Instantaneous gelation and nonexistence for the Oort-Hulst-Safronov coagulation model

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Abstract

The possible occurrence of instantaneous gelation to Oort-Hulst-Safronov (OHS) coagulation equation is investigated for a certain class of unbounded coagulation kernels. The existence of instantaneous gelation is confirmed by showing the nonexistence of mass-conserving weak solutions. Finally, it is shown that for such kernels, there is no weak solution to the OHS coagulation equation at any time interval.

Keywords: Coagulation; Oort-Hulst-Safronov model; Mass-conservation; Gelation; Instantaneous gelation; Nonexistence.

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1 Introduction

Coagulation, a basic kinetic process, represents the dynamics of particle growth in which two or more particles adhere to form a new larger particle. This process may occur in several physical phenomenon such as fluidized bed granulation, planet formation, polymerization etc. A discrete system of differential equations to describe the coagulation of collides travelling in Brownian motion was first developed by Smoluchowski [18] which is known as the Smoluchowski coagulation equation (SCE) and its continuous version was later given by Müller [16]. Each particle is recognized by its size (or volume) and the parameter value of size of each particle is either contained in the set of positive integer numbers \( \mathbb{N} \setminus \{0\} \) (for the discrete case) or in the set of positive real numbers \( \mathbb{R}_{>0} = (0, +\infty) \) (for the continuous case). In a distinct sense, a different coagulation equation was proposed by Oort and Hulst [17] that was adopted in astronomy to describe the coagulation of stellar objects when these objects merge irreversibly via binary interactions to configuration of a bigger objects at a specific moment. However, the tractable form of this equation was later given by Safronov [19]. As a result, it is referred to as the Oort-Hulst-Safronov (OHS) coagulation equation. The discrete version of OHS equation is introduced by Dubvoksi which was later termed as the Safronov-Dubovskii coagulation equation, see [10] [11] [1]. The nonlinear nonlocal OHS coagulation equation for the evolution of the concentration \( \xi(\mu, t) \) of particles of size \( \mu \in \mathbb{R}_{>0} \) at time \( t \geq 0 \) is given by

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\[
\frac{\partial \xi(\mu, t)}{\partial t} = -\frac{\partial}{\partial \mu} \left( \xi(\mu, t) \int_0^{\mu} \nu \Lambda(\mu, \nu) \xi(\nu, t) \, d\nu \right) \\
- \int_{\mu}^{\infty} \Lambda(\mu, \nu) \xi(\mu, t) \xi(\nu, t) \, d\nu, \quad (\mu, t) \in \mathbb{R}^2_{\geq 0},
\]

with initial condition

\[
\xi(\mu, 0) = \xi^{\text{in}}(\mu) \geq 0, \quad \mu \in \mathbb{R}_{> 0}.
\]

The coagulation kernel \( \Lambda(\mu, \nu) \) denotes the intensity force at which particles of size \( \mu \) merge irreversibly with particles of size \( \nu \) to form the larger particles that is assumed to be nonnegative and symmetric, (that is) i.e., \( \Lambda(\mu, \nu) = \Lambda(\nu, \mu) \geq 0 \) for all \( \mu, \nu \in \mathbb{R}^2_{> 0} \). In (1.1), \( \frac{\partial}{\partial t} \) and \( \frac{\partial}{\partial \mu} \) are the partial derivatives with respect to time and space, respectively.

According to [10], in OHS equation (1.1), the aggregation of particles of size \( \mu \) with smaller particles alters the size of particles of size \( \mu \). Furthermore, the coagulation of particle of size \( \mu \) with bigger particles changes the number of particles of size \( \mu \). Therefore, the first term on the right-hand side of (1.1) only changes \( \xi(\mu, t) \) due to coagulation with small particles whereas the last term indicates the decay of particle of size \( \mu \) due to coagulation with bigger particles.

The total mass of the particles in the whole system, at any time \( t \geq 0 \), is given by

\[
\mathcal{M}^1(t) = \int_0^{\infty} \mu \xi(\mu, t) \, d\mu.
\]

Formally, we know that the total mass of particles is neither originated nor ended by any reaction. Therefore, it is expected that the total mass remains conserved throughout time evolution, i.e.,

\[
\int_0^{\infty} \mu \xi(\mu, t) \, d\mu = \int_0^{\infty} \mu \xi^{\text{in}}(\mu) \, d\mu = \varrho_0, \quad \forall \, t \geq 0.
\]

However, for the coagulation kernels growing sufficiently rapidly for large \( \mu, \nu \) such as \( \Lambda(\mu, \nu) = (\mu \nu)^{r/2} \) for \( r \in (1, 2] \), there is a possibility to have a runaway growth that can lead to the formation of particles of infinite mass in finite time. These particles of infinite mass are called infinite gels which are then removed from the system. As a result, we see that the mass conservation breaks down in finite time, i.e.,

\[
\int_0^{\infty} \mu \xi(\mu, t) \, d\mu < \int_0^{\infty} \mu \xi^{\text{in}}(\mu) \, d\mu, \quad T_{\text{gel}} < t,
\]

and this phenomenon is known as gelation. Here, \( T_{\text{gel}} \) is starting time after which the mass conservation breaks down is called the gelation time which can mathematically be defined as

\[
T_{\text{gel}} := \inf \left\{ t \geq 0 \text{ such that } \mathcal{M}^1(t) < \mathcal{M}^1(0) = \varrho_0 \right\}.
\]

If the gelation time \( T_{\text{gel}} = 0 \), then this phenomenon is known as instantaneous gelation.

The gelation phenomenon for the SCE has been significantly discussed in the literature (see [5] and reference therein). Furthermore, the occurrence of instantaneous gelation for the SCE was first investigated by Dongen [9] but the first rigorous mathematical proof was introduced by Carr & da Costa [6]. Recently, Banasiak et al. [5] have supplemented a proof which confirms that the instantaneous gelation takes place for the continuous version of
SCE under certain classes of unbounded coagulation kernels. However, there are very few articles available on the occurrence of gelation for the OHS coagulation equation, see for instance [10], [13]. More recently, Das and Saha [7] studied the instantaneous gelation result to the discrete version of the OHS coagulation equation with a specific class of coagulation kernels. The idea of the their work is mainly motivated from [6]. To the best of our knowledge, the instantaneous gelation result to OHS coagulation equation (1.1)–(1.2) has not been addressed in the literature till date. Therefore, the purpose of this paper is to investigate the occurrence of instantaneous gelation to (1.1)–(1.2) under the class of coagulation kernels mentioned in hypothesis (A) given below. In addition, we also show the nonexistence of weak solutions to (1.1)–(1.2) with the same class of kernels.

Before outlining the results of this paper, let us discuss the mathematical results available on the solutions to the OHS coagulation equation (1.1)–(1.2). Results on the existence, gelation, mass conservation and large time behavior of weak solutions to the OHS equation (1.1)–(1.2) with unbounded coagulation kernels were first established by Lachowicz et al. [13]. Moreover, they also established a deep connection between OHS coagulation equation and the continuous version of Smoluchowski coagulation equation by introducing a generalized coagulation equation. Next, in [1], Bagland proved that a suitable sequence of solutions of discrete version of OHS equation converges towards a solution of OHS equations (1.1)–(1.2). In addition, he has also derived an explicit solution \( \xi(\mu, t) = \frac{2}{M(1+t)}1_{[0,M]}(\mu) \) to the OHS equations (1.1)–(1.2) with the coagulation kernel \( \Lambda(\mu, \nu) = 1 \) and initial data \( \xi(\mu, 0) = \frac{1}{M}1_{[0,M]} \), where \( M \) is a positive constant. Later, in [14] and [15], Laurençot discussed the self similar solution to (1.1)–(1.2) under the coagulation kernels \( \Lambda(\mu, \nu) \equiv 1 \) and \( \Lambda(\mu, \nu) = \mu \nu \), respectively. After these results, Bagland and Laurençot [3] confirmed the presence of self similar solutions to (1.1)–(1.2) under the coagulation kernel \( \Upsilon(\mu, \nu) = \mu^{\alpha} + \nu^{\alpha}, \alpha \in (0,1) \), where the self-similar profiles for these self similar solutions are compactly supported. More recently, Barik et. al. [2] have proved the existence of mass-conserving solutions to (1.1)–(1.2) under the assumptions that the coagulation kernels have an algebraic singularity near zero and that they grow linearly at infinity.

The plan of the paper is the following: Section 2 presents notations of space, definition and hypothesis which are needed in subsequent sections. At the end of this section, the main result of the paper is stated in Theorem 1. However, the proof of Theorem 1 is presented in Section 3. The proof of Theorem 1 depends essentially on Lemmas 2 and 4. These lemmas confirm the non-existence of a mass-conserving solutions to (1.1)–(1.2) at any time. The remaining part of Section 4 is devoted to the mass-conserving solutions to (1.1). In this section, the solution \( \xi \) of (1.1) conserves mass, as shown in Theorem 3, for a special form of unbounded coefficients, if it exists.

2 Preliminaries and main result

In order to study the possible occurrence of instantaneous gelation of solution to (1.1)–(1.2), let us first introduce Banach space \( \mathcal{Y} \) with the norm \( \|\cdot\|_{\mathcal{Y}} \).

\[
\mathcal{Y} = \left\{ \xi \in L^1(0, \infty) : \|\xi\|_{\mathcal{Y}} < \infty \right\} \quad \text{where} \quad \|\xi\|_{\mathcal{Y}} = \int_{0}^{\infty} (1 + \mu)|\xi(\mu, t)| \, d\mu.
\]

We also set

\[
\mathcal{Y}^+ = \left\{ \xi \in \mathcal{Y} : \xi(\mu) \geq 0 \text{ for each } \mu \geq 0 \right\}.
\]
Definition 1. Let us assume that \( \Lambda \) satisfies hypothesis (A) given below. Then, a function \( \xi = \xi(\mu, t) \) is said to be a weak solution to the OHS equation (1.1)–(1.2) with initial condition \( \xi \) in 

\[
0 \leq \xi \in C_w([0, T); L^1(\mathbb{R}_>0)) \cap L^\infty(0, T; \mathcal{Y}^+)
\]

such that

\[
(\mu, \nu, s) \mapsto \mu \Lambda(\mu, \nu) \xi(\mu, s) \xi(\nu, s) \in L^1(\mathbb{R}^2_\times (0, t)),
\]

and

\[
\int_0^\infty \omega(\mu) [\xi(\mu, t) - \xi(\mu, 0)] \, d\mu = \int_0^t \int_0^\infty \int_0^\mu \varpi_1(\mu, \nu) \xi(\mu, s) \xi(\nu, s) \Lambda(\mu, \nu) \, d\nu d\mu ds,
\]

where

\[
\varpi_1(\mu, \nu) = \nu \varpi'(\mu) - \varpi(\nu),
\]

for all \( \varpi \in \mathcal{W}^{1, \infty}(\mathbb{R}_>0) \) and first derivative of \( \varpi \) is compactly supported. Here, the spaces \( \mathcal{W}^{1, \infty} \) and \( C([c, d]; L^1(\mathbb{R}_>0; d\mu)) \) denote the Sobolev space and collection of continuous functions (in time) with respect to weak topology of \( L^1(\mathbb{R}_>0; d\mu) \), respectively.

Now, we state the following hypothesis on the coagulation kernels \( \Lambda \), which is used in the subsequent analysis.

Hypothesis 1. There exists positive constants \( \theta_1, \theta_2 \) and \( 1 < \beta < \gamma \) such that

(A) \( \theta_1 (\mu^\beta + \nu^\beta) \leq \Lambda(\mu, \nu) \leq \theta_2 (1 + \mu)^\gamma (1 + \nu)^\gamma, \quad (\mu, \nu) \in \mathbb{R}^2_>0. \)

Next, we define the moments of the concentration \( \xi \) as follows

\[
\mathcal{M}^r(t) := \int_0^\infty \mu^r \xi(\mu, t) \, d\mu \quad \text{and} \quad \mathcal{M}_m^r(t) := \int_m^\infty \mu^r \xi(\mu, t) \, d\mu, \quad r \in \mathbb{R},
\]

where \( \mathcal{M}^r(t) \) is called \( r^{th} \) moment of the concentration \( \xi \).

We are now in position to state the main result of this paper.

Theorem 1 (Existence of instantaneous gelation). Assume that coagulation kernel \( \Lambda \) satisfies (A). Let \( \xi \) be a weak solution to the OHS equations (1.1)–(1.2) in the sense of Definition 1. Then, for all weak solutions of the OHS equations (1.1)–(1.2), the gelation occurs instantaneously, i.e., \( T_{gel} = 0 \).

To perform the proof of Theorem 1, we follow the arguments developed by Carr and da Costa in [6] and Banasiak et al. in [5]. According to [6] and [5], the proof of Theorem 1 involves two steps. These steps lead to a contradiction if the gelation time is positive. In Section 2, the first step is performed, i.e., if \( T_{gel} > 0 \), then all moments \( \mathcal{M}^r(t) \) are finite for all \( r \geq 1 \) and \( t \in [0, T_{gel}) \). On the other hand, the second step shows that all moments \( \mathcal{M}^r(t) \) are finite only on the time interval \( (0, t_r) \), with \( t_r \to 0 \) as \( r \to \infty \).
3 Existence of instantaneous gelation

Before proving the occurrence of instantaneous gelation for the OHS equations (1.1)–(1.2), we need to show some basic results.

**Theorem 2** (Positivity of first moment). Consider $\xi^{in} \in \mathcal{Y}^+$ such that $\xi^{in} \neq 0$ and $\Lambda > 0$ a.e. in $(0, \infty)^2$. Suppose that $\xi$ is a weak solution of (1.1)–(1.2) on $[0, T)$ in the sense of Definition 1. Then, for all $R > 0$ and $t > 0$,

$$
\int_R^\infty \mu \xi(\mu, t) \, d\mu > 0.
$$

**Proof.** Suppose, for the sake of contradiction, that there exists $t_0 > 0$ such that

$$
R_0 := \inf \left\{ R \geq 0 : \int_R^\infty \mu \xi(\mu, t_0) \, d\mu = 0 \right\} < \infty \quad \text{and} \quad \xi(R_0, s) \neq 0, \quad \forall \ s \in (0, t_0).
$$

(3.1)

Now, for all $\mu \in \mathbb{R}_{>0}$ and $R > R_0$, we set $\varpi(\mu) = \mu \chi_{(R_0, R)}(\mu)$ into (2.1). Then, we obtain

$$
\varpi_1(\mu, \nu) = \begin{cases} 
0, & \text{if } (\mu, \nu) \in (0, R_0] \times (0, \mu), \\
\nu, & \text{if } (\mu, \nu) \in (R_0, R) \times (0, R_0], \\
0, & \text{if } (\mu, \nu) \in (R_0, R) \times (R_0, \mu), \\
-\nu, & \text{if } (\mu, \nu) \in [R, \infty) \times (R_0, R), \\
0, & \text{if } (\mu, \nu) \in [R, \infty) \times [R, \mu).
\end{cases}
$$

Putting the values of $\varpi$ and $\varpi_1$ into (2.1), we have

$$
\int_{R_0}^R \mu \xi(\mu, t_0) \, d\mu \\
= \int_{R_0}^R \mu \xi^{in}(\mu) \, d\mu + \int_0^{t_0} \int_{R_0}^R \nu \Lambda(\mu, \nu) \xi(\mu, s) \xi(\nu, s) \, d\nu \, d\mu \, ds \\
- \int_0^{t_0} \int_{R_0}^R \nu \Lambda(\mu, \nu) \xi(\mu, s) \xi(\nu, s) \, d\mu \, d\nu \, ds \\
= \int_{R_0}^R \mu \xi^{in}(\mu) \, d\mu + \int_0^{t_0} \int_{R_0}^R \mathcal{I}_1(\mu, s) \, d\mu \, ds + \int_0^{t_0} \int_{R_0}^R \mathcal{I}_2(\nu, s) \, d\nu \, ds,
$$

(3.2)

where

$$
\mathcal{I}_1(\mu, s) = \int_0^\nu \nu \Lambda(\mu, \nu) \xi(\mu, s) \xi(\nu, s) \, d\nu \quad \text{and} \quad \mathcal{I}_2(\nu, s) = \int_R^\nu \nu \Lambda(\mu, \nu) \xi(\mu, s) \xi(\nu, s) \, d\mu.
$$

Since $R > R_0$ is arbitrary, we deduce from (3.1), (3.2), Definition 1 the dominated convergence theorem, and the nonnegativity of $\xi^{in}$ and $(\mathcal{I}_1, \mathcal{I}_2)$ that

$$
\xi^{in} = 0 \quad \text{a.e. in } (R_0, \infty), \\
\mathcal{I}_1 = 0 \quad \text{a.e. in } (0, t_0) \times (R_0, \infty), \\
\mathcal{I}_2 = 0 \quad \text{a.e. in } (0, t_0) \times (R_0, \infty).
$$

(3.3)

Now, it follows from (3.3) that

$$
\mu \Lambda(\mu, \nu) \xi(\mu, s) \xi(\nu, s) \chi_{(0, \mu)}(\nu) = 0 \quad \text{a.e. in } (0, t_0) \times (R_0, \infty) \times (0, \infty).
$$
Consequently, an application of Fubini’s theorem yields

\[
0 =: \int_{R_0}^{\infty} \int_0^\mu \mu^2 \xi(\mu, s) \xi(\nu, s) \, d\nu \, d\mu
\]
\[
= \int_{R_0}^{\infty} \int_0^\mu \mu^2 \xi(\mu, s) \xi(\nu, s) \, d\nu \, d\mu + \frac{1}{2} \int_{R_0}^{\infty} \int_0^\mu \mu^2 \xi(\mu, s) \xi(\nu, s) \, d\nu \, d\mu
\]
\[
+ \frac{1}{2} \int_{R_0}^{\infty} \int_\mu^\infty \nu^2 \xi(\mu, s) \xi(\nu, s) \, d\nu \, d\mu.
\]

(3.4)

From (3.4), we infer that

\[
\int_0^{t_0} \left( \int_{R_0}^{\infty} \nu \xi(\nu, s) \, d\nu \right)^2 \, ds = 0
\]

and thus,

\[
\int_{R_0}^{\infty} \nu \xi(\nu, s) \, d\nu = 0, \quad s \in (0, t_0).
\]

(3.5)

Again, we set \( \varpi(\mu) = \mu \chi_{(0,R_0)}(\mu) \) into (2.2) for all \( \mu \in (0, \infty) \). Then, we obtain

\[
\varpi_1(\mu, \nu) = \begin{cases} 
0, & \text{if } (\mu, \nu) \in (0, R_0) \times (0, \mu), \\
-\nu, & \text{if } (\mu, \nu) \in (R_0, \infty) \times (0, R_0), \\
0, & \text{if } (\mu, \nu) \in (R_0, \infty) \times (R_0, \mu).
\end{cases}
\]

Substituting the values of \( \varpi \) and \( \varpi_1 \) into (2.1) gives

\[
\int_0^{R_0} \mu \xi(\mu, t_0) \, d\mu = \int_0^{R_0} \mu \xi^{in}(\mu) \, d\mu - \int_0^{t_0} \int_{R_0}^{\infty} \int_0^\mu \nu \Lambda(\mu, \nu) \xi(\mu, s) \xi(\nu, s) \, d\nu \, d\mu \, ds.
\]

(3.6)

We infer from (3.6) and (3.4) that

\[
\int_0^{R_0} \mu \xi(\mu, t_0) \, d\mu = \int_0^{R_0} \mu \xi^{in}(\mu) \, d\mu.
\]

(3.7)

Now, multiplying the equation (1.1) by \( \mu \) and taking integration with respect to \( \mu \) from 0 to \( R_0 \) gives

\[
\frac{d}{ds} \int_0^{R_0} \mu \xi(\mu, s) \, d\mu = -R_0 \int_0^{R_0} \nu \xi(R_0, s) \xi(\nu, s) \Lambda(R_0, \nu) \, d\nu.
\]

Taking integration with respect to time from 0 to \( t_0 \) yields

\[
\int_0^{R_0} \mu \xi(\mu, t_0) \, d\mu = \int_0^{R_0} \mu \xi^{in}(\mu) \, d\mu - R_0 \int_0^{t_0} \int_0^{R_0} \nu \xi(R_0, s) \xi(\nu, s) \Lambda(R_0, \nu) \, d\nu \, ds.
\]

Now, we conclude from definition (3.1) of \( R_0 \) and (3.7) that

\[
\int_0^{t_0} \int_0^{R_0} \nu \xi(\nu, s) \Lambda(R_0, \nu) \, d\nu \, ds = 0,
\]
from which we readily deduce that
\[ \nu \xi(\nu, s) = 0 \quad \text{a.e. in} \quad (0, t_0) \times (0, R_0) \]
and thus,
\[ \int_0^{t_0} \int_{R_0/2}^{R_0} \nu \xi(\nu, s) d\nu ds \leq \int_0^{t_0} \int_{R_0/2}^{R_0} \nu \xi(\nu, s) d\nu ds = 0. \]
Consequently, it is
\[ \int_{R_0/2}^{R_0} \nu \xi(\nu, s) d\nu = 0, \quad s \in (0, t_0). \quad (3.8) \]
Now, we obtain from (3.8) and (3.5) that
\[ \int_{R_0/2}^{R_0} \nu \xi(\nu, s) d\nu = 0, \quad s \in (0, t_0). \quad (3.9) \]
We conclude, from (3.9) and definition (3.1) of \( R_0 \), that \( R_0 = 0 \). Then \( \xi^{in} \equiv 0 \) according to (3.3). This clearly contradicts our assumption. \( \square \)

**Lemma 1** (Limit behavior for higher moments). Let \( \xi \) be a weak solution to (1.1)–(1.2) with \( \int_R \mu \xi(\mu, t) d\mu > 0 \). Then, we have
\[ \lim_{p \to \infty} \left( \int_0^\infty \mu^p \xi(\mu, t) d\mu \right)^{1/p} = \infty. \]
**Proof.** It is for all \( l \geq 1 \) and \( t \in (0, T) \)
\[ \left( \int_0^\infty \mu^p \xi(\mu, t) d\mu \right)^{1/p} \geq \left( \int_l^\infty \mu^p \xi(\mu, t) d\mu \right)^{1/p} \geq l^{(p-1)/p} \left( \int_l^\infty \mu \xi(\mu, t) d\mu \right)^{1/p}, \]
so that
\[ \lim_{p \to \infty} \left( \int_0^\infty \mu^p \xi(\mu, t) d\mu \right)^{1/p} \geq l. \]
The above inequality is valid for all \( l \geq 1 \). Therefore, we conclude that
\[ \lim_{p \to \infty} (\mathcal{M}^p(t))^{1/p} = \infty. \]

**Lemma 2** (Integrability of all higher moments). Assume that \( \Lambda \) satisfies (A) and that for the initial condition \( 0 \leq \xi^{in} \in \mathcal{Y}^\uparrow \) holds. Assume furthermore that \( \xi \) be a solution to (1.1)–(1.2) on \( [0, T) \) s.t. \( T_{gel} \in (0, T] \). Then, for all integers \( r \geq 1 \) and some \( t_0 \in (0, T_{gel}) \), we have
\[ \sup_{t \in [0, t_0]} (\mathcal{M}^r(t)) < \infty. \]
Proof. For all $\mu \in (0, \infty)$ and $\lambda \geq 1$, set $\varpi(\mu) = \mu \chi_{(0, \lambda)}(\mu)$ into (2.2). Then, we obtain

$$\varpi_1(\mu, \nu) = \begin{cases} 0, & \text{if } (\mu, \nu) \in (0, \lambda) \times (0, \mu), \\ -\nu, & \text{if } (\mu, \nu) \in [\lambda, \infty) \times (0, \lambda), \\ 0, & \text{if } (\mu, \nu) \in [\lambda, \infty) \times [\lambda, \mu). \end{cases}$$

Inserting the value of $\varpi$ and $\varpi_1$ into (2.1), we have for $0 \leq \tau < \varepsilon < t < t_0$ that

$$\int_0^\lambda \mu [\xi(\mu, t) - \xi(\mu, \tau)] \, d\mu = -\int_\tau^t \int_0^\lambda \int_0^\lambda \nu \Lambda(\mu, \nu) \xi(\mu, s) \xi(\nu, s) \, d\nu \, d\mu \, ds.$$

Using the assumptions on $\Lambda$, we arrive at the inequality

$$\Gamma_\lambda(t) - \Gamma_\lambda(\tau) \leq \theta_1 \int_\tau^t \left( \int_0^\lambda \int_0^\lambda \nu \mu \beta \xi(\mu, s) \xi(\nu, s) \, d\nu \, d\mu \right) \, ds \leq -\theta_1 \lambda^{\beta-1} \int_\tau^t \Gamma_\lambda(s) \Theta_\lambda(s) \, ds,$$

where

$$\Gamma_\lambda(t) := \int_0^\lambda \mu \xi(\mu, t) \, d\mu \quad \text{and} \quad \Theta_\lambda(t) := \int_\lambda^\infty \mu \xi(\mu, t) \, d\mu,$$

for $\lambda \geq 1$ and $0 \leq \tau < t < t_0 < T_{gel}$. From $\Gamma_\lambda$ and $\Theta_\lambda$, we have

$$\Gamma_\lambda(t) + \Theta_\lambda(t) = \varrho_0, \quad \forall \ t \in [0, T_{gel}),$$

with $\varrho_0$ defined in (1.3). Now, we infer from Dini’s monotone convergence theorem [12] that $\mathcal{M}^I(t)$ is uniformly convergent in $[0, t_0]$. Therefore, there exist a positive integer $p_0$ s.t.

$$\frac{\varrho_0}{2} \leq \Gamma_\lambda(t), \quad \text{for all } \lambda \geq p_0 \quad \text{and} \quad t \in [0, t_0].$$

Inserting the above lower bound of $\Gamma_\lambda$ into (3.10) leads to

$$U_\lambda(\tau) =: \Theta_\lambda(t) + \frac{1}{2} \varrho_0 \lambda^{\beta-1} \int_\tau^t \Theta_\lambda(s) \, ds \leq \Theta_\lambda(\tau), \quad 0 \leq \tau \leq t \leq t_0 < T_{gel}. \quad (3.11)$$

Setting $C_1 = \frac{1}{2} \varrho_0 \lambda^{\beta-1}$, we infer from (3.11) that

$$\frac{d}{d\tau} U_\lambda(\tau) = C_1 \Theta_\lambda(\tau) \geq C_1 U_\lambda(\tau).$$

Dividing by $U_\lambda(\tau)$, integrating with respect to $\tau$ from $t$ to $t_0$ and then applying (3.11), we obtain

$$\exp(-C_1 t) \Theta_\lambda(t) \leq \exp(-C_1 t_0) U_\lambda(t) \leq \exp(-C_1 t_0) U_\lambda(t_0) \leq \exp(-C_1 t_0) \Theta_\lambda(t_0).$$

Since $\Theta_\lambda(t_0) \leq \varrho_0$, we end up with

$$\Theta_\lambda(t) \leq \varrho_0 \exp(-C_1 (t_0 - t)), \quad (3.12)$$
for all $0 \leq t < \tau < t_0$, and $\lambda \geq p_0$. Next, for fixed $r \geq 2$ and $t \in [0, t_0)$, it is

$$\int_{\lambda_1}^{\lambda} \mu^r \xi(\mu, t) \, d\mu \leq \sum_{k=\lambda_1}^{\lambda-1} (k + 1)^{r-1} \int_k^{k+1} \mu \xi(\mu, t) \, d\mu = \sum_{k=\lambda_1}^{\lambda-1} (k + 1)^{r-1} [\Theta_k(t) - \Theta_{k+1}(t)]$$

$$= \sum_{k=\lambda_1}^{\lambda-1} (k + 1)^{r-1} \Theta_k(t) - \sum_{k=\lambda_1+1}^{\lambda} k^{r-1} \Theta_k(t) \leq (\lambda_1 + 1)^{r-1} \Theta_{\lambda_1}(t) + \sum_{k=\lambda_1+1}^{\lambda-1} ((k + 1)^{r-1} - k^{r-1}) \Theta_k(t)$$

$$\leq (\lambda_1 + 1)^{r-1} \Theta_{\lambda_1}(t) + (r - 1) \sum_{k=\lambda_1+1}^{\lambda-1} (k + 1)^{r-2} \Theta_k(t), \quad \lambda > \lambda_1 \geq p_0. \quad (3.13)$$

Since $\beta > 1$ there is $R \geq p_0$ depending on $t, \tau, C_1, r$ and $\beta$ such that $r \log(k+1) - C_1(\tau-t) \leq 0$. Hence, we get from (3.12) and (3.13) that

$$\int_{\lambda}^{\lambda+1} \mu^r \xi(\mu, t) \, d\mu \leq \lambda^{r-1} \sum_{k=\lambda+1}^{\lambda} \frac{1}{k^{r+1}} \frac{1}{(k + 1)^2}. \quad (3.14)$$

The series on the right-hand side of (3.14) is convergent. Furthermore, we obtain

$$\int_{0}^{R} \mu^r \xi(\mu, t) \, d\mu \leq R^{r-1} \varrho_0. \quad (3.15)$$

Hence, from (3.14) and (3.15), the proof of Lemma 2 is completed. \[\Box\]

**Lemma 3** (Equation for $M^r(t)$). Assume that the coagulation rate $\Lambda$ satisfies (A). Assume further that $\xi$ is a weak solution to (1.1)–(1.2) on $[0, T)$ with $0 \leq \xi_{\text{in}} \in Y^+$ s.t. $T_{\text{gel}} \in (0, T]$. Then, for all $0 \leq \delta \leq t < t_0 < T_{\text{gel}}$ and $r \geq 2$, we have

$$M^r(t) - M^r(\delta) = \int_{\delta}^{t} \int_{0}^{\infty} \int_{0}^{\infty} \omega(\mu, \nu) \xi(\mu, s) \xi(\nu, s) \Lambda(\mu, \nu) \, dv \, d\mu \, ds, \quad (3.16)$$

where $\omega(\mu, \nu) = r^r \mu^{r-1} - \nu^r$. 


Proof. For $\lambda \geq 1$, setting $\varpi(\mu) = \mu^r \chi(0,\lambda)(\mu)$ into (2.2), we obtain

$$\varpi_1(\mu, \nu) = \begin{cases} r\nu \mu^{r-1} - \nu^r, & \text{if } (\mu, \nu) \in (0, \lambda) \times (0, \mu), \\ -\nu^r, & \text{if } (\mu, \nu) \in [\lambda, \infty) \times (0, \lambda), \\ 0, & \text{if } (\mu, \nu) \in [\lambda, \infty) \times [\lambda, \mu). \end{cases}$$

Inserting the value of $\varpi$ and $\varpi_1$ into (2.1), we get

$$\int_0^\lambda \int_0^\mu \left( r\nu \mu^{r-1} - \nu^r \right) \Lambda(\mu, \nu) \xi(\mu, s) \xi(\nu, s) \nu d\nu d\mu = \int_t^\epsilon \int_0^\lambda \int_0^\mu \left( r\nu \mu^{r-1} - \nu^r \right) \Lambda(\mu, \nu) \xi(\mu, s) \xi(\nu, s) \nu d\nu d\mu ds$$

$$- \int_t^\epsilon \int_0^\lambda \int_0^\mu \nu^r \Lambda(\mu, \nu) \xi(\mu, s) \xi(\nu, s) \nu d\nu d\mu ds.$$ (3.17)

Let us estimate both terms on the right-hand side of (3.17), separately. By assumption (A) and Lemma 2, we can simplify the first integral term as

$$\int_0^\lambda \int_0^\mu (r\nu \mu^{r-1} - \nu^r) \Lambda(\mu, \nu) \xi(\mu, s) \xi(\nu, s) \nu d\nu d\mu$$

$$\leq \theta_2 r \int_0^\lambda \int_0^\mu \nu \mu^{r-1} (1 + \mu)^{\gamma_1} (1 + \nu)^{\gamma_2} \xi(\mu, s) \xi(\nu, s) \nu d\nu d\mu$$

$$\leq 2^{\gamma_2} \theta_2 r \int_0^\lambda \int_0^\mu \left( \mu^{r-1} + 2 \mu^{\gamma+r-1} + \mu^{2\gamma+r-1} \right) \nu \xi(\mu, s) \xi(\nu, s) \nu d\nu d\mu$$

$$\leq 2^{\gamma_2} \theta_2 r \phi_0 (\|\xi\|_{r-1} + \|\xi\|_{\gamma+r-1} + \|\xi\|_{2\gamma+r-1})$$

$$\leq \Omega_1 < \infty, \quad \text{for } 0 \leq \delta \leq t < t_0 < T_{\text{gel}},$$ (3.18)

where $\Omega_1$ is a constant independent of $\lambda$ and $s$. Similarly, we conclude from Lemma 2 that

$$\int_0^\lambda \int_0^\mu \nu^r \Lambda(\mu, \nu) \xi(\mu, s) \xi(\nu, s) \nu d\nu d\mu \leq \Omega_1 < \infty, \quad \text{for } 0 \leq \delta \leq t < t_0 < T_{\text{gel}},$$ (3.19)

Thanks to (3.18) and (3.19), we may pass to the limit as $\lambda \to \infty$ in (3.17) and conclude, from the dominated convergence theorem, that (3.16) is valid.

Lemma 4 (Bound of the time interval before instantaneous gelation occurs). Assume that (A) holds and let $\xi$ be a weak solution of (1.1)–(1.2) on $(0, T]$ with $0 \leq \zeta_{\text{in}}^\in \in \mathcal{Y}^+$. Suppose $0 < \delta < t \leq \tau < t_0 < T_{\text{gel}}$ where $\delta$ and $\tau$ are fixed and $T_{\text{gel}} \in (0, T]$. Then, for each $r \geq 2$, we have

$$t \leq \delta + \frac{1}{(\beta - 1) \theta_1 \psi_0^{1-\sigma} \left[ \frac{1}{\mathcal{M}^r(\delta)} \right]^\sigma},$$ (3.20)

where $\sigma := (\beta - 1)/(r - 1)$ and $\beta > 1$.

Proof. We observe, from Lemma 3 and assumption (A), that

$$\mathcal{M}^r(t) - \mathcal{M}^r(\delta) = \int_0^t \int_0^\infty \int_0^\lambda (r\nu \mu^{r-1} - \nu^r) \Lambda(\mu, \nu) \xi(\mu, s) \xi(\nu, s) \nu d\nu d\mu ds$$

$$\geq (r-1) \int_0^t \int_0^\infty \int_0^\lambda \nu \mu^{r-1} \Lambda(\mu, \nu) \xi(\mu, s) \xi(\nu, s) \nu d\nu d\mu ds$$

$$\geq (r-1) \theta_1 \int_0^t \int_0^\infty \int_0^\lambda \nu^{\beta+r-1} \xi(\mu, s) \xi(\nu, s) \nu d\nu d\mu ds,$$ (3.21)
where $0 < \delta < t < t_0 < T_{\text{gel}}$. Let us apply Hölder’s inequality to obtain

$$\mathcal{M}^r(t) = \int_0^\infty \mu^r \xi(\mu, t) \, d\mu$$

$$= \int_0^\infty \mu^{(r+\beta-1)/(\sigma+1)} \mu^{\sigma/(\sigma+1)} \xi(\mu, t)^{1/(\sigma+1)} \xi(\mu, t)^{\sigma/(\sigma+1)} \, d\mu$$

$$\leq \left( \int_0^\infty \left( \mu^{(r+\beta-1)/(\sigma+1)} \xi(\mu, t)^{1/(\sigma+1)} \right)^{\sigma+1} \, d\mu \right)^{1/(\sigma+1)}$$

$$\times \left( \int_0^\infty \left( \mu^{\sigma/(\sigma+1)} \xi(\mu, t)^{\sigma/(\sigma+1)} \right)^{(\sigma+1)/\sigma} \, d\mu \right)^{\sigma/(\sigma+1)}$$

$$\leq \left( \int_0^\infty \mu^{r+\beta-1} \xi(\mu, t) \, d\mu \right)^{1/(\sigma+1)} \left( \int_0^\infty \mu \xi(\mu, t) \, d\mu \right)^{\sigma/(\sigma+1)}$$

$$\leq (\mathcal{M}^{r+\beta-1}(t))^{1/(\sigma+1)}(\theta_0)^{\sigma/(\sigma+1)},$$

where $\sigma = (\beta - 1)/(r - 1)$. We notice, from the above inequality, that

$$(\mathcal{M}^r(t))^{\sigma+1}(\theta_0)^{-\sigma} \leq \mathcal{M}^{r+\beta-1}(t). \quad (3.22)$$

Consequently, from (3.21) and (3.22), we obtain

$$\mathcal{M}^r(t) \geq \mathcal{M}^r(\delta) + (r - 1)\theta_1 \theta_0^{1-\sigma} \int^t_\delta (\mathcal{M}^r(s))^{\sigma+1} \, ds, \quad 0 < \delta < t < t_0 < T_{\text{gel}}. \quad (3.23)$$

Now, we introduce

$$Q^r(t) := \mathcal{M}^r(\delta) + (r - 1)\theta_1 \theta_0^{1-\sigma} \int^t_\delta (\mathcal{M}^r(s))^{\sigma+1} \, ds. \quad (3.24)$$

From (3.23) and (3.24), we get

$$\mathcal{M}^r(t) \geq Q^r(t). \quad (3.25)$$

Next, we infer from (3.24) and (3.25) that

$$\frac{d}{dt}(Q^r(t)) = \theta_1 \theta_0^{1-\sigma} (r - 1)(\mathcal{M}^r(t))^{\sigma+1} \geq \theta_1 \theta_0^{1-\sigma} (r - 1)(Q^r(t))^{\sigma+1}, \quad t \in [\delta, t_0).$$

Now, taking integration with respect to $t$ from $\delta$ to $t_0$, we have

$$\int^t_\delta \frac{1}{(Q^r(t))^{\sigma+1}} \, dQ^r(t) \geq \theta_1 \theta_0^{1-\sigma} (r - 1) \int^t_\delta dt,$$

so that

$$0 \leq (Q^r(t_0))^{-\sigma} \leq (Q^r(\delta))^{-\sigma} + \sigma \theta_1 \theta_0^{1-\sigma} (r - 1)(\delta - t_0). \quad (3.26)$$

The definition of $\sigma$ implies that

$$(Q^r(t))^{-\sigma} \geq \frac{1}{(Q^r(\delta))^{\sigma}} + (\beta - 1)\theta_1 \theta_0^{1-\sigma} (\delta - t) \geq 0.$$ 

Consequently, we obtain

$$t \leq \delta + \frac{1}{(\beta - 1)\theta_1 \theta_0^{(r-\beta)/(r-1)}} \left[ \frac{1}{Q^r(\delta)} \right]^{\sigma},$$

from which (3.20) is easily deduced. □
Proof of the existence of instantaneous gelation: Assume for contradiction that $T_{\text{gel}} \in (0, \infty]$. From Lemma 4, we get an estimate for a solution of (1.1)–(1.2) on $[0, T)$ as
\[ t \leq \delta + \frac{1}{(\beta - 1)\theta_1 \theta_0^{(r-\beta)/(r-1)}} \left[ \frac{1}{M^0(\delta)} \right]^{(\beta-1)/(r-1)}, \]
where $r \geq 2$ and $0 \leq \delta < t < T_{\text{gel}}$. Now, we may pass to the limit as $r \to \infty$ in (3.27) and conclude from Lemma 1 that $t \leq \delta$ for $\delta \in (0, t)$. Hence a contradiction is constructed. Therefore, it is $T_{\text{gel}} = 0$.

In the following section, we show the mass conservation property of the solution to the OHS model if it exists.

4 Mass-conserving solutions

To prove the equality (1.3), we consider the following form of coagulation kernels:
\[ \Lambda(\mu, \nu) = \varphi(\mu) + \varphi(\nu) + \Psi(\mu, \nu), \quad (\mu, \nu) \in (0, \infty)^2, \]
where $\varphi(\mu) = \theta_1 \mu^\beta$ and $0 \leq \Psi(\mu, \nu) \leq K(\mu + \nu)$ for all $\beta > 1$ and some constants $\theta_1, K > 0$. Here, we notice that the class of coagulation kernel (4.1) satisfies hypothesis (A).

Theorem 3 (Mass conservation). Assume that the coagulation kernel $\Lambda$ satisfies (4.1). If $\xi$ is a weak solution to (1.1)–(1.2) with $0 \leq \xi^* \in \mathcal{Y}^+$ on $[0, T)$, where $T \in (0, \infty]$, then $\xi$ satisfies mass conservation property (1.3) for all $0 \leq t < T$.

Proof. First, let us assume that $\xi$ is a weak solution to the coagulation equation (1.1) on $[0, T)$ in the sense of Definition 1. Since $\xi^* \neq 0$ and $\mathcal{M}^0(\xi) \in \mathcal{C}([0, T))$, there are $t_0 \in [0, T)$ and $\varepsilon_1 > 0$ such that
\[ \mathcal{M}^0(\xi) \geq \varepsilon_1, \quad t \in [0, t_0]. \]
From (1.1), (1.2) and Definition 1, we get
\[ \varepsilon_1 \int_0^{t_0} \mathcal{M}^*(\xi(t)) \, dt \leq \int_0^{t_0} \mathcal{M}^0(\xi(t)) \mathcal{M}^*(\xi(t)) \, dt \leq \frac{1}{2} \int_0^{t_0} \int_0^\infty \int_0^\infty \Lambda(\mu, \nu) \xi(t) \xi(t) \, dt < \infty, \]
and this implies
\[ \mathcal{M}^*(\xi) \in L^1(0, t_0) \quad \text{for all } t \geq 0. \]

For all $\mu \in (0, \infty)$, set $\varpi(\mu) = \mu \chi_{(0, \lambda)}(\mu)$ into (2.2) to get
\[ \varpi_1(\mu, \nu) = \begin{cases} 0, & \text{if } (\mu, \nu) \in (0, \lambda) \times (0, \mu), \\ -\nu, & \text{if } (\mu, \nu) \in [\lambda, \infty) \times (0, \lambda), \\ 0, & \text{if } (\mu, \nu) \in [\lambda, \infty) \times [\lambda, \mu]. \end{cases} \]
Inserting the value of \( \varpi \) and \( \varpi_1 \) into (2.1), we end up with

\[
\int_0^\lambda \mu \left[ \xi(\mu, t) - \xi(\mu, 0) \right] \ d\mu = -\int_0^t \int_0^\infty \int_0^\lambda \nu \Lambda(\mu, \nu) \zeta(\mu, s) \xi(\nu, s) \ d\nu \ d\mu \ ds. \tag{4.4}
\]

We simplify the right-hand side of (4.4), using the properties of the coagulation kernel,

\[
\int_0^t \int_0^\infty \int_0^\lambda \nu \Lambda(\mu, \nu) \xi(\mu, s) \xi(\nu, s) \ d\nu \ d\mu \ ds \leq 2(\theta_1 + K) \sup_{t \in [0, T]} \| \xi \|_Y \int_0^t \int_0^\infty \mu^\beta \xi(\mu, s) \ d\mu \ ds. \tag{4.6}
\]

Finally, (4.3), (4.6), and the dominated convergence theorem allow to conclude that the solution \( \zeta \) satisfies (1.3) as \( \lambda \to \infty \) in (4.4). \( \square \)

**Corollary 1** (Nonexistence of a weak solution.) Suppose the coagulation kernel \( \Lambda \) satisfies (4.1). Let \( \xi_{\text{in}} \in Y^+ \) be non-trivial initial data. Then, the equation (1.1)–(1.2) has no weak solution, defined in \([0, T]\), for any \( T > 0 \).

**Proof.** From Theorem 3, we get that the solution \( \xi(t) \) satisfies the mass conserving property (1.3) for the class of coagulation kernels (4.1). The class of coagulation kernels (4.1) satisfies the condition (A). Hence, Theorem 1 implies that \( T_{gel} = 0 \). These both statements are completely opposite to each other. Hence the equation (1.1)–(1.2) with (4.1) has no solution. This completes the proof. \( \square \)

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