Non-Gaussianity in the Squeezed Three-Point Correlation from the Relativistic Effects

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Abstract
Assuming a ΛCDM universe in a single-field inflationary scenario, we compute the three-point correlation function of the observed matter density fluctuation in the squeezed triangular configuration, accounting for all the relativistic effects at the second order in perturbations. This squeezed three-point correlation function characterizes the local-type primordial non-Gaussianity, and it has been extensively debated in literature whether there exists a prominent feature in galaxy clustering on large scales in a single-field inflationary scenario either from the primordial origin or the intrinsic nonlinearity in general relativity. First, we show that theoretical descriptions of galaxy bias are incomplete in general relativity due to ambiguities in spatial gauge choice, while those of cosmological observables are independent of spatial gauge choice. Hence a proper relativistic description of galaxy bias is needed to reach a definitive conclusion in galaxy clustering. Second, we demonstrate that the gauge-invariant calculations of the cosmological observables remain unaffected by extra coordinate transformations like CFC or large diffeomorphism like dilatation. Finally, we show that the relativistic effects associated with light propagation in observations cancel each other, and hence there exists \textit{no} non-Gaussian contribution from the so-called projection effects in the squeezed three-point correlation function.
In the standard model of cosmology, cold dark matter is the majority of the matter content of the Universe, and a cosmological constant dominates the energy content today. The initial condition is set by a slow-rolling single scalar field during the inflationary expansion in the early Universe, and the primordial fluctuations are highly Gaussian. Despite several deficiencies, the standard model of cosmology has been extremely successful in explaining cosmological observations on a wide range of scales, spanning galactic scale to horizon scale (see, e.g., [1–3]). Among the issues in the standard model, the origin of the Universe is particularly interesting and puzzling. The standard single-field
inflationary model is so generic that very little is known about the scalar field and its potential, except that it was slow-rolling and its energy density is dominant in the early Universe (see, e.g., [4, 5] for a review). A further investigation of the standard inflationary model revealed [6] that the primordial fluctuations deviate slightly from a perfect Gaussianity, and the deviation is characterized by a parameter $f_{\text{NL}}$ in the squeezed limit bispectrum. The standard single-field inflationary model predicts negligibly small non-Gaussianity $f_{\text{NL}} \sim (n_s - 1) \sim \epsilon$ [6], where $n_s$ is the spectral index and $\epsilon \sim 0.01$ is the slow-roll parameter. In contrast, other non-standard inflationary models such as multi-field models and models with non-canonical kinetic term predict non-Gaussianity much larger than the standard inflationary model (see, e.g., [7]). Robust detection of the primordial non-Gaussianity is, therefore, one of the important targets in the current and the upcoming large-scale surveys that can reveal the nature of the early Universe.

Recently it was shown [8] that in the presence of a local-type primordial non-Gaussianity the galaxy bias exhibits a strong scale-dependence in the power spectrum on very large scales, where the bias factor is expected to be a constant and the galaxy number density fluctuation is in proportion to the matter density fluctuation. Further investigations show [9–12] that measurements of the galaxy power spectrum in the upcoming surveys would provide constraints on the primordial non-Gaussianity tighter than the constraints obtained in measurements of the cosmic microwave background (CMB) anisotropies, as the upcoming galaxy surveys will have larger three-dimensional volumes, compared to the two-dimensional snapshot from the CMB measurements. Certainly, this would be a promising avenue in cosmology, given the recent developments in large-scale galaxy surveys such as the Dark Energy Spectroscopic Instrument [13] and the Vera C. Rubin Observatory (formerly LSST) [14] and two space missions, Euclid [15] and the Nancy Grace Roman Space Telescope (formerly WFIRST) [16].

The question then arises naturally for the standard inflationary model: Since the primordial non-Gaussianity is small but non-zero, will this induce a scale-dependent bias in the galaxy power spectrum in the standard model? Moreover, the relativistic computation of the matter density fluctuation beyond the linear order in perturbations shows [11, 17–21] that there exist nonlinear relativistic effects in the initial condition, arising from the nonlinear Hamiltonian constraint in general relativity, even when the primordial fluctuation is linear and Gaussian. Will this non-Gaussianity again induce a scale-dependent bias in the galaxy power spectrum in the standard model? It has been argued [22–24] against this implication that for a single-field inflationary model there exists only one degree of freedom and a long-mode fluctuation by this degree can be absorbed into a coordinate transformation, such that the local two-point correlation is not affected by the presence of a long mode fluctuation. This argument can be equally applied to the nonlinear relativistic effects from the Hamiltonian constraint, and no scale-dependent galaxy bias is predicted in the case of a single-field inflationary model.

Given that the expectation for the standard inflationary model is either small ($\sim \epsilon$) or zero if absorbed by a coordinate transformation, an order unity correction from the nonlinear relativistic effects is a significant contamination, if real, for future observations. Furthermore, the power spectrum on large scales or the squeezed limit bispectrum in an infinite hypersurface are not a direct observable we can measure from large-scale surveys; observations are made on a light cone volume in terms of the observed redshift and the observed angular position on our sky, which gives rise to additional relativistic effects in cosmological observations [25–29]. It was argued [9, 11, 22, 30–35] that the relativistic effects associated with the light propagation and observations add extra contributions to measurements of the primordial non-Gaussianity at the level similar to the intrinsic nonlinear relativistic effects, and this contamination is always present, no matter what inflationary models are considered. Therefore, it is of significant interest that we obtain accurate predictions for the primor-
dial non-Gaussianity for each cosmological model that can be measured from large-scale surveys. Here we examine the issue critically, accounting for all the relativistic effects from the Hamiltonian constraint and the light propagation. The past work for or against the extra contributions to the primordial non-Gaussianity is an important first step, but the final answer to the level of observable non-Gaussianities is yet to be derived: While a proper second-order theory is needed to compute the bispectrum or the three-point correlation, most computations [22–24] are based on the linear-order calculations, supplemented by coordinate transformations to absorb long-mode fluctuations. In particular, these calculations are performed in a coordinate system with a finite range of validity, and a Fourier transformation is made to compute the squeezed limit bispectrum, presumably outside the validity range. In other works [31–40], the relativistic effects from the light propagation and observations are also considered, but not in full entirety. For example, the relativistic contributions at the observer position or along the line-of-sight direction are often neglected in the past work, while those individual terms could potentially add an order unity correction to the primordial non-Gaussianity. Here we improve the previous work and provide a complete second-order gauge-invariant calculation of the three-point correlation function.

In fact, we find that there exist several flaws in theoretical descriptions that need to be addressed in full but are less known in literature, before one can reach a definitive answer. These issues are discussed in detail in Section 3. In particular, any computations beyond the linear order in perturbations require a choice of spatial gauge condition, as they change with spatial gauge transformation and so do the bispectrum and the three-point correlation function. We show that when observable quantities are computed, their theoretical descriptions are independent of a choice of spatial gauge. However, the relation between the galaxy and the matter distributions is independent of observations, and it requires a physical explanation for a specific choice of spatial gauge. Moreover, we show that the gauge-invariant calculations of cosmological observables leave no residual degrees of freedom, often employed in a coordinate transformation like dilatation or conformal Fermi coordinates. Adopting a ΛCDM model in the standard inflationary model and using the Einstein equation, we derive exact analytical solutions for the second-order perturbation variables and compute the observed three-point correlation function. We demonstrate that the relativistic effects associated with light propagation and observations generate extra contributions to the primordial non-Gaussianities but they all add up to cancel each other, if all the relativistic effects are properly considered.

The organization of the paper is as follows: In Section 2, we compute the bispectrum of the matter density fluctuation in the standard perturbation theory and the relativistic perturbation theory, and we discuss the implications for the scale-dependent galaxy bias. Theoretical flaws in the previous work are extensively discussed in Section 3: Convention for the primordial non-Gaussianity in Section 3.1, ambiguity in spatial gauge choices in Section 3.2, coordinate dependence of the ensemble average in Section 3.3, possibility of extra symmetry in gauge-invariant calculations in Section 3.4, and gauge choice for galaxy biasing in Section 3.5. In Section 4, we present how such theoretical issues are resolved in the observed matter density fluctuation and compute the observed three-point correlation function in the squeezed limit. Our main findings are summarized in Section 5, and the implication of our work is discussed in Section 6. The details of the second-order relativistic calculations are presented in two Appendices A and B.

2 Second-order matter density fluctuation

In this section we briefly review the theoretical descriptions of the second-order matter density fluctuation in the standard Newtonian perturbation theory and the relativistic perturbation theory. We then
compute the matter bispectra in the squeezed limit for both cases and compare them to the previous work.

2.1 Standard and relativistic perturbation theories

In the standard Newtonian perturbation theory (SPT), the governing equations of the Poisson equation, the Euler equation, and the continuity equation are often solved for the pressureless medium to yield recurrent solutions for the matter density fluctuation (see, e.g., [41–44]). In particular, a simple exact perturbative solution can be derived in a ΛCDM universe [20], and the matter density fluctuation up to the second order in perturbations is

$$\delta_m(x^\mu) = -D_1 \Delta \mathcal{R} + \frac{5}{7} D_A \nabla_\alpha (\mathcal{R}^{\alpha} \Delta \mathcal{R}) + \frac{1}{7} D_B \Delta (\mathcal{R}^{\alpha} \mathcal{R}_{\alpha}) ,$$  \hspace{1cm} (2.1)

where commas represent spatial derivatives, $\Delta = \bar{g}^{\alpha\beta} \nabla_\alpha \nabla_\beta$ is a Laplacian operator, three time-dependent growth functions $D_1(t)$, $D_A(t)$, and $D_B(t)$ are given in Eqs. (A.13) and (A.15), and the comoving-gauge curvature perturbation $\mathcal{R}(x)$ in a hypersurface at early time $t_i$ specifies the initial condition (see Appendix A.1). The first growth function $D_1(t)$ corresponds to the standard linear-order growth factor $D(t)$ in literature, if normalized to unity at present time. So, the first term is just the linear-order matter density fluctuation:

$$\delta_m^{(1)}(x, t_i) = -D_1(t_i) \Delta \mathcal{R}(x) .$$ \hspace{1cm} (2.2)

The other two growth functions are often approximated as $D_A = D_B \approx D^2$ in literature, while this equality is true only in the Einstein-de Sitter universe. Two quadratic terms in Eq. (2.1) comprise the second-order matter density fluctuation often characterized by the Fourier kernel $F_2$ in the standard perturbation theory (SPT), when expressed in terms of the initial density fluctuation $\delta_m(x, t_i)$.

In the relativistic perturbation theory, there exist more degrees of freedom due to the diffeomorphism symmetry in general relativity. In particular, the matter density fluctuation, defined as

$$\delta(x^\mu) := \frac{\rho_m(x^\mu)}{\bar{\rho}_m(t)} - 1 ,$$ \hspace{1cm} (2.3)

is gauge-dependent, as it changes depending on our choice of hypersurface, where $\rho_m(x^\mu)$ is the matter density and the background matter density $\bar{\rho}_m(t)$ is just a function of time. Even at the linear order in perturbations, many different matter density fluctuations exist as the solutions of the Einstein equation in different gauge conditions (see, e.g., [45, 46]). Consequently, we need to choose one gauge condition and its matter density fluctuation among many other choices, when we want to relate it to the galaxy number density fluctuation, which is known as galaxy bias [47–50] (see also [51] for a recent review). Naturally, we demand that the matter density fluctuation at the linear order in relativistic perturbation theory reduces to one in the standard Newtonian perturbation theory. This condition leads to a choice of hypersurface described by the synchronous gauge or the comoving gauge, in which the matter density fluctuations are indeed identical at the linear order to each other and the Newtonian one (not in Newtonian gauge). The hypersurface in those gauge choices represents the proper-time hypersurface of the matter fluid [28, 29, 46, 52].

The situation beyond the linear order in perturbations is somewhat ambiguous, as the matter density fluctuations in the synchronous gauge and the comoving gauge are different [46]. Furthermore, the spatial gauge condition, which is of no relevance for scalar fluctuations at the linear order, starts to affect the matter density fluctuation beyond the linear order. The matter density fluctuation up to the second order in perturbations is then [20]

$$\delta_m(x^\mu) = D_1 \left( -\Delta \mathcal{R} + \frac{3}{2} \mathcal{R}^{\alpha} \mathcal{R}_{\alpha} + 4 \mathcal{R} \Delta \mathcal{R} \right) + \frac{5}{7} D_A \nabla_\alpha (\mathcal{R}^{\alpha} \Delta \mathcal{R}) + \frac{1}{7} D_B \Delta (\mathcal{R}^{\alpha} \mathcal{R}_{\alpha}) ,$$ \hspace{1cm} (2.4)
and it is almost identical to the matter density fluctuation in Eq. (2.1), except two extra terms in the round bracket, arising from the nonlinear constraint equation of general relativity (see also [17, 18] for other derivations in the synchronous gauge). These relativistic contributions are generic in the proper-time hypersurface, and independent of spatial gauge choice, which only affect the Newtonian contributions in Eq. (2.4).

In Section 3 further discussion is presented in regard to the gauge choice and its consequence.

### 2.2 Bispectrum of the matter density fluctuation in the squeezed limit

Using the matter density fluctuation at the second order in perturbations, we compute its three-point correlation function and the bispectrum. In particular, our primary interest is the three-point correlation function \( \xi_m(x_1, x_2, x_3) \) in the squeezed triangular configuration, in which two points \( x_1 = x_2 \) are identical (or close enough) and a third point \( x_3 \) is far away from the two points:

\[
\xi_{\text{squeezed}} := \xi_m(x_1, x_2, x_3) , \quad |x_1 - x_2| \ll |x_1 - x_3| .
\]  

(2.5)

The correlation function in this special triangle is useful in the limit (also known as the squeezed limit):

\[
\xi_{\text{squeezed}}^\text{lim} := \lim_{|x_1 - x_3| \to \infty} \xi_{\text{squeezed}} ,
\]

(2.6)

in which the correlation function \( \xi_{\text{squeezed}}^\text{lim} \) is subject to various consistency relations [6, 53–58], providing clues for the nature of the initial conditions at the early Universe. Moreover, the galaxy two-point correlation function or the power spectrum receives corrections from the squeezed three-point correlation \( \xi_{\text{squeezed}} \) in the presence of non-Gaussianity [59, 60], which is evident for non-vanishing three-point correlation function on large scales. While the non-Gaussian contribution from the Newtonian non-linear evolution in the matter density fluctuation in Eq. (2.1) is negligible on large scales, the non-Gaussian contribution from the initial conditions characterized by

\[
R(x) := R_g(x) + \frac{3}{5} f_{NL} R_g^2(x) \quad \text{at} \quad t = t_i ,
\]

(2.7)

shows prominent features in the galaxy power spectrum on large scales [8, 60, 61], where \( R_g \) represents the linear-order Gaussian fluctuation and \( f_{NL} \) is often assumed to be constant.\(^2\) Note that the numerical factor 3/5 in the convention reflects the linear-order time-evolution of the Newtonian gauge potential \( \varphi_\chi \) in Eq. (A.10) from the radiation dominated era to the matter dominated era.

In the simplest model of galaxy formation, galaxies (or dark matter halos) form in an over-dense region, where the matter density fluctuation is above a threshold \( \delta_m \geq \delta_c \), characterized by a critical density contrast \( \delta_c \) [47, 49, 62]. For the Gaussian distribution of the matter density fluctuation, the galaxy number density is a biased tracer of the matter density fluctuation, and its two-point correlation function can be analytically computed [47–49, 63] as

\[
\xi_g(x_1, x_2) = \nu^2 \frac{\sigma^2}{\sigma_R^2} \xi_m(x_1, x_2) ,
\]

(2.8)

where \( \sigma^2_R \) is the rms matter density fluctuation with smoothing length \( R \) and \( \nu := \delta_c / \sigma_R \). In the presence of non-Gaussianity due to the non-Gaussian initial conditions or the non-linear evolution, the probability distribution of the matter density fluctuation is altered, affecting the number density

\(^2\) Though we already used the subscript \( g \) to represent galaxies in the number density fluctuation \( n_g \), we use the same subscript \( g \) here to indicate the Gaussian field, following the convention.
of the peaks above the threshold. The leading correction to the galaxy two-point correlation function arises from the three-point correlation function \( \xi_{sqz} \) in the squeezed triangular configuration [59, 60]

\[
\Delta \xi_g(\mathbf{x}_1, \mathbf{x}_2) = \frac{\nu^3}{\sigma_R^3} \xi_{sqz}(\mathbf{x}_1, \mathbf{x}_1, \mathbf{x}_2). 
\]

In this context, it is evident that the primordial non-Gaussianity in the initial hypersurface in Eq. (2.7) induces a non-vanishing three-point correlation function, in particular \( \xi_{sqz} \) in the squeezed configuration, and in turn generates the correction to the galaxy two-point correlation function \( \xi_g \) or the power spectrum \( P_g \). Similarly, the non-linear relativistic terms in Eq. (2.4) also contribute to the three-point correlation function \( \xi_{sqz} \) and hence the galaxy two-point correlation function \( \Delta \xi_g \), according to Eq. (2.9).

Here we present the computation of the three-point correlation function \( \xi_{sqz} \) in the squeezed triangular configuration, using the matter density fluctuation in Eq. (2.4). For the leading contribution in the bispectrum or the three-point correlation function, we need to contract one second-order contribution \( \delta_m^{(2)} \) and two linear-order contributions \( \delta_m^{(1)} \) to the matter density fluctuation. The expression of \( \xi_{sqz} \) is derived in Eq. (A.61) as

\[
\xi_{sqz}(\mathbf{x}_1, \mathbf{x}_1, \mathbf{x}_2) = \int \frac{d^3 \mathbf{k}_1}{(2\pi)^3} \int \frac{d^3 \mathbf{k}_2}{(2\pi)^3} e^{i \mathbf{k}_1 \cdot (\mathbf{x}_1 - \mathbf{x}_2)} \left[ B_{112} + B_{211} + B_{121} \right](\mathbf{k}_1, \mathbf{k}_2, -\mathbf{k}_{12}), \quad (2.10)
\]

where \( \mathbf{k}_{12} := \mathbf{k}_1 + \mathbf{k}_2 = -\mathbf{k}_3 \) and three bispectrum contributions \( B_{112}, B_{211}, \) and \( B_{121} \) defined in Eqs. (A.62)–(A.64) are expressed in terms of the Fourier kernels \( F(\mathbf{k}_1, \mathbf{k}_2) \) of the individual components in Eq. (2.4). The detailed computation of the individual Fourier kernels is presented in Appendix B, and all the components in Eq. (2.4) being at the source position are categorized as the contributions at the source position in Appendix B.1. Furthermore, if the correlation function \( \xi_{sqz} \) in the squeezed triangle is treated as a two-point correlation function as in Eq. (2.9) and hence just as a function of its separation, its Fourier transformation yields the power spectrum

\[
\Delta P(k) = \int d^3 L e^{-i \mathbf{k} \cdot \mathbf{L}} \xi_{sqz} = \int \frac{d^3 \mathbf{k}_1}{(2\pi)^3} \left[ B_{112} + B_{211} + B_{121} \right](\mathbf{k}_1, \mathbf{k} - \mathbf{k}_1, -\mathbf{k}), \quad (2.11)
\]

where we defined the separation vector

\[
\mathbf{L} := \mathbf{x}_1 - \mathbf{x}_2. \quad (2.12)
\]

Note that such power spectrum obtained in a hypersurface is not a direct observable, in particular, on large scales, and we present further discussion about this issue later.

To compute \( \xi_{sqz} \) in Eq. (2.10), consider two second-order relativistic contributions to the matter density fluctuation in Eq. (2.4). Two Fourier kernels for the relativistic terms are computed in Appendix B.1 as

- \( \frac{3}{2} D_1(z) R^{\alpha} R_{\alpha} : \quad F_R(\mathbf{k}_1, \mathbf{k}_2) = -\frac{3}{2} D_1(z) \mathbf{k}_1 \cdot \mathbf{k}_2, \quad (2.13) \)
- \( 4D_1(z) R \Delta R : \quad F_R(\mathbf{k}_1, \mathbf{k}_2) = -2D_1 \left( k_1^2 + k_2^2 \right), \quad (2.14) \)

and their one-point ensemble averages are

\[
\left\langle \frac{3}{2} D_1 R^{\alpha} R_{\alpha} \right\rangle = \frac{3}{2} D_1 \sigma_2, \quad \left\langle 4D_1 R \Delta R \right\rangle = -4D_1 \sigma_2. \quad (2.15)
\]
where we defined a dimensionful constant
\[
\sigma_n := \int \frac{d^3k}{(2\pi)^3} k^n P_R(k), \quad [\sigma_n] = L^{-n}. \tag{2.16}
\]

Though not diverging in the infrared, the ensemble averages of the relativistic contributions are non-vanishing \((\sigma_2 \neq 0)\), and so is the ensemble average of the matter density fluctuations \(\delta_m\) in Eq. (2.4). We discuss this point in detail in Section 3.3. After subtracting these non-vanishing constants, their contributions to the bispectrum \(B_{112}\) are
\[
B_{112} = -\frac{3}{2} D_1^3 k_1^2 k_2^2 k_3^2 P_R(k_1) P_R(k_2) \left[ 1 + \frac{5}{3} \left( \frac{k_1^2}{k_3^2} + \frac{k_2^2}{k_3^2} \right) \right], \tag{2.17}
\]
where \(k^3 P_R(k) \propto k^{n_s-1}\) is the primordial curvature power spectrum with the spectral index \(n_s \simeq 1\) and we used
\[
k_1 \cdot k_2 = \frac{1}{2} (k_3^2 - k_1^2 - k_2^2). \tag{2.18}
\]
In the squeezed limit, in which the separation \(L = |x_1 - x_2|\) becomes infinite, the exponential factor in Eq. (2.10) imposes
\[
k_l := k_{12} = k_1 + k_2 \to 0, \quad k_s := k_1 \simeq -k_2, \tag{2.19}
\]
and the bispectrum \(B_{112}\) behaves in this limit as
\[
B_{112} = -\frac{3}{2} D_1^3 k_s^2 k_l^2 P_R^2(k_s) \left[ 1 + \frac{10}{3} \frac{k_s^2}{k_l^2} \right] \propto -\frac{3}{2} \frac{1}{k_s^2} P_m^2(k_s) \left( \frac{k_l^2}{k_s^2} + \frac{10}{3} \right), \tag{2.20}
\]
where the matter density power spectrum scales as \(P_m(k) \propto k^4 P_R(k) \propto k^{n_s}\). The integration over the short mode of the bispectrum \(B_{112}\) yields its contribution to \(\epsilon_{\text{sqz}}^{\text{lim}}\) in the squeezed limit, or its contribution to \(P(k)\) in the squeezed limit, both of which vanish as \(k_l \propto 1/L \to 0\). The contribution of the term with constant \(10/3\) is removed from the tadpole contribution discussed in Appendix A.2.

The other two bispectra can be computed in the same way, and they are identical
\[
B_{211} = B_{121} = -\frac{3}{2} D_1^3 k_1^2 k_2^2 k_3^3 P_R(k_2) P_R(k_3) \left[ 1 + \frac{5}{3} \left( \frac{k_2^2}{k_1^2} + \frac{k_3^2}{k_1^2} \right) \right], \tag{2.21}
\]
and in the squeezed limit they become
\[
B_{211} = B_{121} \propto -\frac{3}{2} \frac{1}{k_l^2} P_m(k_s) P_m(k_l) \left[ \left( 1 + \frac{5}{3} \right) \frac{k_l^2}{k_s^2} + \frac{5}{3} \right]. \tag{2.22}
\]
The long-mode contribution in the round bracket diverges in the limit \(k_l \to 0\), and this contribution is known as the relativistic correction to the primordial non-Gaussianity signal. With the volume factor \(d^3k_l\) in Fourier space, this contribution to \(\epsilon_{\text{sqz}}^{\text{lim}}\) vanishes in fact in the squeezed limit. However, the power spectrum in the hypersurface scales as
\[
\Delta P(k_l) \propto \frac{1}{k_l^2} P_m(k_l), \tag{2.23}
\]
and hence the correction to the galaxy power spectrum would scale in the same way, according to Eq. (2.9):
\[
\Delta P_g(k_l) \propto \frac{1}{k_l^2} P_m(k_l), \tag{2.24}
\]
which should give rise to a prominent feature on large scales, similar to that by the presence of $f_{\text{NL}}$ (see, e.g., [9, 17, 18]).

In essence, the same computation has been performed [60] in the presence of the local-type primordial non-Gaussianity in Eq. (2.7), but without the second-order relativistic contributions. The presence of $f_{\text{NL}} R^2$ term in the initial condition would yield

$$- D_1 \Delta R(x) = - D_1 \left[ \Delta R_g + \frac{6}{5} f_{\text{NL}} \left( R_g^{\alpha \beta} R_{g,\alpha}^{\beta} + R_g \Delta R_g \right) \right], \quad (2.25)$$

whose second-order Fourier kernels are similar to those for the relativistic contributions as

$$\frac{6}{5} f_{\text{NL}} D_1 \left( k_1 \cdot k_2 \right), \quad \frac{3}{5} f_{\text{NL}} D_1 \left( k_1^2 + k_2^2 \right), \quad (2.26)$$

and their contribution to the bispectrum is

$$B_{112} = \frac{6}{5} f_{\text{NL}} D_1 \left[ k_1^2 k_2^2 P_R(k_1) P_R(k_2) \right]. \quad (2.27)$$

In the squeezed limit, the bispectrum $B_{112}$ vanishes, but the other two bispectra scale with $k_l$ as

$$B_{211} = B_{121} \propto \frac{3}{2} \frac{P_m(k_s)}{k^4} P_m(k_l) \left[ - \frac{4}{5} f_{\text{NL}} \frac{k_s^2}{k_l^2} \right]. \quad (2.28)$$

In comparison to Eq. (2.22), one reaches the conclusion [11, 17–19, 21] that the relativistic contributions in the matter density fluctuation (or nonlinearity in general relativity) generate the effective non-Gaussianity

$$\Delta f_{\text{NL}} = - \frac{10}{3}, \quad (2.29)$$

or $\Delta F_{\text{NL}} = -5/3$ for different notation convention in Eq. (3.7). A few remarks are in order, regarding the extra terms in Eq. (2.25) with $f_{\text{NL}}$, compared to the relativistic corrections in Eq. (2.4). With the same coefficients with $f_{\text{NL}}$ for both terms, their one-point ensemble averages cancel each other, as can be inferred in Eq. (2.15). The peculiar scale-dependence $P_m(k_l)/k_l^4$ arises solely from the contribution $\mathcal{R} \Delta \mathcal{R}$, which can originate from the presence of the primordial non-Gaussianity or the Hamiltonian constraint equation in general relativity. While $f_{\text{NL}}$ can be zero in the initial condition, the Hamiltonian constraint is satisfied all the time (hence the constraint). The effective non-Gaussianity can therefore be read off from the coefficient of $\mathcal{R} \Delta \mathcal{R}$ without computing the three-point correlation function or the bispectrum, as shown in Eq. (3.10).

We continue with two Newtonian contributions in Eq. (2.4). Two Fourier kernels of the Newtonian contributions are computed in Eqs. (B.3) and (B.7): 

$$\mathcal{F}_s(k_1, k_2) = \frac{5}{14} D_A(z) \left( k_1 + k_2 \right) \cdot \left( k_1^2 k_2 + k_2^2 k_1 \right), \quad (2.30)$$

$$\mathcal{F}_s(k_1, k_2) = \frac{1}{7} D_B(z) \left( k_1 + k_2 \right)^2 k_1 \cdot k_2. \quad (2.31)$$

In the Einstein-de Sitter Universe, where two growth factors become equivalent, i.e., $D_A = D_B = D_1^2$, two Fourier kernels add up to be the SPT kernel derived in Eq. (B.12):

$$\mathcal{F}_2(k_1, k_2) = D_1^2 \left[ \frac{5}{7} + \frac{1}{2} \frac{k_1 \cdot k_2}{k_1 k_2} \left( \frac{k_1}{k_2} + \frac{k_2}{k_1} \right) + \frac{2}{7} \left( \frac{k_1 \cdot k_2}{k_1 k_2} \right)^2 \right]. \quad (2.32)$$
we present several issues associated with the computation in this section, some of

Naturally, the ensemble averages of two Newtonian contributions vanish. Their contributions to the bispectrum are then computed as

$$B_{112} = D_1^2 k_1^2 k_2^2 P_R(k_1) P_R(k_2) \left\{ \frac{5}{14} D_A \left[ k_2^2 (k_1^2 + k_2^2) - (k_1^2 - k_2^2)^2 \right] + \frac{1}{7} D_B \left[ k_3^2 - k_2^2(k_1^2 + k_2^2) \right] \right\},$$

(2.33)

and two other bispectra can be readily obtained by permutation of its arguments. With $k_3 \to 0$ and $k_1^2 = k_2^2$, all three bispectra naturally vanish in the squeezed limit. It is well established [44] that the nonlinearity in the Newtonian perturbation theory vanishes in the infrared.

2.3 Setting the stage: Relativistic corrections to the primordial non-Gaussian signal?

Up to this point, we have performed a straightforward computation of the three-point correlation function or the bispectrum, given the expression of the matter density fluctuation in Eq. (2.4). It appears that the relativistic contributions inherent in general relativity give rise to correction terms to the standard second-order expression for the matter density fluctuation and the particular term $R \Delta R$ yields unique behavior on large scales in the power spectrum, which is then also related to the galaxy power spectrum. While there is no doubt in the sanity for the matter density fluctuation in Eq. (2.4), it has been intensively debated [17, 18, 21–24] whether the relativistic contributions give rise to a correction in the galaxy power spectrum on large scales in the same way the presence of the primordial non-Gaussianity affects the galaxy power spectrum. For instance, it has been argued [22–24, 35, 37, 64] that an extra coordinate transformation like the dilatation or the conformal Fermi coordinate can remove such contribution, at least, for the single-field inflationary model, and hence no relativistic correction to the primordial non-Gaussian signal on large scales in the galaxy power spectrum.

In Section 3 we present several issues associated with the computation in this section, some of which are largely unknown in literature and some of which are debated in the community. We present solutions to some of the issues, but not all the issues are resolved. In short, we show that the relativistic contributions in the matter density fluctuation cannot be removed by any coordinate transformation. However, there exist gauge ambiguities in relating the matter density fluctuation to the galaxy number density fluctuation. The theoretical description of the observed galaxy number density should be gauge-invariant, and it is proved at the linear order in perturbations [25–29]. However, it becomes tricky [35, 37, 39, 46] in general relativity with many subtle and unresolved issues beyond the linear order in perturbations to relate the matter density fluctuation to the (intrinsic) galaxy number density fluctuation (or galaxy bias), which is not yet an observable.

Furthermore, it is important that in the end we need to provide theoretical predictions for observable quantities such as the observed galaxy correlation function. While the power spectrum or the bispectrum is a useful statistic, these statistics are defined in a hypersurface outside the observed light-cone volume, particularly when our primary interest lies in their signals on large scales. While the (theory) power spectrum in a hypersurface is related to the observed power spectrum [65], a simple way of measuring the power spectrum in observations yields signals very different from the (theory) power spectrum in a hypersurface on large scales due to the wide angle effect, the time-evolution along the line-of-sight direction, the curvature of the sky, and so on (see, e.g., [66–70] for detailed discussion of those complications). For example, note that the correlation function $\xi_{\text{sep}}(x_1, x_2)$ in Eq. (2.10) is not just a function of its separation $L = x_1 - x_2$; it depends on two observed angles $\hat{\mathbf{n}}_1$ and $\hat{\mathbf{n}}_2$ for two positions $x_1$ and $x_2$. Without proper consideration of those issues, a simple computation of the (theory) power spectrum in a hypersurface would lead to conclusions that are highly biased when compared to observations, in particular for the primordial non-Gaussian signature. On the other hand, the correlation function involves no such complication, and we compute the observed matter density correlation function in Section 4.
3 Theoretical issues in the previous calculations

3.1 Convention for the primordial non-Gaussianity: Gaussian vs non-Gaussian?

We first clarify different notation conventions in literature, regarding the primordial non-Gaussianity. While they are just a matter of notational preference, we show that the separation into Gaussian and non-Gaussian fields becomes ambiguous beyond the linear order in perturbations, where things are generally non-Gaussian. Given the general metric representation in Eq. (A.1), the initial condition $R(x)$ is set in terms of the comoving-gauge curvature perturbation $\varphi_v$ in Eq. (A.9):

$$R(x) := \varphi_v(x, t_i),$$

(3.1)

where $t_i$ represents some early time for the initial conditions. The comoving-gauge curvature perturbation $\varphi_v$ is conserved in time outside the horizon to all orders in perturbations, and it is conserved on all scales at the linear order in a universe with pressureless medium. The other popular choice to set up the initial condition is to adopt a different representation of the spatial metric:

$$ds^2 = -a^2 (1 + 2\alpha)\,d\eta^2 - 2a^2\beta_{\alpha \beta}dx^\alpha dx^\beta + a^2 e^{2\zeta}\delta_{\alpha \beta}dx^\alpha dx^\beta,$$

(3.2)

where a rectangular coordinate $\delta_{\alpha \beta}$ is chosen in the three metric and the spatial C-gauge ($\gamma \equiv 0$) in Eq. (A.3) is adopted. By expanding the exponential factor, the relation between two notation conventions is:

$$R = \zeta + \zeta^2 + \frac{2}{3}\zeta^3 + \frac{1}{3}\zeta^4 + \cdots.$$  

(3.3)

Assuming that the same comoving gauge condition in Eq. (A.4) is adopted, both notation conventions yields the same quantity at the linear order:

$$R^{(1)} = \zeta^{(1)},$$

(3.4)

but there exist obvious differences beyond the linear order in perturbations

$$R^{(2)} = \zeta^{(2)} + [\zeta^{(1)}]^2.$$  

(3.5)

Furthermore, while the spatial gauge condition is not relevant at the linear order, it becomes so beyond the linear order, where scalar, vector, and tensor components start to mix with each other. In particular, the presence of traceless transverse component $\gamma_{\alpha \beta}$ (gravitational waves) in the exponential representation

$$a^2 \exp [2(\zeta \delta_{\alpha \beta} + \gamma_{\alpha \beta})] dx^\alpha dx^\beta = a^2 \left[(1 + 2\zeta + 2\zeta^2 + \cdots)\delta_{\alpha \beta} + 2(\gamma_{\alpha \beta} + \gamma_{\alpha \gamma}\gamma_{\gamma \beta} + \cdots)\right] dx^\alpha dx^\beta,$$

(3.6)

generates a non-vanishing off-diagonal scalar component $\gamma$ beyond the linear order (for instance, from $\gamma_{\alpha \gamma}\gamma_{\gamma \beta}$), changing the spatial gauge condition adopted in the exponential metric representation, i.e., it is no longer equivalent to the spatial C-gauge. Therefore, care must be taken in interpreting the calculations in two different notation conventions, in particular for the initial conditions beyond the linear order in perturbations.

Given our convention for the initial conditions in Eq. (2.7), the other convention for the initial conditions in literature is

$$\zeta(x) := \zeta_g(x) + \frac{3}{5} F_{\text{NL}} \zeta_g^2(x),$$

(3.7)

which implies

$$R^{(1)} = \zeta^{(1)} = R_g = \zeta_g,$$

$$R^{(2)} = \frac{3}{5} f_{\text{NL}} R_g^2 = \frac{3}{5} F_{\text{NL}} \zeta_g^2 + \zeta_g^2,$$

(3.8)
and
\[ f_{NL} = F_{NL} + \frac{5}{3}. \]  
\[ (3.9) \]

It is evident that a vanishing non-Gaussianity in one convention \((f_{NL} = 0\) or \(F_{NL} = 0\)) means a non-vanishing non-Gaussianity in the other convention. Note that the parametrization of \(f_{NL}\) or \(F_{NL}\) is a matter of notational preference or representation. This shows that it makes less sense to distinguish Gaussian and non-Gaussian fields beyond the linear order in perturbations, as they are all generically non-Gaussian. Explicitly, the matter density fluctuation in Eq. \((2.4)\) in two different conventions is

\[
\delta_m(x^\mu) = D_1 \left[ -\Delta R_g + \frac{6}{5} \left( \frac{5}{4} - f_{NL} \right) R_{g,\alpha}^\alpha R_{g,\alpha} + \frac{6}{5} \left( \frac{10}{3} - f_{NL} \right) R_g \Delta R_g \right] + \delta^{(2)}_{\text{Newt}}. 
\]

\[
= D_1 \left[ -\Delta \zeta_g - \frac{6}{5} \left( \frac{5}{12} + F_{nl} \right) \zeta_g^\alpha \zeta_{g,\alpha} + \frac{6}{5} \left( \frac{5}{3} - F_{nl} \right) \zeta_g \Delta \zeta_g \right] + \delta^{(2)}_{\text{Newt}}, \tag{3.10}
\]

where we defined the second-order Newtonian contribution to the matter density fluctuation

\[
\delta^{(2)}_{\text{Newt}} := \frac{5}{7} D_A \nabla_\alpha \left( R_{g,\alpha}^\alpha \Delta R_g \right) + \frac{1}{7} D_B \Delta \left( R_{g,\alpha}^\alpha R_{g,\alpha} \right). \tag{3.11}
\]

Equation \((3.10)\) is often expressed in terms of a Newtonian potential \(\phi_g = (3/5)R_g\) in the initial condition as

\[
\delta_m(x^\mu) = \frac{5D_1}{3} \left[ -\Delta \phi_g + 2 \left( \frac{5}{4} - f_{NL} \right) \phi_g^\alpha \phi_{g,\alpha} + 2 \left( \frac{10}{3} - f_{NL} \right) \phi_g \Delta \phi_g \right] + \delta^{(2)}_{\text{Newt}}.
\]

\[
= \frac{5D_1}{3} \left[ -\Delta \phi_g - 2 \left( \frac{5}{12} + F_{nl} \right) \phi_g^\alpha \phi_{g,\alpha} + 2 \left( \frac{5}{3} - F_{nl} \right) \phi_g \Delta \phi_g \right] + \delta^{(2)}_{\text{Newt}}. \tag{3.12}
\]

The matter density fluctuation is generically non-Gaussian beyond the linear order, even in the limit \(t \to 0\). This does not imply that any of the non-Gaussian parametrization is useless. Given a theory for the initial conditions, accurate predictions for \(f_{NL}\) or \(F_{NL}\) can be made, and these numbers can be observationally tested.

3.2 Spatial gauge transformation: pure gauge mode or not?

Diffeomorphism is a symmetry of general relativity, and this symmetry in cosmology is often expressed as gauge freedom in a general coordinate transformation:

\[
\tilde{x}^\mu = x^\mu + \xi^\mu, \tag{3.13}
\]

i.e., the same physical point is described by two different coordinate values in two different coordinate systems. A gauge choice then amounts to completely fixing four degrees of freedom in Eq. \((3.13)\), which are temporal and spatial \([45]\). A temporal gauge choice or time-slicing implies a choice of three-dimensional hypersurface of simultaneity in such a coordinate system, and various temporal gauge choices are discussed in literature (see, e.g., \([71]\)), in conjunction with their physical meaning. However, a spatial gauge choice has received little attention in literature. The main reason is due to the symmetry in a homogeneous and isotropic universe, for which the background quantities are just a function of time only. Since any linear-order perturbation \(\delta T\) of a tensorial quantity \(T\) gauge transforms in terms of Lie derivative \(L\) as

\[
\delta_\xi \delta T = -L_\xi \delta T + \mathcal{O}(2), \tag{3.14}
\]

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the symmetry in the background universe renders the spatial gauge transformation irrelevant at the linear order for scalar fluctuations \( \text{not for general tensors} \). However, beyond the linear order, perturbations gauge transform as
\[
\delta \zeta \delta T = - \mathcal{L}_\zeta \bar{T} + \frac{1}{2} \mathcal{L}_\zeta \mathcal{L}_\zeta \bar{T} - \mathcal{L}_\zeta \delta T + \mathcal{O}(3) ,
\]
and a spatial gauge choice plays a role in determining the perturbation variables uniquely \([72, 73]\), even for the scalar fluctuations. Note that we have used the exponential parametrization of a general coordinate transformation
\[
\tilde{x}^\mu = e^{\zeta^\nu \partial_\nu} x^\mu = x^\mu + \frac{1}{2} \zeta^\mu,\nu \zeta_\nu + \mathcal{O}(3) ,
\]
where two parametrizations are related as
\[
\xi^\mu = \zeta^\mu + \frac{1}{2} \zeta^\mu,\nu \zeta_\nu + \mathcal{O}(3) .
\]

As a simple illustration, consider the matter density fluctuation \( \delta m \) in a given hypersurface, which is fixed by a temporal gauge choice \( (\xi^\eta = 0) \). The matter density fluctuation is uniquely fixed at the linear order, but ambiguities appear beyond the linear order in perturbations as
\[
\delta m(x^\mu) = \tilde{\delta} m(\tilde{x}^\mu) = \tilde{\delta} m(x^\mu) + \xi^\alpha \partial_\alpha \tilde{\delta} m \bigg|_{x^\mu} ,
\]
where the background matter density \( \bar{\rho}_m(t) \) is identical in the same hypersurface. In other words, though the matter density fluctuation at a given physical point is invariant, its functional form in a given hypersurface changes, according to a spatial gauge choice. It was shown \([46]\) that two different spatial gauge choices in a proper-time hypersurface result in different second-order matter density fluctuations. Consequently, the three-point correlation function, whose leading order term depends on the second-order expression, is affected by a spatial gauge choice:
\[
\langle \delta(x_1) \delta(x_2) \delta(x_3) \rangle = \langle \delta(x_1) \delta(x_2) \delta(x_3) \rangle - \left[ \langle \xi^\alpha(x_1) \partial_\alpha \delta(x_1) \delta(x_2) \delta(x_3) \rangle + \text{cycl} \right] + \mathcal{O}(5) .
\]

Unless the gauge choice is fully specified, the three-point correlation function remains ambiguous and gauge-dependent. This result is generic and applicable to any three-point correlation function, e.g., the consistency relation for a single-field inflation in the squeezed limit. In \([58]\), the full three-point correlation function in a single-field inflationary model was computed, and the dependence of a spatial gauge choice was investigated.

We close this section by concluding that spatial gauge choices are \textit{not} pure gauge modes. In Section 4.1 we show that the arbitrariness of a spatial gauge choice is lifted in describing the observable quantities.

### 3.3 Ensemble average: coordinate-dependent?

The ensemble average \( \langle O \rangle \) of a field \( O(x) \) is widely used in cosmology, but often in a way its exact definition or meaning is left ambiguous or implicit. Such ambiguity can then cause inconsistencies in perturbation theory, in particular, beyond the linear order. A field \( O(x) \) at a given spacetime position \( x^\mu \), such as the matter density fluctuation \( \rho_m(x^\mu) \), can be split into the background \( \bar{O}(t) \) and the perturbation \( \delta O \) around it as
\[
O(x^\mu) = \bar{O}(t) \left[ 1 + \delta O(x^\mu) \right] ,
\]
where
and the dimensionless fluctuation $\delta \mathcal{O}$ is often assumed to be Gaussian distributed with zero mean at the linear order. In fact, the correct statement is that at each wave number $k$, the linear-order fluctuation $\delta \mathcal{O}(k; t)$ in Fourier space defined in a given hypersurface set by $t$ is independent with different Fourier modes and Gaussian distributed with its variance specified by the power spectrum $P_{\delta \mathcal{O}}(k; t)$. This applies to any fields $\mathcal{O}$ at the linear order in perturbation theory, because they are all linearly related. To be more specific, consider a Gaussian probability functional $\mathcal{P}[\delta \mathcal{O}]$,

$$
\langle \delta \mathcal{O}_{k; t}^{(1)} \rangle := \int D\delta \mathcal{O} \mathcal{P}[\delta \mathcal{O}] \delta \mathcal{O}_{k; t}^{(1)} = 0 ,
$$

where we defined the ensemble average, the integral is over all values of $\delta \mathcal{O}$, and the isotropy of the power spectrum over the wave vector is assumed, though not needed. Hence, the ensemble average of $\mathcal{O}(x)$ at the linear order yields

$$
\langle \mathcal{O}(x^\mu) \rangle = \bar{\mathcal{O}}(t) ,
$$

where the average over PDF and the Fourier transformation commute. The ensemble average is over multiple realizations of the universe, in which different values of $\delta \mathcal{O}(k; t)$ are given for a fixed $k$ in a $t$-hypersurface (or the same spacetime position). In other words, it is a local process, but with homogeneity it is identical everywhere.

So far, we summarized explicitly the standard procedure for treating the random fluctuations and the ensemble average. It is also common in the standard procedure to assume the Ergodicity of the system — Once the random fluctuations are averaged over a sufficiently large volume, the resulting average is ought to be equivalent to the ensemble average over multiple realizations of the universe:

$$
\langle \mathcal{O}(x; t) \rangle \equiv \lim_{V \to \infty} \frac{1}{V} \int_V d^3x' \mathcal{O}(x'; t) ,
$$

where we made it clear that the $t$-hypersurface is fixed in the local ensemble average or the volume average over the $t$-hypersurface. It is obvious from the definitions that the ensemble average depends on a hypersurface set by a time-coordinate $t$, which changes depending on our choice of gauge conditions. This point is often left implicit or less emphasized in literature. At the linear order in perturbations, however, this gauge dependence matters only for the power spectrum, not for the mean, as the gauge transformation $\mathcal{G}(x)$ at the linear order is also a Gaussian random variable with zero mean, for instance,

$$
\tilde{\delta \mathcal{O}}^{(1)}_x = \delta \mathcal{O}^{(1)}_x + \mathcal{G}^{(1)}_x ,
\langle \delta \mathcal{O}^{(1)}_{k; t} \rangle = 0 ,
\langle \tilde{\delta \mathcal{O}}^{(1)}_{k; t} \rangle = 0 ,
$$

where $P_{\delta \mathcal{O}}(k; t) = P_{\delta \mathcal{O}}(k; t) + P_{\mathcal{G}}(k; t) + 2P_{\delta \mathcal{O}, \mathcal{G}}(k; t) \neq P_{\delta \mathcal{O}}(k; t) ,

Beyond the linear order in perturbations, however, things are not as simple as in the linear-order calculations, and one consequence is that the ensemble average of $\mathcal{O}(x)$ beyond the linear order quite often yields

$$
\langle \mathcal{O}(x^\mu) \rangle \neq \bar{\mathcal{O}}(t) .
$$

Before we proceed, we stress that there exist coordinate systems, in which the relation $\langle \mathcal{O} \rangle = \bar{\mathcal{O}}(t)$ holds beyond the linear order, but there exist a lot more coordinate systems, in which the relation
does not hold beyond the linear order. This is a natural consequence, since the ensemble average in LHS depends on a coordinate system, while the background value \( \bar{O}(t) \) in RHS is just one number at a fixed value of \( t \). For instance, the matter density fluctuation at the second order in perturbations is given in Eq. (2.1), where the first term is the linear-order contribution and the remaining two terms are the Newtonian second-order contributions. The Newtonian second-order contributions are often expressed in terms of their Fourier kernel \( F_2(k_1, k_2) \) in Eq. (2.32). It is well-known that this second-order Newtonian matter density fluctuation vanishes, when the ensemble average is taken. However, there exist extra relativistic contributions at the second order, or the quadratic terms in proportion to \( D_1(t) \) in Eq. (2.4), where their ensemble averages are non-vanishing as computed in Eqs. (B.14) and (B.18). Furthermore, the spatial gauge-transformation changes the second-order Newtonian contributions in Eq. (2.4), according to Eq. (3.18). One of the popular choices is the synchronous gauge at the second order, in which the hypersurface is identical to one for Eq. (2.4), but the spatial gauge is different. Consequently, the second-order Newtonian terms are different from the standard contributions with \( F_2(k_1, k_2) \), while the second-order relativistic contributions are identical. It was shown \([46]\) that the ensemble average of the second-order Newtonian contributions in the synchronous gauge is also non-vanishing.

So, in summary, we have demonstrated that the ensemble average depends on a choice of hypersurface set by time coordinate \( t \) and it also depends on a choice of spatial gridding, i.e., it depends on a choice of full gauge condition. The time-slicing matters already at the linear order, as borne out in Eq. (3.25), and the spatial-gridding becomes relevant beyond the linear order, as shown in Section 3.2. Furthermore, the background solution \( \bar{O}(t) \) in Eq. (3.26), for example the background matter density \( \bar{\rho}_m(t) \), is obtained by solving the Einstein equation under the assumption that the solution is just a function of time due to homogeneity and isotropy, which involves no average of any sort. Hence the equality between \( \langle O \rangle \) and \( \bar{O}(t) \) is not expected to be valid in general. This issue has been extensively discussed in literature (see, e.g., \([74–79]\)) under the name of back-reaction, in which it was shown that the background solutions obtained by ignoring any spatial derivatives in the governing equation are different from those obtained by averaging over a spatial volume in a given hypersurface due to the non-commutativity of two different procedures.

We conclude this subsection with a brief remark about the “average in observation.” It is quite common that the observers measure average quantities of, for example, the luminosity distances, the galaxy number density, the cosmic microwave background temperature, and so on. These observational averages are certainly independent of any coordinate choice we assume to describe observations. Furthermore, these quantities are in fact the average over the observed angle \( \hat{n} \) in the sky at a fixed (observed) hypersurface. For instance, the average of the galaxy number density or the luminosity distance can be obtained by averaging those quantities over the sky at a fixed observed redshift \( z \). The average CMB temperature is also obtained by averaging the observed CMB temperature \( \bar{T}(\hat{n}) \) over the sky at the Earth. These observational averages are naturally different from the ensemble average or the background quantity \( \bar{O}(t) \) in Eq. (3.26). While the observer coordinate is fixed up to a trivial rotation and is independent of coordinate systems in our theoretical calculations, the ensemble average involves a full Euclidean average over the hypersurface. Since the observers can only perform angle average over the sky at one position (or at the Earth), the lack of translation in the observer position results in the cosmic variance or the discrepancy between the observational average and the ensemble average over the observer hypersurface \([83]\), i.e., the ensemble average is not a direct observable.
In the last decade two new types of transformations appeared in the field which do not satisfy the aforementioned condition. The first set of transformations includes the dilatations [84] and special conformal transformations (including tensor analogues) [85, 86] which lie at the heart of the consistency relations of inflationary correlation functions [53, 54, 85, 87, 88]. These transformations arise as a global residual freedom of a fully gauge-fixed metric and do not reduce to the identity at spatial infinity, thus corresponding to so-called “large gauge transformations.” The other type includes the transformations that relate a typical coordinate system of cosmological perturbation theory (e.g., conformal Newtonian gauge) to the conformal Fermi coordinates (CFC) associated with some geodesic world-line [22, 89, 90]. The CFC construction essentially corresponds to a deformed exponential map associated with a particular tetrad along the world-line and is therefore uniquely determined in some surrounding finite world-tube. However, since it is built order by order in a spatial expansion, its asymptotic behavior is obscure, so the corresponding transformation might or might not be a large gauge transformation and it is usually not even defined at infinity. Moreover, since one always stops at finite order, in practice its implementation takes the form of a large gauge transformation.

The utility of these extra types of transformations in cosmology lies in the fact that they affect the first two terms in a spatial expansion of the metric fluctuations and are therefore always used to simplify the metric representation within a finite patch of space. This is in contrast to the standard practice of gauge transformations, in which the scalar-vector-tensor description are globally defined in real space, privileging in particular no point or direction therein. Given these aspects and especially the fact that large gauge transformations arise on top of the standard gauge transformations, there seems to be some confusion in the literature about how the former affect cosmological observables. Indeed, while it is obvious that any physical quantity must be invariant under standard gauge transformations, the use of large gauge transformations is sometimes worded as “going to the physical frame of the observer”, i.e. the coordinates that an observer at $x^i = 0$ would use in practice to map their local patch. Attributing a physical meaning to the transformation might then seem to distinguish the available frames into “right” and “wrong” ones, which leads to the question: what happens to a cosmological observable if one picks a “wrong” frame?

Here we wish to stress that the cosmological observables and their theoretical descriptions are invariant under all of these transformations, simply because they are all just different types of coordinate transformations (diffeomorphism). Indeed, observables can be defined in a completely coordinate-independent way and at the fully non-linear level, i.e. without any reference to the background solution (see, e.g., [91]), so they do not see the difference between standard and large gauge transformations. The fact that some coordinate system can be given a physical interpretation does not mean that one has to commit to that system to obtain the correct answer to a physical question — the essence of general covariance. The content of this subsection is discussed in more detail and depth in a dedicated paper [92], which addresses in particular some more subtle aspects. There we also show
that the relevant property of CFC in cosmology, i.e. that the metric satisfies \( g_{\mu\nu} = a^2 \left( \eta_{\mu\nu} + \mathcal{O}(x^2) \right) \), can be achieved with a standard gauge transformation, which therefore avoids the ambiguity at spatial infinity.

### 3.5 Proper-time hypersurface: unique choice for galaxy bias?

The observed galaxy clustering is described by two physically distinct effects: the volume effect and the source effect [25, 93]. The volume effect can be uniquely determined by solving the geodesic equation for the mismatch between the observed and the physical source positions and volumes occupied by the source galaxies. The source effect deals with the intrinsic properties of the source galaxy sample and its mismatch, compared to the observed properties of the source galaxy sample. More importantly, the most dominant contribution to the source effect and also to galaxy clustering overall is the matter density fluctuation that drives the fluctuation in the observed galaxy number density.

The relation between the galaxy and the matter density distributions, known as galaxy bias [47], is an area of intense research in literature. In the simplest form discussed in Section 2.2, galaxies form in an over-dense region with \( \delta_m \geq \delta_c \), and this simple model yields that the galaxy number density fluctuation \( \delta_g \) is linearly proportional to the matter density fluctuation, i.e.,

\[
n_g(x) = \bar{n}_g(t)[1 + \delta_g(x)] , \quad \delta_g(x) = b \delta_m(x) , \tag{3.27}
\]

or the linear bias relation [47, 48, 63], where the bias factor \( b \) is a constant. Beyond the linear order, the bias relation can be further extended to incorporate the higher-order perturbation contributions by introducing nonlinear bias factors such \( b_n \) with \( n \geq 2 \) [50, 94–96], the tidal gravitational bias factor \( b_t \) [97], the relative velocity bias \( b_r \) between the baryon and the matter distributions [98–101], or effective field descriptions [102–108] (see, e.g., [51] for recent review of galaxy bias on large scales). However, put in the context of general relativity, this linear bias relation is ambiguous and ill-defined. Under a change of coordinates in Eq. (3.13), the galaxy number density fluctuation in Eq. (3.27) transforms at the linear order as

\[
\tilde{\delta}_g(x^\mu) = \delta_g(x^\mu) - \frac{\bar{n}'_g}{\bar{n}_g} T , \tag{3.28}
\]

and for a matter distribution we recover the gauge-transformation relation for the matter density fluctuation

\[
\tilde{\delta}_m(x^\mu) = \delta_m(x^\mu) + 3H T , \quad \bar{\rho}_m \propto a^{-3} . \tag{3.29}
\]

To maintain both the linear bias relation to the matter distribution and the gauge-transformation relation, a strict condition for the galaxy number density is imposed:

\[
b = -\frac{1}{3H} \frac{d \ln \bar{n}_g}{d\eta} = \frac{d \ln \bar{n}_g}{d \ln \bar{\rho}_m} , \tag{3.30}
\]

such that the galaxy bias factor has to be related to the number density evolution. Again, for a matter distribution we recover the consistency relation \( b = 1 \), but for a general galaxy distribution in observations we already know that this relation is not valid (see, e.g., [109, 110]).

Several attempts have been made in literature to generalize the bias relation in general relativity. Noting that the bias relation should reduce to the Newtonian description on small scales, the matter density fluctuation \( \delta_v \) in the comoving-synchronous gauge was advocated in [28] for the linear bias relation:

\[
\delta_g(x^\mu) = b \delta_v(x^\mu) , \tag{3.31}
\]
where the gauge-invariant matter fluctuation $\delta_v$ in both the synchronous gauge and the comoving gauge is equivalent to the Newtonian matter density fluctuation at the linear order. In this bias model, a gauge choice is made by hand, but with the Newtonian correspondence. It was argued [29] that galaxies can only measure their own local time, such that the bias model should be in a hypersurface of a constant-age:

$$n_g(x^\mu) = \bar{n}_g(t_p)(1 + b \delta_v) ,$$

(3.32)

in support of the bias model in [28], where $t_p$ represents the proper-time coordinate (or constant-age) in the rest frame of the source galaxies. Since the proper time of local galaxies is not observable, there exists an extra contribution, when the proper time is expressed in terms of the observed redshift. Hence the most general linear-order expression for galaxy bias in general relativity is then [52]

$$n_g(x^\mu) = \bar{n}_g(z)[1 + b \delta_v - e \delta z_v] ,$$

(3.33)

where two bias parameters are

$$b := \frac{d \ln \bar{n}_g}{d \ln \bar{\rho}_m} \bigg|_{t_p} , \quad e := \frac{d \ln \bar{n}_g}{d \ln (1 + z)} ,$$

(3.34)

and the observed redshift $z$ is related to the redshift $z_p$ at the proper time $t_p$ as $1+z = (1+z_p)(1+\delta z_v)$ at the linear order with $\delta z_v$ in the comoving gauge.

Beyond the linear order in perturbations, the galaxy bias relation poses more challenges in general relativity. With the arguments for a proper-time hypersurface, the galaxy number density can be generically written [111] as

$$n_g(x^\mu) = \bar{n}_g(t_p)\left[1 + \delta_{g\text{int}}(x^\mu)\right] ,$$

(3.35)

where $\delta_{g\text{int}}$ represents the intrinsic (nonlinear) galaxy fluctuation in a proper-time hypersurface. It was argued [111] that the intrinsic fluctuation should vanish upon average over the proper-time hypersurface:

$$\bar{n}_g(t_p) = \langle n_g \rangle_{t_p} , \quad \langle \delta_{g\text{int}} \rangle = 0 .$$

(3.36)

It was shown [46] that both the synchronous gauge in Eqs. (A.6) and (A.7) (denoted as gauge-II)

$$\alpha \equiv 0 , \quad v \equiv 0 , \quad \beta \equiv 0 ,$$

(3.37)

and the comoving gauge in Eqs. (A.4) and (A.3) (denoted as gauge-I)

$$v \equiv 0 , \quad \gamma \equiv 0 ,$$

(3.38)

describe the same proper-time hypersurface, but the matter density fluctuations in two gauges are different, due to the difference in the spatial gauge condition. The difference arises only at the second order. Since only the matter density fluctuation in gauge-I satisfies the condition in Eq. (3.36) and the one in gauge-II has non-vanishing one-point average, the gauge-I was favored for galaxy bias in [46]. However, the matter density fluctuation in gauge-I in fact has non-vanishing one-point average shown in Eq. (2.15), due to the intrinsic relativistic effects in Eq. (2.4), which was missing in [46]. Beyond the linear order in literature, many different attempts have been made. With the focus on the second-order volume effect, a simple model $b = 1$ was assumed in [112]. The matter density fluctuation in the synchronous gauge was chosen [113, 114], while gauge-I was favored in [33, 34]. In both cases, no intrinsic relativistic effects were considered.
In studying the relativistic contributions to the primordial non-Gaussianity, a simple Newtonian biasing $b_1$ and $b_2$ was used \cite{35} in terms of the matter density fluctuation in gauge-II, but without the intrinsic relativistic contribution. Identifying the synchronous gauge (gauge-II) as the Lagrangian frame, the Lagrangian description of galaxy bias was used \cite{37, 39}:

$$1 + \delta _g^{\text{int}} = (1 + \delta _g^L)(1 + \delta _m), \quad \delta _g^L := b_1^L \delta _1^{(1)} + \frac{1}{2} b_2^L [\delta _1^{(1)}]^2 + \cdots , \quad (3.39)$$

where the bias factors $b_i^L$ are defined in Lagrangian space, $\delta _1^{(1)}$ is the linear-order matter density in Lagrangian space, and $\delta _m$ is the nonlinear matter density. Note that the biasing prescription is based on the linear-order matter density fluctuation. The intrinsic relativistic effects like $\mathcal{R}\Delta \mathcal{R}$ arises in this model from the volume fluctuation $(1 + \delta _m)$, while no such terms are multiplied by the bias factors up to the second order.

In summary, when the galaxy bias model is considered, the proper-time hypersurface appears to be the right choice for time slicing, as the proper time is the only clock available in the local galaxy and matter distribution and it provides the right Newtonian correspondence at the linear order. However, beyond the linear order, the spatial gauge choice matters and there is no physical argument to prefer one spatial gauge over the others. For example, while two different gauge choices (gauge-I and-II) can describe the same proper-time hypersurface, the matter density fluctuations in each choice are different.\textsuperscript{3} Only the matter density fluctuation in Eq. (2.4) in gauge-I has the correct correspondence to the Newtonian second-order contributions, though it has extra second-order relativistic contributions.

Therefore, the proper-time hypersurface with spatial C-gauge appears as the best choice for describing galaxy bias, but the choice of spatial gauge remains to be explained in a successful galaxy bias model in general relativity. Keep in mind that the spatial gauge ambiguities are removed, when we compute the observable quantities in terms of the observed redshift and angle, i.e., any choice of spatial gauge would yield the same answer to the observers. However, the relation between the galaxy and the matter distributions (or galaxy bias) should be independent of whether any observers exist, i.e., we need a certain choice of gauge condition (temporal and spatial) with physical explanations. If we take this choice and apply the bias relation developed in \cite{60}, we arrive at the conclusion that the intrinsic nonlinear relativistic effects in general relativity generate corrections to the primordial non-Gaussianity $\Delta f_{NL} = -10/3$ (or $\Delta F_{\text{nl}} = -5/3$). Furthermore, our calculations in Section 4 show that extra relativistic effects associated with the light propagation and observations also generate corrections to the primordial non-Gaussianity, but they cancel together, leaving only the correction from the intrinsic relativistic effects.

In contrast, one can also argue \cite{22} that the galaxy bias models are based on Newtonian descriptions, so that they should be considered only in local coordinates such as CFC, in which the intrinsic relativistic effects are absorbed into the local coordinates in a single-field inflationary scenario. While

\textsuperscript{3}Often in literature, the matter density fluctuations in gauge-I and gauge-II are referred to as the matter fluctuation in the Eulerian and the Lagrangian frames. However, this is a misnomer. Given the exact definition of the Eulerian and the Lagrangian frames in the standard Newtonian perturbation theory, the analogy and correspondence of each gauge choice in general relativity to the Newtonian frames are not exact. For a pressureless medium in a flat Universe, the equations of motion for the matter density fluctuation in two gauges are identical to the Newtonian equations in the Eulerian and the Lagrangian frames, only up to the second order in perturbations \cite{115–117}. Furthermore, the nonlinear constraint equations in general relativity impose extra conditions that are absent in the Newtonian dynamics, which are the origin of the intrinsic relativistic effects in the matter density fluctuation in Eq. (2.4).

The Lagrangian dynamics reproduces the Eulerian dynamics \cite{118}, and this is valid even in GR \cite{119}. The situation of our interest is, however, different. The matter density fluctuation $\delta _m (x^\mu)$ in each gauge choice in the same proper-time hypersurface is described by different functional forms, and in GR there is no preference for one coordinate choice to another.
it provides a useful framework for interpreting Newtonian descriptions in general relativity, it is not clear whether this is enough. With all the relativistic corrections, CFC is not a Newtonian coordinate either. Moreover, such local coordinates as CFC have a finite range of validity, in which Fourier transformation is ill-defined, in particular for long wavelength modes in the squeezed limit. For example, the non-Gaussian correction in Eq. (2.9) needs to be computed in the squeezed limit. Certainly, we need a better and consistent description of galaxy bias in general relativity.

Here we focused on the so-called Eulerian bias, because our goal is to describe the cosmological observables, such that the linear bias factor $b$ is always multiplied by the nonlinear matter density fluctuation.

4 Contribution of the matter density fluctuation to the observed galaxy bispectrum

4.1 Theoretical considerations

Here we compute the major contribution to the observed galaxy three-point correlation function in the squeezed limit, resolving all the issues discussed in Section 3. In particular, we consider the contribution of the matter density fluctuation to the observed galaxy clustering. Given the number counts $dN^\text{obs}_g$ of the observed galaxies in a unit solid angle $d\Omega$ and a unit redshift bin $dz$, the observed galaxy number density is constructed in terms of observable quantities as

$$n^\text{obs}_g(\hat{n}, z) := \frac{dN^\text{obs}_g(\hat{n}, z)}{dV^\text{obs}(\hat{n}, z)} = n^\text{phy}_g(x^\mu_s) \frac{dV^\text{phy}(x^\mu_s)}{dV^\text{obs}(\hat{n}, z)} = n^\text{phy}_g(1 + \delta V),$$

where the observed volume is

$$dV^\text{obs}(\hat{n}, z) := \frac{r^2}{H(z)(1 + z)^3} d\Omega dz,$$

the volume element in the background universe, $n^\text{phy}_g(x^\mu_s)$ is the physical galaxy number density at the source position, and

$$dV^\text{phy}(x^\mu_s) := (1 + \delta V) dV^\text{obs}(\hat{n}, z)$$

is the 3D physical volume in 4D spacetime that appears subtended by the observed redshift bin $dz$ and solid angle $d\Omega$. Note that we used the superscript “phy” for the galaxy number density to contrast with the observed galaxy number density. In Eq. (4.3) we defined the dimensionless and gauge-invariant volume fluctuation $\delta V$. Evident in Eq. (4.1), all the contributions to the observed galaxy clustering can be split into two physically distinct effects [25, 93]: the volume effect $\delta V$ associated with $dV^\text{phy}$ and the source effect associated with $n^\text{phy}_g$.

The former is the ratio of two volume elements in Eq. (4.1), and it includes the redshift-space distortion and the gravitational lensing in addition to other relativistic effects associated with the light propagation (e.g., [120–124]). For instance, the mismatch between the observed angular position and the real position of the source galaxies gives rise to the gravitational lensing effect [120, 123, 124], and the mismatch in volume due to the observed redshift and the real positions gives rise to the redshift-space distortion [122]. These two well-known effects belong to the volume effect. The latter (or the source effect) comes from the physical galaxy number density $n^\text{phy}_g$ (or the source) discussed in Section 3.5, and its main contribution is the matter density fluctuation as galaxies are a biased tracer [47] of the matter density. Additional contributions in the source effect arise from the fact that the physical number density is expressed in terms of the observed redshift and angle, given the observational constraint on the galaxy sample such as the luminosity threshold. For instance, the
magnification bias [125–127] belongs to the source effect, and it arises from the imposed threshold in observation in terms of the inferred luminosity for the galaxy sample. A complete treatment of observed galaxy clustering with full relativistic treatment is first given in [25–29] (see [128] for review), and the formalism was extended to the second order in perturbations [111–113].

As discussed in Section 3.5, the matter density contribution with galaxy bias factor in the source effect is the dominant contribution to the observed galaxy clustering, and more importantly it is a distinct effect that is separable from other contributions due to its unique combination of galaxy bias factor $b$. The intrinsic galaxy fluctuation at the source position in Eq. (3.35) can be written as

$$\delta_g^{\text{int}}(x^\mu_s) = b \delta_m(x^\mu_s) + \cdots,$$

where the galaxy bias factor $b$ is also called the linear bias (sometimes denoted as $b_1$) and we omitted other contributions such as the nonlinear bias factors, the tidal tensor bias, and so on (see, e.g., [50, 51, 97, 99, 129–133]). According to the discussion in Section 3.5, the intrinsic galaxy fluctuation must contain at least the linear bias factor $b$ to reproduce galaxy clustering in the Newtonian limit, and the matter density $\delta_m$ multiplied by $b$ is the nonlinear matter density fluctuation. At the leading order in the bispectrum, this contribution $b\delta_m$ in $\delta_g^{\text{int}}$ is of our primary interest here for the contributions to the observed galaxy bispectrum on large scales. First, the other nonlinear contributions in Eq. (4.4) such as $b^2\delta_m^2$ provide Newtonian contributions to the observed galaxy bispectrum, which is negligible on large scales or in the squeezed triangular configuration [99, 134, 135]. Hence, the relativistic effects from the second-order in the intrinsic fluctuation, i.e.,

$$\langle b\delta_m^{(2)} b\delta_m^{(1)} b\delta_m^{(1)} \rangle,$$

provide the dominant contribution to the observed galaxy bispectrum on large scales. Second, the other important contributions to the observed galaxy bispectrum on large scales arise from the relativistic effects in the volume effect $\delta V$. These contributions at the leading order in bispectrum are

$$\langle b\delta_m^{(1)} b\delta_m^{(1)} \delta V^{(2)} \rangle, \quad \langle b\delta_m^{(1)} \delta V^{(2)} \delta V^{(1)} \rangle, \quad \langle b\delta_m^{(2)} \delta V^{(1)} \delta V^{(1)} \rangle, \quad \langle \delta V^{(2)} \delta V^{(1)} \delta V^{(1)} \rangle,$$

and they are as important on large scales as the contribution in Eq. (4.5). However, as discussed in Section 3.5 these individual contributions are separately gauge-invariant, and they are distinct in terms of scaling with galaxy bias factor $b$. Given the level of difficulties in computing the second-order relativistic effect in $\delta V$, here we focus on the dominant contribution in Eq. (4.5) from the intrinsic matter density fluctuation and call it the observed matter density contribution to the observed galaxy bispectrum. In this way, our calculations are not affected by the uncertainty in galaxy formation theory in general relativity. Calculations of the other contributions in Eq. (4.6) will be performed in future work.

### 4.2 Observed matter density fluctuation

While the proper-time hypersurface is our best physical choice for relating the matter density fluctuation to the intrinsic galaxy fluctuation, the spatial gauge choice remains undetermined as discussed in Section 3.2. Any change in spatial gauge choice alters the prediction of the bispectrum, as illustrated in Eq. (3.19). A natural question arises: What would be the best physical choice for spatial gauge? In fact, there is no convincing physical preference for any spatial gauge, as discussed in Section 3.5. However, this arbitrariness is completely lifted in observable quantities such as galaxy clustering, once the source position is expressed in terms of the observed redshift and angle. To the second order in perturbations, the matter density fluctuation at the observed position is

$$\delta_m(x^\mu_a) = \delta_m(x^\mu_{\bar{a}}) + \Delta x^\mu \partial_\mu \delta_m \bigg|_{x^\mu_{\bar{a}}},$$
where $\Delta x^\mu$ is the spacetime distortion of the source position

$$x_s^\mu =: \bar{x}_z^\mu + \Delta x^\mu,$$

(4.8)

relative to the observed position

$$\bar{x}_z^\mu := (\bar{\eta}_z, \bar{r}_z \hat{n}).$$

(4.9)

For a general coordinate transformation in Eq. (3.13), the observed position $\bar{x}_z^\mu$ in Eq. (4.9) remains unaffected, and hence the distortion of the source position transforms as

$$\tilde{\Delta} x^\mu = \Delta x^\mu + \xi^\mu.$$

(4.10)

As seen in Eq. (3.18), the matter density fluctuation gauge transforms as

$$\tilde{\delta}_m (x_s^\mu) = \delta_m (\bar{x}_z^\mu) - \xi^\mu \frac{\partial}{\partial x^\mu} \delta_m \bigg|_{\bar{x}_z^\mu},$$

(4.11)

and we readily prove that the expression in Eq. (4.7) at the observed position is fully gauge-invariant (temporal and spatial) up to the second order, which states nothing more than the invariance under diffeomorphism of physical quantities [73]. From now on, we call the combination,

$$\delta_{m, \text{obs}} := \delta_m (\bar{x}_z^\mu) + \Delta x^\mu \frac{\partial}{\partial x^\mu} \delta_m \bigg|_{\bar{x}_z^\mu},$$

(4.12)

the “observed” matter density fluctuation and compute its contribution to the observed galaxy bispectrum. Let us emphasize that the observed matter density fluctuation is independent of arbitrariness in choosing a spatial gauge condition, naturally resolving the issue of spatial gauge choice. Furthermore, with galaxy bias factor $b$, this contribution is separable from other relativistic contributions in Eq. (4.6).

Splitting the distortion of the source position in terms of spherical components set by the observed angular direction, the observed matter density fluctuation is written explicitly as

$$\delta (x_s^\mu) = \delta (\bar{x}_z^\mu) + \left( \frac{\partial}{\partial \eta} + \delta_r \frac{\partial}{\partial r} + \delta \theta \frac{\partial}{\partial \theta} + \delta \phi \frac{\partial}{\partial \phi} \right) \delta_m \bigg|_{\bar{x}_z^\mu},$$

(4.13)

where the first term is the matter density fluctuation in Eq. (2.4) at the observed position $\bar{x}_z$. It is clear that we only need the linear-order expressions of $\Delta x^\mu$ for our calculations. The distortion of the source position can be obtained by solving the geodesic equation (see [128, 136] for detailed derivations). The radial distortion of the source position is

$$\delta r = (\chi_o + \delta \eta) - \frac{\delta z}{H_z} + \frac{\delta z}{H_z} [V^{\alpha} \chi - \varphi] - n_\alpha (\delta x^\alpha + G) o - n_\alpha G^\alpha,$$

(4.14)

and the angular distortion is (similarly, $\bar{r}_z \sin \theta \delta \phi$ for the azimuthal distortion)

$$\bar{r}_z \delta \theta = \bar{r}_z \chi o \left[ - V^{\alpha} - \epsilon^{ij}_{\alpha} \eta^j \Omega^i \right] o - \bar{r}_z \chi \left[ (\delta x^\alpha + G^\alpha) o - n_\alpha G^\alpha \right],$$

(4.15)

where various perturbation quantities are defined in Appendix A.1 and $\delta z^\alpha := \delta z + H \chi$. The distortion in the observed redshift is

$$\delta z = -H \chi + (H \delta \eta + H \chi) o + (V^{\parallel} - \chi) o - \bar{r}_z \chi \left[ (\delta x^\alpha + G^\alpha) o - n_\alpha G^\alpha \right]$$

(4.16)
and it is related to the distortion in the time coordinate of the source as

\[ \delta \eta = \frac{\delta z}{\mathcal{H}}. \] (4.17)

Mind that in previous work the distortion in time coordinate of the source was denoted as \( \Delta \eta \), but here we use \( \delta \eta \) for the notational consistency with \( \delta r, \delta \theta, \) and \( \delta \phi \).

For concreteness, we choose the temporal comoving gauge and the spatial C-gauge described in Eq. (A.4) and compute the individual components. It is important to note that our choice of such gauge conditions is driven merely by convenience and the result is independent of our gauge choice. While the expressions in Eqs. (4.14)–(4.17) are general, our interest is the specific prediction in the standard inflationary model. Using the Einstein equation, the observed matter density fluctuation at the second order in perturbations is expressed as

\[ \delta_{\text{obs}}^m = D_1 \left( -\Delta \mathcal{R} + \frac{3}{2} \mathcal{R} \frac{D^\alpha \mathcal{R}}{\partial_{\alpha}} + 4 \mathcal{R} \Delta \mathcal{R} \right) + \frac{5}{4} D_A \nabla_\alpha (\mathcal{R} \mathcal{R} \Delta \mathcal{R}) + \frac{1}{4} D_B \Delta (\mathcal{R} \mathcal{R} \mathcal{R} \mathcal{R}) \] (4.18)

in terms of the initial condition \( \mathcal{R} \), where the detailed derivations for the expression and the definition of the growth factors \( D_V \) and \( D_\Psi \) are presented in Appendix B. The first line is \( \delta_{\text{obs}}^{(2)} \) in Eq. (2.4), and the remaining terms originate from the coupling of \( \Delta x^\mu \) and \( \delta_m \).

### 4.3 One-point ensemble average

Here we present the ensemble average \( \langle \delta_{\text{obs}}^m \rangle \) of the observed matter density fluctuation in Eq. (4.18), and the detailed calculations can be found in Appendix B. As discussed in Section 3.3, the ensemble average depends on a coordinate system, while the observed matter density fluctuation is independent of any gauge choice. This might appear inconsistent at first glance. However, the observed matter density fluctuation is in fact expressed in the hypersurface of the observed redshift with spatial gridding in terms of the observed angle, both of which are independent of our choice of coordinates in theoretical descriptions. The ensemble average of the observed matter density fluctuation is therefore the average over such hypersurface with spatial gridding, and it is also independent of our gauge choice. We emphasize again that such hypersurface is not fully accessible to the observer at one point, as the light-cone surface is limited to a two-dimensional intersection with the hypersurface. However, as discussed in Appendix A.2, fictitious observers besides our vantage point can have access to the full hypersurface. Therefore, while the ensemble average of the observed matter density fluctuation is not a direct observable, it is one of the important quantities in theoretical calculations, in a way that any power spectrum in a hypersurface is not a direct observable, but an important statistic in theoretical calculations.
The ensemble average of the observed matter density fluctuation is derived as

\[
\langle \delta_m^{\text{obs}} \rangle = -\left( \frac{5D_1}{2} + \frac{D_V}{H_z} \right) \sigma_2 + \frac{D_V}{H_z} \sigma_{2,0} + \left( -\frac{D_V D_\psi}{H_z} - 3D_1 D_\psi + \frac{D_1}{H_z} \right) \sigma_{3,1} - \frac{D_1 D_V}{3H_z} \sigma_4
\]

\[
+ \left( \frac{D_1 D_\psi}{3H_z} + \int_0^{\hat{\psi}} d\hat{\psi} D_\psi(\hat{\psi}) \right) \sigma_{4,0} - \frac{2D_1 D_\psi}{3H_z} \sigma_{4,2}
\]

\[
-2 \left( \frac{D_V}{H_z} + 3D_1 \right) \int \frac{d^3k}{(2\pi)^3} \int_0^{r_s} d\bar{r} D_\phi(\bar{r}) j_1(k\Delta r) k^3 P_R(k)
\]

\[
+ \frac{2D_1}{3H_z} \int \frac{d^3k}{(2\pi)^3} \int_0^{r_s} d\bar{r} D_\phi(\bar{r}) \left[ j_0(k\Delta r) - 2j_2(k\Delta r) \right] k^4 P_R(k)
\]

\[
(4.19)
\]

where \( \Delta r = \vec{r}_z - \vec{r} \) and we defined the dimensionless quantity

\[
\sigma_{n,m}(z) := \int \frac{d^3k}{(2\pi)^3} k^n j_m(k\vec{r}_z) P_R(k)
\]

\[
\sigma_{n,m} = L^{-n} .
\]

Given the spectral index \( n_s - 1 \approx 0 \), none of the variances \( \sigma_n \) or \( \sigma_{n,m} \) diverges in the infrared if \( n > 0 \), and the ensemble average of the observed matter density fluctuation is devoid of any infrared divergences. In an Einstein-de Sitter universe, this expression is greatly simplified as

\[
\langle \delta_m^{\text{obs}} \rangle = \eta_2^2 \left( \frac{1}{4} - \frac{1}{10} + \frac{3}{25} + \frac{3}{25} \right) \sigma_2 + \eta_2^2 \left( -\frac{3}{25} + \frac{1}{10} - \frac{3}{25} - \frac{6}{25} \right) \sigma_{2,0}
\]

\[
+ \eta_2^2 \left( -\frac{\eta_0}{50} - \frac{3\eta_z}{50} - \frac{\eta_z}{20} - \frac{\eta_0}{20} \right) \sigma_{3,1} - \frac{\eta_z^4}{300} \sigma_4 + \eta_z^2 \eta_0 \left( \frac{\eta_z}{300} + \frac{\eta_0}{100} \right) \sigma_{4,0} - \frac{\eta_z^2 \eta_0}{150} \sigma_{4,2}
\]

\[
(4.21)
\]

\[
4.4 \; \text{Three-point correlation function in the squeezed limit}
\]

Now we are in a position to compute the observed matter density contribution to the galaxy bispectrum in the squeezed limit. Since Fourier transformation involves integration over an infinite hyper-surface of simultaneity, Fourier quantities like the power spectrum and the bispectrum are less suited for a direct comparison to observations on large scales, in which the geometry of the sky is non-flat and the time evolution along the line-of-sight direction becomes significant. In contrast, being a function of observations at the observed positions only, the correlation function is well defined within the survey region, regardless of the geometry of a survey or the scale of our interest. In particular, we are interested in the three-point correlation function of the observed galaxy fluctuation in the squeezed triangle, in which the separation of two observed positions is negligible compared to the separation to the third position. In the limit the separation goes to infinity, also known as the squeezed limit, this special triangular configuration for the three-point correlation encodes critical information about the primordial non-Gaussianity.

In general, the primordial fluctuation is constrained to be highly Gaussian. However, a slight deviation from the Gaussianity (or the primordial non-Gaussianity) is expected in any inflationary models, and it is often parametrized in terms of \( f_{NL} \) (see, e.g., [7, 137]) in the bispectrum as

\[
B_R(k_1, k_2, k_3) = \frac{6}{5} f_{NL} \left[ P_R(k_1) P_R(k_2) + P_R(k_2) P_R(k_3) + P_R(k_3) P_R(k_1) \right]
\]

\[
(4.22)
\]

where \( f_{NL} \) can be a function of scale and the numerical factor is present for the convention in literature with the bispectrum in terms of Newtonian gauge potential \( \phi_\chi \). A similar relation is also defined for \( \zeta \).
and $F_{NL}$. For the single-field inflationary models, the prediction for the primordial non-Gaussianity in the squeezed limit is slow-roll suppressed [6, 53] as

$$\lim_{k_3 \to 0} B_\zeta(k_1, k_2, k_3) = -(n_s - 1) P_\zeta(k_1) P_\zeta(k_3), \quad F_{NL} = -\frac{5}{12}(n_s - 1), \quad (4.23)$$

and this prediction in the squeezed limit is generic for any scalar-field potential, such that if a non-negligible $F_{NL}$ is observed in the squeezed limit, all classes of single-field inflationary models will be ruled out. As pointed out in Section 3.2, the bispectrum depends on the spatial gauge choice [58], and the single-field consistency relation in Eq. (4.23) is derived with the spatial gauge choice in Eq. (3.2).

Here we compute the three-point correlation function of the observed matter density fluctuation $\mathcal{D} := \delta_m^{\text{obs}} - \langle \delta_m^{\text{obs}} \rangle$, given in Eq. (A.58), in which the non-vanishing ensemble average was subtracted to remove the tadpole contributions to the three-point correlation function. The details of the computation are presented in Appendix A.3. In particular, we consider the squeezed triangular configuration discussed in Section 2.2 and also given in Eq. (A.59) and (A.60), in which two observed positions are identical and the third position is in the opposite side of the sky, but all three positions are at the same redshift:

$$z := z_1 = z_2 = z_3, \quad \hat{n} := \hat{n}_1 = \hat{n}_2 = -\hat{n}_3. \quad (4.24)$$

This is the most squeezed triangular configuration obtainable in observations at a fixed redshift. The true squeezed limit for the consistency relation in Eq. (4.23) could be reached, only when the observed redshift becomes infinite. In fact, two observed positions need not be identical for the squeezed triangular configuration, but we take this triangular configuration for simplicity. Finally, the three-point correlation function of the observed matter density fluctuation in the squeezed triangular configuration is given in Eq. (2.10) and derived in Eq. (A.61) as

$$\xi_{\text{sqz}} = \int \frac{d^3 k_1}{(2\pi)^3} \int \frac{d^3 k_2}{(2\pi)^3} e^{i(k_1 k_2) \cdot (x_1 - x_3)} \left[ B_{112} + B_{211} + B_{121} \right] (k_1, k_2, -k_{12}), \quad (4.25)$$

where the connected bispectra in the integrand are defined in Eqs. (A.62)–(A.64). Compared to the calculations in Section 2.2, the computation of the bispectra is much more involved, as they depend on the line-of-sight integration and the observed angle (see Appendix A.3).

While the expression for $\xi_{\text{sqz}}$ is general for the squeezed triangular configuration, we further simplify the expression by taking the limit, in which the separation between $x_1$ and $x_3$ goes to infinity, i.e., the observed redshift becomes sufficiently large $z \to \infty$. In the squeezed limit, a highly oscillating phase in the exponential factor cancels all the contributions, except for the contribution from the modes:

$$L := |x_1 - x_3|, \quad \frac{1}{L} \simeq k_{12} \to 0, \quad k_{1} \simeq -k_{2}, \quad (4.26)$$

and the connected bispectra are further simplified in Eqs. (A.69)–(A.71), where the connected bispectrum $B_{112}$ is negligible compared to the other two bispectra. In this squeezed limit, the integral over $k_2$ in Eq. (4.25) is essentially removed to satisfy the condition in Eq. (4.26) imposed by the exponential factor with $L \to \infty$, and the three-point correlation function is simplified as

$$\xi_{\text{sqz}} \simeq \int \frac{d^3 k_1}{(2\pi)^3} \left( \frac{B_{211} + B_{121}}{L^3} \right) \times \int d\ln k_1 \int d\mu k_1^3 k_1^3 (B_{211} + B_{121}), \quad (4.27)$$

where we used (mind the dimension)

$$e^{i(k_1 + k_2) \cdot L} \sim \frac{1}{L^3} \delta^D(k_1 + k_2) \sim k_1^3 \delta^D(k_1 + k_2), \quad k_1 := \frac{1}{L}. \quad (4.28)$$
With two dominant bispectra in the squeezed limit given in Eqs. (A.70) and (A.71)

\[
B_{211} = D_{k_1}^2 k_1^3 \mathcal{P}_R(k_1) P_R(k_l) \left[ \mathcal{F}(k_1, k_l; \mathbf{n}) + \mathcal{F}(k_l, k_1; \mathbf{n}) \right],
\]

\[
B_{121} = D_{k_1}^2 k_1^3 \mathcal{P}_R(k_1) P_R(k_l) \left[ \mathcal{F}(k_l, -k_1; \mathbf{n}) + \mathcal{F}(-k_1, k_l; \mathbf{n}) \right],
\]

and the Fourier kernels \( \mathcal{F}(k_1, k_2; \mathbf{n}) \) computed in Appendix B, the important quantity in computing \( \xi_{sqz} \) is the angle average of the Fourier kernels from \( B_{211} \) and \( B_{121} \) in Eq. (4.27),

\[
\int \frac{d\mu_{k_1}}{2} \left[ \mathcal{F}(k_1, k_l) + \mathcal{F}(k_l, k_1) + \mathcal{F}(k_l, -k_1) + \mathcal{F}(-k_1, k_l) \right] \propto k_l^{n_F},
\]

and its dependence on the long mode \( k_l \), where we suppressed the dependence of the Fourier kernels on the observed angle \( \mathbf{n} \) and defined the power-law coefficients \( n_F \). The detailed computation of individual Fourier kernels and their contributions to the sum of two connected bispectra are presented in Appendix B.

In the limit \( k_l \to 0 \), most of the individual components in Eq. (4.18) vanish, because they have \( n_F > 0 \) or their angle average in Eq. (4.31) vanish. Adding up all the surviving contributions in Eq. (4.31), we obtain

\[
\int \frac{d\mu_{k_1}}{2} \left[ \cdots \right] = \frac{0}{\text{SPT}} - \frac{8D_k k_1^2}{\mathcal{H}^2_{\text{obs}}} + \frac{2D_k}{\mathcal{H}^2_{\text{src}}} k_1^2 + \frac{2D_k}{\mathcal{H}^2_{\text{nl}}} k_1^2 + \frac{0}{\mathcal{H}^2_{\text{ini}}} + 4D_1 \frac{3}{5} f_{\text{NL}} k_1^2,
\]

where each contribution is labeled according to its origin. First, the contribution from the standard perturbation theory (SPT) in the matter density fluctuation naturally vanishes in the squeezed limit. Second, the general relativistic effects (GRE) in the matter density fluctuation survive, as their contribution is essentially identical to that from the primordial non-Gaussianity in the initial condition (denoted as “ini”) in proportion to \( f_{\text{NL}} \). The other two surviving contributions are associated with the light propagation, and they cancel each other. The first one arises from the coupling of the Sachs-Wolfe effect (\( \propto R \)) at the source position to the matter density fluctuation (\( \propto \Delta R \)), and the other contribution arises from the same mechanism, but due to the coupling of the Sachs-Wolfe effect at the observer position. Finally, the coupling of the line-of-sight contributions to the matter density fluctuation is denoted as “nl,” and it vanishes. In short, the contributions associated with the light propagation (or the sum of those denoted as src, obs, and nl) completely vanish by cancellation, but the relativistic effects intrinsic to the matter density fluctuation or the primordial non-Gaussian contribution survive, both of which are at the source position, independent of observations or light propagation.

Therefore, with these leading bispectra, we show that the three-point correlation function in Eq. (4.25) scales with the long-mode in the squeezed limit as

\[
\xi_{sqz}^{\lim} \propto D_k^2 k_1^{n_+ + 1} k_l^{n^2 - 2 + n_F} L^{-3} \propto D_k^2 k_1^{n_+ + 1 + n_F},
\]

where \( n_F = 0 \) for the surviving (hence the leading) contributions in long modes. In conclusion, the observed three-point correlation function vanishes in the squeezed limit \( k_l = 1/|x_1 - x_2| \to 0 \), even accounting for all the relativistic corrections from the light propagation and the nonlinearity in the

---

\({}^4\)These effects are often referred to as the projection effects, but it is a misnomer, as they involve contributions at the source position and the observer position, as well as those along the light propagation.
matter density fluctuation due to the Hamiltonian constraint equation. With all the relativistic effects, there exist no contribution that scale as $P_m(k_l)/k_4^4$, if we phrase it in terms of the power spectrum in a hypersurface, as in Section 2.2.

This conclusion is indeed consistent with the single-field consistency relation in Eq. (4.23). In terms of the matter density fluctuation, the consistency relation in Eq. (4.23) can be recast as

$$B_δ \propto k_1^{2}k_2^{2}k_3^{2}B_\mathcal{R} \propto (n_s - 1)k_4^{4+n_s-4}k_2^{2+n_s-4}, \quad (4.34)$$

and its contribution to $\xi_{sqq}$ in Eq. (4.27) corresponds to the leading corrections in $k_l$ we obtained in Eq. (4.33) with $n_\mathcal{F} = 0$. We stress that in deriving the Fourier kernels $\mathcal{F}$ for individual contributions in Eq. (4.18), we have used the Einstein equation in the standard model to relate each component such as $\varphi_v$, $v_\chi$, and so on to the initial condition $\mathcal{R}$. The consistency we obtained is by no chance a coincidence.

5 Summary of new findings

Assuming the standard $\Lambda$CDM model in a single-field inflationary scenario, we have analytically computed the three-point correlation function of the matter density fluctuation in the squeezed triangular configuration, accounting for the intrinsic relativistic effects in the matter density fluctuation and the relativistic effects associated with light propagation and observations. The squeezed three-point correlation function of the matter density fluctuation on large scales is sensitive to the primordial non-Gaussianity and is expected to be the source of a prominent feature in the galaxy power spectrum on large scales. The intrinsic non-Gaussianity in the matter density fluctuation is always present from the Hamiltonian constraint in general relativity, and it has been extensively debated in literature whether such non-Gaussianity in a single-field inflationary model can give rise to signals similar to the presence of primordial non-Gaussianity. In contrast, it is generally accepted in the community that the relativistic effects associated with the light propagation in observations are expected contribute to the non-Gaussian signals, if not directly in the galaxy power spectrum. Our findings are summarized as follows.

- While the linear-order calculations are independent of spatial gauge choice, the calculations beyond the linear order in perturbations are affected by a choice of spatial gauge condition. Consequently, the three-point correlation function depends on a choice of spatial gauge condition (see Section 3.2).

- Since there is no physical argument to prefer one choice to another spatial gauge, any theoretical descriptions beyond the linear order in perturbations have ambiguities in spatial gauge choice, or a choice by hand in such theoretical descriptions would require further physical explanations (see below how these ambiguities are resolved in the theoretical descriptions of cosmological observables). For example, the non-Gaussian correction to the two-point galaxy correlation arises from the three-point matter density correlation [59, 60]. This relation cannot be valid for all spatial gauge choices, because the transformation properties of the two-point and the three-point correlation functions are different, which leaves us two possibilities: This relation is not valid at all in general relativity, or it should be valid for only one specific choice of spatial gauge. Even in the latter case, it still needs a physical explanation behind its choice of spatial gauge.

- If valid for one specific choice of spatial gauge condition, it has to be the spatial C-gauge in Eq. (A.3) in conjunction with the temporal comoving gauge in Eq. (A.4), because only
this gauge choice yields that the matter density fluctuation includes the correct second-order Newtonian contributions in the standard perturbation theory.

- The matter density fluctuation in general relativity exhibits extra relativistic contributions, originating from the nonlinear Hamiltonian constraint in general relativity [11, 17–19, 21]. These intrinsic and nonlinear contributions exist, even if the initial condition is set to be Gaussian at the linear order in perturbations. According to the relation for non-Gaussian correction [59, 60], the intrinsic relativistic effects in the matter density fluctuation generate signals like the primordial non-Gaussianity. In a single-field inflationary scenario, it was argued [20, 22–24, 35, 37] that these contributions can be removed by extra coordinate transformations. However, we showed [92] in Section 3.4 that large diffeomorphisms such as dilatation and special conformal transformation, which are not part of gauge transformation, do not affect the theoretical descriptions of cosmological observables, and coordinate transformations over a finite range of validity regime such as the conformal Fermi coordinate (CFC) can be recast as a gauge transformation over the entire manifold, while matching the CFC coefficients when expanded over the finite validity range. Consequently, there exist no residual symmetries or coordinate transformations that can affect the gauge-invariant calculations of cosmological observables, while they can change the functional form of the expression.

- The matter density fluctuation expressed at the observed position involves extra relativistic contributions associated with light propagation and observation, and this observed matter density fluctuation is independent of spatial gauge choice, which naturally resolves the ambiguities in choosing a spatial gauge condition. However, the theoretical descriptions that relate the matter density fluctuations to the galaxy number density fluctuations are independent of observations, and hence the ambiguities in spatial gauge choice still remain.

- Gauge-invariant calculations of cosmological observables demonstrated (see, e.g., [25, 26, 65, 138–140]) that there exist perturbation contributions at the observer position. Our calculations beyond the linear order in perturbations show that these contributions at the observer position couple to those at the source position and thereby obtain a positional dependence, which cannot be ignored in computing the correlation function or their Fourier counterpart.

- Accounting for all the relativistic effects, we computed the three-point correlation function of the observed matter density fluctuation in the squeezed limit and showed that the relativistic effects associated with the light propagation and observations produce zero non-Gaussian signals like the primordial non-Gaussianity by cancellation, but the intrinsic relativistic effects in the matter density fluctuation are not countered by any relativistic effects in the light propagation.

- The squeezed three-point correlation function receives the dominant contribution from the bispectra in the squeezed triangle in Fourier space, which admit a contribution that scales like the matter-potential cross power spectrum $P_{m\phi} \propto k^{-2}$, but no contribution that scales like the potential power spectrum $P_\phi \propto k^{-4}$. The three-point correlation function vanishes in the squeezed limit, under the standard $\Lambda$CDM model in a single-field inflationary scenario.

6 Discussion

As pointed out in Section 5, the largest uncertainties in the theoretical description of galaxy clustering reside in galaxy bias, or the relation between the galaxy and the matter distributions. Considered in
general relativity, galaxy bias models need a specific choice of gauge condition. The proper-time hypersurface is a preferred choice of temporal gauge, as it is the only hypersurface a local observer can construct without any extra information (see Section 3.5). In contrast, the spatial gauge choice is completely left arbitrary. Given the correspondence to the Newtonian perturbation theory, the spatial C-gauge choice might be preferred, but this is not an explanation for the choice. Nonetheless, if this is the right gauge choice for the bias model [59, 60], the galaxy two-point correlation function receives the non-Gaussian correction from the intrinsic relativistic effects in the matter density fluctuation, even in the absence of the primordial non-Gaussianity. As shown in Section 3.4, this correction cannot be removed by any other subsequent transformations.

On the other hand, it has been argued [20, 22–24, 35, 37] that a long mode in a single-field inflationary model can be absorbed into a local coordinate transformation, and its coupling to short modes can be removed. While we showed in Section 3.4 that any gauge-invariant calculations remain unaffected by extra transformations, it is possible that the theoretical descriptions of galaxy bias might be valid only in such local coordinates, rather than in the entire manifold. Since those models are based on Newtonian descriptions to begin with, there is no reason to be surprised to encounter ambiguities, when they are cast in general relativity. If that is the case, the coupling of long and short modes such as $R\Delta R$ in Eq. (2.4) can be removed in a single-field inflationary scenario due to the consistency relation [22–24, 35, 37], and there is no non-Gaussian correction. It is evident that we need a better theoretical description of galaxy bias, as we can only observe galaxies, not matter. Naturally, the theoretical description of cosmological observables is independent of spatial gauge choice, as the observed redshift and angle fully specifies the observed position of the source. However, note that galaxy bias or the relation between the matter and the galaxy distributions is independent of observations, and any successful model of galaxy bias should address the ambiguities associated with spatial gauge choice.

As opposed to the non-Gaussian contributions from the intrinsic relativistic effects, it has been argued [22, 35, 37] in the community that the light propagation in observations will inevitably generate non-vanishing non-Gaussian signals in the squeezed triangular configuration, even in the case of a single-field inflationary scenario. Indeed, this expectation was based on general arguments, and no accurate calculations have been performed. Here we showed that in the observed three-point correlation function the relativistic contributions from the light propagation in fact cancel each other in a single-field inflationary model, if all the relativistic effects are accounted for. This conclusion is independent of galaxy bias, as it only involves the linear-order matter density fluctuation and the linear-order light propagation. However, we only computed the observed matter density fluctuation. In galaxy clustering, the fluctuation in the physical volume compared to the observed volume also contributes, though this contribution does not mix with the calculations in this work due to the linear-order galaxy bias factor. To be definite about no relativistic contribution from the light propagation in a single-field inflationary scenario, we need to repeat the same computation here with the second-order description of the volume fluctuation $\delta V$. With the complete verification of the gauge-transformation properties of all the second-order expressions in the light propagation and observations [141], this computation will be soon performed in a near future. Furthermore, the cancellation of the relativistic contributions to the primordial non-Gaussian signals takes place in the observed three-point correlation function, while the relativistic effects are in general present in other statistics.

In literature, the power spectrum analysis is often performed to predict the primordial non-Gaussian signals in galaxy clustering and to quantify the detectability in a given survey. Since the local-type primordial non-Gaussianity features strong signals in the squeezed triangle, the power spectrum analysis appears natural. However, a Fourier transformation in a hypersurface as in Eq. (2.11) or Eq. (A.66) yields a prediction that is different from the power spectrum obtainable by integrating
over a light cone volume (see, e.g., [65]), as the correlation function is not just a function of separation in a hypersurface. In particular, on such large scales, where the primordial non-Gaussian signals are strong, the line-of-sight evolution and the geometry of the survey drive the observed power spectrum away from the simple prediction in a hypersurface, which would be accurate enough on small scales.

While the correlation function is free of these issues in the power spectrum analysis, its signal is close to zero and nearly featureless on such large scales, hence making it vulnerable to other systematic errors. The angular power spectrum analysis is also free of the issues associated with the geometry of surveys. However, as it is computed by projecting the observed fluctuation along the line-of-sight, it loses the redshift information in the statistics, and the non-Gaussian signals on large scales show up only at low angular multipoles (see, e.g., [61]), where the cosmic variance is largest and cannot be further reduced by increasing the survey volume. Other methods to quantify the observable signals on such large scales need to be further developed and improved such as the spherical Fourier analysis (see, e.g., [69, 142–145]), which uses the spherical harmonics for angular decomposition and the spherical Bessel function for radial Fourier analysis. Of course, all these two-point statistics are relevant for probing the local-type primordial non-Gaussianity, if the squeezed three-point correlation contributes to the two-point correlation of the galaxy distribution, or if the galaxy bias model is valid.

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A Analytical expressions

A.1 Analytic solutions and simplification in the standard cosmology

Here we adopt the most general representation of a spatially flat Friedmann-Robertson-Walker (FRW) metric and choose a rectangular coordinate:

\[ ds^2 = -a^2 (1 + 2\alpha) d\eta^2 - 2a^2 \beta_\alpha dx^\alpha d\eta + a^2 [(1 + 2\varphi)\delta_{\alpha\beta} + 2\gamma_{\alpha\beta}] dx^\alpha dx^\beta, \]  

(A.1)

where \( \alpha, \beta, \cdots \) represent the spatial indices, \( \eta \) is the conformal time coordinate, and \( a(\eta) \) is the expansion scale factor. Four perturbations \( \alpha, \beta, \gamma, \varphi \) represent the full scalar degrees of freedom in metric tensor. We also assumed that there is no vector or tensor perturbations at the linear order. Though second-order scalar perturbations generate the vector and tensor perturbations at second order, they do not couple to scalar perturbations if there is none at linear order. Furthermore, we use the superscript to indicate the perturbation order of each variable, for instance,

\[ \alpha(x^\mu) = \alpha^{(1)} + \alpha^{(2)} + \cdots. \]  

(A.2)

While our theoretical descriptions of the observable quantities are gauge-invariant as a whole, individual components are gauge-dependent. Furthermore, since their expressions take different forms in different gauges, we introduce two gauge choices convenient for our calculations: conformal Newtonian gauge (\( \chi \equiv 0 \)) and the comoving gauge (\( v \equiv 0 \)). Both gauge conditions fix the spatial gauge condition by setting at all perturbation orders

\[ \gamma \equiv 0 \quad \text{(spatial C gauge)}, \]  

(A.3)
or spatial C-gauge [111, 146]. The temporal gauge conditions are fixed again at all orders in perturbations by

\[
\begin{align*}
\chi & := a \beta + a \gamma' \equiv 0 \quad \text{(Newtonian)}, \\
u_\alpha &= g_{\alpha \mu} u^\mu =: -a v_\alpha \equiv 0 \quad \text{(comoving)},
\end{align*}
\]

where \( u^\mu \) is the four velocity. It is clear from the definition of the comoving gauge that there exist many different choices for comoving gauge, depending on which component’s \( u_\alpha \) is set zero. Here we choose \( v = 0 \) for the matter four velocity \( u^\mu \). Both choices completely fix the gauge symmetry and leave no unphysical degree of freedom. The other popular choice of gauge condition is the synchronous gauge, where the temporal gauge is fixed with vanishing fluctuation in the time component

\[ \alpha \equiv 0 \quad \text{(synchronous)}, \]

combined with the vanishing off-diagonal component

\[ \beta \equiv 0 \quad \text{(spatial B gauge)}. \]

The synchronous gauge conditions leave spatial gauge freedoms (see, e.g., [46, 147])

\[
\begin{align*}
T &= c_1(x) \frac{1}{a}, \\
L &= c_1(x) \int \! \frac{dt}{a^2} + c_2(x),
\end{align*}
\]

in the temporal and spatial gauge, so that an extra temporal comoving condition for the matter velocity \( (v_m \equiv 0) \) is often imposed to fix gauge freedom, which still leaves \( c_2(x) \) arbitrary. According to the convention in [46], this choice of the synchronous gauge \( (\alpha \equiv \beta \equiv v \equiv 0) \) is referred to as gauge-II, while the standard comoving gauge \( (v = 0) \) with spatial C-gauge \( (\gamma = 0) \) is called gauge-I.

Here we consider the standard cosmology, in which the initial condition is set during the single-field inflationary period and the subsequent evolution leads to a \( \Lambda \)CDM universe today. Assuming a pressureless medium, we derive the linear-order analytical relations among the metric perturbation variables by solving the Einstein equation (see [20, 65, 139, 140] for derivations), and they are all related to the initial condition \( R \) characterized by the comoving-gauge curvature perturbation \( \varphi_v \):

\[
\varphi_v := \varphi - H v, \quad \varphi^{(1)}_v = 0, \quad R(x) := \varphi_v(x, t_i),
\]

where the comoving-gauge curvature perturbation \( \varphi_v \) is conserved in time in a \( \Lambda \)CDM universe and we defined the initial condition \( R \) in a hypersurface at some early time \( t_i \). Note that the comoving-gauge curvature perturbation beyond the linear order evolves in time and this second-order growing solution vanishes in the limit \( t_i \to 0 \). So, the initial condition \( R(x) \) includes non-vanishing time-independent solution beyond the linear order in perturbations.

In the conformal Newtonian gauge, three perturbation variables are relevant for our calculations: two gravitational potentials and the scalar velocity potential of the matter four velocity:

\[
\alpha_\chi := \alpha - \dot{\chi}, \quad \varphi_\chi := \varphi - H \chi, \quad v_\chi := v - \frac{1}{a} \chi. \quad \text{(A.10)}
\]

In a \( \Lambda \)CDM universe, two gravitational potentials are identical with different sign:

\[
\Psi := \alpha_\chi^{(1)} = -\varphi_\chi^{(1)} = D_{\Psi} R, \quad D_{\Psi}(t) := \frac{1}{\Sigma} - 1, \quad \Sigma(t) := 1 + \frac{3}{2} \Omega_m(t) \frac{f(t)}{f(t)}, \quad \text{(A.11)}
\]
where we defined the time-dependent growth factor $D_\Psi$ for the Newtonian gauge potential $\Psi$ and $f(t)$ is the standard logarithmic growth rate of structure. The analytic relation to the initial condition is derived [20, 65, 139, 140] from the Einstein equation. Similarly, the scalar velocity potential is then

$$v^{(1)}_\chi = -D_V \mathcal{R}, \quad D_V(t) := \frac{1}{H\Sigma}, \quad u^\alpha := -\frac{1}{a} v^\alpha_\chi. \quad (A.12)$$

In the comoving gauge, the spatial velocity of the matter fluid is zero ($v = 0$), and the comoving-gauge curvature perturbation is conserved at the linear order. The density fluctuation in the comoving gauge describes the growth of structure in the rest frame:

$$\delta_v := \delta + 3Hv, \quad \delta_v^{(1)} = -D_1 \Delta \mathcal{R}, \quad D_1(t) := H \int_0^t \frac{dt'}{H^2(t')} = \frac{1}{H^2 f \Sigma}, \quad (A.13)$$

corresponding to the standard matter density fluctuation in literature. Note that the growth factor $D_1$ is not normalized to unity at the present time $t_0$ and its relation to the logarithmic growth rate is

$$f(t) = \frac{d \ln D_1}{d \ln a}, \quad D'_1 = \mathcal{H} f D_1. \quad (A.14)$$

The second-order growth functions in a $\Lambda$CDM universe were derived in [20], and they are explicitly

$$D_A(t) := \frac{7}{10} H \int_0^t dt' D_1^2 f \left( \Sigma + \frac{1}{2} f + 2 \right), \quad D_B(t) := \frac{7}{4} H \int_0^t dt' D_1^2 f \left( \Sigma - \frac{1}{2} f \right), \quad (A.15)$$

in relation to the second-order solution for the matter density in Eq. (2.1). The growth factors are dimensionful, s.t., the density fluctuation $\delta_v$ is dimensionless: $[D_1] = L^2$ and $[D_A] = [D_B] = L^4$.

Now we use these linear-order relations to derive the analytic expression of the observed matter density fluctuation in Eq. (4.13). We compute the individual components of the analytic expression in the temporal comoving gauge ($v \equiv 0$). Apart from the second-order matter density fluctuation in Eq. (2.4), the individual components need to be computed only up to the linear order in perturbations (see, e.g., [65, 140]) for the individual components associated with the distortion of the source position compared to the observed position). First, the distortion in the observed redshift is

$$\delta z_v := -H_z \chi_v + H_o \chi_v(x_o) + V_\parallel - \chi_\parallel(x_o) - \alpha_\chi + \alpha_\chi(x_o) - \int_0^{\vec{r}_z} d\vec{r} \left( \alpha_\chi - \varphi_\chi \right)' \quad (A.16)$$

where $V_\parallel = -v_{\chi;\alpha} n^\alpha$ is the line-of-sight peculiar velocity set by the observed direction $n^\alpha$, the integration is along the line-of-sight direction, and $x_o$ (or just subscript $o$) represents that quantities are evaluated at the observer position. We also used the subscript $v$ to indicate that the gauge-dependent terms $\delta z$ and $\chi_v$ are evaluated in the comoving gauge ($v = 0$):

$$\chi_v = a \beta_v = -av_\chi = a D_V \mathcal{R}. \quad (A.17)$$

The coordinate time lapse $\delta t_o$ in $\delta z$ vanishes in the comoving gauge, while non-vanishing in the conformal Newtonian gauge (see, e.g., [65]). The line-of-sight integral can be simplified using integration by part

$$-2 \int_0^{\vec{r}_z} d\vec{r} \alpha_\chi' = 2 \int_0^{\vec{r}_z} d\vec{r} \left( \frac{d}{d\vec{r}} [D_\Psi(\vec{r})] \mathcal{R}(\vec{r} \hat{n}) \right) = 2 \left( D_\Psi R - D_{\Psi,\alpha} R_o \right) - 2 \int_0^{\vec{r}_z} d\vec{r} D_{\Psi}(\vec{r}) \partial_\gamma R(\vec{r} \hat{n}) \quad (A.18)$$
Using the analytical relations, we simplify the radial distortion as

\[ \delta z_v = \mathcal{R}_o - \mathcal{R} + n^\alpha \left( D_V \mathcal{R}_{,\alpha} - D_V \mathcal{R}_{,\alpha} \bigg|_o \right) - 2 \int_0^{\bar{r}_v} d\bar{r} \ D_\psi (\bar{r}) \partial_r \mathcal{R}(\bar{r} \hat{n}) , \]

(A.19)

where we used the relation

\[ D_\psi (t) = \mathcal{H} (t) D_V (t) - 1 . \]

(A.20)

Since the distortion in the time coordinate of the source position from the observed redshift is

\[ \delta \eta_s = \frac{\delta z_v}{\mathcal{H}_z} , \]

(A.21)

and the time derivative of the linear-order matter density fluctuation is

\[ \delta'_{\nu} = -D'_1 \Delta \mathcal{R} = -\mathcal{H} f D_1 \Delta \mathcal{R} = -D_V \Delta \mathcal{R} , \]

(A.22)

we can evaluate the first coupling term in Eq. (4.13) from the temporal distortion

\[ \delta \eta_s \delta'_{\nu} = \frac{1}{\mathcal{H}_z} D_V \Delta \mathcal{R} \left[ \mathcal{R} - \mathcal{R}_o - n^\alpha \left( D_V \mathcal{R}_{,\alpha} - D_V \mathcal{R}_{,\alpha} \bigg|_o \right) + 2 \int_0^{\bar{r}_v} d\bar{r} \ D_\psi (\bar{r}) \partial_r \mathcal{R}(\bar{r} \hat{n}) \right] . \]

(A.23)

Mind that we used the notation \( \Delta \eta_s \) for \( \delta \eta_s \) in Eq. (A.21) in previous work \([65, 140]\).

Next we move to the remaining coupling terms in Eq. (4.13) and compute the spatial distortion in the source position. Compared to the observed position, the spatial distortion of the source can be decomposed along and perpendicular to the line-of-sight direction. The radial distortion along the line-of-sight direction is

\[ \delta r_v := \chi_v (x_o) - \frac{\delta z\chi}{\mathcal{H}_z} + \int_0^{\bar{r}_v} d\bar{r} \ (\alpha\chi - \varphi\chi) + n_{\alpha} \delta x^\alpha_v (x_o) , \]

(A.24)

where \( \delta z\chi = \delta z + H \chi \) is a gauge-invariant combination and the spatial shift of the observer position is obtained by integrating the shift until the present time \( \bar{t}_0 \) as

\[ \delta x^\alpha_v = \int_0^{\bar{t}_0} d\eta \beta^\alpha_v = \int_0^{\bar{t}_0} d\eta \ D_V \nabla^\alpha \mathcal{R} \bigg|_{x=0} . \]

(A.25)

Using the analytical relations, we simplify the radial distortion as

\[ \delta r_v = D_V (x_o) \mathcal{R}_o + \frac{1}{\mathcal{H}_z} \left[ -D_\psi \mathcal{R} - \mathcal{R}_o - n^\alpha \left( D_V \mathcal{R}_{,\alpha} - D_V \mathcal{R}_{,\alpha} \bigg|_o \right) + 2 \int_0^{\bar{r}_v} d\bar{r} \ D_\psi (\bar{r}) \partial_r \mathcal{R}(\bar{r} \hat{n}) \right] + 2 \int_0^{\bar{t}_0} d\eta \ D_V \partial_r \mathcal{R} \bigg|_{x=0} , \]

(A.26)

where the integration along the time coordinate in \( \delta x^\alpha_v \) is not to be confused with the line-of-sight integration. The angular distortion of the source position along the polar direction is

\[ \bar{r}_z \delta \theta_\alpha = -\bar{r}_z \theta_\alpha V^\alpha_o - \int_0^{\bar{r}_v} d\bar{r} \ (\bar{r}_z - \bar{r}) \theta_\alpha (\alpha\chi - \varphi\chi)^{-\alpha} + \theta_\alpha \delta x^\alpha_v \]

\[ = -\bar{r}_z \theta^\alpha (D_V \mathcal{R}_{,\alpha}) \bigg|_o - 2 \int_0^{\bar{r}_v} d\bar{r} \ (\bar{r}_z - \bar{r}) D_\psi (\bar{r}) \theta^\alpha \mathcal{R}_{,\alpha} + \int_0^{\bar{t}_0} d\eta \ D_V (\bar{\eta}) \theta^\alpha \mathcal{R}_{,\alpha} \bigg|_{x=0} , \]

(A.27)
where we ignored the orientation $\Omega^i$ of the observer frame in the full expression of $\delta \theta$, as it is not correlated with $\mathcal{R}$. Similarly, the angular distortion along the azimuthal direction is identical with $\vec{r}_z \sin \theta \delta \phi_v$ in the left-hand side and with $\theta_o$ replaced by $\phi_v$ in the right-hand side.

Now that we have expressed all the individual components of the matter density fluctuation in Eq. (4.13) in terms of the initial condition $\mathcal{R}$, we put them together here to show our main analytic equation for the matter density fluctuation in the standard cosmology, and we will use this equation to compute the bispectrum in the squeezed limit:

$$
\delta_{\text{obs}} = \frac{5}{3} D_A \nabla_\alpha (\mathcal{R} \mathcal{R} \mathcal{R}) + \frac{1}{3} \frac{D_B \Delta (\mathcal{R} \mathcal{R})}{D_\text{obs}}
$$

where $\mathcal{R}$ is the angular gradient, the first line is the second-order matter density in Eq. (A.28), and the remaining terms are from $\mathcal{R} \mathcal{R} \mathcal{R}$ in sequential order. Three components from the spatial shift of the observer position are combined into one in the last line, and the product of two angular gradients in the last line represents

$$
2 D_1 \int_0^{r_z} d\vec{r} \left( \frac{\vec{r}_z - \vec{r}}{r_z \vec{r}} \right) D_\Psi (\vec{r}) \left[ \frac{\partial}{\partial \phi_v} \mathcal{R} \frac{\partial}{\partial \phi_v} \mathcal{R} \frac{\partial}{\partial \phi_v} \mathcal{R} \right] + \frac{1}{3} \frac{D_B \Delta (\mathcal{R})}{D_\text{obs}}
$$

Equation (A.28) is the main equation shown in Eq. (4.18) for our computation.

Further simplification can be made in the limit that the Universe is approximated as the Einstein-de Sitter universe, or matter-dominated universe, in which $\Lambda = 0$ and the matter density parameter $\Omega_m = 1$ ($f = 1$). Due to the simplicity, we can derive the analytic solutions, regarding the Hubble parameter and the angular diameter distance

$$
a = \left( \frac{t}{t_o} \right)^{2/3} = \left( \frac{\eta}{\eta_o} \right)^2, \quad \frac{t}{t_o} = \left( \frac{\eta}{\eta_o} \right)^3, \quad \eta_o = 3 t_o,
$$

$$
H = \frac{2}{3 t}, \quad \mathcal{H} = \frac{2}{\eta}, \quad \vec{r}_z = \eta_o - \vec{\eta} = \frac{2}{H_0} \left( 1 - \frac{1}{\sqrt{1 + z}} \right),
$$

and the relation among the perturbation variables

$$
\Sigma = \frac{5}{2}, \quad D_\Psi = -\frac{3}{5}, \quad D_1 = \frac{\eta^2}{10}, \quad D_\mathcal{R} = \frac{\eta}{5}, \quad D_A = D_B = D_1^2 = \frac{\eta^4}{100}.
$$

### A.2 Fourier decomposition and one-point ensemble average

To facilitate the subsequent calculations, we introduce a set of second-order Fourier kernels $F(k_1, k_2)$ that capture various contributions to the matter density fluctuation in Eq. (A.28) at the observed
redshift and describe their time evolution from the initial condition \( \mathcal{R} \). Schematically, the matter density fluctuation at the second order will be expressed as

\[
\delta(x) \propto \int \frac{d^3k_1}{(2\pi)^3} \int \frac{d^3k_2}{(2\pi)^3} e^{ik_1 \cdot x} e^{ik_2 \cdot x} F(k_1, k_2; \hat{n}, \bar{\eta}_z) \mathcal{R}(k_1) \mathcal{R}(k_2),
\]

where the Fourier kernel is dimensionless. As shown in Eqs. (4.13) and (A.28), the observed matter density fluctuation at the second order is coupled with the contributions at the source position, at the observer position, or along the line-of-sight direction. So we discuss three different types of Fourier kernels in turn. The derivation of Fourier kernels for the individual contributions is presented in Section B.

Furthermore, it proves useful for the computational convenience and also conceptual clarity to define Fourier counter parts, given the individual contributions to the matter density fluctuation. These Fourier counter parts in configuration space and Fourier space are defined in a hypersurface set by the observed redshift \( z \) in the usual way:

\[
\delta(x) = \int \frac{d^3k}{(2\pi)^3} e^{ik \cdot x} \delta(k; \hat{n}, \bar{\eta}_z), \quad \delta(k; \hat{n}, \bar{\eta}_z) = \int d^3x e^{-ik \cdot x} \delta(x),
\]

where the volume integral over the position \( x \) is all over the infinite hypersurface, not over the light cone volume. Note that this hypersurface encompasses a volume outside the observed light cone volume and hence these Fourier counter parts are not directly observable, except at the intersection with the light cone volume. However, we can imagine that fictitious observers at different spatial position in the Universe perform the same observations. In other words, for the same observed redshift and source position, these fictitious observers can construct the fluctuation field \( \delta(x) \) with the observed position \( x = \bar{r}_z \hat{n} + x_o \), where \( x_o \) is the spatial position of the fictitious observer and can be set zero for the real observer (us). Noting that our position in the Universe is not special, it is conceptually useful to think of such fictitious observations and to construct the observed matter density field and its Fourier counter part outside our own light cone volume. However, it is noted that since the observed matter density fluctuation depends not only on the redshift, but also on the angle, the observed angle should be specified, when the fictitious observations are considered, and hence the angular dependence in Eq. (A.34). We refer the reader to the work [65] for more detailed discussion and computation.

The Fourier kernels are also useful in computing the one-point ensemble average. The ensemble average \( \langle \delta(x) \rangle \) is an average of the fluctuation \( \delta(x) \) over many realizations of the Universe, as discussed in Section 3.3. With the ergodic theorem, the ensemble average is equivalent to the Euclidean average over the hypersurface, which can be readily computed in our formalism with the observations by the fictitious observers. Note that the cosmic variance in practice arises due to our limitation to one observer position or the lack of average over translation in the Euclidean average (see, e.g., [83]). The average over the hypersurface yields

\[
\langle \delta(x) \rangle = \lim_{V \to \infty} \frac{1}{V} \int d^3x \int \frac{d^3k}{(2\pi)^3} e^{ik \cdot x} \delta(k; \hat{n}, \bar{\eta}_z) = \frac{1}{V} \left. \langle \delta(k = 0; \hat{n}, \bar{\eta}_z) \rangle \right) .
\]

In general, the ensemble average of a fluctuation is zero for a Gaussian distribution. However, at the second order, and in particular with the relativistic contributions, the ensemble average of the matter density fluctuation is non-vanishing, as we show in the following. More importantly, it is apparent by the definition of the average over a hypersurface that the ensemble average depends on a choice of hypersurface.
• **Coupling terms with contributions at the source position.**— First, we consider the contributions \( \delta_s(\mathbf{x}) \) at the source position only, and these terms in Eq. (A.28) can be expressed as

\[
\delta_s(\mathbf{x}) = \int \frac{d^3k_1}{(2\pi)^3} \int \frac{d^3k_2}{(2\pi)^3} \ e^{i\mathbf{k}_1 \cdot \mathbf{x}} e^{i\mathbf{k}_2 \cdot \mathbf{x}} F_s(\mathbf{k}_1, \mathbf{k}_2; \hat{\mathbf{n}}, \bar{\eta}_z) \mathcal{R}(k_1) \mathcal{R}(k_2),
\]

and this yields the Fourier mode according to Eq. (A.34)

\[
\delta_s(\mathbf{k}; \hat{\mathbf{n}}, \bar{\eta}_z) = \int \frac{d^3q}{(2\pi)^3} F_s(\mathbf{q}, \mathbf{k} - \mathbf{q}; \hat{\mathbf{n}}, \bar{\eta}_z) \mathcal{R}(\mathbf{q}) \mathcal{R}(\mathbf{k} - \mathbf{q}),
\]

where the Fourier kernel \( F_s \) is subject to

\[
F_s^*(\mathbf{k}_1, \mathbf{k}_2; \hat{\mathbf{n}}, \bar{\eta}_z) = F_s^*(-\mathbf{k}_1, -\mathbf{k}_2; \hat{\mathbf{n}}, \bar{\eta}_z).
\]

Note that the Fourier kernels include the time-dependence. For example, the linear-order matter density contribution to the Fourier kernel is

\[
\delta_m^{(1)}(\mathbf{x}, t) = -D_1(t) \Delta \mathcal{R}(\mathbf{x}), \quad F_s \ni D_1(t) k_1^2.
\]

For later convenience, we introduce another notation for Fourier kernels \( \mathcal{F} \) that are directly related to the bispectrum, and for the contributions \( \delta_s(\mathbf{x}) \) at the source position we simply have

\[
\mathcal{F}_s(\mathbf{k}_1, \mathbf{k}_2; \hat{\mathbf{n}}, \bar{\eta}_z) := F_s(\mathbf{k}_1, \mathbf{k}_2, \hat{\mathbf{n}}, \bar{\eta}_z).
\]

Finally, the ensemble average of the coupling terms with contributions at the source position is obtained by using Eq. (A.35) as

\[
\Xi := \langle \delta_s(\mathbf{x}) \rangle = \int \frac{d^3q}{(2\pi)^3} F_s(\mathbf{q}, -\mathbf{q}; \hat{\mathbf{n}}, \bar{\eta}_z) P_R(q) = \int d\ln q \ \Delta_{\mathcal{R}}^2(q) \int \frac{d\mu_q}{2} F_s(\mathbf{q}, -\mathbf{q}; \hat{\mathbf{n}}, \bar{\eta}_z),
\]

where \( \mu_q = q \cdot \hat{\mathbf{n}}/|\mathbf{q}| \) is the cosine angle with respect to the observed direction.

• **Coupling terms with contributions at the observer position.**— Next we consider the contributions \( \delta_o(\mathbf{x}) \) to the matter density fluctuation coupled with those at the observer position. To compute the field in a hypersurface outside the light cone, we consider fictitious observers at different spatial positions \( \mathbf{x}_o \neq 0 \). Those coupling terms in Eq. (A.28) can be expressed in terms of Fourier kernel \( F_o \) as

\[
\delta_o(\mathbf{x}) = \int \frac{d^3k_1}{(2\pi)^3} \int \frac{d^3k_2}{(2\pi)^3} \ e^{i\mathbf{k}_1 \cdot \mathbf{x}_o} e^{i\mathbf{k}_2 \cdot \mathbf{x}} F_o(\mathbf{k}_1, \mathbf{k}_2; \hat{\mathbf{n}}, \bar{\eta}_o, \bar{\eta}_z) \mathcal{R}(k_1) \mathcal{R}(k_2),
\]

where the Fourier kernel also has the time-dependence on \( \bar{\eta}_o \) set at the observer position \( \mathbf{x}_o \). Note that since \( \bar{\eta}_z \) and \( \hat{\mathbf{n}} \) are fixed in a hypersurface, the position \( \mathbf{x}_o \) for the fictitious observer is a function of \( \mathbf{x} \) in consideration. Also mind that the Fourier kernel \( F_o \) is not symmetric over the arguments \( \mathbf{k}_1 \) and \( \mathbf{k}_2 \) in our convention, where the wave vector \( \mathbf{k}_1 \) belongs to the Fourier transform of the contribution evaluated at the observer position and \( \mathbf{k}_2 \) belongs to the contribution at the source position. The Fourier counterpart is then

\[
\delta_o(\mathbf{k}; \hat{\mathbf{n}}, \bar{\eta}_o, \bar{\eta}_z) = \int \frac{d^3q}{(2\pi)^3} e^{-i\mathbf{q} \cdot \bar{\mathbf{r}}_o \hat{\mathbf{n}}} F_o(\mathbf{q}, \mathbf{k} - \mathbf{q}; \hat{\mathbf{n}}, \bar{\eta}_o, \bar{\eta}_z) \mathcal{R}(\mathbf{q}) \mathcal{R}(\mathbf{k} - \mathbf{q}),
\]
and its ensemble average is
\[
\Xi = \langle \delta_\omega(x) \rangle = \int \frac{d^3q}{(2\pi)^3} e^{-i\vec{r}_z \cdot \vec{n}} F_o(q, -\vec{q}; \hat{n}, \vec{\eta}_0, \vec{\eta}_z) P_\mathcal{R}(q)
\]
\[
= \int d\ln q \, \Delta^2_\mathcal{R}(q) \int \frac{d\mu_2}{2} e^{-i\vec{r}_z \cdot \vec{n}} F_o(q, -\vec{q}; \hat{n}, \vec{\eta}_0, \vec{\eta}_z).
\]  

We define the Fourier kernel
\[
\mathcal{F}_o(k_1, k_2; \hat{n}, \vec{\eta}_0, \vec{\eta}_z) := e^{-ik_1 \cdot \vec{r}_z} F_o(k_1, k_2; \hat{n}, \vec{\eta}_0, \vec{\eta}_z),
\]  

and note that it includes exponential factor.

**Coupling terms with contributions along the line-of-sight direction.**— Finally, we consider the contributions \( \delta_{nl}(x) \) to the matter density fluctuation coupled with those along the line-of-sight direction:
\[
\delta_{nl}(x) = \int \frac{d^3k_1}{(2\pi)^3} \int \frac{d^3k_2}{(2\pi)^3} \int_0^{\bar{r}_z} d\bar{r} \, e^{i\vec{k}_1 \cdot \hat{n}} e^{i\vec{k}_2 \cdot \hat{n}} F_{nl}(k_1, k_2; \hat{n}, \vec{r}, \vec{\eta}_z) \mathcal{R}(k_1) \mathcal{R}(k_2),
\]  

where the time-dependence of the Fourier kernel is specified by \( \vec{r} \) in the line-of-sight integration. We use the subscript \( nl \) to refer to this as non-local Fourier kernel. The kernel is also not symmetric in the arguments, and the wave vector \( k_1 \) belongs to the contribution evaluated along the line-of-sight, while \( k_2 \) to the one evaluated at the source position. The Fourier counter part is then
\[
\delta_{nl}(k; \hat{n}, \vec{\eta}_z) = \int \frac{d^3q}{(2\pi)^3} \int_0^{\bar{r}_z} d\bar{r} \, e^{-i\Delta r q \cdot \hat{n}} F_{nl}(q, k - q; \vec{r}, \vec{\eta}_z) \mathcal{R}(q) \mathcal{R}(k - q),
\]  

and its ensemble average is
\[
\Xi = \langle \delta_{nl}(x) \rangle = \int \frac{d^3q}{(2\pi)^3} \int_0^{\bar{r}_z} d\bar{r} \, e^{-i\Delta r q \cdot \hat{n}} F_{nl}(q, -q; \vec{r}, \vec{\eta}_z) P_\mathcal{R}(q)
\]
\[
= \int d\ln q \, \Delta^2_\mathcal{R}(q) \int_0^{\bar{r}_z} d\bar{r} \, \int \frac{d\mu_2}{2} e^{-iq\Delta r} F_{nl}(q, -q; \vec{r}, \vec{\eta}_z).
\]

where \( \Delta r := \bar{r}_z - \bar{r} \). The Fourier kernel for the non-local contributions is then defined as
\[
\mathcal{F}_{nl}(k_1, k_2; \hat{n}, \vec{\eta}_z) := \int_0^{\bar{r}_z} d\bar{r} \, e^{-ik_1 \cdot \Delta r \hat{n}} F_{nl}(k_1, k_2; \hat{n}, \vec{r}, \vec{\eta}_z),
\]  

and it includes the line-of-sight integration and the exponential factor in the kernel. While \( F_{nl} \) is dimensionful, the Fourier kernel \( \mathcal{F}_{nl} \) is dimensionless due to the line-of-sight integral.

### A.3 Three-point correlation and the bispectrum

Here we compute the three-point correlation function and its bispectrum. The leading-order contribution to the three-point statistics arises from the contraction of two linear-order contributions and a second-order contribution of the matter density fluctuation. Since the Fourier mode of the observed matter density fluctuation at the second order in perturbations is collectively expressed in terms of Fourier kernels as
\[
\delta(k; \hat{n}, \vec{\eta}_z) = \int \frac{d^3q}{(2\pi)^3} \mathcal{F}(q, k - q; \hat{n}, \vec{\eta}_z) \mathcal{R}(q) \mathcal{R}(k - q),
\]  

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and the linear-order matter density fluctuation is

\[ \delta^{(1)}(k) = D_1(z) k^2 \mathcal{R}(k), \]  

(A.51)

the leading-order contribution to the bispectrum is then

\[
\langle \delta^{(1)}(k_1) \delta^{(1)}(k_2) \delta^{(2)}(k_3) \rangle = D_1^2 k_1^2 k_2^2 P_R(k_1) P_R(k_2)(2\pi)^3 \delta^D(k_1 + k_2 + k_3) \\
\times [\mathcal{F}(-k_1, -k_2; \hat{n}_3) + \mathcal{F}(-k_2, -k_1; \hat{n}_3)] + \langle \delta^{(1)}(k_1) \delta^{(1)}(k_2) \rangle \langle \delta^{(2)}(k_3) \rangle ,
\]  

(A.52)

where we omitted the time dependence in the growth function \( D_1 \) and the Fourier kernels \( \mathcal{F} \). Note that the tadpole term exists because the one-point ensemble average \( \langle \delta^{(2)}(k_3) \rangle \) is not vanishing. Accounting for the permutation of three leading-order contributions and for three different observed positions at the same observed redshift, we derive the observed three-point correlation function

\[
\langle \delta(x_1) \delta(x_2) \delta(x_3) \rangle = \int \frac{d^3k_1}{(2\pi)^3} \int \frac{d^3k_2}{(2\pi)^3} \int \frac{d^3k_3}{(2\pi)^3} e^{i k_1 \cdot x_1} e^{i k_2 \cdot x_2} e^{i k_3 \cdot x_3} \langle \delta(k_1) \delta(k_2) \delta(k_3) \rangle \\
= \int \frac{d^3k_1}{(2\pi)^3} \int \frac{d^3k_2}{(2\pi)^3} e^{i k_1 \cdot (x_1 - x_3)} e^{i k_2 \cdot (x_2 - x_3)} \left\{ D_1^2 k_1^2 k_2^2 P_R(k_1) P_R(k_2) \left[ \mathcal{F}(-k_1, -k_2; \hat{n}_3) + \mathcal{F}(-k_2, -k_1; \hat{n}_3) \right] + \text{perm.} \right\} \\
+ \left[ \xi_m(x_1, x_2) + \xi_m(x_2, x_3) + \xi_m(x_3, x_1) \right] \Xi,
\]  

(A.53)

where the permutation terms contain the Fourier kernels \( \mathcal{F} \) with two other angular positions \( \hat{n}_1 \) and \( \hat{n}_2 \) and the two-point correlation function is the just the linear-order matter correlation function

\[ \xi_m(x_1, x_2) = \langle \delta^{(1)}(x_1) \delta^{(1)}(x_2) \rangle . \]  

(A.54)

The three-point correlation is non-vanishing, even for a configuration \( x_1 = x_2 \) and \( x_3 \to \infty \) due to the non-vanishing constant contribution \( \Xi = \langle \delta(x) \rangle \neq 0 \), or the difference between the ensemble average and the background at a given time.

To address this subtlety, we first define a dimensionless fluctuation

\[ \mathcal{D}(x) := \delta(x) - \Xi , \]

(A.55)

though the correct fluctuation should be with extra constant \((1 + \Xi)\). For instance, the matter density can be split as

\[ \rho(x) = \bar{\rho}(t) (1 + \Xi) \left( 1 + \frac{\mathcal{D}(x)}{1 + \Xi} \right) , \]

(A.56)

The three-point correlation function of \( \mathcal{D}(x) \) is then

\[
\langle \mathcal{D}(x_1) \mathcal{D}(x_2) \mathcal{D}(x_3) \rangle = \langle \delta(x_1) \delta(x_2) \delta(x_3) \rangle - \left[ \langle \delta(x_1) \delta(x_2) \rangle \langle \delta(x_3) \rangle + \text{perm.} \right] + 2 \langle \delta \rangle^3 ,
\]  

(A.57)

where the extra terms in the three-point correlation including the constant contribution ensure that the tadpole terms do not contribute and only the connected contribution to the three-point correlation.
function remains. Computing to the second-order in the matter density fluctuation, we derive the three-point correlation function

\[ \langle \mathcal{D}(x_1) \mathcal{D}(x_2) \mathcal{D}(x_3) \rangle = 2 \Xi^3 + \int \frac{d^3k_1}{(2\pi)^3} \int \frac{d^3k_2}{(2\pi)^3} e^{ik_1 \cdot (x_1 - x_3)} e^{i(k_2 - (x_2 - x_3))} \tag{A.58} \]

where the subscripts represent the order of perturbations, e.g.,

\[ \mathcal{D}_1 \mathcal{D}_2 \mathcal{D}_3 \]

\[ \mathcal{D}_1 \mathcal{D}_2 = \mathcal{D}_1 \mathcal{D}_2 \mathcal{D}_3 \mathcal{D}_4 \]

\[ \mathcal{D}_1 \mathcal{D}_2 \mathcal{D}_3 \mathcal{D}_4 \mathcal{D}_5 \]

\[ \mathcal{D}_1 \mathcal{D}_2 \mathcal{D}_3 \mathcal{D}_4 \mathcal{D}_5 \mathcal{D}_6 \]

\[ \mathcal{D}_1 \mathcal{D}_2 \mathcal{D}_3 \mathcal{D}_4 \mathcal{D}_5 \mathcal{D}_6 \mathcal{D}_7 \]

\[ \mathcal{D}_1 \mathcal{D}_2 \mathcal{D}_3 \mathcal{D}_4 \mathcal{D}_5 \mathcal{D}_6 \mathcal{D}_7 \mathcal{D}_8 \]

\[ \mathcal{D}_1 \mathcal{D}_2 \mathcal{D}_3 \mathcal{D}_4 \mathcal{D}_5 \mathcal{D}_6 \mathcal{D}_7 \mathcal{D}_8 \mathcal{D}_9 \]

\[ \mathcal{D}_1 \mathcal{D}_2 \mathcal{D}_3 \mathcal{D}_4 \mathcal{D}_5 \mathcal{D}_6 \mathcal{D}_7 \mathcal{D}_8 \mathcal{D}_9 \mathcal{D}_{10} \]

where the tadpoles in Eq. (A.53) are all removed. Furthermore, since we only considered the field up to the second order in perturbations or keep terms up to \( P_R \), the constant term \( \Xi^3 \sim P_R^3 \) will be ignored for consistency, as a proper treatment of such term would need one-loop contributions \( P_R^3 \) in the bispectrum.

Having derived the general expression for the three-point statistics, we now consider a special triangular configuration, in which two observed positions are identical and the third position points in the opposite side of the sky\(^5\)

\[ z := z_1 = z_2 = z_3, \quad \hat{n} := \hat{n}_1 = \hat{n}_2 = -\hat{n}_3, \tag{A.59} \]

i.e., one observed redshift \( z \) and one angular vector \( \hat{n} \) for the three points in the sky. In terms of three-dimensional position vectors, the squeezed triangular configuration is represented by

\[ x_2 = x_1, \quad L := x_1 - x_3, \tag{A.60} \]

and our primary interest lies in the squeezed triangular configuration in the limit \( L \to \infty \). The squeezed correlation function can then be further simplified as

\[ \xi_{\text{squeezed}} := \langle \mathcal{D}(x_1) \mathcal{D}(x_2) \mathcal{D}(x_3) \rangle = \int \frac{d^3k_1}{(2\pi)^3} \int \frac{d^3k_2}{(2\pi)^3} e^{i(k_1 + k_2) \cdot L} \left[ B_{112} + B_{211} + B_{121} \right] (k_1, k_2, -k_{12}), \tag{A.61} \]

where \( k_{12} := k_1 + k_2 \). The connected bispectra with dimension \( L^6 \) in the integrand are defined as

\[ B_{112} := D_1^2 D_2^2 D_3^2 P_R(k_1) P_R(k_2) \left[ F(-k_1, -k_2; -\hat{n}) + F(-k_2, -k_1; -\hat{n}) \right], \tag{A.62} \]

\[ B_{211} := D_1^2 D_2^2 D_3^2 P_R(k_2) P_R(k_3) \left[ F(-k_2, -k_3; \hat{n}) + F(-k_3, -k_2; \hat{n}) \right], \tag{A.63} \]

\[ B_{121} := D_1^2 D_2^2 D_3^2 P_R(k_3) P_R(k_1) \left[ F(-k_3, -k_1; \hat{n}) + F(-k_1, -k_3; \hat{n}) \right], \tag{A.64} \]

where the subscripts represent the order of perturbations, e.g.,

\[ B_{112} := \langle \mathcal{D}_{k_1}^{(1)} \mathcal{D}_{k_2}^{(1)} \mathcal{D}_{k_3}^{(2)} \rangle, \tag{A.65} \]

and the dimension of the bispectra is \( L^6 \). The task for computing the squeezed correlation boils down to computing the Fourier kernels for Eqs. (A.62)–(A.64), which are presented in Section B. Despite the simplicity in the notation, it is noted that the kernels \( F_o \) and \( F_{nl} \) for the observer and the non-local contributions include the exponential factors or the line-of-sight integral. Moreover, though

\[ ^5 \text{While these two identical points are then subject to large non-linear corrections, these two points for the squeezed triangular configuration only need to be close to each other, compared to the third position, avoiding any extra complication due to nonlinearity on small scales.} \]
those two types of Fourier kernels are not symmetric over the arguments, their contributions to the squeezed correlation function are made symmetric in the bispectra. However, mind the dependence of \( \hat{n} \) and its sign.

Treating the squeezed three-point correlation function as a two-point correlation or just a function of separation \( L \), the Fourier counter part or the power spectrum can be defined as

\[
P(k) := \int d^3 L e^{-i k L} \xi_{\text{sqz}} \simeq \int \frac{d^3 k_1}{(2\pi)^3} \left[ B_{112} + B_{211} + B_{121} \right] (k_1, k - k_1, -k), \tag{A.66}
\]

where \( k = k_1 + k_2 \). It is shown [60] that this power spectrum is a useful quantity to measure the correction to the standard galaxy power spectrum, arising from the non-Gaussian nature of the matter density fluctuation, or the squeezed three-point correlation \( \xi_{\text{sqz}} \). However, it is important to note that \( \xi_{\text{sqz}} \) here is the observed three-point correlation function, which is not just a function of its separation \( L \) alone. Furthermore, \( \xi_{\text{sqz}} \) is defined on the past light cone, not in an infinite hypersurface of simultaneity at the observed redshift, so that any integration over \( L \) should involves the variation in time or the observed redshift. Therefore, the final equality for the power spectrum in terms of three bispectrum should be taken as a simple theoretical model of non-Gaussianity, rather than a real observed power spectrum that can be obtained in the light cone volume. While one can still define the observed power spectrum as in Eq. (A.66) with the volume integration over the light-cone volume, it is not equal to the expression in the RHS of Eq. (A.66) in terms of the bispectra.

Before we proceed to derive the individual Fourier kernels, we discuss the general scaling of the squeezed correlation function (see also [148]). Though we have computed the three-point correlation function \( \xi_{\text{sqz}} \), this squeezed correlation function contributes to the galaxy two-point correlation function as a non-Gaussian correction discussed in Section 2.2. For such two-point correlation function, in the absence of non-vanishing ensemble average, we expect it to vanish in the limit \( L \to \infty \). Assuming a power-law relation, we expect the power-law index to be at least positive:

\[
\xi_{\text{sqz}} \propto L^{-n}, \quad \lim_{L \to \infty} \xi_{\text{sqz}} = 0, \quad n > 0. \tag{A.67}
\]

This expectation allows the possibility for the power spectrum \( P(k) \propto k^{n-3} \) to scale as \( P(k) \propto P_m/k^2 \propto P_{\delta\phi} \) or the primordial non-Gaussianity signature discussed in Section 2.2.

In the limit \( L \to \infty \), the squeezed triangular configuration takes the form

\[
k_l := k_{12} = k_1 + k_2 \to 0, \quad k_1 \approx -k_2, \quad k_3 = -k_{12} = -k_l, \tag{A.68}
\]

and the connected bispectra in this limit are

\[
B_{112} = D_l^2 k_1^4 P_R(k_1) P_R(k_1) \left[ \mathcal{F}(-k_1, k_l; -\hat{n}) + \mathcal{F}(k_1, -k_l; -\hat{n}) \right], \tag{A.69}
\]

\[
B_{211} = D_l^2 k_1^2 k_2^2 P_R(k_1) P_R(k_1) \left[ \mathcal{F}(k_1, k_l; \hat{n}) + \mathcal{F}(k_1, k_l; \hat{n}) \right], \tag{A.70}
\]

\[
B_{121} = D_l^2 k_1^2 k_2^2 P_R(k_1) P_R(k_1) \left[ \mathcal{F}(k_l, -k_1; \hat{n}) + \mathcal{F}(-k_1, k_l; \hat{n}) \right]. \tag{A.71}
\]

While \( k_l \to 0 \), the other wave vector \( k_1 \) can take any arbitrary value. As the exponential factor in Eq. (A.61) vanishes, the squeezed correlation function is determined by the integration of these connected bispectra. Note that the contribution of \( B_{112} \) is a constant in this limit or independent of \( k_l \), and its ratio to the other contributions is

\[
\frac{B_{112}}{B_{211}} \sim \frac{B_{112}}{B_{121}} \sim \frac{k_1^2 P_R(k_1)}{k_l^2 P_R(k_1)} \sim \left( \frac{k_l}{k_1} \right)^{2-n}, \tag{A.72}
\]

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suppressed at least by $k_l/k_1$ before considering extra suppression factors from the ratio of the Fourier kernels. The calculations in Appendix B show that the ratio of two Fourier kernels vanish due to the symmetry of the kernels in $B_{112}$ or scales with some power of $k_l$. Therefore, we will ignore the contribution from $B_{112}$. The long-mode contributions in $B_{211}$ and $B_{121}$ to the power spectrum are schematically

$$\Delta P(k_1) \propto \int d\ln k_1 k_1 P_m(k_1) P_m(k_l) \int d\mu_1 \frac{[F + \tilde{F}]}{k_i^2}, \quad P_m(k) \propto k^4 P_R(k), \quad (A.73)$$

suggesting that the scaling with $k_l$ for the sum of two kernels in $B_{211}$ or $B_{121}$ after the angular integration should be

$$\Delta P \propto k_l^{n_s-2+n_F}, \quad n_F > -1 - n_s \simeq -2, \quad (A.74)$$

to satisfy the condition in Eq. (A.67), where we approximated

$$\int d\mu_1 [F + \tilde{F}] \propto k_l^{n_F}. \quad (A.75)$$

As demonstrated in Appendix B, the power-law slope is non-negative: $n_F \geq 0$.

**B Detailed calculations of Fourier kernels**

Here we present the detailed calculations of the Fourier kernels for three different types of contributions to the observed matter density fluctuation. With the kernels in hands, the one-point ensemble average and three connected bispectra can be readily computed. Each subsection specifies the calculations according to three different types of Fourier kernels.

The squeezed triangular configuration, which is our primary interest, is specified by only two angular directions $\hat{n}_1$ and $\hat{n}_2 = -\hat{n}_1$, opposite directions in the sky. To facilitate the calculation, we set this direction to be aligned with $z$-direction ($\hat{n} = \hat{n}_1 = \hat{z} = -\hat{n}_2$), in which the azimuthal integration is trivial. We will use the following relations for the polar integration in terms of spherical Bessel functions $j_n(x)$

$$\int \frac{d\mu}{2} \cos(\mu a) = j_0(a), \quad \int \frac{d\mu}{2} \mu \sin(\mu a) = j_1(a), \quad (B.1)$$

$$\int \frac{d\mu}{2} \mu^2 \cos(\mu a) = \frac{j_0(a) - 2j_2(a)}{3}.$$  

**B.1 Contributions at the source position**

The contributions at the source position arise from three different origins. The first is the second-order matter density contributions in the standard perturbation theory shown Eq. (2.1), and the second is the second-order relativistic contribution to the matter density fluctuation in Eq. (2.4). Finally, the last arises from the coupling terms of the linear-order matter density fluctuation and the relativistic contribution at the source position in Eq. (4.13). We present the detailed calculations according to this classification.

*Contributions from the standard perturbation theory*: The second-order contributions in the standard perturbation theory consist of two terms in Eq. (2.1). The Fourier kernel for the first term is

$$\bullet \frac{5}{7} D_A(z) \nabla_\alpha (R^\alpha \Delta R) : \quad F_s(k_1, k_2) = \frac{5}{7} D_A(z) (k_1 + k_2) \cdot k_1 k_2^2, \quad (B.2)$$
and given the structure of the connected bispectra we symmetrize the Fourier kernel

\[ \mathcal{F}_s(k_1, k_2) = \frac{5}{14} D_A(z) (k_1 + k_2) \cdot (k_1^2 k_1 + k_2^2 k_2) . \]  

(B.3)

Having derived the Fourier kernel, the one-point ensemble average in Eq. (A.41) is readily shown to be vanishing

\[ \Xi = 0 , \]  

(B.4)

as the sum of two wave vectors is zero. The contribution to the connected bispectra can be derived by using Eqs. (A.69)–(A.71). The first connected bispectra \( B_{112} \) in Eq. (A.69) is independent of \( k_l \) and is simply zero

\[ B_{112} = 0 \]  

(B.5)

due to symmetry of the wave vectors. Since the pre-factors of three bispectra are the same for different Fourier kernels, we present calculations of the sum of Fourier kernels in the square bracket in Eqs. (A.69)–(A.71). The other two connected bispectra are non-zero,

\[ B_{211} \equiv \frac{5}{7} D_A(z) \left[ 2k_1^2 k_1^2 + (k_1^2 + k_1^2)k_1 \cdot k_1 \right] \to 0 , \]

\[ B_{121} \equiv \frac{5}{7} D_A(z) \left[ 2k_1^2 k_1^2 - (k_1^2 + k_1^2)k_1 \cdot k_1 \right] \to 0 \]  

(B.6)

but they vanish in the limit \( k_l \to 0 \).

The Fourier kernel for the second term of the standard perturbation theory is already symmetric:

- \[ \frac{1}{7} D_B(z) \Delta (\mathcal{R}^\alpha \mathcal{R}_\alpha) : \quad \mathcal{F}_s = F_s(k_1, k_2) = \frac{1}{7} D_B(z) (k_1 + k_2)^2 k_1 \cdot k_2 , \]  

(B.7)

and the one-point ensemble average and the first connected bispectra vanish due to symmetry of the wave vectors

\[ \Xi = 0 , \quad B_{112} = 0 . \]  

(B.8)

The two connected bispectra are obtained by Eqs. (A.70) and (A.71)

\[ B_{211} \equiv \frac{2}{7} D_B(z)(k_1 + k_1)^2 k_1 \cdot k_l \to 0 , \quad B_{121} \equiv -\frac{2}{7} D_B(z)(k_1 - k_l)^2 k_1 \cdot k_l \to 0 \]  

(B.9)

and they also vanish in the limit \( k_l \to 0 \).

In the standard perturbation theory, two growth functions \( D_A(t) \) and \( D_B(t) \) are assumed to be equal to \( D_1^2(t) \), and the equality is only valid in the Einstein-de Sitter universe, while the approximation yields only small errors in the late universe [44]. The first kernel in Eq. (B.3) can then be re-arranged as

\[ \frac{5}{7} D_1^2(t) \left[ 1 + \frac{k_1 \cdot k_2}{2k_1 k_2} \left( \frac{k_1}{k_2} + \frac{k_2}{k_1} \right) \right] , \]  

(B.10)

and the second kernel in Eq. (B.7) is

\[ \frac{1}{7} D_1^2(t) \left[ \frac{k_1 \cdot k_2}{k_3 k_2} \left( \frac{k_1}{k_2} + \frac{k_2}{k_1} \right) + 2 \left( \frac{k_1 \cdot k_2}{k_1 k_2} \right)^2 \right] , \]  

(B.11)

where we scaled each Fourier kernel by \( k_1^2 k_2^2 \) in the denominator, as the Fourier kernels in the standard perturbation theory are expressed in terms of the matter density fluctuation \( \delta(k) = -k^2 \mathcal{R}(k) \). They add up to yield the standard kernel

\[ F_2(k_1, k_2) = D_1^2(t) \left[ \frac{5}{7} + \frac{1}{2} \frac{k_1 \cdot k_2}{k_1 k_2} \left( \frac{k_1}{k_2} + \frac{k_2}{k_1} \right) + \frac{2}{7} \left( \frac{k_1 \cdot k_2}{k_1 k_2} \right)^2 \right] , \]  

(B.12)
but note that the shift term in proportion to \( k_1 \cdot k_2 \) originates from two terms with different time dependence.

**Relativistic contributions in the matter density fluctuation:** At the second order in perturbations, there exist two extra relativistic contributions to the matter density fluctuation in addition to the standard contributions discussed in the context of the standard perturbation theory. They originate from the constraint equation of general relativity. The Fourier kernels for these two terms in Eq. (2.4) are computed here. The Fourier kernel for the first relativistic contribution is

\[
\mathcal{F}_s(k_1, k_2) = -\frac{3}{2} D_1(z) k_1 \cdot k_2 ,
\]

and its one-point ensemble average is non-vanishing

\[
\Xi = \frac{3}{2} D_1(z) \sigma_2 ,
\]

where the variance \( \sigma_n \) is defined in Eq. (2.16). The contributions to the connected bispectra are rather simple

\[
B_{112} \equiv 3D_1 k_1^2 , \quad B_{211} = -B_{121} \equiv -3 D_1 k_1 \cdot k_1 \to 0 .
\]

The Fourier kernel for the second relativistic contribution in Eq. (2.4) is

\[
\mathcal{F}_s(k_1, k_2) = -4D_1(z) k_2^2 ,
\]

and its symmetrized Fourier kernel is

\[
\mathcal{F}_s(k_1, k_2) = -2D_1 \left( k_1^2 + k_2^2 \right) .
\]

For the bispectrum contributions, the kernels \( \mathcal{F}_s \) get symmetrized, so we can use the symmetric kernel \( \mathcal{F}_s \), but for the one-point ensemble average in Eq. (A.41), the kernel \( \mathcal{F}_s \) should be used. The one-point ensemble average of the second relativistic contribution is

\[
\Xi = -4D_1(z) \sigma_2 .
\]

Finally, the contributions to the connected bispectra are then obtained by using Eqs. (A.69)–(A.71)

\[
B_{112} \equiv -8D_1 k_1^2 , \quad B_{211} = B_{121} \equiv -4D_1(k_1^2 + k_2^2) \to -4D_1 k_1^2 ,
\]

and all of them are non-zero in the limit \( k_1 \to 0 \).

**Non-Gaussian contributions in the presence of \( f_{NL} \):** The primordial non-Gaussianity in the initial condition with non-vanishing \( f_{NL} \) in Eq. (2.7) gives rise two extra terms in Eq. (2.25), and their Fourier kernels can be readily read-off from Eqs. (B.13) and (B.16) as

\[
\begin{align*}
\bullet \ -\frac{6}{5} D_1(z) f_{NL} \mathcal{R}^\alpha \mathcal{R}_a : & \quad \mathcal{F}_s = F_s(k_1, k_2) = \frac{6}{5} D_1(z) f_{NL} k_1 \cdot k_2 , \\
\bullet \ -\frac{6}{5} D_1(z) f_{NL} \mathcal{R} \Delta \mathcal{R} : & \quad \mathcal{F}_s(k_1, k_2) = \frac{3}{5} D_1(z) f_{NL} \left( k_1^2 + k_2^2 \right) .
\end{align*}
\]

As opposed to the two relativistic contributions in the matter density fluctuations, the coefficients for these two contributions arrange in a way that the sum of these two contributions in proportion to \( f_{NL} \) is

\[
\mathcal{F}_s(k_1, k_2) = \frac{3}{5} D_1(z) f_{NL} |k_1 + k_2|^2 ,
\]

\[42\]
and the one-point ensemble average vanishes by cancellation of the two terms. The contribution to
the first bispectrum vanishes, and the other two bispectra
\[ B_{211} \equiv \frac{6}{5} D_1 f_{ni} |k_1 + k_l|^2 \rightarrow \frac{6}{5} D_1 f_{ni} k_l^2 \, , \quad B_{121} \equiv \frac{6}{5} D_1 f_{ni} |k_1 - k_l|^2 \rightarrow \frac{6}{5} D_1 f_{ni} k_l^2 \, , \]
survive in the squeezed limit.

**Coupling contributions at the source position:** The linear-order matter density fluctuation is coupled
with the relativistic effects in the light propagation, which appears as \( \Delta x^\mu \partial_\mu \delta \) in Eq. (4.13). This
coupling results in numerous extra terms detailed in Eq. (A.28), as there are many contribution terms
in \( \Delta x^\mu \). Among those, there are four terms, contributing at the source position, and we compute the
Fourier kernels for those four terms.

The first two terms arise from the coupling \( \delta \eta \partial_\eta \delta \), where \( \delta \eta = \delta z / H \) and \( \delta' \propto D_1' = D_V \). The
first of the two contributions is the Sachs-Wolfe contribution at the source, and its Fourier kernel is
\[ F_s(k_1, k_2) = -\frac{D_V(z)}{H_z} k_l^2 \, , \quad F_s(k_1, k_2) = -\frac{D_V(z)}{2H_z} (k_l^2 + k_l^2) \, . \] (B.24)
The one-point ensemble average of this coupling term is
\[ \Xi = -\frac{D_V(z)}{H_z} \sigma_2 \, , \] (B.25)
and the connected bispectra are
\[ B_{112} \equiv -\frac{2D_V(z)}{H_z} k_l^2 \, , \quad B_{211} = B_{121} \equiv -\frac{D_V(z)}{H_z} (k_l^2 + k_l^2) \rightarrow \frac{D_V(z)}{H_z} k_l^2 \, . \] (B.26)
None of the one-point ensemble average or the connected bispectra vanish in the limit \( k_l \rightarrow 0 \).

The second contribution in the coupling \( \delta \eta \partial_\eta \delta \) arises from the line-of-sight velocity contribution
at the source position. The Fourier kernel for the contribution is
\[ \bullet -\frac{D_V}{H_z} \partial_\mu R \Delta R : \quad F_s(k_1, k_2) = \frac{D_V(z)}{H_z} i \mu_1 k_1 k_2^2 \, , \] (B.27)
and its symmetrized Fourier kernel is
\[ F_s(k_1, k_2) = \frac{D_V(z)}{2H_z} i k_1 k_2 (\mu_1 k_2 + \mu_2 k_1) \, . \] (B.28)
The one-point ensemble average and the first connected bispectrum then trivially vanish due to the
symmetry of the wave vectors
\[ \Xi = 0 \, , \quad B_{112} = 0 \, , \] (B.29)
and the remaining two connected bispectra also vanish
\[ B_{211} \equiv \frac{D_V}{H_z} i k_1 k_l (\mu_1 k_l + \mu_1 k_1) \rightarrow 0 \, , \quad B_{121} \equiv \frac{D_V}{H_z} i k_1 k_l (\mu_1 k_1 - \mu_1 k_1) \rightarrow 0 \, , \] (B.30)
upon angle average in the limit.

The remaining two contributions at the source position come from the coupling term \( \delta r \partial_r \delta \) (no
contributions at the source position from the angular distortion). The first term is the coupling of the
matter density fluctuation and the gravitational potential at the source position. The Fourier kernel for this contribution is

\[
F_s(k_1, k_2) = \frac{D_1(z)D_{\Psi}(z)}{\mathcal{H}z} i \mu_2 k_2^3,
\]  

(B.31)

and its symmetrized kernel is

\[
F_s(k_1, k_2) = -\frac{D_1(z)D_{\Psi}(z)}{2\mathcal{H}z} i \left( \mu_1 k_1^3 + \mu_2 k_2^3 \right).
\]  

(B.32)

The one-point ensemble average and the first connected bispectrum vanish

\[
\Xi = 0, \quad B_{112} = 0,
\]  

(B.33)

upon angle average. Mind that the one-point ensemble average is computed by using \( F_s \) in Eq. (A.41), while the connected bispectra are computed by using \( F_s \) in Eq. (A.69). The remaining two connected bispectra

\[
B_{211} \equiv -\frac{D_1D_{\Psi}}{\mathcal{H}z} i \left( \mu_1 k_1^3 + \mu_2 k_2^3 \right) \to 0, \quad B_{121} \equiv -\frac{D_1D_{\Psi}}{\mathcal{H}z} i \left( -\mu_1 k_1^3 + \mu_1 k_2^3 \right) \to 0,
\]  

(B.34)

also vanish upon angle average in the limit \( k_l \to 0 \).

The last contribution is the coupling with the line-of-sight velocity at the source position. The Fourier kernel for the fourth term is

\[
F_s(k_1, k_2) = \frac{D_1(z)D_{V}(z)}{\mathcal{H}z} \mu_1 \mu_2 k_1 k_2 \left( k_1^2 + k_2^2 \right),
\]  

(B.35)

and its symmetrized kernel is

\[
F_s(k_1, k_2) = \frac{D_1(z)D_{V}(z)}{2\mathcal{H}z} \mu_1 \mu_2 k_1 k_2 \left( k_1^2 + k_2^2 \right).
\]  

(B.36)

The one-point ensemble average and the first connected bispectrum are non-zero

\[
\Xi = -\frac{D_1(z)D_{V}(z)}{3\mathcal{H}z} \sigma_4, \quad B_{112} = -\frac{2D_1(z)D_{V}(z)}{\mathcal{H}z} \mu_2 k_1^4 \to -\frac{2D_1(z)D_{V}(z)}{3\mathcal{H}z} k_1^4,
\]  

(B.37)

where the angular dependence is removed upon angular integration. The remaining two connected bispectra, however,

\[
B_{211} = -B_{121} \equiv \frac{D_1(z)D_{V}(z)}{\mathcal{H}z} \mu_1 \mu_2 k_1 k_2 \left( k_1^2 + k_2^2 \right) \to 0,
\]  

(B.38)

vanish, if we take the limit \( k_1 \to 0 \).

### B.2 Coupling contributions from the source and the observer positions

Since cosmological observables are measured by the observer, these observables depend not only on the physical properties of the source, but also on the state of the observer. As the observers perform measurements in the rest frame different from the FRW frame, there exist various contributions in the cosmological observables from the gravitational potential and the line-of-sight velocity at the observer position, as shown in Eqs. (4.14)–(4.15). These contributions contribute only through the
coupling terms in $\Delta x^\mu \partial_\nu \delta$ in Eq. (4.13). As discussed in Appendix A.2, the Fourier kernels $F_o$ for such coupling contributions involve the exponential factor in Eq. (A.45), in addition to the ordinary Fourier kernels $F_o$ from the coupling terms.

There exist six different contributions in Eq. (A.28), involving those at the observer position. The first two terms come from the coupling $\delta \eta \partial_\eta \delta$. The Sachs-Wolfe contribution at the observer position makes the first of such terms, and its Fourier kernel is

$$ F_o(k_1, k_2) = \frac{D_V(z)}{\mathcal{H}_z} k_2^2, \quad (B.39) $$

and its symmetrized kernel with the exponential factor is

$$ F_o = \frac{D_V(z)}{2\mathcal{H}_z} \left( e^{-ik_1 \cdot \hat{r}_s \mu k_1^2} + e^{-ik_2 \cdot \hat{r}_s \mu k_2^2} \right), \quad (B.40) $$

where the subscript $o$ indicates that $\mathcal{R}$ is evaluated at the observer position and there is no time dependence of $\mathcal{R}_o$ on $\bar{\eta}_o$, as the initial condition is constant in time. Furthermore, we keep the convention that the first wave vector $k_1$ describes the quantity at the observer position, while the second wave vector $k_2$ describes one at the source position. The one-point ensemble average and the first connected bispectrum are

$$ \Xi(z) = \frac{D_V(z)}{\mathcal{H}_z} \sigma_{2,0}(z), \quad \quad (B.41) $$

where the time-dependent variance $\sigma_{n,m}(z)$ is defined in Eq. (4.20). Furthermore, in the derivation we used

$$ \int_{-1}^{1} \frac{d\mu}{2} e^{\pm i x \mu} = j_0(x), \quad (B.42) $$

and also for the following derivations we will use the useful relation involving the integration of the exponential factor:

$$ \int_{-1}^{1} \frac{d\mu}{2} \mu e^{\pm i x \mu} = \pm ij_1(x), \quad \quad \quad \quad \quad \int_{-1}^{1} \frac{d\mu}{2} \mu^2 e^{\pm i x \mu} = \frac{j_0(x) - 2j_2(x)}{3}. $$

Two connected bispectra are non-vanishing

$$ B_{211} \equiv \frac{D_V(z)}{\mathcal{H}_z} \left( e^{-ik_1 \cdot \hat{r}_s \mu k_1^2} + e^{-ik_2 \cdot \hat{r}_s \mu k_2^2} \right) \rightarrow \frac{D_V(z)}{\mathcal{H}_z} k_1^2, \quad (B.43) $$

$$ B_{121} \equiv \frac{D_V(z)}{\mathcal{H}_z} \left( e^{-ik_1 \cdot \hat{r}_s \mu k_1^2} + e^{-ik_2 \cdot \hat{r}_s \mu k_2^2} \right) \rightarrow \frac{D_V(z)}{\mathcal{H}_z} k_2^2, \quad (B.44) $$

even in the limit $k_l \rightarrow 0$.

The second term arises from the line-of-sight velocity at the observer position, contributing to $\delta \eta$. The Fourier kernel for this term is

$$ \frac{D_V D_V}{\mathcal{H}} (\partial_\eta \mathcal{R})_o \Delta \mathcal{R} : \quad F_o(k_1, k_2) = -\frac{D_V(z)D_V(\bar{\eta}_o)}{\mathcal{H}_z} i\mu_1 k_1 k_2^2, \quad (B.45) $$

and its symmetrized kernel is

$$ F_o = -\frac{D_V(z)D_V(\bar{\eta}_o)}{2\mathcal{H}_z} i k_1 k_2 \left( e^{-ik_1 \cdot \hat{r}_s \mu_1 k_1^2} + e^{-ik_2 \cdot \hat{r}_s \mu_2 k_2^2} \right). \quad (B.46) $$

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The one-point ensemble average and the first connected bispectrum are

$$\Xi = \frac{D_V(z)D_V(\bar{\eta}_o)}{H_z} \sigma_{3,1}(z), \quad B_{112} \equiv -\frac{2D_V(z)D_V(\bar{\eta}_o)}{H_z} k_1^3 j_1(k_1 \bar{r}_z),$$

non-vanishing, while two remaining connected bispectra

$$B_{211} \equiv -\frac{D_V(z)D_V(\bar{\eta}_o)}{H_z} i k_1 k_l \left( e^{-ik_1 \cdot \bar{r}_z} \mu_1 k_l + e^{-ik_1 \cdot \bar{r}_z} \mu_l k_1 \right) \to 0, \quad (B.48)$$

$$B_{121} \equiv -\frac{D_V(z)D_V(\bar{\eta}_o)}{H_z} i k_1 k_l \left( -e^{ik_1 \cdot \bar{r}_z} \mu_1 k_l + e^{-ik_1 \cdot \bar{r}_z} \mu_l k_1 \right) \to 0, \quad (B.49)$$

vanish in the limit $k_l \to 0$, upon angular integration.

The next two terms come from the coupling term $\delta r \partial_r \delta$, in which the gravitational potential and the line-of-sight velocity terms couple to the derivative of the matter density fluctuation. The Fourier kernel for the gravitational potential contribution at the observer position is

$$\bullet \quad -D_1 \left( D_V \frac{1}{H_z} \right) \mathcal{R}_o \partial_r (\Delta \mathcal{R}) : \quad F_o(k_1, k_2) = D_1(z) \left[ D_V(\bar{\eta}_o) - \frac{1}{H_z} \right] i \mu_2 k_2^3,$$

and its symmetrized kernel is

$$\mathcal{F}_o = \frac{D_1(z)}{2} \left[ D_V(\bar{\eta}_o) - \frac{1}{H_z} \right] i \left( e^{-ik_1 \cdot \bar{r}_z} \mu_2 k_2^3 + e^{-ik_1 \cdot \bar{r}_z} \mu_1 k_1^3 \right). \quad (B.51)$$

Evident in the time-dependent pre-factors, there are multiple contributions from the gravitational potential at the observer position with different pre-factors. The one-point ensemble average and the first connected bispectrum are

$$\Xi(z) = -D_1(z) \left[ D_V(\bar{\eta}_o) - \frac{1}{H_z} \right] \sigma_{3,1}(z), \quad B_{112} \equiv -2D_1(z) \left[ D_V(\bar{\eta}_o) - \frac{1}{H_z} \right] k_1^3 j_1(k_1 \bar{r}_z),$$

and the remaining connected bispectra are

$$B_{211} \equiv D_1(z) \left[ D_V(\bar{\eta}_o) - \frac{1}{H_z} \right] i \left( e^{-ik_1 \cdot \bar{r}_z} \mu_1 k_l^3 + e^{-ik_1 \cdot \bar{r}_z} \mu_l k_1^3 \right) \to 0, \quad (B.53)$$

$$B_{121} \equiv D_1(z) \left[ D_V(\bar{\eta}_o) - \frac{1}{H_z} \right] i \left( e^{ik_1 \cdot \bar{r}_z} \mu_1 k_l^3 - e^{-ik_1 \cdot \bar{r}_z} \mu_l k_1^3 \right) \to 0, \quad (B.54)$$

vanishing in the limit $k_l \to 0$ upon angular integration.

The next term is the line-of-sight velocity contribution to the coupling term at the observer position, and its Fourier kernel is

$$\bullet \quad -\frac{D_1 D_V}{H_z} (\partial_r \mathcal{R}_o) \partial_r (\Delta \mathcal{R}) : \quad F_o(k_1, k_2) = -\frac{D_1(z)D_V(\bar{\eta}_o)}{H_z} \mu_1 \mu_2 k_1 k_2,$$

and its symmetrized kernel is

$$\mathcal{F}_o = -\frac{D_1(z)D_V(\bar{\eta}_o)}{2H_z} \mu_1 \mu_2 k_1 k_2 \left( e^{-ik_1 \cdot \bar{r}_z} \bar{n}_k^2 + e^{-ik_2 \cdot \bar{r}_z} \bar{n}_k^2 \right), \quad (B.56)$$

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The one-point ensemble average and the first connected bispectrum are
\[ \Xi(z) = \frac{D_1(z) D_V(\bar{\eta}_o)}{3H_z} [\sigma_{4,0}(z) - 2\sigma_{4,2}(z)] , \quad B_{112} \equiv \frac{2D_1(z) D_V(\bar{\eta}_o)}{3H_z} k_1^4 [j_0(k_1\bar{r}_z) - 2j_2(k_1\bar{r}_z)] , \] (B.57)
non-vanishing, while the remaining connected bispectra vanish
\[ B_{211} \equiv -\frac{D_1(z) D_V(\bar{\eta}_o)}{H_z} \mu_1 \mu_1 k_1 k_l \left( e^{-ik_1\cdot\bar{r}_z}\hat{n}_{k_1^2} + e^{-ik_2\cdot\bar{r}_z}\hat{n}_{k_1^2} \right) \to 0 , \quad (B.58) \]
\[ B_{121} \equiv \frac{D_1(z) D_V(\bar{\eta}_o)}{H_z} \mu_1 \mu_1 k_1 k_l \left( e^{ik_1\cdot\bar{r}_z}\hat{n}_{k_1^2} + e^{-ik_2\cdot\bar{r}_z}\hat{n}_{k_1^2} \right) \to 0 . \] (B.59)

The other coupling contribution \((\delta \theta \partial_\theta + \delta \phi \partial_\phi)\delta_{im}\) comes from the angular distortions, which includes only the line-of-sight velocity term at the observer position. The Fourier kernel for this term is
\[ \hat{D}_i D^\alpha (\Delta R) \hat{\nabla} \alpha (\Delta R) : \quad F_\alpha(k_1, k_2) = D_1(z) D_V(\bar{\eta}_o) \bar{r}_z (-\mu_1 \mu_2 k_1 k_2 + k_1 \cdot k_2) k_2^2 , \] (B.60)
and its symmetrized kernel is
\[ \mathcal{F}_\alpha = \frac{1}{2} D_1(z) D_V(\bar{\eta}_o) \bar{r}_z (-\mu_1 \mu_2 k_1 k_2 + k_1 \cdot k_2) \left( e^{-ik_1\cdot\bar{r}_z}\hat{n}_{k_2^2} + e^{-ik_2\cdot\bar{r}_z}\hat{n}_{k_2^2} \right) . \] (B.61)
Note that there exist two contributions in total, one from \(\delta \theta\) and one from \(\delta \phi\), which are combined as the angular gradient \(\hat{\nabla}\). The one-point ensemble average and the first connected bispectrum are
\[ \Xi = -2D_1(z) D_V(\bar{\eta}_o) \sigma_{3,1}(z) , \quad B_{112} \equiv -4D_1(z) D_V(\bar{\eta}_o) k_2^2 j_1(k_1\bar{r}_z) , \] (B.62)
non-zero, while the remaining connected bispectra are
\[ B_{211} \equiv D_1(z) \hat{D}_i \bar{r}_z (-\mu_1 \mu_1 k_1 k_l + k_1 \cdot k_l) \left( e^{-ik_1\cdot\bar{r}_z}\hat{n}_{k_l^2} + e^{-ik_2\cdot\bar{r}_z}\hat{n}_{k_l^2} \right) \to 0 , \] (B.63)
\[ B_{121} \equiv D_1(z) \hat{D}_i \bar{r}_z (\mu_1 \mu_1 k_1 k_l - k_1 \cdot k_l) \left( e^{ik_1\cdot\bar{r}_z}\hat{n}_{k_l^2} + e^{-ik_2\cdot\bar{r}_z}\hat{n}_{k_l^2} \right) \to 0 , \] (B.64)
vanishing in the limit \(k_l \to 0\), where we used \(j_1(x) = x(j_0 + j_2)/3\).

One last contribution is from the spatial shift \(\delta x^\alpha\) of the observer position, contained in spatial distortions \(\delta \bar{r}, \delta \theta, \text{and } \delta \phi\). While it contains an integration over time, its spatial position is fixed at the observer position at the perturbation order of our interest, rendering it essentially the same as any other contributions at the observer position in this section. The Fourier kernel for the spatial shift contribution is
\[ \bullet \quad -D_1 \left( \int_0^{\bar{\eta}_o} d\bar{\eta} D_V(\bar{\eta}) \nabla_\alpha (\Delta R) \right) \nabla_\alpha \bar{r}_z : \quad F_\alpha(k_1, k_2) = -D_1(z) \left( \int_0^{\bar{\eta}_o} d\bar{\eta} D_V(\bar{\eta}) \right) (k_1 \cdot k_2) k_2^2 , \] (B.65)
and its symmetrized kernel is
\[ \mathcal{F}_\alpha = -\frac{D_1(z)}{2} \left( \int_0^{\bar{\eta}_o} d\bar{\eta} D_V(\bar{\eta}) \right) (k_1 \cdot k_2) \left( e^{-ik_1\cdot\bar{r}_z}\hat{n}_{k_2^2} + e^{-ik_2\cdot\bar{r}_z}\hat{n}_{k_2^2} \right) , \] (B.66)
where the integration over \(\bar{\eta}\) is along the time coordinate with spatial position fixed (not the line-of-sight integration). The one-point ensemble average and the first connected bispectrum are
\[ \Xi(z) = D_1(z) \sigma_{4,0}(z) \int_0^{\bar{\eta}_o} d\bar{\eta} D_V(\bar{\eta}) , \quad B_{112} \equiv 2D_1(z) \left( \int_0^{\bar{\eta}_o} d\bar{\eta} D_V(\bar{\eta}) \right) k_1^4 j_0(k_1\bar{r}_z) , \] (B.67)
and the remaining connected bispectra are

\[ B_{211} \equiv - D_1(z) \left( \int_0^\eta_0 d\eta D_V(\eta) \right) k_1 \cdot k_l \left( e^{-i k_1 \cdot r_0} k_l^2 + e^{-i k_l \cdot r_0} k_1^2 \right) \rightarrow 0 \, , \quad (B.68) \]

\[ B_{121} \equiv D_1(z) \left( \int_0^\eta_0 d\eta D_V(\eta) \right) k_1 \cdot k_l \left( e^{i k_1 \cdot r_0} k_l^2 + e^{i k_l \cdot r_0} k_1^2 \right) \rightarrow 0 \, . \quad (B.69) \]

In the Einstein-de Sitter universe, the pre-factors are further simplified as

\[ - D_1(z) \left[ D_V(\eta_0) - \frac{1}{H_z} \right] = \frac{\bar{\eta}_z^2}{10} \left( \frac{\eta_0}{5} - \frac{\eta_z}{2} \right) \, , \quad - D_1 \left( \int_0^\eta_0 d\eta D_V(\eta) \right) = - \frac{\bar{\eta}_z^2 \eta_0^2}{100} \, . \quad (B.70) \]

### B.3 Coupling contributions involving the line-of-sight integral

The last type of contributions to the observed matter density fluctuation involves the coupling with contributions from the line-of-sight direction. The well-known components in this category are the contributions of the gravitational lensing and the integrated Sachs-Wolfe effects. The Fourier kernels for such contributions in Eq. (A.49) include not only the line-of-sight integration, but also the exponential factor. There exist four coupling terms arising from the line-of-sight integration.

The first term arises from the coupling term in \( \delta \eta \partial_\mu \delta \), and the integrated Sachs-Wolfe contribution in \( \delta \eta = \delta z/H \) results in the coupling contribution involving the line-of-sight integral. Its Fourier kernel is

\[ \frac{2 D_V(z)}{H_z} \left( \int_0^\bar{\eta} d\bar{\eta} D_\Psi(\bar{\eta}) \partial_\mu R \right) \Delta R : \]

\[ F_{nl}(\bar{r}, k_1, k_2) = - \frac{2 D_V(z)}{H_z} D_\Psi(\bar{r}) i k_1 k_2^3 \, , \quad (B.71) \]

where the first wave vector \( k_1 \) belongs to the contribution along the line-of-sight direction and the second wave vector \( k_2 \) describes the matter fluctuation at the source position. The full and symmetrized Fourier kernel is

\[ F_{nl} = - \frac{D_V(z)}{H_z} \int_0^{\bar{\eta}} d\bar{\eta} D_\Psi(\bar{r}) i k_1 k_2 \left( e^{-i k_1 \cdot r_0} \mu_1 k_2 + e^{-i k_2 \cdot r_0} \mu_2 k_1 \right) \, , \quad (B.72) \]

and its one-point ensemble average is

\[ \Xi(z) = - \frac{2 D_V(z)}{H_z} \int d^3 k \left( \frac{2\pi}{\mu} \right)^3 k^3 P_R(k) \int_0^{\bar{\eta}} d\bar{\eta} D_\Psi(\bar{r}) j_1(k_1 \Delta r) \, , \quad (B.73) \]

where \( \Delta r := r - \bar{r} \). The connected bispectra are then obtained as

\[ B_{112} \equiv - \frac{4 D_V(z)}{H_z} \int_0^{\bar{\eta}} d\bar{\eta} D_\Psi(\bar{r}) k_1^3 j_1(k_1 \Delta r) \, , \quad (B.74) \]

\[ B_{211} \equiv - \frac{2 D_V(z)}{H_z} \int_0^{\bar{\eta}} d\bar{\eta} D_\Psi(\bar{r}) i k_1 k_l \left( e^{-i k_1 \cdot r_0} \mu_1 k_l + e^{-i k_l \cdot r_0} \mu_1 k_1 \right) \rightarrow 0 \, , \quad (B.75) \]

\[ B_{121} \equiv - \frac{2 D_V(z)}{H_z} \int_0^{\bar{\eta}} d\bar{\eta} D_\Psi(\bar{r}) i k_1 k_l \left( - e^{-i k_1 \cdot r_0} \mu_1 k_l + e^{-i k_l \cdot r_0} \mu_1 k_1 \right) \rightarrow 0 \, . \quad (B.76) \]

In the Einstein-de Sitter universe, the gravitational potential is constant in time:

\[ D_\Psi = - \frac{3}{5} \, , \quad (B.77) \]
allowing us to analytically perform the line-of-sight integration by using

\[
k \int_0^{\tilde{r}_s} d\tilde{r} j_1(k\Delta r) = 1 - j_0(k\tilde{r}_s), \qquad \frac{k}{3} \int_0^{\tilde{r}_s} d\tilde{r} \left[ j_0(k\Delta r) - 2j_2(k\Delta r) \right] = j_1(k\tilde{r}_s). \quad (B.78)
\]

The one-point ensemble average and the first connected bispectrum becomes

\[
\Xi(z) = \frac{3\eta^2}{25} [\sigma_2 - \sigma_{2,0}(z)], \qquad B_{112} \equiv \frac{24 k_l^2 [1 - j_0(k_1\tilde{r}_1)]}{25 H_0^2(1+z)}. \quad (B.79)
\]

The second term comes again from the integrated Sachs-Wolfe contribution in \(\delta z\), but through the coupling term \(\delta r \partial_\delta\). Hence the line-of-sight integration is identical to the first term, but with different derivative coupling to the matter density fluctuation. The Fourier kernel for this contribution is

\[
- \frac{2D_1}{H} \left( \int_0^{\tilde{r}_s} d\tilde{r} D_\psi(\tilde{r}) \partial_\tilde{r} R \right) \partial_\tilde{r} (\Delta R) : \quad F_{nl}(\tilde{r}, k_1, k_2) = - \frac{2D_1(z)}{H} D_\psi(\tilde{r}) \mu_1 \mu_2 k_1 k_2^3, \quad (B.80)
\]

and its symmetrized kernel is

\[
\mathcal{F}_{nl} = - \frac{D_1(z)}{H} \int_0^{\tilde{r}_s} d\tilde{r} D_\psi(\tilde{r}) \mu_1 \mu_2 k_1 k_2 \left( e^{-ik_1\Delta r \hat{n} \cdot \hat{k}_2^2} + e^{-ik_2\Delta r \hat{n} \cdot \hat{k}_1^2} \right). \quad (B.81)
\]

The one-point ensemble average is

\[
\Xi(z) = \frac{2D_1(z)}{3H} \int_0^{\tilde{r}_s} d\tilde{r} \int \frac{d^3k}{(2\pi)^3} k^4 P_R(k) \int_0^{\tilde{r}_s} d\tilde{r} D_\psi(\tilde{r}) \left[ j_0(k\tilde{r}) - 2j_2(k\tilde{r}) \right], \quad (B.82)
\]

and the first connected bispectrum is

\[
B_{112} \equiv \frac{4D_1(z)}{3H} \int_0^{\tilde{r}_s} d\tilde{r} D_\psi(\tilde{r}) k_1^4 \left[ j_0(k_1\Delta r) - 2j_2(k_1\Delta r) \right]. \quad (B.83)
\]

Two remaining connected bispectra

\[
B_{211} \equiv \frac{2D_1(z)}{H} \int_0^{\tilde{r}_s} d\tilde{r} D_\psi(\tilde{r}) \mu_1 \mu_1 k_1 k_1 \left( e^{-ik_1\Delta r \hat{n} \cdot \hat{k}_1^2} + e^{-ik_2\Delta r \hat{n} \cdot \hat{k}_1^2} \right) \to 0, \quad (B.84)
\]

\[
B_{121} \equiv \frac{2D_1(z)}{H} \int_0^{\tilde{r}_s} d\tilde{r} D_\psi(\tilde{r}) \mu_1 \mu_1 k_1 k_1 \left( e^{ik_1\Delta r \hat{n} \cdot \hat{k}_1^2} + e^{ik_2\Delta r \hat{n} \cdot \hat{k}_1^2} \right) \to 0, \quad (B.85)
\]

vanish in the limit \(k_1 \to 0\) upon angular integration. In the Einstein-de Sitter universe, the one-point ensemble average and the first connected bispectrum are further simplified as

\[
\Xi = - \frac{3\eta^3}{50} \sigma_{3,1}(z), \qquad B_{112} \equiv \frac{24 k_l^3 j_1(k_1\tilde{r}_1)}{25 H_0^2(1+z)^{3/2}}. \quad (B.86)
\]

The third term arises from the line-of-sight integration of the gravitational potential contribution in the radial distortion \(\delta r\). Its Fourier kernel is

\[
- 2D_1 \left( \int_0^{\tilde{r}_s} d\tilde{r} D_\psi(\tilde{r}) R \right) \partial_\tilde{r} (\Delta R) : \quad F_{nl}(\tilde{r}, k_1, k_2) = 2D_1(z) D_\psi(\tilde{r}) i\mu_2 k_2^3, \quad (B.87)
\]
and its symmetrized kernel is
\[ F_{nl} = D_1(z) \int_0^{\bar{r}_z} \frac{d\bar{r}}{\bar{r}} D_{\Psi}(\bar{r}) \left( e^{-ik_1 \Delta r} \eta_{k_2} e^{ik_2 \Delta r} \eta_{k_1} \right) . \] 
(B.88)

The one-point ensemble average is
\[ \Xi(z) = -2D_1(z) \int_0^{\bar{r}_z} \frac{d\bar{r}}{\bar{r}} D_{\Psi}(\bar{r}) \left( e^{-ik_1 \Delta r} \eta_{k_2} e^{ik_2 \Delta r} \eta_{k_1} \right) \] 
(B.89)
and the first connected bispectrum is
\[ B_{112} \ni -4D_1(z) \int_0^{\bar{r}_z} d\bar{r} D_{\Psi}(\bar{r}) \eta_{k_2} \eta_{k_1} \] 
(B.90)

The remaining connected bispectra are
\[ B_{211} \ni 2D_1(z) \int_0^{\bar{r}_z} d\bar{r} D_{\Psi}(\bar{r}) \left( e^{-ik_1 \Delta r} \eta_{k_2} e^{-ik_2 \Delta r} \eta_{k_1} \right) \to 0 , \] 
(B.91)
\[ B_{121} \ni 2D_1(z) \int_0^{\bar{r}_z} d\bar{r} D_{\Psi}(\bar{r}) \left( e^{ik_1 \Delta r} \eta_{k_2} e^{-ik_2 \Delta r} \eta_{k_1} \right) \to 0 . \] 
(B.92)

In the Einstein-de Sitter universe, we have
\[ \Xi(z) = \frac{3h_2^2}{25} \left[ \sigma_2 - \sigma_{2,0}(z) \right] , \quad B_{112} \ni \frac{24 h_2^2}{25} \int_0^{\bar{r}_z} \eta_{k_2} \eta_{k_1} \] 
(B.93)

The last term comes from the gravitational lensing contribution, which couples to the matter density fluctuation through the angular distortion \((\delta \theta \cdot \partial_{\theta} + \delta \phi \cdot \partial_\phi) \delta_m\). The Fourier kernel for this contribution takes the form
\begin{itemize}
  \item \[ 2D_1 \left( \int_0^{\bar{r}_z} \frac{d\bar{r}}{\bar{r}} D_{\Psi}(\bar{r}) \hat{\nabla}^\alpha \hat{\nabla}^\beta (\Delta R) \right) : 
  \] \[ F_{nl}(\bar{r}, k_1, k_2) = 2D_1(z) \Delta_r D_{\Psi}(\bar{r}) \eta_{k_2} \eta_{k_1} \] 
\end{itemize}
(B.94)
and its symmetrized kernel is
\[ F_{nl} = D_1(z) \int_0^{\bar{r}_z} d\bar{r} D_{\Psi}(\bar{r}) \Delta_r \left( e^{-ik_1 \Delta r} \eta_{k_2} e^{-ik_2 \Delta r} \eta_{k_1} \right) \] 
(B.95)

The one-point ensemble average is
\[ \Xi(z) = -4D_1(z) \int_0^{\bar{r}_z} \frac{d\bar{r}}{\bar{r}} D_{\Psi}(\bar{r}) \left( e^{-ik_1 \Delta r} \eta_{k_2} e^{-ik_2 \Delta r} \eta_{k_1} \right) \] 
(B.96)
and the first connected bispectrum is
\[ B_{112} \ni -8D_1(z) \int_0^{\bar{r}_z} d\bar{r} D_{\Psi}(\bar{r}) \eta_{k_2} \eta_{k_1} \] 
(B.97)
and the remaining connected bispectra are
\[ B_{211} \ni 2D_1(z) \int_0^{\bar{r}_z} d\bar{r} \Delta_r D_{\Psi}(\bar{r}) \left( -\mu_1 \eta_{k_2} \eta_{k_1} + k_1 \cdot k_2 \right) \left( e^{-ik_1 \Delta r} \eta_{k_2} e^{-ik_2 \Delta r} \eta_{k_1} \right) \] 
(B.98)
\[ B_{121} \ni 2D_1(z) \int_0^{\bar{r}_z} d\bar{r} \Delta_r D_{\Psi}(\bar{r}) \left( \mu_1 \eta_{k_2} \eta_{k_1} - k_1 \cdot k_2 \right) \left( e^{ik_1 \Delta r} \eta_{k_2} e^{-ik_2 \Delta r} \eta_{k_1} \right) \to 0 . \] 
(B.99)
In the Einstein-de Sitter universe, the non-vanishing components are further simplified as

\[ \Xi = \frac{6\pi^2}{25} \left[ \sigma_2 - \sigma_{2,0}(z) \right], \quad B_{112} \equiv \frac{48 \, k_1^2 [1 - j_0(k_1 r_z)]}{25 \, H_0^2 (1 + z)}. \]  

(B.100)

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