Diluting Cosmological Constant In Infinite Volume Extra Dimensions

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Abstract

We argue that the cosmological constant problem can be solved in a braneworld model with infinite-volume extra dimensions, avoiding no-go arguments applicable to theories that are four-dimensional in the infrared. Gravity on the brane becomes higher-dimensional at super-Hubble distances, which entails that the relation between the acceleration rate and vacuum energy density flips upside down compared to the conventional one. The acceleration rate decreases with increasing the energy density. The experimentally acceptable rate is obtained for the energy density larger than $(1 \text{ TeV})^4$. The results are stable under quantum corrections because supersymmetry is broken only on the brane and stays exact in the bulk of infinite volume extra space. Consistency of 4D gravity and cosmology on the brane requires the quantum gravity scale to be around $10^{-3} \text{ eV}$. Testable predictions emerging within this approach are: (i) simultaneous modifications of gravity at sub-millimeter and the Hubble scales; (ii) Hagedorn-type saturation in TeV energy collisions due to the Regge spectrum with the spacing equal to $10^{-3} \text{ eV}$. 
1 Introduction

The most severe cosmological problem, from the perspective of an effective quantum field theory, is undoubtedly the one of the cosmological constant. In the nutshell the problem can be outlined as follows. The acceleration rate of the universe, $H$, if nonzero, has to be less than or equal to the present-day value of the Hubble parameter $H_0$:

$$H \lesssim H_0 \sim 10^{-33} \text{eV}.$$  \hspace{1cm} (1)

Recent experimental data appear to confirm a nonzero value of $H$ that nearly saturates the upper bound in (1).

In general relativity $H^2$ determines the scalar curvature of space and is related to the vacuum energy density $\mathcal{E}_4$ as

$$M_{\text{Pl}}^2 H^2 \sim \mathcal{E}_4.$$  \hspace{1cm} (2)

Here $M_{\text{Pl}} \sim 10^{18} \text{GeV}$ is the Planck mass. This relation holds as long as $\mathcal{E}_4 \ll M_{\text{Pl}}^4$. A natural value of $\mathcal{E}_4$ due to the zero-point energies of known elementary particles can be estimated to be

$$\mathcal{E}_4 \sim M_{\text{susy}}^4 \gtrsim (\text{TeV})^4,$$

where $M_{\text{susy}}$ is the supersymmetry (SUSY) breaking scale. Substituting this value of $\mathcal{E}_4$ into (2), one finds $H \gtrsim 10^{-3} \text{eV}$, which is grossly inconsistent with (1).

Since the discrepancy between theory and experiment manifests itself already at enormous distances $\sim H_0^{-1} \simeq 10^{28} \text{cm}$ (i.e., at extremely low energies), it is natural to suspect that the problem is not necessarily due to our lack of knowledge of the underlying short-distance physics. Instead, this may indicate that, in the standard picture of the Universe at cosmic distances, we may be missing an essential part of the puzzle. The problem is so severe, that it seems reasonable to put aside all the other cosmological issues treating them as secondary and focus completely on the cosmological constant problem. This is the approach we take in the present work.

The question we address is essentially as follows: “Can the cosmological constant problem originate from our treatment of space-time as four-dimensional (4D) at cosmic distances, while in fact it is not?” As we shall discuss below the answer can be positive. In particular, we will argue that the problem can be avoided in a model of "brane induced gravity" in which a graviton propagates in extra dimensions that have an infinite volume.

In order to explain our main result, let us set the terminology first. In the brane-world models with finite-volume extra dimensions there exists a normalizable four-dimensional zero-mode graviton mediating 4D gravitational interactions at distances larger than the size of the extra space. The observable 4D Planck mass $M_{\text{Pl}}$ (the

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\footnote{We use the term acceleration rate in order to distinguish it from the inflation rate in the early Universe.}

\footnote{If $\mathcal{E}_4$ is negative it gives rise to anti de Sitter space which is excluded as well.}
inverse square root of Newton’s constant $G_N$) is then determined by the $(4 + N)$-dimensional Planck mass $M_*$, and the volume of extra space $V_N$ as

$$M_{\text{Pl}}^2 = M_*^{2+N} V_N. \quad (3)$$

The same quantity sets the norm (and, thus, the inverse coupling) of a zero-mode graviton. Therefore, in theories with finite-volume extra dimensions, one can give two equivalent definitions of the 4D Planck mass:

(i) As an inverse square root of Newton’s constant, defining the strength of the 4D gravitational potential;

(ii) As the norm (or the inverse coupling) of the massless graviton.

In the latter case the value of $V_N$ is constrained by Eq. (3). In contrast, such a relation is not possible when $V_N$ is infinite. Throughout the paper we shall stick to the first definition of $M_{\text{Pl}}$, since this is the only sensible definition in the models [3, 4] with the infinite-volume extra dimensions. In these theories the 4D zero-mode graviton is non-normalizable and, therefore, it is infinitely weakly coupled because

$$M_{\text{Pl}}^2 \neq M_*^{2+N} V_N \to \infty. \quad (4)$$

In that approach [3] the four-dimensional Plank mass is induced by loops of particles localized on the brane. The absolute value of $M_{\text{Pl}}$ is unrelated to $M_*$. Instead, it is determined by ultraviolet properties of the matter fields on the brane. Hence the name, “brane induced gravity”.

The fact that the value of $V_N$ is not tied to $M_{\text{Pl}}$ will be crucial in what follows. The laws of 4D gravity in this model are obtained only at distances shorter than the Hubble size. At larger distances the effects of infinite-volume extra dimensions take over, and gravity follows higher-dimensional laws [3].

Below we will argue that in the brane induced gravity model with $N$ extra spatial dimensions the relation between $H$ and $E_4$ (for $N \neq 2$) takes the form

$$M_*^{2+N} H^{2-N} \sim E_4. \quad (5)$$

The key point is that for $N > 2$ the acceleration rate $H$ decreases when $E_4$ is increasing,

$$H \sim M_* \left( \frac{M_*^4}{E_4} \right)^{\frac{1}{N-2}}. \quad (6)$$

This is exactly opposite of the 4D result (2), corresponding to $N = 0$, according to which the rate grows as $E_4$ increases. As will be discussed in detail in Sec. 4, the expression in Eq. (6) is valid for

$$M_*^4 \ll E_4 \ll M_{\text{Pl}}^4.$$ 

The expression for $H$ given in Eq. (3) can yield the acceleration rate that is consistent with Eq. (3), as we discuss at length in the bulk of the paper. Our construction is an
example illustrating that the cosmological constant problem can be solved, at least
in principle, in an effective field theory approach, provided that gravity becomes soft
above the scale $M_\ast$ – the property realized in string theory.

The organization of the paper is as follows. In Section 2 we summarize argu-
ments why the braneworld is a right framework for solving the cosmological constant
problem. In Section 3 we setup the model and describe its crucial properties. In
Section 4 we argue that the model gives rise to the acceleration rate in our 4D world
that is consistent with the data. Section 5 is devoted to the issue of UV softening
of the background solution due to higher-derivative terms. In Section 6 we show
that 4D laws of gravity are obtained on a non-zero tension brane in infinite-volume
extra space due to the induced Einstein-Hilbert term. In Section 7 we discuss no-go
theorems on the cancellation of the cosmological constant. We argue that a model
with infinite-volume extra dimensions evades those no-go theorems. Discussions are
given in Section 8.

2 How do extra dimensions help?

The questions that an effective field theorist would ask regarding any potential
solution of the cosmological constant problem are:

(i) Why it is so important to have extra dimensions?

(ii) Can the mechanism be understood in the language of a 4D low-energy ef-
fective field theory?

(iii) How does a prospective solution evade the usual no-go arguments against
the cancellation of the vacuum energy contributions coming from the domain below
the supersymmetry breaking scale?

The content of the paper is devoted to the detailed study of these issues. How-
ever, simple arguments given in this section elucidate the key points.

The answer to (i) is:

Recall that if there is a vacuum energy density $\mathcal{E}_4 \geq \text{TeV}^4$ in a conventional 4D
theory then it unavoidably gives rise to the following scalar curvature of space

$$R \sim \mathcal{E}_4/M_{Pl}^2 \gtrsim (10^{-3} \text{eV})^2.$$  (7)

This can be put in the following simple terms: The vacuum energy density $\mathcal{E}_4$ is a
source of gravity, and, as such, it has to curve the space; the only space in the 4D
theory is the space in which we live. Hence, our space is curved, as in (7), and this
is inconsistent with the data.

However, if there are more than four dimensions the story could be different. In
that case $\mathcal{E}_4$ could curve extra dimensions instead of curving our 4D space [5, 6]. This
idea is particularly transparent for warped geometries [6]. Consider the following
$(4 + N)$-dimensional interval:

$$ds^2 = A^2(p) \bar{g}_{\mu\nu}(x) dx^\mu dx^\nu - B^2(p) d\rho^2 - C^2(p) \rho^2 d\Omega_{N-1}^2,$$  (8)
where $\mu, \nu = 0, 1, 2, 3$, are the indices denoting our 4D world, $\rho \equiv \sqrt{\rho_1^2 + \ldots + \rho_N^2}$, and $\rho_i$’s denote the extra coordinates. The curvature invariants of the metric $\bar{g}_{\mu\nu}(x)$ is what we measure in our 4D world. There can exist solutions to the $(4 + N)$-dimensional Einstein equations in the form of (8) where $E_4$ affects strongly the extra space, i.e., the functions $A, B$ and $C$, while leaving our 4D space almost intact — the 4D metric $\bar{g}_{\mu\nu}(x)$ remains almost flat. In this case the energy density $E_4$ “is spent” totally on curving up the extra space rather than on curving our 4D space.

The answers to (ii) and (iii) are:

The effective field theory arguments are based on the assumption that there is a finite number of 4D degrees of freedom below the scale of the cosmological constant that one wants to neutralize. This condition is not satisfied in the present model — it is a genuinely high-dimensional theory in the far infrared. Therefore, there is an infinite number of degrees of freedom below any non-zero energy scale. As a result, there is no scale below which extra dimensions can be integrated out and the theory reduced to a local 4D field theory with a finite number of degrees of freedom. In order to rewrite the model at hand as a theory of a single 4D graviton, at any given scale, we have to integrate out an infinite number of lighter modes. As usually happens in field theory, integrating out the light states we get non-local interactions. Therefore, the resulting model, rewritten as a theory of a 4D graviton, will contain generally-covariant but non-local terms. The latter dominate the action in the far infrared. Of course, in actuality, the full theory is local — the apparent non-locality is an artifact of integrating out light modes. It tells us that a local $(4 + N)$-dimensional theory can be imitated by a non-local 4D model. The non-local terms modify the effective 4D equations and neutralize a large cosmological constant. As a result, the relation between the acceleration rate and the vacuum energy density is changed, as was mentioned in Sect. 1.

To emphasize the crucial role of the infinite volume, below we would like to summarize no-go arguments on why any 4D theory [7] or a theory that has finite-volume extra dimensions cannot solve the cosmological constant problem [4].

Consider a theory that, below certain scale $M_c (M_c >> H_0)$, flows to a local four-dimensional effective field theory with a finite number of fields. In particular, all Kaluza-Klein (KK) type theories, in which the extra space has a finite volume $\sim 1/M_c^N$, are of this type. Then, below the scale $M_c$, we can integrate out heavy physics and write down an effective low-energy theory, that — by construction — is generally covariant and has a finite number of 4D light degrees of freedom. As is well known, such theories require the existence of a massless spin-2 state, the graviton.

Let us note that these arguments are not applicable to solutions of anthropic type (see discussions in Ref. [8]), nor to any solution based on the cancellation of the cosmological term due to some reason hidden in the ultraviolet (UV) physics. Such a cancellation would look like fine-tuning from the low-energy perspective. We shall not consider this possibility, since it would require the detailed understanding of the UV physics (see, e.g., discussions in Ref. [3]). This is beyond the scope of the present investigation.
The graviton, must be universally coupled.

This fact has an extremely important consequence: it excludes the existence of sectors with unbroken supersymmetry. The reason is simple. Supersymmetry in the Standard Model (SM) sector is broken at the scale $M_{\text{susy}} \sim \text{TeV}$ or higher. Since all states in the theory couple to gravity with the equal strength, the lowest possible SUSY breaking scale in any sector is

$$m_{\text{min}} \sim \frac{\text{TeV}^2}{M_{\text{Pl}}} \sim 10^{-3} \text{ eV}. \quad (9)$$

This is a bad news because of the following two reasons:

(i) Eq. (9) implies that the loop contribution to the cosmological term from each sector is at least as large as

$$\Delta \varepsilon_{\text{min}} \sim \frac{\text{TeV}^4}{M_{\text{Pl}}^2} \Lambda_{\text{cut-off}}^2, \quad (10)$$

where $\Lambda_{\text{cut-off}}$ is the UV cut-off;

(ii) The scalars with masses $\ll 10^{-3} \text{ eV}$ are needed to cancel the cosmological constant (other spin fields, due to the requirement of 4D Lorentz invariance, can only renormalize couplings in the potential). However, because of the above arguments the existence of such light scalars is highly unnatural.

The following condition can be derived. In order to cancel a small change in the 4D curvature to a given accuracy $H$, by re-adjusting the VEV of a scalar field, the latter must be as light as $H$. This is obvious: if the field is much heavier than $H$, it can be integrated out and cannot participate in the low-energy dynamics at this scale. In particular, its VEV cannot be sensitive to a small change in the expansion rate. By the same token, the dynamics of the SM Higgs VEV is completely irrelevant for the present-day cosmological expansion.

Thus, to suppress the acceleration rate down to the observable value $H_0$, the existence of scalars as light as $10^{-33} \text{ eV}$ is required. (This is a necessary but not sufficient condition). Now, as we have argued above, in the theories with finite-volume extra space, all states couple to gravity universally, via $1/M_{\text{Pl}}$, and, therefore, the lightest scalars have mass $\sim 10^{-3} \text{ eV}$. This kills any hope for self-adjustment mechanisms in the theories with finite-volume extra dimensions. It is impossible to cancel cosmological term to accuracy better than $\sim \text{TeV}^4$ in the finite-volume

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There exist two possibilities to have naturally light scalars in a theory, as pseudo Nambu-Goldstone bosons (PNGB), or as composite states. However, neither of these can help to cancel the big cosmological constant. In the case of the PNGB, the potential that corresponds to the explicit symmetry breaking has to be very high in order to cancel the vacuum energy density. If so, the corresponding PNGB is heavy and becomes irrelevant for the low energy dynamics. As to the light composite bosons, they should have the mass of the order of $\sim H_0$. This would imply that the compositeness scale is of the same order, and that the size of those scalars is generically huge, $\sim H_0^{-1}$. However, the latter scalar cannot do the job since they are not composites at the observable distances.

\hfill \{4\}
models, unless a severe fine-tuning is invoked. On the contrary, in the infinite-volume models the higher-dimensional scalar components of the graviton KK modes can and will be arbitrarily light. This will lead to a drastic change, as we shall discuss in the next section.

3 The model: importance of infinite volume

In this section we shall formulate the model, and summarize the main assumptions. The model is the $N \geq 2$ version of Ref. [3] (for generalizations to $N \geq 2$ see [4, 11], and Ref. [12] for a recent summary; for string theory discussions of this model see Refs. [13, 11, 14]).

The theory to be considered is a brane-world model embedded in a space with (asymptotically) flat infinite-volume $N$ extra dimensions. All known SM particles are localized on the brane and obey the conventional 4D laws of gauge interactions up until very high energies, of the order of the GUT scale for instance. The gravitational sector, on the other hand, is spread over the whole $(4 + N)$-dimensional space. The low-energy action of the model is written as follows:

$$S = M_\ast^{2+N} \int d^4x d^N \rho \sqrt{G} \mathcal{R} + \int d^4x \sqrt{g} \left( \mathcal{E}_4 + M_{\text{ind}}^2 \mathcal{R} + \mathcal{L}_{\text{SM}}(\Psi, M_{\text{SM}}) \right). \quad (11)$$

Let us discuss various parts and parameters of the action (11). $\mathcal{L}_{\text{SM}}$ is the Lagrangian for the particle physics including all SM fields $\Psi$. The parameter $M_{\text{SM}}$ denotes the UV cutoff of the SM. Up until that scale the SM obeys the conventional 4D laws. In the present approach $M_{\text{SM}} \gg \text{TeV} \gg M_\ast$. Moreover, $G_{AB}$ stands for a $(4 + N)$-dimensional graviton $(A, B = 0, 1, 2, \ldots, 3 + N)$, while $\rho$ are “perpendicular” coordinates. For simplicity we do not consider brane fluctuations.

Thus, the induced metric on the brane is given by

$$\bar{g}_{\mu\nu}(x) \equiv G_{\mu\nu}(x, \rho_n = 0). \quad (12)$$

Since we discard the brane fluctuations, the brane can be thought of as a boundary of the extra space or an orbifold fixed point (if so, the Gibbons-Hawking surface term is implied in the action hereafter). The brane tension is denoted by $\mathcal{E}_4$.

The first term in (11) is the bulk Einstein-Hilbert action for $(4 + N)$-dimensional gravity, with the fundamental scale $M_\ast$. The expression in (11) has to be understood as an effective low-energy Lagrangian valid for graviton momenta smaller than $M_\ast$. Therefore, in what follows we shall imply the presence of an infinite number of gauge-invariant high-dimensional bulk operators suppressed by powers of $M_\ast$.

\[\text{Footnote 5: For simplicity of notations we use the convention that the whole theory of particles physics including any (SUSY) GUT or other extensions of the Standard Model is denoted by SM.}\]

\[\text{Footnote 6: This limitation could be readily lifted. Indeed, including the brane fluctuations would produce a sterile Goldstone boson, and heavy modes which could manifest themselves only through generation of an extrinsic curvature term on the brane. It is easy to see that the latter can be reduced to an inessential renormalization of constants in the action (11).}\]
The second term in (11) describes the 4D Einstein-Hilbert (EH) term of the induced metric. This term plays the crucial role. It ensures that at observable distances on the brane the laws of 4D gravity are reproduced in spite of the fact that there is no localized zero-mode graviton. Its coefficient $M_{\text{ind}}$ is another parameter of the model. Thus, the low-energy action as it stands is governed by three parameters $M_*$, $M_{\text{ind}}$ and $\mathcal{E}_4$. Let us discuss their natural values separately.

The parameter $M_{\text{ind}}$ gets induced by the SM particle loops localized on the brane. Such corrections are cut-off by the rigidity scale of the SM, $M_{\text{SM}}$, i.e., the scale above which the SM propagators become soft. In the present approach this scale is taken to be very high, $\gg$ TeV. In particular, we will take this scale to be comparable with the GUT or 4D Planck scale. The loops induce the Einstein-Hilbert term in (11),

$$M_{\text{ind}}^2 \sqrt{\bar{g}} \mathcal{R}(\bar{g}),$$

(13)

where the value of the induced constant $M_{\text{ind}}$ is determined by the relation [15, 16],

$$M_{\text{ind}}^2 = i \int d^4x x^2 \langle T(x) T(0) \rangle / 96.$$

The parameter $M_{\text{ind}}$ is proportional to the scale $M_{\text{SM}}$ and to the number of particles in the SM. Since there are about 60 particles in the Weinberg-Salam model, and more are expected in GUT’s, the value of $M_{\text{ind}}$ should be somewhat larger than $M_{\text{SM}}$. In fact, below we define the 4D Planck mass as being completely determined by $M_{\text{ind}}$:

$$M_{\text{Pl}} \equiv M_{\text{ind}}.$$ 

(14)

Thus, the Planck mass is not a fundamental constant in our approach but is rather a derived scale. We see that the SM loop corrections are capable of creating the hierarchy $M_{\text{ind}}/M_*$, even if the initial value of $M_{\text{ind}}/M_*$ was not that large. This hierarchy does not amount to fine tuning, since such a separation of scales is stable under quantum corrections. Indeed, say, $M_*$ gets renormalized by all possible bulk quantum gravity loops. However, there are no SM particles in the bulk the only

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7 We set the thickness of the brane $\Delta$ to be determined by the SM scale, $\Delta \sim M_{\text{SM}}^{-1}$. Therefore, all SM particles live on the brane and obey the 4D laws of non-gravitational interactions, while gravity can propagate into the $(4+N)$-dimensional bulk. In particular, the conventional logarithmic gauge coupling unification is fully preserved, as well as other useful properties of the SM, such as the seesaw mechanism for neutrinos, etc. The fact that all particles other than gravitons and gravitinos are localized on the brane is certainly welcome also from the aesthetic standpoint. We assume for convenience and clarity that above $M_{\text{SM}}$ the SM stops to run logarithmically because of the heavy states with masses $\gtrsim M_{\text{SM}}$ which together with the SM particles complete the multiplets of a certain super-conformal theory.

8 $M_{\text{ind}}$ can certainly contain as well the tree-level terms if these are present in the original action in the first place. We will not discriminate between these and induced terms. $M_{\text{ind}}$ will be regarded as a parameter that stands in (11).

9 The scalars and fermions contribute to $M_{\text{ind}}$ with positive sign while the gauge fields with negative sign.
scale in there is $M_*$. Therefore, any bulk loop gets cut-off at the scale $M_*$, as it is the fundamental gravity scale. While, as we discussed above, the brane SM loops are cut-off by the higher scale $M_{SM}$, and this gives rise to the huge value of $M_{ind}$ on the brane.

Finally, let us discuss the value of the brane tension. To this end, we have to specify our assumptions regarding supersymmetry. We assume that the high-dimensional theory is supersymmetric, and that supersymmetry is spontaneously broken only on the brane (such a scenario with a non-BPS brane was considered in [17]). Non-breaking of supersymmetry in the bulk is only possible due to infinite volume of the extra space; SUSY breaking is not transmitted from the brane into the bulk [18, 19] since the breaking effects are suppressed by an infinite volume factor $10^\infty$. Then, the bulk cosmological term can be set to zero, without any fine-tuning. The natural value of $\mathcal{E}_4$ can be as low as TeV$^4$, since the brane tension can be protected above this value by $\mathcal{N} = 1$ supersymmetry (note that $M_{ind}$ can only be protected by a conformal invariance which we assume is broken at the scale $M_{SM}$). All these properties are summarized in Fig. 1.

Let us now turn to the gravitational dynamics on the brane. This dynamics is quite peculiar. Despite the fact that the volume of extra space is infinite, nevertheless an observer on the brane measures 4D gravitational interaction up to cosmologically large scales [3]. For instance, in the $N \geq 2$ case the static potential between two objects on the brane scales as

$$V(r) = -\frac{1}{16\pi M_{Pl}^2} \frac{1}{r} + ..,$$

for distances in the interval

$$M_*^{-1} \lesssim r \lesssim r_c \sim M_{Pl}/M_*^2.$$  

However, at distances smaller than $M_*^{-1}$ and bigger than $r_c$ the potential changes.

In order for the late-time cosmology to be standard we require that $r_c \sim H_{0}^{-1} \sim 10^{28}$ cm. This unambiguously restricts the value for the quantum gravity scale around the following value:

$$M_* \sim 10^{-3} \text{ eV}.$$  

Small $M_*$ means that gravity in the bulk of extra space is strong. However, the large EH term on the brane “shields” the SM particles on the brane from strong bulk gravity [20]. Small $M_*$ was first discussed in Ref. [21] in a different 4D framework, see also [22]. The case of interest for the cosmological constant problem is a nonzero tension brane, i.e., $\mathcal{E}_4 \neq 0$. For the latter, as we will argue in Section 6, the crossover distance is also comparable with the present-day Hubble size.

\footnote{In general, local SUSY in the bulk does not preclude a negative vacuum energy density of the order of $M_*^{4+N}$. However, the latter can be forbidden by an unbroken $R$ symmetry in the bulk. Such a symmetry is often provided by string theory.}

\footnote{Note that having a small value of $M_*$ does not solve per se the cosmological constant problem since the SM loops are still cutoff by the high scale.}
Figure 1: (A) The worldvolume vacuum diagram of the SM fields that renormalizes the brane tension (i.e., the 4D cosmological constant). These contributions are protected by $\mathcal{N} = 1$ worldvolume supersymmetry. Therefore, they are cutoff by the worldvolume SUSY breaking scale $M_{\text{susy}}$. (B) The worldvolume two-point diagram that renormalizes (induces) the EH term on the brane. These contributions are not protected by $\mathcal{N} = 1$ supersymmetry. They can only be protected by conformal invariance which in our model is broken at the scale $M_{\text{SM}}$ that is close to $M_{\text{Pl}}$. Hence, there can be a hierarchy between (B) and (A). (C) The bulk vacuum diagram. Only the bulk particles (which do not include the SM particles) are running in this loop (bulk particles are denoted by double lines). This diagram is protected by unbroken bulk SUSY. Therefore, the cosmological constant in the bulk is zero. (D) The bulk two-point diagram which renormalizes the bulk EH term. As in (C), only the bulk particles are running in the loop. This diagram is cutoff by the bulk scale $M_s$. Therefore, the natural value of the constant in front of the bulk EH term is $M_s^{2+N}$; there is huge hierarchy between this coefficient and that of the worldvolume EH term coming from (B).
The “shielding” effect plays a crucial role. We will discuss it in more details here. The essence of the shielding is as follows. Due to the presence of the large induced kinetic term on the brane, the wave functions of high-dimensional gravitons with large external momentum are suppressed on the brane. Only the gravitons with extremely low external momenta (smaller than $r_c^{-1}$) can penetrate to the brane and couple to the SM particles. Hence, a brane observer is “shielded” from the strong bulk gravity and detects a weak four-dimensional force up until the distances exceeding the crossover scale $r_c$. The modes participating in gravitational dynamics on the brane at larger distances are not shielded, and as a result gravity becomes high-dimensional beyond the crossover scale.

The effect can be best understood in terms of the four-dimensional mode expansion. From the four-dimensional perspective a high-dimensional graviton represents a continuum of four-dimensional states and can be expanded in these states. Below we shall be interested only in spin-2 components for which the KK decomposition can schematically be written as follows:

$$h_{\mu\nu}(x, \rho_n) = \int d^N m \epsilon^{(m)}_{\mu\nu}(x) \sigma_m(\rho_n), \quad (17)$$

where $\epsilon^{(m)}_{\mu\nu}(x)$ are four-dimensional spin-2 fields of mass $m$ and $\sigma_m(\rho_n)$ are their wave-function profiles in extra dimensions. The strength of the coupling of an individual mode to a brane observer is given by the value of the wave-function at the position of the brane, that is $\sigma_m(0)$. Four-dimensional gravity on the brane is mediated by exchange of all the above modes. Each of these exchanges mediates a Yukawa type gravitational potential. The result is

$$V(r) \propto \frac{1}{M_4^{2+N}} \int_0^\infty dm m^{N-1} |\sigma_m(0)|^2 e^{-rm} r. \quad (18)$$

Four-dimensional gravity on the brane is recovered for $r << r_c$, due to the fact that the modes heavier than $1/r_c$ have suppressed wave-functions and, therefore, the above integral is effectively cut-off at $m \sim 1/r_c$. Most easily this can be seen from the propagator analysis. Gravitational potential (18) on the brane is mediated by an “effective” 4D-graviton which can be defined as:

$$h_{\mu\nu}(x, 0) = \int d^N m \epsilon^{(m)}_{\mu\nu}(x) \sigma_m(0). \quad (19)$$

The Green’s function for this state can be defined in the usual way. Using (19) and orthogonality of the $\epsilon^{(m)}_{\mu\nu}(x)$-states we obtain

$$G(x - x', 0)_{\mu\nu, \gamma\delta} = \langle h_{\mu\nu}(x, 0) h_{\gamma\delta}(x', 0) \rangle = \int d^N m |\sigma_m(0)|^2 \langle \epsilon^{(m)}_{\mu\nu}(x) \epsilon^{(m)}_{\gamma\delta}(x') \rangle. \quad (20)$$

From now on we shall suppress the tensor structure, which is not essential for this discussion. Going into the Euclidean momentum space we get the following expression for the scalar part of the propagator

$$G(p, 0) = \int dm m^{N-1} \frac{|\sigma_m(0)|^2}{m^2 + p^2}. \quad (21)$$
This is the spectral representation for the Green’s function
\[ G(p, 0) = \int ds \frac{\rho(s)}{s + p^2}, \tag{22} \]
with \( s \equiv m^2 \) and
\[ \rho(s) = \frac{1}{2} s^{N-2} |\sigma_m(0)|^2. \tag{23} \]

Therefore, the spectral representation of the effective graviton Green’s function can be simply understood as the KK mode expansion \([17]\). Thus, the wave-function suppressions for the heavy modes can be read-off from Eqs. \((21)\) and \((22)\) by using the explicit form of the propagator \(G(p)\) \([4, 12]\)
\[ G(p, 0) = \frac{1}{M_{Pl}^2 p^2 + M_{*}^{2+N} D^{-1}(p, 0)}, \tag{24} \]

where \( D^{-1}(p, 0) \) is the inverse Green’s function of the bulk theory with no brane. For the purposes of the present discussion it is enough to notice that for \( N > 1 \) and large momenta \( p >> r_c^{-1} = M_{*}^2/M_{Pl} \) the above propagator behaves as \([1, 12]\)
\[ G(p, 0) \sim \frac{1}{M_{Pl}^2 p^2} \tag{25} \]
which is nothing but the propagator of a massless four-dimensional graviton with the \( 1/M_{Pl} \) coupling. Substituting \((23)\) into the left hand side of \((22)\), we find that the function \( \rho(s) \) must be suppressed for \( s >> r_c^{-2} \). If so, the relation \((23)\) implies that the wave-functions of the heavy modes must be vanishingly small as well \([4, 23]\).

For \( N = 1 \) both the propagator \([3]\) and the wave-function profiles \([20]\) can be evaluated analytically and are given by
\[ G(p, 0) = \frac{1}{M_{Pl}^2 p^2 + 2M_{*}^3 p}, \tag{26} \]
and
\[ |\sigma_m(0)|^2 = \frac{4}{4 + m^2 M_{Pl}^4/M_{*}^6}, \tag{27} \]
respectively. This explicitly demonstrates that the modes that are heavier than \( r_c^{-1} = M_{*}^3/M_{Pl} \) are effectively shielded (note the difference in the expression for \( r_c \) from \( N > 1 \) case). Substituting \((27)\) into \([18]\) we derive the usual Newtonian potential \([13]\) at distances \( r < r_c \).

One can interpret the Green’s function as describing a metastable state that decays into the bulk states with the lifetime \( \sim r_c \). The remarkable fact is that the existence of such a metastable state is perfectly compatible with the exact 4D general covariance. At short distances \( r \ll r_c \), the theory reproduces all observed properties of 4D gravity; it dramatically differs, however, at large distances, \( r \gtrsim \)
More detailed analysis indicates that all the predictions of general relativity, are reproduced at short distances. The Schwarzschild type behavior of gravitating bodies at short scales was demonstrated recently in Ref. [24].

The no-go arguments discussed in the previous section are not applicable to the theories with infinite-volume extra dimensions. The crucial property of this class of theories is that despite the unbroken 4D general covariance, there is no 4D zero-mode graviton. 4D gravity on the brane is mediated by a collective mode which cannot be reduced to any 4D state. The fact most important for us is that the 4D general covariance does not require now all states to couple universally to our “graviton”. In fact, the bulk states couple to it infinitely weakly. As a result, there is no universal agent that could mediate supersymmetry breaking from SM to all existing states. Such a situation is impossible in the finite-volume theories where 4D gravity is mediated by a normalizable zero mode, which, by general covariance, must couple universally and, hence, mediates supersymmetry breaking.

Thus, the infinite-volume theories defy both arguments of Sec. 2:

(i) There can be a sector of the theory with unbroken SUSY;
(ii) Arbitrarily light 4D scalars exist that can be exploited to neutralize the cosmological term.

Moreover, the effect of the brane cosmological term is to excite extra components of the bulk graviton and curve the extra space, without inducing a large 4D curvature. We stress again that this is impossible in finite-volume theories (i.e., the theories in which the size of the extra space is smaller than the Hubble size $H_0^{-1}$) because there the extra components of the metric are always heavier than $H_0$.

Before turning to the next section let us reiterate the two main ingredients:

(1) Softness of gravity above $M_*$. If $M_*$ is the scale of quantum gravity in the bulk, we should expect that gravitational amplitudes soften for the graviton momenta above $M_*$. This is certainly supported by string theory, which at present is a primary candidate for quantum gravity. However, the softening of amplitudes does not come for free. The price to pay is the presence of an exponential multiplicity of light Regge states in the bulk. These states have the mass spacing determined by $M_*$ and interact with the SM particles on the brane. The effects of these states on the SM processes, astrophysics, and cosmology on the brane were studied in Ref. [20]. It was found that the Hagedorn-type saturation of scattering processes on the brane will be reached at energies of order $E \sim \sqrt{M_* M_{Pl}} \sim \text{TeV}$. This will manifest itself at these energies in production of states with the Regge spectrum spaced by $M_*$. Therefore, the present model makes a definitive predictions for missing energy signals is accelerator experiments in the TeV range [20].

(2) Unbroken bulk supersymmetry. As we discussed already, this is something that is not possible in any finite-volume theory. The reason being that in any such a theory there is a normalizable zero mode graviton which must couple universally to all the states. Therefore, in an effective low energy theory, below the compactification scale, all the states experience non-zero Fermi-Bose mass splitting. This is true irrespective whether they come from the brane or from the bulk. As we dis-
cussed above, however, the infinite volume theories allow to keep Fermi-Bose mass degeneracy in the bulk while breaking SUSY on the brane.

4 Small cosmological constant

Let us start our discussions in this section with BPS branes. These branes are static even though the 4D cosmological constant on the worldvolume, i.e., the brane tension, takes a nonzero value. For BPS branes this is achieved by tuning the brane tension to the brane charge under the corresponding higher-form antisymmetric fields. The tuning condition is normally nothing but the supersymmetric BPS condition indicating that some part of the original bulk supersymmetry is (un)broken on the brane. However, to describe the real world we would like to break all the supersymmetries on the brane. Hence, the tuning condition is not going to be protected against radiative corrections due to quantum loops and is expected to be violated in general. The result is a non-BPS brane. The key question is whether a non-BPS brane remains static or whether it should start inflating its worldvolume, and if it inflates what should be the inflation rate. These issues will be addressed in the next two subsections.

4.1 Static versus inflating solutions

In this subsection we study the limit $V_N \rightarrow \infty$. There are known non-BPS solutions with a nonzero tension the worldvolume theory of which does not posses 4D Poincare invariance [25]. However, we are after a static solution with a 4D Poincare invariant flat worldvolume. The existence of this solution in the theory with the induced EH term follows from its existence in the theory without the EH term since $\mathcal{R} = 0$ on the solution, by construction. The solution can be parametrized as

$$ds^2 = A^2(\rho) \tilde{g}_{\mu\nu}(x) \, dx^\mu dx^\nu - B^2(\rho) \, d\rho^2 - C^2(\rho) \, \rho^2 \, d\Omega^2_{N-1} ,$$

where $\rho \equiv \sqrt{\rho_1^2 + \ldots + \rho_N^2}$ and the functions $A, B, C$ depend on $\mathcal{E}_4$ and $M_4$. One solution with $\tilde{g}_{\mu\nu}(x) = \eta_{\mu\nu}$, $\mathcal{R}(\tilde{g}) = 0$ is known explicitly [26]. For this solution the functions $A, B, C$ can be written as certain powers of $\left(1 - \rho_N^{N-2}/\rho^{N-2}\right)$ (see Ref. [26]). If this were a unique and physical solution, being supplemented by the present mechanism to obtain 4D gravity on the brane, it would solve the cosmological constant problem. However, it is hard to give physical interpretation to the above
solution since it develops a naked singularity \[27\] in the bulk at \( \rho = \rho_g \), where

\[
\rho_g \sim \left( \frac{\mathcal{E}_s}{M_4^4} \right)^{\frac{1}{N-2}} \frac{1}{M_*}. \tag{29}
\]

In our case the value of \( \rho_g \) is huge, it can be as big as \( H_0^{-1} \). Although this solution can be used in the region \( \rho > \rho_g \), it is hard to continue analytically to the most interesting for us domain \( \rho < \rho_g \). Other solutions of this type can be found in Refs. \[28\]. For these solutions

\[
A^2(\rho) = \left| \frac{1 - (\rho_g/\rho)^{2(N-2)}}{1 + (\rho_g/\rho)^{2(N-2)}} \right|^{1/4},
\]

\[
B^2(\rho) = \left| 1 - (\rho_g/\rho)^{2(N-2)} \right|^{1/(N-2)} \times \left| 1 + (\rho_g/\rho)^{2(N-2)} \right|^{1/(N-2)}, \tag{30}
\]

with \( C^2 = B^2 \) and a nonzero profile of the dilaton

\[
\phi \sim \left( \ln|1 + (\rho_g/\rho)^{2(N-2)}| - \ln|1 - (\rho_g/\rho)^{2(N-2)}| \right).
\]

As in the previous case, the solution has a naked singularity at \( \rho = \rho_g \). At this particular point the Einstein equations are not satisfied. Besides this point, the manifest form of the above solutions have rather limited applicability in the present case. The reason being that the solutions cannot be trusted at distances shorter than certain \( \rho_* \), \( M_*^{-1} \ll \rho_* \ll \rho_g \), since the higher curvature invariants become important in that domain (see detailed discussions in the next section). Because of this, and also in order to emphasize that our discussions below are rather general, we will not use explicit forms of these backgrounds, but instead derive all the expressions for arbitrary \( A, B \) and \( C \). We will utilize the only properties that the naked singularity in these solutions occurs at \( \rho = \rho_g \) and that the UV physics should soften the solutions at the scale \( \rho_* \).

The singularity at \( \rho = \rho_g \) appears because the solutions are highly constrained by the requirement that the worldvolume be \textit{non-inflating} (see also discussions in section 7). Relaxing this constraint and allowing the solution to inflate its world-volume, could smear the singularity. A number of examples are known where the phenomenon of the singularity smearing by inflation takes place. The first example is that of a codimension one object with planar symmetry. Requiring that the

\footnote{It is supposed to be a singularity at the position of the core at \( \rho = 0 \); the latter would be an artifact of the delta-function approximation which we use and could be removed by reintroducing a small width of the brane. However, in the static solution the \( \rho = 0 \) point cannot be approached from the bulk since the singularity at \( \rho = \rho_g \) prevents one to do so. One could cut space at \( \rho = \rho_g \) and place a source at that point \[27\]. However, in this case the matching condition at \( \rho = \rho_g \) require the source to have an equation of state which is different from that of a brane. We would like to thanks R. Emparan for useful discussions on these issues (see also calculations in \[20\]).}

In what follows we will argue that the physical brane inflates in such a way that the singularity at \( \rho = \rho_g \) is removed and the point \( \rho = 0 \) where the brane is located can be approached smoothly.
worldvolume is non-inflating, one obtains a solution that has a singularity at certain distance off the core \([29]\). This singularity is removed when the worldvolume is allowed to inflate. In particular, the singularity is replaced by a horizon in the bulk \([30]\). The distance from the core to the horizon is determined by the curvature radius of the worldvolume de Sitter space \([30]\). The next example is that of a global string in 4D space. The constraint that the solution is non-inflating leads to a naked singularity at a certain distance from the core \([31]\). However, an inflating solution exists in which the singularity is replaced by the horizon \([32]\).

A similar phenomenon takes place in the case of a local string which has the tension that is much larger than the square of the Planck scale \([33]\) (this is similar to our condition \(E_4 \gg M_4^4\)). Finally, a static global string in a 6D theory (a three-brane) has a singularity in the bulk \([34]\) which is removed as the worldvolume is allowed to inflate \([35]\).

Common features in all these cases are that the inflating solution is the only nonsingular solution. However, it is not possible to obtain analytic expressions for these inflating solutions (except for the codimension one case of a domain wall which is not suitable for our purposes). Instead, the proof of the existence and uniqueness of the inflating nonsingular solution is usually given either by the methods of dynamical systems \([32, 35]\) or by numerical studies \([33]\).

Based on these facts we expect that there exists a certain inflating solution in our system,

\[
ds^2 = A_1^2(t, \rho) dt^2 - A_2^2(t, \rho) \gamma_{ij}(x) dx^i dx^j - \gamma_{mn}(t, \rho) d\rho^m d\rho^n,
\]

in which the singularity of the static solution \((28)\) at \(\rho = \rho_g\) is replaced by a horizon.

It is hard to find this solution analytically (numerical results will be discussed elsewhere). Nevertheless, in the next subsection we will find the properties of this solution using the method of a probe-brane. Here, assuming the existence of the solution, we estimate the inflation rate of the worldvolume using simple arguments and postponing more careful discussions of their applicability till the next subsection.

Based on the lower-dimensional examples discussed above, we expect that the worldvolume inflation gives rise to a horizon in the bulk at a distance \(\rho_H\) from the core that is determined by the de Sitter curvature radius of the 4D worldvolume,

\[
\rho_H \sim r_{\text{dS}_4} \equiv \frac{1}{\sqrt{\Lambda_4}} \equiv \frac{1}{H}.
\]

On the other hand, we expect that this horizon is exactly what replaces (smoothes out) the singularity at \(\rho_g\) found in the static solution. Hence, we find

\[
\rho_H \sim \rho_g \sim \left(\frac{E_4}{M_4^4}\right)^{\frac{1}{N-2}} \frac{1}{M_4}.
\]

Substituting this into Eq. \((32)\) we obtain

\[
H \sim M_4 \left(\frac{M_4^4}{E_4}\right)^{\frac{1}{N-2}}.
\]
We would like to make a few comments here. The simple arguments presented above do not use the fact that there is an induced EH term on the brane and that there are higher derivative terms in the bulk. However, as we will see in the next sections, the existence of these terms is absolutely crucial for the above naive arguments to hold. Moreover, in general one should also include the effect of the induced EH term in Eqs. (32, 33). In that case we should make the following substitution in these equations: \( E_4 \rightarrow E_4 - M_{Pl}^2 H^2 \). This gives rise to a second solution (on top of the one in (34)) with the inflation rate \( H^2 \sim E_4/M_{Pl}^2 \). In this solution, the tension curves strongly the 4D worldvolume, but affect only mildly the bulk space. Therefore, the effect is opposite to the one obtained in the solution (34) considered in the present work. A special interesting case of this is a possibility to have “stealth branes” [36, 37] for which \( E_4 - M_{Pl}^2 H^2 \) is zero. On these branes the induces EH term completely neutralizes the brane tensions, so that the brane has no gravitational effect of the outside world. Hence, such a brane would be invisible for a bulk observer. These branes can be produced in a tunneling process via bubble nucleation [36]. Hence, there are three different solutions. These solutions are distinguished by initial and boundary conditions [36, 37]. In the present context we consider the initial and boundary conditions on the brane that are relevant for the solutions of the type (34).

After these comments let us discuss an explicit example of a codimension one object, with the known exact solution [29, 30]. For this solution [29, 30] \( \rho_H \sim 1/H \). Hence, we reproduce the known result \( H \sim E_4/M_{Pl}^3 \).

Therefore, as in four dimensions, the inflation rate in a 5D theory grows when \( E_4 \) is increasing. This is a property of \( N = 0 \) and \( N = 1 \) theories. However, for \( N > 2 \) the exponent on the right-hand side of Eq. (34) is positive, and we get a flipped relation where the inflation rate reduces as \( E_4 \) increases.

### 4.2 The probe-brane method

Let us go back to a static solution discussed in the previous section, that has a singularity at \( \rho_g \). Below we are going to use this singular solution in order to argue that a nonsingular inflating solution should also exist and to estimate its rate of inflation. For this we use the method of a probe-brane. Let us describe this procedure in detail. The singularity at \( \rho = \rho_g \) should be expected to be smoothed out in a physical solution. That is to say, the Einstein equations for the physical solutions should be satisfied everywhere including the point \( \rho = \rho_g \). However, whatever mechanism smoothes out this singularity the very same mechanism can give rise to an additional contribution to the energy density and make the original static solution time dependent [39]. The major goal in that case is to find out how strong is the time dependence of the solution, in particular, what is the inflation rate on the worldvolume. For this we can use the probe-brane method.
Let us compactify extra space at a huge distance \( l = \rho_g \). This is only possible if we place a probe (anti)brane at the point \( l = \rho_g \). The compactification radius is huge, of the order of the Hubble size. Nevertheless, the role of the probe-brane is to enforce that the Einstein equations are satisfied everywhere including the point \( l = \rho_g \). As we argue below, for a generic value of the probe brane tension this procedure deforms the original static solution so that it will start to inflate.

To start with let us neglect the bulk higher derivative terms. The role of those terms, as we will discuss in the next section, is to make the solution sensible in a region close to the brane core. The relevant part of (11) takes the form

\[
L_4 = M_s^{2+N} \int_0^l d^N \rho \sqrt{G} \mathcal{R}(G) + \sqrt{g} \left( M_p^2 \mathcal{R}(\bar{g}) + \mathcal{E}_4' \right) + \int_0^l d^N \rho \sqrt{G} \mathcal{E}_c[l], \tag{35}
\]

where

\[
\mathcal{E}_4' \equiv \mathcal{E}_4 + T_{\text{probe}}, \tag{36}
\]

denotes the sum of the brane tension \( \mathcal{E}_4 \) and the tension of a probe brane \( T_{\text{probe}} \). Furthermore, \( \mathcal{E}_c[l] \) denotes the Casimir energy. The latter arises because of the compactness of extra space. It is clear that \( \mathcal{E}_c[l] \sim 1/l^{4+N} \) and at large \( l \) it vanishes.

Since the extra space is terminated at a finite proper distance \( l \), the spectrum of gravitons consists of a massless zero mode and a tower of massive KK states with the masses \( \sim 1/l \). Let us consider dynamics at very low energies \( E \) that are relevant for the cosmological constant problem. Naively, for \( E \leq l^{-1} \) all the massive KK modes are decoupled, only the zero mode remains. However, this statement needs a more careful justification since its validity is limited, as we shall see now.

Consider an observer who sits in a brane core in the vicinity of the point \( \rho \sim 0 \) and is able to probe distances that are smaller than the brane thickness \( \Delta \). This observer finds him/herself in a \((4+N)\)-dimensional space that has a uniform energy density \( \mathcal{E}_4/\Delta^N \). If the \((4+N)\)-dimensional Hubble distance of this observer, \( H_W^{-1} \), is smaller than the brane width \( \Delta \), the observer would find him/herself in a \((4+N)\)-dimensional space with positive energy density \( \mathcal{E}_4/\Delta^N \) that should inflate along all \((4+N)\)-dimensions (in analogy with the topological inflation of Refs. [40, 11]). This would invalidate our arguments on the low-energy description. In order for this not to happen, the \((4+N)\)-dimensional Hubble distance \( H_W^{-1} \) should be larger than the brane thickness \( \Delta \). This puts a bound on \( \mathcal{E}_4 \). Let us find this bound. A vital role here is played by the induced EH term. Inside the brane core gravity is weak precisely because of this term, and the Hubble distance \( H_W^{-1} \) becomes large. Although the induced term cannot be written as a local term when the \( \Delta \neq 0 \) effects are kept [12], nevertheless, one can think of the induced term as a renormalization of the bulk EH term inside the brane core of the size \( \Delta \). Hence, \( H_W^{-1} \) can be estimated from the

\[ 13 \] If SUSY is broken on the brane but not in the bulk, the SUSY breaking will penetrate to the bulk as long as \( l < \infty \). This will give rise to a contribution to the vacuum energy neglected above which for large \( l \) is suppressed as \( \sim (\mathcal{E}_4/M_s^{2+N} l^N) M_s^N \).
following expression

\[ \frac{M_{\text{Pl}}^2}{\Delta^N} H_w^2 \sim \frac{\mathcal{E}_4}{\Delta^N}. \]

This gives rise to the bound

\[ \mathcal{E}_4 \ll \frac{M_{\text{Pl}}^2}{\Delta^2}. \]

A typical value of the brane thickness, as was discussed in Sect. 3, is \( \Delta \sim M_{\text{SM}}^{-1} \). Hence, we see that the topological inflation does not take place even for the energy density as large as \( M_{\text{SM}}^4 \). This is all we want. If we were to ignore the induced term on the brane, we would find that for a brane the tension of which exceeds TeV\(^4\) the topological inflation would take place unless the brane width was several orders of magnitude smaller than the Planck length \( M_{\text{Pl}}^{-1} \).

After this digression, let us proceed with a brane tension which satisfies the above constraint. We set \( l \) to be somewhat smaller than the Hubble size, and turn to the distances of the order of \( H_0^{-1} \). At those distances all massive KK modes are decoupled. The low energy effective Lagrangian for the zero mode reads

\[ \mathcal{L}_{\text{eff}} = (M_*^2 + l^N + M_{\text{Pl}}^2) \sqrt{g_{zm}} R(\bar{g}_{zm}) + \sqrt{g_{zm}} \left( \mathcal{E}_4' + l^N \mathcal{E}_c[l] \right), \]  

(37)

where \( \bar{g}_{zm} \) denotes the zero mode.

The Einstein equations for this Lagrangian density yield a solution with the following inflation rate:

\[ H^2 \equiv \Lambda_4 \sim \frac{\mathcal{E}_4' + l^N \mathcal{E}_c[l]}{M_*^{2+N} l^N + M_{\text{Pl}}^2}. \]  

(38)

Hence, we see that a solution will inflate in general. One could certainly choose the tension of a probe brane in (36) such that \( H \) vanishes in (38). However, this cannot be considered as a solution to the cosmological constant problem as it is nothing but another fine tuning of the parameters of the theory. Therefore, in what follows we abandon this possibility and consider a case when the tension of a probe brane takes a generic value not necessarily the one that nullifies \( H \). The magnitude of \( T_{\text{probe}} \) can generically be of the order of \( \mathcal{E}_4 \) but otherwise bigger (or smaller) by some factor. From the technical point of view that means that in (38) we can think that \( \mathcal{E}_4' \) takes the value that is within an order of magnitude of the brane tension \( \mathcal{E}_4 \).

If we were confined to a theory in which the conventional KK relation (3) holds we would obtain an unacceptable result from (38)

\[ \Lambda_4 \sim (\mathcal{E}_4' + l^N \mathcal{E}_c)/M_{\text{Pl}}^2 \gtrsim (10^{-3} \text{ eV})^2. \]

However, in the present model the relation (3) does not hold, as we have emphasized in (4). Therefore, we could (and we should) consider a regime where

\[ M_*^{2+N} l^N = M_{\text{g}}^{2+N} \rho_{\text{g}}^N \gg M_{\text{Pl}}^2. \]  

(39)
In this regime, the second terms in the numerator and denominator of Eq. (38) can be neglected compared to the first terms, respectively. Furthermore, the value of $\Lambda_4$ can be made much smaller than $(10^{-3}$ eV$)^2$ by making $l$ large.

How large could $l$ be? Since we terminate extra space at the singularity of the original static solution, $l = \rho_g$. On the other hand, in order for this consideration to be relevant for the cosmological constant problem, $l$ should be somewhat smaller than the present-day Hubble size,

$$l \lesssim H_0^{-1}.$$  

Hence, we get an estimate for the rate of inflation,

$$H^2 \sim \frac{\mathcal{E}_4'}{M_{*}^{2+N} \rho_g^N}.$$  \hspace{1cm} (40)  

Substituting into Eq. (40) the expression

$$\rho_g \sim M_{*}^{-1}(\mathcal{E}_4'/M_{*}^4)^{1/(N-2)}$$

we find the same equation (34) from which we see that the inflation rate $H$ decreases with increasing $\mathcal{E}_4'$ if $N > 2$.

Note that the probe-brane and compactification is used in this subsection as technical tools to argue that the truly physical solution in uncompactified space will inflate with the rate (34). Certainly, if the latter solution is found by some other means, such as, e.g., numerical simulations, the probe-brane and compactification method will just become redundancy.

Finally, as representative examples, we give estimates for $H$ as determined from (34),

$$H \sim 10^{-33} \text{ eV \ for \ } N = 4, \ M_4 \sim 10^{-3} \text{ eV \ , \ } \mathcal{E}_4' \sim (\text{TeV})^4 ;$$  

$$H \sim 10^{-33} \text{ eV \ for \ } N = 6, \ M_4 \sim 10^{-3} \text{ eV \ , \ } \mathcal{E}_4' \sim (M_{Pl})^4.$$ \hspace{1cm} (41)  

These values are consistent with experimental data.

## 5 Softening of the bulk metric

In this section we shall study the behavior of the background solution near the brane. As we shall see, the understanding of this behavior is crucial for determining the crossover scale at which gravity on the brane switches from the 4D regime to a higher-dimensional one.

As we discussed above, the static solution of the type (28) could in general have at least two types of singularities. One is a naked singularity at $\rho = \rho_g$. As was mentioned above, this is an artifact of over-constraining the system at hand, by
requiring the flatness of the 4D metric. This singularity can be smoothed out by relaxing the requirement of the flatness and allowing for an inflating brane.

The second type of singularity occurs at \( \rho = 0 \) and is not related to inflation, but rather to the excessive warping near the origin. Smoothing out the brane (by substituting the delta function warping brane by a smooth sharply localized function of a finite width) would cut-off infinities. Nevertheless, the curvature of the space-time near the brane would still be much higher than the fundamental scale \( M_\ast \) (i.e. the string scale) and an effective field theory approximation would break down. We would like to analyze the later issue in detail in the present section. Before we do so, let us recall that in the previous sections we argued that the physical solution is the one that inflates its worldvolume. However, the inflation rate is tiny. Therefore, for practical calculations it is easier to neglect this weak time-dependence of the metric and to use the static solution (dealing carefully with the singularities). The presence of a small inflation rate is not going to affect any predictions for distances bigger than 0.1 mm and smaller that the present day Hubble size.

Consider, for instance, the behavior of the Riemann tensor \( R_{ABCD} \) in such a background. The bilinear curvature invariant constructed out of this tensor scales as (near the core)

\[
R_{ABCD}R^{ABCD} \sim \frac{\rho_\ast^n}{\rho^{n+4}}, \tag{42}
\]

where \( n \) depends on the precise form of the solution. For the branes of interest the right-hand side in Eq. (42) becomes larger than \( M_\ast^4 \) at a macroscopic distance from the core \( \rho = \rho_\ast \gg M_\ast^{-1} \). This would mean, in particular, that at distances \( \rho \lesssim \rho_\ast \) a freely-falling observer would experience a tidal force exceeding, by many orders of magnitude, the fundamental scale, and, thus, the solution cannot be trusted below \( \rho_\ast \). (Note that because of the nontrivial geometry \( \rho_\ast \) is not a proper distance. The invariant statement is that the solution breaks down when the curvatures becomes of the order of \( M_\ast^2 \).) The question is how can we understand the required smoothness of the solution near the brane from an effective field theory perspective?

The problem that we encounter here is by no means different than that emerging in a problem of a freely-falling observer in the vicinity of an ordinary four-dimensional macroscopic Schwarzschild black hole. At distances

\[
\rho_\ast \sim M_P^{-1} \left( \frac{M}{M_P} \right)^{1/3} \gg M_P^{-1}, \tag{43}
\]

the standard Schwarzschild solution obtained for a point-like object should break down \[\footnote{Modulo the fact that in the interior of the Schwarzschild solution space and time interchange.}\].

In both cases above one can restore the validity of the solutions only by taking account of higher derivative operators (HDO’s) coming from the underlying fundamental theory of gravity. These correction cannot be neglected at distances \( \sim \rho_\ast \).
Thus, one has to take into account all possible higher-dimensional invariants suppressed by the scale $M_*$ in the bulk action,

$$S_{\text{bulk}} = M_*^{N+2} \int d^4x d^N \rho \sqrt{G} \left\{ \mathcal{R} + \frac{\alpha}{M_*^2} \mathcal{R}_{ABCD} \mathcal{R}^{ABCD} + \ldots \right\} , \quad (44)$$

(similar terms added to the brane action are suppressed by powers of $M_{\text{Pl}}$ and are less important). Here $\alpha$ is a numerical coefficient of order one. These terms modify the Einstein equations in the following way:

$$M_*^{2+N} \left[ \mathcal{R}_{AB} - \frac{1}{2} G_{AB} \left( \mathcal{R} + \frac{\alpha}{M_*^2} \mathcal{R}_{CDMN} \mathcal{R}^{CDMN} \right) + \frac{\alpha}{M_*^2} \left( 2 \mathcal{R}_{ACDM} \mathcal{R}_{B}^{CDM} + 4 \nabla^C \nabla^D \mathcal{R}_{ACBD} \right) + \ldots \right] + M_{\text{Pl}}^2 \delta^\mu_A \delta^\nu_B \delta^{(N)}(\rho) \left( \mathcal{R}_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} \mathcal{R} \right) = \mathcal{E}_4 \delta^\mu_A \delta^\nu_B \delta^{(N)}(\rho) . \quad (45)$$

Strictly speaking, one has to include an infinite number of higher-dimensional curvature invariants, and, therefore, finding the exact form of the solution is impossible. Fortunately, the precise form of the metric near the brane is not important for us. What is crucial, however, is the softening of the background metric. The fact that inclusion of higher-derivative terms must lead to the softening can be explicitly seen in the linearized example (note that linearization per se does not make the problem milder. In the Einstein gravity without the higher-curvature terms the ultraviolet singularity in the linearized solution is as severe as in the non-linear one).

For the purpose of a simple and transparent illustration we shall ignore the tensorial structure and consider the linearized scalar gravity. The equation then becomes

$$\left( M_*^{2+N} \partial_A \partial^A + M_{\text{Pl}}^2 \delta^{(N)}(\rho) \partial_\mu \partial^\mu \right) \Phi = \mathcal{E}_4 \delta^{(N)}(\rho) . \quad (46)$$

Here $\Phi$ should be thought of as a counterpart of the classical background. The solution independent of the world-volume space-time coordinates is a massless Euclidean Green’s function in the transverse space

$$\Phi(\rho) = \frac{\mathcal{E}_4}{M_*^{2+N} \rho^{N-2}} . \quad (47)$$

It is obviously singular in the ultraviolet. The singularity is smoothed out by adding higher-derivative terms in the action. The point can be illustrated by adding a single additional power of the operator $\partial_A \partial^A$ in the left-hand side of Eq. (46),

$$M_*^{2+N} \left( \partial_A \partial^A + \left( \frac{\partial_A \partial^A}{M_*^{k-2}} \right)^k \right) \Phi + M_{\text{Pl}}^2 \delta^{(N)}(\rho) \left( \partial_\mu \partial^\mu \Phi \right) = \mathcal{E}_4 \delta^{(N)}(\rho) . \quad (48)$$
The solution to this equation,

\[ \Phi(\rho) = \frac{\mathcal{E}_4}{M_{s+2}^4} \int \frac{d^N q}{(2\pi)^N} \frac{e^{iq\rho}}{q^2 + (q^2)^{k} M_{s}^{2-k}}, \]  

is non-singular at \( \rho = 0 \) for \( 2k > N + 1 \) and \( N \geq 2 \). In fact, \( \Phi(0) \propto \mathcal{E}_4/M_{s}^4 \). Therefore, based on this example, we expect that HDO’s give rise to UV softness of the classical backgrounds at distances \( \rho \lesssim \rho_\ast \). This feature is expected to be true in a theory of quantum gravity.

6 Four-dimensional gravity on a brane

In this section we study whether the laws of 4D gravity are reproduced on a brane at observable distances. The 4D laws are certainly obtained when \( \mathcal{E}_4 = 0 \), as it was shown in Refs. \([3, 4, 20, 12]\). The crucial difference here is the presence of a non-zero brane tension which gives rise to new nontrivial properties of the classical background. For \( N = 2 \) case this was studied in Ref. \([48]\) with the conclusion that the properties obtained in Refs. \([3, 4, 20, 12]\) hold unchanged. In the next two subsections we will perform the analysis for \( N > 2 \) in terms of the Green’s function on the brane following Refs. \([4, 12]\), as well as in terms of the KK modes following the method of Ref. \([20]\). We will show that at observable distances the 4D laws of gravity are indeed reproduced. As before, we will neglect a tiny time dependence of the classical background and will treat it as static.

6.1 Propagator analysis

The nature of gravity on the brane perhaps is simpler understood from the propagator analysis. The equation for the graviton two-point Green’s function (we omit tensorial structures) takes the following form

\[ M_{s+2}^{N-1} \hat{\mathcal{O}}_{4+N} G(x, \rho) + \frac{M_{Pl}^2 \delta(\rho)}{\rho^{N-1} A^2(\rho)} \hat{\mathcal{O}}_{4} G(x, 0) = T \delta^{(4)}(x) \frac{\delta(\rho)}{\rho^{N-1}}, \]  

where \( T \) denotes the source (which will be put equal to 1 below) and

\[ \hat{\mathcal{O}}_{4+N} \equiv \frac{1}{\sqrt{G}} \partial_A \sqrt{G} G^{AB} \partial_B + \text{higher derivatives}, \quad \hat{\mathcal{O}}_{4} \equiv \partial^\mu \partial_\mu. \]  

Using the technique of Ref. \([3]\), the scalar part of the solution in the Euclidean four-momentum space can be written in the form\(^{15}\)

\[ G(p, \rho) = \frac{D(p, \rho)}{M_{Pl}^2 p^2 D(p, 0) + M_{s+2}^{2+N}}, \]

\(^{15}\)Note that in the warped case the scale \( M_{ind} \) differs from that of the flat case by a constant multiplier \( A^2(\Delta) \). For simplicity this won’t be depicted manifestly below.
where $D(p, \rho)$ is the Euclidean 4-momentum Green’s function of the bulk operator $\hat{O}_{4+N}$, that is $\hat{O}_{4+N} D(p, \rho) = \delta(p)/\rho^{N-1}$. What is crucial for us is the behavior of the Green’s function on the brane

$$G(p, \rho = 0) = \frac{1}{M_{\text{Pl}}^2 p^2 + M_4^{2+N} D^{-1}(p, 0)}.$$  \hfill (53)

Let us discuss this expression first. The denominator in (53) consists of two terms. The first term, $M_{\text{Pl}}^2 p^2$, is what gives rise to 4D behavior. The second term in the denominator, $M_4^{2+N} D^{-1}(p, 0)$, sets the deviation from the 4D laws and is due to the infinite-volume extra bulk. Therefore, in the regime when $M_{\text{Pl}}^2 p^2$ dominates over $M_4^{2+N} D^{-1}(p, 0)$ we get 4D laws, while in the opposite case we obtain the higher-dimensional behavior. The question is what is the crossover scale at which this transition occurs. To answer this question we need to know the expression for $D(p, \rho)$. Let us start for simplicity with the case when $E_4 = 0$, i.e., the background metric is flat. We will denote the corresponding Green’s function by $D_0(p, \rho)$ to distinguish it from $D(p, \rho)$. Moreover, let us drop for a moment higher-derivatives in the expression for $\hat{O}_{4+N}$. In this case $D_0(p, \rho)$ is nothing but the Green’s function of the $(4+N)$-dimensional d’Alambertian. Its behavior at the origin is well known:

$$D_0(p, \rho = 0) \sim \frac{1}{\rho^{N-2}}.$$  \hfill (54)

Hence, $D_0(p, 0)$ diverges and therefore the term $M_4^{2+N} D^{-1}(p, 0) = M_4^{2+N} D_0^{-1}(p, 0)$ in Eq. (53) goes to zero. This would indicate that 4D gravity is reproduced on the brane at all distances. However, the UV divergence in (54) is unphysical. This divergence is smoothed out by UV physics \[4, 23, 11, 12\]. In reality the bulk action and the operator $\hat{O}_{4+N}$ contain an infinite number of high-derivative terms that should smooth out singularities in the Green’s function in (54). Since these HDO’s are suppressed by the scale $M_*$, it is natural that the expressions (54) is softened at the very same scale $\rho \sim M_*^{-1}$. As a result one obtains \[11, 12\] $D_0(p, \rho = 0) \sim M_*^{N-2}(1 + O(p/M_*))$. Substituting the latter expression into (53) we find that $M_4^{2+N} D^{-1}(p, 0) = M_4^{2+N} D_0^{-1}(p, 0) \sim M_*^4$. Therefore the crossover scale is $r_c \sim M_{\text{Pl}}/M_* \sim 10^{28}$ cm. At distances shorter than this 4D laws dominate.

Let us now switch on the effects of a non-zero tension $E_4$. The background in this case is highly distorted. The distortion is especially strong near the brane. Let us start again with the case when the HDO’s are neglected and $\hat{O}_{4+N}$ contains only two derivatives at most. Then, the expression for the $D$-function takes the form:

$$D(p, \rho) \sim D_0(p, \rho) F(\rho_g/\rho),$$  \hfill (55)

where, as before, $D_0(p, \rho) \sim 1/\rho^{N-2}$ and $F$ is some function which is completely determined by the background metric (by the functions $A, B$ and $C$) and $F(0) = \text{const}$. In the region where the solution of the Einstein equations can be trusted, $F$ can be approximated as follows, $F(\rho_g/\rho) = (\rho_g/\rho)^{2\gamma} + c$, where $\gamma$ and $c$ are some
constants determined by $N$. If we were to trust this solution all the way down to the point $\rho = 0$ we would obtain again that $M_*^{2+N} D^{-1}(p, 0) = 0$ and that gravity is always four-dimensional on the brane. However, as we discussed above (see also the previous section), the existence of high-derivative terms tells us that the background solution cannot be trusted for distances $\rho \ll \rho_*$. In general, $\rho_* = M_*^{-1}(\rho_g M_*)^\gamma$ with $\gamma \ll 1$ and $\rho_* \ll \rho_g$. Thus, for $\rho \ll \rho_*$ the higher curvature invariants become large in units of $M_*$ and infinite number of them should be taken into account. In order to find the effect of this softening, let us take a closer look at the expression (55). There are two sources of singularities in this expression. The first one emerges on the r.h.s. of (55) as a multiplier, $D_0 \sim 1/\rho^{N-2}$; this singularity was discussed above in (54) and is independent of the background geometry. Instead, it emerges when the operator $\hat{\mathcal{O}}_{4+N}$ is restricted to the quadratic order only. We expect that this singularity, as before, is softened at the scale $M_*^{-1}$ after the higher derivatives are introduced in $\hat{\mathcal{O}}_{4+N}$. Hence, in (55) when we take the limit $\rho \to 0$ we should make a substitution $D_0 \sim 1/\rho^{N-2} \to M_*^{-2}(1 + \mathcal{O}(p/M_*))$. On the other hand, the second source of singularity in (55) is due to the function $\mathcal{F}$. This singularity is directly related to the fact that the background solution breaks down at distances of the order of $\rho_*$. As we discussed in the previous section, the UV completion of the theory by HDO’s should smooth out this singularity in the background solution. In order to get the crossover scale we can use the following procedure which overestimates the value of $M_*^{2+N} D^{-1}(p, 0)$. In the limit $\rho \to 0$ we could make the following substitution in the expression for $\mathcal{F}$ and in (55), $\rho_g/\rho \to \rho_g/\rho_*$. Using these arguments we find $D(p, 0) \lesssim M_*^{-2}[(\rho_g/\rho_*)^\alpha + c]$. Moreover, taking into account that $\rho_* \ll \rho_g$ we get: $M_*^{2+N} D^{-1}(p, 0) \ll M_*^4$. Therefore, we conclude that, as in the zero-tension case, the crossover distance in the non-zero tension case can be of the order of $10^{28}$ cm.

### 6.2 KK mode analysis

The purpose of this section is to study the effect of a nonzero brane tension on 4D gravity in terms of the KK modes. The Einstein equations (with up to two derivatives) that follow from the action (11) take the form

$$
M_*^{2+N} \left( R_{AB} - \frac{1}{2} G_{AB} R \right) + M_*^2 \delta^\mu_A \delta^\nu_B \delta^{(N)}(\rho) \left( \bar{R}_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} \bar{R} \right) = \mathcal{E}_4 \bar{g}_{\mu\nu} \delta^\mu_A \delta^\nu_B \delta^{(N)}(\rho) .
$$

Below we consider fluctuations $h_{\mu\nu}(x, \rho_n)$ that are relevant for 4D interactions on the brane,

$$
ds^2 = A^2(\rho) \left[ \eta_{\mu\nu} + h_{\mu\nu}(x, \rho_n) \right] dx^\mu dx^\nu - B^2(\rho) d\rho^2 - C^2(\rho) \rho^2 d\Omega^2_{N-1}.
$$

---

If we were to assume that the background solution is softened at $M_*^{-1}$ rather than at $\rho_*$, we would obtain even larger value for the crossover scale. This can be turned around to make the following observation. If the background metric softens at $M_*^{-1}$, and/or if $\rho_g \gg \rho_*$, the value of $M_*$ should not necessarily be restricted to $10^{-3}$ eV, but can be much higher. Unfortunately, these properties does not seem to allow further analytic investigation.
In what follows we use the transverse-traceless gauge
\[ \partial_\mu h_{\mu\nu} = 0 = h_0. \]

As typically happens in warped backgrounds, equations for graviton fluctuations are identical to those for a minimally coupled scalar \[42, 43\]. The present case is no exception. Equation (56) on the background defined in Eq. (57) takes the form
\[
M_5^{2+N} \hat{O}_{4+N} h_{\mu\nu}(x, \rho_n) + \frac{M_{Pl}^2 \delta(\rho)}{\rho^{N-1} A^2(\rho)} \hat{O}_4 h_{\mu\nu}(x, 0) = 0, 
\]
where
\[
\hat{O}_{4+N} \equiv \frac{1}{\sqrt{G}} \partial_A \sqrt{G} G^{AB} \partial_B, \quad \hat{O}_4 \equiv \partial^\mu \partial_\mu.
\]

To simplify Eq. (58) we turn to spherical coordinates with respect to \(\rho_n, n = 1, 2, ..., N\), and decompose fluctuations as follows:
\[
h_{\mu\nu}(x, \rho) \equiv \epsilon_{\mu\nu}(x) \sigma(\rho) \phi(\Omega),
\]
where the components in Eq. (60) satisfy the conditions
\[
\hat{O}_{4+N} \epsilon_{\mu\nu}(x) = \frac{1}{A^2} \partial^\mu \partial_\mu \epsilon_{\mu\nu}(x) = -\frac{m^2}{A^2} \epsilon_{\mu\nu}(x),
\]
\[
\hat{O}_{4+N} \phi(\Omega) = \frac{l(l + N - 2)}{C^2 \rho^2} \phi(\Omega).
\]

Using these expressions we rearrange Eq. (58) as follows:
\[
\left\{ \frac{1}{\sqrt{G}} \partial_\rho \sqrt{G} G^{\rho\rho} \partial_\rho + \frac{l(l + N - 2)}{C^2 \rho^2} - \frac{m^2 M_{Pl}^2 \delta(\rho)}{M_5^{2+N} \rho^{N-1} A^2(\rho)} \right\} \sigma = \frac{m^2}{A^2} \sigma.
\]

Our goal is to rewrite this expression in the form of a Schrödinger equation for fluctuations of mass \(m\). We follow the method of Refs. [34, 27]. It is useful to introduce a new function
\[
\chi = \frac{G^{1/4}}{\sqrt{AB}} \sigma,
\]
and a new coordinate
\[
u \equiv \int_0^\rho d\tau \frac{B(\tau)}{A(\tau)}.
\]

In terms of these variables Eq. (58) takes the form
\[
\left\{ -\frac{d^2}{du^2} + V_{\text{eff}}(u) + \frac{A^2 l(l + N - 2)}{C^2 \rho^2} - \frac{m^2 M_{Pl}^2 \delta(\rho)}{M_5^{2+N} \rho^{N-1}} \right\} \chi = m^2 \chi,
\]
where the effective potential $V_{\text{eff}}(u)$ is defined as

$$V_{\text{eff}}(u) = \frac{\sqrt{AB}}{G^{1/4}} \frac{d^2}{du^2} \left( \frac{G^{1/4}}{\sqrt{AB}} \right).$$

(67)

Note that the first two terms in (66) can be rewritten as follows:

$$-\frac{d^2}{du^2} + V_{\text{eff}}(u) = \left( \frac{d}{du} + \frac{d\mathcal{B}}{du} \right) \left( -\frac{d}{du} + \frac{d\mathcal{B}}{du} \right),$$

(68)

where

$$\exp (\mathcal{B}) \equiv \frac{G^{1/4}}{\sqrt{AB}}.$$ 

(69)

With an appropriate physical boundary conditions the operator on the r.h.s. of (68) is self-adjoint positive-semidefinite with a complete set of eigenfunctions of non-negative eigenvalues.

Let us analyze Eq. (66), in particular the properties of the KK modes that follow from it. What is crucial for our purposes is the value of the KK wave functions on the brane, i.e., $|\chi(m, \rho = 0)|^2$. The later determines a potential between two static sources on the brane [20]. We would like to compare the properties of $|\chi(m, \rho = 0)|^2$ which are known [20, 12] only for $\mathcal{E}_4 = 0$, with the properties obtained at $\mathcal{E}_4 \neq 0$.

First, we recall the properties of $|\chi(m, \rho = 0)|^2$ for $N = 1$ and tensionless brane, $\mathcal{E}_4 = 0$. In this case $A = B = 1$, $C = 0$ and $l = 0$. Hence, $u = \rho$ and $V_{\text{eff}}(u) = 0$. Equation (69) takes the form

$$\left\{ -\frac{d^2}{d\rho^2} - \frac{m^2 M_{\text{Pl}}^2}{\mathcal{M}_*^4} \delta(\rho) \right\} \chi = m^2 \chi.$$ 

(70)

For each KK mode of mass $m$ there is a delta-function attractive potential, the strength of which is proportional to the mass of the mode itself. Hence, the higher the mass, the more the influence of the potential is. The attractive potential leads to a suppression of the wave function at the origin (suppression of $|\chi(m, \rho = 0)|^2$). Therefore, the larger the mass of a KK state, the more suppressed is its wave-function at zero.

Simple calculations in this case yield

$$|\chi(m, \rho = 0)|^2 = 4/(4 + m^2 r_{c}^2),$$

where $r_c \sim M_{\text{Pl}}^2/M_\ast^4$. This should be contrasted with the expression for $|\chi(m, \rho = 0)|^2$ in a theory with no brane induced term (i.e., with no potential in Eq. (70)). In that case $|\chi(m, \rho = 0)|^2 = 1$. We see that the KK modes with masses $m \gg r_{c}^{-1}$ are suppressed on the brane. The laws of gravity on the brane are provided by light modes with $m \ll r_{c}^{-1}$. This warrants [3, 20] that at distances $r \ll r_c$ measured along the brane the gravity laws are four-dimensional.
A similar phenomenon takes place for \( N \geq 2 \), with a tensionless brane. Here \( A = B = C = 1 \), \( u = \rho \) and \( V_{\text{eff}}(u) = 0 \). The Schrödinger equation takes the form of Eq. (66) with the above substitutions. The total potential consists of an attractive potential due to the induced term and a centrifugal repulsive potential. Because at \( \rho \to 0 \) the attractive potential is dominant, one finds properties similar to the \( N = 1 \) case. Heavy KK modes are suppressed on the brane — at distances \( r \ll r_c \) the brane-world gravity is four-dimensional. The only difference is that \( r_c \sim M_{\text{Pl}}/M^2 \) for \( N \geq 2 \).

Let us now turn to the discussion of the case of interest when \( \mathcal{E}_4 \neq 0 \) and \( A, B, C \neq 1 \). Here the complete equation (66) must be studied. For the solutions that soften due to the HDO’s close to the brane core we expect that as \( \rho \to 0 \), \( u \sim \rho \).

Hence, to study the suppression of the wave functions on the brane one can replace \( d^2/du^2 \) in (66) by \( d^2/d\rho^2 \). The next step is to clarify the role of the potential \( V_{\text{eff}}(u) \) that is nonzero when we switch on the brane tension \( \mathcal{E}_4 \neq 0 \). Since a positive tension brane should give rise to an additional attractive potential in space with \( N > 2 \), we expect that \( V_{\text{eff}}(u) \) is negative at the origin (it should tend to \(-\infty\) at the origin if the HDO’s are not taken into account).

The warp factors \( A, B \) and \( C \) contain the only dimensionfull parameter, \( \rho_g \). So does the potential \( V_{\text{eff}}(u) \). Therefore, the maximal value of the potential (if any) in the interval \( 0 < \rho < \rho_g \) should be determined by the very same scale, \( \max\{V_{\text{eff}}\} \sim \rho_{\text{g}}^{-2} \).

If the form of the attractive potential were trustable all the way down to small values of the coordinate, then an attractive nature of the potential could make easier to obtain 4D gravity on a brane as compared to the zero tension case. Unfortunately we cannot draw this conclusion since the expression for the potential is not trustable below the distance scales \( \rho < \rho_* \) (see discussions in the previous section). Although \( \rho_* \) is smaller than \( \rho_g \), nevertheless this two scales can have the same order of magnitude. Based on the discussions in the previous section one should expect that the potential in the full theory softens below \( \rho_* \) and does not really give rise to a substantial attraction below that scale. On the other hand, the potential could give rise to some undesirable results. Indeed, it could produce a bump (a potential barrier) at some finite distance from the core somewhere in the interval \( 0 < \rho < \rho_g \).

For a parameter range for which this discussion is applicable (i.e., for \( \rho_g^{-1} \ll M_* \)) the height of the bump can be of the order of \( \max\{V_{\text{eff}}\} \sim \rho_g^{-2} \). A KK mode with the mass \( m \gtrsim \rho_g^{-1} \) will not feel the presence of of such \( V_{\text{eff}} \). Its wave function will have the same properties as in the tensionless brane theory (i.e. the modes with \( m > r_c^{-1} \) will be suppressed on the brane). However, the wave-function of any KK mode with the mass \( m \lesssim \rho_g^{-1} \) will be additionally suppressed on the brane because of the potential barrier in \( V_{\text{eff}} \). The question is whether this effect can alter the laws of 4D gravity on the brane at observable distances. If \( \rho_g \) is small this effect will certainly spoil the emergence of 4D gravity on a brane. The reason is that the KK modes that are lighter than \( \rho_g^{-1} \) will be additionally suppressed on the brane. If these were the “active” modes that participate in the mediation of 4D gravity
at observable distances in the tensionless case, then having them additionally sup-
pressed would change the 4D laws. However, if $\rho_g$ is sufficiently large the modes
which are additionally suppressed are very light $m < \rho_g$, and, if so, “switching off”
these modes won’t be important for 4D gravity. For instance, if $\rho_g \gtrsim 10^{27}$ cm, as
it happens to be the case in the present model, gravity at observable distances will
not be noticeably different from gravity on a tensionless brane.

Therefore, we arrive at the following qualitative conclusion. In the worst case,
gravity on the brane worldvolume is mediated by the KK modes that have masses
in the band $\rho_g^{-1} \lesssim m \lesssim r_c^{-1}$. Hence, at distances $r \lesssim \rho_g$ the effects of the brane
tension are negligible and gravity on a brane reproduces the known four-dimensional
laws. Moreover, in a simple case when $\rho_g^{-1} \sim r_c^{-1}$, one can think, qualitatively, that
gravity on the brane is mediated by a 4D graviton of mass

$$m_g \sim \rho_g^{-1} \sim r_c^{-1}.$$ 

In the present context this value is of the order of the Hubble scale

$$m_g \sim H_0 \sim 10^{-33} \text{eV}.$$ 

A graviton with such a small mass is consistent with observations.

7 Evading no-go arguments

The no-go theorem by Weinberg [7] can rule out solutions that can be entirely un-
derstood from the low-energy point of view and do not require fine-tuning. Here we
shall briefly formulate these arguments and explain how the infinite-volume theories
avoid them.

A mechanism for the cancellation of the cosmological constant might be based
on the idea to use scalar fields which couple to the vacuum energy and dynamical-
ically adjust their expectation values to neutralize it (a lá axion) (see Ref. [7] for
a review and earlier references.). The unbroken Lorentz invariance requires that
whatever number of such fields are introduced, they must transform as scalars from
the standpoint of 4D theory (although they may be high-dimensional components
of higher spin-fields in the underlying UV theory). The cancellation of the cosmo-
logical term in any such theory, with an arbitrarily large but finite number of fields,
requires fine-tuning [7]. Consider a system of $n$ scalar fields $\phi_i$, $i = 1, 2, ..., n$, con-
tributing to the vacuum energy in an arbitrary way. Then, below the scale $M_c$, the
cosmological equations take the conventional 4D form,

\begin{equation}
\frac{d^2 \phi_i}{dt^2} + 3H \frac{d\phi_i}{dt} + \partial_\phi V(\phi) = 0 ,
\end{equation} 

\footnote{The discontinuity which emerges in the theory of massive gravity at the tree level [4] is an
artifact of the tree-level approximation and is absent [4] in a complete nonlinear theory. For
detailed studies of this issue in the context of the present model see Refs. [46, 47, 24].}
and
\[ 6M_{Pl}^2 H^2 = \frac{1}{2} \left( \frac{d\phi_i}{dt} \right)^2 + V(\phi), \]
(72)
where \( V(\phi) \) is the total potential.

Since on the Minkowski background, only scalars are allowed to develop vacuum expectation values (VEV’s), the effect of all other higher-spin fields reduces to renormalizing parameters in \( V(\phi) \) (including the \( \phi \)-independent constant part).

Now, the requirement that this system has a flat space solution, \( H = 0 \), leaves us with \( n + 1 \) algebraic equations,
\[ \partial_{\phi_i} V(\phi) = V(\phi) = 0, \]
(73)
that depend on \( n \) unknowns. This system has no solution in general unless the parameters in \( V(\phi) \) are fine-tuned.

To find a field configuration that yields \( H = 0 \) without fine-tuning (for instance, for an arbitrary \( \phi \)-independent part of \( V(\phi) \)) we have to sacrifice at least one of the equations
\[ \partial_{\phi_i} V(\phi) = 0, \]
(74)
say at \( i = k_0 \). Considering the index \( i \) as a “coordinate” in some discrete internal space, \( i = k_0 \) can be regarded as a singular point in this space. That is to say, we necessarily encounter a “singular point” trying to find a solution with the vanishing \( H \). This fact has a very clear-cut analogue, being considered from the standpoint of the extra-dimensional theories. In that case \( i \) can be literally identified with the physical extra coordinate, and the singularity mentioned above is just a gravitational singularity in the extra space.

Let us show this explicitly. Consider any high-dimensional set-up that allows for such a singular solution with the flat 4D metric in the presence of an arbitrary bulk cosmological term, or the brane tension. The Einstein equation takes the form
\[ M_{*}^{2+N} \left( \mathcal{R}_{AB} - \frac{1}{2} G_{AB} \mathcal{R} \right) = T_{AB}, \]
(75)
where \( T_{AB} \) could contain the bulk cosmological term as well as the brane tension. As before, we denote the 4D coordinates by \( x_\mu \) while the extra ones by \( \rho_m, m = 0, 1, \ldots, N \).

By assumption, the above equation has a solution of the form
\[ ds^2 = A^2(\rho) \eta_{\mu\nu} dx^\mu dx^\nu - B_{mn}(\rho) d\rho^m d\rho^n, \]
(76)
which is singular at some point \( \rho_m = \rho_m(k) \). With such metric the only components that get non-zero VEV’s are the ones that transform as scalars under the 4D coordinate transformations — they have \( x_\mu \)-independent VEV’s. Performing a Fourier expansion of Eq. (76) in terms of these 4D fields — we shall call them \( \phi_i \) — and substituting them in Eq. (73), we get a set of equations similar to (73). The singularity in the metric (76) in the language of these 4D fields is simply a manifestation
of the fact that all the equations cannot be satisfied simultaneously. Thus, the sin-
gularity in the high-dimensional position space is just a “Fourier transform” of the
fact that in the 4D language we cannot satisfy the equation for $\phi_{k_0}$. This, in turn, is
a consequence of the fact that, by requiring flat 4D space, we have over-constrained
the system. The singularity is a response to this over-constraint. Just like in the
4D language the equations cannot be satisfied for all $\phi_i$’s, in the high-dimensional
language the equations cannot be satisfied at every point in the $\rho$-space.

Although one might not be able to get exactly vanishing cosmological term with-
out fine-tuning, one could succeed in getting an acceptably small one. This is in fact
what we observed in the previous sections. There is a pre-condition: this requires an
infinite-volume extra space (or nonlocal interactions from the point of view of
4D theory).

To see that this is the case, return to the 4D system described by Eqs. (71) and
(72). Although the system has no flat solution with $H = 0$, it certainly can have one
if we allow a non-zero curvature, $H \neq 0$, and, possibly, time dependence of some of
the scalar fields $\phi_i$. Then all equations (71) and (72) can be satisfied simultaneously.
In the language of the high-dimensional equation (74), this means that one can
smooth out the singularity, provided that one allows the four-dimensional metric to
be curved. As it was shown in the previous sections, in the brane induced gravity
model, this means that the singularity is smoothed by letting the brane world-
volume to inflate. In order to cancel a small change in the 4D curvature to a given
accuracy $H$, by re-adjusting the VEV of a scalar field, the latter must be as light
as $H$. Now, as we have argued in section 2, in the theories with finite-volume extra
space, all the states that could re-adjust the cosmological constant have masses of
order $\sim 10^{-3}$ eV and cannot do the job. On the contrary, in the infinite-volume
models the higher-dimensional scalar components of the graviton KK are arbitrarily
light and do re-adjust the cosmological constant.

8 Discussions

In this paper we discussed a model where 4D gravity on the brane is obtained
due to an induced 4D Einstein-Hilbert term. However, the arguments about the

cosmological constant in general, and, the formula (6) in particular, are independent
of this mechanism. As long as $M_{Pl}$ does not restrict the value of the volume of extra
space, the arguments of the present paper will apply.

The particular model we suggest has a number of positive features: (i) it provides
a natural explanation of the smallness of the cosmological constant; (ii) it ensures
that all gravity loops are perfectly harmless; (iii) finally, it preserves the logarithmic
gauge coupling unification. The model has testable predictions, for gravity both
at sub-millimeter and Hubble distances, and for accelerator experiments with the
energy in the TeV range.

At this end we would like to give a brief discussion on the question of the cosmo-
logical evolution on the brane (which was not the main focus of the present paper). There are two issues to discuss: (1) How does the inflation proceed in the early universe; (2) What would the 4D Friedmann-Lemaitre-Robertson-Walker (FLRW) evolution look like on the brane. Let us start with (1). We expect that the inflationary paradigm can be incorporated in the present framework by using properties of the bulk. For instance, there is a possibility that in the early universe all the \((4 + N)\)-dimensions inflated to a very large size simultaneously before reaching the state that was considered in the present work. Another possibility might be to use the brane-inflation due to colliding branes \([49]\). The later scenario provides in addition a new mechanism for the baryogenesis \([50]\).

As to the issue (2), it is certainly true that analytic solutions for the cosmology of matter (or radiation) dominated universe is hard to obtain in the present context. However, one could gain some knowledge about these solutions using the arguments of the Newtonian cosmology. In that approach all one needs is the expressions for the potentials and forces on the brane. Since the inflation rate on the brane is tiny, the potentials can be deduced from the corresponding flat space Green’s functions. We will briefly summarize these discussions below.

Consider two points located on the brane. Their coordinates in the worldvolume directions will be denoted by \(x\) and \(x'\), while their \(\rho\) coordinates equal to zero as they are restricted to the brane. We would like to discuss an Euclidean Green function for these two points, \(G(x - x', \rho = 0)\). The latter will tell us about the potential between the two sources. As before, it is convenient to turn to the momentum space with respect to the four worldvolume coordinates while staying in the position space with respect to \(\rho\). We will discuss below the inverse momentum-space Green’s function where we drop the tensorial structure for simplicity

\[
G^{-1}(p, \rho = 0) = M_{Pl}^2 p^2 + M_*^{2+N} D^{-1}(p, \rho = 0).
\]

(77)

The first term on the right-hand side arises due to the induced Ricci term in Eq. (11), while the second term on the right-hand side is due to the bulk EH term in Eq. (11). Moreover, \(D(p, \rho = 0)\) is nothing but the (Fourier transformed) Green function for the bulk action,

\[
D(p, \rho = 0) = \int d^4x \frac{\exp(ip_\mu x^\mu)}{(x^2)^{\frac{2+N}{2}}} = \frac{1}{(p^2)^{\frac{2+N}{2}}} \frac{(2\pi)^2 \Gamma(\frac{2-N}{2})}{\Gamma(\frac{2+N}{2})}.
\]

(78)

Note that for \(N \geq 2\), the momentum-space Green’s function in Eq. (78) diverges. Thus, naively \(D^{-1}(p, \rho = 0) = 0\), and there is no second term on the right-hand side of Eq. (77). However, as we discussed in detail above, this is an UV divergence which is removed by any sensible UV regularization introduced in the theory. This can be done either by reintroducing a nonzero brane width \([4]\), or by introducing a rigid UV cutoff in the bulk \([23]\), or, most conveniently, by taking into account HDO’s in the bulk \([11, 12]\). These details are discussed in the aforementioned works (see Ref. \([12]\) for a summary) and are not important for our present purposes. What is important
here, is the fact that $D^{-1}(p, \rho = 0) \neq 0$ in the regularized theory, and, therefore, there is an unconventional second term on the right-hand side of Eq. (77). Here, we regard Eq. (78) as the definition of $D(p, \rho = 0)$ in dimensional regularization, with $MS$ subtraction scheme, where the regularization parameter $\epsilon$ is introduced as $N \rightarrow N - 2\epsilon$. For large momenta the first term on the right-hand side of Eq. (77) dominates. Hence, the UV behavior of the inverse Green function is

$$G^{-1}(p, \rho = 0) \simeq M^{2}_{Pl} p^{2}.$$  \hspace{1cm} (79)

This is just Green’s function for a conventional 4D theory. It would correspond to the kinetic operator in 4D “effective theory” which is just $M^{2}_{Pl} \partial_{4}^{2}$. The latter would lead to the conventional 4D FLRW evolution on the brane. Hence, we expect that the early cosmology in the model should be indistinguishable from the conventional 4D FLRW cosmology.

Now turn our discussion to the IR region, $p^{2} \ll 1/r_{c}$. In this domain, the interactions on the brane become $(4 + N)$-dimensional. This is reflected in the fact that the second term on the right-hand side of Eq. (77) dominates over the $M^{2}_{Pl} p^{2}$ term. From the 4D point of view this looks as an effect due to non-local operators in the action. Hence, we should expect that the present day evolution will just start to deviate from the conventional 4D FLRW cosmology.

In the present paper we have suggested that a physically acceptable solution with a small four-dimensional inflation rate can exist, despite the large brane tension. This solution represents a “mild” deformation of a singular solution with exactly flat worldvolume metric. Thus, in our inflating solution, as well as in the original undeformed one, the brane tension mostly curves the nearby region of the bulk and only slightly distorts four-dimensional space. The present arguments do not exclude the existence of the other two solutions (discussed at the end of subsection 4.1) with an opposite property such that the brane tension strongly curves the four-dimensional space. However, the important point is that these solutions cannot be smoothly deformed into each other, and their realization in nature must be determined by initial conditions in the early Universe.

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