Optimal Quantization for Some Triadic Uniform Cantor Distributions with Exact Bounds

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Abstract

Let \{S_j : 1 \leq j \leq 3\} be a set of three contractive similarity mappings such that 
\[ S_j(x) = rx + \frac{j-1}{2}(1 - r) \]
for all \( x \in \mathbb{R} \), and \( 1 \leq j \leq 3 \), where \( 0 < r < \frac{1}{3} \).
Let \( P = \sum_{j=1}^{3} \frac{1}{3} P \circ S_j^{-1} \). Then, \( P \) is a unique Borel probability measure on \( \mathbb{R} \)
such that \( P \) has support the Cantor set generated by the similarity mappings \( S_j \) for 
\( 1 \leq j \leq 3 \). Let \( r_0 = 0.1622776602 \), and \( r_1 = 0.2317626315 \) (which are ten digit 
approximations of two real numbers). In this paper, for \( 0 < r \leq r_0 \), we give 
general formula to determine the optimal sets of \( n \)-means and the \( n \)th quantization 
errors for the triadic uniform Cantor distribution \( P \) for all positive integers \( n \geq 2 \).
Previously, Roychowdhury gave an exact formula to determine the optimal sets of 
\( n \)-means and the \( n \)th quantization errors for the standard triadic Cantor distribution, 
i.e., when \( r = \frac{1}{3} \). In this paper, we further show that \( r = r_0 \) is the greatest lower bound, 
and \( r = r_1 \) is the least upper bound of the range of \( r \)-values to which Roychowdhury 
formula extends. In addition, we show that for \( 0 < r \leq r_1 \) the quantization coefficient 
does not exist though the quantization dimension exists.

Keywords Cantor set · Probability distribution · Optimal sets · Quantization error · 
Centroidal Voronoi tessellation

Mathematics Subject Classification 60Exx · 28A80 · 94A34

1 Introduction

Let \( P \) be a Borel probability measure on \( \mathbb{R}^d \), where \( d \geq 1 \). For a finite set \( \alpha \subset \mathbb{R}^d \), write
\[ V(P; \alpha) = \int \min_{a \in \alpha} \|x - a\|^2 dP(x), \text{ and } V_n := V_n(P) \]
\[ = \inf \left\{ V(P; \alpha) : \alpha \subset \mathbb{R}^d, \text{ card}(\alpha) \leq n \right\}, \]

where \(\| \cdot \|\) represents the Euclidean norm on \(\mathbb{R}^d\). Then, \(V(P; \alpha)\) is called the cost or distortion error for \(P\) with respect to the set \(\alpha\), and \(V_n\) is called the \(n\)th quantization error for \(P\) with respect to the squared Euclidean distance. A set \(\alpha \subset \mathbb{R}^d\) is called an optimal set of \(n\)-means for \(P\) if \(V_n(P) = V(P; \alpha)\). It is well-known that for a continuous Borel probability measure an optimal set of \(n\)-means contains exactly \(n\)-elements (see [4]). To see some work in the direction of optimal sets of \(n\)-means, one is referred to [2, 5, 16]. For theoretical results in quantization we refer to [4, 6–8, 11], and for its promising application see [12, 13]. For a finite set \(\alpha \subset \mathbb{R}^d\) and \(a \in \alpha\), by \(M(a|\alpha)\) we denote the set of all elements in \(\mathbb{R}^d\) which are nearest to \(a\) among all the elements in \(\alpha\), i.e.,
\[ M(a|\alpha) = \left\{ x \in \mathbb{R}^d : \|x - a\| = \min_{b \in \alpha} \|x - b\| \right\}. \]

\(M(a|\alpha)\) is called the Voronoi region generated by \(a \in \alpha\). On the other hand, the set \(\{M(a|\alpha) : a \in \alpha\}\) is called the Voronoi diagram or Voronoi tessellation of \(\mathbb{R}^d\) with respect to the set \(\alpha\).

**Definition 1.1** A set \(\alpha \subset \mathbb{R}^d\) is called a centroidal Voronoi tessellation (CVT) with respect to a probability distribution \(P\) on \(\mathbb{R}^d\), if it satisfies the following two conditions:
(i) \(P(M(a|\alpha) \cap M(b|\alpha)) = 0\) for \(a, b \in \alpha\), and \(a \neq b\); (ii) \(E(X : X \in M(a|\alpha)) = a\) for all \(a \in \alpha\),
where \(X\) is a random variable with distribution \(P\), and \(E(X : X \in M(a|\alpha))\) represents the conditional expectation of the random variable \(X\) given that \(X\) takes values in \(M(a|\alpha)\).

A Borel measurable partition \(\{A_a : a \in \alpha\}\) is called a Voronoi partition of \(\mathbb{R}^d\) with respect to the probability distribution \(P\), if \(P\)-almost surely \(A_a \subset M(a|\alpha)\) for all \(a \in \alpha\). Let us now state the following proposition (see [3, 4]).

**Proposition 1.2** Let \(\alpha\) be an optimal set of \(n\)-means, \(a \in \alpha\), and \(M(a|\alpha)\) be the Voronoi region generated by \(a \in \alpha\), i.e., \(M(a|\alpha) = \{ x \in \mathbb{R}^d : \|x - a\| = \min_{b \in \alpha} \|x - b\| \}\). Then, for every \(a \in \alpha\), (i) \(P(M(a|\alpha)) > 0\), (ii) \(P(\partial M(a|\alpha)) = 0\), (iii) \(a = E(X : X \in M(a|\alpha))\).

The number \(D(P) := \lim_{n \to \infty} -\frac{2\log n}{\log V_n(P)}\), if it exists, is called the quantization dimension of the probability measure \(P\). On the other hand, for \(s \in (0, +\infty)\), the number \(\lim_{n \to \infty} n^{\frac{s}{2}} V_n(P)\), if it exists, is called the \(s\)-dimensional quantization coefficient for \(P\). To know details about the quantization dimension and the quantization coefficient one is referred to [4].

Let \(\{S_j : 1 \leq j \leq 3\}\) be a set of three contractive similarity mappings such that \(S_j(x) = rx + \frac{j-1}{2}(1-r)\) for all \(x \in \mathbb{R}\), where \(0 < r < \frac{1}{3}\) and \(1 \leq j \leq 3\). For any
positive integer $n$, if $\sigma := \sigma_1\sigma_2 \cdots \sigma_n \in \{1, 2, 3\}^n$, then we say that $\sigma$ is a word of length $n$. By $\{1, 2, 3\}^*$, we denote the set of all words including the empty word $\emptyset$. The empty word $\emptyset$ has length zero. For $\sigma := \sigma_1\sigma_2 \cdots \sigma_n \in \{1, 2, 3\}^n$, by $S_\sigma$ it is meant that $S_\sigma := S_{\sigma_1} \circ \cdots \circ S_{\sigma_n}$, and by $a(\sigma)$, we mean $a(\sigma) := S_\sigma(\frac{1}{3})$. For the empty word $\emptyset$, by $S_\emptyset$ it is meant the identity mapping on $\mathbb{R}$. For $\sigma := \sigma_1\sigma_2 \cdots \sigma_n \in \{1, 2, 3\}^n$, set $J_\sigma := S_\sigma([0, 1])$. For the empty word $\emptyset$, write $J := J_\emptyset = S_\emptyset([0, 1]) = [0, 1]$. Then, the set $C := \bigcap_{n \in \mathbb{N}} \bigcup_{\sigma \in \{1,2,3\}^n} J_\sigma$ is known as the Cantor set generated by the mappings $S_j$, and equals the support of the probability measure $P$ given by $P = \sum_{j=1}^{3} \frac{1}{3} P \circ S_j^{-1}$. Notice that $C$ satisfies the invariance equality $C = \bigcup_{j=1}^{3} S_j(C)$ (see [10]). In this paper a Cantor set $C$, which is generated by a set of three contractive similarity mappings, is called a triadic Cantor set, and a probability measure $P$ which has support the triadic Cantor set, is called a triadic Cantor distribution. For words $\beta, \gamma, \cdots, \delta$ in $\{1, 2, 3\}^*$, we write

$$a(\beta, \gamma, \cdots, \delta) := E(X | X \in J_\beta \cup J_\gamma \cup \cdots \cup J_\delta) = \frac{1}{P(J_\beta \cup \cdots \cup J_\delta)} \int_{J_\beta \cup \cdots \cup J_\delta} xdP(x),$$

where $X$ is a random variable with probability distribution $P$, and $E(X)$ and $V := V(X)$ represent the expectation and the variance of the random variable $X$. Notice that for any $\omega \in \{1, 2, 3\}^*$, the similarity mapping $S_\omega$ is an injective mapping on $\mathbb{R}$; on the other hand, for any discrete subset $A$ of $\mathbb{R}$, the set $S_\omega(A)$ represents the set of values obtained by applying $S_\omega$ to each of the elements in $A$. Let us now give the following two definitions.

**Definition 1.3** For $n \in \mathbb{N}$ with $n \geq 3$ let $\ell(n)$ be the unique natural number with $3^{\ell(n)} \leq n < 3^{\ell(n)+1}$. Write $\beta_2 := \{a(1), a(2, 3)\}$ and $\beta_3 := \{a(1), a(2), a(3)\}$. For $n \geq 3$, define $\beta_n := \beta_n(I)$ as follows:

$$\beta_n(I) = \left\{ \begin{array}{ll}
\{a(\omega) : \omega \in \{1, 2, 3\}^{\ell(n)} \setminus I\} \cup \bigcup_{\omega \in I} S_\omega(\beta_2) & \text{if } 3^{\ell(n)} \leq n \leq 2 \cdot 3^{\ell(n)}, \\
\{S_\omega(\beta_2) : \omega \in \{1, 2, 3\}^{\ell(n)} \setminus I\} \cup \bigcup_{\omega \in I} S_\omega(\beta_3) & \text{if } 2 \cdot 3^{\ell(n)} < n < 3^{\ell(n)+1},
\end{array} \right.$$ 

where $I \subset \{1, 2, 3\}^{\ell(n)}$ is arbitrary with $\card(I) = n - 3^{\ell(n)}$ if $3^{\ell(n)} \leq n \leq 2 \cdot 3^{\ell(n)}$; and $\card(I) = n - 2 \cdot 3^{\ell(n)}$ if $2 \cdot 3^{\ell(n)} < n < 3^{\ell(n)+1}$.

**Definition 1.4** For $n \in \mathbb{N}$ with $n \geq 3$ let $\ell(n)$ be the unique natural number with $3^{\ell(n)} \leq n < 3^{\ell(n)+1}$. Write $\gamma_2 := \{a(1, 21), a(22, 23, 3)\}$ and $\gamma_3 := \{a(1), a(2), a(3)\}$. For $n \geq 3$, define $\gamma_n := \gamma_n(I)$ as follows:

$$\gamma_n(I) = \left\{ \begin{array}{ll}
\{a(\omega) : \omega \in \{1, 2, 3\}^{\ell(n)} \setminus I\} \cup \bigcup_{\omega \in I} S_\omega(\gamma_2) & \text{if } 3^{\ell(n)} \leq n \leq 2 \cdot 3^{\ell(n)}, \\
\{S_\omega(\gamma_2) : \omega \in \{1, 2, 3\}^{\ell(n)} \setminus I\} \cup \bigcup_{\omega \in I} S_\omega(\gamma_3) & \text{if } 2 \cdot 3^{\ell(n)} < n < 3^{\ell(n)+1},
\end{array} \right.$$ 

where $I \subset \{1, 2, 3\}^{\ell(n)}$ is arbitrary with $\card(I) = n - 3^{\ell(n)}$ if $3^{\ell(n)} \leq n \leq 2 \cdot 3^{\ell(n)}$; and $\card(I) = n - 2 \cdot 3^{\ell(n)}$ if $2 \cdot 3^{\ell(n)} < n < 3^{\ell(n)+1}$.
Remark 1.5 In the paper there are several decimal numbers, they are rational approximations of some real numbers up to ten decimal places.

Roychowdhury showed that if \( r = \frac{1}{3} \), then the sets \( \gamma_n \) given by Definition 1.3, determine the optimal sets of \( n \)-means for all positive integers \( n \geq 2 \) (see [15]). Proposition 2.5 implies that \( \gamma_n \) forms a CVT if \( \frac{1}{79} \left( 21 - 2\sqrt{31} \right) \leq r \leq \frac{1}{41} \left( 2\sqrt{31} - 1 \right) \), i.e., if \( 0.08502712839 \leq r \leq 0.2472080177 \). Thus, we see that the range of \( r \) values for which the sets \( \gamma_n \) form the optimal sets of \( n \)-means is bounded below by \( \frac{1}{79} \left( 21 - 2\sqrt{31} \right) \), and bounded above by \( \frac{1}{41} \left( 2\sqrt{31} - 1 \right) \). But, the greatest lower bound and the least upper bound of the range of \( r \) values for which the sets \( \gamma_n \) form the optimal sets of \( n \)-means were not known. In this paper, in Theorem 5.1 we give an answer of it.

Remark 1.6 Notice that if \( r = 0 \), then \( S_1(x) = 0 \), \( S_2(x) = \frac{1}{2} \), and \( S_3(x) = 1 \) for all \( x \in \mathbb{R} \), and then the probability measure \( P \) becomes a discrete uniform distribution with support \( \{0, \frac{1}{2}, 1\} \). Because of that in our study we are assuming that the contractive ratios \( r \) are positive.

The arrangement of the paper is as follows: In Sect. 2, we give the basic preliminaries. In Sect. 3, we show that the sets \( \beta_n \) form the optimal sets of \( n \)-means if \( r = \frac{1}{25} \). In Sect. 4, we prove the following theorem:

**Theorem 1.7** Let \( \gamma_n := \gamma_n(I) \) be the set for arbitrary \( I \) as defined by Definition 1.4. Let \( r_0, r_1 \in (0, \frac{1}{3}) \) be the unique real numbers satisfying, respectively, the equations

\[
\begin{align*}
-3r^5 + 15r^4 + 6r^3 - 42r^2 + 31r - 13 &= -3r^3 - 3r^2 + r - 1, \\
-3r^5 + 15r^4 + 6r^3 - 42r^2 + 31r - 13 &= -3r^7 + 15r^6 + 60r^5 + 66r^4 + 18r^3 - 324r^2 + 283r - 121.
\end{align*}
\]

Then, \( r_0 = 0.1622776602 \), and \( r_1 = 0.2317626315 \). Then, for all \( n \geq 3 \), the sets \( \gamma_n \) form the optimal sets of \( n \)-means for \( r = r_0 \) and \( r = r_1 \).

In Theorem 5.1, we show that the sets \( \beta_n \) form the optimal sets of \( n \)-means if \( 0 < r \leq r_0 \), and the sets \( \gamma_n \) form the optimal sets of \( n \)-means if \( r_0 \leq r \leq r_1 \). Thus, Theorem 5.1 implies the fact that the greatest lower bound, and the least upper bound of \( r \) for which the sets \( \gamma_n \) form the optimal sets of \( n \)-means are, respectively, given by \( r = r_0 \) and \( r = r_1 \). Notice that for \( r = r_0 \) both the sets \( \beta_n \) and \( \gamma_n \) form the optimal sets of \( n \)-means for \( P \). In addition, in Theorem 5.2, we show that the quantization coefficient for \( 0 < r \leq r_1 \) does not exist though the quantization dimension exists.

**2 Preliminaries**

As defined in the previous section, let \( S_j \) for \( 1 \leq j \leq 3 \) be the contractive similarity mappings on \( \mathbb{R} \) given by \( S_j(x) = rx + \frac{j-1}{2}(1-r) \) for all \( x \in \mathbb{R} \), and \( 1 \leq j \leq 3 \), where \( 0 < r < \frac{1}{3} \). For \( \sigma := \sigma_1\sigma_2\cdots\sigma_k \in \{1, 2, 3\}^k \) and \( \tau := \tau_1\tau_2\cdots\tau_\ell \in \{1, 2, 3\}^\ell \),
Corollary 2.3 Let \( \sigma \tau \) := \( \sigma_1 \cdots \sigma_k \tau_1 \cdots \tau_\ell \) we mean the word obtained from the concatenation of the words \( \sigma \) and \( \tau \). For \( \sigma = \sigma_1 \sigma_2 \cdots \sigma_n \in \{1, 2, 3\}^n \), \( n \geq 0 \), write \( p_\sigma := \frac{1}{\ell^n} \) and \( s_\sigma := \frac{1}{\ell^n} \). Recall that if \( C \) is the Cantor set, then \( C := \bigcap_{n \in \mathbb{N}} \bigcup_{\sigma \in \{1, 2, 3\}^n} J_\sigma \). For \( n \geq 1 \), the intervals \( J_\sigma \), where \( \sigma \in \{1, 2, 3\}^n \), are called the \textit{nth level basic intervals} of the Cantor set \( C \).

The following two lemmas are well-known and easy to prove (see [5, 15]).

**Lemma 2.1** Let \( f : \mathbb{R} \to \mathbb{R}^+ \) be Borel measurable and \( k \in \mathbb{N} \), and \( P \) be the probability measure on \( \mathbb{R} \) given by \( P = \sum_{j=1}^{3} \frac{1}{3} P \circ S_j^{-1} \). Then,

\[
\int f(x) dP(x) = \sum_{\sigma \in \{1, 2, 3\}^k} \frac{1}{3^k} \int f \circ S_\sigma(x) dP(x).
\]

**Lemma 2.2** Let \( X \) be a random variable with the probability distribution \( P \). Then,

\[
E(X) = \frac{1}{2} \text{ and } V := V(X) = \frac{1 - r}{6(r + 1)}, \text{ and }
\]

\[
\int (x - x_0)^2 dP(x) = V(X) + \left(x_0 - \frac{1}{2}\right)^2,
\]

where \( x_0 \in \mathbb{R} \).

The following corollary is useful to obtain the distortion errors.

**Corollary 2.3** Let \( \sigma \in \{1, 2, 3\}^k \) for \( k \geq 1 \), and \( x_0 \in \mathbb{R} \). Then,

\[
\int_{J_\sigma} (x - x_0)^2 dP(x) = \frac{1}{3^k} \left(r^{2k} V + (S_\sigma \left(\frac{1}{2}\right) - x_0)^2\right).
\]

**Proof** By induction, \( P = \frac{1}{3} \sum_{j=1}^{3} P \circ S_j^{-1} \) implies \( P = \sum_{\sigma \in \{1, 2, 3\}^k} p_\sigma P \circ S_\sigma^{-1} \). Using this fact, Lemma 2.1 and Lemma 2.2, the proof of the corollary follows. \( \square \)

**Proposition 2.4** Let \( \beta_n(I) \) be the set given by Definition 1.3. Then, \( \beta_n(I) \) forms a CVT if \( 0 < r \leq 2 - \sqrt{3} \), i.e., if \( 0 < r \leq 0.2679491924 \). Moreover, if \( 3^{\ell(n)} \leq n \leq 2 \cdot 3^{\ell(n)} \), then

\[
V(P, \beta_n(I)) = \frac{1}{3^{\ell(n)}} \cdot r^{2\ell(n)} \left((2 \cdot 3^{\ell(n)} - n)V + (n - 3^{\ell(n)})V(P; \beta_2)\right),
\]

and if \( 2 \cdot 3^{\ell(n)} \leq n < 3^{\ell(n)+1} \), then

\[
V(P, \beta_n(I)) = \frac{1}{3^{\ell(n)}} \cdot r^{2\ell(n)} \left(3^{\ell(n)+1} - n)V(P; \beta_2) + (n - 2 \cdot 3^{\ell(n)})V(P; \beta_3)\right).
\]

**Proof** By the definition, we have \( \beta_2 = \{a(1), a(2, 3)\} \) and \( \beta_3 = \{a(1), a(2), a(3)\} \). Recall that \( \beta_n := \beta_n(I) \) is defined for \( n \geq 3 \), where \( I \subset \{1, 2, 3\}^{\ell(n)} \) with \( \text{card}(I) = n - 3^{\ell(n)} \) if \( 3^{\ell(n)} \leq n \leq 2 \cdot 3^{\ell(n)} \); and \( \text{card}(I) = n - 2 \cdot 3^{\ell(n)} \) if \( 2 \cdot 3^{\ell(n)} < n < 3^{\ell(n)+1} \).
Notice that for $n \geq 3$, if $n \neq 3^{\ell(n)}$ or $n \neq 2 \cdot 3^{\ell(n)}$, the subset $I$ can be chosen more than one way. This leads to the fact that if $n \neq 3^{\ell(n)}$ or $n \neq 2 \cdot 3^{\ell(n)}$, the sets $\beta_n$ can be chosen multiple ways. Let us take

$$
\beta_4 = \{a(1), a(2), a(31), a(32, 33)\} \text{ (by choosing } I = \{3\}),
$$

$$
\beta_5 = \{a(1), a(21), a(22, 23), a(31), a(32, 33)\} \text{ (by choosing } I = \{2, 3\}),
$$

$$
\beta_6 = \{a(11), a(12, 13), a(21), a(22, 23), a(31), a(32, 33)\} \text{ (where } I = \{1, 2, 3\}),
$$

$$
\beta_7 = \{a(11), a(12), a(13), a(21), a(22, 23), a(31), a(32, 33)\}
$$

(by choosing $I = \{1\}).

Since similarity mappings preserve the ratio of the distances of a point from any other two points, $\beta_n(I)$ will form a CVT if we can show that $\beta_2$, $\beta_3$, $\beta_4$, $\beta_5$, $\beta_6$, $\beta_7$ form a CVT. Recall that $a(1) = E(X : X \in J_1)$ and $a(2, 3) = E(X : X \in J_2 \cup J_3)$, and also recall the Definition 1.1. Thus, $\beta_2$ will form a CVT if

$$
P(M(a(1)|\beta_2) \cap M(a(2, 3)|\beta_2)) = 0.
$$

(1)

Since the basic intervals in the first level are $J_1 := [S_1(0), S_1(1)]$, $J_2 := [S_2(0), S_2(1)]$, and $J_3 := [S_3(0), S_3(1)]$, the relation (1) will be true if

$$
S_1(1) \leq \frac{1}{2} (a(1) + a(2, 3)) \leq S_2(0).
$$

Similarly, $\beta_3$ will form a CVT if $S_i(1) < \frac{1}{2}(a(i) + a(i + 1)) < S_{i+1}(0)$ for $i = 1, 2$; $\beta_4$ will form a CVT if

$$
S_1(1) < \frac{1}{2}(a(1) + a(2)) < S_2(0) < S_2(1) < \frac{1}{2}(a(2) + a(31)) < S_{31}(0) < S_{31}(1)
$$

$$
< \frac{1}{2}(a(31) + a(32, 33)) < S_{32}(0).
$$

Similarly, we can obtain the inequalities for which $\beta_5$, $\beta_6$, and $\beta_7$ will form a CVT. Due to similarity, combining all the inequalities, we see that they will be true if the following inequalities are true:

$$
S_1(1) \leq \frac{1}{2} (a(1) + a(2, 3)) \leq S_2(0),
$$

$$
S_1(1) \leq \frac{1}{2} (a(1) + a(21)) \leq S_{21}(0),
$$

$$
S_{13}(1) \leq \frac{1}{2} (a(12, 13) + a(21)) \leq S_{21}(0),
$$

$$
S_{13}(1) \leq \frac{1}{2} (a(13) + a(21)) \leq S_{21}(0).
$$
Upon some simplification, we see that the above inequalities are true if $0 < r \leq 2 - \sqrt{3}$, i.e., if $0 < r \leq 0.2679491924$. If $3^\ell(n) \leq n \leq 2 \cdot 3^\ell(n)$, then

$$V(P; \beta_n(I)) = \sum_{\sigma \in [1, 2, 3]^\ell(n) \setminus I} \int_{J_{\sigma}} (x - a(\sigma))^2 dP + \sum_{\sigma \in I} \min_{a \in S_\ell(\beta_2)} (x - a)^2 dP$$

$$= \frac{1}{3^\ell(n)} r^{2\ell(n)} \left( \sum_{\sigma \in [1, 2, 3]^\ell(n) \setminus I} V + \sum_{\sigma \in I} V(P; \beta_2) \right)$$

$$= \frac{1}{3^\ell(n)} \cdot r^{2\ell(n)} \left( (2 \cdot 3^\ell(n) - n) V + (n - 3^\ell(n)) V(P; \beta_2) \right).$$

Similarly, if $2 \cdot 3^\ell(n) \leq n < 3^\ell(n)+1$, then

$$V(P; \beta_n(I)) = \frac{1}{3^\ell(n)} \cdot r^{2\ell(n)} \left( (3^\ell(n)+1 - n) V(P; \beta_2) + (n - 2 \cdot 3^\ell(n)) V(P; \beta_3) \right).$$

Thus, the proof of the proposition is complete. \qed

**Proposition 2.5** Let $\gamma_n(I)$ be the set given by Definition 1.4. Then, $\gamma_n(I)$ forms a CVT if $\frac{1}{19} \left( 21 - 2 \sqrt{51} \right) \leq r \leq \frac{1}{47} \left( 2 \sqrt{31} - 1 \right)$, i.e., if $0.08502712839 \leq r \leq 0.2472080177$. Moreover, if $3^\ell(n) \leq n \leq 2 \cdot 3^\ell(n)$, then

$$V(P, \gamma_n(I)) = \frac{1}{3^\ell(n)} \cdot r^{2\ell(n)} \left( 2 \cdot 3^\ell(n) - n \right) V(P; \gamma_2),$$

and if $2 \cdot 3^\ell(n) \leq n < 3^\ell(n)+1$, then

$$V(P, \gamma_n(I)) = \frac{1}{3^\ell(n)} \cdot r^{2\ell(n)} \left( (3^\ell(n)+1 - n) V(P; \gamma_2) + (n - 2 \cdot 3^\ell(n)) V(P; \gamma_3) \right).$$

**Proof** By the definition, we have $\gamma_2 = \{a(1, 21), a(22, 23, 3)\}$ and $\gamma_3 = \{a(1), a(2), a(3)\}$. For $n \geq 3$, if $n \neq 3^\ell(n)$ or $n \neq 2 \cdot 3^\ell(n)$, the subset $I$ can be chosen more than one way. This leads to the fact that if $n \neq 3^\ell(n)$ or $n \neq 2 \cdot 3^\ell(n)$, the sets $\delta_n$ can be chosen multiple ways. Proceeding in the similar way, as Proposition 2.4, let us choose

$$\gamma_4 = \{a(1), a(2), a(31, 321), a(322, 323, 33)\}$$
$$\gamma_5 = \{a(1), a(21, 221), a(222, 223, 23), a(31, 321), a(322, 323, 33)\}$$
$$\gamma_6 = \{a(11, 121), a(122, 123, 13), a(21, 221), a(222, 223, 23), a(31, 321), a(322, 323, 33)\}$$
$$\gamma_7 = \{a(11), a(12), a(13), a(21, 221), a(222, 223, 23), a(31, 321), a(322, 323, 33)\}.$$
Due to the same reasoning as described in the proof of Proposition 2.4, to show $\gamma_n(I)$ forms a CVT, it is enough to prove that the following inequalities are true:

\[
S_{21}(1) \leq \frac{1}{2} ((a(1, 21) + a(22, 23, 3)) \leq S_{22}(0),
S_1(1) \leq \frac{1}{2} (a(1) + a(21, 221)) \leq S_{21}(0),
S_{13}(1) \leq \frac{1}{2} (a(122, 123, 13) + a(21, 221)) \leq S_{21}(0),
S_{13}(1) \leq \frac{1}{2} (a(13) + a(21, 221)) \leq S_{21}(0).
\]

Upon some simplification, we see that the above inequalities are true if $\frac{1}{79} (21 - 2\sqrt{5}1) \leq r \leq \frac{1}{41} (2\sqrt{5}1 - 1)$, i.e., if $0.08502712839 \leq r \leq 0.2472080177$. The rest of the proof follows in the similar way as it is given for $V(P; \beta_n)$ in Proposition 2.4. Thus, the proof of the proposition is complete. $\square$

**Definition 2.6** For $n \in \mathbb{N}$ with $n \geq 3$ let $\ell(n)$ be the unique natural number with $3\ell(n) \leq n < 3\ell(n)+1$. Write $\delta_2 := \{a(1, 21, 221), a(222, 223, 23, 3)\}$ and $\delta_3 := \{a(1), a(2), a(3)\}$. For $n \geq 3$, define $\delta_n := \delta_n(I)$ as follows:

\[
\delta_n(I) = \begin{cases} 
\{a(\omega) : \omega \in \{1, 2, 3\}^\ell(n) \setminus I \} \cup \bigcup_{\omega \in I} S_{\omega}(\delta_2) & \text{if } 3\ell(n) \leq n \leq 2 \cdot 3\ell(n), \\
\{S_{\omega}(\delta_2) : \omega \in \{1, 2, 3\}^\ell(n) \setminus I \} \cup \bigcup_{\omega \in I} S_{\omega}(\delta_3) & \text{if } 2 \cdot 3\ell(n) < n < 3\ell(n)+1,
\end{cases}
\]

where $I \subset \{1, 2, 3\}^\ell(n)$ with $\text{card}(I) = n - 3\ell(n)$ if $3\ell(n) \leq n \leq 2 \cdot 3\ell(n)$; and $\text{card}(I) = n - 2 \cdot 3\ell(n)$ if $2 \cdot 3\ell(n) < n < 3\ell(n)+1$.

**Proposition 2.7** Let $\delta_n(I)$ be the set given by Definition 2.6. Then, $\delta_n(I)$ forms a CVT if $0.1845020699 \leq r \leq 0.2705731187$. Moreover, if $3\ell(n) \leq n \leq 2 \cdot 3\ell(n)$, then

\[
V(P, \delta_n(I)) = \frac{1}{3\ell(n)} \cdot r^{2\ell(n)} \left((2 \cdot 3\ell(n) - n)V + (n - 3\ell(n))V(P; \delta_2)\right),
\]

and if $2 \cdot 3\ell(n) \leq n < 3\ell(n)+1$, then

\[
V(P, \delta_n(I)) = \frac{1}{3\ell(n)} \cdot r^{2\ell(n)} \left((3\ell(n)+1 - n)V(P; \delta_2) + (n - 2 \cdot 3\ell(n))V(P; \delta_3)\right).
\]

**Proof** By the definition, we have $\delta_2 = \{a(1, 21, 221), a(222, 223, 23, 3)\}$ and $\delta_3 = \{a(1), a(2), a(3)\}$. For $n \geq 3$, if $n \neq 3\ell(n)$ or $n \neq 2 \cdot 3\ell(n)$, the subset $I$ can be chosen more than one way. This leads to the fact that if $n \neq 3\ell(n)$ or $n \neq 2 \cdot 3\ell(n)$, the sets $\delta_n$ can be chosen multiple ways. Proceeding in the similar way, as Proposition 2.4, let us choose

\[
\delta_4 = \{a(1), a(2), a(31, 321, 3221), a(3222, 3223, 323, 33)\}
\]
\[ \delta_5 = \{ a(1), a(21, 221, 222), a(2222, 2223, 223, 23), \\
\quad a(31, 321, 3221), a(3222, 3223, 323, 33) \} \]
\[ \delta_6 = \{ a(11, 121, 1221), a(1222, 1223, 123, 13), \\
\quad a(21, 221, 2221), a(2222, 2223, 223, 23), \\
\quad a(31, 321, 3221), a(3222, 3223, 323, 33) \} \]
\[ \delta_7 = \{ a(11), a(12), a(13), a(21, 221, 2221), \\
\quad a(2222, 2223, 223, 23), \\
\quad a(31, 321, 3221), a(3222, 3223, 323, 33) \}. \]

Due to the same reasoning as described in the proof of Proposition 2.4, to show \( \delta_n(I) \) forms a CVT, it is enough to prove that the following inequalities are true:

\[ S_{221}(1) \leq \frac{1}{2} (a(1, 21, 221) + a(222, 223, 23)) \leq S_{222}(0), \]
\[ S_1(1) \leq \frac{1}{2} (a(1) + a(21, 221, 2221)) \leq S_2(0), \]
\[ S_{13}(1) \leq \frac{1}{2} (a(1222, 1223, 123, 13) + a(21, 221, 2221)) \leq S_{21}(0), \]
\[ S_{13}(1) \leq \frac{1}{2} (a(13) + a(21, 221, 2221)) \leq S_{21}(0). \]

The above inequalities are true if \( 0.1845020699 \leq r \leq 0.2705731187 \). The rest of the proof follows in the similar way as it is given for \( V(P; \beta_n(I)) \) in Proposition 2.4. Thus, the proof of the proposition is complete. \[ \square \]

The following proposition is useful to establish Lemma 3.1, and Lemma 4.1.

**Proposition 2.8** Let \( \kappa := \{ a_1, a_2 \} \), where \( a_1 := E(X : X \in [0, \frac{1}{2}]) \), and \( a_2 := E(X : X \in [\frac{1}{2}, 1]) \). Then, \( a_1 = \frac{r + 1}{6 - 2r} \), and \( a_2 = \frac{5 - 3r}{6 - 2r} \), and the corresponding distortion error is given by

\[ V(P; \kappa) = \frac{-7r^3 + 13r^2 - 9r + 3}{6(r - 3)^2(r + 1)}. \]

**Proof** By the hypothesis, we have

\[ a_1 = E \left( X : X \in \left[ 0, \frac{1}{2} \right] \right) = E \left( X : X \in J_1 \cup J_{21} \cup J_{221} \cup \ldots \right), \]
\[ a_2 = E \left( X : X \in \left[ \frac{1}{2}, 1 \right] \right) = E \left( X : X \in J_3 \cup J_{23} \cup J_{223} \cup \ldots \right), \]

yielding

\[ a_1 = 2 \sum_{n=1}^{\infty} \frac{1}{3^n} \frac{1}{2} \left( -r^{n-1} + r^n + 1 \right) = \frac{r + 1}{6 - 2r}, \] and
\[ a_2 = 2 \sum_{n=1}^{\infty} \frac{1}{3^n} \frac{1}{2} (r^{n-1} - r^n + 1) = \frac{5 - 3r}{6 - 2r}, \]

and the corresponding distortion error is given by

\[ V(P; \kappa) = 2 \int_{J_1 \cup J_{21} \cup J_{221} \cup J_{2221} \ldots} (x - \frac{r + 1}{6 - 2r})^2 dP \]

implying

\[ V(P; \kappa) = 2 \left( \sum_{n=1}^{\infty} \frac{r^{2n}}{3^n} V + \sum_{n=1}^{\infty} \frac{1}{3^n} \left( \frac{1}{2} (-r^{n-1} + r^n + 1) - \frac{r + 1}{6 - 2r} \right)^2 \right) \]

\[ = \frac{-7r^3 + 13r^2 - 9r + 3}{6(r - 3)^2(r + 1)}. \]

Thus, the proposition is yielded. \( \square \)

3 Optimal Sets of \( n \)-means and the \( n \)th Quantization Errors for \( r = \frac{1}{25} \)

Let \( \beta_n \) be the set given by Definition 1.3. In this section, we show that for all \( n \geq 2 \), the sets \( \beta_n \) form the optimal sets of \( n \)-means for \( r = \frac{1}{25} \). To calculate the distortion errors we will frequently use the formula given by Corollary 2.3. Notice that by Lemma 2.2, in this case, we have \( E(X) = \frac{1}{2} \) and \( V := V(X) = \frac{1 - r}{6(r+1)} = \frac{2}{13} \).

**Lemma 3.1** The set \( \beta := \{a(1), a(2, 3)\} \) forms the optimal set of two-means, and the corresponding quantization error is given by \( V_2 = \frac{314}{8125} = 0.0386462 \).

**Proof** Let \( \beta := \{a_1, a_2\} \) be an optimal set of two-means. Since the points in an optimal set are the conditional expectations in their own Voronoi regions, without any loss of generality, we can assume that \( 0 < a_1 < a_2 < 1 \). Let us consider the set \( \kappa := \{a(1), a(2, 3)\} \). The distortion error due to the set \( \kappa \) is given by

\[ V(P; \kappa) = \int_{J_1} (x - a(1))^2 dP + \int_{J_2 \cup J_3} (x - a(2, 3))^2 dP = 0.0386462. \]

Since \( V_2 \) is the quantization error for two-means, we have \( V_2 \leq 0.0386462 \). Assume that \( 0.38 < a_1 \). Then,

\[ V_2 \geq \int_{J_1} (x - 0.38)^2 dP = 0.0432821 > V_2, \]

which is a contradiction. Hence, \( a_1 \leq 0.38 \). Similarly, \( 0.62 \leq a_2 \). Since \( \frac{1}{2}(a_1 + a_2) \leq \frac{1}{2}(0.38 + 1) = 0.69 < S_3(0) = 0.96 \), the Voronoi region of \( a_1 \) does not contain any
point from $J_3$. Similarly, the Voronoi region of $a_2$ does not contain any point from $J_1$. Since the union of the Voronoi regions of $a_1$ and $a_2$ covers $J_1 \cup J_2 \cup J_3$, without any loss of generality, we can assume that the Voronoi region of $a_2$ contains points from $J_2$, and $\frac{1}{2}(a_1 + a_2) \leq \frac{1}{2}$. If $\frac{1}{2}(a_1 + a_2) = \frac{1}{2}$, then substituting $r = \frac{1}{25}$, by Proposition 2.8, we have

$$V_2 = \frac{866}{17797} = 0.0486599 > V_2,$$

which leads to a contradiction. Hence, we can conclude that $\frac{1}{2}(a_1 + a_2) < \frac{1}{2}$. Using the similar technique as it is given in the proof of Lemma 3.1 in [15], we can show that $S_1(1) \leq \frac{1}{2}(a_1 + a_2) \leq S_2(0)$ yielding the fact that $a_1 = a(1), a_2 = a(2, 3)$, and $V_2 = \frac{314}{8125} = 0.0386462$. Hence, the proof of the lemma is complete. \qed

**Lemma 3.2** The set $\beta := \{a(1), a(2), a(3)\}$ forms an optimal set of three-means, and the corresponding quantization error is given by $V_3 = \frac{2}{8125} = 0.000246154$.

**Proof** Consider the set of three points $\kappa := \{a(1), a(2), a(3)\}$. The distortion error due to the set $\kappa$ is given by

$$V(P; \kappa) = \sum_{j=1}^{3} \int_{J_j} (x - a(j))^2 dP = \frac{2}{8125} = 0.000246154.$$ 

Since $V_3$ is the quantization error for three-means, we have $V_3 \leq 0.000246154$. Let $\beta := \{a_1, a_2, a_3\}$, where $0 < a_1 < a_2 < a_3 < 1$, be an optimal set of three-means. If $S_1(1) = \frac{1}{25} < \frac{1}{23} < a_1$, then

$$V_3 \geq \int_{J_1} (x - \frac{1}{23})^2 dP = \frac{13709}{51577500} = 0.000265794 > V_3,$$

which gives a contradiction. Thus, we can assume that $a_1 \leq \frac{1}{23}$. Similarly, $\frac{22}{23} \leq a_3$. Suppose that $\beta \cap J_1 = \emptyset$. Then, due to symmetry, we can assume that $\beta \cap J_3 = \emptyset$, and then

$$V_3 \geq 2 \int_{J_1} (x - a_1)^2 dP = 2 \int_{J_1} (x - S_1(1))^2 dP = \frac{7}{16250} = 0.000430769 > V_3,$$

which leads to a contradiction. So, we can assume that $\beta \cap J_1 \neq \emptyset$, i.e., $a_1 < S_1(1)$. Similarly, $\beta \cap J_3 \neq \emptyset$, i.e., $S_3(0) < a_3$. Now, we show that $\beta \cap J_2 \neq \emptyset$. Suppose that $\beta \cap J_2 = \emptyset$. Then, either $a_2 < \frac{12}{23} = S_2(0)$, or $\frac{13}{23} = S_2(1) < a_2$. First, assume that $a_2 < S_2(0)$. Then, notice that $S_2(1) = \frac{13}{23} < \frac{1}{2}(S_2(0) + S_3(0)) < S_3(0)$ yielding the fact that the Voronoi region of $S_2(0)$ contains $J_2$. Hence,

$$V_3 \geq \int_{J_2} (x - S_2(0))^2 dP + \int_{J_3} (x - a(3))^2 dP = \frac{29}{97500} = 0.000297436 > V_3,$$

which is a contradiction. Hence, $\beta \cap J_2 \neq \emptyset$. Now, we can assume that $\beta \cap J_2 = \emptyset$. Then, due to symmetry, we can assume that $\beta \cap J_3 = \emptyset$, and then

$$V_3 \geq 2 \int_{J_2} (x - a_2)^2 dP = 2 \int_{J_2} (x - S_2(0))^2 dP = \frac{7}{16250} = 0.000430769 > V_3,$$

which leads to a contradiction. So, we can assume that $\beta \cap J_2 \neq \emptyset$, i.e., $a_2 < S_2(0)$. Similarly, $\beta \cap J_3 \neq \emptyset$, i.e., $S_3(0) < a_3$. Now, we show that $\beta \cap J_3 \neq \emptyset$. Suppose that $\beta \cap J_3 = \emptyset$. Then, either $a_3 < \frac{13}{23} = S_3(0)$, or $\frac{14}{23} = S_3(1) < a_3$. First, assume that $a_3 < S_3(0)$. Then, notice that $S_3(1) = \frac{14}{23} < \frac{1}{2}(S_2(0) + S_3(0)) < S_3(0)$ yielding the fact that the Voronoi region of $S_3(0)$ contains $J_3$. Hence,

$$V_3 \geq \int_{J_3} (x - S_3(0))^2 dP + \int_{J_2} (x - a(3))^2 dP = \frac{29}{97500} = 0.000297436 > V_3,$$

which is a contradiction. Hence, $\beta \cap J_3 \neq \emptyset$. Now, we can assume that $\beta \cap J_3 = \emptyset$. Then, due to symmetry, we can assume that $\beta \cap J_2 = \emptyset$, and then

$$V_3 \geq 2 \int_{J_3} (x - a_3)^2 dP = 2 \int_{J_3} (x - S_3(0))^2 dP = \frac{7}{16250} = 0.000430769 > V_3,$$
which is a contradiction. Similarly, we can show that a contradiction arises if \( \frac{13}{25} = S_2(1) < a_2 \). Thus, we can assume that \( \beta \cap J_2 \neq \emptyset \). Now, if the Voronoi region of \( a_1 \) contains points from \( J_2 \), we have \( \frac{1}{2}(a_1 + a_2) > \frac{12}{25} = S_2(0) \) implying \( a_2 > \frac{24}{25} - a_1 \geq \frac{24}{25} - \frac{1}{25} = \frac{23}{25} > S_2(1) \), which is a contradiction as \( \beta \cap J_2 \neq \emptyset \). Hence, we can assume that the Voronoi region of \( a_1 \) does not contain any point from \( J_2 \), and so from \( J_3 \). Similarly, we can show that the Voronoi region of \( a_2 \) does not contain any point from \( J_1 \) and \( J_3 \), and the Voronoi region of \( a_3 \) does not contain any point from \( J_2 \), and so from \( J_1 \). Thus, by Proposition 1.2, we conclude that \( a_1 = a(1) \), \( a_2 = a(2) \), and \( a_3 = a(3) \), and the corresponding quantization error is given by \( V_3 = \frac{2}{8125} = 0.000246154 \), which is the lemma.

**Proposition 3.3** Let \( \beta_n \) be an optimal set of \( n \)-means for any \( n \geq 3 \). Then, \( \beta_n \cap J_j \neq \emptyset \) for all \( 1 \leq j \leq 3 \), and \( \beta_n \) does not contain any point from the open intervals \((S_1(1), S_2(0))\) and \((S_2(1), S_3(0))\). Moreover, the Voronoi region of any point in \( \beta_n \cap J_i \) does not contain any point from \( J_i \), where \( 1 \leq i \neq j \leq 3 \).

**Proof** By Lemma 3.2, the proposition is true for \( n = 3 \). Let us prove the lemma for \( n \geq 4 \). Let \( \beta_n := \{a_1, a_2, \cdots, a_n\} \) be an optimal set of \( n \)-means for \( n \geq 4 \). Since the points in an optimal set are the conditional expectations in their own Voronoi regions, without any loss of generality, we can assume that \( 0 < a_1 < a_2 < \cdots < a_n < 1 \). Consider the set of four elements \( \kappa := S_1(\beta_2) \cup \{a(2), a(3)\} \). Then,

\[
V(P; \kappa) = \int_{J_1} \min_{a \in S_1(\beta_2)} (x - a)^2 dP + \int_{J_2} (x - a(2))^2 dP + \int_{J_3} (x - a(3))^2 dP
= \frac{938}{5078125} = 0.000184714.
\]

Since \( V_n \) is the quantization error for \( n \)-means for \( n \geq 4 \), we have \( V_n \leq V_4 \leq 0.000184714 \). Suppose that \( S_1(1) \leq a_1 \). Then,

\[
V_n \geq \int_{J_1} (x - S_1(1))^2 dP = \frac{7}{32500} = 0.000215385 > V_n,
\]

which is a contradiction. So, we can assume that \( a_1 < S_1(1) \), i.e., \( \beta_n \cap J_1 \neq \emptyset \). Similarly, \( \beta_n \cap J_3 \neq \emptyset \). We now show that \( \beta_n \cap J_2 \neq \emptyset \). For the sake of contradiction, assume that \( \beta_n \cap J_2 = \emptyset \). Let \( a_j := \max\{a_i : a_i < S_2(0) \text{ for } 1 \leq i \leq n - 1\} \). Then, \( a_j < S_2(0) \). As \( \beta_n \cap J_2 = \emptyset \), we have \( S_2(1) < a_{j+1} \). If \( a_j < \frac{1}{2}(S_1(1) + S_2(0)) = \frac{13}{50} \), then as \( \frac{1}{2}(a_j + a_{j+1}) < \frac{1}{2}(\frac{13}{50} + S_2(1)) = \frac{39}{100} < \frac{12}{25} = S_2(0) \), we have

\[
V_n \geq \int_{J_2} (x - S_2(1))^2 dP = \frac{7}{32500} = 0.000215385 > V_n,
\]

which leads to a contradiction. So, we can assume that \( \frac{13}{50} \leq a_j < S_2(0) \). Then, by Proposition 1.2, we have \( \frac{1}{2}(a_{j-1} + a_j) < \frac{1}{25} \) implying \( a_{j-1} < \frac{2}{25} - a_j \leq \frac{2}{25} - \frac{13}{50} = -\frac{9}{50} < 0 \), which gives a contradiction as \( \beta_n \cap J_1 \neq \emptyset \). Hence, we can conclude that \( \beta_n \cap J_2 \neq \emptyset \). Notice that \( (S_1(1), S_2(0)) = (\frac{1}{25}, \frac{12}{25}) \). Suppose that \( \beta_n \) contains a point
from the open interval \((\frac{1}{25}, \frac{12}{25})\). Let \(a_j := \max\{a_i : a_i < \frac{1}{25} \text{ for } 1 \leq i \leq n - 2\}\). Then, due to Proposition 1.2, \(a_{j+1} \in (\frac{1}{25}, \frac{12}{25})\), and \(a_{j+2} \in J_2\). The following cases can arise:

Case 1. \(\frac{1}{25} < a_{j+1} \leq \frac{13}{50}\).

Then, \(\frac{1}{2}(a_{j+1} + a_{j+2}) > \frac{12}{50}\) implying \(a_{j+2} > \frac{24}{50} - a_{j+1} \geq \frac{24}{50} - \frac{13}{50} = \frac{11}{50} > S_2(1)\), which leads to a contradiction because \(a_{j+2} \in J_2\).

Case 2. \(\frac{13}{50} \leq a_{j+1} < \frac{12}{25}\).

Then, \(\frac{1}{2}(a_{j} + a_{j+1}) < \frac{1}{25}\) implying \(a_{j} \leq \frac{24}{50} - a_{j+1} \leq \frac{24}{50} - \frac{13}{50} = -\frac{9}{50}\), which is a contradiction because \(a_{j} > 0\).

Thus, by Case 1 and Case 2, we can conclude that \(\beta_n\) does not contain any point from the open interval \((S_1(1), S_2(0))\). Reflecting the situation with respect to the point \(\frac{1}{2}\), we can conclude that \(\beta_n\) does not contain any point from the open interval \((S_3(1), S_3(0))\) as well. To prove the last part of the proposition, we proceed as follows:

Let \(a_j := \max\{a_i : a_i < \frac{1}{25} \text{ for } 1 \leq i \leq n - 2\}\). Then, \(a_j\) is the rightmost element in \(\beta_n \cap J_1\), and \(a_{j+1} \in \beta_n \cap J_2\). Suppose that the Voronoi region of \(a_j\) contains points from \(J_2\). Then, \(\frac{1}{2}(a_j + a_{j+1}) > \frac{12}{50}\) implying \(a_{j+1} > \frac{24}{50} - a_j \geq \frac{24}{50} - \frac{1}{25} = \frac{23}{50} > S_2(1)\), which yields a contradiction as \(a_{j+1} \in J_2\). Thus, the Voronoi region of any point in \(\beta_n \cap J_1\) does not contain any point from \(J_2\), and \(J_3\) as well. Similarly, we can prove that the Voronoi region of any point in \(\beta_n \cap J_2\) does not contain any point from \(J_1\) and \(J_3\), and the Voronoi region of any point in \(\beta_n \cap J_3\) does not contain any point from \(J_1\) and \(J_2\). Thus, the proof of the proposition is complete.

The following lemma is a modified version of Lemma 4.5 in [5], and the proof follows similarly. One can also see Lemma 3.5 in [15].

**Lemma 3.4** Let \(n \geq 3\), and let \(\beta_n\) be an optimal set of \(n\)-means such that \(\beta_n \cap J_j \neq \emptyset\) for all \(1 \leq j \leq 3\), and \(\beta_n\) does not contain any point from the open intervals \((S_1(1), S_2(0))\) and \((S_2(1), S_3(0))\). Further assume that the Voronoi region of any point in \(\beta_n \cap J_j\) does not contain any point from \(J_i\), where \(1 \leq i \neq j \leq 3\). Set \(\kappa_j := \beta_n \cap J_j\), and \(n_j := \text{card } (\kappa_j)\) for \(1 \leq j \leq 3\). Then, \(S_j^{-1}(\kappa_j)\) is an optimal set of \(n_j\)-means, and \(V_n = \frac{1}{1875} (V_{n_1} + V_{n_2} + V_{n_3})\).

Let us now state and prove the following theorem which gives the optimal sets of \(n\)-means for all \(n \geq 3\), where \(r = \frac{1}{25}\).

**Theorem 3.5** Let \(P\) be the probability measure on \(\mathbb{R}\) with support the Cantor set \(C\) generated by the three contractive similarity mappings \(S_j\) for \(j = 1, 2, 3\). Let \(n \in \mathbb{N}\) with \(n \geq 3\). Take \(r = \frac{1}{25}\). Then, the sets \(\beta_n := \beta_n(I)\) given by Definition 1.3 form the optimal sets of \(n\)-means for \(P\) with the corresponding quantization error \(V_n := V(P; \beta_n(I))\), where \(V(P; \beta_n(I))\) is given by Proposition 2.4.

**Proof** We will proceed by induction on \(\ell(n)\). If \(n = 3\), then by Lemma 3.2, the theorem is true. Now, we show that the theorem is true if \(n = 4\). Let \(\kappa_j := \beta_n \cap J_j\), and \(n_j := \text{card } (\kappa_j)\) for \(1 \leq j \leq 3\). Since \(S_j^{-1}(\kappa_j)\) is an optimal set of \(n_j\)-means for \(1 \leq j \leq 3\), and for \(n = 4\) the possible choices for the triplet \((n_1, n_2, n_3)\) are \((2, 1, 1)\), \((1, 2, 1)\), and \((1, 1, 2)\), by Proposition 3.3 and Lemma 3.4, the set \(\beta_4\) forms an optimal set of
four-means with quantization error $V(P; \beta_4)$ given by Proposition 2.4. Remember that for a given $n$, among all the possible choices of the triplets $(n_1, n_2, n_3)$, the triplets $(n_1, n_2, n_3)$ which give the smallest distortion error will give the optimal sets of $n$-means. Notice that for $n = 5$, the possible choices of the triplets are $(3, 1, 1), (1, 3, 1), (1, 1, 3), (1, 2, 2), (2, 1, 2), (2, 2, 1)$ among which $(1, 2, 2), (2, 1, 2), (2, 2, 1)$ give the smallest distortion error. Hence, the optimal sets of five-means are $\{a(1)\} \cup S_2(\beta_2) \cup S_3(\beta_2), S_1(\beta_2) \cup \{a(2)\} \cup S_3(\beta_2)$, and $S_1(\beta_2) \cup S_2(\beta_2) \cup \{a(3)\}$ which are the sets $\beta_5$ given by Definition 1.3. Similarly, we can calculate the optimal sets of six- and seven-means. Thus, the theorem is true for all $\ell(n) = 1$. Let us assume that the theorem is true for all $\ell(n) < m$, where $m \in \mathbb{N}$ and $m \geq 2$. We now show that the theorem is true if $\ell(n) = m$. Let us first assume that $3^m \leq n \leq 2 \cdot 3^m$. Let $\beta_n$ be an optimal set of $n$-means for $P$ such that $3^m \leq n \leq 2 \cdot 3^m$. Let $\{\beta_n \cap I_j\} = n_j$ for $j = 1, 2, 3$, and then by Lemma 3.4, we have

$$V_n = \frac{1}{1875} (V_{n_1} + V_{n_2} + V_{n_3}).$$

(3)

Without any loss of generality, we can assume that $n_1 \geq n_2 \geq n_3$. Let $u, v, w \in \mathbb{N}$ be such that

$$3^u \leq n_1 \leq 2 \cdot 3^u, \ 3^v \leq n_2 \leq 2 \cdot 3^v, \ and \ 3^w \leq n_3 \leq 2 \cdot 3^w.$$  

(4)

Proceeding in the similar lines as the proof of Theorem 3.6 in [15], we can show that $u = v = w = m - 1$. Since by Lemma 3.4, for $S_j^{-1}(\beta_n \cap J_j)$ is an optimal set of $n_j$ means where $3^{m-1} \leq n_j \leq 2 \cdot 3^{m-1}$, we have

$$S_j^{-1}(\beta_n \cap J_j) = \{a(\omega): \omega \in \{1, 2, 3\}^{m-1} \setminus I_j\} \cup \bigcup_{\omega \in I_j} S_\omega(\beta_2),$$

where $I_j \subseteq \{1, 2, 3\}^{m-1}$ with card $(I_j) = n_j - 3^{m-1}$ for $1 \leq j \leq 3$. Hence,

$$\beta_n := \beta_n(I) = \bigcup_{j=1}^{3} S_j^{-1}(\beta_n \cap J_j) = \{a(\omega): \omega \in \{1, 2, 3\}^{\ell(n)} \setminus I\} \cup \bigcup_{\omega \in I} S_\omega(\beta_2),$$

where $I \subseteq \{1, 2, 3\}^{m}$ with card $(I) = n - 3^m$, is an optimal set of $n$-means. The corresponding quantization error is

$$V_n = \frac{1}{3^m} r^{2m} \left((2 \cdot 3^m - n)V + (n - 3^m)V_2\right) = V(P; \beta_n(I)),$$

where $V(P; \beta_n(I))$ is given by Proposition 2.4. Thus, the theorem is true if $3^m \leq n \leq 2 \cdot 3^m$. Similarly, we can prove that the theorem is true if $2 \cdot 3^m < n < 3^{m+1}$. Hence, by the induction principle, the proof of the theorem is complete. \qed
4 Optimal Sets of $n$-means and the $n$th Quantization Errors for $r = r_0$ and $r = r_1$

In this section, we give the proof of Theorem 1.7. First, we prove the following two lemmas.

**Lemma 4.1** Let $r_0$ and $r_1$ be the real numbers given by Theorem 1.7. Then, the set $\gamma := \{a(1, 21), a(22, 23, 3)\}$ for $r = r_0$ and $r = r_1$ form the optimal sets of two-means, and the corresponding quantization errors are, respectively, given by $V_2 = 0.0324042$, and $V_2 = 0.026897$.

**Proof** First, we prove that $\gamma$ forms an optimal set of two-means for $r = r_0$. Let $\gamma := \{a_1, a_2\}$ be an optimal set of two-means. Since, the points in an optimal set are the expected values of their own Voronoi regions, without any loss of generality, we can assume that $0 < a_1 < a_2 < 1$. Let us consider the set $\kappa := \{a(1, 21), a(22, 23, 3)\}$. The distortion error due to the set $\kappa$ is given by

$$V(P; \kappa) = \int_{J_1} (x - a(1, 21))^2 dP + \int_{J_2 \cup J_3} (x - a(22, 23, 3))^2 dP = 0.0324042.$$  \hspace{1cm} (5)

Since $V_2$ is the quantization error for two-means, we have $V_2 \leq 0.0324042$. Assume that $0.39 < a_1$. Then,

$$V_2 \geq \int_{J_1} (x - 0.39)^2 dP = 0.0328529 > V_2,$$

which is a contradiction. Hence, $a_1 \leq 0.39$. Similarly, $0.61 \leq a_2$. Since $\frac{1}{2}(a_1 + a_2) \leq \frac{1}{2}(0.39 + 1) = 0.695 < S_3(0) = 0.837722$, the Voronoi region of $a_1$ does not contain any point from $J_3$. Similarly, the Voronoi region of $a_2$ does not contain any point from $J_1$. Since the union of the Voronoi regions of $a_1$ and $a_2$ covers $J_1 \cup J_2 \cup J_3$, without any loss of generality, we can assume that the Voronoi region of $a_2$ contains points from $J_2$, and $\frac{1}{2}(a_1 + a_2) \leq \frac{1}{2}$. If $\frac{1}{2}(a_1 + a_2) = \frac{1}{2}$, then substituting $r = 0.1622776602$, by Proposition 2.8, we have

$$V(P; \kappa) = 0.0329779,$$

which contradicts (5). Hence, we can conclude that $\frac{1}{2}(a_1 + a_2) < \frac{1}{2}$. Using the similar technique as it is given in the proof of Lemma 3.1 in [15], we can show that either $\frac{1}{2}(a_1 + a_2) = \frac{1}{2}(a(1, 21) + a(22, 23, 3)) = 0.466886$, or $\frac{1}{2}(a_1 + a_2) = \frac{1}{2}(a(1) + a(2, 3)) = 0.395285$, i.e., either $S_{21}(1) < \frac{1}{2}(a_1 + a_2) < S_{22}(0)$, or $S_{1}(1) < \frac{1}{2}(a_1 + a_2) < S_{2}(0)$. Notice that if $S_{21}(1) < \frac{1}{2}(a_1 + a_2) < S_{22}(0)$, then $\gamma_2$, given by Definition 1.4, forms the optimal set of two-means. On the other hand, if $S_{1}(1) < \frac{1}{2}(a_1 + a_2) < S_{2}(0)$, then $\beta_2$, given by Definition 1.3, forms the optimal set of two-means. In fact, later we will see that $V(P; \gamma_2) = V(P; \beta_2) = 0.0324042$ for $r = 0.1622776602$. Thus, $\gamma_2$ forms the optimal set of two-means for $r = r_0$ with quantization error $V_2 = 0.0324042$. 


Similarly, we can show that \( \gamma_2 \) forms the optimal set of two-means if \( r = r_1 \) with quantization error \( V_2 = 0.026897 \). Hence, the lemma is yielded. \( \Box \)

The following lemma is true analogously as Lemma 3.3 in [15].

**Lemma 4.2** The set \( \gamma_3 := \{a(1), a(2), a(3)\} \), for \( r = r_0 \), and \( r = r_1 \), form the optimal sets of three-means, and the corresponding quantization errors are, respectively, given by \( V_3 = 0.00316342 \), and \( V_3 = 0.00558347 \).

The following proposition is true analogously as Proposition 3.5 in [15].

**Proposition 4.3** Let \( n \geq 3 \), and let \( \gamma_n \) be an optimal set of \( n \)-means for \( r = r_0 \), and \( r = r_1 \). Then, \( \gamma_n \cap J_j \neq \emptyset \) for all \( 1 \leq j \leq 3 \), and \( \gamma_n \) does not contain any point from the open intervals \( (S_1(1), S_2(0)) \) and \( (S_2(1), S_3(0)) \). Moreover, the Voronoi region of any point in \( \gamma_n \cap J_j \) does not contain any point from \( J_i \), where \( 1 \leq i \neq j \leq 3 \).

The following remark is true due to Proposition 4.3.

**Remark 4.4** Let \( n \geq 3 \), and let \( \gamma_n \) be an optimal set of \( n \)-means for \( r = r_0 \), and \( r = r_1 \). Set \( \kappa_j := \gamma_n \cap J_j \), and \( n_j := \text{card}(\kappa_j) \) for \( 1 \leq j \leq 3 \). Then, \( S_j^{-1}(\kappa_j) \) is an optimal set of \( n_j \)-means, and for \( r = r_0 \) and \( r = r_1 \), respectively, we have \( V_n = \frac{1}{3}r_0^n(V_{n_1} + V_{n_2} + V_{n_3}) \) and \( V_n = \frac{1}{3}r_1^n(V_{n_1} + V_{n_2} + V_{n_3}) \).

**Proof of Theorem 1.7**

We proceed to prove it by induction on \( \ell(n) \). By Lemma 4.2, we see that the theorem is true for \( n = 3 \). Proceeding in the similar way, as mentioned in the proof of Theorem 3.5, we can show that for \( n = 4, 5, 6, 7 \), the sets \( \gamma_n \) form the optimal sets of \( n \)-means for \( r = r_0 \) and \( r = r_1 \). Thus, the theorem is true if \( \ell(n) = 1 \). Let us assume that the theorem is true for all \( \ell(n) < m \), where \( m \in \mathbb{N} \) and \( m \geq 2 \). We now show that the theorem is true if \( \ell(n) = m \). Let us first assume that \( 3^m \leq n \leq 2 \cdot 3^m \). Let \( \gamma_n \) be an optimal set of \( n \)-means for \( P \) such that \( 3^m \leq n \leq 2 \cdot 3^m \). Let \( n_j \) for \( j = 1, 2, 3 \), and then by Remark 4.4, we have

\[
V_n = \frac{1}{3}r_0^n(V_{n_1} + V_{n_2} + V_{n_3}) \quad \text{for} \quad r = r_0, \quad \text{and} \quad V_n = \frac{1}{3}r_1^n(V_{n_1} + V_{n_2} + V_{n_3}) \quad \text{for} \quad r = r_1.
\]

The rest of the proof for \( r = r_0 \) and \( r = r_1 \) follow in the similar way as the proof of Theorem 3.5. Thus, we complete the proof of the theorem. \( \Box \)

**5 Main Results**

The two theorems in this section, state and prove the main results of the paper.

**Theorem 5.1** Let \( r_0, r_1 \in (0, \frac{1}{3}) \) be the unique real numbers satisfying, respectively, the equations

\[
- \frac{3r^5 + 15r^4 + 6r^3 - 42r^2 + 31r - 13}{240(r + 1)} = - \frac{3r^3 - 3r^2 + r - 1}{24(r + 1)}.
\]
Then, \( r_0 = 0.1622776602 \), and \( r_1 = 0.2317626315 \). Let the sets \( \beta_n \) and \( \gamma_n \) be, respectively, given by Definition 1.3, and Definition 1.4. Then, \( \beta_n \) form the optimal sets of \( n \)-means for \( 0 < r \leq r_0 \), and \( \gamma_n \) forms the optimal sets of \( n \)-means for \( r_0 \leq r \leq r_1 \).

**Proof** By Proposition 2.4, Proposition 2.5, and Proposition 2.7, we see that both \( \beta_n \) and \( \gamma_n \) form CVTs if \( 0.08502712839 \leq r \leq 0.2472080177 \); both \( \gamma_n \) and \( \delta_n \) form CVTs if \( 0.1845020699 \leq r \leq 0.2472080177 \); both \( \beta_n \) and \( \delta_n \) form CVTs if \( 0.1845020699 \leq r \leq 0.2679491924 \). Again, \( V(P; \beta_3) = V(P; \gamma_3) = V(P; \delta_3) \). Thus, for any \( 3^\ell(n) \leq n < 3^\ell(n)+1 \), from the aforementioned propositions, in the case of \( V(P; \beta_n(I)) \) and \( V(P; \gamma_n(I)) \), we see that \( V(P; \beta_n(I)) > V(P; \gamma_n(I)) \), \( V(P; \beta_n(I)) = V(P; \gamma_n(I)) \), and \( V(P; \beta_n(I)) < V(P; \gamma_n(I)) \) will be true if \( V(P; \beta_3) > V(P; \gamma_3) \), \( V(P; \beta_3) = V(P; \gamma_3) \), and \( V(P; \beta_3) < V(P; \gamma_3) \), respectively. Similarly, it holds in the case of \( V(P; \beta_n) \) and \( V(P; \delta_n) \), and in the case of \( V(P; \gamma_n) \) and \( V(P; \delta_n) \). Next, we have

\[
V(P; \beta_2) = -\frac{3r^3 - 3r^2 + r - 1}{24(r + 1)},
\]

\[
v(P; \gamma_2) = -\frac{3r^5 + 15r^4 + 6r^3 - 42r^2 + 31r - 13}{240(r + 1)},
\]

\[
v(P; \delta_2) = -\frac{3r^7 + 15r^6 + 66r^5 + 18r^3 - 324r^2 + 283r - 121}{2184(r + 1)}.
\]

After some calculation, we observe that \( V(P; \beta_2) < V(P; \gamma_2) \) is true if \( 0.08502712839 \leq r < 0.1622776602 \); \( V(P; \beta_2) = V(P; \gamma_2) \) if \( r = 0.1622776602 \), and \( V(P; \beta_2) > V(P; \gamma_2) \) if \( 0.1622776602 < r \leq 0.2472080177 \). Again, \( V(P; \beta_2) > V(P; \delta_2) \) if \( 0.1701473031 < r \leq 0.2679491924 \), and \( V(P; \beta_2) = V(P; \delta_2) \) if \( r = 0.1701473031 \). Recall that the sets \( \beta_n \) form CVTs if \( 0 < r \leq 0.2679491924 \). Hence, we can see that the sets \( \beta_n \) do not form the optimal sets of \( n \)-means if \( 0.1622776602 < r \leq 0.2679491924 \). In Theorem 1.7, we have seen that the sets \( \beta_n \) form the optimal sets of \( n \)-means if \( r = \frac{1}{25} \). Using the similar technique, we can show that the sets \( \beta_n \) form the optimal sets of \( n \)-means if \( 0 < r \leq \frac{1}{25} \). Since \( V(P; \beta_2) = V(P; \gamma_2) \) if \( r = r_0 \); and by Theorem 1.7, the sets \( \gamma_n \) form the optimal sets of \( n \)-means if \( r = r_0 \), we can say that the sets \( \beta_n \) also form the optimal sets of \( n \)-means if \( r = r_0 \). Again, \( V(P; \beta_2) \) is strictly decreasing in the closed interval \([0, r_0] \). Hence, the sets \( \beta_n \) form the optimal sets of \( n \)-means for \( 0 < r \leq r_0 \).

To prove the remaining part of the theorem, we see that

(i) \( V(P; \beta_2) < V(P; \gamma_2) \) if \( 0.08502712839 \leq r < 0.1622776602 \); \( V(P; \beta_2) = V(P; \gamma_2) \) if \( r = 0.1622776602 \), and \( V(P; \beta_2) > V(P; \gamma_2) \) if \( 0.1622776602 < r \leq 0.2472080177 \).

(ii) \( V(P; \delta_2) < V(P; \gamma_2) \) if \( 0.2317626315 < r \leq 0.2472080177 \); \( V(P; \delta_2) = V(P; \gamma_2) \) if \( r = 0.2317626315 \), and \( V(P; \delta_2) > V(P; \gamma_2) \) if \( 0.1845020699 \leq r < 0.2317626315 \).
Thus, the sets $\gamma_n$ do not form the optimal sets of $n$-means if $0.08502712839 \leq r < 0.1622776602$, or if $0.2317626315 < r \leq 0.2472080177$; in other words, the range of $r$ values for which the sets $\gamma_n$ form the optimal sets of $n$-means is bounded below by $r_0 = 0.1622776602$ and bounded above by $r_1 = 0.2317626315$. By Theorem 1.7, we see that the sets $\gamma_n$ form the optimal sets of $n$-means if $r = r_0$, and $r = r_1$. Again, $V(P; \gamma_2)$ is strictly decreasing in the closed interval $[r_0, r_1]$. Hence, the precise range of $r$ values for which the sets $\gamma_n$ form the optimal sets of $n$-means is given by $r_0 \leq r \leq r_1$. Thus, the proof of the theorem is complete.

Since the Cantor set $C$ under investigation satisfies the strong separation condition, with each $S_j$ having contracting factor of $r$, the Hausdorff dimension of the Cantor set is equal to the similarity dimension. Hence, from the equation $3^\beta = 1$, we have $\dim_H(C) = \beta = -\frac{\log 3}{\log r}$. By Theorem 14.17 in [4], the quantization dimension $D(P)$ exists and is equal to $\beta$. In Theorem 5.2, we show that $\beta$ dimensional quantization coefficient for $P$ does not exist.

**Theorem 5.2** The $\beta$-dimensional quantization coefficient for $0 < r \leq r_1$ does not exist.

**Proof** We have $3^{\frac{1}{\beta}} = \frac{1}{r}$. Notice that $\left\{\left(3^{\ell(n)}\right)^{\frac{2}{\beta}} V_{3^{\ell(n)}}(P)\right\}$ and $\left\{\left(2 \cdot 3^{\ell(n)}\right)^{\frac{2}{\beta}} V_{2 \cdot 3^{\ell(n)}}(P)\right\}$ are two different subsequences of the sequence $\left\{n^{\frac{2}{\beta}} V_n(P)\right\}$. First, assume that $0 < r \leq r_0$. Then, by Theorem 5.1, $\beta_n$ is an optimal set of $n$-means for $0 < r \leq r_0$. Recall Proposition 2.4. Then, we have

$$\lim_{n \to \infty} \left(3^{\ell(n)}\right)^{\frac{2}{\beta}} V_{3^{\ell(n)}}(P) = \lim_{n \to \infty} \frac{1}{r^{2\ell(n)}} \frac{1}{3^{\ell(n)}} r^{2\ell(n)} 3^{\ell(n)} V = V, \quad (6)$$

and

$$\lim_{n \to \infty} \left(2 \cdot 3^{\ell(n)}\right)^{\frac{2}{\beta}} V_{2 \cdot 3^{\ell(n)}}(P) = \lim_{n \to \infty} \frac{2^{\frac{2}{\beta}}}{r^{2\ell(n)}} \frac{1}{3^{\ell(n)}} r^{2\ell(n)} 3^{\ell(n)} V(P; \beta_2) = 2^{\frac{2}{\beta}} V(P; \beta_2). \quad (7)$$

By (6) and (7), we see that $\left\{n^{\frac{2}{\beta}} V_n(P)\right\}$ has two different subsequences having two different limits, and so $\lim_{n \to \infty} n^{\frac{2}{\beta}} V_n(P)$ does not exist. Due to Theorem 5.1, and Proposition 2.5, similarly, we can show that if $r_0 \leq r \leq r_1$, then $\lim_{n \to \infty} n^{\frac{2}{\beta}} V_n(P)$ does not exist. Thus, we show that the $\beta$-dimensional quantization coefficient for $0 < r \leq r_1$ does not exist, which completes the proof of the theorem.

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