Singularities in isotropic non-minimal scalar field models

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Abstract
Non-minimally coupling a scalar field to gravity introduces an additional curvature term into the action which can change the general behaviour in strong curvature regimes, in particular close to classical singularities. While one can conformally transform any non-minimal model to a minimally coupled one, that transformation can itself become singular. It is thus not guaranteed that all qualitative properties are shared by minimal and non-minimal models. This paper addresses the classical singularity issue in isotropic models and extends singularity removal in quantum gravity to non-minimal models.

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1. Introduction

It is well established that general relativity implies spacetime singularities for most solutions deemed to be relevant for physical situations such as cosmology or black holes [1, 2]. Precise general properties of such singularities, besides the criterion of geodesic incompleteness usually used to prove singularity theorems, are however difficult to find. Even in isotropic models there are several types of singularities in addition to the common big-bang type singularities at the vanishing scale factor $a(t)$ of an FRW model. Other possibilities include singularities where the scale factor diverges in finite proper time, such as in superinflationary solutions of a constant equation of state parameter $w < -1$, or singularities at non-zero, finite $a$ (see, e.g. [3, 4]). These alternatives are rather special: while a big-bang singularity can be shown to arise for isotropic models using only energy conditions and non-zero expansion at one time, singularities at non-zero $a$ require additional conditions to be satisfied by the matter ingredients. Such singularities can therefore be obtained in phenomenological matter models, but not so easily in models sourced by scalar or other fields. In minimally coupled scalar field models, for instance, one has only the freedom to choose a potential which does not allow, e.g. singularities associated with super-inflationary expansion.
One additional freedom for scalar fields is a non-minimal coupling to spacetime curvature by adding a term $-\frac{1}{2} \xi \mathcal{R} \phi^2$ to the Lagrangian. This can change in particular the behaviour close to curvature singularities where $\mathcal{R}$ becomes large. Such a term is also interesting because it is the only possible curvature coupling of any matter field which does not require one to use additional length scales for the correct dimensions. Often, curvature couplings of fundamental fields are considered as effective terms to be derived from a quantum theory of gravity. In such a case, there is an additional length parameter provided by the Planck length $\ell_P = \sqrt{\frac{G}{\hbar}}$ which multiplies effective curvature terms. All these terms thus vanish in the classical limit except for the non-minimal coupling term of a scalar with quadratic coupling function, which is not multiplied by $\ell_P$. Since a classical theory with such a term makes sense, one can also ask whether it would change any results of quantum gravity obtained with minimal coupling. Other curvature couplings, on the other hand, which are multiplied by powers of $\ell_P$, cannot reasonably be quantized because they arise only from an effective description of an already quantized theory of gravity.

One application of quantum gravity is singularity removal which can depend on the matter type. Moreover, with a non-minimal coupling term of a scalar already, the classical behaviour can change close to curvature singularities. We therefore perform in this paper an analysis of isotropic singularities in non-minimal scalar field models based on canonical transformations to a minimally coupled model. We will end with conclusions for singularity removal by quantum gravity.

2. Non-minimal coupling

Non-minimal coupling is usually introduced at the Lagrangian level which makes the properties of Lorentz invariance manifest. At the level of the action, one can also see that any non-minimally coupled model allows a conformal transformation and redefinition of the field variables which brings it into a minimally coupled form [5, 6]. This transformation, however, can be difficult to determine analytically which makes the discussion of singularities complicated. After recalling these properties, we will thus switch to a canonical formulation. This allows us to perform canonical transformations which are more general than transformations allowed at the Lagrangian level. It will turn out that a much simpler transformation to a minimally coupled form is then possible which we will exploit in our discussion of singularities.

The simplest form of a non-minimally coupled scalar field action is given by

$$S[g_{ab}, \phi] = \int d^4x \sqrt{-g} \left( \frac{f(\phi)}{2\kappa} R - \frac{1}{2} g^{ab} \nabla_a \phi \nabla_b \phi - U(\phi) \right),$$

(1)

where $g \equiv \text{det}(g_{ab})$, $\kappa = 8\pi G$, $R$ is the scalar curvature of the metric and $U(\phi)$ is the self-interacting potential of the matter scalar field $\phi$. The simplest non-trivial form of the coupling function $f(\phi)$ corresponding to a curvature coupling quadratic in $\phi$ is

$$f(\phi) = 1 - \sigma \phi^2.$$  

(2)

We will refer to $\sigma = \kappa \xi$ as the coupling strength, vanishing $\sigma$ corresponding to the case of minimal coupling.

A conformal transformation [5] $\tilde{g}_{ab} = f(\phi)g_{ab}$ transforms the action to

$$S[\tilde{g}_{ab}, \phi] = \int d^4x \sqrt{-\tilde{g}} \left( \frac{1}{2\kappa} \tilde{R} - \frac{1}{2} F(\phi)^2 \tilde{g}^{ab} \tilde{\nabla}_a \phi \tilde{\nabla}_b \phi - \tilde{V}(\phi) \right),$$

with

$$F(\phi)^2 = \frac{1 - (1 - 6\xi)\sigma \phi^2}{(1 - \sigma \phi^2)^2} \quad \text{and} \quad \tilde{V}(\phi) = \frac{U(\phi)}{(1 - \sigma \phi^2)^2}$$

(3)
for a coupling function of the form (2). This can be brought to the standard minimally coupled form for the redefined scalar \( \phi = f d\phi F(\phi) \), implying a rather complicated relation \( \phi(\bar{\phi}) \), e.g. for \( \xi < \frac{1}{6} \):

\[
\bar{\phi} = \frac{1}{\kappa} \left( \sqrt{\xi^{-1}(1-6\xi)} \arcsin(\sqrt{\sigma(1-6\xi)} \phi) + \sqrt{\frac{3}{2}} \log \left| \frac{\sqrt{\sigma(1-6\xi)} \phi + \sqrt{1 - \sigma(1-6\xi)} \phi^2}{\sqrt{\sigma(1-6\xi)} \phi - \sqrt{1 - \sigma(1-6\xi)} \phi^2} \right| \right).
\]

Note that this expression significantly simplifies in the case of conformal coupling, when \( \xi = \frac{1}{6} \). As we discuss in the end of this section, this value of the (positive) coupling constant separates two qualitatively different regions of the parameter space: for \( 0 < \xi < \frac{1}{6} \) there exists one additional transformation singularity (as compared to the region \( \xi > \frac{1}{6} \)). In the canonical formulation, however, \( \xi = \frac{1}{6} \) is not special, and there is no additional singularity for any value of \( \xi \).

From the canonical point of view, moreover, a conformal transformation implies a more complicated change in the momenta conjugate to the spatial metric. As we will see below for isotropic models, however, the larger freedom in canonical transformations compared to conformal ones allows a similar transformation to a minimally coupled form which even leads to a simpler redefinition of the scalar. This will be exploited in the rest of this paper to discuss the behaviour of isotropic singularities which will be much more transparent than discussions based on conformal transformations of the spacetime metric.

Upon reducing the action to a flat FRW model, it becomes functional only of the scale factor \( a \) and the homogenenous scalar \( \phi \). In what follows, we will use independent canonical variables \( p \in \mathbb{R} \), such that \( |p| = a^2 \), and \( c = \gamma a \). They have a geometrical meaning as a triad component \( p \) (such that \( \text{sgn}(p) \) determines the orientation of the triad) and connection component \( c = \gamma a \) (or extrinsic curvature), which corresponds to using isotropic Ashtekar instead of metric variables [7–9]. The positive real number \( \gamma \) is the so-called Barbero–Immirzi parameter [8, 10] which we keep for generality, but can be assumed to equal 1 here. These variables are the basis for a loop quantization to be employed in the final section.

With these variables, the action (1) becomes

\[
S_{\text{sym}} = \int dt \left[ \frac{3f(\phi)}{\kappa \gamma} pc + \pi \phi - N \left( -3 \gamma^{-2} f(\phi) \sqrt{|p|} c^2 + \frac{\pi^2}{2|p|^3/2} + |p|^{3/2} U(\phi) \right) \right],
\]

with the lapse function \( N \). The dot denotes a proper time derivative. From (3) we see that the pairs \( (c, f(\phi)p) \) and \( (\phi, \pi) \equiv |p|^{3/2} N^{-1} \phi \) are canonically conjugate variables with non-vanishing Poisson brackets

\[
\{c, f(\phi)p\} = \kappa \gamma / 3, \quad \{\phi, \pi\} = 1.
\]

Note that the Poisson bracket \( \{p, \pi\} = f(\phi)p[f(\phi)^{-1}, \pi] \) is not zero, in contrast to minimal coupling, while \( \{f(\phi), p\} \) is. We thus define

\[
\tilde{c} := c, \quad \tilde{p} := f(\phi)p,
\]

such that \( \{\tilde{c}, \tilde{p}\} = \kappa \gamma / 3 \). Note that this does not correspond to a conformal transformation from the original gravitational pair \( (p, c) \) to a new one \( (\tilde{p}, \tilde{c}) \) because in that case \( c \) would have to change too. The Hamiltonian constraint is particularly simple for a lapse function \( N = \sqrt{|p|} \), corresponding to conformal time, where it takes the form

\[
H = -\frac{3}{\kappa \gamma^2} |\tilde{p}| c^2 + \frac{\pi^2 f(\phi)}{2|\tilde{p}|} + \tilde{p} \frac{\tilde{U}(\phi)}{f(\phi)^2}
\]

in terms of canonical variables \( (\tilde{p}, \tilde{c}, \phi, \pi) \), thus being the starting point for a canonical quantization.
The matter part in these canonical variables is still non-standard, but this can be achieved after redefining the field variables to

\[ \tilde{\pi} := \sqrt{f(\phi)} \pi, \quad \tilde{\phi} := \int_0^\phi \frac{d\phi'}{\sqrt{f(\phi')}} \]  

such that \( \{\phi, \tilde{\pi}\} = 1 \), i.e., the transformation is canonical. This brings the Hamiltonian to a standard minimal form with scalar field potential

\[ \tilde{V}(\tilde{\phi}) := U(\phi(\tilde{\phi})) f(\phi(\tilde{\phi}))^{-2}, \]  

where \( \phi \) is understood as a function of \( \tilde{\phi} \) by inverting (7). Specifically for the coupling function (2), the original and canonically transformed fields are related in a simple manner by

\[ \sqrt{\sigma} \phi = \sin(\sqrt{\sigma} \tilde{\phi}) \quad \text{for} \quad \sigma > 0 \quad \text{and} \quad \sqrt{|\sigma|} \phi = \sinh(\sqrt{|\sigma|} \tilde{\phi}) \quad \text{for} \quad \sigma < 0. \]  

Both the conformal and canonical transformations can fail to be one-to-one and thus become singular. For the conformal transformation, this happens when \( d\tilde{\phi}/d\phi = F(\phi) = 0 \), i.e., at \( \Phi_c = (\sigma(1-6\xi))^{-1/2} \) for \( 0 < \xi < 1/2 \) and when \( F(\phi) \) diverges at \( \phi_c = 1/\sqrt{\sigma} \) for \( \xi > 0 \). While \( \Phi_c \) turns out to be a spurious singularity introduced by the transformation, \( \phi_c \) corresponds to a physical shear singularity at least when anisotropic perturbations are allowed [11, 12].

The latter singularity clearly occurs for the canonical transformation, too, since now \( d\tilde{\phi}/d\phi = f(\phi)^{-1/2} \) diverges at \( \phi_c \). However, this is the only point where the transformation fails to be one-to-one and there is no analogue of \( \Phi_c \). The canonical transformation thus avoids the unphysical regime \( \phi > \Phi_c \) of the conformal transformation.

3. Singularities

We can now investigate what this implies for singularities using the minimally coupled situation realized for the transformed variables. For a regular potential, the singularity corresponds to \( \tilde{p} = 0 \) and \( \tilde{\phi} \to \infty \) for potentials less steep than exponential [13]. For \( \sigma > 0 \), however, the potential can be singular, thus changing properties of spacetime singularities [14]. In the original variables, the behaviour of the system can then be dramatically different for \( \sigma < 0 \) and \( \sigma > 0 \) as is evident from properties of the transformation.

3.1. Negative coupling constant

For \( \sigma < 0 \), the potential \( \tilde{V}(\tilde{\phi}) \) is regular everywhere such that in the transformed variables singularities occur in the usual form, \( \tilde{p} = 0 \) and \( \tilde{\phi} \to \infty \). It will be important later on to know what this implies for the original canonical variables \( \tilde{p} \) and \( \phi \) since these are the variables that will be quantized. It follows from the preceding considerations that the behaviour of \( \phi \) will not be different from that of \( \tilde{\phi} \) since the transformation (9) is one-to-one. In fact, in the vicinity of the singularity we have \( \phi \propto \exp(\sqrt{\sigma} \tilde{\phi}) \), i.e., the original field also blows up. Interestingly, had the coupling function been of the form \( f(\phi) = 1 - \sigma \phi^2 + \epsilon \) for some positive \( \epsilon \), however small, the integral in (7) would have been converging for \( \phi \to \infty \), that is a finite value of \( \tilde{\phi} \) would have corresponded to a diverging \( \phi \) and the possibility of a different type of singularity in the canonical variables would not have been ruled out. The case of a quadratic coupling function, which is preferred by physical arguments since it does not require additional dimensionful constants, thus appears as the limiting case where singularities in non-minimal models occur as in minimal ones.

It will also be of interest to see the behaviour in the geometrical variables \( p \) in addition to \( \phi \). Using the relation between the scale factors (5), we conclude that \( \tilde{p} \to 0 \) implies \( p \to 0 \)
since \( f(\phi) \neq 0 \). Thus, a singularity in the canonical variables implies a singularity in the geometrical variables occurring at the same values of the field and scale factor. The reverse conclusion is not true in general since \( p = 0 \) together with \( \phi \to \infty \) could result in a finite value of \( \tilde{p} \). However, since we already know that the canonical variables are regular except at \( \tilde{p} = 0 \), such cases are ruled out for \( \sigma < 0 \).

In the transformed variables, steepness properties of the potential can change which affects the asymptotic behaviour close to singularities [13]. For instance, for \( U(\phi) = \frac{1}{2}m^2\phi^2 \) or \( U(\phi) = \frac{1}{4}\lambda\phi^4 \), we have

\[
\tilde{V}(\phi) = \frac{m^2\tanh(\sqrt{\sigma}\phi)^2}{2\sigma\cosh(\sqrt{\sigma}\phi)^2} \quad \text{and} \quad \tilde{V}(\phi) = \frac{\lambda}{4\sigma^2}\tanh(\sqrt{\sigma}\phi)^4 \quad (10)
\]

as exact expressions for any value of the field, which is one example for simplifications of the canonical compared to a conformal transformation. When approaching the singularity \( (\tilde{p} \to 0, \sigma\tilde{\phi}^2 \to \infty) \), the first potential vanishes, whereas the second one asymptotically tends to a constant, \( \frac{1}{2}\lambda\sigma^{-2} \). This can be used for an easy demonstration that the asymptotic behaviour is identical to that of a massless scalar. The equations of motion in conformal time \( \eta \) are

\[
\dot{\tilde{p}} = 2\tilde{p}\tilde{c}/\gamma = \sqrt{2\lambda/\gamma}\sqrt{\tilde{p}^2 + 2\tilde{p}^3\tilde{V}(\phi)}, \quad \dot{\tilde{\phi}} = \tilde{\pi}/\tilde{p}, \quad \dot{\tilde{\pi}} = -\tilde{p}^2\tilde{V}, \quad \phi(\eta) \quad (11)
\]

eliminating \( \tilde{c} \) using the constraint \( H \approx 0 \). With asymptotically constant potentials, the field momentum is a constant, \( \tilde{\pi} = \pi_0 = \text{const} \), which will thus dominate over \( \tilde{p}^3\tilde{V}(\phi) \) sufficiently close to \( \tilde{p} = 0 \). We are, therefore, left with just two simple equations \( \tilde{p} = \sqrt{2\lambda/3}\tilde{\pi}_0 \) and \( \dot{\phi} = \pi_0/p \). For \( \tilde{p}(0) = 0 \), we get the solutions \( \tilde{p}(\eta) \propto \eta \) and \( \phi(\eta) \propto \log \eta \). This indeed corresponds to solutions for the massless scalar field independently of which of the two potentials is used. Converting back to the original variables, we have \( \phi \propto -\eta^{-\sqrt{3\lambda}/2} \) and \( p \propto \eta^{1+\sqrt{3\lambda}/2} \).

### 3.2. Positive coupling constant

In this case, the mapping from \( \phi \) to \( \tilde{\phi} \) is not one-to-one. Compared to a conformal transformation, one such singularity at \( \Phi_c \) is removed which thus turns out to be spurious and just introduced by the choice of transformation. The remaining singularity at \( \phi_c \), however, is different because the coupling function vanishes at this value, which thus corresponds to a singular point even in the original system.

As a consequence, the effective potential \( \tilde{V}(\phi) \) diverges at this point. There is thus a potential wall which could simply reflect \( \phi \) back when the critical value is approached and thus restrict the allowed values for \( \phi \) to disconnected regions. Whether or not this may happen in isotropic models, an embedding in anisotropic ones shows that \( \phi_c \) generically corresponds to a shear singularity rather than just a turning point of \( \phi \) [11, 12]. The irregular potential obtained after the canonical transformation thus implies a new type of singularity which occurs at a finite value of \( \phi \) where \( f(\phi) = 0 \). As follows from the considerations in [11, 12], this happens at a non-diverging and possibly non-zero value of \( p \). This is a new type of singularity and presents an interesting test to singularity removal schemes of quantum gravity. Such schemes are usually based on quantum modifications at small scales and thus naturally arise around \( p = 0 \). But if the variables are allowed to be of a larger value at a classical singularity, it would be difficult for quantum gravity to remove such a singularity and at the same time preserve classical behaviour on large scales.

For a canonical quantization such as loop quantum gravity, one uses canonical variables and it is their appearance at classical singularities which is relevant. Thus, we have to determine...
the values of $\tilde{p}$ and $\phi$ at the new type of singularity when $\phi = \phi_c$. While $\phi$ is finite there, the fact that $p$ does not diverge together with $f(\phi_c) = 0$ implies that the canonical variable $\tilde{p}$ must be zero, unlike the geometrical variable $p$. Thus, singularities always occur in regimes of small $\tilde{p}$ where quantum gravity can dramatically change the behaviour and has the potential to remove singularities.

4. Loop quantum cosmology

Loop quantum gravity is based on Ashtekar variables, defined by a densitized triad and a connection, which allow one to set up a kinematical quantum framework in a background-independent manner. For minimal coupling, the densitized triad and connection are conjugates, while for non-minimal coupling the densitized triad has to be multiplied by the coupling function in order to retain canonical variables. Upon reducing to isotropic metrics, these canonical fields directly give the components $\tilde{p}$ and $\tilde{c}$ defined before, in addition to the scalar $\phi$ and its momentum $\pi$. These are the variables that a loop quantization of the model should be based on since any canonical transformation such as that to $\tilde{c}$ and $\tilde{p}$ may change properties and may not be representable by a unitary transformation at the quantum level.

As in models with minimal coupling [9, 15, 16], loop quantum cosmology in the triad representation uses wavefunctions $\psi(\tilde{\mu}, \phi)$ where $\tilde{\mu}$ is a label of eigenvalues of $\tilde{p}$. The classical Friedmann equation for $\tilde{p}$ is then replaced by a difference equation for the wavefunction of the type

$$C_1(\tilde{\mu})\psi(\tilde{\mu} + 1, \phi) - 2C_0(\tilde{\mu})\psi(\tilde{\mu}, \phi) + C_{-1}(\tilde{\mu})\psi(\tilde{\mu} - 1, \phi) \propto \hat{H}_{\text{matter}}(\tilde{\mu})\psi(\tilde{\mu}, \phi), \quad (12)$$

with the matter Hamiltonian operator $\hat{H}_{\text{matter}}(\tilde{\mu})$ which for standard matter is diagonal on states $|\tilde{\mu}\rangle$. In the limit of large scale factors $\tilde{\mu} \gg 1$, this difference equation yields the Wheeler–DeWitt differential equation while at small scales, the discreteness of quantum geometry is essential. The criterion for non-singular behaviour (see also [17] for details) then consists in unique extendability of the wavefunction even across classical singularities for specific values of $\tilde{\mu}$. In the standard case, singularities are reached for $\tilde{\mu} = 0$, where the kinematical structure provides a new branch since there are two sides to the classical singularity corresponding to the opposite orientations of the triad. While they cannot be connected classically, it is possible to extend the wavefunction uniquely. To see this, one has to use a more detailed form of coefficients of the difference equation and discuss when they become zero. It turns out that, although leading coefficients $C_{\pm 1}$ can become zero, this does not spoil the recurrence scheme of the difference equation. The wavefunction is then uniquely extended and quantum gravity can tell us about the other side beyond singularities [18].

For this argument it is important to provide a candidate for the other side, such as $\tilde{\mu} < 0$, and then show that dynamics extends the wavefunction uniquely. Thus, this scheme only works when classical singularities are indeed located at values of vanishing $\tilde{\mu}$ (or degenerate triads) since there would be no new region otherwise. While this is automatically the case in isotropic models with standard types of matter fields, it is non-trivially realized in anisotropic models [19, 22] or even spherical symmetry [20, 23]. A second crucial ingredient is the assumption of a matter Hamiltonian operator diagonal on triad eigenstates. Classically, this corresponds to a matter Hamiltonian which depends only on spatial geometry but not on curvature. If there are curvature terms and $\hat{H}_{\text{matter}}$ is not diagonal in geometric variables, the right-hand side of (12) could also contribute terms proportional to $\psi(\tilde{\mu} \pm 1, \phi)$, changing the recurrence and introducing new places where the leading coefficients can vanish. With such a different recurrence scheme, singularity removal would be less general since the coefficients would depend on the matter field through the Hamiltonian, and not just on $\tilde{\mu}$.
We can now test this scheme with what we have learned about non-minimally coupled isotropic models. First, there are singularities (for positive $\sigma$) where even the geometric triad variable $p$ may not vanish. However, it turned out here that the canonical variable $\tilde{p}$ does vanish. Since the difference equation is based on the quantization of this variable, non-minimally coupled models are included in the non-singularity scheme of loop quantum cosmology. The general mechanism in [18] can in fact be applied without changes after recognizing what the relevant canonical variables and their behaviour at classical singularities are: classical singularities correspond to $\tilde{\mu} = 0$ even in non-minimally coupled models and the difference equation allows one to extend the wavefunction uniquely across this place. As the discussion here illustrates, this happens in a way depending on the dynamical details of the models.

At first sight, one could also expect that curvature couplings of a non-minimal model lead to non-diagonal matter Hamiltonians since there are curvature components in the matter terms. Here, the previous expressions show that the matter Hamiltonian (6) in canonical variables does depend only on $\tilde{p}$ in addition to the matter fields, but not on the connection component $\tilde{c}$ which determines curvature. Even for non-minimal coupling, the matter Hamiltonian is thus a function only of the canonical triad component and the general singularity removal scheme applies. This remains true for positive spatial curvature not spelled out explicitly here, since we would only have to replace $\tilde{c}^2$ by $\tilde{c}^2 + 1$ [21]. Thus, non-minimally coupled isotropic models are non-singular in loop quantum cosmology in the same manner as minimally coupled ones.$^1$

5. Conclusions

Applying a canonical transformation to a non-minimally coupled isotropic cosmological model, we have shown that for classically allowed coupling functions singularities occur as in the minimally coupled case. This has direct implications for the question of singularity removal in quantum gravity. In all cases studied in loop quantum cosmology, it was shown that quantum dynamics provides a well-posed recurrence scheme for wavefunctions across classical singularities. For this, it was important to know at which geometrical configurations a classical singularity occurs, such as at a vanishing scale factor $p = 0$ for isotropic models coupled to a scalar. All these arguments were independent of the form of the matter Hamiltonian and thus present a quantum geometry effect, provided that the classical matter Hamiltonian did not depend on curvature. The only cases not covered explicitly were curvature couplings of matter.

For the purpose of quantum gravity, curvature couplings fall into two different classes, depending on whether they require a factor of the Planck length or not. Most curvature coupling terms in effective matter actions require dimensionful coefficients with a new length scale to compensate the dimensions of curvature factors. Such terms cannot arise in the classical Hamiltonian used to set up a quantization, and such actions with curvature coupling terms will thus not be quantized; they would rather follow from a quantum theory of gravity such as loop quantum gravity in an effective description, see e.g. [24]. The curvature coupling of a non-minimal scalar, on the other hand, is allowed classically provided that the coupling function is quadratic in $\phi$. Such cases were not explicitly covered by previous arguments.

$^1$ This conclusion is independent of the value of the coupling parameter $\xi$, and the value of conformal coupling $\hat{\xi} = \frac{1}{8}$ does not appear special in this application. There are certainly other issues in quantum gravity where conformal coupling may lead to special properties as in the classical case. In particular, in inhomogeneous situations there are gradient terms in the curvature scalar which change in a more complicated way under conformal rescaling and where the value $\hat{\xi} = \frac{1}{8}$ becomes special. Studying these issues would be of interest for understanding the semiclassical limit of the full theory.
With the results of the present paper, we can fill this gap. Classical singularities in canonical variables always occur in the same form as in minimal models. Moreover, the form of quantum dynamics near classical singularities is the same whether a scalar is coupled minimally or non-minimally. Singularity removal thus remains unchanged quite non-trivially. The classically allowed case of a quadratic coupling function is just the limiting case where an infinite scalar $\phi$ is mapped to an infinite $\tilde{\phi}$ by the canonical transformation. For coupling functions of a higher degree, the possibility of singularities at finite values of $\tilde{\phi}$ and non-zero $\tilde{p}$ cannot easily be ruled out. In such a case, the usual singularity removal mechanism and the resulting non-singular picture of quantum geometry would have been very different, if applicable at all. As with calculations of black hole entropy [25] we thus see that, although the non-minimal situation appears initially crucially different from the minimal one, the methods of loop quantum gravity are general enough to encompass automatically also non-minimal situations in quite non-trivial ways.

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