Rotational Surfaces in $S^3$ with Constant Mean Curvature

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Abstract There is a two-parametric family of rotational symmetric CMC surfaces; more precisely, for every real number $H$ and every $C \geq 2(H + \sqrt{1 + H^2})$ there is a rotational symmetry surface $\Sigma_{H,C}$ with mean curvature $H$. Perdomo (Asian J Math 14:73–108, 2010) showed that for every $H$ between $\cot\left(\frac{\pi}{m}\right)$ and $\frac{m^2-2}{2\sqrt{m^2-1}}$, there exists an embedded rotational symmetric example with non-constant principal curvatures that is invariant under the cyclic group $Z_m$. Recently Andrews and Li (J Differ Geom 99:169–189, 2015) showed that these embedded CMC tori are the only embedded genus 1 surfaces with CMC on the sphere. In this paper we complete the study of this family of CMC surfaces and we show that for every integer $m > 2$, there is a properly immersed example in this family that contains a great circle and is invariant under the cyclic group $Z_m$. We will say that these examples contain the axis of symmetry. We also show that every non-isoparametric surface $\Sigma_{H,C}$ is either properly immersed and invariant under the cyclic group $Z_m$ for some integer $m > 1$ or it is dense in the region bounded by two isoparametric tori if the surface $\Sigma_{H,C}$ does not contain the axis of symmetry or it is dense in the region bounded by a totally umbilical surface if the surface $\Sigma_{H,C}$ contains the axis of symmetry.

Keywords Rotational surfaces · Constant mean curvature · Properly immerse · Sphere · Tori

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1 Introduction

We say that a surface $\Sigma$ in the three-dimensional unit sphere $S^3$ is rotational symmetric with respect to the geodesic $\gamma(s) = (\cos s, \sin s, 0, 0)$, if the surface has the form

$$\phi(s, t) = \left( \sqrt{1 - |\alpha(t)|^2} \cos(s), \sqrt{1 - |\alpha(t)|^2} \sin(s), \alpha(t) \right)$$

where $\alpha: \mathbb{R} \to \mathbb{R}^2$ is a regular curve contained in the unit disk. The curve $\alpha(t)$ is called the profile curve of the surface $\phi$ and we will say that the surface contains the axis of symmetry if the curve $\alpha$ passes through the origin. When the profile curve is a segment connecting two points in the unit circle, the surface is a totally umbilical sphere. For sake of simplicity in explaining the results in this paper we will omit totally umbilical spheres from the family of rotational symmetric surfaces. When $\alpha$ is a circle centered at the origin, the principal curvatures of the immersion $\phi$ are constant, that is, the surface is isoparametric. This paper studies rotational symmetric surfaces in $S^3$ with constant mean curvature, that is, surfaces with CMC such that up to a rigid motion are of the form (1.1). From the results in either [11] or [7] we conclude that all the rotational symmetric surfaces can be described with two parameters in the set

$$\{\Sigma_{H, C}: H \in \mathbb{R}, C \geq 2 \left( H + \sqrt{1 + H^2} \right) \}.$$ 

At this point we would like to point out that even though we can pick the Gauss map on $\Sigma_{H, C}$ so that the mean curvature of $\Sigma_{H, C}$ is $H$, $H$ must be viewed as a parameter and not as the mean curvature of the surface. For example, when $C > 2(|H| + \sqrt{1 + H^2})$, $\Sigma_{H, C}$ and $\Sigma_{-H, C}$ are different surfaces even though their Gauss map can be chosen so that they both have constant mean curvature $|H|$.

An interesting property that we show in this paper is the fact that most of these surfaces are not properly immersed. We prove that the properly immersed surfaces are invariant under a cyclic group and also we decide several properties of the surface according to the values of $H$ and $C$; for example, we show that surfaces that satisfy $C = -\frac{1}{H}$ are exactly those that contain the axis of symmetry. For these surfaces we have that for every $m \geq 3$ there is a rotational symmetric surface that contains the axis of symmetry whose profile curve is invariant under the cyclic group $\mathbb{Z}_m$. Figure 1 shows some pictures of these profile curves.

With respect to the problem of deciding if these surfaces are embedded, Furuya [3], and independently Otsuki [5] showed that the only rotational symmetric minimal hypersurfaces in the $n$-dimensional unit sphere are the Clifford tori—the only isoparametric minimal examples with two principal curvatures. In 1986, Ripol [9] showed that there is a non-isoparametric embedded rotational symmetric surface with constant mean curvature $H$, for any $H$ different from 0 and $\pm \frac{1}{\sqrt{3}}$. In 1990, Leite and Brito [2] showed the existence of infinitely many embedded non-isoparametric rotational symmetric hypersurfaces in $S^{n+1}$ with constant mean curvature. Perdomo [7] showed that for every integer $m \geq 2$ and any $H$ between $\cot(\frac{\pi}{m})$ and $\frac{m^2 - 2}{m^2 - 1} \sqrt{n-1}$, there is a non-isoparametric rotational symmetric embedded hypersurface in $S^{n+1}$ whose profile curve is invariant under the cyclic group $\mathbb{Z}_m$. For rotational symmetric surfaces, using complex variables, Li and Andrews [1] showed that the examples found by Perdomo [7] are the only embedded rotational surfaces in the family $\Sigma_{H, C}$ when $H \geq 0$. 
Fig. 1. Profile curve of some CMC rotational symmetric surface that contains the axis of symmetry.
It is not difficult to show that for the immersion (1.1), the vector fields $\frac{\partial \phi}{\partial s}$ and $\frac{\partial \phi}{\partial t}$ define principal directions associated with the principal curvatures $\mu$ and $\lambda$ respectively and that for any $s$, the curve $t \rightarrow \phi(s, t)$ is a geodesic. The curve $\alpha$ can be re-parameterized so that, for any $s$, the curve $\gamma(t) = \phi(s, t)$ is parameterized by arc-length. It is known that CMC rotational symmetry surfaces do not have umbilical points. Besides, assuming that $\frac{\partial \phi}{\partial t}$ has length 1, we will assume that $\lambda(t) - \mu(t)$ is positive. Wei [10] showed that if $g(t) = (\lambda(t) - \mu(t))^2 - 1^2$ then, 

$$g'(t)^2 + g(t)^2 \left(1 + \left(H + g(t)^{-2}\right)^2\right) = C.$$ (2.1)

This differential equation creates a one-to-one correspondence between rotational symmetric surfaces with constant mean curvature $H$ and positive solutions of the differential equation. It is not difficult to see that a positive solution exists if and only if $C \geq 2(H + \sqrt{1 + H^2})$. When $C = 2(H + \sqrt{1 + H^2})$, the solution is constant, more precisely $g = (1 + H^2)^{-\frac{1}{2}}$. In this case the curve $\alpha$ reduces to a circle centered at the origin with radius $\frac{1}{\sqrt{2 + 2H(H - \sqrt{1 + H^2})}}$. These solutions are known as Clifford surfaces. A direct computation shows that when $C > 2(H + \sqrt{1 + H^2})$ 

$$g(t) = \sqrt{\frac{C - 2H}{2 + 2H^2} + \frac{\sqrt{C^2 - 4CH - 4}}{2 + 2H^2}} \sin \left(2\sqrt{1 + H^2} t\right)$$ (2.2)

solves the differential equation (2.1). We have that $g(t)$ is a periodic function with period $T = \frac{\pi}{\sqrt{1 + H^2}}$, that reaches the maximum value $\sqrt{\frac{C - 2H + \sqrt{C^2 - 4CH - 4}}{2 + 2H^2}}$ when $t = \frac{T}{4}$ and the minimum value $\sqrt{\frac{C - 2H - \sqrt{C^2 - 4CH - 4}}{2 + 2H^2}}$ when $t = \frac{3T}{4}$. Figure 2 shows the graph of the function $g$.

The following lemma gives us explicit immersions, the proof is a direct computation and also can be found in ([8], Theorem 2.4)

**Lemma 2.1** If $C > 2(H + \sqrt{1 + H^2})$ and $C \neq -\frac{1}{H}$, then the immersion $\phi$ in (1.1) with 

$$\alpha(t) = \sqrt{\frac{C - g(t)^2}{C}} (\cos(\theta(t)), \sin(\theta(t))) with \theta(t) = \int_{\frac{T}{2}}^{t+\frac{T}{2}} \sqrt{C g(\tau)(H + g(\tau)^{-2})} \frac{d\tau}{C - g(\tau)^2}$$

has constant mean curvature $H$. The function $g$ is given by (2.2). We will denote this surface as $\Sigma_{H,C}$.  

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Fig. 2  Graph of a non-constant positive solution of (2.1)

Remark 2.2 The function θ defined in Lemma 2.1 is not periodic but, for any $t \in [0, T]$ and any integer $q$, it satisfies that $\theta(t + qT) = \theta(t) + q\theta(T)$ where $T = \frac{\pi}{\sqrt{1 + H^2}}$ is the period of the function $g$. Therefore, if

$$K(H, C) = \theta(T) = \int_{T}^{\frac{5T}{4}} \sqrt{C} \frac{g(\tau)(H + g(\tau)^{-2})}{C - g(\tau)^2} d\tau,$$

then the immersion in Lemma 2.1 is invariant under the subgroup of $O(4)$ given by

$$\left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(qK(H, C)) & \sin(qK(H, C)) \\ 0 & 0 & -\sin(qK(H, C)) & \cos(qK(H, C)) \end{pmatrix} : q \in \mathbb{Z} \right\}.$$  

This group is finite if and only if, $\frac{K(H, C)}{\pi}$ is rational.

When $H \geq 0$ the function $\theta(t)$ is increasing, therefore, if $K(H, C) = \frac{2\pi}{m}$ for some integer $m \geq 1$ and some constant $H$ and $C$, then it is easy to see that the profile curve of $\Sigma_{H, C}$ is a simple curve and therefore $\Sigma_{H, C}$ is embedded. Solutions of the equation $K(H, C) = \frac{2\pi}{m}$ were found in [7] and a proof that these solutions were the only ones when $H \geq 0$ was given in [1] using the following lemma.

Lemma 2.3 (Andrews and Li [1]) If $a = \frac{1}{C}$, then

$$T(H, a) = K(H, \frac{1}{a}) = \int_{x_1}^{x_2} \frac{Hu + a}{(1 - u)\sqrt{u}\sqrt{1 + H^2}\sqrt{(u - x_1)(x_2 - u)}} du,$$

where,

$$x_1 = \frac{C - 2H - \sqrt{C^2 - 4CH - 4}}{2(1 + H^2)C} \quad \text{and} \quad x_2 = \frac{C - 2H + \sqrt{C^2 - 4CH - 4}}{2(1 + H^2)C}.$$

Moreover, if $H \geq 0$, then $\frac{\partial T}{\partial a} > 0$. 

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The variation of the angle function $\theta$ over a period of the function $g$ for the profile curve of minimal rotational symmetric hypersurfaces in $S^{n+1}$ was extensively studied by Otsuki. For a paper that reviews the story of this problem we refer to [6]. Otsuki was the first one who provided explicit examples of non-isoparametric rotational symmetric minimal hypersurfaces in $S^{n+1}$ [4].

The main goal of this paper is to study the surfaces $\Sigma_{H,C}$ when $H < 0^2$. We will need to deal with three issues. The first one is that even though the family of surfaces $\Sigma_{H,C}$ varies continuously with the parameters $H$ and $C$, the immersions given in Lemma 2.1 are not well defined when $C = -\frac{1}{H}$. The second issue is that when $H$ is negative the derivative of the function $\theta$ is positive everywhere for some values of $H$ and $C$ and it changes sign for other values of $H$ and $C$. We need to analyze when each case happens and analyze both cases. The third issue is that the function $K(H,C)$ is not continuous when $C = -\frac{1}{H}$, it has a jump discontinuity. We need to analyze the jump of this discontinuity.

3 The Immersions

In this section we provide a formula that parameterizes all the immersions $\Sigma_{H,C}$. As a consequence we obtain the continuity of this family of immersions with respect to the parameters $H$, $C$. It turns out that the reason why the formula given in Lemma 2.1 fails when $C = -\frac{1}{H}$ is because it essentially uses polar coordinates for the profile curve and we can check that when $C = -\frac{1}{H}$ the profile curve passes through the origin. A similar formula for all CMC hypersurfaces in $S^{n+1}$ with rotational symmetry was provided by Wu in [11].

**Lemma 3.1** If $C > 2(H + \sqrt{1 + H^2})$, then the immersion $\phi$ in (1.1) with profile curve

$$\beta = \frac{1}{\sqrt{C g'(\tau)^2 + C g(\tau)^2}} \times \left( -\sqrt{C} g' \cos(\theta) - (H g^2 + 1) \sin(\theta), \sqrt{C} g' \sin(\theta) - (H g^2 + 1) \cos(\theta) \right)$$

with

$$\theta(\tau) = \int_{\tau}^{\tau + t} \frac{\sqrt{C} g(\tau)(H - g(\tau)^{-2})}{g'(\tau)^2 + g(\tau)^2} d\tau$$

has constant mean curvature $H$.

**Proof** The proof is a direct computation. Also we can show that for any $C \neq -\frac{1}{H}$ with $H < 0$, the curve $\alpha$ defined in Lemma 2.1 agrees with the curve $\beta$ defined in this lemma. We can see this by directly checking that

$$|\alpha(\tau)|^2 = |\beta(\tau)|^2 = \frac{C + 2H + 2CH^2 - \sqrt{-4 + C^2 - 4CH^2}}{2C + 2CH^2} \sin(2\sqrt{1 + H^2} \tau)$$

(3.1)
and also checking that if $a(t)$ denotes the angle from the positive direction of the $x$-axis to the position vector $\beta(t)$, then $a'(t) = \frac{\sqrt{C g(t)(H + g(t)^2)}}{C - g(t)^2}$, which agrees with the derivative of the angle from the positive direction of the $x$-axis to the position vector $\alpha(t)$.

**Definition 1** For any $C > 2(H + \sqrt{1 + H^2})$, we will refer to the surface in $S^3$ given by the immersion in Lemma 3.1 as $\Sigma_{H,C}$.

### 4 The Function $K(H, C)$

In this section, using the technique introduced in [1] we will study the function $K(H, C)$ as a function of $C$ for any real number $H$. We have the following proposition.

**Proposition 4.1** For any real number $H$ and any $C > 2(H + \sqrt{1 + H^2})$ different from $-\frac{1}{H}$, let us define $K(C, H)$ as in Lemma 2.3, that is,

$$K(H, C) = \int_{x_1}^{x_2} \frac{H u + C^{-1}}{(1 - u)\sqrt{u\sqrt{1 + H^2}}\sqrt{(u - x_1)(x_2 - u)}} \, du$$

where,

$$x_1 = \frac{C - 2H - \sqrt{C^2 - 4CH - 4}}{2(1 + H^2)C} \quad \text{and} \quad x_2 = \frac{C - 2H + \sqrt{C^2 - 4CH - 4}}{2(1 + H^2)C}$$

The following statements are true:

- The function $C \to K(H, C)$ is decreasing, differentiable and continuous at any value $C \neq -\frac{1}{H}$

- $$\lim_{C\to 2(H + \sqrt{1 + H^2})^+} K(H, C) = \pi \sqrt{2 - \frac{4H}{\sqrt{4 + 4H^2}}}$$

- if $H > 0$ then,
  $$\lim_{C\to \infty} K(H, C) = 2 \arccot(H) \text{ with } 0 < \arccot(H) < \frac{\pi}{2}$$

- if $H < 0$ then,
  $$\lim_{C\to \infty} K(H, C) = 2 \arccot(H) \text{ with } -\frac{\pi}{2} < \arccot(H) < 0$$

- if $H < 0$ then,
  $$\lim_{C\to -\frac{1}{H}^-} K(H, C) = \int_0^1 \frac{-H}{\sqrt{v(1 - v)(H^2 + v)}} \, dv + \pi$$
\begin{align*}
\lim_{C \to -\frac{1}{H}} K(H, C) &= \int_0^1 \frac{-H}{\sqrt{v(1-v)(H^2 + v)}} \, dv - \pi \\
\end{align*}

- The function \( b(H) = \int_0^1 \frac{-H}{\sqrt{v(1-v)(H^2 + v)}} \, dv \) defined in the interval \((-\infty, 0)\) is strictly decreasing. Moreover

\[ \lim_{H \to -\infty} b(H) = \pi \quad \text{and} \quad \lim_{H \to 0} b(H) = 0 \]

**Proof** The proof of the first part of this proposition follows the proof of Lemma 2.3 presented in [1] and we will present the proof here in order to make clear the behavior of the function \( K(H, C) \) when \((H, C)\) is near the hyperbola \( C = -\frac{1}{H} \). Let \( U = \mathbb{C} - \{x + iy : y = 0, x \geq 0\} \). In the set \( U \), let us define the complex function \( sr \) as

\[ sr(z) = \sqrt{|z|} e^{i\theta} \text{ where } z = |z|e^{i\theta} \text{ with } 0 < \theta < 2\pi \]

Clearly the function \( sr \) is an analytic function that satisfies \( sr^2(z) = z \). A direct computation shows that \( 0 < x_1 < x_2 \leq 1 \) and, \( x_2 = 1 \) if and only if \( H < 0 \) and \( C = -\frac{1}{H} \). Let \( l_1 = \{x + iy : y = 0, x_1 < x < x_2\} \) and \( l_2 = \{x + iy : y = 0, x < 0\} \). Since the Möbius transformation \( T(z) = \frac{z - x_1}{x_2 - z} \) sends the segment \( l_1 = \{x + iy : y = 0, x_1 < x < x_2\} \) to the set of positive real numbers, the function \( sr(T(z)) \) is well defined for all \( z \notin l_1 \). We also have that the function \( sr(-z) \) is well defined for all \( z \notin l_2 \). Let \( \Omega \) be the complement of the set \( l_1 \cup l_2 \cup \{0, x_1, x_2\} \) and let \( f : \Omega \to \mathbb{C} \) be the holomorphic function

\[ f(z) = -i \sqrt{1 + H^2} sr(-z) (x_2 - z) sr\left(\frac{z - x_1}{x_2 - z}\right) \]

Let \( \gamma_1, \gamma_2 \) and \( \gamma_3 \) be the curves given in Fig. 3. Using the fact that for every \( x_1 < x < x_2 \) and and \( \epsilon \neq 0 \) we have that

\[ \lim_{\epsilon \to 0^+} sr\left(\frac{(x + i\epsilon) - x_1}{x_2 - (x + i\epsilon)}\right) = \sqrt{\frac{x - x_1}{x_2 - x}} \quad \text{and} \quad \lim_{\epsilon \to 0^-} sr\left(\frac{(x + i\epsilon) - x_1}{x_2 - (x + i\epsilon)}\right) = -\sqrt{\frac{x - x_1}{x_2 - x}} \]

and

\[ \lim_{\epsilon \to 0} sr(-(x + i\epsilon)) = i\sqrt{x} \]
Fig. 3 The closed curves $\gamma_1$, $\gamma_2$ and $\gamma_3$

using these three limits, we can prove (as pointed out in [1]) that,

$$K(H, C) = \int_{x_1}^{x_2} \frac{H u + C^{-1}}{(1 - u)\sqrt{u} \sqrt{1 + H^2 \sqrt{(u - x_1) (x_2 - u)}}} \, du$$

$$= -\frac{1}{2} \int_{\gamma_1} \frac{H z + C^{-1}}{(1 - z) f(z)} \, dz$$

Since the function $\frac{H + C^{-1}}{(1 - z) f(z)}$ is holomorphic, we have that

$$\int_{\gamma_3} \frac{H z + C^{-1}}{(1 - z) f(z)} \, dz + \int_{-\gamma_1} \frac{H z + C^{-1}}{(1 - z) f(z)} \, dz + \int_{-\gamma_2} \frac{H z + C^{-1}}{(1 - z) f(z)} \, dz = 0$$

We can evaluate the limit when $\epsilon \to 0^+$ of the integral above over $\gamma_3$, using the fact

$$f(1) = -i \sqrt{1 + H^2} \sqrt{(1 - x_1)(1 - x_2)} = -i C^{-1} |1 + HC|$$

The last equality follows because we have that $(1 - x_1)(1 - x_2) = \frac{(1 + CH)^2}{C^2(1 + H^2)}$

Therefore,

$$K(H, C) = \begin{cases} 
-\pi - \frac{1}{2} \int_{\gamma_3} \frac{H z + C^{-1}}{(1 - z) f(z)} \, dz & \text{if } C > -\frac{1}{H} \\
\pi - \frac{1}{2} \int_{\gamma_3} \frac{H z + C^{-1}}{(1 - z) f(z)} \, dz & \text{if } C < -\frac{1}{H}
\end{cases} \quad (4.1)$$
A direct computation shows that
\[
\frac{\partial}{\partial C} \left( \frac{Hz + C^{-1}}{(1-z)f(z)} \right) = -\frac{z^2}{C^2 f(z)^3}
\]

Therefore, we obtain that
\[
\frac{\partial K(H, C)}{\partial C} = \frac{1}{2C^2} \int_{\gamma_3} \frac{z^2}{f(z)^3} dx < 0
\]

Taking the limit when the bigger radius in the curve \(\gamma_3\) goes to infinity and the small radius in the curve \(\gamma_3\) goes to 0 we obtain that
\[
\frac{\partial K(H, C)}{\partial C} = \frac{1}{2C^2} \int_{\gamma_3} \frac{z^2}{f(z)^3} dx < 0
\]

A direct computation shows that if \(H < 0\), then
\[
\lim_{C \to -\frac{1}{H}} x_2 = 1 \quad \text{and} \quad \lim_{C \to -\frac{1}{H}} x_1 = \frac{H^2}{1 + H^2}
\]

We also have that, on the curve \(\gamma_3\),
\[
\frac{Hz + C^{-1}}{(1-z)f(z)} \rightarrow G(z) = \frac{H}{i(1-z)\text{sr}(-z) \text{sr} \left( \frac{(1+H^2)z-H^2}{1-z} \right)} \quad \text{as} \quad C \rightarrow -\frac{1}{H}
\]

The function \(G\) is holomorphic in the complement of the set
\[
\left\{ x + iy : y = 0 \text{ and, either } x \leq 0 \text{ or } \frac{H^2}{1 + H^2} \leq x \leq 1 \right\}
\]

If we define \(\gamma_4\) like in Fig. 4, we have that \(\int_{\gamma_3} g(z)dz + \int_{\gamma_4} g(z)dz = 0\), and taking the limit when the radius of the small circle in the curve \(\gamma_4\) goes to zero we obtain that
\[
-\frac{1}{2} \int_{\gamma_3} G(z)dz = \frac{1}{2} \int_{\gamma_4} G(z)dz = \int_{\gamma_4} \frac{-H}{u(1-u)((1+H^2)u-H^2)} du = \int_{0}^{1} \frac{-H}{v(1-v)(H^2 + v)} dv
\]

The last equation follows by doing the substitution \(v = (1 + H^2)u - H^2\). Using the expression for \(K(H, C)\) given in (4.1) we obtain the expressions in the proposition for the limit of \(K(H, C)\) when \(C\) goes to \(-\frac{1}{H}^+\) and \(-\frac{1}{H}^-\). Using the fact that
Fig. 4 The closed curves $\gamma_3$ and $\gamma_4$

Fig. 5 The shaded region shows the possible values of $K(H, C)$ for every $H$.

$$b(H) = \int_0^1 \frac{-H\, dv}{\sqrt{v(1-v)(H^2+v)}}$$

$$y(H) = \text{ArcCot}(H)$$

$$y(H) = \pi - \frac{4H}{\sqrt{4+4H^2}}$$

$$b(H) = -2iH\, \text{EllipticK}(-H^2) + 2H\, \text{EllipticK}(1 + H^2),$$ we obtain that $\lim_{H \to 0^-} b(H) = 0$ and $\lim_{H \to -\infty} b(H) = \pi$.

The limits for $K(H, C)$ when $C$ goes to $2(H + \sqrt{1 + H^2})$ and when $C$ goes to $\infty$ were computed in [7].

Figure 5 shows the limit values of the function $K(H, C)$ as well as all possible values of this function for every $H$. It also shows the discontinuity of this function when $C$ approaches $-\frac{1}{H}$.
5 Fundamental Piece

The profile curve of CMC rotational surface in $S^3$ can be built as the union of rotations of a single piece that we will call the fundamental piece.

**Definition 2** The fundamental piece of the surface $\Sigma_{H,C}$ is defined as the profile curve $\beta(t)$ given in Lemma 3.1 restricted to the interval $\left[\frac{-\pi}{4\sqrt{1+H^2}}, \frac{3\pi}{4\sqrt{1+H^2}}\right]$.

Here are some easy-to-check properties of the fundamental piece of the surfaces $\Sigma_{H,C}$.

**Proposition 5.1** Let $\gamma: \left[\frac{-\pi}{4\sqrt{1+H^2}}, \frac{3\pi}{4\sqrt{1+H^2}}\right] \to \mathbb{R}^2$ be the fundamental piece of the surface $\Sigma_{H,C}$. The following properties hold true:

- The distance from points in the fundamental piece to the origin satisfies the following inequality:
  $$\left|\gamma\left(\frac{\pi}{4\sqrt{1+H^2}}\right)\right| \leq |\gamma(t)| \leq \left|\gamma\left(\frac{3\pi}{4\sqrt{1+H^2}}\right)\right| = \left|\gamma\left(-\frac{\pi}{4\sqrt{1+H^2}}\right)\right|$$

- If $C \neq -\frac{1}{H}$, then, the angle between $\gamma\left(-\frac{\pi}{4\sqrt{1+H^2}}\right)$ and $\gamma\left(-\frac{3\pi}{4\sqrt{1+H^2}}\right)$ is given by $K(H, C)$. If $C = -\frac{1}{H}$, then, the angle between $\gamma\left(-\frac{\pi}{4\sqrt{1+H^2}}\right)$ and $\gamma\left(-\frac{3\pi}{4\sqrt{1+H^2}}\right)$ is given by $b(H, C) = \int_0^1 \frac{-H}{\sqrt{v(v-1)(H^2+v)}} dv$

- The profile curve of $\Sigma_{H,C}$ is the union of rotations of the fundamental piece (Fig. 6).

**Proof** The first item follows from (3.1). The second item follows from Remark 2.2 and the continuity in terms of $H$ and $C$ of the immersions given in Lemma 3.1. The last item again follows from Remark 2.2. \qed

![Fig. 6](image-url) The first picture shows the profile curve and its fundamental piece when $H = -0.2$ and $C = 7.10621080709656$. The second picture shows part of the profile curve when $H = -0.2$ and $C = 6$.

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The next theorem shows a properly immersed vs. dense duality property for the immersions $\Sigma_{H,C}$.

**Theorem 5.2** If $C = -\frac{1}{H}$ with $H < 0$, then the immersion $\Sigma_{H,C}$ is either properly immersed or it is dense in the region

$$\left\{ (x, y, z, w) \in S^3: z^2 + w^2 \leq \frac{1}{1 + H^2} \right\}$$

Moreover, if $C \neq -\frac{1}{H}$, then the immersion $\Sigma_{H,C}$ is either properly immersed or it is dense in the region

$$R_{H,C} = \left\{ (x, y, z, w) \in S^3: m_{H,C} \leq z^2 + w^2 \leq M_{H,C} \right\}$$

where,

$$m_{H,C} = \frac{C + 2H + 2CH^2 - \sqrt{-4 + C^2 - 4CH}}{2C + 2CH^2}$$

$$M_{H,C} = \frac{C + 2H + 2CH^2 + \sqrt{-4 + C^2 - 4CH}}{2C + 2CH^2}$$

**Proof** When $C \neq -\frac{1}{H}$ we have that if $\frac{K(H,C)}{2\pi} = \frac{m}{k}$ with $k$ and $m$ relatively prime integers and $k > 0$, then, it is easy to see that the profile curve is the union of $k$ copies of rotations of the fundamental piece. Therefore the immersion is proper. If $\frac{K(H,C)}{2\pi}$ is an irrational number, for any $r \in [\sqrt{m_{H,C}}, \sqrt{M_{H,C}}]$, using the first item of Proposition 5.1 and the intermediate value theorem, we have that as $t$ goes from 0 to $\infty$, the profile curve $\beta(t)$ of the surface $\Sigma_{H,C}$, hits the circle $C_r$ centered at the origin with radius $r$ at points that differ by a fixed angle $K(H,C)$. The union of all these points in the circle $C_r$ is dense in the $C_r$. The problem of proving this last statement reduces to that of showing that for any irrational number $\tau$, the set $\{n\tau - \lfloor n\tau \rfloor : n \in \mathbb{Z}\}$ is dense in the interval $[0, 1]$, which is a known fact. We therefore have that the profile curve is dense in the annulus, $\{(z, w) \in \mathbb{R}^2 : m_{H,C} \leq z^2 + w^2 \leq M_{H,C}\}$. Looking at the formula for rotational immersion (formula 1.1), we conclude that $\Sigma_{H,C}$ is dense in $R_{H,C}$. The proof of the case $C = -\frac{1}{H}$ is similar. \qed

Let us finish this section by considering in more detail the rotational immersions obtained when $C = -\frac{1}{H}$ with $H < 0$. As pointed out before, for this case the immersions given in Lemma 2.1 are not well defined and therefore the immersions given in Lemma 3.1 are needed.

**Theorem 5.3** The profile curves of all these examples contain the origin and therefore these immersions contain a circle of radius 1. Also, for any $m > 2$, there exists an $H < 0$ such that the profile curve of the rotational immersion $\Sigma_{H,C}$, with $C = -\frac{1}{H}$, is invariant under the cyclic group $\mathbb{Z}_m$. Figure 1 in the Introduction shows some of these immersions.
Proof. Let us start by pointing out that, since $2(H + \sqrt{1 + H^2}) = \frac{2}{\sqrt{1 + H^2 - H}}$, then we have that for every $H < 0$, $2(H + \sqrt{1 + H^2}) < -\frac{1}{H}$. Therefore, for any $H < 0$ there is a rotational surface in $S^3$ such that $C = -\frac{1}{H}$. If $C = -\frac{1}{H}$, then, the values of the function $g$ moves from $g\left(\frac{3\pi}{4\sqrt{1 + H^2}}\right) = \sqrt{\frac{1 - H}{1 + H^2}}$ to $g\left(\frac{\pi}{4\sqrt{1 + H^2}}\right) = \sqrt{-\frac{1}{H}}$. By the continuity with respect to the variables $H$ and $C$ of the immersion given in (3.1) and Proposition 4.1, we have that the angle between the initial and final point of the fundamental piece is given by $b(H) = \int_0^1 \frac{-H}{\sqrt{v(v-1)(H^2+v)}} dv$. We also know from Proposition 4.1 that $b(H)$ takes every value in the open interval $(0, \pi)$. Therefore for any $m > 2$ we can find a $H_m < 0$ such that $b(H_m) = \frac{2\pi}{m}$. Since for this value of $H_m$ the profile curve is the union $m$ fundamental pieces, then the immersion $\Sigma_{H_m, C}$ with $C = -\frac{1}{H_m}$ is properly immersed. These examples are invariant under the cyclic group $Z_m$. This finishes the proof of the theorem.

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