 Locally compact abelian groups admitting non-trivial quasi-convex null sequences*  

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Abstract

In this paper, we show that for every locally compact abelian group $G$, the following statements are equivalent:

(i) $G$ contains no sequence $\{x_n\}_{n=0}^{\infty}$ such that $\{0\} \cup \{\pm x_n \mid n \in \mathbb{N}\}$ is infinite and quasi-convex in $G$, and $x_n \to 0$;

(ii) one of the subgroups $\{g \in G \mid 2g = 0\}$ and $\{g \in G \mid 3g = 0\}$ is open in $G$;

(iii) $G$ contains an open compact subgroup of the form $\mathbb{Z}_2^\kappa$ or $\mathbb{Z}_3^\kappa$ for some cardinal $\kappa$.

1. Introduction

One of the main sources of inspiration for the theory of topological groups is the theory of topological vector spaces, where the notion of convexity plays a prominent role. In this context, the reals $\mathbb{R}$ are replaced with the circle group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, and linear functionals are replaced by characters, that is, continuous homomorphisms to $\mathbb{T}$. By making substantial use of characters, Vilenkin introduced the notion of quasi-convexity for abelian topological groups as a counterpart of convexity in topological vector spaces (cf. \cite{19}). The counterpart of locally convex spaces are the locally quasi-convex groups. This class includes all locally compact abelian groups and locally convex topological vector spaces (cf. \cite{4}).

According to the celebrated Mackey-Arens theorem (cf. \cite{17} and \cite{1}), every locally convex topological vector space $(V, \tau)$ admits a so-called Mackey topology, that is, a locally convex vector space topology $\tau_\mu$ that is finest with respect to the property of having the same set of continuous linear functionals (i.e., $(V, \tau_\mu)^\ast = (V, \tau)^\ast$). Moreover, $\tau_\mu$ can be described as the topology of uniform convergence of the sets of an appropriate family of convex weakly compact sets of $(V, \tau)^\ast$. A counterpart of this notion in the class of locally quasi-convex abelian groups, the so-called Mackey group topology, was proposed in \cite{7}. It seems reasonable to expect to describe the Mackey

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topology of a locally quasi-convex abelian group $G$ as the topology $\tau_\mathcal{G}$ of uniform convergence on members of an appropriate family $\mathcal{G}$ of quasi-convex compact sets of the Pontryagin dual $\hat{G}$, where each set in $\mathcal{G}$ is equipped with the weak topology. This underscores the importance of the compact quasi-convex sets in locally quasi-convex abelian groups. It was proven by Hernández, and independently, by Bruguera and Martín-Peinador, that a metrizable locally quasi-convex abelian group $G$ is complete if and only if the quasi-convex hull of every compact subset of $G$ is compact (cf. [13] and [6]). Lukács extended this result, and proved that a metrizable abelian group $A$ is MAP and has the quasi-convex compactness property if and only if it is locally quasi-convex and complete; he also showed that such groups are characterized by the property that the evaluation map $\alpha_A \colon A \to \hat{A}$ is a closed embedding (cf. [16, I.34]).

Let $\pi \colon \mathbb{R} \to \mathbb{T}$ denote the canonical projection. Since the restriction $\pi|_{[0,1)} : [0,1) \to \mathbb{T}$ is a bijection, we often identify in the sequel, par abus de language, a number $a \in [0,1)$ with its image (coset) $\pi(a) = a + \mathbb{Z} \in \mathbb{T}$. We put $\mathbb{T}_m := \pi([[-\frac{1}{4m}, \frac{1}{4m}]])$ for all $m \in \mathbb{N}\setminus\{0\}$. According to standard notation in this area, we use $\mathbb{T}_+$ to denote $\mathbb{T}_1$. For an abelian topological group $G$, we denote by $\hat{G}$ the Pontryagin dual of a $G$, that is, the group of all characters of $G$ endowed with the compact-open topology.

**Definition 1.1.** For $E \subseteq G$ and $A \subseteq \hat{G}$, the polars of $E$ and $A$ are defined as

$$E^o = \{ \chi \in \hat{G} \mid \chi(E) \subseteq \mathbb{T}_+ \} \quad \text{and} \quad A^o = \{ x \in A \mid \forall \chi \in A, \chi(x) \in \mathbb{T}_+ \}. \quad (1)$$

The set $E$ is said to be quasi-convex if $E = E^{o,q}$. We say that $E$ is $qc$-dense if $G = E^{o,q}$.

Obviously, $E \subseteq E^{o,q}$ holds for every $E \subseteq G$. Thus, $E$ is quasi-convex if and only if for every $x \in G \setminus E$ there exists $\chi \in E^{o,q}$ such that $\chi(x) \notin \mathbb{T}_+$. The set $Q_G(E) := E^{o,q}$ is the smallest quasi-convex set of $G$ that contains $E$, and it is called the quasi-convex hull of $E$.

**Definition 1.2.** A sequence $\{x_n\}_{n=0}^\infty \subseteq G$ is said to be quasi-convex if $S = \{0\} \cup \{ \pm x_n \mid n \in \mathbb{N} \}$ is quasi-convex in $G$. We say that $\{x_n\}_{n=0}^\infty$ is non-trivial if the set $S$ is infinite, and it is a null sequence if $x_n \longrightarrow 0$.

**Example 1.3.** Each of the compact groups $\mathbb{T}$, $\mathbb{J}_2$ (2-adic integers), and $\mathbb{J}_3$ (3-adic integers), and the locally compact group $\mathbb{R}$ admits a non-trivial quasi-convex null sequence (cf. [8, 1.2-1.4] and [9, A-D]).

In this paper, we characterize the locally compact abelian groups $G$ that admit a non-trivial quasi-convex null sequence.

**Theorem A.** For every locally compact abelian group $G$, the following statements are equivalent:

(i) $G$ admits no non-trivial quasi-convex null sequences;
(ii) one of the subgroups $G[2] = \{ g \in G \mid 2g = 0 \}$ and $G[3] = \{ g \in G \mid 3g = 0 \}$ is open in $G$;
(iii) $G$ contains an open compact subgroup of the form $\mathbb{Z}_2^\kappa$ or $\mathbb{Z}_3^\kappa$ for some cardinal $\kappa$.

Furthermore, if $G$ is compact, then these conditions are also equivalent to:

(iv) $G \cong \mathbb{Z}_2^\kappa \times F$ or $G \cong \mathbb{Z}_3^\kappa \times F$, where $\kappa$ is some cardinal and $F$ is a finite abelian group;
(v) one of the subgroups $2G$ and $3G$ is finite.
Theorem A answers a question of L. Aussenhofer in the case of compact abelian groups. The proof of Theorem A is presented in §3. One of its main ingredients is the next theorem, which (together with the results mentioned in Example 1.3) implies that for every prime \( p \), the compact group \( J_p \) of \( p \)-adic integers admits a non-trivial quasi-convex null sequence.

**Theorem B.** Let \( p \geq 5 \) be a prime, \( \underline{a} = \{a_n\}_{n=0}^{\infty} \) an increasing sequence of non-negative integers, and put \( y_n = p^{a_n} \). The set \( L_{\underline{a},p} = \{0\} \cup \{\pm y_n \mid n \in \mathbb{N}\} \) is quasi-convex in \( J_p \).

The proof of Theorem B is presented in §4. Theorem B does not hold for \( p \not\geq 5 \) (see Example 4.9). Dikranjan and de Leo gave sufficient conditions on the sequence \( \underline{a} \) to ensure that \( L_{\underline{a},2} \) is quasi-convex in \( J_2 \) (cf. [8, 1.4]). For the case where \( p = 3 \), Dikranjan and Lukács gave a complete characterization of those sequences \( \underline{a} \) such that \( L_{\underline{a},3} \) is quasi-convex (cf. [9, Theorem D]).

The arguments and techniques developed in the proof of Theorem B are applicable mutandi mutandis for sequences with \( p \)-power denominators in \( \mathbb{T} \). This is done in the next theorem, whose proof is given in §5. Theorem C is a continuation of our study of non-trivial quasi-convex null sequences in \( \mathbb{T} \) (cf. [9]).

**Theorem C.** Let \( p \geq 5 \) be a prime, \( \underline{a} = \{a_n\}_{n=0}^{\infty} \) an increasing sequence of non-negative integers, and put \( x_n = p^{-(a_n+1)} \). The set \( K_{\underline{a},p} = \{0\} \cup \{\pm x_n \mid n \in \mathbb{N}\} \) is quasi-convex in \( \mathbb{T} \).

Theorem C does not hold for \( p \not\geq 5 \) (see Example 5.5). Dikranjan and Lukács gave complete characterizations of those sequences \( \underline{a} \) such that \( K_{\underline{a},2} \) and \( K_{\underline{a},3} \) are quasi-convex in \( \mathbb{T} \) (cf. [9, Theorem A, Theorem C]).

We are leaving open the following three problems. The first two are motivated by the observation that certain parts of Theorem A hold for a class larger than that of (locally) compact abelian groups. Indeed, one can prove, for instance, that items (i), (ii), and (v) of Theorem A remain equivalent for so-called \( \omega \)-bounded abelian groups. (A topological group \( G \) is called \( \omega \)-bounded if every countable subset of \( G \) is contained in some compact subgroup of \( G \).)

**Problem I.** Is it possible to replace the class of locally compact abelian groups in Theorem A with a different class of abelian topological groups that contains all compact abelian groups?

**Problem II.** Is it possible to obtain a characterization for countably compact abelian groups that admit no non-trivial quasi-convex null sequences in terms of their structures, in the spirit of Theorem A?

**Problem III.** Let \( H \) be an infinite cyclic subgroup of \( \mathbb{T} \). Does \( H \) admit a non-trivial quasi-convex null sequence?

2. Preliminaries: Exotic tori and abelian pro-finite groups

In this section, we provide a few well-known definitions and results that we rely on later. We chose to isolate these in order to improve the flow of arguments in §3.

**Definition 2.1.** ([10], [11, p. 141]) A compact abelian group is an exotic torus if it contains no subgroup that is topologically isomorphic to \( J_p \) for some prime \( p \).

The notion of exotic torus was introduced by Dikranjan and Prodanov in [10], who also provided, among other things, the following characterization for such groups.
Theorem 2.2. ([10]) A compact abelian group $K$ is an exotic torus if and only if it contains a closed subgroup $B$ such that

(i) $K/B \cong \mathbb{T}^n$ for some $n \in \mathbb{N}$, and
(ii) $B = \prod_p B_p$, where each $B_p$ is a compact bounded abelian $p$-group.

Furthermore, if $K$ is connected, then each $B_p$ is finite.

Since [10] is not easily accessible for most readers, we provide a sketch of the proof of Theorem 2.2 for the sake of completeness. The main idea of the proof is to pass to the Pontryagin dual, and then to establish that an abelian group is the dual of an exotic torus if and only if it satisfies conditions that are dual to (i) and (ii). To that end, we recall that an abelian group $X$ is strongly non-divisible if and only if its Pontryagin dual $\hat{X}$ contains a subgroup $\hat{F}$ such that there exists a surjective homomorphism $\hat{f}: \hat{X}/\hat{F} \to \mathbb{Z}(p^\infty)$ for some prime $p$ (cf. [10]).

**Proof.** It follows from Pontryagin duality that a compact abelian group $K$ contains no subgroup that is topologically isomorphic to $\mathbb{J}_p$ (i.e., $K$ is an exotic torus) if and only if its Pontryagin dual $X = \hat{K}$ has no quotient that is isomorphic to $\mathbb{Z}(p^\infty) \cong \mathbb{J}_p$ (cf. [18, Theorem 54]), that is, $X$ is strongly non-divisible. Similarly, by Pontryagin duality (cf. [18, Theorem 37 and 54]), a compact abelian group $K$ satisfies (i) and (ii) if and only if its Pontryagin dual $X = \hat{K}$ contains a subgroup $F$ such that

(i') $F \cong \mathbb{Z}^n$ for some $n \in \mathbb{N}$, and
(ii') $X/F \cong \bigoplus_p T_p$, where each $T_p$ is a bounded abelian $p$-group.

Hence, we proceed by showing that a discrete abelian group $X$ is strongly non-divisible if and only if $X$ contains a subgroup $F$ that satisfies (i') and (ii').

Suppose that $X$ is strongly non-divisible. If $X$ contains a free abelian group $F_0$ of infinite rank, then there is a surjective homomorphism $f: F_0 \to \mathbb{Z}(2^\infty)$, and $f$ extends to a surjective homomorphism $\tilde{f}: X \to \mathbb{Z}(2^\infty)$, because $\mathbb{Z}(2^\infty)$ is divisible. Since $X$ is strongly non-divisible, this is impossible. Thus, $X$ contains a free abelian subgroup $F$ of finite rank $n$ such that $X/F$ is a torsion group (i.e., $r_0(X) = n$). Clearly, $F \cong \mathbb{Z}^n$, and so (i') holds. Let $X/F = \bigoplus_p T_p$ be the primary decomposition of $X/F$ (cf. [12, 8.4]), fix a prime $p$, and let $A_p$ be a subgroup of $T_p$ such that $A_p$ is a direct sum of cyclic subgroups, and $T_p/A_p$ is divisible (e.g., take a $p$-basic subgroup of $T_p$; cf. [12, 32.3]). One has $T_p/A_p \cong \bigoplus_{\kappa_p} \mathbb{Z}(p^\infty)$ (cf. [12, 23.1]). Since $T_p/A_p$ is a homomorphic image of the strongly non-divisible group $X$, this implies that $\kappa_p = 0$, and so $T_p = A_p$. Therefore, $T_p$ is a direct sum of cyclic groups. Finally, if $T_p$ is not bounded, then it admits a surjective homomorphism $T_p \to \mathbb{Z}(p^\infty)$. This, however, is impossible, because $T_p$ is a homomorphic image of $X$. Hence, each $T_p$ is a bounded $p$-group, and (ii') holds.

Conversely, suppose that $X$ contains a subgroup $F$ that satisfies (i') and (ii'). Assume that $X$ is not strongly non-divisible, that is, there is a prime $p$ such that there exists a surjective homomorphism $f: X \to \mathbb{Z}(p^\infty)$. Since $\mathbb{Z}(p^\infty)$ is not finitely generated, the finitely generated image $f(F)$ is a proper subgroup of $\mathbb{Z}(p^\infty)$, and so $\mathbb{Z}(p^\infty)/f(F) \cong \mathbb{Z}(p^\infty)$. Thus, by replacing $f$ with its composition with the canonical projection $\mathbb{Z}(p^\infty) \to \mathbb{Z}(p^\infty)/f(F)$, we may assume that $F \subseteq \ker f$. 


Therefore, \( f \) induces a surjective homomorphism \( g: X/F \to \mathbb{Z}(p^\infty) \). Since \( g(\bigoplus_{q \neq p} T_q) = \{0\} \), surjectivity of \( g \) implies that \( g|_{T_p} \) is surjective. This, however, is a contradiction, because by (ii'), \( T_p \) is bounded. Hence, \( X \) is strongly non-divisible.

It remains to be seen that if \( K \) is a connected exotic tori, then each \( B_p \) in (ii) is finite. In terms of the Pontryagin dual \( X = \hat{K} \), this is equivalent to the statement that if \( X \) satisfies (i') and (ii'), and \( X \) is torsion free, then each \( T_p \) is finite (cf. [18, Theorem 46]). We proceed by showing this latter implication. By (ii'), \( X \) contains a subgroup \( F \) such that \( F \cong \mathbb{Z}^n \) for some \( n \in \mathbb{N} \). We have already seen that there is a maximal \( n \) with respect to this property. Since \( X \) is torsion free, maximality of \( n \) implies that \( F \) meets non-trivially every non-zero subgroup of \( X \) (i.e., \( F \) is an essential subgroup). Let \( i: F \to \mathbb{Q}^n \) be a monomorphism such that \( i(F) = \mathbb{Z}^n \). Since \( \mathbb{Q}^n \) is divisible, \( i \) can be extended to a homomorphism \( j: X \to \mathbb{Q}^n \), and \( j \) is a monomorphism, because \( F \cap \ker i = \ker i \) is trivial (and \( F \) is an essential subgroup). Therefore, \( j \) induces a monomorphism \( \bar{j}: X/F \to \mathbb{Q}^n/\mathbb{Z}^n \). Since \( \mathbb{Q}^n/\mathbb{Z}^n \cong \bigoplus_p \mathbb{Z}(p^\infty)^n \), the image \( \bar{j}(T_p) \) is isomorphic to a bounded subgroup of \( \mathbb{Z}(p^\infty)^n \) for every prime \( p \). Hence, each \( T_p \) is finite, because all bounded subgroups of \( \mathbb{Z}(p^\infty)^n \) are finite. This completes the proof.

Recall that a topological group is pro-finite if it is the (projective) limit of finite groups, or equivalently, if it is compact and zero-dimensional. For a prime \( p \), a topological group \( G \) is called a pro-\( p \)-group if it is the (projective) limit of finite \( p \)-groups, or equivalently, if it is pro-finite and \( x^{p^n} \to e \) for every \( x \in G \) (or, in the abelian case, \( p^n x \to 0 \)).

**Theorem 2.3.** ([2], [14, Corollary 8.8(ii)], [11, 4.1.3]) Let \( G \) be a pro-finite group. Then \( G = \prod_p G_p \), where each \( G_p \) is a pro-\( p \)-group.

### 3. LCA groups that admit a non-trivial quasi-convex null sequence

In this section, we prove Theorem A by using two intermediate steps: First, we consider direct products of finite cyclic groups, and then we show that Theorem A holds for pro-finite groups. We start off with a lemma that allows us to relate non-trivial quasi-convex null sequences in closed (and open) subgroups to those in the ambient group.

**Lemma 3.1.** Let \( G \) be a locally compact abelian group, and \( H \) a closed subgroup.

(a) If \( H \) admits a non-trivial quasi-convex null sequence, then so does \( G \).

(b) If \( H \) is open in \( G \), then \( H \) admits a non-trivial quasi-convex null sequence if and only if \( G \) does.

In order to prove Lemma 3.1, we rely on the following general property of the quasi-convexity hull, which will also be used in the proof of Theorem 3.3 below.

**Proposition 3.2.** ([16, I.3(e)], [8, 2.7]) If \( f: G \to H \) is a continuous homomorphism of abelian topological groups, and \( E \subseteq G \), then \( f(Q_G(E)) \subseteq Q_H(f(E)) \).
Proof of Lemma 3.1. (a) Let \( \iota : H \to G \) denote the inclusion. By Proposition 3.2,
\[
Q_H(S) \subseteq \iota^{-1}(Q_G(S)) = Q_G(S) \cap H.
\]
On the other hand, since \( H \) is a subgroup, \( H^\perp = H^\perp \), where \( H^\perp = \{ \chi \in \widehat{G} \mid \chi(H) = \{0\} \} \) is the annihilator of \( H \) in \( \widehat{G} \). Thus, by Pontryagin duality (cf. [18, Theorems 37 and 54]),
\[
Q_G(H) = (H^\perp)^\circ = H^\perp = H,
\]
and so \( Q_G(S) \subseteq Q_G(H) = H \). Finally, \( Q_G(S) \subseteq Q_H(S) \), because by Pontryagin duality, every character of \( H \) extends to a character of \( G \) (cf. [18, Theorem 54]).

(b) The necessity of the condition follows from (a). In order to show sufficiency, let \( \{x_n\}_{n=0}^\infty \) be a non-trivial quasi-convex null sequence in \( G \), and put \( S = \{0\} \cup \{\pm x_n \mid n \in \mathbb{N}\} \). Let \( \iota : H \to G \) denote the inclusion. Then, by Proposition 3.2, \( \iota^{-1}(S) = S \cap H \) is quasi-convex in \( H \). Since \( H \) is a neighborhood of \( 0 \) and \( \{x_n\}_{n=0}^\infty \) is a non-trivial null sequence, the intersection \( S \cap H \) is infinite. Therefore, the subsequence \( \{x_{n_k}\}_{k=0}^\infty \) of \( \{x_n\}_{n=0}^\infty \) consisting of the members that belong to \( H \) is a non-trivial quasi-convex null sequence in \( H \).

Lemma 3.3. Let \( G \) be an abelian topological group of exponent 2 or 3. Then \( G \) admits no non-trivial quasi-convex null sequences. In particular, the groups \( \mathbb{Z}_2^\kappa \) and \( \mathbb{Z}_3^\kappa \) admit no non-trivial quasi-convex null sequences for any cardinal \( \kappa \).

Proof. Observe that if \( x \) is an element of order 2 or 3 in an abelian topological group \( G \), and \( \chi(x) \in T_+ \), then \( \chi(x) = 0 \), and so \( \chi \in \langle x \rangle^\perp \). Thus, if \( S \subseteq G \) and \( S \) consists of elements of order at most 3, then \( S^\circ = S^\perp = \langle S \rangle^\perp \) is a closed subgroup of \( \widehat{G} \). Consequently, \( Q_G(S) = (\langle S \rangle^\perp)^\perp \) is a subgroup of \( G \). Therefore, if \( S \) is a non-trivial null sequence, then \( S \subseteq Q_G(S) \), because every group is homogeneous. Hence, no abelian topological group contains non-trivial quasi-convex sequences consisting of elements of order at most 3. Since each element in \( \mathbb{Z}_2^\kappa \) and \( \mathbb{Z}_3^\kappa \) has order at most 3, this completes the proof.

Lemma 3.3 combined with Lemma 3.1(b) yields the following consequence.

Corollary 3.4. Let \( G \) be a locally compact abelian group, and \( \kappa \) a cardinal. If \( G \) contains an open subgroup that is topologically isomorphic to \( \mathbb{Z}_2^\kappa \) or \( \mathbb{Z}_3^\kappa \), then \( G \) admits no non-trivial quasi-convex null sequences.

Theorem 3.5. Let \( \{m_k\}_{k=1}^\infty \) be a sequence of integers such that \( m_k \geq 4 \) for every \( k \in \mathbb{N} \). Then the product \( P = \prod_{k=0}^\infty \mathbb{Z}_{m_k} \) admits a non-trivial quasi-convex null sequence.

Proof. Let \( \pi_k : P \to \mathbb{Z}_{m_k} \) denote the canonical projection for each \( k \in \mathbb{N} \), and let \( e_n \in P \) be such that \( \pi_k(e_n) = 0 \) if \( k \neq n \), and \( \pi_k(e_k) \) generates \( \mathbb{Z}_{m_k} \). Clearly, \( \{e_n\}_{n=0}^\infty \) is a non-trivial null sequence, and so it remains to be seen that it is quasi-convex. To that end, put \( S = \{0\} \cup \{\pm e_n \mid n \in \mathbb{N}\} \). Since \( \pi_k(S) = \{0, \pm \pi_k(e_k)\} \) is quasi-convex in \( \mathbb{Z}_{m_k} \) for every \( k \in \mathbb{N} \) (cf. [8, 7.8]), by Proposition 3.2,
\[
Q_P(S) \subseteq \bigcap_{k \in \mathbb{N}} \pi_k^{-1}(Q_{\mathbb{Z}_{m_k}}(\pi_k(S))) = \bigcap_{k \in \mathbb{N}} \pi_k^{-1}(\{0, \pm \pi_k(e_k)\}) = \prod_{k \in \mathbb{N}} \{0, \pm \pi_k(e_k)\}.
\]
Every element in the compact group \( P \) can be expressed in the form \( x = \sum_{k \in \mathbb{N}} c_k e_k \), where \( c_k \in \mathbb{Z}_{m_k} \) for every \( k \in \mathbb{N} \). For each \( k \in \mathbb{N} \), let \( \chi_k : P \to \mathbb{T} \) denote the continuous character defined by \( \chi_k(x) = \frac{m_k}{\pi_k} + \mathbb{Z} \), and put \( l_k = \lfloor \frac{m_k}{\pi_k} \rfloor \). (As \( \chi_k \) factors through \( \pi_k \), it is indeed continuous.) Then

\[
l_k \chi_k(e_n) = \begin{cases} \frac{l_k}{m_k} & n = k \\ 0 & n \neq k. \end{cases}
\]

(5)

Consequently, \((l_k \chi_k + l_k \chi_k)(e_n) \in \mathbb{T}_+ \) for every \( k \neq k \), and thus \( l_k \chi_k + l_k \chi_k \in S^\circ \). Let \( x \in Q_p(S) \setminus \{0\} \). By (4), \( x = \sum_{k \in \mathbb{N}} c_k e_k \), where \( c_k \in \{0, 1, -1\} \). Let \( k_1 \in \mathbb{N} \) be the smallest index such that \( c_{k_1} \neq 0 \). By replacing \( x \) with \(-x\) if necessary, we may assume that \( c_{k_1} = 1 \). Let \( k_2 \in \mathbb{N} \) be such that \( k_2 \neq k_1 \). By what we have shown so far, \( l_k \chi_k + c_{k_2} l_k \chi_k \in S^\circ \). Therefore,

\[
l_{k_1} \chi_{k_1} + \frac{l_{k_2} e_{k_2}}{m_{k_2}} + \mathbb{Z} = (l_k \chi_k + c_{k_2} l_k \chi_k)(x) \in \mathbb{T}_+. \quad (6)
\]

Since \( m_k \geq 4 \) for all \( k \in \mathbb{N} \), one has \( l_k > 0 \), and so (6) implies that \( c_{k_2} = 0 \). Hence, \( c_k = 0 \) for all \( k \neq k_1 \). This shows that \( x \in S \), and \( S \) is quasi-convex, as desired.

**Corollary 3.6.** Let \( G \) be a locally compact abelian group that admits no non-trivial quasi-convex null sequences. Then \( G \) has no subgroups that are topologically isomorphic to:

(a) \( \mathbb{J}_p \) for some prime \( p \),

(b) \( \mathbb{Z}_p^\omega \) for some prime \( p \),

(c) \( \mathbb{Z}_p^\omega \) for \( p > 3 \),

(d) \( \mathbb{T} \), or

(e) \( \mathbb{R} \).

**Proof.** By Example 1.3 and Theorem B, for every prime \( p \), the group \( \mathbb{J}_p \) admits a non-trivial quasi-convex null sequence. By Theorem 3.3, for every prime \( p \), the countable product \( \mathbb{Z}_p^\omega \) admits a non-trivial quasi-convex null sequence, and if \( p > 3 \), then so does the group \( \mathbb{Z}_p^\omega \). Finally, by Example 1.3, \( \mathbb{T} \) and \( \mathbb{R} \) admit a non-trivial quasi-convex null sequence. Hence, all five statements follow by Lemma 3.1(a).

**Corollary 3.7.** Let \( p \) be a prime, and \( G \) an abelian pro-\( p \)-group. Then \( G \) admits no non-trivial quasi-convex null sequences if and only if

(i) \( p > 3 \) and \( G \) is finite, or

(ii) \( p \leq 3 \) and \( G \cong \mathbb{Z}^\kappa \times F \) for some cardinal \( \kappa \) and finite group \( F \).

**Proof.** Suppose that \( G \) admits no non-trivial quasi-convex null sequences. Then, by Corollary 3.6(a), \( G \) contains no subgroup that is topologically isomorphic to \( \mathbb{J}_p \), and so \( G \) is an exotic torus. Let \( B \) be a closed subgroup of \( G \) provided by Theorem 2.2. Since \( G \) is a pro-\( p \)-group, it has no connected quotients. So, \( n = 0 \), and \( G = B \). Thus, \( G = B_p \) is a compact bounded \( p \)-group (as \( G \) contains no elements of order coprime to \( p \)). Consequently, \( G \) is topologically isomorphic to a product of finite cyclic groups (cf. [11] 4.2.2)). Hence, \( G \cong \prod_{i=1}^{N} \mathbb{Z}_{p_i}^\kappa_i \) for some cardinals \( \{\kappa_i\}_{i=1}^{N} \). By
Corollary 3.6(b), $G$ contains no subgroup that is topologically isomorphic to $\mathbb{Z}_{p^2}^\omega$. Therefore, $\kappa_i$ is finite for $i \geq 2$, and $G \cong \mathbb{Z}_{p^1}^\times \times F$ for some finite group $F$. Hence, by Corollary 3.6(c), $\kappa < \omega$ or $p \leq 3$, as desired.

Conversely, if $G$ is finite, then it contains no non-trivial sequences, and if (ii) holds, then $G$ contains an open subgroup $H$ that is topologically isomorphic to $\mathbb{Z}_{p^2}^\omega$ or $\mathbb{Z}_3^\kappa$, so the statement follows by Corollary 3.4.

**Proposition 3.8.** Let $G$ be an abelian pro-finite group. Then $G$ admits no non-trivial quasi-convex null sequences if and only if $G \cong \mathbb{Z}_2^\times F$ or $G \cong \mathbb{Z}_3^\kappa \times F$ for some cardinal $\kappa$ and finite abelian group $F$.

**Proof.** Suppose that $G$ admits no non-trivial quasi-convex null sequences. By Lemma 3.1(a), no closed subgroup of $G$ admits a non-trivial quasi-convex null sequence. By Theorem 2.3, $G = \prod_p G_p$, where each $G_p$ is a pro-$p$-group. Since each $G_p$ is a closed subgroup, $G_p$ admits no non-trivial quasi-convex null sequences. Therefore, by Corollary 3.7, for each $p > 3$, the subgroup $G_p$ is finite. Put $F_0 = \prod_{p > 3} G_p$. If $F_0$ is infinite, then there are infinitely many primes $p_k > 3$ such that $G_{p_k} \neq 0$.

Consequently, $F_0$ (and thus $G$) contains a subgroup that is topologically isomorphic to the product $P = \prod_{k=1}^{\infty} \mathbb{Z}_{p_k}$. However, by Theorem 3.5, $P$ does admit a non-trivial quasi-convex null sequence, contrary to our assumption (and Lemma 3.1(a)). This contradiction shows that $F_0$ is finite. Since $G_2$ and $G_3$ contain no non-trivial quasi-convex null sequences, by Corollary 3.7, $G_2 \cong \mathbb{Z}_2^\infty \times F_2$ and $G_3 \cong \mathbb{Z}_3^\kappa_3 \times F_3$ for some cardinals $\kappa_2$ and $\kappa_3$, and finite groups $F_2$ and $F_3$. Thus,

$$G = \prod_p G_p = G_2 \times G_3 \times F_0 \cong \mathbb{Z}_2^\infty \times \mathbb{Z}_3^\kappa_3 \times F_0 \times F_2 \times F_3,$$

and it remains to be seen that at least one of $\kappa_2$ and $\kappa_3$ is finite. Assume the contrary. Then $G$ contains a (closed) subgroup that is topologically isomorphic to $\mathbb{Z}_2^\infty \times \mathbb{Z}_3^\kappa \cong \mathbb{Z}_2^\infty$, which does admit a non-trivial quasi-convex null sequence by Theorem 3.5, contrary to our assumption (and Lemma 3.1(a)). Therefore, at least one of $\mathbb{Z}_2^\kappa_2 \times F_0 \times F_2 \times F_3$ and $\mathbb{Z}_3^\kappa_3 \times F_0 \times F_2 \times F_3$ is finite.

Conversely, if $G \cong \mathbb{Z}_2^\times F$ or $G \cong \mathbb{Z}_3^\kappa \times F$ where $F$ is finite, then $G$ contains an open subgroup $H$ that is topologically isomorphic to $\mathbb{Z}_2^\kappa$ or $\mathbb{Z}_3^\kappa$, so the statement follows by Corollary 3.4.

**Proof of Theorem A.** We first consider the special case where $G$ is compact. Clearly, in this case, (ii) $\Leftrightarrow$ (v) and (iii) $\Leftrightarrow$ (iv).

(i) $\Rightarrow$ (v): Suppose that $G$ admits no non-trivial quasi-convex null sequences. Let $K$ denote the connected component of $G$. By Lemma 3.1(a), $K$ admits no non-trivial quasi-convex null sequences. Thus, by Corollary 3.6(a), $K$ is an exotic torus, and by Theorem 2.2, $K$ contains a closed subgroup $B$ such that $B = \prod_p B_p$, where each $B_p$ is a finite $p$-group, and $K/B \cong \mathbb{T}^n$ for some $n \in \mathbb{T}$. The group $B$ is pro-finite, and by Lemma 3.1(a), it admits no non-trivial quasi-convex null sequences. Therefore, by Proposition 3.8, $B \cong \mathbb{Z}_2^\infty \times F$ or $B \cong \mathbb{Z}_3^\kappa \times F$, where $\kappa$ is some cardinal, and $F$ is a finite abelian group. Since $B_2$ and $B_3$ are finite, this implies that $B$ itself is finite. Consequently, by Pontryagin duality, $\widehat{\mathbb{K}}/B^\perp$ is finite (cf. [18, Theorem 54]), and $B^\perp \cong \mathbb{K}/B = \mathbb{Z}_2^n$.
Proposition 3.8. A direct product of finite cyclic groups (cf. [11, 4.2.2]).

An open compact subgroup admits no non-trivial quasi-convex null sequences. Thus, by what we have shown so far, \( K \) has an open compact subgroup (cf. [18, Theorem 37]). This implies that \( K \) is finitely generated. On the other hand, \( K \) is torsion free, because \( K \) is connected (cf. [18, Example 73]), which means that \( \hat{K} = \mathbb{Z}^n \) and \( K \cong \mathbb{T}^n \). By Corollary 3.6(d), \( K \) contains no subgroup that is topologically isomorphic to \( \mathbb{T} \). Hence, \( n = 0 \), and \( K = 0 \). This shows that (i) implies that \( G \) is pro-finite (i.e., a zero-dimensional compact group). The statement follows now by Proposition 3.8.

(v) \( \Rightarrow \) (iv) is obvious, because (v) implies that \( G \) is a compact bounded group; therefore, \( G \) is a direct product of finite cyclic groups (cf. [11, 4.2.2]).

(iv) \( \Rightarrow \) (i): It follows from (iv) that \( G \) is pro-finite, and thus the statement is a consequence of Proposition 3.8.

Suppose now that \( G \) is a locally compact abelian group.

(i) \( \Rightarrow \) (ii): Suppose that \( G \) admits no non-trivial quasi-convex null sequences. There is \( n \in \mathbb{N} \) and a closed subgroup \( M \) such that \( M \) has an open compact subgroup \( K \), and \( G \cong \mathbb{R}^n \times M \) (cf. [11, 3.3.10]). By Corollary 3.6(e), \( n = 0 \), and so \( G = M \) and \( K \) is open in \( G \). By Lemma 3.1(a), \( K \) admits no non-trivial quasi-convex null sequences. Thus, by what we have shown so far, \( K \) has an open compact subgroup \( O \) that is topologically isomorphic to \( \mathbb{Z}_2^\kappa \) or \( \mathbb{Z}_3^\kappa \) for some cardinal \( \kappa \). Since \( K \) is open in \( G \), it follows that \( O \) is open in \( G \). Therefore, one of \( G[2] \) and \( G[3] \) is open in \( G \), because \( O \subseteq G[2] \) or \( O \subseteq G[3] \).

(ii) \( \Rightarrow \) (iii): Let \( L \) be a bounded locally compact abelian group. Then there is \( n \in \mathbb{N} \) and a closed subgroup \( M \) such that \( M \) has an open compact subgroup \( K \), and \( L \cong \mathbb{R}^n \times M \) (cf. [11, 3.3.10]). Since \( L \) is bounded, \( n = 0 \), and thus \( L \) admits an open compact bounded subgroup \( K \). Consequently, \( K \) is a direct product of finite cyclic groups (cf. [11, 4.2.2]). By this argument, for every locally compact abelian group \( G \), the subgroups \( G[2] \) and \( G[3] \) contain open subgroups \( O_2 \) and \( O_3 \), respectively, such that \( O_2 \cong \mathbb{Z}_2^{\kappa_2} \) and \( O_3 \cong \mathbb{Z}_3^{\kappa_3} \) for some cardinals \( \kappa_2 \) and \( \kappa_3 \). If one of \( G[2] \) and \( G[3] \) is open in \( G \), then \( O_2 \) or \( O_3 \) is open in \( G \), as desired.

(iii) \( \Rightarrow \) (i) follows by Corollary 3.4.

\[ \square \]

4. Sequences of the form \( \{p^{\alpha_n}\}_{n=1}^\infty \) in \( \mathbb{J}_p \)

In this section, we present the proof of Theorem 3. We start off by establishing a few preliminary facts concerning representations of elements in \( \mathbb{T} \), which are recycled and reused in \( \S 5 \) where Theorem C is proven. We identify points of \( \mathbb{T} \) with \((-1/2, 1/2] \). Let \( p > 2 \) be a prime. Recall that every \( y \in (-1/2, 1/2] \) can be written in the form

\[ y = \sum_{i=1}^\infty \frac{c_i}{p^i} = \frac{c_1}{p} + \frac{c_2}{p^2} + \cdots + \frac{c_s}{p^s} + \cdots, \tag{8} \]

where \( c_i \in \mathbb{Z} \) and \( |c_i| \leq \frac{p-1}{2} \) for all \( i \in \mathbb{N} \).

**Theorem 4.1.** Let \( p > 2 \) be a prime, and \( y = \sum_{i=1}^\infty \frac{c_i}{p^i} \in \mathbb{J}_p \), where \( c_i \in \mathbb{Z} \) and \( |c_i| \leq \frac{p-1}{2} \) for all \( i \in \mathbb{N} \).

(a) If \( y \in \mathbb{T}_+ \), then \( |c_1| \leq \left\lfloor \frac{p+1}{4} \right\rfloor \).

(b) If \( my \in \mathbb{T}_+ \) for all \( m = 1, \ldots, \left\lfloor \frac{p}{2} \right\rfloor \), then \( c_1 = 0 \).

(c) If \( my \in \mathbb{T}_+ \) for all \( m = 1, \ldots, \left\lfloor \frac{p}{6} \right\rfloor \), then \( c_1 \in \{-1, 0, 1\} \).
PROOF. Since $|c_i| \leq \frac{p-1}{2}$ for all $i \in \mathbb{N}$, one has

$$
\sum_{i=2}^{\infty} \frac{c_i}{p^i} \leq \sum_{i=2}^{\infty} \frac{|c_i|}{p^i} \leq \frac{p-1}{2} \left( \sum_{i=2}^{\infty} \frac{1}{p^i} \right) = \frac{p-1}{2} \cdot \frac{1}{p^2} \cdot p - 1 = \frac{1}{2p}.
$$

(9)

(a) If $y \in \mathbb{T}_+$, then by (9), one obtains that

$$
\left| \frac{c_1}{p} \right| \leq |y| + \sum_{i=2}^{\infty} \frac{|c_i|}{p^i} \leq \frac{1}{4} + \frac{1}{2p} = \frac{p+2}{4p}.
$$

Therefore, $|c_1| \leq \frac{p+2}{4}$, and hence $|c_1| \leq \frac{p+2}{4}$. 

(b) Put $l = \lceil \frac{p}{2} \rceil$. If $my \in \mathbb{T}_+$ for all $m = 1, \ldots, l$, then $y \in \{1, \ldots, l\} = \mathbb{T}_l$, and so $|y| \leq \frac{1}{4l}$. Since $p$ is odd, $\frac{p}{2} < l$. Thus, $|y| \leq \frac{1}{4l} < \frac{1}{2p}$. Therefore, by (9), one obtains that

$$
\left| \frac{c_1}{p} \right| \leq |y| + \sum_{i=2}^{\infty} \frac{|c_i|}{p^i} < \frac{1}{2p} + \frac{1}{2p} = \frac{1}{p}.
$$

Hence, $c_1 = 0$, as required.

(c) Put $l = \lceil \frac{p}{6} \rceil$. If $my \in \mathbb{T}_+$ for all $m = 1, \ldots, l$, then $y \in \{1, \ldots, l\} = \mathbb{T}_l$, and so $|y| \leq \frac{1}{4l}$. Since $p$ is odd, $\frac{p}{6} < l$. Thus, $|y| \leq \frac{1}{4l} < \frac{3}{2p}$. Therefore, by (9), one obtains that

$$
\left| \frac{c_1}{p} \right| \leq |y| + \sum_{i=2}^{\infty} \frac{|c_i|}{p^i} < \frac{3}{2p} + \frac{1}{2p} = \frac{2}{p}.
$$

Hence, $|c_1| \leq 1$, as required.

Corollary 4.2. Let $p \geq 5$ be a prime such that $p \neq 7$, and $y = \sum_{i=1}^\infty \frac{c_i}{p^i} \in \mathbb{T}$, where $c_i \in \mathbb{Z}$ and $|c_i| \leq \frac{p-1}{2}$ for all $i \in \mathbb{N}$. If $my \in \mathbb{T}_+$ for all $m = 1, \ldots, \lfloor \frac{p}{4} \rfloor$, then $c_1 \in \{-1, 0, 1\}$.

**PROOF.** If $p = 5$, then $y \in \mathbb{T}_+$, and by Theorem 4.1(a), $|c_1| \leq \lfloor \frac{4}{5} \rfloor = 1$. If $p \geq 11$, then $\lfloor \frac{p}{6} \rfloor \leq \lfloor \frac{p}{4} \rfloor$, and the statement follows by Theorem 4.1(c). □

Example 4.3. Let $y = \frac{2}{7} - \frac{3}{10} + \mathbb{Z} = \frac{11}{10} + \mathbb{Z}$. Since $\frac{11}{10} < \frac{12}{18} = \frac{1}{2}$, clearly $y \in \mathbb{T}_+$, and thus $my \in \mathbb{T}_+$ for all $1 \leq m \leq \lfloor \frac{7}{4} \rfloor = 1$. However, $c_1 = 2$ and $c_2 = -3$. This shows that the assumption that $p \neq 7$ cannot be omitted in Corollary 4.2. Nevertheless, a slightly weaker statement does hold for all primes $p \geq 5$, including $p = 7$.

Corollary 4.4. Let $p \geq 5$ be a prime, and $y = \sum_{i=1}^\infty \frac{c_i}{p^i} \in \mathbb{T}$, where $c_i \in \mathbb{Z}$ and $|c_i| \leq \frac{p-1}{2}$ for all $i \in \mathbb{N}$. If $my \in \mathbb{T}_+$ for all $m = 1, \ldots, \lfloor \frac{p}{4} \rfloor$ and $(p-1)y \in \mathbb{T}_+$, then $c_1 \in \{-1, 0, 1\}$.

**PROOF.** In light Corollary 4.2 it remains to be seen that the statement holds for $p = 7$. In this case, it is given that $y, 6y \in \mathbb{T}_+$, which means that

$$
y \in \{1, 6\}^\mathbb{Q} = \mathbb{T}_6 \cup (-\frac{1}{6} + \mathbb{T}_6) \cup (\frac{1}{6} + \mathbb{T}_6).
$$

(13)
Thus, \(|y| \leq \frac{5}{24}\). Therefore, by \((9)\), one obtains that
\[
\left| \frac{c_1}{7} \right| \leq |y| + \left| \sum_{i=2}^{\infty} \frac{c_i}{7^i} \right| \leq \frac{5}{24} + \frac{1}{14} = \frac{47}{168} < \frac{48}{168} = \frac{2}{7}.
\] (14)

Hence, \(|c_1| \leq 1\), as desired. \(\Box\)

We turn now to investigating the set \(L_{\mathbb{Z},p}\), its polar, and its quasi-convex hull. Recall that the Pontryagin dual \(\mathbb{J}_p\) of \(\mathbb{J}_p\) is the Prüfer group \(\mathbb{Z}(p^\infty)\). For \(k \in \mathbb{N}\), let \(\zeta_k: \mathbb{J}_p \to \mathbb{T}\) denote the continuous character defined by \(\zeta_k(1) = p^{-(k+1)}\). For \(m \in \mathbb{N}\), put \(J_{\mathbb{Z},p,m} = \{k \in \mathbb{N} \mid m\zeta_k \in L^*_{\mathbb{Z},p}\}\) and \(Q_{\mathbb{Z},p,m} = \{m\zeta_k \mid k \in J_{\mathbb{Z},p,m}\}\).

**Lemma 4.5.** For \(1 \leq m \leq p - 1\),
\[
J_{\mathbb{Z},p,m} = \begin{cases} \mathbb{N} & \text{if } \frac{m}{p} \in \mathbb{T}_+ \\ \mathbb{N} \setminus \mathfrak{A} & \text{if } \frac{m}{p} \notin \mathbb{T}_+. \end{cases}
\] (15)

**Proof.** For \(1 \leq m \leq p - 1\) and \(i \in \mathbb{Z}\),
\[
m \cdot p^i \in \mathbb{T}_+ \iff i \neq -1 \lor (i = -1 \land \frac{m}{p} \in \mathbb{T}_+). \] (16)
\[
\iff i \neq -1 \lor \frac{m}{p} \in \mathbb{T}_+. \] (17)

For \(k, n \in \mathbb{N}\), one has \(m\zeta_k(x_n) = m \cdot p^{a_n-k-1}\). Thus, by \((16)\) applied to \(i = a_n - k - 1\),
\[
m\zeta_k(x_n) \in \mathbb{T}_+ \iff a_n - k - 1 \neq -1 \lor \frac{m}{p} \in \mathbb{T}_+. \] (18)
\[
\iff k \neq a_n \lor \frac{m}{p} \in \mathbb{T}_+. \] (19)

Consequently, \(m\zeta_k \in L^*_{\mathbb{Z},p}\) if and only if \(k \neq a_n\) for all \(n \in \mathbb{N}\), or \(\frac{m}{p} \in \mathbb{T}_+. \) \(\Box\)

**Theorem 4.6.** If \(p > 2\), then \(Q_{\mathbb{Z},p,1} \cap \cdots \cap Q_{\mathbb{Z},p,p-1} \subseteq \{\sum_{n=0}^{\infty} \varepsilon_n y_n \mid (\forall n \in \mathbb{N})(\varepsilon_n \in \{-1, 0, 1\})\}\).

**Proof.** Recall that every element \(x \in \mathbb{J}_p\) can be written in the form \(x = \sum_{i=0}^{\infty} c_i \cdot p^i\), where \(c_i \in \mathbb{Z}\) and \(|c_i| \leq \frac{p-1}{2}\). For \(x\) represented in this form, for every \(k \in \mathbb{N}\), one has
\[
\zeta_k(x) = \sum_{i=0}^{\infty} c_i \cdot p^{i-k-1} \equiv \sum_{i=0}^{k} c_i \cdot p^{i-k-1} = \sum_{i=1}^{k+1} c_{k-i+1} \cdot p^{i}. \] (20)

Let \(x \in Q_{\mathbb{Z},p,1} \cap \cdots \cap Q_{\mathbb{Z},p,p-1}\). If \(k \neq a_n\) for any \(n \in \mathbb{N}\), then by Lemma 4.5,
\[
k \in \mathbb{N} \setminus \mathfrak{A} = J_{\mathbb{Z},p,1} = \cdots = J_{\mathbb{Z},p,p-1}. \] (21)

Thus, for \(y = \zeta_k(x)\), one has \(my \in \mathbb{T}_+\) for \(m = 1, \ldots, p - 1\). Therefore, by Theorem 4.1(b), the coefficient of \(\frac{1}{p}\) in \((20)\) is zero. Hence, \(c_k = 0\) for every \(k\) such that \(k \neq a_n\) for all \(n \in \mathbb{N}\). For
\( p = 3 \), this already completes the proof, and so we may assume that \( p \geq 5 \). If \( k = a_n \) for some \( n \in \mathbb{N} \), then by Lemma 4.5,

\[
k \in \mathbb{N} = J_{a, 1} = \cdots = J_{a, p-1} = J_{a, p-1}.
\]

Consequently, for \( y = \zeta_k(x) \), one has \( my \in \mathbb{T}_+ \) for \( m = 1, \ldots, \lfloor \frac{p}{4} \rfloor \) and \( (p-1)y \in \mathbb{T}_+ \). Therefore, by Corollary 4.4, the coefficient of \( \frac{1}{p} \) in (20) belongs to \( \{-1, 0, 1\} \). Hence, \( x \) has the form

\[
x = \sum_{n=0}^{\infty} \frac{c_{a_n}}{p^n} \sum_{n=0}^{\infty} \varepsilon_n x_n,
\]

where \( \varepsilon_n = c_{a_n} \in \{-1, 0, 1\} \).

\[\square\]

Put \( L_p = L_{\alpha, p} \), that is, the special case of \( L_{a, p} \), where \( \alpha \) is the sequence \( a_n = n \). First, we show that if \( p \geq 5 \), then \( L_p \) is quasi-convex.

**Lemma 4.7.** Let \( p \geq 5 \) be a prime. Then \( m_1 \zeta_{k_1} + \cdots + m_l \zeta_{k_l} \in L_p^p \) for every \( 0 \leq k_1 < \cdots < k_l \) and \( m_1, \ldots, m_l \) such that \( |m_i| \leq \lfloor \frac{p}{4} \rfloor \) for all \( i \leq l \).

**Proof.** Let \( p^n \in L_p \). We may assume that \( n \leq k_1 \), because \( \eta_{k_i}(p^n) = 0 \) for every \( i \in \mathbb{N} \) such that \( k_i < n \). Since \( 0 \leq k_1 < \cdots < k_l \),

\[
\frac{1}{p^{k_1+1}} + \cdots + \frac{1}{p^{k_l+1}} \leq \frac{1}{p^{k_1+1}} + \frac{1}{p^{k_1+2}} + \cdots + \frac{1}{p^{k_1+l}} < \sum_{i=0}^{\infty} \frac{1}{p^{k_1+i+1}} = \frac{1}{p^{k_1}(p-1)}.
\]

One has \( |m_i| \leq \lfloor \frac{p}{4} \rfloor \leq \frac{p-1}{4} \) for each \( i \), because \( p \) is odd. Thus,

\[
| (m_1 \zeta_{k_1} + \cdots + m_l \zeta_{k_l}) (p^n) | \leq |m_1| |\eta_{k_1}(p^n)| + \cdots + |m_l| |\eta_{k_l}(p^n)| \leq \frac{p-1}{4} \left( \frac{1}{p^{k_1+1}} + \cdots + \frac{1}{p^{k_l+1}} \right) p^n < \frac{p-1}{4} \cdot \frac{1}{p^{k_1}(p-1)} \cdot p^n = \frac{1}{4p^{k_1-n}} \leq \frac{1}{4}.
\]

Therefore, \( (m_1 \zeta_{k_1} + \cdots + m_l \zeta_{k_l})(p^n) \in \mathbb{T}_+ \) for all \( n \in \mathbb{N} \). Hence, \( m_1 \zeta_{k_1} + \cdots + m_l \zeta_{k_l} \in L_p^p \), as desired.

**Proposition 4.8.** Let \( p \geq 5 \) be a prime. Then \( L_p \) is quasi-convex.

**Proof.** Let \( x \in Q_\mathcal{L}(L_p) \setminus \{0\} \). Then \( x \in Q_{\mathbb{N}, 1} \cap \cdots \cap Q_{\mathbb{N}, p-1} \). Consequently, by Theorem 4.6, \( x = \sum_{n=0}^{\infty} \varepsilon_n p^n \), where \( \varepsilon_n \in \{-1, 0, 1\} \) for all \( n \in \mathbb{N} \). Observe that for \( k \in \mathbb{N} \),

\[
\zeta_k(x) = \sum_{n=0}^{\infty} \frac{\varepsilon_n}{p^{n+k+1}} = \sum_{n=0}^{k} \frac{\varepsilon_n}{p^{n+k+1}} = \sum_{i=1}^{k+1} \frac{\varepsilon_{k-i+1}}{p^i}.
\]
Let $k_1$ denote the smallest index such that $\varepsilon_{k_1} \neq 0$, and let $k_2 \in \mathbb{N}$ be such that $k_1 < k_2$. In order to prove that $x \in K_p$, it remains to be seen that $\varepsilon_{k_2} = 0$. In order to simplify notations, we set $\varepsilon_n = 0$ for all $n \in \mathbb{Z} \setminus \mathbb{N}$. Put

$$y_+ = (\zeta_{k_1} + \zeta_{k_2})(x) = \sum_{i=1}^{k_1+1} \frac{\varepsilon_{k_1-i+1}}{p^i} + \sum_{i=1}^{k_2+1} \frac{\varepsilon_{k_2-i+1}}{p^i} = \sum_{i=1}^{k_1+1} \frac{\varepsilon_{k_1-i+1} + \varepsilon_{k_2-i+1}}{p^i}, \quad \text{and} \quad \tag{29}$$

$$y_- = (\zeta_{k_1} - \zeta_{k_2})(x) = \sum_{i=1}^{k_1+1} \frac{\varepsilon_{k_1-i+1}}{p^i} - \sum_{i=1}^{k_2+1} \frac{\varepsilon_{k_2-i+1}}{p^i} = \sum_{i=1}^{k_1+1} \frac{\varepsilon_{k_1-i+1} - \varepsilon_{k_2-i+1}}{p^i}. \quad \tag{30}$$

By Lemma 4.7, $m\zeta_{k_1} + m\zeta_{k_2}, m\zeta_{k_1} - m\zeta_{k_2} \in L_p$ for $m = 1, \ldots, \lfloor \frac{p}{4} \rfloor$. Thus, for $m = 1, \ldots, \lfloor \frac{p}{4} \rfloor$,

$$my_+ = m(\zeta_{k_1} + \zeta_{k_2})(x) \in \mathbb{T}_+, \quad \text{and} \quad \tag{31}$$

$$my_- = m(\zeta_{k_1} - \zeta_{k_2})(x) \in \mathbb{T}_+. \quad \tag{32}$$

One has $|\varepsilon_{k_1-i+1} + \varepsilon_{k_2-i+1}| \leq 2 \leq \frac{p-1}{2}$ and $|\varepsilon_{k_1-i+1} - \varepsilon_{k_2-i+1}| \leq 2 \leq \frac{p-1}{2}$, for all $i \in \mathbb{Z}$, because $|\varepsilon_n| \leq 1$ for all $n \in \mathbb{Z}$. By replacing $x$ with $-x$ if necessary, we may assume that $\varepsilon_{k_1} = 1$. Put $\rho = \varepsilon_{k_2}$. We may apply Corollary 4.2 and 4.4 but we must distinguish between three (overlapping) cases:

1. If $p \neq 7$, then by Corollary 4.2, the coefficients of $\frac{1}{p}$ in (29) and (30) are $-1, 0, or 1$. In other words, $|1 + \rho| \leq 1$ and $|1 - \rho| \leq 1$. Since $\rho \in \{-1, 0, 1\}$, this implies that $\rho = 0$, as required.

2. If $k_1 + 1 < k_2$, then by Lemma 4.7,

$$\begin{cases} (p-1)(\eta_{k_1} + \eta_{k_2}) = \zeta_{k_1-1} + \zeta_{k_2-1} \in L_p, & \text{and} \\ (p-1)(\eta_{k_1} - \eta_{k_2}) = \zeta_{k_1-1} - \zeta_{k_2-1} \in L_p. & \end{cases} \quad \tag{33}$$

(If $k_1 = 0$, then of course, $\zeta_{k_1-1} = \zeta_{-1} = 0$.) Thus, in addition to (31) and (32), one also has

$$\begin{cases} (p-1)y_+ = (p-1)(\zeta_{k_1} + \zeta_{k_2})(x) \in \mathbb{T}_+, & \text{and} \\ (p-1)y_- = (p-1)(\zeta_{k_1} - \zeta_{k_2})(x) \in \mathbb{T}_+. & \end{cases} \quad \tag{35}$$

Therefore, by Corollary 4.4, the coefficients of $\frac{1}{p}$ in (29) and (30) are $-1, 0, or 1$. In other words, $|1 + \rho| \leq 1$ and $|1 - \rho| \leq 1$. Since $\rho \in \{-1, 0, 1\}$, this implies that $\rho = 0$, as required.

3. If $p = 7$ and $k_2 = k_1 + 1$, then by what we have shown so far,

$$x = 7^{k_1} + \rho \cdot 7^{k_1+1} = (1 + 7\rho) \cdot 7^{k_1}, \quad \tag{37}$$

where $\rho \in \{-1, 0, 1\}$. By Lemma 4.7

$$(7\rho + 1)\zeta_{k_1+1} = \rho\zeta_{k_1} + \zeta_{k_1+1} \in L_p. \quad \tag{38}$$

Therefore,

$$\frac{2\rho}{7} + \frac{1}{49} = \frac{14\rho + 1}{49} \equiv_1 \frac{(7\rho + 1)^2}{49} = (7\rho + 1)\zeta_{k_1+1}(x) \in \mathbb{T}_+. \quad \tag{39}$$

Hence, $\rho = 0$, as required. \qed
In particular, \( k \eta \) in this section, we prove Theorem C. Recall that the Pontryagin dual \( Q_\pi(L_{\mathbb{A},p}) \subseteq L_p \). On the other hand, by Theorem 4.6
\[
Q_\pi(L_{\mathbb{A},p}) \subseteq Q_\pi(1 \cap \cdots \cap Q_{\mathbb{A},p,p-1}) \subseteq \left\{ \sum_{n=0}^{\infty} \varepsilon_n y_n \mid (\forall n \in \mathbb{N}) (\varepsilon_n \in \{-1, 0, 1\}) \right\}.
\]
Therefore,
\[
Q_\pi(L_{\mathbb{A},p}) \subseteq L_p \cap \left\{ \sum_{n=0}^{\infty} \varepsilon_n y_n \mid (\forall n \in \mathbb{N}) (\varepsilon_n \in \{-1, 0, 1\}) \right\} = L_{\mathbb{A},p},
\]
as desired.

**Example 4.9.** If \( p = 2 \) or \( p = 3 \), then \( Q_\pi(L_p) = \mathbb{J}_p \), that is, \( L_p \) is qc-dense in \( \mathbb{J}_p \) (cf. 4.6(c)). In particular, \( L_2 \) and \( L_3 \) are not quasi-convex in \( \mathbb{J}_2 \) and \( \mathbb{J}_3 \), respectively. Thus, Proposition 4.8 fails if \( p \not\geq 5 \). Therefore, Theorem B does not hold for \( p \not\geq 5 \).

5. Sequences of the form \( \{p^{-(a_n+1)}\}_{n=1}^{\infty} \) in \( \mathbb{T} \)

In this section, we prove Theorem C. Recall that the Pontryagin dual \( \hat{T} \) of \( T \) is \( \mathbb{Z} \). For \( k \in \mathbb{N} \), let \( \eta_k : T \to \mathbb{T} \) denote the continuous character defined by \( \eta_k(x) = p^k \cdot x \). For \( m \in \mathbb{N} \), put
\[
J_{\mathbb{A},p,m} = \{ k \in \mathbb{N} \mid m \eta_k \in K_{\mathbb{A},p}^c \} \quad \text{and} \quad Q_{\mathbb{A},p,m} = \{ m \eta_k \mid k \in J_{\mathbb{A},p,m} \}^a.
\]

**Lemma 5.1.** For \( 1 \leq m \leq p - 1 \),
\[
J_{\mathbb{A},p,m} = \begin{cases} 
\mathbb{N} & \text{if } \frac{m}{p} \in \mathbb{T}_+ \\
\mathbb{N} \setminus \mathbb{A} & \text{if } \frac{m}{p} \not\in \mathbb{T}_+.
\end{cases}
\]

**Proof.** For \( 1 \leq m \leq p - 1 \) and \( i \in \mathbb{Z} \),
\[
m \cdot p^i \in \mathbb{T}_+ \iff i \not= -1 \lor (i = -1 \land \frac{m}{p} \in \mathbb{T}_+) \iff i \not= -1 \lor \frac{m}{p} \in \mathbb{T}_+.
\]
For \( k, n \in \mathbb{N} \), one has \( m \eta_k(x_n) = m \cdot p^{k-a_n-1} \). Thus, by (44) applied to \( i = k - a_n - 1 \),
\[
m \eta_k(x_n) \in \mathbb{T}_+ \iff k - a_n - 1 \not= -1 \lor \frac{m}{p} \in \mathbb{T}_+ \iff k \not= a_n \lor \frac{m}{p} \in \mathbb{T}_+.
\]
Consequently, \( m \eta_k \in K_{\mathbb{A},p}^c \) if and only if \( k \not= a_n \) for all \( n \in \mathbb{N} \), or \( \frac{m}{p} \in \mathbb{T}_+ \).

**Theorem 5.2.** If \( p > 2 \), then \( Q_{\mathbb{A},p,1} \cap \cdots \cap Q_{\mathbb{A},p,p-1} \subseteq \left\{ \sum_{n=0}^{\infty} \varepsilon_n x_n \mid (\forall n \in \mathbb{N}) (\varepsilon_n \in \{-1, 0, 1\}) \right\} \).
PROOF. Let \( x = \sum_{i=1}^{\infty} \frac{c_i}{p^i} \) be a representation of \( x \in \mathbb{T} \). Observe that for every \( k \in \mathbb{N} \), one has

\[
\eta_k(x) = p^k x = \sum_{i=1}^{\infty} \frac{c_i}{p^i} \equiv 1 \sum_{i=k+1}^{\infty} \frac{c_i}{p^i} = \sum_{i=1}^{\infty} \frac{c_{k+i}}{p^i}.
\]

Let \( x \in Q_{\mathbb{A},p,1} \cap \cdots \cap Q_{\mathbb{A},p,p-1} \). If \( k \neq a_n \) for any \( n \in \mathbb{N} \), then by Lemma 5.1

\[
k \in \mathbb{N} \setminus \mathbb{A} = J_{\mathbb{A},p,1} = \cdots = J_{\mathbb{A},p,p-1}.
\]

Thus, for \( y = \eta_k(x) \), one has \( my \in \mathbb{T}_+ \) for \( m = 1, \ldots, p-1 \). Arguing as in the proof of Theorem 4.6 we conclude that \( c_{k+1} = 0 \) for every \( k \) such that \( k \neq a_n \) for all \( n \in \mathbb{N} \). For \( p = 3 \), this already completes the proof, and so we may assume that \( p \geq 5 \). If \( k = a_n \) for some \( n \in \mathbb{N} \), then by Lemma 5.1

\[
k \in \mathbb{N} = J_{\mathbb{A},p,1} = \cdots = J_{\mathbb{A},p,p-1}. \tag{50}
\]

Arguing further as in the proof of Theorem 4.6 (but making use of the characters \( \eta_k \) instead of \( \zeta_k \)), we conclude that \( c_{k+1} = c_{a_n+1} \in \{-1, 0, 1\} \). Hence, \( x \) has the form

\[
x = \sum_{n=0}^{\infty} \frac{c_{a_n+1}}{p^{a_n+1}} = \sum_{n=0}^{\infty} \varepsilon_n x_n, \tag{51}
\]

where \( \varepsilon_n = c_{a_n+1} \in \{-1, 0, 1\} \).

\[ \square \]

Put \( K_p = K_{\mathbb{A},p} \), that is, the special case of \( K_{\mathbb{A},p} \), where \( \mathbb{A} \) is the sequence \( a_n = n \). First, we show that if \( p \geq 5 \), then \( K_p \) is quasi-convex.

Lemma 5.3. Let \( p \geq 5 \) be a prime. Then \( m_1 \eta_{k_1} + \cdots + m_l \eta_{k_l} \in K_p^0 \) for every \( 0 \leq k_1 < \cdots < k_l \) and \( m_1, \ldots, m_l \) such that \( |m_i| \leq \left\lfloor \frac{p}{4} \right\rfloor \) for all \( i \leq l \).

PROOF. Let \( \frac{1}{p^{n+i}} \in K_p^0 \). We may assume that \( k_l \leq n \), because \( \eta_{k_i}(\frac{1}{p^{n+i}}) = 0 \) for every \( i \in \mathbb{N} \) such that \( n < k_i \). Since \( 0 \leq k_1 < \cdots < k_l \),

\[
p^{k_1} + \cdots + p^{k_l} \leq 1 + p + \cdots + p^{k_l} = \frac{p^{k_l+1} - 1}{p - 1}. \tag{52}
\]

One has \( |m_i| \leq \left\lfloor \frac{p}{4} \right\rfloor \leq \frac{p-1}{4} \) for each \( i \), because \( p \) is odd. Thus,

\[
|(m_1 \eta_{k_1} + \cdots + m_l \eta_{k_l})(\frac{1}{p^{n+i}})| \leq |m_1| |\eta_{k_1}(\frac{1}{p^{n+i}})| + \cdots + |m_l| |\eta_{k_l}(\frac{1}{p^{n+i}})| \leq \frac{p - 1}{4} (p^{k_1} + \cdots + p^{k_l}) \frac{1}{p^{n+1}} \leq \frac{p - 1}{4} \cdot \frac{p^{k_l+1} - 1}{p - 1} \cdot \frac{1}{p^{n+1}} = \frac{p^{k_l+1} - 1}{4p^{n+1}} < \frac{1}{4}. \tag{55}
\]

Therefore, \( (m_1 \eta_{k_1} + \cdots + m_l \eta_{k_l})(\frac{1}{p^{n+i}}) \in \mathbb{T}_+ \) for all \( n \in \mathbb{N} \). Hence, \( m_1 \eta_{k_1} + \cdots + m_l \eta_{k_l} \in K_p^0 \), as desired. \[ \square \]
Proposition 5.4. Let $p \geq 5$ be a prime. Then $K_p$ is quasi-convex.

**Proof.** Let $x \in Q_p(K_p) \setminus \{0\}$. Then $x \in Q_{[\frac{p-1}{2}]} \cap \cdots \cap Q_{[\frac{p-1}{2}]}$. Consequently, by Theorem 5.2, $x = \sum_{n=0}^{\infty} \frac{\varepsilon_n}{p^{n-1}}$, where $\varepsilon_n \in \{-1, 0, 1\}$ for all $n \in \mathbb{N}$. Observe that for $k \in \mathbb{N}$,

$$\eta_k(x) = p^k x = \sum_{n=0}^{\infty} \frac{\varepsilon_n}{p^{n+k-1}} \equiv \sum_{n=k}^{\infty} \frac{\varepsilon_n}{p^{n-k+1}} = \sum_{i=1}^{\infty} \frac{\varepsilon_{k+i-1}}{p^i}. \quad (56)$$

Let $k_1$ denote the smallest index such that $\varepsilon_{k_1} \neq 0$, and let $k_2 \in \mathbb{N}$ be such that $k_1 < k_2$. In order to prove that $x \in K_p$, it remains to be seen that $\varepsilon_{k_2} = 0$. Put

$$y_+ = (\eta_{k_1} + \eta_{k_2})(x) = \sum_{i=1}^{\infty} \frac{\varepsilon_{k_1+i-1}}{p^i} + \sum_{i=1}^{\infty} \frac{\varepsilon_{k_2+i-1}}{p^i} = \sum_{i=1}^{\infty} \frac{\varepsilon_{k_1+i-1} + \varepsilon_{k_2+i-1}}{p^i}, \quad \text{and} \quad (57)$$

$$y_- = (\eta_{k_1} - \eta_{k_2})(x) = \sum_{i=1}^{\infty} \frac{\varepsilon_{k_1+i-1}}{p^i} - \sum_{i=1}^{\infty} \frac{\varepsilon_{k_2+i-1}}{p^i} = \sum_{i=1}^{\infty} \frac{\varepsilon_{k_1+i-1} - \varepsilon_{k_2+i-1}}{p^i}. \quad (58)$$

By Lemma 5.3, $m\eta_{k_1} + m\eta_{k_2}, m\eta_{k_1} - m\eta_{k_2} \in K_p^\circ$ for $m = 1, \ldots, \lceil \frac{p}{4} \rceil$. Thus, for $m = 1, \ldots, \lceil \frac{p}{4} \rceil$,

$$my_+ = m(\eta_{k_1} + \eta_{k_2})(x) \in \mathbb{T}_+; \quad \text{and} \quad (59)$$

$$my_- = m(\eta_{k_1} - \eta_{k_2})(x) \in \mathbb{T}_+. \quad (60)$$

One has $|\varepsilon_{k_1+i-1} + \varepsilon_{k_2+i-1}| \leq 2 \leq \frac{p-1}{2}$ and $|\varepsilon_{k_1+i-1} - \varepsilon_{k_2+i-1}| \leq 2 \leq \frac{p-1}{2}$, for all $i \in \mathbb{N}$, because $|\varepsilon_n| \leq 1$ for all $n \in \mathbb{N}$. By replacing $x$ with $-x$ if necessary, we may assume that $\varepsilon_{k_1} = 1$.

1. If $p \neq 7$, then by Corollary 4.2 the coefficients of $\frac{1}{p}$ in (57) and (58) are $-1, 0, 1$. In other words, $|1 + \rho| \leq 1$ and $|1 - \rho| \leq 1$. Since $\rho \in \{-1, 0, 1\}$, this implies that $\rho = 0$, as required.

2. If $k_1 + 1 < k_2$, then by Lemma 5.3,

$$(p-1)(\eta_{k_1} + \eta_{k_2}) = -\eta_{k_1} + \eta_{k_1+1} - \eta_{k_2} + \eta_{k_2+1} \in K_p^\circ, \quad \text{and} \quad (61)$$

$$(p-1)(\eta_{k_1} - \eta_{k_2}) = -\eta_{k_1} + \eta_{k_1+1} + \eta_{k_2} - \eta_{k_2+1} \in K_p^\circ. \quad (62)$$

Thus, in addition to (59) and (60), one also has

$$(p-1)y_+ = (p-1)(\eta_{k_1} + \eta_{k_2})(x) \in \mathbb{T}_+, \quad \text{and} \quad (63)$$

$$(p-1)y_- = (p-1)(\eta_{k_1} - \eta_{k_2})(x) \in \mathbb{T}_+. \quad (64)$$

Therefore, by Corollary 4.4 the coefficients of $\frac{1}{p}$ in (57) and (58) are $-1, 0, 1$. In other words, $|1 + \rho| \leq 1$ and $|1 - \rho| \leq 1$. Since $\rho \in \{-1, 0, 1\}$, this implies that $\rho = 0$, as required.

3. If $p = 7$ and $k_2 = k_1 + 1$, then by what we have shown so far,

$$x = \frac{1}{7^{k_1+1}} + \frac{\rho}{7^{k_1+2}} = \frac{7 + \rho}{7^{k_1+2}}, \quad (65)$$
where $\rho \in \{-1, 0, 1\}$. By Lemma 5.3

$$(7 + \rho)\eta_k = \rho\eta_{k_1} + \eta_{k_1+1} \in K_7^\circ.$$  \hfill (66)

Therefore,

$$\frac{2\rho}{7} + \frac{\rho^2}{49} = \frac{14\rho + \rho^2}{49} \equiv \frac{(7 + \rho)^2}{49} = (7 + \rho)\eta_{k_1}(x) \in \mathbb{T}.$$  \hfill (67)

Hence, $\rho = 0$, as required. \hfill \Box

**Proof of Theorem C.** By Proposition 5.4, $K_p$ is quasi-convex. Thus, $K_{a,p} \subseteq K_p$ implies that $Q_T(K_{a,p}) \subseteq K_p$. On the other hand, by Theorem 5.2

$$Q_T(K_{a,p}) \subseteq Q_{a,p,1} \cap \cdots \cap Q_{a,p,p-1} \subseteq \left\{ \sum_{n=0}^\infty \varepsilon_n x_n \mid (\forall n \in \mathbb{N})(\varepsilon_n \in \{-1, 0, 1\}) \right\}.$$  \hfill (68)

Therefore,

$$Q_T(K_{a,p}) \subseteq K_p \cap \left\{ \sum_{n=0}^\infty \varepsilon_n x_n \mid (\forall n \in \mathbb{N})(\varepsilon_n \in \{-1, 0, 1\}) \right\} = K_{a,p},$$  \hfill (69)

as desired. \hfill \Box

**Example 5.5.** If $p = 2$ or $p = 3$, then $Q_T(K_p) = \mathbb{T}$, that is, $K_p$ is $qc$-dense in $\mathbb{T}$ (cf. [8, 4.4]). In particular, $K_2$ and $K_3$ are not quasi-convex in $\mathbb{T}$. Thus, Proposition 5.4 fails if $p \not\geq 5$. Therefore, Theorem C does not hold for $p \not\geq 5$.

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