PARTIALLY ADJOINT DISCRETIZATIONS OF ADJOINT OPERATORS AND NONCONFORMING FINITE ELEMENT EXTERIOR CALCULUS

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Abstract. This paper concerns the discretizations in pair of adjoint operators between Hilbert spaces so that the adjoint properties can be preserved. Due to the finite-dimensional essence of discretized operators, a new framework, theory of partially adjoint operators, is motivated and presented in this paper, so that adjoint properties can be figured out for finite-dimensional operators which can not be non-trivially densely defined in other background spaces. A formal methodology is presented under the framework to construct partially adjoint discretizations by a conforming discretization (CD) and an accompanied-by-conforming discretization (ABCD) for each of the operators. Moreover, the methodology leads to an asymptotic uniformity of an infinite family of finite-dimensional operators. A theory of nonconforming finite element exterior calculus (NCFEEC) is developed by constructing a systematic family of nonconforming finite element differential forms with certain structures by the theoretical framework and the formal construction of discretizations; by the NCFEEC, together with the classical FEEC, adjoint properties are reconstructed between discretizations of adjoint operators and between discretizations of variational problems.

The adjoint properties concerned in the paper are the closed range theorem and the strong dualities, whose preservations have not been well studied yet. Quantified versions of the closed range theorem are established for both adjoint operators and partially adjoint discretizations. The notion Poincaré-Alexander-Lefschetz (P-A-L for short) type duality is borrowed for operator theory, and horizontal and vertical P-A-L dualities are figured out for adjoint operators and their analogues are established for partially adjoint discretizations. The ABCD, a new kind of discretization method, is motivated by and plays a crucial role in the construction of partially adjoint discretizations. Besides, it can be used for the discretization of single operators; for example, in this paper, de Rham complexes that start with the Crouzeix-Raviart finite element spaces are constructed by series of ABCDs, including both the discretized operators and the domain spaces; commutative diagrams with minimal request of regularities are constructed thereon. The theory of exterior calculus has shown the importance for mathematics and numerics and contains foundational applications of theory of adjoint operators. Adjoint properties are reconstructed by combining the NCFEEC and the classical FEEC, such as the Poincaré-Lefschetz duality preserved as an identity by the partially adjoint discretizations of the adjoint operators, which was not yet obtained before. Further, equivalences are established between the discretizations for the elliptic variational problems and between the discretizations of the Hodge Laplacian problems. This shows how structure-preserving schemes for equations with linear operators can be lead to by partially adjoint discretizations under the new theoretical framework.

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## 1. Introduction

Let $X$ and $Y$ be two Hilbert spaces with respective inner products $\langle \cdot, \cdot \rangle_X$ and $\langle \cdot, \cdot \rangle_Y$, and let $(T, D) : X \to Y$ be an unbounded linear operator, $D$ being the domain dense in $X$. The adjoint operator of $(T, D)$, denoted by $(T^*, D^*)$, is defined by

$$\langle T^* w^*, v \rangle_X = \langle w^*, T v \rangle_Y, \quad \forall v \in D,$$

and the domain $D^*$ consists of such $w^*$s that there exists an element in $X$ taken as $T^* w^*$ to satisfy (1.1). Then $(T^*, D^*)$ is closed, and if $(T, D)$ is closed, the adjoint operator of $(T^*, D^*)$ is $(T, D)$. For $(T, D)$ closed densely defined from $X$ to $Y$, the pair of adjoint operators bear many important adjoint properties, which can be found in, e.g., [1, 13, 30, 45] and other textbooks and monographs. These properties play important roles in many fields and particularly the theory and numerical methods of partial differential equations. The theory of exterior calculus and finite element exterior calculus are foundational examples. The preservation of these properties by discretizations will be important for practical applications, but is generally not naturally or easily done. Indeed, adjoint operators may each have kinds of discretizations, and we can not expect any two discretizations for each of them can compose a pair with expected adjoint properties. This paper concerns the discretizations of a pair of adjoint operators so that adjoint properties can be preserved by the discretized operators. Schemes of relevant problems can be constructed with structural properties preserved for foundational problems.
1.1. Contents of the paper. In this paper, we will mainly study two types of adjoint properties. As a first and, to our knowledge, a most fundamental type of adjoint property, Banach’s closed range theorem (6) asserts, if a closed densely defined operator has closed range, then its adjoint operator has closed range. This is not only an important issue in functional analysis, but also a fundamental in the theory and numerical analysis of partial differential equations. Though, its preservation or reconstruction by discretizations has not been discussed. The second type is about the Poincaré-Alexander-Lefschetz (P-A-L for short) type dualities of adjoint operators. By the notion P-A-L type duality borrowed from algebraic topology, we mean

\[ C \ominus \bot A = B \ominus \bot D, \]

if \( A \oplus \bot B = C \oplus \bot D \) with \( A \subset C \) and \( D \subset B \) being all closed. Here \( \oplus \) and \( \ominus \) denote orthogonal summation and difference, respectively, whose precise meanings are given later. We will specifically discuss the reconstructions of horizontal and vertical P-A-L dualities, cf. Theorems 2.3 and 2.4, which are not yet well studied. The horizontal type duality was occasionally studied in the formulation of isomorphisms between trace spaces for some specific Sobolev spaces, but the metric properties of the trace spaces are generally very subtle to construct and further to preserve. The Poincaré-Lefschetz duality in the form of (1.2) for exterior differential operators is a typical vertical P-A-L type duality; cf. (4.1) and Lemma 4.3. Its reconstruction can be seen discussed in literature but in a weak formulation; namely, only an isomorphism between the two sides of (1.2) can be obtained, cf. Lemma 4.13; no identity has been seen reconstructed by discretizations.

For the discretization of a pair of adjoint operators, we are constructing a family of pairs of discrete operators with certain domains, to provide consistent approximation to each of the original operators and thus discretization schemes to relevant problems. The significant aspect is that, any of these discrete operators is of finite-dimensional essence, and the whole family which consists of infinitely many operators has to bear some asymptotic properties so that the original infinite-dimensional operator can be approximated. Due to the finite-dimensional essence, these operators can not be non-trivially closed densely defined with respect to other background spaces; the standard theory can not apply directly for them for adjoint properties such as P-A-L type dualities. On the other hand, for any single pair of discrete operators, adjoint properties may be established; note for example that operators of finite-dimensional essence all have closed ranges; however, to describe and establish the adjoint properties of the infinite family of finite-dimensional operators in a uniform way is not that straightforward. A main purpose of this paper has thus to be to construct some new theory serving as a framework to clear up the adjoint properties for finite-dimensional operators, and then to construct discretizations for adjoint operators, so that adjoint properties may be constructed uniformly for an infinite family of finite-dimensional operators. Our concern is focused on the construction of discretizations in pair; this also includes, given a discretization for one operator, to design an accompanied discretization for the other; we may see the adjoint properties bring in convenience in the design and analysis. Also, the structural properties of relevant model problems may this way be preserved by the corresponding discretizations.

The overall approach of this paper is to generalize the relationship “adjoint” between operators to some new relationship like “generalized adjoint”, and then construct pairs of generalized adjoint operators for in-pair discretizations of adjoint operators. There are two key ingredients which allow us to carry out the approach.
First, we introduce a new concept partially adjoint for the relation between operators, and it bears some adjoint properties. Roughly speaking, two operators \((T_X, D_X) : X \rightarrow Y\) and \((T_Y, D_Y) : Y \rightarrow X\) are said to be partially adjoint based on \([([T_X, B_X], [T_Y, B_Y])\), if

\[
D_X = \{v \in B_X : \langle v, T_Y q \rangle_X - \langle T_X v, q \rangle_Y = 0, \ \forall q \in D_Y\},
\]

and

\[
D_Y = \{q \in B_Y : \langle v, T_Y q \rangle_X - \langle T_X v, q \rangle_Y = 0, \ \forall v \in D_X\},
\]

where \([([T_X, B_X], [T_Y, B_Y])\) is a base operator pair formed by \((T_X, B_X) \supset (T_X, D_X)\) and \((T_Y, B_Y) \supset (T_Y, D_Y)\) that satisfy some assumptions. Adjoint properties of the pair \([([T_X, B_X], [T_Y, B_Y])\) are constructed, and these properties rely on the properties of \([([T_X, B_X], [T_Y, B_Y])\) only. The criterion (1.3)-(1.4) looks quite the same as (1.1). Though, the partially adjoint operators are not necessarily densely defined in the base pair, and the criterion only influences the domains \(D_X\) and \(D_Y\), rather than the operations.

Second, we present a general methodology to construct partially adjoint discretizations of adjoint operators. Our 2-step construction is quite formal. Step 1, construct a tensorial base pair using tensor products of cell-wise operators and cell-wise shape functions. The tensorial way brings in the advantage that the features of the global base pair can be completely controlled by the features of cell-wise shape function spaces, and can thus be asymptotically uniform for the infinite family of finite-dimensional operators; see Remark 3.24 and Section 4.2. Step 2, use a conforming discretization (CD) for one operator and an accompanied-by-conforming discretization (ABCD) for the other to form a discrete pair. ABCD consists of an operator with its domain, and is a new kind of discretization of an original operator, developed directed by the theory of partially adjoint operators. A cell-wise-defined projective interpolator can be defined to the tensorial base, and it maps the domain space of the original operator into the ABCFES, the domain space of ABCD. Therefore, those pairs that consists of an optimal CD and an optimal ABCD are called partially adjoint discretizations of the adjoint operators, though various pairs of partially adjoint operators can be constructed based on a same base pair. Adjoint properties including quantified closed range theorem, slice dualities and dual complexes can be established. Further, the ABCDs can themselves be a novel approach as single discretizations. In this paper, though, we focus ourselves on the establishment and analysis of adjoint properties of a pair of operators or variational problems; features of discretization schemes such as the accuracy can be analyzed in standard ways and will not be discussed in details.

Technically, two nonstandard treatments help us circumvent the major obstacles which arise due to the finite-dimensional essence.

The first one is to choose proper formulation of the adjoint properties to be constructed. As aforementioned, for example, any finite-dimensional operator possesses a closed range, and the reconstruction of the closed range theorem for a family of operator pairs needs to find a proper formulation. We introduce the index of closed range (icr) for closed operators, and a closed operator has closed range if and only if its icr is finite. Then a quantified (and preciser) version of the closed range theorem is established, which reads a closed operator and its adjoint operator have the same index of closed range. This quantified property can then be studied for partially adjoint operators and as an asymptotic property for an infinite family of finite-dimensional operators.
Second, we illustrate how the criterion (1.3)-(1.4) can be used to construct a practical discretization of a single original operator. Logically, if a discretization of one operator is established in any way, a discretization of another operator can be simply established in order that (1.3) and (1.4) hold, and a partially adjoint pair follows. This approach is different from classical ones which construct discrete function spaces by matching the cell-wise shape functions and the requirement of continuity across the interfaces of neighborced cells, particularly the standard finite element triple presented in [18] and followed by numerous textbooks of finite element methods. Instead, our approach utilizes a continuity requirement not necessarily restricted on the interface but possibly on a group of cells, and our approach does not generally correspond to a triple as defined by [18]. Beyond the logical definition, we show that the discretization established this way is one practically implementable by figuring out its basis functions. Examples of ABCDs are given. The construction of ABCFES is similar to some nonconforming spline functions ([25, 33, 39, 46, 47]) and with a new way to impose continuities: different from most existing finite element methods, the ABCFES uses a dual way to impose proper continuities. Though, it can be interesting to clarify, these are not mixed element methods. Note that the ABCFESes admit compact-supported basis functions so that they are each a space which can be singly implemented for a discretization scheme for the one-field formulation of the problem, and they themselves can be embedded into exact sequences, such as the complexes which start from the Crouzeix-Raviart element spaces. The ABCD may be further generalized to a wider category, which does not necessarily rely on a base operator pair or be related to partially adjoint discretizations. This can be discussed in future.

The theory of exterior calculus, which has shown the importance for mathematics and numerics, is indeed a foundational examples where theory of adjoint operators and relevant adjoint properties of exterior differential operators play crucial roles. Theory of finite element exterior calculus (FEEC) has been well developed during the past decades and has succeeded to preserve some important structures in exterior calculus. We here are still concerned with some adjoint properties in exterior calculus and try to reconstruct them by discretizations. Therefore, also to illustrate the validities of the theory and the construction of partially adjoint discretizations, nonconforming FEEC associated with the de Rham complex is studied. Directed by the theory and the formal construction, a family of nonconforming Whitney finite element differential forms are given. Further, by combining the NCFEEC and the classical FEEC, some important adjoint properties are reconstructed. Firstly, the conforming and nonconforming Whitney finite element differential forms form well-coupled partially adjoint operator pairs which bear adjoint properties including the Helmholtz decomposition, the Hodge decomposition, and the Poincaré-Lefschetz duality preserved as identities (cf. (4.16)) rather than only isomorphisms. Then, the nonconforming Whitney forms can naturally be used for new discretization schemes, and the equivalence between the discretization schemes of primal and dual formulations of the elliptic source and eigenvalue problems are constructed. Further, as these nonconforming Whitney forms are taken into consideration, several new schemes are constructed for the Hodge Laplace problems, including ones based on new equivalent variational formulations, and the equivalences between those various schemes are constructed. These equivalences are the discrete analogues of the continuous ones.

We also remark that, the conforming and nonconforming Whitney forms provide an alternative way to reconstruct the duality between the trace spaces of the differential forms corresponding to a
horizontal P-A-L type duality between adjoint exterior differential operators. Indeed, for \( H\Lambda^k(\Omega) \) on \( \Omega \subset \mathbb{R}^n \) with Lipschitz boundary, the precise metric of the trace space is subtle for \( 1 \leq k \leq n-1 \) and \( n \geq 3 \). Further, even for the duality between the trace spaces of \( H\Lambda^0(\Omega) \) and \( H^*\Lambda^1(\Omega) \) on \( \Omega \subset \mathbb{R}^3 \), the preservation is not that direct; for example, when we use conforming linear element space to discretize \( H\Lambda^0(\Omega) \) and the lowest-degree Raviart-Thomas space to discretize \( H^*\Lambda^1(\Omega) \), their trace spaces have different dimensions, and no direct isomorphism can be established between them.

This way, we may summarize that, in this paper, motivated by the needs for constructing adjoint-structure-preserving discretizations, a theoretical framework is presented so that adjoint properties can be constructed for finite-dimensional operators which can hardly be done in the classical theory of adjoint operators. This may allow us to see things in a more appropriate way, particularly for problems with discrete essence. Directed by the theory, formal methodologies are given for the construction of pairs of partially adjoint discretizations, including the construction of an accompanied discretization with respect to a given discretization. The validities of the theory and the formal methodology are illustrated by adjoint-structure-preserving discretizations of exterior differential operators by conforming and nonconforming Whitney forms. The constructions of discrete Helmholtz decompositions illustrate that nonconforming finite element spaces are necessary and the internal continuity of the ABCFESes are proper. Further, the Poincaré-Lefschetz duality is reconstructed as identities, for which we know better than the isomorphisms in earlier theory. The nonconforming FECE is illustrated crucial for structure preservation accompanied with the classical FECE for both operator pairs and variational problem pairs. The discussions in the present paper may be generalized to more general settings in the future.

1.2. Bibliographic review and remarks. The discussions in the present paper fall into the category of functional analysis and structure-preserving discretization methods for partially differential equations. For functional analysis, we refer to briefly the monographs [6, 13, 30, 45]; for relevant discussions on discretization methods, as well as structure-preserving issues, we refer to briefly the monographs and survey papers [1, 3, 5, 11, 12, 17, 18, 27]. For the construction of discretized adjoint operators which we are focused on in this paper, we refer to, e.g., the monograph [1] and the references therein for relevant discussions. Within these discussions, a discrete analogue of (1.1) is generally constructed, though, as remarked before, the reconstruction of the closed range theorem and the P-A-L type dualities, which are the main focuses of the present paper, have not drawn much attention. We note that kinds of dualities used to be studied in, e.g., [4, 10, 15, 22, 29, 36, 37, 41, 44]. These works mainly focus on the dual representations for finite element spaces. Some of them discussed dual complexes in the manner that, for a given complex by finite element spaces, they consider how dual representations of these spaces can be constructed and connected to form a complex. The technique of dual/auxiliary grids have usually been used for the construction of discretized dual complexes. Different from them, in this paper, for a given complex composed by primal operators, we study the dual complex composed by adjoint operators (like defined in Section 4.1.2 of [1]). This pair of complexes are discretized by pairs of partially adjoint operators, and a pair of discretized complexes consists of a sequence of complex dualities (see (2.6) and Theorem 2.28 for the so-called complex duality). Note that for here, both complexes are constructed on a same grid, and the reconstruction of Poincaré-Lefschetz
type identity becomes possible.

As a fundamental model problem, the discretizations of the exterior differential operators has been drawing very broad interests during the past decades. It is the central topic of, e.g., the discrete exterior calculus (DEC, [21, 28]) and finite element exterior calculus (FEEC, [1, 3, 5]). These existing works mainly use the adjointness between the differential operator $d$ and the boundary operator $\partial$, say, Stokes theorem, to construct discretizations for an operator. In the present paper, we study the connections between discretizations based on the adjointness between differential and codifferential operators $d$ and $\delta$, and provide a systematic family of discretizations both for a pair of adjoint operators and for a single operator. The emergence of the nonconforming FEEC is illustrated crucial for structure preservation accompanied with the classical FEEC.

The $\text{ABCDs}$ by Whitney forms for exterior differential operators are generalizations of the Crouzeix-Raviart (C-R) finite element ([20]) for $H^1$ problems. The C-R element is one of the most important finite elements for partially differential equations. It possesses a cell-wise defined interpolator for functions with minimal regularities and this characteristic plays crucial roles in many applications, such as the correct computation of convex variational problems ([38]). It is also the starting finite element of the Morley-Wang-Xu family [43], the family of minimal consistent finite elements for $H^m$ problems in arbitrary dimension. It has been far well known that, the C-R finite element space and the Raviart-Thomas (R-T) element space for $H(\text{div})$ problem can form a discrete version of (1.1), cf. (3.23); in this paper, it is easily shown that the criterion (1.3)-(1.4) holds for C-R element space and R-T element space with respect to a tensorial base pair, and a new interpretation of the C-R element interpolator is established. The pair by the C-R element and the R-T element bears some adjoint properties: connections are established in [34] between the C-R element discretization and the R-T element discretization for $H^1$ problem provided the input data is piecewise constants; a discrete Helmholtz decomposition of piecewise constant space is established in [2], corresponding to $L^2A^1$, by C-R element space and the kernel of R-T element space in two dimension, and generalized to three dimension by [35]. In this paper, the C-R element space is generalized to a family for the spaces $H\Lambda^k$ with $0 \leq k \leq n$, the connections between primal and dual schemes for $H\Lambda^k$ problems are established for non-constant input data, and the discrete Helmholtz decompositions are established corresponding to $L^2\Lambda^k$, $0 \leq k \leq n$, $n \geq 2$. For the first time, to our knowledge, discretized de Rham complexes and commutative diagrams that start with the C-R element spaces are constructed.

There have been kinds of interpolators to finite element spaces which work for functions with minimal regularity requirements, such as [16, 19, 23, 24, 26, 31, 32, 42]. For these interpolators, the regularization, smoothing or averaging techniques are usually used based on macroelements consisting of patches of elements. Though, the adjoint projection (see Definition 3.29 and (3.20)) given in this paper is one which can be strictly cell-wise defined and fit for general boundary conditions. It can be quite flexible with respect to the shapes of the cells of the subdivisions.

The examples of $\text{ABCDs}$ given in the present paper are all by piecewise polynomials. We remark that the shape functions and the subdivisions for $\text{ABCD}$ can be flexible. For example,
conforming virtual element space (VEM, cf., e.g., [7-9]) can be adopted for CD for one operator on complicated subdivision, and ABCD with piecewise polynomials can be constructed for the other; also, the utilization of serendipity polynomials and spectral functions for piecewise shape function spaces will be discussed. We emphasize again that, though motivated by the in-pair discretizations of a pair of adjoint operators, the methodology of ABCD, or rather more general as dually consistent schemes, can be fairly used for a single discretization, not necessarily as part of partially adjoint discretizations. This can be illustrated by the Hodge Laplace problem (cf. Section 4.4.2) and other complicated problems. New algorithms and analysis can also be stimulated for eigenvalue computation, Dirichlet-Neumann operator, multigrid methods and so forth.

1.3. Organization of the paper. At the remaining of this section, we introduce some notations and conventions. In Section 2, we present the definition and basic properties of partially adjoint operators. We will firstly revisit some basics of adjoint operators, and then construct the analogue theory for partially adjoint operators. The P-A-L type dualities are presented as Theorems 2.3 and 2.4, and the quantified closed range theorem is established as Theorem 2.10. The definitions of base operator pair and partially adjoint are given as Definition 2.13 and Definition 2.15. Table 1 is given in Section 2.2 for a correspondence between respective adjoint properties for adjoint operators and partially adjoint operators. In Section 3, we present a general methodology for constructing partially adjoint discretizations of adjoint operators by CD and ABCD. The definitions of accompanied-by-conforming discretization and partially adjoint discretizations are given as Definition 3.12 and Definition 3.14. Compact supported basis functions are constructed in Section 3.2, cell-wise-defined interpolations are constructed in Section 3.3, and examples of ABCD are given in Section 3.4. In Section 4, using the methodology of ABCD, we construct a unified family of nonconforming Whitney forms for the exterior differential operators. Adjoint structures are then constructed for the discretizations of adjoint exterior differential operators, for the primal and dual discretizations of elliptic variational problems, and the various discretizations of the Hodge Laplace problems.

1.4. Notations and conventions. In this paper, we usually use a square bracket for a pair of operators, such as \([T, D) : X \rightarrow Y; (T', D') : X \rightarrow Y]\) and \([T, D), (T', D')]\) for short when there is no ambiguity. For an operator \(T : X \rightarrow Y\), we usually use \(H(T : X \rightarrow Y)\) or \(H(T)\) for short to denote the space \(\{v \in X : Tv \in Y\}\). We use \(L^2\) for the Lebesgue space that consists of square integrable functions.

For the readers’ ease, we usually use three capital letters with different mathematical fonts, for example \(X, \mathbb{Y}\) and \(Z\), for three Hilbert spaces, with inner products \(\langle \cdot, \cdot \rangle_X, \langle \cdot, \cdot \rangle_{\mathbb{Y}}\) and \(\langle \cdot, \cdot \rangle_Z\), and norms \(\|\cdot\|_X, \|\cdot\|_{\mathbb{Y}}\) and \(\|\cdot\|_Z\), respectively. We use, e.g., \(v, \mathbb{v}\) and \(\mathbf{v}\) to denote elements in \(X, \mathbb{Y}\) and \(Z\), use \(P, \mathbb{P}\) and \(\mathbf{P}\) to denote subsets of \(X, \mathbb{Y}\) and \(Z\), and use \(T, \mathbb{T}\) and \(\mathbf{T}\) to denote operators defined from \(X, \mathbb{Y}\) and \(Z\), respectively. The fonts of the letters indicate their relations to certain Hilbert spaces.

We use \(\mathcal{N}\) and \(\mathcal{R}\) to denote the null space and range of certain operators. Namely, for example, \(\mathcal{N}(T, D) := \{v \in D : Tv = 0\}\) and \(\mathcal{R}(T, D) := \{Tv : v \in D\}\). We use the overline \(\overline{\cdot}\) for the closure of a set. A capital letter with under tilde \(\tilde{\cdot}\) is generally a subset of the same letter with tilde \(\overline{\cdot}\).

For a Hilbert space \(H\), we use the notations \(\oplus_H^+\) and \(\oplus_H^-\) to denote the orthogonal summation and orthogonal difference; namely, for two spaces \(A\) and \(B\) in \(H\), the presentation \(A \oplus_H^+ B\) implies that
A and B are orthogonal in H, and evaluates as the direct summation of A and B; for A ⊂ B ⊂ H, B ⊥H A evaluates as the orthogonal complementation of A in B. We use the subscripts +⊥ and − to denote the “outer” and “inner” orthogonal complement. For example, A+⊥ to denote the orthogonal complement of a space in H; A− denotes the orthogonal complement of N(T, A) in A. The subscripts H and T can occasionally be dropped. The ring ⊥ generally indicates kernel, and the subscript 0 generally indicates subspace that adjoint-orthogonal to another space. The meanings of ±, ∼, and 0 will be made clear once they are used unless there is no ambiguity.

2. Theory of partially adjoint operators

In this section, we introduce the theory of partially adjoint operators. We first revisit the basics of adjoint operators. Then we present the basics of partially adjoint operators. Some technical proofs are given in a separate subsection. We use X, Y and Z for three Hilbert spaces.

2.1. Basics of adjoint operators revisited. Let [(T, D) : X → Y, (T, D) : Y → X] be a pair of adjoint closed densely defined operators. In this part, we study and present some fundamental properties, known or unknown, of the pair.

2.1.1. Poincaré-Alexander-Lefschetz type dualities. Below is a most fundamental property of adjoint operators, and can be found in every textbook. We call them orthogonal dualities.

Lemma 2.1 (Orthogonal dualities).

\[ N(T, D) = R(T, D)^⊥, \quad R(T, D) = N(T, D)^⊥, \quad \text{and} \quad R(T, D) = N(T, D)^⊥. \]

The Helmholtz decomposition below follows immediately.

Lemma 2.2 (Helmholtz decomposition).

\[ X = N(T, D) ⊕ X R(T, D), \quad \text{and} \quad Y = N(T, D) ⊕ Y R(T, D). \]

Theorem 2.3 (Horizontal Poincaré-Alexander-Lefschetz type duality). For two pairs of adjoint closed operators [(T, D) : X → Y, (T, D) : Y → X] and [(T, D) : X → Y, (T, D) : Y → X] such that \( \overline{D} \supset D \) and, equivalently, \( \overline{D} \subset \overline{D} \),

\[ R(T, D) ⊕ R(T, D) = N(T, D) ⊕ N(T, D), \]

and

\[ R(T, D) ⊕ R(T, D) = N(T, D) ⊕ N(T, D). \]

Proof. The assertion (2.3) follows immediately from that

\[ (R(T, D) ⊕ N(T, D)) = R(T, D) ⊕ N(T, D), \]

\[ R(T, D) ⊊ R(T, D), \quad \text{and} \quad N(T, D) ⊊ N(T, D). \]

Similarly can the assertion (2.4) be proved. \qed
Due to the structural properties (2.5), we borrow the notion from algebraic topology and call the two equalities (2.3) and (2.4) Poincaré-Alexander-Leftschetz (P-A-L) for short) type dualities, particularly of horizontal type, whereas below is a vertical P-A-L type duality.

Let \([((S, E)) : Y \to Z, (S, E) : Z \to Y]\) be a pair of adjoint closed densely defined operators. Then \(Y = N(S, E) \oplus \overline{R(S, E)}\), and \(Z = N(S, E) \oplus \overline{R(S, E)}\). Further,

\[
(2.6) \quad \text{(complex duality)} \overline{R(T, D)} \subset N(S, E) \iff \overline{R(S, E)} \subset N(T, D).
\]

The two sides of (2.6) are the dual complex of each other (cf. [1]). In the sequel, we call the relation (2.6) a complex duality.

**Theorem 2.4** (Vertical P-A-L type duality). Assume \(\overline{R(T, D)} \subset N(S, E)\).

\[
(2.7) \quad N(S, E) \oplus^{\perp} \overline{R(T, D)} = N(T, D) \oplus^{\perp} \overline{R(S, E)}.
\]

**Proof.** The assertion follows by the Helmholtz decompositions (by two ways) of \(Y\). \(\square\)

**Remark 2.5.** In the context of exterior differential calculus, Theorem 2.4 corresponds to the Poincaré-Lefschetz duality; see also Lemma 4.3.

Denote

\[
H_Y := N(S, E) \oplus^{\perp} \overline{R(T, D)} = (\text{by Theorem 2.4}) N(T, D) \oplus^{\perp} \overline{R(S, E)}.
\]

The Hodge decomposition below is well known.

**Lemma 2.6** (Hodge decomposition).

\[
(2.8) \quad Y = \overline{R(T, D)} \oplus^{\perp} H_Y \oplus^{\perp} \overline{R(S, E)}.
\]

2.1.2. Index of closed range and quantified closed range theorem. The Banach closed range theorem below is a fundamental property. We refer to, e.g., Section VII.5 ("The closed range theorem") of [45], for a proof in the context of Banach spaces, and, e.g., Section 3.3 of [1] for a proof in the context of Hilbert spaces.

**Theorem 2.7** (Banach closed range theorem). For \([(T, D), (\mathbb{T}, \mathbb{D})]\) a pair of closed densely defined adjoint operators, \(R(T, D)\) is closed in \(Y\) if and only if \(R(\mathbb{T}, \mathbb{D})\) is closed in \(X\).

**Definition 2.8.** For \((T, D) : X \to Y\) a closed operator, denote

\[
D' := \{ v \in D : \langle v, w \rangle_X = 0, \; \forall w \in N(T, D) \}.
\]

Define the index of closed range of \((T, D)\) as

\[
(2.9) \quad \text{icr}(T, D) := \begin{cases} \sup_{0 \neq v \in D'} \frac{\|v\|_X}{\|Tv\|_Y}, & \text{if } D' \neq \{0\}; \\ 0, & \text{if } D' = \{0\}. \end{cases}
\]

Note that \(\text{icr}(T, D)\) evaluates in \([0, +\infty]\). Evidently, \(\mathcal{R}(T, D) = \mathcal{R}(T, D')\), \(T\) is a bijection from \(D'\) onto \(\mathcal{R}(T, D)\), and \(\text{icr}(T, D) = \text{icr}(T, D')\). Further, \(\text{icr}(T, D')\) plays like the constant for Poincaré inequality in the sense that \(\|v\|_X \leq \text{icr}(T, D')\|Tv\|_Y\) for \(v \in D'\).

Lemma 2.9 below is actually known. We refer to, e.g., Lemma 3.6 of [1] for a proof.
Lemma 2.9. For \((T, D) : X \to Y\) a closed operator, its range \(R(T, D)\) is closed if and only if \(icr(T, D) < +\infty\).

By Theorem 2.7 and Lemma 2.9, for \([\langle T, D \rangle, \langle T \rangle, \langle D \rangle] \) a pair of closed densely defined adjoint operators, \(icr(T, D) < \infty\) if and only if \(icr(T, D) < \infty\). The closed range theorem can this way be viewed quantified. The theorem below gives a preciser quantification.

Theorem 2.10. For \([\langle T, D \rangle, \langle T \rangle, \langle D \rangle] \) a pair of closed densely defined adjoint operators,

\[
(2.10) \quad icr(T, D) = icr(T, D).
\]

Proof. Recalling (2.1), we have

\[
(2.11) \quad D^* = D \cap (\mathcal{N}(T, D))^\perp = D \cap \overline{R(T, D)}.
\]

Therefore, provided that \(0 < icr(T, D) < \infty\) and thus \(\overline{R(T, D)} = R(T, D)\), given \(v \in D^*\), there exists a \(w \in \mathcal{D}^*\), such that \(v = T^*w\), and

\[
\|v\|^2_X = \langle v, v \rangle_X = \langle v, T^*w \rangle_X = \langle Tv, w \rangle_Y \leq \|Tv\|_Y \|w\|_Y \leq icr(T, D)\|Tv\|_Y \|v\|_X.
\]

Therefore, \(\|v\|_X \leq icr(T, D)\|Tv\|_Y\) for any \(v \in D^*\) and \(icr(T, D) \leq icr(T, D) < \infty\). Similarly, \(\infty > icr(T, D) \geq icr(T, D)\); note that \((T, D)\) is the adjoint operator of \((T, D)\). Namely, if one of \(icr(T, D)\) and \(icr(T, D)\) is finitely positive, then \(icr(T, D) = icr(T, D)\).

If \(icr(T, D) = 0\), then \(\overline{R(T, D)} = \{0\}\) and \(D^* = \{0\}\). It follows then \(icr(T, D) = 0\).

Further, if one of \(icr(T, D)\) and \(icr(T, D)\) is \(+\infty\), then so is the other. The proof is completed. \(\square\)

Remark 2.11. Theorem 2.10 is a quantified version of the closed range theorem.

2.2. Theory of partially adjoint operators.

2.2.1. Base operator pair and partially adjoint operators. Let \((T, \bar{M}) : X \to Y\) and \((\bar{T}, \bar{N}) : Y \to X\) be two closed operators, not necessarily densely defined. Denote, for \(v \in \bar{M}\), \(\|v\|_{\bar{T}} := (\|v\|^2_X + \|Tv\|^2_Y)^{1/2}\), and for \(w \in \bar{N}\), \(\|w\|_{\bar{T}} := (\|w\|^2_Y + \|T^*w\|^2_X)^{1/2}\). Denote

\[
(2.12) \quad \bar{M} := \{v \in \bar{M} : \langle v, T^*w \rangle_X - \langle Tv, w \rangle_Y = 0, \forall w \in \bar{N}\},
\]

\[
(2.13) \quad \bar{N} := \{v \in \bar{N} : \langle v, T^*w \rangle_X - \langle Tv, w \rangle_Y = 0, \forall v \in \bar{M}\},
\]

\[
(2.14) \quad M_B := \{v \in \bar{M} : \langle v, w \rangle_X = 0, \forall w \in \mathcal{N}(T, \bar{M}); \langle Tv, T^*w \rangle_Y = 0, \forall w \in \bar{M}\},
\]

\[
\text{and}
\]

\[
(2.15) \quad N_B := \{v \in \bar{N} : \langle v, w \rangle_Y = 0, \forall w \in \mathcal{N}(T, \bar{N}); \langle T^*w, T^*w \rangle_X = 0, \forall w \in \bar{N}\}.
\]

The pair \([M_B, N_B]\) is called a twisted part of \([\bar{M}, \bar{N}]\).

Lemma 2.12. \((T, \bar{M}), (\bar{T}, \bar{N}), (T, M_B)\) and \((\bar{T}, N_B)\) are all closed.
Proof. Let \( \{v_k\}_{k=1,2,\ldots} \subset M \) be such that \( \|v_i - v_j\|_T \to 0 \), as \( i \) and \( j \) tend to infinity. Then, there exists a \( v_\infty \in \widetilde{M} \), such that \( \lim_{k \to \infty} \|v_k - v_\infty\|_T = 0 \). For any \( w \in \widetilde{N} \), \( \langle v_\infty, Tw \rangle_X - \langle Tv_\infty, w \rangle_Y = \lim_{k \to \infty} \langle v_k, Tw \rangle_X - \langle Tv_k, w \rangle_Y = 0 \); namely, \( v_\infty \in M \). Therefore, \( (T, M) \) is closed. The others are proved the same way. The proof is completed. \( \square \)

**Definition 2.13.** A pair of closed operators \( [(T, \widetilde{M}) : X \to Y, (\mathcal{T}, \widetilde{N}) : Y \to X] \) is called a base operator pair, if, with notations (2.12), (2.13), (2.14) and (2.15),

1. \( \mathcal{R}(T, \widetilde{M}), \mathcal{R}(\mathcal{T}, \widetilde{N}), \mathcal{R}(T, M) \) and \( \mathcal{R}(\mathcal{T}, N) \) are all closed;
2. \( \mathcal{N}(T, M_B) \) and \( \mathcal{R}(\mathcal{T}, N_B) \) are isomorphic, and \( \mathcal{N}(\mathcal{T}, N_B) \) and \( \mathcal{R}(T, M_B) \) are isomorphic.

For \( [(T, \widetilde{M}), (\mathcal{T}, \widetilde{N})] \) a base operator pair, for nontrivial \( \mathcal{R}(\mathcal{T}, N_B) \) and \( \mathcal{N}(T, M_B) \), denote

\[
\alpha := \inf_{0 \neq w \in \mathcal{N}(T, M_B)} \sup_{v \in \mathcal{R}(\mathcal{T}, N_B)} \frac{\langle v, w \rangle_X}{\|v\|_T \|w\|_X} = \inf_{0 \neq w \in \mathcal{R}(T, M_B)} \sup_{v \in \mathcal{N}(T, M_B)} \frac{\langle v, w \rangle_X}{\|v\|_T \|w\|_X},
\]

and for nontrivial \( \mathcal{N}(\mathcal{T}, N_B) \) and \( \mathcal{R}(T, M_B) \), denote

\[
\beta := \inf_{0 \neq w \in \mathcal{N}(\mathcal{T}, N_B)} \sup_{v \in \mathcal{R}(T, M_B)} \frac{\langle v, w \rangle_Y}{\|v\|_T \|w\|_Y} = \inf_{0 \neq w \in \mathcal{R}(T, M_B)} \sup_{v \in \mathcal{N}(T, M_B)} \frac{\langle v, w \rangle_Y}{\|v\|_T \|w\|_Y}.
\]

Then \( \alpha > 0 \) and \( \beta > 0 \). We further make a convention that,

\[
\begin{align*}
\alpha &= 1, \quad \text{if } \mathcal{N}(T, M_B) = \mathcal{R}(\mathcal{T}, N_B) = \{0\}; \\
\beta &= 1, \quad \text{if } \mathcal{N}(\mathcal{T}, N_B) = \mathcal{R}(T, M_B) = \{0\}.
\end{align*}
\]

Further,

\[ \mathcal{R}(T, \widetilde{M}) = \mathcal{R}(T, M) \oplus^\perp \mathcal{R}(T, M_B), \quad \mathcal{R}(\mathcal{T}, \widetilde{N}) = \mathcal{R}(\mathcal{T}, N) \oplus^\perp \mathcal{R}(\mathcal{T}, N_B). \]

and

\[ \mathcal{N}(T, \widetilde{M}) = \mathcal{N}(T, M) \oplus^\perp \mathcal{N}(T, M_B), \quad \mathcal{N}(\mathcal{T}, \widetilde{N}) = \mathcal{N}(\mathcal{T}, N) \oplus^\perp \mathcal{N}(\mathcal{T}, N_B). \]

**Lemma 2.14.** For \( [(T, \widetilde{M}), (\mathcal{T}, \widetilde{N})] \) a base operator pair, \( M_B = \{0\} \) if and only if \( N_B = \{0\} \). For \( M_B \neq \{0\}, \)

\[
\inf_{0 \neq w \in M_B} \sup_{v \in N_B} \frac{\langle v, Tw \rangle_X - \langle Tv, w \rangle_Y}{\|v\|_T \|w\|_T} = \inf_{0 \neq w \in N_B} \sup_{v \in M_B} \frac{\langle v, Tw \rangle_X - \langle Tv, w \rangle_Y}{\|v\|_T \|w\|_T} > 0.
\]

**Proof.** If \( \mathcal{N}(T, M_B) = \{0\} \) (or \( \mathcal{R}(T, M_B) = \{0\} \)), then \( \mathcal{R}(\mathcal{T}, N_B) = \{0\} \) (respectively \( \mathcal{N}(\mathcal{T}, N_B) = \{0\} \)), and the assertion follows by Definition 2.13 immediately. We only consider the nontrivial case that \( \mathcal{N}(T, M_B) \neq \{0\} \) and \( \mathcal{R}(T, M_B) \neq \{0\} \).

Denote \( M_B^2 := \{v \in M_B : \langle v, w \rangle_X = 0, \forall w \in \mathcal{N}(T, M_B)\} \) and \( N_B^2 := \{w \in N_B : \langle v, w \rangle_Y = 0, \forall v \in \mathcal{N}(T, N_B)\} \). Decompose \( M_B = \mathcal{N}(T, M_B) \oplus^\perp M_B^2 \), and for any \( v = \tilde{v} + v^2 \) with \( \tilde{v} \in \mathcal{N}(T, M_B) \) and \( v^2 \in M_B^2 \), \( \|\tilde{v}\|_X + \|Tv^2\|_Y \) is a norm of \( v \) equivalent to \( \|v\|_T \) on \( M_B \).

Given \( v \in M_B, v = \tilde{v} + v^2 \), by (2.16) and (2.18), there exists a \( w^2 \in N_B^2 \), such that \( \langle \tilde{v}, Tw^2 \rangle_X = \|\tilde{v}\|_X^2 \), and \( \|w^2\|_T \leq C\|\tilde{v}\|_X \). Let \( w_0 \in \mathcal{N}(T, N_B) \) be such that

\[ \langle \tilde{v}, Tw^2 \rangle_X - \langle T\tilde{v}, w^2 \rangle_Y - \langle T\tilde{v}, w_0 \rangle_Y = (T\tilde{v}, T^2)^{\perp}, \forall \tilde{v} \in M_B^2. \]
Let \( (T, D) \) be partially adjoint based on \( [(T, M), (T, \tilde{N})] \). If
\[
\mathbb{D} = \left\{ v \in \tilde{M} : \langle v, T \bar{w} \rangle_x - \langle T v, \bar{w} \rangle_Y = 0, \; \forall \; \bar{w} \in D \right\},
\]
and
\[
\mathbb{D} = \left\{ v \in \tilde{N} : \langle v, T \bar{w} \rangle_x - \langle T v, \bar{w} \rangle_Y = 0, \; \forall \; \bar{w} \in D \right\}.
\]

In the sequel, we may occasionally omit the base operator pair for partially adjoint operators when there is no ambiguity.

**Lemma 2.16.** Let \( (T, D) \) and \( (T, \mathbb{D}) \) be partially adjoint based on \( [(T, M), (T, \tilde{N})] \). Then both \( (T, D) \) and \( (T, \mathbb{D}) \) are closed.

**Proof.** Let \( \{w_n\}_{n=1}^{\infty} \subset \mathbb{D} \) be such that \( ||w_n - w_n||_T \to 0 \) as \( m \) and \( n \) tend to \( \infty \). Then there exists a \( w_\infty \in \tilde{N} \), such that \( \lim_{k \to \infty} ||w_k - w_\infty||_T = 0 \). It is easy to verify that \( \langle v, T w_n \rangle_x - \langle T v, w_n \rangle_Y = 0 \) for any \( v \in D \), and \( w_n \in D \). Namely \( (T, \mathbb{D}) \) is closed. So is \( (T, D) \). The proof is completed.

**Theorem 2.17.** For \( [(T, M), (T, \tilde{N})] \) a base operator pair, an operator \( (T, D) \) admits a partially adjoint operator \( (T, \mathbb{D}) \) based on \( [(T, M), (T, \tilde{N})] \), if and only if \( (T, D) \) is closed and \( M \subset D \subset \tilde{M} \).

**Proof.** If \( (T, D) \) and \( (T, \mathbb{D}) \) are partially adjoint based on \( [(T, M), (T, \tilde{N})] \), then, by definition, \( M \subset D \) and \( \tilde{N} \subset \mathbb{D} \), and by Lemma 2.16, the two operators are closed. On the other hand, if \( (T, D) \) is closed and \( D \supset M \), denote
\[
\mathbb{D} := \left\{ v \in \tilde{N} : \langle v, T \bar{w} \rangle_x - \langle T v, \bar{w} \rangle_Y = 0, \; \forall \; \bar{w} \in D \right\}.
\]
Then \( \tilde{N} \subset \mathbb{D} \), \( \mathbb{D} = \tilde{N} \oplus (\mathbb{D} \cap N_B) \) and \( (T, \mathbb{D}) \) is closed. It now suffices to show that
\[
\mathbb{D} := \left\{ v \in \tilde{N} : \langle v, T \bar{w} \rangle_x - \langle T v, \bar{w} \rangle_Y = 0, \; \forall \; \bar{w} \in \mathbb{D} \right\}.
\]
which follows from, as \( D = M \oplus (D \cap M_B) \) and \( \{ v \in \tilde{M} : \langle v, T\psi \rangle_x - \langle Tv, \psi \rangle_y = 0, \forall \psi \in D \} = M \oplus \{ v \in M_B : \langle v, T\psi \rangle_x - \langle Tv, \psi \rangle_y = 0, \forall \psi \in D \cap N_B \}, \)
\[
(2.23) \quad D \cap M_B = \{ v \in M_B : \langle v, T\psi \rangle_x - \langle Tv, \psi \rangle_y = 0, \forall \psi \in D \cap N_B \}.
\]
By (2.21),
\[
(2.24) \quad D \cap N_B = \{ v \in N_B : \langle v, T\psi \rangle_x - \langle Tv, \psi \rangle_y = 0, \forall \psi \in D \} = \{ v \in N_B : \langle v, T\psi \rangle_x - \langle Tv, \psi \rangle_y = 0, \forall \psi \in D \cap M_B \}.
\]
Since \((T, N_B)\) is closed, \((T, D \cap N_B)\) is closed. Further, since both \((T, D \cap M_B)\) and \((T, D \cap N_B)\) are closed, (2.23) holds by (2.24) and Lemma 2.14. Then (2.22) follows and the proof is completed. \(\square\)

**Lemma 2.18.** A pair of closed densely defined adjoint operators \([\langle T, D \rangle, \langle T, D \rangle]\) are partially adjoint based on any pair of closed operators \([\langle T, D \rangle, \langle T, D \rangle]\) such that \((T, D) \supset (T, D) \supset (T, D), (T, D) \supset (T, D), (T, D) \supset (T, D)\), and both \(R(T, D)\) and \(R(T, D)\) are closed.

**Proof.** As \((T, D)\) and \((T, D)\) are adjoint operators, we can show by definition that \( D = \{ v \in \tilde{D} : \langle T v, \psi \rangle_y = \langle v, T\psi \rangle_x, \forall \psi \in D \} \) and \( D = \{ v \in \tilde{D} : \langle T v, \psi \rangle_y = \langle v, T\psi \rangle_x, \forall \psi \in D \} \). It remains to show that \( [\langle T, D \rangle, \langle T, D \rangle] \) is a base operator pair.

Let \((T, D)\) and \((T, D)\) be the adjoint operators of \((T, \tilde{D})\) and \((T, \tilde{D})\), respectively. Then \( D = \{ v \in \tilde{D} : \langle T v, \psi \rangle_y = \langle v, T\psi \rangle_x, \forall \psi \in D \}, \) and \( D = \{ v \in \tilde{D} : \langle T v, \psi \rangle_y = \langle v, T\psi \rangle_x, \forall \psi \in D \} \). By closed range theorem, both \( R(T, D) \) and \( R(T, D) \) are closed. By Theorem 2.3, \( R(T, \tilde{D}) \oplus R(T, D) = N(T, \tilde{D}) \oplus N(T, D) \), and \( R(T, \tilde{D}) \oplus R(T, D) = N(T, \tilde{D}) \oplus N(T, D) \). This shows that \( [\langle T, D \rangle, \langle T, D \rangle] \) is a base operator pair. The proof is completed. \(\square\)

In the remaining of this subsection, we present some adjoint properties of partially adjoint operators. They are respective analogues of the properties of adjoint operators; cf. Table 1. Some of the technical proofs are postponed to Section 2.3.

### 2.2.2. Orthogonal dualities

With \([\langle T, M \rangle, \langle T, \tilde{N} \rangle] \) being a base operator pair, denote, for \( M \subset D \subset \tilde{M}, \)
\[
D_B := \{ v \in D : \langle v, w \rangle_x = 0, \forall w \in N(T, M) ; \langle T v, T w \rangle_y = 0, \forall w \in M \}
\]
and, for \( \tilde{N} \subset \tilde{D} \subset \tilde{N}, \)
\[
\tilde{D}_B := \{ v \in \tilde{D} : \langle v, w \rangle_y = 0, \forall w \in N(T, \tilde{N}) ; \langle T \tilde{v}, T w \rangle_x = 0, \forall w \in \tilde{N} \}.
\]
Then \( D_B = D \cap M_B, \) and \( \tilde{D}_B = \tilde{D} \cap N_B ; D = D_B \oplus M, \) and \( \tilde{D} = \tilde{D}_B \oplus \tilde{N}. \) Further, \( N(T, D_B) = N(T, M_B) \cap D, N(T, \tilde{D}_B) = N(T, N_B) \cap \tilde{D}, N(T, D) = N(T, M) \oplus N(T, D_B), \) and \( \overline{R(T, D)} = \overline{R(T, D_B)} \oplus \overline{R(T, M)}. \)
| Property                        | Lemma/Equation | Notes                      |
|--------------------------------|----------------|----------------------------|
| Adjoint properties             |                |                            |
| Orthogonal duality             | Lemma 2.1      | Lemmas 2.20 and 2.19       |
| Complex duality                | Equation (2.6) | Theorem 2.28               |
| Quantified closed range theorem| Theorem 2.10   | Theorem 2.21               |
| Helmholtz decomposition         | Theorem 2.2    | Theorem 2.23               |
| Horizontal P-A-L duality       | Theorem 2.3    | Theorems 2.24, and,        |
|                                |                | 2.22 (slice isomorphism)   |
| Hodge decomposition            | Lemma 2.6      | Theorem 2.31               |
| Vertical P-A-L duality         | Theorem 2.7    | Theorems 2.30, and,        |
|                                |                | 2.29 (slice isomorphism)   |

Table 1. Correspondence between properties of adjoint operators and partially adjoint operators.

**Lemma 2.19.** For \([T, D], (T, D)]\) partially adjoint based on \([T, \overline{M}], (T, \overline{N})\],

1. \(N(T, D_B) = \{v \in N(T, N_B) : \langle v, Tw \rangle_Y = 0, \forall w \in D_B\}\),
2. \(R(T, D_B) = \{p \in R(T, M_B) : \langle p, v \rangle_Y = 0, \forall v \in N(T, D_B)\}\).

The same orthogonalities hold for \(N(T, D_B)\) and \(\overline{R(T, D_B)}\).

**Proof.** Given \(v \in N(T, N_B), v \in N(T, D_B)\) if and only if \(\langle v, Tw \rangle_Y = 0\) for any \(w \in D_B\), which implies that \(v \in D\). This proves the item (1).

Since \(R(T, M_B)\) is closed, we decompose it to \(R(T, M_B) = \overline{R(T, D_B)} \oplus \overline{R(T, D_B)}^c\), where the superscript \(^c\) denotes the orthogonal complement in \(R(T, M_B)\). Denote by \(J\) the isomorphism from \(N(T, N_B)\) onto \(R(T, M_B)\), such that \(\langle v, Jw \rangle_Y = \langle v, w \rangle_Y, w \in N(T, N_B)\), \(\forall v \in R(T, M_B)\).

Then given \(v \in N(T, N_B), v \in N(T, D_B)\) if and only if \(Jv \in R(T, D_B)\), thus \(R(J, N(T, D_B)) = \overline{R(T, D_B)}\). Hence, \(R(T, D_B) = \overline{R(J, N(T, D_B))}^c\) if and only if \(Jv \in R(T, D_B)\), which is equivalent to \(\langle p, Jv \rangle_Y = \langle p, v \rangle_Y = 0, \forall v \in N(T, D_B)\). This proves the item (2).

The proof is completed. \(\square\)

**Lemma 2.20.** For \([T, D], (T, D)]\) partially adjoint based on \([T, \overline{M}], (T, \overline{N})\],

1. \(N(T, D) = \{v \in N(T, \overline{M}) : \langle p, v \rangle_X = 0, \forall p \in R(T, D)\}\),
2. \(R(T, D) = \{p \in R(T, \overline{M}) : \langle p, v \rangle_Y = 0, \forall v \in N(T, D)\}\).

**Proof.** Given \(v \in N(T, \overline{M}) \supset N(T, D), v \in N(T, D)\) if and only if \(v \in D\), namely for any \(w \in D, \langle v, Tw \rangle_X = \langle v, Tw \rangle_X - \langle Tv, w \rangle_Y = 0\). This proves the item (1).

For the item (2),

\[
\overline{R(T, D)} = R(T, M) \oplus \overline{R(T, D_B)}
\]

\[
= R(T, M) \oplus \{p \in R(T, M_B) : \langle p, v \rangle_Y = 0, \forall v \in N(T, D_B)\}
\]

\[
= R(T, M) \oplus \{p \in R(T, M_B) : \langle p, v \rangle_Y = 0, \forall v \in N(T, D)\}
\]

\[
= \{p \in R(T, M) : \langle p, v \rangle_Y = 0, \forall v \in N(T, D)\}.
\]
The proof is completed.

2.2.3. **Quantified closed range theorem for partially adjoint operators.**

**Theorem 2.21.** For \( [(T, D), (T, D)] \) partially adjoint based on \( [(T, \tilde{M}), (T, \tilde{N})] \), with notations given in (2.12), (2.13), (2.16), (2.17) and (2.18), if \( \text{icr}(T, D) < \infty \),

\[
\text{icr}(T, D) \leq (1 + \alpha^{-1}) \cdot \text{icr}(T, \tilde{M}) + \alpha^{-1} \text{icr}(T, D) + \text{icr}(T, \tilde{M});
\]

if \( \text{icr}(T, D) < \infty \),

\[
\text{icr}(T, D) \leq (1 + \beta^{-1}) \cdot \text{icr}(T, \tilde{N}) + \beta^{-1} \text{icr}(T, D) + \text{icr}(T, \tilde{N}).
\]

We postpone the technical proof of Theorem 2.21 to Section 2.3.2.

2.2.4. **Horizontal slice isomorphisms for partially adjoint operators.**

**Theorem 2.22** (Horizontal slice isomorphism). For \( [(T, D), (T, \tilde{D})] \) and \( [(T, \tilde{D}), (T, D)] \) partially adjoint based on \( [(T, \tilde{M}), (T, \tilde{N})] \) such that \( D \subset \tilde{D} \) and \( D \subset \tilde{D} \), denote \( dR := \overline{R(T, \tilde{D})} \oplus \overline{R(T, D)} \).

\[
\inf_{0 \neq w \in dR} \sup_{v \in \tilde{N}} \frac{\langle w, v \rangle_{\tilde{N}}}{\|w\|_{\tilde{Y}} \|v\|_{\tilde{Y}}} = \inf_{0 \neq w \in dR} \sup_{v \in \tilde{N}} \frac{\langle w, v \rangle_{\tilde{N}}}{\|w\|_{\tilde{Y}} \|v\|_{\tilde{Y}}} \geq \beta,
\]

and

\[
\inf_{0 \neq w \in dR} \sup_{v \in \tilde{N}} \frac{\langle w, v \rangle_{\tilde{X}}}{\|w\|_{\tilde{X}} \|v\|_{\tilde{X}}} = \inf_{0 \neq w \in dR} \sup_{v \in \tilde{N}} \frac{\langle w, v \rangle_{\tilde{X}}}{\|w\|_{\tilde{X}} \|v\|_{\tilde{X}}} \geq \alpha.
\]

This slice isomorphism is a weak version of the horizontal **P-A-L** type duality. We postpone the technical proof of Theorem 2.22 to Section 2.3.3.

**Theorem 2.23.** (**Helmholtz decomposition**) Assume \( \mathcal{R}(T, M_B) = \mathcal{N}(T, N_B) \) and \( \mathcal{R}(T, N_B) = \mathcal{N}(T, M_B) \). For \( [(T, D), (T, D)] \) any pair of partially adjoint operators based on \( [(T, \tilde{M}), (T, \tilde{N})] \),

\[
\mathcal{R}(T, \tilde{N}) \oplus \mathcal{N}(T, \tilde{M}) = \overline{\mathcal{R}(T, D)} \oplus \mathcal{N}(T, D),
\]

and

\[
\mathcal{R}(T, \tilde{M}) \oplus \mathcal{N}(T, \tilde{N}) = \overline{\mathcal{R}(T, D)} \oplus \mathcal{N}(T, D).
\]

**Proof.** For any pair \( [(T, D), (T, D)] \) partially adjoint based on \( [(T, \tilde{M}), (T, \tilde{N})] \), since \( \mathcal{R}(T, M_B) = \mathcal{N}(T, N_B) \) and \( \mathcal{N}(T, D_B) \subset \mathcal{R}(T, M_B) \), by Lemma 2.19, we have \( \overline{\mathcal{R}(T, D_B)} = \mathcal{R}(T, M_B) \oplus \mathcal{N}(T, D_B) \); namely \( \mathcal{R}(T, M_B) = \overline{\mathcal{R}(T, D_B)} \oplus \mathcal{N}(T, D_B) \). Therefore

\[
\overline{\mathcal{R}(T, D)} \oplus \mathcal{N}(T, D) = \mathcal{R}(T, M) \oplus \overline{\mathcal{R}(T, D_B)} \oplus \mathcal{N}(T, D_B) \oplus \mathcal{N}(T, N) = \mathcal{R}(T, \tilde{M}) \oplus \mathcal{N}(T, N).
\]

Similarly, (2.29) can be proved. The proof is completed. \( \square \)
**Theorem 2.24.** (Horizontal P-A-L duality) Assume \( \mathcal{R}(T, M_B) = \mathcal{N}(T, N_B) \) and \( \mathcal{R}(T, N_B) = \mathcal{N}(T, M_B) \). Then \( dR = dN \) and \( d\mathcal{R} = d\mathcal{N} \).

**Proof.** By Theorem 2.23, note that \( \mathcal{R}(T, D) \oplus ^\perp N(T, D) = \mathcal{R}(T, D) \oplus \perp N(T, D) \), we obtain \( d\mathcal{R} = d\mathcal{N} \). That \( d\mathcal{R} = d\mathcal{N} \) follows similarly. The proof is completed. \( \square \)

2.2.5. **Vertical slice isomorphism for partially adjoint operators.** According to Definition 2.13, let \( [(S, \overline{U}) : \mathcal{Y} \to Z, (S, \overline{V}) : Z \to \mathcal{Y}] \) be a base operator pair with the notations below:

\begin{align}
\mathbf{U} & := \left\{ p \in \overline{U} : \langle p, Sq \rangle_{\mathcal{Y}} - \langle S p, q \rangle_Z = 0, \forall q \in \overline{V} \right\}; \\
\mathbf{V} & := \left\{ q \in \overline{V} : \langle p, Sq \rangle_{\mathcal{Y}} - \langle S p, q \rangle_Z = 0, \forall p \in \overline{U} \right\}; \\
\mathbf{U}_B & := \left\{ p \in \overline{U} : \langle p, q \rangle_{\mathcal{Y}} = 0, \forall q \in \mathcal{N}(S, \mathbf{U}); \langle S p, q \rangle_Z = 0, \forall q \in \mathbf{U} \right\}; \\
\mathbf{V}_B & := \left\{ q \in \overline{V} : \langle q, p \rangle_Z = 0, \forall p \in \mathcal{N}(S, \mathbf{V}); \langle S p, q \rangle_{\mathcal{Y}} = 0, \forall p \in \mathbf{V} \right\}; \\
\kappa & := \inf_{0 \neq w \in \mathcal{N}(S, \mathbf{U}_B)} \sup_{w \in \mathcal{R}(S, \mathbf{V}_B)} \frac{\langle v, w \rangle_{\mathcal{Y}}}{\|v\|_\mathcal{Y} \|w\|_\mathcal{Y}} = \inf_{0 \neq w \in \mathcal{R}(S, \mathbf{V}_B)} \sup_{w \in \mathcal{N}(S, \mathbf{U}_B)} \frac{\langle v, w \rangle_{\mathcal{Y}}}{\|v\|_\mathcal{Y} \|w\|_\mathcal{Y}}; \\
\sigma & := \inf_{0 \neq w \in \mathcal{N}(S, \mathbf{V}_B)} \sup_{w \in \mathcal{R}(S, \mathbf{U}_B)} \frac{\langle v, w \rangle_Z}{\|v\|_Z \|w\|_Z} = \inf_{0 \neq w \in \mathcal{R}(S, \mathbf{U}_B)} \sup_{w \in \mathcal{N}(S, \mathbf{V}_B)} \frac{\langle v, w \rangle_Z}{\|v\|_Z \|w\|_Z}.
\end{align}

Then \( \kappa > 0 \) and \( \sigma > 0 \). Again, we make the convention that,

\begin{equation}
\begin{cases}
\kappa = 1, \quad \text{if } \mathcal{N}(S, \mathbf{U}_B) = \mathcal{R}(S, \mathbf{V}_B) = \{0\}, \\
\sigma = 1, \quad \text{if } \mathcal{N}(S, \mathbf{V}_B) = \mathcal{R}(S, \mathbf{U}_B) = \{0\}.
\end{cases}
\end{equation}

**Assumption 2.25.** \( \mathcal{R}(T, \overline{M}) \subset \mathcal{N}(S, \overline{U}) \) and \( \mathcal{R}(S, \overline{V}) \subset \mathcal{N}(T, \overline{N}) \).

**Lemma 2.26.** Provided Assumption 2.25,

\( \mathcal{R}(T, \overline{M}) \subset \mathcal{N}(S, \overline{U}), \) and \( \mathcal{R}(S, \overline{V}) \subset \mathcal{N}(T, \overline{N}) \).

**Proof.** Given \( \mathbf{p} \in \overline{M}, T \mathbf{p} \in \mathcal{N}(S, \overline{U}) \). For any \( \mathbf{q} \in \overline{V} \), since \( S \mathbf{q} \in \mathcal{N}(T, \overline{N}) \),

\[ \langle T \mathbf{p}, S \mathbf{q} \rangle_{\mathcal{Y}} - \langle S T \mathbf{p}, \mathbf{q} \rangle_Z = \langle T \mathbf{p}, S \mathbf{q} \rangle_{\mathcal{Y}} = \langle T \mathbf{p}, S \mathbf{q} \rangle_{\mathcal{Y}} - \langle \mathbf{p}, T S \mathbf{q} \rangle_X = 0. \]

Thus \( T \mathbf{p} \in \overline{U} \). This proves \( \mathcal{R}(T, \overline{M}) \subset \mathcal{N}(S, \overline{U}) \). The other part is proved the same way. \( \square \)

Provided Assumption 2.25, denote

\( \overline{H} := \mathcal{N}(S, \overline{U}) \ominus^\perp \mathcal{R}(T, \overline{M}), \) \( \overline{H} := \mathcal{N}(S, \overline{U}) \ominus \mathcal{R}(T, \overline{M}), \)

and

\( \overline{\mathcal{H}} := \mathcal{N}(T, \overline{N}) \ominus^\perp \mathcal{R}(S, \overline{V}), \) \( \overline{\mathcal{H}} := \mathcal{N}(T, \overline{N}) \ominus \mathcal{R}(S, \overline{V}). \)
Assumption 2.27. $\mathcal{H}$ and $\mathcal{D}$ are isomorphic, and $\mathcal{H}$ and $\mathcal{D}$ are isomorphic.

Provided Assumption 2.27, denote

$$\chi := \inf_{0 \neq p \in \mathcal{H}} \sup_{\lambda \in \mathcal{D}} \frac{\langle p, \lambda \rangle}{\|p\| \|\lambda\|} = \inf_{0 \neq \lambda \in \mathcal{D}} \sup_{0 \neq p \in \mathcal{H}} \frac{\langle p, \lambda \rangle}{\|p\| \|\lambda\|},$$

and

$$\epsilon := \inf_{0 \neq p \in \mathcal{H}} \sup_{\lambda \in \mathcal{D}} \frac{\langle p, \lambda \rangle}{\|p\| \|\lambda\|} = \inf_{0 \neq \lambda \in \mathcal{D}} \sup_{0 \neq p \in \mathcal{H}} \frac{\langle p, \lambda \rangle}{\|p\| \|\lambda\|}.$$

Then $\chi > 0$ and $\epsilon > 0$. Particularly, we make the convention that

$$\left\{\begin{array}{ll}
\chi = 1, & \text{if } \mathcal{H} = \mathcal{D} = \{0\}, \\
\epsilon = 1, & \text{if } \mathcal{H} = \mathcal{D} = \{0\}.
\end{array}\right.$$

Theorem 2.28 (Complex duality). Let $[(\mathcal{T}, \mathcal{D}), (T, D)]$ and $[(\mathcal{S}, \mathcal{E}), (S, E)]$ be partially adjoint based on $[(T, \mathcal{M}), (T, \mathcal{N})]$ and $[(S, \mathcal{U}), (S, \mathcal{V})]$, respectively. Provided Assumption 2.25,

$$\mathcal{R}(T, D) \subset N(S, \mathcal{E}) \iff \mathcal{R}(S, E) \subset N(T, \mathcal{D}).$$

Proof. Assuming $\mathcal{R}(T, D) \subset N(S, \mathcal{E})$, given $q \in \mathcal{E}$, we obtain $Sq \in N(T, \mathcal{N})$, and $\langle Sq, q \rangle = 0$ for any $q \in \mathcal{R}(T, D) \subset N(S, \mathcal{E})$. Therefore $Sq \in N(T, D)$; namely $\mathcal{R}(S, E) \subset N(T, \mathcal{D})$. The other direction can be obtained the same way. The proof is completed.

For $[(\mathcal{T}, \mathcal{D}), (T, D)]$ and $[(\mathcal{S}, \mathcal{E}), (S, E)]$ which are partially adjoint based on $[(T, \mathcal{M}), (T, \mathcal{N})]$ and $[(S, \mathcal{U}), (S, \mathcal{V})]$, respectively, assuming $\mathcal{R}(S, E) \subset N(T, D)$, which implies $\mathcal{R}(T, D) \subset N(S, \mathcal{E})$ (by Theorem 2.28), denote

$$\mathcal{H} := N(S, \mathcal{E}) \oplus \mathcal{R}(T, D), \quad \text{and} \quad \mathcal{D} := N(T, D) \oplus \mathcal{R}(S, E).$$

Theorem 2.29 (Vertical slice isomorphism). Assume $\mathcal{R}(S, E) \subset N(T, D)$. Provided Assumptions 2.25 and 2.27, $\mathcal{H}$ and $\mathcal{D}$ are isomorphic. Further, if $\mathcal{H} \neq \{0\}$, with notations as (2.17), (2.36), (2.37), (2.38), (2.39), and (2.40),

$$\inf_{0 \neq p \in \mathcal{H}} \sup_{u \in \mathcal{D}} \frac{\langle p, u \rangle}{\|p\| \|u\|} = \inf_{0 \neq \lambda \in \mathcal{D}} \sup_{0 \neq p \in \mathcal{H}} \frac{\langle p, \lambda \rangle}{\|p\| \|\lambda\|} \geq \max \left[ \min(\beta, \chi), \min(\varpi, \epsilon) \right].$$

This is a weak version of the vertical $P-A-L$ type duality. We postpone the proof to Section 2.3.4.

Theorem 2.30 (Vertical $P-A-L$ duality). With the same assumptions as Theorem 2.28 and provided that $\mathcal{R}(T, \mathcal{M}) = N(T, \mathcal{N})$, that $N(S, \mathcal{U}) = \mathcal{R}(S, \mathcal{V})$, and that $\mathcal{H} = \mathcal{D}$, the vertical $P-A-L$ type duality holds that

$$\mathcal{H} = \mathcal{D}.$$
Proof. By Theorem 2.23,
\[
\overline{\mathcal{R}(T, D)} \oplus^+ N(T, D) = \mathcal{R}(T, \tilde{M}) \oplus^+ N(T, \tilde{N}).
\]
Similar to (2.29), we have
\[
\overline{\mathcal{R}(S, E)} \oplus^+ N(S, E) = \mathcal{R}(S, \tilde{N}) \oplus^+ N(S, \tilde{U}).
\]
Note that
\[
\mathcal{R}(T, \tilde{M}) \oplus^+ N(T, \tilde{N}) = \mathcal{R}(T, \tilde{M}) \oplus^+ \tilde{\mathcal{Y}} \oplus^+ \mathcal{R}(S, \mathcal{V}) = \mathcal{R}(T, \tilde{M}) \oplus^+ \tilde{\mathcal{H}} \oplus^+ \mathcal{R}(S, \mathcal{V})
\]
\[
= N(S, \tilde{U}) \oplus^+ \mathcal{R}(S, \mathcal{V}) = N(S, \tilde{U}) \oplus^+ N(S, \tilde{U}_B) \oplus^+ \mathcal{R}(S, \mathcal{V})
\]
\[
= N(S, \tilde{U}) \oplus^+ \mathcal{R}(S, \tilde{V}_B) \oplus^+ \mathcal{R}(S, \mathcal{V}) = N(S, \tilde{U}) \oplus^+ \mathcal{R}(S, \tilde{V}).
\]
Thus
\[
\overline{\mathcal{R}(T, D)} \oplus^+ \tilde{\mathcal{Y}} \oplus^+ \overline{\mathcal{R}(S, E)} = \mathcal{R}(T, D) \oplus^+ N(T, D) = \mathcal{R}(T, \tilde{M}) \oplus^+ N(T, \tilde{N})
\]
\[
= \mathcal{R}(S, \tilde{N}) \oplus^+ N(S, \tilde{U}) = \overline{\mathcal{R}(S, E)} \oplus^+ N(S, E) = \overline{\mathcal{R}(S, E)} \oplus^+ \tilde{\mathcal{H}} \oplus^+ \overline{\mathcal{R}(T, D)}.
\]
The proof of Theorem 2.30 is completed. □

**Theorem 2.31** (Hodge decomposition). With the assumptions of Theorem 2.30, there holds
(2.43) \[
\mathcal{R}(T, \tilde{M}) \oplus^+ N(T, \tilde{N}) = \overline{\mathcal{R}(T, D)} \oplus^+ \tilde{\mathcal{H}}(= \tilde{\mathcal{Y}}) \oplus^+ \overline{\mathcal{R}(S, E)}.
\]

**Remark 2.32.** With the assumptions of Theorem 2.30, we can prove
\[
\mathcal{R}(T, \tilde{M}) \oplus^+ N(T, \tilde{N}) = \mathcal{R}(T, \tilde{M}) \oplus^+ \mathcal{R}(T, \tilde{N}) = \mathcal{R}(S, \tilde{V}) \oplus^+ N(S, \tilde{U}) = \mathcal{R}(S, \mathcal{V}) \oplus^+ N(S, \tilde{U}).
\]
Hence the Hodge decomposition (2.43) has various equivalent presentations.

According to the proof of Theorem 2.30, provided that \(\mathcal{R}(T, \tilde{M}_B) = N(T, \tilde{N}_B)\) and that \(N(S, \tilde{U}_B) = \mathcal{R}(S, \tilde{V}_B)\), we have
\[
\tilde{\mathcal{H}} = \tilde{\mathcal{Y}} \iff \mathcal{H} = \mathcal{Y}.
\]
Indeed, \(\mathcal{R}(T, \tilde{M}) \oplus^+ N(T, \tilde{N}) = \mathcal{R}(T, \tilde{M}) \oplus^+ \tilde{\mathcal{Y}} \oplus^+ \mathcal{R}(S, \tilde{V})\), and \(N(S, \tilde{U}) \oplus^+ \mathcal{R}(S, \tilde{V}) = N(S, \tilde{U}) \oplus^+ \mathcal{R}(S, \tilde{V})\). That \(\tilde{\mathcal{H}} = \tilde{\mathcal{Y}}\) implies \(\mathcal{R}(T, \tilde{M}) \oplus^+ \tilde{\mathcal{Y}} \oplus^+ \mathcal{R}(S, \tilde{V})\), and that \(\mathcal{H} = \mathcal{Y}\) follows. The reverse direction can be shown the same way.

2.3. Some technical proofs.

2.3.1. An elementary lemma for Hilbert spaces.

**Lemma 2.33.** Let \(\mathcal{A}, \mathcal{B}\) and \(\mathcal{B}_0 \subseteq \mathcal{B}\) be closed subspaces of a Hilbert space \(\mathcal{H}\). Assume
\[
\inf_{\alpha \in \mathcal{A}} \sup_{\beta \in \mathcal{B}_0} \frac{\langle \alpha, \beta \rangle_{\mathcal{H}}}{\|\alpha\|_{\mathcal{H}} \cdot \|\beta\|_{\mathcal{H}}} = \gamma > 0.
\]
(1) Denote $A_{\perp B_0} := \{a \in A : \langle a, b \rangle_H = 0, \forall b \in B_0\}$ the polar set of $B_0$ in $A$, and $A_{|| B_0} := \{a \in A : \langle a, a' \rangle_H = 0, \forall a' \in A_{\perp B_0}\}$. Then

\[
(2.44) \quad \inf_{0 \neq b \in B_0} \sup_{a \in A_{|| B_0}} \frac{\langle a, b \rangle_H}{\|a\|_H \|b\|_H} = \inf_{0 \neq a \in A_{|| B_0}} \sup_{b \in B_0} \frac{\langle a, b \rangle_H}{\|a\|_H \|b\|_H} \geq \gamma.
\]

(2) Let $B_0^c$ be the orthogonal complement of $B_0$ in $B$, and $A_{\perp B_0^c} = \{a \in A : \langle a, b \rangle_H = 0, \forall b \in B_0^c\}$. If further

\[
(2.45) \quad \inf_{0 \neq b \in B_0^c} \sup_{a \in A_{\perp B_0^c}} \frac{\langle a, b \rangle_H}{\|a\|_H \|b\|_H} = \inf_{0 \neq a \in A_{\perp B_0^c}} \sup_{b \in B_0^c} \frac{\langle a, b \rangle_H}{\|a\|_H \|b\|_H} \geq \gamma.
\]

Proof. Given $b \in B_0$, there exists an $a' \in A$, such that $\langle a', b \rangle \geq \gamma \|a'\|_H \|b\|_H$. Let $a$ be the projection of $a'$ into $A_{|| B_0}$, then $\langle a, b \rangle_H = \langle a', b \rangle_H$ and $\|a\|_H \leq \|a'\|_H$. Therefore, $\langle a, b \rangle_H \geq \gamma \|a\|_H \|b\|_H$, and

\[
\inf_{0 \neq a \in A_{|| B_0}} \sup_{b \in B_0} \frac{\langle a, b \rangle_H}{\|a\|_H \|b\|_H} \geq \gamma.\]

On the other hand, given $0 \neq a \in A_{|| B_0}$, there exists a $b \in B_0$, such that $\langle a, b \rangle_H \neq 0$. Thus

\[
\inf_{0 \neq b \in B_0} \sup_{a \in A_{\perp B_0}} \frac{\langle a, b \rangle_H}{\|a\|_H \|b\|_H} = \inf_{0 \neq a \in A_{|| B_0}} \sup_{b \in B_0} \frac{\langle a, b \rangle_H}{\|a\|_H \|b\|_H} \geq \gamma.\]

This proves (2.44).

Now we turn to (2.45). Denote $J : A \rightarrow B$ such that $\langle Ja, b' \rangle_H = \langle a, b' \rangle_H, \forall b' \in B$. Then $J$ is an isomorphism and $R(J, A_{\perp B_0^c}) = B_0$. Thus

\[
\inf_{0 \neq b \in B_0} \sup_{a \in A_{\perp B_0^c}} \frac{\langle a, b \rangle_H}{\|a\|_H \|b\|_H} = \inf_{0 \neq a \in A_{\perp B_0^c}} \sup_{b \in B_0} \frac{\langle a, b \rangle_H}{\|a\|_H \|b\|_H} > 0.
\]

Given $a \in A_{\perp B_0}$, there exists a $b' \in B$, such that $\langle a, b' \rangle_H \geq \gamma \|a\|_H \|b'\|_H$. We further set $b$ the projection of $b'$ into $B_0$, then $\langle a, b \rangle_H \geq \gamma \|a\|_H \|b\|_H$. Thus

\[
\inf_{0 \neq a \in A_{\perp B_0}} \sup_{b \in B_0} \frac{\langle a, b \rangle_H}{\|a\|_H \|b\|_H} \geq \gamma.\]

This proves (2.45), and completes the proof of the lemma. \qed

2.3.2. Proof of Theorem 2.21. Recall that $D^\ast := \{v \in D : \langle v, w \rangle_X = 0, \forall w \in N(T(D))\}$. Denote $D_B^\ast := \{v \in D_B : \langle v, w \rangle_X = 0, \forall w \in N(T(D_B))\}$. Then $D_B^\ast = D^\ast \cap D_B$. Given $v \in D_B^\ast$, there exists a unique decomposition $v = v^* + \hat{v}$, such that $v^* \in M_B^\perp$ and $\hat{v} \in N(T, M_B)$. Note that both $v$ and $v^*$ are orthogonal to $N(T, D_B) \subset N(T, M_B)$ in $X$, and thus $\hat{v} \in N(T, M_B) \ominus N(T, D_B)$. Further,

\[
\langle v, T\hat{v} \rangle_X = \langle Tv, \hat{v} \rangle_X = 0, \forall \hat{v} \in D_B^\ast.
\]

By Lemma 2.33(2), with $\overline{R(T,D_B)} = \{p \in R(T,N_B), \langle p, w \rangle_X = 0, \forall w \in N(T(D_B))\}$, we have

\[
\inf_{0 \neq v \in N(T,M_B) \ominus N(T,D_B)} \sup_{p \in \overline{R(T,D_B)}} \frac{\langle v, p \rangle_X}{\|v\|_X \|p\|_X} \geq \inf_{0 \neq v \in N(T,M_B)} \sup_{w \in R(T,N_B)} \frac{\langle v, w \rangle_X}{\|v\|_X \|w\|_X} = \alpha.
\]
Therefore, as $\mathcal{R}(\mathcal{I}, \mathcal{D}_B)$ is closed,

$$\|\hat{v}\|_X \leq \alpha^{-1} \sup_{v \in \mathcal{D}_0^b} \frac{\langle v^*, \mathcal{I}v \rangle_X - \langle Tv^*, v \rangle_Y}{\|\mathcal{I}v\|_X} \leq \alpha^{-1} \|v^*\|_X + \alpha^{-1} \sup_{v \in \mathcal{D}_0^b} \frac{\|v\|_Y \|Tv^*\|_Y}{\|\mathcal{I}v\|_X} = \alpha^{-1} \|v^*\|_X + \alpha^{-1} \text{icr}(\mathcal{I}, \mathcal{D}) \|Tv^*\|_Y.$$  

Then $\|v\|_X \leq \|v^*\|_X + \|\hat{v}\|_X \leq ((1 + \alpha^{-1}) \cdot \text{icr}(T, \mathcal{M}) + \alpha^{-1} \text{icr}(\mathcal{T}, \mathcal{I}) \|Tv\|_Y$. Thus

$$\text{icr}(T, \mathcal{D}_B) \leq (1 + \alpha^{-1}) \cdot \text{icr}(T, \mathcal{M}) + \alpha^{-1} \text{icr}(\mathcal{T}, \mathcal{I}).$$

Now, given $w \in \mathcal{D}^*$, there exists a unique decomposition $w = w_B^* + w_0$, such that $w_B^* \in \mathcal{D}_B^* \subset \mathcal{D}^*$, $Tw_0 \in \mathcal{R}(T, \mathcal{M})$, and $\langle Tw_0^*, Tw_0 \rangle_Y = 0$. Then $\|w_B^*\|_X \leq \text{icr}(T, \mathcal{D}_B) \|Tw_0^*\|_Y$; further, as $w_0 \in \mathcal{D}^*$ and $Tw_0 \in \mathcal{R}(T, \mathcal{M})$, $\|w_0\|_X \leq \text{icr}(T, \mathcal{M}) \|Tw_0\|_Y$. Thus $\|w\|_X \leq \|w_B^*\|_X + \|w_0\|_X \leq \text{icr}(T, \mathcal{D}_B) \|Tw_B^*\|_Y + \text{icr}(T, \mathcal{M}) \|Tw_0\|_Y \leq (\text{icr}(T, \mathcal{D}_B) + \text{icr}(T, \mathcal{M})) \|Tw\|_Y$. Namely,

$$\text{icr}(T, \mathcal{D}) \leq (1 + \alpha^{-1}) \cdot \text{icr}(T, \mathcal{M}) + \alpha^{-1} \text{icr}(\mathcal{T}, \mathcal{D}) + \text{icr}(T, \mathcal{M}).$$

Similarly, can (2.26) be proved. The proof is completed.

2.3.3. Proof of Theorem 2.22. If $\mathcal{N}(T, \mathcal{M}_B) = \{0\} = \mathcal{R}(\mathcal{T}, \mathcal{N}_B)$ or $\mathcal{N}(\mathcal{T}, \mathcal{N}_B) = \{0\} = \mathcal{R}(T, \mathcal{M}_B)$, the assertion is straightforward. We consider the case that $\mathcal{N}(T, \mathcal{M}_B), \mathcal{R}(\mathcal{T}, \mathcal{N}_B), \mathcal{N}(\mathcal{T}, \mathcal{N}_B)$ and $\mathcal{R}(T, \mathcal{M}_B)$ are all nontrivial. Note that $\mathcal{N}(T, \mathcal{D}) = \mathcal{N}(T, \mathcal{M}) \oplus (\mathcal{N}(T, \mathcal{M}_B) \cap \mathcal{D})$, and $\mathcal{N}(\mathcal{T}, \mathcal{D}) = \mathcal{N}(\mathcal{T}, \mathcal{N}) \oplus (\mathcal{N}(\mathcal{T}, \mathcal{N}_B) \cap \mathcal{D})$. Then

$$\mathcal{N}(\mathcal{T}, \mathcal{N}_B) \cap \mathcal{D} = (\mathcal{N}(\mathcal{T}, \mathcal{N}_B) \cap \mathcal{D}) \oplus (\mathcal{N}(\mathcal{T}, \mathcal{D}) \oplus (\mathcal{N}(T, \mathcal{M}_B) \cap \mathcal{D})) = (\mathcal{N}(\mathcal{T}, \mathcal{N}_B) \cap \mathcal{D}) \oplus \mathcal{D}.$$  

Note also that $\mathcal{R}(T, \mathcal{M}) \perp \mathcal{N}(\mathcal{T}, \mathcal{N})$ in $Y$, “$\perp$” for orthogonal. By Theorem 2.20,

$$\mathcal{R}(T, \mathcal{D}) = \mathcal{R}(T, \mathcal{M}) \oplus \{v \in \mathcal{R}(T, \mathcal{M}_B) : v \perp (\mathcal{N}(\mathcal{T}, \mathcal{N}_B) \cap \mathcal{D}) \text{ in } Y\} =: \mathcal{R}(T, \mathcal{M}) \oplus A_0.$$  

Similarly

$$\mathcal{R}(T, \mathcal{D}) = \mathcal{R}(T, \mathcal{M}) \oplus \{v \in \mathcal{R}(T, \mathcal{M}_B) : v \perp (\mathcal{N}(\mathcal{T}, \mathcal{N}_B) \cap \mathcal{D}) \text{ in } Y\} =: \mathcal{R}(T, \mathcal{M}) \oplus A_s.$$  

By Lemma 2.33(2), we can show

$$\inf_{0 \neq w \in \mathcal{N}(\mathcal{T}, \mathcal{N}_B) \cap \mathcal{D}} \sup_{v \in \mathcal{A}_0} \frac{\langle w, w \rangle_Y}{\|w\|_Y} \geq \inf_{0 \neq w \in \mathcal{N}(\mathcal{T}, \mathcal{N}_B) \cap \mathcal{D}} \frac{\|w\|_Y}{\|w\|_Y} = \beta.$$  

Therefore, as $\mathcal{D}N = (\mathcal{N}(\mathcal{T}, \mathcal{N}_B) \cap \mathcal{D}) \ominus (\mathcal{N}(T, \mathcal{M}_B) \cap \mathcal{D})$, $\inf_{0 \neq w \in \mathcal{D}N} \sup_{v \in \mathcal{A}_0} \frac{\langle w, w \rangle_Y}{\|w\|_Y} \geq \beta.$

Then by Lemma 2.33(1), with $\mathcal{D}R = A_0 \ominus A_s$ and $A_s = \{v \in \mathcal{A}_0 : v \text{ is orthogonal to } \mathcal{D}N\}$,

$$\inf_{0 \neq w \in \mathcal{D}N} \sup_{v \in \mathcal{R}} \frac{\langle w, w \rangle_Y}{\|w\|_Y} = \inf_{0 \neq w \in \mathcal{D}N} \sup_{v \in \mathcal{A}_0} \frac{\langle w, w \rangle_Y}{\|w\|_Y} \geq \inf_{0 \neq w \in \mathcal{D}N} \sup_{v \in \mathcal{A}_0} \frac{\langle w, w \rangle_Y}{\|w\|_Y} \geq \beta.$$  

This proves (2.27). If $\mathcal{D}N = \{0\}$, then easily we know $\mathcal{D}R = \{0\}$, and vice versa. (2.28) can be proved the same way. The proof is completed. $\square$
2.3.4. Proof of Theorem 2.29.

Lemma 2.34. Provided Assumptions 2.25 and 2.27,

\[
\inf_{0 \neq q \in N(S, \mathbb{U})} \sup_{v \in N(T, N_B) \cup R(S, Y)} \frac{\langle q, v \rangle_Y}{\|q\|_Y \|v\|_Y} = \inf_{0 \neq q \in N(S, \mathbb{U})} \sup_{v \in N(T, N_B) \cup R(S, Y)} \frac{\langle q, v \rangle_Y}{\|q\|_Y \|v\|_Y} \geq \max \left[ \min(\beta, \chi), \min(\sigma, \epsilon) \right].
\]

Proof. Direct calculation leads to that

\[
N(T, \mathbb{N}) \oplus^+ \mathcal{R}(S, Y) = N(T, N_B) \oplus^+ \tilde{\mathcal{S}} = \tilde{\mathcal{S}} \oplus^+ \mathcal{R}(S, V_B)
\]

and

\[
N(S, \mathbb{U}) \oplus^+ \mathcal{R}(T, M) = \tilde{\mathcal{H}} \oplus^+ \mathcal{R}(T, M_B) = N(S, \mathbb{U}_B) \oplus^+ \mathcal{H}.
\]

If \( \tilde{\mathcal{H}} = \tilde{\mathcal{S}} = \{0\} \), then \( \chi = 1 \), and

\[
\inf_{0 \neq q \in N(S, \mathbb{U})} \sup_{v \in N(T, N_B) \cup R(S, Y)} \frac{\langle q, v \rangle_Y}{\|q\|_Y \|v\|_Y} = \beta = \min(\chi, \beta).
\]

Otherwise, given \( q_1 \in N(S, \mathbb{U}) \oplus^+ \mathcal{R}(T, M) \), there exists a unique decomposition \( q_1 = q_1^1 + q_1^2 \), such that \( q_1^2 \in \tilde{\mathcal{H}} \) and \( q_1^1 \in \mathcal{R}(T, M_B) \); \( \|q_1^1\|_Y = \|q_1\|_Y^2 + \|q_1^2\|_Y^2 \). Then, by Assumption 2.27, there exist \( v_2 \in \tilde{\mathcal{S}} \), such that \( \langle q_1^2, v_2 \rangle_Y = \|q_1^2\|_Y^2 \), and \( \|v_2\|_Y \leq \chi^{-1}\|q_1^2\|_Y \); further, with \( \beta \) given in (2.17), let \( v_1 \in N(T, N_B) \) be such that \( \langle q_1^1, v_1 \rangle_Y = \|q_1^1\|_Y^2 \), and \( \|v_1\|_Y \leq \beta^{-1}\|q_1^1\|_Y \). Set \( v = v_1 + v_2 \in N(T, \mathbb{N}) \oplus^+ \mathcal{R}(S, Y) \), then \( \langle q_1, v \rangle_Y = \langle q_1, v_1 \rangle_Y \), and \( \|v\|_Y^2 \leq \max(\chi^{-1}, \beta^{-1})^2\|q_1\|_Y^2 \). Hence,

\[
\inf_{0 \neq q \in N(S, \mathbb{U})} \sup_{v \in N(T, N_B) \cup R(S, Y)} \frac{\langle q, v \rangle_Y}{\|q\|_Y \|v\|_Y} \geq \min(\chi, \beta).
\]

Similarly, we can prove

\[
\inf_{0 \neq q \in N(S, \mathbb{U})} \sup_{v \in N(T, N_B) \cup R(S, Y)} \frac{\langle q, v \rangle_Y}{\|q\|_Y \|v\|_Y} \geq \min(\epsilon, \sigma).
\]

The proof is completed. \( \square \)

Proof of Theorem 2.29. With the structure of \( \tilde{V} \),

\[
\mathcal{R}(S, \mathbb{E}) = \mathcal{R}(S, Y) \oplus^+ \mathcal{R}(S, V_B \cap \mathbb{E})
\]

and

\[
N(T, \mathbb{D}) = N(T, \mathbb{N}) \oplus^+ \left[ N(T, N_B) \cap \mathbb{D} \right] = \mathcal{R}(S, Y) \oplus^+ \tilde{\mathcal{S}} \oplus^+ \left( N(T, N_B) \cap \mathbb{D} \right).
\]

Therefore,

\[
\tilde{\mathcal{S}} = N(T, \mathbb{D}) \oplus^+ \mathcal{R}(S, \mathbb{E}) := A_1 \oplus A_2,
\]

where

\[
A_1 = \tilde{\mathcal{S}} \oplus^+ \left[ N(T, N_B) \cap \mathbb{D} \right], \text{ and } A_2 = \mathcal{R}(S, \tilde{V}_B \cap \mathbb{E}).
\]
By Lemma 2.20, noting that \( \mathcal{N}(\mathbf{S}, \overline{\mathbf{U}}) = \mathcal{R}(\mathbf{T}, \mathbf{M}) \oplus \mathcal{R}(\mathbf{T}, \mathbf{M}_B) \oplus \overline{\mathcal{H}} \subset \mathcal{Y} \), and that \( \mathcal{R}(\mathbf{T}, \mathbf{M}) \) is orthogonal to \( \mathcal{N}(\mathbf{T}, \mathbf{N}) \supset \mathcal{R}(\mathbf{S}, \overline{\mathbf{V}}) \), we have
\[
\mathcal{N}(\mathbf{S}, \mathbf{E}) = \left\{ q_1 \in \mathcal{N}(\mathbf{S}, \overline{\mathbf{U}}) : \langle q_1, \mathbf{v} \rangle_{\mathcal{Y}} = 0, \forall \mathbf{v} \in \overline{\mathcal{R}(\mathbf{S}, \mathbf{E})} = \mathcal{R}(\mathbf{S}, \mathbf{V}) \oplus A_2 \right\}
= \mathcal{R}(\mathbf{T}, \mathbf{M}) \oplus A_1 \left\{ q_1 \in \mathcal{R}(\mathbf{T}, \mathbf{M}_B) : \langle q_1, \mathbf{v} \rangle_{\mathcal{Y}} = 0, \forall \mathbf{v} \in A_2 \right\}.
\]

Meanwhile, noting that \( \mathcal{N}(\mathbf{T}, \mathbf{N}) = \overline{\mathcal{R}(\mathbf{S}, \overline{\mathbf{V}})} \oplus \mathfrak{d} \),
\[
\overline{\mathcal{R}(\mathbf{T}, \mathbf{D})} = \left\{ q_1 \in \mathcal{R}(\mathbf{T}, \overline{\mathbf{M}}) : \langle q_1, \mathbf{v} \rangle_{\mathcal{Y}} = 0, \forall \mathbf{v} \in \mathcal{N}(\mathbf{T}, \mathbf{D}) = \mathcal{R}(\mathbf{S}, \mathbf{V}) \oplus A_1 \right\}
= \mathcal{R}(\mathbf{T}, \mathbf{M}) \oplus \left\{ q_1 \in \mathcal{R}(\mathbf{T}, \mathbf{M}_B) : \langle q_1, \mathbf{v} \rangle_{\mathcal{Y}} = 0, \forall \mathbf{v} \in A_1 \right\}.
\]

Therefore,
\[
\mathcal{H} = \mathcal{N}(\mathbf{S}, \mathbf{E}) \oplus \overline{\mathcal{R}(\mathbf{T}, \mathbf{D})} = \mathcal{B}_1 \oplus \mathcal{B}_2,
\]
where
\[
\mathcal{B}_1 = \left\{ q_1 \in \mathcal{R}(\mathbf{T}, \mathbf{M}_B) : \langle q_1, \mathbf{v} \rangle_{\mathcal{Y}} = 0, \forall \mathbf{v} \in A_1 \right\}
\]
and
\[
\mathcal{B}_2 = \left\{ q_1 \in \mathcal{R}(\mathbf{T}, \mathbf{M}_B) : \langle q_1, \mathbf{v} \rangle_{\mathcal{Y}} = 0, \forall \mathbf{v} \in A_2 \right\}.
\]
Denote
\[
\mathcal{B}_{1.5} := \left\{ q_1 \in \mathcal{R}(\mathbf{T}, \mathbf{M}_B) \oplus \overline{\mathcal{H}} : \langle q_1, \mathbf{v} \rangle_{\mathcal{Y}} = 0, \forall \mathbf{v} \in A_1 \right\}.
\]
Then \( \mathcal{B}_2 \subset \mathcal{B}_{1.5} \subset \mathcal{B}_1 \). Further, \( \mathcal{B}_2 = \left\{ q_1 \in \mathcal{R}(\mathbf{T}, \mathbf{M}_B) : \langle q_1, \mathbf{v} \rangle_{\mathcal{Y}} = 0, \forall \mathbf{v} \in \mathcal{N}(\mathbf{T}, \mathbf{N}_B) \cap \mathbf{D} \right\} \), and it can be verified by Assumption 2.27 that \( \mathcal{B}_{1.5} = \mathcal{B}_2 \). Indeed, given \( q_1 \in \mathcal{B}_{1.5} \), there exists a unique decomposition \( q_1 = q_1 + q_2 \) with \( q_1 \in \mathcal{R}(\mathbf{T}, \mathbf{M}_B) \) and \( q_2 \in \overline{\mathcal{H}} \). Then, for any \( \mathbf{v} \in \mathfrak{d} \),
\[
\langle q_1, \mathbf{v} \rangle_{\mathcal{Y}} = \langle q_2, \mathbf{v} \rangle_{\mathcal{Y}} = 0 \text{ and thus } q_1 \in \mathcal{R}(\mathbf{T}, \mathbf{M}_B). \] This way \( \mathcal{B}_{1.5} \subset \mathcal{B}_2 \) and \( \mathcal{B}_{1.5} = \mathcal{B}_2 \). Hence
\[
\mathcal{H} = \left\{ q_1 \in \mathcal{R}(\mathbf{T}, \mathbf{M}_B) : \langle q_1, \mathbf{v} \rangle_{\mathcal{Y}} = 0, \forall \mathbf{v} \in A_2 \right\}
\]
\[
\oplus \left\{ q_1 \in \mathcal{R}(\mathbf{T}, \mathbf{M}_B) : \langle q_1, \mathbf{v} \rangle_{\mathcal{Y}} = 0, \forall \mathbf{v} \in A_1 \right\} := \mathcal{C}_1 \oplus \mathcal{C}_2.
\]
Therefore, that \( \mathfrak{d} = \{0\} \) implies that \( \mathcal{H} = \{0\} \). Similarly, that \( \mathcal{H} = \{0\} \) implies that \( \mathfrak{d} = \{0\} \).
By Lemma 2.33(2),
\[
\inf\limits_{0 \neq \mathbf{v} \in \mathcal{N}(\mathbf{T}, \mathbf{N}_B) \oplus \mathfrak{d} \oplus A_2} \sup_{\mathbf{w} \in \mathcal{C}_1} \frac{\langle \mathbf{v}, \mathbf{w} \rangle_{\mathcal{Y}}}{\|\|\| \mathbf{w} \|\|\|} \geq \inf\limits_{0 \neq \mathbf{v} \in \mathcal{N}(\mathbf{T}, \mathbf{N}_B) \oplus \mathfrak{d}} \sup_{q_1 \in \mathcal{R}(\mathbf{T}, \mathbf{M}_B) \oplus \overline{\mathcal{H}}} \frac{\langle q_1, \mathbf{v} \rangle_{\mathcal{Y}}}{\|q_1\| \|\|\| \mathbf{v} \|\|\|},
\]
As \( \mathcal{C}_2 = \left\{ q_1 \in \mathcal{C}_1 : \langle q_1, \mathbf{v} \rangle_{\mathcal{Y}} = 0, \forall \mathbf{v} \in \mathfrak{d} \right\} \) and \( \mathfrak{d} \subset \mathcal{N}(\mathbf{T}, \mathbf{N}_B) \oplus \mathfrak{d} \oplus A_2 \), by Lemma 2.33(1),
\[
\inf\limits_{0 \neq q_1 \in \mathcal{C}_1} \sup\limits_{\mathbf{v} \in \mathfrak{d}} \frac{\langle q_1, \mathbf{v} \rangle_{\mathcal{Y}}}{\|q_1\| \|\|\| \mathbf{v} \|\|\|} = \inf\limits_{0 \neq q_1 \in \mathcal{C}_1} \sup\limits_{q_1 \in \mathcal{R}(\mathbf{T}, \mathbf{M}_B) \oplus \overline{\mathcal{H}}} \frac{\langle q_1, \mathbf{v} \rangle_{\mathcal{Y}}}{\|q_1\| \|\|\| \mathbf{v} \|\|\|} = \inf\limits_{0 \neq q_1 \in \mathcal{C}_1} \sup\limits_{q_1 \in \mathcal{R}(\mathbf{T}, \mathbf{M}_B) \oplus \overline{\mathcal{H}}} \frac{\langle q_1, \mathbf{v} \rangle_{\mathcal{Y}}}{\|q_1\| \|\|\| \mathbf{v} \|\|\|} \geq \inf\limits_{0 \neq \mathbf{v} \in \mathcal{N}(\mathbf{T}, \mathbf{N}_B) \oplus \mathfrak{d}} \sup\limits_{q_1 \in \mathcal{R}(\mathbf{T}, \mathbf{M}_B) \oplus \overline{\mathcal{H}}} \frac{\langle q_1, \mathbf{v} \rangle_{\mathcal{Y}}}{\|q_1\| \|\|\| \mathbf{v} \|\|\|}.
\]
The proof is then completed by Lemma 2.34. \qed
3. Partially adjoint discretizations of adjoint operators

Let \( \Omega \) be a domain. For \( \Xi \) a subdomain of \( \Omega \), we denote by \( R_{\Omega}^{\Xi} \) and \( E_{\Xi}^{\Omega} \) the restriction to and extension from \( L^1_{\text{loc}}(\Xi) \) from and to \( L^1_{\text{loc}}(\Omega) \), the spaces of locally integrable functions, respectively. Namely,

\[
R_{\Omega}^\Xi : L^1_{\text{loc}}(\Omega) \to L^1_{\text{loc}}(\Xi), \quad R_{\Omega}^\Xi v = v|_{\Xi}, \quad \text{for } v \in L^1_{\text{loc}}(\Omega),
\]

and

\[
E_{\Xi}^\Omega : L^1_{\text{loc}}(\Xi) \to L^1_{\text{loc}}(\Omega), \quad E_{\Xi}^\Omega v = \begin{cases} v, & \text{on } \Xi, \\ 0, & \text{else,} \end{cases} \quad \text{for } v \in L^1_{\text{loc}}(\Xi).
\]

We use the same notation \( L^1_{\text{loc}} \) for both scalar and non-scalar locally integrable functions, and use the same notation \( R_{\Omega}^\Xi \) and \( E_{\Xi}^\Omega \) for both scalar and non-scalar functions.

Let \( X, \, Y \) and \( Z \) be three Hilbert spaces that each consists of locally integrable functions defined on \( \Omega \). Assume,

\[
(3.1) \quad E_{\Xi}^\Omega R_{\Omega}^\Xi v \in X, \quad \forall v \in X, \quad \text{for any subdomain } \Xi \subset \Omega.
\]

Denote \( X_{\Xi} := R(R_{\Omega}^\Xi, X) \). For \( v, w \in X_{\Xi} \), define an inner product \( \langle v, w \rangle_{X_{\Xi}} := \langle E_{\Xi}^\Omega v, E_{\Xi}^\Omega w \rangle_X \). Similar assumptions are imposed for \( Y \) and \( Z \), and spaces \( Y_{\Xi} \) and \( Z_{\Xi} \) and inner products thereon are defined the same way.

Let \( T \) be a linear operator that maps locally integrable functions compatible to \( X \) to locally integrable functions compatible to \( Y \). We assume the operation of \( T \) is local, namely, for \( v, v' \) locally integrable,

\[
R_{\Xi}^\Omega v = R_{\Xi}^\Omega v' \implies R_{\Xi}^\Omega (Tv) = R_{\Xi}^\Omega (Tv'), \quad \text{for any subdomain } \Xi \subset \Omega.
\]

Similarly, \( T \) is a linear operator that maps locally integrable functions compatible to \( Y \) to locally integrable functions compatible to \( X \), and its operation is local. For a pair of closed densely defined adjoint operators \( \{T, D\} : X \to Y, \{(T, D) : Y \to X\} \), this section considers its in-pair discretizations. Particularly, we use a conforming discretization (CD) for one operator and an accompanied-by-conforming discretization (ABCD), constructed below, for the other to construct a pair of partially adjoint discretizations for the adjoint operators.

3.1. Partially adjoint operators on subdivisions and partially adjoint discretizations.

3.1.1. Tensorial base pair and partially adjoint discretized operators. Let \( G \) be a subdivision of \( \Omega \) to cells. Define formally the product of a set of function spaces \( \{\Upsilon(K)\}_{K \in G} \) defined cell by cell such that \( E_{K}^{\Omega} \Upsilon(K) := \mathcal{R}(E_{K}^{\Omega}, \Upsilon(K)) \) for all \( K \in G \) are compatible,

\[
\prod_{K \in G} \Upsilon(K) := \sum_{K \in G} E_{K}^{\Omega} \Upsilon(K),
\]

and the summation is direct. Then \( \nu_{\Omega} \in \prod_{K \in G} \Upsilon(K) \) if and only if there exist (uniquely) a set of functions \( \{\nu_{K}\}_{K \in G} \), with \( \nu_{K} \in \Upsilon(K) \) for any \( K \), such that \( \nu_{\Omega} = \sum_{K \in G} E_{K}^{\Omega} \nu_{K} \). Further, it holds that

\[
X = \prod_{K \in G} X_{K}, \quad Y = \prod_{K \in G} Y_{K}, \quad \text{and } Z = \prod_{K \in G} Z_{K}.
\]
On every $K \in \mathcal{G}$, let $T_K$ and $T_K \in \mathcal{G}$ be two operators that coincide with the local operation of $T$ and $T$ on $K$, respectively, in the sense that 

$$T_K R^K_{\Omega} = R^K_{\Omega} T_K, \quad \text{and} \quad T_K R^K_{\Omega} = R^K_{\Omega} T_K.$$ 

Note again that the same notation $R^K_{\Omega}$ can take different meanings at different places.

Define formally the product of operators $T_\mathcal{G} = \prod_{K \in \mathcal{G}} T_K$ by 

$$T_\mathcal{G} : X \rightarrow Y, \quad T_\mathcal{G} \left( \sum_{K \in \mathcal{G}} E^K_{\mathcal{Q}} v_K \right) = \sum_{K \in \mathcal{G}} \left( E^K_{\mathcal{Q}} T_K v_K \right), \quad \text{with} \ v_K \in X_K, \ \forall K \in \mathcal{G}.$$ 

Then we call $T_\mathcal{G}$ a discretization of $T$. Evidently, $T_\mathcal{G} v = T v$, for $v \in H(T : X \rightarrow Y)$.

We similarly define $T_\mathcal{G} = \prod_{K \in \mathcal{G}} T_K$ and $T_\mathcal{G}$ is a discretization of $T$ on $\mathcal{G}$. Discretizations of $S$ and $S$, say $S_\mathcal{G}$ and $S_\mathcal{G}$, are constructed the same way.

On every $K \in \mathcal{G}$, let there be two finite-dimensional shape function spaces $P(K) \subset H(T_K : X_K \rightarrow Y_K)$ and $P(K) \subset H(T_K : Y_K \rightarrow X_K)$ constructed. Define 

$$\overline{P}(\mathcal{G}) := \prod_{K \in \mathcal{G}} P(K), \quad \text{and} \quad \overline{P}(\mathcal{G}) := \prod_{K \in \mathcal{G}} P(K).$$

Denote 

$$P(\mathcal{G}) := \left\{ v \in \overline{P}(\mathcal{G}) : \langle v, T_\mathcal{G} w \rangle_X = 0, \ \forall w \in \overline{P}(\mathcal{G}) \right\},$$

$$P(\mathcal{G}) := \left\{ v \in \overline{P}(\mathcal{G}) : \langle v, T_\mathcal{G} w \rangle_Y = 0, \ \forall w \in \overline{P}(\mathcal{G}) \right\},$$

$$P_B(\mathcal{G}) := \left\{ v \in \overline{P}(\mathcal{G}) : \langle v, w \rangle_X = 0, \ \forall w \in N(T_\mathcal{G}, P(\mathcal{G})); \langle T_\mathcal{G} v, T_\mathcal{G} w \rangle_Y = 0, \ \forall w \in P(\mathcal{G}) \right\},$$

and 

$$P_B(\mathcal{G}) := \left\{ v \in \overline{P}(\mathcal{G}) : \langle v, w \rangle_Y = 0, \ \forall w \in N(T_\mathcal{G}, P(\mathcal{G})); \langle T_\mathcal{G} v, T_\mathcal{G} w \rangle_X = 0, \ \forall w \in P(\mathcal{G}) \right\}.$$ 

**Assumption 3.1.** $N(T_\mathcal{G}, P_B(\mathcal{G}))$ and $R(T_\mathcal{G}, P_B(\mathcal{G}))$ are isomorphic; $N(T_\mathcal{G}, P_B(\mathcal{G}))$ and $R(T_\mathcal{G}, P_B(\mathcal{G}))$ are isomorphic.

Provided Assumption 3.1, denote 

$$\alpha_\mathcal{G} := \inf_{0 \neq v \in N(T_\mathcal{G}, P_B(\mathcal{G}))} \sup_{w \in R(T_\mathcal{G}, P_B(\mathcal{G}))} \frac{\langle v, w \rangle_X}{\|v\|_X \|w\|_X} = \inf_{0 \neq w \in R(T_\mathcal{G}, P_B(\mathcal{G}))} \sup_{v \in N(T_\mathcal{G}, P_B(\mathcal{G}))} \frac{\langle v, w \rangle_X}{\|v\|_X \|w\|_X}$$

and 

$$\beta_\mathcal{G} := \inf_{0 \neq v \in N(T_\mathcal{G}, P_B(\mathcal{G}))} \sup_{w \in R(T_\mathcal{G}, P_B(\mathcal{G}))} \frac{\langle v, w \rangle_Y}{\|v\|_Y \|w\|_Y} = \inf_{0 \neq w \in R(T_\mathcal{G}, P_B(\mathcal{G}))} \sup_{v \in N(T_\mathcal{G}, P_B(\mathcal{G}))} \frac{\langle v, w \rangle_Y}{\|v\|_Y \|w\|_Y}.$$ 

Then $\alpha_\mathcal{G} > 0$ and $\beta_\mathcal{G} > 0$.

**Remark 3.2.** In this section, we will not discuss in detail the trivial cases such as $N(T_\mathcal{G}, P_B(\mathcal{G})) = \{0\}$ and $R(T_\mathcal{G}, P_B(\mathcal{G})) = \{0\}$, as the relevant discussions are straightforward. Similar trivial cases will be omitted for Assumptions 3.4 and 3.6.

**Lemma 3.3.** Provided Assumption 3.1, $\left[ (T_\mathcal{G}, \overline{P}(\mathcal{G})) : X \rightarrow Y, (T_\mathcal{G}, \overline{P}(\mathcal{G})) : Y \rightarrow X \right]$ is a base operator pair.
All spaces discussed in this section are of finite-dimension, and the closedness can easily be verified. The base operator pair constructed this way is called a tensorial base pair.

Also, on every $K$, let there be $Q(K) \subset H(S_K : \mathcal{Y}_K \rightarrow Z_K)$ and $Q(K) \subset H(S_K : Z_K \rightarrow \mathcal{Y}_K)$ constructed. Define

$$\tilde{Q}(G) := \prod_{K \in G} Q(K),$$

and

$$\tilde{Q}(G) := \prod_{K \in G} Q(K).$$

We similarly introduce the notation $\tilde{Q}(G), Q(G), Q_B(G)$ and $Q_B(G)$ with respect to $S_G$ and $S_{\tilde{G}}$.

**Assumption 3.4.** $N(S_G, Q_B(G))$ and $R(S_G, Q_B(G))$ are isomorphic; $N(S_G, Q_B(G))$ and $R(S_G, Q_B(G))$ are isomorphic.

Provided Assumption 3.4, denote

$$\omega_G := \inf_{0 \neq p \in N(S_G, Q_B(G))} \sup_{q \in R(S_G, Q_B(G))} \frac{\langle p, q \rangle_{\mathcal{Y}}}{\|p\|_\mathcal{Y} \|q\|_\mathcal{Y}} = \inf_{0 \neq q \in R(S_G, Q_B(G))} \sup_{p \in N(S_G, Q_B(G))} \frac{\langle p, q \rangle_{\mathcal{Y}}}{\|p\|_\mathcal{Y} \|q\|_\mathcal{Y}} > 0,$$

and

$$\kappa_G := \inf_{0 \neq p \in N(S_G, Q_B(G))} \sup_{q \in R(S_G, Q_B(G))} \frac{\langle p, q \rangle_{Z}}{\|p\|_Z \|q\|_Z} = \inf_{0 \neq q \in R(S_G, Q_B(G))} \sup_{p \in N(S_G, Q_B(G))} \frac{\langle p, q \rangle_{Z}}{\|p\|_Z \|q\|_Z} > 0.$$

Provided Assumption 3.4, $[(S_G, \tilde{Q}(G)) : \mathcal{Y} \rightarrow Z, (S_G, \tilde{Q}(G)) : Z \rightarrow \mathcal{Y}]$ is a base operator pair.

**Assumption 3.5.** $R(T_G, \tilde{P}(G)) \subset N(S_G, \tilde{Q}(G))$, and $R(S_G, \tilde{Q}(G)) \subset N(T_G, \tilde{P}(G))$.

Provided Assumption 3.5, by Lemma 2.26, $R(T_G, P(G)) \subset N(S_G, Q(G))$, and, $R(S_G, Q(G)) \subset N(T_G, P(G))$. Denote,

$$\tilde{H}_G := N(S_G, \tilde{Q}(G)) \ominus R(T_G, \tilde{P}(G)), \quad \tilde{H}_G := N(S_G, Q(G)) \ominus R(T_G, P(G)),$$

and

$$\tilde{\delta}_G := N(T_G, \tilde{P}(G)) \ominus R(S_G, \tilde{Q}(G)), \quad \tilde{\delta}_G := N(T_G, P(G)) \ominus R(S_G, Q(G)).$$

**Assumption 3.6.** $\tilde{H}_G$ and $\tilde{\delta}_G$ are isomorphic; $\tilde{H}_G$ and $\tilde{\delta}_G$ are isomorphic.

Particularly, denote

$$\chi_G := \inf_{0 \neq p \in \tilde{H}_G} \sup_{v \in \tilde{\delta}_G} \frac{\langle p, v \rangle_{\mathcal{Y}}}{\|p\|_\mathcal{Y} \|v\|_\mathcal{Y}} = \inf_{0 \neq v \in \tilde{\delta}_G} \sup_{p \in \tilde{H}_G} \frac{\langle p, v \rangle_{\mathcal{Y}}}{\|p\|_\mathcal{Y} \|v\|_\mathcal{Y}} > 0$$

and

$$\epsilon_G := \inf_{0 \neq p \in \tilde{H}_G} \sup_{v \in \tilde{\delta}_G} \frac{\langle p, v \rangle_{\mathcal{Y}}}{\|p\|_\mathcal{Y} \|v\|_\mathcal{Y}} = \inf_{0 \neq v \in \tilde{\delta}_G} \sup_{p \in \tilde{H}_G} \frac{\langle p, v \rangle_{\mathcal{Y}}}{\|p\|_\mathcal{Y} \|v\|_\mathcal{Y}} > 0.$$
3.1.2. Adjoint properties of partially adjoint operators. We have immediately the properties below. They are each a correspondence of the results in Section 2.2.

**Theorem 3.7.** For \([(T_g, D_g), (T_g, D_g)]\) partially adjoint based on \([(T_g, \tilde{P}(G)), (T_g, \tilde{P}(G))]\),

\[
icr(T_g, D_g) \leq (1 + \alpha_g^{-1}) \cdot \nicr(T_g, \tilde{P}(G)) + \alpha_g^{-1} \nicr(T_g, D_g) + \nicr(T_g, P(G)).
\]

and

\[
icr(T_g, D_g) \leq (1 + \beta_g^{-1}) \cdot \nicr(T_g, \tilde{P}(G)) + \beta_g^{-1} \nicr(T_g, D_g) + \nicr(T_g, P(G)).
\]

**Theorem 3.8** (Horizontal slice isomorphism). Let \([(T_g, D_g), (T_g, D_g)]\) be partially adjoint based on \([(T_g, \tilde{P}(G)), (T_g, \tilde{P}(G))]\), such that \(D_g \subset \tilde{D}_g\) and \(D_g \subset \tilde{D}_g\). Denote \(dR_g := \mathcal{R}(T_g, \tilde{D}_g) \oplus \mathcal{R}(T_g, D_g),\) \(dR_g := \mathcal{R}(T_g, \tilde{D}_g) \oplus \mathcal{R}(T_g, D_g),\) \(dN_g := N(T_g, D_g) \oplus N(T_g, \tilde{D}_g)\) and \(dN_g := N(T_g, \tilde{D}_g) \oplus N(T_g, D_g)\). Then

\[
\inf_{0 \neq \omega \in dR_g} \sup_{\nu \in dN_g} \frac{\langle w, \nu \rangle_X}{\|w\|_X \|\nu\|_X} = \inf_{0 \neq \omega \in dN_g} \sup_{\nu \in dR_g} \frac{\langle w, \nu \rangle_X}{\|w\|_X \|\nu\|_X} \geq \beta_g > 0,
\]

and

\[
\inf_{0 \neq \omega \in dR_g} \sup_{\nu \in dN_g} \frac{\langle w, \nu \rangle_Y}{\|w\|_Y \|\nu\|_Y} = \inf_{0 \neq \omega \in dN_g} \sup_{\nu \in dR_g} \frac{\langle w, \nu \rangle_Y}{\|w\|_Y \|\nu\|_Y} \geq \alpha_g > 0.
\]

If, further, \(N(T_g, P_B(G)) = \mathcal{R}(T_g, P_B(G))\) and \(N(T_g, \tilde{P}_B(G)) = \mathcal{R}(T_g, P_B(G))\), then

1. (Horizontal P-A-L type duality)

\(dR_g = dN_g,\) and \(dR_g = dN_g;\)

(2) (Helmholtz decomposition) for \((T_g, D_g)\) and \((T_g, D_g)\) any pair of partially adjoint operators based on \([(T_g, \tilde{P}(G)), (T_g, \tilde{P}(G))]\),

\(\mathcal{R}(T_g, \tilde{P}(G)) \oplus N(T_g, P(G)) = \mathcal{R}(T_g, D_g) \oplus N(T_g, D_g),\)

and

\(\mathcal{R}(T_g, \tilde{P}(G)) \oplus N(T_g, P(G)) = \mathcal{R}(T_g, D_g) \oplus N(T_g, \tilde{D}_g).\)

For \([(S_g, E_g), (S_g, E_g)]\) partially adjoint based on \([(S_g, \tilde{Q}(G)), (S_g, \tilde{Q}(G))\)], assuming \(\mathcal{R}(S_g, E_g) \subset N(T_g, D_g),\) which implies \(\mathcal{R}(T_g, D_g) \subset N(S_g, E_g)\) (cf. Theorem 2.28), define

\(H_g := N(S_g, E_g) \oplus \mathcal{R}(T_g, D_g),\) and \(S_g := N(T_g, D_g) \oplus \mathcal{R}(S_g, E_g).\)

**Theorem 3.9** (Vertical slice isomorphism). Provided Assumptions 3.5 and 3.6, if \(\mathcal{R}(S_g, E_g) \subset N(T_g, D_g),\)

\[
\inf_{0 \neq p \in \mathcal{H}_g} \sup_{u \in S_g} \frac{\langle p, u \rangle_Y}{\|p\|_Y \|u\|_Y} = \inf_{0 \neq p \in \mathcal{H}_g} \sup_{u \in S_g} \frac{\langle p, u \rangle_Y}{\|p\|_Y \|u\|_Y} \geq \max[\min(\beta_g, \chi_g), \min(\omega_g, \epsilon_g)].
\]

If further \(\mathcal{R}(T_g, P_B(G)) = N(T_g, P_B(G)),\) \(N(S_g, Q_B(G)) = \mathcal{R}(S_g, Q_B(G)),\) and \(\mathcal{H}_g = S_g,\) then...
\( (1) \) (Vertical \textbf{P-A-L} type duality)
\[ \mathcal{H}_g = \delta_g ; \]

\( (2) \) (Hodge decomposition)
\[ \mathcal{R}(T_g, \overline{P}(G)) \oplus^+ N(T_g, \overline{P}(G)) = \mathcal{R}(T_g, D_g) \oplus^+ \mathcal{H}_g = \delta_g \oplus^+ \mathcal{R}(S_g, E_g) . \]

\textbf{Remark 3.10.} The “Hodge decomposition” here is different from the “discrete Hodge decomposition” given as Equation (5.6) of [1], which reads, using the same notations as defined in the present paper,
\[ \mathbb{E}_g = \mathcal{R}(T_g, D_g) \oplus^+ \mathcal{H}_g \oplus^+ (\mathcal{R}(T_g, D_g) \oplus^+ \mathcal{H}_g)^{\perp}, \]

where \( (\mathcal{R}(T_g, D_g) \oplus^+ \mathcal{H}_g)^{\perp} \) is the orthogonal complementation of \( \mathcal{R}(T_g, D_g) \oplus^+ \mathcal{H}_g \) in \( \mathbb{E}_g \) with respect to the inner product of \( \mathbb{Y} \).

3.1.3. Accompanied-by-conforming discretization and discretized spaces. Among all operator pairs partially adjoint based on \( \big[ (T_g, \overline{P}(G)), (T_g, \overline{P}(G)) \big] \), we pay special attention onto a specific kind that one of the domain is a conforming subspace. As seen below, a specific kind of finite element functions are hinted by this special pair.

On \( K \in G \), denote \( H(T_K) := \{ v \in X_K : T_K v \in \mathbb{Y}_K \} \). Then \( D \subseteq \prod_{K \in G} H(T_K) \). The assumption below, which is mild, is useful.

\textbf{Assumption 3.11.} \( \langle v, T_K v \rangle_{X_K} - \langle T_K v, w_K \rangle_{\mathbb{Y}_K} = 0, \) for any \( w_K \in \mathbb{P}_0(K) \) and any \( v \in H(T_K) \) on every \( K \in G \).

By Assumption 3.11, for any \( w_K \in \mathbb{P}_0(K) \),
\[ \langle v, T_g E^Q_K w_K \rangle_{X} - \langle T v, E^Q_K w_K \rangle_{\mathbb{Y}} = 0, \quad \forall v \in D. \]

Therefore, \( E^Q_K w_K \in D \) and \( T_g E^Q_K w_K = T E^Q_K w_K \). Moreover, \( \prod_{K \in G} \mathbb{P}_0(K) \subset D \).

\textbf{Definition 3.12.} \( (T_g, D_{abc}^g) \) is called an accomplished-by-conforming discretization, \textbf{ABCD} for short, on \( G \) of \( (T, D) \), if there is a \( D_{\mathcal{E}}^g \subset D \) such that \( \big[ (T_g, D_{abc}^g), (T, D_{\mathcal{E}}^g) \big] \) is partially adjoint based on some tensorial base pair \( \big[ (T_g, \overline{P}(G)), (T_g, \overline{P}(G)) \big] \) on \( G \). \( D_{abc}^g \) is called an accompanied-by-conforming finite element space, \textbf{ABCFES} for short, on \( G \) of \( D \).

\textbf{Lemma 3.13.} Provided Assumption 3.11, there exists a pair of operators \( \big[ (T_g, D_g), (\mathcal{T}, D_{\mathcal{E}}) \big] \) partially adjoint based on \( \big[ (T_g, \overline{P}(G)), (T_g, \overline{P}(G)) \big] \), such that \( D_g \subset D \).

\textit{Proof.} We simply set \( D_{\mathcal{E}} := D \cap \mathcal{P}(G) \), and \( D_g := \{ v_g \in P_g : \langle v_g, T_g w_g \rangle_X = \langle T_g v_g, w_g \rangle_{\mathbb{Y}} = 0, \forall w_g \in D_{\mathcal{E}} \} \). Then \( \prod_{K \in G} \mathbb{P}_0(K) \subset D_g \), and \( (T_g, D_g) \) and \( (T, D_{\mathcal{E}}) \) are partially adjoint based on \( \big[ (T_g, \overline{P}(G)), (T_g, \overline{P}(G)) \big] \). The proof is completed. \( \Box \)

\textbf{Definition 3.14.} Let \( \big[ (T_g, D_g), (\mathcal{T}, D_{\mathcal{E}}) \big] \) be partially adjoint based on \( \big[ (T_g, \overline{P}(G)), (T_g, \overline{P}(G)) \big] \) on the subdivision \( G \). If \( D_g = D \cap \mathcal{P}(G) \) or \( D_{\mathcal{E}} = D \cap \mathcal{P}(G) \), then \( \big[ (T_g, D_g), (\mathcal{T}, D_{\mathcal{E}}) \big] \) is called a partially adjoint discretization of \( \big[ (T, D), (\mathcal{T}, D) \big] \).
As will be seen in Section 3.3.3, a partially adjoint discretization combines both an optimal CD and an optimal ABCD.

3.1.4. **ABCD used for variational problems.** Here we consider a typical elliptic variational problem: given \( f \in X \), find \( u \in D \), such that

\[
\langle Tu, Tv \rangle_Y + \langle u, v \rangle_X = \langle f, v \rangle_X, \quad \forall v \in D.
\]

(3.3)

It follows that \( Tu \in D \), and \( f = \mathbb{T}(Tu) + u \).

Let \( [(T_G, D_G^{abc}), (T_G, D_G^c)] \), \( D_G^c \subset D \), be a partially adjoint discretization of \( [(T, D), (\mathbb{T}, D)] \). We here use \( (T_G, D_G^{abc}) \) as a discretization of \( (T, D) \), and consider the discretization of (3.3): to find \( u_G \in D_G^{abc} \), such that

\[
\langle T_G u_G, T_G v_G \rangle_Y + \langle u_G, v_G \rangle_X = \langle f, v_G \rangle_X, \quad \forall v_G \in D_G^{abc}.
\]

(3.4)

Immediately (3.3) and (3.4) are well-posed. Denote \( \|v_G\|_{\mathbb{T}_G} := (\|v_G\|^2_X + \|T_G v_G\|^2_Y)^{1/2} \).

**Theorem 3.15.** Let \( u \) and \( u_G \) be the solutions of (3.3) and (3.4), respectively.

\[
\|u - u_G\|_{\mathbb{T}_G} \leq 2 \inf_{w_G \in D_G^{abc}} \|u - w_G\|_{\mathbb{T}_G} + \sup_{\|v_G\|_{\mathbb{T}_G} = 1} \inf_{w_G \in D_G^{abc}} \|T_G u - w_G\|_{\mathbb{T}_G} \cdot \inf_{\|v_G\|_{\mathbb{T}_G} = 1} \|v_G - w_G\|_{\mathbb{T}_G}.
\]

**Proof.** By Strang’s lemma,

\[
\|u - u_G\|_{\mathbb{T}_G} \leq 2 \inf_{w_G \in D_G^{abc}} \|u - w_G\|_{\mathbb{T}_G} + \sup_{\|v_G\|_{\mathbb{T}_G} = 1} \inf_{w_G \in D_G^{abc}} \|T_G u - w_G\|_{\mathbb{T}_G} \cdot \inf_{\|v_G\|_{\mathbb{T}_G} = 1} \|v_G - w_G\|_{\mathbb{T}_G}.
\]

For the consistency error, with arbitrary \( w_G \in D_G^{abc} \cap D \) and \( w_G \in D_G^c \),

\[
\langle T_G u, T_G v_G \rangle_Y - \langle \mathbb{T} Tu, v_G \rangle_X = \langle T_G u, T_G(v_G - w_G) \rangle_Y - \langle \mathbb{T} Tu, (v_G - w_G) \rangle_X
\]

\[
= \langle (T_G u - w_G), T_G(v_G - w_G) \rangle_Y - \langle \mathbb{T}(Tu - w_G), (v_G - w_G) \rangle_X \leq \|T_G u - w_G\|_{\mathbb{T}_G} \|v_G - w_G\|_{\mathbb{T}_G}.
\]

Therefore

\[
\|u - u_G\|_{\mathbb{T}_G} \leq 2 \inf_{w_G \in D_G^{abc}} \|u - w_G\|_{\mathbb{T}_G} + \inf_{w_G \in D_G^{abc}} \|T_G u - w_G\|_{\mathbb{T}_G} \cdot \inf_{\|v_G\|_{\mathbb{T}_G} = 1} \|v_G - w_G\|_{\mathbb{T}_G}.
\]

The proof is completed.

\[\square\]

3.2. **Characteristics of localization.**

3.2.1. **Structure localization of tensorial pairs.** Using the notations defined in Section 3.1, on every \( K \in \mathcal{G} \), denote with respect to \( T_K \) and \( \mathbb{T}_K \),

- \( \hat{P}(K) := \{ v \in P(K) : T_K v = 0 \} \), \( P^*(K) := \{ v \in P(K) : \langle v, w \rangle_{X_K} = 0, \forall w \in \hat{P}(K) \} \);
- \( P_0(K) := \{ v \in P(K) : \langle v, \mathbb{T}_K w \rangle_{X_K} = 0, \forall w \in \mathbb{P}(K) \} \);
- \( \hat{P}_0(K) := \{ v \in P_0(K) : T_K v = 0 \} \), \( P^*_0(K) := \{ v \in P_0(K) : \langle v, w \rangle_{X_K} = 0, \forall w \in \hat{P}_0(K) \} \);
- \( P_B(K) := \{ v \in P(K) : \langle v, w \rangle_{X_K} = 0, \forall w \in \hat{P}_0(K) \} \), \( \langle T_K v, T_K w \rangle_{Y_K} = 0, \forall w \in P_0(K) \} \);
Lemma 3.17. \( \hat{P}_B(K) := \{ v \in P_B(K) : T_K v = 0 \}, \hat{P}_B^0(K) := \{ v \in P_B(K), \langle v, w \rangle_{X_K} = 0, \forall w \in \hat{P}_B(K) \}. \)

Similarly, denote
- \( \hat{P}(K) := \{ v \in P(K) : T_K v = 0 \}, \hat{P}^0(K) := \{ v \in P(K), \langle v, w \rangle_{X_K} = 0, \forall w \in \hat{P}(K) \}; \)
- \( P_0(K) := \{ w \in P(K) : \langle v, T_K w \rangle_{X_K} - \langle T_K v, w \rangle_{X_K} = 0, \forall v \in P(K) \}; \)
- \( \hat{P}_0(K) := \{ w \in \hat{P}_0(K) : T_K w = 0 \}, \hat{P}_0^0(K) := \{ v \in \hat{P}_0(K), \langle v, w \rangle_{X_K} = 0, \forall w \in \hat{P}_0(K) \}; \)
- \( P_B(K) := \{ v \in P(K) : \langle v, w \rangle_{X_K} = 0, \forall w \in P(B(K)) \}; \)
- \( \hat{P}_B(K) := \{ v \in \hat{P}(K) : T_K v = 0 \}, \hat{P}_B^0(K) := \{ v \in \hat{P}(K), \langle v, w \rangle_{X_K} = 0, \forall w \in \hat{P}_B(K) \}. \)

It can be easily proved by definition that the global spaces are tensor products of certain local spaces. We survey them in the two lemmas below.

**Lemma 3.16.**

1. \( \hat{P}^0(\mathcal{G}) = \prod_{K \in \mathcal{G}} P_0(K), \hat{P}_B(\mathcal{G}) = \prod_{K \in \mathcal{G}} P_B(K); \)
2. \( N(T_G, \hat{P}(\mathcal{G})) = \prod_{K \in \mathcal{G}} \hat{P}(K), N(T_G, \hat{P}_B(\mathcal{G})) = \prod_{K \in \mathcal{G}} \hat{P}_B(K); \)
3. \( \mathcal{R}(T_G, \hat{P}(\mathcal{G})) = \prod_{K \in \mathcal{G}} \mathcal{R}(T_K, P(K)), \mathcal{R}(T_G, \hat{P}_B(\mathcal{G})) = \prod_{K \in \mathcal{G}} \mathcal{R}(T_K, P_B(K)). \)

**Lemma 3.17.**

1. \( \hat{P}^0(\mathcal{G}) = \prod_{K \in \mathcal{G}} P_0(K), \hat{P}_B(\mathcal{G}) = \prod_{K \in \mathcal{G}} P_B(K); \)
2. \( N(T_G, \hat{P}(\mathcal{G})) = \prod_{K \in \mathcal{G}} \hat{P}(K), N(T_G, \hat{P}_B(\mathcal{G})) = \prod_{K \in \mathcal{G}} \hat{P}_B(K); \)
3. \( \mathcal{R}(T_G, \hat{P}(\mathcal{G})) = \prod_{K \in \mathcal{G}} \mathcal{R}(T_K, P(K)), \mathcal{R}(T_G, \hat{P}_B(\mathcal{G})) = \prod_{K \in \mathcal{G}} \mathcal{R}(T_K, P_B(K)). \)

Based on the tensor products, the global structures can be realized locally. We present several lemmas below. They can be actually proved the same way, and we will only present the proof of Lemma 3.18 in details.

**Lemma 3.18.** Assumption 3.1 holds, if and only if, on every \( K \in \mathcal{G} \), there exists an \( \alpha_K > 0 \) such that

\[
\inf_{0 \neq v \in P_B(K)} \sup_{w \in \mathcal{R}(T_K, P_B(K))} \frac{\langle v, w \rangle_{X_K}}{||v||_{X_K} ||w||_{X_K}} = \inf_{0 \neq w \in \mathcal{R}(T_K, P_B(K))} \sup_{v \in P_B(K)} \frac{\langle v, w \rangle_{X_K}}{||v||_{X_K} ||w||_{X_K}} = \alpha_K,
\]

and on every \( K \in \mathcal{G} \), there exists a \( \beta_K > 0 \) such that

\[
\inf_{0 \neq v \in P_B(K)} \sup_{w \in \mathcal{R}(T_K, P_B(K))} \frac{\langle v, w \rangle_{Y_K}}{||v||_{Y_K} ||w||_{Y_K}} = \inf_{0 \neq w \in \mathcal{R}(T_K, P_B(K))} \sup_{v \in P_B(K)} \frac{\langle v, w \rangle_{Y_K}}{||v||_{Y_K} ||w||_{Y_K}} = \beta_K.
\]

Moreover,

\[\alpha_\mathcal{G} = \inf_{K \in \mathcal{G}} \alpha_K, \quad \text{and,} \quad \beta_\mathcal{G} = \inf_{K \in \mathcal{G}} \beta_K.\]

**Proof.** Provided \( \inf_{0 \neq v \in N(T_G, P_B(\mathcal{G}))} \sup_{w \in \mathcal{R}(T_G, P_B(\mathcal{G}))} \frac{\langle v, w \rangle_X}{||v||_X ||w||_X} = \alpha_\mathcal{G} > 0 \), for any \( K \in \mathcal{G} \), given \( \nu_K \in N(T_K, P_B(K)) \), there exists \( v \in \prod_{K \in \mathcal{G}} \mathcal{R}(T_K, P_B(K)) \), such that \( \langle E^0_K v_K, w \rangle_X >= \alpha_\mathcal{G} ||v_K||_{X_K} ||w||_X \).
Then \(\langle v_K, R^K_\Omega w \rangle_{X_K} \geq \alpha_G \|v_K\|_{X_K} \|R^K_\Omega w\|_{X_K} \) with \(R^K_\Omega w \in \mathcal{R}(\mathbb{T}_K, \mathbb{P}_B(K))\). This proves
\[
\alpha_K = \inf_{0 \neq v \in \mathbb{P}_h(K) \cap \mathcal{R}(\mathbb{T}_K, \mathbb{P}_B(K))} \sup_{w \in \mathcal{R}(\mathbb{T}_G, \mathbb{P}_B(G))} \frac{\langle v, w \rangle_{X_K}}{\|v\|_{X_K} \|w\|_{X_K}} \geq \alpha_G, \quad \forall K \in \mathcal{G}.
\]

On the other hand, provided that \(\inf_{0 \neq v \in \mathcal{N}(G, \mathbb{P}_B(G))} \sup_{w \in \mathcal{R}(\mathbb{T}_G, \mathbb{P}_B(G))} \frac{\langle v, w \rangle_{X_K}}{\|v\|_{X_K} \|w\|_{X_K}} = \alpha_K > 0\) on every \(K \in \mathcal{G}\), given \(v \in \mathcal{N}(\mathbb{T}_G, \mathbb{P}_B(G))\), there exists a \(w \in \mathcal{R}(\mathbb{T}_G, \mathbb{P}_B(G))\), such that \(\langle R^K_\Omega v, R^K_\Omega w \rangle_{X_K} \geq \alpha_K \|R^K_\Omega v\|_{X_K} \|R^K_\Omega w\|_{X_K}\) and \(\|R^K_\Omega v\|_{X_K} = \|R^K_\Omega w\|_{X_K}\) on every \(K \in \mathcal{G}\). Further,
\[
\langle v, w \rangle_X = \sum_{K \in \mathcal{G}} \langle R^K_\Omega v, R^K_\Omega w \rangle_{X_K} \geq \sum_{K \in \mathcal{G}} \alpha_K \|R^K_\Omega v\|_{X_K} \|R^K_\Omega w\|_{X_K} \geq \inf_{K \in \mathcal{G}} \alpha_K \|v\|_{X_K} \|w\|_{X_K}.
\]
This proves
\[
\inf_{0 \neq v \in \mathcal{N}(\mathbb{T}_G, \mathbb{P}_B(G))} \sup_{w \in \mathcal{R}(\mathbb{T}_G, \mathbb{P}_B(G))} \frac{\langle v, w \rangle_X}{\|v\|_X \|w\|_X} = \alpha_G \geq \inf_{K \in \mathcal{G}} \alpha_K.
\]

The proof is completed. \(\square\)

**Lemma 3.19.** \(\text{icr}(\mathbb{T}_G, \tilde{\mathbb{P}}(G)) = \sup_{K \in \mathcal{G}} \text{icr}(\mathbb{T}_K, \mathbb{P}(K))\). \(\text{icr}(\mathbb{T}_G, \tilde{\mathbb{P}}(G)) = \sup_{K \in \mathcal{G}} \text{icr}(\mathbb{T}_K, \mathbb{P}_0(K))\).

**Lemma 3.20.** \(\mathcal{N}(\mathbb{T}_G, \mathbb{P}_B(G)) = \mathcal{R}(\mathbb{T}_G, \mathbb{P}_B(G))\) if and only if on every \(K \in \mathcal{G}\), \(\mathbb{P}_B(K) = \mathcal{R}(\mathbb{T}_K, \mathbb{P}_B(K))\).

**Lemma 3.21.** Assumption 3.4 holds, if and only if, on every \(K \in \mathcal{G}\), there exists a \(\sigma_K > 0\) such that
\[
(3.7) \quad \inf_{0 \neq p \in \mathbb{Q}_0(K) \subset \mathcal{R}(\mathbb{T}_K, \mathbb{Q}(K))} \sup_{q \in \mathbb{S}_K, \mathbb{Q}_0(K)} \frac{\langle p, q \rangle_{Y_K}}{\|p\|_{Y_K} \|q\|_{Y_K}} = \inf_{0 \neq q \in \mathcal{S}(\mathbb{S}_K, \mathbb{Q}_0(K))} \sup_{p \in \mathbb{Q}_0(K)} \frac{\langle p, q \rangle_{Y_K}}{\|p\|_{Y_K} \|q\|_{Y_K}} = \sigma_K,
\]

and on every \(K \in \mathcal{G}\), there exists a \(\kappa_K > 0\) such that
\[
(3.8) \quad \inf_{0 \neq p \in \mathbb{Q}_0(K) \subset \mathcal{R}(\mathbb{T}_K, \mathbb{Q}(K))} \sup_{q \in \mathbb{S}_K, \mathbb{Q}_0(K)} \frac{\langle p, q \rangle_{Z_K}}{\|p\|_{Z_K} \|q\|_{Z_K}} = \inf_{0 \neq q \in \mathcal{S}(\mathbb{S}_K, \mathbb{Q}_0(K))} \sup_{p \in \mathbb{Q}_0(K)} \frac{\langle p, q \rangle_{Z_K}}{\|p\|_{Z_K} \|q\|_{Z_K}} = \kappa_K.
\]

Further
\[
\omega_G = \inf_{K \in \mathcal{G}} \sigma_K, \quad \text{and} \quad \kappa_G = \inf_{K \in \mathcal{G}} \kappa_K.
\]

**Lemma 3.22.** Assumption 3.5 holds, if and only if on every \(K \in \mathcal{G}\),
\[
\mathcal{R}(\mathbb{T}_K, \mathbb{P}(K)) \subset \mathcal{N}(\mathbb{S}_K, \mathbb{Q}(K)), \quad \text{and} \quad \mathcal{R}(\mathbb{S}_K, \mathbb{Q}(K)) \subset \mathcal{N}(\mathbb{T}_K, \mathbb{P}(K)).
\]

Provided Assumption 3.5, denote, on \(K \in \mathcal{G}\),
\[
\tilde{\mathbb{H}}_K := \mathcal{N}(\mathbb{S}_K, \mathbb{Q}(K)) \ominus^\perp \mathcal{R}(\mathbb{T}_K, \mathbb{P}(K)), \quad \mathbb{H}_K := \mathcal{N}(\mathbb{S}_K, \mathbb{Q}_0(K)) \ominus^\perp \mathcal{R}(\mathbb{T}_K, \mathbb{P}_0(K)),
\]
and
\[
\tilde{\mathbb{S}}_K := \mathcal{N}(\mathbb{T}_K, \mathbb{P}(K)) \ominus^\perp \mathcal{R}(\mathbb{S}_K, \mathbb{Q}(K)), \quad \mathbb{S}_K := \mathcal{N}(\mathbb{T}_K, \mathbb{P}_0(K)) \ominus^\perp \mathcal{R}(\mathbb{S}_K, \mathbb{Q}_0(K)).
\]

Then
\[
\tilde{\mathbb{H}}_G = \bigsqcup_{K \in \mathcal{G}} \tilde{\mathbb{H}}_K, \quad \mathbb{H}_G = \bigsqcup_{K \in \mathcal{G}} \mathbb{H}_K, \quad \tilde{\mathbb{S}}_G = \bigsqcup_{K \in \mathcal{G}} \tilde{\mathbb{S}}_K, \quad \text{and} \quad \mathbb{S}_G = \bigsqcup_{K \in \mathcal{G}} \mathbb{S}_K.
\]
Lemma 3.23. Assumption 3.6 holds, if and only if, on every \( K \in \mathcal{G} \), there exist constants \( \chi_K, \epsilon_K > 0 \), such that

\[
\inf_{0 \neq p \in \mathcal{H}_K} \sup_{v \in V_K} \frac{\langle p, v \rangle_{Y_K}}{\|p\|_{Y_K} \|v\|_{Y_K}} = \inf_{0 \neq p \in \delta_K} \sup_{v \in V_K} \frac{\langle p, v \rangle_{Y_K}}{\|p\|_{Y_K} \|v\|_{Y_K}} = \chi_K,
\]

and

\[
\inf_{0 \neq p \in \mathcal{H}_K} \sup_{v \in V_K} \frac{\langle p, v \rangle_{Y_K}}{\|p\|_{Y_K} \|v\|_{Y_K}} = \inf_{0 \neq p \in \delta_K} \sup_{v \in V_K} \frac{\langle p, v \rangle_{Y_K}}{\|p\|_{Y_K} \|v\|_{Y_K}} = \epsilon_K.
\]

Moreover,

\[
\chi_G = \inf_{K \in \mathcal{G}} \chi_K, \quad \text{and} \quad \epsilon_G = \inf_{K \in \mathcal{G}} \epsilon_K.
\]

Remark 3.24 (Asymptotics of an infinite family of finite-dimensional operators). Let \( \mathcal{F}^\mathcal{G} \) be a family of subdivisions of \( \Omega \). Denote

\[
\alpha_{\mathcal{F}^\mathcal{G}} := \inf_{G \in \mathcal{F}^\mathcal{G}} \inf_{K \in \mathcal{G}} \alpha_K, \quad \text{and} \quad \beta_{\mathcal{F}^\mathcal{G}} := \inf_{G \in \mathcal{F}^\mathcal{G}} \inf_{K \in \mathcal{G}} \beta_K;
\]

\[
\bar{\rho}_{\mathcal{F}^\mathcal{G}} := \sup_{G \in \mathcal{F}^\mathcal{G}} \inf_{K \in \mathcal{G}} \text{icr}(T_K, P(K)) \quad \text{and} \quad \rho_{\mathcal{F}^\mathcal{G}} := \sup_{G \in \mathcal{F}^\mathcal{G}} \inf_{K \in \mathcal{G}} \text{icr}(T_K, P_0(K));
\]

\[
\sigma_{\mathcal{F}^\mathcal{G}} := \inf_{G \in \mathcal{F}^\mathcal{G}} \inf_{K \in \mathcal{G}} \sigma_K, \quad \text{and} \quad \kappa_{\mathcal{F}^\mathcal{G}} := \inf_{G \in \mathcal{F}^\mathcal{G}} \inf_{K \in \mathcal{G}} \kappa_K;
\]

\[
\chi_{\mathcal{F}^\mathcal{G}} := \inf_{G \in \mathcal{F}^\mathcal{G}} \inf_{K \in \mathcal{G}} \chi_K, \quad \text{and} \quad \epsilon_{\mathcal{F}^\mathcal{G}} := \inf_{G \in \mathcal{F}^\mathcal{G}} \inf_{K \in \mathcal{G}} \epsilon_K.
\]

Let on every \( G \in \mathcal{F}^\mathcal{G} \) be constructed a pair of partially adjoint discretizations of \([T, D], (T, D)\), so that a family of partially adjoint discretizations is formed as \([[T_G, D_G], (T_G, D_G)]\) \( G \in \mathcal{F}^\mathcal{G} \). Then

1. the adjoint properties of the operator pairs in \([[T_G, D_G], (T_G, D_G)]\) \( G \in \mathcal{F}^\mathcal{G} \) can be controlled by the parameters defined by (3.9), (3.10), (3.11), and (3.12);

2. all these parameters can be controlled cell by cell, and subdivision by subdivision.

This way, we can have an infinite family of finite-dimensional operators which bears uniform asymptotic properties based on the cell-wise properties. See Lemma 4.20 for a precise example.

3.2.2. Locality of supports of basis functions. Let \([[T_G, D_G], (T_G, D_G)]\) be a pair of partially adjoint discretizations of \([T, D], (T, D)\). An explicit set of basis functions are usually used in the implementation of the discretization schemes based on \( D_G \) and \( D \). Generally, the basis functions of \( D_G \) are determined by solving a linear system, and it is possible that the supports may be vast. Though, we now show that, the supports of the basis functions of \( D_G \) can be controlled by the supports of the basis functions of \( D_G \).

We use \( \text{LSB}(D_G) = \{I_i\}_{i=1}^{\dim(D_G)} \) to denote a set of linearly independent locally supported basis functions of \( D_G \). By Theorem 2.17, since \( (T_G, D_G) \) and \( (T_G, D_G) \) are partially adjoint, \( \prod_{K \in G} P_0(K) = P(G) \subset D_G \). Therefore, we can particularly set

\[
\text{LSB}(D_G) = \left( \prod_{K \in G} P_0(K) \right) \oplus \text{LSB}(D_G) \cap \prod_{K \in G} P_B(K) = \text{span} \{I_i\}_{i \in I_1} \oplus \text{span} \{I_i\}_{i \in I_2},
\]

where \( I_1 \) and \( I_2 \) are two distinct sets of indices, and \( I_1 \cup I_2 = \{1, \ldots, \dim(D_G)\} \).
Assumption 3.25. On any cell $K$, the set of functions $\{b_i|_K : i \in I_2, \hat{K} \cap \text{supp}(b_i) \neq \emptyset\}$ are linearly independent.

Theorem 3.26. Provided Assumption 3.25, $D_G$ admits a set of locally supported basis functions.

The precise meaning of “locally supported” is given in the end of the proof.

Proof. We prove this theorem by constructing a set of basis functions.

By Assumption 3.25, given $K \in G$, set $I_2^K := \{i \in I_2 : \hat{K} \cap \text{supp}(b_i) \neq \emptyset\}$, and there exist a set of functions $\{v_i^K : i \in I_2^K\} \subset P_B(K)$, such that $\langle v_i^K, \mathbb{T}_K R^K b_j \rangle_{X_K} - \langle T_K v_i^K, R^K b_j \rangle_{Y_K} = \delta_{ij}$, $i, j \in I_2^K$. Denote further $P_B^*(K) := \{v \in P_B(K) : \langle v, \mathbb{T}_K R^K b_j \rangle_{X_K} - \langle T_K v, R^K b_j \rangle_{Y_K} = 0, \forall j \in I_2^K\}$. Note that $\text{supp}(b_i|_K) \supset \{b_i|_K : i \in I_2^K\}$, and they two are equal if and only if $P_B^*(K) = \{0\}$. Therefore, the space $P(K)$ can be decomposed to $P(K) = P_0(K) \oplus \text{span}\{v_i^K : i \in I_2^K\} \oplus P_B^*(K)$.

Now, by definition,

\[
D_G = \{v \in \overline{P}(G) : \langle v, \mathbb{T}_G w \rangle_{X} - \langle T_G v, w \rangle_{Y} = 0, \forall w \in D_G\}
\]

\[
= \overline{P}(G) \oplus \prod_{K \in G} P_B^*(K) \oplus \left\{v \in \sum_{K \in G i \in I_2^K} \text{span}\{E^{\Omega}_{K} v_i^K\} : \langle v, \mathbb{T}_G b_j \rangle_{X} - \langle T_G v, b_j \rangle_{Y} = 0, j \in I_2\right\}
\]

\[
= \overline{P}(G) \oplus \prod_{K \in G} P_B^*(K) \oplus \sum_{j \in I_2} \left\{v \in \sum_{\hat{K} \cap \text{supp}(b_j) \neq \emptyset} \text{span}\{E^{\Omega}_{K} v_i^K\} : \langle v, \mathbb{T}_G b_j \rangle_{X} - \langle T_G v, b_j \rangle_{Y} = 0\right\}.
\]

Therefore, $D_G$ admits two types of basis functions, namely,

Type-I: $E^{\Omega}_{K} P_0(K)$ and $E^{\Omega}_{K} P_B^*(K)$ for all $K \in G$;

Type-II: for $j \in I_2$, such $v \in \sum_{\hat{K} \cap \text{supp}(b_j) \neq \emptyset} \text{span}\{E^{\Omega}_{K} v_i^K\}$ that

\[
(3.14) \quad \sum_{K \in G} \langle v, \mathbb{T}_K R^K b_j \rangle_{X_K} - \langle T_K v, R^K b_j \rangle_{Y_K} = 0.
\]

The supports of basis functions of Type-I are each one cell. The supports of basis functions of Type-II are all contained in the support of $b_j$. The proof is completed. □

Remark 3.27. Given any $b_i$, $i \in I_2$, there are $\#\left\{K \in G : \hat{K} \cap \text{supp}(b_i) \neq \emptyset\right\} - 1$ basis functions of Type-II corresponding to $b_i$. They are each a combination of those $v_i^K$, determined up to one degree of freedom by (3.14).

Remark 3.28. To implement the procedure given in the proof of Theorem 3.26, $[(T_G, D_G), (\mathbb{T}_G, \mathbb{D}_G)]$ is not necessarily a partially adjoint discretization of $[(T, D), (\mathbb{T}, \mathbb{D})]$.

3.3. Cell-wise-defined interpolation and commutative diagram.
3.3.1. Interpolator on one cell. We here use the notations defined in the previous subsections.

**Definition 3.29** (Adjoint projection). Define $\mathbb{T}_K^T : H(T_K) \to P(K)$ by

1. $\langle \mathbb{T}_K^T v, T_K w \rangle_{X_K} = \langle v, T_K w \rangle_{X_K} - \langle T_K v, w \rangle_{Y_K}$, $\forall w \in P_0(K)$;
2. $\langle \mathbb{T}_K^T v, w \rangle_{X_K} = \langle v, w \rangle_{X_K}$, $\forall w \in P_0(K)$;
3. $\langle T_K \mathbb{T}_K^T v, T_K w \rangle_{Y_K} = \langle T_K v, T_K w \rangle_{Y_K}$, $\forall w \in P_0(K)$.

**Remark 3.30.** By Assumption 3.11,

$\langle \mathbb{T}_K^T v, T_K w \rangle_{X_K} = \langle T_K \mathbb{T}_K^T v, w \rangle_{Y_K} = 0 = \langle v, T_K w \rangle_{X_K} - \langle T_K v, w \rangle_{Y_K}$, $\forall w \in P_0(K)$.

Therefore, for $v \in H(T_K)$,

$\langle \mathbb{T}_K^T v, T_K w \rangle_{X_K} = \langle T_K \mathbb{T}_K^T v, w \rangle_{Y_K} = \langle v, T_K w \rangle_{X_K} - \langle T_K v, w \rangle_{Y_K}$, $\forall w \in P(K)$.

This is why the interpolator is called “adjoint projection”.

**Lemma 3.31.** Provided Assumption 3.1, the interpolator $\mathbb{T}_K^T$ is well-defined, and $\mathbb{T}_K^T v = v$ for $v \in P(K)$. Further,

(a) with $\beta_K = \inf_{0 \neq w \in P_0(K)} \sup_{v \in P_0(K)} \frac{\langle w, w \rangle_{Y_K}}{\|v\|_{X_K} \|w\|_{Y_K}}$ as given in (3.6),

$\|T_K \mathbb{T}_K^T v\|_{Y_K} \leq (1 + \beta_K^{-1})\|T_K v\|_{Y_K}$, and $\|T_K(v - \mathbb{T}_K^T v)\|_{Y_K} \leq (2 + \beta_K^{-1}) \inf_{w \in P(K)} \|T_K(v - w)\|_{Y_K}$;

(b) with $\gamma_K := \inf_{0 \neq w \in P_0(K)} \sup_{v \in P_0(K)} \frac{\langle v, T_K w \rangle_{X_K} - \langle T_K v, w \rangle_{Y_K}}{\|v\|_{X_K} \|w\|_{Y_K}}$, and $\rho_K := \text{icr}(T_K, P_0(K))$,

$\|\mathbb{T}_K^T v\|_{T_K} \leq (2 + \rho_K + \gamma_K^{-1} + \beta_K^{-1})\|v\|_{T_K}$, and $\|v - \mathbb{T}_K^T v\|_{T_K} \leq (3 + \rho_K + \gamma_K^{-1} + \beta_K^{-1}) \inf_{w \in P(K)} \|v - w\|_{T_K}$.

**Proof.** Provided Assumption 3.1 and Lemma 3.18 and Lemma 2.14, $\gamma_K > 0$. By the decomposition of $P(K)$ and $P(K)$, there exist uniquely $v_B \in P_B(K)$, $v_0 \in P_0(K)$, and $v_0' \in P_0'(K)$, such that

1. $\langle v_B, T_K w \rangle_{X_K} = \langle T_K v_B, w \rangle_{Y_K} = \langle v, T_K w \rangle_{X_K} - \langle T_K v, w \rangle_{Y_K}$, $\forall w \in P_B(K)$;
2. $\langle v_0, w \rangle_{X_K} = \langle v, w \rangle_{X_K}$, $\forall w \in P_0(K)$;
3. $\langle T_K v_0', T_K w \rangle_{Y_K} = \langle T_K v, T_K w \rangle_{Y_K}$, $\forall w \in P_0'(K)$.

Besides, $\|v_0\|_{X_K} \leq \|v\|_{X_K}$ and $\|T_K v_0'\|_{Y_K} \leq \|T_K v\|_{Y_K}$. Moreover,

1. $\langle v, T_K w \rangle_{X_K} - \langle T_K v, w \rangle_{Y_K} = 0$, $\forall w \in P_B(K)$, if $\tilde{v} \in \{v_0, v_0'\}$;
2. $\langle \tilde{v}, w \rangle_{X_K} = 0$, $\forall w \in P_0(K)$, if $\tilde{v} \in \{v_B, v_0\}$;
3. $\langle T_K \tilde{v}, T_K w \rangle_{Y_K} = 0$, $\forall w \in P_0'(K)$, if $\tilde{v} \in \{v_B, v_0\}$.

Therefore, $\mathbb{T}_K^T v$ is uniquely determined as $v_B + v_0 + v_0'$, and $\mathbb{T}_K^T$ is well defined. Evidently $\mathbb{T}_K^T v = v$ for $v \in P(K)$.

By (3.6), $\|T_K v_B\|_{X_K} \leq \sup_{w \in P_B(K)} \frac{1}{\beta_K} \frac{\langle T_K v, w \rangle_{Y_K}}{\|w\|_{Y_K}} \leq \|T_K v\|_{Y_K}/\beta_K$. Therefore,

$\|T_K \mathbb{T}_K^T v\|_{Y_K} \leq \|T_K v\|_{Y_K}/\beta_K + \|T_K v\|_{Y_K} = (1 + \beta_K^{-1})\|T_K v\|_{Y_K}$.
Similarly, for any $w \in P(K)$, $\|T_K(v - \mathbb{I}_K^T v)\|_{\mathcal{Y}_K} \leq \|T_K(v - w)\|_{\mathcal{Y}_K} + \|T_K^\perp w - \mathbb{I}_K^T(v - w)\|_{\mathcal{Y}_K} \leq \|T_K(v - w)\|_{\mathcal{Y}_K} + (1 + \beta_K^{-1})\|T_K(w - v)\|_{\mathcal{Y}_K}$, and therefore,

$$\|T_K(v - \mathbb{I}_K^T v)\|_{\mathcal{Y}_K} \leq (2 + \beta_K^{-1}) \inf_{w \in P(K)} \|T_K(v - w)\|_{\mathcal{Y}_K}.$$ 

On the other hand, $\gamma_K \|v_B\|_{T_K} \leq \sup_{v \in P_B(K)} \frac{\langle v_B, T_K v \rangle_{X_K} - \langle T_K v_B, v \rangle_{\mathcal{Y}_K}}{\|v\|_{T_K}} \leq \|v\|_{T_K}$. Therefore,

$$\|T_K^T v\|_{X_K} \leq \|v_0\|_{X_K} + \|v_0\|_{X_K} + \|v_B\|_{X_K} \leq \|v\|_{X_K} + \rho_K \|T_K v\|_{\mathcal{Y}_K} + \gamma_K^{-1} \|v\|_{T_K} \leq (1 + \rho_K + \gamma_K^{-1}) \|v\|_{T_K}.$$ 

Therefore,

$$\|T_K^T v\|_{T_K} \leq (2 + \rho_K + \gamma_K^{-1} + \beta_K^{-1}) \|v\|_{T_K}.$$ 

Finally, it holds with any $w \in P(K)$ that

$$\|v - \mathbb{I}_K^T v\|_{T_K} \leq \|v - w\|_{T_K} + \|T_K^T (v - w)\|_{T_K} \leq (3 + \rho_K + \gamma_K^{-1} + \beta_K^{-1}) \|v - w\|_{T_K}.$$ 

The proof is completed. \hfill \Box

**Discrete commutative diagram.** With respect to the spaces $Q(K)$ and $Q(K)$, we can define interpolation $\mathbb{I}_K^S : H(S_K) \to Q(K)$ by

1. $\langle \mathbb{I}_K^S q, S_K q \rangle_{\mathcal{Y}_K} - (S_K \mathbb{I}_K^S q, q)_{T_K} = \langle q, S_K q \rangle_{\mathcal{Y}_K} - (S_K q, q)_{T_K}$, $\forall q \in Q_B(K);
2. \langle \mathbb{I}_K^S q, w \rangle_{\mathcal{Y}_K} = \langle q, w \rangle_{\mathcal{Y}_K}$, $\forall w \in Q_0(K);
3. \langle S_K \mathbb{I}_K^S q, S_K w \rangle_{T_K} = \langle S_K q, S_K w \rangle_{T_K}$, $\forall w \in Q_0(K)$.

The interpolators $\mathbb{I}_K^S$ and $\mathbb{I}_K^T$ commute in the sense below.

**Lemma 3.32.** Provided Assumption 3.5, that $H_K = \mathbb{S}_K$ and that $\mathcal{R}(T_K, H(T_K)) \subset N(S_K, H(S_K))$, $\mathbb{I}_K^S T_K v = T_K \mathbb{I}_K^T v$ for $v \in H(T_K)$.

**Proof.** As, by Assumption 3.5 and Lemma 3.22, $T_K \mathbb{I}_K^T v \in \mathbb{Q}(K)$, we will verify $\mathbb{I}_K^S T_K v = T_K \mathbb{I}_K^T v$ by verifying

(3.15) $\langle T_K^\perp v, q \rangle_{\mathcal{Y}_K} = \langle T_K^T v, q \rangle_{\mathcal{Y}_K}$, $\forall q \in \mathbb{Q}_0(K),$

and

(3.16) $\langle T_K^\perp v, S_K v \rangle_{\mathcal{Y}_K} - (S_K T_K \mathbb{I}_K^T v, v)_{T_K} = \langle T_K v, S_K v \rangle_{\mathcal{Y}_K} - (S_K T_K v, v)_{T_K}$, $\forall v \in Q_B(K)$.

Firstly, by Assumption 3.11,

(3.17) $\langle T_K v, w \rangle_{X_K} - \langle w, T_K v \rangle_{X_K} = 0$, $\forall v \in H(T_K), w \in P_0(K)$.

By the definition of $\mathbb{I}_K^T$,

$$\langle T_K^T v, w \rangle_{X_K} - \langle T_K^\perp v, T_K w \rangle_{X_K} = \langle T_K v, w \rangle_{X_K} - \langle w, T_K v \rangle_{X_K}$$. $\forall v \in H(T_K), w \in P_0(K)$.
Given \( \hat{v} \in \tilde{S}_K \subset \mathbb{P}_B(K) \oplus^\perp \mathbb{P}_0(K) \), decompose \( \hat{v} = \hat{v}_0 + \hat{v}_B \), with \( \hat{v}_0 \in \mathbb{P}_0(K) \) and \( \hat{v}_B \in \mathbb{P}_B(K) \). Then, for any \( v \in H(T_K) \),

\[
\langle T_K \mathbb{I}^T v, \hat{v} \rangle_{Y_K} = \langle T_K \mathbb{I}^T v, \hat{v}_0 + \hat{v}_B \rangle_{Y_K} = (by \ (3.17)) \langle T_K \mathbb{I}^T v, \hat{v}_B \rangle_{Y_K} \]

\[
= \langle T_K \mathbb{I}^T v, \hat{v}_B \rangle_{Y_K} \]

Therefore, for any \( q \in \tilde{Q}_0(K) \), it holds with \( \mathbb{I}_K \mathbb{P}_0(K) = \mathbb{P}_B(K) \oplus \mathbb{I}^\perp \mathbb{P}_0(K) \). Indeed, if \( \hat{v}_B \in \mathbb{P}_B(K) \), by the definition of \( \mathbb{I}^T_K \), \( \langle T_K \mathbb{I}^T v, \hat{v}_B \rangle_{Y_K} = \langle T_K \mathbb{I}^T v, \hat{v}_B \rangle_{Y_K} = 0 \). If \( \hat{v}_0 \in \mathbb{P}_0(K) \), by (3.17),

\[
\langle T_K \mathbb{I}^T v, \hat{v}_0 \rangle_{Y_K} = 0 = \langle T_K \mathbb{I}^T v, \hat{v}_B \rangle_{Y_K} = \langle T_K \mathbb{I}^T v, \hat{v}_B \rangle_{Y_K}.
\]

Therefore, as \( R(S_K, Q(K)) \subset \mathbb{P}(K) \),

\[
\langle T_K \mathbb{I}^T v, S_K q \rangle_{Y_K} = \langle T_K \mathbb{I}^T v, S_K q \rangle_{Y_K} = \langle T_K \mathbb{I}^T v, S_K q \rangle_{Y_K} = \langle T_K \mathbb{I}^T v, S_K q \rangle_{Y_K} = 0 \quad \forall q \in Q(K).
\]

Then (3.16) is verified and the proof is completed. 

3.3.2. Global cell-wise interpolator and commutative diagram.

**Global interpolator and approximation.** Now define

\[
\mathbb{I}^T_G : \bigcap_{K \in \mathcal{G}} H(T_K) \to \bigcap_{K \in \mathcal{G}} \mathbb{P}(K), \text{ by } (\mathbb{I}^T_G v)_K = [\mathbb{I}^T_K v|_K], \forall K \in \mathcal{G}.
\]

Then \( \mathbb{I}^T_G \) is projective onto \( \mathbb{P}(\mathcal{G}) \). Moreover, defined as an interpolator for \( D \), \( \mathbb{I}^T_G \) requires the appropriate regularity and of minimal dependence.

**Lemma 3.33.** Let \( [T_G, \mathbb{D}^{abc}_G], (T_G, \mathbb{D}_G) \) be partially adjoint based on \( [T_G, \mathbb{P}(\mathcal{G})], (T_G, \mathbb{P}(\mathcal{G})) \), and \( \mathbb{D}^c_G \subset \mathbb{D} \). If \( v \in D \), then \( \mathbb{I}^T_G v \in \mathbb{D}^{abc}_G \).

**Proof.** It suffices for proving \( \mathbb{I}^T_G v \in \mathbb{D}^{abc}_G \) to show

\[
\langle \mathbb{I}^T_G v, T_G v \rangle_X - \langle \mathbb{I}^T_G v, T_G v \rangle_X = 0, \forall v \in \mathbb{D}^c_G = \bigcap_{K \in \mathcal{G}} \mathbb{P}_0(K) \oplus (\mathbb{D}^c_G \cap \bigcap_{K \in \mathcal{G}} \mathbb{P}_B(K)).
\]

Given \( v \in \prod_{K \in \mathcal{G}} \mathbb{P}_0(K) \), with \( w_K := w|_K \) and \( v_K := v|_K \),

\[
\sum_{K \in \mathcal{G}} (\mathbb{I}^T_K v_K, T_K w_K)_X - \langle \mathbb{I}^T_K v_K, w_K \rangle_{Y_K} = 0.
\]

Given \( v \in \mathbb{D}^c_G \cap \prod_{K \in \mathcal{G}} \mathbb{P}_B(K) \), it holds with \( w_K := w|_K \) and \( v_K := v|_K \) that

\[
\langle \mathbb{I}^T_K v_K, T_K w_K \rangle_X - \langle \mathbb{I}^T_K v_K, w_K \rangle_{Y_K} = \langle v_K, T_K w_K \rangle_X - \langle v_K, w_K \rangle_{Y_K}, \forall K \in \mathcal{G},
\]
therefore
\[
\sum_{K \in \mathcal{G}} \langle v_K, T_K v_K \rangle_{X_K} - \langle T_K v_K, v_K \rangle_{X_K} = \sum_{K \in \mathcal{G}} \langle v_K, T_K v_K \rangle_{X_K} - \langle T_K v_K, v_K \rangle_{X_K} = 0,
\]
where we have used the fact that \( v \in \mathcal{D}_G^C \subseteq \mathcal{D} \) and thus
\[
\sum_{K \in \mathcal{G}} \langle v_K, T_K v_K \rangle_{X_K} - \langle T_K v_K, v_K \rangle_{X_K} = \langle v, T v \rangle_X - \langle T v, v \rangle_X = 0.
\]
The proof is then completed.

**Remark 3.34.** The interpolation \( \Gamma_G : \mathcal{D} \to \mathcal{D}_G^{abc} \) is both projective and strictly cell-wise defined with the minimal regularity requirement. Here, by “strictly cell-wise defined”, we mean with respect to any \( K \in \mathcal{G} \), \( \langle \Gamma_G v_1 \rangle_K = \langle \Gamma_G v_2 \rangle_K \) for any \( v_1, v_2 \in \mathcal{D} \) such that \( v_1|_K = v_2|_K \).

The global stability and approximation follows directly from Lemma 3.31.

**Theorem 3.35.** Provided Assumption 3.1,

(a) \( \| T_G \Gamma_G v \|_Y \leq (1 + \beta_G^{-1}) \| T v \|_Y \), and \( \| T_G (v - \Gamma_G v) \|_Y \leq (2 + \beta_G^{-1}) \inf_{w \in \mathcal{P}(\mathcal{G})} \| T_G (v - w) \|_Y \);

(b) with \( \gamma_G := \inf_{0 \neq v \in \mathcal{P}(\mathcal{G})} \sup_{v \in \mathcal{P}(\mathcal{G})} \frac{\langle v, T_G v \rangle_X - \langle T_G v, v \rangle_X}{\| v \|_{X_B} \| v \|_{T_G}} \) and \( \rho_G := \max (T_G, \mathcal{P}(\mathcal{G})) \),

\[
\| T_G v \|_{T_G} \leq (2 + \rho_G + \gamma_G^{-1} + \beta_G^{-1}) \| v \|_{T_G}, \quad \text{and} \quad \| v - \Gamma_G v \|_{T_G} \leq (3 + \rho_G + \gamma_G^{-1} + \beta_G^{-1}) \| v - w \|_{T_G}.
\]

**Remark 3.36.** The construction of \( \Gamma_G \) depends on \( (T_G, \mathcal{P}(\mathcal{G})) \) and \( (T_G, \overline{\mathcal{P}}(\mathcal{G})) \), but not on \( \mathcal{D} \) or \( \mathcal{D}_G^c \). Meanwhile, the constants in Theorem 3.35 depend on \( (T_G, \mathcal{P}(\mathcal{G})) \) and \( (T_G, \overline{\mathcal{P}}(\mathcal{G})) \), but not on \( \mathcal{D} \) or \( \mathcal{D}_G^c \), neither on \( \mathcal{D}_G^c \) or \( \mathcal{D}_G^c \).

**Global commutative diagram.** Also, a global interpolation is defined as

\[
(3.21) \quad \Gamma_G^S : \prod_{K \in \mathcal{G}} H(S_K) \to \overline{\mathcal{Q}(\mathcal{G})}, \quad \text{by} \quad \Gamma_G^S (q)|_K = \Gamma_G^S (q)|_K, \quad \forall K \in \mathcal{G}.
\]

**Lemma 3.37.** Provided the assumptions of Lemma 3.32, \( T_G \Gamma_G^S T v = \Gamma_G^S T v \) for \( v \in \prod_{K \in \mathcal{G}} H(T_K) \).

3.3.3. **Optimal ABCD.** Let \( u \) and \( u_G \) be the solutions of (3.3) and (3.4), respectively. Then, by Theorem 3.31, the error estimation reads

\[
\| u - u_G \|_{T_G} \leq C \inf_{w_G \in \mathcal{D}_G^{abc}} \| u - w_G \|_{T_G} + \inf_{w_G \in \mathcal{D}_G^c} \| T_G u - w_G \|_{T_G} + \sup_{v_G \in \mathcal{D}_G^{abc}} \inf_{w_G \in \mathcal{D}_G^c} \| v_G - w_G \|_{T_G}.
\]

In general, \( \mathcal{D}_G^{abc} \subseteq \mathcal{D} \), and

\[
\| u - u_G \|_{T_G} \leq C \inf_{w_G \in \mathcal{P}(\mathcal{G})} \| u - w_G \|_{T_G} + \inf_{w_G \in \mathcal{D}_G^c} \| T_G u - w_G \|_{T_G}.
\]

Namely, up to a constant, the scheme can exploit the best approximation by \( \overline{\mathcal{P}(\mathcal{G})} \) to \( \mathcal{D} \), and the total accuracy depends on the approximation by \( \mathcal{D}_G^c \) to \( \mathcal{D}_G^c \). It is interesting to note by Remark...
3.36 that the constant $C$ does not rely on the choice of $\mathbb{D}_G^c$. For a pair of partially adjoint discretization, where $\mathbb{D}_G^c = \mathbb{D} \cap \overline{\mathbb{D}(G)}$ is the biggest possible conforming subspace, the obtained scheme bears equivalently the best accuracy among all ABCDs. Therefore, a pair of partially adjoint discretization consists of both an optimal CD and an optimal ABCD.

3.4. **Examples of ABCDs.** Let $\Omega \subset \mathbb{R}^2$ be a simply-connected polygon. As usual, we use $\nabla$, curl, div and rot to denote the gradient operator, curl operator, divergence operator and rot operator, respectively. We use $H^1(\Omega)$, $H^1_0(\Omega)$, $H(\text{div}, \Omega)$, $H_0(\text{div}, \Omega)$, $H(\text{rot}, \Omega)$, $H_0(\text{rot}, \Omega)$, and $L^2(\Omega)$ to denote certain Sobolev spaces, and specifically, denote $L^2_0(\Omega)$ := \(\left\{w \in L^2(\Omega) : \int_{\Omega} wd\mathbf{x} = 0\right\}\). For here, we denote vector-valued quantities by “\(\cdot\)”. We use $(\cdot, \cdot)$ with subscripts to represent $L^2$ inner product. We here present some examples of partially adjoint discretizations, by CD and ABCD, to the adjoint operators $[\nabla, H^1(\Omega)], (\text{div}, H_0(\text{div}, \Omega))]$ and $[\nabla, H^1_0(\Omega)], (\text{div}, H(\text{div}, \Omega))$]

Let $\mathcal{T}_h$ be a shape-regular triangular subdivision of $\Omega$ with mesh size $h$, such that $\overline{\Omega} = \bigcup_{T \in \mathcal{T}_h} \overline{T}$, and every boundary vertex is connected to at least one interior vertex. Denote by $\mathcal{E}_h$, $\mathcal{E}_h^i$, $\mathcal{E}_h^b$, $X_h$, $X_h^i$, $X_h^b$ and $X_h^v$ the set of edges, interior edges, boundary edges, vertices, interior vertices, boundary vertices and corners, respectively. We use the subscript “$\cdot_h$” to denote the dependence on triangulation. In particular, an operator with the subscript “$\cdot_h$” indicates that the operation is performed cell by cell.

Denote by $V^1_\cdot_h$ and $V^2_\cdot_h$ the continuous linear and quadratic element spaces, and by $V^\text{RT}_\cdot_h$ and $V^\text{BDM}_\cdot_h$ the Raviart-Thomas [40] and Brezzi-Douglas-Marini [14] element space of lowest degree on $\mathcal{T}_h$. Denote $V^1_{h_0} := V^1_\cdot_h \cap H^1_0(\Omega)$, $V^2_{h_0} := V^2_\cdot_h \cap H^1_0(\Omega)$, $V^\text{RT}_{h_0} := V^\text{RT}_\cdot_h \cap H_0(\text{div}, \Omega)$ and $V^\text{BDM}_{h_0} := V^\text{BDM}_\cdot_h \cap H_0(\text{div}, \Omega)$. We also use $V^\text{RT}_{h_0}$ and $V^\text{BDM}_{h_0}$ for the rotation of $V^\text{BDM}_\cdot_h$ and $V^\text{BDM}_\cdot_h$, respectively, which are conforming subspaces of $H(\text{rot}, \Omega)$ and $H_0(\text{rot}, \Omega)$, respectively.

3.4.1. **A nonconforming $H(\text{div})$ element space by Raviart-Thomas’ shape functions.** On a triangle $T$, denote $\mathcal{R}(T) := \text{span} \{a + b\mathbf{x} : a \in \mathbb{R}^2, b \in \mathbb{R}\}$. Then

$$\mathcal{R}(\text{div}, \mathcal{R}(T)) = \mathbb{R} = N(\nabla, P_1(T)), \ N(\text{div}, \mathcal{R}(T)) = \mathbb{R}^2 = \mathcal{R}(\nabla, P_1(T))$$

Denote

$$\mathcal{R}(\mathcal{T}_h) := \bigcap_{T \in \mathcal{T}_h} \mathcal{R}(T), \text{ and } P_1(\mathcal{T}_h) := \bigcap_{T \in \mathcal{T}_h} P_1(T).$$

Define

$$\mathcal{R}^{\text{sub}}_{\mathcal{T}_h} := \left\{ \mathbf{v}_h \in \mathcal{R}(\mathcal{T}_h) : \sum_{T \in \mathcal{T}_h} (\mathbf{v}_h, \nabla \mathbf{v}_h)_T + (\text{div} \mathbf{v}_h, \mathbf{v}_h)_T = 0, \forall \mathbf{v}_h \in V^1_{h_0} \right\},$$

and

$$\mathcal{R}^{\text{sub}}_{h_0} := \left\{ \mathbf{v}_h \in \mathcal{R}(\mathcal{T}_h) : \sum_{T \in \mathcal{T}_h} (\mathbf{v}_h, \nabla \mathbf{v}_h)_T + (\text{div} \mathbf{v}_h, \mathbf{v}_h)_T = 0, \forall \mathbf{v}_h \in V^1_{h} \right\}.$$
Proposition 3.38. \((1) \{ (\nabla_h, P_1(\mathcal{T}_h)), (\text{div}_h, RT(\mathcal{T}_h)) \} \) is a base operator pair.
\((2) \{ (\nabla, V^1_h), (\text{div}_h, RT^{abc}) \} \) and \(\{ (\nabla, V^1_h), (\text{div}_h, RT^{abc}_{h,0}) \} \) are both partially adjoint operators.
\((3) (\text{div}_h, RT^{abc}) \) is an ABCD of \((\text{div}, H(\text{div}, \Omega)) \) and \((\text{div}_{h,0}, RT^{abc}_{h,0}) \) is an ABCD of \((\text{div}, H_0(\text{div}, \Omega)) \).

Locally-supported basis functions of \(RT^{abc}_h\) and \(RT^{abc}_{h,0}\). Note that the spaces \(V^1_h\) and \(V^1_{h,0}\) each admits a set of locally support basis functions, the supports of which are each a patch around a vertex (for \(V^1_h\)) or around an interior vertex (for \(V^1_{h,0}\)). See Figure 1 for a reference. For \(a \in X_h\), denote by \(\psi_a\) the basis function of \(V^1_h\) supported on the patch around \(a\). By following the procedure of Theorem 3.26, a set of locally supported basis functions can be constructed for \(RT^{abc}_h\) as well as \(RT^{abc}_{h,0}\) by two steps. And we illustrate the supports below as well.

\[ b^{ai}_T := \frac{1}{2S} (x + a_i - a_j - a_k), \quad i = 1, 2, 3, \quad \{i, j, k\} = \{1, 2, 3\}. \]

Then, \(\{b^{ai}_T, i = 1, 2, 3\}\) form a set of linearly independent basis functions of \(RT(T)\). Particularly, \(b_T^{ai} \cdot n_{\mid e_j} = (1 - 2\delta_{ij}) \mid e_j\), and
\[ (b_T^{ai}, \nabla \lambda_j) - (\text{div} b_T^{ai}, \lambda_j) = \delta_{ij}, \quad 1 \leq i, j \leq 3. \]

We note that \(\lambda_j = \psi_{a_j} \mid T\).

Step 1. On a cell \(T \in \mathcal{T}_h\) with vertices \(a_i\) and opposite edges \(e_i\), \(i = 1:3\), let \(\lambda_i\) \(i=1:3\), be the barycentric coordinates, and let \(S\) denote its area; cf. Figure 2, left. Denote

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Illustration of supports of \(\psi_A, \psi_B\) and \(\psi_C\). \(A\) (as well as \(B\)) denotes an interior vertex, and \(C\) denotes a boundary vertex.}
\end{figure}

Step 2. The basis functions on every cell are combined associated with \(\psi_a\) to form a global basis function, with \(a \in E_h\) for \(RT^{abc}_h\) (with \(a \in E_{h,0}\) for \(RT^{abc}_{h,0}\) respectively). Finally, the constructed basis functions fall into two categories.

- Category–I: Supported on successive cells. \(RT^{abc}_{h,0}\) consists of basis functions of this type.

On a patch of a vertex (see Figure 1 for patches of interior vertices \(A\) and \(B\) and a patch of boundary vertex \(C\) which consists of \(m\) cells, there are \((m - 1)\) such basis functions; see Figure 3 for illustrations. Here \(A\) is an interior vertex; we recall Figure 1. The support of
\[
\psi_A = \sum_{i=1}^{6} \gamma_i b_{T_i},
\]
such that
\[
\sum_{i=1}^{6} \left( \gamma_i b_{T_i}, \nabla (\psi_A |_{T_i}) \right)_{T_i} + \left( \gamma_i \operatorname{div} b_{T_i}, \psi_A |_{T_i} \right)_{T_i} = 0.
\]

Then 5 linearly independent basis functions can be solved out, and they are each supported on a two-cell patch contained in the support of \(\psi_A\). See Figure 4(b) for illustration on a patch of a boundary vertex \(C\) (cf. Figure 1).

- **Category–II**: Supported on a single cell. \(\mathbb{RT}_{h}^{abc}\) consists of basis functions of this type added on \(\mathbb{RT}_{h0}^{abc}\).

  On a patch of a boundary vertex (see Figure 1 for a patch of boundary vertex \(C\)) which consists of \(m\) cells, there are \(m\) such basis functions. See Figure 4(a) for illustration on a patch of a boundary vertex \(C\).
Remark 3.39. As shown in Figure 1 (illustrated by Figure 5), a same pair of neighbored triangles can be covered in both the supports of \( \psi_A \) and \( \psi_B \). We have to note that, the basis functions of \( \mathbb{RT}^{abc}_h \) supported thereon associated with \( A \) consists of \( b^A_{\tau_1} \) and \( b^A_{\tau_2} \), while the basis functions associated with \( B \) consists of \( b^B_{\tau_1} \) and \( b^B_{\tau_2} \). Further, basis functions associated with different vertices are all linearly independent.

**Cell-wise-defined global interpolation.** By virtue of Definition 3.29, the cell-wise interpolator \( \mathbb{I}^\text{RT}_T \) is defined by

\[
(\mathbb{I}^\text{RT}_T \tau, \nabla v)_T + (\text{div} \mathbb{I}^\text{RT}_T \tau, v)_T = (\tau, \nabla v)_T + (\text{div} \tau, v)_T, \quad \forall v \in P_1(T), \quad \text{for } T \in \mathcal{H}(\text{div}, T).
\]

By noting the duality (3.22), we have precisely

\[
\mathbb{I}^\text{RT}_T \tau = \sum_{i=1}^{3} \left[ (\tau, \nabla \lambda_i)_T + (\text{div} \tau, \lambda_i)_T \right] b^a_{\tau_i}.
\]

On the triangulation \( T_h \), define the global interpolator by

\[
\mathbb{I}^\text{RT}_h : \prod_{T \in T_h} H(\text{div}, T) \to \mathbb{RT}(T_h), \quad (\mathbb{I}^\text{RT}_h T) = \mathbb{I}^\text{RT}_T |_{T} \tau, \quad \forall \ T \in T_h.
\]
Then
\[ \| \sigma - \|_{h}^{RT} \sigma \|_{\text{div} h} \leq C \inf_{\tau_{h} \in \big{RT}(T_{h})} \| \sigma - \tau_{h} \|_{\text{div} h}, \]

C depending on the regularity of the triangulation only. Further
\[ \mathcal{R}(v_{h}^{RT}, H(\text{div}, \Omega)) \subset \mathbb{R}^{\text{abc}}_{h}, \quad \text{and} \quad \mathcal{R}(v_{h}^{RT}, H_{0}(\text{div}, \Omega)) \subset \mathbb{R}^{\text{abc}}_{h}. \]

**Discrete Helmholtz decomposition.** By Theorem 3.8, a discrete Helmholtz decomposition holds that
\[ \prod_{T \in T_{h}} (P_{0}(T))^{2} = \mathcal{R}(\nabla, \mathcal{V}_{h}^{1}) \oplus^{\perp} \mathcal{N}(\text{div} h, \mathbb{R}^{\text{abc}}_{h}) = \mathcal{R}(\nabla, \mathcal{V}_{h}^{1}) \oplus^{\perp} \mathcal{N}(\text{div} h, \mathbb{R}^{\text{abc}}_{h}). \]

3.4.2. The Crouzeix-Raviart element: reinterpretation. Temporarily in this section, we use the subscript \( \perp \) to denote rotation. Namely, \( x^{\perp} = (-y, x) \). On a triangle \( T \), denote \( \mathbb{R}^{\perp}(T) := \text{span} \{ \alpha + \beta x^{\perp} : \alpha \in \mathbb{R}^{2}, \beta \in \mathbb{R} \} \). Then,
\[ \mathcal{R}(\text{rot}, \mathbb{R}^{\perp}(T)) = \mathcal{N}(\text{curl}, P_{1}(T)), \quad \text{and} \quad \mathcal{N}(\text{rot}, \mathbb{R}^{\perp}(T)) = \mathcal{R}(\text{curl}, P_{1}(T)). \]

Denote \( \mathbb{R}^{\perp}(T_{h}) := \prod_{T \in T_{h}} \mathbb{R}^{\perp}(T) \), and denote
\[ \mathbb{R}^{\perp}_{h} := \mathbb{R}^{\perp}(T_{h}) \cap H(\text{rot}, \Omega), \quad \text{and} \quad \mathbb{R}^{\perp}_{h_{0}} := \mathbb{R}^{\perp}(T_{h}) \cap H_{0}(\text{rot}, \Omega). \]

**Lemma 3.40.** Given \( v_{h} \in \mathbb{P}_{k}^{1}, \) \( v_{h} \in \mathbb{V}^{CR}_{h} \) \( (v_{h} \in \mathbb{V}^{CR}_{h_{0}}, \) respectively) if and only if it holds for \( v_{h} \) and any \( \tau_{h} \in \mathbb{V}^{CR}_{1} \) \( (\tau_{h} \in \mathbb{R}^{\perp}_{h_{0}}) \) respectively that
\[ \sum_{T \in T_{h}} \int_{T} (\text{curl} v_{h} : \tau_{h} - v_{h} \text{rot} \tau_{h}) = 0. \quad (3.23) \]

**Remark 3.41.** The identity (3.23) is usually called a discrete Green formula and is far well known for any \( v_{h} \in \mathbb{V}^{CR}_{h} \) \( (v_{h} \in \mathbb{V}^{CR}_{h_{0}}) \) and any \( \tau_{h} \in \mathbb{R}^{\perp}_{h_{0}} \) \( (\tau_{h} \in \mathbb{R}^{\perp}_{h}) \) respectively. Though, the “if” part of the lemma, which is also evident, has not been seen discussed in literature.

**Proposition 3.42.**
1. \( [(\text{rot}, \mathbb{R}^{\perp}_{h}), (\text{curl} v_{h}, \mathbb{V}^{CR}_{h_{0}})] \) and \( [(\text{rot}, \mathbb{R}^{\perp}_{h_{0}}), (\text{curl} v_{h}, \mathbb{V}^{CR}_{h})] \) are both partially adjoint based on \( [(\text{rot}_{h}, \mathbb{R}^{\perp}(T_{h}))], (\text{curl} v_{h}, \mathbb{P}_{1}(T_{h}))]. \)
2. \( (\text{curl} v_{h}, \mathbb{V}^{CR}_{h_{0}}) \) is an ABCD of \( (\text{curl} H_{1}(\Gamma)) \) and \( (\text{curl} v_{h}, \mathbb{V}^{CR}_{h}) \) is an ABCD of \( (\text{curl} H_{1}(\Omega)). \)

By following the procedure of Theorem 3.26, we can construct a set of locally-supported global basis functions for \( \mathbb{V}^{CR}_{h} \) and \( \mathbb{V}^{CR}_{h_{0}} \). This set of basis functions are identical to the standard basis functions of the Crouzeix-Raviart element spaces. Similarly, by following Definition 3.29 and (3.20), we can define local and global interpolators \( \|_{h}^{CR} \) and \( \|_{h}^{CR} \), they coincide to the standard Crouzeix-Raviart interpolators for \( H_{1}(T) \) and \( H_{1}(\Omega) \) functions, respectively.

Again and in the sequel, for a cell \( T \in \mathcal{T}_{h} \), we use \( a_{i} \) and \( e_{i} \) for the vertices and opposite edges, \( h_{i} \) be the height on \( e_{i}, i = 1 : 3 \). Let \( \lambda_{i} \) be the barycentric coordinates, and let \( |e_{i}| \) and \( |h_{i}| \) denote the length of \( e_{i} \) and \( h_{i} \), respectively, \( , i = 1, 2, 3 \). Denote \( y_{i} := \frac{1}{|h_{i}|}(x^{\perp} - x^{\perp}_{k}), i = 1:3 \). Then \( y_{i}'s \) are the
restrictions of the respective global basis functions of $\mathbb{RT}_h^+$, and $\psi_j \cdot t_j = \delta_{ij}$ on $e_j$, $t_j$ being the unit tangential vector along $e_j$. Denote
\[ b_k := (\lambda_i + \lambda_j - \lambda_k)/|e_k|, \ {i, j, k} = \{1, 2, 3\}, \ k = 1, 2, 3. \]
Then
\[ (\psi_i, \text{curl}b_j)_T - (\text{rot}\psi_j, b_j)_T = \delta_{ij}, \ 1 \leq i, j \leq 3. \]

By virtue of Definition 3.29, the cell-wise interpolator $I_{CR}$ is defined by
\[ (I_{CR}^T v, \text{rot} \eta)_T - (\text{curl} I_{CR}^T v, \eta)_T = (v, \text{rot} \eta)_T - (\text{curl} v, \eta)_T, \ \forall \eta \in \mathbb{RT}^h(T), \ \text{for} \ v \in H^1(T). \]
By noting the duality (3.24), we have precisely
\[ I_{CR}^T v = \sum_{i=1}^{3} [(v, \text{rot} \psi_i)_T - (\text{curl} v, \psi_i)_T] b_i = \sum_{i=1}^{3} (\int_{e_i} v) b_i. \]
This is exactly the classical Crouzeix-Raviart interpolators for $H^1(T)$ functions. The nodal parameter of the C-R element has an interpretation that
\[ \int_{e_i} v = \int_{e_i} v \psi_i \cdot t_i = \sum_{j=1}^{3} \int_{e_i} v \psi_j \cdot t_j = (v, \text{rot} \psi_i)_T - (\text{curl} v, \psi_i)_T. \]
The nodal parameter indeed collects the information of the interpolated function along the whole boundary of $T$ or even the whole cell $T$ by the aid of the “dual” basis $\psi_i$.

By Lemma 3.37, the local and global commutative diagrams hold.

**Lemma 3.43.** Denote by $P_T$ the $L^2$ projection to the space of constants on a triangle $T$, and by $P_h$ the $L^2$ projection to the space of piecewise constants on a grid $T_h$. The below exact complexes and commutative diagrams hold:

\[
\begin{aligned}
\mathbb{R} & \longrightarrow H^1(T) \quad \text{curl} \quad H(\text{div}, T) \quad \text{div} \quad L^2(T) \quad \frac{\int}{f} \quad \mathbb{R} \\
\mathbb{R} & \longrightarrow P_1(T) \quad \text{curl} \quad \mathbb{RT}(T) \quad \text{div} \quad \mathbb{R} \quad \frac{\int}{f} \quad \mathbb{R} \\
\mathbb{R} & \longrightarrow H^1(\Omega) \quad \text{curl} \quad H(\text{div}, \Omega) \quad \text{div} \quad L^2(\Omega) \quad \frac{\int}{f} \quad \mathbb{R} \\
\mathbb{R} & \longrightarrow V_{CR}^h \quad \text{curl}_h \quad \mathbb{RT}_{abc}^h \quad \text{div}_h \quad \prod_{T \in T_h} P_0(T) \quad \frac{\int}{f} \quad \mathbb{R} \\
\{0\} & \longrightarrow H^1_0(\Omega) \quad \text{curl} \quad H_0(\text{div}, \Omega) \quad \text{div} \quad L^2_0(\Omega) \quad \frac{\int}{f} \quad \{0\} \\
\{0\} & \longrightarrow V_{CR}^{h0} \quad \text{curl}_h \quad \mathbb{RT}_{abc}^{h0} \quad \text{div}_h \quad L^2_0(\Omega) \cap \prod_{T \in T_h} P_0(T) \quad \frac{\int}{f} \quad \{0\}.
\end{aligned}
\]
Proof. The commutative diagrams follow by Lemmas 3.32 and 3.37. The exactness of the complexes follow from the exactness of the dual complexes. Particularly, for example, the dual complex of

\begin{equation}
\mathbb{R} \rightarrow V_h^{CR} \xrightarrow{\text{curl}_h} \mathbb{R}T^{abc}_h \xrightarrow{\text{div}_h} \prod_{T \in \mathcal{T}_h} P_0(T) \xrightarrow{\text{f}.} \mathbb{R}
\end{equation}

is

\begin{equation}
\{0\} \xleftarrow{\text{f}.} L_0^2(\Omega) \cap \prod_{T \in \mathcal{T}_h} P_0(T) \xrightarrow{\text{rot}^{T^+}_{h0}} \mathbb{R}T^{rot}_{h0} \xleftarrow{\nabla V_{h0}} \{0\}.
\end{equation}

As \( \Omega \) is simply connected, the latter complex is well known to be exact, and by Theorem 3.9, so is the former one. The proof is completed.

3.4.3. A nonconforming \( H^1 \) element space by enhanced quadratic polynomials. Let \( T \in \mathcal{T}_h \) be a triangle, and denote on \( T \)

\[ \psi_B := \lambda_1 \lambda_2 (\lambda_1 - \lambda_2) + \lambda_2 \lambda_3 (\lambda_2 - \lambda_3) + \lambda_3 \lambda_1 (\lambda_3 - \lambda_1) \]

and

\[ \psi_0 = \lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_1 - \frac{1}{6}. \]

Note that \( \langle \text{curl} \psi_B, \mathbf{n}\rangle_{\mathcal{E}_h} = \lambda_i \lambda_j, \{i, j, k\} = \{1, 2, 3\}, \) and \( \langle \nabla \psi_0, T \rangle_T + \langle \psi_0, \text{div} T \rangle_T = 0, \forall T \in P_1(T). \)

Therefore, with respect to \( P_1(T) \), set \( P_0^{2+}(T) := \text{span } \{\psi_0\} \). Decompose, \( P_1^{2+}(T) = P_0^{2+}(T) \oplus P_B^{2+}(T), \) where \( P_B^{2+}(T) := \{v \in P_2(T) : \langle \nabla v, \nabla \psi_0 \rangle = 0\} \).

Evidently, \( \mathcal{R}(\text{div}, P_1(T)) = \mathcal{N}(\nabla, P_B^{2+}(T)). \) On the other hand, \( \mathcal{N}(\text{div}, P_1(T)) \) and \( \mathcal{R}(\nabla, P_B^{2+}(T)) \) are isomorphic: it is easy to verify that, given \( 0 \neq \tau \in \mathcal{R}(\nabla, P_B^{2+}(T)), \) there exists an \( \eta \in \mathcal{N}(\text{div}, P_1(T)), \) such that \( (\tau, \eta)_T \neq 0; \) given \( \eta \in \mathcal{N}(\text{div}, P_1(T)), \) there exists a \( 0 \neq \tau \in \mathcal{R}(\nabla, P_B^{2+}(T)), \) such that \( (\tau, \eta)_T \neq 0. \)

Define

\[ V^{e\mathcal{FS}}_h := \left\{ v_h \in \prod_{T \in \mathcal{T}_h} P_2^{2+}(T) : \sum_{T \in \mathcal{T}_h} (v_h, \text{rot}_h T) + (\text{curl} v_h, \mathcal{T}_h)_T = 0, \forall \mathcal{T}_h \in V^{e\mathcal{BDM}}_h \right\}, \]

and

\[ V^{e\mathcal{FS}}_{h0} := \left\{ v_h \in \prod_{T \in \mathcal{T}_h} P_2^{2+}(T) : \sum_{T \in \mathcal{T}_h} (v_h, \text{rot}_h T) + (\text{curl} v_h, \mathcal{T}_h)_T = 0, \forall \mathcal{T}_h \in V^e_{h0} \right\}. \]

Note that the finite element spaces possess the same continuity as that of the Fortin-Soulie element space \([25], \) and we thus call them enriched Fortin-Soulie element spaces.

Proposition 3.44. \( (1) \) \( [\text{curl}_h, \prod_{T \in \mathcal{T}_h} P_2^{2+}(T)], [\text{rot}_h, \prod_{T \in \mathcal{T}_h} P_1(T)] \) is a base operator pair.

\( (2) \) \( [\text{curl}_h, V^{e\mathcal{FS}}_h], [\text{rot}_h, V^{e\mathcal{BDM}}_{h0}] \) and \( [\text{curl}_h, V^{e\mathcal{FS}}_{h0}], [\text{rot}_h, V^{e\mathcal{BDM}}_h] \) are partially adjoint based on \( [\text{curl}_h, \prod_{T \in \mathcal{T}_h} P_2^{2+}(T)], [\text{rot}_h, \prod_{T \in \mathcal{T}_h} P_1(T)] \).
(3) \((\text{curl}_h, V^{\text{FS}}_h)\) is an ABCD of \((\text{curl}, H^1(\Omega))\) and \((\text{curl}_h, V^{\text{FS}}_{h_0})\) is an ABCD of \((\text{curl}, H^1_0(\Omega))\).

3.4.4. A nonconforming \(H(\text{div})\) element space by enhanced Brezzi-Douglas-Marini’s shape functions. Let \(T\) be a triangle. Denote

\[ P^{1+}(T) := P_1(T) \oplus \text{span} \{\text{curl}_B\}. \]

Evidently, \(\mathcal{R}(\text{div}, P^{1+}(T)) = \mathbb{R} = N(\nabla, P_2(T))\). On the other hand, it can be shown by elementary calculus that

\[ \inf_{\tau \in \mathcal{N}(\text{div}, P^{1+}(T))} \sup_{\eta \in \mathcal{R}(\nabla, P_2(T))} \frac{(\tau, \eta)_T}{\|\tau\|_{0,T} \|\eta\|_{0,T}} = \inf_{\tau \in \mathcal{N}(\text{div}, P^{1+}(T))} \sup_{\eta \in \mathcal{R}(\nabla, P_2(T))} \frac{(\tau, \eta)_T}{\|\tau\|_{0,T} \|\eta\|_{0,T}} \geq C > 0, \]

with \(C\) depending on the regularity of \(T\) only.

Define

\[ V^{\text{eBDM}}_h := \left\{ \eta_h \in \prod_{T \in T_h} P^{1+}(T) : \sum_{T \in T_h} (\tau_h, \nabla \eta_h)_{T} + (\text{div} \tau_h, \eta_h)_{T} = 0, \forall \eta_h \in \mathcal{V}^2_{h_0} \right\}, \]

and

\[ V^{\text{eBDM}} := \left\{ \eta_h \in \prod_{T \in T_h} P^{1+}(T) : \sum_{T \in T_h} (\tau_h, \nabla \eta_h)_{T} + (\text{div} \tau_h, \eta_h)_{T} = 0, \forall \eta_h \in \mathcal{V}^2_{h_0} \right\}. \]

**Proposition 3.45.**

1. \(\left[ (\text{div}_h, \prod_{T \in T_h} P^{1+}(T)), (\nabla_h, \prod_{T \in T_h} P_2(T)) \right] \) is a base operator pair.

2. \(\left[ (\text{div}_h, V^{\text{eBDM}}_h), (\nabla, \mathcal{V}^2_{h_0}) \right]\) and \(\left[ (\text{div}_h, V^{\text{eBDM}}), (\nabla, \mathcal{V}^2_{h_0}) \right]\) are both partially adjoint based on

\[ \left[ (\text{div}_h, \prod_{T \in T_h} P^{1+}(T)), (\nabla_h, \prod_{T \in T_h} P_2(T)) \right]; \]

3. \((\text{div}_h, V^{\text{eBDM}}_h)\) is an ABCD of \((\text{div}, H(\text{div}, \Omega))\) and \((\text{div}_h, V^{\text{eBDM}})\) is an ABCD of \((\text{div}, H_0(\text{div}, \Omega))\).

By following the procedure of Theorem 3.26, a set of locally supported basis functions can be constructed for \(V^{\text{FS}}_h\) as well as \(V^{\text{FS}}_{h_0}\), and \(V^{\text{eBDM}}_h\) as well as \(V^{\text{eBDM}}_{h_0}\). Similarly, by following Definition 3.29 and (3.20), we can define interpolators \(I^{\text{FS}}_h\) and \(I^{\text{eBDM}}_h\).

Similar to Lemma 3.43, we can prove the local and global exact complexes and commutative diagrams.

**Lemma 3.46.** Denote by \(P_T\) the \(L^2\) projection to the space of constants on a triangle \(T\), and by \(P_h\) the \(L^2\) projection to the space of piecewise constants on a grid \(T_h\). The below exact complexes and commutative diagrams hold:

\[
\begin{align*}
\mathbb{R} & \quad \rightarrow \quad H^1(T) \quad \rightarrow \quad H(\text{div}, T) \quad \rightarrow \quad L^2(T) \quad \rightarrow \quad \mathbb{R} \\
\mathbb{R} & \quad \rightarrow \quad P^*_2(T) \quad \rightarrow \quad P^{1+}(T) \quad \rightarrow \quad \mathbb{R} \quad \rightarrow \quad \mathbb{R} \quad ;
\end{align*}
\]

\[(3.30)\]
4.1. Some preliminaries.

4.1.1. \textit{L}^2 theory of exterior calculus revisited. Following [1], we denote by \( \Lambda^k(\Xi) \) the space of differential \( k \)-forms on an \( n \)-dimensional domain \( \Xi \), and \( L^2\Lambda^k(\Xi) \) consists of differential \( k \)-forms with coefficients in \( L^2(\Xi) \) component by component. \( L^2\Lambda^k(\Xi) \) is a Hilbert space with inner product \( \langle \cdot, \cdot \rangle_{L^2\Lambda^k(\Xi)} \). The exterior differential operator \( \mathbf{d}^k : \Lambda^k(\Xi) \rightarrow \Lambda^{k+1}(\Xi) \) is an unbounded operator from \( L^2\Lambda^k(\Xi) \) to \( \Lambda^{k+1}(\Xi) \). Denote, for \( 0 \leq k \leq n-1 \),

\[
H \Lambda^k(\Xi) := H(\mathbf{d}^k : L^2\Lambda^k(\Xi) \rightarrow L^2\Lambda^{k+1}(\Xi)) = \left\{ \omega \in L^2\Lambda^k(\Xi) : \mathbf{d}^k \omega \in L^2\Lambda^{k+1}(\Xi) \right\},
\]

and by \( H_0\Lambda^k(\Xi) \) the closure of \( C^0_0\Lambda^k(\Xi) \) in \( H \Lambda^k(\Xi) \).

The Hodge star operator \( \star \) maps \( L^2\Lambda^k(\Xi) \) isomorphically to \( L^2\Lambda^{n-k}(\Xi) \) for each \( 0 \leq k \leq n \). The codifferential operator \( \delta_{\mu} = (-1)^k \mathbf{d}^{n-k} \star \mu \) is unbounded from \( \Lambda^k(\Xi) \) to \( \Lambda^{k-1}(\Xi) \). Denote

\[
H^*\Lambda^k(\Xi) := H(\delta_{\mu} : L^2\Lambda^k(\Xi) \rightarrow L^2\Lambda^{k-1}(\Xi)) = \left\{ \mu \in L^2\Lambda^k(\Xi) : \delta_{\mu} \in L^2\Lambda^{k-1}(\Xi) \right\},
\]

and \( H^*_0\Lambda^k(\Xi) \) the closure of \( C^0_0\Lambda^k(\Xi) \) in \( H^* \Lambda^k(\Xi) \). Then \( H^*\Lambda^k(\Xi) = \star H \Lambda^{n-k}(\Xi) \), and \( H^*_0\Lambda^k(\Xi) = \star H_0\Lambda^{n-k}(\Xi) \). Further \( N(\mathbf{d}^k, H \Lambda^k) = \star N(\delta_{n-k}, H^* \Lambda^{n-k}) \), and \( \mathcal{R}(\mathbf{d}^{k-1}, H \Lambda^{k-1}) = \star \mathcal{R}(\delta_{n-k+1}, H^* \Lambda^{n-k+1}) \).

\textbf{Lemma 4.1.} \textit{[1]} The adjoint of

1. \( (\mathbf{d}^k, H \Lambda^k(\Xi)) \) is \( (\delta_{k+1}, H_0 \Lambda^{k+1}(\Xi)) \);
2. \( (\mathbf{d}^k, H_0 \Lambda^k(\Xi)) \) is \( (\delta_{k+1}, H^* \Lambda^{k+1}(\Xi)) \);
3. \( (\delta_{k+1}, H^* \Lambda^{k+1}(\Xi)) \) is \( (\mathbf{d}^k, H_0 \Lambda^k(\Xi)) \);
4. \( (\delta_{k+1}, H_0 \Lambda^{k+1}(\Xi)) \) is \( (\mathbf{d}^k, H \Lambda^k(\Xi)) \).

It is well known that \( \mathcal{R}(\mathbf{d}^k, H \Lambda^k(\Xi)) \) is closed in \( L^2\Lambda^{k+1}(\Xi) \), \( 0 \leq k \leq n-1 \). Meanwhile, \( \mathcal{R}(\mathbf{d}^k, H_0 \Lambda^k(\Xi)) \), \( \mathcal{R}(\delta_{k+1}, H^* \Lambda^{k+1}(\Xi)) \), and \( \mathcal{R}(\delta_{k+1}, H^* \Lambda^{k+1}(\Xi)) \) are all closed. In the sequel, we will drop the domain \( \Xi \) when there is no ambiguity brought in.

Lemmas 4.2 and 4.3 follow from Lemma 4.1 immediately.
Lemma 4.2 (Helmholtz decomposition). For $1 \leq k \leq n$

$$L^2 \Lambda^k = \mathcal{R}(d^{k-1}, H\Lambda^{k-1}) \oplus^\perp N(\delta_k, H_0\Lambda^k) = \mathcal{R}(d^{k-1}, H_0\Lambda^{k-1}) \oplus^\perp N(\delta_k, H\Lambda^k),$$

and for $0 \leq k \leq n - 1$

$$L^2 \Lambda^k = N(d^k, H\Lambda^k) \oplus^\perp \mathcal{R}(\delta_{k+1}, H_0^*\Lambda^{k+1}) = N(d^k, H_0\Lambda^k) \oplus^\perp \mathcal{R}(\delta_{k+1}, H\Lambda^{k+1}).$$

The complex structures hold that

$$\mathcal{R}(d^{k-1}, H\Lambda^{k-1}) \subset N(d^k, H\Lambda^k), \text{ and } \mathcal{R}(d^{k-1}, H_0\Lambda^{k-1}) \subset N(d^k, H_0\Lambda^k).$$

Denote,

(4.1) $\mathcal{H}_k := N(d^k, H\Lambda^k) \oplus^\perp \mathcal{R}(d^{k-1}, H\Lambda^{k-1})$, and $\mathcal{H}_0^k := N(d^k, H_0\Lambda^k) \oplus^\perp \mathcal{R}(d^{k-1}, H_0\Lambda^{k-1})$, and similarly

(4.2) $\mathcal{H}^*_k := N(\delta_k, H^*\Lambda^k) \oplus^\perp \mathcal{R}(\delta_{k+1}, H\Lambda^{k+1})$, and $\mathcal{H}_0^*_k := N(\delta_k, H_0^*\Lambda^k) \oplus^\perp \mathcal{R}(\delta_{k+1}, H_0\Lambda^{k+1})$.

Then

$$\mathcal{H}_k = \star \mathcal{H}_0^{n-k}, \mathcal{H}_0^k = \star \mathcal{H}_0^k\Lambda^{n-k}.$$

Lemma 4.3 (Poincaré-Lefschetz duality). $\mathcal{H}_k = \star \mathcal{H}_0^{n-k}$.

Moreover we pay attention on the isomorphisms between border spaces (4.3) which are not well presented yet. Denote $\hat{H}_k := \{\omega \in H\Lambda^k : d^k\omega = 0\}$, $\hat{H}_0^k := \{\omega \in H_0\Lambda^k : d^k\omega = 0\}$, and, $H^*\Lambda^k := \{\omega \in H\Lambda^k : \langle \omega, \mu \rangle_{L^2\Lambda^k} = 0, \forall \mu \in \hat{H}_0^k\}$. Define the border space by

(4.3) $H_B\Lambda^k := \{\omega \in H\Lambda^k : \langle d^k\omega, d^k\mu \rangle_{L^2\Lambda^{k+1}} = 0, \forall \mu \in H_0\Lambda^k, \langle \omega, \mu \rangle_{L^2\Lambda^k} = 0, \forall \mu \in \hat{H}_0^k\}.$

Then by the closedness of $d^k$ and $\delta_k$,

$$\mathcal{R}(d^k, H_B\Lambda^k) = \mathcal{R}(d^k, H\Lambda^k) \oplus^\perp \mathcal{R}(d^k, H_0\Lambda^k), \text{ and } N(d^k, H_B\Lambda^k) = N(d^k, H\Lambda^k) \oplus^\perp N(d^k, H_0\Lambda^k).$$

We can define $H_B^*\Lambda^k$ by an analogue of (4.3) with respect to $\delta_k$. Note that $H_B^*\Lambda^k = \star H_B^*\Lambda^{n-k}$.

The lemma below holds as a horizontal P-A-L type duality by Theorem 2.3 immediately.

Lemma 4.4. $\mathcal{R}(d^{k-1}, H_B\Lambda^{k-1}) = N(\delta_k, \star H_B^*\Lambda^{n-k})$; $N(d^k, H_B\Lambda^k) = \mathcal{R}(\delta_{k+1}, \star H_B^*\Lambda^{n-k-1})$.

Further, denote

(4.4) $\hat{H}_B\Lambda^k := \{\omega \in \hat{H}\Lambda^k : \langle \omega, \mu \rangle_{L^2\Lambda^k} = 0, \forall \mu \in \hat{H}_0\Lambda^k\},$

and

(4.5) $H_B^*\Lambda^k := \{\omega \in H^*\Lambda^k : \langle d^k\omega, d^k\mu \rangle_{L^2\Lambda^{k+1}} = 0, \forall \mu \in H_0\Lambda^k\}.$

The following two lemmas hold immediately.

Lemma 4.5. $H_B\Lambda^k = \hat{H}_B\Lambda^k \oplus^\perp H_B^*\Lambda^k$, orthogonal in $L^2\Lambda^k$.

Lemma 4.6. $d^{k-1}$ is an isomorphism from $H_B^*\Lambda^{k-1}$ onto $\star \hat{H}_B^*\Lambda^{n-k}$, and $\delta_k$ is an isomorphism from $\star H_B^*\Lambda^{n-k}$ onto $\hat{H}_B\Lambda^{k-1}$. 
By direct calculation, we have the lemma below.

**Remark 4.7.** The spaces $H_B\Lambda^{k-1}$ and $\star H_B \Lambda^{n-k}$ are isomorphic to the trace spaces of $H\Lambda^{k-1}$ and $\star H\Lambda^{n-k}$, respectively. Though, the intrinsic metric of the trace spaces can be subtle, while the metrics of $H_B\Lambda^{k-1}$ and $\star H_B \Lambda^{n-k}$ are clear. We illustrate Lemmas 4.4 and 4.6 with the decomposition below:

$$H_B\Lambda^{k-1} = \delta_k (d^{k-1})^{-1} \star H_B \Lambda^{n-k}$$

(4.6)

$$\hat{H}_B\Lambda^{k-1} = (d^{k-1})^{-1} \delta_k \star H_B \Lambda^{n-k}$$

4.1.2. **Basics of Whitney $k$-forms.** Denote the set of $k$-indices as

$$\mathcal{IX}_{k,n} := \{\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{N}^k : 1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_k \leq n, \mathbb{N} \text{ the set of integers}\}.$$  

For $\alpha \in \mathcal{IX}_{k,n},$

$$d^{k-1}(\kappa(dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_k})) = k dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_k},$$

and

$$\delta_{k+1}(\star \kappa \star (dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_k})) = (-1)^{(n-1)(n-k)}(dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_k}),$$

where $\kappa$ is the Koszul operator

$$\kappa(dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_k}) = \sum_{j=1}^{k} (-1)^{j+1} x_j \alpha_j dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_{j-1}} \wedge dx^{\alpha_{j+1}} \wedge \cdots \wedge dx^{\alpha_k}.$$  

Following [3], denote by $\mathcal{P}_r \Lambda^k$ the space of differential $k$-forms with polynomial coefficients of degree of at most $r$. The space of Whitney forms, the lowest-degree trimmed polynomial $k$-forms, associated with the operator $d^{k}$ is denoted by ([1, 3, 5])

$$\mathcal{P}_1^{-} \Lambda^k = \mathcal{P}_0 \Lambda^k + \kappa(\mathcal{P}_0 \Lambda^{k+1}).$$

We denote the space associated with the operator $\delta_k$ by

$$\mathcal{P}_1^{-} \Lambda^k := \star(\mathcal{P}_1^{-} \Lambda^{n-k}) = \mathcal{P}_0 \Lambda^k + \star \kappa \star (\mathcal{P}_0 \Lambda^{k-1}).$$

**Lemma 4.8.** $\mathcal{R}(d^{k}, \mathcal{P}_1^{-} \Lambda^k) = \mathcal{P}_0 \Lambda^{k+1} ; \mathcal{N}(d^{k}, \mathcal{P}_1^{-} \Lambda^k) = \mathcal{P}_0 \Lambda^k$.

Given $T$ a simplex, denote on $T$ $\tilde{x}^j = x^j - c_j$ where $c_j$ is a constant such that $\int_T \tilde{x}^j = 0$. Denote a simplex dependent Koszul operator

$$\kappa_T(dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_k}) := \sum_{j=1}^{k} (-1)^{(j+1)} \tilde{x}^j \alpha_j dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_{j-1}} \wedge dx^{\alpha_{j+1}} \wedge \cdots \wedge dx^{\alpha_k}, \text{ for } \alpha \in \mathcal{IX}_{k,n}.$$  

Then

$$d^{k-1} \kappa_T(dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_k}) = k dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_k}.$$  

Note that $\mathcal{P}_1^{-} \Lambda^k(T) = \mathcal{P}_0 \Lambda^k(T) \oplus \kappa_T(\mathcal{P}_0 \Lambda^{k+1}(T))$, and $\mathcal{P}_1^{-} \Lambda^k(T) = \mathcal{P}_0 \Lambda^k(T) \oplus \star \kappa_T \star (\mathcal{P}_0 \Lambda^{k-1}(T))$. By direct calculation, we have the lemma below.
Lemma 4.9. There exists a constant $C_{k,n}$, depending on the regularity of $T$, such that
\begin{equation}
\| \mu \|_{L^2(A^k(T))} \leq C_{k,n}h_T\| d^k \mu \|_{L^2(A^{k+1}(T))}, \quad \text{for } \mu \in \kappa_T(P_0^k\Lambda^{k+1}(T)),
\end{equation}
and
\begin{equation}
\| \mu \|_{L^2(A^k(T))} \leq C_{k,n}h_T\| \delta^k \mu \|_{L^2(A^{k+1}(T))}, \quad \text{for } \mu \in \kappa_T* (P_0^k\Lambda^{k-1}(T)).
\end{equation}

Denote
\[ P_0^k\Lambda^k(T) = \prod_{T \in \mathcal{G}_h} P_0^k(T), \quad P_1^k\Lambda^k(T) := \prod_{T \in \mathcal{G}_h} P_1^k(T), \quad \text{and} \quad P_1^{k-}\Lambda^k(T) := \prod_{T \in \mathcal{G}_h} P_1^{k-}\Lambda^k(T). \]

The conforming finite element spaces with Whitney forms are
\[ W_h\Lambda^k := P_1^-(\mathcal{G}_h) \cap H\Lambda^k, \quad W_{h0}\Lambda^k := P_1^-(\mathcal{G}_h) \cap H_0\Lambda^k, \]
and
\[ W^*_h\Lambda^k := P_1^{*+}\Lambda^k(T) \cap H^\star \Lambda^k, \quad W^*_{h0}\Lambda^k := P_1^{*+}\Lambda^k(T) \cap H_0^\star \Lambda^k. \]

Then
\[ W^*_h\Lambda^k = \star W_h\Lambda^{n-k}, \quad \text{and} \quad W^*_{h0}\Lambda^k = \star W_{h0}\Lambda^{n-k}. \]

These spaces are identical to the standard spaces of Whitney forms presented by the continuity of nodal parameters as in, e.g., [1].

Lemma 4.10. [3] There exists a constant $C_{k,n}$ uniform for $\mathcal{T}^G$, such that
\[ \text{icr}(d^k, W_h\Lambda^k) \leq C_{k,n}, \quad \text{and} \quad \text{icr}(d^k, W_{h0}\Lambda^k) \leq C_{k,n}. \]

Remark 4.11. We can find Lemma 4.10 proved in Section 5 of [3] but in the formulation of discrete Poincaré inequalities. We further note that, for any $k$ and $n$, there is a constant $C'_{k,n} > 0$, such that $\text{icr}(d^k, W_h\Lambda^k) \geq C'_{k,n}$ and $\text{icr}(d^k, W_{h0}\Lambda^k) \geq C'_{k,n}$ when $h$ is small enough.

Lemma 4.12. [1] $\mathcal{R}(d^k, W_h\Lambda^k) \subset N(d^{k+1}, W_h\Lambda^k)$ and $\mathcal{R}(d^k, W_{h0}\Lambda^k) \subset N(d^{k+1}, W_{h0}\Lambda^k)$.

Anallogues of Lemmas 4.10 and 4.12 hold for $(\delta^k, W^*_h\Lambda^k)$.

Denote
\[ \mathcal{H}_h\Lambda^k := N(d^k, W_h\Lambda^k) \ominus \mathcal{R}(d^{k-1}, W_h\Lambda^{k-1}), \quad \text{and} \quad \mathcal{H}_{h0}\Lambda^k := N(d^k, W_{h0}\Lambda^k) \ominus \mathcal{R}(d^{k-1}, W_{h0}\Lambda^{k-1}). \]

Lemma 4.13. ([1]) $\mathcal{H}_h\Lambda^k$ and $\mathcal{H}_{h0}\Lambda^k$ are isomorphic to $\mathcal{H}\Lambda^k$ and $\mathcal{H}_{0}\Lambda^k$, respectively.

Denote
\[ \mathcal{H}^*_h\Lambda^k := N(\delta^k, W^*_h\Lambda^k) \ominus \mathcal{R}(\delta^{k+1}, W^*_h\Lambda^{k+1}), \quad \text{and} \quad \mathcal{H}^*_{h0}\Lambda^k := N(\delta^k, W^*_{h0}\Lambda^k) \ominus \mathcal{R}(\delta^{k+1}, W^*_{h0}\Lambda^{k+1}). \]

Then
\[ \mathcal{H}_h\Lambda^k = \star \mathcal{H}^*_h\Lambda^{n-k}, \quad \text{and} \quad \mathcal{H}_{h0}\Lambda^k = \star \mathcal{H}^*_{h0}\Lambda^{n-k}. \]

4.2. Partially adjoint discretizations of adjoint exterior differential operators.
4.2.1. ABCDx of exterior differential operators by Whitney k-forms. On a subdivision $\mathcal{G}_h$, $d^k$ and $\delta_{k+1,h}$ denote the cell wise operations by $d^k$ and $\delta_{k+1}$, respectively. We follow the lines in Section 3.1 for adjoint operator pairs. Particularly, with $\mathcal{P}(\mathcal{G}) = \mathcal{P}_1^k \Lambda^k(\mathcal{G}_h)$ and $\mathcal{P}(\mathcal{G}) = \mathcal{P}_1^{-1} \Lambda^{k+1}(\mathcal{G}_h)$, we have $\mathcal{P}(\mathcal{G}) = \{0\}$ and $\mathcal{P}(\mathcal{G}) = \{0\}$, which implies that the respective indices of closed range are zero. By Lemma 4.8, Assumptions 3.1, 3.4, 3.5, and 3.6 can be verified. The relevant parameters evaluate as $\alpha_G = \beta_G = 1$, $\sigma_G = \kappa_G = 1$, and $\chi_G = \varepsilon_G = 1$ where as $\mathcal{H}_G = \mathcal{P}_G = \{0\}$ and $\mathcal{H}_G = \mathcal{P}_G = \{0\}$. For simplicity and for the ease of readers, we just use here the notations defined in Section 3.1. All results in the remaining of this subsection follow by Section 3.1 directly, except Lemma 4.20 for which we present a detailed proof.

**Lemma 4.14.** $\left[(d^k_h, \mathcal{P}_1^k(\mathcal{G}_h)), (\delta_{k+1,h}, \mathcal{P}_1^{-1} \Lambda^{k+1}(\mathcal{G}_h))\right]$ is a base operator pair.

Define

\[(4.9) \quad W_{h_0}^{abc} \Lambda^k := \{\omega_h \in \mathcal{P}_1^k (\mathcal{G}_h) : \langle \omega_h, \delta_{k+1} \eta_h \rangle_{L^2} = -\langle d^k_h \omega_h, \eta_h \rangle_{L^2}, \forall \eta_h \in W_{h_0}^* \Lambda^{k+1} \}, \]

and

\[(4.10) \quad W_{h_0}^{abc} \Lambda^k := \{\omega_h \in \mathcal{P}_1^k (\mathcal{G}_h) : \langle \omega_h, \delta_{k+1} \eta_h \rangle_{L^2} = -\langle d^k_h \omega_h, \eta_h \rangle_{L^2}, \forall \eta_h \in W_{h_0}^* \Lambda^{k+1} \}. \]

Similarly, define

\[(4.11) \quad W_{h_0}^{ab} \Lambda^k := \{\mu_h \in \mathcal{P}_1^{-1} \Lambda^k (\mathcal{G}_h) : \langle \delta_{k+1} \mu_h, \tau_h \rangle_{L^2} = -\langle d^k_h \mu_h, \tau_h \rangle_{L^2}, \forall \tau_h \in W_{h_0} \Lambda^k \}, \]

and

\[(4.12) \quad W_{h_0}^{ab} \Lambda^k := \{\mu_h \in \mathcal{P}_1^{-1} \Lambda^k (\mathcal{G}_h) : \langle \delta_{k+1} \mu_h, \tau_h \rangle_{L^2} = -\langle d^k_h \mu_h, \tau_h \rangle_{L^2}, \forall \tau_h \in W_{h_0} \Lambda^k \}. \]

Then

\[W_{h}^{abc} \Lambda^k = \star W_{h}^{abc} \Lambda^{n-k}, \quad \text{and} \quad W_{h_0}^{ab} \Lambda^k = \star W_{h_0}^{ab} \Lambda^{n-k}. \]

Note that $W_{h_0}^{abc} \Lambda^0$ and $W_{h_0}^{ab} \Lambda^n$ are basically the famous lowest-degree Crouzeix-Raviart element spaces [20]. Besides, we have, for example,

\[W_{h}^{abc} \Lambda^k = \{\mu_h \in \mathcal{P}_1^{-1} \Lambda^k (\mathcal{G}_h) : \langle \delta_{k+1} \mu_h, \tau_h \rangle_{L^2} = -\langle d^k_h \mu_h, \tau_h \rangle_{L^2}, \forall \tau_h \in W_{h_0} \Lambda^k \}. \]

We then present the properties for $W_{h}^{abc} \Lambda^k$ versus $W_{h}^{ab} \Lambda^k$. The properties for $W_{h_0}^{ab} \Lambda^k$ versus $W_{h_0} \Lambda^k$ are exactly the same.

**Proposition 4.15.** $W_{h}^{abc} \Lambda^k$ and $W_{h_0}^{ab} \Lambda^k$ each admits a set of locally supported basis functions.

Actually, as $W_{h_0} \Lambda^k$ each has locally supported nodal basis functions, locally supported basis functions of $W_{h_0}^{ab} \Lambda^k$ follow by the general procedure of Section 3.2.2, particularly Theorem 3.26. We refer to Section 3.4.1 for an example with precise constructions of basis functions.

**Lemma 4.16.** $\mathcal{R}(d^k_h, W_{h}^{abc} \Lambda^k) \subset N(d^k_{h+1}, W_{h}^{abc} \Lambda^{k+1})$ and $\mathcal{R}(d^k_h, W_{h_0}^{abc} \Lambda^k) \subset N(d^k_{h+1}, W_{h_0}^{abc} \Lambda^{k+1})$. 

Following Definition 3.29, we define a local interpolator $\mathbb{I}^d_T : H^k(\mathcal{T}) \to \mathcal{P}_1^\perp \Lambda^k(\mathcal{T})$ by

$$\langle \mathbb{I}^d_T \omega, \delta_{k+1}\mu \rangle_{L^2(\mathcal{T})} - \langle \mathbb{I}^d_T \omega, \mu \rangle_{L^2(\mathcal{T})} = \langle \omega, \delta_{k+1}\mu \rangle_{L^2(\mathcal{T})} - \langle \mathbb{I}^d_T \omega, \mu \rangle_{L^2(\mathcal{T})}, \quad \forall \mu \in \mathcal{P}_1^\perp \Lambda^{k+1}(\mathcal{T}),$$

and define a global interpolator, following (3.20),

$$\mathbb{I}^d_h : \bigcap_{T \in \mathcal{G}_h} H^k(\mathcal{T}) \to \mathcal{P}_1^\perp \Lambda^k(\mathcal{G}_h), \text{ by } (\mathbb{I}^d_h \omega)|_T = \mathbb{I}^d_T (\omega|_T), \quad \forall T \in \mathcal{G}_h.$$

**Lemma 4.17.** For $\omega \in \bigcap_{T \in \mathcal{G}_h} H^k(\mathcal{T})$, with $C_{k,n}$ uniform for $\mathcal{F}^\mathcal{G}$,

$$\|\omega - \mathbb{I}^d_h \omega\|_{d} \leq C_{k,n} \inf_{\mu_h \in \mathcal{P}_1^\perp \Lambda^k(\mathcal{G}_h)} \|\omega - \mu_h\|_{d}, \quad \text{on } \mathcal{G}_h.$$

**Theorem 4.18.** $\mathbb{I}^d_h H^k \subset W^{abc}_h \Lambda^k$ and $\mathbb{I}^d_h H_0^k \subset W^{abc}_{h0} \Lambda^k$. Further, the commutative diagrams below hold:

\[
\begin{array}{cccccccc}
\mathcal{R} & \rightarrow & \mathbb{I}^d_h H^0 & \rightarrow & H^1 & \rightarrow & \ldots & \rightarrow & H^n & \rightarrow & \mathcal{R} \\
\downarrow & & \downarrow & & \downarrow & & \ldots & & \downarrow & & \downarrow \\
\mathcal{R} & \rightarrow & W^{abc}_h H^0 & \rightarrow & W^{abc}_h H^1 & \rightarrow & \ldots & \rightarrow & W^{abc}_h H^n & \rightarrow & \mathcal{R} \\
0 & \rightarrow & H_0^0 & \rightarrow & H_0^1 & \rightarrow & \ldots & \rightarrow & H_0^n & \rightarrow & 0 \\
0 & \rightarrow & W^{abc}_{h0} H^0 & \rightarrow & W^{abc}_{h0} H^1 & \rightarrow & \ldots & \rightarrow & W^{abc}_{h0} H^n & \rightarrow & 0 \\
\end{array}
\]

Moreover, if the de Rham complex below is exact:

\[
\mathcal{R} \rightarrow H^0 \rightarrow H^1 \rightarrow \ldots \rightarrow H^n \rightarrow \mathcal{R},
\]

all the complexes aforementioned are exact.

4.2.2. Adjoint properties between conforming and nonconforming Whitney forms.

**Lemma 4.19.** Both $\left[ (d_h^k, W^{abc}_h \Lambda^k), (\delta_{k+1}, W^{*}_{h0} \Lambda^{k+1}) \right]$ and $\left[ (d_h^k, W^{abc}_{h0} \Lambda^k), (\delta_{k+1}, W^{*}_{h} \Lambda^{k+1}) \right]$ are partially adjoint based on $\left[ (d_h^k, \mathcal{P}_1^\perp \Lambda^k(\mathcal{G}_h)), (\delta_{k+1}, \mathcal{P}_1^\perp \Lambda^{k+1}(\mathcal{G}_h)) \right]$.

The proof follows directly by the definitions of $W^{abc}_h \Lambda^k$ and $W^{abc}_{h0} \Lambda^k$, and the definition of partially adjoint operators.

**Lemma 4.20.** There exists a constant $C_{k,n}$ dependent on the shape regularity of $\mathcal{G}_h$, such that

$$\text{icr}(d_h^k, W^{abc}_h \Lambda^k) \leq C_{k,n}, \quad \text{and} \quad \text{icr}(d_h^k, W^{abc}_{h0} \Lambda^k) \leq C_{k,n}.$$
Proof. By Lemma 4.9, $\text{icr}(d_h^k, \mathcal{P}_1^k \Lambda^k(T)) \leq \tilde{C}_{k,n} \cdot h_T, h_T$ the size of $T$. It then follows that
$$\text{icr}(d_h^k, \mathcal{P}_1^k \Lambda^k(G_h)) \leq C_{k,n}, \quad \text{and } \text{icr}(\delta_{k+1,h}, \mathcal{P}_1^{n-\Lambda^{k+1}}(G_h)) \leq C_{n-(k+1),n},$$
and if further $G_h$ is quasi-uniform,
$$\text{icr}(d_h^k, \mathcal{P}_1^k \Lambda^k(G_h)) \leq \tilde{C}_{k,n} \cdot h, \quad \text{icr}(\delta_{k+1,h}, \mathcal{P}_1^{n-\Lambda^{k+1}}(G_h)) \leq \tilde{C}_{n-(k+1),n} \cdot h.$$
Submit all relevant parameters into Theorem 3.7, and we obtain
$$\text{icr}(d_h^k, W_{h0}^{abc} \Lambda^k) \leq C_{k,n}, \quad \text{and } \text{icr}(d_h^k, W_{h0}^{abc} \Lambda^k) \leq C_{k,n},$$
and if further $G_h$ is quasi-uniform,
$$\text{icr}(\delta_{k+1}, W_{h0}^{*\Lambda^{k+1}}) - \tilde{C}_{k,n} h \leq \text{icr}(d_h^k, W_{h0}^{abc} \Lambda^k) \leq \text{icr}(\delta_{k+1}, W_{h0}^{*\Lambda^{k+1}}) + \tilde{C}_{k,n} h,$$
and
$$\text{icr}(\delta_{k+1}, W_{h0}^{*\Lambda^{k+1}}) - \tilde{C}_{k,n} h \leq \text{icr}(d_h^k, W_{h0}^{abc} \Lambda^k) \leq \text{icr}(\delta_{k+1}, W_{h0}^{*\Lambda^{k+1}}) + \tilde{C}_{k,n} h.$$
By noting the upper and lower bounds of $\text{icr}(\delta_{k+1}, W_{h0}^{\Lambda^{k+1}})$ and $\text{icr}(\delta_{k+1}, W_{h0}^{*\Lambda^{k+1}})$, we obtain the assertion of Lemma 4.20. The proof is completed. \qed

The lemma reveals that, the partially adjoint operators $(\delta_{k+1}, W_{h0}^{*\Lambda^{k+1}})$ and $(d_h^k, W_{h0}^{abc} \Lambda^k)$ have asymptotically the same indices of closed range up to a high order term as $h$ tends to zero. The Poincaré inequality follows for $(d_h^k, W_{h0}^{abc} \Lambda^k)$, with asymptotically a same constant as $(\delta_{k+1}, W_{h0}^{*\Lambda^{k+1}})$.

The results in the remaining of this subsection follow by Section 3.1 directly.

**Theorem 4.21** (Discrete Helmholtz decomposition). For $1 \leq k \leq n$,
$$\mathcal{P}_0 \Lambda^k(G_h) = R(d_h^{k-1}, W_{h0}^{abc} \Lambda^k) \oplus \perp N(\delta_{k}, W_{h0}^{abc} \Lambda^k),$$
for $0 \leq k \leq n - 1$,
$$\mathcal{P}_0 \Lambda^k(G_h) = N(d_h^k, W_{h0}^{abc} \Lambda^k) \oplus \perp \mathcal{R}(\delta_{k+1}, W_{h0}^{*\Lambda^{k+1}}) = N(d_h^k, W_{h0}^{abc} \Lambda^k) \oplus \perp \mathcal{R}(\delta_{k+1}, W_{h0}^{*\Lambda^{k+1}}).$$

**Remark 4.22.** The constructions of discrete Helmholtz decomposition illustrate that nonconforming finite element spaces are necessary and the internal continuity of the ABCDFES are proper.

Denote
$$\mathcal{H}_{h}^{abc} \Lambda^k := N(d_h^k, W_{h0}^{abc} \Lambda^k) \ominus \mathcal{R}(d_h^{k-1}, W_{h0}^{abc} \Lambda^{k-1}), \quad \text{and } \mathcal{H}_{h0}^{abc} \Lambda^k := N(d_h^k, W_{h0}^{abc} \Lambda^k) \ominus \mathcal{R}(d_h^{k-1}, W_{h0}^{abc} \Lambda^{k-1}).$$
Then by the discrete Helmholtz decomposition,
$$\mathcal{H}_{h}^{abc} \Lambda^k = \mathcal{H}_{h0}^{abc} \Lambda^k, \quad \text{and } \mathcal{H}_{h0}^{abc} \Lambda^k = \mathcal{H}_{h0}^{abc} \Lambda^k.$$

Denote
$$\mathcal{H}_h^{*,abc} \Lambda^k := N(\delta_{k,h}, W_{h0}^{*,abc} \Lambda^k) \ominus \mathcal{R}(\delta_{k+1,h}, W_{h0}^{*,abc} \Lambda^{k+1})$$
and
$$\mathcal{H}_{h0}^{*,abc} \Lambda^k := N(\delta_{k,h}, W_{h0}^{*,abc} \Lambda^k) \ominus \mathcal{R}(\delta_{k+1,h}, W_{h0}^{*,abc} \Lambda^{k+1}).$$
Then
$$\mathcal{H}_h^{abc} \Lambda^k = \mathcal{H}_{h0}^{abc} \Lambda^{n-k}, \quad \text{and } \mathcal{H}_{h0}^{abc} \Lambda^k = \mathcal{H}_{h0}^{abc} \Lambda^{n-k}.$$
**Theorem 4.23** (Discrete Poincaré-Lefschetz duality).

\[ \mathcal{H}^{abc}_h \Lambda^k = * \mathcal{H}^0 \Lambda^{n-k}, \text{ and } \mathcal{H}^{abc}_h \Lambda^k = * \mathcal{H}_h \Lambda^{n-k}. \]

**Theorem 4.24** (Discrete Hodge decomposition). For \(1 \leq k \leq n-1\),

\[ P_0 \Lambda^k(\mathcal{G}_h) = \mathcal{R}(d^{-1}_h, W^{abc}_h \Lambda^{k-1}) \oplus \mathcal{H}^{abc}_h \Lambda^k(= \mathcal{H}_0^* \Lambda^{n-k}) \oplus \mathcal{R}(\delta_{k+1}, W^*_h \Lambda^{k+1}) \]

Denote

\[ W^{abc}_{h0} \Lambda^k := \{ \omega_h \in W^{abc}_h \Lambda^k : d^h \omega_h = 0 \}, \text{ and } W^*_h \Lambda^k := \{ \omega_h \in W^*_h \Lambda^k : \delta_k \omega_h = 0 \}. \]

Denote accordingly

\[ W^{abc}_{h} \Lambda^k := \{ \omega_h \in W^{abc}_h \Lambda : \langle \omega_h, \mu_h \rangle_{L^2 \Lambda^k} = 0, \forall \mu_h \in \mathbb{W}^*_0 \Lambda^k \}, \text{ and } \]

\[ W^{abc}_{h0} \Lambda^k := \{ \omega_h \in W^{abc}_h \Lambda : \langle \omega_h, \mu_h \rangle_{L^2 \Lambda^k} = 0, \forall \mu_h \in \mathbb{W}^*_0 \Lambda^k \}. \]

**Theorem 4.25** (Horizontal P-A-L type duality).

\[ \mathcal{R}(d^k_h, W^{abc}_h \Lambda^k) = N(\delta_{k+1}, W^{abc}_{h} \Lambda^{k-1}) \text{, and } \mathcal{R}(\delta_{k+1}, W^*_h \Lambda^{k+1}) = N(d^k_h, W^{abc}_h \Lambda^k). \]

4.3. Equivalences between discretizations of elliptic variational problems.

4.3.1. Primal and dual discretizations of the elliptic source problem. For a model problem, we consider the source problem: given \(f \in L^2 \Lambda^k\), find \(\omega \in H \Lambda^k\), such that

\[ \langle d^k \omega, d^k \mu \rangle_{L^2 \Lambda^k} + \langle \omega, \mu \rangle_{L^2 \Lambda^k} = \langle f, \mu \rangle_{L^2 \Lambda^k}, \forall \mu \in H \Lambda^k. \]

Evidently, \(d^k \omega \in H^0 \Lambda^{k+1}\).

The dual formulation of the problem is then to find \((\xi, \overline{\omega}) \in H^0_0 \Lambda^{k+1} \times L^2 \Lambda^k\), such that

\[ \left\{ \begin{array}{l}
\langle \xi, \eta \rangle_{L^2 \Lambda^k} + \langle \overline{\omega}, \delta_{k+1} \eta \rangle_{L^2 \Lambda^k} = 0, \forall \eta \in H^0_0 \Lambda^{k+1} \\
\langle \delta_{k+1} \xi, \overline{\mu} \rangle_{L^2 \Lambda^k} + \langle \omega, \mu \rangle_{L^2 \Lambda^k} = \langle f, \mu \rangle_{L^2 \Lambda^k}, \forall \mu \in L^2 \Lambda^k.
\end{array} \right. \]

Then \(\overline{\omega} = \omega\), and \(\xi = d^k \omega = d^k \omega\).

Let \(P^k_h\) denote the \(L^2\) projection onto \(P_0 \Lambda^k(\mathcal{G}_h)\). We consider respective discretizations:

- for (4.18): find \(\omega_h \in W^{abc}_h \Lambda^k\), such that

\[ \langle d^k_h \omega_h, d^k_h \mu_h \rangle_{L^2 \Lambda^k} + \langle \omega_h, \mu_h \rangle_{L^2 \Lambda^k} = \langle f, \mu_h \rangle_{L^2 \Lambda^k}, \forall \mu_h \in W^{abc}_h \Lambda^k; \]

- for (4.19): find \((\xi_h, \overline{\omega}_h) \in W^*_0 \Lambda^{k+1} \times P_0 \Lambda^k(\mathcal{G}_h)\), such that

\[ \left\{ \begin{array}{l}
\langle P^k_h \xi_h, P^k_h \eta_h \rangle_{L^2 \Lambda^k} + \langle \overline{\omega}_h, \delta_{k+1} \eta_h \rangle_{L^2 \Lambda^k} = 0, \forall \eta_h \in W^*_0 \Lambda^{k+1}, \\
\langle \delta_{k+1} \xi_h, \overline{\mu}_h \rangle_{L^2 \Lambda^k} + \langle \omega_h, \mu_h \rangle_{L^2 \Lambda^k} = \langle f, \mu_h \rangle_{L^2 \Lambda^k}, \forall \mu_h \in P_0 \Lambda^k(\mathcal{G}_h).
\end{array} \right. \]

Namely, for here, we use a nonconforming finite element space to discretize (4.18), and a conforming pair for (4.19). Note that we here use low order numerical quadrature for the inner products denoted by a projection \(P^k_h\) or \(P^{k+1}_h\). By discrete Poincaré inequalities, (4.21) is wellposed by standard way.
Lemma 4.26. Let $\omega_h$ and $(\xi_h, \overline{\omega}_h)$ be the solutions of (4.20) and (4.21), respectively. Then
\[
\overline{\omega}_h = P_h^0 \omega_h, \quad P_h^k \omega_h + \delta_{k+1} \xi_h = P_h^k f, \quad \text{and} \quad d_h^k \omega_h = P_h^{k+1} \xi_h.
\]

Proof. Let $\omega_h$ be the solution of (4.20). For any $\mu_h \in P_1^k (G_h)$ and $\overline{\eta}_h \in W_h^\ast \Lambda^{k+1}$, with $\xi_h \in W_h^\ast \Lambda^{k+1}$,
\[
(4.22)
\begin{align*}
\{ & \langle d_h^k \omega_h, d_h^k \mu_h \rangle_{L^2 \Lambda^{k+1}} + \langle P_h^k \omega_h, P_h^k \mu_h \rangle_{L^2 \Lambda^k} + \langle \delta_{k+1} \xi_h, \mu_h \rangle_{L^2 \Lambda^k} - \langle \xi_h, d_h^k \mu_h \rangle_{L^2 \Lambda^{k+1}} = \langle f, P_h^k \mu_h \rangle_{L^2 \Lambda^k} = 0,
\end{align*}
\]
Choosing $\mu_h$ to be any in $P_0 \Lambda^k (G_h) \subset P_1^k (G_h)$, we obtain
\[
P_h^k \omega_h + \delta_{k+1} \xi_h = P_h^k f.
\]
Thus
\[
\langle d_h^k \omega_h, d_h^k \mu_h \rangle_{L^2 \Lambda^{k+1}} - \langle \xi_h, d_h^k \mu_h \rangle_{L^2 \Lambda^{k+1}} = 0, \forall \mu_h \in P_1^k (G_h),
\]
and therefore, $d_h^k \omega_h = P_h^{k+1} \xi_h$. Further, by the second equation of (4.22),
\[
\langle \overline{\eta}_h, P_h^{k+1} \xi_h \rangle_{L^2 \Lambda^{k+1}} - \langle \delta_{k+1} \overline{\eta}_h, P_h^k \omega_h \rangle_{L^2 \Lambda^k} = 0, \forall \overline{\eta}_h \in W_h^\ast \Lambda^{k+1}.
\]
We have thus
\[
\{ \langle P_h^{k+1} \xi_h, P_h^{k+1} \overline{\eta}_h \rangle_{L^2 \Lambda^{k+1}} - \langle P_h^k \omega_h, \delta_{k+1} \overline{\eta}_h \rangle_{L^2 \Lambda^k} = 0, \forall \overline{\eta}_h \in W_h^\ast \Lambda^{k+1},
\end{align*}
\]
Namely $(\xi_h, P_h^k \omega_h)$ solves (4.21). The proof is completed. \qed

Remark 4.27. Lemma 4.26 reconstructs the equivalence between the primal and dual formulations. Given the solution of one of (4.20) and (4.21), the solution of the other can be computed cell by cell by Lemma 4.26.

Remark 4.28. We here focus on the adjoint equivalence between schemes. The accuracies of the schemes, numerical quadratures taken into consideration, can be studied in standard ways.

4.3.2. Primal and dual discretizations of elliptic eigenvalue problems. We consider the eigenvalue problem: find $\omega \in H^k$ and $\lambda \neq 0$, such that
\[
(4.23) \quad \langle d^k \omega, d^k \mu \rangle_{L^2 \Lambda^{k+1}} = \lambda \langle \omega, \mu \rangle_{L^2 \Lambda^k}, \forall \mu \in H^k.
\]
Provided that $\lambda \sim \omega$ is such an eigenpair, denote $\xi = d^k \omega$, then $\xi \in H^k \Lambda^{k+1}$, and
\[
(4.24) \quad \langle \delta_{k+1} \xi, \delta_{k+1} \eta \rangle_{L^2 \Lambda^k} = \lambda \langle \xi, \eta \rangle_{L^2 \Lambda^{k+1}}, \forall \eta \in H^k \Lambda^{k+1}.
\]
(4.23) and (4.24) are the primal and dual formulations for the nonzero eigenvalues of $\delta_{k+1} d^k$, and the two have the same nonzero eigenvalues.

Lemma 4.29. Let $P_h^k$ denote the $L^2$ projection onto $P_0 \Lambda^k (G_h)$. Let $\omega_h \in W^{abc} \Lambda^k$ and $\lambda_h \neq 0$ be such that
\[
(4.25) \quad \langle d_h^k \omega_h, d_h^k \mu_h \rangle_{L^2 \Lambda^{k+1}} = \lambda_h \langle P_h^k \omega_h, P_h^k \mu_h \rangle_{L^2 \Lambda^k}, \forall \mu_h \in W^{abc} \Lambda^k.
\]
Then there exists a unique \( \zeta_h \in W^*_h \Lambda^{k+1} \), such that
\[
P_h^{k+1} \zeta_h = d_h^k \omega_h, \quad \delta_h^{k+1} \zeta_h = P_h^k \omega_h,
\]
and
\[
\langle \delta_h^{k+1} \zeta_h, \delta_h^{k+1} \eta_h \rangle_{L^2 \Lambda^k} = \lambda_h \langle P_h^{k+1} \zeta_h, P_h^{k+1} \eta_h \rangle_{L^2 \Lambda^{k+1}}, \quad \forall \eta_h \in W^*_h \Lambda^{k+1}.
\]
(4.26)

The proof is the same as that of Lemma 4.26, and we omit it here.

4.4. Equivalences between discretizations of the Hodge Laplace problem. In this section, we consider kinds of primal or dual discretizations of the Hodge Laplace problem: given \( f \in L^2 \Lambda^k \), with \( P^d_h \), the \( L^2 \) projection to \( \mathcal{H} \Lambda^k \), find
\[
\omega \in \mathcal{H} \Lambda^k(\Omega), \quad \text{and} \quad \delta_h^k \omega + d_h^k \omega = f - P^d_h f,
\]
(4.27)
with boundary conditions
\[
\omega \in H^k(\Omega) \cap H^1_0 \Lambda^k(\Omega), \quad d_h^k \omega \in H^1_0 \Lambda^{k+1}(\Omega), \quad \delta_h^k \omega \in H^1 \Lambda^{k-1}(\Omega).
\]
(4.28)
The primal weak formulation is: find \( \omega \in H \Lambda^k \cap H^1_0 \Lambda^k \), such that
\[
\left\{ \begin{array}{l}
\langle \omega, \zeta \rangle_{L^2 \Lambda^k} = 0, \\ 
\langle d_h^k \omega, d_h^k \mu \rangle_{L^2 \Lambda^{k+1}} + \langle \delta_h^k \omega, \delta_h^k \mu \rangle_{L^2 \Lambda^{k-1}} = \langle f - P^d_h f, \mu \rangle_{L^2 \Lambda^k}
\end{array} \right., \quad \forall \zeta \in \mathcal{H} \Lambda^k,
\]
(4.29)
Standard mixed formulations based on that \( \omega \in H \Lambda^k \) or \( \omega \in H^1_0 \Lambda^k \) are generally used; and we here simply call them “primal” and “dual” mixed formulations. They are respectively
- “primal” mixed formulation: to find \( (\omega^p, \sigma^p, \vartheta^p) \in H^k \Lambda^k \times H^{k-1} \Lambda^k \times \mathcal{H} \Lambda^k \), such that, for \( (\mu, \tau, \varsigma) \in H^k \Lambda^k \times H^{k-1} \Lambda^k \times \mathcal{H} \Lambda^k \),
\[
\left\{ \begin{array}{l}
\langle \omega^p, \tau \rangle_{L^2 \Lambda^{k+1}} = 0 \\ 
\langle \sigma^p, \tau \rangle_{L^2 \Lambda^k} - \langle \omega^p, d_h^k \tau \rangle_{L^2 \Lambda^k} = 0
\end{array} \right.
\]
(4.30)
- “dual” mixed formulation: to find \( (\omega^d, \zeta^d, \vartheta^d) \in H^1_0 \Lambda^k \times H^1_0 \Lambda^{k+1} \times \mathcal{H}^*_0 \Lambda^k \), such that, for \( (\mu, \eta, \varsigma) \in H^1_0 \Lambda^k \times H^1_0 \Lambda^{k+1} \times \mathcal{H}^*_0 \Lambda^k \),
\[
\left\{ \begin{array}{l}
\langle \zeta^d, \eta \rangle_{L^2 \Lambda^{k+1}} = 0 \\ 
\langle \delta_h^k \omega^d, \mu \rangle_{L^2 \Lambda^k} + \langle \delta_h^k \sigma^d, \mu \rangle_{L^2 \Lambda^{k-1}} = \langle f, \mu \rangle_{L^2 \Lambda^k}
\end{array} \right.
\]
(4.31)

As nonconforming Whitney forms are constructed for \( (d^k, \mathcal{H} \Lambda^k) \) and \( (\delta^k, H^1 \Lambda^k) \), some new discretizations are constructed for the Hodge Laplace problem. In this section, we study the primal and mixed formulations, and present lowest-degree discretizations of them respectively, namely (4.34), (4.36), (4.37) and (4.38); the schemes (4.34), (4.36), and (4.37) are new ones hinted by nonconforming finite element exterior calculus. Particularly, the equivalences among schemes (4.34), (4.36), (4.37) and (4.38) are constructed by Lemmas 4.33 and 4.34, in the sense that, given the solution of one of them four, the solutions of the others can be computed cell by cell. This way, the equivalences among the primal and different dual formulations are inherited by their respective discretizations.
4.4.1. A “completely” mixed element scheme. We begin with a “completely” mixed formulation: to find \( (\omega^c, \zeta^c, \sigma^c, \vartheta^c) \in L^2 \Lambda^k \times H_0^\Lambda^{k+1} \times H\Lambda^{k-1} \times \mathcal{H}\Lambda^k \), such that, for \( (\mu, \eta, \tau, \varsigma) \in L^2 \Lambda^k \times H_0^\Lambda^{k+1} \times H\Lambda^{k-1} \times \mathcal{H}\Lambda^k \),

\[
\begin{align*}
\langle \zeta^c, \eta \rangle_{L^2 \Lambda^{k+1}} &= 0 \\
-\langle \omega^c, \delta_{k+1} \eta \rangle_{L^2 \Lambda^k} &= 0 \\
\langle \sigma^c, \tau \rangle_{L^2 \Lambda^{k-1}} &= 0 \\
-\langle \omega^c, d^{-1} \tau \rangle_{L^2 \Lambda^k} &= 0.
\end{align*}
\]

(4.32)

Lemma 4.30. For \( f \in L^2 \Lambda^k \), the problem (4.32) admits a unique solution \( (\omega^c, \zeta^c, \sigma^c, \vartheta^c) \), and

\[
\|\omega^c\|_{L^2 \Lambda^k} + \|\zeta^c\|_{\delta_{k+1}} + \|\sigma^c\|_{d^{-1}} + \|\vartheta^c\|_{L^2 \Lambda^k} \leq C \|f\|_{L^2 \Lambda^k}.
\]

Further, \( \zeta^c = d^k \omega^c, \sigma^c = \delta_k \omega^c, \) and \( \omega^c \) solves (4.29).

Proof. For (4.33), we only have to verify Brezzi’s conditions, which hold by the orthogonal Hodge decomposition

\[ L^2 \Lambda^k = \mathcal{R}(d^{k-1}, H\Lambda^{k-1}) \oplus \mathcal{H}\Lambda^k \oplus \mathcal{R}(\delta_{k+1}, H_0^\Lambda^{k+1}), \]

with the closeness of \( \mathcal{R}(d^{k-1}, H\Lambda^{k-1}) \) and \( \mathcal{R}(\delta_{k+1}, H_0^\Lambda^{k+1}) \). The remaining assertions are straightforward. The proof is completed. \( \square \)

A lowest-degree stable discretization of (4.32) is: find \( (\omega_h^c, \zeta_h^c, \sigma_h^c, \vartheta_h^c) \in \mathcal{P}_0 \Lambda^k(\mathcal{G}_h) \times W_{h_0}^\Lambda^{k+1} \times W_{h}^{abc} \Lambda^{k-1} \times \mathcal{H}_{h}^{abc} \Lambda^k \), such that, for \( (\mu_h, \eta_h, \tau_h, \varsigma_h) \in \mathcal{P}_0 \Lambda^k(\mathcal{G}_h) \times W_{h_0}^\Lambda^{k+1} \times W_{h}^{abc} \Lambda^{k-1} \times \mathcal{H}_{h}^{abc} \Lambda^k \),

(4.34)

\[
\begin{align*}
\langle P_{h}^{k+1} \omega_h^c, P_{h}^{k+1} \eta_h \rangle_{L^2 \Lambda^{k+1}} &= 0 \\
-\langle \omega_h^c, \delta_{k+1} \eta_h \rangle_{L^2 \Lambda^k} &= 0 \\
\langle P_{h}^{k-1} \sigma_h^c, P_{h}^{k-1} \tau_h \rangle_{L^2 \Lambda^{k-1}} &= 0 \\
-\langle \omega_h^c, d^{-1} \tau_h \rangle_{L^2 \Lambda^k} &= 0.
\end{align*}
\]

Lemma 4.31. Given \( f \in L^2 \Lambda^k \), the problem (4.34) admits a unique solution \( (\omega_h^c, \zeta_h^c, \sigma_h^c, \vartheta_h^c) \), and

\[
\|\omega_h^c\|_{L^2 \Lambda^k} + \|\zeta_h^c\|_{\delta_{k+1}} + \|\sigma_h^c\|_{d^{-1}} + \|\vartheta_h^c\|_{L^2 \Lambda^k} \leq C \|f\|_{L^2 \Lambda^k}.
\]

The constant \( C \) depends on \( \text{incr}(\delta_{k+1}, W_{h_0}^\Lambda^{k+1}) \) and \( \text{incr}(d^{-1}, W_{h}^{abc} \Lambda^{k-1}) \).

Remark 4.32. The well-posedness of (4.34) depends on the discrete Hodge decomposition (4.17), which comes true by the aid of the nonconforming space \( W_{h}^{abc} \Lambda^k \). We thus recognize that (4.34) is a new scheme hinted in nonconforming finite element exterior calculus.

4.4.2. A theoretically lowest-degree primal discretization. Denote, on any simplex \( T \),

\[
P_{d^k}^m \Lambda^k(T) := \mathcal{P}_0 \Lambda^k(T) \oplus \kappa_T(\mathcal{P}_0 \Lambda^{k+1}(T)) \oplus \kappa_T \ast (\mathcal{P}_0 \Lambda^{k-1}(T))
\]

and, on the subdivision \( \mathcal{G}_h \),

\[
P_{d^k}^m \Lambda^k(\mathcal{G}_h) = \bigoplus_{T \in \mathcal{G}_h} P_{d^k}^m \Lambda^k(T).
\]
Define

\begin{equation}
V_{d=\delta}^m \Lambda^k := \left\{ \mu_h \in P_{d=\delta}^m \Lambda^k(G_h) : \langle d_h^k \mu_h, \eta_h \rangle_{L^2 \Lambda^{k+1}} - \langle \mu_h, \delta_{k+1,h} \eta_h \rangle_{L^2 \Lambda^k} = 0, \forall \eta_h \in W_{h_0}^* \Lambda^{k+1}, \right. \\
\left. \text{and } \langle \delta_{k,h} \mu_h, \tau_h \rangle_{L^2 \Lambda^{k-1}} - \langle \mu_h, d_h^{k-1} \tau_h \rangle_{L^2 \Lambda^k} = 0, \forall \tau_h \in W_{h}^{abc} \Lambda^{k-1} \right\}.
\end{equation}

By the discrete Hodge decomposition,

\[ \mathcal{H}_h \lambda^k = \left\{ \mu_h \in P_0 \Lambda^k(G_h) : \langle \mu_h, \delta_{k+1,h} \eta_h \rangle_{L^2 \Lambda^k} = 0, \forall \eta_h \in W_{h_0}^* \Lambda^{k+1}, \right. \\
\left. \text{and } \langle \mu_h, d_h^{k-1} \tau_h \rangle_{L^2 \Lambda^k} = 0, \forall \tau_h \in W_{h}^{abc} \Lambda^{k-1} \right\}.
\]

Hence, we consider the lowest-degree finite element scheme: find \( \omega_h \in V_{d=\delta}^m \Lambda^k \), such that

\begin{equation}
\begin{cases}
\langle d_h^k \omega_h, \eta_h \rangle_{L^2 \Lambda^{k+1}} + \langle \delta_{k,h} \omega_h, \delta_{k,h} \mu_h \rangle_{L^2 \Lambda^{k-1}} = \langle f - P_{\mathcal{H}_h \Lambda^k} f, \mu_h \rangle_{L^2 \Lambda^k}, & \forall \mu_h \in V_{d=\delta}^m \Lambda^k. \\
\end{cases}
\end{equation}

\( P_{\mathcal{H}_h \Lambda^k} \) denotes the projection to \( \mathcal{H}_h \Lambda^k \).

**Lemma 4.33.** *The system (4.36) admits a unique solution \( \omega_h \). Particularly, let \( (\omega_h^c, \zeta_h^c, \sigma_h^c, \phi_h^c) \) be the solution of (4.34), then, with \( P_T^k \) the L2 projection to \( P_0 \Lambda^k(T) \),

\[ \text{on every } T, \quad d^k \omega_h = P_T^k \zeta_h^c, \delta_T \omega_h = P_T^k \sigma_h^c, \text{ and } P_T^k \omega_h = P_T^k \phi_h^c. \]

**Proof.** Evidently, the solution of (4.36), if exists, is unique. Now, for \( (\omega_h^c, \zeta_h^c, \sigma_h^c, \phi_h^c) \) the solution of (4.34), there exists a unique \( \omega_h \in V_{d=\delta}^m \Lambda^k \), such that

\[ \text{on every } T, \quad d^k \omega_h = P_T^k \zeta_h^c, \delta_T \omega_h = P_T^k \sigma_h^c, \text{ and } P_T^k \omega_h = P_T^k \phi_h^c. \]

Then we have

\[ \langle d_h^k \omega_h, \eta_h \rangle_{L^2 \Lambda^{k+1}} - \langle \omega_h, \delta_{k+1,h} \eta_h \rangle_{L^2 \Lambda^k} = 0, \forall \eta_h \in W_{h_0}^* \Lambda^{k+1}, \]

and

\[ \langle \delta_{k,h} \omega_h, \tau_h \rangle_{L^2 \Lambda^{k-1}} - \langle \omega_h, d_h^{k-1} \tau_h \rangle_{L^2 \Lambda^k} = 0, \forall \tau_h \in W_{h}^{abc} \Lambda^{k-1}. \]

Therefore \( \omega_h \in V_{d=\delta}^m \Lambda^k \), and is orthogonal to \( \mathcal{H}_h \Lambda^k \). Now, given \( \mu_h \in V_{d=\delta}^m \Lambda^k \),

\[ \langle d_h^k \omega_h, d_h^k \mu_h \rangle_{L^2 \Lambda^{k+1}} + \langle \delta_{k,h} \omega_h, \delta_{k,h} \mu_h \rangle_{L^2 \Lambda^{k-1}} \]

\[ = \langle \zeta_h, d_h^k \mu_h \rangle_{L^2 \Lambda^{k+1}} + \langle \sigma_h, \delta_{k,h} \mu_h \rangle_{L^2 \Lambda^{k-1}} + \langle \delta_{k+1,h} \zeta_h, \mu_h \rangle_{L^2 \Lambda^k} + \langle d_h^{k-1} \sigma_h, \mu_h \rangle_{L^2 \Lambda^k} \]

\[ = \langle \delta_{k+1,h} \zeta_h, P_T^k \mu_h \rangle_{L^2 \Lambda^k} + \langle d_h^{k-1} \sigma_h, P_T^k \mu_h \rangle_{L^2 \Lambda^k} = \langle f - P_{\mathcal{H}_h \Lambda^k} f, P_T^k \mu_h \rangle_{L^2 \Lambda^k}. \]

Namely, \( \omega_h \) is the solution of (4.36). The proof is completed. \( \square \)

4.4.3. *Equivalences among lowest-degree mixed element schemes.* We here consider the respective discretizations of the classical mixed formulations. We use nonconforming and conforming finite element space pairs for them, respectively:
for (4.30): to find \((\omega_h^p, \sigma_h^p, \vartheta_h^p) \in W_{h}^{abc} \Lambda^k \times W_{h}^{abc} \Lambda^{k-1} \times \mathcal{H}_{h}^{abc} \Lambda^k\), such that, for \((\mu_h, \tau_h, \zeta_h) \in W_{h}^{abc} \Lambda^k \times W_{h}^{abc} \Lambda^{k-1} \times \mathcal{H}_{h}^{abc} \Lambda^k\),

\[
\begin{aligned}
\langle \vartheta_h^p, \mu_h \rangle_{L^2(\Omega)} & - \Delta_h \langle \omega_h^p, \sigma_h^p \rangle_{L^2(\Omega)} + \langle \delta_h \omega_h^p, \sigma_h^p \rangle_{L^2(\Omega)} = 0, \\
0 & = -\langle \omega_h^p, \vartheta_h^p \rangle_{L^2(\Omega)} + \langle \delta_h \omega_h^p, \mu_h \rangle_{L^2(\Omega)} + \langle \delta_h \omega_h^p, \delta_h \mu_h \rangle_{L^2(\Omega)} = \langle f, \vartheta_h^p \rangle_{L^2(\Omega)}.
\end{aligned}
\]

(4.37)

- for (4.31): to find \((\omega_h^d, \zeta_h^d, \vartheta_h^d) \in W_{h_0}^s \Lambda^k \times W_{h_0}^{s} \Lambda^{k+1} \times \mathcal{H}_{h_0}^{s} \Lambda^k\), such that, for \((\mu_h, \eta_h, \zeta_h) \in W_{h_0}^s \Lambda^k \times W_{h_0}^{s} \Lambda^{k+1} \times \mathcal{H}_{h_0}^{s} \Lambda^k\),

\[
\begin{aligned}
\langle \vartheta_h^d, \mu_h \rangle_{L^2(\Omega)} & - \Delta_h \langle \omega_h^d, \sigma_h^d \rangle_{L^2(\Omega)} + \langle \delta_h \omega_h^d, \sigma_h^d \rangle_{L^2(\Omega)} + \langle \delta_h \omega_h^d, \delta_h \mu_h \rangle_{L^2(\Omega)} = \langle f, \vartheta_h^d \rangle_{L^2(\Omega)}.
\end{aligned}
\]

(4.38)

Note that (4.37) which uses nonconforming finite element spaces is a new scheme.

**Lemma 4.34.** Let \((\omega_h^c, \zeta_h^c, \sigma_h^c, \vartheta_h^c), (\omega_h^p, \sigma_h^p, \vartheta_h^p)\), and \((\omega_h^d, \zeta_h^d, \vartheta_h^d)\) be the solutions of (4.34), (4.37) and (4.38), respectively. Then

\[
\begin{aligned}
\vartheta_h^d = \vartheta_h^c, & \quad \zeta_h^d = \zeta_h^c, & \quad \vartheta_h^d = \vartheta_h^c, & \quad \omega_h^d = \omega_h^c, & \quad \delta_h \omega_h^d = \delta_h \omega_h^c, & \quad \delta_k \zeta_h^d = \delta_k \zeta_h^c = \vartheta_h^d - \vartheta_h^c, \\
\vartheta_h^p = \vartheta_h^c, & \quad \sigma_h^p = \sigma_h^c, & \quad \vartheta_h^p = \vartheta_h^c, & \quad \omega_h^p = \omega_h^c, & \quad \delta_h \omega_h^p = \delta_h \omega_h^c, & \quad \delta_k \zeta_h^d = \delta_k \zeta_h^c = \vartheta_h^p - \vartheta_h^c, \\
\vartheta_h^d = \vartheta_h^p, & \quad \vartheta_h^d = \vartheta_h^c, & \quad \delta_k \zeta_h^d = \delta_k \zeta_h^c, & \quad \delta_k \zeta_h^d = \delta_k \zeta_h^c = \vartheta_h^d - \vartheta_h^c.
\end{aligned}
\]

**Proof.** Let \((\omega_h^d, \zeta_h^d, \vartheta_h^d)\) be the solution of (4.38). Then, with a \(\overline{\vartheta}_h \in W_{h}^{abc} \Lambda^{k-1}\),

\[
\begin{aligned}
\langle \vartheta_h^d, \mu_h \rangle_{L^2(\Omega)} + \langle \delta_h \zeta_h^d, \mu_h \rangle_{L^2(\Omega)} + \langle \delta_h \omega_h^d, \delta_h \mu_h \rangle_{L^2(\Omega)} - \langle \overline{\vartheta}_h, \delta_h \mu_h \rangle_{L^2(\Omega)} = \langle f, \vartheta_h^d \rangle_{L^2(\Omega)},
\end{aligned}
\]

for any \(\mu_h \in P_{0}^{-1} \Lambda^k(\mathcal{G}_h)\). Choose particularly \(\mu_h \in P_{0}^{-1} \Lambda^k(\mathcal{G}_h)\), we have

\[
\vartheta_h^d = \vartheta_h^c, \quad \vartheta_h^d = \vartheta_h^c, \quad \delta_k \zeta_h^d = \delta_k \zeta_h^c = \vartheta_h^d - \vartheta_h^c.
\]

(4.42)

and

\[
\langle \delta_h \omega_h^d, \delta_h \mu_h \rangle_{L^2(\Omega)} = 0, \quad \forall \mu_h \in P_{0}^{-1} \Lambda^k(\mathcal{G}_h),
\]

which leads to that \(\delta_h \omega_h^d = \vartheta_h^d - \vartheta_h^c\). Further, noting that \(\langle \delta_h \omega_h^d, \tau_h \rangle_{L^2(\Omega)} = \langle \omega_h^d, \vartheta_h^d - \vartheta_h^c \rangle_{L^2(\Omega)}\) for \(\tau_h \in W_{h}^{abc} \Lambda^{k-1}\), we obtain \((P_{h}^{k} \omega_h^d, \vartheta_h^d, \tau_h)_{L^2(\Omega)} = 0\) for \(\tau_h \in W_{h}^{abc} \Lambda^{k-1}\).

In all, \((P_{h}^{k} \omega_h^d, \vartheta_h^d, \tau_h, \vartheta_h^d) \in P_{0}^{-1} \Lambda^k(\mathcal{G}_h) \times W_{h_0}^{s} \Lambda^{k+1} \times W_{h}^{abc} \Lambda^{k-1} \times \mathcal{H}_{h_0}^{abc} \Lambda^k\) satisfies the system (4.34), and thus \((P_{h}^{k} \omega_h^d, \vartheta_h^d, \vartheta_h^d, \vartheta_h^d) = (\omega_h^c, \zeta_h^c, \sigma_h^c, \vartheta_h^c)\). This proves (4.39), and (4.40) can be proved the same way. (4.41) follows by (4.39) and (4.40). The proof is completed.

**Remark 4.35.** All these schemes each uses an \(L^2\) projection to piecewise constants in the inner products, which corresponds to a numerical quadrature of lowest-degree. Though, by a standard analysis, particularly by Lemma 4.34 and by the stability of the schemes, no lost of order of convergence rate will be caused by the \(L^2\) projections.
References

[1] Douglas Arnold. *Finite element exterior calculus*. SIAM, 2018.

[2] Douglas Arnold and Richard Falk. A uniformly accurate finite element method for the Reissner–Mindlin plate. *SIAM Journal on Numerical Analysis*, 26(6):1276–1290, 1989.

[3] Douglas Arnold, Richard Falk, and Ragnar Winther. Finite element exterior calculus, homological techniques, and applications. *Acta Numerica*, 15:1–155, 2006.

[4] Douglas Arnold, Richard Falk, and Ragnar Winther. Geometric decompositions and local bases for spaces of finite element differential forms. *Computer Methods in Applied Mechanics and Engineering*, 198(21-26):1660–1672, 2009.

[5] Douglas Arnold, Richard Falk, and Ragnar Winther. Finite element exterior calculus: from hodge theory to numerical stability. *Bulletin of the American Mathematical Society*, 47(2):281–354, 2010.

[6] Stefan Banach. Théorie des opérations linéaires. 1932.

[7] L Beirão da Veiga, Franco Brezzi, Andrea Cangiani, Gianmarco Manzini, L Donatella Marini, and Alessandro Russo. Basic principles of virtual element methods. *Mathematical Models and Methods in Applied Sciences*, 23(01):199–214, 2013.

[8] L Beirão da Veiga, Franco Brezzi, Franco Dassi, L Donatella Marini, and Alessandro Russo. Serendipity virtual elements for general elliptic equations in three dimensions. *Chinese Annals of Mathematics, Series B*, 39(2):315–334, 2018.

[9] L Beirão da Veiga, Franco Brezzi, L Donatella Marini, and Alessandro Russo. H(div) and H(curl)-conforming virtual element methods. *Numerische Mathematik*, 133(2):303–332, 2016.

[10] Yakov Berchenko-Kogan. Duality in finite element exterior calculus and Hodge duality on the sphere. *Foundations of Computational Mathematics*, 21(5):1153–1180, 2021.

[11] Daniele Boffi, Franco Brezzi, and Michel Fortin. *Mixed finite element methods and applications*. Springer, 2013.

[12] Susanne C Brenner and L Ridgway Scott. *The mathematical theory of finite element methods*. Springer, 2008.

[13] Haim Brezis. *Functional Analysis, Sobolev Spaces and Partial Differential Equations*. Springer Science & Business Media, 2010.

[14] Franco Brezzi, Jim Douglas, and L Donatella Marini. Two families of mixed finite elements for second order elliptic problems. *Numerische Mathematik*, 47(2):217–235, 1985.

[15] Annalisa Buffa and Snorre Christiansen. A dual finite element complex on the barycentric refinement. *Mathematics of Computation*, 76(260):1743–1769, 2007.

[16] Snorre Christiansen and Ragnar Winther. Smoothed projections in finite element exterior calculus. *Mathematics of Computation*, 77(262):813–829, 2008.

[17] Snorre Christiansen, Hans Z Munthe-Kaas, and Brynjulf Owren. Topics in structure-preserving discretization. *Acta Numerica*, 20:1–119, 2011.

[18] Philippe G Ciarlet. *The finite element method for elliptic problems*. North-Holland, Amsterdam, 1978.

[19] Ph Clément. Approximation by finite element functions using local regularization. *Revue française d’automatique, informatique, recherche opérationnelle. Analyse numérique*, 9(2):77–84, 1975.

[20] Michel Crouzeix and P-A Raviart. Conforming and nonconforming finite element methods for solving the stationary Stokes equations I. *Revue française d’automatique informatique recherche opérationnelle. Mathématique*, 7(R3):33–75, 1973.

[21] Mathieu Desbrun, Anil N Hirani, Melvin Leok, and Jerrold E Marsden. Discrete exterior calculus. *arXiv preprint math/0508341*, 2005.

[22] Paweł Dłotko and Ruben Specogna. Physics inspired algorithms for (co) homology computations of three-dimensional combinatorial manifolds with boundary. *Computer Physics Communications*, 184(10):2257–2266, 2013.

[23] Alexandre Ern and Jean-Luc Guermond. Finite element quasi-interpolation and best approximation. *ESAIM: Mathematical Modelling and Numerical Analysis*, 51(4):1367–1385, 2017.

[24] Richard Falk and Ragnar Winther. Local bounded cochain projections. *Mathematics of Computation*, 83(290):2631–2656, 2014.
[25] M. Fortin and M. Soulie. A non-conforming piecewise quadratic finite element on triangles. *International Journal for Numerical Methods in Engineering*, 19(4):505–520, 1983.

[26] Evan Gawlik, Michael J. Holst, and Martin W. Licht. Local finite element approximation of Sobolev differential forms. *ESAIM: Mathematical Modelling and Numerical Analysis*, 55(5):2075–2099, 2021.

[27] Ralf Hiptmair. Finite elements in computational electromagnetism. *Acta Numerica*, 11:237–339, 2002.

[28] Anil Nirmal Hirani. *Discrete exterior calculus*. PhD thesis, 2003. Caltech.

[29] Varun Jain, Yi Zhang, Artur Palha, and Marc Gerritsma. Construction and application of algebraic dual polynomial representations for finite element methods on quadrilateral and hexahedral meshes. *Computers & Mathematics with Applications*, 95:101–142, 2021.

[30] Tosio Kato. *Perturbation theory for linear operators*, volume 132. Springer Science & Business Media, 2013.

[31] Martin Licht. Smoothed projections and mixed boundary conditions. *Mathematics of Computation*, 88(316):607–635, 2019.

[32] Martin Licht. Smoothed projections over weakly Lipschitz domains. *Mathematics of Computation*, 88(315):179–210, 2019.

[33] Wenjia Liu and Shuo Zhang. A lowest-degree strictly conservative finite element scheme for incompressible stokes problem on general triangulations. *Journal of Scientific Computing*, accepted, 2022.

[34] Luisa Donatella Marini. An inexpensive method for the evaluation of the solution of the lowest order Raviart–Thomas mixed method. *SIAM Journal on Numerical Analysis*, 22(3):493–496, 1985.

[35] Peter Monk. A mixed method for approximating Maxwell’s equations. *SIAM Journal on Numerical Analysis*, 28(6):1610–1634, 1991.

[36] Yosuke Nakata, Yoshiro Urade, and Toshihiro Nakanishi. Geometric structure behind duality and manifestation of self-duality from electrical circuits to metamaterials. *Symmetry*, 11(11):1336, 2019.

[37] J. T. Oden. Generalized conjugate functions for mixed finite element approximations of boundary value problems. *The Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations*, Academic Press, New York, pages 629–667, 1972.

[38] Christoph Ortner. Nonconforming finite-element discretization of convex variational problems. *IMA Journal of Numerical Analysis*, 31(3):847–864, 2011.

[39] Chunjae Park and Dongwoo Sheen. P1-nonconforming quadrilateral finite element methods for second-order elliptic problems. *SIAM Journal on Numerical Analysis*, 41(2):624–640, 2003.

[40] Pierre-Arnaud Raviart and Jean-Marie Thomas. A mixed finite element method for 2nd order elliptic problems. In *Mathematical aspects of finite element methods*, pages 292–315. Springer, 1977.

[41] Joachim Schöberl. A posteriori error estimates for Maxwell equations. *Mathematics of Computation*, 77(262):633–649, 2008.

[42] L. Ridgway Scott and Shangyou Zhang. Finite element interpolation of nonsmooth functions satisfying boundary conditions. *Mathematics of Computation*, 54(190):483–493, 1990.

[43] Ming Wang and Jinchao Xu. Minimal finite element spaces for 2m-th-order partial differential equations in \( \mathbb{R}^n \). *Mathematics of Computation*, 82(281):25–43, 2013.

[44] Christian Wieners and Barbara Wohlmuth. A primal-dual finite element approximation for a nonlocal model in plasticity. *SIAM Journal on Numerical Analysis*, 49(2):692–710, 2011.

[45] Köksu Yōsida. *Functional analysis*. Springer Science & Business Media, 2012.

[46] Shuo Zhang. Minimal consistent finite element space for the biharmonic equation on quadrilateral grids. *IMA Journal of Numerical Analysis*, 40(2):1390–1406, 2020.

[47] Shuo Zhang. An optimal piecewise cubic nonconforming finite element scheme for the planar biharmonic equation on general triangulations. *Science China Mathematics*, 64(11):2579–2602, 2021.