \( A \) and \( \Psi \) of the system (14), (15) are in canonical form at the same time, because the matrices \( S \) and \( P \) are related by (17) and the matrices \( T \) and \( Q \) are related by (20). However we can calculate the invariants \( d_\psi, x_\psi, y_\psi \) for \( \Psi \) in the three cases of Theorem 2.

In Case 1 of Theorem 2 using the explicit form of \( \Psi \) (26), we conclude that

\[ d_\psi = 0, \quad x_\psi = 0, \quad y_\psi = 1. \]

In Case 2 of Theorem 2 we have \( \Psi \neq 0 \) and from (28) it follows that \( \text{rank}(\Psi) \neq 1 \). Thus \( \text{rank}(\Psi) = 2 \). We get \( x_\psi + y_\psi + d_\psi = 2 \). Using the explicit form of \( \Psi \) (27) and the conditions (28), we get

\[ \Psi^\dagger \omega \Psi = \begin{pmatrix} \tilde{G} & -i\tilde{D} & \tilde{K} & -i\tilde{N} \\ -i\tilde{G} & \tilde{D} & i\tilde{K} & \tilde{N} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} G & iG \\ iD & D \\ K & iK \\ iN & N \end{pmatrix} = \begin{pmatrix} 2(|D|^2 - |K|^2) & 2(|N|^2 - |K|^2) \\ -2i(|N|^2 - |K|^2) & 2(|D|^2 - |K|^2) \end{pmatrix}. \]

The determinant of this matrix equals

\[ (|D|^2 - |K|^2)^2 (|N|^2 - |K|^2)^2 = (|D|^2 - |N|^2)^2(|D|^2 + |N|^2 - 2|K|^2) = (|D|^2 - |N|^2)(|G|^2 - |K|^2). \]

The eigenvalues of the matrix \( \Psi^\dagger \omega \Psi \) are \((|D|^2 - |N|^2)\) and \((|G|^2 - |K|^2)\).

The cases \( d_\psi = 2, x_\psi = y_\psi = 0 \) are realized when \( |D| = |K| = |N| = |G| \). For example, we have the solution

\[ D = G = \sqrt{\frac{a_3}{2}}, \quad K = iD, \quad N = -iD. \]

The case \( d_\psi = 0, x_\psi + y_\psi = 2 \) is realized when \( |D| \neq |N| \) and \( |G| \neq |K| \). For example, we have the solution with \( x_\psi = y_\psi = 1 \):

\[ D = K = 0, \quad N = \sqrt{\frac{a_3}{2}}, \quad G = i\sqrt{\frac{a_3}{2}}. \]

Also we have solutions with \( x_\psi = 2, y_\psi = 0 \) or \( x_\psi = 0, y_\psi = 2 \) of the following type

\[ K = N, \quad G = -D, \quad \text{Im}(\tilde{K}G) = \frac{a_3}{2}. \]

The case \( d_\psi = 1, x_\psi + y_\psi = 1 \) is realized when only one of the following conditions is satisfied: \( |D| = |N| \) or \( |G| = |K| \). For example, we have the solutions

\[ K = \sqrt{\frac{a_3}{\sqrt{2}}}, \quad N = \sqrt{2a_3}, \quad G = iK, \quad D = 0. \]
In Case 3 of Theorem 2 using the explicit form of \( \Psi \) (30) and the conditions (31), we get

\[
\Psi^t \omega \Psi = \begin{pmatrix} \bar{G} & \bar{C} & \bar{G} & \bar{G} \\ \bar{B} & \bar{D} & \bar{D} & \bar{B} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} G \\ B \\ C \\ D \end{pmatrix} = 0.
\]

Thus \( x_\Psi = y_\Psi = 0 \) and \( d_\Psi = \text{rank}(\Psi) \). We consider nonzero \( \Psi \neq 0 \) in this case and it can be verified that \( \text{rank}(\Psi) \neq 1 \) because of the conditions (31). We obtain \( d_\Psi = \text{rank}(\Psi) = 2 \).

We summarize the results on all types of constant solutions of the SU(2) Yang–Mills–Dirac equations in Table IV.

| Case | \( d_A \) | \( x_A \) | \( y_A \) | \( d_J \) | \( x_J \) | \( y_J \) | \( d_\Psi \) | \( x_\Psi \) | \( y_\Psi \) | \( m \) | \( F^2 \) | Expl. Form |
|------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 1    | 1   | 0   | 0   | 0   | 0   | 0   | 2   | 0   | 0   | 0   | 0   | 0   | 30  |
| 2    | 0   | 1   | 2   | 0   | 1   | 0   | 0   | 0   | 0   | 1   | \( m \neq 0 \) | \( F^2 \neq 0 \) | 26  |
| 3    | 0   | 0   | 2   | 0   | 0   | 0   | 2   | \( d_\Psi + x_\Psi + y_\Psi = 2 \) | \( m \neq 0 \) | \( F^2 \neq 0 \) | 27  |
| 4    | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | \( \gamma m \) | \( F^2 = 0 \) | 55  |
| 5    | 1   | 0   | 0   | 0   | 0   | 0   | 0   | \( \gamma m \) | \( F^2 = 0 \) | 56  |
| 6    | 0   | 1   | 0   | 0   | 0   | 0   | 0   | \( \gamma m \) | \( F^2 = 0 \) | 57  |
| 7    | 0   | 0   | 1   | 0   | 0   | 0   | 0   | \( \gamma m \) | \( F^2 = 0 \) | 58  |
| 8    | 0   | 0   | 0   | 0   | 0   | 0   | 2   | 0   | 0   | \( m \neq 0 \) | \( F^2 = 0 \) | 59  |
| 9    | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 2   | 0   | \( m = 0 \) | \( F^2 = 0 \) | 60  |
| 10   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 2   | \( m \neq 0 \) | \( F^2 = 0 \) | 61  |

### IV. ON NONCONSTANT SOLUTIONS IN THE FORM OF PERTURBATION THEORY SERIES

In previous section of this paper, we present explicit form of all constant solutions of the SU(2) Yang–Mills–Dirac equations. Let us consider nonconstant solutions of the system of SU(2) Yang–Mills–Dirac equations

\[
\partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu] =: F_{\mu \nu},
\]

\[
\partial_\mu F^{\mu \nu} - [A_\mu, F^{\mu \nu}] =: J^\nu := i \Psi^t D^0 \gamma^\nu \Psi - \frac{1}{2} \bar{\Psi} (i \Psi^t D^0 \gamma^\nu \Psi) J^2,
\]

\[
i \gamma^\mu (\partial_\mu \Psi + \Psi A_\mu) - m \Psi = 0, \quad m \geq 0,
\]

in the form of series of perturbation theory near the constant solutions.

Denote the left-hand part of the system (52) by

\[
H^\nu = H^\nu (A) := \partial_\mu (\partial^\mu A^\nu - \partial^\nu A^\mu - [A^\mu, A^\nu]) - [A_\mu, \partial_\nu A^\nu - \partial^\nu A^\mu - [A^\mu, A^\nu]]
\]

and the left-hand part of the equation (53) by

\[
T = T (\Psi, A) := i \gamma^\mu (\partial_\mu \Psi + \Psi A_\mu) - m \Psi.
\]

Using these notations, we can rewrite (52), (53) in the following way

\[
H^\nu (A) = J^\nu (\Psi), \quad T (\Psi, A) = 0.
\]

Denote the known constant solutions by \( A^\mu \) and \( \Psi \). Taking the small parameters \( \epsilon << 1 \), \( \lambda << 1 \), we get

\[
A^\mu = \sum_{k=0}^{\infty} \epsilon^k A^\mu = A^\mu + \epsilon A^\mu + \epsilon^2 A^\mu + \cdots,
\]

\[
\Psi = \sum_{k=0}^{\infty} \lambda^k \Psi = \Psi + \lambda \Psi + \lambda^2 \Psi + \cdots
\]

Substituting these expressions into \( \text{(56)} \), we get respectively

\[
H^\nu = \sum_{k=0}^\infty e^k H^\nu = \sum_{k=0}^\infty \lambda^k J^\nu = J^\nu, \quad T = \sum_{k=0}^\infty \sum_{l=0}^\infty e^k \lambda^l T = 0, \quad (59)
\]

where

\[
H^\nu = H^\nu (A^\mu_1, \ldots, A^\mu_i), \quad J^\nu = J^\nu (\Psi, \ldots, \Psi), \quad T = T (A^\mu_1, \ldots, A^\mu_i, \Psi, \ldots, \Psi)
\]

are some fixed expressions.

The small parameters \( \epsilon \) and \( \lambda \) can be connected in different ways, for example, in the way \( \lambda = \epsilon^r \) with some natural number \( r \).

Let us consider the case \( \epsilon = \lambda \) (with \( r = 1 \)) further. We use the notation \( T = T_k \) in this case. We get the equations

\[
H^\nu (A^\mu_1, \ldots, A^\mu_i) = J^\nu (\Psi, \ldots, \Psi), \quad k = 0, 1, 2, \ldots, \quad (60)
\]

\[
T (A^\mu_1, \ldots, A^\mu_i, \Psi, \ldots, \Psi) = 0, \quad k = 0, 1, 2, \ldots \quad (61)
\]

The conditions \( H^\nu (A^\mu_1, \ldots, A^\mu_i) = J^\nu (\Psi, \ldots, \Psi) \) and \( T (A^\mu, \Psi) = 0 \) hold automatically because \( (A^\mu, \Psi) \) are constant solutions of the system \( \text{(52)}, \text{(53)}. \)

For the first approximation, we get the system of linear partial differential equations with constant coefficients

\[
H^\nu (A^\mu_1, A^\mu_2) = J^\nu (A^\mu_1, \Psi), \quad T (A^\mu_1, A^\mu_2, \Psi, \Psi) = 0 \quad (62)
\]

for the variables \( \Psi \) and \( A^\mu_1 \). The explicit form of the system \( \text{(62)} \) is

\[
\partial_\mu \partial^\nu A^\nu - \partial_\mu A^\nu = i(\Psi^\dagger \gamma^\nu \gamma^\mu \Psi - \Psi^\dagger \gamma^\mu \gamma^\nu \Psi) - \frac{1}{2} \text{tr}(i(\Psi^\dagger \gamma^\nu \gamma^\mu \Psi - \Psi^\dagger \gamma^\mu \gamma^\nu \Psi), \quad (63)
\]

\[
i \gamma^\mu (\partial_\mu \Psi + \Psi A^\mu_1 + \Psi A^\mu_2 - m \Psi) = 0. \quad (64)
\]

We can take some solution of this system \( (\Psi, A^\mu_1) \) and substitute it and the constant solution \( (\Psi, A^\mu_2) \) into

\[
H^\nu (A^\mu_1, A^\mu_2, A^\mu_3) = J^\nu (A^\mu_1, A^\nu_2, \Psi), \quad T (A^\mu_1, A^\mu_2, A^\nu_3, \Psi, \Psi) = 0. \quad (65)
\]

We get a system of linear partial differential equations with variable coefficients (dependent on \( x \in \mathbb{R}^{1,3} \)) for the variables \( \Psi \) and \( A^\mu_1 \). We can take one of solutions of this system and substitute it into the next equation. In the same way, we can get \( \Psi \) and \( A^\mu_2 \) for any \( k = 0, 1, 2, \ldots \) This procedure allows us to obtain approximate solutions of the SU(2) Yang–Mills–Dirac system of equations up to terms of order \( k \).

For the solutions 4–7 from Table \( \text{(1)} \) the system for the first approximation \( \text{(63)} \) takes the form

\[
\partial_\mu \partial^\nu A^\nu - \partial_\mu A^\nu = i(\Psi^\dagger \gamma^\nu \gamma^\mu \Psi - \Psi^\dagger \gamma^\mu \gamma^\nu \Psi) - \frac{1}{2} \text{tr}(i(\Psi^\dagger \gamma^\nu \gamma^\mu \Psi - \Psi^\dagger \gamma^\mu \gamma^\nu \Psi), \quad (66)
\]

\[
i \gamma^\mu (\partial_\mu \Psi + \Psi A^\mu_1 + \Psi A^\mu_2 - m \Psi) = 0. \quad (67)
\]

For the solutions 8–10 from Table \( \text{(1)} \) the system for the first approximation \( \text{(63)} \) takes the form

\[
\partial_\mu \partial^\nu A^\nu - \partial_\mu A^\nu = i(\Psi^\dagger \gamma^\nu \gamma^\mu \Psi - \Psi^\dagger \gamma^\mu \gamma^\nu \Psi) - \frac{1}{2} \text{tr}(i(\Psi^\dagger \gamma^\nu \gamma^\mu \Psi - \Psi^\dagger \gamma^\mu \gamma^\nu \Psi), \quad (68)
\]

\[
i \gamma^\mu (\partial_\mu \Psi + \Psi A^\mu_1 + \Psi A^\mu_2 - m \Psi) = 0. \quad (69)
\]

The presented systems can be further studied using various methods of the theory of linear partial differential equations with constant coefficients.
V. CONCLUSIONS

In this paper, we present explicit form of all constant solutions of the Yang–Mills–Dirac equations with SU(2) gauge symmetry in Minkowski space \( \mathbb{R}^{1,3} \) (see Theorems 2, 3, and 5). The solutions are written out with an appropriate choice of the coordinate system and gauge. The complete classification of all constant solutions is presented in Table IV. We present explicit formulas for the spinor \( \Psi \), the potential \( A \), the current \( J \), and the invariant \( F^2 = F_{\mu \nu}F^{\mu \nu} \). Note that the invariant \( F^2 \) does not depend on the gauge fixing and the coordinate system; it is used in the Lagrangian of the Yang–Mills field. Note that solutions of different types from our classification can not be related by a gauge transformation by construction. Nonconstant solutions are considered in the form of series of perturbation theory using constant solutions as a zeroth approximation. The presented systems for the first approximation can be studied using different methods of the theory of linear partial differential equations with constant coefficients or numerical analysis. For the second and subsequent approximations, the problem reduces to solving the systems of linear partial differential equations with variable coefficients. The results can be used further to obtain a local classification of all (nonconstant) solutions of the classical Yang–Mills–Dirac equations. The results can be used to describe physical vacuum and better understand quantum gauge theory.

Note the papers [12,13] on instability of the constant Yang–Mills fields. Problem of stability of the constant Yang–Mills–Dirac fields presented in this paper is a task for further research. Another task for further research is to generalize the results of this paper to the case of the Lie group SU(3), which is important for quantum chromodynamics, or the more general case of the special unitary group SU(\( N \)). Another important task is to generalize results to the case of an arbitrary, possibly globally hyperbolic, Lorentzian manifold.

ACKNOWLEDGMENTS

This work is supported by the Russian Science Foundation (project 23-71-10028), https://rscf.ru/en/project/23-71-10028/.

DATA AVAILABILITY STATEMENT

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

Appendix A: The proof of Theorem 2

Cases 1, 2, and 3. Let us consider the first, the second, and the third cases (24) together supposing \( a_1 \geq 0 \) and \( a_2 \geq 0 \). We have

\[
A^0 = A_0 = A^1 = -A_1 = \tau^1 = \frac{1}{2i} \sigma^3, \quad A^2 = -A_2 = a_1 \tau^2 = \frac{a_1}{2i} \sigma^1, \quad A^3 = -A_3 = a_2 \tau^3 = \frac{a_2}{2i} \sigma^2.
\]

Substituting these expressions into (15), we get

\[
\left( \begin{array}{cc} I & 0 \\ 0 & -I \end{array} \right) \left( \begin{array}{c} \Psi_1 \\ \Psi_2 \end{array} \right) \frac{\sigma^3}{2} - \frac{1}{2} \left( \begin{array}{cc} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{array} \right) \left( \begin{array}{c} \Psi_1 \\ \Psi_2 \end{array} \right) \frac{\sigma^3}{2} - \frac{1}{2} \left( \begin{array}{cc} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{array} \right) \left( \begin{array}{c} \Psi_1 \\ \Psi_2 \end{array} \right) \frac{a_1 \sigma^1}{2} - \frac{1}{2} \left( \begin{array}{cc} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{array} \right) \left( \begin{array}{c} \Psi_1 \\ \Psi_2 \end{array} \right) \frac{a_2 \sigma^2}{2} \right) m \left( \begin{array}{c} \Psi_1 \\ \Psi_2 \end{array} \right) = \left( \begin{array}{c} 0 \\ 0 \end{array} \right).
\]

We obtain the following two equations

\[
\Psi_1 \sigma^3 - \sigma^1 \Psi_2 \sigma^3 - a_1 \sigma^2 \Psi_2 \sigma^1 + a_2 \sigma^3 \Psi_2 \sigma^2 - 2m \Psi_1 = 0, 
\]

\[
-\Psi_2 \sigma^3 + \sigma^1 \Psi_1 \sigma^3 + a_1 \sigma^2 \Psi_1 \sigma^1 + a_2 \sigma^3 \Psi_1 \sigma^2 - 2m \Psi_2 = 0.
\]

In Case 1, we have \( a_1 = a_2 = 0 \) and get

\[
\Psi_1 \sigma^3 - \sigma^1 \Psi_2 \sigma^3 - 2m \Psi_1 = 0, 
\]

\[
-\Psi_2 \sigma^3 + \sigma^1 \Psi_1 \sigma^3 - 2m \Psi_2 = 0.
\]

Multiplying (A3) on the right by \( \sigma^3 \) and on the left by \( \sigma^1 \) and using \( (\sigma^1)^2 = (\sigma^3)^2 = I_2 \), we get

\[
\Psi_2 = \sigma^1 \Psi_1 - 2m \sigma^1 \Psi_1 \sigma^3.
\]
Substituting (A5) into (A4) and using \((\sigma^3)^2 = I\), we get \(m^2\Psi_1 = 0\). If \(\Psi_1 = 0\), then using (A5), we get \(\Psi_2 = 0\) and \(\Psi = 0\), i.e. a contradiction. If \(m = 0\), then we get \(\Psi_2 = \sigma^1 \Psi_1\). We have \([A_{\mu}, [A^\nu, A^\gamma]] = 0\) in this case. Substituting

\[
\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \frac{\Psi_1}{\sigma^1 \Psi_1} = \begin{pmatrix} G & B \\ C & D \end{pmatrix}
\]

into (A4), we get

\[
J^0 = J^1 = \frac{1}{2} \left( \begin{array}{cc} |G|^2 + |C|^2 - |B|^2 - |D|^2 \\ 2(BG + DC) \end{array} \right) \begin{array}{cc} 2(GB + CD) \\ |B|^2 + |D|^2 - |G|^2 - |C|^2 \end{array}, \quad J^2 = J^3 = 0.
\]

We obtain the solutions (30).

In Cases 2 and 3, from (A1) and (A2), we get

\[
\Psi_1(\sigma^3 - 2mI) = \sigma^1 \Psi_2 \sigma^3 + a_1 \sigma^2 \Psi_2 \sigma^1 + a_2 \sigma^3 \Psi_2 \sigma^2,
\]

(A6)

\[
\Psi_2(\sigma^3 + 2mI) = \sigma^1 \Psi_1 \sigma^3 + a_1 \sigma^2 \Psi_1 \sigma^1 + a_2 \sigma^3 \Psi_1 \sigma^2.
\]

(A7)

If \(m \neq \frac{1}{2}\), then the matrices \(\sigma^3 - 2mI\) and \(\sigma^3 + 2mI\) are invertible:

\[
(\sigma^3 - 2mI)(\sigma^3 + 2mI) = (1 - 4m^2)I.
\]

Taking \(\Psi_1\) from (A6) and substituting into (A7), we get a homogeneous system of four linear equations for the four variables \(K, L, M,\) and \(N\). The matrix of this homogeneous system

\[
\begin{vmatrix}
4m^2(-2m + 1) + (a_1^2 + a_2^2)(-2m + 1) & -4ia_1m \\
4ia_1m & 4m^2(-2m + 1) + (a_1^2 + a_2^2)(-2m + 1) \\
2a_1a_2(-2m + 1) & -4ia_2m \\
-4ia_2m & 2a_1a_2(-2m + 1) \\
4m^2(-2m + 1) + (a_1^2 + a_2^2)(-2m + 1) & 4ia_1m \\
-4ia_1m & 4m^2(-2m + 1) + (a_1^2 + a_2^2)(-2m + 1) \\
2a_1a_2(-2m + 1) & 4ia_2m \\
4ia_2m & 2a_1a_2(-2m + 1) \\
\end{vmatrix}
\]

has nonzero determinant

\[
(1 + 2m)^2(1 - 2m)^2((a_1 - a_2)^2 + 4m^2)((a_1 + a_2)^2 + 4m^2) > 0,
\]

thus \(K = L = M = N = 0\). We conclude that \(\Psi_2 = 0\), \(\Psi_1 = 0\), and \(\Psi = 0\), i.e. we get a contradiction. We have no solutions of this type.

Let us consider the case of \(m = \frac{1}{2}\). Substituting

\[
\Psi_1 = \begin{pmatrix} G & B \\ C & D \end{pmatrix}, \quad \Psi_2 = \begin{pmatrix} K & L \\ M & N \end{pmatrix}
\]

into (A6) and (A7), we get the system

\[
\begin{align*}
\begin{pmatrix} 0 & -2B \\
0 & -2D \end{pmatrix} &= \begin{pmatrix} M & -N \\
K & -L \end{pmatrix} + a_1 \begin{pmatrix} -Ni & -Mi \\
Li & Ki \end{pmatrix} + a_2 \begin{pmatrix} Li & -Ki \\
-Ni & Mi \end{pmatrix}, \\
\begin{pmatrix} 2K & 0 \\
2M & 0 \end{pmatrix} &= \begin{pmatrix} C & -D \\
G & -B \end{pmatrix} + a_1 \begin{pmatrix} -Di & -Ci \\
Bi & Gi \end{pmatrix} + a_2 \begin{pmatrix} Bi & -Gi \\
-Di & Ci \end{pmatrix}.
\end{align*}
\]

Eliminating \(B, D, K,\) and \(M\), we get a homogeneous system of four linear equations for the four variables \(G, C, L,\) and \(N\) with the matrix

\[
\begin{vmatrix}
1 - a_1^2 - a_2^2 & 2a_1a_2 & 2ia_2 & 2ia_1 \\
2a_1a_2 & 1 - a_1^2 - a_2^2 & -2ia_1 & -2ia_2 \\
2ia_2 & -2ia_1 & 1 - a_1^2 - a_2^2 & -2a_1a_1 \\
2ia_1 & -2ia_2 & -2a_1a_2 & 1 - a_1^2 - a_2^2 \\
\end{vmatrix}.
\]
The determinant of this matrix is nonzero

\[(1 + (a_1 - a_2)^2)(1 + (a_1 + a_2)^2) > 0,\]

thus \(G = C = L = N = 0\) and then \(M = K = D = B = 0\). We get a contradiction, we have no solutions in this case.

**Cases 4, 5, and 6.** Let us consider the fourth, the fifth, and the sixth cases (24) together supposing \(a_1 > 0, a_2 \geq 0,\) and \(a_3 \geq 0\).

We have

\[A^0 = A_0 = a_1 \tau^1 = \frac{a_1}{2i}\sigma^3, \quad A^1 = A_1 = a_2 \tau^2 = \frac{a_2}{2i}\sigma^1, \quad A^2 = A_2 = a_3 \tau^3 = \frac{a_3}{2i}\sigma^2, \quad A^3 = A_3 = 0.\]  

(A8)

Substituting these expressions into (15), we get

\[
\begin{pmatrix}
I & 0 \\
0 & -I
\end{pmatrix}
\begin{pmatrix}
\Psi_1 \\
\Psi_2
\end{pmatrix}
= \begin{pmatrix}
\frac{a_1 \sigma^3}{2} - \frac{a_2 \sigma^1}{2} - \frac{a_3 \sigma^2}{2} \\
\frac{a_1 \sigma^1}{2} + \frac{a_2 \sigma^2}{2} + \frac{a_3 \sigma^3}{2}
\end{pmatrix}
= m
\begin{pmatrix}
\Psi_1 \\
\Psi_2
\end{pmatrix}.
\]

We obtain the following two equations

\[a_1 \Psi_1 \sigma^3 - a_2 \sigma^1 \Psi_1 \sigma^1 - a_3 \sigma^2 \Psi_2 \sigma^2 = 2m \Psi_1,\]  

(A9)

\[-a_1 \Psi_2 \sigma^3 + a_2 \sigma^1 \Psi_1 \sigma^1 + a_3 \sigma^2 \Psi_1 \sigma^2 = 2m \Psi_2.\]  

(A10)

In Case 4, we have \(a_2 = a_3 = 0\) and get the system

\[a_1 \Psi_1 \sigma^3 = 2m \Psi_1, \quad -a_1 \Psi_2 \sigma^3 = 2m \Psi_2.\]

Thus

\[\Psi_1 (a_1 \sigma^3 - 2ml_2) = 0, \quad \Psi_2 (a_1 \sigma^3 + 2ml_2) = 0.\]

If \(a_1 \neq 2m\), then \(\Psi_1 = \Psi_2 = 0\), i.e. \(\Psi = 0\) and we obtain a contradiction. Let us consider the case \(a_1 = 2m > 0\). For

\[\Psi = \begin{pmatrix}
\Psi_1 \\
\Psi_2
\end{pmatrix} = \begin{pmatrix}
G & B \\
C & D
\end{pmatrix}
\begin{pmatrix}
0 & 0 \\
0 & -4m
\end{pmatrix}, \quad \begin{pmatrix}
K & L \\
M & N
\end{pmatrix}
\begin{pmatrix}
4m & 0 \\
0 & 0
\end{pmatrix},
\]

we obtain

\[
\begin{pmatrix}
G & B \\
C & D
\end{pmatrix}
\begin{pmatrix}
0 & 0 \\
0 & -4m
\end{pmatrix} = 0, \quad \begin{pmatrix}
K & L \\
M & N
\end{pmatrix}
\begin{pmatrix}
4m & 0 \\
0 & 0
\end{pmatrix} = 0,
\]

i.e. \(B = D = K = M = 0\). In this case, we have \([A_{\mu}, [A^\mu, A^\nu]] = 0\). We calculate the current

\[f^0 = i \left(\begin{array}{c}
|G|^2 + |C|^2 - |L|^2 - |N|^2 \\
|L|^2 + |N|^2 - |G|^2 - |C|^2
\end{array}\right), \quad f^1 = i \left(\begin{array}{c}
0 \\
|\bar{N}G + LC|
\end{array}\right),\]

\[f^2 = i \left(\begin{array}{c}
0 \\
i(\bar{N}G - LCN)
\end{array}\right), \quad f^3 = i \left(\begin{array}{c}
0 \\
0
\end{array}\right),
\]

and obtain

\[|G|^2 + |C|^2 = |L|^2 + |N|^2, \quad \bar{C}L - \bar{C}N = 0, \quad \bar{C}L = \bar{G}N = 0,
\]

i.e. \(G = C = L = N = 0\). We get \(\Psi = 0\), i.e. a contradiction. We have no solutions in this case.

Let us consider Case 5 (we have \(a_1 > 0, a_2 > 0, a_3 = 0\)). Multiplying (A9) on the right and on the left by \(\sigma^1\) and using \((\sigma^1)^2 = I\), we get

\[\Psi_2 = \frac{a_1}{a_2} \sigma^1 \Psi_1 \sigma^3 - \frac{2m}{a_2} \sigma^1 \Psi_1 \sigma^1.\]  

(A11)

Substituting this expression into (A10), using \((\sigma^1)^2 = I\) and \(\sigma^1 \sigma^3 = -\sigma^3 \sigma^1\), we get

\[\Psi_1 ((a_1^2 + a_2^2 + 4m^2)I - 4a_1m \sigma^3) = 0.
\]
The determinant of the matrix \((a_1^2 + a_2^2 + 4m^2)I - 4a_1m\sigma^3\) is nonzero
\[
((a_1 - 2m)^2 + a_3^2)((a_1 + 2m)^2 + a_3^2) > 0,
\]
thus \(\Psi_1 = 0\). Then from (A11), we get \(\Psi_2 = 0\), and \(\Psi = 0\), i.e. we obtain a contradiction.

Now let us consider Case 6 (with \(a_1, a_2, a_3 > 0\)). From (A9) and (A10), we get
\[
\begin{align*}
\Psi_1 & = \begin{pmatrix} a_1 - 2m & 0 \\ 0 & -2m - a_1 \end{pmatrix} = a_2\sigma^3\Psi_2\sigma^3 + a_3\sigma^2\Psi_2\sigma^2, \\
\Psi_2 & = \begin{pmatrix} -a_1 - 2m & 0 \\ 0 & -2m + a_1 \end{pmatrix} = -a_2\sigma^3\Psi_1\sigma^3 - a_3\sigma^2\Psi_1\sigma^2.
\end{align*}
\]
(A12) (A13)

In the case \(a_1 \neq 2m\), we can express \(\Psi_1\) through \(\Psi_2\) from (A12) and substitute into (A13). Denoting
\[
\Psi_2 = \begin{pmatrix} K \\ M \end{pmatrix} \gamma \begin{pmatrix} L \\ N \end{pmatrix}, \quad K, L, M, N \in \mathbb{C},
\]
we get the following equations for the variables \(K, L, M,\) and \(N\):
\[
\begin{align*}
K(-2m + a_1)((2m + a_1)^2 + (a_2 + a_3)^2) = 0, \\
L(2m + a_1)((-2m + a_1)^2 + (a_2 - a_3)^2) = 0, \\
M(-2m + a_1)((2m + a_1)^2 + (a_2 - a_3)^2) = 0, \\
N(2m + a_1)((-2m + a_1)^2 + (a_2 + a_3)^2) = 0,
\end{align*}
\]
i.e. \(K = L = M = N = 0\). If \(\Psi_2 = 0\), then \(\Psi_1 = 0\) and \(\Psi = 0\), and we obtain a contradiction.

Let us consider the case \(a_1 = 2m > 0\). We get from (A12) and (A13) the equations
\[
\begin{align*}
-4m\begin{pmatrix} 0 & B \\ 0 & D \end{pmatrix} = \begin{pmatrix} a_2 & N \\ M & K \end{pmatrix} + a_3\begin{pmatrix} N & -M \\ -L & K \end{pmatrix}, \\
-4m\begin{pmatrix} K & 0 \\ M & 0 \end{pmatrix} = -a_2\begin{pmatrix} D & C \\ B & G \end{pmatrix} - a_3\begin{pmatrix} D & -C \\ -B & G \end{pmatrix}.
\end{align*}
\]

Solving the eight equations, we obtain \(a_2 = a_3\) and \(G = D = K = N = B = M = 0\). Substituting (A8) into (14), we get the following expressions on the left side of the equations
\[
\begin{align*}
J^0 = \begin{pmatrix} a_1, [A^1, A^0] \end{pmatrix} + \begin{pmatrix} a_2, [A^2, A^0] \end{pmatrix} = 4((A^1)^2 + (A^2)^2), A^0 = \frac{-ia_1(a_1^2 + a_2^2)}{2}\sigma^3, \\
J^1 = \begin{pmatrix} a_0, [A^0, A^1] \end{pmatrix} + \begin{pmatrix} a_2, [A^2, A^1] \end{pmatrix} = 4((A^0)^2 - (A^2)^2), A^1 = \frac{ia_2(a_1^2 - a_2^2)}{2}\sigma^3, \\
J^2 = \begin{pmatrix} a_0, [A^0, A^2] \end{pmatrix} + \begin{pmatrix} a_1, [A^1, A^2] \end{pmatrix} = 4((A^0)^2 - (A^1)^2), A^2 = \frac{ia_3(a_1^2 - a_2^2)}{2}\sigma^3, \\
J^3 = 0.
\end{align*}
\]
(A14)

On the right side of (14), we obtain
\[
\begin{align*}
J^0 = \frac{i}{2}\left(\begin{pmatrix} \bar{C}L \end{pmatrix}^2 - |L|^2 \right) \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} L \end{pmatrix}^2 - |C|^2 \end{pmatrix} + \begin{pmatrix} \bar{C}L \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right), \\
J^1 = \begin{pmatrix} 0, \bar{C}L \end{pmatrix} \begin{pmatrix} 0 \\ LC \end{pmatrix}, \\
J^2 = \begin{pmatrix} 0, \bar{C}L \end{pmatrix} \begin{pmatrix} 0 \\ LC \end{pmatrix}, \\
J^3 = 0.
\end{align*}
\]
(A15)

Equating (A14) and (A15), we get \(a_1 = a_2, C = 0\), and the solutions (20).

**Cases 7, 8, and 9.** Let us consider the seventh, the eighth, and the ninth cases (24) together supposing \(a_1 > 0, a_2 \geq 0,\) and \(a_3 \geq 0\).

We have
\[
\begin{align*}
A^0 = A_0 = 0, \\
A^1 = -A_1 = a_1\tau^1 = \frac{a_1}{2i}\sigma^3, \\
A^2 = -A_2 = a_2\tau^2 = \frac{a_2}{2i}\sigma^1, \\
A^3 = -A_3 = a_3\tau^3 = \frac{a_3}{2i}\sigma^2.
\end{align*}
\]
(A16)

Substituting these expressions into (15), we get
\[
\begin{pmatrix} 0 & \sigma^3 \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} \frac{a_1\sigma^3}{2} + \begin{pmatrix} 0 & \sigma^2 \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} \frac{a_2\sigma^1}{2} + \begin{pmatrix} 0 & \sigma^3 \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} \frac{a_3\sigma^2}{2} = m \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}.
\]

We obtain the following two equations
\[
\begin{align*}
a_1\sigma^1\Psi_2\sigma^3 + a_2\sigma^2\Psi_2\sigma^1 + a_3\sigma^3\Psi_2\sigma^2 = -2m\Psi_1, \\
a_1\sigma^1\Psi_1\sigma^3 + a_2\sigma^2\Psi_1\sigma^1 + a_3\sigma^3\Psi_1\sigma^2 = 2m\Psi_2.
\end{align*}
\]
(A17) (A18)
Let us consider Case 7. We have $a_1 > 0$, $a_2 = a_3 = 0$, and get the system

$$a_1 \sigma^1 \Psi_2 \sigma^3 = -2m \Psi_1, \quad a_1 \sigma^1 \Psi_1 \sigma^3 = 2m \Psi_2.$$  

From the first equation, we get $\Psi_2 = -\frac{2m}{a_1} \sigma^1 \Psi_1 \sigma^3$. Substituting this expression into the second equation, we get $(a_1^2 + 4m^2) \Psi_1 = 0$, i.e. $\Psi_1 = 0$. Thus $\Psi_2 = 0$, $\Psi_3 = 0$, and we get a contradiction.

Let us consider Cases 8 and 9 (we have $a_1 > 0$, $a_2 > 0$, $a_3 \geq 0$). If $m \neq 0$, then from the equation \((A17)\), we get $\Psi_1 = -\frac{1}{2m} (a_1 \sigma^1 \Psi_2 \sigma^3 + a_2 \sigma^2 \Psi_2 \sigma^1 + a_2 \sigma^3 \Psi_2 \sigma^3)$. Substituting this expression into \((A18)\), we get

$$(a_1^2 + a_2^2 + a_3^2 + 4m^2) \Psi_2 + 2a_1 a_2 \sigma^1 \Psi_2 \sigma^1 \sigma^3 + 2a_1 a_3 \sigma^1 \sigma^3 \Psi_2 \sigma^2 \sigma_3^2 + 2a_2 a_3 \sigma^2 \sigma^3 \Psi_2 \sigma^2 \sigma^1 = 0.$$  

For $\Psi_2 = \begin{pmatrix} K & L \\ M & N \end{pmatrix}$, we obtain the equation

$$(a_1^2 + a_2^2 + a_3^2 + 4m^2) (K L) + 2a_1 a_2 \begin{pmatrix} iL & -iM \\ -iN & iK \end{pmatrix} + 2a_1 a_3 \begin{pmatrix} -iN & iM \\ iL & -iK \end{pmatrix} + 2a_2 a_3 \begin{pmatrix} M & -N \\ K & -L \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{A19}$$

which can be rewritten in the form of a homogeneous system of four linear equations with the matrix

$$\begin{pmatrix} a_1^2 + a_2^2 + a_3^2 + 4m^2 & 2a_1 a_2 & 2a_1 a_3 & -2a_1 a_2 \\ -2a_1 a_2 & a_1^2 + a_2^2 + 4m^2 & 2a_2 a_3 & -2a_2 a_3 \\ 2a_1 a_3 & 2a_2 a_3 & a_1^2 + a_2^2 + a_3^2 + 4m^2 & -2a_1 a_2 \\ 2a_2 a_3 & -2a_1 a_3 & 2a_1 a_2 & a_1^2 + a_2^2 + a_3^2 + 4m^2 \end{pmatrix}.$$  

The determinant of this matrix is nonzero

$$(a_1 + a_2 + a_3)^2 + 4m^2) (a_1 - a_2 + a_3)^2 + 4m^2) (a_1 + a_2 - a_3)^2 + 4m^2) (a_1 - a_2 + a_3)^2 + 4m^2) > 0,$$

thus $\Psi_2 = 0$, $\Psi_1 = 0$ and $\Psi_0 = 0$, and we obtain a contradiction. We have no solutions of this type.

If $m = 0$, then the system \((A17), (A18)\) takes the form

$$a_1 \begin{pmatrix} M & -N \\ K & -L \end{pmatrix} + a_2 \begin{pmatrix} -iN & -iM \\ iL & -iM \end{pmatrix} + a_3 \begin{pmatrix} M & -N \\ K & -L \end{pmatrix} = 0. \tag{A19}$$

In Case 8 ($a_3 = 0$), we get

$$\Psi = \begin{pmatrix} G & iG \\ iD & D \\ K & iK \\ iN & N \end{pmatrix}, \quad a_1 = a_2.$$

Equating the left side of \((A14)\)

$$J^0 = 0, \quad J^1 = [A_2, [A^2, A^1]] = -4(A^2)^2 A^1 = a_2^2 a_1 \tau^1 = \frac{a_2^2 a_1}{2i} \sigma^3, \quad J^3 = 0,$$

and the right side of \((A14)\)

$$J^0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad J^1 = i\begin{pmatrix} -iNG + iKD + iDG + GN \\ -iNG - iKD + iDG + GN \\ iNG + iKD + iDG + GN \end{pmatrix}, \quad J^2 = i\begin{pmatrix} -iNG + iKD + iDG + GN \\ iNG - iKD + iDG - iGN \\ iNG + iKD - iDG + iGN \end{pmatrix}, \quad J^3 = i\begin{pmatrix} 0 \\ i(\bar{KG} + \bar{ND} + \bar{GK} + \bar{DN}) \end{pmatrix},$$
we get

$$|G|^2 + |K|^2 = |D|^2 + |N|^2, \quad \Re(\bar{K}G + \bar{N}D) = 0, \quad \Re(\bar{N}G + \bar{K}D) = 0, \quad \Im(\bar{N}G - \bar{K}D) = a_1^2a_2 = a_2^2a_1.$$  

We obtain $a_1 = a_2$ and the solutions \(^{(27)}\).

In Case 9 ($a_3 \neq 0$), we get from \((A19)\)

$$L = \frac{i}{a_1}(a_2K + a_3M), \quad N = -\frac{i}{a_1}(a_2M + a_3K), \quad K = \frac{a_2^2 - a_2^2 - a_3^2}{2a_2a_3}, \quad (a_1^2 - (a_2 + a_3)^2)(a_1^2 - (a_2 - a_3)^2)M = 0.$$  

If $M \neq 0$, then $a_1 = a_2 + a_3$, because we can always take the hyperbolic singular values in decreasing order $a_1 > a_2 > a_3 > 0$. If $M = 0$, then $K = 0$, and $L = N = 0$, thus $\Psi_2 = 0$. We get the same for the variables $G, B, C, D$ from \((A19)\). We obtain

$$\Psi = \begin{pmatrix} G \\ C \\ K \\ M \end{pmatrix} = \begin{pmatrix} a_2G + a_3C \\ a_2C + a_3G \\ a_2K + a_3M \\ a_2M + a_3K \end{pmatrix}.$$  

On the left side of \((14)\), we get

$$J^0 = 0, \quad J^1 = [A_2, [A^2,A^1]] + [A_3, [A^3,A^1]] = (a_2^2 + a_3^2)a_1\tau^1, \quad J^2 = [A_1, [A^1,A^2]] + [A_3, [A^3,A^2]] = (a_1^2 + a_3^2)a_2\tau^2, \quad J^3 = [A_1, [A^1,A^3]] + [A_2, [A^2,A^3]] = (a_1^2 + a_2^2)a_3\tau^3.$$  

Calculating $J^0$ on the right side of \((14)\), we get

$$a_2(a_2 + a_3)^2(|G|^2 + |C|^2 + |K|^2 + |M|^2) = (a_2G + a_3C)^2 + (a_2C + a_3G)^2 + (a_2K + a_3M)^2 + (a_2M + a_3K)^2, \quad a_2(|G|^2 - |C|^2 + |K|^2 - |M|^2) + a_3(GC - CG + KM - MK) = 0.$$  

Simplifying the first equation, we get $|G - C|^2 + |K - M|^2 = 0$, i.e. $G = C$ and $K = M$. We conclude that

$$\Psi = \begin{pmatrix} G \\ G \\ K \\ M \end{pmatrix} = \begin{pmatrix} iG \\ -iG \\ iK \\ -iK \end{pmatrix}.$$  

Equating

$$J^1 = i\begin{pmatrix} 2(\bar{K}G + \bar{G}K) \\ 0 \\ -2(\bar{K}G + \bar{G}K) \end{pmatrix}, \quad J^2 = i\begin{pmatrix} 0 \\ -2(\bar{K}G + \bar{G}K) \\ 0 \end{pmatrix}, \quad J^3 = i\begin{pmatrix} 0 \\ 2i(\bar{K}G + \bar{G}K) \\ 0 \end{pmatrix},$$

and \((A20)\), we get

$$4(\bar{K}G + \bar{G}K) = -a_1(a_2^2 + a_3^2) = a_2(a_1^2 + a_3^2) = a_3(a_1^2 + a_2^2),$$

i.e. a contradiction. We have no solutions in this case.

The theorem is proved.

**Appendix B: The proof of Theorem 5**

**Case 1.** In the first case \((47)\), we calculate $J^\mu, \mu = 0, 1, 2, 3$:

$$J^0 = \bar{\rho}\Psi^\dagger\Psi - \frac{1}{2}\text{tr}(\bar{\rho}\Psi^\dagger\Psi) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i\sigma^3 = -2\tau^1,$$

$$J^\mu = \bar{\rho}\gamma^\mu\Psi - \frac{1}{2}\text{tr}(\bar{\rho}\gamma^\mu\Psi^\dagger\Psi) = 0, \quad \mu = 1, 2,$$

$$J^3 = \bar{\rho}\gamma^3\Psi - \frac{1}{2}\text{tr}(\bar{\rho}\gamma^3\Psi^\dagger\Psi) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i\sigma^3 = -2\tau^1.$$
We get the nonzero matrix

$$J = (J^\mu_a) = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

with $d_J = 1$, $x_J = y_J = 0$.

**Case 2.** In the second case (47), we calculate $J^\mu$, $\mu = 0, 1, 2, 3$. In the case of signs $+$ and $+$, we get

$$J^0 = 0, \quad J^1 = -4\tau^2, \quad J^2 = -4\tau^3, \quad J^3 = -4\tau^4$$

i.e. the nonzero matrix

$$J = (J^\mu_a) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{pmatrix}$$

with $d_J = 0$, $x_J = y_J = 3$.

In the cases of signs $+$ and $-$; $-$ and $+$, we get $J^0 = J^1 = J^2 = J^3 = 0$, i.e. the zero matrix $J = 0$.

In the case of signs $-$ and $-$, we get

$$J^0 = 0, \quad J^1 = 4\tau^2, \quad J^2 = 4\tau^3, \quad J^3 = 4\tau^4$$

i.e. the nonzero matrix

$$J = (J^\mu_a) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

with $d_J = 0$, $x_J = y_J = 3$.

**Case 3.** In the third case (47), we calculate $J^\mu$, $\mu = 0, 1, 2, 3$:

$$J^0 = (2 - \psi_1^2)\tau^1, \quad J^1 = 2\tau^1, \quad J^2 = 0, \quad J^3 = -2\psi_1\tau^2.$$  

We get the nonzero matrix

$$J = (J^\mu_a) = \begin{pmatrix} 2 - \psi_1^2 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

**Case 4.** In the forth case (47), we calculate $J^\mu$, $\mu = 0, 1, 2, 3$:

$$J^0 = (2 - \psi_1^2)\tau^1, \quad J^1 = 2\tau^1, \quad J^2 = 0, \quad J^3 = -2\psi_1\tau^2.$$  

We get the nonzero matrix

$$J = (J^\mu_a) = \begin{pmatrix} 2 - \psi_1^2 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

**Cases 5 and 6.** Let us consider the fifth and the sixth cases (47) together supposing $\psi_1 > 0$ and $\psi_2 \in \mathbb{R}$. We get

$$J^0 = i \left( \frac{\psi_1^2 - \psi_2^2}{2} \frac{0}{\psi_2^2 - \psi_1^2} \right) = i \frac{\psi_1^2 - \psi_2^2}{2} \sigma^3 = (\psi_2 - \psi_1^2)\tau^1, \quad J^\mu = 0, \quad \mu = 1, 2, 3.$$
We obtain the matrix

\[ J = \begin{pmatrix} \psi_2^2 - \psi_1^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \]

This matrix is the zero matrix in the cases \( \psi_1 = \psi_2 > 0 \) and \( \psi_1 = -\psi_2 > 0 \).

**Cases 7 and 8.** Let us consider the seventh and the eighth cases (47) together with \( \psi_1 > 0, \psi_2 \in \mathbb{R} \). We get

\[ J^0 = i \left( \frac{\psi_1^2 - \psi_2^2}{2} \right) = i \left( \psi_1^2 - \psi_2^2 \right) \sigma^3 = (\psi_2^2 - \psi_1^2) \tau^1, \quad J^0 = 0, \quad \mu = 1, 2, 3. \]

We obtain the matrix

\[ J = \begin{pmatrix} \psi_2^2 - \psi_1^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \]

This matrix is the zero matrix in the cases \( \psi_1 = \psi_2 > 0 \) and \( \psi_1 = -\psi_2 > 0 \).

**Case 9.** In the ninth case (47), we calculate \( J^\mu, \mu = 0, 1, 2, 3 \):

\[ J^0 = (\psi_2^2 - \psi_1^2) \tau^1, \quad J^1 = J^2 = 0, \quad J^3 = -2 \psi_1 \psi_2 \tau^2. \]

We get the nonzero matrix

\[ J = (J^\mu_a) = \begin{pmatrix} \psi_2^2 - \psi_1^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \psi_1 \psi_2 \end{pmatrix}. \]

The theorem is proved.

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