EQUITY WARRANT PRICING UNDER SUBDIFFUSIVE FRACTIONAL BROWNIAN MOTION OF THE SHORT RATE

FOAD SHOKROLLAHI

Department of Mathematics and Statistics, University of Vaasa, P.O. Box 700, FIN-65101 Vaasa, FINLAND

MARCIN MAGDZIARZ

Hugo Steinhaus Center, Department of Applied Mathematics, Wroclaw University of Science and Technology, Wyspianskiego 27, Wroclaw, 50-370, Poland

Abstract. In this paper we propose an extension of the Merton model. We apply the subdiffusive mechanism to analyze equity warrant in a fractional Brownian motion environment, when the short rate follows the subdiffusive fractional Black-Scholes model. We obtain the pricing formula for zero-coupon bond in the introduced model and derive the partial differential equation with appropriate boundary conditions for the valuation of equity warrant. Finally, the pricing formula for equity warrant is provided under subdiffusive fractional Black-Scholes model of the short rate.

1. Introduction

Analysis of financial data displays that various processes viewed in finance show certain periods in which they are constant [14]. Analogous property is observed in physical system with subdiffusion. The constant periods of financial processes correspond to the trapping event in which the subdiffusive particle is motionless [18, 19, 6]. The mathematical interpretation of subdiffusion is in terms of Fractional Fokker-Planck equation (FFPE). This equation was introduced from the continuous time random walk (CTRW) strategy with fat tail waiting times [18, 19], later used as a substantial tool to evaluate complex system with slow dynamics. In this paper we use the fractional Black-Scholes (FBS) model and the subdiffusive mechanism to better describe the dynamics observed in financial markets. We use similar strategy as in [17, 25], where the objective time $t$ was replaced by the inverse $\alpha$-stable subordinator $T_\alpha(t)$ in the FBS model. $T_\alpha(t)$ corresponds to the fat-tailed waiting times in the underlying CTRW and adds the constant periods to the dynamics of financial assets. Then, the dynamic of asset price $V(t)$ is given by the following subdiffusive FBS
\begin{equation}
  dV(T_\alpha(t)) = \mu_v V(T_\alpha(t))d(T_\alpha(t)) + \sigma_v V(T_\alpha(t))dB^H(T_\alpha(t)),
\end{equation}

where $\mu_v, \sigma_v$ are constant, $B^H_1$ is the fractional Brownian motion (FBM) with Hurst parameter $H \in \left[\frac{1}{2}, 1\right]$. $T_\alpha(t)$ is the inverse $\alpha$-stable subordinator with $\alpha \in (0, 1)$ defined as

\begin{equation}
  T_\alpha(t) = \inf\{\tau > 0 : U_\alpha(\tau) > t\},
\end{equation}

Here $\{U_\alpha(t)\}_{t \geq 0}$ is a $\alpha$-stable Lévy process with nonnegative increments and Laplace transform: $E(e^{-uU_\alpha(t)}) = e^{-tu^\alpha}$ \cite{15, 8, 28, 10}. When $\alpha \uparrow 1$, $T_\alpha(t)$ degenerates to $t$.

\begin{figure}[h]
  \centering
  \includegraphics[width=\textwidth]{figure}
  \caption{Typical trajectory of the asset price from formula (1.1). The parameters are: $\mu_v = \sigma_v = V(0) = 1, H = 0.7, \alpha = 0.9$.}
\end{figure}

However, all the above mentioned papers assume that the short rate is constant during the life of an option. But in reality the short rate evolves randomly over time. Hence, in order to take into account the stochastic short rate, we assume in this paper that the short rate follows the subdiffusive equation:

\begin{equation}
  dr(T_\alpha(t)) = \mu_r d(T_\alpha(t)) + \sigma_r dB^H_2(T_\alpha(t)).
\end{equation}

Here $\mu_r, \sigma_r$ are some constants, $B^H_2$ is a FBM with Hurst parameter $H \in \left[\frac{1}{2}, 1\right]$ and $T_\alpha(t)$ is assumed to be independent of $B^H_2$. Moreover, $B^H_2$ and $B^1_1$ are two dependent FBM\textsc{s} with correlation coefficient $\rho$. Additionally $T_\alpha(t)$ is assumed to be independent of $B^H_1$ and $B^H_2$. 
The ongoing financial empirical evidences show that the mostly used Gaussian or, more general, Markov models may not be sufficient to capture the market structure for short interest rates \([27, 3, 24, 13, 7, 2]\). One reason for this may be the fact that short rates, which are driven by macroeconomic variables, like domestic gross products, supply and demand rates or volatilities exhibit long range dependence, cannot be captured by Markov models \([5, 26, 12]\). Motivated by the fact that the bond market and interest rates often, exhibit long-range dependence, in this section, we incorporate the long memory nature of the short rate in our valuation model and derive explicit formulas for equity warrants when the short rate follows the subdiffusive fractional Brownian process.

Since the subdiffusive fractional Brownian process is a well-developed model, it is an efficient tool to capture the behavior of interest rates \([23, 20, 21, 22]\). In what follows, we state some basic assumptions that will be used in this paper. Given a risk neutral probability measure \((\Omega, \mathcal{F}, Q)\) on which the fractional Brownian motion is defined, we present some “ideal conditions” in the market for the firm value and for the equity warrant:

(i) There are no transaction costs or taxes and all securities are perfectly divisible;
(ii) Dividends are not paid during the lifetime of the outstanding warrants, and the sequential exercise of the warrants is not optimal for warrant holders;
(iii) The warrant-issuing firm is an equity firm with no outstanding debt;
(iv) There are no signaling effects associated with issuing warrants. The market neither reacts positively nor negatively to the information that the firm will issue (or has issued, respectively) warrants;

(v) Suppose that the firm has \(N\) shares of common stocks and \(M\) shares of equity warrants outstanding. Each warrant entitles the owner to receive \(k\) shares of stocks at time \(T\) upon payment of \(X\).

Assumptions (i)–(ii) are the standard assumptions in the Black–Scholes environment. Assumption (iii) implies that stocks and equity warrants are the only sources of financing that are issued by the firm. Assumption (iv) is a necessary condition for our analysis. This assumption implies that both the stock price and the firm’s value remain unaffected by warrants issuance. Actually, this condition could be achieved, see [11]. Assumption (v) sets some notations for the pricing model. Throughout this research, we assume that \(\alpha \in \left(\frac{1}{2}, 1\right)\) and \(\alpha + \alpha H > 1\) (see, [8] and [21]).

2. Pricing model for a zero-coupon bond

In this section, we assume that the short rate \(r(t)\) satisfy Equation (1.3). Then, we obtain the pricing formula for zero-coupon bond \(P(r, t, T)\). Here, \(P(r, T, T) = 1\), that is, the zero coupon bond will pay 1 dollar at expiry date \(T\).

**Theorem 2.1.** The price of a zero-coupon bond with maturity \(t \in [0, T]\) in the fractional Black-Scholes model is given by

\[
P(r, t, T) = e^{-\tau f_2(\tau) + f_1(\tau)},
\]

where \(\tau = T - t\) and

\[
f_1(\tau) = \frac{H \sigma_r^2}{(\Gamma(\alpha))^{2H}} \int_0^\tau (T - v)^{(\alpha-1)2H+2H-1} v^2 dv
\]

\[
f_2(\tau) = \tau.
\]

**Proof.** By the Taylor series expansion, we can get

\[
P(r + \Delta r, t + \Delta t) = P(r, t, T) + \frac{\partial P}{\partial r} \Delta r + \frac{\partial P}{\partial t} \Delta t + \frac{1}{2} \frac{\partial^2 P}{\partial r^2} (\Delta r)^2 + \frac{1}{2} \frac{\partial^2 P}{\partial r \partial t} \Delta r (\Delta t) + \frac{1}{2} \frac{\partial^2 P}{\partial t^2} (\Delta t)^2 + O(\Delta t).
\]

From Eq. (1.3) and [8] and [21], we have

\[
\Delta r = \mu_r(\Delta T_\alpha(t)) + \sigma_r B_2^H(T_\alpha(t))
\]

\[
= \mu_r \left(\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right)(\Delta t) + \sigma_r \Delta B_2^H(T_\alpha(t)) + O(\Delta t).
\]

\[
(\Delta r)^2 = \sigma_r^2 \left(\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right)^{2H} (\Delta t)^2 + O(\Delta t).
\]

\[
\Delta r(\Delta t) = O(\Delta t).
\]
Hence

\[ dP(r, t, T) = \left[ \frac{\mu r^{\alpha-1}}{\Gamma(\alpha)} \frac{\partial P}{\partial r} + \sigma_r^2 H t^{2H-1} \left( \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right)^{2H} \frac{\partial^2 P}{\partial r^2} + \frac{\partial P}{\partial t} \right] dt + \sigma_r \frac{\partial P}{\partial t} dB_2^H(T_\alpha(t)). \]  

(2.8)

Putting

\[ \mu = \frac{1}{P} \left[ \frac{\mu r^{\alpha-1}}{\Gamma(\alpha)} \frac{\partial P}{\partial r} + \sigma_r^2 H t^{2H-1} \left( \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right)^{2H} \frac{\partial^2 P}{\partial r^2} + \frac{\partial P}{\partial t} \right], \]

(2.9)

\[ \sigma = \frac{1}{P} \left( \frac{\partial P}{\partial r} \right), \]

and letting the local expectations hypothesis holds for the term structure of interest rates (i.e. \( \mu = r \)), we have

\[ \frac{\partial P}{\partial t} + \frac{\mu r^{\alpha-1}}{\Gamma(\alpha)} \frac{\partial P}{\partial r} + H t^{2H-1} \sigma_r^2 \left( \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right)^{2H} \frac{\partial^2 P}{\partial r^2} - r P = 0. \]  

(2.10)

Then, zero-coupon bond \( P(r, t, T) \) with boundary condition \( P(r, t, T) = 1 \) satisfies the following partial differential equation

\[ \frac{\partial P}{\partial t} + \frac{\mu r^{\alpha-1}}{\Gamma(\alpha)} \frac{\partial P}{\partial r} + H t^{2H-1} \sigma_r^2 \left( \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right)^{2H} \frac{\partial^2 P}{\partial r^2} - r P = 0. \]  

(2.11)

To solve Equation (2.11) for \( P(r, t, T) \), let \( \tau = T - t \), \( P(r, t, T) = \exp\{f_1(\tau) - rf_2(\tau)\} \), then we can get

\[ \frac{\partial P}{\partial \tau} = P \left( \frac{\partial f_1(\tau)}{\partial \tau} + r \frac{\partial f_2(\tau)}{\partial \tau} \right), \]  

(2.12)

\[ \frac{\partial P}{\partial r} = -P f_2(\tau), \]  

(2.13)

\[ \frac{\partial^2 P}{\partial \tau^2} = P f_2^2(\tau). \]  

(2.14)

Replacing Equations (2.13) and (2.14) into Equation (2.12) and simplifying Equation (2.11) we get

\[ P \left[ H t^{2H-1} \sigma_r^2 f_2(\tau)^2 \left( \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right)^{2H} - \mu_r f_2(\tau) \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right] - \frac{\partial f_1(\tau)}{\partial \tau} + r \left( \frac{\partial f_2(\tau)}{\partial \tau} - 1 \right) = 0. \]  

(2.15)
From Equation (2.15), we obtain
\[
\frac{\partial f_1(\tau)}{\partial \tau} = \sigma_r^2 H t^{2H-1} \left( \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right)^{2H} f_2(\tau)^2 - \mu_r \frac{t^{\alpha-1}}{\Gamma(\alpha)} f_2(\tau),
\]
(2.16)
\[
\frac{\partial f_2(\tau)}{\partial \tau} = 1.
\]
Then,
\[
f_1(\tau) = H \sigma_r^2 \left( \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right)^{2H} \int_0^\tau (T - v)^{2\alpha H - 1} v^2 dv - \mu_r \frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_0^\tau (T - v)^{\alpha-1} v dv,
\]
(2.17)
\[
f_2(\tau) = \tau.
\]
This ends the proof. \(\square\)

Figure 3. Bond price as a function of maturity time \(T\) for different values of \(H\), see formula (2.1). The parameters are: \(\mu_r = \sigma_r = r(0) = 1, \alpha = 0.9, t = 0\).

**Corollary 2.1.** When \(\alpha \uparrow 1\), Equations (1.1) and (1.3) reduce to the FBS. Then we obtain
\[
f_1(\tau) = H \sigma_r^2 \int_0^\tau (T - v)^{2H - 1} v^2 dv - \mu_r \int_0^\tau v dv,
\]
(2.19)
specially, if \(t = 0\)
\[
f_1(\tau) = \sigma_r^2 \frac{T^{2H+2}}{(2H+1)(2H+2)} - \frac{\mu_r T^2}{2},
\]
(2.20)
\begin{equation}
P(r, t, T) = \exp \left\{ -rT + \frac{\sigma_r^2 T^{2H+2}}{(2H+1)(2H+2)} - \frac{\mu_r T^2}{2} \right\}.
\tag{2.21}
\end{equation}

**Corollary 2.2.** If $H = \frac{1}{2}$, from Equation (2.17), we obtain

\begin{equation}
f_1(\tau) = \frac{1}{2} \sigma_r^2 \int_0^\tau (T-v)^{\alpha-1} v^2 dv
- \mu_r \int_0^\tau (T-v)^{\alpha-1} vdv,
\tag{2.22}
\end{equation}

then the result is consistent with the result in [9]. Further, if $\alpha \uparrow 1$ and $H = \frac{1}{2}$, Equations (1.1) and (1.3) reduce to the geometric Brownian motion, then we have

\begin{equation}
f_1(\tau) = \frac{1}{6} \sigma_r^2 \tau^3 - \frac{1}{2} \mu_r \tau^2,
\tag{2.23}
\end{equation}

then

\begin{equation}
P(r, t, T) = e^{-rT + \frac{1}{6} \sigma_r^2 \tau^3 - \frac{1}{2} \mu_r \tau^2},
\tag{2.24}
\end{equation}

which is consistent with the result in [16, 4].

### 3. Fractional Black-Scholes equation

This section provides corresponding FBS equation for equity warrants when the stock price and short rate satisfy Eqs. (1.1) and (1.3), respectively. Recall that $B^H_1$ and $B^H_2$ are two dependent FBM with Hurst parameter $H \in [\frac{1}{2}, 1)$ and correlation coefficient $\rho$.

**Theorem 3.1.** Let assumptions (i)–(v) hold and $W_t$ be the valuation of the equity warrant. Then the valuation of equity warrant at time $t \in [0, T]$ satisfies the following PDE and the following boundary condition

\begin{equation}
\frac{\partial W}{\partial t} + \tilde{\sigma}_r^2(t)V^2 \frac{\partial^2 W}{\partial V^2} + \tilde{\sigma}_v^2(t) \frac{\partial^2 W}{\partial r^2} + 2\rho \tilde{\sigma}_r(t)\tilde{\sigma}_v(t) \frac{\partial^2 W}{\partial V \partial r} + \mu_r \frac{\Gamma(\alpha)}{\Gamma(\alpha)} \frac{\partial W}{\partial r} + rV \frac{\partial W}{\partial V} - rW = 0,
\tag{3.1}
\end{equation}

with boundary condition

\begin{equation}
W_T = \frac{1}{N + M k} (k V_T - N X)^+.
\tag{3.2}
\end{equation}

where

\begin{equation}
\tilde{\sigma}_v^2(t) = H t^{2H-1} \sigma_v^2 \left( \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right)^{2H},
\tag{3.3}
\end{equation}

\begin{equation}
\tilde{\sigma}_r^2(t) = H t^{2H-1} \sigma_r^2 \left( \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right)^{2H}.
\tag{3.4}
\end{equation}

$\sigma_v, \sigma_r, \mu_r, \mu_v$ are constant, $H \in [\frac{1}{2}, 1)$ and $\alpha \in (\frac{1}{2}, 1)$. 

\[ W_T = \frac{1}{N + M k} (k V_T - N X)^+, \]

which is the boundary condition of Eq \[3.1\]. Consider a portfolio with one share of warrants, \( D_1 t \) shares of stock and \( D_2 t \) shares of zero-coupon bond \( P(r, t, T) \). Then the value of the portfolio \( \Pi \) at current time \( t \) is

\[ \Pi_t = W_t - D_1 t V_t - D_2 t P_t. \]

Then, from \[9\] we have

\[
d\Pi_t = dW_t - D_1 t dV_t - D_2 dP_t
\]

\[= \left[ \frac{\partial W}{\partial t} dt + H t^{2H-1} \sigma^2 V^2 \left( \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right) ^{2H} \frac{\partial^2 W}{\partial V^2} + H t^{2H-1} \sigma^2 \left( \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right) ^{2H} \frac{\partial^2 W}{\partial r^2} + 2 H t^{2H-1} \rho \sigma \sigma_v V \left( \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right) ^{2H} \frac{\partial^2 W}{\partial V \partial r} \right] dt + \left[ \frac{\partial W}{\partial r} - D_1 t \right] dV_t
\]

\[= \left[ \frac{\partial W}{\partial t} dt + H t^{2H-1} \left( \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right) ^{2H} \left( \sigma^2 V^2 \frac{\partial^2 V}{\partial V^2} + \sigma^2 \frac{\partial^2 W}{\partial r^2} + 2 \rho \sigma \sigma_v V \frac{\partial^2 W}{\partial V \partial r} \right) \right] dt
\]

\[+ \frac{\partial W}{\partial r} \left[ \frac{\partial P}{\partial r} - D_2 t \frac{\partial P}{\partial r} \right] dt.
\]

Set \( D_1 t = \frac{\partial W}{\partial t} \), \( D_2 t = \frac{\partial W}{\partial r} \), to eliminate the stochastic noise. Then

\[d\Pi_t = \left[ \frac{\partial W}{\partial t} dt + H t^{2H-1} \left( \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right) ^{2H} \left( \sigma^2 V^2 \frac{\partial^2 V}{\partial V^2} + \sigma^2 \frac{\partial^2 W}{\partial r^2} + 2 \rho \sigma \sigma_v V \frac{\partial^2 W}{\partial V \partial r} \right) \right] dt
\]

\[+ \frac{\partial W}{\partial r} \left[ \frac{\partial P}{\partial r} - \mu_r \frac{t^{\alpha-1}}{\Gamma(\alpha)} \frac{\partial P}{\partial r} \right] dt.
\]

The return of the amount \( \Pi_t \) invested in bank account is equal to \( r(t) \Pi_t dt \) at time \( dt \), \( \mathbb{E}(d\Pi_t) = r(t) \Pi_t dt = r(t) (W_t - D_1 t V_t - D_2 t P_t) \), hence from Equation \[3.8\] we have

\[
\frac{\partial W}{\partial t} + H t^{2H-1} \left( \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right) ^{2H} \left( \sigma^2 V^2 \frac{\partial^2 C}{\partial S^2} + \sigma^2 \frac{\partial^2 W}{\partial r^2} + 2 \rho \sigma \sigma_v V \frac{\partial^2 W}{\partial V \partial r} \right) + \mu_r \frac{t^{\alpha-1}}{\Gamma(\alpha)} \frac{\partial W}{\partial r} + r v \frac{\partial W}{\partial V} - r W = 0.
\]

Let

\[ \sigma^2_v(t) = H t^{2H-1} \sigma^2 \left( \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right) ^{2H}, \]

\[ \sigma^2_r(t) = H t^{2H-1} \sigma^2 \left( \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right) ^{2H} .\]
Then
\[
\frac{\partial W}{\partial t} + \tilde{\sigma}_v^2(t)V_t^2 \frac{\partial^2 W}{\partial V_t^2} + \tilde{\sigma}_r^2(t) \frac{\partial^2 W}{\partial r^2} + 2\rho \tilde{\sigma}_r(t)\tilde{\sigma}_v(t) \frac{\partial^2 W}{\partial V \partial r} + \mu_r \frac{t^{\alpha-1}}{\Gamma(\alpha)} \frac{\partial W}{\partial r} + r V \frac{\partial W}{\partial V} - rW = 0,
\]
and the proof is completed. 

**Theorem 3.2.** Let assumptions (i)--(v) hold and \( W_t \) be the valuation of the equity warrant. Then the valuation of an equity warrant at time \( t \in [0, T] \) is given by

\[
W_t = \frac{1}{N + M k} \left( k V_t \phi(d_1) - N X e^{-r(T-t)} P(r, t, T) \phi(d_2) \right),
\]

where

\[
d_1 = \frac{\ln \frac{k V_t}{N X} - \ln P(r, t, T) + \frac{H}{(\Gamma(\alpha))^2} \int_t^T \tilde{\sigma}^2(v) v^{2\alpha H - 1} dv}{\sqrt{\frac{2H}{(\Gamma(\alpha))^2} \int_t^T \tilde{\sigma}^2(v) v^{2\alpha H - 1} dv}},
\]

\[
d_2 = d_1 - \sqrt{\frac{2H}{(\Gamma(\alpha))^2} \int_t^T \tilde{\sigma}^2(v) v^{2\alpha H - 1} dv},
\]

\[
\tilde{\sigma}^2(t) = \sigma_v^2 + 2\rho \sigma_r \sigma_v(T-t) + \sigma_r^2(T-t)^2.
\]

\( P(r, t, T) \) is given by Eq (2.1) and \( \phi(\cdot) \) is the cumulative standard normal distribution function.

**Proof.** Consider the partial differential equation (3.1) of the equity warrants with boundary condition \( W_T = \frac{1}{N + M k} (k V_T - N X)^+ \)

\[
\frac{\partial W}{\partial t} + \tilde{\sigma}_v^2(t)V_t^2 \frac{\partial^2 W}{\partial V_t^2} + \tilde{\sigma}_r^2(t) \frac{\partial^2 W}{\partial r^2} + 2\rho \tilde{\sigma}_r(t)\tilde{\sigma}_v(t) \frac{\partial^2 W}{\partial V \partial r} + \mu_r \frac{t^{\alpha-1}}{\Gamma(\alpha)} \frac{\partial W}{\partial r} + r V \frac{\partial W}{\partial V} - rW = 0,
\]

Denote

\[
z = \frac{V}{P(r, t, T)}, \quad \Theta(z, t) = \frac{W(V, r, t)}{P(r, t, T)}.
\]

Then we get
\[
\frac{\partial W}{\partial t} = \Theta \frac{\partial P}{\partial t} + P \frac{\partial \Theta}{\partial t} - z \frac{\partial \Theta}{\partial z}, \\
\frac{\partial W}{\partial r} = \Theta \frac{\partial P}{\partial r} - z \frac{\partial \Theta}{\partial z}, \\
\frac{\partial W}{\partial V} = \frac{\partial \Theta}{\partial z}, \\
\frac{\partial^2 W}{\partial r^2} = \Theta \frac{\partial^2 P}{\partial r^2} - z \frac{\partial \Theta}{\partial z} \frac{\partial^2 P}{\partial r^2} + z^2 \frac{\partial^2 \Theta}{\partial z^2} \left( \frac{1}{P} \frac{\partial P}{\partial r} \right)^2, \\
\frac{\partial^2 W}{\partial r \partial V} = -z \frac{\partial \Theta}{\partial z} \frac{\partial^2 P}{\partial r^2}, \\
\frac{\partial^2 W}{\partial V^2} = \frac{1}{P} \frac{\partial^2 \Theta}{\partial z^2}.
\]

Inserting Equation (3.20) into Equation (3.18)

\[
\frac{\partial \Theta}{\partial t} + \frac{\partial^2 \Theta}{\partial z^2} \left[ \tilde{\sigma}^2_v(t) \frac{V^2}{P^2} + 2 \rho z^2 \tilde{\sigma}_r(t) \tilde{\sigma}_v(t) \frac{1}{P} \frac{\partial P}{\partial r} + \tilde{\sigma}^2_r(t) z^2 \left( \frac{1}{P} \frac{\partial P}{\partial r} \right)^2 \right]
- \frac{z}{P} \left[ \frac{\partial P}{\partial t} + \tilde{\sigma}^2_r(t) \frac{\partial^2 P}{\partial r^2} + \mu r \left( \frac{\partial \tilde{\sigma}_r(t)}{\partial r} - \frac{\mu}{z} \frac{V}{P} \right) \right] \\
\frac{\partial \Theta}{\partial r} + \frac{\partial^2 \Theta}{\partial z^2} \left[ \tilde{\sigma}^2_r(t) \frac{\partial^2 P}{\partial r^2} + \mu \frac{\partial \tilde{\sigma}_r(t)}{\partial r} - \frac{\tilde{\sigma}_r(t)}{P} \right]
= 0.
\]

From Equation (2.11), we can obtain

\[
\frac{\partial \Theta}{\partial t} + \sigma^2(t) z^2 \frac{\partial^2 \Theta}{\partial z^2} = 0,
\]
with boundary condition \( \Theta(z, T) = (z - K)^+ \),

where

\[
\tilde{\sigma}^2(t) = \tilde{\sigma}^2_v(t) + 2 \rho \tilde{\sigma}_r(t) \tilde{\sigma}_v(t)(T - t) + \tilde{\sigma}_r(t)^2 (T - t)^2.
\]

The solution of partial differential Equation (3.22) with boundary condition \( \Theta(z, T) = \frac{1}{N+Mk} (kz - NX)^+ \), is given by

\[
\Theta(z, t) = k z \phi(\hat{d}_1) - N X \phi(\hat{d}_2),
\]

where

\[
\hat{d}_1 = \frac{\ln \frac{kz}{NX} + \int_t^T \tilde{\sigma}^2(v) dv}{\sqrt{2 \int_t^T \tilde{\sigma}^2(v) dv}}, \\
\hat{d}_2 = \hat{d}_1 - \sqrt{2 \int_t^T \tilde{\sigma}^2(v) dv}.
\]

Thus, from Equations (3.19) and (3.24)–(3.26) we obtain

\[
W(V, r, t) = \frac{1}{N + Mk} \left( kV \phi(\hat{d}_1) - N X \exp(-r(T - t)) P(r, t, T) \phi(\hat{d}_2) \right)
\]
where
\begin{align}
    d_1 &= \frac{\ln \frac{W_t}{N} - \ln P(r, t, T) + \frac{H}{(\Gamma(\alpha))^{2H}} \int_t^T \tilde{\sigma}_v^2(v) v^{2\alpha H - 1} dv}{\sqrt{\frac{2H}{(\Gamma(\alpha))^{2H}} \int_t^T \tilde{\sigma}_v^2(v) v^{2\alpha H - 1} dv}}, \\
    d_2 &= d_1 - \sqrt{\frac{2H}{(\Gamma(\alpha))^{2H}} \int_t^T \tilde{\sigma}_v^2(v) v^{2\alpha H - 1} dv}.
\end{align}

\[\boxed{}

\textbf{Acknowledgements}

The research of MM was partially supported by NCN SONATA BIS-9 grant number 2019/34/E/ST1/00360.

\textbf{References}

[1] F. Biagini, Y. Hu, B. Øksendal, and T. Zhang, \textit{Stochastic calculus for fractional Brownian motion and applications}, Springer Science & Business Media, 2008.

[2] G. Chacko and S. Das, \textit{Pricing interest rate derivatives: a general approach}, The Review of \textit{F}.

[3] J. C. Cox, J. E. Ingersoll Jr, and S. A. Ross, \textit{A theory of the term structure of interest rates}, in Theory of Valuation, World Scientific, 2005, pp. 129–164.

[4] Z. Cui and D. Mcleish, \textit{Comment on “option pricing under the Merton model of the short rate” by kung and lee} [Math. Comput. Simul. \textit{80} (2009) 378–386], Mathematics and Computers in Simulation, \textit{81} (2010), pp. 1–4.
[5] Z. Ding, C. W. Granger, and R. F. Engle, *A long memory property of stock market returns and a new model*, Journal of Empirical Finance, 1 (1993), pp. 83–106.

[6] I. Eliazar and J. Klafter, *Spatial gliding, temporal trapping, and anomalous transport*, Physica D: Nonlinear Phenomena, 187 (2004), pp. 30–50.

[7] R. J. Elliott and R. S. Mamon, *An interest rate model with a Markovian mean reverting level*, Quantitative Finance, 2 (2002), pp. 454–458.

[8] H. Gu, J.-R. Liang, and Y.-X. Zhang, *Time-changed geometric fractional Brownian motion and option pricing with transaction costs*, Physica A: Statistical Mechanics and its Applications, 391 (2012), pp. 3971–3977.

[9] Z. Guo, *Option pricing under the merton model of the short rate in subdiffusive Brownian motion regime*, Journal of Statistical Computation and Simulation, 87 (2017), pp. 519–529.

[10] R. J. Elliott and R. S. Mamon, *An interest rate model with a Markovian mean reverting level*, Quantitative Finance, 2 (2002), pp. 454–458.

[11] H. Gu, J.-R. Liang, and Y.-X. Zhang, *Time-changed geometric fractional Brownian motion and option pricing with transaction costs*, Physica A: Statistical Mechanics and its Applications, 391 (2012), pp. 3971–3977.

[12] J. Hull and A. White, *Pricing interest-rate-derivative securities*, The Review of Financial Studies, 3 (1990), pp. 573–592.

[13] J. Janczura and A. Wyłomańska, *Subdynamics of financial data from fractional Fokker-Planck equation*, Acta Physica Polonica B, 40 (2009), pp. 1341–1351.

[14] A. Janicki and A. Weron, *Simulation and chaotic behavior of alpha-stable stochastic processes*, vol. 178, CRC Press, 1993.

[15] J. J. Kung and L.-S. Lee, *Option pricing under the Merton model of the short rate*, Mathematics and Computers in Simulation, 80 (2009), pp. 378–386.

[16] M. Magdziarz, *Black–Scholes formula in subdiffusive regime*, Journal of Statistical Physics, 136 (2009), pp. 553–564.

[17] R. Metzler, E. Barkai, and J. Klafter, *Anomalous diffusion and relaxation close to thermal equilibrium: A fractional Fokker-Planck equation approach*, Physical Review Letters, 82 (1999), p. 3563.

[18] B.-N. Huang and C. W. Yang, *The fractal structure in multinational stock returns*, Applied Economics Letters, 2 (1995), pp. 67–71.

[19] R. Metzler and J. Klafter, *The random walk's guide to anomalous diffusion: a fractional dynamics approach*, Physica Reports, 339 (2000), pp. 1–77.

[20] F. Shokrollahi, *Subdiffusive fractional Black–Scholes model for pricing currency options under transaction costs*, Cogent Mathematics & Statistics, 5 (2018), p. 1470145.

[21] F. Shokrollahi, *The valuation of European option under subdiffusive fractional Brownian motion of the short rate*, International Journal of Theoretical and Applied Finance, (2020), p. 2050022.

[22] F. Shokrollahi, A. Kilicman, and M. Magdziarz, *Pricing european options and currency options by time changed mixed fractional Brownian motion with transaction costs*, International Journal of Financial Engineering, 3 (2016), p. 1650003.

[23] T. K. Siu, *Bond pricing under a markovian regime-switching jump-augmented vasicek model via stochastic flows*, Applied Mathematics and Computation, 216 (2010), pp. 3184–3190.

[24] I. M. Sokolov, *Solutions of a class of non-Markovian Fokker-Planck equations*, Physical Review E, 66 (2002), p. 041101.

[25] B. M. Tabak and D. O. Cajueiro, *The long-range dependence behavior of the term structure of interest rates in japan*, Physica A: Statistical Mechanics and its Applications, 350 (2005), pp. 418–426.

[26] O. Vasicek, *An equilibrium characterization of the term structure*, Journal of Financial Economics, 5 (1977), pp. 177–188.

[27] J. Wang, J.-R. Liang, L.-J. Lv, W.-Y. Qiu, and F.-Y. Ren, *Continuous time Black–Scholes equation with transaction costs in subdiffusive fractional Brownian motion regime*, Physica A: Statistical Mechanics and its Applications, 391 (2012), pp. 750–759.