Note on counterterms in asymptotically flat spacetimes

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Abstract

We consider in more detail the covariant counterterm proposed by Mann and Marolf in asymptotically flat spacetimes. With an eye to specific practical computations using this counterterm, we present explicit expressions in general $d$ dimensions that can be used in the so-called ‘cylindrical cut-off’ to compute the action and the associated conserved quantities for an asymptotically flat spacetime. As applications, we show how to compute the action and the conserved quantities for the NUT-charged spacetime and for the Kerr black hole in four dimensions.

1 Introduction

Over the years many expressions have been proposed for computing conserved quantities in asymptotically flat spacetimes. The general idea in such constructions is to study the asymptotic values of the gravitational field, far away from an isolated object, and compare them with those corresponding to a gravitational field in the absence of the respective object \cite{[1]}. However, most of these proposals will provide results that are relative to the choice of

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a reference background — they are usually called background subtraction methods. The background must be chosen such that its topological properties match the solution whose action and conserved charges we want to compute. However, this does not unequivocally fix the choice of the background \[2\] and moreover, there might be cases in which the topological properties of the solution rule out any natural choice of the background.

Inspired by the AdS/CFT correspondence, an alternative procedure — referred to as the ‘counterterm method’ — has been proposed \[3, 4, 5\]. In this approach one supplements the action by including suitable boundary counterterms. These counterterms are functionals only of curvature invariants of the induced metric on the boundary and so they do not affect the equations of motion. By choosing appropriate counterterms that cancel the divergences, one can then obtain finite expressions for the action. Unlike the background subtraction methods, this counterterm procedure is intrinsic to the spacetime of interest and it is unambiguous once the counterterm action is specified. However, while there is a general algorithm for generating the counterterms for asymptotically AdS spacetimes \[3\], the asymptotically flat case has been considerably less-explored. Early proposals \[4, 5, 6\] engendered study of proposed counterterm expressions for a class of \(d\)-dimensional asymptotically flat solutions whose boundary topology is \(S^n \times R^{d-1-n}\) \[3\].

Recently there have been interesting developments in this area. Astefanesei and Radu proposed a renormalized stress-tensor for a particular family of locally asymptotically flat spacetimes \[7\] — it was computed by varying the total action (including the counterterms) with respect to the boundary metric. Conserved quantities can be constructed from this stress-tensor via the algorithm of Brown and York \[8\]. Subsequently, Mann and Marolf have generalized this method to arbitrary asymptotically locally flat spacetimes \[9\]. By using a new local, covariant counterterm they obtained conserved quantities that agree with older definitions known in literature. In particular, they constructed a boundary stress-tensor similar with the one used in quasi-local context \[8, 2\], which in the so-called ‘hyperbolic cut-off’ has been shown to lead to conserved quantities that are built out of the electric part of the Weyl tensor, similar to the Ashtekar-Hansen expressions in four dimensions \[10\]. The connection with the Ashtekar-Hansen conserved quantities has been examined in more detail in \[11\], in which it was argued that the canonical form of the action built using the Mann-Marolf (MM) counterterm reduces to the ADM action \[12\]. Hence the conserved quantities should agree with the ADM conserved charges.

This method was also used in computing the gravitational energy of the Kaluza Klein monopole \[13\] and of \((2 + 1)\) Minkowski space \[14\], the mass of Kaluza-Klein black holes with squashed horizons \[15, 16, 17, 18\], and the mass of non-uniform black string solutions \[19\].

The main motivation for the present work is to investigate more closely the local MM counterterm prescription for computing the action and conserved charges associated with asymptotically flat spaces. We will specifically focus on the so-called ‘cylindrical cut-offs’ of the spacetime. We exhibit explicit expressions for the boundary stress-tensor and we show how to compute the general renormalised action in \(d\)-dimensions. As an example of this counterterm technique, we compute the action and the conserved quantities for NUT-charged spaces in four dimensions. Using also a distinct proposal for the boundary counterterm, namely the counterterm proposed by Lau \[4\] and Mann \[5\] we find agreement in both cases.
with previous results in literature. We consider also the conserved charges and action for a rotating black hole in four dimensions.

The structure of this paper is as follows: in the next section we review the MM counterterm prescription and show how to compute the boundary stress-tensor and action in cylindrical cut-offs. In Section 3 we briefly review other counterterm proposals for asymptotically flat spacetimes, while in Section 4 we apply them to computation of the action and conserved quantities for NUT-charged spacetimes. In Section 5 we compute the action and conserved charges for the Kerr solution. The last section is dedicated to conclusions.

2 The Mann-Marolf counterterm

For asymptotically flat spacetimes, the gravitational action consists of the bulk Einstein-Hilbert term, supplemented by the boundary Gibbons-Hawking term in order to have a well-defined variational principle [20]. In general \( d \)-dimensions, the gravitational action for an asymptotically flat spacetime is then taken to be:

\[
I_B + I_{\partial B} = -\frac{1}{16\pi G} \int_M d^d x \sqrt{-g} R - \frac{1}{8\pi G} \int_{\partial M} d^{d-1} x \sqrt{-h} K.
\]  

(1)

Here \( M \) is a \( d \)-dimensional manifold with metric \( g_{\mu\nu} \), \( K \) is the trace of the extrinsic curvature \( K_{ij} = \frac{1}{2} h^k_i \nabla_k n_j \) of the boundary \( \partial M \) with unit normal \( n^i \) and induced metric \( h_{ij} \).

When evaluated on non-compact solutions of the field equations, it turns out that both terms in (1) diverge. The general remedy for this situation is to add a counterterm, i.e. a coordinate invariant functional of the intrinsic boundary geometry that is specifically designed to cancel out the divergencies.

In [9], Mann and Marolf put forward a new counterterm that is also given by a local function of the boundary metric and its curvature tensor. The new counterterm is taken to be the trace \( \hat{K} \) of a symmetric tensor \( \hat{K}_{ij} \) that is defined implicitly in terms of the Ricci tensor \( R_{ij} \) of the induced metric on the boundary via the relation

\[
\mathcal{R}_{ik} = \hat{K}_{ik} \hat{K} - \hat{K}_i^m \hat{K}_{mk}.
\]  

(2)

In solving (2), one chooses a solution \( \hat{K}_{ij} \) that asymptotes to the extrinsic curvature of the boundary of Minkowski space as \( \partial M \) is taken to infinity. Therefore, in contrast to previous counterterm proposals this new counterterm assigns an identically zero action to the flat background in any coordinate systems, while giving finite values for asymptotically flat backgrounds. The renormalized action leads to the usual conserved quantities that can also be expressed in terms of a boundary stress-tensor whose leading-order expression involves the electric part of the Weyl tensor:

\[
T_{ij} = \frac{1}{8\pi G} \frac{\rho E_{ij}}{d-3} + \mathcal{O}(\rho^{-(d-3)}).
\]  

(3)

Here \( E_{ij} \) is the pull-back to the boundary of the contraction of the bulk Weyl tensor \( C_{\mu\nu\rho\tau} \) with the induced metric \( h^{\mu\nu} \). More precisely, introducing the unit normal vector \( n^\mu \) to the boundary \( \partial M \) then the electric part of the bulk Weyl tensor is defined by
\[ E_{\mu\nu} = C_{\mu\nu\rho} n^{\rho} n^{\tau} = -C_{\mu\nu\rho} h^{\rho\tau}, \]

and \( E_{ij} \) is simply the pull-back to the boundary of the above tensor. Notice that we are using here the so-called ‘hyperbolic cut-off’, in which the line-element is taken to admit the following asymptotic expansion at spatial infinity:

\[
ds^2 = \left( 1 + \frac{2\sigma}{\rho^{d-3}} + \mathcal{O}(\rho^{-(d-2)}) \right) d\rho^2 + \rho^2 \left( h_0^{ij} + \frac{h_1^{ij}}{\rho^{d-3}} + \mathcal{O}(\rho^{-(d-2)}) \right) d\eta^i d\eta^j + \rho \left( \mathcal{O}(\rho^{-(d-2)}) \right) d\rho d\eta^j. \tag{4} \]

Here \( h_0^{ij} \) and \( \eta^i \) are a metric and the associated coordinates on the unit \((d - 2, 1)\) hyperboloid \( \mathcal{H}^{d-1} \), while \( \sigma \) and \( h_1^{ij} \) are smooth tensorial fields defined on \( \mathcal{H}^{d-1} \). Moreover, \( \rho \) plays here the role of a ‘radial’ coordinate such that spacelike infinity is reached in the \( \rho \to \infty \) limit with fixed \( \eta \) and the symbols \( \mathcal{O}(\rho^{-(d-2)}) \) refer to terms that fall-off at least as fast as \( \rho^{-(d-2)} \) in the \( \rho \to \infty \) limit.

While the hyperbolic cut-off is more natural for covariant investigations, its use in practical applications is cumbersome since in order to apply the above results one has to transform every solution into a form that is asymptotically of the form (4). Such coordinate transformations can be very involved. Moreover, even if the conserved charges can be found by using the boundary stress-tensor to leading order, to evaluate the finite regularised action one has to find the next-to-leading terms in \( \hat{K}_{ij} \). For these reasons, it is desirable to find directly the corresponding expressions for the regularised action and conserved charges in the ‘cylindrical cut-offs’. In the remainder of this section we will focus on the so-called ‘cylindrical cut-off’, in which the asymptotic form of the metric at spatial infinity takes the form

\[
ds^2 = -\left( 1 + \mathcal{O}(\rho^{-(d-3)}) \right) dt^2 + \left( 1 + \mathcal{O}(\rho^{-(d-3)}) \right) dr^2 + r^2 \left( \omega_{IJ} + \mathcal{O}(\rho^{-(d-3)}) \right) d\theta^I d\theta^J + \mathcal{O}(\rho^{-(d-4)}) d\rho d\theta^I dt, \tag{5} \]

where \( \omega_{IJ} \) and \( \theta^I \) are the metric and coordinates on the unit \((d - 2)\)-sphere.

For vacuum metrics the Einstein-Hilbert action vanishes on solutions. We are therefore interested in computing the boundary terms only

\[
I_{\partial B} = -\frac{1}{8\pi G} \int_{\partial M} d^{d-1}x \sqrt{-h} (K - \hat{K}).
\]

where we note that the extrinsic curvature satisfies the Gauss-Codazzi relation

\[
\mathcal{R}_{ik} = R_{ikjl} h^{kl} + K_{ik} K - K^m_i K_{mk},
\]

which is very similar to \( \square \). The only difference is the presence of the term \( R_{ikjl} h^{kl} \), where \( R_{ikjl} \) is the pullback to the boundary \( \partial M \) of the bulk Riemann tensor. The leading terms of \( K_{ij} \) and \( \hat{K}_{ij} \) are given by the extrinsic curvature of the standard cylinder of radius \( r \) in Minkowski spacetime.
\( K_{ij} \sim \hat{K}_{ij} = r \omega_{ij} + \mathcal{O}(\rho^{-(d-4)}) \). \hspace{1cm} (6)

Here, \( \omega_{ij} \) is the pull-back to \( \partial M \) of the round metric \( \omega_{IJ} \) on the unit sphere \( S^{d-2} \). Furthermore, one has to solve the linearised Gauss-Codazzi relation:

\[
R_{ikjl} h^{kl} = -\Delta klh_{kl} \hat{K}_{ij} - \Delta ij \hat{K}_{ij} + \Delta ilh_{lj} + \Delta jlh_{il} \hspace{1cm} (7)
\]

to compute the difference \( \Delta_{ij} \equiv K_{ij} - \hat{K}_{ij} \) to first order \([9, 11]\). It is convenient at this stage to define \( \mu_{ij} = h_{ij} + u_iu_j \), where \( u^i \) is a unit future-directed timelike vector on \( \partial M \), associated with a foliation \( \Sigma_t = \{ t = \text{const.} \} \) of the spacetime \( M \). Then \( \tilde{K}_{ij} = \frac{\mu_{ij}}{r} + \mathcal{O}(\rho^{-(d-4)}) \) and replacing it in (7) one obtains:

\[
R_{ikjl} h^{kl} = -\frac{\Delta (h_{ij} + u_iu_j)}{r} - \frac{d-4}{r} \Delta_{ij} + \frac{\Delta_{il}u^lu^i + \Delta_{jl}u^lu^i}{r}. \hspace{1cm} (8)
\]

In the following we shall denote \( \tilde{\Delta}_i = \Delta_{ij}u^j \) and \( \tilde{\Delta} = \Delta_{ij}u^iu^j \). More generally we shall indicate by a tilded quantity the presence of at least one contraction of that quantity with \( u^i \).

Note that for \( d = 4 \) one cannot find the general expression of \( \Delta_{ij} \). However, as we shall subsequently show, when computing the action or the conserved quantities one does not actually need \( \Delta_{ij} \) but its various contractions with the vector \( u^i \). Now, contracting the above relation with \( u^j \) we obtain

\[
\frac{d-3}{r} \tilde{\Delta}_i = -R_{ikjl} h^{kl} u^j + \frac{\tilde{\Delta} u^i}{r}
\]

and further contracting with \( u^i \) we obtain

\[
\tilde{\Delta} = -\frac{r}{d-2} R_{ikjl} h^{kl} u^iu^j. \hspace{1cm} (9)
\]

Recall that we are interested in Ricci-flat metrics and therefore we can effectively replace in the above equations the curvature tensor in the bulk with the bulk Weyl tensor. Using the earlier definition of the pull-back \( E_{ij} \) to the boundary of the electric part of the bulk Weyl tensor we obtain the following simple result

\[
\Delta = \frac{\tilde{\Delta}}{d-3}
\]

upon taking the trace of (8) with \( h^{ij} \).

In order to compute the boundary action we only have to evaluate the difference \( K - \hat{K} = h^{ij} \Delta_{ij} = \Delta \) and so we obtain

\[
K - \hat{K} = -\frac{r}{(d-2)(d-3)} R_{ikjl} h^{kl} u^iu^j = \frac{r}{(d-2)(d-3)} E_{ij} u^iu^j \hspace{1cm} (10)
\]
Finally, the regularized action is given by

$$I_{BB} = -\frac{1}{8\pi G} \int_{\partial M} d^{d-1} x \sqrt{-h} \frac{r}{(d-2)(d-3)} E_{ij} u^i u^j. \quad (11)$$

Now, in order to evaluate the conserved charges we evaluate the boundary stress-tensor using (see eq. (4.2) of [9])

$$T_{ij} = \frac{1}{8\pi G} (\Delta_{ij} - h_{ij} \Delta),$$

from which the conserved charges associated with a boundary Killing vector $\xi$ are found by direct integration:

$$Q = \oint_{\Sigma} d^{d-1} S^i T_{ij} \xi^j. \quad (12)$$

Recall that in $d = 4$ we cannot directly evaluate $\Delta_{ij}$ so we cannot find the explicit expression of $T_{ij}$. Fortunately to compute $Q$ we only need the contraction of $T_{ij}$ with $u^j$

$$T_{ij} u^j = \frac{1}{8\pi G} (\tilde{\Delta}_i - \Delta u_i) = \frac{1}{8\pi G} \frac{r E_{ij} u^j}{d-3}, \quad (13)$$

such that the conserved charge associated with a Killing vector $\xi$ of the boundary metric $h_{ij}$ is given by

$$Q = \frac{1}{8\pi G} \int_{\Sigma} d^{d-2} \sigma \frac{r E_{ij} u^i \xi^j}{d-3}. \quad (14)$$

Here, $\Sigma$ is a closed surface with unit normal $u^i$ and volume element $d^{d-2} \sigma$. This completes our derivation of the regularized action and conserved quantities defined using the Mann-Marolf counterterm in the cylindrical cut-off of the spacetime. For $d = 4$ the relation (13) has been used previously in [13] to compute the total mass associated with the Kaluza-Klein magnetic monopole in four dimensions.

### 3 Other counterterm prescriptions

For asymptotically flat 4-dimensional spacetimes the counterterm

$$I_{ct} = \frac{1}{8\pi G} \int d^3 x \sqrt{-h} \sqrt{2R} \quad (15)$$

was proposed [4, 5] to eliminate divergences that occur in (11). An analysis of the higher dimensional case [3] suggested in $d$-dimensions the counterterm:

$$I_{ct} = \frac{1}{8\pi G} \int d^{d-1} x \sqrt{-h} \frac{R^2}{\sqrt{R^2 - R_{ij} R^{ij}}} \quad (16)$$
where $\mathcal{R}_{ij}$ is the Ricci tensor of the induced metric $h_{ij}$ and $\mathcal{R}$ is the corresponding Ricci scalar. This counterterm removes divergencies in the action for a general class of asymptotically flat spacetimes with boundary topologies $S^n \times \mathbb{R}^{d-1-n}$. There also exists another simpler counterterm that removes the divergencies in the action for the general class of asymptotically flat spacetimes with boundary topologies $S^n \times \mathbb{R}^{d-1-n}$:

$$I_{ct} = \frac{1}{8\pi G} \int d^{d-1}x \sqrt{-h} \sqrt{\frac{n\mathcal{R}}{n-1}}.$$ (17)

By taking the variation of the total action (including the counterterm (17)) with respect to the boundary metric $h_{ij}$ the boundary stress-tensor is given by:

$$T_{ij} = \frac{1}{8\pi G} (K_{ij} - Kh_{ij} - \Psi (\mathcal{R}_{ij} - \mathcal{R} h_{ij}) - h_{ij} \Box \Psi + \Psi_{;ij}),$$ (18)

where we denoted $\Psi = \sqrt{\frac{n}{(n-1)\mathcal{R}}}$. If the boundary geometry has an isometry generated by a Killing vector $\xi^i$, then $T_{ij} \xi^j$ is divergence free, from which it follows that the quantity

$$Q = \oint_{\Sigma} d^{d-2}\sigma T_{ij} u^i \xi^j,$$ (19)

associated with a closed surface $\Sigma$, is conserved. Physically, this means that a collection of observers on the boundary with the induced metric $h_{ij}$ measure the same value of $Q$, provided the boundary has an isometry generated by $\xi$. In particular, if $\xi^i = \partial/\partial t$ then $Q$ is the conserved mass $\mathcal{M}$.

### 4 Thermodynamics of NUT-charged spaces

In this section we shall apply the results from the previous sections to derive the action and the conserved charges associated with NUT-charged spaces in four dimensions.

We start with the Euclidean form of the Taub-NUT solution:

$$ds^2 = F_E(r)(d\chi - 2n \cos \theta d\varphi)^2 + F^{-1}_E(r)dr^2 + (r^2 - n^2)d\Omega^2,$$

where

$$F_E(r) = \frac{r^2 - 2mr + n^2}{r^2 - n^2}.$$ (20)

In general, the $U(1)$ isometry generated by the Killing vector $\frac{\partial}{\partial \chi}$ (that corresponds to the coordinate $\chi$ that parameterises the fibre $S^1$) can have a zero-dimensional fixed point set (referred to as a ‘Nut’ solution) or a two-dimensional fixed point set (correspondingly referred to as a ‘Bolt’ solution). The regularity of the Euclidean Taub-Nut solution requires that the period of $\chi$ be $\beta = 8\pi n$ (to ensure removal of the Dirac-Misner string singularity), $F_E(r = n) = 0$ (to ensure that the fixed point of the Killing vector $\frac{\partial}{\partial \chi}$ is zero-dimensional) and also $\beta F'_E(r = n) = 4\pi$ in order to avoid the presence of the conical singularities at $r = n$. With these conditions we obtain $m = n$, yielding

$$F_E(r) = \frac{r - n}{r + n}.$$ (21)
The other possibility to explore is the Taub-Bolt solution in four-dimensions [26]. In this case the Killing vector \( \frac{\partial}{\partial \chi} \) has a two-dimensional fixed point set in the four-dimensional Euclidean sector. The regularity of the solution is then ensured by demanding that \( r \geq 2n \), while the period of the coordinate \( \chi \) is \( 8\pi n \) and for the bolt solution we obtain (with \( m = 5n/2 \)):

\[
F_E(r) = \frac{(r - 2n) \left( r - \frac{1}{2}n \right)}{r^2 - n^2}.
\]  

(22)

Using the results derived using the Mann-Marolf prescription we obtain:

\[
\Delta = -\frac{m}{r^2} + O(r^{-3})
\]  

(23)

while to leading order the contracted boundary stress-tensor [13] has the following components:

\[
8\pi G T_{\chi u} u^i = \frac{2m}{r^2} + O(r^{-3}),
\]

\[
8\pi G T_{\phi u} u^i = \frac{6mn \cos \theta}{r^2} + O(r^{-3}).
\]

(24)

Then the total mass and action are found to be:

\[
\mathcal{M} = \frac{m}{G}, \quad I_{\partial B} = \frac{4\pi mn}{G}.
\]  

(25)

Using now the counterterm [15] one finds the following components of the boundary stress-tensor [5]:

\[
8\pi G T_{\chi} = \frac{2m}{r^2} + O(r^{-3}),
\]

\[
8\pi G T_{\phi} = \frac{4mn \cos \theta}{r^2} + O(r^{-3}),
\]

\[
8\pi G T_{\theta} = 8\pi G T_{\phi} = \frac{2n^2 - m^2}{2r^3} + O(r^{-4}).
\]

(26)

Therefore we find \( \mathcal{M} = \frac{m}{G} \), while the action is given by \( I = \frac{4\pi mn}{G} \). By making use of the quantum statistical relation the entropy is simply given by:

\[
S = \beta\mathcal{M} - I = \frac{4\pi mn}{G}.
\]

For the Nut solution \( m = n \) while for the bolt solution \( m = 5n/2 \) and our results are consistent with those obtained previously [5].

We can also compute directly the thermodynamic quantities and the action for the Taub-NUT space in the Lorentzian section. The metric can be written as:

\[
d s^2 = -F(r)(d t - 2N \cos \theta d \varphi)^2 + F^{-1}(r)d r^2 + (r^2 + N^2)d \Omega^2,
\]

where

\[
F(r) = \frac{r^2 - 2mr - N^2}{r^2 + N^2}
\]  

(27)
and in order to eliminate the Misner string singularities one has to periodically identify the coordinate $t$ with period $\beta = 8\pi N$. In the Mann-Marolf prescription we obtain the mass and the action:

$$M = \frac{m}{G}, \quad I_{\beta B} = \frac{4\pi mN}{G}. \quad (28)$$

Using now the counterterm (15) one finds the following components of the boundary stress-tensor:

$$8\pi GT^t_t = \frac{2m}{r^2} + O(r^{-3}),$$
$$8\pi GT^t_\phi = \frac{4mN \cos \theta}{r^2} + O(r^{-3}),$$
$$8\pi GT^\theta_\theta = 8\pi GT^\phi_\phi = \frac{2N^2 + m^2}{2r^3} + O(r^{-4}). \quad (29)$$

such that the conserved mass is $M = m/G$ and the action is again given by $I = 4\pi mN/G$. The same results have been obtained in [27] by other means.

5 Rotating black holes in four dimensions

In this section we consider 4-dimensional asymptotically flat rotating black holes. Using the Mann-Marolf counterterm we shall study the thermodynamic properties of the Kerr black hole and we shall compare our results against the conserved charges and action as computed using the counterterm (15).

To apply the counterterm-prescription it is more convenient to use the expression of the Kerr metric in Boyer-Lindquist coordinates as in this case there are no cross-terms between $dr$ and the other coordinate differentials. This simplifies the analysis of the event horizons and the causal structure of the metrics. Moreover Boyer-Lindquist coordinates are valid either outside the horizon or inside the horizon. In these coordinates, the Kerr metric is:

$$ds^2 = -\frac{(\delta - a^2 \sin^2 \theta)}{\sigma} dt^2 - 2a \sin^2 \theta \frac{(r^2 + a^2 - \delta)}{\sigma} dt d\phi$$
$$+ \left[ \frac{(r^2 + a^2)^2 - \delta a^2 \sin^2 \theta}{\sigma} \right] \sin^2 \theta d\phi^2 + \frac{\sigma}{\delta} dr^2 + \sigma d\theta^2, \quad (30)$$

where

$$\sigma = r^2 + a^2 \cos^2 \theta, \quad \delta = r^2 - 2Mr + a^2.$$  

The two parameters describing the metric are $M$ and $a$.

Using the Mann-Marolf prescription we obtain from (13):

$$8\pi GT_{xi} u^i = \frac{2m}{r^2} + O(r^{-3}),$$
$$8\pi GT_{\phi i} u^i = -\frac{3am \sin^2 \theta}{r^2} + O(r^{-3}). \quad (32)$$
while $\Delta = -\frac{m}{r^2} + {\mathcal O}(r^{-3})$. We find then the total mass $M = m/G$ and the action $I_{\partial B} = \frac{\beta m}{2G}$, where $\beta$ is the inverse temperature. It is worth mentioning that the stress-tensor can be computed on either the Lorentzian or the Euclidean sections. However to compute the action we used the real Euclidean section of the Kerr black hole. Notice that we can also compute the angular momentum $J$ using in (14) the axial Killing vector $\xi = \frac{\partial}{\partial \phi}$. By simple integration, using the components listed in (32), we find the angular momentum $J = -\frac{ma}{G} = -Ma$.

Using now the counterterm (15) we find the following components of the boundary stress-tensor:

\begin{align}
8\pi G T_{t}^{t} &= \frac{2m}{r^2} + {\mathcal O}(r^{-3}), \\
8\pi G T_{\phi}^{t} &= -\frac{3am}{r^2} \sin^2 \theta + {\mathcal O}(r^{-3}), \\
8\pi G T_{t}^{\phi} &= \frac{3am}{r^4} + {\mathcal O}(r^{-5}), \\
8\pi G T_{\phi}^{\phi} &= \frac{3a^2 \cos^2 \theta + a^2 - m^2}{2r^3} + {\mathcal O}(r^{-4}), \\
8\pi G T_{\theta}^{\theta} &= \frac{9a^2 \cos^2 \theta - 5a^2 - m^2}{2r^3} + {\mathcal O}(r^{-4}).
\end{align} 

Also, using (19) we find that the total mass is $M = m/G$ while the total angular momentum is $J = -Ma$. Finally, a direct computation reveals the total action to be $I = \beta M/2$, i.e. the same results as the ones obtained by the Mann-Marolf prescription. We also note that the conserved charges and the action are in agreement with previous results in the literature (see for instance [21] and references therein).

6 Conclusions

The main motivation for this work was to obtain explicit expressions in the so-called ‘cylindrical cut-off’ for the boundary stress-tensor and the renormalised action defined by using the boundary counterterm recently proposed by Mann and Marolf for asymptotically flat spacetimes. In particular, we showed that, even if in four dimensions one cannot find directly the expression $\hat{K}_{ij}$ of the counterterm to first order, one is still able to compute the conserved quantities and the renormalised action. As with the results obtained previously in the ‘hyperbolic cut-off’, we found that the final action and the conserved charges in the ‘cylindrical cut-offs’ are essentially constructed using the electric part of the Weyl tensor.

As an example of this technique we computed the action and the conserved quantities for NUT-charged spaces in four dimensions (both in Euclidian and Lorentzian sections) and for the Kerr solution. We showed explicitly our results are consistent with that previously known in literature and overcome difficulties inherent in the background subtraction approach [28]. In particular, we compared the results with the ones derived from a counterterm proportional with the square-root of the boundary Ricci scalar proposed by [4, 5].

In general, different counterterms can lead to different results when computing the energy and the total action, seriously constraining the various choices of the boundary counterterms.
(see for instance \cite{29, 30} for a general study of the counterterm charges and a comparison with charges computed by other means in AdS context). While it is clear that the distinct choices of counterterms yield the same mass and action for the asymptotically flat spaces considered here, some of the components of the boundary stress-energy tensors obtained using various counterterms have slightly different coefficients (for example compare the second equations in \eqref{24} and \eqref{26} respectively; alternatively, see the stress-energy components computed for the Kaluza-Klein monopole in \cite{13} using different counterterms). These components are important if one considers for instance the conserved charges associated with bubbles of nothing. At first sight therefore, different counterterms might lead to different results for such spaces.

While our discussion has focussed on a particular class of stationary solutions in four dimensions, namely on the NUT-charged spaces and on the Kerr solution, using the results from this work it is also possible to investigate more general stationary backgrounds that are asymptotically flat, such as general rotating black objects in higher dimensions. Work on this is in progress and it will be reported elsewhere \cite{31}.

Finally, we believe that our results warrant further study of the counterterm method in asymptotically flat spacetimes.

Acknowledgements

DA was supported by the Department of Atomic Energy, Government of India and the visitor programme of ICTP, Trieste. The work of RBM and CS was supported by the Natural Sciences and Engineering Research Council of Canada.

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