Fluctuation-dissipation relations under Lévy noises

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received 9 January 2012; accepted in final form 15 May 2012
published online 13 June 2012

PACS 05.10.Gg – Stochastic analysis methods (Fokker-Planck, Langevin, etc.)
PACS 05.70.Ln – Nonequilibrium and irreversible thermodynamics
PACS 05.40.Fb – Random walks and Lévy flights

Abstract – For systems close to equilibrium, the relaxation properties of measurable physical quantities are described by the linear response theory and the fluctuation-dissipation theorem (FDT). Accordingly, the response or the generalized susceptibility, which is a function of the unperturbed equilibrium system, can be related to the correlation between spontaneous fluctuations of a given conjugate variable. There have been several attempts to extend the FDT far from equilibrium, introducing new terms or using effective temperatures. Here, we discuss applicability of the generalized FDT to out-of-equilibrium systems perturbed by time-dependent deterministic forces and acting under the influence of white Lévy noise. For the linear and Gaussian case, the equilibrium correlation function provides a full description of the dynamic properties of the system. This is, however, no longer true for non-Gaussian Lévy noises, for which the second and sometimes also the first moments are divergent, indicating absence of underlying physical scales. This self-similar behavior of Lévy noises results in violation of the classical dissipation theorem for the stability index $\alpha < 2$. We show that by properly identifying appropriate variables conjugated to external perturbations and analyzing time-dependent distributions, the generalized FDT can be restored also for systems subject to Lévy noises. As a working example, we test the use of the generalized FDT for a linear system subject to Cauchy white noise.

Introduction. – The fluctuation-dissipation theorem (FDT) connects correlation functions to linear response functions and constitutes a useful tool in investigations of physical properties of systems at thermodynamic equilibrium [1]. By virtue of the FDT, measurable macroscopic physical quantities like specific heats, susceptibilities or compressibilities can be related to correlation functions of spontaneous fluctuations. For systems weakly displaced from equilibrium, the FDT allows one to express the linear response of physical observables to time-dependent external fields in terms of time-dependent correlation functions. Accordingly, departures from the FDT can be expected for far-from-equilibrium situations and have been demonstrated in various aging, glassy and biological media [2–5].

On the other hand, the wealth of theoretical, experimental and numerical research indicate that the FDT is a special case of more general fluctuation relations that remain valid also in a specific class of non-equilibrium systems [5–13]. Following a former generalization of FDT [14] based on the identity derived by Hatano and Sasa [15], we discuss here an extension of the fluctuation theorem to stochastic models obeying Markovian dynamics and driven by white $\alpha$-stable noises. We apply the generalized fluctuation-response theorem to this case and analyze the regime in which linear response theory becomes invalidated. We illustrate our results with the simple example of an oscillator coupled to a non-equilibrium bath whose action is represented by a Cauchy white noise.

Let us first review the generalized FDT introduced in [14] and extended to arbitrary observables in [7]. The theorem applies to any Markov process $x(t)$ whose
dynamics depends on a set of parameters $\vec{\lambda}$ and for which a well-defined (non-equilibrium) stationary state exists. We study the linear response of the system to perturbations $\vec{\lambda}(t) = \vec{\lambda}_0 + \delta\vec{\lambda}(t)$ around a reference stationary state corresponding to constant parameters $\vec{\lambda}_0$ and resulting probability density function (PDF) $\rho_{ss}(x; \vec{\lambda}_0)$. Given an arbitrary observable $A(x)$, the response (evaluated to first order in the perturbation) can be written as

$$\langle A(t) \rangle - \langle A \rangle_0 \simeq \int_0^T \chi_{A, \gamma}(t-t') \delta\lambda_{\gamma}(t') dt',$$

where $A(t) \equiv A(x(t))$ and the brackets $\langle \ldots \rangle_0$ indicate an average over the reference state $\rho_{ss}(x; \vec{\lambda}_0)$; summation over the repeated index $\gamma$ is assumed and $\chi_{A, \gamma}$ is the time-dependent susceptibility of variable $A$ with respect to variations of $\lambda_{\gamma}$ (i.e., perturbations of the $\gamma$ component of $\vec{\lambda}$). The FDT relates this susceptibility to correlations measured in the reference unperturbed state [7,14]

$$\chi_{A, \gamma}(t-t') = \frac{d}{dt} \langle A(t) X_{\gamma}(t') \rangle_0,$$

where $X_{\gamma}(x)$ is the variable conjugate to the perturbation $\lambda_{\gamma}$ and is defined as

$$X_{\gamma}(x) = -\frac{\partial \ln \rho_{ss}(x; \vec{\lambda})}{\partial \lambda_{\gamma}} \Bigg|_{\vec{\lambda} = \vec{\lambda}_0} = \frac{\partial \phi}{\partial \lambda_{\gamma}}.$$

In this definition $\phi \equiv -\ln \rho_{ss}$ stands for a non-equilibrium potential [7,14]. If the reference state is the Gibbs equilibrium state corresponding to a temperature $kT = \beta^{-1}$ and a Hamiltonian $H(x; \vec{\lambda})$, the stationary PDF $\rho_{ss}(x; \vec{\lambda})$ assumes the form $\rho_{ss}(x; \vec{\lambda}) = \exp[-\beta H(x; \vec{\lambda})/Z(\beta; \vec{\lambda})]$ and the conjugate variable reads

$$X_{\gamma}(x) = \frac{1}{kT} \frac{\partial}{\partial \lambda_{\gamma}} \left[ \frac{H(x; \vec{\lambda})}{Z(\beta; \vec{\lambda})} \right] \Bigg|_{\vec{\lambda} = \vec{\lambda}_0},$$

where $F = -kT \ln Z$ stands for the free energy. Accordingly, $X_{\gamma}$ can be interpreted as the fluctuation of the quantity

$$\frac{\partial F(x; \vec{\lambda})}{\partial \lambda_{\gamma}} \equiv \frac{\partial F(x; \vec{\lambda})}{\partial \lambda_{\gamma}} \Bigg|_{\vec{\lambda} = \vec{\lambda}_0}.$$

$$X_{\gamma}(x) = \frac{1}{kT} \left[ \frac{\partial H(x; \vec{\lambda}_0)}{\partial \lambda_{\gamma}} - \left\langle \frac{\partial H(x; \vec{\lambda}_0)}{\partial \lambda_{\gamma}} \right\rangle_0 \right].$$

For instance, if $\lambda_{\gamma}$ is a force coupled to a coordinate $x_{\gamma}$, i.e., if the control parameter appears in the Hamiltonian as $-\lambda_{\gamma} x_{\gamma}$, then the conjugate variable $X_{\gamma} = -\left(x_{\gamma} - \langle x_{\gamma} \rangle\right)/(kT)$ represents fluctuations of $x_{\gamma}$.

In general, if the reference state $\rho_{ss}(x; \vec{\lambda}_0)$ is not an equilibrium state, the conjugate variables defined by eq. (3) do not have any straightforward physical interpretation [7,14]. In this letter, we examine the generalized FDT, eqs. (1) and (2), for a system obeying non-equilibrium Markovian dynamics and driven by Lévy white noise. The system of this type may be conceived as a generalization of Brownian motion: the particle undergoing Lévy superdiffusion is performing motion with random jumps and step lengths following a power-law distribution. As a result, the width of the distribution of particles grows superlinearly with time [16,17] signaling anomalous dynamics. Notably, unlike in case of a standard Brownian motion, in Lévy superdiffusion large fluctuations of the position occur with probability higher than for linear systems subjected to Gaussian uncorrelated noise. Consequently, it is rather counterintuitive to expect linear response of the system, even for weak perturbations. The divergence of first and second moments of some Lévy PDFs indicates absence of underlying physical scales and is usually interpreted as the scale invariance, characteristic of self-similar (or fractal) behavior. For the standard Fokker-Planck equation (FPE) equivalent to the Langevin equation driven by Gaussian white noise, as well as for the subdiffusive fractional FPE one finds the generalized Einstein relation, connecting the first moment in the presence of the perturbing force to the second moment in the absence of the force [18,19]. This is no longer true for the Lévy flight [19], when only in the Brownian limit $\alpha = 2$, this relation is satisfied (provided that the proper amplitude of the noise interpreted as thermal fluctuations is considered). In general though, due to a diverging mean square displacement, the generalized Einstein relation does not hold leading to a violation of the classical fluctuation-dissipation theorem.

Signatures of Lévy noise and anomalous transport have been found ubiquitous in nature [16,20] and serve as suitable models describing atmospheric turbulence [21], transport in turbulent plasmas [22], activation kinetics by non-thermal baths [23], transport in fractured materials [24], epidemic spreading [25], dispersal of banknotes [26] or light scattering in heterogeneous dielectric media [27]. In what follows we address foundations of linear response and FDT in systems perturbed by Lévy noises.

**Linear system driven by Lévy white noises.** – We proceed to discuss response properties of an overdamped Lévy-Brownian particle moving in a parabolic potential that is subject to a deterministic time-dependent force $f(t)$ and a white Lévy noise $\zeta(t)$ resulting from the fluctuating environment. The corresponding Langevin equation reads

$$\begin{cases} \dot{x}(t) = -ax + f(t) + \zeta(t), \\ x(0) = x_0. \end{cases}$$

The white Lévy noise $\zeta(t)$ is defined as the time derivative of a stationary Lévy process $\zeta(t)$, i.e., the integral over time

$$L_{\alpha,\beta}(t) \equiv \int_0^t \zeta(s) ds = z(t)$$

represents a stochastic process with independent increments whose probability density $p_{\alpha,\beta}(z,t)$ is a stable Lévy distribution. Consequently, the Fourier transform of
the probability density (characteristic function) \( \varphi(k, t) = e^{ikz(t) \xi(t)/\sigma_\alpha (z, t; \alpha, \mu, \sigma)}dz \) reads

\[
\varphi(k, t) = \exp \left[ ik \mu t - \sigma_\alpha^2 |k|^{\alpha t} \left( 1 - i \beta \text{sign}(k) \tan \frac{\pi \alpha}{2} \right) \right]
\] (8)

for \( \alpha \neq 1 \) and

\[
\varphi(k, t) = \exp \left[ ik \mu t - \sigma_0 |k| t \left( 1 + i \beta \frac{2}{\pi} \text{sign}(k) \ln |k| \right) \right]
\] (9)

for \( \alpha = 1 \) [28]. Here \( \alpha \in (0, 2] \) is the Lévy (stability) index, \( \beta \in [-1, 1] \) is the skewness parameter (for \( \beta = 0 \) the distributions are symmetric), \( \sigma_0 > 0 \) represents the noise intensity, and \( \mu_0 \in \mathbb{R} \) is a location (shift) parameter. For \( \alpha = 2 \) and \( \beta = 0 \) a standard Gaussian distribution is recovered with \( \mu_0 \) indicating the mean of the random variable \( z(t) \) and \( \sigma_0^2 t \) staying for its variance. For \( \alpha < 2 \), stable probability densities exhibit heavy tails and divergent moments: the asymptotic (large \( t \)) behavior of the corresponding PDF is then characterized by a power law \( p_{\alpha, \beta}(z, t; \sigma_0, \mu_0) \propto |z|^{-(1+\alpha)}. \) Under those circumstances, eq. (6) is associated with the space-fractional Fokker-Planck-Smoluchowski equation (FFPE) [18,29]:

\[
\frac{\partial p(x, t)}{\partial t} = -\frac{\partial}{\partial x} \left[ \mu_0 ax + f(t) \right] p(x, t)
+ \sigma_0^2 \frac{\partial^\alpha}{\partial |x|^\alpha} p(x, t) + \sigma_0^2 \beta \frac{\pi \alpha}{2} \frac{\partial}{\partial x} \frac{\partial^{\alpha-1}}{\partial |x|^{\alpha-1}} p(x, t).
\] (10)

Here, the fractional (Riesz-Weyl) derivative is defined by its Fourier transform \( \mathcal{F} \left[ \frac{\partial^\alpha}{\partial |x|^\alpha} f(x) \right] = -|k|^\alpha \mathcal{F} [f(x)] \) [18,29]. Accordingly, eq. (10) has the following Fourier representation:

\[
\frac{\partial \hat{p}(k, t)}{\partial t} = -ak \frac{\partial}{\partial k} \hat{p}(k, t) + ik \left[ \mu_0 f(t) + f(t) \right] \hat{p}(k, t)
- \sigma_0^2 |k|^{\alpha t} \left[ 1 - i \beta \text{sign}(k) \tan \frac{\pi \alpha}{2} \right] \hat{p}(k, t)
\] (11)

where \( \hat{p}(k, t) = \mathcal{F} [p(x, t)] \). In what follows, we adhere to the analysis of strictly \( \alpha \)-stable random variables [28], i.e., those for which \( \mu_0 = 0 \), and additionally \( \beta = 0 \) if \( \alpha = 1 \).

Since our original Langevin equation (6) is linear, its solution depends linearly on the stable process \( L_{\alpha, \beta}(t) \). Accordingly, the probability density of the solution, \( p(x, t|x_0, 0) \), has the form of an \( (\alpha, \beta) \)-stable Lévy distribution with time-dependent location \( \mu(t) \) and scale \( \sigma(t) \) parameters [19]. By analogy, its characteristic function is given by (cf. eqs. (8) and (9))

\[
\hat{p}(k, t) = \exp \left[ ik \mu(t) - \sigma_0 |k|^{\alpha t} \left( 1 - i \beta \text{sign}(k) \tan \frac{\pi \alpha}{2} \right) \right].
\]

We insert this ansatz into FFPE (11). Since the derivative with respect to \( k \) appears multiplied by \( k \) in (11), the non-analyticity of \( |k|^{\alpha t} \) at \( k = 0 \) does not create any singularity in the equation. The real part of eq. (11) yields the following evolution equation for the scale parameter \( \sigma(t) \):

\[
-\alpha \sigma^{\alpha-1} \ddot{\sigma} + a \sigma - \sigma_0^\alpha,
\] (12)

whereas the imaginary part gives

\[
[\mu + a \mu - f(t)] k = \left[ -a \alpha^{\alpha-1} \dot{\sigma} - a \alpha \sigma^\alpha + \sigma_0^\alpha \right] \times \beta \tan \frac{\pi \alpha}{2} |k|^{\alpha t} \text{sign}(k).
\] (13)

The RHS of eq. (13) vanishes due to eq. (12). From LHS one gets the evolution equation for the location parameter:

\[
\dot{\mu} = -a \mu + f(t).
\] (14)

The evolution equations (12) and (14) are completed with the initial conditions \( \mu(0) = x_0 \) and \( \sigma(0) = 0 \) (we are calculating probability densities conditioned to \( x(0) = x_0 \)). The solution of these differential equations are

\[
\mu(t) = e^{-at} x_0 + e^{-at} \int_0^t e^{as} f(s) ds
\] (15)

and

\[
\sigma(t) = \sigma_0 \left[ \frac{1}{a \alpha} \left( 1 - e^{-a \sigma} \right) \right]^{1/\alpha},
\] (16)

where \( \sigma_0 \) is the scale parameter of the corresponding \( \alpha \)-stable density. For a constant force \( f(t) \equiv f \), the long time asymptotics of the above equations are \( \lim_{t \to \infty} \mu(t) = f/a \) and \( \lim_{t \to \infty} \sigma(t) = \sigma_0/(a \alpha)^{1/\alpha} \).

The conjugate variable. – To determine the conjugate variable to the external force, we need the stationary distribution in position space for a constant force \( f \). Despite the characteristic functions of stable distributions assume closed expressions, the corresponding PDFs have a known simple analytical form [28,30] only in a few cases: For \( \alpha = 2 \) and \( \beta = 0 \) the resulting distribution is Gaussian; for \( \alpha = 1 \), \( \beta = 0 \) one gets the Cauchy distribution; finally, for \( \alpha = 1/2, \beta = 1 \) the Lévy-Smirnoff distribution is obtained. Here, we derive explicit expressions for the conjugate variable for these three cases.

For \( \alpha = 2 \) and \( \beta = 0 \), the time-dependent solution of the corresponding Langevin equation (6) is

\[
p_{2,0}(x, t|x_0, 0) = \frac{1}{\sqrt{2a^2(t)}} \exp \left[ -\frac{(x - \mu(t))^2}{2a^2(t)} \right]
\] (17)

with \( \mu(t) \) and \( \sigma(t) \) given by eqs. (15) and (16). The stationary solution \( p_{\alpha, \beta}(x) \) for a constant force \( f \) is obtained by replacing \( \mu(t) \) and \( \sigma^2(t) \) by their stationary values, \( \mu/f \) and \( \sigma^2_0/(2a) \), respectively. We then get the non-equilibrium potential \( \phi \equiv -\ln p_{\alpha, \beta}(x) \) and the conjugate variable can be easily derived as

\[
X_G = -\frac{\partial \ln p_{\alpha, \beta}(x)}{\partial f} \bigg|_{f=0} = \frac{\partial \phi}{\partial f} \bigg|_{f=0} = \frac{2x}{\sigma_0^2},
\] (18)

which is proportional to \( x \), as expected, since the Gaussian case corresponds to a Brownian particle in equilibrium.

For \( \alpha = 1 \) and \( \beta = 0 \), the time-dependent solution of the corresponding Langevin equation (6) is the Cauchy distribution

\[
p_{1,0}(x, t|x_0, 0) = \frac{\sigma(t)}{\pi} \frac{1}{[x - \mu(t)]^2 + \sigma^2(t)}
\] (19)
and the stationary solution for a constant force \( f \) is obtained replacing \( \mu(t) \) and \( \sigma(t) \) by their stationary values, \( f/\alpha \) and \( \sigma_0/\alpha \), respectively. The corresponding conjugate variable takes now the form

\[
X_C = -\frac{2x}{\alpha [x^2 + (\sigma_0/\alpha)^2]} \tag{20}
\]

which is proportional to \( x \) only for small values of \( x \) and becomes proportional to \( 1/x \) for large \( x \). This large \( x \) behavior ensures the convergence of all the moments of \( X_C \), whereas for the Cauchy case \( |x|^\nu \) exists only if \( \nu < 1 \), see [16,28].

Finally, for \( \alpha = 1/2 \) and \( \beta = 1 \) the solution to eq. (6) is the Lévy-Smirnoff PDF

\[
p_{1/2,1}(x,t|x_0,t_0) = \frac{\sqrt{\sigma(t)}}{2\pi [x - \mu(t)]^3} \exp \left[ -\frac{\sigma(t)}{2(x - \mu(t))} \right] \tag{21}
\]

for \( x > \mu(t) \) and \( p_{1/2,1}(x,t|x_0,0) \equiv 0 \) for \( x \leq \mu(t) \). The stationary values of \( \mu(t) \) and \( \sigma(t) \) are in this case \( f/\alpha \) and \( 4\sigma_0/\alpha^2 \), respectively. Inserting these values to eq. (21), one can easily obtain \( p_{st}(x) \) and the conjugate variable

\[
X_{L-S} = \frac{4\sigma_0 - 3\sigma_0^2 x}{2\sigma_0^2 x^2}, \quad x > \mu(t). \tag{22}
\]

**Susceptibility and response.** – The main objective of the current work is to compare the response of the system to external perturbation as calculated directly from the definition

\[
\langle X(t) \rangle = \int_{-\infty}^\infty X(x)p(x,t)dx, \tag{23}
\]

or, otherwise determined by the generalized susceptibility \( \chi(t) = \frac{d}{dt} \langle X(t)X(0) \rangle_0 \) within linear response theory:

\[
\langle X(t) \rangle_{LR} = \int_0^t \chi(t-s)f(s)ds. \tag{24}
\]

For that purpose, we restrict our analysis to the fully analytically solvable Cauchy case, \( \alpha = 1, \beta = 0 \), and identify the conjugate variable as \( X \equiv X_C \), with \( X_C \) given by eq. (20). In this case, the time-dependent average (23) can be calculated exactly with the probability density:

\[
p(x,t) = \int_{-\infty}^\infty p(x,t|x_0,0)p(x_0)dx_0, \tag{25}
\]

where

\[
p(x_0) = \frac{\sigma_0}{\alpha \pi} \frac{1}{\sqrt{\sigma_0^2 + (\sigma_0/\alpha)^2}} \tag{26}
\]

and \( p(x,t|x_0,0) \) is given by eq. (19).

On the other hand, the FDT relates the susceptibility with the autocorrelation of the conjugate variables in the reference state, i.e., for \( f = 0 \). The autocorrelation is defined as

\[
\langle X(t)X(0) \rangle_0 = \int \frac{2x}{\alpha [x^2 + (\sigma_0/\alpha)^2]} \frac{2y}{\pi |x - \mu(t)|^2 + \sigma^2(t)} \times \frac{\sigma(t)}{\alpha \pi} |y + (\sigma_0/\alpha)^2| \; dx \; dy
\]

where \( \mu(t) = e^{-\nu t} \) and \( \sigma(t) = \sigma_0[(1 - e^{-\nu t})/\alpha] \). The final result is surprisingly simple:

\[
\langle X(t)X(0) \rangle_0 = \frac{1}{2\sigma_0^2} e^{-\nu t}. \tag{27}
\]

From the above, the generalized susceptibility can be derived by differentiation with respect to time (see eq. (2)):

\[
\chi(t) = \frac{d}{dt} \langle X(t)X(0) \rangle_0 = \frac{a}{2\sigma_0^2} e^{-\nu t}. \tag{28}
\]

In further calculations, for the sake of simplicity, it is assumed that \( a = 1 \) and \( \sigma_0 = 1 \), so that \( \langle X(t)X(0) \rangle_0 = \frac{1}{2} e^{-t} \) and \( \chi(t) = -\frac{1}{2} e^{-t} \).

In order to test the linear response theory for our dynamic Markov system subjected to Lévy white noise, we calculate the response of the conjugate variable \( X \) to two different time dependent perturbations: the sum of a small periodic and a linearly increasing force, \( f_1(t) = \sin(t)/10 + t/100 \); and a periodic force with increasing amplitude, \( f_2(t) = t \sin(t)/100 \). Figure 1 displays the exact evolution of \( \langle X(t) \rangle \) and the result obtained from the linear response theory. For small perturbations, (i.e., short times, \( t \leq 50 \)) in both cases, linear response theory yields an accurate estimation of the response. In the case of \( f_1(t) \), for large times the (exact) response \( \langle X(t) \rangle \) is insensitive to the sinusoidal component of the force, which is small compared with the linear part. This is due to the peculiar form of the conjugate variable \( X \) given by eq. (20). For a constant force \( f \), the mean value of \( X \) is

\[
\langle X \rangle = -\frac{\sigma_0}{\alpha \pi} \int_{-\infty}^\infty \frac{dx}{|x - f/\alpha|^2 + (\sigma_0/\alpha)^2} \frac{2x}{\alpha^2 x^2 + (\sigma_0/\alpha)^2}
\]

which yields \( \langle X \rangle = -0.5 \) for \( \sigma_0 = 1 \) and \( f = 2 \) (at \( t = 200 \), \( f_1(t) \approx 2 \)). A similar saturation effect is not observed for the sinusoidal force \( f_2(t) \).

We can apply the generalized FDT to any function \( A(x) \) with finite average. Due to the heavy tails of stable distributions, only moments \( \langle |x|^\nu \rangle \) with \( \nu < \alpha \) converge (\( \nu < 1 \) in the case of Cauchy distributions). Moreover, those moments are even functions of \( x \) and, for symmetry reasons, the correlation with \( X \) vanishes: \( \langle |x|^\nu \rangle \langle X(0) \rangle_0 = 0 \). Consequently, the deviation of \( \langle |x(t)|^\nu \rangle \) with respect to its reference value is non-linear in the perturbation \( f(t) \). On the other hand, we can obtain non-trivial results for fractional moments \( A(x) = \text{sign}(|x|)|x|^\nu \), whose
average in the reference state vanishes \( \langle A \rangle_0 = 0 \). The corresponding correlation function reads

\[
\langle A(x(t)) X(0) \rangle = - \frac{\nu}{\sin \left( \frac{\pi \nu}{2} \right)} e^{-t},
\]

and the generalized susceptibility is given by

\[
\chi_A(t) = \frac{\nu}{\sin \left( \frac{\pi \nu}{2} \right)} e^{-t}.
\]

In the spirit of the former definition, see eq. (23), the exact value of \( \langle A(x(t)) \rangle \) can be calculated as

\[
\langle A(x(t)) \rangle = \int_{-\infty}^{\infty} A(x(t))p(x, t)dx
= \int_{-\infty}^{\infty} A(x(t))p(x, t|x_0, 0)p(x_0)dx,
\]

where \( p(x_0) \) and \( p(x, t|x_0, 0) \) are given by eqs. (19) and (26), respectively. Figure 2 displays the comparison of the exact evolution \( \langle \text{sgn}[x(t)]|x(t)|^{1/2} \rangle \) with the linear response approximation \( \int_0^t \chi_A(t-s)f(s)ds \).

As previously, linear response theory is valid for weak perturbation up to \( f \approx 0.5 \). However, for the fractional moment and the linearly increasing force (upper plot), we do not observe saturation.

**Summary and conclusions.** – We have shown that the generalized FDT can be applied to linear systems driven by Lévy noise. The FDT allows one to calculate the susceptibility of any observable and then the response to any small time-dependent perturbation. For a noise distributed according to the Cauchy distribution, we have calculated the susceptibility of the conjugate variable \( X_C \) and the susceptibility of odd fractional moments \( \langle \text{sgn}[x(t)]|x(t)|^\nu \rangle \), which have a simple exponential behavior. From these susceptibilities it is easy to get simple analytical expressions for the response of the system using eq. (24). We have to notice that, although the exact response can be calculated analytically using eq. (23), the corresponding integrals are cumbersome and can be only solved numerically in the simplest cases. Therefore, the generalized FDT is shown to be a useful analytical tool to deal with these types of systems.

It is still not obvious whether the conjugate variables that we have calculated for the Cauchy and Lévy-Smirnoff noises have any physical meaning, besides the one provided by the generalized FDT itself. The generalized FDT shows that these conjugate variables represent the change in the probability distribution of the system under the
perturbation. In equilibrium, this change is also related
with the energy that the system absorbs from the per-
turbation. On the other hand, for non-equilibrium systems,
the lack of conserved quantities prevents such an inter-
pretation. For instance, in the case of the harmonic oscillator
driven by a Cauchy-Lévy noise, \( \langle x \rangle \) and higher moments
diverge\(^1\). Consequently, both the potential energy of the
system in the harmonic potential and the work done by the
external force \( f(t) \) also diverge. The system is plagued by
divergent quantities and exhibits anomalous work fluctu-
ations [8,9] which imply that large positive fluctuations of
work are asymptotically as likely to be observed as nega-
tive fluctuations of equal magnitude. However, the conju-
gate variable \( X_C \) given by eq. (20) has finite moments and
still captures the dynamical response of the Lévy particle.
Summarizing, although systems driven by \( \alpha \)-stable noises
might significantly differ from their Brownian (equilib-
rium) counterparts [31–33] due to their heavy tail asymp-
totics, we have shown that in such far-from-equilibrium
situations some concepts from weakly perturbed equilib-
rium systems can still be used. One of the drawbacks of the
generalized FDT derived in [14] is the difficulty to find the
conjugate variable, since it requires the knowledge of the
stationary state. We have been able to find this stationary
state for a linear system. An interesting open question is
whether this state, or some slight modification, can still be
used to calculate susceptibilities in the presence of weak
non-linearities.

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We are grateful to JORDAN HOROWITZ for discussions
and useful comments. The authors acknowledge the
support by the European Science Foundation through
EPSD/PESC program. JMRP acknowledges financial
support from grants MOSAICO (Spanish Government)
and MODELICO (Comunidad de Madrid). BD acknow-
edges the Danish National Research Foundation for
financial support through the Center for Models of Life.

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\(^1\)With a more general definition of integrals, i.e., by using a
“principal value” integral, the first moment of a Cauchy distribution
can be evaluated as \( \langle x \rangle = \lim_{\alpha \to \infty} \frac{1}{\pi} \int_{-\alpha}^{\alpha} \frac{1}{1+x^2} \, dx \)
and is equal to zero.