Exploring the Level of Urbanization Based on Zipf’s Scaling Exponent

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Abstract: The rank-size distribution of cities follows Zipf’s law, and the Zipf scaling exponent often tends to a constant 1. This seems to be a general rule. However, a recent numerical experiment shows that there exists a contradiction between the Zipf exponent close to 1 and the urbanization level of a large population country. Based on logical reasoning, computational analysis, and empirical evidence, this paper is devoted to exploring the relationships between urbanization level and Zipf scaling exponent. Euler’s formula and Huygens series are employed to make mathematical transform. Research on the old problem results in new findings. (1) If Zipf scaling exponent equals 1, the level of urbanization is a logarithmic function of city number. For a large populous country, urbanization rate cannot exceed 50%. (2) If Zipf scaling exponent is less than 1, there is approximate scaling relation between urbanization level and city number. Supposing the top-level city is large enough, the urbanization level of large populous countries can exceeds 80%. Conclusions can be reached as follows. First, due to the competition between urbanization level and city number in large populous country, the rank-size scaling exponent will be less than 1, or the scaling range breaks into two parts. Second, due to competition between city number and city size of the largest city, the rank-size distribution of cities in medium and small countries may change into primate distribution. This study shows new way of looking at Zipf’s law of city-size distribution and urbanization dynamics.

Key words: Zipf’s law; allometric scaling; level of urbanization; Euler’s formula; Huygens series; Chinese cities

1. Introduction

Urbanization is a process of nonlinear dynamics in self-organized evolution of human settlements.
Urbanization result in rank-size distribution at macro level and complex form and growth of cities at micro level. In intra-urban geography, urban form and growth can be modeled by fractal geometry (Batty and Longley, 1994; Frankhauser, 1994; Frankhauser, 1998). In interurban geography, the city size distribution can be described with Zipf’s law, which is also termed rank-size rule in the literature on cities. The rank-size rule represents the pure Zipf distribution with scaling exponent equal to 1 (Knox and Marston, 2009). The rule states that the product of a city’s size \( P(k) \) and its rank \( k \) is a constant, which equals the size of the largest city \( P_1 \), i.e., \( kP(k) = P_1 \). This rule is simple, graceful, easy to understand, and can be found everywhere in urban studies (Batty and Longley, 1994; Gabaix, 1999a; Gabaix, 1999b; Jiang et al, 2015; Jiang and Jia, 2011; Krugman, 1996). For a long time, Zipf’s distribution of city sizes were regarded as super-stable distribution (Buchanan, 2000; Knox and Marston, 2009; Madden, 1956; Zhou, 1995). The distribution models and the parameter values used to be regarded as depending on the development degree of a country (Berry, 1961; Roehner, 1991). However, in the real world, the Zipf scaling exponent of city size distribution in some countries such as China and the United States of America deviates significantly from 1 (Chen, 2012a; Chen, 2012b). New evidences show that the size distribution models of cities and the related scaling exponent are not always decided by the extent of economic development of a country. On the one hand, Zipf exponent value depends on the definition of city; on the other hand, Zipf exponent relies on population size of a country. The common definition include city’s proper, urbanized area, metropolitan area, and urban agglomeration (Davis, 1978; Knox and Marston, 2009; Rubenstein, 1999; Zhou, 1995). The problem is that if one of the city’s definitions is closely related to the level of urbanization, the logical contradiction can be deduced from the pure Zipf’s law. Suppose that the population size of a country is more than 500 million, and the Zipf exponent is equal to 1, we can theoretically draw the following inference: the proportion of urban population in the total population of the country will hardly exceed 50%. In other word, based on pure Zipf size distribution of cities, it is hard to reach the level of urban majority for a large populous country.

If the above inference is true, the urbanization dynamics of a large population country will inevitably lead to the deviation of the scaling exponent of rank size distribution of cities from 1. The relationships between urbanization level and city size distribution is an old-fashioned problem in urban studies (Berry, 1961). The rank-size rule is one of the three basic laws in urban geography. Another two ones are distance-decay law and allometric growth law, respectively. The history of
Zipf law in urban research can be traced back to Auerbach’s law of population concentration (Auerbach, 1913). After that, Jefferson (1939) proposed the law of the primate city, which resulted in the concepts of the primate city and primate rule. Thus city size distribution were divided into two typical categories: rank-size type and primate type (Berry, 1961). Since then, a great many studies have been devoted to the field of city-size distributions, and many interesting findings emerged (Carroll, 1982; Gabaix and Ioannides, 2004; Zhou, 1995). However, long-term and numerous studies do not guarantee that the basic theoretical issues can be clarified. What is mathematical relationships between urbanization level and Zipf scaling exponent? What is physical mechanism causing the differentiation between the rank-size distribution and primate distribution? Based on mathematical derivation, computational analyses, and empirical evidence, this paper aims at the dynamical mechanism of urbanization resulting in variation of Zipf scaling exponent of rank-size distribution of cities. The rest parts are arranged as follows. In Section 2, the mathematical equations are derived to describe the relationships between urbanization level, city number, national population, and Zipf scaling exponent. In Section 3, a series of computational analyses based on related mathematical models are made to testify and complement the derived theoretical results in Section 2. In Section 4, several related questions are responded and discussed, and finally, the discussion are concluded by summarizing the main points of this work.

2. Models

2.1 A formula of urbanization level based on pure Zipf’s law

The well-known Zipf’s law is the starting point of this theoretical exploration. A set of basic formula can be derived for computational and empirical analyses. Suppose that there are \( N \) cities and towns in a geographical region, say, a country, and the rank-size distribution of these human settlements follows Zipf’s law. The sizes of cities and towns are measured by urban population. The general form of Zipf’s law can be expressed as

\[
P(k) = P_1 k^{-q},
\]

where \( k \) denotes the rank of a city/town, \( P(k) \) is the city population size of the \( k \)th city/town, \( P_1 \) refers to the size of the largest city, i.e., the primacy city, and \( q \) can be termed Zipf scaling exponent. Equation (1) is the well-known two-parameter Zipf model of city size distributions. It was
demonstrated to be equivalent to Pareto’s cumulative distribution function (CDF) (Chen et al, 1993). In the simplest case, the total urban population of a country is

\[ U = \sum_{k=1}^{N} P(k) = P \sum_{k=1}^{N} k^{-q}, \]

where \( U \) refers to the total urban population in a region. For the pure Zipf’s distribution, the Zipf exponent \( q=1 \). The corresponding scaling exponent of Pareto’s density distribution function (DDF) is 2 (Manrubia and Zanette, 1998; Rozenfeld et al, 2011; Zanette and Manrubia, 1998). The two-parameter model is reduced to a one-parameter model, \( P(k) = P_1/k \), which describes the inverse relationship between city rank and size. According to Euler’s formula for the sum of harmonic series (Appendix 1), we have

\[ \lim_{N \to \infty} \left( \sum_{k=1}^{N} \frac{1}{k} \right) = \ln N \to C = 0.577216 \cdots, \]

where \( C=0.577216 \cdots \) denotes one of Euler’s constant (Havil, 2003). The total urban population is

\[ U = \sum_{k=1}^{N} P(k) = P_1(C + \ln N). \]

This is the discrete expression of urban total population. The level of urbanization of a country is defined as \( L = U/P_T \times 100\% \), where \( P_T \) represents entire population of a region, including urban population and rural population (United States, 1980). So we have

\[ L = \frac{U}{P_T} \times 100 = \frac{100P_1(\ln(N) + C)}{P_T}. \]

The differential of \( L \) with respect to \( N \) is \( dL/dN=100P_1/(P_TN) \), which suggests that if \( N \) is large enough, new increasing towns will lead to very little increase of urbanization level. Equation (5) can be rewritten as

\[ N = \exp\left( \frac{LP_T}{100P_1} - E \right) = N_0 \exp\left( \frac{LP_T}{100P_1} \right) = N_0 \exp\left( \frac{U}{P_1} \right), \]

where the coefficient \( N_0 = \exp(-E) = \exp(-0.577216 \cdots) = 0.561459 \cdots \). This means that the number of cities increases exponentially as the level of urbanization goes up, and the size of the largest city is just the characteristic value of total population of cities and towns. Using equation (6), we can reveal the relationships between the level of urbanization, the population of the largest city, the number of cities and towns, the total urban population, and the total population of a country.
2.2 Zipf’s scaling exponent and total urban population

The rank-size rule is regarded as a solution to the scaling functional equation, but the pure Zipf distribution does not reflect a strict scaling relation. For a scaling function, both its differential and integral results obey scaling law (Chen, 2015). However, the integral of pure Zipf’s law does not satisfy the scaling relation. It is actually the intersection of inverse hyperbolic function and power law. A hyperbolic is not really a scaling relation, but it treated as linear scaling in literature. In mathematics, Zipf’s distribution can be abstracted as simple problem of series. The changing regularity of a sequence can be judged through generalized integral. If one is familiar with advanced mathematics, he can easily understand the following analysis. For the case \( q=1 \), we have

\[
T = \int_{k=1}^{N} P(k)dk = P_1 \int_{k=1}^{N} \frac{1}{k} dk = P_1 \left[ \ln k \right]_1^N = P_1 \ln N ,
\]  

where \( T \) denotes the total urban population in a region. This is the continuous expression of urban total population. The integral result is a logarithmic function rather than a power function. This suggests that the pure Zipf’s law is a semi-scaling relation. The population size of the largest city, \( P_1 \), is the characteristic value of the total urban population, \( T \). The difference between the sum of discrete sequences, \( U \), and the integration of continuous variable, \( T \), is

\[
U - T = P_1 \left( \sum_{k=1}^{N} \frac{1}{k} \right) - \int_{k=1}^{N} \frac{1}{k} dk = CP_1 .
\]  

This implies \( U = T + CP_1 \). The error comes from approximate processing and does not reflect the essence of the problem.

The reasoning results based on discrete variables are more accurate, but the reasoning processes are often more difficult. The analysis process based on continuous variables is simpler than that based on discrete variables. Generally speaking, theoretical studies are always based on continuous variable, and calculus is utilized, while application studies are usually based on discrete variables, and the methods of difference are adopted. If the exponent \( q \neq 1 \), the model returns to its general form. In this case, we have

\[
T = \int_{k=1}^{N} P(k)dk = P_1 \int_{k=1}^{N} k^{-q}dk = P_1 \left[ \frac{k^{1-q}}{1-q} \right]_1^N .
\]  

Suppose that the city number, \( N \), is large enough. The \( N \) value can be infinite in theory. Then the total urban population is
\[ T = P_1 \lim_{N \to \infty} \left( \frac{k^{1-q}}{1-q} \right)^N = P_1 \lim_{N \to \infty} \frac{N^{1-q} - 1}{1-q} = \begin{cases} \frac{P_1}{q-1}, & q > 1 \\ \infty, & q < 1 \end{cases}. \] (10)

This implies that different \( q \) values lead to different results. If \( q > 1 \), the total urban population is limited, while if \( q \leq 1 \), the total urban population is not limited. For \( q > 1 \), the level of urbanization is

\[ L = \frac{100T}{P_T} = \frac{100P_1}{P_T} \lim_{N \to \infty} \frac{N^{1-q} - 1}{1-q} = \frac{100P_1}{(q-1)P_T}, \] (11)

which suggests that it is impossible to promote the level of urbanization of a large populous country by adding more towns, unless the property of rank-size distribution is changed. The case of exponent \( q > 1 \) is suitable for the smaller countries, and the population size of the largest city determine the urbanization level. In extreme cases, the law of primate city will replace the rank-size law to dominate the city size distribution of a country. Maybe to a degree equation (11) can account for the law of primate city, which was proposed by Jefferson (1939). In contrast, for \( q < 1 \), we have

\[ L = \frac{100T}{P_T} = \frac{100(N^{1-q} - 1)P_1}{(1-q)P_T}. \] (12)

If city number \( N \) is large enough, then an approximate power law can be derived as

\[ L = \frac{100T}{P_T} = \lim_{N \to \infty} \frac{100(N^{1-q} - 1)P_1}{(1-q)P_T} = \frac{100P_1N^{1-q}}{(1-q)P_T}. \] (13)

This case is suitable for the largely populous nations such as China and India. In this case, the level of urbanization depends heavily on city number and the population size of the largest city. As for \( q = 1 \), the level of urbanization can be formulated by equation (5), which can be approximated to \( (100P_1/P_T)\ln N \). For given population size of the largest city, equation (13) suggests an approximate power law relation between city number \( N \) and total urban population \( T \), that is

\[ T = \frac{P_1}{1-q} N^{1-q}, \] (14)

which differs the exponential relation, equation (6). This indicates an allometric relation between city number and total urban population. For given city number \( N \), total urban population is proportional to the population size of the largest city. For a country, equation (14) can be used to describe the cumulative distribution of city sizes based on general Zipf distribution; for different regions (e.g., a system of provinces, a system of states) in a country, equation (14) can be used to
describe the aggregate relation between city number and total urban population.

2.3 The function of the largest city for urbanization

The population size of the largest city influence the level of urbanization of a nation through city-size distribution mode. It is necessary to examine the mathematical principle of the relationship between Zipf’s law and urbanization level. Suppose the rank \( k \) becomes a continuous variable. Differentiating equation (1) with respect to \( k \) yields

\[
\frac{dP(k)}{dk} = -qP(k)^{q-1},
\]

which give the density distribution of city size. Then, generally, discretizing equation (15) yields

\[
\Delta U = \sum_{k=1}^{N-1} (P(k) - P(k+1)) = P_1 \sum_{k=1}^{N-1} \left( \frac{1}{k^q} - \frac{1}{(k+1)^q} \right) = P_1 \left( 1 - \frac{1}{N^q} \right),
\]

in which the summation is

\[
\left( \frac{1}{1^q} - \frac{1}{2^q} \right) + \left( \frac{1}{2^q} - \frac{1}{3^q} \right) + \cdots + \left( \frac{1}{(N-1)^q} - \frac{1}{N^q} \right) = 1 - \frac{1}{N^q}.
\]

For the pure rank-size distribution, \( q = 1 \), the scaling exponent of the density distribution is 2. Then, equation (16) becomes

\[
\frac{\Delta P(k)}{\Delta k} = \frac{P(k) - P(k+1)}{(k+1) - 1} = P_1 \left( \frac{1}{k} - \frac{1}{k+1} \right) \propto k^{-2},
\]

which can be readily testified by mathematical experiment. In this case, \( \Delta k = 1 \). Thus we have

\[
\Delta U = \sum_{k=1}^{N-1} (P(k) - P(k+1)) = P_1 \sum_{k=1}^{N-1} \left( \frac{1}{k} - \frac{1}{k+1} \right) = P_1 \sum_{k=1}^{N-1} \left( \frac{1}{k} \right).
\]

where \( \Delta U \) denotes the change rate of urban population along the rank-size distribution. In terms of Gaussian summation formula, \( 1+2+3+\ldots+k = k(k+1)/2 \), so equation (19) can be expressed as

\[
\Delta U = \frac{1}{2} P_1 \sum_{k=1}^{N-1} \frac{1}{k} \left( \frac{1}{k} \right) = \frac{1}{2} P_1 \sum_{k=1}^{N-1} \frac{1}{1+2+\ldots+k} = \frac{1}{2} P_1 H.
\]

Here \( H \) refers to Huygens’ series (Appendix 2), that is

\[
H = \sum_{k=1}^{N-1} \frac{1}{1+2+\ldots+k} = \sum_{k=1}^{N-1} \left( \frac{1}{k} \sum_{i=1}^{k} i \right) = \frac{2(N-1)}{N}.
\]

Thus the change rate of urban population is
\[
\Delta U = \frac{1}{2} P_1 H = P_1 \left( \frac{N-1}{N} \right). \tag{22}
\]

If city number \( N \) is large enough, we will have \( \Delta U = P_1 \). This suggests that, for \( q=1 \), the change rate of urban population depends mainly on the population size of the largest city, unless the regularity of city-size distribution is broken.

### 3. Computational and empirical analyses

#### 3.1 Computational results and analyses

Mathematical experiment based on the above-shown formulae can be employed to explore the relationships between urbanization and Zipf’s law. Mathematical experiment is in fact a type of computational method. Based on certain scientific principle, a series of numerical computation can be conducted by means of basic assumptions and mathematical models (Chen, 2012). New findings can be made by comparing the computational results with the observational phenomena and mathematical reasoning results. The following computational process consists of four parts. (1) Computational urbanization levels for populous nations. (2) Computing city numbers for populous nations. (3) Computing urbanization levels for different sizes of nations. (4) Computing urbanization levels for populous nations in terms of general Zipf’s law. The first three parts are based on pure Zipf’s law, and the final part is based on general Zipf’s law. Let us make computation step by step.

**Step 1: computational urbanization levels for populous nations.** Based on pure Zipf’s rank-size distribution of cities, the levels of urbanization are derived for a country with a large population. Suppose a country has a population of 1.4 billion \((P_T = 1.4 \times 10^9)\). Two approaches can be used to predict the levels of urbanization based on different numbers of cities and towns \((N)\) and different population sizes of the largest city \((P_1)\). One is the brute force method based on equation (1), and the other is the simple method based on equation (5). The results show that for such a large population, the level of urbanization is difficult to exceed 50\% (Table 1). If the population of the largest city is twenty million \((P_1 = 2 \times 10^7)\), then 500,000 cities and towns can only accommodate about 19.57\% of the population \((L < 20\%)\). In this case \((N = 500,000)\), if the urbanization rate is expected to exceed 50\%, the population size of the largest city should reach more than 50 million. The size of the largest city is over large. There will be about 60 cities with a population of more
than 1 million.

Table 1 The levels of urbanization based on different city numbers and different population sizes of the largest city

| City number, N | Level of urbanization, L (%) |
|---------------|-----------------------------|
|               | $P_1=1000$                  | $P_2=2000$                  | $P_3=3000$                  | $P_4=4000$                  | $P_5=5000$                  | $P_6=6000$                  |
| 1000          | 5.3468                      | 10.6935                     | 16.0403                     | 21.3871                     | 26.7338                     | 32.0806                     |
| 5000          | 6.4961                      | 12.9922                     | 19.4882                     | 25.9843                     | 32.4804                     | 38.9765                     |
| 10000         | 6.9911                      | 13.9823                     | 20.9734                     | 27.9646                     | 34.9557                     | 41.9469                     |
| 50000         | 8.1407                      | 16.2814                     | 24.4222                     | 32.5629                     | 40.7036                     | 48.8443                     |
| 100000        | 8.6358                      | 17.2716                     | 25.9075                     | 34.5433                     | 43.1791                     | 51.8149                     |
| 200000        | 9.1309                      | 18.2618                     | 27.3928                     | 36.5237                     | 45.6546                     | 54.7855                     |
| 300000        | 9.4205                      | 18.8411                     | 28.2616                     | 37.6822                     | 47.1027                     | 56.5232                     |
| 400000        | 9.6260                      | 19.2521                     | 28.8781                     | 38.5041                     | 48.1301                     | 57.7562                     |
| 500000        | 9.7854                      | 19.5708                     | 29.3562                     | 39.1417                     | 48.9271                     | 58.7125                     |

Note: (1) The symbols are as follows: $N$—number of cities and towns; $L$—level of urbanization (%); $P_i$—urban population of the largest city. (2) The national total population is assumed to be $P_T=14\times10^8$, i.e., 1 billion 400 million. (3) The population unit of $P_i$ is 10 thousand.

**Step 2: computational city numbers for populous nations.** For given level of urbanization, the number of cities and towns can be worked out. Two approaches can be employed to predict the city number based on different levels of urbanization ($L$) and different population sizes of the largest city ($P_i$). One is the brute force method by using equation (1), and the other is the simple method by means of equation (6). For a country bearing a population of 1.4 billion, the results are tabulated as below (Table 2). Suppose the population size of the largest city is twenty million ($P_i=2\times10^7$). If the urbanization level reach 80%, it will need $1.1744\times10^{24}$ cities and towns. The number of cities and towns is $8.39\times10^{18}$ times the national population. The small towns increased afterwards can accommodate less than one person. The result is absurd. The reason is that the standard Zipf distribution is not suitable for a country with a very large population such as China and India. So, is the pure Zipf law invalid? Of course not. For countries with small populations, the law works.

Table 2 The city number based on different levels of urbanization and different population sizes of the largest city

| $L$            | Number of total cities and towns, $N$ |
|----------------|---------------------------------------|
|                |                                       |

9
Step 3: computational urbanization levels for different sizes of nations. Let us consider countries with smaller populations. Suppose that the population of the countries ranges from 100 million to 500 million. Using equation (1) or equation (5), we can calculate the levels of urbanization based on different numbers of cities and towns and national population sizes (Table 3). For a country with a population of 100 million, if the size of the largest city is \( P_1 = 10 \text{ million} \), 2000 cities and towns can accommodate 81.78% of the population. In other words, the level of urbanization can easily exceed 80% if its level of industrialization allows. On the other hand, for countries with a population of 100 million, the population size of the largest cities should not exceed 10 million. If the largest city size reaches 20 million, either the total number of cities is small, or the rank-size distribution of cities breaks. In light of equation (6), based on pure Zipf distribution, about 83 cities can accommodate the whole population of the country. In this case, if the national territory is not large enough, its city size may follow the primate distribution. The largest city is very large, but the second largest city is far smaller than the largest one. In other words, the primate degree is significantly greater than 2, say, \( P_1/P_2 > 3 \), where \( P_1 \) and \( P_2 \) denote the population sizes of the first and second largest cities (Appendix 3).

Note: (1) The symbols are as follows: \( L \)—level of urbanization (%); \( N \)—number of cities and towns; \( P_1 \)—urban population of the largest city. (2) The national total population is assumed to be \( P_T = 14 \times 10^8 \). (3) The population unit of \( P_1 \) is 10 thousand.

| City | \( P_1=1000 \) | \( P_1=2000 \) | \( P_1=3000 \) | \( P_1=4000 \) | \( P_1=5000 \) | \( P_1=6000 \) |
|------|----------------|----------------|----------------|----------------|----------------|----------------|
| 10   | 675213         | 616            | 60             | 19             | 9              | 6              |
| 20   | 8.1201E+11     | 675213         | 6349           | 616            | 152            | 60             |
| 30   | 9.7653E+17     | 7.4046E+08     | 675213         | 20390          | 2497           | 616            |
| 40   | 1.1744E+24     | 8.1201E+11     | 7.1804E+07     | 675213         | 41060          | 6349           |
| 50   | 1.4123E+30     | 8.9048E+14     | 7.6358E+09     | 2.2360E+07     | 675213         | 65477          |
| 60   | 1.6985E+36     | 9.7653E+17     | 8.1201E+11     | 7.4046E+08     | 1.1104.E+07    | 6.7521.E+05    |
| 70   | 2.0426E+42     | 1.0709E+21     | 8.6352E+13     | 2.4521E+10     | 1.8260.E+08    | 6.9630.E+06    |
| 80   | 2.4564E+48     | 1.1744E+24     | 9.1829E+15     | 8.1201E+11     | 3.0027.E+09    | 7.1804.E+07    |
| 90   | 2.9541E+54     | 1.2879E+27     | 9.7653E+17     | 2.6890E+13     | 4.9379.E+10    | 7.4046.E+08    |
| 100  | 3.5526E+60     | 1.4123E+30     | 1.0385E+20     | 8.9048E+14     | 8.1201.E+11    | 7.6358.E+09    |

Table 3 The level of urbanization based on different city numbers, different population sizes of the largest city, and different national total population
Step 4: computational urbanization levels for populous nations based on general Zipf’s law.

For given national population in a country, the level of urbanization depends on city number, the population size of largest city, and Zipf scaling exponent. This can be reflected by equations (5) and (12). As indicated above, equation (5) is available for the case of $q=1$, while equation (12) is available for the case of $q\neq 1$. According to above analysis, the first situation is not suitable for a big country like China. Let us look at the second situation. Suppose a country has 1.4 billion people. The population of the largest city is considered to be two cases: one is 10 million, and the other, 20 million. Mathematical experiments show that with the increase of the largest city size and the decrease of Zipf scaling exponent, the level of urbanization rises rapidly (Table 4). The impact of the number of cities and towns on the level of urbanization is not particularly significant. If $P_1=20$ million and $q=0.8$, then 300 thousand cities and towns can hold about 82.64% of the national population.

Table 4 The level of urbanization based on different city numbers, different population sizes of the largest city, and different Zipf scaling exponent values

| City number, $N$ | $P_1=1000$ | $P_1=2000$ |
|-----------------|-----------|-----------|
|                 | $q=0.8$   | $q=0.9$   | $q=1$     | $q=1.1$   | $q=1.2$   | $q=0.8$   | $q=0.9$   | $q=1$     | $q=1.1$   | $q=1.2$   |
| 1000            | 11.0499   | 7.5168    | 5.3468    | 3.9806    | 3.0970    | 22.0997   | 15.0336   | 10.6935   | 7.9612    | 6.1939    |
### 3.2 Empirical analysis of Chinese cities

China is famous for its large population, so Chinese cities can be employed to test the results of theoretical derivation and calculation analyses. In fact, the problems explored in this paper are just caused by the theoretical dilemma of China’s urbanization and city size distribution analysis. Whether the size distribution of Chinese cities follows Zipf’s law is a controversial issue. Some scholars examined China’s urban rank-size patterns by means of Zipf’s law (Chen et al., 1993; Chen and Zhou, 2008; Gangopadhyay and Basu, 2009; Ye and Xie, 2012), and others regarded China’s city size distribution as non-Zipf’s distribution (Anderson and Ge, 2005; Benguigui and Blumenfeld-Lieberthal, 2007a; Benguigui and Blumenfeld-Lieberthal, 2007b). In fact, the rank-size distribution of Chinese cities can be described by the three-parameter Zipf model (Chen, 2016). If we use the one-parameter Zipf’s law (pure model) or the two-parameter Zipf model (general model) to describe China’s city size distribution, the results seem to be ambiguous and thus inconclusive (Table 5). The reason lies in that China’s urban area and size threshold are of great administrative significance. Urban census data only include officially approved cities. A large number of important cities lack census data because they are not included in the official city list.

#### Table 5 The characters of rank-size distribution of Chinese cities and related behaviors

| Level          | Item                      | Actual model                  | Expected model              | Consistency |
|----------------|---------------------------|-------------------------------|-----------------------------|-------------|
| Whole country  | Rank-size distribution    | Zipf’s law, equation (1)     | Zipf’s law, equation (1)    | True        |
|                | Cumulative size distribution | Logarithmic function, equation (7) | Power function, equation (14) | False       |
By using the data of four times population censuses, we can thoroughly investigate the basic characteristics and changing trends of rank size distribution of cities in China. Four datasets of city sizes include the observations of the third census (1982), the fourth census (1990), the fifth census (2000), and the sixth census (2010) (Table 6). Draw the data points on a double logarithmic plot for rank-size relation. If the points form a straight line, they can be regarded as following Zipf law. The results show that not all the data points in each year take on straight trend on the rank-size log-log plot. The straight segments can be treated as a scaling range of Zipf distribution. Fitting the data points within the scaling range to Zipf’s law, we can estimate the Zipf model parameters by the regression coefficients. The intercept represents the theoretical value of the population size of the largest city, and the slope represents Zipf scaling exponent (Table 7). The \( P_1 \) value in the model is very large, and the largest city in reality is far from reaching this size. This shows that the urbanization dynamics needs the primacy city to increase to such as size, but the actual environmental and economic conditions cannot support it. All the Zipf scaling exponent values are less than 1, but from 1990 to 2010, the exponent values went up and up. Of course, this is only an approximate estimation. China’s city size distributions were not well fitted by the conventional Zipf model (Chen, 2016).

### Table 6 The basic data of Chinese cities from four times of population census

| Year | Total population \( P_T \) | Urban population \( U \) | City number \( N \) | Largest city size \( P_1 \) | Urbanization level \( L \) |
|------|-----------------|-----------------|--------------|-----------------|-----------------|
| 1982 | 100818.0000     | 21082.000       | 238          | 6320829         | 20.9109         |
| 1990 | 113368.0000     | 29971.000       | 460          | 7469509         | 26.4369         |
| 2000 | 126583.0000     | 45844.000       | 666          | 12720701        | 36.2166         |
| 2010 | 133971.9546     | 66557.000       | 654          | 17640842        | 49.6796         |
Table 7 The parameter values of two-parameter Zipf modeling for Chinese cities and related numbers and statistics

| Year | Scaling range N' | Coefficient $P_{1}^{*}$ | Zipf scaling exponent $q$ | Goodness of fit $R^{2}$ |
|------|-----------------|--------------------------|--------------------------|-------------------------|
| 1982 | 200             | 19080947.5526            | 0.9518                   | 0.9504                  |
| 1990 | 400             | 17271080.7380            | 0.8705                   | 0.9819                  |
| 2000 | 600             | 34743311.2477            | 0.8955                   | 0.9865                  |
| 2010 | 600             | 56265884.6457            | 0.9453                   | 0.9842                  |

Note: (1) The scaling range means that the $N'$ largest cities approximately form a straight line on a log-log plot. (2) The model’s proportionality coefficient $P_{1}^{*}$ represents the population size of the largest city.

Zipf’s law is associated in internal logic with allometric scaling law. From Zipf’s law we can derive allometric scaling law (Chen, 2012). The cross-sectional relationships between city number and total urban population of different regions follows allometric scaling law. So does the relationships between the largest city’s population and total urban population. Using the cross-sectional data of China’s different provinces and autonomous regions, we can explore the influence of city number and the population size of the largest city on urbanization. As shown above, equation (14) is derived from equation (13). This suggests that the relationships between city number, the largest city’s population, and total urban population can reflect the relationships between city number, the largest city’s population, and the level of urbanization. Equation (14) can be generalized to the form of Cobb-Douglas function, that is

$$U = f(N, P_{1}) = kN^{\alpha}P_{1}^{\beta},$$

(23)

where $k$ denotes the proportionality constant, and $\alpha$, $\beta$ are two partial scaling exponent. The logarithmic linear expression of equation (23) is

$$\ln U = A + \alpha N + \beta \ln P_{1},$$

(24)

where the constant $A=\ln k$. Multivariate linear regression can be utilized to estimate model parameters of equations (23) and (24). The results display that the Cobb-Douglas model can be used to well depict the relationships between city number $N$, the largest city’s population $P_{1}$, and total urban population, $U$ (Table 8). From 2000 to 2020, the partial scaling exponent value of the number of cities and towns decreased from 1.0199 to 0.9584, while the partial scaling exponent of the population sizes of largest cities increased from 0.4225 to 0.5263. This suggests that the number of cities and towns contributes more to the total urban population and thus to the level of urbanization.
than the population of the largest cities. However, the former \((N)\) is weakening while the latter \((P_1)\) is strengthening over time. This judgment is consistent with the previous theoretical inference.

Table 8 The parameters and statistics of Cobb-Douglas model on the relationships between city number, the largest city’s population, and total urban population

| Level        | 2000           | 2010           |
|--------------|----------------|----------------|
|              | Coefficients   |     P-value     | Coefficients   |     P-value     |
| Local parameters and statistics | | | | |
| \(A\)        | 6.6804         | 4.0164E-07     | 5.4511         | 6.3130E-04     |
| \(\alpha\)   | 1.0199         | 1.5487E-12     | 0.9584         | 6.4422E-09     |
| \(\beta\)    | 0.4225         | 1.7832E-05     | 0.5263         | 8.1033E-05     |
| Global statistics | | | | |
| \(R^2\)      | 0.9769         | 0.9622         |
| \(F\)        | 507.9787       | 305.2483       |

Note: The \(P\)-value is equivalent to the \(t\)-statistics, but it does not need to look up the table.

4. Discussion

The mathematical reasoning and computational analyses indicate that urbanization dynamics may change the scaling character of city-size distribution. The pure Zipf distribution is suitable for the countries with smaller population, but not suitable for the countries with large population. For countries with a population of no more than 100 million, the Zipf scaling exponent of city size distribution is always close to 1. However, if the population of a country is more than 500 million, the Zipf scaling exponent of rank-size distribution of cities will be less than 1, otherwise the level of urbanization will hardly exceed 50%. Zipf scaling exponent of city size distribution can be equal to or even greater than 1 for countries with medium and small populations \((q \geq 1)\). For large population countries, the scaling exponent of city size distribution will so decrease because of urbanization that its value is less than 1 \((q<1)\). In short, countries of different population sizes may exhibit different characteristics of size distribution of cities (Table 9). For \(q \geq 1\), if the gap between \(P_1\) and \(P_T\) is smaller, the rank-size distribution may change into the primate distribution. In particular, for the case \(q>1\), urbanization is mainly determined by the largest city, and if the largest city attract too many population, the primate distribution will probably come into being (Figure 1).

Table 9 The relationships between total urban population and the Zipf’s scaling exponent

| Zipf | Total urban | Theoretical | Urbanization level | Country | Urbanization |
|------|------------|------------|--------------------|---------|-------------|

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### Table

| exponent | population | limit | size            |
|----------|-----------|-------|-----------------|
| $q > 1$  | $\frac{P_1(N^{1-q} - 1)}{1 - q}$ | $\frac{P_1}{q - 1}$ | $\frac{100P_1}{(q - 1)P_T}$ | Small | Low |
| $q = 1$  | $P_1(C + \ln N)$ | $\infty$ | $\frac{100P_1(\ln(N) + C)}{P_T}$ | Small and medium-sized | Low or high |
| $q < 1$  | $\frac{P_1(N^{1-q} - 1)}{1 - q}$ | $\infty$ | $\frac{100P_1N^{1-q}}{(1 - q)P_T}$ | Large | High |

### Figure 1

The antecedents and consequences of the macro-level correlation between city size distributions and urbanization dynamics

**Note:** For given national total population and Zipf scaling exponent, the level of urbanization depends heavily on the number of cities and towns and the population of the largest city. The interaction between different elements leads to three possible results: (1) The Zipf exponent reduce to $q < 1$; (2) The scaling range breaks into two parts; (3) The rank-size distribution evolve into primate distribution.

The mathematical model of a system reflects the system’s structure at macro level, while the
model parameters reflect the distribution and relationships of elements at micro level. The Zipf scaling exponent is a parameter of the rank-size distribution model. Sometimes, the model may change in structure. The rank-size distribution will turn into the primate distribution. The rank-size distribution is based on Zipf’s law (Zipf, 1949), which came from the law of population concentration (Auerbach, 1913). In contrast, the primate distribution is based on the law of the primate city (Jefferson, 1939). Based on Zipf’s law and primate city law, city size distributions were divided into three categories: rank size distribution, primate distribution, and intermediate distribution (Berry, 1961). For a long time, geographers and economists have tried to explain the context of the different types of size distribution of cities. The rank-size distribution and primate distribution used to be regarded as two extreme types. According to the current evidences, that may not be the case. The primate distribution may be a local disturbance phenomenon of the rank-size distribution (Table 6). The above mathematical derivations and computational analyses provide new understanding for this traditional academic problem. In fact, it is impossible for a large population country to have the primate size distribution of cities. The necessary conditions for the primate distribution of city sizes are as follows. First, small national territory and small population; second, urbanization causes rapid growth of the primacy city; third, the primacy city transcends national boundaries and becomes a member of international city network. Urbanization leads to the population expansion of the cities with the best conditions (generally the national capital). If the rest cities grow according to Zipf's law, the country has neither enough population supplement nor enough resource support. In this way, the primacy cities form a shadow effect in the hierarchy. Just as a tall tree deprives the surrounding plants of sunlight and thus inhibit the growth of the nearby plants, the second, third… or even tenth largest cities are covered and limited by the first largest cities. An inevitable result is the primate distribution of size of cities in a country.

| Category    | Population law                  | Distribution rule              | Country type       | Rule type          | Origin                   |
|-------------|---------------------------------|---------------------------------|--------------------|--------------------|--------------------------|
| Rank size   | Law of population concentration | Zipf’s law, rank-size rule      | Countries of any size | Global rule        | The most probable distribution |
| Primate     | Law of the primate city         | Primate rule                    | Smaller countries  | Local rule         | Hierarchical shadow effect |
Figure 2 The antecedents and consequences of the micro-level correlation between allometric growth and Zipf scaling exponent

Note: At the micro level, urban elements involve population, land, infrastructure, and so on. The relationships between these elements follow allometric scaling law. There are positive and negative forces in these relationships. The push-pull effect of positive and negative forces makes the Zipf exponent tend to the appropriate value.

In theory, Zipf’s size distribution of cities is associated with urban allometric growth. To explain the Zipf scaling exponent, a $q$-equation of urban hierarchy based on Zipf’s law and generalized allometric model was constructed (Chen, 1995; Chen, 2014). Starting from the general system theory of Bertalanffy (1968), a general allometric growth equation can be derived as follows (Chen, 1995; Chen, 2014)

$$M(k) = \eta P(k)^p,$$  \hspace{1cm} (21)

where $P(k)$ denotes the city size of the $k$th city, and $M(k)$ refers to some related response such as economic output, land use quantity, energy consumption, water consumption, and so on, $\eta$ is a proportionality coefficient, and $p$ is an allometric scaling exponent. Rewrite equation (2) as below
\[ P_i = U \sum_{k=1}^{N} k^{-q}. \]  

(22)

The sum of the above responses \( M(k) \) is

\[ M(p, q) = \sum_{r=1}^{n} M(r) = \eta \sum_{r=1}^{n} P(r)^p. \]

(23)

where \( M(p, q) \) refers to the total response of a system of cities and towns. Substituting equations (1) and (22) into equation (23) yields the \( q \)-equation as follows

\[ M(p, q) = \eta \sum_{r=1}^{n} (P_r r^{-q})^p = K \sum_{r=1}^{N} (r^{-q} / \sum_{r=1}^{N} r^{-q})^p. \]

(24)

where \( K = \eta U^p \). If \( p > 1 \), \( M(p, q) \) proved to be the increasing function of \( q \), that is

\[ \frac{dM(p, q)}{dq} > 0; \]

(25)

In contrast, if \( p < 1 \), \( M(p, q) \) proved to be the decreasing function of \( q \), that is

\[ \frac{dM(p, q)}{dq} < 0. \]

(26)

Empirical analyses shows that, if the response \( M(k) \) represents urban economic output value, the scaling exponent \( p > 1 \) (Chen, 1995; Chen and Liu, 1998; Chen and Zhou, 2003). This indicates increasing return. In contrast, if the response \( M(k) \) represents urban land use quantity, the scaling exponent \( p < 1 \) (Chen, 1995; Chen and Liu, 1998; Lee, 1989). This implies agglomeration effect and scale economics. What is more, if the response \( M(k) \) represents urban energy and water consumption, the allometric scaling exponent \( p > 1 \) in China (Chen, 1995; Chen and Liu, 1998). This suggests that the Zipf exponent, the \( q \) value, cannot be too high or too low. If \( q > 1 \), the total economic output of an urban system will be high and total urban land will be less. Meanwhile, the total quantity of energy and water consumption will also be high. If \( q < 1 \), the reverse is true: the total economic output of an urban system will be low and total urban land will be high. At the meantime, the total quantity of energy and water consumption will also be low. Zipf law indicates that cities bear no characteristic scale. In other words, cities have no typical size (Buchanan, 2000). However, the \( q \)-equation suggests that although cities do not have the best size, the urban system has the best size distribution (Figure 2). Where population is concerned, for small and medium-sized countries, the pure Zipf distribution represents the best size distribution of cities (\( q = 1 \)). However, for large
population countries, this balance will lose, or the symmetry of the rank size distribution will break. The urbanization level will do not go up, or the Zipf scaling exponent will be less than 1, or even the Zipf distribution will have scaling breaking.

The law of allometric growth is one of basic mathematical models in urban geography. It was introduced into urban studies early by Naroll and Bertalanffy (1956). Because the allometric scaling exponent cannot be reasonably explained by Euclidean geometry, the related research once declined for a time. Due to introduction of fractal geometry, allometric studies on cities revived at the turn of the century (Batty and Longley, 1994; Chen, 1995; Chen and Xu, 1999; Lo, 2002). Based on equation (23), a number of interesting allometric scaling analyses have been made recent years (Arcaute et al., 2015; Bettencourt, 2013; Bettencourt et al., 2007; Chen, 1995; Lobo et al., 2013; Louf and Barthelemy, 2014a; Louf and Barthelemy, 2014b). A revealing finding is that the calculated values of the scaling exponent comes between 2/3 and 4/3 (Bettencourt, 2013). Some calculated values are less 2/3 (close to 1/2), while the other calculated values are greater than 4/3. The threshold value seems to be \( b=1 \), which forms a dividing line between two urban economic processes: increasing returns \( (b>1) \) and economies of scale \( (b<1) \) (Bettencourt et al., 2007). Arcaute et al. (2015) found that the scaling exponent values of the allometric relation between patents and city population sizes relies on the cut-offs of city sizes. Louf and Barthelemy (2014a; 2014b) discovered that the scaling exponent values of the allometric relation between urban CO\(_2\) emissions and city population sizes depend on the definition of urban area. A typical result is the allometric scaling exponent of the allometric relation between urban area and population. The expected value is about 0.85 (Chen, 2008a; Chen, 2010), and the empirical values are close to 0.85 (Louf and Barthelemy, 2014a; Chen and Xu, 1999). These studies not only lead to new achievements (Arcaute et al., 2015; Bettencourt, 2013; Bettencourt et al., 2007; Chen, 2014; Lobo et al., 2013; Louf and Barthelemy, 2014a), but also to confusing problems (Arcaute et al., 2015; Chen and Lin, 2009; Louf and Barthelemy, 2014a; Louf and Barthelemy, 2014b). The puzzling problems may in turn result in new researching results. Bettencourt (2013) and his co-workers have develop new models of urban scaling, which may cause a new theory of city size in the future (Batty, 2013). Based on the results from Bettencourt (2013) and Bettencourt et al. (2007), the \( q \)-equation can be further developed to explain the Zipf scaling exponent on city-size distribution and the dynamics of city development from a novel angle of view.

There are numerous papers on Zipf's law and numerous works on urbanization. The relationships
between city size distribution and urbanization has been researched for many years. Compared with previous studies, the significant novelty of this work is as follows. First, new models and formulae are derived to reveal the relationships between urbanization level, city number, the population size of largest city, national population, and Zipf scaling exponent. Second, the spheres of application of pure Zipf law, general Zipf law, and the law of primate city are made clear. Third, new understanding about Zipf law and city size distribution are obtained from the mathematical reasoning and computational analyses (Table 11). The main disadvantages of this study are as below. First, due to the lack of continuous time series data of cities and urbanization, dynamic analysis and verification cannot be carried out based on the mathematical models proposed in this paper. Second, due to the limitation of the length of the paper, the comparative analysis of urbanization between China and the west cannot be implemented by means of the abovementioned models.

| Number | New discoveries and understandings |
|--------|-----------------------------------|
| 1      | If Zipf scaling $q=1$, Zipf distribution can be abstracted as a harmonic sequence, and its difference sequence is a Huygens sequence |
| 2      | If Zipf scaling $q=1$, the entire urban population can be theoretically calculated by Euler’s formula of summation of harmonic sequence |
| 3      | If Zipf scaling $q=1$, urbanization level is a logarithmic function of city number |
| 4      | If Zipf scaling $q=1$, the population size of the largest city is the characteristic value of entire urban population |
| 5      | The pure Zipf distribution is suitable for medium and small countries |
| 6      | The competition between the largest city size and city number may lead to primate distribution, and the Zipf exponent $q$ becomes a local exponent |
| 7      | For large population countries, the strong increase of urbanization level will lead to Zipf scaling exponent $q<1$ |

5. Conclusions

This is a theoretical study on the relationships between the level of urbanization and Zipf’s city-
size distribution. Although the problem is old, there are new discoveries and understandings in this work. Previous studies on the size distribution of cities were majorly aimed at the countries with small population size. China, India, or even the future United States of America are different because of their large population. Based on the theoretical derivations, computational analysis, and empirical evidence, the main conclusions can be reached as follows. (1) The pure Zipf size distribution of cities is suitable for median and small countries, not suitable for large populous countries. If the city size distribution of a populous country follows the pure Zipf’s law (scaling exponent $q=1$), the level of urbanization will be limited under 50%. If urbanization level goes up strongly, urban dynamics will force the Zipf scaling exponent $q$ to depart from 1 and become less than 1. Another possible case is that the scaling of rank-size distribution will break into two parts, and thus two scaling ranges will appear on a log-log plot for rank-size distribution. (2) The macro influencing factors of urbanization level include the number of cities and towns, the population of the largest city, the total population of the region, and the scaling exponent of rank-size distribution. For given national total population and Zipf exponent $q$, the city number and the population of the largest city play an important part. In the early stage of urbanization, the increase of the number of cities and towns plays a leading role in the process of urbanization. When the number of cities and towns increases to a certain extent, the population of the largest cities plays a leading role. The population of the primacy city is the characteristic value of total urban population. (3) The spatial competition between city number and the population of largest city may result in primate distribution of city sizes. Under the condition that the population of a country is not too large, and the primary city participate in the evolution of the international urban network, the largest city may stand out from the rest and become very protruding in size. In this way, the largest city will form a shadow effect in the hierarchy and restrict the growth of other larger cities. Therefore, the local destruction of the city rank-size pattern appears. The primate distribution of city sizes is a local disturbance of the global rank-size distribution of cities in a country.

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Appendices

Appendix 1: Euler’s formula for summation of pure Zipf sequence

The pure Zipf distribution can be abstracted as harmonic sequence as follows: 1, 1/2, 1/3, ..., 1/k, ... In theory, the entire population of all cities and towns can be given by

\[ U = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \sum_{k=1}^{N} \frac{1}{k} \, . \quad (A1) \]

Euler once proved a formula as below

\[ C = \lim_{N \to \infty} \left( \sum_{k=1}^{N} \frac{1}{k} - \ln N \right) = 0.577216 \cdots . \quad (A2) \]

Thus the total urban population based on pure Zipf distribution is

\[ U = \lim_{N \to \infty} \sum_{k=1}^{N} \frac{1}{k} = \ln N + C \, . \quad (A3) \]

where \( C=0.5772156649015328606059335 \) is termed Euler’s constant in literature. See: Leonhard Euler’s paper titled “De progressionibus harmonicus observationes”, which was published in 1735.

Appendix 2: Huygens series and difference of pure Zipf sequence

The general Zipf sequence is a \( p \)-sequence \((p=q)\), and the pure Zipf sequence is a harmonic sequence \((p=q=1)\). The difference sequence of the pure Zipf sequence can be proved to be a Huygens sequence, which indicates the role of the largest city. Based on the difference sequence of the pure Zipf sequence, the sum of the first \( N+1 \) items is as follows

\[ h = \frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \cdots + \frac{1}{N \times (N+1)} \]

\[ = \frac{1}{[N+1-N]} \left[ (\frac{1}{1} - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) + \cdots + (\frac{1}{N} - \frac{1}{N+1}) \right] . \quad (B1) \]

\[ = \frac{N}{N+1} \]

The well-known Gaussian formula of the sum of natural number series is as below

\[ S_N = 1 + 2 + 3 + 4 + \cdots + N = \frac{1}{2} \times N \times (1 + N) = \frac{1}{2} N(N+1) \, . \quad (B2) \]

In light of equation \((B2)\), the difference sequence of the pure Zipf sequence can be converted into
Huygens’ sequence. Thus the sum of the first \( N+1 \) items is as below:

\[
H = 2h = \frac{1}{2 \times 1 \times 2} + \frac{1}{2 \times 1 \times 3} + \frac{1}{2 \times 1 \times 4} + \cdots + \frac{1}{2 \times N \times (N+1)} \\
= \frac{1}{1+2} + \frac{1}{1+2+3} + \frac{1}{1+2+3+4} + \cdots + \frac{1}{1+2+\cdots+N} \\
= \frac{2N}{N+1}.
\]  \( \text{(B3)} \)

In fact, the transform of the first \( N+1 \) items of the Huygens series is as follows

\[
H = 2h = 1+ \frac{1}{1+2} + \frac{1}{1+2+3} + \frac{1}{1+2+3+4} + \cdots + \frac{1}{1+2+\cdots+N} \\
= \frac{1}{2 \times 1 \times 2} + \frac{1}{2 \times 1 \times 3} + \frac{1}{2 \times 1 \times 4} + \cdots + \frac{1}{2 \times N \times (N+1)} \\
= 2 \left[ \frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \cdots + \frac{1}{N \times (N+1)} \right] \\
= 2 \left[ \frac{1}{(N-1)-N} \times \left[ (\frac{1}{1} - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) + \cdots + \frac{1}{N-1} - \frac{1}{N} \right] \right] \\
= 2 \left( 1 - \frac{1}{N+1} \right) = \frac{2N}{N+1}.
\]  \( \text{(B4)} \)

The Huygens series suggests that if city number is big enough, new increasing cities and towns have negligible influence on the level of urbanization.

**Appendix 3: Rank-size distribution and primate distribution**

The rank-size distribution and primate distribution represent two typical size distributions of cities. The former is well-known to urban scientists, the latter is familiar to urban geographers, but maybe not familiar to economic and social physicists. The cities of the United Kingdom (UK) in 1981 and cities of the United States of America (USA) in 1990 and 2000 can be used to illustrate the two types of size distributions. Three urban indexes are always utilized to characterize city size distributions, that is, two-city index (primacy or primacy ratio) \( S^{(2)} \), four-city index \( S^{(4)} \), and eleven-city index \( S^{(11)} \). The formula are as follows

\[
S^{(2)} = \frac{P_1}{P_2},
\]  \( \text{(C1)} \)

\[
S^{(4)} = \frac{P_1}{P_2 + P_3 + P_4},
\]  \( \text{(C2)} \)

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\( S^{(11)} = \frac{2P_1}{P_2 + P_3 + \cdots + P_{11}}, \) (C3)

where \( P_k \) denotes the city size of rank \( k (k=1, 2, 3, \ldots, 11) \). For the standard city rank-size distribution dominated by Zipf’s law, the expected values are \( S^{(2)}=2, S^{(4)}=0.9231 \approx 1, S^{(11)}=0.9902 \approx 1 \). If the \( S^{(2)} \) value is significantly greater than 3, the \( S^{(4)} \) and \( S^{(11)} \) values are greater than 2, the city size distribution can be treated as the primate distribution, which is governed by the law of primate city. In literature, UK’s city-size distribution belongs to primate-type distribution, and USA’s size distribution of cities belongs to rank-type distribution (Table A1).

| Index                  | UK (1981) | USA (1990) | USA (2000) | Standard value |
|------------------------|-----------|------------|------------|----------------|
| Two-city index (primacy ratio) | 6.5207    | 2.1009     | 2.1674     | 2              |
| Four-city index        | 2.8611    | 0.9191     | 0.9372     | 0.9231         |
| Eleven-city index      | 2.5511    | 0.9468     | 0.9549     | 0.9902         |

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