Shear viscosity and entropy production related to viscous process in hot QGP at finite density

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Abstract

In the framework of irreversible thermodynamics, we studied the transport properties of QGP. Shear viscosity and non-equilibrium entropy density related to viscous process at finite density has been investigated in weakly coupled limit by using kinetic theory. The results show that the chemical potential increases viscosity yet decreases the non-equilibrium entropy density and thus contributes positively to their ratio compared to the pure temperature case. As to the temperature dependence of the ratio, the ratio first decreases rapidly and then increases in the physical region, presenting a minimum value of 0.4 at the temperature around 182MeV.

1 Introduction

Recently scientists believe that the quark-gluon plasma(QGP) found at relativistic heavy ion collider(RHIC) is strongly coupled, contrasting with the weakly coupled as expected before, which is the so-called sQGP [1–3]. They also believe when the temperature goes higher, for example $T > 2 - 4T_c$, the hot medium would be expected to become a weakly coupled system with some dissipative structures. Additionally, investigations on compact star demonstrate that the viscous properties are significant in explaining many of their
behaviors [4, 5]. Therefore dissipative structure of strong interaction system, especially
the sQGP, is a remarkable topic.

In irreversible thermodynamics, the discussions on the dissipative properties of a sys-
tem are focused on the entropy production in an unit time

$$\Delta S = \sum_i T_i X_i$$  \hspace{1cm} (1)

where $X_i$ is the thermal force which is determined by the gradients of energy, temperature,
chemical potential etc. $T_i$ is the corresponding flow driven by $X_i$ which can be written in
the linear response approximation as

$$T_i = \sum_j L_{ij} X_j$$  \hspace{1cm} (2)

where $L_{ij}$ are the transport coefficients. Thus it can be seen clearly that the entropy
production is determined by two factors: one is the thermal force $X_i$ which is the the
external cause describing the environment; the other factor is the transport coefficients
$L_{ij}$ which are the intrinsic causes reflecting the responsibility of the system driven by
the thermal force. Generally speaking, the entropy variation $\Delta S$ in an unit time can
reflect the evolution of the dissipative non-equilibrium state. Therefore the Eqs. (1) and (2)
provide us the basic evolulional information of a dissipative system theoretically.

Inserting Eq. (2) into Eq. (1), one could also analyze the state, or namely the thermal
force, of a dissipative system through the ratio of transport coefficients to the entropy
production which might be inversely obtained through fitting the experimental data. As
for the QGP produced at RHIC, this general discussion is focalized on the shear viscosity
to entropy density ratio which determines whether it is feasible to describe the hot and
dense medium by the ideal energy-momentum tensor $T_{\mu\nu}^0$. Besides the shear and bulk
viscosity induced by the velocity gradient, this tensor is also relevant to the thermal
conductivity induced by the energy gradient, which is zero with Landau and Lifshitz’s
definition on the hydrodynamic velocity [6].

In heavy ion physics, the evolution of the QGP density is almost longitudinal, with ve-
locity gradient along the transverse direction. Thereby bulk viscosity is usually neglected
due to the domination of shear viscosity. More proof from recent lattice simulation favors
the ignorance of the bulk viscosity since it is much smaller than the shear viscous one [7].
As a result, the shear viscosity and entropy density ratio becomes a very important means
of understanding the dissipative properties of the QGP produced in heavy ion collision.
As far as the sQGP is concerned, Ref. [7] demonstrated a result from lattice Monte Carlo simulation, pointing out the ratio of the shear viscosity to the entropy density is smaller than unit but most probably larger than the universal bound $1/4\pi$ which is obtained from the gauge theory/gravity duality [8–11] in equilibriate superstring theory. While one must notice the entropy density in this ratio is supposed to be an equilibrium one, which is not exactly the one appears in the Navier-Stocks equation. Some other estimations [12], e.g. from the Heisenberg uncertainty principle, has given a result as $\frac{1}{15}$.

Studies on this strong coupled mechanism are going on since it is desirable to understand the perfect behavior of the production at RHIC. As to wQGP(weakly coupled QGP), where the perturbative expansion works, people knew much more about its viscous properties than those of the sQGP. The transport coefficients, especially the shear viscosity, were discussed by many authors in weakly coupled limit in the kinetics theory [13–21] and in the thermal field theory via the Kubo formulae [22–33]. From the theoretical points of view based on the previous analysis, the ratio of shear viscosity to entropy density is also the foundation of understanding the wQGP’s dissipative properties. However most of those investigations mentioned above are usually focused on the extremely high temperature but zero chemical potential environment. Further studies on the ratio of shear viscosity to entropy density at finite density should be carried on, which is one of our motivations. In addition, as for the denomination of the ratio, only the equilibrium entropy density [8–11,34] and the entropy production in phase transition [12] are discussed. The entropy production induced by the dissipative forces should be included. In this paper, considering these two factors we calculate the shear viscosity of the wQGP at finite chemical potential and the corresponding entropy production in dissipative processes self-consistently so as to obtain the viscosity to entropy density ratio, representing a upper bound in the weak coupling limit.

The paper is arranged as following: shear viscosity and entropy density are evaluated in the framework of kinetic theory in Sec. 2 and 3 respectively. Summary and discussions will be presented in Sec. 4.

2 Shear viscosity

In this section, we derive an expression for shear viscosity $\eta$ of hot QGP at finite chemical potential in leading logarithm order in the framework of kinetics theory by using
variational approach, following Arnold et al [19] and our previous paper [21].

### 2.1 Formalism and definition

In a state that is departed not far from the equilibrium, the energy momentum tensor can be decomposed into ideal and dissipative parts as

\[ T^{\mu\nu} = T_0^{\mu\nu} + \pi^{\mu\nu} = (\epsilon + P)u^\mu u^\nu - Pg^{\mu\nu} + \pi^{\mu\nu} \]  

(3)

where \( \epsilon, P, \pi^{\mu\nu} \) are the energy density, pressure, and viscous shear stress, respectively. The four-velocity \( u^\mu(x) \) in the local rest frame is \((1, 0, 0, 0)\). In the linear response theory with Landau-Lifshitz convention, the fluctuation of spatial stress tensor is proportional to the first order of velocity gradients with neglected bulk viscosity by defining the coefficient \( \eta \) as shear viscosity,

\[ \delta \langle \pi_{ij} \rangle = -\eta(\partial_i u_j + \partial_j u_i - \frac{2}{3} \delta_{ij} \partial_k u^k) \equiv -\eta X_{ij}, \]  

(4)

where \( X_{ij} \) is the so-called driving force.

On one side, in kinetic theory the energy momentum tensor is the secondary moment of the distribution function,

\[ \langle T_{ij} \rangle = \int_{\mathbf{p}} \frac{p_i p_j}{p^0} \left[ g_f f(t, \mathbf{x}; \mathbf{p}) + g_f \bar{f}(t, \mathbf{x}; \mathbf{p}) + g_b b(t, \mathbf{x}; \mathbf{p}) \right], \]  

(5)

where the momentum space integration \( \int_{\mathbf{p}} \) is a shorthand for \( \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \), and \( f(x; \mathbf{p}), \bar{f}(x; \mathbf{p}) \) and \( b(x; \mathbf{p}) \) are one particle fermion, anti-fermion and boson distribution function respectively, with \( g_f, g_{\bar{f}} \) and \( g_b \) of their degeneration degrees.

On the other side, when one decomposed the one particle distribution function as a local equilibrium part plus a fluctuant part, namely \( f_s = n_s + \delta f_s \) where \( n_s \) is the equilibrium distribution function and \( s \) denotes for species of particles, the fluctuation of the energy moment tensor, which is exactly contributed by the fluctuation of the stress tensor, is

\[ \delta \langle T_{ij} \rangle = \int_{\mathbf{p}} \frac{p_i p_j}{p^0} (g_f \delta f + g_{\bar{f}} \delta \bar{f} + g_b \delta b) = \delta \langle \pi_{ij} \rangle = -\eta(\partial_i u_j + \partial_j u_i - \frac{2}{3} \delta_{ij} \partial_k u^k). \]  

(6)

In principle, one could evaluate the shear viscous coefficient as long as the distribution functions are known. While it is not an easy task because the one particle distribution function satisfies the Boltzmann equation of the usual form

\[ \left( \frac{\partial}{\partial t} + \hat{\mathbf{p}} \cdot \frac{\partial}{\partial \mathbf{x}} + \mathbf{F}_\text{ext} \cdot \frac{\partial}{\partial \mathbf{p}} \right) f_s(t, \mathbf{x}; \mathbf{p}) = -(\mathcal{C} f_s)(t, \mathbf{x}; \mathbf{p}), \]  

(7)
where $F_{\text{ext}}$ is the external field and the $C$ is the collision operator. Even in a simple steady system without external field, this evolution equation contains both differential and integral terms, where the collision integral might be rather complicated in different processes. In our paper, for a upper limit estimation, we just consider $2 \rightarrow 2$ elastic collisions to obtain the leading logarithm result, although for the relaxation and the shear viscosity, the inelastic scattering might be significant especially in the initial stage of QGP formation. Under this assumption, the collision operator is defined as

$$\langle C f \rangle(x; p) = \frac{1}{2} \int_{p', k, k'} \frac{|M(P, K; P', K')|^2}{(2p_0')(2p'_0)(2k_0)(2k'_0)} (2\pi)^4 \delta^{(4)}(P + K - P' - K') \times \{ f_s(p)f_s(k)[1 \pm f_s(p')][1 \pm f_s(k')] - f_s(p')f_s(k')[1 \pm f_s(p)][1 \pm f_s(k)] \},$$

where $p, k, p'$ and $k'$ denote the momenta of the incoming and outgoing particles respectively. $|M|^2$ is the two-body scattering amplitude. The $1 \pm f_s$ factor is the final state statistical weight, for boson with the upper sign and for fermion with the down sign.

For convenience, one can rewrite the fluctuation of distribution function in the form

$$\delta f_s = n_s(p)[1 \pm n_s(p)] \varphi_s(x; p) \tag{8}$$

Inserting the decomposed distribution function into the right hand side of the Boltzmann equation (7), one notices that $\langle C n_s \rangle(t; x, p) = 0$ when $n_s$ takes the form of Jüttner distribution as

$$n_s(t; x, p) = \frac{1}{e^{\beta(p \cdot u - \mu_s)} + 1} \tag{9}$$

with $\mu_s = \pm \mu$ for both fermion and anti-fermion. Notice that in the local rest frame, this one particle distribution function is degenerated into the ordinary fermion and boson distributions,

$$n_f(p) = \frac{1}{e^{\beta(p - \mu)} + 1}, \quad n_f(p) = \frac{1}{e^{\beta(p + \mu)} + 1}, \quad n_b(p) = \frac{1}{e^{\beta p} - 1} \tag{10}$$

Linearizing the collision term, one can obtain

$$\langle C f \rangle(x; p) = \frac{1}{2} \int_{p', k, k'} \sum_{bcd} |M_{abc}(P, K; P', K')|^2 \times (2\pi)^4 \delta^{(4)}(P + K - P' - K') \times n_s(p)n_b(k)[1 \pm n_c(p')][1 \pm n_d(k')] \times [\varphi_a(x; p) + \varphi_b(x; k) - \varphi_c(x; p') - \varphi_d(x; k')], \tag{11}$$
where $a b c d$ represent the species of the particles and $|M_{ab}^{cd}(P, K; P', K')|^2$ denotes for $\frac{|M_{ab}^{cd}(P, K; P', K')|^2}{(2\pi)^3(2\pi)^3(2\pi)^3(2\pi)^3}$. The sum in front of the matrix element means all possible collision processes relevant to the leading-log contribution are involved and properly treated without double counting or multi-counting.

As for the left hand side of the Boltzmann equation, the gradients acting on $\delta f_s$ give higher order in the departure from equilibrium, so that in the first order approximation, only $n_s$ should be considered on this side, namely,

$$LHS = \beta n_s(p)[1 \pm n_s(p)]I_{ij}(\hat{p})X_{ij}^s(x)$$

with $I_{ij}(\hat{p}) = \frac{1}{2}(\hat{p}_i\hat{p}_j - \frac{1}{3}\delta_{ij})$, where the time derivative and the the external field terms vanish for viscosity and the spatial tensor $X_{ij}^s(x)$ is just the driving force defined in Eq. (11).

Comparing both sides of the Boltzmann equation, one finds that $\varphi(x; p)$ on the right hand side must have the same angular dependence as the driving term, i.e.

$$\varphi_s(x; p) = \beta^2 I_{ij}(\hat{p})X_{ij}(x)\chi_s(p)$$

where $\chi_s(p)$ is rotationally invariant function depending only on the amplitude of momentum.

With equations (11), (12) and (13), one cancels the driving field on both sides and recasts the Boltzmann equation into

$$S_{ij}^s(p) = (C\chi_{ij}^s)(p)$$

where

$$S_{ij}^s(p) = -Tp n_s(p)[1 \pm n_s(p)]I_{ij}(\hat{p}),$$

$$\chi_{ij}^s(p) = I_{ij}(\hat{p})\chi_s(p).$$

To solve this equation, we introduced the variational approach following the basic steps in reference [19]. First, one defines an inner product as

$$(f, g) \equiv \beta^3 \int f(p)g(p).$$

Then taking product on both sides of the Boltzmann Eq. (14) and defining the functional

$$Q[\chi] \equiv (\chi_{ij}, S_{ij}) - \frac{1}{2}(\chi_{ij}, C_{ij})$$

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Figure 1: The possible processes which contribute to the leading-log in the collision term in QCD plasma. The solid line is for quark and the wiggly line is for gluon. The matrix elements arise from all the five diagrams are $16g^4d_A C_A^2 \left(3 - \frac{8u}{t^2} - \frac{8t}{u^2} - \frac{8s}{t^2}\right)$, $8g^4d_F C_F C_A \left(\frac{s^2+u^2}{t^2}\right)$, $8g^4d_F C_F^2 \left(\frac{s}{t} + \frac{u}{s}\right)$ and $-8g^4d_F C_F^2 \left(\frac{s}{t} + \frac{u}{s}\right)$, respectively, with $d_A = 8$, $d_F = C_A = 3$ and $C_F = \frac{4}{3}$ for SU(3) group.

With explicit form of each term as

$$\langle \chi_{ij}, S_{ij} \rangle = -\beta^2 \sum_a \int_p p f^a_0(p)[1 \pm f^a_0(p)]\chi^a(p), \quad (19)$$

$$\langle \chi_{ij}, C\chi_{ij} \rangle = \frac{\beta^2}{8} \sum_{p,k,k'} |M^{gh}_{abcd}|^2 (2\pi)^4 \delta^{(4)}(P + K - P' - K') \times n_a(p)n_b(k)[1 \pm n_c(p')][1 \pm n_d(k')] \times \left[\chi^a_{ij}(p) + \chi^b_{ij}(k) - \chi^c_{ij}(p') - \chi^d_{ij}(k')\right]^2, \quad (20)$$

one may relate the maximal value of $Q$ with the viscous coefficient through Eqs. (6), (8) and (13) with the fact that the maximum value of $Q[\chi]$ occurs when $\chi(p)$ satisfies Eq. (14), i.e.,

$$\eta = \frac{2}{15} Q_{\text{max}} \bigg|_{\chi = \chi_{\text{max}}}. \quad (21)$$

## 2.2 Collision terms

The collision terms of the Boltzmann equation are associated with different interaction processes. In the leading logarithm order, only five diagrams contribute, which are presented in Fig. (1).
Before starting calculation of these collision terms, some arguments and kinematic relations should be manifested first.

- **Approximations**

  Two approximations have been employed in our further evaluation. First, the high temperature approximation. We assume the temperature $T$ of the QGP is extremely high which is the only 'hard scale' in the system, with other quantities like chemical potential $\mu$, thermal mass etc. indicated by $gT$, are much less than the hard scale in weakly coupled theory. With this high temperature approximation, the fermion(anti-fermion) distribution function can be expanded in terms of $\mu/T$, neglecting the thermal mass term directly [21]. The second important approximation is the forward scattering approximation, i.e. the momentum transfer between the incident particles $q \sim gT$ is rather small so that the momentum difference between the incoming and outgoing particles on the same interaction vertex can be ignored in the distribution functions.

- **Kinematics**

  We mark the momenta of the particles as shown in Fig.1(c). The three usual Mandelstam variables are defined as $s = (P + K)^2$, $t = (P - P')^2$ and $u = (P - K')^2$. With the forward scattering approximation, the incident angle between $p$ and $k$ can be described by $\theta$ and $\phi$, which are the angle between $p$ and $q$ and the angle between the $pq$ plane and $p'q$ plane,

  \[
  \cos \theta_{pk} = 1 + (1 - \cos^2 \theta)(1 - \cos \phi). \tag{22}
  \]

  Now let us turn back to the Eq. (20). Performing the integral over $dk'$ with the help of $\delta^3(p + k + p' + k')$, one may transform the equation into

  \[
  (\chi_{ij}, C\chi_{ij}) = \frac{\beta^3}{(4\pi)^6} \int_0^\infty dq \int_{-q}^q dq \int_0^\infty d\omega \int_0^\infty dp \int_0^\infty dk \int_0^{2\pi} d\phi \sum_{abcd} |M_{abcd}|^2 n_a(p)n_b(k)[1 \pm n_c(p)][1 \pm n_d(k)]
  \]

  \[
  \times \left[\chi_{ij}^a(p) + \chi_{ij}^b(k) - \chi_{ij}^c(p') - \chi_{ij}^d(k')\right]^2. \tag{23}
  \]

  where $\omega$ is a dummy integration variable first introduced by Baym [17] with $p' = p + \omega$ and $k' = k - \omega$. 


It is easy to discover that the integrand of Eq. (23) is composed by three parts: the matrix element, the distribution functions and the $\chi$ term. According to the exchanged particle, the $\chi$ term can be sorted into two classes: Fig.1(a-c) belong to one category where the on-shell particles interact with each other through exchanging a boson, namely, they have the same species of incoming and outgoing particles on one interaction vertex; Fig.1(d) and (e) belong to another category where they contain off-shell fermions, i.e., different species of on-shell particles are bounded to the same interaction vertex. For the first category, the $\chi$ term contributes a small $q^2$ in the forwarding scattering approximation, which softens the infrared singularity and gives the leading-log form of viscosity [19]. For the two on-shell line of quarks, the $\chi$ term thus is specified as

$$\big[\chi_{ij}^q(p) - \chi_{ij}^q(p')\big]^2 \approx \big[q \cdot \nabla \chi_{ij}^q(p)\big]^2 = \omega^2 [\chi^q(p')^2] + 3 \frac{q^2 - \omega^2}{p^2} [\chi^q(p)]^2.$$  (24)

where $\chi^q(p') = d\chi^q(p)/dp$. We assumed that the quark and anti-quark have the same departure behavior from the equilibrium state which is denoted by $\chi^q$, since nothing but the viscous process is involved. The same trick will be performed on $\chi^g$, the gluon $\chi$ function, and leads to similar result. One can prove that the interference term like $[\chi^q(p) - \chi^q(p')][\chi^g(k) - \chi^g(k')]$ vanishes when carrying out the $d\omega$ and $d\phi$ integration. Following the above discussion, we can list the all the $\chi$ terms for different diagrams in Table 1, where the only survived formats are $(\chi^q - \chi^q)^2$, $(\chi^q - \chi^g)^2$ and $(\chi^g - \chi^g)^2$.

There are two facts that should be clarified. One is that all possible channels for one diagram are not to be added directly. In fact for each channel, one must perform a convolution integral for the exact $\chi$ term from the very channel with the corresponding distribution function. The other fact to be noticed is the relationship between the Mandelstam variables under the forward scattering approximation. For instance, as to the massless on-shell particles, $s = (P + K)^2 = 2P \cdot K \approx -(P - K')^2 = -u$ in $u$-channel. Similarly, one has $s \approx -t$ in $t$-channel. With this convenience, combining with Eq. (22) one finds out that all the matrix elements reduce to only two forms,

$$\left(\frac{s^2 + t^2}{u^2}\right)_{u\text{-channel}} = \left(\frac{s^2 + u^2}{t^2}\right)_{t\text{-channel}} \approx \frac{8p^2k^2}{q^4}(1 - \cos \phi)^2,$$  (25)

$$\left(\frac{t}{u}\right)_{u\text{-channel}} = \left(\frac{u}{t}\right)_{t\text{-channel}} \approx \frac{2pk}{q^2}(1 - \cos \phi).$$  (26)

The last part of collision integrand is the distribution function, which is the main distinguished feature for each diagram and channel. Analyzing each possible process
Table 1: The $\chi$ terms for five diagrams. The Fig.1(b) has two sets of $\chi$ functions because it involves different channels which bring on different momentum dependence of $\chi^q$ and $\chi^g$. 

| Processes   | $\chi$ functions                                                |
|------------|-----------------------------------------------------------------|
| Fig.1(a)   | $[\chi^q(p) - \chi^q(p')]^2 + [\chi^g(k) - \chi^g(k')]^2$     |
| Fig.1(b1)  | $[\chi^q(p) - \chi^q(p')]^2 + [\chi^g(k) - \chi^g(k')]^2$     |
| Fig.1(b2)  | $[\chi^q(p) - \chi^q(p')]^2 + [\chi^g(k) - \chi^g(k')]^2$     |
| Fig.1(c)   | $[\chi^q(p) - \chi^q(p')]^2 + [\chi^g(k) - \chi^g(k')]^2$     |
| Fig.1(d)(e)| $[\chi^q(p) - \chi^q(p')]^2 + [\chi^g(k) - \chi^g(k')]^2$     |
| Processes | Distribution functions |
|-----------|------------------------|
| Fig.1(a)  | 2n_b(p)n_b(k)[1 + n_b(p)][1 + n_b(k)] |
| Fig.1(b1) | 2N_f\{n_f(k)[1 - n_f(k)] + n_f(k)[1 - n_f(k)]\}n_b(p)[1 + n_b(p)] |
| Fig.1(b2) | 2N_fn_b(k)[1 + n_b(k)]\{n_f(p)[1 - n_f(p)] + n_f(p)[1 - n_f(p)]\} |
| Fig.1(c)  | 2N_f^2\{n_f(p)n_f(k)[1 - n_f(p)][1 - n_f(k)] + n_f(p)n_f(k)[1 - \bar{n}_f(k)][1 - n_f(k)] + n_f(p)n_f(k)[1 - n_f(p)][1 - n_f(k)]\} |
| Fig.1(d+e)| 2N_f\{n_f(p)n_b(k)[1 - n_f(k)][1 + n_b(p)] + n_f(p)n_b(k)[1 - n_f(k)][1 + n_b(p)] + n_b(p)n_f(k)[1 - n_f(p)][1 + n_b(k)] + n_b(p)n_f(k)[1 - n_f(p)][1 + n_b(k)]\} |

Table 2: The distribution function terms for the five processes, where N_f is the quark flavor. The factors in front of the distribution functions are the freedom of degeneration, which are related to the distinguished reaction channels. For example, q\bar{q} \leftrightarrow q\bar{q} appears 4N_f times in the sum over species.

Carefully, one can obtain the distribution function term for each diagram in Fig. 1, which is listed in Table 2.

Inserting the expressions in the three tables into Eq. (23) and carrying on the integration over dk, d\phi and d\omega, one can finally obtain the full collision term,

\begin{align}
(\chi_{ij}, C\chi_{ij}) &= (\chi_{ij}, C\chi_{ij})^{(a)} + (\chi_{ij}, C\chi_{ij})^{(b1)} + (\chi_{ij}, C\chi_{ij})^{(b2)} \\
&+ (\chi_{ij}, C\chi_{ij})^{(c)} + (\chi_{ij}, C\chi_{ij})^{(d+e)} \tag{27}
\end{align}

where

\begin{align}
(\chi_{ij}, C\chi_{ij})^{(a)} &= \frac{\alpha_s^2 d_A C_A^2}{3\pi} \int_T^\infty \frac{dq}{q} \int_0^\infty dp \ n_b(p)[1 + n_b(p)] \{p^2 [\chi^g(p)]^2 + 6 [\chi^g(p)]^4 \}, \tag{28}
\end{align}
\[
(\chi_{ij}, C_{\chi_{ij}})_{(b1)} = \frac{2\alpha_s^2 d_F N_f C_F C_A}{3\pi} \int_{\alpha_s T}^{\infty} dq \int_0^\infty dp \left\{ p^2 [\chi^q(p)]^2 + 6 [\chi^q(p)]^2 \right\} \\
\quad \times \left\{ n_f(p) [1 - n_f(p)] + n_f(p) [1 - n_f(p)] \right\}, 
\]

\[
(\chi_{ij}, C_{\chi_{ij}})_{(b2)} = \frac{2\alpha_s^2 d_F N_f C_F C_A}{3\pi} \left(1 + \frac{3}{\pi^2 T^2}\right) \int_{\alpha_s T}^{\infty} dq \int_0^\infty dp \left\{ n_b(p) [1 + n_b(p)] \right\} \\
\quad \times \left\{ p^2 [\chi^q(p)]^2 + 6 [\chi^q(p)]^2 \right\}, 
\]

\[
(\chi_{ij}, C_{\chi_{ij}})_{(c)} = \frac{2\alpha_s^2 d_F N_f C_F^2}{3d_A \pi} \int_{\alpha_s T}^{\infty} dq \int_0^\infty dp \left\{ p^2 [\chi^q(p)]^2 + 6 [\chi^q(p)]^2 \right\} \\
\quad \times \left\{ n_f(p) [1 - n_f(p)] + n_f(p) [1 - n_f(p)] \right\}, 
\]

\[
(\chi_{ij}, C_{\chi_{ij}})_{(d+c)} = \frac{2\alpha_s^2 d_F N_f C_F^2 \beta}{8d_A \pi^3} \int_{\alpha_s T}^{\infty} dq \int_0^\infty dp \left\{ p^2 [\chi^q(p) - \chi^q(p)]^2 \right\} \\
\quad \times \left\{ n_f(p) [1 + n_b(p)] \left( \frac{\pi^2}{8} - 0.616 \frac{\mu}{T} + \frac{\mu^2}{8T^2} \right) \\
+ n_f(p) [1 + n_b(p)] \left( \frac{\pi^2}{8} + 0.616 \frac{\mu}{T} + \frac{\mu^2}{8T^2} \right) \\
+ b_f(p) [1 - n_f(p)] \left( \frac{\pi^2}{8} + 0.616 \frac{\mu}{T} + \frac{\mu^2}{8T^2} \right) \\
+ n_b(p) [1 - n_f(p)] \left( \frac{\pi^2}{8} - 0.616 \frac{\mu}{T} + \frac{\mu^2}{8T^2} \right) \right\}, 
\]

with \(\alpha_s\) as the fine structure constant. We replace the limits of \(dq\) integration by the hard and the soft scale \(T\) and \(\alpha_s T\) due to the leading-log treatment, which includes the self-energy effect of the exchange line [19].

### 2.3 Variational method

So far we have obtained the right hand side of the Boltzmann equation, and the other side can be written down directly as

\[
(\chi_{ij}, S_{ij}) = -\frac{\beta^2}{\pi^2} \int_0^\infty dp \left\{ d_F N_f \left\{ n_f(p) [1 - n_f(p)] + n_f(p) [1 - n_f(p)] \right\} \chi^q(p) \\
+ d_A n_b(p) [1 + n_b(p)] \chi^q(p) \right\} 
\]

In this subsection, we expand the trial function \(\chi(p)\) by a finite set of basis with independent variational parameters to maximize the functional \(Q[\chi]\).
As we have known that in the case of viscosity, the $\chi$ function has two components, $\chi^q$ and $\chi^g$,

$$\chi(p) = \begin{pmatrix} \chi^g(p) \\ \chi^q(p) \end{pmatrix}. \quad (34)$$

Expanding the two components by the same basis

$$\chi^g(p) = \sum_{m=1}^{N} a_m \phi_m(p), \quad \chi^q(p) = \sum_{m=1}^{N} a_{N+m} \phi_m(p), \quad (35)$$

one could read out the basis-set components of $\tilde{S}$ and $\tilde{C}$

$$(S_{ij}, \chi_{ij}) = \sum_{m} a_m \tilde{S}_m, \quad (\chi_{ij}, C \chi_{ij}) = \sum_{mn} a_m \tilde{C}_{mn} a_n. \quad (36)$$

When the Boltzmann equation is satisfied, the coefficients in front of the bases will be expressed as $a = \tilde{C}^{-1} \tilde{S}$. Accordingly, the shear viscosity becomes

$$\eta = \frac{2}{15} Q_{\text{max}} = \frac{1}{15} a \cdot \tilde{S} = \frac{1}{15} \tilde{S}^t \tilde{C}^{-1} \tilde{S} \quad (37)$$

With the natural one function ansatz $\phi_1(p) = p^2/T$, one can evaluate the integral of Eqs. (27) and (33) analytically,

$$\tilde{S} = -\frac{\beta^3}{\pi^2} \int_0^\infty dp \, p^5 \begin{pmatrix} d_A n_b(p) [1 + n_b(p)] \\ d_F n_f \{ n_f(p) [1 - n_f(p)] + n_f(p) [1 - \bar{n}_f(p)] \} \end{pmatrix} \quad (38)$$

$$\tilde{C} = \frac{16 \pi^3 \alpha_s^2 T^3 \ln \alpha_s^{-1}}{9 d_A} \left[ \begin{array}{c} d_A C_A [d_A C_A] \\ \frac{7}{4} N_f d_F C_F [d_A C_A (1 + \frac{30}{7 \pi^2} \mu^2)] \\ 0 \end{array} \right] + \frac{9 \pi^2}{128} N_f d_F C_F^2 d_A \left( 1 - 0.013 \frac{\mu^2}{T^2} \right) \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}. \quad (39)$$
where \(\zeta(x)\) is the Riemann \(\zeta\)-function. Then one can compute the coefficient vector and the shear viscosity of 2-flavor QGP at finite temperature and chemical potential as,

\[
a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = (g^4 \ln \alpha_s^{-1})^{-1} \begin{pmatrix} -3.281 + 0.940 \frac{\mu^2}{T^2} \\ -6.792 + 0.199 \frac{\mu^2}{T^2} \end{pmatrix}
\]

(40)

and

\[
\eta \approx \frac{1.09 T^3}{\alpha_s^2 \ln \alpha_s^{-1}} \left( 1 + 0.25 \frac{\mu^2}{T^2} \right)
\]

(41)

which recovers the result in Ref. [19] when the chemical potential vanishes.

3 Entropy production in viscous process

From the non-equilibrium thermodynamical points of view, the entropy \(S\) symbols a scale that how far the system is apart from the equilibrium state where the entropy takes the maximum value. Other than the viscosity which reflects the intrinsic property of a system, the entropy depends further on the environment the system resides in, or the velocity gradient in our case. It is the velocity gradient that determined the evolution from non-equilibrium to equilibrium in a given system. Unfortunately, we know little about the velocity distribution which depends on the experimental environment and should be obtained by fitting data. In Sec. 4, a maximum estimation of the velocity gradient will be employed so as to demonstrate a maximum entropy production which leads to an upper bound of viscosity to entropy density ratio.

We introduce two different schemes to calculate the entropy production and find out that their results are comparable qualitatively.

3.1 Scheme I

In relativistic kinetic theory, the entropy density of a system is defined as

\[
s = - \sum_s \int_p g_s \left\{ f_s(x; p) \ln f_s(x; p) \mp [1 \pm f_s(x; p)] \ln [1 \pm f_s(x; p)] \right\}
\]

(42)

where \(g_s\) is the degeneration degree of each particle species.

Decomposing the distribution functions and inserting Eq. (8) into Eq. (42), one finds that only the second order in \(\varphi\) survives while the first order in \(\varphi\) vanishes due to the
rotational invariance of $n_s(p)$. The entropy density thus becomes

$$s = s_0 - \int \sum s g_s n_s(p) \left[1 \pm n_s(p)\right] \phi_s^2(\mathbf{x}; \mathbf{p})$$

where $s_0$ is the entropy density in the equilibrium state,

$$s_0 = -\int \left\{ g_f n_f \ln n_f + (1 - n_f) \ln (1 - n_f) \right\} + g_f n_f \ln n_f$$

$$+ (1 - n_f) \ln (1 - n_f) \} + g_b \left[ n_b \ln n_b - (1 + n_b) \ln (1 + n_b) \right] \right\}$$

$$= 16.22T^3 \left(1 + 0.123 \frac{\mu^2}{T^2} \right)$$

with $g_f = g_f = 2N_f d_F$ and $g_b = 2d_A$ and the integral evaluated by expanding the fermion distribution functions in terms of small $\mu/T$. It is clear that the second term of Eq. (43) denotes for the departure of the entropy density from its maximum value, i.e., the entropy production. As general, the non-equilibrium entropy density has the structure of $s = s_0 - \Delta s$, as we expect.

In this scheme we are going to directly evaluate the second term of Eq. (43) by adopting the solutions of the Boltzmann equation in Eq. (40), leaving only the velocity gradient to be estimated. Inserting expression (13) into Eq. (43) with the solved $\chi(p)$ in the variational approach, we can obtain

$$s = s_0 - \frac{\beta^6}{120\pi^2} \int_0^\infty dp \, \rho^6 \left\{ g_f [n_f (1 - n_f) + n_f (1 - n_f)] a_1^2 \right\}$$

$$+ g_b n_b (1 + n_b) a_2^2 \right\} \left( \frac{\partial u_z}{\partial x} \right)^2$$

$$= s_0 - \frac{9.75 T}{(\alpha_s^2 \ln(\alpha_s^{-1}))^2} \left( 1 + 0.284 \frac{\mu^2}{T^2} \right) \left( \frac{\partial u_z}{\partial x} \right)^2. \tag{46}$$

It’s obvious that the second term of Eq. (45) is the entropy production relative to viscous process when the system evolving to its equilibrium state, which depends on the kinetic parameter $\alpha_s$, the thermodynamic parameters $T$ $\mu$ and the transport environment $\partial u_z/\partial x$. To understand further on their physical meaning, we introduce a second scheme to evaluate the entropy produced in the viscous process.
3.2 Scheme II

From Scheme I, one can see the non-equilibrium entropy density is obtained by subtracting the entropy production from the equilibrium entropy density, which is denoted by

\[ s = s_0 - \int dX \partial_\mu s^\mu = s_0 - \int dt \partial_t s^0 \]

(47)

where \( s^\mu = su^\mu \) is the four entropy flow. The last equality is based on the description of longitudinal expansion model in local rest frame.

As we stated in Sec. 1, the entropy production could be obtained by the product of the driving force and corresponding thermodynamic flux. Generally speaking, the entropy produced in unit phase space \( \partial_\mu s^\mu \) takes a bilinear form: it is the sum of all kinds of transport flows, each multiplied by a characteristic thermodynamic driving force. Furthermore, under the linear law, the transport flow is proportional to the driving force with the coefficient of viscosity, conductivity, etc. Therefore, the universal expression for the entropy production should be the combination of squared driving forces coming from various transport processes, weighted by the corresponding transport coefficients.

In our case, we would like to know the entropy production just from the viscous process, which is [13]

\[ \partial_\mu s^\mu = \partial_\mu s^0 = \frac{\eta}{T} X_{ij}^2 = \frac{2\eta}{T} \left( \frac{\partial u_z}{\partial x} \right)^2 \]

(48)

where \( X_{ij} \) is defined in expression(4). Noticing that the integration variable \( t = \tau \cosh y \), where \( \tau \) describes the time scale of system departure, one could replace it by the relaxation time \( \tau_\eta \) for the maximum estimation. With this approximation, one can perform the integration on the pseudorapidity plateau,

\[ s \leq s_0 - \frac{4\eta \tau_\eta}{TR_A^2} \int_1^{\cosh y_0} (\sinh y)^2 \cosh y \, d \cosh y \]

(49)

\[ = s_0 - \frac{1.875 T}{(\alpha_s^2 \ln \alpha_s^{-1})^2} \left( 1 + 0.25 \frac{\mu^2}{T^2} \right)^2 \left( \frac{\partial u_z}{\partial x} \right)^2 \]

(50)

where \( y_0 = 2 \) is the edge value of the pseudorapidity plateau and \( \tau_\eta \) will be calculated in the appendix.

We argue that this estimation is consistent, for the relaxation time defined in the Boltzmann equation describes the one particle property in the proper reference frame. From Eq. [48] one can see clearly that the entropy produced in viscous process depends not only on the intrinsic property \( \eta \) but also on the exterior transport environment of
the system. These internal and external factors determine the dependence of the entropy density on the kinetic and thermodynamic parameters, and also the velocity gradient.

4 The ratio of $\eta$ to $s$

Now we are going to turn to a significant quantity $\eta/s$ which has been mentioned in the first section. In a pure viscous transport process, the state deviating from the equilibrium is solely stimulated by the spacial derivative of velocity field $\partial_{i}u_{j}$. For a rough estimation of the velocity gradient for a longitudinal Lorentz boost invariant system, we adopt Brioken’s assumption $u^{\mu} = x^{\mu}/\tau$ [35], where $x^{\mu}$ is the space-time point and $\tau$ is the proper time. In this scenario, the shear velocity is just along longitudinal direction $\hat{z}$ which is perpendicular to the velocity gradient direction $\hat{x}$. Thereby $\partial_{i}u_{j}$ has just one non-zero component $\partial u_{z}/\partial x$. Noticing that in the light-cone coordinate, $u_{z} = x_{z}/\tau = \sinh y$ where $y$ denotes the pseudo-rapidity and the maximum of velocity gradient is $\partial u_{z}/\partial x \leq u_{z}/R_{A} = \sinh y/R_{A}$ when $y$ takes the edge value of the pseudo-rapidity plateau. Here $R_{A}$ is the nucleus radius and the system is considered rotational invariant around the z-axis. In Au+Au 200GeV collision at RHIC, the pseudo-rapidity plateau is from $-2 \sim 2$ [36] in the most central collision, and the radius of gold nucleus is about 35.381GeV$^{-1}$.

With this maximal velocity gradient estimation, the ratio of viscosity to entropy density obtained in the two schemes are

$$\left(\frac{\eta}{s}\right)_{I} \geq \frac{\eta}{s_{0}} \left[1 - \frac{0.013GeV^{2}}{(T\alpha_{s}^{2}\ln\alpha_{s}^{-1})^{2}} \left(1 + 0.16\frac{\mu^{2}}{T^{2}}\right)\right]^{-1}, \quad (51)$$

$$\left(\frac{\eta}{s}\right)_{II} \geq \frac{\eta}{s_{0}} \left[1 - \frac{0.23GeV^{2}}{(T\alpha_{s}^{2}\ln\alpha_{s}^{-1})^{2}} \left(1 + 0.12\frac{\mu^{2}}{T^{2}}\right)\right]^{-1}, \quad (52)$$

respectively. These two results are qualitatively comparable for they have exactly the same dependence on dynamic and thermodynamic parameters except some differences on the factors.

We manage to demonstrate the basic features of the dependence of the ratio on the thermal parameters by adopting the result from scheme I. As their chemical potential dependence is quite clear, we here just present their variation on the temperature. To
Figure 2: The shear viscosity to entropy density ratio of 2-flavor QGP.

isolate the temperature dependence, we use a simple form of the running coupling constant [7]
\[ \alpha_s = \frac{2\pi}{11} \left( \ln \frac{4T}{1.5T_c} \right)^{-1}, \]  
(53)
where \( T_c \) is the critical temperature of QGP phase transition. We assume the chemical potential has negligible effect on the constant since it is small compared to the temperature above \( T_c \). Inserting Eq. (53) into Eq. (51), we can obtain Fig. 2 with \( T_c = 175 MeV \) and \( \mu = 46 MeV \).

It should be noticed here that in Fig. 2 the temperature dependence of the ratio has been extrapolated into the strongly coupled region where \( \alpha_s > 1 \) to see where is the physical boundary and the whole variety of the ratio in the physical region. One may find that the structure of the viscosity to entropy ratio is rather complicated due to the appearance of two singularities, which are contributed by the logarithm and the denominator of the enhancement factor. To distinguish the physical curve from the complex structure, we naturally introduce a physical criterion: both the ratio in equilibrium and the enhancement factor, which are presented in Figs. 2 and 3, should be positive.

It is clear in Figs. 2 and 3 that the physical curve is the one on the right side of the right singularity, i.e., the physical region is bounded by a temperature, which varies according to the constant over the fine structure constant under the logarithm. With detailed observation on the physical curve, one finds that it does not decrease monotonically with the temperature but presents a minimum value of about 0.4 at \( T = 182.1 MeV \) instead, which is twice greater than the phenomenological estimation [37]. This minimum value, seeing from mathematical point of view, is owing to the competition between the
Figure 3: Ratio of shear viscosity to entropy density in equilibrium.

Figure 4: Enhancement factor.
weakening running coupling constant and the rising temperature itself.

5 Summary

In the framework of irreversible thermodynamics, the transport properties of so-called QGP produced in heavy ion collision have been studied. Based on the Boltzmann equation, we calculated the shear viscosity of two-flavor QGP at high temperature and finite density in weakly coupled limit in kinetic theory. The result shows the finite density effect provides positive contribution to the shear viscosity by adding a quadratic term of $\mu/T$ to the pure temperature case. Furthermore, we calculated the non-equilibrium entropy density at finite chemical potential through two different schemes by subtracting the entropy production due to shear viscous process from the final equilibrium entropy density. In our evaluation, the thermal force in the shear viscous process, namely, the velocity gradient, was estimated by its maximum value considering the longitudinal expansion in heavy ion collision. Finally the viscosity to entropy density ratio was demonstrated in an extensive temperature regime including both physical and unphysical regions, which are identified by two natural conditions, to see clearly the temperature boundary. In the physical region the ratio appears a minimum value of 0.4 which is two times larger than that expected from the ideal behavior of the elliptic flow. The chemical potential effect, distinguished from the viscosity case, decreases the entropy density and thus contributes a final positive effect on the ratio.

At last, we emphasize here that one should pay attention not only to the ratio of transport coefficients to the equilibrium entropy density but also to the ratio of transport coefficients to the non-equilibrium entropy density when to understand the transport or dissipative properties of QGP. Generally speaking, the entropy production $dS$ comes either from the external sources $d_{ex}S$ and the internal dissipative and viscous processes, which is expressed as

$$dS = d_{in}S + d_{ex}S.$$  \hspace{1cm} (54)

As to the external sources, two basic aspects should be taken into account. One is the energy loss of parton when it is passing through the medium. The other might be the increase of the degree of freedom excited by the phase transition. Actually, these problems are really open and far from known, which need further study.
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Appendix: Relaxation time

In this appendix, we first define the relaxation time in Boltzmann equation, and then relate it with the shear viscosity which has been already obtained in the variational approach.

The Boltzmann equation without time derivative and external force terms in the relaxation time approximation is to substitute the collision term on the right hand side for the fluctuation of the distribution function scaled by the relaxation time $\tau_\eta$.

$$\hat{p} \cdot \frac{\partial f^0_s(x; p)}{\partial x} = -\frac{\delta f_s}{\tau_\eta},$$  \hspace{1cm} (55)

where $f^0(x; p) = [\exp(-\beta P u^\nu \mp \mu) \pm 1]^2$ is the Jüttner distribution function [6]. With this definition and Eq. (6), one could find

$$\delta \langle T_{xx} \rangle = \tau_\eta \beta \int_p \left( \frac{p_x p_z}{p^2} \right)^2 \left\{ g f n_f(p)[1 - n_f(p)] + g b n_b(p)[1 + n_b(p)] \right\} \left( \frac{\partial u_z}{\partial x} \right)$$  \hspace{1cm} (56)

$$= \eta \frac{\partial u_z}{\partial x}$$  \hspace{1cm} (57)

where the last line is the linear law.

Noticing the velocity gradient is independent with momentum thus can be cancelled on both side, we can find out the relation between the relaxation time and the shear viscosity by trivially evaluating the momentum integral,

$$\tau_\eta = \eta \left[ 3.246T^4 \left( 1 + 0.246 \frac{\mu^2}{T^2} \right) \right]^{-1}.$$  \hspace{1cm} (58)
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