J. Rysti · J. Tuoriniemi

Quartz Tuning Forks and Acoustic Phenomena in Superfluid Helium

Abstract Mechanical resonators are well suited for probing the properties of liquid helium, since the resonant characteristics are sensitive to the fluid properties. Quartz tuning forks have gained much popularity in recent years as the resonators of choice. They have many superior properties when compared to other resonators, such as vibrating wires. However, the intricate geometry of a tuning fork represents a challenge for analyzing their behavior in a fluid environment; analytical approaches cannot be employed. In this article the characteristics of quartz tuning fork resonators in superfluid helium are studied by numerical simulations. We account for the compressibility of the medium, that is, acoustic phenomena and neglect viscosity. We compute the influence to the oscillator response when the fork is immersed in an inviscid compressible fluid. The significance of different tuning fork shapes is studied. Acoustic emission in infinite medium and acoustic resonances in a confined space are investigated. The results can aid in choosing a quartz tuning fork with suitable properties for experiments, as well as understanding measured data.

Keywords Quartz tuning fork · superfluid helium · acoustic emission
PACS 47.80.-v · 67.90.+z · 85.50.-n

1 Introduction

Quartz tuning fork resonators have become popular tools for studying superfluid helium [1, 2, 3, 4]. To a great extent, they have replaced other mechanical resonators, such as vibrating wires. Their main advantages are availability through mass production, ease of use, supreme stability, and high quality factor. A major disadvantage is a tricky geometry, which restricts analytical studies of their resonant behavior in a fluid environment. All quartz tuning forks have the same...
two-pronged shape, but they come in a variety of sizes, relative dimensions, and resonant frequencies. The most common room temperature resonant frequency of a quartz tuning fork in vacuum is designed to be $2^{15} = 32768$ Hz, but other available frequencies allow experiments to be done in a range of acoustic wavelengths. Mass produced quartz tuning forks are encapsulated in hermetically sealed cylindrical metal cans or parallelepiped caskets. To make contact between the resonator and fluid, either holes must be created in the capsule or the container be completely removed.

The resonant frequency $f_0$ of a mechanical oscillator immersed in inviscid fluid medium is given by [2]

$$f_0 = f_{0\text{vac}} \left(1 + B \frac{\rho_F}{\rho_S}\right)^{-1/2},$$

where $f_{0\text{vac}}$ is the resonant frequency in vacuum, $B$ is a geometry-dependent dimensionless factor of order one, and $\rho_F$ and $\rho_S$ are the densities of the fluid and the oscillator (solid), respectively. The density of quartz is $\rho_S = 2659$ kg/m$^3$. Generally for a compressible medium, the factor $B$ depends on the wavelength of sound in the fluid. If the wavelength is much larger than the relevant oscillator dimensions, the medium is often assumed to be incompressible. Under this approximation, $B$ is a constant, which depends only on the geometry of the system. In precise measurements this assumption cannot be made even if the wavelength is an order of magnitude larger than the dimensions. The situation becomes even worse if acoustic resonances in the cavity around the oscillator, or within the oscillator geometry, are present. In the vicinity of an acoustic resonance, $B$ changes rapidly as a function of wavelength. The geometrical factor can be calculated analytically only for some simple geometries, such as an infinitely long circular or elliptic cylinder. It is one goal of this paper to determine, how $B$ depends on various tuning fork properties and configurations.

In this paper, we examine the resonant behavior of quartz tuning forks by numerical simulations using the finite element method (FEM). We study the properties of tuning forks with various prong shapes and compute the resonant frequency change due to the surrounding compressible fluid. We consider the oscillators in infinite medium, as well as in a confined cylindrical cavity. In infinite medium we compute the emission of acoustic radiation and in confined space we consider the acoustic resonant modes within the fluid cavity. The influence of acoustic impedance on the surfaces is also discussed. The significance of different shapes of tines are studied by assuming the oscillator is infinitely long, which reduces the calculation to two dimensions. These are computationally much less demanding than the full 3D models. The full fork geometry is simulated to make contact with experiments and to see the difference between 2D and 3D models. We have previously reported the results of similar simulations for various 2D geometries [5].
2 Computational Model

2.1 Harmonic Oscillator

For small oscillations of a resonator, we can use the equation of motion of a driven damped harmonic oscillator

\[ F_e = m \frac{d^2 x}{dt^2} + m \gamma \frac{dx}{dt} + k_S x, \]

where \( F_e \) is the excitation force, \( m \) is the oscillator mass, \( x \) is the displacement, \( k_S \) is the spring constant, and \( \gamma \) represents damping. If the oscillator is in vacuum, \( \gamma \) stands for internal damping of the oscillator. A finite value for \( \gamma \) is required for the computational method in question to work. We seek time-harmonic solutions of Eq. (2), where all the variables are \( \propto e^{i \omega t} \). That is, they all oscillate sinusoidally at a frequency \( \omega \). This eliminates time from the equations. We denote \( x = X e^{i \omega t}, F_e = F_E e^{i \omega t}, \) and so on. Separating the real and imaginary components of the maximum amplitude, \( X = a + bi \), gives

\[ a = -\frac{F_e \omega^2 - \omega_0^2}{m} \left( 1 + \left( \frac{\omega^2 - \omega_0^2}{\gamma \omega} \right)^2 \right)^{-1} \]

and

\[ b = -\frac{F_e}{m \gamma \omega} \left( 1 + \left( \frac{\omega^2 - \omega_0^2}{\gamma \omega} \right)^2 \right)^{-1}. \]

Here \( \omega_0^2 \equiv k_S / m \) is the resonant frequency. We can turn these equations into a "single frequency measurement method", also used experimentally [6], by solving the width \( \Delta f = \gamma / 2\pi \) and resonant frequency \( f_0 = \omega_0 / 2\pi \) as functions of frequency \( f = \omega / 2\pi \) and the real and imaginary parts of the amplitude:

\[ \Delta f = -\frac{F_e}{4\pi^2 m} \frac{b}{a^2 + b^2} \]

\[ f_0 = \sqrt{f \left( f - \frac{\Delta f}{a^2} \right)} \]

We can now vary some properties of the system, for example, the speed of sound in the fluid, and simply use a single "measurement frequency" \( f \) to determine the resonant frequency. Non-linear frequency dependence of the oscillator response on the fluid motion causes a slight computational inconvenience. If one uses Eqs. (5) and (6), the "measurement frequency" should be close to the resonance. This is ensured by iterating \( f \) a few times and having it depend on the system property being varied, in our case the speed of sound. The necessity of having \( f \) close to \( f_0 \) is particularly profound if the oscillator is coupled to an acoustic resonance.

We model the surrounding fluid as an acoustic medium, which obeys the Helmholtz equation

\[ \nabla^2 P - k^2 P = 0, \]
where \( P \) is the acoustic pressure (deviation from some equilibrium value) and \( k = \omega / c = 2\pi / \lambda \) is the acoustic wave vector corresponding to wavelength \( \lambda \). Incompressible fluid is obtained in the limit \( k \to 0 \) (\( \lambda \to \infty \)). Speed of first sound in helium liquids vary between 100 m/s and 400 m/s, depending on temperature, pressure, and concentration in mixtures. A 32.8 kHz quartz tuning fork can therefore cover a wave vector range between \( k = 500 \text{ m}^{-1} \) and \( k = 2000 \text{ m}^{-1} \). Gaseous helium extends the range to even higher values. Higher-frequency oscillators allow experiments to be performed at wave vectors in the several thousands.

The oscillating object directs a normal acceleration on the fluid in the direction of its motion. The boundary condition is thus

\[
\hat{n} \cdot \nabla P = \rho F \ddot{X},
\]  

where \( \hat{n} \) is the unit normal vector of the boundary and \( \ddot{X} \) is the "acceleration amplitude" defined through \( d^2x / dt^2 = \dot{X} e^{i\omega t} \). The reaction force \( F_z \) of the medium on the oscillator is obtained by integrating the pressure field of the fluid over the oscillator surface. The acceleration of the resonator \( \ddot{X} \) is solved from the harmonic oscillator equation, Eq. (2), by including \( F_z \) in the equation of motion:

\[
F_e = \left( m - i \Delta f_{\text{vac}} - \frac{k_S}{(2\pi f)^2} \right) \ddot{X} - F_z. \tag{9}
\]

One should note that now \( \Delta f \), which is solved from Eq. (5), can be different from \( \Delta f_{\text{vac}} \), which is given in as a parameter describing internal losses in the oscillator.

The displacement is simply given by

\[
X = \frac{\ddot{X}}{(2\pi f)^2}. \tag{10}
\]

The boundary condition on the container walls and non-accelerating surfaces of the oscillator is

\[
\hat{n} \cdot \nabla P = -\rho F \frac{i\omega}{Z}, \tag{11}
\]

where \( Z \) is the surface acoustic impedance. The limit \( Z \to \infty \) corresponds to the perfectly reflecting hard boundary. The surface impedance is given by

\[
Z = \rho_F c_F \frac{1 + R}{1 - R}, \tag{12}
\]

where \( R \) is the amplitude reflection coefficient. For a completely absorbent interface, \( R = 0 \) and the surface impedance equals the characteristic impedance of the medium \( \rho_F c_F \). In the case of an interface of two bulk media, it is given by

\[
R = \frac{\rho_S c_S - \rho_F c_F}{\rho_S c_S + \rho_F c_F}. \tag{13}
\]

Infinite medium is simulated by applying a second order expansion cylindrical wave (also spherical wave in 3D) radiation boundary condition at the artificial boundary. This type of boundary condition tends to minimize reflections at the boundaries, thus mimicking infinite medium.
Fig. 1 Microscope image of a Fox Electronics NC38 quartz tuning fork, similar to the one used in the experiments of references [10] and [11]. The lower image shows the side-view of the tuning fork, but the oscillator has not been photographed along its entire length from this direction. Length of the tines is \( L = 3.76 \) mm, width of the tuning fork \( W = 0.34 \) mm, tines in the other direction \( T = 0.60 \) mm, base \( B = 1.51 \) mm, and the total height \( H = 6.00 \) mm. The electrodes used for excitation and detection of the resonance are visible on the surface of the quartz. Electrical contact is achieved with wires soldered to the electrodes on both sides of the base.

2.2 Quartz Tuning Fork

Experimental data of the geometrical factor \( B \) for a quartz tuning fork exist [10, 11]. The oscillator used in these experiments was a Fox Electronics NC38 quartz tuning fork. Unfortunately the individual used in the experiments was not available for inspection, so the exact dimensions and configuration of the system could not be determined. We did obtain, however, similar devices and measured their properties. One NC38 quartz tuning fork was removed completely from its casing in order to find the dimensions of the oscillator. A microscope image of the tuning fork and the measured dimensions are shown in Fig. 1.

The inner dimensions of the cylindrical container were measured by puncturing a small hole into the container of another NC38 and filling the volume with epoxy, after which the metallic can was corroded by acid. It was then possible to use a microscope to determine the dimensions. The cavity diameter was found to be 2.63 mm and the distance between the tine ends and the container “ceiling” 1.12 mm. The space between the tuning fork base and the container “floor” could not be determined very accurately, but was of the order of 0.1 mm. The total cavity height used in the simulations was 7.12 mm. The small space between the fork base and the container was neglected in the simulations, as its effect is negligible and meshing such a small volume with FEM elements would require a rather fine mesh and thus increase the number of degrees of freedom to little avail.

By filling the cavity with epoxy, we could also see how the tuning fork was positioned inside the container. The epoxy-filled tuning fork had a clearly observable tilt of about 5 degrees in the direction perpendicular to the oscillation. Any possible tilt in the other direction was impossible to observe, as the epoxy cast caused too much distortion. In its normal operation, the quartz tuning fork is in vacuum and thus its orientation inside does not matter. When the container is breached
and fluid fills the cavity, orientation makes a considerable difference, as will be demonstrated in Sec. 3.3.

The fact that the tines of a quartz tuning fork are effectively cantilever beams, complicates its analysis to some extent. The tines can oscillate in a great variety of different resonant modes with increasing frequency. The lowest-frequency mode, which can be excited by the electrodes on the surface of the fork, is such that the tines oscillate in antiphase toward each other and no nodes exist along the tines. This is also the typical mode used in most applications. Experiments with higher-frequency modes of quartz tuning forks have also been performed, but here we consider only the lowest usable mode. The fork can be approximated with an effective harmonic oscillator model using Eq. (2) by replacing the mass of the oscillator (one tine) with an effective mass given by [12]

\[ m^* = 0.2427m. \]  

This applies to a uniform rectangular cantilever beam. If the shape or density distribution of the tine is something else, the effective mass must be recalculated. The spring constant in the harmonic oscillator equation is given by

\[ k_S = \frac{E}{4W} \left( \frac{T}{L} \right)^3, \]  

where the dimensions are as in Fig. 1 and \( E \) is the elastic modulus, which for quartz is \( E = 7.87 \cdot 10^{10} \) Pa. These, together with \( f_0 = \sqrt{k_S/m^*} / 2\pi \), give a vacuum resonant frequency for the NC38 as \( f_\text{vac} = 37300 \) Hz. This is 14 \% larger than the measured value (32770 Hz) at room temperature. This discrepancy has been attributed to additional mass of the electrodes, difference in the elastic modulus, and deviation from ideal geometry [2]. It seems, however, that the discrepancy is in fact mostly due to the simplifying assumptions of the effective model. Fixing the cantilever beams from one end raises the resonant frequency compared to a full quartz tuning fork model, where the oscillator is fixed at the base. We determined the eigenfrequencies of the complete tuning fork geometry by using a linear elastic material model and Comsol Multiphysics software. Thus obtained resonant frequency is 30790 Hz, which is 6 \% smaller than the measured value and 18 \% smaller than the one given by the effective cantilever model. By fixing the tines of the full tuning fork model similarly as in the effective model at the base connection, the resonant frequency of the effective model was reproduced. If we take into account the actual circular shape between the tines, the obtained frequency is 31770 Hz, which brings the difference between measured and calculated values to 3 \%. This can be attributed with confidence to the other effects mentioned before.

In the effective model the excitation force is applied to the tips of the tines. Correspondingly the acoustic force due to pressure of the surrounding fluid must be scaled. The correct scaling factor is the well-known result for the displacement \( x \) of a cantilever beam at height \( z \) from the tine’s fixing point [13]

\[ x/x_0 = \frac{1}{3} (z/z_0)^4 - \frac{4}{3} (z/z_0)^3 + 2 (z/z_0)^2. \]  

Here \( z \) obtains values between 0 and \( z_0 \), and \( x \) correspondingly between 0 and \( x_0 \). The normal acceleration applied to the fluid is also evidently scaled by this along the fork tines, as is seen from Eq. (10).
This effective model of a quartz tuning fork was compared to a "full model", where we meshed the tuning fork as well and treated the system as an acoustic-solid interaction problem in the frequency domain. The solid was modeled as linear elastic material. Even without damping ($\gamma = 0$) we can use the real part of the oscillation amplitude, Eq. (3), to determine the resonant frequency by calculating the response at a few frequencies. That is, we fit the oscillation amplitude data to

$$a = \frac{C}{f^2 - f_0^2},$$  \hspace{1cm} (17)

where $C$ is a constant. After obtaining the constant, we can use a single frequency to determine the resonant frequency. The calculated geometrical factors $B$ of the effective and the full model agreed to an accuracy of about 5%. The difference is likely due to the fixed-cantilever-beam approximation mentioned above. This will be discussed more later.

The coupled equations were solved using the finite element method, which is practical for systems with complicated geometry. The numerical computations were performed with Comsol. A desktop computer with 8 Gb of random access memory was used for the simulations, which restricted the number of elements, especially in the 3D models. In particular, the "full model" and the 3D "infinite medium" proved to be on the limits of our computational resources. Even with this restriction, the results obtained are quite accurate. Symmetries were utilized throughout when possible to decrease the size of elements and to speed up the simulations. The maximum number of elements in the 3D simulations was about 300,000.

3 Results

3.1 Geometrical Factor B

The geometrical factor $B$ is obtained from the simulated frequency data through Eq. (1). We begin by reporting the results for oscillators of infinite length. These reduce to two dimensions and are computationally much easier. We study the effects of different prong shapes and separations on the geometrical factor. Fig. 2 shows $B$ between different relative prong dimensions of infinite oscillators in infinite medium as functions of wave vector. At lower values of $k$, $B$ does not change very rapidly, but after $k \sim 1000 \text{ m}^{-1}$, it begins to increase faster. It reaches a maximum, after which the factor decreases with an even faster rate. The wave vector of the maximum and its value clearly depend on the geometry. It seems that larger $T/W$ ratio results in a more constant $B$ over the wave vector values. Similar studies varying, for example, the separation $d$ between the oscillators have been reported in ref. [5] and are not repeated here in detail. Comparing to ref. [5], we note that decreasing $T$ has a similar effect to $B$ as decreasing $d$.

Fig. 3 contains the results of the 3D tuning fork simulations for $B$ in a cylindrical cavity and in infinite medium, and the experimental data of refs [10] and [11]. The tuning fork is positioned symmetrically inside the cavity. For comparison, the results for 2D and the full model are also given. The simulated value of the geometrical factor and the general dependence of $B$ on the wave vector are more
Fig. 2 Geometrical factors $B$ of a pair of infinite oscillators in infinite medium oscillating in antiphase. The geometry is depicted in the upper-left corner. The relative dimensions of the oscillators are varied by changing $T$ as the separation $d = 0.30\, \text{mm}$ and $W = 0.34\, \text{mm}$ are kept constant. The case $T = 0.60\, \text{mm}$ is the 2D version of the NC38 quartz tuning fork shown in Fig. 1.

or less consistent with experiment. At lower wave vectors, $B$ increases at a slow rate and begins to grow faster as a function of $k$. In infinite medium, the increase of $B$ is significantly more moderate. As noted in ref. [5], the steep increase of $B$ inside the container compared to infinite medium is due to the impending acoustic resonance in the fluid-filled cavity. We see that the simulated value of $B$ at low values of $k$ is about 7% lower than the measured and about 45% larger at large $k$. When the tuning fork is tilted relative to the container, it can excite new acoustic resonant modes within the cavity. These modes also affect $B$ and at least most of the discrepancy between simulation and experiment can be explained by acoustic resonances in the container due to asymmetric positioning of the tuning fork. This is discussed further in Sec. 3.3. We now consider other effects, which change $B$ from the ideal perfectly symmetrical situation.

The apparent imperfections in the tuning fork seen in Fig. 1 do not explain the difference between measurement and simulation. Even the clear imperfection of the tuning fork tips, where one would expect strongest effects, result in surprisingly small change in the geometrical factor. Many other possible effects, which change $B$, can also be thought of, but none of them are realistically strong enough to explain the observed difference. Impedance of bulk steel, $Z \approx 3 \cdot 10^7\, \text{Pa}\cdot\text{s/m}$, is so large that the surface is practically fully reflecting. There is no observable effect due to the impedance unless the system is very close to acoustic resonance of the cavity and the impedance is significantly smaller. If $Z$ is two orders of magnitude smaller, then $B$ changes from 1.71 to 1.66 at $k = 2055$. In the experiments, the top of the can was partially open to allow fluid to enter the cavity. Some acoustic
radiation unavoidably also escaped the container through these holes. This leads to small decreases in $B$. If the top is completely open, then $B \approx 1.68$. Only if large parts of the container walls were removed, exposing the tuning fork, would there be a significant decrease in $B$ at large values of $k$, although still not large enough to reproduce the experimental values. Changing the radius of the container by a factor $0.98$ changed $B$ to $1.64$. One may also wonder about the effect of the electrodes on the surface of the quartz. Density and elastic modulus of the tines are not uniform because of them. Density variations change the effective mass, but these variations should be rather large to change the results in a significant way. Different effective masses approximately just scale $B$ over the $k$-values, at least if the mass differences are small. Smaller effective mass yields larger $B$.

The results obtained by the "full calculation" are persistently approximately 5 % smaller than the ones obtained from the effective model at all values of $k$. The absolute difference in $B$ grows from $0.055$ at small wave vectors to $0.067$ at about $k \approx 2100 \text{ m}^{-1}$, but the relative difference decreases at the same time from approximately 5.5 % to 3.8 %. The number of elements could have been inadequate in the "full model", as our computational power did not permit a finer mesh, but the disagreement did not increase significantly by making the mesh even sparser. As was discussed earlier, the discrepancy is likely due to the effective
model assuming the tines being fixed at the base end. The full model disagrees with experiment more than the effective model at low \( k \). This conclusion might be a bit hasty, however, since tilting of the tuning fork inside the capsule tends to correct the too-low values at low-\( k \) and the too-high values at large-\( k \), as is demonstrated in the final subsection of results.

3.2 Acoustic Emission

If the oscillator is placed inside a perfectly reflecting fluid-filled container, the resonant frequency changes according to Eq. (1), but the width of the resonance \( \Delta f \) remains the same as in vacuum. If, however, dissipation of acoustic energy is present in the system, the width increases compared to the vacuum value. Dissipation can exist, for example, in the form of absorbing boundaries or lack of boundaries all together (acoustic energy does not return to the oscillator). In infinite medium we can study the acoustic emission by determining \( \Delta f \). The width is related to dissipation in the system and in this case to the emitted acoustic power. An example of the connection between the geometrical factor \( B \) and the width \( \Delta f \) is shown in Fig. 4. The figure shows the case of two infinite square-shaped oscillators separated by 1.0125 mm in infinite medium oscillating in antiphase. Rather large separation between the oscillators has been used here to make acoustic resonances appear at lower values of \( k \). The undulation of \( B \) and \( \Delta f \) is partly due to resonant modes in the fluid between the tines. The first peak is a general property of all oscillators, and probably reflects some kind of optimal wavelength relative to...
Fig. 5 Width of the resonance of infinite oscillators in infinite medium. The left figure shows the variation of $T$ holding $d = 0.30$ mm and $W = 0.34$ mm constant. The right figure shows the variation of $d$ with $T = W = 0.34$ mm constant. Power law fit to the low-$k$ data is also plotted in the right figure (dashed line). A small offset (prefactor adjustment) has been applied to show the fit more clearly. The width of a single square-shaped oscillator is for reference. The geometries are depicted in the figures. The vacuum width used in the calculations (20 Hz) has been subtracted from $\Delta f$.

to the oscillator dimensions. The subsequent peaks appear only when acoustic modes between two oscillators exist. Fig. 4 demonstrates an apparent connection between the changes in $B$ and $\Delta f$. This is not coincidental. It seems that the two quantities are related through the Kramers-Kronig relation. By using a cutoff frequency, at least the wave vector values of the minima and maxima agree well between the actual quantity and the one calculated from the Kramers-Kronig relation. More accurate analysis is difficult due to finite range of simulated data and acoustic resonances.

Fig. 5 illustrates how the width of the resonance depends on the oscillator geometry in 2D. In the left figure, the relative dimensions are varied by changing $T$ and keeping other dimensions constant. In the right figure, the separation between the oscillating rods is varied. As a comparison, the case of a single oscillating rod is also shown in the right figure. A power law $\Delta f \propto k^p$ fits the data well at small wave vectors with an exponent $p = 3.94$. The exponent does not depend on the distance or relative dimensions of the oscillator, but the prefactor does. The exponent for a single oscillator is clearly different, showing the significance of two “tines” in acoustic phenomena as opposed to just one.

Resonance width for the 3D tuning fork geometry in infinite medium is represented in Fig. 6. The general behavior is the same as in 2D, which is also given in the figure. A power law again fits the simulated data well at low values of $k$. The obtained exponent is $p = 4.87$. A large exponent, such as this, is typical for acoustic emission. After about $k \approx 1500$ m$^{-1}$, the width begins to deviate from the power law, leveling off and eventually beginning to decrease. Schmoranzer et al. fitted a power law to their experimental data of various different quartz tuning forks with different resonant frequencies and found an exponent $p = 5.63$. 
The width of the resonance $\Delta f$ as a function of $k$ for a quartz tuning fork in infinite medium. The dimensions are as in Fig. 1. Power law fit to the low-$k$ data is plotted as a dashed line. Result for 2D corresponding to the tuning fork dimensions from Fig. 5 is shown for reference. The vacuum width used in the calculation (20 Hz) has been subtracted from $\Delta f$. The scatter in their data does not rule out an exponent $p \lesssim 5$. They also constructed several semianalytical models for the acoustic emission of a quartz tuning fork. By modeling the emission as 3D and 2D quadrupole sources and by two infinite cylinders, they predicted $p = 6$ (3D), $p = 5$ (2D), and $p = 5$ (two cylinders). These models are somewhat crude approximations for the quartz tuning fork and the numerical results of this paper are expected to give a more realistic value for the exponent. We should note that the actual values of $\Delta f$ from our simulations are much larger than the experimental results of ref. [14]. If part of the acoustic energy is reflected back to the oscillator, the width decreases, possibly explaining the difference. In our simulations, the tuning fork is placed in infinite medium and no energy emitted by the oscillator is returned.

### 3.3 Acoustic Resonances

When boundaries are added to an acoustic system, new phenomena arise. The mechanical oscillator can couple strongly to acoustic resonances in the fluid cavity. The resonances significantly affect the resonant behavior of the oscillator. In Fig. 7 the geometrical factor of a quartz tuning fork positioned symmetrically inside a perfectly reflecting cylindrical cavity has been calculated to larger values of $k$ (smaller wavelengths) than in Fig. 3 so that acoustic resonances in the cavity are excited by the tuning fork. In this figure, the data have been plotted as a function of wavelength instead of wave vector to ease comparison with experimental data of Gritsenko et al. [15]. Applying the impedance of bulk steel on the walls
Fig. 7 Geometrical factor $B$ for a quartz tuning fork inside a perfectly reflecting cylindrical container as a function of wavelength calculated over an acoustic resonance in the cavity. The corresponding wave vector values are indicated on top of the figure. Wavelength $2r_c = 2.63$ mm (container diameter) is indicated by the vertical dashed line. The tuning fork dimensions are as in Fig. 1. The horizontal dashed line is the limit of incompressible fluid, i.e. $\lambda \to \infty$. When the resonance is approached from the large wavelength side, $B$ increases steeply and reappears from the negative side. Another resonance is reached soon, causing $B$ to grow rapidly again.

would not change the results noticeably. We see that the first acoustic resonance appears close to $\lambda \approx 2r_c$, where $r_c$ is the inner radius of the cylindrical container, as one might expect. Close to the resonance, $B$ grows fast and reappears from below, actually achieving negative values. Negative value for the factor means that the resonant frequency of the tuning fork is larger in the fluid than in vacuum. This has been observed experimentally by Gritsenko et al. [15]. When the system is close to a resonance, it has two different resonant frequencies and the behavior of $B$, as seen in Fig. 7, is due to the cross-over of the two modes. As $B$ is recovering from the negative side, the system simulated in Fig. 7 reaches yet another resonance immediately, and $B$ grows again. The “density of modes” increases as $\lambda$ decreases and the behavior of $B$ becomes very erratic. As the fork response is this “chaotic” around acoustic resonances, it is clear that using the tuning fork for many purposes becomes near impossible. Depending on the application, considering the possible acoustic resonances is worthwhile. If one wants to avoid acoustic resonances, removing the container all together may work as a remedy, as long as the tuning fork does not couple too strongly to acoustic modes in the cell geometry.

To see how many modes exist and how strongly they couple to the tuning fork, it is useful to determine the eigenmodes in the fluid-filled cavity. The eigenmodes
(pressure field and frequency) are found by solving Eq. (7) directly as an eigenfrequency problem with \( k^2 \) as the eigenvalue. In order to compare different modes with each other, the eigenvectors (pressure fields) have been normalized to the same RMS-value. The relative coupling strength of a particular mode is found by performing a surface integration over the tine surfaces perpendicular to the motion and weighing the integral by Eq. (16). The obtained couplings for the two tines are subtracted, which reflects the demand that the tines are moving in antiphase.

Fig. 8 plots all the acoustic eigenmodes of the cavity up to wave vector \( k = 4000 \, \text{m}^{-1} \). It shows how the modes and the coupling change when the tuning fork is tilted four degrees in the direction perpendicular to the oscillation and two degrees in the other direction. In the symmetric case the first modes are not excited until \( k \geq 2250 \, \text{m}^{-1} \), slightly before the container diameter, as was already seen in Fig. 7. When the fork is tilted, some modes are already excited at \( k = 1250 \, \text{m}^{-1} \). The pressure fields of the lowest-frequency modes (\( k < 2250 \, \text{m}^{-1} \)) are such that the nodes are either along the container or in the radial direction perpendicular to the fork oscillation. These are not excited by a symmetrically positioned tuning fork.
fork. When the fork is tilted in both directions, the symmetries are broken and some additional modes are excited. The tilt also shifts some of the modes slightly at larger values of $k$.

We now compare the simulated results of a tilted tuning fork inside a cylindrical cavity using the effective model and the experimental data of Fig. 3. We use the same amount of tilt as in Fig. 8. In the asymmetric case we must solve the equations of motion for both tines with a common frequency because the tines can experience different forces from the fluid and can have different amplitudes. Due to the asymmetry, we cannot exploit any symmetry conditions, and the entire geometry must be meshed. This increases the size of elements because of limitations in computing power, and some loss of acoustic energy occurs even though the system should be completely lossless. This is seen as an increase in the resonance width, even though it should remain at the vacuum value. This does not affect the simulated value of $B$ noticeably. The result is plotted in Fig. 9. The geometrical factor has not been calculated between $k = 1250 \text{ m}^{-1}$ and $1750 \text{ m}^{-1}$, and beyond $k = 2100 \text{ m}^{-1}$. The dashed line is the symmetric case given in Fig. 3. The positions of eigenmodes, whose coupling is strengthened by the tilting, are indicated by vertical dotted lines.

Fig. 9 Comparison between simulation and experimental data of the geometrical factor $B$. The solid line is the simulated value of a quartz tuning fork, which is tilted by $\alpha = 4^\circ$ in the direction perpendicular to the tine motion and $\beta = 2^\circ$ in the other direction. The simulation has not been carried out between wave vectors $k = 1250 \text{ m}^{-1}$ and $1750 \text{ m}^{-1}$, and beyond $k = 2100 \text{ m}^{-1}$. The dashed line is the symmetric case given in Fig. 3. The system behaves in an unstable manner and the measurement frequency in the “single frequency method” must be very close to the resonant frequency. The weaker the coupling, the closer one
must be to the eigenmode for it to have any significant effect. When the system is close to a resonance, \( B \) behaves divergently, as can be seen in the figure. We note that the resonance at \( k = 1882 \text{ m}^{-1} \) does not seem to couple to the oscillator very strongly, even though the coupling strength given in Fig. 8 suggest stronger coupling than for the \( k = 1002 \text{ m}^{-1} \) mode, for example. By running the simulation with very small wave vector steps in the vicinity of this eigenfrequency, its coupling to the oscillator can be noticed. The reason for the seemingly large coupling may be that the normalization used for the eigenvectors is not perfect and for some pressure fields the procedure of calculating the coupling overestimates the actual situation. Similarly the coupling for some modes may be underestimated. If a mode is such that the pressure field tends to move the tines in the same direction, which is the case with the \( k = 1882 \text{ m}^{-1} \) mode, the computed coupling term becomes a difference between two rather large terms. In this case mesh quality, for example, begins to have greater importance. Our computed coupling term takes into account only how the fluid modes affect the tines, but may not take correctly into consideration how the mode is excited by the tuning fork in the first place. Since the tines must oscillate in antiphase, a mode in which the tines should move in the same direction, cannot be excited very effectively. If the fluid forces of such a mode acting on the tines are different in magnitude, the oscillator has two resonance modes nearby, which differ in phase by 180 degrees. The small bump in the experimental data at \( k \approx 1025 \text{ m}^{-1} \) could be the first additional mode to be excited by the tilted fork, which in our simulated configuration appears at \( k = 1002 \text{ m}^{-1} \). We see that other modes exist around \( k = 2100 \text{ m}^{-1} \), where the \(^4\text{He} \) vapor data exist. The increased proximity of the cylinder walls due to tilting of the oscillator is apparently the reason for the increase in \( B \) at small wave vectors compared to the symmetrical case. At larger \( k \) the factor seems to be closer to that of a symmetrical positioning, unless a resonance is nearby. One can imagine, that by tuning the modeled system by varying the fork position and tilt, simulated data could be made to fit experiment even better. Taking into account that no other plausible effect has been discovered and the one studied NC38 quartz tuning fork had a clearly observable tilt, we conclude that the discrepancy between the symmetrical simulation and experiment is mainly due to the tilted fork exciting additional modes inside the cavity. It would be interesting to see experimental data in the wave vector range between \( k = 1150 \text{ m}^{-1} \) and \( k = 2050 \text{ m}^{-1} \). This could be covered, for example, with pure \(^3\text{He} \) between 1 K and 3 K, where the speed of sound changes between 180 m/s and 100 m/s. At these temperatures, viscosity of \(^3\text{He} \) is not excessive.

### 4 Conclusions

We have performed numerical simulations to study the characteristics of quartz tuning fork resonators in acoustic medium. We have verified that the "effective model" described in Sec. 2.2 is accurate to within at least a few percent compared to a "full model". The effective model offers computationally cheaper way to simulate the tuning fork in fluid. It can be extended to other vibrational modes of the fork besides the lowest-frequency mode by determining the displacement as a function of height for the mode in question, similar to Eq. (16). A small
discrepancy between the two models remains and cannot be explained with absolute confidence at this moment. A reasonable agreement with available set of experimental data has also been achieved. Given the uncertainties in the exact experimental setup, the agreement could even be considered excellent. More experimental data with different oscillators would be interesting. Acoustic emission has been shown to be an important effect with quartz tuning forks in helium, when the wave vector becomes large enough (small $\lambda$), since the emitted acoustic power is proportional to $k^{4.86}$.

Our results can help choosing appropriate quartz tuning forks depending on the experiment and decide how to position them. In many applications, an oscillator with a constant geometrical parameter would be desirable. Our 2D simulations indicate that the tines should be as "narrow" as possible to achieve this. That is, the ratio $T/W$ should be large. At the same time the emitted acoustic power increases at lower wave vectors (decreases at larger $k$), however. The exponent in the acoustic energy power law remains constant. We have also demonstrated that acoustic resonances in the container cavity play a significant role, even at wavelengths much larger than the container diameter. To avoid low-$k$ acoustic resonances in the cavity, the quartz tuning fork should be installed symmetrically in the container. We note that in addition to first sound, second sound can also couple to a resonator, especially in helium mixtures. Since the speed of second sound is an order of magnitude smaller than that of first sound, these resonances appear at lower oscillator frequencies than first sound modes. Second sound effects have not been considered in this paper at all.

Our results apply only to inviscid fluid. Viscosity introduces another geometrical parameter, which is added to Eq. (1). In the present treatment, viscosity could be included as a linear approximation by adding complex-valued terms into the wave equation, Eq. (7). In some cases, such as those presented in refs. [10] and [11], experimental data can be extrapolated to zero viscosity, and Eqs. (1) and (7) are valid as they are.

Acknowledgements We thank F. B. Rasmussen for useful comments. This work has been supported in part by the EU 7th Framework Programme (FP7/2007-2013, Grant No. 228464 Microkelvin) and by the Academy of Finland through its LTQ CoE grant (project no. 250280). We also thank the National Doctoral Programme in Materials Physics for financial support.

References

1. D. O. Clubb, O. V. L. Buu, R. M. Bowley, R. Nyman, and J. R. Owerson-Bradley, J. Low Temp. Phys. 136, 1 (2004).
2. R. Blaauwgeers, M. Blazkova, M. Clovecko, V. Eltsov, R. de Graaf, J. Hosio, M. Krusius, D. Schmoranzer, W. Schoepe, L. Skrbek, P. Skyba, R. Solntsev, and D. Zmeev, J. Low Temp. Phys. 146, 537 (2007).
3. M. Blažková, M. Človečko, E. Gažo, L. Skrbek, and P. Skyba, J. Low Temp. Phys. 148, 305 (2007).
4. M. Blažková, M. Človečko, V. Eltsov, E. Gažo, R. de Graaf, J. Hosio, M. Krusius, D. Schmoranzer, W. Schoepe, L. Skrbek, P. Skyba, R. Solntsev, and W. Vinen, J. Low Temp. Phys. 150, 525 (2008).
5. J. Rysti and J. Tuoriniemi, J. Low Temp. Phys. 171, 273 (2013).
6. E. Pentti, J. Rysti, A. Salmela, A. Sebedash, and J. Tuoriniemi, J. Low Temp. Phys. 165, 132 (2011).
7. H. Kuttruff, Room Acoustics Fifth Edition (Spon Press, 2009).
8. L. Landau and E. Lifshitz, Fluid Mechanics Second Edition (Pergamon Press, 1987).
9. A. Bayliss, M. Gunzburger, and E. Turkel, SIAM J. Appl. Math. 42, 430 (1982).
10. E. Pentti, J. Tuoriniemi, A. Salmela, and A. Sebedash, J. Low Temp. Phys. 150, 555 (2008).
11. E. M. Pentti, J. T. Tuoriniemi, A. J. Salmela, and A. P. Sebedash, Phys. Rev. B 78, 064509 (2008).
12. K. Karrai and R. Grober, Proc. SPIE 2535, 69 (1995).
13. J. Tuoriniemi, J. Rysti, A. Salmela, and M. Manninen, J. Phys.: Conf. Ser. 400, 012077 (2012).
14. D. Schmoranzer, M. La Mantia, G. Sheshin, I. Gritsenko, A. Zadorozhko, M. Rotter, and L. Skrbek, J. Low Temp. Phys. 163, 317 (2011).
15. I. Gritsenko, A. Zadorozhko, and G. Sheshin, J. Low Temp. Phys. 171, 194 (2013).