Abstract. This paper studies coherent quantization, the way operators in the quantum space of a coherent space – defined in the recent book ‘Coherent Quantum Mechanics’ by the first author – can be studied in terms of objects defined directly on the coherent space. The results may be viewed as a generalization of geometric quantization, including the non-unitary case.

Care has been taken to work with the weakest meaningful topology and to assume as little as possible about the spaces and groups involved. Unlike in geometric quantization, the groups are not assumed to be compact, locally compact, or finite-dimensional. This implies that the setting can be successfully applied to quantum field theory, where the groups involved satisfy none of these properties.

The paper characterizes linear operators acting on the quantum space of a coherent space in terms of their coherent matrix elements. Coherent maps and associated symmetry groups for coherent spaces are introduced, and formulas are derived for the quantization of coherent maps.

The importance of coherent maps for quantum mechanics is due to the fact that there is a quantization map that associates homomorphically with every coherent map a linear operator from the quantum space into itself. The quantization map generalizes the second quantization procedure for free classical fields to symmetry groups of general coherent spaces. Field quantization is obtained by specialization to Klauder spaces, whose quantum spaces are the bosonic Fock spaces. Implied by the new approach is a short, coordinate-free derivation of all basic properties of creation and annihilation operators in Fock spaces.

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1 Introduction

This paper studies coherent quantization, the way operators in the quantum space of a coherent space – defined in the recent book 'Coherent Quantum Mechanics' [23] by the first author – can be studied in terms of objects defined directly on the coherent space. The results may be viewed as a generalization of geometric quantization, including the non-unitary case.

Care has been taken to work with the weakest meaningful topology and to assume as little as possible about the groups involved. In particular, unlike in geometric quantization (BATES & WEINSTEIN [5], WOODHOUSE [35]), we do not assume the groups to be compact, locally compact, or finite-dimensional. This implies that the setting can be successfully applied to quantum field theory, where the groups involved satisfy none of these properties.

More specifically, we characterize linear operators acting on the quantum space of a coherent space in terms of their coherent matrix elements. We discuss coherent maps and associated symmetry groups for the coherent spaces introduced in NEUMAIER [23], and derive formulas for the quantization of coherent maps.

An early paper by Itô [18] describes unitary group representations in terms of what are now called (generalized) coherent states. Group theoretic work on the subject was greatly extended by PERELOMOV [27,28], GILMORE [11], and others; see, e.g., the survey by ZHANG et al. [36]. The present setting may be viewed as a generalization of this and of vector coherent states (ROWE et al. [31]) to the non-unitary case.

The importance of coherent maps for quantum mechanics is due to the fact proved in Theorem 3.11 below that there is a quantization map $\Gamma$ that associates homomor-
A linear operator $\Gamma(A)$ on the augmented quantum space $Q^+(Z)$ that maps the quantum space $Q(Z)$ into itself. In the special case (discussed in Section 6 below) where $Z$ is a Klauder space, the quantum space $Q(Z)$ is a dense subspace of a bosonic Fock space and the quantization map is the restriction of the second quantization map of Dereziński & Gérard [8] to $Q(Z)$. Thus the quantization map generalizes the second quantization procedure of free classical fields to symmetry groups of general coherent spaces.

Contents. In the present section we review notation, terminology, and some results on coherent spaces and their quantum spaces, introduced in Neumaier [23], on which the present paper is based. Section 2 provides fundamental but abstract necessary and sufficient conditions for recognizing when a kernel, i.e., a map from a coherent space into itself is a shadow (i.e., definable in terms of coherent matrix elements), and hence determines an operator on the corresponding quantum space.

Section 3 discusses symmetries of a coherent space, one of the most important concepts for studying and using coherent spaces. Indeed, most of the applications of coherent spaces in quantum mechanics and quantum field theory rely on the presence of a large symmetry group. The main reason is that there is a quantization map that furnishes a representation of the semigroup of coherent maps on the quantum space, and thus provides easy access to a class of very well-behaved linear operators on the quantum space. Section 4 looks at self-mappings of coherent spaces satisfying homogeneity or separability properties. These often give simple but important coherent maps. In Section 5 we prove quantization theorems for a restricted class of coherent spaces for which many operators on a quantum space have a simple description in terms of normal kernels. These generalize the normal ordering of operators familiar from quantum field theory.

In the final Section 6 we discuss in some detail the coherent quantization of Klauder spaces, a class of coherent spaces with a large semigroup of coherent maps, introduced in Neumaier [23]. The corresponding coherent states are closely related to those introduced by Schrödinger [33] and made prominent in quantum optics by Glauber [12]. The quantum spaces of Klauder spaces are the bosonic Fock spaces, which play a very important role in quantum field theory (Baez et al. [2], Glimm & Jaffe [13]), and the theory of Hida distributions in the white noise calculus for classical stochastic processes (Hida & Sti [15], Hida & Streit [16], Obata [25]). In particular, we give a coordinate-free derivation of the basic properties of creation and annihilation operators in Fock spaces.

1.1 Euclidean spaces

In this paper, we use the notation and terminology of Neumaier [23], quickly reviewed here.

We write $\mathbb{C}$ for the field of complex numbers, $\mathbb{C}^\times$ for the multiplicative group of nonzero complex numbers, and $\overline{\alpha} = \alpha^*$ for the complex conjugate of $\alpha \in \mathbb{C}$. $\mathbb{C}^X$ denotes the vector space of all maps from a set $X$ to $\mathbb{C}$. 
A (complex) **Euclidean space** is a complex vector space \( \mathbb{H} \) with a Hermitian form that assigns to \( \phi, \psi \in \mathbb{H} \) the **Hermitian inner product** \( \langle \phi, \psi \rangle \in \mathbb{C} \), antilinear in the first and linear in the second argument, such that

\[
\overline{\langle \phi, \psi \rangle} = \langle \psi, \phi \rangle, \tag{1}
\]
\[
\langle \psi, \psi \rangle > 0 \quad \text{for all } \psi \in \mathbb{H} \setminus \{0\}. \tag{2}
\]

Associated with \( \mathbb{H} \) is the triple of spaces

\[
\mathbb{H} \subseteq \overline{\mathbb{H}} \subseteq \mathbb{H}^\times, \tag{3}
\]

where \( \overline{\mathbb{H}} \) is a Hilbert space completion of \( \mathbb{H} \), and \( \mathbb{H} \) is dense in the vector space \( \mathbb{H}^\times \) of all antilinear functionals on \( \mathbb{H} \), with

\[
\psi(\phi) := \langle \phi, \psi \rangle \quad \text{for } \phi \in \mathbb{H}.
\]

\( \mathbb{H}^\times \) carries a Hermitian partial inner product \( \phi^* \psi \) with

\[
\phi^* \psi = \langle \phi, \psi \rangle \quad \text{for } \phi, \psi \in \mathbb{H}. \tag{4}
\]

and is a PIP space in the sense of Antoine & Trapani [1]. \( \mathbb{H} \) also serves as a definition of the linear functional \( \phi^* \) on \( \overline{\mathbb{H}} \) for \( \phi \in \mathbb{H} \).

Let \( U \) and \( V \) be (complex) topological vector spaces. We write \( \text{Lin} (U,V) \) for the vector space of all continuous linear mappings from \( U \) to \( V \), \( \text{Lin} U \) for \( \text{Lin} (U,U) \), and \( U^\times \) for the **antidual** of \( U \), the space of all continuous antilinear mappings from \( U \) to \( \mathbb{C} \). We identify \( V \) with the space \( \text{Lin} (\mathbb{C}, V) \). If \( U, V \) are Euclidean spaces and \( A \in \text{Lin} (U,V) \), the adjoint operator \( A^* \in \text{Lin} (V^\times, U^\times) \) is defined by

\[
(A^* \phi)(\psi) := \phi(A \psi) \quad \text{for } \phi \in V^\times, \psi \in U. \tag{5}
\]

We write \( \text{Lin}^\times \mathbb{H} := \text{Lin} (\mathbb{H}, \mathbb{H}^\times) \) for the vector space of linear operators from a Euclidean space \( \mathbb{H} \) to its antidual. Note that the adjoint of an operator in \( \text{Lin}^\times \mathbb{H} \) is again in \( \text{Lin}^\times \mathbb{H} \).

As in [23], continuity in a Euclidean vector space \( U \) is always understood in the finest locally convex topology, and continuity in its antidual \( U^\times \) is always understood in the weak-* topology. In particular, if \( U, V \) are Euclidean spaces, all linear and antilinear functionals on \( U \) and all linear mappings from \( U \to V^\times \) are continuous.

### 1.2 Coherent spaces

The notion of a coherent space is a nonlinear version of the notion of a complex Euclidean space: The vector space axioms are dropped while the notion of inner product is kept. Coherent spaces provide a setting for the study of geometry in a different direction than traditional metric, topological, and differential geometry. Just as it pays to study the properties of manifolds independently of their embedding into a Euclidean space, so it appears fruitful to study the properties of coherent spaces independent of their embedding into a Hilbert space.
A coherent space is a nonempty set $Z$ with a distinguished function $K : Z \times Z \to \mathbb{C}$ of positive type called the coherent product. Thus
\[ K(z, z') = K(z', z), \] (6)
and for all $z_1, \ldots, z_n \in Z$, the $n \times n$ matrix $G$ with entries $G_{jk} = K(z_j, z_k)$ is positive semidefinite.

The coherent space $Z$ is called nondegenerate if
\[ K(z'', z') = K(z, z') \quad \forall \; z' \in Z \quad \Rightarrow \quad z'' = z. \]

The coherent space $Z$ is called projective of degree $e \in \mathbb{Z} \setminus \{0\}$ if there is a scalar multiplication that assigns to each $\lambda \in \mathbb{C}$ and each $z \in Z$ a point $\lambda z \in Z$ such that
\[ K(z', \lambda z) = \lambda^e K(z', z). \] (7)
Equivalently,
\[ |\lambda z\rangle = \lambda^e |z\rangle. \]

For any coherent space $Z$, the projective extension of $Z$ of degree $e$ (a nonzero integer) is the coherent space $PZ := \mathbb{C}^\times \times Z$ with coherent product
\[ K_{pe}((\lambda, z), (\lambda', z')) := \sum K(z, z') \lambda^e \] (8)
and scalar multiplication $\lambda'(\lambda, z) := (\lambda' \lambda, z)$, defined in [23, Proposition 5.8.5], with the same quantum space as $Z$.

Throughout the paper, $Z$ is a fixed coherent space with coherent product $K$. A quantum space $Q(Z)$ of $Z$ is a Euclidean space spanned (algebraically) by a distinguished set of vectors $|z\rangle$ ($z \in Z$) called coherent states satisfying
\[ \langle z|z'\rangle = K(z, z') \quad \text{for} \; z, z' \in Z, \] (9)
where $\langle z| := |z\rangle^*$. The associated augmented quantum space $Q^\times(Z)$, the antidual of $Q(Z)$, contains the completed quantum space $\overline{Q}(Z)$, the Hilbert space completion of $Q(Z)$. By [23, Section 4.3], any linear or antilinear map from a quantum space of a coherent space into $\mathbb{C}$ is continuous, and by the above convention for the topology of Euclidean spaces, any linear or antilinear map from a quantum space of a coherent space into its antidual is continuous, too.

2 Quantization through admissibility conditions

We regard the quantization of a coherent space $Z$ as the problem of describing interesting classes of linear operators from $\text{Lin}^\times Q(Z)$ and their properties in terms of objects more tangibly defined on $Z$. The key to coherent quantization is the observation that one can frequently define and manipulate operators on the quantum space in terms of their coherent matrix elements, without needing a more explicit description in terms of differential or integral operators on a Hilbert space of functions.
A kernel on \( Z \) is a map \( X \in \mathbb{C}^{Z \times Z} \). The shadow of a linear operator \( X \in \text{Lin}^\times \mathbb{Q}(Z) \) is the kernel \( \text{Sh} X \in \mathbb{C}^{Z \times Z} \) defined by (cf. Klauder [19])

\[
\text{Sh} X(z, z') := \langle z | X | z' \rangle \quad \text{for} \quad z, z' \in Z.
\]

Thus shadows represent the information in the coherent matrix elements \( \langle z | X | z' \rangle \) of an operator \( X \).

This section discusses admissibility conditions. They provide fundamental but abstract necessary and sufficient conditions for recognizing when a kernel is a shadow and hence determines an operator \( X \in \text{Lin}^\times \mathbb{Q}(Z) \). Later sections then provide applications to more concrete situations.

The admissibility conditions are infinite generalizations of the simple situation when \( Z \) is finite. In this case we may w.l.o.g. take \( Z = \{1, 2, \ldots, n\} \) and regard kernels as \( n \times n \) matrices. Then the coherent product is just a positive semidefinite matrix \( K = R^* R \), and the shadow of an operator \( X \) is \( X = R^* X R \). Admissibility of \( X \), here equivalent with strong admissibility, is the condition that for any column vector \( c \), \( Rc = 0 \) implies \( Xc = 0 \) and \( X^* c = 0 \), which forces \( X \) to have at most the same rank as \( K \). It is not difficult to see (and follows from the results below) that this condition implies that \( X \) has the form \( X = R^* X R \) for some matrix \( X \), so that an admissible \( X \) is indeed a shadow.

### 2.1 Admissibility

Let \( Z \) be a coherent space with the coherent product \( K \). We want to characterize the functions \( f : Z \to \mathbb{C} \) for which there is an antilinear mapping \( \psi : \mathbb{Q}(Z) \to \mathbb{C} \) such that

\[
f(z) = \langle z | \psi := \psi(\cdot) \rangle \quad \text{for all} \quad z \in Z.
\]

We call a function \( f : Z \to \mathbb{C} \) admissible if for arbitrary finite sequences of complex numbers \( c_k \) and points \( z_k \in Z \),

\[
\sum c_k K(z_k, z) = 0 \quad \forall z \in Z \quad \Rightarrow \quad \sum c_k f(z_k) = 0.
\]

For example, for \( K(z, z') = 0 \) for all \( z, z' \) one gets a trivial coherent space whose quantum space is \( \{0\} \), and only the zero function is admissible. On the other hand, a condition guaranteeing that every map is admissible is given in Proposition 5.1 below.

### 2.1 Theorem

Let \( Z \) be a coherent space. For a quantum space \( \mathbb{Q}(Z) \) of \( Z \), the following conditions on a function \( f : Z \to \mathbb{C} \) are equivalent.

(i) There is an antilinear functional \( \psi : \mathbb{Q}(Z) \to \mathbb{C} \) (i.e., a \( \psi \in \mathbb{Q}^\times(Z) \)) such that (10) holds.

(ii) For arbitrary finite sequences of complex numbers \( c_k \) and points \( z_k \in Z \),

\[
\sum c_k | z_k \rangle = 0 \quad \Rightarrow \quad \sum c_k f(z_k) = 0.
\]

(iii) \( f \) is admissible.

Moreover, in (i), \( \psi \) is uniquely determined by \( f \).
Proof. (ii)⇔(i): Let \( f : Z \to \mathbb{C} \) be a function satisfying (12). We define the antilinear functional \( \psi \in \mathbb{Q}(Z) \to \mathbb{C} \) by

\[
\psi\left( \sum c_k|z_k\rangle \right) := \sum \overline{c}_k f(z_k) \quad \text{for all } \sum c_k|z_k\rangle \in \mathbb{Q}(Z).
\]

Because of (12), \( \psi \) is well-defined; it is clearly antilinear. Thus, \( \psi \) defines an antilinear functional on the quantum space \( \mathbb{Q}(Z) \). Specializing (13) to the case of a sum containing a single term only gives

\[
\langle z| \psi = \psi(|z\rangle) = f(z) \quad \text{for } z \in Z,
\]

so that \( \psi \) satisfies (10). If (10) also holds for \( \psi' \) in place of \( \psi \) then \( \psi = \psi' \) since the coherent states span \( \mathbb{Q}(Z) \). This shows that \( \psi \) is uniquely determined by \( f \) and (10).

Conversely, let \( f : Z \to \mathbb{C} \) be a function that satisfies (10) for some antilinear functional \( \psi \) on \( \mathbb{Q}(Z) \). If the left hand side of (12) holds then

\[
\sum \overline{c}_k f(z_k) = \sum \overline{c}_k \langle z_k| \psi = \psi\left( \sum c_k|z_k\rangle \right) = 0.
\]

(iii)⇔(ii): Clearly (11) is equivalent to

\[
\sum \overline{c}_k K(z_k, z) = 0 \quad \forall \ z \in Z \quad \Rightarrow \quad \sum \overline{c}_k f(z_k) = 0,
\]

The left hand side of (12) is equivalent to

\[
0 = \sum \overline{c}_k \langle z_k| z \rangle = \sum \overline{c}_k K(z_k, z') \quad \text{for } z \in Z.
\]

Since

\[
\langle z| \sum c_k|z_k\rangle = \sum c_k \langle z| z_k\rangle = \sum c_k K(z, z_k) = \sum \overline{c}_k K(z_k, z),
\]

this is equivalent to (12). \( \square \)

The admissibility space of \( Z \) is the set \( \mathbb{A}(Z) \) of all admissible functions over the coherent space \( Z \). It is easy to see that \( \mathbb{A}(Z) \) is a vector space with respect to pointwise addition of functions and pointwise multiplication by complex numbers.

2.2 Theorem. Let \( Z \) be a coherent space and let \( \mathbb{Q}(Z) \) be a quantum space of \( Z \).

(i) For every admissible function \( f : Z \to \mathbb{C} \),

\[
\theta_f\left( \sum c_k|z_k\rangle \right) := \sum \overline{c}_k f(z_k) \quad \text{for } \sum c_k|z_k\rangle \in \mathbb{Q}(Z),
\]

defines a continuous antilinear functional on \( \mathbb{Q}(Z) \).

(ii) The identification map \( \Theta : \mathbb{A}(Z) \to \mathbb{Q}(Z)^\times \) given by

\[
\Theta(f) := \theta_f
\]

is a vector space isomorphism. In particular, the admissibility space \( \mathbb{A}(Z) \) can be equipped with a locally convex topology such that the linear map \( \Theta \) is a homeomorphism.
Proof. (i) Let \( f : Z \rightarrow \mathbb{C} \) be an admissible function. By Theorem 2.1, \( \theta_f = \psi \) is the unique vector in \( \mathbb{Q}(Z)^\times \) satisfying (10).

(ii) By (i), the linear map \( \Theta : A(Z) \rightarrow \mathbb{Q}(Z)^\times \) given by \( \Theta(f) := \theta_f \) is a vector space homomorphism. Let \( \psi \in \mathbb{Q}(Z)^\times \) be a given antilinear functional and define \( f : Z \rightarrow \mathbb{C} \) via \( f(z) := \langle z|\psi \rangle \), for all \( z \in Z \). Then, it is easy to check that \( f \in A(Z) \) and \( \theta_f = \psi \). Thus \( \Theta \) is an isomorphism.

\[ \Box \]

2.3 Corollary. Let \( Z \) be a coherent space. The admissible spaces \( A(Z) \), \( A(PZ) \), and \( A([Z]) \) are canonically isomorphic as topological vector space.

2.2 Kernels and shadows

For any kernel \( X \) we define the related kernels \( X^T \), \( \overline{X} \), and \( X^* \) by

\[ X^T(z,z') := X(z',z), \quad \overline{X}(z,z') := \overline{X(z,z')}, \quad X^*(z,z') := \overline{X(z',z)}. \]

Clearly,

\[ X^{TT} = \overline{X} = X^{**} = X, \quad X^* = \overline{X^T} = \overline{X^T}. \]

For example, any coherent product is a kernel \( K \); it is Hermitian iff \( K^T = K \). Given a kernel \( X \) and \( z \in Z \), we define the functions \( X(z,\cdot) \), \( X(\cdot,z) \) and \( X(\cdot,\cdot) \) by

\[ X(\cdot,z)(z') := X(z',z), \quad X(z,\cdot)(z') := X(z,z') \quad \text{for } z' \in Z. \]

2.4 Proposition.

(i) The shadow of the identity operator \( 1 \) is \( \text{Sh} 1 = K \).

(ii) For \( X \in \text{Lin}^\times \mathbb{Q}(Z) \), the adjoint \( X^* \in \text{Lin}^\times \mathbb{Q}(Z) \), defined by

\[ X^* \psi(\phi) := \overline{X(\phi(\psi))}, \]

satisfies

\[ \langle z|X^*|z' \rangle = \overline{\langle z'|X|z \rangle} \quad \text{for all } z,z' \in Z, \]

\[ (\text{Sh} X)^* = \text{Sh} X^*. \]

Proof. (i) holds since \( \text{Sh} 1(z,z') = \langle z|1|z' \rangle = \langle z|z' \rangle = K(z,z') \) for all \( z,z' \in Z \).

(ii) Linearity implies already \( X^* \in \text{Lin}^\times \mathbb{Q}(Z) \). Then, for \( z,z' \in Z \),

\[ \langle z|X^*|z' \rangle = \overline{X^*(|z')\langle z | \rangle} = \overline{X(\overline{\langle z |} z)} = \overline{\langle z'|X|z \rangle}, \]

\[ \text{Sh} X^*(z,z') = \langle z|X^*|z' \rangle = \overline{\langle z'|X|z \rangle} = (\text{Sh} X)^*(z,z'). \]

\[ \Box \]

The following characterization of shadows is the fundamental theorem on which all later quantization results are based.
2.5 Theorem. Let $Z$ be a coherent space. A kernel $X \in C^{Z \times Z}$ is a shadow iff $X(z, \cdot)$ and $\overline{X}(\cdot, z)$ are admissible for all $z \in Z$. In this case there is a unique operator $X \in \text{Lin}^{\times} Q(Z)$ whose shadow is $X$, i.e.,

$$\langle z|X|z' \rangle = X(z, z') \quad \text{for} \quad z, z' \in Z. \quad (16)$$

Proof. Let $z \in Z$, $X \in \text{Lin}^{\times} Q(Z)$, and $X := \text{Sh} \ X$. Then $X|z \rangle \in Q(Z)^{\times}$, and for $z_1, \ldots, z_n \in Z$ and $c_1, \ldots, c_n \in \mathbb{C}$, we have

$$\overline{X}(z_\ell, z) = \overline{\text{Sh} \overline{X}(z_\ell, z)} = \overline{\langle z_\ell|X|z \rangle},$$

hence

$$\sum_\ell z_\ell \overline{X}(z_\ell, z) = \sum_\ell c_\ell \overline{\langle z_\ell|X|z \rangle} = \overline{\left( \sum_\ell c_\ell \langle z_\ell \rangle \right) \langle X|z \rangle}.$$ 

By Theorem 2.1 this implies that $\overline{X}(\cdot, z)$ is admissible. For $z, z' \in Z$, we have by Proposition 2.4(ii),

$$X(z, z') = \langle z|X|z' \rangle = \langle z'|X^*|z \rangle = \text{Sh} \overline{X}^*(z', z).$$

Hence $X(z, \cdot) = \text{Sh} \overline{X}^*(\cdot, z)$ for all $z \in Z$. This implies that $X(z, \cdot)$ is admissible as well.

Conversely, let $X$ be a kernel such that $X(z, \cdot)$ and $\overline{X}(\cdot, z)$ are admissible for all $z \in Z$. Then for fixed $z_\ell$,

$$\sum_k c_k \langle z_\ell^k \rangle = 0 \quad \Rightarrow \quad \sum_k c_k X(z_\ell, z_k^\ell) = 0,$$

and for fixed $z_k^\ell$,

$$\sum_\ell c_\ell \langle z_\ell \rangle = 0 \quad \Rightarrow \quad \sum_\ell \overline{c_\ell} X(z_\ell, z_k^\ell) = 0.$$

Therefore, for given vectors $\phi = \sum_k c_k \langle z_k^\ell \rangle$ and $\psi = \sum_\ell c_\ell \langle z_\ell \rangle \in Q(Z)$, the double sum

$$(\psi, \phi)_X := \sum_\ell \sum_k \overline{c_\ell} c_k X(z_\ell, z_k)$$

is independent of the representation of $\phi$ and $\psi$, hence defines a sesquilinear form. Thus

$$\psi \rightarrow \psi^* X \phi := (\psi, \phi)_X, \quad \text{for all} \quad \psi \in Q(Z)$$

defines an antilinear functional $X \phi : Q(Z) \rightarrow \mathbb{C}$. Thus $X \phi \in Q(Z)^{\times}$. Clearly, $\phi \rightarrow X \phi$ defines a linear map $X : Q(Z) \rightarrow Q(Z)^{\times}$. Therefore $X \in \text{Lin}^{\times} Q(Z)$. It is easy to check that $\langle z|X|z' \rangle = X(z, z')$ for all $z, z' \in Z$. Thus $\text{Sh} \ X = X$. Finally, it can be readily checked that $X$ is the unique operator which satisfies $X = \text{Sh} \ X$. \qed
3 Coherent maps and their quantization

This section discusses symmetries of a coherent space, one of the most important concepts for studying and using coherent spaces. Indeed, most of the applications of coherent spaces in quantum mechanics and quantum field theory rely on the presence of a large symmetry group. The main reason is that – as we show in Theorem 3.11 below – there is a quantization map that furnishes a representation of the semigroup of coherent maps on the quantum space, and thus provides easy access to a class of very well-behaved linear operators on the quantum space.

Let \( Z, Z' \) be coherent spaces. Recall from Neumaier [23, Section 5.3] that a morphism from \( Z \) to \( Z' \) is a map \( \rho : Z \to Z' \) such that

\[
K'(\rho z, \rho w) = K(z, w) \quad \text{for } z, w \in Z;
\]  

(17)

if \( Z' = Z \), \( \rho \) is called an endomorphism. Two coherent spaces \( Z \) and \( Z' \) are called isomorphic if there is a bijective morphism \( \rho : Z \to Z' \). In this case we write \( Z \cong Z' \) and we call the map \( \rho : Z \to Z' \) an isomorphism of the coherent spaces. Clearly, \( \rho^{-1} : Z' \to Z \) is then also an isomorphism.

In the spirit of category theory one should define the symmetries of a coherent space \( Z \) in terms of its automorphisms, i.e., isomorphisms from \( Z \) to itself. Remarkably, however, coherent spaces allow a significantly more general concept of symmetry, based on the notion of a coherent map.

3.1 Coherent maps

Let \( Z \) and \( Z' \) be coherent spaces with coherent products \( K \) and \( K' \), respectively. A map \( A : Z' \to Z \) is called coherent if there is an adjoint map \( A^* : Z \to Z' \) such that

\[
K(z, Az') = K'(A^*z, z') \quad \text{for } z \in Z, \ z' \in Z';
\]  

(18)

If \( Z' \) is nondegenerate, the adjoint is unique, but not in general. A coherent map \( A : Z' \to Z \) is called an isometry if it has an adjoint satisfying \( A^*A = 1 \). A coherent map on \( Z \) is a coherent map from \( Z \) to itself.

A symmetry of \( Z \) is an invertible coherent map on \( Z \) with an invertible adjoint. We call a coherent map \( A \) unitary if it is invertible and \( A^* = A^{-1} \). Thus unitary coherent maps are isometries.

3.1 Example. An orbit of a group \( G \) acting on a set \( S \) is a set consisting of all images \( Ax \ (A \in G) \) of a single vector. The group is transitive on \( S \) if \( S \) is an orbit. The orbits of groups of linear self-mappings of a Euclidean space give coherent spaces with predefined transitive symmetry groups. Indeed, in the coherent space formed by an arbitrary subset \( Z \) of a Euclidean space \( U \) with coherent product \( K(z, z') := z^*z' \), all linear operators \( A : U \to U^* \) such that \( A \) and \( A^* \) map \( Z \) into itself are coherent maps, and all such operators mapping \( Z \) bijectively onto itself are symmetries. This is the reason why coherent spaces are important in the theory of group representations.
For example, the symmetric group Sym(5) acts as a group of Euclidean isometries on the 12 points of the icosahedron in $\mathbb{R}^3$. The coherent space consisting of these 12 points with the induced coherent product therefore has Sym(5) as a group of unitary symmetries. The skeleton of the icosahedron is a distance-regular graph, here a double cover of the complete graph on six vertices. Many other interesting examples of finite coherent spaces are related to Euclidean representations of distance regular graphs (Brouwer et al. [7]) and other highly symmetric combinatorial objects.

3.2 Proposition.
(i) Every unitary coherent map is a symmetry.

(ii) An invertible map $A : Z \to Z$ is a unitary coherent map iff

$$K(Az, Az') = K(z, z') \quad \text{for all } z, z' \in Z.$$ 

Proof. (i) $A^{-1}$ exists and is coherent by the preceding since $A^{-1} = A^*$. 

(ii) Replace $z$ in (18) by $Az$. \hfill $\square$

3.3 Proposition.
(i) Every morphism $A$ with right inverse $A'$ is coherent, with adjoint $A^* = A'$.

(ii) Every isometry is a morphism.

(iii) A map $A : Z \to Z$ is an automorphism of $Z$ iff it is a unitary coherent map.

Proof. (i) Put $A^* := A'$. Then $AA^* = 1$, and we have $K(z, Az') = K(AA^*z, Az') = K(A^*z, z')$, proving the claim.

(ii) Let $A : Z \to Z'$ be an isometry. Then, for $z, z' \in Z$,

$$K'(Az, Az') = K(z, z').$$ 

(iii) Let $A : Z \to Z$ be an automorphism of $Z$. Since $A$ is a morphism and invertible, for $z, z' \in Z$, we get

$$K(z, Az') = K(AA^{-1}z, Az') = K(A^{-1}z, z').$$ 

This implies that $A$ is coherent with $A^* := A^{-1}$ with $A^*A = AA^* = 1$. Hence $A$ is unitary. Conversely, assume that $A : Z \to Z$ is a unitary coherent map. Then Proposition 3.2(ii) implies that $A$ is a morphism. Since $A$ is bijective, it is an automorphism of $Z$ as well. \hfill $\square$

3.4 Proposition. Let $Z$ be a coherent space and $A : Z \to Z$ be a coherent map. Then for $z, z' \in Z$,

$$K(Az, z') = K(z, A^*z'), \quad (19)$$

$$\langle z | Az' \rangle = \langle A^*z | z' \rangle, \quad \langle Az | z' \rangle = \langle z | A^*z' \rangle. \quad (20)$$
Proof. For \( z, z' \in Z \), (6) implies
\[
\langle Az, z' \rangle = K(Az, z') = K(z', Az) = K(A^* z', z) = K(z, A^* z') = \langle z | A^* z' \rangle.
\]
This proves both (19) and the second half of (20). The first half of (20) follows directly from (18). \( \square \)

3.5 Theorem. Let \( Z \) be a coherent space. Then the set \( \text{Coh} Z \) consisting of all coherent maps is a semigroup with identity. Moreover:

(i) Any adjoint \( A^* \) of \( A \in \text{Coh} Z \) is coherent.

(ii) For any invertible coherent map \( A : Z \rightarrow Z \) with an invertible adjoint, the inverse \( A^{-1} \) is coherent.

Proof. The identity map \( I : Z \rightarrow Z \) is trivially coherent. Let \( A, B \in \text{Coh} Z \). Then, for \( z, z' \in Z \),
\[
K(z, ABz') = K(A^* z, Bz') = K(B^* A^* z, z'),
\]
which implies that \( AB \) is coherent with adjoint \( (AB)^* = B^* A^* \).

(i) Using Proposition 3.4, we can write
\[
K(z, A^* z') = K(Az, z'),
\]
which implies that \( A^* \) is coherent with \( A^{**} = A \).

(ii) Let \( A : Z \rightarrow Z \) be a coherent map with an adjoint \( A^* \) such that \( A \) and \( A^* \) are invertible with the inverses \( A^{-1} \) and \( (A^*)^{-1} \). Then, for \( z, z' \in Z \),
\[
K(A^{-1} z, z') = K(A^{-1} z, A^* (A^*)^{-1} z') = K(AA^{-1} z, (A^*)^{-1} z') = K(z, (A^*)^{-1} z'),
\]
which implies that \( A^{-1} \) is coherent with \( (A^{-1})^* = (A^*)^{-1} \). \( \square \)

3.6 Corollary. Let \( Z \) be a nondegenerate coherent space. Then \( \text{Coh} Z \) is a \(*\)-semigroup with identity, i.e.,
\[
1^* = 1, \quad A^{**} = A, \quad (AB)^* = B^* A^* \quad \text{for} \, A, B \in \text{Coh} \, Z.
\]
Moreover, the set \( \text{Sym}(Z) \) of all invertible coherent maps with invertible adjoint is a \(*\)-group, and
\[
A^{-*} := (A^{-1})^* = (A^*)^{-1} \quad \text{for} \, A \in \text{Sym}(Z).
\]

Proof. If \( Z \) is nondegenerate then the adjoint is unique. Therefore the claim follows from the preceding result. \( \square \)

As a consequence, the vector space spanned by all coherent maps of a nondegenerate coherent space has a natural \(*\)-algebra structure in the sense of Powers \[29\].

For any coherent space \( Z \), \([Z]\) denotes the nondegenerate coherent space defined in \[23\] Proposition 5.8.7, with the same quantum space as \( Z \).
3.7 Proposition. Let $Z$ be a coherent space. Then

$$\text{Coh}[Z] = \{ [A] \mid A \in \text{Coh } Z \}. $$

Proof. Let $A \in \text{Coh } Z$. Then $[A] \in \text{Coh}[Z]$ by [23] Theorem 5.8.9. Thus $\{ [A] : A \in \text{Coh } Z \} \subseteq \text{Coh}[Z]$. Let $\iota : [Z] \to Z$ be a choice function, that is a function which satisfies $[\iota[z]] = [z]$ for all $z \in Z$. For any coherent map $A : [Z] \to [Z]$, define $A : Z \to Z$ by $z \to Az := \iota(A[z])$. Then $A : Z \to Z$ is a well-defined map. Thus, for $z, z' \in Z$,

$$K(Az, z') = K(\iota(A[z]), z') = K(\iota(A[z]), [z']) = K(A[z], [z']) = K([z], A^*[z']) = K([z], A^*[z']).$$

This implies that $A$ is a coherent map with an adjoint $A^* : Z \to Z$ given by $A^*z = \iota(A^*[z'])$. For $z, z' \in Z$, we have

$$K([A][z], [z']) = K([Az], [z']) = K([\iota(A[z])], [z']) = K(A[z], [z']),$$

implying that $[A] = A$.

3.8 Theorem. Let $PZ$ be the projective extension of degree 1 of the coherent space $Z$.

(i) Let $A : Z \to Z$ be a map with the property

$$K(z, Az')v(z') = \overline{w(z)}K(A^*z, z') \quad \text{for } z, z' \in Z,$$

for suitable $v, w : Z \to \mathbb{C}$ and $A^* : Z \to Z$. Then

$$[\alpha, A](\lambda, z) := (\alpha v(z)\lambda, Az), \quad [\alpha, A]^*(\lambda, z) := (\overline{\alpha}w(z)\lambda, A^*z)$$

define a coherent map $[\alpha, A]$ of $PZ$ and its adjoint $[\alpha, A]^*$.

(ii) For every coherent map $A : Z \to Z$ and every $\alpha \in \mathbb{C}$, the map $[\alpha, A] : PZ \to PZ$ defined by

$$[\alpha, A](\lambda, z) := (\alpha \lambda, Az) \quad \text{for all } (\lambda, z) \in PZ,$$

is coherent.

Proof. Let $(\lambda, z), (\lambda', z') \in PZ$. Then

$$K_{pe}((\lambda, z), (\lambda', z')) = \overline{\lambda} K(z, z') \lambda'.$$

Therefore,

$$K_{pe}((\lambda, z), [\alpha, A](\lambda', z')) = K_{pe}((\lambda, z), (\alpha v(z')\lambda, A^*z')) = \overline{\lambda} K(z, Az') \alpha v(z') \lambda'$$

$$= \overline{\lambda} \overline{\alpha} w(z) K(A^*z, z') \lambda' = K_{pe}((\lambda \overline{\alpha} w(z), A^*z), (\lambda', z')) = K_{pe}([\alpha, A]^*(\lambda, z), (\lambda', z')).$$

This proves (i), and (ii) is the special case of (i) where $v$ and $w$ are identically 1.

Something similar can be shown for projective extensions of any integral degree $e \neq 0$. Condition (21) appears first in a paper by BERTRAM & HILGERT [6] on reproducing kernels invariant under an involutive semigroup.
3.2 Some examples

Coherent spaces with the same quantum space can have very different symmetry groups. Typically, the largest groups are associated with projective coherent spaces. We illustrate this here with two simple coherent spaces. Another large class of examples of this situation is treated extensively in Section 6.

3.9 Example. (Szegő [32], 1911) The Szegő space is the coherent space $Z$ defined on the open unit disk in $\mathbb{C}$,

$$D(0,1) := \{ z \in \mathbb{C} \mid |z| < 1 \},$$

by the coherent product

$$K(z,z') := (1 - zz')^{-1},$$

the inverse is defined since $|zz'| < 1$. A corresponding quantum space is the Hardy space of power series

$$f(x) = \sum_{\ell=0}^{\infty} f_\ell x^\ell$$

such that

$$\|f\| := \sqrt{\sum |f_\ell|^2} < \infty,$$

describing analytic functions on $Z$ that are square integrable over the positively oriented boundary $\partial Z$ of $Z$, with inner product

$$f^\ast g := \sum f_\ell g_\ell = \int_0^{2\pi} d\phi f(e^{i\phi}) g(e^{i\phi}) = \int_{\partial Z} |dz| f(z) g(z).$$

The associated coherent states are the functions

$$k_z(x) = (1 - zx)^{-1}$$

with $(k_z)_\ell = z^\ell$, since

$$k_z^\ast k_{z'} = \sum z^\ell (z')^\ell = \frac{1}{1 - \overline{zz'}} = K(z,z').$$

Clearly, the scalar multiplication maps $z \to \lambda z$ for $\lambda \in D(0,1)$ are coherent, with the complex conjugate as adjoint. There are no other coherent maps with adjoints. Indeed, suppose $A : Z \to Z$ is a coherent map with adjoint $A^\ast$. Then for every $z, z' \in D(0,1)$,

$$A \overline{z} z' = 1 - K(A z, z')^{-1} = 1 - K(z, A^\ast z')^{-1} = \overline{z} A^\ast z'.$$

Thus $A z = \lambda z$ for $\lambda := A^\ast z'/z'$ with any fixed nonzero $z'$.

3.10 Example. The Möbius space $Z := \{ z \in \mathbb{C}^2 \mid |z_1| > |z_2| \}$ is a coherent space with coherent product

$$K(z,z') := (\overline{z}_1 z'_1 - \overline{z}_2 z'_2)^{-1}.$$
with the same quantum spaces as the Szegö space. Indeed, the functions

$$f_z(x) = (z_1 - z_2 x)^{-1}$$  \hspace{1cm} (22)

from the Szegö space from Example 3.9 are associated Möbius coherent states. The Möbius space is a projective coherent space of degree $-1$; indeed, with the scalar multiplication induced from $C^2$, we have

$$K(z, \lambda z') = (\overline{z}_1 \lambda z'_1 - \overline{z}_2 \lambda z'_2)^{-1} = \lambda^{-1} (\overline{z}_1 z'_1 - \overline{z}_2 z'_2)^{-1} = \lambda^{-1} K(z, z')$$

for all $z, z' \in Z$. It is now easy to see that the projective completion of the Szegö space for this degree is isomorphic to the Möbius space.

Unlike the Szegö space, the Möbius space has a large symmetry group. Indeed, for any $A \in C^{2 \times 2}$ we have

$$|(A z)_1|^2 - |(A z)_2|^2 = \alpha |z_1|^2 + 2 \Re (\beta \overline{z}_1 z_2) - \gamma |z_2|^2$$  \hspace{1cm} (23)

with

$$\alpha := |A_{11}|^2 - |A_{21}|^2, \quad \beta := \overline{A}_{11} A_{12} - \overline{A}_{21} A_{22}, \quad \gamma := |A_{22}|^2 - |A_{12}|^2,$$

If $z \in Z$ and

$$\alpha > 0, \quad |\beta| \leq \alpha, \quad \gamma \leq \alpha - 2|\beta|$$  \hspace{1cm} (24)

we write $\beta = |\beta| \delta$ with $|\delta| = 1$ and obtain from (23)

$$|(A z)_1|^2 - |(A z)_2|^2 = \alpha |z_1|^2 + 2 |\beta| \Re (\delta \overline{z}_1 z_2) - \gamma |z_2|^2 \geq \alpha |z_1|^2 + 2 |\beta| \Re (\delta \overline{z}_1 z_2) + (2 |\beta| - \alpha) |z_2|^2 = |\beta| |z_1 + \delta z_2|^2 + (|\alpha - |\beta|| |z_1|^2 - |z_2|^2|) \geq 0.$$

Equality in the last step is possible only if $|\beta| = \alpha > 0$ and $z_1 + \delta z_2 = 0$, contradicting $|z_1| > |z_2|$. Hence $A z \in Z$. Thus $A$ maps $Z$ into itself whenever (24) holds. Now

$$K(z, A z') = (\overline{z}_1 (A_{11} z'_1 + A_{12} z'_2) - \overline{z}_2 (A_{21} z'_1 + A_{22} z'_2))^{-1} = (A_{11} \overline{z}_1 - A_{21} \overline{z}_2) z'_1 - (A_{12} \overline{z}_1 + A_{22} \overline{z}_2) z'_2)^{-1} = K(A^* z, z'),$$

where

$$A^* = \left( \begin{array}{cc} \overline{A}_{11} & -\overline{A}_{21} \\ -\overline{A}_{12} & \overline{A}_{22} \end{array} \right).$$

Thus every linear mapping $A \in C^{2 \times 2}$ satisfying (24) is coherent, with adjoint $A^*$ given by $A^*$ rather than by the standard matrix adjoint. These mappings form a semigroup, a homogeneous version of an Olshanski semigroup of compressions (Olishanski [26]). By (24), $A$ preserves the Hermitian form $|z_1|^2 - |z_2|^2$ up to a positive scaling factor iff $\beta = 0$ and $\gamma = \alpha > 0$. This implies (24) and is equivalent with

$$|A_{11} A_{12} = |A_{21} A_{22}, \quad |A_{22}| = |A_{11}| > |A_{21}|.$$  \hspace{1cm} (25)

Indeed, the first equation and the inequality follow from $\beta = 0$ and $\alpha > 0$. From the first equation,

$$\alpha = \gamma = |A_{22}|^2 - |A_{12}|^2 = |A_{22}|^2 - \left( |A_{21}| |A_{22}| / |A_{11}| \right)^2 = \alpha \left( |A_{22}| / |A_{11}| \right)^2,$$

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giving \(|A_{22}| = |A_{11}|\). Thus the group \(GU(1, 1)\) of all matrices satisfying \((22)\) is a group of symmetries of \(Z\). This fact is relevant for applications to quantum systems with a dynamical symmetry group \(SU(1, 1)\) or the closely related groups \(SO(2, 1), SL(2, \mathbb{R})\), which are subgroups of \(GU(1, 1)\).

### 3.3 Quantization of coherent maps

#### 3.11 Theorem. Let \(Z\) be a coherent space, \(\mathcal{Q}(Z)\) a quantum space of \(Z\), and let \(A\) be a coherent map on \(Z\).

(i) There is a unique linear map \(\Gamma(A) \in \text{Lin} \mathcal{Q}(Z)\) such that

\[
\Gamma(A)|z\rangle = |Az\rangle \quad \text{for all } z \in Z.
\]

(ii) For any adjoint map \(A^*\) of \(A\),

\[
\langle z|\Gamma(A) = \langle A^*z| \quad \text{for all } z \in Z,
\]

\[
\Gamma(A)^*|_{\mathcal{Q}(Z)} = \Gamma(A^*).
\]

(iii) The definition \(\Gamma(A) := \Gamma(A^*)^*\) extends \(\Gamma(A)\) to a linear map \(\Gamma(A) \in \text{Lin} \mathcal{Q}^*(Z)\).

We call \(\Gamma(A)\) and its extension the quantization of \(A\) and \(\Gamma\) the quantization map. In the special case (discussed in Section 6 below) where \(Z\) is a Klauder space, the quantum space \(\mathcal{Q}(Z)\) is a dense subspace of a bosonic Fock space and the quantization map is the restriction of the second quantization map of Dereziński & Gérard \[8\] to \(\mathcal{Q}(Z)\). This terminology goes back to Fock \[10\].

**Proof.** (i) Let \(A : Z \to Z\) be a coherent map and \(S : Z \times Z \to \mathbb{C}\) be the kernel given by

\[
S(z, z') := K(z, Az') \quad \text{for all } z, z' \in Z.
\]

We first show that for all \(z \in Z\), \(S(\cdot, z)\) and \(\overline{S}(z, \cdot)\) are admissible functions. Suppose that \(\sum_\ell c_\ell|z_\ell\rangle = 0\). Then

\[
\sum c_\ell S(z_\ell, z') = \sum c_\ell K(z_\ell, Az') = \left\langle \sum c_\ell z_\ell \right| Az' \right\rangle = 0,
\]

proving that \(S(\cdot, z)\) is admissible. Similarly,

\[
\sum c_\ell \overline{S}(z, z_\ell) = \sum c_\ell K(z, A\overline{z_\ell}) = \sum c_\ell K(A\overline{z_\ell}, z) = \left\langle \sum c_\ell \overline{z_\ell} \right| A^*z \right\rangle = 0,
\]

proving that \(\overline{S}(z, \cdot)\) is admissible. By Theorem 2.5, there is a unique linear operator \(\Gamma(A) : \mathcal{Q}(Z) \to \mathcal{Q}(Z)^*\) satisfying

\[
S(z, z') = \langle z|\Gamma(A)|z'\rangle \quad \text{for all } z, z' \in Z,
\]

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and it is automatically continuous. To prove the theorem we need to show that the images are actually in \(Q(Z)\). Using (26), we have
\[
\langle z|Az' \rangle = K(z, Az') = S(z, z') = \langle z|\Gamma(A)|z' \rangle \quad \text{for all } z, z' \in Z.
\]
which implies that \(\Gamma(A)|z' = |Az'\) for all \(z' \in Z\). We conclude that \(\Gamma(A)\) maps \(Q(Z)\) already into the smaller space \(Q(Z)\). Hence \(\Gamma(A) \in \text{Lin} \ Q(Z)\).

(ii) Let \(z \in Z\) and \(\phi = \sum c_k|z_k \rangle \in Q(Z)\). Then (27) follows from
\[
\left(\Gamma(A)^* \langle z \right| \phi \rangle = \langle z|\Gamma(A)\phi = \langle z|\Gamma(A) \sum c_k|z_k \rangle = \langle z| \sum c_k\Gamma(A)|z_k \rangle
\]
\[
= \langle z| \sum c_k|Az_k \rangle = \sum c_k \langle z|Az_k \rangle = \sum c_k \langle A^* z|z_k \rangle
\]
\[
= \langle A^* z| \sum c_k|z_k \rangle = \langle A^* z|\phi
\]
By Theorem 3.5(i), the map \(A^*\) is coherent as well. Thus we have
\[
\langle z|\Gamma(A)^*|z' \rangle = \langle z'|\Gamma(A)\rangle = \langle z'|Az \rangle = \overline{K(z', Az)} = K(Az, z') = K(z, A^* z') = \langle z|A^* z' \rangle = \langle z|\Gamma(A^*)|z' \rangle,
\]
which implies that the restriction of \(\Gamma(A)^*\) to \(Q(Z)\) is precisely \(\Gamma(A^*)\), as claimed.

(iii) is a simple consequence of (i) and (ii).

We now show that the quantization map \(\Gamma\) furnishes a representation of the semigroup of coherent maps on \(Z\) in the quantum space of \(Z\).

3.12 Theorem. The quantization map \(\Gamma\) has the following properties.

(i) The identity map \(1\) on \(Z\) is coherent, and \(\Gamma(1) = 1\).

(ii) For any two coherent maps \(A, B\) on \(Z\),
\[
\Gamma(AB) = \Gamma(A)\Gamma(B).
\]

(iii) For any invertible coherent map \(A : Z \rightarrow Z\) with an invertible adjoint, \(\Gamma(A)\) is invertible with inverse
\[
\Gamma(A)^{-1} = \Gamma(A^{-1}).
\]

(iv) For a coherent map \(A : Z \rightarrow Z\), \(A\) is unitary iff \(\Gamma(A)\) is unitary.

Proof. (i) is straightforward.

(ii) Let \(A, B\) be coherent maps and \(z, z' \in Z\). Then we have
\[
\langle z|\Gamma(AB)|z' \rangle = K(z, ABz') = \langle z|\Gamma(A)|Bz' \rangle = \langle z|\Gamma(A)\Gamma(B)|z' \rangle,
\]
which implies that \(\Gamma(AB) = \Gamma(A)\Gamma(B)\).
(iii) follows from $\Gamma(1) = 1$ and the fact that $AA^{-1} = A^{-1}A = 1$. Indeed, using Theorem 3.5(ii), $A^{-1}$ is coherent and we get
\[\Gamma(A)\Gamma(A^{-1}) = \Gamma(AA^{-1}) = \Gamma(1) = \Gamma(A^{-1}A) = \Gamma(A^{-1})\Gamma(A),\]
which implies that $\Gamma(A)$ is invertible with $\Gamma(A)^{-1} = \Gamma(A^{-1})$.

(iv) Let $A$ be a coherent map. Also, suppose that $A$ is unitary as well. Then, $A$ is invertible with the inverse $A^{-1} = A^*$. Thus, $A$ and $A^*$ are invertible. Then, we get
\[\Gamma(A)\Gamma(A^*) = \Gamma(A)\Gamma(A^*) = \Gamma(AA^*) = \Gamma(1) = 1,\]
and also
\[\Gamma(A)^*\Gamma(A) = \Gamma(A^*)\Gamma(A) = \Gamma(A^*A) = \Gamma(1) = 1.\]
Hence, we deduce that $\Gamma(A)$ is a unitary linear operator. Conversely, assume that $\Gamma(A)$ is a unitary linear operator. Then we get $AA^* = 1$ and also $A^*A = 1$, which means that $A$ is unitary.

The quantization map is important as it reduces many computations with coherent operators in the quantum space of $Z$ to computations in the coherent space $Z$ itself. By Theorem 3.12, large semigroups of coherent maps $A$ produce large semigroups of coherent operators $\Gamma(A)$, which may make complex calculations much more tractable.

### 3.4 Geometric quantization and quantum field theory

In this subsection we discuss informally connections between coherent spaces and topics from geometric quantization and quantum field theory. Later, we give full details for one particular case, the case of Klauder spaces and their connection to bosonic Fock spaces. Details regarding the other issues will be discussed elsewhere (Neumaier [24]).

**Line bundles and central extensions.** Example 3.10 generalizes to central extensions of other semisimple Lie groups and associated line bundles over symmetric spaces. This follows from the material on the corresponding coherent states discussed in detail in Perelomov [28] from a group theoretic point of view, and in Zhang et al. [36] in terms of applications to quantum mechanics. Coherent spaces with the structure of a vector bundle also accommodate the vector coherent states of Rowe et al. [31] and Bertram & Hilgert [6]. Other related material is in the books by Faraut & Korányi [9], Neeb [21], and Neretin [22].

In the applications, a group $G$ of quantum symmetries is typically first defined classically on a symmetric space. In a quantization step, it is then represented by a unitary representation on a Hilbert space. Typically, unitary representations of the symmetry groups of symmetric spaces are only projective representations, defined in terms of a family of multipliers satisfying a cocycle condition. This central extension (defined through the respective cocycle) is represented linearly in the Hilbert space defined through the geometric quantization procedure.
Kähler potentials. Therefore, in geometric quantization (Woodhouse [35]), the symmetric space (typically a Kähler manifold) needs to be extended to a line bundle on which a central extension of the group acts classically. This line bundle can be turned in various ways into coherent spaces by defining the coherent product as the exponential of suitable Kähler potentials. For the symmetric spaces associated to finite-dimensional semisimple Lie groups, the corresponding unitary representations and their Kähler potentials are constructed in papers by Bar Moshe & Marinov [3, 4].

In the coherent space setting, the coherent product defined on an orbit $Z$ of $G$ on the symmetric space $Z$ via the coherent states available from geometric quantization leads in these cases to a coherent space. However, on this space, most elements of $G$ are not represented coherently since they only satisfy a relation (21) with multipliers that are not constant. Theorem 3.8 shows that the projective extension $\mathcal{P}Z$ of degree 1 represents the central extension coherently. This shows that projective coherent spaces are the natural starting point for coherent quantization since they represent all classically visible symmetries in a coherent way. The projective property is therefore typically needed whenever one has a quantum system given in terms of a coherent space and wants to describe all symmetries of the quantum system through coherent maps.

Bosonic Fock spaces. A very important class of Hilbert spaces is the family of bosonic Fock spaces (Fock [10]). They are indispensable in quantum field theory (Baez et al. [2], Dereziński & Gérard [8], Glimm & Jaffe [13]) and the theory of Hida distributions in the white noise calculus for classical stochastic processes (Hida & Si [15], Hida & Streit [16], Obata [25]). In Section 6 we discuss in some detail Klauder spaces, a class of coherent spaces whose completed quantum spaces of Klauder coherent spaces are shown in Section 6.3 to be the bosonic Fock spaces. We identify operators on these quantum spaces corresponding to creation and annihilation operators in Fock space, and prove their basic properties. In particular, we prove the Weyl relations, the canonical commutation relations, without the need to know a particular realization of the quantum space. We also show that the abstract normal ordering, introduced in Subsection 5.3 below, reduces for Klauder spaces to that familiar from traditional second quantization. Klauder spaces have a large semigroup of coherent maps; their quantization defines normally ordered exponentials of certain inhomogeneous quadratics in $a^*$ and $a$ corresponding to a particular class of Bogoliubov transformations in quantum field theory.

Squeezed states and metaplectic groups. Bosonic Fock spaces are also completed quantum spaces of a class of coherent spaces containing the labels for all squeezed states (cf. Zhang et al. [36] for the case of finitely many modes, and Várilly & Gracia-Bondía et al. [34] for the case of infinitely many modes). Geometrically, these coherent spaces are complex line bundles over symmetric spaces carrying a unitary representation of a metaplectic group. Therefore the metaplectic groups are realized by coherent maps on these coherent spaces. Their quantization leads to normally ordered exponentials of all sufficiently regular homogeneous quadratics in $a^*$ and $a$. Even larger coherent spaces contain coherent maps realizing the semidirect product of the metaplectic group with certain Heisenberg groups. Their quantization leads to normally ordered exponentials of all sufficiently regular inhomogeneous quadratics in $a^*$ and $a$. 

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**Fermionic Fock spaces and spin groups.** Statements analogous to those for meta-plecic groups and bosonic Fock spaces can be proved for spin groups and fermionic Fock spaces. Now the coherent spaces, which we call Hua spaces for reasons discussed in a moment, are complex line bundles over symmetric spaces carrying a unitary representation of a spin group. Therefore the spin groups are realized by coherent maps on Hua spaces. Their quantization again leads to normally ordered exponentials of all sufficiently regular homogeneous quadratics in $a^*$ and $a$.

Traditionally, the fermionic Fock space construction is based on a distinguished vacuum state – the 0-particle state. However, in quantum field theory, many vectors in a fermionic Fock space may serve as a potential vacuum state; the vacuum of a Fock space depends on the Hamiltonian under consideration. In time-dependent systems, the Hamiltonian and hence the corresponding vacuum state changes with time. Thus the structure of interest in quantum field theory is not the Fock space itself but a mathematically precise version of “what is left from Fock space when no vacuum state is distinguished”. This Fock space stripped of a special vacuum state is, as a vector space, the same object as the Fock space itself, but each choice of a vacuum defines a different associated multiparticle structure, anticommuting algebra, and exterior algebra, related to each other by Bogoliubov transformations.

This becomes transparently encoded in terms of the associated Hua spaces. Seen abstractly, these are line bundles of symmetric spaces without any distinguished origin. This like using an affine space where all points are on equal footing, as compared to a vector space that has a distinguished origin. Just as in the case of affine spaces, its coordinatization requires the choice of an origin. Its stabilizer defines (in the simplest case of Fock spaces with only finitely many modes) a particular group $U(n)$ that gives rise to an identification of the symmetric space with a homogeneous space $SO(2n)/U(n)$ in the coordinates used by Zhang et al.. However, just as in the case of affine spaces, this choice of origin is immaterial due to the presence of coherent maps that move the origin to an arbitrary point. In particular, the associated quantum space has no distinguished state vector – only the whole class of coherent states is distinguished, and any of them may serve as a potential vacuum state.

In the case of charged matter in a time-dependent external electromagnetic field (the only case whose quantization is truly understood, Gracia-Bondía & Várilly et al. [14]), the single-particle dynamics is given by a linear Dirac equation in this Euclidean space with (in general) time-dependent Hamiltonian $H(t)$, called a Dirac operator. In the language of second quantization, this Euclidean space is called the 1-particle space, although its classical meaning is that of a space of half-densities for charged fermionic matter. As is well-known, a Dirac operator has a real spectrum unbounded in both directions. At all times where this Dirac operator has no generalized eigenstate with zero energy, the invariant subspace spanned by the positive eigenfunctions of the Dirac operator is a maximal isotropic subspace of the classical symplectic form defined by the imaginary part of the Hilbert inner product. Unless the Dirac operator happens to be time independent, this subspace changes with time, and defines the coherent state specifying the correct vacuum state at each time $t$.

Therefore the coherent rays $\mathbb{C}|z\rangle$ of a Hua space are in one-to one correspondence with
the orthogonal projectors $P$ to a maximal isotropic subspace of the Euclidean space. $1 - P$ projects to a complementary maximal isotropic subspace. A description of the symmetric space associated with $SO(2n)/U(n)$ in terms of such pairs of complementary maximal isotropic subspaces was first given by Hua [17, Section II], who showed that they have the structure of a metric space with a distance function taking integer values only. Thus the points form an infinite bipartite graph, in modern terminology (see Brouwer et al. [7]) called the dual polar graph $D_{n}(C)$, and the $SO(2n)$ acts as a distance transitive group of automorphism on each bipartite half.

A Hua space is a coherent space defined (in the finite mode case) on the canonically associated complex line bundle via a canonical Kähler structure. In the applications to quantum field theory, a nontrivial limit $n \to \infty$ must be taken, and one must consider infinite-dimensional Hua spaces. This limit $n \to \infty$ involves renormalization issues. With the current knowledge, one can do nonperturbative renormalization only in 2-dimensional quantum field theories, where normal ordering is all that is needed – see Pressley & Segal [30].

4 Homogeneous and separable maps

In this section we look at self-mappings of coherent spaces satisfying homogeneity or separability properties. These often give simple but important coherent maps.

4.1 Homogeneous maps and multipliers

Let $Z$ be a coherent space. We say that a function $m : Z \to \mathbb{C}$ is a multiplier for the map $A : Z \to Z$ if

$$|z'| = \lambda |z\rangle \Rightarrow m(z') |Az\rangle = \lambda m(z) |Az\rangle;$$

(31)

for all $\lambda \in \mathbb{C}$ and $z, z' \in Z$, equivalently if

$$K(w, z') = \lambda K(w, z) \forall w \in Z \Rightarrow m(z') K(w, Az') = \lambda m(z) K(w, Az) \forall w \in Z.$$  

A function $m : Z \to \mathbb{C}$ is called homogeneous if

$$|z'| = \lambda |z\rangle, \lambda \neq 0 \Rightarrow m(z') = m(z);$$

(32)

this is the case iff it is a multiplier for the identity map.

We call a map $A : Z \to Z$ homogeneous if

$$|z'| = \lambda |z\rangle \Rightarrow |Az'\rangle = \lambda |Az\rangle;$$

(33)

this is the case iff $m = 1$ is a multiplier for $A$. We write $\text{Hom} Z$ for the set of all homogeneous maps $A : Z \to Z$.  

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4.1 Theorem. Let $Z$ be a coherent space. Then,

(i) each coherent map is homogeneous.

(ii) the composition of any two homogeneous maps is homogeneous.

Proof. (i) Let $A$ be a coherent map with an adjoint $A^*$. Suppose that $z, z' \in Z$ and $\lambda \in \mathbb{C}^\times$ with $|z'| = \lambda |z|$. Then, for $z'' \in Z$, we get

$$\langle z'' | Az' \rangle = \langle A^* z'' | z' \rangle = \lambda \langle A^* z'' | z \rangle = \lambda \langle z'' | Az \rangle.$$  

Thus $|Az'| = \lambda |Az|$. Therefore, $m = 1$ is a multiplier for $A$ and hence $A$ is homogeneous.

(ii) Let $A, B \in \operatorname{Hom}(Z)$. Suppose that $z, z' \in Z$ and $\lambda \in \mathbb{C}^\times$ with $|z'| = \lambda |z|$. Since $B$ is homogeneous, we have $|Bz'| = \lambda |Bz|$. Then applying homogeneity of $A$, we have $|ABz'| = \lambda |ABz|$. Therefore, $m = 1$ is a multiplier for $AB$ and hence $AB$ is homogeneous.$\quad \Box$

4.2 Theorem. Let $Z$ be a projective coherent space. We then have

$$K(z, \lambda z') = K(\lambda z, z'),$$

for all $z, z' \in Z$ and $\lambda \in \mathbb{C}^\times$. In particular, if $Z$ is a nondegenerate and projective coherent space the scalar multiplication map $\lambda : Z \to Z$ is coherent, with unique adjoint $\lambda^* = \overline{\lambda}$.

Proof. Let $\lambda \in \mathbb{C}^\times$ be given. Then, for $z, z' \in Z$, we have

$$K(\lambda z, z') = K(z', \lambda z) = \lambda \lambda^* K(z', z) = \lambda^e K(z, z') = K(\lambda z, z').$$

In particular, if $Z$ is nondegenerate then the multiplication map $\lambda$ is coherent with the unique adjoint $\overline{\lambda}$. $\quad \Box$

4.3 Proposition. Let $Z$ be a projective and non-degenerate coherent space. Then:

(i) $m : Z \to \mathbb{C}$ is a multiplier for $A : Z \to Z$ iff

$$m(\mu z) | A \mu z \rangle = m(z) | \mu Az \rangle \quad \text{for all } \mu \in \mathbb{C}^\times.$$

(ii) A map $A : Z \to Z$ is homogeneous iff $A \mu = \mu A$ for all $\mu \in \mathbb{C}^\times$.

(iii) A map $m : Z \to \mathbb{C}$ is homogeneous iff $m \mu = \mu m$ for all $\mu \in \mathbb{C}^\times$.

Proof. In a projective coherent space of degree $e$, $|\lambda z\rangle = \lambda^e |z\rangle$. Nondegeneracy therefore implies that $|z'| = \lambda |z\rangle$ iff $z' = \mu z$ for some choice of the $e$th root $\mu = \lambda^{1/e}$. The definition of a multiplier now gives (i), and a straightforward specialization gives (ii) and (iii). $\quad \Box$
4.2 Factorizing maps

Let $Z$ be a coherent space. We call a map $\alpha : Z \to Z$ factorizing if there is a number $\chi(\alpha) \in \mathbb{C}$, called a factorization constant, such that

$$K(z, \alpha z') = \chi(\alpha)K(z, z') \quad \text{for} \quad z, z' \in Z. \quad (34)$$

4.4 Proposition. Let $Z$ be a coherent space and $\alpha : Z \to Z$ be a map. Then, $\alpha$ is factorizing iff there exists a complex constant $\lambda_{\alpha}$, such that for any quantum space $Q(Z)$ of $Z$ we have

$$|\alpha z\rangle = \lambda_{\alpha}|z\rangle \quad \text{for all} \quad z \in Z. \quad (35)$$

In this case, $\chi(\alpha) = \lambda_{\alpha}$.

Proof. Let $Q(Z)$ be a quantum space of $Z$ and $z, z' \in Z$. If $\alpha$ is factorizing with the factorization constant $\chi(\alpha)$, then

$$\langle z'|\alpha z \rangle = K(z', \alpha z) = \chi(\alpha)K(z', z) = \chi(\alpha)\langle z'|z \rangle = \langle z'|\chi(\alpha)|z\rangle.$$ 

Hence $|\alpha z\rangle = \chi(\alpha)|z\rangle$ and (35) holds with $\lambda_{\alpha} := \chi(\alpha)$. Conversely, suppose that (35) holds for some complex number $\lambda_{\alpha}$. Then, for $z, z' \in Z$, we get

$$K(z', \alpha z) = \langle z'|\alpha z \rangle = \langle z'|\lambda_{\alpha}|z\rangle = \lambda_{\alpha}\langle z'|z \rangle = \lambda_{\alpha}K(z', z).$$

This implies that $\alpha$ is a factorizing map with the factorization constant $\chi(\alpha) := \lambda_{\alpha}$. \qed

4.5 Proposition.
(i) Every factorizing map $\alpha : Z \to Z$ satisfies

$$K(\alpha z, z') = \overline{\chi(\alpha)}K(z, z') \quad \text{for} \quad z, z' \in Z. \quad (36)$$

(ii) Every factorizing map $\alpha$ with $\chi(\alpha) = 1$ is coherent, with adjoint 1.

(iii) Every factorizing map $\alpha : Z \to Z$ satisfies

$$K(\alpha z, \alpha z') = |\chi(\alpha)|^2K(z, z') \quad \text{for all} \quad z, z' \in Z.$$

(iv) Every factorizing map is homogeneous.

Proof. (i) and (ii) are straightforward.

(iii) Let $\alpha \in \text{Fac} Z$ and $z, z' \in Z$. Then (36) implies

$$K(\alpha z, \alpha z') = \overline{\chi(\alpha)}K(z, z') = \chi(\alpha)\overline{\chi(\alpha)}K(z, z') = |\chi(\alpha)|^2K(z, z').$$

(iv) Let $\alpha : Z \to Z$ be a factorizing map with the factorization constant $\chi(\alpha)$. Suppose that $z, z' \in Z$ and $\lambda \in \mathbb{C}^\times$ with $|z'\rangle = \lambda|z\rangle$. Then, for $z'' \in Z$,

$$\langle z''|\alpha z' \rangle = K(z'', \alpha z') = \chi(\alpha)K(z'', z') = \chi(\alpha)\lambda\langle z''|z \rangle = \lambda\langle z''|\alpha z \rangle.$$
Thus $|\alpha z'| = \lambda |\alpha z\rangle$. Therefore, $m = 1$ is a multiplier for $\alpha$ and hence $\alpha$ is homogeneous.

We denote the set of all factorizing maps by $\text{Fac} Z$ and the set of all factorizing maps with nonzero factorization constants by $\text{Fac}_\times Z$. It is easy to check that any invertible factorizing map has a nonzero factorization constant.

**4.6 Proposition.** Let $Z$ be a coherent space. Then:

(i) The identity $1$ is a factorizing map with $\chi(1) = 1$.

(ii) The composition of factorizing maps is factorizing.

(iii) Any adjoint $\alpha^*$ of a coherent and factorizing map $\alpha$ is factorizing with $\chi(\alpha^*) = \overline{\chi(\alpha)}$.

(iv) The inverse $\alpha^{-1}$ of any invertible factorizing map is factorizing with $\chi(\alpha^{-1}) = \overline{\chi(\alpha)}^{-1}$.

**Proof.** (i) and (ii) are straightforward.

(iii) Let $\alpha : Z \to Z$ be a coherent and factorizing map and let $\alpha^*$ be an adjoint for $\alpha$. Using (36) we find for $z, z' \in Z$,

$$K(\alpha^* z, z') = K(z, \alpha z') = \chi(\alpha) K(z, z').$$

This implies that $\alpha^*$ is factorizing with $\chi(\alpha^*) := \overline{\chi(\alpha)}$.

(iv) Let $\alpha \in \text{Fac} Z$ be invertible with the inverse $\alpha^{-1}$. Since $\chi(\alpha) \neq 0$, for $z, z' \in Z$, we have

$$K(\alpha^{-1} z, z') = \overline{\chi(\alpha^{-1})} K(\alpha \alpha^{-1} z, z') = \overline{\chi(\alpha)^{-1}} K(z, z'),$$

which implies that $\alpha^{-1}$ is factorizing with factorization constant $\chi(\alpha^{-1}) := \overline{\chi(\alpha)}^{-1}$.

**4.7 Proposition.** Let $Z$ be a coherent space. Then,

(i) $\text{Fac} Z$ is a semigroup with identity.

(ii) $\text{Fac} Z \cap \text{Coh} Z$ is $\ast$-semigroup.

(iii) The factorizing maps $\alpha$ with $\chi(\alpha) = 1$ form a subsemigroup $\text{Fac}_1(Z)$ of $\text{Fac} Z$.

(iv) In the nondegenerate case, $\chi$ is an injective multiplicative homomorphism into $\mathbb{C}$ and $\text{Fac}_1(Z)$ consists of the identity only.

(v) Each factorizing map $\alpha$ with $|\chi(\alpha)| = 1$ preserves the coherent product. In particular, elements of $\text{Fac}_1(Z)$ preserves the coherent product.

**Proof.** Straightforward.  

□
4.8 Theorem. Let $Z$ be a coherent space. Then $Z_\times := (\text{Fac } Z) \times Z$ with the coherent product
\[ K_\times((\alpha, z); (\alpha', z')) := K(\alpha z, \alpha' z') \quad \text{for all } (\alpha, z), (\alpha', z') \in Z_\times \]
is a coherent space.

Proof. Let $\alpha_1, \ldots, \alpha_n \in \text{Fac } Z$ and $z_1, \ldots, z_n \in Z$. Then, for all $c_1, \ldots, c_n \in \mathbb{C}$, we have
\[
\sum_{j,k} c_j c_k K_\times((\alpha_j, z_j); (\alpha_k, z_k)) = \sum_{j,k} c_j c_k K(\alpha_k z_j, \alpha_j z_k) = \sum_{j,k} c_j c_k \chi(\alpha_k) K(z_j, z_k) = \sum_{j,k} d_j d_k K(z_j, z_k) \geq 0,
\]
where $d_\ell := c_\ell \chi(\alpha_\ell)$ for $1 \leq \ell \leq n$. □

4.9 Theorem. Let $Z$ be a coherent space. Then, for any $A : Z \to Z$ and $f : Z \to \mathbb{C}^*$, the map $\mathcal{H}_{(\alpha, A)} : PZ \to PZ$ defined on the projective extension $PZ$ via
\[
\mathcal{H}_{(f, A)}(\lambda, z) := (f(z)\lambda, A z) \quad \text{for all } (\lambda, z) \in PZ,
\]
is homogeneous in $\lambda$.

Proof. Let $\lambda, \lambda' \in \mathbb{C}^*$ and $z \in Z$. Then, we have
\[
\mathcal{H}_{(f, A)}(\lambda' \lambda, z) = (f(z)\lambda' \lambda, A z) = (\lambda' f(z)\lambda, A z) = \lambda' (f(z)\lambda, A z) = \lambda' \mathcal{H}_{(f, A)}(\lambda, z).
\]
□

4.10 Proposition. The factorizing maps on a projective and nondegenerate coherent space of degree $e = \pm 1$ are precisely the multiplication maps.

Proof. Clearly each multiplication map on a projective and non-degenerate coherent space is factorizing. Conversely, let $Z$ be such a coherent space and let $\alpha$ be a factorizing map with factorization constant $\chi(\alpha)$. Then, for $z, z' \in Z$,
\[
K(z, \alpha z') = \chi(\alpha) K(z, z') = K(z, \chi(\alpha)^e z')
\]
since $\alpha$ is factorizing and $Z$ is projective. Since $Z$ is nondegenerate we conclude $\alpha z = \chi(\alpha)^e z$. □

For any coherent space $Z$, $PZ$ denotes the projective extension defined in [23 Proposition 5.8.5], with the same quantum spaces as $Z$.
4.11 Theorem. Let \( Z \) be a coherent space, \( S : Z \to Z \) be a factorizing map with factorization constant \( \chi(S) \in \mathbb{C} \). Then, the maps \( A_S : PZ \to PZ \) and \( B_S : PZ \to PZ \) defined via

\[
A_S(\lambda, z) := (\lambda, S z) \quad \text{for all } (\lambda, z) \in PZ,
\]

\[
B_S(\lambda, z) := (\chi(S)\lambda, z) \quad \text{for all } (\lambda, z) \in PZ,
\]

are coherent with \( A_S^* = B_S \) and \( B_S^* = A_S \).

Proof. Let \((\lambda, z), (\lambda', z') \in PZ\). Then, we have

\[
\text{Kpe}(A_S(\lambda, z), (\lambda', z')) = \text{Kpe}((\lambda, S z), (\lambda', z')) = \chi K(S z, z') \lambda' = \chi K(z, z') \chi(S) \lambda' = \text{Kpe}((\lambda, z), B_S(\lambda', z')).
\]

Thus, \( A_S \) is coherent with \( A_S^* = B_S \). This also implies that \( B_S \) is coherent with \( B_S^* = A_S \). \( \square \)

4.12 Proposition. Let \( Z \) be a coherent space. Then:

(i) The map \( P : \mathbb{C} \times \text{Coh } Z \to \text{Coh } PZ \) given by \((\alpha, A) \to [\alpha, A]\) is an anti-homomorphism of \(*\)-semigroups.

(ii) The map \( A : \text{Fac } Z \to \text{Coh } PZ \) given by \( S \to A_S \) is a homomorphism of semigroups.

(iii) The map \( B : \text{Fac } Z \to \text{Coh } PZ \) given by \( S \to B_S \) is a homomorphism of semigroups.

Proof. (i) Let \((\alpha, A), (\beta, B) \in \mathbb{C}^\times \times \text{Coh } Z\). Then, for \((\lambda, z) \in PZ\), we have

\[
(\alpha, A)P(\beta, B)(\lambda, z) = P(\beta, B)(\alpha\lambda, Az) = P(\beta, B)(\alpha\lambda, BAz) = P(\beta, B)(\alpha, A)(\lambda, z).
\]

(ii) Let \( S, S' \in \text{Fac } Z \). Then, for \((\lambda, z) \in PZ\), we have

\[
A_{SS'}(\lambda, z) = (\lambda, SS'z) = A_S(\lambda, S'z) = A_S A_{S'}(\lambda, z).
\]

(iii) Let \( S, S' \in \text{Fac } Z \). Then, for \((\lambda, z) \in PZ\), we have

\[
B_{SS'}(\lambda, z) = (\chi(SS')\lambda, z) = (\chi(S)\chi(S')\lambda, z) = B_S(\chi(S')\lambda, z) = B_S B_{S'}(\lambda, z).
\]

\( \square \)

4.13 Corollary. Let \( Z \) be a coherent space. Then

\[
\text{Fac } [PZ] \cong \text{Fac } P[Z] \cong \mathbb{C}^\times.
\]

In particular, the map \( \chi : \text{Fac } [PZ] \cong \text{Fac } P[Z] \to \mathbb{C}^\times \) is a group isomorphism.
A map $A : Z \to Z$ is called **strongly homogeneous** if $A\alpha = \alpha A$ for all factorizing maps $\alpha \in \text{Fac } Z$. We write $\text{Hom}_s Z$ for the set of all strongly homogeneous maps over $Z$. It can be readily checked that $\text{Fac } Z \subseteq \text{Hom}_s Z$ and $\text{Hom}_s Z \subseteq \text{Hom}(Z)$.

A function $f : Z \to \mathbb{C}$, or a kernel $X : Z \times Z \to \mathbb{C}$ is called **strongly homogeneous** if 

$$f(\alpha z) = f(z) \quad \text{for } \alpha \in \text{Fac } Z, \ z \in Z,$$

or

$$X(\alpha z, \alpha' z') = X(z, z') \quad \text{for } \alpha, \alpha' \in \text{Fac } Z, \ z, z' \in Z,$$

respectively.

**4.14 Proposition.** Let $Z$ be a coherent space.

(i) Any adjoint of a coherent and strongly homogeneous map is homogeneous.

(ii) The set $\text{Coh } Z \cap \text{Hom}(Z)$ is $*$-subsemigroup of $\text{Coh } Z$.

**Proof.** (i) Let $A : Z \to Z$ be a strongly homogeneous coherent map with an adjoint $A^*$. Then, for all $\alpha \in \text{Fac } Z$, we have

$$K(A^*(\alpha z), z') = K(\alpha z, A z') = \chi(\alpha)K(z, A z') = \chi(\alpha)K(A^*z, z') = K(\alpha A^*z, z'),$$

for all $z, z' \in Z$. Thus, $A^*$ is strongly homogeneous.

(ii) is straightforward. \(\square\)

**4.15 Proposition.** Let $Z$ be a nondegenerate coherent space. Then,

(i) each coherent map is strongly homogeneous.

(ii) $\text{Fac } Z$ is in the center of $\text{Coh } Z$.

(iii) For $z \in Z$, $\alpha \in \text{Fac } Z$, and $A \in \text{Coh } Z$ we have $|A\alpha z\rangle = \chi(\alpha)|Az\rangle$.

**Proof.** (i) Let $A : Z \to Z$ be a coherent map. Then, for all $z, z' \in Z$ and $\alpha \in \text{Fac } Z$, we have

$$K(A\alpha z, z') = K(\alpha z, A^* z') = \overline{\chi(\alpha)}K(z, A^* z') = \overline{\chi(\alpha)}K(Az, z') = K(\alpha Az, z').$$

Since $K$ is nondegenerate over $Z$, we get $A \circ \alpha = \alpha \circ A$ for all $\alpha \in \text{Fac } Z$.

(ii) Let $\alpha \in \text{Fac } Z$ with the factorization constant $\chi(\alpha)$. Also, let $A \in \text{Coh } Z$ be given. Using (i), $A$ is strongly homogeneous as well. Thus, by definition of strongly homogeneous we have $A\alpha = \alpha A$. Hence $\alpha$ belongs to the center of $\text{Coh } Z$.

(iii) By (ii), $|A\alpha z\rangle = |\alpha Az\rangle = \chi(\alpha)|Az\rangle$. \(\square\)
4.16 Proposition. Let $Z$ be a coherent space and $z, z' \in Z$. If there exists a factorizing map $\alpha \in \text{Fac} \ Z$ such that $\alpha z = z'$ then the coherent states $|z\rangle, |z'\rangle$ are parallel. In this case, we have $|z'\rangle = \chi(\alpha)|z\rangle$.

Proof. Suppose that there exists a factorizing map $\alpha \in \text{Fac} \ Z$ such that $\alpha z = z'$. Then, for $w \in Z$, we have
\[
\langle w | z' \rangle = K(w, z') = K(w, \alpha z) = \chi(\alpha) K(w, z) = \chi(\alpha) \langle w | z \rangle.
\]
Thus we get $|z\rangle = \chi(\alpha)|z'\rangle$. \hfill $\Box$

4.17 Remark. If $Z$ is a projective and nondegenerate coherent space then $\text{Hom} \ Z = \text{Hom}_s \ Z$. Indeed, a function $f : Z \to \mathbb{C}$, or a kernel $X : Z \times Z \to \mathbb{C}$ is homogeneous iff
\[
f(\alpha z) = f(z) \quad \text{for } \alpha \in \mathbb{C}^\times, \ z \in Z,
\]
or
\[
X(\alpha z, \alpha' z') = X(z, z') \quad \text{for } \alpha, \alpha' \in \mathbb{C}^\times, \ z, z' \in Z,
\]
respectively.

The next result shows that each coherent map over a projective coherent space is automatically homogeneous as well.

4.18 Corollary. Let $Z$ be a projective and nondegenerate coherent space. Then,

(i) every coherent map is homogeneous.

(ii) $\mathbb{C}^\times$ is in the center of $\text{Coh} \ Z$.

Proof. The results follow directly from Propositions 4.15 and 4.10. \hfill $\Box$

4.19 Corollary. Let $Z$ be a coherent space. Then

(i) $\text{Coh} \ [PZ] \subseteq \text{Hom} \ [PZ]$ and $\text{Coh} \ P[Z] \subseteq \text{Hom} \ P[Z]$.

(ii) $\text{Fac} \ [PZ]$ is in the center of $\text{Coh} \ [PZ]$.

(iii) $\text{Fac} \ P[Z]$ is in the center of $\text{Coh} \ P[Z]$.

Proof. Apply Corollary 4.18 to the projective and non-degenerate spaces $Z' := [PZ]$ and $Z'' := P[Z]$. \hfill $\Box$

4.20 Proposition. Let $Z$ be a coherent space and $z \in Z$. Then

(i) For $\alpha \in \text{Fac} \ Z$ and $A \in \text{Hom}_s \ Z$ we have
\[
|A\alpha z\rangle = \chi(\alpha)|Az\rangle.
\]
For \( A \in \text{Coh} \ Z \) and \( \alpha \in \text{Fac} \ Z \) we have
\[
|A\alpha z\rangle = \chi(\alpha) \Gamma(A) |z\rangle = |\alpha Az\rangle.
\]

Proof. Straightforward.

## 5 Slender coherent spaces

In this section we prove quantization theorems for the class of slender coherent spaces for which the admissibility of functions, defined in Section 2, is particularly easy to check. For slender coherent spaces, many operators on a quantum space have a simple description in terms of normal kernels. These generalize the normal ordering of operators familiar from quantum field theory.

### 5.1 Slender coherent spaces

**5.1 Proposition.** For a coherent space \( Z \), the following are equivalent:

(i) Every function \( f : Z \to \mathbb{C} \) is admissible.

(ii) Every finite set of distinct coherent states is linearly independent.

Proof. If any finite set of distinct coherent states is linearly independent then the hypothesis of (i) implies that all \( c_k \) vanish. Thus each function \( f : Z \to \mathbb{C} \) is admissible. Hence \( A(Z) = \mathbb{C}^Z \). Conversely, suppose that \( A(Z) = \mathbb{C}^Z \), and \( \sum c_\ell |z_\ell\rangle = 0 \) with distinct \( z_\ell \). Then every \( f = \delta_{z_k} \) is admissible and
\[
0 = \sum c_\ell f(z_\ell) = \sum c_\ell \delta_{z_k}(z_\ell) = c_k,
\]
which implies that \( c_k = 0 \). This holds for all \( k \), whence any finite set of distinct coherent states is linearly independent.

The most interesting cases are covered by a slightly more general class of coherent spaces. We call a coherent space **slender** if any finite set of linearly dependent, nonzero coherent states in a quantum space \( \mathbb{Q}(Z) \) of \( Z \) contains two parallel coherent states. Clearly, every subset of a slender coherent space is again a slender coherent space.

**5.2 Proposition.** Let \( S \) be a subset of the Euclidean space \( \mathbb{H} \) such that any two elements of \( S \) are linearly independent. Then the set \( Z = \mathbb{C}^\times \times S \) with the coherent product
\[
K((\lambda, s); (\lambda', s')) := \overline{\lambda} \lambda' s^* s' \quad \text{for all } (\lambda, s), (\lambda', s') \in Z
\]
and scalar multiplication \( \alpha(\lambda, s) := (\alpha \lambda, s) \) is a slender, projective coherent space of degree 1.
Proof. It is easy to see that \( \mathcal{Q}(Z) := \text{Span } S \) is a quantum space of \( Z \). Let the \( z_k \in Z \) be such that \( \sum c_k |z_k\rangle = 0 \) with \( c_k \neq 0 \) for all \( k \). We then have \( z_k = (\lambda_k, z'_k) \) with \( \lambda_k \in \mathbb{C}^\times \) and \( z'_k \in S \), hence

\[
\sum c_k \lambda_k z'_k = \sum c_k |(\lambda_k, z'_k)\rangle = \sum c_k |z_k\rangle = 0.
\]

But the \( z'_k \) are linearly independent, hence \( c_k \lambda_k = 0 \) for all \( k \), and since \( \lambda_k \neq 0 \), all \( c_k \) vanish. Thus \( Z \) is slender. Projectivity is obvious.

Thus slender coherent spaces are very abundant. However, proving slenderness for a given coherent space is a nontrivial matter once \( Z \) contains infinitely many elements.

5.3 Theorem. The Möbius space defined in Example 3.10 is a slender coherent space.

Proof. Suppose that there is a nontrivial finite linear dependence \( \sum c_k |z_k\rangle = 0 \) such that no two \( |z_k\rangle \) are parallel. By definition of \( Z \), the numbers \( \mu_k := z_{k2}/z_{k1} \) satisfy \( |\mu_k| < 1 \). Moreover, \( z_k = z_{k1}(\frac{1}{\mu_k}) \), and since \( Z \) is projective of degree \(-1\),

\[
|z_k\rangle = z_{k1}^{-1}|\left(\frac{1}{\mu_k}\right)\rangle.
\]

Since no two \( |z_k\rangle \) are parallel, the \( \mu_k \) are distinct. Since \( x = \left(\frac{1}{\mu}\right) \in Z \) for \( |\mu| < 1 \), we have

\[
0 = \langle x| \sum c_k |z_k\rangle = \sum c_k K(x, z_k) = \sum \frac{c_k}{x_1z_{k1} - x_2z_{k2}} = \sum \frac{c_k z_{k1}^{-1}}{1 - \mu_k} \text{ for } |\mu| < 1.
\]

The right hand side is the partial fraction decomposition of a rational function of \( \mu \) vanishing in an open set. Since the partial fraction decomposition is unique, each term vanishes. Therefore \( c_k z_{k1}^{-1} = 0 \) for all \( k \), which implies that all \( c_k \) vanish, contradiction. Thus \( Z \) is slender.

5.4 Proposition.

(i) A projective coherent space is slender iff \( \sum_{k \in I} |z_k\rangle = 0 \) implies that there exist distinct \( j, k \in I \) such that \( |z_k\rangle = \alpha |z_j\rangle \) for some \( \alpha \in \mathbb{C} \).

(ii) A nondegenerate projective coherent space is slender iff \( \sum_{k \in I} |z_k\rangle = 0 \) implies that there exist distinct \( j, k \in I \) such that \( z_k = \alpha z_j \) for some \( \alpha \in \mathbb{C} \).

(iii) A coherent space \( Z \) is slender iff its projective extension \( PZ \) is slender.

Proof. In the projective case, \( \sum \alpha_k |z_k\rangle = 0 \) implies \( \sum |\beta_k z_k\rangle = 0 \) with \( \beta_k := \alpha_k^{1/e} \). Thus we may assume w.l.o.g. that the linear combination in the definition of slender is a sum. Hence (i) holds. (ii) is straightforward.
(iii) Let $Z$ be a slender coherent space with a quantum space $Q(Z)$, and let $PZ$ be a projective extension of $Z$ of degree $e$ with the same quantum space $Q(PZ) = Q(Z)$. Let \( \sum_k |(\lambda_k, z_k)\rangle = 0 \) in $Q(PZ)$. Then \( \sum_k \lambda_k^e |z_k\rangle = 0 \) in $Q(Z)$, and we may assume that the sum extends only over the nonzero $\lambda_k$. Since $Z$ is slender, there exists distinct $j, k$ with $\lambda_k \neq 0$ such that $|z_j\rangle = \alpha|z_k\rangle$ for some $\alpha \in \mathbb{C}$. But then

\[
|(\lambda_j, z_j)\rangle = \lambda_j^e |z_j\rangle = \lambda_j^e \alpha |z_k\rangle = \left(\frac{\lambda_j}{\lambda_k}\right)^e \alpha |(\lambda_k, z_k)\rangle.
\]

Thus $PZ$ is slender. The converse is obvious. \( \square \)

5.5 Proposition. Let $Z$ be a slender coherent space and let $Q(Z)$ be a quantum space of $Z$. Let $I$ be a finite index set. If the $z_k \in Z$ ($k \in I$) satisfy \( \sum_{k \in I} c_k |z_k\rangle = 0 \) then there is a partition of $I$ into nonempty subsets $I_t$ ($t \in T$) such that $k \in I_t$ implies $|z_k\rangle = \alpha_k |z_t\rangle$ with \( \sum_{k \in I_t} c_k \alpha_k = 0 \) for all $t \in T$.

Proof. It is easy to see that it is enough to consider the case where none of the $c_k |z_k\rangle$ vanishes, as the general case can be reduced to this case by removing zero contributions to the sum. Let $T$ be a maximal subset of $I$ with the property that no two coherent states $|z_t\rangle$ are multiples of each other. For each $t \in T$, let $I_t$ be the set of $k \in I$ such that $|z_k\rangle$ is a multiple of $|z_t\rangle$, say, $|z_k\rangle = \alpha_k |z_t\rangle$. Then the $I_t$ ($t \in T$) form a partition of $I$. If we define for $t \in T$ the numbers

\[
a_t := \sum_{k \in I_t} c_k \alpha_k
\]

we have

\[
\sum_{t \in T} a_t |z_t\rangle = \sum_{t \in T} \left( \sum_{k \in I_t} c_k \alpha_k \right) |z_t\rangle = \sum_{k \in I} c_k |z_k\rangle = 0.
\]

Since $Z$ is a slender coherent space and no two of the $|z_t\rangle$ ($t \in T$) are parallel, the $|z_t\rangle$ ($t \in T$) are linearly independent. We conclude that all $a_t$ vanish. Therefore \( \sum_{k \in I_t} c_k \alpha_k = 0 \) for all $t \in T$. \( \square \)

5.6 Corollary. Let $Z$ be a slender coherent space, projective of degree $e$ and let $Q(Z)$ be a quantum space of $Z$. Let $I$ be a finite index set. If $z_k \in Z$ ($k \in I$) satisfies \( \sum_{k \in I} |z_k\rangle = 0 \) then there is a partition of $I$ into nonempty subsets $I_t$ ($t \in T$) such that $k \in I_t$ implies $z_k = \alpha_k z_t$ with $\sum_{k \in I_t} \alpha_k^e = 0$, for all $t \in T$. 

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5.2 Quantization theorems

5.7 Theorem. Let $Z$ be a slender coherent space and suppose that $m : Z \to \mathbb{C}$ is a multiplier map for the map $A : Z \to Z$. Then there exists a unique linear operator $\Gamma_m(A) : Q(Z) \to Q(Z)$, the quantization of $A$ relative to $m$, such that

$$\Gamma_m(A)|z\rangle = m(z)|Az\rangle \quad \text{for all } z \in Z.$$ 

Proof. Let $m : Z \to \mathbb{C}$ be a multiplier for the map $A : Z \to Z$. We then define $\Gamma_m(A) : Q(Z) \to Q(Z)$ by

$$\Gamma_m(A)\left( \sum_k c_k|z_k\rangle \right) := \sum_k c_k m(z_k)|Az_k\rangle \quad \text{for all } \sum_k c_k|z_k\rangle \in Q(Z).$$ 

Let $\sum c_k|z_k\rangle = 0$. Then, we have $\sum |c_kz_k\rangle = 0$. Hence, using Proposition 5.5, there is a partition of $I := \{ k : c_k \neq 0 \}$ into nonempty subsets $I_t$ ($t \in T$) such that $k \in I_t$ implies $|z_k\rangle = \alpha_k|z_t\rangle$ with $\sum_{k \in I_t} c_k\alpha_k = 0$, for all $t \in T$. Since $\alpha$ is a multiplier for $A$, we have for $t \in T$ and $k \in I_t$,

$$m(z_k)|Az_k\rangle = \alpha_k m(z_t)|Az_t\rangle.$$ 

Thus, using (38), we get

$$\sum_k c_k m(z_k)|Az_k\rangle = \sum_{t \in T} \sum_{k \in I_t} c_k m(z_k)|Az_k\rangle = \sum_{t \in T} \sum_{k \in I_t} c_k \alpha_k m(z_t)|Az_t\rangle = \sum_{t \in T} \left( \sum_{k \in I_t} c_k \alpha_k \right) m(z_t)|Az_t\rangle = 0.$$

Therefore, $\Gamma_m(A) : Q(Z) \to Q(Z)$ is a well-defined linear map. In particular, we have

$$\Gamma_m(A)|z\rangle = m(z)|Az\rangle \quad \text{for } z \in Z.$$

5.8 Corollary. Let $Z$ be a slender, projective, and non-degenerate coherent space, and let $Q(Z)$ be a quantum space of $Z$. Then for every homogeneous map $A : Z \to Z$, there is a unique linear operator $\Gamma(A) : Q(Z) \to Q(Z)$, the quantization of $A$, such that

$$\Gamma(A)|z\rangle = |Az\rangle \quad \text{for } z \in Z.$$ 

Proof. We define $\Gamma(A) : Q(Z) \to Q(Z)$ by $\Gamma(A) := \Gamma_1(A)$. Then, $\Gamma(A)$ satisfies (39).

Note that the results just proved assume that $Z$ is slender but need minimal assumption about $A$ In contrast, Theorem 5.11 holds for arbitrary coherent spaces, but it assumes
that $A$ is a coherent map. The simple relationship (28) between $\Gamma(A)^*$ and $\Gamma(A^*)$, valid for coherent maps $A$, does not generalize to the situation discussed in the present section.

5.9 Proposition. Let $Z$ be a slender, projective and non-degenerate coherent space. The quantization map $\Gamma : \text{Hom} Z \rightarrow \text{Lin} \mathbb{Q}(Z)$ is a semigroup homomorphism,

$$\Gamma(AB) = \Gamma(A)\Gamma(B) \quad \text{for } A, B \in \text{Hom} Z.$$  \hspace{1cm} (40)

Proof. It is straightforward to check that $AB \in \text{Hom} Z$. By Theorem 5.8,

$$\Gamma(AB)|z\rangle = |ABz\rangle = \Gamma(A)|Bz\rangle = \Gamma(A)\Gamma(B)|z\rangle \quad \text{for all } z \in Z.$$

Thus (40) holds. \hfill \Box

5.10 Theorem. Let $Z$ be a slender coherent space and let $\mathbb{Q}(Z)$ be a quantum space of $Z$. Then for every homogeneous function $m : Z \rightarrow \mathbb{C}$ there is a unique linear operator $a(m) : \mathbb{Q}(Z) \rightarrow \mathbb{Q}(Z)$ such that

$$a(m)|z\rangle = m(z)|z\rangle \quad \text{for } z \in Z.$$  \hspace{1cm} (41)

Proof. We define $a(m) : \mathbb{Q}(Z) \rightarrow \mathbb{Q}(Z)$ by $a(m) := \Gamma_m(1)$. Then (41) follows easily. \hfill \Box

This generalizes the property of traditional coherent states to be eigenstates of annihilator operators. Indeed, in the special case of Klauder spaces treated in Subsection 6.4, the $a(m)$ are found to be the smeared annihilator operators acting on a Fock space.

$a(m)$ is a linear function of $m$. To preserve this property in the adjoint, we define

$$a'(m) := a(\overline{m})^*,$$

the analogues of smeared creation operators. Here $\overline{m}$ is the function defined by

$$\overline{m}(z) := \overline{m(z)},$$

which is homogeneous since $|z'\rangle = \lambda |z\rangle$ implies $m(z') = m(z)$ by homogeneity of $m$, hence $\overline{m}(z') = \overline{m(z')} = \overline{m(z)} = \overline{m(z)}$.

5.3 Normal ordering

A kernel $X : Z \times Z \rightarrow \mathbb{C}$ on a coherent space $Z$ is called homogeneous if, for all $z \in Z$, the functions $X(\cdot, z)$ and $X(z, \cdot)$ are homogeneous in the sense defined in Subsection 4.1. The homogeneous kernels on $Z$ form a vector space $\mathbf{X}(Z)$.
5.11 Theorem. Let $Z$ be a slender coherent space and $Q(Z)$ be a quantum space of $Z$. Then, for every homogeneous kernel $X$, there is a unique linear operator $N(X)$ from $Q(Z)$ to $Q^\times(Z)$, called the normal ordering of $X$, such that
\[
\langle z|N(X)|z' \rangle = X(z,z')K(z,z') \quad \text{for } z, z' \in Z.
\] Equivalently, $N(X)$ defines a Hermitian form on $Q(Z)$.

The name indicates a relation to the normal ordering prescription for operators in quantum field theory; cf. Theorem 6.6 below.

Proof. This follows from Theorem 2.5(ii) and slenderness. To see this, we define, for any two vectors $\phi = \sum_k c'_k|z'_k\rangle$ and $\psi = \sum_\ell c_\ell|z_\ell\rangle$ from $Q(Z)$, the complex number
\[
(\psi, \phi)_X := \sum_\ell \sum_k \overline{c_\ell} c'_k X(z_\ell,z'_k)K(z_\ell,z'_k).
\]

We first claim that $(\psi, \phi) \to (\psi, \phi)_X$ is well-defined. Because $(.,.)_X$ is a Hermitian form in the $c_\ell$ and the $c'_k$, it is enough to show that $\phi = 0$ implies $(\psi, \phi)_X = 0$. By Proposition 5.5 if $\phi = \sum_k c'_k|z'_k\rangle = 0$, there is a partition of $I := \{k : c'_k \neq 0\}$ into nonempty subsets $I_t \ (t \in T)$ such that $k \in I_t$ implies $|z'_k\rangle = \alpha_k|z'_t\rangle$ with $\sum_{k \in I_t} c'_k \alpha_k = 0$ for all $t \in T$. Using the homogeneity assumption of $X(z_\ell, \cdot)$ we find for each $t \in T$ and each $k \in I_t$,
\[
X(z_\ell, z'_t)K(z_\ell, z'_t) = \alpha_k X(z_\ell, z'_t)K(z_\ell, z'_t).
\]

Therefore
\[
\sum_\ell \sum_k \overline{c_\ell} c'_k X(z_\ell,z'_k)K(z_\ell,z'_k) = \sum_\ell \sum_{t \in T} \sum_{k \in I_t} \overline{c_\ell} c'_k X(z_\ell,z'_k)K(z_\ell,z'_k)
\]
\[
= \sum_\ell \sum_{t \in T} \sum_{k \in I_t} \overline{c_\ell} c'_k \alpha_k X(z_\ell,z'_t)K(z_\ell,z'_t)
\]
\[
= \sum_\ell \sum_{t \in T} \overline{c_\ell} \left( \sum_{k \in I_t} c'_k \alpha_k \right) X(z_\ell,z'_t)K(z_\ell,z'_t) = 0.
\]

Using the homogeneity assumption of $X(\cdot, z'_k)$, a similar argument shows that if $\psi = \sum c_\ell|z_\ell\rangle = 0$ then $(\psi, \phi)_X = 0$. Hence, $(\psi, \phi) \to (\psi, \phi)_X$ defines a well-defined Hermitian form on $Q(Z)$. \hfill \Box

5.12 Proposition. Let $Z$ be a slender coherent space whose coher ent product vanishes nowhere. Then any linear operator $X : Q(Z) \to Q(Z)^\times$ is the normal ordering of a unique homogeneous kernel $X$.

Proof. The kernel $X$ defined by
\[
X(z,z') := \frac{\langle z|X|z' \rangle}{K(z,z')}
\]
is homogeneous and satisfies $[33]$. Hence $N(X) = X$ by Theorem $[34]$. □

**5.13 Proposition.** If $A : Z \to Z$ is coherent, homogeneous and invertible then $\Gamma(A)$ is invertible, and for every homogeneous kernel $X$, the kernel $AX$ defined by

$$AX(z, z') := X(A^*z, A^{-1}z') \quad \text{for } z, z' \in Z$$

is homogeneous, and

$$N(AX) = \Gamma(A)N(X)\Gamma(A)^{-1}.$$  

Moreover, if $B : Z \to Z$ is also coherent, homogeneous and invertible then

$$(AB)X = A(BX).$$

**Proof.** This follows from $[34]$ since

$$\langle z | N(AX)\Gamma(A) | z' \rangle = \langle z | N(AX)|Az' \rangle = AX(z, Az')K(z, Az') = X(A^*z, z')K(A^*z, z') = \langle A^*z | N(X) | z' \rangle = \langle z | \Gamma(A)N(X) | z' \rangle$$

and

$$(AB)X(z, z') = X((AB)^*z, (AB)^{-1}z') = X(B^*A^*z, B^{-1}A^{-1}z') = BX(A^*z, A^{-1}z') = A(BX)(z, z').$$

□

Define for $f, g : Z \to C$,

$$(fX)(z, z') := f(z)X(z, z'), \quad (Xf)(z, z') := X(z, z')f(z'),$$

Then

$$(fX)^* = X^*f^*, \quad (Xf)^* = f^*X^*.$$  

We write $\lambda$ for a **constant kernel** with constant value $\lambda \in \mathbb{C}$. Note that $f1$ and $1f$ are different homogeneous kernels!

**5.14 Proposition.**

(i) The normal ordering operator $N : X(Z) \to \text{Lin}^\ast \mathbb{Q}(Z)$ is linear.

(ii) If $X$ is homogeneous then $X^\ast$ is homogeneous and

$$N(X^\ast) = N(X)^\ast.$$ 

(iii) Any constant kernel $\lambda$ is homogeneous, and $N(\lambda) = \lambda$.

(iv) If $N$ is homogeneous then $mNm'$ is homogeneous for all homogeneous $m, m'$, and

$$N(mXm') = a^*(m)N(X)a(m').$$

(v) If $X_t \to X$ pointwise and all $X_t$ are homogeneous then $X$ is homogeneous, and

$$N(X_t) \to N(X).$$
Proof. Statements (i)–(iii) are straightforward.

(iv) Using \(41\) and \(42\), we find
\[
\langle z|ax(m)N(X)ax(m')|z'\rangle = \langle z|a(m*)N(X)a(m')|z'\rangle = \langle z|m(z)N(X)m'(z')|z'\rangle = m(z)\langle z|N(X)|z'\rangle m'(z') = m(z)X(z,z')K(z,z')m'(z').
\]

(v) Let \(X_\ell \to X\) pointwise with all \(X_\ell\) homogeneous. Then \(X\) is homogeneous. Indeed, for \(z, z' \in Z\) and \(c, c' \in \mathbb{C}\), we have
\[
X(cz, c'z') = \lim_\ell X_\ell(cz, c'z') = \lim_\ell X_\ell(z, z') = X(z, z').
\]
We then have
\[
\lim_\ell \langle z|N(X_\ell)|z'\rangle = \left(\lim_\ell X_\ell(z, z')\right)K(z, z') = X(z, z')K(z, z') = \langle z|N(X)|z'\rangle,
\]
for all \(z, z' \in Z\). \(\square\)

5.15 Theorem. Let \(Z\) be a slender coherent space. Let \(S\) be a set, \(d\mu\) a measure on \(S\). Suppose that the \(f_\ell, g_\ell : S \times Z \to \mathbb{C}\) are measurable in the first argument and homogeneous in the second argument, and
\[
X(z, z') := \lim_\ell \int d\mu(s)g_\ell(s, z)f_\ell(s, z')
\]
exists for all \(z, z' \in Z\). Then \(X\) is homogeneous and, with notation as in \(41\),
\[
N(X) = \lim_\ell \int d\mu(s)a(g_\ell(s, \cdot))^*a(f_\ell(s, \cdot)). \tag{46}
\]

Proof. Let \(z, z' \in Z\) and \(\alpha \in \mathbb{C}\) such that \(|z'| = \alpha|z|\). Using the homogeneity of the \(f_\ell\), we find for each \(w \in Z\)
\[
X(w, z')|z'\rangle = \lim_\ell \int d\mu(s)g_\ell(s, w)f_\ell(s, z')|z'\rangle = \alpha \lim_\ell \int d\mu(s)g_\ell(s, w)f_\ell(s, z)|z\rangle = \alpha X(w, z)|z\rangle.
\]
This implies that \(X(w, \cdot)\) is a homogeneous function. A similar argument, using homogeneity assumption of each \(g_\ell\), guarantees that \(X(\cdot, w)\) is a homogeneous function as well. Thus \(X\) is homogeneous and \(N(X)\) is defined. Now for \(z, z' \in Z\),
\[
\langle z| \lim_\ell \int d\mu(s)a(g_\ell(s, \cdot))^*a(f_\ell(s, \cdot))|z'\rangle = \lim_\ell \int d\mu(s)\left(a(g_\ell(s, \cdot))^*(z)\right)a(f_\ell(s, \cdot))|z\rangle = \lim_\ell \int d\mu(s)g_\ell(s, z)f_\ell(s, z')K(z, z') = X(z, z')K(z, z') = \langle z|N(X)|z'\rangle.
\]
Thus \(46\) holds by Theorem 5.11 \(\square\)
6 Klauder spaces and bosonic Fock spaces

6.1 Klauder spaces

We recall from Neumaier [23, Example 5.2.2] that the Klauder space \( Kl(V) \) over the Euclidean space \( V \) is defined by the set
\[
z := [z_0, z] \in \mathbb{C} \times V
\]
with the coherent product
\[
K(z, z') := e^{z_0 + z_0' + z^* z'}.
\]
(47)

Klauder spaces are degenerate since
\[
|[z_0 + 2\pi ik, z]| = |[z_0, z]| \quad \text{for} \quad k \in \mathbb{Z}.
\]
Thus it is enough to specify \( z_0 \) modulo \( 2\pi i \).

6.1 Proposition. With the scalar multiplication
\[
\alpha [z_0, z] := [z_0 + \log \alpha, z],
\]
using an arbitrary but fixed branch of \( \log \), Klauder spaces are projective of degree 1. The factorizing maps are precisely the multiplication maps \( z \rightarrow \alpha z \), with \( \chi(\alpha) = \alpha \).

Proof. Using the definition of the scalar multiplication, one finds
\[
K(z, \lambda z') = \lambda K(z, z').
\]
The second statement can be verified directly; Proposition 4.10 is not applicable. \( \square \)

6.2 Theorem. Klauder spaces are slender.

Proof. Suppose that there is a nontrivial finite linear dependence \( \sum c_k |z_k\rangle = 0 \) such that no two \( |z_k\rangle \) are parallel. Since \( z_0 \) only contributes a scalar factor to \( |z\rangle \), we may assume w.l.o.g. that \( z_k = [0, z_k] \) and conclude that the \( z_k \) are distinct. Now let \( v \in V \) and \( z = [0, nv] \) for some nonnegative integer \( n \). Then, with \( \xi_k := e^{v^* z_k} \),
\[
0 = \langle z | \sum c_k z_k \rangle = \sum c_k \langle z | z_k \rangle = \sum c_k e^{nv^* z_k} = \sum c_k \xi_k^n \quad \text{for} \quad n = 0, 1, 2, \ldots.
\]
Since the sum has finitely many terms only, we find a homogeneous linear system with a Vandermonde coefficient matrix having a nontrivial solution. So the matrix is singular, and we conclude that two of the \( \xi_k \) must be identical. Thus for every \( v \in V \) there are indices \( j < k \) such that \( e^{v^* z_j} = e^{v^* z_k} \), hence, with \( z_{jk} := z_j - z_k \neq 0 \),
\[
v^* z_{jk} \equiv 0 \mod 2\pi i.
\]
Now let \( u \in V \). If \( u^* z_{jk} \neq 0 \) for all \( j < k \) then picking \( v = \lambda u \) with sufficiently many different \( \lambda \in \mathbb{R} \) gives a contradiction. Thus for every \( u \in V \) there are indices \( j < k \) such that
\[
v^* z_{jk} = 0.
\]
(48)
Since \( u \in V \) was arbitrary and the \( z_{jk} \) are nonzero, this implies that \( V \) is the union of finitely many hyperplanes (13), which is impossible.

Therefore the assumed nontrivial finite linear dependence does not exist. This proves that the Klauder space \( Z = Kl(V) \) is slender. \( \square \)

### 6.2 Oscillator groups

Klauder spaces have a large semigroup of coherent maps, which contains a large unitary subgroup. We call \( A \in \text{Lin}^\times V \) horizontal if both \( A \) and \( A^* \) map \( V \) into \( V \) and \( \overline{V} \) into \( \overline{V} \). We write \( \text{Hor} V \) for the vector space of all horizontal linear maps \( A \in \text{Lin}^\times V \). The oscillator semigroup over \( V \) is the semigroup \( Os(V) \) of matrices

\[
A = [\rho, p, q, A] := \begin{pmatrix} 1 & p^* & \rho \\ 0 & A & q \\ 0 & 0 & 1 \end{pmatrix} \in \text{Lin} (\mathbb{C} \times V \times \mathbb{C})
\]

with \( \rho \in \mathbb{C}, p, q \in V, \) and \( A \in \text{Hor} V \). One easily verifies the formula for the product

\[
[\rho, p, q, A][\rho', p', q', A'] = [\rho' + \rho + p^*q', A'^*p + p', q + Aq', AA']
\] (49)

and the identity \( 1 = [0, 0, 0, 1] \). Writing

\[
[A] := [0, 0, 0, A]
\]

we find

\[
[B][\alpha, p, q, A][B'] = [\alpha, B'^*p, Bq, BAB'].
\] (50)

\( Os(V) \) turns elements \( z \in Z \) written in the projective form

\[
z = [z_0, z] = \begin{pmatrix} z_0 \\ z \\ 1 \end{pmatrix} \in \mathbb{C}^\times \times V \times \mathbb{C}
\]

into elements the same form, corresponding to the action of \( Os(V) \) on \( [z_0, z] \in Kl(V) \) as

\[
[\rho, p, q, A][z_0, z] := [\rho + z_0 + p^*z, q + Az].
\] (51)

### 6.3 Proposition. \( Os(V) \) is a *-semigroup of coherent maps of \( Kl(V) \), with adjoints defined by

\[
[\rho, p, q, A]^* = [\overline{\rho}, q, p, A^*].
\] (52)

**Proof.** We have

\[
K(Az, z') = K([\rho, p, q, A]z, z') = e^{p^*z_0 + p^*z + z_0^* + (q + Az)^*z'} = e^{z_0^* + \overline{p} + z_0^* + q^*z' + z^* + (p + A^*z') + K(z, [\overline{\rho}, q, p, A^*]z')}.
\]
Hence the elements of $Os(V)$ are coherent maps, with the stated adjoints. □

The **linear oscillator group** $LOs(V)$ over $V$ consists of the elements $[\rho, p, q, A]$ with invertible $A$. One easily checks that the inverse is given by

$$[\rho, p, q, A]^{-1} = [p^* A^{-1} q - \rho, -A^{-*} p, -A^{-1} q, A^{-1}],$$

where

$$A^{-*} = (A^{-1})^* = (A^*)^{-1}.$$

The **unitary oscillator group** $UOs(V)$ over $V$ consists of the unitary elements of $LOs(V)$.

### 6.4 Proposition.

(i) $UOs(V)$ consists of the coherent maps of the form

$$[\alpha, q, A] := \left[\frac{i}{2}(i\alpha - q^* q), -A^* q, q, A\right]$$

with unitary $A \in \text{Lin}_V$, $q \in V$, and $\alpha \in \mathbb{R}$.

(ii) Product, inverse, and adjoint of unitary elements are given by

$$[\alpha, q, A][\alpha', q', A'] = [\alpha + \alpha' - 2\text{Im} q^* A q', q' + A q, A A'],$$

$$[\alpha, q, A]^{-1} = [\alpha, q, A]^* = [-\alpha, -A^{-1} q, A].$$

Moreover,

$$[B][\alpha, q, A][B'] = [\alpha, B q, B A B'].$$

**Proof.** (i) Equating (52) and (53) gives the unitarity conditions

$$\overline{\rho} = p^* A^{-1} q - \rho, \quad q = -A^{-*} p, \quad p = -A^{-1} q, \quad A^* = A^{-1}.$$

Thus $A$ must be unitary and $p = -A^{-1} q = -A^* q$. In this case, $q = -A^{-*} p$ and

$$p^* A^{-1} q = -q^* A^{-*} A^{-1} q = -q^* q,$$

hence the unitarity conditions reduce to $\overline{\rho} = -q^* q - \rho$, i.e., $2\text{Re} \rho = -q^* q$. Writing $\alpha = 2\text{Im} \rho$, (i) follows.

(ii) (56) follows from the preceding using (52), and (57) follows from (50). To obtain the multiplication law we note that

$$[\alpha, q, A][\alpha', q', A'] = \left[\frac{i}{2}(i\alpha - q^* q), -A^* q, q, A\right][\frac{i}{2}(i\alpha' - q'^* q'), -A'^* q', q', A']$$

$$= \left[\frac{i}{2}(i\alpha' - q'^* q') + \frac{i}{2}(i\alpha - q^* q) - q^* A q', -A^* A^* q - A'^* q', q + A q, A A'\right]$$

$$= [\alpha + \alpha' - 2\text{Im} q^* A q', q' + A q, A A'].$$

Indeed, since $A^* A = 1$, we have $-A^* A q - A'^* q' = -(A A')^*(q + A q')$ and

$$\frac{i}{2}(i\alpha' - q'^* q') + \frac{i}{2}(i\alpha - q^* q) - q^* A q' = \frac{i}{2} \left( i\beta - (q + A q')^*(q + A q') \right),$$

40
where
\[ i\beta := i\alpha' - q^*q' + i\alpha - q^*q - 2q^*Aq' + (q + Aq')^*(q + Aq') \]
\[ = i(\alpha + \alpha') - q^*Aq' + q^*A^*q = i(\alpha + \alpha' - 2 \text{Im} q^*Aq'). \]

The unitary coherent maps of the form
\[ W_\alpha(q) := [\alpha, q, 1] \quad (q \in V, \alpha \in \mathbb{R}) \quad (58) \]
form the Heisenberg group \( H(V) \) over \( V \). The \( n \)-dimensional Weyl group is the subgroup of \( H(\mathbb{C}^n) \) consisting of the \( W_\alpha(q) \) with real \( q \) and \( \alpha \).

**6.5 Proposition.** With the symplectic form
\[ \sigma(q, q') := -2 \text{Im} q^*q', \]
we have
\[ W_\alpha(q)W_{\alpha'}(q') = W_{\alpha + \alpha' + \sigma(q, q')}(q + q'), \]
\[ W_\alpha(q)^{-1} = W_\alpha(q)^* = W_{-\alpha}(-q), \]
\[ [B]W_\alpha(q)[B]^{-1} = W_\alpha(Bq) \quad \text{if } B \text{ is invertible.} \]

**Proof.** Specialize Proposition 6.4. \( \square \)

By (51), the action of the Heisenberg group on \( Z \) is given by
\[ W_\alpha(q)[z_0, z] = \left[ \frac{i}{2}(i\alpha - q^*q), -q, q, 1 \right][z_0, z] = \left[ \frac{i}{2}(i\alpha - q^*q) + z_0 - q^*z, q + z \right]. \]

### 6.3 Bosonic Fock spaces

A **bosonic Fock space** is a quantum space of a Klauder space \( Kl(V) \). The quantization map on a Klauder space defines on the corresponding Fock spaces both a representation of the linear oscillator group and a unitary representation of the unitary oscillator group \( UOs(V) \). The quantization of the coherent maps in the linear oscillator semigroup leads to linear operators \( \Gamma([\rho, p^T, q, A]) \in \text{Lin} Q(Z) \). Since Klauder spaces are slender, additional linear operators \( \in \text{Lin}^* Q(Z) \) come from the quantization of homogeneous kernels.

In the following, we work with an arbitrary quantum space \( Q(Z) \), to demonstrate that everything of interest follows on this level, without any need to use any explicit integration. However, to connect to tradition, we note that for \( V = \mathbb{C}^n \), an explicit completed quantum space is the space \( L^2(\mathbb{R}^n, \mu) \) of square integrable functions of \( \mathbb{R} \) with respect to the measure \( \mu \) given by \( d\mu(x) = (2\pi)^{-n/2}e^{-\frac{1}{2}x^Tx}dx \) and the inner product
\[ f^*g := \int d\mu(x)\overline{f(x)}g(x). \]
To check this we show that the functions
\[ f_z(x) := e^{z_0 - \frac{1}{2}(x-z)^2} \]
constitute the coherent states of finite-dimensional Klauder spaces. Indeed, using definition (47), we have
\[ f^*_zf_z' = \int d\mu(x)e^{z_0 - \frac{1}{2}(x-z)^2 + z_0' - \frac{1}{2}(x-z')^2} = e^{z_0 + z_0' + z'z'} \int d\mu(x)e^{-\frac{1}{2}(x-z)^2 - \frac{1}{2}(x-z')^2 - z'z}. \]
Expanding into powers of \( x \) and using the Gaussian integration formula
\[ \int \frac{dx}{(2\pi)^{n/2}} e^{-\frac{1}{2}(x-u)^*(x-u)} = 1 \text{ for } u \in \mathbb{C}^n \]
with \( u = z + z' \), the last integral can be evaluated to 1, hence \( f^*_zf_z' = K(z, z') \). This proves that \( K \) is a coherent product and the \( f_z \) are a corresponding family of coherent states.

In the special case \( n = 1 \), we find for \( z = [i\omega \tau - \frac{1}{2}\omega^2, \tau + i\omega] \) that
\[ f_z(t) = e^{i\omega \tau - \frac{1}{2}\omega^2 + \frac{1}{2}(t-\tau-i\omega)^2} = e^{i\omega t}e^{-\frac{1}{2}(t-\tau)^2} \]
is the time-frequency shift by \( (\tau, \omega) \in \mathbb{R}^2 \) of the standard Gaussian \( e^{-\frac{1}{2}t^2} \). Thus the general coherent state is a scaled time-frequency shifted standard Gaussian.

In any quantum space \( \mathbb{Q}(Z) \) of a Klauder space, we write
\[ |z\rangle := |[0, z]\rangle \]
and find from (47) that
\[ \langle z|z'\rangle = e^{z'z'}, \quad (60) \]
\[ |z\rangle = e^{z_0}|z\rangle. \quad (61) \]
Because of (61), the coherent subspace \( Z_0 \) consisting of the \([0, z]\) with \( z \in V \) has the same quantum space as \( KL(V) \). We call the coherent spaces \( Z_0 \) **Glauber spaces** since the associated coherent states (originally due to Schrödinger [33]) were made prominent in quantum optics by Glauber [12]. Glauber spaces give a more parsimonious coherent description of the corresponding Fock space, but Klauder spaces are much more versatile since they have a much bigger symmetry group, with corresponding advantages in the applications.

### 6.4 Lowering and raising operators

In the quantum space of a Klauder space, we introduce an abstract **lowering symbol** \( a \) and its formal adjoint, the abstract **raising symbol** \( a^* \). For \( f : V \rightarrow \mathbb{C} \), we define the formal functions \( f(a) \) and \( f(a^*) \) to be the operators defined by
\[ f(a) := a(\tilde{f}), \quad f(a^*) := a^*(\tilde{f}), \]
where $a$ and $a^*$ are given by (41) and (42) and $\tilde{f} : Z \rightarrow \mathbb{C}$ is the homogeneous map defined by

$$\tilde{f}(z) := f(z) \quad \text{for } z \in Z.$$  

Clearly $f = \tilde{f}$, hence (42) gives $f(a)^* = a(\tilde{f})^* = a(\tilde{f}) = \tilde{f}(a^*)$, so that

$$f(a)^* = \tilde{f}(a^*).$$  

For any map $F : V \times V \rightarrow \mathbb{C}$, we define the homogeneous kernel $\tilde{F} : Z \times Z \rightarrow \mathbb{C}$ with $\tilde{F}(z, z') := F(z, z')$, and put

$$:F(a^*, a) := N(\tilde{F}) \in \text{Lin}^\times \mathbb{Q}(Z).$$  

(The pair of colons is the conventional notation for normal ordering in quantum field theory [8].)

6.6 Theorem. Let $Z = Kl(V)$. Then:

(i) Every linear operator $A \in \text{Lin}^\times \mathbb{Q}(Z)$ can be written uniquely in normally ordered form $A = :F(a^*, a):$.

(ii) The map $F \rightarrow :F:$ is linear, with $:1: = 1$ and

$$:f(a)^* F(a^*, a)g(a) : = f(a)^* :F(a^*, a): g(a);$$

in particular,

$$:f(a)^* g(a) : = f(a)^* g(a).$$  

(iii) The quantized coherent maps satisfy

$$\Gamma(A) = :e^{p^* a + a^* q + a^*(A - 1)a} : \quad \text{for } A = [\rho, p, q, A] \in Os(V).$$  

(iv) We have the Weyl relations

$$e^{p^* a} e^{a^* q} = e^{p^* q} e^{a^* q} e^{p^* a}$$

and the canonical commutation relations

$$(p^* a)(q^* a) = q^* (a)(p^* a), \quad (a^* p)(a^* q) = (a^* q)(a^* p),$$

$$(p^* a)(a^* q) - (a^* q)(p^* a) = \sigma(p, q)$$

hold, with the symplectic form (59).

Proof. (i) follows from Proposition 5.12 since the coherent product vanishes nowhere and homogeneous kernels are independent of $z_0$ and $z'_0$.

(ii) Let $F, G : V \times V \rightarrow \mathbb{C}$. We then have

$$:F + G : = N(\tilde{F} + \tilde{G}) = N(\tilde{F}) + N(\tilde{G}) = :F: + :G:,$$
which implies that the map \( F \rightarrow :F: \) is linear.

(iii) holds since (47) implies

\[
\langle z|A|z' \rangle = \langle z|[\rho^*, q, A]|z' \rangle = e^{z_0 + p^* z' + z^* (q + A z')} = X(z, z') K(z, z')
\]

with \( X(z, z') := e^{p^* z' + z^* q + z^* (A - 1) z'} \).

(iv) (64) follows directly from the definition of the \( f(a) \) and \( f(a^*) \). The Weyl relations follow from

\[
e^{p^* a} e^{a^* q} = :e^{p^* a}; e^{a^* q}; = \Gamma([0, p, 0, 1]) \Gamma([0, 0, q, 1]) = \Gamma([p^* q, p, q, 1]) = :e^{p^* q} e^{a^* q} e^{p^* a}.
\]

Here the first and the last equality are due to (62). The canonical commutation relations (65) are obtained from the Weyl relations by replacing \( p \) and \( q \) by \( \varepsilon p \) and \( \varepsilon q \) with \( \varepsilon > 0 \), expanding their exponentials to second order in \( \varepsilon \), and comparing the coefficients of \( \varepsilon^2 \).

If \( V = \mathbb{C}^n \) we define

\[
a_k := e_k(a), \quad a_k^* := e_k(a^*),
\]

where \( e_k \) maps \( z \) to \( z_k \). Thus formally, \( a \) is a symbolic column vector with \( n \) symbolic lowering operators \( a_k \), also called annihilation operators. Similarly, \( a^* \) is a symbolic row vector with \( n \) symbolic raising operators \( a_k^* \), also called creation operators. They satisfy the standard canonical commutation relations

\[
a_j a_k = a_k a_j, \quad a_j a_k^* = a_k^* a_j^*, \\
a_j a_k^* - a_k^* a_j = \delta_{jk}
\]

following from (64) and (65).

### 6.5 Oscillator algebras

In the applications one often needs formulas for the Lie products in the Lie algebras generating the groups discussed in this section, and formulas for the resulting representations on Fock space.

The oscillator algebra over \( V \) is the Lie algebra \( os(V) \) of infinitesimal transformations

\[
X_{\rho, p, q, A} := \lim_{\varepsilon \to 0} \frac{[\varepsilon \rho, \varepsilon p, \varepsilon q, 1 + \varepsilon A] - 1}{\varepsilon} = \begin{pmatrix} 0 & p^* & \rho \\ 0 & A & q \\ 0 & 0 & 0 \end{pmatrix}
\]

of \( Os(V) \). From (49) we find

\[
X_{\rho, p, q, A} X_{\rho', p', q', A'} = [p^* q', A'^* p, A q', A A'],
\]
whence the Lie product is

\[ [X_{\rho,p,q},X_{\rho',p',q',A'}] = [p^* q' - p'^* q, A^* p - A^* p', A q' - A' q, A A' - A' A]. \tag{66} \]

The corresponding representation on Fock space follows from (63) and is given by

\[ d \Gamma(X_{\rho,p,q,A}) := \lim_{\epsilon \to 0} \frac{\Gamma([\epsilon \rho, \epsilon p, \epsilon q, 1 + \epsilon A]) - 1}{\epsilon} = \rho + p^* a + a^* q + a^* A a. \tag{67} \]

The unitary oscillator algebra over \( V \) is the Lie algebra \( uos(V) \) of infinitesimal transformations

\[ X_{\alpha,q,A} := \lim_{\epsilon \to 0} \frac{[\epsilon \alpha, \epsilon q, 1 + \epsilon A] - 1}{\epsilon} = X_{\frac{1}{2} i \alpha, -q,q,A} \]

of \( UOs(V) \). By specializing (66), the Lie product is found to be

\[ [X_{\alpha,q,A},X_{\alpha',q',A'}] = X_{i \sigma(q,q'),A q' - A' q,A A' - A' A}. \tag{68} \]

The corresponding representation on Fock space follows from (67) and is given by

\[ d \Gamma(X_{\alpha,q,A}) := \lim_{\epsilon \to 0} \frac{\Gamma([\epsilon \alpha, \epsilon q, 1 + \epsilon A]) - 1}{\epsilon} = \frac{1}{2} i \alpha - q^* a + a^* q + a^* A a. \tag{69} \]

The Heisenberg algebra over \( V \) is the Lie algebra \( h(V) \) of infinitesimal transformations

\[ X_{\alpha,q} := \lim_{\epsilon \to 0} \frac{W_{\alpha}(\epsilon q) - 1}{\epsilon} = X_{\alpha,q,0} \]

of \( H(V) \). By specializing (68), the Lie product is found to be

\[ [X_{\alpha,q},X_{\alpha',q'}] = X_{\sigma(q,q'),0}. \tag{70} \]

The corresponding representation on Fock space follows from (69) and is given by

\[ d \Gamma(X_{\alpha,q}) := \lim_{\epsilon \to 0} \frac{\Gamma(W_{\alpha}(\epsilon q)) - 1}{\epsilon} = \frac{1}{2} i \alpha - q^* a + a^* q. \tag{71} \]

7 Declarations

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Not applicable

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Not applicable

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