Prediction and theoretical study for the experimentally detectable Sachdev-Ye-Kitaev physics in multilayered type-II Weyl semimetal WTe$_2$ through optical SHG technique

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We propose a proposal for the easier experimental realization of Sachdev-Ye-Kitaev model in multilayered type-II Weyl semimetal WTe$_2$ in $T_d$ phase, through the optical second harmonic generation (SHG) technique. As the WTe$_2$ is a promising candidate to show strong nonlinear optical characters, including the SHG, we consider the effects come from the random interactions between the nondispersive bosonic modes induced by the SHG response (with two photon resonance) in each layer. For normally incident laser at proper frequency with ignorable extinction, the induced bosons can be regarded as the same energies, and randomly interact with each other due to the existence of two-photon linear response, which has a much small weight compared to the SHG. In addition to the approximate zero-dimensional SYK model, we also consider the effects of the one-dimensional SYK model (in the direction perpendicular to the layer surface) as a correction to our results. The finite spacial effect are studied in terms of the energy current in the perpendicular direction, and is base on the broadened range of zero-dimensional SYK interactions in real material at finite temperature by introducing a UV cutoff which is the interlayer spacing instead of the lattice constant. The numerical simulation and exact diagonalization show that the system exhibiting a maximally chaotic character and the spectral is characterized by many-body statistic of Gaussian symplectic ensemble. Our result shows that, different to the fermionic SYK models, the many-body statistic of bosonic SYK system does not depend on the value of $N$, and the conserved charge operator also exhibiting the holographic duality properties. For numerical simulation, we use the data of experimentally detected electronic structure of WTe$_2$ with spin-orbit coupling, and picking the photon energy at near-infrared range. A finite deviation of the probability distribution of random SYK couplings from the standard Gaussian distribution, which is due to the finite spacial effect, is being considered and we prove that this deviation (although small) is important for the experimental observable SYK effects. A detailed proposal of experimental detection for this finite deviation is also presented, which is through the measurement of surface tunneling conductivity using the STM. Our results show that, the SYK effects can be experimentally detected in multilayered WTe$_2$, for a certain range of layer number, photon energy, and coupling strength which can be tuned through the technique of magnetic Feshbach resonance. Besides, the SYK effects in multilayer materials will make a difference to the nonlinear optical properties of the material itself, thus our result can help to further improve the reliability of nonlinear multilayer materials used in optical devices.

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1 Introduction

The Sachdev-Ye-Kitaev (SYK) physics emerges usually in the non-Fermi liquid phase with zero-dimensional couplings much larger than the fermionic frequency while the fermionic frequency much larger than the coherence scale (incoherent critical metal phase). In such a regime, the quantum critical behaviors can be found in itinerant electron system due to the quantum fluctuation induced quantum phase transition between disordered state and the Fermi liquid state. The SYK physics is usually be considered in zero-dimensional space to remove the momentum-dependence, i.e., modeling the flat band system. Thus the experimental realization of SYK physics has been proposed to be under the strong magnetic field\[^{12}\] (to obtain the flat Landau levels) or in an artificial Kagome-type optical lattice\[^{51}\]. Similar consideration also applied to the topological flat band systems\[^{58}\], like the twisted bilayer graphene near the magic angle\[^{1}\]. In this article, we try to propose a new route of possibly realization of SYK physics, which is through the detection of second-order nonlinear optical response (the second-harmonic generation; SHG) on the surface of simple which is circularly polarized and follows a Gaussian profile. The realization of SYK physics in this scheme relys on the generation of photoelectrical current generated by the photogalvanic effect, and we choose to utilizing the SHG effect of WTe\(_2\) here which is to the second-order response with triplet resonance while the response to higher order can be ignored.

In contrast to the current-induced SHG through the bias and gate voltage, the circular photovoltage response is more easy to control experimentally, which is vital in keeping the equilibrium distribution of electrons. For example, by selecting a proper low-frequency of light (at near-infrared range and satisfying the optical resonance), we can greatly avoid the optical absorption as well as the off-resonance excitations. Note that the optical absorption is a linear response, and will be enhanced with increasing layer number in a layered crystal, and thus reduces the intensity of SHG. Besides, the optical absorption is possibly leads to photodamages. For WTe\(_2\) in\(T_d\) phase which is noncentrosymmetric, its two-fold rotational symmetry with a mirror plane \(M_{yz}\) and a glide mirror plane \(M_{xz}\) results in an in-plane inversion symmetry, and thus leads to the cancellation of of any in-plane components of photocurrent induced by a normally incident circularly polarized light\[^{65}\]. When considering the spin and Weyl cone tilting, there is actually a compound symmetry comprising both the time-reversal and two-fold rotation where the time-reversal process here has two effects: reverse the tilting direction and flip the spin. Experimental result\[^{65}\] shows that for WTe\(_2\) under obliquely incident light, the in-plane nonzero polarization-dependent current is only possible in the presence of a \(z\) polarization, while the in-plane nonzero spin current is only possible in the presence of nonzero electromagnetic potential components in three directions, i.e., the propagating direction of photon cannot be parallel to any crystal axises. Thus for a normally incident light, we can efficiently avoid the disturbations from in-plane photocurrents, and the detecting SHG intensity is purely origin from induced current in \(z\)-direction, and thus reduces the system to 1D, and for such a configuration, even approximately the 0D. That also makes the helicity-dependence as well as the photon drag effect can be ignored here. The avoided spin-flipping also provide a possibility to construct a conserved U(1) spin charge, which is vital to perform the exact diagonalization. In the mean time, a linear response induced by a reflected radiation with twice of the frequency of the incident one can possibly happen although has a much lower weight compares to the nonlinear one (SHG), and such a linear response plays the role of disorder to the interactions between bosons created by SHG. The boson mode induced by such a linear response has almost the equal energy with that induced by SHG, the only difference is that the in-plane inversion symmetry no more prohibits the photocurrent induced by linear response, but this effect can not be accounted due to its small weight.

The SYK model considered in this paper has three degrees of freedom, and we derive its
Hamiltonian as well as the Lagrangian using the supersymmetry theory by defining a supercharge. The results exhibit that, the random bosonic interactions are relevant as long as some certain ordering of the boson indices exist, and the resulting boson mode has the velocities obeying the chiral law. Note that the chirality here (or the interaction relevance) originates from the finite UV cutoff in 1D space, which inevitably exists for a real material and this also induce a small deviation for the distribution of the couplings from the standard Gaussian one. We also prove that such deviation is essential for the observation of SYK characters.

2 Theoretical preparation for SHG

For a material lacking the inversion symmetry, like the most typical one, type-II Weyl semimetals including the MoS$_2$, hBN, and 1T$_d$ WTe$_2$, the SHG effect and the shift current are available without needing the material junction or bias voltage. By choosing the velocity gauge for the applied laser electric field with an electromagnetic perturbation, and the induced current for the applied laser electric field with an electromagnetic perturbation, and the induced current through the unitary transformation, the perturbation from electric field reads, i.e., the commutator containing the second-order covariant derivative.

Note that to overcome the invalidity of velocity gauge when calculating the nonlinear optical coefficients with finite set of bands close to fermi level, the perturbation term as well as the single-particle velocity gauge Hamiltonian as derived from the length gauge one through the unitary transformation. The perturbation from electric field reads. Where in terms of single-component and the power series, we have $v_{nm}^a = -i\varepsilon_{nm}A_{nm}^a$ and $E(\Omega) = i\Omega A(\Omega)$, respectively. Thus make the velocity gauge be exactly valid for finite band

$$\sigma_{abc}(2\Omega; \Omega, \Omega) = -\frac{\varepsilon^3}{2\hbar^2 \Omega^2} \sum_{abc} \int \left[ \frac{v_{mn}^a v_{nm}^b v_{ln}^c (2N_F(\varepsilon_{ml}) + N_F(\varepsilon_{nl}) + N_F(\varepsilon_{mn}))}{\Omega - \varepsilon_{mn}} + \frac{N_F(\varepsilon_{mn}) v_{mn}^b v_{nm}^c}{\Omega - \varepsilon_{mn}} + \frac{N_F(\varepsilon_{mn}) v_{mn}^a v_{nm}^b}{\Omega - \varepsilon_{mn}} \right]$$

$$+ \frac{N_F(\varepsilon_{mn}) v_{mn}^b v_{nm}^c}{\Omega - \varepsilon_{mn}} + \frac{N_F(\varepsilon_{mn}) v_{mn}^a v_{nm}^b}{\Omega - \varepsilon_{mn}}$$

$$= -\frac{\varepsilon^3}{2\hbar^2 \Omega^2} \sum_{abc} \int \left[ \frac{v_{mn}^a (v_{ln}^c v_{ml}^b + v_{ml}^c v_{ln}^b)}{\varepsilon_{mn} + \varepsilon_{ln}} + \frac{2N_F(\varepsilon_{ml})}{\Omega - \varepsilon_{lm}} + \frac{N_F(\varepsilon_{mn}) v_{mn}^b v_{nm}^a}{\Omega - \varepsilon_{mn}} \right]$$

$$+ \frac{N_F(\varepsilon_{mn}) (v_{mn}^b v_{nm}^c + v_{nm}^c v_{mn}^b)}{\Omega - \varepsilon_{mn}} + \frac{N_F(\varepsilon_{mn}) v_{mn}^a v_{nm}^b}{\Omega - \varepsilon_{mn}}$$

where the generalized derivative is defined as (see Appendix.B for detail derivation)

$$v_{mn}^{bc} = \frac{\partial v_{mn}^b}{\partial k^c} - i(A_{mn}^c - A_{nm}^c) v_{mn}^b$$

$$= -\frac{v_{mn}^b (v_{ln}^c v_{ml}^b + v_{ml}^c v_{ln}^b)}{\varepsilon_{mn} + \varepsilon_{ln}} - i \sum_{l} (v_{ml}^b v_{ln}^c - v_{ml}^c v_{ln}^b)$$

with $A_{mn} = i\langle u_{mk} | \partial_k u_{nk} \rangle$ the Berry connection defined through Bloch functions, and $v_{mn}^{bc} = v_{mn}^{cb}$ only when the two income bosonic frequencies are connected in the same point of the loop, otherwise it becomes $v_{mn}^{bc} - v_{mn}^{cb} = -i \sum_{l} (v_{ml}^b v_{ln}^c - v_{ml}^c v_{ln}^b)$, i.e., the commutator containing the second-order covariant derivative.
model, it requires an expansion to infinite order, i.e., \( n^* \rightarrow \infty \), otherwise, if the truncation is performed to a certain order, then the accuracy depends on how large the next-order coefficient is.

We define \( N_F(\varepsilon_{mn}) = N_F(\varepsilon_m) - N_F(\varepsilon_n) \) here. Similarly, for shift current, the SHG response tensor reads,\(^{32, 31}\)

\[
\sigma_{abb}(0; \Omega, -\Omega) = \frac{e^3}{\hbar^2 \Omega^2} \sum_{ab} \int \left[ \frac{N_F(\varepsilon_{mn}) v_{mn}^b v_{mn}^a}{\varepsilon_{ml} + i\eta} \right] \left( P\left[ \frac{1}{\Omega - \varepsilon_{mn}} \right] + P\left[ \frac{1}{-\Omega - \varepsilon_{mn}} \right] + 2i\pi \delta(\Omega - \varepsilon_{mn}) \right)
\]

\[
+ \frac{N_F(\varepsilon_{mn}) v_{mn}^b v_{mn}^a}{\Omega - \varepsilon_{mn}} + \frac{1}{2} \frac{N_F(\varepsilon_{mn}) v_{mn}^b v_{mn}^a}{\varepsilon_{nm}} + (\Omega \leftrightarrow -\Omega)\]

\[
= \frac{e^3}{\hbar^2 \Omega^2} \sum_{ab} \int \left[ \frac{N_F(\varepsilon_{mn}) v_{mn}^b v_{mn}^a}{\varepsilon_{ml} + i\eta} \right] \left( P\left[ \frac{1}{\Omega - \varepsilon_{mn}} \right] + P\left[ \frac{1}{-\Omega - \varepsilon_{mn}} \right] + 2i\pi \delta(\Omega - \varepsilon_{mn}) \right)
\]

\[
+ \frac{N_F(\varepsilon_{mn}) v_{mn}^b v_{mn}^a}{\Omega - \varepsilon_{mn}} + \frac{N_F(\varepsilon_{mn}) v_{mn}^b v_{mn}^a}{-\Omega - \varepsilon_{mn}}
\]

\[
+ \frac{N_F(\varepsilon_{mn}) v_{mn}^b v_{mn}^a}{\varepsilon_{nm}}\right] .
\]

In terms of the functional derivatives, the SHG tensor (second order nonlinear conductivity) becomes

\[
\chi_{abc}(2\Omega; \Omega, \Omega) = \int \frac{dt_1}{2\pi} e^{2i\Omega t_1} \int \frac{dt_2}{2\pi} e^{2i\Omega t_2} \int \frac{dt_3}{2\pi} e^{2i\Omega t_3} \frac{\delta^2 J_z(t)}{\delta E_0(t_1) \delta E_0(t_2)} .
\]

Then it is obvious that the diagrammatic approach here is a generalization of the Kubo formula for the linear conductivity,\(^{31}\) and the validity of perturbation theory here precludes the case of strong laser field, e.g., with the extreme-ultraviolet source, which could create excitons recombination and drive the electrons back to the parent core to recombine into ground state as widely investigated in High-order harmonic generation (HHG) system. In the latter case, the energy conservation and equilibrium states of electrons are absent, the observable is no longer the induced polarized electrical currents but the optical response, like the shift current in SHG or the harmonic emissions in HHG. Thus to experimentally realize the SYK physics, we need to apply the laser-induced (normal incidence) electric field well beyond the strong field approximation, and the frequency of pump pulse should be much lower than that of the interband excitation. During HHG exploration, by applying the laser electric field following Gaussian profile with relatively large beam waist on sample surface,\(^{33}\) \( E(x, y) = \frac{E_0}{\pi w^2} e^{-\frac{(x^2+y^2)}{w^2}} \), where \( w \) is the laser beam waist, and the the current in z direction is \( J_z(t) = \int_0^1 d\alpha \frac{1}{\alpha} J(\alpha E_0, t) \), with the corresponding detectable intensity \( I_z(2\Omega) \sim 4\Omega^2 \int dt J_z(t) e^{2i\Omega t} \).

Different to that, for circularly polarized light or linearly polarized light, the evaluation of current replies on the electric polarization

\[
P_z(t) = \chi_{xyz}(2\Omega; \Omega, \Omega) E_y(\Omega) E_z(\Omega) \left( e^{-2i\Omega t} + e^{2i\Omega t} \right) ,
\]

which depends on the point group of the material. For type-II Weyl semimetal WTe\(_2\) in \( T_d \) phase, the point group is \( Pm \),\(^{16}\) and thus there are 10 nonzero elements in SHG tensor; they are \( \chi_{xxx}, \chi_{xyy}, \chi_{xxz}, \chi_{xzy}, \chi_{zzx}, \chi_{zzy}, \chi_{xzy}, \chi_{xyy}, \chi_{zzx}, \chi_{zzy} \). Then for in-plane laser electric field Using the 3 x 6 second-rank tensor \( d_{ij} \) (see Ref.\(^{45, 46}\)), the induced current can be obtained as

\[
J(2\Omega) = \begin{pmatrix}
\chi_{xxx} E_x^2(\Omega) + \chi_{xyy} E_y^2(\Omega) \\
2\chi_{xyz} E_x(\Omega) E_y(\Omega) \\
\chi_{zzx} E_z^2(\Omega) + \chi_{zzy} E_y^2(\Omega)
\end{pmatrix} .
\]
Next we will calculate the nonzero SHG response tensors in WTe$_2$.

3 SHG response tensors for Weyl semimetals

3.1 general calculation of SHG response in Weyl system

Firstly we focus on the general Weyl semimetals, whose energy band structure can be described by the following Hamiltonian

$$H = \chi \varepsilon_0 (k_x \sigma_x + k_y \sigma_y + k_z \sigma_z) - \mu \sigma_0,$$

(8)

where in a lattice model $\varepsilon_0 = \hbar v = ta$ where $t$ is the hopping integral and $a$ is the lattice constant. For such linearized Hamiltonian, the generalized derivatives $v_{mn}^{ab}$ can be ignored. Then for convenience during the calculation, we use the following coordinate transformation

$$
\begin{align*}
  k_x &= \rho \sin \theta \cos \phi, \\
  k_y &= \rho \sin \theta \sin \phi, \\
  k_z &= \rho \cos \theta.
\end{align*}
$$

(9)

Thus we have

$$dk_x dk_y dk_z = \rho^2 \sin \theta d\rho d\theta d\phi,$$

(10)

and the eigenvalues of the above Hamiltonian reads $\varepsilon_{m=0,1} = -\mu \pm \chi \rho$ for a two-band model. Then the velocity operators $v^a$ can be obtained as

$$
\begin{align*}
  v^x &= \chi \left( \frac{\cos 2\theta \csc \theta \sec \theta \sec \phi}{(1 - \cot \phi)(2i + \cot \phi)\tan \phi} \right), \\
  v^y &= \chi \left( \frac{\cos 2\theta \csc \theta \csc \phi \sec \theta}{-3i + 2\cot \phi - \tan \phi} \right), \\
  v^z &= \chi \left( \frac{\cos 2\theta \csc \theta \sec \theta (-\cos \phi - i\sin \phi)}{2 - \cos 2\theta \csc \theta \sec \theta \cos \phi + i\sin \phi} \right),
\end{align*}
$$

(11)

and the matrix elements can be obtained as

$$
\begin{align*}
  v_{mn}^x &= \langle m | v^x | n \rangle \\
  &= \delta_{mn} \langle v^x \rangle \left( \frac{k_x}{\rho} \right) (-\delta_{m,0} + \delta_{m,1}) + (1 - \delta_{mn}) \langle v^x \rangle \left( \frac{k_x}{\rho} \frac{k_z}{\sqrt{k_x^2 + k_y^2}} - i \frac{k_y}{\sqrt{k_x^2 + k_y^2}} \right), \\
  v_{mn}^y &= \delta_{mn} \langle v^y \rangle (\sin \theta \cos \phi (-\delta_{m,0} + \delta_{m,1}) + (1 - \delta_{mn}) \langle v^y \rangle (\cos \theta \cos \phi - i \sin \phi), \\
  v_{mn}^z &= \delta_{mn} \langle v^z \rangle \cos \theta (-\delta_{m,0} + \delta_{m,1}) + (1 - \delta_{mn}) \langle v^z \rangle \left( \frac{\sin \theta}{\rho} \right),
\end{align*}
$$

(12)

where the additional term $(-\delta_{m,0} + \delta_{m,1})$ consider the two-band model in which case the intraband velocity depends on the band index $m, n = 0, 1$. $\langle v^a \rangle$ denotes the eigenvalues of matrix $v^a$.

To calculate the required SHG response tensors, we introduce a third level, which is a virtual level between the two real bands close to the Weyl node, with energy $\varepsilon_n$. A three-level model is required instead of two-band model, thus we define the following energies

$$
\begin{align*}
  \varepsilon_m &= -\mu - \chi \varepsilon_0 \rho, \\
  \varepsilon_n &= -\mu + \chi \varepsilon_0 \rho, \\
  \varepsilon_l &= -\mu + \chi \varepsilon_0 \rho,
\end{align*}
$$

(13)
which implies that total incoming external bosonic frequency $2\Omega$ increases the eigenenergy by $\chi \rho$. Since for intraband transition, the transfer matrix element depends only on the single-band energy, while for interband transition, the transfer matrix element depends on both the energy difference and the derivation of a polar angle-dependent phase factor with respect to momentum\textsuperscript{35} \textsuperscript{36} \textsuperscript{37}, we can obtain the following (approximate) relations: $v_{nm} = v_{nl} = \frac{1}{2}v_{ml}$, $v_{nm}^z = (v_{mn}^z)^*$, $v_{mm} = -v_{ll} < 0$. A disadvantage of introducing the third virtual level is that the term $(\varepsilon_{ml} + \varepsilon_{ln})$ in the denominator of expression of $\sigma_{abc}(2\Omega; \Omega, \Omega)$ will becomes zero and thus leads to singularity result, that is why we add an small energy shift $\chi \delta \rho$ to $\varepsilon_{l}$. In the following we will consider only the $\chi = +1$ Weyl node and pick the positive eigenvalue of velocity operators.

By solving the above velocity matrices, the obtained eigenvalues are quite lengthiness, and in fact the effect of angle-dependence of eigenvalues is much lower than that of the momentum

\[ m \]

leads to the result (consider the two-band model fwith band indices $m, l = 0, 1$), the above evaluation can be applied here\textsuperscript{39}.

To obtain the analytical result, we consider the zero temperature case, and turning the chemical potential to zero, where $N_F(\varepsilon_m) = 1, N_F(\varepsilon_n) = 1/2, N_F(\varepsilon_l) = 0$. Under such approximations, we found that $\chi_{xxx} = \chi_{xyy} = \chi_{yx} = 0$, and $\chi_{xxy} = \chi_{zyy}$ which read

\[ \chi_{xxx} = -\frac{e^3}{2\hbar^2\Omega^2(2\pi)^3} \int_0^{\rho_c} dp \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \rho^2 \sin \theta \rho \left[ \frac{v_{ml}^z(v_{mn}^z v_{nl}^z + v_{mn}^z v_{ml}^z)}{\varepsilon_{mn} + \varepsilon_{ln} + 2N_F(\varepsilon_{ml}) + \frac{N_F(\varepsilon_{nl})}{\Omega - \varepsilon_{ln}} + \frac{N_F(\varepsilon_{nm})}{\Omega - \varepsilon_{mn}}} \right] \approx 3\pi^2 \rho (-\delta \rho^2 + (\Omega - \rho)^2 + \delta \rho (-\Omega + \rho)) v_x^2 v_z, \]

\[ \chi_{zxy} = \frac{3\pi^2 \rho (\Omega - \rho)(\delta \rho + \Omega - \rho)(\delta \rho - \Omega + \rho)}{4\hbar^2(\Delta\Omega + \rho)^2} \]

That means in low-enough temperature, the detected SHG intensity will nearly totally comes from the $z$-component of the induced current.

The above result is indeed under the relaxation free approximation, to consider the finite scattering rate, the analytical continuation need to be performed again. There is another route which does not assuming a third virtual level but incorporate it into the conduction band or valence band as introduced in Ref.\textsuperscript{39}, which consider also the intraband process. This will leads to the result (consider the two-band model fwith band indices $m = 0, l = 1$)

\[ \chi_{xxx} = \int_k \frac{v_{ml}^z (2v_{mn}^z v_{ml}^z - 2v_{ml}^z v_{tm}^z)}{\varepsilon_{ml}} \left[ \frac{e^3}{2\hbar^2\Omega^2} \frac{2N_F(\varepsilon_{lm})}{\Omega - \varepsilon_{ml}} + \frac{e^3}{4\hbar^2\Omega^2} \frac{2N_F(\varepsilon_{lm})}{\Omega - \varepsilon_{ml}} \right] \]

\[ + \int_k \frac{e^3}{2\hbar^2\Omega^2} \frac{v_{ll}^z (2v_{lm}^z v_{ml}^z - 2v_{ml}^z v_{ml}^z v_{lm}^z)}{2\varepsilon_{ml}} \]

\[ = \chi_{zyy}. \]

### 3.2 Application to type-II Weyl semimetal WTe\textsubscript{2} in $T_d$ phase

We note that, when the effect of spin-orbit coupling (SOC) is considered, there are gapless Weyl nodes in type-II Weyl semimetal WTe\textsubscript{2}. Similar to graphene, when the electron move from the linear valence band to the linear conduction band near a Weyl node, or vice versa, there will be a strong instantaneous acceleration when the electron close the node, and leads to strong
emission of detectable radiation in the mean time. That means the detected SHG intensity will mostly come from the interband transition, although the electron will turning back to the equilibrium position after the perturbation. But the broad peak at finite temperature implies the existence of relaxations, due to, e.g., the impurity or phonon scatterings. This part of SHG intensity cannot be applied to investigate the possible SYK behaviors.

For WTe$_2$ with $T_d$ structure in $Pmn2_1$ space group, both the time-reversals symmetry and the inversion symmetry are broken, which give rise to nonzero Berry curvature. When the spin-orbit coupling (SOC) is being considered to make the interband transition is possible\cite{footnote1}, the layer number indeed affects littlely on the band structure as well as the valence band SOC splitting\cite{footnote2} of WTe$_2$. Thus the contribution from the interlayer hopping to the induced current in $z$-direction can be disregarded, which guarantees the $z$-distance-independence of the random currents $J^z_{ij}$.

Next we use the experimentally obtained parameters of Ref.\cite{footnote3}, where the Hamiltonian of type-II Weyl semimetal WTe$_2$ reads

$$H = Ak_x + Bk_y + a k_x \sigma_y + b k_x \sigma_z + d k_y \sigma_z + e k_z \sigma_x,$$

with $A = -2.7, B = 0.6, a = 1, b = 1.1, d = 0.27, e = 0.184$. Following the above coordinate transformation, the matrix form of the Hamiltonian is

$$H = \begin{pmatrix} (A\rho + b\rho)\cos\phi \sin\theta + (B\rho + d\rho)\sin\phi \sin\theta & \epsilon \rho \cos\theta - i\lambda \rho \cos\phi \sin\theta \\ \epsilon \rho \cos\theta + i\lambda \rho \cos\phi \sin\theta & (A\rho - b\rho)\cos\phi \sin\theta + (B\rho - d\rho)\sin\phi \sin\theta \end{pmatrix}.$$  

After substituting the above parameters, we obtain the eigenvalue of the Hamiltonian as $\varepsilon_+ = 1.06 \rho \hbar v_0$ for the conduction band, and $\varepsilon_- = 0.7 \rho \hbar v_0$ for valence band. Here we select the points at $|k| = \pi a$ away from the Weyl node, and setting the chemical potential as $\mu \approx (\varepsilon_+ + \varepsilon_-)/2$ to prevent the optical absorption when the incoming photon energies is higher than the band gap. Thus for WTe$_2$, by turning the chemical potential to $\mu = \rho \hbar v_0$, we redefinen the energies of the three levels as $\varepsilon_m = -0.3 \rho \hbar v_0, \varepsilon_n = -0.15 \rho \hbar v_0, \varepsilon_l = 1.06 \rho \hbar v_0$. Both of these two energies are positive, indicating that one of them is electron band while the other is hole band, with the same topologically protected chirality.

Since the type-II Weyl point requires kinetic energy dominates\cite{footnote4}, and thus $\phi$ should be in the range $0.58\pi > \phi > \pi/2$ or $1.58\pi > \phi > 3\pi/2$. In this case, through the same procedure shown in above and by taking $\phi = 0.55\pi$, we obtain the eigenvalues of velocity operators as $\langle v_x \rangle \approx -12.3 \epsilon_0 / \hbar, \langle v_y \rangle \approx -4.9 \epsilon_0 / \hbar$ and $\langle v_z \rangle \approx 0.75 \epsilon_0 / \hbar$.

We plot the results of SHG response tensor in Fig.1. Consistent with our analytical result in above subsection, although the tensors $\chi_{xxx}, \chi_{xyy}, \chi_{xzy}$ are nonzero, they are vanishly small compared to $\chi_{zzz}$ and $\chi_{zxy}$. This result guarantees that the experimentally detected SHG intensity will most contribute to the $z$-component of the induced current, i.e., for isotropic laser electric field, we have $I_{SHG} = J_z E^2_{planal}$. By substituting the specific parameter of WTe$_2$, our results show that, the SHG happen at $\Omega = \mu$, i.e., the total incoming bosonic frequencies $\Omega_{in} = 2\mu$. Besides, as we have made the comparison between the methods of three-level and two-level model, we found that, the two-photon resonance, which corresponds to the main peak, can be seen in both methods, while the one-photon resonance, which corresponds to the smaller peak, can be seen only in the two band model.

4 Possibility of experimental realization and theoretical analysis

As a layered transition-metal dichalcogenide, the type-II Weyl semimetal WTe$_2$ in $T_d$ phase is a noncentrosymmetric material, which is different to the WTe$_2$ in $H$ or $T'$ phases. The stacked layers of bulk WTe$_2$ are connected by Van der Waals forces, and in orthorhombic
phase, it has large interlayer spacing\[^{17}\] which lighten the effect of interlayer hopping. Due to its broken inversion symmetry and time-reversal symmetry, the effects of interlayer coupling (interlayer cancel effect) and optical absorption to the SHG intensity can be disregarded. Otherwise, for some other materials with certain symmetries, e.g., the MoS\(_2\), hBN, and PdSe\(_2\), the SHG exhibits an even-odd oscillation for different layer numbers\[^{49}\], and can be suppressed by the increasing absorption when the photon energy of SHG is higher than the band gap\[^{52}\]. Besides, the SU(2) symmetry is also broken by the SOC in WTe\(_2\). Thus for the materials with central symmetry, the SHG decreases with increasing \(N\) (or the increased absorption). While for noncentrosymmetric materials, like the \(\varepsilon\)-GaSe and \(Td\) WTe\(_2\), the SHG intensity will increase with increasing \(N\) (or optical absorption) nonlinearly\[^{52}\]. For example, the SHG signal of \(\varepsilon\)-GaSe exhibits a cubic dependence on \(N\) when above five layers and a quadratic dependence on \(N\) when lower than five layers\[^{52}\]. We will show that this change of the dependence is also in consistent with our theoretical prediction. Similar experiment has been done on Zr(HOPO\(_3\))\(_2\)\[^{43}\] under different polarization, where the SHG intensity is found to be quadratically enhanced with the increasing layer number (optical density of the film). So it seems for noncentrosymmetric materials, the SHG intensity will enhanced by increasing layer number with a proportionality depends on the effective nonlinearity of the material itself.

We assume the laser is well inside the multilayer \(Td\)-WTe\(_2\), and the induced unit currents \(J_{z0}\) in each larye are all with the same frequency, i.e., the single band case in semiclassical approximation. This is an important precondition to realize SYK physics, which is to eliminate the momentum (or space) dependence of each mode. Now that each \(J_{z0}\) is independent of the coordinates in the \(x - y\) plane, we also assume the distance in \(z\)-direction is small enough to ignore its effect on the generated current. For WTe\(_2\), since its refractive index along the direction of \(J_0\) (i.e., the \(c\)-axis) is much smaller than that along \(a\)- or \(b\)-axis for wavelength \(\lambda \gg 200\) nm\[^{44}\], and its extinction coefficient is vanishingly small in \(c\)-axis for this range which guarantees large enough light penetration depth across the \(N\) layers. Besides, the existing SYK behaviors can also be verified through the relation between the variance \(\sigma^2\) (which is \(N\)-dependent due to finite mean value of SYK interactions \(g_{ij}\)) and the detected slope of SHG intensity \(I_{SHG}\), by artificially changing the layer number.

To guarantees low enough extinction coefficient (imaginary part of refractive index) and long enough light penetration depth along the \(z\)-axis, the wavelength of the \(p\)-polarized laser can be chosen around 800nm (357 THz; 1.55 eV). The SHG susceptibility is related to the SHG response tensor (conductivity) by \(\chi_s^{abc} = \chi^{abc}/(2i\omega\epsilon_0)\), with \(\epsilon_0\) the vacuum permittivity. Consider the inevitable reflection effect, the corrected SHG susceptibility reads\[^{50}\]

\[
\chi_R^{abc} = \frac{\chi_s^{abc}}{\epsilon^{1/2}(2\Omega) + \epsilon^{1/2}(\Omega)}\left[\epsilon^{1/2}(2\Omega) + 1\right]\left(\frac{2}{n_R(\Omega) + 1}\right)^2, (18)
\]

where \(\epsilon\) is the relative dielectric constant and \(n_R\) is the reflective index. Since the interlayer cancel effect is absent in multilayer WTe\(_2\), and the layer spacing is large, we can write the detected SHG intensity as (proportional to the induced current) as

\[
J_{bulk}^z = \chi_R^{bulk} E_{planar}^2 = NJ_{sheet} + J_{SYK}. (19)
\]

That is to say, when the interlayer hopping is ignored, the increase of the slope of \(J_{bulk}\) with respect to layer number \(N\) is due to the current-current interaction effect which can be obtained by the SYK model. This part of contribution to the \(J_{bulk}\) is denoted by \(J_{SYK}\), which is finite only when the layer number is large. The interlayer hopping effect is not being considered here to avoid the spacial effect, although the minor finite \(z\)-distance will be considered in the following sections.

Note that the intensity here, observed by experiments, is related to the induced current...
through the following relations

\[ I_{2\Omega} = \frac{c}{2\pi} |E_{2\Omega}|^2, \]
\[ |E_{2\Omega}| = \frac{2\pi}{c} J_z(2\Omega), \]
\[ J_z(2\Omega) = E_{2\Omega}^2 \chi, \]

where \( E_{2\Omega} \) is the effective electric field regarding to the effect of induced current, and \( c \) is the speed of light. Thus we obtain

\[ I_{2\Omega} = \frac{2\pi}{c} |E_{\Omega}|^2 \chi^2 = \left( \frac{2\pi}{c} \right)^2 \chi^2 I_{\Omega}^2, \tag{20} \]

i.e., the detected SHG signal is

\[ I_{\text{bulk}} = \frac{I_{2\Omega}}{I_{\Omega}^2} = \left( \frac{2\pi}{c} \right)^2 \chi^2. \tag{21} \]

5 Zero-dimensional SYK effects

5.1 SYK model: antisymmetry treatment and field theory analysis

The SYK\(_2\times\) SYK\(_2\) model considering the random interactions between photon-excited boson modes in different layers can be written as constructed as

\[ H_{\text{SYK}} = \sum_{ij} g_{ij} u_i^\dagger u_j, \tag{23} \]

where \( u_i \) is the boson excitation induced by the external electromagnetic radiation and it is be regarded as a unit mode in \( i \)-layer. Note that the above Hamiltonian is not in a fermion bilinear form (which is forbidden by the \( \mathbb{Z}_2 \) symmetry in SYK model), but the SYK\(_2\times\) SYK\(_2\) one, which can be rewritten in the form \( H_{\text{SYK}} = \sum_{ij} g_{ij} b_i^\dagger b_j^\dagger b_i b_j \), with \( i(j) \) independent with \( i'(j') \) but the selection of a combination of \( i \) and \( j \) is constrained by that of \( i' \) and \( j' \), and vice versa, which means there are actually three degrees of freedom instead of four. However, as will be seen in the next section, we can applying a ultraviolet cutoff for each boson, and results in simplification \( i(j) = i'(j') \), although that will leads to scaling dimensions mismatch between kinetic and interacting terms, but the results can be correctly solved by the replica procedure and the mismatch can be removed by introducing the normal ordered kinetic term in the presence of finite UV cutoff in that order. That will be a necessary treatment in studying the SYK model beyond zero-dimension.

For such a SYK\(_2\times\) SYK\(_2\) model, due to the special relation between bosons \( b_i b_{i'} \) and \( b_i^\dagger b_j \), there are only three degrees of freedom instead of two or four. Thus to obtain the corresponding Lagrangian, we firstly define a supercharge according to the supersymmetry theory:\[67\]

\[ Q_b = \sum_{ij} g_{ij} b_i^\dagger b_j. \tag{24} \]

Note that in order to satisfy the supersymmetry requirement, the supercharge must equivalents to a fully antisymmetry tensor.

Although the supersymmetry theory has the antisymmetry requirement, we will summing over all the cases when dealing with SYK Hamiltonians, which is due to a simple reason, the original SYK Hamiltonian in Eq.\((24)\) is not in a normal ordering, i.e., it contains all the cases: \( i < j, i = j, i > j \), and we found that, as shown in the following calculations throughout the
whole paper, this summation will leads to correct results and the supersymmetry theory works. Thus the above SYK Hamiltonian can be written as

$$H_{SYK} = Q_b^i Q_b^j$$

$$= \sum_{i,j} g_{ij}^2 b_i^j b_j^i + \sum_{i,j,k \neq j} g_{ij,k} b_i^j b_j^k b_k^i + \sum_{i,j,l \neq i} g_{ij,l} b_i^j b_l^i b_i^l + \cdots,$$

where $\cdots$ is the antisymmetry-allowed term. Note that the boson index $l(k)$ is be certain once a pair of values of $i, j, k(i, j, l)$ are given.

Note that, in this configuration, there is an important factor, which is the ratio between the cases of full summation and summation constrained by a certain normal ordering,

$$\eta = \frac{\sum_{i<j} g_{ij}}{\sum_{i<j} g_{ij}} = 2.$$

Thus the above Hamiltonian can indeed be represented as

$$H_{SYK} = Q_b^i Q_b^j$$

$$= \eta \sum_{i<j} g_{ij}^2 b_i^j b_j^i + \eta \sum_{i<j,k \neq j} g_{ij,k} b_i^j b_j^k + \cdots.$$

While since $g_{ij}$ is a Gaussian variable $g_{ij}^2$ has an index-independent mean value (the variance), the first term in above Hamiltonian can be reduced to $\sum_{i<j} g_{ij}^2 n_i n_j$, where $n_i = b_i^i b_i^i$ is the single bosonic number operator. The corresponding Lagrangian reads

$$\mathcal{L} = \sum_i b_i^i \partial b_i^i + \sum_i \frac{i}{2} g_{ij} b_i^j b_i^j + \sum_{i,j} g_{ij} Q_b^i Q_b^j,$$

where the last term be equivalently replaced by $\eta \sum_{i<j} g_{ij} Q_b^i Q_b^j$. Note that here the derivative operator should be in the matrix form $\partial \sigma_0$ which plays the role of boson propagator, but for simplicity we omit the Pauli matrix in the following. In this antisymmetry configuration, the summation in the second term of the above Lagrangian cannot simply be regarded as the total number $N$ times the bosonic filling, it has another form, which turns out to be $\eta$:

$$\sum_i b_i^i b_i^i = \sum_{i<j} b_i^j b_j^i - \sum_{i<j} b_j^i b_i^j + \sum_{i>j} b_i^j b_i^j - \sum_{i>j} b_j^i b_i^i = \eta = 2.$$

This antisymmetrically defined conserved bosonic charge is similar to (but no exactly the same) that defined in terms of fermions, e.g., Ref. [59], and such an antisymmetrically constructed conserved quantity will be used below when dealing with the many-body statistic using the exactly diagonalization. Note that this antisymmetry treatment is only valid in a supersymmetry method in order to verify the validity of the Lagrangian we need in the following calculation. Otherwise the term $\sum_i b_i^i b_i^i$ should be directly related to the total boson number. Most importantly, the antisymmetry treatment here sometimes gives the interacting system more than three degree-of-freedom. We can expand the third term of Hamiltonian Eq. (28) to see this point, where we use the following form of supercharge $Q_b = \sum_{kl} g_{xk} b_k b_l$

$$\sum_{ij} g_{ij} Q_b^i Q_b^j = Q_b^2 = \sum_{ij} (g_{ij} b_i^j b_j^i)^2 + \sum_{ij} g_{ij} (\sum_{k} g_{k} b_k^l b_l^i b_i^l b_k^i) + \sum_{ij} g_{ij} (\sum_{k} g_{kl} b_k^l b_l^i b_i^l b_k^i).$$
By comparing to the original form of SYK Hamiltonian \( H = \sum_{ij} g_{ij} b_i^\dagger b_j b_j^\dagger \), it is easy to see that the first term corresponds \( i(j) = i'(j') \), the last term corresponds \( i, j \neq i', j' \). Since the same mapping (like time evolution) from \( i(j) \) to \( i'(j') \) requires the case like \( i = i' \) but \( j \neq j' \) cannot be exist, which is also the origin of three degrees of freedom, but the antisymmetry treatment indeed allow such occasion to exist, which can be seen from the second and third term of above equation. That only in the short time case, where the relation \( b_i^\dagger b_j - b_j^\dagger b_i = 1 \) exist. The second term and third term can be written as

\[
\sum_{ij} g_{ij} (\sum_d g_d b_i^\dagger b_j^\dagger b_j b_d) = \sum_{ij} g_{ij} (\sum_d g_d b_i^\dagger b_j)(1 + b_j^\dagger b_i) = \sum_{ij} g_{ij} (\sum_d g_d b_i^\dagger b_d b_i) = (\sum_i g_{ij} b_j^\dagger)(\sum_l g_l b_l \eta, \tag{31}\]

\[
\sum_{ij} g_{ij} (\sum_k g_{kj} b_k b_j^\dagger b_j) = \sum_{ij} g_{ij} (\sum_k g_{kj} b_j^\dagger)(1 + b_j^\dagger b_i) = \sum_{ij} g_{ij} (\sum_k g_{kj} b_j^\dagger b_j) = (\sum_i g_{ij} b_j^\dagger)(\sum_k g_{kj} b_k)(\eta + N). \]

Thus for the first term and last terms, \( Q_b \) is commute with the \( b_i^\dagger b_j \), but is no longer the case in the second and third terms, which is related to the number of degree-of-freedom.

The inverse supercharge here is indeed a second rank tensor, with its real part be the symmetry result and imaginary part be the antisymmetry result (which are coincident here),

\[
Q_b^{-1} = (\sum_{ij} g_{ij} b_i^\dagger b_j)^{-1} = x_{ij} = 1 + \frac{i}{2} \eta = 1 + i, 
\]

\[
\text{Re} Q_b^{-1} = 1, \quad \text{Im} Q_b^{-1} = \frac{1}{2} \eta = 1. \tag{32}\]

By inserting this result into the above Lagrangian, we obtain

\[
\mathcal{L} = \sum_i b_i^\dagger \partial b_i + \sum_{ij} g_{ij} b_i^\dagger b_j, \tag{33}\]

with the derivation operator within the first term reads \( \partial = (\partial_\tau - i \partial_z) \), i.e., considering the time variance and possible \( z \)-direction movement, and this derivation term can indeed be expanded as

\[
\sum_i b_i^\dagger \partial b_i = -Q_b^2 - \sum_i b_i^\dagger (\frac{i}{2} Q_b^2) b_i = -Q_b = - \sum_{ij} g_{ij} b_i b_j = \sum_i b_i^\dagger (\partial_\tau - i \partial_z) b_i, \tag{34}\]

which obeys the law of Euler-Lagrange equation of motion. By comparing the Lagrangians Eq.\,(28) and Eq.\,(33), we can obtain that, in this system, the SYK\(_2\) model indeed has the same Lagrangian with the SYK\(_2\times\text{SYK}_2\) model. The situation change only in the case when we applying the UV cutoff in space, which is \( i(j) = i'(j') \) (see Eq.\,(52)). Under this UV approximation, which is more close to the behavior in reality, the imaginary part of supercharge vanishes and it reduces to \( Q_b^{-1} = 1\).

But the imaginary part can be preserved even in this case by extending the scaling dimension of kinetic term as clarify in below. In construct
with this, in zero-temperature limit, where the UV cutoff does not exist since the effective momentum in z-direction becomes infinite, and in this case the real part of $Q_b$ vanishes, and $Q_b^2$ becomes ill-defined. This special "speed of light" case will be mentioned again in Sec.8.

The latter case is also the pure gauge transformation of free particles in the model of Ref.[71]. It is obvious that under this approximation, the interacting term (or the SYK Hamiltonian which becomes $\sum_{ij} (g_{ij} b_i^\dagger b_j)^2$) is irrelevant as the interaction $g_{ij}^2$ vanishes in large $N$ limit. However, the resulting Lagrangian can still be smoothly deformed into the above ones, which reads

$$\mathcal{L}(z) = \sum_i b_i^\dagger (-i\partial_z) b_i + \sum_{ij} g_{ij} b_i^\dagger b_j$$

$$= \frac{\pi}{N} \sum_{ij} b_i^\dagger b_j b_j^\dagger b_i - \frac{1}{2\pi(z - z')^2} + \sum_{ij} g_{ij} b_i^\dagger b_j$$

$$= \frac{\pi}{N} \sum_{ij} b_i^\dagger b_i b_j^\dagger b_j + \sum_{ij} (g_{ij} b_i^\dagger b_j)^2. \quad (35)$$

Note that due to the supersymmetry requirements, the first term in the right-hand side of above equation can not be simply written as

$$\frac{\pi}{N} \sum_{ij} b_i^\dagger b_i b_j^\dagger b_j = \frac{\pi}{N} (\sum_i b_i^\dagger b_i)(\sum_{ij} b_j^\dagger b_j). \quad (36)$$

In fact, due to the special three-degree-of-freedom configuration of this model we have the following relation

$$\frac{\pi}{N} \sum_{ij} b_i^\dagger b_i b_j^\dagger b_j = \frac{\pi}{N} (\sum_i b_i^\dagger b_i)(\sum_{k} b_k^\dagger b_k) = \sum_{i} b_i^\dagger (-i\partial_z) b_i,$$ \quad (37)

where again the index $l$ is be uniquely determined once a pair of $(i,j,k)$ is given. That also means the correlation term scaling as

$$\frac{1}{2\pi(z - z')^2} = -(N^2 - N)Q_b. \quad (38)$$

Thus the above Lagrangian becomes

$$\mathcal{L}(z) = \sum_i b_i^\dagger (-i\partial_z) b_i + \sum_{ij} (g_{ij} b_i^\dagger b_j)^2. \quad (39)$$

This derivation use the relations Eq.(63), and will be use in Eq.(94). The only deviation from this and the Eq.(94) is that we omit the term $\frac{1}{2\pi(z - z')^2}$ in the second line of above equation, which is due to the UV cutoff in the condensate system, and we denote it as $\mathcal{L}^{(2)}(z)$ to indicate this point. The deviation here will be corrected by defining a normal ordering for boson indices, although we still sum over all the possible $i$ and $j$ to consider both of the two opposite orderings. In other word, the deviation caused by the normal ordering performed in kinetic term compensate with that induced by the UV cutoff approximation in second term for SYK$_2 \times$SYK$_2$ model, which is necessary to investigating the SYK behavior (in a real material) which is beyond zero-dimension (due to, e.g., the temperature and disorder-type scattering effects). Otherwise, if such an UV cutoff is absent, the resulting SYK model will be instable[70], and the preserved $\frac{1}{2\pi(z - z')^2}$ term is indeed the two point correlation as in a fermi liquid system.
To this stage, we can obtain that, although the interacting term with \( i(j) = i'(j') \) seems to be subleading in large-\( N \) limit and thus be irrelevant, but the replica procedure is still validated by the extended range of boson index in kinetic term to \( N^2 \) instead of \( N \). This will be proved below, and the obtained results are in consistent with those obtained by introducing a Hubbard-Stratonovich antisymmetric field[67], which is time-independent (nondynamic) and in the form of \( Q_b b^\dagger_k \) \((k \neq i)\). But note that in the presence of finite UV cutoff, the supersymmetry method by introducing a nondynamic Hubbard-Stratonovich field fails for an initially interacting system, like the quenched one[71].

Also, similar to the treatment of Ref.[67], by defining a new Hubbard-Stratonovich-like field which can be in the form of \( d^\dagger_i = Q_b b^\dagger_i \), then regular replica procedure leads to the result \( u_d = \Pi_d \), where \( u_d \) and \( \Pi_d \) are the propagator and self-energy of boson field \( d^\dagger_i d_i \). Using the power law ansatz \( u(\tau) = 1/\tau^\Delta \) where \( \Delta \) the corresponding scaling dimension, it is straightforward to obtain that \( \Delta_b + \Delta_d = 1 \), while \( \Delta_{Q_b^2} = \Delta_d - \Delta_b \), where \( \Delta_b \) and \( \Delta_{Q_b^2} \) are the scaling dimensions of boson field \( b^\dagger_i b_i \) and operator \( Q_b^2 \), respectively. Note that the power law ansatz here requires a long time limit, i.e., \( b^\dagger_i(\tau) b_j(\tau') \) with \( |\tau - \tau'| \to \infty \), which does not contradict with the short time condition \( (b^\dagger_i(\tau) b_j(\tau^\dagger)) \) required by the above antisymmetry results.

5.2 zero-dimensional approximation

As \( u \) is defined as the boson propagator regarding to the induced current in \( z \)-direction, its bosonic self-energy \( \Pi_0(2\Omega) \) (in the absence of SYK interactions) induced by laser is related to the SHG response tensor through

\[
\chi(2\Omega) = \frac{\Pi_0(2\Omega)}{2\Omega} = I_{SHG}(\frac{c}{2\pi})^3.
\]

(40)

Note that here the \( \chi \) is not the three-component one as appear above, but the single-component one regarding to the 1D survival current in \( z \) direction. For simplicity, we rewrite the boson propagator as

\[
u(\Omega) = \frac{1}{(2\Omega)^2 - \Pi_0(2\Omega)} = \frac{1}{\tilde{\Omega}^2}.
\]

(41)

We also define

\[
u(\tau, \tau') = \frac{1}{N} \sum_i b_i^\dagger(\tau') b_i(\tau), \]

\[
u_i = b_i^\dagger b_i,
\]

(42)

where \( b \) is the boson operator. In approximated \( 0+1 \) space-time dimension, each \( b \) has a scaling dimension \( 1/2 \), and each derivative operator acting on it has scaling dimension 1. Each unit mode in different layers has the same energy \( \tilde{\Omega} \). The coupling between arbitrarily two such boson operator is described by \( g_{ij}(i, j = 1 \cdots N) \), whose distribution is Gaussian with nonzero mean

\[
p(g_{ij}) = \frac{1}{\sqrt{\pi \sigma}} e^{-\frac{1}{\sigma^2}|g_{ij}|^2}.
\]

(43)

Considering the finite spacial effect, the distribution of \( g_{ij} \) has a little deviation from the standard Gaussian distribution especially for large value of \( g_{ij} \). Thus we write the variance as \( \sigma^2 = g_{ij}^2 = \frac{g^2}{2N} - g_m^2 \) where \( g_m^2 \) denotes the deviation from standard result in 0D system, and both \( g \) and \( g_m \) have the dimension of energy. We will see that, \( g_m \) is important for observability of SYK effects. By this way, we can artificially change the number of \( N \) and produce the fluctuation to the stable conformal point, and then relating the theoretically results to the
experimental observation. In this SYK model, the disorder come from the laser field applied in $z$-direction, while the interactions $g_{ij}$ does not depend on the difference of layer indices $|i - j|$.

Note that the effective boson-boson correlation $\langle u_i u_j \rangle$ cannot be simply regarded as the current-current correlation function $\langle J_i^x J_j^x \rangle$ as appears in the Kubo formula for linear conductivity. For Kubo formula in semiclassical limit $\Omega \to 0$, the remaining conductivity is mainly the Hall conductivity which is induced by the Berry curvature. While the Berry curvature is obviously momentum-dependent (supported by Weyl nodes), it will not be suitable to calculate a randomly distributed variable. In the case of shift current, in which case the observed phenomena should be the optical response instead of the electrical current, the measured SHG intensity may be large due to the divergent enhancement origin from the Drude weight.

5.3 Replica procedure

Each unit mode has the same energy, e.g., for shift current case, all of them have energy $\sim 0$, and for finite induced current, all of them have energy $\sim 2\Omega$. So their interactions could be randomly distributed under a certain disorder (see Sec.9). Using the relation

$$
\int dg_{ij} dg'_{ij} \exp \left[ \frac{-1}{\sigma^2} (g_{ij} + \sigma^2 b_i^\dagger b_i)(g'_{ij} + \sigma^2 b'_i b'_j) \right] \exp [\sigma^2 b_j^\dagger b'_j b_j] = \exp [\sigma^2 b_j^\dagger b_j^\dagger b_j],
$$

we can obtain the replicated partition function for each SYK$_2$ mode

$$
Z = \int \mathcal{D}[u] e^{-S(u)},
$$

$$
S(u) = - \int d\tau d\tau' \sum_{ij} b_i^\dagger(\tau)(\partial_\tau^2 \delta_{\tau,\tau'}) b_i(\tau') + \int d\tau d\tau' \sum_{ij} \left( \frac{g_{ij}^2}{2N} - g_{ij}^m \right) b_j^\dagger(\tau') b_i(\tau) b_i^\dagger(\tau) b_j(\tau).
$$

Notice again that the derivative operator in first term should be in the matrix form. Note that unlike the SYK$_4$ model, the scaling dimensions of kinetic term and the interacting term in the actions of SYK$_2$ or SYK$_2 \times$SYK$_2$ models will no be equal, and the scaling dimension of kinetic term is always twice of that of the interacting term. This can be seen from the above SYK$_2$ action where the scaling dimensions of kinetic term and the interacting term are 2 and 1, respectively, or the SYK$_2 \times$SYK$_2$ one,

$$
S(u) = - \int d\tau d\tau' \sum_i b_i^\dagger(\tau)(\partial_\tau \delta_{\tau,\tau'}) b_i(\tau')
$$

$$
+ \int d\tau d\tau' \sum_{ij} \left( \frac{g_{ij}^2}{2N} - g_{ij}^m \right) b_j^\dagger(\tau') b_i(\tau) b_i^\dagger(\tau') b_j(\tau),
$$

where the scaling dimensions of kinetic term and the interacting term are 4 and 2, respectively. The reason for this mismatch is that the kinetic terms indeed equivalents to the two independent particles. But this will not affect the final results around the conformal point, where the self-energy and the Green’s function still have the scaling dimensions reciprocal to each other.

For convenient, we consider only the SYK$_2$ action. By inserting the integral

$$
1 = \int \mathcal{D}[u] \mathcal{D}[\Pi] \exp \left[ \Pi_{\tau,\tau'} (u(\tau', \tau) - \frac{1}{N} \sum_i b_i^\dagger(\tau) b_i(\tau')) \right],
$$

which
we obtain
\[ Z = \int \mathcal{D}[u] e^{-S(u)}, \]
\[ S(u) = -\int d\tau d\tau' \sum_i b_i^\dagger(\tau) (\partial_\tau \delta_{\tau,\tau'} - \Pi(\tau, \tau')) b_i(\tau') + N \int d\tau d\tau' \left( \frac{g^2}{2} - N g_m \right) u^2(\tau', \tau) \] (48)
\[ - N \int d\tau d\tau' \Pi(\tau, \tau') u(\tau', \tau). \]

For long-wavelength limit, we have
\[ u_i(\tau', \tau) = u_i(\tau, \tau'), \]
\[ \Pi(\tau, \tau') = \Pi(\tau', \tau), \]
where \( |\tau - \tau'| \to \infty \) (i.e., for the case \( \tilde{\Omega}^2 \ll \Pi \)); otherwise for short-time limit it has \( u_i(\tau, \tau') = b_i^\dagger(\tau') b_i(\tau) = 1 + b_i(\tau') b_i^\dagger(\tau) = 1 + u_i(\tau', \tau) \), by solving saddle-point equations \( \partial S/\partial \Pi(\tau', \tau) = 0, \partial S/\partial u(\tau', \tau) = 0 \), the conformal symmetry leads to the results
\[ u(\tilde{\Omega}) = \frac{1}{\tilde{\Omega}^2 - \Pi(\tilde{\Omega})}, \]
\[ \Pi(\tau, \tau') = (\frac{g^2}{2} - N g_m^2) u(\tau', \tau), \]
\[ \Pi(\tilde{\Omega}) = (\frac{g^2}{2} - N g_m^2) u^*(\Omega) = (\frac{g^2}{2} - N g_m) \frac{1}{\tilde{\Omega}^2 - \Pi^*(\tilde{\Omega})}, \] (49)

thus the bosonic self energy (current-current correlation effect) can be solved as \( \Pi(\tilde{\Omega}) = \frac{1}{2} (\tilde{\Omega}^2 \pm \sqrt{-2g^2 + 4g_m^2 N + \tilde{\Omega}^4}) \), and in the \( g \gg \tilde{\Omega} \) limit, it reduce to \( \Pi = \pm \sqrt{-\frac{g^2}{T} + g_m^2 N} \). This is the case of 0 + 1 dimensional SYK model, where we set the layer number \( N \) properly so that the laser inside the material creates boson modes with the same energies and the interactions between them is independent of the spacial distance in \( z \)-direction.

Thus the contribution of (0 + 1)-D SYK interaction to the current is
\[ J_{SYK} = \frac{\Pi(\tilde{\Omega})}{\tilde{\Omega}^2} E^*. \] (50)

Here the effective electric field reads \( E^* = -iq^* \phi^* \), where \( -i\phi^* \sim (\Delta N_F)^{-1} \) is related to the inverse gradient (fractional change) of particle number in real space thus should be taken as 1. \( q^* \) is the small variance of momentum which should be taken as \( q^* = \Omega^2/v_b \) where \( v_b \) is the bosonic velocity in the absence of SYK effects. Thus we have
\[ J_{SYK} = \frac{\Pi(\tilde{\Omega})}{v_b}. \] (51)

In experiment, \( v_b \) can be evaluated by \( \frac{\Delta n}{L} \) where \( \Delta n \) is the difference of local density-of-states (LDOS) between top layer and bottom layer, and \( L \) is the sample thickness. Then it is easy to check that the SYK effect-induced current is purely imaginary if \( g_m = 0 \), and thus will not affect the detected SHG signal.

6 Effect of finite \( z \)-distance on SYK results

Next we consider the effects of finite thickness of layered WTe\(_2\). We apply the (1 + 1) dimensional SYK model proposed in Ref.\,[23\]., In this case, the single bosonic excitation (electron-hole
pair) should not be regarded as zero-dimensional operator anymore, thus we rewrite it as

$$u_i = \lim_{\delta z \to 0} b_i^\dagger(z)b_i(z + \delta z),$$

$$u_j = \lim_{\delta z' \to 0} b_j^\dagger(z')b_j(z' + \delta z').$$

(52)

For each boson excitation, since the process of excitation is very soon, we consider the $z$ coordinates of bosonic creation and annihilation operators are very close, i.e., the ultraviolet (UV) cutoff in the $z$-direction.

The Lagrangian density of single SYK\(_2\) mode reads

$$\mathcal{L}(\tau, z) = \sum_i b_i^\dagger(\partial_\tau - i\partial_z)b_i + \sum_{ij} g_{ij} b_i^\dagger b_j.$$

(53)

The first term of this Lagrangian contains both the $z$-dependence and $\tau$-depende of boson mode $u$, but we can see that its interacting term has only half of the scaling dimension of that of first term. As stated in above section, this is to validate the replica procedure for the SYK\(_2\) model. The $z$-dependent part of Lagrangian, which is what we focusing on, reads

$$\mathcal{L}(z) = \sum_i b_i^\dagger(-i\partial_z)b_i + \sum_{ij} g_{ij} b_i^\dagger b_j,$$

(54)

and this expression will used below in studying the SYK\(_2\)-SYK\(_2\) correlation effects. So the above Lagrangian does not describe the kinetuc term and interacting potential term of boson modes $u$, but the kinetic term and the correlation effect of a scalar bosonic degree of freedom. But for the SYK\(_2\)×SYK\(_2\) model, and we rewrite the kinetic term in normal ordering, $\sum_i b_i^\dagger(-i\partial_z)b_i = \sum_i b_i^\dagger b_i - \sum_i b_i^\dagger b_i$, then the scaling dimension of kinetic term is the same with the interacting term, $\sum_{ij} g_{ij} b_i^\dagger b_j$, which is 2. This is different to the case of SYK\(_4\), and can also be understood in another way: In normal ordering, if we view the $g_{ij}$ as a constant energy of the boson mode $\langle b_i^\dagger b_j \rangle$, the Lagrangian becomes $\mathcal{L}'(z) = i\sum_{ij} g_{ij} b_i^\dagger b_j + \frac{i}{N^2} \sum_{ijkl} b_i^\dagger b_j^\dagger b_k b_l$, i.e., $g_{ij} = \frac{i}{N} \sum_{kl} b_k^\dagger b_l$, and the interaction between $\langle b_i^\dagger b_j \rangle$ and itself is $\frac{i}{N}$. We can see that now the single boson term and the interacting term have the same scaling dimension 2, and the boson velocity is $\partial \mathcal{L}'(z)/\partial (\sum_{ij} b_i^\dagger b_j) = (\pi + 1) g_{ij}$.

By solving the Euler-Lagrange equation of motion of Eq.(53)

$$(\partial_\tau - i\partial_z)(b_i^\dagger + b_i) + \sum_j g_{ij}^2 (b_i^\dagger - b_i)b_j^\dagger b_j = 0,$$

(55)

which turns to be

$$b_i^\dagger(\partial_\tau - i\partial_z)b_i + \sum_{ij} g_{ij} b_i^\dagger b_j = 0,$$

(56)

by letting the $b_i^\dagger$ operator acting in both sides, and using the result of Eq.(55), where we discarding the ill-defined Cooper channel terms. The detail of the derivation of above equation of motion is presented in Appendix.

The effective SYK interaction can only happen between two particles with a $z$-distance in UV cutoff. Firstly, we make the following expansion (for $z \neq z'$),

$$b_i^\dagger(z)b_i(z') = \frac{i}{2\pi(z-z'+i\eta)} + b_i^\dagger(z)b_i(z) + O((z-z')^2),$$

(57)

and since $[b_i^\dagger(z), b_i(z')] = 0$, the above expression is completely equivalents to

$$b_i(z')b_i^\dagger(z) = -\frac{i}{2\pi(z-z'+i\eta)} + b_i^\dagger(z)b_i(z) + O((z-z')^2).$$

(58)
As will be seen from the following calculations, the finite correlation between two SYK\(_2\) modes in different \(z\), simply comes from the UV cutoff in space, i.e., we ignore the part of the order of \((z-z')^2\) or that of higher orders. This is wrong in perspective of mathematics, but it will be a good approximation for a physical system in reality, since the SYK interaction is not strictly a delta-function in space coordinate, but a broadened one.

The correlations can be expressed through the following commutators (after necessary basis transformations)

\[
[\hat{b}_i^\dagger(z)\hat{b}_i(z), \hat{b}_j^\dagger(z')\hat{b}_j(z')] = \delta_{ij}\delta(z-z')\hat{b}_i^\dagger(z')\hat{b}_i(z') - \delta_{ij}\hat{b}_i^\dagger(z')\delta(z-z')\hat{b}_i(z') = \frac{-i}{2\pi}\delta_{ij}\partial_z\delta(z-z'),
\]

\[
[\hat{b}_i^\dagger(z), \hat{b}_j(z')] = \delta_{ij}\hat{b}_i^\dagger(z), \delta(z-z') = \delta(z-z') + O((z-z')^2)
\]

\[
\begin{align*}
&= \frac{-i}{2\pi(z-z' - i\eta)} + \frac{-i}{2\pi z - z' - i\eta} + O((z-z')^2), \\
&[\hat{b}_i^\dagger(z)\hat{b}_i(z), \hat{b}_j(z')] = -\delta_{ij}\delta(z-z')\hat{b}_i(z'), \\
&[\hat{b}_i^\dagger(z)\hat{b}_i(z), \hat{b}_j^\dagger(z')] = \delta_{ij}\delta(z-z')\hat{b}_i^\dagger(z').
\end{align*}
\]

In the vanishing limit of \(\delta z\) in each boson \(u_i\), the full correlation between two SYK modes with different \(z\) can be calculated through the replica method with two replicas, otherwise it requires four replicas.

### 6.1 Commutator between two interacting term

The commutator between two interacting term is \([\hat{b}_i^\dagger(z)\hat{b}_j(z), \hat{b}_j^\dagger(z')\hat{b}_i(z')]\), i.e., the correlation between \(\hat{b}_i^\dagger(z)\hat{b}_j(z)\) and its replica (complex conjugation) at \(z'\):

\[
[\hat{b}_i^\dagger(z)\hat{b}_j(z), \hat{b}_j^\dagger(z')\hat{b}_i(z')] = \hat{b}_i^\dagger(z)[\hat{b}_j(z), \hat{b}_j^\dagger(z')][\hat{b}_i(z'), \hat{b}_i(z)] + \hat{b}_j^\dagger(z')[\hat{b}_i^\dagger(z), \hat{b}_i(z')][\hat{b}_j(z), \hat{b}_j(z)]
\]

\[
= 2\hat{b}_i^\dagger(z)[\hat{b}_j(z), \hat{b}_j^\dagger(z')][\hat{b}_i(z'), \hat{b}_i(z)]
\]

\[
= 2\hat{b}_i^\dagger(z)[\hat{b}_j(z), \hat{b}_j^\dagger(z')]\delta(z-z')
\]

\[
= 2\left(\frac{i}{2\pi(z-z' + i\eta)} + \frac{i}{2\pi(z'-z + i\eta)}\right)\hat{b}_i^\dagger(z)\hat{b}_i(z').
\]

Using the relation in Eq. (57), the above can be written as

\[
[\hat{b}_i^\dagger(z)\hat{b}_j(z), \hat{b}_j^\dagger(z')\hat{b}_i(z')] = 2\left(\frac{-i}{2\pi(z-z' + i\eta)}\right)\left(\frac{-i}{2\pi(z'-z + i\eta)}\right)\hat{b}_i^\dagger(z)\hat{b}_i(z)
\]

\[
+ \frac{-1}{2\pi(z'-z - i\eta)}\[\frac{2\pi(z'-z + i\eta)}{2\pi(z'-z + i\eta)}\] + \hat{b}_i^\dagger(z)\hat{b}_i(z)
\]

\[
= 2\left[\frac{1}{4\pi^2(z-z')^2}z' + \frac{1}{4\pi^2(z'-z)^2} - \frac{1}{4\pi^2(z-z')^2}z' - \frac{1}{4\pi^2(z-z)^2}z' - z
\]

\[
+ \hat{b}_i^\dagger(z)\hat{b}_i(z)\delta(z-z')\]

\[
= 2\left[\frac{-i}{2\pi}\hat{b}_i^\dagger(z)\partial_z\hat{b}_i(z) + \frac{-i}{2\pi}\hat{b}_j^\dagger(z')\partial_z\hat{b}_j(z')
\]

\[
- \hat{b}_i^\dagger(z)\hat{b}_i(z')\hat{b}_j(z)\delta(z-z') - \hat{b}_j^\dagger(z')\hat{b}_j(z)\hat{b}_i(z')
\]

\[
+ \hat{b}_i^\dagger(z)\hat{b}_i(z)\delta(z-z')\]

\[
= \frac{-i}{\pi}\hat{b}_i^\dagger(z)\partial_z\hat{b}_i(z) - \frac{-i}{\pi} :\hat{b}_i^\dagger(z)\partial_z\hat{b}_i(z) : + \frac{-i}{\pi}\hat{b}_j^\dagger(z')\partial_z\hat{b}_j(z') - \frac{-i}{\pi} :\hat{b}_j^\dagger(z')\partial_z\hat{b}_j(z') :.
\]
Here we use the expansion

\[
\frac{(z - z')^2}{(z - z' + i\eta)^2} = 1 - \frac{2i\eta}{z - z'} + O((z - z')^2),
\]

(62)

and omit the \(O((z - z')^2)\) part. Another mathematically correct relation is also used here

\[
\frac{-1}{4\pi^2(z - z')^2} = -\frac{i}{2\pi} b_i^\dagger(z) \partial_z b_i(z) - \frac{-1}{2} \frac{b_i^\dagger(z) b_j(z') b_j(z)}{i} \quad \frac{-1}{2\pi} b_j^\dagger(z') \partial_z b_j(z') - \frac{-1}{2} \frac{b_j^\dagger(z') b_j(z) b_i^\dagger(z)}{i} \quad \frac{-1}{2\pi} b_j^\dagger(z') \partial_z b_j(z') - \frac{-1}{2} \frac{b_j^\dagger(z') \partial_z b_j(z')}{i}
\]

(63)

where \(\cdots\) denotes the normal order.

In the last step of the above commutator, by taking the normal order (omit the vacuum terms), we obtain

\[
[b_i^\dagger(z) b_j(z), b_j^\dagger(z') b_i(z')] = \frac{i}{\pi} : b_i^\dagger(z) \partial_z b_i(z) : + \frac{i}{\pi} : b_j^\dagger(z') \partial_z b_j(z') : .
\]

(64)

The notion of normal order will be omitted in the following calculations. Note that although the normal ordering is sometimes used to replace the kinetic term in this paper, the summation is over all the possible values of \(i\) and \(j\) when we are considering the collective effect of both two orderings \((i > j\) and \(i < j\)). Specific ordering will be needed only when we are considering the relative effects between \(i\) and \(j\), e.g., the individual boson mode velocities.

### 6.2 commutator between kinetic term and the interacting term

The commutator between kinetic term and the interacting term can be obtained base on the above results

\[
[b_i^\dagger(z)(-i\partial_z) b_i(z), b_j^\dagger(z') b_j(z')] = \left[\frac{-1}{2\pi(z - z')^2} - \pi b_i^\dagger(z) b_i(z') b_j^\dagger(z') b_j(z), b_i^\dagger(z') b_j(z')\right]
\]

\[
= -\pi [b_i^\dagger(z) b_i(z') b_j^\dagger(z') b_j(z), b_i^\dagger(z') b_j(z')]
\]

\[
= -\pi ([b_i^\dagger(z) b_i(z'), b_i^\dagger(z')] b_j(z) b_j(z') b_j^\dagger(z') b_j(z) b_i^\dagger(z') b_i(z') [b_j^\dagger(z') b_j(z), b_j(z')]).
\]

(65)
Next we replace the \( b_i(z') \) and \( b_j'(z') \) within the brackets by \( b_i(z) \) and \( b_j'(z) \), respectively. We will replace it back (for their conjugate operators) in the final step. Then we arrive at

\[
[b_i'(z)(-i\partial_z)b_i(z), b_j'(z')b_j(z')]
\]

\[
= \pi \{ b_i'(z)[b_i(z), b_j'(z')b_j(z)] \}
\]

\[
+ b_i'(z')b_i'(z)b_i(z)\}
\]

\[
= \pi \{ b_i'(z)[b_i(z), b_j'(z')b_j(z)] \}
\]

\[
+ b_i'(z')b_i'(z)b_i(z)\}
\]

\[
= \pi \{ b_i'(z)[b_i(z), b_j'(z')b_j(z)] \}
\]

\[
+ b_i'(z')b_i'(z)b_i(z)\}
\]

\[
(66)
\]

Now replacing the \( b_i'(z) \) and \( b_j'(z) \) within the brackets back to \( b_i'(z') \) and \( b_j'(z') \), we obtain

\[
[b_i'(z)(-i\partial_z)b_i(z), b_j'(z')b_j(z')]
\]

\[
= \pi \left[ \frac{1}{2} b_i'(z')b_j(z) + \frac{1}{2} b_i'(z)b_j(z) \right].
\]
6.3 commutator between two kinetic terms

Using the Eq. (63), the commutator between two kinetic terms reads

$$[b_i^1(z)(-i\partial z)b_i(z), b_j^1(z')(-i\partial z')b_i(z')]$$

$$= \frac{-1}{2\pi(z-z')^2} + \pi b_i^1(z)b_j^1(z')b_j^1(z)b_j(z), \frac{-1}{2\pi(z-z')^2} + \pi b_i^1(z')b_i(z)b_j^1(z)b_j(z')]$$

$$= \pi^2 \{b_i^1(z)b_i(z')b_j^1(z'), b_i^1(z')b_i(z)b_j^1(z)b_j(z') + b_i^1(z')b_i(z)b_i^1(z')b_j^1(z')b_j^1(z)\}$$

$$= \pi^2 \{b_i^1(z)b_i(z')b_j^1(z')b_j^1(z)\}$$

$$= \pi^2 \{b_i^1(z)b_i(z')b_j^1(z')b_j^1(z)\}$$

$$+ \frac{i}{2\pi(z'-z+i\eta) - b_i^1(z')b_i(z')b_j^1(z)\}}\}$$

$$+ \frac{i}{2\pi(z'-z+i\eta) - b_i^1(z')b_i(z')b_j^1(z)\}}\}$$

$$+ \frac{i}{2\pi(z'-z+i\eta) - b_i^1(z')b_i(z')b_j^1(z)\}}\}$$

$$+ \frac{i}{2\pi(z'-z+i\eta) - b_i^1(z')b_i(z')b_j^1(z)\}}\}$$

$$= \pi^2 \{b_i^1(z)b_i(z')b_j^1(z')b_j^1(z)\}$$

$$= \pi^2 \{b_i^1(z)b_i(z')b_j^1(z')b_j^1(z)\}$$

In the last step of above equation, for kinetic terms (with scaling dimension 2), only the term of the order of $O((z-z')^2)$ preserved, while other terms which are of the order of $O((z-z')^3)$ cancel each other out. Thus we obtain

$$[b_i^1(z)(-i\partial z)b_i(z), b_j^1(z')(-i\partial z')b_i(z')]$$

$$= -2i\pi b_i^1(z)\partial z\partial b_i(z)]$$

6.4 energy current

Defining the energy current as $\rho_z = \int dz'[L(z), L(z')]$, it can be written in terms of the above obtained commutators

$$\rho_z = 2\pi \sum_i b_i^1(-i\partial z)b_i(z) + 2\pi \sum_{ij} b_i^1 b_j + (g_{ij} - g_{mj})^2 \frac{i}{\pi} \sum_{ij} (\partial_i b_j - \partial_j b_i).$$
Using Eq. (36) the above can be reduced to

\[
\rho_z = 2\pi \sum_i^N b_i^\dagger (-\partial_z \phi_i(z)) + \left( \frac{g_2^2}{2N} - g_m^2 \right) \frac{N}{\pi} \sum_i^N (b_i^\dagger (-i \partial_z \phi_i)
\]

\[
= 2\pi \sum_i^N b_i^\dagger (-\partial_z - \frac{g_2^2}{2N} - g_m^2) \frac{N}{2\pi} \partial_z b_i(z).
\]

(71)

6.5 Energy current in boson representation

By treating \( b_i \) as hard boson operators, we can use the boson field representation through the Jordan Wigner transformation,

\[
b_i^\dagger = c_i^\dagger e^{i\phi},
\]

\[
b_i = e^{-i\phi} c_i,
\]

\[
\phi = 2\pi \sum_{k=0}^{z-1} c_i^\dagger c_k.
\]

(72)

Thus we have

\[
\partial_z \phi_i = \frac{\partial 2\pi \sum_{k=0}^{z-1} c_i^\dagger c_k}{\partial z}|_{z=i} = 2\pi b_i^\dagger b_i = 2\pi c_i^\dagger c_i.
\]

(73)

Then the energy current can be rewritten as

\[
\rho_z = i \partial_z \phi_i \partial_z \phi_i - \frac{N}{2\pi} \left( \frac{g_2^2}{2N} - g_m^2 \right) (\partial_z \phi_i)^2 + (i \leftrightarrow j).
\]

(74)

By defining \( \phi_\pm := \phi_i \pm \phi_j = 2\pi \sum_{k=0}^{z-1} c_i^\dagger c_k \pm 2\pi \sum_{k=0}^{z-1} c_j^\dagger c_k \), we can change the energy current into the pseudospin basis \( \phi_{i/j} = \phi_{i\pm} / 2 \phi_\pm \),

\[
\rho_z = \sum_{\sigma = \pm} \left[ i \partial_z \phi_{i\sigma} \partial_z \phi_{i\sigma} \right] - \left( \frac{g_2^2}{2N} - g_m^2 \right) \frac{N}{2\pi^2} (\partial_z \phi_{i\sigma})^2.
\]

(75)

In this expression, the relation between operators \( \partial_z \) and \( \partial_\phi \) can be understood through Eq. (36). Firstly we rewrite the Eq. (36) in terms of SYK_2 x SYK_2 form,

\[
\sum_i^{N^2} b_i^\dagger (-i \partial_z) b_i + \sum_{ij}^{N^2} g_{ij}^2 b_i^\dagger b_i b_j^\dagger b_j = \sum_i^{N^2} b_i^\dagger (-\partial_\tau) b_i.
\]

(76)

This can be represented in boson representation as

\[
\left[ \frac{\partial_\tau \phi_i}{2\pi} (-1) \partial_\tau \phi_i \right]^2 + g_{ij}^2 \left[ \frac{\partial_\tau \phi_i}{2\pi} \frac{\partial_\tau \phi_j}{2\pi} \right] = \left[ \frac{\partial_\tau \phi_i}{2\pi} (i) \partial_\tau \phi_i \right]^2.
\]

(77)

Then by defining a bosonic number ratio

\[
n_{ij} = \frac{b_i^\dagger b_i}{b_j^\dagger b_j} = \frac{\partial_\tau \phi_i}{2\pi} / \frac{\partial_\tau \phi_j}{2\pi},
\]

(78)

we can easily obtain

\[
[\partial_\tau \phi_i]^2 + g_{ij}^2 n_{ij}^{-1} = -[\partial_\tau \phi_i]^2,
\]

(79)

and similarly,

\[
[\partial_\tau \phi_j]^2 + g_{ij}^2 n_{ij} = -[\partial_\tau \phi_j]^2.
\]

(80)
Then in terms of pseudospin basis, the energy current in Eq. (74) can be rewritten as

\[
\rho_z = \sum_{\sigma=\pm} \frac{i}{2} \partial_z \phi_\sigma \partial_\sigma \phi_\sigma + \left( \frac{g^2}{2N} - g_m^2 \right) \frac{N}{2\pi^2} \frac{1}{2} (\partial_\sigma \phi_\sigma)^2 + \left( \frac{g^2}{2N} - g_m^2 \right) \frac{N}{2\pi^2} \left[ \frac{\partial_\sigma \phi_+ + \partial_\sigma \phi_-}{\partial_\sigma \phi_+ - \partial_\sigma \phi_-} \right]^2. \tag{81}
\]

This can also be rewritten as

\[
\rho_z = \sum_{\sigma=\pm} \frac{i}{2} \partial_z \phi_\sigma \partial_\sigma \phi_\sigma - \left( \frac{g^2}{2N} - g_m^2 \right) \frac{N}{2\pi^2} \frac{i}{2} \partial_\sigma \phi_\sigma \partial_\sigma \phi_\sigma + \frac{1}{2} \partial_z (\phi_\sigma (-1)^\sigma) \left( \frac{g^2}{2N} - g_m^2 \right)^{3/2} \frac{N}{2\pi^2} \left[ \frac{\partial_\sigma \phi_+ + \partial_\sigma \phi_-}{\partial_\sigma \phi_+ - \partial_\sigma \phi_-} \right]^{-\sigma/2} \tag{82}
\]

\[
+ \frac{1}{2} \partial_z \phi_\sigma \left( \frac{g^2}{2N} - g_m^2 \right)^{3/2} \frac{N}{2\pi^2} \left[ \frac{\partial_\sigma \phi_+ + \partial_\sigma \phi_-}{\partial_\sigma \phi_+ - \partial_\sigma \phi_-} \right]^{\sigma/2}.
\]

Then the dependence of the modes \( \phi_\pm \) on SYK coupling can be clearly seen.

The temperature effects are considered in Ref. [23], where the energy current is consist of the vacuum expectation and a temperature-dependent term \( \sim T^2 \). The vacuum term which is of the order of \( (z - z')^2 \) is being ignored, but experimentally, since the properly selected multilayered sample (with thickness small enough compares to the penetration depth of the laser), we can define \( (z - z') \) as the thickness of the multilayered sample, in which case the SYK coupling \( g \) is not stringently a zero-dimensional interaction, i.e., the delta function \( \delta(z - z') \) can be replaced by a smeared one, which has a finite broaded peak. Note that this broadening is only due to the small energy dissipation between the induced bosons in the first layer and the bottom layer, but not the effects of phonon and impurity scatterings, which will affect the accuracy of the experimental result, and thus the temperature (in the second term) should in fact as low as possible. The small change of coupling \( g \) is indeed related to the change of bosonic frequencies of modes \( u_{i=1} \) and \( u_{j=N} \). As the change of coupling \( \delta g \) is small enough for properly selected thickness \( \langle z - z' \rangle \), we can see that the energy current is proportional to the SYK coupling strength \( g^4 \), which means for stronger \( g \), the experimental result will be more accurate (the errors origin from the distance in \( z \)-direction will be smaller).

7 SYK-induced energy current and the particle current

From the above results, we know that, as long as \( n_{ij} \neq 1 \), which is due to the weak nonequilibrium of energy distribution in \( z \)-direction, the energy current difference \( \Delta \rho_z \) between modes \( \phi_+ \) and \( \phi_- \) depends only on the SYK interactions (for a detailed derivation, see Appendix.C):

\[
\Delta \rho_z = \left( \frac{g^2}{2N} - g_m^2 \right)^2 \frac{N}{2\pi^2} (n_{ij}^{-1} - n_{ij}). \tag{83}
\]

But how to relate this energy current to the particle current in reality? This can be done by considering imaginary particle velocity in \( z \)-direction. Since there is no interlayer particle hopping, we will apply the intraband current derived by the linear response of Kubo formula, with a planar effective electric field \( E_\parallel = \lim_{q_\parallel \to 0} (-iq_\parallel \Phi_{q_\parallel}) \) \( (q_\parallel = k_\parallel - k'_\parallel \) which is vanishingly small in long wavelength limit and \( \Phi_{q_\parallel} \) is the field potential)

\[
J_z^{\text{SYK}} = \frac{ie^2}{\omega + i\eta} \frac{\partial \varepsilon_{k_z}}{\partial k_z} \left( -\frac{\partial N_F}{\partial k_\parallel} \right) E_\parallel. \tag{84}
\]

We define the \( z \)-component of energy term in momentum space as \( \varepsilon_{k_z} = \varepsilon_0 n_{k_z} \). \( n_{k_z} \) is the particle number in momentum space, which can be obtained through the Fourier transform of
\[
\partial_z \phi
\]

\[n_{kz} = \int e^{-ikz}(b_i^\dagger(z)b_i(z))dz = \int e^{-ikz}\frac{\partial_z \phi}{2\pi}dz. \tag{85}\]

where we assuming the following identity

\[e^{ikz} = \frac{\phi}{2\pi} = \sum_{k=0}^{z-1} c_k^\dagger c_k = \sum_{k=0}^{z-1} b_k^\dagger b_k = \sum_{k=0}^{\infty} \frac{(ikz)^k}{k!}. \tag{86}\]

Then the above particle number can be written as

\[n_{kz} = \int \partial_z \ln \left[ \frac{\partial_z \phi}{2\pi} \right] dz, \tag{87}\]

by inserting the the relation

\[\frac{\phi}{2\pi} = \sum_{k=0}^{z-1} b_k^\dagger b_k = \sum_{k=0}^{z-1} n_{kz}^2 = \frac{n_{ij}^2 - 1}{n_{ij} - 1} = e^{ikz}, \tag{88}\]

we have

\[n_{kz} = \ln(n_{ij}^L - 1) - \ln(n_{ij} - 1), \tag{89}\]

where we integrate \(z\) from 1 to \(L\) (the thickness of multilayer sample). Here we define the ratio of particle numbers in two adjacent indices \((i - j = \pm 1)\)

\[n_{ij} = \frac{b_i^\dagger b_i}{b_j^\dagger b_j}, \tag{90}\]

which is a constant and can only takes two possible values depending on the ordering of summation. Once modes \(\phi_i\) and \(\phi_j\) are given in the system, both the \(n_{ij}\) and \(n_{ij}^{-1}\) have only one value. That is to say, \(n_{ij}\) does not depends on the detail values of \(i\) and \(j\), and this need to be remembered to prevent confusing when reading is article. In the mean time, we do not restrict the relation between \(i\) and \(j\), i.e., we never take the normal ordering, thus it is possible to set \(i = j\) \((z = z')\). This would make the analysis more convient. Unlike \(\phi_i\) or \(\phi_j\), the ratio \(n_{ij}\) is a dimensionless quantity, so it can appears in the expression of momentum space.

Then the velocity in momentum space can be obtained as

\[v_z = \varepsilon_0 \frac{\partial N}{\partial k_z} \left[ \frac{n_{ij}^L - 1}{n_{ij} - 1} \right] = i\varepsilon_0 L, \tag{91}\]

which can only be imaginary due to the absence of interlayer hopping. The frequency term (at resonance position) in the denominator of Eq.(93) is given by the inverse velocity-difference between \(\phi_i\) and \(\phi_j\)

\[\omega = \Delta v_{ij} = \frac{\Delta \rho_z}{n_i - n_j} = \frac{g^2}{2N} - \frac{g_m^2}{2N} \frac{N}{n_i + n_j}. \tag{92}\]

It is also easy to see that the term \(\frac{\partial N}{\partial k_z}\) cancels out with the electric field term. So the final result of particle current reads (using Sokhotski-Plemelj theorem)

\[J_{z}^{\text{SYK}} = e^2 \varepsilon_0 L(-i\pi \delta(\eta) - \frac{g^2}{2N} - \frac{g_m^2}{2N} \frac{N}{n_i + n_j})^{-1}. \tag{93}\]

It is easy to know that, when layer number \(N\) increase, the thickness \(L\) increases linearly, and the factor \(\frac{N}{2\pi^2 n_i n_j}\) will also increases. Experimentally, we shoule use the STM to measure
the local density-of-state (LDOS) in $k_z$ direction, and the values of LDOS in the top layer and bottom layer correspond to $n_i$ and $n_j$ respectively. Since $J_z^{SYK} < 0$, it will reduces the observed current induced by $(0 + 1)$-D SYK system (Eq. (51)).

For SYK$_2$×SYK$_2$ model, the boson velocities of $\phi_i$ and $\phi_j$ modes can be obtained by the derivation of corresponding Lagrangians. As we stated in Sec.6, for SYK$_2$×SYK$_2$ model, the Lagrangian is

$$L^{(2)}(z) = \sum_{i}^{N^2} b^\dagger_i (-i \partial_z) b_i + \sum_{ij} g_{ij}^2 b^\dagger_i b_j b^\dagger_j b_i$$

$$= \pi \sum_{ij} b^\dagger_i b_j b^\dagger_j b_i,$$

(94)

thus the velocities are

$$\sum_i v_i^{(2)} = - \sum_{ij} \frac{\partial L^{(2)}_{ij}(z)}{\partial (b^\dagger_i b_j)^2}$$

$$= - \sum_{ij} \left( \pi n_{ij} - |ij| \right) + \pi n_i \frac{n_{ij} - |ij| n_R - \varepsilon_0^2}{\Omega^2 N_{ij}}$$

$$+ g_{ij}^2 n_{ij} - |ij| + \frac{n_{ij}}{\Omega^2 N_{ij}}$$

$$+ g_{ij}^2 n_{ij} - |ij| + \frac{n_{ij}}{\Omega^2 N_{ij}}$$

$$= - \frac{n_{ij}^N - 1}{n_{ij} - 1} \left( \pi + \frac{g_i^2}{2N} - \frac{g_j^2}{2N} \right) - \frac{n_{ij}^N - 1}{(n_{ij} - 1) N_{ij}} \left( \pi + \frac{g_i^2}{2N} - \frac{g_j^2}{2N} \right),$$

(95)

where we use the estimation of $n_i$ in next section (Eq. (102)). One may suspect the correctness of these two velocities as the normal ordering is being taken in each velocity and momentum operator ($n_i^2$ or $n_j^2$) but not being taken in the Lagrange as well as the summation over boson indices. In fact, the Lagrangian within the energy current (Lagrangian correlation) is comprehensive, i.e., it contains both orderings (of boson $b_i b_j$), but we do not constrain the ordering of $i,j$ to the relation between $z, z'$ as well as the energies, thus the $u_i$ and $u_j$ only play the roles of two correlated subsystems, and the only invariant one (with constant speed of light) is the comprehensive one ($L^{(2)}_{ij}(z)$ or $L_{ij}(z)$), that is why the individual velocities for $u_i$ an $u_j$ (relative to each other) can be correctly obtained through above expressions.

When $i = j$, the second term of the velocity summation vanishes, and $\sum_i v_i^{(2)} = \sum_{i,j} v_j^{(2)}$. The comprehensive Lagrangian has an interacting term shared by two correlated SYK$_2$ modes, and this can be proved by the fact that the disorder average over the product of two SYK couplings $g_{ij} g_{ij'}$ is nonzero only when $i = i', j = j'$ (otherwise the two SYK$_2$ models commutate with each other).
Since \( n_{ij} = n_R^{(i-j)/2} \), the modes \( u_i^2 \) and \( u_j^2 \) always have the positive velocities, which is similar to the electron band and hole band around a type-II Weyl node. Note that this system satisfies the time-reversal symmetry, as when the particle number ordering \( n \) becomes negative (e.g., when the multilayer sample is inverted), the velocities of modes \( u_i^2 \) and \( u_j^2 \) also becomes negative. Note that the time-reversal symmetry here for the bosons near Weyl node relies on the specially constructed basis, where the particle number operator is being viewed as effective momentum such that different boson modes can be in a flat band. Otherwise, it is indeed the spacial parity (inversion) symmetry in \( z \)-direction and thus will not breaks the chiral (unpaired) topological order here. The only way to deform this robust topological ordered state to the trivial one is by rebuilding the true time-reversal symmetry (or other \( \mathbb{Z}_2 \) discrete symmetries) or by enclosing the bulk gap, i.e., making the collective energy vanish.

Note that here the energy term \( L^{(2)}_{ij} \) is not the energies of the individual bosons \( u_i^2 \) and \( u_j^2 \), but the collective effect of them, which assuming boson in each layer has the same kinetic energy \( \varepsilon_k = \sum b_i^\dagger b_i \), that is why velocities of the two modes will inversely proportional to the corresponding momenta (equivalents to particle number). Different to this constant velocity, the layer-dependent velocities can be regarded as the response velocities. In this way, the boson modes in different layers can be regarded as degenerate localized states (or degenerate ground states during experimental manipulation by using a UV angular momentum cutoff as we explain in next section).

Thus the energy current difference can be written as

\[
\Delta \rho_z = \sum_{ij} \left( \frac{\partial L_{ij}(z)}{\partial n_i} L(z) - \frac{\partial L_{ij}(z)}{\partial n_j} L(z) \right)
= \sum_{ij} (v_i L_{ij}(z) - v_j L_{ij}(z))
= \sum_{ij} (v_i - v_j) L_{ij}(z),
\]

where \( v_i \) and \( L_{ij}(z) \) are the velocity and collective energy term of single SYK\(_2\) model. That is why the energy difference between \( u_i \) and \( u_j \) (not the collective one) Eq.(92) can be expressed as

\[
\omega = (v_i - v_j) k_{max},
\]

where the maximum momentum corresponds to minimum spacial distance here (UV cutoff in condensed system), i.e., the interlayer spacing, which reads

\[
k_{max} = \frac{L_{ij}(z)}{n_i - n_j}.
\]

Here \( n_j \) is the number operator with layer index next to \( n_i \), thus \( k_{max} \) corresponds to the maximum speed in this system (speed of light). Note that here we cannot take \( i = j \) due to the definition of constant ratio \( n_{ij} \). In this system, \( k_{max} \) is invariant, thus it will not breaks the Lorentz symmetry.

Here we also obtain the more accurate estimation of bosonic velocity in Eq.(51), i.e., the 0D SYK-induced current is \( J_{SYK} = \Pi(\Omega)/N \sum_j v_j^{(2)} \), when \( n_{ij} \gtrsim 1 \).

8 Estimation of \( g_m \): the deviation from standard Gaussian distribution

To observe the SYK effect, i.e., the nonlinear enhancement of SHG intensity in multilayered WTe\(_2\), firstly it require large boson-boson interaction \( g_{ij} \), and secondly the layer number need to
be selected properly such that the LDOS in the top layer is close to that in bottom layer, which is to make sure the energies of boson modes in top layer and bottom layer are almost equal. From the zero-dimensional SYK and one-dimensional SYK results obtained above, we know that if the deviation from standard Gaussian distribution $g_m$ is zero, then the zero-dimensional SYK interaction’s contribution to current is purely imaginary, while the one-dimensional SYK’s contribution is also independent of layer number $N$. That implies the finite deviation from standard Gaussian distribution of random interactions are necessary for estimation of SYK effect in multilayer system.

From the expressions of $J_{SYK}^2$ and $J_{SYK}^2$, we know the particle numbers in top layer and bottom layer are needed. This is to evaluate the energy difference of excited bosons (the SHG resonance; which is quasiparticle-type as we explained above) by laser in the top layer and bottom layer. However, using the STM tip to detect the LDOS in direction parallel to the sample surface is difficult in reality, thus here we propose a new method.

Firstly we consider the refractive index $n_R$ as nearly constant. The intensity of incoming laser is $I_0^2$ (we do not consider the refraction of light in top layer), when it crosses $(N-1)$ layers and reach the bottom layer, the intensity becomes $I_0^2 n_R^{2(N-1)}$, we assume the corresponding bosonic energy as $\Omega'$, then the induced SHG signal has the intensity $I_0^2 n_R^{2(N-1)(2\pi)^3} (2\Omega'; \Omega'; \Omega')$. When the energy of boson in bottom layer transfer to the surface, it reduces again, and become $I_0^2 n_R^{2(N-1)(2\pi)^3} \chi^2 (2\Omega'; \Omega'; \Omega') n_R^{-1}$. Then the contribution (other that the SYK part) to the detected SHG signal will be

$$I_{2\Omega'}^{tot} = \sum_{k=0}^{N-1} I_{2\Omega'}^{k+1} n_R^k,$$

$$I_{2\Omega'}^{k+1} = I_0^2 n_R^{2k} (2\pi)^3/\epsilon^2 \chi^2 (2\Omega'; \Omega', \Omega'),$$

which is nearly linear with $N$. The Boson frequency in bottom layer can be obtained by solving

$$I_0^2 n_R^{2N-2} = \left(\frac{c}{2\pi} |E_{\Omega}|^2 \right)^2 n_R^{2N-2} \approx \left(\frac{c}{2\pi} \left(\frac{\Omega}{2}\right)^2 \right)^2 n_R^{2N-2} = \left(\frac{c}{2\pi} \left(\frac{\Omega'}{2}\right)^2 \right)^2,$$

which turns out to be $\Omega' = \Omega n_R^{(N-1)/2}$. Unlike the centrosymmetric materials, there are not interlayer cancellation effect in multilayer WTe$_2$, and since the extinction coefficient $k(\lambda)$ is vanishingly small in $z$-direction, the optical absorption $a(\lambda) = 4\pi k(\lambda)/\lambda$ should also small, and the refraction of substrate (like SiO$_2$/Si) which related to variance of $n_R$ by $\delta n_R = 4\alpha(\lambda)/(\eta_R^{4/2} - 1)$.[52] should also has a small effect on the experimental results. In fact, the wavelength of laser should be carefully selected to avoid the optical absorption, i.e., it must be make sure the excited electrons will go back to the original level after releasing the SHG signal. For this purpose, we can measuring the energy dispersion (along $k_z$ direction) of the sample before and after the illumination of laser.

According to Sec.5, the detected frequency of boson (with SYK interaction effect) from $(k+1)$th-layer can be obtained through

$$\tilde{\Omega}'^2 = (2\Omega')^2 - \Pi_0 (2\Omega'),$$

$$\Pi_0 (2\Omega') = 2\Omega' \chi(2\Omega').$$

Then the SYK bosonic self-energy $\Pi(\tilde{\Omega}')$ and the corresponding $J_{SYK}$ and $J_{SYK}^2$ can be obtained, where the particle number difference between top layer and bottom layer can be estimated by

$$n_i - n_j = \frac{\Omega (1 - n_R^{(N-1)/2})}{\varepsilon_0},$$

26
where $\varepsilon_0$ is the single particle energy.

The Bosonic spectral function, which coincides with LDOS, can be obtained as

$$A(\omega) = \frac{-1}{\pi} \text{Im} u(\tilde{\Omega}' \rightarrow -i\omega + \eta).$$

(103)

$A(\omega) = \delta(\omega^2 + \text{Im} \Pi(\tilde{\Omega}'))$ if we do not consider the phonon or impurity effects. Thus the square of peak position of spectral function corresponds to the value of imaginary part of SYK bosonic self-energy, which is

$$-\omega^2 = \frac{1}{2} \sqrt{| -2g^2 + 4g_m^2N + \omega^4 |}.$$  

(104)

Once this peak position is known, we can change the layer number $N$ and measure the gradient of $-\omega^2$ with increasing $N$, which is $g_m^2/(-2\omega^2)$, then the $g_m$ can be identified. The LDOS of surface can be obtained by measuring the surface tunneling conductivity $dI/dV$ using the STM (at low-temperature).

9 Existence of linear response as the disorder which induces Random interaction

To make sure the interactions between bosons $u_i$ and $u_j$ are random such that the strength of coupling $g_{ij}$ is independent of the set of $(i, j)$, the existence of disorder is necessary. For this system, besides the usual SHG process as discuss above, it is also possible for the two-photon (TP) shift current to emerge, whose expression is

$$\chi_{\alpha\alpha}^{TP} = \frac{-e^3}{2\hbar\Omega^2} \int d\rho \int d\theta \int d\phi \rho^2 \sin\theta \sum_{ml} N_F(\varepsilon_{ml}) \frac{e^{\alpha\alpha} u_i^z v_j^z}{2\Omega - \varepsilon_{ml}}.$$  

(105)

where $\alpha = x, y$. However, as the resulting current is a DC type, and cannot be detected by the photodetector, this will not be the disorder in this system and it also has a minor effect on our results.

The disorder should be the two photon-assisted linear response (TPLR). The number of bosons induced by TPLR is much smaller than that induced by SHG, but they have almost the same energies (2\Omega) especially in the conformal limit. Thus the current induced by the linear response can be treated as another ostensible second-harmonic beam, which can still be detected, although it has a much smaller weight. The two kinds of second-harmonic beam generation mechanisms are shown in Fig.2, where the outcoming frequencies are all 2\Omega.

The bosons induced by the TPLR is in fact highly possible to exist. Since the photodetector is right above the top layer, any bosons (i.e., the zero dimensional photocurrent response-induced SHG signal with intensity $\sim \Omega^4$) need to go through layers above it to be detected, and the process of TPLR may happen at an arbitrary layer. For an illustration, we show this process in Fig.2(c), where the disorders at an arbitrary position stopping the free movement of boson created in somewhere others, although it will not change the intrinsic properties of each boson (like the energy). We do not need to add another disorder term to the previous SYK Hamiltonian $H_{SYK}$ to incorporating this disorder-induced process, since the effect of bosons induced by these two mechanisms have not difference. But the existence of minor amount of bosons induced by TPLR can efficiently make the SYK couplings between two bosons more random and unpredictable, which, in other word, making the value of coupling $|g_{ij}|$ independent of the selection of $ij$. Besides, unlike the other kinds of disorders like the lattice vacancies or on-site impurities\textsuperscript{51}, the disorder induced by TPLR will not reduces the number of eigenstates in flat band or partially changing the energy of states in flat band. In fact, it is possible for the
emergence of disorders induced by higher-order nonlinear processes, for example, the fourth-order harmonic generation induced by two bosons with frequencies $2\Omega$. But this has a lower probability and we do not consider this here.

### 10 Numerical simulation and exact diagonalization

We treating the refraction-related factor $n_R$ as a constant in above derivations, but in fact there exist the second-order nonlinearity which depends on the light intensity, which means the refraction decrease for lower intensity of light. We will consider this minor effect during the numerical simulations. As shown in Fig.3, the intensity of SHG $I_{2\Omega}$ initially grows linearly with layer number, and the effects of SYK interactions appear latter. Due to the collective effect of $g$ and $g_m$, the SYK-induced intensity will tends to stabilize in a range of $N$. For layer number exceeds a certain range, the 0D SYK effect will vanishes and the SHG $I_{2\Omega}$ will no more affected by it. Note that experimentally the three intensities $I_{2\Omega}$, $I_{SYK}$ and $I_{zSYK}$ will all be detected by the photomultiplier photodetector since they are all of the order of $\Omega^4$, while the higher-order component as well as the reflected fundamental light will not be detected.

For noncentrasymmetry material without the interlayer cancellation for SHG singals, the almost linear results has been obtained by Ref.[43] for layer number up to 30. Similar linear relations also shown in the numerical simulation of Ref.[66] which consider the multilayer SHG. In these works, the SYK effects are not been considered, instead, absorption and partially (or multiple) reflection and transmission, laser-induced oxidation, bulk quadrupolar source from Si substrate, as well as the optical interferences are considered, whose effects may dominate over the SYK effects as long as the SYK coupling is not strong enough.

As we stated in above section, if we prevent $i = j$ artificially by setting a UV cutoff in real space $|k_z| \gtrsim \Delta L^{-1}$ where $\Delta L$ is the interlayer spacing, we can prevent a infinite-velocity singularity, and thus the relativistic effect, as well as the induced strong nonlinearity of reflection of light[54] does not need to be considered. In the other hand, if we apply strong magnetic field, the large magnetic-optical conductivity will enlarging the nonlinearity of reflection of light and possibly leads to chiral anomaly of type-II Weyl semimetal (when the direction is within the cone). Considering this, instead of applying a strong magnetic field to enhancing the Coulomb interaction between charged bosons, we think it will better to using the technic of magnetic Feshbach resonance. Since the interacting boson modes in SYK system are nonlocal in $k_x - k_y$ plane, and are all of $d$-wave type with the similar energies, it will be easy to creating an effective pairing degree of freedom that connecting two bosons with distinct spin or angular momenta. While the singularity induced by high spin angular momentum can be got rid of by applying the UV cutoff $|k_z| \gtrsim (\Delta L)^{-1}$, then the phenomenon that pairing interaction leads to higher spin angular momentum of each boson can be avoided even for large boson number as proved by Ref.[53]. The enlarged (although slightly) momenta and energy in fact origin from the higher-order process which contains the nonperturbative effects. That means some electrons absorpt the energy of light and the chemical potential is changed, which also leads to bosons with higher or lower frequencies. If this happen, higher-order nonlinearity should be taken into account as a correction. Ref.[51] also notice this problem, which suggests a Feshbach resonance-enhanced $p$-wave pseudopotential. While in this paper, the $s$-wave potential between bosons is considered, besides, the deviation from the standard Gaussian distribution as well as the one-dimensional spacial effect are considered, to increasing the accuracy of SYK results.

#### 10.1 Exact diagonalization results base on U(1) symmetry in boson spin channel

As we state in above section, the Feshbach resonance-enhanced coupling consider the spin angular momentum of bosons, which can be treated as a U(1) symmetry in this system. This
is because we are considering the excitation processes that will not induce spin flipping, thus the total spin angular momentum can be conserved and be independent of the boson’s creation and annihilation.

The U(1) charge in spin channel \( Q_\alpha \) in sector \( \alpha \) has the following relation

\[
\sum_\alpha Q_\alpha = \sum_{ij} b_i^\dagger b_j,
\]

(106)

which is a conserved quantity and thus validate the block diagonalization of the Hamiltonian. By defining the SO(N) rotational operator \( O_{ij} \) (\( i \neq j \); which is an orthogonal matrix \( O_{ij}^\dagger O_{ij} = \mathbf{O}_{ji}O_{ij} = 1 \)), we have

\[
B_j^\dagger = b_i^\dagger O_{ij},
\]

\[
B_j = O_{ji}b_j.
\]

(107)

The real skew-symmetry \( g_{ij} \) becomes

\[
g'_{ij} = \mathcal{O}_{kli}^\dagger g_{kl} \mathcal{O}_{ij} = \mathcal{O}_{ik}g_{kl} \mathcal{O}_{ij} = \begin{pmatrix}
0 & g_{12}' & 0 & \cdots & 0 \\
g_{21}' & 0 & g_{34}' & \cdots & 0 \\
0 & g_{34}' & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & g_{N(N-1)}' & 0 & \cdots & 0
\end{pmatrix}.
\]

(108)

The corresponding SYK \(_2\) Lagrange reads (\( \mathcal{O}_{ij}^\dagger b_i^\dagger b_k \mathcal{O}_{ki} = B_i^\dagger B_j \))

\[
\mathcal{L}(z) = \sum_i B_i^\dagger (-i\partial_z) B_i + g_{12}B_1^\dagger B_2 + g_{34}B_3^\dagger B_4 + \cdots + g_{(N-1)N}B_{N-1}^\dagger B_N
\]

\[
= \sum_i B_i^\dagger (-i\partial_z) B_i + \sum_{\alpha} g_{\alpha}B_{2\alpha-1}^\dagger B_{2\alpha},
\]

(109)

where we use the simplified notation for the couplings \( g_{\alpha} := g'_{(2\alpha-1)2\alpha} \). This procedure reduces the \( N^2 \) values of \( g_{ij} \) to the \( N \) values of \( g'_{ij} \). It is easy to see that the two elements in the first block of \( g'_{ij} \) with \( i, j = 1, 2 \) correspond to four elements in \( g_{ij} \) \( (i, j = 1, 2) \).

10.2 Many-body level statistic

The above boson spin charge conservation validate the exact diagonalization for the above-mentioned SYK Hamiltonian, which means the \( N \) interacting bosons \( (b_i^\dagger b_i; i = 1 \cdots N) \) with equal energies can be transformed to another basis with \( N/2 \) interacting bosons \( (B_{2\alpha-1}^\dagger B_{2\alpha}; \alpha = 1 \cdots N/2) \). The corresponding conserved (relative to the Hamiltonian in new basis) spin charge operator is

\[
Q_{s}^{\alpha=k,l} = \sum_{ij} \epsilon_{ijkl}b_i^\dagger b_j,
\]

(110)

where \( \epsilon_{ijkl} \) is the Levi-Civita symbol. In terms of this basis, there are \( N/2 \) states in a flat band and their interactions are still random and be Gaussian. The disorder can be interpreted as that the existence of TPSC SHG makes \( Q_s \) no more be a conserved quantity, and then slightly increasing or decreasing the energies of the states in flat band.
For different $\alpha$, we have $[H', Q^\alpha_s] = 0$, and $[H', g_\alpha] = 0$ which means the couplings $g_\alpha$ are mutually independent for different $\alpha$’s. To perform the exact diagonalization, we rewrite the above Hamiltonian as

$$H' = \sum_{\alpha}^{N/2} g_\alpha B_{2\alpha-1}^\dagger B_{2\alpha}. \quad (111)$$

In this basis, the term $\frac{g_\alpha}{p(g_\alpha)}$ has the same effect with a boson, and it can be expressed as $\frac{g_\alpha}{p(g_\alpha)} = B_{2\alpha}^\dagger B_{2\alpha'}$ with $\alpha'$ must be independent of $\alpha$. This is because, unlike the bosonized representation, the term $\frac{g_\alpha}{p(g_\alpha)}$ is randomly distributed and independent of subscript $\alpha$ (that is why this term could be commuting with $H'$). In other word, although both $\frac{g_\alpha}{p(g_\alpha)}$ and $B_{2\alpha'}^\dagger B_{2\alpha'-1}$ commute with $H'$, the former has an additional degree of freedom compares to the latter one, which is the value of $g_\alpha$ (independent of subscript $\alpha$).

Note that the Gaussian probability distribution has $\int p(g_\alpha) = 1$. The term in $B_{2\alpha-1}^\dagger B_{2\alpha} \frac{g_\alpha}{p(g_\alpha)}$ in above Hamiltonian $H'$ is indeed a conserved quantity relative to $H'$ (or in other word, commute to $H'$), which we define as $R_\alpha$:

$$R_\alpha := B_{2\alpha-1}^\dagger B_{2\alpha} \frac{g_\alpha}{p(g_\alpha)}$$

$$= Q_s^\alpha \frac{g_\alpha}{p(g_\alpha)} + \sum_{\alpha' \neq \alpha}^{N/2-1} \frac{\sqrt{p(g_\alpha) p(g_{\alpha'})} \ g_{\alpha'}}{p(g_{\alpha'}) - p(g_\alpha)} B_{2\alpha'-1}^\dagger B_{2\alpha}$$

$$= \sum_{\alpha' \neq \alpha}^{N/2-1} \frac{\sqrt{p(g_\alpha) p(g_{\alpha'})} \ g_\alpha}{p(g_{\alpha'}) - p(g_\alpha)} B_{2\alpha'-1}^\dagger B_{2\alpha} \frac{g_{\alpha'}}{p(g_{\alpha'})}$$

$$+ \sum_{\alpha' \neq \alpha}^{N/2-1} \frac{\sqrt{p(g_\alpha) p(g_{\alpha'})} \ g_{\alpha'}}{p(g_{\alpha'}) - p(g_\alpha)} B_{2\alpha'-1}^\dagger B_{2\alpha} \frac{g_{\alpha'}}{p(g_{\alpha'})}$$

In this expansion, the first term is in fact a commutator ($\alpha = k, l; \alpha' = k', l'$)

$$Q_s^\alpha \frac{g_\alpha}{p(g_\alpha)} = [B_{2\alpha-1}^\dagger B_{2\alpha}, \frac{g_{\alpha'}}{p(g_{\alpha'})}] \delta_{\alpha,\alpha'}$$

$$= B_{2\alpha-1}^\dagger B_{2\alpha} \frac{g_\alpha}{p(g_\alpha)} - \frac{g_\alpha}{P(g_\alpha)} B_{2\alpha-1}^\dagger B_{2\alpha}$$

$$= B_{2\alpha-1}^\dagger B_{2\alpha} \frac{g_\alpha}{p(g_\alpha)} - (B_{2\alpha}^\dagger B_{2\alpha-1} (\frac{g_\alpha}{p(g_\alpha)} + 1) + 1)$$

$$= (B_{2\alpha-1}^\dagger B_{2\alpha} - B_{2\alpha}^\dagger B_{2\alpha-1}) \frac{g_\alpha}{p(g_\alpha)} - B_{2\alpha}^\dagger B_{2\alpha-1} - 1 \quad (114)$$

$$= \frac{g_\alpha}{p(g_\alpha)} - B_{2\alpha}^\dagger B_{2\alpha-1} - 1.$$
The third line uses the relation 
\[ B_{2α}^1 B_{2α-1}^1 \ell = \frac{gα}{p(gα)} + 1 = \frac{gα}{p(gα)} B_{2α-1}^1 B_{2α}^1, \]
which is valid as long as \( N/2 \gg 1 \).

The commutator \( Q_{s}^α \frac{gα}{p(gα)} \) is essential for the following derivations of eigenvalue and eigenstate. The above commutator also has the following properties,

\[
\begin{align*}
[Q_s^α \frac{gα}{p(gα)}, B_{2α-1}^1 B_{2α}^1] &= \frac{gα}{p(gα)} B_{2α}^1 B_{2α-1}^1 B_{2α} - [B_{2α-1}^1 B_{2α}^1, B_{2α-1}^1 B_{2α}^1] \\
&= Q_s^α \frac{gα}{p(gα)},
\end{align*}
\]

\[ [Q_s^α \frac{gα}{p(gα)} , B_{2α-1}^1 B_{2α}^1] = [-B_{2α}^1 B_{2α-1}^1, Q_s^α \frac{gα}{p(gα)},]
\]

\[ = -Q_s^α \frac{gα}{p(gα)}. \]

That means the conserved term \( Q_{s}^α \frac{gα}{p(gα)} \) has the property of self-duality (or holographic duality).

There is a great difference between the two basises, whose conserved quantities and the corresponding Hamiltonians are \( Q_{s}^α \) and \( Q_{s}^α \), \( H' \) and \( H \), respectively. Their are totally \( N/2 \) mutually independent \( Q_{s}^α \) (\( α = 1 \cdots N/2 \)), and \( N \) mutually independent \( n_i = b_i^1 b_i^1 (i = 1 \cdots N) \) which form the conserved bosonic spin charge \( Q_{s} \). We collect the related commutators in \( SO(N) \) rotated basis (in the mean time mapping to the Richardson-Gaudin model) below, and some of them will also be used in the following derivation,

\[
\begin{align*}
[R_s^α, H'] &= 0, \ [R_s^α, N/2] = 0, \\
[Q_s^α, Q_{s}^α] &= 0, \ [Q_s^α, H'] = 0, \ [Q_s^α, H'] = 0, \\
[Q_s^α gα, H'] &\neq 0, \ [p(gα) B_{2α}^1 B_{2α-1}^1, H'] \neq 0, \ [p(gα) B_{2α}^1 B_{2α-1}^1, H'] \neq 0, \\
\sum_{α=1}^{N/2} Q_s^α, H' &\neq 0, \ [\sum_{α=1}^{N/2} Q_s^α, N/2] \neq 0, \\
[Q_s^α \frac{gα}{p(gα)}, H'] &= 0, \ [\frac{gα}{p(gα)}, H'] = 0, \ [Q_{s}^α \frac{gα}{p(gα)}, N/2] = 0, \ [\frac{gα}{p(gα)}, N/2] = 0, \\
[H', N/2] &\neq 0, \ [Q_s^α, \sum_{α=1}^{N/2} Q_s^α] = 0, \ [B_{2α-1}^1 B_{2α}^1, H'] = 0, \ [B_{2α-1}^1 B_{2α}^1, N/2] = 0.
\end{align*}
\]

While in the old basis (with conserved quantity \( Q_{c} = \sum_i^N b_i^1 b_i^1 = \sum_i^N n_i \)) we have

\[
[Q_{c}, H] = 0, \ [Q_{c}, N] = 0, \ [H, N] = 0, \ [n_i, n_j] = 0, \ [n_i, H] \neq 0, \ [n_i, N] \neq 0, \ [n_i, Q_{c}] \neq 0. \tag{117}
\]

In this basis, the eigenvalues of \( R_s^α \) can be obtained as

\[
\begin{align*}
E_{s} = &Q_s^α \frac{gα}{p(gα)} + \sum_{β=1}^{N/2-1} \frac{1}{p(gα)} - \frac{1}{p(gβ)} \left[ Q_s^α \frac{gα}{p(gα)} \right] \frac{Q_s^α \frac{gα}{p(gα)}}{p(gα)} \frac{1}{p(β)} - (Q_{s}^α gβ)^{-1}, \\
&= Q_s^α \frac{gα}{p(gα)} + \sum_{β=1}^{N/2-1} \frac{1}{p(gα)} - \frac{1}{p(gβ)} \left[ Q_s^α \frac{gα}{p(gα)} \right] \frac{Q_s^α \frac{gα}{p(gα)}}{p(gα)} \frac{1}{p(β)} - (Q_{s}^α gβ)^{-1},
\end{align*}
\]

where the first term corresponds to the eigenvalue of Eq. (123) which is the case that the second term sums up to zero, and it is an unconserved term as mentioned below. Note that although
\( \beta \) can only takes \((N/2 - 1)\) values, it does not have to be different from \(\alpha \). While \(\beta\) has totally \(N/2\) possible values, the random summation in second term over \((N/2 - 1)\) of them will leads to \(N/2\) possible results, including zero, that is why \(R_\alpha\) has \(N/2\) nondegenerated eigenstates and eigenvalues.

This maximal number of \(\alpha\) is also related to the above commutator by

\[
\sum_{\alpha} Q_s^{\alpha} \frac{g_\alpha}{p(g_\alpha)} = \sum_{\alpha} \left( \frac{g_\alpha}{p(g_\alpha)} - B_{2n}^\dagger B_{2n-1} - 1 \right) = \frac{N}{2},
\]

thus we have

\[
\sum_{\alpha} \left( \frac{g_\alpha}{p(g_\alpha)} - B_{2n}^\dagger B_{2n-1} \right) = 0
\]
as long as the summation over \(Q_s^{\alpha} \frac{g_\alpha}{p(g_\alpha)}\) is \(N/2\) (as setted above by us) which may change for other blocks of the diagonalized Hamiltonian.

For probability distribution \(p(g_\beta)\), it must be guarantees that \(g_\beta \neq g_\beta'\) where \(g_\beta'\) satisfies the following Bethe-Ansat equation\(^{55, 56}\) (which is valid for all \(\beta' = 1, \cdots, N/2 - 1\))

\[
\sum_{\alpha} \frac{Q_s^{\alpha} \frac{g_\alpha}{p(g_\alpha)}}{Q_s^{\alpha} g_\beta' - Q_s^{\alpha} g_\beta} = -\sum_{\beta' \neq \beta} \frac{N/2 - 1}{Q_s^{\beta'} g_\beta' - Q_s^{\beta'} g_\beta}.
\]

The left-hand-side of the above equation has the following relation

\[
\sum_{\alpha} \frac{Q_s^{\alpha} g_\alpha}{p(g_\alpha)} \frac{1}{Q_s^{\alpha} g_\beta' - Q_s^{\alpha} g_\beta} - \sum_{\alpha} \frac{N/2 - 1}{Q_s^{\alpha} g_\beta' - Q_s^{\alpha} g_\beta} p(g_\alpha) B_{2n}^\dagger B_{2n-1} B_{2n}^\dagger B_{2n-1} B_{2n}
\]

In the limit of large \(Q_s^{\alpha} g_\beta\), the above commutator reduces to of the order of 1, in which case the Hamiltonian reduces to

\[
H' = \sum_{\alpha} p(g_\alpha) B_{2n}^\dagger B_{2n-1} B_{2n}^\dagger B_{2n-1}.
\]

In this way, the eigenvalue of \(H'\) can be obtained as

\[
\langle H' \rangle = \sum_{\alpha} \sum_{\beta} p(g_\alpha) Q_s^{\alpha} g_\alpha \frac{1}{p(g_\alpha)} - \sum_{\beta} \sum_{\alpha} Q_s^{\alpha} \frac{1}{p(g_\alpha)} - \sum_{\beta' \neq \beta} \frac{N/2 - 1}{Q_s^{\beta'} g_\beta' - Q_s^{\beta'} g_\beta}
\]

\[
= \sum_{\alpha} Q_s^{\alpha} g_\alpha - \sum_{\beta} Q_s^{\beta} g_\beta + \sum_{\beta' \neq \beta} \frac{1}{Q_s^{\beta'} g_\beta'}
\]

\[
= \sum_{\alpha} Q_s^{\alpha} g_\alpha - \sum_{\beta} Q_s^{\beta} g_\beta + Q_s^{\beta'} g_\beta'.
\]
Obviously, although we have the commutation relations $[Q^a_s, H'] = 0$ and $[Q^a_s g_a^s, H'] = 0$, when there is a single SYK coupling (without divided by its corresponding probability distribution function), it turns to an unconserved quantity for $H'$, i.e., $[Q^a_s g_a^s, H'] \neq 0$. Inserting Eq.(115) to the above result, we obtain (still setting $\alpha = k, l$)

$$
\sum_{\alpha}^{N/2} Q^a_s g_\alpha = \sum_{\alpha=k,l}^{N/2} \epsilon_{kij}p(g_\alpha)\left(\frac{g_\alpha}{p(g_\alpha)} - 1 - B_{2a}^\dagger B_{2a-1}\right) \\
= \sum_{i,j}^{N} (g_{ij} - p(g_{ij}) - p(g_{ij})b_j^\dagger b_i) \\
= -1 - \sum_{i,j}^{N} p(g_{ij})b_j^\dagger b_i \\
= \sum_{\beta}^{N/2} Q^\beta_s g_\beta.
$$

(125)

The summation of $Q^a_s g_\alpha$ over $\alpha$ in the new basis is equivalents to the summation over all $i, j (= 1, \cdots, N)$ in the old basis, and they all commute with $H'$. Thus it is easy to know that the eigenvalue of $H'$ is

$$
\langle H' \rangle = Q^\alpha_s g_\alpha.
$$

(126)

There are $N/2$ eigenvalues for $H'$, which are mutually independent, corresponding to $\beta' = 1, \cdots, N/2$ (since $\beta$ does not need to be different from $\alpha$), and this is the precondition for many-body Hamiltonian $H'$ to exhibiting level spacing characterized by random matrix theory.

We also notice that, Eq.(125) has the following property

$$
\sum_{\alpha}^{N/2} Q^\alpha_s g_{\beta'} = \sum_{\alpha}^{N/2} \sqrt{p(g_\alpha)}B_{2\alpha}^\dagger B_{2\alpha-1}, \sum_{\alpha}^{N/2} \frac{g_\alpha}{\sqrt{p(g_\alpha)}}], \\
\sum_{\alpha}^{N/2} Q^\alpha_s g_{\beta'}, \sum_{\alpha}^{N/2} \sqrt{p(g_\alpha)}B_{2\alpha}^\dagger B_{2\alpha-1}B_{2\alpha} = 1, \\
\sum_{\alpha}^{N/2} Q^\alpha_s g_{\beta'}, \sum_{\alpha}^{N/2} \frac{g_\alpha}{\sqrt{p(g_\alpha)}} = -1,
$$

(127)

which are validated by the term $\delta_{\alpha, \alpha'}$ in first line of Eq.(115).

Now we can perform the many-body level statistic base on the result of Eq.(126). By collecting the eigenvalues of $H'$ in ascending order, the probability distribution $p(r)$ of level spacing $r = \frac{E_{n+1} - E_n}{E_{n+1} - E_n}$ is shown in Fig.5. Different to the the exact diagonalization base on conservation of U(1) charge\cite{12}, the distribution of level spacing obtained here does not following different Gaussian ensembles depending on different $N$. Instead, we found that for different values of $N$ ($N$ mod 2; or $N/2$ mod 1), the results will not change. Besides the cases of $N/2 = 20$ and $N/2 = 25$, we also used to setting the $N/2$ as a half-integer, and the result shows no difference with the integer one. This is because the eigenvalues of $Q^a_s$ (which is the conserved quantity here) does not depend on $N$, while the eigenvalue of fermion charge $Q_c$ depends on $N$, e.g., the charge eigenvalue $\langle Q_c \rangle = \sum_i^{N} n_i - \frac{N}{2}$ will be integer (half-integer) for even (odd) $N$. Thus the many-body statistic here will not be affected by $N$, which is the biggest difference from the fermionic SYK system. Since the conserved boson spin charge $Q^a_s$ is independent of $N$, it is indeed not a $\mathbb{Z}_2$ operator, that means it will not breaks the $\mathbb{Z}_2$ discrete symmetries (if exist) even in the large-$N$ limit, which is different to the fermion charge case\cite{50}. In the mean time, the conservation of $Q^a_s$ is robust since it is related to the spin angular momentum conservation instead of the particle number conservation.
In fact, by defining a spin charge conjugation operator, \( C = e^{i\pi S_z} \), which fliping the spin by angle \( \pi \), the charge \( Q^\alpha_s \) satisfies the time-reversal symmetry,

\[
\mathcal{T}Q^\alpha_s \mathcal{T}^{-1} = SCQ^\alpha_s K^{-1}C^{-1}S^{-1} \\
= SCQ^\alpha_s C^{-1}S^{-1} \\
= S(-Q^\alpha_s)S^{-1} \\
= Q^\alpha_s,
\]

where \( S \) is the chiral symmetry operator, and \( K \) is the complex conjugation which makes no effect on \( Q^\alpha_s \) as it is a real matrix (and its corresponding Hamiltonian \( H' \) is a real quaternion matrix satisfying GSE; see below). The third line in above equation is deduced by

\[
S(b_i^\dagger_\downarrow b_j^\uparrow - b_i^\dagger_\uparrow b_j^\downarrow)S^{-1} = b_i^\dagger_\downarrow b_j^\uparrow - b_i^\dagger_\uparrow b_j^\downarrow \\
= b_i^\dagger_\downarrow b_j^\uparrow - 1 - b_i^\dagger_\uparrow b_j^\downarrow + 1 \\
= -(b_i^\dagger_\uparrow b_j^\downarrow - b_i^\dagger_\downarrow b_j^\uparrow).
\]

Similarly, the Hamiltonian \( H' \) also satisfies TRS, which is invariant under the individual operations of \( SC \), and \( K \). For \((H')^s\) follows GSE level statistic, we have \( \mathcal{T}^2 = -1 \). The preserved TRS of bosonized \( H' \) and spin charge \( Q^\alpha_s \) is in consistent with the above-mentioned TRS for modes \( \phi_i \) and \( \phi_j \) in Sec.7, but note that the time-reversal process for modes \( \phi_i \) and \( \phi_j \) is indeed the spacial parity inversion in \( z \) direction if out of the basis used in Sec.7. The presence of TRS also implies that there will be not degenerate states at low energy, which means the energies of modes \( u_i \) and \( u_j \) can not be equal (due to the definition of SO(N) rotational operator).

As the term \( Q^\alpha_s g_{ij} \) does not commute with \( H' \) while \( Q^\alpha_s \) commute with \( H' \), we can simply seting \( Q^\alpha_s g_{ij} = 1 \) even away from the large-\( N \) limit, and then the eigenvalue only depends on the random couplings. By selecting only the interger eigenvalues, the distribution of level spacing becomes

\[
p(r) = \sum_{g=0}^{N/2-(r+1)x} \sum_{x=1}^{[\frac{N/2}{x+1}]} p(g)p(g+x)p(g+(r+1)x),
\]

where \([.]\) denotes taking the integer part. For comparasion, we also present the Wigner-Dyson level statistics in ETH phase with

\[
p(r) = \frac{1}{Z} \frac{(r + r^2)^{\beta/2}}{(1 + r + r^2)^{1+3\beta/2}},
\]

where the parameters are: \( \beta = 1, Z = 8/27 \) for Gaussian orthogonal ensemble (GOE), \( \beta = 2, Z = 4\pi/81\sqrt{3} \) for Gaussian unitary ensemble (GUE), \( \beta = 4, Z = 4\pi/729\sqrt{3} \) for Gaussian symplectic ensemble (GSE), and the Poisson distribution \( p(r) = 1/(1 + r)^2 \). To make the contrast more clear, we also plot the \( p(lnr) = p(r)r \) in Fig.4(b).

The result shows that the block-diagonalized Hamiltonian has a level spacing follows the distribution of GSE, and characterized by the random matrix theory in SYK model, although it does not depends on the value of \( N \). Thus we can suspect that, the formation of conserved \( Q^\alpha_s \) must breaks some certain symmetry and leads to asymptotic degeneracy in low energy in the spectral of Hamiltonian. However, some robust symmetries in WTe2 may leads to the existence of nonergodicity for some eigenstates [57], which may recover some localization and can help us to better recognize the disorders in this SYK system.

In conclusion, the relation between \( H \) and \( H' \) is

\[
H = \bigotimes_{i<j}^{N(N-1)/2} H_{ij} = \bigoplus_{\gamma=0}^{N/2} H^\gamma,
\]
where $H'_\gamma$ denotes the sector containing $\gamma$ unit sectors (whose size is the same with the sector $\alpha$ as we define above), i.e., $H'_{\gamma=N/2} = H'$. Each sector has the Hilbert dimension

$$\dim H'_\gamma = \frac{N!}{\gamma!(\frac{N}{2} - \gamma)!}$$  \hspace{1cm} (133)$$

which sum up to $2^{N/2}$.

### 10.3 Extension to SYK$_4$ and the corresponding bosonic velocities

Now we can write the SYK$_4$ in the new basis as

$$\mathcal{L}^{(4)}(z) = \sum_{\alpha}^{(N/2)^2} B_{2\alpha-1}^\dagger (-i\partial_z) B_{2\alpha} + \sum_{\alpha\alpha'}^{N/2} g_{\alpha\alpha'} B_{2\alpha-1}^\dagger B_{2\alpha} B_{2\alpha'-1}^\dagger B_{2\alpha'}$$

$$= \pi \sum_{\alpha\alpha'}^{N/2} B_{2\alpha-1}^\dagger B_{2\alpha} B_{2\alpha'-1}^\dagger B_{2\alpha'} + \sum_{\alpha,\alpha'}^{N/2} g_{\alpha,\alpha'} B_{2\alpha-1}^\dagger B_{2\alpha} B_{2\alpha'-1}^\dagger B_{2\alpha'}$$  \hspace{1cm} (134)$$

The corresponding $N/2 \times N/2$ paired boson velocity matrix can be obtained similar to the procedure shown in Eq.(95)

$$v^{(4)}_{\alpha\alpha'} = \begin{pmatrix} -\pi + g_{2\alpha\alpha'}^2 & -\pi + g_{2\alpha\alpha'}^2 & \cdots \\ -\pi + g_{2\alpha\alpha'}^2 & -\pi + g_{2\alpha\alpha'}^2 & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix},$$  \hspace{1cm} (135)$$

where the $\frac{N(N-1)}{2}$ elements in upper triangle are

$$v^{(4)}_{\alpha} = \frac{\partial \mathcal{L}^{(4)}(z)}{\partial (B_{2\alpha-1}^\dagger B_{2\alpha})^2},$$  \hspace{1cm} (136)$$

and can be summed to

$$\sum_{\alpha}^{N/2} v^{(4)}_{\alpha} = -\left(\frac{N}{2} - 1\right) \frac{n_{aa'} - n_{2aa'}^{1-(\frac{N}{2}-1)}}{n_{aa'} - 1} (\pi + (\frac{g^2}{N} - g_m^2))$$

$$- \frac{\left(\frac{N}{2} - 1\right) \frac{n_{aa'} - n_{2aa'}^{1-(\frac{N}{2}-1)}}{n_{aa'} - 1} \ln n_{aa'}}{\ln R} (\pi + (\frac{g^2}{N} - g_m^2));$$  \hspace{1cm} (137)$$

while the $\frac{N(N-1)}{2}$ elements in upper triangle are

$$v^{(4)}_{\alpha'} = \frac{\partial \mathcal{L}^{(4)}(z)}{\partial (B_{2\alpha'-1}^\dagger B_{2\alpha'})^2},$$  \hspace{1cm} (138)$$

and can be summed to

$$\sum_{\alpha'}^{N/2} v^{(4)}_{\alpha'} = -\left(\frac{N}{2} - 1\right) \frac{n_{aa'} - n_{2aa'}^{1-(\frac{N}{2}-1)}}{n_{aa'} - 1} (\pi + (\frac{g^2}{N} - g_m^2))$$

$$- \frac{\left(\frac{N}{2} - 1\right) \frac{n_{aa'} - n_{2aa'}^{1-(\frac{N}{2}-1)}}{n_{aa'} - 1} \ln n_{aa'}}{\ln R} (\pi + (\frac{g^2}{N} - g_m^2));$$  \hspace{1cm} (139)$$
where the ratio $n_{aa'}$ is defined as (similar to the definition of $n_{ij}$)

$$n_{aa'} = \frac{n_a}{n_{a'}} (\delta_{a,a+1} + \delta_{a,a-1}) = n_R^{\pm 1/2}. \quad (140)$$

11 Conclusion

As a type-II Weyl semimetal, WTe$_2$ has a $Pm$ point group symmetry, which leads to vanishingly small responses (induced current density) in directions other than $z$ under a normally incident circularly polarized light (equivalents to an in-plane electric field). That makes the dimensionless responses in each layer nearly the same energy, despite of the spatial inequivalence in $x - y$ plane. And we only need to consider the finite spacial effects of 1D SYK system by aid of the UV cutoff (a finite point splitting in $z$-direction), which is the interlayer spacing (about 0.7 nm [62] for WTe$_2$) here insteads of the lattice constant (about 0.3 nm [61] for WTe$_2$) like in other lattice model in condensate matter physics which are focusing on the contact-type fermion interaction.

Another advantage for choosing WTe$_2$ in $T_d$ phase as an experimental candidate is that, in the presence of SOC, there are not Weyl nodes at $k_z \neq 0$ [41], and the tilting is along the $k_y$ direction ($b$-axis of WTe$_2$) [64], which makes the chiral photocurrent response origining from different tilting of Weyl cones is impossible to exist for a normally incident light. Such a chiral photocurrent [63] is possible only by a circularly polarized light which could induce spin-flipping and be irradiated along the tilting direction ($b$-axis), due to the intrinsic noncentrasymmetry properties where both the time-reversal symmetry and inversion symmetry are broken.

In fact, although the inversion symmetry is absence in Weyl semimetal WTe$_2$, the in-plane two-fold rotational symmetry is preserved [63, 64] as indicated by the mirror symmetry plane $M_{yz}$ and glide mirror symmetry plane $M_{xz}$. This in-plane symmetry will leads to cancellation of any in-plane optical response induced by normally-incident circularly-polarized light. That also agree with our calculation results that the SHG response tensors (nonlinear AC phot conductivity tensor) are vanishingly small other than the $\chi_{zzz}$ and $\chi_{zxy}$ (which are directly related to the induced current in $z$-direction which does not affected by the twofold rotational symmetry), base on the experimentally obtained WTe$_2$ band structure near a Weyl cone. Such an in-plane responses cancellation will vanish for third-order Harmonic generation.

Similarly, the chiral anomaly-induced linear or nonlinear Hall current is only possible to realized by a pair of non-orthogonal magnetic field and electric field [61], applied in the direction of tilting. Thus a vertical magnetic field can still be applied to enhance the SYK coupling. That is also why we consider only one Weyl node, as the internode scattering or the internode current (like chiral anomaly) are absent without the strong magnetic field or light field along $b$-aixs.

Another difference to the photocurrent process is that, for circularly polarized light-driven photocurrent, the excitation usually has a long relaxation time due to the large weight of optical absorption which may even increased for larger layer number and thus reduce the SHG intensity. While for the SHG configuration described in this paper, which is another type of circular photagalvanic effect, the weight of effects of optical absorption is vanishingly small, as the relaxation time for a perturbed electron back to equilibrium is very short. That is why the SHG singal origin more from an optical response instead of an induced polarized electrical current. Thus the absence of in-plane current and optical absorption-induced TRS (between Weyl fermions) and the controllable boson interactions, are the essential factors to making the SYK effects detectable.
12 Appendix A: High-order response function with and without momentum-dependence

To calculate the response function in first (free-particle) to third order (note that the third order case here also corresponds to the well-known second-harmonic generation (SHG) in the momentum-dependent case), we need to evaluate the following integrals

\[ I_1 = \int d\omega G_m(\omega), \]
\[ I_2 = \int d\omega G_m(\omega)G_n(\omega + \Omega_1), \]
\[ I_3 = \int d\omega G_m(\omega)G_n(\omega + \Omega_1)G_l(\omega + \Omega_2). \]

by using the contour integral technique.

12.1 Without momentum-dependence: SYK

Firstly we discuss the case with zero kinetic term, in which case the subscripts of Green’s functions can be ignored. After performing the analytical continuation to the fermionic Matsubara frequencies \( i\omega \rightarrow \omega + i\eta \equiv z \), we know that in this case, the poles of Green’s functions only come from the Fermi-Dirac distribution, and each with residue \(-1/\beta\) where \( \beta \) is the inverse temperature. All these poles locate in the imaginary axis of complex \( z \) plane, and thus the circuits in contour integral enclosing the poles can be extended to the form consisting of a large (with infinite radius) loop and the transverse lines along the branch cuts. Then for \( I_1 \) we obtain \( I_1 = \int_C \frac{dz}{2\pi i} dG(z) = 0 \); for \( I_2 \), we obtain

\[ I_2 = \int_C \frac{dz}{2\pi i} dG(z)G(z + i\Omega_1)N_F(z) \]
\[ = -\int_{-\infty}^{\infty} \frac{d\xi}{2\pi i} N_F(\xi)[G(\xi - i\eta) - G(\xi + i\eta)][G(\xi + i\Omega_1) + G(\xi - i\Omega_1)]. \]  

Performing analytical continuation to \( i\Omega_1 \) and replacing \( \xi \) to \( \xi + \Omega_1 \) in some terms, we have

\[ I_2 = -\int_{-\infty}^{\infty} \frac{d\xi}{2\pi i} [N_F(\xi + \Omega_1)G(\xi + \Omega_1 - i\eta)G(\xi - i\eta) - N_F(\xi)G(\xi + i\eta)G(\xi + i\Omega_1)] \]
\[ - (N_F(\xi + \Omega_1) - N_F(\xi))G(\xi + \Omega_1 + i\eta)G(\xi - i\eta)]. \]  

For \( I_3 \), we obtain

\[ I_3 = \int_C \frac{dz}{2\pi i} dG(z)G(z + i\Omega_1)G(z + i\Omega_1 + i\Omega_2)N_F(z) \]
\[ = -\int_{-\infty}^{\infty} \frac{d\xi}{2\pi i} N_F(\xi)[G(\xi + i\eta) - G(\xi - i\eta)][G(\xi - i\Omega_1)G(\xi - i\Omega_1 - i\Omega_2) + G(\xi + i\Omega_1)G(\xi - i\Omega_2)] \]
\[ + G(\xi + i\Omega_1)G(\xi + i\Omega_1 + i\Omega_2)]. \]  

Similar to the procedure we presented in Ref.[?], there are six terms in above expression, to simplify the calculation we again shift the real \( \xi \) to be complex \( \xi + i\eta \rightarrow i\xi \), then as we assume \( \text{sgn}[\Omega_1] = \text{sgn}[\Omega_2] \) and \( |\Omega_1,2| \gg \eta \), there are only two terms leave that contain the poles locate in opposite sides of real axis of \( i\xi \) plane. Thus \( I_3 \) can be simplified as (analytically continuing back)

\[ I_3 = -\int_{-\infty}^{\infty} \frac{d\xi}{2\pi i} N_F(\xi)[G(\xi + i\eta) - G(\xi - i\eta)]G(\xi + i\Omega_1)G(\xi - i\Omega_2), \]
where each term has two poles $-i\Omega_1$ and $i\Omega_2$ with different signs.

### 12.2 With momentum-dependence: non-SYK

In the presence of finite kinetic term, the subscripts of Green’s function becomes matter, and in addition to the poles that locate in imaginary axis, there are poles locate in the real axis. Using the fact that for contour with infinite radius we have

$$ \oint_C \frac{dz}{2\pi i} N_F(z)f(z), $$

(146)

the $I_1$ can be obtained as

$$ I_1 = \frac{1}{\beta} \sum_\alpha \frac{1}{i\omega_\alpha - \epsilon_m} \frac{1}{i\omega_\alpha + i\Omega_1 - \epsilon_n} = \int_C \frac{dz}{2\pi i} dz G(z - \epsilon_m)N_F(z) $$

(147)

$$ = -\sum_{z_0} \text{Res}[N_F(z_0)G(z_0 - \epsilon_m)] $$

$$ = N_F(\epsilon_m), $$

where $z_0$ are the poles locate in imaginary axis. For $I_2$,

$$ I_2 = \frac{1}{\beta} \sum_\alpha \frac{1}{i\omega_\alpha - \epsilon_m} \frac{1}{i\omega_\alpha + i\Omega_1 - \epsilon_n} = \int_C \frac{dz}{2\pi i} dz G(z - \epsilon_m)G(z + i\Omega_1 - \epsilon_n)N_F(z) $$

$$ = N_F(\epsilon_m) + \frac{N_F(\epsilon_n - i\Omega_1)}{i\Omega_1 + \epsilon_m - \epsilon_n} $$

(148)

$$ = \frac{N_F(\epsilon_m) - N_F(\epsilon_n)}{i\Omega_1 + \epsilon_m - \epsilon_n}. $$

Note that for bosonic $i\Omega$, we have $N_F(\epsilon - i\Omega) = N_F(\epsilon)$. For $I_3$,

$$ I_3 = \frac{1}{\beta} \sum_\alpha \frac{1}{i\omega_\alpha - \epsilon_m} \frac{1}{i\omega_\alpha + i\Omega_1 - \epsilon_n} \frac{1}{i\omega_\alpha + i\Omega_1 + i\Omega_2 - \epsilon_n} $$

$$ = \int_C \frac{dz}{2\pi i} dz G(z - \epsilon_m)G(z + i\Omega_1 - \epsilon_n)G(z + i\Omega_1 + i\Omega_2 - \epsilon_l)N_F(z) $$

$$ = \frac{N_F(\epsilon_m)}{(i\Omega_1 + \epsilon_m - \epsilon_l)(\epsilon_m + i\Omega_1 + i\Omega_2 - \epsilon_l)} + \frac{N_F(\epsilon_n)}{(\epsilon_n - \epsilon_m - i\Omega_1)(\epsilon_n - \epsilon_l + i\Omega_2)} $$

(149)

$$ + \frac{N_F(\epsilon_l)}{(\epsilon_l - \epsilon_m - i\Omega_1 - i\Omega_2)(\epsilon_l - \epsilon_n - i\Omega_2)}. $$

### 13 Appendix.B

We choosing $b,c = x,y$ as an example to show the derivation of $v_{mn}^{bc}$ in Eq. (150):

$$ v_{mn}^{xy} = \frac{\partial v_{mn}^x}{\partial k^y} - i(A_{mn}^y - A_{nm}^y)v_{mn}^x, $$

(150)
where the Berry connection is $A^x_{mn} = \frac{\gamma_{xm}}{\varepsilon_{mn}}$. The first term of above equation can be

$$\frac{\partial v^x_{mn}}{\partial k^y} = -\frac{v^x_{mn}}{\int_y \partial_y v^x_{mn}} \frac{\partial (\int_y \partial_y v^x_{mn})}{\partial k^y}$$

$$= -\frac{v^x_{mn}}{\int_y \partial_y v^x_{mn}} \frac{(\int_y \partial_y v^x_{mn})m - (\int_y \partial_y v^x_{mn})n}{k^y_m - k^y_n}$$

$$= -\frac{v^x_{mn}}{\int_y \partial_y v^x_{mn}} \frac{v^y_m - v^y_n}{\varepsilon_m - \varepsilon_n}$$

For second term, we only discuss the simplest case: $\varepsilon_{ml} = \varepsilon_{ln}$ and $\vec{m}l = \vec{l}n$, which leads to

$$i(A^y_{mn} - A^y_{ml})v^x_{mn} = \frac{v^x_{mn}v^y_{ln} - v^x_{mn}v^y_{ml}}{\varepsilon_{mn}v^x_{mn}}$$

$$= \frac{v^x_{ml}v^x_{ln}}{\varepsilon_{ln}v^x_{ml}} - \frac{v^x_{ml}v^x_{ln}}{\varepsilon_{ml}v^x_{ml}}$$

$$= v^x_{ml}v^x_{ln} \left( \frac{1}{\varepsilon_{ln}} + \frac{1}{\varepsilon_{ml}} \right) - v^x_{ml}v^x_{ml} \left( \frac{1}{\varepsilon_{ln}} + \frac{1}{\varepsilon_{ml}} \right)$$

$$= v^x_{ml}(v^x_{ln} - v^x_{ml}) + \frac{v^x_{ln}}{\varepsilon_{ln}}(v^y_{ml} - v^y_{ml})$$

$$= v^x_{ln}2v^y_{ln}$$

$$= 2iA^x_{ln}v^y_{ln}.$$  (152)

For arbitrary levels $l$, the above procedure can be done with integral over any possible $\langle v \rangle$ and $\theta, \phi$.

### 14 Appendix.C

According to energy current in Eq.(74), the difference of energy currents between $\phi_i$ and $\phi_j$ can be obtained as

$$\Delta \rho_z = i\partial_z \phi_i \partial_z \phi_i - \frac{N}{2\pi^2} \frac{g^2_{ij}}{2N} \frac{g^2_{m}}{g^2_{m}} (\partial_z \phi_i)^2 - (i \leftrightarrow j).$$  (153)

To estimate the experimental error, we must set $\phi_i$ as the boson number of first layer and $\phi_j$ the last one ($i = 1, j = N$). We assume the small energy loss for SHG resonance frequency through the sample is $\delta \tilde{\Omega} \ll 0$, then we have

$$\partial_z \phi_i = 2\pi z \tilde{\Omega}^2,$$

$$\partial_z \phi_j = 2\pi z (\tilde{\Omega} + \delta \tilde{\Omega})^2 \approx 2\pi z \tilde{\Omega}^2 + (\delta \tilde{\Omega})^2,$$  (154)

thus the ratio $n_{ij}$ is

$$n_{ij} = \frac{\tilde{\Omega}^2}{\Omega^2 + (\delta \tilde{\Omega})^2}.$$  (155)

The effect of velocity difference induced by $\delta \tilde{\Omega}$ is much lighter than that of the difference of boson distribution, thus using Eq.(179-80), we can approximately write

$$(\partial_z \phi_i)^2 - (\partial_z \phi_j)^2 = g^2_{ij}(n_{ij} - n_{ij}^{-1}).$$  (156)
Thus the current difference will be (in momentum space)

$$\Delta \rho_z = 2\pi \partial_{\tilde{\Omega}^2} n_{ij}^L - \frac{1}{n_{ij}} g_{ij} (n_{ij} - n_{ij}^{-1})^{1/2} + \frac{N}{2\pi^2} \left( \frac{g^2}{2N} - g_m^2 \right) (n_{ij}^{-1} - n_{ij}). \quad (157)$$

Since the emergence of SYK physics requires strong random interactions $g_{ij} \gg \tilde{\Omega}^2$. Under this condition, the energy current difference reduces to

$$\Delta \rho_z = -\frac{N}{2\pi^2} \left( \frac{g^2}{2N} - g_m^2 \right)^2 (n_{ij}^{-1} - n_{ij}). \quad (158)$$

## 15 Appendix

To solving the Euler-Lagrange equation of motion, we firstly transform the hard core boson operators back to the fermion representation, and then using the chiral Majorana fields which are defined by the following relations

$$\psi_i = \frac{c_i + c_i^\dagger}{\sqrt{2}},$$
$$\psi_j = \frac{(-i)(c_i - c_i^\dagger)}{\sqrt{2}},$$
$$c_i = \frac{\psi_i + i\psi_j}{\sqrt{2}},$$
$$c_i^\dagger = \frac{\psi_i - i\psi_j}{\sqrt{2}},$$
$$-ic\partial_c = \frac{-i}{2} \psi_i \partial \psi_i + \frac{-i}{2} \psi_j \partial \psi_j,$$
$$c_i^\dagger c_i = i\psi_i \psi_j.$$

Base on these relations, the Eq.(53) can be rewritten as

$$\mathcal{L}(\tau, z) = \sum_i \left[ \frac{1}{2} (\psi_i \partial_\tau \psi_i + \psi_j \partial_\tau \psi_j) - \frac{i}{2} (\psi_i \partial_z \psi_i + \psi_j \partial_z \psi_j) \right] - \frac{1}{N^2} \sum_{ijkl} g_{ij} g_{kl} \psi_i \psi_j \psi_k \psi_i. \quad (160)$$

Using the relation

$$\frac{\partial \mathcal{L}}{\partial \psi_i} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\psi}_i} + \frac{d}{dz} \frac{\partial \mathcal{L}}{\partial \dot{\psi}_i},$$
$$\frac{\partial \mathcal{L}}{\partial \psi_j} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\psi}_j} + \frac{d}{dz} \frac{\partial \mathcal{L}}{\partial \dot{\psi}_j}, \quad (161)$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{d\psi}{dt} \frac{\partial}{\partial \psi} + \frac{d^2\psi}{dt^2} \frac{\partial}{\partial \psi},$$

we can obtain

$$\sum_i \left( \partial_\tau - i \partial_z \right) \psi_i - \frac{1}{N^2} \sum_{jkl} g_{ij} g_{kl} \psi_j \psi_k \psi_i = 0, \quad (163)$$

which is equivalents to Eq.() in the main text

$$\sum_i \left( \partial_\tau - i \partial_z \right) \frac{b_i + b_i^\dagger}{\sqrt{2}} + i \sum_{ij} g_{ij}^2 \frac{b_i^\dagger b_j - b_i b_j^\dagger}{\sqrt{2}} = 0. \quad (164)$$
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Figure 1: Real and imaginary part of SHG response tensors $\chi_{zzx}$ and $\chi_{zzy}$ in unit of $-e^3/2\hbar^2$ calculated based on the data of energy dispersion of WTe$_2$ [41] near one of the Weyl cone (at finite momentum for nonzero band gap). The panels (a) and (c) are calculated by using Eq. (14), while panels (b) and (d) are calculated by using Eq. (15). The horizontal axis is the photon energy in unit of 5 $\mu$. The chemical potential is being set at the middle between the conduction band and valence band to prevent the optical absorption when the incoming photon energies is higher than the band gap. $a$ and $b$ denote the main peak (at $\omega = \mu$) and second (smaller) peak (at $\omega = 2\mu$) which are contributed by two photon and one photon resonances, respectively.
Figure 2: Feynmann diagrams for SHG (a) and TPLR (b). The latter process TPLR is much harder to happen comparing to the SHG in a nonlinear optical material WTe$_2$ but plays the role of disorder that introducing the randomness. (c) illustrates the process of TPLR (dashed line) which plays the role of disorder as the boson $u_1$ is possible to interacting with boson $u_2$ or $u_2'$ which are created in third and second layers, respectively. (d) and (e) show the diagrams of one-photon and two-photon resonances as can be seen from Fig.1. Both of them are of the usual SHG mechanism, i.e., the special cases of the triangle diagram shown in (a).
Figure 3: Panels in first column ((a), (d), (g)) show the SHG intensities in the presence (orange line) and absence (blue line) of SYK effect. Panels in second ((b), (e), (h)) and third ((c), (f), (i)) columns show the part of SHG intensities induced solely by zero-dimensional SYK effect (Eq.(51)) and one-dimensional SYK effect (Eq.(93)), respectively. Panels in first ((a), (b), (c)), second ((d), (e), (f)) and third ((g), (h), (i)) rows correspond to SYK coupling $g = 10, 12, 8 \text{ eV}$, respectively. The continue line and dashed line correspond to photon energies $\omega = 1.55 \text{ eV}$ and $\omega = 1.45 \text{ eV}$, respectively.
Figure 4: Color plot of the SHG intensities induced solely by zero-dimensional and one-dimensional SYK effects (the different between the orange and blue lines in panels of first column in Fig.2). For (a) (c) we set the layer number as $N = 10$, for (b) (d) we set the layer number as $N = 15$. The vertical and horizontal axises are the SYK coupling $g$ and photon energy, respectively.
Figure 5: Many-body level statistic in terms of the level spacing $r$ and $\ln r$ distribution. The red star and black rhombus correspond to the $N/2 = 20$ and $N/2 = 25$, respectively. The result shows that the spectral of the block-diagonalized Hamiltonian is characterized by GSE and independent of number of $N$ (but will depend on the eigenvalue of spin charge $Q^\alpha_s$ in sector $\alpha$).