Capacity of an infinite family of networks related to the diamond network for fixed alphabet sizes

Sascha Kurz
Mathematisches Institut, Universität Bayreuth, D-95440 Bayreuth, Germany,
sascha.kurz@uni-bayreuth.de

Abstract

We consider the problem of error correction in a network where the errors can occur only on a proper subset of the network edges. For a generalization of the so-called Diamond Network we consider lower and upper bounds for the network’s (1-shot) capacity for fixed alphabet sizes.

Keywords network coding capacity adversarial network single-error correction codes
Mathematics Subject Classification (2000) 94B65

1 Introduction

The correction of errors introduced by noise or adversaries in networks have been studied in a number of papers, see e.g. [8, 17]. Here we assume that the adversary can manipulate a subset of the edges of a given network. If the adversary is unrestricted, i.e., all edges can be manipulated, then it is well known that the 1-shot capacity of the network can be achieved, see e.g. [11]. In [1, 3] specific networks are considered where the actual 1-shot capacity cannot be achieved using linear operations at intermediate nodes, so that network decoding is required in order to achieve the capacity. For a unified combinatorial treatment of adversarial network channels we refer to [15].

The aim of this paper is to study the capacity of a generalization of the diamond network, see [1, 3] for the diamond network and Figure 1 for the generalized case. This family of networks is one of five infinite families studied in [2], more precisely family B. The dashed edges $e_0, \ldots, e_{s+1}$ can be manipulated by the adversary and the solid edges $f_0, \ldots, f_s$ are those that cannot be manipulated. Node $S$ is the source, $T$ the terminal, and $V_1, V_2$ are the intermediate nodes. We assume that the adversary can manipulate exactly one dashed edge and we are interested in the 1-shot capacity, i.e., where the edges of the network are just used once.

The possible mappings from the input to the output space of the nodes are numerous. So, after stating the necessary preliminaries in Section 2 we study the structure of “good” network codes in the single-error correction setting for generalized diamond networks $\mathcal{N}_s$ in Section 3. Our main structure result, see Theorem 12 states that we may assume that node $V_1$ just forwards its received information and node $V_2$ performs a partial error-correction and sends sets of codeword candidates as states, so that the design of an optimal network code becomes a covering problem for sets. Based on these insights we determine lower and upper bounds for the 1-shot capacity of $\mathcal{N}_s$ in Sections 4 and 5 respectively. We close with a brief conclusion in Section 6.
While the family of networks $N_s$ seems to be rather specific and non-general, there are a few reasons to study this specific class in detail. It is one of three families of so-called 2-level networks whose capacity couldn’t be determined in [2]. If we remove vertex $V_1$, then we are in the classical situation of coding along a single edge, so that this class is somehow the smallest possible example where we can see network (coding) effects. Even though $N_s$ is very well structured, the exact determination of its 1-shot capacity for a given parameter $s$ and a given alphabet size seems to be a combinatorial challenge. In [2] reductions for general networks to 3-level networks and then to 2-level networks were presented, so that it makes sense to study the smaller and more structured networks first. Nevertheless there is some hope that some of the presented techniques may generalize to different networks too.

2 Preliminaries

In this section we formally define communication networks, network codes, and the channels they induce cf. [2, 15]. Due to a more restricted setting we use slightly different notation and refer the interested reader to [15] for more details.

Definition 1. A (single-source communication) network is a 4-tuple $N = (V, E, S, T)$, where $(V, E)$ is a finite, directed, acyclic multigraph $S \in V$ is the source, and $\emptyset \neq T \subseteq V \backslash \{S\}$ is the set of terminals. Additionally we assume that there exists a directed path from $S$ to any $T \in T$ and for every $V \in V \backslash (\{S\} \cup T)$ there exist directed paths from $S$ to $V$ and from $V$ to some $T \in T$.

The elements of $V$ are called nodes, the elements of $E$ are called edges, and the elements of $V \backslash (\{S\} \cup T)$ are called intermediate nodes. The set of incoming and outgoing edges for a node $V \in V$ are denoted by $\text{in}(V)$ and $\text{out}(V)$, respectively. As alphabet we choose an arbitrary finite set $A$ with at least two elements, e.g. $A = \{0, 1, \ldots, |A| - 1\}$.

In order to describe the transmission across the network we need functions $F_V : A^{\text{in}(V)} \to A^{\text{out}(V)}$ for each intermediate node $V$. We write $A^B := \{\varphi : B \to A\}$ for the set of all functions from $B$ to $A$. For the ease of notation we associate the elements in $\varphi \in A^B$, for arbitrary subsets $B \subseteq E$, with the vectors in $A^{|B|}$. This is made precise by fixing an ordering “<” of the elements of $E$ and replacing $\varphi$ by the vector of images $(\varphi(b_1), \ldots, \varphi(b_{|B|})) \in A^{|B|}$, where $\{b_1, \ldots, b_{|B|}\} = B$ and $b_1 < \cdots < b_{|B|}$. For our network depicted in Figure 1, see Definition 6 for a formalization, we assume the order $e_0 < \cdots < e_{s+1} < f_0 < \cdots < f_s$ for the edges. E.g., if $B \subseteq E$ contains $e_0$, then the corresponding symbol on that edge is the first coordinate of each vector $x \in A^{|B|}$.

Definition 2. Let $N = (V, E, S, T)$ be a network and $A$ an alphabet. A network code $F$ for $(N, A)$ is
a family of functions \( \{ F_V : V \in \mathcal{V}(\{S\} \cup T) \} \), where \( F_V : \mathcal{A}^{\text{in}(V)} \to \mathcal{A}^{\text{out}(V)} \) for all intermediate nodes \( V \).

**Definition 3.** An (outer) code for a network \( \mathcal{N} = (\mathcal{V}, \mathcal{E}, S, T) \) and an alphabet \( \mathcal{A} \) is a subset \( C \subseteq \mathcal{A}^{\text{out}(S)} \) with \( |C| \geq 1 \).

The elements of an outer code \( C \) are called codewords. For any two \( c, c' \in \mathcal{A}^{|B|} \), where \( B \subseteq \mathcal{E} \) is arbitrary, we define the Hamming distance \( d(c, c') \) as the number of differing coordinates, i.e., \( d(c, c') := \{ 1 \leq i \leq |B| : c_i \neq c'_i \} \). Note that the Hamming distance is a metric. For each subset \( C \subseteq \mathcal{A}^{\text{out}(S)} \) we write \( d(C) \) for the minimum Hamming distance \( d(c, c') \) for any pair of different elements \( c, c' \in C \), where we formally set \( d(C) := \infty \) if \( |C| \leq 1 \).

In this paper we consider networks that are affected by potentially restricted adversarial noise. More precisely, we assume that at most \( t \) of the alphabet symbols on a given edge set \( U \subseteq \mathcal{E} \) can be changed into any other alphabet symbols.

**Definition 4.** Let \( \mathcal{N} = (\mathcal{V}, \mathcal{E}, S, T) \) be a network, \( \mathcal{A} \) an alphabet, \( T \in \mathcal{T} \) a terminal, \( F \) a network code for \((\mathcal{N}, \mathcal{A})\), \( U \subseteq \mathcal{E} \) a subset of the edges, and \( t \geq 0 \) an integer. We denote by \( \Omega[\mathcal{N}, \mathcal{A}, F, S \rightarrow T, U, t] : \mathcal{A}^{\text{out}(S)} \to \mathcal{A}^{\text{in}(T)} \) the (adversarial) channel representing the transfer from \( S \) to terminal \( T \in \mathcal{T} \), when the network code \( F \) is used by the vertices and at most \( t \) edges in \( U \) are manipulated. In this context, we call \( t \) the adversarial power.

Let \( \Omega[\mathcal{N}, \mathcal{A}, F, S \rightarrow T, U, t] \) be an adversarial channel and \( C \subseteq \mathcal{A}^{\text{out}(S)} \) be an outer code for \( \mathcal{N} \). We say that \( C \) is good for \( \Omega \) if \( \Omega(c) \neq \Omega(c') \) for all codewords \( c, c' \in C \) with \( c \neq c' \). The (1-shot) capacity of \((\mathcal{N}, \mathcal{A}, U, t)\) is the largest real number \( \kappa \) for which there exists an outer code \( C \subseteq \mathcal{A}^{\text{out}(S)} \) and a network code \( F \) for \((\mathcal{N}, \mathcal{A})\) with \( \kappa = \log_{|\mathcal{A}|}(|C|) \) such that \( C \) is good for each adversarial channel \( \Omega[\mathcal{N}, \mathcal{A}, F, S \rightarrow T, U, t] \) for all \( T \in \mathcal{T} \). The notation for this largest \( \kappa \) is \( C_1(\mathcal{N}, \mathcal{A}, U, t) \).

In the general model of Kschischang and Ravagnani, see [15], an adversary can modify \( t \) edges from a given subset \( U \) of all edges. Here we restrict ourselves to the situation where \( U = \text{out}(S) \), i.e., the set of all outgoing edges of the unique source. Additionally, we assume that the adversary is omniscient, i.e., he can see all the sent symbols. With this, given a network \( \mathcal{N} = (\mathcal{V}, \mathcal{E}, S, T) \) and an alphabet \( \mathcal{A} \) we say that a pair of an outer code \( C \subseteq \mathcal{A}^{\text{out}(S)} \) and a network code \( F \) for \( \mathcal{N} \) is \( t \)-error correcting if \( C \) is good for each adversarial channel \( \Omega[\mathcal{N}, \mathcal{A}, F, S \rightarrow T, \text{out}(S), t] \) for all \( T \in \mathcal{T} \). We also say that \( C \) can be \( t \)-error corrected on \( \mathcal{N} \) if a network code \( F \) exists such that \( (C, F) \) is \( t \)-error correcting for \( \mathcal{N} \).

**Lemma 5.** Let \( \text{out}(S) \subseteq U \). If an outer code \( C \) can be \( t \)-error corrected for a network \( \mathcal{N} = (\mathcal{V}, \mathcal{E}, S, T) \) and an arbitrary alphabet \( \mathcal{A} \), then we have \( d(C) \geq 2t + 1 \).

**Proof.** Assume that \( d(C) \leq 2t \) and let \( c, c' \) be two codewords with \( d(c, c') \leq 2t \). Then there exists a vector \( x \in \mathcal{A}^{\text{out}(S)} \) with \( d(c, x) \leq t \) and \( d(x, c') \leq t \). So, for each of the codewords \( c \) and \( c' \) the adversary can modify the values of the edges in \( \text{out}(S) \subseteq U \) to \( x \), so that \( \Omega(c) = \Omega(c') \) while \( c \neq c' \) – contradiction.

If \( U \not\supseteq \text{out}(S) \) then Lemma 5 does not need to be valid in general.

In this paper we will consider a specific parameterized family of networks:
Assume correcting for a given outer code. The aim of this section is to obtain some structure results on network codes.

Proof. Let \( C_1 \) be the set of all possible manipulations of a codeword and the codewords themselves.

**Definition 6.** For each positive integer \( s \) the network \( N_s \) has one source \( S \), one terminal \( T \), and two intermediate nodes \( V_1, V_2 \). There is exactly one directed edge from \( S \) to \( V_1 \), one directed edge from \( V_1 \) to \( T \), \( s + 1 \) parallel edges from \( S \) to \( V_2 \), and \( s \) parallel edges from \( V_2 \) to \( T \), see Figure 1.

We remark that the notation \( B_s \) was used in [2] for \( N_s \) and that \( N_1 = B_1 \) is the original diamond network, see [3]. In \( N_s \) we label the edge from \( S \) to \( V_1 \) as \( e_0 \) and the \( s + 1 \) edges from \( S \) to \( V_2 \) by \( e_1, \ldots, e_{s+1} \). The unique edge from \( V_1 \) to \( T \) is denoted by \( f_0 \) and the \( s \) edges from \( V_2 \) to \( T \) are labeled by \( f_1, \ldots, f_s \), cf. Figure 1.

**Example 7.** For alphabet \( A = \{0,1,2\} \), \( t = 1 \), and \( s = 2 \) a code of size \( s \) for \( N_s \) is given by \( C = \{0000,0111,1012,1120,2021\} \). We easily determine \( d(C) = 3 \) matching the lower bound in Lemma 5.

Since we are interested in fixed alphabet sizes in the following we will directly consider the maximum cardinality \( \sigma(N_s,|A|) \) of an outer code \( C \) that can be 1-error corrected for \((N_s,A)\), i.e., \( \log_{|A|} \sigma(N_s,|A|) = C_1(N_s,A,\text{out}(S),1) \). In Example 13 we will continue Example 7 and show that the stated code can be 1-error corrected, i.e., we have \( \sigma(N_s,3) \geq 5 \).

As an abbreviation we set \( C := \{c' \in A^{s+2} \mid \exists c \in C : d(c,c') \leq 1\} \) (1) for the set of all possible manipulations of a codeword and the codewords themselves.

### 3 Structure of network codes for generalized diamond networks \( N_s \)

The aim of this section is to obtain some structure results on network codes \( F \) that are 1-error correcting for a given outer code \( C \) on the network \( N_s \) as well as on outer codes \( C \) that can be 1-error corrected. Lemma 5 gives the lower bound \( \log_{|A|} \sigma(N_s,|A|) = C_1(N_s,A,\text{out}(S),1) \). For each \( c \in A^{s+2} \) we write \( c = (c_1,c_2) \), where \( c_1 \in A \). Given our ordering of the edges, the symbol sent along edge \( e_0 \) (before the possible modification of the adversary) is \( c_1 \). As an abbreviation we set \( C_2 := \{c_2 \mid (c_1,c_2) \in C\} \).

Since \( C_2 \) is just a truncation of \( C \) we have \( d(C_2) \geq 2 \) in our 1-error correcting setting.

**Definition 8.** For each \( c_2 \in A^{s+1} \) we define

\[
\tau(c_2) := \{(a,c'_2) \in C \mid a \in A, d(c_2,c'_2) \leq 1\}.
\]

In words, if \( F_{V_2} \) receives the input \( c_2 \), then the originally sent codeword \( c' \in C \) has to be contained in \( \tau(c_2) \). Moreover, the adversary can ensure that \( F_{V_2} \) receives the input \( c_2 \) for all codewords \( c' \in \tau(c_2) \). Clearly, we have \( |\tau(c_2)| = 1 \) if \( c_2 \in C_2 \).

**Lemma 9.** Let \( C \) be a code that can be 1-error corrected for the network \( N_s \) and alphabet \( A \). For each \( c_2 \in A^{s+1} \) we have \( |\tau(c_2)| \leq \min\{|A|, s + 1\} \).

**Proof.** For \( |\tau(c_2)| \leq 1 \) the statement is true due to our assumptions \( s \geq 1 \) and \(|A| \geq 2\), so that we assume \( |\tau(c_2)| \geq 2 \) in the following. Since \( d(C_2) \geq 2 \) and \( d(c_2,c'_2) \leq 1 \) for each codeword \((c'_1,c'_2) \in C \) with \( c'_1 \in A \) we have \( d(c_2,c'_2) = 1 \) (using \( |\tau(c_2)| > 1 \)). I.e. \( c'_2 \) differs from \( c_2 \) in exactly one of the \( s + 1 \) coordinates. Since no two codewords can differ in the same coordinate due to \( d(C_2) \geq 2 \) we have \( |\tau(c_2)| \leq s + 1 \). Moreover, \( d(C) \geq 3 \) implies that all codewords in \( \tau(c_2) \) pairwise differ in their first coordinate, so that \( |\tau(c_2)| \leq |A| \). ☐
**Theorem 12.** Let \((C, F)\) be 1-error correcting for \(N_s\). For each pair of different elements \(c_2, c'_2 \in A^{s+1}\) with \(F V_2(c_2) = F V_2(c'_2)\) we have that \(\tau(c_2) \cup \tau(c'_2)\) does not contain two different elements with coinciding first coordinate.

**Proof.** Let \(c_2, c'_2 \in A^{s+1}\) with \(F V_2(c_2) = F V_2(c'_2)\). Assume that there exist \(a \in A, b, b' \in C_2\) with \(b \neq b'\), and \((a, b), (a, b') \in \tau(c_2) \cup \tau(c'_2)\). So, there exists an adversary channel \(\Omega\) that changes the codeword \(x := (a, b)\) to either \((a, c_2)\) or \((a, c'_2)\) and also the codeword \(x' := (a, b')\) to either \((a, c_2)\) or \((a, c'_2)\). Using \(F V_2(c_2) = F V_2(c'_2)\) we compute \(\Omega(x) = (F V_1(a), F V_2(c_2)) = \Omega(x')\) while \(x \neq x'\) – contradiction. \(\square\)

**Lemma 10.** Let \((C, F)\) be 1-error correcting for \(N_s\). Let \(c := (c_1, c_2) \in C\) be a codeword and \(c' := (c'_1, c'_2) \in \overline{C}\) with \(c_1, c'_1 \in A, c \neq c'\) and \(F V_2(c_2) = F V_2(c'_2)\). Then, we have \(\tau(c'_2) = \{c\}\).

**Proof.** Assume that there exists a codeword \(\tilde{c} := (\tilde{c}_1, \tilde{c}_2) \in \tau(c'_2)\) with \(\tilde{c} \neq c\) and \(\tilde{c}_1 \in A\). So, there exists an adversary channel \(\tilde{\Omega}\) that changes the codeword \(c\) to \((\tilde{c}_1, c_2)\) and the codeword \(\tilde{c}\) to \((\tilde{c}_1, c'_2)\). Using \(F V_2(c_2) = F V_2(c'_2)\) we compute \(\tilde{\Omega}(c) = (F V_1(\tilde{c}_1), F V_2(c_2)) = (F V_1(\tilde{c}_1), F V_2(c'_2)) = \tilde{\Omega}(\tilde{c})\) while \(c \neq \tilde{c}\) – contradiction. \(\square\)

Based on Lemma 10 and Lemma 11 we can state a characterization of all outer codes that can be 1-error corrected for \(N_s\):

**Theorem 12.** An outer code \(\emptyset \neq C \subseteq A^{s+2}\) can be 1-error corrected for \(N_s\) iff there exists a set \(B\) consisting of subsets of \(C\) of cardinality \(|B| \leq |A|^s - |C|\) such that the elements within each subset have pairwise different first coordinates and for each \(c_2 \in A^{s+1}\) with \(|\tau(c_2)| > 1\) there exists an element \(B \in B\) with \(\tau(c_2) \subseteq B\).

**Proof.** Let \((C, F)\) be 1-error correcting for \(N_s\) and define

\[ S_b = \bigcup_{c_2 \in A^{s+1}}: F V_2(c_2) = b \tau(c_2) \subseteq C \]

for all \(b \in A^s\). From Lemma 10 we conclude that the elements in \(S_b\) have pairwise different first coordinates for each \(b \in A^s\). For

\[ B := \{S_b : b \in A^s, |S_b| \geq 2\} \]

we have that for each \(c_2 \in A^{s+1}\) with \(|\tau(c_2)| > 1\) there exists an element \(B \in B\) with \(\tau(c_2) \subseteq B\). For each codeword \(c = (c_1, c_2)\) with \(c_1 \in A\) we have \(S_{F V_2(c_2)} = \{c\}\) due to Lemma 11. So using Lemma 10 at least \(|C|\) of the sets \(S_b\) have cardinality 1 and \(|B| \leq |A|^s - |C|\).

For the other direction let a set \(B\) satisfying the stated properties be given. We set \(B' := \{c \in C\} \) and \(B'' := B \cup B'\), so that \(|B''| \leq |A|^s\). Let \(\pi : B'' \rightarrow A^s\) be an arbitrary injection. With this we define the network code \(F\) for \(N_s\) as follows. We choose \(F V_1\) as the identity mapping, i.e., \(F V_1(x) = x\) for all \(x \in A\). For each \(c_2 \in A^{s+1}\) with \(|\tau(c_2)| > 1\) we set \(F V_2(c_2)\) to the lexicographically minimal element in \(\{\pi(B) : B \in B, \tau(c_2) \subseteq B\}\). If \(|\tau(c_2)| = 1\) we set \(F V_2(c_2)\) to \(\pi(B)\) for the unique element \(B \in \tau(c_2) \subseteq B'\) and to \(\pi(B)\) for an arbitrary element \(B \in B'\) if \(\tau(c_2) = \emptyset\). With this \(F\) is well defined. In order to show that \((C, F)\) is 1-error correcting for \(N_s\) we construct a “decoder” \(F_T : A^{[\text{in}(T)]} \rightarrow A^{[\text{out}(S)]}\) with \(F_T(\Omega(c)) = c\) for every codeword \(c \in C\) and every adversary channel \(\Omega\). For a given \(y := (y_1, y_2) \in A^{s+1}\) with \(y_1 \in A\) let \(B \in B''\) be the unique element with \(\pi(B) = y_2\) or \(B = \emptyset\) if no such element \(B\) exists. If \(|B| = 0\), then we set \(F_T(y)\) to an arbitrary element in
\(A^{s+2}\) since this case cannot occur if a codeword is sent across the network. If \(|B| = 1\) we set \(F_T(y)\) to the unique codeword contained in \(B\). If \(|B| \geq 2\), then the adversary has not modified the value on edge \(e_0\), so that we can set \(F_T(y)\) to the unique codeword in \(B\) whose first coordinate equals \(y_1 = F_{V_1}(y_1)\).

**Example 13.** As in Example 7 we consider the network \(N_2\), the alphabet \(A = \{0, 1, 2\}\), and the outer code \(C\) given by the codewords \(c_1 = (0, 0, 0, 0), \ c_2 = (0, 1, 1, 1), \ c_3 = (1, 0, 1, 2), \ c_4 = (1, 1, 2, 0), \) and \(c_5 = (2, 0, 2, 1)\). It can be easily checked that the different values of \(\tau(c'_2)\) for all \(c'_2 \in A^{s+1}\) are given by

\[
\{\emptyset, \{c_1\}, \{c_2\}, \{c_3\}, \{c_4\}, \{c_5\}, \{c_2, c_3, c_5\}, \{c_1, c_3\}, \{c_1, c_4, c_5\}, \{c_2, c_4, c_5\}\}.
\]

For e.g. \(c'_2 = (0, 1, 1)\) we have \(\tau(c'_2) = \{c_2, c_3, c_5\}\). For the set \(B\) in Theorem 12 we may choose

\[
B = \{\{c_2, c_3, c_5\}, \{c_1, c_3\}, \{c_1, c_4, c_5\}, \{c_2, c_4, c_5\}\}.
\]

Note that the elements of \(B\) are given by \(\tau(c'_2)\) with \(|\tau(c'_2)| > 1\) and no two elements of \(B\) can be joined since the elements have cardinality at most \(|A| = 3\). Thus, all \(3^2 = 9 = 5 + 4\) different output vectors of the intermediate node \(V_2\) have to be used. As injection \(\pi\) we may choose e.g. \(\pi(\{c_1\}) = (0, 0), \ \pi(\{c_2\}) = (0, 1), \ \pi(\{c_3\}) = (0, 2), \ \pi(\{c_4\}) = (1, 0), \ \pi(\{c_5\}) = (1, 1), \ \pi(\{c_2, c_3, c_5\}) = (1, 2), \ \pi(\{c_1, c_3\}) = (2, 0), \ \pi(\{c_1, c_4, c_5\}) = (2, 1), \) and \(\pi(\{c_2, c_4, c_5\}) = (2, 2)\). With this we e.g. have \(F_{V_2}(0, 1, 2)\) = \((0, 2)\) and \(F_{V_2}(0, 1, 1)\) = \((1, 2)\).

**Example 14.** Again we consider the network \(N_2\) and the alphabet \(A = \{0, 1, 2\}\). Now let the outer code \(C\) be given by the codewords \(c_1 = (0, 0, 0, 0), \ c_2 = (0, 1, 1, 1), \ c_3 = (0, 2, 2, 2), \ c_4 = (1, 0, 1, 2), \) and \(c_5 = (2, 0, 2, 1)\). Here the different values of \(\tau(c'_2)\) for all \(c'_2 \in A^{s+1}\) are given by

\[
\{\emptyset, \{c_1\}, \{c_2\}, \{c_3\}, \{c_4\}, \{c_5\}, \{c_1, c_4\}, \{c_1, c_5\}, \{c_2, c_4, c_5\}, \{c_3, c_4, c_5\}\}.
\]

The sets \(\{c_1, c_4\}\) and \(\{c_1, c_5\}\) can be joined to \(\{c_1, c_4, c_5\}\), so that we set

\[
B = \{\{c_1, c_4, c_5\}, \{c_2, c_4, c_5\}, \{c_3, c_4, c_5\}\}
\]

in Theorem 12 and only \(|C| + |B| = 8 < 3^2\) different output states are used in the corresponding function \(F_{V_2}\).

As a byproduct of the proof of Theorem 12 we see that it does not make a difference in terms of the capacity if the adversary can manipulate edge \(e_0\) or \(e_0\) and \(f_0\). We may even remove the intermediate vertex \(V_1\) and replace the two edges \(e_0\) and \(f_0\) by a direct edge between \(S\) and \(T\) that can be manipulated by the adversary. Going over the proofs of Lemmas 5 and 11 we see that the number of outgoing edges of \(V_2\) does not play a role. So, denoting the network where we replace the \(s\) outgoing edges of \(V_2\) to \(T\) by \(s'\) such edges by \(N_{s,s'}\) we can prove a variant of Theorem 12 where we replace the upper bound on the cardinality by \(|B| \leq |A|^{1′} - |C|\). Cf. the code of Example 14 in the situation where we replace the two outgoing edges of \(V_2\) in \(N_2\) by three outgoing edges, which however can only transmit symbols from the restricted alphabet \(\{0, 1\}\). i.e., a rather specific case of the more general situation where the allowed alphabets may differ on the different edges.

In other words, the conditions of Theorem 12 say that the elements of \(B\) cover all \(\tau(c_2)\) with \(c_2 \in A^{s+1}\) and \(|\tau(c_2)| > 1\). The underlying cover problem can be solved by a straightforward integer
linear programming (ILP) formulation, as it is usually the case for cover problems. This yields a first algorithm to determine the largest possible size $\sigma(\mathcal{N}_s, |A|)$ of a code that can be 1-error corrected for $\mathcal{N}_s$ and alphabet $A$ for any parameter $s \in \mathbb{N}_{\geq 1}$. Due to Lemma 5 we just have to loop over all codes with minimum Hamming distance at least 3 and use Theorem 12 to check whether $C$ can be 1-error corrected for $\mathcal{N}_s$ via an ILP computation.

We have applied this approach to the list of classified optimal binary one-error-correcting codes. These are denoted as $(n, |C|, 3)$ codes, where the length is given by $n = s + 2$ in our situation and $|C|$ is the maximum possible code size. For lengths 4, . . . , 11 the codes were classified in [13], for lengths 12, 13 in [12], and for lengths 14, 15 in [14]. The corresponding numbers are summarized in Table 1 and the codes can be downloaded at http://pottonen.kapsi.fi/codes. Since those enumerations only consider one-error correcting codes up to symmetry we have to loop over the coordinates and decide which one should be the one corresponding to $e_0$. Interestingly enough, for all $n \in \{3, \ldots, 15\} \setminus \{3, 7, 15\}$ all choices for $e_0$ and all choices for $C$ lead to a code that can be 1-error corrected for $\mathcal{N}_{n-2}$. Note that for $n \in \{3, 7, 15\}$ we deal with the parameters of the Hamming codes. Here no choice of $C$ or the coordinate of $e_0$ leads to a code that can be 1-error corrected for $\mathcal{N}_{n-2}$.

Thus, we have $\sigma(\mathcal{N}_1, 2) = 1$, $\sigma(\mathcal{N}_2, 2) = 2$, $\sigma(\mathcal{N}_3, 2) = 4$, $\sigma(\mathcal{N}_4, 2) = 8$, $\sigma(\mathcal{N}_5, 2) \leq 15$, $\sigma(\mathcal{N}_6, 2) = 20$, $\sigma(\mathcal{N}_7, 2) = 40$, $\sigma(\mathcal{N}_8, 2) = 72$, $\sigma(\mathcal{N}_9, 2) = 144$, $\sigma(\mathcal{N}_{10}, 2) = 256$, $\sigma(\mathcal{N}_{11}, 2) = 512$, $\sigma(\mathcal{N}_{12}, 2) = 1024$, and $\sigma(\mathcal{N}_{13}, 2) \leq 2047$.

In the next section we will determine $\sigma(\mathcal{N}_5, 2) = 14$ and $\sigma(\mathcal{N}_s, a)$ for other small parameters $s$ and $a \geq 3$. We remark that $\sigma(\mathcal{N}_1, a) = a - 1$ was determined in [1, 3].

## 4 Exact values and lower bounds for the maximum code sizes

As mentioned in the previous section, $\sigma(\mathcal{N}_1, a) = a - 1$ was determined in [1, 3]. Here an element $\tilde{a} \in A$ is chosen as a special symbol. With this a suitable code $C$ is a threefold repetition code of the symbols in $A \setminus \{\tilde{a}\}$. If $|\tau(c_2)| > 1$, then $V_2$ outputs $\tilde{a}$ with the corresponding set $B = C$. Here we will determine a few exact values and lower bounds for $\sigma(\mathcal{N}_s, a)$ for small parameters. Upper bounds are discussed in Section 5.

One of the most simple algorithmic approaches is to loop over all $|A|$-ary codes of length $s + 2$ and minimum Hamming distance at least three with increasing sizes. For each candidate code $C$ we can use Theorem 12 and a small ILP computation to decide whether $C$ can be 1-error corrected. For alphabet size $|A| = 3$ and $s = 2$ two codes that can be 1-error corrected and have cardinality 5 are given in Example 13 and Example 14. By exhaustive search we have verified that no 3-ary code with length 4, minimum Hamming distance at least 3, and size 6 can be 1-error corrected, so that $\sigma(\mathcal{N}_2, 3) = 5$.\footnote{Here by an optimal code we understand a code of the maximum possible size given block length, alphabet size, and minimum distance.}

| $n$ | 3 4 5 6 7 8 9 10 11 12 13 14 15 |
|-----|-------------------------------|
| max $|C|$ | 2248162040 72 144 256 512 10242048 |
| number | 12 11 1 5 15627398237610117832384085983 |

Table 1: Number of non-isomorphic optimal binary $(n, |C|, 3)$ codes.
In the following we will only state the codes $C$ since a suitable collection of sets $\mathcal{B}$ can be easily computed using Theorem 12 and a small ILP computation. For $|\mathcal{A}| = 3$ and $s=3$ a code of size 14 is given by

$$C = \{0000, 0111, 0022, 0101, 01120, 02201, 10021, 11102, 11211, 12010, 20210, 21001, 22022, 22100\}.$$ 

For $|\mathcal{A}| = 3$ and $s=4$ a code of size 35 is given by

$$C = \{00000, 212000, 102021, 122210, 012122, 120121, 202201, 211220, 110211, 121022, 001120, 202110, 002212, 020012, 021102, 011001, 111100, 200122, 200011, 121201, 022111, 102102, 010110, 211112, 000221, 101111, 220200, 110020, 201002, 010202, 210101, 022020, 221010, 112012, 222222\}.$$ 

By exhaustive search we have verified $\sigma(\mathcal{N}_3, 3) = 14$. For $|\mathcal{A}| = 4$ and $s=2$ a code of size 9 is given by

$$C = \{0000, 0111, 0222, 1012, 1103, 2021, 2130, 3201, 3310\}.$$ 

For $|\mathcal{A}| = 4$ and $s=3$ a code of size 31 is given by

$$C = \{00000, 32222, 21202, 31010, 31133, 22311, 20323, 01320, 03131, 30120, 11100, 00232, 12021, 23233, 11331, 32101, 22132, 13201, 23020, 02303, 33330, 01213, 10033, 02012, 21121, 33003, 12230, 13322\}.$$ 

**Example 15.** For $|\mathcal{A}| = 5$ and $s=2$ a code of size 15 is given by the codewords $c_0 = 0000, c_1 = 4341, c_2 = 2410, c_3 = 2143, c_4 = 3211, c_5 = 1421, c_6 = 2232, c_7 = 4104, c_8 = 3024, c_9 = 1244, c_{10} = 1012, c_{11} = 3302, c_{12} = 0433, c_{13} = 4220,$ and $c_{14} = 1330$. A set $\mathcal{B}$ for the construction of a suitable network code is given by

$$\begin{align*}
\{c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9\}, \{c_0, c_3, c_7, c_8, c_9\}, \{c_0, c_3, c_7, c_8, c_9\}, \{c_0, c_3, c_7, c_8, c_9\}, \{c_0, c_3, c_7, c_8, c_9\}, \\
\{c_0, c_3, c_7, c_8, c_9\}, \{c_0, c_3, c_7, c_8, c_9\}, \{c_0, c_3, c_7, c_8, c_9\}, \{c_0, c_3, c_7, c_8, c_9\}, \{c_0, c_3, c_7, c_8, c_9\}, \\
\{c_0, c_3, c_7, c_8, c_9\}, \{c_0, c_3, c_7, c_8, c_9\}, \{c_0, c_3, c_7, c_8, c_9\}, \{c_0, c_3, c_7, c_8, c_9\}, \{c_0, c_3, c_7, c_8, c_9\}.
\end{align*}$$

For $|\mathcal{A}| = 2$ we have that $|\tau(c_2)| > 1$ is equivalent to $|\tau(c_2)| = 2$ and the two codewords in $\tau(c_2)$ are at Hamming distance 2. Moreover, for each pair of codewords $a, b \in C_2$ with $d(a, b) = 2$ there are exactly two vectors $x \in \mathcal{A}^{s+1}$ with $d(a, x) = d(x, b) = 1$. This characterization allows to determine $\sigma(\mathcal{N}_s, 2)$ via the following ILP formulation. For each $v \in \{0, 1\}^{s+2}$ we introduce binary variables $x_v$ and binary variables $y_{v'}$ for each $v' \in \{0, 1\}^{s+1}$. The meaning of the $x_v$'s is given by $x_v = 1$ iff $v \in C$. The meaning of the $y_{v'}$'s is given by $y_{v'} = 1$ iff $|\tau(v')| > 1$. We minimize the code size $|C| = \sum_{v \in \{0, 1\}^{s+2}} x_v$ subject to the constraints

$$\sum_{v \in \{0, 1\}^{s+2}} x_v + \frac{1}{2} \cdot \sum_{v' \in \{0, 1\}^{s+1}} y_{v'} \leq 2^s \quad (6)$$

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modeling the condition from Theorem \[12\]

\[ y_m \geq \sum_{i \in A} x_{(i,a)} + \sum_{j \in A} x_{(j,b)} - 1 \]  \hspace{1cm} (7)

for all \(a, b \in \{0,1\}^{s+1}\) with \(d(a, b) = 2\) and all \(m \in \{0,1\}^{s+1}\) with \(d(a, m) = d(m, b) = 1\), which ensures that the pair of codewords \((i, a)\) and \((j, b)\) can only be taken if \(y_m = 1\), and

\[ \sum_{v \in \{0,1\}^{s+2} : v|S = u} x_v \leq 1 \]  \hspace{1cm} (8)

for all \(u \in \{0,1\}^s\) and all \(S \subseteq \{1, \ldots, s + 2\}\) with \(|S| = s\), which models \(d(C) \geq 3\), where \(v|S\) is the restriction of the vector \(v\) to the coordinates in \(S\).

For \(s = 5\) a code of size 14 is given by

\[ C = \{0001101, 0011110, 0100100, 0101011, 0110111, 0111000, 1000111, 1001000, 1010100, 1011011, 1100001, 1101110, 1110010, 1111101\} \]

and using the above ILP we could verify that there is no such code of size 15, so that \(\sigma(N_5, 2) = 14\). We remark that the bound \(|C| + |B| = |A|^s\) is met with equality in our example.

\section{Upper bounds for the maximum code sizes}

In \[2\] the two bounds \(\sigma(N_s, a) \leq a^s\) and \(\sigma(N_s, a) \geq a^{s-1}\), if \(a\) is sufficiently large, were shown. Both have an easy explanation. For the lower bound we may restrict to network codes where \(V_1\) just forwards and the terminal \(T\) ignores the output from \(V_1\), i.e., the entire decoding is actually performed in intermediate node \(V_2\). Let \(A_q(n,d)\) denote the maximum number of vectors in a \(q\)-ary code of word length \(n\) and with Hamming distance \(d\), so that \(\sigma(N_s, a) \geq A_a(s + 1, 3)\). Since for a fixed length and minimum distance MDS codes exist for all sufficiently large alphabet sizes \(a\) this implies \(\sigma(N_s, a) \geq a^{s-1}\). For other known lower bounds for \(A_q(n,d)\) we refer to e.g. \[3, 5, 7\] and especially the webpage \url{https://www.win.tue.nl/~aeb/} For upper bounds we can use Lemma \[5\] to conclude \(\sigma(N_s, a) \leq A_a(s + 2, 3)\), so that the Singleton bound gives \(\sigma(N_s, a) \leq a^s\) for all parameters \(s\) and \(a\). In Table \[3\] we summarize some known upper bounds for \(A_a(s + 2, 3)\) from the mentioned website.

| \(s\) | 2 | 3 | 4 | 5 | 6 | 2 | 3 | 4 | 5 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| \(a\) | 3 | 3 | 3 | 4 | 4 | 4 | 4 | 4 | 5 | 5 | 5 | 5 | 5 |
| \(\leq\) | \(9\) | \(18\) | \(38\) | \(111\) | \(333\) | \(16\) | \(64\) | \(176\) | \(596\) | \(25\) | \(125\) | \(625\) | \(2291\) |

Table 2: Upper bounds for \(A_a(s + 2, 3)\).

In \[2\] Remark 5.10 the “conjecture” \(\sigma(N_s, a) = \frac{a^s + a}{2} - 1\) was mentioned. For \((\mathcal{N}, a) \in \{(\mathcal{N}_1, *), (\mathcal{N}_2, 2), (\mathcal{N}_3, 2), (\mathcal{N}_4, 2), (\mathcal{N}_2, 3), (\mathcal{N}_3, 3), (\mathcal{N}_2, 4)\}\) code sizes \(\frac{a^s + a}{2} - 1\) can indeed be attained. However, in \[10\] the upper bound

\[ A_a(n, 3) \leq \frac{a^n \cdot (hn - x(a - x))}{(hn + x) \cdot (hn + x - a)}. \]

\footnote{Cf. \url{https://mathscinet.ams.org/mathscinet-getitem?mr=484738}}
where \( h = a - 1, n \equiv x \pmod{a} \), and \( 1 \leq x \leq a \), was concluded for \( a > 2 \) and sufficiently large \( n \) from Delsarte’s linear programming method. Thus, we have \( A_3(n, 3) < \frac{a^n}{(a-2)n} \) for \( a > 2 \) and sufficiently large \( n \), i.e., if the conjecture is true, then \( a \) has to be sufficiently large for a given parameter \( s \). For \( s = 4 \) and alphabet size \( 3 \) we have \( 35 \leq \sigma(N_4, 3) \leq A_3(6, 3) = 38 < 41 \) and also \( \sigma(N_s, 2) < \frac{2^n+2}{2} - 1 \) for all \( s \geq 5 \). We remark that the code of Example 15 shows \( \sigma(N_2, 5) \geq 15 > \frac{3^5+3}{2} - 1 \).

While Lemma 5 implies \( d(C_2) \geq 2 \) only, we can use the number of ordered pairs of codewords of \( C_2 \) at distance 2

\[
\Lambda := \left| \left\{ (a, b) \in C_2^2 : d(a, b) = 2 \right\} \right|
\]

(9)
to lower bound \( |B| \) in Theorem 12 and to derive the following necessary criterion:

**Lemma 16.** Let \( C \subseteq \mathcal{A}^{s+2} \) be a code that can be 1-error corrected for \( N_s \). Then, we have

\[
|C| + \frac{\Lambda}{|\mathcal{A}|(|\mathcal{A}| - 1)} \leq |\mathcal{A}|^s.
\]

**Proof.** Let \( B \) as in Theorem 12. If \( a, b \in C_2 \) are two codewords with \( d(a, b) = 2 \), then there exists a vector \( x \in \mathcal{A}^{s+1} \) with \( d(a, x) = d(x, b) = 1 \), so that \( a, b \in \tau(x) \). Thus, there exists a set \( B \in \mathcal{B} \) with \( \tau(x) \subseteq B \). Since \( |B| \leq |\mathcal{A}| \) at most \( |\mathcal{A}|(|\mathcal{A}| - 1) \) ordered pairs counted in \( \Lambda \) can yield the same set \( B \in \mathcal{B} \), so that \( |B| \geq \frac{\Lambda}{|\mathcal{A}|(|\mathcal{A}| - 1)} \) and the stated inequality follows from Theorem 12. \( \square \)

A first, rather weak, lower bound for \( \Lambda \) can be obtained easily. For network \( N_s \) the Singleton bound implies \( |C_2| \leq |\mathcal{A}|^{s-1} \) if \( d(C_2) \geq 3 \), so that \( C_2 \) contains at least one pair of codewords at Hamming distance 2. Iteratively removing a codeword in such a pair, we can conclude \( \Lambda \geq |C| - |\mathcal{A}|^{s-1} \) for \( |C| \geq |\mathcal{A}|^{s-1} \). In Subsection 5.1 we will utilize the linear programming method for codes to obtain lower bounds for \( \Lambda \) and state the corresponding upper bounds for \( \sigma(N_s, a) \) for small parameters.

**5.1 Upper bounds for the code sizes for \( N_s \) based on the linear programming method**

We can also use Delsarte’s linear programming method to derive lower bounds for \( \Lambda \) and then apply Lemma 16 to upper bound the code size \( |C| \). To this end we mention that for integers \( n \geq 1 \) and \( q \geq 2 \) the Krawtchouk polynomials are defined as

\[
K_i^{(n,q)}(z) := \sum_{j=0}^{i} (-1)^j(q-1)^{i-j}q^j \binom{n-j}{n-i} \binom{z}{j}
\]

(10)
for all \( i \geq 0 \), where \( \binom{z}{j} := z(z-1) \cdots (z-j+1)/j! \) for all \( z \in \mathbb{R} \). The vector \( B(C) = (B_0, B_1, \ldots, B_n) \), where

\[
B_i = \frac{1}{|C|} \cdot \left| \left\{ (a, b) \in C^2 : d(a, b) = i \right\} \right|, \ i = 0, 1, \ldots, n
\]

(11)
is called the distance distribution of \( C \). Clearly, \( B_0 = 1 \) and \( B_i = 0 \) for all \( 1 \leq i \leq d(C) - 1 \). Moreover, \( \Lambda = |C| \cdot B_2 \). The vector \( B'(C) = (B'_0, B'_1, \ldots, B'_n) \), where

\[
B'_i = \frac{1}{|C|} \sum_{j=0}^{n} B_j K_i^{(n,q)}(j), \ i = 0, 1, \ldots, n
\]

(12)
is called the dual distance distribution of \( C \). Obviously we have \( B'_0 = 1 \).
Theorem 17. \((9, 10)\) The dual distance distribution of \(C\) satisfies \(B'_i \geq 0\) for all \(0 \leq i \leq n\).

In Table 3 we have listed a few explicit upper bounds for \(\sigma(N_s, a)\), where \(2 \leq s \leq 5\) and \(3 \leq a \leq 5\), based on Lemma 16 and the linear programming method (using \(B_1 = 0\) and minimizing \(B_2\)). For more details we refer to the recent survey [6].

|   |   |   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|---|---|
| s | 2 | 3 | 4 | 5 | 2 | 3 | 4 | 5 |
| a | 3 | 3 | 3 | 3 | 4 | 4 | 4 | 4 |
|   | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |
|   | 6 | 15 | 42 | 108 | 137 | 133 | 484 | 17 |
|   | 76 | 337 | 1512 |

Table 3: Upper bounds for \(\sigma(N_s, a)\) based on the linear programming method.

We remark that the upper bounds in Table 3 seem to be better than those in Table 2 if \(s\) is rather small. E.g. the approach from this subsection implies \(\sigma(N_4, 3) \leq 42\), while we have \(\sigma(N_4, 3) \leq A_3(6, 3) = 38\). On the other hand we have \(\sigma(N_3, 3) \leq A_3(5, 3) = 18\) while Table 3 states \(\sigma(N_3, 3) \leq 15\) noting that we have determined \(\sigma(N_3, 3) = 14\) by exhaustive enumeration in Section 4. We give a summary of the best known bounds for small parameters in the subsequent conclusion.

6 Conclusion

In this paper we have considered the problem of determining the maximum size \(\sigma(N_s, a)\) of a code that can be 1-error corrected for the network \(N_s\) and a given alphabet of size \(a\). The considered family of networks is one of five infinite families of so-called 2-level networks studied in [2], more precisely family B. Bounds for such 2-level networks can be used to conclude upper bounds for more general networks as demonstrated in [2]. Besides that they form a starting point for first exact computations there is some hope that the obtained insights might be generalized. In this context we mention that determining the maximum size of a code that can be \(t\)-error corrected for the most simple “network” consisting just of a source \(S\), a terminal \(T\), and \(n\) parallel edges between \(S\) and \(T\) over an alphabet of size \(q\) corresponds to the determination of \(A_q(n, 2t + 1)\), i.e., the maximum number of vectors in a \(q\)-ary code of word length \(n\) and with Hamming distance \(2t + 1\), which triggered a lot of research and turned out to be a quite intricate problem that does not seem to admit an easy closed-form solution.

Our main result is a characterization result for outer codes that can be 1-error corrected for network \(N_s\) in Theorem 12. On the computational side the problem of deciding whether a given code \(C\) can be 1-error corrected (and the construction of an achieving network code) is reduced to a covering problem. Via exhaustive enumeration of codes we have determined a few exact values of \(\sigma(N_s, a)\) in Section 3 and Section 4. For the special case of a binary alphabet we also stated a direct integer linear programming formulation for \(\sigma(N_s, 2)\) which was used to compute \(\sigma(N_5, 2) = 14\). With respect to upper bounds for the code size \(|C|\) the easy observation \(d(C) \geq 3\) for the minimum Hamming distance yields \(\sigma(N_s, a) \leq A_3(s + 2, 3)\), so that known results from the literature can be applied. We e.g. remark that the “conjecture” \(\sigma(N_s, a) = \frac{a^2 + a}{2} + 1\) from [2, Remark 5.10] is wrong if one does not assume that the alphabet size \(a\) is sufficiently large. Also based on the characterization result in Theorem 12 we have utilized the number of codewords of Hamming distance 2 to upper bound the maximum possible code size, see Lemma 16 in Section 5. An application of the linear...
programming method for codes yields numerical improvements on $\sigma(\mathcal{N}_s, a) < a^s$, see Subsection 5.1. We summarize our numerical results for $\sigma(\mathcal{N}_s, a)$ and small parameters in Table 4.

|   | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|----|----|----|
| $s$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
| $a$ | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2   | 2   | 2   |
| $\sigma(\mathcal{N}_s, a)$ | 1 | 2 | 4 | 8 | 12| 20| 40| 72| 144| 256| 512| 1024|

Table 4: Best known bounds for $\sigma(\mathcal{N}_s, a)$ and small parameters.

We propose the determination of tighter bounds for $\sigma(\mathcal{N}_s, a)$ as an interesting open problem. In our opinion also the two infinite families $A$ and $E$ from [2] deserve a tailored study for fixed alphabet sizes. For family $E$ one can use a subset of a repetition code and a suitable network code to achieve code sizes that grow linearly in the alphabet size. For network $A_t$, where the adversary can manipulate $t$ of the outgoing edges of the source $S$, a similar result as Theorem 12 can be shown. Without going into details we just state that one can assume that intermediate vertex $V_1$ forwards the information from its $t$ incoming edges and for intermediate vertex $V_2$ we can define

$$\tau(c_2) := \left\{ c' \in C \mid \exists c_1 \in A^t; d(c', (c_1, c_2)) = 1 \right\}, \ldots,$$

$$\left\{ c' \in C \mid \exists c_1 \in A^t; d(c', (c_1, c_2)) = t \right\}$$

for all $c_2 \in A^{2t}$. The “analog” of $B$ would be a set whose elements consist of lists of length $t$ of subsets of the outer code $C$. The interpretation is that those lists contain candidates for the originally sent codeword assuming that a certain number of incoming edges of $V_2$ was attacked. Together with the information from $V_1$ a unique decoding should be possible. Of course everything gets more technical and we currently do not know for which classes of 2-level networks a similar approach might work or what a good generalization would be.

Acknowledgements

The author would like to thank the organizers of the Sixth Irsee Conference on Finite Geometries for their invitation. Further thanks go to Altan Kılıç who caught the authors interest by talks on the mentioned conference and the 25th International Symposium on Mathematical Theory of Networks and Systems (MTNS) 2022 as well as several discussions during the coffee breaks and afterwards. Last but not least the author thanks the two anonymous referees for their careful reading and their valuable comments that helped to improve the presentation of this paper.

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