On certain rings of differentiable type and finiteness properties of local cohomology

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Abstract

Let $R$ be a commutative $F$-algebra, where $F$ is a field of characteristic 0, satisfying the following conditions: $R$ is equidimensional of dimension $n$, every residual field with respect to a maximal ideal is an algebraic extension of $F$, and $\text{Der}_F(R)$ is a finitely generated projective $R$-module of rank $n$ such that $R_m \otimes_R \text{Der}_F(R) = \text{Der}_F(R_m)$. We show that the associated graded ring of the ring of differentiable operators, $D(R, F)$, is a commutative Noetherian regular with unity and pure graded dimension equal to $2 \dim(R)$. Moreover, we prove that $D(R, F)$ has weak global dimension equal to $\dim(R)$ and that its Bernstein class is closed under localization at one element. Using these properties of $D(R, F)$, we show that the set of associated primes of every local cohomology module, $H^i_I(M)$, is finite. If $(S, m, K)$ is a complete regular local ring of mixed characteristic $p > 0$, we show that the localization of $S$ at $p$, $S_p$, is such a ring. As a consequence, the set of associated primes of $H^i_I(S)$ that does not contain $p$ is finite. Moreover, we prove this finiteness property for a larger class of functors.

1 Introduction

Let $R$ denote a commutative Noetherian ring with unity. If $M$ is an $R$-module and $I \subset R$ is an ideal, we denote the $i$-th cohomology of $M$ with support in $I$ by $H^i_I(M)$. The structure of these modules has been widely studied by several authors. Among the results obtained, one encounters the following finiteness properties for certain regular rings.

(1) The set of associated primes of $H^i_I(R)$ is finite;
(2) The Bass numbers of $H^i_I(R)$ are finite;
(3) $\text{inj.dim} H^i_I(R) \leq \text{dimSupp} H^i_I(R)$. 

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Lyubeznik showed (1), (2) and (3) hold in the local case of equal characteristic 0 \[\text{Ly1}\]. His technique relies in the use of \(D\)-modules in a power series ring with coefficients over a field of characteristic 0. Later, in his work on local cohomology for unramified mixed characteristic \(p > 0\) \[\text{Ly2}\], Lyubeznik used rings of differentiable type to prove that the set of associated primes not containing \(p\) is finite. Our goal is to develop the theory of \(D\)-modules in a more general setting in order to prove a similar result for any regular local ring of mixed characteristic, namely:

**Theorem 1.1.** Let \(R\) be a regular commutative Noetherian ring with unity that contains a field, \(F\), of characteristic 0 satisfying the following conditions:

1. \(R\) is equidimensional of dimension \(n\);
2. every residual field with respect to a maximal ideal is an algebraic extension of \(F\);
3. \(\text{Der}_F(R)\) is a finitely generated projective \(R\)-module of rank \(n\) such that \(R_m \otimes_R \text{Der}_F(R) = \text{Der}_F(\mathfrak{m})\).

Then, the ring of \(F\)-linear differential operators \(D(R, F)\) is a ring of differentiable type of weak global dimension equal to \(\dim(R)\). Moreover, the Bernstein class of \(D(R, F)\) is closed under localization at one element.

This theorem generalizes some of the results of Mebkhout and Narváez-Macarro about certain rings of differentiable type \[\text{MeNa}\]. There, \(R\) is a commutative Noetherian regular ring that contains a field, \(F\), of characteristic zero satisfying (1), (2), but instead of (3) in Theorem 1.1 there exist \(F\)-linear derivations \(\partial_1, \ldots, \partial_n \in \text{Der}_F(R)\) and \(a_1, \ldots, a_n \in R\) such that \(\partial_i a_j = 1\) if \(i = j\) and 0 otherwise.

**Theorem 1.2.** Let \((R, m, K)\) be a regular commutative Noetherian local ring of mixed characteristic \(p > 0\). Then, the set of associated primes of \(H^i_I(R)\) that do not contain \(p\) is finite for every \(i \in \mathbb{N}\) and every ideal \(I \subset R\).

In fact, we prove this finiteness property for a larger class of rings and functors.

This manuscript is organized as follows. In section 2, we study certain rings of differentiable type, and we prove Theorem 1.1. Later, in section 3, we generalize some results about properties of the Bernstein class of rings of differentiable type obtained by Mebkhout and Narváez-Macarro \[\text{MeNa}\]. Finally, in section 4, we show the finiteness of the associated primes of the local cohomology for certain rings in characteristic 0; moreover, we prove this result for a larger class of functors. As a consequence, we obtain Theorem 1.2.
2 Rings of differentiable type

We start by recalling a theorem from Matsumura’s book:

**Theorem 2.1** (Teo. 99 in [Ma]). Let \((R, m, F)\) be a regular local commutative Noetherian ring with unity of dimension \(n\) containing a field \(F_0\). Suppose that \(F\) is an algebraic separable extension of \(F_0\). Let \(\hat{R}\) denote the completion of \(R\) with respect to \(m\). Let \(x_1, \ldots, x_n\) be a regular system of parameters of \(R\). Then, \(\hat{R} = F[[x_1, \ldots, x_n]]\) the power series ring with coefficients in \(F\), and \(\text{Der}_{F} \hat{R}\) is a free \(\hat{R}\)-module with basis \(\partial/\partial x_1, \ldots, \partial/\partial x_n\). Moreover, the following conditions are equivalent:

- \(\partial/\partial x_i\) (\(i = 1, \ldots, n\)) maps \(R\) into \(R\), i.e. \(\partial/\partial x_i \in \text{Der}_{F_0}(R)\);
- there exist \(D_1, \ldots, D_n \in \text{Der}_{F_0}(R)\) and \(a_1, \ldots, a_n \in R\) such that \(D_ia_j = 1\) if \(i = j\) and 0 otherwise;
- there exist \(D_1, \ldots, D_n \in \text{Der}_{F_0}(R)\) and \(a_1, \ldots, a_n \in R\) such that \(\det(D_ia_j) \notin m\);
- \(\text{Der}_{F_0}(R)\) is a free module of rank \(n\);
- \(\text{rank}(\text{Der}_{F_0}(R)) = n\).

**Hypothesis 2.2.** From now on, we will consider a commutative Noetherian regular ring \(R\) with unity that contains a field, \(F\), of characteristic zero satisfying:

1. \(R\) is equidimensional of dimension \(n\);
2. every residual field with respect to a maximal ideal is an algebraic extension of \(F\);
3. \(\text{Der}_{F}(R)\) is a finitely generated projective \(R\)-module of rank \(n\) such that \(R_m \otimes_R \text{Der}_{F}(R) = \text{Der}_{F}(R_m)\).

This hypothesis is inspired by the properties (i), (ii) and (iii) (1.1.2) in [MeNa]. There, \(R\) is a commutative Noetherian regular ring that contains a field, \(F\), of characteristic zero satisfying (1), (2), but instead of (3) in Hypothesis 2.2, there exist \(F\)-linear derivations \(\partial_1, \ldots, \partial_n \in \text{Der}_{F_0}(R)\) and \(a_1, \ldots, a_n \in R\) such that \(\partial_ia_j = 1\) if \(i = j\) and 0 otherwise. In our hypothesis, part (3) includes more rings; for instance, Remark 2.6 gives an example of a ring that satisfies Hypothesis 2.2 but not (1.1.2) in [MeNa]. However, when \(R\) a local ring the properties are the same by Theorem 2.1.

**Remark 2.3.** Every regular finitely generated algebra over the complex numbers, \(R\), satisfies Hypothesis 2.2. This is because, by Theorem 8.8 [La], \(\text{Der}_{C}(R) = \text{Hom}_R(\Omega_{R/C}, R)\) and \(\Omega_{R/C}\) is a projective module such that \(\text{rank}(\Omega_{R_m/C}) = \dim(R)\) for every maximal ideal \(m \subset R\).
Proposition 2.4. Let $R$ be a commutative Noetherian regular ring that contains a field, $F$, of characteristic zero satisfying (1), (2), and such that there exist $F$-linear derivations $\partial_1, \ldots, \partial_n \in \text{Der}_F(R)$ and $a_1, \ldots, a_n \in R$ such that $\partial_i a_j = 1$ if $i = j$ and 0 otherwise. Then, $R$ satisfies Hypothesis 2.2.

Proof. This follows from Theorem 2.1. \qed

A proof of Proposition 2.4 along with several consequences, is contained in Remark 2.2.5 in [MeNa].

Theorem 2.5. Let $S$ be a commutative Noetherian regular ring that contains a field, $F$, of characteristic zero satisfying Hypothesis 2.2. If there is an element $f \in S$ such that $R = S/fS$ is a regular ring, then $R$ satisfies Hypothesis 2.2.

Proof. We have that property (1) holds because $\dim S - 1 = \dim S_\eta - 1 = \dim R_m$ for every maximal ideal $m = \eta R \subset R$, where $\eta \subset S$ is a maximal ideal of $S$ containing $fS$. In addition, property (2) holds because every residual field of $R$ is a residual field of $S$.

We only need to prove property (3). Let $n = \dim(S)$. For every maximal ideal $\eta \subset S$ containing $fS$, we may pick a regular system of parameters, $y_1, \ldots, y_n$, for $S_\eta$ such that $y_1 = f$. Then, by Theorem 2.1, there exist $\delta_i \in \text{Der}_F(S_\eta)$ such that $\delta_i(y_j) = 1$ if $i = j$ and zero otherwise; moreover, $\text{Der}_F(S_\eta)$ is a free $S_\eta$-module of rank $n$ generated by $\delta_1, \ldots, \delta_n$.

Let $\varphi_f : \text{Der}_F(S) \to R$ be the morphism defined by $\varphi : \partial \to [\partial(f)]$, where $[\partial(f)]$ represents the class of $\partial(f)$ in $R$. Then, $S_\eta \otimes_R \text{Ker}(\varphi_f)$ is isomorphic to $\{ \delta \in \text{Der}_F(S_\eta) : \delta(f) \in f\cdot S_\eta \} = S_\eta \otimes_R \delta_1 \oplus \cdots \oplus S_\eta \delta_n$.

Noticing that $f \cdot \text{Der}_F(S) \subset \text{Ker}_f$, we define $N = \text{Ker}_f/\{ f \cdot \text{Der}_F(S) \}$ and point out that it is a finitely generated $R$-module. Let $m = \eta R$. Then, $R_m \otimes_R N = R_m \delta_2 \oplus \cdots \oplus R_m \delta_n = \text{Der}_F(R_m)$, where the last equality uses Theorem 2.1.

We have a morphism $\psi : N \to \text{Der}_F(R)$ defined by taking $\psi(\partial(r)) = [\partial(r)]$, which is well defined by the definition of $N$. For every maximal ideal $m \subset R$, there is a natural morphism $i_m : R_m \otimes_R \text{Der}_F(R) \to \text{Der}_F(R_m)$. We notice $(i_m \circ 1_{R_m} \otimes \psi)$ is an isomorphism between $R_m \otimes_R N$ and $\text{Der}_F(R_m)$ for all maximal $m \subset R$. Therefore, $N_m \cong R_m \otimes \text{Der}_F(R) \cong \text{Der}_F(R_m)$ for all maximal $m \subset R$. Hence, $\psi$ is an isomorphism. \qed

Remark 2.6. It is worth pointing out that there are examples were $R$ satisfies Hypothesis 2.2 but $\text{Der}_F(R)$ is not free. Let $S = \mathbb{R}[x, y, z]$ be the polynomial ring in three variables and coefficients over $\mathbb{R}$. Let $f = x^2 + y^2 + z^2 - 1 \in S$. Then, $R = S/fS$, the coordinate ring associated to the sphere, satisfies Hypothesis 2.2 but $\text{Der}_R(R)$ is not
free. Let \( \phi : R^3 \to R \) be the morphism given by \( (a, b, c) \to (ax, by, cz) \).
Thus, \( \text{Der}_R(R) = Ker(\phi) \) by the proof of Theorem 2.5. Therefore, 
\( \text{Der}_R(R) \) is the projective module corresponding to the tangent bundle 
of the sphere, and so it is not free. This example also shows that the conclusion of Theorem 2.5 does not hold for properties (i), (ii) and (iii) 
(1.1.2) in \( \text{MeNa} \). In that sense, Hypothesis 2.2 behaves better under 
regular subvarieties.

**Main Example.** Let \( (V, \pi V, K) \) be a DVR of mixed characteristic \( p > 0 \), and let \( F \) denote its fraction field. Let \( S = V[[x_1, \ldots, x_n]] \otimes_V F \) 
be the tensor product of the power series ring with coefficients in \( V \) and \( F \). Let \( R = S/(f)S \) be a regular ring where \( f = \pi - h \) for an element \( h \) in the square of maximal ideal of \( V[[x_1, \ldots, x_n]] \). Then, \( R \) satisfies Hypothesis 2.2

**Proof.** Since \( S \) is as in Proposition 2.4 \( \text{Ly2} \) pages 5880–5881) and 
\( \pi - h \in S \) is a regular element, we have that \( R \) satisfies Hypothesis 2.2 
by Theorem 2.5. \( \square \)

Let \( F \) be a field of characteristic 0 and \( R \) a commutative Noetherian 
ring with unity containing \( F \). We denote by \( D(R, F) \) the ring of \( F \)-
linear differential operators of \( R \). This is a subring of \( \text{Hom}_F(R, R) \) defined 
inductively as follows. The differential operators of order zero are 
the morphisms induced by multiplying by elements in \( R \). An element 
\( \theta \in \text{Hom}_F(R, R) \) is a differential operator of order less than or equal to 
\( j + 1 \) if \([\theta, r] := \theta \cdot r - r \cdot \theta \) is a differential operator of order less than or equal to \( j \). We have an induced filtration \( \Gamma = (\Gamma^j) \) on \( D(R, F) \) given by 
\( \Gamma^j = \{ \theta \in D(R, F) \mid \text{ord}(\theta) \leq j \} \). As a consequence of the definition, we have that \( \Gamma^j \Gamma^i \subset \Gamma^{j+i} \) and that \( \text{gr}^F(D(R, F)) = \oplus_{j=0}^{\infty} \Gamma^j / \Gamma^{j-1} \) 
is a commutative ring.

An example is given by a commutative Noetherian regular ring \( R 
\) with unity that contains a field, \( F \), of characteristic 0, as in Proposition 
2.4. In this case, \( D(R, F) = R[\partial_1, \ldots, \partial_n] \subset \text{Hom}_F(R, R) \); moreover, 
\( \text{gr}^F(D(R, F)) = R[y_1, \ldots, y_n] \) the polynomial ring with coefficients on 
\( R \) and variables \( y_1, \ldots, y_n \) and \( w.g.l.\text{dim}(D(R, F)) = \text{dim}(R) \) (cf. Main 
Theorem in \( \text{Bj2} \) (1.1.3) and Theorem 1.1.4 in \( \text{MeNa} \), and Theorem 
2.17 in \( \text{Na} \)). We would like to have similar properties for \( D(R, F) \) 
and \( \text{gr}^F(D(R, F)) \) when \( R \) satisfies Hypothesis 2.2.

We will denote by \( D \) the subalgebra of \( \text{Hom}_F(R, R) \) 
generated by \( R \) and \( \text{Der}_F(R) \), where \( R = \text{Hom}_F(R, R) \subset \text{Hom}_F(R, R) \). We define 
an ascending filtration \( \Gamma^j_D \) of \( R \)-modules in \( D \) inductively as follows. 
\( \Gamma^0_D = R \). Given \( \Gamma^j_D \), we take \( \Gamma^j_{D+1} \) as the Abelian additive group 
generated by \( \{ \Gamma^j_{D+1} : \text{Der}_F(R) \cdot \Gamma^j_D \} \). Since \( \Gamma^j_{D+1} \) is generated by multiplying derivations, we have that for every \( \delta \in \Gamma^j_D \) and \( f \in R \), \( [\delta, f] = f\delta - \delta f \in \Gamma^j_{D+1} \). Therefore, \( \Gamma^j_{D+1} \) is an \( R \)-submodule of \( D \) with respect to the structures 
induced by multiplication by the left or by the right. Additionally,
$D \subset D(R, F)$ and $\Gamma_j' \subset \Gamma_j$ because $\text{Der}_F(R) \subset \Gamma_1$.

We have that for every $s \in R$, $\text{Adj}_s : D(R, A) \to D(R, A)$, defined by $\text{Adj}_s(\delta) = s\delta - \delta s$, is nilpotent. Let $m \subset R$ be a maximal ideal and $S = R \setminus m$ be the induced multiplicative system. Then, $S$ is a multiplicative set satisfying the Ore condition on the left and on the right in $D(R, A)$ and, as a consequence, in $D$. Hence, $S^{-1}D(R, F)$ and $S^{-1}D$ exist as filtered rings.

**Proposition 2.7.** With the same notation as above, $D(R, F) = D$ as filtered rings.

**Proof.** Let $m \subset R$ be a maximal ideal and $S = R \setminus m$ be the induced multiplicative system. We have that $S^{-1}\Gamma_j = S^{-1}\Gamma_j'$ by condition (3) in Hypothesis 2.2. Therefore, $S^{-1}D = R_m[\Omega_{R_m, F}] = D(R_m, F) = S^{-1}D(R, F)$ as filtered rings. □

We recall the definition of a ring of differentiable type (cf. (1.1) in [MeNa]).

**Definition 2.8.** A filtered ring $A$ is a ring of differentiable type if its associated graded ring is commutative Noetherian regular with unity and pure graded dimension.

**Theorem 2.9.** $(D, \Gamma)$ is a ring of differentiable type such that $\text{gr}^F(D)$ is a ring of pure graded dimension $2n$.

**Proof.** Let $\text{gr}^F(D)$ be the associated graded ring. We will prove the proposition by parts.
gr^Γ(D) is commutative: This follows from the definition of the filtration Γ on D = D(R, F).

gr^Γ(D) is Noetherian: Let ∂_1, ..., ∂_m be a set of generators for Der_F(R).

Let φ : R[z_1, ..., z_m] → gr^Γ(D) be the morphism of commutative R-algebras defined by z_i → [∂_i]. We have, by the definition of Γ', that φ is surjective. Hence gr^Γ(D) is Noetherian.

gr^Γ(D) is regular: Let Q ⊂ gr^Γ(D) be a prime ideal and m ⊂ R be a maximal ideal that contains Q ∩ R. Then gr^Γ(D)_Q = (gr^Γ(D)_m)_Q which is regular because gr^Γ(D)_m is a polynomial ring over R.

gr^Γ(D) has pure graded dimension 2n: Let η be a maximal homogeneous ideal of gr^Γ(D). We claim that m = η ∩ R is a maximal ideal of R. If not, there exist a maximal ideal m' ⊂ R strictly containing m. Then, m' + η would be a proper ideal of gr^Γ(D) that strictly contains η. Hence, gr^Γ(D)_η is the localization of gr^Γ(D)_m at a maximal homogeneous ideal, then, dim(gr^Γ(D)_η) = 2n because gr^Γ(D)_m is a ring of pure graded dimension 2n.

Remark 2.10. Narváez-Macarro [Na] showed that if S is a ring containing a field, F, of characteristic 0 and Der_F(S) is a projective S-modules of finite rank, then gr(D(S, F)) ∼= Sym(Der_F(S)). Hence, we have that gr(D) ∼= Sym(Der_F(R)) by Hypothesis 2.2.

Corollary 2.11. D is left and right Noetherian.

Proof. This follows from Proposition 6.1 in [Bj1].

Proposition 2.12. w.gl.dim(D) = dim(R)

Proof. Since D is left and right Noetherian, w.gl.dim(D) = l.pd(D) = r.pd(D) by Theorem 8.27 in [Ro]. The value to this dimension is equal to the maximum integer j such that Ext^j_D(M, R) ≠ 0 for some finitely generated D-module M because D is of differentiable type. As R_m ⊗_R Ext^j_D(M, D) = Ext^n_{D_m}(M_m, D_m) = 0 for every maximal ideal m ⊂ R and integer j > n, we have that Ext^j_D(M, D) = 0 for every D-module M and for j > n. Hence, w.gl.dim(D) ≤ n. Likewise,

R_m ⊗_R Ext^j_D(R, D) = Ext^n_{D_m}(R_m, D_m) ≠ 0

for any m ⊂ R, so, Ext^j_D(R, D) ≠ 0. Hence w.gl.dim(D) ≥ n.

3 The theory of the Bernstein-Sato polynomial and the Bernstein class of D

Throughout this section we are adapting the results of Mebkhout and Narváez-Macarro to R and D [MeNa]. In particular, we show that
the existence of the Bernstein-Sato polynomial and that the Bernstein class of $D$ is closed under localization at one element.

**Definition 3.1.** Let $A$ be a ring of differentiable type. Let $M \neq 0$ be a finitely generated left or right $A$-module. We define

$$\text{grade}_A(M) = \inf\{j : \text{Ext}_A^j(M, A) \neq 0\}.$$

**Proposition 3.2.** Let $A$ be a ring of differentiable type. Let $M \neq 0$ be a finitely generated left or right $A$-module. Then,

$$\dim(M) + \text{grade}_A(M) = \dim(\text{gr}(A)) \geq \dim(gr(A)) - \text{w.gl.dim}(A).$$

In particular, $\dim(M) \geq \dim(\text{gr}(A)) - \text{w.gl.dim}(A)$. Moreover, we have that

$$\text{codim}_A(\text{Ext}_A^i(M, A)) \geq i$$

for all $i \geq 0$ such that $\text{codim}_A(\text{Ext}_A^i(M, A)) \neq 0$.

**Proof.** This is a generalized form of Theorem 7.1 of section 2 in [Bj1] given by Gabber [Ga]. The proposition is stated in this form in Mebkhout and Narváez-Macarro’s article as Theorem 1.2.2 [MeNa].

**Definition 3.3.** Let $A$ be a ring of differentiable type. Let $M$ be a finitely generated left or right $A$-module. We say that $M$ is in the left or right Bernstein class if it has minimal dimension, i.e. $\dim(M) = \dim(\text{gr}(A)) - \text{w.gl.dim}(A)$.

This class is closed under submodules, quotients and extensions. The functor that sends $M$ to $\text{Ext}_A^i(M, A)$ is an exact contravariant functor that interchanges the left Bernstein class and the right Bernstein class. Moreover, $M = \text{Ext}_A^i(M, A)$ naturally if $M$ is in either the left or the right Bernstein class, so that we have an anti-equivalence of categories. In consequence, the modules in the Bernstein class have finite length as $A$-modules because it is a left and right Noetherian ring [MeNa, Prop. 1.2.7].

**Proposition 3.4** (Prop. in 1.2.7 [MeNa]). Let $A$ be a ring of differentiable type and let $f$ be an element in $A_0$. Let $M$ be an $A_f$-module finitely generated, such that $\text{Ext}_A^i(M, A_f) = 0$ if $i \neq \text{w.gl.dim}(A)$. Then, there exists a submodule $M' \subset M$ over $A$ such that $M'$ is finitely generated with minimal dimension and $M'_f = M$.

Through the rest of this section, $F(s)$ denotes the fraction field of the polynomial ring $F[s]$ over the field $F$, and $D(s)$ denotes the ring $F(s) \otimes_F D$ with the filtration given by $F(s) \otimes_F \Gamma^i$. By $R(s)$, we mean the $F(s)$-algebra $F(s) \otimes_F R$. Similarly, $D[s]$ denotes $F[s] \otimes_F D$ and $R[s]$ denotes $F[s] \otimes_F R$. 

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Proposition 3.5. $R(s)$ is an $F(s)$-algebra equidimensional of dimension $\dim(R)$.

Proof. This is an immediate consequence of Theorem 2.1.1 in [MeNa].

Proposition 3.6. $D(s)$ is a ring of differentiable type with the filtration $F(s) \otimes F \Gamma$ such that $\text{gr}^{F(s) \otimes F \Gamma}(D(s))$ is a ring of pure graded dimension $2\dim(R)$.

Proof. Since $D$ is a ring of differentiable type, $\text{gr}^{F(s) \otimes F \Gamma}(D(s)) = F(s) \otimes F \Gamma[D(s)]$ is commutative, Noetherian and regular. For the sake of simplicity, we will omit the filtration. We claim that $\text{gr}(D(s))$ has pure graded dimension $2\dim(R) = 2n$. Let $\eta \subset \text{gr}(D(s))$ be a maximal homogeneous ideal and $P = \eta \cap R$. Let $m \subset R$ be a maximal ideal containing $P$. We have that the ideal $\eta_m$, induced by $\eta$, is a maximal homogeneous ideal of

$$(R \setminus m)^{-1}\text{gr}(D(s)) = F(s) \otimes F \Gamma(D_m) = (F(s) \otimes F R_m)[y_1, \ldots, y_n],$$

the polynomial ring with coefficients on $F(s) \otimes F R_m$ and variables $y_1, \ldots, y_n$. Then, $\text{ht}(\eta) = \text{ht}(\eta_m) = 2n$ because $F(s) \otimes F R_m$ is equidimensional of dimension $n$ by Theorem 2.1.4 in [MeNa].

Let $M$ be a left $D(s)$-module in the Bernstein class of $D(s)$. Let $N$ be a $D$-module in the Bernstein class of $D$ such that $F(s) \otimes F N = M$. For every $\ell \in F$, the $D$-module $M_\ell := N/(s-\ell)N$ is the Bernstein class of $D$.

Proposition 3.7. With the same notation as above, we have that

$$\dim_{D(s)}(M) \geq \dim_D(N_\ell),$$

for all but finitely many $\ell \in F$.

Proof. This is analogous to the proof of Theorem 2.2.1 in [MeNa].

Proposition 3.8. $\text{w.gl.dim}(D(s)) = \dim(R) = n$.

Proof. This is analogous to the proof of Theorem 2.2.3 in [MeNa].

Let $N[s]$ be the free $R_f[s]$-module generated by a symbol $f^s$ and let $N(s) = F(s) \otimes F N[s]$. We give to $N[s]$ (resp. $N(s)$) a structure of a left $D_f[s]$-module (resp. $D_f(s)$-module) as follows,

$$\partial_g f^s = (\partial_g + sg f^{-1}) f^s$$
for every \( \partial \in \text{Der}_F(R) \) and every \( g \in R_f[s] \) (resp. \( g \in R_f(s) \)). If \( M \) is a left \( D \)-module, we define \( M_f[s]F^* := M_f[s] \otimes_{R_f[s]} N[s] = M \otimes_R N[s] \) and \( M_f(s)F^* := N(s) \otimes_{R_f[s]} M_f(s) = M \otimes_R N(s) \). This is a left \( D_f[s] \)-module (\( D_f(s) \)-module), and clearly, \( M_f[s]F^* (M_f(s)F^*) \) is a finitely generated \( D_f[s] \)-module (\( D_f(s) \)-module) if \( M \) is.

**Proposition 3.9.** Let \( M \) be a left \( D \)-module in the Bernstein class and let \( u \in M \). Then, there exists a nonzero polynomial \( b(s) \in F[s] \) and an operator \( P(s) \in D[s] \) that satisfies the equation

\[
b(s)(u \otimes f^s) = P(s)f(u \otimes f^s)
\]

in \( M[s]F^* \).

**Proof.** This is analogous to 3.1.1 in \[MeNa\]. \( \square \)

**Corollary 3.10.** If \( M \) is a left \( D \)-module in the Bernstein class, the \( M_f \) is a finitely generated \( D \)-module.

**Proof.** For \( \ell \in \mathbb{Z} \), we define a morphism of specialization \( \phi_{\ell} : M_f[s]F^* \to M_f \) by \( \phi_{\ell}(us^\ell \otimes f^s) = \ell f^\ell u \), such that \( \phi_{\ell}(P(s)v) = P(\ell)v \). Then, by applying this morphism to the result of Proposition 3.9, we have \( b(\ell)f^\ell u = P(\ell)f^\ell u \) and the conclusion follows. \( \square \)

**Corollary 3.11.** Let \( M \) be a left \( D \)-module in the Bernstein class. Then, \( M_f \) is also in the Bernstein class for all \( f \in R \).

**Proof.** Since \( M_f \) is finitely generated as a \( D \)-module, it suffices to show that \( \dim_{gr(D)}(\text{gr}(M_f)) = n \). Since \( R_m \otimes_R M \) is in the Bernstein class of \( D_m \), we have that \( M_f \) is in the Bernstein class of \( D_m \) for every maximal ideal \( m \subset R \) by Theorem 2.3 in \[MeNa\]. Thus, \( \dim_{gr(D_m)}(\text{gr}((M_m)_f)) = n \) and, therefore \( \dim_{gr(D)}(\text{gr}(M_f)) = n \). \( \square \)

**Proof of Theorem 1.1** This is a consequence of Theorem 2.9 Proposition 2.12 and Corollary 3.11 \( \square \)

### 4 Local cohomology

Let us recall the family of functors introduced by Lyubeznik \[LY1\]. If \( Z \subset \text{Spec}(R) \) is a closed subset and \( M \) is an \( R \)-module, we denote by \( H^i_Z(M) \) the \( i \)-th local cohomology module of \( M \) with support in \( Z \). This can be calculated via the Čech complex as follows:

\[
0 \to M \to \oplus_i M_{f_i} \to \ldots \to \oplus_{i_1 \ldots f_t} M_{f_1 \ldots f_t} \to M_{f_1 \ldots f_t} \to 0
\]  \( (1) \)
where \( Z = \mathcal{V}(f_1, \ldots, f_\ell) = \{ P \in \text{Spec}(R) : (f_1, \ldots, f_\ell) \subset P \} \).

For two closed subsets of \( \text{Spec}(R) \), \( Z_1 \subset Z_2 \), there is a long exact sequence of functors. In particular, \( H^i_{Z_2}(M) = H^i_{Z_1}(M) \).

\[
\ldots \to H^i_{Z_1} \to H^i_{Z_2} \to H^i_{Z_1/Z_2} \to \ldots
\]

(2)

**Definition 4.1.** We say that \( \mathcal{T} \) is a Lyubeznik functor if has the form \( \mathcal{T} = \mathcal{T}_1 \circ \cdots \circ \mathcal{T}_t \), where every functor \( \mathcal{T}_j \) is either \( H^i_{Z_1} \), \( H^i_{Z_1/Z_2} \), or the kernel, image or cokernel of some arrow in the previous long exact sequence for closed subsets \( Z_1, Z_2 \) of \( \text{Spec}(R) \) such that \( Z_2 \subset Z_1 \).

**Lemma 4.2.** Let \( M \) be a left \( D \)-module in the Bernstein class. Then, \( \mathcal{T}(M) \) has a natural structure of \( D \)-module such that it belongs to the Bernstein class for every functor \( \mathcal{T} \) as in Definition 4.1.

**Proof.** \( M_f \) has a structure of \( D \)-module given by

\[
\partial \cdot m/f^\ell = (f^\ell \delta \cdot m - \delta(f^\ell m))/f^{2\ell}
\]

for every \( \delta \in \text{Der}_F(R) \). Then, \( \mathcal{T}(M) \) is a \( D \)-modules by Examples 2.1 in [Ly1]. Since \( M \) is in the Bernstein class, \( M_f \) is in the Bernstein class and \( M \rightarrow M_f \) is a morphism of \( D \)-modules by Corollary 3.11.

Since the Bernstein class is closed under extension, submodules and quotients, every element in the complexes (1) and (2) as well as the kernels, images and homology groups are in the same class, and the result follows.

**Lemma 4.3.** Let \( M \) be a left \( D \)-module in the Bernstein class. Then, \( \text{Ass}_R(M) \) is finite.

**Proof.** Suppose \( M \neq 0 \). Let \( M_1 = M \) and \( P_1 \) be a maximal element in the set of the associated primes of \( M_1 \). Then, \( N_1 = H^0_{P_1}(M_1) \) a nonzero \( D \)-submodule of \( M_1 \), and it has only one associated prime. Given \( N_j \) and \( M_j \), set \( M_{j+1} = M_j/N_j \). If \( M_{j+1} \neq 0 \), let \( P_{j+1} \) be a maximal element in the set of the associated primes of \( M_{j+1} \). Then \( N_j = H^0_{P_j}(M_{j+1}) \) has only one associated prime. If \( M_{j+1} = 0 \), set \( N_{j+1} = 0 \). Since \( M_1 = M \) has finite length as a \( D(R, A) \)-module, there exist \( \ell \in \mathbb{N} \) such that \( M_j = 0 \) for \( j \geq \ell \), and then \( \text{Ass}(M) \subset \{ P_1, \ldots, P_\ell \} \).

**Theorem 4.4.** Let \( R \) be a ring that satisfies Hypothesis 2.2 and let \( M \) be an \( D \)-module in its left Bernstein class. Then, \( \text{Ass}_R(\mathcal{T}(M)) \) is finite for every functor \( \mathcal{T}(-) \) as in Definition 4.1. In particular, this holds for \( H^i_{Z_1}(R) \) for every \( i \in \mathbb{N} \) and ideal \( I \subset R \).

**Proof.** This follows immediately from Lemmas 1.2 and 1.3.
**Corollary 4.5.** Let $(R, m, K)$ be a regular local ring of mixed characteristic $p > 0$. Then, the set of associated primes of $\mathcal{I}(R)$ that does not contain $p$ is finite for every functor $\mathcal{I}$ as in Definition 4.1.

**Proof.** Let $\hat{R}$ be the completion of $R$ with respect to the maximal ideal. Then, the set of associated primes of $\mathcal{I}(R)$ that does not contain $p$ is finite if the set of associated primes of $\mathcal{I}(\hat{R}) = \hat{R} \otimes_R \mathcal{I}(R)$ that does not contain $p$ is finite. We can assume without loss of generality that $R$ is complete. Thus, $R = V[[x_1, \ldots, x_{n+1}]]/(p - h)V[[x_1, \ldots, x_{n+1}]]$ where $(V, pV, K)$ is a DVR of unramified mixed characteristic $p > 0$ and $h$ is an element in the square of maximal ideal of $V[[x_1, \ldots, x_{n+1}]]$ by Cohen Structure Theorems. Let $F$ be the fraction field of $V$. It suffices to show that $\text{Ass}_R(F \otimes_V \mathcal{I}(R)) = \text{Ass}_R(\mathcal{I}(F \otimes_V R)$ is finite, which follows from our main example and Theorem 4.4.

**Proof of Theorem 1.2.** This is an immediate consequence of Corollary 4.5.

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