Renormalization-group improved effective potential for interacting theories with several mass scales in curved spacetime

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Abstract

The renormalization group (RG) is used in order to obtain the RG improved effective potential in curved spacetime. This potential is explicitly calculated for the Yukawa model and for scalar electrodynamics, i.e. theories with several (namely, more than one) mass scales, in a space of constant curvature. Using the $\lambda \phi^4$-theory on a general curved spacetime as an example, we show how it is possible to find the RG improved effective Lagrangian in curved spacetime. As specific applications, we discuss the possibility of curvature induced phase transitions in the Yukawa model and the effective equations (back-reaction problem) for the $\lambda \phi^4$-theory on a De Sitter background.
1 Introduction

It is commonly accepted nowadays that the early universe experienced an inflationary epoch \[1, 2\] during its evolution (for an introduction and a review of different models of inflationary universe, see \[3-6\]). Some of the models of inflationary universes are based on the effective potential of the scalar field. As a rule, a flat space effective potential is used in such models for the description of the effective cosmological constant. However, such an approach cannot be completely consistent, since the curvature was not so small at this epoch.

Thus, the necessity of calculating the effective potential (or, more generally, the effective action \[7\]) in curved spacetime appears naturally motivated by cosmological considerations. However, the problem of obtaining the effective action in interacting theories in curved spacetime can currently be solved only in part, owing to the fact that even one-loop calculations can be done just on some very specific backgrounds and for quite simple interacting theories (see, for instance, \[8-14\]; for a general review and an extensive list of references see \[8\]). (Note that the effective action can be also found as an expansion on terms of fixed order of the curvature tensors; however, such technique can be applied only in spaces with small, slowly varying curvature.) It is quite an old idea \[15\] that of using a kind of effective action formalism for investigating the quantum-corrected Einstein equations (in Refs. \[15\] this approach has been discussed for conformally invariant scalar fields mainly). However, this program is not so easy to realize.

Given this situation, where even the one-loop effective action in general curved spacetime is hard to find, some indirect methods may prove of great importance. In particular, the flat space effective potential \[16, 17\] can be found in the leading (or even subleading) logarithmic approximation using the renormalization group (RG) equations. Such RG improved effective potential \[16, 18, 19\] yields the sum of the leading logarithms of the whole perturbation theory and actually provides the extension for the range of the scalar field, if we compare it with the one-loop effective potential. Recently, the RG improved effective potential for massive theories in flat spacetime has been discussed \[20-22\]. It has been found in these references that such effective potential (which may be important for the discussion of stability of the electroweak vacuum in the standard model) acquires several new features when compared with the one for the massless theory \[16\].

First of all, the effective cosmological constant (or running vacuum energy) which is obtained from the transformation of the trivial \(V(0)\) into a \(\Phi\)-dependent quantity by the RG, appears already in flat space \[21-22\]. As we will see below, a most consistent and natural interpretation of this effective cosmological constant among the other running couplings is obtained in curved space, where it is not necessary to introduce any artificial functions \[21\] which make the effective potential satisfy the RG equations.

Second, in theories which have several mass scales there appears a problem connected with the choice of the RG parameter, which is not unique there. Some solution of this problem — based on the use of the decoupling theorem \[23\] and of the effective field theory \[24\] — has been given in Ref. \[22\] using the Yukawa model as an example.

The purpose of the present work is to study the RG improvement of the effective potential (or, more generally, of the effective Lagrangian) in massive theories in curved spacetime. The method of Ref. \[22\] can be extended to curved spacetime and provides a rigorous treatment for the RG improved effective potential when there are a few effective mass scales in the theory. (Notice that, generally speaking, the RG in curved spacetime has been constructed
in Refs. [25]-[27] (see [8] for a review), but this version of the RG is convenient for the study of the high-energy asymptotics mainly and not in the context under discussion here).

The paper is organized as follows. In the next section the RG improved effective potential is obtained for the Yukawa model in curved spacetime for the case of constant curvature. In section 3 we present the same calculation using scalar electrodynamics as an example. The RG improved effective Lagrangian for the \( \lambda \Phi^4 \)-theory in a general, slowly varying spacetime is found in section 4. In section 5 we investigate the curvature-induced phase transitions for the Yukawa model in the region where scalar particles are very heavy. Section 6 is devoted to the discussion of the effective equations on a De Sitter background, using the \( \lambda \phi^4 \)-theory as an example. Finally, Sect. 7 includes conclusions and some additional remarks.

2 The RG improved effective potential for the Yukawa model in curved spacetime

Let us consider the following Lagrangian for the Yukawa model in curved spacetime

\[
L_m = \frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - \frac{1}{2} m^2 \varphi^2 + \frac{1}{2} \xi R \varphi^2 - \frac{\lambda}{4!} \varphi^4 + \overline{\psi} (i \gamma^\mu (x) \nabla_\mu - h \varphi) \psi,
\]

where \( \psi \) is a massless, \( N \)-component Dirac spinor and \( \varphi \) is a massive scalar.

In what follows we will restrict ourselves to spacetimes of constant curvature, and we will find the RG improved effective potential for the theory (1). As usually, in order to obtain from (1) a multiplicatively renormalizable theory, one has to add to (1) the action of the external gravitational field, i.e.

\[
L_{\text{ext}} = \Lambda + \kappa R + a_1 R^2 + a_2 C_{\mu \nu \alpha \beta}^2 + a_3 G,
\]

where \( C_{\mu \nu \alpha \beta} \) is the Weyl tensor and \( G \) is the Gauss-Bonnet invariant. Note that the \( \square R \)-term is absent from (2), because of the restriction to constant curvature spaces.

Now, since the theory is multiplicatively renormalizable, its effective potential satisfies the standard RG equation:

\[
DV \equiv \left( \mu \frac{\partial}{\partial \mu} + \beta_i \frac{\partial}{\partial \lambda_i} - \gamma \frac{\partial}{\partial \Phi} \right) V(\mu, \lambda, \Phi) = 0,
\]

where \( \lambda_i = (\xi, \lambda, h^2, m^2, \Lambda, \kappa, a_1, a_2, a_3) \), \( \beta_i \) are the corresponding \( \beta \)-functions, and \( \Phi \) is the background scalar. The only difference with respect to the analogue of (3) in flat space is that, among the constants \( \lambda_i \) we have some gravitational coupling constants which do not appear in flat space. (In this case \( L_{\text{ext}} = \Lambda \).)

The solution of Eq. (3) by the method of the characteristics gives

\[
V(\mu, \lambda, \Phi) = V(\mu e^t, \lambda_i(t), \Phi(t)),
\]

where

\[
\frac{d\lambda_i(t)}{dt} = \beta_i(\lambda_i(t)), \quad \lambda_i(0) = \lambda_i,
\]

\[
\frac{d\Phi(t)}{dt} = -\gamma(t) \Phi(t), \quad \Phi(0) = \Phi.
\]
As usually, the physical meaning of (4) is that the effective potential is found provided its functional form at some certain \( t \) is known.

The one-loop \( \beta \)-functions of the theory can be calculated using the standard background field method. Of course, the \( \beta \)-functions corresponding to \( \lambda, h, m^2, \Lambda \) and \( \Phi \) have the same form as in flat space. Let us write all \( \beta \)-functions explicitly in the notations of Ref. [22] which were introduced for flat space

\[
\beta_\lambda = \frac{1}{(4\pi)^2} (3\lambda^2 + 8N\lambda h^2 - 48Nh^4) \equiv \beta_{\lambda\lambda}\lambda^2 + \beta_{\lambda h}\lambda h^2 + \beta_{h h}h^4,
\]

\[
\beta_h = \frac{1}{(4\pi)^2} (4N + 6)h^4 \equiv \beta_{h1}h^4,
\]

\[
\beta_{m^2} = \frac{1}{(4\pi)^2} (\lambda + 4Nh^2)m^2 \equiv (\gamma_{m\lambda}\lambda + \gamma_{m h}h^2)m^2, \quad \gamma = \frac{2Nh^2}{(4\pi)^2} = \gamma_{1}h^2,
\]

\[
\beta_\xi = (\gamma_{m\lambda}\lambda + \gamma_{m h}h^2)(\xi - 1/6), \quad \beta_\Lambda = \frac{m^4}{2(4\pi)^2}, \quad \beta_\kappa = \frac{m^2(\xi - 1/6)}{(4\pi)^2},
\]

\[
\beta_{a_1} = \frac{(\xi - 1/6)^2}{2(4\pi)^2}, \quad \beta_{a_2} = \frac{1}{120(4\pi)^2}(1 + 6N), \quad \beta_{a_3} = -\frac{1}{360(4\pi)^2}(1 + 11N). \quad (6)
\]

The solution of the one-loop RG equations for all coupling constants can be found in the form

\[
h^2(t) = \frac{h^2}{1 - \beta_{h1}h^2 t} \equiv \frac{h^2}{B(t)}, \quad \Phi(t) = \Phi B(t)^{\gamma_{1}/\beta_{h1}},
\]

\[
\lambda(t) = h^2 \frac{a(\lambda - bh^2)B(t)^{a_{\beta_{\lambda 1}}/\beta_{h1} - 1} - b(\lambda - ah^2)B(t)^{b_{\beta_{\lambda 1}}/\beta_{h1} - 1}}{(\lambda - bh^2)B(t)^{a_{\beta_{\lambda 1}}/\beta_{h1}} - (\lambda - ah^2)B(t)^{b_{\beta_{\lambda 1}}/\beta_{h1}}},
\]

\[
m^2(t) = m^2B(t)^{-\gamma_{m h1}/\beta_{h1}} \left( \frac{\left(\frac{\gamma_{m h1}\lambda}{\beta_{h1}}\right) - \left(\frac{\gamma_{m h1}}{\beta_{h1}}\right)}{(a - b)h^2} \right),
\]

\[
\xi(t) = \frac{1}{6} + \left(\frac{\xi - 1}{6}\right) B(t)^{-\gamma_{m h1}/\beta_{h1}} \left( \frac{\left(\frac{\gamma_{m h1}\lambda}{\beta_{h1}}\right) - \left(\frac{\gamma_{m h1}}{\beta_{h1}}\right)}{(a - b)h^2} \right),
\]

\[
a_2(t) = a_2 + \frac{t(1 + 6N)}{120(4\pi)^2}, \quad a_3(t) = a_3 - \frac{t(1 + 11N)}{360(4\pi)^2},
\]

\[
\Lambda(t) = \Lambda + \frac{m^4}{2(4\pi)^2} \int_0^t dt B(t)^{-2\gamma_{m h1}/\beta_{h1}}
\]

\[
\left( \frac{\left(\frac{\gamma_{m h1}\lambda}{\beta_{h1}}\right) - \left(\frac{\gamma_{m h1}}{\beta_{h1}}\right)}{(a - b)h^2} \right)^{-2\gamma_{m h1}/\beta_{h1}} \equiv \Lambda + \frac{m^4}{2(4\pi)^2}A(t),
\]

\[
\kappa(t) = \kappa + \frac{m^2}{(4\pi)^2} \left( \frac{\gamma}{6} \right) A(t), \quad a_1(t) = a_1 + \frac{1}{2(4\pi)^2} \left( \frac{\gamma_{1}}{6} \right)^2 A(t). \quad (7)
\]

Here \( a \) and \( b \) are the roots of the equation

\[
\beta_{\lambda 1}y^2 + (\beta_{\lambda h} - \beta_{h1})y + \beta_{h h} = 0
\]

(see also [22]).
Now the question appears of what is the choice of the RG parameter $t$ which leads to the summation of all the logarithms to all orders. Working in the one-loop approximation we have two logarithms appearing in the one-loop effective potential, namely

$$
\log \frac{\frac{1}{2} \lambda \Phi^2 + m^2 - (\xi - 1/6)R}{\mu^2}, \quad \log \frac{\frac{1}{4} h^2 \Phi^2 - \frac{1}{4} R}{\mu^2}.
$$  \hfill (8)

(Notice that in the problem under discussion we consider the effective potential as an expansion on the curvature invariants. Hence, $\lambda \Phi^2$ and $h^2 \Phi^2$ are dominant when compared with the $R$-terms in (8). Note also that in massless theories the choice of $t$ is actually unique: $t = \frac{1}{2} \log(\Phi^2/\mu^2)$ \cite{28}.)

Hence, we have two effective masses in the theory, and there seems no way open to a choice of $t$ which eliminates the leading logarithms to all orders —as a consequence of the existence of these two different mass scales. However, there exists some solution to this problem, that has been suggested in flat space \cite{22}. Such solution is based on the decoupling theorem \cite{23} and in the effective field theory \cite{24}. It can actually be generalized to curved spacetime, as we will see below.

Indeed, let us consider the region $h^2 \Phi^2 \geq m^2$. There,

$$
\log \frac{\frac{1}{2} \lambda \Phi^2 + m^2 - (\xi - 1/6)R}{\mu^2} \simeq \log \frac{h^2 \Phi^2 - \frac{1}{4} R}{\mu^2} + \log \frac{\frac{1}{2} \lambda \Phi^2 + m^2 - (\xi - 1/6)R}{h^2 \Phi^2} \\
\simeq \log \frac{h^2 \Phi^2 - \frac{1}{4} R}{\mu^2} + \log \left( \frac{\lambda}{2h^2} + \frac{m^2}{h^2 \Phi^2} - \frac{(\xi - 1/6)R}{h^2 \Phi^2} \right).
$$  \hfill (9)

One can see that the second term on the rhs in (9) is small if compared with the first one (note that $h^2 > \lambda$), i.e., in the region $h^2 \Phi^2 \geq m^2$ we effectively have just a single mass scale which is the natural one to be used as the RG parameter $t$.

The standard procedure, in which we employ the tree-level effective potential as boundary function, gives us the following leading log approximation for the effective potential in the region $h^2 \Phi^2 \geq m^2$:

$$
V = \frac{1}{2} m^2(t)\Phi^2(t) - \frac{1}{2} \xi(t)\Phi^2(t)R + \frac{1}{4!} \lambda(t)\Phi^4(t) - A(t) - \kappa(t)R - a_1(t)R^2 - a_2(t)C^2_{\mu\nu\alpha\beta} - a_3(t)G,
$$  \hfill (10)

where the effective couplings in (10) are given by Eqs. (7) and

$$
t = \frac{1}{2} \log \frac{h^2 \Phi^2 - \frac{1}{4} R}{\mu^2}.
$$  \hfill (11)

Thus, we have been able to construct the RG improved effective potential for a quite complicated Yukawa theory in curved spacetime (for a discussion of the RG improved potential in curved spacetime in the massless case, see \cite{28}). This potential performs the summation of all the leading logarithms in perturbation theory and, in this sense, it is much more exact than the one-loop effective potential, because it takes into account all orders of the perturbation theory. Notice that even the calculation of the one-loop effective potential for the theory (10) in curved spacetime is very hard to do. From our result (11) we can get the one-loop effective potential very easily as an expansion of (10) when $|t| \ll 1$ (with suitable renormalization conditions, which must be imposed after expanding). This ends the analysis of the region $h^2 \Phi^2 \geq m^2$. 

5
In order to investigate the region \( m^2 \geq h^2 \Phi^2 \), we adopt the method developed in Ref. [22] for the same theory in flat space. According to this method, the scalar particle \( \Phi \) is treated as a very heavy particle of mass \( m \). In such a formulation, the effects of a heavy particle lead to a shift of the parameters of the low-energy theory. We will choose only logarithmic terms to be responsible for this shift, because in a direct calculation of the one-loop effective potential the non-logarithmic terms depend on the scheme (or, in other words, on the choice of renormalization conditions). The parameters of the low-energy theory are defined by

\[
\bar{\lambda} = \lambda + \frac{3\lambda^2}{2(4\pi)^2} \log \frac{m^2}{\mu^2}, \quad \bar{\Phi} = \Phi, \quad \bar{m}^2 = m^2 + \frac{\lambda m^2}{2(4\pi)^2} \log \frac{m^2}{\mu^2},
\]

\[
\bar{\kappa} = \kappa + \frac{m^2(\xi - 1/6)}{2(4\pi)^2} \log \frac{m^2}{\mu^2}, \quad \bar{\xi} = \xi + \frac{\lambda(\xi - 1/6)}{2(4\pi)^2} \log \frac{m^2}{\mu^2}, \quad \bar{\Lambda} = \Lambda + \frac{m^4}{4(4\pi)^2} \log \frac{m^2}{\mu^2},
\]

\[
\bar{a}_1 = a_1 + \frac{(\xi - 1/6)^2}{4(4\pi)^2} \log \frac{m^2}{\mu^2}, \quad \bar{a}_2 = a_2 + \frac{1}{240(4\pi)^2} \log \frac{m^2}{\mu^2}, \quad \bar{a}_3 = a_3 - \frac{1}{720(4\pi)^2} \log \frac{m^2}{\mu^2}.
\]

(12)

Notice that \( h^2 \) gets also shifted, but from two-loop corrections [22]:

\[
h^2 = h^2 + \left[ 3h^4/(4\pi)^2 \right] \log (m^2/\mu^2).
\]

Using Eqs. (12) we may transform the RG operator acting on the effective potential (3) as follows

\[
D = (D\mu) \frac{\partial}{\partial \mu} + (D\bar{\lambda}) \frac{\partial}{\partial \bar{\lambda}_i} + (D\bar{\Phi}) \frac{\partial}{\partial \bar{\Phi}} = \mu \frac{\partial}{\partial \mu} + \bar{\beta}_\lambda \frac{\partial}{\partial \bar{\lambda}_i} + \bar{\gamma} \bar{\Phi} \frac{\partial}{\partial \bar{\Phi}},
\]

(13)

where from (12) and (3) it follows that (in the one-loop approximation)

\[
\bar{\beta}_\lambda = D\bar{\lambda} = \frac{1}{(4\pi)^2} (8N\lambda h^2 - 48Nh^4), \quad \bar{\beta}_h = D\bar{h}^2 = \frac{4Nh^4}{(4\pi)^2},
\]

\[
\bar{\beta}_m^2 = D\bar{m}^2 = \frac{4Nh^2}{(4\pi)^2} m^2, \quad \gamma = \bar{\gamma}, \quad \bar{\beta}_\xi = D\bar{\xi} = \frac{4Nh^2}{(4\pi)^2} \left( \xi - \frac{1}{6} \right),
\]

\[
\bar{\beta}_\Lambda = \bar{\beta}_\kappa = \bar{\beta}_a_1 = 0, \quad \bar{\beta}_a_2 = \frac{6N}{120(4\pi)^2}, \quad \bar{\beta}_a_3 = -\frac{11N}{360(4\pi)^2}.
\]

(14)

Solving the RG equations for the effective couplings with tilde, we find

\[
\bar{h}^2(t) = h^2 \left( 1 - \frac{4N\bar{h}^2}{(4\pi)^2} t \right)^{-1/2}, \quad \bar{\Phi}(t) = \bar{\Phi} \left( 1 - \frac{4N\bar{h}^2}{(4\pi)^2} t \right)^{1/2}, \quad \bar{\beta}_{h1} = \frac{4N}{(4\pi)^2},
\]

\[
\bar{\lambda}(t) = \left( \bar{\lambda} - \frac{\beta_{\lambda h}}{\beta_{h1} - \beta_{\lambda h}} \bar{h}^2 \right) \left( 1 - \frac{4N\bar{h}^2}{(4\pi)^2} t \right)^{-\beta_{\lambda h}/\beta_{h1}} + \frac{\beta_{\lambda h} \bar{h}^2(t)}{\beta_{h1} - \beta_{\lambda h}},
\]

\[
\bar{m}^2(t) = \bar{m}^2 \left( 1 - \frac{4N\bar{h}^2}{(4\pi)^2} t \right), \quad \bar{\Lambda}(t) = \bar{\Lambda}, \quad \bar{\kappa}(t) = \bar{\kappa}, \quad \bar{\xi}(t) = \frac{1}{6} + \left( \frac{\bar{\xi} - \frac{1}{6}}{\bar{h}^2(t)} \right) \left( 1 - \frac{4N\bar{h}^2}{(4\pi)^2} t \right),
\]

\[
\bar{a}_1(t) = a_1, \quad \bar{a}_2(t) = a_2 + \frac{6Nt}{120(4\pi)^2}, \quad \bar{a}_3(t) = a_3 - \frac{11Nt}{360(4\pi)^2}.
\]

(15)

Summing all leading logarithms, the RG improved effective potential is now given by

\[
V = \frac{1}{2} \bar{m}^2(t) \bar{\Phi}^2(t) - \frac{1}{2} \bar{\xi}(t) \bar{\Phi}^2(t) R + \frac{1}{4!} \bar{\lambda}(t) \bar{\Phi}^4(t) - \bar{\Lambda}(t) - \bar{\kappa}(t) R - \bar{a}_1(t) R^2 - \bar{a}_2(t) C_{\mu \nu \alpha \beta}^2 - \bar{a}_3(t) G, \quad t = \frac{1}{2} \log \frac{\bar{h}^2 \bar{\Phi}^2 - R/4}{\mu^2}.
\]

(16)
It is interesting to note that the effective conformal coupling \( \tilde{\xi}(t) \) grows with \( t \) as in the case of the asymptotically free models of Refs. [8, 26, 27]. Agreement between the potentials (10) and (16) can be easily checked [22]. Indeed, both of them satisfy the same RG equation and are independent on \( \mu \). Choosing \( \mu = m \) and considering the region \( g\Phi \approx \tilde{g}\Phi = m \), one sees that \( t \approx 0 \) both in (10) and in (16), and hence the effective potentials reduce to the same boundary function (tree potential) and here coincide.

To summarize, in this section we have generalized the approach of Ref. [22] to curved spacetime, and found the RG improved effective potential for the Yukawa theory.

3 RG improved potential for scalar electrodynamics in curved spacetime

We now consider the simple model of an abelian gauge theory—scalar electrodynamics—in curved spacetime. We are going to show how the method developed in the previous section can be easily applied to gauge theories, considering scalar electrodynamics as an example. In other words, we will find the RG improved effective potential for scalar electrodynamics in a curved spacetime of constant curvature.

The classical Lagrangian for the theory is given by

\[
L = L_m + L_{\text{ext}},
\]

where

\[
L_m = \frac{1}{2} (\partial_{\mu} \Phi_1 - e A_{\mu} \Phi_2)^2 + \frac{1}{2} (\partial_{\mu} \Phi_2 + e A_{\mu} \Phi_1)^2 - \frac{1}{2} m^2 \Phi_2^2 + \frac{1}{2} \xi R \Phi_2^2 - \frac{1}{4!} \lambda \Phi_2^4 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu},
\]

\[L_{\text{ext}}\] is given by [2], and \( \Phi^2 = \Phi_1 \Phi_1 + \Phi_2 \Phi_2 \).

We will adopt the Landau gauge, in which the one-loop \( \beta \)-function for the gauge parameter is zero, and the RG equation for the scalar effective potential has the same form (3) as in the Yukawa theory (except for the change \( h^2 \to e^2 \) in \( \lambda_i \)). The solution of the RG equation (3) has the same formal structure (4), being now the one-loop \( \beta \)-functions given by

\[
\beta_{\lambda} = \frac{1}{(4\pi)^2} \left( \frac{10}{3} \lambda^2 - 12e^2 \lambda + 36e^4 \right), \quad \beta_{e_2} = \frac{2e^4}{3(4\pi)^2}, \quad \beta_{m^2} = \frac{m^2}{(4\pi)^2} \left( \frac{4}{3} \lambda - 6e^2 \right),
\]

\[
\gamma = -\frac{3e^2}{(4\pi)^2}, \quad \beta_{\xi} = \frac{\xi - 1/6}{(4\pi)^2} \left( \frac{4}{3} \lambda - 6e^2 \right), \quad \beta_{\lambda} = \frac{m^4}{(4\pi)^2}, \quad \beta_{\kappa} = \frac{2m^2(\xi - 1/6)}{(4\pi)^2},
\]

\[
\beta_{a_1} = \frac{\xi - 1/6}{(4\pi)^2}, \quad \beta_{a_2} = \frac{7}{60(4\pi)^2}, \quad a_3(t) = a_3 - \frac{8t}{45(4\pi)^2}.
\]

The solutions of the RG equations for the coupling constants are

\[
e^2(t) = e^2 \left( 1 - \frac{2e^2 t}{3(4\pi)^2} \right)^{-1}, \quad \Phi^2(t) = \Phi^2 \left( 1 - \frac{2e^2 t}{3(4\pi)^2} \right)^{-9},
\]

\[
\lambda(t) = \frac{1}{10} e^2(t) \left( \sqrt{719} \tan \left( \frac{1}{2} \sqrt{719} \log e^2(t) + C \right) + 19 \right),
\]
We see again that there naturally appears the effective cosmological constant $\Lambda(t)$. Notice that the flat-space coupling constants (20) have been obtained in the classical work [16]. In this region the natural choice for the RG parameter $\xi(t)$ is $\xi = \frac{1}{6} + \left( \xi - \frac{1}{6} \right) \left( \frac{e^2(t)}{e^2} \right)^{\frac{-26}{5}} \frac{\cos^{2/5} \left( \frac{1}{2} \sqrt{719} \log e^2 + C \right)}{\cos^{2/5} \left( \frac{1}{2} \sqrt{719} \log e^2(t) + C \right)}$.

Now the question of the choice of the RG parameter $t$ appears. Here the situation is even more complicated, as compared with the case of the Yukawa model, because after diagonalization of the mass matrix we get the following effective masses in the theory:

$$m_1^2 = m^2 + \frac{\lambda}{2} \Phi^2 - (\xi - 1/6)R, \quad m_2^2 = m^2 + \frac{\lambda}{6} \Phi^2 - (\xi - 1/6)R, \quad m_{\mu\nu}^2 = \delta_{\mu\nu}(e^2 \Phi^2 - R/4). \quad (21)$$

Using the same techniques as in the previous section, we may consider first the region $e^2 \Phi^2 > m^2 + (\lambda/2) \Phi^2$. In this region the natural choice for the RG parameter $t$ is

$$t = \frac{1}{2} \log \frac{e^2 \Phi^2 - R/4}{\mu^2}. \quad (22)$$

Substituting (20), with $t$ as in (22), into the RG improved effective potential defined by the tree potential, we have the same form (10) for the RG improved potential.

In the region $m^2 > e^2 \Phi^2$, when the scalar particles are very heavy, we may again define the low-energy theory parameters. Then, as in the previous section, the coupling constants corresponding to the effective theory can be cast into the form

$$\tilde{\alpha}(t) = \tilde{a}_1, \quad \tilde{a}_2(t) = \tilde{a}_2 + \frac{t}{10(4\pi)^2}, \quad \tilde{a}_3(t) = \tilde{a}_3 - \frac{31t}{180(4\pi)^2}. \quad (23)$$
As we see, the conformal effective coupling constant does not show asymptotically a conformally invariant behavior, as happened in the Yukawa model above.

The RG improved effective potential is given by

\[
V = \frac{1}{2} m^2(t) \tilde{\Phi}^2(t) - \frac{1}{2} \tilde{\zeta}(t) \tilde{\Phi}^2(t) R + \frac{1}{4!} \tilde{\lambda}(t) \tilde{\Phi}^4(t) - \tilde{\Lambda}(t) R - \tilde{a}_1(t) R^2 \\
- \tilde{a}_2(t) C^2_{\mu\nu\alpha\beta} - \tilde{a}_3(t) G, \quad t = \frac{1}{2} \log \frac{\tilde{e}^2 \Phi^2 - R/4}{\mu^2},
\]

with the effective couplings being given by (23).

Thus, we have been able to construct the RG improved effective potential for scalar electrodynamics. Proceeding in a similar way, one can in principle construct it for more complicated theories (as GUTs), where more effective mass scales are present.

Another important point concerns now the possibility to generalize the above results to non-constant curvature spaces. In the next section we will demonstrate that this is indeed feasible.

4 RG improved effective Lagrangian in the $\lambda\Phi^4$-theory in curved spacetime

Let us consider the $\lambda\Phi^4$-theory with the Lagrangian

\[
L = L_m + L_{\text{ext}}, \quad L_m = \frac{1}{2} g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - \frac{1}{2} (m^2 - \xi R) \phi^2 - \frac{\lambda}{24} \phi^4 + a_5 \Box \phi^2,
\]

in general curved spacetime. Now curvature will not be constant, in general, but we shall make the assumption that the gravitational field is slowly varying. In this case, $L_{\text{ext}}$ as given by (2) contains also the term $a_4 \Box R$ which is absent when the curvature is constant. Notice that we have also added to $L_m$ the total derivative term $a_5 \Box \phi^2$ which becomes important at the quantum level, as we will see below.

Using the same technique as in Sect. 2 and the explicit form of the one-loop $\beta$-functions of the $\lambda\Phi^4$-theory (notice however that here the coupling constants $a_4$ and $a_5$ are to be considered together with the other coupling constants of the theory), we get the RG improved effective Lagrangian under the following form

\[
L_{\text{eff}} = \frac{1}{2} g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - \frac{1}{2} [m^2(t) - \xi(t) R] \Phi^2 - \frac{\lambda(t)}{24} \Phi^4 + a_5(t) \Box \Phi^2 + \Lambda(t) \\
+ \kappa(t) R + a_1(t) R^2 + a_2(t) C^2_{\mu\nu\alpha\beta} + a_3(t) G + a_4(t) \Box R.
\]

Here

\[
t = \frac{1}{2} \log \frac{m^2 - (\xi - 1/6) R + \lambda \Phi^2/2}{\mu^2},
\]

and

\[
\lambda(t) = \lambda \left(1 - \frac{3\lambda t}{(4\pi)^2}\right)^{-1}, \quad m^2(t) = m^2 \left(1 - \frac{3\lambda t}{(4\pi)^2}\right)^{-1/3},
\]
However, the RG improved effective Lagrangian gives now some additional information, because $\Phi$ and $R$ in (26) are not constant but may depend on spacetime coordinates.

In the same way, we can construct the RG improved effective Lagrangian for more complicated theories, like the Yukawa model, scalar electrodynamics, or some other theories. The approach developed above still works in those cases and the only supplementary difficulties that appear are of technical nature. Such difficulties are connected with the more involved structure of the RG parameter $t$ — owing to the fact that we must then diagonalize a more involved effective mass matrix with non-constant (i.e., coordinate dependent) elements. Notice also the new terms of type $\Box R$ and $\Box \Phi^2$.

For the sake of conciseness, we are not going to discuss in the present case the RG improved effective Lagrangian for theories with several mass scales. This effective Lagrangian will be however extremely useful in the discussion of dynamical, quantum corrected Einstein equations (for an introduction, see [15]), which are very difficult to solve due to the high non-linearity of the problem. In what follows we will restrict ourselves to the discussion of some cosmological applications of the RG improved effective potential above.

5 Curvature induced phase transitions in the Yukawa model in curved spacetime

It is generally accepted nowadays that the very early universe experienced several phase transitions before it could reach its present state. It is also very possible that one of those phase transitions could be induced by the external gravitational field existing at an early epoch [10, 12]. We will here investigate this possibility using as main ingredient the RG improved effective potential for the case of the Yukawa model. For simplicity, we will just consider the region $m^2 \geq \hbar^2 \Phi^2$ and restrict ourselves to the linear curvature approximation. We shall also put $\Lambda = \kappa = 0$, since this just amounts to a simple rescaling of the vacuum energy. Implementing these conditions in the RG improved effective potential (15), we get

\[
V = \frac{1}{2} \tilde{m}^2 \left( 1 - \frac{4N\tilde{h}^2}{(4\pi)^2} t \right)^2 \tilde{\Phi}^2 - \frac{1}{2} \left[ \frac{1}{6} + \left( \tilde{\xi} - \frac{1}{6} \right) \left( 1 - \frac{4N\tilde{h}^2}{(4\pi)^2} t \right) \right] \tilde{\Phi}^2 \left( 1 - \frac{4N\tilde{h}^2}{(4\pi)^2} t \right) R
\]
\[ + \frac{1}{4!} \bar{\Phi}^4 \left[ \bar{\lambda} - 12\bar{\h}^2 + 12\bar{\h}^2 \left( 1 - \frac{4N\bar{h}^2}{(4\pi)^2} \right) \right], \]  

(28)

with

\[ t = \frac{1}{2} \log \frac{\bar{h}^2 \tilde{\Phi}^2 - R/4}{\mu^2}. \]

We can now start with the analysis of critical points corresponding to the potential (28) both for the massless and for the massive case. Calling in general \( x \equiv \bar{\Phi}^2 / \mu^2 \) and \( y \equiv R / \mu^2 \), so that \( V = V(x,y) \), let us remember that the critical parameters, \( x_c, y_c \), corresponding to the first-order phase transition are found from the conditions

\[ V(x_c, y_c) = 0, \quad \left. \frac{\partial V}{\partial x} \right|_{x_c,y_c} = 0, \quad \left. \frac{\partial^2 V}{\partial x^2} \right|_{x_c,y_c} > 0. \]  

(29)

For the RG improved potential they lead to some transcendental equations which cannot be solved analytically. However, we do will obtain some particular solutions. We shall be concerned with first-order phase transitions where the order parameter \( \bar{\Phi}^2 \) experiences a quick change for some critical value, \( R_c \), of the curvature.

The first two equations (29) for the potential (28) are (after some immediate recombination and having eliminated already the trivial solution at \((0,0)\)):

\[ 2\bar{m}^2 C(t) - R \left[ \frac{1}{6} + \left( \tilde{\xi} - \frac{1}{6} \right) C(t) \right] + \frac{x}{12} \left[ (\bar{\lambda} - 12\bar{\h}^2) C(t)^{-1} + 12\bar{h}^2 \right] = 0, \]

\[ -2\bar{m}^2 C(t) + R \left[ \frac{1}{6} + 2 \left( \tilde{\xi} - \frac{1}{6} \right) C(t) \right] + \frac{(4\pi)^2}{24N\bar{h}^2} \left[ \bar{\lambda} - 12\bar{h}^2 + 12\bar{h}^2 C(t) \right] - \bar{h}^2 x = 0, \]

\[ C(t) \equiv 1 - \frac{4N\bar{h}^2}{(4\pi)^2} t. \]  

(30)

Two particular choices of \( \tilde{\xi} \) are interesting [29], in each of the following two cases.

(a) Massless case: \( \bar{m}^2 = 0 \), with \( \bar{\h}^2 < 1 \) and \( \bar{\lambda} < 1 \).

We have carried out two different analysis, corresponding to \( \tilde{\xi} = 1/6 \) and \( \tilde{\xi} = 10^4 \), respectively. The result in both cases is the same: Eqs. (30) are incompatible, so that the only critical point is obtained at \( \bar{\Phi}_c^2 = 0 \), \( R_c = 0 \).

(b) Massive case: \( \bar{m}^2 \neq 0 \), with \( \bar{\h}^2 < 1 \) and \( \bar{\lambda} < 1 \).

Now the result changes completely, a non-trivial critical point exists, and its location and properties depend clearly on the value of \( \tilde{\xi} \) considered. For \( \tilde{\xi} = 1/6 \), we obtain

\[ \bar{\Phi}_c^2 \simeq \frac{4!N\bar{h}^4\bar{m}^2}{(4\pi)^2} \left( \frac{N\bar{h}^4}{(4\pi)^2} - \bar{\lambda} \right)^{-1} \simeq \frac{48\bar{m}^2}{\bar{h}^2}, \quad R_c \simeq 12\bar{m}^2. \]  

(31)

On the other hand, for \( \tilde{\xi} = 10^4 \), the result is

\[ \bar{\Phi}_c^2 \simeq \frac{4!N\bar{h}^4\bar{m}^2}{(4\pi)^2} \left( \frac{N\bar{h}^4}{(4\pi)^2} - \bar{\lambda} \right)^{-1} \simeq \frac{96\xi\bar{m}^2}{\bar{h}^2}, \quad R_c \simeq \frac{2\bar{m}^2}{\bar{\xi}} \].  

(32)

Apart from the additional factor of 2 in the expression for \( \bar{\Phi}_c^2 \), notice the specific behavior of \( R_c \) in terms of \( \tilde{\xi} \) obtained from the last case. We observe that for small values of \( \bar{\lambda} \),
namely \( \bar{\lambda} < N \bar{h}^4/(4\pi)^2 \), an admissible, positive solution is obtained in this massive case. Moreover, it is immediate to check the consistency of such solution of the quite involved, transcendental equations (30). In fact, it is obtained for \( t \) of the order of 1, a condition that is easily matched \textit{a posteriori}:

\[
t \sim \frac{1}{2} \log \frac{[48 - 1/(4\bar{\xi})]\bar{m}^2}{\mu^2} \sim 1.
\]

We finish this section by pointing out that what we have found above is a very interesting example of curvature-induced phase transition which has the property that it takes place only at non-zero mass and it is absent in the massless case.

6 Effective equations in De Sitter space

Let us discuss now the effective equations for the \( \lambda \varphi^4 \)-theory in a De Sitter background. This may be viewed as an attempt at studying the qualitative features of quantum corrected gravitational equations in a static geometry (the back reaction problem).

It is rather natural to start from some curved background in this problem, since flat spacetime is unstable as a background [30]. However, it is well known [15] that this problem is very difficult to solve technically, due to the appearance of higher derivatives, of its high non-linearity, etc. Recently some interesting approach has been discussed [31] that reduces the problem to second-order equations. This technique may prove to be useful also in our context, however, notice that we have here many more non-linearities, connected with the several logarithms.

Our Lagrangian is chosen in the form (in this section Euclidean notations will be used)

\[
L = \frac{1}{2} (\partial_\mu \varphi)^2 + \frac{1}{2} m^2 \varphi^2 + \frac{\lambda}{4!} \varphi^4 + \frac{1}{2} \xi R \varphi^2 + \Lambda_0 + \kappa R + \bar{a}_1 R^2.
\]

(Notice that \( G = R^2/6 \) in such a background, and hence only the \( R^2 \) term is induced.) We have put a subindex to \( \Lambda \) in (33).

Repeating the same procedure of Sect. 4, with some minor sign modifications, we can easily get the RG improved effective Lagrangian on the De Sitter background

\[
L_{\text{eff}} = \frac{1}{2} m^2(t) \Phi^2 + \frac{\lambda(t)}{4!} \varphi^4 + \frac{1}{2} \xi(t) R \varphi^2 + \Lambda_0(t) + \kappa(t) R + \bar{a}_1(t) R^2,
\]

where

\[
\lambda(t) = \lambda \left(1 - \frac{3\lambda t}{(4\pi)^2}\right)^{-1}, \quad m^2(t) = m^2 \left(1 - \frac{3\lambda t}{(4\pi)^2}\right)^{-1/3}, \quad \xi(t) = \frac{1}{6} \left(\xi - \frac{1}{6}\right) \left[1 - \frac{3\lambda t}{(4\pi)^2}\right]^{-1/3},
\]

\[
\Lambda_0(t) = \Lambda_0 - \frac{m^4}{2\lambda} \left(1 - \frac{3\lambda t}{(4\pi)^2}\right)^{-1/3} + \frac{m^4}{2\lambda}, \quad \kappa(t) = \kappa - \frac{m^2}{\lambda} \left(\xi - \frac{1}{6}\right) \left[1 - \frac{3\lambda t}{(4\pi)^2}\right]^{1/3} - 1,
\]

\[
\bar{a}_1(t) = \bar{a}_1 - \frac{1}{2\lambda} \left(\xi - \frac{1}{6}\right) \left[1 - \frac{3\lambda t}{(4\pi)^2}\right]^{1/3} - 1 - \frac{t}{2160(4\pi)^2},
\]

(35)
and being
\[ t = \frac{1}{2} \log \frac{m^2 + (\xi - 1/6)R + \lambda \Phi^2/2}{\mu^2}. \]
Using the properties of the De Sitter space:
\[ R = 4\Lambda, \quad \int d^4x \sqrt{g} = \frac{24\pi^2}{\Lambda^2}, \]
we may study the effective equations along the line of Ref. [14] (see also [35]). First of all, let us introduce the variables \( y = 1/\Lambda \) and \( x = \Phi^2/\Lambda \). Then the effective action is given by
\[ S_{\text{eff}} = \frac{1}{24\pi^2} \left( \frac{\lambda(t)}{4!} x^2 + 2\xi(t)x + \Lambda_0(t)y^2 + 4\kappa(t)y + 16\bar{a}_1(t) \right), \]
with
\[ t = \frac{1}{2} \log \frac{m^2y + 4(\xi - 1/6) + \lambda x/2}{\mu^2y}. \]
For simplicity, let us consider only the massless case: \( m^2 = 0 \). Presumably, the introduction of mass will not change the results of the following qualitative analysis. Then,
\[ S_{\text{eff}} = \frac{\lambda(t)}{4!} x^2 + 2\xi(t)x + \Lambda_0 y^2 + 4\kappa y + 16\bar{a}_1(t). \]
The variational equations obtained from this effective action are
\[ \frac{\lambda(t)}{12} + 2\xi(t) + \frac{B(x)}{2(4\pi)^2} \left( x + \frac{8(\xi - 1/6)}{\lambda} \right)^{-1} = 0, \]
\[ 2y\Lambda_0 + 4\kappa - \frac{B(x)}{2(4\pi)^2y} + \phi^2 \left[ \frac{\lambda(t)x}{12} + 2\xi(t) + \frac{B(x)}{2(4\pi)^2} \left( x + \frac{8(\xi - 1/6)}{\lambda} \right)^{-1} \right] = 0, \]
where
\[ B(x) = \frac{3\lambda x^2}{4!} F(t)^{-2} + 2\lambda x \left( \xi - \frac{1}{6} \right) F(t)^{-4/3} + 8 \left( \xi - \frac{1}{6} \right)^2 F(t)^{-2/3} - \frac{1}{135}, \quad F(t) \equiv 1 - \frac{3\lambda t}{(4\pi)^2}. \]
For \( \Phi = 0, \Lambda_0 \neq 0 \), then as in Ref. [14] we get a perturbative solution of the kind:
\[ \Lambda \simeq -\frac{\Lambda_0}{2\kappa} + \mathcal{O}(\hbar), \quad \kappa < 0. \]
If \( \Lambda_0 = 0 \), we get the same quantum solution as in Ref. [32], namely, for \( \xi = 1/6 \)
\[ \Lambda = -1080(4\pi)^2 \kappa. \]
For \( \xi \neq 1/6 \) we get iterative corrections to (39).
For \( \Phi \neq 0 \) the system (39) should be analyzed more carefully. We get two solutions of (39), perturbatively in \( \xi - 1/6 \), in the following way (plus and minus signs go together)
\[ \phi^2 = \frac{4\Lambda_0 F(t)^2}{\kappa \lambda \left[ F(t) + \frac{3\lambda}{4(4\pi)^2} \right]^{1/2}} \left\{ 1 \pm \left[ 1 + \frac{\Lambda_0 F(t)^2}{\kappa^22(4\pi)^2 \left[ F(t) + \frac{3\lambda}{4(4\pi)^2} \right]^2} \right]^{1/2} \right\}^{-1}, \]
\[ \Lambda = -\frac{\Lambda_0}{\kappa} \left\{ 1 \pm \left[ 1 + \frac{\Lambda_0 F(t)^2}{\kappa^22(4\pi)^2 \left[ F(t) + \frac{3\lambda}{4(4\pi)^2} \right]^2} \right]^{1/2} \right\}^{-1}. \]
\[ (41) \]
Of course these are not explicit expressions, since \( t \) involves the variable \( \Phi^2 \) itself (notice that solving the transcendental equations (39) explicitly is an impossible task). Eqs. (41) are to be used recurrently, in the ordinary way: for \( \lambda \) small and \( \xi \neq 1/6 \) (\( \xi - 1/6 \) small), starting from

\[
t_0 = \frac{1}{2} \log \frac{4(\xi - 1/6)\Lambda_0}{\mu^2}, \quad \Phi_0 = \Phi(t_0), \quad \Lambda_{(0)} = \Lambda(t_0),
\]

one defines

\[
t_1 = t_0 + \frac{1}{2} \log \left(1 + \frac{\lambda \Phi_0^2}{2(\xi - 1/6)\Lambda_{(0)}}\right), \quad \Phi_1 = \Phi(t_1), \quad \Lambda_1 = \Lambda(t_1),
\]

and so on (do not confuse the \( \Lambda_{(0)} \) here with the \( \Lambda_0 \) above). In general,

\[
t_n = t_0 + \frac{1}{2} \log \left(1 + \frac{\lambda \Phi_{n-1}^2}{2(\xi - 1/6)\Lambda_{n-1}}\right), \quad \Phi_n = \Phi(t_n), \quad \Lambda_n = \Lambda(t_n).
\]

It is interesting to see explicitly the results that one gets after the first step of the iteration:

\[
\frac{1}{\Lambda} \approx -\frac{\kappa}{\Lambda_0} \left(1 \pm \sqrt{1 + \frac{\Lambda_0}{\kappa^2 2(4\pi)^2}}\right),
\]

\[
\Phi^2 \approx \frac{4\Lambda_0}{\kappa \lambda} \left(1 - \frac{3\lambda}{2(4\pi)^2} \log \frac{4(\xi - 1/6)\Lambda}{\mu^2}\right) \left(1 \pm \sqrt{1 + \frac{\Lambda_0}{\kappa^2 2(4\pi)^2}}\right)^{-1}.
\]

These expressions already provide a good approximation for the assumed conditions.

The case when \( m^2 = 0 \) and simultaneously \( \xi = 1/6 \), is a special one. It is certainly not obtained as a limiting case of (45), even if this was obtained perturbatively in \( \xi - 1/6 \), since the starting point (42) is now different. In fact, in this situation, provided

\[
t = \frac{1}{2} \log \left(\frac{\lambda \Phi^2}{2\mu^2}\right) \sim 1,
\]

—so that, as a consequence, \( F(t) \sim 1 \) (this may be viewed as a condition to define \( \mu^2 \))— we also obtain sensible expressions for the approximate solution to first order in the recurrence, of the kind

\[
\frac{1}{\Lambda} \approx -\frac{\kappa}{\Lambda_0} \left(1 \pm \sqrt{1 + \frac{\Lambda_0}{\kappa^2 2(4\pi)^2}}\right),
\]

\[
\Phi^2 \approx \frac{4\Lambda_0}{\kappa \lambda} \left(1 \pm \sqrt{1 + \frac{\Lambda_0}{\kappa^2 2(4\pi)^2}}\right)^{-1}.
\]

We observe here a difference with respect to the situation when \( \Lambda_0 = 0 \) from the beginning, Eq. (40), that we will now comment on. As a general observation, we see that in all these cases we get a reasonable solution (with positive sign) which can be consistently given a perturbative form, in the sense that quantum corrections adopt the form of small corrections to the classical solutions (as it should be, in accordance with our approach). However, we
must also point out that when we have a negative sign in (45) or (46) (which can be obtained for reasonable values of the parameters, too), we get a new quantum solution

\[ \Lambda \simeq 4(4\pi)^2\kappa, \]

which has the same structure as in (40). Actually, the solution (40) is also obtained in a different limit, in which the term \(-1/135\) in the definition of \(B(x)\) becomes dominant, and it goes down in the square root of (46) (as \(1 - \Lambda_0/\kappa^2540(4\pi)^2\)).

It is interesting to note that for this quantum solution of the effective equations the cosmological constant \(\Lambda_0\) is arbitrary but the curvature does not depend on it. Of course this solution is not realistic, as it is indicated by the transition that takes place to anti-De Sitter space \((\kappa < 0, \Lambda < 0)\), and our starting proposal was \(\Lambda > 0\).

Summing up, we have shown explicitly that it is possible, in principle, to apply the RG improved effective potential to the back-reaction problem. It is clear, however, that already for the case of a static geometry the system of equations coming from the effective potentials is quite involved.

7 Conclusions

In this paper we have constructed a RG improved effective potential for theories which involve a few mass scales in curved spacetime. Among the examples that have been discussed there are the Yukawa model, scalar electrodynamics and the \(\lambda\phi^4\)-theory. This last example has been employed to show that the method is very general and can be used to carry out the full program of construction of a RG improved effective Lagrangian. Moreover, the procedure can be extended also to the case of quantum gravity.

Indeed, let us consider the theory of quantum \(R^2\)-gravity with the following Lagrangian,

\[ L = \Lambda - \frac{1}{\kappa^2} R + a \left( R_{\mu\nu}^2 - \frac{1}{3} R^2 \right) + \frac{1}{3} b R^2. \]  

(47)

It is well known that this theory is multiplicatively renormalizable and asymptotically free (for a general review, see [8]). The RG equations in a gauge of Landau type is given by [33]

\[
\begin{align*}
\frac{da}{dt} &= \frac{13.3}{(4\pi)^2} a, \quad \frac{du}{dt} = -\frac{1}{(4\pi)^2 a} \left( \frac{10}{3} u^2 + \frac{183}{10} u + \frac{5}{12} \right), \\
\frac{d\kappa^2}{dt} &= \frac{\kappa^2}{(4\pi)^2 a} \left( \frac{10}{3} u - \frac{1}{3} \right), \\
\frac{d\Lambda}{dt} &= \frac{1}{(4\pi a)^2 \kappa^4} \left( \frac{5}{2} + \frac{1}{8u^2} \right) + \frac{\Lambda}{2a} \left( 10 + \frac{1}{u} \right), \quad u \equiv -\frac{b}{a}. 
\end{align*}
\]

(48)

Notice that, as should be the case, the RG equations for the coupling constants \(a\) and \(u\) are gauge independent, while those for \(\Lambda\) and \(\kappa^2\) are gauge dependent. We expect that in a gauge of Landau type (in which the RG equations for \(\Lambda\) and \(\kappa^2\) are written) there is no one-loop \(\beta\)-function for the gauge parameter in Eq. (3), at least in the case of a background of constant curvature.

Concerning now the choice of the RG parameter \(t\) (and restricting ourselves to a background of constant curvature), we observe that we have a few different masses in our theory.
We may consider the (non-physical) region where, for instance, $\Lambda$ is dominant as compared with $|R|/\kappa^2$. Then, naturally, $t = \frac{1}{2} \log(\Lambda/\mu^2)$, and

$$L_{\text{eff}} = \Lambda(t) - \frac{1}{\kappa(t)} R + \mathcal{O}(R^2).$$  \hspace{1cm} (49)

Solving now Eqs. (48) we obtain $L_{\text{eff}}$ explicitly. In a similar way we can consider other regions. Our main purpose here has been to show that, in fact, the above considerations can be applied as well to the theory of quantum gravity itself. We shall not continue here the discussion of the properties of $R^2$-gravity in any more detail since, as is known, such a theory is apparently not consistent (an unitarity problem exists).

Note, finally, that in this paper we have limited ourselves to investigate only one application of the RG in curved spacetime to the effective Lagrangian. Some other applications, very well studied in the literature, include the calculation of multiloop $\beta$-functions [34], asymptotics of vertex functions for strong curvature [8, 25, 26, 27], etc.

Acknowledgments

SDO would like to thank the members of the Dept. ECM, Barcelona University, for the kind hospitality. This work has been partly supported by DGICYT (Spain) and by CIRIT (Generalitat de Catalunya).
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