Design of time-varying input-to-state stability tracking control Lyapunov functions for disturbance attenuation

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ABSTRACT
In this study, we address limitations of the existing methods of disturbance attenuation for nonlinear control systems. We focus on scenarios with trajectory tracking in differentially flat systems. In this context, we propose a framework of a time-varying input-to-state stability tracking control Lyapunov function (time-varying ISS-TCLF) and prove that a Sontag-type time-varying ISS-CLF based controller compensates for a predefined ISS-gain performance. As an example, we design a time-varying ISS-TCLF for trajectory tracking control of a vessel. The effectiveness of our proposed method is confirmed by computer simulation.

ARTICLE HISTORY
Received 29 October 2020
Accepted 4 May 2021

KEYWORDS
Input-to-state stability; disturbance rejection; trajectory tracking; ISS Lyapunov function; time-varying systems; uncertain systems; robust control

1. Introduction

Disturbance attenuation for nonlinear control systems is an important design criterion for trajectory tracking control systems, such as autopilots of automobiles, vessels and aircraft [1].

In this paper, we consider disturbance attenuation with internal stability, which works with various disturbance estimation methods [2]. Disturbance attenuation control problem can be formulated by using an evaluation criterion that $L_2$-gain is bounded from above by a constant peak value of the output/input $L_2$-norm [3,4]. The solution and attenuation level for linear systems are characterized by algebraic Riccati equations (AREs) that ensures a constant $H_\infty$-gain induced by $L_2$-norm [5]. On the contrary, the control design for nonlinear systems is a difficult problem. The difficulties are associated with solving the Hamilton–Jacobi–Issacs (HJI) partial differential equations associated with AREs for linear systems [6,7].

Mylvaganam and Sassano [8] have proposed an alternative approach in which solutions of PDEs were not required. The method relies on solutions of algebraic equations relating to $H_\infty$ control. The result is based on the introduction of dynamic extension variables that enable mapping a nonlinear system to a linearized system defined on an augmented space. Then, the problem of local disturbance attenuation with internal stability is solved by dynamic output feedback.

However, disturbance attenuation for trajectory tracking control has not yet been proposed, unlike the case of local stabilization of a stationary equilibrium.

Differentially flat systems are a subclass of nonlinear systems that can address trajectory tracking problems [9,10]. Ikeda et al. [11] proposed a time-varying trajectory tracking control Lyapunov function (TCLF) design method for differentially flat systems. Furthermore, Fujii et al. [12] proposed an input-to-state stability control Lyapunov function (ISS-CLF) design method for stabilization control. However, input-to-state stability for a trajectory tracking control system has not been defined.

In this study, we define a time-varying ISS-TCLF for differentially flat systems. Moreover, we design a time-varying ISS-TCLF gain to track trajectory of a vessel with a circular reference trajectory.

The conference version of this paper [13] omitted the proof of Theorem 4.2. As a new theoretical result, we provide the detailed proof of Theorem 4.2 based on newly provided Lemmas 4.1 and 4.3. Furthermore, we show computer simulation results with different disturbances not presented in the conference paper.

The remainder of this paper is structured as follows. Section 2 introduces basic definitions and properties. Sections 3–5 present the problem statement, main theoretical results, and controller design. Section 6 describes the conditions and results of computer simulations. Finally, Section 7 concludes this paper.

2. Preliminaries

This study investigates disturbance attenuation control of differentially flat systems. First, we introduce basic definitions and properties used in this study.
2.1. Nonlinear control system

We consider the following nonlinear control system:

\[ \dot{x} = f(x) + g(x)u, \quad y = \xi(x), \]

where \( x \in \mathbb{R}^n \), \( u \in \mathbb{R}^m \) and \( y \in \mathbb{R}^p \) are a state, input, and output, respectively. Then, we can obtain the following continuous and mapping \( \mathbb{R} \rightarrow \mathbb{R} \):

\[ z = \xi(x), \]

\[ y = \xi(x), \]

Differentially flat systems have been defined strictly by Murray et al. [9,14], Fliess et al. [15], and Levine [10].

Hence, we define the differentially flat system as follows:

Definition 2.1 (Differentially Flat System [10]): System (1) is said to be a differentially flat system if there exist input (3)–(4) and coordinates transformation (6) such that system (5) can be transformed into the following linear system:

\[ \begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} f(x) + g(x)b(x, z, v) \\ a(x, z, v) \end{bmatrix}, \]

where \((x, z) = (0, 0)\) is origin. In this paper, we consider the following diffeomorphism \( \mathbb{E} : \mathbb{R}^{n+l} \rightarrow \mathbb{R}^{n+l} \):

\[ \xi = \mathbb{E}(x, z), \quad \mathbb{E}(0, 0) = 0. \]

2.2. Linear augmented system for nonlinear control system

Consider the system represented in augmented space \( \mathbb{R}^{n+l} \). We introduce the following dynamic compensator and dynamic state feedback:

\[ \begin{aligned} \dot{z} &= a(x, z, v), \\ u &= b(x, z, v), \end{aligned} \]

where \( z \in \mathbb{R}^l \) and \( v \in \mathbb{R}^m \) are a virtual state and a virtual input, respectively. Then, we can obtain the following augmented system:

\[ \begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} f(x) + g(x)b(x, z, v) \\ a(x, z, v) \end{bmatrix}, \]

where \((x, z) = (0, 0)\) is origin. In this paper, we consider the following diffeomorphism \( \mathbb{E} : \mathbb{R}^{n+l} \rightarrow \mathbb{R}^{n+l} \):

\[ \xi = \mathbb{E}(x, z), \quad \mathbb{E}(0, 0) = 0. \]

2.3. Differentially flat systems

Differentially flat systems have been defined strictly by Murray et al. [9,14], Fliess et al. [15], and Levine [10]. An important advantage of the differentially flat system is characterized by transformation to linear systems. Hence, we define the differentially flat system as follows according to [10, Corollary 6.2].

Definition 2.1 (Differentially Flat System [10]): System (1) is said to be a differentially flat system if there exist input (3)–(4) and coordinates transformation (6) such that system (5) can be transformed into the following linear system:

\[ \begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A \xi + Bv, \end{bmatrix} \]

where \( A \in \mathbb{R}^{(n+l)\times(n+l)} \) and \( B \in \mathbb{R}^{(n+l)\times m} \) are constant matrices.

2.4. Control Lyapunov function

Definition 2.2 (Control Lyapunov Function (CLF)): Consider a nonlinear control system (1). Then, function \( V(x) : \mathbb{R}^n \rightarrow \mathbb{R} \) is said to be a control Lyapunov function for system (1) if the following three conditions hold.

(B1) \( V(x) \) is a proper function: \( \{ x \in \mathbb{R}^n | V(x) \leq L \} (L > 0) \) is compact.

(B2) \( V(x) \) is a finite function: \( V(0) = 0 \), and \( V(x) > 0 \) for each \( x \in \mathbb{R}^n \) \( \backslash \{0\} \).

(B3) \( V(x) \) satisfies the following inequality:

\[ \inf_{u \in \mathbb{R}^m} \{ LfV + LgV \cdot u \} < 0, \quad \forall x \in \mathbb{R}^n \backslash \{0\}, \]

where \( LfV = (\partial V/\partial x)f(x) \), \( LgV = (\partial V/\partial x)g(x) \), \( LfV \) and \( LgV \) are Lie derivatives.

2.5. Minimum projection method

We use the following theorem provided by Yamazaki et al. [16].

Theorem 2.3 (Minimum projection method [16]): Let \( \tilde{V}(x, z) \) be a dynamic CLF that includes a virtual state \( z \) for system (5). Then, the following function \( V(x) \) is a static CLF excluding virtual state \( z \) for the original nonlinear control system (1):

\[ V(x) = \min_{z \in \mathbb{R}^l} \tilde{V}(x, z). \]

2.6. Design of reference state and reference input

For a differentially flat system, reference state \( x_r(t) \) and reference input \( u_r(t) \) can be derived from reference virtual state \( \xi_r(t) \). In [11], \( \xi_r(t) \) is calculated as follows:

\[ \xi_r(t) = [y_{r1}(t), \ldots, y_{r1}^{(\mu_1)}(t), \ldots, y_{rm}(t), \ldots, y_{rm}^{(\mu_m)}(t)]^T. \]

Then, we design a reference state \( x_r(t) \), reference virtual state \( z_r(t) \), and reference input \( u_r(t) \) by diffeomorphisms \( \mathbb{E} \) and \( \Phi_2 \) as follows:

\[ \begin{bmatrix} x_r(t) \\ z_r(t) \end{bmatrix} = \mathbb{E}^{-1}(\xi_r(t)), \]

\[ \begin{aligned} x_r &= \Phi_1 \left( y_{r1}(t), \ldots, y_{r1}^{(\mu_1)}(t), \ldots, y_{rm}(t), \ldots, y_{rm}^{(\mu_m)}(t) \right), \\ y_{r1}(t), \ldots, & \ldots, y_{rm}(t), \ldots, y_{rm}^{(\mu_m)}(t) \end{aligned} \]

\[ \begin{aligned} u_r &= \Phi_2 \left( y_{r1}(t), \ldots, y_{r1}^{(\mu_1+1)}(t), \ldots, y_{rm}(t), \ldots, y_{rm}^{(\mu_m+1)}(t) \right). \]

2.7. Nonlinear error system with disturbance

We consider the following disturbed nonlinear control system and the reference system:

\[ \dot{x} = f(x) + g(x)u + h(x)w, \]

\[ \dot{x}_r(t) = f(x_r(t)) + g(x_r(t))u_r(t), \]
where \( t \in \mathbb{R} \) is time, \( x_r : \mathbb{R} \to \mathbb{R}^n \) is a reference state, and \( u_r : \mathbb{R} \to \mathbb{R}^m \) is a reference input. Let \( e = x - x_r \) be an error. Thus, we obtain the following error system:

\[
\dot{e} = \tilde{f}(e, t) + \tilde{g}(e, t)\tilde{u} + \tilde{h}(e, t)w, \quad (16)
\]

\[
y_e = \tilde{\zeta}(e), \quad (17)
\]

where \( w \in \mathbb{R}^r \) is a bounded disturbance, \( \tilde{f}(0, t) \equiv 0 \), and mapping \( \tilde{h} : \mathbb{R}^{n+1} \to \mathbb{R}^{n \times r} \) is locally Lipschitz continuous with respect to \( e \in \mathbb{R}^n, y_e \in \mathbb{R}^p \) is an output, and \( \tilde{\zeta} : \mathbb{R}^n \to \mathbb{R}^p \) is a function. Note that functions \( \tilde{f} \) and \( \tilde{g} \) are represented as follows:

\[
\tilde{f}(e, t) = f(e + x_r(t)) - f(x_r(t)) + \left[ g(e + x_r(t)) - g(x_r(t)) \right] u_r(t), \quad (18)
\]

\[
\tilde{g}(e, t) = g(e + x_r(t)), \quad (19)
\]

\[
\tilde{u} = u - u_r(t), \quad (20)
\]

\[
\tilde{h}(e, t) = h(e + x_r(t)). \quad (21)
\]

Substituting \( w = 0 \in \mathbb{R}^r \), nonlinear error system (16) is represented as follows:

\[
\dot{e} = \tilde{f}(e, t) + \tilde{g}(e, t)\tilde{u}. \quad (22)
\]

### 2.8. Tracking control Lyapunov function

We define a tracking control Lyapunov function as follows.

**Definition 2.4 (Tracking Control Lyapunov Function (TCLF) [11]):** Consider a nonlinear error control system (22). Then, function \( V(e, t) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \) is said to be a tracking control Lyapunov function for system (22) if the following conditions hold.

\((C1)\) There exist two proper positive-definite functions \( V(e) \) and \( \overline{V}(e) \) such that \( \overline{V}(e) \leq V(e, t) \leq V(e) \).

\((C2)\) There exists a positive-definite function \( S(e) \) such that:

\[
\inf_{u \in \mathbb{R}^m} \left( \frac{\partial V}{\partial t} + L_f V + L_g V \cdot \tilde{u} \right) \\
< -S(e), \ \forall e \in \mathbb{R}^n \setminus \{0\}, \quad (23)
\]

where \( L_f V = (\partial V/\partial e)\tilde{f}(e, t), \ L_g V = (\partial V/\partial e)\tilde{g}(e, t), \ L_f V \) and \( L_g V \) are Lie derivatives.

**Theorem 2.5 (Design of TCLF for Differentially Flat System [11]):** Consider system (22), (17), and TCLF \( \bar{V}(e, \tilde{z}, t) : \mathbb{R}^{n+l} \to \mathbb{R} \) including virtual states. Then, the following function \( V(e, t) \) is the TCLF without virtual states for system (22):

\[
V(e, t) = \min_{\tilde{z}} \bar{V}(e, \tilde{z}, t) \quad (24)
\]

2.9. Input-to-State stability

We define \( L_2 \) norm and input-to-state stability as follows.

**Definition 2.6 (\( L_2 \) norm):** Consider function \( u : [0, \infty) \to \mathbb{R}^m \) defined on \([0, \infty)\). \( L_2 \) space is defined as \( L_2 := \{u|\int_0^\infty \|u(t)\|^2 dt < +\infty\} \). For each element \( u(t) \) of \( L_2 \) spaces, the following norm is called \( L_2 \) norm:

\[
\|u\|_2 = \sqrt{\int_0^\infty \|u(t)\|^2 dt}. \quad (27)
\]

**Definition 2.7 (Input-to-state stability (ISS) [17,18]):** System (14) is said to be input-to-state stable (ISS) if there exists a controller that guarantees the condition

\[
\|x(t)\| \leq \beta (\|x(0)\|, t) + \chi \left( \sup_{0 \leq r \leq t} \|w(r)\| \right), \quad (28)
\]

where \( \beta \in \mathcal{K}\mathcal{L}, \ \chi \in \mathcal{K} \). Note that we refer to [19] for comparison functions \( \mathcal{K}, \mathcal{K}\mathcal{L}, \mathcal{K}\mathcal{C} \).

**Remark 2.8:** An ISS-gain \( \chi \) is one of the upper bound for (28), and it is not unique.

**Definition 2.9 (ISS-Lyapunov Function [1,20]):** Consider a nonlinear system:

\[
\dot{x} = f(x, u), \quad (29)
\]

where mapping \( f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \) is supposed to be locally Lipschitz on \( \mathbb{R}^n \times \mathbb{R}^m \) and satisfies \( f(0, 0) = 0 \). Moreover, we suppose that external input \( u \in \mathbb{R}^m \) is a function that includes \( \mathcal{L}^1_2 \) disturbances.

A \( C^1 \) function \( V : \mathbb{R}^2 \to \mathbb{R} \) is said to be an ISS-Lyapunov function for system (29) if there exist a smooth positive-definite radially unbounded function \( V(x) \) and functions \( \nu \in \mathcal{K}_{\infty} \) and \( \mu \in \mathcal{K} \) such that the following dissipation inequality holds:

\[
\dot{V}(x) \leq -\mu(\|x\|) + \nu(\|u\|). \quad (30)
\]

### 2.10. Sontag type disturbance attenuation controller

**Proposition 2.10 (Disturbance Attenuation Controller via ISS-TCLF [7]):** Let a function \( V(e, t) \) be a
time-varying ISS-TCLF for system (16). Then, the following controller input-to-state stabilizes origin $e = 0$ of system (16).

$$\ddot{u} = \alpha(e, t)$$

$$\alpha(e, t) = \begin{cases} 
- \left( \frac{\sqrt{\alpha + \|\dot{L}_g V\|}}{\|L_g V\|} \right) (L_g V)^T (L_g V \neq 0) \\
0 (L_g V = 0) 
\end{cases}$$

(31)

Function $\omega(e, t) : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is represented as follows:

$$\omega(e, t) = \frac{\partial V}{\partial t} + L_g V + \epsilon \left| L_h V \right| \rho^{-1} \rho \in \mathcal{K}_\infty,$$

(32)

where $\epsilon \in (0, 1)$ is a switch parameter to describe ON/OFF of disturbance attenuation control. The controller at $\epsilon = 0$ is an original Sontag-type controller.

3. Problem statement

We have introduced disturbed error systems of differentially flat systems in Section 2; however, a framework of a time-varying ISS-TCLF has never been discussed. We define a time-varying ISS-TCLF as follows.

**Definition 3.1 (Time-varying ISS-TCLF):** Function $V(e, t)$ is said to be a time-varying input-to-state stability control Lyapunov function (ISS-TCLF) if the following conditions hold for all $t \in \mathbb{R}$.

1. There exist two proper positive-definite functions $V(e)$ and $\overline{V}(e)$ such that $V(e) \leq \overline{V}(e)$.
2. There exists a class $\mathcal{K}_\infty$ function $\rho : \mathbb{R} \to \mathbb{R}$, a positive-definite function $\tilde{S}(e)$, and $V(e, t)$ that satisfies the following implication for all $e \neq 0$ and all $w \in \mathbb{R}^2$:

$$\|e(t)\| \geq \rho(\|w\|), \quad \rho \in \mathcal{K}_\infty$$

$$\implies \inf_{w \in \mathbb{R}^m} \left\{ \frac{\partial V}{\partial t} + L_g V + L_g V u + L_h V w \right\}$$

$$\leq -\tilde{S}(e).$$

(34)

The above definition satisfies the general definition of a time-invariant ISS-TCLF with $\partial V/\partial t = 0$ and $x_i(t) \equiv 0$.

A disturbed time-varying differentially flat system (16) formulated as a tracking error system is transformed into a linearized system as follows.

**Definition 3.2 (Disturbed Time-Varying Differentially Flat System):** We introduce the following dynamic compensator (35), dynamic state feedback (36), and diffeomorphism (37) $\Xi : \mathbb{R}^{n+l+1} \to \mathbb{R}^{n+l+1}$:

$$\tilde{z} = a(e, \tilde{z}, \tilde{v}, t)$$

$$\ddot{u} = b(e, \tilde{z}, \tilde{v}, t),$$

$$\Xi(e, \tilde{z}, t) = (\xi_e, t),$$

(36)

(37)

where $\tilde{z} = z - z_r$ and $\tilde{v} = v - v_r$. We consider a graph space $G \subset \mathbb{R}^{n+l+1}$ such that $(\xi_e, t) \in G$. Let $\Xi_e(e, \tilde{z}, t) = \{\xi_e | (\xi_e, t) \in G\}$. If we can transform system (16) into the following disturbed linear system, system (16) is called a disturbed time-varying differentially flat system:

$$\dot{\xi}_e = A\xi_e + Bv + Hw,$$

$$y_e = C\xi_e,$$

(38)

(39)

where $A \in \mathbb{R}^{(n+l) \times (n+l)}$, $B \in \mathbb{R}^{(n+l) \times m}$, $H \in \mathbb{R}^{(n+l) \times r}$, and $C \in \mathbb{R}^{(n+l) \times (n+l)}$. We suppose that system (38) is controllable and observable. Besides, we can obtain the following augmented system by dynamic compensator (35), dynamic state feedback (36), and diffeomorphism (37):

$$\begin{bmatrix} \tilde{e} \\ \tilde{z} \end{bmatrix} = \begin{bmatrix} \tilde{f}(e, t) + \tilde{g}(e, \tilde{z}, \tilde{v}, t) + \tilde{h}(e)w \\ \alpha(e, \tilde{z}, \tilde{v}, t) \end{bmatrix}.$$}

(40)

**Assumption 3.3:** A mapping $\Xi_e$ can be written as:

$$\Xi_e(e, \tilde{z}, t) = \alpha(e, t)\tilde{z} + \beta(e, t),$$

(41)

where $\alpha : \mathbb{R}^{n+1} \to \mathbb{R}^{(n+l) \times l}$ and $\beta : \mathbb{R}^{n+1} \to \mathbb{R}^{n+l}$ are smooth mappings, and $\alpha(e, t)$ is a full column rank. If $e = 0$ and $\tilde{z} = 0$, then $\Xi_e = 0$. Moreover, for all $t$ and $e$, there exist upper norms of $\alpha$ and $\beta$ satisfying the following inequalities:

$$\|\alpha(e, t)\| \geq \|\alpha(e)\| > 0,$$

$$\|\beta(e, t)\| \geq \|\beta(e)\| > 0,$$

(42)

(43)

where $\alpha : \mathbb{R}^{n \to \mathbb{R}^{(n+l) \times l}}$ and $\beta : \mathbb{R}^{n \to \mathbb{R}^{n+l}}$ are smooth mappings, and $\alpha(e)$ is a full column rank.

**Assumption 3.4:** There exists a matrix $H \in \mathbb{R}^{(n+l) \times r}$ in (38) that satisfies the following equation:

$$H = \frac{\partial \Xi_e}{\partial e} (e, \tilde{z}, t) \tilde{h}(e, t).$$

(44)

**Remark 3.5:** Assumption 3.3 implies that mapping $\Xi_e$ is regular, and an ISS-TCLF $V$ is radially unbounded at each time $t$. The system satisfying Assumption 3.4 requires that disturbance $w$ can affect finite-order derivatives of flat output $y$.

In this study, we design an ISS-TCLF and disturbance attenuation tracking controller that excludes the virtual state from system (16) and satisfies Definition 3.1, Definition 3.2, Assumption 3.3, and Assumption 3.4.
4. Main results

In this study, we design a Sontag-type disturbance attenuation controller via time-varying ISS-TCLF for a disturbed time-varying differentially flat system. We propose a time-varying ISS-TCLF design method for a disturbed differentially flat system in this section.

4.1. Design of time-varying ISS-TCLF

We can easily design an ISS-TCLF for system (16). We apply the minimum projection method and $H_\infty$ optimal control theory to design an ISS-TCLF $\tilde{V}$ for linearized system (38)–(39).

Lemma 4.1 (ISS-TCLF design for linear system and augmented system): Consider system (38)–(39). Let $P$ be a positive-definite symmetric solution of the following Riccati equation:

$$A^TP + PA + P\left(\frac{1}{\gamma^2}HH^T - BR^{-1}B^T\right)P = Q,$$

where $\gamma > 0$ is constant, and $Q := C^TC \in \mathbb{R}^{(n+1) \times (n+1)}$ and $R \in \mathbb{R}^{m \times m}$ are positive symmetric matrices.

Then, the following function $\tilde{V}(\xi_e) : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ is an ISS-TCLF for the linearized system:

$$\tilde{V}(\xi_e) = \xi_e^TP\xi_e.$$  

Moreover, with a mapping $\Xi_e$ and a diffeomorphism $\Xi, \tilde{V}(e, \tilde{z}, t) : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ is an ISS-TCLF, including a virtual state for the augmented system:

$$\tilde{V}(e, \tilde{z}, t) = (\Xi_e)^TP\Xi_e.$$  

Note that $\gamma$ and $Q$ can be considered as design parameters.

Proof: The Riccati equation satisfies the following condition:

$$A^TP + PA + P\left(\frac{1}{\gamma^2}HH^T - BR^{-1}B^T\right)P = -\tilde{Q},$$

where a positive matrix $\tilde{Q}$ is represented as follows:

$$\tilde{Q} = Q + \frac{1}{\gamma^2}PHH^TP.$$  

Then, according to [11], the following function $\tilde{V}(\xi_e) : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ is a TCLF for the linearized system (38):

$$\tilde{V}(\xi_e) = \xi_e^TP\xi_e.$$  

The time derivative for the linearized system along the solution is represented as follows:

$$\dot{\tilde{V}}(\xi_e) \leq -\xi_e^TQ\xi_e - \frac{1}{\gamma^2}\|HT^TP\xi_e\|^2 + \|R^{-1}\|\|B^TP\xi_e\|^2 + 2\xi_e^TPBv + 2\xi_e^TPHw,$$

where we applied (45). For any state $\xi_e$ and any disturbance $w$, the following inputs input-to-state stabilizes the origin of linearized system (38):

$$v = -\frac{1}{2}R^{-1}B^TP\xi_e.$$  

Therefore, Equation (51) is represented as follows:

$$\dot{\tilde{V}}(\xi_e) \leq -\xi_e^TQ\xi_e - \|yw - \frac{1}{\gamma}HT^TP\xi_e\|^2 + \gamma^2\|w\|^2,$$

$$\leq -\lambda_{\text{min}}(Q)\|\xi_e\|^2 + \gamma^2\|w\|^2,$$

where we use the following inequalities:

$$\|yw - \frac{1}{\gamma}HT^TP\xi_e\|^2 = \gamma^2\|w\|^2 - 2\xi_e^TPHw + \frac{1}{\gamma^2}\|HT^TP\xi_e\|^2 \geq 0,$$

$$-\lambda_{\text{max}}(Q)\|\xi_e\|^2 \leq -\xi_e^TQ\xi_e \leq -\lambda_{\text{min}}(Q)\|\xi_e\|^2.$$  

Here, $\lambda_{\text{min}}(Q)$ is a minimum eigenvalue of matrix $Q$ of (45). Note that inequality (55) satisfies (30).

Definition 3.1 implies that we can choose a function $\rho \in K_{\infty}$ of (34). In fact, for $\epsilon$ such that $\lambda_{\text{min}} > \epsilon > 0$, the following inequality holds:

$$\|\xi_e\| \geq \sqrt{\frac{\gamma^2\|w\|^2}{\lambda_{\text{min}}(Q) - \epsilon}} = \rho(\|w\|)$$

$$\Rightarrow \dot{\tilde{V}}(\xi_e) \leq -\epsilon\|\xi_e\|^2.$$  

(58)

where (58) is obtained by substituting the antecedent of (58) into (55).

By the inequality (55) and the antecedent of (58), $\tilde{V}(\xi_e)$ is calculated as follows:

$$\dot{\tilde{V}}(\xi_e) \leq -\lambda_{\text{min}}(Q)\left(\frac{\gamma^2\|w\|^2}{\lambda_{\text{min}}(Q) - \epsilon}\right) + \gamma^2\|w\|^2$$

$$\leq -\left(\frac{\epsilon}{\lambda_{\text{min}}(Q) - \epsilon}\right)\cdot\gamma^2\|w\|^2.$$  

(60)

Therefore, $\tilde{V}(\xi_e)$ is an ISS-TCLF for linearized system.

Moreover, with a mapping $\Xi_e$ and a diffeomorphism $\Xi, \tilde{V}(e, \tilde{z}, t) : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ for the augmented system is represented as follows:

$$\tilde{V}(e, \tilde{z}, t) = (\Xi_e)^TP\Xi_e(e, \tilde{z}, t).$$  

(61)
Similarly, for any time \( t \in \mathbb{R} \) and \( \tilde{z} \in \mathbb{R}^l \), there exists constant \( \varepsilon \) such that the following implication holds:

\[
\| \mathcal{E}_e(e, \tilde{z}, t) \| \geq \sqrt{\frac{\nu^2}{\lambda_{\min}(Q)} - \varepsilon} = \rho(\|w\|) \\
\Rightarrow \dot{V}(e, \tilde{z}, t) \leq -\tilde{S}(\mathcal{E}_e(e, \tilde{z}, t)),
\]

where we applied the following implication by Assumption 3.3:

\[
\tilde{S}(\mathcal{E}_e(e, \tilde{z}, t)) = \varepsilon \cdot \mathcal{E}_e(e, \tilde{z}, t)^T \| \mathcal{E}_e(e, \tilde{z}, t) \|^2 \\
\geq \varepsilon \cdot \left( \| g(e) \tilde{z} \|^2 + \| \beta(e) \|^2 \right) > 0.
\]

Therefore, \( \dot{V}(e, \tilde{z}, t) \) is an ISS-TCLF for the augmented system.

**Theorem 4.2:** We consider a disturbed linearized system (38)–(39) and a disturbed differentially flat system (16) rewritten by an augmented system (40). If there exists an ISS-TCLF \( \dot{V}(\xi_e) : \mathbb{R}^{n+1} \to \mathbb{R} \) for linearized system, the following function \( V(e, t) : \mathbb{R}^{n+1} \times \mathbb{R} \to \mathbb{R} \) is an ISS-TCLF for the differentially flat system (16):

\[
V(e, t) := \tilde{V}(e, \tilde{z}_0, t), \quad \tilde{z}_0 \in \arg\min_{\tilde{z}} \tilde{V}(e, \tilde{z}, t).
\]

**Proof:** By the definition of TCLF \( \dot{V}(e, t) \), for \( \tilde{V}(e, \tilde{z}(e, t), t), \tilde{z} \in \mathbb{R}^l \), the following equation and inequality hold:

\[
\dot{V}(e, t) \leq \dot{\tilde{V}}(e, \tilde{z}_0, t), \quad \tilde{z}_0 \in \arg\min_{\tilde{z}} \tilde{V}(e, \tilde{z}, t).
\]

Then, for any time \( t \in \mathbb{R} \) and constant \( \varepsilon > 0 \) such that \( \lambda_{\min} > \varepsilon > 0 \), the following implication holds:

\[
\| \mathcal{E}_e(e, \tilde{z}_0, t) \| \geq \sqrt{\frac{\nu^2}{\lambda_{\min}(Q)} - \varepsilon} = \rho(\|w\|) \\
\Rightarrow \dot{V}(e, t) \leq \dot{\tilde{V}}(e, \tilde{z}_0, t) \\
\leq \dot{\tilde{V}}(\xi_e) \\
\leq -\varepsilon \cdot \mathcal{E}_e(e, \tilde{z}, t)^T \| \mathcal{E}_e(e, \tilde{z}, t) \|^2 < 0.
\]

Therefore, \( V(e, t) \) is an ISS-TCLF for the differentially flat system.

We can use the following Sontag-type controller to input-state-stabilize the origin of time-varying error system (16).

**Lemma 4.3 (Disturbance attenuation controller via time-varying ISS-TCLF):** Let a function \( V(e, t) \) be a time-varying ISS-TCLF for system (16). Then, according to Proposition 2.10, disturbance attenuation controller (32) with \( \varepsilon = 1 \) input-to-state stabilizes the origin \( e = 0 \) of system (16).

**5. Controller design**

We consider modelling of vessel trajectory tracking control in time-varying differentially flat systems. Then, we design a time-varying ISS-CLF to provide a disturbance attenuation controller.

**5.1. Modeling of vessel trajectory tracking control system**

We consider a time-varying ISS-TCLF design problem for a vessel trajectory control system. The coordinate system is defined in Figure 1. We assume the following simple manoeuvring model:

\[
\begin{aligned}
\dot{x}_1 &= u_1 \cos x_3 + w_x \\
\dot{x}_2 &= u_1 \sin x_3 + w_y \\
\dot{x}_3 &= u_2
\end{aligned}
\]
where \( x_1, x_2 \in \mathbb{R} \), \( x_3 \in \mathbb{S} \), and \( u = [u_1, u_2]^T \) denote states and control input. \( x_1 \) and \( x_2 \) are positions. \( x_3 \) is an attitude angle. \( u_1 \) and \( u_2 \) are the input of the forward velocity and the yaw rate, respectively. \( w_x(t), w_y(t) : \mathbb{R} \rightarrow \mathbb{R} \) are functions of time \( t \) to represent total disturbance forces; wind force and wave drift force are such examples.

### 5.2. Design of reference states and reference input

To track a vessel to a time-varying curve trajectory, we give the following reference states of class \( C^2 \) and reference input:

\[
\begin{align*}
x_{r1}(t) &= R \sin(x_3(t)),\\
x_{r2}(t) &= -R \cos(x_3(t)),\\
x_{r3}(t) &= \arctan2(x_r(2)(t), \dot{x}_r(1)(t)),\\
\dot{u}_r(1) &= \sqrt{\dot{x}_r(1)(t)^2 + \dot{x}_r(2)(t)^2},\\
\dot{u}_r(2) &= -\ddot{x}_r(1)(t) \sin x_3(t) + \ddot{x}_r(2)(t) \cos x_3(t) / u_r(1). \\
\end{align*}
\]

Furthermore, we design a controller to input-to-state stabilize the origin of the tracking error system and asymptotically follow the time-varying reference output \( y_r(t) \) as follows:

\[
y_r(t) = \begin{bmatrix} x_{r1}(t) \\ x_{r2}(t) \end{bmatrix}.
\]  

### 5.3. ISS-TCLF design for vessel trajectory control

We design the tracking error system as follows.

**Lemma 5.1:** Consider tracking error \( e := x - x_r \). Then, the tracking error system for (79) is represented as the following disturbed error system:

\[
\begin{align*}
\dot{e} &= \tilde{f}(e, t) + \tilde{g}(e, t)u + \tilde{h}(e, t)w, \\
\end{align*}
\]

where \( \tilde{u} = u - u_r(t) \) and \( w = [w_x(t), w_y(t)]^T \). Functions \( \tilde{f}(e, t), \tilde{g}(e, t), \) and \( \tilde{h}(e) \) are represented as follows:

\[
\begin{align*}
\tilde{f}(e, t) &= f(e + x_r(t)) - f(x_r(t)) \\
&+ \left[ g(e + x_r(t)) - g(x_r(t)) \right] u_r(t), \\
\tilde{g}(e, t) &= \begin{bmatrix} u_r(1) \left[ \cos(e_3 + x_3(t)) - \cos(x_3(t)) \right] \\
&\quad + \ddot{x}_r(1)(t) \sin x_3(t) - \sin x_3(t) \end{bmatrix}, \\
\tilde{h}(e, t) &= \begin{bmatrix} \cos(e_3 + x_3(t)) \\
&\sin(e_3 + x_3(t)) \end{bmatrix}.
\end{align*}
\]

Next, we design the virtual linearized error system as follows.

**Lemma 5.2:** We consider the coordinate transformation \( \bar{z}_1 = \hat{u}_1 \) and give the following diffeomorphism:

\[
\mathcal{E}_e(e, \bar{z}_1, t) = \begin{bmatrix} \xi_{e1}(t), \xi_{e2}(t), \xi_{e3}(t), \xi_{e4}(t), t \end{bmatrix}^T.
\]

Moreover, the mapping to linearize tracking error system is represented as follows:

\[
\begin{align*}
\xi_e &= \begin{bmatrix} \xi_{e1}, \xi_{e2}, \xi_{e3}, \xi_{e4} \end{bmatrix}^T = \mathcal{E}_e(e, \bar{z}_1, t) \\
\end{align*}
\]

Then, tracking error system (86) is linearized as follows:

\[
\begin{align*}
\dot{\xi}_e &= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{bmatrix} \xi_e + \begin{bmatrix} 0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
\end{bmatrix} \tilde{v}_1 + \begin{bmatrix} 0 & 0 \\
1 & 0 \\
0 & 0 \\
0 & 1 \\
\end{bmatrix} \tilde{w}_x \\
&= A \xi_e + B \tilde{v} + Hw, \\
\end{align*}
\]

where \( \tilde{v} := [v_1, \tilde{v}_2]^T = [\xi_{e2}, \xi_{e4}]^T \) is virtual input.

According to Theorem 2.3 and [21], we can obtain the following lemma:

**Lemma 5.3:** Consider matrix \( P \) having the form with positive constants \( A_{11}, A_{12}, A_{22} \):

\[
P \in \begin{bmatrix} A_{11} & A_{12} & 0 & 0 \\
A_{12} & A_{22} & 0 & 0 \\
0 & 0 & A_{13} & A_{12} \\
0 & 0 & A_{12} & A_{22} \\
\end{bmatrix}.
\]
6. Computer simulations

We conform the effectiveness of a time-varying ISS-TCLF design method by computer simulations.

6.1. Simulation conditions

We consider a trajectory tracking control problem to follow reference states represented as (80)−(82). The circle radius is R = 20, and the reference velocity input is ur = [3, 0.15]T. The initial state of system (79) is x(0) = [20.1, 0, π/2]T, and we assume the bounded random disturbance as in Figure 2.

To see the effectiveness of disturbance attenuation control using the proposed ISS-TCLF (101), we use four types of controller (32). Note that these controllers have different solutions P1, P2, P3, and P4; however, the design parameter of Riccati equation (100) except for γ has the same value at R = I2 and Q = 0.01I4.

(E1) Non  \( H_\infty \) designed TCLF based controller (\( \epsilon = 0 \)) Solution P1 of Riccati equation (100) at \( H = 0 \) and \( C = 0.1I_4 \) is calculated as follows:

\[
P_1 = \begin{bmatrix}
0.046 & 0.1 & 0 & 0 \\
0.1 & 0.458 & 0 & 0 \\
0 & 0 & 0.046 & 0.1 \\
0 & 0 & 0.1 & 0.458
\end{bmatrix}.
\]

(E2)  \( H_\infty \) designed ISS-TCLF based controller (\( \epsilon = 0 \)) To achieve the best disturbance attenuation level in (100), we calculate minimum γ by γ-iteration. The minimum γ at \( C = 0.1I_4 \) is γmin = 0.325. Then, solution P2 of Riccati equation (100) is calculated as follows:

\[
P_2 = \begin{bmatrix}
6.856 & 21.095 & 0 & 0 \\
21.095 & 65.233 & 0 & 0 \\
0 & 0 & 6.856 & 21.095 \\
0 & 0 & 21.095 & 65.233
\end{bmatrix}.
\]

(E3)  \( H_\infty \) designed ISS-TCLF based controller with function \( \rho^{-1} \) (\( \epsilon = 1 \) and P2) According to [12], we transform angle error \( e_3 \) to \( \tilde{e} := [e_1, e_2, e_\psi] \), \( e_\psi := \tan(e_3 + x_3) \in \mathbb{R} \). Then, an explicit function \( \rho^{-1} : \mathbb{R}^3 \times \mathbb{R}^2 \to \mathbb{R} \) is derived as follows:

\[
\rho^{-1}(\tilde{e}, x_3, u_r) = \begin{bmatrix}
\max \left( \frac{e_3^2 + 1}{\tilde{e}_1}, 1 \right) \\
\frac{\gamma_{\min}}{\sigma_{\min}(C)}
\end{bmatrix}^{-1} \cdot \|\tilde{e}\|,
\]

where \( e_r := \tan(x_3) \), γmin = 0.325, and a minimum singular value \( \sigma_{\min}(C) = 0.1 \).

(E4)  \( H_\infty \) designed ISS-TCLF based controller with function \( \rho^{-1} \) (\( \epsilon = 1 \) and P4) To obtain restrained input with controller (E3), we redesign a controller with \( \gamma = 0.4 \) increased over γmin. Then, the solution P4 of Riccati equation (100) is calculated as follows:

\[
P_4 = \begin{bmatrix}
0.110 & 0.291 & 0 & 0 \\
0.291 & 1.060 & 0 & 0 \\
0 & 0 & 0.110 & 0.291 \\
0 & 0 & 0.291 & 1.060
\end{bmatrix}.
\]

6.2. Simulation results

Figure 3 represents vessel trajectories to compare the results using controller (E1)−(E3). Figure 4 represents vessel trajectories to compare the results using controller (E4).

The trajectory of controller (E1) deviates in the direction of disturbance w. The trajectory of controller
(E2), which has higher gain than controller (E1), is still affected by the amplitude of disturbance $w$. The trajectory of controller (E3) illustrates that $H_\infty$ designed ISS-TCLF based controller with function $\rho^{-1}$ has achieved constant disturbance attenuation levels based on a predefined ISS-gain. On the other hand, controller (E4) is a little more sensitive to disturbances than the optimally designed controller (E3) in Figure 4.

Disturbance attenuation controllers $u = \tilde{u} + u_r$ give the surge input and the yaw rate input in Figures 5 and 6. Figure 5 illustrates that controller (E3) acts as a high gain controller when the system is highly disturbed. The third term of (33) at $\epsilon = 1$ contributes to this performance unlike in controller (E2). Figure 6 illustrates that controller (E3) reduces the yaw rate input of the closed-loop system on steady-state. Therefore, controller (E3) has a positive effect on the yaw rate input.

On the other hand, optimal designed controller (E3) has a disadvantage that the input is too high at the initial time. Figure 8 illustrates that controller (E4) adequately
Figure 8. Simulation results: input velocity (Yaw rate) with controller (E4).

Figure 9. Simulation results: disturbance attenuation level with controller (E3).

reduces the input giving by a lower sensitivity than controller (E3).

Figures 9 and 10 illustrate the disturbance attenuation levels with controller (E1)–(E4). We use the following equation to evaluate the disturbance attenuation levels:

$$\bar{\gamma} := \frac{\| (e_1, e_2) \|_2}{\| w \|_\infty} \quad (107)$$

We can find that controller (E3) has the most reduced sensitivity of the error to disturbances in all time intervals. On the other hand, in Figure 10, controller (E4) is not optimally designed; therefore, its sensitivity increases in a time interval from 40 to 45 s.

The details of states and errors are shown in Appendix.

In summary, these results demonstrate that disturbance attenuation requires an implicit design of function $\rho^{-1}$. The proposed method enabled us to find suitable function $\rho^{-1}$, which is the important design factor to design the attenuation level systematically.

7. Conclusion

In this study, we proposed a framework of a time-varying input-to-state stability tracking control Lyapunov function. Furthermore, we applied the proposed framework to design disturbance attenuation controllers for trajectory tracking of a vessel. The effectiveness of our approach was confirmed by computer simulations. Implementing control considering sideslips on a real vessel and giving an ISS-gain design method for general disturbed differentially flat systems remain as future work.

Acknowledgments

This work is the result of joint research with BEMAC Corporation.

Disclosure statement

No potential conflict of interest was reported by the author(s).

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Appendix. Figures of simulation results

**Figure A1.** Simulation results: states with controller (E3)

**Figure A2.** Simulation results: deviation with controller (E3)
Figure A3. Simulation results: states with controller (E4)

Figure A4. Simulation results: deviation with controller (E4)

Figure A5. Simulation Condition: Disturbance norm $\|w\|_2$