Optimal Control Strategy for Linear Systems with Time Varying Random Plant Parameters

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Abstract—We propose an open loop control scheme for linear systems with time-varying random elements in the plant's state matrix. This paper focuses on joint chance constraints for potentially time-varying target sets. Under assumption of finite and known expectation and variance, we use the one-sided Vysochanskij–Petunin inequality to reformulate joint chance constraints into a tractable form. We demonstrate our methodology on two resource allocation problems. One involving a two-bus power system with stochastic load and wind power generation and the second allocation of labor hours to control an invasive plant population. We compare our method with situation approach and particle control. We show that in both situations the proposed method had superior solve times and favorable optimally considerations.

I. INTRODUCTION

In much of the linear controls literature, stochasticity is regarded as a factor external to the system modeling process. As evidenced by the common assertion of additive noise, we have failed to account for much of the systemic stochasticity that is found in natural processes. For example, wind speeds can effect the output of a wind turbine in a local grid, yet state-of-the-art models have considerable difficulty in making accurate predictions of their power output [1]. New control techniques that can incorporate this stochasticity systemically have the potential to enable more efficient controllers that can be robust to natural phenomena. In this paper, we develop an optimal control derivation scheme for discrete time linear systems with time-varying stochastic elements in the state matrix subject to joint chance constraints.

Early work in the 1960s and 1970s illuminated the need for incorporating random elements into the plant with applications in industrial manufacturing, communications systems, and econometrics [2], [3], [4]. Several works considered minimization strategies for linear quadratic regulator (LQR) problems. Without the addition of joint chance constraints, dynamic programming techniques [5] can easily be employed to find optimal controllers [6], [7], [8]. These works have been extended to account for unknown distributions associated with the random parameters. Sampling techniques and feedback mechanisms have been used to overcome these hurdles [9], [10]. These regulation problems are limited in scope and cannot solve the problem presented in this research. Random plants with more complex structure have been investigated [11], but have typically been limited to Gaussian disturbances. Since the late 1970s research in this area has been sparse, appearing only occasionally in econometric literature [12], [13] where plant uncertainty has been used to model economic trends.

A similar problem where the uncertainty in the plant is modeled either by bounded parameterization or a bounded column space has been extensively studied in the robust model predictive control community [14], [15], [16]. By exploiting the bounded parameter and column spaces, estimation techniques [17], [18] and stability results [19], [20] allow for closed loop computation of controllers. We note that while several of these techniques can address uncertainty in the plant, in many cases the uncertainty addressed is not random in nature [21], [22] as we address in this work. Further, these methods can address uncertainty that result from bounded random variables, such as discrete distributions with finite outcomes, and uniform or beta distributions, but cannot address random variables on semi-infinite or infinite supports.

Our approach is to employ a convex approximation of the joint chance constraint and solve the problem via convex optimization techniques. To achieve this, we use Boole’s inequality [23] to bound the joint chance constraint by a series of individual chance constraints. To transform the probabilistic constraint into a linear or biconvex constraint, we use the one-sided Vysochanskij–Petunin inequality [24]. While the one-sided Vysochanskij–Petunin inequality introduces conservatism, it enables optimization under a wide range of distributional assumptions. Using the one-sided Vysochanskij–Petunin can result in a bi-convex optimization problem. We discuss the alternate convex search method to solve the problem when this occurs. The main contribution of this paper is the construction of a tractable optimization problem that solves for convex joint chance constraints in the presence of random elements in the state matrix.

The paper is organized as follows. Section II provides mathematical preliminaries and formulates the optimization problem. Section III derives the reformulation of the chance constraints with Boole’s inequality and the one-sided Vysochanskij–Petunin inequality. Section IV demonstrates our approach on two problems involving power generation and labor allocation, and Section V provides concluding remarks.
II. PRELIMINARIES AND PROBLEM FORMULATION

We denote the interval that enumerates all natural numbers from \( a \) to \( b \), inclusively, as \( \mathbb{N}_{[a,b]} \). Random vectors and matrices will be denoted with bold case, \( \mathbf{x} \) and \( \mathbf{A} \), respectively. Non-random vectors and matrices are denoted with an overline, \( \bar{x} \) and \( \bar{A} \), respectively. We use the notation \( a_{ij} \) to denote the \((i,j)\)th element of the matrix \( \mathbf{A} \). For a random variable \( x \), we denote the \( n \)th moment as \( \mathbb{E}[x^n] \), and variance as \( \text{Var}(x) \). For sum and multiplication functions, \( \sum_{i=a}^{b} \) and \( \prod_{i=a}^{b} \), respectively, when \( a > b \) the index decreases from \( a \) to \( b \) by \(-1\).

A. The Vysochanskij–Petunin Inequality

For reference, we state the one-sided Vysochanskij-Petunin inequality.

**Theorem 1.** (The one-sided Vysochanskij–Petunin inequality [24]). Let \( x \) be a real valued unimodal random variable with finite expectation \( \mathbb{E}[x] \) and finite, non-zero variance \( \text{Var}(x) \). Then, for \( \lambda > \sqrt{5/3} \),

\[
P \left\{ x - \mathbb{E}[x] \geq \lambda \sqrt{\text{Var}(x)} \right\} \leq \frac{4}{9(\lambda^2 + 1)} \tag{1}
\]

The one-sided Vysochanskij–Petunin inequality is a refinement of the Chebyshev-Cantelli inequality for unimodal distributions. The one-sided Vysochanskij–Petunin inequality upper bounds one-sided tail probabilities for the deviation of unimodal univariate random variables from their mean. The two-sided Vysochanskij–Petunin inequality [25] is commonly cited as the foundation for the \(3\sigma\) rule in statistics. The one-sided variant can be used to find similar results for each tail.

B. Problem Formulation

We consider a discrete-time linear system given by

\[
x(k+1) = A(k)x(k) + B\bar{u}(k) \tag{2}
\]

with state \( x(k) \in \mathcal{X} \subseteq \mathbb{R}^n \), input \( \bar{u}(k) \in \mathcal{U} \subseteq \mathbb{R}^m \), and time index \( k \in \mathbb{N}_{[0,N]} \). We presume initial conditions, \( \bar{x}(0) \), are known, and the set \( \mathcal{U} \) is convex. The state matrix \( A(k) \) contains real valued random variables, \( a_{ij} \), each with probability space \( (\Omega, B(\Omega), \mathbb{P}_{a_{ij}}) \) with outcomes \( \Omega \), Borel \( \sigma \)-algebra \( B(\Omega) \), and probability measure \( \mathbb{P}_{a_{ij}} \) [23]. Each random variable is potentially time-varying.

**Assumption 1.** All random elements are mutually independent within their matrix. Further, the matrices at each time step are mutually independent.

**Assumption 2.** Each random element \( a_{ij}(k) \) has a finite expectation and variance.

Both assumptions are easily met in most scenarios. We would expect the parameters to be independent in many biological and physical processes, and most distributional assumptions would provide for finite expectation and variance. Of notable exception are certain parameterizations of the \( t \), the Pareto, and the inverse-Gamma distributions.

We presume desired polytopic sets, represented by the linear inequalities \( G_i(k)x(k) \leq h_i(k) \), that the state must reach at each time step with a desired likelihood

\[
P \left\{ \bigcap_{i=1}^{q} G_i(k)x(k) \leq h_i(k) \right\} \geq 1 - \alpha \tag{3}
\]

where \( q_k \) is the number of linear inequalities at time \( k \). We presume convex, compact, and polytopic sets \( \{x(k) \mid \cap_{i=1}^{q} G_i(k)x(k) \leq h_i(k) \} \subseteq \mathbb{R}^n \), and probabilistic violation threshold \( \alpha < 1/6 \).

**Assumption 3.** Each probabilistic constraint, \( \bar{G}_i(k)x(k) \leq h_i(k) \), marginally follows a unimodal distribution.

This is likely to be the most restrictive assumption as verifying unimodality can be challenging in cases where the distributional assumptions are not strongly unimodal [26]. For a thorough review of unimodality in distributions and strong unimodality, we recommend [27]. The primary concern for unimodality within this framework is maintaining unimodality through both additive and multiplicative operations. As the terminal time increases the more likely a non-unimodal distribution can arise from the complex and intricate interactions of the random state and the random plant parameters.

We seek to minimize a convex performance objective \( J : \mathcal{X}^N \times \mathcal{U}^N \to \mathbb{R} \).

\[
\begin{align*}
\text{minimize} & \quad J(\bar{x}(1), \ldots, \bar{x}(N), \bar{u}(0), \ldots, \bar{u}(N-1)) \\
\text{subject to} & \quad \bar{u}(0), \ldots, \bar{u}(N-1) \in \mathcal{U}, \\
& \quad \text{Dynamics (2) with } \bar{x}(0) \\
& \quad \text{Probabilistic constraint (3) (4d)} \\
& \quad \text{Assumptions 1, 2, and 3 (4e)}
\end{align*}
\]

**Problem 1.** Solve the stochastic optimization problem (4) with open loop control \( \bar{u}(0), \ldots, \bar{u}(N-1) \in \mathcal{U} \), and probabilistic violation threshold \( \alpha \).

The main challenge in solving Problem 1 is assuring (4d).

III. METHODS

To solve problem one, we reformulate the joint chance constraint (3) into a sum of individual chance constraints. Then this series of constraints is transformed into constraints that are affine in the constraint’s expectation and variance. Vysochanskij–Petunin’s inequality guarantees the generated controller satisfies the probabilistic constraint. The reformulation results in an easy to solve convex optimization problem.

A. Constraint Reformulation

We first address the polytopic constraint. Without loss of generality, we drop the time index \( k \) for the vectors and scalars defining the polytope. We take the complement and employ Boole’s inequality to convert the joint chance constraint into a sum of individual chance constraints,

\[
P \left\{ \bigcup_{i=1}^{q} \bar{G}_i(k)x(k) \geq h_i \right\} \leq \sum_{i=1}^{q} P(\bar{G}_i(k)x(k) \geq h_i) \tag{5}
\]
Using the approach in [28], we introduce risk allocation variables \( \omega_i \) for each of the individual chance constraints and bound the sum of risk allocation variables,

\[
\Pr \{ \tilde{G}_i x(k) \geq h_i \} \leq \omega_i \quad \forall i \in \mathbb{N}_{[1,q]},
\]

(6a)

\[
\sum_{i=1}^{q} \omega_i \leq \alpha
\]

(6b)

where \( \omega_i \) is a non-negative real number.

We add an additional constraint to (6) in the form of

\[
\mathbb{E}[\tilde{G}_i x(k)] + \lambda_i \sqrt{\text{Var}(\tilde{G}_i x(k))} \leq h_i
\]

(7)

with a non-negative optimization parameter \( \lambda_i \). Here, the enforcement of (7) implies

\[
\Pr \{ \tilde{G}_i x(k) \geq h_i \} \\
\leq \Pr \left\{ \tilde{G}_i x(k) \geq \mathbb{E}[\tilde{G}_i x(k)] + \lambda_i \sqrt{\text{Var}(\tilde{G}_i x(k))} \right\}
\]

(8)

We add constraints (7)-(8) to (6) allowing us can write the constraints as

\[
\Pr \left\{ \tilde{G}_i x(k) \geq \mathbb{E}[\tilde{G}_i x(k)] + \lambda_i \sqrt{\text{Var}(\tilde{G}_i x(k))} \right\} \leq \omega_i
\]

(9a)

\[
\mathbb{E}[\tilde{G}_i x(k)] + \lambda_i \sqrt{\text{Var}(\tilde{G}_i x(k))} \leq h_i
\]

(9b)

\[
\sum_{i=1}^{q} \omega_i \leq \alpha
\]

(9c)

where (9a) is a simplification of (6a) and (8), and (9a)-(9b) are iterated for all \( i \). Next, we observe

\[
\Pr \left\{ \tilde{G}_i x(k) \geq \mathbb{E}[\tilde{G}_i x(k)] + \lambda_i \sqrt{\text{Var}(\tilde{G}_i x(k))} \right\} \leq \frac{4}{9(\lambda_i^2 + 1)}
\]

(10a)

\[
\sum_{i=1}^{q} \omega_i \leq \alpha
\]

(10b)

where (10b) results from (1) and Assumption 3. By substituting (10) into (9), we can establish the relationship between \( \lambda_i \) and \( \omega_i \). Hence, (9) simplifies to

\[
\mathbb{E}[\tilde{G}_i x(k)] + \lambda_i \sqrt{\text{Var}(\tilde{G}_i x(k))} \leq h_i \quad \forall i \in \mathbb{N}_{[1,q]}
\]

(11a)

\[
\sum_{i=1}^{q} \frac{4}{9(\lambda_i^2 + 1)} \leq \alpha
\]

(11b)

for non-negative real parameter(s) \( \lambda_i \).

**Lemma 1.** For the controller \( \tilde{u}(0), \ldots, \tilde{u}(N-1) \), if there exists risk allocation variables \( \lambda_i \) satisfying (11) for constraints in the form of (3), then \( \tilde{u}(0), \ldots, \tilde{u}(N-1) \) satisfy (4d).

**Proof.** Satisfaction of (11a) implies (8) holds. Vysochanskij–Petunin’s inequality upper bounds (8) via (10). Boole’s inequality and De Morgan’s law [23] guarantee that if (11b) holds then (4d) is satisfied. \( \square \)

We formally define Problem 2.

\[
\begin{aligned}
\text{minimize} & \quad J(x(1), \ldots, x(N), \tilde{u}(0), \ldots, \tilde{u}(N-1)) \\
\text{subject to} & \quad \tilde{u}(0), \ldots, \tilde{u}(N-1) \in U, \\
\text{Expectation and variance derived from dynamics (2) with } \tilde{x}(0) \\
\text{Constraint (11) } & \quad (12b) \\
\text{Assumptions 1, 2, and 3 } & \quad (12c)
\end{aligned}
\]

(12a)

(12b)

(12c)

**) Problem 2.** Solve the stochastic optimization problem (12) with open loop control \( \tilde{u}(0), \ldots, \tilde{u}(N-1) \in U \), and probabilistic violation threshold \( \alpha \).

**Lemma 2.** Any solution to Problem 2 is a conservative solution to Problem 1.

**Proof.** By Lemma 1, (12c)-(12d) satisfy (4d). Here, (12c) replaces (4c) as we only need the expectation and variance derived from the dynamics. All other elements remain unchanged. Conservatism is introduced from Boole’s inequality as equality is only achieved when constraints are independent. Similarly, the Vysochanskij-Petunin inequality only achieves equality only when

\[
\Phi(x) = \begin{cases} 
\frac{4}{9(|x - \mathbb{E}[x]|)^2 + 1} & \text{if } x < \mathbb{E}[x] \\
\frac{2}{9} & \text{if } x = \mathbb{E}[x] \\
\frac{4}{9(|x - \mathbb{E}[x]|)^2 + 1} & \text{if } x > \mathbb{E}[x]
\end{cases}
\]

(13)

where \( \Phi(x) \) is the cumulative distribution function. This does not correspond to any named distribution. In any other scenario, the use of the Vysochanskij-Petunin inequality will introduce conservatism. \( \square \)

Here, Problem 2 is a conservative but tractable reformulation of Problem 1. While we cannot guarantee a solution exists to Problem 2, we can guarantee any solution to Problem 2 is a solution to Problem 1, if one exists.

**B. Solving Problem 2**

We note that while (11a) will elicit a closed form due to Assumptions 1 and 2, and linear dynamics, deriving this expression may be difficult. This is particularly true for longer time horizons. We start by observing that the linear dynamics allow us to write the state at time step \( k \) as the summation,

\[
x(k) = \left( \prod_{j=k-1}^{0} A(j) \right) \tilde{x}(0) + \sum_{i=0}^{k-1} \left( \prod_{j=k-1}^{i+1} A(j) \right) \bar{B} \tilde{u}(i)
\]

(14)

For the constraint \( \tilde{G}_i x(k) \leq h_i \), the affine form of (14) allows us to easily compute the expectation for an individual
constraint via the linearity of the expectation operator,
\[
\mathbb{E}[\tilde{G}_i x(k)] = \tilde{G}_i \left( \prod_{j=k-1}^{0} \mathbb{E}[A(j)] \right) \bar{x}(0) \\
+ \tilde{G}_i \sum_{i=0}^{k-1} \left( \prod_{j=k-1}^{i+1} \mathbb{E}[A(j)] \right) \bar{B}u(i)
\]

However, the main challenge will be deriving the variance term. The variance can be written as
\[
\text{Var}(\tilde{G}_i x(k)) = \tilde{G}_i \text{Var} \left( \left( \prod_{j=k-1}^{0} A(j) \right) \bar{x}(0) \right) \\
+ \tilde{G}_i \sum_{i=0}^{k-1} \left( \prod_{j=k-1}^{i+1} A(j) \right) \bar{B}u(i) \tilde{G}_i^T
\]

In this form, we cannot easily expand over the summations as the random matrices create complex interactions that can lead to non-zero covariance measures between any two terms. We recommend writing out each equation in scalar form and using the variances properties as it pertains to linear forms.

By inserting (15)-(16) into (11a), we see that \( \lambda \) and the inclusion of \( \bar{u}(k) \) in \( \text{Var}(\tilde{G}_i x(k)) \) form a biconvex constraint [29]. A biconvex problem has the following form:
\[
\min_{x,y} f(x,y) \\
\text{s.t. } g_i(x,y) \leq 0 \quad \forall i \in \mathbb{N}
\]

where \( x \in X \subseteq \mathbb{R}^n, y \in Y \subseteq \mathbb{R}^m, f(\cdot, \cdot) : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R} \) and \( g_i(\cdot, \cdot) : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R} \) are convex when optimizing over one parameter while holding the other constant [29]. For completeness, we include Algorithm 1 to demonstrate one method of solving (17) via the well known alternate convex search method [30]. While this method cannot guarantee global optimality, Lemma 2 guarantees any solution will be a feasible solution to Problem 1. We note that this method can be sensitive to chosen initial conditions. However, users may opt to utilize a grid search approach to find the initial conditions that produce the most optimal solution.

**Algorithm 1** Computing solutions to (17) with alternate convex search

**Input:** Feasible initial condition for \( y \), denoted \( y^* \), maximum number of iterations \( n_{\text{max}} \).

**Output:** Solution to (17), \((x^*,y^*)\)

1. for \( i = 1 \) to \( n_{\text{max}} \) do
2. \hspace{0.5cm} Solve (17) assuming \( y = y^* \)
3. \hspace{0.5cm} Set \( x^* = x \)
4. \hspace{0.5cm} Solve (17) assuming \( x = x^* \)
5. \hspace{0.5cm} Set \( y^* = y \)
6. \hspace{0.5cm} if Solutions converged then
7. \hspace{1cm} break
8. \hspace{0.5cm} end if
9. end for

We consider two scenarios: 1) a two-bus electric grid with two thermal generation units, one stochastic wind power plant, and a stochastic load, and 2) allocation of labor to combat an invasive plant species. All computations were done on a 1.80GHz i7 processor with 16GB of RAM, using MATLAB, CVX [31] and Mosek [32]. All code is available at https://github.com/unm-hscl/shawnpriore-time-varying-plant.

### A. Power Generation

Operation of power grids with high penetration of renewable energy sources are known to be a challenging task, largely because of the stochasticity inherent to intermittent sources of power generation (e.g., solar and wind generation units). Although thermal generation trends are predictable [33], state-of-the-art modeling efforts are often unable to effectively capture the fundamentally erratic and stochastic nature of wind energy production [1]. Indeed, this is an active area of research in power grids, because of the challenges that these intermittent sources of power can create for the reliable and stable operation of power grids.

Figure 1 shows a sketch of the prototypical system we consider, a small, two bus network with both wind and thermal generation, and a single load.

We model the system with the LTV dynamics [34]:

\[
x(k+1) = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
A(k)
\begin{bmatrix}
P_1 \\
P_2 \\
P_W \\
C_L \\
L_k \\
\gamma(k)
\end{bmatrix}
+ \begin{bmatrix}
\bar{u}_1(k) \\
\bar{u}_2(k)
\end{bmatrix}
\]

where \( P_1, P_2 \) are power generated by the thermal generators connected to Bus 1 and 2, respectively, \( C_W \) is a multiplier to convert cubed wind speed in \( m^3 \cdot s^{-3} \) to MW, \( P_W \) is the actual power generated from the wind farm, \( C_L \) is the maximum load requirement in MW, and \( L_k \) is the actual load. We presume \( C_L = 1,600 \) MW, and the wind farm houses 100 wind turbine generators with a blade length of 65 m. For standard air density of 1.225 kg·m\(^{-3} \), \( C_{\text{wind}} = 0.8130 \) MW·s\(^3\)·m\(^{-3} \). All other initial conditions are set to 0 without loss of generality. Here, the random variables are \( \gamma(k) \sim \text{Weibull}(5,30) \), and \( \beta(k) \sim \text{Beta}(50,50) \), and are presumed independent. Here, \( \gamma(k) \) represents the wind speed in m·s\(^{-1} \) at time \( k \) and is presumed to be consistent for all wind turbines.
The two thermal generators have a maximum nominal injection of 600 MW and must maintain at least 10% of the maximum nominal injection to remain on, hence,

\[ U = \{ \bar{u}(k) | -I_2 \bar{u}(k) \leq \begin{bmatrix} 60 & 1 \\ 600 & 1 \end{bmatrix} \} \] (19)

We presume that supply must meet demand and the power transmission line between buses has a maximum rating of 900 MW. Hence, the target set for each time step is defined by the inequality

\[ \begin{bmatrix} -1 & -1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \bar{x}(k) \leq \begin{bmatrix} 0 \\ 900 \end{bmatrix} \] (20)

Since the wind speed and load are stochastic, we consider the target constraint in a probabilistic manner and require they must hold with probability \( 1 - \alpha \). The optimization cost is presumed to be

\[ J(\bar{u}(1), \ldots, \bar{u}(N)) = \sum_{k=1}^{N} \bar{u}^T(k) \begin{bmatrix} 0.05 & 0 \\ 0 & 0.10 \end{bmatrix} \bar{u}(k) + \begin{bmatrix} 30 \\ 60 \end{bmatrix} \bar{u}(k). \] (21)

Here, the cost of running the thermal generator on Bus 2 is more costly than Bus 1.

From this construction, we observe that

\[ \prod_{i=k}^{0} A(i) = A(k) \] (22a)
\[ A(k)\bar{B} = \bar{0}_{n \times m} \quad \forall k \in \mathbb{N}_{[0,N]} \] (22b)

Hence, the expectation and variance terms, (15)-(16), simplify to

\[ \mathbb{E}[\bar{G}_i \bar{x}(k + 1)] = \bar{G}_i \mathbb{E}[A(k)] \bar{x}(0) + \bar{G}_i \bar{B}(k) \bar{u}(k) \] (23a)
\[ \text{Var}(\bar{G}_i \bar{x}(k + 1)) = \bar{G}_i \text{Var}(A(k)\bar{x}(0)) \bar{G}_i^T \] (23b)

The constraint (11a) simplifies to a linear constraint for all time steps. Further, since neither term changes as a function of the time step, the optimal solution will have the same controller for all time steps. Thus, we only need to solve Problem 2 for one time step. From (23), we can easily find the expectation and variance of our constraints:

\[ \mathbb{E}[\bar{G}_1 \bar{x}(k + 1)] = 0.5 \cdot C_L - 118.9188 \cdot C_W \] (24a)
\[ - \bar{u}_1(k) - \bar{u}_2(k) \]
\[ \mathbb{E}[\bar{G}_2 \bar{x}(k + 1)] = 118.9188 \cdot C_W + \bar{u}_1(k) \] (24b)

and

\[ \text{Var}(\bar{G}_1 \bar{x}(k)) = 204.6946 \cdot C_W^2 + 0.0025 \cdot C_L^2 \] (25a)
\[ \text{Var}(\bar{G}_2 \bar{x}(k)) = 204.6946 \cdot C_W^2 \] (25b)

It is easy to show that the probability density function of \( \gamma(k)^3 \) is log-concave via substitution. By [26], \( \gamma(k)^3 \) is strongly unimodal. Beta distributions with both parameters \( \geq 1 \) are also strongly unimodal. As strong unimodal distributions are closed under convolution, we guarantee the constraints are unimodal [27].

We compare the proposed methodology with scenario approach [35]. As the scenario approach relies on samples of the random state matrix, it can only guarantee constraint satisfaction up to a set confidence level. For fair comparison between methods, we set the confidence level, \( 1 - \beta \), to 0.999. We compute the number of samples, \( N_S \), required to achieve this confidence level as [35]

\[ N_S \geq \frac{2}{\alpha} \left( \log \left( \frac{1}{\beta} \right) + 2 \right) \] (26)

corresponding to 112 samples for \( 1 - \alpha = 0.84 \) and 1,781 samples for \( 1 - \alpha = 0.99 \).

In Figures 2 and 3, we compare the optimal cost and solve time of our approach to the scenario approach. We consider discrete values of \( 1 - \alpha \in [0.84, 0.99] \), and evaluate each approach for each value. As shown in Figure 2, for lower safety probabilities, our approach has a lower cost, however, as the safety probability increases, the conservatism of our proposed approach is evident in the higher cost. (We note the proposed method was not able to find a solution at \( 1 - \alpha = 0.99 \) as the admissible input set was too constraining to find a solution.) However, the solve time of our method is superior to the scenario approach for all safety probabilities for which a feasible solution was found. In contrast, the solve time seems of the scenario approach appears to grow exponentially as the safety probability increases. In considering between the two methods at high safety thresholds, the tradeoff between cost and solve time may inform choice of method.

Lastly, we note that potential extension of this approach to more complex grid architectures could exploit the fact that the Problem 2 can be solved via a sequence of linear programs, meaning that efficient scaling would be possible. Additionally, modeling choices in which hard constraints are cast as probabilistic constraints with high safety likelihoods may incur feasibility issues due to the conservatism inherent to the Vysochanski–Petunin inequality.
years will be dictated by the number of seeds dropped in
life cycle in one growing season, the population in future
simultaneously.

Both methods will require labor as well as a herbicide that inhibits the ability for the plants
their disposal is manually removing the plant from area, as
the next several years to combat the spread of this plant. At
conservation group is planning how to allocate resources over
servation area in rural Pennsylvania. A local government run
persicaria longiseta
B. Invasive Plant Population Control

Consider a scenario in which the invasive plant species
bristled knotweed (persicaria longiseta) has invaded a con-
servation area in rural Pennsylvania. A local government run
conservation group is planning how to allocate resources over
the next several years to combat the spread of this plant. At
their disposal is manually removing the plant from area, as
well as a herbicide that inhibits the ability for the plants
seeds to germinate [36]. Both methods will require labor
hours, \( \bar{u}(k) \), to perform the action and cannot be performed simultaneously.

To derive a model of population growth we make the
following assumptions. First, since the plant completes its
life cycle in one growing season, the population in future
years will be dictated by the number of seeds dropped in
previous years and the current year. Second, we limit the
number of previous years populations to three even though
seeds can stay dormant for several years before sprouting
[36]. Third, we assume the number of seeds that sprout is a random multiplier of the previous seasons respective
populations and that this multiplier generally decreases over
time. In summation, we will use an autoregressive model of
order 3 with random coefficients,

\[
x(k + 1) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\
\rho(k-2) \\ \rho(k-1) \\ \rho(k) \end{bmatrix} A(k) 
+ \begin{bmatrix} 0 & 0 \\ -0.0025 & 0 \\
-0.0015 & -0.003 \end{bmatrix} \begin{bmatrix} u_1(k) \\ u_2(k) \end{bmatrix}
\]

where \( \beta_i(k) \sim Gamma(i^4, 81) \) is the growth multiplier, \( \rho(k) \) is the area coverage of the plant (in 100 square
kilometers), \( u_1(k) \) and \( u_2(k) \) are the number of labor
hours spent removing plants and spraying herbicide, respec-
tively. Initial conditions were chosen randomly as \( \bar{x}(0) = \begin{bmatrix} .958 & 1.063 & 1.196 \end{bmatrix}^\top \). Control authority was limited to

\[
\mathcal{U} = \left\{ \bar{u}(k) \left| \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 1 \end{bmatrix} \bar{u}(k) \leq \begin{bmatrix} 0 \\ 0 \\ 1000 \end{bmatrix} \right. \right\}
\]

The terminal sets are chosen to be \( \rho(k) \leq 1 - .1k \), hence,
the set is defined by the inequality

\[
\begin{bmatrix} 0 & 0 & 1 \\ \bar{g}(k) \end{bmatrix} x(k) \leq 1 - .1k \\
\bar{h}(k)
\]

The safety threshold is chosen to be \( 1 - \alpha = 0.85 \). Strong
unimodality of the Gamma distribution guarantees the ter-
mary set at \( k = 1 \) elicits a unimodal distribution [26]. For
\( k = 2 \), unimodality was verified via random sampling.

Suppose the planning committee is planning two years in
advance and intends to minimize the squared labor hours
required. Hence,

\[
J(\bar{u}(0), \bar{u}(1)) = \bar{u}(0)\top \bar{u}(0) + \bar{u}(1)\top \bar{u}(1)
\]

For each constraint, we can easily find the expectation with
(15). Then we compute the variance for each time step,

\[
\text{Var}(\bar{G}x(1)) = 0.0206
\]

\[
\text{Var}(Gx(2)) = \begin{bmatrix} 0.222 & -7.556 \times 10^{-5} & -1.183 \times 10^{-4} \\ 0.1932 \times 10^{-4} & 9.755 \times 10^{-5} & 2.960 \times 10^{-4} \end{bmatrix} \begin{bmatrix} 1 \\ u_1(0) \\ u_2(0) \end{bmatrix}^2
\]

To accommodate the form (17), we add \( \sum \lambda_i \) to the
cost function. We optimize by alternating solving for \( \lambda_i \) and
\( \bar{u}(k) \). We instantiate by setting all \( \lambda_i \) equal. We define the
convergence criteria to be the norm of the differences in
These new constraints typically result in a biconvex optimization problem and outlined the alternate convex search approach to solve them. We demonstrated our method for an auto-regressive population model and compared our results with particle control. We showed that our method performed two orders of magnitude faster and met the required probabilistic threshold.

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