ON THE IMAGE OF POLYNOMIALS EVALUATED ON INCIDENCE ALGEBRAS: A COUNTER-EXAMPLE AND A SOLUTION

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Abstract. In this paper, we investigate the subset obtained by evaluations of a fixed multilinear polynomial on a given algebra. We provide an example of a multilinear polynomial, whose image is not a vector subspace; namely, the product of two commutators need not to be a subspace whenever evaluated on certain subalgebras of upper triangular matrices (the so-called incidence algebras).

In the last part of the paper, given that the field is infinite, we reduce the problem of the description of the image of a polynomial evaluated on an incidence algebra to the study of evaluations of a certain family of polynomials on its Jacobson radical. In particular, we are able to describe the image of multilinear polynomials evaluated on the algebra of upper triangular matrices.

1. Introduction

We let $F$ be an arbitrary field. We denote the algebra of $n \times n$ matrices having entries in $F$ by $M_n(F)$. Recall that, if $p = p(x_1, \ldots, x_m)$ is a polynomial (in non-commutative variables), then we define

$$\text{Im } p = \{ r \in M_n(F) \mid \exists a_1, \ldots, a_m \in M_n(F), p(a_1, \ldots, a_m) = r \}.$$ 

Recently, some researches are conducted in the direction to answer the following question, attributed to Kaplansky:

Question. Given a multilinear polynomial $p$, is $\text{Im } p$, evaluated on $M_n(F)$, a subspace?

It is not hard to see that, if $p$ is multilinear, then the subspace of $M_n(F)$ generated by $\text{Im } p$ is a Lie ideal. Thus, the subspace generated by $\text{Im } p$ is either 0, $K$ (the scalar matrices), $\mathfrak{sl}_n(F)$, or $M_n(F)$. So, the question reduces to prove that the image of a multilinear polynomial is one of these subspaces, or to find a counter-example. Clearly $\text{Im } p = M_n(F)$ is possible when $p(x) = x$. Moreover, $\text{Im } p = \{0\}$ if, and only if, $p$ is a polynomial identity of $M_n(F)$: $\text{Im } p = F$ if and only if $p$ is a central polynomial for $M_n(F)$ (see, for instance, [8, 14]). Hence, this question is related to the theory of algebras satisfying polynomial identities.

As an example, it is natural to conjecture that the image of the commutator $p(x_1, x_2) = [x_1, x_2] := x_1x_2 - x_2x_1$ is all $\mathfrak{sl}_n(F)$. However, less obvious is to prove that every trace zero matrix is a commutator of two matrices. This is indeed the case: Shoda proved that, over a field of characteristic zero, every matrix with zero trace can be expressed as a commutator of two matrices [13]. Latter, Albert and

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Muckenhoupt extended the result for arbitrary fields [1]. In other words, they proved that $\text{Im} [x_1, x_2] = \mathfrak{sl}_n(F)$.

This question also has a parallel in group theory. We refer to [9] for the historical background and an account of the problem.

Among the recent papers in the subject, a positive answer involving multilinear polynomials is given for $M_2(F)$; moreover, the result is positive for the more general situation of the so-called semi-homogeneous polynomials, and easy counter-examples are known for non-multilinear polynomials (see [9] for quadratically closed fields, and [13] for real closed and arbitrary fields). The algebra $M_3(F)$ has also an almost complete answer [10]. In [17], the question is resolved positively for multilinear Lie polynomials of degree at most 4. Further developments were achieved in [5], studying polynomials of degree 3; and in [11], where the authors investigate the so-called power-central polynomials, in particular, obtaining interesting results concerning the image of multilinear polynomials.

One can also ask what is the image of a multilinear polynomial, evaluated on other algebras. Some authors investigated this analogous problem. For instance, we cite that positive answers were obtained, given some restrictions, to Jordan algebras [12], simple classical Lie algebras [3, 2], and upper-triangular matrices [6]. For strict upper triangular matrices, a complete positive answer was achieved in [7]. Thus, it is interesting to investigate other algebras, and in particular, which are not necessarily associative.

However, for the best of our knowledge, no counter-examples were provided for multilinear polynomials; that is, no multilinear polynomial is presented such that its image is not a subspace. The closest multilinear counter-example is given in Remark 4 of [11]: the image of $f : (a, b) \in M_{1\times 2}(F)^2 \mapsto ab \in M_2(F)$ is not a vector space, where $M_{1\times 2}(F)$ is the vector space of $1 \times 2$ matrices. This counter-example is a multilinear function, but it is not a multilinear polynomial, in the sense of an element of the free associative algebra. But in our present paper, working with the so-called incidence algebras, we were able to provide a such multilinear polynomial counter-example.

2. THE COUNTER-EXAMPLE IN ITS SIMPLEST FORM

Let $F$ be an infinite field, and let

$$A = \left\{ \begin{pmatrix} * & 0 & * & * \\ * & * & * & * \\ * & * & * & * \\ * & 0 & * \end{pmatrix} \right\}. $$

Let $J(A)$ be its Jacobson radical, that is,

$$J(A) = \left\{ \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix} \right\}. $$
Note that the left and right annihilators of \( J(A) \) in itself are, respectively,

\[
\text{Ann}^l_{J(A)} J(A) = \begin{pmatrix}
0 & * & * \\
0 & * & * \\
* & * & *
\end{pmatrix},
\]

\[
\text{Ann}^r_{J(A)} J(A) = \begin{pmatrix}
* & * & * \\
* & * & * \\
0 & 0 & 0
\end{pmatrix}.
\]

Also, note that (see the proof of [6, Lemma 10])

\[
\text{Im} \{ x, y \} = J(A).
\]

Let \( p = [x_1, x_2][x_3, x_4] \). Then, using (3), to compute \( \text{Im} \{ x_1, x_2 \}[x_3, x_4] \), it is sufficient to compute \( r_1r_2 \), where \( r_1, r_2 \in J(A) \). Using (12), we see that \( r_1r_2 = 0 \), if either \( r_1 \) or \( r_2 \) belong to the annihilators above. Thus, we have:

\[
\text{Im} p = \{ rs \mid r, s \in J(I(X)) \}
\]

\[
= \{ (ae_{13} + be_{23})(ce_{34} + de_{35}) \mid a, b, c, d \in K \}
\]

\[
= \{ a(e_{14} + e_{15}) + b(e_{24} + e_{25}) \mid a, b, c, d \in K \}.
\]

In particular, \( e_{14} + e_{15} + e_{24} + e_{25} \) and \( 0e_{14} - e_{15} + 0e_{24} + e_{25} \) are both in \( \text{Im} p \), but their sum is not in \( \text{Im} p \). Thus \( \text{Im} p \) is not a vector subspace of \( A \).

2.1. Alternative definition: Incidence algebras. The algebra presented in the previous section is a particular example of the so-called incidence algebras. The incidence algebras have an interesting geometric flavor, and at the same time, a rich combinatorial nature. We shall define these objects in this section. A good source for the theory is the book [10].

Let \( (X, \leq_X) \) be a partially ordered set (poset, for short). Assume that \( X \) is locally finite, that is, for all \( x, y \in X \), there exists only a finite number of \( z \in X \) satisfying \( x \leq_X z \leq_X y \). We let

\[
I(X) = \{ f : X \times X \to F \mid f(x, y) = 0 \text{ whenever } x \not\leq_X y \}.
\]

Using pointwise addition and scalar multiplication, we see that \( I(X) \) is a vector space. Moreover, given \( f, g \in I(X) \), define their product by

\[
f g(x, y) = \sum_{z \in X} f(x, z) g(z, y).
\]

Since \( X \) is locally finite, it is readily seen that the above sum is well defined. Moreover, \( f g \in I(X) \), and the defined operation is an associative product on \( I(X) \). Thus, \( I(X) \) is an associative algebra, called the incidence algebra of \( X \).

Now, assume that \( X = \{ x_1, x_2, \ldots, x_n \} \) is finite. For the particular case where \( X \) is a chain, we obtain that \( I(X) \) coincides with the upper triangular matrix algebra of order \( n \), denoted by \( UT_n(F) \). For the general case, we can rename the elements of \( X \), say \( X = \{ 1, 2, \ldots, n \} \) in such a way that \( i \leq_X j \) implies \( i \leq j \) (in the usual ordering of the integers). Using such identification, we see that \( I(X) \subset UT_n(F) \), that is, \( I(X) \) is a subalgebra of \( UT_n(F) \). Hence, usually one uses the notation of
matrix units to designate the elements of $I(X)$. For instance, a (vector space) basis of $I(X)$ constitutes of elements $e_{xy}$, $x, y \in X$, $x \leq_X y$.

To represent a locally finite poset $X$, we can make use of the Hasse diagram. Given $x, y \in X$, we say that $y$ is a cover of $x$ if $x <_X y$ and $x <_X z \leq_X y$ implies $z = y$. Thus, we draw a vertex for each element of $x \in X$, and draw a (directed) edge between $x$ and $y$ if $y$ is a cover of $x$. We obtain then a (directed) graph, which is called the Hasse diagram of $X$.

If $X$ is finite, then it is easy to see that the Jacobson radical $J(I(X))$ of $I(X)$ is spanned by $e_{xy}$, where $x, y \in X$, and $x < y$. With the aid of the Hasse diagram, $J(I(X))$ is spanned by the elements $e_{xy}$ where $x, y$ are (different elements) linked by a path (respecting the directions of the edges). Moreover, the powers of $J(I(X))^m$ are spanned by the $e_{xy}$, where $x$ and $y$ are connected by a (directed) path of, at least, $m$ edges. Thus, the Hasse diagram helps in the intuition of making computations in $I(X)$, translating its combinatorial complexity to the algebra structure.

In our counter-example above, we consider the poset $X = \{1, 2, 3, 4, 5\}$, where $1, 2 \leq_X 3$, and $3 \leq_X 4, 5$. In the Hasse diagram, we have:

```
  4  5
   \  /
    3
  /  \
1   2
```

Note that both $J(I(X))$ and $J(I(X))^2$ are spanned by 4 elements. The key point is that $J(I(X))$ constitutes by 4 chains, each of them contained in exactly 2 chains of $J(I(X))^2$. Thus, in making the product of two elements of $J(I(X))$, we are not completely free to generate all elements of $J(I(X))^2$.

3. Positive solution for $UT_n(F)$

We assume that $F$ is an infinite field.

**Lemma 1** (Theorem 5.2.1(ii) of [4]). The relatively free algebra in the variety $V(UT_n(F))$ has a basis consisting of all products

$$(4) \quad x_1^{a_1} \cdots x_m^{a_m} [x_{i_1}, \ldots, x_{i_{p_1}}] \cdots [x_{i_r}, \ldots, x_{i_{p_r}}],$$

where the number $r$ of participating commutators if less or equal than $n - 1$, and the indices in each commutator $[x_{i_1}, \ldots, x_{i_{p_s}}]$ satisfy

$$i_{1s} > i_{2s} \leq i_{3s} \leq \ldots \leq i_{p_s}.$$

\[\square\]

In other words, if $p$ is a multilinear polynomial, then there exist unique multilinear polynomials $q_i$ as in (4), unique coefficients $\alpha_i \in F$, and a unique (multilinear) polynomial identity $\tilde{p}$ of $UT_n(F)$ such that

$$(5) \quad p = \tilde{p} + \sum_i \alpha_i q_i.$$

In our previous discussion, if we have an incidence algebra $I(X) \subseteq UT_n$, then $I(X)$ belongs to the variety $V(UT_n(F))$. In other words, every polynomial identity of
$UT_n(F)$ obviously is a polynomial identity of $I(X)$ as well. Thus, given a multilinear polynomial $p$, and using the notation of equation (5), we have

$$\text{Im } p = \text{Im } \sum_i \alpha_i q_i.$$ 

As a conclusion, to investigate $\text{Im } p$ evaluated on a finite-dimensional incidence algebra, it is sufficient to understand the image of linear combination of the elements given by (4). We are particularly interested in the situation where the linear combination is a multilinear polynomial.

From now on, we let $A = UT_n(F)$, and we shall describe $\text{Im } p$ evaluated on $A$. Our result does not hold for arbitrary incidence algebras, as our previous counter-example suggests. However, the constructions below work for some posets $X$. We were not able to find a nice property to describe any poset satisfying that condition. So, to make the arguments simpler, we restrict ourselves to the case $A = UT_n(F)$.

Given a multilinear polynomial $p$, let $e$ be the maximum integer such that every polynomial of the kind (4) appearing with nonzero coefficient in (5) has a product of at least $e$ long commutators. We call $e$ the index of $p$, and denote $e$ by $\text{ind } p$. Then clearly $\text{Im } p \subseteq J(A)^{\text{ind } p}$. Note that ind $p$ can be zero; and we define $J(A)^0 = A$.

We are going to prove the following result, which completes the answer of the main result of [4], for infinite fields:

**Theorem 2.** Let $F$ be an infinite field, and $p$ a multilinear polynomial. Consider the algebra of upper triangular matrices $UT_n(F)$. Then $\text{Im } p = J(UT_n(F))^m$, where $m$ is the non-negative integer $\text{ind } p$. In particular, the image of a multilinear polynomial evaluated on the algebra of upper triangular matrices is a vector subspace.

First, we deal with a particular case of the problem. We introduce a structure of a grading to facilitate notation. Let $S = \{1, s\}$ be the semigroup with 2 elements such that 1 is the unit and $s^2 = s$; that is, the multiplication table given by:

$$
\begin{align*}
1 &\ast 1 = 1, & 1 \ast s = s \\
1 \ast 1 = 1, & s \ast s = s.
\end{align*}
$$

Let $A_1$ be the set of diagonal matrices and $A_s = J(A)$ the Jacobson radical. Note that $A_1$ is a subalgebra, and $A_s$ is a two-sided ideal. Thus $A = A_1 \oplus A_s$ is a $S$-grading on $A$.

Consider the free $S$-graded algebra with free generators $\{x_1^{(1)}, x_1^{(s)}, x_2^{(1)}, x_2^{(s)}, \ldots\}$, where $\text{deg } S x_i^{(g)} = g$. For every $i \in \mathbb{N}$, denote

$$
\begin{align*}
y_i := x_i^{(1)}, & z_i := x_i^{(s)}, \\
x_i := y_i + z_i.
\end{align*}
$$

Thus $[y_1, y_2] = 0$ and $z_1 z_2 \cdots z_n = 0$ are $S$-graded polynomial identities of $A$.

Now, fix positive integers $r, m$, such that $m \geq r$. Call a set $I$ adequate if $I = \{I_1, \ldots, I_r\}$, where $I_1, \ldots, I_r$ are non-empty pairwise disjoint sets, and $I_1 \cup \ldots \cup I_r = \{1, 2, \ldots, m\}$. Given a permutation $\tau \in S_r$, let $q_{\tau, I}$ be the polynomial

$$q_{\tau, I} = [z_{\tau(1)} y_{i_1}, \ldots, y_{i_{t_1}}] \cdots [z_{\tau(r)} y_{i_r}, \ldots, y_{i_{t_r}}],$$

where $I_j = \{i_{1j}, \ldots, i_{t_j}\}$. The elements $q_{\tau, I}$ are linearly independent (see, for instance, the proof of Theorem 5.2.1(ii) of [4]).
Consider the commutative polynomial algebra $F[\eta_{ij}^{(\ell)} \mid \ell = 1, 2, \ldots, r, 1 \leq i < j \leq n]$. Any unital homomorphism of algebras $\psi : F[\eta_{ij}^{(\ell)}] \to F$ is uniquely determined by a choice of evaluations of the variables $\eta_{ij}^{(\ell)}$. Moreover, $\psi$ determines a homomorphism of algebras $F[\eta_{ij}^{(\ell)}] \otimes A \to A$, which also is denoted by $\psi$, in such a way that $\psi(f \otimes a) = \psi(f)a$.

**Proposition 3.** Let $\emptyset \neq U \subseteq S_r$ be any subset, and $\lambda_{ij}^{(\ell, \tau)} \in F \setminus \{0\}$ be arbitrary nonzero elements, for $1 \leq i < j \leq n$, $\ell = 1, 2, \ldots, r$, $\tau \in U$. In the algebra $F[\eta_{ij}^{(\ell)}] \otimes A$, let

$$z_{\tau}^{(\ell)} = \sum_{1 \leq i < j \leq n} \eta_{ij}^{(\ell)} \lambda_{ij}^{(\ell, \tau)} e_{ij}, \quad \ell = 1, 2, \ldots, r.$$

Then, for any $u \in J(A)^r$, there exists a unital homomorphism $\psi : F[\eta_{ij}^{(\ell)}] \to F$ such that

$$\psi \left( \sum_{\tau \in U} z_{\tau}^{(1)} \cdots z_{\tau}^{(r)} \right) = u.$$

**Proof.** Renaming the variables, we can assume that $U$ contains the identity. We can evaluate $\eta_{ij}^{(\ell)} = 0$, for all $i + 1 < j$, and $\ell = 2, 3, \ldots, r$. So, the proof follows the same steps of the proof of the main result of [2, Theorem 3]. We include the argument for completeness.

Let $U' \subseteq U$ be the set of all permutations $\tau \in U$ such that $\tau(1) = 1$. Then we have

$$\sum_{\tau \in U} z_{\tau}^{(\tau(1))} \cdots z_{\tau}^{(\tau(r))} =$$

$$\sum_{k=1,2,\ldots,n-r-1}^{l=1,2,\ldots,n-k-r} \lambda_{l,k}^{(\tau(1))} \lambda_{l+k}^{(\tau(2))} \cdots \lambda_{l+k+r-1,l+k+r}^{(\tau(r))} n_{l+k,r+1,l+k+r}^{(\tau(1))} \cdots n_{l+k+l+k+r}^{(\tau(1))}$$

$$+ f_{l,k}(n_{l+1,l+k+1}^{(1)} \cdots n_{r+1,l+r+1+k}^{(1)} | \ell = 2, 3, \ldots, r) e_{l+k+r-1,1},$$

where $f_{l,k}$ are polynomials (possibly zero) not depending on $n_{ij}^{(1)}$, unless $i > l$, and $j = i + k$. Let $g_{l,k}(n_{i,i+k+1}^{(\ell')} \mid \ell' = 2, 3, \ldots, r)$ be the (nonzero) polynomial multiplying $n_{l+k+1}^{(1)}$. Since the field is infinite, we can find an evaluation on the variables $n_{i,i+1}^{(\ell')}$, for $\ell' > 1$, such that $g_{l,k} \neq 0$, for all $l, k$. So, with this evaluation, each $f_{l,k}$ becomes a polynomial depending only on $n_{i,i+1}^{(1)}, \ldots, n_{r+1,l+r+1+k}^{(1)}$. Given

$$u = \sum_{k=1,2,\ldots,n-r-1}^{l=1,2,\ldots,n-k-r} b_{l+k+r-1,l+k+r-1}^{(1)} e_{l+k+r-1,1} \in J(A)^r,$$

we need to solve in $n_{ij}^{(1)}$ the following system of equations

$$b_{l+k+r-1}^{(1)} = g_{l,k}(n_{l+1,l+k+1}^{(1)} \cdots n_{r+1,l+r+1+k}^{(1)}), \quad \forall l, k.$$  

Note that two expressions of $F$, given by pairs $(l_1, k_1)$ and $(l_2, k_2)$, can have common variables only if $k_1 = k_2$. Thus, for each fixed $k = 1, 2, \ldots, n - r - 1$, we can solve the system of equations separately.
Let \( U \) and (9), for a fixed \( I \) where clearly the \( f \) for any evaluation \( \phi \) can find an evaluation \( \phi \). Furthermore, we obtain values for \( \eta_{l-1,l-1+k}, \eta_{l-2,l-2+k}, \ldots, \eta_{l,1+k} \). This concludes the proof. □

Now, consider a multilinear polynomial \( q \) in the variables \( z_1, \ldots, z_r, y_1, \ldots, y_m \), which is a linear combination of polynomials \( q_{\tau,l} \), that is,

\[
q = \sum_{\tau \in S, \text{adequate } l} \sum_{i<j} \alpha_{\tau,l} q_{\tau,l}.
\]

Let \( \xi^{(l)}_{k} \), for \( k = 1, 2, \ldots, n-1, l = 1, 2, \ldots, m \) be commutative algebraically independent variables. For short, for \( i < j \), denote \( \xi_{ij} = \xi_{i1}^{(l)} + \xi_{i1}^{(l)} + \ldots + \xi_{ij}^{(l)} \). Let

\[
y_{\ell} = \xi_{1:1:1}^{(l)}e_{22} + \xi_{1:2:1}^{(l)}e_{33} + \ldots + \xi_{1:n-1}^{(l)}e_{nn}, \quad \ell = 1, 2, \ldots, m.
\]

Let \( \mathcal{U} \) be the set of \( \tau \in S \), such that at least one \( \alpha_{\tau,l} \) is nonzero in the expression (8). Combining the evaluations

\[
z_{\tau} = \sum_{1 \leq i < j \leq n} \eta_{ij}^{(l)} e_{ij},
\]

and (3), for a fixed \( \tau \in \mathcal{U} \), we have

\[
\sum_{\text{adequate } l} \alpha_{\tau,l} [z_{\tau(l)}^{(l)}, y_{l1}, \ldots, y_{l\ell}] = \sum_{1 \leq i < j \leq n} \left( \sum_{\text{adequate } l} \alpha_{\tau,l} \xi_{i1}^{(l)} \cdots \xi_{ij}^{(l)} \right) \eta_{ij}^{(l)} e_{ij} = \sum_{1 \leq i < j \leq n} f_{ij}^{(\tau,l)}(\xi_{k}^{(l)}) \eta_{ij}^{(l)} e_{ij},
\]

where clearly the \( f_{ij}^{(\tau,l)}(\xi_{k}^{(l)}) \) are nonzero polynomials. Since the field is infinite, we can find an evaluation \( \varphi : F[\xi_{k}^{(l)}] \to F \) such that \( \varphi(f_{ij}^{(\tau,l)}(\xi_{k}^{(l)})) = \lambda_{ij}^{(\tau,l)} \neq 0 \), for all \( f_{ij}^{(\tau,l)} \). So, we find that

\[
\varphi(q) = \sum_{\tau \in \mathcal{U}} z_{\tau(1)}^{(l)} \cdots z_{\tau(r)}^{(l)},
\]

where each \( z_{\tau}^{(l)} \) is the expression given by (9). Thus, applying Proposition 3 we obtain:

**Lemma 4.** Let \( q \) be a multilinear polynomial in the variables \( z_1, \ldots, z_r, y_1, \ldots, y_m \), where \( r \) and \( m \) are arbitrary positive integers, which is a linear combination of polynomials \( q_{\tau,l} \). Then, the image of \( q \) obtained by \( S \)-graded evaluations is \( \text{Im } q = J(A)^r \).

Now, we go back to our general non-graded context. Let \( p \) be an arbitrary multilinear polynomial. Write \( p \) as in (3). Consider the elements of the kind (4) having \( r = \text{ind } p \) product of commutators. Among them, choose one such that \( \max\{i_1, i_2, \ldots, i_r\} \) is minimal. Choose the evaluation \( x_{i\ell} = z_{\ell} \), for \( \ell = 1, \ldots, r \), and the remaining variables to some \( y_{j} \) (obtaining a multilinear polynomial). Thus, we obtain a new nonzero polynomial, where all the elements of kind (4) vanish,
except the ones having exactly \( r \) commutators, and such that all the commutators have some \( z_i \) in its first position. Evaluating the possibly variables appearing outside any commutator to the identity, we obtain a new (nonzero) polynomial \( \hat{p} \), which is \( S \)-graded and it is a linear combination of polynomials \( q_{r,p} \). Since \( \hat{p} \) was obtained by partial evaluations of \( p \), clearly \( \text{Im} \hat{p} \subseteq \text{Im} p \). Now, using Lemma 4 we conclude that \( J(A)^{\text{ind}p} = \text{Im} \hat{p} \subseteq \text{Im} p \subseteq J(A)^{\text{ind}p} \). This is the proof of our main theorem of this section:

**Corollary 5.** \( \text{Im} p = J(A)^{\text{ind}p} \).

**Remark 6.** It is worth mentioning that the arguments after Proposition 3 remain valid for an arbitrary incidence algebra \( I(X) \). So, if Proposition 3 is true for \( I(X) \), then it will also be true that \( \text{Im} p = J(I(X))^{\text{ind}p} \), where \( p \) is any multilinear polynomial.

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