Minimizing geodesic nets and critical points of distance

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Abstract
In this paper we establish a relationship between geodesic nets and critical points of the distance function. We bound the number of balanced points for certain minimizing geodesic nets on manifolds homeomorphic to the $n$-sphere. We also bound the length of certain minimizing geodesic nets.

1. Introduction

A net is a (finite) multigraph embedded into a Riemannian manifold. The vertices of a net are partitioned into balanced and boundary vertices. A number of adjectives are often added to the definition of a net, c.f. Nabutovsky and Parsch [10]. A geodesic net is a net such that each edge is a geodesic segment, and a stationary geodesic net is a geodesic net such that at each balanced vertex, the sum of the unit tangent vectors to the incident edges equals zero. As such, stationary geodesic nets are critical points of the length/energy functional on the space of graphs with prescribed boundary vertex set embedded in a Riemannian manifold. In this paper we work with a new, more restrictive class of nets:

Definition 1.1. A minimizing geodesic net is a stationary geodesic net such that each edge realizes the distance between its vertices.

We provide some examples. In flat Euclidean space, every stationary geodesic net is a minimizing geodesic net, as geodesics in Euclidean space minimize between any pair of their points. There are open questions about stationary geodesic nets even in the Euclidean setting, c.f. Nabutovsky and Parsch [10].

The theta-graph is a graph with three edges between two vertices. Such a theta-graph can be realized as a minimizing geodesic net on the round 2-sphere by taking the poles as the vertices, and three great semicircles of longitude as the edges, each meeting at angle $2\pi/3$ at the poles (see Figure 1). Hass and Morgan [8] provide one of the only known existence results on geodesic nets, demonstrating that convex metrics on the 2-sphere sufficiently close to the round metric necessarily contain a stationary geodesic net modeled on the theta-graph.

An important question in the study of geodesic nets is whether one can bound the number of balanced vertices in terms of the other geometric properties of the net. For example, Parsch [13] has recently proved that a geodesic net with
three unbalanced (boundary) vertices in the Euclidean plane can admit at most one balanced vertex, occurring at the Fermat point for the triangle determined by the three unbalanced vertices. Gromov [6] has conjectured that the number of balanced vertices of a geodesic net in the Euclidean plane can be bounded above in terms of the number of unbalanced vertices and the total imbalance, see also [10, Conjecture 3.4.1].

Our main result provides an upper bound on the number of balanced points on minimizing geodesic nets modeled on a certain graphs embedded in manifolds homeomorphic to the $n$-sphere.

**Theorem 1.2.** Let $M$ be a complete Riemannian manifold with sectional curvature $k \geq 1$ and injectivity radius $> \pi/2$. Let $\Gamma$ be a star multigraph, i.e. a multigraph with radius one whose underlying graph is acyclic. Then any minimizing geodesic net modeled on $\Gamma$ whose image contains a simple closed geodesic admits at most one non-central balanced vertex.

The proof of the theorem relies on the notion of a Grove-Shiohama critical point of distance [Definition 2.1]. For a minimizing geodesic net modeled on $\Gamma$, the unique central vertex together with the halfway point along the assumed closed geodesic form a pair of mutual Grove-Shiohama critical points of distance. Under the assumed lower bounds on sectional curvature and injectivity radius, we show that mutual Grove-Shiohama critical points of distance are unique [Proposition 2.3]. It follows that the net cannot admit additional balanced vertices; full details are provided in Section 2.

This theorem has a nice reformulation in the setting of minimizing geodesic flowers. A *geodesic flower* is a bouquet of (finitely many) geodesic loops (petals) based at the same vertex and satisfying the stationary condition at the vertex. We define a *minimizing* geodesic flower to be a geodesic flower that realizes the distance from the vertex to the halfway point along each of its petals. If a minimizing geodesic flower contains a simple closed geodesic, then the vertex together with the halfway point form a mutual Grove-Shiohama critical pair, and uniqueness [Proposition 2.3] yields the following:
Theorem 1.3. Let $M$ be a complete Riemannian manifold with sectional curvature $k \geq 1$ and injectivity radius $> \pi/2$. Then any minimizing geodesic flower on $M$ that contains a simple closed geodesic must have exactly one petal, namely the simple closed geodesic.

Rotman [15] (also [11] and [13]) has proved the existence of a constant $c(n)$ such that each closed Riemannian manifold $M^n$ admits a geodesic flower with length bounded above by $c(n)\text{vol}(M)^{\frac{1}{n}}$, alternatively by $c(n)\text{diam}(M)$. In the compact setting, any minimizing geodesic flower with $m$ petals must have length bounded above by $(2m)\text{diam}(M)$. In the non-compact setting such a universal upper bound does not exist, but we provide the following pointwise result.

Theorem 1.4. Let $M$ be a complete non-compact Riemannian manifold with sectional curvature $k \geq 0$. Then for every $p \in M$ there exists a constant $R_p > 0$ such that every minimizing geodesic flower with vertex $p$ and $m$ petals has length bounded above by $(2m)R_p$.

Note that unlike in [15], our result does not provide an upper bound on the length of the shortest geodesic flower. We do not prove the existence of a minimizing geodesic flower; we merely provide a pointwise upper bound on the length should such a minimizing geodesic flower exist.

The paper proceeds as follows. In Section 2 we introduce the relevant ideas from critical point theory. We prove the fact that mutual Grove-Shiohama critical points of distance are unique in our setting [Proposition 2.3]. We show how Theorems 1.2 and 1.3 follow from this proposition. Finally, we show how Theorem 1.4 follows from a lemma due to Gromov. For more on geodesic nets in this setting see Croke [3], Heppes [9], Rotman [16], and [1].

2. Proof of the Theorems

We first introduce the relevant ideas from critical point theory.

Definition 2.1. A Grove-Shiohama critical point of $d_p: M \to \mathbb{R}$ is a point $q \in M$ such that for any $v \in T_qM$ there exists a minimizing geodesic from $q$ to $p$ with initial velocity vector $w \in T_qM$ such that $\angle(v, w) \leq \pi/2$.

The original application of this critical point definition is the celebrated Grove-Shiohama diameter sphere theorem [7].

Theorem 2.2 (Grove-Shiohama). Let $M$ be a complete Riemannian manifold with sectional curvature $k \geq 1$ and diameter $> \pi/2$. Then $M$ is homeomorphic to the sphere.

On Riemannian manifolds we know diameter $\geq$ injectivity radius, so that by the above result, all manifolds considered by Theorems 1.2 and 1.3 are homeomorphic to the sphere. The theorem of Bonnet-Myers states that a complete Riemannian manifold with sectional curvature $k \geq 1$ has an upper bound of $\pi$ on the diameter, with Cheng [5] proving that only the round sphere realizes this diameter upper bound.
At the other extreme of the manifolds considered by Theorems 1.2 and 1.3, we note that the standard metric on $\mathbb{R}P^n$ has sectional curvature $k = 1$ and diameter = injectivity radius = $\pi/2$. This projective space demonstrates that the bounds in Theorems 1.2 and 1.3 are sharp, as $\mathbb{R}P^n$ admits minimizing geodesic flowers with any number of petals (any number of projected great circles through a central vertex).

Theorems 1.2 and 1.3 rely on the following proposition. Here we say that a pair $p, q \in M$ are mutually Grove-Shiohama critical if $q$ is a Grove-Shiohama critical point of $d_p$ and $p$ is a Grove-Shiohama critical point of $d_q$.

**Proposition 2.3.** Let $M$ be a complete Riemannian manifold with sectional curvature $k \geq 1$ and injectivity radius $> \pi/2$. If a pair $p, q \in M$ are mutually Grove-Shiohama critical, then neither $d_p$ nor $d_q$ admits additional Grove-Shiohama critical points of distance.

**Proof.** Let $p$ and $q$ be mutually Grove-Shiohama critical and assume by contradiction that $x \neq p, q \in M$ is a Grove-Shiohama critical point of $d_p$. Let $\gamma_2$ be a minimal geodesic from $q$ to $x$. We know there exists a minimal geodesic $\gamma_0$ from $x$ to $p$ such that $\angle(-\dot{\gamma}_2(d(q, x)), \dot{\gamma}_0(0)) \leq \pi/2$. Since $p$ and $q$ are mutually critical there exist minimal geodesics $\gamma_1$ and $\tilde{\gamma}_1$ from $p$ to $q$ such that $\angle(\dot{\gamma}_1(0), \dot{\gamma}_0(d(p, x))) \leq \pi/2$ and $\angle(-\dot{\tilde{\gamma}}_1(d(p, q)), \dot{\gamma}_2(0)) \leq \pi/2$. Now apply the triangle version of Toponogov’s theorem to both $\{\gamma_0, \gamma_1, \gamma_2\}$ and $\{\gamma_0, \tilde{\gamma}_1, \gamma_2\}$ yielding comparison triangles in the unit 2-sphere. Because triangles in the sphere are determined up to congruence by side lengths we get a unique comparison triangle, each of whose angles is $\leq \pi/2$. This implies that the comparison triangle is completely contained in an octant of the sphere, hence has side lengths $\leq \pi/2$. This is a contradiction, as $q$ a Grove-Shiohama critical point of $d_p: M \to \mathbb{R}$ implies that $d(p, q) \geq \text{injrad}(M) > \pi/2$. \[\square\]

We now provide short proofs of Theorems 1.2 and 1.3.

**Proof of Theorem 1.2.** Assume that $M$ admits a minimizing geodesic net modeled on the graph $\Gamma$. Note that the assumed simple closed geodesic must contain the unique central vertex $p$ for the net. The minimizing geodesic net conditions imply that this central vertex together with the halfway point along the closed geodesic are mutually Grove-Shiohama critical. Any other balanced vertex will be a Grove-Shiohama critical point of $d_p: M \to \mathbb{R}$, and the result then follows from Proposition 2.3. \[\square\]

**Proof of Theorem 1.3.** Assume that $M$ admits a minimizing geodesic flower that contains a simple closed geodesic. The minimizing geodesic flower conditions imply that the vertex $p$ together with the halfway point along the closed geodesic are mutually Grove-Shiohama critical. The halfway point along any other petal will be a Grove-Shiohama critical point of $d_p: M \to \mathbb{R}$, and the result then follows from Proposition 2.3. \[\square\]

Theorem 1.4 follows from this lemma due to Gromov [4, Corollary 2.9].
Lemma 2.4 (Gromov). Let $M$ be a complete non-compact Riemannian manifold with sectional curvature $k \geq 0$. Then for every $p \in M$ the distance function $d_p : M \to \mathbb{R}$ has no critical points outside of some ball $B(p, R_p)$. In particular, $M$ is homeomorphic to the interior of a compact manifold with boundary.

Proof of Theorem 1.4. Consider a minimizing geodesic flower with $m$ petals and vertex $p$. The halfway point along each petal is a Grove-Shiohama critical point of $d_p : M \to \mathbb{R}$, and by Lemma 2.4 must lie in $B(p, R_p)$. We conclude that the total length of the minimizing geodesic flower is bounded above by $(2m)R_p$. \qed

See also [2] where minimizing closed geodesics are studied via Grove-Shiohama critical points of distance, a version of Proposition 2.3 and Lemma 2.4.

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