Constrained WZWN Models on $G/[S \otimes U(1)^n]$ and Exchange Algebra of $G$-Primaries

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Abstract

Consistently constrained WZWN models on $G/[S \otimes U(1)^n]$ is given by constraining currents of the WZWN models with $G$. Poisson brackets are set up on the light-like plane. Using them we show the Virasoro algebra for the energy-momentum tensor of constrained WZWN models. We find a $G$-primary which satisfies a classical exchange algebra in an arbitrary representation of $G$. The $G$-primary and the constrained currents are also shown to obey the conformal transformation with respect to the energy-momentum tensor. It is checked that conformal weight of the constrained currents is 0. This is necessary for the consistency for our formulation of constrained WZWN models.

1. Introduction

The AdS/CFT correspondence between the type IIB string theory with the AdS background and the $N = 4$ supersymmetric QCD was studied with great interest. A remarkable relationship between the $N = 4$ supersymmetric QCD and a spin-chain system was discovered\[1, 2\]. Integrability of the latter system provided a powerful tool for studying the AdS/CFT correspondence. Namely the S-matrix of the latter system was constructed by assuming algebraic structure with the group symmetry $PSU(2, 2|4)$ or its variants\[3\]. But the construction was not based on the Poisson structure of some underlying world-sheet theory. To explain it, we consider a tensor product chain of quantum operators $\Psi$. We exchange two of them in a adjacent position, say, $\Psi(x)$ and $\Psi(y)$. Then the S-matrix defines a quantum exchange algebra as

$$S_{xy}\Psi(x) \otimes \Psi(y) = \Psi(y) \otimes \Psi(x).$$

(1)

The invariant S-matrix under generators $T^A$ of the above symmetry group defined as

$$[T^A \otimes 1 + 1 \otimes T^A, S] = 0,$$

(2)

was explicitly given by requiring that

$$S_{xy}S_{yx} = 1,$$

(3)

$$S_{x}yS_{z}S_{y}z = S_{y}zS_{x}S_{y}z,$$

(Yang Baxter equation).

(4)

The R-matrix is defined by $S = PR$ with a permutation operator of the position. Then the relations (2) and (4) form the axioms of the quasi-triangular Hopf algebra\[4\], when supplemented by two more conditions\[5, 6\]: existence of $R^{-1}$ and

$$(S \otimes 1)R = R^{-1}, \quad (1 \otimes S^{-1})R = R^{-1},$$

Evaluation of $S$ using the $R$-matrix is

$$S_{xy} = (R^{-1})_{xy}, \quad (1 \otimes R^{-1})S_{xy} = (S \otimes 1)(R^{-1})_{xy},$$

(5)

Similarly $R$ is evaluated using $S$.

$$R_{xy} = (S^{-1})_{xy}, \quad (1 \otimes S^{-1})R_{xy} = (S \otimes 1)(S^{-1})_{xy},$$

(6)

In this paper, we consider string theories whose world-sheet is a CFT with the symmetries $G$. Poisson brackets are set up on the light-like plane. Using them we show the Virasoro algebra for the energy-momentum tensor of constrained WZWN models. We find a $G$-primary which satisfies a classical exchange algebra in an arbitrary representation of $G$. The $G$-primary and the constrained currents are also shown to obey the conformal transformation with respect to the energy-momentum tensor. It is checked that conformal weight of the constrained currents is 0. This is necessary for the consistency for our formulation of constrained WZWN models.
with a map $S$, called antipode. In physical terms the last relation is interpreted as crossing symmetry between scattering of particles and one replaced by anti-particles\cite{5}.

Let us rewrite the quantum exchange algebra (1) as

$$R_{yx} \Psi(y) \otimes \Psi(x) = P \Psi(y) \otimes \Psi(x),$$

(5)

multiplying the above permutation operator $P$ on both sides. Suppose the R-matrix to be a quantum deformation of a certain classical r-matrix as

$$R_{xy} = 1 + h r_{xy} + O(h^2),$$

with an infinitesimal parameter $h$. Then (5) and (4) respectively become a classical exchange algebra

$$[\Psi(x) \otimes \Psi(y)] = -h r_{xy} \Psi(x) \otimes \Psi(y),$$

(6)

and the classical Yang-Baxter equation

$$[r_{xy}, r_{xz}] + [r_{xy}, r_{yz}] + [r_{xz}, r_{yz}] = 0.$$  

(7)

The quantity in the l.h.s. of (6) is a Poisson bracket which was substituted for the commutator $[\Psi(x) \otimes \Psi(y)]$. The discovered S-matrix in [3] was of a new type. It did not take the difference form of a spectral parameter\cite{7}. Hence the corresponding classical r-matrix was studied with interest as much\cite{8}. But the study was not based on the Poisson structure of some underlying world-sheet theory.

In the previous paper\cite{9} we have studied constrained WZWN models and found the classical r-matrix by studying the Poisson structure. Namely we have discussed constrained WZWN models with a symmetry group $SL(N)$ based on the coset spaces

$$SL(N)/[SL(N-M) \otimes SL(M) \otimes U(1)], \quad SL(N)/[U(1)^{N-1}],$$

(8)

We have found $\Psi$ transforming as the fundamental representation of $SL(N)$, simply called a $SL(N)$ primary. Then it was shown to satisfy the classical exchange algebra (6) and transform as a conformal primary with respect of the energy-momentum tensor. The r-matrix of the classical exchange algebra for this case does not take the classical limit\cite{8} of the S-matrix discovered in [3]. It does not take the difference form either\cite{7}. But its existence naturally leads us to consider constrained WZWN models on the coset spaces (8) as an underlying world-sheet field theory for spin-chain systems. Indeed in [10] constrained WZWN models were studied as a generalized Toda theory. The works on the ordinary WZWN models, i.e., without constraints, were done in quest of an underlying world-sheet field theory describing spin-chain systems\cite{11}.

So the aim of this paper is to generalize the arguments to the most extent in this line of thoughts. In section 2 we formulate constrained WZWN models with any simple group $G$ based on the coset space

$$G/[S \otimes U(1)^n], \quad n = rank G - rank S,$$

with an arbitrarily chosen simple or semi-simple subgroup $S(\subset G)$. It is done by constraining currents of the ordinary WZWN model in a specific way. In section 3 Poisson brackets are consistently set up on the light-like plane. The Virasoro algebra for the energy-momentum tensor is shown by means of them. In section 4 we construct a $G$ primary $\Psi$ in arbitrary representations of $G$, which satisfies the classical exchange algebra (6). In section 5 they are shown to obey the conformal transformation law. We also show that the constrained currents transform as conformal primaries with weight 0. It guarantees the consistency of constrained WZWN models.
2. Constrained WZWN models

Consider a simple group \( G \) on the base of a subgroup \( H \). Generators of \( G \) are decomposed into ones of \( H \) and remaining ones as \( \{ T_G/H, T_H \} \). Suppose that \( H = S \otimes U(1)^n \) with a simple or semi-simple group \( S \) and \( n \leq \text{rank } G \), and write the generators as
\[
\{ T_H \} = \{ T_i^S, Q_1^H, Q_2^H, \ldots, Q^n_H \}.
\]
We define a \( Y \)-charge such as
\[
T^Y = \sum_{\mu=1}^n Q^\mu.
\]
Then the coset generators are decomposed as \( \{ T_G/H \} = \{ T^{-\alpha}_L, T^\alpha_R \} \), since they have positive or negative definite \( Y \)-charges as
\[
[ T^Y, T^{-\alpha}_L ] = -y(\alpha) T^{\alpha}_R,
\]
\[
[ T^Y, T^\alpha_R ] = y(\alpha) T^\alpha_R,
\]
where \( y(\alpha) > 0 \). They are further decomposed as
\[
\{ T^{-\alpha}_L \} = \{ T^{-\alpha_1}_L, T^{-\alpha_2}_L, \ldots \},
\]
\[
\{ T^\alpha_R \} = \{ T^{\alpha_1}_R, T^{\alpha_2}_R, \ldots \},
\]
according to the \( Y \)-charge \( Y_d = y(\alpha_d), d = 1, 2, \ldots \). This is nothing but an irreducible decomposition of the coset generators under the subgroup \( S \). In other words, \( \{ T^\alpha_R \} \) transform under \( S \) as a reducible representation as
\[
N^{\alpha_1} \oplus N^{\alpha_2} \oplus \cdots,
\]
while \( \{ T^{-\alpha}_L \} \) as its complex conjugate. Here \( \alpha_d \) is a set of weights denoting the representation \( N^{\alpha_d}, d = 1, 2, \ldots \).

A constrained WZWN model with a symmetry group \( G \) is given by the action
\[
S = -\frac{k}{4\pi} S_{\text{WZWN}} - \frac{k}{2\pi} \int d^2 x \text{Tr} [A_-(g^{-1} \partial_+ g - e)],
\]
in which
\[
g \in G, \quad A_- = \sum_{\gamma(\alpha) > 0} a^\alpha T^\alpha_R \equiv a_- \cdot T_R,
\]
and \( e \) is a constant matrix in the representation space of \( G \). Here \( A_- \) is a gauge field and the action is invariant under local variations
\[
\delta g \rightarrow g v, \quad \delta A_- \rightarrow -\partial_+ v - [A_-, v],
\]
with an infinitesimal parameter \( v = v(x^+, x^-) \cdot T_R \). The action (10) is different from the one of the ordinary gauged WZWN models with the gauge field \( A = a_+ \cdot T_H \), which was given in (12). The latter is invariant under vector-like variations as
\[
\delta g \rightarrow g v - v g, \quad \delta A_+ \rightarrow -\partial_+ v - [A_+, v].
\]
with \( v = v(x^+, x^-) \cdot T_H \).

The equation of motion for \( A_\alpha \) provides the constraint for the current

\[
J_\alpha^+ = \text{Tr}[g^{-1} \partial_+ g T_\mu^\alpha] = \text{const}. \tag{13}
\]

We parametrize \( g \in G \) by the Gauss decomposition as \( g = g_L g_R g_H \) with

\[
g_L = \exp(i \sum_{y(\alpha) > 0} G^\alpha T_L^\alpha) \equiv \exp(i G \cdot T_L),
\]

\[
g_R = \exp(i \sum_{y(\alpha) > 0} F^{-\alpha} T_R^\alpha) \equiv \exp(i F \cdot T_R),
\]

\[
g_H = \exp(i \sum I T_H^i + i \sum \lambda^\mu Q^\mu) \equiv \exp(i \lambda \cdot T_H).
\]

In this paper we consider \( G \) as a compact group so that \( T_L^i = T_R, T_H^i = T_H \). Hence the variables of parametrization \( G^\alpha, F^{-\alpha}, \lambda^i, \lambda^\mu \) are constrained by the unitary condition \( g^1 g = 1 \), i.e.,

\[
\exp(-i \lambda^i \cdot T_H) \exp(G^\ast \cdot T_R) \exp(G \cdot T_L) \exp(i \lambda \cdot T_H) = \exp(F^\ast \cdot T_L) \exp(-F \cdot T_R).
\]

Here \( G^\ast, F^\ast, \lambda^i, \lambda^\mu \) are complex conjugates of \( G^\alpha, F^{-\alpha}, \lambda^i, \lambda^\mu \). We choose the gauge \( F^{-\alpha} = F^\ast = 0 \) and denote the remaining variables by \( \mathcal{G} \). The gauge-fixed transformation of \( g(\mathcal{G}) = g_L(G^\alpha) g_H(\lambda^i, \lambda^\mu) \) is given by

\[
g(\mathcal{G}) \longrightarrow e^{i \mathbf{e}^T} g(\mathcal{G}) e^{-\mathbf{e}^T} = g(\mathcal{G}^\ast), \tag{14}
\]

with a unitary element \( e^{i \mathbf{e}^T} \in G \) and an appropriate compensator \( e^{-\mathbf{e}^T} \) of which generator is

\[
u = \sum_{y(\alpha) > 0} u^{-\alpha} T_R^\alpha. \tag{15}
\]

This is nothing but a symmetry transformation on the coset space \( G/\hat{H} \) with the complex sub-group \( \hat{H} \) generated by \( T_L^\alpha \) and \( T_H \). Soon later we show that the action \( (10) \) is indeed invariant by this transformation even after the gauge-fixing. This symmetry plays a crucial role in the subsequent discussions. However \( g(\mathcal{G}^\ast) \) in \( (14) \) loses the unitarity because the compensator never satisfies \( e^{i \mathbf{e}^T} e^{-\mathbf{e}^T} = 1 \). Hence the gauge-fixed action is not real. It is a natural consequence because the action \( (10) \) is complex from the beginning. At this point our WZWN models differ from \( SL(N) \) WZWN models in \( [12] \), where \( g = g_L g_R \) merely satisfies \( \det g = 1 \) and the gauge-fixing does not break the reality of the action. Losing the unitarity by the gauge-fixing we have no reason to impose a further constraint as \( (J_\alpha^+)^\ast = J_\alpha^{-\ast} \). Indeed such a constraint holds no longer since we have \( J_\alpha^{-\ast} = \text{Tr}[g^{-1}(\mathcal{G}) \partial_- g(\mathcal{G}) T_L^\alpha] = 0 \).

Let us assume that \( e^{i \mathbf{e}^T} \) depends on \( x^- \) alone. Then the transformation \( (14) \) induces

\[
\delta(g^{-1} \partial_+ g) = -\partial_+ u - [g^{-1} \partial_+ g, u]. \tag{16}
\]

We understand the transformation \( (14) \) with this assumption hereinafter all the time. Under this the currents defined in \( (13) \) transform as

\[
\delta J_\alpha^+ = -\text{Tr}(g^{-1} \partial_+ g[u, T_R^\alpha]), \tag{17}
\]
by the constraints (13). We put the $Y$-charges of the generators $T_R^a$ in an increasing order as $0 < Y_1 < Y_2 < \cdots$. We take the constraints (13) on $J'_\mu$ in the specific form

$$\text{Tr}[g^{-1}\partial_* g T_R^a] = \begin{cases} \text{const}, & \text{for } Y_1 = y(\alpha), \\ 0, & \text{for other } y(\alpha)'s. \end{cases}$$ (18)

Then the variation (17) is vanishing because $u$ takes the form (15) and $[u, T_R^a]$ is valued in the Lie algebra of $G$ with the $Y$-charge larger than the lowest $Y_1$. This guarantees the self-consistency of the constraints (13). Next we consider the energy-momentum tensor $T_{\mu\nu}$. The one of the naive form is no longer invariant under the gauge-fixed transformation (14), so that we improve it as

$$T_{\mu\nu} = \left( \frac{1}{2} \text{Tr}[(g^{-1}\partial_* g)^2] + \frac{1}{Y_1} \partial_\mu \text{Tr}[g^{-1}\partial_* g T^1] \right).$$ (19)

This is invariant under the gauge-fixed transformation as

$$\delta T_{\mu\nu} = -k \text{Tr}[(g^{-1}\partial_* g) \partial_* u] - \frac{1}{Y_1} k \partial_\mu \text{Tr}([g^{-1}\partial_* g] [u, T^1])$$

$$= -k \text{Tr}[(g^{-1}\partial_* g) \sum_{a_1} \partial_\mu u^{-a_1} T_{R, a_1}^a] - k \partial_\mu \frac{1}{Y_1} \text{Tr}([g^{-1}\partial_* g] \sum_{a_1} u^{-a_1} T_{R, a_1}^a, T^1) = 0,$$

using (16), (15) and the above constraints. Similarly we can show that the action (10) is invariant under the transformation (13), even after the gauge is fixed.

We study the gauge-fixed transformation (14) at two steps such as

$$g_L(G^\alpha) \mapsto e^{i\epsilon T} g_L(G^\alpha) e^{-i\rho \cdot \hat{H}} = g_L(G_0^\alpha),$$ (20)

$$g_H(\lambda', \lambda''\mu') \mapsto e^{\rho \cdot \hat{H}} g_H(\lambda', \lambda''\mu') e^{-i\mu''} = g_0(\lambda''\mu'),$$ (21)

in which

$$\epsilon \cdot T = \sum_{y(\alpha) > 0} \epsilon_k^a T_{R, a}^a + \sum_{y(\alpha) > 0} \epsilon_i^a T_{R, a}^a + \sum_{i} \epsilon_i T^i_H + \sum_{\mu} \epsilon^\mu Q^\mu,$$

$$\rho \cdot \hat{H} = \rho_L^a T_{R, a} + \sum_{i} \rho_i T^i_H + \sum_{\mu} \rho^\mu Q^\mu.$$

$\rho \cdot \hat{H}$ is the generator of the compensator for the transformation at the first step. Both transformations are written in the infinitesimal forms

$$\delta G^\alpha = \text{Tr}[(\epsilon \cdot T g_L - g_L \rho \cdot \hat{H}) T_{R, a}^a] \equiv t_{AB} \epsilon^A \delta^B G^\alpha,$$

$$\delta \lambda^i = \text{Tr}[(\rho \cdot \hat{H} g_H - g_H \mu u) T^i_H] \equiv t_{AB} \epsilon^A \delta^B \lambda^i,$$

$$\delta \lambda''^\mu = \text{Tr}[(\rho \cdot \hat{H} \lambda''^\mu - \lambda''^\mu u) Q^\mu \equiv t_{AB} \epsilon^A \delta^B \lambda''^\mu,$$

which define the Killing vectors $\delta^A G, A = 1, 2, \cdots, \text{dim } G$, which realize the symmetry of the coset space $G/S \otimes U(1)^\eta$. The variables $G^\alpha$ are coordinates of the coset space, while $\lambda^i, \lambda''^\mu$ auxiliary. An important point is that owing to $\delta J'_{\mu} = 0$ the constraints (13) do not reduce the symmetry of the coset space.
3. The Virasoro transformation

We shall set up Poisson brackets for the group variables \( g' = (G^s, \lambda', A') \). The guiding principle to do this is that they satisfy the Jacobi identities and are able to reproduce the Virasoro algebra for the energy-momentum tensor \( (19) \). They are given by

\[
\{ g'(x) \otimes g'(y) \} = \frac{2\pi}{k} (0(x-y)\tau_{AB} \delta^A g'(x) \otimes \delta^B g'(y) - \theta(y-x)\tau_{AB} \delta^A g'(y) \otimes \delta^B g'(x)).
\]

(22)

at \( x^- = y^- \). The Killing vectors \( \delta^A g \) satisfy the Lie algebra of \( G \)

\[
\delta^{[A} \delta^{B]} g(x) = f^{AB} c \delta^C g(x),
\]

(23)

by the construction. \( \tau_{AB} \) is a modified Killing metric such as defining the classical r-matrix

\[
r^\pm = \tau_{AB} T^A \otimes T^B
\]

\[
= \sum_{y(\alpha) > 0} T^y_R \otimes T^\alpha_L + \sum_{\alpha > 0} T^\alpha_S \otimes T^\alpha_L - \sum_{y(\alpha) > 0} T^\alpha_R \otimes T^\alpha_L - \sum_{\alpha > 0} T^\alpha_S \otimes T^\alpha_R
\]

\[
\pm \tau_{AB} T^A \otimes T^B,
\]

(24)

with \( \tau_{AB} = -\tau_{BA} \). For \( r^+ \) it reads

\[
r^+ = 2 \sum_{y(\alpha) > 0} T^y_R \otimes T^\alpha_L + \sum_{\alpha > 0} T^\alpha_S \otimes T^\alpha_L + \sum_{\mu=1}^{\text{dim} S} Q^\mu \otimes Q^\mu + \sum_{\mu=1}^{n} Q^\mu \otimes Q^\mu.
\]

It satisfies the classical Yang-Baxter equation \( (7) \). Here the generators of the subgroup \( S \) were further decomposed in the Cartan-Weyl basis as

\[
\{ T_S \} = \{ T^\alpha_S, T^\alpha_R, Q^\mu \}, \quad \mu = 1, 2, \cdots, \text{rank} \, S,
\]

with the normalization \( \text{Tr} T^A T^B = \frac{1}{2} \delta^{AB} \equiv \frac{1}{2} \tau^{AB} \). \( \tau \) The Jacobi identities for the Poisson brackets \( (22) \) can be shown in the same way as in the previous paper \( (19) \). Namely the proof there can be straightforwardly generalized to the case in this paper. The Virasoro algebra for the energy-momentum tensor \( (19) \) follows by means of these Poisson brackets. It can be also shown in the same way as in \( (19) \). Here it suffices to outline the demonstration of the Virasoro algebra referring to the details to \( (19) \). At \( x^- = y^- \) we have

\[
\{ T_{++}(x) \otimes T_{++}(y) \} = k \left( \text{Tr}[\partial_x g g^{-1}] \partial_x [g \otimes T_{++}(y)] g^{-1} \right) + \frac{1}{y^1} \partial_x \text{Tr}[\partial_x (g \otimes T_{++}(y)) g^{-1} g T^y g^{-1}] \}
\]

(25)

with the help of the formula for a generic variation

\[
\delta(g^{-1} \partial_x g) = g^{-1} \partial_x (\delta g g^{-1}) g.
\]

(26)
We further calculate the Poisson bracket \( \{g(x) \circ T_+(y)\} \) in the r.h.s. to find
\[
\{g(x) \circ T_+(y)\} = k \left[ \text{Tr}[\partial_x (|g(x) \circ g| g^{-1}) (\partial_x g g^{-1})] + \frac{1}{Y_1} \partial_y \text{Tr}[\partial_y (|g(x) \circ g| g^{-1}) g T^Y g^{-1}] \right]. \tag{27}
\]

Finally we have to calculate the Poisson bracket \( \{g(x) \circ g\} \). To this end we have recourse to the formula
\[
\{g(x) \circ g(y)\} = \frac{\partial g(x)}{\partial G'}, \{g(y) \circ G'\} \frac{\partial g(y)}{\partial G'}, \tag{28}
\]

By means of the Poisson brackets (22) it reads
\[
\{g(x) \circ g(y)\} = \frac{2\pi}{k} \left[ \delta(x - y) \iota_{AB}^* \delta^A g(x) \otimes \delta^B g(y) - \delta(y - x) \iota_{AB}^* \delta^A g(y) \otimes \delta^B g(x) \right]. \tag{29}
\]

Plug this Poisson bracket into the r.h.s. of (27). First of all note that (27) may be put into a simplified form
\[
\{g(x) \circ T_+(y)\} = k \left[ \text{Tr}[\partial_x (|g(x) \circ g| g^{-1}) (\partial_x g g^{-1})] + \frac{1}{Y_1} \partial_y \text{Tr}[\partial_y (|g(x) \circ g| g^{-1}) g T^Y g^{-1}] \right]. \tag{30}
\]
as follows. The quantity \( \delta g g^{-1} \) by the transformation (14) is valued in the subalgebra formed by \( (T_{L}^{-}) , \Theta) \). So is the Poisson bracket \( \{g(x) \circ g\} \) from (29). Therefore we can simplify the second term of (27) as
\[
\partial_y \text{Tr}[\partial_y (|g(x) \circ g| g^{-1}) g T^Y g^{-1}] = \partial^2_y \text{Tr}[\partial_y (|g(x) \circ g| g^{-1}) g T^Y g^{-1}], \tag{31}
\]
to find (30). Owing to \( \delta^B T_+(y) = 0 \) the r.h.s. of (30) is vanishing except when the derivative \( \partial_y \) acts on the step functions \( \theta(x - y) \) and \( \theta(y - x) \) in calculating \( \partial_y \{g(x) \circ g\} \). Hence picking up both contributions we get
\[
\{g(x) \circ T_+(y)\} = 4\pi \left[ \partial_y \theta(x - y) \iota_{AB}^* \delta^A g(x) \otimes \text{Tr}[\delta^B g g^{-1}] \right] + \frac{1}{Y_1} \partial^2_y \theta(x - y) \iota_{AB}^* \delta^A g(x) \otimes \text{Tr}[\delta^B g g^{-1}] T^Y \]
\[
+ \frac{2}{Y_1} \partial_y \theta(x - y) \iota_{AB}^* \delta^A g(x) \otimes \partial_y \text{Tr}[\delta^B g g^{-1}] T^Y \right]. \tag{32}
\]

Here note that \( \iota_{AB}^* \) could be changed to the usual Killing metric \( \iota_{AB} \). Finally we evaluate the Poisson bracket (28) by plugging this expression for \( \{g(x) \circ T_+(y)\} \). For the rest of the calculations the reader may refer to (19). It leads us to the Virasoro transformation
\[
\frac{1}{2\pi} \int d\eta(x) \{T_+(x) \circ T_+(y)\}
\]
\[
= \eta(y) \partial_x \{T_+(y) \} + \frac{2}{Y_1} \partial_y \eta(y) \partial^2_y T_+(y) - \frac{k}{Y_1} \text{Tr}[T^Y T^Y] \partial^3_y \eta(y). \tag{33}
\]
with an infinitesimal parameter \( \eta(y) \). Here use was made of the formula
\[
\text{Tr} O_1 O_2 = 2i \text{Tr} (\text{Tr} O_1 T^A)(\text{Tr} O_2 T^B),
\]
for quantities \( O_1 \) and \( O_2 \) valued in the Lie algebra of \( G \), with the normalization given below (24).
So far our arguments are irrelevant to the representation of \( G \). Let us choose it to be an \( N \)-dimensional irreducible representation. Under the subgroup \( S \) it is decomposed into irreducible ones as
\[
N = N^{w_1} \oplus N^{w_2} \oplus \cdots \oplus N^{w_a} \oplus N^{w_e}.
\]
Here \( w_d, d = 1, 2, \cdots, a \), is a set of weights denoting the \( N_d \)-plet representation. An element \( g \in G \) is represented by a \( N \times N \) matrix \( D(g) \). For \( g_H \in H \) it is decomposed as
\[
D(g_H) = \begin{pmatrix}
D_{w_1}(g_H) & 0 & \cdots & 0 \\
0 & D_{w_2}(g_H) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & D_{w_a}(g_H)
\end{pmatrix},
\]
in which \( D_{w_d}(g_H), d = 1, 2, \cdots, a, \) is a \( N_d \times N_d \) matrix. Each \( N^{w_a} \) have a definite \( Y \)-charge \( y_d \). Putting them in order as \( y_1 < y_2 < \cdots < y_{a-1} < y_a \) we have
\[
D(T^Y) = \sum_{\mu=1}^{N} D(Q^\mu) = \begin{pmatrix}
(y_1)_{N_1 \times N_1} & 0 & \cdots & 0 \\
0 & (y_2)_{N_2 \times N_2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & (y_a)_{N_a \times N_a}
\end{pmatrix},
\]
with \( (y_d)_{N_d \times N_d} \equiv y_d \cdot (1)_{N_d \times N_d} \). The central charge of (33) is given by using this as
\[
c = 12 \frac{k}{Y_1} \text{Tr}[T^YT^Y] = 12 \frac{k}{Y_1} \text{Tr}[D(T^Y)D(T^Y)] = 12 \frac{k}{Y_1} \sum_{d=1}^{a} N_d y_d^2.
\]

4. The classical exchange algebra

We are in a position to discuss the classical exchange algebra in arbitrary representation of \( G \), say, \( N \). We shall find a primary \( \Psi(x) \) which is a composite of \( G(= \{G(x), A(x), A^d(x)\}) \) and satisfies
\[
[\Psi(x) \otimes \Psi(y)] = \frac{2\pi}{k} [\theta(x-y)\rho(r^+)+\theta(y-x)\rho(r^-)]\Psi(x) \otimes \Psi(y).
\]
Here \( D(r^\pm) \) are \( r \)-matrices given by (24) in the representation \( N \). By the identification
\[
r_{xy} \equiv \frac{2\pi}{k} [\theta(x-y)\rho(r^+)+\theta(y-x)\rho(r^-)],
\]
The linear transformation law (41) is shown as follows. The gauge-fixed transformation (14) is dimensional vector which linearly transforms by the gauge-fixed transformation (14) as

\[\Psi(x) = D(\epsilon \cdot T)\Psi(x), \quad (41)\]

which is called \(G\)-primary. When we write \(g = g_L g_R\) in the representation \(N\) as

\[D(g) = \begin{pmatrix}
(g)_{\alpha_1 \beta_1} & (g)_{\alpha_1 \beta_2} & \cdots & (g)_{\alpha_1 \beta_n} \\
(g)_{\alpha_2 \beta_1} & (g)_{\alpha_2 \beta_2} & \cdots & (g)_{\alpha_2 \beta_n} \\
\vdots & \vdots & \ddots & \vdots \\
(g)_{\alpha_n \beta_1} & (g)_{\alpha_n \beta_2} & \cdots & (g)_{\alpha_n \beta_n}
\end{pmatrix}, \quad (42)\]

Then such a \(G\)-primary is given by

\[\Psi(x) = \begin{pmatrix}
(g)_{\alpha_1 \beta_1} \\
(g)_{\alpha_2 \beta_1} \\
\vdots \\
(g)_{\alpha_n \beta_1}
\end{pmatrix}, \quad (43)\]

The linear transformation law (41) is shown as follows. The gauge-fixed transformation (14) is written in the infinitesimal form

\[D(\delta g) = D(\epsilon \cdot Tg) - D(\epsilon u) = D(gu), \quad (44)\]

\(D(g)\) transforms as a tensor product \(N \otimes \bar{N}\), which is decomposed by (45) as

\[N \otimes \bar{N} = [N^{\alpha_1} \oplus N^{\alpha_2} \oplus \cdots \oplus N^{\alpha_{s-1}} \oplus N^{\alpha_s}] \otimes [\bar{N}^{\beta_1} \oplus \bar{N}^{\beta_2} \oplus \cdots \oplus \bar{N}^{\beta_{s-1}} \oplus \bar{N}^{\beta_s}].\]

So does \(D(g)D(u)\). However \(u\) is given by (45) and \(D(u)\) transforms as

\[N^{\alpha_1} \oplus N^{\alpha_2} \oplus \cdots \oplus N^{\alpha_{s-1}} \oplus N^{\alpha_s}.\]

The matrix multiplication \(D(g)D(u)\) implies that the tensor product

\[N \otimes \left[ \bar{N} \otimes [N^{\alpha_1} \oplus N^{\alpha_2} \oplus \cdots \oplus N^{\alpha_{s-1}} \oplus N^{\alpha_s}] \right],\]

is decomposed into a tensor product

\[N \otimes \left[ \bar{N}^{\alpha_1} \oplus \bar{N}^{\alpha_2} \oplus \cdots \oplus \bar{N}^{\alpha_{s-1}} \oplus \bar{N}^{\alpha_s} \right].\]

Here the component \(N \otimes \bar{N}^{\alpha_s}\) with the lowest \(Y\)-charge can never appear because of the positivity of the \(Y\)-charge of \(D(u)\). Therefore multiplying \(D(g)\) by \(D(u)\) in the r.h.s. of (44) annihilates the far right column vectors of \(D(g)\), denoted as (43). Hence we have the transformation law (41).
5. Conformal transformation of $\Psi$ and $J_+^a$

In the previous section we have shown that the $G$-primary $\Psi$ linearly transforms as (41) under the gauge-fixed transformation (14) and satisfies the classical exchange algebra (39). In this section we will examine its transformation property with respect to the energy-momentum tensor $T_{++}$. We show that it indeed transforms as a conformal primary such that

$$\frac{1}{2\pi} \int dx \eta(x)[T_{++}(x) \otimes \Psi(y)] = \eta(y)\partial_y \Psi(y) + \left(\gamma_uN_{\alpha} \sum_{d=1}^{a} N_d \gamma_d \right) \partial_y \eta(y) \Psi. \quad (45)$$

In section 2 we have also seen that the constrained currents $J_+^a$, given by (13), are invariant by the gauge-fixed transformation (14). For imposing the constraints (13) consistently it is crucially important that the constrained currents are conformal primaries with weight 0 as

$$\frac{1}{2\pi} \int dx \eta(x)[T_{++}(x) \otimes J_+^a(y)] = \eta(y)\partial_y J_+^a(y). \quad (46)$$

We shall show these conformal transformation laws in this section. To this end we give here the useful formulae in connection with the compensator $e^{-u}$ for the gauge-fixed transformation (14)

$$\text{Tr}[gug^{-1}(\partial_y g^{-1})] = \frac{1}{Y_1} \partial_y \text{Tr}[(\delta gg^{-1})T], \quad (47)$$

$$\text{Tr}[T_{-\alpha}^a g^{-1} T^\beta g] = u^\beta - u^\alpha. \quad (48)$$

with $g = g_L g_H$. The first formula can be shown as follows. Note that

$$\delta \text{Tr}[(g^{-1} \partial_y g)T] = - \text{Tr}[(g^{-1} \partial_y g)[u, T]] = Y_1 \text{Tr}[(g^{-1} \partial_y g)u],$$

by the successive use of (15), (16), (9) and the constraints (18) for $J_+^a$. This variation can be calculated also as

$$\delta \text{Tr}[(g^{-1} \partial_y g)T] = \text{Tr}[(\partial_y (g^{-1})g T^\gamma) \epsilon] = \partial_y \text{Tr}[(\delta gg^{-1})T^\gamma],$$

by using (26) and the same argument as for (31). Equating both variations yields the formula (47). The second formula can be also shown by the argument for (31). Namely we have

$$0 = \text{Tr}[T_{-\alpha}^a g^{-1} (\delta gg^{-1})g] = \text{Tr}[T_{-\alpha}^a g^{-1} (\epsilon \cdot T)g] - \sum_{y(y) > 0} \epsilon^\gamma_L u^\beta - u^\alpha,$$

in which (44) was substituted for $\delta gg^{-1}$ and (15) was further decomposed as

$$u = \sum_{y(y) > 0} u^\gamma T^\gamma_K = \sum_{y(y) > 0, y(y) > 0} \epsilon^\gamma_K u^\beta - u^\alpha. \quad (49)$$

This gives the relation (48).

i. Conformal transformation of $\Psi$
Using the Poisson bracket (42) we have

\[
[\Psi(x) \neq T_{++}(y)] = 4\pi \left( \partial_y \theta(x-y) T_{AB} \delta^A \Psi(x) \otimes \text{Tr}[\delta^B g g^{-1}(\partial_y g g^{-1})] \right.
\]

\[
+ \frac{1}{Y_1} \partial^y \theta(x-y) T_{AB} \delta^A \Psi(x) \otimes \text{Tr}[\delta^B g g^{-1}) T^T_{++}] 
\]

\[
+ \frac{2}{Y_1} \partial_y \theta(x-y) T_{AB} \delta^A \Psi(x) \otimes \partial_y \text{Tr}[(\delta^B g g^{-1}) T^T_{++}] \right).
\]

(50)

The first term of the r.h.s. is reduced to

\[
4\pi \partial_y \theta(x-y) \left( t_{AB} T^A \Psi(x) \otimes \text{Tr}[T^B \partial_y g g^{-1}] - \frac{1}{Y_1} t_{AB} T^A \Psi(x) \otimes \partial_y \text{Tr}[(\delta^B g g^{-1}) T^T_{++}] \right).
\]

by (41), (44) and (47). Plugging this expression into (50) we find

\[
\frac{1}{2\pi} \int dx \eta(x)[\Psi(x) \neq T_{++}(y)] 
\]

\[
= 2 \left( - \frac{1}{Y_1} \partial_y \eta(x) T_{AB} \delta^A \Psi(x) - \frac{1}{Y_1} \partial_y \eta(x) T_{AB} \delta^A \Psi \text{Tr}[(\delta^B g g^{-1}) T^T_{++}] \right).
\]

(51)

This becomes (45) owing to a remarkable relation such that

\[
t_{AB} T^A \Psi(x) \text{Tr}[(\delta^B g g^{-1}) T^T_{++}]) = \left( \langle y_a \rangle_{N_a \times N_a} \sum_{d=1}^a N_d N^d_a \right) \Psi.
\]

(52)

with the \(Y\)-charge defined by (37). It is shown as follows. We manipulate the l.h.s. by (44), (48) and (49) as

\[
-t_{AB} T^A \Psi \text{Tr}[(\delta^B g g^{-1}) T^T_{++}] + t_{AB} T^A \Psi \text{Tr}[T^B T^T_{++}] 
\]

\[
= \sum_{y(\beta \neq 0), \; \gamma(\beta \neq 0)} \text{Tr}[T^B \Psi \cdot \partial^B \gamma \text{Tr}[T^B g g^{-1} T^T_{++} \gamma]] = \sum_{y(\beta \neq 0), \; \gamma(\beta \neq 0)} \text{Tr}[T^B \Psi \text{Tr}[T^B \gamma \cdot T^B g g^{-1} \gamma \text{Tr}[T^B g g^{-1} T^T_{++} \gamma]]] 
\]

\[
= \sum_{y(\beta \neq 0)} \sum_{\gamma(\beta \neq 0)} t_{AB} T^A \Psi \text{Tr}[T^B \gamma \cdot T^B g g^{-1} \gamma \text{Tr}[T^B g g^{-1} T^T_{++} \gamma]] 
\]

\[
= g(t_{AB} T^A \Psi) \text{Tr}[(\delta^B g g^{-1}) T^T_{++}] - g T^B \gamma \cdot \text{Tr}[T^B g g^{-1} T^T_{++} \gamma].
\]

Here the calculation of the last two lines was proceeded with \(g = g_{L R H}\). The trace formula (34) reduces this relation to

\[
t_{AB} T^A \Psi \text{Tr}[(\delta^B g g^{-1}) T^T_{++}] = g T^B \gamma \cdot \text{Tr}[T^B g g^{-1} T^T_{++} \gamma].
\]

(53)

For the quantities of the r.h.s. we make the following calculations

\[
g T^B \gamma \cdot \text{Tr}[T^B g g^{-1} T^T_{++} \gamma] = \left( g T^B \gamma \right)_{N_a \times N_a} \Psi.
\]

(11)
\[ \text{Tr}[g^{T'}g^{-1}T'] = \text{Tr}[gL^{T'}g^{-1}L'T'] = \text{Tr}[T'^{T'}] = \sum_{d=1}^{a} N_d y_d^2. \]

The first equation was calculated by using the form (43) for \( \Psi \) and (37), while the second equation by \( g = g_L g_R \) and (38). Putting them into the r.h.s. of (53) we find (52).

ii. Conformal transformation of \( J_+^a \)

We start by calculating the Poisson bracket using (46)

\[ \{ J_+^a(x) \circ T_{++}(y) \} = \text{Tr}[\partial_{x}((g(x) \circ T_{++}(y))g^{-1})g^{T'}g^{-1}]. \] (54)

Plug (52) into the r.h.s. Keep only the terms with \( \theta(x-y) \) differentiated by \( x \) because other terms drop out due to the invariance \( \delta^4 J_+^a(x) = 0 \). Then (54) becomes

\[
\{ J_+^a(x) \circ T_{++}(y) \} = 4\pi \left[ \partial_x \partial_y (x-y) \mu_{AB} \text{Tr}[\delta^4 g g^{-1}]g^{T'} \otimes \text{Tr}[\delta^B g g^{-1}] \right] + 1 \frac{1}{y_1} \partial_x \partial_y (x-y) \mu_{AB} \text{Tr}[\delta^4 g g^{-1}]g^{T'} \otimes \text{Tr}[\delta^B g g^{-1}] T' \] (55)

To simplify the expression of the r.h.s. we rewrite the first term as

\[
4\pi \left[ \partial_x \partial_y (x-y) \mu_{AB} \text{Tr}[\delta^4 g g^{-1}]g^{T'} \otimes \text{Tr}[\delta^B g g^{-1}] \right] - \partial_x \partial_y (x-y) \mu_{AB} \text{Tr}[\delta^B g g^{-1}] g^{T'} \otimes \text{Tr}[\delta^4 g g^{-1}] \right] + \frac{1}{y_1} \partial_x \partial_y (x-y) \mu_{AB} \text{Tr}[\delta^4 g g^{-1}]g^{T'} \otimes \text{Tr}[\delta^B g g^{-1}] T' \right]. \] (56)

by (44) and (47). Plug this result into (55). Multiplying \( \eta(y) \) on both sides we integrate (55) over \( y \). The integration is reduced to

\[
\frac{1}{2\pi} \int \eta(y) \{ J_+^a(x) \circ T_{++}(y) \} \]

\[
= -\eta(x) \partial_x \partial_y J_+^a  - 2\partial_y \eta \left[ t_{AB} \text{Tr}[\delta^4 g g^{-1}]g^{T'} \otimes \text{Tr}[\delta^B g g^{-1}] \right] + 2\partial_y \eta \left[ \frac{1}{y_1} t_{AB} \text{Tr}[\delta^4 g g^{-1}]g^{T'} \otimes \text{Tr}[\delta^B g g^{-1}] T' \right]. \] (57)

The first term of the r.h.s. was obtained from the second term in (56) by using the trace formula (44) and (46). The second and third terms are respectively vanishing as follows. For the third term we have

\[ t_{AB} \text{Tr}[\delta^4 g g^{-1}]g^{T'} \otimes \text{Tr}[\delta^B g g^{-1}] T' \]

\[ = \frac{1}{2} \text{Tr}[g^{T'}g^{-1}T'] \sum_{y(\beta)>0, \eta(y)>0} \text{Tr}[T^{-\beta}g g^{-1} \text{Tr}[g^{T'}g^{-1}T']]. \] (58)
by (44) and (49). The sum over $\beta$ of the second term in the r.h.s. is calculated as
\[
\sum_{\beta, \gamma > 0} \text{Tr}[T^{-\beta}_L g T^{-\gamma}_R g^{-1} T^\beta_L g] = \sum_{\beta, \gamma > 0} \text{Tr}[T^{-\beta}_L g T^{-\gamma}_R g^{-1} \text{Tr}[T^{-\gamma}_L g^{-1} T^\beta_L g]]
\]
\[
= i\epsilon_{AB} \text{Tr}[T^A g T^{-\gamma}_R g^{-1} \text{Tr}[T^{-\gamma}_L g^{-1} T^B g]] = \frac{1}{2} \delta^{\beta, \gamma},
\]
by means of (48) and the trace formula (54). Due to this formula the r.h.s. of (58) vanishes. So does the third term of the r.h.s. in (57). Similarly we can show that the second term in (57) is vanishing. Thus the conformal transformation (46) was shown.

6. Conclusions

In this paper we have given a general account of constrained WZWN models on $G/(SU(1)^r)$ to the most extent. We have found the $G$-primary in an arbitrary representation of $G$, which satisfies the classical exchange algebra. It was shown to have conformal weight given by (45) with respect to $i$, i.e.,
\[
\delta(\beta, \gamma).
\]
Imposed so that the constrained currents are invariant under the gauge-fixed transformation (14),
\[
\sigma
\]
which satisfies (40) with (24), does not have such parameter dependence, but a position dependence on the light-like plane through $\theta(x - y)$. So our exchange algebra (46) was shown.

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both of (60) and (61). These arguments about non-linear \(\sigma\)-models on \(G/H\) were also extended by taking \(G\) to be the supergroup \(PSU(2, 2|4)\) or its variants\(^{16, 17}\). Then there appears a two-dimensional topological term which descends from the WZ term. In that case the Poisson structure is modified by first-class constraints such as one encountered in the covariant formalism of the Green-Schwarz superstring. A careful study is needed for disentangling them from second-class constraints.

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