THE LEMMENS-SEIDEL CONJECTURE FOR BASE SIZE 5

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Abstract. In 2020, Lin and Yu claimed to prove the so-called Lemmens-Seidel conjecture for base size 5. However, their proof has a gap, and in fact, some set of equiangular lines found by Greaves et al. in 2021 is a counterexample to one of their claims. In this paper, we give a proof of the conjecture for base size 5. Also, we answer in the negative a question of Greaves et al. in 2021 whether some sets of 57 equiangular lines with common angle \( \arccos(1/5) \) in dimension 18 are contained in a unique set of 276 equiangular lines with common angle \( \arccos(1/5) \) in dimension 23. In addition, we answer in the negative a question of Cao et al. in 2021 whether a strongly maximal set of equiangular lines with common angle \( \arccos(1/5) \) exists except the set of 276 equiangular lines with common angle \( \arccos(1/5) \) in dimension 23.

1. Introduction

A set of lines through the origin in a Euclidean space is **equiangular** if any pair from these lines forms the same angle. The problem of determining the maximum cardinality of a set of equiangular lines in a Euclidean space dates back to the result of Haantjes [9]. Denote by \( N(d) \) the maximum cardinality of a set of equiangular lines in dimension \( d \). The values of \( N(d) \) are known for \( d \leq 43 \) with \( d \neq 18, 19, 20, 42 \) [1, 7, 8, 12, 14]. Also, Gerzon proved the so-called absolute bound \( N(d) \leq d(d+1)/2 \) [12, Theorem 3.5]. If equality holds, then \( d+2 \) is 4, 5 or the square of an odd integer at least 3.

For a fixed angle, sets of equiangular lines have been studied. Denote by \( N_\alpha(d) \) the maximum cardinality of a set of equiangular lines with common angle \( \arccos(\alpha) \) in dimension \( d \). Lemmens and Seidel proved that for a set of \( n \) equiangular lines with common angle \( \arccos(\alpha) \) in dimension \( d \), \( 1/\alpha \) is an odd integer if \( n > 2d \) [12, Theorem 3.4]. In low dimensions, they proved \( N_\alpha(d) \leq d(1-\alpha^2)/(1-d\alpha^2) \) for \( d < 1/\alpha^2 \). In high dimensions, Jiang et al. proved for every integer \( k \geq 2 \), \( N_1/\alpha_2(kd-1)(d) = [k(d-1)/(k-1)] \) for all sufficiently large \( d \) [10, Corollary 1.3].

In the case where the common angle is \( \arccos(1/3) \) or \( \arccos(1/5) \), sets of equiangular lines have been investigated precisely. Lemmens and Seidel introduced the pillar method, and determined the values of \( N_1/3(d) \) for all \( d \) [12, Theorem 3.6]. Their pillar method is the main tool to prove our result, and will be given in Definition 2.1. In another way by using root lattices, Cao et al. [4] investigated sets of equiangular lines with common angle \( \arccos(1/3) \) more precisely. Also, Lemmens and Seidel raised the following, which is the so-called Lemmens-Seidel conjecture.

**Theorem 1.1** (The Lemmens-Seidel conjecture). For \( d \geq 23 \), \( N_1/5(d) = \max \{276, [(3d-3)/2]\} \).

Here a set of 276 equiangular lines with common angle \( \arccos(1/5) \) in dimension 23 is known to be unique [6, Theorem A]. In order to prove the Lemmens-Seidel conjecture, we need to show it for base sizes 3, 4, 5 and 6, where the base size will be given in Definition 2.1. In 1973, Lemmens and Seidel proved it for base size 6 [12, Theorem 5.7]. In 2020, Lin and Yu proved the conjecture for base size 3 with a computer [13, Theorem 4.3], and claimed to prove it for base size 5 [13, Theorem 4.6]. However, there is a gap in [13, Proof of Theorem 4.6 (1)]. In 2022, Cao et al. proved the conjecture for base sizes 3 and 4 without a computer [5, Theorems 6.1, 7.5 and 9.3]. Hence, to complete a proof of the conjecture, we give Theorem 1.2, which immediately implies the conjecture for base size 5.

**Theorem 1.2** (The Lemmens-Seidel conjecture for base size 5). A set of \( n \) equiangular lines with common angle \( \arccos(1/5) \) with base size 5 in dimension \( d \) satisfies \( n \leq \max \{276, [(4d+36)/3]\} \).

The gap in [13, Proof of Theorem 4.6 (1)] is in claiming that a set of equiangular lines with common angle \( \arccos(1/5) \), base size 6 and at least two pillars having edges is contained in a unique set of 276 equiangular lines in dimension 23. Since a set of equiangular lines with base size 5 without \((5,1)\)-pillars can be regarded as one with base size 6 by adding some extra line, they discussed sets of equiangular lines with base size 6. However, the four sets of 57 equiangular lines with common angle \( \arccos(1/5) \) in dimension 18 induced by the Seidel matrices written as

\[ A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \ 0 & 1 & 0 & 0 & 0 \ 0 & 0 & 1 & 0 & 0 \ 0 & 0 & 0 & 1 & 0 \ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \]
$S_1, S_2, S_3$ and $S_4$ in [7] are counterexamples to their claim. To show it, we will answer the following question in the negative in Proposition 7.3, and verify an easy fact in Proposition 7.2.

**Question 1.3 ([7] Question 2.1).** Can the two sets of 57 equiangular lines in dimension 18 corresponding to the Seidel matrices $S_1$ and $S_2$ in [7] be found inside the set of 276 equiangular lines in dimension 23?

Although Greaves et al. found more sets of 57 equiangular lines in dimension 18 [7], they posed a question about only two Seidel matrices $S_1$ and $S_2$. Note that we treat the four Seidel matrices $S_1, S_2, S_3$ and $S_4$, which are explicitly given in [7]. Also, we will show that the sets of equiangular lines induced by the Seidel matrices $S_1, S_2, S_3$ and $S_4$ are strongly maximal in Proposition 7.3. As a result, we answer the following question in the negative.

**Question 1.4 ([4] Question 5.7).** Is the set of 276 equiangular lines with common angle $\arccos(1/5)$ in dimension 23 a unique strongly maximal set of equiangular lines with common angle $arccos(1/5)$?

Here a set $U$ of equiangular lines is said to be *strongly maximal* if there is no set of equiangular lines properly containing $U$. Note that the concept of strong maximality was defined for graphs and Seidel matrices in [4]. It is well known that there exists a set of equiangular lines with common angle $arccos(\alpha)$ satisfying the absolute bound for each $1/\alpha \in \{2, \sqrt{3}, 3\}$. Cao et al. showed that a set of equiangular lines is strongly maximal if it satisfies the absolute bound [4, Theorem 5.5]. In addition, they proved uniqueness of strongly maximal sets of equiangular lines with common angle $arccos(\alpha)$ for $1/\alpha \in \{2, \sqrt{3}, 3\}$, and posed Question 1.4.

This paper is organized as follows. In Section 2 we introduce some concepts in connection with equiangular lines, and explain some notations. In Section 3 we rewrite Lemmens and Seidel’s results for sets of equiangular lines with common angle $arccos(1/5)$ and base size 5. In Section 4 we prove a key theorem to show the Lemmens-Seidel conjecture for base size 5. In Section 5 we give an upper bound on the order of $(5,2)$-pillars under some assumptions. In Section 6 we prove the Lemmens-Seidel conjecture for base size 5. In Section 7 we show some properties of sets of 57 equiangular lines with common angle $arccos(1/5)$ in dimension 18 found by Greaves et al. [7], and answer Questions 1.3 and 1.4 in the negative.

## 2. Notations

Throughout this paper, we will consider undirected graphs, without loops and multiedges. Let $H$ be a graph. Denote by $V(H)$ the set of vertices, and by $E(H)$ the set of edges. Denote by $N(x) = N_H(x)$ the set of neighbors of a vertex $x$ in $H$. We write $G + H$ for the disjoint union of two graphs $G$ and $H$. For a non-negative integer $m$, we write $mH$ for the disjoint union of $m$ copies of $H$. A clique in $H$ is an induced subgraph isomorphic to a complete graph, and a maximum clique is a clique such that there is no clique with more vertices. We will identify a clique with its vertex set. The clique number $\chi(H)$ is the maximum value of the orders of cliques in $H$.

Denote by $I$ and $J$ the identity matrix and the all-ones matrix, respectively. If the size of each matrix is not clear, then we will indicate its size by a subscript.

A **Seidel matrix** is a symmetric matrix with zero diagonal and all off-diagonal entries $\pm 1$. Two Seidel matrices $S$ and $S'$ are said to be switching equivalent if there exist a permutation matrix $P$ and a diagonal matrix $D$ with diagonal entries $\pm 1$ such that $(PD)^\top S (PD) = S'$. For a graph $H$, denote by $A(H)$ the adjacency matrix of $H$, and define $S(H) := J - I - 2A(H)$. Note that for any Seidel matrix $S$, there exists a graph $H$ such that $S = S(H)$. The eigenvalues of $S(H)$ are called the **Seidel eigenvalues** of $H$. Two graphs $G$ and $H$ are said to be switching equivalent if $S(G)$ and $S(H)$ are switching equivalent.

Fix a set $U$ of $n$ equiangular lines with common angle $arccos(\alpha)$. Then we may take unit vectors $u_1, \ldots, u_n$ such that $U = \{ \mathbb{R} u_1, \ldots, \mathbb{R} u_n \}$. There exists a Seidel matrix $S$ such that $I + \alpha S$ equals the Gram matrix of $u_1, \ldots, u_n$. Hence the set $U$ equiangular lines induces the Seidel matrix $S$ up to switching. In addition, it induces the graph $H$ with $S = S(H)$ up to switching. Note that the smallest Seidel eigenvalue of $H$ is at least $-1/\alpha$. Conversely, we can recover $U$ from the Seidel matrix $S$ or the graph $H$.

**Definition 2.1.** Let $U$ be a set of equiangular lines. The maximum value of clique numbers of graphs induced by $U$ is called the base size of $U$.

Denote by $\langle v_1, \ldots, v_n \rangle$ the linear space generated by vectors $v_1, \ldots, v_n$. Also denote by $(u, v)$ the inner product of two vectors $u$ and $v$.

**Definition 2.2.** Let $H$ be a graph, and $B$ be a maximum clique. A **pillar** $P_{B, U}$ with respect to $B$ for a subset $U \subseteq B$ is defined to be the induced subgraph in $H$ on

$$\{ x \in V(H) \setminus B \mid N(x) \cap B = U \}.$$
Moreover, this is called a \(|B|, |U|\)-pillar. Also, assume that the smallest Seidel eigenvalue of \(H\) is at least \(-5\). Then denote by \(\tilde{x}\)'s the vectors such that
\[
(\tilde{x}, \tilde{y}) = (S(H) + 5I)_{xy} \quad (x, y \in V(H)).
\]
Denote by \(\tilde{x}\) the orthogonal projection of \(\tilde{x}\) onto the subspace \(\tilde{b} : b \in B \)\(\perp\).

The connected graphs with largest eigenvalue at most 2 are enumerated in Figure 1 (cf. [3] Theorem 3.13]).

**Figure 1.** The connected graphs having largest eigenvalue at most 2, where the colors of vertices will be used in Lemma 5.1.

### 3. Lemmens and Seidel’s results

The proof of the following lemma is essentially contained in [13] Proof of Theorem 4.6.

**Lemma 3.1.** Let \(H\) be a graph with smallest Seidel eigenvalue at least \(-5\) having a maximum clique \(B = \{b_1, \ldots, b_6\}\). Assume that the only pillars in \(H\) with respect to \(B\) having at least one vertex are \((5, 2)\)-pillars. Then there exist a supergraph \(G\) of \(H\) with smallest Seidel eigenvalue \(-5\) and a vertex \(b_6 \in V(G)\) satisfying the following.

(i) \(V(H) \cup \{b_6\} = V(G)\), and \(B \cup \{b_6\}\) is a maximum clique.

(ii) The \((5, 2)\)-pillars in \(H\) with respect to \(B\) coincide with the \((6, 3)\)-pillars adjacent to \(b_6\) in \(G\) with respect to \(B \cup \{b_6\}\).

**Proof.** Define \(G\) by taking a vertex \(b_6\) such that \(\tilde{b}_6 = -b_1 - b_2 - b_3 - b_4 - b_5\).

By this lemma, we can rewrite some results in [12] on graphs with Seidel smallest eigenvalue at least \(-5\) and clique number 6 as follows. We remark that if the base size is 6 and common angle \(\arccos(1/5)\), then the only pillars with respect to some clique of size 6 are \((6, 3)\)-pillars.

**Theorem 3.2 ([12] Theorem 5.2]).** Let \(H\) be a graph with smallest Seidel eigenvalue at least \(-5\) and clique number 5. Let \(P\) be a \((5, 2)\)-pillar. If another \((5, 2)\)-pillar has at least one vertex, then the following hold.

(i) If an induced subgraph of \(P\) is isomorphic to \(\tilde{A}_t\), then \(t + 1 \mod 3 = 0\).

(ii) If an induced subgraph of \(P\) is isomorphic to \(\tilde{D}_t\), then \(t + 1 \mod 3 = 2\).

**Theorem 3.3 ([12] Theorems 5.3, 5.4 and 5.5]).** Let \(H\) be a graph with smallest Seidel eigenvalue at least \(-5\) and clique number 5. Assume a \((5, 2)\)-pillar is isomorphic to \(mK_2 + nK_1\) for some non-negative integers \(m\) and \(n\).

(i) \(2m + n \leq 18\) if another \((5, 2)\)-pillar has an edge.

(ii) \(2m + n \leq 24\) if another \((5, 2)\)-pillar has non-adjacent vertices.

(iii) \(2m + n \leq 36\) if another \((5, 2)\)-pillar has a vertex.

**Theorem 3.4 ([12] Proof of Theorem 5.6 with Theorems 5.3, 5.4 and 5.5]).** Let \(H\) be a graph with smallest Seidel eigenvalue at least \(-5\) and clique number 5. Let \(P\) be a \((5, 2)\)-pillar.

(i) \(|V(P)| \leq 27\) if another \((5, 2)\)-pillar has an edge.
(ii) \(|V(P)| \leq 36\) if another \((5, 2)\)-pillar has non-adjacent vertices.

(iii) \(|V(P)| \leq 54\) if another \((5, 2)\)-pillar has a vertex.

4. \((5, 2)\)-PILLARS ISOMORPHIC TO \(mK_2\) AND \((5, 1)\)-PILLARS

The following theorem plays a key role in proving the main result Theorem \[\text{[12]}\]. It is proved at the end of this section.

**Theorem 4.1.** Let \(H\) be a graph with smallest Seidel eigenvalue at least \(-5\) having a maximum clique \(B = \{b_1, \ldots, b_5\}\). Assume that the \((5, 2)\)-pillar \(P_{B,\{b_1,b_2\}}\) is isomorphic to \(mK_2\), for some integer \(m\), and assume one of the following.

(I) The \((5, 1)\)-pillar \(P_{B,\{b_1\}}\) has non-adjacent vertices.

(II) Both \((5, 1)\)-pillars \(P_{B,\{b_1\}}\) and \(P_{B,\{b_2\}}\) have at least one vertex.

If the \((5, 2)\)-pillar \(P_{B,\{b_3,b_4\}}\) contains at least one edge, then \(m \leq 8\).

By \[\text{[11]}\ Proposition 3.10\], we have the following lemma.

**Lemma 4.2.** Let \(H\) be a graph with smallest Seidel eigenvalue at least \(-5\) having a maximum clique \(B = \{b_1, \ldots, b_5\}\). For \(\{i, j, k, l, m\} = \{1, \ldots, 5\}\) the following hold.

(i) For a vertex \(x\) in the \((5, 2)\)-pillar \(P_{B,\{b_1,b_2\}}\), the orthogonal projection of \(x\) onto \((b : b \in B)\) is \(\frac{1}{3} \left(b_k + b_l + b_m\right)\).

(ii) For a vertex \(x\) in the \((5, 1)\)-pillar \(P_{B,\{b_1\}}\), the orthogonal projection of \(x\) onto \((b : b \in B)\) is \(\frac{1}{3} \left(b_i + 2b_j + 2b_k + 2b_l + 2b_m\right)\).

In particular, we have the following inner products.

(iii) For \(x, y \in P_{B,\{b_1,b_2\}}\) and \(z, w \in P_{B,\{b_1\}}\), the following hold.

\[
\frac{15}{2} \cdot (\bar{x}, \bar{y}) = \begin{cases} 
6 & \text{if } x = y, \\
-3 & \text{if } x \sim y, \\
0 & \text{if } x \not\sim y.
\end{cases}
\]

\[
\frac{15}{2} \cdot (\bar{z}, \bar{w}) = \begin{cases} 
4 & \text{if } z = w, \\
-2 & \text{if } z \not\sim w.
\end{cases}
\]

(iv) For \(x \in P_{B,\{b_1,b_2\}}, y \in P_{B,\{b_1,b_4\}}, z \in P_{B,\{b_1\}}, w \in P_{B,\{b_2\}}\), the following hold.

\[
\frac{15}{2} \cdot (\bar{x}, \bar{y}) = \begin{cases} 
-1 & \text{if } x \sim y, \\
2 & \text{if } x \not\sim y.
\end{cases}
\]

\[
\frac{15}{2} \cdot (\bar{z}, \bar{z}) = \begin{cases} 
-3 & \text{if } x \sim z, \\
0 & \text{if } x \not\sim z.
\end{cases}
\]

\[
\frac{15}{2} \cdot (\bar{y}, \bar{z}) = \begin{cases} 
-2 & \text{if } y \sim z, \\
1 & \text{if } y \not\sim z.
\end{cases}
\]

\[
\frac{15}{2} \cdot (\bar{z}, \bar{w}) = \begin{cases} 
-4 & \text{if } z \sim w, \\
-1 & \text{if } z \not\sim w.
\end{cases}
\]

Let \(s, t\) and \(r\) be positive integers, and \(Z\) and \(Y\) be sets of numbers. Denote by \(M_{r,s}(Z)\) the set of \(r \times s\)-matrices all of whose entries are in \(Z\), and write \(M_r(Z)\) for \(M_{r,r}(Z)\) if \(r = s\). Denote by \(M_{r,s,t}(Z,Y)\) the set of \(r \times (s+t)\)-matrices obtained by joining a matrix in \(M_{r,s}(Z)\) and one in \(M_{r,t}(Y)\) horizontally. For example,

\[
M_{2,1,2}(\{0,1\}, \{2,3\}) = \left\{ \begin{bmatrix} i & e & f \\ j & g & h \end{bmatrix} : i, j \in \{0,1\}, e, f, g, h \in \{2,3\} \right\}.
\]

To prove Theorem 4.1, we prepare some matrices as follows. Let \(r\) be a positive integer. Let \(M\) be a finite set of \(2 \times r\) matrices, and \(a : M \to \mathbb{Z}_{\geq 0}\) a function. Define

\[
Q_{21} = Q_{21}(a)
\]

as the matrix obtained by joining all \(a(A)\) copies of \(A \in M\) vertically. Let \(m\) be the sum of the images of the function \(a\), and define

\[
Q_{22} = Q_{22}(m) := (9I_2 - 3J_2)^{\oplus m} = \begin{bmatrix} 6 & -3 \vspace{1em} \\ -3 & 6 \end{bmatrix}^{\oplus m}.
\]

In addition, let \(Q_{11}\) be an \(r \times r\) matrix, and define

\[
Q = Q \left( Q_{11}; a \right) := \begin{bmatrix} Q_{11} & Q_{21} \\ Q_{21} & Q_{22} \end{bmatrix}.
\]
For example, we let \( a : M_2(\{2, -1\}) \to \mathbb{Z}_{\geq 0} \) be a function such that \( a \left( \begin{bmatrix} -1 & -1 \\ 2 & -1 \end{bmatrix} \right) = 1, a \left( \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix} \right) = 2, \) and \( a \) takes 0 on the other matrices. Then

\[
Q_{21}(a) = \begin{bmatrix}
-1 & -1 \\
2 & -1 \\
-1 & 2 \\
-1 & 2 \\
2 & -1 \\
2 & -1
\end{bmatrix}
\quad \text{and} \quad
Q(5I_2 - J_2; a) = \begin{bmatrix}
4 & -1 & -1 & 2 & 2 & 2 & 2 \\
-1 & 4 & -1 & -1 & 2 & 2 & 2 \\
-1 & -1 & 6 & -3 & 0 & 0 & 0 \\
2 & -1 & -3 & 6 & 0 & 0 & 0 \\
2 & 2 & 0 & 0 & 6 & -3 & 0 & 0 \\
2 & -1 & 0 & 0 & -3 & 6 & 0 & 0 \\
2 & 2 & 0 & 0 & 0 & 6 & -3 & 0 & 0 \\
2 & -1 & 0 & 0 & 0 & -3 & 6 & 0 & 0
\end{bmatrix}.
\]

For each \( 2 \times t \) matrix \( A \), let \( \bar{A} \) be the matrix obtained from \( A \) by exchanging the first and second row. Although Theorem 4.4 has been proved in [12], Proof of Theorem 5.3], we give a proof for the convenience of the readers. Let

\[
\mathcal{M} := \left\{ A, \bar{A} : A \in\left\{ \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & -1 \\ -1 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 2 \\ -1 & -1 \end{bmatrix} \right\} \right\}.
\]

**Lemma 4.3** (Theorem 2.7.1). Let \( C \) be a positive definite matrix. Then a symmetric matrix \( \begin{bmatrix} A & B \\ B^\top & C \end{bmatrix} \) is positive semidefinite if and only if \( A - BC^{-1}B^\top \) is positive semidefinite.

**Theorem 4.4.** Let \( a : M_2(\{1, 2\}) \to \mathbb{Z}_{\geq 0} \) be a function. Let \( m \) be the sum of images of \( a \). If \( Q(9I_2 - 3J_2; a) \) is positive semidefinite, then \( m \leq 9 \). Furthermore, if equality holds, then \( a(A) = 0 \) for \( A \not\in \mathcal{M} \).

**Proof.** Since \( Q(9I_2 - 3J_2; a) \) is positive semidefinite, Lemma 4.3 implies that

\[
\Delta := (9I_2 - 3J_2) - \sum_{A \in M_2(\{-1, 1, 2\})} a(A) \cdot A^\top (9I_2 - 3J_2)^{-1} A
\]

is positive semidefinite. We have

\[
0 \leq 3 \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 1 \end{bmatrix}^\top \Delta \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}
= 36 - 4 \sum_{A \in \mathcal{M}} a(A) - 2a \left( \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \right)
= 36 - 4 \sum_{A \in \mathcal{M}} a(A) - 2a \left( \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \right)
\]

\[
- 16 \left( a \left( \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} \right) + a \left( \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix} \right) + a \left( \begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix} \right) + a \left( \begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix} \right) + a \left( \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \right) \right)
\]

\[
- 8 \left( a \left( \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} \right) + a \left( \begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix} \right) + a \left( \begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix} \right) + a \left( \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix} \right) \right) \]

\[
\leq 36 - 4m.
\]

Hence \( m \leq 9 \). Moreover, if \( m = 9 \) then \( a(A) \) is equal to 0 for every \( A \not\in \mathcal{M} \). This is the desired condition. \( \square \)

To prove Theorem 4.1, we prepare some more matrices as follows.

\[
B_{11}^{(1)} := \begin{bmatrix} 4 & -2 \\ -2 & 4 \end{bmatrix}, \quad B_{11}^{(2)} := \begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix}, \quad B_{22} := \begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix}.
\]

We let

\[
B_{21}^{(1)} := \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad B_{21}^{(2)} := \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}, \quad B_{21}^{(3)} := \begin{bmatrix} -2 & -2 \\ -2 & 1 \end{bmatrix}, \quad B_{21}^{(4)} := \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix},
\]

\[
B_{21}^{(5)} := \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix}, \quad B_{21}^{(6)} := \begin{bmatrix} -2 & -2 \\ -2 & 1 \end{bmatrix}, \quad B_{21}^{(7)} := \begin{bmatrix} -2 & -2 \\ -2 & -2 \end{bmatrix}, \quad B_{21}^{(8)} := \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix},
\]

and set

\[
B(X, i) := \begin{bmatrix} B_{11}^{(X)} & B_{21}^{(i)} \\ B_{21}^{(i)} & B_{22} \end{bmatrix}.
\]
Lemma 4.5. Let \( a : M_{2,2,2}((-3,0),(-1,2)) \to \{0\} \) be the zero map. If \((X,i) \in \{(I,6),(I,7),(II,7)\}\), then \(Q(B(X,i);a)\) is not positive semidefinite.

Proof. This follows from direct calculation. \(\square\)

Lemma 4.6. Let \( a : M_{2,2,2}((-3,0),(-1,2)) \to \mathbb{Z}_{\geq 0} \) be a function. Assume \((X,i) \in \{(I,1),(I,3),(I,4)\} \cup \{(II,1),(II,3),(II,4),(II,6)\}\). The sum of images of \(a\) is at most 6 if \(Q(B(X,i),a)\) is positive semidefinite.

Proof. By Lemma 4.3 we see that
\[
\Delta := B(X,i) - \sum_{A \in M_{2,2,2}((-3,0),(-1,2))} a(A) \cdot A^\top (9I_2 - 3J_2)^{-1}A
\]
is positive semidefinite. We let
\[
x := \begin{cases} 
[1 \ 1 \ -1 \ -1]^\top & \text{if } (X,i) \in \{(I,1),(II,1)\}, \\
[1 \ 1 \ 1 \ 0]^\top & \text{if } (X,i) \in \{(I,3),(II,3)\}, \\
[1 \ 0 \ 1 \ 1]^\top & \text{if } (X,i) \in \{(I,4),(II,4)\}, \\
[1 \ 1 \ 1 \ 1]^\top & \text{if } (X,i) = (II,6). 
\end{cases}
\]
We regard \(x^\top \Delta x\) as a linear polynomial with variables \(a(A)\)'s. The constant term satisfies
\[
x^\top B(X,i)x = \begin{cases} 
2 & \text{if } (X,i) \in \{(I,1),(I,3),(I,4)\} \cup \{(II,4),(II,6)\}, \\
4 & \text{if } (X,i) \in \{(II,1),(II,3)\}.
\end{cases}
\]
Also, the coefficients satisfy
\[
\min \left\{ (Ax)^\top (9I_2 - 3J_2)^{-1}Ax : A \in M_{2,2,2}((-3,0),(-1,2)) \right\} = \frac{2}{3}.
\]

Since \(x^\top \Delta x \geq 0\), the sum of images of \(a\) is at most \(4/(2/3) = 6\). \(\square\)

Lemma 4.7. Let \( a : M_{2,2,2}((-3,0),(-1,2)) \to \mathbb{Z}_{\geq 0} \) be a function. Assume \(X \in \{I,II\}\). The sum of images of \(a\) is at most 8 if \(Q(B(X,5),a)\) is positive semidefinite.

Proof. Let \(m\) be the sum of images of \(a\). By Theorem 4.4 \(m \leq 9\) holds. By way of contradiction, we assume \(m = 9\). By applying Theorem 4.4 again,
\[
a \begin{bmatrix} A_1 & A_2 \end{bmatrix} = 0
\]
holds for every \(A_1 \in M_2((-3,0))\) and \(A_2 \notin M\). We define a positive semidefinite matrix \(\Delta\) as \(4.1\). Let
\[
x_1 := [1 \ 0 \ -1 \ -1]^\top, \quad x_2 := [1 \ 1 \ 1 \ 1]^\top, \quad x_3 := [0 \ 1 \ -1 \ 1]^\top.
\]
We regard \(\sum_{i=1}^3 x_i^\top \Delta x_i\) as a linear polynomial with variables \(a(A)\)'s. The constant term satisfies
\[
\sum_{i=1}^3 x_i^\top B(X,5)x_i = \begin{cases} 
38 & \text{if } X = I \\
40 & \text{if } X = II
\end{cases}
\]
Also, the coefficients satisfy
\[
\min \left\{ \sum_{i=1}^3 \left[ \begin{bmatrix} A_1 & A_2 \end{bmatrix} x_i \right]^\top (9I_2 - 3J_2)^{-1} \begin{bmatrix} A_1 & A_2 \end{bmatrix} x_i : A_1 \in M_2((-3,0)), A_2 \in M \right\} = \frac{14}{3}.
\]

Hence \(m \leq \left[40/\left(14/3\right)\right] = 8\). This is a contradiction. We have \(m \leq 8\). \(\square\)

Let \(M_I'\) be the set of
\[
\begin{bmatrix} 0 & 0 & -1 & -1 \\
0 & 0 & -1 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 & -1 \\
0 & 0 & 2 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 & 2 \\
0 & 0 & 2 & -1 \end{bmatrix}, \begin{bmatrix} -3 & 0 & 2 & -1 \\
0 & -3 & -1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & -3 & -1 & -1 \end{bmatrix}.
\]

Let \(M_{II}'\) be the union of \(M_I'\) and the set of
\[
\begin{bmatrix} -3 & 0 & 2 & -1 \\
0 & -3 & -1 & -1 \end{bmatrix}, \begin{bmatrix} -3 & 0 & 2 & -1 \\
0 & 0 & -1 & 2 \end{bmatrix}, \begin{bmatrix} -3 & 0 & 2 & -1 \\
0 & 0 & -1 & 2 \end{bmatrix}, \begin{bmatrix} -3 & 0 & 2 & -1 \\
0 & 0 & -1 & 2 \end{bmatrix}.
\]
For $X \in \{I, II\}$, we let

$$\mathcal{M}_X := \mathcal{M}'_X \cup \{ A : A \in \mathcal{M}'_X \}.$$ 

**Lemma 4.8.** Let $a : M_{2,2,2}(\{-3,0\}, \{-1,2\}) \to \mathbb{Z}_{\geq 0}$ be a function. Let $X \in \{I, II\}$. If $Q(B(X,2), a)$ is positive semidefinite and the sum of images of $a$ equals 9, then $a(A) = 0$ holds for every $A \notin \mathcal{M}_X$.

**Proof.** Let $m$ be the sum of images of $a$. Assume $m = 9$. We define a positive semidefinite matrix $\Delta$ as (4.42). By applying Theorem 1.2.3 we have $a([A_1 \ A_2]) = 0$ for any $A_1 \in M_2(\{-3,0\})$ and $A_2 \notin \mathcal{M}$.

We let

$$x_1 := [0 \ -1 \ 1 \ 1]^\top, \quad x_2 := [1 \ 0 \ 1 \ 0]^\top, \quad x_3 := [1 \ 1 \ 0 \ -1]^\top.$$ 

We regard $x_i^\top \Delta x_i$ as a linear polynomial with variables $x_i$'s. The constant term satisfies $x_i^\top B(X,2)x_i = 0$ for each $(X,i) \in \{(I,1), (I,2), (I,3)\} \cup \{(II,1), (II,2)\}$. Also, the coefficients satisfy

$$\min \left\{ \left( \begin{array}{c} A_1 \ A_2 \end{array} \right) (9I_2 - 3J_2)^{-1} \left( \begin{array}{c} A_1 \ A_2 \end{array} \right) x_i : A_1 \in M_2(\{-3,0\}), A_2 \in \mathcal{M} \right\} = \frac{2}{3}$$ 

for $i \in \{1, 2, 3\}$. Hence for $(X,i) \in \{(I,1), (I,2), (I,3)\} \cup \{(II,1), (II,2)\}$,

$$0 \leq x_i^\top \Delta x_i = x_i^\top B(X,2)x_i - a(A) \cdot (x_i^\top (9I_2 - 3J_2)^{-1}Ax_i)$$

$$= x_i^\top B(X,2)x_i - \sum_{A_1 \in M_2(\{-3,0\}), A_2 \in \mathcal{M}} a ([A_1 \ A_2]) \cdot \left( \begin{array}{c} A_1 \ A_2 \end{array} \right) (9I_2 - 3J_2)^{-1} \left( \begin{array}{c} A_1 \ A_2 \end{array} \right) x_i$$

$$= \sum_{A_1 \in M_2(\{-3,0\}), A_2 \in \mathcal{M}} a ([A_1 \ A_2]) \cdot \left( \begin{array}{c} A_1 \ A_2 \end{array} \right) (9I_2 - 3J_2)^{-1} \left( \begin{array}{c} A_1 \ A_2 \end{array} \right) x_i - \frac{2}{3} x_i.$$ 

We may verify by direct calculation that

$$\left( \begin{array}{c} A_1 \ A_2 \end{array} \right) (9I_2 - 3J_2)^{-1} \left( \begin{array}{c} A_1 \ A_2 \end{array} \right) x_i > \frac{2}{3},$$

for some $i$ if $[A_1 \ A_2] \notin \mathcal{M}_X$. Hence $a([A_1 \ A_2]) = 0$ holds for every $A \notin \mathcal{M}_X$.

**Proof of Theorem 4.7** Let $x_{i,1}$ and $x_{i,2}$ be adjacent vertices of $P_{B,(i,b_1,b_2)}$ for $i \in \{1, \ldots, m\}$. Let $y_1$ and $y_2$ be adjacent vertices of $P_{B,(b_1,b_2)}$. In the case of (I), we set $X := I$, and let $z_1$ and $z_2$ be vertices of $P_{B,(b_1)}$.

In the case of (II), we set $X := II$. Let $z_1$ be a vertex of $P_{B,(b_1)}$, and $z_2$ one in $P_{B,(b_2)}$. Then $z_1$ and $z_2$ are not adjacent. Indeed, if $z_1$ and $z_2$ are adjacent, then we have by Lemma 4.2 $(\bar{z}_1, \bar{z}_2) = (\bar{z}_2, \bar{z}_2) = 8/15$, and $(\bar{z}_1, \bar{z}_2) = -8/15$. These imply $\bar{z}_1 = -\bar{z}_2$. Also Lemma 4.2 asserts $(\bar{z}_1, \bar{y}_1), (\bar{z}_2, \bar{y}_2) \in \{-2/15, 4/15\}$. Hence we have a contradiction, and see that $z_1$ and $z_2$ are not adjacent.

The Gram matrix $G$ of $z_1, \bar{z}_2, \bar{y}_1, \bar{y}_2, \bar{x}_{1,1}, \bar{x}_{1,2}, \ldots, \bar{x}_{m,1}, \bar{x}_{m,2}$ is positive semidefinite. By Lemma 4.2 we see that $(15/2)G$ is

$$Q \left( \begin{array}{cc} P_{11}^{(X)} & B_{21}^\top \\ B_{21} & B_{22}^\top \end{array} \right) : a$$

for some $B_{21} \in M_2(\{-1,2\})$ and some function $a : M_{2,2,2}(\{-3,0\}, \{-1,2\}) \to \mathbb{Z}_{\geq 0}$. Let $m$ be the sum of images of $a$. By exchanging $y_1$ and $y_2$, or $z_1$ and $z_2$ if necessary, we may assume $B_{21} = B_{21}^{(i)}$ for some $i \in \{1, \ldots, 7\}$. By Lemmas 4.5, 4.6 and 4.7 we have $m \leq 8$ if $i \neq 2$. Hence we consider the case of $i = 2$.

We write $C_1, \ldots, C_5$ for the matrices in (4.2) in order from left to right. In addition, we write $C_6, C_7$ and $C_8$ for the three matrices in (4.3) in order from left to right. Note that

$$\mathcal{M}'_I = \{C_1, C_2, C_3, C_4, C_5\} \quad \text{and} \quad \mathcal{M}'_II = \mathcal{M}'_I \cup \{C_6, C_7, C_8\}.$$

We have

$$C_1^\top (9I_2 - 3J_2)^{-1}C_1 = C_2^\top (9I_2 - 3J_2)^{-1}C_2 = C_3^\top (9I_2 - 3J_2)^{-1}C_3 = \frac{1}{9} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$
and

\[
\begin{align*}
C_4^T (9I_2 - 3J_2)^{-1} C_4 &= \frac{1}{9} \cdot \begin{bmatrix} 18 & 0 & -9 & 9 \\ 0 & 0 & 0 & 0 \\ -9 & 0 & 6 & -3 \\ 9 & 0 & -3 & 6 \end{bmatrix}, & C_5^T (9I_2 - 3J_2)^{-1} C_5 &= \frac{1}{9} \cdot \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 18 & 0 & 9 \\ 0 & 0 & 6 & -3 \\ 0 & 9 & -3 & 6 \end{bmatrix}, \\
C_6^T (9I_2 - 3J_2)^{-1} C_6 &= \frac{1}{9} \cdot \begin{bmatrix} 18 & 0 & -9 & 0 \\ 0 & 0 & 0 & 0 \\ -9 & 0 & 6 & -3 \\ 0 & 0 & -3 & 6 \end{bmatrix}, & C_7^T (9I_2 - 3J_2)^{-1} C_7 &= \frac{1}{9} \cdot \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 18 & 0 & 9 \\ 0 & 9 & 6 & -3 \\ 0 & 0 & -3 & 6 \end{bmatrix}, \\
C_8^T (9I_2 - 3J_2)^{-1} C_8 &= \frac{1}{9} \cdot \begin{bmatrix} 18 & 9 & -9 & 9 \\ 9 & 18 & 0 & 9 \\ -9 & 0 & 6 & -3 \\ 9 & 9 & -3 & 6 \end{bmatrix}.
\end{align*}
\]

Theorem 4.3 asserts \( m \leq 9 \). In order to prove \( m \leq 8 \) by way of contradiction, we assume \( m = 9 \). By Lemma 4.8 we have \( a(A) = 0 \) for \( A \notin \mathcal{M}_X \). Also, we may assume that \( a(A) = 0 \) for each \( A \in \mathcal{M}_X \setminus \mathcal{M}_X' \) by exchanging \( x_{i,1} \) and \( x_{i,2} \) (\( i \in \{1, \ldots, m\} \)) if necessary. Below we consider the values of \( a(A) \) with \( A \in \mathcal{M}_X' \) for \( \Delta \) to be positive semidefinite. We have

\[
\Delta = B(X, 2) - \sum_{A \in \mathcal{M}_{2,2,2}(-3,0,\{\{-1,2\}\})} a(A) \cdot A^T (9I_2 - 3J_2)^{-1} A
\]

\[
= B(X, 2) - \sum_{A \in \mathcal{M}_X} a(A) \cdot A^T (9I_2 - 3J_2)^{-1} A
\]

\[
= B(X, 2) - \sum_{i=1}^{8} a(C_i) \cdot C_i^T (9I_2 - 3J_2)^{-1} C_i.
\]

Noting that \( a(C_1) + \cdots + a(C_8) = m = 9 \), we obtain

\[
\Delta = \begin{bmatrix}
B_{11}^{(X)} & -2 & 1 \\
-2 & 1 & 1 \\
1 & 1 & 1
\end{bmatrix} - a(C_4) \cdot \begin{bmatrix} 2 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} - a(C_5) \cdot \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}
\]

\[
- a(C_6) \cdot \begin{bmatrix} 2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - a(C_7) \cdot \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - a(C_8) \cdot \begin{bmatrix} 2 & 1 & -1 & 1 \\ 1 & 2 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}.
\]

Here note that \( a(C_6) = a(C_7) = a(C_8) = 0 \) if \( X = I \). However, if \( X = I \), then the principal submatrix of \( \Delta \) indexed by \( \{2, 3\} \) has negative determinant. This is a contradiction, and \( m \leq 8 \) holds.

Next we consider the case of \( X = \Pi \). Since every principal submatrix of order 2 has non-negative determinant, the submatrix of \( \Delta \) indexed by \( \{1, 2\} \times \{3, 4\} \) is zero. Thus, we have \( (a(C_4), a(C_5), a(C_6), a(C_7), a(C_8)) \in \{(1, 1, 1, 1, 0), (0, 0, 1, 1, 1)\} \). Then the principal submatrix indexed by \( \{1, 2\} \) is \( 2I_2 - 2J_2 \) or \( 3I_2 - 3J_2 \). Since these two matrices are not positive semidefinite, we obtain a contradiction. Therefore \( m \leq 8 \).

\section{(5, 2)-Pillars and (5, 1)-Pillars}

The following lemma is obtained by slightly improving [12] Proof of Theorem 5.6).

\textbf{Lemma 5.1.} Let \( G \) be a connected graph with largest eigenvalue at most 2. Assume that \( G \) is not isomorphic to \( \tilde{A}_t \) \((t+1 \equiv 0 \pmod{3}) \) or \( D_t \) \((t+1 \equiv 2 \pmod{3}) \). Then there exist non-negative integers \( n \) and \( m \) such that \( G \) contains an induced subgraph isomorphic to \( nK_1 + mK_2 \) and the order of \( G \) is at most

\[
\frac{4n}{3} + 3m.
\]

\textbf{Proof.} The graph \( G \) is isomorphic to one of the graphs in Figure [1] (cf. [3] Theorem 3.1.3). If \( G \) is not isomorphic to \( D_t \), then the graph obtained from \( G \) by removing the white vertices in Figure [1] is the desired induced subgraph. Hence
we consider the case where $G$ is isomorphic to $D_t$ for some $t \geq 4$. Here we may assume that the vertices of $G$ are indexed as in Figure 1. Let $H$ be the graph obtained from $G$ by removing the following vertices.

$$\begin{align*}
\{1\} & \cup \{i \in \{2, \ldots, t-1\} : i \mod 3 = 0\} \text{ if } t \mod 3 = 0, \\
\{2\} & \cup \{i \in \{3, \ldots, t-1\} : i \mod 3 = 1\} \text{ if } t \mod 3 = 1, \\
\{i \in \{1, \ldots, t-1\} : i \mod 3 = 2\} & \text{ if } t \mod 3 = 2.
\end{align*}$$

Then $H$ is isomorphic to

$$\begin{align*}
\begin{cases}
mK_2 & \text{if } t \mod 3 = 0, \\
3K_1 + (m-1)K_2 & \text{if } t \mod 3 = 1, \\
2K_1 + mK_2 & \text{if } t \mod 3 = 2,
\end{cases}
\end{align*}$$

where $m := \lfloor t/3 \rfloor$. We see that $H$ is the desired induced subgraph. \hfill \square

**Corollary 5.2.** Let $H$ be a graph with smallest Seidel eigenvalue at least $-5$ having a maximum clique $B = \{b_1, \ldots, b_5\}$. Assume one of the following.

(i) The $(5, 1)$-pillar $P_{B, \{b_1\}}$ has non-adjacent vertices.

(ii) Both $(5, 1)$-pillars $P_{B, \{b_1\}}$ and $P_{B, \{b_2\}}$ have at least one vertex.

If the $(5, 2)$-pillar $P_{B, \{b_3, b_4\}}$ contains at least one edge, then the $(5, 2)$-pillar $P_{B, \{b_3, b_2\}}$ is of order at most 26.

**Proof.** Set $G := P_{B, \{b_1, b_2\}}$. By Lemma 4.2, the graph $G$ has largest eigenvalue at most 2. Hence, by Lemma 5.1 there exists an induced subgraph $H'$ of $G$ isomorphic to $nK_1 + mK_2$ for some non-negative integers $n$ and $m$ such that $|V(G)| \leq 4n/3 + 3m$. Also Theorem 3.3 asserts $2m + n \leq 18$, and Theorem 4.1 asserts $m \leq 8$. Therefore,

$$|V(G)| \leq \frac{4n}{3} + 3m = \frac{4}{3} \cdot (2m + n) + \frac{1}{3} \cdot m \leq 24 + \frac{8}{3} < 27.$$ 

\hfill \square

6. A Proof of the Lemmens-Seidel Conjecture for Base Size 5

In this section, we prove the main result Theorem 1.2. First, we provide an upper bound on the sum of orders of $(5, 1)$-pillars, which is smaller than the upper bound in Lemma D.1.

**Lemma 6.1.** Let $H$ be a graph with smallest Seidel eigenvalue at least $-5$ having a maximum clique $B$ of size 5. Then the sum of orders of $(5, 1)$-pillars with respect to $B$ in $H$ is at most 5.

**Proof.** If the tuple of orders of $(5, 1)$-pillars is $(4, 0, 0, 0, 0), (3, 1, 0, 0, 0), (2, 2, 1, 0, 0)$ or $(2, 1, 1, 1, 0)$ up to permutation, then we see by direct calculation that the smallest Seidel eigenvalue of the graph obtained from $H$ by removing all $(5, 2)$-pillars is less than $-5$. Here note that there is no edge in each $(5, 1)$-pillar. Thus the sum of orders of $(5, 1)$-pillars is at most 5. \hfill \square

**Theorem 6.2 [13, Theorem 4.6 (2)].** Let $U$ be a set of $n$ equiangular lines with common angle $\arccos(1/5)$ and base size 5 in dimension $d$. Let $H$ be a graph induced by $U$ with maximal clique $B$ of size 5. If at most one $(5, 2)$-pillar with respect to $B$ in $H$ has a vertex, then

$$n \leq \left\lfloor \frac{4d + 36}{3} \right\rfloor. \quad (6.1)$$

**Proof of Theorem 1.2** Let $U$ be a set of $n$ equiangular lines with common angle $\arccos(1/5)$ and base size 5 in dimension $d$. Fix a graph $H$ induced by $U$ such that $H$ has a maximal clique $B = \{b_1, \ldots, b_5\}$ of size 5. Below we consider pillars with respect to $B$ in $H$. If at most one $(5, 2)$-pillar has a vertex, then Theorem 6.2 gives (6.1). Thus we may assume that at least two $(5, 2)$-pillars have vertices. Also, by Lemma 6.1 the sum of orders of $(5, 1)$-pillars is at most 5.

We may assume that there is a $(5, 2)$-pillar of order at least 2. First, we assume that every $(5, 2)$-pillar has no edge. Then by Theorem 3.3 we have

$$n \leq 5 + 5 + 9 \cdot 24 + 36 = 262.$$ 

Secondly we assume that only one $(5, 2)$-pillar has edges. Then by Theorems 3.3 and 3.4 we have

$$n \leq 5 + 5 + 9 \cdot 18 + 54 = 226.$$
Thirdly we assume that at least two \((5, 2)\)-pillars have edges, and that at least one \((5, 2)\)-pillar has no edge. Then by Theorem \ref{thm:base_size_18} we have

\[
  n \leq 5 + 5 + 9 \cdot 27 + 18 = 271.
\]

Below we assume that every \((5, 2)\)-pillar has at least one edge. We consider the case where a \((5, 1)\)-pillar is of order at least 2. Without loss of generality we may assume that \(P_{B_i}^{(b_i)}\) is of order at least 2. Since the base size of \(U\) is 5, we see that every \((5, 1)\)-pillar has no edges. Hence Corollary \ref{cor:base_size_18} implies that \(P_{B_i}^{(b_i)}\) \((i = 2, 3, 4, 5)\) are of order at most 26. In addition, Theorem \ref{thm:base_size_18} implies that the other \((5, 2)\)-pillars are of order at most 27. Hence

\[
  n \leq 5 + 5 + 6 \cdot 27 + 4 \cdot 26 = 276.
\]

Next we consider the other case, where every \((5, 1)\)-pillar is of order at most 1. Let \(k\) be the number of \((5, 1)\)-pillars of order 1. Without loss of generality we may assume that \(P_{B_i}^{(b_i)}\) \((i = 1, \ldots, k)\) is of order 1. If \(k = 1\), then

\[
  n \leq 5 + 1 + 10 \cdot 27 = 276.
\]

Otherwise by Corollary \ref{cor:base_size_18} \((5, 2)\)-pillars \(P_{B_i}^{(b_i)}\) \((1 \leq i < j \leq k)\) are of order at most 26. Then we have

\[
  n \leq 5 + k + \left(10 - \left(\frac{k}{2}\right)\right) \cdot 27 + \frac{k}{2} \cdot 26 = 275 + k - \frac{k}{2} \leq 276.
\]

This ends the proof. \(\square\)

7. Some properties of sets of 57 equiangular lines with common angle \(\arccos(1/5)\) in dimension 18 found by Greaves et al. \cite{Greaves}

In this section, we answer Questions 1.3 and 1.4 in the negative with the aid of a computer. For each \(i \in \{1, \ldots, 4\}\), write \(F_i\) for the \(10 \times 57\) matrix in \cite{Greaves} Figures 1-4. Let \(S_i := F_i^T F_i/2 - 5I\). Let \(L_i\) be the lattice generated by the 57 columns of \(F_i/\sqrt{2}\). Let \(f_i\) be the \(i\)-th column of \(F_i/\sqrt{2}\), and write \(L_G := L_1\).

Proposition 7.1. The four lattices \(L_1, L_2, L_3\) and \(L_4\) are pairwise isometric, and their minimum norms are at most 4. In particular, the four sets of 57 equiangular lines with common angle \(\arccos(1/5)\) in dimension 18 induced by \(S_1, S_2, S_3\) and \(S_4\) are not contained in the set of 276 equiangular lines with common angle \(\arccos(1/5)\) in dimension 23.

Proof. First, we can verify that \(L_1, L_2, L_3\) and \(L_4\) are pairwise isometric by software such as Magma \cite{Magma}. Next the vector

\[
  [0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ -1 \ -1 \ -1 \ 1 \ 0 \ -1 \ 0 \ 0 \ 1]^T/\sqrt{2}
\]

has norm 4, and is represented as

\[
  f_{44} - f_{48} - f_{49} + f_{51} - f_{52} + f_{53}.
\]

This means that the minimum norm of \(L_G\) is at most 4.

Let \(S_W\) be the Seidel matrix with smallest eigenvalue 5 corresponding to the set of 276 equiangular lines with common angle \(\arccos(1/5)\) in dimension 23, and let \(L_W\) be the lattice with Gram matrix \(5I + S_W\). If the set of equiangular lines corresponding to \(S_i\) is contained in the set of 276 equiangular lines in dimension 23 for some \(i \in \{1, 2, 3, 4\}\), then \(L_G\) is a sublattice of \(L_W\) up to isometry. However, we can verify that the minimum norm of \(L_W\) equals 5 by a computer. Hence \(L_G\) is not a sublattice of \(L_W\) up to isometry. Therefore, the four sets of equiangular lines corresponding to the Seidel matrices \(S_1, S_2, S_3\) and \(S_4\) are not contained in the set of 276 equiangular lines in dimension 23. \(\square\)

Recall that the gap in \cite{Dinke} Proof of Theorem 4.6 (1)] is in claiming that a set of equiangular lines with common angle \(\arccos(1/5)\), base size 6 and at least two pillars having edges is contained in a unique set of 276 equiangular lines in dimension 23. The following together with Proposition 7.1 implies that the four sets of equiangular lines induced by \(S_1, S_2, S_3\) and \(S_4\) are counterexamples to their claim.

Proposition 7.2. The sets of 57 equiangular lines with common angle \(\arccos(1/5)\) in dimension 18 induced by \(S_1, S_2, S_3\) and \(S_4\) have base size 6 and at least two pillars with edges.

Proof. Let \(G\) be the graph induced by the Seidel matrix \(S_1\) with vertex set \(V(G) = \{1, \ldots, 57\}\). Then we can easily check that \(B := \{9, 13, 16, 17, 18, 28\}\) is a maximum clique, and edges \(\{1, 54\}\) and \(\{5, 8\}\) are contained in two distinct pillars with respect to \(B\), respectively. Similarly, we may find a desired clique and edges for each of \(S_2, S_3\) and \(S_4\). \(\square\)
Finally we answer Question 1.4 in the negative as follows.

**Proposition 7.3.** The sets of 57 equiangular lines with common angle $\arccos(1/5)$ in dimension 18 induced by $S_1, S_2, S_3$ and $S_4$ are strongly maximal.

**Proof.** Recall that $L_G$ is generated by $f_1, \ldots, f_{57}$, and $5I + S_1$ equals the Gram matrix of the vectors $f_1, \ldots, f_{57}$. We see that the set is not strongly maximal if and only if there is a non-zero vector $u \in L_G^*: = \{v \in \mathbb{Q}L_G : (v, w) \in \mathbb{Z} \text{ for every } w \in L_G\}$ of norm at most 5 such that $(u, f_i) \in \{1, -1\}$ for every $i \in \{1, \ldots, 57\}$. With a computer, we can verify that such a vector does not exist. Hence we see that the set of equiangular lines corresponding to $S_1$ is strongly maximal. Similarly, we may obtain the desired result for each of $S_2$, $S_3$ and $S_4$. □

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