Limits on the Robustness of MIMO Joint Source-Channel Codes

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Abstract—In this paper, the theoretical limits on the robustness of MIMO joint source channel codes is investigated. The case in which a single joint source channel code is used for the entire range of SNRs and for all levels of required fidelity is considered. Limits on the asymptotic performance of such a system are characterized in terms of upper bounds on the diversity-fidelity tradeoff, which can be viewed as an analog version of the diversity-multiplexing tradeoff. In particular, it is shown that there is a considerable gap between the diversity-fidelity tradeoff of robust joint source-channel codes and the optimum tradeoff (without the constraint of robustness).

I. INTRODUCTION

Many applications call for the transmission of analog sources over wireless channels. Results of research during the past decade have shown that using multiple-antenna systems can substantially improve the rate and the reliability of communications in wireless fading environments. Most research on multiple-antenna systems has focused on the transmission of digital data over multiple-input multiple-output (MIMO) channels, and the study of analog source transmission over such channels is still in its early stages. In [1], [2] and [3] some digital and hybrid digital-analog techniques are examined for joint source-channel coding over MIMO channels, and some bounds on the asymptotic exponents of the average distortion are presented. In [4], the asymptotic exponents of the probability of having a large distortion is studied. This measure, which is called the diversity-fidelity tradeoff, can be seen as an analog version of the well-known diversity-multiplexing tradeoff which has proven to be very useful in evaluating various digital space-time coding schemes. In [4], also some semi-robust joint-source channel codes were proposed which can use the same joint source-channel mapping for different ranges of SNR and different ranges of desired resolution. However it was observed that there is a gap between the optimum diversity-fidelity tradeoff and the performance of those semi-robust codes. In this paper, we investigate bounds on the robustness of MIMO joint source-channel codes.

II. SYSTEM MODEL

We consider a communication system in which an analog source of Gaussian independent samples with variance \(\sigma_s^2\) is to be transmitted over an \((N_t, N_r)\) block fading MIMO channel where \(N_t\) and \(N_r\) are the number of transmit and receive antennas respectively. Each sequence of \(m\) samples of the source, represented by a vector \(x_s\), is transmitted over \(n\) channel uses. We assume a quasi-static fading channel in which the channel matrix \(H\) is fixed during these \(n\) channel uses and changes independently for the next \(n\) channel uses. We call the ratio \(\eta = \frac{n}{m}\) the expansion/contraction factor of the system. In a general setting, the communication strategy consists of source/channel coding and source/channel decoding. As a result of source channel coding, \(x_s\) is mapped into an \(N_t \times n\) space-time matrix \(X\) which in turn is received at the receiver side as an \(N_r \times n\) matrix \(Y\) given by

\[
Y = HX + \sqrt{\frac{N_t}{\text{SNR}}} \mathbf{W}
\]

in which SNR is the average signal to noise ratio at each receive antenna, and \(\mathbf{W}\) is the normalized additive noise matrix at the receiver whose entries are taken to be \(\mathcal{CN}(0, 1)\) (the real variance of the noise is \(\sigma^2 = \frac{N_r}{\text{SNR}}\)). At the receiver side, the source/channel decoder yields an estimate of \(x_s\) from \(Y\) as \(\hat{x}_s\). For a specific channel realization \(H\), the distortion measure is

\[
D(H) = E_{x_s}\{\|x_s - \hat{x}_s\|^2|H\}. \tag{1}
\]
For any specific strategy, we define the $f-$fidelity event as $A(f) = \{H : D(H) > \text{SNR}^{-f}\}$ and we call $f$ the fidelity exponent. For specific values of $\eta$, $N_t$ and $N_r$, we define

$$d(f) = \lim_{\text{SNR} \to \infty} \frac{-\log \Pr\{A(f)\}}{\log \text{SNR}}.$$  

We call $d(f)$ the diversity, and denote its maximum (over all possible source-channel coding schemes) as $d^*(f)$.

In [4], it is shown that the optimal diversity (if we can use different source-channel codes for different SNR values and different fidelity exponents) can be characterized as

$$d^*(f) = \left( \frac{N_t}{2} - \frac{f}{2\eta} \right) \left( \frac{N_r}{2} - \frac{f}{2\eta} \right)$$

for integer values of $\frac{f}{2\eta}$.

III. BOUNDS ON THE DIVERSITY-FIDELITY TRADEOFF OF A SINGLE MIMO SOURCE-CHANNEL MAP

In this paper we investigate upper bounds on the diversity-fidelity tradeoff, when the joint source-channel code is fixed. In the general case, this joint source-channel code is a mapping from the set of all $m$-tuples of source samples to $F$, the set of transmitted vectors (or the modulation set), which is a subset of $\mathbb{R}^{2nN_t}$ (or indeed $\mathbb{C}^{nN_t}$). We assume that $N_r \geq N_t$. Also, we focus on the case in which the source is uniformly distributed on $[0,1]$, which has variance $\frac{1}{12}$.

To obtain bounds on the diversity-fidelity tradeoff of a single MIMO source-channel map, we use the concept of box-counting dimension [5]. If we partition the space into a grid of cubic boxes of size $\sigma$, and consider $N_\sigma$ as the number of boxes that intersect the set $F$, the box-counting dimension of $F$ is defined as

$$\text{Dim}(F) \triangleq \lim_{\sigma \to 0} \frac{\log N_\sigma}{\log \frac{1}{\sigma}}.$$  

We modify this definition and define the $c$-effective box-counting dimension (for $0 < c \leq 1$) of a modulation set as

$$\text{Dim}_c(F) \triangleq \lim_{\sigma \to 0} \frac{\log N'_{c,\sigma}}{\log \frac{1}{\sigma}}$$

where $N'_{c,\sigma}$ is the minimum number of those boxes whose total probability of containing the modulated signal is at least $c$.

**Theorem 1** Consider a space-time joint source-channel coding with modulation set $F$ (mapping $m$-dimensional source vectors to $2nN_t$-dimensional transmitted vectors). If for every $c > 0$, the $c$-effective box-counting dimension of $F$ is at least $2n\beta$ and at most $2n\beta'$, then for any $0 \leq f \leq 2n\eta \beta'$, we have

$$d(f) \leq (N_r - \beta + 1) (N_t - \beta + 1) \left( 1 - \frac{f}{2\eta \beta'} \right).$$

**Proof:** For any positive numbers $0 < c_1 < c_2 < 1$, if $\text{Dim}_c(F) = \beta_1$ and $\text{Dim}_{c_2}(F) = \beta_2$, then $2n\beta' \geq \beta_1 \geq \beta_2 \geq 2n\beta$, and for any $\sigma$ and for any of the boxes corresponding to $c_1$, the probability of containing the modulated signal is at least in the order of $\sigma^\beta_2$ and their number is of the order of $\sigma^{-\beta_1}$. Based on the monotonicity of $\text{Dim}_c(F)$, we can find $c_1$ and $c_2$ such that $\beta_1$ and $\beta_2$ are arbitrarily close to each other.

Now we look at the received modulation set $H_F$. We denote the nonzero eigenvalues of $HH^H \sigma^\beta$ by $0 < \lambda_1 \leq \ldots \leq \lambda_N$, and consider $\alpha_i = \frac{\log \lambda_i}{\log \sigma}$. If $\alpha_i \geq 1$ (for $1 \leq i \leq N_t - \beta + 1$), then $H_F$ (and all the boxes corresponding to $c_1$) will be inside a $2nN_t$-dimensional orthotope whose volume is less than $\sigma^{-2n(N_t+1-\beta)}$. In this case, because the order of the number of the boxes corresponding to $c_1$ is greater than $\sigma^{-2n\beta}$, the majority of them (with their portion approaching to $1$) become adjacent to other boxes (with a distance less than $\sigma$). Also, because of the isotropy of the channel distribution and the eigenvectors of $H$, with probability approaching to $1$, this also includes boxes containing the mapping of distant sub-segments of the source. Therefore, in this case, the distortion becomes lower bounded by a positive number, and hence the fidelity exponent will be $f = 0$.

Thus, to bound $d(0)$ we need only to bound $\Pr\{\alpha_i \geq 1 | 1 \leq i \leq N_t - \beta + 1\}$. Similarly to [6], we can bound it as $\frac{1}{\text{SNR}^{-N_t}} \left( 1 - \frac{f}{2\eta \beta'} \right)$.

$$d(0) \leq (N_t - \beta + 1) (N_r - \beta + 1).$$

For $f > 0$, we use a similar approach, by considering the effect of the channel on the boxes of size $\sigma$ inside larger boxes of size $\sigma^{1-\beta}$ (containing at least approximately $\sigma^{-\beta}$ smaller boxes). Consider $\alpha' = \frac{\log \lambda_i}{\log \sigma^{1-\beta}}$. Now if $\alpha' \geq 1$ (for $1 \leq i \leq N_t - \beta + 1$),

1In this paper, we use $a \triangleq b$ to denote that $a$ and $b$ are asymptotically equivalent.
similarly to the case of \( f = 0 \), we can show that the distortion will be at least on the order of \( \sigma^{2f} \) (or \( \text{SNR}^{-f} \)). Therefore, we have

\[
d(f) \leq (N_r - \beta + 1)(N_t - \beta + 1)
\left(1 - \frac{f}{2\eta\beta}\right), \quad (8)
\]

\[\blacksquare\]

**Theorem 2** Consider a space-time joint source-channel coding with modulation set \( \mathcal{F} \) (mapping \( m \)-dimensional source vectors to \( 2nN_t \)-dimensional transmitted vectors). If for some \( c > 0 \) the \( c \)-effective box-counting dimension of \( \mathcal{F} \) is at most \( 2n\beta \), then the coding scheme cannot achieve a fidelity exponent larger than \( 2\eta\beta \).

*Proof:* If we divide the signal space into boxes of size \( \sigma \), the number of boxes corresponding to \( c \) is bounded by the order of \( \sigma^{-2n\beta} \). If we divide the subset of source vectors (that are mapped into these boxes) to \( \sigma^{2n(-\beta-\varepsilon)} \) sub-cubes whose size is on the order of \( \sigma^{2n(\beta+\varepsilon)} \), a large portion of them will be adjacent to each other (with a distance less than \( \sigma \)), hence \( \Pr \left\{ D > \text{SNR}^{-2n(\beta+\varepsilon)} \right\} \) can be lower bounded by a positive number. This argument is valid for any small \( \varepsilon \). Thus, the fidelity exponent cannot be larger than \( 2\eta\beta \). \[\blacksquare\]

**Corollary 1** No single joint source-channel mapping can achieve any point on the optimum diversity-fidelity curve, other than the two extreme points, \( d = 0 \) or \( f = 0 \).

Theorems 1 and 2 show that the effective dimensionality of the analog modulation set is a key factor in determining its asymptotic performance. While low-dimensional mappings are incapable of achieving a high fidelity exponent, high-dimensional mappings cannot achieve a high diversity order. This is totally different from the case of digital space-time coding, in which many full-rank lattice codes can be used to construct diversity-multiplexing-tradeoff-achieving space-time codes (assuming that maximum-likelihood decoding is performed at the receiver).

### IV. Conclusions

In this paper, we have introduced an upper bound on the diversity-fidelity tradeoff of single-mapping MIMO source-channel codes. This result shows that, unlike the case of a single-input/single-output (SISO) channel (in which we can achieve the optimum signal-to-distortion-ratio (SDR) scaling by using a single mapping [7]), in the MIMO case there is a considerable gap between the asymptotic performance of a single robust mapping and the optimum tradeoff.

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