CONVERGENCE OF POSITIVE OPERATOR SEMIGROUPS

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Abstract. We present new conditions for semigroups of positive operators to converge strongly as time tends to infinity. Our proofs are based on a novel approach combining the well-known splitting theorem by Jacobs, de Leeuw and Glicksberg with a purely algebraic result about positive group representations. Thus we obtain convergence theorems not only for one-parameter semigroups but for a much larger class of semigroup representations.

Our results allow for a unified treatment of various theorems from the literature that, under technical assumptions, a bounded positive $C_0$-semigroup containing or dominating a kernel operator converges strongly as $t \to \infty$. We gain new insights into the structure theoretical background of those theorems and generalise them in several respects; especially we drop any kind of continuity or regularity assumption with respect to the time parameter.

As applications we derive, inter alia, a generalisation of a famous theorem by Doob for operator semigroups on the space of measures and a Tauberian theorem for positive one-parameter semigroups under rather weak continuity assumptions. We also demonstrate how our results are useful to treat semigroups that do not satisfy any irreducibility conditions.

1. Introduction

One of the most important aspects in the study of linear autonomous evolution equations is the long term behaviour of their solutions. Since these solutions are usually described by one-parameter operator semigroups, it is essential to have affective tools for the analysis of their asymptotic behaviour available. In many applications the underlying Banach space is a function space and thus exhibits some kind of order structure. In such a situation it is often reasonable to assume that a positive initial value of the evolution equation leads to a positive solution, meaning that the solution semigroup consists of positive operators.

Because of the prevalence of such positive one-parameter semigroups in applications, various approaches were developed to study their long term behaviour and, more specifically, to prove convergence of such semigroups as time tends to infinity. See the end of the introduction for a very brief overview of the theory and for several references.

Semigroups of kernel operators. Consider a one-parameter semigroup $(T_t)_{t \in (0, \infty)}$ (meaning that $T_{t+s} = T_t T_s$ for all $t, s > 0$) of positive operators on a Banach lattice $E$. In concrete applications it happens frequently that the operators $T_t$ are kernel operators (see below for a precise definition of this notion) or that, at least, they dominate a non-zero kernel operator. Under mild additional assumptions this property is already sufficient for $(T_t)_{t \in (0, \infty)}$ to converge strongly as $t \to \infty$. A classical example for such a convergence result reads as follows:

**Theorem.** Let $\mathcal{T} = (T_t)_{t \in [0, \infty)}$ be a positive and contractive $C_0$-semigroup on an $L^p$-space over a $\sigma$-finite measure space where $1 \leq p < \infty$. Assume that $\mathcal{T}$ possesses a fixed vector $h$ which fulfils $h > 0$ almost everywhere and that $T_{t_0}$ is a kernel operator for some $t_0 \geq 0$.

*Date: May 5, 2017.*
Then \((T_t)_{t \in [0, \infty)}\) converges strongly as \(t \to \infty\).

In this version the theorem is due to Greiner [31, Kor 3.11] who published it in the early 1980s. It was observed only recently that this result can also be used to give an analytic proof of a famous theorem by Doob about the asymptotic behaviour of one-parameter Markov semigroups, see [30] and [29, Sec 4]. During the last two decades several new versions of the above theorem were discovered. If one assumes the semigroup \(\mathcal{T}\) to be irreducible, then it suffices that \(T_{t_0}\) dominates a non-trivial kernel operator instead of being a kernel operator itself. For one-parameter Markov semigroups on \(L^1\)-spaces this was observed by Pichóř and Rudnicki in [56, Thm 1]. This result was applied to study the asymptotic behaviour of numerous models, in particular from mathematical biology; several examples can be found, for instance, in [60, 11, 49, 19, 8, 50] and in the survey article [61]. In fact, the same result remains true not only on \(L^1\)-spaces but on Banach lattices with order continuous norm, as observed by the first of the present authors in [24, Thm 4.2]. Greiner’s original theorem was also revisited in a paper by Arendt [3], where an application to elliptic differential operators is given. Another related result due to Mokhtar-Kharroubi [53] focusses on perturbed one-parameter semigroups on \(L^1\)-spaces and was successfully employed in the study of the linear Boltzmann equation in [46]. A series of related papers deals with the more specific topic of positive \(C_0\)-semigroups on atomic Banach lattices, see [14, 39, 69].

Contributions of this article. In its current stage of development the theory described above has proven useful in the asymptotic analysis of evolution equations from various branches of analysis. Yet, it still seems to have a lot of unexplored potential, both with regard to applications and from a theoretical point of view. At the same time, the theory currently appears to lack a certain degree of cohesion.

In this paper we present a new and very algebraic approach which allows us to achieve the following three goals:

1. We prove much more general versions of the above mentioned results which considerably broadens their range of applicability.
2. We unify a large part of the theory.
3. We settle a couple of theoretical questions, thus giving new insight into the foundations of the theory.

Let us elaborate on these points in more detail.

1. As a first major generalisation we drop any continuity (or measurability) assumption on the semigroup parameter. Actually one encounters several classes of one-parameter semigroups in analysis which have rather disparate continuity properties. For instance, dual semigroups of \(C_0\)-semigroups on non-reflexive spaces are weak∗-continuous but not strongly continuous in general. Also one-parameter semigroups on spaces of measures which appear in probability theory are rarely strongly continuous with respect to the total variation norm but often stochastically continuous, i.e. weakly continuous in duality with the continuous functions, cf. [37, 43]. Furthermore, there are also one-parameter semigroups which are strongly continuous or even analytic on the time interval \((0, \infty)\) but not \(C_0\), for instance the Gaussian semigroup on \(C_0(\mathbb{R}^d)\) or semigroups arising from parabolic equations on domains with certain non-local boundary conditions, see [7]. Our results can be applied to all these one-parameter semigroups without any regard to continuity.

A second important generalisation is that, in case where \(T_{t_0}\) only dominates a kernel operator, we can replace the irreducibility of the one-parameter semigroup from [56, Thm 1] and [24, Thm 4.2] with a much weaker assumption. The usefulness of this will become apparent in Example 5.2 and in the proof of Theorem 5.4. Third, we state our main results not only for one-parameter semigroups of the type...
(T_t)_{t \in (0, \infty)}$ but for a much more general class of positive semigroup representations. This yields, for instance, also convergence results for multi-parameter semigroups of the type $(T_t)_{t \in (0, \infty)^d}$.

Besides this, our results include some further minor generalisations. For instance, Greiner’s original theorem that we stated above remains true on general Banach lattices with order continuous norm and for one-parameter semigroups which are merely bounded. Apparently, this has never been stated explicitly in the literature, so we mention it as a corollary of one of our main results.

(2) The essence of the present paper are two main results (Theorems 3.7 and 3.11). We shall see that one can obtain large parts of the theory described above as simple consequences of these two theorems. In our opinion this brings the theory, which currently consists of a variety of disparate results, into a much more cohesive state.

(3) It is an interesting question what special property of kernel operators enforces the convergence of operator semigroups they are contained in. Our approach sheds new light onto this question: it is well-known that, under mild assumptions on the underlying space, a kernel operator is a so-called AM-compact operator, i.e. it maps order intervals to relatively compact sets. It turns out that this property suffices to develop the entire theory, i.e. we can prove our main results for AM-compact operators instead of kernel operators. Surprisingly, this leads at the same time to significant simplifications of the proofs. Another interesting theoretical question is what special role is played by the time interval $(0, \infty)$. Simple finite-dimensional counterexamples show that our results do not remain true for operator semigroups indexed over the discrete set $\mathbb{N}$, i.e. for powers of single operators. We shall see that this is not because of different topological properties of $(0, \infty)$ and $\mathbb{N}$ but rather due to their different algebraic properties. In fact, our main results hold for operator semigroups indexed over the positive rational numbers, while they fail for operator semigroups indexed over the positive dyadic numbers (although these two sets are order isomorphic and thus homeomorphic with respect to the topology inherited from $\mathbb{R}$). We refer to the discussion after Proposition 3.10 for more details.

How the article is organised. In the rest of this introduction we briefly comment on different approaches to study the long term behaviour of positive one-parameter semigroups and we fix some notation and recall a bit of terminology. In Section 2 we study positive group representations on so-called atomic Banach lattices and give sufficient conditions for them to be trivial. This is the key to prove our main results about convergence of positive semigroup representations in Section 3. In the subsequent Section 4 we demonstrate that many known theorems are consequences of our main results. We state many special cases of our results explicitly as their assumptions are particularly easy to check in applications. Besides this, we derive two consequences for one-parameter semigroups on spaces of measures: a Tauberian theorem and a generalised version of a convergence theorem by Doob. At the end of Section 4 we demonstrate how our theorems can be used to derive spectral theoretic information about positive semigroups. In Section 5 we sketch several applications of our results. For instance, we prove convergence results for transport semigroups whose generator is perturbed by a kernel operator and we briefly consider one-parameter semigroups on $L^p$-spaces over measure spaces containing an atom, a situation which occurs for instance in the analysis of queuing systems. Moreover, we show how our theorems can be used to derive spectral results for semigroups on spaces of continuous functions. Yet, throughout the section we do not go into too much detail and leave the study of more advanced examples to future articles.

Related results and further literature. Our general assumption in this paper, which is frequently fulfilled in applications, is that the semigroup representation
under consideration contains – or dominates – a kernel operator. Yet, it is worthwhile pointing out that there are also several other approaches to the asymptotic analysis of positive one-parameter semigroups. For instance, one can employ results relying on constrictivity properties (see \cite{21} for an overview), one can use so-called lower-bound methods (see for instance \cite{15} Thm 5.6.2 and Thm 7.4.1 and the recent paper \cite{24}) or one can employ (quasi-)compactness properties \cite{47} Thm 4. Another important concept is the so-called spectral theoretic approach which uses spectral properties of the generator of positive $C_0$-semigroups; we refer to the monographs \cite{0} \cite{9} for an overview and to the article \cite{52} for some recent contributions to the theory. The asymptotic behaviour of one-parameter Markov semigroups, which are used to describe Markov processes, can for example be studied by employing the theory of Harris operators (see e.g. \cite{23} Ch V) or a well-known theorem by Doob (see for instance \cite{13} Sec 4.2). The latter two approaches are rather closely related to the topic of the present paper: on the one hand, Pichó and Rudnicki proved the following result in \cite{50} Thm 1 by using the theory of Harris operators. On the other hand, we will derive a version of Doob’s theorem from one of our main results in Section 4.

**Preliminaries.** Let $E, F$ be Banach spaces. We denote by $\mathcal{L}(E,F)$ the space of bounded linear operators from $E$ to $F$ and write $\mathcal{L}(E)$ shorthand for $\mathcal{L}(E,E)$. The dual spaces of $E$ and $F$ are denoted by $E'$ and $F'$ and the adjoint of an operator $T \in \mathcal{L}(E,F)$ by $T' \in \mathcal{L}(F',E')$.

Throughout, we assume the reader to be familiar with the theory of Banach lattices; here we only recall a few things in order to fix the terminology. Unless stated otherwise, all Banach lattices in this article are real, meaning that the underlying scalar field is $\mathbb{R}$. Let $E$ and $F$ be Banach lattices. A vector $f \in E$ is called ***positive*** if $f \geq 0$. We write $f > 0$ if $f \geq 0$ but $f \neq 0$. The positive cone in $E$ is denoted by $E_+$. For each $f \in E_+$ the ***principal ideal*** generated by $f$ is the set $E_f := \{ g \in E : \exists c \geq 0, |g| \leq cf \}$. A vector $f \in E_+$ is called a ***quasi-interior point*** of $E_+$ if $E_f$ is dense in $E$. If $E = L^p(\Omega, \mu)$ for a $\sigma$-finite measure space $(\Omega, \mu)$ and $p \in [1, \infty]$ then a vector $f \in E$ is a quasi-interior point of $E_+$ if and only if $f(\omega) > 0$ for almost all $\omega \in \Omega$. For instance, on a finite measure space the principal ideal $E_1$, generated by the constant function of value 1, equals $L^\infty(\Omega, \mu)$. Given any subset $A \subseteq E$, we denote by $A^\perp := \{ g \in E : |g| \land |f| = 0 \forall f \in A \}$ the disjoint complement of $A$.

We shall often consider Banach lattices with order continuous norm. For a definition and many important properties of these spaces we refer to \cite{62} Sec II.5 or to \cite{71} Sec 2.4; here, we only recall that every $L^p$-space (over an arbitrary measure space) has order continuous norm provided that $p \in [1, \infty)$. A Banach lattice $E$ is called an **AL-space** if the norm is additive on the positive cone, meaning that $\|f + g\| = \|f\| + \|g\|$ for all $f, g \in E_+$. Kakutani’s representation theorem for AL-spaces asserts that every AL-space is isometrically lattice isomorphic to $L^1(\Omega, \mu)$ for an appropriate (and not necessarily $\sigma$-finite) measure space $(\Omega, \mu)$; see \cite{71} Thm 2.7.1.

The dual space $E'$ of a Banach lattice $E$ is a Banach lattice, too. For every $\varphi \in E'$ we have $\varphi \in E'_+$ if and only if $\langle \varphi, f \rangle \geq 0$ for all $f \in E_+$. A functional $\varphi \in E'$ is called ***strictly positive*** if $\langle \varphi, f \rangle > 0$ for all $0 < f \in E$.

Let $E$ and $F$ be Banach lattices. An operator $T \in \mathcal{L}(E,F)$ is called ***positive*** if $TE_+ \subseteq F_+$; it is called ***strictly positive*** if $Tf > 0$ whenever $f > 0$. We write $\mathcal{L}^+(E,F)$ for the regular operators from $E$ to $F$, i.e. the linear span of all positive operators in $\mathcal{L}(E,F)$, and we set $\mathcal{L}^0(E) := \mathcal{L}^0(E,E)$. If $F$ is order complete, then $\mathcal{L}^+(E,F)$ is itself an order complete vector lattice \cite{51} Thm 1.3.2. An operator $T \in \mathcal{L}(E)$ on an AL-space $E$ is called a **Markov operator** if $T$ is positive and
norm-preserving on the positive cone, meaning that $Tf \in E_+$ and $\|Tf\| = \|f\|$ for all $f \in E_+$.

By a semigroup $S = (S, \cdot)$ we simply mean an algebraic semigroup, i.e. a set $S$ equipped with a associative binary operation $\cdot$. Let $E$ be a Banach space. Then the space $\mathcal{L}(E)$ is a semigroup when endowed with the operator multiplication as binary operation. For a semigroup $S$ a semigroup homomorphism $S \to \mathcal{L}(E)$ is called a representation of $S$ on $E$ or simply a semigroup representation on $E$ and is denoted by $(T_t)_{t \in S} \subseteq \mathcal{L}(E)$. A semigroup representation is said to be commutative if the semigroup $S$ is commutative; in this case we denote the semigroup operation on $S$ by “+”. In the case where $S = ((0, \infty), +)$ we call a representation $(T_t)_{t \in (0, \infty)} \subseteq \mathcal{L}(E)$ a one-parameter semigroup. A one-parameter semigroup $(T_t)_{t \in (0, \infty)}$ is called a $C_0$-semigroup if $\lim_{t \downarrow 0} T_t x = x$ for all $x \in E$. A semigroup representation $(T_t)_{t \in S}$ is said to be bounded if $\sup_{t \in S} \|T_t\| < \infty$. A subset $M \subseteq E$ is called invariant under the semigroup representation $\mathcal{T} := (T_t)_{t \in S}$ if $T_t M \subseteq M$ for all $t \in S$. A vector $x \in E$ is called a fixed point or a fixed vector of $\mathcal{T}$ if $T_t x = x$ for all $t \in S$; the fixed space of $\mathcal{T}$ is the vector subspace of $E$ that consists of all fixed vectors of $\mathcal{T}$.

Let the space $E$ be a Banach lattice and let $\mathcal{T} := (T_t)_{t \in S}$ be a semigroup representation on $E$. We say that “$\mathcal{T}$ possesses a quasi-interior fixed point” if $\mathcal{T}$ has a fixed point which is a quasi-interior point of $E_+$. We say that the semigroup representation $\mathcal{T}$ is positive if each operator $T_t$ is positive. A positive semigroup representation $\mathcal{T} = (T_t)_{t \in (0, \infty)}$ on $E$ is called irreducible if $[0] = [0 \infty]$ and $E$ are the only closed ideals in $E$ which are invariant under $\mathcal{T}$.

If $(G, \cdot)$ is a group (and thus, in particular, a semigroup), then a semigroup representation $(T_t)_{t \in G}$ on a Banach space $E$ is called a group representation of $G$ on $E$ if each operator $T_t$ is invertible in $\mathcal{L}(E)$. In this case, the mapping $t \mapsto T_t$ is automatically a group homomorphism from $G$ into the group of all invertible operators in $\mathcal{L}(E)$.

Given a commutative semigroup representation $\mathcal{T} := (T_t)_{t \in S}$ on a Banach space $E$, we consider $\mathcal{T}$ as a net where $S$ is endowed with the preorder

\begin{equation}
\text{If for given } x, y \in E \text{ the net } (T_t x)_{t \in S} \text{ converges to } y, \text{ we denote this by } \lim_{t \in S} T_t x = y. \text{ In other words, } \lim_{t \in S} T_t x = y \text{ if and only for every } \varepsilon > 0 \text{ there exists } s \in S \text{ such that } \|T_t x - y\| \leq \varepsilon \text{ for every } t \in S. \text{ If the net } (T_t x)_{t \in S} \text{ is convergent in } E \text{ for any } x, \text{ we call the representation } \mathcal{T} \text{ strongly convergent.}
\end{equation}

In many of our results we assume a commutative semigroup $S$ to generate a group, which shall be understood as follows. A semigroup $S$ is embeddable into a group if there exists an injective semigroup homomorphism from $S$ into a group. If $S$ is commutative, then this is the case if and only if $S$ is cancellative, meaning that $st = sr$ implies $t = r$ for any $s, t, r \in S$ \cite[§1.10]{[Ref]}. However, since we are primarily interested in applications to representations of subsemigroups of $((0, \infty), +)$, that trivially embed into the group $(\mathbb{R}, +)$, we do not elaborate more on this condition.

If a semigroup $S$ is embeddable into a group $G$, we may consider $S$ as a subset of $G$ and we can define the group generated by $S$, denoted by $\langle S \rangle$, as the smallest subgroup of $G$ containing $S$. If $S$ is in addition commutative, then the group $\langle S \rangle$ is, up to group isomorphism, independent of the choice of $G$ and we say that $S$ generates $(S)$.

In our main results we also require the generated group to have the following property: A group $(G, \cdot)$ is called divisible if for every $t \in G$ and every $n \in \mathbb{N}$ there exists an element $s \in G$ such that $s^n = t$. For example, the additive group of the rational numbers is divisible and so is the additive group of real numbers. For this reason, our main results are in particular applicable to one-parameter semigroups.
2. Group Representations on Atomic Banach Lattices

In this section we study group representations on atomic Banach lattices with order continuous norm. Our main result characterises groups whose bounded positive representations possessing a quasi-interior fixed point are trivial. In particular, the additive group $\mathbb{R}$ has this property which yields surprising new insights into one-parameter semigroups.

Positive group representations and groups of positive operators have appeared occasionally in the literature but it seems that they have not been studied as intensively as, for instance, unitary group representations on Hilbert spaces. Some spectral theoretic results about positive group representations can be found in [68, 32] and some structural results about groups of positive operators in [63, Sec 3]. A systematic investigation of positive group representations was just initiated in the recent series of papers [16, 17, 15]. While all of these references focus on strongly continuous representations of certain topological groups, we do not require any topological assumptions in the following. The main result of this section, Theorem 2.5, is therefore not only a tool for our investigation of the asymptotic behaviour of semigroups in Section 3 but also a new contribution to the theory of positive group representation and interesting in its own right.

We start by recalling some basic properties of atomic Banach lattices. Let $E$ be a Banach lattice. An element $a \in E_+$ is called an atom if the principal ideal $E_a$ is one-dimensional. The Banach lattice $E$ is called atomic if the band generated by all atoms equals $E$. Typical examples of atomic Banach lattices are real-valued $\ell^p$-spaces (over an arbitrary index set) for $p \in [1, \infty]$, as well as the space $c$ of real-valued convergent sequences endowed with the supremum norm and its subspace $c_0$ of sequences which converge to 0. In these spaces the atoms of norm 1 are exactly the canonical unit vectors.

It is immediate from the definition that for any two atoms $a, b \in E_+$ the infimum $a \wedge b$ is a multiple of $a$ and $b$; hence $a$ and $b$ are either disjoint or linearly dependent. In particular, two different atoms of norm 1 are disjoint.

Now let $E$ be an atomic Banach lattice and let $a \in E_+$ be an atom of norm 1. Then there exists a unique order continuous positive functional of norm 1 on $E$, which we denote by $\langle a', f \rangle$, such that for every normalized atom $b$ we have $\langle a', b \rangle = 1$ if $b = a$ and $\langle a', b \rangle = 0$ else. The functional $a'$ can be constructed as follows: by [2, Thm 2.16] there exists a band projection $P_a \colon E \to E$ onto $E_a = \{a\}^\perp$. For a fixed $\varphi \in E_+$ with $\langle \varphi, a \rangle > 0$ we may define $a' := \langle \varphi, a \rangle^{-1} \cdot (\varphi \circ P_a)$. Note that $P_a f = \langle a', f \rangle a$ for every $f \in E$.

We now collect a few results about atomic Banach lattices with order continuous norm. As prototypical examples for those one should always have the spaces $\ell^p$ for $p \in [1, \infty)$ in mind.

**Lemma 2.1.** Let $E$ be an atomic Banach lattice with order continuous norm and let $A \subseteq E_+$ be the set of all atoms of norm 1. For every $f \in E_+$ and every $\varepsilon > 0$ there are at most finitely many $a \in A$ for which $\langle a', f \rangle \geq \varepsilon$.

**Proof.** Fix $f \in E_+$ and let $(a_k) \subseteq A$ be an arbitrary infinite sequence of pairwise disjoint atoms. Since the norm on $E$ is order continuous, this implies that

$$\langle a_k', f \rangle = \| \langle a_k', f \rangle a_k \| = \| P_{a_k} f \| \to 0$$

as $k \to \infty$, see [61, Thm 2.4.2]. Hence for any fixed $\varepsilon > 0$, $(a', f) \geq \varepsilon$ can only hold for finitely many $a \in A$. □
Proposition 2.4. Let $G$ be a group and consider the following assertions:

(i) The group $G$ is divisible.

(ii) Every normal subgroup of $G$, except for $G$ itself, has infinite index.
Then (i) always implies (ii). If \( G \) is commutative, then (ii) implies (i), too.

This shows that, for instance, the additive groups \( \mathbb{R} \) and \( \mathbb{Q} \) fulfil the equivalent assertions (i)–(iii) in Lemma 2.3.

Proof of Proposition 2.4

(i) \( \Rightarrow \) (ii): Let \( G \) be divisible and assume that \( H \) is a normal subgroup of \( G \) which has finite index \( n \in \mathbb{N} \). Let \( t \in G \) and pick \( s \in G \) with \( s^n = t \). Since the order of every element in the factor group \( G/H \) is a divisor of \( n \), we have \( tH = s^nH = H \). Since \( t \in G \) was arbitrary, this shows that \( G = H \).

(ii) \( \Rightarrow \) (i): Now assume that \( G = (G,+ ) \) is commutative and that \( G \) is not divisible. Then we can find an integer \( n \geq 2 \) and an element of \( G \) which cannot be divided by \( n \). After a moment’s reflection we see that we can then even find a prime \( p \) such that at least one element of \( G \) cannot be divided by \( p \). Therefore, the set \( H = \{ pt : t \in G \} \) is not equal to \( G \). Since \( G \) is commutative, it follows that \( H \) is a subgroup of \( G \) and at that normal.

Since \( p \) is prime and the order of every element of \( G/H \) is a divisor of \( p \), every non-trivial element of \( G/H \) thus has order \( p \). Hence, the mapping

\[
\mathbb{Z}/p\mathbb{Z} \times G/H \rightarrow G/H, \quad (n + p\mathbb{Z}, t + H) \mapsto nt + H.
\]

is well-defined and it easy to see that \( G/H \) is a vector space over the field \( \mathbb{Z}/p\mathbb{Z} \) with respect to this scalar multiplication and the addition on \( G/H \). Since \( H \neq G \), the dimension of this space is at least 1 but can possibly be infinite. If we choose a subspace \( V \) of \( G/H \) of co-dimension 1, then \( V \) is in particular a subgroup of \( G/H \) and the factor group \((G/H)/V\) is isomorphic to \( \mathbb{Z}/p\mathbb{Z} \). Hence, there exists a surjective group homomorphism \( G \rightarrow G/H \rightarrow \mathbb{Z}/p\mathbb{Z} \). Thus, we obtain from Lemma 2.3 that \( G \) contains a normal subgroup which has finite index and is distinct from \( G \).

For more information (in case that \( G \) is commutative) we refer for instance to [59, Exer 10.25].

Now we have all results at hand that we need to prove the main result of this section. We call a group representation \((T_t)_{t \in G}\) of a group \( G \) on a Banach space \( E \) trivial if \( T_t = \text{id}_E \) for all \( t \in G \).

**Theorem 2.5.** For every group \( G \) the following assertions are equivalent:

(i) Every normal subgroup of \( G \), except for \( G \) itself, has infinite index.

(ii) On every atomic Banach lattice \( E \) with order continuous norm every positive and bounded group representation \((T_t)_{t \in G} \subseteq \mathcal{L}(E) \) possessing a quasi-interior fixed point is trivial.

Proof. (i) \( \Rightarrow \) (ii): Assume that (i) holds and let \( E \) and \( \mathcal{F} := (T_t)_{t \in G} \) be as in (ii).

Let \( y \in E \) denote a fixed point of \( \mathcal{F} \) which is a quasi-interior point of \( E_+ \) and let \( A \subseteq E_+ \) be the set of all atoms of norm 1.

Since every operator \( T_t \) is a lattice isomorphism, it maps maps atoms to atoms. Hence, for every \( a \in A \) and \( t \in G \) there exists a uniquely determined atom \( \sigma_t(a) \in A \) of norm 1 such that \( T_ta = \langle \sigma_t(a), T_ta \rangle \sigma_t(a) \). Let \( \mathcal{F}(A) \) be the group of all bijections \( A \rightarrow A \). For every \( t \in G \), the mapping \( \sigma_t : A \rightarrow A \) is a bijection and the map \( G \rightarrow \mathcal{F}(A), t \mapsto \sigma_t \) is a group homomorphism.

Now fix an atom \( a \in A \). We are going to show that the orbit \( \{ \sigma_t(a) : t \in G \} \subseteq A \) is finite. After rescaling \( y \), we may assume that \( a \leq y \). Aiming for a contradiction, suppose that there is in infinite set \( J \subseteq G \) such that all the atoms \( \sigma_t(a) \) for \( t \in J \) are distinct. Let \( \varepsilon > 0 \). According to Lemma 2.1 we have \( \langle \sigma_t(a), y \rangle < \varepsilon \) for some \( t \in J \). Since \( y \) is a fixed point of \( \mathcal{F} \), it follows from \( T_t a \leq T_t y = y \) that \( T_t a = \langle \sigma_t(a), T_t a \rangle \sigma_t(a) < \varepsilon \sigma_t(a) \). This shows that \( \| T_t a \| \leq \varepsilon \) and hence
\[ \|T_\cdot f\| \geq \frac{1}{2}, \] which contradicts the boundedness of \( \mathcal{S} \). We thus proved that the orbit \( O := \{ \sigma_t(a) : t \in G \} \subseteq A \) is indeed finite.

Clearly, \( O \) is invariant under each mapping \( \sigma_t : A \to A \) and the restriction of any of the maps \( \sigma_t \) to \( O \) is bijective on \( O \). Moreover, the mapping \( t \mapsto \sigma_t|_O \) from \( G \) to the group \( \mathcal{S}(O) \) of all bijections on \( O \) is a group homomorphism and thus trivial by assumption (i) and Lemma \[ \ref{lem:trivial_homo} \]. In particular, \( \sigma_t(a) = \sigma_t|_O(a) = a \) for every \( t \in G \). Since \( a \in G \) was arbitrary, we proved that \( \sigma_t = \text{id}_A \) for every \( t \in G \).

Let \( t \in G \) and \( a \in A \). Since \( T_1 \) is invertible, we have \( 0 \neq T_1a = \langle a', T_1a \rangle a \). Moreover, as \( (T_t)_{t \in G} \) is bounded, the sequence \( (T_t^*a)_{a \in \mathbb{Z}} \) is bounded. Hence, the positive number \( \langle a', T_1a \rangle \) is equal to 1, which proves that \( T_1a = a \). Since \( E \) has order continuous norm, the set \( A \) is linearly dense in \( E \) by Lemma \[ \ref{lem:linear_density} \] and therefore \( T_1f = f \) for all \( f \in E \), as claimed.

(ii) \( \Rightarrow \) (i): Assume that (ii) holds and let \( N \subseteq G \) be a normal subgroup of finite index; we have to prove that \( N = G \).

The factor group \( G/N \) operates transitively on itself via, say, left-translation and thus \( G \) operates transitively on \( G/N \) via the canonical mapping \( G \to G/N \). This group action induces a group homomorphism \( \psi : G \to \mathcal{S}(G/N) \), where \( \mathcal{S}(G/N) \) denotes the group of all bijections on \( G/N \). For every \( \pi \in \mathcal{S}(G/N) \), let \( T_\pi \) be the Koopman operator induced by \( \pi \) on the finite dimensional space \( \ell^2(G/N) \). Then \( \pi \mapsto T_\pi^{-1} \) is a group homomorphism from \( \mathcal{S}(G/N) \) into the group of lattice isomorphisms on \( \ell^2(G/N) \) and the mapping \( g \mapsto T_{(\psi(g))^{-1}} \) is a bounded and positive group representation of \( G \) on the finite dimensional Banach lattice \( \ell^2(G/N) \); the constant function with value 1 on \( G/N \) is a fixed-point of this group representation and a quasi-interior point of \( \ell^2(G/N)_+ \). Hence, the representation \( (T_{(\psi(g))^{-1}})_{g \in G} \) is trivial according to (ii). This implies that \( \psi(g) = \text{id}_{G/N} \) for each \( g \in G \). Since the action of \( G \) on \( G/N \) is transitive, this implies that \( G/N \) is a singleton, i.e. \( N = G \).

The implication “(i) \( \Rightarrow \) (ii)” in the above theorem fails if one considers group representations that do not possess a quasi-interior fixed point: let \( G \) be the additive group \( \mathbb{R} \) and let \( E \) be the \( \ell^p \)-space over the index set \( \mathbb{R} \) for some \( p \in [1, \infty) \), then a counterexample is given by the shift group \( (T_t)_{t \in \mathbb{R}}(\omega) = f(\omega - t) \) for \( f \in E \) and \( \omega \in \mathbb{R} \). In the following we construct a counterexample on an \( \ell^p \)-space over a countable index set.

**Example 2.6.** Let \( E := \ell^p(\mathbb{Q}) \) for some \( 1 \leq p < \infty \). Then there exists a positive group representation \( (T_t)_{t \in \mathbb{R}} \subseteq \mathcal{L}(E) \) of \( (\mathbb{R}, +) \) on \( E \) which is positive and bounded, but not trivial (and which does not possess a quasi-interior fixed point according to Theorem \[ \ref{thm:trivial_representation} \]).

In order to see this, we first construct a surjective group homomorphism \( \varphi : \mathbb{R} \to \mathbb{Q} \). Let \( \{ v_j : j \in J \} \) denote a basis of \( \mathbb{R} \) as a vector space over \( \mathbb{Q} \) and fix some \( i \in J \). Then \( V := \text{span}_{\mathbb{Q}} \{ v_j : j \neq i \} \) is a subspace of \( \mathbb{R} \) of codimension 1. In particular, \( V \) is a normal subgroup of \( \mathbb{R} \) and the quotient group \( \mathbb{R}/V \) is isomorphic to \( \mathbb{Q} \). The composition of the canonical epimorphism \( \mathbb{R} \to \mathbb{R}/V \) with the last-mentioned isomorphism from \( \mathbb{R}/V \) to \( \mathbb{Q} \) is the desired surjective group homomorphism \( \varphi \).

For \( t \in \mathbb{R} \) we now define \( T_t \in \mathcal{L}(E) \) as the Koopman operator associated with the translation by \( \varphi(t) \), i.e. \( T_t(x_q)_{q \in \mathbb{Q}} := (x_{q+\varphi(t)})_{q \in \mathbb{Q}} \) for any \( (x_q) \in \ell^p(\mathbb{Q}) \). Then \( (T_t)_{t \in \mathbb{R}} \) has all the desired properties.

It is worth noting that on an atomic Banach lattice \( E \) with order continuous norm every positive and bounded group representation \( (T_t)_{t \in \mathbb{R}} \in \mathcal{L}(E) \) which is strongly continuous is trivial. This follows for instance from \[ \text{[69 Prop 2.3]} \]. In this case one does not need to know a priori that \( (T_t)_{t \in \mathbb{R}} \) possesses a quasi-interior fixed point in contrast to Theorem \[ \ref{thm:trivial_representation} \].
3. Convergence of Positive Semigroup Representations

In this section we prove our main results, convergence theorems for positive semigroup representations. Our main tools are Theorems 2.5 and the well-known splitting theorem by Jacobs, de Leeuw and Glicksberg; we start by providing a version of the latter for positive semigroup representations. For this purpose, we need further simple results from group theory, given in Lemma 3.3 and Lemma 3.2, as well as a result from Banach lattice theory which we state in Proposition 3.1. The latter is essentially contained in [62 Prop III.11.5] but we give a bit more explicit information here.

**Proposition 3.1.** Let \((E, \| \cdot \|)\) be a Banach lattice and let \(P \in \mathcal{L}(E)\) be a positive projection. Then \(\|x\|_{PE} := \|P|x\|\) defines a norm on \(PE\) that is equivalent to \(\| \cdot \|\) and \((PE, \| \cdot \|_{PE})\) is a Banach lattice with respect to the order induced by \(E\) and with the modulus \(x|_{PE} := P|x|\). Moreover, if \(y \in E_+\) is a quasi-interior point of \(E_+\), then \(Py\) is a quasi-interior point of \((PE)_+\).

**Proof.** Let \(x \in PE\). It follows from \(\pm x \leq |x|\) that \(\pm x = \pm Px \leq P|x|\). On the other hand for every \(z \in (PE)_+\) with \(\pm z \leq z\) we have \(|x| \leq z\) by the definition of the modulus \(|x|\) in \(E\). This implies that \(P|x| \leq Pz = z\). Hence \(P|x|\) is the modulus of \(x\) in \(PE\) and we proved that \(PE\) is a vector lattice.

It is obvious that \(\| \cdot \|_{PE}\) is a seminorm. Since \(P|x| \geq |x|\), we have that \(P|x| \geq |x|\). This implies that the seminorm \(\| \cdot \|_{PE}\) is equivalent to \(\| \cdot \|\) on \(PE\). In particular, \(\| \cdot \|_{PE}\) is a norm on \(PE\) and \((PE, \| \cdot \|_{PE}\) is a Banach lattice.

Finally, let \(y \in E_+\) be a quasi-interior point of \(E_+\). For a fixed \(x \in F\) we find a sequence \((x_n)\) in \(E_y\) such that \(x_n \to x\). By continuity of \(P\) we have that \(Px_n \to Px\) as \(n \to \infty\). Since each \(Px_n\) belongs to \(F_{Py}\), the principal ideal generated by \(Py\) in \(F\), this shows that \(Py\) is a quasi-interior point of \((PE)_+\).  

As said in the preliminaries, for a group \(G\) and \(A \subseteq G\) we denote by \(\langle A \rangle\) the smallest subgroup of \(G\) containing \(A\). Note that if all elements of \(A\) commute, then \(\langle A \rangle\) is commutative. This follows from the fact that if \(a \in G\) commutes with \(b \in G\), then \(a\) commutes with \(b^{-1}\) and \(a^{-1}\) commutes with \(b\). We adopt the convention of denoting the binary operation in commutative groups by \(\cdot\).

**Lemma 3.2.** Let \(S\) be a commutative semigroup that generates a group \(G := \langle S \rangle\) and let \(s \in S\). Then \(S' := \{s + t : t \in S\}\) also generates \(G\).

**Proof.** Clearly \(\langle S' \rangle \subseteq G\). Since for \(s + t \in S'\) we also have \(s + s + t \in S'\), it follows that \(s = (2s + t) - (s + t) \in \langle S' \rangle\). Now it follows from

\[
G = \langle S \rangle = \{t_1 - t_2 : t_1, t_2 \in S\}
\]

that for any \(t \in G\) we have

\[
t = t_1 - t_2 = s + t_1 - (s + t_2) \in \langle S' \rangle
\]

for certain \(t_1, t_2 \in S\). This shows that \(G \subseteq \langle S' \rangle\).  

**Lemma 3.3.** Let \(G, H\) be groups where \(G\) is commutative. Let \(\emptyset \neq S \subseteq G\) be a subsemigroup of \(G\) with \(\langle S \rangle = G\) and let \(\varphi : S \to H\) be a semigroup homomorphism. Then there exists a unique group homomorphism \(\psi : G \to H\) which extends \(\varphi\).

**Proof.** Since \(G\) is commutative, every \(t \in G\) can be represented in the form \(t = s - r\) where \(s, r \in S\). Now define \(\psi(t) = \varphi(s) - \varphi(r)\). Using that \(G\) and \(\langle \varphi(S) \rangle\) are commutative one easily checks that this is well-defined, i.e. \(\psi(t)\) does not depend on the representation of \(t\), and that \(\psi\) is a group homomorphism. The uniqueness follows readily from the fact that each \(t \in G\) is of the form \(s - r\) for some \(s, r \in S\).  

Theorem 3.4 (Jacobs–de Leeuw–Glicksberg decomposition). Let $S$ be a commutative semigroup that generates a group $G$ and let $\mathcal{F} := (T_t)_{t \in S} \subseteq \mathcal{L}(E)$ be a positive and bounded representation of $S$ on a Banach lattice $E$ such that the orbits $(T_t x : t \in S)$ are relatively weakly compact for all $x \in E$.

Then there exists a positive projection $P \in \mathcal{L}(E)$ that commutes with each $T_t$ and preserves all fixed points of $\mathcal{F}$ such that the following assertions hold:

(a) There exists a bounded and positive representation $(S_t)_{t \in G} \subseteq \mathcal{L}(PE)$ of $G$ on $PE$ such that $S_t = T_{t|PE}$ for each $t \in S$.

(b) $0$ is a weak accumulation point of $\{T_t x : t \in S\}$ for any $x \in \ker P$.

Proof. First note that $\mathcal{F} \subseteq \mathcal{L}(E)$ is a positive and bounded semigroup of operators which is relatively compact with respect to the weak operator topology, cf. [20, Lem 16.16]. Thus, its closure in this topology, denoted by $\overline{\mathcal{F}} \subseteq \mathcal{L}(E)$, is a compact and commutative semitopological semigroup in the sense of [20, Def 16.2]. By [20, Thm 16.5] there exists a projection $P \in \overline{\mathcal{F}}$ such that $P\overline{\mathcal{F}}$ is a commutative group with neutral element $P$. This implies that the subspaces $PE$ and $\ker P$ are invariant under each $T \in \overline{\mathcal{F}}$. Since $\overline{\mathcal{F}}$ consists of positive operators, also $P$ is positive. Moreover, each fixed point of $\mathcal{F}$ is a fixed point of any operator in $\overline{\mathcal{F}}$ and thus $P$ leaves the fixed points of $\mathcal{F}$ invariant.

By Lemma 3.3 the semigroup homomorphism $S \rightarrow P\overline{\mathcal{F}}$, $t \mapsto PT_t$, uniquely extends to a group homomorphism $\hat{T} : G \rightarrow P\overline{\mathcal{F}}$ and we define $S_t := \hat{T}_{t|PE} \in \mathcal{L}(PE)$ for any $t \in G$. Since the operators in $P\overline{\mathcal{F}}$ are positive and uniformly bounded, the representation $(S_t)_{t \in G}$ of $G$ on $PE$ is positive and bounded, which proves property (a).

Finally, it follows from [20, Prop 16.27] that $0$ is a weak accumulation point of $\{T_t x : t \in S\}$ for any $x \in \ker P$, which proves (b). $\square$

Now we turn to the first of our convergence theorems for certain positive semigroup representations. Our main assumption is that the semigroup representation contains a so-called AM-compact operator, which is defined as follows:

Definition 3.5. Let $E$ be a Banach lattice and $Y$ be a Banach space. A linear operator $T : E \rightarrow Y$ is called AM-compact if $T$ maps order intervals in $E$ to relatively compact subsets in $Y$.

Remark 3.6. It follows from [2, Cor 21.13] that a Banach lattice $E$ is atomic with order continuous norm if and only if every order interval is compact. Since a bounded operator maps relatively compact sets to relatively compact sets, we thus obtain that on an atomic Banach lattice with order continuous norm every bounded operator is AM-compact.

Moreover, every (abstract) kernel operator on a large class of Banach lattices is AM-compact as we explain in Section 4. This is why the following theorem, our first main result, has a broad range of applications.

Theorem 3.7. Let $S$ be a commutative semigroup that generates a divisible group $G := \langle S \rangle$ and let $\mathcal{F} := (T_t)_{t \in S} \subseteq \mathcal{L}(E)$ be a positive and bounded representation of $S$ on a Banach lattice $E$ such that $T_s$ is AM-compact for some $s \in S$. Assume in addition that $\mathcal{F}$ possesses a quasi-interior fixed point.

Then $\mathcal{F}$ is strongly convergent.

Proof. Consider the subsemigroup $S' := \{s + t : t \in S\}$ and its positive and bounded representation $\mathcal{F}' := (T_{t|S'})$. We show that $\mathcal{F}'$ is strongly convergent which implies that $\mathcal{F}$ is strongly convergent (although the order on $S'$ does not necessarily coincide with the order inherited from $S$).
We start by showing that the set \( \{ T_t x : t \in S' \} \subseteq E \) is relatively compact for every \( x \in E \). Let \( x \in E_0 \). Then \( |x| \leq c y \) for some \( c > 0 \) and therefore \( |T_t x| \leq c y \) for all \( t \in S \). Hence, \( T_{t+\epsilon} x = T_t T_{\epsilon} x \) is contained in the relatively compact set \( T_{[c\epsilon, c\epsilon]} \) for each \( t \in S \). This shows that \( \{ T_t x : t \in S' \} \) is relatively compact. Since \( E_0 \) is dense in \( E \) and the representation \( \mathcal{J} \) is bounded, this implies that the orbit \( \{ T_t x : t \in S' \} \) is relatively compact for any \( x \in E \).

Therefore, we can apply the Jacobs–de Leeuw–Glicksberg decomposition, Theorem 3.4 to the representation \( \mathcal{J} \) of \( S' \). Let \( P \) be the projection from that theorem and consider the decomposition \( E = PE \oplus \ker P \). We show next that \( \lim_{t \in S'} T_t x = 0 \) for all \( x \in \ker P \). So fix \( x \in \ker P \). By assertion (b) of Theorem 3.4 there exists a net in \( \{ T_t x : t \in S' \} \) that converges weakly to 0 which in turn has a subnet that converges to 0 in norm as the orbit of \( x \) is relatively compact. In particular, for \( M := \sup_{t \in S'} \| T_t \| \) and an arbitrary \( \varepsilon > 0 \) there exists \( r \in S' \) such that \( \| T_r x \| < \varepsilon \), which implies that \( \| T_{t+\epsilon} x \| \leq M \varepsilon \) for all \( t \in G \). This shows that \( \lim_{t \in S'} T_t x = 0 \).

To complete the proof it suffices to show that \( T_t|PE = \text{id}_{PE} \) for any \( t \in S' \). We obtain from Proposition 3.1 that \( F := (PE, \| \cdot \|_{PE}) \) is itself a Banach lattice with respect to the order of \( E \) and an equivalent norm \( \| \cdot \|_{PE} \) and that \( y = Py \) is a quasi-interior point of \( F_{+} \). Since \( S' \) also generates the group \( G \) by Lemma 3.2, we obtain from assertion (a) of Theorem 3.4 that there exists a bounded and positive representation \( (S_t)_{t \in S} \subseteq \mathcal{L}(F) \) of \( G \) on \( F \) such that \( S_t = T_t|F \) for each \( t \in S' \).

Finally, we show that \( F \) is atomic with order continuous norm. It follows from [2] Cor 21.13 that, for any Banach lattice, this is equivalent to the fact that any order interval in \( F \) is compact. Let \( [x, z] \subseteq F \). Then \( S_{2s} = T_{2s}|F \) as \( 2s \in S' \) and thus we have

\[
[x, z] = T_{2s}S_{-2s}[x, z] \subseteq T_{2s}[S_{-2s}x, S_{-2s}z] \subseteq T_{2s}[S_{-2s}x, S_{-2s}z] = F.
\]

The latter set is relatively compact in \( E \) because \( T_{2s} \) is AM-compact. Hence, the closed set \( [x, z] \) is compact in \( E \) and thus also in \( F \) as the norms are equivalent.

In summary, we proved that there is a positive and bounded representation of \( G \) on \( F \) that possesses a fixed point \( y \) which is a quasi-interior point of \( F_{+} \). Since \( F \) is atomic with order continuous norm, it follows from Theorem 2.2 that \( S_{t} \) is the identity on \( F \). In particular, \( T_t|F = \text{id}_F \) for every \( t \in S' \) which completes the proof.

\[ \square \]

**Remark 3.8.** The assertion of Theorem 3.7 also holds if not the entire representation \( \mathcal{J} := (T_t)_{t \in S} \) but merely \( \mathcal{J} := \{ T_{r+t} : t \in S \} \) is assumed to be bounded for some \( r \in S \). Indeed, by Lemma 3.2 the semigroup \( S' := \{ r + t : t \in S \} \) also generates \( G \) and \( \mathcal{J} \) is a representation of \( S' \) that fulfills all conditions of the theorem since \( T_{r+t} \in \mathcal{J} \) is AM-compact. Hence, \( \mathcal{J} \) is strongly convergent by Theorem 3.7. A moment of reflection shows that this also implies strong convergence of \( \mathcal{J} \), although the order on the semigroup \( S' \) does not coincide with the order inherited from the semigroup \( S \).

The assumption of Theorem 3.7 that the representation \( (T_t)_{t \in S} \subseteq \mathcal{L}(E) \) possesses a quasi-interior fixed point is crucial. Let us elaborate a bit more on this point:

**Remarks 3.9.** (a) Let \( \mathcal{J} = (T_t)_{t \in [0, \infty]} \) be the Gaussian semigroup on \( L^1(\mathbb{R}) \). Then each operator \( T_t \) is AM-compact, as follows for instance from Proposition A.3 in the appendix. Hence, \( \mathcal{J} \) fulfills all assumptions of Theorem 3.7 except that it does not possess a quasi-interior fixed point. Since \( \mathcal{J} \) is not strongly convergent, this shows that the existence of a quasi-interior fixed point cannot in general be omitted in Theorem 3.7.

(b) On the other hand, the Gaussian semigroup is not mean ergodic on \( L^1(\mathbb{R}) \). In fact, using Theorem 3.7 it is not difficult to show the following Tauberian result:
Theorem. Let $E$ be an AL-space and let $\mathcal{F} = (T_t)_{t \in (0, \infty)}$ be a mean ergodic $C_0$-semigroup on $E$ which is positive and bounded. If $T_{t_0}$ is AM-compact for some $t_0 > 0$, then $\mathcal{F}$ is strongly convergent.

Sketch of proof. Let $\text{Fix}(\mathcal{F})_+$ denote the set of all positive fixed points of $\mathcal{F}$ and consider the band $B$ generated by $\text{Fix}(\mathcal{F})_+$. Then $B$ is $\mathcal{F}$-invariant and one can conclude from Theorem 3.7 that the restriction of $\mathcal{F}$ to $B$ is strongly convergent.

For semigroups on spaces of measures we will show in Theorem 3.10 that a similar result holds under weaker regularity and ergodicity assumptions.

(c) The above observation gives rise to the conjecture that the existence of a quasi-interior fixed point in Theorem 3.7 can be omitted in case that $G$ is mean ergodic with mean ergodic projection 0. Hence, the additivity of the norm on the positive cone implies that this induced semigroup converges strongly to 0 in $E/B$.

□

For semigroups on spaces of measures we will show in Theorem 3.10 that a similar result holds under weaker regularity and ergodicity assumptions.

(d) If a one-parameter semigroup $(T_t)_{t \in (0, \infty)}$ does not satisfy any time regularity, then strong convergence as $t \to \infty$ cannot be guaranteed without assuming the existence of a quasi-interior fixed point, even if the space $E$ is reflexive. Recall from Example 2.6 that there exists a non-trivial bounded and positive one-parameter group $(T_t)_{t \in \mathbb{R}}$ on an $\ell^p$-space without any quasi-interior fixed point. Then every operator $T_t$ is AM-compact by Remark 3.6, but Proposition 3.10 below shows that the semigroup representation $(T_t)_{t \in (0, \infty)}$ is not strongly convergent.

Proposition 3.10. Let $G$ be a commutative group and let $(T_t)_{t \in G}$ be a bounded group representation on a Banach space $E$. Consider a subsemigroup $S$ of $G$ with $(S) = G$. If the semigroup representation $(T_t)_{t \in S}$ is strongly convergent, then $T_t = \text{id}_E$ for all $t \in G$.

Proof. Assume that $(T_t)_{t \in S}$ is strongly convergent. Let $x \in E$, $t \in S$ and $M := \sup_{t \in G} \|T_t\|$. For any $\varepsilon > 0$ we find $s \in S$ such that

$$\|x - T_t x\| = \|T_{-s}(T_s x - T_{t+s} x)\| \leq M \varepsilon.$$ 

This shows that $T_t = \text{id}_E$ for all $t \in S$. Since each $t \in G$ is the difference of two elements in $S$, the assertion follows.

□

Before we proceed to our second main result, let us point out once again the crucial role of the algebraic structure of the semigroup $S$ in Theorem 3.7. The additive semigroup $(0, \infty)$ is in our context certainly the most important example of a semigroup which generates a divisible group. Other simple examples are constituted by the additive semigroup of strictly positive rational numbers and by the additive semigroup $(0, \infty)^d$ for any $d \in \mathbb{N}$. On the other hand, the additive semigroup of strictly positive dyadic numbers $D_{>0} := \{k2^{-n} : k \in \mathbb{N}, n \in \mathbb{N}_0\}$ generates the group $D := \{k2^{-n} : k \in \mathbb{Z}, n \in \mathbb{N}_0\}$ which is not divisible. Hence, according to Theorem 2.5 there exists a non-trivial bounded and positive group representation $\mathcal{F} = (T_t)_{t \in D}$ of $D$ on an atomic Banach lattice with order continuous norm and $\mathcal{F}$ can be chosen to possess a quasi-interior fixed point. The restriction of this group representation to $D_{>0}$ fulfills the assumptions of Theorem 3.7 but it is not strongly convergent by Proposition 3.10. Hence, Theorem 3.7 fails for semigroup representations of $D_{>0}$. Since $D_{>0}$ is homeomorphic to the strictly positive rationals, this
stresses that the crucial assumption for our results is the algebraic structure of the semigroup and not any kind of topological structure.

Now we come to our second main result: we consider the case where the representation of the semigroup does not necessarily contain an AM-compact operator but where one operator dominates a non-trivial AM-compact operator. Although this condition looks rather technical at a first sight, it appears frequently in models from mathematical biology (see the introduction for references). In Section 3.11 we demonstrate that it is in particular fulfilled for $C_0$-semigroups whose generators are subject to certain perturbations. To ensure convergence in this case, we need further requirements on the fixed space of the representation and on the dominated AM-compact operator that we discuss subsequent to the proof.

We use the following terminology: a vector $x$ in a Banach lattice $E$ is said to be a super fixed point of a family of operators $\mathcal{T} \subseteq \mathcal{L}(E)$ if $x \geq 0$ and if $Tx \geq x$ for all $T \in \mathcal{T}$.

**Theorem 3.11.** Let $S$ be a commutative semigroup that generates a divisible group $G := \langle S \rangle$ and let $\mathcal{T} := \{T_t\}_{t \in S} \subseteq \mathcal{L}(E)$ be a positive and bounded representation of $S$ on a Banach lattice $E$ with order continuous norm. Assume that $\mathcal{T}$ possesses a quasi-interior fixed point $y \in E_+$ and that $\mathcal{T}$ has the following two properties:

(a) Every super fixed point of $\mathcal{T}$ is a fixed point of $\mathcal{T}$.

(b) For every fixed point $x > 0$ of $\mathcal{T}$ there exists $s \in S$ and an AM-compact operator $K \geq 0$ such that $T_s \geq K$ and $Kx > 0$.

Then $\mathcal{T}$ is strongly convergent.

The key to the proof of Theorem 3.11 is the following lemma.

**Lemma 3.12.** Under the assumptions of Theorem 3.11 the following holds: for every $\varepsilon > 0$ there exists $t \in S$ and an AM-compact operator $0 \leq K_t \leq T_t$ such that $\|(T_t - K_t)y\| < \varepsilon$.

**Proof.** Since $E$ has order complete norm, the AM-compact operators form a band in $\mathcal{L}^r(E)$, see [31] Prop. 3.7.2, which we denote by $\mathcal{K}$. Then for every $t \in S$ we have $T_t = K_t + R_t$ for certain uniquely determined positive operators $K_t \in \mathcal{K}$ and $R_t \in \mathcal{K}^\perp$. Moreover,

$$T_{t+r} = K_t T_r + R_t T_r = K_t K_r + R_t K_r + R_t R_r$$

for all $t, r \in S$. It follows immediately from Definition 3.8 that the composition of an AM-compact and a regular operator is again AM-compact, i.e. that $\mathcal{K}$ is an ideal in the algebra $\mathcal{L}^r(E)$. Therefore, the operator $K_t T_r + R_t K_r$ is AM-compact and dominated by $T_{t+r}$, so $K_{t+r} \geq K_t T_r + R_t K_r$. This in turn implies that $R_{t+r} \leq R_t R_r$. In particular, $R_{t+r} y \leq R_t T_r y = R_t y$ for all $t, r \in S$, i.e. the net $(R_t y)_{t \in S} \subseteq E_+$ is decreasing and thus convergent as the norm is order continuous. Its limit $z := \lim_{t \in S} R_t y$ fulfills

$$R_t z = \lim_{r \in S} R_t R_r y \geq \lim_{r \in S} R_{t+r} y = z$$

and hence $T_t z \geq R_t z \geq z$ for every $t \in S$. By assumption (a) this implies that $T_t z = z$ and therefore $K_t z = 0$ for every $t \in S$. Assumption (b) implies that $z = 0$.

We have thus shown that $\lim_{t \in S} R_t y = 0$, which proves the lemma. \hfill \Box

**Proof of Theorem 3.11** We first show that $\{T_t x : t \in S\} \subseteq E$ is relatively weakly compact for every $x \in E$. First let $x \in E_y$, i.e. $x \in [-cy, cy]$ for some $c > 0$. Then $T_t x \in [-cy, cy]$ for all $t \in S$ since $y$ is a fixed point of $\mathcal{T}$. By the order continuity of the norm every order interval in $E$ is weakly compact and therefore $\{T_t x : t \in S\}$, the orbit of $x$, is relatively weakly compact. Since $E_y$ is dense in
E and the representation is bounded, it follows that the orbit of any \( x \in E \) is relatively weakly compact.

We can thus apply Theorem 3.12 let \( P \) be the projection given by this theorem. We are going to show first that \( \lim T_t x = 0 \) for any \( x \in \ker P \) and second that \( T_t|_{P E} = \text{id}_{P E} \) for all \( t \in S \). By combining these two assertions we obtain the theorem.

Let \( z \in [0, y] \) and consider the vector \( x := z - Pz \in \ker P \). Since \( Pz \leq Py = y \) we have \( x \in [-y, y] \), i.e. \( |x| \leq y \). By assertion (b) of Theorem 3.4 there exists a net \( \{x_\alpha\}_{\alpha \in \Lambda} \subseteq \{T_t x : t \in S\} \) such that \( \{x_\alpha\}_{\alpha \in \Lambda} \) converges weakly to 0. Let \( \varepsilon > 0 \). By Lemma 3.12 we find \( r \in S \) and an AM-compact operator \( 0 \leq K_r \leq T_r \) such that \( ||T_r y - K_r y|| < \varepsilon \). For every \( \alpha \in \Lambda \) we thus have

\[
||T_r x_\alpha|| \leq ||K_r x_\alpha|| + ||T_r - K_r|| x_\alpha || = ||K_r x_\alpha || + \varepsilon,
\]

where the second inequality follows from the fact that \( ||(T_r - K_r)x_\alpha|| \leq (T_r - K_r)y \).

The net \( \{K_r x_\alpha\} \) is contained in the relatively compact set \( K_r [-y, y] \), so there exists a subnet of \( \{K_r x_\alpha\}_{\alpha \in \Lambda} \) that converges in norm; since it also converges weakly to 0, it follows that \( ||K_r x_\alpha|| < \varepsilon \) for some \( \alpha \in \Lambda \). In summary, this shows that \( ||T_r x|| < 2\varepsilon \) for some \( t \in S \). Since \( \mathcal{F} \) is bounded and \( y \) is a quasi-interior point of \( E_+ \), this readily implies that \( \lim_{t \in S} T_t x = 0 \) for all \( x \in \ker P = (\text{id}_E - P)E \).

Now we show that each \( T_t \) acts trivially on \( P E \). By Proposition 3.11 \( F := (P E, ||\cdot||_{P E}) \) is itself a Banach lattice with respect to the order of \( E \) and an equivalent norm \( ||\cdot||_{P E} \); moreover, \( y = Py \) is a quasi-interior point of \( F_+ \). We obtain from assertion (a) of Theorem 3.3 that there exists a bounded and positive representation \( (S_t)_{t \in G} \subseteq \mathcal{L}(F) \) of \( G \) on \( F \) such that \( S_t = T_t|_F \) for each \( t \in S \).

Let \( u, v \in F \) such that \( u \leq v \). We show that \( [u, v]_F \) is totally bounded and thus compact in \( F \). Given \( \varepsilon > 0 \), first choose \( c > 0 \) such that \( [u, v]_F \subseteq [-cy, cy]_F + B_E(0, \varepsilon) \), where \( B_E(0, \varepsilon) \) denotes the ball in \( E \) of radius \( \varepsilon \) centered at 0. Then, by Lemma 3.12 we find \( r \in S \) and an AM-compact operator \( 0 \leq K_r \leq T_r \) such that \( ||(T_r y - K_r y)|| < \varepsilon \). Thus we have

\[
[u, v]_F \subseteq [-cy, cy]_F + B_E(0, \varepsilon) = T_r [-cy, cy]_F + B_E(0, \varepsilon)
\]

so is \( [u, v]_F \). Now it follows from that fact that the norm on \( E \) is equivalent to the norm on \( E \) that the closed set \( [u, v]_F \) is compact in \( F \). Since a Banach lattice is atomic with order continuous norm if and only if all order intervals are compact, cf. [2 Cor 21.13], this shows that \( F \) is an atomic Banach lattice with order continuous norm.

As \( G \) is divisible and the quasi-interior point \( y \in F_+ \) is a fixed vector of the group representation \( (S_t)_{t \in G} \), it follows from Theorem 3.5 that every operator \( S_t \) is the identity on \( F \). In particular, \( T_t|_{P E} = \text{id}_{P E} \) for every \( t \in S \) which completes the proof.

We close the section with a discussion of the technical requirements (a) and (b) of Theorem 3.12. The following proposition provides sufficient conditions for assumption (a) to hold.

**Proposition 3.13.** Let \( E \) be a Banach lattice and let \( \mathcal{F} = (T_t)_{t \in S} \) be a positive and bounded representation of a semigroup \( S \) on \( E \). Each of the following conditions implies that every super fixed point of \( \mathcal{F} \) is in fact a fixed point of \( \mathcal{F} \) and that the fixed space of \( \mathcal{F} \) is a sublattice of \( E \).
(a) $E$ has strictly monotone norm, meaning that $\|f\| < \|g\|$ whenever $0 \leq f < g$, and each operator in $\mathcal{T}$ is contractive.

(b) There exists a strictly positive $\varphi \in E'_+^+$ such that $T'_t \varphi \leq \varphi$ for all $t \in S$.

(c) The representation $\mathcal{T}$ is irreducible.

(d) The space $E$ has order continuous norm and the adjoint $T'_t$ of each operator $T_t$ is a lattice homomorphism.

Proof. First note that if $x \in E$ is a fixed point of $\mathcal{T}$, then $|x| = |T_t x| \leq T_t |x|$ for every $t \in S$, i.e. $|x|$ is a super fixed point of $\mathcal{T}$. This shows that the fixed space of $\mathcal{T}$ is a sublattice of $E$ if any super fixed point of $\mathcal{T}$ is a fixed point.

(a) Fix $t \in S$ and let $x \in E_+$ such that $T_t x \geq x$. Since $T_t$ is contractive we have $\|T_t x\| = \|x\|$ and as the norm is strictly monotone, this readily implies that $T_t x = x$.

(b) Fix $t \in S$ and let $x \in E_+$ such that $T_t x \geq x$; then $\langle T_t x - x, \varphi \rangle \geq 0$. On the other hand, $\langle T_t x - x, \varphi \rangle = \langle x, T'_t \varphi - \varphi \rangle \leq 0$ and thus $\varphi$ vanishes on the positive vector $T_t x - x$. Since $\varphi$ is strictly positive, this implies that $T_t x - x = 0$.

(c) Let $x \in E_+$ be a non-zero super-fixed point of $\mathcal{T}$ and fix a functional $\alpha \in E'_+$ such that $\langle \alpha, x \rangle > 0$. Since every commutative semigroup is amenable [18, p. 178], there exists a positive functional $\psi \in C(S; \mathbb{R})'$ which maps the constant function with value 1 to 1 and which is invariant with respect to translations on $S$. Now, we define a functional $\varphi \in E'$ by the formula

$$\langle \varphi, z \rangle = \left\langle \psi, (\langle \alpha, T_t z \rangle)_{t \in S} \right\rangle$$

for each $z \in E$. The functional $\varphi$ is positive and, as $\psi$ is translation invariant, a fixed point of the adjoint semigroup $\mathcal{T}' := (T'_t)_{t \in S}$. Since $\langle \alpha, T_t x \rangle \geq \langle \alpha, x \rangle$ for each $s \in S$, we conclude moreover that $\varphi$ is non-zero; as $\mathcal{T}$ is irreducible, $\varphi$ is even strictly positive. Hence, assumption (b) is fulfilled and this proves the assertion.

(d) Fix $t \in S$ and let $x \in E_+$ such that $T_t x \geq x$. Since $T_t$ is a lattice homomorphism and the norm on $E$ is order continuous, it follows from [51, Exer 1.4.E2] that $T_t$ is interval preserving. Thus, we have $T_t [0, x] = [0, T_t x] \supseteq [0, x]$ and hence we find $x_1 \in [0, x]$ such that $T_t x_1 = x$. Now we construct recursively a decreasing sequence $x_n \in E_+$ such that $T_t x_{n+1} = x_n$ and $x_0 = x$. Since $E$ has order continuous norm, the sequence $(x_n)$ converges and $z := \lim x_n$ fulfils $T_t z = \lim T_t x_n = \lim x_{n-1} = z$. Now it follows from

$$\|z - x\| = \|T_t^n z - T_t^n x_n\| \leq \sup_{s \in S} \|T_s\| \cdot \|z - x_n\| \to 0 \text{ as } n \to \infty$$

that $x = z$ is a fixed point of $T_t$. $\square$

Remark 3.14. Regarding assumption (b) of Theorem 3.11 it is quite obvious that in order to enforce convergence it cannot be sufficient that some operator $T_t$ simply dominates a non-trivial AM-compact operator. Instead, one has to ensure that the dominated AM-compact operators interact, in a sense, with the entire semigroup. Assumption (b) in Theorem 3.15 shows that this “interaction condition” is as weak as one could possibly hope for: it suffices that the family of dominated AM-compact operators sees every positive fixed vector of the semigroup.

Let us describe two situations more explicitly in which assumption (b) of Theorem 3.11 is fulfilled.

(1) If for every $0 < x \in E$ there exists $s \in S$ and an AM-compact operator $0 \leq K \leq T_s$ such that $K x > 0$, then assumption (b) in Theorem 3.11 is obviously fulfilled. In particular, assumption (b) holds if some operator $T_s$ dominates an AM-compact operator $K > 0$ that is strictly positive, meaning that $K x > 0$ for all $0 < x \in E$. 

(2) If \( \mathcal{T} \) is irreducible, then every non-zero positive fixed vector of \( \mathcal{T} \) is a quasi-interior point of \( E_+ \) (the fixed space of \( \mathcal{T} \) is actually one-dimensional). Hence, assumption (b) is automatically fulfilled if some \( T_s \) dominates a non-zero AM-compact operator.

4. A Bunch of Consequences

In the following we present some special cases of Theorems 3.7 and 3.11 which are most important for applications or interesting in their own right. Many of them are generalisations of known theorems in that any continuity requirement is completely eliminated.

For the sake of simplicity and in order to keep this section as comprehensible as possible, we formulate all the following consequences of our results from Section 3 for one-parameter semigroups, i.e. in the case where \( S \) is the additive semigroup \((0, \infty)\) that generates the divisible group \( G = \mathbb{R} \).

Semigroups of kernel operators. In the following we consider semigroups which contain a so-called kernel operator. Let us first recall the definition of this notion in the general setting of Banach lattices; afterwards, we proceed with a discussion on \( L^p \)-spaces.

Definition 4.1. Let \( E \) and \( F \) be Banach lattices such that \( F \) is order complete. We denote by \( E' \otimes F \) the space of all finite rank operators from \( E \) to \( F \). The elements of \( (E' \otimes F)^{\perp \perp} \), the band generated by \( E' \otimes F \) in \( \mathcal{L}^r(E, F) \), are called kernel operators.

While this definition is rather abstract, there exists a concrete description of kernel operators on concrete function spaces. In fact, each positive kernel operator on an \( L^p \)-space can be represented as an integral operator. Let us first make precise what we mean by this notion:

Definition 4.2. Let \( p,q \in [1, \infty) \), let \((\Omega_1, \mu_1)\) and \((\Omega_2, \mu_2)\) be \( \sigma \)-finite measure spaces and let \((\Omega, \mu)\) denote their product space. A bounded linear operator \( T: L^p(\Omega_1, \mu_1) \to L^q(\Omega_2, \mu_2) \) is called an integral operator if there exists a measurable function \( k: \Omega \to \mathbb{R} \) such that for every \( f \in L^p(\Omega_1, \mu_1) \) and for \( \mu_2 \)-almost every \( y \in \Omega_2 \) the function \( f(\cdot)k(\cdot, y) \) is contained in \( L^1(\Omega_1, \mu_1) \) and the equality

\[
Tf = \int_{\Omega_1} f(x)k(x, \cdot) \, d\mu_1(x).
\]

holds in \( L^q(\Omega_2, \mu_2) \). In this case the function \( k \) (which is uniquely determined up to a nullset) is called the integral kernel of \( T \).

The relation between kernel operators and integral operators is described by the following proposition. It shows that kernel operators can be viewed as an abstract analogue if integral operators.

Proposition 4.3. Let \( p,q \in [1, \infty) \), let \((\Omega_1, \mu_1)\) and \((\Omega_2, \mu_2)\) be \( \sigma \)-finite measure spaces and let \((\Omega, \Sigma, \mu)\) denote their product space. A linear operator \( T: L^p(\Omega_1, \mu_1) \to L^q(\Omega_2, \mu_2) \) is a positive kernel operator if and only if it is an integral operator whose integral kernel is positive almost everywhere on \( \Omega_1 \times \Omega_2 \).

Proof. See [62] Prop IV 9.8. \( \square \)
In [62, Prop IV 9.8] the reader can also find a similar result about non-positive kernel operators and information about the case where \( p \) or \( q \) equals \( \infty \). For more information about kernel operators and integral operators we refer to [63, 66].

The following proposition explains why our main results are applicable to semigroups containing a kernel operator.

**Proposition 4.4.** Let \( E \) and \( F \) be Banach lattices where the norm on \( F \) is order continuous. Then every kernel operator \( T \in (E \otimes F)^{\perp \perp} \) is AM-compact.

**Proof.** See [51, Cor 3.7.3]. \( \square \)

Propositions [4.3] and [4.3] together show that every positive integral operator on \( L^p \) is AM-compact. For the convenience of the reader we include a more direct proof of this statement in Proposition A.1 in the appendix.

Using the notation introduced above, we can now derive a generalisation of Greiner’s theorem quoted in the introduction.

**Theorem 4.5.** Let \( E \) be a Banach lattice with order continuous norm and let \( T = (T_t)_{t \in (0, \infty)} \) be a bounded and positive one-parameter semigroup on \( E \). Assume that for some \( s > 0 \) the operator \( T_s \) is a kernel operator and that \( T \) possesses a quasi-interior fixed point.

Then \( T \) is strongly convergent.

**Proof.** This follows readily from Theorem 3.7 and Proposition 4.4. \( \square \)

As explained in the introduction, Greiner proved this result for the special case of contractive \( C_0 \)-semigroups on \( L^p \)-spaces in [31, Kor 3.11] (see also [14, Thm 12] for a related spectral result) and deduced it from a certain 0-2-law ([31, Thm 3.7]; see also [24, Thm 5.1]), which has itself a very technical proof. By contrast, our proof of Theorem 4.5 only relies on the fact that every kernel operator is AM-compact, on the Jacobs–de Leeuw–Glicksberg decomposition and on our result on group representations, Theorem 2.5.

On atomic Banach lattices with order continuous norm every positive operator is a kernel operator (see e.g. [25, Lem 4.1.5]). Hence, we obtain the following special case of Theorem 4.5.

**Theorem 4.6.** Let \( E \) be an atomic Banach lattice with order continuous norm and let \( T = (T_t)_{t \in (0, \infty)} \) be a bounded and positive one-parameter semigroup on \( E \). If \( T \) possesses a quasi-interior fixed point, then \( T \) is strongly convergent.

**Proof.** This follows from Theorem 4.5 since every positive operator on \( E \) is a kernel operator by [25, Lem 4.1.5]. Alternatively, we can derive the result directly from Theorem 3.7 since every operator on \( E \) is AM-compact according to Remark 4.6. \( \square \)

For \( C_0 \)-semigroups Theorem 4.6 was proved by Keicher [39, Cor 3.8]; see also [14] and [69] for related results about positive \( C_0 \)-semigroups on atomic Banach lattices.

**Semigroups of partial kernel operators.** Now we consider semigroups which do not necessarily contain a kernel operator but which at least dominate a kernel operator. On \( L^1 \)-spaces such semigroups occur frequently in applications (see the introduction for references). To keep the terminology as concise as possible, we use the notion of a partial kernel operator which is defined as follows.

**Definition 4.7.** Let \( E \) and \( F \) be Banach lattices such that \( F \) is order complete. A positive operator \( T : E \to F \) is called a partial kernel operator if there exists a non-zero kernel operator \( K : E \to F \) such that \( 0 \leq K \leq T \).
We note in passing that $T$ is a partial kernel operator if and only if it is not orthogonal to all finite rank operators in the vector lattice $L^r(E,F)$, i.e. $T \notin (E' \otimes F)^\perp$.

Using Theorem 3.11 and Proposition 4.4 we immediately obtain the following convergence result for one-parameter semigroups of partial kernel operators.

**Theorem 4.8.** Let $E$ be a Banach lattice with order continuous norm and let $\mathcal{T} = (T_t)_{t \in (0, \infty)}$ be a bounded and positive one-parameter semigroup on $E$. Assume that $\mathcal{T}$ possesses a quasi-interior fixed point and that the following two assumptions are fulfilled:

1. Every super fixed point of $\mathcal{T}$ is a fixed point.
2. For every fixed point $x > 0$ of $\mathcal{T}$ there exists $s > 0$ and a kernel operator $0 \leq K \leq T_s$ which fulfils $Kx > 0$.

Then $\mathcal{T}$ is strongly convergent.

**Proof.** This follows from Theorem 3.11 and Proposition 4.4. $\square$

Note that the theorem remains true if the notion “kernel operator” in assumption (b) is replaced with the notion “compact operator” because every compact operator is obviously $\text{AM}$-compact. Sufficient conditions for the assumptions (a) and (b) in the theorem can be found in Proposition 3.13 and Remark 3.14.

In the subsequent corollaries we list a few special cases of Theorem 4.8 which also demonstrate that assumption (b) is fulfilled in various situations. Let us begin with the case of irreducible semigroups.

**Corollary 4.9.** Let $E$ be a Banach lattice with order continuous norm and let $\mathcal{T} = (T_t)_{t \in (0, \infty)}$ be a bounded and positive one-parameter semigroup on $E$. Assume that $\mathcal{T}$ has a non-zero fixed point.

If $\mathcal{T}$ is irreducible and if $T_s$ is a partial kernel operator for some $s > 0$, then $\mathcal{T}$ is strongly convergent.

**Proof.** By Proposition 3.13, the irreducibility of $\mathcal{T}$ implies that the fixed space of $\mathcal{T}$ is a sublattice of $E$. Hence, $\mathcal{T}$ also possesses a positive non-zero fixed vector $y$. Employing again the irreducibility assumption we see that $y$ is even a quasi-interior point of $E_+$. Thus, the assertion follows from Theorem 4.8 where assumption (a) is fulfilled by Proposition 3.13(c) and assumption (b) by Remark 3.14(2). $\square$

For $C_0$-semigroups the above result was proved by the first author in [24, Thm 4.2] by using Greiner’s 0-2-law. For strongly continuous Markov semigroups on $L^1$-spaces, Corollary 4.9 was proved earlier by Pichór and Rudnicki [55, Thm 1] who used the theory of Harris operators on $L^1$-spaces which is, for instance, presented in [23, Ch V].

If the semigroup is not irreducible, a similar conclusion still holds in case that the dominated kernel operators have, in a sense, sufficiently large support. This is the content of the next corollary.

**Corollary 4.10.** Let $E$ be a Banach lattice with order continuous norm and let $\mathcal{T} = (T_t)_{t \in (0, \infty)}$ be a bounded and positive one-parameter semigroup on $E$. Assume that $\mathcal{T}$ possesses a quasi-interior fixed point and that every super fixed point of $\mathcal{T}$ is even a fixed point.

Then each of the following conditions implies that $\mathcal{T}$ is strongly convergent:

1. For every $0 < x \in E$ there exists $t \in (0, \infty)$ and a kernel operator $0 \leq K \leq T_t$ such that $Kx > 0$.
2. There exists $t \in (0, \infty)$ and a kernel operator $0 \leq K \leq T_t$ such that $Kx > 0$ whenever $x \in E_+$ and $T_t x > 0$. 

For $C_0$-semigroups the above result was proved by the first author in [24, Thm 4.2] by using Greiner’s 0-2-law. For strongly continuous Markov semigroups on $L^1$-spaces, Corollary 4.9 was proved earlier by Pichór and Rudnicki [55, Thm 1] who used the theory of Harris operators on $L^1$-spaces which is, for instance, presented in [23, Ch V].
Proof. This is an immediate consequence of Theorem 4.8. □

A condition very similar to assumption (a) of Corollary 4.10 was in the focus of two recent papers of Pichó r and Rudnicki [57, 55], where they study the long term behaviour of substochastic operators and semigroups on $L^1$-spaces; see the conditions (K) in [57, pp. 308 and 309] and in the introduction of [55].

The fact that $Kx$ only needs to be non-zero in case that $T_tx$ is non-zero in assumption (b) of Corollary 4.10 accounts for situations where the semigroup vanishes on a part of the space $E$.

**Semigroups on the space of measures.** We now turn to positive one-parameter semigroups on spaces of measures as they appear naturally in the context of Markov chains and processes. When endowed with the total variation norm, the space of signed measures on a measurable space $(\Omega, \Sigma)$ is an AL-space and therefore a Banach lattice with order continuous norm. Yet, operator semigroups on this space are hardly ever $C_0$; in fact, even the Gaussian semigroup on the space of signed Borel measures over $\mathbb{R}$ fails to be strongly continuous at 0. Hence, there is a specific interest in results on semigroups that satisfy some weaker or no continuity condition.

However, the positive cone of the space of measures does in general not contain a quasi-interior point, which is why one also has to treat orbits of measures that are disjoint from any fixed point in order to apply our main results. This can be achieved by either assuming the semigroup to map every measure eventually to one which is absolutely continuous with respect to an invariant measure or by assuming the semigroup to be weakly ergodic. In the following, we elaborate these two approaches.

Let $\Omega$ be a measurable space whose $\sigma$-algebra we denote by $\Sigma(\Omega)$. Let $\mathcal{M}(\Omega)$ and $B_b(\Omega)$ denote the spaces of signed (finite) measures, endowed with the total variation norm, and the space of bounded measurable functions on $\Omega$, respectively.

A transition kernel on $\Omega$ is a map $k: \Omega \times \Sigma(\Omega) \to \mathbb{R}$ such that

(a) $A \mapsto k(x, A)$ is a signed measure for every $x \in \Omega$ and
(b) $x \mapsto k(x, A)$ is a measurable function for every $A \in \Sigma(\Omega)$.

Writing $|k|(x, \cdot)$ for the total variation of the measure $k(x, \cdot)$, a transition kernel $k$ is called **bounded** if $\sup_{x \in \Omega} |k|(x, \Omega) < \infty$. We recall that the total variation $|\mu|$ of a measure $\mu \in \mathcal{M}(\Omega)$ coincides with the modulus of $\mu$ in the Banach lattice $\mathcal{M}(\Omega)$.

To each bounded transition kernel $k$ on $\Omega$ we can associate an operator $T \in \mathcal{L}(\mathcal{M}(\Omega))$ by setting

$$ (T\mu)(A) := \int_{\Omega} k(x, A) \, d\mu(x) $$

for every $\mu \in \mathcal{M}(\Omega)$ and every measurable set $A \subseteq \Omega$. It follows from [44, Prop 3.1 and 3.5] that for any $T \in \mathcal{L}(\mathcal{M}(\Omega))$ the following conditions are equivalent:

(i) There exists a bounded transition kernel $k$ such that $T$ is given by (4.2).
(ii) The norm adjoint $T'$ of $T$ leaves $B_b(\Omega)$ invariant.
(iii) The operator $T$ is continuous in the $\sigma(\mathcal{M}(\Omega), B_b(\Omega))$ topology.

We denote by $\mathcal{L}(\mathcal{M}(\Omega), \sigma)$ the subspace of $\mathcal{L}(\mathcal{M}(\Omega))$ of those operators that satisfy the equivalent conditions (i)–(iii) above.

Recall once again that the Banach lattice $\mathcal{M}(\Omega)$ is an AL-space, meaning that the norm is additive on the positive cone. Let $T \in \mathcal{L}(\mathcal{M}(\Omega), \sigma)$ be positive and let $k$ be the transition kernel associated to $T$. Then $T$ is a Markov operator if and only if $k(x, \cdot)$ is a probability measure for each $x \in \Omega$.

It is worthwhile pointing out that an operator $T \in \mathcal{L}(\mathcal{M}(\Omega), \sigma)$ need not be a kernel operator in the sense of Definition 4.1). Yet, under appropriate conditions...
it follows that the restriction of $T$ to a certain invariant ideal is a kernel operator; this observation is employed in the proof of the subsequent theorem.

**Theorem 4.11.** Let $\Omega$ be a measurable space whose $\sigma$-algebra $\Sigma(\Omega)$ is countably generated and let $\mathcal{T} = (T_t)_{t \in (0, \infty)} \subseteq \mathcal{L}(\mathcal{M}(\Omega), \sigma)$ be a semigroup of Markov operators on $\mathcal{M}(\Omega)$ with associated transition kernels $k_t$.

If $\mu \in \mathcal{M}(\Omega)$ is a $\mathcal{T}$-invariant probability measure such that for some $s > 0$ and for every $x \in \Omega$ the measure $k_s(x, \cdot)$ is absolutely continuous with respect to $\mu$, then $T_t\nu$ converges with respect to the total variation norm as $t \to \infty$ for every $\nu \in \mathcal{M}(\Omega)$.

**Proof.** First note that the band $\{\mu\}^{\perp\perp}$ generated by $\mu$ in $\mathcal{M}(\Omega)$, which is invariant under $\mathcal{T}$, consists exactly of those measures which are absolutely continuous with respect to $\mu$, cf. [11 Thm 10.61].

It follows from formula (4.2) that $T_t\nu$ is absolutely continuous with respect to $\mu$ for every $\nu \in \mathcal{M}(\Omega)$, so the range of $T_t$ is contained in $\{\mu\}^{\perp\perp}$ for all $t \geq s$. Hence, it suffices to show that the restriction of $\mathcal{T}$ to $\{\mu\}^{\perp\perp}$ converges strongly.

The band $\{\mu\}^{\perp\perp}$ is isometrically Banach lattice isomorphic to $L^1(\Omega, \mu)$ via the mapping $\Phi: \{\mu\}^{\perp\perp} \to L^1(\Omega, \mu)$ which maps each $\nu \in \{\mu\}^{\perp\perp}$ to its Radon-Nikodym derivative with respect to $\mu$, see [1 Cor 13.19]. If the operator induced by $T_{t_s}\{(\mu)\}^{\perp\perp}$ on $L^1(\Omega, \mu)$ via $\Phi$ is a kernel operator, then the assertion follows from Theorem 4.3.

Since $\Sigma(\Omega)$ is countably generated and since $k_s(x, \cdot)$ is absolutely continuous with respect to $\mu$ for each $x \in \Omega$, there exists, according to [58 Cor 5.4 and Exa 1.4(i) in Ch 1], a measurable function $h: \Omega \times \Omega \to \mathbb{R}$ such that $k_s(x, A) = \int_A h(x, y) \mu(dy)$ for all $x \in \Omega$ and all $A \in \Sigma(\Omega)$. Clearly, $h$ can be chosen to be positive (replace $h$ with $h^+$ if necessary) and we have $T_t\Phi(\nu)(A) = \int_{A} \int_{\Omega} h(x, y) \mu(dy) \nu(dx)$ for all $A \in \Sigma(\Omega)$. Hence, we obtain

$$T_{t_s}\{(\mu)\}^{\perp\perp} \Phi^{-1} f(A) = \int_{\Omega} \int_{A} h(x, y) f(x) \mu(dy) \nu(dx)$$

$$= \int_{A} \int_{\Omega} h(x, y) f(x) \mu(dy) \mu(dx)$$

for all $f \in L^1(\Omega, \mu)$ and all $A \in \Sigma(\Omega)$, where we used the definition of $\Phi$ for the first equality and Tonelli’s theorem for the second. Employing again the definition of $\Phi$ we conclude that $\Phi T_{t_s}\{(\mu)\}^{\perp\perp} \Phi^{-1} f = \int_{A} h(x, \cdot) f(x) \mu(dx)$ for all $f \in L^1(\Omega, \mu)$, so $\Phi T_{t_s}\{(\mu)\}^{\perp\perp} \Phi^{-1}$ is indeed an integral operator, whose integral kernel is obtained by switching the arguments of $h$. Thus, it is a kernel operator on $L^1(\Omega, \mu)$ according to Proposition 4.3.

Note that in Theorem 4.11 the fixed space of $\mathcal{T}$ can be of arbitrary dimension. If, however, the measures $k_s(x, \cdot)$ are even mutually absolutely continuous, then the fixed space of $\mathcal{T}$ is always one-dimensional; this is discussed in more detail in the following corollary. To this end, we call two positive measures $\nu_1$ and $\nu_2$ equivalent if $\nu_1$ is absolutely continuous with respect to $\nu_2$ and vice versa.

**Corollary 4.12.** Let $\Omega$ be a measurable space whose $\sigma$-algebra $\Sigma(\Omega)$ is countably generated and let $\mathcal{T} = (T_t)_{t \in (0, \infty)} \subseteq \mathcal{L}(\mathcal{M}(\Omega), \sigma)$ be a semigroup of Markov operators on $\mathcal{M}(\Omega)$ with associated transition kernels $k_t$ that possesses an invariant probability measure $\mu$.

If for some $s > 0$ all the measures $k_s(x, \cdot)$ are mutually equivalent, then $\mu$ is the only $\mathcal{T}$-invariant probability measure and $T_t\nu \to \nu(\Omega)\mu$ with respect to the total variation norm as $t \to \infty$ for every $\nu \in \mathcal{M}(\Omega)$.

**Proof.** For any $0 < \nu \in \mathcal{M}(\Omega)$ it follows from formula (4.2) that the pairwise equivalent measures $k_s(x, \cdot)$ are also equivalent to $T_t\nu$ which in turn implies that $T_t\nu$ is
equivalent to \( \mu = T_\mu \). In particular, any two \( \mathcal{T} \)-invariant non-zero positive measures are equivalent. We show that this readily implies that \( \mu \) is the only invariant probability measure: Let \( \nu \in \mathcal{M}(\Omega) \) be an \( \mathcal{T} \)-invariant probability measure and for \( c \in \mathbb{R} \) define \( \nu_c := \nu + c\mu \). Since \( \mathcal{T} \) is contractive and the total variation norm is strictly monotone, it follows from Proposition 3.13 that the fixed space of \( \mathcal{T} \) is a sublattice of \( \mathcal{M}(\Omega) \) and therefore, both \( \nu_c^+ \) and \( \nu_c^- \) are \( \mathcal{T} \)-invariant and hence equivalent to \( \mu \). This is only possible if either \( \nu_c^+ = 0 \) or \( \nu_c^- = 0 \). As \( c \in \mathbb{R} \) was arbitrary, this implies that \( \nu = \mu \).

Now, let \( \nu \in \mathcal{M}(\Omega)_+ \) be arbitrary. According to Theorem 4.11 \( (T_t \nu)_{t \in (0, \infty)} \) converges in norm as \( t \to \infty \). Since \( \mu \) is the only invariant probability measure of \( \mathcal{T} \), we have \( \lim_{t \to \infty} T_t \nu = c\mu \) for a number \( c \geq 0 \) and the Markov property implies that \( c = \nu(\Omega) \). By linearity we obtain the assertion for all \( \nu \in \mathcal{M}(\Omega) \).

An earlier version of Corollary 4.12 where only a weaker mode of convergence is considered can be found in [13, Sec 4.2] and goes originally back to Doob [18]. Convergence with respect to the total variation norm was considered in [35]. Purely analytic proofs for the convergence with respect to the total variation norm were recently given in [30] and [29, Sec 4]. Yet, all these results only deal with the special case where \( \Omega \) is a Polish space and \( \Sigma(\Omega) \) is the Borel-\( \sigma \)-algebra on \( \Omega \). The more general case of an arbitrary measurable space with a countably generated \( \sigma \)-algebra was recently studied in [41] by means of couplings. In fact, the authors of [41] mainly deal with the time-discrete case and then derive the convergence result on the time interval \((0, \infty)\) as a corollary, [41, Rem 2]. They also note that no regularity of the semigroup with respect to the time parameter is needed.

Our more general result from Theorem 4.11 seems to be new and we stress that, in contrast to Corollary 4.12 a time discrete analogue of Theorem 4.11 does not hold. Indeed, let \( \Omega := \{1, 2\} \) be endowed with the discrete \( \sigma \)-algebra and consider the Markov operator

\[
T := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]

on \( \mathcal{M}(\Omega) \). Then all the assumptions of Theorem 4.11 are fulfilled for \( \mu(\{1\}) := \mu(\{2\}) := \frac{1}{2} \) and the discrete semigroup \( (T^n)_{n \in \mathbb{N}} \), which is yet not strongly convergent.

Next, we prove convergence of certain weakly ergodic semigroups, i.e. we prove a Tauberian theorem. To this end, we slightly restrict our setting and assume from now on that \( \Omega \) is a Polish space which is endowed with its Borel \( \sigma \)-algebra \( \mathcal{B}(\Omega) \). We denote by \( C_b(\Omega) \) the space of bounded continuous functions on \( \Omega \).

A Markov operator \( T \in \mathcal{L}(\mathcal{M}(\Omega), \sigma) \) whose adjoint fulfills \( T^*C_b(\Omega) \subseteq C_b(\Omega) \) is called a Markov-Feller operator. We call a one-parameter semigroup \( (T_t)_{t \in (0, \infty)} \subseteq \mathcal{L}(\mathcal{M}(\Omega), \sigma) \) of Markov-Feller operators weakly measurable if \( t \mapsto \langle T_t \mu, f \rangle \) is measurable for any \( \mu \in \mathcal{M}(\Omega) \) and \( f \in C_b(\Omega) \). For such a semigroup we obtain the following proposition, which is essentially a patchwork of results and arguments from [44].

**Proposition 4.13.** Let \( \Omega \) be a Polish space and let \( \mathcal{T} = (T_t)_{t \in (0, \infty)} \subseteq \mathcal{L}(\mathcal{M}(\Omega), \sigma) \) be a weakly measurable one-parameter semigroup of Markov-Feller operators. Then for every \( t > 0 \) there exists a Markov-Feller operator \( A_t \in \mathcal{L}(\mathcal{M}(\Omega), \sigma) \) such that

\[
\langle A_t \mu, f \rangle = \frac{1}{t} \int_0^t \langle T_s \mu, f \rangle \, ds
\]

holds for any \( \mu \in \mathcal{M}(\Omega) \) and \( f \in B_b(\Omega) \).
Proof. Fix $t > 0$. We show in the following that for every $f \in B_b(\Omega)$ and $\mu \in \mathcal{M}(\Omega)$ there exists $Rf \in B_b(\Omega)$ and $S\mu \in \mathcal{M}(\Omega)$ such that

\begin{equation}
\langle Rf, \mu \rangle = \int_0^t (T_s f, \mu) \, ds = \int_0^t \langle f, T'_s \mu \rangle \, ds = \langle f, S\mu \rangle.
\end{equation}

From this it follows easily that $R$ and $S$ are linear positive operators on $B_b(\Omega)$ and $\mathcal{M}(\Omega)$, respectively, and clearly $\langle S\mu, 1 \rangle = t\mu(\Omega)$ for every $\mu \in \mathcal{M}(\Omega)$. Since $(B_b(\Omega), \mathcal{M}(\Omega))$ is a norming dual pair, we may also conclude that $R = S'|_{B_b(\Omega)}$, which implies that $S \in \mathcal{L}($ $\mathcal{M}(\Omega), \sigma)$. It also follows from the construction that $RC_b(\Omega) \subseteq C_b(\Omega)$, which is why the operator $A_t := \frac{1}{t}S$ has all desired properties.

Let us briefly recall the definition of the strict topology on $C_b(\Omega)$. Denote by $\mathcal{F}_0(\Omega)$ the space of all bounded functions $f$ on $\Omega$ that vanish at infinity, i.e. for every $\varepsilon > 0$ there is a compact set $K \subseteq \Omega$ such that $|f(x)| < \varepsilon$ for all $x \in \Omega \setminus K$. The strict topology $\beta$ on $C_b(\Omega)$ is the locally convex topology generated by the set of seminorms $\{q_\varphi : \varphi \in \mathcal{F}_0(\Omega)\}$ where $q_\varphi(f) := \|\varphi f\|_\infty$. Since $\Omega$ is assumed to be metrizable, the strict topology is consistent with the duality, i.e. $(C_b(\Omega), \beta)' = \mathcal{M}(\Omega)$, and complete, see [42, Appx B].

By [14] Thm 6.5 there exists a countable set $F \subseteq C_b(\Omega)$ that separates the points of $\mathcal{M}(\Omega)$. Hence, it follows from the Hahn-Banach theorem applied on the locally convex space $(C_b(\Omega), \beta)$ that the span $F$ is $\beta$-dense in $C_b(\Omega)$. Since clearly the $\beta$-closure of span $F$ and span$_{\mathcal{M}}F$ coincide, this shows that $C_b(\Omega)$ is $\beta$-separable. In particular, for every $f \in C_b(\Omega)$ the $\|\cdot\|_\infty$-bounded function $t \mapsto T_t f$ is $\beta$-separable valued. It thus follows from [14] Thm 4.4 that for every $f \in C_b(\Omega)$ there exists an element $Rf \in C_b(\Omega)$ such that the first equality of (4.3) holds for all $\mu \in \mathcal{M}(\Omega)$.

Now fix $\mu \in \mathcal{M}(\Omega)$. It follows from [14] Lem 6.1 that even for every $f \in B_b(\Omega)$ the mapping $t \mapsto \langle T_t f, \mu \rangle$ is measurable and that there exists a function $Rf \in B_b(\Omega)$ satisfying the first equality of (4.3). Furthermore, we conclude from [14] Lem 4.6 that the linear functional $\varphi \in B_b(\Omega)'$ given by

\[ \varphi(f) := \int_0^t \langle f, T'_s \mu \rangle \, ds \]

is sequentially $\sigma(B_b(\Omega), \mathcal{M}(\Omega))$-continuous. Consequently, $(S\mu)(A) := \varphi(1_A)$ defines a measure on $\mathcal{F}(\Omega)$ and $\varphi(f) = \langle f, S\mu \rangle$ holds for any $f \in B_b(\Omega)$. This shows the last equality of (4.3) and completes the proof. \qed

We say that a weakly measurable one-parameter semigroup $T = (T_t)_{t \in (0,\infty)}$ of Markov-Feller operators is $B_t$-ergodic (or $C_t$-ergodic) if $P\mu := \lim_{t \to \infty} A_t \mu$ exists in the $\sigma(\mathcal{M}(\Omega), B_t(\Omega))$-topology (the $\sigma(\mathcal{M}(\Omega), C_t(\Omega))$-topology) for all $\mu \in \mathcal{M}(\Omega)$. In both cases it follows from [28] Lem 4.5 that $P$ defines a projection onto the fixed space of $T$, called the mean ergodic projection, and we refer to [28] for further reading on weakly ergodic semigroups.

In order to obtain a one-parameter semigroup of kernel operators we have to impose the extra condition that one operator is strong Feller, which is defined as follows.

**Definition 4.14.** Let $\Omega$ be a Polish space. A Markov operator $T \in \mathcal{L}($ $\mathcal{M}(\Omega), \sigma)$ is called

(a) **strong Feller** if $T' B_0(\Omega) \subseteq C_b(\Omega)$.

(b) **ultra Feller** if $\{ T' f : f \in B_0(\Omega), \|f\|_\infty \leq 1 \}$ is equi-continuous.

The notations of strong Feller, ultra Feller and kernel operator are related as follows:

**Remark 4.15.** It is well-known that the product of two strong Feller operators is ultra Feller, see [55] Thm 5.9. Moreover, as proven in [26] Thm 2.6, every ultra
Feller operator is a kernel operator. Therefore, if \( T \in \mathcal{L}(\mathcal{M}(\Omega,\sigma)) \) is a strong Feller operator, then \( T^2 \) is a kernel operator, i.e. \( T^2 \in (\mathcal{M}(\Omega) \otimes \mathcal{M}(\Omega))^\perp \).

We now obtain the following generalisation of a result by the first author from [26, Cor 3.7], where the one-parameter semigroup under consideration was assumed to be stochastically continuous.

We would like to point out that the definition of a Markov semigroup in [26, Sec 3] lacks erroneously the requirement that each operator is a Markov-Feller operator; if the semigroup merely consist of Markov operators, it does not follow from the arguments in [44] that the semigroup is integrable in the sense of [44, Def. 5.1], not even if it is stochastically continuous.

**Theorem 4.16.** Let \( \Omega \) be a Polish space and let \( \mathcal{F} = (T_t)_{t \in (0,\infty)} \subseteq \mathcal{L}(\mathcal{M}(\Omega),\sigma) \) be a weakly measurable one-parameter semigroup of Markov-Feller operators. Assume that for some \( s > 0 \) the operator \( T_s \) is strong Feller. Then the following assertions are equivalent:

(i) The fixed space of \( \mathcal{F} \) separates the fixed space of \( \mathcal{F}' \) in \( C_0(\Omega) \), i.e. for any \( f \in C_0(\Omega) \setminus \{0\} \) such that \( T_t f = f \) for all \( t > 0 \) there exists an \( \mathcal{F} \)-invariant measure \( \mu \in \mathcal{M}(\Omega) \) such that \( \langle \mu, f \rangle \neq 0 \).

(ii) \( \mathcal{F} \) is \( C_0 \)-ergodic.

(iii) \( \mathcal{F} \) is \( B_0 \)-ergodic.

(iv) \( \mathcal{F} \) converges strongly to its mean ergodic projection.

**Proof.** We are precisely in the situation of [26, Cor 3.7] except that \( \mathcal{F} \) is merely assumed to be weakly measurable instead of stochastically continuous. Inspecting the proof carefully shows that this assumption is only used twice in [26]: once to construct the averaging operator \( A_t \), which is now done by Proposition 4.13 and once in [26, Prop 3.5] to guarantee that restriction of \( \mathcal{F} \) to principal bands \( \{\zeta_k\}^\perp \) generated by certain invariant probability measures \( \zeta_k \) is strongly continuous in order to apply [26, Thm 3.3] to this restriction. Using Theorem 4.5 in place of the last mentioned theorem, this already proves the assertion.

It is also worth noting that the proof of [26, Prop 3.5] can further be simplified: one can omit the decomposition of the space into irreducible components by [26, Thm 3.4] since in contrast to [26, Thm 3.3], Theorem 4.5 does not require irreducibility.

**Spectral theoretical consequences.** In this subsection we are concerned with one-parameter semigroups of (partial) kernel operators that do not necessarily possess a quasi-interior fixed point. As we have seen in Remarks 3.9 we cannot expect them to be strongly convergent in general. Instead, we show that they satisfy a certain spectral condition which is necessary for strong convergence. To make this precise, we introduce the following terminology.

Let \( \mathcal{F} = (T_t)_{t \in (0,\infty)} \) be a one-parameter semigroup on a complex Banach space \( E \). A function \( (0,\infty) \ni t \mapsto \lambda_t \in \mathbb{C} \), also denoted by \( (\lambda_t)_{t \in (0,\infty)} \), is called an **eigenvalue** of \( \mathcal{F} \) if there exists a vector \( z \in E \setminus \{0\} \), called an **eigenvector**, such that \( T_t x = \lambda_t x \) for all \( t \in (0,\infty) \). An eigenvalue \( (\lambda_t)_{t \in (0,\infty)} \) of the semigroup \( \mathcal{F} \) is called **unimodular** if \( |\lambda_t| = 1 \) for all \( t \in (0,\infty) \). Now, let \( \mathcal{F} = (T_t)_{t \in (0,\infty)} \) be a one-parameter semigroup on a real Banach lattice \( E \) and denote by \( E_C \) the complexification of \( E \). Each operator \( T_t \) admits a canonical \( \mathbb{C} \)-linear extension \( E_C \to E_C \) which we denote again by \( T_t \) for simplicity. We say that \( (\lambda_t)_{t \in (0,\infty)} \subseteq \mathbb{C} \) is an eigenvalue of \( \mathcal{F} \) if it is an eigenvalue of the one-parameter semigroup consisting of the complex extensions of the operators \( T_t \).

In the following remark we explain how this general notion corresponds for \( C_0 \)-semigroup to the eigenvalues of the semigroup generator.
Remark 4.17. Let $\mathcal{T} = (T_t)_{t \in (0, \infty)}$ be a $C_0$-semigroup with generator $A$ on a complex Banach space $E$ and let $\lambda = (\lambda_t)_{t \in (0, \infty)}$ be a complex-valued function. Then $\lambda$ is an eigenvalue for $\mathcal{T}$ if and only if there exists an eigenvalue $\mu$ of $A$ such that $\lambda_t = e^{it\mu}$ for all $t \in (0, \infty)$.

Proof. $\Leftarrow$: If $0 \neq x \in \ker(\mu - A)$, then $(z - A)^{-1}x = (z - \mu)^{-1}x$ for all $z$ within the resolvent set of $A$. By the Euler formula for $C_0$-semigroups [22 Cor III.5.5] this implies that $T_t x = e^{it\mu}x$ for all $t \in (0, \infty)$.

$\Rightarrow$: Let $x \in E \setminus \{0\}$ such that $T_t x = \lambda_t x$ for all $t \in (0, \infty)$. With the definition $\lambda_0 := 1$ it follows from the semigroup law that $\lambda_{t+s} = \lambda_t \lambda_s$ for all $t, s \in [0, \infty)$. Moreover, the mapping $[0, \infty) \ni t \mapsto \lambda_t \in \mathbb{C}$ is continuous since $\mathcal{T}$ is a $C_0$-semigroup. According to [22 Thm I.1.4] this implies that there exists a number $\mu \in \mathbb{C}$ such that $\lambda_t = e^{it\mu}$ for all $t \in [0, \infty)$. It now follows from the very definition of the generator $A$ that $\mu$ is an eigenvalue of $A$ with eigenvector $x$. □

The present subsection is devoted to proving criteria for the absence of non-trivial unimodular eigenvalues. This property is of interest as it is closely related to the asymptotic behaviour of the semigroup. Let us briefly illustrate this with the following proposition:

Proposition 4.18. Let $\mathcal{T} = (T_t)_{t \in (0, \infty)}$ be a one-parameter semigroup in a real or complex Banach space $E$ which does not possess any unimodular eigenvalue except possibly $(1)_{t \in (0, \infty)}$

(a) If $\mathcal{T}$ has relatively compact orbits, then $\mathcal{T}$ is strongly convergent.
(b) If the mapping $[0, \infty) \ni t \mapsto T_t \in \mathcal{L}(E)$ is strongly continuous, then $\mathcal{T}$ does not admit non-trivial periodic orbits.

Here, way say that $\mathcal{T}$ admits a non-trivial periodic orbit if there exists a vector $x \in E$ and a time $t_0 > 0$ such that $T_t x = T_{t_0+t} x$ for all $x \in (0, \infty)$ while the mapping $[0, \infty) \ni t \mapsto T_t x \in E$ is not constant; in this case, $t_0$ is called a period of this orbit.

Proof of Proposition 4.18. We may assume throughout the proof that the scalar field is complex; if this is not the case we can replace $E$ with a complexification.

(a) This is a consequence of [10] Thm 4.5 in §2.

(b) Suppose that the orbit $[0, \infty) \ni t \mapsto T_t x \in E$ is not constant but periodic with $t_0 > 0$ as a period. Denote by $F \subseteq E$ the closed linear span of $\{T_t x : t \in (0, \infty)\}$. Then $F$ is non-zero and invariant with respect to $\mathcal{T}$. The restriction $\mathcal{T} := (T_t|_F)_{t \in (0, \infty)}$ of $\mathcal{T}$ to $F$ is periodic with $t_0$ as a period, but it is not constant since $t \mapsto T_t|_F x$ is not constant. Moreover, $T_{t_0}|_F$ is the identity operator on $F$, so $\mathcal{T}$ extends to a periodic $C_0$-semigroup on $F$. Hence, we conclude from [22 Thm IV.2.26] that the generator of $\mathcal{T}$ possesses an eigenvalue $i\beta \in i\mathbb{R} \setminus \{0\}$. It now follows from Remark 4.17 that $(e^{it\beta})_{t \in (0, \infty)}$ is an eigenvalue of $\mathcal{T}$ and hence of $\mathcal{F}$.

We point out that assertion (b) of the above proposition fails if one drops the strong continuity assumption. A counterexample is again provided by the shift semigroup on an $L^p$-space over the index set $\mathbb{R}$.

Now we give sufficient criteria for a positive semigroup to have no non-trivial unimodular eigenvalue. Recall that a Banach lattice $E$ is called a KB-space if every norm-bounded increasing net (or equivalently: sequence) in $E$ converges. Important examples of KB-spaces are reflexive Banach lattices and AL-spaces; in particular, each $L^p$-space is a KB-space for $1 \leq p < \infty$.

Theorem 4.19. Let $\mathcal{T} := (T_t)_{t \in (0, \infty)}$ be a bounded and positive one-parameter semigroup on a Banach lattice $E$ such that $T_s$ is AM-compact for some $s \in S$. 

If every super fixed point of $\mathcal{T}$ is a fixed point of $\mathcal{T}$ or if $E$ is a KB-space, then $(1)_{t \in (0, \infty)}$ is the only possible unimodular eigenvalue of $\mathcal{T}$.

**Proof.** Assume that $(\lambda_t)_{t \in (0, \infty)}$ is a unimodular eigenvalue for $\mathcal{T}$. Denote by $z \in E_C \setminus \{0\}$ a corresponding eigenvector, where $E_C$ is the Banach lattice complexification of $E$. We have to show that $\lambda_t = 1$ for all $t \in (0, \infty)$.

We first prove that there exists a fixed point $f \in E_+^+$ for $\mathcal{T}$ which fulfills $f \geq |z|$. Since $|z| = |\lambda_t z| = |T_t z| \leq T_t |z|$ for each $t \in (0, \infty)$, $|z|$ is a super-fixed point of $\mathcal{T}$. Hence, under the assumption that every super fixed point of $\mathcal{T}$ is a fixed point we have found our fixed point $f := |z|$. If instead $E$ is a KB-space, the increasing and norm bounded net $(T_t |z|)_{t \in (0, \infty)} \subseteq E$ converges to a vector $f \in E_+^+$, which is clearly a fixed point of $\mathcal{T}$ and dominates $|z|$. Hence, we have found a fixed point with the desired properties in each of both cases.

Let $F$ denote the closure of the principal ideal $E_f$ in $E$ and let $F_C := F + iF \subseteq E_C$. Since $E_f$ is $\mathcal{T}$-invariant, so are $F$ and $F_C$. The restriction of $\mathcal{T}$ to $F$ has $f$ as a quasi-interior fixed point and the restriction of $T_s$ to $F$ is clearly AM-compact. Thus, $\mathcal{T}$ fulfills all assumptions of Theorem 5.7 which implies that that $\lim_{t \to \infty} T_t g$ exists for all $g \in F$ and hence also for all $g \in F_C$. In particular, $(T_t z)$ converges as $t$ tends to infinity. Hence, if we fix $t \in (0, \infty)$, then the sequence $(T_t |z|) = (\lambda_t^n z)$ converges. Since the complex number $\lambda_t$ has modulus 1 and $z \neq 0$, this implies that $\lambda_t = 1$. □

If the semigroup only dominates an AM-compact operator, we have the following result, which is a consequence of Theorem 3.11. The proof is very similar to that of Theorem 4.19.

**Theorem 4.20.** Let $\mathcal{T} := (T_t)_{t \in (0, \infty)}$ be a bounded and positive one-parameter semigroup on a Banach lattice $E$ with order continuous norm. Assume that $\mathcal{T}$ has the following two properties:

(a) Every super fixed point of $\mathcal{T}$ is a fixed point of $\mathcal{T}$.

(b) For every fixed point $x > 0$ of $\mathcal{T}$ there exists $s \in (0, \infty)$ and an AM-compact operator $K \geq 0$ such that $T_s \geq K$ and $K x > 0$.

Then $(1)_{t \in S}$ is the only possible unimodular eigenvalue of $\mathcal{T}$.

**Proof.** Assume that $(\lambda_t)_{t \in (0, \infty)}$ is a unimodular eigenvalue for $\mathcal{T}$, denote by $E_C$ the Banach lattice complexification of $E$ and let $z \in E_C \setminus \{0\}$ be an eigenvector corresponding to $(\lambda_t)_{t \in (0, \infty)}$. As in the proof of Theorem 4.19 $|z|$ is a fixed point of $\mathcal{T}$. Let $F$ denote the closure of the principal ideal $E_{|z|}$ and set $F_C := F + iF$. Then $F$, and thus $F_C$, is $\mathcal{T}$-invariant and it is easy to verify that the restriction of $\mathcal{T}$ to $F$ satisfies all assumptions of Theorem 3.11. Thus, $\lim_{t \to \infty} T_t g$ exists for all $g \in F$ and hence also for all $g \in F_C$. In particular, $(T_t z)$ converges as $t$ tends to infinity.

As in the proof of Theorem 4.19 this implies that $\lambda_t = 1$ for all $t \in (0, \infty)$. □

**Remark 4.21.** Theorems 4.19 and 4.20 remain true if we replace $(T_t)_{t \in (0, \infty)}$ with a representation of an arbitrary commutative semigroup (where one has to use the obvious generalisation of the notion eigenvalue). The proofs are the same except for the last step where one has to employ a slightly more involved argument since a commutative semigroup need not be Archimedean, in general. For instance, the following observation does the job:

If $t \mapsto \lambda_t$ is a semigroup homomorphism from a commutative semigroup $S$ into the complex unit circle and if the net $(\lambda_t)_{t \in S}$ converges, then $\lambda_t = 1$ for all $t \in S$.

To see this, first note that $\lim_{t \to \infty} \lambda_t = 1$. Let $\varepsilon > 0$ and choose $t_0 \in S$ such that $|\lambda_t - 1| < \varepsilon$ for all $t \geq t_0$. Then, $|\lambda_{t_0} - 1| < \varepsilon$ and $|\lambda_{t_0+t} - 1 < \varepsilon|$ for all $t \in S$. We
thus conclude that
\[ |\lambda_t - 1| = |\lambda_{t_0 + t} - \lambda_{t_0}| < 2\varepsilon \]
for all \( t \in S \). Since \( \varepsilon > 0 \) was arbitrary, all \( \lambda_t \) are equal to 1.

5. Examples and Applications

Many consequences of our main results have already been discussed in Section 4. Now we briefly present a few consequences of more concrete type. First we present two situations in which a perturbed \( C_0 \)-semigroups consists of partial kernel operators. In doing so, we restrict ourselves to simple toy examples and leave the study of more sophisticated models from the applied sciences to future articles. We conclude this section with two more abstract implications of our results for one-parameter semigroups over measure spaces with atoms and for one-parameter semigroups on spaces of continuous functions.

Transport processes and perturbations by kernel operators. If the generator of a positive \( C_0 \)-semigroup is perturbed by a kernel operator, then it follows from the Dyson–Phillips series representation that the perturbed semigroup dominates a kernel operator. This has already been observed by Pichór and Rudnicki in \[56\] See 2.3. In the subsequent theorem we recall this argument in a slightly more general form and we demonstrate by an example of a simple transport equation how this observation can be used in conjunction with our results from Section 4 in order to prove convergence of the perturbed semigroup without irreducibility assumptions.

We call a bounded operator \( M \) on an \( L^p \)-space over a \( \sigma \)-finite measure space a multiplication operator if there exists a function \( m \in L^\infty \) such that \( Mf = mf \) for all \( f \in L^p \). In this case, the operator norm of \( M \) equals the \( L^\infty \)-norm of \( m \) and therefore \( -\|M\| \text{id} \leq M \leq \|M\| \text{id} \).

**Theorem 5.1.** Let \( E \) be a real-valued \( L^p \)-space over a \( \sigma \)-finite measure space for \( p \in [1, \infty) \) and let \( \mathcal{F} = (T_t)_{t \in [0, \infty)} \) be a positive \( C_0 \)-semigroup on \( E \). Let \( B \in \mathcal{L}(E) \) be a positive kernel operator and let \( M \in \mathcal{L}(E) \) be a multiplication operator; denote by \( \mathcal{F} := (S_t)_{t \in [0, \infty)} \) the \( C_0 \)-semigroup generated by \( A + B + M \). Then \( \mathcal{F} \) is positive and for every \( t > 0 \) the operator \( S_t \) dominates a kernel operator \( K_t \in \mathcal{L}(E) \) given by

\[ K_t f = e^{-\|M\|t} \int_0^t T_{t-s} B T_s f \, ds \]

for all \( f \in E \).

**Proof.** Let \( \mathcal{F} = (R_t)_{t \geq 0} \) denote the \( C_0 \)-semigroup on \( E \) generated by \( A + B + M + \|M\| \text{id} \). Since the operator \( M + \|M\| \text{id} \) is positive, it follows from the Dyson–Phillips series representation \[22\] Thm III.1.10 that

\[ R_t f \geq \int_0^t T(t-s)(B + M + \|M\| \text{id})T(s)f \, ds \geq \int_0^t T(t-s)BT(s)f \, ds \]

for all \( f \in E \) and all \( t > 0 \). This shows that \( S_t = e^{-\|M\|t}R_t \) is positive and dominates the operator \( K_t \) for each \( t > 0 \). Moreover, since \( B \) is a kernel operator so is \( T(t-s)BT(s) \) for every \( t \geq 0 \) and every \( s \in [0, t] \) according to \[5\] Prop 1.12(b)]. Hence, it follows from \[5\] Thm 2.1] that \( K_t \) is a kernel operator.

If for some \( t > 0 \) the kernel operator \( K_t \) in the above theorem is non-zero, then \( T_t \) is a partial kernel operator and we may apply our results from Section 4.

It should be pointed out that if \( T_t \) is not a kernel operator, then the perturbed operator \( S_t \) is not a kernel operator, either. This follows since \( T_t \) occurs as a
summand in the Dyson–Phillips series for $R_t$ in the above proof. Hence, in case that $K_t \neq 0$ we have a simple example here where the semigroup consists of partial kernel operators but not of kernel operators.

Let us now demonstrate how the above observation can be combined with Corollary 4.10 to prove strong convergence of a perturbed transport semigroup.

The long term behaviour of various kinds of kinetic and transport equations is an important subject of research. Apparently, those applications were a motivation for the development of the main result in [56] which is one of the predecessors of Theorem 3.11 above. We also refer to [111, 113, 53] for more recent articles dealing with this subject.

Although we shall not engage in a thorough study of this subject here, we present at least a simple instance of a transport equation as an interesting toy example. The equation can be thought of as modeling a population which is subject to a local death rate, a non-local reproduction behaviour and a certain transport process. From a mathematical viewpoint, the probably most interesting feature of the model is that the semigroup which describes the time evolution is not in general irreducible. Hence, we cannot employ results from the literature such as [56, Thm 1] and [24, Thm 4.2] to study its long term behaviour. Instead, we use Corollary 4.10 for this purpose.

**Example 5.2.** Let $\emptyset \neq \Omega \subseteq \mathbb{R}^d$ be a bounded and open set and endow its closure $\overline{\Omega}$ with the Lebesgue measure. We intend to describe the time evolution of a population on $\overline{\Omega}$ whose reproduction is described by a certain non-local operator $B$ and which is subject to a constant death rate. More precisely, we model the population density at time $t$ as a function $u(t) \in L^1(\overline{\Omega})$ and we consider the evolution equation

$$\frac{d}{dt} u(t) = Bu(t) - mu(t),$$

where $m \in (0, \infty)$ is a constant which describes the death rate of the population and $B \in \mathcal{L}(L^1(\overline{\Omega}))_+$ is a kernel operator. We assume that $B'1 = m1$, meaning that the total birth rate of the population equals its death rate, i.e., the integral over the population is constant over time. To keep the example as simple as possible, we also assume that $B1$ is a multiple of $1$, meaning that a population which is equi-distributed in space also leads to a birth rate which is equi-distributed in space; given our assumption on $B'$, this implies that $B1 = m1$.

Denote by $M \in \mathcal{L}(L^1(\overline{\Omega}))$ the multiplication operator with symbol $m$, meaning that $Mf = mf$ for all $f \in L^1(\overline{\Omega})$. The operator $B - M$ is bounded and thus generates a norm-continuous $C_0$-semigroup $\mathcal{T} = (T_t)_{t \in [0, \infty)}$ on $L^1(\overline{\Omega})$. According to our assumptions, $\mathcal{T}$ is positive and contractive (in fact, it even consists of Markov operators) and possesses $1$ as a fixed point. It follows from the well-known spectral theory of positive semigroups that $(T_t)_{t \in [0, \infty)}$ converges strongly as $t \to \infty$, without using that $B$ is a kernel operator. Indeed, the spectrum of the generator $B - M$ intersects the imaginary axis only in $0$ according to [111] Cor C-III.2.13. Since $\mathcal{T}$ possesses a quasi-interior fixed point and $L^1(\overline{\Omega})$ has order continuous norm, the semigroup $\mathcal{T}$ has weakly relatively compact orbits and is thus totally ergodic. Hence, $(T_t)_{t \in [0, \infty)}$ converges strongly as $t \to \infty$ by [111] Theorems 5.5.5 and 5.4.6. So far, we only used classical results and did not have to employ our theory.

Now, we extend the model in the following way: we assume that the medium on which the population lives (which could, for instance, be water) is subject to a transport process while the conditions that influence the reproduction of the population are not, which means that they do not move together with the medium. We model the transport process as a family of continuous mappings $\varphi_t: \overline{\Omega} \to \overline{\Omega}$, $t \in \mathbb{R}$, such that $\varphi_0 = \text{id}_\Omega$, $\varphi_{s+t} = \varphi_s \circ \varphi_t$ for all $s, t \in \mathbb{R}$ and such that $t \mapsto \varphi_t(\omega)$
is continuous for each $\omega \in \Omega$. Provided that the medium is not compressible, we may also assume that each mapping $\varphi_t$ is measure preserving. The transport of the population induced by the flow $(\varphi_t)_{t \in \mathbb{R}}$ can be described by a family of linear operators $\mathcal{U} := (U_t)_{t \in \mathbb{R}}$ which are given by $U_t u(\omega) = u(\varphi_t(\omega))$ for all $t \in \mathbb{R}$ and all $u \in L^1(\Omega)$. By [38, Lem B-II.3.2], we see that $\mathcal{U}$ is a $C_0$-group on $L^1(\Omega)$. The function $1$ is clearly a fixed point of $\mathcal{U}$ and, since each mapping $\varphi_t$ is measure preserving, the group is isometric.

Since the population is moved by the flow $(\varphi_t)_{t \in \mathbb{R}}$ on $\Omega$ while the conditions for reproduction are not, the evolution of the population can now be described by the equation

$$\frac{d}{dt} u(t) = Au(t) + Bu(t) - mu(t),$$

where $A$ denotes the generator of $\mathcal{U}$. By Theorem 5.1 the operator $A + B - M$ generates a positive $C_0$-semigroup $\mathcal{S} := (S_t)_{t \in [0,\infty)}$ on $L^1(\Omega)$ and for each $t > 0$ the operator $S_t$ dominates the positive kernel operator $K_t$ given by

$$K_t f = e^{-mt} \int_0^t U_{t-s} B U_s f \, ds$$

for all $f \in L^1(\Omega)$. By our assumptions on $B$ and $M$, the semigroup $\mathcal{S}$ is isometric on the positive cone and possesses $1$ as a fixed point. Moreover, note that $K_t 1 = te^{-mt} 1$ for every $t > 0$, and therefore $K_t f > 0$ whenever $f > 0$. Hence the condition (b) of Corollary 4.10 is fulfilled and we conclude that $(S_t)_{t \in (0,\infty)}$ converges strongly as $t \to \infty$.

**Perturbations by lattice homomorphisms.** In Theorem 5.1 we observed that perturbing the generator of a semigroup by a kernel operator results in a semigroup which dominates a kernel operator. This observation is, as useful as it is, perhaps not particularly surprising. In what follows, we briefly discuss another more unexpected phenomenon. It was already shown in [56, p. 677] that a perturbed semigroup can contain partial kernel operators even if neither the original semigroup nor the perturbation is a partial kernel operator. This phenomenon occurred in several applications in [56, Sec 3]. Here, we give another example of this type, which is not immediately motivated by an application, but it is very elementary and quite illustrating.

Throughout this subsection let $\mathbb{T}$ denote the complex unit circle endowed with the Lebesgue measure and fix $p \in [1,\infty)$. Let $\mathcal{F} = (T_t)_{t \in [0,\infty)}$ be the rotation semigroup on $L^p(\mathbb{T})$, meaning that $T_t f(\cdot) = f(e^{it \cdot})$ for all $f \in L^p(\mathbb{T})$ and all $t \in (0,\infty)$, and denote its generator by $A$. Now define $\varphi: \mathbb{T} \to \mathbb{T}$ by $\varphi(z) := z^2$ and let $B_\varphi \in \mathcal{F}(L^p(\mathbb{T}))$ denote the Koopman operator induced by $\varphi$, i.e. $B_\varphi f := f \circ \varphi$ for all $f \in L^p(\mathbb{T})$. Note that both the semigroup operators $T_t$ and the perturbation $B_\varphi$ are lattice homomorphisms and thus disjoint from the band of kernel operators (see [38, Sec 4.3]). Yet, we are going to show that the $C_0$-semigroup $\mathcal{S} := (S_t)_{t \in (0,\infty)}$ generated by $A + B_\varphi$ contains a partial kernel operator.

By considering the first coefficient of the Dyson–Phillips series expansion of $\mathcal{S}$, cf. [22, Thm III.1.10], we obtain for all $t > 0$ and $f \in L^p(\mathbb{T})_+$ that

$$(5.1) \quad S_t f \geq K_t f := \int_0^t T_{t-s} B_\varphi T_s f \, ds.$$  

Now we choose $t = 2\pi$ and show that $K_{2\pi}$ is a kernel operator. To this end, pick a continuous function $f \in C(\mathbb{T})$. Since the mapping

$$s \mapsto T_{2\pi-s} B_\varphi T_s f = f(e^{i(2\pi-s)}(\cdot)) = f(e^{i(4\pi-s)}(\cdot))$$


is continuous from \([0,2\pi]\) to \(C(\mathbb{T})\), the integral in \([53,1]\) coincides with the \(C(\mathbb{T})\)-valued Riemann integral and therefore
\[
K_{2\pi}f(z) = \int_0^{2\pi} f(e^{i(t-s)}z) \, ds = \int_0^{2\pi} f(e^{is}z) \, ds = \int_0^{2\pi} f(e^{is}) \, ds = \int_T f(w) \, dw
\]
for almost all \(z \in \mathbb{T}\). Since \(C(\mathbb{T})\) is dense in \(L^p(\mathbb{T})\), we obtain that \(K_{2\pi}f = \langle f, 1 \rangle \mathbb{1}\) for all \(f \in L^p(\mathbb{T})\). Thus, \(S_{2\pi}\) dominates the non-zero kernel operator \(K_{2\pi}\).

A similar argument also works for the mapping \(\varphi(z) := z^n\) for any fixed \(n \in \mathbb{Z}\). It would be interesting to understand this phenomenon at a more general level, i.e. to find sufficient (and abstract) conditions for a perturbed semigroup to contain a partial kernel operator even if neither the original semigroup nor the perturbation is a partial kernel operator.

**Semigroups over measure spaces with atoms.** In the following we briefly discuss semigroups over \(L^p\)-spaces in case that the underlying measure space contains an atom. This simple assumption has astonishing consequences for the asymptotic behaviour of the semigroup and is fulfilled in several applications.

Let \((\Omega, \mu)\) be a \(\sigma\)-finite measure space. A point \(\omega_0 \in \Omega\) is called an atom if \(\{\omega_0\}\) is measurable and \(\mu(\{\omega_0\}) > 0\). Note that \(\mu(\{\omega_0\}) < \infty\) since \((\Omega, \mu)\) is \(\sigma\)-finite.

**Theorem 5.3.** Let \((\Omega, \mu)\) be a \(\sigma\)-finite measure space with an atom \(\omega_0 \in \Omega\). Let \(p \in [1,\infty)\) and let \(\mathcal{T} := (T_t)_{t \in (0,\infty)}\) be a bounded and positive semigroup on \(E := L^p(\Omega, \mu)\). If \(\mathcal{T}\) is irreducible and has a non-zero fixed-point, then \(\mathcal{T}\) is strongly convergent.

**Proof.** The assertion is obvious if \(\dim E = 1\), so assume that \(\dim E \geq 2\). Let \(e: \Omega \to \mathbb{R}\) be the indicator function of the singleton \(\{\omega_0\}\). Denote by \(B := \{e\}^\perp\) the band generated by \(e\) and by \(P \in \mathcal{Z}(E)\) the band projection onto \(B\). Since \(B\) is one-dimensional and \(\dim E \geq 2\), the orthogonal band \(B^\perp\) is non-zero and can therefore not be invariant under the semigroup \(\mathcal{T}\). Thus, there exists \(t \in (0,\infty)\) such that \(PT_t \neq 0\). On the other hand, \(PT_t\) has rank 1 and is thus a kernel operator. Since \(T_t\) dominates \(PT_t\), the assertion follows from Corollary 5.9. □

Note that in the important case of \(C_0\)-semigroups of Markov operators on \(L^1\)-spaces, the above theorem also follows from [56, Thm 1] although this was not explicitly stated in [56].

The assumption that the underlying measure space contains an atom is frequently fulfilled in models from queueing and reliability theory. The state spaces in such models are often isometrically lattice isomorphic to an \(L^1\)-space over a measure space containing an atom, and the time evolution of the system is described by means of a \(C_0\)-semigroup consisting of Markov operators; see e.g. [39] for an introduction. For several of these models and/or under sufficiently strong assumptions, one can prove convergence of the semigroup with respect to the operator norm as e.g. in [71, 57, 70]. On the other hand, there also exist models for which one can only show strong stability; in [55, 54, 56] such results are proved by employing spectral analysis of the generator and a version of the ABLV-theorem. Theorem 5.3 allows for a simplification of several of these arguments.

**Semigroups on spaces of continuous functions.** We complete our discussion with an analysis of one-parameter semigroups on spaces of continuous functions or, more generally, on AM-spaces. Let \(L\) be a locally compact Hausdorff space and let \(C_0(L)\) denote the space of real-valued continuous functions on \(L\) that vanish at infinity. We would like to apply our results to prove strong convergence of positive semigroups on such spaces. Unfortunately though, the norm of the space \(C_0(L)\) is in general not order continuous, so we cannot apply Theorem 8.11 nor...
any of its consequences from Section 4. On the other hand, Theorem 5.4 and its consequences are not particularly well-suited for this type of spaces, either: in the important special case where \( L \) is compact, each AM-compact operator is automatically compact and for positive semigroups containing a compact operator one even has convergence in operator norm under very weak assumptions, cf. [17, Thm 4].

Still, we can derive new and non-trivial results for positive semigroups on AM-spaces by duality arguments. Here, we benefit from the fact that our general theory works without continuity assumptions on the time parameter since the dual of a semigroup \( \mathcal{S} \) on an AM-space is not strongly continuous in general, even if \( \mathcal{S} \) itself is a \( C_0 \)-semigroup.

Recall that prototypical examples for AM-spaces are constituted by \( C(K) \) for a compact Hausdorff space \( K \) and, more generally, by \( C_0(L) \) for a locally compact Hausdorff space \( L \). Also recall the notion of an eigenvalue of a general one-parameter semigroup, as defined before Theorem 4.10.

**Theorem 5.4.** Let \( \mathcal{S} = (T_t)_{t \in (0, \infty)} \) be a bounded, positive and irreducible semigroup on an AM-space \( E \). If there exists a time \( s \in (0, \infty) \) and a compact operator \( K \in \mathcal{L}(E) \) such that \( 0 < K \leq T_s \), then \((1)_{t \in (0, \infty)}\) is the only possible unimodular eigenvalue of \( \mathcal{S} \).

Recall from Proposition 4.18 that non-existence of unimodular eigenvalues as given in the above theorem is related to the long term behaviour of the semigroup; in particular, it implies non-existence of periodic orbits under appropriate regularity assumptions. In order to prove the theorem, we make use of the following lemma.

**Lemma 5.5.** Let \( \mathcal{S} = (T_t)_{t \in (0, \infty)} \) be a bounded one-parameter semigroup on a complex Banach space \( E \). Then any unimodular eigenvalue of \( \mathcal{S} \) is also an eigenvalue of the dual semigroup \( \mathcal{S}' := (T'_t)_{t \in (0, \infty)} \subseteq \mathcal{L}(E') \).

**Proof.** Let \( \langle \lambda_t \rangle_{t \in (0, \infty)} \) be an unimodular eigenvalue of \( \mathcal{S} \) with a corresponding eigenvector \( z \in E \setminus \{0\} \) and fix a functional \( z' \in E' \) such that \( \langle z', z \rangle = 1 \). The additive semigroup \((0, \infty)\) is commutative and thus amenable [18, p. 178], so we find a shift invariant positive functional \( \psi \in E'((0, \infty); \mathbb{C})' \) which maps the constant function with value 1 to 1. Now, define \( z' \in E' \) by

\[
\langle z', f \rangle = \psi((z', \lambda_t^{-1}T_t f))_{t \in (0, \infty)}
\]

for all \( f \in E \). We have \( \langle z', z \rangle = 1 \), so \( z' \neq 0 \). Moreover, for every \( s \in (0, \infty) \) and every \( f \in E \) we obtain

\[
\langle T'_sz', f \rangle = \langle z', T_s f \rangle = \psi((z', \lambda_t^{-1}T_t f))_{t \in (0, \infty)}
\]

\[
= \lambda_s \psi((z', \lambda_s^{-1}T_{s+s} f))_{t \in (0, \infty)} = \lambda_s \langle z', f \rangle,
\]

where the last equality follows from the shift invariance of \( \psi \). Hence, \( T'_sz' = \lambda_s z' \) for every \( s \in (0, \infty) \), which proves the assertion. \( \square \)

**Proof of Theorem 5.4.** Let \( \langle \lambda_t \rangle_{t \in (0, \infty)} \) be a unimodular eigenvalue of \( \mathcal{S} \). Then it follows from Lemma 5.5 that \( \langle \lambda_t \rangle_{t \in (0, \infty)} \) is also an eigenvalue of the dual semigroup \( \mathcal{S}' := (T'_t)_{t \in (0, \infty)} \subseteq \mathcal{L}(E') \). We check that \( \mathcal{S}' \) satisfies all assumptions of Theorem 4.10.

Since \( E' \) is an AL-space, its norm is order continuous; see Corollary 2.4.13 and the discussion below Definition 2.4.11 in [51]. To see that every super fixed point of \( \mathcal{S}' \) is a fixed point, one argues as follows. Let \( z \in E_C \setminus \{0\} \) be an eigenvector of \( \mathcal{S} \) for the eigenvalue \( \langle \lambda_t \rangle_{t \in (0, \infty)} \), where \( E_C \) denotes the Banach lattice complexification of \( E \). It is easy to see that the modulus \( |z| \) is a super fixed point of \( \mathcal{S} \). Since \( \mathcal{S} \)
is irreducible, $|z|$ is a fixed point of $\mathcal{T}$ by Proposition 5.13(c) and a quasi-interior point of $E_+$. Hence, we have $\langle \varphi, |z| \rangle > 0$ for all $\varphi \in E'_+ \setminus \{0\}$, so $|z| \in E \subseteq E''$ acts as a strictly positive functional on $E'$. Since $|z|$ is clearly a fixed point of $(T_t')_{t \in (0, \infty)} \subseteq \mathcal{L}(E'')$, it follows from Proposition 5.13(b) that every super fixed point of $\mathcal{T}'$ is indeed a fixed point.

It remains to show that assumption (b) of Theorem 4.20 is fulfilled for $\mathcal{T}'$, so let $0 < \varphi \in E'$ be a fixed point of $\mathcal{T}'$. Then $\varphi$ is strictly positive since $\mathcal{T}$ is irreducible. The operator $K' \in \mathcal{L}(E')$ is compact, thus AM-compact, and fulfills $0 \leq K' \leq T'$. Since $K > 0$, there exists $0 < f \in E$ such that $Kf > 0$ and hence $(K'\varphi, f) = \langle \varphi, Kf \rangle > 0$. Therefore, $K'\varphi > 0$ and all assumptions of Theorem 4.20 are fulfilled. This implies that $\lambda_t = 1$ for all $t \in (0, \infty)$. \hfill $\square$

We have already stressed above that, even in case that $\mathcal{T}$ is a $C_0$-semigroup, the preceding proof requires results for semigroups with less time regularity since the dual semigroup $\mathcal{T}'$ is in general not strongly continuous. The proof of Theorem 5.3 also demonstrates another advantage of our approach: since the dual space of an AM-space is, in general, quite large we cannot expect the dual semigroup $\mathcal{T}'$ to be irreducible, even though the semigroup $\mathcal{T}$ itself is. Hence, the above proof only works because Theorem 4.20 (respectively, the underlying Theorem 3.11) is formulated in a very general version which does not require irreducibility.

Let us close the article with the following consequence of Theorem 5.4. If a locally compact Hausdorff space $L$ contains an isolated point and if $E = C_0(L)$, then the assumption in Theorem 5.4 that $T_s$ dominate a non-zero compact operator is automatically fulfilled and we obtain the following corollary.

**Corollary 5.6.** Let $L$ be a locally compact Hausdorff space which contains an isolated point and let $\mathcal{T} = (T_t)_{t \in (0, \infty)}$ be a bounded, positive and irreducible semigroup on $E := C_0(L)$. Then $(1)_{t \in (0, \infty)}$ is the only possible unimodular eigenvalue of $\mathcal{T}$.

**Proof.** We may assume that $\dim E \geq 2$. Let $\omega$ be an isolated point in $L$. Then the indicator function $1_{\{\omega\}}$ is continuous from $L$ to $\mathbb{R}$ and thus, $Pf := 1_{\{\omega\}}f$ for all $f \in E$ defines an operator from $E$ to $E$; clearly, $P$ is a band projection and compact. The subspace of $E$ spanned by $1_{\{\omega\}}$ is a non-trivial closed ideal in $E$, so it cannot be $\mathcal{T}$-invariant. In particular, there exists a time $s \in (0, \infty)$ such that $T_s 1_{\{\omega\}} > 0$.

Now, define $K := T_s P$. Then $K$ is a compact operator, fulfills $0 \leq K \leq T_s$ and $K$ is non-zero since $K 1_{\{\omega\}} = T_s 1_{\{\omega\}} > 0$. Hence, the assumptions of Theorem 5.3 are fulfilled and the assertion follows. \hfill $\square$

**Appendix A. AM-Compactness of Integral Operators**

In order to improve the readability of the article, we use this appendix to give a direct proof of the fact that every integral operator on $L^p$ is AM-compact. This fact was already mentioned in Section 4 as a consequence Propositions 4.3 and 4.4. However, the references we gave for these propositions are rather abstract while the proof below is of a more concrete nature.

**Proposition A.1.** Let $(\Omega_1, \mu_1)$ and $(\Omega_2, \mu_2)$ be $\sigma$-finite measure spaces and $p, q \in [1, \infty)$. Let $T : L^p(\Omega_1, \mu_1) \to L^q(\Omega_2, \mu_2)$ be a positive integral operator with integral kernel $k : \Omega_1 \times \Omega_2 \to \mathbb{R}$ according to Definition 4.3. Then $T$ is AM-compact.

**Proof.** First note that since $T$ is positive, the kernel function $k$ is positive almost everywhere by [51, Prop. 3.3.1]. Let $0 \leq f \in L^p(\Omega_1, \mu_1)$ and $(f_n) \subseteq [0, f]$. After passing to a subsequence, we may assume that $(f_n)$ converges weakly to some $g \in [0, f]$ by the order continuity of the $L^p$-norm. As $\Omega_1 \times \Omega_2$ is $\sigma$-finite, we find a monotonically increasing sequence $(k_m)$ of positive and simple functions on $\Omega_1 \times \Omega_2$
such that \((\mu_1 \otimes \mu_2)(\{k_m > 0\}) < \infty\) and \(\lim k_m(x, y) = k(x, y)\) almost everywhere. Since, in addition, each \(k_m\) is bounded, we have that \(k_m(\cdot, y) \in L^p(\Omega_1, \mu_1)\) for each \(m \in \mathbb{N}\) and almost every \(y \in \Omega_2\), where \(\frac{1}{p} + \frac{1}{p'} = 1\). Hence,

\[
\lim_{n \to \infty} \int_{\Omega_1} f_n(x)k_m(x, y) \, d\mu_1(x) = \int_{\Omega_1} g(x)k_m(x, y) \, d\mu_1(x)
\]

for almost every \(y \in \Omega_2\) by the definition of \(g\).

Now fix \(y \in \Omega_2\) such that \(k_m(\cdot, y) \in L^{p'}(\Omega_1, \mu_1)\), \(\lim k_m(x, y) = k(x, y)\) for \(\mu_1\)-almost every \(x \in \Omega_1\) and such that \((Tg)(y), (Tf)(y)\) and all \((Tf_n)(y)\) are given according to \([11]\). Let \(\varepsilon > 0\). By dominated convergence we find \(m \in \mathbb{N}\) such that

\[
0 \leq \int_{\Omega_1} f(x)(k(x, y) - k_m(x, y)) \, d\mu_1(x) \leq \varepsilon
\]

and hence, since \(g, f_n \in [0, f]\), this implies that

\[
0 \leq \int_{\Omega_1} g(x)(k(x, y) - k_m(x, y)) \, d\mu_1(x) \leq \varepsilon
\]

and

\[
0 \leq \int_{\Omega_1} f_n(x)(k(x, y) - k_m(x, y)) \, d\mu_1(x) \leq \varepsilon
\]

for all \(n \in \mathbb{N}\). Now pick \(n_0 \in \mathbb{N}\) such that \(|(f_n - g, k_m(\cdot, y))| < \varepsilon\) for all \(n \geq n_0\). Then we obtain that

\[
|(Tf_n)(y) - (Tg)(y)| = \left| \int_{\Omega_1} (f_n(x) - g(x))k_m(x, y) \, d\mu_1(x) \right|
\]

\[
= \left| \int_{\Omega_1} f_n(x)(k(x, y) - k_m(x, y)) \, d\mu_1(x) - \int_{\Omega_1} g(x)(k(x, y) - k_m(x, y)) \, d\mu_1(x) \right|
\]

\[
\leq \int_{\Omega_1} f_n(x)(k(x, y) - k_m(x, y)) \, d\mu_1(x) + \int_{\Omega_1} g(x)(k(x, y) - k_m(x, y)) \, d\mu_1(x) 
\]

\[
+ \int_{\Omega_1} g(x)(k(x, y) - k_m(x, y)) \, d\mu_1(x) \leq 3\varepsilon
\]

for all \(n \geq n_0\). Since this holds for almost all \(y \in \Omega_2\), we actually proved that \((Tf_n)\) converges to \(Tg\) pointwise almost everywhere. As \((Tf_n) \subseteq [0, Tf]\) it follows from the dominated convergence theorem that \((Tf_n)\) converges to \(Tg\) in \(L^p\)-norm. This readily shows that \(T[0, f]\) is compact. Since \(f\) was arbitrary, \(T\) is AM-compact. \(\square\)

References

[1] C. Aliprantis and K. Border. *Infinite dimensional analysis: a hitchhiker’s guide*. Springer Verlag, 2006.

[2] C. D. Aliprantis and O. Burkinshaw. *Locally solid Riesz spaces*. Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1978. Pure and Applied Mathematics, Vol. 76.

[3] W. Arendt. Positive semigroups of kernel operators. *Positivity*, 12(1):25–44, 2008.

[4] W. Arendt, C. J. K. Batty, M. Hieber, and F. Neubrander. *Vector-valued Laplace transforms and Cauchy problems*, volume 96 of *Monographs in Mathematics*. Birkhäuser/Springer Basel AG, Basel, second edition, 2011.

[5] W. Arendt and A. V. Bukhvalov. Integral representations of resolvents and semigroups. *Forum Math.*, 6(1):111–135, 1994.

[6] W. Arendt, A. Grabosch, G. Greiner, U. Groh, H. P. Lotz, U. Moustakas, R. Nagel, F. Neubrander, and U. Schlotterbeck. *One-parameter semigroups of positive operators*, volume 1184 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1986.
The ergodic theory of Markov processes

S. R. Foguel.

K. Engel and R. Nagel.
One-parameter semigroups for linear evolution equations

M. Gerlach.
Semigroups of Kernel Operators

M. Gerlach.
On the peripheral point spectrum and the asymptotic behavior of irreducible

G. Gupur.
Functional analysis methods for reliability models

M. Gerlach and M. Kunze.
On the lattice structure of kernel operators.

A. Haji and A. Radl.
A semigroup approach to the Gnedenko system with single vacation of a repairman.
Semigroup Forum, 86(1):41–58, 2013.

S. C. Hille and D. T. H. Worm.
Continuity properties of Markov semigroups and their restrictions to invariant $L^1$-spaces. Semigroup Forum, 79(3):575–600, 2009.

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[7] W. Arendt, S. Kunkel, and M. Kunze. Diffusion with nonlocal boundary conditions. J. Funct. Anal., 270(7):2483–2507, 2016.

[8] J. Banasiak, K. Pichór, and R. Rudnicki. Asynchronous exponential growth of a general structured population model. Acta Appl. Math., 119:149–166, 2012.

[9] A. Bátkai, M. Kramar-Fijavž, and A. Rhandi. Positive Operator Semigroups. From Finite to Infinite Dimensions, volume 257 of Operator Theory: Advances and Applications. Birkhäuser, 2017.

[10] É. Bernard and F. Salvarani. On the convergence to equilibrium for degenerate transport problems. Arch. Ration. Mech. Anal., 208(3):977–984, 2013.

[11] A. Bobrowski, T. Lipniacki, K. Pichór, and R. Rudnicki. Asymptotic behavior of distributions of mRNA and protein levels in a model of stochastic gene expression. J. Math. Anal. Appl., 333(2):753–769, 2007.

[12] A. H. Clifford and G. B. Preston. The algebraic theory of semigroups. Vol. I. Mathematical Surveys, No. 7. American Mathematical Society, Providence, R.I., 1961.

[13] G. Da Prato and J. Zabczyk. Ergodicity for infinite-dimensional systems, volume 229 of London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 1996.

[14] E. B. Davies. Triviality of the peripheral point spectrum. J. Evol. Equ., 5(3):407–415, 2005.

[15] M. de Jeu and J. Rozendaal. Disintegration of positive isometric group representations on $L^p$-spaces. Preprint, available from arxiv.org/abs/1502.00755, 2015.

[16] M. de Jeu and M. Wortel. Positive representations of finite groups in Riesz spaces. Internat. J. Math., 23(7):1250076, 28, 2012.

[17] M. de Jeu and M. Wortel. Compact groups of positive operators on Banach lattices. Indag. Math. (N.S.), 25(2):186–205, 2014.

[18] J. L. Doob. Asymptotic properties of Markoff transition probabilities. Trans. Amer. Math. Soc., 63:393–421, 1948.

[19] N. H. Du and N. H. Dang. Dynamics of Kolmogorov systems of competitive type under the telegraph noise. J. Differential Equations, 250(1):386–409, 2011.

[20] T. Eisner, B. Farkas, M. Haase, and R. Nagel. Operator Theoretic Aspects of Ergodic Theory, volume 272 of Graduate Texts in Mathematics. Springer-Verlag, New York, 2015.

[21] E. Y. Emel’yanov. Non-spectral asymptotic analysis of one-parameter operator semigroups, volume 173 of Operator Theory: Advances and Applications. Birkhäuser Verlag, Basel, 2007.

[22] K. Engel and R. Nagel. One-parameter semigroups for linear evolution equations, volume 194 of Graduate Texts in Mathematics. Springer-Verlag, New York, 2000.

[23] S. R. Foguel. The ergodic theory of Markov processes. Van Nostrand Mathematical Studies, No. 21. Van Nostrand Reinhold Co., New York-Toronto, Ont.-London, 1969.

[24] M. Gerlach. On the peripheral point spectrum and the asymptotic behavior of irreducible semigroups of Harris operators. Positivity, 17(3):875–898, 2013.

[25] M. Gerlach. Semigroups of Kernel Operators. PhD thesis, Universität Ulm, 2014.

[26] M. Gerlach. A Tauberian theorem for strong Feller semigroups. Arch. Math. (Basel), 102(3):245–255, 2014.

[27] M. Gerlach and J. Glück. Lower Bounds and the Asymptotic Behaviour of Positive Operator Semigroups. To appear in Ergod. Th. & Dynam. Sys.

[28] M. Gerlach and M. Kunze. Mean ergodic theorems on norming dual pairs. Ergod. Th. & Dynam. Sys., 34(4):1210–1229, 8 2014.

[29] M. Gerlach and M. Kunze. On the lattice structure of kernel operators. Math. Nachr., 288(5-6):584–592, 2015.

[30] M. Gerlach and R. Nittka. A new proof of Doob’s theorem. J. Math. Anal. Appl., 388(2):763–774, 2012.

[31] G. Greiner. Spektrum und Asymptotik stark stetiger Halbgruppenpositiver Operatoren. Sitzungsber. Heidelb. Akad. Wiss. Math.-Natur. Kl., pages 55–80, 1982.

[32] G. Greiner and U. Groh. A Perron Frobenius theory for representations of locally compact abelian groups. Math. Ann., 262(4):517–528, 1983.

[33] G. Gupur. Functional analysis methods for reliability models, volume 6 of Pseudo-Differential Operators. Theory and Applications. Birkhäuser/Springer Basel AG, Basel, 2011.

[34] G. Gupur and M. W. Wong. On a dynamical system for a reliability model. J. Pseudo-Differ. Oper. Appl., 2(4):509–542, 2011.

[35] A. Haji and A. Radl. A semigroup approach to queueing systems. J. Math. Anal. Appl., 388(3):610–624, 2007.

[36] A. Haji and A. Radl. Asynchronous exponential growth of a general structured population model. Acta Appl. Math., 262(4):393–421, 1988.

[37] A. Haji and A. Radl. A semigroup approach to the Gnedenko system with single vacation of a repairman. Semigroup Forum, 86(1):41–58, 2013.
[68] M. Wolff. Group actions on Banach lattices and applications to dynamical systems. In Toeplitz centennial (Tel Aviv, 1981), volume 4 of Operator Theory: Adv. Appl., pages 501–524. Birkhäuser, Basel-Boston, Mass., 1982.

[69] M. P. H. Wolff. Triviality of the peripheral point spectrum of positive semigroups on atomic Banach lattices. Positivity, 12(1):185–192, 2008.

[70] F. Zheng and B.-Z. Guo. Quasi-compactness and irreducibility of queueing models. Semigroup Forum, 91(3):560–572, 2015.

[71] F. Zheng, G. Zhu, and C. Gao. Well-posedness and stability of the repairable system with N failure modes and one standby unit. J. Math. Anal. Appl., 375(1):174–184, 2011.

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