ABSTRACT. We study the behavior of blocks in flat families of finite-dimensional algebras. In a general setting we construct a finite directed graph encoding a stratification of the base scheme according to the block structures of the fibers. We furthermore show that all block structures are determined by “atomic” ones living on the components of a Weil divisor. A byproduct is that the number of blocks of fibers defines a lower semicontinuous function on the base scheme. We prove that this semicontinuity also holds in more general settings. We furthermore discuss how to obtain information about the simple modules in the blocks by generalizing and establishing several properties of decomposition matrices by Geck and Rouquier.

Introduction
It is a classical fact in ring theory that a non-zero noetherian ring $A$ can be decomposed as a direct product $A = \prod_{i=1}^{n} B_i$ of indecomposable rings $B_i$. Such a decomposition is unique up to permutation and isomorphism of the factors. Let us denote by $\mathcal{B}(A)$ the set of the $B_i$, called the blocks of $A$. The decomposition of $A$ into blocks induces a decomposition $A\text{-Mod} = \bigoplus_{i=1}^{n} B_i\text{-Mod}$ of the category of (left) $A$-modules. In particular, a simple $A$-module is a simple $B_i$-module for a unique block $B_i$ and so we get an induced decomposition $\text{Irr} A = \bigcup_{i=1}^{n} \text{Irr} B_i$ of the set of simple modules. Let us denote by $\mathcal{Fam}(A)$ the set of the $\text{Irr} B_i$, called the families of $A$. The blocks and families of a ring are important invariants which help to organize and simplify its representation theory. The aim of this paper is to investigate how these invariants vary in a flat family of finite-dimensional algebras.

More precisely, we consider a finite flat algebra $A$ over an integral domain $R$, i.e., $A$ is finitely generated and flat as an $R$-module. This yields a family of algebras parametrized by $\text{Spec}(R)$ consisting of the specializations (or fibers)

$$A(p) := k(p) \otimes_R A \simeq A_p/p_p A_p,$$

where $k(p) = \text{Frac}(R/p)$ is the residue field of $p \in \text{Spec}(R)$ in $R$ and $A_p$ is the localization of $A$ in $p$. Note that the fiber $A(p)$ is a finite-dimensional $k(p)$-algebra. Now, the primary goal would be to describe for any $p$ the blocks of $A(p)$, e.g., the number of blocks, and to describe the simple modules in each block, e.g., the number of such modules and their dimensions. It is clear that there will be no general theory giving the precise solutions to these problems for arbitrary $A$. For example, we can take the group ring $A = \mathbb{Z}S_n$ of the symmetric group. The fibers of $A$ are precisely the group rings $\mathbb{Q}S_n$ and $\mathbb{F}_p S_n$ for all primes $p$, and the questions above are still unanswered. Nonetheless, and this is the point of this paper, there are some general phenomena, some patterns in the behavior of blocks and simple modules along the fibers, which are true quite generally. The best situation turns out to be when $R$ is noetherian and normal, and the generic fiber $A^K$ is
a split $K$-algebra, where $K$ is the fraction field of $R$ (we will shortly address the case of non-split generic fiber). This setting includes many examples in representation theory like Brauer algebras, Hecke algebras, and (restricted) rational Cherednik algebras. We show (see Theorem 2.2, Lemma 2.6, and Corollary 2.9) that in this case we can always construct a finite directed graph encoding the block structures of all fibers and giving a complete overview about what happens to blocks under specialization. In Figure 2 below we give an example of such a graph in case of a generic Brauer algebra. The partitions on

![Diagram of a finite directed graph]

**Figure 1.** Block graph for the Brauer algebra over $\mathbb{Z}[[\delta]]$ for $n = 3$. The numbers $1, 2, 3, 4$ label blocks of the generic fiber. Equally, they label cell modules. Even though not important for the example we note that in terms of partitions we have $1 \simeq (0, (1, 1, 1)), 2 \simeq (0, (3)), 3 \simeq (0, (2, 1)), 4 \simeq (1, 1)$.

the top of each vertex describe the generic block structure on the zero locus of the ideal at the bottom by showing which blocks of the generic fiber $A^K$ will "glue" when specializing to the corresponding zero locus (we will make this precise in §2A). If at any given vertex we remove the zero loci at all vertices below, we obtain a locally closed subset on which the block structure is always equal to the one described by the vertex. So, this graph encodes a stratification of the base scheme, in this case the two-dimensional scheme $\text{Spec}(\mathbb{Z}[[\delta]])$. We want to point out that it is central for us to work with (affine) schemes. For example in Figure 2 we have one vertex with zero locus $(2)$, i.e., we consider the Brauer algebra in characteristic two. Now, we do not only have the case $\delta \in \{0, 1\} = F_2$, which is described by the two vertices below $(2)$, but we also have a generic characteristic two case, described by the generic point of $F_2[[\delta]]$, and this is really different from the case of specialized $\delta$. There is one further aspect visible in Figure 2. Namely, in the middle row of the graph we have four subschemes of codimension one on which the block structure is different from the generic one, i.e., the one of $A^K$. And the block structures on these components have an "atomic" character, i.e., any other block structure is obtained by gluing "atomic" ones. We can thus say that the block structures are governed by "atomic" block structures living on the components of a Weil divisor of the base scheme. This Weil divisor should be considered as a sort of new *discriminant* of $A$. Note that the values occurring in this discriminant in Figure 2 are precisely the parameters where the Brauer algebra is not semisimple anymore (the precise parameters have been determined by Rui [39] for all $n \in \mathbb{N}$). In Lemma 4.7 we prove why this must be the case.

Now, as already mentioned, our aim is clearly not to derive new results about Brauer algebras. Our intention is to show that the kind of behavior just described is actually a very general phenomenon. It also holds for group algebras, Hecke algebras, (restricted) rational Cherednik algebras, etc.—we can always draw such a graph with "atomic" block structures on a Weil divisor.

We can of course collapse the above graph by just considering the number of blocks and not their actual block structure in comparison to the generic one. What we obtain is a stratification of the base scheme by the number of blocks of the fibers. In other words, the map $\text{Spec}(R) \to \mathbb{N}, p \mapsto \#\text{Bl}(A(p))$, is lower semicontinuous. We show that this property in
fact also holds in cases where we do not have split generic fiber—as long as we restrict to a “nice” enough subset of \( \text{Spec}(R) \). More precisely, we show in Corollary 2.13 that \( \operatorname{Max}(R) \to \mathbb{N}, m \mapsto \# \text{Bl}(A(m)) \), is lower semicontinuous whenever \( R \) is a finite type algebra over an algebraically closed field, where \( \operatorname{Max}(R) \) is the subset of closed points of \( \text{Spec}(R) \). This establishes the lower semicontinuity of blocks (in closed points) for example also for quantized enveloping algebras of semisimple Lie algebras at roots of unity, enveloping algebras of semisimple Lie algebras in positive characteristic, and quantized function algebras of semisimple groups at roots of unity. More generally, this also applies to Hopf PI triples as introduced by Brown–Goodearl [6] (see also Brown–Gordon [7] and Gordon [18]), where questions about blocks in closed points have been raised and studied. We note that the number of blocks will in general not be lower semicontinuous on the whole of \( \text{Spec}(R) \), see Example 2.14.

In §2 we discuss the construction of the block graph and the corresponding stratification as illustrated above. The main results here are Theorem 2.2, Lemma 2.6, Corollary 2.9, Lemma 2.7, and Corollary 2.13. Before we start, we review a few standard facts in §1, essentially to fix notations. We have also included an appendix with some more elementary results we use throughout the paper. Even though partially standard, we feel that there are several results which are not mentioned in the literature. In §3 we establish a relationship between blocks of fibers and reductions of central characters. The main result here is Theorem 3.9 which essentially says that once we know the central characters of simple \( A^K \)-modules, we can compute the whole block graph. This fact is very useful for explicit computations. In §4 we address questions about the simple modules in a block. The main tool here are the decomposition matrices introduced by Geck and Rouquier. In Theorem 4.2 we show that they satisfy Brauer reciprocity in a quite general setting in which it was not known to hold before. In §4B we contrast the preservation of simple modules with the preservation of blocks under specialization, and show in Theorem 4.3 that preservation of simple modules implies preservation of blocks. It is an interesting question to ask when the converse holds. We show in Example 4.5 that in general we do not have an equivalence, but in Lemma 4.7 we establish one context where this is true (this context includes the Brauer algebras and explains why our Weil divisor is given by the non-semisimple parameters). Finally, in §4C we generalize the concept of Brauer graphs and show how these relate to blocks. In §5 we mention some open problems we encountered. In §6 we give a short overview on results in the literature which address our questions in some or the other form.

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§1. Notations

To fix notations and to recall some standard facts, we begin with a short review of basic block theory. For us, a ring is always a ring with identity and a module is always a left module unless we explicitly say it is a right module.

§1A. Block decompositions

For a ring $A$ we denote by $\text{Idem}(A)$ the set of non-zero idempotents of $A$ and by $\mathcal{E}(A)$ we denote the set of finite sets $\{e_i\}_{i \in I}$ of pairwise orthogonal non-zero idempotents satisfying $1 = \sum_{i \in I} e_i$. We similarly define the sets $\text{Idem}_p(A)$ and $\mathcal{E}_p(A)$ using primitive idempotents. If $e$ is an idempotent, then $Ae$ is a projective left ideal of $A$, and this yields a bijection between $\mathcal{E}(A)$ and direct sum decompositions of the $A$-module $A$ into non-zero left ideals up to permutation of the summands. The idempotent $e$ is primitive if and only if the $A$-module $Ae$ is indecomposable.

Let us now concentrate on idempotents in the center $Z := Z(A)$ of $A$. To simplify notations, we set $\text{Idem}_c(A) := \text{Idem}(Z)$, $\text{Idem}_p(Z) := \text{Idem}_p(Z)$, $\mathcal{E}(A) := \mathcal{E}(Z)$, and $\mathcal{E}_p(A) := \mathcal{E}_p(Z)$. Primitive idempotents of $Z$ are also called centrally-primitive idempotents of $A$. If $c$ is a central idempotent of $A$, then $Ac = cA$ is a two-sided ideal of $A$ and at the same time a ring with identity element equal to $c$ (hence not a subring). This yields a bijection between $\mathcal{E}_c(A)$ and direct sum decompositions of the ring $A$ into non-zero two-sided ideals of $A$ up to permutation of the summands, and such decompositions are in turn in bijection with direct product decompositions of the ring $A$ into non-zero rings up to permutation and isomorphism of the factors. A central idempotent $c$ is centrally-primitive if and only if $Ac$ is an indecomposable ring. It is a standard fact—and the starting point of block theory—that if $\mathcal{E}_p(A)$ is not empty, then it contains exactly one element, namely $\text{Idem}_c(A)$ itself, and that any central idempotent of $A$ is a sum of a subset of $\text{Idem}_c(A)$. We then say that $A$ has a block decomposition, call the centrally-primitive idempotents of $A$ also the block idempotents, and call the corresponding rings $Ac$ the blocks of $A$. In this case we prefer to write $\text{Bl}(A) := \text{Idem}_c(A)$. To avoid pathologies we set $\text{Bl}(0) := \emptyset$ for the zero ring 0.

§1B. Families of simple modules

Let $\mathcal{E} := \{e_i\}_{i \in I} \in \mathcal{E}(A)$ be some decomposition, not necessarily a block decomposition. Let $B_i := Ac_i$. If $V$ is a non-zero $A$-module, then $V = \bigoplus_{i \in I} c_i V$ as $A$-modules and each summand $c_i V$ is a $B_i$-module. In this way we obtain a decomposition $A \text{-Mod} = \bigoplus_{i \in I} B_i \text{-Mod}$ of module categories, which also restricts to a decomposition of the category of finitely generated modules. If a non-zero $A$-module $V$ is under this decomposition obtained from a $B_i$-module, then $V$ is said to belong to $B_i$. This is equivalent to $c_i V = V$ and $c_j V = 0$ for
Assume now that another class of rings having block decompositions are semiperfect. Recall that

\[ \text{Lemma 1.1.} \]

In each of the following cases the map \( \phi_V : V \rightarrow V^S \) is injective:

(a) \( \phi \) is injective and \( V \) is \( R \)-projective.

(b) \( \phi \) is faithfully flat.

(c) \( \phi \) is the localization morphism for a multiplicatively closed subset \( \Sigma \subset R \) and \( V \) is \( \Sigma \)-torsion-free.

\[ \text{§1C. Linkage relation and noetherian rings} \]

For a general ring \( A \) the \textit{linkage relation} on \( \text{Idem}_p(A) \) is the relation \( \sim \) defined by \( e \sim e' \) if and only if there is \( f \in \text{Idem}_p(A) \) and non-zero \( A \)-module morphisms \( Af \rightarrow Ae \) and \( Af \rightarrow Ae' \). The equivalence classes of the equivalence relation generated by \( \sim \) are called the \textit{linkage classes}. It is now a standard fact that if \( \{e_i \}_{i \in I} \in \mathcal{E}_p(A) \) and \( \{e'_i \}_{i \in I'} \) is the set of linkage class sums, then \( \{c_i \}_{i \in I} \in \mathcal{E}_p(A) \), so \( A \) has a block decomposition. In this case the indecomposable projective modules \( Ae \) and \( Ae' \) belong to the same block if and only if \( e \) and \( e' \) lie in the same linkage class. If \( A \neq 0 \) is noetherian, then \( \mathcal{E}_p(A) \neq \emptyset \), so noetherian rings have a block decomposition.

\[ \text{§1D. Semiperfect rings} \]

Another class of rings having block decompositions are \textit{semiperfect} rings. Recall that one of the many properties characterizing a ring \( A \) as semiperfect is that every finitely generated (or, equivalently, just every simple) \( A \)-module has a projective cover. Another characterization is that there exists a decomposition \( \{e_i \}_{i \in I} \in \mathcal{E}(A) \) with \( e_i \) being \textit{local}, i.e., \( \text{End}_A(Ae_i) \) is local. A module with local endomorphism ring is also called \textit{strongly indecomposable} as this property is stronger than being indecomposable. Hence, by the preceding paragraph, a semiperfect ring has a block decomposition. Recall that every artinian ring, and thus every finite-dimensional algebra over a field, is semiperfect. Assume now that \( A \) is semiperfect. If \( e_i \) is a primitive idempotent of \( A \), it is already \textit{local} and \( Ae_i \) is the projective cover of its head, which is a simple \( A \)-module. In fact, if \( \{e_i \}_{i \in I} \in \mathcal{E}_p(A) \), then there is a subset \( I' \subseteq I \) such that \( \{(P_i)_{i \in I'} \} \) is a system of representatives of the isomorphism classes of projective indecomposable \( A \)-modules, and their heads \( S_i := \text{Hd}(P_i) = P_i/\text{Rad}(P_i) \) give a system of representatives of the isomorphism classes of simple \( A \)-modules. The projective class group \( K_0(A) := K_0(A\text{-proj}) \) is a free abelian group with basis formed by the isomorphism classes of the \( (P_i)_{i \in I} \) and we have a natural isomorphism \( K_0(A) \cong G_0(A) := K_0(A\text{-mod}) \) mapping \( P_i \) to \( S_i \).

\[ \text{§1E. Base change of blocks} \]

Let \( \phi : R \rightarrow S \) be a morphism of commutative rings. If \( V \) is an \( R \)-module, we write \( V^S := \phi^*V := S \otimes_R V \) for the scalar extension of \( V \) to \( S \) and by \( \phi_V : V \rightarrow V^S \) we denote the canonical map \( v \mapsto 1 \otimes v \).

\[ \text{Lemma 1.1.} \] In each of the following cases the map \( \phi_V : V \rightarrow V^S \) is injective:

(a) \( \phi \) is injective and \( V \) is \( R \)-projective.

(b) \( \phi \) is faithfully flat.

(c) \( \phi \) is the localization morphism for a multiplicatively closed subset \( \Sigma \subset R \) and \( V \) is \( \Sigma \)-torsion-free.
We call the \( \phi \)-Jacobson radical of \( A \) the sum of a subset of the block idempotents of \( A \). Let \( \{ c_i \}_{i \in I} \) be the block idempotents of \( A \) and let \( \{ c'_j \}_{j \in J} \) be the block idempotents of \( A^S \). Since \( \phi_A \) is idempotent stable, we have \( \text{Bl}_\phi(A^S) := \{ c_i \}_{i \in I} \subseteq \mathcal{E}_\phi(A^S) \). We call the \( \phi \)-blocks of \( A^S \) and call the corresponding families as defined in §1B the \( \phi \)-families of \( A^S \), denoted \( \text{Fam}_\phi(A^S) \). As explained in §1B each \( \phi \)-block \( \phi_A(c_i) \) is a sum of a subset of the block idempotents of \( A^S \) and the \( \phi \)-families are coarser than the

**Proof.** The first case follows from [4, II, §5.1, Corollary to Proposition 4], the second follows from [3, I, §3.5, Proposition 8(i,iii)], and the last case follows from the fact that \( \phi \) is flat in conjunction with [3, I, §2.2, Proposition 4].

If \( A \) is an \( R \)-algebra, then the \( S \)-module \( A^S \) is naturally an \( S \)-algebra and the map \( \phi_A : A \to A^S \) is a ring morphism. Moreover, if \( V \) is an \( A \)-module, then the underlying \( S \)-module of \( A^S \otimes_A V \) is simply \( V^S \). Our aim is to study the behavior of blocks under the morphism \( \phi_A : A \to A^S \). Clearly, if \( e \in A \) is an idempotent, also \( \phi_A(e) \in A^S \) is an idempotent, and if \( e \) is central, so is \( \phi_A(e) \) by the elementary fact that

\[
\phi_A(Z(A)) \subseteq Z(A^S).
\]

To describe some properties of \( \phi_A \) with respect to idempotents and blocks, we introduce the following notations.

**Definition 1.2.** We say that \( \phi_A \) is:

(a) idempotent stable if \( \phi_A(e) \neq 0 \) for any non-zero idempotent \( e \) of \( A \),

(b) central idempotent stable if \( \phi_A(c) \neq 0 \) for any non-zero central idempotent \( c \) of \( A \),

(c) idempotent surjective if for each idempotent \( e' \in A^S \) there is an idempotent \( e \in A \) with \( \phi_A(e) = e' \),

(d) central idempotent surjective if for each central idempotent \( c' \in A^S \) there is a central idempotent \( c \in A \) with \( \phi_A(c) = c' \),

(e) primitive idempotent bijective if \( \phi_A \) induces a bijection between the isomorphism classes of primitive idempotents of \( A \) and the isomorphism classes of primitive idempotents of \( A^S \),

(f) block bijective if \( \phi_A \) induces a bijection between the centrally-primitive idempotents of \( A \) and the centrally-primitive idempotents of \( A^S \).

Recall that central idempotents are isomorphic if and only if they are equal, so we do not have to consider isomorphism classes in the definition of “block bijective”. Note that in case \( \phi_A \) is idempotent stable, respectively central idempotent stable, it induces a map between the sets of decompositions \( \mathcal{E}(A) \) and \( \mathcal{E}(A^S) \), respectively between \( \mathcal{E}_c(A) \) and \( \mathcal{E}_c(A^S) \), as defined in §1A. The following lemma reveals two opposing situations in which \( \phi_A \) is idempotent stable (and thus central idempotent stable). We denote by \( \text{Rad}(A) \) the Jacobson radical of \( A \).

**Lemma 1.3.** If \( \text{Ker}(\phi_A) \subseteq \text{Rad}(A) \), then \( \phi_A \) is idempotent stable. This holds in the following two cases:

(a) \( \phi_A \) is injective (see Lemma 1.1),

(b) \( \phi \) is surjective, \( \text{Ker}(\phi) \subseteq \text{Rad}(R) \), and \( A \) is finitely generated as an \( R \)-module.

**Proof.** If \( e \in A \) is an idempotent contained in \( \text{Rad}(A) \), then by a well-known characterization of the Jacobson radical (see [11, 5.10]) we conclude that \( e^\uparrow = 1 - e \in A^\ast \) is a unit, and since \( e^\uparrow \) is also an idempotent, we must have \( e^\downarrow = 1 \), implying that \( e = 0 \). If \( \phi_A \) is injective, the condition clearly holds. In the second case we have \( \text{Ker}(\phi_A) = \text{Ker}(\phi)A \subseteq \text{Rad}(R)A \subseteq \text{Rad}(A) \), where the last inclusion follows from [27, Corollary 5.9].

Now, suppose that \( \phi_A \) is idempotent stable and that both \( A \) and \( A^S \) have block decompositions. Let \( \{ c_i \}_{i \in I} \) be the block idempotents of \( A \) and let \( \{ c'_j \}_{j \in J} \) be the block idempotents of \( A^S \). Since \( \phi_A \) is idempotent stable, we have \( \text{Bl}_\phi(A^S) := \{ c_i \}_{i \in I} \in \mathcal{E}_\phi(A^S) \).
families in the sense that each \( \phi \)-family is a union of \( A^S \)-families. In particular, we have

\[
\#\text{Bl}(A) = \#\text{Bl}_\phi(A^S) \leq \#\text{Bl}(A^S).
\]

The following picture illustrates this situation:

\[
\begin{array}{ccc}
c_1 & \cdots & c_m \\
\downarrow & \cdots & \downarrow \\
\phi_A(c_1) & \cdots & \phi_A(c_m) \\
\end{array}
\]

\[
\begin{array}{ccc}
c_1 & \cdots & c_n \\
\downarrow & \cdots & \downarrow \\
A \text{-blocks} & \cdots & A \text{-blocks}
\end{array}
\]

In Appendix A we have collected several further facts about base change of blocks. We will use these results in the sequel.

§2. Semicontinuity of blocks

In this section we construct the stratifications and graphs mentioned in the introduction. The main results are Theorem 2.2, Lemma 2.6, Corollary 2.9, Lemma 2.7, and Corollary 2.13.

Throughout this paragraph, we assume that \( A \) is a finite flat algebra over an integral domain \( R \) with fraction field \( K \).

§2A. Blocks of localizations

Before we consider blocks of specializations, we first take a look at blocks of localizations of \( A \) as these are much easier to control and are still strongly related to blocks of specializations as we will see in the next paragraph.

It follows from Corollary A.2 that \( A \) and any localization \( A_p \) for \( p \in \text{Spec}(R) \) has a block decomposition, even if \( A \) is not necessarily noetherian. Since the canonical map \( \phi_p : A_p \to A^K \) is injective by Lemma 1.1, we have the notion of \( \phi_p \)-blocks and \( \phi_p \)-families of \( A^K \) as defined in §1E. To shorten notations, we call them the \( p \)-blocks and \( p \)-families, and write \( \text{Fam}_p(A^K) \) for the \( p \)-families. Recall that we have a natural bijection

\[
\text{Bl}(A_p) = \text{Fam}_p(A^K).
\]

There is also the following more concrete view of \( p \)-blocks. Let \((c_i)_{i \in I}\) be the block idempotents of \( A^K \). If \( c \in A_p \) is any block idempotent, we know from §1A that there is \( I' \subseteq I \) with \( c = \sum_{i \in I'} c_i \) in \( A^K \). Hence, to any block idempotent of \( A_p \) we can associate a subset of \( I \), and if we take all block idempotents of \( A_p \) into account, we get a partition \( \gamma_A(p) \) of the set \( I \), from which we can recover the block idempotents of \( A_p \) by taking sums of the \( c_i \) over the members of \( \gamma_A(p) \). Hence, we get a map

\[
\gamma_A : \text{Spec}(R) \to \text{Part}(I)
\]

to the set of partitions of the set \( I \). If \( q \subseteq p \), then we have an embedding \( A_p \hookrightarrow A_q \) and by the same argumentation as above the block idempotents of \( A_p \) are obtained by summing up block idempotents of \( A_q \). Hence, the map \( \gamma_A \) is actually a morphism of posets if we equip \( \text{Spec}(R) \) with the partial order \( \leq \) defined by \( p \leq q \) if \( q \subseteq p \) (i.e., \( V(p) \subseteq V(q) \)) and we equip \( \text{Part}(I) \) with the partial order \( \leq \) defined by \( \mathcal{P} \leq \mathcal{Q} \) if \( \mathcal{P} \) is a coarser partition than \( \mathcal{Q} \), i.e., the members of \( \mathcal{P} \) are unions of members of \( \mathcal{Q} \). We consider the image \( \Gamma_A \) of \( \gamma_A \) as a sub-poset of \( \text{Part}(I) \) with the induced order \( \leq \) and call its elements the local block structures of \( A \). To \( \mathcal{P} \in \Gamma_A \) we attach the \( \mathcal{P} \)-stratum

\[
\Gamma_A(\mathcal{P}) := \gamma_A^{-1}(\mathcal{P}) \subseteq \text{Spec}(R),
\]
and the $\mathcal{P}$-skeleton
\begin{equation}
\Gamma_A^\mathcal{P} := \bigcup_{\mathcal{P} \leq \mathcal{P}'} \Gamma_A(\mathcal{P}') .
\end{equation}

We clearly have a finite decomposition
\begin{equation}
\text{Spec}(R) = \prod_{\mathcal{P} \in \Gamma_A} \Gamma_A(\mathcal{P})
\end{equation}
and the relation
\begin{equation}
\Gamma_A(\mathcal{P}) = \Gamma_A^\mathcal{P}(\mathcal{P}) \setminus \bigcup_{\mathcal{P}' \in \mathcal{P}} \Gamma_A^\mathcal{P}(\mathcal{P}') ,
\end{equation}
The set $\Gamma_A^\mathcal{P}$ of all skeleta is naturally in bijection with $\Gamma_A$ since from any skeleton $\Gamma_A^\mathcal{P}(\mathcal{P})$ we can recover $\mathcal{P}$ as the unique maximal local block structure in the points of $\Gamma_A^\mathcal{P}(\mathcal{P})$. Moreover, we have $\mathcal{P}' \leq \mathcal{P}$ if and only if $\Gamma_A^\mathcal{P}(\mathcal{P}') \subseteq \Gamma_A^\mathcal{P}(\mathcal{P})$. Hence, when considering $\Gamma_A^\mathcal{P}$ as a poset ordered by inclusion, then in fact $\Gamma_A \cong \Gamma_A^\mathcal{P}$ as posets. The local block graph of $A$ is the finite directed graph defined by the poset $\Gamma_A$ together with the skeleta $\Gamma_A^\mathcal{P}(\mathcal{P})$ as vertex labels. This graph encodes all all information about local block structures of $A$.

In Figure 2 we have given an example of such a graph in the case of a Brauer algebra as mentioned in the introduction.

![Figure 2. The local block graph for the Brauer algebra over $\mathbb{Z}[\delta]$ for $n = 3$. Here, $V$ denotes the zero locus. The vertices are the local block structures with the corresponding skeleton as labels (written beneath the block structure). To find $\gamma_A((2))$ for example, we just have to find the smallest skeleton which contains (2) and then $\gamma_A((2))$ is equal to $V(6) \setminus (V(\delta - 1, 6) \cup V(3) \cup V(\delta + 2, 6)) = V(2) \setminus (V(\delta - 1, 2) \cup V(\delta, 2))$. So, whenever $p \in V(2)$ and $p \notin V(\delta - 1, 2) \cup V(\delta, 2)$, then $\gamma_A(p) = [(2), (3), (4)]$. We note that this graph has a unique sink, namely $(1, 2, 3, 4)$, meaning that the Brauer algebra is indecomposable on the corresponding stratum. It would be interesting to know if this happens for any $n \in \mathbb{N}$.](image)

The poset $\Gamma_A$ clearly has a unique maximal element, namely the block structure $\gamma_A(\bullet)$ of $A$ in the generic point $\bullet$ of $\text{Spec}(R)$, i.e., $\gamma_A(\bullet) = \{(i) \mid i \in I\}$ is the block structure of the generic fiber $A^K = A$. The deviation of block structures from the generic one thus takes place on the set
\begin{equation}
\text{BIEx}^{\text{loc}}(A) := \{p \in \text{Spec}(R) \mid \gamma_A(p) < \gamma_A(\bullet)\} = \bigcup_{\mathcal{P} \in \text{Max}(\Gamma_A)} \Gamma_A^\mathcal{P}(\mathcal{P}) .
\end{equation}
We call this set the local block divisor of $A$ for reasons to become apparent soon. The generic local block structure occurs precisely on the set
\begin{equation}
\text{BIGen}^{\text{loc}}(A) := \text{Spec}(R) \setminus \text{BIEx}^{\text{loc}}(A) = \{p \in \text{Spec}(R) \mid \gamma_A(p) = \gamma_A(\bullet)\} .
\end{equation}
Our aim is to show that the skeleta are in fact closed subsets of \( \text{Spec}(R) \) and that (10) is a stratification of the scheme \( \text{Spec}(R) \). In Figure 2 we can see this in the example already. The key ingredient in proving this is the following general proposition, which is essentially due to Bonnafé and Rouquier [2, Proposition C.2.11]. We give a slightly more general version here.

**Proposition 2.1.** Let \( R \) be an integral domain with fraction field \( K \), let \( A \) be a finite flat \( R \)-algebra, and let \( \mathcal{F} \subseteq \Lambda^K \) be a finite set. Then

\[
\text{Gen}_A(\mathcal{F}) := \{ p \in \text{Spec}(R) \mid \mathcal{F} \subseteq A_p \}
\]

is a neighborhood of the generic point of \( \text{Spec}(R) \). If \( A \) is finitely presented flat, then \( \text{Gen}_A(\mathcal{F}) \) is an open subset of \( \text{Spec}(R) \), and if moreover \( R \) is a Krull domain, the complement \( \text{Ex}_A(\mathcal{F}) \) of \( \text{Gen}_A(\mathcal{F}) \) in \( \text{Spec}(R) \) is a reduced Weil divisor, i.e., it is either empty or pure of codimension one with finitely many irreducible components.

**Proof.** Let us first assume that \( A \) is actually \( R \)-free. For an element \( \alpha \in K \) we define \( I_\alpha := \{ r \in R \mid r\alpha \in R \} \). This is a non-zero radical ideal in \( R \), and it has the property that \( \alpha \in R_p \) if and only if \( I_\alpha \not\subseteq p \). To see this, suppose that \( \alpha \in R_p \). Then we can write \( \alpha = \xi x \) for some \( x \in R \setminus \mathfrak{p} \). Hence, \( \alpha x = x \in R \) and therefore \( x \in I_\alpha \). Since \( x \not\in \mathfrak{p} \), it follows that \( I_\alpha \not\subseteq \mathfrak{p} \).

Conversely, if \( I_\alpha \not\subseteq \mathfrak{p} \), then there exists \( x \in I_\alpha \) with \( x \not\in \mathfrak{p} \). By definition of \( I_\alpha \) we have \( xx = x \in R \) and since \( x \not\in \mathfrak{p} \), we can write \( \alpha = \xi x \in R_p \). Now, let \( (a_1, \ldots, a_n) \) be an \( R \)-basis of \( A \). Then we can write every element \( f \in \mathcal{F} \) as \( f = \sum_{i=1}^n a_{f,i} a_i \) with \( a_{f,i} \in K \). Let

\[
I := \prod_{f \in \mathcal{F}, i=1,\ldots,n} I_{a_{f,i}} \subseteq R.
\]

By the properties of the ideals \( I_\alpha \) we have the following logical equivalences:

\[
(\mathcal{F} \subseteq A_p) \iff (a_{f,i} \in R_p \ \forall f \in \mathcal{F}, i=1,\ldots,n) \iff (I_{a_{f,i}} \not\subseteq \mathfrak{p} \ \forall f \in \mathcal{F}, i=1,\ldots,n) \iff (I \not\subseteq \mathfrak{p}),
\]

the last equivalence following from the fact that \( \mathfrak{p} \) is prime. Hence,

\[
\text{Ex}_A(\mathcal{F}) = \text{Spec}(R) \setminus \text{Gen}_A(\mathcal{F}) = \text{V}(I) = \bigcup_{f \in \mathcal{F}, i=1,\ldots,n} \text{V}(I_{a_{f,i}}),
\]

implying that \( \text{Gen}_A(\mathcal{F}) \) is an open subset of \( \text{Spec}(R) \).

Next, still assuming that \( A \) is \( R \)-free, suppose that \( R \) is a Krull domain. To show that \( \text{Ex}_A(\mathcal{F}) \) is either empty or pure of codimension 1 in \( \text{Spec}(R) \) with finitely many irreducible components, it suffices to show this for the closed subsets \( \text{V}(I_\alpha) \). If \( \alpha \in R \), then \( I_\alpha = R \) and therefore \( \text{V}(I_\alpha) = \emptyset \). So, let \( \alpha \in R \). Let \( \text{V}(I_\alpha) = \bigcup_{\lambda \in \Lambda} \text{V}(q_\lambda) \) be the decomposition into irreducible components. Note that this decomposition is unique and contains every irreducible component of \( \text{V}(I_\alpha) \) since \( \text{V}(I_\alpha) \) is a sober topological space. The inclusion \( \text{V}(I_\alpha) \supseteq \text{V}(q_\lambda) \) is equivalent to \( I_\alpha = \sqrt{I_\alpha} \supseteq \sqrt{q_\lambda} = q_\lambda \). Since an irreducible component is a maximal proper closed subset, we see that the \( q_\lambda \) are the minimal prime ideals of \( \text{Spec}(R) \) containing \( I_\alpha \). Let \( q = q_\lambda \) for an arbitrary \( \lambda \in \Lambda \). We will show that \( \text{ht}(q) = 1 \). Since \( I_\alpha \subseteq q \), we have seen above that \( \alpha \in R_q \). As \( R \) is a Krull domain, also \( R_q \) is a Krull domain by [30, Theorem 12.1]. By [3, VII, §1.6, Theorem 4] we have

\[
R_q = \bigcap_{q' \in \text{Spec}(R), \text{ht}(q') = 1} (R_q)_{q'} = \bigcap_{q' \in \text{Spec}(R), \text{ht}(q') = 1} R_{q'}.
\]

Since \( \alpha \not\in R_q \), this shows that there exists \( q' \in \text{Spec}(R) \) with \( q' \subseteq q \), \( \text{ht}(q') = 1 \) and \( \alpha \in R_{q'} \). The last property implies \( I_\alpha \subseteq q' \) and now the minimality in the choice of \( q \) implies that \( q' = q \). Hence, \( \text{ht}(q) = 1 \) and this shows \( \text{V}(I_\alpha) \) is pure of codimension 1. Since \( I_\alpha \neq 0 \), there is some \( 0 \neq r \in I_\alpha \). This element is contained in all the height one prime ideals \( q_\lambda \). As \( R \)
is a Krull domain, a non-zero element of \( R \) can only be contained in finitely many height one prime ideals (see [24, 4.10.1]), so \( \Lambda \) must be finite.

Now, assume that \( R \) is an arbitrary integral domain and that \( A \) is finite flat. Then Grothendieck’s generic freeness lemma [21, Lemme 6.9.2] shows that there exists a non-zero \( f \in R \) such that \( A_f \) is a free \( R_f \)-module. Note that \( \text{Spec}(R_f) \) can be identified with the distinguished open subset \( D(f) \) of \( \text{Spec}(R) \). We obviously have

\[
\text{Gen}_{A_f}(\mathcal{F}) = \text{Gen}_A(\mathcal{F}) \cap D(f) .
\]

By the arguments above, \( \text{Gen}_{A_f}(\mathcal{F}) \) is an open subset of \( D(f) \), and thus of \( \text{Spec}(R) \). This shows that \( \text{Gen}_A(\mathcal{F}) \) is a neighborhood in \( \text{Spec}(R) \).

Next, let \( R \) be arbitrary and assume that \( A \) is finitely presented flat. It is a standard fact (see [41, Tag 00NX]) that the assumptions on \( A \) imply that \( A \) is already finite locally free, i.e., there exist a family \( (f_i)_{i \in I} \) of elements of \( R \) such that the standard open affines \( D(f_i) \) cover \( \text{Spec}(R) \) and \( A_{f_i} \) is a finitely generated free \( R_{f_i} \)-module for all \( i \in I \). Since \( \text{Spec}(R) \) is quasi-compact, see [19, Proposition 2.5], we can assume that \( I \) is finite. Again note that \( \text{Spec}(R_{f_i}) \) can be identified with \( D(f_i) \) and that

\[
\text{Gen}_{A_{f_i}}(\mathcal{F}) = \text{Gen}_A(\mathcal{F}) \cap D(f_i) .
\]

By the above, the set \( \text{Gen}_{A_{f_i}}(\mathcal{F}) \) is open and since the \( D(f_i) \) cover \( \text{Spec}(R) \), it follows that \( \text{Gen}_A(\mathcal{F}) \) is open. Now, suppose that \( R \) is a Krull domain. Similarly as in (15) we have

\[
\text{Ex}_{A_{f_i}}(\mathcal{F}) = \text{Ex}_A(\mathcal{F}) \cap D(f_i) .
\]

Suppose that \( \text{Ex}_A(\mathcal{F}) \) is not empty and let \( Z \) be an irreducible component of \( \text{Ex}_A(\mathcal{F}) \).

There is an \( i \in I \) with \( Z \cap D(f_i) \neq \emptyset \). The map \( T \rightarrow T' \) defines a bijection between irreducible closed subsets of \( D(f_i) \) and irreducible closed subsets of \( \text{Spec}(R) \) which meet \( D(f_i) \), see [19, §1.5]. This implies that \( Z \cap D(f_i) \) is an irreducible component of \( \text{Ex}_A(\mathcal{F}) \cap D(f_i) = \text{Ex}_{A_{f_i}}(\mathcal{F}) \).

It follows from the above that \( Z \cap D(f_i) \) is of codimension 1 in \( D(f_i) \). Hence, \( Z \) is of codimension 1 in \( \text{Spec}(R) \) by [41, Tag 021A]. All irreducible components of \( \text{Ex}_A(\mathcal{F}) \) are thus of codimension 1 in \( \text{Spec}(R) \). Since each set \( \text{Ex}_{A_{f_i}}(\mathcal{F}) \) has only finitely many irreducible components and since \( I \) is finite, also \( \text{Ex}_A(\mathcal{F}) \) has only finitely many irreducible components.

For \( p \in \text{Spec}(R) \) let us denote by \( \mathcal{B}_A(p) \subseteq A^K \) the set of block idempotents of \( A_p \). Clearly, \( \mathcal{B}_A(p) \) and \( \gamma_A(p) \) are in bijection by taking sums of the \( c_i \) over the subsets in \( \gamma_A(p) \). Note that \( \mathcal{B}_A(p) \) is constant on \( \Gamma_A(\mathcal{P}) \) for any \( \mathcal{P} \). We can thus define \( \text{Gen}_A(\mathcal{P}) := \text{Gen}_A(\mathcal{B}_A(p)) \) where \( p \in \Gamma_A(\mathcal{P}) \) is arbitrary. We are now ready to prove our first main result.

**Theorem 2.2.** Suppose that \( A \) is finitely presented as an \( R \)-module. Then the map \( \gamma_A : \text{Spec}(R) \rightarrow \Gamma_A \) is lower semicontinuous, i.e., each \( \Gamma_A^\leq(\mathcal{P}) \) is closed in \( \text{Spec}(R) \). In particular, \( \Gamma_A(\mathcal{P}) \) is open in \( \Gamma_A^\leq(\mathcal{P}) \), thus locally closed in \( \text{Spec}(R) \).

**Proof.** Since \( \text{Spec}(R) = \bigsqcup_{\mathcal{P} \in \Gamma_A} \Gamma_A(\mathcal{P}) \), we have \( \text{Spec}(R) \setminus \Gamma_A^\leq(\mathcal{P}) = \bigsqcup_{\mathcal{P}' \not\subseteq \mathcal{P}} \Gamma_A(\mathcal{P}') \). Let \( \mathcal{P}' \not\subseteq \mathcal{P} \) and \( p' \in \text{Gen}_A(\mathcal{P}') \). Then \( \mathcal{P}' \not\subseteq \gamma_A(p') \). But this implies that \( \gamma_A(p') \not\subseteq \mathcal{P} \) since otherwise \( \mathcal{P}' \subseteq \gamma_A(p') \subseteq \mathcal{P} \). Hence, \( \text{Gen}_A(\mathcal{P}') \subseteq \text{Spec}(R) \setminus \Gamma_A^\leq(\mathcal{P}) \). Conversely, we clearly have \( \Gamma_A(\mathcal{P}') \subseteq \text{Gen}_A(\mathcal{P}') \). This shows that

\[
\text{Spec}(R) \setminus \Gamma_A^\leq(\mathcal{P}) = \bigcup_{\mathcal{P}' \not\subseteq \mathcal{P}} \Gamma_A(\mathcal{P}') = \bigcup_{\mathcal{P}' \not\subseteq \mathcal{P}} \text{Gen}_A(\mathcal{P}')
\]

is open, so

\[
\Gamma_A^\leq(\mathcal{P}) = \bigcap_{\mathcal{P}' \subseteq \mathcal{P}} \text{Ex}_A(\mathcal{P}')
\]

is closed. Using (11) we now see that \( \Gamma_A(\mathcal{P}) \) is locally closed.

Theorem 2.2 implies in particular that \( \text{BlGen}^\text{loc}(A) \) is a dense open subset of \( \text{Spec}(R) \).
Corollary 2.3. Suppose that $A$ is finitely presented as an $R$-module and that $R$ is a Krull domain. Then $\text{Bl}E\text{Ex}^{\text{loc}}(A)$ is a reduced Weil divisor.

Proof. This follows directly from Proposition 2.1 since $\text{Bl}E\text{Ex}^{\text{loc}}(A) = E\text{X}_A(P_A \bullet)$.

Remark 2.4. We note that $A$ is finitely presented flat if and only if it is finite projective, see [28, Theorem 4.30] or [41, Tag 058R]. Hence, we could have equally assumed that $A$ is finite projective in Theorem 2.2 but we preferred the seemingly more general notion.

We assume for the rest of this paragraph that $R$ is noetherian (which implies that $A$ is finitely presented as an $R$-module).

Lemma 2.5. For any $P \in \Gamma_A$ we have $\overline{\Gamma_A(P)} \subseteq \Gamma_A^\gamma(P)$. In particular, the partition (10) is a stratification of the scheme $\text{Spec}(R)$.

Proof. Since $\text{Spec}(R)$ is noetherian, the locally closed set $\Gamma_A(P)$ has only finitely many irreducible components $Z_1, \ldots, Z_n$. We then have $\Gamma_A(P) = \bigcup_{i=1}^n Z_i$. If $\xi_i$ denotes the generic point of $Z_i$ (note that any irreducible locally closed set has a unique generic point), then $\xi_i$ is also the generic point of $Z_i$. Since $\xi_i \in Z_i \subseteq \Gamma_A(P)$, we have $\gamma_A(\xi_i) = P$, so $\xi_i \in \Gamma_A^\gamma(P)$. We thus obtain $\Gamma_A(P) = \bigcup_{i=1}^n V(\xi_i) \subseteq \Gamma_A^\gamma(P)$.

In general it is not true that we have equality $\Gamma_A(P) = \Gamma_A^\gamma(P)$, so the stratification (10) is in general not a so-called good stratification. For example, in Figure 2 we have $P' := \gamma_A(3) = \{(1, 2, 3), (4)\} \subseteq \{(1, 2), (3), (4)\} = \gamma_A(2) =: P$, so $(3) \in \Gamma_A(P)$, but $(3)$ is not contained in $\Gamma_A(P) = V(2))$. The problem here is that the skeleton $\Gamma_A^\gamma(P)$ has an irreducible component on which the maximal local block structure is strictly smaller than the maximal one on the entire skeleton. To overcome this defect, we construct a refinement of the stratification which gives a much better overview of how the local block structures are formed.

For any two-element subset $\{i, j\} \subseteq I$ we define the corresponding gluing locus as

$$\Gamma_A^\gamma(\{i, j\}) := \{p \in \text{Spec}(R) \mid c_i \text{ and } c_j \text{ lie in the same block of } A_p\}.$$  

It is not hard to see that

$$\Gamma_A^\gamma(\{i, j\}) = E\text{X}_A(c_i) \cap E\text{X}_A(c_j) \cap \bigcap_{\ell \leq I, i,j \notin \ell} E\text{X}_A(c_i + \sum_{k \in \ell} c_k) \cap E\text{X}_A(c_j + \sum_{k \in \ell} c_k),$$

so $\Gamma_A^\gamma(\{i, j\})$ is closed in $\text{Spec}(R)$ by Proposition 2.1. We denote by $\Xi_A^{(1)}(P)$ the set of irreducible components of gluing loci and we call these the atomic gluing loci. On each $Z \in \Xi_A^{(1)}(P)$ there is clearly a unique maximal local block structure $\gamma_A(Z)$, namely the one in the generic point of $Z$. We denote by $\text{At}(\Gamma_A)$ the set of these block structures and call them atomic local block structures. The important point is now that for any $p \in \text{Spec}(R)$ we can determine $\gamma_A(p)$ simply by determining the atomic gluing loci containing $p$. More precisely, we have $\gamma_A(p) = I/\sim_p$, where $\sim_p$ is the equivalence relation on $I$ generated by $i \sim_p j$ if and only if $p \in \Gamma_A^\gamma(\{i, j\})$. From this it is clear that

$$\gamma_A(p) = \bigwedge_{Z \in \Xi_A^{(1)}(P), p \in Z} \gamma_A(Z),$$

where $\bigwedge$ denotes the meet in the lattice $\text{Part}(I)$, i.e., $\mathcal{P} \wedge \mathcal{P}'$ for $\mathcal{P}, \mathcal{P}' \in \text{Part}(I)$ is the finest partition of $I$ being coarser than both $\mathcal{P}$ and $\mathcal{P}'$, and this is obtained by joining members with non-empty intersection. Hence, any local block structure of $A$ is a meet of atomic local block structures (whence, the prefix “atomic”). But we note that not all
meets must actually occur as block structures since atomic gluing loci might have empty intersection. The point is that once we know the atomic gluing loci and their maximal local block structures, we essentially know the complete local block graph by analyzing intersections of atomic gluing loci. To this end, we inductively define the sets $\Xi_A$ for $r \in \mathbb{N}$ as being the set of irreducible components of intersections of any two elements of $\Xi^{(r-1)}_A$, with $\Xi^{(1)}_A$ being defined above already as the set of atomic gluing loci. This yields an increasing sequence $\Xi^{(1)}_A \subseteq \Xi^{(2)}_A \subseteq \ldots$ of irreducible closed subsets of $\text{Spec}(R)$ which will eventually become stationary since $\text{Spec}(R)$ is noetherian. We add $\text{Spec}(R)$ to this maximal set and denote the resulting set by $\Xi_A$. We consider it as a poset ordered by inclusion. Let $\text{At}(\Xi_A) := \Xi^{(1)}_A$ and for $Z \in \Xi_A$ we denote by $\text{At}(Z)$ the set of all $T \in \text{At}(\Xi_A)$ containing $Z$. For any $Z \in \Xi_A$ we define

$$\Xi_A(Z) := Z \setminus \bigcup_{Z < Z'} Z'. \tag{21}$$

Note that $\Xi_A(\text{Spec}(R)) = \text{BlGen}(A)$ and that $\text{BEx}(A) = \bigcup_{T \in \text{At}(\Xi_A)} T = \bigcup_{Z \in \Xi_A} Z$.

**Lemma 2.6.** We have $\text{Spec}(R) = \bigsqcup_{Z \in \Xi_A} \Xi_A(Z)$, and this is a good stratification of the scheme $\text{Spec}(R)$. Moreover, for any $Z \in \Xi_A$ we have

$$\gamma_A(Z) = \bigwedge_{T \in \text{At}(Z)} \gamma_A(T). \tag{22}$$

and

$$\Xi_A(Z) \subseteq \Gamma_A(\gamma_A(Z)). \tag{23}$$

Hence, for any $\mathcal{P} \in \Gamma_A$ we have

$$\Gamma_A(\mathcal{P}) = \bigsqcup_{Z \in \Xi_A \atop \gamma_A(Z) = \mathcal{P}} \Xi_A(Z), \tag{24}$$

so the stratification of $\text{Spec}(R)$ by the $\Xi_A(Z)$ is a refinement of the stratification (10).

**Proof.** Suppose that $\Xi_A(Z) \cap \Xi_A(Z') \neq \emptyset$ with $Z \neq Z'$. Then there is $p \in Z \cap Z'$. Let $Z''$ be an irreducible component of $Z \cap Z'$ containing $p$. If $Z'' = Z$, then $Z \cap Z' = Z$, so $Z < Z'$. But this contradicts $p \in \Xi_A(Z') \subseteq Z' \setminus Z$. Hence, $Z'' < Z$. But since $Z'' \in \Xi_A$ by construction of $\Xi_A$ and $p \in \Xi_A(Z) \subseteq Z \setminus Z''$, this is again a contradiction. Hence, we have a partition $\text{Spec}(R) = \bigsqcup_{Z \in \Xi_A} \Xi_A(Z)$. It is clear that each $\Xi_A(Z)$ is locally closed in $\text{Spec}(R)$. By definition we have $\Xi_A(Z) = Z \setminus \bigcup_{Z < Z'} Z'$, thus $Z = \Xi_A(Z) \cup \bigcup_{Z < Z} Z'$. From this we obtain inductively that $Z = \bigcup_{Z < Z} \Xi_A(Z')$. Since the closure of $\Xi_A(Z)$ is obviously equal to $Z$ (recall that $Z$ is irreducible), it follows that $\text{Spec}(R) = \bigsqcup_{Z \in \Xi_A} \Xi_A(Z)$ is a good stratification. That the local block structure in the generic point of $Z \in \Xi_A(Z)$ is given by (22) follows directly from (20) and $\Xi_A(Z) \subseteq \Gamma_A(\gamma_Z(A))$ is now clear. \[\square\]

**Lemma 2.7.** Suppose that $R$ is normal. Then the maximal atomic gluing loci are the irreducible components of the local block divisor $\text{BEx}^{\text{loc}}(A)$ and are thus of codimension one in $\text{Spec}(R)$.

**Proof.** We have $\text{BEx}^{\text{loc}}(A) = \bigcup_{T \in \text{At}(\Xi_A)} T = \bigcup_{T \in \text{Max}(\Xi_A)} T$. It is now clear that the $T \in \text{Max}(\Xi_A)$ are precisely the irreducible components of $\text{BEx}^{\text{loc}}(A)$. The claim thus follows from Corollary 2.3. \[\square\]

Again we can consider $\Xi_A$ as a directed graph and attach the corresponding maximal local block structure $\gamma_A(Z)$ to each vertex $Z$. We call this the atomic local block graph of $A$. It not only gives us complete information about local block structures of $A$ but at once also an overview of how these block structures are formed. In Figure 3 we repeat the example from the introduction.
We define \( \beta \) and thus define for arbitrary structures of specializations. We can, however, compare numerical invariants in general to blocks of localizations there is in general no possibility to compare the actual block good stratification, however.

\[ \text{Figure 3. Atomic local block graph for the Brauer algebra over } \mathbb{Z}[\delta] \text{ for } n = 3. \]

The nice fact about the gluing loci and their block structures is that we can describe them rather explicitly, see Theorem 3.9.

Instead of considering a refinement of the stratification (10) we will now construct an interesting coarsening. For the moment we can drop the assumption about \( R \) being noetherian and just assume that \( A \) is finitely presented flat as an \( R \)-module. For \( n \in \mathbb{N} \) we define

\[ B_{\leq n}^{\text{loc}}(A) := \{ p \in \text{Spec}(R) \mid \# B(A_p) \leq n \}. \]

It is clear that

\[ B_{\leq n}^{\text{loc}}(A) = \bigcup_{\mathcal{P} \in \Gamma_A} \Gamma_A(\mathcal{P}), \]

so \( B_{\leq n}^{\text{loc}}(A) \) is closed in \( \text{Spec}(R) \) by Theorem 2.2. This means nothing else than the map \( \text{Spec}(R) \to \mathbb{N}, p \mapsto \# B(A_p), \) being lower semicontinuous. Consequently,

\[ B_{n}^{\text{loc}}(A) := B_{\leq n}^{\text{loc}}(A) \setminus B_{\leq n-1}^{\text{loc}}(A) = \{ p \in \text{Spec}(R) \mid \# B(A_p) = n \} \]

is locally closed in \( \text{Spec}(R) \) and we have a partition

\[ B_{\leq n}^{\text{loc}}(A) = \bigsqcup_{n \in \mathbb{N}} B_{n}^{\text{loc}}(A). \]

Note that

\[ B_{\leq n}^{\text{loc}}(A) = \{ p \in \text{Spec}(R) \mid \# \gamma_A(p) < \# \gamma_A(\bullet) \} = B_{\leq \gamma_A(\bullet)-1}^{\text{loc}}(A). \]

If \( R \) is noetherian, it follows from Lemma 2.5 that

\[ B_{n}^{\text{loc}}(A) \subseteq \bigcup_{m < n} B_{m}^{\text{loc}}(A), \]

so the partition (28) is in fact a stratification of \( \text{Spec}(R) \). Again, in general it will not be a good stratification, however.

\section{Blocks of specializations}

We finally turn to our actual problem, namely blocks of specializations of \( A \). Compared to blocks of localizations there is in general no possibility to compare the actual block structures of specializations. We can, however, compare numerical invariants in general and thus define for arbitrary \( A \):

\[ B_{\leq n}(A) := \{ p \in \text{Spec}(R) \mid \# B(A_p) \leq n \}, \]

\[ B_n(A) := B_{\leq n}(A) \setminus B_{\leq n-1}(A) = \{ p \in \text{Spec}(R) \mid \# B(A_p) = n \}, \]

\[ \beta(A) := \max(\# B(A(p)) \mid p \in \text{Spec}(R)). \]
(34) \[ \text{BlEx}(A) := \text{Bl}_{\beta(A) - 1}(A), \]
(35) \[ \text{BlGen}(A) := \text{Spec}(R) \setminus \text{BlEx}(A). \]

In general, these invariants will be distinct from the corresponding ones for blocks of localizations, see Example 2.14. This is why we attached the superscript “loc” to these invariants in the preceding paragraph. There is, however, a rather general setting where blocks of specializations are naturally identified with blocks of localizations, namely when \( R \) is normal and \( A^K \) splits. In this case not only the above sets are equal to their local versions but we can also compare the actual block structures of specializations and all results from the preceding paragraph are actually also results about blocks of specializations (we can thus remove the superscript “loc” and the prefix “local” everywhere under these assumptions). The key ingredient to establish this natural correspondence is the next proposition. To formulate it more generally, we use the property block-split introduced in Definition A.4 but note that the reader might just simply replace it by the more special property split. Moreover, we recall that a local integral domain \( R \) is called unibranch if its henselization \( R^h \) is again an integral (local) domain. This is equivalent to the normalization of \( R \) being again local (see [37, IX, Corollaire 1]). This clearly holds if \( R \) is already normal. Examples of non-normal unibranch rings are the local rings in ordinary cusp singularities of curves.

**Proposition 2.8.** Let \( R \) be an integral domain and let \( A \) be a finite flat \( R \)-algebra with block-split generic fiber \( A^K \) (e.g., if \( A^K \) splits). Let \( p \in \text{Spec}(R) \) and suppose that \( R_p \) is unibranch (e.g., if \( R_p \) is normal). Then the quotient morphism \( A_p \to A(p) \) is block bijective.

**Proof.** By assumption, \( R_p \) and its henselization \( R_p^h \) are integral domains. Since \( A \) is \( R \)-flat, it follows that \( A_p = R_p \otimes R A \) is \( R_p \)-flat and that \( A^h_p := R^h_p \otimes_{R_p} A_p \) is \( R^h_p \)-flat. Hence, both \( A_p \) and \( A^h_p \) have block decompositions by Lemma A.2. Let \( p^h_p \) be the maximal ideal of \( R^h_p \). The henselization morphism \( R_p \to R^h_p \) is local and faithfully flat by [23, Théorème 18.6.6(iii)]. We now have a commutative diagram

\[
\begin{array}{ccc}
A_p & \to & A^h_p \\
\downarrow & & \downarrow \\
A(p) = A_p/p_p A_p & \to & A^h_p/p^h_p A_p
\end{array}
\]

of idempotent stable morphisms. We know from Lemma A.15(b) and Lemma A.13 that \( A^h_p \to A^h_p/p^h_p A_p^h \) is block bijective. Since \( A \) has block-split generic fiber and \( R_p \to R^h_p \) is a faithfully flat morphism of integral domains, we can use Theorem A.12 to deduce that \( A_p \to A^h_p \) is block bijective. In [23, Théorème 18.6.6(iii)] it is proven that \( R_{p}/p_p \cong R^h_{p}/p^h_p \). Hence, the map \( A_p/p_p A_p \to A^h_p/p^h_p A_p^h \) is an isomorphism and so in particular block bijective. We thus have

\[ \#\text{Bl}(A^h_p) = \#\text{Bl}(A_p) \leq \#\text{Bl}(A(p)) = \#\text{Bl}(A^h_p/p^h_p A^h_p) = \#\text{Bl}(A^h_p) \]

by equation (4). Hence, \( \#\text{Bl}(A_p) = \#\text{Bl}(A(p)) \), so \( A_p \to A(p) \) is block bijective. \( \blacksquare \)

**Corollary 2.9.** Suppose that \( R \) is normal and \( A^K \) splits. Then \( A_p \to A(p) \) is block bijective for all \( p \in \text{Spec}(R) \). Hence, all results from §2A can be used to study blocks of specializations of \( A \).

**Corollary 2.10.** Assume that \( R \) is normal and that \( A^K \) splits. Then, the map \( \text{Spec}(R) \to \mathbb{N}, \ p \mapsto \#\text{Bl}(A(p)), \) is lower semicontinuous and \( \text{Spec}(R) = \bigsqcup_{n \in \mathbb{N}} \text{Bl}(A^n) \) is a partition into locally closed subsets. Moreover, \( \beta(A) = \#\text{Bl}(A^K) \) and \( \text{BlEx}(A) \) is a reduced Weil divisor in \( \text{Spec}(R) \). If \( R \) is also noetherian, then \( \text{Spec}(R) = \bigsqcup_{n \in \mathbb{N}} \text{Bl}(A^n) \) is a stratification.
Even though this setting is restrictive, we still include a lot of important examples in representation theory like Brauer algebras, Hecke algebras, restricted rational Cherednik algebras, etc. But clearly there are also many interesting examples which are not included, like restricted quantized enveloping algebras, mainly because they do not have a split generic fiber. However, even in these cases our results can be applied when we restrict to a certain subset of Spec(R). We discuss a general strategy.

Assume that $R'$ is an integral extension of $R$ which is also an integral domain. Let $K'$ be the fraction field of $R'$ and let $\psi : \text{Spec}(R') \to \text{Spec}(R)$ be the morphism induced by $R \subseteq R'$. The scalar extension $A' := R' \otimes_R A$ is again a finitely presented flat $R'$-algebra (using Remark 2.4). For any $p \in \text{Spec}(R)$ and any $p' \in \text{Spec}(R')$ lying over $p$ we have a diagram

\[
\begin{array}{c}
A_p' \\
\downarrow
\end{array} \quad \begin{array}{c}
A(p) = A_p/p_pA_p \quad \longrightarrow \quad A'(p') = A_{p'}/p_{p'}A_{p'}
\end{array}
\]

and it then follows from (4) that

\[
\# \text{Bl}(A(p)) \leq \# \text{Bl}(A'(p')) \geq \# \text{Bl}(A_{p'}').
\]

Let $X$ be a set contained in

\[
X_{R'}(A) := \{ p \in \text{Spec}(R) \mid \# \text{Bl}(A(p)) = \# \text{Bl}(A'(p')) = \# \text{Bl}(A_{p'}') \text{ for all } p' \in \psi^{-1}(p) \}.
\]

We have seen in Corollary 2.9 that in case $R$ is normal and $A^K$ splits we can choose $R = R'$ and have $X = \text{Spec}(R)$. In general $X$ will be a proper subset of Spec($R$) and we have to choose $R'$ appropriately to enlarge it a bit more. Let us first concentrate on what we can say when restricting to $X$. We introduce the following restricted versions of our invariants:

\[
\begin{align*}
\text{Bl}^X_{\text{loc}}(A) & := \text{Bl}_{\text{loc}}(A) \cap X = \psi(\text{Bl}^X_{\text{loc}}(A')) \cap X, \\
\text{Bl}^X_{\text{gen}}(A) & := \text{Bl}_{\text{gen}}(A) \cap X = \psi(\text{Bl}^X_{\text{gen}}(A')) \cap X, \\
\beta^X(A) & := \max \{ \# \text{Bl}(A(p)) \mid p \in X \}, \\
\text{BlEx}^X(A) & := \text{Bl}^X_{\leq \beta^X(A)-1}, \\
\text{BlGen}^X(A) & := X \setminus \text{BlEx}^X(A).
\end{align*}
\]

**Corollary 2.11.** The map $X \to \mathbb{N}$, $p \mapsto \# \text{Bl}(A(p))$, is lower semicontinuous on $X$ and $X = \bigsqcup_{n \in \mathbb{N}} \text{Bl}_{n}(A)$ is a partition into locally closed subsets. Moreover,

\[
\beta^X(A) \leq \# \text{Bl}(A^{K'}).
\]

If $R$ is noetherian, then $X = \bigsqcup_{n \in \mathbb{N}} \text{Bl}^X_{n}(A)$ is a stratification of $X$.

**Proof.** Since $\psi$ is a closed morphism and $\text{Bl}^X_{\text{loc}}(A')$ is closed in Spec($R'$) by (26), it follows that $\psi(\text{Bl}^X_{\text{loc}}(A'))$ is closed in Spec($R$), hence $\text{Bl}^X_{n}(A)$ is closed in $X$ by (39). Since $\text{Bl}^X_n(A) = \text{Bl}^X_{\text{loc}}(A) \setminus \text{Bl}^X_{n-1}(A)$, it is clear that $\text{Bl}^X_n(A)$ is locally closed in $X$. We have shown in (30) that $\text{Bl}^X_{n}(A') = \bigcup_{m \leq n} \text{Bl}^X_{m}(A')$. Hence, since $\psi$ is closed, we obtain

\[
\text{Bl}^X_n(A) = \psi(\text{Bl}^X_{n}(A')) \cap X = \bigcup_{m \leq n} \psi(\text{Bl}^X_{m}(A')) \cap X = \bigcup_{m \leq n} \text{Bl}^X_{m}(A).
\]

Note that in (44) we could only bound $\beta^X(A)$ above by $\text{Bl}(A^{K'})$, and not by $\text{Bl}(A^K)$. In fact, we will see in Example 2.14 that we may indeed have $\beta^X(A) > \# \text{Bl}(A^K)$ in general. This is an important difference to blocks of localizations where we always have the
maximal number of blocks in the generic point. In the following lemma we describe a situation where we have $\beta^X(A) = \#\text{Bl}(A^K)$. We recall that $X$ being very dense means that the embedding $X \hookrightarrow \text{Spec}(R)$ is a quasi-homeomorphism, i.e., the map $Z \rightarrow Z \cap X$ is a bijection between the closed (equivalently, open) subsets of the two spaces. This notion was introduced by Grothendieck [22, §10].

**Lemma 2.12.** Suppose that $X$ is very dense in $\text{Spec}(R)$, that $R$ is noetherian, and that $\psi$ is finite. Then $\beta^X(A) = \#\text{Bl}(A^K)$, thus $\text{Bl}^X(A) = \psi(\text{Bl}^\text{loc}(A')) \cap X$. If moreover $R'$ is normal and $R$ is universally catenary, then $\text{Bl}^X(A)$ is a reduced Weil divisor in $X$.

**Proof.** The assumption imply that $R'$ is noetherian, too. We know from Theorem 2.2 that $\text{Bl}^\text{loc}(A')$ is a non-empty open subset of $\text{Spec}(R')$. In particular, it is constructible. Since $\text{Spec}(R)$ is quasi-compact, also $\psi$ is quasi-compact by [19, Remark 10.2.(1)]. It thus follows from Chevalley’s constructibility theorem, see [19, Corollary 10.71], that $\psi(\text{Bl}^\text{loc}(A'))$ is constructible in $\text{Spec}(R)$. Since $X$ is very dense in $\text{Spec}(R)$, we conclude that $\psi(\text{Bl}^\text{loc}(A')) \cap X \neq \emptyset$ by [22, Proposition 10.1.2]. Hence, there is $p \in X$ and $p' \in \text{Bl}^\text{loc}(A')$ with $\psi(p') = p$. But then we have $\#\text{Bl}(A(p)) = \#\text{Bl}(A(p')) = \#\text{Bl}(A^K)$, so $\beta^X(A) = \#\text{Bl}(A^K)$. Now, assume that $R'$ is normal and $R$ is universally catenary. We know that $\text{Bl}^\text{loc}(A')$ is either empty or pure of codimension one in $\text{Spec}(R')$ by Corollary 2.3. In [24, Theorem B.5.1] it is shown that the extension $R \subseteq R'$ satisfies the dimension formula, hence $\psi(\text{Bl}^\text{loc}(A'))$ is either empty or pure of codimension one. Since $X$ is very dense in $\text{Spec}(R)$, the same is also true for $X \cap \psi(\text{Bl}^\text{loc}(A')) = \text{Bl}^X(A)$. □

**Corollary 2.13.** Suppose that $R$ is a finite type algebra over an algebraically closed field. Let $X$ be the set of closed points of $\text{Spec}(R)$. Then the map $X \rightarrow \mathbb{N}$, $m \mapsto \#\text{Bl}(A(m))$, is lower semicontinuous and $X = \bigsqcup_{n \in \mathbb{N}} \text{Bl}^X(A)$ is a stratification of $X$. Moreover, $\beta^X(A) = \#\text{Bl}(A^\overline{K})$, where $\overline{K}$ is an algebraic closure of $K$. If $R$ is also universally catenary, then $\text{Bl}^X(A)$ is a reduced Weil divisor in $X$.

**Proof.** Let $K'$ be a finite extension of $K$ such that $A^{K'}$ splits (this is always possible, see [11, Proposition 7.13]) and let $R'$ be the integral closure of $R$ in $K'$. Now, $\#\text{Bl}(A(p')) = \#\text{Bl}(A_p^\overline{K})$ for all $p' \in \text{Spec}(R)$ by Proposition 2.8. Since $R$ is a finite type algebra over an algebraically closed field $k$, the residue field in a closed point $m$ of $\text{Spec}(R)$ is just $k$. Hence, the specialization $A(m)$ is a finite-dimensional algebra over an algebraically closed field, thus splits and we therefore have $\#\text{Bl}(A(m)) = \#\text{Bl}(A'(m'))$ for any $m' \in \psi^{-1}(m)$ by Lemma A.5. Hence, $X \subseteq X_R(A)$. The claim about semicontinuity and the stratification thus follows from Corollary 2.11. It is shown in [19, Proposition 3.35] that $X$ is very dense in $\text{Spec}(R)$. Since $R$ is a finite type algebra over a field, it is japanese, so $\psi$ is a finite morphism. Hence, $\beta^X(A) = \#\text{Bl}(A^K) = \#\text{Bl}(A^\overline{K})$ by Lemma 2.12. Also the claim that $\text{Bl}^X(A)$ is a reduced Weil divisor if $R$ is universally catenary follows from Lemma 2.12. □

**Example 2.14.** The following example due to K. Brown shows that in the setting of Lemma 2.13 we may indeed have $\beta^X(A) > \#\text{Bl}(A^K)$ so that the map $p \mapsto \#\text{Bl}(A(p))$ will not be lower semicontinuous on the whole of $\text{Spec}(R)$. Let $k$ be an algebraically closed field of characteristic zero, let $X$ be an indeterminate over $k$, let $R := k[X^n]$ for some $n > 1$, and let $A := k[X]$. Let $C_n$ be the cyclic group of order $n$. We fix a generator of $C_n$ and let it act on $X$ by multiplication with a primitive $n$-th root of unity. Then $R = k[X]^{C_n}$, so $A$ is free of rank $n$ over $R$. Moreover, $\text{Frac}(A) = k(X)$ is a Galois extension of degree $n$ of $K := \text{Frac}(R)$ by [1, Proposition 1.1.1], so in particular $K \neq k(X)$ since $n > 1$. By [17, Ex. 6R] we have

$$A^K = A \otimes_R K = A[(R \setminus \{0\})^{-1}] = \text{Frac}(A) = k(X),$$
so the $K$-algebra $A^K = Z(A^K)$ is not split (and thus also not block-split by Lemma A.5). It is clear that
\begin{equation}
\#B(A^K) = 1.
\end{equation}
Now, let $m := (X^n - 1) \in \text{Max}(R)$. Then $k(p) = k$ and since $k$ is algebraically closed, we have $A(m) = A/mA = k^n$ as $k$-algebras. In particular,
\begin{equation}
\#B(A(m)) = n > 1 = \#B(A^K).
\end{equation}

Finally, we want to provide a setting where our base ring is not necessarily normal but we still get a global result on $\text{Spec}(R)$.

**Lemma 2.15.** Suppose that $A$ has split fibers, i.e., $A(p)$ splits for all $p \in \text{Spec}(R)$. Then the map $\text{Spec}(R) \rightarrow \mathbb{N}$, $p \mapsto \#B(A(p))$, is lower semicontinuous and $\text{Spec}(R) = \bigsqcup_{n \in \mathbb{N}} \text{Bl}_n(A)$ is a partition into locally closed subsets. Moreover, $\beta(A) = \#B(A^K)$. If $R$ is also universally catenary, japanese, and noetherian, then $\text{BlEx}(A)$ is a reduced Weil divisor in $\text{Spec}(R)$.

**Proof.** Let $R'$ be the integral closure of $R$ in $K$. Then $\#B(A'(p')) = \#B(A'_p)$ for all $p' \in \text{Spec}(R')$ by Proposition 2.8. Since $A(p)$ splits, we moreover have $\#B(A(p)) = \#B(A'(p'))$ for all $p \in \text{Spec}(R)$ $p' \in \psi^{-1}(p)$ by Lemma A.5. Hence, $X_k(A) = \text{Spec}(R)$. The claim about semicontinuity and the partition follows from Corollary 2.11. Now, assume that $R$ is universally catenary, japanese, and noetherian. Since $R$ is japanese, it follows by definition that $\psi$ is finite. The claim about $\text{BlEx}(A)$ being a reduced Weil divisor now follows from 2.12. 

### §3. Blocks via central characters

The main result in this section is Theorem 3.9 which gives an explicit description of the gluing loci introduced in §2A via zero loci of central characters of simple modules of the generic fiber. This allows us (in principle) to construct the whole (atomic) block graph once we know the central characters. Parts of the argumentation are due to Bonnafé and Rouquier [2, Appendice C].

### §3A. Müller’s theorem

The central ingredient to establish a relationship between blocks and central characters is the general Lemma 3.6 below, which is usually referred to as Müller’s theorem. We were not able to find a proof of it in this generality in the literature, so we include a proof here but note that this is known. The main ingredient is an even more general result by B. Müller [32] about the fibration of cliques of prime ideals in a noetherian ring over its center, see Lemma 3.5. We will recall only a few basic definitions from the excellent exposition in [17, §12] and refer to loc. cit. for more details.

Throughout this paragraph, we assume that $A$ is a noetherian ring.

If $p, q$ are prime ideals of $A$, we say that there is a link from $p$ to $q$, written $p \rightsquigarrow q$, if there is an ideal $a$ of $A$ such that $p \cap q \supseteq a \supseteq pq$ and $(p \cap q)/a$ is non-zero and torsion-free both as a left $(A/p)$-module and as a right $(A/q)$-module. The bimodule $(p \cap q)/a$ is then called a linking bimodule between $q$ and $p$. The equivalence classes of the equivalence relation on $\text{Spec}(A)$ generated by $\rightsquigarrow$ are called the cliques of $A$. We write $\text{Cl}(A)$ for the set of cliques of $A$ and $\text{Cl}(p)$ for the unique clique of $A$ containing $p$. For the proof of Lemma 3.6 we will need a few preparatory lemmas.

We call the supremum of lengths of chains of prime ideals in $A$ the classical Krull dimension of $A$. The following lemma is standard.
Lemma 3.1. Suppose that $A$ is noetherian and of classical Krull dimension zero. Then there is a canonical bijection

$$\text{Bl}(A) \xrightarrow{\sim} \text{Clq}(A)$$

where $c^\dagger = 1 - c$. If moreover $A$ is commutative, then the cliques are singletons, i.e., there is a unique $m_c \in \text{Max}(A)$ with $c^\dagger \in m_c$. Hence, in this case we have $\text{Bl}(A) = \text{Max}(A) = \text{Spec}(A)$.

Proof. The first assertion is proven in [17, Corollary 12.13]. In a commutative noetherian ring the cliques are singletons (see [17, Exercise 12F]), and this immediately implies the second assertion.

Lemma 3.2. Let $p$ be a prime ideal of a noetherian ring $A$ and let $V$ be a non-zero $A$-module with $p \subseteq \text{Ann}(V)$. If $V$ is torsion-free as an $(A/p)$-module, then $p = \text{Ann}(V)$.

Proof. Suppose that $p \subset \text{Ann}(V)$. Then $\text{Ann}(V)/p$ is a non-zero ideal of the noetherian prime ring $A/p$ and thus contains a regular element $\overline{x}$ by [25, Corollary 2.3.11]. But then $\overline{x}V = 0$, contradicting the assumption that $V$ is a torsion-free $(A/p)$-module.

Lemma 3.3. The following holds:

(a) If $p$ and $q$ are prime ideals of $A$ and if $b$ is an ideal of $A$ with $b \subseteq p \cap q$ such that $p/b \twoheadrightarrow q/b$ in $A/b$, then $p \twoheadrightarrow q$ in $A$.

(b) Let $p$ and $q$ be two prime ideals of $A$ with $p \twoheadrightarrow q$ and let $b$ be an ideal of $A$. If there exists a linking ideal $a$ from $p$ to $q$ with $b \subseteq a$, then $p/b \twoheadrightarrow q/b$ in $A/b$.

Proof.

(a) We can write a linking ideal from $p/b$ to $q/b$ as $a/b$ for an ideal $a$ containing $b$. By definition, we have

$$(p \cap q)/b = (p/b) \cap (q/b) \supseteq a/b \supseteq (p/b) \cdot (q/b) = (pq)/b,$$

implying that $p \cap q \supseteq a \supseteq pq$. Moreover, we have

$$(p \cap q)/b \cdot (a/b) \cong (p \cap q)/a$$

as $(A/b)$-bimodules. By definition, $(p \cap q)/a$ is torsionfree as a left module over the ring $(A/b)/(p/b) \cong A/p$.

Similarly, it follows that $(p \cap q)/a$ is torsionfree as a right module over the ring $A/q$. Hence, $a$ is a linking ideal from $p$ to $q$.

(b) We have

$$p/b \cap q/b = (p \cap q)/b \supseteq a/b \supseteq (pq + b)/b = (p/b) \cdot (q/b).$$

Since

$$((p \cap q)/b)/(a/b) \cong (p \cap q)/a, \quad (A/b)/(p/b) \cong A/p, \quad (A/b)/(q/b) \cong A/q,$$

it follows that $a/b$ is a linking ideal from $p/b$ to $q/b$.

Lemma 3.4. Let $p$ and $q$ be distinct prime ideals of a noetherian ring $A$ with $p \twoheadrightarrow q$. If $\overline{z}$ is a centrally generated ideal of $A$ with $\overline{z} \subseteq p$ or $\overline{z} \subseteq q$, then $\overline{z} \subseteq p \cap q$ and $p/\overline{z} \twoheadrightarrow q/\overline{z}$ in $A/\overline{z}$.

Proof. This is proven in [33] but we also give a proof here for the sake of completeness. First note that since $\overline{z}$ is centrally generated and $p \twoheadrightarrow q$, it follows from [17, Lemma 12.15] that already $\overline{z} \subseteq p \cap q$. Let $a$ be a linking ideal from $p$ to $q$. We claim that $\overline{z}$ is contained in $a$. To show this, suppose that $\overline{z}$ is not contained in $a$. Then $(a + \overline{z})/a$ is a non-zero submodule of $(p \cap q)/a$ which is torsionfree as a left $(A/p)$-module and as a right $(A/q)$-module. In conjunction with the fact that $\overline{z}$ is centrally generated it now follows from Lemma 3.2. that

$$p = \text{Ann}(A((a + \overline{z})/a)) = \text{Ann}(((a + \overline{z})/a)_A) = q,$$
contradicting the assumption \( p \neq q \). Hence, we must have \( \mathfrak{z} \subseteq \mathfrak{a} \) and it thus follows from Lemma 3.3(b) that \( p/\mathfrak{z} \sim q/\mathfrak{z} \).

Lemma 3.5. Let \( \mathfrak{z} \) be a centrally generated ideal of a noetherian ring \( A \). Let \( p \) be a prime ideal of \( A \) with \( \mathfrak{z} \subseteq p \). Then all prime ideals in \( \text{Clq}(p) \) contain \( \mathfrak{z} \) and the map

\[
\text{Clq}(p) \longrightarrow \text{Clq}(p/\mathfrak{z})
\]

\[
q \longmapsto q/\mathfrak{z}
\]

is a bijection between a clique of \( A \) and a clique of \( A/\mathfrak{z} \).

Proof. It follows immediately from [17, Lemma 12.15] that all prime ideals in \( \text{Clq}(p) \) contain \( \mathfrak{z} \). If \( q \in \text{Clq}(p) \), then there exists a chain \( p = p_0, p_1, \ldots, p_r = q \) of prime ideals of \( A \) with \( p_i \sim p_{i+1} \) or \( p_{i+1} \sim p_i \) for all indices \( i \). An inductive application of Lemma 3.4 shows now that \( p_i/\mathfrak{z} \sim p_{i+1}/\mathfrak{z} \) or \( p_{i+1}/\mathfrak{z} \sim p_i/\mathfrak{z} \) for all \( i \). Hence, \( p/\mathfrak{z} \) and \( q/\mathfrak{z} \) lie in the same clique of \( A/\mathfrak{z} \) so that the map \( \text{Clq}(p) \rightarrow \text{Clq}(p/\mathfrak{z}) \) is well-defined. On the other hand, similar arguments and Lemma 3.3(a) show that if \( q/\mathfrak{z} \in \text{Clq}(p/\mathfrak{z}) \), then also \( q \in \text{Clq}(p) \), so that we also have a well-defined map \( \text{Clq}(p/\mathfrak{z}) \rightarrow \text{Clq}(p) \). It is evident that both maps defined are pairwise inverse thus proving the first assertion. The second assertion is now obvious.

Lemma 3.6 (B. Müller). Let \( A \) be a ring with center \( Z \) such that \( Z \) is noetherian and \( A \) is a finite \( Z \)-module. If \( \mathfrak{z} \) is a centrally generated ideal of \( A \) such that \( A/\mathfrak{z}A \) is of classical Krull dimension zero, then the inclusion \( (Z + \mathfrak{z})/\mathfrak{z} \hookrightarrow A/\mathfrak{z}A \) is block bijective. In other words, the block idempotents of \( A/\mathfrak{z}A \) are already contained in the central subalgebra \( (Z + \mathfrak{z})/\mathfrak{z} \).

Proof. Let \( \widetilde{A} := A/\mathfrak{z} \) and let \( \widetilde{Z} := (Z + \mathfrak{z})/\mathfrak{z} \). Then \( \widetilde{A} \) is a finitely generated \( \widetilde{Z} \)-module since \( A \) is a finitely generated \( Z \)-module. Hence, \( \widetilde{Z} \subseteq \widetilde{A} \) is a finite centralizing extension and now it follows from going up in finite centralizing extensions [31, Theorem 10.2.9] that the classical Krull dimension of \( Z \) is equal to that of \( \widetilde{A} \), which is zero by assumption. Hence, by Lemma 3.1 we have \( \text{Bl}(\widetilde{Z}) = \text{Clq}(\widetilde{Z}) \) and \( \text{Bl}(\widetilde{A}) = \text{Clq}(\widetilde{A}) \). Since \( \#\text{Bl}(\widetilde{Z}) \leq \#\text{Bl}(\widetilde{A}) \), the claim is thus equivalent to the claim that over each clique of \( \widetilde{Z} \), there is just one clique of \( \widetilde{A} \). So, let \( X, Y \in \text{Clq}(\widetilde{A}) \) be two cliques. We pick \( \mathfrak{M}/\mathfrak{z} \in X \) and \( \mathfrak{N}/\mathfrak{z} \in Y \) with \( \mathfrak{M}, \mathfrak{N} \) maximal ideals of \( A \). Assume that \( X \) and \( Y \) lie over the same clique of \( \widetilde{Z} \). Since \( \widetilde{Z} \) is commutative, we know from Lemma 3.1 that all cliques are singletons and so the assumption implies that \( \mathfrak{M}/\mathfrak{z} \) and \( \mathfrak{N}/\mathfrak{z} \) lie over the same maximal ideal of \( \widetilde{Z} \), i.e.,

\[
(\mathfrak{M}/\mathfrak{z}) \cap ((Z + \mathfrak{z})/\mathfrak{z}) = (\mathfrak{N}/\mathfrak{z}) \cap ((Z + \mathfrak{z})/\mathfrak{z})
\]

hence

\[
\mathfrak{M} \cap (Z + \mathfrak{z}) = \mathfrak{N} \cap (Z + \mathfrak{z})
\]

Since \( Z \subseteq Z + \mathfrak{z} \), we thus get

\[
\mathfrak{M} \cap Z = \mathfrak{M} \cap Z \cap (Z + \mathfrak{z}) = \mathfrak{N} \cap Z \cap (Z + \mathfrak{z}) = \mathfrak{N} \cap Z
\]

Now, Müller’s theorem [17, Theorem 13.10] implies that \( \mathfrak{M} \) and \( \mathfrak{N} \) lie in the same clique of \( A \). An application of Lemma 3.5 thus implies that \( \mathfrak{M}/\mathfrak{z} \) and \( \mathfrak{N}/\mathfrak{z} \) lie in the same clique of \( A/\mathfrak{z} \), so \( X = Y \).

§3B. Blocks as fibers of a morphism

We assume that \( A \) is a finite flat algebra over a noetherian integral domain \( R \).

By Lemma B.2 the morphism

\[
(48) \quad Y : \text{Spec}(Z) \to \text{Spec}(R)
\]
induced by the canonical morphism from \( R \) to the center \( Z \) of \( A \) is finite, closed, and surjective. The center \( Z \) of \( A \) is naturally an \( R \)-algebra and so we can consider its fibers

\[
Z(p) = k(p) \otimes_R Z/pZ = Z/pZ_p
\]

in prime ideals \( p \) of \( R \). On the other hand, the image of \( Z_p = Z(A_p) \) under the canonical (surjective) morphism \( A_p \twoheadrightarrow A(p) \) yields a central subalgebra

\[
Z_p(A) := (Z_p + p_pA_p)/p_pA_p
\]

of \( A(p) \). In general this subalgebra is not equal to the center of \( A(p) \) itself. We have a surjective morphism

\[
\varphi_p : Z(p) \twoheadrightarrow Z_p(A)
\]

of finite-dimensional \( k(p) \)-algebras. This morphism is in general not injective—it is if and only if

\[
p_pA_p \cap Z_p = p_pZ_p \iff \text{Rad}(A_p) \cap Z_p = \text{Rad}(Z_p).
\]

Nonetheless, we have the following result.

**Lemma 3.7.** The map \( \varphi_p : Z(p) \to Z_p(A) \) in (51) is block bijective.

**Proof.** Since \( \varphi_p \) is surjective, the induced map \( a_p \varphi_p : \text{Spec}(Z_p(A)) \to \text{Spec}(Z(p)) \) is injective, so \( \# \text{Bl}(Z_p(A)) \leq \# \text{Bl}(Z(p)) \) by Lemma 3.1. Now we just need to show that \( \varphi_p \) does not map any non-trivial idempotent to zero. Since \( R_p \) is noetherian, also \( A_p \) is noetherian. The Artin–Rees lemma [30, Theorem 8.5] applied to the \( R_p \)-module \( A_p \), the submodule \( Z_p \) of \( A_p \), and the ideal \( p_p \) of \( R_p \) shows that there is an integer \( k \in \mathbb{N}_{>0} \) such that for any \( n > k \)

\[
p^n_p A_p \cap Z_p = p^n_p (p^k_p A_p) \cap Z_p.
\]

In particular, there is \( n \in \mathbb{N}_{>0} \) such that \( p^n_p A_p \cap Z_p \subseteq p_pZ_p \). Now, let \( \overline{e} \in Z(p) = Z/p_pZ_p \) be an idempotent with \( \varphi_p(\overline{e}) = 0 \). By assumption, \( \overline{e} \in \ker(\varphi_p) = (p_pA_p \cap Z_p)/p_pZ_p \). Hence, if \( e \in Z_p \) is a representative of \( \overline{e} \), we have \( e \in p_pA_p \cap Z_p \). We have \( e^n \in p^n_p A_p \cap Z_p \subseteq p_pZ_p \), so already \( \overline{e} = 0 \). \( \blacksquare \)

**Theorem 3.8.** For any \( p \in \text{Spec}(R) \) there are canonical bijections

\[
\text{Bl}(A(p)) \cong \text{Bl}(Z_p(A)) \cong \text{Bl}(Z(p)) \cong Y^{-1}(p).
\]

The first bijection \( \text{Bl}(A(p)) \cong \text{Bl}(Z_p(A)) \) is induced by the embedding \( Z_p(A) \hookrightarrow A(p) \). In other words, all block idempotents of \( A(p) \) are already contained in the central subalgebra \( Z_p(A) \) of \( A(p) \). The second bijection is the bijection from Lemma 3.7. The last bijection \( \text{Bl}(Z(p)) \cong Y^{-1}(p) \) maps a block idempotent \( c \) of \( Z(p) \) to the (by the theorem unique) maximal ideal \( m_c \) of \( Z \) lying above \( p \) such that \( c^\dagger \in (m_c + p_pZ_p)/p_pZ_p \), where \( c^\dagger = 1 - c \).

**Proof.** The first bijection follows directly from Lemma 3.6 applied to \( A_p \) and the centrally generated ideal \( \mathfrak{z} := p_pA_p \). Let \( Y_p : \text{Spec}(Z(p)) \to \text{Spec}(R_p) \) be the morphism induced by the canonical map \( R_p \to Z_p \). Recall from Lemma B.2 that \( R_p \subseteq Z_p \) is a finite extension so that \( Y_p \) is surjective. We have

\[
Y_p^{-1}(p_p) = (\Omega \in \text{Spec}(Z_p) \mid \Omega \cap R_p = p_p) = (\Omega \in \text{Spec}(Z_p) \mid p_p \subseteq \Omega) = (\Omega \in \text{Spec}(Z_p) \mid p_p\Omega \subseteq \Omega) = \text{Spec}(Z(p)).
\]

In the second equality we used the fact that \( R_p \to Z_p \) is a finite morphism and \( R_p \) is local with maximal ideal \( p_p \). The identification with \( \text{Spec}(Z(p)) \) is canonical since \( Z(p) = Z_p/p_pZ_p \). The morphism \( \Theta_p : \text{Spec}(Z_p) \to \text{Spec}(Z) \) induced by the localization map \( Z \to Z_p \) is injective by [13, Proposition 2.2(b)]. We claim that this map induces \( \Theta_p : Y_p^{-1}(p_p) \to Y^{-1}(p) \). If \( \Omega \in Y_p^{-1}(p_p) \), then clearly \( (\Omega \cap Z) \cap R = \Omega \cap R \subseteq R \cap p_p = p \) and therefore \( \Theta_p \)
induces an injective map $Y^{-1}(p) \to Y^{-1}(p)$. If $\Omega \in Y^{-1}(p)$, then, since $\Omega \cap R = p$, we have $\Omega \cap (R \setminus p) = \emptyset$ so that $\Omega \in \text{Spec}(Z(p))$ and clearly $p = \Omega_{p}$, implying that $\Omega \in Y^{-1}(p)$. The map $Y^{-1}(p) \to Y^{-1}(p)$ is thus bijective. Hence, we have a canonical bijection $\text{Spec}(Z(p)) \cong Y^{-1}(p)$. Now, recall from Lemma 3.1 that $\text{Spec}(Z(p)) \cong \text{Bl}(Z(p))$. 

§3C. Blocks and gluing loci via central characters

We assume that $R$ is noetherian and normal, and that $A$ is a finite flat $R$-algebra with split generic fiber $A^{K}$.

Recall from Corollary 2.9 that the quotient map $A_{p} \to A(p)$ induces $\text{Bl}(A_{p}) \cong \text{Bl}(A(p))$, so together with Theorem 3.8 we have a canonical bijection

\begin{equation}
\text{Bl}(A_{p}) \cong Y^{-1}(p).
\end{equation}

Recall from §2A that $\text{Fam}_{p}(A^{K})$ is the partition of $\text{Irr} A^{K}$ induced by the blocks of $A_{p}$ and that we naturally have $\text{Bl}(A_{p}) \cong \text{Fam}_{p}(A^{K})$. Altogether, we now have canonical bijections

\begin{equation}
\text{Fam}_{p}(A) \cong \text{Bl}(A_{p}) \cong Y^{-1}(p) \cong \text{Bl}(A(p)).
\end{equation}

Since $A$ has split generic fiber $A^{K}$, we have a central character $\Omega_{S} : Z(A^{K}) \to K$ for every simple $A^{K}$-module $S$. Recall that $\Omega_{S}(z)$ is the scalar by which $z \in Z(A^{K})$ acts on $S$. Since $R$ is normal, the image of the restriction of $\Omega_{S}$ to $Z(A) \subseteq Z(A^{K})$ is contained in $R \subseteq K$. We thus get a well-defined $R$-algebra morphism

\begin{equation}
\Omega_{S} : Z(A) \to R.
\end{equation}

It is a classical fact that $S, T \in \text{Irr} A^{K}$ lie in the same family if and only if $\Omega_{S} = \Omega_{T}$. We can thus label the central characters of $A^{K}$ as $\Omega_{S}$ with $S$ a family (block) of $A^{K}$. Using Theorem 3.8 this description generalizes modulo $p$ so that we get an explicit description of the $p$-families, and thus of the block stratification.

**Theorem 3.9.** Under the bijection $Y^{-1}(p) \cong \text{Fam}_{p}(A)$ from (55) the $p$-family of a simple $A^{K}$-module $S$ corresponds to $\text{Ker} \Omega_{S}^{p}$. Hence, two simple $A^{K}$-modules $S$ and $T$ lie in the same $p$-family if and only if $\Omega_{S}^{p}(z) \equiv \Omega_{T}^{p}(z) \mod p$ for all $z \in Z(A)$. So, if $z_{1}, \ldots, z_{n}$ is an $R$-algebra generating system of $Z(A)$ and $\mathscr{F}, \mathscr{F}'$ are two distinct $A^{K}$-families, then the corresponding gluing locus is given by

\begin{equation}
\Gamma_{A}((\mathscr{F}, \mathscr{F}')) = V(|\Omega_{S}(z_{i}) - \Omega_{T}(z_{i})| \mod p, i = 1, \ldots, n)).
\end{equation}

**Proof.** Considering the explicit form of the bijection given in Theorem 3.8 we see that the bijection (54) maps a block idempotent $c$ of $A_{p}$ to the (by the theorem unique) maximal ideal $\Delta_{c}$ of $Z$ lying above $p$ and satisfying $c + \Delta_{c} \in (\text{Irr}_{p})$. Let $c_{\Omega}$ be the block idempotent of $A_{p}$ corresponding to $\Omega \in Y^{-1}(p)$.

For $S \in \text{Irr} A^{K}$ let $\Omega_{S}^{p} : Z \to R/p$ be the composition of $\Omega_{S}'$ and the quotient morphism $R \to R/p$. It is clear that $\text{Ker}(\Omega_{S}^{p}) \in Y^{-1}(p)$. Note that $\Omega_{S}'(z) = \Omega_{T}'(z) \mod p$ for all $z \in Z(A)$ if and only if $\Omega_{S}' = \Omega_{T}'.\Theta_{A}$ has an exact sequence

\begin{equation}
0 \to \text{Ker}(\Omega_{S}') \to Z \xrightarrow{\Omega_{S}'} R \to 0
\end{equation}

of $R$-modules. Since $\Omega_{S}'$ is an $R$-algebra morphism, the canonical map $R \to Z$ is a section of $\Omega_{S}'$ and therefore $Z = R \oplus \text{Ker}(\Omega_{S}')$ as $R$-modules. Similarly, we have $Z = R \oplus \text{Ker}(\Omega_{T}')$. Since $\text{Ker}(\Omega_{S}') \subseteq \text{Ker}(\Omega_{S}^{p})$ and $\text{Ker}(\Omega_{T}') \subseteq \text{Ker}(\Omega_{T}^{p})$, this implies that $\Omega_{S}^{p} = \Omega_{T}'$ if and only if $\text{Ker}(\Omega_{S}^{p}) = \text{Ker}(\Omega_{T}^{p})$.

Now, suppose that $\text{Ker}(\Omega_{S}^{p}) = \text{Ker}(\Omega_{T}^{p})$. Denote this common kernel by $\Omega$. Clearly, $\Omega \in Y^{-1}(p)$. We know that the corresponding block idempotent $c_{\Omega}$ of $A_{p}$ has the property
that $c_\Omega^\dagger \in \Omega_p$. Since $\text{Ker}(\Omega'_S) \subseteq \text{Ker}(\Omega'^p_S) = \Omega = \text{Ker}(\Omega'^p_T) \supseteq \text{Ker}(\Omega'_T)$, this certainly implies that $c_\Omega^\dagger S = 0 = c_\Omega^\dagger T$. Hence, $S$ and $T$ lie in the same $p$-family.

Conversely, suppose that $S$ and $T$ lie in the same $p$-family. We can write the corresponding block idempotent of $A_p$ as $c_\Omega$ for some $\Omega \in \mathcal{Y}^{-1}(p)$. By definition, $c_\Omega^\dagger S = 0 = c_\Omega^\dagger T$. We know that $c_\Omega \in \Omega_p$ and $c_\Omega \notin \Omega_p$ and therefore $\text{Ker}((\Omega'_S)_p) = \Omega_p = \text{Ker}((\Omega'_T)_p)$. Hence, $\Omega \subseteq \text{Ker}(\Omega'_S) \subseteq \text{Ker}(\Omega'^p_S)$ and $\Omega \subseteq \text{Ker}(\Omega'_T) \subseteq \text{Ker}(\Omega'^p_T)$. Since $\Omega_p, \text{Ker}(\Omega'^p_S), \text{Ker}(\Omega'^p_T) \in \mathcal{Y}^{-1}(p)$ and all prime ideals in $\mathcal{Y}^{-1}(p)$ are incomparable, we thus conclude that $\text{Ker}(\Omega'^p_S) = \text{Ker}(\Omega'^p_T)$.

The equation for the gluing locus is now clear.

§4. Blocks and decomposition maps

To obtain information about the actual members of the $A(p)$-families we use decomposition maps as introduced by Geck and Rouquier [16] (see also [15] and [42]). For the theory of decomposition maps we need the following (standard) assumption:

$A$ is finite free with split generic fiber and for any non-zero $p \in \text{Spec}(R)$ there is a discrete valuation ring $\mathcal{O}$ with maximal ideal $m$ in $K$ dominating $R_p$ such that the canonical map $G_0(A(p)) \to G_0(A(\mathcal{O})(m))$ of Grothendieck groups is an isomorphism.

Here, $G_0$ denotes the Grothendieck group, i.e., the zeroth $K$-group of the category of finitely generated modules. We call a ring $\mathcal{O}$ as above a perfect $A$-gate in $p$. We refer to [42] for more details. The following lemma lists two standard situations in which the above assumptions hold. Part (a) is obvious and part (b) was proven in [42, Theorem 1.22].

**Lemma 4.1.** A finite free $R$-algebra $A$ with split generic fiber satisfies the above assumptions in the following two cases:

(a) $R$ is a Dedekind domain.
(b) $R$ is noetherian and $A$ has split fibers.

If $\mathcal{O}$ is a perfect $A$-gate in $p$, then there is a group morphism

$$d_A^{p, \mathcal{O}} : G_0(A^K) \to G_0(A(p))$$

between Grothendieck groups generalizing reduction modulo $p$. In case $R$ is normal, it was proven by Geck and Rouquier [16] that this map is independent of the choice of $\mathcal{O}$ and in this case we just write $d_A^p$. We note that in case $R$ is noetherian and $A$ has split fibers, any decomposition map in the sense of Geck and Rouquier can be realized by a perfect $A$-gate, see [42, Theorem 1.22].

§4A. Brauer reciprocity

An important tool for relating decomposition maps and blocks is the so-called Brauer reciprocity which we prove in Theorem 4.2 below in our general setup (this was known to hold before only in special settings). Recall that the intertwining form for a finite-dimensional algebra $B$ over a field $F$ is the $Z$-linear pairing $\langle \cdot, \cdot \rangle_B : K_0(B) \times G_0(B) \to Z$ uniquely defined by

$$\langle [P], [V] \rangle := \dim_F \text{Hom}_B(P, V)$$

for a finite-dimensional projective $B$-module $P$ and a finite-dimensional $B$-module $V$, see [16, §2]. Here, $K_0(B)$ is the zeroth $K$-group of the category of finite-dimensional projective $B$-modules. The intertwining form is always non-degenerate, see Lemma A.8. Due to the non-degeneracy of $\langle \cdot, \cdot \rangle_A^K$ there is at most one adjoint

$$e_A^{p, \mathcal{O}} : K_0(A(p)) \to K_0(A^K)$$
of \(d_A^{\mathcal{O}}: G_0(A^K) \to G_0(A(p))\) with respect to \(\langle \cdot, \cdot \rangle_{A(p)}\), characterized by the relation

\[
\langle e_A^{\mathcal{O}}([\mathcal{P}]), [V] \rangle_{A^K} = \langle [\mathcal{P}], d_A^{\mathcal{O}}([V]) \rangle_{A(p)}.
\]

for all finitely generated \(A^K\)-modules \(V\) and all finitely generated projective \(A(p)\)-modules \(\mathcal{P}\), see Lemma A.8. Brauer reciprocity is about the existence of this adjoint.

**Theorem 4.2.** The (unique) adjoint \(e_A^{\mathcal{O}}\) of \(d_A^{\mathcal{O}}\) exists. Moreover, the diagram

\[
\begin{array}{ccc}
K_0(A^K) & \xrightarrow{\zeta_{A^K}} & G_0(A^K) \\
\downarrow e_A^{\mathcal{O}} & & \downarrow \quad d_A^{\mathcal{O}} \\
K_0(A(p)) & \xrightarrow{\zeta_{A(p)}} & G_0(A(p))
\end{array}
\]

commutes, where the horizontal morphisms are the canonical ones (Cartan maps) mapping a class \([P]\) of a projective module \(P\) to its class \([P]\) in the Grothendieck group. If \(R\) is normal, the morphism \(e_A^{\mathcal{O}}\) does not depend on the choice of \(\mathcal{O}\) and we denote it by \(e_A^{\mathcal{O}}\).

**Proof.** Since \(\langle \cdot, \cdot \rangle_{A^K}\) is non-degenerate by Lemma A.8, it follows that \(d_A^{\mathcal{O}}\) has at most one adjoint \(e_A^{\mathcal{O}}\), characterized by equation (61), see [40, Satz 78.1]. By assumption there is a perfect \(A\)-gate \(\mathcal{O}\) in \(p\). Let \(m\) be the maximal ideal of \(\mathcal{O}\). Since \(A^K\) splits by assumption, Corollary A.17 implies that \(A^{\mathcal{O}}\) is semiperfect. The morphism \(K_0(A^{\mathcal{O}}) \to K_0(A^{\mathcal{O}}(m))\) induced by the quotient map \(A^{\mathcal{O}} \to A^{\mathcal{O}}(m)\) is thus an isomorphism by lifting of idempotents. Furthermore, by assumption the morphism \(d_A^{m, \mathcal{O}}: G_0(A(p)) \to G_0(A^{\mathcal{O}}(m))\) is an isomorphism and then the proof of Theorem A.12 shows that the canonical morphism \(e_A^{m, \mathcal{O}}: K_0(A(p)) \to K_0(A^{\mathcal{O}}(m))\) is also an isomorphism. We can thus define a morphism \(e_A^{\mathcal{O}}: K_0(A(p)) \to K_0(A^K)\) as the following composition

\[
\begin{array}{ccc}
K_0(A(p)) & \xrightarrow{\sim} & K_0(A^{\mathcal{O}}(m)) & \xrightarrow{\sim} & K_0(A^{\mathcal{O}}) & \xrightarrow{\sim} & K_0(A^K) \\
\downarrow e_A^{m, \mathcal{O}} & & & & & & \downarrow e_A^{\mathcal{O}}
\end{array}
\]

We will now show that \(e_A^{\mathcal{O}}\) is indeed an adjoint of \(d_A^{\mathcal{O}}\). The arguments in the proof of [11, 18.9] can, with some refinements, be transferred to our more general situation and this is what we will do. Let \(\mathcal{P}\) be a finitely generated projective \(A(p)\)-module and let \(V\) be a finitely generated \(A^K\)-module. Since \(K_0(A^{\mathcal{O}}) \simeq K_0(A^{\mathcal{O}}(m))\), there exists a finitely generated projective \(A^{\mathcal{O}}\)-module \(P\) such that \((e_A^{m, \mathcal{O}})^{-1}([P/mP]) = [\mathcal{P}]\) and then we have \(e_A^{\mathcal{O}}([\mathcal{P}]) = [P^K]\). Let \(\mathcal{V}\) be an \(A^{\mathcal{O}}\)-lattice in \(V\). Then by definition of \(d_A^{\mathcal{O}}\), see [42, Corollary 1.14], we have \(d_A^{\mathcal{O}}([V]) = (d_A^{m, \mathcal{O}})^{-1}([\mathcal{V}(m)])\). We denote by \(\mathcal{V}\) a representative of \(d_A^{\mathcal{O}}([V])\).

Since \(P\) is a finitely generated projective \(A^{\mathcal{O}}\)-module, we can write \(P \oplus Q = (A^{\mathcal{O}})^n\) for some finitely generated projective \(A^{\mathcal{O}}\)-module \(Q\) and some \(n \in \mathbb{N}\). Since \(\text{Hom}_{A^{\mathcal{O}}}\) is additive, we get

\[
\text{Hom}_{A^{\mathcal{O}}}(P, \mathcal{V}) \oplus \text{Hom}_{A^{\mathcal{O}}}(Q, \mathcal{V}) = \text{Hom}_{A^{\mathcal{O}}}(P \oplus Q, \mathcal{V}) = \text{Hom}_{A^{\mathcal{O}}}(A^{\mathcal{O}})^n, \mathcal{V})
\]

\[= (\text{Hom}_{A^{\mathcal{O}}}(A^{\mathcal{O}}), \mathcal{V}))^n \simeq \mathcal{V}^n.\]

This shows that \(\text{Hom}_{A^{\mathcal{O}}}(P, \mathcal{V})\) is a direct summand of \(\mathcal{V}^n\) and as \(\mathcal{V}^n\) is \(\mathcal{O}\)-free, we conclude that \(\text{Hom}_{A^{\mathcal{O}}}(P, \mathcal{V})\) is \(\mathcal{O}\)-projective and thus even \(\mathcal{O}\)-free since \(\mathcal{O}\) is a discrete valuation ring. Since \(P\) is a finitely generated projective \(A^{\mathcal{O}}\)-module, it follows from Lemma B.3 that there is a canonical \(K\)-vector space isomorphism

\[K \otimes_{\mathcal{O}} \text{Hom}_{A^{\mathcal{O}}}(P, \mathcal{V}) \simeq \text{Hom}_{A^{\mathcal{O}}}(P^K, \mathcal{V})\]

and a canonical \(k(m)\)-vector space isomorphism

\[k(m) \otimes_{\mathcal{O}} \text{Hom}_{A^{\mathcal{O}}}(P, \mathcal{V}) \simeq \text{Hom}_{A^{\mathcal{O}}(m)}(P/mP, \mathcal{V}/m\mathcal{V}).\]
Combining all results and the fact that both $e_{A}^{\rho,\sigma}$ and $d_{A}^{\rho,\sigma}$ preserve dimensions by construction, we can now conclude that
\[
\langle e_{A}^{\rho,\sigma}(P), [V] \rangle_{A^{\rho}} = \dim_K \text{Hom}_{A^{\rho}}(P^K, V) = \dim_K \text{Hom}_{A^{\rho}}(P, \tilde{V}) = \dim_K \text{Hom}_{A^{\rho}}(P/mP, \tilde{V}/m\tilde{V}) = \dim_K \text{Hom}_{A^{\rho}}(P, \tilde{V}) = \langle \tilde{P}, e_{A}^{\rho,\sigma}(\{V\}) \rangle_{A(p)}.
\]

Proving the commutativity of diagram (62) amounts to proving that $c_{A(p)}(\tilde{P}) = d_{A}^{\rho,\sigma} \circ e_{A}^{\rho,\sigma}(\tilde{P})$ for every finitely generated projective $A(p)$-module $\tilde{P}$. To prove this, note that the diagram
\[
\begin{array}{ccc}
K_0(A^{\rho}(m)) & \xrightarrow{c_{A^{\rho}(m)}} & G_0(A^{\rho}(m)) \\
\downarrow e_{A}^{p,m} & & \uparrow d_{A}^{p,m} \\
K_0(A(p)) & \xrightarrow{c_{A(p)}} & G_0(A(p))
\end{array}
\]

commutes. As above we know that there exists a finitely generated projective $A^{\rho}$-module $P$ such that $(e_{A}^{p,m})^{-1}(P/mP) = \tilde{P}$ and $e_{A}^{\rho,\sigma}(\tilde{P}) = P^K$. Since $P$ is a finitely generated projective $A^{\rho}$-module and $A$ is a finite $\mathcal{O}$-module, it follows that $P$ is also a finitely generated projective $\mathcal{O}$-module. As $\mathcal{O}$ is a discrete valuation ring, we conclude that $P$ is actually $\mathcal{O}$-free of finite rank. Hence, $P$ is an $\mathcal{O}$-lattice in $P^K$ and therefore
\[
d_{A}^{\rho,\sigma} \circ c_{A^{\rho}} \circ e_{A}^{\rho,\sigma}(\tilde{P}) = d_{A}^{\rho,\sigma}(P^K) = (d_{A}^{p,m})^{-1}(P/mP) = (d_{A}^{p,m})^{-1} \circ c_{A^{\rho}(m)}(P/mP) = c_{A(p)} \circ (e_{A}^{p,m})^{-1}(P/mP) = c_{A(p)}(\tilde{P}).
\]

If $R$ is normal, then the independence of $e_{A}^{p,\sigma}$ from the choice of $\sigma$ follows from the independence of $d_{A}^{p,\sigma}$ from the choice of $\sigma$ and the fact that $d_{A}^{p,\sigma}$ has at most one adjoint.

\section{Preservation of simple modules vs. preservation of blocks}

In [42] we studied the set
\[
\text{DecGen}(A) := \{ p \in \text{Spec}(R) \mid d_{A}^{p,\sigma} \text{ is trivial for any } A\text{-gate in } p \}.
\]

where $d_{A}^{p,\sigma}$ being \textit{trivial} means that it induces a bijection between simple modules. We have proven in [42, Theorem 2.3] that DecGen$(A)$ is open if $R$ is noetherian and $A$ has split fibers. Using Brauer reciprocity we thus deduce that in this case the locus of all $p$ such that $e_{A}^{p,\sigma}$ is trivial for any $\sigma$ is an open subset of Spec$(R)$.

If $p \in \text{DecGen}(A)$, then the simple modules of $A^K$ and $A(p)$ are “essentially the same”, in particular their dimensions are the same. This is why explicit knowledge about DecGen$(A)$ is quite helpful to understand the representation theory of the fibers of $A$, see [42]. So far, we do not have an explicit description of DecGen$(A)$, however. Brauer reciprocity enables us to prove the following relation between decomposition maps and blocks.

\textbf{Theorem 4.3. We have an inclusion}
\[
\text{DecGen}(A) \subseteq \text{BlGen}(A).
\]

\textbf{Proof.} Let $p \in \text{Spec}(R)$ be non-zero. By assumption there is a perfect $A$-gate $\sigma$ in $p$. If $p \in \text{DecGen}(A)$, then by definition $d_{A}^{p,\sigma}$ is trivial, so the matrix $D_{A}^{p,\sigma}$ of this morphism in bases given by isomorphism classes of simple modules of $A^K$ and $A(p)$, respectively, is equal to the identity matrix when ordering the bases appropriately. It now follows from Brauer reciprocity, Theorem 4.2, that $C_{A(p)} = C_{A^{\rho}}$ in appropriate bases, where $C_{A(p)}$ is
the matrix of the Cartan map $c_{A(p)}$ and $C_A^K$ is the matrix of the Cartan map $c_{A^K}$. Due to the
linkage relation explained in §1D, the families of $A^K$ and of $A(p)$ are determined by
the respective Cartan matrices. Since $C_{A(p)} = C_A^K$, it follows that $#\mathcal{B}(A(p)) = #\mathcal{B}(A^K)$, so $p \in BlGen(A)$. ■

**Remark 4.4.** Suppose that $A$ has split fibers and that $R$ is noetherian. Then the fact that
$#\mathcal{B}(A(p)) = #\mathcal{B}(Z(p))$ by Theorem 3.8 together with the Lemma A.11 yields the following equivalence:

$$p \in BlGen(A) \iff \dim_K(Z^K + \text{Rad}(A^K)) = \dim_{k(p)}(Z(p) + \text{Rad}(A(p))).$$

Let $\mathcal{O}$ be a perfect $\mathcal{A}$-gate in $\mathcal{P}$. This exists by Lemma 4.1(b). Suppose that $p \in \text{DecGen}(A)$. In [42, Theorem 2.2] we have proven that this implies that

$$\dim_K \text{Rad}(A^K) = \dim_{k(p)}(\text{Rad}(A(p))).$$

Let $X := Z + J$, where $J := \text{Rad}(A^K) \cap A^\mathcal{O}$. The arguments in [42] show that $X$ is an $A^\mathcal{O}$-lattice of $Z^K + \text{Rad}(A^K)$ and that the reduction in the maximal ideal $m$ of $\mathcal{O}$ is equal to $Z^\mathcal{O}(m) + \text{Rad}(A^\mathcal{O}(m))$. We thus have $\dim_K(Z^K + \text{Rad}(A^K)) = \dim_{k(m)}(Z^\mathcal{O}(m) + \text{Rad}(A^\mathcal{O}(m)))$.

Since $A(p)$ splits, the $k(m)$-dimension of $Z^\mathcal{O}(m) + \text{Rad}(A^\mathcal{O}(m))$ is equal to the $k(p)$-dimension of $Z(p) + \text{Rad}(A(p))$. Hence, we have $p \in BlGen(A)$ by (66). This yields another proof of the inclusion $\text{DecGen}(A) \subseteq BlGen(A)$ in case $A$ has split fibers.

**Example 4.5.** The following example due to C. Bonnafé shows that in the generality of
Theorem 4.3 we do not have equality in (65). Let $R$ be a discrete valuation ring with
fraction field $K$ and uniformizer $\pi$, i.e., $p := (\pi)$ is the maximal ideal of $R$. Denote by $k := R/p$ the residue field in $p$. Let

$$A := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Mat}_2(R) \mid b, c \in p \right\}. $$

This is an $R$-subalgebra of $\text{Mat}_2(R)$ and it is $R$-free with basis

$$e := E_{11}, f := E_{22}, x := \pi E_{12}, y := \pi E_{21},$$

where $E_{ij} = (\delta_{i,k} \delta_{j,l})_{kl}$ is the elementary matrix. Clearly, $A^K = \text{Mat}_2(K)$, so the generic fiber of $A$ is split semisimple. In particular, $A^K$ has just one block, and this block contains just one simple module we denote by $S$. Now, consider the specialization $\overline{A} := A(p) = A/pA$. We know from Corollary A.17 that the quotient map $A \to \overline{A}$, $a \to \overline{a}$, is block bijective, so we must have $#\mathcal{B}(A(p)) \leq #\mathcal{B}(A^K)$ and therefore $#\mathcal{B}(A(p)) = 1$, so $p \in BlGen(A)$. Let $\mathcal{J}$ be the $k$-subspace of $\overline{A}$ generated by $\overline{x}$ and $\overline{y}$. This is in fact a two-sided ideal of $\overline{A}$ since it is stable under multiplication by the generators (67). Moreover, we have $\overline{x}^2 = 0 = \overline{y}^2$, so $\mathcal{J}$ is a nilpotent ideal of $\overline{A}$. Hence, $\dim_k \text{Rad}(\overline{A}) \geq 2$. The number of simple modules of $\overline{A}$ is by [27, Theorem 7.17] equal to $\dim_{k(\overline{A})}(\text{Rad}(\overline{A}) + [\overline{A}, \overline{A}])$, so $#\text{Ir}r\overline{A} \leq 2$ since $\dim_k \overline{A} = \dim_K A^K = 4$. The two elements $\overline{x}$ and $\overline{y}$ are orthogonal idempotents and so the constituents of the two $\overline{A}$-modules $\overline{A} \overline{x}$ and $\overline{A} \overline{y}$ are not isomorphic. So, we have $#\text{Ir}r\overline{A} \geq 2$ and due to the aforementioned we conclude that $#\text{Ir}r\overline{A} = 2$. Let $\overline{S}_1$ and $\overline{S}_2$ be these two simple modules. Since $R$ is a discrete valuation ring, reduction modulo $p$ yields the well-defined decomposition map $d^A_p : G_0(A^K) \to G_0(A(p))$, see [42, Corollary 1.14]. It is an elementary fact that the all simple $\overline{A}$-modules must be constituents of $d^A_p((S)) = [S/pS]$. Since $\dim_K S = 2$, the only possibility is that $d^A_p((S)) = [\overline{S}_1] + [\overline{S}_2]$ and $\dim_k \overline{S}_1 = 1$. In particular, $p \notin \text{DecGen}(A)$, so $p \in BlGen(A) \setminus \text{DecGen}(A)$. Finally, we note that $\overline{A}$ also splits since $#\text{Ir}r(\overline{A}) = 2$ implies by the above formula that $\dim_k \text{Rad}(\overline{A}) = 2$ and we have $\dim_k \overline{A} = \dim_k \text{Rad}(\overline{A}) + \sum_{i=1}^2 (\dim_k \overline{S}_i)^2$, so $\overline{A}$ is split by [27, Corollary 7.8].
Lemma 4.6. Assume that the $A^K$-families are singletons, and that $\#\lrr A(p) \leq \#\lrr A^K$ for all $p \in \Spec(R)$. Then 
$$\BlGen(A) \setminus \DecGen(A) = \{ p \in \Spec(R) \mid D^p_A \text{ is diagonal but not the identity} \}.$$ 

Proof. Since the $A^K$-families are singletons, we have $\#\lrr(A^K) = \#\Bl(A^K)$. We clearly have $\#\lrr(A(p)) \leq \#\Bl(A(p))$ for all $p \in \Spec(R)$. Assume that $p \in \BlGen(A)$. Then we have $\#\lrr A(p) \geq \#\lrr A^K$, so $\#\lrr A(p) = \#\lrr A^K$ by our assumption. Hence, the decomposition matrix $D^p_A$ is quadratic. By Theorem 4.9 the $p$-families are equal to the Brauer $p$-families. Since $p \in \BlGen(A)$ and the $A^K$-families are singletons, it follows that $D^p_A$ is a diagonal matrix. The claim is now obvious. ■

Lemma 4.7. Let $R$ be a noetherian integral domain with fraction field $K$ and let $A$ be a cellular $R$-algebra of finite dimension such that $A^K$ is semisimple. Then $\DecGen(A) = \BlGen(A)$.

Proof. First of all, specializations of $A$ are again cellular by [20, 1.8]. Moreover, it follows from [20, Proposition 3.2] that $A$ has split fibers, so $A$ satisfies Lemma 4.1(b) and therefore our basic assumption in this paragraph. Let $\Lambda$ be the poset of the cellular structure of $A^K$. Since $A^K$ is semisimple, each cell module $M_\lambda$ has simple head $S_\lambda$ and $\#\lrr A^K = \#\Lambda$. Let $p \in \Spec(R)$. The poset for the cellular structure of $A(p)$ is again $\Lambda$. Denote by $M^p_\lambda$ the corresponding cell modules of $A(p)$. There is a subset $\Lambda'$ of $\Lambda$ such that $M^p_\lambda$ has simple head $S^p_\lambda$ for all $\lambda \in \Lambda'$ and that these heads are precisely the simple $(p)$-modules. In particular, we have $\#\lrr A(p) \leq \#\lrr A^K$. Now, assume that $p \in \BlGen(A)$. By Lemma 4.6 we just need to show that the decomposition matrix $D^p_A$, which is square by the proof of Lemma 4.6, cannot be a non-identity diagonal matrix. By [20, Proposition 3.6] we know that $[M_\lambda : S_\lambda] = 1$ and $[M^p_\lambda : S^p_\lambda] = 1$. By construction, it is clear that $d^p_\lambda([M_\lambda]) = [M^p_\lambda]$. Hence, if $d^p_\lambda([S_\lambda]) = n_\lambda[S^p_\lambda]$, we have $n_\lambda = [M^p_\lambda : S^p_\lambda] = 1$. Hence, $D^p_A$ is the identity matrix, so $p \in \BlGen(A)$. ■

§4C. The Brauer graph

Geck and Pfeiffer [15] have introduced the so-called Brauer $p$-graph of $A$ in our general context but assuming that $A^K$ is semisimple so that the $A^K$-families are singletons. For general $A$ this definition seems not to be the correct one. We introduce the following generalization of this concept.

Definition 4.8. Suppose that $R$ is normal so that we have unique decomposition maps. The Brauer $p$-graph of $A$ is the graph with vertices the simple $A^K$-modules and an edge between $S$ and $T$ if and only if in the $A^K$-family of $S$ there is some $S'$ and in the $A^K$-family of $T$ there is some $T'$ such that $d^p_A([S'])$ and $d^p_A([T'])$ have a common constituent. The connected components of this graph are called the Brauer $p$-families of $A$.

If the $A^K$-families are singletons, we have an edge between $S$ and $T$ if and only if $d^p_A([S])$ and $d^p_A([T])$ have a common constituent, so this indeed generalizes the Brauer $p$-graph from [15] for $A^K$ semisimple. Our final theorem shows that decomposition maps are compatible with $p$-families and $A(p)$-families, and relates the Brauer $p$-families to the $p$-families.

Theorem 4.9. Assume that $R$ is normal. The following holds:

(a) A finite-dimensional $A^K$-module $V$ belongs to a $p$-block of $A$ if and only if $d^p_A([V])$ belongs to a block of $A(p)$.

(b) Two finite-dimensional $A^K$-modules $V$ and $W$ lie in the same $p$-block if and only if $d^p_A([V])$ and $d^p_A([W])$ lie in the same block of $A(p)$.
(c) If $\mathcal{F} \in \text{Fam}_p(A)$ is a $p$-family, then
\[
d^p_A(\mathcal{F}) := \{T | T \text{ is a constituent of } d^p_A([S]) \text{ for some } S \in \mathcal{F}\}
\]
is a family of $A(p)$, and all families of $A(p)$ are obtained in this way.

(d) The Brauer $p$-families are equal to the $p$-families.

Proof.

(a) By assumption there is a perfect $A$-gate $\mathcal{O}$ in $p$. Let $m$ be the maximal ideal of $\mathcal{O}$. We have the following commutative diagram of canonical morphisms which are all idempotent stable:

\[
\begin{array}{ccc}
A_p & \longrightarrow & A^\mathcal{O} \\
\downarrow & & \downarrow \\
A(p) & \longrightarrow & A^\mathcal{O}(m)
\end{array}
\]

Since $R$ is assumed to be normal, it follows from Proposition 2.8 that $A_p \rightarrow A(p)$ is block bijective. By assumption the morphism $d^p_{A,m}: G_0(A(p)) \rightarrow G_0(A^\mathcal{O}(m))$ is an isomorphism and therefore $A(p) \rightarrow A^\mathcal{O}(m)$ is block bijective by Theorem A.12. Furthermore, by assumption the generic fiber $A^K$ is split and therefore $A^\mathcal{O} \rightarrow A^\mathcal{O}(m)$ is block bijective by Corollary A.17. Because of (4) it thus follows that $A_p \rightarrow A^\mathcal{O}$ is block bijective.

Now, let $V$ be a finite-dimensional $A^K$-module and let $\tilde{V}$ be an $A^\mathcal{O}$-lattice of $V$. Suppose that $V$ belongs to an $A_p$-block of $A^K$. Since $A_p \rightarrow A^\mathcal{O}$ is block bijective, the $A_p$-blocks of $A^K$ coincide with the $A^\mathcal{O}$-blocks of $A^K$ and therefore $V$ belongs to an $A^\mathcal{O}$-block of $A^K$. Since $\tilde{V}$ is $\mathcal{O}$-free, it follows from Lemma A.3 that $\tilde{V}$ belongs to a block of $A^\mathcal{O}$. Again by Lemma A.3 and the fact that $A^\mathcal{O} \rightarrow A^\mathcal{O}(m)$ is block bijective, it follows that $\tilde{V}/m\tilde{V}$ belongs to a block of $A^\mathcal{O}(m)$. Since $A(p) \rightarrow A^\mathcal{O}(m)$ is block bijective, Lemma A.3 shows that $d^p_A([V])$ belongs to a block of $A(p)$.

Conversely, suppose that $d^p_A([V])$ belongs to a block of $A(p)$. Then $\tilde{V}/m\tilde{V}$ belongs to a block of $A^\mathcal{O}(m)$ and therefore $\tilde{V}$ belongs to a block of $A^\mathcal{O}$ by Lemma A.3. But then $V$ belongs to an $A^\mathcal{O}$-block of $A^K$ and thus to an $A_p$-block of $A^K$ by Lemma A.3.

(b) This follows now from part (a).

(c) Fix a $p$-family $\mathcal{F}$ of $A^K$. If $S \in \mathcal{F}$, then $d^p_A([S])$ belongs to an $A(p)$-block by (a) and therefore all constituents of $d^p_A([S])$ belong to a fixed family $\mathcal{F}_S$. If $S' \in \mathcal{F}$ is another simple module, then by (b) the constituents of $d^p_A([S'])$ also lie in $\mathcal{F}_S$. Hence, $d^p_A(\mathcal{F})$ is contained in a fixed $A(p)$-family $\mathcal{F}_S$. Let $T \in \mathcal{F}$ be arbitrary. Due to the properties of decomposition maps there is some $S \in \text{irr} A^K$ such that $T$ is a constituent of $d^p_A([S])$. Since $T$ and $d^p_A([S])$ lie in the same $A(p)$-block by (a) and (b), we must have $S \in \mathcal{F}$ by (b). Hence, $\mathcal{F} = d^p_A(\mathcal{F})$ is an $A(p)$-family. Since every simple $A(p)$-module is a constituent of $d^p_A([S])$ for some simple $A^K$-module $S$, it is clear that any $A(p)$-family is of the form $d^p_A(\mathcal{F})$ for a $p$-family $\mathcal{F}$.

(d) Let $S$ and $T$ be simple $A^K$-modules contained in the same Brauer $p$-family, i.e., in the $A^K$-family of $S$ there is some $S'$ and in the $A^K$-family of $T$ there is some $T'$ such that $d^p_A([S'])$ and $d^p_A([T'])$ have a common constituent. It follows from part (b) that $S'$ and $T'$ lie in the same $p$-family of $A^K$. Since $S'$ is in the same $A^K$-family as $S$, it is also in the same $p$-family as $S$ because the $p$-families are unions of $A^K$-families. Similarly, $T'$ is in the same $p$-family as $T$. Hence, $S$ and $T$ lie in the same $p$-family.

Conversely, suppose that $S$ and $T$ lie in the same $p$-family. We have to show that they lie in the same Brauer $p$-family. Let $(S_i)_{i=1}^n$ be a system of representatives of the isomorphism classes of simple $A^K$-modules and let $(U_j)_{j=1}^m$ be a system of representatives of the isomorphism classes of simple $A(p)$-modules. Let $\mathcal{Q} := (Q_i)_{i=1}^n$ with $Q_i$ being the
projective cover of $S_i$, and let $\mathcal{P} := (P_j)^{m}_{j=1}$ with $P_j$ being the projective cover of $U_j$. Let $C_{A(p)}$ be the matrix of the Cartan map $C_{A(p)}$ with respect to the chosen bases, and similarly let $C_{A^K}$ be the matrix of $C_{A^K}$. Furthermore, let $D_A^p$ be the matrix of $d_A^p$ with respect to the chosen bases. Since $C_{A(p)} = D_A^p C_{A^K} (D_A^p)^T$ by Brauer reciprocity, Theorem 4.2, we have

$$ (C_{A(p)})_{p,q} = (D_A^p C_{A^K} (D_A^p)^T)_{p,q} = \sum_{k,l=1}^{n} (D_A^p)_{p,k} (C_{A^K})_{k,l} (D_A^p)^{l,q} $$

for all $p,q$. Let $U$ be a constituent of $d_A^p ([S])$ and let $V$ be a constituent of $d_A^p ([T])$. Since $S$ and $T$ lie in the same $p$-family of $A^K$, both $d_A^p ([S])$ and $d_A^p ([T])$ lie in the same block of $A(p)$ by (b), and therefore $U$ and $V$ lie in the same family of $A(p)$. As the families of $A(p)$ are equal to the $\mathcal{P}$-families of $A(p)$ by §1D, there exist functions $f : [1,r] \to [1,m]$, $g : [1,r-1] \to [1,m]$ with the following properties: $U_{f(1)} = U$, $U_{f(r)} = V$, and for any $j \in [1,r-1]$ both $P_{f(j)}$ and $P_{f(j+1)}$ have $U_{g(j)}$ as a constituent. We can visualize the situation as follows:

$$
\begin{array}{c}
P_{f(j)} \\
\downarrow \\
U_{f(j)} \quad U_{g(j)} \quad U_{f(j+1)}
\end{array}
$$

where an arrow $U \to P$ signifies that $U$ is a constituent of $P$. For any $j \in [1,r-1]$ we have $(C_{A(p)})_{g(j),f(j)} \neq 0$ and so it follows from (69) that there are indices $k(j)$ and $l(j)$ such that

$$(D_A^p)_{g(j),k(j)} \neq 0, \quad (C_{A^K})_{k(j),l(j)} \neq 0, \quad (D_A^p)_{f(j),l(j)} \neq 0.$$ 

Similarly, since $(C_{A(p)})_{g(j),f(j+1)} \neq 0$, there exist indices $k'(j)$ and $l'(j)$ such that

$$(D_A^p)_{g(j),k'(j)} \neq 0, \quad (C_{A^K})_{k'(j),l'(j)} \neq 0, \quad (D_A^p)_{f(j+1),l'(j)} \neq 0.$$ 

This can be visualized as follows:

$$
\begin{array}{c}
d_A^p ([S_{l(j)}]) \quad d_A^p ([S_{h(j)}]) \\
\downarrow \\
U_{f(j)} \quad U_{g(j)} \quad U_{f(j+1)}
\end{array}
$$

Here, the dashed edges in the upper row signify that the respective simple $A^K$-modules lie in the same $A^K$-family. Since $U_{f(1)} = U$ and $U_{f(r)} = V$, this shows that $S$ and $T$ lie in the same Brauer $p$-family of $A^K$.

§5. Questions and open problems

(a) Are all atomic gluing loci already maximal atomic gluing loci? A related question is: if a gluing locus $Z$ is properly contained in another locus $Z'$, is then $Z$ an irreducible component of $Z'$?

(b) Are the block structures in all vertices of the atomic block graph distinct?

(c) Are the intersections of atomic gluing loci always irreducible?

(d) What graph theoretic properties can be proven about the atomic block graph?

(e) Are there “nice” conditions on $A$ implying that the irreducible components of $\text{BlGen}(A)$ are normal or even smooth?

(f) How can we explicitly describe the complement $\text{BlGen}(A) \setminus \text{DecGen}(A)$? Which conditions ensure that it is empty?
§6. Notes

The behavior of blocks under specialization has been studied in several situations already. All of our results are well-known in modular representation theory of finite groups since the work of R. Brauer and C. Nesbitt [5]. Our Corollary 2.9 and Theorem 4.9 generalize results by S. Donkin and R. Tange [12] about algebras over Dedekind domains. Our results about lower semicontinuity of the number of blocks generalize a result by P. Gabriel [14] to mixed characteristic and non-algebraically closed settings, see also the corresponding result by I. Gordon [18]. In general, K. Brown and I. Gordon [7, 8] used Muller’s theorem [32] to study blocks under specialization. Theorem 3.8 has been treated in a more special setting by K. Brown and K. Goodearl [6]. The codimension one property in Corollary 2.10 and Theorem 3.9 were proven by C. Bonnafé and R. Rouquier [2] in a more special setting. Their work is without doubt one of the main motivations for this paper. Blocks and decomposition matrices of generically semisimple algebras over discrete valuation rings have been studied by M. Geck and G. Pfeiffer [15], and more generally by M. Chlouveraki [9]. Brauer reciprocity has been studied more generally by M. Geck and R. Rouquier [16], and by M. Neunhöffer [35]. M. Neunhöffer and S. Scherotzke [36] have shown generic triviality of $e^p_A$ over Dedekind domains.

§A. Base change of blocks

In this appendix we collect several facts about base change of blocks. Some results here should also be of independent interest.

§AA. Existence of block decompositions

**Lemma A.1.** Let $\phi : R \to S$ be a morphism of commutative rings and let $A$ be an $R$-algebra. Suppose that $\phi_A : A \to A^S$ is central idempotent stable. If $A^S$ has a block decomposition, also $A$ has a block decomposition.

**Proof.** If $A$ does not contain any non-trivial central idempotent, then $A$ is indecomposable and thus has a block decomposition. So, assume that $A$ is not indecomposable and let $c$ be a non-trivial central idempotent. Then $A = Ac \oplus Ac^\perp$. We can now continue this process to get finer and finer decompositions of $A$ as a ring. Since $\phi_A$ is central idempotent stable, we get decompositions of the same size of $A^S$. As $A^S$ has a block decomposition, this process has to end after finitely many steps. We thus arrive at a ring decomposition of $A$ with finitely many and indecomposable factors, hence, at a block decomposition of $A$. ■

**Corollary A.2.** A non-zero finite flat algebra over an integral domain has a block decomposition.

**Proof.** Let $R$ be an integral domain with fraction field $K$, let $\phi : R \to K$ be the embedding, and let $A$ be a finite flat $R$-algebra. Since $A$ is $R$-torsion-free, it follows from Lemma 1.1(c) that $\phi_A$ is injective and so $\phi_A$ is idempotent stable by Lemma 1.3(a). Since $\phi_A^* A = A^K$ is a finite-dimensional algebra over a field, it has a block decomposition. Hence, $A$ has a block decomposition by Lemma A.1. ■

The important point of the corollary above is that we do not have to assume $R$ to be noetherian—otherwise $A$ is noetherian and we already know it has a block decomposition.

§AB. Block compatibility of scalar extension of modules

Recall the decomposition of the module category of a ring $A$ relative to a decomposition in $\mathcal{C}(A)$ described in §1B. We have the following compatibility.
Lemma A.3. Let $\phi : R \rightarrow S$ be a morphism of commutative rings and let $A$ be an $R$-algebra. Suppose that $\phi_A$ is central idempotent stable and let $V$ be a non-zero $A$-module. In any of the following cases the $A$-module $V$ belongs to the block $c_i$ if and only if the $A^S$-module $V^S$ belongs to the $\phi$-block $\phi_A(c_i)$:

(a) $\phi$ is injective and $V$ is $R$-projective.
(b) $\phi$ is faithfully flat.
(c) $R$ is local or a principal ideal domain and $V$ is $R$-free.

Proof. As $c_j V$ is a direct summand of $V$, it follows that we have a canonical isomorphism $\text{ext}_V^S(c_j V) \simeq \phi_A^*(c_j V)$. By definition, it is clear that $\text{ext}_V^S(c_j V) = \phi_A(c_j)\phi_A^* V$ and so we have a canonical isomorphism $\phi_A^*(c_j V) \simeq \phi_A(c_j)\phi_A^* V$ of $A^S$-modules for all $j$. The claim thus holds if we can show that no non-zero direct summand $V'$ of $V$ is killed by $\phi_A^*$, i.e., $\phi_A^* V' \neq 0$. But this is implied by the assumptions in each case. Namely, in the first two cases it follows from Lemma 1.1 that $\phi_V$ is injective, which implies that $\phi_V'$ is also injective, so $\phi_A^* V'$ cannot be zero for non-zero $V'$. In the third case neither $\phi$ nor $\phi_V$ need to be injective, so this needs extra care. First of all, since $V$ is $R$-free, the assumptions on $R$ imply that a direct summand $V'$ of $V$, which a priori is only $R$-projective, is already $R$-free, too. In case $R$ is local, this follows from Kaplansky’s theorem [26] and in case $R$ is a principal ideal domain, this is a standard fact. Now, if $V$ is $R$-free with basis $(v_1)_{i \in \mathbb{N}}$, then it is a standard fact (see [4, II, §5.1, Proposition 4]) that $\phi_A^* V$ is $S$-free with basis $(\phi_V(v_1))_{i \in \mathbb{N}}$. This shows that $\phi_A^* V \neq 0$ for any non-zero $R$-free $A$-module $V$. This applied to direct summands of $V$, which are $R$-free as shown, proves the claim.

§AC. Field extensions

Throughout this paragraph let $A$ be a finite-dimensional algebra over a field $K$. From (4) we know that $\#Bl(A) \leq \#Bl(A^L)$ for any extension field $L$ of $K$.

Definition A.4. We say that $A$ is block-split if $\#Bl(A) = \#Bl(A^L)$ for any extension field $L$ of $K$.

Our aim is to show the following lemma.

Lemma A.5. If $Z(A)$ is a split $K$-algebra (e.g., if $A$ itself splits), then $A$ is block-split. The converse holds if $K$ is perfect.

The first assertion of the lemma is essentially obvious since $Z(A)$ is semiperfect and therefore

$$
\#Bl(A) = \#Bl(Z(A)) = rk_{\mathbb{Z}} K_0(Z(A)) = \#rk_{\mathbb{Z}} G_0(Z(A)) = \#1rr Z(A),
$$

where the second equality follows from the fact that idempotents in a commutative ring are isomorphic if and only if they are equal, see [27, Ex. 22.2]. The same equalities of course also hold for $Z(A)^L = Z(A^L)$, where $L$ is an extension field of $K$. Hence, if $Z(A)$ is split, then $A$ is block-split. If $A$ itself is split, it is a standard fact that its center splits, so $A$ is block-split.

We will prove the converse (assuming that $K$ is perfect) from a more general point of view as the results might be of independent interest and we re-use some of them in the last section. First of all, the field extension $K \subseteq L$ induces natural group morphisms

$$
d_A^L : G_0(A) \rightarrow G_0(A^L) \quad \text{and} \quad e_A^L : K_0(A) \rightarrow K_0(A^L).
$$

Without any assumptions on the field $K$ we have the following property.

Lemma A.6. The morphisms $d_A^L$ and $e_A^L$ are injective.
Proof. Let \((S_i)_{i \in I}\) be a system of representatives of the isomorphism classes of simple \(A\)-modules. For each \(i\) let \((T_{ij})_{j \in J_i}\) be a system of representatives of the isomorphism classes of simple \(A^L\)-modules which occur as constituents of \(S_i^L\). Then by [27, Proposition 7.13] the set \((T_{ij})_{i \in I, j \in J_i}\) is a system of representatives of the isomorphism classes of simple \(A^L\)-modules. Hence, the matrix \(D^L_A\) of \(d^L_A\) in bases given by the isomorphism classes of simple modules is in column-echelon form, has no zero columns, and no zero rows. In particular, \(d^L_A\) is injective.

For each \(i \in I\) let \(P_i\) be the projective cover of \(S_i\) and for each \(j \in J_i\) let \(Q_{ij}\) be the projective cover of \(T_{ij}\). By the above, \((Q_{ij})_{i \in I, j \in J_i}\) is a system of representatives of the isomorphism classes of projective indecomposable \(A^L\)-modules. We claim that in the direct sum decomposition of the finitely generated projective \(A^L\)-module \(P_i^L\) into projective indecomposable \(A^L\)-modules only the \(Q_{ij}\) with \(j \in J_i\) occur. With the same argument as above, this implies that \(e^{L}_{\lambda}\) is injective. So, let us write \(P_i^L = \bigoplus_{\lambda \in \Lambda} U_{\lambda}\) for (not necessarily non-isomorphic) projective indecomposable \(A^L\)-modules \(U_{\lambda}\). The \(U_{\lambda}\) are the up to isomorphism unique projective indecomposable \(A^L\)-modules occurring as direct summands of \(P_i^L\). As the radical is additive by [27, Proposition 24.6(ii)], we have

\[
S_i^L = (P_i / \text{Rad}(P_i))^L = P_i^L / \text{Rad}(P_i)^L = \bigoplus_{\lambda \in \Lambda} U_{\lambda} / (\text{Rad}(P_i)^L \cap U_{\lambda}).
\]

Moreover, we have \(\text{Rad}(P_i)^L \subseteq \text{Rad}(P_i^L)\). This follows from the fact that \(\text{Rad}(A)^L \subseteq \text{Rad}(A^L)\) by [27, Theorem 5.14] and the fact that \(\text{Rad}(P_i) = \text{Rad}(A)P_i\) and \(\text{Rad}(P_i^L) = \text{Rad}(A^L)P_i^L\) by [27, Theorem 24.7] since \(P_i\) and \(P_i^L\) are projective. For each \(\lambda \in \Lambda\) the radical of \(U_{\lambda}\) is a proper submodule of \(U_{\lambda}\) and therefore

\[
\text{Rad}(P_i)^L \cap U_{\lambda} \subseteq \text{Rad}(P_i^L) \cap U_{\lambda} = \text{Rad}(U_{\lambda}) \subseteq U_{\lambda}.
\]

Hence, the head of \(U_{\lambda}\) is a constituent of \(U_{\lambda} / (\text{Rad}(P_i^L) \cap U_{\lambda})\), and since all constituents of the latter are constituents of \(S_i^L\), we must have \(Hd(U_{\lambda}) = S_{ij}\), for some \(j \in J_i\) by the above. This implies that \(U_{\lambda} = Q_{ij}\), thus proving the claim.

**Lemma A.7.** The following holds:

(a) The morphism \(d_i^L\) is an isomorphism if and only if it induces a bijection between isomorphism classes of simple modules. Similarly, the morphism \(e^{L}_{\lambda}\) is an isomorphism if and only if it induces a bijection between isomorphism classes of projective indecomposable modules.

(b) If \(d_i^L\) is an isomorphism, so is \(e^{L}_{\lambda}\). The converse holds if \(K\) is perfect.

For the proof of Lemma A.7 we will need the following well-known elementary lemma that is also used in the last section. Recall from (59) the intertwining form \(\langle \cdot, \cdot \rangle_A\) of \(A\).

**Lemma A.8.** Let \(P\) be a projective indecomposable \(A\)-module and let \(V\) be a finitely generated \(A\)-module. Then

\[
\langle [P], [V]\rangle_A = [V : \text{Hd}(P)] \cdot \dim_K \text{End}_A(\text{Hd}(P)),
\]

where \(\text{Hd}(P) = P / \text{Rad}(P)\) is the head of \(P\). In particular, \(\langle \cdot, \cdot \rangle_A\) is non-degenerate.

**Proof.** We first consider the case \(V = \text{Hd}(P)\). Let \(f \in \text{Hom}_A(P, \text{Hd}(P))\) be non-zero. Since \(\text{Hd}(P)\) is simple, this morphism is already surjective and thus induces an isomorphism \(P / \text{Ker}(f) \cong \text{Hd}(P)\). But as \(\text{Rad}(P)\) is the unique maximal submodule of \(P\), we must have \(\text{Ker}(f) = \text{Rad}(P)\) and thus get an induced morphism \(\text{Hd}(P) \to \text{Hd}(P)\). This yields a \(K\)-linear morphism \(\Phi : \text{Hom}_A(P, \text{Hd}(P)) \to \text{End}_A(\text{Hd}(P))\). On the other hand, if \(f \in \text{End}_A(\text{Hd}(P))\), then composing it with the quotient morphism \(P \to P / \text{Rad}(P) = \text{Hd}(P)\) yields a morphism \(P \to \text{Hd}(P)\). In this way we also get a \(K\)-linear morphism
\[ \Psi : \text{End}_A(\text{Hd}(P)) \to \text{Hom}_A(P, \text{Hd}(P)). \] By construction, \( \Phi \) and \( \Psi \) are pairwise inverse, hence \( \langle P, [\text{Hd}(P)] \rangle_A = \text{dim}_K(\text{Hom}_A(P, \text{Hd}(P))) = \text{dim}_K \text{End}_A(\text{Hd}(P)) \) as claimed.

Now, suppose that \( V \) is a simple \( A \)-module not isomorphic to \( \text{Hd}(P) \). We can write \( P = Ae \) for some primitive idempotent \( e \in A \). Since \( A \) is artinian, \( e \) is already local and now it follows from [27, 21.19] that \( \text{Hom}_A(Ae, V) \) is non-zero if and only if \( V \) has a constituent isomorphic to \( \text{Hd}(Ae) \). This is not true by assumption, and therefore \( \text{Hom}_A(P, V) = 0 \), so \( \langle P, [V] \rangle_A = 0 \).

Finally, for \( V \) general we have \( \langle V \rangle = \sum_{S \in \text{Irr}_A} \langle V : S \rangle \langle S \rangle \) in \( G_0(A) \). By the above we get
\[
\langle P, [V] \rangle_A = \sum_{S \in \text{Irr}_A} \langle V : S \rangle \langle [P], [S] \rangle_A = \langle V : \text{Hd}(P) \rangle \langle [P], [\text{Hd}(P)] \rangle_A
\]
as claimed. It follows that the Gram matrix \( \langle \cdot, \cdot \rangle \) with respect to the basis \( (P(S))_{S \in \text{Irr}_A} \) of \( G_0(A) \) and the basis \( (S)_{S \in \text{Irr}_A} \) of \( G_0(A) \) is diagonal with positive diagonal entries. The determinant of \( \langle \cdot, \cdot \rangle \) is thus a non-zero divisor on \( Z \) and since both \( K \) and \( G_0(A) \) are \( Z \)-free of the same finite dimension, it follows that \( \langle \cdot, \cdot \rangle \) is non-degenerate, see [40, Satz 70.5].

**Proof of Lemma A.7.** We use the same notations as in the proof of Lemma A.6. Since \( A \mathbb{A} \) is a projective \( A \)-module, there is a decomposition \( A \mathbb{A} = \bigoplus_{i \in I} P_i^L \) for some \( r_i \in \mathbb{N} \). Using Lemma A.8 we see that
\[
\text{dim}_K \text{Hd}(P_j) = \langle [A \mathbb{A}], [\text{Hd}(P_j)] \rangle_A = \sum_{i \in I} r_i \langle [P_i], [\text{Hd}(P_j)] \rangle_A = r_j \langle [P_j], [\text{Hd}(P_j)] \rangle_A
\]
Hence, \( r_i = \frac{n_i}{m_i} \), where \( n_i := \text{dim}_K S_i \) and \( m_i := \text{dim}_K \text{End}_A(S_i) \). In particular,
\[
(74) \quad \text{dim}_K A = \sum_{i \in I} \frac{n_i}{m_i} \text{dim}_K P_i.
\]

Now, suppose that \( \text{d}_A^L \) is an isomorphism. Then clearly \# \text{Irr} \( A \mathbb{A} = \# \text{Irr} \mathbb{A}^L \). The properties of the matrix \( \text{D}_A^L \) of the morphism \( \text{d}_A^L \) derived in the proof of Lemma A.6 immediately imply that \( \text{D}_A^L \) is diagonal. Since it is invertible with natural numbers on the diagonal, it must already be the identity matrix, i.e., \( \text{d}_A^L \) induces a bijection between the isomorphism classes of simple modules. In particular, \( (S_i^L)_{i \in I} \) is a system of representatives of the isomorphism classes of simple \( \mathbb{A}^L \)-modules. The properties of the matrix \( \text{E}_A^L \) of \( \text{e}_A^L \) derived in the proof of Lemma A.6 now imply that we must have \( P_i^L \cong Q_i^{n_i} \) for some \( s_i \in \mathbb{N} \). We argue that \( s_i = 1 \). This shows that \( \text{e}_A^L \) is an isomorphism inducing a bijection between the isomorphism classes of projective indecomposable modules. In the same way we deduced equation (74) we now get
\[
(75) \quad \text{dim}_K A = \text{dim}_L A^L = \sum_{i \in I} \frac{n_i}{m_i} \text{dim}_L Q_i
\]
with
\[
n_i' = \text{dim}_L \text{Hd}(Q_i) = \text{dim}_L S_i^L = \text{dim}_K S_i = n_i
\]
and
\[
(76) \quad m_i' = \text{dim}_L \text{End}_A(\text{Hd}(Q_i)) = \text{dim}_L \text{End}_A(S_i^L) = \text{dim}_K \text{End}_K(S_i) = m_i,
\]
using the fact that \( L \otimes_K \text{End}_A(S_i) \cong \text{End}_A(L(S_i^L)) \), see Lemma [38, Theorem 2.38]. Since \( \text{dim}_L Q_i \leq \text{dim}_L P_i^L = \text{dim}_K P_i \), equations (74) and (75) imply that \( \text{dim}_L Q_i = \text{dim}_K P_i \), so \( Q_i = P_i^L \).

Conversely, suppose that \( \text{e}_A^L \) is an isomorphism. With the properties of the matrix \( \text{E}_A^L \) of \( \text{e}_A^L \) established in the proof of Lemma A.6 we see similarly as above that \( \text{e}_A^L \) already induces a bijection between the projective indecomposable modules. In particular,
$P_i^L \cong Q_i$. Due to the properties of the matrix $D_A^L$ of $d_A^L$ established in the proof of Lemma A.6 the only constituent of $S_i^L$ is $T_i$. Since $P_i$ is the projective cover of $P_i$, we have a surjective morphism $\phi : P_i \rightarrow S_i$ with $\text{Ker}(\phi) = \text{Rad}(P_i)$. Scalar extension induces a surjective morphism $\phi^L : P_i^L \rightarrow S_i^L$ with $\text{Ker}(\phi^L) = \text{Rad}(P_i)^L \subseteq \text{Rad}(P_i^L)$. It thus follows from [11, Corollary 6.25(i)] that $P_i^L$ is the projective cover of $S_i^L$. Now, we assume that $K$ is perfect. Then by [11, Theorem 7.5] all simple $A$-modules are separable, so $S_i^L = T_i^L$ for some $s_i$. Since projective covers are additive, we get $P_i^L = Q_i^L$. As $P_i^L = Q_i$, this implies that $s_i = 1$, so $S_i^L = T_i$ is simple. Hence, $d_A^L$ induces a bijection between the isomorphism classes of simple modules.

**Remark A.9.** With the same arguments as in the proof of Lemma A.7 we can show that the converse in Lemma A.7(b) still holds when we only assume that all simple $A$-modules are separable, i.e., they remain semisimple under field extension. This holds for example when $A$ splits or if $A$ is a group algebra (over any field). We do not know whether it holds more generally.

**Proof of Lemma A.5.** Let $Z := Z(A)$. Suppose that $L$ is an extension field of $K$ with $\# \text{Bl}(A) = \# \text{Bl}(A^L)$. By (70) we know that $\# \text{tr} Z = \# \text{tr} Z^L$. The arguments in the proof of Lemma A.6 thus imply that the matrix $D_A^L$ of the morphism $d_A^L : G_0(Z) \rightarrow G_0(Z^L)$ must be a diagonal matrix. We claim that it is the identity matrix. Since this holds for any $L$, it means that the simple modules of $Z$ remain simple under any field extension, so $Z$ splits. Our assumption implies that $\# \text{Idem}_p(Z) = \# \text{Idem}_p(Z^L)$, so every primitive idempotent $e \in Z$ remains primitive in $Z^L$. This shows that $e_A^L : K_0(Z) \rightarrow K_0(Z^L)$ induces a bijection between projective indecomposable modules. In particular, it is an isomorphism. Now, Lemma A.7 shows that also $d_A^L$ is an isomorphism. Since its matrix $D_A^L$ is invertible with natural numbers on the diagonal, it must be the identity.

**Remark A.10.** In the proof of Lemma A.5 we have deduced that for a commutative finite-dimensional $K$-algebra $Z$ the condition $rk_{Z^L} K_0(Z) = rk_{Z^L} K_0(Z^L)$ already implies that $e_A^L$ induces a bijection between projective indecomposable modules. This follows from the fact that idempotents in a commutative ring are isomorphic if and only if they are equal. This is not true for a non-commutative ring $A$. Here, we can have $rk_{Z^L} K_0(A) = rk_{Z^L} K_0(A^L)$ but still a primitive idempotent $e \in A$ can split into a sum of isomorphic orthogonal primitive idempotents of $A^L$. Then the matrix $E_A^L$ of $e_A^L$ is diagonal but not the identity.

Let us record the following additional fact:

**Lemma A.11.** If $Z(A)$ splits, then

$\# \text{Bl}(A) = \dim_K Z(A) - \dim_K \text{Rad}(Z(A)) = \dim_K Z(A) - \dim_K (Z(A) \cap \text{Rad}(Z(A)))$.

**Proof.** This follows immediately from (70) and the fact that that $\text{Rad}(Z(A)) = Z(A) \cap \text{Rad}(A)$ since $Z(A) \subseteq A$ is a finite normalizing extension, see [29, Theorem 1.5].

§AD. Faithfully flat extensions

We will need the following general result.

**Theorem A.12.** Let $\phi : R \rightarrow S$ be a faithfully flat morphism of integral domains and let $A$ be a finite flat $R$-algebra. Let $K$ and $L$ be the fraction field of $R$ and $S$, respectively. If $\# \text{Bl}(A^L) = \# \text{Bl}(A^S)$, then the morphism $\phi_A : A \rightarrow A^S$ is block bijective.

**Proof.** Recall from Corollary A.2 that both $A$ and $A^S$ have block decompositions. The map $\phi_A : A \rightarrow A^S$ is injective by Lemma 1.1(b) since $\phi$ is faithfully flat. Hence, $\phi_A$ is idempotent stable by Lemma 1.3(a) and therefore $\# \text{Bl}(A) \leq \# \text{Bl}(A^S)$ by (4). We thus have to show that $\# \text{Bl}(A) \geq \# \text{Bl}(A^S)$. We split the proof of this fact into several steps.
The case $R = K$ and $S = L$ holds by assumption. Now, assume that still $R = K$ but that $S$ is general as in the theorem. Since $A$ is $R$-flat, the extension $A^S$ is $S$-flat and thus $S$-torsionfree. Hence, the map $A^S \to A^L$ is injective by Lemma 1.1(c). In particular, it is idempotent stable by Lemma 1.3(a) and so $\#\text{Bl}(A^S) \leq \#\text{Bl}(A^L)$ by (4). In total, we have $\#\text{Bl}(A) \leq \#\text{Bl}(A^S) \leq \#\text{Bl}(A^L) = \#\text{Bl}(A^K) = \#\text{Bl}(A)$. Hence, $\#\text{Bl}(A) = \#\text{Bl}(A^S)$.

Finally, let both $R$ and $S$ be general as in the theorem. Let $\Sigma := R \setminus \{0\}$ and $\Omega := S \setminus \{0\}$. Then $K = \Sigma^{-1}R$ and $L = \Omega^{-1}S$. Set $T := \Sigma^{-1}S$. Since $R$ and $S$ are integral domains, we can naturally view all rings as subrings of $L$ and so we get the two commutative diagrams

\begin{equation}
\begin{array}{c}
L \\
\uparrow
\\
K \\
\leftarrow \\
T
\end{array}
\quad
\begin{array}{c}
A^L \\
A^T \\
A^S
\end{array}
\end{equation}

the right one being induced by the left one. All morphisms in the left diagram are clearly injective. We claim the same holds for the right diagram. We have noted at the beginning that the map $A \to A^S$ is injective. Since $A$ is $R$-flat, it is $R$-torsionfree and so the map $A \to A^K$ is injective by Lemma 1.1(c). We have argued above already that the map $A^S \to A^L$ is injective. Since $S \to T$ is a localization map, the induced scalar extension functor is exact so that $A^T$ is a flat $T$-module. In particular, $A^T$ is $T$-torsionfree and so $A^T \to A^L$ is injective by 1.1(c). The map $A^K \to A^L$ is injective by Lemma 1.1(a). Due to the commutativity of the diagram, the remaining maps must be injective, too. We can thus view all scalar extensions of $A$ naturally as subsets of $A^L$. We claim that

\begin{equation}
A = A^K \cap A^S
\end{equation}

as subsets of $A^L$. Because of the commutative diagram above, this intersection already takes place in $A^T$. Consider $A^K$ as an $R$-module now. We have a natural identification $\phi^*(A^K) = S \otimes_R A^K = S \otimes_R (\Sigma^{-1}A) = (\Sigma^{-1}S) \otimes_R A = T \otimes_R A = A^T$ as $S$-modules by [3, II, §2.7, Proposition 18]. Note that the map $A^K \to A^T$ in the diagram above is the map $\phi_{A^K}$, when considering $A^K$ as an $R$-module. The $R$-submodule $A$ of $A^K$ is now identified with $\phi_{A^K}(A)$ and $\text{ext}^n_{A^K}(A)$ is the $S$-submodule of $A^T$ generated by $A \subseteq A^T$, which is precisely $A^S$. Since $\phi$ is faithfully flat, it follows from [3, I, §3.5, Proposition 10(ii)] applied to the $R$-module $A^K$ and the submodule $A$ that

\begin{equation}
A = \phi_{A^K}(A) = \phi_{A^K}(A^K) \cap \text{ext}^n_{A^K}(A) = A^K \cap A^S
\end{equation}

inside $A^T$. Let $(c_i)_{i \in I} \in \mathcal{O}^{\text{cp}}(A^K)$ and let $(d_j)_{j \in J} \in \mathcal{O}^{\text{cp}}(A^K)$. By assumption the morphism $A^K \to A^L$ is block bijective, which means that $(d_j)_{j \in J} \in \mathcal{O}^{\text{cp}}(A^L)$. Since $A^S \to A^L$ is idempotent stable, there exists by the arguments preceding (4) a partition $(J_i)_{i \in I}$ of $J$ such that the non-zero central idempotent $c_i$ can in $A^L$ be written as $c_i = \sum_{j \in J_i} d_j$. But this shows that $c_i \in A^K \cap A^S$, hence $c_i \in A$ and so $(c_i)_{i \in I} \in \mathcal{O}^{\text{cp}}(A)$ by (78). Hence, $\#\text{Bl}(A) = \#\text{Bl}(A^S)$.

§AE. Reductions

Now, we consider a situation which in a sense is opposite to the one considered in the last paragraph, namely we consider the quotient morphism $\phi : R \to R/m =: S$ for a local commutative ring $R$ with maximal ideal $m$ and a finitely generated $R$-algebra $A$. By Lemma 1.3(b) the morphism $\phi_A : A \to A^S = A/mA =: \overline{A}$ is idempotent stable. The question whether $\phi_A$ is idempotent surjective is precisely the question whether idempotents of
A can be lifted to $A$, and this is a classical topic in ring theory. The following lemma is standard, we omit the proof.

**Lemma A.13.** If $\phi_A : A \rightarrow \overline{A}$ is idempotent surjective, it is primitive idempotent bijective and block bijective.

The next theorem was proven by M. Neunhöffer [35, Proposition 5.10].

**Theorem A.14 (M. Neunhöffer).** The morphism $\phi_A : A \rightarrow \overline{A}$ is idempotent surjective if and only if $A$ is semiperfect.

We recall two standard situations of idempotent surjective reductions.

**Lemma A.15.** In the following two cases the morphism $\phi_A : A \rightarrow \overline{A}$ is idempotent surjective:

(a) $R$ is noetherian and $m$-adically complete.

(b) $R$ is henselian.

**Proof.** For a proof of the first case, see [27, Proposition 21.34]. For a proof of the second case assuming that $A$ is commutative, see [37, I, §3, Proposition 2]. To give a proof for non-commutative $A$ let $\overline{e} \in \overline{A}$ be an idempotent. Let $k := R/m$ and let $\overline{B} := k[\overline{e}]$ be the $k$-subalgebra of $\overline{A}$ generated by $\overline{e}$. Since $A$ is a finite-dimensional $k$-algebra, also $\overline{B}$ is finite-dimensional. Moreover, $\overline{B}$ is commutative. Let $e \in A$ be an arbitrary element with $\phi_A(e) = \overline{e}$. Let $B := R[e]$, a commutative subalgebra of $A$. Note that $\overline{B} = B/mB$. Since $A$ is a finitely generated $R$-module, the Cayley–Hamilton theorem implies that $B$ is a finitely generated $R$-algebra. Now, by the commutative case, the map $\phi_B : B \rightarrow \overline{B}$ is idempotent surjective and so there is an idempotent $e' \in B \subseteq A$ with $\phi_A(e') = \phi_B(e') = \overline{e}$. This shows that $\phi_A$ is idempotent surjective.

The next theorem was again proven by M. Neunhöffer [35, Proposition 6.2]. It is one of our key ingredients in proving Brauer reciprocity for decomposition maps in a general setting.

**Theorem A.16 (M. Neunhöffer).** Suppose that $R$ is a valuation ring with fraction field $K$ and that $A$ is a finite flat $R$-algebra with split generic fiber $A^K$. If $\hat{R} \otimes_R A$ is semiperfect, where $\hat{R}$ is the completion of $R$ with respect to the topology defined by a valuation on $K$ defining $R$, then also $A$ is semiperfect.

**Corollary A.17 (J. Müller, M. Neunhöffer).** Suppose that $R$ is a discrete valuation ring and that $A$ is a finite flat $R$-algebra with split generic fiber. Then $A$ is semiperfect. In particular, $\phi_A : A \rightarrow \overline{A}$ is primitive idempotent bijective and block bijective.

**Proof.** Since $R$ is a discrete valuation ring, its valuation topology coincides with its $m$-adic topology so that the topological completion $\hat{R}$ is $\hat{m}$-adically complete, where $\hat{m}$ denotes the maximal ideal of $\hat{R}$ and $\hat{m}$ denotes the maximal ideal of $\hat{R}$. Hence, $\hat{R} \otimes_R A$ is semiperfect by Lemma A.15(a) and Theorem A.14. Now, Theorem A.16 shows that $A$ is also semiperfect.

**Remark A.18.** One part of Corollary A.17, the fact that idempotents lift, was also stated earlier by C. Curtis and I. Reiner [11, Exercise 6.16] in an exercise in the special case where $A^K$ is assumed to be semisimple. The semisimplicity assumption was later removed by J. Müller in his PhD thesis [34, Satz 3.4.1] using the Wedderburn–Malcev theorem (this can be applied without perfectness assumption on the base field if $A^K$ splits since then $A^K/\text{Rad}(A^K)$ is separable, see [10, Theorem 72.19]).
§B. Three further elementary facts

Lemma B.1. A finitely generated module $M$ over an integral domain $R$ is flat if and only if it is faithfully flat. In particular, if $M \neq 0$, we have $0 \neq k(p) \otimes_R M = M(p)$ for all $p \in \text{Spec}(R)$.

Proof. We can assume that $M \neq 0$. Since $M$ is flat, it is torsion-free and so the localization map $M \to M_p$ is injective, see Lemma 1.1(c). Hence, $M_p \neq 0$. Since $M$ is a finitely generated $R$-module, also $M_p$ is a finitely generated $R_p$-module and now Nakayama’s lemma implies that $0 \neq M_p/p_pM_p = k(p) \otimes_R M$. Hence, $M$ is faithfully flat by [30, Theorem 7.2].

Lemma B.2. Let $A$ be a finite flat algebra over an integral domain $R$. Then the structure map $R \to A$, $r \mapsto r \cdot 1_A$, is injective. Hence, we can identify $R \subseteq Z(A)$. If $R$ is noetherian, the induced map $\text{Spec}(Z(A)) \to \text{Spec}(R)$ is finite, closed, and surjective.

Proof. It follows from Lemma B.1 that $A$ is already faithfully flat. Let $\phi : R \to A$ be the structure map. This is an $R$-module map and applying $- \otimes_R A$ yields a map

$$A \simeq R \otimes_R A \xrightarrow{\phi \otimes A} A \otimes_R A$$

of right $A$-modules, mapping $a$ to $1 \otimes a$. This map has an obvious section mapping $a \otimes a'$ to $aa'$, hence it is injective. Since $A$ is faithfully flat, the original map $\phi$ has to be injective, too. As the image of $\phi$ is contained in the center $Z$ of $A$, the structure map is actually an injective map $R \to Z$. Now, assume that $R$ is noetherian. Since $A$ is a finitely generated $R$-module, also $Z$ is a finitely generated $R$-module. Hence, $R \subseteq Z$ is a finite ring extension and now it is an elementary fact that $Y$ is closed and surjective.

The following lemma about base change of homomorphism spaces is well known but we could not find a reference in this generality (see [3, II, §5.3] for a proof in case of a commutative base ring).

Lemma B.3. Let $A$ be an algebra over a commutative ring $R$ and let $\phi : R \to S$ be a morphism into a commutative ring $S$. Let $V$ and $W$ be $A$-modules. If $V$ is finitely generated and projective as an $A$-module, then there is a canonical $S$-module isomorphism.

$$S \otimes_R \text{Hom}_A(V, W) \simeq \text{Hom}_{A^S}(V^S, W^S).$$

(79)

Proof. We can define a map $\gamma : S \otimes_R \text{Hom}_A(V, W) \to \text{Hom}_{A^S}(V^S, W^S)$ by mapping $s \otimes f$ with $s \in S$ and $f \in \text{Hom}_A(V, W)$ to $s_r \otimes f$, where $s_r$ denotes right multiplication by $s$. It is a standard fact that this is an $S$-module morphism, see [38, (2.36)]. Recall that $\text{Hom}_A(-, W)$ commutes with finite direct sums by [4, II, §1.6, Corollary 1 to Proposition 6]. This shows that the canonical isomorphism $\text{Hom}_A(A, W) \simeq W$ induces a canonical isomorphism $\text{Hom}_A(A^n, W) \simeq W^n$ for any $n \in \mathbb{N}$ and now we conclude that there is a canonical isomorphism

$$S \otimes_R \text{Hom}_A(A^n, W) \simeq (S \otimes_R W)^n \simeq \text{Hom}_{A^S}((A^S)^n, W^S),$$

which is easily seen to be equal to $\gamma$. The assertion thus holds for finitely generated free $A$-modules. Now, the assumption on $V$ allows us to write without loss of generality $A^n = V \oplus X$ for some $A$-module $X$. It is not hard to see that we get a commutative diagram

$$\begin{array}{ccc}
S \otimes_R \text{Hom}_A(A^n, W) & \xrightarrow{=} & (S \otimes_R \text{Hom}_A(V, W)) \oplus (S \otimes_R \text{Hom}_A(X, W)) \\
= & \downarrow & \downarrow \\
\text{Hom}_{A^S}((A^S)^n, W^S) & \xrightarrow{=} & (\text{Hom}_{A^S}(V^S, W^S)) \oplus (\text{Hom}_{A^S}(X^S, W^S))
\end{array}$$

where the horizontal morphisms are obtained by the projections and the vertical morphisms are the morphisms $\gamma$ in the respective situation. The commutativity of this
diagram implies that the morphism $S \otimes_R \text{Hom}_A(V, W) \to \text{Hom}_A(V^S, W^S)$ also has to be an isomorphism.

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