CYCLIC AND SUPERCYCLIC WEIGHTED COMPOSITION OPERATORS ON THE FOCK SPACE

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ABSTRACT. We study the cyclic and supercyclic dynamical properties of weighted composition operators on the Fock space $F_2$. A complete characterization of cyclicity which depends on the derivative of the symbol for the composition operator and zeros of the weight function is provided. It is further shown that the space fails to support supercyclic weighted composition operators. As a consequence, we also noticed that the space supports no cyclic multiplication operator.

1. Introduction

For a pair of entire functions $(u, \psi)$ on the complex plane $\mathbb{C}$, the induced weighted composition operator $W_{(u, \psi)}$ maps $f$ to $uf(\psi)$. If $u = 1$, then $W_{(u, \psi)}$ is just the composition map $C_\psi : f \mapsto f(\psi)$. On the other hand, if $\psi$ is the identity map, then $W_{(u, \psi)}$ reduces to the multiplication operator $M_u : f \mapsto uf$. Thus, $W_{(u, \psi)}$ generalizes the two operators and can be also written as a product $M_u C_\psi$. The theory of weighted composition operators traces back to the sixties in the work of Forelli [4] where it was shown that the isometries in the Hardy spaces $H^p$ whenever $1 < p < \infty, p \neq 2$ are weighted composition operators. De Leeuw [9] later showed that the same holds true on the space $H^1$ as well. Since then the operator became a natural object of study and its investigations has rapidly evolved in function related operator theory. A number of researchers have studied the operator over various settings with the aim to express its spectral, topological and dynamical properties in terms of the function theoretic properties of the inducing pairs of symbols $(u, \psi)$. See for example [3, 11, 12, 15] and the references therein.

In this note we study the cyclic and supercyclic dynamical properties of the operators on the classical Fock space $F_2$. Recall that $F_2$ consists of square integrable analytic functions in $\mathbb{C}$ with respect to the Gaussian measure $d\mu(z) = \frac{1}{\pi}e^{-|z|^2}dA(z)$ where $dA$ is the Lebesgue measure in the complex plane $\mathbb{C}$. It is a reproducing kernel Hilbert space with inner product

$$\langle f, g \rangle = \int_{\mathbb{C}} f(z)\overline{g(z)}d\mu(z),$$

norm $\|f\|_2 := \sqrt{\langle f, f \rangle}$, and kernel function $K_w(z) = e^{(z,w)}$.

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A great deal of the studies on weighted composition operators has been devoted
to characterizing boundedness and compactness spectral properties over various
functional spaces. On Fock spaces, these properties have been well understood
and described for example in [11, 12, 15] and expressed in different conditions
among which following [11], \( W_{(u,\psi)} \) is bounded on \( F_2 \) if and only if

\[
\sup_{z \in \mathbb{C}} |u(z)| e^{\frac{1}{2}(|\psi(z)|^2 - |z|^2)} < \infty. \tag{1.1}
\]

Furthermore, it was proved that (1.1) implies \( \psi(z) = az + b, |a| \leq 1 \) and whenever
\(|a| = 1\), the multiplier function has the special form

\[
u(z) = u(0)K_{-m_b}(z) \tag{1.2}\]

for all \( z \) in \( \mathbb{C} \). Similarly, compactness has been described by the additional con-
dition that \(|a| < 1\). An interesting feature here is that there exists an interplay
between \( u \) and \( \psi \) such that \( W_{(u,\psi)} = M_u C_\psi \) is bounded (compact) on \( F_2 \) while
both \( C_\psi \) and \( u \) fail to be. As pointed earlier, the purpose of this note is to study
the effect of this interplay on the dynamical structures of \( W_{(u,\psi)} \) on \( F_2 \). Recall
that a bounded linear operator \( T \) on a separable Banach space \( \mathcal{H} \) is said to be
cyclic if there exists a vector \( f \) in \( \mathcal{H} \) for which the span of the orbit
\( \text{Orb}(T,f) = \{f, Tf, T^2f, T^3f, \ldots\} \)
is dense in \( \mathcal{H} \). Such a vector is called cyclic for \( T \). The operator is hypercyclic if
the orbit itself is dense. \( T \) is supercyclic with vector \( f \) if the projective orbit
\( \mathbb{C} \). \( \text{Orb}(T,f) = \{\lambda T^n f, \ \lambda \in \mathbb{C}, \ n = 0, 1, 2, \ldots\} \)
is dense. These dynamical properties of \( T \) depend on the behaviour of its iterates
\( T^n = T \circ T \circ T \circ \ldots \circ T \) \( n \) times. For detailed backgrounds, one may consult the
monographs [1, 5].

It is worth noting that identifying cyclic and hypercyclic operators have been a
subject of high interest partly because they play central rolls in the study of other
operators. More specifically, it is known that every bounded linear operator on an
infinite dimensional complex separable Hilbert space is the sum of two hypercyclic
operators [1, p. 50]. Interestingly, this result holds true with the summands being
cyclic operators [16]. In [7, 13] it was reported that there exists no supercyclic
(and hence hypercyclic) composition operator on Fock spaces. On the other hand,
the orbit of any vector \( f \) under \( W_{(u,\psi)} \) has elements of the form

\[
W_{(u,\psi)}^n f = f(\psi^n) \prod_{j=0}^{n-1} u(\psi^j) \tag{1.3}
\]

for all nonnegative integers \( n \) and \( \psi^0 \) is the identity map. This shows that the
product of weighted composition operators is another weighted composition op-
erator with symbol \( (U_n, \psi^n) \) where

\[
U_n = \prod_{j=0}^{n-1} u(\psi^j).
\]
The formula in (1.3) further displays a kind of interplay between the functions \( \psi \) and \( u \) and generates interest to ask whether the interplay results in hypercyclic, supercyclic or cyclic weighted composition operators on the Fock space. In fact, we can make the following simple observation right away. Boundedness of \( W_{(u,\psi)} \) implies that \( \psi(z) = az + b, |a| \leq 1 \). If \( a \neq 1 \), then \( \psi \) fixes the point \( z_0 = b/(1-a) \) in \( \mathbb{C} \). A computation with adjoint property gives

\[
W_{(u,\psi)}^* K_{z_0} = \overline{u(z_0)} K_{\psi(z_0)} = \overline{u(z_0)} K_{z_0}.
\]

Since \( K_{z_0} \) is a nonzero vector, it follows that \( u(z_0) \) is an eigenvalue for the adjoint operator \( W_{(u,\psi)}^* \) which contradicts the general fact that the adjoint of a hypercyclic operator can not have eigenvalue (see Proposition 1.17 in [1]). This shows there exists no hypercyclic weighted composition operator on \( \mathcal{F}_2 \) in this case. Here the fixed point behaviour of the composition operator decisively determined the absence of hypercyclicity for the weighted operator. Later, we will in fact show that the same conclusion applies when \( a = 1 \) as well. Composition operators on Fock spaces are among the groups of non-hypercyclic operators. It turns out that the weighted composition operators exhibit the exact same hypercyclic phenomena as the unweighted ones. The relation in (1.3) is weak enough to make any orbit as big as the whole space.

Given the above observation on the absence of hypercyclic weighted composition operator \( W_{(u,\psi)} \), the natural question is what happens to the weaker cyclicity and supercyclicity properties. Clearly, hypercyclicity is a much stronger property than cyclic and every hypercyclic operator is cyclic. Thus, hypercyclic operators enjoy richer operator-theoretic properties than cyclic ones. One interesting difference between the two properties is that if an operator has a hypercyclic vector \( f \), then it has a dense subset of hypercylic vectors because every element in its orbit is also hypercyclic. This fails to hold for cyclic operators: see [2] for an example. On the other hand, supercyclicity is a property which is intermediate between the two.

2. MAIN RESULTS

We now state our first main result on the cyclic behavior of \( W_{(u,\psi)} \).

**Theorem 2.1.** Let \( u \) and \( \psi(z) = az + b, |a| \leq 1 \) be analytic maps on \( \mathbb{C} \) such that \( W_{(u,\psi)} \) is bounded on \( \mathcal{F}_2 \). Then \( W_{(u,\psi)} \) is cyclic on \( \mathcal{F}_2 \) if and only

(i) \( u \) fails to vanish on \( \mathbb{C} \) and

(ii) \( a^k \neq a \) for all positive integer \( k \geq 2 \).

The cyclic property of \( W_{(u,\psi)} \) depends on the size of the power of the derivative of the symbol \( \psi(z) = az + b \) and the existence of zeros of the symbol \( u \). Part b) is the condition required for the cyclicity of the composition operator [7, 13]. It means that the absence of zeros for the multiplier function makes it possible for the weighted operator to inherit the same condition. Notice that the cyclicity condition clearly restricts \( \psi \) to be a nonconstant function.
Proof. For the sufficiency, we may exhibit that $K_z$ is a cyclic vector for $W_{(u,ψ)}$. Since condition ii) implies that $0 \neq a \neq 1$, observe that using (1.3) we may rewrite

$$W^m_{(u,ψ)}K_z(w) = K_z\left(a^mw + \frac{1 - a^m}{1 - a} \prod_{j=0}^{m-1} u(ψ^j(w))\right)$$

$$= e^{(a^mw + \frac{1 - a^m}{1 - a})w} \prod_{j=0}^{m-1} u(ψ^j(w)) = \sigma_m K_{\overline{a}^m z}(w) \prod_{j=0}^{m-1} u(ψ^j(w)),$$

where

$$\sigma_m = e^{\left(\frac{1 - a^m}{1 - a}\right)w}$$

is independent of the point $w$. Now, fix a vector $f \in F_2$ that is orthogonal to the orbit of $K_z$ under $W_{(u,ψ)}$. We claim that $f$ should be the zero function. To see this, note that for every $m$

$$0 = \langle f, W^m_{(u,ψ)}K_z \rangle = \sigma_m \left\langle f, \prod_{j=0}^{m-1} u(ψ^j)K_{\overline{a}^m z} \right\rangle$$

$$= \sigma_m \left\langle \prod_{j=0}^{m-1} u(ψ^j)f, K_{\overline{a}^m z} \right\rangle = \sigma_m \prod_{j=0}^{m-1} u(ψ^j(\overline{a}^m z))f(\overline{a}^m z). \quad (2.1)$$

By assumption, since $u$ posses no zero so does the quantity

$$\prod_{j=0}^{m-1} u(ψ^j(\overline{a}^m z)).$$

If $|a| < 1$, then $\sigma_m$ is different from zero for every $m$, and since $\overline{a}^m z \to 0$ as $m \to \infty$ for each fixed point $z$. Thus, the relation in (2.1) holds only if $f$ vanishes on the null sequence $\overline{a}^m z$. It follows that $f$ is the zero function as asserted.

For the case $|a| = 1$, and $a^m \neq a$ for each $m \geq 2$, we may argue as follows to arrive at the same conclusion, namely that $f$ is still the zero function. For each fixed $z$ and all $m \geq 0$, we have

$$f(\overline{a}^m z) = 0. \quad (2.2)$$

Set in particular $z = 1$ and consider the Taylor series expansion of $f$ at it;

$$f(w) = \sum_{n=0}^{\infty} a_n(w - 1)^n. \quad (2.3)$$

We need to show that $a_n = 0$ for all $n$. Applying (2.3) and (2.2) for $m = 0$, gives that $a_0 = 0$. Using this and similarity considering $m = 1, 2, 3, 4, 5, \ldots$ we get an
infinite system of linear equations $VA = 0$ where

$$V = \begin{pmatrix} 1 & \varpi - 1 & (\varpi - 1)^2 & (\varpi - 1)^3 & \cdots \\ 1 & \varpi^2 - 1 & (\varpi^2 - 1)^2 & (\varpi^2 - 1)^3 & \cdots \\ 1 & \varpi^3 - 1 & (\varpi^3 - 1)^2 & (\varpi^3 - 1)^3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ 1 & \varpi^{n-1} - 1 & (\varpi^{n-1} - 1)^2 & (\varpi^{n-1} - 1)^3 & \cdots \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \end{pmatrix}$$

Observe that this is a matrix with the terms of a geometric progression in each row, and restricting to finite $n \times n$ actually gives the well-known Vandermonde matrix

$$V_n = \begin{pmatrix} 1 & \varpi - 1 & (\varpi - 1)^2 & (\varpi - 1)^3 & \cdots & (\varpi - 1)^{n-1} \\ 1 & \varpi^2 - 1 & (\varpi^2 - 1)^2 & (\varpi^2 - 1)^3 & \cdots & (\varpi^2 - 1)^{n-1} \\ 1 & \varpi^3 - 1 & (\varpi^3 - 1)^2 & (\varpi^3 - 1)^3 & \cdots & (\varpi^3 - 1)^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \varpi^{n-1} - 1 & (\varpi^{n-1} - 1)^2 & (\varpi^{n-1} - 1)^3 & \cdots & (\varpi^{n-1} - 1)^{n-1} \end{pmatrix}$$

with determinant

$$\det V_n = \prod_{1 \leq i < j \leq n} (\varpi^j - \varpi^i). \quad (2.4)$$

Since $a^m \neq a$ for all $m > 1$, we have that $\varpi^m - 1 \neq 0$ for all $m > 0$. In fact for two different positive integers $p$ and $q$, $\varpi^p \neq \varpi^q$. This shows that (2.4) is non-zero. Thus, the Vandermonde matrix is invertible and the solution of the above mentioned system of linear equations is unique for any dimension $n$ of the system. But $n$ is arbitrary above, therefore, the infinite system $VA = 0$ holds only if the system has a trivial solution $A = 0$ as claimed and completes the proof of the sufficiency.

We next proof the necessity of the conditions, and assume on the contrary that $u$ vanishes at the point $z_0$ in $\mathbb{C}$. Then, for any possible cyclic vector $f$ again

$$W_{(u,\psi)}^n f(z_0) = \prod_{j=0}^{n-1} u\left(e^{ij}(z_0)\right) f\left(e^{jn}(z_0)\right) = u(z_0) \prod_{j=1}^{n-1} u\left(e^{ij}(z_0)\right) f\left(e^{jn}(z_0)\right) = 0$$

which shows that all the functions in the orbit vanish at the point $z_0$. This extends to all functions $g$ in the closed linear span of the orbit of $f$ under $W_{(u,\psi)}$ which is not the case.

For part b), we consider first the case when $a = 1$ and hence $\psi^j(z) = z + jb$ for all $j \geq 0$. Then for any possible cyclic vector $f$

$$W_{(u,\psi)}^m f(z) = f(z + mb)u(0)^m \prod_{j=0}^{m-1} K_b(z + jb) = f(z + mb)u(0)^m e^{bm(z)},$$
where
\[ h_m(z) := -b \sum_{j=0}^{m-1} (z + jb) = -bmz - \frac{|b|^2}{2}m(m-1) = \frac{|b|^2}{2}m(m-1)K_{-mb}(z). \]

First observe that if \( b = 0 \), then the relations above give
\[ W_{(u, \psi)}^m f = fu(0)^m \]  
(2.5)
asserting that all the elements in the orbit are scalar multiplies of the cyclic vector \( f \). Any vector \( g \) in \( \mathcal{F}_2 \) orthogonal to \( f \) is also orthogonal to the closed linear span of its orbit. Thus, it suffices to show that there exists such a non-zero \( g \). To this end, since \( S = \{ f \} \) is a closed subspace of \( \mathcal{F}_2 \), the projection operator \( P : H \rightarrow S \) is continuous. Then for \( f_1 \in \mathcal{F}_2, f_1 \neq f \), the function
\[ g = f_1 - Pf_1 = f_1 - f \in S^\perp \]  
and orthogonal to the linear span of the orbit.

Next, we assume \( b \neq 0 \), and consider \( g \in \mathcal{F}_2 \) such that
\[ \langle g, W_{(u, \psi)}^m f \rangle = 0, \]  
(2.6)
for all \( m \). Our proof will be completed if we manage to construct a nonzero function \( g \in \mathcal{F}_2 \) satisfying condition (2.6). Taking the condition further
\[ 0 = \langle g, W_{(u, \psi)}^m f \rangle = u(0)^m e^{-\frac{|b|^2}{2}m(m-1)} \langle gf(\psi^m), K_{-mb} \rangle \]  
\[ = u(0)^m e^{-\frac{|b|^2}{2}m(m-1)}g(-mb)f(\psi^m(-mb)) \]  
\[ = u(0)^m e^{-\frac{|b|^2}{2}m(m-1)}g(-mb)f(0) \]
which holds only if \( g \) vanishes on the sequence \( \{ -bm : m \in \mathbb{N} \} \). A good candidate is to set \( g \) to be the product
\[ g(z) = z \prod_{m=1}^{\infty} \left( 1 + \frac{z}{bm} \right) e^{-\frac{z}{bm}}. \]
Accordingly, it suffices to show that the product above converges and belongs to the space \( \mathcal{F}_2 \). But it is know that the function \( \sin z \) has the product expansion
\[ \sin z = z \prod_{m=1}^{\infty} \left( 1 - \frac{z}{m} \right) e^{\frac{z}{m}} \]
from which we may rewrite \( g \) as
\[ g(z) = \frac{-b}{\pi} \sin \left( \frac{-\pi z}{b} \right), \]
and observe that \( g \) belongs to the space with norm \( \|g\|_2 < |b|/\pi \).

It remains to show the case when \( \psi(z) = az + b, |a| = 1 \) and \( a \neq 1 \). In this case \( \psi \) fixes the point \( z_0 = \frac{b}{1-a} \). A simple modification of the arguments in the proof of Lemma 3 of \cite{6} gives that the adjoint operator \( W_{(u, \psi)}^* \) has the following set of eigenvalues
\[ \left\{ u(z_0), au(z_0), a^2u(z_0), a^3u(z_0), \ldots, a^m u(z_0) \right\}, \]  
(2.7)
where $m$ is the smallest positive integer such that $m \geq 2$ and $a^m = a$. Note that such an $m$ exists by our assumption. On the other hand, $u(z_0) \neq 0$ and hence the set in (2.7) has at least two elements. By Proposition 2.7 of [2], the adjoint of a cyclic operator can not have multiple eigenvalues from which our assertion follows and completes the proof of the first result. \hfill \Box

Unlike cyclicity, our next result shows that the space $\mathcal{F}_2$ fails to support supercyclic weighted composition operators.

**Theorem 2.2.** Let $u$ and $\psi$ be analytic maps on $\mathbb{C}$ such that $W(u, \psi)$ is bounded on $\mathcal{F}_2$. Then $W(u, \psi)$ can not be supercyclic on $\mathcal{F}_2$.

The result extends earlier work obtained for the unweighted composition operators on the space [13]. The interplay between the additional multiplier function $u$ and $\psi$ has failed to make any projective orbit dense in $\mathcal{F}_2$. It is natural to compare this result with the analogues space over the unit disc where it was reported that there exists hypercyclic and hence supercyclic weighted composition operators, see for example [14]. It seems these properties are less attainable on functional spaces defined on unbounded domains than bounded ones.

**Proof.** Since $W(u, \psi)$ is bounded, we set $\psi(z) = az + b$, $|a| \leq 1$. As supercyclicity implies cyclicity; the proof for the case when $a^m = a$ for some $m > 1$ follows from Theorem 2.1. Therefore, we plan to prove the case for $a^m \neq a$ for all $m \geq 2$. Assume first that $|a| < 1$. Then $W(u, \psi)$ is a compact operator. For this we rely on some spectral properties of supercyclic operators. The space $\mathcal{F}_2$ is an infinite dimensional complex Banach space, and if a compact operator $W(u, \psi)$ is supercyclic on $\mathcal{F}_2$, then its spectrum $\sigma(W(u, \psi))$ contains only the zero element: see [1, p.29]. Thus, it suffices to show that $\sigma(W(u, \psi))$ contains at least two elements. Since the point $z_0 = b/(1-a)$ is fixed by $\psi$, we simply change the kernel functions and repeat the arguments in the proof of Theorem 1 in [6] to deduce

$$\sigma(W(u, \psi)) = \{0, u(z_0), au(z_0), a^2u(z_0), a^3u(z_0), \ldots\}. \quad (2.8)$$

Observe that if $u(z_0) = 0$, then the operator is not supercyclic. If not, $W(u, \psi)$ becomes cyclic and contradicts Theorem 2.1. On the other hand, if $a = 0$, then $\psi(z) = b$ and the relation in (1.3) implies

$$W^n(u, \psi)f(z) = f(b)u(b)^n$$

This means that the orbit of $f$ contains only constant functions, and hence $W(u, \psi)$ is not supercyclic here again. Therefore, both $a$ and $u(z_0)$ are nonzero and the set in (2.8) contains infinitely many elements.

It remains to show the case for $|a| = 1$ and $a^m \neq a$ for all $m > 1$. This is rather immediate as $u(z) = u(0)K_b(z)$, by Theorem 3.2 of [11], $W(u, \psi)$ is a constant multiple of a unitary operator and hence normal. Then, our conclusion follows from a result of Hilden and Wallen [8]. \hfill \Box

The following corollary is now immediate.

**Corollary 2.3.**

(i) Let $C_\psi$ be bounded on $\mathcal{F}_2$, that is $\psi(z) = az + b, |a| \leq 1$ and $b = 0$ whenever $|a| = 1$. Then $C_\psi$ is cyclic if and only if $a^k \neq a$ for
each positive integer \( k \geq 2 \). In this case, the reproducing kernel \( K_z \) is a cyclic vector.

(ii) \( C_\psi \) is not supercyclic on \( \mathcal{F}_2 \).

(iii) There exists no cyclic multiplication operator on \( \mathcal{F}_2 \).

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