ON THE CHOW GROUPS OF SOME HYPERKÄHLER FOURFOLDS WITH A NON–SYMPLECTIC INVOLUTION II

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ABSTRACT. This article is about hyperkähler fourfolds $X$ admitting a non–symplectic involution $\iota$. The Bloch–Beilinson conjectures predict the way $\iota$ should act on certain pieces of the Chow groups of $X$. The main result is a verification of this prediction for Fano varieties of lines on certain cubic fourfolds. This has some interesting consequences for the Chow ring of the quotient $X/\iota$.

1. INTRODUCTION

For a smooth projective variety $X$ defined over $\mathbb{C}$, let $A^i(X) := CH^i(X)_{\mathbb{Q}}$ denote the Chow groups of $X$ (i.e. the groups of codimension $i$ algebraic cycles on $X$ with $\mathbb{Q}$–coefficients, modulo rational equivalence). As explained for instance in [13] or [32] or [20], the Bloch–Beilinson conjectures form a beautiful crystal ball, allowing to make strikingly concrete predictions about Chow groups. In this article, we consider one instance of such a prediction, concerning non–symplectic involutions on hyperkähler varieties.

Let $X$ be a hyperkähler variety (that is, a projective irreducible holomorphic symplectic manifold, cf. [1], [2]), and assume that $X$ admits an anti–symplectic involution $\iota$. The action of $\iota$ on the subring $H^{*,0}(X)$ is well–understood: one has

$$\iota^* = -\text{id} : H^{2i,0}(X) \to H^{2i,0}(X) \quad \text{for } i \text{ odd},$$

$$\iota^* = \text{id} : H^{2i,0}(X) \to H^{2i,0}(X) \quad \text{for } i \text{ even}.$$

The action of $\iota$ on the Chow ring $A^*(X)$ is more mysterious. To state the conjectural behaviour, we will now assume the Chow ring of $X$ has a bigraded ring structure $A^*_X$, where each $A^i(X)$ splits into pieces

$$A^i(X) = \bigoplus_j A^i_{(j)}(X),$$

and the piece $A^i_{(j)}(X)$ is isomorphic to the graded $Gr_j^A A^i(X)$ for the Bloch–Beilinson filtration that conjecturally exists for all smooth projective varieties. (It is expected that such a bigrading $A^*_{(\cdot)}$ exists for all hyperkähler varieties [4].)

Since the pieces $A^i_{(\cdot)}(X)$ and $A_{(\cdot)}^\dim(X)$ should only depend on the subring $H^{*,0}(X)$, we arrive at the following conjecture:

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Conjecture 1.1. Let $X$ be a hyperkähler variety of dimension $2m$, and let $\iota \in \text{Aut}(X)$ be a non–symplectic involution. Then

\begin{align*}
\iota^* &= (-1)^i \text{id}: A^{2i}_{(2i)}(X) \to A^{2i}(X), \\
\iota^* &= (-1)^i \text{id}: A^{2m}_{(2m)}(X) \to A^{2m}(X).
\end{align*}

This conjecture is studied (and proven in some favourable cases) in [14], [15], [16], [17], [18]. The aim of this article is to provide more examples where conjecture 1.1 is verified, by considering Fano varieties of lines on cubic fourfolds. The main result is as follows:

Theorem (=theorem 3.5). Let $Y \subset \mathbb{P}^5(\mathbb{C})$ be a smooth cubic fourfold defined by an equation

\begin{equation*}
(X_0)^2\ell_0(X_3, X_4, X_5) + (X_1)^2\ell_1(X_3, X_4, X_5) + (X_2)^2\ell_2(X_3, X_4, X_5) + X_0X_1\ell_3(X_3, X_4, X_5) + X_0X_2\ell_4(X_3, X_4, X_5) + X_1X_2\ell_5(X_3, X_4, X_5) + g(X_3, \ldots, X_5) = 0,
\end{equation*}

where the $\ell_i$ are linear forms and $g$ is a homogeneous degree 3 polynomial. Let $X = F(Y)$ be the Fano variety of lines in $Y$. Let $\iota \in \text{Aut}(X)$ be the anti–symplectic involution induced by

\begin{equation*}
[\mathbb{P}^5(\mathbb{C})] \to \mathbb{P}^5(\mathbb{C}), \quad [X_0, X_1, \ldots, X_5] \mapsto [-X_0, -X_1, -X_2, X_3, X_4, X_5].
\end{equation*}

Then

\begin{align*}
\iota^* &= - \text{id}: A^i_{(2i)}(X) \to A^i_{(2i)}(X) \text{ for } i = 2, 4; \\
\iota^* &= \text{id}: A^j_{(j)}(X) \to A^j_{(j)}(X) \text{ for } j = 0, 4.
\end{align*}

Here, the notation $A^*_i(X)$ refers to the Fourier decomposition of the Chow ring of $X$ constructed by Shen–Vial [23]. (We mention in passing that for $X$ as in theorem 3.5 it is unfortunately not yet known whether $A^*_i(X)$ is a bigraded ring, cf. remark 2.6 below.)

To prove theorem 3.5 we exploit the fact that the family of cubics under consideration is sufficiently large for the method of “spread” as developed by Voisin [29], [30] to apply. There is only one other family of cubic fourfolds with a polarized involution that is anti-symplectic on the Fano variety; this other family has been treated in [17], using arguments very similar to those of the present article. The action of polarized symplectic automorphisms on Chow groups of Fano varieties of cubic fourfolds has already been studied by L. Fu [8], similarly using Voisin’s method of “spread”.

Theorem 3.5 has some rather striking consequences for the Chow ring of the quotient (this quotient is a slightly singular Calabi–Yau fourfold):

Corollary (=corollaries 4.1 and 4.3). Let $(X, \iota)$ be as in theorem 3.5 and let $Z := X/\iota$ be the quotient. Then the images of the intersection product maps

\begin{align*}
A^2(Z) \otimes A^2(Z) &\to A^4(Z), \\
A^3(Z) \otimes A^1(Z) &\to A^4(Z)
\end{align*}

are of dimension 1.
In particular, this means that for any two cycles $b, c \in A^2(Z)$ (or $b \in A^3(Z)$ and $c \in A^1(Z)$), the 0–cycle $b \cdot c$ is rationally trivial if and only if it has degree 0. This is similar to results for Calabi–Yau complete intersections obtained in [28], [7].

**Corollary (=corollary 4.4).** Let $(X, \iota)$ be as in theorem 3.5 and let $Z := X/\iota$ be the quotient. Then the image of the intersection product map

$$\text{Im}(A^2(Z) \otimes A^1(Z) \to A^3(Z))$$

injects into $H^6(Z)$ under the cycle class map.

Corollaries 4.1 and 4.4 provide some support in favour of the conjecture that $A^2_{hom}(Z) \equiv 0$.

(The Bloch–Beilinson conjectures would imply that $A^2_{AJ}(M) = 0$ for any Calabi–Yau variety $M$ of dimension $> 2$. As far as I know, there is not a single Calabi–Yau variety $M$ for which this is known to be true.)

**Conventions.** In this article, the word variety will refer to a reduced irreducible scheme of finite type over $\mathbb{C}$. A subvariety is a (possibly reducible) reduced subscheme which is equidimensional.

All Chow groups will be with rational coefficients: we will denote by $A_j(X)$ the Chow group of $j$–dimensional cycles on $X$ with $\mathbb{Q}$–coefficients; for $X$ smooth of dimension $n$ the notations $A_j(X)$ and $A^{n-j}(X)$ are used interchangeably.

The notations $A^j_{hom}(X)$, $A^j_{AJ}(X)$ will be used to indicate the subgroups of homologically trivial, resp. Abel–Jacobi trivial cycles. For a morphism $f: X \to Y$, we will write $\Gamma_f \in A_*(X \times Y)$ for the graph of $f$. The category of Chow motives (i.e., pure motives with respect to rational equivalence as in [22], [20]) will be denoted $M_{\text{rat}}$.

We will write $H^j(X)$ to indicate singular cohomology $H^j(X, \mathbb{Q})$.

## 2. Preliminaries

### 2.1. MCK decomposition.

**Definition 2.1 (Murre [19]).** Let $X$ be a smooth projective variety of dimension $n$. We say that $X$ has a CK decomposition if there exists a decomposition of the diagonal

$$\Delta_X = \pi_0 + \pi_1 + \cdots + \pi_{2n} \text{ in } A^n(X \times X),$$

such that the $\pi_i$ are mutually orthogonal idempotents and $(\pi_i)_* H^*(X) = H^i(X)$.

(NB: “CK decomposition” is shorthand for “Chow–Künneth decomposition”.)

**Remark 2.2.** The existence of a CK decomposition for any smooth projective variety is part of Murre’s conjectures [19], [13].

**Definition 2.3 (Shen–Vial [23]).** Let $X$ be a smooth projective variety of dimension $n$. Let $\Delta_X^{\text{sm}} \in A^{2n}(X \times X \times X)$ be the class of the small diagonal

$$\Delta_X^{\text{sm}} := \{(x, x, x) \mid x \in X\} \subset X \times X \times X.$$
An MCK decomposition is a CK decomposition \( \{ \pi^X_i \} \) of \( X \) that is multiplicative, i.e. it satisfies
\[ \pi^X_k \circ \Delta^m_X \circ (\pi^X_i \times \pi^X_j) = 0 \text{ in } A^{2n}(X \times X \times X) \text{ for all } i + j \neq k. \]
(NB: “MCK decomposition” is shorthand for “multiplicative Chow–Künneth decomposition”.)
A weak MCK decomposition is a CK decomposition \( \{ \pi^X_i \} \) of \( X \) that satisfies
\[ \left( \pi^X_k \circ \Delta^m_X \circ (\pi^X_i \times \pi^X_j) \right)_* (a \times b) = 0 \text{ for all } a, b \in A^*(X). \]

**Remark 2.4.** The small diagonal (seen as a correspondence from \( X \times X \) to \( X \)) induces the multiplication morphism
\[ \Delta^m_X : h(X) \otimes h(X) \to h(X) \text{ in } \mathcal{M}_{\text{rat}}. \]
Suppose \( X \) has a CK decomposition
\[ h(X) = \bigoplus_{i=0}^{2n} h^i(X) \text{ in } \mathcal{M}_{\text{rat}}. \]
By definition, this decomposition is multiplicative if for any \( i, j \) the composition
\[ h^i(X) \otimes h^j(X) \to h(X) \otimes h(X) \xrightarrow{\Delta^m_X} h(X) \text{ in } \mathcal{M}_{\text{rat}} \]
factors through \( h^{i+j}(X) \).
If \( X \) has a weak MCK decomposition, then setting
\[ A^i_{(j)}(X) := (\pi^X_{2i-j})_* A^i(X), \]
one obtains a bigraded ring structure on the Chow ring: that is, the intersection product sends
\[ A^i_{(j)}(X) \otimes A^j_{(j')} (X) \to A^{i+j}_{(j+j')} (X). \]

It is expected (but not proven !) that for any \( X \) with a weak MCK decomposition, one has
\[ A^i_{(j)}(X) \overset{?}{=} 0 \text{ for } j < 0, \quad A^i_{(0)}(X) \cap A^i_{\text{hom}}(X) \overset{?}{=} 0; \]
this is related to Murre’s conjectures B and D, which have been formulated for any CK decomposition [19].
The property of having an MCK decomposition is severely restrictive, and is closely related to Beauville’s “(weak) splitting property” [4]. For ample discussion, and examples of varieties with an MCK decomposition, we refer to [23, Section 8], as well as [26], [24], [11].

In what follows, we will make use of the following:

**Theorem 2.5** (Shen–Vial [23]). Let \( Y \subset \mathbb{P}^5(\mathbb{C}) \) be a smooth cubic fourfold, and let \( X := F(Y) \) be the Fano variety of lines in \( Y \). There exists a CK decomposition \( \{ \pi^X_i \} \) for \( X \), and
\[ (\pi^X_{2i-j})_* A^i(X) = A^i_{(j)}(X), \]
where the right–hand side denotes the splitting of the Chow groups defined in terms of the Fourier transform as in [23, Theorem 2]. Moreover, we have
\[ A^i_{(j)}(X) = 0 \text{ for } j < 0 \text{ and for } j > i. \]

In case \( Y \) is very general, the Fourier decomposition \( A^*_{(0)}(X) \) forms a bigraded ring, and hence \( \{ \pi^X_i \} \) is a weak MCK decomposition.
Proof. (A remark on notation: what we denote $A_{(j)}^i(X)$ is denoted $CH^i(X)$ in [23].)

The existence of a CK decomposition $\{\pi_i^X\}$ is [23, Theorem 3.3], combined with the results in [23, Section 3] to ensure that the hypotheses of [23, Theorem 3.3] are satisfied. (Alternatively, the existence of a CK decomposition is also established in [10, Proposition A.6].) According to [23, Theorem 3.3], the given CK decomposition agrees with the Fourier decomposition of the Chow groups. The “moreover” part is because the $\{\pi_i^X\}$ are shown to satisfy Murre’s conjecture B [23, Theorem 3.3].

The statement for very general cubics is [23, Theorem 3]. □

Remark 2.6. Unfortunately, it is not yet known that the Fourier decomposition of [23] induces a bigraded ring structure on the Chow ring for all Fano varieties of smooth cubic fourfolds. For one thing, it has not yet been proven that

$$A_{(0)}^2(X) \cdot A_{(0)}^2(X) \subseteq A_{(0)}^4(X)$$

for the Fano variety of a given (not necessarily very general) cubic fourfold (cf. [23, Section 22.3] for discussion).

To prove that $A_{(0)}^*(X)$ is a bigraded ring for all Fano varieties of smooth cubic fourfolds, it would suffice to construct an MCK decomposition for the Fano variety of the very general cubic fourfold.

2.2. A multiplicative result. Let $X$ be the Fano variety of lines on a smooth cubic fourfold. As we have seen (Theorem 2.5), the Chow ring of $X$ splits into pieces $A_{(j)}^i(X)$. The work [23] contains a thorough analysis of the multiplicative behaviour of these pieces. Here are the relevant results we will be needing:

Theorem 2.7 (Shen–Vial [23]). Let $Y \subset \mathbb{P}^5(\mathbb{C})$ be a smooth cubic fourfold, and let $X := F(Y)$ be the Fano variety of lines in $Y$.

(i) There exists $\ell \in A_{(0)}^2(X)$ such that intersecting with $\ell$ induces an isomorphism

$$\ell : A_{(2)}^2(X) \xrightarrow{\cong} A_{(4)}^4(X).$$

(ii) Intersection product induces a surjection

$$A_{(2)}^2(X) \otimes A_{(2)}^2(X) \twoheadrightarrow A_{(4)}^4(X).$$

Proof. Statement (i) is [23, Theorem 4]. Statement (ii) is [23, Proposition 20.3]. □

2.3. The involution.

Lemma 2.8. Let $\iota_\mathbb{P} \in \text{Aut}(\mathbb{P}^5(\mathbb{C}))$ be the involution defined as

$$[X_0, X_1, \ldots, X_5] \mapsto [-X_0, -X_1, -X_2, X_3, X_4, X_5].$$

The cubic fourfolds invariant under $\iota_\mathbb{P}$ are exactly those defined by an equation

$$(X_0)^2\ell_0(X_3, X_4, X_5) + (X_1)^2\ell_1(X_3, X_4, X_5) + (X_2)^2\ell_2(X_3, X_4, X_5) + X_0X_1\ell_3(X_3, X_4, X_5) + X_0X_2\ell_4(X_3, X_4, X_5) + X_1X_2\ell_5(X_3, X_4, X_5) + g(X_3, \ldots, X_5) = 0,$$

where the $\ell_i$ are linear forms and $g$ is a homogeneous degree 3 polynomial.
Let \( Y \subset \mathbb{P}^5(\mathbb{C}) \) be a smooth cubic invariant under \( \iota_{P} \), and let \( \iota_{Y} \in \text{Aut}(Y) \) be the involution induced by \( \iota_{P} \). Let \( X = F(Y) \) be the Fano variety of lines in \( Y \), and let \( \iota \in \text{Aut}(X) \) be the involution induced by \( \iota_{Y} \). The involution \( \iota \) is anti–symplectic.

**Proof.** The only thing that needs explaining is the last phrase; this is proven in [5, Section 7]. The idea is that there is an isomorphism of Hodge structures, compatible with the involution

\[
H^2(X) \cong H^4(Y)
\]

The action of \( \iota_{Y} \) on \( H^{3,1}(Y) \) is minus the identity, because \( H^{3,1}(Y) \) is generated by the meromorphic form

\[
\sum_{i=0}^{5} (-1)^i X_i dX_0 \wedge \ldots \wedge d\hat{X}_i \wedge \ldots \wedge dX_5 / f^2,
\]

where \( f \) is an equation for \( Y \).

\[\square\]

2.4. Spread.

**Lemma 2.9** (Voisin [29], [30]). Let \( M \) be a smooth projective variety of dimension \( n+1 \), and \( L \) a very ample line bundle on \( M \). Let

\[\pi: \mathcal{X} \to B\]

denote a family of hypersurfaces, where \( B \subset |L| \) is a Zariski open. Let

\[p: \tilde{\mathcal{X}} \times_B \mathcal{X} \to \mathcal{X} \times_B \mathcal{X}\]

denote the blow–up of the relative diagonal. Then \( \tilde{\mathcal{X}} \times_B \mathcal{X} \) is Zariski open in \( V \), where \( V \) is a projective bundle over \( \tilde{M} \times \tilde{M} \), the blow–up of \( M \times M \) along the diagonal.

**Proof.** This is [29] Proof of Proposition 3.13 or [30] Lemma 1.3. The idea is to define \( V \) as

\[V := \left\{ ((x, y, z), \sigma) \mid \sigma|_z = 0 \right\} \subset \tilde{M} \times M \times |L| .\]

The very ampleness assumption ensures \( V \to \tilde{M} \times \tilde{M} \) is a projective bundle.

\[\square\]

This is used in the following key proposition:

**Proposition 2.10** (Voisin [30]). Assumptions as in lemma 2.9. Assume moreover \( M \) has trivial Chow groups. Let \( R \in A^n(V) \). Suppose that for all \( b \in B \) one has

\[H^n(X_b)_{prim} \neq 0 \quad \text{and} \quad R|_{\tilde{X}_b \times X_b} = 0 \in H^{2n}(\tilde{X}_b \times \tilde{X}_b) .\]

Then there exists \( \gamma \in A^n(M \times M) \) such that

\[(p_b)_*(R|_{\tilde{X}_b \times X_b}) = \gamma|_{X_b \times X_b} \in A^n(X_b \times X_b)\]

for all \( b \in B \). (Here \( p_b \) denotes the restriction of \( p \) to \( \tilde{X}_b \times X_b \), which is the blow–up of \( X_b \times X_b \) along the diagonal.)

**Proof.** This is [30] Proposition 1.6.

\[\square\]

The following is an equivariant version of proposition 2.10:
Proposition 2.11 (Voisin [30]). Let $M$ and $L$ be as in proposition 2.10. Let $G \subset \text{Aut}(M)$ be a finite group. Assume the following:

(i) The linear system $|L|^G := \mathbb{P}(H^0(M, L)^G)$ has no base–points, and the locus of points in $\widetilde{M} \times \widetilde{M}$ parametrizing triples $(x, y, z)$ such that the length 2 subscheme $z$ imposes only one condition on $|L|^G$ is contained in the union of (proper transforms of) graphs of non–trivial elements of $G$, plus some loci of codimension $> n + 1$.

(ii) Let $B \subset |L|^G$ be the open parametrizing smooth hypersurfaces, and let $X_b \subset M$ be a hypersurface for $b \in B$ general. There is no non–trivial relation

$$\sum_{g \in G} c_g \Gamma_g + \gamma = 0 \quad \text{in } H^{2n}(X_b \times X_b),$$

where $c_g \in \mathbb{Q}$ and $\gamma$ is a cycle in $\text{Im}(A^n(M \times M) \to A^n(X_b \times X_b))$.

Let $R \in A^n(X \times_B X)$ be such that

$$R|_{X_b \times X_b} = 0 \quad \text{in } H^{2n}(X_b \times X_b) \quad \forall b \in B.$$

Then there exists $\gamma \in A^n(M \times M)$ such that

$$R|_{X_b \times X_b} = \gamma|_{X_b \times X_b} \in A^n(X_b \times X_b) \quad \forall b \in B.$$

Proof. This is not stated verbatim in [30], but it is contained in the proof of [30, Proposition 3.1 and Theorem 3.3]. Let us very briefly review the argument. One considers the variety

$$V := \left\{ \left( (x, y, z), \sigma \right) \mid \sigma|_z = 0 \right\} \subset \widetilde{M} \times \widetilde{M} \times |L|^G.$$

The problem is that $V$ is no longer a projective bundle over $\widetilde{M} \times \widetilde{M}$. However, as explained in the proof of [30, Theorem 3.3], hypothesis (i) ensures that one can obtain a projective bundle after blowing up the graphs $\Gamma_g, g \in G$ plus some loci of codimension $> n + 1$. Let $M' \to \widetilde{M} \times \widetilde{M}$ denote the result of these blow–ups, and let $V' \to M'$ denote the projective bundle obtained by base–change.

Analyzing the situation as in [30, Proof of Theorem 3.3], one obtains

$$R|_{X_b \times X_b} = R_0|_{X_b \times X_b} + \sum_{g \in G} \lambda_g \Gamma_g \quad \text{in } A^n(X_b \times X_b),$$

where $R_0 \in A^n(M \times M)$ and $\lambda_g \in \mathbb{Q}$ (this is [30, Equation (15)]). By assumption, $R|_{X_b \times X_b}$ is homologically trivial. In view of hypothesis (ii), this implies that all $\lambda_g$ must be 0. \qed

3. MAIN RESULT

This section contains the proof of the main result of this note, theorem 3.5. The proof is split in two parts. In the first part, we prove a statement (theorem 3.1) about the action of the involution on 1–cycles on the cubic $Y$. The proof is an application of the technique of “spread” of cycles in a family, as developed by Voisin [29, 30, 31, 32] (more precisely, the results recalled in subsection 2.4).
In the second part, we deduce from this our main result, theorem 3.5. This second part builds on the structural results of Shen–Vial [23] (notably the results recalled in subsections 2.1 and 2.2).

3.1. First part.

**Theorem 3.1.** Let $Y \subset \mathbb{P}^5(\mathbb{C})$ be a smooth cubic fourfold defined by an equation
\[(X_0)^2\ell_0(X_3, X_4, X_5) + (X_1)^2\ell_1(X_3, X_4, X_5) + (X_2)^2\ell_2(X_3, X_4, X_5) + X_0X_1\ell_3(X_3, X_4, X_5)
\quad + X_0X_2\ell_4(X_3, X_4, X_5) + X_1X_2\ell_5(X_3, X_4, X_5) + g(X_3, \ldots, X_5) = 0,
\]where the $\ell_i$ are linear forms and $g$ is a homogeneous degree 3 polynomial.

Let $\iota_Y \in \text{Aut}(Y)$ be the involution of lemma 2.8. Then
\[\left(\iota_Y\right)^* = -\text{id}: A^3_{\text{hom}}(Y) \to A^3(Y).\]

**Proof.** We have seen (proof of lemma 2.8) that
\[\left(\iota_Y\right)^* = -\text{id}: H^{3,1}(Y) \to H^{3,1}(Y).\]

Let $H^4_{tr}(Y)$ denote the orthogonal complement (under the cup–product pairing) of $N^2H^4(Y)$ (which coincides with $H^{2,2}(Y, \mathbb{Q})$ since the Hodge conjecture is true for $Y$). Since $H^4_{tr}(Y) \subset H^4(Y)$ is the smallest Hodge substructure containing $H^{3,1}(Y)$, we must also have
\[1)
\left(\iota_Y\right)^* = -\text{id}: H^4_{tr}(Y) \to H^4_{tr}(Y).
\]

This implies that there is a decomposition
\[2)
\iota_Y^* = -\Delta_Y + \gamma \quad \text{in } H^8(Y \times Y),
\]
where $\gamma \in A^4(Y \times Y)$ is a “completely decomposed” cycle, i.e.
\[\gamma = \gamma_0 + \gamma_2 + \gamma_4 + \gamma_6 + \gamma_8,
\]and $\gamma_2i$ has support on $V_i \times W_i \subset Y \times Y$ with $\dim V_i = i$ and $\dim W_i = 4 - i$. (Indeed, the cycle $\gamma$ is obtained by considering
\[\gamma_{2i} := (\iota_Y^* + \Delta_Y) \circ \pi_{2i} \in H^8(Y \times Y),
\]where $\pi_i$ denotes the Künneth component. For $i \neq 4$, the claimed support condition is obviously satisfied since it is satisfied by $\pi_i$. For $i = 4$, one uses (11) to see that $\gamma_4$ is supported on $N^2H^4(Y) \otimes N^2H^4(Y) \subset H^8(Y \times Y)$.)

We now consider things family-wise. Let
\[\mathcal{Y} \to B
\]denote the universal family of all smooth cubic fourfolds defined by an equation as in theorem 3.1. Let $Y_b \subset \mathbb{P}^5(\mathbb{C})$ denote the fibre over $b \in B$.

The involution $\iota_\mathcal{Y}$ defines an involution $\iota_Y \in \text{Aut}(\mathcal{Y})$ by restriction. Let $\Delta_\mathcal{Y} \in A^4(\mathcal{Y} \times_B \mathcal{Y})$ denote the relative diagonal. Obviously the argument leading to the decomposition (2) applies to each fibre $Y_b$. This means that for each $b \in B$, there exists a completely decomposed cycle $\gamma_b \in A^4(Y_b \times Y_b)$ such that
\[\left(\iota_{\mathcal{Y}}^* + \Delta_\mathcal{Y}\right)|_{Y_b \times Y_b} = \gamma_b \quad \text{in } H^8(Y_b \times Y_b).\]
Applying the “spread” result \cite[Proposition 3.7]{29}, we can find a “completely decomposed” relative correspondence \( \gamma \in A^4(Y \times_B Y) \) such that
\[
\left( t_{\Gamma_{Y \times Y}} + \Delta_Y - \gamma \right)|_{Y_b \times Y_b} = 0 \quad \text{in} \quad H^8(Y_b \times Y_b) \quad \forall b \in B.
\]
(By this, we mean the following: there exist subvarieties \( \mathcal{V}_i, \mathcal{W}_i \subset Y \) for \( i = 0, 2, 4, 6, 8 \) with
\[
\text{codim}\mathcal{V}_i + \text{codim}\mathcal{W}_i = 4,
\]
and such that the cycle \( \gamma \) is supported on
\[\bigcup_i \mathcal{V}_i \times_B \mathcal{W}_i \subset Y \times_B Y.\]

Actually, for \( i \neq 4 \) this is obvious since the \( \pi_i, i \neq 4 \) obviously exist relatively. The recourse to \cite[Proposition 3.7]{29} can thus be limited to \( i = 4 \).

That is, the relative correspondence
\[
\Gamma := t_{\Gamma_{Y \times Y}} + \Delta_Y - \gamma \in A^4(Y \times_B Y)
\]
is fibrewise homologically trivial:
\[
\Gamma|_{Y_b \times Y_b} = 0 \quad \text{in} \quad H^8(Y_b \times Y_b) \quad \forall b \in B.
\]

At this point, we note that the family \( Y \to B \) is large enough to verify the hypotheses of proposition \ref{proposition:2.11}, this will be proven in lemma \ref{lemma:3.2} below. Applying proposition \ref{proposition:2.11} to the relative correspondence \( \Gamma \), we find that there exists \( \delta \in A^4(P^5 \times P^5) \) such that
\[
\Gamma|_{Y_b \times Y_b} + \delta|_{Y_b \times Y_b} = 0 \quad \text{in} \quad A^4(Y_b \times Y_b) \quad \forall b \in B.
\]

But
\[
(\delta|_{Y_b \times Y_b})_* = 0 : \quad A^4_{\text{hom}}(Y_b) \to A^4(Y_b) \quad \forall b \in B
\]

(indeed, the action factors over \( A^4_{\text{hom}}(P^5) \) which is 0). Also, we have
\[
(\gamma|_{Y_b \times Y_b})_* = 0 : \quad A^3_{\text{hom}}(Y_b) \to A^3(Y_b) \quad \text{for general} \ b \in B
\]

(indeed, for general \( b \in B \) the restriction \( \gamma|_{Y_b \times Y_b} \) is a completely decomposed cycle; such cycles do not act on \( A^3_{\text{hom}} \) for dimension reasons).

By definition of \( \Gamma \), this means that
\[
\left( t_{\Gamma_{Y \times Y}} + \Delta_Y \right)_* = 0 : \quad A^3_{\text{hom}}(Y_b) \to A^3(Y_b) \quad \text{for general} \ b \in B.
\]

This proves theorem \ref{theorem:3.1} for general \( b \in B \). To extend to all \( b \in B \), one can reason as in \cite[Lemma 3.1]{8}.

It only remains to check that the hypotheses of Voisin’s result are satisfied:

**Lemma 3.2.** Let \( Y \to B \) be the family of smooth cubic fourfolds as in theorem \ref{theorem:3.1} i.e.
\[
B \subset \left( \mathbb{P} H^0(\mathbb{P}^5, O_{\mathbb{P}^5}(3)) \right)^G
\]
is the open subset parametrizing smooth \( G \)-invariant cubics, where \( G = \{ \text{id}, t_\varphi \} \subset \text{Aut}(\mathbb{P}^5) \) is as above. This set–up verifies the hypotheses of proposition \ref{proposition:2.77}.
Proof. Let us first prove that hypothesis (i) of proposition \ref{proposition2.11} is satisfied.

To this end, we consider the quotient morphisms
\[ p: \mathbb{P}^5 \rightarrow P := \mathbb{P}^5/G \rightarrow P' := \mathbb{P}(2^3, 1^3) = \mathbb{P}^5/(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) , \]
where \( P' := \mathbb{P}(2^3, 1^3) \) denotes a weighted projective space.

Let us write \( \iota_0, \iota_1, \iota_2 \) for the involutions of \( \mathbb{P}^5 \)
\[ \iota_0[X_0: \ldots : X_5] := [-X_0 : X_1 : \ldots : X_5] , \]
\[ \iota_1[X_0: \ldots : X_5] := [X_0 : -X_1 : X_2 : \ldots : X_5] , \]
\[ \iota_2[X_0: \ldots : X_5] := [X_0 : X_1 : -X_2 : X_3 : X_4 : X_5] . \]

(We note that \( \iota_2 = \iota_0 \circ \iota_1 \circ \iota_2 \), and the weighted projective space \( P' \) is \( \mathbb{P}^5/\langle \iota_0, \iota_1, \iota_2 \rangle \).)

The sections in \( (\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(3)) \)^\(G\) are in bijection with sections coming from \( P \), and contain the sections coming from \( P' \):
\[ (\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(3))^{\mathbb{P}^5} H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(3))^{\Gamma}\subset \mathbb{P}^5 H^0(P', \mathcal{O}_{P'}(3)). \]

Let us now assume \( x, y \in \mathbb{P}^5 \) are two points such that
\[ (x, y) \notin \Delta_{\mathbb{P}^5} \cup \bigcup_{0 \leq r_0, r_1, r_2 \leq 1} \Gamma_{(r_0)^\iota_0 (r_1)^\iota_1 (r_2)^\iota_2} . \]
Then
\[ p(x) \neq p(y) \quad \text{in} \quad P' , \]
and so (using lemma \ref{lemma3.3} below) there exists \( \sigma \in H^0(P', \mathcal{O}_{P'}(3)) \) containing \( p(x) \) but not \( p(y) \). The pullback \( p^*(\sigma) \) contains \( x \) but not \( y \), and so these points \( (x, y) \) impose 2 independent conditions on \( (\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(3))^{\Gamma} \).

It remains to check that a generic element
\[ (x, y) \in \bigcup_{0 \leq r_0, r_1, r_2 \leq 1} \Gamma_{(r_0)^\iota_0 (r_1)^\iota_1 (r_2)^\iota_2} \setminus \Gamma_\iota \]
also imposes 2 independent conditions. Let us first assume \((x, y)\) is generic on \( \Gamma_{\iota_0} \). Let us write \( x = [a_0 : a_1 : \ldots : a_5] \). By genericity, we may assume all \( a_i \) are \( \neq 0 \) (intersections of \( \Gamma_{\iota_0} \) with a coordinate hyperplane have codimension \( n + 1 \) and so need not be considered for hypothesis (i) of proposition \ref{proposition2.11}). We can thus write
\[ x = [a_0 : a_1 : a_2 : a_3 : a_4 : a_5] , \quad y = [-a_0 : a_1 : a_2 : a_3 : a_4 : a_5] , \quad a_i \neq 0 . \]
The cubic
\[ (a_1)^3 (X_0)^3 - (a_0)^3 (X_1)^3 = 0 \]
is \( G \)-invariant and contains \( x \) while avoiding \( y \), and so the element \((x, y)\) again imposes 2 independent conditions.

The argument for the other \( r_i \) is similar: consider for instance a generic element \((x, y)\) in \( \Gamma_{\iota_0 \iota_1} \). By genericity, we can write
\[ x = [a_0 : a_1 : a_2 : a_3 : a_4 : a_5] , \quad y = [-a_0 : -a_1 : a_2 : a_3 : a_4 : a_5] , \quad a_i \neq 0 . \]
The cubic
\[(a_2)^3(X_0)^3 - (a_0)^3(X_2)^3 = 0\]
is \(G\)-invariant and contains \(x\) while avoiding \(y\), and so the element \((x, y)\) again imposes 2 independent conditions.

This proves hypothesis (i) of proposition 2.11 is satisfied.

To establish hypothesis (ii) of proposition 2.11, let \(Y = Y_b\) be a cubic as in theorem 3.1, and let us suppose there is a relation
\[c \Delta_Y + d \Gamma_x + \delta = 0 \quad \text{in} \quad H^8(Y \times Y),\]
where \(c, d \in \mathbb{Q}\) and \(\delta \in \text{Im} \left( A^4(\mathbb{P}^5 \times \mathbb{P}^5) \rightarrow A^4(Y_b \times Y_b) \right)\). Looking at the action on \(H^{3,1}(Y)\), we find that necessarily \(c = d\) (indeed, \(\delta\) does not act on \(H^{3,1}(Y)\), and \(\iota\) acts as minus the identity on \(H^{3,1}(Y)\)).

On the other hand, looking at the action on \((H^4(Y)_{\text{prim}})^\dagger\) (which is non-zero thanks to lemma 3.4), we find that \(c = -d\). We conclude that \(c = d = 0\), and so hypothesis (ii) is satisfied.

Lemma 3.3. Let \(P' = \mathbb{P}(2^3, 1^3)\). Let \(r, s \in P'\) and \(r \neq s\). Then there exists \(\sigma \in \mathbb{P}H^0(P', \mathcal{O}_{P'}(3))\) containing \(r\) but avoiding \(s\).

Proof. It follows from Delorme’s work [6, Proposition 2.3(iii)] that the locally free sheaf \(\mathcal{O}_{P'}(2)\) is very ample. This means there exists \(\sigma' \in \mathbb{P}H^0(P, \mathcal{O}_P(2))\) containing \(r\) but avoiding \(s\). Taking the union of \(\sigma'\) with a hyperplane avoiding \(s\), one obtains \(\sigma\) as required.

Lemma 3.4. Let \(Y \subset \mathbb{P}^5(\mathbb{C})\) be a smooth cubic as in theorem 3.1. Then
\[\dim H^4(Y)^{\text{inv}} > 1.\]

Proof. Griffiths’ description of the cohomology of a hypersurface [27, §18] implies there is an isomorphism, given by the residue map
\[H^5(\mathbb{P}^5 \setminus Y) \supset H_{\leq 3} \xrightarrow{\cong} H^{2,2}(Y)_{\text{prim}},\]
where \(H_{\leq 3}\) is by definition the subspace of meromorphic forms with poles of order \(\leq 3\) along \(Y\). To prove the lemma, it thus suffices to exhibit a \(\iota_P\)-invariant meromorphic 5-form with a pole of order 3. Let \(f = f(X_0, \ldots, X_5)\) be an equation defining \(Y\). The meromorphic form
\[\frac{(X_0)^3}{f^3} \sum_{j=0}^5 X_j dX_0 \wedge \cdots \wedge dX_j \wedge \cdots \wedge dX_5 \in H_{\leq 3}\]
doesthe job.

Another proof of lemma 3.4 (suggested by the referee, whom I thank) is as follows: the cubic \(Y\) contains the plane \(P := \{x_3 = x_4 = x_5 = 0\}\). The plane \(P\) is not proportional to the class \(h^2\) where \(h \in A^1(Y)\) is a hyperplane class (indeed: if \(P\) were equal to \(mh^2\) in \(H^4(Y)\) for some integer \(m\), the intersection \(P \cdot h^2\) would be a multiple of 3, whereas \(P \cdot h^2 = 1\)). Since both \(P\) and \(h^2\) are \(\iota_Y\)-invariant, this proves the lemma.

This closes the proof of lemma 3.2 and hence of theorem 3.1.
3.2. Second part.

**Theorem 3.5.** Let $Y \subset \mathbb{P}^5(\mathbb{C})$ be a smooth cubic fourfold defined by an equation

$$(X_0)^2\ell_0(X_3, X_4, X_5) + (X_1)^2\ell_1(X_3, X_4, X_5) + (X_2)^2\ell_2(X_3, X_4, X_5) + X_0X_1\ell_3(X_3, X_4, X_5) + X_0X_2\ell_4(X_3, X_4, X_5) + X_1X_2\ell_5(X_3, X_4, X_5) + g(X_3, \ldots, X_5) = 0,$$

where the $\ell_i$ are linear forms and $g$ is a homogeneous degree 3 polynomial.

Let $X = F(Y)$ be the Fano variety of lines in $Y$, and let $\iota \in \text{Aut}(X)$ be the anti–symplectic involution of lemma 2.8. Then

$$\iota^* = -\text{id} : A_i^{(2)}(X) \rightarrow A_i^{(2)}(X) \text{ for } i = 2, 4;$$

$$\iota^* = \text{id} : A_i^{(4)}(X) \rightarrow A_i^{(4)}(X) \text{ for } j = 0, 4.$$

**Proof.** First, we note that

$$A_2^{(2)}(X) = I_*A_{\text{hom}}^4(X),$$

where $I \subset X \times X$ is the incidence correspondence [23, Proof of Proposition 21.10]. On the other hand,

$$I = (\iota P) \circ P \text{ in } A^2(X \times X),$$

where $X \leftarrow P \rightarrow Y$ denotes the universal family of lines on $Y$ [23, Lemma 17.2]. Hence,

$$A_2^{(2)}(X) = (\iota P)_*P_*A_{\text{hom}}^4(X).$$

But $P_* : A_{\text{hom}}^4(Y) \rightarrow A_{\text{hom}}^3(Y)$ is surjective [21], and so

$$A_2^{(2)}(X) = (\iota P)_*A_{\text{hom}}^3(Y).$$

It is readily checked that the diagram

$$\begin{array}{ccc}
A_{\text{hom}}^3(Y) & \xrightarrow{(\iota P)_*} & A^2(X) \\
\downarrow (\text{sy})^* & & \downarrow \iota^* \\
A_{\text{hom}}^3(Y) & \xrightarrow{(\iota P)_*} & A^2(X)
\end{array}$$

is commutative (this is because the involution extends to an involution on $P$). Using this diagram, theorem 3.1 implies that $\iota$ acts as minus the identity on $(\iota P)_*A_{\text{hom}}^3(Y) = A_2^{(2)}(X)$.

Because the intersection product induces a surjection

$$A_2^{(2)}(X) \otimes A_2^{(2)}(X) \rightarrow A_i^{(4)}(X)$$

(themorem 2.7(ii)), it follows that $\iota$ acts as the identity on $A_i^{(4)}(X)$.

Next, we want to exploit the fact that there is an isomorphism

$$\cdot \ell : A_2^{(2)}(X) \xrightarrow{\cong} A_2^{(2)}(X)$$

(themorem 2.7(i)). Since $\iota^*(\ell) = \ell$ (proposition 3.6 below), this implies that $\iota$ acts as minus the identity on $A_2^{(2)}(X)$. 
Proposition 3.6. Let $X$ be the variety of lines on a smooth cubic fourfold $Y \subset \mathbb{P}^5$ (C), and let $\iota \in \text{Aut}(X)$ be an involution induced by an involution $\iota_Y \in \text{Aut}(Y)$. Let $\ell \in A^2(X)$ be the class of theorem [27]i). Then

$$\iota^*(\ell) = \ell \quad \text{in} \quad A^2(X).$$

Proof. We give two proofs of this fact. The first proof has the benefit of brevity; the second proof will be useful in proving another result (lemma [3.7] below).

First proof: It is known that $\ell = \frac{5}{6}c_2(X)$ in $A^2(X)$ (where the right-hand side denotes the second Chern class of the tangent bundle $T_X$ of $X$) [23, Equation (108)]. Since $\iota^*c_2(X) = c_2(\iota^*T_X) = c_2(X)$ in $A^2(X)$, this proves the proposition.

Second proof: Shen–Vial define the class $L \in A^2(X \times X)$ (lifting the Beauville-Bogomolov class $B \in H^4(X \times X)$) as

$$L := \frac{1}{3}((g_1)^2 + \frac{3}{2}g_1g_2 + (g_2)^2 - c_1 - c_2) - I \quad \text{in} \quad A^2(X \times X)$$

[23, Equation (107)]. Here $I$ is the incidence correspondence, and

$$g := -c_1(E_2) \quad \in \quad A^1(X),$$
$$c := c_2(E_2) \quad \in \quad A^2(X),$$
$$g_i := (p_i)^*(g) \quad \in \quad A^1(X \times X) \quad (i = 1, 2),$$
$$c_i := (p_i)^*(c) \quad \in \quad A^2(X \times X) \quad (i = 1, 2),$$

where $E_2$ is the rank 2 vector bundle coming from the tautological bundle on the Grassmannian, and $p_i : X \times X \to X$ denote the two projections.

Clearly one has

$$(\iota \times \iota)^*(I) = I, \quad (\iota \times \iota)^*(c) = c_i, \quad (\iota \times \iota)^*(g) = g_i.$$

In view of the definition of $L$, it follows that

$$(\iota \times \iota)^*(L) = L \quad \text{in} \quad A^2(X \times X).$$

Using Lieberman’s lemma [25, Lemma 3.3], plus the fact that $^t\Gamma_\iota = \Gamma_\iota$, this means there is a commutativity relation

$$(3) \quad L \circ \Gamma_\iota = \Gamma_\iota \circ L \quad \text{in} \quad A^2(X \times X).$$

The class $\ell$ is defined as $\ell := (i_\Delta)^*(L) \in A^2(X)$. We now find that

$$\iota^*(\ell) = \iota^*(i_\Delta)^*(L)$$

$$= (i_\Delta)^*(\iota \times \iota)^*(L)$$

$$= (i_\Delta)^*(\iota \times \iota)^*(L)$$

$$= (i_\Delta)^*(\Gamma_\iota \circ L \circ \Gamma_\iota)$$

$$= (i_\Delta)^*(L) = \ell \quad \text{in} \quad A^2(X).$$
Here the second equality is by virtue of the commutative diagram

$$
\begin{array}{c}
X \\
\downarrow \iota \\
X \\
X \xrightarrow{\iota \times \iota} X \times X
\end{array}
$$

The third equality is again Lieberman’s lemma, plus the fact that \( \iota \Gamma_i = \Gamma_i \). The last equality is (3).

It only remains to prove theorem 3.5 is true for \((i, j) = (4, 0)\). This follows from the fact that \( A^i_4(X) \) is generated by \( \ell^2 \) [23], plus the fact that \( \ell \) is \( \iota \)-invariant (proposition 3.6). Theorem 3.5 is now proven.

For later use, we remark that the argument of proposition 3.6 also proves the following compatibility statement:

**Lemma 3.7.** Let \( X \) be the variety of lines on a smooth cubic fourfold \( Y \subset \mathbb{P}^5(\mathbb{C}) \), and let \( \iota \in \text{Aut}(X) \) be an involution induced by an involution \( \iota_Y \in \text{Aut}(Y) \). Then

\[ \iota^* A^i_{(j)}(X) \subset A^i_{(j)}(X) \quad \forall i, j. \]

**Proof.** Let \( L \in A^2(X \times X) \) be the Shen–Vial class as above. We observe that equality (3) also implies

\[ (\ell \times \ell)^*(L^r) = ((\ell \times \ell)^*(L))^r = L^r \quad \text{in} \quad A^4(X \times X) \quad \forall r \in \mathbb{N}. \]

Using Lieberman’s lemma, this is equivalent to the commutativity relation

\[ \Gamma_\iota \circ L^r = L^r \circ \Gamma_\iota \quad \text{in} \quad A^{2r}(X \times X). \]

Since the Shen–Vial Fourier transform \( \mathcal{F} : A^*(X) \to A^*(X) \) is defined by a polynomial in \( L \) [23, Section C.1], we find that

\[ \mathcal{F}(\iota^*(a)) = \iota^* \mathcal{F}(a) \quad \forall a \in A^i(X). \]

This proves the lemma, for the decomposition of [23] is defined as

\[ A^i_{(j)}(X) := \{ a \in A^i(X) \mid \mathcal{F}(a) \in A^{4-i+j}(X) \}. \]

4. **Corollaries**

In this last section, we consider the quotient \( Z := X/\iota \), for \((X, \iota)\) as in theorem 3.5. The variety \( Z \) is a slightly singular Calabi–Yau variety. As is well–known, Chow groups with \( \mathbb{Q} \)-coefficients of quotient varieties such as \( Z \) still have a ring structure [12, Examples 8.3.12 and 17.4.10]. For this reason, we will write \( A^i(Z) \) for the Chow group of codimension \( i \) cycles on \( Z \) (just as in the smooth case).

**Corollary 4.1.** Let \((X, \iota)\) be as in theorem 3.5. Let \( Z := X/\iota \) be the quotient. Then the image of the intersection product map

\[ A^2(Z) \otimes A^2(Z) \to A^4(Z) \]

has dimension 1.
Proof. We start by establishing a lemma:

**Lemma 4.2.** Let \((X, \iota)\) be as in theorem 3.5. Then

\[ A^2(X)^\iota \subset A^2_{(0)}(X) . \]

**Proof.** Let \(c \in A^2(X)^\iota\), and let us write

\[ c = c_0 + c_2 \quad \text{in} \quad A^2_{(0)}(X) \oplus A^2_{(2)}(X) , \]

where \(c_j \in A^2_{(j)}(X)\). Since \(c\) is \(\iota\)-invariant, we also have

\[ c = \iota^*(c) = \iota^*(c_0) + \iota^*(c_2) = \iota^*(c_0) - c_2 \quad A^2(X) \]

(here we have used theorem 3.5 to conclude that \(\iota^*(c_2) = -c_2\)). But we know that \(\iota^*(c_0) \in A^2_{(0)}(X)\) (lemma 3.7), and so (by unicity of the decomposition \(c = c_0 + c_2\)) we must have

\[ \iota^*(c_0) = c_0, \quad -c_2 = c_2. \]

\(\square\)

Now, let \(p: X \to Z\) denote the quotient morphism. Thanks to lemma 4.2, one has

\[ p^* A^2(Z) \subset A^2_{(0)}(X) . \]

It follows that

\[ p^* \text{Im} \left( A^2(Z) \otimes A^2(Z) \to A^4(Z) \right) \subset \text{Im} \left( p^* A^2(Z) \otimes p^* A^2(Z) \to A^4(X) \right) \]

\[ \subset \text{Im} \left( A^2_{(0)}(X) \otimes A^2_{(0)}(X) \to A^4(X) \right) \]

\[ \subset A^4_{(0)}(X) \oplus A^4_{(2)}(X) . \]

(Here for the last inclusion we have used [23, Proposition 22.8].)

On the other hand, one has

\[ p^* \text{Im} \left( A^2(Z) \otimes A^2(Z) \to A^4(Z) \right) \subset A^4(X)^\iota , \]

and so (by combining with the above inclusion) we find that

\[ p^* \text{Im} \left( A^2(Z) \otimes A^2(Z) \to A^4(Z) \right) \subset \left( A^4_{(0)}(X) \oplus A^4_{(2)}(X) \right) \cap A^4(X)^\iota . \]

But we have seen (lemma 3.7) that \(\iota\) respects the Fourier decomposition, and so

\( \left( A^4_{(0)}(X) \oplus A^4_{(2)}(X) \right) \cap A^4(X)^\iota = A^4_{(0)}(X)^\iota \oplus A^4_{(2)}(X)^\iota . \)

But \(A^4_{(2)}(X)^\iota = 0\) (theorem 3.5), and so

\[ A^4(X)^\iota = A^4_{(0)}(X)^\iota . \]

Since \(\ell^2\) generates \(A^4_{(0)}(X)\) and is \(\iota\)-invariant (proposition 3.6), we conclude that

\[ A^4(X)^\iota = A^4_{(0)}(X)^\iota = A^4_{(0)}(X) \cong \mathbb{Q} . \]

\(\square\)
Corollary 4.3. Let \((X, \iota)\) be as in theorem 3.5. Let \(Z := X/\iota\) be the quotient. Then the image of the intersection product map

\[
\text{Im}(A^3(Z) \otimes A^1(Z) \to A^4(Z))
\]

has dimension 1.

Proof. As above, let \(p: X \to Z\) denote the quotient morphism. It will suffice to show that

\[
\text{Im}(A^3(X)^{\iota} \otimes A^1(X)^{\iota} \to A^4(X)^{\iota})
\]

is of dimension 1.

There is a decomposition

\[
A^3(X)^{\iota} = A^3_{(0)}(X)^{\iota} \oplus A^3_{(2)}(X)^{\iota}
\]

(this follows from lemma 3.7). Moreover, it is known that

\[
A^3_{(2)}(X) \cdot A^1(X) \subset A^4_{(2)}(X)
\]

[23, Proposition 22.6]. But we know (theorem 3.5) that \(A^4_{(2)}(X)^{\iota} = 0\), and so

\[
A^3_{(2)}(X)^{\iota} \otimes A^1(X)^{\iota} \to A^4(X)^{\iota}
\]

is the zero map. It follows that

\[
\text{Im}(A^3(X)^{\iota} \otimes A^1(X)^{\iota} \to A^4(X)^{\iota}) = \text{Im}(A^3_{(0)}(X)^{\iota} \otimes A^1(X)^{\iota} \to A^4(X)^{\iota})
\]

To analyze the right–hand side, we observe that

\[
A^3_{(0)}(X) = A^1(X)^{\text{prim}} \cdot A^2_{(0)}(X)
\]

(Indeed, the inclusion “\(\supset\)” is proven in [23, Proposition 22.7]; the inclusion “\(\subset\)” follows from [23, Remark 4.7]). This implies that

\[
\text{Im}(A^3_{(0)}(X)^{\iota} \otimes A^1(X)^{\iota} \to A^4(X)^{\iota}) \subset \left( A^1(X)^{\text{prim}} \cdot A^2_{(0)}(X) \cdot A^1(X) \right) \cap A^4(X)^{\iota}
\]

\[
\subset \left( A^4_{(0)}(X) \oplus A^4_{(2)}(X) \right) \cap A^4(X)^{\iota}
\]

\[
= A^4_{(0)}(X).
\]

Here, the second equality is [23, Equation (118)], and the third equality is theorem 3.5 combined with lemma 3.7. □

There is a similar statement for 1–cycles on the quotient:

Corollary 4.4. Let \((X, \iota)\) be as in theorem 3.5. Let \(Z := X/\iota\) be the quotient. Then the image of the intersection product map

\[
\text{Im}(A^2(Z) \otimes A^1(Z) \to A^3(Z))
\]

injects into \(H^6(Z)\) under the cycle class map.
Proof. As above, let \( p: X \to Z \) denote the quotient morphism. We have seen (lemma 4.2) that
\[
p^* A^2(Z) \subset A^2_{(0)}(X),
\]
and so
\[
p^* \text{Im}(A^2(Z) \otimes A^1(Z) \to A^3(Z)) \subset \text{Im}(A^2_{(0)}(X) \otimes A^1(X) \to A^3(X)).
\]
It is known [10, Proposition A.7] that
\[
\text{Im}(A^2_{(0)}(X) \otimes A^1(X) \to A^3(X)) \subset A^3_{(0)}(X),
\]
and so it follows that
\[
p^* \text{Im}(A^2(Z) \otimes A^1(Z) \to A^3(Z)) \subset A^3_{(0)}(X).
\]
But \( A^3_{(0)}(X) \) injects into cohomology under the cycle class map [23]. (A quick way of proving this injectivity can be as follows: let \( \mathcal{F} \) be the Fourier transform of [23]. We have that \( a \in A^3(X) \) is in \( A^3_{(0)}(X) \) if and only if \( \mathcal{F}(a) \in A^1_{(0)}(X) = A^1(X) \) [23, Theorem 2]. Suppose \( a \in A^3_{(0)}(X) \) is homologically trivial. Then also \( \mathcal{F}(a) \in A^1(X) \) is homologically trivial, hence \( \mathcal{F}(a) = 0 \) in \( A^1(X) \). But then, using [23, Theorem 2.4], we find that
\[
\frac{25}{2} a = \mathcal{F} \circ \mathcal{F}(a) = 0 \quad \text{in} \quad A^3(X).
\]
\( \square \)

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