DEGREE LIMIT THEOREMS FOR P.A RANDOM GRAPHS WITH EDGE-STEPs

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Abstract. In this work we investigate a random graph model that combines preferential attachment and edge insertion between previously existing vertices. The probabilities of adding either a new vertex or a new connection between previously added vertices are time dependent and given by a function \( f \) called the edge-step function. We prove convergence theorems and a CLT for the properly scaled maximum degree, as well as the degree of any given vertex. Our results state that under a summability condition, the maximum degree grows linearly in time and sub-linearly if this condition is dropped. Our condition for linearity is sharp when \( f \) is a regularly varying function at infinity. Moreover, as a byproduct of our analysis, we prove that there exists true competition for the leadership only during a finite number of steps, i.e., after a certain point a single vertex becomes the one with maximum degree and maintains this predominance forever. These results also relate to Pólya urns with immigration. We also explore our knowledge about the maximum degree in order to understand infectious process over the graphs generated by this model. We show that for some choices of the parameters the graphs are highly susceptible to the spread of infections, which requires only 4 steps to infect a positive fraction of the whole graph.

1. Introduction

The random graph theory started with work of P. Erdős and A. Rényi \cite{12} in the 1960’s. The original model generates homogeneous graphs, i.e., graphs whose vertices are statistically indistinguishable. However, in the late 90’s, the capacity of analyzing large amount of data brought to light many characteristics of concrete networks, which were not captured by the graphs generated by the ER-model. The empirical works of S. H. Strogatz and D. J. Watts in \cite{19} and of A-L. Barabási and R. Álbert in \cite{4} showed that real-life networks are not homogeneous. Their works coined the concepts of scale-free (graphs with degree distribution following power-laws), small-worlds (small diameter graphs), hubs (the existence of very high degree vertices) and boost the investigation of non-homogeneous random graph models.

Since then, many non-homogeneous random graph models have been proposed, many of them are generalizations of the Barabási-Álbert model, which proposed the preferential attachment rule as mechanism for breaking vertices homogeneity present in ER-model and variants. The works on the area have been devoted to understand the degree distribution \cite{6}, the asymptotic behavior of the maximum degree \cite{15,16}, the diameter order \cite{21} and other graph observables. We refer the reader to the book of R. van der Hofstad \cite{20} and of R. Durrett \cite{10} for a wide and rigorous introduction to many important random graph models.

Furthermore, since non-homogeneous graphs represent well real-life phenomena, they also become the natural environment to consider some kind of random process, for instance, infectious process like bootstrap percolation. See the recent works of M. Amin Abdullah and N. Fountoulakis \cite{3} for example of infectious process on graphs generated by the preferential
attachment mechanism and how the spread relates to the degree distribution and maximum degree.

It is also noteworthy that the degree analysis on such class of evolving graph models is closely related to generalizations of the classical bin-and-ball model proposed by G. Póya and F. Eggenberger in [11], since the evolution of the degree does not require any information of the graph’s geometry, thus they may be seen as a kind of Póya-Eggenberguer urn scheme. See Section 2 where we discuss how the graph model investigated here relates to recent works on Póya-Eggenberguer urn models.

The first part of this work is devoted to investigate (non)-asymptotic behavior of the maximum degree of a random graph model. In the second part we investigate how the maximum degree impacts on the geometry of the graph. More precisely, we study a bootstrap percolation model over the random graphs. In order to proper state and discuss our results, we will define formally our random graph model.

1.1. The model. The model depends on a real non-negative function $f$, called edge-step function, with domain given by the semi-line $[1, \infty)$ such that $\|f\|_\infty \leq 1$. Without loss of generality, we will start the process from an initial graph $G_1$ which is the graph with one vertex and one loop. As the process evolves, one of the two graph stochastic operations below may be performed on the present graph $G$:

- **Vertex-step** - Add a new vertex $v$ and add an edge $\{u, v\}$ by choosing $u \in G$ with probability proportional to its degree. More formally, conditionally on $G$, the probability of attaching $v$ to $u \in G$ is given by
  \[
  P(v \rightarrow u | G) = \frac{\text{degree}(u)}{\sum_{w \in G} \text{degree}(w)}.
  \]  

- **Edge-step** - Add a new edge $\{u_1, u_2\}$ by independently choosing vertices $u_1, u_2 \in G$ according to the same rule described in the vertex-step. We note that both loops and parallel edges are allowed.

We let $\{Z_t\}_{t \geq 1}$ be independent sequence of random variables such that $Z_t \overset{d}{=} \text{Ber}(f(t))$. We then define a markovian random graph process $\{G_t(f)\}_{t \geq 1}$ as follows: begin with initial state $G_1$. Given $G_t(f)$, obtain $G_{t+1}(f)$ by either performing a vertex-step on $G_t(f)$ when $Z_t = 1$ or performing an edge-step on $G_t(f)$ when $Z_t = 0$.

1.2. Regularity conditions. Again, in order state and discuss properly our results we will need introduce some regularity conditions for the functions $f$. Some of these conditions have the objective of preventing pathological examples, for instance if one drop monotonicity is possible to construct $f$ such that the sequence of graphs $\{G_t(f)\}_{t \in \mathbb{N}}$ has two sub sequences of graphs: one similar to the BA random tree and the other one a quasi-complete graph, see the discussion in Section 8 of [diameter].

Some of our results will require information about the asymptotic behavior of $f$ and for this reason a wide class of functions will play important role: the regularly varying functions. We say that a positive function $f$ is a regularly varying function (r.v.f for short) at infinity with index of regular variation $-\gamma$, for some $\gamma > 0$, if

\[
\lim_{t \to \infty} \frac{f(at)}{f(t)} = a^{-\gamma},
\]

for all $a > 0$. The special case $\gamma = 0$ is called slowly varying function. Below we define the conditions over $f$ that will be useful to our purposes. For $p \in [0, 1]$, we define two conditions

\[
\text{(Cons}_p) \quad f \equiv p;
\]

\[
\text{(D}_p) \quad f \text{ decreases to } p.
\]
The two summability conditions below will be important to us

$$\sum_{s=1}^{\infty} f(s) = \infty. \quad (V_\infty)$$

The above condition relates to expected number of vertices. Since at each step $s$ we add a vertex with probability $f(s)$, the above condition tells us that the process keeps introducing new vertices into the graph.

$$\sum_{s=1}^{\infty} \frac{f(s)}{s} < \infty. \quad (S)$$

Whenever $(S)$ does not hold for a specific $f$, we say that $f$ satisfies $(S)^c$. For any $\gamma \in [0, 1]$, we let $\text{RES}(-\gamma)$ be the following class of functions

$$\text{RES}(-\gamma) := \{ f : [1, \infty] \rightarrow [0, 1] | f \text{ satisfies } (D)_0 \text{ and is r.v.f with index } -\gamma \}. \quad (\text{RES})$$

1.3. **Main results.** In this paper we investigate the evolution of degree of a fixed vertex and of the maximum degree. We prove non-asymptotic estimates and limit theorems: almost surely convergence and CLT. We also study the implications of the emergence of vertices with extremely high degree in the geometry of the graphs and in the spread of infectious over them.

1.3.1. **Limit Theorems for the Maximum Degree.** Beyond the degree distribution, the maximum degree has been studied in the context of non-homogeneous random graphs. In [15], Y. Malyshkin and E. Paquette prove almost surely convergence of a evolving tree model which combines the PA-rule with local choice. Their result shows that by some choices of the parameter the maximum degree is linear in $t$. This contrasts with the results of T. Móri, [16], for the affine version of the BA-model, which shows that scaled by $t^{1/(2+\beta)}$, where $\beta$ is the attractiveness constant, the maximum degree converges to a strictly positive random variable $\mu$.

In our settings, the right order of the maximum degree depends on $f$ and it is related to the summability condition $(S)$. Our results state that $(S)$ is a sharp condition for linearity, i.e., the maximum degree is of linear order if and only if $f$ satisfies $(S)$. The dependence of $f$ is expressed in the normalizing factor $\phi$ defined below:

$$\phi(t) = \phi(t, f) := \frac{t}{\prod_{s=1}^{t-1} \left( 1 + \frac{1}{s} - \frac{f(s+1)}{2s} \right)}.$$  \hspace{1cm} (2)

Then, our limit theorems regarding the max. degree are the following ones

**Theorem 1.1** (Convergence of the Maximum degree). Let $f$ be an edge-step function such that either $f$ belongs to $\text{RES}(-\gamma)$ with $\gamma \in [0, 1]$, or $f$ is identically equal to $p \in (0, 1]$. Then for every $q \geq 1$, there exists a finite random variable $\mu$ such that

$$\lim_{t \to \infty} \frac{\text{maxdegree} \{ G_t(f) \}}{\phi(t)} = \mu, \text{ P-a.s. and in } L_q(\mathbb{P}).$$ \hspace{1cm} (3)

Furthermore, $\mu$ is P-a.s. strictly positive.

The limit distribution $\mu$ assumes a specific form: it is the supremum of a sequence of distributions $\zeta_i$, where $\zeta_i$ is the limit of the properly scaled degree of the $i$-th vertex added by the process.

Our next result states that condition $(S)$ is a *sharp* condition in order to obtain a maximum degree which grows linearly in time:

**Corollary 1.2** (Sharp condition for linear max. degree). Let $f$ be an edge-step function in $\text{RES}(-\gamma)$, with $\gamma \in [0, 1]$. Then, there exists a strictly positive random variable $\nu$ such that

$$\lim_{t \to \infty} \frac{\text{maxdegree} \{ G_t(f) \}}{t} = \nu, \text{ P-a.s.}$$

if and only if $f$ satisfies condition $(S)$.

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Finally, we also prove a Central Limit Theorem for the maximum degree when \( f \) is a decreasing regularly varying function at infinity, here we use the notation \( X_n \implies X \) for convergence in distribution.

**Theorem 1.3** (CLT for the maximum degree). Let \( f \) be an edge-step function such that either \( f \) belongs to \( \text{RES}(-\gamma) \) with \( \gamma \in [0, 1) \) or \( f \) is constant and equal to \( p \in (0, 1] \). Denoting by \( N \) a normal random variable, independent from \( \mu \), with expectation 0 and variance 1, we have

(a) if \((S)^c\) holds, then

\[
\sqrt{\phi(s)} \left( \frac{\max\{G_t(f)\}}{\phi(t)} - \mu \right) \implies \sqrt{\pi} N;
\]

(b) if \((S)\) holds, then, denoting

\[
\xi(\infty) = \lim_{s \to \infty} \phi(s)s^{-1},
\]

we have

\[
\sqrt{\phi(s)} \left( \frac{\max\{G_t(f)\}}{\phi(t)} - \mu \right) \implies \sqrt{\mu \left( 1 - \frac{\xi(\infty)}{2} \right)} N.
\]

We note that (1.3) also applies when the hypotheses of (1.3) are assumed, since in that case \( \xi(\infty) = 0 \). We chose to state the Theorem above separating in the different cases in order to highlight the effect that the convergence of \( \sum_{s \geq 1} f(s)/s \) has on the result.

1.3.2. The Persistent Leadership. As part of our analysis, we investigate the competition for degree. Our result shows that true competition holds only for a finite number of steps and eventually a fixed vertex takes the leadership over becoming the only one with maximum degree.

**Theorem 1.4** (The Persistent Leadership). Consider a edge-step function \( f \) such that either \( f \) belongs to \( \text{RES}(-\gamma) \) with \( \gamma \in [0, 1) \) or \( f \) is constant and equal to \( p \in (0, 1] \). Denote by \( U_{\max}(s, N) \) the event where at time \( s \) there exists a unique vertex with maximum degree and all other vertices have degree smaller than the maximum minus \( N \in \mathbb{N} \). Then \( U_{\max}(s, N) \) occurs for all but a finite number of integers \( s \) almost surely.

Besides being a key step in the proof of Theorem 1.3, the above theorem is interesting by itself, since it also states that once a vertex becomes the leader, it becomes greater than its adversaries “by far”, i.e., the difference between the leader’s degree and the second place is greater than large constants after certain point.

1.3.3. Application: Bootstrap Percolation. As an application of our results regarding the degree, we analyze the Bootstrap Percolation model over the graphs generated by \( f \in \text{RES}(-\gamma) \), with \( \gamma \in [0, 1) \) and under condition \((S)\). Our estimates allow us to construct structures on the graph \( G_t(f) \) that make it highly susceptible to the spread of infections.

More rigorously, given a finite graph \( G = (V(G), E) \), a number \( a \in [0, |V(G)|] \), and an integer \( r \geq 2 \), we define the bootstrap percolation measure \( Q_{G,a,r} \) on \( G \) with threshold \( r \) and rate of infection \( a \) in the following manner:

- Each vertex \( v \in V(G) \) is infected at round 0 independently of the others with probability \( a|V(G)|^{-1} \). The collection of all infected vertices at round 0 is denoted by \( I_0 \).
- At round \( s \in N \), every vertex sending at least \( r \) edges to some vertex( or vertices) in \( I_{s-1} \) becomes infected.
- We let \( I_\infty \) be the set of all infected vertices when the process stabilizes, that is,

\[
I_\infty = \cup_{s \geq 0} I_s.
\]

In [3] the authors study the bootstrap percolation process on the preferential attachment random graph where each vertex has \( m \in \mathbb{N} \) outgoing edges with end vertices chosen according to an affine preferential attachment rule, that is, the PA rule in (1) but with a \( \delta > -m \) summed in both the numerator and denominator. There they prove the existence of a critical function \( \alpha_\gamma^f : \mathbb{N} \to \mathbb{R}_+ \)
such that the bootstrap percolation process on this random graph at time $t$ with threshold $r \leq m$ and rate $a_t \gg a_t^c$ infects the whole graph with high probability, but the same process with rate $a_t' \ll a_t^c$ dies out without before infecting a positive proportion of the graph, also with high probability.

In our context it is not possible for the infection to spread to the whole graph due to the existence of many vertices with degree smaller than $r$ at all times. We therefore turn the problem into asking if $\mathcal{I}_\infty$ eventually encompasses a set with positive density in the vertex set.

In this sense, we show in our next result that the bootstrap percolation is always supercritical for the graph $G_t(f)$, with $f$ under (S), i.e., every unbounded rate sequence gives rise to set of infected vertices with positive density:

**Theorem 1.5** (The outbreak phenomenon). Let $f$ be an edge-step function in $\text{RES}(-\gamma)$, with $\gamma \in [0,1]$, and satisfying summability condition (S). Then, for any sequence $(a_t)_{t \in \mathbb{N}}$ increasing to infinity and all $t$ sufficiently large, there exists a collection of graphs $G_t$ and a $f$ dependent constant $C > 0$ such that,

$$P\left(G_t(f) \in G_t\right) = 1 - o(1)$$

and for all $G \in G_t$ the bootstrap percolation process on $G$ with parameters $a_t$ and $r \geq 2$ satisfies

$$\mathbb{Q}_{G,a_t,r}(|\mathcal{I}_\infty| \geq C|V(G)|) = 1 - o(1).$$

The above theorem illustrates how interconnected $G_t(f)$ is. From the definition of the bootstrap percolation on $G_t(f)$ follows that the initial infected set $\mathcal{I}_0$ is essentially $a_t$. Thus, by Theorem 1.5 one can choose $a_t$ of order arbitrarily smaller than the expected number of vertices in $G_t(f)$, as long as it increases to infinity, even so the infection manages to spread to a positive fraction of the whole graph, with high probability.

1.4. **Notation and conventions.**

1.4.1. **Graph theory.** We will use integer letters like $i$ and $j$ to denote the $i$-th and $j$-th vertices added by the process. And will denote their degrees in $G_t(f)$ by

$$d_t(i) := \text{degree at time } t \text{ of vertex } i.$$  

It will be important to understand the evolution of increment process, so we define the following notation

$$\Delta x_t := x_{t+1} - x_t.$$  

For $r \in \mathbb{R}$, we also let $V_r(f)$ denote the number of vertices of the process up to time $\lfloor r \rfloor$, that is,

$$V_r(f) := |V(G_{\lfloor r \rfloor}(f))|.$$  

Note that $V_r(f)$ is the sum of $\lfloor r \rfloor$ independent random variables.

1.4.2. **Special functions and asymptotics.** We define the functions $F : \mathbb{R} \to \mathbb{R}$ and $F^{-1} : \mathbb{R} \to \mathbb{R}$ by

$$F(r) := \sum_{1 \leq s \leq r} f(s); \quad F^{-1}(r) := \inf_{s \in \mathbb{R}}\{F(s) \geq r\}.\quad (8)$$

Notice that $\mathbb{E}[V_r(f)] = F(r)$, and since we are assuming $f(s)$ to be strictly positive for all $s$, and since $\|f\|_\infty \leq 1$, we have that

$$r \leq F(F^{-1}(r)) \leq r + 1, \quad \text{and} \quad r \leq F^{-1}(F(r)) \leq r + 1.\quad (9)$$

We will make use of the Bachman-Landau notation $o/O/\omega$, which in general assumes some asymptotic in time $t$. When the asymptotic is on another parameter we will indicate it on the sub index, for instance $o_n(1)$.

We will write $a_t \sim b_t$ if $a_t/b_t$ converges to 1 as $t$ goes to infinity. The arrows $a_t \downarrow a$ and $a_t \searrow a$ will mean that $(a_t)_t$ decreases to $a$ as $t$ goes to infinity.

1.4.3. **General notation.** Throughout the text $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots$ will stand for the natural filtration, i.e., $\mathcal{F}_t$ contains all the information about the process up to time $t$. 

1.5. **Organization of the paper.** In Section 2 we discuss the main ideas behind the proof of the CLT for the maximum degree and how the analysis of the degree of a vertex relates to new kinds of Pólya-Eggenberger urn models. In Section 3 we prove useful non-asymptotic bounds for the degree of a vertex and the maximum degree, as well. We then move to Section 4 where we prove convergence theorems for the scaled degree and maximum degree. We also prove some results for the limit distribution. Section 5 is crucial for the proof of the CLT for the maximum degree. In this section we study how evolves the competition for degree. We prove that only one vertex becomes the one with maximum degree. In Section 6 we prove CLT using the results developed in the previous section. Finally, in Section 7 we investigate the bootstrap percolation process over the random graphs.

2. **Generalized Pólya Urn Models and Technical Ideas**

In order to prove convergence theorems for the properly scaled maximum degree an intermediate step is to prove convergence theorems for the degree of a fixed vertex. In this intermediate step the convergence of the scaled degree comes from martingale results, i.e., we prove that degree properly scaled is a martingale. The convergence in $L_q$ is proven by looking to a family of martingales involving the degree of a vertex which gives moments bounds. The proofs of this intermediate steps follow the arguments present in [16], however, the presence of the edge-step and the degree of generality we work on let our expressions more involved, preventing a straightforward application of the results of T. Móri. The convergence of the maximum degree properly scaled is then obtained using the moment bounds and the fact that being the maximum of martingales, it is a submartingale.

The proof for the Central Limit Theorem for the normalized maximum degree is more involved in our settings. Again an intermediate step is to prove CLT for the properly scaled degree of a fixed vertex. And again, the edge step makes the computations complicated and a particular analysis is required. Once we have the CLT for the degree of a fixed vertex, the key step is to prove the persistent leadership, i.e., after a certain random time (which is not a stopping time) the maximum degree behaves like the degree of a single vertex and then on this event the analysis done for the degree of a vertex may be generalized to the maximum degree.

The proof of the persistent leadership makes clear the difference between the model with edge-step functions and the other preferential attachment ones in which a single vertex of degree one is added every step. In the latter case the persistent leadership may be obtained applying classical results from the Pólya-Eggenberger theory. Suppose one desires to keep track on the evolution of the degree of a fixed vertex $j$. Then, its possible to ignore the graph geometry entirely and see the degree from a balls-and-bins perspective: see the degree of $j$ as the red balls and the degree of the remaining vertices as the blue ones. Or, if one just wants to compare the degree of fixed vertices, consider the evolution of the degrees only at times one of them has been chosen (observe that in this case only and only can be chosen) then we obtain a Pólya urn scheme and by classical results it follows that the proportion of red/blue balls converges to an absolute continuous distribution, thus they cannot be equal.

In our case, the urn schemes that emerge are somewhat related to the recent work of E. Pekşis, A. Röllin and N. Ross in [17] which accommodates the possibility of immigration (as they have called) at random times, i.e., at random times we simply add some number of blue balls to the urn. More precisely, the balls-and-bins approach for the degree of a fixed vertex yields to the following balls and bills model. We begin with a bin containing $r$ red balls and $b$ blue balls. Then, the process evolves from time $t$ to $t + 1$ in the following way:

- **Immigration step:** Perform a regular Pólya-Eggenberger urn step and add a new blue ball;
- **Non-immigration step:** Perform two regular Pólya-Eggenberger urn steps independently.

Then, with probability $f(t+1)$ we perform an immigration step and with probability $1 - f(t+1)$ we perform a non-immigration step. This way, the proportion of red balls is the degree of $j$
divided by the total number of edges. But different from [17], in our case the time increment between the addition of a foreign blue ball and another one is not identically distributed.

Just to point out, our results show that for $f$ under condition (S) the proportion of red balls converges (a.s. and in $L_q$) to a strictly positive random variable $ζ_j$. We also prove that there is no dominance, i.e., the probability of $ζ_j$ being equal 1 is zero. On the other hand, if $f$ does not satisfy (S), there is dominance of the blue ones. Thus, in this case the question is the right order of magnitude of number of red balls.

Thus, in order to prove the persistent leadership (Theorem 1.4), which is crucial for the CLT for maximum degree, we first prove Lemma 5.1 which is a Law of the Iterated Logarithm for order of magnitude of number of red balls.

And this Lemma is crucial to prove that eventually there exists only one vertex having the maximum degree.

Finally, since our analysis require information on the asymptotic behavior of $f$, part of our results relies on theorems of the Karamata’s Theory.

3. Bounds for the degrees

In this section we prove useful bounds for the degree of a fixed vertex. Since the $i$-th is added at a random time, it will be needed to control this random time. We define the variable $τ(i)$

$$τ(i) := \inf\{ s ≥ 1 : V_s(f) = i \},$$

i.e., $τ(i)$ is the time the $i$-th vertex was added by the process. Recall that the increment of the degree of a vertex at each step may be one or two. In the first case we may either add a new vertex to the graph and connect it to $i$, or we may add a new edge and choose $i$ to be one of its endvertices. In the later case, we add a new edge to the graph and choose $i$ twice, adding a loop to it. This yields the formula, on the event where $s > τ_i$,

$$\mathbb{E}[\Delta d_s(i)|F_s] = f(s + 1) \frac{d_s(i)}{2s} + 2(1 - f(s + 1)) \left(1 - \frac{d_s(i)}{2s}\right) + 2(1 - f(s + 1)) \left(\frac{d_s(i)}{2s}\right)^2,$$

which in turn implies

$$\mathbb{E}[d_{s+1}(i) \mathbb{1}\{τ(i) = t_i\}|F_s] = \left(1 + \frac{1}{s} - \frac{f(s + 1)}{2s}\right) d_s(i) \mathbb{1}\{τ(i) = t_i\}. \quad (11)$$

We let $d_s(i) ≡ 0$ in the event where $s < τ_i$. We then have that

$$X_s = X_{s,i,t_i} := \frac{d_s(i)}{φ(s)} \mathbb{1}\{τ(i) = t_i\} \quad (12)$$

is a martingale for $s ≥ t_i$ with expectation $φ(t_i)^{-1} \cdot P[τ(i) = t_i]$. We now apply Freedman’s inequality (Theorem 3.2) to a related martingale in order to obtain an upper bound for the degree that holds with very high probability.
Lemma 3.1. Let $f$ be an edge-step function such that $f(t) \searrow 0$ as $t$ goes to infinity. Then there exists a $f$-dependent constant $C_f > 0$ such that, for every $\alpha > 1$,

$$
P \left( \exists s \in \mathbb{N} \text{ such that } d_s(i) \geq \frac{\phi(s)}{\phi(t_i)} \phi(t_i) \right) \leq \exp \left( -C_f \cdot \alpha \right).$$

(13)

Proof. As stated before, we use Freedman’s inequality. Given some real number $\lambda > \phi(t_i)^{-1}$, we define the stopping time

$$
\eta = \eta(\lambda) := \inf_{s \in \mathbb{N}} \{ X_s \geq \lambda \}.
$$

Consider the measure $\mathbb{P}_{i,t_i}$, the probability measure associated with the process conditioned on the event $\{ \tau(i) = t_i \}$, and let $\mathbb{E}_{i,t_i}$ be the associate expectation. Under this measure we will bound the increment and the sum of the conditional expectation of the squares of the increments of $X_{\omega | \eta}$, which continues to be a martingale under $\mathbb{P}_{i,t_i}$. We have, by (11), on the event $\{ \tau(i) = t_i \}$,

$$
|\Delta X_{\omega | \eta}| \geq \mathbb{1} \{ \lambda > s \} \left| \frac{d_{s+1}(i)}{\phi(s+1)} - \frac{d_s(i)}{\phi(s)} \right|.
$$

(14)

Furthermore, since for any $a, b \in \mathbb{R}$ it holds that $(a + b)^2 \leq 2a^2 + 2b^2$, and since

$$
\Delta d_s(i) \leq 2 \cdot \mathbb{1} \{ \text{the degree of } i \text{ is increases at time } s \},
$$

we have

$$
\mathbb{E}_{i,t_i} \left[ \Delta X_{\omega | \eta}^2 | \mathcal{F}_s \right] \leq \mathbb{1} \{ \lambda > s \} \cdot \mathbb{E}_{i,t_i} \left[ \frac{8 \Delta d_s(i)}{\phi(s+1)^2} + \frac{d_s(i)^2}{s^2 \phi(s+1)^2} | \mathcal{F}_s \right] \leq \mathbb{1} \{ \lambda > s \} \cdot \left( \frac{8d_s(i)}{s \phi(s+1)^2} + \frac{d_s(i)^2}{s^2 \phi(s+1)^2} \right)
$$

(15)

Consider the function $\xi : \mathbb{N} \rightarrow \mathbb{R}$ is defined by

$$
\xi(s) := \frac{\phi(s)}{s} = \prod_{r=1}^{s-1} \left( 1 + \frac{1}{r} - \frac{f(r+1)}{2r} \right) = \prod_{r=1}^{s-1} \left( 1 - \frac{f(r+1)}{2(r+1)} \right).
$$

(16)

We may write $\xi$ as

$$
\xi(s) = \exp \left\{ \sum_{r=1}^{s-1} \log \left( 1 - \frac{f(r+1)}{2(r+1)} \right) \right\} = \exp \left\{ \sum_{r=1}^{s-1} \left( -\frac{f(r+1)}{2(r+1)} + O(r^{-2}) \right) \right\}.
$$

(17)
Therefore \( \xi \) is a slowly varying function as long as \( f(t) \) goes to zero as \( t \) goes to infinity, since for any \( a \in \mathbb{R}_+ \),

\[
\frac{\xi(as)}{\xi(s)} = \exp \left\{ \sum_{r=|as|}^{s-1} \left( \frac{f(r+1)}{2(r+1)} + O(r^{-2}) \right) \right\} \\
\leq \exp \left\{ C \cdot as^{-1} + \frac{\inf_{r \geq as} f(r)}{2} (\log s - \log as) \right\} \\
\leq C \exp \left\{ -\frac{\inf_{r \geq as} f(r)}{2} (\log a) \right\} \\
as \to \infty \to 1.
\]

This fact together with Karamata’s Theorem (B.3) (Proposition 1.5.8 of [5]) implies

\[
\sum_{s=1}^{t-1} \mathbb{E}_{i,t_i} \left[ \Delta X_{s \wedge \eta}^2 | \mathcal{F}_s \right] \leq \sum_{s \geq t_i} \mathbb{E}_{i,t_i} \left[ \Delta X_{s \wedge \eta}^2 | \mathcal{F}_s \right] \leq C \sum_{s \geq t_i} \frac{\lambda}{s^2 \xi(s)} \leq C \frac{\lambda}{t_i \xi(t_i)} = C \frac{\lambda}{\phi(t_i)},
\]

where the constant above is \( f \)-dependent. Since \( s \wedge \eta \) is a finite stopping time, by the Optional Stopping Theorem we have \( \mathbb{E}[X_{s \wedge \eta}] = \mathbb{E}[X_{t_i}] = \phi(t_i)^{-1} \). Applying Freedman’s inequality and using the above bound together with (14), we obtain, for any \( A > 0 \),

\[
\mathbb{P}_{i,t_i} \left( X_{s \wedge \eta} - \frac{1}{\phi(t_i)} \geq A \right) \leq \exp \left\{ \frac{-A^2}{C \lambda + 2A} \right\} \leq \exp \left\{ \frac{-A^2 \phi(t_i)}{C \lambda + 2A} \right\}.
\]

Letting now \( A = \lambda - \phi(t_i)^{-1} \), recalling that \( \lambda > \phi(t_i)^{-1} \), we obtain, setting \( \alpha = \lambda \phi(t_i) \),

\[
\mathbb{P}_{i,t_i} \left( \exists s \in \mathbb{N} \text{ such that } d_s(i) \geq \frac{\phi(s)}{\phi(t_i)} \right) \leq \mathbb{P}_{i,t_i} (\eta < \infty) = \lim_{s \to \infty} \mathbb{P}_{i,t_i} (\eta \leq s) = \lim_{s \to \infty} \mathbb{P}_{i,t_i} \left( X_{s \wedge \eta} \geq \frac{\alpha}{\phi(t_i)} \right) \leq \exp \{-C \alpha\},
\]

finishing the proof of the result.

We now work in the direction of obtaining an analogous bound to the above result without the dependence on \( t_i \). The idea is elementary: the variable \( \tau(i) \) concentrates strongly around its mean, so paying a small price we may ignore all possible times \( t_i \) that are far away from said mean. Our next result uses this fact in order to obtain a lower bound for \( \tau_i \) with high probability.

**Lemma 3.2.** Consider \( f \) either in \( \operatorname{RES}(\gamma) \) with \( \gamma \in [0,1) \) or identically equal to \( p \in (0,1] \). Then there exists a constants \( C_f > 0 \) such that for \( i \in \mathbb{N} \)

\[
\mathbb{P} \left( \tau(i) \leq \frac{F^{-1}(i)}{2} \right) \leq C_f \exp \{-C_f \cdot i\}.
\]

**Proof.** If \( f \equiv p \), then the result follows by an elementary Bernstein bound. Assume then that \( f \in \operatorname{RES}(\gamma) \). By Corollary [1.2] we know that \( f(t) = \ell(t)^{-\gamma} \) with \( \ell \) being a slowly varying function. Theorem [1.3] implies that as \( r \to \infty \), using the monotonicity of \( f \) to approximate the integral by the sum,

\[
F(r/2) = \sum_{s=1}^{[r/2]} f(s) = \frac{1}{2} \sum_{s=1}^{r} \ell(s)r^{-\gamma} \sim \frac{1}{1-\gamma} \ell(r/2)^{r(1-\gamma)/2} \sim \frac{F(r)}{2^{r(1-\gamma)}},
\]
where we also used the fact that \( \ell \) is slowly varying to obtain \( \ell(r/2) \sim \ell(r) \). Letting

\[
i_*= \frac{F^{-1}(i)}{2}
\]

and recalling that \( V_*(f) \) is a sum of independent random variables, we may use the above equation, \( [7] \), and an elementary Chernoff bound to obtain

\[
P \left( \tau(i) \leq \frac{F^{-1}(i)}{2} \right) = P \left( V_*(f) \geq i \right)
\]

\[
= P \left( V_*(f) \geq \left( 1 + \left( \frac{F^{-1}(i)}{2} \right)^{-1} \right) \right) \cdot \mathbb{E} \left[ V_*(f) \right]
\]

\[
\leq \exp \left\{ -\frac{1}{3} \left( \frac{F^{-1}(i)}{2} \right)^{-1} - 1 \right\} F \left( \frac{F^{-1}(i)}{2} \right)
\]

\[
\leq \exp \left\{ -\frac{1}{3} \left( (3/2)^{1-\gamma} - 1 \right) \frac{i}{2^{1-\gamma}} \right\},
\]

for sufficiently large \( i \). By adjusting the constant we can then show \( (22) \) for every \( i \geq 1 \). \( \square \)

We can now prove the next result:

**Theorem 3.3.** Consider \( f \) either in \( \text{RES}(-\gamma) \) with \( \gamma \in [0,1) \) or identically equal to \( p \in (0,1) \). Then there exists a constant \( C_f > 0 \) such that, for every \( \alpha > 1 \),

\[
P \left( \exists s \in \mathbb{N} \text{ such that } d_s(i) \geq \alpha \frac{\phi(s)}{\phi(F^{-1}(i))} \right) \leq C_f^{-1} \left( \exp \left\{ -C_f \cdot i \right\} + \exp \left\{ -C_f \cdot \alpha \right\} \right). \tag{25}
\]

**Proof.** We first assume that \( f \in \text{RES}(\gamma) \).

\[
\frac{\phi \left( \frac{F^{-1}(i)}{2} \right)}{\phi \left( F^{-1}(i) \right)} = \prod_{s=F^{-1}(i)}^{F^{-1}(i)-1} \left( 1 + \frac{1}{s} - \frac{f(s+1)}{2s} \right)^{-1} \geq Ce^{-\log 2} > 0 \tag{26}
\]

uniformly in \( i \). By monotonicity and the union bound we obtain

\[
P \left( \exists s \in \mathbb{N} \text{ such that } d_s(i) \geq \alpha \frac{\phi(s)}{\phi(F^{-1}(i))} \right)
\]

\[
\leq P \left( \tau(i) \leq \frac{F^{-1}(i)}{2} \right) \tag{27}
\]

\[
+ P \left( \exists s \in \mathbb{N} \text{ such that } d_s(i) \geq \alpha \frac{\phi \left( \frac{F^{-1}(i)}{2} \right)}{\phi \left( F^{-1}(i) \right)} \frac{\phi(s)}{\phi \left( \frac{F^{-1}(i)}{2} \right)} \mid \tau(i) = \frac{F^{-1}(i)}{2} \right),
\]

The result for \( f \in \text{RES}(\gamma) \) then follows after using lemmas \( 3.1 \) and \( 3.2 \) taking the smallest among the constants there defined and absorbing into it the lower bound in \( (26) \). For \( f \equiv p \), Theorem 3 of \( [1] \) already implies the result. \( \square \)

**Remark 3.4.** We note that the bound in \( (25) \) is sharp in the sense that the order of \( \mathbb{E}[d_s(i)] \) is already \( \phi(s)/\phi(F^{-1}(i)) \).

We end the section with results on lower bounds for the degree. This type of result cannot be as strong as the upper bounds discussed above, since we may pay a relatively small price in order to make a vertex \( i \) behave like it was born much after its mean appearance time \( \mathbb{E}[\tau(i)] \).

We can however obtain stronger results when dealing with the total degree of a collection of vertices. This in turn gives us good lower bounds for the degree of some random vertex, by the pigeonhole Principle. This is the subject of our next result.
Theorem 3.5. Let $f$ be an edge-step function such that either $f(t)$ goes to zero as $t$ goes to infinity or $f(t) \equiv p \in (0,1]$. Then, for every $N \in \mathbb{N}$, there exists $C_f > 0$ depending on $f$ only such that

$$
P \left( \forall t \in \mathbb{N}, \exists i \in \{1, 2, \cdots, N\} \text{ such that } d_t(i) \geq \frac{\phi(t)}{\phi(N)} \right) \geq 1 - \exp\{-C_f N\}. \tag{28}$$

Furthermore, there exists almost surely a random integer $N_0 \geq 0$ such that for every $t \geq N_0$, there exists at least one vertex $i$ in $G_t(f)$ such that

$$d_t(i) \geq \frac{\phi(t)}{\phi(N_0)}. \tag{29}$$

Proof. Fix $N \in \mathbb{N}$ and denote by $\{\tilde{X}_{N,s}\}_{s \geq N}$ the process

$$\tilde{X}_{N,s} := \sum_{i=1}^{N} d_s(i) \mathbb{1}\{\tau(i) \leq N\}, \tag{30}$$

i.e., $\tilde{X}_{N,s}$ denotes the sum of the degree of all vertices added by the process up to time $N$ normalized by $\phi(s)$. By an argument analogous to the one following equation (11), it follows that $\{\tilde{X}_{N,s}\}_{s \geq N}$ is a positive martingale such that

$$\mathbb{E} \tilde{X}_{N,s} = \tilde{X}_{N,N} \equiv \frac{2N}{\phi(N)}, \tag{31}$$

since at time $N$ the sum of the degree of all vertices added up to this time equals twice the number of edges, which is $N$. Moreover, since we can update the degree of at most two vertices by an amount of at most 2, it follows that $\{\tilde{X}_{N,s}\}_{s \geq N}$ also has bounded increments. More precisely, proceeding as in equation (11) we obtain

$$\left| \Delta \tilde{X}_{N,s} \right| \leq \frac{3}{\phi(s + 1)}. \tag{32}$$

If $f(t)$ goes to zero as $t$ goes to infinity, then, by a similar argument as in (12), using the fact that $\xi^{-2}$ is also a slowly varying function, we obtain

$$\sum_{s=N}^{t} \left| \Delta \tilde{X}_{N,s} \right|^2 \leq C \sum_{s=N}^{t} s^{-2} \xi(s)^{-2} \leq \frac{C}{N \xi(N)^2} = \frac{C}{\phi(N) \xi(N)}. \tag{33}$$

If $f(t) \equiv p \in (0,1]$, then, by (14),

$$\xi(s) = \exp \left\{ \sum_{r=1}^{s-1} \left( -\frac{f(r+1)}{2(r+1)} + O(r^{-2}) \right) \right\} \geq Cs^{-p/2}. \tag{34}$$

Therefore, in this case,

$$\sum_{s=N}^{t} \left| \Delta \tilde{X}_{N,s} \right|^2 \leq C \sum_{s=N}^{t} s^{-2+p} \leq \frac{C}{N^{1-p}}. \tag{34}$$

Now we proceed as in the proof of the upper bound, this time stopping our martingale when it becomes unexpectedly small. For this, let $\eta$ be the stopping time below

$$\eta := \inf_{s \geq N} \left\{ \tilde{X}_{N,s} \leq \frac{N}{\phi(N)} \right\}. \tag{35}$$

Again by the Optional Stopping Theorem, it follows that $\{\tilde{X}_{N,s \wedge \eta}\}_{s \geq N}$ is a martingale having the same expected value as $\{\tilde{X}_{N,s}\}_{s \geq N}$. Thus, by Azuma’s inequality, we obtain

(i) if $\lim_{t \to \infty} f(t) = 0,$

$$\mathbb{P} \left( \tilde{X}_{N,s \wedge \eta} \leq \frac{2N}{\phi(N)} - \frac{N}{\phi(N)} \right) \leq \exp \left\{-C \frac{N^2}{\phi(N)^2} \phi(N) \xi(N) \right\} \leq \exp \{-CN\}; \tag{36}$$
(ii) if \( f(t) \equiv p \),
\[
\mathbb{P}\left( \tilde{X}_{N,s} \leq \frac{2N}{\phi(N)} - \frac{N}{\phi(N)} \right) \leq \exp\left\{ -C \frac{N^2}{(N^{1-p}/2)^2} N^{1-p} \right\} \leq \exp\left\{ -CN \right\}.
\] (37)

Using the above inequalities, the definition of \( \eta \), and arguing as in \([24]\), we obtain
\[
\mathbb{P}\left( \exists t \geq N, \tilde{X}_{N,t} \leq \frac{N}{\phi(N)} \right) = \mathbb{P}(\eta < \infty) \leq \exp\left\{ -CN \right\}.
\] (38)

By the pigeonhole Principle the above inequality implies \([25]\) since
\[
\tilde{X}_{N,t} > \frac{N}{\phi(N)} \iff \sum_{i=1}^{N} d_i(i) \mathbb{I}\{\tau(i) \leq N\} > \frac{N\phi(t)}{\phi(N)},
\] (39)

implying the existence of at least one vertex with degree \( \phi(t)/\phi(N) \) among the \( N \) first vertices. The remainder of the Theorem follows from \([38]\) together with the first Borel-Cantelli Lemma. \( \square \)

4. Convergence of the normalized degree

In Section \([3]\) it was important to define the process \( d_s(i)\phi(s)^{-1} \), which is a martingale when the appropriate measure is considered. We did so in order to apply martingale concentration inequalities, but there is still a lot of information that can be extracted from this process. In this section we focus on moment bounds and convergence theorems for this martingale. We will again use the measure \( \mathbb{P}_{i,k} \). The ideas in this section were inspired by \([10]\).

Given \( i, t, k \in \mathbb{N} \), define the process
\[
Y_{k,i}(i) = Y_{k,i}(i, t) := \left( d_s(i) + k - 1 \right) \mathbb{I}\{\tau_i = t_i\}.
\] (40)

The properties of the binomial coefficients then imply, on the event \( \{\tau_i = t_i\} \),
\[
Y_{s+1,k}(i) = \left( d_s(i) + \Delta d_s(i) + k - 1 \right)
= \left( d_s(i) + k - 1 \right) \mathbb{I}\{\Delta d_s(i) = 0\} + \left( d_s(i) + 1 + k - 1 \right) \mathbb{I}\{\Delta d_s(i) = 1\}
+ \left( d_s(i) + 2 + k - 1 \right) \mathbb{I}\{\Delta d_s(i) = 2\}
\] (41)

\[
= \left( d_s(i) + k - 1 \right) \mathbb{I}\{\Delta d_s(i) = 0\}
+ \left( \left( \frac{d_s(i) + k - 1}{k} \right) + \left( \frac{d_s(i) + 1}{k} \right) \right) \mathbb{I}\{\Delta d_s(i) = 1\}
+ \left( \left( \frac{d_s(i) + 1}{k} \right) + 2 \left( \frac{d_s(i) + 1}{k} \right) + \left( \frac{d_s(i) + k - 1}{k - 2} \right) \right) \mathbb{I}\{\Delta d_s(i) = 2\}
\]

\[
= Y_{s,k}(i) \left( 1 + \frac{k}{d_s(i)} \mathbb{I}\{\Delta d_s(i) = 1\} + \left( \frac{2k}{d_s(i)} + \frac{k(k-1)}{d_s(i)(d_s(i) + 1)} \right) \mathbb{I}\{\Delta d_s(i) = 2\} \right)
= Y_{s,k}(i) \left( 1 + \frac{k}{d_s(i)} \Delta d_s(i) + \frac{k(k-1)}{d_s(i)(d_s(i) + 1)} \mathbb{I}\{\Delta d_s(i) = 2\} \right).
\]
Using the process’ definition and (15), we then obtain
\[
\mathbb{E}_{i,t_i}[Y_{s+1,k}(i)|\mathcal{F}_s]
\]
\[
= Y_{s,k}(i) \left( 1 + \frac{k}{d_s(i)} \frac{d_s(i)}{s} \left( 1 - \frac{f(s+1)}{2} \right) + \frac{k(k-1)}{d_s(i)(d_s(i)+1)}(1-f(s+1)) \frac{d_s(i)^2}{4s^2} \right)
\]
\[
\leq Y_{s,k}(i) \left( 1 + \frac{k}{s} \left( 1 - \frac{f(s+1)}{2} \right) + \frac{k(k-1)}{4s^2} (1-f(s+1)) \left( 1 - \frac{1}{d_s(i)+1} \right) \right).
\]

Defining then
\[
\phi_k(t) := \prod_{s=1}^{t-1} \left( 1 + \frac{k}{s} \left( 1 - \frac{f(s+1)}{2} \right) + \frac{k(k-1)}{4s^2} (1-f(s+1)) \right),
\]
we obtain that \(\{Y_{t,k}(i)\phi_k(t)^{-1}\}_{t \geq t_i}\) is a positive supermartingale and therefore it converges \(\mathbb{P}_{i,t_i}\)-a.s.

The following relation between the normalization factor \(\phi_k\) and \(\phi\) will be important: there exists \(C \equiv C_{k,f} > 0\) independent from \(i\) and \(t_i\) such that
\[
\phi_k(t) \leq C \prod_{s=1}^{t-1} \left( 1 + \frac{1}{s} \left( 1 - \frac{f(s+1)}{2} \right) \right)^k = C \phi(t)^k,
\]
\[
C^{-1} \phi(t)^k \leq \phi_k(t).
\]

This is an important ingredient in the proof of our next result.

**Proposition 4.1.** Let \(f\) be any edge-step function such that \((V)_\infty\) holds. Then there exists a non-negative random variable \(\tilde{\zeta}_{i,t_i}\) in the same probability space as the process \(\{G_i(f)\}_{t \geq 1}\) such that, for any \(q \geq 1\),
\[
\lim_{t \to \infty} \frac{d_t(i)}{\phi(t)} = \tilde{\zeta}_{i,t_i}, \quad \mathbb{P}_{i,t_i}\text{-a.s. and in } L_q(\mathbb{P}_{i,t_i}).
\]

Furthermore, defining the mixture
\[
\zeta_i = \sum_{n=1}^{\infty} \mathbb{I}\{\tau_i = n\} \tilde{\zeta}_{i,n},
\]
where \(\tilde{\zeta}_{i,n}\) is arbitrarily defined outside of \(\{\tau_i = n\}\), we obtain
\[
\lim_{t \to \infty} \frac{d_t(i)}{\phi(t)} = \zeta_i, \quad \mathbb{P}\text{-a.s. and in } L_q(\mathbb{P}).
\]

**Proof.** Since \(d_t(i)\phi(t)^{-1}\) is a positive martingale with respect to \(\mathbb{P}_{i,t_i}\), it converges \(\mathbb{P}_{i,t_i}\)-a.s. to a finite random variable, which we can denote by \(\tilde{\zeta}_{i,t_i}\). Furthermore, since \(Y_{t,k}\phi_k(t)^{-1}\) is bounded in \(L_1(\mathbb{P}_{i,t_i})\), we have by (14), for any \(k \in \mathbb{N}\) and uniformly in \(i\) and \(t_i\),
\[
\mathbb{E}_{i,t_i} \left[ d_s(i)^k \phi(s)^{-k} \right] \leq C k! \mathbb{E}_{i,t_i} \left[ Y_{s,k}\phi_k(s)^{-1} \right] < C.
\]
This implies the \(L_q\) convergence in (15). Since \(\sum_{s=1}^{\infty} f(s) = \infty\), we have \(\mathbb{P}(\tau_i = \infty) = 0\), and therefore, almost surely,
\[
\lim_{t \to \infty} \frac{d_t(i)}{\phi(t)} = \lim_{t \to \infty} \sum_{n=1}^{\infty} \mathbb{I}\{\tau_i = n\} \frac{d_t(i)}{\phi(t)} = \lim_{t \to \infty} \sum_{n=1}^{\infty} \mathbb{I}\{\tau_i = n\} \tilde{\zeta}_{i,n} = \zeta_i, \quad \mathbb{P}\text{-a.s.}
Recalling that we define $d_t(i) \equiv 0$ for $t < \tau_i$, we obtain
\[
\mathbb{E} \left[ \frac{d_{t+1}(i)}{\phi(t+1)} \bigg| \mathcal{F}_t \right] = \mathbb{E} \left[ \sum_{n=1}^{\infty} 1\{\tau_i = n\} \frac{d_{t+1}(i)}{\phi(t+1)} \bigg| \mathcal{F}_t \right]
\]
\[
= \sum_{n=1}^{t} 1\{\tau_i = n\} \frac{d_t(i)}{\phi(t)} + \frac{1}{\phi(t+1)} \mathbb{P} (\tau_i = t+1|\mathcal{F}_t) \quad (49)
\]
\[
= \sum_{n=1}^{\infty} 1\{\tau_i = n\} \frac{d_t(i)}{\phi(t)} + \frac{1}{\phi(t+1)} \mathbb{P} (\tau_i = t+1|\mathcal{F}_t),
\]
implying that $\{d_t(i)\phi(t)^{-1}\}_{t \geq 1}$ is a submartingale. Now, the uniformity of the constant in \( \frac{1}{\phi(t+1)} \) over $t_i$ gives us, for $k \geq 1$,
\[
\mathbb{E} \left[ \frac{d_t(i)^k}{\phi(t)^k} \right] = \mathbb{E} \left[ \sum_{n=1}^{\infty} 1\{\tau_i = n\} \frac{d_t(i)^k}{\phi(t)^k} \right] = \sum_{n=1}^{\infty} \mathbb{P} (\tau_i = n) \mathbb{E}_{i,n} \left[ \frac{d_t(i)^k}{\phi(t)^k} \right] < C; \quad (50)
\]
yielding that $\{d_t(i)\phi(t)^{-1}\}_{t \geq 1}$ is bounded in $L_k$ for any $k \in \mathbb{N}$, which implies $L_q(\mathbb{P})$ convergence for any $q \geq 1$, finishing the proof of the result. \( \square \)

The next natural step is to see whether the process $\{d_t(i)\}_{t \geq 1}$ goes to infinity at $\phi(t)$ speed, which means proving that $\zeta_i$ is strictly positive almost surely. Moreover, this result by itself will be necessary for the proof of Central Limit Theorem for the maximum degree. Then, we the next result we prove $\tau(t)$ is the right order for the degree.

**Proposition 4.2.** Let $i$ be the $i$-th vertex added by the process and $\zeta_i$ as in Proposition 4.1. Then $\zeta_i$ is strictly positive $\mathbb{P}$-almost surely.

**Proof.** As we have seen in Proposition 4.1, $\zeta_i$ may be written as
\[
\zeta_i = \sum_{n=i}^{\infty} \tilde{\zeta}_{i,n} 1\{\tau(i) = n\} \quad (51)
\]
and it is enough to proof that each $\tilde{\zeta}_{i,n}$ is strictly positive $\mathbb{P}_{i,n}$-a.s. This means, we can focus on the process conditioned to $i$ has been added at time $n$. Since $n$ is going to be fixed throughout this proof, we will simply write $\{X_s\}_{s \geq n}$ for $\{X_{s,n}\}_{s \geq n}$ and simply $\mathbb{P}$ for the conditioned measure $\mathbb{P}_{i,n}$. Following this convention, we denote by $g_s$ and $g$ the Laplace transform of $X_s$ and $\zeta_{i,n}$ respectively. It will be useful to recall a fact from real analysis that there exist a positive $c$, which we may assume smaller than one, such that, for all $x \in [0,c]$,
\[
e^{-x} \leq 1 - x + \frac{x^2}{2}. \quad (52)
\]
For each $s \in \mathbb{N}$, put $\lambda_s := c \phi(s)/2$. We will follow the argument given in [nadia18] in the context of Generalized Polya Urns. The proof will follow from the relation below
\[
g_t(\lambda_s) \leq g_{t-r} \left( \lambda_s - \lambda_s^2 \sum_{u=t-r+1}^{t} \frac{1}{\phi(u)^2} \right), \quad (53)
\]
which we prove holds for all $s$, $t > s$ and $0 \leq r \leq t - s$. In order to prove it, we make an induction argument on $r$.

**Base case $r = 1$.** Fix $\lambda \in [0, c \phi(s)/2]$ (it is necessary in our case to take half of the interval taken in 18, since the degree’s increment can be 2) and recall that
\[
X_{t+1} = \frac{\phi(t)}{\phi(t+1)} X_t + \frac{\Delta d_t(i)}{\phi(t+1)}. \quad (54)
\]
Using \(52\) and Equation \(11\), we have that
\[
\mathbb{E} \left[ \exp \left\{ -\frac{\lambda \Delta d_i(t)}{\phi(t+1)} \right\} \mid \mathcal{F}_t \right] \leq \mathbb{E} \left[ 1 - \frac{\lambda \Delta d_i(t)}{\phi(t+1)} + \frac{\lambda^2 \Delta^2 d_i(t)}{2\phi(t+1)^2} \right] \mid \mathcal{F}_t \]
\[
= 1 - \frac{\lambda}{\phi(t+1)} \left( \frac{2 - f(t)}{2t} \right) \frac{(2 - f(t))d_i(t)}{2t} + \frac{\lambda^2}{2\phi(t+1)^2} \frac{2(1 - f(t))d_i^2(t)}{4t^2} \tag{55}
\]
\[
\leq 1 - \frac{\lambda}{\phi(t+1)} \left( \frac{2 - f(t)}{2t} \right) + \frac{\lambda^2}{2\phi(t+1)^2} \frac{2(2 - f(t))d_i(t)}{2t}
\]

It will be useful to recall that
\[
\phi(t + 1) = \phi(t) \left( 1 + \frac{2 - f(t)}{2t} \right). \tag{56}
\]

Using \(53\) and that \(\log(1 + x) \leq x\), we obtain that
\[
\mathbb{E} \left[ e^{-\lambda X_{t+1}} \mid \mathcal{F}_t \right] \leq
\exp \left\{ -\frac{\lambda \phi(t)}{\phi(t+1)} X_t \log \left( 1 - \frac{\lambda}{\phi(t+1)} \left( \frac{2 - f(t)}{2t} \right) \frac{(2 - f(t))d_i(t)}{2t} + \frac{\lambda^2}{2\phi(t+1)^2} \frac{2(2 - f(t))d_i(t)}{2t} \right) \right\}
\leq \exp \left\{ -\frac{\lambda \phi(t)}{\phi(t+1)} X_t - \frac{\lambda \phi(t)}{\phi(t+1)} \left( \frac{2 - f(t)}{2t} \right) \frac{(2 - f(t))d_i(t)}{2t} + \frac{\lambda^2 \phi(t)}{2\phi(t+1)^2} \frac{2(2 - f(t))d_i(t)}{2t} \right\}
\tag{57}
\]

Taking the expected value both sides, leads to
\[
g_{t+1}(\lambda) \leq g_t \left( \lambda - \frac{\lambda^2}{2\phi(t+1)^2} \right), \tag{58}
\]
for all \(\lambda \in [0, c\phi(s)/2]\), which proves the base case of our induction.

Inductive step. From here the proof is similar to that given in \(18\) setting \(\sigma_t = 1\) and \(\tau_t = \phi(t)\). Our inductive hypothesis, tells us that
\[
g_t(\lambda_s) \leq g_{t-r} \left( \lambda_s - \lambda_s^2 \sum_{u=t-r+1}^{t} \frac{1}{\phi(u)^2} \right) \tag{59}
\]
holds for \(0 \leq r \leq t - s - 1\). Now, observe that
\[
\lambda_s - \lambda_s^2 \sum_{u=t-r+1}^{t} \frac{1}{\phi(u)^2} \leq \lambda_s,
\]
thus, combining the above inequality with \(15\) and recalling that \(g_t\) is decreasing, we have that
\[
g_{t-r} \left( \lambda_s - \lambda_s^2 \sum_{u=t-r+1}^{t} \frac{1}{\phi(u)^2} \right) \leq g_{t-r-1} \left( \lambda_s - \lambda_s^2 \sum_{u=t-r+1}^{t} \frac{1}{\phi(u)^2} \right) - \left( \lambda_s - \lambda_s^2 \sum_{u=t-r+1}^{t} \frac{1}{\phi(u)^2} \right)^2 \frac{1}{\phi(t-r)^2}
\leq g_{t-r-1} \left( \lambda_s - \lambda_s^2 \sum_{u=t-r}^{t} \frac{1}{\phi(u)^2} \right) \tag{60}
\]
which proves the induction. Taking \( r = t - s \) on (68) and noticing that
\[
\lambda_s - \lambda_s^2 \sum_{u=s+1}^{t} \frac{1}{g^2(u)} \geq \lambda_s - \lambda_s^2 \int_{\phi(s)}^{\infty} \frac{dx}{x^2} = \lambda_s (1 - c/2) \geq 0
\]
we obtain that
\[
g_t(\lambda_s) \leq g_s((1 - c/2)\lambda_s) = \mathbb{E} \left[ e^{-(1-c/2)c \phi(s) X_s/2} \right] = \mathbb{E} \left[ e^{-(1-c/2)c \phi_d(i)/2} \right].
\]
Taking the limit on \( t \), the Dominated Convergence Theorem gives us that
\[
g(\lambda_s) \leq \mathbb{E} \left[ e^{-(1-c/2)c \phi_d(i)/2} \right].
\]
Finally, another application of the Dominated Convergence Theorem combined with the fact that \( \{ d_s(i) \}_{s \geq n} \) goes to infinity \( \mathbb{P}_{i,n} \)-a.s. as \( s \) goes to infinity yields \( g(\lambda_s) \to 0 \) as \( s \) goes to infinity. The Proposition follows by the fact that
\[
\mathbb{P}_{i,n}(\zeta_{i,n} = 0) \leq g(\lambda),
\]
for any \( \lambda > 0 \).

The family of supermartingale \( \{ Y_{t,k}(i) \}_{t \geq i} \) defined in (40) allows us to obtain upper bounds for all the moments of the limit random variables \( \zeta_i \).

**Lemma 4.3 (Moment bounds for the limit distribution).** Assume \( f \) either belongs to \( \text{RES}(-\gamma) \) with \( \gamma \in [0,1) \) or is identically equal to \( p \in (0,1] \). Then given \( k \in \mathbb{N} \), there exists constants \( C, C' > 0 \) independent from \( i \) such that
\[
\mathbb{E} \left[ \tilde{c}_k^i \right] \leq C'^{-1} \left( \exp \{ -C \cdot i \} + \frac{1}{\phi(F^{-1}(i))} \right) \leq C' \cdot i^{-\frac{k}{2}}.
\]

**Proof.** Since \( \{ d_s(i) \}_{s \geq n} \) goes to infinity, we have that, on \( \{ \tau_i = t_i \} \),
\[
Y_{s,k}(i) = \binom{d_s(i) + k - 1}{k} = \frac{(d_s(i) + k - 1) \cdots (d_s(i) + 1)}{k!} \to_{s \to \infty} \tilde{c}_k^{\tau_i,t_i}, \quad \mathbb{P}\text{-a.s.}
\]
Now, since \( \{ Y_{t,k}(i) \phi_k(t)^{-1} \}_{t \geq \tau_i} \) is a positive supermartingale in \( \mathbb{P}_{i,t_i} \), we have, by (44),
\[
\mathbb{E}_{i,t_i} \left[ \tilde{c}_k^{\tau_i,t_i} \right] \leq k! \frac{C}{\phi(t_i)^k}.
\]
By definition of \( \zeta_i \), we then obtain
\[
\mathbb{E} \left[ \tilde{c}_k^i \right] = \sum_{n=1}^{\infty} \mathbb{P} (\tau_i = n) \mathbb{E}_{i,t_i} \left[ \tilde{c}_k^{\tau_i,t_i} \right] \leq C \mathbb{P} (\tau_i \leq F^{-1}(i)/2) + \frac{C}{\phi(F^{-1}(i)/2)^k},
\]
By Lemma 3.2 and Equation (20), the right hand side of the above equation is bounded from above by
\[
C \cdot C_f^{-1} \exp \{ -C_f \cdot i \} + \frac{C}{\phi(F^{-1}(i))}.
\]
To prove the second inequality in (64), we note that, since \( \| f \|_\infty \leq 1 \), for \( s \in \mathbb{N} \),
\[
\phi(s) = \prod_{n=1}^{t-1} \left( 1 + \frac{1}{s} - \frac{f(s+1)}{2s} \right) \geq C \sqrt{s}.
\]
If \( f \equiv p > 0 \), then \( F^{-1}(i) = i/p \) and the result follows. Assume now that \( f \in \text{RES}(\gamma) \). By the monotonicity of \( F \), Theorem B.3 and Equation (29), we have for sufficiently large \( s \), using the
representation $f(t) = t^{-\gamma} \ell(t)$ with $\ell : \mathbb{N} \to \mathbb{R}$ being a slowly varying function,

$$F(s) \leq 2^{1-\gamma} \frac{s^{1-\gamma}}{1-\gamma} \ell(s) \quad s \leq F^{-1}(i^{1-2^{-1\gamma}}) \quad \text{(66)}$$

where in the second inequality we used the fact that, if $\gamma = 0$, then $f \equiv \ell$ converges to 0. These inequalities then imply

$$\phi \left( F^{-1}(i) \right)^k \geq C i^{\frac{k}{\gamma}},$$

which finishes the proof of the lemma. \hfill \square

4.1. Convergence of the scaled maximum degree. In this part we are going to extend the convergence result proven for a fixed vertex $i$ to the maximum degree process. We begin by introducing new notation and recalling others. Given $s, t \in \mathbb{N}$, with $s \leq t$, we define the variables

$$X_{s,i} := \frac{d_s(i)}{\phi(s)}, \quad M[s, i] := \max_{1 \leq k \leq i} \{ X_{s,k} \}, \quad M_s := \max_{k \geq 1} \{ X_{s,k} \}, \quad \mu(i) := \max_{1 \leq k \leq i} \{ \zeta_k \}, \quad \mu := \sup_{k \geq 1} \{ \zeta_k \}.$$

(67)

Now we have all the tools needed for proof Theorem 1.1.

Proof of Theorem 1.1. Equation (49) yields that $\{ X_{s,i} \}_{s \geq 1}$ is a submartingale, and for $q \geq 1$ the function that maps $x \in \mathbb{R}$ to $x^q$ is convex. Jensen’s inequality then implies, together with Lemma 4.3,

$$\mathbb{E} [M_s^q] \leq \sum_{j=1}^{\infty} \mathbb{E} \left[ X_{s,j}^q \right] \leq \sum_{j=1}^{\infty} \mathbb{E} \left[ \zeta_j^q \right] \leq C \sum_{j=1}^{\infty} j^{-\frac{q}{\mu}} < \infty,$$

for $q > 2$. But $M_s = M[s, s]$, since $X_{s,k} = 0$ for $k > s$. Being the maximum of submartingales implies that $\{ M_s \}_{s \geq 1}$ is itself a submartingale, and the above inequality shows that it is $L_q$-bounded for any $q > 2$. This implies that it converges almost surely and in $L_q$ to some finite random variable. Again with $q > 2$, we have that

$$\mathbb{E} [(M_s - M[s, i])^q] \leq \sum_{j=i+1}^{\infty} \mathbb{E} \left[ X_{s,j}^q \right] \frac{i}{i+1} \to 0.$$

But by Proposition 4.1, $M[s, i]$ converges to $\mu(i)$ $\mathbb{P}$-a.s. and in $L_q$ as $s \to \infty$ and therefore the limit of the LHS is equal to

$$\lim_{i \to \infty} \mathbb{E} \left[ \left( \lim_{s \to \infty} M_s - \mu(i) \right)^q \right] = 0,$$

and $\lim_{i} \mu(i) = \mu$ a.s. and in $L_q$ by monotone convergence. This implies (3). And finally, the fact that $\mu > 0, \mathbb{P}$-a.s. then comes as consequence of Proposition 4.2. \hfill \square

Now, the sharpness of the condition (S) when $f$ belongs to RES$(-\gamma)$, with $\gamma \in [0, 1)$, comes naturally from our results.

Proof of Corollary 1.2. By Theorem 1.1 under RES$(-\gamma)$, with $\gamma \in [0, 1)$ the maximum degree scaled by $\phi$ converges to a finite and positive limit. On the other hand, by Equation (16), it follows that $\phi(t) = \Theta(t)$ if, and only if, condition (S) is satisfied. \hfill \square
5. The Persistent Leadership

This section is devoted to the proof of Theorem 1.4, which states that eventually only one vertex becomes the one with maximum degree and stay in this position forever. The key step is to prove that for any pair of vertices $i$ and $j$ we have that $\zeta_i \neq \zeta_j$ almost surely. As discussed in Section 2, in the case of the BA-model and its affine version the absence of the edge-step rule implies $\zeta_i \neq \zeta_j$ in a straightforward way, since for this class of models a natural Pólya urn scheme emerges from the degree process when we look to the degrees of $i$ and $j$ only at the random times one of them has increased. Thus, from Polya urn results follows that each $\zeta_i$ is a continuous random variable.

Since in our settings, we cannot rely on direct Polya Urn results, we prove this key step dealing directly with the process $\{D_s\}_{s \geq \tau_j}$ defined as

$$D_s := |d_s(i) - d_s(j)|.$$  

In particular, we prove that $D_s$ goes to infinity almost surely. In order to do so, it will be useful to look to $\{D_s\}_{s \geq \tau_j}$ only at the times at least one of the vertices has been chosen. Thus, we introduce the following sequence of stopping times:

$$\sigma_0 := \tau_j,$$

$$\sigma_k := \inf_{s \geq \sigma_{k-1}} \{\Delta d_s(i) > 0 \text{ or } \Delta d_s(j) > 0\},$$

which are finite since both degrees go to infinity. We also let $\{\tilde{D}_k\}_{k \geq 0}$ to be

$$\tilde{D}_k := D_{\sigma_k}.$$  

Our proves then require an intermediate result, which needs extra notation to be stated properly. We say a finite (multi)graph $\tilde{G}$ is $(i, j, t, t, f)$-admissible whenever

$$P(\tau_i = t_i, \tau_j = t_j, G_{t_i}(f) = \tilde{G}) > 0.$$  

We denote by $P_{\tilde{G}}$ the probability measure $P$ conditioned on the event

$$\{\tau_i = t_i, \tau_j = t_j, G_{t_i}(f) = \tilde{G}\}.$$

**Lemma 5.1 (Law of Iterated Logarithm for $\tilde{D}_k$).** Assume $f$ either belongs to RES($-\gamma$) with $\gamma \in [0, 1)$ or is identically equal to $p \in (0, 1)$. Let $i, j \in \mathbb{N}$ denote two distinct vertices that were added to the graph at times $i$ and $j$ respectively. Then, there exist a strictly positive random variable $\delta$ such that

$$\limsup_{k \to \infty} \frac{\tilde{D}_k - \sum_{n=0}^{k-1} E_{\tilde{G}} \left[ \frac{\Delta \tilde{D}_n}{F_{s_n}} \right]}{\sqrt{\delta k \log \log (\delta k)}} \geq 1, \text{ } P_{\tilde{G}}\text{-a.s.},$$  

for any admissible graph $\tilde{G}$.

The reason why we need such lemma is the following: once one of the degrees is sufficiently larger than the other it hard for the smallest one to catch up its adversary. Thus, we may bootstrap the above lemma by proving that actually the whole process $\{D_s\}_{s \geq \tau_j}$ goes to infinity.

**Proof of Lemma 5.1.** We will apply Theorem 6.3 of [13] which gives LIL for submartingales satisfying certain conditions. Thus, the proof consist of showing that the stopped process $\{\tilde{D}_k\}_k$ is a submartingale and proving estimates for the variance of its increments.

We begin by showing that $\{\tilde{D}_k\}_k \geq 1$ is a submartingale. Note that, by the same reasoning as in [12], the process

$$\left\{ \frac{d_s(i) - d_s(j)}{\phi(s)} \right\}_{s \geq t_j}$$

is a martingale in $P_{\tilde{G}}$. Therefore, by Jensen’s inequality,

$$E_{\tilde{G}} \left[ \left| \frac{d_{s+1}(i) - d_{s+1}(j)}{\phi(s+1)} \right| F_s \right] \geq \left| \frac{d_s(i) - d_s(j)}{\phi(s)} \right|,$$

for any admissible graph $\tilde{G}$.  

(73)
and we obtain
\[
\mathbb{E}_\tilde{G} [ |d_{s+1}(i) - d_{s+1}(j)| |\mathcal{F}_s] \geq |d_s(i) - d_s(j)| \frac{\phi(s+1)}{\phi(s)} \geq |d_s(i) - d_s(j)|. \quad (74)
\]

Now, since
\[
|d_{s\wedge \sigma_{k+1}}(i) - d_{s\wedge \sigma_{k+1}}(j)| \leq k + 1 + d_{\ell_s}(i) + 1,
\]
the process \{\{d_{s\wedge \sigma_{k+1}}(i) - d_{s\wedge \sigma_{k+1}}(j)\}\}_s \text{ is uniformly integrable in } \mathbb{P}_\tilde{G} \text{ and the submartingale}\]
\text{optional stopping theorem (see Theorem 5.7.4 of } \text{[8]}) \text{ shows that } \{\tilde{D}_k\}_{k \geq 1} \text{ is a submartingale.}

We will need lower bounds on the conditional variance of \(\Delta \tilde{D}_k\). We begin by noting that
\[
\text{Var}_\tilde{G} \left[ \Delta \tilde{D}_k | \mathcal{F}_{\sigma_k} \right] = \mathbb{E}_\tilde{G} \left[ \left( \Delta \tilde{D}_k - \mathbb{E}_\tilde{G} \left[ \Delta \tilde{D}_k | \mathcal{F}_{\sigma_k} \right] \right)^2 | \mathcal{F}_{\sigma_k} \right] 
\geq \left( 1 - \mathbb{E}_\tilde{G} \left[ \Delta \tilde{D}_k | \mathcal{F}_{\sigma_k} \right] \right)^2 \mathbb{P}_\tilde{G} \left[ \Delta \tilde{D}_k = 1 | \mathcal{F}_{\sigma_k} \right] 
+ \left( -1 - \mathbb{E}_\tilde{G} \left[ \Delta \tilde{D}_k | \mathcal{F}_{\sigma_k} \right] \right)^2 \mathbb{P}_\tilde{G} \left[ \Delta \tilde{D}_k = -1 | \mathcal{F}_{\sigma_k} \right] 
\geq 4 \mathbb{P}_\tilde{G} \left[ \Delta \tilde{D}_k = 1 | \mathcal{F}_{\sigma_k} \right] \mathbb{P}_\tilde{G} \left[ \Delta \tilde{D}_k = -1 | \mathcal{F}_{\sigma_k} \right] 
\geq 2 \mathbb{P}_\tilde{G} \left[ \Delta \tilde{D}_k = 1 | \mathcal{F}_{\sigma_k} \right] \mathbb{P}_\tilde{G} \left[ \Delta \tilde{D}_k = -1 | \mathcal{F}_{\sigma_k} \right], \quad (75)
\]
where we have used that \((1-x)^2 + (1+x)^2 \geq 4ab/(a+b)\) for all values of \(x\). Assume \(\tilde{D}_k \neq 0\).

In order for \(\Delta \tilde{D}_k\) to be 1, resp. \(-1\), the vertex with highest, resp. lower degree must receive a connection at time \(\sigma_{k+1}\). We have, by the definition of the process,
\[
\mathbb{P}_\tilde{G} \left( \Delta d_s(i) + \Delta d_s(j) \geq 1 | \mathcal{F}_s \right) 
= f(s+1) \frac{d_s(i) + d_s(j)}{2s} + 2 \left( 1 - f(s+1) \right) \frac{d_s(i) + d_s(j)}{2s} \left( 1 - \frac{d_s(i) + d_s(j)}{2s} \right) 
+ \left( 1 - f(s+1) \right) \left( \frac{d_s(i) + d_s(j)}{2s} \right)^2 
= \frac{d_s(i) + d_s(j)}{s} \left( 1 - \frac{f(s+1)}{2} - (1 - f(s+1)) \frac{d_s(i) + d_s(j)}{4s} \right). \quad (76)
\]

At the same time,
\[
\mathbb{P}_\tilde{G} \left( \Delta d_s(i) = 1, \Delta d_s(j) = 0 | \mathcal{F}_s \right) 
= f(s+1) \frac{d_s(i)}{2s} + 2 \left( 1 - f(s+1) \right) \frac{d_s(i)}{2s} \left( 1 - \frac{d_s(i) + d_s(j)}{2s} \right) 
= \frac{d_s(i)}{s} \left( 1 - \frac{f(s+1)}{2} - (1 - f(s+1)) \frac{d_s(i) + d_s(j)}{2s} \right). \quad (77)
\]
Now, by the two above equations, in the event where $\sigma_k \leq s$ we have
\[
\mathbb{P}_G(\Delta d_s(i) = 1, \Delta d_s(j) = 0 | \mathcal{F}_{\sigma_k}, \sigma_{k+1} = s + 1) = \mathbb{E}_G[\mathbb{P}_G(\Delta d_s(i) = 1, \Delta d_s(j) = 0, \sigma_{k+1} = s + 1 | \mathcal{F}_s) | \mathcal{F}_{\sigma_k}] \]
\[
= \mathbb{E}_G[\mathbb{P}_G(\Delta d_s(j) = 0, \sigma_{k+1} = s + 1 | \mathcal{F}_s) | \mathcal{F}_{\sigma_k}] \]
\[
= \mathbb{E}_G[1 \{ \sigma_{k+1} > s \} \mathbb{P}_G(\Delta d_s(i) = 1 | \mathcal{F}_s) | \mathcal{F}_{\sigma_k}] \]
\[
= \mathbb{E}_G[\mathbb{P}_G(\Delta d_s(i) = 1 | \mathcal{F}_s) | \mathcal{F}_{\sigma_k}] \]
\[
= \frac{d_{\sigma_k}(i)}{d_{\sigma_k}(i) + d_{\sigma_k}(j)} \left( 1 - f(s + 1) \right) \frac{\phi(\sigma_k)}{2} \overset{(78)}{=}
\]
\[
\mathbb{P}_G(\Delta d_s(i) = 1, \Delta d_s(j) = 0 | \mathcal{F}_{\sigma_k}) \]
\[
\leq (1 - f(s + 1)) \frac{d_{\sigma_k}(i) + d_{\sigma_k}(j)}{2s} \leq \frac{2 (s - d_s(j) + 1)}{2s \xi(s)} \overset{(79)}{=}
\]
\[
(1 - f(s + 1)) \frac{(d_{\sigma_k}(i) + d_{\sigma_k}(j)) \xi(\sigma_k)}{\phi(\sigma_k)} \]
\[
\overset{(80)}{=}
\frac{2}{\xi(\infty)} - \zeta_{j+1} < \frac{2}{\xi(\infty)}. \]

Denote by $\xi(\infty)$ the limit of $\xi(s)$ as $s$ tends to infinity. It exists by Equation (10), and is only nonzero if $1 \sum_{s \geq 1} f(s)/s = \infty$. In the event where $\sigma_k \leq s$, we have
\[
(1 - f(s + 1)) \frac{d_{\sigma_k}(i) + d_{\sigma_k}(j)}{2s} \leq (1 - f(s + 1)) \frac{d_{\sigma_k}(i) + d_{\sigma_k}(j)}{2s} \frac{\phi(\sigma_k)}{2} \overset{(79)}{=}
\]
\[
(1 - f(s + 1)) \frac{(d_{\sigma_k}(i) + d_{\sigma_k}(j)) \xi(\sigma_k)}{\phi(\sigma_k)} \]
\[
\overset{(80)}{=}
\frac{2}{\xi(\infty)} - \zeta_{j+1} < \frac{2}{\xi(\infty)}. \]

Proposition 4.2 yields that the RHS of (79) is strictly less the 1 almost surely. Furthermore,
\[
\lim_{k \to \infty} \frac{d_{\sigma_k}(i)}{d_{\sigma_k}(i) + d_{\sigma_k}(j)} = \frac{\zeta_i}{\zeta_i + \zeta_j}, \overset{(81)}{=}
\]
and since $\zeta_i$ and $\zeta_j$ are strictly positive and finite almost surely, the RHS of the above equation is also strictly positive. By integrating over the possible values of $\sigma_{k+1}$, we obtain from the above equations that there exists an a.s. positive random variable $\tilde{\delta}_1$ such that, uniformly over $k$,
\[
\mathbb{P}_G(\Delta d_{\sigma_{k+1}-1}(i) = 1, \Delta d_{\sigma_{k+1}-1}(j) = 0 | \mathcal{F}_{\sigma_k}) \geq \tilde{\delta}_1. \overset{(82)}{=}
\]
This is possible since, by the above discussion, for each $k$ the LHS of the above inequality is bounded from below by a $k$-dependent random variable whose limit in $k$ is strictly positive $\mathbb{P}_G$ a.s. Notice that analogous inequalities are valid for $d_s(j)$ in place of $d_s(i)$.

Equation (75) then implies that there exists an a.s. positive random variable $\tilde{\delta}_2$ such that when $\tilde{D}_k \neq 0$, uniformly over $k$,
\[
\text{Var}_G[\Delta \tilde{D}_k | \mathcal{F}_{\sigma_k}] \geq 2\tilde{\delta}_2. \overset{(83)}{=}
\]
By the Azuma-Höffding inequality we obtain

\[ \mathbb{P}_G \left( \sum_{k=1}^{N} \mathbb{1}\{ \tilde{D}_k \neq 0 \} - \mathbb{P}_G \left( \tilde{D}_k \neq 0 \mid \mathcal{F}_{\sigma_{k-1}} \right) \right) \leq \exp \left\{ -4^{-1}N^{3/2} \cdot N^{-1} \right\} \]

\[ \leq \exp \left\{ -4^{-1}N^{1/2} \right\}. \]  

Together with Equation (82) this implies, almost surely, the existence of a finite random \( N_0 \in \mathbb{N} \) such that, for \( N \geq N_0 \),

\[ \sum_{k=1}^{N} \mathbb{1}\{ \tilde{D}_k \neq 0 \} \geq \sum_{k=1}^{N} \mathbb{P}_G \left( \tilde{D}_k \neq 0 \mid \mathcal{F}_{\sigma_{k-1}} \right) - N^{3/4} \]

\[ \geq \sum_{k=1}^{N} \mathbb{P}_G \left( \Delta \tilde{D}_{k-1} = 1 \mid \mathcal{F}_{\sigma_{k-1}} \right) - N^{3/4} \]

\[ \geq N(\delta_1 - N^{-1/4}). \]

Equation (83), which holds on \( \tilde{D}_k \neq 0 \), then yields, for \( N \) larger than \( N_0 \),

\[ \sum_{k=1}^{N} \text{Var}_G \left[ \Delta \tilde{D}_k \mid \mathcal{F}_{\sigma_k} \right] \geq 2\delta_2 \sum_{k=1}^{N} \mathbb{1}\{ \tilde{D}_k \neq 0 \} \geq N\delta_1 \delta_2. \]  

From now on, we denote \( \delta := \delta_1 \delta_2 \). Consider now the Doob decomposition of the submartingale \( \{ \tilde{D}_k \}_{k \geq 0} \).

\[ \tilde{D}_k = \tilde{D}_0 + \sum_{n=0}^{k-1} \left( \tilde{D}_{n+1} - \mathbb{E}_G \left[ \tilde{D}_{n+1} \mid \mathcal{F}_{\sigma_n} \right] \right) + \sum_{n=0}^{k-1} \mathbb{E}_G \left[ \Delta \tilde{D}_n \mid \mathcal{F}_{\sigma_n} \right]. \]

Then

\[ S_k := \tilde{D}_k - \sum_{n=0}^{k-1} \mathbb{E}_G \left[ \Delta \tilde{D}_n \mid \mathcal{F}_{\sigma_n} \right] \]

is a mean zero martingale, and the predictable process being subtracted from \( \tilde{D}_k \) above is always nonnegative, since \( \tilde{D}_k \) is a submartingale. Furthermore,

\[ \Delta \left( \tilde{D}_k - \sum_{n=0}^{k-1} \mathbb{E}_G \left[ \Delta \tilde{D}_n \mid \mathcal{F}_{\sigma_n} \right] \right) = \Delta \tilde{D}_k - \mathbb{E}_G \left[ \Delta \tilde{D}_k \mid \mathcal{F}_{\sigma_k} \right], \]

and therefore the conditional second moment of the increment of \( S_k \) equals the conditional variance of \( \Delta \tilde{D}_k \). We can finally apply Theorem 6.3 of [13] to \( S_n \) with \( V_k := \text{Var}_G \left[ \Delta \tilde{D}_k \mid \mathcal{F}_{\sigma_k} \right] \) and \( T_n = \sum_{k=1}^{n} V_k \), which gives the LIL for the martingale component of the submartingale \( \{ \tilde{D}_k \}_{k \in \mathbb{N}} \). The result extends to \( \{ \tilde{D}_k \}_{k \in \mathbb{N}} \) itself since its predictable component is positive.

We will need another lemma before proving Theorem 1.4.

**Lemma 5.2.** Assume \( f \) either belongs to \( \text{RES}(-\gamma) \) with \( \gamma \in [0,1) \) or is identically equal to \( p \) in \( [0,1] \). Let \( i, j \in \mathbb{N} \) denote two distinct vertices that were added to the graph at times \( i \) and \( j \) respectively. Then,

\[ \lim_{s \to \infty} |d_s(i) - d_s(j)| = \infty, \quad \mathbb{P}\text{-a.s.} \]

**Proof.** The proof strategy we adopt here will be to bootstrap the LIL proved in Lemma 5.1 i.e., we use that at some random times the difference between the degrees of \( i \) and \( j \) are large enough
in order to show that this fact is “irreversible. Thus, by Lemma 5.1 we know that there exists a.s. an increasing random sequence \((T_m)_{m \geq 1}\) of stopping times where

\[
\hat{D}_{T_m} \geq \hat{D}_{T_m} - \sum_{n=0}^{T_m-1} \mathbb{E}_G \left[ \Delta \hat{D}_n \mid \mathcal{F}_{\sigma_n} \right] \geq \sqrt{\delta T_m \log \log (\delta T_m)}.
\] (91)

Given such a \(T_m\) and some number \(\varepsilon > 0\), consider the stopping time

\[
\eta_1 := \inf \left\{ s \geq T_m; \hat{D}_s < (1 - \varepsilon)\sqrt{\delta T_m \log \log (\delta T_m)} \right\}.
\]

Assume, w.l.o.g. that \(d_{T_m(i)} > d_{T_m(j)}\). Then, for \(k \geq T_m\), we have by a similar argument as in (78) and by Equations (3) and (76),

\[
\mathbb{E}_G \left[ \Delta \hat{D}_{k \wedge \eta_1} \mid \mathcal{F}_{\sigma_k}, \sigma_{k+1} = s + 1 \right]
\]

\[
= 1\{ \eta_1 > k \} \mathbb{E}_G \left[ \mathbb{E}_G \left[ \mathbb{E}_G \left[ \Delta (d_s(i) - d_s(j)) \mathbbm{1}\{ \sigma_{k+1} = s + 1 \} \mid \mathcal{F}_s \} \mid \mathcal{F}_{\sigma_k} \right] \right] \right]
\]

\[
= 1\{ \eta_1 > k \} \frac{d_{\sigma_k}(i) - d_{\sigma_k}(j)}{d_{\sigma_k}(i) + d_{\sigma_k}(j)} (1 - f(s + \frac{1}{2}) \right)
\]

\[
(1 - f(s + \frac{1}{2}) - (1 - f(s + 1)) \frac{d_{\sigma_k}(i) + d_{\sigma_k}(j)}{4s}).
\]

If \(f \equiv p\), Stirling’s approximation gives

\[
\phi(t) = \prod_{s=1}^{t-1} \left( \frac{s + 1 - p/2}{s} \right) = \Gamma(t + 1 - p/2) = \frac{1 + o(1)}{t^{1-p/2}}.
\] (93)

and therefore the fraction

\[
\frac{\left(1 - \frac{f(s + 1)}{2}\right)}{\left(1 - f(s + \frac{1}{2}) - (1 - f(s + 1)) \frac{d_{\sigma_k}(i) + d_{\sigma_k}(j)}{4s}\right)}
\]

converges to 1 a.s. as \(k \to \infty\) since \((d_s(i) + d_s(j))s^{-1+p/2}\) converges a.s. to a finite random variable as \(s\) goes to infinity by Proposition 1.1. If \(f(s) \downarrow 0\), then the right hand side of (76) is bounded from below by

\[
1\{ \eta_1 > k \} \frac{\hat{D}_k}{k + t_j} \cdot \left(1 - \frac{f(s + 1)}{2}\right),
\] (94)

since the total number of connections added to \(i\) or \(j\) by time \(\sigma_k\) is at most \(k + t_j\). Then for \(k\) sufficiently large, integrating over the possible values of \(\sigma_{k+1}\), we obtain

\[
\mathbb{E}_G \left[ \Delta \hat{D}_{k \wedge \eta_1} \mid \mathcal{F}_{\sigma_k} \right] \geq 1\{ \eta_1 > k \} \frac{3\hat{D}_k}{4k}.
\] (95)

This in turn yields, for large enough \(m\),

\[
\sum_{k=T_m}^{(1+\varepsilon)T_m \wedge \eta_1} \mathbb{E}_G \left[ \Delta \hat{D}_k \mid \mathcal{F}_{\sigma_k} \right] \geq 1\{ \eta_1 > (1 + \varepsilon)T_m \} \varepsilon T_m \frac{3(1+\varepsilon)\sqrt{\delta T_m \log \log (\delta T_m)}}{4(1+\varepsilon)T_m}
\]

\[
\geq 1\{ \eta_1 > (1 + \varepsilon)T_m \} \frac{\varepsilon (1+\varepsilon)}{4(1+\varepsilon)} \sqrt{\delta T_m \log \log (\delta T_m)}.
\]

We now use the Azuma–Höffding inequality for the martingale defined in (88) between times \(T_m\) and \(1 + \varepsilon)T_m\) stopped in \(\eta_1\). Denote by \(A_\delta\) the event where \(\delta > \delta\) for some given \(\delta > 0\). Since

\[
\left| \Delta \hat{D}_k - \mathbb{E}_G \left[ \Delta \hat{D}_k \mid \mathcal{F}_{\sigma_k} \right] \right| \leq 4,
\]

(97)
we obtain, using the fact that $m \leq T_m$,

$$\mathbb{P}_\tilde{G} \left(D_{(1+\varepsilon)T_m} \wedge \eta_1 - \sum_{n=\eta_1}^{(1+\varepsilon)T_m \wedge \eta_1-1} \mathbb{E}_\tilde{G} \left[ \Delta \tilde{D}_n | F_{\sigma_n} \right] - \tilde{D}_{\eta_1} \leq -\varepsilon \sqrt{\Delta T_m \log \log (\delta T_m), A_\delta} \right)
\leq \mathbb{E}_\tilde{G} \left[ \exp \left\{ -\frac{\varepsilon^2 \delta T_m \log \log (\delta T_m)}{16 \varepsilon T_m} \right\} \mathbb{1}_{A_\delta} \right]
\leq \exp \left\{ -\frac{\varepsilon^2 \delta \log \log (\delta m)}{16 \varepsilon} \right\}, \quad (98)$$

yielding

$$\mathbb{P}_\tilde{G} (\eta_1 \leq (1+\varepsilon)T_m, A_\delta) \leq \exp \left\{ -\frac{\varepsilon^2 \delta \log \log (\delta m)}{16 \varepsilon} \right\}. \quad (99)$$

Replacing $\varepsilon$ by $\varepsilon^2$ multiplying the log log factor, we have

$$\mathbb{P}_\tilde{G} \left(\tilde{D}_{(1+\varepsilon)T_m} - \sum_{n=\eta_1}^{(1+\varepsilon)T_m} \mathbb{E}_\tilde{G} \left[ \Delta \tilde{D}_n | F_{\sigma_n} \right] - \tilde{D}_{\eta_1} \leq -\varepsilon^2 \sqrt{\delta T_m \log \log (\delta T_m), A_\delta} \right)
\leq \exp \left\{ -\frac{\varepsilon^4 \delta \log \log (\delta m)}{16 \varepsilon} \right\}. \quad (100)$$

Now, define the event

$$B_1(\delta, \varepsilon, m) \triangleq A_\delta \cap \left\{ \tilde{D}_{(1+\varepsilon)T_m} \geq \left( 1 + \frac{3 \varepsilon (1-\varepsilon)}{4 (1+\varepsilon)} - \varepsilon^2 \right) \sqrt{\delta T_m \log \log (\delta T_m)} \right\}. \quad (101)$$

By equations \[96, 100, 101\], we obtain

$$\mathbb{P}_\tilde{G} (B_1(\delta, \varepsilon, m)^c) \leq \mathbb{P}_\tilde{G}(A_\delta^c) + 2 \exp \left\{ -\frac{\varepsilon^3 \delta \log \log (\delta m)}{16} \right\}. \quad (102)$$

Now, define

$$\varrho_\varepsilon \triangleq 1 + \frac{3 \varepsilon (1-\varepsilon)}{4 (1+\varepsilon)} - \varepsilon^2,$$

We repeat the same proof on the event $B_1(\delta, \varepsilon, m)$ now for time steps between $(1+\varepsilon)T_m$ and $(1+\varepsilon)^2T_m$, defining again a stopping time

$$\eta_2 \triangleq \inf \left\{ s \geq (1+\varepsilon)T_m; \tilde{D}_s < (1-\varepsilon)\varrho_\varepsilon \sqrt{\delta T_m \log \log (\delta T_m)} \right\},$$

and proving inequalities analogous to \[96, 100, 101\]. In this way, we can define the event

$$B_2(\delta, \varepsilon, m) \triangleq B_1(\delta, \varepsilon, m) \cap \left\{ \tilde{D}_{(1+\varepsilon)^2T_m} \geq \varrho_\varepsilon^2 \sqrt{\delta T_m \log \log (\delta T_m)} \right\} \quad (103)$$

and show that

$$\mathbb{P}_\tilde{G} (B_2(\delta, \varepsilon, m)^c) \leq \mathbb{P}_\tilde{G}(A_\delta^c) + 2 \exp \left\{ -\frac{\varepsilon^3 \delta \log \log (\delta m)}{16} \right\} + \exp \left\{ -\frac{\varepsilon^3 \delta \varrho_\varepsilon^2 \log \log (\delta T_m)}{16 (1+\varepsilon)} \right\}. \quad (104)$$

We define inductively, for $n \in \mathbb{N}$,

$$B_{n+1}(\delta, \varepsilon, m) \triangleq B_n(\delta, \varepsilon, m) \cap \left\{ \tilde{D}_{(1+\varepsilon)^{n+1}T_m} \geq \varrho_\varepsilon^{n+1} \sqrt{\delta T_m \log \log (\delta T_m)} \right\}, \quad (105)$$

and prove in the same way that

$$\mathbb{P}_\tilde{G} (B_n(\delta, \varepsilon, m)^c) \leq \mathbb{P}_\tilde{G}(A_\delta^c) + 2 \sum_{k=1}^{n} \exp \left\{ -\frac{\varepsilon^3 \delta \varrho_\varepsilon^{2(k-1)} \log \log (\delta T_m)}{16 (1+\varepsilon)^{k-1}} \right\}. \quad (106)$$
We note that \( \{ \tilde{D}_k \}_{k \geq 1} \) is transient on \( \bigcap_{n \geq 1} B_n(\delta, \varepsilon, m) \). This implies
\[
P_G \left( \{ \tilde{D}_k \}_{k \geq 1} \text{ is transient} \right) \geq 1 - \exp \left\{ \frac{-\varepsilon^3 \delta^2}{16(1 + \varepsilon)^{k-1}} \right\}.
\] (107)

We note however that for \( \varepsilon \) sufficiently small, \( \delta^2 > (1 + \varepsilon) \). Choosing such an \( \varepsilon \), we can then choose \( \delta \) arbitrarily small, then \( m \) arbitrarily large depending on \( \delta \) in order to make the above right hand side as close as we want to 1. This implies the result.

We now have all the tools needed for the proof of the main result of this section, which states that there exist only one leader, vertex with maximum degree, and its degree is much larger than the degree of its adversaries. In general lines, we first prove that the set of candidates for the leadership is finite almost surely. Then, Lemma 5.2 gives us that only one of them can have the largest degree.

**Proof of Theorem 6.1.** If \( f \in \text{RES}(-\gamma) \) with \( \gamma \in [0,1) \), by Lemma 3.1, the monotonicity of \( \phi \), the fact that \( \tau_i \geq i \), and Equation (65), we know that
\[
P \left( \exists s \geq 1, \frac{d_s(i)}{\phi(s)} \geq \frac{1}{\sqrt{\phi(i)}} \right) \leq \exp \left\{ -C \sqrt{\phi(\tau_i)} \right\} \leq \exp \left\{ -C_i^{1/4} \right\}.
\] (108)

For \( f \equiv p \), Theorem 3.3 and the fact that \( F^{-1}(i) = i/p \) imply an analogous inequality. By the Borel-Cantelli Lemma, there exists a random integer \( M > 0 \) such that, for every \( i \geq M \),
\[
d_s(i) \leq \frac{\phi(s)}{\phi(i)}.
\] (109)

Then, by Theorem 3.5, there exists \( N_0 \) such that, for every \( s \geq N_0 \),
\[
\max_{i \in V(G_s(f))} d_s(i) \geq \frac{\phi(s)}{\phi(N_0)}.
\] (110)

But
\[
\frac{\phi(s)}{\sqrt{\phi(i)}} \geq \frac{\phi(s)}{\phi(N_0)} \implies \phi(i) \leq \phi(N_0)^2 \implies i \leq \phi(N_0)^4,
\] (111)

since \( \phi(i) \geq \sqrt{i} \) by the definition of \( \phi \) and the fact that \( f(s) \leq 1 \). Defining the random variable
\[
i_0 := \max \{ i \in \mathbb{N} ; i \leq \phi(N_0)^4 \},
\] (112)

we obtain that only the first \( M + i_0 \) vertices can be a candidate for the vertex with maximum degree. At the same time, we know by Lemma 5.2 that the distance between the degrees of any finite number of given vertices goes to infinity almost surely. This finishes the proof of the result.

\[ \square \]

**6. Central Limit Theorems**

In this section we prove a Central Limit Theorem for the scaled maximum degree process \( \{ M_t \}_{t \geq 1} \). We begin by taking a step back and proving a CLT for the degree of a fixed vertex. Then, this result combined with Theorem 4.4 allows us to extend the CLT for the degree of a single vertex to the maximum degree.

**Theorem 6.1 (CLT for the degree of a vertex).** Let \( f \) be a monotone non-increasing edge-step function such that \( (V)_{\infty} \) holds. Denoting by \( N \) a normal random variable, independent from \( \zeta_i \), with expectation 0 and variance 1, we have

- if \( (S)^c \) holds and if either \( (\text{Const})_p \) or \( (D)_0 \) holds, then
\[
\sqrt{\phi(s)} \left( \frac{d_s(i)}{\phi(s)} - \zeta_i \right) \implies \sqrt{\zeta_i} \mathcal{N} ;
\] (113)
• if (S) holds, then denoting

$$\xi(\infty) = \lim_{s \to \infty} \phi(s)s^{-1} > 0,$$

we have

$$\sqrt{\phi(s)} \left( \frac{d_s(i)}{\phi(s)} - \zeta_i \right) \Rightarrow \sqrt{\zeta_i} \left( 1 - \frac{\xi(\infty)}{2} \zeta_i \right) \mathcal{N}.$$  \hfill (114)

**Proof.** In the beginning it will again be important to use the conditioned measure \(\mathbb{P}_{i,t_i}\). We will use Corollary 4.2.1 from [7] (cf. the proof of Theorem 4.1 (i) of [16], or Corollary 3.5 of [14] together with the discussion after Equation (3.7) in the same book). Start by defining the martingale difference array \((\kappa^n_s)_{n \in \mathbb{N}, s \geq n}^n:

$$\kappa^n_s := \sqrt{\phi(n)} \left( \frac{d_s+1(i)}{\phi(s+1)} - d_s(i) \phi(s) \right)
= \sqrt{\phi(n)} \left( \frac{\Delta d_s(i)}{\phi(s+1)} - \frac{d_s(i)}{s \phi(s+1)} \left( 1 - \frac{f(s+1)}{2} \right) \right),$$  \hfill (115)

the second equality following by the same reasoning as in (3). We note that, by Proposition 4.3 \(\mathbb{P}_{i,t_i} \text{-a.s.},

$$\sum_{s=n}^\infty \kappa^n_s = \sqrt{\phi(n)} \left( \tilde{c}_{i,t_i} - \frac{d_n(i)}{\phi(n)} \right).$$  \hfill (116)

We will need to study the quantity

$$\sum_{s=n}^\infty \mathbb{E}_{i,t_i} \left[ (\kappa^n_s)^2 \mathcal{F}_s \right] = \phi(n) \sum_{s=n}^\infty \mathbb{E}_{i,t_i} \left[ \left( \frac{\Delta d_s(i)}{\phi(s+1)} - \frac{d_s(i)}{s \phi(s+1)} \left( 1 - \frac{f(s+1)}{2} \right) \right)^2 \mathcal{F}_s \right].$$

Note that, by the same reasoning as in (3),

$$\mathbb{P}_{i,t_i}(\Delta d_s(i) = 1|\mathcal{F}_s) = \frac{f(s+1)\frac{d_s(i)}{2s} + 2(1 - f(s+1)) \frac{d_s(i)}{2s} \left( 1 - \frac{d_s(i)}{2s} \right)}{\frac{1}{2s}},$$

$$\mathbb{P}_{i,t_i}(\Delta d_s(i) = 2|\mathcal{F}_s) = (1 - f(s+1)) \frac{d_s(i)^2}{4s^2},$$

which yields

$$\mathbb{E}_{i,t_i} \left[ (\Delta d_s(i))^2 \mathcal{F}_s \right] = \frac{f(s+1)\frac{d_s(i)}{2s} + 2(1 - f(s+1)) \frac{d_s(i)}{2s} \left( 1 - \frac{d_s(i)}{2s} \right) + 4(1 - f(s+1)) \frac{d_s(i)^2}{4s^2}}{2s},$$

$$\mathbb{E}_{i,t_i} \left[ (\Delta d_s(i))^2 \mathcal{F}_s \right] = \frac{d_s(i)}{s} \left( 1 - \frac{f(s+1)}{2} \right) + 2(1 - f(s+1)) \frac{d_s(i)^2}{4s^2}.$$  \hfill (118)

Using the above equation and (3), we obtain,

$$\mathbb{E}_{i,t_i} \left[ \left( \frac{\Delta d_s(i)}{\phi(s+1)} - \frac{d_s(i)}{s \phi(s+1)} \left( 1 - \frac{f(s+1)}{2} \right) \right)^2 \mathcal{F}_s \right]$$

$$\mathbb{E}_{i,t_i} \left[ \left( \frac{\Delta d_s(i)}{\phi(s+1)} - \frac{d_s(i)}{s \phi(s+1)} \left( 1 - \frac{f(s+1)}{2} \right) \right)^2 \mathcal{F}_s \right]$$

$$\mathbb{E}_{i,t_i} \left[ \left( \Delta d_s(i)^2 \mathcal{F}_s \right) - \frac{d_s(i)^2}{s \phi(s+1)^2} \left( 1 - \frac{f(s+1)}{2} \right)^2 + \frac{d_s(i)^2}{s^2 \phi(s+1)^2} \left( 1 - \frac{f(s+1)}{2} \right)^2 \right]$$

$$\mathbb{E}_{i,t_i} \left[ \left( \Delta d_s(i)^2 \mathcal{F}_s \right) - \frac{d_s(i)^2}{s \phi(s+1)^2} \left( 1 - \frac{f(s+1)}{2} \right)^2 + \frac{d_s(i)^2}{s^2 \phi(s+1)^2} \left( 1 - \frac{f(s+1)}{2} \right)^2 \right]$$

$$= \frac{d_s(i)}{s \phi(s+1)^2} \left( 1 - \frac{f(s+1)}{2} \right) + (1 - f(s+1)) \frac{d_s(i)^2}{2s^2 \phi(s+1)^2} - \frac{d_s(i)^2}{s^2 \phi(s+1)^2} \left( 1 - \frac{f(s+1)}{2} \right)^2.$$
Using Proposition 4.2, define the variable
\[ \epsilon_{i,t}(s) := 1 - \frac{d_s(i)}{\phi(s + 1) \xi_{i,t}}, \]
which goes to 0 as \( s \) goes to infinity \( \mathbb{P}_{i,t} \)-a.s. by Proposition 4.1. The above equation then yields
\[
\mathbb{E}_{i,t} \left[ \left( \frac{\Delta d_s(i)}{\phi(s + 1)} - \frac{d_s(i)}{s \phi(s + 1)} \left( 1 - f(s + 1) \right) \right)^2 \right] \]
\[ = \frac{\xi_{i,t} (1 - \epsilon_{i,t}(s))}{s \phi(s + 1)} \left( 1 - f(s + 1) \right) + (1 - f(s + 1)) \frac{\xi_{i,t}^2 (1 - \epsilon_{i,t}(s))^2}{2s^2} \]
\[ - \frac{\xi_{i,t}^2 (1 - \epsilon_{i,t}(s))^2}{2s^2} \left( 1 - f(s + 1) \right)^2. \]
For \( f \) such that \( \sum_{s=1}^{\infty} f(s)/s = \infty \), we have that \( \phi(n)/n \to 0 \) as \( n \to \infty \) and therefore
\[
\phi(n) \sum_{s=n}^{\infty} \left( 1 - f(s + 1) \right) \frac{\xi_{i,t}^2 (1 - \epsilon_{i,t}(s))^2}{2s^2} \to 0 \quad \text{as} \quad n \to \infty.
\]
If \( f \equiv p \) equations (119) and (120) imply
\[
\sum_{s=n}^{\infty} \mathbb{E}_{i,t} \left[ (\kappa_s^n)^2 | \mathcal{F}_s \right] = o_n(1) + \phi(n)(1 + o_n(1))(1 - p/2) \xi_{i,t} \sum_{s=n}^{\infty} s^{-2+p/2}
\]
\[ = o_n(1) + \xi_{i,t} \phi(n)n^{-1+p/2} \]
\[ \to 0 \quad \text{as} \quad n \to \infty. \]
If \( f(t) \downarrow 0 \) but \( \sum_{s=1}^{\infty} f(s)/s = \infty \), then, recalling the definition of the slowly varying function \( \xi \) from (10), and using Theorem B.3 we obtain
\[
\sum_{s=n}^{\infty} \mathbb{E}_{i,t} \left[ (\kappa_s^n)^2 | \mathcal{F}_s \right] = o_n(1) + \phi(n)(1 + o_n(1)) \xi_{i,t} \sum_{s=n}^{\infty} s^{-2} \xi(s)^{-1}
\]
\[ = o_n(1) + \xi_{i,t} \phi(n)(n \xi(n))^{-1} \]
\[ \to 0 \quad \text{as} \quad n \to \infty. \]
Finally, if \( \sum_{s=1}^{\infty} f(s)/s^{-1} < \infty \), we recall that the limit of \( \phi(s)/s \) as \( s \) goes to infinity exists, and again by Theorem B.3
\[
\sum_{s=n}^{\infty} \mathbb{E}_{i,t} \left[ (\kappa_s^n)^2 | \mathcal{F}_s \right]
\]
\[ = \phi(n) \xi_{i,t} \sum_{s=n}^{\infty} \left( (1 + o_n(1))s^{-2} \xi(s)^{-1} + (1 + o_n(1)) \xi_{i,t} 2^{-1}s^{-2} - (1 + o_n(1)) \xi_{i,t} s^{-2} \right)
\]
\[ = (1 + o_n(1)) \xi_{i,t} \phi(n)(n \xi(n))^{-1} + (1 + o_n(1)) 2^{-1} \xi_{i,t} \phi(n)n^{-1} - (1 + o_n(1)) \xi_{i,t} \phi(n)n^{-1}
\]
\[ \to 0 \quad \text{as} \quad n \to \infty. \]
We note that in every case, by (114) and the fact that \( d_s(i) \leq 2s \) and \( \Delta d_s(i) \leq 2 \),
\[
\sup_{s \geq n} |\kappa_s^n| \leq \sqrt{\phi(n)} \cdot \frac{4}{\phi(n + 1)} \to 0 \quad \text{as} \quad n \to \infty,
\]
and therefore the following Lindenberg-type condition is satisfied of every \( \varepsilon > 0 \):
\[
\sum_{n=1}^{\infty} \mathbb{E}_{i,t} \left[ (\kappa_s^n)^2 \mathbb{1}[|\kappa_s^n| > \varepsilon] | \mathcal{F}_s \right] \to 0 \quad \text{as} \quad n \to \infty.
\]
since the sum is identically equal to 0 for large enough \( n \). Equations (113) and (114) imply that we are then under the hypotheses of Corollary 4.2.1 of from [7], which in turn implies, together with the symmetry of the normal distribution, that (113) and (114) hold true in \( P_{i,t} \). To end the proof, we assume that we are in the case of Equation (113), the other case being proven in the exact same manner. Given some number \( a \in \mathbb{R} \),

\[
\begin{align*}
\mathbb{P} \left( \sqrt{\phi(s)} \left( \frac{d_s(i)}{\phi(s)} - \zeta_i \right) \leq a \right) &= \sum_{t_i=1}^{\infty} \mathbb{P}(\tau_i = t_i) \mathbb{P}_{i,t_i} \left( \sqrt{\phi(s)} \left( \frac{d_s(i)}{\phi(s)} - \zeta_{i,t_i} \right) \leq a \right) \\
&= \sum_{t_i=1}^{\infty} \mathbb{P}(\tau_i = t_i) \left[ \mathbb{P}_{i,t_i} \left( \sqrt{\zeta_{i,t_i} N} \leq a \right) + o_{s,t_i}^{i,t_i}(1) \right] \\
&= \mathbb{P} \left( \sqrt{\zeta_i N} \leq a \right) + \sum_{t_i=1}^{\infty} \mathbb{P}(\tau_i = t_i) o_{s,t_i}^{i,t_i}(1),
\end{align*}
\]

where \( o_{s,t_i}^{i,t_i}(1) \) is a function of \( s \) depending on \( i \) and \( t_i \) that goes to 0 as \( s \to \infty \) and is uniformly bounded by 2 in the \( \ell_\infty \)-norm, since it is the difference between two probabilities. Now, for some \( N \in \mathbb{N} \), we have

\[
\left| \mathbb{P} \left( \sqrt{\phi(s)} \left( \frac{d_s(i)}{\phi(s)} - \zeta_i \right) \leq a \right) - \mathbb{P} \left( \sqrt{\zeta_i N} \leq a \right) \right| \leq \sum_{t_i=1}^{N} o_{s,t_i}^{i,t_i}(1) + 2\mathbb{P}(\tau_i > N). \tag{127}
\]

Since \( \sum_{s \geq 1} f(s) = \infty \), \( \mathbb{P}(\tau_i > N) \) goes to 0 with \( N \). Choosing \( N \) large and then choosing \( s \) sufficiently large, depending on \( N \), we show that the right hand side of the above equation can be made arbitrarily small, finishing the proof of the theorem. \( \square \)

We can then prove the maximum degree CLT. But before, we recall some definitions

\[
\begin{align*}
X_{s,i} &:= \frac{d_s(i)}{\phi(s)}, \quad M[s,i] := \max_{1 \leq k \leq i} \{ X_{s,k} \}, \quad M_s := \max_{k \geq 1} \{ X_{s,k} \}, \\
\mu(i) &:= \max_{1 \leq k \leq i} \{ \zeta_k \}, \quad \mu := \sup_{k \geq 1} \{ \zeta_k \}.
\end{align*}
\]

And also recall from the Persistent Leadership Theorem 1.4 the definition of \( U_{\max}(s,N) \), which is the event where there exist only one vertex with maximum degree at time \( s \) and all the remaining vertices have degree smaller than the maximum minus \( N \).

**Proof of Theorem 1.3:** CLT for the maximum degree. The proof is mostly analogous to that of Theorem 0.1. The main issue is that \( M_s \) is usually not a martingale, but it does behave like one if we know that there exists only one vertex with the maximum degree.

We will prove the result for the process \( M_s \mathbb{1}\{U_{\max}(s,3)\} \), since \( M_s \mathbb{1}\{U_{\max}(s,3)^c\} \) is 0 at all but a finite number of times a.s., this will imply the theorem. Define

\[
\tilde{\kappa}_s^n := \sqrt{\phi(n)} (M_{s+1} \mathbb{1}\{U_{\max}(s+1,3)\} - M_s \mathbb{1}\{U_{\max}(s,3)\}) \tag{128}
\]

By Theorem 1.1 we have, \( P_{i,t_i} \)-a.s. for all but finitely many integers \( n \),

\[
\sum_{s=n}^{\infty} \tilde{\kappa}_s^n = \sqrt{\phi(n)} (\mu - M_n \mathbb{1}\{U_{\max}(s,3)\}) \tag{129}.
\]
We have
\[ \mathbb{E}
\left[
\left(
\hat{\kappa}_n^s
\right)^2 \mid F_s
\right]
\]
\[ = \phi(n) \mathbb{E}
\left[
\left(M_{s+1}\mathbb{1}\{U_{\max}(s+1,3)\} - M_s\mathbb{1}\{U_{\max}(s,3)\}\right)^2 \mid F_s
\right]
\]
\[ = \phi(n) \mathbb{E}
\left[
\left(M_{s+1}\mathbb{1}\{U_{\max}(s+1,3)\} - \mathbb{1}\{U_{\max}(s,3)\}\right) + (M_{s+1} - M_s)\mathbb{1}\{U_{\max}(s,3)\}\right)^2 \mid F_s
\]
\[ = \phi(n) \mathbb{E}
\left[
M_{s+1}^2\mathbb{1}\{U_{\max}(s+1,3)\} - \mathbb{1}\{U_{\max}(s,3)\}\right)^2 \mid F_s
\]
\[ + 2\phi(n)\mathbb{1}\{U_{\max}(s,3)\} \mathbb{E}
\left[
M_{s+1}(M_{s+1} - M_s)(\mathbb{1}\{U_{\max}(s+1,3)\} - \mathbb{1}\{U_{\max}(s,3)\})\mid F_s
\right]
\]
\[ + \phi(n)\mathbb{1}\{U_{\max}(s,3)\} \mathbb{E}
\left[
(M_{s+1} - M_s)^2 \mid F_s
\right].
\]

Observe that in order for \(\mathbb{1}\{U_{\max}(s+1,3)\} - \mathbb{1}\{U_{\max}(s,3)\}\) to be different than 0, either \(U_{\max}(s,3)\) does not take place, or \(U_{\max}(s,3)\) happens but at time \(s+1\) the degree of some vertex gets within distance 3 of the maximum degree. Since, a vertex can only increase its degree by 1 or 2 at each step, we obtain
\[ |\mathbb{1}\{U_{\max}(s+1,3)\} - \mathbb{1}\{U_{\max}(s,3)\}| \leq \mathbb{1}\{U_{\max}(s,5)^c\} + \mathbb{1}\{U_{\max}(s,3)^c\}. \]

Together we Lemma 1.3, the above equations imply that, almost surely, for every \(n\) and all but a finite number of integers \(s \in \mathbb{N}\),
\[ \mathbb{E}
\left[
\left(\kappa_n^s\right)^2 \mid F_s
\right] = \phi(n)\mathbb{1}\{U_{\max}(s,3)\} \mathbb{E}
\left[
(M_{s+1} - M_s)^2 \mid F_s
\right].
\]

In the event \(U_{\max}(s,3)\), there exists at times \(s\) and \(s+1\) a unique vertex \(\tilde{v}\) with the maximum degree, and the only way for the maximum degree to increase at time \(s+1\) is if the degree of \(\tilde{v}\) is increased. The right hand side of the above equation can then be studied in the same way as in Theorem 5.1 substituting \(\tilde{\zeta}_{s,t}\) by \(\mu\) and \(d_s(i)\phi(s)^{-1}\) by \(M_s\). Equations analogous to (122 123 124 125) are derived and the proof of the result follows from Corollary 4.2.1 from [7].

\[ \square \]

7. Application: Bootstrap percolation

This section is devoted to understand the evolution of the Bootstrap Percolation process on \(G_t(f)\), when \(f\) satisfies condition (S) and belongs to \(\text{RES}(-\gamma)\), with \(\gamma \in (0,1)\). We are going to prove Theorem 1.5 which essentially states that, over the typical graph, the bootstrap percolation process have the outbreak phenomenon for any initial amount of infected vertices which grows to infinity and threshold \(r \leq 2\), i.e., even for small initial infected vertices the infectious spreads to a positive fraction of the whole graph. The reader may recall the definition of the bootstrap percolation at page 4.

Proof of Theorem 1.5. We begin by recalling some properties of \(G_t(f)\) and \(f\) when \(f\) satisfies the summability condition (S) and belongs to \(\text{RES}(-\gamma)\), for \(\gamma \in [0,1)\). First, from (S) and the fact that \(f \in \text{RES}(-\gamma)\) we have that
\[ \lim_{t \to \infty} f(t) \log t = 0. \]

And by the definition of the normalizing factor \(\phi\), recall (\ref{phi-def}) which implies that under (S) there exist two \(f\)-dependent constants \(C_1\) and \(C_2\) such that
\[ C_1 t \leq \phi(t) \leq C_2 t. \]

Moreover, by the Karamata’s Theorem (Theorem 13) we have that
\[ \mathbb{E}V_t(f) \sim f(t)t \implies \mathbb{E}V_t(f) \leq \frac{5}{4} f(t)t, \]

for large enough \(t\). In the light of the above relation, we also point out that we may suppose that the sequence \(\{a_t\}_{t \in \mathbb{N}}\) is given in such way that \(a_t \ll f(t)t\). Otherwise, by the definition of the Bootstrap percolation process \(|I_0| \approx a_t \geq cf(t)t\), i.e., if \(a_t\) is too large, the initial set of infected vertices already represents a positive proportion of the total number of vertices, \(w.h.p\), and we have nothing to prove.
Now, let \( g \) and \( h \) be the following functions

\[
g(t) := \frac{1}{f(t)}; \quad h(t) := \frac{C_1}{2r^2C_2} \cdot \frac{g(t)}{\log(t)}. \tag{134}
\]

And define an important property we expect the typical graph generated by the graph process satisfies. Let \( \mathcal{P}_1^t \) be the following class of (multi)graphs

\[
\mathcal{P}_1^t := \left\{ G = (V, E) : |V| \leq \frac{3}{2} f(t) t \text{ and } \exists v^* \in V(G), \text{ with } d_G v^* \geq \frac{C_1 t}{C_2 h(a_t)} \right\}. \tag{135}
\]

In words, \( \mathcal{P}_1^t \) is the collection of multigraphs with \( O(f(t)t) \) vertices and containing a vertex \( v^* \) whose degree is at least \( C_1 C_2^{-1} t/h(a_t) \), where \( C_1 \) and \( C_2 \) are the \( f \)-dependent constants in (132).

We say that a multigraph in \( \mathcal{P}_1^t \) has property 1 and call the distinguishable vertex \( v^* \) a star.

In the next lines we are going to prove that \( \text{w.h.p } G_t(f) \) has the property 1. Now, by Theorem 3.5, choosing \( N = h(a_t) \), we have that

\[
P \left( \exists i \in \{1, 2, \ldots, h(a_t)\}, \text{ such that } d_t(i) \geq \frac{C_1}{C_2 h(a_t)} t \right) \geq 1 - \exp\{-C_f h(a_t)\} \tag{136}
\]

for another \( f \)-dependent constant \( C_f \). Since \( V_t(f) \) is sum of independent Bernoulli random variables, Chernoff bound yields

\[
P \left( V_t(f) \geq \frac{3}{2} f(t) t \right) \leq \exp \left\{ -\frac{12 \mathbb{E} V_t(f)}{25} \right\}. \tag{137}
\]

which proves that \( G_t(f) \) has property 1 w.h.p.

Now, we require another property of the typical multigraph.

\[
\mathcal{P}_2^t := \left\{ G = (V, E) : |V| = O(f(t)t) \text{ and } \exists H \subset G, \text{ connected and with } d_G H \geq \frac{t}{2} \right\}. \tag{138}
\]

In words, \( \mathcal{P}_2^t \) represents those (multi)graphs on \( O(f(t)t) \) vertices containing a subgraph whose degree is at least a constant times \( t \). We claim that \( G_{2t}(f) \) also has property 2 w.h.p. Suppose for a moment that \( G_t(f) \) has property 1 and let \( v^* \) be its star. Also let \( j \) be a vertex whose degree is at least \( g(t) \). Since \( f \) is regularly varying function with index \( -\gamma \in (-1, 0] \), it follows \( g(t) = o(\sqrt{t}) \), thus there exists such \( j \) w.h.p.

Now, fix \( k \in \{2, 3, \ldots, r+1\} \) and note that for all times \( s \in [(k-1)t, kt] \) we have

\[
P \left( j \leftrightarrow v^* \text{ at step } s+1 \mid G_s(f), d_t(j) \geq g(t), v^* \text{ is a star} \right) = (1 - f(s+1)) \frac{d_s(j) d_s v^*}{2 s^2} \geq \frac{C_1}{4(r+1)^2 C_2 t^2 h(a_t)} g(t) t \tag{139}
\]

for sufficiently large \( t \), since the degree of a vertex is increasing in \( s, s \leq (r+1) t \) and \( f \searrow 0 \). Thus, for each \( k \), and recalling that \( a_t < t \), for all \( t \), our choice of \( h \) leads to

\[
P \left( j \leftrightarrow v^* \text{ in } G_{(k-1)t}, d_{(k-1)t}(j) \geq g(t), v^* \text{ is a star} \right) \leq \left( 1 - \frac{C_1}{4(r+1)^2 C_2 t h(a_t)} g(t) t \right)^t \leq e^{-2 \log t}
\]

Recalling that \( G_t(f) \) has at most \( t \) vertices deterministically, by an union bound argument the above inequality allows us to say that \( \text{w.h.p} \) in time interval \( [(k-1)t, kt] \) all the vertices in \( G_t(f) \) having degree at least \( g(t) \) end up sending at least one edge to the star in \( G_{kt}(f) \). Therefore, \( \text{w.h.p} \) all vertices in \( G_t(f) \) having degree at least \( g(t) \) actually send at least \( r \) edges to the star in \( G_{(r+1)t}(f) \).

Finally, let \( H_t \) be the subgraph of \( G_t(f) \) composed by all vertices whose degree is at least \( g(t) \). Conditioning that \( V_t(f) \leq 3f(t)t/2 \) vertices, the sum of the degree of all vertices not in \( H_t \) satisfies

\[
\sum_{v \not\in H_t} d_t v < \frac{3}{2} g(t) f(t) t = \frac{3t}{2}. \tag{140}
\]
Since the sum of all degrees equals in $G_t(f)$ equals $2t$, by the above inequality, $d_tH_t \geq t/2$, whenever $V_t(f) \leq 3f(t)/2$, which occurs w.h.p. This shows that whenever $G_t(f)$ has property 1, $G_{(t+1)}$ has properties 1 and 2 with the extra feature that each vertex in $H$ shares at least $r$ edges with $v^*$. w.h.p. Therefore,

$$
\mathbb{P} \left( G_t \notin \mathcal{P}_t^2 \right) \leq \mathbb{P} \left( G_t \notin \mathcal{P}_t^1, G_{t/r} \in \mathcal{P}_{t/r}^1 \right) + \mathbb{P} \left( G_t \notin \mathcal{P}_t^1 \right) = o(1).
$$

In the next lines we show that if $G_t(f)$ has property 2 then the subgraph $H_t$, seen as a subgraph of $G_{2t}$, has at least $c_t(t)$ vertices sending $r$ edges to it, for a constant $c$, w.h.p. The way we prove this is forcing a positive proportion of vertices of degree 1 in $G_t(f)$ to connect in $H_t$ at time $2t$ and iterating the process $r$ times, similarly to what we have done for $H_t$. For this, we need some definitions. Denote by $N_t(1)$ the random variable which counts the number of vertices having degree 1 at time $t$. Also, let $A_t$ and $C_t$ be the following random sets

$$
A_t := \{ v \in N_t(1) \mid v \to H_t \text{ up to time } s \}; \quad C_t = \{ v \in N_t(1) \mid v \to H_t \text{ up to time } s \}.
$$

Observe that for all $s \in [t, 2t]$ we have $N_t(1) = A_t \cup C_t$. In words, $A_t$ is the set of those vertices of $N_t(1)$ which did not connect to $H_t$ at none of the edges created at time between $t$ and $s$. In this sense, we say a vertex in $A_t$ is available to connect to $H_t$. On the other hand, $C_t$ contains exactly those that connected to $H_t$ between time $t$ and $s$.

At each step, we try to connect an available vertex to $H_t$ using an edge-step. We are going to prove that w.h.p after $t$ steps $|C_{2t}|$ is a positive proportion of $N_t(1)$. So, let $\{Y_s\}_{s \geq t}$ be the following process

$$
Y_s := \{ \text{some vertex } v \text{ in } A_s \text{ connects to } H_t \text{ at time } s \}.
$$

Note that $|C_{2t}| = \sum_{s=t}^{2t} Y_s$. And consider the following stopping time

$$
\eta := \inf \{ s \geq t : |C_s| \geq N_t(1)/2 \},
$$

To simplify our writing, we will denote by $\mathbb{P}_{G_t}$ the law of the graph process conditioned on a graph $G_t$ with property 2. With this notation in mind, observe that, for $s \in [t, 2t]$, there exists a constant $c$ such that

$$
\mathbb{P}_{G_t} \left( Y_{s+1} = 1 \mid G_s(f) \right) \geq (1 - f(s + 1)) \frac{|A_s| d_s H_t}{2s^2} \geq \frac{c|A_s|}{t},
$$

since $d_s H_t \geq c't$. Then, by the definition of $\eta$, it also follows that

$$
\mathbb{P}_{G_t} \left( Y_{s+1} = 1, \eta > s \mid G_s(f) \right) \geq \frac{cN_t/2}{t} \mathbb{1}_{\{\eta > s\}} =: q \mathbb{1}_{\{\eta > s\}}.
$$

The above inequality implies that on the event $\{\eta > 2t\}$ the sum $\sum_{s=t}^{2t} Y_s$ dominates a r.v. distributed as $\text{bin}(t, q)$. Then,

$$
\mathbb{P}_{G_t} \left( |C_{2t}| \leq \varepsilon N_t(1) \right) \leq \mathbb{P}_{G_t} \left( \text{bin}(t, q) \leq \varepsilon N_t(1), \eta > 2t \right) + \mathbb{P}_{G_t} \left( |C_{2t}| \leq \varepsilon N_t(1), \eta \leq 2t \right).
$$

Choosing $\varepsilon < \min\{c/2, 1/2\}$ and using Chernouff bounds, we obtain that

$$
\mathbb{P}_{G_t} \left( |C_{2t}| \leq \varepsilon N_t(1) \right) \leq \mathbb{P}_{G_t} \left( \text{bin}(t, q) \leq \varepsilon N_t(1) \right) \leq e^{-cN_t(1)},
$$

for some positive constant $c$, since $|C_{2t}| \geq N_t(1)/2$ on $\{\eta \leq 2t\}$ and $\text{bin}(t, q)$ has mean $cN_t/2$. The above upper bound combined with inequality (141) and Theorem 2 of [2], which guarantees that $N_t(1)$ is large enough, yields $\mathbb{P} \left( |C_{2t}| \leq \varepsilon N_t(1) \right)$ is bounded from above by

$$
\mathbb{P} \left( |C_{2t}| \leq \varepsilon N_t(1), G_t(f) \in \mathcal{P}_t^2, N_t(1) \geq c f(t) t \right) + \mathbb{P} \left( G_t(f) \notin \mathcal{P}_t^2 \text{ or } N_t(1) \leq c f(t) t \right) = o(1).
$$

Now we can iterate this argument with the set $C_{2t}$ instead of $N_t(1)$ to guarantee w.h.p that there exist at least $c f(t) t$ vertices sending at least two edges to $H_t$ in $G_{3t}(f)$. This shows that there exists a constant $c$ such that w.h.p $G_t(f)$ belongs to the family of (multi)graphs below

$$
P_t^3 := \{ G \in \mathcal{P}_t^2 : \exists S \subseteq G \text{ s.t. } |S| \geq c f(t) t \text{ and all } v \in S \text{ connects } r \text{ times to } H_t \}.
$$
Finally, fixed a sequence \( \{a_t\}_{t \in \mathbb{N}} \) and \( r \geq 2 \), we let the family \( \mathcal{P}_t^3 \) constructed as above be our family \( \mathcal{G}_t \). Now we are left to prove that the bootstrap percolation process on a (multi)graph in the collection \( \mathcal{G}_t \) and parameters \( a_t \) and \( r \cdot w.h.p \) has an outbreak of infection.

Summarizing all we have proven about \( G_t(f) \) so far, a (multi)graph in \( \mathcal{G}_t \) has the following structure:

1. \( G_t(f) \) has at most \( cf(t)t \) vertices;
2. There exists a star \( v^* \in G_t(f) \) such that
   \[
   d_t v^* \geq \frac{C_1}{C_2} \cdot \frac{t}{h(a_t)}.
   \]
   And consequently
   \[
   \Gamma_t(v^*) \geq C \frac{f(t)t}{h(a_t)},
   \]
   where \( \Gamma_t(v^*) \) denotes the number of neighbors of \( v^* \). We explain the above lower bound later.
3. All vertices of degree at least \( g(t) \) are connected \( r \) times to \( v^* \). This is the subgraph \( H_t \);
4. There exists a subset \( S \subset G_t \) of order \( f(t)t \) whose elements are connected to \( H_t \) at least \( r \) times.

To see why (150) holds, observe that if \( v^* \) has degree at least \( Ct/h(a_t) \) at time \( t \), then for \( s \in [t, 2t] \), we have that
\[
\mathbb{P} \left( \Delta_G s(v^*) \geq 1, |\mathcal{F}_s|, d_t v^* \geq C \cdot \frac{t}{h(a_t)} \right) \geq C f(s) \cdot \frac{t}{2sh(a_t)} \geq \frac{C' f(2t)}{h(a_t)}.
\]

Thus, the number of neighbors at time \( 2t \) dominates a binomial random variable with parameters \( t \) and \( C' f(2t)/h(a_t) \).

Having all the properties above in mind, we start the bootstrap percolation process. Observe that the number of infected neighbors of \( v^* \) dominates a binomial random variable with parameters \( C f(t)t/h(a_t) \) and \( a_t/f(t) \). Thus, at the initial round, \( w.h.p \), \( v^* \) has at least \( r \) infected neighbors. At the second round the star gets infected and this is enough to spread the infection. At the third round all the subgraph \( H_t \) gets infected and finally at the next round \( S \) gets infected. This proves that in four steps a positive proportion of \( G_t(f) \) gets infected and proves our theorem.

\[ \square \]

### Appendix A. Martingale Concentration Inequalities

For the sake of completeness we state here two useful concentration inequalities for martingales which are used throughout the paper.

**Theorem A.1** (Azuma-Höfeding Inequality - see [3]). Let \( (M_n, \mathcal{F}_n)_{n \geq 1} \) be a martingale satisfying
\[
|M_{i+1} - M_i| \leq a_i
\]
Then, for all \( \lambda > 0 \) we have
\[
\mathbb{P} (|M_n - M_0| > \lambda) \leq \exp \left( -\frac{\lambda^2}{\sum_{i=1}^{n} a_i^2} \right).
\]

**Theorem A.2** (Freedman’s Inequality - see [13]). Let \( (M_n, \mathcal{F}_n)_{n \geq 1} \) be a (super)martingale. Write
\[
V_n := \sum_{k=1}^{n-1} \mathbb{E} \left[ (M_{k+1} - M_k)^2 | \mathcal{F}_k \right]
\]
and suppose that \( M_0 = 0 \) and
\[
|M_{k+1} - M_k| \leq R, \text{ for all } k.
\]
Then, for all $\lambda > 0$ we have
\[
P\left(M_n \geq \lambda, V_n \leq \sigma^2, \text{ for some } n\right) \leq \exp\left(-\frac{\lambda^2}{2\sigma^2 + 2R\lambda/3}\right).
\]

**Appendix B. Karamata theory**

The three following results are used throughout the paper.

**Corollary B.1** (Representation theorem - Theorem 1.4.1 of [5]). Let $f$ be a continuous regularly varying function with index of regular variation $\gamma$. Then, there exists a slowly varying function $\ell$ such that
\[
f(t) = t^{\gamma}\ell(t),
\]
for all $t$ in the domain of $f$.

**Corollary B.2.** Let $f$ be a continuous regularly varying function with index of regular variation $\gamma < 0$. Then,
\[
f(x) \to 0,
\]
as $x$ tends to infinity. Moreover, if $\ell$ is a slowly varying function, then for every $\varepsilon > 0$
\[
x^{-\varepsilon}\ell(x) \to 0 \text{ and } x^{\varepsilon}\ell(x) \to \infty
\]
\[\text{(153)}\]

**Proof.** Comes as a straightforward application of Theorem 1.3.1 of [5] and Corollary B.1. □

**Theorem B.3** (Karamata’s theorem - Proposition 1.5.8 of [5]). Let $\ell$ be a continuous slowly varying function and locally bounded in $[x_0, \infty)$ for some $x_0 \geq 0$. Then
\begin{enumerate}[(a)]
    \item for $\alpha > -1$
\[
\int_{x_0}^{x} t^{\alpha}\ell(t)dt \sim \frac{x^{1+\alpha}\ell(x)}{1+\alpha}.
\]\[\text{(154)}\]
    \item for $\alpha < -1$
\[
\int_{x}^{\infty} t^{\alpha}\ell(t)dt \sim \frac{x^{1+\alpha}\ell(x)}{1+\alpha}.
\]\[\text{(155)}\]
\end{enumerate}

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