Type IIB Matrix Theory at Two Loops

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Abstract

The IKKT matrix model was proposed to be a non-perturbative formulation of type IIB superstring theory. One of its important consistency criteria is that the leading one-loop $1/r^8$ effective interaction between a cluster of type IIB D-objects should not receive any corrections from higher loop effects for it to describe accurately the type IIB supergravity results. In analogy with the BFSS matrix model versus the eleven-dimensional supergravity example, we show in this work that the one-loop effective potential in the IKKT matrix model is also not renormalized at the two-loop order.

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1 Introduction

One of the remarkable consequences of the open string theory description of D-branes [1] is the existence of a close correspondence between the supergravity and supersymmetric Yang-Mills theory (SYM) results for certain interactions of D-branes. This surprising idea emerges in its simplest form from the examination of the low-energy dynamics of parallel Dp-branes [2] which is found to be described by a $U(N)$ maximally supersymmetric Yang-Mills theory on the $p+1$-dimensional worldvolume of the Dp-barnes. To understand better this correspondence the authors of [3] described the interactions that arise between two D-branes in two ways. Either by considering the sum of all closed string exchanges, or using the modular properties of string theory, as the sum of one-loop amplitudes in the open strings ending on the branes in the spirit of Bachas’s D-brane dynamics calculation [4,5]. In general this constitutes a relation between two different description within string theory, and requires keeping all the string modes for its validity.

As articulated first in [3], for large D-brane separations $r >> l_s$, where $l_s = \sqrt{\alpha'}$ is the string scale, the velocity-dependent interaction between the D-branes is most easily described by a supergravity theory in terms of the massless closed string exchange. Whereas for substringy separations $r << l_s$, the effective interaction is best described by the dynamics of the lightest open strings stretching between the D-branes which is encoded in the SYM on the brane worldvolume[1]. This is not surprising since the truncation of the sum over the string states in each description to the lightest modes of each type must be valid in very different regimes. In some special situations, however, with some residual supersymmetry (left unbroken by the velocity), an approximate cancellation between the bosons and the fermions persists [6] allowing for the decoupling of the massive string states, and hence a correspondence between the supergravity results based on the masless closed string exchange and the SYM theory on the brane worldvolume based on the lightest open strings stretching between the branes. A prototype for this correspondence and most important in what follows is the computation of the leading order velocity-dependent potential $v^4/r^7$ between two D0-branes in [4,7]. One other interesting follow-up to this correspondence is the realization that at substringy distances the classical geometry of spacetime is identified with the quantum moduli space of gauge inequivalent configurations. See [8], however, for other examples in connection with this phenomenon.

Using these observations along with the description of the supermembrane worldvolume action in terms of a supersymmetric quantum mechanical system as found in [9] and which was reinterpreted later in [10] as being the maximally supersymmetric quantum mechanical system describing N D0-branes in the large N limit, Banks, Fischler, Shenker and Susskind (BFSS) put forward the more far-reaching Matrix theory conjecture [11]. Simply put the conjecture states that M-theory, in the light-cone frame, is exactly described by the

\[1\text{In the supergravity picture, the massive closed string modes induces exponentially falling additional interactions. Whereas in the SYM picture, the massive open string states contribute create higher derivative interactions on the brane worldvolume field theory.}\]
large N limit of the supersymmetric matrix quantum mechanics of N D0-branes \[^2\]. This conjecture came as an attempt to describe the short-distance limit of M-theory. M-theory made its first entry in the string web of dualities as the strong coupling limit of type IIA string theory \[^13\]. Before the BFSS conjecture, very little was known about this theory except that at low energies and large distances M-theory is described by the eleven-dimensional supergravity. The BFSS conjecture (if correct) seems to indicate that all of the eleven-dimensional physics in the infinite momentum frame is contained in the maximally supersymmetric gauge theory reduced to the quantum mechanics on the worldline of the D0-branes.

Here is now that come the relevance of the correspondence between supergravity and SYM theory explained above. The exact equivalence (due to supersymmetry) between the leading long-distance supergravity interaction \( v^4 / r^7 \) between D0-branes governed by a single supergraviton exchange and the one-loop matrix theory result becomes an important consistency criteria for the BFSS conjecture. In other words for the BFSS conjecture to hold it is important that the \( v^4 / r^7 \) potential receives no contribution beyond one-loop on the matrix theory side. The authors of \[^11\] suspected the existence of some non-renormalization theorem in the context of a supersymmetric quantum mechanics model with 16 supercharges to protect this term from higherloops contribution. This belief is strengthened by the existence in the litterature of a similar non-renormalization theorem for the \( F_{\mu
u}^4 \) term in the action of the ten dimensional string theory \[^4, 14\].

This question was undertaken first in \[^15\] and later in \[^16\] where using the background field method they showed that at two-loops the \( v^4 / r^7 \) term is robust. Very recently in \[^17\] the robustness of this term was completely established when the proof of the non-renormalization theorem alluded to above was found. The status of the BFFS matrix theory as a candidate for the nonperturbative formulation of of M-theory has also improved \[^3\] after the work of the authors in \[^19\] where they ruled out any discrepancy between eleven-dimensional supergravity and the BFSS matrix theory for three-body scattering as reported earlier in \[^20\]. For a more complete description of the matrix theory and its status we refer the reader to the excellent reviews \[^21\].

The BFSS matrix theory describes naturally the ten-dimensional IIA superstring theory since the latter is the limit of M-theory at weak string couling. In \[^22\] Ishibashi, Kawai, Kitazawa and Tsuchiya (IKKT) proposed another matrix model associated with the IIB superstring theory, which is in the spirit of the Eguchi-Kawai \[^23\] large-N reduced ten-dimensional SYM theory. This non-perturbative formulation of IIB superstring theory is called the IKKT matrix model. In analogy with the BFSS matrix model where the theory is expressed in terms of D0-branes, in the IKKT model one expects that the non-perturabtive formulation of type IIB superstring theory is described in terms of a supresymmetric Yang-Mills gauge theory of N D-instantons derived upon reduction to a point \[^4\]. The connection

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\[^2\] See however \[^12\] for the new version of this conjecture at finite \( N \). For the purpose of our work here it suffices to deal with the original BFSS Matrix theory of \[^11\].

\[^3\] See also the work in \[^18\].

\[^4\] This interpretation is not totally exact since the action describing the IKKT matrix model contains
of the IKKT matrix model with the one of BFSS was demonstrated in [22] by considering the action of the former on some special point of its moduli space of degenerate vacua. Contrary to the BFSS case, however, the IKKT matrix model has the manifest Lorentz invariance in ten dimensions and so does not present us with the awkwardness of the light-cone frame.

The validity of the IKKT matrix model relies so far on its ability to describe the classical D-brane configurations of type IIB superstring and their interactions. In [22] a one-loop computation in the background of operator-like solutions corresponding to a cluster of IIB D-objects with relative motion and occupying some region of spacetime has revealed a leading $1/r^8$ behavior in the potential between two D-block objects. This result manifestly agrees with the long-range potential obtained in the supergravity calculation based on the massless closed string exchange. As argued in [22], the $1/r^8$ behavior of the effective interaction in the IKKT matrix model ensures the cluster property among the D-objects which is important to the N=2 supersymmetry of the IKKT action and hence to the dynamical generation of the spacetime coordinates which constitutes one of the nicest feature of the IKKT matrix model.

Furthermore and as in the BFSS case, the exact agreement between the IKKT matrix theory and the supergravity result is an important consistency criterion for the IKKT matrix model to describe type IIB superstring theory. In particular, higher loop effects on the IKKT matrix side had better not to spoil this correspondence. So for these reasons the question of the non-renormalization of the one-loop result becomes also important here. This article is an investigation along this line. We start in Section (2), with some preliminaries about the IKKT matrix model. In Section (3), we deal with the one-loop computation of the effective action in the general background of multi-D-objects with very large separations from each other. This background is represented by a block-diagonal operator-like solutions. The one-loop computation is not new but was considered previously in [21,24] but we give it here just to fix conventions and introduce notations and definitions that set the ground for the two-loop evaluation of the effective action in Section (4). Our basic tool for the two-loop computation will be the background field method and we find in analogy with the investigation of in the BFFS case [15] that the one-loop result is not renormalized at this order. Section (5) contains some concluding remarks where we argue for a possible extension of the non-renormalization theorem applicable to the D0-branes [17] to include the IKKT model on a point. In the Appendix, we have gathered some technical details that arise in the two-loop computation of Section (4).

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an extra term that does not arise from any dimensional reduction and was tinterpreted in [22] as a kind of a chemical potential.
2 Some Preliminaries

The IKKT matrix model is defined by the partition function

\[ Z_0 = \sum_{n=1}^{\infty} \int dA_\mu d\Phi_a e^{-S}, \] (1)

which is a second quantized Euclidean field theory, with action\footnote{In what follows, we set \( \alpha' = 1 \) and \( g_s = 1 \).}

\[ S = \frac{1}{g_s (\alpha')^2} \left( -\frac{1}{4} Tr \left( [A_\mu, A_\nu]^2 \right) - \frac{1}{2} Tr \left( \bar{\Phi} \Gamma^\mu \left[ A_\mu, \Phi \right] \right) \right) + \beta n. \] (2)

Here \( i^j A_\mu \) and \( i^j \Phi_a \) are \( n \times n \) Hermitian bosonic and fermionic matrices, respectively\footnote{Our notation follows the one of reference [22].}. The vector index \( \mu \) runs from 0 to 9 and the spinor index \( a \) runs from 1 to 32. The fermion \( \Phi \) is a Majorana-Weyl spinor which satisfies the condition \( \Gamma_{11} \Phi = \Phi \).

The action (2) is invariant under the \( \mathcal{N} = 2 \) supersymmetry transformations

\[ \delta^{(1)} (i^j \Phi_a) + \delta^{(2)} (i^j \Phi_a) = \frac{i}{2} \left( i^j [A_\mu, A_\nu] \right) (\Gamma^\mu \epsilon)_a + \xi_a \delta^j, \]

\[ \delta^{(1)} (i^j A_\mu) + \delta^{(2)} (i^j A_\mu) = i \bar{\epsilon} \Gamma_\mu (i^j \Phi), \] (3)

where \( \epsilon \) and \( \xi \) are the supersymmetry parameters, as well as under the gauge transformation

\[ \delta_{\text{gauge}} A_\mu = i [A_\mu, \omega], \]

\[ \delta_{\text{gauge}} \Phi_a = i [\Phi_a, \omega]. \] (4)

The formulas (3) and (4) look like as if 10D SYM theory is reduced to a point. For instance all the spacetime derivatives drop out from the non-Abelian field strength \( F_{\mu\nu} = i [A_\mu, A_\nu] \). However, the action (2) coincides with the one of 10D SYM theory in the zero volume limit only if \( \beta = \) and \( n \) fixed. In [22], the \( \beta \) parameter was interpreted as a kind of chemical potential generated form the one-loop renormalization of the action (2) even if initially \( \beta \) is set to zero.

So the action (2) is, up to the \( \beta \)-term, the low-energy effective action of a D-instanton of charge \( n \) \cite{2}. The other higher dimensional branes are represented by the solutions of the classical equations of motion

\[ [A_\mu, [A_\mu, A_\nu]] = 0, \quad [A_\mu, (\Gamma^\mu \Phi)_a] = 0, \] (5)

which are to be solved by \( n \times n \) matrices \( A_\mu \) at infinite \( n \). A general solution has a block-diagonal form

\[ A_\mu = \begin{pmatrix} 1 y_\mu & 2 y_\mu & 3 y_\mu & \cdots \\ y_\mu & 2 y_\mu & 3 y_\mu & \cdots \\ 2 y_\mu & 3 y_\mu & \cdots & \cdots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix}, \quad \Phi_a = 0, \] (6)
where $^i y_\mu$ ($i = 1, 2, 3, \ldots$) is a non-diagonal $n_i \times n_i$ matrix. We may regard $^i y_\mu$ as a D-object occupying some region of spacetime. A much simpler solution corresponds to the case where the $^i y_\mu$'s become diagonal matrices. Using this fact, we can decompose $^i y_\mu$ in the general solution (3) as

\begin{equation}
^i y_\mu = ^i d_\mu 1_{n_i} + ^i p_\mu,
\end{equation}

\begin{equation}
Tr ^i p_\mu = 0,
\end{equation}

where $^i d_\mu$ is a real number representing the center of mass coordinate of the $i$-th block. More on the notations we are adopting for the block elements $^i y_\mu$, $^i p_\mu$ and $^i d_\mu$ in the next section. For the purpose of our work here, we assume through out all the paper that the blocks are separated far enough from each other so that for all $i$ and $j$'s, $(^i d_\mu - ^j d_\mu)^2$ are large.

It was argued in [22,25] that for the classical solutions representing BPS states the field strength should be proportional to the unit matrix, that is,

\begin{equation}
^i f_{\mu \nu} = i [^i p_\mu, ^i p_\nu] = c_{\mu \nu} 1_{n_i},
\end{equation}

or since $f_{\mu \nu}$ has a block-diagonal form we can also write

\begin{equation}
f_{\mu \nu} = \begin{pmatrix}
1 f_{\mu \nu} & 2 f_{\mu \nu} & 3 f_{\mu \nu} \\
& & \\
& & \\
& & 
\end{pmatrix}.
\end{equation}

The classical equations (5) are in this case automatically satisfied. Since D-branes are BPS states [1], the classical solutions of the matrix model which correspond to D-branes should have this property.

### 3 The One-Loop Effective Action

In this section, we calculate the one-loop effective action for the interaction between many diagonal blocks. The calculation is similar to the one performed in [22] and we repeat here only to fix notation and prepare the ground for the derivation of the two-loop effective action in Section (4). Using the background field method, we decompose the matrices $A_\mu$ and $\Phi$ into a general background having a block-diagonal form plus the quantum fluctuations. Namely,

\begin{equation}
^i j A_{\mu}^{qr} = ^i d_\mu \delta^{ij} \delta^{qr} + ^i p_{\mu}^{qr} \delta^{ij} + ^i j a_{\mu}^{qr},
\end{equation}

\begin{equation}
^i j \phi_a^{qr} = ^i j \phi_a^{qr} + ^i j \varphi_a^{qr},
\end{equation}

where the matrix elements $^i j a_{\mu}^{qr}$ and $^i j \varphi_a^{qr}$ are the bosonic and the fermionic quantum fluctuations, respectively.
Since these quantum fluctuations will come often in this paper, a word on the notation we are adopting for them is in order. This proves also useful later when we come to the derivation of the Feynman rules involved in the two-loop computation. The quantum fields matrices \( a_\mu \) and \( \varphi_a \) belong to the general space of matrices. A general matrice \( X \) on this space can be denoted by its \((i,j)\) blocks as an \( n_i \times n_j \) matrix \( i^j X \). So the block elements of a block-diagonal matrix \( X \) satisfy \( i^j X = i^i X \delta^{ij} \), where \( i^i X \) is an \( n_i \times n_i \) matrix. The matrix elements of a block element matrix \( i^j X \) are given by \( i^j_{qr} X \). For instance the matrix \( d_\mu \) representing the center of mass coordinates of some \( i\)-th block has a block diagonal form and we denote its elements as \( i^i d_\mu_{qr} \). Now, we move to the derivation of the one-loop effective action. As is usual in a gauge theory (which is the case here) one should add both the gauge fixing and the ghost terms to the action (2), such terms are given by

\[
S_{gf} = -\frac{1}{2} \text{Tr} \left( [d_\mu + p_\mu, a_\nu]^2 \right) - \frac{1}{2} \text{Tr} \left( [d_\mu + p_\mu, b] [d_\mu + p_\mu, c] \right),
\]

where the matrices \( c \) and \( \bar{b} \) represent the ghosts and anti-ghosts, respectively. In the following, adding \( S_{gf} \) to the action (2) expanded in the quantum fluctuations \( a_\mu \) and \( \varphi_a \) and using the classical equations of motion (5) after setting the fermionic background \( \phi \) to zero, we find

\[
S + S_{gf} = S_2 + S_4^B + S_3^B + S_3^F + S_3^{\text{ghost}}.
\]

The action \( S_2 \) is obtained by keeping the quantum fluctuations up to second order such as

\[
S_2 = \frac{1}{2} \text{Tr} \left( a_\mu \left[ p_\mu, \left[ p_\mu, a_\nu \right] \right] \right) + \text{Tr} \left( a_\mu \left[ \left[ p_\mu, p_\nu \right], a_\nu \right] \right) - \frac{1}{2} \text{Tr} \left( \bar{\varphi} \Gamma^\mu \left[ p_\mu, \varphi \right] \right)
\]

\[
+ \text{Tr} \left( \bar{b} \left[ p_\mu, \left[ p_\mu, c \right] \right] \right).
\]

\( S_4^B \) involves the quartic interactions among the bosonic quantum fluctuations \( a_\mu \) and \( S_3^B \) the cubic ones

\[
S_4^B = -\frac{1}{4} \text{Tr} \left( [a_\mu, a_\nu]^2 \right) = A_{\mu\nu\lambda\rho} \text{Tr} \left( a_\mu a_\nu a_\lambda a_\rho \right),
\]

\[
S_3^B = -\text{Tr} \left( [d_\mu + p_\mu, a_\nu] [a_\mu, a_\nu] \right) = B_{\mu\nu\lambda\rho} \text{Tr} \left( (d_\mu + p_\mu) a_\nu a_\lambda a_\rho \right),
\]

where \( A_{\mu\nu\lambda\rho} \) and \( B_{\mu\nu\lambda\rho} \) are given by

\[
A_{\mu\nu\lambda\rho} = \frac{1}{2} (\delta_{\mu\nu} \delta_{\lambda\rho} - \delta_{\mu\lambda} \delta_{\nu\rho}) ,
\]

\[
B_{\mu\nu\lambda\rho} = (\delta_{\mu\nu} \delta_{\lambda\rho} + \delta_{\mu\rho} \delta_{\nu\lambda} - 2 \delta_{\mu\lambda} \delta_{\nu\rho}) .
\]

\( S_3^F \) is the action involving cubic interactions of both fermionic \( \varphi \) and bosonic \( a_\mu \) fluctuations and is given by

\[
S_3^F = -\frac{1}{2} \text{Tr} \left( \bar{\varphi} \Gamma^\mu \left[ a_\mu, \varphi \right] \right).
\]
Similarly $S_{3}^{\text{ghost}}$ includes the cubic ghost interactions

$$S_{3}^{\text{ghost}} = - Tr \left( \left[ p_\mu, \bar{b} \right] \left[ a_\mu, c \right] \right). \tag{20}$$

In order to compute the one-loop effective action, it suffices to consider the quantum fluctuations in the action only up to the second order. The action $S_{2}$ contains all such terms. To perform the functional integration involved in the evaluation of the one-loop effective action

$$\mathcal{W}^{(1)} = - \log \int \mathcal{D}a_\mu \mathcal{D}\varphi_a \mathcal{D}c \mathcal{D}\bar{b} \ e^{-S_{2}}, \tag{21}$$

it will be convenient to introduce the notation in which the the adjoint operators $P_\mu$ and $F_{\mu\nu}$ act on the space of matrices as

$$P_\mu X = \left[ d_\mu + p_\mu + a_\mu, X \right], \tag{22}$$

$$F_{\mu\nu} X = \left[ f_{\mu\nu}, X \right] = \left[ i \left[ p_\mu, p_\nu \right], X \right]. \tag{23}$$

Using this notation, we can write the action $S_{2}$ as

$$S_{2} = \frac{1}{2} Tr \left[ a_\mu \left( P^{2} \delta_{\mu\nu} - 2 i F_{\mu\nu} \right) a_\nu \right] - \frac{1}{2} Tr \left( \bar{\varphi} \Gamma^{\mu} P_\mu \varphi \right) + Tr \left( \bar{b} P^{2} c \right). \tag{24}$$

From this the one-loop effective action $\mathcal{W}^{(1)}$ in (21) is evaluated easily and is given by

$$\mathcal{W}^{(1)} = \frac{1}{2} Tr \left( P^{2} \delta_{\mu\nu} - 2 i F_{\mu\nu} \right) - \frac{1}{4} Tr \left[ \left( P^{2} + \frac{i}{2} F_{\mu\nu} \Gamma^{\mu\nu} \right) \left( \frac{1 + \Gamma_{11}}{2} \right) \right]$$

$$- Tr \left( P^{2} \right). \tag{25}$$

The cases where $^{ij}f_{\mu\nu}^{qr} = c_{\mu\nu} \delta^{ij} \delta^{qr}$ have special meaning. These correspond to BPS-saturated states backgrounds [22,24,26]. Since $F_{\mu\nu} = 0$ in these cases, we have

$$\mathcal{W}^{(1)} = \left( \frac{1}{2} \cdot 10 - \frac{1}{4} \cdot 16 - 1 \right) Tr \left( P^{2} \right) = 0, \tag{26}$$

which means that the one-loop quantum corrections vanish due to supersymmetry. This is consistent with the well known fact that the BPS-saturated states have no quantum corrections which also ensures their stability. The simplest example is the BPS plane vacuum for which the background matrix elements are given by $^{ij}f_{\mu\nu}^{qr} = i d_\mu \delta^{ij} \delta^{qr}$ and in this case $F_{\mu\nu} = 0$ trivially.

For a general non-BPS background such as $F_{\mu\nu} \neq 0$, the one-loop effective action (25) expanded in the inverse power of $(i d_\mu - j d_\mu)^2$ is given by

$$\mathcal{W}^{(1)} = - Tr \left( \frac{1}{P^{2}} F_{\mu\nu} \frac{1}{P^{2}} F_{\nu\lambda} \frac{1}{P^{2}} F_{\lambda\rho} \frac{1}{P^{2}} F_{\rho\mu} \right).$$
\[-2 \text{Tr} \left( \frac{1}{p_2^2} F_{\mu \nu} \frac{1}{p_2^2} F_{\nu \lambda} \frac{1}{p_2^2} F_{\mu \rho} \frac{1}{p_2^2} F_{\rho \lambda} \right) \]
\[+ \frac{1}{2} \text{Tr} \left( \frac{1}{p_2^2} F_{\mu \nu} \frac{1}{p_2^2} F_{\nu \rho} \frac{1}{p_2^2} F_{\lambda \rho} \frac{1}{p_2^2} F_{\lambda \rho} \right) \]
\[+ \frac{1}{4} \text{Tr} \left( \frac{1}{p_2^2} F_{\mu \nu} \frac{1}{p_2^2} F_{\lambda \rho} \frac{1}{p_2^2} F_{\mu \nu} \frac{1}{p_2^2} F_{\lambda \rho} \right)\]
\[+ \mathcal{O} \left( F^5 \right), \tag{27} \]

where we have used the following identities for the Dirac matrices

\[ \{ \Gamma^\mu, \Gamma^\nu \} = -2 \delta^{\mu \nu}, \quad \Gamma^{\mu \nu} = \frac{1}{2} \left[ \Gamma^\mu, \Gamma^\nu \right], \quad \text{Tr} \left( \Gamma^\mu \Gamma^\nu \right) = -32 \delta^{\mu \nu}, \]
\[ \text{Tr} \left( \Gamma^\mu \Gamma^\nu \Gamma^\lambda \Gamma^\rho \right) = 32 \left( \delta^{\mu \nu} \delta^{\lambda \rho} - \delta^{\mu \lambda} \delta^{\nu \rho} + \delta^{\mu \rho} \delta^{\nu \lambda} \right). \tag{28} \]

Although this background is not BPS-saturated there is still some left asymptotic residual $N=2$ supersymmetry which ensures the cancellation between the bosonic and the fermionic contribution up to the third order in $F_{\mu \nu}$. This feature is very reminiscent of the results of many previous investigations [3,4,14] with $N=2$ supersymmetry. It is not surprising that we recover here the same kind of cancellation since as argued originally in [22] the IKKT matrix model is $T$-dual to the BFFS matrix model where this phenomenon appear also. In fact one of the goals of Section (4) is to show that this cancellation continues to hold even at two-loops.

To simplify further the expression of (27), we need to introduce more notations such as

\[ i^j (P_\mu X) = i^j d_\mu i^j X + i^j p_\mu i^j X \equiv i^j P_\mu i^j X, \]
\[ i^j d_\mu i^j X = \left( i^j d_\mu - j d_\mu \right) i^j X, \quad \left( i^j d_\mu - j d_\mu \right) \text{act as real numbers}, \]
\[ i^j p_\mu i^j X = i^j p_\mu i^j X - i^j X j p_\mu. \tag{29} \]

Using these definitions, it is easy to show that

\[ i^j (P_2^2 X) = i^j (P_2^2) i^j X = \left( i^j d_2 \right) i^j X + 2 i^j d. i^j p. i^j X + \left( i^j p_2^2 \right) i^j X, \tag{30} \]

where

\[ \left( i^j p_2^2 \right) i^j X = \left( i^j p_2 \right) i^j X + i^j X \left( i^j p_2 \right) - 2 i^j p_\mu i^j X j p_\mu, \]
\[ i^j d. i^j p. i^j X = \left( i^j d_\mu - j d_\mu \right) \left( i^j p_\mu i^j X - i^j X j p_\mu \right), \]
\[ \left( i^j d_2^2 \right) i^j X = \left( i^j d_2 - j d_2 \right)^2 i^j X. \tag{31} \]

In the same way, we can decompose $F_{\mu \nu}$ as

\[ i^j (F_{\mu \nu} X) = i^j f_{\mu \nu} i^j X \equiv i^j F_{\mu \nu} i^j X, \]
\[ i^j f_{\mu \nu} i^j X = i^j f_{\mu \nu} i^j X - i^j X j f_{\mu \nu}. \tag{32} \]
It is clear from the above expressions that not only does $P_\mu$ and $F_{\mu\nu}$ operate on each block $i_jX$ independently but also their action on the left or on the right of $i_jX$ are totally independent. It follows then that the trace of an operator $O$ such as the one appearing in $\mathcal{W}^{(1)}$ in (27) and which consists of $P_\mu$ and $F_{\mu\nu}$ is evaluated using this formula

$$\text{Tr} (O) = \sum_{i,j=1}^{n} \text{Tr}^{ij} O_L \text{Tr}^{ij} O_R.$$  \hspace{1cm} (33)

Using these definitions we can easily evaluate the one-loop effective action $\mathcal{W}^{(1)}$ and the result is a sum of contributions from each $(i, j)$ block $\mathcal{W}^{(i,j)}$ which describes the effective interactions between the $i$-th and $j$-th blocks, that is,

$$\mathcal{W}^{(1)} = \sum_{i,j=1}^{n} \mathcal{W}^{(i,j)},$$  \hspace{1cm} (34)

where $\mathcal{W}^{(i,j)}$ are given to leading order for large separation $(i^2 - j^2)^{\frac{1}{2}}$ by

$$\mathcal{W}^{(i,j)} = \frac{1}{4 (i^2 j^2)^{\frac{1}{2}}} \left[ -4 n_j \text{Tr} \left( i f_{\mu\nu}^j f_{\nu\lambda}^i f_{\lambda\rho}^i f_{\rho\mu}^j \right) - 8 n_j \text{Tr} \left( i f_{\mu\nu}^i f_{\nu\lambda}^i f_{\mu\rho}^j f_{\rho\lambda}^i \right) \\
+ 2 n_j \text{Tr} \left( i f_{\mu\nu}^i f_{\mu\rho}^i f_{\lambda\rho}^j f_{\nu\lambda}^j \right) + n_j \text{Tr} \left( i f_{\mu\nu}^i f_{\mu\rho}^i f_{\rho\lambda}^j f_{\nu\lambda}^j \right) \\
- 4 n_i \text{Tr} \left( i f_{\mu\nu}^i f_{\nu\lambda}^j f_{\lambda\rho}^i f_{\rho\mu}^j \right) - 8 n_i \text{Tr} \left( i f_{\mu\nu}^i f_{\lambda\rho}^j f_{\mu\rho}^j f_{\nu\lambda}^i \right) \\
+ 2 n_i \text{Tr} \left( i f_{\mu\nu}^i f_{\nu\lambda}^j f_{\lambda\rho}^j f_{\rho\mu}^i \right) + n_i \text{Tr} \left( i f_{\mu\nu}^i f_{\lambda\rho}^j f_{\mu\rho}^j f_{\nu\lambda}^j \right) \\
- 48 \text{Tr} \left( i f_{\mu\nu}^i f_{\nu\lambda}^i \right) \text{Tr} \left( i f_{\mu\rho}^j f_{\rho\lambda}^j \right) + 6 \text{Tr} \left( i f_{\mu\nu}^i f_{\mu\nu}^i \right) \text{Tr} \left( i f_{\lambda\rho}^j f_{\lambda\rho}^j \right) \\
+ \mathcal{O} \left( 1 / (i^2 j^2)^{\frac{1}{2}} \right).$$  \hspace{1cm} (35)

In [22] the last two terms were identified with the exchange of the graviton and the scalar dilaton. After this review, we are now ready to move to the two-loop computation.

## 4 The Two-Loop Effective Action

In this section we shall carry out the derivation of the two-loop effective action of the IKKT matrix gauge system described by the total action in (13). A similar calculation was performed for the BFFS matrix model in [15,16,19] using the background field method. Following the same approach [27], we treat the background field $i^2 d_\mu + j p_\mu$ in (10) exactly so that it enters in the propagators and vertices of the theory. Therefore to compute the gauge invariant background field effective action at two-loops one has to sum only over all 1PI vacuum diagrams (without external lines) involving quartic and cubic vertices as dictated by the action (13). The two-loop vacuum graphs in question are displayed in Fig(1) and are adapted for our purposes to describe the IKKT matrix model [22]. We shall
return back to explaining our representation of these graphs in Fig (1) after introducing below the propagators for the bosonic, fermionic and ghost fields.

Knowing that at two-loops the effective action is given by the four vacuum graphs in Fig (1) allows us to represent it as a sum of contributions arising from each of the interactions $S_B^4$, $S_B^3$, $S_F^3$ and $S_3^{ghost}$, that is,

$$W^{(2)} = W_B^4 + W_B^3 + W_F^3 + W_3^{ghost},$$  \hspace{1cm} (36)

where $W_B^4$ and $W_B^3$ account for the graphs in Fig (1) involving the quartic and the cubic vertices involving the bosonic fluctuations $a_\mu$. $W_F^3$ accounts for the cubic vertex graph involving the fermionic fluctuations $\varphi_a$ and $W_3^{ghost}$ describe the cubic vertex graph with ghost fields. The computation of $W^{(2)}$ will boil down then to evaluating the terms $W_B^4$, $W_B^3$, $W_F^3$, $W_3^{ghost}$ individually. For this we need to know the Green’s functions of the theory under consideration. It is a well known result of quantum field theory that the effective action is expressed as a product of the Green’s functions of the theory with the appropriate ‘contractions’. Next we shall describe such Green’s functions.

### 4.1 Feynman Rules and Diagrammatics

The Green’s functions of interest are determined by taking the functional derivatives with respect to the different bosonic and fermionic source functions in the whole generating functional of the IKKT matrix system (21) considered after adding to $S_2$ the interacting part $S_{int}$ given by

$$S_{int} = S_B^4 + S_B^3 + S_F^3 + S_3^{ghost}.$$

(37)

To escape unnecessary details we display below only the key formulas used to evaluate the different terms in the two-loop effective action (36). After adding the source functions $J_\mu$, $\bar{\eta}$, $\eta$, $\bar{\psi}$ and $\chi$ the complete free generating functional is given as a product

$$Z_0 [J, \bar{\eta}, \eta, \bar{\psi}, \chi] = Z^B_0 [J] \ Z^F_0 [\bar{\eta}, \eta]\ Z_0^{ghost} [\bar{\psi}, \chi],$$  \hspace{1cm} (38)

where

$$Z^B_0 [J] = \int D a_\mu e^{-\frac{1}{2} \text{Tr} (a_\mu A_{\mu \nu} a_\nu) + \text{Tr} (J_\mu a_\mu)} = e^{\frac{1}{2} \text{Tr} (J_\mu G_{\mu \nu} J_\nu)},$$  \hspace{1cm} (39)

$$Z^F_0 [\bar{\eta}, \eta] = \int D \bar{\eta} D \eta e^{\frac{1}{2} \text{Tr} (\bar{\varphi} S \varphi) + \text{Tr} (\bar{\varphi} \eta) + \text{Tr} (\eta \varphi)} = e^{-2 \text{Tr} (\bar{\eta} H \eta)},$$  \hspace{1cm} (40)

$$Z_0^{ghost} [\bar{\psi}, \chi] = \int D \bar{b} D c e^{-\text{Tr} (\bar{b} T \bar{c}) + \text{Tr} (\bar{b} \chi) + \text{tr} (\bar{\psi} c)} = e^{\text{Tr} (\bar{\psi} E \chi)},$$  \hspace{1cm} (41)

and

$$A_{\mu \nu} = P^2 \delta_{\mu \nu} - 2 i F_{\mu \nu}, \quad S = \Gamma^\mu P_\mu, \quad T = P^2.$$  \hspace{1cm} (42)

In the functional approach, the Green’s functions are given by the inverse of the operators appearing in the quadratic part of the action. From the formulas above (39), (40) and
we can straightforwardly read the Green’s functions of our theory. For the bosonic quantum fluctuation we have

\[ G_{\mu\nu} = A^{-1}_{\mu\nu} = \left( P^2 \delta_{\mu\nu} - 2iF_{\mu\nu} \right)^{-1}, \]

\[ = \frac{1}{P^2} \delta_{\mu\nu} + 2i \frac{1}{P^2} \frac{1}{P^2} F_{\mu\nu} - 4 \frac{1}{P^2} \frac{1}{P^2} F_{\mu\nu} \frac{1}{P^2} F_{\alpha\nu} + \mathcal{O} \left( F^3, 1/P^8 \right), \]

(43)

where we have expanded the propagator in the inverse power of $1/P^2$ since eventually we are interested (as in the one-loop) only in the long range contributions of the vacuum graphs emerging at two-loops. Such interactions will in principle add up to the one-loop result in (27) and (35) and constitute higher order corrections. The fermionic quantum fluctuations on the other hand involves the Green’s function

\[ H = S^{-1} = (\Gamma^{\mu} P_{\mu})^{-1} = -\frac{P}{P^2} \left( 1 + i \frac{1}{2} \Gamma^{\mu} \Gamma^{\nu} \frac{1}{P^2} F_{\mu\nu} \right)^{-1}, \]

\[ = -\frac{P}{P^2} + \frac{i}{2} P \Gamma^{\mu} \frac{1}{P^2} P \Gamma^{\nu} \frac{1}{P^2} \frac{1}{P^2} F_{\mu\nu} + \frac{1}{4} P \Gamma^{\mu} \Gamma^{\nu} \frac{1}{P^2} \frac{1}{P^2} F_{\mu\nu} \frac{1}{P^2} F_{\alpha\beta} + \mathcal{O} \left( F^3, 1/P^7 \right). \]

(44)

For the ghost fields the Green’s function is simply

\[ E = \mathcal{T}^{-1} = \frac{1}{P^2}. \]

(45)

In the above formulas what we mean by our notation $\text{Tr} \left( J_\mu G_{\mu\nu} J_\nu \right)$ is the following

\[ \text{Tr} \left( J_\mu G_{\mu\nu} J_\nu \right) = \sum_{i,j=1}^{n} \sum_{p=1}^{n_i} \sum_{q=1}^{n_j} \sum_{r=1}^{n_i} \sum_{s=1}^{n_j} (ij, pq) \left( ji, G_{\mu\nu} qr \right) \left( ji, J_{\mu}^{rs} \right), \]

\[ \equiv \sum_{i,j} \sum_{p,q,r,s} (ij, pq) \left( ji, H_{\mu\nu} qr \right) \left( ji, J_{\mu}^{rs} \right), \]

(46)

where we have used our observation of Section (3) which indicates that an operator such as $G_{\mu\nu}$ consisting of only $P_{\mu}$ and $F_{\mu\nu}$ (since it is the inverse of $A_{\mu\nu}$) must act on the blocks sources $ij, J_\mu$ independently. Moreover it should act both on the left and on the right of $ij, J_\mu$ according to the rules (29), (30), (31) and (32) derived in Section (3). The different sums in (46) are easy to understand recalling that the block source $ij, J_\mu$ is a $n_i \times n_j$ block matrix and that the order and the range over which the indices $p, q, r, s$ are summed is in such a way to respect the usual matrix product and the trace cyclic property. Note also that the block indices $(i, j)$ are arranged so as to respect the matrix product and the trace formula. The sum over the repeated spacetime indices $\mu, \nu$ is also understood. Applying the same remarks above to the fermionic and ghost terms, we have readily

\[ \text{Tr} \left( \bar{\eta}, H, \eta \right) = \sum_{i,j=1}^{n} \sum_{p=1}^{n_i} \sum_{q=1}^{n_j} \sum_{r=1}^{n_i} \sum_{s=1}^{n_j} (ij, \bar{\eta}_a pq) \left( ji, H_{ab} qr \right) \left( ji, \eta_b^{rs} \right), \]

\[ \equiv \sum_{i,j} \sum_{p,q,r,s} (ij, \bar{\eta}_a pq) \left( ji, H_{ab} qr \right) \left( ji, \eta_b^{rs} \right), \]

(47)
where the sum over the repeated Dirac indices \(a, b\) above is understood. For the ghosts we have

\[
Tr \left( \bar{\psi}.E.\chi \right) = \sum_{i,j=1}^{n} \sum_{p=1}^{n_i} \sum_{q=1}^{n_j} \sum_{r=1}^{n_j} \sum_{s=1}^{n_i} \left( ij \bar{\psi}^{pq} \right) \left( ji E^{qr}_{sp} \right) \left( ji \chi^{rs} \right),
\]

\[\equiv \sum_{i,j} \sum_{p,q,r,s} \left( ij \bar{\psi}^{pq} \right) \left( ji E^{qr}_{sp} \right) \left( ji \chi^{rs} \right).\]  

(48)

Using these properties, we shall illustrate in the Appendix using some examples that arises in the computation of \(W_B^4, W_B^3, W_F^3\) and \(W_{\text{ghost}}^3\) below how in the trace operation we take into account the fact that the operators \(ij G_{\mu\nu pq}^r, ij H_{ab pq}^r\) and \(ij E_{sp}^r\) act on both the left and the right of the block matrices. Finally, it is clear that the above remarks carry over similarly to our notation of the quadratic terms \(Tr (a_{\mu}.A_{\mu\nu}.a_{\nu}), Tr (\bar{\phi}.S.\phi)\) and \(Tr (\bar{b}.T.c)\) in (38), (40) and (41).

We are now in a position to derive the explicit expressions of each of the two-loop vacuum graphs in Fig (1). By taking into account the matrix nature of the quantum fluctuations on which the propagators act we have indicated bosonic propagator \(ij G_{\mu\nu pq}^r\) by two wavy lines where we put on each the appropriate indices as dictated by the formulas in (46). The fermionic propagator \(ij H_{ab pq}^r\) is indicated by two solid lines with the indices as in (47) and the ghost propagator \(ij E_{sp}^r\) by two dashed lines and with the indices as in (48). To calculate \(W^{(2)}\), we proceed perturbatively treating \(exp(-S_{\text{int}})\) as a power series in the formula

\[
Z \left[ J, \bar{\eta}, \eta, \bar{\psi}, \chi \right] = e^{-S_{\text{int}}}^{\frac{1}{2}} Z_B^0 \left[ J \right] Z_F^0 \left[ \bar{\eta}, \eta \right] Z_{\text{ghost}}^0 \left[ \bar{\psi}, \chi \right],
\]

(49)

where we have make the following substitutions in \(S_B^4, S_B^3, S_F^3\) and \(S_{\text{ghost}}^3\)

\[
\frac{\delta}{\delta ij J_{\mu}^{pq}} \rightarrow ji A_{\mu}^{qp},
\]

(50)

\[
\frac{\delta}{\delta ij H_{ab}^{pq}} \rightarrow - ji \bar{\phi}_{a}^{qp},
\]

(51)

\[
\frac{\delta}{\delta ij \bar{\eta}_{a}^{pq}} \rightarrow ji \phi_{a}^{qp},
\]

(52)

\[
\frac{\delta}{\delta ij \chi^{pq}} \rightarrow - ji b^{qp},
\]

(53)

\[
\frac{\delta}{\delta ij \bar{\psi}^{pq}} \rightarrow ji c^{qp}.
\]

(54)

Escaping further details since at this point our treatment becomes very similar to the usual steps encountered in perturbative field theory [28], we can evaluate the different terms constituting the two-loop effective action \(W^{(2)}\) in (36). We find that the quartic and cubic bosonic interactions contribute respectively the following terms

\[
W_B^4 = - \frac{1}{2} B_{\mu\nu\lambda\rho} \sum_{i,j,k} \sum_{p_1,p_2,p_3,p_4} \left[ ij G_{\mu\nu p_1 p_2} \right] \left[ ik G_{\lambda\rho p_3 p_4} \right],
\]

(55)
\[ W^B_3 = \frac{1}{2} B_{\mu\nu\lambda\rho} B_{\alpha\beta\gamma\delta} \sum_{i,j,k} \sum_{p_1-p_2,p_3-p_4} \sum_{q_1-q_2,q_3-q_4} \left( \left( i d_{\mu} \delta^{p_1 p_2} + i P_{p_1 p_2} \right) \left( i d_{\nu} \delta^{q_1 q_2} + i P_{q_1 q_2} \right) \left[ ij G_{\mu\nu}^{p_3 q_4} \right] \left[ jk G_{\lambda\rho}^{p_4 q_3} \right] \left[ ki G_{\nu\delta}^{p_4 q_3} \right] \left[ jk G_{\lambda\rho}^{p_4 q_3} \right] \right) \]

\[ \left( i d_{\mu} \delta^{p_1 p_2} + i P_{p_1 p_2} \right) \left( i d_{\nu} \delta^{q_1 q_2} + i P_{q_1 q_2} \right) \left( jk G_{\lambda\rho}^{p_2 q_3} \right) \left( ki G_{\mu\nu}^{p_4 q_3} \right) \left[ ji G_{\mu\nu}^{p_4 q_3} \right] \left[ ki G_{\mu\nu}^{p_4 q_3} \right] \left[ jk G_{\lambda\rho}^{p_4 q_3} \right] \left[ jk G_{\lambda\rho}^{p_4 q_3} \right] . \]

Our notation above for \( [ij G_{\mu\nu}^{p_1 p_2}] \) stands for the symmetrised bosonic Green’s function

\[ [ij G_{\mu\nu}^{p_1 p_2}] = \frac{1}{2} \left( ij G_{\mu\nu}^{p_1 p_2} + ji G_{\nu\mu}^{p_1 p_2} \right) , \]

and the coefficient \( B_{\mu\nu\lambda\rho} \) is given in (55). The sum over the indices \((p_1, p_2, p_3, p_4)\) in (55) and (56) must respect of course the left-right multiplication property of \( [ij G_{\mu\nu}^{p_1 p_2}] \) and the order of the block indices \((i, j, k)\). The same field theory techniques lead to the evaluation of the cubic fermionic and ghost interactions which are respectively found to be

\[ W^F_3 = - \frac{1}{2} \sum_{i,j,k} \sum_{p_1-p_2,p_3-p_4} \sum_{q_1-q_2,q_3-q_4} \sum_{a,b,c,d} \left( [ij G_{\mu\nu}^{p_3 q_1}] \Gamma_{\mu}^{\alpha} \left( k H_{\nu\alpha}^{p_2 q_3} \right) \Gamma_{\nu}^{\beta} \left( k H_{\lambda\rho}^{d q_1 p_1} \right) \right) \]

\[ + \left[ ij G_{\mu\nu}^{p_3 q_1} \right] \Gamma_{\mu}^{\alpha} \left( k H_{\nu\alpha}^{q_2 p_1} \right) \Gamma_{\nu}^{\beta} \left( k H_{\lambda\rho}^{d q_1 p_1} \right) ; \]

\[ W^{g\text{host}}_3 = - \sum_{i,j,k} \sum_{p_1-p_2,p_3-p_4} \sum_{q_1-q_2,q_3-q_4} \left( \left( i d_{\mu} \delta^{p_1 p_2} + i P_{p_1 p_2} \right) \left( i d_{\nu} \delta^{q_1 q_2} + i P_{q_1 q_2} \right) \left[ kj G_{\mu\nu}^{p_3 q_4} \right] \left[ ji G_{\rho\lambda}^{p_4 q_3} \right] \left[ ki G_{\mu\nu}^{p_4 q_3} \right] \left[ jk G_{\rho\lambda}^{p_4 q_3} \right] \right) \]

\[ \left( i d_{\mu} \delta^{p_1 p_2} + i P_{p_1 p_2} \right) \left( i d_{\nu} \delta^{q_1 q_2} + i P_{q_1 q_2} \right) \left[ kj G_{\mu\nu}^{p_3 q_4} \right] \left[ ji G_{\rho\lambda}^{p_4 q_3} \right] \left[ ki G_{\mu\nu}^{p_4 q_3} \right] \left[ jk G_{\rho\lambda}^{p_4 q_3} \right] \right) \]

\[ - 2 \left( i d_{\mu} \delta^{p_1 p_2} + i P_{p_1 p_2} \right) \left( i d_{\nu} \delta^{q_2 q_3} + i P_{q_2 q_3} \right) \left[ kj G_{\mu\nu}^{p_3 q_4} \right] \left[ ji G_{\rho\lambda}^{p_4 q_3} \right] \left[ ki G_{\mu\nu}^{p_4 q_3} \right] \left[ jk G_{\rho\lambda}^{p_4 q_3} \right] . \]

As for the bosonic case, we need to keep track also here of the order of the block indices \((i, j, k)\) as well as the left-right multiplication of the propagators while summing over the \((p_1, p_2, p_3, p_4)\) and \((q_1, q_2, q_3, q_4)\) indices. The sum over the \((a, b, c, d)\) indices in (58) will turn into a trace over the product of the Dirac gamma matrices where we can use the identities in (58).
4.2 Comparison to the One-Loop Result

To compare the contribution of the two-loop effective action to the one-loop result in (35) we proceed to replacing in the expressions of $\mathcal{W}^{B}_4$, $\mathcal{W}^{B}_3$, $\mathcal{W}^{F}_3$ and $\mathcal{W}^{\text{ghost}}_3$ above by the long-range expansion of the propagators as given by (43), (44) and (45). Using the rules of (29), (30), (31) and (32) and performing the sums over different indices involved in the expressions of $\mathcal{W}^{B}_4$, $\mathcal{W}^{B}_3$, $\mathcal{W}^{F}_3$ and $\mathcal{W}^{\text{ghost}}_3$ above we can classify our results from the different contributions in the increasing power of $1/(i^d - j^d)^2$ as follows:

$$\mathcal{W}^{B}_4 = \sum_{i,j,k} \left[ -\frac{45 n_i n_j n_k}{(ijd)^2 (iku)^2} - \frac{36 n_j n_k}{(ijd)^2 (iku)^2} \right] \text{tr} \left( f_{i\mu} f_{j\nu} \right) - \frac{36 n_j n_i}{(ijd)^2 (iku)^2} \text{tr} \left( f_{j\mu} f_{k\nu} \right) - \frac{6 n_j n_k}{(ijd)^4 (iku)^4} \text{tr} \left( f_{j\mu} f_{k\nu} \right) + O \left( \frac{1}{d^9} \right) \right),$$

(60)

$$\mathcal{W}^{B}_3 = \sum_{i,j,k} \left[ \frac{27 n_i n_j n_k}{2 (ijd)^2 (iku)^2} + \frac{2 n_j n_i}{(ijd)^4 (iku)^4} \right] \text{tr} \left( f_{i\mu} f_{j\nu} \right) + O \left( \frac{1}{d^9} \right),$$

(61)

$$\mathcal{W}^{F}_3 = \sum_{i,j,k} \left[ \frac{128 n_i n_j n_k}{(ijd)^2 (iku)^2} + \frac{768 n_j n_k}{(ijd)^2 (iku)^4 (jkd)^4} \right] \text{tr} \left( f_{i\mu} f_{j\nu} \right) + O \left( \frac{1}{d^9} \right),$$

(62)

and finally the expression of $\mathcal{W}^{\text{ghost}}_3$ is also calculated up to the $1/d^8$ order and is exactly found to be given by: $\mathcal{W}^{\text{ghost}}_3 = -\mathcal{W}^{B}_4 - \mathcal{W}^{B}_3 - \mathcal{W}^{F}_3 + O(1/d^9)$. This establishes our claim in this paper.

5 Conclusions

In this paper we derived the effective action for the IKKT matrix model up to two loops for the scattering of an arbitrary number D-brane objects of the type IIB string theory. The one-loop computation in [22] revealed the $F^4/r^8$ behavior of the effective action. Our calculation at two-loops showed that no renormalization of the $1/r^8$-term in the effective potential occur.

These results are in agreement with the arguments made in [22] that $1/r^8$ behavior in the effective interactions ensures the cluster property among the D-objects which is important to the $N = 2$ supersymmetry of the IKKT action and also to the dynamical generation of the spacetime coordinates. Furthermore, the exact agreement between the IKKT matrix model and the supergravity results (long-range interactions) is an important consistency check criterion for the IKKT matrix model to describe type IIB superstrings.
6 Appendix

As advertised in Section (4.1), we shall below present some examples where we show how we take into account the fact that the propagators \( G_{\mu\nu qr} \), \( H_{ab qr} \) and \( E_{qr} \) act from the left and from the right while summing over the different indices appearing in \( W_B^4 \), \( W^3_3 \), \( W^F_3 \) and \( W^ghots_3 \) given in (55), (56), (58) and (59), respectively. Since in this paper, we are only interested in the long-range contributions of the two-loop vacuum graphs we can proceed by replacing \( 1/P^2 \) by \( 1/d^2 \) in all the propagators. After doing this, the \( 1/d^2 \) will act simply as an overall real number factor but the field strength \( F_{\mu\nu} \) keeps its action from the left and from the right while summing over the indices. To illustrate this more, we start by putting indices on the bosonic Green’s function which are given in (43) as they appear in the term \( W_B^4 \) of the two-loop effective action. For example, we have for \( G_{\mu\nu qr} \)

\[
G_{\mu\nu qr}^{ij} = \frac{1}{(ij)^2} \delta_{\mu\nu} \delta^{qr} \delta^{sp} + 2 \frac{1}{(ij)^4} \left[ i f_{\mu\nu}^{qr} \delta^{sp} - j f_{\mu\nu}^{sp} \delta^{qr} \right]
- 2 \frac{1}{(ij)^6} \left[ \left( i f_{\mu\alpha}^{qr} j f_{\alpha\nu}^{sp} \right) \delta^{qr} + \left( j f_{\mu\alpha}^{qr} i f_{\alpha\nu}^{sp} \right) \delta^{qr} \right]
- \frac{1}{(ij)^8} \left[ \left( i f_{\mu\alpha}^{qr} j f_{\alpha\nu}^{sp} \right) - \left( j f_{\mu\alpha}^{qr} i f_{\alpha\nu}^{sp} \right) \right] + O\left(F^3, 1/d^8\right).
\]  
\( (63) \)

If we set \( q = r \) and sum over \( q \) using the fact that \( \text{tr}(i f_{\mu\nu}) = 0 \), the above expression simplifies to

\[
\sum_q G_{\mu\nu qr}^{qq} = \frac{1}{(ij)^2} \delta_{\mu\nu} n_i \delta^{sp} - 2 \frac{1}{(ij)^4} n_i j f_{\mu\nu}^{sp}
- 4 \frac{1}{(ij)^6} \left[ \text{tr} \left( i f_{\mu\alpha}^{qr} j f_{\alpha\nu}^{sp} \right) \delta^{qr} + n_i \left( j f_{\mu\alpha}^{qr} i f_{\alpha\nu}^{sp} \right) \right].
\]  
\( (64) \)

Similarly, if we set \( s = p \) and sum over \( p \) using \( \text{tr}(j f_{\mu\nu}) = 0 \), we have

\[
\sum_p G_{\mu\nu qr}^{pp} = \frac{1}{(ij)^2} \delta_{\mu\nu} n_i \delta^{qr} + 2 \frac{1}{(ij)^4} n_i j f_{\mu\nu}^{qr}
- 4 \frac{1}{(ij)^6} \left[ \text{tr} \left( i f_{\mu\alpha}^{qr} j f_{\alpha\nu}^{sp} \right) \delta^{qr} + n_i \left( i f_{\mu\alpha}^{qr} j f_{\alpha\nu}^{sp} \right) \right].
\]  
\( (65) \)

Further simplifications occur if we set \( q = r \) and \( s = p \) in (64) and summing over \( q \) and \( p \) afterwards. The end result is then

\[
\sum_q \sum_p G_{\mu\nu qr}^{qp} = \frac{n_i n_j \delta_{\mu\nu}}{(ij)^2} - 4 \frac{1}{(ij)^6} \left[ n_j \text{tr} \left( i f_{\mu\alpha}^{qr} j f_{\alpha\nu}^{sp} \right) \delta^{qr} + n_i \text{tr} \left( i f_{\mu\alpha}^{qr} j f_{\alpha\nu}^{sp} \right) \right].
\]  
\( (66) \)

From the above remarks, the expression of \( W^B_4 \) in (50) follows readily since it is a simple product of two Green’s functions of the type (64) and (65) with the appropriate sum over
the block indices as required by the trace operation. The evaluation of $W_{3}^{B}$ uses the same manipulations as $W_{4}^{B}$ except that it is more tedious since it involves the contraction of three Green's functions. The other expressions of $W_{3}^{F}$ and $W_{3}^{ghots}$ are also easy to derive once we know how to put and contract the indices in the fermionic and ghost propagators. As for the bosonic propagator above, we display here only the main formulas that are needed. For the ghost propagator they are

\[
\begin{align*}
ij E_{sp}^{qr} &= \frac{\delta^{qr} \delta^{sp}}{(ijd)^2}, \\
\sum_{q} ij E_{sp}^{qq} &= \frac{n_{i} \delta^{sp}}{(ijd)^2}, \\
\sum_{p} ij E_{pp}^{qr} &= \frac{n_{j} \delta^{qr}}{(ijd)^2}, \\
\sum_{q} \sum_{p} ij E_{pp}^{qr} &= \frac{n_{i} n_{j}}{(ijd)^2},
\end{align*}
\]

and for the fermionic propagator they are

\[
\begin{align*}
ij H_{absp}^{qr} &= -ij \delta^{qr} \delta^{sp} \frac{1}{(ijd)^2} + \frac{i}{2} \frac{1}{(ijd)^4} \left( ij \frac{d}{\Gamma^{\mu\nu}} \right)_{ab} \left[ f_{\mu\nu}^{qr} \delta^{sp} - i f_{\mu\nu}^{sp} \delta^{qr} \right] \\
&\quad + \frac{1}{4} \frac{1}{(ijd)^6} \left( ij \frac{d}{\Gamma^{\mu\nu} \Gamma^{\alpha\beta}} \right)_{ab} \left[ \left( i f_{\mu\nu}^{ij} f_{\alpha\beta}^{qr} \right) \delta^{sp} + \left( i f_{\mu\nu}^{ij} f_{\alpha\beta}^{sp} \right) \delta^{qr} \\
&\quad - i f_{\mu\nu}^{qr} f_{\alpha\beta}^{sp} - j f_{\mu\nu}^{sp} i f_{\alpha\beta}^{qr} \right] + O \left( \frac{F^{3}}{1/d^{7}} \right),
\end{align*}
\]

\[
\begin{align*}
\sum_{q} ij H_{absp}^{qq} &= -ij \delta^{qq} \delta^{sp} \frac{1}{(ijd)^2} - \frac{i}{2} \frac{1}{(ijd)^4} \left( ij \frac{d}{\Gamma^{\mu\nu}} \right)_{ab} n_{i} j f_{\mu\nu}^{sp} \\
&\quad + \frac{1}{4} \frac{1}{(ijd)^6} \left( ij \frac{d}{\Gamma^{\mu\nu} \Gamma^{\alpha\beta}} \right)_{ab} \left[ \text{tr} \left( i f_{\mu\nu}^{ij} f_{\alpha\beta}^{qq} \right) \delta^{sp} + n_{i} \left( i f_{\mu\nu}^{ij} f_{\alpha\beta}^{sp} \right) \delta^{qr} \\
&\quad - i f_{\mu\nu}^{qr} f_{\alpha\beta}^{sp} - j f_{\mu\nu}^{sp} i f_{\alpha\beta}^{qr} \right],
\end{align*}
\]

\[
\begin{align*}
\sum_{p} ij H_{abpp}^{qr} &= -ij \delta^{qr} \delta^{sp} \frac{1}{(ijd)^2} + \frac{i}{2} \frac{1}{(ijd)^4} \left( ij \frac{d}{\Gamma^{\mu\nu}} \right)_{ab} n_{j} i f_{\mu\nu}^{qr} \\
&\quad + \frac{1}{4} \frac{1}{(ijd)^6} \left( ij \frac{d}{\Gamma^{\mu\nu} \Gamma^{\alpha\beta}} \right)_{ab} \left[ \text{tr} \left( j f_{\mu\nu}^{ij} f_{\alpha\beta}^{qq} \right) \delta^{sp} + n_{j} \left( i f_{\mu\nu}^{ij} f_{\alpha\beta}^{sp} \right) \delta^{qr} \\
&\quad - i f_{\mu\nu}^{qr} f_{\alpha\beta}^{sp} - j f_{\mu\nu}^{sp} i f_{\alpha\beta}^{qr} \right],
\end{align*}
\]

\[
\begin{align*}
\sum_{q} \sum_{p} ij H_{abpp}^{qq} &= -ij \delta^{qq} \delta^{sp} \frac{1}{(ijd)^2} + \frac{ij \frac{d}{\Gamma^{\mu\nu} \Gamma^{\alpha\beta}}}{4} \left( \frac{1}{(ijd)^6} \right)_{ab} \left[ n_{i} \text{tr} \left( i f_{\mu\nu}^{ij} f_{\alpha\beta}^{qq} \right) + n_{j} \text{tr} \left( j f_{\mu\nu}^{ij} f_{\alpha\beta}^{qq} \right) \right].
\end{align*}
\]

The calculation of $W_{3}^{F}$ involves an extra step which consists of tracing over the Dirac matrices using the identities in (28).
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Figure 1: The standard four two-loop vacuum graphs. Wavy lines are gauge field matrix propagators. Solid lines are the propagators for the fermionic quantum fluctuation matrices and the dashed lines are the matrix ghost propagators.
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