INCONGRUENT EQUIPARTITIONS OF THE PLANE

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Abstract. R. Nandakumar asked whether there is a tiling of the plane by pairwise incongruent triangles of equal area and equal perimeter. Recently a negative answer was given by Kupavskii, Pach and Tardos. Still one may ask for weaker versions of the problem, or for the analogue of this problem for quadrangles, pentagons, or hexagons. Several answers were given by the first author in a previous paper. Here we solve three further cases. In particular, our main result shows that there are vertex-to-vertex tilings by pairwise incongruent triangles of unit area and bounded perimeter.

1. Introduction

R. Nandakumar posed several intriguing questions about tilings [N2] that have triggered a lot of research recently [BBS, BZ, KHA, KPT2, NR, PT, RW]. Two of his problems are these [N1]:

Question 1. “Can the plane be tiled by triangles of same area and perimeter such that no two triangles are congruent to each other?” Here congruence is meant with respect to Euclidean isometries including reflections.

This question was answered in [KPT1] by showing that no such tiling exists. Nandakumar also asked for a weaker version

Question 2. “Can the plane be tiled by triangles of same area, and with uniformly bounded perimeter, such that no two triangles are congruent to each other?” Here congruence is meant with respect to Euclidean isometries including reflections.

Weakening the problem by dropping any requirement on the perimeter makes the problem easy: it is not hard to find tilings of the plane by triangles of unit area with unbounded perimeter, see [N1]. The second question was answered in [F] (partly), and in [KPT2].

Theorem 1 ([F, KPT2]). There are tilings of the plane by pairwise incongruent triangles of unit area and bounded perimeter.

Once this is settled, one may ask the same question for convex n-gons. Moreover, since the tilings in [F, KPT2] are not vertex-to-vertex (vtv), one may ask whether there are vtv tilings fulfilling the properties of Theorem 1. Table 1 provides an overview of several variants of the problems, together with the known answers.

This paper is devoted to solve further instances of the problem (shown in black in Table 1). In the remainder of this section we introduce some notations and basic ideas. Along the way we show that there are vertex-to-vertex tilings of the plane by pairwise incongruent convex pentagons of unit area and bounded perimeter (Theorem 2). The main result of this paper is Theorem 3. It says that there are vertex-to-vertex tilings of the plane by pairwise incongruent triangles of unit area and bounded perimeter. Section 2 is devoted to the proof of this result. In the last section we construct a tiling of the plane by pairwise incongruent convex hexagons of unit area and bounded perimeter that are not vertex-to-vertex. Note that usually the requirement of being vertex-to-vertex is more restrictive than the requirement being not vertex-to-vertex, but for convex hexagons it is the other way around.

Date: May 21, 2019.
Triangles | vtv | not vtv
---|---|---
bounded perimeter | Yes | ?
tiling a tile | ? | ?
equal perimeter | No | No

Pentagons | vtv | not vtv
---|---|---
bounded perimeter | Yes | ?
tiling a tile | Yes | ?
equal perimeter | ? | ?

Quadrangles | vtv | not vtv
---|---|---
bounded perimeter | Yes | ?
tiling a tile | Yes | ?
equal perimeter | ? | ?

Hexagons | vtv | not vtv
---|---|---
bounded perimeter | Yes | ?
tiling a tile | Yes | ?
equal perimeter | ? | ?

Table 1. Several instances of the problem “tiling the plane with pairwise incongruent convex \(n\)-gons of unit area” plus some further conditions. Grey entries marked 1 were solved in [F]. The grey entry marked 1, 2 was solved in [F] (partly) and in [KPT2]. The grey entry marked 3 was solved in [KPT1]. Black entries “Yes” are solved in this paper.

Figure 1. Partitioning a convex pentagon into six distinct convex pentagons of equal area. A white dot (resp., a dashed line) indicates two degrees (resp., one degree) of freedom for moving the corresponding vertex. The numbers give the order of choosing free vertices. Unmarked vertices are not free. The resulting tiling is vertex-to-vertex.

1.1. Notation. A tiling of a set \(A \subseteq \mathbb{R}^2\) is a collection \(\{T_1, T_2, \ldots\}\) of compact sets \(T_i \subseteq \mathbb{R}^2\) (the tiles) that is a packing (i.e., the interiors of distinct tiles are disjoint) as well as a covering of \(A\) (i.e., the union of the tiles equals \(A\)). In general, shapes of tiles may be pretty complicated, but for the purpose of this paper tiles are always convex polygons. A tiling is called vertex-to-vertex (vtv), if the intersection of any two tiles is either an entire edge of both tiles, or a vertex of both tiles, or empty. An equipartition of the plane is a tiling of \(\mathbb{R}^2\) such that each tile has unit area. We refer to [GS] as a standard reference work on tilings.

1.2. Tiling a tile. One possible approach to find a solution for an entry in Table 1 is the following. If one can partition a set \(P \subseteq \mathbb{R}^2\) into convex \(n\)-gons of unit area such that (i) \(P\) tiles the plane (i.e., \(\mathbb{R}^2\) can be tiled by congruent copies of \(P\)), and (ii) all \(n\)-gons in \(P\) can be distorted continuously, in a way such that any two \(n\)-gons in \(P\) are incongruent (but still have unit area), then this yields a solution to the problem under consideration. Let us illustrate this concept for pentagons. Figure 1 shows a large pentagon \(P\) that tiles the plane. This large pentagon is divided into six smaller pentagons of unit area (left). Four of the five inner vertices of this subdivision can be wiggled continuously without destroying the unit area property: As indicated in the image, there
are two continuous degrees of freedom for the choice of the first vertex, marked by 1 in the image. I.e., vertex 1 can be moved within some ball of small radius. Once this vertex is fixed, there is just one line segment representing the possible positions of vertex 2 such that the area of the lower left pentagon remains one. That is, vertex 2 will still have one degree of freedom (indicated by a dashed line). In a similar manner now vertex 3 needs to be adjusted such that the area of the pentagon left of it remains one. This vertex still has one degree of freedom: it can be shifted in direction of the dashed line without affecting the area of the pentagon left of it. The same is true for vertex 4: it will be affected by the choice of vertex 3 in order to keep the area of the upper pentagon to one, but still has one degree of freedom, indicated by a dashed line. When all these four vertices are chosen, the last interior vertex is fixed by the condition that both the two right-hand interior pentagons need to have area one: it is the intersection point of two segments representing admissible positions such that the areas of the upper right and the lower right interior pentagons are one, respectively. The central pentagon will automatically have unit area, since the areas of the six small pentagons add up to 6.

Hence we may partition the large pentagon (of area 6, say) in uncountably many ways into pairwise incongruent pentagons of unit area. This yields the desired equipartition of the plane: In the first large pentagon we will distort the pentagons in a way such that all of them are incongruent. This can be achieved easily. In the second large pentagon we use the uncountably many possible choices in order to achieve that all small pentagons in the second large pentagon are (i) incongruent to all pentagons in the first large pentagon and (ii) pairwise incongruent to each other. And so on. The large pentagons will be arranged as on the right in Figure 1. At each stage there are only finitely many shapes of small pentagons to be avoided. Because we may choose from uncountably many pentagons, this procedure yields the desired equipartition of the plane into pentagons. Clearly these equipartitions are vertex-to-vertex. This way tiling a tile gives the following result.

Theorem 2. There is a vertex-to-vertex tiling of the plane by pairwise incongruent convex pentagons of unit area and uniformly bounded perimeter.

In the next section we will prove a similar result for triangles, but the proof requires more effort.

2. VTV equipartitions of the plane into triangles

The argument that we can use continuous degrees of freedom in order to avoid finitely many (or countably many) shapes will also be used in the proof of the following result.

Theorem 3. There is a vertex-to-vertex tiling of the plane by pairwise incongruent triangles of unit area and uniformly bounded perimeter.

Proof. The general idea of the construction is the following: Consider the strip $S = \mathbb{R} \times [-1, 1]$. Tile $S$ by (pairwise congruent) triangles of unit area with edge lengths $\sqrt{2}$, $\sqrt{2}$ and 2, see Figure 2. Distort the tiling of $S$ by moving the vertex at $(0, 0)^T$ to $(0, y_0)^T$ for $0 < y_0 < 1$, see Figure 3.

Under the conditions that (i) the topology of the tiling is unchanged, (ii) the new tiling is still a tiling of $S$, (iii) the new tiling is mirror symmetric with respect to the vertical axis $x = 0$, and (iv) the tiles of the new tiling stay triangles of unit area, the value of $y_0$ determines all other vertices of the tiling. See Figure 3 for the situation where $y_0 = \frac{3}{4}$.

In the sequel the strategy of the proof is as follows: first we obtain recursive formulas for the coordinates of the triangles in the tiling of the strip when $y_0$ varies, in order to control the amount...
of distortion of the triangles (Lemma 4). This ensures in particular that the perimeter of the triangles stays uniformly bounded (Lemma 5). Then we study the tiling for \( y_0 = \frac{1}{\sqrt{3}} \), having the particular property that it contains many pairwise congruent triangles \( T \cong T' \); and even stronger: it contains many triangles \( T, T' \) such that \( \pm T \) is a translate of \( T' \) (see Figure 3); this property is denoted by \( T \simeq T' \). This is Lemma 6. This can be used in Lemmas 7 and 8 to show that there are only countably many \( y_0 \) such that the corresponding tiling contains triangles \( T, T' \) such that \( T \simeq T' \). This in turn enables us to pick a tiling \( \mathcal{T} \) of the strip \( S \) such that for all \( T, T' \in \mathcal{T} \) holds: \( T \not\simeq T' \) (Corollary 9). Then again a countability argument allows us to find sheared copies \( \frac{1}{0,1} \mathcal{T} \) of \( \mathcal{T} \) such that no pair of congruent tiles occur within them, nor in between them (Lemma 10, Corollary 11). The tilings \( \frac{1}{0,1} \mathcal{T} \) (of the strip \( S \)) can be stacked in order to obtain the desired vertex-to-vertex tiling of the plane.

Let us start by considering the tilings of the strip \( S \). Since we have mirror symmetry with respect to the vertical axis, we first study the situation within the right half \( S^+ = [0, \infty) \times [-1, 1] \) of \( S \). We need some notation, see Figure 3. Let \((x_i, y_i)^T\) denote the coordinates of the vertices along the central (distorted) line, separating the upper layer of triangles from the lower layer of triangles. Let \( a_i \) denote the \( x \)-coordinate of the vertices along the upper boundary of the strip \( S \) (the \( y \)-coordinate is always 1), and let \( b_i \) denote the \( x \)-coordinate of the vertices along the lower boundary of the strip \( S \) (the \( y \)-coordinate is always \(-1 \)). Based on the parameter \( y_0 \), let

\[
(1) \quad x_0 = 0, \quad y_0 = y_0, \quad a_1 = \frac{1}{1 - y_0}, \quad b_1 = \frac{1}{1 + y_0}
\]

and, for \( i = 1, 2, \ldots \),

\[
(2) \quad x_i = x_{i-1} + 2 - \frac{2(a_i - b_i)y_{i-1}}{(a_i - x_{i-1})(1 + y_{i-1}) + (b_i - x_{i-1})(1 - y_{i-1})},
\]

\[
(3) \quad y_i = y_{i-1} - \frac{4a_i - 2(a_i - x_{i-1})(1 + y_{i-1}) + (b_i - x_{i-1})(1 - y_{i-1})}{4y_{i-1}},
\]

\[
(4) \quad a_{i+1} = a_i + \frac{2}{1 - y_i},
\]

\[
(5) \quad b_{i+1} = b_i + \frac{2}{1 + y_i}.
\]

The choice of \( a_1 \) and \( b_1 \) ensures that the triangles \( T_0^1 \) and \( T_0^4 \) have area 1. Formulas (2), (3), (4) and (5) show that \( T_0^1, T_0^2, T_0^3 \) and \( T_0^4 \) are of unit area: a simple computation yields that they imply

\[
(6) \quad 1 = \frac{1}{2} \det \left( (x_i, y_i)^T - (x_{i-1}, y_{i-1})^T, (a_i, 1)^T - (x_{i-1}, y_{i-1})^T \right),
\]

\[
(7) \quad 1 = \frac{1}{2} \det \left( (b_i, -1)^T - (x_{i-1}, y_{i-1})^T, (x_i, y_i)^T - (x_{i-1}, y_{i-1})^T \right),
\]

\[
(8) \quad 1 = \frac{1}{2} (a_{i+1} - a_i)(1 - y_i),
\]

\[
(9) \quad 1 = \frac{1}{2} (b_{i+1} - b_i)(1 + y_i)
\]
for $i = 1, 2, \ldots$. Induction shows that
\begin{equation}
4i - 3 = \frac{1}{2} ((x_{i-1} + a_i)(1 - y_{i-1}) + (x_{i-1} + b_i)(1 + y_{i-1}))
\end{equation}
for $i = 1, 2, \ldots$. Indeed, (10) gives (10) for $i = 1$, and adding (6), (7), (8) and (9) to (10) yields (10) with $i$ replaced by $i+1$. By (10), the denominator in (2) and (3) coincides with $2(-4i + 3 + a_i + b_i)$. Thus
\begin{equation}
x_i = x_{i-1} + 2 - \frac{(a_i - b_i)y_{i-1}}{-4i + 3 + a_i + b_i}, \quad y_i = y_{i-1} - \frac{2y_{i-1}}{-4i + 3 + a_i + b_i}.
\end{equation}
For the sake of simplicity let $\alpha_i = a_i - (2i - 1)$, $\beta_i = b_i - (2i - 1)$, $\xi_i = x_i - 2i$ denote the deviations of $a_i, b_i, x_i$ in the distorted tiling from the corresponding values in the undistorted situation. Then
\begin{equation}
\xi_0 = 0, \quad y_0 = y_0, \quad \alpha_1 = \frac{y_0}{1 - y_0}, \quad \beta_1 = -\frac{y_0}{1 + y_0}
\end{equation}
and, for $i = 1, 2, \ldots$,
\begin{align}
\xi_i &= \xi_{i-1} - \frac{(\alpha_i - \beta_i)y_{i-1}}{1 + \alpha_i + \beta_i}, \\
y_i &= y_{i-1} - \frac{2y_{i-1}}{1 + \alpha_i + \beta_i}, \\
\alpha_{i+1} &= \alpha_i + \frac{2y_i}{1 - y_i}, \\
\beta_{i+1} &= \beta_i - \frac{2y_i}{1 + y_i}.
\end{align}
Formulas (12), (15) and (16) show that
\[ \alpha_i + \beta_i = 2 \frac{y_0^2}{1 - y_0} + 4 \left( \frac{y_1^2}{1 - y_1} + \cdots + \frac{y_{i-1}^2}{1 - y_{i-1}} \right). \]
For the sake of brevity let
\begin{equation}
h_i = 1 + \alpha_{i+1} + \beta_{i+1} = 1 + 2 \frac{y_0^2}{1 - y_0} + 4 \left( \frac{y_1^2}{1 - y_1} + \cdots + \frac{y_i^2}{1 - y_i} \right)
\end{equation}
for $i = 0, 1, \ldots$. Then (14) becomes
\[ y_{i+1} = \left( 1 - \frac{2}{h_i} \right) y_i \quad \text{with} \quad h_{i+1} = h_i + 4 \frac{y_{i+1}^2}{1 - y_{i+1}^2}, \quad h_0 = 1 + 2 \frac{y_0^2}{1 - y_0}. \]

**Lemma 4.** If
\begin{equation}
\frac{1}{\sqrt{3}} < y_0 < \frac{3}{\sqrt{19}}
\end{equation}
then, for all $i \geq 0$,
\begin{itemize}
  \item[(a_i)] $0 < y_i < 1$,
  \item[(b_i)] $y_i < \frac{1}{2} y_{i-1}$ (no claim if $i = 0$),
  \item[(c_i)] $2 < h_{i-1} < h_i$ (the claim means only $h_0 > 2$ if $i = 0$),
  \item[(d_i)] $h_i \leq 1 + \frac{10^{-4^{i-1}}}{3} \frac{y_0^2}{1 - y_0^2}$.
\end{itemize}
In particular, the sequence $(y_i)_{i \geq 0}$ is positive, strictly decreasing with $\lim_{i \to \infty} y_i = 0$ and
\begin{equation}
0 < y_i < 2^{-1} y_{i-1} < 2^{-2} y_{i-2} < \cdots < 2^{-i} y_0 < y_0 < 1 \quad \text{for} \quad i = 1, 2, \ldots
\end{equation}

**Proof.** The claims are proved by induction over $i$. Base case:
\begin{itemize}
  \item[(a_0)] $0 < y_0 < 1$ by (18).
  \item[(b_0)] There is nothing to show. ((b_0) is not needed in the sequel.)
  \item[(c_0)] $h_0 = 1 + 2 \frac{y_0^2}{1 - y_0} = 1 + 2 \frac{1}{\sqrt{3} - 1} > 1 + 2 \frac{1}{\sqrt{3} - 1} = 2$, because $\frac{1}{\sqrt{3}} < y_0 < 1$.
\end{itemize}
Proof. Since the triangles of the undistorted tiling from Figure 2 have constant perimeter, it is

Lemma 5. Let \( y_0 \in \left( \frac{1}{\sqrt{3}}, \frac{3}{\sqrt{19}} \right) \). Then the perimeters of the triangles in the distorted tiling \( T = T(y_0) \) of \( S \) given by \( y_0 \) are bounded by some common constant.

Remark 1. One may also choose other values \( 0 < y_0 < 1 \) as the initial value. Numerical computations suggest that in the case \( 0 < y_0 < \frac{1}{\sqrt{3}} \) the recursion formulas (1)-(5) yield values \( y_i \) that converge to 0 alternatingly. A corresponding tiling is shown in Figure 4. This one may serve as well as a starting point for constructing a vertex-to-vertex equipartition of the plane into triangles. However, the mix of signs will make the computations more tedious. For \( \frac{3}{\sqrt{19}} \leq y_0 < 1 \), numerical evidence shows that the behaviour of the resulting tilings is similar to Figure 3, but the proof of Lemma 4 does not work in the same way as above. The critical case \( y_0 = \frac{3}{\sqrt{19}} \) is considered separately in Lemma 6 and Figure 5 below.

Lemma 5. Let \( y_0 \in \left( \frac{1}{\sqrt{3}}, \frac{3}{\sqrt{19}} \right) \). Then the perimeters of the triangles in the distorted tiling \( T = T(y_0) \) of \( S \) given by \( y_0 \) are bounded by some common constant.

Proof. Since the triangles of the undistorted tiling from Figure 2 have constant perimeter, it is enough to show that the deviations of the coordinates of the vertices of the triangles of \( T \) from the respective coordinates from the undistorted tiling are uniformly bounded. I.e., we have to show uniform boundedness of the values \( \xi_i, y_i, \alpha_i, \text{ and } \beta_i \).

Lemma 4 (a\(_i\)) settles the claim for \( y_i \). For \( \alpha_i, \beta_i, \text{ and } \xi_i, i \geq 1 \), we estimate

\[
|\alpha_i| \leq \frac{\sum_{j=0}^{i-1} 2y_j}{1 - y_j} \quad |\beta_i| \leq \frac{\sum_{j=0}^{i-1} \frac{2y_j}{1 + y_j}}{1 - \frac{y_0}{1 + y_0}} \quad |\xi_i| \leq \frac{i}{1 + \alpha_i + \beta_i} \quad (\xi_i) \leq \frac{\sum_{j=1}^{i} \frac{(\alpha_j - \beta_j)y_{j-1}}{h_{j-1}}}{2} \leq \frac{\sum_{j=1}^{i} \frac{(\alpha_j + |\beta_j|)y_{j-1}}{|h_{j-1}|}}{2} \quad (\beta_0) \leq \frac{\sum_{j=1}^{i} \frac{(\alpha_j + \beta_j)2^{-j-i-1}y_0}{2}}{(C_\alpha + C_\beta)y_0}.
\]

\( (d_0) \): \( h_0 = 1 + 2 \frac{y_0^2}{1 - y_0} = 1 + \frac{10 - 4^{1-i} y_0^2}{3} \frac{1 - y_0}{1 - y_0} \)
Next we study congruence relations between triangles from tilings $T = T(y_0)$ of the strip $S$. Let $\cong$ denote congruence with respect to the group of Euclidean isometries (including reflections). We write $\simeq$ for congruence under the subgroup of all translations and all rotations by an angle of $180^\circ$. That is, two sets $A, B \in \mathbb{R}^2$ satisfy $A \simeq B$ if and only if there exist $s \in \{\pm 1\}$ and $t_1, t_2 \in \mathbb{R}$ such that $B = sA + (t_1, t_2)^T$.

We start with an observation on the tiling $T^* = T(\frac{1}{\sqrt{3}})$ of $S$ based on the parameter $y_0 = \frac{1}{\sqrt{3}}$. The respective triangles are denoted by $T^*_i = T_i(\frac{1}{\sqrt{3}})$, see Figure 5. Here $T_{i,j}^*$ denotes the image of $T_{i,j}$ under reflection in the vertical axis.

**Lemma 6.** The coordinates of the tiling $T^*$ are $x_0 = 0$, $y_0 = \frac{1}{\sqrt{3}}$ and $x_i = 2i - \frac{1}{2}$, $y_i = 0$, $a_i = 2i + \frac{1}{\sqrt{3}}$, $b_i = 2i - \frac{1}{\sqrt{3}} + 1$ for $i \geq 1$. Every triangle of $T^*$ is congruent to one of $T_0^1$, $T_1^1$, $T_1^2$, $T_3^2$, $T_0^{34}$ and $T_1^{34}$. Moreover,

$$T_i^{*j} \not \simeq T_j^{*i} \quad \text{for} \quad i = 1, 2, \ldots, j = 1, 2, 3, 4.$$

**Proof.** This is a direct consequence of (1). $\square$

**Lemma 7.** For $y_0 \in (0, 1)$ we denote the triangles from the tiling $T = T(y_0)$ of $S$ by $T_i^j = T_i^j(y_0)$, $(i, j) \in I = ((\mathbb{Z} \setminus \{0\}) \times \{1, 2, 3, 4\}) \cup \{(0, 1), (0, 4)\}$. If $(i, j), (i', j') \in I$ are such that $T_i^{*j} \not \simeq T_{i'}^{*j'}$, then the set

$$F(i, j, i', j') = \left\{ y_0 \in (0, 1) \mid T_i^j(y_0) \simeq T_{i'}^{j'}(y_0) \right\}$$

is finite.

**Proof.** We describe the triangles by their vertices, i.e., $T_i^j = \triangle((v_1^i, v_2^i)^T, (v_1^j, v_2^j)^T, (v_3^i, v_2^j)^T)$ and $T_i^{j'} = \triangle((v_1^j, v_2^j)^T, (v_1^k, v_2^k)^T, (v_3^k, v_2^j)^T)$. Assume that $T_i^j \simeq T_i^{j'}$. Then there are $s \in \{\pm 1\}$ and $t_1, t_2 \in \mathbb{R}$ such that $T_i^{j'} = sT_i^j + (t_1, t_2)^T$. The corresponding map $\varphi((x, y)^T) = s(x, y)^T + (t_1, t_2)^T$ induces a permutation $\pi$ of $\{1, 2, 3\}$ via $\varphi((v_1^i, v_2^j)^T) = (v_1^{\pi(k)}, v_2^{\pi(k)})^T$. Thus

$$s(v_1^k, v_2^k)^T + (t_1, t_2)^T = (v_1^{\pi(k)}, v_2^{\pi(k)})^T \quad \text{for} \quad k = 1, 2, 3.$$

Now we distinguish 12 situations depending on the choice of $s \in \{\pm 1\}$ and the permutation $\pi$.

**Case 1:** $s = 1$ and $\pi$ is the identity. From (21) with $k = 1$ we obtain $t_1 = v_1^i - v_1^j$ and $t_2 = v_2^i - v_2^j$. Substituting these into (21) gives

$$(v_1^i, v_2^i)^T + (v_1^j - v_1^i, v_2^j - v_2^i)^T = (v_1^k, v_2^k)^T \quad \text{for} \quad k = 1, 2, 3.$$
We obtain only finitely many elements of and follow the same arguments as above. In the same way we see that each of the 12 cases yields only finitely many elements of $F(i, j, i', j')$.

**Lemma 8.** The set

$$F = \left\{ y_0 \in \left( \frac{1}{\sqrt{3}}, \frac{3}{\sqrt{19}} \right) \mid \text{There are distinct triangles } T, T' \in T(y_0) \text{ such that } T \simeq T' \right\}$$

is at most countable.

**Proof.** Using notation from Lemma 7 we have

$$F = \bigcup_{(i, j), (i', j') \in I, (i, j) \neq (i', j')} F(i, j, i', j') \cap \left( \frac{1}{\sqrt{3}}, \frac{3}{\sqrt{19}} \right).$$

We shall see that all the sets $F(i, j, i', j') \cap \left( \frac{1}{\sqrt{3}}, \frac{3}{\sqrt{19}} \right)$ are finite.

Let $y_0 \in \left( \frac{1}{\sqrt{3}}, \frac{3}{\sqrt{19}} \right)$. The vertical width $vw(T)$ of a triangle $T$ is the maximal distance between second coordinates of vertices of $T$. By Lemma 4, $(y_i)_{i \geq 0}$ is positive and strictly decreasing. We obtain

$$vw \left( T_j^i \right) = \left\{ \begin{array}{ll}
1 - y_{|i|}, & j \in \{1, 2\}, \\
1 + y_{|i|-1}, & j = 3, \\
1 + y_{|i|}, & j = 4.
\end{array} \right.$$  

See Figure 3 for an illustration. Note that $T_j^i \simeq T_j'^{i'}$ implies $vw \left( T_j^i \right) = vw \left( T_j'^{i'} \right)$.

Now assume that $y_0 \in F(i, j, i', j')$, i.e., $T_j^i \simeq T_j'^{i'}$.

**Case 1:** $j \in \{1, 2\}$ and $j' \in \{3, 4\}$ (resp. $j' \in \{1, 2\}$ and $j \in \{3, 4\}$). We see that $vw \left( T_j^i \right) \neq vw \left( T_j'^{i'} \right)$, which contradicts $T_j^i \simeq T_j'^{i'}$. Hence $F(i, j, i', j') \cap \left( \frac{1}{\sqrt{3}}, \frac{3}{\sqrt{19}} \right)$ is empty.

**Case 2:** $\{j, j'\} = \{1, 2\}$ or $\{j, j'\} = \{3, 4\}$. Then one of $T_j^i$ and $T_j'^{i'}$ has a horizontal edge and the other one has not, again a contradiction to $T_j^i \simeq T_j'^{i'}$. So $F(i, j, i', j') \cap \left( \frac{1}{\sqrt{3}}, \frac{3}{\sqrt{19}} \right)$ is empty, too.

**Case 3:** $j = j'$ and $|i| \neq |i'|$. We can argue as in Case 1.

**Case 4:** $j = j'$ and $|i| = |i'|$. Since $(i, j) \neq (i', j')$, we have $i' = -i \neq 0$. By 20 and Lemma 7, $F(i, j, i', j')$ is finite. This completes the proof.

Picking $y_0 \in \left( \frac{1}{\sqrt{3}}, \frac{3}{\sqrt{19}} \right) \setminus F$ and using Lemma 5 yields the following result.

**Corollary 9.** There is a vertex-to-vertex tiling $\mathcal{T}$ of the strip $S$ by triangles of unit area and uniformly bounded perimeter such that $T \not\simeq T'$ for all $T, T' \in \mathcal{T}$ with $T \not\equiv T'$.

The final tiling of $\mathbb{R}^2$ will be obtained by stacking sheared copies $(\frac{1}{2} \delta_k) \mathcal{T}$ of $\mathcal{T}$; compare Figure 6. In order to make sure that almost every shear mapping of $\mathcal{T}$ produces (i) mutually incongruent triangles that are (ii) different from countably many prescribed shapes, we need the following result.

**Lemma 10.** Let $T$ and $T'$ be triangles.

(a) If $T \not\simeq T'$ then the set $\{ \delta \in \mathbb{R} \mid (\frac{1}{2} \delta) T \cong (\frac{1}{2} \delta) T' \}$ is finite.

(b) The set $\{ \delta \in \mathbb{R} \mid (\frac{1}{2} \delta) T \cong T' \}$ is finite.
We construct the tilings

**Corollary 11.** There exist sheared images \( T_n = (1, \delta_n) \tilde{T} \) of the tiling \( \tilde{T} \), \( n = 1, 2, \ldots \), such that

(a) \(|\delta_n| < 1\) for all \( n \geq 1\),

(b) for all \( n \geq 1 \), \( T_n \) does not contain two distinct congruent triangles,

(c) for all \( 1 \leq n' < n \), there are no congruent triangles \( T \in T_n \) and \( T' \in T_{n'} \).

**Proof.** We construct the tilings \( T_n \) by induction over \( n \).
Figure 7. Copies of these four clusters are used to tile the plane. In each copy the hexagons will be distorted in a different way.

Base case (construction of $T_1$): By Corollary 9 and Lemma 10 (a), the set

$$A = \bigcup_{T, T' \in \mathcal{T}, T \neq T'} \{ \delta \in \mathbb{R} | (\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix})^T \delta (\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix})^T \neq \mathbb{T} \}$$

is at most countable, because $\mathcal{T}$ is countable. We pick $\delta_1 \in (-1, 1) \setminus A$ and obtain $T_1 = (\begin{smallmatrix} 1 & \delta_1 \\ 0 & 1 \end{smallmatrix})^T$. The choice of $\delta_1$ implies claims (a) and (b) for $n = 1$.

Step of induction (construction of $T_n$, $n \geq 2$): By Lemma 10 (b), the set

$$B = \bigcup_{T, T' \in \mathcal{T}, \mathcal{T}_1 \cup \ldots \cup \mathcal{T}_{n-1}} \{ \delta \in \mathbb{R} | (\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix})^T \delta (\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix})^T \neq \mathbb{T} \}$$

is at most countable, since $\mathcal{T}$ and $\mathcal{T}_1 \cup \ldots \cup \mathcal{T}_{n-1}$ are countable. We fix some $\delta_n \in (-1, 1) \setminus (A \cup B)$ this way defining $T_n = (\begin{smallmatrix} 1 & \delta_n \\ 0 & 1 \end{smallmatrix})^T$. We obtain the respective items of condition (a) by $\delta_n \in (-1, 1)$, of (b) by $\delta_n \notin A$, and of (c) by $\delta_n \notin B$.

All tilings $\overline{T}_n = (\begin{smallmatrix} 1 & \delta_n \\ 0 & 1 \end{smallmatrix})^T$ of $S$ have the same mutual distances $a_{i+1} - a_i$ between adjacent vertices at the upper boundary $y = 1$ as $\overline{T}$. The same applies to the lower boundary $y = -1$. $S$ is tiled by $\overline{T}_1$. Therefore the parallel strip $S + (0, 2)^T$ can be tiled by a suitable image $T_2$ of $T_2$ under a reflection with respect to the horizontal axis and some translation such that $\mathcal{T}_1 \cup T_2$ is vertex-to-vertex. Similarly, we tile $S + (0, -2)^T$ and $S + (0, -4)^T$ by suitable translates $T_3$ and $T_5$ of $\mathcal{T}_4$ and $\mathcal{T}_5$ etc. This way we obtain the desired vertex-to-vertex tiling $\overline{T}_1 \cup T_2 \cup T_3 \cup \ldots$ of $\mathbb{R}^2$, see Figure 6. The perimeters of all triangles of that tiling are uniformly bounded because of the respective property of $\overline{T}$ (see Corollary 9) and by Corollary 11 (a).

The proof of Theorem 3 is complete.

3. Non-vtv equipartitions of the plane into hexagons

Theorem 12. There is a non-vertex-to-vertex tiling of the plane by pairwise incongruent convex hexagons of unit area and uniformly bounded perimeter.

Proof. The construction uses a variant of the tiling-a-tile method from Subsection 1.2. Here we need four different clusters $C_1, C_2, C_3, C_4$ that (i) can tile the plane, and (ii) can be dissected into convex hexagons of unit area in uncountably many ways. Figure 7 shows how each of the four clusters is dissected into copies of altogether eight different hexagons $H'_1, H'_2, H'_3, H_2, \ldots, H_6$, each hexagon having unit area (unit area means here: 72 small boxes). Figure 8 illustrates the
degrees of freedom for distortions within each cluster. In order to see that the distortions act as desired one needs to consider the dependencies in each cluster. For instance, in $C_1$, vertex 1 can be shifted continuously along the dashed line, hence produces uncountably many distinct versions of the lower left hexagon. Once vertex 1 is fixed, the position of the vertex marked by $a$ is determined uniquely by the requirements that hexagons I and II both need to have unit area. Independently from this choice, vertex 2 can be shifted along the dashed line, fixing vertex $b$ by the requirement that hexagons III and IV both have unit area, too. Independent of these choices vertex 3 can be shifted along the approximately vertical dashed line. Vertex $c$ is then determined uniquely by the requirement that hexagons V and VI have unit area. Note also that indeed all hexagons will be continuously distorted under these transformations.

In a similar manner one may shift vertex 1 in cluster $C_2$ by some small amount along the dashed line, preserving the area of hexagon $I'$. Once vertex 1 is fixed this determines the vertical position of the dashed line at vertex 2 by the requirement that hexagon $II'$ has unit area. Still vertex 2 can be shifted along that line in a continuous manner. The choice of 2 determines the exact position of the dashed line at vertex 3. Still vertex 3 has one degree of freedom along this line. The position of 3 determines the position of vertex $a$ uniquely. Again note that all six hexagons will be distorted continuously under these transformations. In a similar manner the reader can convince oneself that the same is true for clusters $C_3$ and $C_4$.

The desired tiling is sketched in Figure 9. It can be obtained in a similar manner inductively as in the last section: Start with the central cluster $C_1$, distort the hexagons such that no two of them are congruent to each other. Add the next cluster, distort the hexagons such that no pairwise congruent hexagons occur. In each step there are only finitely many shapes to avoid, whereas there are uncountably many hexagon shapes available.

Note that the cluster $C_1$ is used only once in the tiling. This cluster contains the only points where the tiling is not vertex-to-vertex.

\[\square\]

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Figure 9. The image illustrates how copies of the four clusters are used to tile the plane.