EXPLICIT FORMULA FOR THE AVERAGE OF GOLDBACH AND PRIME TUPLES REPRESENTATIONS

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ABSTRACT. Let $\Lambda(n)$ be the Von Mangoldt function, let

$$r_G(n) = \sum_{m_1, m_2 \leq n \atop m_1 + m_2 = n} \Lambda(m_1) \Lambda(m_2),$$

$$r_{PT}(N, h) = \sum_{n=0}^{N} \Lambda(n) \Lambda(n + h), \quad h \in \mathbb{N}$$

be the counting function of the Goldbach numbers and the counting function of the prime tuples, respectively. Let $N > 2$ be an integer. We will find the explicit formulae for the averages of $r_G(n)$ and $r_{PT}(N, h)$ in terms of elementary functions, the incomplete Beta function $B_z(a, b)$, series over $\rho$ that, with or without subscript, runs over the non-trivial zeros of the Riemann Zeta function and the Dilogarithm function. We will also prove the explicit formulae in an asymptotic form and a truncated formula for the average of $r_G(n)$. Some observation about these formulae and the average with Cesàro weight

$$\frac{1}{\Gamma(k+1)} \sum_{n \leq N} r_G(n) (N - n)^k, \quad k > 0$$

are included.

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1. Introduction

In this paper we prove an explicit formula and an asymptotic formula for the average of the functions $r_G(n)$ and $r_{PT}(N, h)$, which are the counting function of the Goldbach numbers and the counting function of the prime tuples, respectively. This type of research is classical; the first result for the average of counting function of the Goldbach numbers was proved in 1991 by Fujii in a series of paper [9, 10, 11] writing two terms of the asymptotic expansion with an error of $O\left((N \log (N))^{4/3}\right)$. Then Granville [12, 13] gave the same result with a different technique and Bhowmik and Schlage-Puchta [2] improved the error term to $O\left(N \log^5 (N)\right)$. Finally, Languasco and Zaccagnini [18] were able to reach the error term to $O\left(N \log^3 (N)\right)$. In recent years there has been some papers analyzing the weighed average with Cesàro weight

$$(1) \quad \frac{1}{\Gamma(k+1)} \sum_{n \leq N} r_G(n) (N - n)^k, \quad k > 0,$$

see [7], [14] and [16]. Even if the technique of Languasco and Zaccagnini, developed to study (1), can be applied to various problems (see [5, 6, 17]) in all of these papers there are some limitations over the parameter $k$ due to some convergence problems. In a very recent paper Brüdern, Kaczorowski and Perelli [4] were able to find an explicit formula which holds for all $k > 0$. We present an approach that analyzes the pure average form or, in other words, the case $k = 0$. We will find an explicit formula and we will prove that it is possible write it as an asymptotic formula with
three terms and an error term $O(N)$ without the assumption of the Riemann hypothesis (RH for brevity). We will prove the following

**Theorem 1.** Let $N > 2$ be an integer. Then

$$
\sum_{n \leq 2N} r_G(n) = 2N^2 - 2 \sum_{\rho} \frac{(2N - 2)^{\rho + 1}}{\rho (\rho + 1)}
+ 2 \sum_{\rho_1} 
(2N)^{\rho_1} \left( \Gamma (\rho_1) \sum_{\rho_2} \frac{(2N)^{\rho_2} \Gamma (\rho_2)}{\Gamma (\rho_1 + \rho_2 + 1)} 
- \sum_{\rho_2} \frac{(2N)^{\rho_2}}{\rho_2} (B_{1/N} (\rho_2 + 1, \rho_1) + B_{1/2} (\rho_1, \rho_2 + 1)) \right)
+ F(N)
$$

where

$$
\sum_{n \leq 2N} r_G(n) = \sum_{n \leq 2N} r_G(n) - \frac{r_G(2N)}{2}
$$

$B_z(a,b)$ is the incomplete Beta function, $\rho = \beta + i \gamma$, with or without subscript, runs over the non-trivial zeros of the Riemann Zeta function $\zeta(s)$ and $F(N)$ is a function that can be explicitly calculated in terms of elementary functions, series over non-trivial zeros, Dilogarithm and incomplete Beta functions and with

$$
F(N) = O(N)
$$

as $N \to \infty$. Furthermore for all $T', T'' > 2$ we have

$$
\sum_{n \leq 2N} r_G(n) = 2N^2 - 2 \sum_{\rho : |\gamma| \leq T'} \frac{(2N - 2)^{\rho + 1}}{\rho (\rho + 1)}
+ 2 \sum_{\rho_1 : |\gamma_1| \leq T'} 
(2N)^{\rho_1} \left( \Gamma (\rho_1) \sum_{\rho_2 : |\gamma_2| \leq T''} \frac{(2N)^{\rho_2} \Gamma (\rho_2)}{\Gamma (\rho_1 + \rho_2 + 1)} 
- \sum_{\rho_2 : |\gamma_2| \leq T''} \frac{(2N)^{\rho_2}}{\rho_2} (B_{1/N} (\rho_2 + 1, \rho_1) + B_{1/2} (\rho_1, \rho_2 + 1)) \right)
+ 2 \sum_{\rho_1 : |\gamma_1| \leq T'} \sum_{\rho_2 : |\gamma_2| \leq T''} \frac{N^{\rho_1}}{\rho_1} \sum_{\rho_2 : |\gamma_2| \leq T''} \frac{N^{\rho_2}}{\rho_2} - 2 \left( \sum_{\rho_1 : |\gamma_1| \leq T'} \frac{N^{\rho_1}}{\rho_1} \right)^2
+ F(N, T', T'') + O \left( \frac{N \log^2 (NT'') T' \log (T') G(N)}{T''} + \frac{N^2 \log^2 (T'N)}{T'} \right)
$$

where

$$
G(N) = \begin{cases} 
N \exp \left( -C \sqrt{\log(N)} \right) & \text{without RH} \\
\sqrt{N \log^2 (N)} & \text{with RH}
\end{cases}
$$

and $F(N, T', T'')$ is a function that can be explicitly calculated in terms of elementary functions, series over non-trivial zeros, Dilogarithm and incomplete Beta functions and with the property

$$
F(N, T', T'') \ll N
$$

where the implicit constant does not depend on $T'$ and $T''$. 

Note that the term
\[ \sum_{\rho_1} (2N)^{\rho_1} \Gamma(\rho_1) \sum_{\rho_2} \frac{(2N)^{\rho_2} \Gamma(\rho_2)}{\Gamma(\rho_1 + \rho_2 + 1)} \]
is what we expect considering the formula in [4] and taking \( k = 0 \). It is interesting to note that if we assume the third term of the explicit formula in Theorem 1 grows in a suitable way as \( N \to \infty \) then we can prove that every interval \([2N, 2N + 2H]\), where \( H = H(N) \) is a function of \( N \) that grows in a suitable way, contains a Goldbach number. More precisely, we propose the following conjecture

**Conjecture 2.** Under RH we have the estimation
\[ \sum_{\rho_1} (2N)^{\rho_1} \left( \Gamma(\rho_1) \sum_{\rho_2} \frac{(2N)^{\rho_2} \Gamma(\rho_2)}{\Gamma(\rho_1 + \rho_2 + 1)} - \sum_{\rho_2} \frac{(2N)^{\rho_2}}{\rho_2} \left( B_{1/N} (\rho_2 + 1, \rho_1) + B_{1/2} (\rho_1, \rho_2 + 1) \right) \right) \ll N. \]

If Conjecture 2 holds then we get the following

**Theorem 3.** Assume that Conjecture 2 holds. Then in every interval \([2N, 2N + 2H]\) where \( H = H(N) \) is \((2N)^{H(N)} = C \log \log \omega(N)\), \( C > 0 \) is a sufficiently large constant contains a Goldbach number.

**Proof.** From Theorem 1 and Conjecture 2 we get, for every \( H > 0 \), that
\[ \sum_{n=2N+1}^{2N+2H} r_G(n) = 4NH + 2H^2 - 2 \sum_{\rho} \frac{(2N + 2H - 2)^{\rho+1}}{\rho(\rho + 1)} + 2 \sum_{\rho} \frac{(2N - 2)^{\rho+1}}{\rho(\rho + 1)} \]
\[ + \frac{r_G(2N + 2H)}{2} - \frac{r_G(2N)}{2} + O(N) \]
then we can observe that
\[ -2 \sum_{\rho} \frac{(2N + 2H - 2)^{\rho+1}}{\rho(\rho + 1)} + 2 \sum_{\rho} \frac{(2N - 2)^{\rho+1}}{\rho(\rho + 1)} = -2 \int_{2N-2}^{2N+2H-2} \sum_{\rho} \frac{\rho^\rho}{\rho} dt \]
since we know we can switch the integral and the series over the non-trivial zeros (see Lemma 3). So, by (31), we get
\[ -2 \sum_{\rho} \frac{(2N + 2H - 2)^{\rho+1}}{\rho(\rho + 1)} + 2 \sum_{\rho} \frac{(2N - 2)^{\rho+1}}{\rho(\rho + 1)} \ll H \sqrt{N} \log^2 (2N + 2H) \]
and since
\[ r_G(M) \ll M \mathcal{G}(M) \ll M \log \log (M), \]
where
\[ \mathcal{G}(M) = 2 \prod_{p > 2} \left( 1 - \frac{1}{(p-2)^2} \right) \prod_{p | M; p > 2} p - 1, p \text{ prime number}, \]
if \( M \) is even and vanishes if \( M \) is odd (see for example [20], Theorem 3.13), then if we take \( H = H(N) \) as in [2] we get the thesis.

Probably, with a more accurate analysis, it is possible to obtain \( H(N) \) as a large constant but it is not the aim of this paper.

We need some comments about the truncated formula. This form is interesting since allows to work with finite sums instead of series and so it is reasonable to think that, with a clever choice of \( T' \) and \( T'' \), we can estimate the double sums efficiently. As we will see in the proof the error term
in the formula strictly depends on the choice of some parameters and, probably, is not optimized; we expect that a better analysis can be done and this will be the subject of future research.

In Section 4 we will also talk about the possibility to use our method to calculate the weighed form (1).

Let us talk about the average of prime tuples. We can recall Bombieri and Davenport [3], Maier and Pomerance [19] and Balog [1], which obtained a “Bombieri-Vinogradov type results”. We will prove the following

**Theorem 4.** Let $N > 2$ and $0 \leq M \leq N$ be integers. Then

$$
\sum_{h=0}^{M} r_{PT}(N, h) = N M + \sum_{\rho} \frac{N^{\rho+1}}{\rho (\rho + 1)} - \sum_{\rho} \frac{(N + M)^{\rho+1}}{\rho (\rho + 1)} + \sum_{\rho} \frac{(2 + M)^{\rho+1}}{\rho (\rho + 1)}
$$

where

$$1(M) = \begin{cases} 
0, & M = 0 \\
1, & M > 0,
\end{cases}
$$

$$\bar{1}(M) = \begin{cases} 
2, & M = 0 \\
1, & M > 0,
\end{cases}
$$

$$
\sum_{h=0}^{M} r_{PT}(N, h) = \sum_{h=0}^{M} r_{PT}(N, h) - \frac{r_{PT}(N, M)}{2} - \frac{r_{PT}(N, 0)}{2},
$$

and $G(N, M)$ is a function that can be explicitly calculated in terms of special functions like the incomplete Beta function and with the property

$$G(N, M) \ll \begin{cases} 
N(M + 1) \exp \left( -C \sqrt{\log(N)} \right), & \text{without RH} \\
\sqrt{N} (M + 1) \log^2(N), & \text{with RH}
\end{cases}
$$

and $C > 0$ is a real constant and the implicit constant does not depend on $M$.

Again we can observe that a precise control of the series in (3) and (4) allows us to obtain information on the sum $r_{PT}(N, h)$ with a fixed $M$.

I thank my mentor Alessandro Zaccagnini for a discussion on this topic.

2. **Lemmas**

We recall a Lemma that we use several times.
Lemma 5. Let $g$ be a continuously differentiable function on $[a,b]$ with $2 \leq a \leq b < \infty$ and $\psi(t)$ the Chebyshev psi function. We have

$$
\int_a^b \psi(t) g(t) \, dt = \int_a^b t g(t) \, dt - \sum_{\rho} \frac{1}{\rho} \int_a^b t^\rho g(t) \, dt - \int_a^b \left( \frac{\zeta'}{\zeta}(0) + \frac{\log (1 - 1/t^2)}{2} \right) g(t) \, dt.
$$

The proof can be found in [22], Lemma 4. The formula can be extended to $b = \infty$ assuming that $g(t)$ decays at $+\infty$ sufficiently fast (an example is present in [7]). Also, from the proof of the Lemma, it is clear that the formula holds even if $g(t) \in \mathbb{C}$ with the hypothesis $\int_a^b |g(t)| \, dt < \infty$.

Now we present our fundamental lemma.

Lemma 6. Let $x, y \in \mathbb{R}$, $3 \leq x \leq y$ and $\alpha \in \mathbb{C}$, $\text{Re}(\alpha) > 0$. Then

$$
\sum_{n \leq [x]} \Lambda(n) \left( 1 - \frac{n}{y} \right)^\alpha = \frac{y}{\alpha + 1} \left( \left( 1 - \frac{2}{y} \right)^{\alpha+1} - \left( 1 - \frac{|x|}{y} \right)^{\alpha+1} \right) + 2 \left( 1 - \frac{2}{y} \right)^\alpha 
$$

$$
- \sum_{\rho} y^\rho \frac{\Gamma(\rho + 1) \Gamma(\alpha + 1)}{\Gamma(\rho + 1 + \alpha)} + \alpha \sum_{\rho} \frac{y^\rho}{\rho} \left( B_{2/y} (\rho + 1, \alpha) + B_{(y-[x])/y} (\alpha, \rho + 1) \right) 
$$

$$
- \left( 1 - \frac{|x|}{y} \right)^\alpha \sum_{\rho} \frac{|x|^\rho}{\rho} - \frac{\zeta'}{\zeta}(0) \left( 1 - \frac{2}{y} \right)^\alpha - \frac{1}{2} \log \left( \frac{3}{4} \right) \left( 1 - \frac{2}{y} \right)^\alpha 
$$

$$
+ \omega(\alpha, y, x) + \frac{\Lambda([x])}{2} \left( 1 - \frac{|x|}{y} \right)^\alpha
$$

where $[x]$ is the floor function, $\psi(s)$ is the Chebyshev psi function,

$$
\omega(\alpha, y, x) = - \frac{(y - 1)^\alpha}{2y^\alpha} \left( B_{(y-2)/(y-1)} (\alpha + 1, 0) - B_{(y-[x])/(y-1)} (\alpha + 1, 0) \right) 
$$

$$
- \frac{(y + 1)^\alpha}{2y^\alpha} \left( B_{(y-2)/(y+1)} (\alpha + 1, 0) - B_{(y-[x])/(y+1)} (\alpha + 1, 0) \right) 
$$

$$
+ B_{(y-2)/y} (\alpha + 1, 0) - B_{(y-[x])/y} (\alpha + 1, 0)
$$
and \( B_z(a,b) \) is the incomplete Beta function, with the convention \( B_0(a,b) = 0 \). Furthermore if \( y > x \) then for all \( T > 2 \) we have

\[
\sum_{n \leq |x|} \Lambda(n) \left( 1 - \frac{n}{y} \right)^\alpha = \frac{y}{\alpha + 1} \left( \left( 1 - \frac{2}{y} \right)^{\alpha + 1} - \left( 1 - \frac{|x|}{y} \right)^{\alpha + 1} \right) + 2 \left( 1 - \frac{2}{y} \right) \alpha - \sum_{\rho : |\gamma| \leq T} y^\rho \frac{\Gamma(\rho) \Gamma(\alpha + 1)}{\Gamma(\rho + 1 + \alpha)} + \alpha \sum_{\rho : |\gamma| \leq T} y^\rho \left( B_{2/y}(\rho + 1, \alpha) + B_{(y-|x|)/y}(\alpha, \rho + 1) \right)
\]

\[
- \left( 1 - \frac{|x|}{y} \right)^\alpha \sum_{\rho : |\gamma| \leq T} \frac{|x|^\rho}{\rho} - \frac{\zeta^\prime}{\zeta}(0) \left( 1 - \frac{2}{y} \right)^\alpha - \frac{1}{2} \log \left( \frac{3}{4} \right) \left( 1 - \frac{2}{y} \right) \alpha
\]

\[
+ \omega(\alpha, y, x) + \left( 1 - \frac{|x|}{y} \right)^\alpha \frac{\Lambda(|x|)}{2} + O\left( M(\alpha, y, x) \frac{|\alpha|^2 \log^2(|x| T)}{y T} + \frac{|x| \log^2(|x| T)}{T} \right)
\]

where the implicit constant in the error term does not depend on \( \alpha, y \) and \( x \) and

\[
M(\alpha, y, x) = \begin{cases} 1, & \text{Re}(\alpha) = 1 \\ \left( 1 - \frac{|x|}{y} \right)^{\text{Re}(\alpha) - 1}, & 0 < \text{Re}(\alpha) < 1. \end{cases}
\]

**Proof.** By the Abel summation formula we have

\[
(6) \quad \sum_{n \leq |x|} \Lambda(n) \left( 1 - \frac{n}{y} \right)^\alpha = \psi(|x|) \left( 1 - \frac{|x|}{y} \right)^\alpha + \frac{\alpha}{y} \int_2^{|x|} \psi(t) \left( 1 - \frac{t}{y} \right)^{\alpha - 1} dt
\]

where \(|x|\) is the floor function, so by the explicit formula for \( \psi_0(t) = \psi(t) - \frac{\Lambda(t)}{2} \) (or by Lemma [5]) we have

\[
\sum_{n \leq |x|} \Lambda(n) \left( 1 - \frac{n}{y} \right)^\alpha = \psi(|x|) \left( 1 - \frac{|x|}{y} \right)^\alpha + \frac{\alpha}{y} \int_2^{|x|} \left( t - \sum_{\rho} \frac{t^\rho}{\rho} - \frac{\zeta^\prime}{\zeta}(0) - \frac{1}{2} \log \left( 1 - \frac{1}{t^2} \right) \right) \left( 1 - \frac{t}{y} \right)^{\alpha - 1} \frac{1}{y} dt
\]

\[
= \psi(|x|) \left( 1 - \frac{|x|}{y} \right)^\alpha + \frac{\alpha}{y} \sum_{w=1}^4 \int_2^{|x|} g_w(t) \left( 1 - \frac{t}{y} \right)^{\alpha - 1} dt
\]

where \( g_w(t) \) are the terms of the explicit formula of \( \psi_0(t) \).

**Integral of \( g_1(t) \)**

We have to calculate

\[
\frac{\alpha}{y} \int_2^{|x|} g_1(t) \left( 1 - \frac{t}{y} \right)^{\alpha - 1} dt = \frac{\alpha}{y} \int_2^{|x|} t \left( 1 - \frac{t}{y} \right)^{\alpha - 1} dt
\]

then taking \( t/y = u \) and integrating by parts we get

\[
\frac{\alpha}{y} \int_2^{|x|} g_1(t) \left( 1 - \frac{t}{y} \right)^{\alpha - 1} dt = y \alpha \int_{2/y}^{|x|/y} u (1 - u)^{\alpha - 1} du
\]
\[
\frac{y}{\alpha + 1} \left( \left( 1 - \frac{2}{y} \right)^{\alpha + 1} - \left( 1 - \frac{|x|}{y} \right)^{\alpha + 1} \right) - |x| \left( 1 - \frac{|x|}{y} \right)^\alpha + 2 \left( 1 - \frac{2}{y} \right)^\alpha.
\]

**Integral of \( g_2(t) \)**

We have to estimate
\[
\frac{\alpha}{y} \int_2^{|x|} g_2(t) \left( 1 - \frac{t}{y} \right)^{\alpha - 1} dt = \frac{\alpha}{y} \int_2^{|x|} \sum_{\rho} \frac{t^\rho}{\rho} \left( 1 - \frac{t}{y} \right)^{\alpha - 1} dt
\]
then by Lemma 5 we we know that we can exchange the integral with the series so
\[
\frac{\alpha}{y} \int_2^{|x|} g_2(t) \left( 1 - \frac{t}{y} \right)^{\alpha - 1} dt = -\frac{\alpha}{y} \sum_{\rho} \frac{1}{\rho} \int_2^{|x|} t^\rho \left( 1 - \frac{t}{y} \right)^{\alpha - 1} dt = -\alpha \sum_{\rho} \frac{y^\rho}{\rho} \int_{2/y}^{|x|/y} u^\rho (1 - u)^{\alpha - 1} du
\]
where \( B(a, b) \) is the incomplete Beta function (for details see for example [21], chapter 8.17). Note that the last identity is valid since \( \sum_{\rho} y^\rho \frac{\Gamma(\rho)}{\Gamma(\rho + 1 + \alpha)} \) is absolutely and compactly convergent for \( \text{Re}(\alpha) > 0 \) (see [4]).

**Integral of \( g_3(t) \)**

Trivially we have
\[
\frac{\alpha}{y} \int_2^{|x|} g_3(t) \left( 1 - \frac{t}{y} \right)^{\alpha - 1} dt = -\frac{\zeta'(0)}{\zeta(0)} \frac{\alpha}{y} \int_2^{|x|} \left( 1 - \frac{t}{y} \right)^{\alpha - 1} dt = \frac{\zeta'(0)}{\zeta(0)} \left( 1 - \frac{|x|}{y} \right)^\alpha - \frac{\zeta'(0)}{\zeta(0)} \left( 1 - \frac{2}{y} \right)^\alpha.
\]

**Integral of \( g_4(t) \)**

Integrating by parts we have that
\[
\frac{\alpha}{y} \int_2^{|x|} g_4(t) \left( 1 - \frac{t}{y} \right)^{\alpha - 1} dt = -\frac{\alpha}{2y} \int_2^{|x|} \log \left( 1 - \frac{1}{t^2} \right) \left( 1 - \frac{t}{y} \right)^{\alpha - 1} dt
\]
\[
= \frac{1}{2} \log \left( 1 - \frac{1}{|x|^2} \right) \left( 1 - \frac{|x|}{y} \right)^\alpha - \frac{1}{2} \log \left( \frac{3}{4} \right) \left( 1 - \frac{2}{y} \right)^\alpha
\]
(8) \quad + \frac{1}{y^\alpha} \int_2^{|x|} t^{-1} (y - t)^{\alpha} dt - \frac{1}{2y^\alpha} \int_2^{|x|} (t + 1)^{-1} (y - t)^{\alpha} dt - \frac{1}{2y^\alpha} \int_2^{|x|} (t - 1)^{-1} (y - t)^{\alpha} dt.
\]

Now we calculate explicitly only the first integral of (8) since the other are similar. Taking \( \frac{y - t}{y} = s \) we get
\[
\frac{1}{y^\alpha} \int_2^{|x|} t^{-1} (y - t)^{\alpha} dt = \int_{(y-|x|)/y}^{(y-2)/y} (1 - s)^{-1} s^\alpha ds
\]
\[
= B_{(y-2)/y} (\alpha + 1, 0) - B_{(y-|x|)/y} (\alpha + 1, 0)
\]
then, arguing in this way for all the integrals in (8) and expanding \( \psi(|x|) \) with its explicit formula, we get the thesis.

For the proof of the truncated version we use the formula
(9)
\[
\psi_0(x) = x - \sum_{\rho: |\gamma| \leq T} \frac{x^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} \log \frac{1}{2} - O \left( \frac{x \log^2(Tx)}{T} + \log(x) \min \left( 1, \frac{x}{T \langle x \rangle} \right) \right), \quad T > 2, \quad x > 1
\]
(see for example [3], formulae (9) and (10) at page 109) where \( \langle x \rangle \) is the distance of \( x \) to the nearest prime power other than \( x \), then, substituting (9) in (6), we can see that the problem boils down to evaluate

(10) \[
\frac{|\alpha|}{yT} \int_2^{|x|} t \log^2 (tT) \left( 1 - \frac{t}{y} \right)^{\text{Re}(\alpha)-1} \, dt
\]

and

(11) \[
\frac{|\alpha|}{y} \int_2^{|x|} \log (t) \min \left( 1, \frac{t}{T \langle t \rangle} \right) \left( 1 - \frac{t}{y} \right)^{\text{Re}(\alpha)-1} \, dt.
\]

Let us define

\[ M(\alpha, y, x) = \begin{cases} 1, & \text{Re} (\alpha) \geq 1 \\ \left( 1 - \frac{|x|}{y} \right)^{\text{Re}(\alpha)-1}, & 0 < \text{Re} (\alpha) < 1. \end{cases} \]

Then

(12) \[
\ll \frac{|\alpha|}{y T} \int_2^{|x|} t \log^2 (tT) \left( 1 - \frac{t}{y} \right)^{\text{Re}(\alpha)-1} \, dt \ll \frac{|\alpha|}{y T} M(\alpha, y, x) \int_2^{|x|} t \log (t) \, dt
\]

\[
\ll M(\alpha, y, x) \frac{|\alpha|}{y T} \frac{|x|^2 \log (|x|)}{T}.
\]

We now analyze (11). Let us define

\[ \Omega_1 = \{ t \in (2, |x|) : \langle t \rangle \geq 1 \} \]

then we can easily see that

(11) \[
\frac{|\alpha|}{y} \int_{\Omega_1} \log (t) \min \left( 1, \frac{t}{T \langle t \rangle} \right) \left( 1 - \frac{t}{y} \right)^{\text{Re}(\alpha)-1} \, dt \ll \frac{|\alpha|}{y T} M(\alpha, y, x) \int_2^{|x|} t \log (t) \, dt
\]

\[
\ll M(\alpha, y, x) \frac{|\alpha|}{y T} \frac{|x|^2 \log (|x|)}{T}.
\]

Now we consider the set

\[ \Omega_2 = \bigcup_{p, m : p^m \leq |x|} \left\{ t \in (2, |x|) : \frac{p^m T}{T + 1} \leq t \leq \frac{p^m T}{T - 1}, t \neq p^m \right\} \]

where \( p \) runs over primes and \( m \geq 1 \) are integers. We also assume that, if \( p^m = 2 \) or \( p^m = |x| \), then the intervals to consider are

\[
2 < t \leq \frac{2T}{T - 1}, \quad \frac{|x| T}{T + 1} \leq t < |x|,
\]

respectively. So we can observe that

\[
t \in \Omega_2 \iff \frac{t}{T |t - p^m|} \geq 1
\]

then

(11) \[
\frac{|\alpha|}{y} \int_{\Omega_2} \log (t) \min \left( 1, \frac{t}{T \langle t \rangle} \right) \left( 1 - \frac{t}{y} \right)^{\text{Re}(\alpha)-1} \, dt \ll \frac{|\alpha|}{y T} M(\alpha, y, x) \sum_{p, m : p^m \leq |x|} \int_{p^m T/(T - 1)}^{p^m T/(T + 1)} \log (t) \, dt
\]

\[
\ll \frac{|\alpha|}{y T} M(\alpha, y, x) \sum_{p, m : p^m \leq |x|} p^m \log (p^m) \ll M(\alpha, y, x) \frac{|\alpha|}{y T} \frac{|x|^2}{T}.
\]
Now we take
\[ \Omega_3 = (2, |x|)^* \setminus \{\Omega_1 \cup \Omega_2\} \]
where
\[ (2, |x|)^* = (2, |x|) \setminus \{m \geq 1, p \text{ prime} : 2 < p^m < |x|\}. \]
Obviously we can observe that if \( t \in \Omega_3 \) then we have to consider the intervals
\[ p^m - 1 < t < \frac{p^m T}{T + 1}, \]
\[ \frac{p^m T}{T - 1} < t < p^m + 1 \]
if \( p^m - 1 \) or \( p^m + 1 \) are not prime powers, respectively and
\[ \frac{(p^m - 1) T}{T - 1} < t < \frac{p^m T}{T + 1}, \]
\[ \frac{p^m T}{T - 1} < t < \frac{(p^m + 1) T}{T + 1} \]
if \( p^m - 1 \) or \( p^m + 1 \) are prime powers, respectively. If (13) holds then
\[ \frac{|\alpha|}{yT} \int_{p^m - 1}^{p^m + 1} t \log (t) dt \ll \frac{|\alpha|}{yT} p^m \log (p^m) \log \left( \frac{p^m}{T} \right) \]
and the same bound holds for (14). If (15) holds then
\[ \frac{|\alpha|}{yT} \int_{(p^m - 1) T/(T+1)}^{p^m T/(T+1)} t \log (t) dt \ll \frac{|\alpha|}{yT} p^m \log (p^m) \log \left( \frac{p^m}{T} \right) \]
and the same holds for (16).
Summing up, we get
\[ \frac{|\alpha|}{y} \int_{\Omega_3} \log (t) \min \left( 1, \frac{t}{T \langle t \rangle} \right) \left( 1 - \frac{t}{y} \right)^{\Re(\alpha) - 1} dt \ll M (\alpha, y, x) \frac{|\alpha| |x| ^2 \log ^2 (|x| T)}{yT} \]
so finally we can write
\[ |\alpha| \int_{2}^{\lfloor x \rfloor} \log (t) \min \left( 1, \frac{t}{T \langle t \rangle} \right) \left( 1 - \frac{t}{y} \right)^{\Re(\alpha) - 1} dt \ll M (\alpha, y, x) \frac{|\alpha| |x| ^2 \log ^2 (T \lfloor x \rfloor)}{yT}. \]
To finish the proof we have only to substitute the \( \psi (\lfloor x \rfloor) \) term in (7) with (9), recalling that, if \( x \) is an integer, then the error term in (9) can be written as \( O (x \log ^2 (x T) / T) \) since \( \langle x \rangle \geq 1 \). \( \square \)

3. PROOF OF THEOREM 1
Let \( N > 2 \) be an integer and let \( \psi (N) = \sum_{n \leq N} \Lambda (n) \) the Chebyshev psi function. From the identity
\[ \left( \sum_{m=0}^{k} a_m \right) \left( \sum_{m=0}^{k} b_m \right) = \sum_{m=0}^{2k} \sum_{h=0}^{m} a_h b_{m-h} - \sum_{m=0}^{k-1} \left( a_m \sum_{h=k+1}^{2k-m} b_h + b_m \sum_{h=k+1}^{2k-m} a_h \right), \]
which can be proved observing that the set of lattice points
\[ \{(i, j) : 0 \leq i + j \leq 2N, 0 \leq i \leq N, 0 \leq j \leq N\} \]
forms a triangle that can be seen as a \( N \times N \) square joint the two triangles
\[ \{(i, j) : 0 \leq i \leq N - 1, N + 1 \leq j \leq 2N - i\} \]
and

\[ \{ (i, j) : N + 1 \leq i \leq 2N, 0 \leq j \leq 2N - i \}, \]

we have that

\[ \sum_{n \leq N} \Lambda (n) \sum_{m \leq N} \Lambda (m) = \sum_{n \leq 2N} \sum_{m \leq n} \Lambda (m_1) \Lambda (m_2) - 2 \sum_{n \leq N - 1} \left( \Lambda (n) \sum_{m = N + 1}^{2N - n} \Lambda (m) \right) \]

then

\[ \sum_{n \leq 2N} r_G (n) = \psi^2 (N) + 2 \sum_{n \leq N - 1} \Lambda (n) (\psi (2N - n) - \psi (N)) \]

(18)

\[ = 2 \sum_{n \leq N} \Lambda (n) \psi (2N - n) - \psi (N)^2 \]

so we will find the explicit formula for \( \sum_{n \leq 2N} r_G (n) \) using the classical explicit formula of \( \psi_0 (N) = \psi (N) - \frac{\Lambda (N)}{2} \) (for a reference see [8], chapter 17). It is quite simple to observe that the most delicate term to evaluate is

(19)

\[ 2 \sum_{n \leq N} \Lambda (n) \psi (2N - n). \]

From the explicit formula of \( \psi_0 (N) \) we have that

\[ \psi (2N - n) = 2N - n - \sum_{\rho} \frac{(2N - n)^\rho}{\rho} - \frac{\zeta' (0)}{\zeta (0)} - \frac{1}{2} \log \left( 1 - \frac{1}{(2N - n)^2} \right) + \frac{\Lambda (2N - n)}{2} \]

so we now evaluate (19) term by term.

3.1. The main term. From the Abel summation formula we have

(20)

\[ 2 \sum_{n \leq N} \Lambda (n) (2N - n) = 2N \psi (N) + 2 \psi_1 (N) \]

where

(21)

\[ \psi_1 (N) = \int_0^N \psi (t) \, dt \]

so from Theorem 28 of [15] we get

(22)

\[ 2 \sum_{n \leq N} \Lambda (n) (2N - n) = 2N \psi (N) + N^2 - 2 \sum_{\rho} \frac{N^{\rho + 1}}{\rho (\rho + 1)} - 2N \frac{\zeta' (0)}{\zeta (0)} + 2 \frac{\zeta' (-1)}{\zeta (-1)} - \sum_{r \geq 1} \frac{N^{-r - 1}}{r (2r - 1)}. \]

Furthermore it is not difficult to see that, expanding \( 2N \psi (N) \), we have the asymptotic formula

(23)

\[ 2 \sum_{n \leq N} \Lambda (n) (2N - n) = 3N^2 - 2 \sum_{\rho} \frac{N^{\rho + 1}}{\rho (\rho + 1)} + 2 \sum_{\rho} \frac{N^{\rho + 1}}{\rho} + N \Lambda (N) + O (N) \]

as \( N \to \infty. \)
3.2. The term involving the series over the non-trivial zeros of $\zeta (s)$. We now consider

$$-2 \sum_{n \leq N} \Lambda (n) \sum_{\rho} \frac{(2N-n)^\rho}{\rho} = -2 \sum_{\rho} \frac{(2N)^\rho}{\rho} \sum_{n \leq N} \Lambda (n) \left( 1 - \frac{n}{2N} \right)^\rho$$

then from Lemma 6, taking $x = N, y = 2N, \alpha = \rho$, we get

(24)

$$-2 \sum_{n \leq N} \Lambda (n) \sum_{\rho} \frac{(2N-n)^\rho}{\rho} = -2 \sum_{\rho} \frac{(2N-2)^{\rho+1}}{\rho (\rho+1)} + 2 \sum_{\rho} \frac{N^{\rho+1}}{\rho (\rho+1)} - 4 \sum_{\rho} \frac{(2N-2)^\rho}{\rho}$$

(25)

$$+ 2 \sum_{\rho_1} (2N)^{\rho_1} \left( \Gamma (\rho_1) \sum_{\rho_2} \frac{(2N)^{\rho_2}}{\Gamma (\rho_1 + \rho_2 + 1)} \right)$$

(26)

$$- \sum_{\rho_2} \frac{(2N)^{\rho_2}}{\rho_2} \left( B_{1/N} (\rho_2 + 1, \rho_1) + B_{1/2} (\rho_1, \rho_2 + 1) \right)$$

(27)

$$+ 2 \sum_{\rho_1} \frac{N^{\rho_1}}{\rho_1} \sum_{\rho_2} \frac{N^{\rho_2}}{\rho_2} + 2 \sum_{\rho} \frac{\zeta' (0)}{\zeta (0)} \frac{(2N-2)^\rho}{\rho}$$

(28)

$$+ \log \left( \frac{3}{4} \right) \sum_{\rho} \frac{(2N-2)^\rho}{\rho} - 2 \sum_{\rho} \frac{(2N)^\rho}{\rho} \omega (\rho, 2N, N) - \Lambda (N) \sum_{\rho} \frac{N^\rho}{\rho}$$

where

$$\omega (\rho, 2N, N) = - \frac{1}{2} \left( 1 - \frac{1}{2N} \right)^\rho \left( B_{(2N-2)/(2N-1)} (\rho + 1, 0) - B_{N/(2N-1)} (\rho + 1, 0) \right)$$

$$- \frac{1}{2} \left( 1 + \frac{1}{2N} \right)^\rho \left( B_{(2N-2)/(2N+1)} (\rho + 1, 0) - B_{N/(2N+1)} (\rho + 1, 0) \right)$$

$$+ B_{(2N-2)/2N} (\rho + 1, 0) - B_{1/2} (\rho + 1, 0).$$

We can observe that the rearrangement in (24), (25), (26), (27) and (28) is legitimate: in (24), (27) and (28) the series are convergent by the explicit formula of $\psi (N)$ (that is, in the sense $\sum_\rho x^\rho / \rho = \lim_{T \to \infty} \sum_{|\rho| \leq T} x^\rho / \rho$) and $\sum_\rho \frac{(2N)^\rho}{\rho} \omega (\rho, 2N, N)$ is convergent since, integrating by parts, we have, for all $0 \leq h < 1$, that

(29)

$$\sum_{\rho} \frac{(2N)^\rho}{\rho} \left( 1 - \frac{1}{2N} \right)^\rho B_h (\rho + 1, 0) = \sum_{\rho} \frac{(2N)^\rho}{\rho} \left( 1 - \frac{1}{2N} \right)^\rho \int_0^h \frac{t^\rho}{1-t} dt$$

(30)

$$= \sum_{\rho} \frac{(2N-1)^\rho}{\rho (\rho+1)} \frac{t^{\rho+1}}{1-t} \bigg|_0^h + \sum_{\rho} \frac{(2N-1)^\rho}{\rho (\rho+1)} \int_0^h \frac{t^{\rho+1}}{(1-t)^2} dt$$

and so the convergence. This allow us to conclude that the double series in (25) and (26) is convergent.

Now we want to give an estimation of some terms of (24), (25), (26), (27) and (28). We start from the term

$$\sum_{\rho} \frac{(2N-2)^\rho}{\rho}.$$
Then, by the well known asymptotic
\[ (31) \quad \psi(x) - x \ll \begin{cases} x \exp \left( -C \sqrt{\log(x)} \right) & \text{without RH} \\ \sqrt{x} \log^2(x) & \text{with RH} \end{cases} \quad \text{with } C > 0, \quad x > 1 \]
(see [3], chapter 18) we obtain
\[ \sum \rho \frac{(2N - 2)^\rho}{\rho} \ll \begin{cases} N \exp \left( -C \sqrt{\log(N)} \right) & \text{without RH} \\ \sqrt{N} \log^2(N) & \text{with RH} \end{cases} \]
where \( C > 0 \) is a real number.

Now let us consider the terms
\[ -2 \frac{\xi'}{\zeta}(0) \sum \rho \frac{(2N - 2)^\rho}{\rho} + \log \left( \frac{3}{4} \right) \sum \rho \frac{(2N - 2)^\rho}{\rho}. \]
We can easily see that the estimation
\[ -2 \frac{\xi'}{\zeta}(0) \sum \rho \frac{(2N - 2)^\rho}{\rho} + \log \left( \frac{3}{4} \right) \sum \rho \frac{(2N - 2)^\rho}{\rho} \ll \begin{cases} N \exp \left( -C \sqrt{\log(N)} \right) & \text{without RH} \\ \sqrt{N} \log^2(N) & \text{with RH} \end{cases} \]
holds.

We now estimate the series
\[ \sum \rho \frac{(2N)^\rho}{\rho} \omega(\rho, 2N, N). \]
We will consider only
\[ \sum \rho \frac{(2N)^\rho}{\rho} \left( 1 - \frac{1}{2N} \right)^\rho \left( B_{(2N-2)/(2N-1)} (\rho + 1, 0) - B_{N/(2N-1)} (\rho + 1, 0) \right) \]
since the other calculations are essentially the same. From the definition of incomplete Beta function we have
\[ \sum \rho \frac{(2N)^\rho}{\rho} \left( 1 - \frac{1}{2N} \right)^\rho \left( B_{(2N-2)/(2N-1)} (\rho + 1, 0) - B_{N/(2N-1)} (\rho + 1, 0) \right) \]
\[ = \sum \rho \frac{(2N - 1)^\rho}{\rho} \int_{N/(2N-1)}^{(2N-2)/(2N-1)} \frac{t^\rho}{1 - t} \, dt = \int_{N/(2N-1)}^{(2N-2)/(2N-1)} \frac{1}{1 - t} \sum \rho \frac{(2N - 1)^\rho}{\rho} \, dt \]
since we know that we can exchange the series over the non-trivial zeros and the integral. Now from (31) we get
\[ (32) \quad \sum \rho \frac{(2N)^\rho}{\rho} \left( 1 - \frac{1}{2N} \right)^\rho \left( B_{(2N-2)/(2N-1)} (\rho + 1, 0) - B_{N/(2N-1)} (\rho + 1, 0) \right) \]
\[ \ll \begin{cases} (2N - 1) \int_{N/(2N-1)}^{(2N-2)/(2N-1)} t \exp \left( -C \sqrt{\log(t \cdot (2N - 1))} \right) / (1 - t) \, dt & \text{without RH} \\ \sqrt{(2N - 1)} \int_{N/(2N-1)}^{(2N-2)/(2N-1)} \sqrt{t} \log^2(t \cdot (2N - 1)) / (1 - t) \, dt & \text{with RH} \end{cases} \]
\[ \ll \begin{cases} N \exp \left( -C \sqrt{\log(N)} \right) & \text{without RH} \\ \sqrt{N} \log^3(N) & \text{with RH} \end{cases} \]
where $C > 0$ is a real constant. Hence we can conclude that
\[
-2 \sum_{n \leq N} \Lambda(n) \sum_{\rho} \frac{(2N-n)^{\rho}}{\rho} = -2 \sum_{\rho} \frac{(2N-2)^{\rho+1}}{\rho (\rho+1)} + 2 \sum_{\rho} \frac{N^{\rho+1}}{\rho (\rho+1)} + 2 \sum_{\rho_1, \rho_2} \frac{N^{\rho_1+\rho_2}}{\rho_1 \rho_2} - \Lambda(N) \sum_{\rho} \frac{N^\rho}{\rho}
\]
(33)
\[
+2 \sum_{\rho_1} (2N)^{\rho_1} \left( \Gamma(\rho_1) \sum_{\rho_2} \frac{(2N)^{\rho_2} \Gamma(\rho_2)}{\Gamma(\rho_1+\rho_2+1)} \right)
\]
(34)
\[
-\sum_{\rho_2} \frac{(2N)^{\rho_2}}{\rho_2} \left( B_{1/N}(\rho_2+1, \rho_1) + B_{1/2}(\rho_1, \rho_2+1) \right)
\]
(35)
\[
+O \left( \begin{cases} N \exp \left( -C \sqrt{\log(N)} \right) & \text{without RH} \\ \sqrt{N} \log^3(N) & \text{with RH} \end{cases} \right)
\]
(36)

3.3. The constant term. Trivially
\[
-2 \sum_{n \leq N} \Lambda(n) \frac{\zeta'}{\zeta}(0) = -2 \frac{\zeta'}{\zeta}(0) \psi(N).
\]

3.4. The term involving the logarithmic function. By the Abel summation formula we have
\[
- \sum_{n \leq N} \Lambda(n) \log \left( 1 - \frac{1}{(2N-n)^2} \right) = - \psi(N) \log \left( 1 - \frac{1}{N^2} \right) - \int_2^N \frac{2\psi(t)}{(2N-t)(2N-t-1)(2N-t+1)} dt
\]
and now using again the explicit formula for $\psi(t)$ we can evaluate term by term. Trivially we have that
\[
-2 \int_2^N \frac{tdt}{(2N-t)(2N-t-1)(2N-t+1)} = \int_2^N \frac{2t}{2N-t} dt - \int_2^N \frac{t}{2N-t-1} dt - \int_2^N \frac{t}{2N-t+1} dt
\]
\[
= 4 + 8N \arccoth(1 - 2N) + 2N \log(4) - 1
\]
\[
+ 2N - 2 + (4N + 2) \arccoth \left( \frac{3N}{2 - N} \right)
\]
\[
- 2(2N-1) \log \left( \frac{N-1}{2N-3} \right).
\]

For the second term, by Lemma 5 we observe that we can switch the integral with the series over the non-trivial zeros so
\[
\int_2^N \frac{2t}{(2N-t)(2N-t-1)(2N-t+1)} \sum_{\rho} \frac{t^\rho}{\rho} dt = \sum_{\rho} \frac{1}{\rho} \int_2^N dt \left( \frac{-2t^\rho}{2N-t} + \frac{t^\rho}{2N-t-1} - \frac{t^\rho}{2N-t+1} \right)
\]
(37)
\[
= -2 \sum_{\rho} \frac{(2N)^\rho}{\rho} \int_{1/N}^{1/2} \frac{u^\rho}{1-u} du
\]
(38)
\[
+ \sum_{\rho} \frac{(2N-1)^\rho}{\rho} \int_{2/(2N-1)}^{N/(2N-1)} \frac{u^\rho}{1-u} du
\]
(39)
and the integrals in (37), (38) and (39) are difference of two incomplete Beta functions. Observe that this arrangement is legitimate since, arguing as in (32) we can prove that the series in (37), (38) and (39) converges absolutely. Then we obviously get

$$-2 \zeta' (0) \int_2^N \frac{dt}{(2N-t) (2N-t-1) (2N-t+1)} = \frac{\zeta'}{\zeta} (0) \left( \log \left( 1 - \frac{1}{(2N-2)^2} \right) - \log \left( 1 - \frac{1}{N^2} \right) \right).$$

It remains to evaluate

$$\int_2^N \frac{\log (1 - 1/t^2)}{(2N-t) (2N-t-1) (2N-t+1)} dt = \int_2^N \frac{2 \log (t) - \log (t-1) - \log (t+1)}{(2N-t) (2N-t-1) (2N-t+1)} dt = -2 \int_2^N \frac{2 \log (t) - \log (t-1) - \log (t+1)}{2N-t} dt + \int_2^N \frac{2 \log (t) - \log (t-1) - \log (t+1)}{2N-t-1} dt + \int_2^N \frac{2 \log (t) - \log (t-1) - \log (t+1)}{2N-t+1} dt.$$  

(40) 

(41) 

(42)

We will show only a single evaluation since the others are essentially the same thing. We have that

$$\int_2^N \frac{\log (t)}{2N-t} dt = \int_{1/N}^{1/2} \frac{\log (2N) + \log (u)}{1-u} du = \log (2N) \left( \log \left( \frac{1}{2} \right) - \log \left( 1 - \frac{1}{N} \right) \right) + \text{Li}_2 \left( \frac{1}{2} \right) - \text{Li}_2 \left( 1 - \frac{1}{N} \right)$$

where \( \text{Li}_2 (x) \) is the Dilogarithm function. Using this strategy we will get, for all integrals in (40), (41) and (42), a combination of elementary functions and Dilogarithms.

Finally we note that

$$- \sum_{n \leq N} \Lambda (n) \log \left( 1 - \frac{1}{(2N-n)^2} \right) \ll \frac{1}{N^2} \sum_{n \leq N} \Lambda (n) \ll \frac{1}{N}.$$

3.5. The term involving the Von Mangoldt function. Lastly we have

$$\sum_{n \leq N} \Lambda (n) \Lambda (2N-n) = \frac{1}{2} \sum_{n \leq 2N} \Lambda (n) \Lambda (2N-n) = \frac{1}{2} \sum_{m_1, m_2 \leq 2N \atop m_1 + m_2 = 2N} \Lambda (m_1) \Lambda (m_2) = \frac{r_G (2N)}{2}.$$

3.6. Put together all the pieces. Finally we can rearrange all the parts. Expanding \( \psi^2 (N) \) in (18) with its explicit formula and observing that some terms cancel each other out (see, for example, (23) and (24)) we get that

$$\sum_{n \leq 2N} ' r_G (n) = 2N^2 - 2 \sum_{\rho} \frac{(2N-2)_{\rho+1}}{\rho (\rho + 1)} + 2 \sum_{\rho_1} (2N)^{\rho_1} \left( \Gamma (\rho_1) \sum_{\rho_2} \frac{(2N)^{\rho_2} \Gamma (\rho_2)}{\Gamma (\rho_1 + \rho_2 + 1)} \right) - \sum_{\rho_2} \frac{(2N)^{\rho_2}}{\rho_2} \left( B_{1/N} (\rho_2 + 1, \rho_1) + B_{1/2} (\rho_1, \rho_2 + 1) \right) + F (N)$$

where \( F (N) \) can be explicitly calculated in terms of special functions like the incomplete Beta function and the Dilogarithm and \( F (N) = O (N) \) as \( N \to \infty \).
3.7. The truncated formula. We want to prove the truncated version of the formula. We start taking \( T_1 > 2 \) and substituting the formula

\[
\psi (2N - n) = 2N - n - \sum_{\rho : \gamma | \leq T_1} \frac{(2N - n)^\rho}{\rho} - \frac{\zeta'(0)}{\zeta} - \frac{1}{2} \log \left( 1 - \frac{1}{(2N - n)^2} \right) \\
+ \frac{\Lambda (2N - n)}{2} + O \left( \frac{N \log^2 (NT_1)}{T_1} \right)
\]

in \((19)\). We will evaluate the sum term by term. Again we recall that, if \( x \) is an integer, then the error term in \((9)\) can be written as \( O \left( x \log \frac{2}{xT} \right) \) since \( \langle x \rangle \geq 1 \).

3.8. The main term of the truncated formula. Following the 3.1 section we get

\[
2 \sum_{n \leq N} \Lambda (n) (2N - n) = 2N \psi (N) + 2 \psi_1 (N) .
\]

From \((9)\) we get

\[
2N \psi (N) = 2N^2 - 2 \sum_{\rho : \gamma | \leq T_2} \frac{N^{\rho + 1}}{\rho} - 2N \frac{\zeta'(0)}{\zeta} - N \log \left( 1 - \frac{1}{N^2} \right) + N \Lambda (N) + O \left( \frac{N^2 \log^2 (NT_2)}{T_2} \right)
\]

where \( T_2 > 2 \) will be chosen later. For the evaluation of \( \psi_1 (N) \) we observe that

\[
2 \psi_1 (N) = 2 \int_0^N \psi (t) \, dt
\]

\[
= 2 \int_2^N \left( \frac{t^\rho}{\rho} - \frac{\zeta'(0)}{\zeta} - \frac{\log (1 - 1/t^2)}{2} + O \left( \frac{t \log^2 (T_3 t)}{T_3} + \log (t) \min \left( 1, \frac{t}{T_3 \langle t \rangle} \right) \right) \right) \, dt
\]

\[
= N^2 - 2 \sum_{\rho : \gamma | \leq T_3} \frac{N^{\rho + 1}}{\rho (\rho + 1)} - 2N \frac{\zeta'(0)}{\zeta} - \sum_{r \geq 1} \frac{N^{-2r + 1}}{r (2r - 1)} + O \left( \frac{N^2 \log^2 (T_3 N)}{T_3} \right)
\]

where \( T_3 > 2 \) and, for the integration of the error term we used the same strategy of \((12)\) and \((17)\).

3.9. The term involving the series over the non-trivial zeros of \( \zeta (s) \) of the truncated formula. Following the 3.2 section we have

\[
-2 \sum_{n \leq N} \Lambda (n) \sum_{\rho : \gamma | \leq T_1} \frac{(2N - n)^\rho}{\rho} = -2 \sum_{\rho : \gamma | \leq T_1} \frac{(2N)^\rho}{\rho} \sum_{n \leq N} \Lambda (n) \left( 1 - \frac{n}{2N} \right)^\rho
\]

then from Lemma \([6]\) taking \( x = N, y = 2N, \alpha = \rho, T = T_4 > 2 \) and observing that, in this case, we have

\[
M (\rho, 2N, N) \ll 1
\]
where the implicit constant does not depend on \( N \) or \( \rho \), we get

\[
-2 \sum_{n \leq N} \Lambda(n) \sum_{\nu : |\gamma| \leq T_1} \frac{(2N - n)^\rho}{\rho} = -2 \sum_{\nu : |\gamma| \leq T_1} \frac{(2N - 2)^{\rho+1}}{\rho(\rho + 1)} + 2 \sum_{\nu : |\gamma| \leq T_1} \frac{N^{\rho+1}}{\rho(\rho + 1)} - 4 \sum_{\nu : |\gamma| \leq T_1} \frac{(2N - 2)^\rho}{\rho}
\]

\[
+ 2 \sum_{\nu_1 : |\gamma| \leq T_1} (2N)^{\rho_1} \left( \Gamma(\rho_1) \sum_{\nu_2 : |\gamma| \leq T_4} \frac{(2N)^{\rho_2}}{\rho_2} \frac{\Gamma(\rho_2)}{\Gamma(\rho_1 + \rho_2 + 1)} \right)
\]

\[- \sum_{\nu_2 : |\gamma| \leq T_4} \frac{(2N)^{\rho_2}}{\rho_2} (B_{1/N} (\rho_2 + 1, \rho_1) + B_{1/2} (\rho_1, \rho_2 + 1)) \]

\[
+ 2 \sum_{\nu_1 : |\gamma| \leq T_1} \frac{N^{\rho_1}}{\rho_1} \sum_{\nu_2 : |\gamma| \leq T_4} \frac{N^{\rho_2}}{\rho_2} + 2 \frac{\zeta' (0)}{\zeta} \sum_{\nu : |\gamma| \leq T_1} \frac{(2N - 2)^\rho}{\rho}
\]

\[
+ \log \left( \frac{3}{4} \right) \sum_{\nu : |\gamma| \leq T_1} (2N - 2)^\rho - 2 \sum_{\nu : |\gamma| \leq T_1} \frac{(2N)^\rho}{\rho} \omega (\rho, 2N, N)
\]

\[
(43)
\]

where \( T_4 > 2 \) will be choose later. It is clear that the size of (43) changes if we assume RH or not.

3.10. The constant term of the truncated formula. Again we observe that

\[
-2 \sum_{n \leq N} \Lambda(n) \frac{\zeta'}{\zeta} (0) = -2 \frac{\zeta'}{\zeta} (0) \psi (N)
\]

and so we can substitute (9) in (44) with some \( T_5 > 2 \) that will be choose later.

3.11. The term involving the logarithmic function of the truncated formula. By section 3.4 we know that By the Abel summation formula we have

\[
- \sum_{n \leq N} \Lambda(n) \log \left( 1 - \frac{1}{(2N - n)^2} \right) = -\psi(N) \log \left( 1 - \frac{1}{N^2} \right) - \int_2^N \frac{2\psi(t)}{(2N - t)(2N - t - 1)(2N - t + 1)} dt.
\]

We fix \( T_6 > 2 \). Then we can expand the term

\[
\psi(N) \log \left( 1 - \frac{1}{N^2} \right)
\]

with (9). The integral

\[
\int_2^N \frac{2\psi(t)}{(2N - t)(2N - t - 1)(2N - t + 1)} dt
\]

will be treated as in 3.4 with but, fixing \( T_7 > 2 \), we will get the extra terms

\[
\frac{1}{T_7} \int_2^N \frac{t \log^2(tT_7)}{(2N - t)(2N - t - 1)(2N - t + 1)} dt \ll \frac{\log^2(NT_7)}{NT_7}
\]

and

\[
\int_2^N \frac{\log(t) \min(1, t/(T_7(t)))}{(2N - t)(2N - t - 1)(2N - t + 1)} dt \ll \frac{\log^2(N)}{T_7}
\]

arguing as in (17).
3.12. **The error term and the Von Mangoldt term.** Trivially we have

\[
\frac{1}{T_1} \sum_{n \leq N} \Lambda(n) (2N - n) \log^2(T_1(2N - n)) \ll \frac{N^2 \log^2(T_1N)}{T_1}
\]

and the “Von Mangoldt term” is exactly as in 3.5.

3.13. **Put together all the pieces of the truncated formula.** Now it remains to expand the term \( \psi^2(N) \) in (18) with (9) fixing some \( T_8 > 2 \). We want to exploit the cancellation of this formula so we have to choose carefully the \( T_j \) terms. Obviously if we take \( T_j \to \infty \) in a suitable order we can recognize the previous formula. The choice of \( T_j \) is very delicate; we must take advantage of the cancellation effectively but we do not want to take too large parameters. To finish our version of the formula we have to impose the condition

\[
T_j = T', \quad j = 1, \ldots, 8, \quad j \neq 4;
\]

this assumption guarantees, for example, the cancellation of sums like

\[
-2 \sum_{\rho : |\gamma| \leq T_3} \frac{N^{\rho+1}}{\rho(\rho+1)} + 2 \sum_{\rho : |\gamma| \leq T_1} \frac{N^{\rho+1}}{\rho(\rho+1)}
\]

(see section 3.8 and 3.9) and

\[
-2 \sum_{\rho : |\gamma| \leq T_2} \frac{N^{\rho+1}}{\rho} + 2 \sum_{\rho : |\gamma| \leq T_8} \frac{N^{\rho+1}}{\rho}
\]

(for the first sum see section 3.8, for the second sum just expand \( \psi^2(N) \) with its truncated formula). Then we take

\[
T_4 = T''
\]

and we estimate the sum in (13) with the Riemann - Von Mangoldt formula and the zero free region of \( \zeta(s) \).

4. SOME REMARKS

We now present some remark of this result. The first remark is that this result can be improved: with a bit of work it is not difficult to extract the term \( N \cdot \text{constant} \) explicitly and give a lower error of the asymptotic (which will depends on the RH assumption). The second remark is that from Lemma 6 we can, in principle, find the explicit formula for the Cesàro average of Goldbach numbers

\[
\frac{1}{\Gamma(k+1)} \sum_{n \leq N} r_G(n)(N-n)^k, \quad k > 0.
\]

We use the words “in principle” because we will expect a lot of terms to calculate. The idea is the following: from the identity

\[
\frac{1}{\Gamma(k+1)} \sum_{n \leq N} r_G(n)(N-n)^k = \frac{N^k}{\Gamma(k+1)} \sum_{n \leq N} \Lambda(n) \left(1 - \frac{n}{N}\right)^k \sum_{m \leq N-n} \Lambda(m) \left(1 - \frac{m}{N-n}\right)^k
\]

(observed in [4]) we can easily see that the problem boils down to evaluate the combination of sums involving the Von Mangoldt function with a Cesàro weight. So we can substitute in (15) the explicit formula with \( y = N - n, \quad x = N - n - 1 \) and \( \alpha = k \) and evaluate the sum term by term. For example the main term (that we know is \( \frac{N^{k+2}}{\Gamma(k+3)} \) from [4]) will be from

\[
\frac{N^k}{\Gamma(k+1)} \sum_{n \leq N} \Lambda(n) \left(1 - \frac{n}{N}\right)^k \frac{N-n}{k+1} = \frac{1}{\Gamma(k+2)} \sum_{n \leq N} \Lambda(n)(N-n)^{k+1}.
\]
To confirm our claim note that by the Abel summation formula we find that

\[(46) \quad \frac{N^k}{\Gamma (k + 1)} \sum_{n < N} \Lambda (n) \left(1 - \frac{n}{N}\right)^k = \frac{1}{\Gamma (k + 2)} \int_2^N \psi (t) (N - t)^k \, dt\]

then substituting the explicit formula for \(\psi (t)\) in \((46)\) we will find the main term of the explicit formula plus other terms. In fact we can see that

\[
\frac{k + 1}{\Gamma (k + 2)} \int_2^N t (N - t)^k \, dt = \frac{N^{k + 2}}{\Gamma (k + 3)} + H_k (N)
\]

where \(H_k (N) = O_k (N^{k+1})\), as expected.

5. Proof of Theorem 4

Now we show that a very similar approach to the previous one can be used also to find the explicit form of the average of primes in tuples. We start again with a summation identity:

\[
\sum_{h=0}^M \sum_{n=0}^N a_n b_{n+h} = \left( \sum_{n=0}^N a_n \right) \left( \sum_{n=0}^{N+M} b_n \right) - \sum_{n=0}^{N-1} \left( b_n \sum_{m=n+1}^N a_m + a_n \sum_{m=n+M+1}^{N+M} b_m \right)
\]

where \(M, N \geq 0\) are integers, which can be proved observing that the set of lattice points \(\{(i, i+j) : 0 \leq i \leq N, 0 \leq j \leq M\}\) forms a parallelogram, which can be seen as a \(N \times (N + M)\) rectangular minus the triangles

\[\{(i, j) : 0 \leq j \leq N - 1, j + 1 \leq i \leq N\}, \{(i, j) : 0 \leq i \leq N - 1, i + 1 + M \leq j \leq M + N\}\]

We now fix \(N > 2\) and \(0 \leq M \leq N\) and define

\[r_{PT} (N, h) = \sum_{n=0}^N \Lambda (n) \Lambda (n + h)\]

We have that

\[
\sum_{h=0}^M r_{PT} (N, h) = \left( \sum_{n=0}^N \Lambda (n) \right) \left( \sum_{n=0}^{N+M} \Lambda (n) \right) - \sum_{n=0}^{N-1} \left( \Lambda (n) \sum_{m=n+1}^N \Lambda (m) + \Lambda (n) \sum_{m=n+M+1}^{N+M} \Lambda (m) \right)
\]

\[= \psi (N) \psi (N + M) - \sum_{n=0}^{N-1} \Lambda (n) (\psi (N) - \psi (n)) - \sum_{n=0}^{N-1} \Lambda (n) (\psi (N + M) - \psi (n + M))
\]

\[
= \sum_{n \leq N} \Lambda (n) \psi (n + M) + \sum_{n \leq N} \Lambda (n) \psi (n) - \psi^2 (N).
\]

Again we will consider

\[
\sum_{n \leq N} \Lambda (n) \psi (n + M) + \sum_{n \leq N} \Lambda (n) \psi (n)
\]

and we will substitute \(\psi (x)\) with its explicit formula.
5.1. **The main term.** Substituting \( \psi(x) \) with \( x \) we get

\[
\sum_{n \leq N} \Lambda(n)(n+M) + \sum_{n \leq N} n\Lambda(n) = 2 \sum_{n \leq N} n\Lambda(n) + M\psi(N).
\]

By the Abel summation formula we have that

\[
2 \sum_{n \leq N} n\Lambda(n) = 2\psi(N)N - 2\psi_1(N),
\]

where \( \psi_1(N) \) is [21]. So, expanding \( \psi(N) \) and \( \psi_1(N) \) with their explicit formulae, we obtain

\[
2 \sum_{n \leq N} n\Lambda(n) + M\psi(N) = N^2 + NM - 2 \sum_{\rho} \frac{N^{\rho+1}}{\rho} + 2 \sum_{\rho} \frac{N^{\rho}}{\rho - 1} - M\sum_{\rho} \frac{N^{\rho}}{\rho},
\]

\[
\quad \quad = -M\frac{C}{\zeta}(0) - 2\frac{C}{\zeta}(-1) - \left( N + \frac{M}{2} \right) \log \left( 1 - \frac{1}{N^2} \right) - \sum_{r \geq 1} \frac{N^{-2r+1}}{r(2r-1)} + \left( N + \frac{M}{2} \right) \Lambda(N).
\]

Obviously, from [31], we can also see that

\[
2 \sum_{n \leq N} n\Lambda(n) + M\psi(N) = N^2 + NM - 2 \sum_{\rho} \frac{N^{\rho+1}}{\rho} + 2 \sum_{\rho} \frac{N^{\rho+1}}{\rho(\rho+1)} + \left( N + \frac{M}{2} \right) \Lambda(N) + O(E(M,N))
\]

(48)

where

\[
E(M,N) = \begin{cases} 
N(M+1) \exp \left( -C \sqrt{\log(N)} \right) & \text{without RH} \\
\sqrt{N} (M+1) \log^2(N) & \text{with RH}
\end{cases}, \quad C > 0,
\]

and where the implicit constant in (48) does not depend on \( M \).

5.2. **The term involving the series over the non-trivial zeros of \( \zeta(s) \).** We have now to evaluate

\[
- \sum_{n \leq N} \Lambda(n) \sum_{\rho} \frac{(n+M)^{\rho}}{\rho} - \sum_{n \leq N} \Lambda(n) \sum_{\rho} \frac{n^{\rho}}{\rho}.
\]

We can consider only the sum

\[
\sum_{n \leq N} \Lambda(n) \sum_{\rho} \frac{(n+M)^{\rho}}{\rho} = -\sum_{\rho} \frac{1}{\rho} \sum_{n \leq N} \Lambda(n)(n+M)^{\rho}
\]

since the other is the same sum with the assumption \( M = 0 \). Again from the Abel summation formula we obtain

\[
\sum_{n \leq N} \Lambda(n)(n+M)^{\rho} = \psi(N)(N+M)^{\rho} - \rho \int_{2}^{N} \psi(t)(t+M)^{\rho-1} dt.
\]

(49)
Substituting the main term of the explicit formula of $\psi(t)$ in (49) we obtain

$$-\rho \int_2^N t (t + M) \rho^{-1} dt = - N (N + M)^{\rho} + 2 (2 + M)^{\rho}$$

$$+ \frac{(N + M)^{\rho+1}}{\rho + 1} - \frac{(2 + M)^{\rho+1}}{\rho + 1}.$$ 

Obviously if $M = 0$ we can make the same calculations. Now we consider the sum over the non-trivial zeros. Assume that $M > 0$. Then by Lemma 5 we have

$$\rho_1 \int_2^N \sum_{\rho_2} \frac{t^\rho_2}{\rho_2} (t + M)^{\rho_1 - 1} dt = \rho_1 \int_2^N \frac{1}{\rho_2} \int_2^N t^\rho_2 (t + M)^{\rho_1 - 1} dt$$

$$= \rho_1 \sum_{\rho_2} \frac{M^{\rho_1 + \rho_2}}{\rho_2} \int_{2/M}^{N/M} u^\rho_2 (1 + u)^{\rho_1 - 1} du$$

$$= \rho_1 \sum_{\rho_2} \frac{M^{\rho_1 + \rho_2} (1)^{\rho_2 + 1}}{\rho_2} (B_{-N/M} (\rho_2 + 1, \rho_1) - B_{-2/M} (\rho_2 + 1, \rho_1)).$$

(50)

where in (50) we extended the definition of incomplete Beta function to a negative integration domain (or, if we prefer, we can write the integral in terms of the Gauss Hypergeometric function $\, _2F_1(a, b; c; z)$). In the other case (or if $M = 0$) we get

$$\rho_1 \int_2^N \sum_{\rho_2} \frac{t^\rho_2}{\rho_2} t^{\rho_1 - 1} dt = \rho_1 \sum_{\rho_2} \frac{1}{\rho_2} \int_2^N t^{\rho_1 + \rho_2 - 1} dt$$

$$= \rho_1 \sum_{\rho_2} \frac{N^{\rho_1 + \rho_2} - 2^{\rho_1 + \rho_2}}{\rho_2 (\rho_1 + \rho_2)}.$$

Then we have to consider the constant term

$$\rho \frac{\zeta'}{\zeta} (0) \int_2^N (t + M)^{\rho - 1} dt = \frac{\zeta'}{\zeta} (0) (N + M)^{\rho} - \frac{\zeta'}{\zeta} (0) (2 + M)^{\rho}$$

and lastly

$$\frac{\rho}{2} \int_2^N \log \left(1 - \frac{1}{t^2}\right) (t + M)^{\rho - 1} dt = \frac{1}{2} \log \left(1 - \frac{1}{N^2}\right) (N + M)^{\rho}$$

$$- \frac{1}{2} \log \left(1 - \frac{1}{4}\right) (2 + M)^{\rho}$$

(51)

$$+ \int_2^N \frac{(t + M)^{\rho - 1}}{t} dt - \frac{1}{2} \int_2^N \frac{(t + M)^{\rho - 1}}{t - 1} dt$$

(52)

$$- \frac{1}{2} \int_2^N \frac{(t + M)^{\rho - 1}}{t + 1} dt$$

(53)

and the integrals in (52) and (53) can be evaluated as a difference of two incomplete Beta functions with negative integration domain. For example

$$\int_2^N \frac{(t + M)^{\rho}}{t} dt = M^\rho (-1)^{\rho} \int_{-2/M}^{-N/M} (1 - u)^{\rho} du$$

$$= M^\rho (-1)^{\rho} \lim_{\epsilon \to 0^+} (B_{-2/M} (\epsilon, \rho + 1) - B_{-N/M} (\epsilon, \rho + 1))$$
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(if one prefer this integral can be written as a combination of Gauss Hypergeometric function). If $M = 0$ we can do a similar calculation. It is more interesting to note that, in the form of (52) and (53), we can easily evaluate the integral since, summing up, we have

$$\sum_{\rho} \frac{1}{2} \int_{-\frac{1}{2}}^{N} \log \left(1 - \frac{1}{t^2}\right) (t + M)^{\rho-1} dt = \frac{1}{2} \log \left(1 - \frac{1}{N^2}\right) \sum_{\rho} \frac{(N + M)^{\rho}}{\rho}$$

(54)

$$-\frac{1}{2} \log \left(1 - \frac{1}{4}\right) \sum_{\rho} \frac{(2 + M)^{\rho}}{\rho}$$

(55)

$$-\int_{\frac{1}{2}}^{N} \sum_{\rho} \frac{(t + M)^{\rho}}{\rho} \cdot \frac{1}{t (t - 1) (t + 1)} dtt$$

(56)

and the implicit constant does not depend on $M$. So, expand $\psi(N)$ in (49) with its explicit formula and observing that some terms cancel each other out, we finally get

$$-\sum_{n \leq N} \Lambda(n) \sum_{\rho} \frac{(n + M)^{\rho}}{\rho} - \sum_{n \leq N} \Lambda(n) \sum_{\rho} \frac{n^{\rho}}{\rho} = -\sum_{\rho} \frac{(N + M)^{\rho+1}}{\rho (\rho + 1)} + \sum_{\rho} \frac{(2 + M)^{\rho+1}}{\rho (\rho + 1)} - 2 \sum_{\rho} \frac{(2 + M)^{\rho}}{\rho} - \sum_{\rho} \frac{N^{\rho+1}}{\rho (\rho + 1)}$$

$$- F_1(N, M) - F_2(N, M)$$

$$+ \sum_{\rho_1} \frac{N^{\rho_1}}{\rho_1} \sum_{\rho_2} \frac{(N + M)^{\rho_2}}{\rho_2} + \sum_{\rho_1} \frac{N^{\rho_1}}{\rho_1} \sum_{\rho_2} \frac{N^{\rho_2}}{\rho_2}$$

$$- \frac{\Lambda(N)}{2} \left( \sum_{\rho} \frac{(N + M)^{\rho}}{\rho} + \sum_{\rho} \frac{N^{\rho}}{\rho} \right) + F_3(N, M)$$

where:

$$F_1(N, M) = \left( \sum_{\rho_1} \sum_{\rho_2} \frac{M^{\rho_1 + \rho_2} (-1)^{\rho_2+1}}{\rho_2} \left( B_{-N/M} (\rho_2 + 1, \rho_1) - B_{-2/M} (\rho_2 + 1, \rho_1) \right) \right) \cdot 1(M),$$

$$F_2(N, M) = \left( \sum_{\rho_1} \sum_{\rho_2} \frac{N^{\rho_1 + \rho_2} - 2^{\rho_1 + \rho_2}}{\rho_2 (\rho_1 + \rho_2)} \right) \cdot \tilde{1}(M),$$

$$1(M) = \begin{cases} 0, & M = 0 \\ 1, & M > 0 \end{cases}$$

$$\tilde{1}(M) = \begin{cases} 2, & M = 0 \\ 1, & M > 0 \end{cases}$$

and $F_3(N, M)$ can be explicitly calculated in terms of the incomplete Beta function and with the property

$$F_3(N, M) \ll \begin{cases} N \exp\left(-C \sqrt{\log(N)}\right), & \text{without RH} \\ \sqrt{N} \log^2(N), & \text{with RH} \end{cases}$$
Note that, arguing analogously to 3.2, we can conclude that the rearrangement is legitimate and the double series in $F_1(N, M)$ and $F_2(N, M)$ converges.

5.3. **The constant term.** Trivially we have

$$-2 \sum_{n \leq N} \Lambda(n) = -2 \sum_{n \leq N} \psi(n)$$

5.4. **The term involving the logarithmic function and the Von Mangoldt function.** We have now to evaluate

$$-\frac{1}{2} \sum_{n \leq N} \Lambda(n) \left( \log \left(1 - \frac{1}{(n + M)^2}\right) + \log \left(1 - \frac{1}{n^2}\right) \right)$$

$$= -\frac{\psi(N)}{2} \left( \log \left(1 - \frac{1}{(N + M)^2}\right) + \log \left(1 - \frac{1}{N^2}\right) \right)$$

$$+ \int_2^N \frac{\psi(t)}{(M + t) (M + t + 1) (M + t - 1)} dt + \int_2^N \frac{\psi(t)}{t (t + 1) (t - 1)} dt$$

(57)

which can be evaluated again integrating term by term the explicit formula of $\psi(t)$. Arguing as in the previous sections, it is possible to calculate the integrals in (57) in terms of elementary functions, incomplete Beta functions and Dilogarithms. Furthermore

$$-\frac{1}{2} \sum_{n \leq N} \Lambda(n) \left( \log \left(1 - \frac{1}{(n + M)^2}\right) + \log \left(1 - \frac{1}{n^2}\right) \right) \ll \sum_{n \leq N} \frac{\Lambda(n)}{n^2}$$

uniformly in $M$. Obviously we will have also the “Von Mangoldt terms”

$$\frac{1}{2} \sum_{n \leq N} \Lambda(n) \Lambda(n + M) + \frac{1}{2} \sum_{n \leq N} \Lambda(n)^2 = \frac{r_{PT}(N, M)}{2} + \frac{r_{PT}(N, 0)}{2}$$

5.5. **Put together all the pieces.** Expanding $\psi^2(N)$ in (47) and observing that some terms cancel each other out (for example the term $N^2$ in Section 5.1 or the double series $\sum_{\rho_1} \frac{N^{\rho_1}}{\rho_1} \sum_{\rho_2} \frac{N^{\rho_2}}{\rho_2}$ in Section 5.2) we finally get

$$\sum_{h=0}^{M} r_{PT}(N, h) = NM + \sum_{\rho} \frac{N^{\rho+1}}{\rho (\rho + 1)} - \sum_{\rho} \frac{(N + M)^{\rho+1}}{\rho (\rho + 1)} + \sum_{\rho} \frac{(2 + M)^{\rho+1}}{\rho (\rho + 1)}$$

$$- \left( \sum_{\rho_1} \sum_{\rho_2} \frac{M^{\rho_1+\rho_2} (\rho_2 + 1)^{\rho_2+1}}{\rho_2} (B_{-N/M} (\rho_2 + 1, \rho_1) - B_{-2/M} (\rho_2 + 1, \rho_1)) \right) \cdot 1 (M)$$

$$- \left( \sum_{\rho_1} \sum_{\rho_2} \frac{N^{\rho_1+\rho_2} - 2^{\rho_1+\rho_2}}{\rho_2 (\rho_1 + \rho_2)} \right) \cdot \tilde{\Gamma}(M) + \sum_{\rho_1} \frac{N^{\rho_1}}{\rho_1} \sum_{\rho_2} \frac{(N + M)^{\rho_2}}{\rho_2}$$

$$+ \frac{\Lambda(N)}{2} \sum_{\rho} \frac{N^{\rho}}{\rho} - \frac{\Lambda(N)}{2} \sum_{\rho} \frac{(N + M)^{\rho}}{\rho}$$

$$+ G(N, M)$$

where

$$G(N, M) \ll \begin{cases} N (M + 1) \exp \left(-C \sqrt{\log(N)}\right), & \text{without RH} \\ \sqrt{N} (M + 1) \log^2(N), & \text{with RH} \end{cases}$$

as claimed.
It is reasonable to think that a truncated version of this formula can be done with the same strategy we used in Sections 3.7 – 3.13; it will be the subject of future research.

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