Renormalization-group approach to the stochastic Navier–Stokes equation:
Two-loop approximation

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The field theoretic renormalization group is applied to the stochastic Navier–Stokes (NS) equation subject to an external random force that models the energy injection by the large-scale modes; see, e.g., \cite{11–15}. The aim of the theory is to verify the basic principles of the celebrated Kolmogorov–Obukhov phenomenological theory, study deviations from this theory, determine the dependence of various correlation functions on the times, distances, external (integral) and internal (viscous) turbulence scales, and derive the corresponding scaling dimensions. Most results of this kind were obtained within the framework of numerous semiphenomenological models that cannot be considered to be the basis for construction of a regular expansion in certain small (at least formal) parameter \cite{1–2}. An important exception is provided by the renormalization group (RG) method that was earlier successfully applied in the theory of critical behavior to explain the origin of critical scaling and to calculate universal quantities (critical dimensions and scaling functions) in the form of the \( \varepsilon \) expansions \cite{1–2}.

The RG approach to the stochastic NS equation, pioneered in \cite{5–8}, allows one to prove the existence of the infrared (IR) scale invariance with exactly known “Kolmogorov” dimensions and the independence of the correlation functions of the viscous scale (the second Kolmogorov hypothesis), and calculate a number of representative constants in a reasonable agreement with experiment. Detailed review of the RG theory of turbulence and more references can be found in \cite{5–8}.

I. INTRODUCTION

One of the oldest open problems in theoretical physics is that of describing fully developed fluid turbulence on the basis of a microscopic model. The latter is usually taken to be the stochastic Navier–Stokes (NS) equation subject to an external random force that models the energy injection by the large-scale modes; see, e.g., \cite{11–15}. The aim of the theory is to verify the basic principles of the celebrated Kolmogorov–Obukhov phenomenological theory, study deviations from this theory, determine the dependence of various correlation functions on the times, distances, external (integral) and internal (viscous) turbulence scales, and derive the corresponding scaling dimensions. Most results of this kind were obtained within the framework of numerous semiphenomenological models that cannot be considered to be the basis for construction of a regular expansion in certain small (at least formal) parameter \cite{1–2}. An important exception is provided by the renormalization group (RG) method that was earlier successfully applied in the theory of critical behavior to explain the origin of critical scaling and to calculate universal quantities (critical dimensions and scaling functions) in the form of the \( \varepsilon \) expansions \cite{1–2}.

The RG approach to the stochastic NS equation, pioneered in \cite{5–8}, allows one to prove the existence of the infrared (IR) scale invariance with exactly known “Kolmogorov” dimensions and the independence of the correlation functions of the viscous scale (the second Kolmogorov hypothesis), and calculate a number of representative constants in a reasonable agreement with experiment. Detailed review of the RG theory of turbulence and more references can be found in \cite{5–8}.

The standard RG formalism is applicable to the stochastic NS equation if the correlation function of the random force is chosen in the form \( \propto k^{4-d-2\varepsilon} \), where \( k \) is the momentum (wave number), \( d \) is the space dimension and the exponent \( \varepsilon \) plays the part analogous to that played by \( 4-d \) in the RG theory of critical behavior. Although the results of the RG analysis are reliable and internally consistent for small \( \varepsilon \), the possibility of their extrapolation to the real value \( \varepsilon = 2 \) (see below) and thus their relevance for the real fluid turbulence was called in question in a number of studies, e.g. \cite{1–3}.

The crucial role in recent studies of intermittency and anomalous scaling of fluid turbulence was played by a simple model of a passive scalar quantity advected by a random Gaussian field, white in time and self-similar in space, the so-called Kraichnan’s rapid-change model \cite{16}. There, for the first time the existence of anomalous scaling was established on the basis of a microscopic model \cite{17} and the corresponding anomalous exponents were calculated within controlled approximations \cite{18–19} and numerical experiments \cite{20}. Detailed review of the recent theoretical research on the passive scalar problem and more references can be found in \cite{20}.

The field theoretic RG and the operator product expansion (OPE) were applied to the rapid-change model in \cite{22}. The RG allows one to construct a systematic perturbation expansion for the anomalous exponents, analogous to the \( \varepsilon \) expansion in models of critical behavior, and to calculate the exponents to the second \cite{24} and third \cite{25} orders. For passively advected vector fields the exponents for higher-order correlation functions were derived by means of the RG techniques to the leading order in \( \varepsilon \) in Refs. \cite{24}. It was shown that the knowledge of three terms allows one to obtain reasonable predictions for finite \( \varepsilon \sim 1 \); even the plain \( \varepsilon \) expansion captures some subtle qualitative features of the anomalous exponents established in numerical experiments \cite{23}.

In contrast to standard \( \phi^4 \) model of critical behavior \cite{24}, where the critical exponents are known up to the order \( \varepsilon^5 \) (five-loop approximation), and to the rapid-change model, where the anomalous exponents are known up to \( \varepsilon^3 \),
all the calculations in the RG approach to the stochastic NS equation have been confined with the simplest one-loop approximation. The reason for this distinction is twofold. First, any multiloop calculation for this dynamical model is a demanding job: one can say that the two-loop calculation for the stochastic NS equation is as cumbersome as the four-loop calculation for the conventional \(\phi^4\) model. Second, the critical dimensions for the most important physical quantities (velocity and its powers, frequency, energy dissipation rate and so on) are given by the one-loop approximation exactly (the corresponding \(\varepsilon\) series terminate at first-order terms) and the higher-order calculations for them are not needed.

However, the \(\varepsilon\) series for other important quantities do not terminate and the calculation of the higher-order terms for them is of great interest. In this paper, we present the results of the two-loop calculation for a number of such quantities: the \(\beta\) function, the coordinate of the RG fixed point, the ultraviolet (UV) correction exponent \(\omega\), the Kolmogorov constant \(C_K\) and the inertial-range skewness factor \(S\). The knowledge of higher-order terms is also important to judge about the validity and convergence properties of the \(\varepsilon\) expansions in the models of turbulence on the whole.

The plan of the paper is as follows. In Sec. II we introduce the stochastic NS equation paying special attention to the choice of the random force correlator and the physical value of the parameter \(\varepsilon\). In Sec. III we recall the field theoretic formulation of the model, diagrammatic technique, renormalization and RG equations. Since the “ideology” of the RG and OPE approach to the stochastic NS model is explained in Refs. [9,10,25] in detail, here we confine ourselves to only the necessary information. In Sec. IV we present in detail the two-loop calculation of the renormalization constant, RG functions, coordinate of the fixed point and the UV correction exponent. We explain some tricks (e.g. an appropriate choice of the IR cutoff) that essentially simplify the calculation and make it feasible.

In Sec. V we calculate two orders of the \(\varepsilon\) expansion for the pair correlation function of the velocity field (this accuracy is consistent with the two-loop calculation of the RG functions in Sec. IV). They are used later in Sec. VI in the calculation of the skewness factor and the Kolmogorov constant.

In Sec. VII we discuss on the example of the pair correlation function the possibility of the extrapolation of the \(\varepsilon\) expansion to finite values of \(\varepsilon\). We explain an important distinction that exists between the RG technique and the traditional approach based on diagrammatic self-consistency equations and argue that, contrary to what is sometimes claimed, the RG is applicable beyond the threshold where the so-called sweeping effects become important, but one should combine the RG with the OPE and go beyond the plain \(\varepsilon\) expansions.

In Sec. VIII we perform the two-loop calculation of the Kolmogorov constant \(C_K\) and the inertial-range skewness factor \(S\). The new point is not only the inclusion of the second-order correction, but also the derivation of \(C_K\) through an universal (in the sense of the theory of critical behavior) quantity. This allows us to circumvent the main drawback of earlier calculations of \(C_K\): the intrinsic ambiguities in the corresponding \(\varepsilon\) expansions. To the best of our knowledge, the inertial-range skewness factor \(S\) has never been calculated earlier within the RG approach.

The experience on the RG theory of critical behavior shows that for dynamical models the higher-order corrections are not small, and in order to obtain reasonable predictions, say, for critical exponents, one should augment the plain \(\varepsilon\) expansions by additional information obtained from the instanton calculus and Padé–Borel summations; the situation for the amplitudes in scaling laws is even worse (see e.g. [3]). Surprisingly enough, in our case already the first-order results are in a reasonable agreement with experiment, which is not destroyed by the inclusion of the second-order corrections. It turns out that the experimental value of \(C_K\) and \(S\) lie in between of the two consecutive approximations, like for the exactly solvable Heisenberg model [28].

The main conclusion of the paper is that the renormalization group and the \(\varepsilon\) expansion allow one to derive a number of important characteristics of the real fluid turbulence beyond the simplest first-order approximations and in a good agreement with the experiment.

II. STOCHASTIC NAVIER–STOKES EQUATION AND THE CHOICE OF THE RANDOM FORCE

As the microscopic model of the fully developed, homogeneous, isotropic turbulence of an incompressible viscous fluid one usually takes the stochastic NS equation with a random driving force

\[
\nabla_t \varphi_i = \nu_0 \partial^2 \varphi_i - \partial_t \mathcal{P}_i + f_i, \quad \nabla_t \equiv \partial_t + (\varphi \partial_x).
\]

Here \(\varphi_i\) is the transverse (due to the incompressibility) vector velocity field, \(\mathcal{P}\) and \(f_i\) are the pressure and the transverse random force per unit mass (all these quantities depend on \(x \equiv (t, \mathbf{x})\)), \(\nu_0\) is the kinematical viscosity coefficient, \(\partial^2\) is the Laplace operator and \(\nabla_t\) is the Lagrangian derivative. The problem (2.1) is studied on the entire \(t\) axis and is augmented by the retardation condition and the condition that \(\varphi_i\) vanishes for \(t \to -\infty\). We assume for \(f\) a Gaussian distribution with zero average and correlator
\[ \langle f_i(x) f_j(x') \rangle = \frac{\delta(t-t')}{(2\pi)^d} \int dk \, P_{ij}(k) \, d_f(k) \, \exp \left[ i k \cdot (x - x') \right], \]  

where \( P_{ij}(k) = \delta_{ij} - k_i k_j / k^2 \) is the transverse projector, \( d_f(k) \) is some function of \( k \equiv |k| \) and model parameters, and \( d \) is the dimension of the \( x \) space. The time decorrelation of the random force guarantees Galilean invariance of the full stochastic problem \( (2.1), (2.2) \).

Let us specify the form of the function \( d_f(k) \) in the correlator \( (2.2) \) used in the RG theory of turbulence; more detailed discussion is given in Refs. [11–15]. Physically, the random force models the injection of energy to the system owing to interaction with the large-scale eddies. Idealized injection by infinitely large eddies corresponds to \( d_f(k) \propto \delta(k) \), more precisely,

\[ d_f(k) = 2(2\pi)^d \bar{\varepsilon} \delta(k)/(d-1), \]  

where \( \bar{\varepsilon} \) is the average power of the injection (equal to the average dissipation rate) and the amplitude factor comes from the exact relation

\[ \bar{\varepsilon} = \frac{(d-1)}{2(2\pi)^d} \int dk \, d_f(k). \]  

On the other hand, for the use of the standard RG technique it is important that the function \( d_f(k) \) have a power-law behavior at large \( k \). This condition is satisfied if \( d_f(k) \) is chosen in the form \( (2.5) \)

\[ d_f(k) = D_0 k^{4-d-2\varepsilon}, \]  

where \( D_0 > 0 \) is the amplitude factor and \( \varepsilon > 0 \) is the exponent with the physical value \( \varepsilon = 2 \) (see below).

Let us recall the well-known power-law representation of the \( d \)-dimensional \( \delta \) function,

\[ \delta(k) = \lim_{\varepsilon \to 2} \frac{1}{(2\pi)^d} \int d\lambda \, (\Lambda x)^{2\varepsilon-4} \exp[i(k \cdot x)] = S_d^{-1} k^{-d} \lim_{\varepsilon \to 2} [(4 - 2\varepsilon)(k/\Lambda)^{4-2\varepsilon}], \]  

with some ultraviolet (UV) momentum scale \( \Lambda \). Here and below we denote

\[ S_d = 2\pi^{d/2}/\Gamma(d/2), \quad \bar{S}_d = S_d/(2\pi)^d, \]  

where \( S_d \) is the surface area of the unit sphere in \( d \)-dimensional space and \( \Gamma(\cdots) \) is Euler’s Gamma function. It then follows that for \( \varepsilon \to 2 \), the function \( (2.5) \) turns to the ideal injection \( (2.3) \) if the amplitude \( D_0 \) is related to \( \bar{\varepsilon} \) as

\[ D_0 \to \frac{4(2 - \varepsilon) \Lambda^{2\varepsilon-4}}{S_d(d-1) \bar{\varepsilon}} \bar{\varepsilon} \quad \text{for } \varepsilon \to 2. \]  

A more realistic model, used e.g. in [6,25], is

\[ d_f(k) = D_0 k^{4-d-2\varepsilon} h(m/k), \quad h(0) = 1, \]  

where \( m = 1/L \) is the reciprocal of the integral turbulence scale \( L \) and \( h(m/k) \) is some well-behaved function that provides the IR regularization. Its specific form can be chosen to simplify the practical calculation in higher orders; see Sec. [17].

In the RG approach to the problem \( (2.1), (2.2), (2.9) \), the exponent \( \varepsilon \) plays the part analogous to that played by \( 4 - d \) in Wilson’s theory of critical phenomena [8,11]. All the results of the RG analysis are reliable and internally consistent for small \( \varepsilon \); the possibility of their extrapolation to the real value \( \varepsilon = 2 \) was called in question in a number of studies, e.g. [8,11,15]. We shall return to this important issue later, and here we only note that the replacement of the ideal injection \( (2.3) \) by the power-law model \( (2.4) \) followed by the use of the \( \varepsilon \) expansion was confirmed by the example of the exactly solvable Heisenberg model [20].

**III. FIELD THEORETIC FORMULATION AND RENORMALIZATION OF THE MODEL**

Detailed exposition of the RG theory of turbulence and the bibliography can be found in [8,11]; below we restrict ourselves to only the necessary information.
Stochastic problem \(\{2.1\}, \{2.2\}\) is equivalent to the field theoretic model of the doubled set of transverse vector fields \(\Phi \equiv \{\varphi, \varphi'\}\) with action functional
\[
S(\Phi) = \varphi' D_f \varphi'/2 + \varphi'[-\partial_i \varphi + \nu_0 \partial^2 \varphi - (\varphi \partial)\varphi],
\]
where \(D_f\) is the random force operator with \(\nu_0\) and \(\bar{\nu}\) and operate on both sides of that relation with it. This gives the basic differential RG equation:
\[
\epsilon(\varphi'\varphi')_0 = \langle \varphi'\varphi' \rangle_0 \equiv (i\omega + \nu_0 k^2)^{-1}, \quad \langle \varphi\varphi' \rangle_0 = 0, \quad \langle \varphi\varphi' \rangle_0 = d_f(k)/(\omega^2 + \nu_0^2 k^4)
\]
in the frequency-momentum \((\omega-k)\) representation or
\[
\langle \varphi(t)\varphi'(t') \rangle_0 = \theta(t-t') \exp[-\nu_0 k^2(t-t')], \quad \langle \varphi'(t)\varphi'(t') \rangle_0 = 0, \quad \langle \varphi\varphi' \rangle_0 = d_f(k) \exp[-\nu_0 k^2|t-t'|]/2\nu_0 k^2
\]
in the time-momentum \((t-k)\) representation; the common factor \(P_{ij}(k)\) in \(\{3.2\}, \{3.3\}\) is understood. The interaction in \(\{3.3\}\) corresponds to the triple vertex \(-\varphi'(\varphi\varphi) = \varphi'[V_{ij}\varphi_i\varphi_j]/2\) with vertex factor \(V_{ij} = i(k_j\delta_{ij} + k_i\delta_{ij})\), where \(k\) is the momentum of the field \(\varphi'\). The part of the coupling constant \((\epsilon = \text{dimensionless})\) at \(\epsilon = 0\) and the UV divergences have the form of the poles in \(\epsilon\) in the correlation functions of the fields \(\Phi \equiv \{\varphi, \varphi'\}\). Superficial UV divergences, whose removal requires counterterms, are present only in the 1-irreducible function \(\langle \varphi'\varphi \rangle\), and the corresponding counterterm reduces to the form \(\varphi' \partial^2 \varphi\). In the special case \(d = 2\) a new UV divergence appears in the 1-irreducible function \(\langle \varphi'\varphi' \rangle\). We shall study this important case in a separate paper, and for now we always assume \(d > 2\). Then for the complete elimination of the UV divergences it is sufficient to perform the multiplicative renormalization of the parameters \(\nu_0\) and \(g_0\) with the only independent renormalization constant \(Z_\nu\):
\[
\nu_0 = \nu Z_\nu, \quad g_0 = g\mu^{2\varepsilon} Z_g, \quad Z_g = Z_\nu^{-3}, \quad (D_0 = g_0\nu_0^3 = g\mu^{2\varepsilon}\nu^3).
\]
Here \(\mu\) is the reference mass in the minimal subtraction (MS) scheme, which we always use in what follows, and \(g\) and \(\nu\) are renormalized analogs of the bare parameters \(g_0\) and \(\nu_0\), and \(Z = Z(g, \varepsilon, d)\) are the renormalization constants. No renormalization of the fields and the “mass” \(m_0 = m\) is needed, i.e., \(Z_\Phi = 1\) for all \(\Phi\) and \(Z_m = 1\).

In the MS scheme the renormalization constants have the form “\(1 + \text{only poles in } \varepsilon\)” in particular,
\[
Z_\nu = 1 + \sum_{k=1}^{\infty} a_k(g)\varepsilon^{-k} = 1 + \sum_{n=1}^{\infty} g^n \sum_{k=1}^{n} a_{nk}\varepsilon^{-k},
\]
where the coefficients \(a_{nk}\) depend only on \(d\). The one-loop result
\[
a_{11} = -(d - 1)\bar{S}_d/8(d + 2),
\]
with \(\bar{S}_d\) from \(\{2.7\}\), was presented in \(\{3\}\).

Since the fields are not renormalized, their renormalized correlation functions \(W_R\) coincide with their unrenormalized analogs \(W = \langle \Phi \ldots \Phi \rangle\); the only difference is in the choice of variables and in the form of perturbation theory (in \(g\) instead of \(g_0\)): \(W_R(g, \nu, \mu, m, \ldots) = W(g_0, \nu_0, m_0, \ldots)\). Here the dots stand for other arguments like coordinates, times, momenta and so on. We use \(\bar{D}_\mu\) to denote the differential operator \(\mu \partial_{\mu}\) for fixed bare parameters \(g_0, \nu_0, m_0\) and operate on both sides of that relation with it. This gives the basic differential RG equation:
\[
D_{RG} W_R(g, \nu, \mu, m, \ldots) = 0, \quad D_{RG} \equiv \bar{D}_\mu + \beta(g)\partial_g - \gamma_\nu(g) D_\nu,
\]
where \(D_{RG}\) is the operation \(\bar{D}_\mu\) expressed in renormalized variables, \(D_x \equiv x \partial_x\) for any variable \(x\), and the RG functions (the anomalous dimension \(\gamma_\nu\) and the \(\beta\) function) are defined as
\[
\gamma_\nu(g) \equiv \bar{D}_\mu \ln Z_\nu = -2D_g a_1(g), \quad \beta(g, \varepsilon) \equiv \bar{D}_\mu g = g [-2\varepsilon + 3\gamma_\nu(g)]
\]
with \(a_1(g)\) from \(\{3.5\}\). In the MS scheme only the residues at the first-order poles in \(\varepsilon\), that is, only the coefficients \(a_{k1}\) contribute to the RG functions owing to the UV finiteness of the latter; this explains the last relation for \(\gamma_\nu\).

The relation between \(\beta(g, \varepsilon)\) and \(\gamma_\nu\) results from the definitions and the last relation in \(\{3.4\}\).

From the relations \(\{3.8\}\) using Eq. \(\{3.6\}\) we find the first-order expressions for the RG functions:
\[
\gamma_\nu(g) = (d - 1)\tilde{S}_d g/4(d + 2) + O(g^2), \quad \beta(g, \varepsilon) = g\left[-2\varepsilon + 3(d - 1)\tilde{S}_d g/4(d + 2)\right] + O(g^3).
\] (3.9)

From (3.9) it follows that an IR-attractive fixed point
\[
g_* = 8(d + 2)\varepsilon / 3(d - 1)\tilde{S}_d + O(\varepsilon^2), \quad \beta(g_*) = 0, \quad \omega \equiv \beta'(g_*) = 2\varepsilon + O(\varepsilon^2) > 0
\] (3.10)
of the RG equation (3.7) exists in the physical region \( g > 0 \) for \( \varepsilon > 0 \). The value of \( \gamma_\nu(g) \) at the fixed point is found exactly using relations (3.8):
\[
\gamma_\nu^* \equiv \gamma_\nu(g_*) = 2\varepsilon/3,
\] (3.11)
without corrections of order \( \varepsilon^2, \varepsilon^3 \), and so on.

For definiteness, consider the solution of the RG equation (3.7) on the example of a correlation function \( W^R \) that involves \( n \) fields \( \varphi \) and depends on a single coordinate difference \( r = |r| \); the extension to the general case is straightforward. We shall omit the superscript \( R \) in what follows. In renormalized variables, dimensionality considerations give:
\[
W = (\nu/r)^n R(s, u, g), \quad s \equiv \mu r, \quad u \equiv mr,
\] (3.12)
where \( R(\cdot, \cdot, \cdot) \) is a function of completely dimensionless arguments (the dependence on \( d \) and \( \varepsilon \) is understood).

From the RG equation the identical representation follows,
\[
W = (\nu/r)^n R(s, u, g) = (\nu/r)^n R(1, u, \bar{g}),
\] (3.13)
where the invariant variables \( \bar{e} = \bar{e}(s, \varepsilon) \) satisfy the differential equation \( D_{RG} \bar{e} = 0 \) with the operator \( D_{RG} \) from (3.7) and the normalization conditions \( \bar{e} = e \) at \( s = 1 \) (we have used \( e \equiv \{\nu, g, m\} \) to denote the full set of renormalized parameters). The identity \( \bar{u} \equiv u \) is a consequence of the absence of \( D_m \) in the operator \( D_{RG} \) owing to the fact that \( m \) is not renormalized. Equation (3.13) is valid because both sides of it satisfy the RG equation and coincide for \( s = 1 \) owing to the normalization of the invariant variables.

The relation between the bare and invariant variables has the form
\[
\nu_0 = \bar{\nu} Z_{\nu}(\bar{g}), \quad g_0 = \bar{g} r^{-2\varepsilon} Z_g(\bar{g})
\] (3.14)
with \( Z_{\nu,g} \) from (3.4). Equation (3.14) determines implicitly the invariant variables as functions of the bare parameters; it is valid because both sides of it satisfy the RG equation, and because Eq. (3.14) at \( s = \mu r = 1 \) coincides with (3.4) owing to the normalization conditions.

It is well known that the behavior of the invariant charge \( \bar{g} \) at large \( \mu r \) is governed by the IR stable fixed point: \( \bar{g} \to g_* \) with \( g_* \) from (3.10). Then the large-\( \mu r \) behavior of the invariant viscosity \( \bar{\nu} \) is found explicitly from Eq. (3.14) and the last relation in (3.4):
\[
\bar{\nu} = \left[D_0 r^{2\varepsilon} / \bar{g}\right]^{1/3} \to \left[D_0 r^{2\varepsilon} / g_*\right]^{1/3}.
\]
Then for \( s \to \infty \) and any fixed \( u \equiv mr \) we obtain
\[
W = (D_0/g_*)^{n/3} r^{-n\Delta_\varphi} f(u), \quad \Delta_\varphi \equiv 1 - 2\varepsilon/3, \quad f(u) \equiv R(1, u, g_*).
\] (3.15)
From Eq. (3.15) it follows that in the IR range (large \( \mu r \) and arbitrary \( mr \)) the parameters \( g_0 \) and \( \nu_0 \) enter into the correlation functions only in the form of the combination \( D_0 = g_0 \nu_0^3 \), a fact first established in Ref. [4]. For \( \varepsilon = 2 \) this proves the Second Kolmogorov hypothesis: owing to the relation (2.8), the correlation functions depend on \( \varphi \) but do not depend on the viscosity coefficient \( \nu_0 \) and the UV scale \( \Lambda \sim g_0^{1/2 \varepsilon} \); see also the discussion in [10,25].

Representation (3.15) for any scaling function \( f(u) \) describes the behavior of the correlation functions for \( s = \mu r \gg 1 \) and any fixed value of \( u \equiv mr \); the inertial range corresponds to the additional condition \( u \ll 1 \). The form of the function \( f(mr) \) is not determined by the RG equation (3.7). Calculating the function \( R \) in (3.12) within the renormalized perturbation theory, \( R = \sum_{n=0}^\infty g^n R_n \), substituting \( g \to g_* \) and expanding \( g_* \) and \( R_n \) in \( \varepsilon \), one obtains the \( \varepsilon \) expansion for the scaling function:
\[
f(u) = \sum_{n=0}^\infty \varepsilon^n f_n(u).
\] (3.16)

Although the coefficients \( f_n \) in our model are finite at \( u = 0 \), this does not prove the finiteness of \( f(u) \) beyond the \( \varepsilon \) expansion: one can show that for any arbitrarily small value of \( \varepsilon \) there are diagrams that diverge at \( m \propto u \to 0 \).
As a result, the coefficients $f_a$ contain IR singularities of the form $u^p \ln^q u$, these “large IR logarithms” compensate for the smallness of $\varepsilon$, and the actual expansion parameter appears to be $\varepsilon \ln u$ rather than $\varepsilon$ itself. Thus the plain expansion (3.10) is not suitable for the analysis of the small-$u$ behavior of $f(u)$.

The formal statement of the problem is to sum up the expansion (3.10) assuming that $\varepsilon$ is small with the additional condition that $\varepsilon \ln u \sim 1$. By analogy with the theory of critical behavior [14], the desired solution can be obtained using the well-known operator product expansion (OPE); see [11,25]. The OPE shows that for small $u$ the function $f(u)$ has the form

$$f(u) = \sum_F C_F (u) u^{\Delta_F},$$

(3.17)

where $C_F$ are coefficients analytical in $u^2$, the summation runs over all local composite operators $F(x)$ consistent with the symmetries of the model and the left-hand side, and $\Delta_F$ are their scaling dimensions calculated as series in $\varepsilon$.

If all $\Delta_F$ are non-negative (as in models of critical behavior), expression (3.17) is finite at $u = 0$ and the $\varepsilon$ expansion can be used for the calculation of $f(0)$. The feature specific to statistical models of fluid turbulence is the existence of composite operators with negative dimensions $\Delta_F$. Such operators, termed “dangerous” in [9,10,25], give rise to strong IR singularities in the correlation functions and to their divergence at $m \propto u \to 0$.

The problem is that dangerous operators in our model are absent in the $\varepsilon$ expansions and can appear only at finite values of $\varepsilon$. This means that they can be reliably identified only if their dimensions are derived exactly with the aid of Schwinger equations or Ward identities that express Galilean symmetry. Due to the nonlinear nature of the problem, they enter the corresponding OPE’s as infinite families whose spectra of dimensions are not bounded from below, and in order to find the small-$m \varepsilon$ behavior one has to sum up all their contributions in the representation (3.17). The needed summation of the most singular contributions, related to the operators of the form $\varphi^n$ with known dimensions, was performed in [25] using the so-called infrared perturbation theory for the case of the different-time pair correlation functions; see also [11,10]. It has revealed their strong dependence on $m$, which physically can be explained by the infamous “sweeping effects.” This demonstrates that, contrary to the existing opinion [11–15], the sweeping effects can be properly described within the RG approach, but one should combine the RG with the OPE and go beyond the plain $\varepsilon$ expansions.

In Secs. IV–VII we shall be interested in the Galilean invariant objects like the equal-time structure functions. For such objects, the singular contributions mentioned above drop out from the representation (3.17) in agreement with the fact that Galilean invariant quantities are not affected by the sweeping. Only the dimensions of Galilean invariant operators can appear in (3.17), and the singular behavior can be related to invariant operators with $\Delta_F < 0$. No such operator, however, has been presented in any study we know of; see Refs. [11,27,28]. Although the problem should be considered open, this fact suggests that Galilean invariant objects may have a finite limit at $u \to 0$. Below in Sec. VII we shall calculate the Kolmogorov constant within the assumption that the second-order structure function has a finite limit at $u \to 0$ for the physical value $\varepsilon = 2$. Such an assumption can be justified by the real experiments, which indicate that the corresponding exponent is hardly distinguishable from the Kolmogorov value 1, or by the example of the Kraichnan model, in which the second-order function is not anomalous [19].

IV. TWO-LOOP CALCULATION OF THE RENORMALIZATION CONSTANT, RG FUNCTIONS, FIXED POINT AND THE UV CORRECTION EXPONENT

Let us turn to the calculation of the renormalization constant $Z_\nu$ in Eq. (3.5) with the accuracy $O(g^2)$ (two-loop approximation).

The renormalization constants in the MS scheme are independent of the value of the “mass” $m$ and the specific form of the function $h(m/k)$ in the correlator (2.3), and the practical calculations are usually performed in the “massless” model (2.3) with $m = 0$ and $h(m/k) = 1$. It turns out, however, that for the multiloop calculations it is more convenient to use the model (2.3) with $m \neq 0$ and $h(m/k) = \theta(k - m)$. We shall determine the constant $Z_\nu$ from the requirement that the 1-irreducible correlation function $\langle \varphi' \varphi \rangle$ at zero frequency ($\omega = 0$) and in the limit $p \to 0$ be UV finite (that is, be finite at $\varepsilon \to 0$) when expressed in renormalized variables using relations (3.4). More precisely, we shall consider the scalar dimensionless quantity

$$\Gamma = \lim_{p \to 0} \frac{\langle \varphi' \varphi \rangle_{1-ir} (p; \omega = 0)}{\nu p^2 (d - 1)},$$

(4.1)

which depends only on $g$ and $m/\mu$ and in the loopless (tree) approximation equals $\Gamma^{(0)} = -Z_\nu$. The IR regularization is provided by the sharp cutoff $h(m/k) = \theta(k - m)$ in Eq. (2.9).
Let us illustrate this scheme by revisiting the one-loop calculation. To first order in \( g \), the quantity (4.1) is determined by the only diagram
\[
\Gamma^{(1)} = -\frac{g^2 Z^{-2}}{4(d-1)(2\pi)^d} \int \frac{dk}{k^{d+2\varepsilon}} h(m/k) [d - 3 + (9 - d)z^2 - 6z^4],
\]
(4.2)
where \( z \equiv (\mathbf{n} \cdot \mathbf{k})/k \) is the cosine between \( \mathbf{k} \) and \( \mathbf{n} \equiv \mathbf{p}/p \), the direction of the external momentum. The quantity \( \Gamma \) does not depend on the direction \( \mathbf{n} \) and we can average (4.3) with respect to it using the relations
\[
\langle z^{2n} \rangle = \frac{(2n-1)!!}{d(d+2)\ldots(d+2n-2)}, \quad \langle z^{2n+1} \rangle = 0; \quad (4.4)
\]
the brackets denote the averaging over the unit sphere in \( d \) dimensions. This gives
\[
\Gamma^{(1)} = -\frac{g^2 S_{d}(d-1)}{4(d+2)Z^2_{\nu}} \int_{m}^{\infty} \frac{dk}{k^{1+2\varepsilon}} = -\frac{(d-1)(\mu/m)^{2\varepsilon}gS_{d}}{8(d+2)\varepsilon Z^2_{\nu}},
\]
(4.5)
with \( S_{d} \) from (2.7). In the expansion in the number of loops,
\[
\Gamma = -Z_{\nu} + \Gamma^{(1)} + \Gamma^{(2)} + \ldots, \quad (4.6)
\]
the \( n \)-loop term \( \Gamma^{(n)} \) is the quantity of order \( O(g^n) \), so that in the \( O(g) \) approximation in (4.4) it is sufficient to keep the term \( \Gamma^{(1)} \) (and put \( Z_{\nu} = 1 \) in it) and the linear in \( g \) contribution in the first term. The residue at the pole in \( \varepsilon \) in (4.5) is found by the simple substitution \( (\mu/m)^{2\varepsilon} \to 1 \). From the requirement that the function \( \Gamma(g, m/\mu) \) be UV finite (that is, finite at \( \varepsilon \to 0 \)) we thus find \( a_{11} = -(d-1)S_{d}/(d+2) \) for the coefficient \( a_{11} \) in agreement with the well-known result (3.6).

The two-loop contribution \( \Gamma^{(2)} \) in (4.6) is given by the sum of the following eight diagrams (normalization according to (4.1) is implied):

\[
\Gamma^{(2)}_1 = \quad \Gamma^{(2)}_2 =
\Gamma^{(2)}_3 = \quad \Gamma^{(2)}_4 = \frac{1}{2}
\Gamma^{(2)}_5 = \quad \Gamma^{(2)}_6 =
\Gamma^{(2)}_7 = \quad \Gamma^{(2)}_8 =
\]
(4.7)

After two simple integrations over the frequencies, these diagrams are represented as integrals over two independent momenta \( \mathbf{k}_1, \mathbf{k}_2 \) that can be assigned to the two \( \langle \varphi \varphi \rangle_0 \) lines, while the external momentum \( \mathbf{p} \) flows only via the \( \langle \varphi \varphi' \rangle_0 \) lines (such choice of the integration contours is possible for any of the diagrams in (4.7)). The way in which
the momenta \( \mathbf{k}_1 \) and \( \mathbf{k}_2 \) are assigned to the \( \langle \varphi \varphi \rangle_0 \) lines is explicitly shown in (4.7); additional symmetrization in \( \mathbf{k}_1, \mathbf{k}_2 \) is understood for the diagrams \( \Gamma_i^{(2)} \) with \( i = 5, \ldots, 8 \). The diagram \( \Gamma_8^{(2)} \) is automatically symmetric.

After the contractions of the tensor indices and the limit transition \( p \to 0 \) have been performed, the resulting integrands take on the form of polynomials in the cosines \( z_1 = (\mathbf{n} \cdot \mathbf{k}_1)/k_1 \) and \( z_2 = (\mathbf{n} \cdot \mathbf{k}_2)/k_2 \). The integrands are further simplified after averaging over the direction \( \mathbf{n} \equiv \mathbf{p}/p \) using (4.4), which is possible owing to the independence of the quantities \( \Gamma_i^{(2)} \) of \( \mathbf{n} \). In the variables \( \mathbf{k}_1, \mathbf{k}_2 \) and \( z \equiv (\mathbf{k}_1 \cdot \mathbf{k}_2)/k_1 k_2 \) they can be written in the form

\[
\Gamma_i^{(2)} = g^2 s_0^2 \mu^{4\varepsilon} Z_{\nu}^{-5} \int_{\infty}^{0} \frac{dk_1}{k_1^{1+2\varepsilon}} \int_{\infty}^{0} \frac{dk_2}{k_2^{1+2\varepsilon}} \int_0^1 dz f_i(z, k_1/k_2). \tag{4.8}
\]

The lower limits in the integrals over \( k_1, k_2 \) are due to the choice \( h(m/k) = \theta(k - m) \). For convenience reasons, here and in analogous formulas below we restrict the integration over \( z \) to the half-interval \([0, 1]\) instead of the full interval \([-1, 1]\) and simultaneously replace the integrands by their doubled even-in-\( z \) parts. The expressions for the functions \( f_i \) are given in the Appendix. Within our accuracy, it is necessary to replace \( Z_{\nu} \to 1 \) in (4.8).

In terms of the dimensionless variables \( \kappa_i \equiv k_i/m \) one can write

\[
\Gamma_i^{(2)} = g^2 s_0^2 (\mu/m)^{4\varepsilon} A_i(\varepsilon) \tag{4.9}
\]

with

\[
A_i(\varepsilon) = \int_{1}^{\infty} \frac{dk_1}{\kappa_1^{1+2\varepsilon}} \int_{1}^{\infty} \frac{dk_2}{\kappa_2^{1+2\varepsilon}} \int_0^1 dz f_i(z, \kappa_1/\kappa_2). \tag{4.10}
\]

We are interested in the coefficients \( a_i, b_i \) in the pole part of the quantities \( A_i(\varepsilon) \):

\[
a_i(\varepsilon) = \frac{a_i}{\varepsilon} + \frac{b_i}{\varepsilon} + c_i + O(\varepsilon). \tag{4.11}
\]

The subdiagram in \( \Gamma_4^{(2)} \) does not contain an UV divergence, so that \( f_4(z, 0) = f_4(z, \infty) = 0 \), the integrals over \( k_1 \) and \( k_2 \) are separately finite, and the divergence at \( \varepsilon \to 0 \) comes only from the region where \( k_1, k_2 \) tend to infinity simultaneously. As a result, the second-order pole is absent: \( a_4 = 0 \). The subdiagrams in \( \Gamma_5^{(2)}, \ldots, \Gamma_8^{(2)} \) contain UV divergences, but in the sum \( \Gamma_0^{(2)} = \sum_{i=5}^{8} \Gamma_i^{(2)} \) they cancel each other (a consequence of the absence of divergence in the 1-irreducible function \( \langle \varphi' \varphi \rangle \)), so that \( a_0 = \sum_{i=5}^{8} a_i = 0 \).

For the diagrams \( \Gamma_i \) with \( i = 1, 2, 3 \) one still has \( f_i(z_0, 0) = 0 \), but the second-order poles exist due to the relations \( f_i(z, \infty) = \text{const} \neq 0 \). Expressions (4.10) can be simplified using the identity

\[
D_m \Gamma_i^{(2)} = -4\varepsilon g^2 s_0^2 (\mu/m)^{4\varepsilon} A_i(\varepsilon) \tag{4.12}
\]

that follows from (4.9). The operation \( D_m \equiv m\partial/\partial m \) reduces the number of integrations in (4.10) and explicitly isolates the pole factor \( \varepsilon^{-1} \): calculating the left-hand side of (4.12) using (4.8) and changing to dimensionless variables again, one obtains

\[
A_i(\varepsilon) = \frac{1}{4\varepsilon} \int_0^1 dz [f_i(z, \kappa) + f_i(z, 1/\kappa)]. \tag{4.13}
\]

For \( i = 0, 4 \) the integral in (4.13) is finite at \( \varepsilon = 0 \) and determines the residue at the first-order pole:

\[
a_i = 0, \quad b_i = \frac{1}{4} \int_1^{\infty} \frac{dk}{\kappa} \int_0^1 dz [f_i(z, \kappa) + f_i(z, 1/\kappa)], \quad i = 0, 4. \tag{4.14}
\]

For the quantities \( A_i(\varepsilon) \) with \( i = 1, 2, 3 \) the residue at the second-order pole is obtained if the value of the function \( [f_i(z, \kappa) + f_i(z, 1/\kappa)] \) in (4.13) is substituted by its limit value \( f_i(z, \infty) + f_i(z, 0) = f_i(z, \infty) \) [we recall that \( f_i(z, 0) = 0 \)]; then the integration over \( \kappa \) becomes simple and gives

\[
a_i = \frac{1}{8} \int_0^1 dz f_i(z, \infty), \quad i = 1, 2, 3. \tag{4.15}
\]

The remaining integral with the replacement \( f_i(z, \kappa) \to [f_i(z, \kappa) - f_i(z, \infty)] \) is finite at \( \varepsilon = 0 \) and determines the residue at the first-order pole:
\[
\begin{align*}
&b_i = \frac{1}{4} \int_1^\infty \frac{d\kappa}{\kappa} \int_0^1 dz \left[ f_i(z, \kappa) - f_i(z, \infty) + f_i(z, 1/\kappa) \right], \quad i = 1, 2, 3. \\
&\text{According to (4.13), (4.11), the summed values} \\
&a \equiv \sum_{i=0}^4 a_i, \quad b \equiv \sum_{i=0}^4 b_i
\end{align*}
\]

of the coefficients \(a_i, b_i\) from (4.14)–(4.16) determine the needed residues for \(\Gamma^{(2)} = \sum_{i=0}^4 \Gamma_{1}^{(2)}\). Analysis shows that all integrations over \(z\) in (4.14)–(4.16) can be performed in terms of elementary functions for any \(d\), which gives simple expressions for \(a_i\). For their summed value \(a\) in (4.17) this gives:

\[
a = \frac{(d-1)^2}{64(d+2)^2} = \bar{S}_d^{-2} a_{11}^2
\]

with \(\bar{S}_d\) from Eq. (2.7) and \(a_{11}\) from (3.3).

It turns out, however, that the most convenient way to find the residues \(b_i\) is the direct numerical calculation of the double integrals in (4.14)–(4.16), which can be performed for any given value of \(d\). Below we give the results of such calculation for the most interesting case \(d = 3\); it is convenient to represent them in terms of the quantities \(b_0' \equiv 10^3 \cdot b_i\) and \(b' \equiv 10^3 \cdot b = \sum_{i=0}^4 b_i'\):

\[
\begin{align*}
&b_0' = -1.34, \quad b_1' = 0.16, \quad b_2' = 1.21, \quad b_3' = 0.006, \quad b_4' = -4.17, \quad b' = -4.13.
\end{align*}
\]

Now let us write down the condition of the UV finiteness of the right-hand side of Eq. (1.6) in the second order in \(g\). In Eq. (4.18) it was sufficient to replace \(Z_\nu \rightarrow 1\), while in the expression (4.15) for \(\Gamma^{(1)}\) one should keep in the expansion \(Z_\nu^{-1} = 1 - 2a_{11}g/\varepsilon + \ldots\) the term linear in \(g\). Then using Eq. (3.5) the condition of the UV finiteness is written in the form

\[
-(a_{22} \varepsilon^{-2} + a_{21} \varepsilon^{-1}) - 2a_{11} \varepsilon^{-2} \left(1 + 2\varepsilon \ln \frac{\mu}{m} \right) + \bar{S}_d^2 \varepsilon^{-2} \left(1 + 4\varepsilon \ln \frac{\mu}{m} \right) + \bar{S}_d^2 \varepsilon^{-1} = 0.
\]

From Eq. (4.18) it is easily seen that the terms with \(\ln(\mu/m)\) in (4.20) cancel each other (a consequence of the renormalizability of the model in the MS scheme). For the sought coefficients \(a_{22}\) and \(a_{21}\) in (3.3) from Eqs. (4.20) and (4.18) we thus obtain

\[
a_{22} = -a_{11}^2, \quad a_{21} = \bar{S}_d^2 b
\]

with \(a_{11}\) from Eq. (8.8) and \(b\) from (4.17). This completes the calculation of the renormalization constant \(Z_\nu\) in the two-loop approximation.

The knowledge of the renormalization constant \(Z_\nu\) to order \(O(g^2)\) allows for the calculation of the anomalous dimension \(\gamma_\nu\) in (3.3) to order \(O(g^3)\):

\[
\gamma_\nu(g) = -2g \partial_\nu a_1(g) = -2 \left( a_{21} g + 2a_{21} g^2 \right) + O(g^3)
\]

then the \(\beta\) function is found from (3.8) with the proper accuracy. The coordinate of the fixed point in (3.10) is then determined to order \(O(\varepsilon^2)\). From Eqs. (4.21) and (4.22) we obtain:

\[
g_* \bar{S}_d = \alpha(1 + \lambda \varepsilon) + O(\varepsilon^3), \quad \alpha \equiv -\bar{S}_d/3a_{11} = 8(d+2)/3(d-1), \quad \lambda \equiv 2a_{21}/3a_{11}^2 = 2b[8(d+2)/(d-1)]^2/3
\]

with the coefficient \(b\) from (4.17) and \(\bar{S}_d\) from (2.7). In particular, for \(d = 3\) using Eq. (4.19) we obtain:

\[
g_* = (40\pi^2 \varepsilon/3)(1 + \lambda \varepsilon) + O(\varepsilon^3), \quad \lambda \simeq -1.101.
\]

The case of general \(d\) dimensions is also of interest and will be discussed in Sec. VII. Results for the coefficient \(\lambda\) for different values of \(d\) are given in Table I along with other quantities that will appear later on.

The correction exponent \(\omega = \beta'(g_*)\) in (3.10), determined by the slope of the \(\beta\) function at the fixed point, is found from (4.22) and (3.10) to second order of the \(\varepsilon\) expansion:

\[
\omega = 2\varepsilon(1 - \lambda \varepsilon) + O(\varepsilon^3)
\]

From Table it follows that, for any \(d\), the two-loop correction makes the value of \(\omega\) “more positive,” that is, it enhances the IR stability of the fixed point.
V. PAIR CORRELATION FUNCTION: ε EXPANSION

In this section we shall discuss the equal-time pair correlation function of the velocity field in the momentum representation,

\[ G_{ij}(p) = P_{ij}(p) G(p). \]  

\[ G(p) = g^2 p^{-d+2} R(s, u, g) \simeq D_0^{2/3} \frac{g^1/3}{p^{-d+2}} \varepsilon R(1, u, g), \quad s \equiv p/\mu, \quad u \equiv m/p, \]  

where the second relation holds in the IR asymptotic region and one factor \( g \) is explicitly isolated such that the expansion in \( g \) of the dimensionless function \( R(1, u, g) \) begins with \( O(g^0) \).

Our aim is twofold. Below in this section we shall calculate two orders \((1 \text{ and } \varepsilon)\) of the \( \varepsilon \) expansion for the scaling function \( R(1, u, g) \) at \( u = 0 \) (this accuracy is consistent with the two-loop calculation of the RG functions in Sec. IV). The coefficients of the \( \varepsilon \) expansions in our model have a finite limit at \( u = 0 \) (see the end of Sec. II) so we can perform this calculation in the “massless” model \((2.5)\) with \( m = 0 \) and \( h(m/k) = 1 \), which is always assumed below unless stated to be otherwise. These results will be used later in Sec. VI in the calculation of the skewness factor and the Kolmogorov constant. In Sec. VII, we shall use the example of the pair correlator to discuss the existence of the limit \( u \to 0 \) for finite \( \varepsilon \), the possibility of the extrapolation of the \( \varepsilon \) expansion beyond the threshold where the sweeping effects become important (see the remark in the end of Sec. II), and the relation between the RG approach and other approaches to the stochastic NS equation.

In order to calculate two terms of the \( \varepsilon \) expansion of the amplitude \( R(1, 0, g) \) we need to find the coefficients \( c_1 \) and \( c_2 \) in the expansion

\[ R(1, 0, g) = G(p)|_{p=\mu, u=0} + g \tilde{S}_d c_2 + O(g^2) \]  

(the factor \( \tilde{S}_d \) from Eq. \((2.7)\) is isolated in the second term for convenience). We shall see below that \( c_1 = 1/2 \), while the coefficient \( c_2 = c_2(\varepsilon) \) should be found to the leading order in \( \varepsilon \), that is, at \( \varepsilon = 0 \).

With the needed accuracy of \( O(g^2) \), the function \( G(p) \) is given by the sum of the loopless (tree) and one-loop diagrams. The first (tree) contribution to the correlator \((5.1)\) is simply given by the bare correlator \( \langle \varphi \varphi \rangle_0 \) from \((3.4)\) at \( t = t' \) and expressed in renormalized variables using Eq. \((3.4)\). The corresponding contribution \( R_0 \) to the dimensionless quantity \((5.3)\) has the form

\[ R_0 = Z_\nu^{-1}/2 = \left[ 1 - a_{11} g/\varepsilon + O(g^2) \right]/2 \]  

with the coefficient \( a_{11} \) from \((3.6)\). This gives the exact result \( c_1 = 1/2 \) for the first coefficient in \((5.3)\).

The one-loop contribution to the function \( G(p) \) is given by the sum of the following diagrams

\[ \frac{1}{2} \begin{array}{c} \begin{array}{c} \text{k} \\ \text{p} \end{array} \end{array} + \begin{array}{c} \begin{array}{c} \text{k} \\ \text{q} \end{array} \end{array} + \begin{array}{c} \begin{array}{c} \text{p} \\ \text{q} \end{array} \end{array} \]  

\[ \begin{array}{c} \begin{array}{c} \text{p} \\ \text{q} \end{array} \end{array} + \begin{array}{c} \begin{array}{c} \text{k} \\ \text{p} \end{array} \end{array} \]  

\[ \begin{array}{c} \begin{array}{c} \text{k} \\ \text{q} \end{array} \end{array} + \begin{array}{c} \begin{array}{c} \text{q} \\ \text{k} \end{array} \end{array} \]  

the diagrammatic notation was explained below Eq. \((4.2)\). We shall denote their contributions to the dimensionless function \((5.3)\) as \( R_{1,2,3} \) (from the left to the right). They have the forms

\[ R_1 = \frac{g}{8(d-1)} \int \frac{d\kappa}{(2\pi)^d} U(\kappa, \varphi) Q_1(\kappa) \kappa^{2-d-2\varepsilon} q^{d-2\varepsilon}, \]  

\[ R_2 = \frac{g}{8(d-1)} \int \frac{d\kappa}{(2\pi)^d} U(\kappa, \varphi) Q_2(\kappa) \kappa^{2-d-2\varepsilon}, \]  

\[ R_3 = \frac{g}{8(d-1)} \int \frac{d\kappa}{(2\pi)^d} U(\kappa, \varphi) Q_2(\kappa) q^{2-d-2\varepsilon}, \]  

where we included the symmetry coefficient \( 1/2 \) for \( R_1 \) and denoted

\[ Q_1(\kappa) = d - 1 - 2d(\mathbf{n} \cdot \kappa) + 2(d - 2)\kappa^2 + 4(\mathbf{n} \cdot \kappa)^2, \]  

\[ Q_2(\kappa) = 1 - d + 2(d - 1)(\mathbf{n} \cdot \kappa) - (d - 3)\kappa^2 - 2\kappa^2(\mathbf{n} \cdot \kappa), \]  

\[ U(\kappa, \varphi) = [\kappa^2 - (\mathbf{n} \cdot \kappa)^2]/\kappa^2 q^2(1 + \kappa^2 + q^2). \]  

(5.7)
In Eqs. (5.6) and (5.7) we use only dimensionless variables, namely, the momenta divided by the modulus $p$ of the external momentum; $\mathbf{n} \equiv \mathbf{p}/p$ is the direction of the external momentum, $\kappa \equiv \mathbf{k}/p$ is the momentum flowing via the $\langle \varphi \varphi' \rangle_0$ line for $R_3$, the $\langle \varphi \varphi' \rangle_0$ line for $R_2$ and any one of the two equivalent $\langle \varphi \varphi \rangle_0$ lines for $R_1$, and $\mathbf{q} = \mathbf{n} - \kappa$ is the momentum flowing via the remaining line. In order to isolate the scalar factor $G(p)$ we contracted the full expression \[\bar{S}(\tau, \kappa)\] with the transverse projector $P_\perp(p)$ and divided the result by its trace; this explains the origin of the factor $P_\perp(p) = (d - 1)$ in the denominator.

It is easily checked that $Q_1(\kappa) = Q_1(\mathbf{q})$ and $\kappa^2 - (\mathbf{n} \cdot \kappa)^2 = g^2 - (\mathbf{n} \cdot \mathbf{q})^2$. From the last relation it follows that $U(\kappa, \mathbf{q}) = U(\mathbf{q}, \kappa)$ and therefore $R_2 = R_3$. Another consequence of the last relation is that $U(\kappa, \mathbf{q})$ remains finite at $\kappa \to 0$ and $\mathbf{q} \to 0$, so that the IR convergence of the integrals (5.6) at $\kappa \to 0$ and $\mathbf{q} \to 0$ are determined by the factors $\kappa^{2-d-2\varepsilon}$ and $q^{2-d-2\varepsilon}$, respectively. The IR divergences in all three integrals (5.6) arise for $\varepsilon \geq 1$, but in their sum they cancel each other: the singularities at $\kappa = 0$ (when $q^2 = 1$) in the integrands (5.6a) and (5.6c) cancel each other due to the relation

$$Q_1(\kappa)|_{\kappa=0} = (d - 1) = -Q_2(\kappa)|_{\kappa=0},$$

while the singularities at $\mathbf{q} = 0$ (when $\kappa^2 = 1$) in the integrands (5.6a) and (5.6c) cancel out due to the relation

$$Q_1(\kappa)|_{\mathbf{q}=0} = Q_1(\mathbf{q})|_{\mathbf{q}=0} = (d - 1) = -Q_2(\mathbf{q})|_{\mathbf{q}=0},$$

where we have used the identity $Q_1(\kappa) = Q_1(\mathbf{q})$. Thus the IR divergences that appear in the individual diagrams $R_{1,2,3}$ at $\varepsilon = 1$ cancel each other in their sum, and the latter becomes IR divergent only for $\varepsilon = 2$. These general facts will be illustrated by explicit formulas for the limit $d \to \infty$ in Sec. VI.

Formulas (5.6) are written for the “massless” model (2.3) that is, $h = 1$ in Eq. (2.9). In the “massive” model with $h \neq 1$ additional factors $h(u/\kappa)h(u/\mathbf{q})$, $h(u/\kappa)h(u)$, and $h(u/\mathbf{q})h(u)$ appear in the integrands (5.6a), (5.6c), and (5.6d), respectively. Then the divergence of the integrals (5.6) for $\varepsilon \geq 1$ is manifest in the form of the contributions $m^{2-2\varepsilon}$, divergent as $m \to 0$ for $\varepsilon \geq 1$.

As the independent variables in the integrands in (5.6) one can take the modulus $\kappa$ and $z \equiv (\mathbf{n} \cdot \kappa)/\kappa$, the cosine between the directions of the momenta $\mathbf{p}$ and $\mathbf{k}$. Then the integrals over $\mathbf{k}$ in (5.6) in the spherical coordinates are written as

$$\int d\kappa \ldots = S_0 \int_0^{\infty} d\kappa \kappa^{d-1} \langle \ldots \rangle = S_{d-1} \int_0^{\infty} d\kappa \kappa^{d-1} \int_{-1}^{1} dz (1 - z^2)^{(d-3)/2} \ldots$$

with $S_0$ from Eq. (2.7). Like in Eq. (4.4), the brackets denote the averaging over the unit sphere in $d$ dimensions, while the second relation is obtained by the integration over all angles in the $(d - 1)$-dimensional subspace orthogonal to the vector $\mathbf{p}$.

Now we turn to the calculation of the coefficient $c_2 = c_2|_{\varepsilon=0}$ in (5.3). The quantity $R_1$ is UV finite and it is sufficient to calculate its contribution at $\varepsilon = 0$. We put $\varepsilon = 0$ in (5.6a) and rewrite the integral in variables $z, \kappa$. This gives:

$$R_1|_{\varepsilon=0} \equiv g\bar{S}_d D = \frac{gS_{d-1}}{16(d-1)(2\pi)^d} \int_0^{\infty} d\kappa \int_{-1}^{1} dz (1 - z^2)^{(d-1)/2} \frac{\kappa(d - 1 - 4\kappa^2 + 2d\kappa^2 - 2d\kappa - 2z\kappa + 4z^2\kappa^2)}{(1 - 2z\kappa + \kappa^2)(1 - z\kappa + \kappa^2)^2}. \quad (5.9)$$

The integrals $R_2 = R_3$ contain a pole in $\varepsilon$ (manifestation of the UV divergence in the 1-irreducible function $\langle \varphi' \varphi \rangle$; see Sec. IV) and can be written as

$$R_2 = R_3 = g\bar{S}_d [A/\varepsilon + B + O(\varepsilon)]. \quad (5.10)$$

Expression (5.6b) can be written in the form

$$R_2 = \frac{gS_{d-1}}{16(d-1)(2\pi)^d} \int_0^{\infty} d\kappa \frac{\int_{-1}^{1} dz (1 - z^2)^{(d-1)/2} \psi(z, \kappa)}{\kappa^{1+2\varepsilon}}, \quad (5.11)$$

where

$$\psi(z, \kappa) \equiv \frac{(1 - d + 3z^2 - 6d\kappa^2 - 2d\kappa - 2z\kappa - 2z\kappa^2)\kappa^2}{(1 - 2z\kappa + \kappa^2)(1 - z\kappa + \kappa^2)}. \quad (5.12)$$

The integration region over $\kappa$ can be split in two parts: $[0,1]$ and $[1, +\infty)$; the pole in $\varepsilon$ comes only from the second part. The expansion of $\psi(z, \kappa)$ at large $\kappa$ has the form $\psi(z, \kappa) = -2z\kappa + (3 - d - 6z^2) + O(1/\kappa)$. The first term is odd in $z$ and vanishes after the integration over $z$; the second term completely determines the residue at the pole:
where the brackets denote the averaging over the unit sphere in $d$ dimensions and Eqs. (4.4) and (5.8) have been used. One can see that in representation (4.3), the total part 2$A/\varepsilon$ of the diagrams $R_{2,3}$ and the pole that comes from expression (5.4) cancel each other (a consequence of the renormalizability). Thus the total contribution of the diagrams $R_{1,2,3}$ into Eq. (5.3) has the form

$$c_2 \equiv c_2|_{\varepsilon=0} = 2B + D$$

(5.14)

with $B$ from (5.10) and $D$ from (5.9), and for the first two terms of the $\varepsilon$ expansion of the amplitude $R(1,0,g_\ast)$ using (4.4) and (5.3) we obtain

$$R(1,0,g_\ast) = (1/2) \left[ 1 + 2\alpha c_2 \varepsilon + O(\varepsilon^2) \right],$$

(5.15)

with $\alpha = (d+2)/(d-1)$ from (4.4).

The quantity $B$ can be found as follows. The integrals that remain in (5.11) after the pole part has been subtracted (that is, the integral over $[0,1]$ and the integral over $[1,\infty]$ with the substitution $\psi(z,\kappa) \to \psi(z,\kappa)+2z\kappa-(3-d-6\varepsilon^2)$ in the integrand) are UV finite and we may set $\varepsilon = 0$ in them. This gives:

$$\frac{16(d-1)S_d}{S_{d-1}} B = \int_0^1 \frac{d\kappa}{\kappa} \int_{-1}^1 dz \left( 1-z^2 \right)^{(d-1)/2} \psi(z,\kappa) + \int_{1}^\infty \frac{d\kappa}{\kappa} \int_{-1}^1 dz \left( 1-z^2 \right)^{(d-1)/2} \left[ \psi(z,\kappa)+2z\kappa-(3-d-6\varepsilon^2) \right].$$

(5.16)

The integrals (5.9) and (5.10) converge and can be evaluated numerically for any given value of $d$. For $d=3$ one obtains:

$$B = -0.000057, \quad D = 0.06699, \quad c_2 = 0.0669.$$  

(5.17)

The quantities (5.17) for other values of $d$ are given in Table I; we shall discuss them later in Sec. VI.

**VI. FINITE $\varepsilon$: NONLOCAL INTERACTIONS, INFRARED SINGULARITIES, AND GALILEAN SYMMETRY**

In a number of studies, the results of the RG approach to model (2.1), (2.2), (2.9) were interpreted and criticized in the language traditional for the classical theory of turbulence, comparison with the well-known direct interaction approximation (DIA) was made and the validity of the $\varepsilon$ expansion for finite $\varepsilon \sim 1$ was called into question; see e.g. (11, 13).

Let us discuss the problem on the example of the pair correlation function in the one-loop approximation (1.3). The corresponding analytical expressions (5.6) involve three vectors $p, k, q$ subject to the restriction $p = k + q$ (here, the momenta are not divided by $p = |p|$). In the following, the external momentum $p$ will be taken to lie in the inertial range. Then relevant contributions to (5.6), roughly speaking, can come from the three regions: $k \sim q \gg p$, $p \sim q \gg k$, where $k$ is the momentum flowing via the $\langle \phi \phi \rangle_0$ line (any one of the two equivalent $\langle \phi \phi \rangle_0$ lines for the diagram $R_1$), and $p \sim k \sim q$ (all three momenta lie in the inertial range). We shall refer to the corresponding contributions as being determined by UV-nonlocal, IR-nonlocal, and local (in the momentum space) interactions, respectively.

The typical argument against the use of the $\varepsilon$ expansion for finite $\varepsilon$ can be formulated as follows. The diagrams $R_{2,3}$ have poles in $\varepsilon$ as a manifestation of the UV divergence at $\varepsilon = 0$. Thus for small $\varepsilon$ the leading contribution to those diagrams is determined by UV-nonlocal interactions. For $\varepsilon \gtrsim \varepsilon_c$ with certain $\varepsilon_c = O(1)$ the diagrams become IR divergent and do not exist without the IR cutoff $m$ [ $\varepsilon_c = 1$ for all diagrams $R_{1,2,3}$ in (5.3)]. In the “massless” model (2.3) with $m = 0$ they have the form of the poles in $\varepsilon_c - \varepsilon$. Physically such IR divergences are explained by the so-called sweeping effects, that is, the transport of small eddies as a whole by large ones. This purely kinematic phenomenon has no effect on shaping the energy flow over the spectrum and is irrelevant in determining the exponents and amplitudes in power laws for Galilean invariant quantities (e.g. equal-time structure functions). According to the classical phenomenology the latter are formed by the interactions of the eddies in the inertial range, that is, by local interactions (in the above sense); see e.g. (4). One may thus believe that the extrapolation of the $\varepsilon$ expansion
The problem (2.1), (2.2), (2.9) is Galilean invariant for any perturbation expansion. The use of the RG and OPE implies some infinite resummations of the original perturbation series within controlled approximations and therefore does not violate the Galilean symmetry of the original DIA equations. Although the \( \varepsilon \) expansion fails to reproduce this solution, no definitive conclusion can be made at this point: for the different-time pair correlation function, neither of the aforementioned solutions is correct for \( \varepsilon \geq \varepsilon_c \).

Indeed, it has long been realized that the DIA misrepresents the contributions of the IR-nonlocal interactions, at least in the energy spectrum (it is determined by the equal-time pair correlation function and is Galilean invariant); see Ref. 29. Although the original Dyson–Wyd equations for the correlation and response functions are exact, they involve infinite series of skeleton diagrams; the practical solution implies a truncation (the one-loop truncation gives the DIA) followed by the substitution of a scaling Ansatz. As a result of the approximation, the Galilean symmetry is violated, and the IR singularities, related to the sweeping effects, do not cancel out in Galilean invariant quantities like the energy spectrum. This leads to spurious strong dependence of the latter on the IR scale and erroneous value (3/2 instead of 5/3) for the corresponding exponent. No closed equation, however, can be written for the equal-time correlation function, and the problem cannot be solved by taking into account more diagrams.

The key difference between the RG and the self-consistency approaches is that the former is based on the ordinary perturbation theory, that is, expansion in the nonlinearity in (2.1), which involves a formal expansion parameter \( g_0 \). The problem (2.1), (2.2), (2.9) is Galilean invariant for any \( g_0, \varepsilon \) and \( d \), so that the perturbation theory in \( g_0 \) is manifestly Galilean invariant in any given order for any value of \( \varepsilon \) and \( d \); this is equally true for the renormalized perturbation theory (in \( g \)). The use of the RG and OPE implies some infinite resummations of the original perturbation series within controlled approximations and therefore does not violate the Galilean symmetry of the original perturbation expansion.

The plain \( \varepsilon \) expansion is indeed not suitable for the description of the IR singularities related to the sweeping. Even in the standard \( \phi^4 \) model of critical behavior, where the IR singularities are weaker, \( \varepsilon \) expansions of the form (2.16) cannot be used for the analysis of the asymptotical behavior at \( m \to 0 \). As already discussed in the end of Sec. II, this fact does not hinder the use of the RG technique, which should be combined with the operator product expansion to derive resummed representations of the form (3.17); see e.g. 3.1. The distinguishing feature of our model (2.1), (2.2), (2.9) is the existence in Eq. (3.17) of negative dimensions \( \Delta_F < 0 \). The summation of the most singular contributions coming from the operators \( \varphi^n \) was performed in Ref. 29, the generalization to the case of a time-dependent large-scale field is given in 28. This gives the adequate description of the sweeping effects within the RG formalism; see also Refs. 30.

Admittedly, negative dimensions can be reliably identified, and their contributions can be summed up, only with the aid of nonperturbative methods that allow one to go beyond the plain \( \varepsilon \) expansion: functional Schwinger equations, Ward identities that express Galilean symmetry, and infrared perturbation theory. These techniques are discussed in Refs. 31 in detail. At the moment, however, we are interested in the equal-time pair correlation function (1.3).

As already mentioned in the end of Sec. II, the operators \( \varphi^n \), as well as other noninvariant operators, give no contribution to the corresponding representation (3.17) in agreement with the fact that Galilean invariant quantities are not affected by the sweeping. Only the dimensions of Galilean invariant operators can appear in (3.17), and the singular behavior can be related to invariant operators with \( \Delta_F < 0 \). No such operator, however, has been presented in any study we know of, and for the time being we can assume that all dimensions \( \Delta_F \) are non-negative. This fact suggests that the scaling function \( R(1, u, g_\ast) \) (5.3) in Sec. IV is finite at \( u = 0 \) and the value of \( R(1, 0, g_\ast) \) can be calculated within the framework of the “massless” model (2.3) and the \( \varepsilon \) expansion.

The cancellation of the IR divergences for \( 1 < \varepsilon < 2 \) in the sum of diagrams (5.6) was demonstrated in Sec. V for all \( d \). Below we consider them in the limit \( d \to \infty \), where explicit expressions for the integrals entering Eq. (5.6)

\[1\] The DIA is not a given-order approximation to the exact Dyson–Wyd equations in any small, at least formal, expansion parameter. It can be understood as the leading approximation for \( N \to \infty \) in certain extension of the original NS equation to the case of \( N \) interacting velocity fields 32, but such an extension is not Galilean invariant unless \( N = 1 \).
can be written for all $0 < \varepsilon < 2$ (in Sec. V, only two terms of the $\varepsilon$ expansion were calculated).

In the diagrams $R_2 = R_3$ nontrivial dependence on $d$ comes only from the angular averaging. It follows from Eqs. (3.4) that for $d \to \infty$, the angular averages behave as $(z^{2n}) \propto d^{-n}$. Thus in order to find the leading term of the large-$d$ behavior, it is sufficient to discard all such averages with $n > 1$, or, equivalently, to put $z = (n \cdot k)/k = 0$ in the integrand (6.6a). In the variables $\kappa$, $\varepsilon$ introduced above Eq. (5.8) this leads to the replacements $(1 - \varepsilon^2) \to 1$, $q^2 \to (1 + \kappa^2)$, $1 + \kappa^2 + q^2 \to 2(1 + \kappa^2)$, which along with the asymptotic relation $Q_2 = -d(1 + \kappa^2) + O(d^0)$ for the quantity $Q_2$ in (5.7) gives:

$$R_2 + R_3 = 2R_2 \simeq \frac{-g}{8(2\pi)^d} \int d\kappa \frac{\kappa^2 - 2\varepsilon}{(1 + \kappa^2)} = \frac{-g}{8} \int_0^\infty d\kappa \frac{\kappa^1 - 2\varepsilon}{(1 + \kappa^2)} = \frac{-gS_d}{16\varepsilon} \Gamma(1 + \varepsilon) \Gamma(1 - \varepsilon). \tag{6.1}$$

The latter equality in (6.1) holds for $\varepsilon < 1$; for $\varepsilon \geq 1$ the integral over $\kappa$ diverges at small $\kappa$.

In the integrand (6.6a) for diagram $R_1$, nontrivial dependence on $d$ comes also from the factor $q^{-d}$ and the procedure described above cannot be applied. In the variables $\kappa$, $\varepsilon$ expression (5.6a) takes on the form

$$R_1 = \frac{gS_{d-1}}{8(d - 1)(2\pi)^d} \int_0^\infty d\kappa \int_{-1}^{1} dz \frac{(1 - z^2)(d - 1)/2 Q_1 \kappa^1 - 2\varepsilon q^{-d - 2\varepsilon}}{(1 + \kappa^2 + q^2)} \tag{6.2}$$

with $q^2 = 1 + \kappa^2 - 2\varepsilon$ and $Q_1$ from (5.7). For large $d$ one can write

$$(1 - z^2)^{d/2} q^{-d} = (1 - z^2)^{d/2} (1 + \kappa^2 - 2\varepsilon) \to (1 + \kappa^2 - 2\varepsilon)^{-d/2} = \left\{ 1 + \frac{(\kappa - z)^2}{(1 - z^2)} \right\}^{-d/2} \simeq \exp \left\{ -\frac{d}{2} \frac{(\kappa - z)^2}{(1 - z^2)} \right\}.$$

Along with the relation $S_{d - 1}/S_d \simeq \sqrt{d/2\pi} [1 + O(1/d)]$, which follows from the Stirling formula for the $\Gamma$ function that enters Eq. (2.7), this gives:

$$(1 - z^2)^{(d - 1)/2} q^{-d} S_{d - 1} \simeq S_d \delta(\kappa - z).$$

We substitute this relation into (5.2), retain only intervals $[0, 1]$ in the integrals, use the relation $Q_1|_{k = z} = d + O(d^0)$ for the quantity $Q_1$ in (5.7) and perform the integration over any one of the variables; this gives:

$$R_1 \simeq \frac{gS_d}{8(d - 1)} \int_0^1 d\kappa \int_0^1 dz Q_1 \frac{\kappa^1 - 2\varepsilon q^{-2\varepsilon}}{(1 + \kappa^2 + q^2)} \Delta(\kappa - z) = \frac{gS_d}{16} \int_0^1 d\kappa \frac{\kappa^1 - 2\varepsilon}{(1 - \kappa^2)^2} = \frac{-gS_d}{32} \frac{\Gamma^2(1 - \varepsilon)}{\Gamma(2 - 2\varepsilon)} \tag{6.3}.$$

Like for the integral (6.1), the latter equality holds for $\varepsilon < 1$; for $\varepsilon \geq 1$ the integral over $\kappa$ diverges at small $\kappa \to 0$ and $\kappa \to 1$. Adding together the expressions (6.1), (6.3) with the loopless contribution (5.4), where the limit $d \to \infty$ of the explicit form (5.6) should be used for the coefficient $a_{11}$, gives:

$$R(1, 0, g) = \frac{1}{2} + \frac{gS_d}{32} R(\varepsilon), \quad R(\varepsilon) = \left\{ \begin{array}{ll} \frac{2}{\varepsilon} - \frac{2}{\varepsilon} \frac{\Gamma(1 - \varepsilon) \Gamma(1 + \varepsilon)}{\Gamma(2 - 2\varepsilon)} & \\
\frac{g^2(1 - \varepsilon)}{\Gamma(2 - 2\varepsilon)} & \end{array} \right\} \tag{6.4}$$

(the product $gS_d$ at the fixed point has a finite limit for $d = \infty$; see Eq. (3.10)). The sum of the integrals (6.1), (6.3) converges for all $0 < \varepsilon < 2$: the divergences in the integrals (6.1) and (6.3) at $\kappa \to 0$ cancel each other, while the cancellation of the divergence in the integral (6.3) at $\kappa \to 1$ with the divergence in the diagram $R_3$ becomes obvious after the replacement $\kappa \to 1 - \kappa$ in expression (6.1) for $R_2$. Thus the expression (6.4) for the sum of the integrals holds for the whole interval $0 < \varepsilon < 2$.

In Eq. (6.4), the cancellation of the poles at $\varepsilon = 0$ (a consequence of the renormalizability) and $\varepsilon = 1$ (a consequence of the Galilean symmetry) are obvious; the renormalizability and the Galilean invariance of the model guarantee similar cancellations in all the higher orders of the (renormalized) perturbation theory.

In models of quantum field theory, the elimination of UV divergences is needed to ensure existence of meaningful (finite) results at the physical value $\varepsilon = 0$. In our case it is not necessary to take the limit $\varepsilon \to 0$ and ensure that it exists, that is, there is no need to renormalize the model. However, it turns out that the possibility itself of doing this implicitly contains valuable information about the original model with finite $\varepsilon$: it is this possibility that ensure the validity of the RG equations (2.7) and the operator product expansion (3.17).

Physically, elimination of the UV singularities (poles in $\varepsilon$) removes the contributions of the UV-nonlocal interactions that would obscure relevant local interactions. Coefficients of the renormalized perturbation series for the correlation functions have a finite limit at $\varepsilon \to 0$: they are determined by the local interactions, while the UV-nonlocal interactions are taken into account in the formulas of multiplicative renormalization (3.4). A fine correlation that exists between
different orders of the perturbation theory is expressed by the RG equations (3.4); solving them allows one to perform certain infinite resummation of the perturbation series for (renormalized) correlation functions.

Expression (3.4) contains a pole at \( \varepsilon \to 2 \): \( R(\varepsilon) \propto 6/(2 - \varepsilon) \) for general \( d \) we would obtain \( R(\varepsilon) \simeq 2(3d^2 + d - 12)/(d(d + 2)(2 - \varepsilon)) \); higher-order poles will necessarily appear in the higher orders of the perturbation theory. Such singularities are necessary to ensure the existence of the finite limit for the pair correlation function at \( \varepsilon \to 2 \) in terms of the physical parameter \( \overline{\varepsilon} \) from (2.1). For the correlation function of \( n \) fields \( \varphi \), they must combine into the singularity \( (2 - \varepsilon)^{-n/3} \) that will cancel the vanishing factor \( D_0^{n/3} \propto \overline{\varepsilon}^{n/3}(2 - \varepsilon)^{n/3} \) in representation (3.13); see Eq. (2.8). The conjecture that the singularities at \( \varepsilon \to 2 \) indeed combine into the needed expressions is confirmed by two examples: the pair correlation function in the exactly solvable Heisenberg model \([26]\) and the triple correlation function in our case; see the exact expression (7.11) in Sec. VII.

VII. TWO-LOOP CALCULATION OF THE KOLMOGOROV CONSTANT

The Kolmogorov constant \( C_K \) can be defined as the (dimensionless) coefficient in the inertial-range expression \( S_2(r) = C_K(\overline{\varepsilon}r)^{2/3} \) for the second-order structure function, predicted by the Kolmogorov–Obukhov theory and confirmed by experiment \([27]\). Here \( \overline{\varepsilon} \) is the mean energy dissipation rate and the \( n \)-th order (longitudinal, equal-time) structure function is defined as

\[
S_n(r) \equiv \langle [\varphi_r(t, \mathbf{x} + \mathbf{r}) - \varphi_r(t, \mathbf{x})]^n \rangle, \quad \varphi_r \equiv (\varphi_i \cdot \mathbf{r}_i)/r, \quad r \equiv |\mathbf{r}|. \tag{7.1}
\]

Alternatively, the Kolmogorov constant \( C'_K \) can be introduced via the relation \( E(k) = C'_K(\overline{\varepsilon}r)^{2/3}k^{-5/3} \), where the energy spectrum \( E(k) \) is related to the equal-time correlation function (5.1) as \( E(k) = \overline{\varepsilon}(d-1)k^{d-1}G(k)/2 \). From the definitions one can derive the following relation between these two constants:

\[
C_K = \frac{3 \cdot 2^{1/3}\Gamma(2/3)\Gamma(d/2)}{(d + 2/3)\Gamma(d/2 + 1/3)} C'_K, \tag{7.2}
\]

which for \( d = 3 \) gives (cf. 1)

\[
C_K = \frac{9 \cdot 2^{1/3}\sqrt{\pi}\Gamma(2/3)}{22\Gamma(11/6)} C'_K = \frac{27}{55}\Gamma(1/3)C'_K. \tag{7.3}
\]

Using the exact relation \( S_2(r) = -12\overline{\varepsilon}r/(d + 2) \) that follows from the energy balance equation (see e.g. 12 for \( d = 3 \)), the constant \( C_K \) can be related to the inertial-range skewness factor \( S \):

\[
S \equiv S_3/S_2^{3/2} = -[12/(d + 2)]C_K^{-3/2}. \tag{7.4}
\]

All these relations refer to real physical quantities in the inertial range, which in the stochastic model (2.2), (2.2) corresponds to \( \varepsilon = 2 \) and \( m = 0 \) in the random force correlator (2.4).

Many studies have been devoted to the derivation of \( C_K \) within the framework of the RG approach; see Refs. 1-14, 21, 23, 27. In order to obtain \( C_K \), it is necessary to augment the solution of the RG equation for \( S_2 \) by some formula that relates the amplitude \( D_0 \) in the random force correlator (2.4) to the physical parameter \( \overline{\varepsilon} \). In particular, in the first-order term of the \( \varepsilon \) expansion for the pair correlator was combined with the so-called eddy-damped quasinormal Markovian approximation for the energy transfer function, taken at \( \varepsilon = 2 \). More elementary derivation, based on the exact relation (2.4) between \( \overline{\varepsilon} \) and the function \( d_f(k) \) from (2.8) was given in 25; see also 21, 23. In spite of the reasonable agreement with the experiment, such derivations are not immaculate from the theoretical viewpoints. Their common flaw is that the relation between \( \overline{\varepsilon} \) and \( D_0 \) is unambiguous only in the limit \( \varepsilon \to 2 \) (see Eq. (2.8) in Sec. II), so that the coefficients of the corresponding \( \varepsilon \) expansions can, in fact, be made arbitrary; see the discussion in Ref. 13 and Sec. 2.10 of 1. The ambiguity is a consequence of the fact that the notion itself of the Kolmogorov constant has no unique extension to the nonphysical range \( \varepsilon < 2 \).

The experience on the RG theory of critical behavior shows that well-defined \( \varepsilon \) expansions can be written for universal quantities, such as critical exponents, normalized scaling functions and ratios of amplitudes in scaling laws 21, 24. The constant \( C_K \) extended to the range \( \varepsilon < 2 \) as in 24, 25 involves a bare parameter, \( D_0 \), and hence is not universal in the above sense.

To circumvent this difficulty, we propose below an alternative derivation that relates \( C_K \) to an universal quantity and thus does not involve any relation between \( D_0 \) and \( \overline{\varepsilon} \). Consider the ratio
\[ Q(\varepsilon) \equiv \mathcal{D}_r S_2(r)/|S_3(r)|^{2/3} = \mathcal{D}_r S_2(r)/(-S_3(r))^{2/3} \] (7.5)

with \( \mathcal{D}_r \equiv r \partial / \partial r \). As we shall see below, the quantity \( Q(\varepsilon) \) is universal and can be calculated in the form of a well-defined \( \varepsilon \) expansion. On the other hand, its value at \( \varepsilon = 2 \) determines the Kolmogorov constant and the skewness factor through the exact relations

\[ C_K = [3Q(2)/2]^{12/(d+2)}^{2/3}, \quad S = -[3Q(2)/2]^{-3/2} \] (7.6)

that follow from the definitions, relation (7.4) and the identity \( \mathcal{D}_r r^\alpha = \alpha r^\alpha \) for any \( \alpha \).

Solving the RG equations for the quantities in \( Q(\varepsilon) \) in the inertial range \( (\Lambda r \to \infty \text{ and } mr \to 0) \) for general \( 0 < \varepsilon < 2 \) gives (see Sec. III):

\[ S_3(r) = D_0 r^{-3\Delta \varepsilon} f_3(\varepsilon), \quad \mathcal{D}_r S_2(r) = D_0^{2/3} r^{-2\Delta \varepsilon} f_2(\varepsilon), \quad \Delta \varepsilon = 1 - 2\varepsilon / 3. \] (7.7)

It is important here that the operation \( \mathcal{D}_r \) kills the constant contribution \( \langle \varphi^2 \rangle \) in \( S_2 \) that diverges as \( \Lambda \to \infty \) for \( \varepsilon < 3/2 \) (see below); in \( S_3 \) such constant contributions are absent.

It follows from Eq. (7.7) that the ratio \( Q(\varepsilon) = f_2(-f_3)^{2/3} \), in contrast to its numerator and denominator, does not depend on the amplitude \( D_0 \), it is universal (in the above sense) and can be calculated in the form of a well-defined \( \varepsilon \) expansion. The latter has the form \( Q(\varepsilon) = \varepsilon^{1/3} p(\varepsilon) \), where \( p(\varepsilon) \) is a power series in \( \varepsilon \) (see below). We shall find the first two terms of \( p(\varepsilon) \), which corresponds to the two-loop accuracy in representations (1.23) and (5.15). In this sense, one can speak about the two-loop approximation for the Kolmogorov constant (previous attempts have been confined with the first order).

To avoid possible misunderstandings, we stress again that we shall not try to extend the definition of the physical quantities \( C_K \) and \( S \) to the whole interval \( 0 < \varepsilon < 2 \) and to construct the corresponding \( \varepsilon \) expansions from the known expansion for \( Q(\varepsilon) \). Instead, the latter is used to give the value of \( Q(2) \), which, in its turn, determines \( C_K \) and \( S \) through the relations (7.6) that make sense only for \( \varepsilon = 2 \).

Using the definition (7.1), the function \( S_2 \) can be related to the momentum-space pair correlation function (5.1) as follows:

\[ S_2(r) = 2 \int \frac{dk}{(2\pi)^d} G(k) \left[ 1 - (k \cdot r)^2/(kr)^2 \right] \left\{ 1 - \exp \{i(k \cdot r)\} \right\}. \] (7.8)

Applying the operation \( \mathcal{D}_r \equiv r \partial / \partial r \) gives:

\[ \mathcal{D}_r S_2(r) = 2 \int \frac{dk}{(2\pi)^d} G(k) \left[ 1 - (k \cdot r)^2/(kr)^2 \right] (k \cdot r) \sin(k \cdot r). \] (7.9)

In order to obtain the inertial-range form of the function \( \mathcal{D}_r S_2(r) \) it is sufficient to substitute the asymptotic expression (7.2) into (7.3). A straightforward calculation gives:

\[ \mathcal{D}_r S_2(r) = \frac{2(d-1)\Gamma(2-2\varepsilon/3)}{(4\pi)^{d/2} \Gamma(d/2 + 2\varepsilon/3)} g_{4s}^{1/3} R(1, 0, g_s) D_0^{2/3} (r/2)^{-2\Delta \varphi} \] (7.10)

with the amplitude \( R(1, 0, g_s) \) from (5.2). It is important here that the resulting integral exists for all \( 0 < \varepsilon < 2 \). It is the operation \( \mathcal{D}_r \) that ensures the convergence of the integral (7.9) with the function (7.2): the original integral (7.8) would be UV divergent for \( 0 < \varepsilon < 3/2 \) while the analogous expression for the pair correlator (that is, the plain Fourier transform of (7.2)) would be IR divergent for \( \varepsilon > 3/2 \).

The needed terms of the \( \varepsilon \) expansion for \( f_3 \) can be obtained not only from the direct perturbative calculation, but also from the exact expression

\[ S_3(r) = -\frac{3(d-1)\Gamma(2-\varepsilon)}{(4\pi)^{d/2} \Gamma(d/2 + \varepsilon)} D_0 (r/2)^{-3\Delta \varphi} \] (7.11)

that follows from the energy balance equation and in the limit \( \varepsilon \to 2 \), along with the formula (7.8), reproduces the correct coefficient \(-12/(d+2)\).

The pole \( \sim 1/(2 - \varepsilon) \) cancels the vanishing factor \( \sim (2 - \varepsilon) \) in (7.8) and ensures the finiteness of \( S_3 \) at \( \varepsilon = 2 \) when expressed in terms of the physical parameter \( \varepsilon \). In the ratio (7.3) this pole must cancel with the (hypothesized) singularity in the pair correlator: see the discussion in the end of Sec. VI.

Thus from Eqs. (7.10) and (7.11) for the ratio \( Q(\varepsilon) \) one obtains
\[ Q(\varepsilon) = \left[4(d-1)g_*S_d/9\right]^{1/3} A(\varepsilon, d)R(1, 0, g_*), \]  
(7.12)

where the coefficient
\[ A(\varepsilon, d) \equiv \frac{\Gamma(2-2\varepsilon/3)}{\Gamma(2-2\varepsilon/3)\Gamma(2+\varepsilon/3)} = 1 + O(\varepsilon^2) \]  
(7.13)

has no term of order \( O(\varepsilon) \) and in our approximation can be replaced by unity. Substituting the two-loop result \((4.23)\) for the coordinate of the fixed point \(g_*\) and the first two terms \((7.15)\) of the \(\varepsilon\) expansion of the amplitude \(R(1, 0, g_*\) into \((7.12)\) gives
\[ Q(\varepsilon) = (1/3)[4\varepsilon(d+2)]^{1/3} \left[1 + \varepsilon (\lambda/3 + 2\alpha c_2) + O(\varepsilon^2)\right] \]  
(7.14)

with \(\alpha\) and \(\lambda\) from Eq. \((8.23)\) and \(c_2\) from \((5.14)\). For \(d = 3\) this gives
\[ Q(\varepsilon) = (1/3)(20\varepsilon)^{1/3} \left[1 + 0.525\varepsilon + O(\varepsilon^2)\right]. \]  
(7.15)

The value of \(Q(\varepsilon)\) at \(\varepsilon = 2\) determines the Kolmogorov constant and skewness factor through the exact relations \((7.6)\), which for \(d = 3\) have the form \(C_K = 6 \cdot 10^{-2/3} Q(2)\) and \(S = -[1.5 \cdot Q(2)]^{-3/2}\). Let us denote as \(C^{(0)}_K\) and \(S^{(0)}\) the results obtained using the \(n\)-loop approximation for the RG functions and the corresponding approximations for the scaling functions. Substituting the value of \((7.15)\) with \(\varepsilon = 2\) into these relations we obtain
\[ C^{(2)}_K = 3.02, \quad S^{(2)} = -0.15. \]

If we had retained only the first-order term in \((7.15)\) we would have obtained
\[ C^{(1)}_K = 1.47, \quad S^{(1)} = -0.45. \]

We also recall the experimental estimates recommended in \[11]\:
\[ C_K \approx 1.9, \quad S \approx -0.28; \]

it is worth noting that they lie in between of the first-order and second-order approximations.

Let us conclude this section with a brief discussion of the \(d\)-dimensional case. Results of the two-loop calculation for several values of \(d\), including the limits \(d \to \infty\) and \(d \to 2\), are given in Table \[11\]. It includes: the parameter \(\lambda\) that determines the second-order corrections to the coordinate of the fixed point \(g_*\) from Eq. \((4.23)\) and the exponent \(\omega\) from Eq. \((4.22)\), the parameter \(c_2 = c_2|_{\varepsilon=0}\) that determines the second-order correction to the scaling function of the equal-time pair correlator in Eq. \((5.15)\), the contributions \(B\) and \(D\) from the diagrams \(R_{2,3}\) and \(R_1\) that determine \(c_2\) via Eq. \((5.14)\), and the first-order and second-order approximations for the Kolmogorov constant \(C^{(n)}_K\) \((n = 1, 2)\).

One can see that, for \(d \to \infty\), the two-loop contributions become relatively smaller for all these quantities. This gives some quantitative support to the old idea that \(1/d\) can be a promising expansion parameter for the stochastic NS problem \[11\]. However, in contrast to the Kraichnan model, where the \(O(1/d)\) approximation for the anomalous exponents is a function linear in \(\varepsilon\) \[13\], in our case the second-order relative corrections have a finite limit at \(d \to \infty\), so that the \(O(1/d)\) approximation for \(g_*, \omega, Q(\varepsilon)\) are infinite series in \(\varepsilon\). On the other hand, like for the Kraichnan model, the calculation of the diagrams drastically simplifies for \(d \to \infty\): the leading terms can be calculated analytically. We believe that the three-loop calculation is a feasible task at \(d = \infty\); this work is now in progress.

For \(d \to \infty\), the constant \(C_K\) decreases as \(C_K \propto 1/d\); see Table \[11\]. From Eq. \((7.2)\) one then obtains \(C^{(n)}_K \propto d^{1/3}\) for the coefficient in the energy spectrum, in agreement with the earlier results obtained within the DIA \[10\] or the RG \[11\].

For \(d \to 2\), the two-loop contributions diverge; this is a manifestation of the additional UV divergence that emerges in the 1-irreducible function \(\langle \phi' \phi' \rangle\) at \(d = 2\). In this region our results cannot be trusted, and additional renormalization should be performed in order to remove the divergences at \(d = 2\); see e.g. Sec. 3.10 of Ref. \[10\] and references therein. This is necessary, in particular, in the discussion of possible crossovers in the inertial-range behavior that can occur between \(d = 3\) and \(2\); see Refs. \[11\]. It is worth noting that, for all \(d\), the main contribution to the coefficient \(c_2 = D + 2B\) in \((5.14)\) comes from the term \(D\), that is, from the diagram \(R_1\). What is more, the contribution from \(B\) totally vanishes at \(d = \infty\) while the contribution from \(D\) diverges at \(d = 2\). It thus might happen that the aforementioned additional renormalization is needed to obtain more accurate numerical predictions in three dimensions; we shall consider this important problem elsewhere.
We have accomplished the complete two-loop calculation of the renormalization constant and RG functions for the stochastic problem (2.1)–(2.9) and derived the coordinate of the fixed point, the UV correction exponent \( \omega \), the Kolmogorov constant \( C_K \) and the inertial-range skewness factor \( S \) to the second order of the corresponding \( \varepsilon \) expansions. The new point is not only the inclusion of the second-order correction, but also the derivation of \( C_K \) through the universal (in the sense of the theory of critical behavior) quantity (7.3).

Of course, one should have not expected that the second-order terms of the \( \varepsilon \) expansions would be small in comparison to the first-order terms. The experience on the RG theory of critical behavior shows that such corrections are not small for dynamical models (in contrast to static ones) and for amplitudes (in contrast to exponents); see \([4,3]\). It is thus rather surprising that in our case the account of the two-loop contributions leads to reasonable changes in the results.

Although the \( \varepsilon^2 \) correction to \( \omega \) in \([4,2,3]\) is rather large, it does not change its sign and hence does not destroy the IR stability of the fixed point for the real \( \varepsilon = 2 \).

The first-order approximation \( C^{(1)}_K = 1.47 \) underestimates, and the second-order approximation \( C^{(2)}_K = 3.02 \) overestimates the conventional experimental value of the Kolmogorov constant \( C_K \approx 1.9 \) \([3]\). Thus the experimental value of \( C_K \) (and hence for \( S \)) lies in between of the two consecutive approximations. A similar situation is encountered for the well-known Heisenberg model \([3]\), where the analog of the Kolmogorov constant is known exactly and lies between the first-order and second-order approximations given by the corresponding \( \varepsilon \) expansion \([2]\). If we assume, by the analogy with the Heisenberg model, that the (unknown) exact predictions for \( C_K \) and \( S \) lie between the first two approximations, we may conclude that our calculation has given satisfactory estimates for these quantities.

The two-loop corrections become relatively small for \( d \to \infty \), which confirms the relevance of the \( 1/d \) expansion for the issue of inertial-range scaling behavior.

One may thus conclude that our results confirm the applicability of the RG formalism and the \( \varepsilon \) expansion to the calculation of important characteristics of the real fluid turbulence. The next important steps should be the derivation of the two-loop results for the case of a passive scalar advected by stochastic NS equation; inclusion of the additional renormalization near two dimensions, and the three-loop calculation in the limit \( d \to \infty \); this work is now in progress.

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**APPENDIX:**

Below we give explicit expressions for the integrands \( f_0 – f_4 \) in the expressions \([4,8]\) for the two-loop diagrams of the 1-irreducible correlation function \( \langle \phi^2 \rangle \). They have the form

\[
\psi_i(z, x) = x^2(1 - z^2)^{(d-1)/2} \frac{S_d-1}{64(d-1)d(d+2)S_d} [\psi_i(z, x) + \psi_i(-z, x)]
\]

with \( S_d \) from \([2,7]\) and \( x \equiv k_1/k_2 \). Note that the functions \( f_i(z, x) \) are even in \( z \) [see the remark below Eq. \([4,8]\)]. The functions \( \psi_i \equiv \psi_i(z, x) \) in \( d \) dimensions have the forms (the function \( \psi_0 \) is even in \( z \), while the other functions \( \psi_i \) are not)

\[
\psi_0 = \{ (x^{20} + 1) \left[ (-3d^3 + 8d^2 - 7d - 16) + (-14d^2 + 100d + 40)z^2 - 120dz^4 \right] \\
+ (x^{18} + x^2) \left[ (-30d^3 + 76d^2 - 78d - 64) + (12d^3 + 24d^2 + 552d - 144)z^2 \\
+ (-184d^2 - 76d + 184)z^4 - 440dz^6 \right] \\
+ (x^{16} + x^4) \left[ (-135d^3 + 328d^2 - 395d + 48) + (96d^3 + 522d^2 + 1080d - 1272)z^2 \\
+ (63d^3 - 1796d^2 + 1495d + 1728)z^4 + (1054d^3 - 2984d + 576)z^6 + 816dz^8 \right] \\
+ (x^{14} + x^6) \left[ (-360d^3 + 848d^2 - 1160d + 768) + (336d^3 + 1952d^2 - 88d - 2752)z^2 \\
+ (294d^2 - 420d - 1568)z^4 - 672dz^6 \right] \}
\]
\[
\begin{align*}
&+ (378 d^3 - 7348 d^2 + 5918 d + 3136)z^4 + (-312 d^3 + 6124 d^2 - 5320 d - 1232)z^6 \\
&+ (-1432 d^2 - 208 d + 224)z^8 + 1248dz^{10} \\
&+ (x^2 + x^6)\left[-(630 d^3 + 1456d^2 - 2158d + 2016) + (672d^3 + 384d^2 - 3740d - 2864)z^2 \\
&+ (945d^3 - 16172d^2 + 10945d - 64)z^4 + (-1248d^3 + 15362d^2 - 2296d + 1408)z^6 \\
&+ (240d^3 - 5376d^2 - 4224d - 708)z^8 + (832d^3 + 1792d + 128)z^{10} - 640dz^{12}\right] \\
&+ 2z^{10}\left[-(378d^3 + 868d^2 - 1322d + 1344) + (420d^3 + 2376d^2 - 3024d - 1200)z^2 \\
&+ (630d^3 - 10436d^2 + 6478d - 1656)z^4 + (-936d^3 + 10292d^2 + 416d + 2064)z^6 \\
&+ (240d^3 - 3912d^2 - 3360d - 480)z^8 + (832d^3 + 544d)z^{10} + (-128d^2 + 256d)z^{12}\right] \\
&\bigg/ \left[(x^2 + 2x + 1)^3(x^2 + x + 1)^3(x^2 - x + 1)^3\right] ; \\
\psi_1 &= 2\left\{x^2z(-2d + 2d) + x^6[(d^3 - 4d^2 + d + 6) + (-6d^2 + 6d - 12)z^2] + x^7z[(5d^3 - 20d^2 + 7d - 4) \\
&+ (-4d^3 + 4d - 16)x^2] + x^4[(3d^3 - 10d^2 + 3d) + (8d^3 - 26d^2 + 10d - 28)x^2] \\
&+ x^5z[(10d^3 - 28d^2 + 10d - 20) + (4d^3 - 8d^2 + 4d)z^2] + x^6[(3d^3 - 8d^2 + 3d - 6) \\
&+ (8d^3 - 16d^2 + 8d)x^2] + xz(5d^3 + 10d^2 + 5d) + x^3 - 2d^2 \bigg/ \\
&\bigg/ \left[(x^2 + 2x + 1)^2(x^2 + x + 1)^2\right] ; \\
\psi_2 &= 2\left\{x^4(2d^2 - 2d + 2) + x^3z(-4d^2 + 4d - 4) + x^2[(5d^3 - 5d + 6) + (2d^2 - 2d + 2)z^2] \\
&+ xz(-5d^2 + 5d - 6) + 3d^2 - 3d + 6] [2x^3z + x^2(d - 3) + xz(-2d + 2) + d - 1] \\
&\bigg/ \left[(x^2 - 2x + 1)(x^2 + x + 1)^3\right] ; \\
\psi_3 &= 2d(d - 1) [2x^3z + x^2(d - 3) + xz(-2d + 2) + d - 1] / \left[(x^2 - 2x + 1)(x^2 - x + 1)^3\right] ; \\
\psi_4 &= \left\{ (x^4 + 1)(3d^2 - 3d + 2) + (x^4 + x)z(-7d^2 + 7d - 8) + 2x^2[(3d^2 - 3d + 2) + (2d^2 - 2d + 4)z^2] \\
&\bigg/ \left[(x^2 - 2x + 1)^3(x^2 - x + 1)^3\right] .
\end{align*}
\]

[1] A. S. Monin and A. M. Yaglom, *Statistical Fluid Mechanics* (MIT Press, Cambridge, Mass., 1975), Vol.2.
[2] U. Frisch, *Turbulence: The Legacy of A. N. Kolmogorov* (Cambridge University Press, Cambridge, 1995).
[3] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena* (Clarendon, Oxford, 1989).
[4] A. N. Vasil’ev, *Quantum-Field Renormalization Group in the Theory of Critical Phenomena and Stochastic Dynamics* (St Petersburg Institute of Nuclear Physics, St Petersburg, 1998) [in Russian; English translation: Gordon and Breach, in press].
[5] D. Forster, D. R. Nelson, and M. J. Stephen, Phys. Rev. Lett. 36, 867 (1976); Phys. Rev. A 16, 732 (1977).
[6] C. De Dominicis and P. C. Martin, Phys. Rev. A 19, 419 (1979); Progr. Theor. Phys. Suppl. No 64, 108 (1978).
[7] J. D. Fournier and U. Frisch, Phys. Rev. A 19, 1000 (1983).
[8] L. Ts. Adzhemyan, A. N. Vasil’ev, and Yu. M. Pis’mak, Theor. Math. Phys. 57, 1131 (1983).
[9] L. Ts. Adzhemyan, N. V. Antonov, and A. N. Vasil’ev, Usp. Fiz. Nauk, 166, 1257 (1996) [Phys. Usp. 39, 1193 (1996)].
[10] L. Ts. Adzhemyan, N. V. Antonov, and A. N. Vasiliev, *The Field Theoretic Renormalization Group in Full Developed Turbulence* (Gordon & Breach, London, 1999).
[11] R. H. Kraichnan, Phys. Fluids A 30, 2400 (1987).
[12] S. Chen and R. H. Kraichnan, Phys. Fluids A 1, 2019 (1989).
[13] É. V. Teodorovich, Izv. Akad. Nauk SSSR, Ser. Mekh. Zhidk. Gaza, No 4, 29 (1987) [in Russian]; Sov. Phys. Doklady 33, 247 (1988); Izv. Akad. Nauk, Fiz. Atmosf. Okeana, 29, 149 (1993) [in Russian].
[14] É. V. Teodorovich, *On the infrared divergences and the role of local and nonlocal interactions in forming developed turbulence*. Preprint No. 388, IPM, USSR Academy of Science (Moscow, 1989) [in Russian].
[15] S. L. Woodburn, Phys. Fluids A 4(5), 1077 (1992); Phys. Fluids A 6(9), 3051 (1994).
[16] R. H. Kraichnan, Phys. Fluids 11, 945 (1968).
[17] R. H. Kraichnan, Phys. Rev. Lett. 72, 1016 (1994); 78, 4922 (1997).
[18] K. Gawędzki and A. Kupiainen, Phys. Rev. Lett. 75, 3834 (1995); D. Bernard, K. Gawędzki, and A. Kupiainen, Phys. Rev. E 54, 2564 (1996).
TABLE I. Results of the two-loop calculation for different values of $d$.

| $d$  | $\lambda$ | $B$ | $D$ | $c_2$ | $C_2(1)$ | $C_2(2)$ |
|------|-----------|-----|-----|-------|----------|----------|
| 2 + 2$\delta$ | $-1/3\delta + O(\delta^0)$ | $-2.296$ | $-1.101$ | $-0.560$ | $0.0013$ | $0.000057$ |
| 2.5 | $O(\delta^0)$ | $0.0013$ | $-0.000057$ | $-0.00194$ | $0.098$ | $0.0669$ |
| 3 | $1/64\delta + O(\delta^0)$ | $0.0999$ | $0.06699$ | $0.0436$ | $1.35$ | $5.24/d + O(1/d^2)$ |
| 5 | $1/64\delta + O(\delta^0)$ | $0.103$ | $0.0669$ | $0.0397$ | $1.35$ | $5.24/d + O(1/d^2)$ |
| $\rightarrow \infty$ | $-1/3 + O(1/d)$ | $-0.560$ | $0.00194$ | $0.098$ | $0.0669$ | $0.0397$ |