A Practical Algorithm for Intersecting Weighted Context-free Grammars with Finite-State Automata

Thomas Hanneforth
Linguistics Dept.
University of Potsdam, Germany
Thomas.Hanneforth@uni-potsdam.de

Abstract

It is well known that context-free parsing can be seen as the intersection of a context-free language with a regular language (or, equivalently, the intersection of a context-free grammar with a finite-state automaton). The present article provides a practical efficient way to compute this intersection by converting the grammar into a special finite-state automaton (the GLR(0)-automaton) which is subsequently intersected with the given finite-state automaton. As a byproduct, we present a generalisation of Tomita’s algorithm to recognize several inputs simultaneously.

1 Introduction

At least since the paper of Billot and Lang (1989) which defined parsing as the intersection of a context-free language (given by a grammar) with a regular language (the “input”, which can be seen as a simple finite-state automaton) the importance of the notion of intersection for parsing purposes became apparent. Intersection is first of all defined on the language level: Intersect the (possibly infinite) set of strings which the grammar generates with the (possibly infinite) set of strings the finite-state automaton accepts. Since this may not be done effectively due to the infiniteness of the involved sets, the operation has to be lifted to the more compact level of the devices generating the sets.

In principle, there are at least two ways to intersect a context-free grammar $G$ with a finite-state automaton $A$:

1. Convert $G$ into a suitable pushdown automaton (PDA) $M$ and intersect $A$ with $M$ to get $M'$. Then extract the result grammar $G'$ from $M'$.

2. Intersect $G$ and $A$ directly.

With respect to 1., it is well known that the class of pushdown automata is closed under intersection with finite-state automata (henceforth FSA) (cf. Hopcroft and Ullman (1979)). The standard construction assumes a deterministic FSA (see next section) and operates both automata in parallel: Whenever the PDA makes a move on a certain input symbol $a$, this move is combined with a corresponding move of the FSA on $a$ (see Hopcroft and Ullman (1979) for details of the construction).

Method 1 works most efficiently in case of grammars in Greibach normal form (GNF). Remember that a context-free grammar is in GNF if each of its rules conforms to the format $A \rightarrow aB_1 \ldots B_k$ where $a$ is an alphabet symbol, $A$ is a nonterminal (phrase) symbol and $B_1 \ldots B_k$ is a possibly empty sequence of nonterminals. Given a grammar in GNF, it is very easy to construct a pushdown automaton from it: when the PDA has $A$ on its stack and next reads an $a$, it replaces $A$ by $B_1 \ldots B_k$. Every context-free grammar can be converted into a weakly equivalent grammar in GNF (cf. Hopcroft and Ullman (1979)), but this changes the trees generated by the grammar and may lead to a substantial increase in grammar size.

Method 2 was presented in Bar-Hillel et al. (1964) and works in the following way: Consider a grammar rule $X_0 \rightarrow X_1X_2 \ldots X_k$. Then choose an arbitrary state sequence $p_0 \ldots p_k$ from the state set $Q$ of the FSA and create a new grammar rule

$$
\langle p_0, X_0, p_k \rangle \rightarrow \langle p_0, X_1, p_1 \rangle \langle p_1, X_2, p_2 \rangle \ldots \langle p_{k-1}, X_k, p_k \rangle.
$$

(1)
Leaving the grammar fixed, the time complexity of this method is in $\mathcal{O}(|Q|^{|q|+1})$ where $r$ is the length of the longest right-hand side of a grammar rule. The drawback of the method is that it creates a great amount of useless nonterminals and rules. Nederhof and Satta (2008) improved the algorithm by adding simultaneous top-down and bottom-up filtering mechanisms to prevent the creation of useless symbols which of course does not change its worst-case complexity. This upper bound can be reduced to $\mathcal{O}(|Q|^3)$ by binarising the grammar rules (which naturally changes the generated trees). However, it is not clear how the algorithm behaves in practice on grammars with several thousands of rules ubiquitous in natural language processing.

In the present paper, we propose another algorithm based on the creation of a GLR(0) automaton which is subsumed by method 1 above. The rest of the paper is organised as follows: Section 2 briefly defines the relevant notions. Section 3 defines GLR(0) automata and presents an algorithm how to intersect them with FSAs. In Section 4 we report some experiments conducted with a big grammar extracted from a treebank.

## 2 Preliminaries

An alphabet $\Sigma$ is a finite set of symbols. A string $x = a_1 \ldots a_n$ over $\Sigma$ is a finite concatenation of symbols $a_i$ taken from $\Sigma$. The length of a string $x = a_1 \ldots a_n$ is symbolically $|x|$ is $n$. The empty string is denoted by $\varepsilon$ and has length zero. Let $\Sigma^*$ denote the set of all finite-length strings (including $\varepsilon$) over $\Sigma$.

A finite-state automaton (FSA) $A$ is a 5-tuple $\langle Q, \Sigma, q_0, \delta, F \rangle$ with $Q$ being a finite set of states; $\Sigma$, an alphabet, $q_0 \in Q$, the start state; $\delta : Q \times \Sigma \rightarrow 2^Q$, the transition function; and $F \subseteq Q$, the set of final states.

Given two states $p, q \in Q$, a path from $p$ to $q$ in $A$ is symbolically $p \xrightarrow{A} q$ is a sequence $s_0 s_1 \ldots s_k$ of states, such that $s_0 = p$, $s_k = q$, and for all $1 \leq i \leq k : \exists a \in \Sigma : s_i \in \delta(s_{i-1}, a)$. Given a path $\pi = p \xrightarrow{A} q$, define labels($\pi$) as the concatenation of the symbols labeling the transitions along $\pi$. For an empty path $\pi = s_0$, labels($\pi$) = $\varepsilon$. The length of path $\pi = s_0 s_1 \ldots s_k$ is symbolically $|\pi|$ is $k$. We use $p \xrightarrow{A} q$ to denote a path from $p$ to $q$ of length $k$ in $A$.

An FSA is called deterministic if for all symbol-state pairs $q, a$, $|\delta(q, a)| \leq 1$. For a deterministic FSA, we may modify $\delta$ to be a partial function $Q \times \Sigma \rightarrow Q$. When appropriate, we use $\delta$ as a total function by adding a special element $\perp$ to $Q$ denoting failure. For deterministic FSA, we use sometimes the notation $p \xrightarrow{a} q$ to denote transitions: $p \xrightarrow{a} q$ if $\delta(p, a) = q$.

Define $\delta^* : Q \times \Sigma \rightarrow G$ as the reflexive and transitive closure of $\delta$: $\forall q \in Q, \delta^*(q, \varepsilon) = q$ and $\forall q \in Q, a \in \Sigma, w \in \Sigma^* : \delta^*(q, aw) = \delta^*(\delta(q, a), w)$.

Given a (deterministic) FSA $A = \langle Q, \Sigma, q_0, \delta, F \rangle$, the language $L(A)$ of $A$ is defined as: $L(A) = \{ x \in \Sigma^* \mid \delta^*(q_0, x) \in F \}$.

A semiring $\mathcal{K}$ is a 5-tuple $(W, \oplus, \otimes, \bar{0}, \bar{1})$ such that $1$. $W$ is a non-empty set, the carrier set of the semiring, 2. $(W, \oplus, \bar{0})$ is a commutative monoid, 3. $(W, \otimes, \bar{1})$ is a monoid, 4. $\otimes$ distributes over $\oplus$, and 5. $\bar{0}$ is an annihilator for $\otimes$: $\forall x \in W : x \otimes \bar{0} = \bar{0} \otimes x = \bar{0}$.

In the following, we will identify a semiring $\mathcal{K}$ with its carrier set $W$. Common semirings are the tropical semiring $\mathcal{T} = \langle \mathbb{R}, \min, +, \infty, 0 \rangle$ and the probabilistic semiring $\mathcal{P} = \langle \mathbb{R}, +, \cdot, 0, 1 \rangle$.

A weighted context-free grammar (WCFG) $G$ over a semiring $\mathcal{K}$ is a 4-tuple $\langle N, \Sigma, S, P \rangle$: $N$ is a finite set, the non-terminals, $\Sigma$ is an alphabet, $S \in N$ the start term, and $P$ a finite set of pairs $\langle A \rightarrow \beta, c \rangle \in (N \times (\Sigma \cup N^*)^*) \times \mathcal{K}$, the set of weighted rules. A WCFG without rule weights is called a context-free grammar (CFG).

In particular, if $\mathcal{K}$ is the probabilistic semiring and if we define an additional condition on $\sigma$:

$$\forall A \in N : \sum_{\{ A \rightarrow \beta, c \} \in P} c = 1 \quad (2)$$

then $G$ is called a probabilistic context-free grammar (PCFG) (see also Nederhof and Satta (2008)). Fig. 1 shows a toy PCFG.

| 1: S $\rightarrow$ NP VP / 1.0 | 2: N $\rightarrow$ DET N / 0.6 |
| 3: NP $\rightarrow$ NE / 0.3 | 4: NP $\rightarrow$ NP PP / 0.1 |
| 5: PP $\rightarrow$ P NP / 1.0 | 6: VP $\rightarrow$ V / 0.5 |
| 7: VP $\rightarrow$ V NP / 0.4 | 8: VP $\rightarrow$ VP PP / 0.1 |

Figure 1: A toy PCFG with numbered rules. Probabilities are stated after $/$.
Let $G = \langle \Sigma, N, S, P \rangle$ be a context-free grammar, and let $V$ be $N \cup \Sigma$. Define a relation $\Rightarrow \subseteq V^* \times V^*$ as follows: $\alpha \Rightarrow \beta$ if $\alpha = xA\gamma, \beta = x\psi\gamma, x \in \Sigma^*, \gamma \in \Sigma^*$ and $A \rightarrow \psi \in P$.

Let $\Rightarrow$ be equal to $\bigcup_{k \geq 0} \Rightarrow_k$. A leftmost derivation of a string $w \in \Sigma^*$ is a sequence of elements from $\Rightarrow$ such that $S \Rightarrow X_1 \ldots X_k \Rightarrow \ldots \Rightarrow w$. We abbreviate this to $S \Rightarrow^n w$. The language of a CFG $G$ symbolically $L(G)$ – is defined as $L(G) = \{ w \in \Sigma^* \mid S \Rightarrow^n w \}$. Finally, given a CFG $G$ and an FSA $A$, define the intersection of $G$ and $A$ as follows: $G_\cap = G \cap A$ if $L(G) \cap L(A) = L(G_\cap)$.

The notion of a derivation and the language of a CFG carry over to WCFGs, as well as the notion of intersection. See Nederhof and Satta (2008) for details.

### 3 The Intersection Algorithm

The main idea to compute the intersection of a weighted context-free grammar $G$ with a finite-state automaton $A$ is stated in Algorithm 1.

#### Algorithm 1: Intersection of a WCFG and an FSA

Input: WCFG $G = \langle N, \Sigma, S, P \rangle$ over semiring $\mathcal{K}$

Input: FSA $A = \langle Q, \Sigma, q_0, F, \delta \rangle$

Output: WCFG $G_\cap = \langle N_\cap, \Sigma_\cap, S_\cap, P_\cap \rangle$ over semiring $\mathcal{K}$ with $N_\cap \subseteq N \times Q$ and $P_\cap \subseteq (N_\cap \times (N_\cap \cup \Sigma)^*) \times \mathcal{K}$

1. Construct GLR(0) automaton $M$ for $G$

2. Compute $M_\cap$, the intersection of $M$ and $A$

3. Extract $G_\cap$ from $M_\cap$

In line 1, the WCFG is converted into a GLR(0) automaton. Given a WCFG $G = \langle N, \Sigma, S, P \rangle$ over $\mathcal{K}$, a GLR(0) automaton (for Generalised LR) $M = \langle Q, \Delta, q_0, F, \delta, \tau \rangle$ over $\mathcal{K}$ is a finite-state automaton with $Q$, $F$ and $q_0$ defined as for FSAs; $\Delta$ is $N \cup \Sigma$. Since $M$ is required to be deterministic, $\delta : Q \times \Delta \rightarrow Q$ is a partial transition function which maps — when defined — a state $q$ and a symbol $a \in \Delta$ to a follow state. $\tau : Q \rightarrow 2^P$ is a mapping from states to subsets of grammar rules (indices).

GLR(0) automata are computed from grammars by an algorithm adapted from a standard algorithm (cf. Aho et al. (1986, p. 216ff.)) which will be explained in greater detail in Section 3.1.

Fig. 2 gives an example GLR(0) automaton for the grammar from Fig. 1. A GLR(0) recognizer is controlled by a GLR(0) automaton $M$. Given some input string $w$, it creates another GLR(0) automaton $M'$ as a result by repeatedly applying its two main operations:

- **Shift**: When the recognizer reads an input symbol $a \in \Sigma$ in state $q$, it adds a transition $q \overset{a}{\rightarrow} \delta(q, a)$ to $M'$.

- **Reduce**: When while processing an input string, the parser reaches a state $q$ for which $\tau(q)$ is defined, for each rule $A \rightarrow \alpha \in \tau(q)$, find all predecessor states $p$ such that $labels(p \rightarrow q) = \alpha$. Then add a transition $p \overset{A}{\rightarrow} \delta(p, A)$ to $M'$ (which may be interpreted as a nonterminal shift). $\delta(p, A)$ is also called a GOTO state.

The GLR(0) recognizer starts in state $q_0$ and scans the input string from left to right. It applies the operations shift and reduce until either an accepting state

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1. In the original definition of LR(k) automata, $\tau$ is a partial function $Q \rightarrow P$. The presence of multiple reduce actions would indicate an ambiguity (a reduce/reduce conflict) which entails that the language of the underlying grammar $G$ is not a LR(k)-language. See Aho and Ullman (1972).

2. Actually, the LR(k) method was invented for parsing deterministic languages like programming languages. In these cases it is not necessary to create an output automaton. Instead, a simple stack is used on which state symbols are pushed and from which they are popped during a reduction.

3. We momentarily disregard the rule weight here.
f ∈ F is reached (in which case f is also marked as final in M′) or the GLR(0) automaton blocks which causes an error to be signaled.

Continuing with Algorithm 1, line 2 intersects the GLR(0) automaton M with the FSA A, resulting in a GLR(0) automaton M′.

Finally, in line 3, the WCFG G containing only the LR(0) item S is extracted from M′.

The following subsections explain all three steps in greater detail.

### 3.1 Efficient Construction of GLR(0) Automata

The construction of a GLR(0) automaton M is based on the computation of the collection of LR(0) item sets. Here, the crucial notion is that of a dotted rule. A dotted rule is a WCFG rule with a dot somewhere in its right-hand side. This dot indicates which part of the rule was already successfully applied and which part has yet to be matched. An example with respect to the grammar in Fig. 1 is: NP → NP • PP. A dotted rule is also called a LR(0) item. To compute M, we start with a set containing only the LR(0) item S′ → • S where S′ is a new super start symbol. Then the main operation of the algorithm – closure – is applied to it. Basically, given a grammar G = ⟨N, Σ, S, P⟩, closure( {A → α • Bβ} ) (with A, B ∈ N and α, β ∈ (N ∪ Σ)*) computes on the basis of G the symbols which are expected next given that the automaton’s expectation is to read a B.4

In our example case,

\[ q_0 = \text{closure}( \{ S' \rightarrow \bullet S \} ) = \{ S' \rightarrow \bullet S, S \rightarrow \bullet NP \text{ VP}, NP \rightarrow \bullet \text{ DET N}, NP \rightarrow \bullet \text{ NE} \} \]

The δ-function of M is computed as follows:

\[ \delta(q, B) = \text{closure}( \{ A \rightarrow \alpha B \bullet \beta \mid A \rightarrow \alpha B \beta \in q \} ) . \]

For example, \( \delta(q_0, \text{NE}) = \text{closure}( \{ \text{NP} \rightarrow \text{NE} \bullet \} ) = \{ \text{NP} \rightarrow \text{NE} \bullet \} . \)

Let the state set Q of M be the set of all LR(0) item sets that can be reached by recursively applying δ and closure to \( q_0 \) and all item sets originating from it.

If a state \( q \) contains an item \( A \rightarrow \alpha \bullet \), the rule \( A \rightarrow \alpha \) is added to \( \tau(q) \). M reaches an accepting state \( f \) if the item set contains the LR(0) item \( S' \rightarrow S \bullet \).

For grammars not having the LR(0) property (for example, ambiguous grammars), the construction introduces conflicts, see Aho and Ullman (1972). Nevertheless, the algorithm leads to deterministic GLR(0) automata for all grammars.

The naive approach representing LR(0) states as sets of dotted rules leads to increased computation times for bigger grammars, for example those extracted from treebanks. For example, the grammar extracted from the TiGer treebank (Brants et al. (2002)) has over 14,300 rules.

A better approach is replacing the dotted rule by a pair consisting of the rule index and the current position of the dot. But even then quite big item sets may result since treebank grammars often have several thousand rules for expanding a single nonterminal symbol.5 Since the right hand sides of grammar rules expanding a given nonterminal symbol often share common prefixes, left-factoring the grammar (cf. Aho et al. (1986)) is an option when the structure of the parse tree is not of concern. In general, this is not the case in using (weighted) grammars for parsing natural languages.

Instead of altering the grammar, we prefer a more sophisticated representation of the dotted rules. All right hand sides \( \alpha_i \) of a set of rules \( \{ B \rightarrow \alpha_i \} \) expanding B can be combined into a disjunctive regular expression \( r = \alpha_1 + \alpha_2 + \ldots + \alpha_k \). \( r \) can be converted into a deterministic weighted finite-state machine by standard techniques (cf. Hopcroft and Ullman (1979)). The result is a trie-like left-factored automaton \( A_B \) representing the right hand sides of the rules for B. For the final states of \( A_B \), we define a function \( \rho : Q \rightarrow I \), where I is the set of rule indices of the WCFG. For a given final state \( q \) of \( A_B \) (representing a fully found right hand side \( \alpha \) of a rule expanding B), \( \rho(q) = i \) if \( \exists c \in K : \langle B \rightarrow \alpha, c \rangle \) is the \( i^{th} \) rule of the grammar.

Fig. 3 shows the rule FSA \( A_{VP} \) for the toy grammar shown in Fig. 1.

An LR(0) item is then represented as a pair \( \langle B, q \rangle \)

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4 Due to space limitations, we cannot state the definition of the closure algorithm. Please refer to Aho et al. (1986, p. 223) for the details.

5 For example, the TiGer grammar used in Section 4 contains 2,378 rules for prepositional phrases and 3,475 rules for verbal phrases.
where $B$ is a nonterminal and $q$ a state in the FSA $A_B$ associated with $B$. When during the closure operation a pair $⟨B, q⟩$ is processed, the transitions $q \xrightarrow{X_i} p$ leaving $q$ in $A_B$ are enumerated and new items $⟨X_i, q_{0_i}⟩$ (if $X_i \in N$) are added to the closure items set.

### 3.2 Intersecting LR(0) Automata with Finite-state Automata

Algorithm 2 computes the intersection of $M$ and $A$. The algorithm maintains a breadth-first queue $L$ of state pairs in $Q_M \times Q_A$. In line 4, a pair consisting of the two start states is inserted into $L$. In the while-loop between lines 5 and 31, a state pair $⟨q_M, q_A⟩$ is removed from $L$. Then, two types of actions are applied to the current state pair $⟨q_M, q_A⟩$: Reductions and shifts. Line 9 checks whether $M$ defines reductions for state $q_M$. If true, the for-loop between lines 10 and 23 considers each rule $B \rightarrow α$ with weight $c$. In line 11, the set of all ancestor nodes for current state $⟨q_M, q_A⟩$ is computed for which there are paths $π$ of length $|α|$ to $⟨q_M, q_A⟩$ (simultaneously, we also record the labels of the paths between an ancestor state and $⟨q_M, q_A⟩$). By definition of the construction of a GLR(0) automaton, labels$(π)$ equals $α$ (disregarding the indices).

Then, the for-loop between lines 12 and 23 operates over each ancestor state $⟨q, q’⟩$ of $⟨q_M, q_A⟩$ and constructs new states and transitions which correspond to the GOTO-actions of the GLR(0) recognizer for nonterminal symbols. Before doing that, a rule $⟨B_qA \rightarrow α’, c⟩$ is added to the reduce actions of state $⟨q_M, q_A⟩$ in line 13. Note that $α’$ differs from the original $α$ in rule $B \rightarrow α$ in the indices carried by the nonterminals. The subsequent steps are:

- In line 14, a new state $⟨δ_M(q, B), q_A⟩$ is created

\[\text{Algorithm 2: Intersection of a GLR(0) Automaton and an FSA}\]

Input: GLR(0) automaton $\begin{align*} M &= (Q_M, δ_M, q_0_M, F_M, δ_M, τ_M) \text{ over a semiring } K \end{align*}$

Input: Deterministic FSA $A = (Q_A, Σ, q_0_A, δ_A, F_A)$

Output: GLR(0) automaton $\begin{align*} M' &= (Q_M', δ_M', q_0_M', F_M', δ_M', τ_M') \text{ over } K \end{align*}$

#### 6.1.1 Pseudocode:

```plaintext
Algorithm 2: Intersection of a GLR(0) Automaton and an FSA

Input: GLR(0) automaton $M = (Q_M, δ_M, q_0_M, F_M, δ_M, τ_M)$ over a semiring $K$

Input: Deterministic FSA $A = (Q_A, Σ, q_0_A, δ_A, F_A)$

Output: GLR(0) automaton $M' = (Q_M', δ_M', q_0_M', F_M', δ_M', τ_M')$ over $K$

1. $q_0_M' \leftarrow ⟨q_0_M, q_0_A⟩$
2. $Q_M' \leftarrow \{q_0_M'\}$
3. $F_M' \leftarrow ∅$
4. Enqueue($q_0_M', L)$

while $L \neq ∅$ do

6. $(q_M, q_A) \leftarrow$ Dequeue($L$)

7. if $q_M \in F_M \land q_A \in F_A$ then

8. $F_M' \leftarrow F_M' \cup \{(q_M, q_A)\}$

9. if $τ_M(q_M) \neq ⊥$ then

// Perform reductions

10. for $(B \rightarrow α, c) \in τ_M(q_M)$ do

11. $V_α \leftarrow \{(q, q', \text{labels}(π)) | (q, q') \in Q_M \land π = (q, q') \mid α \mid_M \not\in Q_M \land q_A\}$

12. for $(q, q', α') \in V_α$ do

13. $δ_M'(⟨q_M, q_A⟩) \leftarrow δ_M'(⟨q_M, q_A⟩) \cup \{δ_{B_qA} \leftarrow α', c\}$

14. if $δ_M(q, B), q_A \not\in Q_M'$ then

15. $Q_M' \leftarrow Q_M' \cup \{δ_M(q, B), q_A\}$

16. $δ_M'(q, q') \leftarrow (δ_M(q, B), q_A)$

17. $Δ_M' \leftarrow Δ_M' \cup \{B_qA\}$

18. Enqueue($δ_M(q, B), q_A), L$)

19. else if $δ_M(q, q'), B_qA \not\in L$ then

20. $δ_M'(q, q'), B_qA \leftarrow (δ_M(q, B), q_A)$

21. if $δ_M(q, B), q_A \not\in L$ then

22. Enqueue($δ_M(q, B), q_A), L$)

23. // Perform shifts

24. for $α \in Σ$ do

25. if $δ_M(q_M, α) \not\in Δ_M' \land δ_A(q_A, α) \not\in Δ_A'$ then

26. $Δ_M' \leftarrow Δ_M' \cup \{α\}$

27. $p \leftarrow (δ_M(q_M, α), δ_A(q_A, α))$

28. $δ_M'(q_M, q_A), α \leftarrow p$

29. if $p \not\in Q_M'$ then

30. $Q_M' \leftarrow Q_M' \cup \{p\}$

31. Enqueue($p, L$)

32. return $M'$
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and checked whether it is present in the state set (and inserted if it is not). Here, $δ_M(q, B)$ denotes the GOTO-state $M$ defines for nonterminal $B$. Note that the second component $q_A$ of $⟨δ_M(q, B), q_A⟩$ is copied from the current state pair $⟨q_M, q_A⟩$ (the input index does not “move” on after a reduction, so to speak).

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*We will discuss the necessity of indexed nonterminals $B_qA$ below.*
In line 20, it is checked whether a transition leaving ancestor state \(\langle q, q' \rangle\) with symbol \(B_{qA}\) already exists. Here, there are two subcases to consider:

1. \(\langle \delta_M(q, B), q_A \rangle\) is a new state which entails that the transition \(\langle q, q' \rangle \xrightarrow{B_{qA}} \langle \delta_M(q, B), q_A \rangle\) also does not exist (lines 15–18). This happens when \(\langle \delta_M(q, B), q_A \rangle\) is encountered for the first time. \(\langle \delta_M(q, B), q_A \rangle\) is added to the state set (line 15) and a new transition leading to \(\langle \delta_M(q, B), q_A \rangle\) is inserted into the queue (line 18).

2. \(\langle \delta_M(q, B), q_A \rangle\) already existed, but not \(\langle q, q' \rangle \xrightarrow{B_{qA}} \langle \delta_M(q, B), q_A \rangle\) (lines 21–23). Here, we repeatedly encountered \(\langle \delta_M(q, B), q_A \rangle\) during processing. This happens in case of local ambiguities where there exist multiple trees for some subpart of \(A\) headed by the same nonterminal. In terms of the graph structure of \(M'\), we create a reentrant node \(\langle \delta_M(q, B), q_A \rangle\) with more than one incoming transition (line 21). Since \(\langle \delta_M(q, B), q_A \rangle\) may trigger further reductions (its ancestor set was changed), it is reinserted into the queue (when not already present).

The shift operations performed in the for-loop between lines 24 and 31 are similar to the case of the intersection of two FSAs: For every transition leaving \(q_A\) labeled \(a\) it is tried to find a corresponding transition leaving \(q_M\). If this transition exists, a new state pair \(\langle \delta_M(q_M, a), \delta_A(q_A, a) \rangle\) is added to \(Q_M'\) and \(L\), if not already present (line 30). In addition, a transition \(\langle q_M, q_A \rangle \xrightarrow{a} \langle \delta_M(q_A, a), \delta_A(q_A, a) \rangle\) is created (line 28).

Unsurprisingly, Algorithm 2 is simply a generalisation of Tomita’s GLR algorithm (cf. Tomita and Ng (1991)) to the recognition of the language of an FSA instead of the recognition of a single sentence (which can be seen as a simple, linear FSA with a single final state). In the general case treated here, \(A\) may have several final states and may contain an infinite number of paths. Because of that, the non-terminal symbols of \(M'\) are indexed with the current state \(q_A\) of \(A\). In that way, \(M'\) keeps track of the different reductions made for different paths in \(A\).

Fig. 4(b) shows the result of applying Algorithm 2 to the GLR(0) automaton of Fig. 2 and the FSA of Fig. 4(a).

Figure 4: (a) A deterministic FSA representing three sentences of the PCFG of Fig. 1. (b) The result of Algorithm 2 applied to the automata in Fig. 2 and Fig. 4(a). States are labeled with pairs \(\langle q_M, q_A \rangle\).

### 3.2.1 Complexity.

The outer while-loop of Algorithm 2 is bounded by \(O(|Q_M \times Q_A|)\). Since \(M\) and \(A\) are both deterministic, only a small subset of \(Q_M \times Q_A\) will be actually created in the average case. Concerning the shift-actions (for-loop lines 24–31), each pair is inserted exactly once into the queue. Looking at the reduce-actions (lines 9–23), a state pair may be reinserted into the queue in case a new incoming transition is added for \(\langle \delta_M(q, B), q_A \rangle\) (line 23). The number of reinsertions is bounded by the number of pos-
\[ S' \rightarrow S_A / 1 \quad S' \rightarrow S_5 / 1 \]
\[ S_A \rightarrow NP_1 \ VP_4 / 1 \quad S_A \rightarrow NP_3 \ VP_4 / 1 \]
\[ S_5 \rightarrow NP_1 \ VP_5 / 1 \quad NP_1 \rightarrow NE / 0.3 \]
\[ NP_3 \rightarrow DET \ N / 0.6 \quad NP_4 \rightarrow NE / 0.3 \]
\[ VP_4 \rightarrow V \ NP_4 / 0.4 \quad VP_4 \rightarrow V / 0.5 \]

Figure 5: Rule set of the grammar extracted from Fig. 4(b).

Disregarding discontinuous constructions like verb–particle rules which are not directly representable in the context-free grammar format.

The most expensive step is found in line 11. Let \( r \) be the length of the longest right-hand side of a rule in \( G \). The number of state pairs \( \langle q, q' \rangle \) created in line 11 and subsequently considered in the for-loop lines 12–23 is bounded by \( O(|Q_M \times Q_A|^r) \) (the number of ancestors increases exponentially with respect to the distance \( r \)). By applying the memoisation techniques proposed in Kipps (1991), this bound can be strengthened to \( O(|Q_M \times Q_A|^3) \). Putting everything together, the overall complexity of Algorithm 2 is in \( O(|Q_M \times Q_A|^3) \).

### 3.3 Extracting Grammar Rules

The last step, the extraction of \( G_{\cap} = G \cap A \) is easy and stated in Algorithm 3.

**Algorithm 3: Extracting \( G_{\cap} \).**

**Input:** A GLR(0) automaton \( M = (Q_M, \Delta_M, \emptyset_0 M, F_M, \delta_M, \tau_M) \) over \( K \).

**Output:** A WCFG \( G_{\cap} = (N, \Delta_M \setminus N, S', P) \) over \( K \).

1. \( N \leftarrow \{ X_i \in \Delta_M \mid i \in N \} \cup \{ S' \} \)
2. \( P \leftarrow \{ \langle S' \rightarrow X_i, \tau \rangle \mid \exists X_i \in N : \delta_M(\emptyset_0 M, X_i) \in F_M \} \cup \bigcup_{q \in Q_M \setminus \tau_M(q) \neq \bot} \tau_M(q) \)

Algorithm 3 simply extracts the grammar rules from the states \( q \) for which \( \tau_M(q) \) is defined. Additionally, a new start symbol \( S' \) is introduced and trivially weighted rules \( S' \rightarrow X_i \) are added to the rule set such that \( X_i \) is labeling a transition from \( \emptyset_0 M \) to a final state. Fig. 5 shows the grammar extracted from the automaton shown in Fig. 4(b).

| Automaton | \( A_1 \) | \( A_2 \) | \( A_3 \) |
|-----------|--------|--------|--------|
| \( |Q| \) | 8      | 56     | 173    |
| \( |\delta| \) | 9      | 64     | 259    |

Table 1: Sizes of the input FSA.

| Phase | Operation | Time (ms) | Avg. time per sent | \( |Q| \) | \( |\delta| \) |
|-------|-----------|-----------|--------------------|-------|-------|
| 1     | Constr. of \( M \) | 3,520     | -                  | 17,279| 1,226,776|
| 2     | \( M \cap A_1 \) | 47        | 47                 | 1,841 | 14,005|
| 2     | \( M \cap A_2 \) | 3,198     | 320                | 18,355| 389,140|
| 2     | \( M \cap A_3 \) | 17,847    | 178                | 50,332| 1,097,708|

Table 2: Results of the experiments with the TiGer grammar.

### 4 Experiments

We implemented the algorithm from the last section in the C++ programming language within the fsm2 framework (see Hanneforth (2009)). The grammar used for the experiments has been extracted from the TiGer treebank (cf. Brants et al. (2002)). It contains 14,379 rules\(^7\), and the sizes of the alphabet and the nonterminal sets are 51 and 25, resp. The length of the longest right-hand side of a rule is 17. For the FSA operand of the intersection algorithm, we created three minimal FSA accepting 1, 10 and 100 sentences (tag sequences) randomly taken from the TiGer corpus. The sizes of the automata are summarised in Table 1.

Table 2 shows the results of the experiments which were carried out on a 2.8 GHz CPU. The columns \( |Q| \) and \( |\delta| \) contain the sizes of the automata resulting from the operation mentioned before.

Since treebank grammars tend to avoid recursive rules and therefore assign flat structures to input strings, they create a lot of readings with spurious ambiguities. Johnson (1998) reports that approximately 9% of the rules of the Penn treebank are never used in a maximum likelihood setting since these rules are subsumed by combinations of other rules with a higher combined probability. We expect even better intersection timings in the face of more linguistically realistic grammars.

\(^7\)Disregarding discontinuous constructions like verb–particle rules which are not directly representable in the context-free grammar format.
5 Conclusion

Above, we presented a theoretical and practical algorithm to intersect weighted grammars with FSAs which can be used for parsing or language model training purposes (cf. Nederhof (2005)). No grammar transformation (for example, binarisation) is necessary to achieve optimal cubic complexity. Instead, the binarisation is implicit by using the dotted rule technique. However, the algorithm may suffer from a big grammar dependent constant for artificial grammars (see Johnson (1991) for details). This is due to the implicit subset construction present in the construction of $M$’s $\delta$-function in Eq. (4). An option to investigate for these artificial grammars would be considering the construction of non-deterministic GLR(0) automata.

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