Research Article

On a Sum Involving the Sum-of-Divisors Function

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1. Introduction

As usual, denote by \(\varphi(n)\) the Euler function and by \([r]\) the integral part of real \(r\), respectively. Recently, Bordellès et al. [1] studied the asymptotic behaviour of the quantity

\[
S_\varphi(x) = \sum_{n \leq x} \varphi\left(\left\lfloor \frac{x}{n} \right\rfloor \right),
\]

for \(x \to \infty\). By exponential sum technique, they proved that

\[
\left(\frac{2629}{4009} \cdot \frac{6}{\pi^2} + o(1)\right) x \log x \leq S_\varphi(x)
\]

and conjectured that

\[
S_\varphi(x) \sim \frac{6}{\pi^2} x \log x,
\]

as \(x \to \infty\). Very recently, Wu [2] improved (2) and Zhai [3] resolved conjecture (3) by showing

\[
S_\varphi(x) = \frac{6}{\pi^2} x \log x + O\left( x (\log x)^{2/3} (\log_2 x)^{4/3} \right),
\]

and also proved that the error term in (4) is \(\Omega(x)\), where \(\log_2\) denotes the iterated logarithm. Some related works can be found in [4, 5]. Since the sum-of-divisors function \(\sigma(n) = \sum_{d \mid n} d\) has similar properties as the Euler function \(\varphi(n)\) in many cases, it seems natural and interesting to consider its analogy of (3).

Our result is as follows.

Theorem 1

(i) For \(x \to \infty\), we have

\[
S_\sigma(x) = \sum_{n \leq x} \sigma\left(\left\lfloor \frac{x}{n} \right\rfloor \right) = \frac{\pi^2}{6} x \log x
\]

\[+ O\left( x (\log x)^{2/3} (\log_2 x)^{4/3} \right). \tag{5}\]

(ii) Let \(E(x)\) be the error term in (5). Then, for \(x \to \infty\), we have

\[
E(x) = \Omega(x), \quad \text{i.e. } \limsup_{x \to \infty} \frac{|E(x)|}{x} > 0. \tag{6}\]

Let \(\mu(n)\) be the Möbius function and define \(\text{id}(n) = n\) and \(1(n) = 1\) for all integers \(n \geq 1\). Then, \(\varphi = \text{id} \ast \mu\) and \(\sigma = \text{id} \ast 1\). In Zhai’s approach proving (4), the inequality

\[
\frac{6}{\pi^2} x \log x \leq S_\sigma(x)
\]

for \(x \to \infty\) was obtained. To this end, we introduce the following notations.
\[ \sum_{n \leq x} \mu(n) \ll x \exp \left\{ -c \sqrt{\log x} \right\}, \quad (x \geq 1), \]  

plays a key role, where \( c > 0 \) is a positive constant. Clearly, such a bound is not true for 1. By refining Zhai’s approach, we shall prove our result.

### 2. Preliminary Lemmas

As in [3], we need some bounds on exponential sums of the type \( \sum_{N \leq n \leq N'} e(T/n) \) where \( N < N' \leq 2N \). For large values of \( N \), Zhai used the theory of exponent pair, and for smaller ones the Vinogradov method. Both estimates are contained in the following general theorem of Karatsuba [6, Theorem 1], which will be a key tool for proving Theorem 1.

**Lemma 1.** Let \( k \geq 2 \) and \( M \) and \( P \) be integers, \( P \) being positive. Let \( f \in \mathbb{R}^{k+1}([M, M+P]; \mathbb{R}) \). Suppose that there exist positive absolute constants \( c_0, c_1, c_2, c_3 \), and \( c_4 \) such that \( c_0 < 1, c_1 < 1, \) and \( c_2 + c_3 < c_1 \); an integer \( r \) such that \( c_0/k \leq r \leq k \); and distinct numbers \( s_j \geq 2 \) (\( j = 1, \ldots, r \)) not exceeding \( k \), such that for \( M \leq t \leq M + P \) the following inequalities are satisfied:

(i) \( |f(t)/(k+1)| \leq P^{-c_1} \),

(ii) \( P^{-c_2} \leq |f(s_j)/(s_j)| \leq P^{-c_3} \), (\( j = 1, \ldots, r \)).

Then, for each positive integer \( P \), not exceeding \( P \), we have

\[ \left| \sum_{M \leq m < M+P} e(f(m)) \right| \leq A P^{1-c/k^2}, \]  

where \( e(t) := e^{2\pi t} \) and \( A > 0, c > 0 \) are absolute constants.

The next two lemmas are essentially a special case of [7, Lemmas 2.5 and 2.6] with \( a = 1 \). The only difference is that the ranges of \( T \) and \( N \) here are slightly larger than those of [7, Lemmas 2.5 and 2.6] (\( T \geq N^2 \) in place of \( T \geq N^{(3/2)} \) and \( N \leq x^{(2/3)} \) in place of \( N \leq x^{(1/2)} \), respectively). Although the proof is completely similar, for the convenience of readers, we still reproduce a proof here.

**Lemma 2.** Let \( e^{100} \leq N < N' \leq 2N \) and \( T \geq N^{(3/2)} \). Then, there exists an absolute positive constant \( c_5 \) such that

\[ \sum_{N \leq n < N'} e\left( \frac{T}{n} \right) \ll N \exp \left\{ \left( \frac{c_5 \log^3 N}{\log^2 T} \right)^2 \right\}, \quad (9) \]

where the implied constant is absolute.

**Proof.** We apply Lemma 1 to \( f(t) = (T/t) \) with \( M = N, P = N, P_1 = N' - N \). For this, we choose

\[ c_0 = \frac{1}{100}, \quad c_1 = \frac{99}{100}, \quad c_2 = \frac{87}{100}, \quad c_3 = \frac{3}{4}, \quad c_4 = \frac{1}{100}, \]

and take the \( s_j \) to be all integers \( s \) such that

\[ \frac{4 \log(T/n)}{\log N} < s < 5 \frac{\log(T/n)}{\log N}. \]  

Obviously the number \( r \) of \( s_j \) is between \( c_0 k \) and \( k \). Next we shall verify that \( f(t) \) satisfies the conditions (i) and (ii) of Lemma 1 with the parameters chosen above.

For \( N \leq t \leq 2N \), we have

\[ \left| \frac{f^{(k+1)}(t)}{(k+1)!} \right| = T^{1-k-2} \leq T N^{1-k} = N^{-\eta_1}, \]  

where

\[ \eta_1 := k + 1 - \frac{\log(T/n)}{\log N} \geq k + 1 - \frac{1}{100} k \geq \frac{99}{100} (k + 1) = c_1 (k + 1). \]

Similarly for \( N < t \leq 2N \), we find the inequality

\[ \left| f^{(s_j)}(t) / s_j ! \right| \leq N^{-\eta_2}, \]

where

\[ \eta_2 := \frac{\log(T/n)}{\log N} - \frac{3}{4} s_j = c_3 s_j. \]

For the lower bound of (ii), we have

\[ \left| \frac{f(t)}{s_j !} \right| = T T^{1-s_j} \geq T (2N)^{1-s_j} = N^{-\eta_3}, \]

where

\[ \eta_3 := \frac{s_j}{5} + \frac{\log 2}{\log N} \left( s_j + 1 \right) \leq \frac{87}{100} s_j = c_3 s_j. \]

From Lemma 1, there exist two positive constants \( c \) and \( A \) such that
\[
\left| \sum_{N \leq n < N'} e^{it/n} \right| \leq AN^{1-(\varepsilon x)^2} \leq AN \exp \left( -\frac{c_2 \log^3 N}{\log^2 (T/n)} \right),
\]
(17)

with \(c_2 = 10^{-4}c\). This completes the proof of Lemma 2. \(\square\)

Lemma 3. Define \(\psi(t) = t - [t] - (1/2)\). Let \(c_5\) be the constant defined by Lemma 2 and \(c_6 = (8/9)^2 c_5\), \(c^* = (3/5)c_5^{-(1/3)}\). Then, we have

\[
\sum_{N \leq n < N'} \frac{1}{n} \psi \left( \frac{x}{n} \right) \ll e^{-c_6 (\log N)^i/(\log x)^2} \frac{(\log N)^3}{(\log x)^2},
\]
uniformly for \(x \geq 10\), \(\exp \left[ c^* (\log x)^{(2/3)} \right] \leq N \leq x^{(2/3)}\) and \(N < N' \leq 2N\).

Proof. By invoking a classical result on \(\psi(t)\) (see 8, page 39), we can write, for any \(H \geq 1\),

\[
\sum_{N \leq n < N'} \psi \left( \frac{x}{n} \right) \ll NH^{-1} + \sum_{1 \leq b \leq H} \left| \sum_{N \leq n < N'} e^{ibx/n} \right|.
\]
(19)

An application of Lemma 2 with \(T = hx \geq x \geq N^{(3/2)}\) yields

\[
\sum_{N \leq n < N'} \psi \left( \frac{x}{n} \right) \ll N \left( H^{-1} + e^{-c_6 (\log N)^i/(\log x)^2} (\log x)^2 \log H \right).
\]
(20)

Taking \(H = \exp \left[ (\log N)^3/(\log x)^2 \right] \leq x^{(9/27)}\), we easily deduce that

\[
\sum_{N \leq n < N'} \psi \left( \frac{x}{n} \right) \ll N \left( e^{-c_6 (\log N)^i/(\log x)^2} + e^{-c_6 (\log N)^i/(\log x)^2} (\log N)^3 \right) / (\log x)^2.
\]
(21)

The first term can be absorbed by the second, since \(c_5\) can be chosen small enough to ensure that \(c_5 < 1\) and since \(\exp \left( c^* (\log x)^{(2/3)} \right) \leq N\) implies \( (\log N)^3/(\log x)^2 \geq c^*\). Hence,

\[
\sum_{N \leq n < N'} \psi \left( \frac{x}{n} \right) \ll Ne^{-c_6 (\log N)^i/(\log x)^2} \frac{(\log N)^3}{(\log x)^2},
\]
(22)

and an Abel summation produces the required result. \(\square\)

Lemma 4. Let \(2 \leq z_1 < z_2 \leq x\) and \(F_x(t) = (1/t)\psi(x/t)\). Denote by \(V_{F_x}[z_1, z_2]\) the total variation of \(F_x\) on \([z_1, z_2]\). Then,

\[
V_{F_x}[z_1, z_2] \ll \frac{x}{z_1^{3/2}} + \frac{1}{z_1},
\]
(23)

where the implied constant is absolute.

Proof. If \(z_1 = t_0 < t_1 < \cdots < t_n = z_2\) is a partition of the interval \([z_1, z_2]\), then

\[
\sum_{k=1}^n \left| F_x(t_k) - F_x(t_{k-1}) \right| \ll \sum_{k=1}^n \left| \frac{1}{t_k} \psi \left( \frac{x}{t_k} \right) - \frac{1}{t_{k-1}} \psi \left( \frac{x}{t_{k-1}} \right) \right| \leq \sum_{k=1}^n \left| t_{k-1} - t_k \right| \psi \left( \frac{x}{t_k} \right) + \sum_{k=1}^n \left| t_k - t_{k-1} \right| \psi \left( \frac{x}{t_{k-1}} \right) + \left( \frac{x}{t_{k-1}} \right) \frac{1}{N} \leq \frac{2}{z_1} \left( \frac{x}{z_1} + 1 \right) V_{\psi}[0, 1] \leq \frac{2}{z_1} \left( \frac{x}{z_1} + 1 \right).
\]
(24)

Since \(|\psi(t)| \leq 1\) for all \(t\), we have

\[
\sum_{k=1}^n \left| \frac{1}{t_k} - \frac{1}{t_{k-1}} \right| \psi \left( \frac{x}{t_k} \right) \leq \frac{1}{z_1} - \frac{1}{z_2} \leq \frac{1}{z_1} - \frac{1}{z_2}.
\]
(25)

On the other hand, since \(\psi(t)\) is of period 1, we have

\[
\sum_{k=1}^n \left| \frac{1}{t_k} - \frac{1}{t_{k-1}} \right| \psi \left( \frac{x}{t_k} \right) \leq \frac{1}{z_1} - \frac{1}{z_2} \leq \frac{1}{z_1} - \frac{1}{z_2}.
\]
(26)

Inserting these two bounds into (24), we obtain the required result. \(\square\)

3. Proof of Theorem 1

3.1. A Formula on the Mean Value of \(\sigma(n)\)

Lemma 5

(i) For \(x \geq 2\) and \(1 \leq z \leq x^{(1/3)}\), we have

\[
\sum_{n \leq x} \sigma(n) = \frac{\pi^2}{12} x^2 - x \left( \frac{z - [z]^2}{z} + [z] \right) + O \left( \frac{x}{z} \right) - \Delta(x, z),
\]
(27)

where

\[
\Delta(x, z) = \sum_{d \leq \sqrt{x}} \frac{x}{d} \psi \left( \frac{x}{d} \right) \psi \left( \frac{x}{d} \right).
\]
(28)

(ii) For \(x \rightarrow \infty\), we have

\[
\sum_{n \leq x} \sigma(n) = \frac{\pi^2}{12} x^2 + O(x \log x).
\]
(29)

Proof. Using \(\sigma(n) = \sum_{d | n} m\), the hyperbola principle of Dirichlet allows us to write

\[
\sum_{n \leq x} \sigma(n) = \sum_{d \leq x} m = S_1 + S_2 - S_3,
\]
(30)

where
\[ S_1 := \sum_{d \leq x} \sum_{m \leq \nu(d)} m, \]
\[ S_2 := \sum_{m \leq z} \sum_{d \leq \nu(m)} m, \]
\[ S_3 := \sum_{d \leq \nu(z)} \sum_{m \leq z} m. \]

Firstly we have
\[
S_2 = \sum_{m \leq z} \left[ \frac{m}{m} \right] = x[z] + O(z^2), \tag{32}\]
\[
S_3 = \left[ \frac{x}{z} \right] \left( \left[ \frac{z}{z} \right] + 1 \right) \left( \frac{x}{z} \right) = x \left[ \frac{z}{z} \right] \left( \frac{x}{z} \right) + O(z^2). \tag{33}\]

Secondly we can write
\[
S_1 = \frac{1}{2} \sum_{d \leq \nu(z)} \left( \frac{x}{d} - \psi \left( \frac{x}{d} \right) \right) - \frac{1}{2} \left( \frac{x}{d} - \psi \left( \frac{x}{d} \right) \right) + \frac{x}{d} = \frac{1}{2} \sum_{d \leq \nu(z)} \left( \frac{x^2}{d^2} - \frac{2x}{d^2} \psi \left( \frac{x}{d} \right) + \psi \left( \frac{x}{d} \right) \right) \tag{34}\]
\[
= \frac{\pi^2}{12} x^2 - \frac{1}{2} x \log x - \Delta(x, z) + O(x/z),
\]
where \( \Delta(x, z) \) is as in (28). Inserting (32), (33), and (34) into (30) and using \( z^2 \leq (x/z) \), we get (27).

Taking \( z = 1 \) in (27) and noticing that
\[
\sum_{d \leq x} \frac{1}{d^2} = \frac{\pi^2}{6} + O \left( \frac{1}{x} \right), \tag{35}\]
\[
\sum_{d \leq x} \frac{x}{d^2} \psi \left( \frac{x}{d} \right) \ll x \log x,
\]
we obtain the required bound. This completes the proof. \( \square \)

3.2. Estimates of Error Terms

**Lemma 6.** Let \( N_0 := \exp \left\{ (6/c_6) \log x \left( \log \log x \right)^{1/3} \right\} \), where \( c_6 \) is given as in Lemma 3. Let \( \Delta(x, z) \) be defined by (28). Then, for \( x \geq 10 \) and \( 2 \leq z \leq \sqrt{N_0} \), we have
\[
\left| \sum_{N_0 \leq n \leq \sqrt{x}} \Delta \left( \frac{x}{n}, z \right) \right| + \left| \sum_{N_0 \leq n \leq \sqrt{x}} \Delta \left( \frac{x}{n}, 1, z \right) \right| \ll \left( \frac{1}{\log x} + \frac{\log x}{z} \right). \tag{36}\]

**Proof.** Denote by \( \Delta_1(x, z) \) and \( \Delta_2(x, z) \) two sums on the left-hand side of (36), respectively. By (28) of Lemma 5, we can write
\[
\Delta_1(x, z) = x \sum_{N_0 \leq n \leq \sqrt{x}} \sum_{d \leq x} \frac{1}{dn} \psi \left( \frac{x}{dn} \right),
\]
\[
= x \sum_{d \leq x} \frac{1}{dn} \sum_{N_0 \leq n \leq \sqrt{x}} \frac{1}{n} \psi \left( \frac{x}{dn} \right),
\]
\[
= x \Delta_1^*(x, z) + x \Delta_2^*(x, z), \tag{37}\]
where
\[
\Delta_1^*(x, z) := \sum_{d \leq x} \frac{1}{d} \sum_{N_0 \leq n \leq \sqrt{x}(d/n)} \frac{1}{n} \psi \left( \frac{x}{dn} \right),
\]
\[
\Delta_2^*(x, z) := \sum_{d \leq x} \frac{1}{d} \sum_{N_0 \leq n \leq \sqrt{x}(d/n)} \frac{1}{n} \psi \left( \frac{x}{dn} \right). \tag{38}\]

For \( 0 \leq k \leq (\log((x/d)^{1/3})/n_0)/\log 2 \), let \( N_k := 2^k N_0 \) and define
\[
\mathcal{G}_k(d) := \sum_{N_0 \leq n \leq 2N_k} \frac{1}{n} \psi \left( \frac{x}{dn} \right). \tag{39}\]

Noticing that \( N_0 \leq N_k \leq (x/d) \), we can apply Lemma 3 to derive that
\[
\mathcal{G}_k(d) \ll e^{-\theta \left( \log N_k \right)^{1/3} \left( \log(x/d)^{1/3} \right)}, \tag{40}\]
with \( \theta(t) = c_6 t - \log t \). It is clear that \( \theta(t) \) is increasing on \( [c_6, \infty) \). On the other hand, for \( k \geq 0 \) and \( d \geq 1 \), we have
\[
\left( \log N_k \right)^{1/3} \left( \log(x/d)^{1/3} \right) \geq \left( \log N_0 \right)^{1/3} \left( \log x \right)^{1/3} = (6/c_6) \log_2 x.
\]

Thus,
\[
\theta \left( \left( \log N_k \right)^{1/3} \right) \geq \theta \left( \left( \frac{6}{c_6} \right) \log_2 x \right)
\]
\[
= 6 \log_2 x - \log \left( \left( \frac{6}{c_6} \right) \log_2 x \right) \geq 5 \log_2 x,
\]
which implies that \( \mathcal{G}_k(d) \ll (\log x)^{-5} \). Inserting this into the expression of \( \Delta_1^*(x, z) \), we get
\[
\Delta_1^*(x, z) \ll \sum_{d \leq x} \frac{1}{d^2} \sum_{N_0 \leq n \leq \sqrt{x}(d/n)} \left| \mathcal{G}_k(d) \right| \ll (\log x)^{-3}. \tag{43}\]

Next we bound \( \Delta_2^*(x, z) \). Let \( F(t) \) be a function of bounded variation on \( [n, n+1] \) for each integer \( n \) and let \( V_F[n, n+1] \) be the total variation of \( F \) on \( [n, n+1] \). Integrating by parts, we have
\[ \int_0^{n+1} \left( t - n - \frac{1}{2} \right) dF(t) = \frac{1}{2} (F(n + 1) + F(n)) - \int_0^{n+1} F(t) \, dt. \] (44)

From this, we can derive that
\[ \frac{1}{2} (F(n + 1) + F(n)) = \int_0^{n+1} F(t) \, dt + O(V_F [n, n + 1]), \] (45)

for \( n \geq 1 \). Summing over \( n \), we find that
\[ \sum_{N_1 < n \leq N_2} F(n) = \int_{N_1}^{N_2} F(t) \, dt \]
\[ + \frac{1}{2} (F(N_1) + F(N_2)) + O(V_F [N_1, N_2]). \] (46)

We apply this formula to
\[ F_{(x/d)}(t) = \frac{1}{t} \psi \left( \frac{x}{d} \right), \]
\[ N_1 = \left\lceil \frac{x}{d} \right\rceil, \]
\[ N_2 = \left\lceil \frac{x}{d} \right\rceil, \] (47)

According to Lemma 4, we have
\[ V_F \left( [N_1, N_2] \right) \ll \left( x/d \right)^{-(1/3)}, \] and thus by putting
\[ u = (x/d)t, \] we obtain, with the notation
\[ x_{d,1} = \max \left( \frac{\sqrt{x}}{d}, tz \right) \] and \( x_{d,2} = (x/d)^{(1/3)}, \)
\[ \sum_{(x/d)^{(2/3)} < n \leq \min \left( \sqrt{x}, \left\lceil \frac{x}{d} \right\rceil \right)} \frac{1}{n} \psi \left( \frac{x}{dn} \right) = \int_{x_{d,1}}^{x_{d,2}} \frac{\psi(u)}{u} \, du + O \left( \left( \frac{x}{d} \right)^{-(1/3)} \right) \]
\[ \ll z^{-1} \left( \frac{x}{d} \right)^{-(1/3)} \ll \frac{1}{z}, \] (48)

where we have used the fact that \( z \leq \sqrt{N_0} \) and \( d \leq (x/\sqrt{N_0}) \Rightarrow z \leq (x/d)^{(1/3)} \) and the bound
\[ \int_{x_{d,1}}^{x_{d,2}} \frac{\psi(u)}{u} \, du = \int_{x_{d,1}}^{x_{d,2}} \frac{\psi(t)}{u} \, du - \int_{x_{d,2}}^{x_{d,2}} \frac{\psi(t)}{u} \, dt \]
\[ \ll x_{d,1}^{-1} + (x_{d,2})^{-(2/3)} \ll \frac{1}{z} + (x/d)^{-(2/3)} \ll \frac{1}{z}. \] (49)

Using (48), a simple partial integration allows us to derive that
\[ \Delta_1^t(x, z) \ll z^{-1} \sum_{d \leq x} (N_{d,z}) d^{-1} \ll z^{-1} \log x. \] (50)

Combining (43) and (50), it follows that
\[ \Delta_1(x, z) \ll x \left( \log x \right)^{-3} + xz^{-1} \log x. \] (51)

Similarly, we can prove the same bound for \( \Delta_2(x, z) \). This completes the proof. \( \square \)

3.3. End of the Proof of Theorem 1. Let \( c_0 \) be the constant given as in Lemma 3 and \( N_0 = \exp \left( \left( 6/c_0 \right) \left( \log x \right)^{2/3} \right) \).

Let \( z \in \left[ 2, \sqrt{N_0} \right) \) be a parameter to be chosen later.

Putting \( d = [x/n] \), we have \( (x/n) - 1 < d \leq (x/n) \) and \( x/(d + 1) < n \leq (x/d) \). We have, with the convention \( \sigma(0) = 0 \),
\[ S_\sigma(x) = \sum_{d \leq x} \sigma(d) \sum_{(x/d+1) \leq n \leq x} 1 \]
\[ = \sum_{d \leq x} \sigma(d) - \sum_{d \leq x} \sigma(d - 1) \] (52)
\[ = \sum_{d \leq x} \sigma(d) - \sigma(d - 1). \]

By the hyperbole principle of Dirichlet, we can write
\[ S_\sigma(x) = S_1(x, \sigma) + S_2(x, \sigma) - S_3(x, \sigma), \] (53)
where
\[ S_1(x, \sigma) = \sum_{d \leq \sqrt{x}, \sigma \leq x} \left( \sigma(d) - \sigma(d - 1) \right); \]
\[ S_2(x, \sigma) = \sum_{d \leq \sqrt{x}, \sigma \leq x} \left( \sigma(d) - \sigma(d - 1) \right); \] (54)
\[ S_3(x, \sigma) = \sum_{d \leq \sqrt{x}, \sigma \leq x} \left( \sigma(d) - \sigma(d - 1) \right). \]

With the help of the bound \( \sigma(n) \ll n \log_4 n \), we can derive that
\[ S_3(x, \sigma) = [\sqrt{x}] \sigma([\sqrt{x}]) \ll x \log_4 x. \] (55)

For evaluating \( S_1(x, \sigma) \), we write
\[ S_1(x, \sigma) = \sum_{d \leq \sqrt{x}} \left( \sigma(d) - \sigma(d - 1) \right) \left( \frac{x}{d} \right) \]
\[ = x \sum_{d \leq \sqrt{x}} \sigma(d) \sigma(d - 1) \frac{x}{d} + O \left( \sum_{d \leq \sqrt{x}} |\sigma(d) - \sigma(d - 1)| \right). \] (56)

With the help of Lemma 5 (ii), a simple partial integration gives us
\[
\sum_{d \leq x} \frac{\sigma(d) - \sigma(d-1)}{d} = \sum_{d \leq x} \frac{\sigma(d)}{d} - \sum_{d \leq x} \frac{\sigma(d)}{d^2 (d+1)}
\]

\[
= \sum_{d \leq x} \frac{\sigma(d)}{d^2} - \sum_{d \leq x} \frac{\sigma(d)}{d^2 (d+1)}
\]

\[
= \int_1^x t^{-2} \left( \frac{\pi^2}{12} + O(t \log t) \right) + O(1)
\]

\[
= \frac{\pi^2}{12} \log x + O(1),
\]

\[
\sum_{d \leq x} |\sigma(d) - \sigma(d-1)|
\]

\[
\ll \sum_{d \leq x} \sigma(d) \ll x.
\]

(57)

Inserting these estimates into (56), we find that

\[
S_1(x, \sigma) = \frac{\pi^2}{12} x \log x + O(x).
\]

(58)

Finally, we evaluate \( S_2(x, \sigma) \). For this, we write

\[
S_2(x, \sigma) = S_2^1(x, \sigma) + S_2^2(x, \sigma),
\]

(59)

where

\[
S_2^1(x, \sigma) := \sum_{\substack{n \leq x \atop n \in \mathbb{N}}} \left( \frac{\pi^2}{6} \cdot \frac{x}{n} - \Delta \left( \frac{x}{n}, z \right) + \Delta \left( \frac{x}{n} - 1, z \right) + O(\frac{x}{n}) \right)
\]

\[
= \frac{\pi^2}{12} x \log x + O(x (\log x)^{2/3} (\log_2 x)^{1/3} + xz^{-1} \log x) - \Delta_1(x, z) + \Delta_2(x, z),
\]

(63)

\[
\sum_{d | p} (\sigma(d) - \sigma(d-1)) = S_p(p) - S_{\sigma}(p-1)
\]

\[
= \frac{\pi^2}{6} (\log p - \log (p-1)) + E(p) - E(p-1)
\]

\[
\geq E(p) - E(p-1) \geq 2E^*(p),
\]

(66)

where \( E^*(p) := \max\{|E(p)|, |E(p-1)|\} \). On the other hand, we have

\[
\sum_{d | p} (\sigma(d) - \sigma(d-1)) = \sigma(p) - \sigma(p-1) + 1
\]

\[
\leq p + 1 - \left( p - 1 + \frac{1}{2}(p - 1) + 2 + 1 \right) + 1 \leq \frac{1}{4} p.
\]

(67)

Thus, \( E^*(p) \geq (1/8)p \) for all odd primes.

3.4. Proof of Theorem 1. (ii) For any odd prime \( p \), (52) allows us to write

\[
\frac{\pi^2}{12} x \log x + O(x (\log x)^{2/3} (\log_2 x)^{1/3} + xz^{-1} \log x),
\]

(65)

Now (5) follows from (53), (55), (58), and (66) with the choice of \( z = (\log x)^{1/3} \).

Data Availability

No data were used to support this study.
Conflicts of Interest

The authors declare that they have no conflicts of interest.

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