Strongly ergodic equivalence relations: spectral gap and type III invariants

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Abstract. We obtain a spectral gap characterization of strongly ergodic equivalence relations on standard measure spaces. We use our spectral gap criterion to prove that a large class of skew-product equivalence relations arising from measurable 1-cocycles with values in locally compact abelian groups are strongly ergodic. By analogy with the work of Connes on full factors, we introduce the Sd and τ invariants for type III strongly ergodic equivalence relations. As a corollary to our main results, we show that for any type III1 ergodic equivalence relation \( \mathcal{R} \), the Maharam extension \( c(\mathcal{R}) \) is strongly ergodic if and only if \( \mathcal{R} \) is strongly ergodic and the invariant \( \tau(\mathcal{R}) \) is the usual topology on \( \mathcal{R} \). We also obtain a structure theorem for almost periodic strongly ergodic equivalence relations analogous to Connes’ structure theorem for almost periodic full factors. Finally, we prove that for arbitrary strongly ergodic free actions of bi-exact groups (e.g. hyperbolic groups), the Sd and τ invariants of the orbit equivalence relation and of the associated group measure space von Neumann factor coincide.

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1. Introduction and statement of the main results

Following [Sc79], a non-singular action $\Gamma \acts (X, \mu)$ of a countable discrete group $\Gamma$ on a standard probability space $(X, \mu)$ is strongly ergodic if any sequence of $\Gamma$-almost invariant measurable subsets is trivial, i.e. for any sequence of measurable subsets $A_n \subset X$ such that $\lim_n \mu(A_n \triangle A_n) = 0$ for every $\gamma \in \Gamma$, we have $\lim_n \mu(A_n)(1 - \mu(A_n)) = 0$. Likewise, a non-singular equivalence relation $\mathcal{R}$ on a standard probability space $(X, \mu)$ is strongly ergodic if for any sequence of measurable subsets $A_n \subset X$ such that $\lim_n \mu(A_n \triangle g A_n) = 0$ for every $g \in [\mathcal{R}]$, we have $\lim_n \mu(A_n)(1 - \mu(A_n)) = 0$. It is easy to see that the notion of strong ergodicity only depends on the measure class of $\mu$. By Rokhlin’s lemma, a hyperfinite or equivalently amenable (by [CFW81]) equivalence relation on a diffuse standard probability space is never strongly ergodic (see [Sc79, Proposition 2.2]). In particular, a non-singular action $\Gamma \acts (X, \mu)$ of an amenable countable discrete group on a diffuse standard probability space is never strongly ergodic.

The notion of strong ergodicity is related to the notion of fullness for von Neumann factors. Following [Co74], a factor $M$ with separable predual is full if any uniformly bounded centralizing sequence is trivial, i.e. for any uniformly bounded sequence $x_n \in M$ such that $\lim_n \|x_n \psi - \psi x_n\| = 0$ for every $\psi \in M_*$, there exists a bounded sequence $\lambda_n \in \mathbb{C}$ such that $x_n - \lambda_n 1 \to 0$ $\ast$-strongly. By the work of Connes [Co74, Co75b] and Haagerup [Ha85], a hyperfinite or equivalently amenable diffuse factor with separable predual is never full. Following [FM75], denote by $L(\mathcal{R})$ the factor associated with the ergodic equivalence relation $\mathcal{R}$ on the standard probability space $(X, \mu)$. If $L(\mathcal{R})$ is full then $\mathcal{R}$ is strongly ergodic. However, the converse is not true, as demonstrated by Connes and Jones [CJ81].

For non-singular equivalence relations $\mathcal{R}(\Gamma \acts X)$ arising from non-singular actions of the countable discrete group $\Gamma \acts (X, \mu)$ on standard probability spaces, strong ergodicity of the action $\Gamma \acts (X, \mu)$ is equivalent to strong ergodicity of the orbit equivalence relation $\mathcal{R}(\Gamma \acts X)$. As observed by Schmidt in [Sc79, Proposition 2.10], for any countable discrete group $\Gamma$ and any probability measure preserving (pmp) action $\Gamma \acts (X, \mu)$, if the Koopman unitary representation $\pi : \Gamma \rightarrow \mathcal{U}(L^2(X, \mu) \otimes \mathbf{C}^1_X)$ has spectral gap in the sense that $\pi$ does not weakly contain the trivial representation, then the orbit equivalence relation $\mathcal{R}(\Gamma \acts X)$ is strongly ergodic. However, the converse is not true, as explained in [Sc80, Example 2.7]. Any non-amenable countable discrete group $\Gamma$ admits a strongly ergodic free pmp action, namely the plain Bernoulli action $\Gamma \acts ([0, 1]^\mathbb{Z}, \text{Leb} \otimes \Gamma)$. We refer the reader to [HI15, Oz16] and the references therein for various examples of type III strongly ergodic free actions associated with free groups [HI15] and lattices in simple connected Lie groups with finite center [Oz16].

Our first main result provides a spectral gap characterization of strongly ergodic equivalence relations of type II$_1$ and of type III$\lambda$ for $\lambda \in (0, 1]$. Recall in that respect that a strongly ergodic equivalence relation can never be of type III$_0$ (see, for example, [Sc79, Remark 2.9]). Before stating Theorem A, we need to introduce some terminology. A non-singular automorphism $\theta$ of a standard probability space $(X, \mu)$ is said to be $\mu$-bounded if the function $\log(d(\mu \circ \theta)/d\mu)$ is bounded. In this case, the map $L_\theta : f \mapsto \theta(f)$ defines a bounded operator in $B(L^2(X, \mu))$. 






























































































































































































































































































































































































































**Theorem A.** Let $\mathcal{R}$ be any ergodic equivalence relation on a standard probability space $(X, \mu)$. Assume that $\mathcal{R}$ either preserves $\mu$ or is of type III. Then the following assertions are equivalent:

(i) $\mathcal{R}$ is strongly ergodic.

(ii) There exist a constant $\kappa > 0$ and a finite family $\theta_1, \ldots, \theta_n$ of $\mu$-bounded elements in $[\mathcal{R}]$ such that

$$\text{for all } f \in L^2(X, \mu), \quad \|f - \mu(f)\|_2^2 \leq \kappa \sum_{k=1}^n \|\theta_k(f) - f\|_2^2. \tag{1.1}$$

When $\mathcal{R}$ preserves $\mu$, the inequality in (1.1) shows that the Koopman unitary representation $\pi : [\mathcal{R}] \to \mathcal{U}(L^2(X, \mu) \otimes \mathbb{C}1_X)$ has a spectral gap. Thus, in the pmp case and working inside the ambient full group $[\mathcal{R}]$, Theorem A provides a converse to Schmidt’s observation [Sc79, Proposition 2.10]. As we will see later, the main interest of Theorem A is also to provide a spectral gap characterization for arbitrary strongly ergodic equivalence relations of type III. Note that the inequality in (1.1) can be reformulated by saying that the eigenvalue 0 is a simple isolated point in the spectrum of the positive self-adjoint bounded operator $\sum_{k=1}^n |L_{\theta_k} - 1|^2$. Let us point out that the spectral gap characterization of strongly ergodic equivalence relations in Theorem A is analogous to Connes’ spectral gap characterization of full factors of type II_1 (see [Co75b, Theorem 2.1]). We refer the reader to [Ma16, Theorem A] and [HMV16, Theorem 3.2] for very recent spectral gap characterizations of full factors of type III.

We next use our spectral gap criterion to prove that a large class of skew-product equivalence relations are strongly ergodic. Before stating Theorem B, we need to introduce some more terminology. Let $\mathcal{R}$ be any non-singular equivalence relation on a standard measure space $(X, \mu)$, $G$ any locally compact second countable abelian group and $\Omega \in Z^1(\mathcal{R}, G)$ any measurable 1-cocycle. On $X \times G$, put the measure class obtained as the product of the measure class of $X$ and the Haar measure class of $G$. The skew-product of $\mathcal{R}$ by $\Omega$ is the equivalence relation $\mathcal{R} \times_\Omega G$ on $X \times G$ defined by

$$((x, g), (y, h)) \in \mathcal{R} \times_\Omega G \quad \text{if and only if} \quad (x, y) \in \mathcal{R} \quad \text{and} \quad \Omega(x, y) = gh^{-1}$$

for all $x, y \in X$ and all $g, h \in G$. Let $\hat{\mathcal{G}}$ be the Pontryagin dual of $G$ and introduce the map $\widehat{\Omega} : \hat{\mathcal{G}} \to Z^1(\mathcal{R}, \mathbb{T})$ defined by $\widehat{\Omega}(p)(x, y) = \langle p, \Omega(x, y) \rangle$ for almost every (a.e.) $(x, y) \in \mathcal{R}$, where $\langle \cdot, \cdot \rangle : \hat{\mathcal{G}} \times G \to \mathbb{T}$ is the duality pairing. Note that $\widehat{\Omega}$ is a consistent group homomorphism. We also introduce the continuous homomorphism $\widehat{\Omega} : \hat{\mathcal{G}} \to H^1(\mathcal{R}, \mathbb{T})$ which sends $p \in \hat{\mathcal{G}}$ to the cohomology class $[\widehat{\Omega}(p)] \in H^1(\mathcal{R}, \mathbb{T})$.

**Theorem B.** Let $\mathcal{R}$ be any ergodic equivalence relation on a standard measure space $X$. Let $\Omega \in Z^1(\mathcal{R}, G)$ be any measurable 1-cocycle with values in a locally compact second countable abelian group $G$. Consider the following assertions.

(i) The skew-product equivalence relation $\mathcal{R} \times_\Omega G$ is strongly ergodic.

(ii) The equivalence relation $\mathcal{R}$ is strongly ergodic and the map $[\widehat{\Omega}] : \hat{\mathcal{G}} \to H^1(\mathcal{R}, \mathbb{T})$ is a homeomorphism onto its range.

Then (i) $\Rightarrow$ (ii), and if $G$ contains a lattice, we also have (ii) $\Rightarrow$ (i).
It is unclear whether the assumption that $G$ contains a lattice (i.e. a discrete cocompact subgroup) is really needed, but it already covers all the most common cases: compact groups, discrete groups, connected groups and all their direct products.

**Problem 1.** Prove that the equivalence (ii) $\Leftrightarrow$ (i) holds for all locally compact abelian groups.

We also point out the fact that $Z^1(R, \mathbb{T})$ can be identified with the group $\text{Aut}(L(R)/L^\infty(X))$ of all automorphisms of $L(R)$ that fix $L^\infty(X)$ pointwise (see Lemma 2.7). Hence the homomorphism $\hat{\Omega}$ defines an action of $\hat{G}$ on $L(R)$ and this allows us to identify $L(R \times \Omega G)$ with the crossed product $L(R) \rtimes \hat{\Omega} G$. Therefore, in the case where $G$ is compact (or equivalently $\hat{G}$ is discrete), Theorem B can be seen as an analogue of a theorem of Jones on fullness of crossed products [Jo81] and its recent generalization to arbitrary factors [Ma16, Theorem B]. These results inspired Theorem B.

We now apply Theorem B to the structure of type III strongly ergodic equivalence relations and we introduce two new orbit equivalence invariants, namely the $Sd$ and $\tau$ invariants. Let $R$ be any type III ergodic equivalence relation on a standard measure space $(X, \mu)$. Denote by $\delta_\mu \in Z^1(R, \mathbb{R}_0^+)$ the Radon–Nikodym 1-cocycle. Its cohomology class $[\delta_\mu] \in H^1(R, \mathbb{R}_0^+)$ only depends on the measure class of $\mu$. The skew-product equivalence relation $c(R) = R \times \delta_\mu, \mathbb{R}_0^+$ is called the Maharam extension of $R$. Then $c(R)$ is an infinite measure preserving equivalence relation that only depends on the measure class of $\mu$ up to canonical isomorphism. Moreover, $R$ is of type $III_1$ if and only if $c(R)$ is ergodic. Let us now assume that $R$ is strongly ergodic. Then $B^1(R, \mathbb{T}) \subset Z^1(R, \mathbb{T})$ is a closed subgroup and therefore $H^1(R, \mathbb{T})$ is a Hausdorff Polish group (see [Sc79, Proposition 2.3]). By analogy with the work of Connes on full factors of type III [Co74], we introduce the invariant $\tau(R)$ as the weakest topology on $\mathbb{R}$ that makes the map $[\delta_\mu] : \mathbb{R} \to H^1(R, \mathbb{T}) : t \mapsto [\delta_\mu^t]$ continuous. Applying Theorem B to the locally compact abelian group $G = \mathbb{R}_0^+$ and to the Radon–Nikodym 1-cocycle $\Omega = \delta_\mu \in Z^1(R, \mathbb{R}_0^+)$, we obtain the following characterization of strong ergodicity for the Maharam extension of any type $III_1$ ergodic equivalence relation.

**Corollary C.** Let $R$ be any type $III_1$ ergodic equivalence relation on a standard measure space. Then the Maharam extension $c(R)$ is strongly ergodic if and only if $R$ is strongly ergodic and $\tau(R)$ is the usual topology on $\mathbb{R}$.

Let us point out that Corollary C is analogous to [Ma16, Theorem C] where it was shown that for any factor $M$ with separable predual, the continuous core $c(M)$ is full if and only if $M$ is full and $\tau(M)$ is the usual topology on $\mathbb{R}$.

We next turn to investigate a large class of type III strongly ergodic equivalence relations for which the $\tau$ invariant is not the usual topology on $\mathbb{R}$. Let $R$ be any type III ergodic equivalence relation on a standard measure space $(X, \mu)$. By analogy with the work of Connes on almost periodic factors of type III [Co74], we say that $R$ is *almost periodic* if there exists a $\sigma$-finite measure $\nu$ on $X$ that is equivalent to $\mu$ and for which the Radon–Nikodym 1-cocycle $\delta_\nu \in Z^1(R, \mathbb{R}_0^+)$ essentially takes values in a countable subset of $\mathbb{R}_0^+$ that we denote by $\text{Range}(\delta_\nu)$. Such a measure $\nu$ is called *almost periodic* and the set of all almost periodic measures is denoted by $\mathcal{M}_{ap}(X, R)$. We then introduce the invariant
Sd(\mathcal{R}) as the intersection of Range(δ_ν) over all measures ν ∈ \mathcal{M}_{ap}(X, \mathcal{R}). The invariant Sd(\mathcal{R}) is a countable subgroup of \mathbb{R}_0^+ (see Proposition 5.2). Our next main result provides a structure theorem for type III strongly ergodic almost periodic equivalence relations which is analogous to Connes’ structure theorem for almost periodic full factors of type III [Co74].

**THEOREM D.** Let \mathcal{R} be a type III strongly ergodic almost periodic equivalence relation on a standard measure space X. Put Γ = Sd(\mathcal{R}). Then the following assertions hold true.

(i) If \mathcal{R} is of type III_λ for λ ∈ (0, 1), then Γ = \{λ^n | n ∈ \mathbb{Z}\}. If \mathcal{R} is of type III_1, then Γ is a dense countable subgroup of \mathbb{R}_0^+.

(ii) For any sequence \( (t_n)_{n∈\mathbb{N}} \) in \mathbb{R}, we have \( t_n \to 0 \) with respect to τ(\mathcal{R}) if and only if \( γ^{t_n} \to 1 \) for every γ ∈ Γ.

(iii) There exists a measure ν ∈ \mathcal{M}(X, \mathcal{R}) such that Range(δ_ν) = Γ.

(iv) For any measure ν as in (iii), the measure preserving subequivalence relation \mathcal{R}_ν ⊂ \mathcal{R} defined by \( \mathcal{R}_ν := \ker(δ_ν) \) is strongly ergodic.

(v) For any infinite measures ν_1, ν_2 as in (iii), there exist α > 0 and θ ∈ [\mathcal{R}] such that \( θ^\nu_1 ν_1 = αν_2 \).

In §6, we use generalized Bernoulli equivalence relations to construct families of strongly ergodic equivalence relations with prescribed Sd and τ invariants. For the class of plain Bernoulli equivalence relations arising from non-amenable groups, the corresponding factor is full and the Sd and τ invariants of the equivalence relation and of the factor coincide. However, we also construct a family of generalized Bernoulli equivalence relations with prescribed Sd and τ invariants and for which the corresponding factor is not full. The following open question is interesting in this respect.

**Problem 2.** Construct an example of an ergodic equivalence relation \mathcal{R} on a standard measure space X such that the associated factor L(\mathcal{R}) is almost periodic but \mathcal{R} is not almost periodic. What if we require L(\mathcal{R}) to be full?

As we already pointed out, the Sd and τ invariants of a strongly ergodic equivalence relation \mathcal{R} may differ from the Sd and τ invariants of the corresponding factor L(\mathcal{R}). Using [HI15, Theorems A], we obtain the following rigidity result for arbitrary strongly ergodic free actions Γ ↾ (X, μ) of bi-exact countable discrete groups (e.g. hyperbolic groups) on standard measure spaces: the Sd and τ invariants of the orbit equivalence relation \mathcal{R}(Γ ↾ X) and of the factor L(\mathcal{R}(Γ ↾ X)) coincide (note that L(\mathcal{R}(Γ ↾ X)) is full by [HI15, Theorem C]). We refer to [BO08, Definition 15.1.2] and §6 for the definition of bi-exact groups.

**THEOREM E.** Let Γ be any bi-exact countable discrete group and Γ ↾ (X, μ) any strongly ergodic free action on a standard measure space. Then L(\mathcal{R}(Γ ↾ X)) is a full factor and

\[ τ(\mathcal{R}(Γ ↾ X)) = τ(L(\mathcal{R}(Γ ↾ X))). \]

If, moreover, \mathcal{R}(Γ ↾ X) is almost periodic, then L(\mathcal{R}(Γ ↾ X)) is almost periodic and

\[ Sd(\mathcal{R}(Γ ↾ X)) = Sd(L(\mathcal{R}(Γ ↾ X))). \]
In [HI15, §6], for every \( \lambda \in (0, 1) \), a concrete example of a strongly ergodic free action \( \mathbb{F}_\infty \curvearrowright (X_\lambda, \mu_\lambda) \) on a standard measure space was constructed using the main result of [BISG15]. The following open question is interesting in this respect.

**Problem 3.** Construct examples of strongly ergodic free actions of free groups \( \mathbb{F}_n \curvearrowright (X, \mu) \) on a standard measure space for which the induced orbit equivalence relation \( \mathcal{R}(\mathbb{F}_n \curvearrowright X) \) (and hence the factor \( L(\mathcal{R}(\mathbb{F}_n \curvearrowright X)) \) by Theorem E) has prescribed \( \text{Sd} \) and \( \tau \) invariants.

2. **Preliminaries**

2.1. **Notation.** Let \((X, \mathcal{B}, \mu)\) be a standard \( \sigma \)-finite measure space. We will often omit the \( \sigma \)-algebra \( \mathcal{B} \) as well as the measure \( \mu \) when considering objects that only depend on the measure class of \( \mu \). In particular, for every Polish space \( E \), we denote by \( L^0(X, E) \) the set of all equivalence classes of measurable functions from \( X \) to \( E \), for the equivalence relation of equality almost everywhere. Then \( L^0(X, E) \) is a Polish space for the topology of convergence in measure (which does not depend on the choice of the measure within the same measure class). When \( E = \mathbb{C} \), we will use the shorthand notation \( L^0(X) = L^0(X, \mathbb{C}) \).

If \( f \in L^0(X, E) \), then the measure class of the push-forward measure \( f_* \mu \) is well defined and only depends on the measure class of \( \mu \). The topological support of \( f_* \mu \) (i.e. the smallest closed subset of \( E \) on which it is supported) is called the essential range of \( f \) and is denoted \( \text{Range}(f) \). If \( f_* \mu \) is atomic, we say that \( f \) is a step function, and in this case we denote by \( \text{Range}(f) \) the set of atoms of \( f_* \mu \). Note that in this case \( \text{Range}(f) \) is a countable subset and its closure is precisely \( \text{Range}(f) \), by definition.

Note that \( L^0(X, \{0, 1\}) \) can be identified with the set \( \mathcal{P}(X) \) of all equivalence classes of measurable subsets of \( X \). We will often abuse notation and identify a measurable subset \( A \subset X \) with its equivalence class in \( \mathcal{P}(X) \). We will need the crucial fact that the boolean algebra \( \mathcal{P}(X) \) is complete: every increasing net has a supremum.

We will denote by \( \mathcal{M}(X) \) the set of all \( \sigma \)-finite measures in the measure class of \( X \) (note that this is smaller than the usual set of all Borel measures on \( X \) which is also often denoted by \( \mathcal{M}(X) \) in the literature). If \( \mu, \nu \in \mathcal{M}(X) \), then their Radon–Nikodym derivative is a well-defined element \( d\mu/d\nu \in L^0(X, \mathbb{R}_+^\times) \).

2.2. **Equivalence relations, full groups and strong ergodicity.** Let \( X \) be a standard measure space. Let \( \mathcal{R} \) be an equivalence relation on \( X \). In this article, we will always assume that \( \mathcal{R} \) is measurable as a subset of \( X \times X \), that it has countable classes and that it is non-singular, i.e. the \( \mathcal{R} \)-saturation of every null set of \( X \) is again a null set. As a measurable space, \( \mathcal{R} \) is equipped with a canonical measure class for which a subset \( A \subset \mathcal{R} \) is a null set if and only if one of its coordinate projections on \( X \) is a null set. We refer the reader to [FM75] for general background on non-singular equivalence relations.

We define the full pseudo-group of \( \mathcal{R} \), denoted by \([\mathcal{R}]\), as the set of all partial isomorphisms \( \theta : \text{dom}(\theta) \rightarrow \text{ran}(\theta) \) where \( \text{dom}(\theta), \text{ran}(\theta) \in \mathcal{P}(X) \) such that \((x, \theta(x)) \in \mathcal{R}\) for a.e. \( x \in \text{dom}(\theta) \). If \( \theta, \theta' \in [\mathcal{R}] \), then we can naturally compose them to obtain a new element \( \theta' \circ \theta \in [\mathcal{R}] \) with \( \text{dom}(\theta' \circ \theta) = \theta^{-1}(\text{dom}(\theta) \cap \text{ran}(\theta)) \) and \( \text{ran}(\theta' \circ \theta) = \theta'(\text{ran}(\theta) \cap \text{dom}(\theta')) \). The set of invertible elements of \([\mathcal{R}]\) is called the full group of \( \mathcal{R} \) and is denoted by \([\mathcal{R}]\).
For every element $\theta \in [[R]]$, we let $\text{graph}(\theta) = \{(x, \theta(x)) \mid x \in \text{dom}(\theta)\} \in \mathfrak{P}(R)$ be the graph of $\theta$. Then we have

$$R = \bigcup_{\theta \in [R]} \text{graph}(\theta),$$

and any subset $A \in \mathfrak{P}(R)$ can be written as a disjoint union

$$A = \bigsqcup_{n \in \mathbb{N}} \text{graph}(\theta_n)$$

for some elements $\theta_n \in [[R]]$. We have that $[R]$ is a Polish group for the topology induced by the metric

$$d_{\mu}(\theta, \theta') := \mu(\{x \in X \mid \theta(x) \neq \theta'(x)\}),$$

where $\mu \in \mathcal{M}(X)$ is any probability measure (the induced topology does not depend on $\mu$).

Let $E$ be a Polish space. For every function $f \in L^0(X, E)$, we define two functions $f^\ell$ and $f^r$ in $L^0(R, E)$ by

$$f^\ell(x, y) = f(x),$$

$$f^r(x, y) = f(y), \quad \text{for a.e. } (x, y) \in R.$$

We say that $f$ is invariant by $R$ if $f^\ell = f^r$, i.e. we have $f(x) = f(y)$ for a.e. $(x, y) \in R$. This is equivalent to the property that $\theta(f) = f$ for every $\theta \in [R]$, where $\theta(f) := f \circ \theta^{-1}$. We denote by $L^0(X, E)^R$ the subset of all $R$-invariant functions. We say that $R$ is ergodic when $L^\infty(X)^R = \mathbb{C}$. We say that a sequence $f_n \in L^\infty(X)$, $n \in \mathbb{N}$ is almost $R$-invariant if $f^\ell_n - f^r_n \to 0$ in the measure topology. This property is equivalent to $\theta(f_n) - f_n \to 0$ for all $\theta \in [R]$. An almost $R$-invariant sequence $(f_n)_{n \in \mathbb{N}}$ is called trivial if there exists a sequence $\lambda_n \in \mathbb{C}$ such that $f_n - \lambda_n \to 0$ in the measure topology. We say that $R$ is strongly ergodic if every almost $R$-invariant sequence in $L^\infty(X)$ is trivial.

### 2.3. First cohomology of equivalence relations

Let $R$ be an equivalence relation on a standard measure space $X$ and $G$ a locally compact second countable abelian group. Put $R^{(2)} = \{(x, y, z) \in X^3 \mid (x, y) \in R \text{ and } (y, z) \in R\}$. As a measurable subset of $X^3$, $R^{(2)}$ is again equipped with a canonical measure class for which a subset $A \subset R^{(2)}$ is a null set if and only if one (hence any) of its coordinate projections is a null set.

A measurable $G$-valued 1-cocycle for $R$ is a function $\Omega \in L^0(R, G)$ satisfying $\Omega(x, y)\Omega(y, z) = \Omega(x, z)$ for a.e. $(x, y, z) \in R^{(2)}$. We denote by $Z^1(R, G)$ the set of all such cocycles. It is a closed subgroup of the Polish abelian group $L^0(R, G)$.

**Remark 2.1.** Given any 1-cocycle $\Omega \in Z^1(R, G)$, one may always choose a representing 1-cocycle such that $\Omega(x, y)\Omega(y, z) = \Omega(x, z)$ holds for every $(x, y, z) \in R^{(2)}$ (see, for example, [Zi84, Theorem B.9] and [FM75, Theorem 1]).

We define a boundary map $\partial : L^0(X, G) \ni f \mapsto \partial f \in Z^1(R, G)$ by $(\partial f)(x, y) = f(x)f(y)^{-1}$ for a.e. $(x, y) \in R$. Note that $\partial$ is a continuous group homomorphism. The image of $\partial$ is denoted by $B^1(R, G)$ and its elements are called 1-coboundaries.
The 1-cohomology of $\mathcal{R}$ with coefficients in $G$ is the quotient group $H^1(\mathcal{R}, G) = Z^1(\mathcal{R}, G)/B^1(\mathcal{R}, G)$ equipped with the quotient topology (which is not necessarily Hausdorff). If $\Omega \in Z^1(\mathcal{R}, G)$ is a 1-cocycle, we denote by $[\Omega] \in H^1(\mathcal{R}, G)$ its cohomology class.

**Proposition 2.2.** [Sc79, Proposition 2.3] Let $\mathcal{R}$ be an equivalence relation on a standard measure space $X$. Then $\mathcal{R}$ is strongly ergodic if and only if $B^1(\mathcal{R}, \mathbb{T})$ is closed in $Z^1(\mathcal{R}, \mathbb{T})$. In that case, $H^1(\mathcal{R}, \mathbb{T})$ is a Hausdorff Polish group.

From now on, if $G = \mathbb{T}$, we will use the shorthand notation $Z^1(\mathcal{R}) = Z^1(\mathcal{R}, \mathbb{T})$, $B^1(\mathcal{R}) = B^1(\mathcal{R}, \mathbb{T})$ and $H^1(\mathcal{R}) = H^1(\mathcal{R}, \mathbb{T})$.

**Definition 2.3.** Let $\mathcal{R}$ and $\mathcal{S}$ be two equivalence relations on two standard measure spaces $X$ and $Y$. A morphism from $\mathcal{R}$ to $\mathcal{S}$ is a non-singular measurable map $f : X \to Y$ such that $(f(x), f(y)) \in \mathcal{S}$ for a.e. $(x, y) \in \mathcal{R}$. Two such morphisms $f$ and $g$ are said to be equivalent if $(f(x), g(y)) \in \mathcal{S}$ for a.e. $x \in X$.

It is easy to see that if $f : X \to Y$ is a morphism from $\mathcal{R}$ to $\mathcal{S}$, then the natural map $f^* : Z^1(\mathcal{S}) \ni \Omega \mapsto \Omega \circ (f \times f) \in Z^1(\mathcal{R})$ is a continuous group homomorphism which sends 1-coboundaries to 1-coboundaries. Hence it induces a continuous group homomorphism $[f^*] : H^1(\mathcal{S}) \to H^1(\mathcal{R})$. The next proposition shows that this induced map $[f^*]$ only depends on the equivalence class of $f$.

**Proposition 2.4.** Let $\mathcal{R}$ and $\mathcal{S}$ be two equivalence relations on two standard measure spaces $X$ and $Y$. And let $f, g : X \to Y$ be two morphisms from $\mathcal{R}$ to $\mathcal{S}$. If $f$ and $g$ are equivalent, then $[f^*] = [g^*]$.

**Proof.** Let $c \in Z^1(\mathcal{S})$. Since $f \sim g$, we can define an element $u \in L^0(X, \mathbb{T})$ by

$$u(x) = c(f(x), g(x)) \quad \text{for a.e. } x \in X.$$ 

Then, by using the cocycle identity, we compute

$$\partial u(x, y) = c(f(x), g(x)) \cdot \overline{c(f(y), g(y))} = c(f(x), f(y)) \cdot \overline{c(g(x), g(y))}$$

for a.e. $(x, y) \in \mathcal{R}$, and this shows exactly that $[f^*(c)] = [g^*(c)]$. \hfill $\Box$

**Corollary 2.5.** Let $\mathcal{R}$ be an ergodic equivalence relation on a standard measure space $X$. Let $Y \subset X$ be any non-zero measurable subset and let $\iota : Y \to X$ be the inclusion map. Then the map $[\iota^*] : H^1(\mathcal{R}) \to H^1(\mathcal{R}_Y)$ is an isomorphism of topological groups.

**Proof.** Since $\mathcal{R}$ is ergodic, we can find a family $(\theta_i)_{i \in I}$ of elements of the full pseudogroup of $\mathcal{R}$ such that ran($\theta_i$) $\subset Y$ for all $i \in I$ and the family $(\text{dom}(\theta_i))_{i \in I}$ forms a partition of $X \setminus Y$. Define a map $r : X \to Y$ by $r(x) = x$ if $x \in Y$ and $r(x) = \theta_i(x)$ if $x \in \text{dom}(\theta_i)$. By construction, $r$ is a morphism from $\mathcal{R}$ to $\mathcal{R}_Y$. Moreover, we have $r \circ \iota = \text{id}_Y$ and $\iota \circ r \sim \text{id}_X$ (i.e. $r$ is a retraction). Hence, at the cohomological level, $[r^*]$ is an inverse of $[\iota^*]$.
2.4. Skew-product equivalence relations. Let $\mathcal{R}$ be an equivalence relation on a standard measure space $X$, $G$ a locally compact second countable abelian group and $\Omega \in Z^1(\mathcal{R}, G)$ a 1-cocycle. On $X \times G$, put the measure class obtained as the product of the measure class of $X$ and the Haar measure class of $G$. The skew-product of $\mathcal{R}$ by $\Omega$ is the equivalence relation $\mathcal{R} \times_\Omega G$ on $X \times G$ defined by
\[(x, g), (y, h)) \in \mathcal{R} \times_\Omega G \quad \text{if and only if} \quad (x, y) \in \mathcal{R} \quad \text{and} \quad \Omega(x, y) = gh^{-1}\]
for all $x, y \in X$ and all $g, h \in G$.

Let $\Omega' \in Z^1(\mathcal{R}, G)$ be another 1-cocycle in the same cohomology class of $\Omega$. Take a function $\alpha \in L^0(X, G)$ such that $\Omega' = (\partial \alpha) \Omega$. Define an element $T_\alpha \in \text{Aut}(X \times G)$ by
\[T_\alpha(x, g) = (x, \alpha(x)g) \quad \text{for a.e.} \quad (x, g) \in X \times G.\]
Then $T_\alpha$ is an isomorphism from $\mathcal{R} \times_\Omega G$ to $\mathcal{R} \times_{\Omega'} G$.

2.5. Maharam extension and type classification of equivalence relations. Let $\mathcal{R}$ be an equivalence relation on a standard measure space $X$. Let $\mu \in \mathcal{M}(X)$. On $\mathcal{R} \subset X \times X$, define two measures $\mu_\ell, \mu_r \in \mathcal{M}(\mathcal{R})$ as follows:
\[\mu_\ell(W) = \int_X \mathbb{1}_{\{y \in X \mid (x, y) \in W\}} \, d\mu(x),\]
\[\mu_r(W) = \int_X \mathbb{1}_{\{x \in X \mid (x, y) \in W\}} \, d\mu(y), \quad \text{for } W \in \mathcal{B}(\mathcal{R}).\]

We say that $\mu$ is $\mathcal{R}$-invariant if $\mu_\ell = \mu_r$. In general, one can define the modulus of $\mu$ (with respect to $\mathcal{R}$) by
\[\delta_\mu := \frac{d\mu_\ell}{d\mu_r} \in Z^1(\mathcal{R}, \mathbb{R}_0^+).\]

If $\nu \in \mathcal{M}(X)$ is another measure, then we have
\[\delta_\nu \delta_\mu^{-1} = \partial \left( \frac{d\nu}{d\mu} \right) \in B^1(\mathcal{R}, \mathbb{R}_0^+).\]
Hence we have a canonical cohomology class
\[\delta := [\delta_\mu] \in H^1(\mathcal{R}, \mathbb{R}_0^+),\]
and the skew-product equivalence relation $c(\mathcal{R}) = \mathcal{R} \times_{\delta_\mu} \mathbb{R}_0^+$ does not depend on the choice of $\mu$ up to canonical isomorphism. We call $c(\mathcal{R})$ the Maharam extension of $\mathcal{R}$.

Now suppose that the equivalence relation $\mathcal{R}$ is ergodic. We say that $\mathcal{R}$ is of type
I \quad \text{if} \quad \mathcal{R} \text{ has only one equivalence class, up to measure zero;}
II_1 \quad \text{if} \quad \mathcal{R} \text{ is not of type I and if there exists a probability measure } \mu \in \mathcal{M}(X) \text{ that is } \mathcal{R} \text{-invariant;}
II_\infty \quad \text{if} \quad \mathcal{R} \text{ is not of type I and if there exists an infinite measure } \mu \in \mathcal{M}(X) \text{ that is } \mathcal{R} \text{-invariant;}
III \quad \text{otherwise.}

By analogy with [Co72] (see also [Kr67, Kr75]), for any ergodic equivalence relation $\mathcal{R}$, we define the S invariant by the formula
\[S(\mathcal{R}) = \bigcap_{\mu \in \mathcal{M}(X)} \overline{\text{Range}(\delta_\mu)}.\]
where $\text{Range}(\delta_\mu)$ denotes the essential range of $\delta_\mu$ in $\mathbb{R}^+$. The $S$ invariant can be computed by using a single measure $\mu \in \mathcal{M}(X)$ thanks to the formula

$$S(\mathcal{R}) = \bigcap_{U \in \Psi(X), U \neq \emptyset} \text{Range}(\delta_\mu |_{\mathcal{R}_U}),$$

where $\mathcal{R}_U = \mathcal{R} \cap (U \times U)$ and $\delta_\mu |_{\mathcal{R}_U}$ is the restriction of $\delta_\mu$ to $\mathcal{R}_U$. If, moreover, the $\mu$-preserving subequivalence relation $\mathcal{R}_\mu := \ker(\delta_\mu) \subseteq \mathcal{R}$ is ergodic, then we actually have

$$S(\mathcal{R}) = \text{Range}(\delta_\mu).$$

Note that $\mathcal{R}$ is of type I or II if and only if $S(\mathcal{R}) = \{1\}$. Assume now that $\mathcal{R}$ is a type III ergodic equivalence relation. Consider the translation action of $\mathbb{R}_0^+$ on $X \times \mathbb{R}_0^+$ and note that this action preserves $c(\mathcal{R})$. Hence it induces an action of $\mathbb{R}_0^+$ on $L^\infty(X \times \mathbb{R}_0^+)^{c(\mathcal{R})}$. Then $S(\mathcal{R}) \cap \mathbb{R}_0^+$ is the kernel of this action and thus $S(\mathcal{R}) \cap \mathbb{R}_0^+$ is a closed subgroup of $\mathbb{R}_0^+$ (see [Ta03b, Theorem XIII.2.23]). We say that $\mathcal{R}$ is of type

- $\text{III}_0$ if $S(\mathcal{R}) = \{0, 1\};$
- $\text{III}_\lambda$, $\lambda \in (0, 1)$ if $S(\mathcal{R}) = \{0\} \cup \lambda \mathbb{Z};$
- $\text{III}_1$ if $S(\mathcal{R}) = \mathbb{R}^+.$

Finally, by analogy with [Co74], we introduce the $\tau$ invariant.

**Definition 2.6.** Let $\mathcal{R}$ be a strongly ergodic equivalence relation on a standard measure space $X$. The $\tau$ invariant of $\mathcal{R}$, denoted by $\tau(\mathcal{R})$, is defined as the weakest topology on $\mathbb{R}$ that makes the map $\mathbb{R} \to H^1(\mathcal{R}) : t \mapsto [\delta_t^\mathcal{R}]$ continuous.

### 2.6. Von Neumann algebras associated with equivalence relations

Let $\mathcal{R}$ be an equivalence relation on a standard measure space $X$. The von Neumann algebra associated with $\mathcal{R}$, denoted by $L(\mathcal{R})$, is generated by a copy of $L^\infty(X)$ and a set of unitaries $u_\theta$, $\theta \in [\mathcal{R}]$ which satisfy the following three conditions.

- $u_\theta u_\phi = u_{\theta \circ \phi}$ for all $\theta, \phi \in [\mathcal{R}].$
- $u_\theta u_\theta^* = \theta(f)$ for all $\theta \in [\mathcal{R}]$ and $f \in L^\infty(X)$, where $\theta(f) = f \circ \theta^{-1}.$
- There exists a faithful normal conditional expectation

$$E_{L^\infty(X)} : L(\mathcal{R}) \to L^\infty(X)$$

that is characterized by $E_{L^\infty(X)}(u_\theta) = 1_{\{x \in X | \theta(x) = x\}}$ for every $\theta \in [\mathcal{R}]$.

**Pick** $\mu \in \mathcal{M}(X)$. We define $L^2(\mathcal{R}, \mu_r)$ to be the Hilbert space of quadratic integrable functions with respect to $\mu_r$. Then $L(\mathcal{R})$ is naturally represented on $L^2(\mathcal{R}, \mu_r).$ For each $\theta \in [\mathcal{R}]$, the unitary $u_\theta$ acts on $L^2(\mathcal{R}, \mu_r)$ by

$$(u_\theta \xi)(x, y) = \xi(\theta^{-1}(x), y) \text{ for } \xi \in L^2(\mathcal{R}, \mu_r), (x, y) \in \mathcal{R},$$

and $L^\infty(X)$ acts on $L^2(\mathcal{R})$ by $(f \xi)(x, y) = f(x) \xi(x, y)$ for $f \in L^\infty(X), \xi \in L^2(\mathcal{R}, \mu_r), (x, y) \in \mathcal{R}.$

On $L(\mathcal{R})$, define the natural faithful normal semifinite weight $\varphi = \tau_\mu \circ E_{L^\infty(X)}$, where $\tau_\mu = \int_X \cdot d\mu$. Then $L^2(\mathcal{R}, \mu_r)$ can be identified with the Gelfand–Naimark–Segal construction of $L(\mathcal{R})$ with respect to $\varphi$. The modular operator $\Delta_\varphi$ of $\varphi$ is then determined by $(\Delta^t_\varphi \xi)(x, y) = \delta_\mu(x, y)^t \xi(x, y)$, for $t \in \mathbb{R}, \xi \in L^2(\mathcal{R}, \mu_r), (x, y) \in \mathcal{R}.$

We recall the following two well-known facts. For an inclusion of von Neumann algebras $A \subset M$, we denote by $\text{Aut}(M/A)$ the group of automorphisms of $M$ that fix $A$ pointwise.
LEMMA 2.7. Let $\mathcal{R}$ be a non-singular equivalence relation on a standard measure space $X$. The following statements hold true.

- For every 1-cocycle $c \in Z^1(\mathcal{R}) \subset L^\infty(\mathcal{R})$, the map $\text{Ad}(c) : L(\mathcal{R}) \to B(L^2(\mathcal{R}))$ is an automorphism of $L(\mathcal{R})$ that fixes $L^\infty(X)$ pointwise, i.e. $\text{Ad}(c) \in \text{Aut}(L(\mathcal{R})/L^\infty(X))$.
- The map $c \mapsto \text{Ad}(c)$ is a topological isomorphism between $Z^1(\mathcal{R})$ and $\text{Aut}(L(\mathcal{R})/L^\infty(X))$ that sends $B^1(\mathcal{R})$ onto $\text{Ad}(\mathcal{U}(L^\infty(X)))$.

Proof. See [Po04, Proposition 1.5.1] and [Ta03b, Theorem XIII.2.21]. \qed

Note that if $c \in Z^1(\mathcal{R})$ and $\theta \in [\mathcal{R}]$, then we have $\text{Ad}(c)(u_{\theta}) = c(\theta)u_{\theta}$ where $c(\theta) \in L^0(X, \mathbb{T})$ is defined by

$$c(\theta)(x) = c(\theta^{-1}(x), x)$$

for a.e. $x \in X$.

Finally, we will later need the following lemma.

LEMMA 2.8. [Ta03b, Theorem XIII.2.13] Let $\mathcal{R}$ be an ergodic equivalence relation on a standard measure space $X$. Then $S(\mathcal{R}) = S(L(\mathcal{R}))$.

3. Spectral gap for strongly ergodic equivalence relations

We start this section with the following definition.

Definition 3.1. Let $(X, \mu)$ be a standard probability space. A non-singular partial isomorphism $\theta$ of $(X, \mu)$ is said to be $\mu$-bounded if the function $\log(d(\mu \circ \theta)/d\mu)$ is bounded on the support of $\theta$. In this case, the map $L_{\theta} : f \mapsto \theta(f)$ defines a bounded operator in $B(L^2(X, \mu))$, where $\theta(f) = f \circ \theta^{-1}$.

In Appendix A, we show that such $\mu$-bounded partial isomorphisms are abundant in the full pseudo-group of any type III ergodic equivalence relation $\mathcal{R}$ on a probability space $(X, \mu)$.

Our first main result in this section, Theorem 3.2 below, provides equivalent characterizations of the spectral gap property for finite sets of $\mu$-bounded automorphisms of $(X, \mu)$.

THEOREM 3.2. Let $(X, \mu)$ be a standard probability space. Let $\theta_1, \ldots, \theta_n$ be a finite family of $\mu$-bounded automorphisms of $X$. Then the following assertions are equivalent.

(i) There exists a constant $\kappa > 0$ such that for all measurable subsets $A \subset X$ we have

$$\mu(A)(1 - \mu(A)) \leq \kappa \sum_{k=1}^n \mu(A \Delta \theta_k(A)).$$

(ii) There exists a constant $\kappa > 0$ such that for all $f \in L^2(X, \mu)$ we have

$$\|f - \mu(f)\|_2 \leq \kappa \sum_{k=1}^n \|\theta_k(f) - f\|_2.$$

(ii') There exists a constant $\kappa > 0$ such that for all $f \in L^2(X, \mu)$ we have

$$\|f - \mu(f)\|_2^2 \leq \kappa \sum_{k=1}^n \|\theta_k(f) - f\|_2^2.$$
The eigenvalue $0$ is a simple isolated point in the spectrum of

$$T = \sum_{k=1}^{n} |L_{\theta_k} - 1| \in \mathcal{B}(L^2(X, \mu)).$$

The eigenvalue $0$ is a simple isolated point in the spectrum of

$$T' = \sum_{k=1}^{n} |L_{\theta_k} - 1|^2 \in \mathcal{B}(L^2(X, \mu)).$$

When these conditions are satisfied, we say that the family $\theta_1, \ldots, \theta_n$ has a spectral gap.

Theorem 3.2 relies on the following lemma which is inspired by the trick of Namioka that was used in [Co75b, Sc80]. Our new input is item (iii) which is crucial in the proof of Theorem 3.2.

**Lemma 3.3.** Let $(X, \mu)$ be a probability space. For $a \geq 0$ and $f$ a positive measurable function on $X$, we use the notation $e_a(f) := 1_{[a, \infty)}(f)$.

(i) For every positive function $f \in L^1(X, \mu)$, we have

$$\mu(f) = \int_0^{\infty} \mu(e_a(f)) \, da.$$ 

(ii) For every positive function $f, g \in L^1(X, \mu)$, we have

$$\|f - g\|_1 = \int_0^{\infty} \mu(e_a(f) \Delta e_a(g)) \, da.$$ 

(iii) For every positive function $f \in L^2(X, \mu)$ we have

$$\|f - \mu(f)\|^2 \leq \int_0^{\infty} \mu(e_a(f^2)) (1 - \mu(e_a(f^2))) \, da.$$ 

**Proof.** (i) By Fubini’s theorem, we have

$$\int_0^{\infty} \mu(e_a(f)) \, da = \int_0^{\infty} \int_X e_a(f(x)) \, d\mu(x) \, da = \int_X \int_0^{\infty} e_a(f(x)) \, da \, d\mu(x) = \int_X f(x) \, d\mu(x).$$

(ii) By Fubini’s theorem, we have

$$\int_0^{\infty} \mu(e_a(f) \Delta e_a(g)) \, da = \int_X \int_0^{\infty} |e_a(f(x)) - e_a(g(x))| \, da \, d\mu(x)$$

$$= \int_X |f(x) - g(x)| \, d\mu(x).$$

(iii) First, note that $\|f - \mu(f)\|^2 = \mu(f^2) - \mu(f)^2$ and that

$$\mu(f^2) = \int_0^{\infty} \mu(e_a(f^2)) \, da.$$ 

Therefore, we only have to show that

$$\int_0^{\infty} \mu(e_a(f^2))^2 \, da \leq \mu(f)^2.$$
On $(X, \mu) \otimes (X, \mu)$, we have $e_a(f^2) \otimes e_a(f^2) \leq e_a(f \otimes f)$. Hence, by applying $\mu \otimes \mu$ we get

$$\mu(e_a(f^2))^2 \leq (\mu \otimes \mu)(e_a(f \otimes f)),$$
and thus, after integrating over $a$ and using (i), we finally get

$$\int_0^\infty \mu(e_a(f^2))^2 \, da \leq \int_0^\infty (\mu \otimes \mu)(e_a(f \otimes f)) \, da \leq (\mu \otimes \mu)(f \otimes f) = \mu(f)^2. \quad \square$$

Proof of Theorem 3.2. The equivalences (ii) $\Leftrightarrow$ (ii') $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iii') are clear. On the other hand, (ii') applied to the indicator function $1_A$ gives (i). Hence we only have to prove that (i) $\Rightarrow$ (ii). Assume that (i) is satisfied for some constant $\kappa > 0$. Take a positive function $f \in L^2(X, \mu)$. Lemma 3.3 gives

$$\|f - \mu(f)\|_2^2 \leq \kappa \sum_{k=1}^n \|f^2 - \theta_k(f^2)\|_1.$$

Let $C = \max_k \|1 + L\theta_k\|$. Then for all $k$, we have

$$\|f^2 - \theta_k(f^2)\|_1 \leq \|f + \theta_k(f)\|_2 \cdot \|f - \theta_k(f)\|_2 \leq C \|f\|_2 \|f - \theta_k(f)\|_2,$$

hence we obtain

$$\|f - \mu(f)\|_2^2 \leq \kappa C \|f\|_2 \sum_{k=1}^n \|f - \theta_k(f)\|_2.$$

Now let $f \in L^2(X, \mu)$ be a real-valued function with $\mu(f) = 0$. Write $f = f_+ - f_-$ where $f_+$ and $f_-$ are the positive and the negative part of $f$. Then we have

$$\|f\|_2^2 \leq 2(\|f_+ - \mu(f_+)\|_2^2 + \|f_- - \mu(f_-)\|_2^2),$$

hence

$$\|f\|_2 \leq 2\kappa C \sum_{k=1}^n (\|f_+ - \theta_k(f_+)\|_2 + \|f_- - \theta_k(f_-)\|_2),$$

and since $\|f_\pm - \theta_k(f_\pm)\|_2 \leq \|f - \theta_k(f)\|_2$, we obtain

$$\|f\|_2 \leq 4\kappa C \sum_{k=1}^n \|f - \theta_k(f)\|_2.$$

Finally, by applying this inequality to $f - \mu(f)$ we obtain (2) for all real-valued functions $f \in L^2(X, \mu)$. Finally, for complex-valued functions, one can just decompose over the real and imaginary part to get the desired result. \square

As we saw, the proof of Theorem 3.2 is very elementary. Moreover, it provides a short proof of the following well-known result (see the proof of [Sc80, Proposition 2.3]).

Corollary 3.4. Let $\Gamma$ be any countable discrete group and $\Gamma \curvearrowright (X, \mu)$ any strongly ergodic pmp action. The following conditions are equivalent.

(i) The action $\Gamma \curvearrowright (X, \mu)$ does not have spectral gap.

(ii) There exists an I-sequence in the sense of [Sc80, §2], i.e. there exists a sequence $(A_i)_{i \in \mathbb{N}}$ of proper measurable subsets of $X$ such that $\mu(A_i) \to 0$ and $\mu(A_i \Delta \gamma A_i)/\mu(A_i) \to 0$ for all $\gamma \in \Gamma$. 

Proof. We only have to prove (i) \(\Rightarrow\) (ii). Since the action \(\Gamma \acts (X, \mu)\) does not have a spectral gap, the equivalence (i) \(\Leftrightarrow\) (ii) in Theorem 3.2 implies the existence of a sequence \((A_i)_{i \in \mathbb{N}}\) of proper measurable subsets of \(X\) such that

\[
\frac{\mu(A_i \triangle \gamma(A_i))}{\mu(A_i)(1 - \mu(A_i))} \to 0
\]

for all \(\gamma \in \Gamma\). Since the action \(\Gamma \acts (X, \mu)\) is strongly ergodic, we infer that \(\mu(A_i)(1 - \mu(A_i)) \to 0\). Up to replacing \(A_i\) by \(X \setminus A_i\) for each \(i \in \mathbb{N}\), we may assume that \(\mu(A_i) \to 0\) and hence \((A_i)_{i \in \mathbb{N}}\) is an \(I\)-sequence.

The second main result in this section, Theorem 3.5 below, gives a spectral gap characterization of strongly ergodic equivalence relations.

**Theorem 3.5.** (Theorem A) Let \(\mathcal{R}\) be a strongly ergodic equivalence relation on a standard probability space \((X, \mu)\) that either preserves \(\mu\) or is of type III. Then there exists a finite family of \(\mu\)-bounded elements \(\theta_1, \ldots, \theta_n \in [\mathcal{R}]\) that has a spectral gap.

Before proving Theorem 3.5, we need to introduce some terminology.

**Definition 3.6.** Let \(\mathcal{R}\) be an ergodic equivalence relation on a probability space \((X, \mu)\). We say that \(\mathcal{R}\) admits small almost invariant sets (with respect to \(\mu\)) if for any \(\varepsilon > 0\) and any finite family of \(\mu\)-bounded elements \(\theta_1, \ldots, \theta_n \in [\mathcal{R}]\), we can find a non-zero measurable subset \(A \subset X\) such that \(\mu(A) < \varepsilon\) and

\[
\mu(A \triangle \theta_k(A)) \leq \varepsilon \mu(A).
\]

The following lemma will be a crucial step in the proof of Theorem 3.5.

**Lemma 3.7.** Let \(\mathcal{R}\) be an ergodic equivalence relation on a probability space \((X, \mu)\) that either preserves \(\mu\) or is of type III\(\lambda\), \(0 < \lambda \leq 1\). Suppose that \(\mathcal{R}\) admits small almost invariant sets. Then for any non-zero measurable subset \(Y \subset X\) and any finite family of \(\mu\)-bounded elements \(\theta_1, \ldots, \theta_n \in [\mathcal{R}]\), we can find a non-zero measurable subset \(A \subset Y\) such that \(\mu(A) < \varepsilon\) and

\[
\mu(A \cap \theta_k(Y \setminus A)) + \mu((Y \setminus A) \cap \theta_k(A)) \leq \varepsilon \mu(A).
\]

In particular, the reduced equivalence relation \(\mathcal{R}_Y\) admits small almost invariant sets.

**Proof.** Since \(\mathcal{R}\) admits small almost invariant sets, we can find a net \((A_i)_{i \in I}\) of measurable subsets of \(X\) with \(\mu(A_i) > 0\) for all \(i\), such that \(\mu(A_i) \to 0\) and

\[
\frac{\mu(A_i \triangle \theta(A_i))}{\mu(A_i)} \to 0
\]

when \(i \to \infty\) for all \(\mu\)-bounded \(\theta \in [\mathcal{R}]\). Up to extracting a subnet, we may also assume that the net of probability measures \(\mu_i := (1/\mu(A_i))\mu(A_i \cap \cdot)\) converges in the weak* topology to some positive linear functional \(\varphi\) in the unit ball of \(L^\infty(X, \mu)^*\). Now, let \(B_i = A_i \cap Y\). We want to show that \(\mu(B_i) > 0\) for \(i\) large enough and that

\[
\frac{1}{\mu(B_i)}(\mu(B_i \cap \theta(Y \setminus B_i)) + \mu((Y \setminus B_i) \cap \theta(B_i))) \to 0
\]

as \(i \to \infty\) for all \(\mu\)-bounded \(\theta \in [\mathcal{R}]\).
In order to show this, note that for every $\theta \in [\mathcal{R}]$ we have
\[
\mu(B_i \cap \theta(Y \setminus B_i)) + \mu((Y \setminus B_i) \cap \theta(B_i)) \leq \mu(A_i \Delta \theta(A_i)).
\]
Hence, it will be enough to show that $\mu(B_i)/\mu(A_i)$ converges to some positive number, i.e. that $\varphi(1_Y) > 0$.

If $\mathcal{R}$ preserves $\mu$, we know by construction that $\varphi$ is invariant by $[\mathcal{R}]$. Hence we have $\varphi(1_{\theta(Y)}) = \varphi(1_Y)$ for all $\theta \in [\mathcal{R}]$. Since we can find finitely many $\theta_1, \ldots, \theta_n \in [\mathcal{R}]$ such that $1_X \leq \sum_{k=1}^{n} 1_{\theta_k(Y)}$, we conclude that $1 = \varphi(1_X) \leq n \varphi(1_Y)$. This shows that $\varphi(1_Y) > 0$.

If $\mathcal{R}$ is of type III$_\lambda$, $0 < \lambda \leq 1$, we can construct, by Theorem A.1 in Appendix A, a $\mu$-bounded element $\theta \in [\mathcal{R}]$ such that $\theta(X \setminus Y) \subset Y$. Since $\theta$ is $\mu$-bounded, we can find $C > 0$ such that $\mu \leq C(\mu \circ \theta)$. Then, since the net $(A_i)_{i \in I}$ is almost invariant by $\theta$, we can see that $\varphi \leq C(\varphi \circ \theta)$. Hence $\varphi(1_{X \setminus Y}) \leq C \varphi(1_Y)$. This again shows that $\varphi(1_Y) > 0$. □

**Proof of Theorem 3.5.** We suppose that such a family does not exist and we contradict the strong ergodicity of $\mathcal{R}$. Using the negation of Theorem 3.2(i), there exists a net $(A_i)_{i \in I}$ of proper measurable subsets of $X$ such that
\[
\frac{\mu(A_i \Delta \theta(A_i))}{\mu(A_i)(1 - \mu(A_i))} \to 0
\]
for all $\mu$-bounded elements $\theta \in [\mathcal{R}]$. Since the set of $\mu$-bounded elements is dense in $[\mathcal{R}]$ by Corollary A.3, we infer that $\mu(A_i \Delta \theta(A_i)) \to 0$ for all $\theta \in [\mathcal{R}]$. Hence, by the strong ergodicity of $\mathcal{R}$, we know that $\mu(A_i)(1 - \mu(A_i)) \to 0$. Then, by replacing the set $A_i$ by $X \setminus A_i$ for each $i$ if necessary, we can assume that $\mu(A_i) \to 0$. Hence $\mathcal{R}$ admits small almost invariant sets. Now fix $\theta_1, \ldots, \theta_n$ a finite set of $\mu$-bounded elements in $[\mathcal{R}]$ and take $\varepsilon > 0$. Consider the set $\Lambda$ of all elements $A \in \mathcal{B}(X)$ such that $\mu(A) \leq \frac{1}{2}$ and
\[
\mu(A \Delta \theta_k(A)) \leq \varepsilon \mu(A).
\]
Since $\Lambda$ is closed in the complete boolean algebra $\mathcal{B}(X)$, it is inductive as a poset. Let $A \in \Lambda$ be a maximal element. Suppose that $\mu(A) < \frac{1}{2}$. Let $Y = X \setminus A$. By Lemma 3.7, we can find $B \subset Y$ such that $0 < \mu(B) < \frac{1}{2} - \mu(A)$ and
\[
\mu(B \cap \theta_k(Y \setminus B)) + \mu((Y \setminus B) \cap \theta_k(B)) \leq \varepsilon \mu(B).
\]
Then for $C = A \cup B$, we easily check that
\[
\mu(C \cap \theta_k(X \setminus C)) + \mu((X \setminus C) \cap \theta_k(C)) \leq \varepsilon \mu(C)
\]
and $\mu(C) = \mu(A) + \mu(B) < \frac{1}{2}$. Therefore $C \in \Lambda$ and this contradicts the maximality of $A$. Hence we must have $\mu(A) = \frac{1}{2}$. We have proved that for any finite family of $\mu$-bounded elements in $[\mathcal{R}]$ and any $\varepsilon > 0$, we can find $A \subset X$ such that $\mu(A) = \frac{1}{2}$ and
\[
\mu(A \Delta \theta_k(A)) \leq \varepsilon.
\]
This contradicts the strong ergodicity of $\mathcal{R}$. □

**Remark 3.8.** Let us point out that in the case where $\mathcal{R}$ is pmp and ergodic, the existence of small almost invariant sets for $\mathcal{R}$ does not guarantee the existence of an $I$-sequence for the full group $[\mathcal{R}]$ (in the sense of [Sc80, §2]), but rather an $I$-net. This is because the full group $[\mathcal{R}]$ is uncountable. For that reason, we cannot use [Sc80, Proposition 2.3] to prove Theorem 3.5 even in the case where $\mathcal{R}$ is pmp.
4. Strong ergodicity of skew-product equivalence relations

Let $G$ be a second countable locally compact abelian group. Let $\mathcal{R}$ be a non-singular equivalence relation on a standard measure space $X$ and $\Omega \in \mathbb{Z}^1(\mathcal{R}, G)$ a measurable 1-cocycle. Our goal in this section is to give a criterion for the strong ergodicity of the skew-product equivalence relation $\mathcal{R} \times \Omega G$. For this, we let $\widehat{G}$ be the Pontryagin dual of $G$ and we introduce the map $\widehat{\Omega} : \widehat{G} \to \mathbb{Z}^1(\mathcal{R})$ defined by $\widehat{\Omega}(p)(x, y) = (p, \Omega(x, y))$ for a.e. $(x, y) \in \mathcal{R}$, where $(\cdot, \cdot) : \widehat{G} \times G \to \mathbb{T}$ is the duality pairing. Note that $\widehat{\Omega}$ is a continuous group homomorphism. We also introduce the continuous homomorphism $[\widehat{\Omega}] : \widehat{G} \to H^1(\mathcal{R})$ which sends $p \in \widehat{G}$ to the cohomology class $[\widehat{\Omega}(p)] \in H^1(\mathcal{R})$.

The main theorem of this section is the following result.

**Theorem 4.1.** (Theorem B) Let $\mathcal{R}$ be an ergodic equivalence relation on a standard measure space $(X, \mu)$. Let $\Omega \in \mathbb{Z}^1(\mathcal{R}, G)$ be any measurable 1-cocycle with values in a locally compact second countable abelian group $G$. Consider the following assertions.

(i) The skew-product equivalence relation $\mathcal{R} \times \Omega G$ is strongly ergodic.

(ii) The equivalence relation $\mathcal{R}$ is strongly ergodic and the map $[\widehat{\Omega}] : \widehat{G} \to H^1(\mathcal{R})$ is a homeomorphism onto its range.

Then (i) $\Rightarrow$ (ii), and if $G$ contains a lattice, we also have (ii) $\Rightarrow$ (i).

We first prove the following proposition which deals with the ergodicity of the skew-product equivalence relation. It gives a hint for the proof of Theorem 4.1. Note that this proposition is closely related to [Zi76, Corollary 3.8]. Indeed, if $\Omega$ is cohomologous to some other 1-cocycle $\Omega'$ such that $\text{Range}(\Omega')$ is contained in some closed subgroup $H \subset G$ then $H^1 \subset \ker([\widehat{\Omega}])$.

**Proposition 4.2.** Let $\mathcal{R}$ be a non-singular equivalence relation on a standard measure space $X$. Let $\Omega \in \mathbb{Z}^1(\mathcal{R}, G)$ be a measurable 1-cocycle with values in a second countable locally compact abelian group $G$. Consider the following assertions.

(i) The skew-product equivalence relation $\mathcal{R} \times \Omega G$ is ergodic.

(ii) The relation $\mathcal{R}$ is ergodic and the map $[\widehat{\Omega}] : \widehat{G} \to H^1(\mathcal{R})$ is injective.

Then (i) $\Rightarrow$ (ii), and if $G$ is compact we also have (ii) $\Rightarrow$ (i).

**Proof.** (i) $\Rightarrow$ (ii). Suppose that $\mathcal{R} \times \Omega G$ is ergodic. Since $\mathcal{R} \times \Omega G$ clearly surjects onto $\mathcal{R}$, we know that $\mathcal{R}$ is also ergodic. Moreover, this surjection induces a natural inclusion $\iota : Z^1(\mathcal{R}) \to Z^1(\mathcal{R} \times \Omega G)$. Let $p \in \widehat{G}$ and suppose that $\widehat{\Omega}(p) = \delta u$ for some $u \in L^0(X, \mathbb{T})$. By definition of the skew-product construction, we have the relation $\delta(1 \otimes p) = \iota(\widehat{\Omega}(p))$ where we view $1 \otimes p$ as an element of $L^0(X \times G, \mathbb{T})$. Therefore we have $\delta(1 \otimes p) = \delta(u \otimes 1)$. Since $\mathcal{R} \times \Omega G$ is ergodic, this means that $1 \otimes p = \lambda(u \otimes 1)$ for some $\lambda \in \mathbb{T}$ and this easily implies that $p = 1$. Hence $[\widehat{\Omega}]$ is injective.

(ii) $\Rightarrow$ (i) when $G$ is compact. Let $\Gamma = \widehat{G}$ be the dual discrete group. Let $f \in L^\infty(X \times G)^{\mathcal{R} \times \Omega G}$. Consider its Fourier decomposition given by the formal sum

$$f = \sum_{\gamma \in \Gamma} f_{\gamma} \otimes \gamma,$$
where \( f_\gamma \in L^\infty(X) \) for all \( \gamma \in \Gamma \). Then, since \( f \) is invariant, we have

\[
0 = f^\ell - f^r = \sum_{\gamma \in \Gamma} \Omega(\gamma) f^\ell_\gamma - f^r_\gamma \otimes \gamma^r,
\]

where we used the relation \((1 \otimes \gamma)^\ell = \iota(\Omega(\gamma))(1 \otimes \gamma)^r\). This shows that \( f^\ell_\gamma = \Omega(\gamma) f^\ell_\gamma \) for all \( \gamma \in \Gamma \). In particular, we have \(|f_\gamma|^\ell = |f_\gamma|^r\), and since \( \mathcal{R} \) is ergodic, this means that \(|f_\gamma|\) is a constant for all \( \gamma \in \Gamma \). Suppose that for some \( \gamma \), we have \( f_\gamma \neq 0 \). Let \( u_\gamma \) be the polar part of \( f_\gamma \). Then we have \( \Omega(\gamma) = \partial u_\gamma \). Since \( \hat{\Omega} \) is injective, this implies that \( \gamma = e \). We then have \( f = f_e \otimes e \) and \( \partial f_e = 1 \). Since \( \mathcal{R} \) is ergodic, this means that \( u_e \in \mathcal{T} \) and therefore \( f_e \) is constant. We conclude that \( f \) is constant. Hence \( \mathcal{R} \times \Omega \mathcal{G} \) is ergodic.

For the proof of Theorem 4.1, we will need the following proposition. We prove it by using the ultraproduct construction.

**Proposition 4.3.** Let \( \mathcal{R} \) be an ergodic equivalence relation on a standard measure space \( X \). Let \( \Gamma \rhd X \) be a non-singular action of an amenable countable discrete group by automorphisms of \( \mathcal{R} \). Then \( \mathcal{R} \) is strongly ergodic if and only if the equivalence relation generated by \( \mathcal{R} \) and \( \mathcal{R}(\Gamma \rhd X) \) is strongly ergodic.

**Proof.** Clearly, if \( \mathcal{R} \) is strongly ergodic then a fortiori the equivalence relation generated by \( \mathcal{R} \) and \( \mathcal{R}(\Gamma \rhd X) \) is strongly ergodic. Now suppose that \( \mathcal{R} \) is not strongly ergodic. Choose a probability measure \( \mu \) in the measure class of \( X \) and put \( \tau(f) = \int_X f d\mu \) for every \( f \in L^\infty(X) \). Fix a non-principal ultrafilter \( \omega \in \beta(N) \setminus N \) and consider the ultraproduct von Neumann algebra \( L^\infty(X)^\omega \) together with its canonical faithful normal trace \( \tau^\omega \) defined by \( \tau^\omega((f_n)^\omega) = \lim_{n \to \omega} \tau(f_n) \) for every \( (f_n)^\omega \in L^\infty(X)^\omega \). Define the (abelian) von Neumann subalgebra \( \mathcal{A} \subset L^\infty(X)^\omega \) by

\[
\mathcal{A} := \left\{ (f_n)^\omega \in L^\infty(X)^\omega : \lim_{n \to \omega} \|\theta(f_n) - f_n\|_2 = 0, \forall \theta \in [\mathcal{R}] \right\}.
\]

By assumption, we have \( \mathcal{A} \neq C1 \). We claim that \( \mathcal{A} \) is diffuse. Indeed, following the proof of [Co74, Corollary 3.8], let \( p \in \mathcal{A} \) be any projection such that \( p \neq 0, 1 \). Write \( p = (p_n)^\omega \) and \( \alpha = \tau^\omega(p) \in (0, 1) \). By [Co75a, Proposition 1.1.3 (a)], we may further assume that \( p_n \in L^\infty(X) \) is a projection for every \( n \in N \). Since \( \mathcal{R} \) is ergodic and since \( \lim_{n \to \omega} \|\theta(p_n) - p_n\|_2 = 0 \), it follows that \( p_n \to \alpha 1 \) weakly as \( n \to \omega \). Fix a countable dense subset \( \{\theta_n : n \in N\} \subset [\mathcal{R}] \). For every \( n \in N \), there exists \( k_n \in N \) large enough such that

for all \( 0 \leq j \leq n \), \( \|\theta_j(p_{k_n}) - p_{k_n}\|_2 \leq \frac{1}{n+1} \) and \( |\tau(p_j p_{k_n}) - \alpha \tau(p_j)| \leq \frac{1}{n+1} \).

Then \( q = (p_n p_{k_n})^\omega \in \mathcal{A} \), \( q \leq p \) and \( \tau^\omega(q) = \alpha^2 \). This shows that \( \mathcal{A} \) has no minimal projection and hence is diffuse.

Observe that the natural action \( \Gamma \rhd L^\infty(X)^\omega \) defined by \( \gamma \cdot (f_n)^\omega = (\gamma(f_n))^\omega \) for every \( (f_n)^\omega \in L^\infty(X)^\omega \) and every \( \gamma \in \Gamma \) leaves the von Neumann subalgebra \( \mathcal{A} \subset L^\infty(X)^\omega \) globally invariant. Since \( \mathcal{A} \) is diffuse and \( \Gamma \) is countable, we may find a diffuse von Neumann subalgebra \( \mathcal{D} \subset \mathcal{A} \) with separable predual that is globally invariant under the
action $\Gamma \curvearrowright \mathcal{A}$. Write \( \mathcal{D} = L^\infty(Y, \eta) \) where \((Y, \eta)\) is a diffuse standard probability space. Since $\Gamma$ is amenable and \((Y, \eta)\) is diffuse, the action $\Gamma \curvearrowright (Y, \eta)$ is not strongly ergodic (see, for example, [Se79, Proposition 2.2]). Thus, there exists a $\Gamma$-almost invariant sequence of measurable subsets $U_k \subset Y$ such that $\inf_{k \in \mathbb{N}} \eta(U_k)(1 - \eta(U_k)) > 0$. For every $k \in \mathbb{N}$, write \( L^\infty(Y, \eta) \ni 1_{U_k} = (p_n^k)^\omega \in \mathcal{D} \subset \mathcal{A} \) where \((p_n^k)\) is a sequence of projections in \( L^\infty(X) \). By diagonal extraction, we may find a sequence of projections \((q_m)_m \in L^\infty(X) \) of the form $q_m = p_m^{Q_m}$ such that $\inf_{m \in \mathbb{N}} \tau(q_m(1 - \tau(q_m))) > 0$, $\lim_m \|\gamma(q_m) - q_m\|_2 = 0$ for every $\gamma \in \mathbb{N}$ and $\lim_m \|\theta(q_m) - q_m\|_2 = 0$ for every $\theta \in [\mathbb{R}]$. For every $m \in \mathbb{N}$, write $q_m = 1_{V_m}$ where $V_m \subset X$ is a measurable subset. Then \((V_m)_m \) is a non-trivial sequence of measurable subsets that are almost invariant under both \( \mathcal{R} \) and \( \mathcal{R}(\Gamma \curvearrowright X) \). Therefore \( \mathcal{R} \) and \( \mathcal{R}(\Gamma \curvearrowright X) \) generate an equivalence relation that is not strongly ergodic. \( \square \)

The next lemma will be the crucial ingredient in the proof of Theorem 4.1. It gives a spectral gap control for the topology of \( H^1(\mathcal{R}) \) when \( \mathcal{R} \) is a strongly ergodic equivalence relation. Of course, it relies on Theorem 3.5.

If $c \in Z^1(\mathcal{R})$ and $\theta \in [\mathcal{R}]$, we introduce the function \( c(\theta) \in L^0(X, \mathbb{T}) \) defined by $c(\theta)(x) = c(\theta^{-1}(x), x)$ for a.e. $x \in X$. Note that a sequence $c_n \in Z^1(\mathcal{R})$ converges to $1$ in the measure topology if and only if $c_n(\theta)$ converges to $1$ in the measure topology for all $\theta \in [\mathcal{R}]$.

**Lemma 4.4.** Let $\mathcal{R}$ be a strongly ergodic equivalence relation on a standard probability space $(X, \mu)$. Assume that $\mathcal{R}$ either preserves $\mu$ or is of type III. Let $\mathcal{V}$ be a neighborhood of the identity in $H^1(\mathcal{R})$. Then there exist a constant $\kappa > 0$ and a finite family of $\mu$-bounded elements $\theta_1, \ldots, \theta_n \in [\mathcal{R}]$ such that

\[
\text{for all } f \in L^2(X, \mu), \quad \| f - \mu(f) \|_2 \leq \kappa \sum_{k=1}^n \| f - \theta_k(f) \|_2,
\]

and such that for all cocycles $c \in Z^1(\mathcal{R})$ with $[c] \notin \mathcal{V}$, we have

\[
\text{for all } f \in L^2(X, \mu), \quad \| f \|_2 \leq \kappa \sum_{k=1}^n \| c(\theta_k) f - \theta_k(f) \|_2.
\]

**Proof.** Using Theorem 3.5, we may choose a finite family $\theta_1, \ldots, \theta_n$ of $\mu$-bounded elements in $[\mathcal{R}]$ with a constant $\kappa > 0$ such that for all $f \in L^2(X, \mu)$ we have

\[
\| f - \mu(f) \|_2 \leq \kappa \sum_{k=1}^n \| \theta_k(f) - f \|_2,
\]

and for all $c \in Z^1(\mathcal{R})$ with $[c] \notin \mathcal{V}$ we have

\[
\frac{1}{\kappa} \leq \sum_{k=1}^n \| c(\theta_k) - 1 \|_2.
\]

Now suppose by contradiction that there exist a sequence $(f_i)_{i \in \mathbb{N}}$ in $L^2(X, \mu)$ with $\| f_i \|_2 = 1$ for all $i \in \mathbb{N}$ and a sequence of measurable 1-cocycles $c_i \in Z^1(\mathcal{R})$ with $[c_i] \notin \mathcal{V}$ for all $i \in \mathbb{N}$ such that $\lim_i \| c(\theta_k) f_i - \theta_k(f_i) \|_2 = 0$ for all $k \in \{1, \ldots, n\}$.
Then, in particular, we have \( \lim_i \| f_i - \theta_k(f_i) \|_2 = 0 \) for all \( k \in \{1, \ldots, n\} \). Therefore, we have \( \lim_i \| f_i - \mu(f_i) \|_2 = 0 \). Since \( \| f_i \|_2 = 1 \) for all \( i \in \mathbb{N} \), this means that

\[
\lim_i \| f_i - 1 \|_2 = 0.
\]

Thus, we can find a sequence \( u_i \in L^0(X, \mathbb{T}) \) such that \( \lim_i \| f_i - u_i \|_2 = 0 \). Since \( \theta_k \) is \( \mu \)-bounded for all \( k \in \{1, \ldots, n\} \), we then obtain

\[
\lim_i \| c_i(\theta_k)u_i - \theta_k(u_i) \|_2 = 0
\]

for all \( k \in \{1, \ldots, n\} \). Let \( \partial u_i \in B^1(\mathcal{R}) \) be the measurable 1-coboundary associated with \( u_i \) and let \( c_i = (\partial u_i)^{-1}c_i \in Z^1(\mathcal{R}) \). Then we have

\[
\lim_i \| c_i(\theta_k) - 1 \|_2 = 0.
\]

But, since \( \{c_i\} \neq \emptyset \), we also have

\[
\sum_{k=1}^n \| c_i(\theta_k) - 1 \|_2 \geq \frac{1}{\kappa}
\]

for all \( i \in \mathbb{N} \) and this is a contradiction. \( \square \)

**Proof of Theorem 4.1.** (i) \( \Rightarrow \) (ii). Suppose that \( \mathcal{R} \times_\Omega G \) is strongly ergodic. Since \( \mathcal{R} \times_\Omega G \) clearly surjects onto \( \mathcal{R} \), we know that \( \mathcal{R} \) is also strongly ergodic. Moreover, we have a natural embedding of topological groups \( \iota : Z^1(\mathcal{R}) \to Z^1(\mathcal{R} \times_\Omega G) \). Now we have to show that if we have a sequence \( (p_i)_{i \in \mathbb{N}} \) of elements in \( \widehat{G} \) and a sequence \( (u_i)_{i \in \mathbb{N}} \) of elements in \( L^0(X, \mathbb{T}) \) such that \( \lim_i (\partial u_i)^{-1}\widehat{\Omega}(p_i) = 1 \) in \( Z^1(\mathcal{R}) \) then \( \lim_i p_i = 1 \) in \( \widehat{G} \). By definition of the skew-product construction, it is easy to check that \( i((\partial u_i)^{-1}\widehat{\Omega}(p_i)) = \partial(u_i^{-1} \otimes p_i) \) where we view \( u_i^{-1} \otimes p_i \) as an element of \( L^0(X \times \mathbb{T}) \). Since \( i \) is an embedding of topological groups, we have \( \lim_i \partial(u_i^{-1} \otimes p_i) = 1 \) in \( Z^1(\mathcal{R} \times_\Omega G) \). Since \( \mathcal{R} \times_\Omega G \) is strongly ergodic, this means that there exists a sequence \( z_i \in \mathbb{T} \) such that \( \lim_i z_i(u_i^{-1} \otimes p_i) = 1 \) and this easily implies that \( \lim_i p_i = 1 \) in \( \widehat{G} \).

We now prove (ii) \( \Rightarrow \) (i). First we deal with the case where \( G \) is compact because we will need it for the more general case where \( G \) contains a lattice.

(ii) \( \Rightarrow \) (i) when \( G \) is compact. Assume that (ii) holds. Then, thanks to Corollary 2.5, we know that (ii) also holds for the relation \( \mathcal{R}_Y \) and the cocycle \( \Omega|_{\mathcal{R}_Y} \) where \( Y \subset X \) is any non-zero measurable subset. Moreover, by Proposition 4.2, we know that \( \mathcal{R} \times_\Omega G \) is ergodic which means that \( \mathcal{R} \times_\Omega G \) is strongly ergodic if and only if \( (\mathcal{R} \times_\Omega G)|_{Y \times G} \) is strongly ergodic (again by Corollary 2.5). This shows that it is enough to prove the desired result for \( \mathcal{R}_Y \) instead of \( \mathcal{R} \). In particular, we can assume that \( \mathcal{R} \) is either of type II \( 1 \) (in which case we choose \( \mu \) to be the unique invariant probability measure) or of type III.

Now let \( \Gamma = \widehat{G} \) be the dual discrete group. Then for every \( f \in L^2(X, \mu) \otimes L^2(G, \nu) \) we have

\[
f^\ell - f^r = \sum_{\gamma \in \Gamma} (\widehat{\Omega}(\gamma)f_\gamma^\ell - f_\gamma^r) \otimes \gamma^r.
\]

Therefore, by restricting to the graph of an element \( \theta \in [\mathcal{R}] \subset [\mathcal{R} \times_\Omega G] \) we obtain

\[
\| \theta(f) - f \|^2_2 = \sum_{\gamma \in \Gamma} \| \theta(f_\gamma) - \overline{\Omega(\gamma)(\theta)}f_\gamma \|^2_2.
\]
By assumption, we can take \( \mathcal{V} \) to be a neighborhood of the identity in \( H^1(\mathcal{R}) \) such that \( \pi_{\mathcal{R}}^{-1}(\mathcal{V}) \cap \hat{\Omega}(\Gamma) = \{1\} \). Take \( \theta_1, \ldots, \theta_n \in [\mathcal{R}] \) and \( \kappa > 0 \) as in Lemma 4.4. Since we have

\[
\sum_{k=1}^{n} \|\theta_k(f) - f\|_2^2 = \sum_{k=1}^{n} \|\theta_k(f_e) - f_e\|_2^2 + \sum_{\gamma \neq e}^{n} \sum_{k=1}^{n} \|\theta_k(f_\gamma) - \hat{\Omega}(\gamma)(\theta_k)f_\gamma\|_2^2,
\]

we obtain

\[
\kappa \sum_{k=1}^{n} \|\theta_k(f) - f\|_2^2 \geq \|f_e - \mu(f_e)\|_2^2 + \sum_{\gamma \neq e} \|f_\gamma\|_2^2 = \|f - (\mu \otimes v)(f)\|_2^2.
\]

This shows that \( \theta_1, \ldots, \theta_k \) has a spectral gap in \( \mathcal{R} \times_\Omega G \) and, in particular, that \( \mathcal{R} \times_\Omega G \) is strongly ergodic.

(ii) \( \Rightarrow \) (i) when \( G \) contains a lattice. By assumption, \( G \) contains a discrete subgroup \( H \) such that the quotient group \( K = G/H \) is compact. Let \( \Theta \in \mathbb{Z}_1(\mathcal{R}, K) \) be the measurable 1-cocycle obtained by composing \( \Omega \) with the quotient map \( G \to K \). Then \( \hat{\Theta} = \hat{\Omega} |_{\hat{\mathcal{R}}} \) so that \( \Theta \) also satisfies assumption (ii). Since \( K \) is compact, we can apply the first step and therefore we know that \( \mathcal{R} \times_\Theta K \) is strongly ergodic. Now let \( q : X \times G \to X \times K \) be the quotient map and consider the lifted equivalence relation \( Q = q^*(\mathcal{R} \times_\Theta K) \). Then \( Q \) is strongly ergodic because it is isomorphic to the product equivalence relation \( (\mathcal{R} \times_\Theta K) \times S_H \) on \( (X \times K) \times H \) where \( S_H \) is the type I transitive equivalence relation on \( H \). Moreover, \( Q \) is generated by \( \mathcal{R} \times_\Omega G \) and the orbit equivalence relation \( \mathcal{R}(H \sim X \times G) \) of the translation action \( H \sim X \times G \) on the second coordinate. By Proposition 4.3, we conclude that \( \mathcal{R} \times_\Omega G \) is strongly ergodic.

A direct application of Theorem 4.1 gives the following characterization of the strong ergodicity of the Maharam extension.

**Corollary 4.5.** (Corollary C) Let \( \mathcal{R} \) be a type III\(_1\) ergodic equivalence relation on a standard measure space \( X \). Then the Maharam extension \( c(\mathcal{R}) \) is strongly ergodic if and only if \( \mathcal{R} \) is strongly ergodic and \( \tau(\mathcal{R}) \) is the usual topology on \( \mathbb{R} \).

### 5. Almost periodic equivalence relations

Let \( \mathcal{R} \) be an ergodic equivalence relation on a standard measure space \( X \). We say that a measure \( \mu \in \mathcal{M}(X) \) is *almost periodic* for \( \mathcal{R} \) if \( \delta_\mu \) is a step function. We say that \( \mu \) is \( \Lambda \)-almost periodic for a countable subgroup \( \Lambda \subset \mathbb{R}_0^+ \) if, moreover, \( \text{Range}(\delta_\mu) \subset \Lambda \). It is easy to check that a measure \( \mu \) is almost periodic if and only if the image of the homomorphism

\[
\hat{\delta}_\mu : \mathbb{R} \to \mathbb{Z}_1(\mathcal{R}) : t \mapsto \delta_\mu^t
\]

has a compact closure. It is \( \Lambda \)-almost periodic if and only if \( \hat{\delta}_\mu \) extends to a continuous homomorphism from the compact group \( \hat{\Lambda} \) to \( \mathbb{Z}_1(\mathcal{R}) \).

We denote by \( \mathcal{M}_{\text{ap}}(X, \mathcal{R}) \) the set of all almost periodic measures on \( X \). We say that \( \mathcal{R} \) is almost periodic if \( \mathcal{M}_{\text{ap}}(X, \mathcal{R}) \neq \emptyset \).

By analogy with [Co74], we introduce the Sd invariant for ergodic equivalence relations. It is a discrete version of the S invariant.
Definition 5.1. Let $\mathcal{R}$ be an almost periodic ergodic equivalence relation on $X$. Define

$$\text{Sd}(\mathcal{R}) = \bigcap_{\mu \in \mathcal{M}_{ap}(X, \mathcal{R})} \text{Range}(\delta_\mu).$$

Observe that when $\mathcal{R}$ is ergodic and of type I or II, then $\mathcal{R}$ is almost periodic and $\text{Sd}(\mathcal{R}) = 1$. More generally, we show that $\text{Sd}(\mathcal{R}) < \mathbb{R}_0^+$ is a countable subgroup.

Proposition 5.2. Let $\mathcal{R}$ be an almost periodic ergodic equivalence relation on $X$. Then $\text{Sd}(\mathcal{R}) < \mathbb{R}_0^+$ is a countable subgroup.

Proof. We may assume that $\mathcal{R}$ is of type III. It is easy to see that $\text{Sd}(\mathcal{R})$ is stable under taking inverses. It remains to show that $\text{Sd}(\mathcal{R})$ is stable under taking products. Let $\lambda_1, \lambda_2 \in \text{Sd}(\mathcal{R})$. Fix a measure $\mu \in \mathcal{M}_{ap}(X, \mathcal{R})$. Since $\lambda_1 \in \text{Sd}(\mathcal{R})$, we have $\lambda_1 \in \text{Range}(\delta_\mu)$. Then there exists a non-zero $\theta \in [[\mathcal{R}]]$ such that $\delta_\mu(\theta(x), x) = \lambda_1$ for a.e. $x \in \text{dom}(\theta)$. Since $\mathcal{R}$ is of type III, there exists $\phi \in [[\mathcal{R}]]$ such that $\text{dom}(\phi) = \text{ran}(\theta)$ and $\text{ran}(\phi) = X$. Then $\nu = \phi_\ast \mu|_{\text{dom}(\phi)} \in \mathcal{M}_{ap}(X, \mathcal{R})$. Since $\lambda_2 \in \text{Sd}(\mathcal{R})$, we have $\lambda_2 \in \text{Range}(\delta_\nu)$. Then there exists $\psi \in [[\mathcal{R}]]$ such that $\delta_\psi(\psi(y), y) = \lambda_2$ for a.e. $y \in \text{dom}(\psi)$. Then $\varphi = \phi^{-1} \psi \phi \psi \in [[\mathcal{R}]]$ is non-zero and satisfies

$$\delta_\mu(\varphi(x), x) = \delta_\mu(\phi^{-1} \psi \phi \theta(x), \theta(x)) \delta_\mu(\theta(x), x)$$

$$= \delta_\psi(\psi \phi \theta(x), \phi \theta(x)) \delta_\mu(\theta(x), x)$$

$$= \lambda_2 \lambda_1$$

for a.e. $x \in \text{dom}(\theta)$. This shows that $\lambda_2 \lambda_1 \in \text{Range}(\delta_\mu)$. Since this holds true for every $\mu \in \mathcal{M}_{ap}(X, \mathcal{R})$, we obtain that $\lambda_2 \lambda_1 \in \text{Sd}(\mathcal{R})$. \hfill $\Box$

We obtain an analogue of [Co74, Theorem 4.1].

Theorem 5.3. Let $\mathcal{R}$ be an almost periodic ergodic equivalence relation on $X$. Assume that $\mathcal{R}$ is strongly ergodic. Then, for any $\mu \in \mathcal{M}_{ap}(X, \mathcal{R})$, we have

$$\text{Sd}(\mathcal{R}) = \bigcap_{U \in \mathcal{P}(X), U \neq \emptyset} \text{Range}(\delta_\mu |_{\mathcal{R}_U}),$$

where $\mathcal{R}_U = \mathcal{R} \cap (U \times U)$ is the restricted equivalence relation.

Before proving Theorem 5.3, we need some preparation.

Lemma 5.4. Let $\Lambda < \mathbb{R}_0^+$ be any countable subgroup. Let $\nu_1, \nu_2 \in \mathcal{M}_{ap}(X, \mathcal{R})$ be any $\Lambda$-almost periodic measures. Then there exists $h \in L^0_1(X, \Lambda)$ such that

$$\delta_{\nu_2} \delta_{\nu_1}^{-1} = \partial h \in B_1(\mathcal{R}, \Lambda).$$

Proof. Without loss of generality, we may assume that $\Lambda \neq 1$. Regarding $\Lambda$ as a discrete group, we denote by $G = \hat{\Gamma}$ the Pontryagin dual of $\Gamma$. Then $G$ is a second countable compact abelian group and we regard $\mathbb{R} = \mathbb{R}_0^+ \times \hat{\Gamma} = G$ as a dense subgroup. Let $\nu \in \mathcal{M}_{ap}(X, \mathcal{R})$ be any $\Lambda$-almost periodic measure. Then the map $\delta_\nu : \mathbb{R} \to Z_1^0(\mathcal{R}) : t \mapsto \delta_\mu^t$ uniquely extends to a continuous group homomorphism $\Delta_\nu : G \to Z_1^0(\mathcal{R})$ such that $\Delta_\nu(t) = \delta_\mu^t$ for every $t \in \mathbb{R}$. 
For almost every \((x, y) \in \mathcal{R}\), we have 
\[
\delta_{v_2}(x, y) = \frac{dv_2}{dv_2, r}(x, y) = \frac{dv_2}{dv_1}(x)\frac{dv_1}{dv_1, r}(x, y) \frac{dv_1}{dv_2}(y) = \frac{dv_2}{dv_1}(x)\delta_{v_1}(x, y) \frac{dv_1}{dv_2}(y).
\]
This implies that \(\Delta_{v_2}(t)/\Delta_{v_1}(t) = \delta_{v_2}^t/\delta_{v_1}^t \in \mathcal{B}^1(\mathcal{R})\) for every \(t \in \mathbb{R}\). By density and since \(\mathcal{B}^1(\mathcal{R}) \subset \mathcal{Z}^1(\mathcal{R})\) is closed, it follows that \(\Delta_{v_2}(g)/\Delta_{v_1}(g) \in \mathcal{B}^1(\mathcal{R})\) for every \(g \in G\).

Since \(\mathcal{T} \subset L^0(\Lambda, \mathcal{T})\) is a closed subgroup and since \(\mathcal{B}^1(\mathcal{R}) = L^0(\Lambda, \mathcal{T})/\mathcal{T}\), there exist a Borel map \(w : G \to L^0(\Lambda, \mathcal{T})\) such that \(\Delta_{v_2}(g)/\Delta_{v_1}(g) = \partial(w(g))\) for every \(g \in G\) and a Borel map \(\sigma : G \times G \to \mathcal{T}\) such that \(\sigma \in Z^2_m(G, \mathcal{T})\) and \(w(gh) = \sigma(g, h)w(g)w(h)\) for all \(g, h \in G\), where \(Z^2_m(G, \mathcal{T})\) denotes the group of all measurable scalar 2-cocycles on \(G\). By [AM10, Proof of Proposition 5.8], we have \(Z^2_m(G, \mathcal{T}) = B^2_m(G, \mathcal{T})\) and hence there exists a measurable map \(\nu : G \to \mathcal{T}\) such that \(\sigma(g, h) = \nu(g)\nu(h)\nu(gh)\) for all \(g, h \in G\) (see [Mo75, Theorem 5] for the fact that we may choose \(\nu : G \to \mathcal{T}\) so that the 2-coboundary relation holds everywhere). Define \(u : G \to L^0(\mathcal{T})\) : \(g \mapsto \nu(g)w(g)\). Then \(u : G \to L^0(\Lambda, \mathcal{T})\) is a measurable group homomorphism and hence is continuous. Moreover, we have \(\Delta_{v_2}(g)/\Delta_{v_1}(g) = \partial(w(g)) = \partial(u(g))\) for every \(g \in G\).

Since \(\hat{\mathcal{G}} = \Lambda\), we may regard the continuous group homomorphism \(u : G \to L^0(\Lambda, \mathcal{T})\) as a measurable map \(h : X \to \Lambda\) such that \(\Delta_{v_2}(g)/\Delta_{v_1}(g) = \partial(u(g)) = \langle \partial h, g \rangle\) for every \(g \in G\). This implies that \(\delta_{v_2}/\delta_{v_1} = \partial h \in \mathcal{B}^1(\mathcal{R}, \Lambda)\). 

\[\text{Proof of Theorem 5.3.} \quad \text{Let } \nu \in M_{ap}(X, \mathcal{R}) \text{ be any measure and } U \in \mathcal{P}(X), U \neq \emptyset. \text{ If } \mathcal{R} \text{ is of type I or II, we have } Sd(\mathcal{R}) = 1 \subset \text{Range}(\delta_{v|\mathcal{R}_U}). \text{ If } \mathcal{R} \text{ is of type III, there exists } \theta \in [\mathcal{R}] \text{ such that } \text{dom}(\theta) = U \text{ and } \text{ran}(\theta) = X. \text{ Then } \theta_*\nu \in M_{ap}(X, \mathcal{R}) \text{ and } Sd(\mathcal{R}) \subset \text{Range}(\delta_{\theta_*\nu}) = \text{Range}(\delta_{v|\mathcal{R}_U}). \text{ This shows that in all cases, we have}
\]
\[
\text{Sd}(\mathcal{R}) \subset \bigcap_{U \in \mathcal{P}(X), U \neq \emptyset} \text{Range}(\delta_{v|\mathcal{R}_U}).
\]
On the other hand, we have
\[
\bigcap_{v \in M_{ap}(X, \mathcal{R})} \bigcap_{U \in \mathcal{P}(X), U \neq \emptyset} \text{Range}(\delta_{v|\mathcal{R}_U}) \subset \text{Sd}(\mathcal{R}).
\]
Therefore, in order to show the reverse inclusion of (5.1), it suffices to prove that for any measures \(v_1, v_2 \in M_{ap}(X, \mathcal{R})\), we have
\[
\Lambda_1 := \bigcap_{U \in \mathcal{P}(X), U \neq \emptyset} \text{Range}(\delta_{v_1|\mathcal{R}_U}) = \bigcap_{U \in \mathcal{P}(X), U \neq \emptyset} \text{Range}(\delta_{v_2|\mathcal{R}_U}) =: \Lambda_2.
\]
By contradiction, assume that \(\Lambda_1 \neq \Lambda_2\). Without loss of generality, we may assume that there exists \(\lambda \in \Lambda_1 \setminus \Lambda_2\). By Lemma 5.4, there exists \(h \in L^0(\Lambda, \Lambda)\) such that \(\delta_{v_1}/\delta_{v_1} = \partial h\). Since \(\lambda \notin \Lambda_2\), there exists \(U \in \mathcal{P}(X)\) such that \(\lambda \notin \text{Range}(\delta_{v_2|\mathcal{R}_U})\) and, up to shrinking \(U \in \mathcal{P}(X)\) if necessary, we may assume that \(h|_U\) is constant. This means that \(\delta_{v_1|\mathcal{R}_U} = \delta_{v_2|\mathcal{R}_U}\). Since \(\lambda \in \Lambda_1\), we have \(\lambda \in \text{Range}(\delta_{v_1|\mathcal{R}_U})\). This, however, contradicts the fact that \(\lambda \notin \text{Range}(\delta_{v_2|\mathcal{R}_U})\). 

\[\text{We next obtain an analogue of [Co74, Theorem 4.7].}\]
THEOREM 5.5. Let $\mathcal{R}$ be an almost periodic strongly ergodic equivalence relation on $X$. Put $\Gamma = \text{Sd}(\mathcal{R})$. Then there exists a measure $\nu \in \mathcal{M}_{ap}(X, \mathcal{R})$ that is $\Gamma$-almost periodic. Moreover, for any $\Gamma$-almost periodic infinite measures $\nu_1, \nu_2 \in \mathcal{M}_{ap}(X, \mathcal{R})$, there exist $\alpha \in \mathbb{R}_0^+$ and $\theta \in [\mathcal{R}]$ such that $\theta \nu_1 = \alpha \nu_2$.

Before proving Theorem 5.5, we need some preparation.

LEMMA 5.6. Let $\mathcal{R}$ be an almost periodic strongly ergodic equivalence relation on $X$. Put $\Gamma = \text{Sd}(\mathcal{R})$. Let $\nu \in \mathcal{M}_{ap}(X, \mathcal{R})$ be any measure. The following assertions are equivalent.

(i) $\nu$ is $\Gamma$-almost periodic.

(ii) The subequivalence relation $\mathcal{R}_\nu \subset \mathcal{R}$ defined by $\mathcal{R}_\nu := \ker(\delta_\nu)$ is ergodic.

Proof. (i) $\Rightarrow$ (ii). By Theorem 5.3, we have $\Gamma = \text{Sd}(\mathcal{R}) \subset \text{Range}(\delta_\nu|_{\mathcal{R}_\nu})$ for every non-zero $U \in \mathcal{P}(X)$. Since $\nu$ is $\Gamma$-almost periodic, we also have $\text{Range}(\delta_\nu|_{\mathcal{R}_\nu}) \subset \Gamma$ for every non-zero $U \in \mathcal{P}(X)$. This implies that $\text{Range}(\delta_\nu|_{\mathcal{R}_\nu}) = \Gamma$ for every non-zero $U \in \mathcal{P}(X)$. In order to show that $\mathcal{R}_\nu$ is ergodic, it suffices to show that for any non-zero $U$, $V \in \mathcal{P}(X)$, there exists a non-zero element $\theta \in [\mathcal{R}_\nu]$ such that $\text{dom}(\theta) \subset U$ and $\text{ran}(\theta) \subset V$. Since $\mathcal{R}$ is ergodic, we can find a non-zero $\phi \in [[\mathcal{R}]]$ such that $\text{dom}(\phi) \subset U$, $\text{ran}(\phi) \subset V$ and $\delta_\nu(\phi(x), y) = \lambda \in \Gamma$ for a.e. $y \in \text{dom}(\phi)$. Since $\lambda^{-1} \in \mathcal{R}$, we can find a non-zero $\psi \in [[\mathcal{R}]]$ such that $\text{dom}(\psi) \subset \text{dom}(\phi)$, $\text{ran}(\psi) \subset \text{dom}(\phi)$ and $\delta_\nu(\psi(x), x) = \lambda^{-1}$ for a.e. $x \in \text{dom}(\psi)$. Then $\theta = \phi \psi$ is a non-zero element of $[[\mathcal{R}_\nu]]$ such that $\text{dom}(\theta) \subset U$ and $\text{ran}(\theta) \subset V$. This shows that $\mathcal{R}_\nu$ is ergodic.

(ii) $\Rightarrow$ (i). By Theorem 5.3, we know that

$$\Gamma = \bigcap_{U \in \mathcal{P}(X), U \neq \emptyset} \text{Range}(\delta_\nu|_{\mathcal{R}_\nu}).$$

Hence, in order to show that $\nu$ is $\Gamma$-almost periodic, it suffices to prove that $\text{Range}(\delta_\nu|_{\mathcal{R}_\nu}) = \text{Range}(\delta_\nu|_{\mathcal{R}_\nu})$ for every non-zero $U \in \mathcal{P}(X)$. First, we always have $\text{Range}(\delta_\nu|_{\mathcal{R}_\nu}) \subset \text{Range}(\delta_\nu)$. Next, let $\lambda \in \text{Range}(\delta_\nu)$. Then there exists a non-zero $\phi \in [[\mathcal{R}]]$ such that $\delta_\nu(\phi(x), x) = \lambda$ for a.e. $x \in \text{dom}(\phi)$. Since $\mathcal{R}_\nu$ is ergodic, we can find a non-zero $\psi \in [[\mathcal{R}_\nu]]$ such that $\text{dom}(\psi_1) \subset U$ and $\text{ran}(\psi_1) \subset \text{dom}(\phi)$ and a non-zero $\psi_2 \in [[\mathcal{R}]]$ such that $\text{dom}(\psi_2) \subset \phi(\text{ran}(\psi_1))$ and $\text{ran}(\psi_2) \subset U$. Then $\theta = \phi \psi_1 \psi_2$ is a non-zero element of $[[\mathcal{R}_\nu]]$ such that $\delta_\nu(\theta(x), x) = \lambda$ for a.e. $x \in \text{dom}(\theta)$. This shows that $\lambda \in \text{Range}(\delta_\nu|_{\mathcal{R}_\nu})$. Therefore, $\text{Range}(\delta_\nu|_{\mathcal{R}_\nu}) = \text{Range}(\delta_\nu)$ and hence $\text{Range}(\delta_\nu) = \Gamma$.

Proof of Theorem 5.5. First, we prove that there exists a measure $\nu \in \mathcal{M}_{ap}(X, \mathcal{R})$ that is $\Gamma$-almost periodic. We may assume without loss of generality that $\mathcal{R}$ is of type III. Let $\eta \in \mathcal{M}_{ap}(X, \mathcal{R})$ be any $\Lambda$-almost periodic measure for some countable subgroup $\Lambda < \mathbb{R}_0^+$. By viewing $\delta_\eta$ as an element of $Z^1(\mathcal{R}, \Lambda)$, we can consider the skew-product equivalence relation $d(\mathcal{R}) = \mathcal{R} \times \delta_\eta \Lambda$ on $X \times \Lambda$. Observe that the translation action $\Lambda \acts X \times \Lambda$ acts by automorphisms of the equivalence relation $d(\mathcal{R})$. Moreover, the equivalence relation generated by $d(\mathcal{R})$ and $\mathcal{R}(X \acts X \times \Lambda)$ coincides with the equivalence relation $\mathcal{S}$ defined by

$$((x, g), (y, h)) \in S \quad \text{if and only if} \quad (x, y) \in \mathcal{R},$$

for a.e. $(x, y) \in \mathcal{R}$ and all $g, h \in \Lambda$. Observe that $\mathcal{S}$ is nothing but an amplification of $\mathcal{R}$ and hence $\mathcal{S}$ is strongly ergodic.
We show that $d(\mathcal{R})$ has a completely atomic ergodic decomposition. Since the translation action $\Lambda \acts X \times \Lambda$ acts by automorphisms of the equivalence relation $d(\mathcal{R})$, it induces an action on $L^\infty(X \times \Lambda)$ which globally preserves the algebra of $d(\mathcal{R})$-invariant functions $L^\infty(X \times \Lambda)^{d(\mathcal{R})}$. Since $S$ is ergodic, the action $\Lambda \acts L^\infty(X \times \Lambda)^{d(\mathcal{R})}$ is ergodic. So $L^\infty(X \times \Lambda)^{d(\mathcal{R})}$ is either discrete (completely atomic) or diffuse. Assume by contradiction that it is diffuse. Since $\Lambda$ is abelian, hence amenable, the action $\Lambda \acts L^\infty(X \times \Lambda)^{d(\mathcal{R})}$ is not strongly ergodic (see, for example, [Sc79, Proposition 2.2]). Hence, we can find a non-trivial almost $\Lambda$-invariant sequence in $L^\infty(X \times \Lambda)^{d(\mathcal{R})}$. But this provides a non-trivial almost $S$-invariant sequence in $L^\infty(X \times \Lambda)$. This, however, contradicts the fact that $S$ is strongly ergodic.

Since $\mathcal{R}_\eta \cong d(\mathcal{R})_{X \times \{1\}}$, it follows that $\mathcal{R}_\eta$ also has a completely atomic ergodic decomposition. Let $U \in \mathcal{F}(X)$ be any non-zero subset such that $\mathcal{R}_\eta \cap (U \times U)$ is ergodic. Since $\mathcal{R}$ is of type III, there exists $\theta \in [[\mathcal{R}]]$ such that $\ker(\theta) = U$ and ran$(\theta) = X$. Put $v = \theta_\eta \in \mathcal{M}_{ap}(X, \mathcal{R})$. Then $\mathcal{R}_\nu$ is ergodic and hence $v$ is $\Gamma$-almost periodic by Lemma 5.6.

Secondly, let $v_1, v_2 \in \mathcal{M}_{ap}(X, \mathcal{R})$ be any $\Gamma$-almost periodic infinite measures. By Lemma 5.4, there exists $h \in L^0(X, \Gamma)$ such that $\delta_v = \partial h$. For almost every $(x, y) \in \mathcal{R}$, we have

$$h(x)h(y)^{-1} = \frac{\delta_v(x, y)}{\delta_{v_1}(x, y)} = \frac{dv_2}{dv_1}(x) \frac{dv_1}{dv_2}(y)$$

and hence

$$h(x) \frac{dv_1}{dv_2}(x) = h(y) \frac{dv_1}{dv_2}(y).$$

Since $\mathcal{R}$ is ergodic, it follows that the function $h(dv_1/dv_2)$ is constant and equal to some $\alpha \in \mathbb{R}_0^+$. Consider the amplified equivalence relation $\mathcal{T} = \mathcal{R} \otimes S_2$ on $X \times \{0, 1\}$. Since $\mathcal{T}$ is isomorphic to $\mathcal{R}$, we know that $\mathcal{T}$ is strongly ergodic, almost periodic and that $Sd(\mathcal{T}) = Sd(\mathcal{R}) = \Gamma$. Define the measure $\nu = v_1 \otimes \delta_0 + \alpha v_2 \otimes \delta_1$ on $X \times \{0, 1\}$. For almost every $(x, y) \in \mathcal{R}$, we have $\delta_v(x, 0, (y, 0)) = \delta_{v_1}(x, y), \delta_v(x, 1, (y, 1)) = \delta_{v_2}(x, y)$ and $\delta_v(x, 0, (y, 1)) = \delta_{v_1}(x, 0, (y, 1))\delta_{v_2}(x, 1, (y, 1)) = h(x)^{-1}\delta_{v_1}(x, y)$. This implies that $\nu$ is $\Gamma$-almost periodic. Hence, by Lemma 5.6, $\mathcal{T}_\nu$ is ergodic. Since $\nu(X \times \{0\}) = v_1(X) = +\infty = \alpha v_2(X) = \nu(X \times \{1\})$, there exists a partial isomorphism $\Theta \in [[\mathcal{T}_\nu]]$ such that $\ker(\Theta) = X \times \{0\}$ and ran$(\Theta) = X \times \{1\}$. Define $\theta \in [\mathcal{R}]$ by the formula $(\theta(x), 1) = \Theta(x, 0)$ for a.e. $x \in X$. Then $\alpha v_2 = \theta_\eta v_1$. \hfill \Box

**Theorem 5.7.** Let $\mathcal{R}$ be an almost periodic strongly ergodic equivalence relation on $X$. Put $\Gamma = Sd(\mathcal{R})$. Then for any $\Gamma$-almost periodic measure $\nu \in \mathcal{M}_{ap}(X, \mathcal{R})$, the subequivalence relation $\mathcal{R}_\nu \subset \mathcal{R}$ defined by $\ker(\delta_v)$ is strongly ergodic.

**Proof.** As in the first part of the proof of Theorem 5.5, consider the skew-product equivalence relation $d(\mathcal{R}) = \mathcal{R} \times \delta$, $\Gamma$ on $X \times \Gamma$. The equivalence relation generated by $d(\mathcal{R})$ and $\mathcal{R}(\Gamma \acts X \times \Gamma)$ coincides with the equivalence relation $S$ defined by

$$(x, g), (y, h) \in S \iff (x, y) \in \mathcal{R}$$

for a.e. $(x, y) \in \mathcal{R}$ and all $g, h \in \Gamma$. Observe that $S$ is nothing but an amplification of $\mathcal{R}$ and hence $S$ is strongly ergodic. By Lemma 5.6, we know that $d(\mathcal{R})_{X \times \{1\}} \cong \mathcal{R}_\nu$ is ergodic.
Since \( \Gamma \curvearrowright X \times \Gamma \) preserves \( d(\mathcal{R}) \), we have that \( d(\mathcal{R})_{X \times \{\gamma\}} \) is ergodic for every \( \gamma \in \Gamma \). It follows that the ergodic component of \( X \times \{1\} \) is of the form \( X \times S \) for some subset \( S \subset \Gamma \). But, by definition of \( d(\mathcal{R}) \), this means that \( S = \text{Range}(\delta_\nu) = \Gamma \). Hence \( d(\mathcal{R}) \) is ergodic. We conclude, by Proposition 4.3, that \( d(\mathcal{R}) \) is strongly ergodic. Hence \( d(\mathcal{R})_{X \times \{1\}} \cong \mathcal{R}_\nu \) is also strongly ergodic. □

Proof of Theorem D. (iii) and (v) follow from Theorem 5.5 and (iv) follows from Theorem 5.7.

(i) Choose \( \nu \in \mathcal{M}_{ap}(X, \mathcal{R}) \) as in the first part of Theorem 5.5. Then \( \mathcal{R}_\nu \) is ergodic by Lemma 5.6. Hence we have \( S(\mathcal{R}) = \overline{\text{Range}(\delta_\nu)} \). We also have \( \text{Sd}(\mathcal{R}) = \text{Range}(\delta_\nu) \). Hence \( S(\mathcal{R}) = \text{Sd}(\mathcal{R}) \).

(ii) Choose \( \nu \in \mathcal{M}_{ap}(X, \mathcal{R}) \) as in the first part of Theorem 5.5. Let \( (t_n)_{n \in \mathbb{N}} \) be any sequence in \( \mathbb{R} \). First, assume that \( \gamma^{t_n} \to 1 \) for every \( \gamma \in \text{Sd}(\mathcal{R}) \). Since \( \text{Range}(\delta_\nu) = \text{Sd}(\mathcal{R}) \), we then have \( \delta^{t_n}_\nu \to 1 \) in \( Z^1(\mathcal{R}) \) for the convergence in measure. Conversely, assume that \( t_n \to 0 \) with respect to \( \tau(\mathcal{R}) \). Then there exists a sequence \( u_n \in L^0(X, \mathcal{T}) \) such that \( (\partial t_n) \delta^{t_n}_\nu \to 1 \) in \( Z^1(\mathcal{R}) \) for the convergence in measure. Then we have that \( \partial u_n \to 1 \) in \( Z^1(\mathcal{R}_\nu) \) for the convergence in measure. Since \( \mathcal{R}_\nu \) is strongly ergodic by Theorem 5.7, there exists a sequence \( z_n \in \mathcal{T} \) such that \( z_n u_n \to 1 \) in \( L^0(X, \mathcal{T}) \) for the convergence in measure. This implies that \( \delta^{t_n}_\nu \to 1 \) in \( Z^1(\mathcal{R}) \) for the convergence in measure. Since \( \text{Sd}(\mathcal{R}) = \text{Range}(\delta_\nu) \), this further implies that \( \gamma^{t_n} \to 1 \) for every \( \gamma \in \text{Sd}(\mathcal{R}) \). □

6. Explicit computations of \( \text{Sd} \) and \( \tau \) invariants

6.1. Equivalence relations arising from actions of bi-exact groups. Following [BO08, Definition 15.1.2], we say that a discrete group \( \Gamma \) is bi-exact if \( \Gamma \) is exact and if there exists a map \( \mu : \Gamma \to \text{Prob}(\Gamma) \) such that \( \lim_{x \to \infty} \| \mu(gxh) - g_*\mu(x) \| = 0 \) for all \( g, h \in \Gamma \). The class of bi-exact discrete groups includes amenable groups, free groups [AO74], discrete subgroups of simple connected Lie groups of real rank one [Sk88], Gromov word-hyperbolic groups [Oz03], wreath product groups \( H \wr \Lambda \) where \( H \) is amenable and \( \Lambda \) is bi-exact [Oz04] and the group \( \mathbb{Z}^2 \rtimes \text{SL}_2(\mathbb{Z}) \) [Oz08]. We refer the reader to [BO08, Chapter 15] for more information on bi-exact discrete groups.

It was shown in [HI15, Theorem C] that for any bi-exact countable discrete group \( \Gamma \) and any strongly ergodic free non-singular action on a standard measure space \( \Gamma \curvearrowright (X, \mu) \), the group measure space factor \( L^\infty(X) \rtimes \Gamma \) is full. The following rigidity result shows that in that case, the \( \text{Sd} \) and \( \tau \) invariants of the orbit equivalence relation \( \mathcal{R}(\Gamma \curvearrowright X) \) coincide with those of the corresponding factor \( L(\mathcal{R}(\Gamma \curvearrowright X)) = L^\infty(X) \rtimes \Gamma \).

**THEOREM 6.1. (Theorem E)** Let \( \Gamma \) be any bi-exact countable discrete group and \( \Gamma \curvearrowright X \) any strongly ergodic free action on a standard measure space. Then \( L(\mathcal{R}(\Gamma \curvearrowright X)) \) is a full factor and

\[
\tau(\mathcal{R}(\Gamma \curvearrowright X)) = \tau(L(\mathcal{R}(\Gamma \curvearrowright X))).
\]

If, moreover, \( \mathcal{R}(\Gamma \curvearrowright X) \) is almost periodic, then \( L(\mathcal{R}(\Gamma \curvearrowright X)) \) is almost periodic and

\[
\text{Sd}(\mathcal{R}(\Gamma \curvearrowright X)) = \text{Sd}(L(\mathcal{R}(\Gamma \curvearrowright X))).
\]
Proof. Write \( \mathcal{R} = \mathcal{R}(\Gamma \curvearrowright X) \), \( A = L^\infty(X) \) and \( M = L(\mathcal{R}) \). Denote by \( E_A : M \to A \) the unique faithful normal conditional expectation and put \( \varphi = \tau_\mu \circ E_A \in M_* \) where \( \mu \in \mathcal{M}(X) \) is any probability measure. By [HI15, Theorem C], \( M = L^\infty(X) \times \Gamma \) is a full factor.

We first prove that \( \tau(\mathcal{R}) = \tau(M) \). Let \( (t_n)_n \) be any sequence in \( \mathbb{R} \). It is clear that if \( t_n \to 0 \) with respect to \( \tau(\mathcal{R}) \) then \( t_n \to 0 \) with respect to \( \tau(M) \). Conversely, assume that \( t_n \to 0 \) with respect to \( \tau(M) \). Then there exists a sequence of unitaries \( (u_n)_n \in \mathcal{U}(M) \) such that \( \text{Ad}(u_n) \circ \sigma_{t_n}^\varphi \to \text{id}_M \) with respect to the \( u \)-topology in \( \text{Aut}(M) \). Write \( c(M) = M \rtimes_\alpha \mathbb{R} \) for the continuous core of \( M \). Observe that \( c(M) = L^\infty(X \times \mathbb{R}) \rtimes \Gamma \) where \( \Gamma \curvearrowright X \times \mathbb{R} \) is the Maharam extension of \( \Gamma \curvearrowright X \). Denote by \( \mathcal{E}_{L^\infty(X \times \mathbb{R})} : c(M) \to L^\infty(X \times \mathbb{R}) \) the unique faithful normal conditional expectation. Denote by \( (\lambda_{\varphi}(t))_{t \in \mathbb{R}} \) the canonical unitaries in \( c(M) \) implementing the modular flow \( \sigma^\varphi \). By [Ta03a, Lemma XII.6.14], we have that

\[
\text{Ad}(u_n \lambda_{\varphi}(t_n)) \to \text{id}_{c(M)}
\]

with respect to the \( u \)-topology in \( \text{Aut}(c(M)) \). Since \( M \) is a non-amenable factor, \( c(M) \) has no amenable direct summand (see, for example, [BHR12, Proposition 2.8]). Then [HI15, Theorem A] implies that \( \mathcal{E}_{L^\infty(X \times \mathbb{R})}((u_n \lambda_{\varphi}(t_n)) - u_n \lambda_{\varphi}(t_n)) \to 0 \) with respect to the \( \ast \)-strong topology. Since \( L^\infty(X \times \mathbb{R}) = A \overline{\otimes} L(\mathbb{R}) \) (with \( \overline{\otimes} = \mathbb{R} \)) and \( \lambda_{\varphi}(t_n) \in \mathcal{U}(L(\mathbb{R})) \) for every \( n \in \mathbb{N} \), it follows that \( E_A(u_n) - u_n \to 0 \) with respect to the \( \ast \)-strong topology. We can then find a sequence of unitaries \( (v_n)_n \in \mathcal{U}(A) \) such that \( v_n - u_n \to 0 \) with respect to the \( \ast \)-strong topology. This further implies that \( \text{Ad}(v_n) \circ \sigma_{t_n}^\varphi \to \text{id}_M \) with respect to the \( u \)-topology in \( \text{Aut}(M) \). Since \( \sigma_{t_n}^\varphi|_A = \text{id}_A \) for every \( t \in \mathbb{R} \), we infer that \( t_n \to 0 \) with respect to \( \tau(\mathcal{R}) \) by Lemma 2.7. Therefore, we have \( \tau(\mathcal{R}) = \tau(M) \).

Next, assume that \( \mathcal{R} \) is almost periodic. Then \( M \) is almost periodic as well. It is clear that \( \text{Sd}(M) \subset \text{Sd}(\mathcal{R}) \). By Theorem 5.5, choose a probability measure \( \nu \in \mathcal{M}_\text{ap}(X, \mathcal{R}) \) that is \( \text{Sd}(\mathcal{R}) \)-almost periodic. Put \( \psi = \tau_\nu \circ E_A \in M_* \). By Lemma 5.6, \( \mathcal{R}_\psi \) is ergodic and hence \( M_\psi = L(\mathcal{R}_\psi) \) is a factor. Then [Co74, Lemma 4.8] implies that the point spectrum of \( \Delta_\psi \) coincides with \( \text{Sd}(M) \). Since \( \psi = \tau_\nu \circ E_A \) and since \( \text{Range}(\delta_\nu) = \text{Sd}(\mathcal{R}) \), it follows that the point spectrum of \( \Delta_\psi \) is \( \text{Sd}(\mathcal{R}) \). Therefore, we have \( \text{Sd}(\mathcal{R}) = \text{Sd}(M) \).

\[ \square \]

6.2. Generalized Bernoulli equivalence relations. Let \( \Lambda \) be any countable discrete group, \( I \) any non-empty countable set and \( \Lambda \curvearrowright I \) any action. Let \( (Y, \eta) \) be any non-trivial standard probability space and \( S \) any equivalence relation defined on \( (Y, \eta) \). Define the product standard probability space \( (X, \mu) = (Y, \eta)^I \) and the product equivalence relation \( S^\otimes I \) on \( (X, \mu) \) by

\[
((x_i)_{i \in I}, (y_i)_{i \in I}) \in S^\otimes I \iff \text{there exists } F \subset I \text{ finite subset such that}
\begin{align*}
\text{for all } i \in F, & \quad (x_i, y_i) \in S \\
\text{for all } i \in I \setminus F, & \quad x_i = y_i.
\end{align*}
\]

Define the generalized Bernoulli action \( \Lambda \curvearrowright X \) by \( \lambda \cdot (x_i)_{i \in I} = (x_{\lambda^{-1}i})_{i \in I} \) and denote by \( \mathcal{R}(\Lambda \curvearrowright X) \) the corresponding orbit equivalence relation.

Definition 6.2. We define the generalized Bernoulli equivalence relation \( \mathcal{R} = S^\otimes I \times \Lambda \) on \( (X, \mu) \) by

\[
((x_i)_{i \in I}, (y_i)_{i \in I}) \in \mathcal{R} \iff \text{there exists } \lambda \in \Lambda \text{ such that } (\lambda \cdot (x_i)_{i \in I}, (y_i)_{i \in I}) \in S^\otimes I.
\]

In other words, \( \mathcal{R} \) is the equivalence relation generated by \( S^\otimes I \) and \( \mathcal{R}(\Lambda \curvearrowright X) \).
From now on we assume that the action $\Lambda \rtimes I$ has no invariant mean. Then the generalized Bernoulli action $\Lambda \rtimes (X, \mu)$ has a spectral gap and therefore is strongly ergodic (see, for example, [KT06, Theorem 1.2]). Since $R(\Lambda \rtimes X) \subset R_\mu \subset R$, the generalized Bernoulli equivalence relation $R$ is strongly ergodic as well as $R_\mu := \ker(\delta_\mu)$.

In this subsection, we first show that almost periodic generalized Bernoulli equivalence relations can have prescribed $Sd$ invariant. Let $\Gamma < \mathbb{R}_0^+$ be any non-trivial countable subgroup. Take an enumeration $\Gamma \cap (0, 1) = \{\gamma_n \mid n \geq 1\}$. Consider $Y = \bigsqcup_{n \geq 1} (0, 1]$ and $\eta$ to be the probability measure on $Y$ defined by $\eta = \sum_{n \geq 1} (1/2^n (1 + \gamma_n)) (\delta^{(n)}_0 + \gamma_n \delta^{(n)}_1)$. Denote by $S_2$ the transitive equivalence relation defined on $\{0, 1\}$ and consider the equivalence relation $S = \bigsqcup_{n \geq 1} S^{(n)}_2$ on the standard measure space $(Y, \eta)$.

**Theorem 6.3.** Keep the same setup as above. Then the generalized Bernoulli equivalence relation $R = S^{\otimes I} \rtimes \Lambda$ is almost periodic, strongly ergodic and $\text{Sd}(R) = \Gamma$.

**Proof.** It is clear that the product measure $\mu$ is almost periodic for $R$ and that $\text{Range}(\delta_\mu) = \Gamma$. Since $R_\mu$ is strongly ergodic, Lemma 5.6 shows that $\text{Sd}(R) = \text{Range}(\delta_\mu) = \Gamma$. $\square$

We next show that generalized Bernoulli equivalence relations can have prescribed $\tau$ invariant. Following [Co74, §5], let $\xi$ be any non-zero finite Borel measure on $\mathbb{R}_0^+$ with a finite first moment, that is, $\int_{\mathbb{R}_0^+} x \, d\xi(x) < +\infty$. We normalize $\xi$ so that $\int_{\mathbb{R}_0^+} (1 + x) \, d\xi(x) = 1$. Consider the unitary representation $\rho_\xi : \mathbb{R} \to \mathcal{U}(L^2(\mathbb{R}_0^+, \xi))$ defined by

$$(\rho_\xi(t) f)(x) = x^t f(x)$$

for all $t \in \mathbb{R}$, $x \in \mathbb{R}_0^+$ and $f \in L^2(\mathbb{R}_0^+, \xi)$. Denote by $\tau(\xi)$ the weakest topology on $\mathbb{R}$ that makes $\rho_\xi$ continuous. Consider $Y = \mathbb{R}_0^+ \times \{0, 1\}$ and $\eta$ the unique probability measure on $Y$ that satisfies

$$\int_Y f(x, i) \, d\eta(x, i) = \int_{\mathbb{R}_0^+} f(x, 0) \, d\xi(x) + \int_{\mathbb{R}_0^+} x f(x, 1) \, d\xi(x)$$

for any non-negative Borel function $f : Y \to \mathbb{R}_0^+$. Denote by $S$ the equivalence relation defined on $(Y, \eta)$ by

$$(x_1, i_1), (x_2, i_2)) \in S \iff x_1 = x_2.$$

**Theorem 6.4.** Keep the same setup as above. Then the generalized Bernoulli equivalence relation $R = S^{\otimes I} \rtimes \Lambda$ is strongly ergodic and $\tau(R) = \tau(\xi)$.

**Proof.** For every $x \in \mathbb{R}_0^+$, we have $\delta_\eta((x, 1), (x, 0)) = x$. This implies that the weakest topology on $\mathbb{R}$ that makes the map $\mathbb{R} \to Z^1(S) : t \mapsto \delta_\eta^t$ continuous is $\tau(\xi)$. By construction of $R = S^{\otimes I} \rtimes \Lambda$, it follows that the weakest topology on $\mathbb{R}$ that makes the map $\mathbb{R} \to Z^1(R) : t \mapsto \delta_\mu^t$ continuous is $\tau(\xi)$ as well. In order to prove that $\tau(R) = \tau(\xi)$, it suffices to show that for any sequence $(t_n)_n$ in $\mathbb{R}$ such that $t_n \to 0$ with respect to $\tau(R)$, we have $\delta_\eta^{t_n} \to 1$ in $Z^1(S)$ for the convergence in measure. Assume that $t_n \to 0$ with respect to $\tau(R)$. Then there exists a sequence $u_n \in L^0(X, \mathbb{T})$ such that $(\partial u_n)^{\delta_\mu^{t_n}} \to 1$ in $Z^1(R)$ for the convergence in measure. In particular, we have $\partial u_n \to 1$ in $Z^1(R_\mu)$ for the convergence in measure. Since $R_\mu$ is strongly ergodic, there exists a sequence $z_n \in \mathbb{T}$
such that $z_n c_n \to 1$ in $L^0(\mathcal{R})$ for the convergence in measure. This finally shows that $\delta_{\mu_n} \to 1$ in $Z^1(\mathcal{R})$ for the convergence in measure. \hfill\blacktriangle

Finally, we point out that we can construct examples of strongly ergodic generalized Bernoulli equivalence relations $\mathcal{R}$ with prescribed $\text{Sd}$ and $\tau$ invariants for which the associated factor $L(\mathcal{R})$ is not full. For this, we need the following group-theoretic result.

**Proposition 6.5.** Write $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}_3 = \mathbb{Z}/3\mathbb{Z}$. Put $\Gamma = \mathbb{Z}_2 \ast \mathbb{Z}_3$, $\Lambda = \Gamma \ast \mathbb{N}$, $H = \mathbb{Z}_2^\mathbb{N}$ and $I = \Lambda / H$. Then for every $\lambda \in \Lambda \setminus \{1\}$, there are infinitely many $i \in I$ such that $\lambda \cdot i \neq i$ and the action $\Lambda \curvearrowleft I$ has no invariant mean.

**Proof.** Observe that for every $g \in \Gamma \setminus \{1\}$, the conjugacy class $\{g \gamma g^{-1} : \gamma \in \Gamma\}$ is infinite. This implies that for every $\lambda \in \Lambda \setminus \{1\}$, there are infinitely many $i \in I$ such that $\lambda \cdot i \neq i$. Since $\Lambda$ is non-amenable and $H$ is amenable, the action $\Lambda \curvearrowleft I$ has no invariant mean. \hfill\blacktriangle

**Corollary 6.6.** Let $\Lambda \curvearrowleft I$ be any action as in Proposition 6.5. Then for any non-trivial standard probability space $(Y, \eta)$, the generalized Bernoulli equivalence relation $\mathcal{R} = S^{\otimes I} \rtimes \Lambda$ is strongly ergodic and the corresponding factor $L(\mathcal{R})$ is not full.

**Proof.** We already know that the generalized Bernoulli equivalence relation $\mathcal{R} = S^{\otimes I} \rtimes \Lambda$ is strongly ergodic. Since for every $\lambda \in \Lambda \setminus \{1\}$, there are infinitely many $i \in I$ such that $\lambda \cdot i \neq i$, the action $\Lambda \curvearrowleft X$ is essentially free. We then have the crossed product decomposition $L(\mathcal{R}) = L(S)^I \rtimes \Lambda$. Write $\mathbb{Z}_2 = \{1, s\}$. For every $n \in \mathbb{N}$, let $h_n = 1 \oplus \cdots \oplus 1 \oplus s \in H$ where the element $s$ occupies the $n$th position. Then the sequence $(h_n)_n$ is eventually central in $\Lambda$. Thus, the sequence $(u h_n)_n$ is a non-trivial uniformly bounded central sequence in the group von Neumann algebra $L(\Lambda)$ such that for every $i \in I$, we have $u h_n \in L(\text{Stab}(i))$ eventually. Then [VV14, Lemma 2.7] implies that $L(\mathcal{R})$ is not full. \hfill\blacktriangle

**Remark 6.7.** We point out that for plain Bernoulli equivalence relations $\mathcal{R} = S^{\otimes \Lambda} \rtimes \Lambda$ arising from non-amenable groups $\Lambda$, the corresponding factor $L(\mathcal{R})$ is full and $\tau(\mathcal{R}) = \tau(L(\mathcal{R}))$ (see [VV14, Lemma 2.7]). If, moreover, $S$ is as in Theorem 6.3, then we have $\text{Sd}(\mathcal{R}) = \text{Sd}(L(\mathcal{R}))$.

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**Note added in proof.** Since this paper was posted on the arXiv in April 2017, Problem 3 above has been solved in [VV17]. Indeed, it is shown in [VV17, §7] that for a large class of non-singular Bernoulli actions of the free groups $\mathbb{F}_n$ with $n \geq 3$, the induced orbit equivalence relation is strongly ergodic and has prescribed $\text{Sd}$ and $\tau$ invariants.

**A. Appendix. Bounded elements in the full pseudo-group**

**Theorem A.1.** Let $(X, \mu)$ be a standard probability space and $\mathcal{R} \subset X \times X$ an ergodic equivalence relation of type $\text{III}_\lambda$ with $0 < \lambda \leq 1$. Then for any non-zero subsets $A, B \in $
\[ \Psi(X), \text{ we can find a } \mu\text{-bounded partial isomorphism } \theta : A \rightarrow B \text{ in } [\mathcal{R}]. \] More precisely, if \( 0 < C < \lambda \), we can choose \( \theta \) so that for a.e. \( x \in A \) we have

\[
\frac{C \mu(B)}{\mu(A)} \leq \frac{d(\mu \circ \theta)}{d\mu}(x) \leq \frac{C^{-1} \mu(B)}{\mu(A)}.
\]

The proof relies on a maximality argument based on the following lemma.

**Lemma A.2.** Let \((X, \mu)\) be a standard probability space and \( \mathcal{R} \subset X \times X \) an ergodic equivalence relation of type III\( \lambda \) with \( 0 < \lambda \leq 1 \). Let \( A, B \in \Psi(X) \) be two non-zero subsets and \( c_1, c_2 > 0 \) two constants such that \( c_1 < \lambda c_2 \). Then we can find a non-zero element \( \pi \in [\mathcal{R}] \) such that \( \text{dom}(\pi) \subset A, \text{ran}(\pi) \subset B \) and

\[
c_1 \leq \frac{d(\mu \circ \pi)}{d\mu}(x) \leq c_2
\]

for a.e. \( x \in \text{dom}(\pi) \).

**Proof.** First we can find a non-zero \( \psi \in [\mathcal{R}] \) such that \( U := \text{dom}(\psi) \subset A \) and \( \text{ran}(\psi) \subset B \). Moreover, by shrinking \( \psi \) if necessary, we can assume that

\[
b \leq \frac{d(\mu \circ \psi)}{d\mu}(x) \leq b \sqrt{\lambda c_2/c_1}
\]

for some constant \( b > 0 \) and for a.e. \( x \in \text{dom}(\psi) \).

Since \( c_1/b < \lambda c_2(b \sqrt{\lambda c_2/c_1})^{-1} \) we know that

\[
S(\mathcal{R}) \cap [c_1/b, c_2(b \sqrt{\lambda c_2/c_1})^{-1}] \neq \emptyset.
\]

But we also know that

\[
S(\mathcal{R}) \subset \text{Range}(\delta_\mu|_{\mathcal{R}_U}).
\]

Hence, we obtain that

\[
\delta_\mu^{-1}([c_1/b, c_2(b \sqrt{\lambda c_2/c_1})^{-1}]) \in \Psi(\mathcal{R})
\]

has a non-trivial intersection with \( U \times U \). Therefore, we can find a non-zero \( \phi \in [\mathcal{R}] \) such that \( \text{dom}(\phi), \text{ran}(\phi) \subset U \) and

\[
c_1/b \leq \frac{d(\mu \circ \phi)}{d\mu}(x) \leq c_2(b \sqrt{\lambda c_2/c_1})^{-1}
\]

for a.e. \( x \in \text{dom}(\phi) \). Hence \( \pi = \psi \phi \in [\mathcal{R}] \) is a non-zero element that satisfies the condition we wanted. \[\square\]

**Proof of Theorem A.1.** Take any constant \( c \in ]\lambda^{-1}C, 1[ \). Consider the set \( \Omega \) of all elements \( \theta \in [\mathcal{R}] \) such that for a.e. \( x \in \text{dom}(\theta) \) we have

\[
\frac{C \mu(B)}{\mu(A)} \leq \frac{d(\mu \circ \theta)}{d\mu}(x) \leq \frac{C^{-1} \mu(B)}{\mu(A)}
\]

and

\[
c \frac{\mu(B)}{\mu(A)} \mu(A \setminus \text{dom}(\theta)) \leq \mu(B \setminus \text{ran}(\theta)) \leq c^{-1} \frac{\mu(B)}{\mu(A)} \mu(A \setminus \text{dom}(\theta)).
\]

The poset \( \Omega \) is closed in \([\mathcal{R}]\), hence it is inductive. Let \( \theta \in \Omega \) be a maximal element and let us show that \( \mu(A \setminus \text{dom}(\theta)) = 0 \) and \( \mu(B \setminus \text{ran}(\theta)) = 0 \).
Suppose that
\[ c \frac{\mu(B)}{\mu(A)} \mu(A \setminus \text{dom}(\theta)) < \mu(B \setminus \text{ran}(\theta)). \]
Then we will contradict the maximality of $\theta$. Since $\mathcal{R}$ is of type $\text{III}_\lambda$ and $c^{-1} < \lambda C^{-1}$, then by Lemma A.2, we can find a non-zero $\pi \in [\mathcal{R}]$ with $\text{dom}(\pi) \subseteq A \setminus \text{dom}(\theta)$ and $\text{ran}(\pi) \subseteq B \setminus \text{ran}(\theta)$ such that for a.e. $x \in \text{dom}(\pi)$ we have
\[ c^{-1} \frac{\mu(B)}{\mu(A)} \leq d(\mu \circ \pi) \leq C^{-1} \frac{\mu(B)}{\mu(A)}. \]
Moreover, since
\[ c \frac{\mu(B)}{\mu(A)} \mu(A \setminus \text{dom}(\theta)) < \mu(B \setminus \text{ran}(\theta)), \]
we can choose $\pi$ to be small enough so that
\[ c \frac{\mu(B)}{\mu(A)} \mu(A \setminus \text{dom}(\theta)) \leq \mu(B \setminus \text{ran}(\theta)), \]
where $\theta' = \theta + \pi$. Then it is easy to check by construction that $\theta' \in \Omega$, and this contradicts the maximality of $\theta$. Hence we must have
\[ c \frac{\mu(B)}{\mu(A)} \mu(A \setminus \text{dom}(\theta)) = \mu(B \setminus \text{ran}(\theta)). \]
Similarly, by exchanging the roles of $\theta$ and $\theta^{-1}$, we can show that
\[ \mu(B \setminus \text{ran}(\theta)) = c^{-1} \frac{\mu(B)}{\mu(A)} \mu(A \setminus \text{dom}(\theta)). \]
Since $c < 1$, this implies that
\[ \mu(B \setminus \text{ran}(\theta)) = \mu(A \setminus \text{dom}(\theta)) = 0, \]
as we wanted. \qed

**Corollary A.3.** Let $(X, \mu)$ be a standard probability space and $\mathcal{R} \subset X \times X$ an ergodic equivalence relation of type $\text{III}_\lambda$ with $0 < \lambda \leq 1$. Then the $\mu$-bounded elements are dense in $[\mathcal{R}]$.

**Proof.** Take $\theta \in [\mathcal{R}]$. Take $A$ such that the partial isomorphism $\theta|_{X \setminus A}$ is $\mu$-bounded and let $B := \theta(A)$. By Theorem A.1, we can find a partial isomorphism $\pi : A \to B$ which is $\mu$-bounded. Then $\theta' = \theta|_{X \setminus A} + \pi \in [\mathcal{R}]$ is $\mu$-bounded. Since we can choose $A$ and $B = \theta(A)$ to be arbitrarily small, we see that $\theta'$ is arbitrarily close to $\theta$. Hence the $\mu$-bounded elements are dense in $[\mathcal{R}]$. \qed

**Corollary A.4.** Let $(X, \mu)$ be a standard probability space and $\mathcal{R} \subset X \times X$ an ergodic equivalence relation of type $\text{III}_\lambda$ with $0 < \lambda \leq 1$. If $\nu$ is another probability measure on $X$ which is equivalent to $\mu$, then for any $C < \lambda$ we can find $\theta \in [\mathcal{R}]$ such that
\[ C \mu \leq \nu \circ \theta \leq C^{-1} \mu. \]
**Proof.** Let $S_2$ be the transitive equivalence relation on the set $\{0, 1\}$ and consider the product equivalence relation $R \otimes S_2$ on $X \times \{0, 1\}$. On $X \times \{0, 1\}$, put the measure $\omega = \frac{1}{2}(\mu \otimes \delta_0 + \nu \otimes \delta_1)$. Let $A = X \times \{0\}$ and $B = X \times \{1\}$. Then $\omega(A) = \omega(B) = \frac{1}{2}$. Thus, Theorem A.1 provides us with a partial isomorphism $\theta : A \rightarrow B$ in $[[R \otimes S_2]]$ such that
\[
C \leq \frac{d(\omega \circ \theta)}{d\omega}(x) \leq C^{-1}
\]
for a.e. $x \in A$. Viewing $\theta$ as an element of $[R]$, we obtain
\[
C \leq \frac{d(\nu \circ \theta)}{d\mu} \leq C^{-1}.
\]

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