REDUCED PHASE SPACE QUANTIZATION OF
SPHERICALLY SYMMETRIC EINSTEIN-MAXWELL THEORY
INCLUDING A COSMOLOGICAL CONSTANT

T. Thiemann*†
Institut für Theoretische Physik, RWTH Aachen
D-52074 Aachen, Germany

Abstract

We present here the canonical treatment of spherically symmetric (quantum) gravity coupled to spherically symmetric Maxwell theory with or without a cosmological constant. The quantization is based on the reduced phase space which is coordinatized by the mass and the electric charge as well as their canonically conjugate momenta, whose geometrical interpretation is explored.

The dimension of the reduced phase space depends on the topology chosen, quite similar to the case of pure (2+1) gravity.

We also compare the reduced phase space quantization to the algebraic quantization.

Altogether, we observe that the present model serves as an interesting testing ground for full (3+1) gravity.

We use the new canonical variables introduced by Ashtekar which simplifies the analysis tremendously.

The present article summarizes the work of the papers dealing with the quantization of pure gravity and gravity coupled to a spherically symmetric Maxwell field with or without a cosmological constant, depending on the topology of the the initial data hypersurface.

It is self-evident that we can give here only the results, for details the reader is encouraged to refer to.

Throughout we assume that the reader is familiar with the Ashtekar-formulation of gravity. Also we use the abstract index formalism and the conventions of.

To reduce full gravity including matter to spherical symmetry, we require that the 3-metric, the Maxwell electric (\(\epsilon^a\)) and magnetic fields (\(\mu^a\)) are Lie annihilated by the generators of the SO(3) Killing group. That leaves the freedom free that a rotation of the triads with respect to the tangent bundle is compensated by a rotation with respect to the SO(3) bundle. The result of these Killing-reduction
prescriptions is the following:

Denoting the (local) coordinates on the sphere by \( \theta, \phi \) and the radial variable by \( x \) as in \([\hat{1}]\), we have for the gravitational sector \((A_I, E^I, I = 1, 2, 3 \text{ are angle-independent})\)

\[
(E^x_i, E^\theta_i, E^\phi_i) = (E^1 n^x_i \sin(\theta), \frac{\sin(\theta)}{\sqrt{2}}(E^2 n^\theta_i + E^3 n^\phi_i), \frac{1}{\sqrt{2}}(E^2 n^\phi_i - E^3 n^\theta_i)), \tag{1}
\]

\[
(A^x_i, A^\theta_i, A^\phi_i) = (A^1 n^x_i, \frac{1}{\sqrt{2}}(A^2 n^\theta_i + (A^3 - \sqrt{2}) n^\phi_i), \frac{\sin(\theta)}{\sqrt{2}}(A^2 n^\phi_i - (A^3 - \sqrt{2}) n^\theta_i)) \tag{2}
\]

and for the Maxwell sector we obtain \((n^a \text{ is the standard orthonormal base on the sphere})\)

\[
(\epsilon^x, \epsilon^\theta, \epsilon^\phi) := (\epsilon(x, t), 0, 0), \quad (\mu^x, \mu^\theta, \mu^\phi) := (\mu(x, t), 0, 0). \tag{3}
\]

The Maxwell potential is thus given by

\[
(\omega^x, \omega^\theta, \omega^\phi) = (\omega(x, t), 0, 0) + (\Omega_a(x, t, \theta, \phi)), \tag{4}
\]

where \(\Omega_a\) is a monopole solution with charge \(\mu\). The cosmological constant will be labelled by the (real) parameter \(\lambda\) and by performing a 'duality rotation' we get rid of the magnetic charge.

The model has thus 4 canonical pairs \((\omega, p; A^I, E^I)\) and is subject to the 4 constraints (so that the reduced phase space is finite dimensional), defined by the following 4 constraint functionals:

\[
\begin{align*}
M_G &= \ p' \text{ Maxwell-Gauss constraint,} \tag{5} \\
E_G &= \ (E^1)' + A^2 E^3 - A^3 E^2 \text{ Einstein Gauss constraint,} \tag{6} \\
V &= \ B^2 E^3 - B^3 E^2 \text{ Vector constraint,} \tag{7} \\
C &= \ (B^2 E^2 + B^3 E^3)E^1 + \frac{1}{2}((E^2)^2 + (E^3)^2)(B^1 + \kappa \frac{p^2}{2E^1} + \kappa \lambda E^1) \tag{8}
\end{align*}
\]

: Scalar constraint

where we have abbreviated the components of the magnetic fields as \(B^I\) whose definition is analogous to \((0.1)\) and \(E := (E^2)^2 + (E^3)^2\) (\(\kappa\) is the gravitational constant).

As usual, one has to add the ADM energy and the electric charge to the constraint generators in order to make these functionals well-defined.

For spherically symmetric systems, the topology of the 3 manifold is necessarily of the form \(S^3 = S^2 \times \Sigma\), where \(\Sigma\) is a 1-dimensional manifold. We will deal here only with asymptotically flat topologies. As is motivated in \([\hat{1}]\), we choose \(\Sigma = \Sigma_n, \Sigma_n \cong K \cup \bigcup_{A=1}^n \Sigma_A\), i.e. the hypersurface is the union of a compact set \(K\) (diffeomorphic to a compact interval) and a collection of ends (each of which is diffeomorphic to the positive real line without the origin) i.e. asymptotic regions with outward orientation and all of them are joined to \(K\). We want to point out here that the compactum \(K\) has nothing to do with a horizon, it is just a tool to glue the
various ends together and thus is a kinematical, fixed ingredient of the canonical formalism, whereas the location of a horizon will depend on the mass of the system which is a dynamical object. Thus, although it is appropriate to draw the spacetime pictures which one can find in textbooks for, say, the Schwarzschild configuration with parameter m, the lines $x = m$ which separate the 4 Kruskal regions do not, in general, coincide with the (time evolution of the) compactum K. Boundary conditions for the fields can be derived as in 5 for the asymptotic regions. In the interior, the compactum K, we adapt the support of the fields in such a way that observables are well-defined. This point is subtle and is explained in more detail in 1.

For a review of the method of symplectic reduction the reader is referred to 3, for Hamilton-Jacobi methods to 6.

Choosing 'cylinder coordinates’ ($A_2, A_3 = \sqrt{A} (\cos(\alpha), \sin(\alpha))$, $(E^2, E^3) = \sqrt{E} (\cos(\beta), \sin(\beta))$) it is easy to see that the symplectic potential becomes

$$i \kappa \Theta \partial_t \gamma = \int_\Sigma dx (\dot{\gamma} \pi_\gamma + \dot{B}_1 \pi_1 + \dot{\omega} p(i \kappa)),$$

where $\gamma := A_1 + \alpha'$, $\pi_\gamma := E^1$, $B_1 := \frac{1}{2} (A - 2)$ and $\pi_1 := \sqrt{E/A} \cos(\alpha - \beta)$.

In the following $p$ will already be taken as a constant. Also we will deal with an arbitrary cosmological constant for the sake of generality. We take then the following linear combinations of the vector and the scalar constraint functional $E^1 E^2 V + E^3 C$ and $- E^1 E^3 V + E^2 C$ and set these expressions strongly zero. We then obtain 2 possible solutions:

Case I: $E = 0$ corresponds to degenerate metrics and thus to an unphysical sector.

Case II: $E \neq 0$ (nondegenerate case)

We now conclude

$$E^{2/3} = -\frac{2(E^1)^2}{\kappa (p^2/2 + \lambda (E^1)^2) + B_1 E^1} B^{2/3}$$

and insert this into the Gauss constraint which can be solved as follows:

$$[\kappa (-p^2 + \lambda (E^1)^2/3) + B_1 E^1] = m^2 E^1.$$  

The integration constant, $m$, is real and can be shown to coincide with the gravitational mass up to a factor.

Equation (0.11) is an algebraic equation of fourth order in terms of $E^1$ and, although algebraically solvable, becomes unpractical to handle in the process of symplectic reduction. The idea is to change the polarization and to chose $B_1$ as a momentum. Then (0.11) can be easily solved for $B_1$ and the vector constraint for $\gamma$. We can then apply the theorem proved in 1 and solve the Hamilton Jacobi equation by quadrature techniques. The result for the reduced symplectic potential is given by (modulo a total differential)

$$(\iota^* \Theta) \partial_\gamma = p \int_\Sigma dx (-i \frac{p \pi_1}{\pi_\gamma} - \omega) + \dot{m} \int_\Sigma dx (-i/\kappa) \frac{\pi_1}{\sqrt{\pi_\gamma}} =: \Phi + \dot{m} T$$
where we have assumed that the cosmological constant is time-independent.

The integral expressions for \( T \) and \( \Phi \) turn out not to be well-defined yet, one has to add a certain linear combination of constraints in order to achieve this off the constraint surface. This is satisfactory because a Dirac observable is anyway only unique up to a weakly vanishing expression. In it is proved that the function \( \gamma \) is (weakly) imaginary, hence \( \pi_1 \) is imaginary while \( \pi_\gamma = E^1 \) is real. Accordingly, the volume parts of \( T \) and \( \Phi \) are both real.

The resulting reduced phase space can thus be described as follows: in every asymptotic end \( A \) we have a cotangent bundle over \( \mathbb{R}^2 \). The situation for \( K \) can be handled in a similar manner.

Computing Poisson brackets among the observables and between observables and symmetry generators reaffirms the canonical structure that has been formally derived above and that the observables are really gauge invariant when choosing the Lagrange-multipliers of compact support. For symmetry transformations on the other hand we obtain

\[
\{m_A, G\} = 0, \quad \{T_A, G\} = N_A, \quad \{p_A, G\} = 0, \quad \{\Phi_A, G\} = U_A
\]

(13)

where we have defined \( N_A(t) := N(x = \partial\Sigma_A, t) \), \( U_A(t) := U(x = \partial\Sigma_A, t) \) where \( N := \text{det}(q)^{1/2} N \) is the lapse function, \( G := \int dx [\Lambda(E_G) + U(M_G) + N^x V + \tilde{N} C] \) being the constraint generator. Hence the observables are invariant under radial translations and \( O(2) \)-rotations at spatial infinity while they react nontrivially under time-translations and phase transformations at spatial infinity.

For field theories it is obvious that the equations of motion (0.13) indeed coincide with the equations of motion that follow from the reduced Hamiltonian \( \bar{\Gamma} \) is the constraint surface)

\[
H_{\text{red}}[m, T, p, \Phi] := G[A, N^x, M, U]_{|\bar{\Gamma}} = \sum_{A=1}^n m_A N_A + p_A U_A.
\]

(14)

It is easy to solve the equations of motion (0.13) : introduce functions \( \tau_A(t) \) and \( \phi_A(t) \) defined by \( \frac{d}{dt} \tau_A = N_A \) and \( \frac{d}{dt} \phi_A = U_A \). Then the solution can be written

\[
m_A(t) = \text{const.}, \quad T_A(t) = \text{const.} + \tau_A(t),
\]

\[
p_A(t) = \text{const.}, \quad \Phi_A(t) = \text{const.} + \phi_A(t) \quad A = 1..n
\]

(15)

i.e. the reduced system adopts the form of an integrable system whereby the role of the action variables is played by the masses and the charges whereas their conjugate variables take the role of the angle variables.

What now is the interpretation of this second set of conjugate variables? The interpretation of \( m \) and \( p \) follows simply from the fact that they can be derived from the reduced Hamiltonian, i.e. they are the well-known surface integrals ADM-energy and Maxwell-charge. However, their conjugate partners are genuine volume
integrals and we are not able to write them as known surface integrals. Nevertheless it is possible to give an interpretation: In \cite{1} it is shown that $T$ is the eigentime at spatial infinity while $\Phi$ plays the same role as the variable conjugate to the electric charge of 1+1 Maxwell theory.

Two objections have occurred in the past:
1) the variable $T$ vanishes for a Reissner-Nordstrom foliation and by an extension of Birkhoff’s theorem this foliation can be achieved always. But the evolution law (0.13) contradicts Birkhoff’s theorem.
2) The space of gauge-inequivalent solutions of the Euler-Lagrange equations is labelled by mass and charge only and thus it is surprising that the observables $\Phi$ and $T$ exist at all.

Both objections can be resolved by the observation that the notions of gauge are different when one looks at a system either from the Hamiltonian or Lagrangian viewpoint. This interesting item is analyzed in much more detail in \cite{1}.

We finally come to the quantization of the system. We follow the group theoretical quantization scheme (see ref. \cite{5}). The phase space for every end is just the cotangent bundle over the two-plane so the unique Hilbert-space is the usual one: $L^2(R^2, d^2x)$. The Schroedinger-equation in the polarization in which eigentime and the flux act by multiplication and the mass and the charge by differentiation becomes unambiguously

$$i\hbar \frac{\partial}{\partial t} \Psi(t; \{T_A\}, \{\Phi_A\}) = (-i\hbar \sum_{A=1}^{n}(N_A(t)\frac{\partial}{\partial T_A} + U_A(t)\frac{\partial}{\partial \Phi_A}))\Psi(t; \{T_A\}, \{\Phi_A\}).$$

(16)

It can be solved trivially by separation: $\Psi(t; \{m_A\}, \{\Phi_A\}) := \prod_{A=1}^{n} \psi_A(t, m_A, \Phi_A)$ and by introducing the functions defined by integrating $\dot{\tau}_A(t) := N_A(t), \dot{\phi}_A(t) := U_A(t)$. We then find as the general solution

$$\psi_A(t, T_A, \phi_A) = C_A \exp(k_A \frac{i}{\hbar}[T_A - \tau_A(t)]) \times \exp(l_A \frac{i}{\hbar}[\Phi_A - \phi_A(t)])$$

(17)

where $C_A$ is a complex number, whereas $k_A, l_A$ must be real because the spectrum of the momenta, which are self-adjoint, is real. These solutions of the time-dependent Schroedinger equation are obviously peaked at an instant of ‘time’ $t$ around the classical solutions (see (0.15)) in the sense that they are strongly oscillating off the classical trajectory.

Let us compare with the operator constraint method:
We multiply the scalar constraint with a factor of $E^1$ so that in the ordering in which all momenta (for the Ashtekar polarization) stand to the right, the scalar constraint becomes

$$[(B^2 \frac{\delta}{\delta A_2} + B^3 \frac{\delta}{\delta A_3}) \frac{\delta^2}{\delta A_1^2} + \frac{1}{2}(B^1 \frac{\delta}{\delta A_1} + \kappa(-\frac{1}{2} \frac{\delta^2}{\delta \omega^2} + \lambda \frac{\delta^2}{\delta A_1^2}))(\frac{\delta^2}{\delta A_2^2} + \frac{\delta^2}{\delta A_3^2})] \Psi(A_I, \omega) = 0$$

(18)
which is a 4th order functional differential equation. We have here no space to dwell on the issues of ordering and regularization but it turns out that in the polarization in which the Ashtekar-connection becomes a momentum, the quantization schemes of Dirac and symplectic reduction actually coincide.

Final remark:
It is interesting to express the observables found in terms of the Ashtekar-connection : one has simply to solve equation (0.11) for $E^1$ in terms of $B^1$, $p$ and $\lambda$ and plug this into the expressions for the observables. We restrict for the sake of brevity to the case $\lambda = 0$ and obtain for the integrand of the variables $T$ and $\Phi + \int_\omega d\omega$
respectively

$$
-2A_1 + [\text{arctan}(A_2)]' [p^2\kappa + \frac{m^2}{2}\pi^2 + \sqrt{[p^2\kappa + \frac{m^2}{2}\pi^2]^2 - [p^2\kappa]^2}]^{2-n} \\
\frac{(2B^1)^{2-n}}{p^2\kappa + 1/2(\frac{m^2}{2} + \sqrt{[p^2\kappa + \frac{m^2}{2}\pi^2]^2 - [p^2\kappa]^2})}
$$

where $n = 1/2$ and $n = 1$ respectively and $B^1 = 1/2((A_2)^2 + (A_3)^2 - 2)$. This expression is much more complicated than the one in (0.12) in terms of $\pi_1$ and $\pi_2$ and it is thus suggested that in general the polarization that one starts with will not turn out to be the natural one for the problem at hand.

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