Quantum generalized fluctuation-dissipation relations in terms of time-distributed observations

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Abstract. New formulations of quantum generalized fluctuation-dissipation relations in terms of characteristic and probabilistic functionals of continuous observations are suggested and discussed. It is shown that control of entropy production in quantum system turns any measurement in it to a source of its extra perturbations, and because of this effect relations between probabilities of mutually time-reversed processes become definitely non-local in their functional space.

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1. Introduction

This paper continues recent preprint [1] and my earlier work [2] (see also notes [3]) devoted to quantum formulations of the “generalized fluctuation-dissipation relations” (FDR) [4, 5, 6, 7] (see also references in [2, 7, 8]).

In classical statistical mechanics (CSM) many different modifications of the FDR may be unified into one formula [9]

\[ P(\Pi_+) \exp \left[ -\Delta S(\Pi_+) \right] = P(\Pi_-) \]

connecting probability \( P(\Pi_+) \) of realization and observation of a “forward” process \( \Pi_+ \) with probability \( P(\Pi_-) \) of its time reversion \( \Pi_- \) and change \( \Delta S(\Pi_+) = -\Delta S(\Pi_-) \) of entropy of a system under consideration in these processes (for details please see [7] or [9]). The exponential factor here may be moved from left to right-hand side, - as \( \exp \left[ -\Delta S(\Pi_-) \right] \), - or somehow split between two sides,

\[ P(\Pi_+) \exp \left[ -z \Delta S(\Pi_+) \right] = P(\Pi_-) \exp \left[ -(1-z) \Delta S(\Pi_-) \right] \]

with some \( z \), and, of course, this equality is quantitatively equivalent to Eq.1. In quantum statistical mechanics (QSM), however, such the equivalence generally is true in
qualitative sense only, since measurements of $\Delta S$ do not commute with that of $\Pi_{\pm}$ and thus may influence upon quantities $P(\Pi_{\pm})$. Our purpose below will be to show that this non-commutativity is actually significant at least for time-distributed ("continuous") observations of system’s evolution under external perturbations, and that best (most similar) quantum analogue of Eqs[1][2] appears under symmetric splitting, i.e. at $z = 1/2$. Simultaneously, our consideration will throw more light on physical meaning of the entropy production operator $\Delta \dot{S}(t)$ suggested in [1]. As the result, formally exact quantum FDR for generating - characteristic and probabilistic - functionals obtained in [2] now will be reformulated in more comfortable fashion allowing to extend Eq[2] with $z = 1/2$, to continuous observations in QSM.

2. Basic quantum FDR

In many theoretical models and physical applications of CSM and QSM it seems reasonable (or necessary [5, 7]) to divide Hamiltonian $H(x)$ ($x = x(t)$) of an externally driven system into two parts:

$$H(x) = H_0 - h(x) \quad (H_0 = H(0), \quad h(0) = 0),$$

- with $H_0$ representing system’s internal energy while $-h(x)$ its interaction with sources of its perturbations, - and focus on changes of just the internal energy in their connection with entropy production in the system. Corresponding treatment of FDR [4] later [10, 11] was termed “exclusive” (on its motivations see e.g. [7]). For brevity, in the present paper we confine ourselves just by it. Therefore we can base directly on quite general quantum FDR (for Gibbs statistical ensembles) which were written out already in [4] in Eqs.17-24. Namely, in slightly different but transparent notations,

$$\langle A_1(t_1) \ldots A_n(t_n) e^{-H_0(t)/T} e^{H_0(t)/T} \rangle_{H_0, h(x(t))} =$$

$$= \langle \overline{A}_n(t - t_n) \ldots \overline{A}_1(t - t_1) \rangle_{H_0, h(x(t))},$$

Here $A_j(t)$ are arbitrary operators in the Heisenberg representation,

$$A(t) = U^\dagger(t, 0) A U(t, 0) \equiv U^\dagger(t) A U(t),$$

$$U(t, 0) = \exp\left[ -\frac{i}{\hbar} \int_{t_0}^{t} H(x(\tau)) d\tau \right];$$

the angle brackets denote average over (trace with) normalized canonical density matrix (statistical operator) $\rho_0 = \exp\left[ (F_0 - H_0)/T \right]$ at given “eigen” Hamiltonian $H_0$, intercations Hamiltonian $h(x)$ and given external driving “forces” (Hamiltonian parameters) $x(t)$; the over-line denotes transposition of operators, $\overline{A} = A^T$. Thus, on right-hand side of Eq[3] evolution of all (transposed) operators $\overline{A}(\tau)$ is governed by

‡ In addition to historical remarks in [7] we would like to notice that some of scientists interpret results of their precursors in as broad sense as possible while another in maximally narrow sense. We believe to be in the first group, at that naturally ascribing ourselves to our precursors.

§ Or, according to remarks on p.129 in [4], more general transform, $\overline{A} = \Theta A^T \Theta^\dagger$ with some unitary $\Theta$ such that square of this transform is identical one, $\overline{A} = A$, that is $\Theta^T \Theta = \Theta^* \Theta = 1$. 

Hamiltonian $\mathcal{H}(x(t-\tau))$. For brevity below without loss of generality (which is trivially restorable within final formulae) we assume that $\mathcal{H}_0 = H_0$ and therefore $H_0$ may be removed from the angle brackets’ subscripts.

It should be underlined that, first, arbitrary operators $A$ in Eq.3 - as well as in Eqs.17-19 from [4], - may be non-Hermitian [3]. Second, of course, some of them may have form $\exp(c H_0/T)$ with suitable numbers $c$. Third, clearly, Eq.3 holds true if we replace the product of $n$ operators by arbitrary linear combination of products with various $n$. Exploiting this freedom and general properties of the trace operation, we get rights to write, instead of Eq.3

$$\langle e^{H_0/2T} B\{V(\tau)\} e^{-H_0(t)/2T} A\{V(\tau)\} e^{H_0/2T} \rangle_{h(x(\tau))} =$$

$$= \langle B\{V(t-\tau)\} \frac{e}{e} A\{V(t-\tau)\} \rangle_{\mathcal{H}(x(t-\tau))} ,$$

where $A\{V(\tau)\}$ and $B\{V(\tau)\}$ are some operator-valued expressions (functionals) composed by arbitrary collection of quantum variables (Heisenberg operators) $V(t)$. Or, making replacements

$$A\{V(\tau)\} \Rightarrow e^{(1-z) H_0(t)/2T} A\{V(\tau)\} e^{-(1-z) H_0(t)/2T},$$

$$B\{V(\tau)\} \Rightarrow e^{-(1-z) H_0(t)/2T} B\{V(\tau)\} e^{(1-z) H_0(t)/2T},$$

formally equivalently we can write

$$\langle e^{z H_0/2T} B\{V(\tau)\} e^{-z H_0(t)/2T} A\{V(\tau)\} e^{z H_0/2T} \rangle_{h(x(\tau))} =$$

$$= \langle e^{(1-z) H_0(t)/2T} B\{V(t-\tau)\} e^{-(1-z) H_0(t)/2T} A\{V(t-\tau)\} \rangle_{\mathcal{H}(x(t-\tau))} ,$$

$$\times e^{-(1-z) H_0(t)/2T} A\{V(t-\tau)\} e^{(1-z) H_0(t)/2T} \rangle_{\mathcal{H}(x(t-\tau))}.$$  

Evidently, Eqs.4,5 are direct quantum analogues of classical generalized FDR like Eq.25 from [7] (for $a = b = 0$), and Eq.5 certainly is valid at least at $0 \leq \Re z \leq 1$.

Similarly to the classical case, one can say that Eqs.4,5 compare observations of mutually time-reversed processes conditioned (at least on one of two sides) by two measurements of system’s internal energy, at beginning and end of observation time interval. Then difference of two measured quantities can be interpreted as result of measurement of system’s internal energy change, $E(t) = H_0(t) - H_0(0)$, although $E(t)$ is not quantum observable in literal sense [11]. Under such treatment, in view of arbitrariness of $A\{V(\tau)\}$ and $B\{V(\tau)\}$, FDR (4)-(5) form clear ground for construction (definition) of quantum analogues of probabilistic FDR (2)-(3), with quantity $\Delta S$ identified with $E(t)/T$.

3. Time-distributed energy exchange observations and their influence upon system’s evolution

Just mentioned treatment of the FDR (4)-(5) is intelligent when addressed to small enough quantum system (or, to be precise, statistical ensembles of small systems) but

|| In particular, some of $A$’s may look like $|\mu\rangle\langle\nu|$ with $|\mu\rangle$ and $|\nu\rangle$ being various pure quantum states (e.g. eigenstates of $H_0$).
seems rather doubtful in respect to large ones, in particular, consisting of a small subsystem in contact with thermal bath (thermostat). If external perturbations are applied to the subsystem then observation of its behavior is at once (indirect) continuous observation of energy flow through it from external “work sources” to thermostat, even though the latter eventually may accept all the work. Therefore it would be more adequate approach to practice if we reformulated FDR (4)-(5) in terms of time-local “internal energy change rate” (IECR) [1] and related entropy production.

This task all the more is actual since in quantum theory changes of system’s entropy naturally are ordered in time, we have to resort to a kind of chronological ordering of quantum variables (operators) under statistical averaging. For certainty, we may choose the well grounded “Jordan-symmetrized chronological” operator ordering rule (see e.g. [1, 2, 3] and references in [2]). Then the functionals $A\{V(\tau)\}$ and $B\{V(\tau)\}$ can be chosen e.g. in the form

$$A\{V(\tau)\} = \exp[\frac{1}{2} \int_0^t a(\tau) \cdot V(\tau) d\tau] ,$$

$$B\{V(\tau)\} = \exp[\frac{1}{2} \int_0^t a(\tau) \cdot V(\tau) d\tau] ,$$

where $V(t) = U_t^\dagger V U_t$ is some collection of variables (operators), $a(t)$ are $c$-number valued (generally complex) probe functions, and $\cdot$ means summation (“scalar product” or “convolution” of two arrays).

First, consider left and right-hand sides of Eq.5 after substitution of expressions (6). On the left, let us make transformations as follows,

$$\langle e^{\frac{z}{2}H_0/2T} B\{V(\tau)\} e^{-\frac{z}{2}H_0(0)/T} A\{V(\tau)\} e^{\frac{z}{2}H_0/2T} \rangle_{h(x(\tau))} =$$

$$= \langle e^{\frac{z}{2}H_0/2T} \exp \left\{ \int_0^t \left[ \frac{i}{\hbar} H(x(\tau)) + \frac{a(\tau)}{2} \cdot V \right] d\tau \right\} e^{-\frac{z}{2}H_0/2T} \times$$

$$\times e^{-\frac{z}{2}H_0/2T} \exp \left\{ \int_0^t \left[ -\frac{i}{\hbar} H(x(\tau)) + \frac{a(\tau)}{2} \cdot V \right] d\tau \right\} e^{\frac{z}{2}H_0/2T} \rangle_{h(x(\tau))} =$$

$$= \langle \exp \left\{ \int_0^t \left[ \frac{i}{\hbar} e_z H(x(\tau)) e_z^{-1} + \frac{a(\tau)}{2} \cdot e_z V e_z^{-1} \right] d\tau \right\} \times$$

$$\times \exp \left\{ \int_0^t \left[ -\frac{i}{\hbar} e_z^{-1} H(x(\tau)) e_z + \frac{a(\tau)}{2} \cdot e_z^{-1} V e_z \right] d\tau \right\} \rangle_{h(x(\tau))} ,$$

with $e_z \equiv e^{\frac{z}{2}H_0/2T}$ and $e_z^{-1} \equiv e^{-\frac{z}{2}H_0/2T}$.

Quite similarly, but with $e_{1-z}$ in place of $e_z$, transforms right side of Eq.5.

Second, notice that instead of Hermitian operator $H(x(t))$ in Eq.7 two generally non-Hermitian operators have appeared, $e_z^{-1} H(x(\tau)) e_z$ and its conjugation [4]. To feel ¶ Here and below we neglect differences between concepts of Hermitian, self-conjugated, self-adjoint,
meaning of such transformations, - which already were in use in [2], - let us set there \( a = 0 \) and confine our present consideration by case of \( \text{real} \) \( z \). Then, after simple algebraic manipulations, expression (7) turns to

\[
\langle e^{z H_0/2T} e^{-z H_0(t)/T} e^{z H_0/2T} \rangle_{h(x(\tau))} =
\]

\[
= \langle \exp \left\{ \int_0^t \left[ \frac{i}{\hbar} H(z, x(\tau)) - \frac{1}{2} G(z, x(\tau)) \right] d\tau \right\} \times
\]

\[
\times \exp \left\{ \int_0^t \left[ -\frac{i}{\hbar} H(z, x(\tau)) - \frac{1}{2} G(z, x(\tau)) \right] d\tau \right\} \rangle_{h(x(\tau))},
\]

with operators \( H(z, x) \) and \( G(z, x) \) defined by

\[
H(z, x) = H_0 - h(z, x),
\]

\[
h(z, x) = \frac{1}{2} \left[ e_z h(x) e_z^{-1} + e_z^{-1} h(x) e_z \right],
\]

\[
G(z, x) = \frac{i}{\hbar} \left[ e_z h(x) e_z^{-1} - e_z^{-1} h(x) e_z \right]
\]

Thus, \( H(z, x) = H_0 - h(z, x) \) and \( G(z, x) \) both are Hermitian operators formed by Hermitian and anti-Hermitian components of \( e_z h(x) e_z^{-1} \) respectively. It will be comfortable to characterize such operators by their matrix elements in basis formed by eigenstates of \( H_0 \). Then Eqs.9-10 read

\[
h_{\mu\nu}(z, x) = h_{\mu\nu}(x) \cosh \frac{z E_{\mu\nu}}{2T},
\]

\[
G_{\mu\nu}(z, x) = \frac{2i}{\hbar} h_{\mu\nu}(x) \sinh \frac{z E_{\mu\nu}}{2T},
\]

with \( E_{\mu\nu} = E_\mu - E_\nu \) and \( E_\mu \) being \( H_0 \)'s eigenvalues: \( H_0 |\mu\rangle = E_\mu |\mu\rangle \).

Eq.8 evidently prompts to rewrite it in the spirit of Jordan-symmetrized chronological ordering rule, as

\[
\langle e^{z H_0/2T} e^{-z H_0(t)/T} e^{z H_0/2T} \rangle_{h(x(\tau))} =
\]

\[
= \text{Tr} \exp \left\{ \int_0^t \left[ -\frac{i}{\hbar} H(z, x(\tau)) - \frac{1}{2} G(z, x(\tau)) \right] d\tau \right\} \times
\]

\[
\times \exp \left\{ \int_0^t \left[ \frac{i}{\hbar} H(z, x(\tau)) - \frac{1}{2} G(z, x(\tau)) \right] d\tau \right\} \rangle_{h(x(\tau))},
\]

with \( U_t \rangle G(z, x(t)) U_t \) playing role of continuously measured quantum variable (observable) and unitary evolution operator \( U_t \) now being determined by modified (effective) Hamiltonian \( H(z, x) = H_0 - h(z, x) \).

Before writing out result of analogous transformation of right-hand side of Eq.5, for brevity and visuality let us assume, as usually, that the external forces possess definite parities \( \epsilon = \pm 1 \) in respect to time reversal, that is

\[
\bar{h}(x) = h(\epsilon x)
\]

etc., operators and everywhere exploit mere term “Hermitian”. 
Then at $\alpha(t) = 0$, i.e. at $A\{\cdot\} = B\{\cdot\} = 1$, on the right in Eq\textsuperscript{[5]} we have

$$
\langle e^{(1-z) H_0/2T} e^{-(1-z) H_0(t)/T} e^{(1-z) H_0/2T} h(\epsilon x(t-\tau)) \rangle = 
\langle \exp \left[ - \int_0^t U^{\dagger}_\tau G(1-z, \epsilon x(t-\tau)) U_\tau d\tau \right] \rangle h(1-z, \epsilon x(t-\tau)) \ ,
$$

(14)

now with evolution governed by effective Hamiltonian $H(1-z, \epsilon x)$.

Hence, the particular case of Eq\textsuperscript{[5]}

$$
\langle e^{z H_0/2T} e^{-z H_0(t)/T} e^{z H_0/2T} h(x(t)) \rangle =
\langle e^{(1-z) H_0/2T} e^{-(1-z) H_0(t)/T} e^{(1-z) H_0/2T} h(\epsilon x(t-\tau)) \rangle 
$$

(15)

can be expressed, - for real $z$, - also by equality

$$
\langle \exp \left[ - \int_0^t U^{\dagger}_\tau G(z, x(\tau)) U_\tau d\tau \right] \rangle h(z, x(\tau)) = 
\langle \exp \left[ - \int_0^t U^{\dagger}_\tau G(1-z, \epsilon x(t-\tau)) U_\tau d\tau \right] \rangle h(1-z, \epsilon x(t-\tau)) 
$$

(16)

This equality is formally identical to Eq\textsuperscript{[15]} but, in contrast to it, describes continuous observations of system’s energy exchange with sources of external perturbations. At that, variables $G(z, x)$ and $G(1-z, x)$ delegate the IEC’s rate (IECR) but, clearly, not in literal sense. Naive direct interpretation of $G(z, x)$ is possible in the classical limit only when, according to Eqs\textsuperscript{[9]-[10]}

$$
h(z, x) \rightarrow h(x) \ , \ G(z, x) \rightarrow \frac{z}{T} L_0 h(x) = z \Delta \dot{S}(x) 
$$

$$
\Delta \dot{S}(x) = \frac{1}{T} \frac{i}{\hbar} [H_0, h(x)] = \frac{1}{T} \frac{i}{\hbar} [H(x), H_0]
$$

with $\Delta \dot{S}(x)$ having sense of entropy production per unit time, or simply entropy production (EP), and $L_0$ being unperturbed Liouville super-operator defined by

$$
L_0 A = \frac{i}{\hbar} [H_0, A] \ , \ (L_0 A)_{\mu\nu} = \frac{i}{\hbar} E_{\mu\nu} A_{\mu\nu}
$$

In essentially quantum situations, however, $h(z, x) \neq h(x)$ and $G(z, x)$ is not proportional to $z$, which means that in fact EP depends on conditions, or “intensity”, of its measurements.

Moreover, relation (16), together with (9)- (11), shows that in general its left and right sides concern, strictly speaking, two different systems with different interaction Hamiltonians. But practical control of external parameters of interaction Hamiltonian (IH) usually in no way means control of its detail structure (all matrix elements). Therefore it seems reasonable to restrict further consideration by special choices of the “observation parameter” $z$. Clearly, such are first of all $z = 1$ and $z = 1/2$.

At $z = 1$, Eq\textsuperscript{[16]} reduces to

$$
\langle \exp \left[ - \int_0^t U^{\dagger}_\tau G(1, x(\tau)) U_\tau d\tau \right] \rangle h(1, x(\tau)) = 1
$$

(17)
From viewpoint of this relation in itself, the effective, - “renormalized by observation”, - IH \( h(1, x) \) is nothing but system’s actual IH. Therefore, making redesignation \( h(1, x) \Rightarrow h(x) \), we can rewrite Eq.17 in the form

\[
\langle \exp \left[ - \int_0^t U^\dagger_\tau \Delta \hat{S}(x(\tau)) U_\tau \, d\tau \right] \rangle_{h(x(\tau))} = 1 ,
\]

where EP \( \Delta \hat{S}(x) \) is defined by

\[
\Delta \hat{S}_{\mu\nu}(x) \equiv G_{\mu\nu}(1, x) = \frac{2i}{\hbar} h_{\mu\nu}(1, x) \tanh \frac{E_{\mu\nu}}{2T} \Rightarrow \frac{2i}{\hbar} h_{\mu\nu}(x) \tanh \frac{E_{\mu\nu}}{2T}.
\]

This (above mentioned) statistical equality was derived originally in [2] and in different way recently in [1].

4. Time-symmetric observations and entropy production operator

At \( z = 1/2 \), in framework of analogous treatment of renormalized IH \( h(1/2, x) \) as factual IH, i.e. after redesignation \( h(1/2, x) \Rightarrow h(x) \), Eq.16 yields

\[
\langle \exp \left[ - \int_0^t U^\dagger_\tau \frac{1}{2} \Delta \hat{S}(x(\tau)) U_\tau \, d\tau \right] \rangle_{h(x(\tau))} = \langle \exp \left[ - \int_0^t U^\dagger_\tau \frac{1}{2} \Delta \hat{S}(\epsilon x(t-\tau)) U_\tau \, d\tau \right] \rangle_{h(\epsilon x(t-\tau))} ,
\]

where now by definition

\[
\frac{1}{2} \Delta \hat{S}_{\mu\nu}(x) \equiv G_{\mu\nu}(1/2, x(t)) = \frac{2i}{\hbar} h_{\mu\nu}(1/2, x) \tanh \frac{E_{\mu\nu}}{4T} \Rightarrow \frac{2i}{\hbar} h_{\mu\nu}(x) \tanh \frac{E_{\mu\nu}}{4T}.
\]

This is quantum version of classical equality following from FDR (2) after its integration over all possible processes (observations).

Notice that Eq.13 together with Eq.21 or 19 implies anti-symmetry property

\[
\Delta \hat{S}(\epsilon x) = - \Delta \hat{S}(x) , \quad \Delta \hat{S}_{\mu\nu}(\epsilon x) = - \Delta \hat{S}_{\nu\mu}(x) ,
\]

ensuring that changes of system’s entropy in mutually time-reversed processes (under time-reversed external conditions) differ by their signs only.

Now, let us consider at \( z = 1/2 \) simultaneously both sides of FDR (5) with functionals \( B\{V(\tau)\} \) and \( A\{V(\tau)\} \) taken in the form (6). It reads

\[
\langle \exp \left\{ \int_0^t \left[ \frac{i}{\hbar} e H(x(\tau)) e^{-1} + \frac{a(\tau)}{2} e V e^{-1} \right] d\tau \right\} \times \exp \left\{ \int_0^t \left[ - \frac{i}{\hbar} e^{-1} H(x(\tau)) e + \frac{a(\tau)}{2} e^{-1} V e \right] d\tau \right\} \rangle = \langle \exp \left\{ \int_0^t \left[ \frac{i}{\hbar} e H(\epsilon x(t-\tau)) e^{-1} + \frac{a(t-\tau)}{2} e^{-1} V e \right] d\tau \right\} \times \exp \left\{ \int_0^t \left[ - \frac{i}{\hbar} e^{-1} H(\epsilon x(t-\tau)) e + \frac{a(t-\tau)}{2} e^{-1} V e \right] d\tau \right\} \rangle ,
\]

\[
(23)
\]
where \( e \equiv e_{1/2} = \exp(H_0/4T) \) and \( e^{-1} \equiv e_{1/2}^{-1} = \exp(-H_0/4T) \), we used our conventions (13) and \( \overline{H}_0 = H_0 \), and omitted the angle brackets’ subscripts because they are superfluous here.

Then perform at Eq.23 transformations like (9)-(11), assuming that variables under consideration possess definite parities,

\[
\nabla = \varepsilon V \ (\varepsilon \pm 1),
\]

and introducing short notations

\[
\bar{x}(\tau) = \varepsilon x(t-\tau), \quad \bar{a}(\tau) = \varepsilon a(t-\tau),
\]

(24)

\[
V_+ = \frac{1}{2} [e^{-1}V e + e V e^{-1}],
\]

(25)

\[
V_- = \frac{1}{2} [e^{-1}V e - e V e^{-1}] \quad (e \equiv e^{H_0/4T})
\]

After that Eq.23 turns to

\[
\langle \hat{\exp} \{ \int_0^t [\frac{i}{\hbar} H'(x(\tau),a(\tau)) - \frac{1}{2} G(1/2,x(\tau)) + \frac{a(\tau)}{2} \cdot V_+] d\tau \} \times \hat{\exp} \{ \int_0^t [\frac{i}{\hbar} H'(x(\tau),a(\tau)) - \frac{1}{2} G(1/2,x(\tau)) + \frac{a(\tau)}{2} \cdot V_+] d\tau \} \rangle
\]

\[
= \langle \hat{\exp} \{ \int_0^t [\frac{i}{\hbar} H'(\bar{x}(\tau),\bar{a}(\tau)) - \frac{1}{2} G(1/2,\bar{x}(\tau)) + \frac{\bar{a}(\tau)}{2} \cdot V_+] d\tau \} \times \hat{\exp} \{ \int_0^t [\frac{i}{\hbar} H'(\bar{x}(\tau),\bar{a}(\tau)) - \frac{1}{2} G(1/2,\bar{x}(\tau)) + \frac{\bar{a}(\tau)}{2} \cdot V_+] d\tau \} \rangle
\]

(26)

with the effective “renormalized” Hamiltonian \( H' \) is expressed by formulae

\[
H'(x,a) = H_0 - h'(x,a),
\]

\[
h'(x,a) = h_+(x) + \frac{\hbar}{2i} a \cdot V_-, \quad (27)
\]

\[
h_+(x) = \frac{1}{2} \left[ e h(x) e^{-1} + e^{-1} h(x) e \right],
\]

\[
h_{\mu\nu}^'(x,a) = h_{\mu\nu}(x) \cosh \frac{E_{\mu\nu}}{4T} + \frac{i\hbar}{2} a \cdot V_{\mu\nu} \sinh \frac{E_{\mu\nu}}{4T}.
\]

(28)

Thus, naturally, continuous observations of arbitrary variables \( V \) also influence upon system’s behavior, - by changing its effective IH, \( h(x) \Rightarrow h'(x,a) \), - so that probe functions \( a(t) \) acquire role of additional external forces.

It is important to underline that left and right-hand effective IHs in Eq.4 are not mutually transposed, i.e. \( h'(ex,ea) \neq \overline{h'}(x,a) \). In fact, \( \overline{h'}(x,a) = h'(ex,-ea) \), because \( V_- \)’s parity is opposite to that of \( V \). In this sense, observations (as well as perturbations) violate time symmetry of observed processes, even in spite of symmetry due to choice \( z = 1/2 \).

5. Time symmetrized quantum characteristic and probabilistic functionals

In order to reformulate Eq.26 in more plausible and standard form, it is reasonable to make “back renormalization” of both the IH and observed variables, by replacements
and introduce variables with operator $\Delta \dot{\tau}$ defined by

$$h_{\mu\nu}(x) \cosh \frac{E_{\mu\nu}}{4T} \Rightarrow h_{\mu\nu}(x) , \ V_{\mu\nu} \cosh \frac{E_{\mu\nu}}{4T} \Rightarrow V_{\mu\nu} ,$$

Besides, it is comfortable to introduce super-operator $T$ defined by

$$(\mathcal{T}A)_{\mu\nu} = A_{\mu\nu} \tanh \frac{E_{\mu\nu}}{4T} \equiv \mathcal{T}_{\mu\nu} A_{\mu\nu} \quad (29)$$

After that effective Hamiltonian becomes

$$H'(x, a) \Rightarrow H(x, a) = H_0 - h(x, a) , \ h'(x, a) \Rightarrow h(x, a) ,$$

$$h(x, a) \equiv h(x) + \frac{i\hbar}{2} a \cdot \mathcal{T} V , \quad (30)$$

and Eq. (26) can be rewritten as

$$h_{\mu\nu}(x, a) = h_{\mu\nu}(x) + \frac{i\hbar}{2} a \cdot V_{\mu\nu} \tanh \frac{E_{\mu\nu}}{4T} , \quad (31)$$

and Eq. (26) can be written as

$$\langle \exp \{ \int_0^t \left[ -\frac{i}{\hbar} H(x(\tau), a(\tau)) - \frac{1}{4} \Delta \dot{S}(x(\tau)) + \frac{a(\tau)}{2} \cdot V(\tau) \right] d\tau \} \times$$

$$\times \exp \{ \int_0^t \left[ -\frac{i}{\hbar} H(x(\tau), a(\tau)) - \frac{1}{4} \Delta \dot{S}(x(\tau)) + \frac{a(\tau)}{2} \cdot V(\tau) \right] d\tau \} =$$

$$\langle \exp \{ \int_0^t \left[ \frac{i}{\hbar} H(x(\tau), a(\tau)) - \frac{1}{4} \Delta \dot{S}(x(\tau)) + \frac{a(\tau)}{2} \cdot V(\tau) \right] d\tau \} \times$$

$$\times \exp \{ \int_0^t \left[ \frac{i}{\hbar} H(x(\tau), a(\tau)) - \frac{1}{4} \Delta \dot{S}(x(\tau)) + \frac{a(\tau)}{2} \cdot V(\tau) \right] d\tau \} \rangle , \quad (32)$$

with operator $\Delta \dot{S}(x)/2$ already introduced in (31):

$$\frac{1}{2} \Delta \dot{S}(x) = \frac{2i}{\hbar} \mathcal{T} h(x) \quad (33)$$

Further, for more visibility and without loss of generality (see e.g. remarks in [71]), let us choose the “seed” IH $h(x)$ in “bilinear” form

$$h(x) = x \cdot Q , \quad (34)$$

and introduce variables

$$I = \mathcal{L}Q , \ J = \mathcal{L}V \quad \left( \mathcal{L} \equiv \frac{4iT}{\hbar} \mathcal{T} = \frac{4iT}{\hbar} \tanh \frac{\hbar \mathcal{L}_0}{4iT} \right) \quad (35)$$

Then the “half of EP” operator $\Delta \dot{S}$ and effective IH $h(x, a)$ determined by (30) become

$$\frac{1}{2} \Delta \dot{S}(x) = \frac{1}{2T} x \cdot I , \quad (36)$$

$$h(x, a) = x \cdot Q + \frac{\hbar^2}{8T} a \cdot J \quad (37)$$

Taking in mind conventions of the Jordan-symmetrized chronological operator ordering rule, instead of Eq. (32) we can write also simply

$$\langle \exp \{ \int_0^t \left[ -\frac{1}{2T} x(\tau) \cdot I(\tau) + a(\tau) \cdot V(\tau) \right] d\tau \} \rangle_{h(x(\tau), a(\tau))} =$$

$$\langle \exp \{ \int_0^t \left[ -\frac{1}{2T} \bar{x}(\tau) \cdot I(\tau) + \bar{a}(\tau) \cdot V(\tau) \right] d\tau \} \rangle_{h(\bar{x}(\tau), \bar{a}(\tau))} , \quad (38)$$
where $I(\tau)$ and $V(\tau)$ are meant as commutative (c-number valued) random processes imaging *Heisenberg* operator variables $U_{\tau}^{\dagger}IU_{\tau}$ and $U_{\tau}^{\dagger}VU_{\tau}$, with evolution operator $U_{\tau}$ concretized by angle brackets’ subscripts. It is easy to extend FDR (20)-(48) to operator variables $V$ which are time dependent already in Schrödinger representation, so that then time argument of classical image variable $V(t)$ in Eq.48 delegates complete “double” time dependency of its quantum originals.

The characteristic functional FDR (48) in essence is symmetrized version of relation (20)-(21) from [2]. These relations show that under time-distributed observations the perturbation parameters (external forces) and observation parameters (probe functions) inevitably get entangled and partially swap their roles. In contrast to the classical limit, the entangling is mutual and not exact since it is accompanied by “deformation” of operator variables due to factors implied by the super-operator $T$ (29). In particular, clearly, variables $I$ and $J$ (35) are nothing but (unperturbed) time derivatives of $Q$ and $V$, respectively, deformed by smoothing over time intervals $\sim h/T$.

Notice, besides, that at imaginary probe functions, $a(t) = i\xi(t)$, or complex ones the effective IH $h(x,a)$ becomes non-Hermitian. But this one more unpleasant peculiarity of quantum case in principle does not prevent transformation of FDR (48) for characteristic functionals (CF) into FDR for probabilistic functionals (PF).

With such purpose, introduce auxiliary effective IH

$$h'(x,y) \equiv x \cdot Q + y \cdot J,$$

with $J$ defined in (35), so that

$$h(x,a) = h'(x, \frac{h^2}{8T} a),$$

and functional of four (sets of) variables as follows,

$$\Xi\{a(\tau), b(\tau) \mid x(\tau), y(\tau)\} =$$

$$= \langle \exp \left\{ \int_{0}^{t} [a(\tau) \cdot V(\tau) + b(\tau) \cdot I(\tau)] d\tau \right\} \rangle_{h'(x(\tau), y(\tau))}$$

Evidently, this is joint CF of variables $V$ and $I$ evolving under perturbations described by effective IH $h'(x,y)$ with arbitrary additional forces $y(t)$ in place of forces $(h^2/8T) a(t)$ from (37) related to probe functions. In terms of new CF (40), FDR (48) reads

$$\Xi\{a, -\frac{x}{2T} \mid x, \frac{h^2}{8T} a\} = \Xi\{\bar{a}, -\frac{x}{2T} \mid \bar{x}, \frac{h^2}{8T} \bar{a}\},$$

thus reducing CF (40) of four independent (collections of) arguments to functional of twice lesser number of arguments and entangling perturbations and observations.

Then, consider joint PF of classical images of $V(t)$ and $I(t)$,

$$P\{V(\tau), I(\tau) \mid x(\tau), y(\tau)\} =$$

$$= \int_{\xi} \int_{\eta} \exp \left\{ -\int_{0}^{t} [i\xi(\tau) \cdot V(\tau) + i\eta(\tau) \cdot I(\tau)] d\tau \right\} \times$$

$$\times \Xi\{i\xi(\tau), i\eta(\tau) \mid x(\tau), y(\tau)\},$$
where \( \int \xi \int \eta \ldots \) means functional Fourier transform. In terms of this PF, for left side of Eq.\( \text{II} \) we have
\[
\Xi \{ i\xi(\tau), \frac{x(\tau)}{2T} | x(\tau), \frac{h^2}{8T} i\xi(\tau) \} = \int_{V} \int_{I} \exp \left\{ \int [i\xi \cdot V - \frac{h^2}{8T} i\xi \cdot \delta \frac{\delta}{\delta y}] d\tau \right\} \times \Xi \{ i\xi, \frac{x}{2T} | x, y \} ||_{y=0} = \int \exp \left[-i\int x \cdot I \, d\tau \right] P\{V - \frac{h^2}{8T} \delta \frac{\delta}{\delta y}, I \mid x, y \} ||_{y=0} = \\
= \exp \left[- \int \frac{h^2}{8T} \delta \frac{\delta}{\delta y} \int \exp \left[- \frac{1}{2T} \int x \cdot I \, d\tau \right] P\{V, I \mid x, y \} ||_{y=0} = \int \exp \left[- \frac{1}{2T} \int \tilde{x} \cdot I \, d\tau \right] P\{\tilde{V} - \frac{h^2}{8T} \delta \frac{\delta}{\delta y}, I \mid \tilde{x}, \tilde{y} \} ||_{\tilde{y}=0}
\] (43)

with \( \tilde{V}(\tau) = \varepsilon V(t-\tau) \) and \( \tilde{y}(\tau) = \varepsilon y(t-\tau) \). This is symmetrized analogue of relations (30)-(31) from [2].

The coefficient \( \frac{h^2}{8T} \) in these relations is the only factor evidently including the Planck constant and therefore determines most significant differences from classical limit. Under the latter, \( I \rightarrow L_{0}Q, J \rightarrow L_{0}V \), and Eq.\( \text{II} \) reduces to
\[
\int \exp \left[- \frac{1}{2T} \int x \cdot I \, d\tau \right] P\{V, I \mid x, 0 \} = \int \exp \left[- \frac{1}{2T} \int \tilde{x} \cdot I \, d\tau \right] P\{\tilde{V}, I \mid \tilde{x}, 0 \}
\] (44)

thus canceling the auxiliary forces \( y \).

Probabilistic quantum FDR (45) strongly simplifies under choice \( V \Rightarrow I = LQ = (4iT/\hbar)TQ \), when \( V \) become (time-smoothed) time derivatives of the variables \( Q \) conjugated with actual “seed” external forces \( x \). In this interesting special case, clearly, \( \varepsilon = -\varepsilon \), and Eq.\( \text{II} \) shortens to relation
\[
\Xi \{ a - \frac{x}{2T} | x, \frac{h^2}{8T} a \} = \Xi \{ \tilde{a} - \frac{\tilde{x}}{2T} | \tilde{x}, \frac{h^2}{8T} \tilde{a} \}
\] (47)

for functional of three arguments,
\[ \Xi \{ a(\tau) \mid x(\tau), y(\tau) \} = \langle \exp \left[ \int_0^t a(\tau) \cdot I(\tau) \, d\tau \right] \rangle_{h'(x(\tau), y(\tau))} , \]

with \( I = \mathcal{L}Q = (4iT/\hbar) \mathcal{T}Q \), \( \tilde{a}(\tau) = -\epsilon a(t-\tau) \), and effective IH
\[ h'(x, y) = x \cdot Q + y \cdot \mathcal{L}^2 Q \]

Introducing PF \( P\{I \mid x, y\} \) generated by CF \( (48) \) and repeating above manipulations, one can come to relation
\[ \exp \left[ -\int d\tau \frac{h^2}{8T} \frac{\delta}{\delta I} \cdot \frac{\delta}{\delta y} \right] P_0\{I \mid x, y\} \|_{y=0} = \]
\[ = \exp \left[ -\int d\tau \frac{h^2}{8T} \frac{\delta}{\delta I} \cdot \frac{\delta}{\delta y} \right] P_0\{\tilde{I} \mid \tilde{x}, \tilde{y}\} \|_{\tilde{y}=0} , \]

where
\[ P_0\{I \mid x, y\} \equiv P\{I \mid x, y\} \exp \left[ -\frac{1}{2T} \int x \cdot I \, d\tau \right] \]
and \( \tilde{I}(\tau) = -\epsilon I(t-\tau) \), \( \tilde{y}(\tau) = -\epsilon y(t-\tau) \).

This is quantum generalization of classical FDR for probabilistic functional of “currents” (or “flows”, or “velocities”) \( I(t) = dQ(t)/dt \) conjugated with external forces, which for the first time were derived in \[4, 5\]. It is easy to generalize relations \( (47) \) and \( (49)-(50) \) to cases when along with \( I \) some other variables \( V \) are under observations as in CF \( (40) \).

In comparison with classical case, Eq \( (49) \) looks unclosed since involves virtual excess arguments, - i.e. forces \( y(t) \), - which eventually are frozen at zero. But we can make Eq \( (49) \) formally closed if sufficiently expand collection of quantum variables under observation. For instance, if take
\[ Q = \{Q_{\mu\nu}^+, Q_{\mu\nu}^-\} , \]
\[ Q_{\mu\nu}^+ = q_{\mu\nu} \frac{\vert \mu \rangle \langle \nu \vert + \vert \nu \rangle \langle \mu \vert}{\sqrt{2}} , \quad Q_{\mu\nu}^- = q_{\mu\nu} \frac{\vert \mu \rangle \langle \nu \vert - \vert \nu \rangle \langle \mu \vert}{i\sqrt{2}} , \]
with some real symmetric \( q_{\mu\nu} \), and, respectively, \( x = \{x_{\mu\nu}^+, x_{\mu\nu}^-\} \) and
\[ I = \{I_{\mu\nu}^+, I_{\mu\nu}^-\} = \frac{4iT}{\hbar} \mathcal{T}_{\mu\nu} \{ -Q_{\mu\nu}^-, Q_{\mu\nu}^+ \} , \]
\[ J = \{J_{\mu\nu}^+, J_{\mu\nu}^-\} = -\left( \frac{4T}{\hbar} \mathcal{T}_{\mu\nu} \right)^2 \{Q_{\mu\nu}^+, Q_{\mu\nu}^-\} \]

Clearly, now
\[ \frac{\delta}{\delta y_{\mu\nu}} = -\left( \frac{4T}{\hbar} \mathcal{T}_{\mu\nu} \right)^2 \frac{\delta}{\delta x_{\mu\nu}} , \]
and Eq \( (49) \) simplifies to
\[ \exp \left[ 2T \int d\tau \sum \frac{\delta}{\delta I_{\mu\nu}^\pm} \mathcal{T}_{\mu\nu}^2 \frac{\delta}{\delta x_{\mu\nu}^\pm} \right] P_0\{I \mid x\} = \]
\[ = \exp \left[ 2T \int d\tau \sum \frac{\delta}{\delta I_{\mu\nu}^\pm} \mathcal{T}_{\mu\nu}^2 \frac{\delta}{\delta x_{\mu\nu}^\pm} \right] P_0\{\tilde{I} \mid \tilde{x}\} , \]
where $\tilde{I}_{\mu\nu}(\tau) = \mp I_{\mu\nu}(t-\tau)$, $\tilde{x}^{\pm\mu\nu}(\tau) = \pm x^{\pm\mu\nu}(t-\tau)$, and, as before,

$$P_0\{I \mid x\} \equiv P\{I \mid x\} \exp\left[-\frac{1}{2T} \int \sum x^{\pm\mu\nu} I^{\pm\mu\nu} d\tau\right]$$  \hspace{1cm} (52)

Notice that all the variables $Q$, $I$, $J$ have finite classical limits at $\hbar \to 0$, therefore role of common characteristic parameter of their “quantumness” now is played by factor $T^2_{\mu\nu}$ in Eq.51.

These relations possibly may be better visualized with the help of easy derivable operator equality

$$e^{A\nabla_x \nabla_y} e^{B XY} = \frac{1}{1-AB} \times$$

$$\times \left\{ \exp \left[ \frac{B XY}{1-AB} + \frac{AB}{1-AB} (X\nabla_x + Y\nabla_y) + \frac{A \nabla_x \nabla_y}{1-AB} \right] \right\},$$

where the braces mean normal ordering of operators (all differentiation, or “annihilation”, operators are on the right from the multiplication, or “creation”, ones). Correspondingly,

$$e^{A\nabla_x \nabla_y} e^{B XY} P(X,Y) = \frac{1}{1-AB} \exp \left( \frac{B XY}{1-AB} \right) \times$$

$$\times \exp \left[ A (1-AB) \nabla_x \nabla_y \right] P \left( \frac{X}{1-AB} \bigg| \frac{Y}{1-AB} \right)$$

with formally arbitrary function $P(X,Y)$. Applying these formulae to Eqs.51, 52, we can write, symbolically but quite transparently,

$$\exp\left[-\frac{1}{2T} \int \frac{x I}{1+T^2} d\tau\right] \times$$

$$\times \exp\left[2T \int d\tau \frac{\delta}{\delta I} T^2 (1+T^2) \frac{\delta}{\delta x} \right] P \left( \frac{I}{1+T^2} \bigg| \frac{x}{1+T^2} \right) =$$

$$= \exp\left[-\frac{1}{2T} \int \frac{\tilde{x} I}{1+T^2} d\tau\right] \times$$

$$\times \exp\left[2T \int d\tau \frac{\delta}{\delta \tilde{x}} T^2 (1+T^2) \frac{\delta}{\delta \tilde{x}} \right] P \left( \frac{\tilde{I}}{1+T^2} \bigg| \frac{\tilde{x}}{1+T^2} \right)$$

(55)

Other, more physically meaningful, representations of the probabilistic FDR (51) = (55) will be considered separately.

In terms of CF, FDR (51) = (55) become much more nicely: Eq.47 takes form

$$\Xi \left\{ a - \frac{x}{2T} \bigg| x - 2T T^2 a \right\} = \Xi \left\{ \tilde{a} - \frac{\tilde{x}}{2T} \bigg| \tilde{x} - 2T T^2 \tilde{a} \right\}$$

(56)

where parities of $a$’s are opposite to that of $x$’s. If we are interested in FDR for various many-time statistical moments (or cumulants) and response functions of $I$’s and related variables, the Eq.56 may be good start point.

6. Discussion and resume

Our above results show that analysis of differences between quantum FDR for continuous observations and similar classical FDR can be reduced to analysis of specific symmetry
relations like Eq\[49\] - involving virtual auxiliary forces \(y(t)\), - in comparison with their classical limit at \(\hbar^2 \to 0\), for instance,
\[
P_0\{I \mid x, 0\} = P_0\{\tilde{I} \mid \tilde{x}, 0\}
\] (57)

Indeed, according to Eq\[50\] in both quantum and classical theories we have
\[
\frac{P\{I \mid x\}}{P\{\tilde{I} \mid \tilde{x}, 0\}} = \frac{P_0\{I \mid x\}}{P_0\{\tilde{I} \mid \tilde{x}\}} \exp\left[\frac{1}{T} \int x \cdot I d\tau\right],
\] (58)

where in quantum case \(P\{I \mid x\} \equiv P\{I \mid x, y = 0\}\) and \(P_0\{I \mid x\} \equiv P_0\{I \mid x, y = 0\}\), and we took into account that
\[
\int_0^t \tilde{x} \cdot \tilde{I} d\tau = - \int_0^t x \cdot I d\tau
\]
because of opposite parities of the forces and currents. In classical case, evidently, right-hand fraction in Eq\[58\] is identically unit, \(P_0\{I \mid x\}/P_0\{\tilde{I} \mid \tilde{x}\} \Rightarrow 1\). In quantum theory this equality, that is relation (57), seemingly does not contradict Eq\[49\] but at the same time does not follow from it. What is for the Eq\[51\], it is certainly inconsistent with (57).

Thus, in the framework of continuous quantum observations there is no general “one-to-one” correspondence between probabilities of mutually time-reversed processes. Instead of simple \(c\)-number exponential proportionality coefficient, they are connected through definite integral operators acting in functional space of observations and perturbations.

This is not surprising, for anybody knows that in quantum mechanics any measurements causes more or less unpredictable excess perturbation of observed system. Therefore, naturally, comparison of observations of mutually time-reversed processes can not be as literal as in classical theory. It requires control of total change in system’s (internal) energy or entropy. But if such the control is included into consideration then the artificial excess perturbation manifests itself evidently as effective renormalization of system’s Hamiltonian. This results in mutual entanglement of perturbations and observations and thus in the mentioned complication of probabilistic FDR.

To resume, we expounded several different forms of quantum generalized fluctuation-dissipation relations (FDR) which express symmetry of laws of quantum dynamics in respect to time reversal. Then we focused on formulation of FDR for continuously time-distributed observations of energy exchange and entropy production rates and other variables in externally driven systems.

It is demonstrated that one and the same statistical ensemble of quantum processes requires different Hamiltonian and entropy production operators to be described in terms of discrete measurements or continuous measurements, because of their influence upon evolution of quantum system. Naturally, this effect is significant at high frequencies \(\gtrsim T/\hbar\) only, and just they determine peculiarities of quantum FDR in comparison with the classical ones.

In particular, operator \(\Delta \dot{S}\) of entropy production (EP) per unit time differs from operator of external work per unit time (divided by temperature \(T\)) by suppressing contributions of high-frequency quantum transitions. This result coincides with one
obtained in another way in [1] and means merely that at high frequencies the EP operator counts number of quanta rather than their energy.

Maximum visual similarity of quantum FDR for continuous observations to their classical prototypes is achieved when mutually and time-reversed processes symmetrically involve EP measurements and therefore equally renormalize system’s Hamiltonian. In such representation, differences of quantum from classical FDR reduce to definite entanglement of observations and perturbations at high frequencies, which is expressed by rather simple formulae.

Nevertheless, the obtained probabilistic quantum FDR still are not quite comfortable for heuristic or analytical exploitation and need in more careful mathematical investigations.

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