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To cite this article: Francesco Calogero (2017) New C-integrable and S-integrable systems of nonlinear partial differential equations, Journal of Nonlinear Mathematical Physics 24:1, 142–148, DOI: https://doi.org/10.1080/14029251.2017.1287387

To link to this article: https://doi.org/10.1080/14029251.2017.1287387

Published online: 04 January 2021
New C-integrable and S-integrable systems of nonlinear partial differential equations

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Received 10 November 2016
Accepted 27 November 2016

A technique to identify new C-integrable and S-integrable systems of nonlinear partial differential equations is reported, with two representative examples displayed and tersely discussed.

Keywords: systems of integrable partial differential equations, C-integrable PDEs, S-integrable PDEs

1. Introduction

The main tool used in this paper are the nonlinear reversible relations—by definition, algebraic—among the $N$ coefficients of a monic polynomial of degree $N$ in the (complex) variable $z$ and its $N$ zeros. The approach based on these relations allowed over time to identify many dynamical systems solvable by algebraic operations, including many-body problems characterized by Newtonian equations of motion (“accelerations equal forces”) [1], and also several solvable/integrable systems of nonlinear Partial Differential Equations (PDEs) [2]. These developments were until recently mainly restricted to the consideration of nonlinear evolutions satisfied by the zeros of a time-dependent polynomial the coefficients of which evolve according to linear systems of Ordinary Differential Equations (ODEs) [1] or of PDEs [2]. Recently a convenient way to relate the time-evolution of the zeros of a time-dependent polynomial to the time-evolution of its coefficients has been noted [3], and this development has allowed the identification and investigation of several new solvable dynamical systems and many-body problems characterized by the time-evolution of the zeros of polynomials the coefficients of which evolve in a nonlinear but solvable/integrable manner [3, 4]. In the present paper we show how this development can be as well employed to identify new systems of solvable/integrable nonlinear PDEs. Since our main goal in this paper is to introduce this approach we limit its application herein to the exhibition of just two new systems of integrable PDEs in 1+1 dimensions, the first of which is associated to the evolution of the $N$ zeros of a polynomial the coefficients of which evolve according to the Burgers PDE—perhaps the most elementary C-integrable nonlinear PDE in 1+1 dimensions, being solvable by a Change of dependent variables—and the second of which is associated with the KdV PDE—perhaps the most famous of the nonlinear S-integrable PDEs in 1+1 dimensions, since the discovery half a century ago of its integrability via the Spectral (or Scattering) Transform opened the way to a major development in pure and applied mathematics. [5]
Notation 1.1. Hereafter we always refer to monic polynomials of arbitrary order $N$ ($N \geq 2$),

\[ P_N (z; \bar{\phi}(x,t), \psi(x,t)) = z^N + \sum_{m=1}^{N} [\phi_m(x,t) z^{N-m}] = \prod_{n=1}^{N} [z - \psi_n(x,t)] ; \quad (1.1) \]

the complex variable $z$ is the argument of the polynomial, indices such as $n, m$ run throughout from 1 to $N$, the $N$-vector $\bar{\phi}(x,t)$ has the $N$ coefficients $\phi_m$ of the polynomial (1.1) as its $N$ components, $\psi(x,t)$ denotes the unordered set of the $N$ zeros $\psi_n(x,t)$ of the polynomial (1.1), and we generally assume all these dependent variables to be complex (this of course does not exclude that they might be real, see indeed the examples below). We instead assume the independent variables $x$ ("space") and $t$ ("time") to be real numbers; and we indicate partial differentiations with respect to these variables by appending them as subscripts preceded by commas, so for instance $\phi_{m,t}(x,t) \equiv \partial \phi_m(x,t)/\partial t$, $\psi_{n,x}(x,t) \equiv \partial^2 \psi_n(x,t)/\partial x^2$. We generally focus on generic polynomials the coefficients and zeros of which are generic complex numbers, and which in particular feature zeros all different among themselves, $\psi_n(x,t) \neq \psi_m(x,t)$ if $n \neq m$. Hereafter we often omit the explicit indication of the dependent variables $x$ and $t$ when this can be done without causing confusion. Note that the notation $P_N (z; \bar{\phi}, \psi)$ is somewhat redundant, since this monic polynomial of degree $N$ in $z$ can be identified by assigning either its $N$ coefficients $\phi_m$ or its $N$ zeros $\psi_n$; indeed the $N$ coefficients $\phi_m$ can be expressed in terms of the $N$ zeros $\psi_n$ via the standard formula

\[ \phi_m = (-1)^m \sum_{1 \leq n_1 < n_2 < \ldots < n_m \leq N} (\psi_{n_1} \psi_{n_2} \cdots \psi_{n_m}) ; \quad (1.2a) \]

so that

\[ \phi_1 = - (\psi_1 + \psi_2 + \ldots + \psi_N) ; \quad (1.2b) \]

\[ \phi_2 = (\psi_1 \psi_2 + \psi_1 \psi_3 + \ldots + \psi_1 \psi_N) + (\psi_2 \psi_3 + \psi_2 \psi_4 + \ldots + \psi_2 \psi_N) + \ldots + (\psi_{N-2} \psi_{N-1} + \psi_{N-2} \psi_N) + \psi_{N-1} \psi_N ; \quad (1.2c) \]

and so on. On the other hand, while the assignment of the $N$ coefficients $\phi_m$ determines uniquely, up to permutations, the $N$ zeros $\psi_n$, of course explicit formulas in terms of elementary functions (including radicals) expressing the zeros of a polynomial of degree $N$ in terms of its coefficients are generally only available for $N \leq 4$. Finally let us note that hereafter we adopt the standard convention according to which a void sum vanishes, and a void product equals unity.

In the following Section 2 we report and discuss our main findings, which are then proven in the following Section 3. A terse Section 4 outlines possible future developments.
2. Results

Proposition 2.1. The following system of $N$ coupled nonlinear PDEs in 1+1 variables is $C$-integrable:

$$
\psi_{n,t} + \psi_{n,xx} = \sum_{\ell=1, \ell \neq n}^{N} \left( \frac{2 \psi_{n,x} \psi_{\ell,x}}{\psi_n - \psi_\ell} \right) - \left[ \prod_{\ell=1, \ell \neq n}^{N} (\psi_n - \psi_\ell) \right]^{-1} \sum_{m=1}^{N} \left[ a_m \, \varphi_{m,x} \, \varphi_m \, (\psi_n)^{N-m} \right],
$$

where the parameters $a_m$ are $N$ arbitrary (complex) numbers, the $N$ (complex) functions $\psi_n \equiv \psi_n(x,t)$ are the dependent variables, and the $N$ (complex) functions $\varphi_m \equiv \varphi_m(x,t)$ are expressed in terms of the dependent variables $\psi_n \equiv \psi_n(x,t)$ by the formulas (1.2), implying of course

$$
\varphi_{1,x} = - \sum_{n=1}^{N} (\psi_{n,x}) ,
$$

$$
\varphi_{m,x} = (-1)^m \sum_{s=1}^{N} \left[ \psi_{n,s} \cdot \sum_{1 \leq n_1 < n_2 < \ldots < n_{s-1} < n_{s+1} < \ldots < n_N \leq N} \left( \psi_{n_1} \psi_{n_2} \cdots \psi_{n_{s-1}} \psi_{n_{s+1}} \cdots \psi_{n_N} \right) \right] ,
\quad m = 2, \ldots, N .
$$

This means that the initial-value problem—to compute the $N$ functions $\psi_n(x,t)$ for all time $t > 0$ from given initial data $\psi_n(x,0)$—can be solved by algebraic operations (including changes of variables from the coefficients to the zeros of a polynomial of degree $N$ such as (1.1)) and quadratures. The procedure to do so is detailed in the following Section 3, and this implies the validity of the solutions reported below.

For $N = 2$ this system, (2.1), of 2 coupled nonlinear PDEs reads as follows:

$$
\psi_{n,t} + \psi_{n,xx} = (\psi_n - \psi_{n+1})^{-1} \left\{ \psi_{n,x} \psi_{n+1,x} + \left[ a_1 \, (\psi_{n,x} + \psi_{n+1,x}) \psi_n \psi_{n+1} \right] - a_2 \, (\psi_{n,x} \psi_{n+1} + \psi_n \psi_{n+1,x}) \psi_n \psi_{n+1} \right\} , \quad n = 1, 2 \mod (2) .
$$

An example of specific solution of this system of 2 coupled nonlinear PDEs, (2.3), reads as follows:

$$
\psi_n(x,t) = - \frac{1 + (-1)^n \left\{ 1 - 4 \left[ f_1(x,t) \right]^2 / f_2(x,t) \right\}^{1/2}}{2 \, f_1(x,t)} , \quad n = 1, 2 ,
$$

$$
f_n(x,t) = - \frac{\alpha_n}{2 \gamma_n} + \beta_n \exp \left[ -\gamma_n (x - \gamma_n t) \right] , \quad n = 1, 2 ,
$$

where the 2 parameters $\alpha_n$ are those appearing in the PDEs (2.3) and the 4 (nonvanishing) parameters $\beta_n$ and $\gamma_n$ can be arbitrarily assigned. Note that if the 6 parameters $\alpha_n, \beta_n, \gamma_n$ are all real numbers, then the 3 inequalities $a_1 \beta_1 \gamma_1 < 0$, $a_2 \gamma_2 > 0$, $\beta_2 < 0$ are sufficient to guarantee that for all real values of the independent variables $x$, $t$ these solutions (2.4) are real and nonsingular. Also note that if $\gamma_1 = \gamma_2 = \gamma$ this solution has the "single soliton" feature to depend on the space and time coordinates only via their combination $x - \gamma t$.
Proposition 2.2. The following system of $N$ coupled nonlinear PDEs in $1+1$ variables is $S$-integrable:

$$ \psi_{n,t} + \psi_{n,xxx} = 3 \sum_{\ell=1, \ell \neq n}^{N} \left( \frac{\psi_{n,xx}}{\psi_{n} - \psi_{\ell}} \right) \left( \psi_{n} - \psi_{\ell} \right) - 3 \sum_{\ell_1, \ell_2 = 1; \ell_1 \neq \ell_2, \ell_1, \ell_2 \neq n}^{N} \left( \frac{\psi_{n,x}}{\psi_{n} - \psi_{\ell_1}} \right) \left( \psi_{n} - \psi_{\ell_2} \right) + \left[ \prod_{\ell=1, \ell \neq n}^{N} (\psi_{n} - \psi_{\ell}) \right]^{-1} \sum_{m=1}^{N} a_m \varphi_{m,x} \varphi_{m} (\psi_{n})^{N-m} , $$

(2.5)

where the parameters $a_m$ are $N$ arbitrary (complex) numbers, the $N$ (complex) functions $\psi_{n} \equiv \psi_{n}(x,t)$ are the dependent variables, and the $N$ (complex) functions $\varphi_{m} \equiv \varphi_{m}(x,t)$ respectively $\varphi_{m,x} \equiv \varphi_{m,x}(x,t)$ are expressed in terms of the dependent variables $\psi_{n} \equiv \psi_{n}(x,t)$ and their $x$-derivatives $\psi_{n,x} \equiv \psi_{n,x}(x,t)$ by the formulas (1.2) respectively (2.2).

This means that the initial-value problem—to compute the $N$ functions $\psi_{n}(x,t)$ for all time $t > 0$ from given initial data $\psi_{n}(x,0)$—can be solved by algebraic operations (including changes of variables from the coefficients to the zeros of a polynomial of degree $N$ such as (1.1)) and via the standard Spectral Transform technique. The procedure to do so is detailed in the following Section 3, and this implies the validity of the solutions reported below.

For $N = 2$ this system of 2 coupled nonlinear PDEs reads as follows:

$$ \psi_{n,t} + \psi_{n,xxx} = (\psi_{n} - \psi_{n+1})^{-1} \left( 3 (\psi_{n,xx} \psi_{n+1,x} + \psi_{n,x} \psi_{n+1,xx}) - [a_1 (\psi_{n,x} + \psi_{n+1,x}) (\psi_{n} + \psi_{n+1}) \psi_{n}] + a_2 (\psi_{n,x} \psi_{n+1} + \psi_{n} \psi_{n+1,x}) \psi_{n} \psi_{n+1} \right) , \quad n = 1, 2 \mod (2) . $$

(2.6)

An example of specific solution of this system of 2 coupled nonlinear PDEs, (2.6), reads as follows:

$$ \psi_{n}(x,t) = \left\{ \frac{-6 \beta_1 (\gamma_1)^2}{a_1 \cosh^2 [\gamma_1 (x - 4 \gamma_1 t)]} \right\} \cdot \left\{ 1 + (-1)^n \left[ 1 - \frac{(a_1)^2 \beta_2 (\gamma_2)^2 \cosh^4 [\gamma_1 (x - 4 \gamma_1 t)]}{3 a_2 (\beta_1)^2 (\gamma_1)^4 \cosh^2 [\gamma_2 (x - 4 \gamma_2 t)]} \right]^{1/2} \right\} , $$

(2.7)

where the $2$ parameters $a_n$ are those appearing in the system (2.6) and the $4$ (nonvanishing) parameters $\beta_n$ and $\gamma_n$ can be arbitrarily assigned. Note that if these $4$ parameters are all real it is then sufficient that the parameter ratio $\beta_2/a_2$ be negative, $\beta_2/a_2 < 0$, for this solution to be real and nonsingular for all real values of the dependent variables $x$ and $t$. Also note that if $\gamma_1 = \gamma_2 = \gamma$ this solution has the "single soliton" feature to depend on the space and time coordinates only via their combination $x - 4\gamma t$.

For $N = 3$ this system of 3 coupled nonlinear PDEs, (2.5), reads as follows:

$$ \psi_{n,t} + \psi_{n,xxx} = 3 \sum_{s=1,2} \left( \frac{\psi_{n,xx}}{\psi_{n} - \psi_{n+s}} \right) \left( \psi_{n} - \psi_{n+s} \right) + \left[ (\psi_{n} - \psi_{n+1}) (\psi_{n} - \psi_{n+2}) \right]^{-1} \left\{ -6 \left[ \psi_{n,x} \psi_{n+1,x} \psi_{n+2,x} \right] + \sum_{m=1}^{3} a_m \varphi_{m,x} \varphi_{m} (\psi_{n})^{N-m} \right\} , \quad n = 1, 2, 3 \mod (3) , $$

(2.8)
where of course \( \varphi_n \) respectively \( \varphi_{m,x} \) are given by (1.2) respectively (2.2) (with \( N = 3 \)).

3. Proofs

The proofs of the above two Propositions are actually quite easy. The starting point are the 3 identities [3]

\[
\psi_{n,t} = - \left[ \prod_{\ell = 1, \ell \neq n}^{N} (\psi_n - \psi_\ell) \right]^{-1} \sum_{m=1}^{N} \left[ \varphi_{m,t} (\psi_n)^{N-m} \right], \tag{3.1a}
\]

\[
\psi_{n,xx} = \sum_{\ell = 1, \ell \neq n}^{N} \left( \frac{2 \psi_{n,x} \psi_{\ell,x}}{\psi_n - \psi_\ell} \right) - \left[ \prod_{\ell = 1, \ell \neq n}^{N} (\psi_n - \psi_\ell) \right]^{-1} \sum_{m=1}^{N} \left[ \varphi_{m,xx} (\psi_n)^{N-m} \right], \tag{3.1b}
\]

\[
\psi_{n,xxx} = 3 \sum_{\ell = 1, \ell \neq n}^{N} \left( \frac{\psi_{n,xx} \psi_{\ell,x} + \psi_{n,x} \psi_{\ell,xx}}{\psi_n - \psi_\ell} \right)
- 3 \sum_{\ell_1, \ell_2 = 1, \ell_1 \neq \ell_2, \ell_1, \ell_2 \neq n}^{N} \left[ \frac{\psi_{n,x} \psi_{\ell_1,x} \psi_{\ell_2,x}}{(\psi_n - \psi_{\ell_1}) (\psi_n - \psi_{\ell_2})} \right]
- \left[ \prod_{\ell = 1, \ell \neq n}^{N} (\psi_n - \psi_\ell) \right]^{-1} \sum_{m=1}^{N} \left[ \varphi_{m,xxx} (\psi_n)^{N-m} \right], \tag{3.1c}
\]

that relate the \( N \) zeros \( \psi_n \) and the \( N \) coefficient \( \varphi_n \) of a polynomial such as (1.1).

We now note that the sum of the first two of these identities imply the identity

\[
\psi_{n,t} + \psi_{n,xx} = \sum_{\ell = 1, \ell \neq n}^{N} \left( \frac{2 \psi_{n,x} \psi_{\ell,x}}{\psi_n - \psi_\ell} \right)
- \left[ \prod_{\ell = 1, \ell \neq n}^{N} (\psi_n - \psi_\ell) \right]^{-1} \sum_{m=1}^{N} \left\{ \varphi_{m,t} + \varphi_{m,xx} \right\} (\psi_n)^{N-m}; \tag{3.2a}
\]

and likewise the sum of the first and third of the identities (3.1) implies the identity

\[
\psi_{n,t} + \psi_{n,xxx} = 3 \sum_{\ell = 1, \ell \neq n}^{N} \left( \frac{\psi_{n,xx} \psi_{\ell,x} + \psi_{n,x} \psi_{\ell,xx}}{\psi_n - \psi_\ell} \right)
- 3 \sum_{\ell_1, \ell_2 = 1, \ell_1 \neq \ell_2, \ell_1, \ell_2 \neq n}^{N} \left[ \frac{\psi_{n,x} \psi_{\ell_1,x} \psi_{\ell_2,x}}{(\psi_n - \psi_{\ell_1}) (\psi_n - \psi_{\ell_2})} \right]
- \left[ \prod_{\ell = 1, \ell \neq n}^{N} (\psi_n - \psi_\ell) \right]^{-1} \sum_{m=1}^{N} \left\{ \varphi_{m,t} + \varphi_{m,xxx} \right\} (\psi_n)^{N-m}; \tag{3.2b}
\]

Now assume that the \( N \) functions \( \varphi_n \equiv \varphi_n (x,t) \) satisfy the Burgers equations

\[
\varphi_{m,t} + \varphi_{m,xx} = \alpha_m \varphi_{m,x} \varphi_m; \tag{3.3}
\]

it is plain, see (3.2a), that this implies that the \( N \) functions \( \psi_n \equiv \psi_n (x,t) \) satisfy the system of PDEs (2.1). Proposition 2.1 is thereby proven. Indeed this implies that the solution of the initial-value problem for this system of PDEs, (2.1), is yielded by the following procedure. Step (i): from
the initial data $\psi_n(x,0)$ compute the corresponding functions $\varphi_m(x,0)$ (via the formulas (1.2)).

**Step (ii):** solve the C-integrable PDEs (3.3) with these initial data $\varphi_m(x,0)$, obtaining thereby the functions $\varphi_m(x,t)$ for all time $t > 0$. **Step (iii):** the solutions $\psi_n(x,t)$ of the system of PDEs (2.1) are then provided by the $N$ zeros of the polynomial (1.1) with coefficients $\varphi_m(x,t)$. And of course the explicit solution (2.4) is manufactured using the single-soliton solutions of the Burgers equations (3.3).

The proof of **Proposition 2.2**, and the procedure to solve the system of PDEs (2.5), are quite analogous, except that the role of the identity (3.3a) is now played by the identity (3.2a), and the role played by the C-integrable Burgers PDEs (3.3) is now played by the S-integrable KdV PDEs

$$\varphi_{m,t} + \varphi_{m,xxx} + a_m \varphi_{m,x} \varphi_m = 0.$$  \hspace{1cm} (3.4)

**4. Outlook**

It is plain that the approach employed in this paper provides the possibility to identify a large universe of new integrable/solvable systems of nonlinear PDEs; the two PDEs specifically discussed above are merely examples of the vistas opened by this methodology to identify new integrable/solvable systems of nonlinear PDEs. Note for instance that the assumptions made above—that all the coefficients $\varphi_m(x,t)$ satisfy the same integrable PDE—are not quite necessary; for instance in the case of **Proposition 2.1** some of the coefficients $\varphi_{m}(x,t)$ might satisfy the C-integrable Kundu-Eckhaus PDE [6] and in the case of **Proposition 2.2** some of the coefficients $\varphi_{m}(x,t)$ might satisfy the S-integrable Modified KdV PDE... Moreover, all the novel integrable/solvable PDEs identified via this approach can themselves be subsequently interpreted as characterizing the evolution of the coefficients of a polynomial, hence as inputs for the generation of new systems of integrable/solvable PDEs via this approach [4]; and it is also possible to extend this approach to a multidimensional context (beyond the 1+1 context of the present paper) [2] and to more general auxiliary functions than polynomials [7].

This approach might moreover open the way to the identification and investigation of new integrable/solvable systems of nonlinear PDEs which are of interest because of their universality hence possible wide applicability (for these notions see for instance [8] and references therein).

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