The unirationality of the moduli spaces of 2-elementary K3 surfaces

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with an appendix by Ken-Ichi Yoshikawa

ABSTRACT

We prove that the moduli spaces of K3 surfaces with non-symplectic involutions are unirational. As a by-product, we describe configuration spaces of $5 \leq d \leq 8$ points in $\mathbb{P}^2$ as arithmetic quotients of type IV.

1. Introduction

K3 surfaces with non-symplectic involutions were classified by Nikulin [31], and Yoshikawa [36] showed that their moduli spaces are Zariski open sets of certain modular varieties of orthogonal type. In this paper, we prove that these moduli spaces are unirational. This work was inspired by a recent result of Yoshikawa on the Kodaira dimensions of those spaces, which is presented by him in Appendix A of this paper. Let us begin by recalling basic definitions.

Let $X$ be a complex K3 surface with an involution $\iota$. When $\iota$ acts non-trivially on $H^0(K_X)$, $\iota$ is called non-symplectic, and the pair $(X, \iota)$ is called a 2-elementary K3 surface. In this case, the lattice $L_+ = H^2(X, \mathbb{Z})^\iota$ of $\iota$-invariant cycles is a hyperbolic lattice with 2-elementary discriminant form $D_{L_+}$. The main invariant of $(X, \iota)$ is the triplet $(r, a, \delta)$, where $r$ is the rank of $L_+$, $a$ is the length of $D_{L_+}$, that is, $D_{L_+} \cong (\mathbb{Z}/2\mathbb{Z})^a$, and $\delta$ the parity of $D_{L_+}$. Nikulin [31] proved that the deformation type of $(X, \iota)$ is determined by the main invariant $(r, a, \delta)$, and he enumerated all main invariants of 2-elementary K3 surfaces, which are 75 in number.

By the theory of period mapping, 2-elementary K3 surfaces of a fixed main invariant $(r, a, \delta)$ are parametrized by the Hermitian symmetric domain associated to a certain lattice $L_-$ of signature $(2, 20 - r)$. Yoshikawa [36, 38] determined the correct monodromy group as the orthogonal group $O(L_-)$ of $L_-$. Consequently, he constructed the moduli space $\mathcal{M}_{r,a,\delta}$ of those pairs $(X, \iota)$ as a Zariski open set of the modular variety defined by $O(L_-)$.

The principal result of the present paper is the following.

THEOREM 1.1. For every main invariant $(r, a, \delta)$, the moduli space $\mathcal{M}_{r,a,\delta}$ of 2-elementary K3 surfaces of type $(r, a, \delta)$ is unirational.

We recall that the 2-elementary K3 surfaces in $\mathcal{M}_{1,1,1}$ are double planes ramified over smooth sextics so that $\mathcal{M}_{1,1,1}$ is birational to the orbit space $O_{\mathfrak{P}^2}(6)/\text{PGL}_3$, which is unirational. This fact is a prototype of Theorem 1.1. Kondo [21] proved the rationality of $\mathcal{M}_{10,2,0}$ and $\mathcal{M}_{10,10,0}$, the latter being isomorphic to the moduli of Enriques surfaces. Shepherd-Barron [34] practically established the rationality of $\mathcal{M}_{5,5,1}$ in the course of proving that of the moduli
of genus 6 curves. Matsumoto–Sasaki–Yoshida [26] constructed general members of \( \mathcal{M}_{16,6,1} \) starting from six lines on \( \mathbb{P}^2 \). A similar idea was used by Koike–Shiga–Takayama–Tsutsui [20] to obtain general members of \( \mathcal{M}_{14,8,1} \) from four bidegree \((1,1)\) curves on \( \mathbb{P}^1 \times \mathbb{P}^1 \). In particular, \( \mathcal{M}_{16,6,1} \) and \( \mathcal{M}_{14,8,1} \) are also unirational.

Yoshikawa studied the birational type of \( \mathcal{M}_{r,a,\delta} \) in a systematic way by using a criterion of Gritsenko [12] and Borcherds products. He found that \( \mathcal{M}_{r,a,\delta} \) has Kodaira dimension \(-\infty\) when \( 13 \leq r \leq 17 \) and when \( r + a = 22 \), where \( r \leq 17 \). Then he suggested to the author to study the birational type of \( \mathcal{M}_{r,a,\delta} \) through a geometric approach. The present work grew out of this suggestion. After Theorem 1.1 was proved, Yoshikawa and the author decided to write both approaches in this paper. Yoshikawa’s work is presented in Appendix A. Now the Kodaira dimensions of some of \( \mathcal{M}_{r,a,\delta} \) may be calculated by two methods: by modular forms on the moduli spaces and by the geometry of 2-elementary K3 surfaces.

We will prove Theorem 1.1 by using certain Galois covers of \( \mathcal{M}_{r,a,\delta} \) and isogenies between them. The strategy is as follows.

1. Let \( \tilde{\mathcal{M}}_{r,a,\delta} \) be the modular variety associated to the group \( \tilde{O}(L_-) \) of isometries of \( L_- \) which act trivially on the discriminant form. The variety \( \mathcal{M}_{r,a,\delta} \) is a Galois cover of \( \tilde{\mathcal{M}}_{r,a,\delta} \).
2. Construct an isogeny \( \tilde{\mathcal{M}}_{r,a,\delta} \to \tilde{\mathcal{M}}_{r,a',\delta'} \) when \( a' < a, \delta = 1 \), and when \( a' < a, \delta = \delta' \).
3. For each fixed \( r \), choose a large \( a \) and find a moduli interpretation of (an open set of) \( \mathcal{M}_{r,a,\delta} \). Then prove that \( \mathcal{M}_{r,a,\delta} \) is unirational using that interpretation. By step (2), follow the unirationality of \( \tilde{\mathcal{M}}_{r,a',\delta'} \) for \( a' < a \).
4. The remaining moduli spaces \( \mathcal{M}_{r,a'',\delta''} \) with \( a'' > a \), if any, are also proved to be unirational in some way.

One of the advantages of studying the covers \( \tilde{\mathcal{M}}_{r,a,\delta} \) is that we have isogenies between them so that the problem is reduced to fewer modular varieties. These isogenies admit geometric interpretation in terms of twisted Fourier–Mukai partners. By this strategy, we will derive the unirationality of \( 70 \tilde{\mathcal{M}}_{r,a,\delta} \) by studying just \( 22 \tilde{\mathcal{M}}_{r,a,\delta} \). The remaining five moduli spaces \( \mathcal{M}_{r,a,\delta} \), for which we do not know whether the covers \( \mathcal{M}_{r,a,\delta} \) are unirational, are treated in step (4) or already settled [21]. In step (3), we often identify \( \mathcal{M}_{r,a,\delta} \) with the moduli of certain plane sextics endowed with a labelling of the singularities. We can attach such geometric interpretations to \( \tilde{\mathcal{M}}_{r,a,\delta} \) in a fairly uniform manner: this is another virtue of studying \( \tilde{\mathcal{M}}_{r,a,\delta} \).

We shall explain a general idea of such procedures (Section 3.4), discuss few cases in detail as models (Sections 4 and 5), and for other cases omit some detail.

Let us comment on other possible approaches for Theorem 1.1. Firstly, as explained by Alexeev–Nikulin [1], 2-elementary K3 surfaces with \( r + a \leq 20 \) are related to log del Pezzo surfaces of index \( \leq 2 \). Thus, one might study \( \mathcal{M}_{r,a,\delta} \) via the moduli of such surfaces, using the explicit description of log del Pezzo surfaces of index 2 given by Nakayama [29]. Secondly, by using singular curves on \( \mathbb{P}^2 \) and \( \mathbb{F}_n \) as branches (as in this paper), for most \( (r,a,\delta) \) we can actually find a unirational parameter space that dominates \( \mathcal{M}_{r,a,\delta} \).

In [25], these will be developed further to derive the rationality of \( 67 \mathcal{M}_{r,a,\delta} \). Hence, one may establish Theorem 1.1 also by just studying the remaining moduli spaces. However, the proof of rationality is delicate and ad hoc, so that the whole proof of unirationality would be lengthy if we do so. We here prefer the proof using \( \mathcal{M}_{r,a,\delta} \) because it is more systematic, short, and self-contained.

Our descriptions of the covers \( \tilde{\mathcal{M}}_{r,a,\delta} \) (for a relatively large \( a \)) are rather systematic. When \( r = a, \delta = 1 \), we relate \( \tilde{\mathcal{M}}_{r,a,\delta} \) with Severi varieties of plane sextics. On the other hand, when \( r + a = 22, r \geq 12 \), we relate \( \tilde{\mathcal{M}}_{r,a,\delta} \) with configuration spaces of points in \( \mathbb{P}^2 \). As a by-product, we describe those configuration spaces as arithmetic quotients of type IV. To be more precise, let \( U_d \subset (\mathbb{P}^2)^d \) (resp. \( V_d \subset (\mathbb{P}^2)^d \)) be the variety of \( d \) ordered points of which no three are...
collinear (resp. only the first three are collinear). Let \( U_d/G \) and \( V_d/G \) denote the quotient varieties for the diagonal actions of \( G = \text{PGL}_3 \). Let \( L_n \) be the lattice \((2)^2 \oplus (-2)^n\).

**Theorem 1.2.** Let \( 5 \leq d \leq 8 \). For each \( 1 \leq n \leq 8 \), there exists an arithmetic group \( \Gamma_n \subset \text{O}(L_n) \) such that one has birational period maps

\[
U_d/G \longrightarrow \mathcal{F}(\Gamma_{2d-8}), \quad V_d/G \longrightarrow \mathcal{F}(\Gamma_{2d-9}),
\]

where \( \mathcal{F}(\Gamma_n) \) is the modular variety associated to \( \Gamma_n \). One has \( \Gamma_n = \tilde{\text{O}}(L_n) \) for \( 1 \leq n \leq 6 \), and for \( n = 7,8 \) one has \( \Gamma_n \supset \tilde{\text{O}}(L_n) \) with \( \Gamma_n/\tilde{\text{O}}(L_n) \simeq \mathfrak{S}_{n-5} \), where \( \mathfrak{S}_N \) is the symmetric group on \( N \) letters.

When \( d \leq 6 \), we recover some results of Matsumoto–Sasaki–Yoshida [26]. They constructed a period map for \( U_6 \) and then obtained lower-dimensional period maps by degeneration. The novel part of Theorem 1.2 is the construction of the period maps for \( d = 7,8 \) points. Also our period maps for \( d \leq 6 \) are derived from those for \( d = 7,8 \), and are not identical to the ones of [26]. It is a future task to study the whole boundary behaviour of the period maps.

Dolgachev et al. [10] and Kondō [23, 24] described the spaces \( U_d/G \) for \( 5 \leq d \leq 7 \) as ball quotients. It is also known [11] that \( U_7/G \) can be described as a Siegel modular variety. Thus, these spaces \( U_d/G \) admit (birationally) the structure of an arithmetic quotient in more than one way. After suitable compactifications, they may provide examples of ‘Janus-like’ varieties (cf. [17]). In view of the relation with the moduli of del Pezzo surfaces, it would also be interesting to study the Weyl group action on \( \mathcal{F}(\Gamma_{2d-8}) \) induced by the period map.

The rest of the paper is structured as follows. In Section 2, we review the necessary facts concerning lattices, modular varieties, and invariant theory. In Section 3, we gather basic results on 2-elementary \( K3 \) surfaces with particular attention to the relation with singular sextic curves. The proof of Theorem 1.1 will be developed from Section 4 to Section 9. Theorem 1.2 will be proved in Sections 7–9. In Section 10, we deduce the unirationality of the moduli spaces of Borcea–Voisin threefolds as a consequence of Theorem 1.1. In Appendix A written by Yoshikawa, the approach by modular forms is presented.

Otherwise stated, we work in the category of algebraic varieties over \( \mathbb{C} \).

## 2. Preliminaries

### 2.1. Lattices

Let \( L \) be a **lattice**, that is, a free \( \mathbb{Z} \)-module of finite rank endowed with a non-degenerate integral symmetric bilinear form \((,\)\). The orthogonal group of \( L \) is denoted by \( \text{O}(L) \). For an integer \( n \neq 0 \), \( L(n) \) denotes the scaled lattice \((L, n(,))\). The lattice \( L \) is even if \((l,l) \in 2\mathbb{Z} \) for all \( l \in L \), and **odd** otherwise. The dual lattice \( L^\vee = \text{Hom}(L, \mathbb{Z}) \) of \( L \) is canonically embedded in \( L \otimes \mathbb{Q} \) and contains \( L \). On the finite abelian group \( D_L = L^\vee/L \), we have the \( \mathbb{Q}/\mathbb{Z} \)-valued bilinear form \( b_L \) defined by \( b_L(x + L, y + L) = (x,y) + \mathbb{Z} \). We denote by \( \tilde{\text{O}}(L) \subset \text{O}(L) \) the group of isometries of \( L \) which act trivially on \( D_L \). When \( L \) is even, \( b_L \) is induced by the quadratic form \( q_L : D_L \to \mathbb{Q}/2\mathbb{Z}, q_L(x + L) = (x,x) + 2\mathbb{Z} \), which is called the **discriminant form** of \( L \). We denote by \( r_L : \text{O}(L) \to \text{O}(D_L, q_L) \) the natural homomorphism.

**Proposition 2.1** (Nikulin [30]). Let \( \Lambda \) be an even unimodular lattice and \( L \) be a primitive sublattice of \( \Lambda \) with the orthogonal complement \( M \). Then one has a natural isometry \( \lambda : (D_L, q_L) \simeq (D_M, -q_M) \) defined by the relation \( x + \lambda(x) \in \Lambda, x \in D_L \). For two isometries
\[ \gamma_L \in O(L) \text{ and } \gamma_M \in O(M), \text{ the isometry } \gamma_L \oplus \gamma_M \text{ of } L \oplus M \text{ extends to that of } \Lambda \text{ if and only if } r_L(\gamma_L) = \lambda^{-1} \circ r_M(\gamma_M) \circ \lambda. \]

A lattice \( L \) is called 2-elementary if \( D_L \) is 2-elementary, that is, \( D_L \simeq \mathbb{Z}/2\mathbb{Z}^a \) for some \( a \geq 0 \). The main invariant of an even 2-elementary lattice \( L \) is the quadruplet \((r_+, r_-, a, \delta)\), where \((r_+, r_-)\) is the signature of \( L \), \( a \) is the length of \( D_L \) as above, and \( \delta \) is defined by \( \delta = 0 \) if \( q_L(D_L) \subset \mathbb{Z}/2\mathbb{Z} \) and \( \delta = 1 \) otherwise. By Nikulin [30], the isometry class of \( L \) is uniquely determined by the main invariant if \( L \) is indefinite. When \( L \) is hyperbolic, we also call the triplet \((1 + r_-, a, \delta)\) the main invariant of \( L \). In this paper, we often use the following 2-elementary lattices with basis:

\[
M_n = \langle 2 \rangle \oplus (-2)^{n-1} = \langle h, e_1, \ldots, e_{n-1} \rangle, \quad (2.1)
\]
\[
U(2) = \langle u, v \rangle, \quad (2.2)
\]

where \( \{h, e_1, \ldots, e_{n-1}\} \) are orthogonal basis with \( \langle h, h \rangle = 2 \) and \( \langle e_i, e_i \rangle = -2 \), and \( \{u, v\} \) are basis with \( \langle u, u \rangle = \langle v, v \rangle = 0 \) and \( \langle u, v \rangle = 2 \). Let

\[
\Lambda_{K3} = U^3 \oplus E_8^2
\]

be the even unimodular lattice of signature \((3, 19)\), where \( U \) is the hyperbolic plane (the scaling of \( U(2) \) by \( \frac{1}{2} \)) and \( E_8 \) is the rank-8 even negative-definite unimodular lattice. The following assertion is due to Nikulin.

**Proposition 2.2** (Nikulin [30, 31]). *Let \( L \) be an even hyperbolic 2-elementary lattice. If a primitive embedding \( L \hookrightarrow \Lambda_{K3} \) exists, then it is unique up to the action of \( O(\Lambda_{K3}) \).*

### 2.2. Orthogonal modular varieties

Let \( L \) be a lattice of signature \((2, r_-)\) and let \( \Gamma \subset O(L) \) be a finite-index subgroup. The group \( \Gamma \) acts properly discontinuously on the complex manifold

\[
\Omega_L = \{ C\omega \in \mathbb{P}(L \otimes \mathbb{C}) \mid \langle \omega, \omega \rangle = 0, \langle \omega, \overline{\omega} \rangle > 0 \}. \quad (2.4)
\]

The domain \( \Omega_L \) has two connected components, say \( \Omega_L^+ \) and \( \Omega_L^- \). We denote by \( \Gamma^+ \) the group of those isometries in \( \Gamma \) which preserve \( \Omega_L^+ \). The quotient space

\[
\mathcal{F}_L(\Gamma^+) = \Gamma^+ \backslash \Omega_L^+ \quad (2.5)
\]

is a normal quasi-projective variety of dimension \( r_- \), by [2], called the modular variety associated to \( \Gamma^+ \). When the lattice \( L \) is understood from the context, we abbreviate \( \mathcal{F}_L(\Gamma^+) \) as \( \mathcal{F}(\Gamma^+) \).

**Proposition 2.3.** *Let \( L \) be a finite-index sublattice of a lattice \( M \) of signature \((2, r_-)\). Then there exists a finite surjective morphism \( \mathcal{F}(\hat{O}(L)^+) \to \mathcal{F}(\hat{O}(M)^+) \).*

**Proof.** We have the sequence \( L \subset M \subset M^\vee \subset L^\vee \) of inclusions. If we regard the finite groups \( G_1 = M/L \) and \( G_2 = M^\vee /L \) as subgroups of \( D_L \), then we have \( G_2 = \{x \in D_L, b_L(x, G_1) \equiv 0 \} \) and the induced bilinear form \( (G_2/G_1, b_L) \) is canonically isometric to \( (D_M, b_M) \). Since the isometries in \( O(L) \) act trivially on both \( G_1 \) and \( G_2 \), they preserve the overlattice \( M \) of \( L \), and as isometries of \( M \) act trivially on \( D_M \). Thus, we have a finite-index embedding \( \hat{O}(L) \hookrightarrow \hat{O}(M) \) of groups. Via the natural identification \( \Omega_L = \Omega_M \subset \mathbb{P}(L \otimes \mathbb{C}) = \mathbb{P}(M \otimes \mathbb{C}) \), this embedding induces a finite morphism \( \mathcal{F}(\hat{O}(L)^+) \to \mathcal{F}(\hat{O}(M)^+) \). \( \square \)
The following proposition was used by Kondō [21] to prove the rationality of the moduli space of Enriques surfaces.

**Proposition 2.4.** Let \( L \) be an even 2-elementary lattice of signature \((2, r_-)\). Then the lattice \( M = L^{\vee}(2) \) is 2-elementary and we have \( \mathcal{F}(O(L)^+) \simeq \mathcal{F}(O(M)^+) \).

**Proof.** Since \( L(2) \subset M \subset L^{\vee}(2) = M^{\vee} \), we see that \( M \) is 2-elementary. We have the coincidence \( O(L) = O(L^{\vee}) \) in \( O(L \otimes \mathbb{Q}) \) because of the double dual relation \( L^{\vee \vee} = L \). Thus, we have \( \mathcal{F}_L(O(L)^+) \simeq \mathcal{F}_{L^{\vee}}(O(L^{\vee})^+) \simeq \mathcal{F}_M(O(M)^+) \). \( \square \)

### Geometric Invariant Theory

We review some facts from Geometric Invariant Theory. Throughout this section, let \( X \) be a variety acted on by a reductive algebraic group \( G \). We will apply the machinery of Geometric Invariant Theory to plane sextic curves [32], bidegree \((4, 4)\) curves on \( \mathbb{P}^1 \times \mathbb{P}^1 \) [33], and point sets in \( \mathbb{P}^2 \) [11, 28].

**Theorem 2.5** (Mumford [28]). Let \( X, G, L \) be as above. Then a geometric quotient \( X^s(L)/G \) of \( X^s(L) \) exists and is a quasi-projective variety.

**Lemma 2.6.** Let \( f : X \rightarrow Y \) be a \( G \)-equivariant finite morphism of \( G \)-varieties. Suppose we have an ample \( G \)-linearized line bundle \( L \) on \( Y \) such that \( Y = Y^s(L) \). Then we have \( X = X^s(f^*L) \). In particular, we have a geometric quotient \( X/G \).

**Proof.** Note that \( f^*L \) is ample and naturally \( G \)-linearized. For every \( x \in X \), the stabilizer \( G_x \) is a subgroup of \( G_{f(x)} \) and hence is finite. For an invariant section \( s \in H^0(Y, L^{\otimes n})^G \) with \( s(f(x)) \neq 0 \) and with a closed \( G \)-action on \( Y_s \), we have \( f^*s(x) \neq 0 \) and the \( G \)-action on \( X_{f^*s} = f^{-1}(Y_s) \) is also closed. \( \square \)

We will apply the machinery of Geometric Invariant Theory to plane sextic curves [32], bidegree \((4, 4)\) curves on \( \mathbb{P}^1 \times \mathbb{P}^1 \) [33], and point sets in \( \mathbb{P}^2 \) [11, 28].

**Definition 1.** Let \( C \subset S \) be a reduced curve on a smooth surface \( S \). A singular point \( p \in C \) is a simple singularity if (i) \( p \) is a double or triple point and (ii) the strict transform of \( C \) in the blow-up of \( S \) at \( p \) does not have triple point over \( p \).

See [3, II.8] for the A-D-E classification of the simple singularities. In this paper, we will deal mainly with nodes (\( A_1 \)-points) and ordinary triple points (\( D_4 \)-points). In some literatures, the condition (ii) above is stated in the form ‘\( C \) has no consecutive triple point’ [32] or ‘\( C \) has no infinitely near triple point’ [16].
We consider the PGL$_3$-action on the linear system $|O_{P^2}(6)|$ of plane sextic curves, which is endowed with a natural linearized ample line bundle.

**Proposition 2.7 (Shah [32]).** A reduced plane sextic is PGL$_3$-stable if and only if it has only simple singularities.

We also need a stability criterion for the PGL$_2 \times$ PGL$_2$-action on the linear system $|O_{P^1 \times P^1}(4, 4)|$ endowed with the naturally linearized $O(1)$.

**Proposition 2.8 (Shah [33]).** Let $C \subset P^1 \times P^1$ be a reduced curve of bidegree $(4, 4)$. If $C$ has only nodes as singularities, then $C$ is PGL$_2 \times$ PGL$_2$-stable.

Finally, we consider the diagonal action of PGL$_3$ on the product $(P^2)^d$. Let $U_d \subset (P^2)^d$ be the open set of ordered points $(p_1, \ldots, p_d)$ such that no three of $\{p_i\}_{i=1}^d$ are collinear, and let $V_d \subset (P^2)^d$ be the variety of ordered points $(p_1, \ldots, p_d)$ such that $\{p_1, p_2, p_3\}$ are collinear and no other three of $\{p_i\}_{i=1}^d$ are collinear.

**Proposition 2.9 (cf. [11], [28]).** For $d \geq 4$ (resp. $d \geq 5$), a geometric quotient $U_d/PGL_3$ (resp. $V_d/PGL_3$) exists and is a quasi-projective rational variety of dimension $2d - 8$ (resp. $2d - 9$).

*Proof.* For the assertion for $U_d$, see [11, Chapter II]. The variety $V_d$ is contained in the stable locus with respect to the SL$_3$-linearized line bundle $O_{P^2}(1) \otimes \cdots \otimes O_{P^2}(1)$ so that a geometric quotient exists by Theorem 2.5. For $d \geq 7$, the rationality of $V_d/PGL_3$ follows from the birational equivalence $V_d/PGL_3 \sim V_{d-4}$. The remaining $V_5/PGL_3$ and $V_6/PGL_3$ are also clearly rational. $\square$

3. 2-Elementary K3 surfaces

3.1. Basic properties

We recall basic facts on 2-elementary K3 surfaces following [1, 31]. Let $(X, \iota)$ be a 2-elementary K3 surface, that is, a pair of a complex K3 surface $X$ and a non-symplectic involution $\iota$ on $X$. The surface $X$ is always algebraic due to the presence of $\iota$. The invariant and anti-invariant lattices

$$L_\pm = L\pm(X, \iota) = \{l \in H^2(X, \mathbb{Z}) \mid \iota^*l = \pm l\}$$

are even 2-elementary lattices of signature $(1, r - 1)$ and $(2, 20 - r)$, respectively, where $r$ is the rank of $L_\pm$. Note that $L_-$ is the orthogonal complement of $L_+$ in $H^2(X, \mathbb{Z})$ and hence we have a natural isometry $(D_{L_+}, q_{L_+}) \simeq (D_{L_-}, -q_{L_-})$. The main invariant $(r, a, \delta)$ of $L_+$ is also called the main invariant of $(X, \iota)$ and may be calculated geometrically as follows.

**Proposition 3.1 (Nikulin [31]).** Let $(X, \iota)$ be a 2-elementary K3 surface of type $(r, a, \delta)$. Let $X^\iota$ be the fixed locus of $\iota$.

(i) If $(r, a, \delta) = (10, 10, 0)$, then $X^\iota = \emptyset$.

(ii) If $(r, a, \delta) = (10, 8, 0)$, then $X^\iota$ is a union of two elliptic curves.
(iii) In other cases, $X^t$ is decomposed as $X^t = C^g \sqcup E_1 \sqcup \cdots \sqcup E_k$, where $C^g$ is a genus $g$ curve and $E_1, \ldots, E_k$ are $(-2)$-curves with

$$g = 11 - \frac{r + a}{2}, \quad k = \frac{r - a}{2}. \quad (3.2)$$

One has $\delta = 0$ if and only if the class of $X^t$ is divisible by 2 in $NS_X$.

Let $f : X \to Y = X/\langle \iota \rangle$ be the quotient morphism and $B = f(X^t)$ be the branch curve of $f$. If $X^t \neq \emptyset$, $Y$ is a smooth rational surface and $B$ is a smooth member of $\{-2K_Y\}$. Following [1], we call such a pair $(Y, B)$ a right DPN pair. The 2-elementary $K3$ surface $(X, \iota)$ is recovered from $(Y, B)$ as the double cover of $Y$ branched over $B$. In this way, 2-elementary $K3$ surfaces with a non-empty fixed locus are in canonical correspondence with right DPN pairs. The invariant $(r, a)$ of $(X, \iota)$ can be read off from the topology of $B$ by Proposition 3.1. We also have

$$r = \rho(Y). \quad (3.3)$$

For the parity $\delta$, if $B = \sum_i B_i$ is the irreducible decomposition of $B$, then we have $\delta = 0$ if and only if the class $\sum_i (-1)^{n_i} [B_i]$ is divisible by 4 in $NS_Y$ for some $n_i \in \{0, 1\}$. The lattice $L_+(X, \iota)$ may be obtained as follows.

**Proposition 3.2.** Let $(Y, B)$ be a right DPN pair and $(X, \iota)$ be the associated 2-elementary $K3$ surface with the quotient morphism $f : X \to Y$. Then the invariant lattice $L_+ = L_+(X, \iota)$ is generated by the sublattice $f^* NS_Y$ and the classes of irreducible components of $X^t$.

**Proof.** Let $B = \sum_{i=0}^k B_i$ be the irreducible decomposition and let $C_i = f^{-1}(B_i)$. We have $X^t = \sum_{i=0}^k C_i$ and $C_i \sim \frac{1}{2} f^* B_i$. According to Kharlamov [19, p. 304], the relation $\sum_{i=0}^k C_i \sim -f^* K_Y$ is the only relation among $\{C_i\}_{i=0}^k$ in $L_+/f^* NS_Y$. Since the lattice $f^* NS_Y \simeq NS_Y(2)$ is of index $2^{1/2(r-a)} = 2^k$ in $L_+$, this proves the assertion. \qed

3.2. Right resolutions of plane sextics

We explain a relationship between 2-elementary $K3$ surfaces and plane sextics with only simple singularities. The topic is classical as it goes back to Horikawa [16] and Shah [32]. Here we develop the argument in more generality in the framework of Alexeev–Nikulin [1]. Recall from [1] that a DPN pair is a pair $(Y, B)$ of a smooth rational surface $Y$ and an anti-bicanonical curve $B \in \{-2K_Y\}$ with only simple singularities.

**Definition 2.** A right resolution of a DPN pair $(Y_0, B_0)$ is a triplet $(Y, B, \pi)$ such that $(Y, B)$ is a right DPN pair and $\pi : Y \to Y_0$ is a birational morphism with $\pi(B) = B_0$. By abuse of terminology, we also call $(Y, B, \pi)$ a right resolution of $B_0$ when $Y_0$ is obvious from the context.

**Proposition 3.3 (cf. [1]).** A right resolution of a DPN pair $(Y_0, B_0)$ exists and is unique up to isomorphism.

**Proof.** Let $S \to Y_0$ be the double cover branched over $B_0$. As $B_0$ has only simple singularities, $S$ is a normal surface with only A-D-E singularities (corresponding to those of $B_0$) and with a trivial canonical divisor. The minimal resolution $X$ of $S$ is a $K3$ surface, and the covering transformation of $S \to Y_0$ induces a non-symplectic involution $\iota$ on $X$. If $(Y, B)$ is the right DPN pair associated to $(X, \iota)$, then by the universality of the quotient $X \to Y$,
we have a birational morphism \( \pi: Y \to Y_0 \) with \( \pi(B) = B_0 \). This proves the existence. For any other right resolution \((Y', B', \pi')\) with the associated 2-elementary \(K3\) surface \((X', \iota')\), let \(X' \to S' \to Y_0\) be the Stein factorization of the morphism \(X' \to Y' \to Y_0\). Then \(S' \to Y_0\) is a double cover branched over \(B_0\) and thus is isomorphic to \(S \to Y_0\). It follows that \(X' \to Y_0\) is isomorphic to \(X \to Y_0\) and we have \((Y, B, \pi) \simeq (Y', B', \pi')\).

In [1] right resolution is constructed explicitly as follows. Let

\[
\cdots \xrightarrow{\pi_{i+1}} (Y_i, B_i) \xrightarrow{\pi_i} (Y_{i-1}, B_{i-1}) \xrightarrow{\pi_{i-1}} \cdots \xrightarrow{\pi_1} (Y_0, B_0)
\]

be the successive blow-ups defined inductively by

\[
Y_{i+1} = bl_{E_i} Y_i, \quad B_{i+1} = \hat{B}_i + \sum_{k=1}^N (m_k - 2)E_k,
\]

where \(\Sigma_i = \{p_k\}_{k=1}^N\) is the singular locus of \(B_i\), \(B_i\) is the strict transform of \(B_i\), \(E_k\) is the \((-1)\)-curve over \(p_k\), and \(m_k\) is the multiplicity of \(B_i\) at \(p_k\). Each \((Y_i, B_i)\) is also a DPN pair. This process will terminate and we finally obtain a right DPN pair \((Y, B) = (Y_N, B_N)\).

In this way, one can associate a 2-elementary \(K3\) surface \((X, \iota)\) to a DPN pair \((Y_0, B_0)\) by taking its right resolution \((Y, B, \pi)\). Composing \(\pi\) with the quotient map \(X \to Y\), we have a natural generically two-to-one morphism \(g: X \to Y_0\) branched over \(B_0\). In this paper, we will deal only with the following simple situations.

**Example 1.** When \(B_0\) has only nodes \(p_1, \ldots, p_n\) as the singularities, then \(E_i = g^{-1}(p_i)\) is a \((-2)\)-curve on \(X\), and each component of the fixed curve \(X^i\) is mapped by \(g\) birationally onto a component of \(B_0\). By Proposition 3.2, the lattice \(L_+(X, \iota)\) is generated by the sublattice \(g^*NS_{Y_0}\), the classes of the \((-2)\)-curves \(E_1, \ldots, E_n\), and of the components of \(X^i\). In particular, we have \(r = \rho(Y_0) + n\).

**Example 2.** As a slight generalization, suppose that \(\text{Sing}(B_0)\) consists of nodes \(p_1, \ldots, p_n\) and ordinary triple points \(q_1, \ldots, q_m\). Then the curve \(g^{-1}(q_j)\) is decomposed as \(g^{-1}(q_j) = G_j + \sum_{k=1}^3 E_{jk}\) such that \(G_j\) is a rational component of \(X^i\), and \(E_{jk}\) are the \((-2)\)-curves over the infinitely near points of \(q_j\) given by the branches of \(B_0\). We have \((G_j \cdot E_{jk}) = 1\) and \((E_{jk} \cdot E_{jk'}) = -2\delta_{kk'}\). Other components of \(X^i\) than \(G_1, \ldots, G_m\) are mapped by \(g\) birationally onto the components of \(B_0\). The lattice \(L_+(X, \iota)\) is generated by \(g^*NS_{Y_0}\), the classes of the \((-2)\)-curves \(g^{-1}(p_i), E_{jk}, G_j\), and of those components of \(X^i\). In particular, we have \(r = \rho(Y_0) + n + 4m\).

When \(Y_0 = \mathbb{P}^2\) or \(\mathbb{P}^1 \times \mathbb{P}^1\), for which \(B_0\) is a sextic or a bidegree \((4,4)\) curve, respectively, we have the following useful property.

**Lemma 3.4.** Let \((Y_0, B_0)\) be a DPN pair with \(Y_0\) being either \(\mathbb{P}^2\) or a smooth quadric in \(\mathbb{P}^3\). Let \((X, \iota)\) be the associated 2-elementary \(K3\) surface with the natural projection \(g: X \to Y_0\). Then the morphism \(g: X \to Y_0 \subset \mathbb{P}^d\) can be identified with the morphism \(\phi_H: X \to |H|^\vee\) associated to the bundle \(H = g^*\mathcal{O}_{Y_0}(1)\).

**Proof.** The bundle \(H\) is nef and big. Use the Riemann–Roch formula and the vanishing \(h^i(H) = 0\) for \(i > 0\) to see that \(|H| = g^*|\mathcal{O}_{Y_0}(1)|\). \(\square\)
3.3. Classification and the moduli spaces

2-Elementary K3 surfaces were classified by Nikulin in terms of the main invariants.

**Theorem 3.5 (Nikulin [31]).** The deformation type of a 2-elementary K3 surface \((X, \iota)\) is determined by the main invariant \((r, a, \delta)\). All possible main invariants of 2-elementary K3 surfaces are shown in Figure 1 which is identical to the table in p. 31 of [1].

A moduli space of 2-elementary K3 surfaces of main invariant \((r, a, \delta)\) is constructed as follows. We fix an even 2-elementary lattice \(L\) of main invariant \(2, 20 - r, a, \delta\), which is isometric to the anti-invariant lattice of every 2-elementary K3 surface of type \((r, a, \delta)\). Let \(\mathcal{F}(O(L)^+) = O(L)^+ \setminus \Omega^+_L\) be the modular variety associated to \(O(L)^+\). The divisor \(\sum \delta^+ \subset \Omega^+_L\), where \(\delta\) are \((-2)\)-vectors in \(L\), is the inverse image of an algebraic divisor \(D \subset \mathcal{F}(O(L)^+)\). Let \(M_{r, a, \delta}\) be the variety

\[
M_{r, a, \delta} = \mathcal{F}(O(L)^+) - D, \tag{3.6}
\]

which is normal, irreducible, quasi-projective, and of dimension \(20 - r\). For a 2-elementary K3 surface \((X, \iota)\) of type \((r, a, \delta)\), we may choose an isometry \(\Phi: L^{-}(X, \iota) \to L\) with \(\Phi(H^{2,0}(X)) \in \Omega^+_L\). Then we define the period of \((X, \iota)\) by

\[
P(X, \iota) = [\Phi(H^{2,0}(X))] \in M_{r, a, \delta}, \tag{3.7}
\]

which is independent of the choice of \(\Phi\).

**Theorem 3.6 (Yoshikawa [36, 38]).** The variety \(M_{r, a, \delta}\) is a moduli space of 2-elementary K3 surfaces of type \((r, a, \delta)\) in the following sense.

(i) For a complex analytic family \((X \to U, \iota)\) of such 2-elementary K3 surfaces, the period map \(P: U \to M_{r, a, \delta}, u \mapsto P(X_u, \iota_u)\), is holomorphic. When the family is algebraic, \(P\) is a morphism of algebraic varieties.
(ii) Via the period mapping, the points of $\mathcal{M}_{r,a,\delta}$ are in one-to-one correspondence with the isomorphism classes of 2-elementary K3 surfaces of type $(r,a,\delta)$.

3.4. The discriminant covers

Let $L$ be the lattice used in the definition \((3.6)\) and $\tilde{M}_{r,a,\delta}$ be the modular variety

$$\tilde{M}_{r,a,\delta} = \mathcal{F}(\tilde{\Omega}(L)^+),$$

\((3.8)\)

which is a Galois cover of $\mathcal{F}(\Omega(L)^+)$ with the Galois group $\text{O}(D_L, q_L)$. We call $\tilde{M}_{r,a,\delta}$ the discriminant cover of $\mathcal{M}_{r,a,\delta}$. Since $\Omega(L)^+ \neq \tilde{\Omega}(L)$, we may identify $\tilde{M}_{r,a,\delta} = \tilde{\Omega}(L) \setminus \Omega_L$. The next proposition is a key for our proof of Theorem 1.1.

**Proposition 3.7.** Let $(r,a,\delta)$ and $(r,a',\delta')$ be main invariants of 2-elementary K3 surfaces. Assume that either (i) $\delta = 1, a > a'$ or (ii) $\delta = \delta', a > a'$. Then one has a finite surjective morphism $\varphi: \tilde{M}_{r,a,\delta} \to \tilde{M}_{r,a',\delta'}$.

**Proof.** Let $L$ and $L'$ be even 2-elementary lattices of main invariant $(2,20-r,a,\delta)$ and $(2,20-r,a',\delta')$, respectively. Calculating the discriminant form $(D_L, q_L)$ explicitly, one can find an isotropic subgroup $G \subset D_L$ such that the 2-elementary quadratic form $(G^+ / G, q_L)$ has the invariant $(a',\delta')$. By the coincidence of main invariant, the overlattice of $L$ defined by $G$ is isometric to $L'$. Hence the assertion follows from Proposition 2.3. \hfill \Box

The relationship between the modular varieties is as follows:

$$\begin{align*}
\tilde{M}_{r,a,\delta} - H &\xrightarrow{\varphi} \tilde{M}_{r,a',\delta'} - H' \\
\downarrow &\quad \downarrow \\
\mathcal{M}_{r,a,\delta} &\quad \mathcal{M}_{r,a',\delta'}
\end{align*}$$

\((3.9)\)

Here $H$ and $H'$ are appropriate Heegner divisors.

**Remark 1.** When $a' = a - 2$, $\varphi$ admits the following geometric interpretation. For an $\omega \in \tilde{M}_{r,a,\delta}$, let $(X,i) \in \mathcal{M}_{r,a,\delta}$ and $(X',i') \in \mathcal{M}_{r,a',\delta'}$ be the 2-elementary K3 surfaces given by the images of $\omega$ and $\varphi(\omega)$, respectively. Then $X$ is derived equivalent to the twisted K3 surface $(X',\alpha')$ for a Brauer element $\alpha' \in \text{Br}(X')$ of order at most 2. Indeed, we have a Hodge embedding $T_X \hookrightarrow T_{X'}$ of the transcendental lattices of index at most 2 so that the twisted derived Torelli theorem \([18]\) applies.

General points of $\tilde{M}_{r,a,\delta}$ may be obtained as follows (cf. \([1, 9]\)). We fix an even hyperbolic 2-elementary lattice $L_+$ of main invariant $(r,a,\delta)$, a primitive embedding $L_+ \subset \Lambda_{K3}$, and an isometry $(L_+) \cap \Lambda_{K3} \to L$. Let $(X,i) \in \mathcal{M}_{r,a,\delta}$ and $j: L_+ \to L_+(X,i)$ be a given isometry. By Proposition 2.2, the isometry $j$ can be extended to an isometry $\Phi: \Lambda_{K3} \to H^2(X,\mathbb{Z})$, which in turn induces the isometry $\Phi|_L: L \to L_+(X,i)$. By Proposition 2.1, the isometry $\Phi|_L$ is determined from $j$ up to the action of $\tilde{\Omega}(L)$. Then we define the period of the lattice-marked 2-elementary K3 surface $(X,i,j)$ by

$$\tilde{P}((X,i),j) = [\Phi|_L^{-1}(H^{2,0}(X))] \in \tilde{M}_{r,a,\delta}.$$  

\((3.10)\)

If we define equivalence of two such objects $((X,i),j)$ and $((X',i'),j')$ by the existence of a Hodge isometry $\Psi: H^2(X,\mathbb{Z}) \to H^2(X',\mathbb{Z})$ with $j' = \Psi \circ j$, then via the period mapping $\tilde{P}$, the open set of $\tilde{M}_{r,a,\delta}$ over $\mathcal{M}_{r,a,\delta}$ parametrizes the equivalence classes of such objects $((X,i),j)$. The assignment $((X,i),j) \mapsto (X,i)$ gives the projection $\mathcal{M}_{r,a,\delta} \to \mathcal{M}_{r,a,\delta}$. 


This interpretation of $\tilde{\mathcal{M}}_{r,a,\delta}$ using lattice-marked 2-elementary $K3$ surfaces is useful, but not so geometric. In the rest of this paper, using this interpretation temporarily, we will seek for more geometric interpretations for some of $\mathcal{M}_{r,a,\delta}$.

Here is a general strategy. We define a space $U$ parametrizing certain plane sextics $B$ (or bidegree $(4,4)$ curves on $\mathbb{P}^1 \times \mathbb{P}^1$) which are endowed with some labelling of their singularities and components. The 2-elementary $K3$ surface $(X,\iota)$ associated to the right resolution of $B$ has main invariant $(r,a,\delta)$. The point is that the labelling for $B$ induces an isometry $j: L_+ \to L_+(X,\iota)$. Actually, an argument as in Examples 1 and 2 will suggest an appropriate definition of the reference lattice $L_+$, and then $j$ will be obtained naturally. Considering the period of $((X,\iota),j)$ as defined above, we obtain a morphism $p: U \to \tilde{\mathcal{M}}_{r,a,\delta}$. We will prove that $p$ descends to an open immersion $U/G \to \mathcal{M}_{r,a,\delta}$, where $G = \text{PGL}_3$ (or $\text{PGL}_2 \times \text{PGL}_2$). This amounts to showing that $\dim(U/G) = 20 - r$ and that the $p$-fibres are $G$-orbits. The latter property is verified using the Torelli theorem and that the curve $B$ with its labelling may be recovered from $((X,\iota),j)$ via Lemma 3.4.

In this way, some of $\mathcal{M}_{r,a,\delta}$ will be birationally identified with the moduli of certain curves with labelling. Such geometric interpretations vary according to $\mathcal{M}_{r,a,\delta}$ and are out of single formulation. However, the processes by which we attach them to $\mathcal{M}_{r,a,\delta}$ are largely common, as suggested above. Then, in order to avoid repetition, we will discuss such processes in detail for only few cases (Section 4.1). For other cases, we omit some detail and refer to Section 4.1 as a model.

Now our geometric descriptions will imply that those $\tilde{\mathcal{M}}_{r,a,\delta}$ are often unirational. With the aid of Proposition 3.7, we will finally obtain the following.

**Theorem 3.8.** The discriminant covers $\tilde{\mathcal{M}}_{r,a,\delta}$ are unirational except possibly for $(r,a) = (10,10), (11,11), (12,10), (13,9)$.

Sometimes our interpretations of $\tilde{\mathcal{M}}_{r,a,\delta}$ using sextics are translated into yet another geometric interpretations, such as configuration spaces of points in $\mathbb{P}^2$.

4. **The case $r \leq 9$**

In this section, we prove that $\tilde{\mathcal{M}}_{r,a,\delta}$ are unirational for $r \leq 9$. We first prove in Section 4.1 that $\tilde{\mathcal{M}}_{r,r,1}$ with $r \leq 9$ are unirational using the Severi varieties of nodal plane sextics. This case is a model for the subsequent sections and hence discussed in detail. From Proposition 3.7 and Figure 1 follows the unirationality of $\mathcal{M}_{r,a,\delta}$ with $r \leq 9$ and $(r,a,\delta) \neq (2,2,0)$. In Section 4.2, we treat $\tilde{\mathcal{M}}_{2,2,0}$.

4.1. **$\tilde{\mathcal{M}}_{r,r,1}$ and the Severi varieties of nodal sextics**

For $r \leq 11$, let $V_{r-1} \subset |\mathcal{O}_{\mathbb{P}^2}(6)|$ be the variety of irreducible plane sextics with $r-1$ nodes and with no other singularity. The variety $V_{r-1}$, known as a Severi variety, is smooth, of dimension $28 - r$, and irreducible [15]. By endowing the sextics with markings of the nodes, we have the following $\mathcal{S}_{r-1}$-cover of $V_{r-1}$:

$$ \tilde{V}_{r-1} = \{(C,p_1,\ldots,p_{r-1}) \in V_{r-1} \times (\mathbb{P}^2)^{r-1}, \text{ Sing}(C) = \{p_i\}_{i=1}^{r-1}\}. \quad (4.1) $$

By Lemma 2.6 and Proposition 2.7, we have a geometric quotient $\tilde{V}_{r-1}/\text{PGL}_3$.

**Proposition 4.1.** For $r \leq 9$, the variety $\tilde{V}_{r-1}$ is rational. In particular, the quotient $\tilde{V}_{r-1}/\text{PGL}_3$ is a unirational variety of dimension $20 - r$. 
Proof. We consider the nodal map
\[ \kappa : \tilde{V}_{r-1} \longrightarrow (\mathbb{P}^2)^{r-1}, \quad (C, p_1, \ldots, p_{r-1}) \mapsto (p_1, \ldots, p_{r-1}). \]
(4.2)
For a general \( p = (p_1, \ldots, p_{r-1}) \), the fibre \( \kappa^{-1}(p) \) may be identified with an open set of \( |-2K_{Y'}| \), where \( Y' \) is the blow-up of \( \mathbb{P}^2 \) at \( \{p_i\}_{i=1}^{r-1} \). Since \( Y' \) is a del Pezzo surface, we have \( \dim |-2K_{Y'}| \geq 3 \) so that \( \kappa \) is dominant. As \( \kappa^{-1}(p) \) is an open set of a linear subspace of \( |O_{\mathbb{P}^2}(6)| \), we see that \( \tilde{V}_{r-1} \) is birationally equivalent to the projective space bundle associated to a locally free sheaf on an open set of \( (\mathbb{P}^2)^{r-1} \).

We shall construct a period map \( \tilde{p} : \tilde{V}_{r-1} \to \tilde{\mathcal{M}}_{r,r,1} \) for \( r \leq 11 \). For a sextic with labelling \( (C, p) = (C, p_1, \ldots, p_{r-1}) \) in \( \tilde{V}_{r-1} \), let \((X, i)\) be the 2-elementary \( K3 \) surface associated to the right resolution of \( C \), and \( g : X \to \mathbb{P}^2 \) be the natural projection branched over \( C \). The quotient \( X/\langle i \rangle \) is the blow-up of \( \mathbb{P}^2 \) at \( p_1, \ldots, p_{r-1} \). On \( X \) we have the line bundle \( H = g^*O_{\mathbb{P}^2}(1) \) and the \(-2\)-curves \( E_i = g^{-1}(p_i) \). Let \( M_r = \langle h, e_1, \ldots, e_{r-1} \rangle \) be the lattice defined in (2.1). By Example 1, the classes of \( H \) and \( E_1, \ldots, E_{r-1} \) define an isometry of lattices \( j : M_r \to L_+ \) by \( h \mapsto [H] \) and \( e_i \mapsto [E_i] \). We thus associate a lattice-marked 2-elementary \( K3 \) surface \( ((X, i), j) \) to \((C, p)\). Fixing a primitive embedding \( M_r \hookrightarrow \Lambda_{K3} \) and considering the period of \((X, i), j\) as defined in (3.10), we then obtain a point \( \tilde{p}(C, p) \) of \( \tilde{\mathcal{M}}_{r,r,1} \).

**Proposition 4.2.** Let \( r \leq 11 \). Two sextics with labelling \( (C, p), (C', p') \in \tilde{V}_{r-1} \) are \( \text{PGL}_3\)-equivalent if and only if \( \tilde{p}(C, p) = \tilde{p}(C', p') \).

Proof. It suffices to prove the ‘if’ part. Let \( X, j, H, \ldots \) (resp. \( X', j', H', \ldots \)) be the objects constructed from \((C, p)\) (resp. \((C', p')\)) as above. If \( \tilde{p}(C, p) = \tilde{p}(C', p') \), we have a Hodge isometry \( \Phi : H^2(X', \mathbb{Z}) \to H^2(X, \mathbb{Z}) \) with \( j = \Phi \circ j' \). This equality means that \( \Phi([H']) = [H] \) and \( \Phi([E'_i]) = [E_i] \). Since \( \Phi \) maps the ample class \( 4H' - \sum_{i=1}^{r-1} E'_i \) to the ample class \( 4H - \sum_{i=1}^{r-1} E_i \), by the strong Torelli theorem there exists an isomorphism \( \varphi : X \to X' \) with \( \varphi^* = \Phi \). Then we have \( \varphi(E_i) = E'_i \) and \( \varphi^*H' \cong H \). By Lemma 3.4, we obtain an automorphism \( \psi : \mathbb{P}^2 \to \mathbb{P}^2 \) with \( \varphi \circ \varphi = \psi \circ g \). Since \( p_i = g(E_i) \) and \( p_i' = g'(E'_i) \), we have \( \psi(p_i) = p_i' \). Since \( C \) and \( C' \) are, respectively, the branches of \( g \) and \( g' \), we also have \( \psi(C) = C' \).

**Theorem 4.3.** Let \( r \leq 11 \). The period map \( \tilde{p} : \tilde{V}_{r-1} \to \tilde{\mathcal{M}}_{r,r,1} \) is a morphism of varieties and induces an open immersion \( \tilde{V}_{r-1}/\text{PGL}_3 \to \tilde{\mathcal{M}}_{r,r,1} \).

Proof. We repeat the above construction for families. Let \( \tilde{\mathcal{C}}_{r-1} \subset \tilde{V}_{r-1} \times \mathbb{P}^2 \) be the universal marked nodal sextic over \( \tilde{V}_{r-1} \) (which may be obtained from the universal sextic over \( \mathcal{V}_{r-1} \)). We have the sections \( s_i : \tilde{V}_{r-1} \to \tilde{\mathcal{C}}_{r-1} \) defined by \((C, p) \mapsto ((C, p), p_i)\), where \( p = (p_1, \ldots, p_{r-1}) \).

There is an open set \( \mathbb{V} \subset \tilde{V}_{r-1} \) such that the divisor \( \tilde{\mathcal{C}} = \tilde{\mathcal{C}}_{r-1}|_{\tilde{V}} \) is linearly equivalent to \( \pi_2^*O_{\mathbb{P}^2}(6) \), where \( \pi_2 : \tilde{V} \times \mathbb{P}^2 \to \mathbb{P}^2 \) is the projection. We denote \( W_i = s_i(\mathbb{V}) \). Let \( \mathcal{Y} \) be the blow-up of \( \hat{V} \times \mathbb{P}^2 \) along \( \bigcup_{i=1}^{r-1} W_i \) and \( \mathcal{D} \subset \mathcal{Y} \) be the exceptional divisor over \( W_i \). Since the strict transform \( \mathcal{B} \subset \mathcal{Y} \) of \( \mathcal{C} \) is a smooth divisor linearly equivalent to \( \pi_2^*O_{\mathbb{P}^2}(6) - 2 \sum_{i=1}^{r-1} D_i \), we may take a double cover \( f : \mathcal{X} \to \mathcal{Y} \) branched over \( \mathcal{B} \). The natural projection \( \pi : \mathcal{X} \to \hat{V} \) is a family of \( K3 \) surfaces. Let \( \iota \) be the covering transformation of \( f \) and \( L_+ \) be the local system \( (R^2\pi_*\mathcal{Z})^! \) over \( V \). Then the divisors \( \{f^{-1}(\mathcal{D}_i)\}_i \) and the pullback of \( \pi_2^*O_{\mathbb{P}^2}(1) \) define a trivialization \( L_+ \to M_+ \times V \). This means that the monodromy group of the local system \( L_- = (L_+)^\perp \cap R^2\pi_*\mathcal{Z} \) is contained in \( \hat{O}(L_r) \), where \( L_r = (M_r)^\perp \cap \Lambda_{K3} \). Considering the local system \( L_- \), we see that the period map \( \tilde{p}|_{\mathcal{V}} : \tilde{V} \to \tilde{\mathcal{M}}_{r,r,1} \) is a locally liftable holomorphic map. By Borel’s extension theorem [7], \( \tilde{p}|_{\mathcal{V}} \) is a morphism of algebraic varieties. This implies that \( \tilde{p} \)
is a morphism of varieties. By the PGL$_3$-invariance, $\tilde{p}$ induces a morphism $\tilde{\mathcal{P}}: \tilde{V}_{r-1}/\text{PGL}_3 \to \tilde{\mathcal{M}}_{r,r,1}$. Proposition 4.2 implies the injectivity of $\tilde{\mathcal{P}}$. Then $\tilde{\mathcal{P}}$ is dominant because we have $\dim(\tilde{V}_{r-1}/\text{PGL}_3) = 20 - r$ and $\mathcal{M}_{r,r,1}$ is irreducible. Thus, $\tilde{\mathcal{P}}$ is an open immersion by the Zariski’s Main Theorem.

**Corollary 4.4.** If $r \leq 9$ and $(r,a,\delta) \neq (2,2,0)$, then $\tilde{\mathcal{M}}_{r,a,\delta}$ is unirational.

**Proof.** By Proposition 4.1 and Theorem 4.3, $\tilde{\mathcal{M}}_{r,r,1}$ is unirational for $r \leq 9$. Then the assertion follows from Proposition 3.7 and Figure 1.

**Remark 2.** Morrison–Saitō [27] constructed an open immersion $V_{r-1}/\text{PGL}_3 \to \mathcal{F}(\Gamma_r)$ for a certain arithmetic group $\Gamma_r \subset \text{O}(L_r)^+$. Our idea to relate $\mathcal{M}_{r,r,1}$ with $V_{r-1}$ was inspired by their argument.

**Remark 3.** In fact, $\tilde{V}_{r-1}/\text{PGL}_3$ is rational when $2 \leq r \leq 9$. For $r \geq 5$, this may be seen by fixing the first four nodes. For $r \leq 4$, we need invariant-theoretic techniques. In the rest of the paper, one would find that several $\mathcal{M}_{r,a,\delta}$ are rational as well.

### 4.2. $\tilde{\mathcal{M}}_{2,2,0}$ and bidegree $(4,4)$ curves

Let $Q = \mathbb{P}^1 \times \mathbb{P}^1$ be a smooth quadric embedded in $\mathbb{P}^3$. The group $G = \text{PGL}_2 \times \text{PGL}_2$ acts naturally on $Q$. Let $U \subset |\mathcal{O}_Q(4,4)|$ be the open set of smooth bidegree $(4,4)$ curves. By Proposition 2.8, we have a geometric quotient $U/G$ as an affine unirational variety of dimension 18.

For a curve $C \in U$, let $(X,\iota)$ be the 2-elementary $K3$ surface associated to the right DPN pair $(Q,C)$ and $f: X \to Q$ be the quotient morphism. The lattice $L_+(X,\iota)$ is equal to $f^*\text{NS}_Q$ by Proposition 3.2, and thus isometric to the lattice $U(2)$. In fact, using the basis $\{u,v\}$ of $U(2)$ defined in (2.2), we have an isometry $j: U(2) \to L_+(X,\iota)$ by $u \mapsto [f^*\mathcal{O}_Q(1,0)]$ and $v \mapsto [f^*\mathcal{O}_Q(0,1)]$. Here it is important to distinguish the two rulings of $Q$. In this way, we obtain a lattice-marked 2-elementary $K3$ surface $((X,\iota),j)$ from $C$. We then obtain a point $\tilde{p}(C)$ in $\tilde{\mathcal{M}}_{2,2,0}$ as the period of $((X,\iota),j)$ as before.

In this construction, one may recover the morphism $f: X \to Q$ (and hence its branch $C$) from the class $j(u+v)$ by Lemma 3.4. Via $f$, the two rulings $|\mathcal{O}_Q(1,0)|$, $|\mathcal{O}_Q(0,1)|$ of $Q$ may, respectively, be recovered from the elliptic fibrations on $X$ given by the classes $j(u)$, $j(v)$.

**Theorem 4.5.** The period map $\tilde{p}: U \to \tilde{\mathcal{M}}_{2,2,0}$ is a morphism of varieties and induces an open immersion $U/G \to \tilde{\mathcal{M}}_{2,2,0}$. In particular, $\mathcal{M}_{2,2,0}$ is unirational.

**Proof.** Basically one may apply a similar argument as for Proposition 4.2 and Theorem 4.3. In the present case, one should note that $G$ is the group of automorphisms of $Q$ preserving the two rulings. This ensures the $G$-invariance of $\tilde{p}$ for its definition involves the distinction of the two rulings. The recovery of the morphisms $f$, the curves $C$, and the two rulings of $Q$ as explained above implies the injectivity of the induced morphism $U/G \to \tilde{\mathcal{M}}_{2,2,0}$. Here one may apply the strong Torelli theorem by using the ample classes $j(u+v)$.
In this section, we prove that \( \mathcal{M}_{10, a, \delta} \) are unirational. Kondō [21] proved the rationality of \( \mathcal{M}_{10,10,0} \), the moduli of Enriques surfaces, and of \( \mathcal{M}_{10,2,0} \). We study the remaining moduli spaces. In Sections 5.1 and 5.2, we prove the unirationality of \( \mathcal{M}_{10,8,0} \) and \( \mathcal{M}_{10,8,1} \), respectively, which implies that \( \tilde{\mathcal{M}}_{10,a,\delta} \) are unirational for \( a \leq 8 \). The unirationality of \( \mathcal{M}_{10,10,1} \) is proved in Section 5.3.

5.1. \( \tilde{\mathcal{M}}_{10,8,0} \) and cubic pairs

Let \( U \subset |O_{\mathbb{P}^2}(6)| \times (\mathbb{P}^2)^8 \) be the space of pointed sextics \( (C_1 + C_2, p) = (C_1 + C_2, p_1, \ldots, p_8) \) such that \( C_1 \) and \( C_2 \) are smooth cubics transverse to each other and that \( p_1, \ldots, p_8 \) are distinct points contained in \( C_1 \cap C_2 \). The variety \( U \) is unirational. Indeed, if we denote by \( V \subset |O_{\mathbb{P}^2}(3)| \times (\mathbb{P}^2)^8 \) the locus of \( (C_1, p_1, \ldots, p_8) \) such that \( \{p_i\}_{i=1}^8 \subset C \), then \( U \) is dominated by the fibre product \( V \times (\mathbb{P}^2)^8 \). As the projection \( V \to (\mathbb{P}^2)^8 \) is dominant with a general fibre being a line in \( |O_{\mathbb{P}^2}(3)| \), the variety \( V \times (\mathbb{P}^2)^8 \) is rational, and so \( U \) is unirational. By Proposition 2.7 and Lemma 2.6, the natural projection \( U \to |O_{\mathbb{P}^2}(6)| \) shows that we have a geometric quotient \( U/PGL_3 \) as a unirational variety of dimension 10.

For a pointed sextic \( (C_1 + C_2, p) \in U \), we denote by \( p_9 \) the ninth intersection point of \( C_1 \) and \( C_2 \). This gives a complete labelling of the nodes of \( C_1 + C_2 \). Let \( (X, \iota) \) be the 2-elementary K3 surface associated to \( C_1 + C_2 \) and \( g : X \to \mathbb{P}^2 \) be the natural projection branched over \( C_1 + C_2 \). The quotient \( X/\iota \) is the blow-up of \( \mathbb{P}^2 \) at \( p_1, \ldots, p_9 \) and is a rational elliptic surface. We have the decomposition \( X' = F_1 + F_2 \) such that \( g(F_i) = C_i \). By Example 1, the lattice \( L_+((X, \iota)) \) is generated by the classes of the bundle \( H = g^*O_{\mathbb{P}^2}(1) \), the \(-2\)-curves \( F_i = g^{-1}(p_i) \) for \( i \leq 9 \), and the elliptic curves \( F_1 \sim F_2 \). This suggests to define a reference lattice \( L_+ \) as follows. Let \( M_{10} = \langle h, e_1, \ldots, e_9 \rangle \) be the lattice defined in (2.1) and \( v \in M_{10} \) be the vector defined by \( 2v = 3h - \sum_{i=1}^8 e_i \). The even overlattice \( L_+ = (M_{10}, v) \) is 2-elementary of main invariant \((10, 8, 0)\). Then we have a natural isometry \( j : L_+ \to L_+((X, \iota)) \) by sending \( h \mapsto [H], e_i \mapsto [E_i] \), and \( v \mapsto [F_j] \). Therefore, we obtain a point \( \bar{p}(C_1 + C_2, p) \) in \( \tilde{\mathcal{M}}_{10,8,0} \) as the period of \(( (X, \iota), j) \) as before.

As in Section 4.1, one may recover the morphism \( g : X \to \mathbb{P}^2 \) from the class \( j(h) \) by Lemma 3.4, the points \( p_i = g(E_i) \) from the classes \( j(e_i) \), and the sextic \( C_1 + C_2 \) from \( g \) as the branch locus. Also one has the ample class \( j(h + v) \) on \( X \) defined in terms of \( j \). Hence, one may proceed as Section 4.1 to see the following.

**Theorem 5.1.** The period map \( \bar{p} : U \to \tilde{\mathcal{M}}_{10,8,0} \) is a morphism of varieties and descends to an open immersion \( U/PGL_3 \to \mathcal{M}_{10,8,0} \).

**Corollary 5.2.** If \( a \leq 8 \), then \( \tilde{\mathcal{M}}_{10,a,0} \) is unirational.

5.2. \( \tilde{\mathcal{M}}_{10,8,1} \) and bidegree \((3, 2)\) curves

Let \( Q = \mathbb{P}^1 \times \mathbb{P}^1 \) be a smooth quadric in \( \mathbb{P}^3 \) and let \( G = PGL_2 \times PGL_2 \). Let \( U \subset |O_Q(4, 4)| \times Q^8 \) be the variety of pointed bidegree \((4, 4)\) curves \( (C + D, p) = (C + D, p_1, \ldots, p_8) \) such that (i) \( C \) is smooth of bidegree \((3, 2)\), (ii) \( D \) is smooth of bidegree \((1, 2)\) and transverse to \( C \), and (iii) \( C \cap D = \{p_1, \ldots, p_8\} \). The space \( U \) is an \( \mathfrak{S}_8 \)-cover of an open set of \( |O_Q(3, 2)| \times |O_Q(1, 2)| \). By Proposition 2.8 and Lemma 2.6, we have a geometric quotient \( U/G \) as a ten-dimensional variety.

**Lemma 5.3.** The variety \( U \) is rational.
Proof. Let \( V \) be the linear system \( |\mathcal{O}_Q(1,2)| \) and \( X \subset V \times Q \) be the universal curve over \( V \). The projection \( \pi_1: X \to V \) is birationally equivalent to the natural projection \( \mathbb{P}^1 \times V \to V \) for bidegree \((0,1)\) curves on \( Q \) give sections of \( \pi_1 \). This implies that the fibre product \( Y = X \times_\mathcal{O}_Y X \times_\mathcal{O}_Y X \) (eight times) is rational. We have a morphism \( \pi_2: U \to Y \) defined by \((C + D, p) \mapsto (D, p)\). Then \( \pi_2 \) is dominant. Indeed, for every smooth \( D \in V \), the restriction map \(|\mathcal{O}_Y(3,2)| \to |\mathcal{O}_D(8)|\) is dominant by the vanishing of \( H^1(\mathcal{O}_Q(2,0)) \). Since a general \( \pi_2\)-fibre is an open set of a linear subspace of \(|\mathcal{O}_Q(3,2)|\), this proves the rationality of \( U \). \( \square \)

For a curve with labelling \((C + D, p) \in U\), let \((X, \iota)\) be the 2-elementary \( K3 \) surface associated to the DPN pair \((Q, C + D)\) and \( g: X \to Q \) be the natural projection branched over \( C + D \). The fixed curve \( X^* \) is decomposed as \( X^* = F_1 + F_2 \) such that \( g(F_1) = C \) and \( g(F_2) = D \). In this case, a reference lattice \( L_+ \) should be defined as follows. Let \( M \) be the lattice \( U(2) \oplus (-2)^8 = \langle u, v, e_1, \ldots, e_8 \rangle \), where \( \{u, v\} \) is the basis of \( U(2) \) defined in (2.2) and \( \{e_1, \ldots, e_8\} \) is a natural basis of \((-2)^8\). Let \( f_1, f_2 \in M^* \) be the vectors defined by \( f_1 = 3u + 2v - \sum e_i \) and \( f_2 = u + 2v - \sum e_i \). The overlattice \( L_+ = \langle M, f_1, f_2 \rangle \) is even and 2-elementary of main invariant \((10,8,1)\). Then, by Example 1, we have a natural isometry \( j: L_+ \to L_+(X, \iota) \) by sending \( u \mapsto [g^*\mathcal{O}_Q(1,0)], v \mapsto [g^*\mathcal{O}_Q(0,1)], e_i \mapsto [g^{-1}(p_i)] \), and \( f_j \mapsto [F_j] \). In this way, we associate to \((C + D, p)\) a lattice-marked 2-elementary \( K3 \) surface \(((X, \iota), j)\), and hence a point \( \tilde{p}(C + D, p) \) in \( \tilde{M}_{10,8,1} \).

As in Section 4.2, the morphism \( q: X \to Q \), the curve \( C + D \), and the two rulings of \( Q \) are recovered from \( j \). The points \( p_i \) are recovered from the classes \( j(e_i) \). Therefore, we have the following theorem.

**Theorem 5.4.** The period map \( \tilde{p}: U \to \tilde{M}_{10,8,1} \) is a morphism of varieties and descends to an open immersion \( U/G \to \tilde{M}_{10,8,1} \).

**Corollary 5.5.** If \( a \leq 8 \), then \( \tilde{M}_{10, a, 1} \) is unirational.

5.3. The unirationality of \( M_{10,10,1} \)

By Theorem 4.3, general members of \( M_{10,10,1} \) are obtained from Halphen curves, irreducible nine-nodal sextics. However, since the nodal map \( \tilde{V}_9 \to (\mathbb{P}^2)^9 \) for Halphen curves is not dominant (see [8, pp. 389–391]), our proof of Proposition 4.1 does not apply to \( \tilde{V}_9 \). Here we instead prove the unirationality of \( M_{10,10,1} \) using the description as a modular variety.

**Theorem 5.6.** The moduli space \( M_{10,10,1} \) is unirational.

**Proof.** Recall that \( M_{10,10,1} \) is an open set of the arithmetic quotient \( \mathcal{F}(\mathcal{O}(L_1)^+) \) for the lattice \( L_1 = U \oplus \langle 2 \rangle \oplus \langle -2 \rangle \oplus E_8(2) \). By Proposition 2.4, we have an isomorphism \( \mathcal{F}(\mathcal{O}(L_1)^+) \simeq \mathcal{F}(\mathcal{O}(L_2)^+) \) for the odd lattice \( L_2 = U(2) \oplus \langle 1 \rangle \oplus \langle -1 \rangle \oplus E_8 \). Let \( L_3 \) be the lattice \( U(2)^2 \oplus E_8 \) and \( \{u, v\} \) be the basis of its second summand \( U(2) \) as defined in (2.2). Then \( L_2 \) is isometric to the overlattice \( \langle L_3, \frac{1}{2}(u + v) \rangle \) of \( L_3 \). Thus, \( \mathcal{F}(\mathcal{O}(L_2)^+) \) is dominated by \( \mathcal{F}(\mathcal{O}(L_3)^+) \) by Proposition 2.3. The variety \( \mathcal{F}(\mathcal{O}(L_3)^+) = \mathcal{M}_{10,4,0} \) is unirational by Corollary 5.2. Hence \( \mathcal{F}(\mathcal{O}(L_1)^+) \) is unirational. \( \square \)

**Remark 4.** Alternatively, considering morphisms to \( \mathbb{P}^2 \) of genus 1 and degree 6, one can prove that \( V_9 \) is unirational using, for example, the relative Poincaré bundle for a rational elliptic surface with a section.
6. The case $r = 11$

In this section, we prove that $\mathcal{M}_{11,11,1}$ is unirational (Section 6.1) and that the covers $\tilde{\mathcal{M}}_{11,a,\delta}$ are unirational for $a \leq 9$ (Section 6.2).

6.1. $\mathcal{M}_{11,11,1}$ and Coble curves

Let $\tilde{V}_{10}$ be the variety defined in (4.1). By Theorem 4.3, we have an open immersion $\tilde{V}_{10}/\mathrm{PGL}_3 \to \mathcal{M}_{11,11,1}$ and hence a dominant morphism $\mathcal{P}: \tilde{V}_{10}/\mathrm{PGL}_3 \to \mathcal{M}_{11,11,1}$. Clearly, $\mathcal{P}$ descends to a morphism $V_{10}/\mathrm{PGL}_3 \to \mathcal{M}_{11,11,1}$. The Severi variety $V_{10}$ is dense in the variety of rational plane sextics (cf. [15]). As the latter is dominated by the variety of morphisms $\mathbb{P}^1 \to \mathbb{P}^2$ of degree 6, which is obviously rational, we have the following.

**Theorem 6.1.** The moduli space $\mathcal{M}_{11,11,1}$ is unirational.

6.2. $\tilde{\mathcal{M}}_{11,9,1}$ and degenerated cubic pairs

Let $U \subset |\mathcal{O}_{\mathbb{P}^2}(6)| \times (\mathbb{P}^2)^8$ be the variety of pointed sextics $(C_1 + C_2, p) = (C_1 + C_2, p_1, \ldots, p_8)$ such that $C_1$ is a smooth cubic, that $C_2$ is an irreducible cubic with a node and transverse to $C_1$, and that $p_1, \ldots, p_8$ are distinct points contained in $C_1 \cap C_2$. Letting $p_9$ be the remaining intersection point of $C_1$ and $C_2$, and $p_{10}$ be the node of $C_2$, we have the complete labelling $(p_1, \ldots, p_{10})$ of the nodes of $C_1 + C_2$. As in Section 5.1, we have a geometric quotient $U/\mathrm{PGL}_3$ as a nine-dimensional variety.

**Lemma 6.2.** The variety $U$ is unirational.

**Proof.** Let $V$ denote the variety of irreducible cubics with nodes and $\mathcal{C} \subset V \times \mathbb{P}^2$ be the universal curve over $V$. Let $X = \mathcal{C} \times_V \mathcal{C} \cdots \mathcal{C} \times_V \mathcal{C}$ (eight times). We have a morphism $\pi: X \to \mathcal{C}$ defined by $(C_1 + C_2, p) \mapsto (C_2, p)$. A general $\pi$-fibre is an open set of a line in $|\mathcal{O}_{\mathbb{P}^2}(3)|$, namely the linear system $|\mathcal{O}_{\mathbb{P}^2}(3)|$ for the blow-up $Y$ of $\mathbb{P}^2$ at $\{p_i\}_{i=1}^8$. Therefore, $U$ is birational to $X \times \mathbb{P}^1$. Take a nodal cubic $[C] \in V$. Since $\mathrm{PGL}_3 \cdot [C] = V$, we have $\mathrm{PGL}_3 \cdot (C)^8 = X$ and hence $X$ is unirational.

For a pointed sextic $(C_1 + C_2, p) \in U$, the 2-elementary $K3$ surface $(X, \iota)$ associated to $C_1 + C_2$ has the main invariant $(11,9,1)$. As before, the above labelling of the nodes induces a natural isometry $j: L_+ \to L_+(X, \iota)$ from a reference lattice $L_+$, and this defines a morphism $\tilde{\rho}: U \to \tilde{\mathcal{M}}_{11,9,1}$. Then we see the following.

**Theorem 6.3.** The period map $\tilde{\rho}$ descends to an open immersion $U/\mathrm{PGL}_3 \to \tilde{\mathcal{M}}_{11,9,1}$.

**Corollary 6.4.** For $a \leq 9$, the covers $\tilde{\mathcal{M}}_{11,a,\delta}$ are unirational.

7. The case $r = 12$

In this section, we study the case $r = 12$. In Section 7.1, we construct a birational map from the configuration space of eight general points in $\mathbb{P}^2$ to a certain cover of $\mathcal{M}_{12,10,1}$, which, in particular, implies that $\mathcal{M}_{12,10,1}$ is unirational. In Section 7.2, we prove that the covers $\tilde{\mathcal{M}}_{12,a,\delta}$ for $a \leq 8$ are unirational.
7.1. $\mathcal{M}_{12,10,1}$ and eight general points in $\mathbb{P}^2$

We begin by preparing lattices and an arithmetic group. Let $M_{12} = \langle h, e_1, \ldots, e_{11} \rangle$ be the lattice defined in (2.1). Let $f_1, f_2 \in M_{12}'$ be the vectors defined by $2f_i = 3h - 2e_i - \sum_{j=3}^{11} e_j$, $i = 1, 2$. Then the overlattice $L_+ = \langle M_{12}, f_1, f_2 \rangle$ is even and 2-elementary of the main invariant $(12,10,1)$. We fix a primitive embedding $L_+ \subset \Lambda_{K3}$, which exists by Figure 1, and set $L_-(L_+) = (L_+)^\perp \cap \Lambda_{K3}$. The lattice $L_-$ is isometric to $(2)^2 \oplus (-2)^8$. We let the symmetric group $S_3$ act on the set $\{e_0, e_{10}, e_{11}\}$ by permutation, and on the set $\{h, e_1, \ldots, e_8\}$ trivially. This defines an action $i: S_3 \to O(L_+)$ of $S_3$ on the lattice $L_+$. Let $r_\pm: O(L_+) \to O(D_{L_\pm})$ be the natural homomorphisms and let $\lambda: O(D_{L_\pm}) \to O(D_{L_-})$ be the isomorphism induced by the relation $L_- = (L_+)^\perp$. Then we define a subgroup of $O(L_-)$ by $\Gamma = r_\pm^{-1}(\lambda \circ r_+(i(S_3)))$. By Proposition 2.1, an isometry $\gamma$ of $L_-$ is contained in $\Gamma$ if and only if there exists a $\sigma \in S_3$ such that $i(\sigma) \oplus \gamma$ extends to an isometry of $\Lambda_{K3}$. We have $\tilde{O}(L_-) \subset \Gamma$ with $\Gamma^+/\tilde{O}(L_-)^+ \simeq S_3$. Hence the modular variety $F_{L_-}(\Gamma^+)$ is a quotient of $\mathcal{M}_{12,10,1}$ by $S_3$. The moduli space $\mathcal{M}_{12,10,1}$ is dominated by $F_{L_-}(\Gamma^+)$. We shall define a parameter space. First, we note that for seven general points $q_1, \ldots, q_7$ in $\mathbb{P}^2$ there uniquely exists an irreducible nodal cubic $C$ passing $q_1, \ldots, q_7$ with $\text{Sing}(C) = q_1$. Indeed, the blow-up $Y$ of $\mathbb{P}^2$ at $q_2, \ldots, q_7$ is a cubic del Pezzo surface. We embed $Y$ in $\mathbb{P}^3$ naturally. Then the intersection of $Y$ with the tangent plane of $Y$ at $q_1$ is an irreducible nodal $-K_Y$-curve, whose image in $\mathbb{P}^2$ gives the desired cubic $C$. Now let $U \subset (\mathbb{P}^2)^8$ be the open set of eight distinct points $p = (p_1, \ldots, p_8)$ such that there exist irreducible nodal cubics $C_1, C_2$ which pass $p_3, \ldots, p_8$ with $\text{Sing}(C_i) = p_i$ and which are transverse to each other (Figure 2). The finite morphism $U \to |O_{\mathbb{P}^2}(6)|$, $p \mapsto C_1 + C_2$, shows that we have a geometric quotient $U/\Gamma_{12}$ as an eight-dimensional variety, which is rational by Proposition 2.9.

For $p = (p_1, \ldots, p_8) \in U$, the associated sextic $C_1 + C_2$ is endowed with the partial labelling $(p_1, \ldots, p_8)$ of its nodes. The remaining three nodes $S = C_1 \cap C_2 \backslash \{p_i\}_{i=3}^{8}$ are not marked. We temporarily choose a bijection $S \simeq \{9, 10, 11\}$ and accordingly denote $S = \{p_9, p_{10}, p_{11}\}$. Then let $(X, i)$ be the 2-elementary $K3$ surface associated to $C_1 + C_2$. If $g: X \to \mathbb{P}^2$ is the natural projection branched over $C_1 + C_2$, then we have an isometry $L_- = L_-(X, i)$ defined by $h \mapsto [g^*O_{\mathbb{P}^2}(1)], e_i \mapsto [g^{-1}(p_i)]$ for $i \leq 11$, and $f_j \mapsto [F_j]$ where $F_j$ is the component of $X^i$ with $g(F_j) = C_j$. Then the period of $(X, i, j)$ is determined as a point in $\mathcal{M}_{12,10,1}$. We consider the image of that point in $F_{L_-}(\Gamma^+)$ and denote it by $\mathcal{P}(p) \in F_{L_-}(\Gamma^+)$. Theorem 7.1. The map $\mathcal{P}: U \to F_{L_-}(\Gamma^+)$ is well-defined. It is a morphism of varieties and induces an open immersion $U/\Gamma_{12} \to F_{L_-}(\Gamma^+)$. Proof. For the first assertion, it suffices to show that $\mathcal{P}(p)$ is independent of the choice of a labelling $S = \{p_9, p_{10}, p_{11}\}$. For another labelling $S = \{p_9', p_{10}', p_{11}'\}$, we have $p_{\sigma(i)} = p_i'$ for a $\sigma \in S_3$, $9 \leq i \leq 11$. Then the isometry $j': L_+ \to L_-(X, i)$ associated to $(p_9', p_{10}', p_{11}')$ is given

\[\text{Figure 2. Sextic curve for (r, a, \delta) = (12,10,1).}\]
with points, and consider the space

For the proof of unirationality, it is convenient to reduce sextics with labellings to such cubics

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Figure 3. Sextic curve for \((r, a, δ) = (12, 8, 1)\).

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\(j' = j \circ i(σ)\). If \(Φ, Φ' : Λ_{K3} \to H^2(X, Z)\) are extensions of \(j\) and \(j'\), respectively, then \(Φ|_{L-}\) is \(Γ\)-equivalent to \(Φ'|_{L-}\).

The map \(P\) is obviously \(PGL_3\)-invariant. Conversely, suppose that \(P(p) = P(p')\) for two \(p, p' \in U\). We choose labellings of the three nodes for \(p\) and \(p'\), respectively, and let \((X, j)\) and \((X', j')\) be the associated marked \(K3\) surfaces. Then the equality \(P(p) = P(p')\) means that we have a Hodge isometry \(Φ : H^2(X, Z) \to H^2(X', Z)\) with \(Φ \circ j = j' \circ i(σ)\) for some \(σ \in Γ_3\).

In particular, we have \(Φ(j(h)) = j'(h), Φ(j(f_j)) = j'(f_j), \) and \(Φ(j(ϵ)) = j'(ϵ_i)\) for \(i ≤ 8\). As before, we deduce that \(p\) and \(p'\) are \(PGL_3\)-equivalent. This concludes the proof.

**Corollary 7.2.** The variety \(F_{L-}(Γ^+)\) is rational. Hence \(M_{12,10,1}\) is unirational.

**Remark 5.** The space \(U/PGL_3\) is birationally identified with the moduli of marked del Pezzo surfaces of degree 1. It would be interesting to study the rational action of the Weyl group on \(F_{L-}(Γ^+)\) induced by the above immerssion. Kondō [22] described the moduli of del Pezzo surfaces of degree 1 as a ball quotient.

7.2. The unirationality of \(M_{12,8,1}\)

Let \(U \subset |O_{P2}(3)| \times (P^2)^8\) be the locus of cubics with points \((C, p) = (C, p_1, \ldots, p_8)\) such that (i) \(p_1, \ldots, p_8\) are distinct, (ii) \(C\) is smooth and passes \(\{p_i\}_{i=6}\), (iii) \(p_1, \ldots, p_6\) lie on a smooth conic \(Q\), (iv) \(p_6, p_7, p_8\) lie on a line \(L\), and (v) \(C + Q + L\) has only nodes as singularities. The sextic \(C + Q + L\) is uniquely determined by \((C, p)\) (Figure 3). By setting \(p_9 = L \cap C\{p_7, p_8\}, p_{10} = L \cap Q\{p_6\}, p_{11} = Q \cap C\{p_i\}_{i=1}^5\), we have a complete marking of the nodes of \(C + Q + L\).

For the proof of unirationality, it is convenient to reduce sextics with labellings to such cubics with points, and consider the space \(U\) of the latter.

**Lemma 7.3.** The variety \(U\) is unirational.

**Proof.** Let \(V \subset (P^2)^6\) be the locus of six points \((p_1, \ldots, p_6)\) lying on some conic and let \(W \subset (P^2)^3\) be the locus of three collinear points \((q_1, q_2, q_3)\). The fibre product \(V ×_{P^2} W\) over \(P^2 = \{p_6 ∈ P^2\} = \{q_1 ∈ P^2\}\) is birational to the image of the projection \(U → (P^2)^8, (C, p) → p\). As a general fibre of the projection \(U → V ×_{P^2} W\) is an open set of a plane in \(|O_{P2}(3)|\), it suffices to prove the unirationality of \(V ×_{P^2} W\), which is easily reduced to that of \(V\). Let \(p_1, \ldots, p_4 ∈ P^2\) be four general points and let \(S\) be the blow-up of \(P^2\) at \(\{p_i\}_{i=1}^4\). The conic pencil determined by \(\{p_i\}_{i=1}^4\) defines a morphism \(S → P^1\). We have a birational map \(PGL_3 × (S × P^1) → V\). Then the existence of sections of \(S → P^1\) implies the rationality of \(S × P^1\).

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For a \((C, p) \in U\), the 2-elementary \(K3\) surface \((X, i)\) associated to the sextic \(C + Q + L\) has main invariant \((12, 8, 1)\). As before, our labelling for \(C + Q + L\) will induce an isometry \(j: L_+ \to L_+(X, i)\) from an appropriate reference lattice \(L_+\). This defines a morphism \(\tilde{p}: U \to \mathcal{M}_{12,8,1}\), and we have the following.

\[ \text{THEOREM 7.4.} \quad \text{The period map } \tilde{p} \text{ descends to an open immersion } U/PGL_3 \to \mathcal{M}_{12,8,1} \text{ from a geometric quotient } U/PGL_3. \]

\[ \text{COROLLARY 7.5.} \quad \text{For } a \leq 8, \text{ the covers } \mathcal{M}_{12,a,\delta} \text{ are unirational.} \]

8. The case \(r = 13\)

In this section, we study the case \(r = 13\). In Section 8.1, we construct a birational map from a configuration space of eight special points in \(\mathbb{P}^2\) to a certain cover of \(\mathcal{M}_{13,9,1}\) in a similar way as in Section 7.1. In Section 8.2, we prove that the covers \(\mathcal{M}_{13,a,\delta}\) are unirational for \(a \leq 7\).

8.1. \(\mathcal{M}_{13,9,1}\) and eight special points in \(\mathbb{P}^2\)

Let \(M_{13} = (h, e_1, \ldots, e_{12})\) be the lattice defined in (2.1). We define the vectors \(f_1, f_2, f_3 \in M_{13}^\vee\) by \(2f_3 = 3h - 2e_1 - \sum_{i=3}^{11} e_i, 2(f_1 + f_2) = 3h - 2(e_2 + e_{12}) - \sum_{i=3}^{11} e_i,\) and \(2f_2 = 2h - (e_2 + e_{12}) - \sum_{i=3}^{10} e_i\). The overlattice \(L_+ = \langle M_{13}, f_1, f_2, f_3 \rangle\) is 2-elementary of the main invariant \((13, 9, 1)\). We let \(\mathfrak{S}_2\) act on \(L_+\) by the permutation on \(\{e_9, e_{10}\}\). We fix a primitive embedding \(L_+ \subset \Lambda_{K3}\) and set \(L_-(L_+) = (L_+) \cap \Lambda_{K3}.\) The lattice \(L_-\) is isometric to \(\langle 2 \rangle^2 \oplus \langle -2 \rangle^7.\) Then let \(\Gamma \subset O(L_-)\) be the group \(\mathfrak{r}^{-1}(\lambda \circ r_+(\mathfrak{S}_2)),\) where \(r_\pm: O(L_{\pm}) \to O(D_{L_{\pm}})\) and \(\lambda: O(D_{L_+}) \to O(D_{L_-})\) are defined as in Section 7.1. The arithmetic quotient \(F_{L_-}(\Gamma^+)\) is a quotient of \(\mathcal{M}_{13,9,1}\) by \(\mathfrak{S}_2\) and dominates \(\mathcal{M}_{13,9,1}\).

Let \(V \subset (\mathbb{P}^2)^8\) be the codimension 1 locus of eight distinct points \(p = (p_1, \ldots, p_8)\) such that (i) there exists an irreducible nodal cubic \(C\) passing \(p_i, i \neq 2\) with \(\text{Sing}(C) = p_1, (ii) p_2\) lies on the line \(L = \overline{p_3p_4}, (iii)\) there exists a smooth conic \(Q\) passing \(p_2 \cup \{p_3, \ldots, p_8\}\), and (iv) the sextic \(C + Q + L\) has only nodes as singularities (Figure 4). We shall denote \(p_{11} = L \cap C \setminus \{p_3, p_4\}\) and \(p_{12} = L \cap Q \setminus p_2.\) In this way, we obtain from \(p\) the sextic \(C + Q + L\) and the partial labelling \((p_1, \ldots, p_8, p_{11}, p_{12})\) of its nodes. The remaining two nodes \(S = Q \cap C \setminus \{p_i\}_{i=5}^{8}\) are not naturally marked. We have a geometric quotient \(V/PGL_3\) as a seven-dimensional variety, which is rational by Proposition 2.9.

For a \(p \in V,\) let \((X, i)\) be the 2-elementary \(K3\) surface associated to the sextic \(C + Q + L\). A temporary choice of a labelling \(S = \{p_9, p_{10}\}\) induces a natural isometry \(j: L_+ \to L_+(X, i)\), which defines a point in \(\mathcal{M}_{13,9,1}\) as the period of \((X, i), j).\) Considering the image in \(F_{L_-}(\Gamma^+)\) of
the period of \(((X,\iota),j)\), we obtain a well-defined morphism \(\mathcal{P}: V \rightarrow \mathcal{F}_{L_+}(\Gamma^+)\) as in Section 7.1. Then we have the following.

**Theorem 8.1.** The period map \(\mathcal{P}\) descends to an open immersion \(V/\text{PGL}_3 \rightarrow \mathcal{F}_{L_-}(\Gamma^+)\). In particular, \(\mathcal{F}_{L_-}(\Gamma^+)\) is rational and \(\mathcal{M}_{13,9,1}\) is unirational.

8.2. \(\mathcal{M}_{13,7,1}\) and pointed cubics

Let \(U \subset |\mathcal{O}_P(3)| \times (\mathbb{P}^2)^6\) be the space of pointed cubics \((C, p) = (C, p_{1+}, p_{1-}, \ldots, p_{3-})\) such that (i) \(C\) is smooth, (ii) \(p_{1+}, \ldots, p_{3-}\) are distinct points on \(C\), and (iii) if we denote \(L_i = \overline{p_i+p_{i-}}\), the sextic \(C + \sum L_i\) has only nodes as singularities. The variety \(U\) is rational, for the natural projection \(U \rightarrow (\mathbb{P}^2)^6\) is birational to the projectivization of a vector bundle on an open set. For a pointed cubic \((C, p) \in U\), we set \(p_i = L_i \cap C \backslash \{p_{i+}, p_{i-}\}\) and \(q_i = L_j \cap L_k\), where \(\{i, j, k\} = \{1, 2, 3\}\). Thus, we associate to \((C, p)\) the nodal sextic \(C + \sum L_i\) with the labelling \((p_i, q_i)\) of its nodes. As before, from these we will obtain a lattice-marked 2-elementary \(K3\) surface \(((X,\iota),j)\) of type \((13,7,1)\). This defines a morphism \(\tilde{p}: U \rightarrow \mathcal{M}_{13,7,1}\), and we have the following.

**Theorem 8.2.** The period map \(\tilde{p}\) descends to an open immersion \(U/\text{PGL}_3 \rightarrow \mathcal{M}_{13,7,1}\) from a geometric quotient \(U/\text{PGL}_3\).

**Corollary 8.3.** The covers \(\tilde{M}_{13,a,\delta}\) for \(a \leq 7\) are unirational.

9. The case \(r \geq 14\)

Let \(U_d, V_d \subset (\mathbb{P}^2)^d\) be the loci defined in Section 2.3. By Proposition 2.9, when \(d \geq 5\), we have geometric quotients \(U_d/\text{PGL}_3\) and \(V_d/\text{PGL}_3\) as rational varieties of dimension \(2d - 8\) and \(2d - 9\), respectively. In this section, we prove the following.

**Theorem 9.1.** One has birational period maps \(U_d/\text{PGL}_3 \rightarrow \tilde{M}_{28-2d,2d-6,\delta}\) and \(V_d/\text{PGL}_3 \rightarrow \tilde{M}_{29-2d,2d-7,1}\) for \(5 \leq d \leq 7\).

By Proposition 3.7 and Figure 1, we have the following corollary, which completes the proof of Theorem 1.1.

**Corollary 9.2.** The covers \(\tilde{M}_{r,a,\delta}\) are unirational for \(r \geq 14\).

Our constructions of the period maps are similar to those for eight points (Sections 7.1 and 8.1): we draw a sextic from a given point set, label its singularities in a natural way, and then associate a lattice-marked 2-elementary \(K3\) surface. Unlike the eight-point cases, our labellings for \(d \leq 7\) leave no ambiguity, and so we obtain points in \(\mathcal{M}_{r,22-r,\delta}\). One will find that these period maps may be derived from the ones for eight points by degeneration: as we specialize a configuration of points, the resulting sextic gets more degenerate, and the period goes to a Heegner divisor.

Theorem 9.1 for \(U_6\) was first found by Matsumoto–Sasaki–Yoshida [26]. Considering degeneration, they essentially obtained the assertion also for \(V_6, U_5, V_5\) with \(\delta = 1\). The novelty
Let and the complete labelling of Theorem 9.1 is the constructions for Theorem 9.2. ˜cubic \( C \) by direct calculations in \( M \) as explained in Section 3.4 makes it easier to derive the monodromy groups, which were found from the ones in [26]. Specifically, from a given point set, we draw lines on the same plane, while in [26] the point set is regarded as a set of lines on the dual plane (see Remark 6). Our argument as explained in Section 3.4 makes it easier to derive the monodromy groups, which were found by direct calculations in [26].

9.1. \( \tilde{M}_{14,8,1} \) and seven general points in \( \mathbb{P}^2 \)

Let \( U \subset (\mathbb{P}^2)^7 \) be the open set of seven distinct points \( p = (p_1, \ldots, p_7) \) such that (i) there exists an irreducible nodal cubic \( C \) passing \( p_1, \ldots, p_7 \) with \( \text{Sing}(C) = p_7 \) and (ii) if we denote \( L_i = p_i p_{i+3} \) for \( i \leq 3 \), the sextic \( C + \sum_i L_i \) has only nodes as singularities (Figure 5). We put \( q_i = L_i \cap C \setminus \{p_i, p_{i+3}\} \) and \( q_{ij} = L_i \cap L_j \). We thus obtain from \( p \) the nodal sextic \( C + \sum_i L_i \) and the complete labelling \( (p_i, q_{ij})_{i,\mu} \) of its nodes. The components of \( C + \sum_i L_i \) are also labelled obviously. Taking the right resolution of \( C + \sum_i L_i \) and using these labellings, we obtain a lattice-marked 2-elementary \( K3 \) surface \( ((X,\iota),\jmath) \) of type \( (14,8,1) \) as before. This defines a morphism \( \tilde{p}: U \to \tilde{M}_{14,8,1} \), and we will see the following.

**Theorem 9.3.** The period map \( \tilde{p} \) descends to an open immersion \( U/\text{PGL}_3 \to \tilde{M}_{14,8,1} \) from a geometric quotient \( U/\text{PGL}_3 \).

In the next section, we degenerate the points \( p_5, p_6, p_7 \) to collinear position. This forces the cubic \( C \) to degenerate to the union of a conic and a line.

9.2. \( \tilde{M}_{15,7,1} \) and seven special points in \( \mathbb{P}^2 \)

Let \( V \subset (\mathbb{P}^2)^7 \) be the codimension 1 locus of seven distinct points \( p = (p_1, \ldots, p_7) \) such that (i) \( p_5, p_6, p_7 \) lie on a line \( L_0 \), (ii) \( p_1, \ldots, p_4, p_7 \) lie on a smooth conic \( Q \), and (iii) if we put \( L_i = p_i p_{i+3} \) for \( 1 \leq i \leq 3 \), the sextic \( Q + \sum_{i=0}^3 L_i \) has only nodes as singularities (Figure 6). We set \( q_0 = L_0 \cap Q \setminus p_7, q_i = L_i \cap Q \setminus p_i \) for \( i = 2, 3 \), and \( q_{ij} = L_i \cap L_j \) when \( q_{ij} \neq p_k \) for some \( k \). In this way, we obtain from \( p \) the sextic \( Q + \sum_i L_i \), the labelling \( (p_i, q_{ij})_{i,\mu} \) of its nodes, and also the obvious labelling of its components. As before, from these, we obtain a lattice-marked 2-elementary \( K3 \) surface of type \( (15,7,1) \). This defines a morphism \( \tilde{p}: V \to \tilde{M}_{15,7,1} \), and we have the following.

**Theorem 9.4.** The period map \( \tilde{p} \) descends to an open immersion \( V/\text{PGL}_3 \to \tilde{M}_{15,7,1} \) from a geometric quotient \( V/\text{PGL}_3 \).
In the next section, we degenerate $p_7$ on $p_1$. Then $p_7$ is determined as $p_7 = p_1 \cap p_5$, so that the parameters are reduced to six points (we make renumbering).

9.3. $\tilde{M}_{16,6,1}$ and six general points in $\mathbb{P}^2$

Let $U \subset (\mathbb{P}^2)^6$ be the open set of six distinct points $p = (p_1, \ldots, p_6)$ such that if we draw six lines by $L_1 = \overline{p_1p_2}$, $\ldots$, $L_5 = \overline{p_5p_6}$, and $L_6 = \overline{p_6p_1}$, then the sextic $\sum_i L_i$ has only nodes as singularities (Figure 7). Since the nodes of $\sum_i L_i$ are the intersections of the lines $L_i$, the labelling $(L_1, \ldots, L_6)$ of the lines induces that of the nodes, for example, by setting $p_{ij} = L_i \cap L_j$. Hence from $p$, we obtain the sextic $\sum_i L_i$ with a labelling of its nodes and components. This defines a lattice-marked 2-elementary $K3$ surface of type $(16, 6, 1)$. Thus, we obtain a morphism $\tilde{p}: U \to \tilde{M}_{16,6,1}$ and deduce the following.

**Theorem 9.5.** The period map $\tilde{p}$ descends to an open immersion $\tilde{P}: U/PGL_3 \to \tilde{M}_{16,6,1}$ from a geometric quotient $U/PGL_3$.

**Remark 6.** If we identify $\mathbb{P}^2 \simeq |O_{\mathbb{P}^2}(1)|$, the assignment $p \mapsto (L_1, \ldots, L_6)$ induces a Cremona transformation $w$ of $U/PGL_3$. The period map of [26] is written as $P \circ w^{-1}$. One sees that $w^2$ is the cyclic permutation $(654321)$ on $U/PGL_3$.

9.4. $\tilde{M}_{17,5,1}$ and six special points in $\mathbb{P}^2$

Let $V \subset (\mathbb{P}^2)^6$ be the codimension 1 locus of six distinct points $p = (p_1, \ldots, p_6)$ such that (i) $p_3, p_4, p_6$ are collinear and (ii) if we draw lines by $L_1 = \overline{p_1p_2}$, $\ldots$, $L_5 = \overline{p_5p_6}$, and $L_6 = \overline{p_6p_1}$, then any singularity of the sextic $\sum_i L_i$ other than $p_6$ is a node (Figure 8). The point $p_6$ is...
Figure 8. Sextic curve for $(r, a, \delta) = (17, 5, 1)$.

an ordinary triple point of $\sum_i L_i$. As in Section 9.3, we obtain a labelling of the nodes of $\sum_i L_i$ from the obvious one of the lines $L_i$. Denoting by $q_i$ the infinitely near point of $p_6$ given by $L_i$ for $i = 3, 5, 6$, we also obtain a labelling of the branches of $\sum_i L_i$ at $p_6$. The 2-elementary $K3$ surface $(X, \iota)$ associated to the sextic $\sum_i L_i$ has the main invariant $(17, 5, 1)$. Here, we encounter a triple point for the first time, but we can proceed as before referring to Example 2: if $g: X \to \mathbb{P}^2$ is the natural projection branched over $\sum_i L_i$, then the curve $g^{-1}(p_6)$ over $p_6$ consists of four labelled $(-2)$-curves, namely the $(-2)$-curves over $q_i$ and a component of $X'$. Together with the above labelling for the nodes and the lines, this induces an isometry $j: L_+ \to L_+(X, \iota)$ from a reference lattice $L_+$. This defines a morphism $\tilde{\rho}: V \to \tilde{\mathcal{M}}_{17,5,1}$, and we have the following.

**Theorem 9.6.** The period map $\tilde{\rho}$ descends to an open immersion $V/\text{PGL}_3 \to \tilde{\mathcal{M}}_{17,5,1}$ from a geometric quotient $V/\text{PGL}_3$.

Degenerating $p_2, p_4, p_5$ to collinear position produces a period map for $\tilde{\mathcal{M}}_{18,4,0}$ (Section 9.5), while degenerating $p_4$ to $p_3$ produces that for $\tilde{\mathcal{M}}_{18,4,1}$ (Section 9.6).

9.5. $\tilde{\mathcal{M}}_{18,4,0}$ and five general points in $\mathbb{P}^2$

Let $U \subset (\mathbb{P}^2)^5$ be the open set of five distinct points $p = (p_1, \ldots, p_5)$ such that no three of $p_1, \ldots, p_5$ other than $\{p_1, p_2, p_3\}$ are collinear. For a $p \in U$, we draw six lines by $L_i = \overline{p_ip_4}$ for $1 \leq i \leq 3$ and $L_i = \overline{p_{i-3}p_2}$ for $4 \leq i \leq 6$ (Figure 9). Then the sextic $\sum_{i=1}^6 L_i$ has ordinary triple points at $p_4$ and $p_5$, nodes at $L_i \cap L_j$ for $i \leq 3$ and $j \geq 4$, and no other singularity. The obvious labelling of the lines $L_i$ induces that of the nodes and the branches at the triple points of $\sum_i L_i$. The 2-elementary $K3$ surface $(X, \iota)$ associated to $\sum_i L_i$ has invariant $(r, a) = (18, 4)$. We have to identify its parity $\delta$. Let $(Y, B, \pi)$ be the right resolution of $\sum_i L_i$. We have the decomposition $B = \sum_{i=0}^7 B_i$ such that $\pi(B_i) = L_i$ for $1 \leq i \leq 6$ and $\pi(B_0) = p_5$, $\pi(B_7) = p_4$. One checks that the divisor $\left(\sum_{i=0}^3 B_i\right) - \left(\sum_{i=4}^7 B_i\right)$ is in $4NS_Y$. Hence $(X, \iota)$ has parity $\delta = 0$. Using our labelling for $\sum_i L_i$, we will obtain a morphism $\tilde{\rho}: U \to \tilde{\mathcal{M}}_{18,4,0}$. Then we have the following.

**Theorem 9.7.** The period map $\tilde{\rho}$ descends to an open immersion $U/\text{PGL}_3 \to \tilde{\mathcal{M}}_{18,4,0}$ from a geometric quotient $U/\text{PGL}_3$ (Figure 9).
9.6. \( \tilde{\mathcal{M}}_{18,4,1} \) and five general points in \( \mathbb{P}^2 \)

Let \( U_5 \subset (\mathbb{P}^2)^5 \) be the open set defined in Section 2.3. To a point \( p = (p_1, \ldots, p_5) \) in \( U_5 \), we associate six lines by \( L_1 = p_2 p_3 \), \( L_i = p_1 p_{i+2} \) for \( i = 2, 3 \), \( L_4 = p_5 p_{i-2} \) for \( i = 4, 5 \), and \( L_6 = p_4 p_5 \) (Figure 10). The sextic \( \sum_i L_i \) has ordinary triple points at \( p_4 \) and \( p_5 \). Any other singularity of \( \sum_i L_i \) is a node. The 2-elementary \( K3 \) surface \( (X, \iota) \) associated to \( \sum_i L_i \) has the invariant \( (r, a) = (18, 4) \). In order to determine its parity \( \delta \), let \( g: X \to \mathbb{P}^2 \) be the natural projection branched over \( \sum_i L_i \), and let \( E_{ij} \) be the \((-2)\)-curves \( g^{-1}(L_i \cap L_j) \) for \( i, j \leq 3 \). Then the \( \mathbb{Q} \)-divisor \( D = \frac{1}{2}(E_{12} + E_{23} + E_{31}) \) is in \( L_+(X, \iota) \) by Proposition 3.2. Since \( (D.D) = -\frac{3}{2} \), \( (X, \iota) \) has the parity \( \delta = 1 \). Using the obvious labelling of the lines \( L_i \), we obtain a morphism \( \tilde{p}: U_5 \to \tilde{\mathcal{M}}_{18,4,1} \) as before. Then we see the following.

**Theorem 9.8.** The period map \( \tilde{p} \) descends to an open immersion \( U_5/\text{PGL}_3 \to \tilde{\mathcal{M}}_{18,4,1} \).

9.7. \( \tilde{\mathcal{M}}_{19,3,1} \) and five special points in \( \mathbb{P}^2 \)

Let \( V_5 \subset (\mathbb{P}^2)^5 \) be the codimension 1 locus defined in Section 2.3. Given a point \( p = (p_1, \ldots, p_5) \) in \( V_5 \), for which \( p_1, p_2, p_3 \) are collinear, we define six lines in the same way as in Section 9.6: \( L_1 = p_2 p_3 \), \( L_i = p_1 p_{i+2} \) for \( i = 2, 3 \), \( L_4 = p_5 p_{i-2} \) for \( i = 4, 5 \), and \( L_6 = p_4 p_5 \) (Figure 11). Then the points \( p_1, p_4, p_5 \) are ordinary triple points of the sextic \( \sum_i L_i \), and any other singularity of \( \sum_i L_i \) is a node. As before, by taking the right resolution of the sextic \( \sum_i L_i \) and using the labelling \( (L_1, \ldots, L_6) \) of the lines, we obtain a lattice-marked 2-elementary \( K3 \) surface of type \( (19, 3, 1) \). This defines a morphism \( \tilde{p}: V_5 \to \tilde{\mathcal{M}}_{19,3,1} \). Then we have the following.

**Theorem 9.9.** The period map \( \tilde{p} \) descends to an open immersion \( V_5/\text{PGL}_3 \to \tilde{\mathcal{M}}_{19,3,1} \).
10. Moduli of Borcea–Voisin threefolds

The unirationality of $\mathcal{M}_{r,a,\delta}$ implies that of the moduli of Borcea–Voisin threefolds. Let $(X,\iota)$ be a 2-elementary $K3$ surface and $E$ be an elliptic curve. The involution $(\iota, -1_E)$ of $X \times E$ extends to an involution $j$ of the blow-up $\widetilde{X} \times E$ of $X \times E$ along the fixed curve of $(\iota, -1_E)$. The quotient $Z = \widetilde{X} \times E/(j)$ is a smooth Calabi–Yau threefold [4, 35]. The projection $\widetilde{X} \times E \to X$ (resp. $\widetilde{X} \times E \to E$) induces a fibration $\pi_1: Z \to Y = X/\langle \iota \rangle$ (resp. $\pi_2: Z \to E/(1_E)$) with constant $E$-fibres (resp. $X$-fibres), whose discriminant locus is the branch locus of the quotient morphism $X \to Y$ (resp. $E \to E/(1_E)$). Following [37], we call the triplet $(Z, \pi_1, \pi_2)$ the Borcea–Voisin threefold associated to $(X, \iota)$ and $E$. Two Borcea–Voisin threefolds are isomorphic if and only if the corresponding 2-elementary $K3$ surfaces and elliptic curves are, respectively, isomorphic [37]. The data $(\pi_1, \pi_2)$ may be regarded as a kind of polarization of $Z$, as the following remark shows.

**Lemma 10.1.** Let $(Z, \pi_1, \pi_2)$, $(Z', \pi'_1, \pi'_2)$ be Borcea–Voisin threefolds, and let $\Lambda$ (resp. $\Lambda'$) be the primitive closure of $\pi_1^* \text{Pic} Y$ in $\text{Pic} Z$ (resp. $(\pi'_1)^* \text{Pic} Y'$ in $\text{Pic} Z'$). Then we have $(Z, \pi_1, \pi_2) \simeq (Z', \pi'_1, \pi'_2)$ if and only if we have $(Z, \Lambda) \simeq (Z', \Lambda')$.

**Proof.** It suffices to prove the ‘if’ part. Let $f: Z \to Z'$ be an isomorphism with $f^* \Lambda' = \Lambda$. There exist a very ample line bundle $H$ on $Y$ and a line bundle $H'$ on $Y'$ with $\pi_1^* H \simeq f^* (\pi'_1)^* H'$. Since $|H| \cong |\pi_1^* H| \cong |(\pi'_1)^* H'| \cong |H'|$, we see that $H'$ has no base point. Via the projective morphisms $Z \to |\pi_1^* H'|$ and $Z' \to |(\pi'_1)^* H'|$, we obtain a morphism $g: Y' \to Y$ with $g \circ \pi_1' = \pi_1 \circ f^{-1}$. One checks that $g$ is bijective and hence is isomorphic. Considering the fibres and the discriminant loci of $\pi_1$ and $\pi'_1$, we obtain $E \simeq E'$ and $(X, \iota) \simeq (X', \iota')$. \hfill \Box

The main invariant of a Borcea–Voisin threefold is defined as that of the associated 2-elementary $K3$ surface. Obviously, two Borcea–Voisin threefolds are deformation-equivalent if and only if they have the same main invariant. Let $X(1) = \text{SL}_2(\mathbb{Z}) \backslash \mathbb{H}$ be the moduli space of elliptic curves.

**Theorem 10.2 ([37]).** The variety $\mathcal{M}_{r,a,\delta} \times X(1)$ is a coarse moduli space of Borcea–Voisin threefolds of the main invariant $(r, a, \delta)$.

By Theorem 1.1, we have the following.

**Theorem 10.3.** The moduli spaces of Borcea–Voisin threefolds are unirational.
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Appendix A
by Ken-Ichi Yoshikawa

In this section, we give a proof of the following result using automorphic forms.

**Theorem A.1.** The moduli space $M_{r,a,\delta}$ has Kodaira dimension $-\infty$ if either $13 \leq r \leq 17$ or $r + a = 22$, where $r \leq 17$.

This is a consequence of the following criterion due to Gritsenko [12] (the idea first appeared in [14]).

**Theorem A.2** (Gritsenko). Let $L$ be a lattice of signature $(2, n)$ with $n \geq 3$ and $\Gamma \subset O(L)^+$ be a subgroup of finite index. Following [13], let $R \subset \Omega_+^L$ denote the ramification divisor of the projection $\pi: \Omega_+^L \rightarrow F_L(\Gamma)$. Suppose we have an integer $\nu \geq 0$ and an automorphic form $F_k$ on $\Omega_+^L$ for $\Gamma$ of weight $k$ such that $k \geq \nu n$ and that $\nu R - \text{div}(F_k)$ is an effective divisor. If $k > \nu n$ or $\nu R - \text{div}(F_k) \neq 0$, then

$$\kappa(F_L(\Gamma)) = -\infty.$$  

**Proof.** When $\nu = 1$, the result is exactly [12, Theorem 1.5]. When $\nu > 1$, the same proof works after replacing $F_{nm}/F_k^m$ by $F_{nm}^{\nu}/F_k^m$ in the proof of [12, Theorem 1.5]. For the convenience of the reader, we give some detail. Assume $\omega \in H^0(F_L(\Gamma), mK_{F_L(\Gamma)})$, $m > 0$. Regard $\Omega_+^L$ as a tube domain of $\mathbb{C}^n$. Then $\pi^* \omega = F_{nm}(z)(dz_1 \wedge \cdots \wedge dz_n)^\otimes m$, where $F_{nm}(z)$ is a non-zero automorphic form on $\Omega_+^L$ for $\Gamma$ of weight $mn$. Since $\omega$ is holomorphic on $F_L(\Gamma)$, $F_{nm}$ must vanish on $R$ at least of order $m$ (cf. [13]). Hence $\text{div}(F_{nm}) - mR \geq 0$. Then $F_{nm}^{\nu}/F_k^m$ is an automorphic form for $\Gamma$ of weight $-m(k - \nu n) \leq 0$ with the effective divisor

$$\text{div}(F_{nm}^{\nu}/F_k^m) \geq m(\nu R - \text{div}(F_k)) \geq 0.$$  

Since $n \geq 3$, $F_{nm}^{\nu}/F_k^m$ must be a constant. Hence $k = \nu n$ and $\nu R = \text{div}(F_k)$, which contradicts the assumption.  

As an application of his criterion, Gritsenko gives several examples of orthogonal modular varieties with Kodaira dimension $-\infty$. See [12] for those examples. We thank Professor V.A. Gritsenko, whose lecture in the conference 'Moduli and Discrete Groups' at RIMS, Kyoto (2009), inspired this note and who kindly showed his paper [12] when we wrote this note.

A.1. The case $13 \leq r \leq 17$

**Theorem A.3.** If $13 \leq r \leq 17$, then $\kappa(M_{r,a,\delta}) = -\infty$.

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Proof. Let \( L_\ast \) be the anti-invariant lattice of a 2-elementary \( K3 \) surface of type \((r,a,\delta)\) with \( r \geq 11\). We denote \( g = 11 - \frac{1}{2}(r + a)\). By [38, Theorem 8.1], there exists an automorphic form \( \Psi_{L_\ast} \) for \( O(L_\ast)^+ \) of weight \( k = (r - 6)(2^g + 1) \) with the divisor \( \text{div}(\Psi_{L_\ast}) = D'_{L_\ast} + (2^g + 1)D''_{L_\ast} \), where \( D'_{L_\ast} \) and \( D''_{L_\ast} \) are reduced divisors given by

\[
D'_{L_\ast} := \sum_{\lambda \in L_\ast, \lambda^2 = -2, \lambda \neq 0} \lambda^1, \quad D''_{L_\ast} := \sum_{\lambda \in L_\ast, \lambda^2 = -2, \lambda \neq 0} \lambda^1.
\]

By definition, \( D := D'_{L_\ast} + D''_{L_\ast} \) is the discriminant divisor of \( \Omega_{L_\ast}^+ \). Let \( R \subset \Omega_{L_\ast}^+ \) be the ramification divisor of the projection \( \Omega_{L_\ast}^+ \to \mathcal{F}(O(L_\ast)^+) \). We set \( \nu = 2^g + 1 \) in Theorem A.2. Since \( n = 20 - r \) and \( r \geq 13 \), we obtain \( k - \nu n = 2\nu(r - 13) \geq 0 \). Since \( R \supset D \) by [13, Proof of Theorem 1.1.], we obtain \( \nu R - \text{div}(\Psi_{L_\ast}) \geq (\nu - 1)D'_{L_\ast} \geq 0 \). When \( r > 13 \) or \( D'_{L_\ast} \neq 0 \), the result follows from Theorem A.2. When \( r = 13 \) and \( D'_{L_\ast} = 0 \), then \( L_\ast = U(2) \oplus M_7 \). Let \( r \in L_\ast \) be a vector with \( r^2 = -4 \). Since the reflection with respect to \( r \) is an element of \( O(L_\ast)^+ \), we obtain \( r^1 \in R \) and \( r^1 \notin D \), which implies that \( \nu R - \text{div}(\Psi_{L_\ast}) \neq 0 \). The result follows again from Theorem A.2.

\( \square \)

A.2. The case \( r + a = 22 \) and \( r \leq 17 \)

We construct an automorphic form for \( O(L_\ast)^+ \) satisfying the conditions in Theorem A.2 as a Borcherds product [5]. For this, we first construct a modular form of type \( \rho_{L_\ast} \) with those properties required in [5, Theorem 13.3]. In what follows, we write \( r_\ast = r(L_\ast), a_\ast = a(L_\ast) \), \( \sigma_\ast = 4 - r_\ast \). Let \( \text{Mp}_2(\mathbb{Z}) \) be the metaplectic double cover of \( \text{SL}_2(\mathbb{Z}) \), which is generated by \( S := (\left( \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} \right), \sqrt{\tau}) \) and \( T := (\left( \begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix} \right), 1) \). See [5, Section 2] for more about \( \text{Mp}_2(\mathbb{Z}) \).

A.2.1. Elliptic modular forms. We set \( q = e^{2\pi i \tau} \) for \( \tau \in \mathbb{H} \) and

\[
\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \quad \theta(2) = \sum_{n \in \mathbb{Z}} q^{n^2}, \quad \theta(2 + 1/2) = \sum_{n \in \mathbb{Z}} q^{(n + 1/2)^2}.
\]

Set \( \mathbb{M}_0(4) := \{ (\left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right), \sqrt{\tau} + d) \in \text{Mp}_2(\mathbb{Z}); c \equiv 0 \mod 4 \} \). By [6, Lemma 5.2], there exists a character \( \chi_0 \) such that \( \theta(2) \) is a modular form for \( \mathbb{M}_0(4) \) of weight 1/2 with character \( \chi_0 \).

Set \( \eta_{1 - 2s} \times \eta_{1 - 2s}^\ast (\tau) := \eta(\tau)^{-8} \eta(2\tau)^8 \eta(4\tau)^{-8} \) and define \( \psi_m(\tau), m \in \mathbb{Z}, \) by

\[
\psi_m(\tau) := \eta_{1 - 2s} \times \eta_{1 - 2s}^\ast (\tau)^2 \theta(2\tau)^{2 + m} - 2(m + 16) \eta_{1 - 2s} \times \eta_{1 - 2s}^\ast (\tau) \theta(2\tau)^m.
\]

Since \( \eta_{1 - 2s} \times \eta_{1 - 2s}^\ast (\tau) \) is a modular form for \( \mathbb{M}_0(4) \) of weight \(-4\) with trivial character, \( \psi_m(\tau) \) is a modular form for \( \mathbb{M}_0(4) \) of weight \((m - 8)/2\) with character \( \chi_0^m \). Since \( \eta_{1 - 2s} \times \eta_{1 - 2s}^\ast (\tau) = q^{-1} + 8 + 36q + O(q^2) \) and \( \theta(2\tau) = 1 + 2q + O(q^3) \), we obtain

\[
\psi_m(\tau) = q^{-2} + 2(-m^2 - 9m + 124) + O(q).
\]

Write \( \psi_m(\tau) = \sum_{l \in \mathbb{Z}} d_m(l) q^l \) and define \( h_m^{(i)}(\tau), i \in \mathbb{Z}/4\mathbb{Z}, \) as the series

\[
h_m^{(i)}(\tau) := \sum_{l \equiv i \mod 4} d_m(l) q^{l/4}.
\]

Then we have \( \sum_{i \in \mathbb{Z}/4\mathbb{Z}} h_m^{(i)}(\tau) = \psi_m(\tau/4) \).

A.2.2. Vector-valued elliptic modular forms. Let \( \mathbb{C}[D_{L_\ast}] \) be the group ring of the discriminant group \( D_{L_\ast} \) with the standard basis \( \{ e_{\gamma} \}_{\gamma \in D_{L_\ast}} \). The Weil representation \( \rho_{L_\ast} : \text{Mp}_2(\mathbb{Z}) \to \text{GL}(\mathbb{C}[D_{L_\ast}]) \) is defined as follows (cf. [5, Section 2]):

\[
\rho_{L_\ast}(T) e_{\gamma} := e^{2\pi i \gamma} e_{\gamma}, \quad \rho_{L_\ast}(S) e_{\gamma} := \frac{i^{-\sigma_\ast/2}}{|D_{L_\ast}|^{1/2}} \sum_{\delta \in D_{L_\ast}} e^{-2\pi i (\gamma, \delta)} e_{\delta}.
\]
We use the notion of modular forms of type $\rho_{L_-}$, for which we refer to [5, Section 2].
Our construction is based on the following observation due to Borcherds.

**Proposition A.4.** If $\phi(\tau)$ is a modular form for $\Gamma_0(4)$ with character $\chi^\rho_\sigma$, then

$$B_{L_-}[\phi](\tau) := \sum_{g \in \Gamma_0(4) \backslash \Gamma_2} \phi(g) \rho_{L_-}^{-1}(g^{-1}) e_0$$

is a modular form for $\Gamma_0(4)$ of the same weight as that of $\phi(\tau)$, where $\phi(g) := \phi((\text{ad } b)/(\text{ct } d))(\text{ct } d)^{-2l}$ for $g = (a \ b \ c \ d)^{-1}$.

**Proof.** See, for example, [38, Proposition 7.1].

Set $V := S^{-1}T^2S = \left( \begin{smallmatrix} 1 & 0 \\ 0 & -2 \tau + 1 \end{smallmatrix} \right)$. The coset $\Gamma_0(4) \backslash \Gamma_2$ is represented by $\{1, S, ST, ST^2, ST^3, V\}$. We define $v_k := \sum_{\delta \in D_{L_-}, \delta^2 \equiv k \mod 2} e_5 \in \mathbb{C}[D_{L_-}]$ for $k \in \mathbb{Z}/4\mathbb{Z}$. Let $1_{L_-} \in D_{L_-}$ be the unique element such that $(1_{L_-}, \gamma) = \gamma^2$ for all $\gamma \in D_{L_-}$. By [38, Proof of Lemma 7.5], we obtain the following relations:

$$\rho_{L_-}((ST^l)^{-1}) e_0 = i^{\sigma_-/2}2^{2-a_-/2} \sum_{k=0}^3 i^{-k} v_k, \quad \rho_{L_-}(V^{-1}) e_0 = e_{1_{L_-}},$$

$$\eta_{1-s} e_4 \cdot ST^i(\tau) = 2^4 \eta_{1-s} e_4 \cdot \left( \frac{\tau + l}{4} \right), \quad \eta_{1-s} e_4 \cdot |V(\tau)| = -16 \eta(2\tau)^{-16} \eta(4\tau)^8,$$

$$\theta_{(2)} |ST^i(\tau) = (2i)^{-1/2} \theta_{(2)} \left( \frac{\tau + l}{4} \right), \quad \theta_{(2)} |V(\tau) = \theta_{(2)+1/2}(\tau).$$

Then we obtain

$$\psi_m |ST^i(\tau) = 2^{(8-m)/2} i^{-m/2} \psi_m \left( \frac{\tau + l}{4} \right).$$

Since $\eta(2\tau)^{-16} \eta(4\tau)^8 = 1 + O(q)$ and $\theta_{(2)+1/2}(\tau) = 2q^{1/4} + O(q^{5/4})$, we obtain

$$\psi_m |V(\tau) = O(q^{m/4}).$$

In what follows, we assume $r_- < 12$ and $m = 8 + \sigma_-$. Then

$$\sum_{l=0}^3 \psi_m |ST^i(\tau) \rho_{L_-}((ST^l)^{-1}) e_0 = 2^{-(\sigma_- + a_-)/2} \sum_{j=0}^3 \sum_{s=Z/4Z} h_{m,j}^{(s)} (\tau + s) i^{-l} v_j$$

$$= 2^{(r_- - a_-)/2} \sum_{j=0}^3 h_{m,j}^{(j)} (\tau) v_j.$$

By Proposition A.4, $B_{L_-}[\psi_{8+\sigma_-}]$ is a modular form of weight $\sigma_-/2$. By the definition of $B_{L_-}[\psi_{8+\sigma_-}]$ and the expansion of $h_{m,j}^{(j)}$, we obtain the expansion

$$B_{L_-}[\psi_{8+\sigma_-}](\tau) = \psi_{8+\sigma_-}(\tau) e_0 + 2^{(r_- - a_-)/2} \sum_{l=0}^3 \psi_{8+\sigma_-}^{(l)} (\tau) v_l + \psi_{8+\sigma_-} |V(\tau) e_{1_{L_-}}$$

$$= \{q^{-2} + 2(-m^2 - 9m + 124) + O(q)\} e_0$$

$$+ 2^{(r_- - a_-)/2} \{2(-m^2 - 9m + 124) + O(q)\} v_0 + O(q^{1/4}) v_1$$

$$+ 2^{(r_- - a_-)/2} \{2^{-1/4} + O(q^{1/2})\} v_2 + O(q^{3/4}) v_3 + O(q^{m/4}) e_{1_{L_-}}.$$
obtain a zero divisor which implies

We have an explicit expression

We set

If

Theorem A.5. If

Proof. By the conditions

Thus we obtain

We set

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