Breaking the degeneracy barrier for coloring graphs with no $K_t$ minor

Sergey Norin*  
Zi-Xia Song†

October 22, 2019

Abstract

Kostochka and Thomason in the 1980s independently proved that every graph with no $K_t$ minor has average degree $O(t \sqrt{\log t})$, which implies that every such graph is $O(t \sqrt{\log t})$-colorable. We show that every graph with no $K_t$ minor is $O(t \log t^{0.354})$-colorable, making the first improvement on the order of magnitude of the Kostochka-Thomason bound.

1 Introduction

All graphs in this paper are finite and simple. Given graphs $H$ and $G$, we say that $G$ has an $H$ minor if a graph isomorphic to $H$ can be obtained from a subgraph of $G$ by contracting edges. We denote the complete graph on $t$ vertices by $K_t$.

In 1943 Hadwiger made the following famous conjecture.

Conjecture 1.1 (Hadwiger’s conjecture [Had43]). For every integer $t \geq 0$, every graph with no $K_{t+1}$ minor is $t$-colorable.

---

*Department of Mathematics and Statistics, McGill University. Email: sergey.norin@mcgill.ca. Supported by an NSERC grant.
†Department of Mathematics, University of Central Florida, Orlando, FL 32816, USA. Email: Zixia.Song@ucf.edu. Supported by the National Science Foundation under Grant No. DMS-1854903.
Hadwiger’s conjecture is widely considered among the most important problems in graph theory and has motivated numerous developments in graph coloring and graph minor theory. We briefly overview major progress towards the conjecture below, and refer the reader to a recent survey by Seymour [Sey16] for further background.

Hadwiger [Had43] has shown that Conjecture 1.1 holds for \( t \leq 3 \). Wagner [Wag37] proved that for \( t = 4 \) the conjecture is equivalent to the Four Color Theorem, which was subsequently proved by Appel and Haken [AH77] using extensive computer assistance. Robertson, Seymour and Thomas [RST93] went one step further and proved Hadwiger’s conjecture for \( t = 5 \), also by reducing it to the Four Color Theorem. Setting the conjecture for \( t \geq 6 \) appears to be extremely challenging, perhaps in part due to absence of a transparent proof of the Four Color Theorem.

Another notable challenging case of Hadwiger’s conjecture is the case of graphs with no independence set of size three. If \( G \) is such a graph on \( n \) vertices then properly coloring \( G \) requires at least \( n/2 \) colors, and so Hadwiger’s conjecture implies that \( G \) has a \( K_{\lceil n/2 \rceil} \) minor. This is still open. In fact, as mentioned in [Sey16], it is not known whether there exists any \( c > 1/3 \) such that every graph \( G \) as above has a \( K_t \) minor for some \( t \geq cn \).

The following natural weakening of Hadwiger’s conjecture, which has been considered by several researchers, sidesteps the above challenges.

**Conjecture 1.2** (Linear Hadwiger’s conjecture [RS98, Kaw07, KM07a]). There exists \( C > 0 \) such that for every integer \( t \geq 1 \), every graph with no \( K_t \) minor is \( Ct \)-colorable.

In this paper we take a step towards Conjecture 1.2 by improving for largest \( t \) the upper bound on the number of colors needed to color graphs with no \( K_t \) minor. Prior to our work, the best bound was \( O(t\sqrt{\log t}) \). It has been obtained independently by Kostochka [Kos82, Kos84] and Thomason [Tho84] in the 1980s. The only improvement [Tho01, Woo13, KP19] since then has been in the constant factor.

The results of [Kos82, Kos84, Tho84] bound the “degeneracy” of graphs with no \( K_t \) minor. Recall that a graph \( G \) is \( d \)-degenerate if every non-null subgraph of \( G \) contains a vertex of degree at most \( d \). A standard inductive argument shows that every \( d \)-degenerate graph is \((d + 1)\)-colorable. Thus the following bound on the degeneracy of graphs with no \( K_t \) minor gives a corresponding bound on their chromatic number.
Theorem 1.3 ([Kos82, Kos84, Tho84]). Every graph with no $K_t$ minor is $O(t^{\sqrt{\log t}})$-degenerate.

Kostochka [Kos82, Kos84] and de la Vega [FdlV83] have shown that there exist graphs with no $K_t$ minor and minimum degree $\Omega(t^{\sqrt{\log t}})$. Thus the bound in Theorem 1.3 is tight, and it is natural to consider the possibility that coloring graphs with no $K_t$ minor requires $\Omega(t^{\sqrt{\log t}})$ colors. In fact, Reed and Seymour [RS98] refer to this assertion as “a commonly expressed counter-conjecture” to Conjecture 1.1.

We disprove the above “counter-conjecture” by proving the following main result.

Theorem 1.4. For every $\beta > \frac{1}{2} - \frac{\log(3/2)}{4\log 2} = 0.3537\ldots$, every graph with no $K_t$ minor is $O(t(\log t)^\beta)$-colorable.

The proof of Theorem 1.4 occupies the rest of the paper. In Section 2 we outline our proof and introduce the necessary tools. In Section 3 we adapt an argument of Thomason [Tho01] to show that every sufficiently well-connected graph containing a large number of disjoint dense subgraphs has a $K_t$ minor. In Section 4 we finish the proof of Theorem 1.4 by using a density increment argument to find a relatively small and dense subgraph in a graph with no $K_t$ minor. We end the paper with the concluding remarks given in Section 5.

Notation

We use largely standard graph-theoretical notation. We denote by $v(G)$ and $e(G)$ the number of vertices and edges of a graph $G$, respectively, and denote by $d(G) = e(G)/v(G)$ the density of a non-null graph $G$. We use $\chi(G)$ to denote the chromatic number of $G$, and $\kappa(G)$ to denote the (vertex) connectivity of $G$. We write $H \prec G$ if $G$ has an $H$ minor. We denote by $G[X]$ the subgraph of $G$ induced by a set $X \subseteq V(G)$.

For a positive integer $n$, let $[n]$ denote $\{1, 2, \ldots, n\}$. The logarithms in the paper are natural unless specified otherwise.

We say that vertex-disjoint subgraphs $H$ and $H'$ of a graph $G$ are adjacent if there exists an edge of $G$ with one end in $V(H)$ and the other in $V(H')$, and $H$ and $H'$ are non-adjacent, otherwise.

A collection $\mathcal{X} = \{X_1, X_2, \ldots, X_h\}$ of pairwise disjoint subsets of $V(G)$ is a model of a graph $H$ in a graph $G$ if $G[X_i]$ is connected for every $i \in [h]$, and there exists a bijection $\phi : V(H) \to [h]$, such that $G[X_{\phi(u)}]$ and $G[X_{\phi(v)}]$
are adjacent for every $uv \in E(H)$. It is well-known and not hard to see that $G$ has an $H$ minor if and only if there exists a model of $H$ in $G$. We say that a model $\mathcal{X}$ as above is $S$-rooted for $S \subseteq V(G)$ if $|S| = h$ and $|X_i \cap S| = 1$ for every $i \in [h]$.

2 Outline of the proof

The proof of Theorem 1.4 uses different methods depending on the magnitude of $v(G)$. In the case when $v(G)$ is small our argument is based on the following classical bound due to Duchet and Meyniel [DM82] on the independence number of graphs with no $K_t$ minor.

**Theorem 2.1** ([DM82]). Every graph $G$ with no $K_t$ minor has an independent set of size at least $v(G)/(2(t-1))$.

Theorem 2.1 implies that every graph with no $K_t$ minor contains a $t$-colorable subgraph on a constant proportion of vertices. Seymour [Sey16] observed that the proof of Theorem 2.1 in [DM82] yields the following stronger result.

**Theorem 2.2** ([Sey16]). Let $G$ be a graph with no $K_t$ minor. Then there exists $X \subseteq V(G)$ with $|X| \geq v(G)/2$ such that $\chi(G[X]) \leq t - 1$.

Theorem 2.2 straightforwardly implies the following bound on the chromatic number of graphs with no $K_t$ minor.

**Corollary 2.3.** Let $G$ be a graph with no $K_t$ minor then

$$\chi(G) \leq (\log_2(v(G)/t) + 2)t.$$  

**Proof.** By Theorem 2.2 for every integer $s \geq 0$ there exist disjoint $X_1, X_2, \ldots, X_s \subseteq V(G)$ such that $|V(G) - \cup_{i=1}^s X_i| \leq v(G)/2^s$ and $\chi(G[X_i]) \leq t$. Let $s = \lceil \log_2(v(G)/t) \rceil$, then $v(G)/2^s \leq t$, and so $\chi(G \setminus (\cup_{i=1}^s X_i)) \leq t$. It follows that $\chi(G) \leq t + \sum_{i=1}^s \chi(G[X_i]) \leq (s + 1)t$, implying the corollary.

By Corollary 2.3 we may assume that $v(G)$ is large. Given a graph $G$ with $\chi(G) = \Omega(t(\log t)^\beta)$, where $\beta$ is as in Theorem 1.4 we find a $K_t$ minor in $G$ by adapting the following strategy employed by Thomason [Tho01]: We construct a large collection of disjoint dense subgraphs $H_1, \ldots, H_r$ of $G$. 

find a model of a smaller complete graph in each $H_i$, and link these models together to build a model of $K_t$.

In the first step of implementing this strategy we construct the small dense subgraphs $H_1, \ldots, H_r$ of $G$, one by one. Finding one such subgraph (for appropriate notion of “dense” and “small”) is the most challenging part of the proof, occupying Section 4 where we employ a density increment argument. That is, we prove that either we can find the required subgraph, or we can replace $G$ by a minor of significantly higher density. Applying this result repeatedly we either obtain a graph dense enough that it necessarily has a $K_t$ minor, or find the required subgraph after a small number of replacements. In the last case the resulting subgraph can be traced back to a suitable subgraph of the original graph.

Assuming that $H_1, \ldots, H_r$ are found we need to implement the rest of the strategy. This is done in Section 3 using the following tools.

Theorem 1.3 provides a bound on density of a graph that guarantees existence of a complete minor of a given size, which is optimum up to a constant factor. We use the following explicit form of Theorem 1.3 from [Kos82] to guarantee existence of the models of small complete graphs in $H_i$’s.

**Theorem 2.4 ([Kos82]).** Let $s \geq 2$ be an integer. Then every graph $G$ with $d(G) \geq 3.2s\sqrt{\log s}$ has a $K_s$ minor.

Linking the resulting models requires a bound on the connectivity of $G$. We say that a graph $G$ is contraction-critical if $\chi(H) < \chi(G)$ for every $H \prec G$ such that $H$ is not isomorphic to $G$. Clearly we may assume that $G$ is contraction-critical. This allows us to use the connectivity bound established by Kawarabayshi [Kaw07].

**Theorem 2.5 ([Kaw07]).** Let $G$ be a contraction-critical graph with $\chi(G) \geq k$. Then $\kappa(G) \geq 2k/27$.

Additionally, we need each $H_i$ to be not only dense, but highly-connected. This is not hard to guarantee using a classical result of Mader [Mad72] which ensures that every dense graph contains a highly-connected subgraph.

**Lemma 2.6 ([Mad72]).** Every graph $G$ contains a subgraph $G'$ such that $\kappa(G') \geq d(G)/2$.

The final technical part of our argument involves linking the models we constructed. To accomplish this we employ a toolkit introduced by Bollobás and Thomason [BT96] for finding rooted models in highly connected graphs.
Lemma 2.7 ([BT96]). Let $G$ be a graph $G$ with $d = d(G) \geq 3$. Then $G$ has a minor $H$ such that $v(H) \leq d + 2$ and $2\delta(H) \geq v(H) + 0.3d - 2$.

Lemma 2.8 ([BT96]). Let $n \geq 0, k \geq 2$ and $h \geq n + 3k/2$ be integers. Let $G$ be a graph with $\kappa(G) \geq k$ containing vertex-disjoint non-empty connected subgraphs $C_1, \ldots, C_h$ such that each of them is non-adjacent to at most $n$ others. Let $S = \{s_1, \ldots, s_k\} \subseteq V(G)$. Then $G$ contains vertex-disjoint non-empty connected subgraphs $D_1, \ldots, D_m$ where $m = h - \lfloor k/2 \rfloor$, such that $s_i \in V(D_i)$ for each $i \in [k]$ and every element of $\{D_1, \ldots, D_m\}$ is non-adjacent to at most $n$ subgraphs among $D_{k+1}, \ldots, D_m$.

It is worth noting that Lemma 2.8 corresponds to [BT96, Lemma 2], where the last condition is only stated for subgraphs $D_1, \ldots, D_k$, but the family $D_1, \ldots, D_m$ constructed in the proof has the stronger condition claimed in Lemma 2.8.

In addition to the above lemmas, we also use one of the main results of [BT96].

Theorem 2.9 ([BT96]). There exists $C = C_{2.9} > 0$ satisfying the following. Let $s$ be a positive integer, let $G$ be a graph with $\kappa(G) \geq Cs$, and let $S_1, S_2, \ldots, S_k$ be non-empty disjoint subsets of $V(G)$ such that $\sum_{i=1}^{k} |S_i| \leq s$. Then there exist vertex-disjoint connected subgraphs $C_1, \ldots, C_k$ of $G$ such that $S_i \subseteq V(C_i)$ for every $i \in [k]$.

The value of $C_{2.9}$ is not explicitly given in [BT96], but it is not hard to see that $C_{2.9} = 22$ suffices. Thomas and Wollan [TW05] improve the bounds from [BT96], and the results of [TW05] directly imply that $C_{2.9} = 10$ satisfies Theorem 2.9. The exact value of $C_{2.9}$ does not substantially affect our bounds.

3 Building a $K_t$ minor

We use the following helpful technical definition throughout our proof of Theorem 1.4. We say that a pair of real numbers $(n, d)$ is $(D, t)$-forced if every graph $G$ with $d(G) \geq D$ and no $K_t$ minor has a subgraph $H$ with $v(H) \leq n$ and $d(H) \geq d$.

In this section we implement part of the strategy presented in Section 2 and prove the following.
Theorem 3.1. There exists $C = C_{3.1}$ satisfying the following. Let $t$ be a positive integer, and let $k, n$ be real numbers such that

$$k > Ct \cdot \max \left\{ \log \left( \frac{2n \sqrt{\log t}}{t} \right), (\log t)^{1/4} \right\}$$

(1)

and $(n, Ct(\log t)^{1/4})$ is a $(k/4, t)$-forced pair. Then every graph $G$ with $\chi(G) \geq k$ has a $K_t$ minor.

In Section 4 we finish the proof of Theorem 1.4 by finding forced pairs with parameters sufficient to derive Theorem 1.4 from Theorem 3.1.

We start with a lemma which will be used to construct the pieces of our model of $K_t$. Given a collection $S = \{(s_i, t_i)\}_{i \in \mathcal{I}}$ of pairs of vertices of a graph $G$ an $S$-linkage $\mathcal{P}$ is a collection of vertex-disjoint paths $\{P_1, \ldots, P_l\}$ in $G$ such that $P_i$ has ends $s_i$ and $t_i$ for every $i \in \mathcal{I}$.

Lemma 3.2. There exists $C = C_{3.2} > 0$ satisfying the following. Let $G$ be a graph, let $l \geq s \geq 2$ be positive integers. Let $s_1, \ldots, s_l, t_1, \ldots, t_l, r_1, \ldots, r_s \in V(G)$ be distinct, except possibly $s_i = t_i$ for some $i \in \mathcal{I}$. If

$$\kappa(G) \geq C \max\{l, s \log s\}$$

then there exists a $K_s$ model $\mathcal{M}$ in $G$ rooted at $\{r_1, \ldots, r_s\}$ and an $\{(s_i, t_i)\}_{i \in \mathcal{I}}$-linkage $\mathcal{P}$ in $G$ such that $\mathcal{M}$ and $\mathcal{P}$ are vertex-disjoint.

Proof. Our choice of $C$ will be implicit, i.e. we assume that it is chosen to satisfy the inequalities appearing throughout the proof.

Let $d = d(G) \geq \kappa(G)/2 \geq Cl/2$. By Lemma 2.7 there exists a model $\mathcal{H}$ of a graph $H$ in $G$ such that $v(H) \leq d + 2$, and every vertex in $H$ has at most $v(H)/2 - d/10$ non-neighbors.

Let $h = v(H)$, $n = h/2 - d/10$, and let $k = 2l + s \leq 3l \leq d/150$. (The last inequality assumes $C \geq 900$.) We have $h \geq n + 3k/2$, and so by Lemma 2.8 applied to the elements of $\mathcal{H}$ and $S = \{s_1, \ldots, s_l, t_1, \ldots, t_l, r_1, \ldots, r_s\}$, there exist a collection $\mathcal{D} = \{D_1, \ldots, D_m\}$ vertex-disjoint non-empty connected subgraphs $D_1, \ldots, D_m$ of $G$ where $m = h - \lceil k/2 \rceil$, such that $D_1, \ldots, D_k$ each contain exactly one vertex from $S$, and every element of $\mathcal{D}$ is non-adjacent to at most $n$ subgraphs in $\mathcal{D}' := \{D_{k+1}, \ldots, D_m\}$. We may assume that $r_i \in V(D_i)$ for every $i \in [s]$. As $|\mathcal{D}'| \geq h - 3k/2$, every two elements of $\mathcal{D}$ have at least $|\mathcal{D}'| - 2n - 2 \geq d/5 - 3k - 2 \geq d/6$ common neighbors in $\mathcal{D}'$. 

\[ \Box \]
Let $\mathcal{D}' \subseteq \mathcal{D}$ be chosen by selecting each element of $\mathcal{D}$ independently at random with probability $1/2$. Then by Chernoff bound the probability that a given pair of elements of $\mathcal{D}$ have fewer than $d/24$ common neighbors in $\mathcal{D}'$ is at most $e^{-d/100}$. For sufficiently large $C$ we have $(d+2)^2e^{-d/100} < 1/2$, and thus by linearity of expectation there exists $\mathcal{D}' \subseteq \mathcal{D}$ such that $|D''| \leq h/2$ and every pair of elements of $\mathcal{D}$ have at least $d/24 \geq k$ common neighbors in $\mathcal{D}'$. Let $\mathcal{M}' = \mathcal{D}' - \mathcal{D}'$, then $|\mathcal{M}'| \geq h/2 - 3k/2$. As every element of $\mathcal{M}'$ is non-adjacent to at most $n$ other elements of $\mathcal{M}'$, it is adjacent to at least

$$|\mathcal{M}'| - n \geq \frac{d}{10} - \frac{3k}{2} \geq \frac{d}{12} \geq \frac{C}{24}\sqrt{\log s}$$

others. By Theorem 2.4, we can find a model $\mathcal{M}' = \{M'_1, \ldots, M'_s\}$ of $K_s$ in $G$, such that elements of $\mathcal{M}'$ are unions of vertex sets of elements of $\mathcal{M}'$. By the choice of $\mathcal{D}'$, there exists $\{D'_1, \ldots, D'_s\} \subseteq \mathcal{D}'$ such that $D'_i$ is adjacent to $D_i$ and $G[M'_i]$ for every $i \in [s]$. Let $\mathcal{M} = \{M_i \cup V(D_i) \cup V(D'_i)\}_{i \in [s]}$, then $\mathcal{M}$ is a model of $K_s$ in $G$, rooted at $\{r_1, \ldots, r_s\}$. Similarly, using $l$ elements of $\mathcal{D}'$ - $\{D'_1, \ldots, D'_s\}$, we find an $\{(s_i, t_i)\}_{i \in [l]}$-linkage $\mathcal{P}$ in $G$, such that $\mathcal{P}$ is vertex-disjoint from $\mathcal{M}$, as desired. $\blacksquare$

In the next lemma we build a model of $K_t$ from the pieces constructed in Lemma 3.2.

**Lemma 3.3.** There exists $C = C[3.3] > 1$ satisfying the following. Let $G$ be a graph with $\kappa(G) \geq Ct(\log t)^{1/4}$, and let $r \geq \sqrt{\log t}/2$ be an integer. If there exist pairwise disjoint $X_1, X_2, \ldots, X_r \subseteq V(G)$ such that $d(G[X_i]) \geq Ct(\log t)^{1/4}$ for every $i \in [r]$ then $G$ has a $K_t$ minor.

**Proof.** Once again we will choose $C$ implicitly, sufficiently large with respect to $C[2.9]$ and $C[3.2]$.

By Lemma 2.9, replacing each $X_i$ by a subset, we may assume that

$$\kappa(G[X_i]) \geq \frac{C}{2} t(\log t)^{1/4},$$

instead of $d(G[X_i]) \geq Ct(\log t)^{1/4}$. Let $y = \lfloor (\log t)/4 \rfloor$, and let $x = \lfloor t/y \rfloor$, thus $xy \geq t$ and it suffices to show that $G \supset K_{xy}$. We reindex the sets $X_1, \ldots, X_{\lfloor y \rfloor + 1}$ to $X_0$ and $\{X_{i,j}\}_{i, j \in [y]}$. By choosing $C$ appropriately, we may assume $\kappa(G) \geq xy(y-1)$. Thus there exist vertex disjoint linkages $Q_{(i,j)}$ for all $i, j \in [y]$, $i \neq j$, such that each $Q_{(i,j)}$ consists of $x$ paths $Q^1_{(i,j)}, \ldots, Q^x_{(i,j)}$.
each starting in \( X_{i,j} \), ending in \( X_0 \) and otherwise disjoint from \( X_{i,j} \cup X_0 \). Let \( Q = \bigcup_{i,j \in [y], i \neq j} Q_{i,j} \).

We now apply Lemma 3.2 consecutively to each of subgraph \( H = G[X_{i,j}] \) with \( s = 2x \) and \( l \leq xy(y-1) - 2x \) equal to the number of paths in \( Q - Q_{i,j} - Q_{j,i} \) which intersect \( H \). The vertices \( \{(s_i, t_i)\}_{i \in [l]} \) are then chosen to be the first and last vertex of these paths in \( H \), while the vertices \( r_1, r_2, \ldots, r_s \) are the ends of the paths \( Q_{i,j} \cup Q_{j,i} \) in \( H \). By using the linkage \( \mathcal{P} \) given by Lemma 3.2 to reroute the paths in \( Q - Q_{i,j} - Q_{j,i} \) within \( H \), we may assume that \( H \) contains a \( K_{2x} \) model \( M_{i,j} \) rooted at \( \{r_1, r_2, \ldots, r_s\} \subseteq V(Q_{i,j}) \cup V(Q_{j,i}) \), which is otherwise disjoint from \( Q \).

Finally we need to join the ends of paths in \( Q \) in \( X_0 \). By Theorem 2.9 there exist vertex-disjoint connected subgraphs \( \{C_a\}_{a \in [x]} \) of \( G[X_0] \) such that \( V(C_a) \) contains the ends of paths \( Q_{a,i,j} \) for all \( j \in [y] - \{i\} \), and is otherwise disjoint from \( Q \). These \( xy \) connected subgraphs together with the paths of \( Q \) ending in them, and the elements of the \( K_{2x} \) models containing the second ends of these paths now form the elements of a \( K_{xy} \) model in \( G \), as desired.

We end this section with the proof of Theorem 3.1 using Corollary 2.3 to show that Lemma 3.3 is applicable.

**Proof of Theorem 3.1.** We show that \( C = 14C_3 \) satisfies the theorem.

Suppose for a contradiction that there exists a graph \( G \) with \( \chi(G) \geq k \) and no \( K_t \) minor. We may choose such \( G \) to be contraction-critical. Thus, by Theorem 2.5 and the choice of \( C_3 \), we have

\[
\kappa(G) \geq \frac{2k}{27} \geq C_3 \frac{t \log t}{t^{1/4}}.
\]

Let \( X_1, X_2, \ldots, X_r \subseteq V(G) \) be a maximal collection of pairwise disjoint subsets of \( V(G) \) such that \( |X_i| \leq n \) and \( d(G[X_i]) \geq Ct \log t / t^{1/4} \).

By Lemma 3.3 we have \( r \leq \sqrt{\log t / 2} \). Let \( X = \bigcup_{i=1}^r X_i \). Then \( |X| \leq n \sqrt{\log t / 2} \). By Corollary 2.3

\[
\chi(G[X]) \leq \frac{t \log (2n \sqrt{\log t / t})}{\log 2}.
\]

As \((n, d)\) is \((k/4, t)\)-forced, the maximality of the collection we have chosen above implies that the graph \( G \setminus X \) contains no subgraph \( G' \) with \( d(G') \geq k/4 \). It follows that \( \chi(G \setminus X) \leq k/2 \), and so \( \chi(G[X]) \geq k/2 \), which contradicts \( (1) \) by \( (1) \), as \( C \geq 2 / \log 2 \). \( \square \)
4 Density increment

In this section we deduce Theorem 1.4 from Theorem 3.1. The main ingredient of our proof is Theorem 4.1. Its statement requires one additional definition. We say that a graph $H$ is a $k$-bounded minor of a graph $G$ if there exists a model $X$ of $H$ in $G$ such that $|X| \leq k$ for every $X \in \mathcal{X}$. That is, $H$ can be obtained from a subgraph of $G$ by contracting sets of size at most $k$.

**Theorem 4.1.** Let $0 < \varepsilon < 1$, $K > 1$ be real. Let $G$ be a graph with $d = d(G) \geq 2/\varepsilon$. Then $G$ contains either

(i) a subgraph $H$ of $G$ with $v(H) \leq 4Kd$ and $e(H) \geq \varepsilon^2d^2/2$, or

(ii) a 2-bounded minor $H$ with $d(H) \geq \frac{3}{2}\frac{K(1-4\varepsilon)}{K+3}d$, or

(iii) a 3-bounded minor $H$ with $d(H) \geq 2\frac{K(1-10\varepsilon)}{K+4}d$.

**Proof.** We assume without loss of generality that $d(G') < d$ for every proper subgraph $G'$ of $G$. In particular, $\deg(v) \geq d$ for every $v \in V(G)$. We say that $v \in V(G)$ is big if $\deg(v) \geq Kd$, and small, otherwise. Let $A$ denote the set of all small vertices of $G$, and let $B = V(G) - A$ be the set of big vertices. We say that two distinct small vertices $u, v$ are mates if $u$ and $v$ have at least $\varepsilon d$ common neighbors.

Assume first that some $u \in A$ has at least $\varepsilon d$ mates. Let $X \subseteq V(G)$ consist of $\lceil \varepsilon d \rceil$ mates of $u$ and all neighbors of $u$, and let $H = G[X]$ then $H$ satisfies (i). Thus we may assume that

(C1) Every vertex in $A$ has at most $\varepsilon d$ mates.

We now define the main technical concept considered in the proof. We say that $\mathcal{S} = \{T_1, T_2, \ldots, T_m\}$ is a shrubbery in $G$ if

- $T_i \subseteq G$ is a tree with $|V(T_i)| \in \{2, 3\}$ for all $i \in [m]$,
- $V(T_i) \cap V(T_j) = \emptyset$ for all $i, j \in [m]$ with $i \neq j$,
- no two vertices of $T_i$ are mates for all $i \in [m]$.
The proof is divided into two major cases depending on whether there exists a shrubbery $S$ in $G$ such that $A \subseteq V(S)$. Suppose first such a shrubbery exists.

Let $n = |V(G)|$, then $|B| \leq 2n/K$. Let $S' \subseteq S$ consist of all trees $T \in S$ such that $V(T) \subseteq A$. As $|V(S')| \geq n - 3|B|$, we have $|S'| \geq n/3 - 2n/K$. Let a matching $M \subseteq E(G)$ be obtained by selecting one edge from each tree in $S'$, let $m = |M|$ and $W = V(M)$. Thus $m = |S'| \geq n/3 - 2n/K$, $W \subseteq A$, and the ends of every edge $uv \in M$ are not mates. By deleting at most $2\varepsilon dm$ edges of $G$ we obtain a subgraph $G'$ of $G$ such that no pair of adjacent vertices in $G'[W]$ are mates. We say that a pair of edges $\{e_1, e_2\} \in M$ is an $M$-bad pair if there exists a cycle $C$ of length four in $G'$ with $e_1, e_2 \in E(C)$.

We may further assume that $M$ is chosen among all perfect matchings in $G'[W]$ so that $M$ has the minimum number of bad pairs.

Suppose that there exist $b \geq \varepsilon d$ and distinct $u_0v_0, u_1v_1, \ldots, u_bv_b \in M$ such that $\{u_0v_0, u_iv_i\}$ is a bad pair for every $i \in [b]$. We may further assume that $u_0u_i, v_0v_i \in E(G)$ for every $i \in [b]$. Let $M'_i$ be obtained from $M$ by replacing the edges $u_0v_0$ and $u_iv_i$ with edges $u_0u_i$ and $v_0v_i$. By the choice of $M$ the number of $M'_i$-bad pairs is at least as large as the number of $M$-bad pairs, and so the edges $u_0u_i$ and $v_0v_i$ are involved in at least $b$ $M_i$-bad pairs. Let $X_0$ be the vertex set of all edges $e \in M$ such that some end of $e$ is adjacent to $u_0$ or $v_0$. As $u_0$ and $v_0$ are small we have $|X_0| \leq 4Kd$. However by observation above $G[X_0]$ has at least $b$ edges incident to either $u_i$ or $v_i$. It follows that $e(G[X_0]) \geq b^2/2 \geq \varepsilon^2d^2/2$, and so (i) holds.

Thus we may assume that every edge of $M$ belongs to at most $\varepsilon d$ bad pairs, and so there are at most $\varepsilon dm/2$ bad pairs in total.

Let $H$ be a minor of $G'$ obtained by contracting all edges of $M$. It is not hard to verify that $e(G) - e(H) \leq 4\varepsilon dn$. Thus $e(H) \geq (1 - 4\varepsilon)dn$, while $v(H) = n - m \leq 2n/3 + 2n/K$. Thus $d(H) \geq \frac{3K(1-4\varepsilon)}{2(K+3)}d$ and (ii) holds.

It remains to consider the case when no shrubbery covers $A$. Let a shrubbery $S$ in $G$ be chosen so that $|A \cap V(S)|$ is maximum, and subject to this $|B \cap V(S)|$ is minimum.

For a tree $T \in S$ we say that $v \in V(T)$ is the center of $T$ if $v$ has degree two in $T$ (and so $|V(T)| = 3$), and we say that $v \in V(G)$ is a center of $S$ if $v$ is the center of some tree in $S$.

It follows from the choice of $S$, that if $u \in B \cap V(T)$ for some $T \in S$ with $|V(T)| = 3$ then $u$ is the center of $T$. We say that a path $P$ in $G$ with vertices $(v_0, u_1, v_1, \ldots, u_{l-1}, v_{l-1}, u_l)$ in order is an $S$-alternating $(v_0, u_l)$-path.
if

• $v_0 \in A - V(S)$,

• for every $i \in [l-1]$ there exists $T_i \in S$ such that $u_i$ is the center of $T_i$ and $v_i \in V(T_i)$,

• for every $i \in [l-1]$ the vertex $v_{i-1}$ has no mate in $V(T_i)$,

• $v_{l-1}$ is not a mate of $u_l$, and, if $u_l \in V(T)$ for some $T \in S$, then $v_{l-1}$ has no mate in $V(T)$.

• Furthermore, $u_l \not\in V(T_i)$ for any $i \in [l-1]$.

We claim that

$$(C2) \text{ Every } S\text{-alternating path ends in a center of } S.$$ 

Suppose not. Let $P$ be an $S$-alternating $(v_0, u_l)$-path with vertices labelled as in the definition above such that $u_l$ is not a center of $S$. We construct a shrubbery $S'$ from $S$ with

$$V(S') \cap A = (V(S) \cap A) \cup \{v_0\},$$

(3)

contradicting the choice of $S$, as follows. For every $i \in [l-1]$ let $T'_i$ be obtained from $T_i$ by deleting $v_i$ and adding the edge $v_{i-1}u_i$. The trees $T'_1, \ldots, T'_{l-1}$ then cover $v_0$ and all the vertices of $T_1, \ldots, T_{l-1}$, except $v_{l-1}$.

It remains to cover $v_{l-1}$. Let $T_l$ be a tree in $G$ defined as follows: if $u_l \in V(T)$ for some $T \in S$ then $T_l = T$, and otherwise, $T_l$ is a tree with $V(T_l) = \{u_l\}$. If $|V(T_l)| \leq 2$ then we add to $S'$ the tree $T'_l$ obtained from $T_l$ by adding the edge $v_{l-1}u_l$. Otherwise, $|V(T_l)| = 3$, and we add the trees $T'_l$ with vertex set $\{v_{l-1}, u_l\}$ and $T''_l = T_l \setminus u_l$ to $S'$. Note that $T''_l$ is indeed a tree, as $u_l$ is not the center of $T_l$. It is easy to check that the collection of trees $S'$ we obtained is a shrubbery satisfying (3), yielding the claimed contradiction and finishing the proof of (C2).

By our assumptions there exists $v_0 \in A - V(S)$. Let $U$ be the set of all vertices $u \in V(G)$ such that $u$ is a center of $S$ and there exists an $S$-alternating $(v_0, u)$-path. For each $u \in U$, let $T_u$ be the tree in $S$ such that $u$ is the center of $T_u$, and let $W$ be the set of leaves of the trees $\{T_u\}_{u \in U}$. Then $W \subseteq A$. 

12
(C3) For every $u \in U$, every $v \in V(T_u) - \{u\}$, and every $w \in V(G) - U - V(T_u)$ such that $vw \in E(G)$, either $w$ belongs to a tree in $S$ containing a mate of $v$, or $w$ has a mate in $T_u$.

Suppose for a contradiction that $u, v$ and $w$ violate (C3). Then $w \neq v_0$, as otherwise, $\{v, w\}$ induces a one edge $S$-alternating $(v_0, v)$-path contradicting (C2). Let $P$ be an $S$-alternating $(v_0, u)$-path. If $w$ does not satisfy the conditions above, then either $w \in V(T_u')$ for some $u' \in V(P) \cap U$, or appending $v$ and $w$ to $P$ we obtain an $S$-alternating $(v_0, w)$-path, contradicting (C2).

Thus we assume that $w \in V(T_u')$ for some $u' \in V(P) \cap U$. In this case, $P$ contains an $S$-alternating $(v_0, u')$-path $P'$ and appending $w$ and $v$ to $P'$ we obtain an $S$-alternating $(v_0, v)$-path, once again contradicting (C2). This finishes the proof of (C3).

It follows from (C1) and (C3) that every $v \in W$ has at most $5\varepsilon d$ neighbors outside of $U$, and thus the following holds.

(C4) Every $v \in W$ has at least $(1 - 5\varepsilon)d$ neighbors in $U$.

By (C4) it is possible to find a bipartite subgraph $G'$ of $G$ with bipartition $(U, W)$, such that $T_u \subseteq G'$ for every $u \in U$, and $(1 - 5\varepsilon)d \leq \deg_{G'}(v) \leq d$ for every $v \in W$. Let $m = |U|$. Then $|W| = 2m$ and $2(1 - 5\varepsilon)dm \leq \varepsilon(G') \leq 2dm$.

In the rest of the proof we analyze the graph $G'$ and obtain an outcome (i) or (iii) using an argument which parallels the first case, but requires several modifications. Let $B'$ be the set of vertices $u \in U$ such that $\deg_{G'}(u) \geq K d$. Then $|B'| \leq 2m/K$. Let $U' = U - B'$, and let $W' = W - \cup_{u \in B'} V(T_u)$. A tree $T$ with $V(T) = \{u, w_1, w_2\}$ in $G'$ is good if $u \in U'$, $w_1, w_2 \in W'$ and $w_1, w_2$ are not mates.

A collection $\mathcal{T}$ of vertex-disjoint good trees in $G'$ is a hedge if $V(\mathcal{T}) = U' \cup W'$. Note that $\{T_u\}_{u \in U'}$ is a hedge. Given distinct trees $T_1, T_2 \in \mathcal{T}$ and edges $u_1 w_1 \in E(T_1)$ and $u_2 w_2 \in E(T_2)$ we say that $\{u_1 w_1, u_2 w_2\}$ is a $\mathcal{T}$-bad pair if $u_1 w_2, u_2 w_1 \in E(G')$.

Let a hedge $\mathcal{T}$ be chosen so that the number of $\mathcal{T}$-bad pairs is minimum. We now bound the maximum number of $\mathcal{T}$-bad pairs that an edge of $\mathcal{T}$ can belong to, as we did in the first case.

Let $T \in \mathcal{T}$ be such that $V(T) = \{u, w, w'\}$ with $u \in U'$ and suppose that $\{(uw, u_i w_i)\}_{i=1}^b$ are $b := \lceil 3\varepsilon d \rceil$ bad pairs containing $uw$ such that the vertices $u_1, \ldots, u_b \in U'$ are pairwise distinct. Let $T_i \in \mathcal{T}$ such that $u_i \in V(T_i)$. There are at most $2\varepsilon d$ trees $T'$ in $\mathcal{T}$ such that either $w$ or $w'$ has a mate in $T'$, and so we may assume that neither $w$ nor $w'$ has a mate in $T_i$ for
$1 \leq i < b - 2\varepsilon d$. For each such $i$, let $T'_i$ be obtained from $T$ by deleting $w$ and adding the edge $uw_i$, and let $T''_i$ be obtained from $T_i$ by deleting $w_i$ and adding the edge $u_iw$. Then $T'_i$ and $T''_i$ are good trees by our assumption, and so $T_i = (T - \{T, T_i\}) \cup \{T'_i, T''_i\}$ is a hedge.

Let $X$ be the union of vertex set of all trees $T - \{T\}$ containing a neighbor of $u$ or $w$. Then $|X| \leq 3(K + 1)d$. Let $H = G'[X]$. It follows from the choice of $T$ that for every $i < b - 2\varepsilon d$ there are at least $b T_i$-bad pairs, which are not $T$-bad. Each such pair must contain one of the edges $uw_i$ and $u_iw$. It follows that $\deg_H(u_i) + \deg_H(w_i) \geq b$, and consequently $e(H) \geq b(b - 2\varepsilon d - 1)/2 \geq \varepsilon^2 d^2/2$. Thus (i) holds.

Therefore we may assume that every edge of every tree in $T$ belongs to at most $3\varepsilon d$ bad pairs, implying that the total number of bad pairs is at most $3\varepsilon dm$. We now obtain a minor $H$ of $G'$ by contracting every edge of $T$. We have $v(H) = 3m - 2|U'| \leq (1 + 4/K)m$. Every pair of edges of $G'$ that become parallel after the contraction correspond to a bad pair or a common neighbor of two leaves of some tree in $T$. It follows that $e(H) \geq e(G') - 2m - \varepsilon dm - 3\varepsilon dm \geq 2(1 - 10\varepsilon)dm$. Thus $d(H) \geq 2K(1 - 10\varepsilon)/(K + 4)$ and (iii) holds.

Taking $K = 1/\varepsilon$ in Theorem 4.1 we obtain the following.

**Corollary 4.2.** Let $0 < \varepsilon < 1$ be real. Let $G$ be a graph with $d = d(G) \geq 2/\varepsilon$. Then $G$ contains either

(i) a subgraph $H$ such that $|V(H)| \leq 4d/\varepsilon$ and $d(H) \geq \varepsilon^3 d/8$, or

(ii) a 2-bounded minor $H$ of $G$ such that $d(H) \geq \frac{3}{2}(1 - 7\varepsilon)d$.

(iii) a 3-bounded minor $H$ of $G$ such that $d(H) \geq 2(1 - 14\varepsilon)d$.

**Corollary 4.3.** For $0 < \varepsilon < 1/30$, let

$$\lambda = \max \left\{ \frac{\log 2}{\log(3(1 - 7\varepsilon)/2)}, \frac{\log 3}{\log(2(1 - 14\varepsilon))} \right\}. \tag{4}$$

Let $t$ be a positive integer and let $D = D(t)$ be such that every graph with $d(G) \geq D$ has a $K_t$ minor. Then $(4r^\lambda D/\varepsilon, \varepsilon^3 r^{-\lambda} D/8)$ is $(D/r, t)$-forced for every $1 \leq r \leq \varepsilon D/2$.

**Proof.** Suppose for a contradiction that the corollary fails for some $1 \leq r_0 \leq \varepsilon D/2$, but holds for all $1 \leq r \leq \frac{30}{\varepsilon} r_0$. Thus there exists a graph $G$ with
\(G \not\simeq K_t, \ d(G) \geq D/r_0\) such that no subgraph \(J\) of \(G\) satisfies \(v(J) \leq 4r_0^\lambda D/\varepsilon\), and \(d(J) \geq \varepsilon^3 D r_0^{-\lambda}/8\). Then \(G\) contains a minor \(H\) satisfying Corollary 4.2 (ii) or (iii).

Suppose first that \(H\) satisfies (ii). Let \(r = D/d(H)\) then \(r_0 \geq \frac{2}{3}(1-7\varepsilon)r\), and in particular \(r_0 \geq 31r/30\). Moreover, \(H \not\simeq K_t\) and so \(r \geq 1\). Thus by the choice of \(r_0\) there exists a subgraph \(J'\) of \(H\) such that \(v(J') \leq 4r^\lambda D/\varepsilon\) and \(d(J') \geq \varepsilon^3 Dr_0^{-\lambda}/8\). As \(H\) is a 2-bounded minor of \(G\), there exists a subgraph \(J\) of \(G\) corresponding to \(J'\) such that \(v(J) \leq 2v(J')\) and \(d(J) \geq d(J')/2\).

Thus we have
\[
v(J) \leq 2r^\lambda \frac{4D}{\varepsilon} \leq \frac{2}{(3(1-7\varepsilon)/2)^\lambda} \frac{4Dr_0^\lambda}{\varepsilon} \leq \frac{4Dr_0^\lambda}{\varepsilon},
\]
where the last inequality holds by the choice of \(\lambda\). Similarly, \(d(J) \geq \varepsilon^3 Dr_0^{-\lambda}/8\). This contradicts the choice of \(G\) and finishes the proof in this case.

The case, when a minor \(H\) of \(G\) satisfies Corollary 4.2 (iii), is completely analogous.

**Proof of Theorem 1.4.** Let \(\alpha = \frac{4 \log 2}{\log(3/2)}\). It suffices to show that for every \(\delta > 0\) there exists \(t_0\) such that for every \(t \geq t_0\) and every graph \(G\) with no \(K_t\) minor we have \(\chi(G) < k\), where \(k = t(\log t)^{1/2-1/\alpha+\delta}\).

Choose \(\varepsilon = \varepsilon(\delta) > 0\) such that if \(\lambda = \lambda(\varepsilon)\) is given by (4) then \(\lambda > \frac{1}{\alpha} - \delta/2\). By definition of \(\alpha\) such a choice is possible.

Let \(G\) be a graph such that \(G \not\simeq K_t\). Let \(C = C_{3,1}\) we assume without loss of generality that \(C \geq 1\), and assume further that \(t_0\) is chosen large enough so that
\[
k \geq Ct \cdot \max\{t(\log t)^{1/4}, \frac{3}{2} \log \log t\}. \tag{5}
\]
for all \(t \geq t_0\). Let \(D = 4t\sqrt{\log t}\). By Theorem 2.4 every graph \(G\) with \(d(G) \geq D\) has a \(K_t\) minor. Let \(r = D/k\) and so
\[
r = \frac{4t\sqrt{\log t}}{t(\log t)^{1/2-1/\alpha+\delta}} = 4(\log t)^{1/\alpha-\delta} \leq 4(\log t)^{1/(4\lambda)-\delta/2} \tag{6}
\]
By Corollary 4.3 there exists \(C_0 > 0\) depending only on \(\varepsilon\) such that \((C_0 r^\lambda D, r^{-\lambda} D/C_0)\) is \((k/4, t)\)-forced. Thus by (6) the pair
\[
(C_1 t(\log t)^{3/4-\lambda\delta/2}, t(\log t)^{1/4+\lambda\delta/2}/C_1)
\]

15
is \((k/4, t)\)-forced, where \(C_1 = C_0 4^{\lambda}\). Let \(n = t \log t/2\). If \(t_0\) is chosen so that

\[
(\log t_0)^{\min(1/4, \lambda \delta/2)} > 2C_1 C
\]

then \((n, Ct(\log t)^{1/4})\) is \((k/4, t)\)-forced for all \(t \geq t_0\). Moreover, \(\Box\) holds by \(\Box\). Thus Theorem 3.1 implies that \(\chi(G) < k\), as desired. \(\Box\)

5 Concluding remarks

Further improvements.

Our proof of the upper bound on the chromatic number of graphs with no \(K_t\) minor consists of two disjoint components, a procedure for obtaining a \(K_t\) minor by linking several smaller pieces in Section 3 and a density increment argument in Section 4. Improving either of these components would yield an improvement of the final bound. The bounds in Section 4 are likely far from tight, and it is possible that Corollary 4.3 can be improved so that \(\lim_{\varepsilon \to 0} \lambda(\varepsilon) = 1\), which in turn would imply that graphs with no \(K_t\) minor are \(O(t(\log t)^{\beta})\)-colorable for every \(\beta > 1/4\). The next major challenge on the way to the Linear Hadwiger’s conjecture appears to be in reducing the bound to \(o(t(\log t)^{1/4})\), which would require improving or replacing the argument in Section 8.

Answering the following question would help determine the limits of our current approach.

Question 5.1. Does there exist \(C > 0\) such that for every integer \(t \geq 1\) the following holds?

If \(G\) is a graph with \(\kappa(G) \geq Ct\), \(X_1, X_2, \ldots, X_r\) are disjoint subsets of \(V(G)\) for some \(r \geq (\log t)^C\), and \(\kappa(G[X_i]) \geq Ct\) for every \(i \in [r]\), then \(G\) has a \(K_t\) minor.

Note that Böhme et al. \cite{BKMM09} have shown that for every integer \(t \geq 1\) there exists \(N(t)\) such that every graph \(G\) with \(\kappa(G) \geq 31(t + 1)/2\) and \(v(G) \geq N(t)\) has a \(K_t\) minor. Their result implies that if we replace the requirement \(r \geq (\log t)^C\) in Question 5.1 by \(r \geq N(t)\), then the modified question has a positive answer.
List coloring.

A graph $G$ is said to be $k$-list colorable if for every assignment of lists $\{L(v)\}_{v \in V(G)}$ to vertices of $G$ such that $|L(v)| \geq k$ for every $v \in V(G)$, there is a choice of colors $\{c(v)\}_{v \in V(G)}$ such that $c(v) \in L(v)$, and $c(v) \neq c(u)$ for every $uv \in E(G)$. Clearly every $k$-list colorable graph is $k$-colorable, but the converse implication does not hold. Voigt [Vo93] has shown that there exist planar graphs which are not 4-list colorable. Generalizing the result of [Vo93], Barát, Joret and Wood [BJW11] constructed graphs with no $K_{3t+2}$ minor which are not $4t$-list colorable for every $t \geq 1$. These results leave open the possibility that the Linear Hadwiger’s Conjecture holds for list coloring, as conjectured by Kawarabayashi and Mohar [KM07b].

**Conjecture 5.2** ([KM07b]). There exists $C > 0$ such that for every integer $t \geq 1$, every graph with no $K_t$ minor is $Ct$-list colorable.

Theorem 1.3 implies that every graph with no $K_t$ minor is $O(t^{\sqrt{\log t}})$-list colorable, which is still the best known upper bound for general $t$. (Our methods do not extend to list coloring.) It would be of interest to resolve a “counter-conjecture” to Conjecture 5.2.

**Question 5.3.** Is every graph with no $K_t$ minor $o(t^{\sqrt{\log t}})$-list colorable?

Acknowledgement.

The research presented in this paper was completed during the visit of the second author to McGill University. Z-X. Song thanks the Department of Mathematics and Statistics, McGill University for its hospitality.

References

[AH77] K. Appel and W. Haken. Every planar map is four colorable. I. Discharging. *Illinois J. Math.*, 21(3):429–490, 1977.

[AHK77] K. Appel, W. Haken, and J. Koch. Every planar map is four colorable. II. Reducibility. *Illinois J. Math.*, 21(3):491–567, 1977.

[BJW11] János Barát, Gwenaël Joret, and David R. Wood. Disproof of the list Hadwiger conjecture. *Electron. J. Combin.*, 18(1):Paper 232, 7, 2011.
[BKMM09] Thomas Böhme, Ken-ichi Kawarabayashi, John Maharry, and Bojan Mohar. Linear connectivity forces large complete bipartite minors. *J. Combin. Theory Ser. B*, 99(3):557–582, 2009.

[BT96] Béla Bollobás and Andrew Thomason. Highly linked graphs. *Combinatorica*, 16(3):313–320, 1996.

[DM82] Pierre Duchet and Henri Meyniel. On hadwiger’s number and the stability number. In *North-Holland Mathematics Studies*, volume 62, pages 71–73. Elsevier, 1982.

[FdlV83] W. Fernandez de la Vega. On the maximum density of graphs which have no subcontraction to $K^s$. *Discrete Math.*, 46(1):109–110, 1983.

[Had43] H. Hadwiger. Über eine Klassifikation der Streckenkomplexe. *Vierteljschr. Naturforsch. Ges. Zürich*, 88:133–142, 1943.

[Kaw07] Ken-ichi Kawarabayashi. On the connectivity of minimum and minimal counterexamples to hadwiger’s conjecture. *Journal of Combinatorial Theory, Series B*, 97(1):144–150, 2007.

[KM07a] K. Kawarabayashi and Bojan Mohar. Some recent progress and applications in graph minor theory. *Graphs Combin.*, 23(1):1–46, 2007.

[KM07b] Ken-ichi Kawarabayashi and Bojan Mohar. A relaxed Hadwiger’s conjecture for list colorings. *J. Combin. Theory Ser. B*, 97(4):647–651, 2007.

[Kos82] A. V. Kostochka. The minimum Hadwiger number for graphs with a given mean degree of vertices. *Metody Diskret. Analiz.*, (38):37–58, 1982.

[Kos84] A. V. Kostochka. Lower bound of the Hadwiger number of graphs by their average degree. *Combinatorica*, 4(4):307–316, 1984.

[KP19] Tom Kelly and Luke Postle. A local epsilon version of reed’s conjecture. to appear in *Journal of Combinatorial Theory, Series B*, 2019.
[Mad72] W Mader. Existence of n-times connected subgraphs in graphs having large edge density. In Essays from the Mathematical Seminar of the University of Hamburg, volume 37, pages 86–97, 1972.

[RS98] Bruce Reed and Paul Seymour. Fractional colouring and Hadwiger’s conjecture. J. Combin. Theory Ser. B, 74(2):147–152, 1998.

[RST93] Neil Robertson, Paul Seymour, and Robin Thomas. Hadwiger’s conjecture for $K_6$-free graphs. Combinatorica, 13(3):279–361, 1993.

[Sey16] Paul Seymour. Hadwiger’s conjecture. In Open problems in mathematics, pages 417–437. Springer, 2016.

[Tho84] Andrew Thomason. An extremal function for contractions of graphs. Math. Proc. Cambridge Philos. Soc., 95(2):261–265, 1984.

[Tho01] Andrew Thomason. The extremal function for complete minors. J. Combin. Theory Ser. B, 81(2):318–338, 2001.

[TW05] Robin Thomas and Paul Wollan. An improved linear edge bound for graph linkages. European J. Combin., 26(3-4):309–324, 2005.

[Voi93] Margit Voigt. List colourings of planar graphs. Discrete Math., 120(1-3):215–219, 1993.

[Wag37] K. Wagner. Über eine Eigenschaft der ebenen Komplexe. Mathematische Annalen, 114:570–590, 1937.

[Woo13] David R Wood. A note on hadwiger’s conjecture. manuscript, arXiv:1304.6510, 2013.