SU(2) WZW theory at higher genera

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Abstract

We compute, by free field techniques, the scalar product of the SU(2) Chern-Simons states on genus $>1$ surfaces. The result is a finite-dimensional integral over positions of “screening charges” and one complex modular parameter. It uses an effective description of the CS states closely related to the one worked out by Bertram [1]. The scalar product formula allows to express the higher genus partition functions of the WZW conformal field theory by finite-dimensional integrals. It should provide the hermitian metric preserved by the Knizhnik-Zamolodchikov-Bernard connection describing the variations of the CS states under the change of the complex structure of the surface.

1. Introduction

As noted in [2], there exists a close relation between the Chern-Simons (CS) topological gauge theory in 3D and the Wess-Zumino-Witten (WZW) model of conformal field theory in 2D. The (fixed time) quantum states of the CS theory on a Riemann surface $\Sigma$ without boundary are solutions of the current algebra Ward identities of the WZW theory. The states of the CS theory are holomorphic functionals $\Psi$ on the space $A^{01}$ of (smooth) 0,1-forms $A^{01}$ with values in the complexified Lie algebra $g^C$ of a compact Lie group $G$. The functionals $\Psi$ are required to be invariant under the complex (chiral) gauge transformations $\Psi \mapsto h\Psi$, where

$$h\Psi(A^{01}) = e^{-kS(h,A^{01})}\Psi((h^{-1}A^{01})$$

(1.1)

for $h: \Sigma \rightarrow G^C$. In the above formula, $S(h,A^{01})$ denotes the action of the WZW model in the external gauge field $A^{01}$. For a general gauge field, it takes the form

$$S(h,A^{10} + A^{01}) = -\frac{i}{4\pi} \int_{\Sigma} \text{tr} (h^{-1}\partial h) \land (h^{-1}\bar{\partial} h) - \frac{i}{12\pi} \int_{\Sigma} d^{-1} \text{tr} (h^{-1} dh)^{\land 3}$$

$$+ \frac{i}{2\pi} \int_{\Sigma} \text{tr} [(h\partial h^{-1}) \land A^{01} + A^{10} \land (h^{-1}\bar{\partial} h) + h A^{10} h^{-1} \land A^{01} - A^{10} \land A^{01}] .$$

(1.2)
The non-negative integer \( k \) is called the level of the model. The invariance \( \Psi = h \Psi \) is exactly the chiral gauge symmetry Ward identity for the WZW partition function. Moreover, adding static Wilson lines in the CS theory, one obtains the chiral Ward identities for the Green functions of the primary fields of the WZW theory. For the sake of simplicity, we shall concentrate here on the case of the WZW partition function and we shall take \( G = SU(2) \).

The space of CS states has finite dimension. The CS states \( \Psi \) may be viewed as sections of a complex line bundle over the orbit space \( \mathcal{A}^{01}/G^C \) of the group \( G^C \) of complex gauge transformations. The orbits \( G^C \mathcal{A}^{01} \) are in one to one correspondence with the isomorphism classes of holomorphic vector bundles (h.v.b.) \( E \) of rank 2, with trivial determinant, given by the \( SL(2, \mathbb{C}) \)-valued holomorphic 1-cocycles \( g_{\alpha \beta} \),

\[
    g_{\alpha \beta} g_{\beta \gamma} = g_{\alpha \gamma,}
\]

where \( A^{01} = g_\alpha^{-1} \partial g_\alpha \) locally and \( g_{\alpha \beta} = g_\alpha g_\beta^{-1} \). If, for the genus \( g \) of \( \Sigma > 1 \), one limits oneself to the open dense (in the \( C^\infty \) topology) subset \( \mathcal{A}^{01}_s \subset \mathcal{A}^{01} \) corresponding to the stable bundles, then the orbit space \( \mathcal{A}^{01}_s/G^C \) becomes a complex variety \( \mathcal{N}_s \) of dimension \( 3g - 3 \equiv N \). Besides, \( \mathcal{N}_s \) possesses a natural compactification \( \mathcal{N}_{ss} \), the Seshadri moduli space of semi-stable bundles [4][5]. The CS states coincide with the holomorphic sections of the \( k^{th} \) power of the natural determinant bundle \( D \) over \( \mathcal{N}_{ss} \). The spaces \( H^0(D^k) \) of such sections have dimensions given by the Verlinde formula [6]. They form a holomorphic vector bundle \( W_k \) over the moduli space \( \mathcal{M} \) of complex curves. This bundle may be equipped with a projectively flat “heat kernel” connection first described by Bernard [7], see also [2][8][9][10], which generalizes the Knizhnik-Zamolodchikov connection [11] to the higher genus situation.

The partition function of the WZW model is formally given by the functional integral

\[
    Z(A^{10} + A^{01}) = \int \exp \left( kS(g, A^{10} + A^{01}) \right) Dg
\]

where \( Dg \) stands for the formal product \( \prod_{x \in \Sigma} dg(x) \) of the Haar measures on \( G \). It has been argued in [13] that the solution of (1.4) is given by

\[
    Z(A^{10} + A^{01}) = \sum_r \Psi_r(- (A^{10})^\dagger) \Psi_r(A^{10}) e^{\frac{ik}{2\pi} \int_\Sigma \tr A^{10} \cdot A^{01}},
\]

for an arbitrary basis \( (\Psi_r) \) of the CS states orthonormal with respect to the scalar product corresponding to the norm

\[
    \lVert \Psi \rVert^2 = \int \lvert \Psi(A^{01}) \rvert^2 e^{\frac{ik}{2\pi} \int_\Sigma \tr A^{10} \cdot A^{01}} DA .
\]

The functional integral in (1.6) is over the unitary gauge fields \( A = A^{10} + A^{01} \) with \( A^{10} = -(A^{01})^\dagger \). This way the calculation of the partition functions in the WZW theory is reduced to that of the scalar product of the CS states.

Formal arguments show that the scalar product (1.6) should supply the bundle \( W_k \) of CS states with a hermitian structure preserved by the Knizhnik-Zamolodchikov-Bernard
(KZB) connection, see [1]. Proving rigorously the metricity of the KZB connection is an important mathematical problem in conformal field theory still left open. The purpose of this work is to provide a candidate for its (constructive) solution by computing exactly the functional integral Eq. (1.6). The result will have a form of an explicit finite-dimensional integral of a positive measure. Similar work has been done in the case of genus zero with insertions in [12][13] for $G = SU(2)$ and in [14] for a general simple $G$. The integral formulae for the scalar product are the dual versions of the expressions for conformal blocks of the WZW theory in terms of the generalized hypergeometric integrals [15][16][17], at the core of the relations between the WZW models and the quantum groups and of the recently discovered relation between the WZW model and the Bethe ansatz [18][19][20][21]. The extension to higher genera has required a nontrivial generalization of the low genus arguments and has taken some time.

The paper is organized as follows. In Sect. 2 we describe a slice in the space of gauge fields, transversal to generic $G^C$-orbits. It corresponds to realizing generic rank 2 determinant 0 holomorphic vector bundles as extensions of a one-parameter family of degree $g - 1$ line bundles. In Sect. 3, we examine the restrictions of CS states to the slice. They become sections of the $k^{th}$ power of the determinant bundle of the family of extensions. This picture of the higher genus CS states is closely related to the one worked out in [1], see also [22], based on considering the extensions of fixed degree $g$ line bundle. The relation between the two presentations is the subject of Sect. 4. Sect. 5 describes a projective version of the scalar product formula, from which the surface dependent constants were omitted. It has been extracted from the full-fledged formula discussed later for the sake of a moderately interested reader who would not like to dwell into the details of the functional integration which occupies most of the rest of the paper. And so, in Sect. 6, using the slice of the space $A^{01}$, we decompose the functional integral (1.6) to the one over $G^C$ and over the orbit space. The Jacobian of the relevant change of variables is computed in Sect. 6 by free field functional integration. The crucial Sect. 7 performs the integration over $G^C$ by reducing it to an iterative Gaussian integral. Finally, Sect. 8 assembles the complete formula for the scalar product. In Appendices, besides treating a number of technical points, we work out the details of the relation of our description of CS states to that of [1] (Appendix C) and submit the scalar product formula to simple consistency checks (Appendix F). What remains to be proven, however, is that the finite-dimensional integrals appearing in the formula actually always converge resulting in a hermitian metric on the bundle $W_k$ of state spaces which is preserved by the KZB connection. What is also missing is an interpretation of the formula in terms of modular geometry, providing a counterpart of the analysis of the KZB connection carried out in [1]. As the first step in this direction one could try to simplify the formal arguments given below. Numerous cancellations occurring in intermediate steps of the calculation suggest that such simplification should be possible.

This is a non-rigorous work in its manipulation of formal functional integrals which lead, in the end, to a chain of Gaussian integrations. Handling these integrals required, nevertheless, careful treatment. As a result, the paper employs relatively sophisticated mathematical tools. It may be viewed as a piece of “theoretical mathematics” in the sense

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1We thank B. Feigin and S. Ramanan for attracting our attention to [1] and [22].
of [23]: it uses formal functional integral to extract an interesting mathematical structure which should be submitted now to rigorous analysis. Care was taken to clearly mark the non rigorous steps in the discussion. The main result of the calculation performed here was announced in the note [24]. The case with insertion points will be treated in a separate publication.

2. Space of orbits

We shall need below an effective description of generic orbits $G^C A^1$. It will be based on the fact that every h.v.b. $E$ has a line subbundle $L^{-1} \subset E$ or, equivalently, that the cocycle $(g_{\alpha \beta})$ of $E$ may be always chosen in the triangular form

$$g_{\alpha \beta} = \left( \begin{array}{cc} a_{\alpha \beta} & b_{\alpha \beta} \\ 0 & a_{\alpha \beta} \end{array} \right),$$

(2.1)

where $(a_{\alpha \beta})$ is a 1-cocycle of a holomorphic line bundle (h.l.b.) $L^s$. t. the dual bundle $L^{-1} \subset E$. $(b_{\alpha \beta})$ satisfy the twisted cocycle condition

$$a_{\alpha \beta}^{-1} b_{\beta \gamma} + b_{\alpha \beta} a_{\beta \gamma} = b_{\alpha \gamma}$$

(2.2)

which means that they define a holomorphic 1-cocycle with values in the h.l.b. $L^{-2}$ (this may be better seen by rewriting Eq. (2.2) as $a_{\alpha \beta}^{-2} b'_{\beta \gamma} + b'_{\alpha \beta} = b'_{\alpha \gamma}$, where $b'_{\alpha \beta} \equiv a_{\alpha \beta}^{-1} b_{\alpha \beta}$). The corresponding cohomology class $[(b_{\alpha \beta})]$ in $H^1(L^{-2})$ describes $E$ as an extension

$$0 \rightarrow L^{-1} \rightarrow E \rightarrow L \rightarrow 0$$

(2.3)

of the line bundle $L$ by $L^{-1}$. Proportional $[(b_{\alpha \beta})]$ give rise to isomorphic bundles (but the converse may be not true). By the Riemann-Roch Theorem,

$$\dim(H^1(L^{-2})) = g - 1 + 2\deg(L) \quad \text{for} \quad \deg(L) > 0.$$  

(2.4)

Let $L(\pm x)$ denote the h.l.b. $LO(\pm x)$ (omitting the sign of the tensor product between the bundles), where $O(\pm x)$ is the degree $\pm 1$ h.l.b. with divisor $\pm x$. We shall fix for the rest of the paper a h.l.b. $L$ of degree $g$. For later convenience, we shall assume that this is done so that $L(-x)^2$ never coincides with the canonical line bundle $K$ of $\Sigma$. For any rank 2 h.v.b. bundle $E$ with trivial determinant, there exists $x \in \Sigma$ and a non-trivial homomorphism

$$\phi : L(-x)^{-1} \rightarrow E,$$

(2.5)

see [3], Lemma 5.4. If $\phi$ has zeros (counted with multiplicities) at $x_1, \ldots, x_r$ then $\phi$ induces an embedding of $L(-x - x_1 \ldots - x_r)^{-1}$ into $E$ or, in other words, $E$ is an extension of $L(-x - x_1 \ldots - x_r)$. Notice that $\deg(L(-x - x_1 \ldots - x_r)) = g - 1 - r$. If $E$ is stable then in can have only negative degree subbundles so that, necessarily, $r < g - 1$ and, moreover, the extension has to be nontrivial.

\[2\]as opposed to the choice of $L$ employed in [24]
The above discussion gives rise to the following description of the groups \( G^C A^{01} \), related to the picture of the moduli space \( N \) discussed in the papers [1][22]. For \( 0 \leq r < g-1 \), consider a holomorphic family \((L(-x-x_1 \ldots -x_r))\) of line bundles. By definition, it is a holomorphic line bundle \( \mathcal{L}_r \) over \( \Sigma^{r+1} \times \Sigma \) whose restriction to the fiber \( \text{pr}_1^{-1}(\{x, x_1, \ldots, x_r\}) \) of the projection on the first factor gives \( L(-x-x_1 \ldots -x_r) \equiv L(-X_r) \). \( \mathcal{L}_r \) is not unique: for each h.l.b. \( M \) over \( \Sigma^{r+1} \), we may take \( \text{pr}_1(M) \mathcal{L}_r \) instead of \( \mathcal{L}_r \). Let \( W_r \) denote the first direct image \( R^1\text{pr}_1_*(\mathcal{L}_r^{-2}) \) of \( \mathcal{L}_r^{-2} \) by \( \text{pr}_1 \) (\( W_r \) is a h.v.b. of dimension \( N - 2r \) \( N \equiv 3g - 3 \) over \( \Sigma^{r+1} \) with fibers \( H^1(L(-X_r)) \)). Let \( \mathbb{P}W_r \) denote the corresponding holomorphic bundle of projective spaces \( \mathbb{P}H^1(L(-X_r)^{-2}) \). The total dimension of the compact complex manifold \( \mathbb{P}W_r \) is \( N - r \). Now, each element \( w \in \mathbb{P}W_r \), with the base point \( \{x, x_1, \ldots, x_r\} \equiv X_r \), defines (uniquely up to isomorphisms) a holomorphic bundle \( E \) of rank 2 and trivial determinant which is an extension of \( L(-X_r) \). (Many \( w \)'s may define the same \( E \).) The dimensions imply that, generically, \( r = 0 \) (recall that \( \text{dim}(N) = N \)). From this and the analysis of [1] and [22], the following picture of the orbit space emerges:

an open dense subset of \( \mathbb{P}W_0 \) corresponding to stable bundles is a ramified (2-fold) cover of a dense subset of the stable moduli space \( \mathcal{N}_s \). The rest of \( \mathcal{N}_s \) is in the image of subsets of \( \mathbb{P}W_r \). In particular, the union of the \( G^C \)-orbits corresponding to h.v.b.'s \( E \) obtained from \( W_0 \) is dense in \( A^{01} \).

Other details of that geometry may be found in [1][22].

We shall construct gauge fields corresponding to points in \( \mathbb{P}W_0 \) (a slice \( s : \mathbb{P}W_0 \rightarrow A^{01} \)). Let us start by an explicit construction of a family \((L(-x))\) of line bundles on \( \Sigma \) and of the corresponding bundle \( W_0 \). A more natural but less explicit construction will be discussed in the next section and, a somewhat pedantic, distinction between different realizations of the family \((L(-x))\) will later play an important role. In order to describe the first construction, fix a point \( x_0 \in \Sigma \) and denote \( L_0 \equiv L(-x_0) \). The family \((L(-x))\) will be obtained by twisting the \( \bar{\partial} \)-operator in \( L_0 \). Let \((\omega^i)_i \equiv \text{basis of holomorphic 0,1-forms on } \Sigma \) adapted to a marking of \( \Sigma \) i.e. to a choice of a standard homology basis \((a_i, b_j)\). \( \int_{a_i} \omega^j = \delta^{ij} \) and \( \int_{b_j} \omega^j = \tau^{ij} \) were \( \tau \) is the period matrix. Define a 0,1-form

\[
a = \pi \sum_{i,j=1}^{g} x_0^x (\int_{x_0} \omega^i) \left( \frac{1}{\text{Im} \tau} \right)_{ij} \bar{\omega}^j \equiv \pi (\int_{x_0} \omega) (\text{Im} \tau)^{-1} \bar{\omega} . \tag{2.6}
\]

Notice, that \( a \equiv a_x \) depends on the lift \( x \) of the point \( x \) to the covering space \( \tilde{\Sigma} \) of \( \Sigma \) (with the base point \( x_0 \)). Denote by \( L_x \) the line bundle \( L_0 \) with \( \bar{\partial} \) replaced by \( \partial_{L_x} \equiv \bar{\partial} + a_x \). It is a standard fact that all \( L_x \) corresponding to the same \( x \) are isomorphic to \( L(-x) \). Consider the holomorphic bundle \( \tilde{\Sigma} \times L_0 \) over \( \tilde{\Sigma} \times \Sigma \) with the antiholomorphic derivative \( \delta + \bar{\partial} \) where \( \delta \) differentiates in the trivial direction of \( \tilde{\Sigma} \). We shall twist \( \tilde{\Sigma} \times L_0 \) by replacing its antiholomorphic derivative by \( \delta + \bar{\partial} + a \). Denote the resulting h.l.b. over \( \tilde{\Sigma} \times \Sigma \) by \( L_0 \). It gives a specific realization of a holomorphic family \((L_x)\). The action of the fundamental group \( \Pi_1(\Sigma, x_0) \equiv \Pi_1 \) on \( \tilde{\Sigma} \) lifts to an action on \( L_0 \) preserving the

\footnote{We denote by \( \Sigma^n \) the symmetrized \( n \)-fold Cartesian product of \( \Sigma \).}
structure of the h.l.b. The lifted action of \( p \in \Pi_1 \) is
\[
(x, l_y) \mapsto (px, c_p(y)^{-1} l_y)
\] (2.7)
for \( l_y \) in the fiber of \( L_x \) over \( y \in \Sigma \), where
\[
c_p(y) = e^{2\pi i \text{Im}[(\int_y \omega)(\text{Im} \tau)^{-1}(\int_{x_0} \tilde{\omega})]}. \tag{2.8}
\]
Note that \( c_p \) is a function on \( \Sigma \) (it does not depend on the integration path from \( x_0 \) to \( y \)). Dividing \( \tilde{L}_0 \) by the action of \( \Pi_1 \), we obtain a h.l.b. \( L_0 \) over \( \Sigma \times \Sigma \), the first explicit realization of the holomorphic family \( (L(-x)) \).

For a line bundle \( M \), we shall denote by \( \Gamma(M) \) the space of smooth sections of \( L \) and by \( \wedge^01(M) \) the space of smooth 0,1-forms with values in \( M \). The bundle \( \tilde{W}_0 = R^1 pr_1^* (\tilde{L}_0^{-2}) \) (the first direct image of \( \tilde{L}_0^{-2} \) under \( pr_1 \)) may be viewed as the quotient of the infinite-dimensional trivial bundle \( \Sigma \times \wedge^01(L_0^{-2}) \) by the subbundle whose fiber over \( x \) is the image by \( \partial_{L_x^{-2}} \equiv \tilde{\partial} - 2a_x \) of \( \Gamma(L_0^{-2}) \). Indeed,
\[
\wedge^01(L_0^{-2})/\partial_{L_x^{-2}}(\Gamma(L_0^{-2})) = \wedge^01(L_0^{-2})/\tilde{\partial}_{L_x^{-2}}(\Gamma(L_0^{-2})) \cong H^1(L_0^{-2}), \tag{2.9}
\]
which is the Dolbeault realization of \( H^1(L_0^{-2}) \). Division by the action of \( \Pi_1 \) gives an explicit construction of the fiber bundle \( \tilde{W}_0 \).

Let us construct now a gauge field \( A^{01} \) whose \( G^C \)-orbit corresponds to a given point \( w \in W_0 \). To this end, we shall choose a smooth isomorphism \( U \) of rank 2 vector bundles over \( \Sigma \) with trivial determinants,
\[
U : L_0^{-1} \oplus L_0 \longrightarrow \Sigma \times \mathbb{C}^2. \tag{2.10}
\]
Let us twist the holomorphic structure of the vector bundle \( E_0 \equiv L_0^{-1} \oplus L_0 \) by replacing its \( \tilde{\partial} \)-operator by
\[
\tilde{\partial} + \begin{pmatrix} -a_x & b \\ 0 & a_x \end{pmatrix} \equiv \tilde{\partial} + B^{01}_{x,b}, \tag{2.11}
\]
where \( b \in \wedge^01(L_0^{-2}) \). We shall denote the twisted bundle by \( E \). Note that \( E \) is an extension of the line bundle \( L_x \) by \( L_x^{-1} \). We may use the smooth isomorphism \( U \) of (2.10) to transport the holomorphic structure from \( E \) to the trivial bundle where we get the \( \tilde{\partial} \)-operator
\[
\tilde{\partial} + U B^{01}_{x,b} U^{-1} + U \tilde{\partial} U^{-1} \equiv \tilde{\partial} + A^{01}_{x,b}. \tag{2.12}
\]
Let \( c \) be a constant \( \neq 0 \) or \( c = c_p \), see Eq. (2.8), and let \( v \in \Gamma(L_0^{-2}) \).
\[
g_{c,v} \equiv \begin{pmatrix} c^{-1} & cv \\ 0 & c \end{pmatrix} \tag{2.13}
\]
is a smooth section of the bundle \( \text{Aut}(L_0^{-1} \oplus L_0) \) of automorphisms of \( L_0^{-1} \oplus L_0 \). The gauge transformation \( B^{01}_{x,b} \mapsto g_{c,v} B^{01}_{x,b} g_{c,v}^{-1} B^{01}_{x,b} g_{c,v} + g_{c,v}^{-1} \tilde{\partial} g_{c,v} \) preserves the form of the gauge field \( B^{01}_{x,b} \) shifting
\[
a_x \mapsto a_x + c^{-1} \tilde{\partial} c \quad \text{and} \quad b \mapsto c^2(b + (\tilde{\partial} - 2a_x)v). \tag{2.14}
\]
In particular, for \( c = c_p \), \( a_x \mapsto a_x + c_p^{-1} \partial c_p = a_{px} \). The corresponding fields \( A^0_{x,b} \) are gauge-related by

\[
h_{c,v} \equiv U g_{c,v} U^{-1} \in \mathcal{G}^C, \tag{2.15}\]

so that they lie in the same \( \mathcal{G}^C \)-orbit. Taking constant \( c \neq 0 \), we see that \( b \) leading to the same class in the projective space

\[
\mathbb{P}^1(\wedge^0(L_0^{-2})/\partial L_x^2(\Gamma(L_0^{-2}))) \equiv \mathbb{P}H^1(L_x^{-2}) \tag{2.16}\]

give gauge fields \( A^0_{x,b} \) in the same \( \mathcal{G}^C \)-orbit. The class of \( b \) in \( \mathbb{P}H^1(L_x^{-2}) \) is exactly the one describing the rank 2 h.v.b. \( E \), an extension of \( L_x \), associated to the orbit \( \mathcal{G}^C A^0_{x,b} \).

Choosing \( x \) in a fundamental domain in \( \tilde{\Sigma} \) and one \( b \) in each class of \( \mathbb{P}H^1(L_x^{-2}) \), we obtain a slice \( s : \mathbb{P}W_0 \to A^0 \) which cuts a generic orbit a finite number (\( = 2g \)) of times. One may take \( s \) to be piecewise holomorphic.

### 3. Determinant bundle

Let us fix a hermitian structure on the h.l.b. \( L_0 \). It induces a metric connection on \( L_0 \) whose covariant derivative in the antiholomorphic direction coincides with the \( \bar{\partial} \) operator. Let \( F_0 \) denote the curvature form of this connection (normalized so that \( \int_{\Sigma} \frac{i}{2\pi} F_0 = \deg(L_0) = g - 1 \)). The hermitian metric on \( L_0 \) induces a hermitian structure and a connection on \( L_0^{-1} \) and, putting both together, a hermitian structure and a connection on \( L_0^{-1} \oplus L_0 \). Let us denote by \( \nabla \) the holomorphic covariant derivative in \( L_0^{-1} \oplus L_0 \). The complete covariant derivative is \( \nabla + \bar{\partial} \). Clearly, its curvature form

\[
\text{curv}(\nabla + \bar{\partial}) = \begin{pmatrix}
-F_0 & 0 \\
0 & F_0
\end{pmatrix}. \tag{3.1}
\]

Let us assume that the smooth isomorphism \( U : L_0^{-1} \oplus L_0 \to \Sigma \times \mathbb{C}^2 \) maps the hermitian metric of \( L_0^{-1} \oplus L_0 \) into the one coming from the standard scalar product of \( \mathbb{C}^2 \). Using \( U \), we may transport the connection on \( L_0^{-1} \oplus L_0 \) to the trivial bundle \( \Sigma \times \mathbb{C}^2 \):

\[
U(\nabla + \bar{\partial})U^{-1} = d + U\nabla U^{-1} + U\bar{\partial}U^{-1} \equiv d + A_0^{10} + A_0^{01} \equiv d + A_0. \tag{3.2}
\]

The right hand side gives a unitary connection, so that \( A_0^{10} = -(A_0^{01})^\dagger \). The curvature forms are related by

\[
\text{curv}(U(\nabla + \bar{\partial})U^{-1}) = F(A_0) \equiv dA_0 + A_0 \wedge A_0 = U \begin{pmatrix}
-F_0 & 0 \\
0 & F_0
\end{pmatrix} U^{-1}. \tag{3.3}
\]

We shall represent the CS states by holomorphic sections of a line bundle over \( \mathbb{P}W_0 \). To this end, let us define, for a CS state \( \Psi \), a holomorphic function

\[
\psi(x,b) \equiv \exp\left[\frac{ik}{2\pi} \int_{\Sigma} \text{tr} A_0^{10} \wedge A_0^{01} \right] \Psi(A_0^{01}) \tag{3.4}
\]
of $x \in \bar{\Sigma}$ and $b \in \wedge^0 (L_0^{-2})$. As is shown in Appendix F, only the normalization of $\psi$ depends on the choice of the hermitian structure on $L_0$ and of the isomorphism $U$. Since the $G^C$-orbits of the chiral gauge fields $A_{x,b}^{01}$ form a dense subset of $A^{01}$, $\Psi$ is uniquely determined by the function $\psi$. The gauge relations between the forms $A_{x,b}^{01}$ induce constraints for functions $\psi$, due to the gauge invariance of CS states $\Psi$. In particular,

$$\psi(x, b + (\bar{\partial} - 2a_x) v) = \exp[kS(h_v, A_0^{10} + A_{x,b}^{01})] \psi(x, b), \quad (3.5)$$

$$\psi(x, c^2 b) = \exp[kS(h_c, A_0^{10} + A_{x,b}^{01})] \psi(x, b), \quad (3.6)$$

$$\psi(px, c^2 b) = \exp[kS(h_{cp}, A_0^{10} + A_{x,b}^{01})] \psi(x, b), \quad (3.7)$$

where $h_v$ is given by Eq. (2.13) with $c = 1$, $h_c$ by the same formula with $v = 0$ and $c \in \mathbb{C} \equiv \mathbb{C} \setminus \{0\}$ and $h_{cp}$ by setting $v = 0$ and $c = c_p$, see (2.8).

Let us study these transformation properties in greater detail. The first two equations become much more transparent if we rewrite them in the infinitesimal form. Since \( \frac{\delta}{\delta h_i} |_{h=1} S(h, A) = \frac{i}{\pi} \sigma^+ F(A) \), we obtain

$$\frac{\delta}{\delta c} |_{c=1} S(h, A_0^{10} + A_{x,b}^{01}) = \frac{i}{\pi} \sigma^+ F(A_0^{10} + A_{x,b}^{01}) U$$

$$= \frac{i}{\pi} \sigma^+ \text{curv}(\nabla + \bar{\partial})(\nabla (B_{x,b}^{01})) = \frac{i}{\pi} \text{tr} \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -F_0 & 0 \\ 0 & F_0 \end{pmatrix} + \frac{i}{\sigma_x} \text{tr}(\nabla(b)) = 0, \quad (3.8)$$

where we have used the identity: \( \text{curv}(\nabla + \bar{\partial}) + \nabla(B_{x,b}^{01}) \). Above, $\nabla(b)$ stands for the holomorphic covariant derivative of the $L_0^{-2}$-valued 0,1-form $b$. Similarly,

$$\frac{\delta}{\delta c} |_{c=1} S(h, A_0^{10} + A_{x,b}^{01}) = \frac{1}{2\pi} \int_{\Sigma} \text{tr} \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} -F_0 & 0 \\ 0 & F_0 \end{pmatrix} + \frac{i}{\sigma_x} \text{tr}(\nabla(b))$$

$$= \frac{i}{\pi} \int_{\Sigma} F_0 = 2(g - 1). \quad (3.9)$$

As for the relation (3.7), first notice that we may reduce the calculation to the case $b = 0$ since for $h = h_{cp}$,

$$S(h, A_0^{10} + A_{x,b}^{01}) = S(h, A_0^{10} + A_{x,0}^{01}) + \frac{i}{2\pi} \int_{\Sigma} \text{tr} [(\bar{\partial}h^{-1} + hA_0^{10} h^{-1} - A_0^{10}) \wedge U \sigma^+ b U^{-1}]$$

$$= S(h, A_0^{10} + A_{x,0}^{01}) + \frac{i}{2\pi} \int_{\Sigma} \text{tr} [\sigma^+ c_p^{-1} \partial c_p \wedge b] = S(h, A_0^{10} + A_{x,0}^{01}).$$

We shall show in Appendix A that for $c$ a non-vanishing function on $\Sigma$, for $c = \begin{pmatrix} c^{-1} & 0 \\ 0 & c \end{pmatrix}$ and for $h_c = U g_c U^{-1}$,

$$\exp[S(h_c, A_0^{10} + A_{x,0}^{01})] = e^{\frac{i}{2\pi} \int_{\Sigma} c^{-1} \partial c \wedge (c^{-1} \partial c + 2a_x)} \nu(c). \quad (3.10)$$

Note that, except for the last factor, the expression on the right hand side is $e^{S(g_c, -\sigma^3 a_x)}$ where $g_c$ is viewed as a standard $SL(2, \mathbb{C})$-valued field. The correction term is

$$\nu(c) = e^{\frac{i}{2} \int_{\Sigma} F_0 \text{ln} c} \prod_{j=1}^g \left( W_{a_j}^{-1} f_{a_j} c^{-1} b_{c} W_{b_j}^{-1} f_{b_j} c^{-1} b_{c} \right), \quad (3.11)$$
where $W_{a_j}$ ($W_{b_j}$) stand for the holonomy of the metric connection on $L_0$ along the $a_j$ ($b_j$) cycle. Inc$(x) = \int_{x_0}^x e^{-1} dc$, where the integration path is taken inside a fundamental domain of $\Sigma$ obtained by cutting the surface along the cycles $a_j$, $b_j$ starting at $x_0$. Altogether,

$$\exp[S(h_{c_p}, A_0^{10} + A_{x, b}^{01})] = e^{\pi(\int_P \omega) \Im \tau^{-1} (\int_P \omega) + 2\pi(\int_P \omega) \Im \tau^{-1} (\int_0^x \omega) \nu(c_p)}$$

$$\equiv \mu(p, x) \nu(c_p).$$

(3.12)

Gathering Eqs. (3.3) and (3.8), (3.9) and (3.7), (3.7) and (3.12), we obtain

**PROPOSITION. Holomorphic functions $\psi$ possess the following transformation properties:**

$$\psi(x, \lambda b + (\partial - 2a_x)v) = \lambda^{k(g-1)} \psi(x, b),$$

$$\psi(px, c_p^2 b) = \mu(p, x) \nu(c_p)^k \psi(x, b).$$

(3.13)

(3.14)

In particular, for fixed $x$, $\psi(x, \cdot)$ is a homogeneous polynomial of degree $k(g-1)$ on $H^1(L_x^{-2})$. Note that the factor $\mu(p, x)$ appears in the transformation property for the square of Riemann’s theta function:

$$e^{\pi k(\int_{x_0}^{x} \omega) \Im \tau^{-1} (\int_{x_0}^{x} \omega) \partial(\int_0^x \omega | \tau)^2 = \mu(p, x) e^{\pi k(\int_{x_0}^{x} \omega) \Im \tau^{-1} (\int_{x_0}^{x} \omega) \partial(\int_0^x \omega | \tau)^2.}$$

(3.15)

The map $x \mapsto \partial(\int_{x_0}^{x} \omega | \tau)^2$ defines a holomorphic section of the bundle $K(2x_0)$. The map $\Pi_1 \ni p \mapsto \nu(c_p)$ is a character of the fundamental group $\Pi_1$. We show in Appendix A that it defines the flat bundle $L(-gx_0)^2$. Hence a holomorphic function $\phi(x)$ s.t.

$$\phi(px) = \mu(p, x) \nu(c_p) \phi(x)$$

(3.16)

defines (upon multiplication by $e^{-\pi k(\int_{x_0}^{x} \omega) \Im \tau^{-1} (\int_{x_0}^{x} \omega)}$) a section of the line bundle $L^2K((2-2g)x_0)$. The transformation properties (3.13) and (3.14) may be rephrased by saying that $\psi$ is a holomorphic section of the $k^{th}$ power of a h.l.b., which we shall denote DET, over the total space of the fiber bundle $\mathbb{P}W_0$. Explicitly,

$$\text{DET} = \varpi^*(L^2K((2-2g)x_0)) \text{Hf}(W_0)^{(1-g)}$$

(3.17)

where $\varpi$ is the bundle projection of $\mathbb{P}W_0$ and $\text{Hf}(W_0)$ is the tautological bundle over $\mathbb{P}W_0$. In particular, the restriction of the h.l.b. DET to the fiber $\mathbb{P}H^1(L(-x)^{-2})$ of $\mathbb{P}W_0$ over $x \in \Sigma$ is the $(1-g)^{th}$ power of the tautological bundle over $\mathbb{P}H^1(L(-x)^{-2})$. As we shall see in the next section, the h.l.b. $L^2K((2-2g)x_0)$ is isomorphic to (the $\Pi_1$ quotient of) the determinant bundle of the family $(\partial - \sigma^3 a_x)$ of $\partial$-operators in $L_0^{-1} \oplus L_0$. In turn, the line bundle DET is isomorphic to (the $\Pi_1$ quotient of) the determinant bundle for the family $(\partial + B_{x, b}^{01})$ of $\partial$-operators in $L_0^{-1} \oplus L_0$. Note the simple way by which the addition of the upper-diagonal gauge field $b$ in the $\partial$-operator manifests itself in the
determinant bundle. The map \( \Psi \rightarrow \psi \) embeds the space of CS states onto a subspace \( \mathcal{H} \subset H^0(\text{DET}) \). The space \( H^0(\text{DET}) \) of the holomorphic sections of a line bundle over the compact space \( W_0 \) is finite dimensional so that the finite-dimensionality of the space of CS states follows. Given \( \psi \in H^0(\text{DET}) \), the question whether it comes from a CS state should be determined by its behavior at the codimension one subvariety defined by \( \mathbb{P} \mathbb{P} W_1 \) in the moduli space \( \mathcal{N}_s \) of stable bundles. We shall return below to this question which requires some refinement of the analysis of [1][22].

For the later use, it will be convenient to write the homogeneous polynomial \( \psi(x, \cdot) \) on \( H^1(L^{-2}) \) in an integral form following from the Serre duality:

\[
\psi(x, b) = \int_{\Sigma^{k(g-1)}} \chi(x; x_1, \ldots, x_{k(g-1)}) b(x_1) \cdots b(x_{k(g-1)})
\]  

for \( b \in \wedge^1(L_0) \) and \( \chi(x; \cdot) \in H^0(S^{k(g-1)}(L_2^2K)) \), where, for a h.l.b. \( M \) over \( \Sigma \), \( S^nM \) stands for the \( n \)-fold (symmetrized) tensor product with the bases space \( \Sigma^n \), the \( n \)-fold symmetric Cartesian product of \( \Sigma \). The \( x \)-dependence of \( \chi \) again gives rise to a section of \( L_{2}^{K}(2(g-2)x_0) \) so that we may view \( \chi(\cdot; \cdot) \) as a holomorphic, \( pr_2 \)-horizontal \( k(g-1), 0 \)-form on \( \Sigma \times \Sigma^{k(g-1)} \) with values in the h.l.b.

\[
pr_1^*(L^2K((2-2g)x_0))^k S^{k(g-1)}(L_0^2) \equiv B_k. \]  

4. Relation to Bertram’s picture

Paper [1], see also [22], describes a somewhat different, simpler, construction of the space of CS states. It is based on the realization of a generic rank 2 determinant 0 bundle as an extension of a fixed degree \( g \) line bundle \( L \). Taking

\[
B^{01}_{b'} \equiv \begin{pmatrix} 0 & \psi \\ 0 & 0 \end{pmatrix},
\]  

where \( b' \in \wedge^1(L^{-2}) \), we shall put

\[
A^{01}_{b'} = U'^* A^{01}_{b'} U'^{-1} + U'^* \bar{\partial} U'^{-1}
\]

where \( U : L^{-1} \oplus L \rightarrow \Sigma \times \mathbb{C}^2 \) is a fixed smooth isometric isomorphism. Defining

\[
\psi'(b') = \exp\left[ \frac{i k}{2\pi} \int_\Sigma \text{tr} A'^0_0 \wedge A'^0_{b'} \right] \Psi(A'^0_{b'}) ,
\]

we infer in the same way as above that

\[
\psi'(\lambda(b' + \bar{\partial}v')) = \lambda^{kg} \psi'(b') ,
\]

compare (3.13). Hence each \( \psi' \) is a degree \( kg \) homogeneous polynomial on \( H^1(L^{-2}) \). The latter space has dimension \( 3g-1 \) and the number of independent homogeneous polynomials of degree \( kg \) on it is \( \binom{kg+3g-2}{3g-2} \). By the Serre duality, we may write

\[
\psi'(b') = \int_{\Sigma^{kg}} \chi'(x_1, \ldots, x_{kg}) b'(x_1) \cdots b'(x_{kg})
\]
where \( \chi' \in H^0(S^{kg}(L^2K)) \).

**THEOREM 2a of [1]** The CS states correspond exactly to the polynomials \( \psi' \) s. t. \( \chi' \in H^0(S^{kg}(L^2K)) \) vanish whenever \( k+1 \) of their arguments \( x_n \) coincide.

The dimension of the space of such polynomials was shown in [22] to be

\[
\frac{(k+2)^{g-1}}{2} \sum_{j=0, \frac{1}{2}, \ldots, \frac{k}{2}} (\sin \frac{\pi(2j+1)}{k+2})^{2-2g},
\]

in agreement with the Verlinde formula [3] for the dimensions of the spaces of CS states. Let us set

\[
\chi(x; x_1, \ldots, x_{k(g-1)}) \equiv \chi'_{\text{times}}(x_1, \ldots, x_{k(g-1)}) .
\]

Note that

\[
\chi(x; x_1, \ldots, x_{k(g-1)}) = 0 \quad \text{and} \quad \frac{\partial}{\partial x_1}\big|_{x_1=x} \chi(x; x_1, \ldots, x_{k(g-1)}) = 0
\]

where the second equality is obtained by differentiating the first one over \( x \). In particular, for fixed \( x \), \( \chi(x; \cdot) \in (L^2K)^k|_x \otimes H^0(S^{kg-1}(L^2(-2x))) \). We may also interpret \( \chi(x; \cdot) \) as a holomorphic \( k(g-1), 0 \)-form on \( \Sigma^{g-1} \) with values in the line bundle \( (L^2K)^k|_x \otimes S^{kg-1}(L^2(-2x)) \).

We shall show that giving \( \chi \) is equivalent to giving \( \psi \) in the description of the CS states of the previous section. For this purpose, let us consider a line bundle \( O(-\Delta) \) over \( \Sigma \times \Sigma \) where \( \Delta \) denotes the diagonal. \( \mathcal{L}_0' \equiv pr_2^*(L)O(-\Delta) \equiv pr_1^*(L)(-\Delta) \) is another realization of a family \( (L(-x)) \) of the h.l.b.'s, different one from \( \mathcal{L}_0 \) described before. It is not difficult to see that

\[
\mathcal{L}_0' \equiv pr_1^*(O(-x_0)) \mathcal{L}_0 .
\]

An explicit isomorphism is given in Appendix B. If \( W_0' \equiv R^3pr_1^*(\mathcal{L}_0'^{-2}) \) then

\[
W_0' \cong O(2x_0)W_0, \quad \mathcal{P}W_0' \cong \mathcal{P}W_0 \quad \text{and} \quad Hf(W_0') \cong \psi^*(O(2x_0)) Hf(W_0) .
\]

Now \( \chi \), with its \( x \)-dependence taken into account, may be viewed as a holomorphic \( (pr_2\text{-horizontal}) k(g-1), 0 \)-form on \( \Sigma \times \Sigma^{g-1} \) with the values in the h.l.b.

\[
B_k' \equiv pr_1^*(L^2K)^k S^{kg-1}(\mathcal{L}_0'^2) .
\]

Since the h.l.b.'s \( B_k' \) of Eq. (4.11) and \( B_k \) of (3.19) are isomorphic due to (4.9), \( \chi \) introduced by formula (4.7) is the same type of object as \( \chi \) considered in Sect. 3, see Eq. (B.18). Indeed, in Appendix C we show that the two \( \chi \)'s coincide completely under an explicit isomorphism. This will establish the precise relation between the two descriptions

\[\text{We have learned this reformulation of the result of [1] from B. Feigin who discussed in [23] its generalization to the case with insertions and with arbitrary simple groups.}\]
of the CS states: by functions $\psi$ which we shall employ in this work and by polynomials $\psi'$. Let us briefly sketch here the geometric picture due to \cite{[1][22]} which is at the core of the detailed analysis of Appendix C. One may embed the curve $\Sigma$ into $\mathbb{P}H^1(L^{-2})$ by

$$\Sigma \ni x \mapsto [b_x] \in \mathbb{P}H^1(L^{-2}) ,$$

(4.12)

where, for $\eta' \in H^0(L^2K)$, $\int_\Sigma \eta'b'_x = \eta'(x)$ in some trivialization of $L^2K$ around $x$. We shall see in Appendix C that $[b_x]$ corresponds to an extension of $L$ which, as a rank two bundle, is isomorphic to the split bundle $L(-x)^{-1} \oplus L(-x)$. The condition of Theorem 2a of \cite{[1]} means\cite{[1]} that $\psi'$ vanishes to order $k(g - 1) - 1$ on (the image of) $\Sigma$. In this description, the CS states are thus realized as homogeneous polynomials $\psi'$ on $(3g - 1)$-dimensional vector space which vanish to some order on the explicit embedding of $\Sigma$ into the space. $\psi$’s are given by the $k(g - 1)^{th}$ Taylor coefficients (the first non-trivial order) of $\psi'$ on $\Sigma$.

Similarly, one may embed $\Sigma$ into $\mathbb{P}H^1(L(-x)^{-2})$ by mapping $y$ to $[b_y]$ s.t. $\int_\Sigma \eta b_y = \eta(y)$ for each $\eta \in H^0(L(-x)^2K)$. $[b_y]$ defines an extension of $L(-x)$ which, as a h.v.b., is isomorphic to $L(-x - y)^{-1} \oplus L(-x - y)$. Changing also $x$, we get an embedding of $\Sigma^2$ into the bundle $\mathbb{P}W_0$ considered in Sect. 2. Replacement of the image of $\Sigma^2$ in $\mathbb{P}W_0$ by $\mathbb{P}W_1$ is the second blow-up step of \cite{[1][22]}. Their analysis shows that, again, $\chi(x; y_1, \ldots, y_{k(g - 1)})$ vanishes whenever $k + 1$ of $y_n$’s coincide. We may then equate $k$ of $y_n$’s and continue the process including higher and higher $\mathbb{P}W_i$’s into the game. It is interesting to know whether the vanishing of $\chi(x; x_1, \ldots, x_{k(g - 1)})$ at $k + 1$ coincident points characterizes completely the sections $\psi \in H^0(\text{DET})$ coming from the CS states.

Let us remark, in the end, that the h.l.b.’s isomorphisms $\det Rpr_{1*}(L_0'') \cong L^{-1}$ and $\det Rpr_{1*}(L_0''^{-1}) \cong (LK)^{-1}$ following from the exact sequence

$$0 \longrightarrow pr_2^*(L)(-\Delta) \longrightarrow pr_2^*(L) \longrightarrow pr_2^*(L)|_\Delta \longrightarrow 0$$

and the relation (4.9) imply that the determinant bundle $\det^{-1} R pr_{1*}(L_0^{-1} \oplus L_0)$ of the family $(\partial - \sigma^2 a x)$ of $\partial$-operators in $L_0^{-1} \oplus L_0$ is isomorphic to the h.l.b. $L^2K((2 - 2g)x_0)$. This provides an interpretation of the first factor on the right hand side of Eq. (3.17).

5. Projective formula

The outcome of the calculation performed in this paper is much simpler than the calculation itself. We shall start by describing a softened version of its result. It will give the scalar product of CS states up to a $\Sigma$-dependent constant. Such data are enough to generate a projective connection on the bundle $\mathcal{W}_k$ of state spaces. This should coincide with the projective class of the KZB connection and hence be flat. Sect. 9 contains a more detailed scalar product formula with the normalization fixed, up to an overall constant depending only on the level $k$ and genus $g$ (which could be traced through the calculation). The detailed formula gives also the dependence of the scalar product on the metric of the surface (i.e., in particular, on its complex structure).

---

\[ \text{this was the original formulation of Theorem 2a of [1]} \]
In the simplified formula, we shall use, as the geometric input, the representation of the CS states by the holomorphic $k(g-1),0$-forms $\chi$ with values in the line bundle $B'_k$, see (4.11), as well as hermitian structures on these bundles. It will be convenient to choose the latter in a specific way. Following [31], we shall call a hermitian metric on a h.l.b. $M$ on $\Sigma$ admissible, if the curvature form of the induced connection is proportional to the 2-form $\alpha \equiv \frac{i}{2g}(\text{Im} \tau)^{-1}\wedge \tilde{\omega}$. Admissible hermitian structures exist and are unique up to normalization. We shall call a Riemannian metric on $\Sigma$ admissible if it induces an admissible hermitian structure on the holomorphic tangent bundle of $\Sigma$. Let $G(x,y)$ denote the Green function of the scalar Laplacian $\Delta$ on $\Sigma$ chosen so that $\int_{\Sigma} G(\cdot,y) \alpha(y) = 0$. $G(x,y)$ has a logarithmic singularity at coinciding points. We shall define

$$: G(x,x) := \lim_{\epsilon \to 0} \left( G(x,x') - \frac{1}{2\pi} \ln \epsilon \right),$$

with the distance $d(x,x') = \epsilon$. An admissible Riemannian metric on $\Sigma$ normalized so that $G(x,x) = 0$ is called the Arakelov metric [32]. It will be also convenient to fix the dependence of the hermitian structures on the parameter for families of h.l.b.’s on $\Sigma$. In particular, the line bundle $\mathcal{O}(\Delta)$ over $\Sigma \times \Sigma$ may be provided with a hermitian metric by setting

$$|1(x,y)|^2 = e^{4\pi G(x,y)}$$

for its canonical section $1$. Fixing also an admissible hermitian structure on the h.l.b. $L$, we obtain this way a hermitian metric on $L' = pr_2^*(L)(-\Delta)$ which may be viewed as a family of admissible hermitian structures on the family $(L(-x))$ of h.l.b.’s realized as $L'_0$. The above choices determine a hermitian metric on the h.l.b. $B'_k$ of (4.11) in which forms $\chi$ of Eq. (4.7) take values.

Another geometric input in the scalar product formula comes from the linear map

$$H^0(K) \ni \nu \frac{l(x,b'')}{\nu} b'' \in H^1(L(-x)^{-2}K) \cong H^0(L(-x)^2)^*$$

defined for each $b'' \in \wedge^{01}(L(-x)^2)$ and depending only on the class of $b''$ in $H^1(L(-x)^{-2})$. As we shall see in Sect. 6.3, the rank of this map controls the (local) regularity of the projection from $\mathcal{A}^{01}$ into the orbit space $\mathcal{A}^{01}/G^\Sigma$. $l(x,b'')$ may be viewed as an element of the vector space $H^0(K)^* \otimes H^0(L(-x)^2)^*$. Since

$$\wedge^{g-1} H^0(K)^* \cong H^0(K) \otimes \det^{-1} H^0(K),$$

one may consider $\wedge^{g-1} l(\cdot,b'')$ as a holomorphic 1,0-form on $\Sigma$ with values in the bundle $\det^{-1} H^0(K) \otimes \det^{-1} R^0 pr_1_* \mathcal{L}_0^2$. Since it depends homogeneously on $[b''] \in H^1(L(-x)^{-2})$, we may write

$$\wedge^{g-1} l(x,b'') = \int_{\Sigma^{g-1}} \phi(x;x_1,\ldots,x_{g-1}) b''(x_1) \cdots b''(x_{g-1})$$

where $\phi$ is a holomorphic $g,0$-form on $\Sigma \times \Sigma^{g-1}$ with values in

$$\det^{-1} H^0(K) \otimes pr_1_* (\det^{-1} R^0 pr_1_* \mathcal{L}_0^2) S^{g-1}(\mathcal{L}_0^2)^*.$$
Note that the choices of the metric on $\Sigma$ and of the hermitian structure on $L_0'$ described above induce a hermitian metric on the h.l.b. (5.4).

The functional integral calculation which we describe in this paper implies the following scalar product formula for the CS states:

$$\|\Psi\|^2 = \text{const.} \cdot i^{-1-M} \int \det'(\bar{\partial}^2_{L_{(x)}^z} \partial_{L_{(x)}^z}) \cdot |\mathcal{S}_M(\phi(x; x_1, \ldots, x_{g-1}) \chi(x; x_g, \ldots, x_M))|^2 e^{-\frac{4\pi}{k+2} G(x_{m_1}, x_{m_2})} (5.7)$$

Above, $M \equiv (k+1)(g-1)$, $\mathcal{S}_M$ stands for the symmetrizer of $(x_m)_{m=1}^M$, $| \cdot |^2$ denotes the $(1+M)$, $(1+M)$-form obtained by pairing a $(1+M)$, $0$-form with values in a h.l.b. with itself using the hermitian structures described above. The prefactor $i^{-1-M}$ assures the positivity of the integrated form. The determinant of the operator $\bar{\partial}_{L_{(x)}^z} \partial_{L_{(x)}^z}$ restricted to the subspace orthogonal to the zero modes should be zeta-function regularized.

Notice that the product on the right hand side of Eq. (5.7) has a form of the Boltzmann factor for a gas of two-dimensional particles interacting with attractive Coulomb forces. Appearance of such “Coulomb (or, more properly, Newton) gas representation” was a characteristic feature of the genus zero scalar product formula e, see [13][14]. As mentioned above, we have not proven that the above equation defines the scalar product of CS states which induces (projectivized) KZB connection. The first thing which remains to be shown is that the integral on the right hand side of (5.7) (over the modular parameter $x$ and over $M$ positions of “screening charges” at points $x_m$) converges for $\chi$ corresponding to the CS states. Although not proven in general, the convergence seems very plausible in view of the analysis of Sect. 9 below. In particular, it is evident for genus 2. Appendix F discusses other consistency checks of the complete scalar product formula worked out in Sect. 9.

6. Change of variables

As we have mentioned, the main idea of this work is a brute-force calculation of the functional integral (1.13) giving the formal scalar product of the CS states. This will be a long process in which the first step is the change of variables

$$A^{01} = h^{-1} A^{01}(n)$$

where $n \mapsto A^{01}(n)$ parametrizes holomorphically a $(3g-3) \equiv N$-dimensional slice of $A^{01}$ (generically) transversal to the chiral gauge orbits. The reparametrization (6.1) permits to transform the formal scalar product formula into

$$\|\Psi\|^2 = \int |\Psi(h^{-1} A^{01}(n))|^2 e^{-\frac{4\pi}{k+2} \int_L \text{tr} (h^{-1} A^{01}(n)) \chi(h^{-1} A^{01}(n)) \frac{\partial (h^{-1} A^{01}(n))}{\partial (h, n)} |^2 D_h \prod \alpha d^2 n_\alpha (6.2)$$

where $D_h = \prod \alpha dh(x)$ is a formal local product of the Haar measures on $SL(2, \mathbb{C})$. We have to compute the Jacobian $|\frac{\partial (h^{-1} A^{01}(n))}{\partial (h, n)}|^2$ of the change of variables. Notice that under
an infinitesimal variation of $h$ and $n$, 

$$\delta (h^{-1}A^{01}(n)) = h^{-1}D_n(h^{-1}\delta h) + h^{-1} \frac{\partial A^{01}(n)}{\partial n_\alpha} \delta n_\alpha h \quad (6.3)$$

where $D_n \equiv \partial + [A^{01}(n), \cdot]$ and $h^{-1}D_n \equiv Ad_{h^{-1}}D_n Ad_h$. Assume that $A^{01}(n)$ defines a stable vector bundle. Then, for $n$ close to $n_0$ one may choose a basis $(\omega^\alpha(n))_{\alpha=1}^N$ of 1,0-forms with values in $sl(2,\mathbb{C})$ such that $D_n \omega^\alpha(n) = 0$ and $\omega^\alpha(n)$ depends holomorphically on $n$. Notice that the relation $\overline{\delta} (\omega^\alpha(n))_{\alpha=1}^N$ for all (smooth) $sl(2,\mathbb{C})$-valued function $\Lambda$ so that $\omega^\alpha(n)$ may be viewed as covectors tangent to the orbit space $A^{01}/G^C$ since they vanish on the variations $D_n \Lambda$ tangent to the orbit of $G^C$ through $A^{01}(n)$. The space $A^{01}$ has a natural scalar product corresponding to the norm 

$$\|A^{01}\|^2 = i \int_\Sigma \text{tr} (A^{01})^\dagger \wedge A^{01}. \quad (6.5)$$

Using this scalar product, we may decompose 

$$A^{01} = \text{im}(h^{-1}D_n) \oplus (\text{im}(h^{-1}D_n))^\perp \quad (6.6)$$

Notice that the subspace $(\text{im}(h^{-1}D_n))^\perp$ orthogonal to the image of $h^{-1}D_n$ is spanned by the forms $(h^{-1}\omega^\alpha(n) h)^\dagger$. The (holomorphic) derivative of the change of variables (6.1) is 

$$\frac{\delta (h^{-1}A^{01}(n))}{\delta (h, n)} = \left( \begin{array}{ccc} h^{-1}D_n & (h^{-1}D_n \frac{\partial A^{01}(n)}{\partial n_\alpha} h) \end{array} \right) = \left( \begin{array}{ccc} h^{-1}D_n & 0 \\ 0 & (h^{-1}D_n \frac{\partial A^{01}(n)}{\partial n_\alpha} h) \end{array} \right), \quad (6.7)$$

where 

$$(h^{-1} \frac{\partial A^{01}(n)}{\partial n_\alpha} h) = i (h^{-1} \omega^\gamma(n) h)^\dagger (\Omega(h h^\dagger, n)^{-1})_{\gamma\beta} \int_\Sigma \text{tr} \omega^\beta(n) \wedge \frac{\partial A^{01}(n)}{\partial n_\alpha} \quad (6.8)$$

with $\Omega(h h^\dagger, n)^{\beta\gamma} = i \int_\Sigma \text{tr} h^{-1} \omega^\beta(n) h \wedge (h^{-1} \omega^\gamma(n) h)^\dagger$. It follows that the Jacobian of the change of variables (6.1) is 

$$\left| \frac{\partial (h^{-1}A^{01}(n))}{\partial (h, n)} \right|^2 = \det (h^{-1}D_n)^\dagger h^{-1}D_n \det (\Omega(h h^\dagger, n))^{-1} \cdot \left| \int_\Sigma \text{tr} \omega^\beta(n) \wedge \frac{\partial A^{01}(n)}{\partial n_\alpha} \right|^2 \quad (6.9)$$

Of course, $\det (h^{-1}D_n)^\dagger h^{-1}D_n$ has to be regularized, e.g. by the zeta-function prescription. The chiral anomaly permits to compute the $h$-dependence of the regularized Jacobian:

$$\det (h^{-1}D_n)^\dagger h^{-1}D_n \det (\Omega(h h^\dagger, n))^{-1}$$

$$= e^{4 S(h h^\dagger, A(n))} \det (\overline{D}_n D_n) \det (\Omega(1, n))^{-1}, \quad (6.10)$$
where \( A(n) \equiv -(A^{01}(n))^\dagger + A^{01}(n) \). Also a short calculation using the transformation properties\([\ref{10}]\) shows that

\[
|\Psi^{(h^{-1}A^{01}(n))}|^2 e^{-\frac{ik}{2} \int_S \text{tr} (h^{-1}A^{01}(n))^\dagger \wedge h^{-1}A^{01}(n)} = |\Psi(A^{01}(n))|^2 e^{-\frac{ik}{2} \int_S \text{tr} (A^{01}(n))^\dagger \wedge A^{01}(n)} e^{kS(hh^\dagger, A(n))} .
\]  

(6.11)

Inserting Eqs. (6.9), (6.10) and (6.11) into the functional integral (6.2), we obtain

\[
\|\Psi\|^2 = \int |\Psi(A^{01}(n))|^2 e^{-\frac{ik}{2} \int_S \text{tr} (A^{01}(n))^\dagger \wedge A^{01}(n)} e^{(k+4)S(hh^\dagger, A(n))} \det(D_n^\dagger D_n) \cdot \det(\Omega(1,n))^{-1} \left| \det \left( \int_S \omega^{\beta}(n) \wedge \frac{\partial A^{01}(n)}{\partial n}\alpha \right) \right|^2 D(hh^\dagger) \prod_\alpha d^2 n_\alpha ,
\]

(6.12)

where we have used the fact that the integrand depends on \( h \) only through \( hh^\dagger \), related to the gauge invariance of the original integral \([\ref{10}]\), to reduce the \( h \)-integration to that over the \( hh^\dagger \) fields effectively taking values in the hyperbolic space \( SL(2,C)/SU(2) \). \( D(hh^\dagger) \) should then be interpreted as the local formal product \( \prod_\alpha d(hh^\dagger)(x) \) of \( SL(2,C) \)-invariant measures on \( SL(2,C)/SU(2) \).

Formula (6.12) decomposes the original functional integral (6.6) over \( \mathcal{A}^{01} \) into the one along the orbits of the chiral gauge transformations, which has the form of the partition function of an \( SL(2,C)/SU(2) \)-valued WZW model\([\ref{20}, \ref{21}]\)

\[
\int e^{(k+4)S(hh^\dagger, A(n))} D(hh^\dagger) ,
\]

(6.13)

and the integral along a slice \( n \mapsto A^{01}(n) \) of \( \mathcal{A}^{01} \) which we shall parametrize by the complex bundle \( \mathbb{P}W_0 \). More exactly, as discussed in Sect. 2, we shall consider the map

\[
(x, b) \mapsto A^{01}_{x,b}
\]

(6.14)

with \( x \) running through a fundamental domain of \( \Pi_1 \) in \( \bar{\Sigma} \) and one \( b \) in each class of \( \mathbb{P}H^1(L^{-2}_x) \). Such a map gives a multiply parametrized slice of \( \mathcal{A}^{01} \) (as we have seen in Section 2, the induced map from \( \mathbb{P}W_0 \) is essentially a multiple covering of \( \mathcal{A}^{01}/\mathcal{G}^C \); one may show that multiplicity is equal to \( 2g \)). Let \( (\eta^X)^{\alpha}_{\alpha=1} \) be a basis of \( H^0(L^2_xK) \). \( \eta^X \) may be chosen locally as depending holomorphically on \( x \). The integrals

\[
z^\alpha_x = \int_\Sigma \eta^\alpha_x \wedge b
\]

(6.15)

provide coordinates on \( H^1(L^{-2}_x) \) (homogeneous coordinates on \( \mathbb{P}H^1(L^{-2}_x) \)) and a local (holomorphic) trivialization of \( W_0 \). We shall have to find explicit expressions for various terms under the integral (6.12).

6.1. Term \( |\Psi(A^{01}(n))|^2 e^{-\frac{ik}{2} \int_S \text{tr} (A^{01}(n))^\dagger \wedge A^{01}(n)} \)

Using Eq. (3.4), we obtain

\[
|\Psi(A^{01}_{x,b})|^2 e^{-\frac{ik}{2} \int_{\bar{\Sigma}} \text{tr} (A^{01}_{x,b})^\dagger \wedge A^{01}_{x,b}} = |\psi(x, b)|^2
\]

(6.15)
Recall that \( A_{x,b}^{01} = UB_{x,b}U^{-1} + U\bar{\partial}U^{-1} \), \( A_{00}^{10} = U\nabla U^{-1} = -(U\bar{\partial}U^{-1})^\dagger = -(A_{00}^{01})^\dagger \) with \( B_{x,b}^{01} = (-a^x \bar{b}^{01} + b^{01} a^x) \). Hence \(-A_{00}^{10} + A_{x,b}^{01} = UB_{x,b}U^{-1} \) and \( A_{00}^{10} + (A_{x,b}^{01})^\dagger = U(B_{x,b}^{01})^\dagger U^{-1} \), where \( (B_{x,b}^{01})^\dagger = (\frac{\overline{x}}{b} b^x - \overline{a} b^x) \) with the \( L_0^{-2} \)-valued 1,0-form \( b^\dagger \) being the conjugate of \( b \) with respect to the hermitian metric \( \langle \cdot, \cdot \rangle \) of \( L_0^{-2} \). It follows that \( \int_{\overline{x}}(A_{00}^{10} + (A_{x,b}^{01})^\dagger) \wedge (-A_{00}^{10} + A_{x,b}^{01}) = 2\int_{\overline{x}}\overline{a^x} \wedge (b \wedge b) \). Consequently,

\[
|\Psi(A_{x,b}^{01})|^2 e^{-\frac{i}{2\pi} \frac{1}{2\pi} \int_{\overline{x}}(A_{x,b}^{01})^\dagger A_{x,b}^{01}} = e^{-\frac{i}{2\pi} \frac{1}{2\pi} \int_{\overline{x}} A_{00}^{10} A_{00}^{01} - 2\pi k (\overline{\nabla} (\overline{b^0} \omega)) (\overline{b^0} (\omega))} |\psi(x, b)|^2. \tag{6.17}
\]

\[
S(hhh^\dagger, A^{01}_{x,b}) = \frac{i}{2\pi} \int_{\overline{x}} \frac{\partial}{\partial \varphi} (\bar{\partial} \varphi - 2F_0) \tag{6.19}
\]

Using the formula (A.2) of Appendix A, we see that

\[
\delta S(hhh^\dagger, A^{01}_{x,b}) = \frac{i}{2\pi} \int_{\overline{x}} (A^{01}_{x,b})^\dagger (h_{hh^\dagger})^{-1} \delta h_{hh^\dagger} - F((A^{01}_{x,b})^\dagger + (hh^\dagger((A^{01}_{x,b})^\dagger)) = \frac{i}{2\pi} \int_{\overline{x}} \frac{\partial}{\partial \varphi} (\bar{V} - (B_{x,b}^{01})^\dagger + (gg^\dagger)^{-1}(\bar{\partial} + B_{x,b}^{01})(gg^\dagger)) \tag{6.20}
\]

By a straightforward computation, under holomorphic variations of \( v \)

\[
\delta S(hhh^\dagger, A^{01}_{x,b}) = -\frac{i}{2\pi} \int_{\overline{x}} \delta \varphi (\bar{\partial} + 2a^x) \left( e^{-2\varphi} (b + (\bar{\partial} - 2a^x) v) \right)^\dagger \tag{6.21}
\]

and under antiholomorphic ones

\[
\delta S(hhh^\dagger, A^{01}_{x,b}) = \frac{i}{2\pi} \int_{\overline{x}} \delta \varphi (\bar{\nabla} + 2\overline{a^x}) \left( e^{-2\varphi} (b + (\bar{\partial} - 2a^x) v) \right) \tag{6.22}
\]

which coincides with the \( v \)-variations of the right hand side of (6.19). Thus we may assume that \( v = 0 \). Then the variation of \( S \) with respect to \( \varphi \) becomes

\[
\delta S(hhh^\dagger, A^{01}_{x,b}) = \frac{i}{\pi} \int_{\overline{x}} \delta \varphi \left( \bar{\partial} \varphi - F_0 + e^{-2\varphi} b^\dagger \wedge b \right) \tag{6.23}
\]

(recall that \( \bar{\partial} a^x = 0 \)) which is also the variation of the right hand side of Eq. (6.19) for \( v = 0 \). This ends the proof of (6.19) since \( S(hhh^\dagger, A^{01}_{x,b}) = 0 \) for \( hh^\dagger = 1 \).
6.3. Term \( \det(\Omega(1, n))^{-1} \left| \det \left( \int_{\Sigma} \text{tr} \, \omega^\beta(n) \wedge \frac{\partial A_{01}(n)}{\partial \alpha} \right) \right|^2 \prod_\alpha d^2n_\alpha \)

We have to look for \( sl(2, \mathbb{C}) \)-valued 1,0-forms \( \omega(x, b) \equiv \omega \) such that \( \bar{\partial} \omega + A_{01,x,b}^0 \omega + \omega A_{01,x,b}^0 = 0 \). Such forms represent vectors cotangent to the orbit space \( \mathcal{A}^{01}/\mathcal{G}^\mathbb{C} \). Writing

\[
\omega = U \left( \begin{array}{c} -\mu \\ \lambda \\ \mu \end{array} \right) U^{-1} \equiv U \rho U^{-1},
\]

where \( \eta \in \wedge^{10}(L_0^2) \), \( \mu \in \wedge^{10} \) and \( \lambda \in \wedge^{10}(L_0^{-2}) \), the condition for \( \omega \) becomes \( \bar{\partial} \rho + B_{01}^0 \rho + \rho B_{01}^0 = 0 \) or, in components,

\[
(\bar{\partial} + 2a_x) \eta = 0, \quad \bar{\partial} \mu = -\eta \wedge b, \quad (\bar{\partial} - 2a_x) \lambda = 2\mu \wedge b.
\]

The first of these equations requires that \( \eta \in H^0(L_x^{-2}K) \) which has dimension \( N \). The second one has a solution if and only if \( \int_{\Sigma} \eta \wedge b = 0 \) which, for \( b \) corresponding to a non-zero element in \( H^1(L_x^{-2}) \), defines a \( N - 1 \) dimensional subspace in \( H^0(L_x^2K) \). For \( \eta \) in this subspace,

\[
\mu = 2i \bar{\partial} \int G(\cdot, y) (\eta \wedge b)(y) + \nu \equiv \mu^0(\eta) + \nu,
\]

where \( G(x, y) \) is a Green function of the Laplacian on \( \Sigma \) and \( \nu \) is an arbitrary holomorphic 1,0-form on \( \Sigma \). Finally, let \( (\kappa^r)_{r=1}^{g-1} \) be a basis of \( H^0(L_x^2) \). \( \kappa^r \) may be chosen locally depending holomorphically on \( x \). The third of the equations (6.25) has a solution for \( \lambda \) if and only if

\[
\int_{\Sigma} \kappa^r \mu \wedge b = \int_{\Sigma} \kappa^r \mu^0(\eta) \wedge b + \int_{\Sigma} \kappa^r \nu \wedge b = 0
\]

for each \( r \). Then the solution for \( \lambda \) is unique since \( H^0(L_x^{-2}K) = \{0\} \) since we have chosen \( L \) so that \( L_x^2 \) is never isomorphic to \( K \). Let us consider more carefully the condition (6.27). Notice that the exterior multiplication by \( b \) induces a linear map

\[
l(x, b): H^0(K) \rightarrow H^1(L_x^{-2}K).
\]

\( l(x, b) \) depends only on the class \([b]\) of \( b \) in \( H^1(L_x^{-2}) \). The dimensions of the spaces are \( \dim(H^0(K)) = g \) and \( \dim(H^1(L_x^{-2}K)) = \dim(H^0(L_x^2)) = g - 1 \). If \( l(x, b) \) maps onto then, for fixed \( \mu^0(\eta) \), there exists \( \nu \) solving (6.27) and it is unique up to the addition of \( \nu \) from the one-dimensional kernel of \( l(x, b) \). Altogether, the space of solutions of Eqs. (6.25) is then \( N \)-dimensional: \( N - 1 \) dimensions of the freedom to choose \( \eta \) and 1 dimension in the choice of \( \nu \). Let us examine the condition of surjectivity of \( l(x, b) \) which guarantees that the the space tangent to the \( \mathcal{G}^\mathbb{C} \)-orbit through \( A_{x,b} \) is of maximal codimension \( (= N) \). Taking the standard basis \( (\omega^i)_{i=1}^g \) of \( H^0(K) \), this condition means that the matrix

\[
\left( \int_{\Sigma} \kappa^r \omega^i \wedge b \right)
\]

has rank \( = g - 1 \). The \([b]\)'s in \( H^1(L_x^{-2}) \) for which this fails are common zeros of \( g \) homogeneous polynomials giving the \( (g-1) \times (g-1) \) minors of the matrix (6.29). If these
equations are non-trivial, it follows that \( l(x, b) \) is surjective except for a subvariety of positive codimension. To see their non-triviality notice that \( l(x, b) \) fails to be surjective if and only if for some \( 0 \neq \kappa \in H^0(L^2_X) \)

\[
[b] \in B_\kappa \equiv \{ [b] \mid \sum \kappa \nu \wedge b = 0 \text{ for all } \nu \in H^0(K) \}.
\]

But \( \cup_{\kappa} B_\kappa \) is at most \( 3g - 5 \) dimensional (\( \dim(B_\kappa) = 2g - 3 \) and \( B_\kappa \) depends only on the class of \( \kappa \) in the \((g - 2)\)-dimensional projective space \( \mathbb{P}H^0(L^2_X) \)).

For given \( b \in \wedge^0(L^{-2}_X) \) corresponding to a non-trivial element in \( H^1(L^{-2}_X) \), we may choose the basis \( (\eta^\alpha)_{\alpha = 0}^3 \) of \( H^0(L^2_X K) \) so that \( z^1(\equiv \int_X \eta^1 \wedge b) \neq 0 \) and \( z^\alpha = 0 \) for \( \alpha > 1 \). Suppose also that the non-zero \( (g - 1) \times (g - 1) \) minor of the matrix \((6.29)\) corresponds to \( i < g \). Then, we may take

\[
\eta = 0, \quad \mu^1 = \omega^g - \omega^j M_{ir}(\sum \kappa^r \omega^g \wedge b), \quad \lambda^1 = 2 \bar{\partial}^{-1}(\mu^1 \wedge b), \quad (6.30)
\]

where \((M_{ir})\) is the matrix inverse to \((\int_X \kappa^r \omega^i \wedge b)_{i < g}\), and, for \( \alpha > 1 \),

\[
\eta^\alpha, \quad \mu^\alpha = \mu^0(\eta^\alpha) - \omega^i M_{ir}(\sum \kappa^r(\mu^0(\eta^\alpha)) \wedge b), \quad \lambda^\alpha = 2 \bar{\partial}^{-1}(\mu^\alpha \wedge b) \quad (6.31)
\]

as giving a basis of solutions of Eqs. \((6.25)\) and, consequently, a basis \( (\omega^\alpha(x, b)) \) of the \( sl(2, \mathbb{C}) \)-valued 1, 0-forms representing covectors tangent to the orbit space \( A^0_1/G^\mathbb{C} \) at the orbit passing through \( A^0_1(x, b) \). Above \( \bar{\partial}^{-1} \equiv (\bar{\partial}_{L^{-2}_X} K)^{-1} \). With this choice,

\[
\det(\Omega(1, n)) \equiv \det(\Omega(1, x, b)) = \det(\int_X \frac{1}{n} (2 \mu^{\alpha^2} \wedge \mu^\beta + \langle \eta^\alpha, \eta^\beta \rangle + \langle \lambda^\alpha, \lambda^\beta \rangle)) \quad (6.32)
\]

(\( \eta^1 \) should be replaced by zero). Moreover, since \( \int_X \sum \omega^\beta(x, b) \delta A^0_1 = \int_X (2 \mu^{\beta} \delta a_\kappa + \eta^\beta b) \),

\[
\left| \det(\int_X \sum \omega^\beta(n) \frac{\partial A^0_1(n)}{\partial a_\alpha}) \right|^2 \prod_{\alpha} d^2 n_\alpha = \frac{8 \pi^2}{i} |\omega^g(x) - \omega^j(x) M_{ir}(\sum \kappa^r \omega^g \wedge b)|^2 N \prod_{\alpha=2}^{\infty} d^2 z_\alpha, \quad (6.33)
\]

where, for a form \( \omega \), \( |\omega|^{2n} \) denotes the form \( \bar{\omega} \wedge \omega \). A simple algebra shows that

\[
\omega^g(x) - \omega^j(x) M_{ir}(\sum \kappa^r \omega^g \wedge b) = \det(\int_X \omega^j \wedge b)_{i < j} \sum_{j=1}^g (-1)^{g-j} \det(\int_X \omega^j \wedge b)_{i \neq j} \omega^j(x). \quad (6.34)
\]

It will be convenient to represent \( \det(\Omega(1, x, b)) \) as a finite dimensional integral. To this end, consider a linear map \( B : H^0(L^2_X K) \oplus H^0(K) \to \mathbb{C}^g \) given by

\[
B(\eta, \nu) = (\sum \eta \wedge b, \sum \kappa^r(\mu^0(\eta) + \nu) \wedge b) \quad (6.35)
\]

and another linear map \( C : \ker(B) \to \wedge^{10}(\operatorname{End}(L_0^{-1} \oplus L_0)) \) s.t.

\[
C(\eta, \nu) = \left( -\mu^0(\eta) - \nu \cdot \frac{2 \bar{\partial}^{-1}(\mu^0(\eta) + \nu) \wedge b)}{\mu^0(\eta) + \nu} \right), \quad (6.36)
\]

\[\text{We thank J.-B. Bost for this argument.}\]
(with \( \tilde{\partial}^{-1} \equiv (\tilde{\partial}_{L^2 K})^{-1} \)). A straightforward calculation shows that with \( V \equiv (\eta, \nu) \) and \( DV \) standing for the volume element of \( H^0(L^2 K) \oplus H^0(K) \) coming from the scalar product induced by the hermitian metric of \( L_0 \) and the metric of \( \Sigma \),

\[
\int \delta(BV) \ e^{-\|CV\|^2} \ DV = |z|^2 - 2 |\text{det}(\int \kappa^r \omega^i \wedge b)_{<g}|^{-2} \ \text{det}(H_0) \\
\cdot \text{det(Im } \tau) \ \text{det}(\Omega(1, x, b))^{-1},
\]

where

\[
(H_0)^{\alpha\beta} \equiv \frac{1}{i} \int_{\Sigma} \langle \eta^\alpha, \eta^\beta \rangle.
\]

Putting together Eqs. (6.33), (6.34) and (6.37), we obtain

\[
\text{det}(\Omega(1, n))^{-1} \int_{\Sigma} \left( \text{det}\left( \int_{\Sigma} \text{tr} \ \omega^\beta(n) \wedge \frac{2\xi_{i, (a)}}{\eta_{i, a}} \right) \right)^2 \prod_{\alpha} d^2n_{\alpha} \\
= \text{const. } i^{-N} \ \text{det}(H_0)^{-1} \ \text{det(Im } \tau)^{-1} \ \left( \int \delta(BV) \ e^{-\|CV\|^2} \ DV \right) \\
\cdot \ |\epsilon_{\alpha_1, \ldots, \alpha_N} z^{\alpha_1}dz^{\alpha_2} \wedge \ldots \wedge dz^{\alpha_N}|^2 \ |\sum_{j=1}^{g} (-1)^j \ \text{det}(\int_{\Sigma} \kappa^r \omega^i \wedge b)_{\neq j} \ \omega^j(x)|^2
\]

(7.3)

(with a numerical, easy to trace, \( g \)-dependent positive constant in front; the power of \( i \) makes the right hand side a positive measure). We have given the term \( |z^1 dz^2 \wedge \ldots \wedge dz^N|^2 \) a form independent of the assumed relations \( z^1 \neq 0, \ z^\alpha = 0 \) for \( \alpha > 1 \).

### 7. Calculation of \( \text{det}(\tilde{D}_n^\dagger \tilde{D}_n) \)

Let \( \Lambda \) be an \( \mathfrak{sl}(2, \mathbb{C}) \) valued function on \( \Sigma \). Writing

\[
\Lambda = U \left( \begin{array}{cc} -X & Y \\ Z & X \end{array} \right) U^{-1},
\]

where \( X \) is a function, \( Y \in \Gamma(L_0^{-2}) \) and \( Z \in \Gamma(L_0^2) \), we obtain

\[
\tilde{D}_n \Lambda = U \left( \begin{array}{cc} -\bar{\partial}X + Zb & (\bar{\partial} - 2a_k)Y + 2Xb \\ (\bar{\partial} + 2a_k)Z & \bar{\partial}X - Zb \end{array} \right) U^{-1}
\]

(7.2)

It follows that

\[
(\Lambda, \tilde{D}_n^\dagger \tilde{D}_n \Lambda) \equiv i \int_{\Sigma} \text{tr} (\tilde{D}_n \Lambda)^\dagger \wedge \tilde{D}_n \Lambda = i \int_{\Sigma} \left( 2(\bar{\partial}X - Zb) \wedge (\bar{\partial}X - Zb) + (\bar{\partial}Y + 2Xb) \wedge (\bar{\partial}Y + 2Xb) + (\bar{\partial}Z, \wedge \bar{\partial}Z) \right),
\]

(7.3)

where in the last line \( \bar{\partial} \equiv \bar{\partial}_{L^2 K} \) (\( \bar{\partial}_{L^2} \)) when acting on \( Y (Z) \). Formally,

\[
\text{det}(\tilde{D}_n^\dagger \tilde{D}_n)^{-1} = \int e^{-i(\Lambda, \tilde{D}_n^\dagger \tilde{D}_n \Lambda)} \ DA \\
= \int e^{-i \int_{\Sigma} (2(\bar{\partial}X - Zb) \wedge (\bar{\partial}X - Zb) + (\bar{\partial}Y + 2Xb) \wedge (\bar{\partial}Y + 2Xb) + (\bar{\partial}Z, \wedge \bar{\partial}Z))} \ DY \ DX \ DZ
\]

(7.4)
and we shall compute the latter Gaussian integral iteratively, first integrating over $Y$ then over $X$ and in the end over $Z$. As we shall see, this procedure requires a correction if we want to assure that the final result gives the zeta-function regularized determinant of $\bar{D}_n^\dagger \bar{D}_n$. It would be more natural to consider $\Lambda$ as anticommuting ghost field rather than the commuting one. Indeed, this is $\det(\bar{D}_n^\dagger \bar{D}_n)$ and not its inverse which appears in the expression for the scalar product of the CS states. The choice of commuting fields $\Lambda$ in this calculation is purely a matter of convenience.

7.1. Integral over $Y$

\[ I_Y \equiv \int e^{-i \int_\Sigma \langle \partial Y + 2Xb, \partial Y + 2Xb \rangle} \ D Y = \det(\partial^\dagger_{L_x} \partial_{L_x})^{-1} e^{-4i \int_\Sigma \langle (Xb)^\perp, (Xb)^\perp \rangle}, \quad (7.5) \]

where $(Xb)^\perp$ is the component of $Xb$ orthogonal to $\partial (\Gamma(L_{\Sigma}^{-2}) \subset \wedge^{01}(L_{\Sigma}^{-2})$. Recall that the scalar product in the spaces of sections is induced by the fixed hermitian structure of $L_0$ and the metric on $\Sigma$. Explicitly,

\[ (Xb)^\perp = i(\eta^\alpha)^\dagger (H_0^{-1})_{\beta\alpha} \int_\Sigma \eta^\beta \wedge Xb, \quad (7.6) \]

where $(H_0)^{\alpha\beta}$ is given by Eq. (6.38). We have

\[ i \int_\Sigma \langle (Xb)^\perp, (Xb)^\perp \rangle = \int_\Sigma \eta^\alpha \wedge Xb \ (H_0^{-1})_{\beta\alpha} \int_\Sigma \eta^\beta \wedge Xb. \quad (7.7) \]

It will be convenient to express $e^{-4i \int_\Sigma \langle (Xb)^\perp, (Xb)^\perp \rangle}$ as a $6(g-1)$-dimensional Gaussian integral. Namely

\[ e^{-4i \int_\Sigma \langle (Xb)^\perp, (Xb)^\perp \rangle} = \pi^{-N} \det(H_0) \int e^{2i c_\alpha \int_\Sigma \eta^\alpha \wedge Xb + 2i \bar{c}_\alpha \int_\Sigma \eta^\alpha \wedge Xb - \bar{c}_\alpha \cdot \prod_\alpha d^2 c_\alpha}. \quad (7.8) \]

7.2. Integral over $X$

We have to calculate

\[ I_X \equiv \int e^{-i \int_\Sigma 2(\partial X - Zb) \wedge (\partial X - Zb) + 2i c_\alpha \int_\Sigma \eta^\alpha \wedge Xb + 2i \bar{c}_\alpha \int_\Sigma \bar{\eta}^\alpha \wedge Xb} \ D X. \quad (7.9) \]

We have to fix the constant mode $X_0 \equiv (\int X \text{vol})/(\text{area})^{1/2}$ which corresponds to the flat direction in the above Gaussian integral. $\text{vol}$ is the Riemannian volume form of $\Sigma$ and $\text{area} \equiv \int_\Sigma \text{vol}$. Let us multiply $I_X$ by $1 = \text{area} \cdot \int \delta(X_0 - a \text{area}^{1/2}) \ d^2 a$. Changing the order of integration and shifting $X$ to $X + a$, we obtain

\[ I_X = \text{area} \cdot \int \delta(X_0 - a \text{area}^{1/2}) \ d^2 a \quad I_X = \text{area} \cdot \int e^{-i \int_\Sigma 2(\partial X - Zb) \wedge (\partial X - Zb)} \]
Gathering Eqs. (7.11), (7.12) and (7.14), we obtain
\[ I_X = \text{const.} \left( \frac{\text{det}'(-\Delta)}{\text{area}} \right)^{-1} \cdot e^{-2i \int_\Sigma (\omega^i \wedge Zb) - 4 \int_\Sigma (\mu^0(c_0 \eta^a) \wedge Zb) - 2i \int_\Sigma (\mu^0(c_0 \eta^a) \wedge \mu^0(c_0 \eta^a)) \delta(c_0 \int \eta^a \wedge b) \right), \quad (7.11) \]
where \( \text{det}' \) denotes the determinant of the operator restricted to the subspace orthogonal to its kernel, \( G(x,y) \) is a Green function of the Laplacian \( \Delta \) on \( \Sigma \) and \( \mu^0(\eta) \equiv 2i \partial \int G(-, y)(\eta \wedge b)(y) \), as in Eq. (6.26). It is easy to see that
\[ e^{-2i \int_\Sigma (\omega^i \wedge Zb) - 4 \int_\Sigma (\mu^0(\eta^a) \wedge Zb) \delta(\mu^0(\eta^a)) \delta(c_0 \int \eta^a \wedge b), \quad (7.12) \]
where \( (Zb)^\perp \) is the component of \( Zb \) orthogonal to the image of \( \bar{\partial} \) acting on functions on \( \Sigma \). Explicitly,
\[ (Zb)^\perp = \frac{i}{2} \omega^i \left( \frac{1}{\text{Im} \tau} \right)_{ij} \int_\Sigma \omega^j \wedge Zb, \]
\[ 2i \int_\Sigma (Zb)^\perp \wedge (Zb)^\perp = (\int_\Sigma (\omega^i \wedge Zb) \left( \frac{1}{\text{Im} \tau} \right)_{ij} (\int_\Sigma \omega^j \wedge Zb). \quad (7.13) \]
We shall rewrite the exponential of the latter expression as a 2g-dimensional Gaussian integral:
\[ e^{-2i \int_\Sigma (Zb)^\perp \wedge (Zb)^\perp} = \pi^{-g} \text{det} \left( \text{Im} \tau \right) \int e^{-4 \epsilon_i (\text{Im} \tau)^{ij} \epsilon_j - 2i \epsilon_i \int_\Sigma \omega^i \wedge Zb - 2i \epsilon_i \int_\Sigma \omega^i \wedge Zb} \prod_i d^2 \epsilon_i. \quad (7.14) \]
Gathering Eqs. (7.11), (7.12) and (7.14), we obtain
\[ I_X = \text{const.} \left( \frac{\text{det}'(-\Delta)}{\text{area}} \right)^{-1} \cdot e^{-2i \int_\Sigma (\omega^i \wedge Zb) - 4 \int_\Sigma (\mu^0(\eta^a) \wedge Zb) - 2i \int_\Sigma (\mu^0(\eta^a) \wedge \mu^0(\eta^a)) \delta(c_0 \int \eta^a \wedge b) \right). \quad (7.15) \]
\[ . \ e^{2i c_0 \int_\Sigma \eta^a \wedge Xb + 2i c_0 \int_\Sigma \eta^a \wedge Xb + 2i a \int_\Sigma \eta^a \wedge b + 2i \bar{a} \int_\Sigma \bar{\eta}^a \wedge \bar{b}} \delta(X_0) \ DX \ d^2 a. \quad (7.10) \]
7.3. Integral over $Z$

The integral to calculate is

$$I_Z \equiv \int e^{-i \int \langle \partial Z, \partial \delta Z \rangle - 2i \int \Sigma Z (\mu^0(c_\alpha \eta^\alpha) + e_i \omega^i) \wedge b - 2i \int \Sigma \bar{Z} (\mu^0(c_\alpha \eta^\alpha) + e_i \omega^i) \wedge \bar{b}} \ DZ \ . \quad (7.16)$$

Let us decompose $Z$ into the part $Z_0$ in the kernel of $\bar{\partial}$ (i.e. in $H^0(L^n_\mathbb{Z})$) and the part $Z'$ orthogonal to $H^0(L^n_\mathbb{Z})$. Writing $Z_0 = f_r \kappa^r$, where $(\kappa^r)^{\alpha-1}_r$ is a basis of $H^0(L^n_\mathbb{Z})$, we obtain

$$I_Z = \int e^{-i \int \langle \bar{\partial}Z', \bar{\partial}\delta Z' \rangle - 2i \int \Sigma Z' (\mu^0(c_\alpha \eta^\alpha) + e_i \omega^i) \wedge b - 2i \int \Sigma \bar{Z}' (\mu^0(c_\alpha \eta^\alpha) + e_i \omega^i) \wedge \bar{b}} \cdot \text{const.} \ \det (K_0) \ \int e^{-2i \int \Sigma \kappa^r (\mu^0(c_\alpha \eta^\alpha) + e_i \omega^i) \wedge b - 2i \int \Sigma \kappa^r (\mu^0(c_\alpha \eta^\alpha) + e_i \omega^i) \wedge \bar{b}} \prod \ DZ'$$

$$= \text{const.} \ \det (K_0) \ \det' (\partial^{L_\mathbb{Z}}_{L_\mathbb{Z}})^{-1} \prod \delta (\int \kappa^r (\mu^0(c_\alpha \eta^\alpha) + e_i \omega^i) \wedge b) \ e^{4i \int \Sigma \langle \delta^{-1}((\mu^0(c_\alpha \eta^\alpha) + e_i \omega^i) \wedge b) , \delta^{-1}((\mu^0(c_\alpha \eta^\alpha) + e_i \omega^i) \wedge b) \rangle} \ , \quad (7.17)$$

where

$$(K_0)^{\alpha^2} \equiv \int \langle \kappa^r , \kappa^s \rangle \text{ vol} \quad (7.18)$$

and, in the last line of (7.17), $\delta^{-1}$ stands for the inverse of $\partial_{L_\mathbb{Z}}^{-2} K$. We may finally collect Eqs. (7.3), (7.8), (7.9), (7.15), (7.16) and (7.17):

$$I_{X Y Z} \equiv e^{-i \int \Sigma (2(\bar{\partial}X - Zb) \wedge (\bar{\partial}X - Zb) + (\bar{\partial}Y + 2Xb) \wedge (\bar{\partial}Y + 2Xb)) + (\bar{\partial}Z, \wedge \bar{\partial}Z))} \ D Y \ D X \ D Z$$

$$= \text{const.} \ \det (H_0) \ \det (\text{Im} \tau) \ \det (K_0) \ \det (\partial^{L_\mathbb{Z}}_{L_\mathbb{Z}})^{-1} \left( \frac{\det' (\Delta)}{\text{area}} \right)^{-1} \ \det' (\partial^{L_\mathbb{Z}}_{L_\mathbb{Z}})^{-1} \ \cdot \ \int \delta (\int c_\alpha \eta^\alpha \wedge b) \prod \delta (\int \kappa^r (\mu^0(c_\alpha \eta^\alpha) + e_i \omega^i) \wedge b) \ e^{-c_\alpha (H_0)_{\alpha\beta} c_\beta + 2i \int \Sigma \mu^0(c_\alpha \eta^\alpha) \wedge \mu^0(c_\alpha \eta^\alpha)} \ \cdot \ e^{4i \int \Sigma \langle \delta^{-1}((\mu^0(c_\alpha \eta^\alpha) + e_i \omega^i) \wedge b) , \delta^{-1}((\mu^0(c_\alpha \eta^\alpha) + e_i \omega^i) \wedge b) \rangle} \prod \alpha \ D^2 c_\alpha \ \prod \ D^2 e_i$$

$$= \text{const.} \ \det (K_0) \ \det (\partial^{L_\mathbb{Z}}_{L_\mathbb{Z}})^{-1} \left( \frac{\det' (\Delta)}{\text{area}} \right)^{-1} \ \det' (\partial^{L_\mathbb{Z}}_{L_\mathbb{Z}})^{-1} \ \cdot \ \left( \int \delta (B V) \ e^{-\|C V\|^2} \ D V \right) \ , \quad (7.19)$$

where the last integral is the same as the one introduced in Section 4.3. Notice that, with the use of Eq. (6.37), one obtains then

$$I_{X Y Z} \det (\Omega(1, n)) = \text{const.} \ |z|^2 \left| \langle \delta (\int \kappa^r \omega^i \wedge b) \rangle \right|^{-2} \det (H_0) \ \det (\text{Im} \tau) \ \det (K_0)$$

$$\det (\partial^{L_\mathbb{Z}}_{L_\mathbb{Z}})^{-1} \left( \frac{\det' (\Delta)}{\text{area}} \right)^{-1} \ \det' (\partial^{L_\mathbb{Z}}_{L_\mathbb{Z}})^{-1} \ , \quad (7.20)$$

i.e. the $V$-integrals cancel. This ends the formal calculation of $\det (\partial^{L_\mathbb{Z}}_{L_\mathbb{Z}})$.

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Clearly, the determinants appearing on the right hand side of the expression (7.19) need regularization. If we use the zeta-function procedure to give sense to them, it is not guaranteed that the result will coincide with $\det(\bar{D}_1 D_n)$ regularized by the zeta-function prescription. Indeed, the latter should satisfy the chiral anomaly relation (6.11) but for $h = U \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} U^{-1}$, we obtain

$$\det((h^{-1} \bar{D}_n)^\dagger h^{-1} D_n) \det(\Omega(hh^\dagger, n))^{-1} = \det(D_n^\dagger \bar{D}_n) \det(\Omega(1, n))^{-1}, \quad (7.21)$$

if we use for $\det(\bar{D}_n D_n)^{-1}$ the expression on the right hand side of Eq. (7.19), instead of

$$\det((h^{-1} \bar{D}_n)^\dagger h^{-1} D_n) \det(\Omega(hh^\dagger, n))^{-1}$$

$$= e^{-\frac{2i}{\pi} \int_{\Sigma} \langle b + \partial \nu, \wedge (b + \partial \nu) \rangle + \frac{2i}{\pi} \int_{\Sigma} (b, \wedge b) \det(\bar{D}_n^\dagger \bar{D}_n) \det(\Omega(1, n))^{-1} \quad (7.22)$$

given by (6.10) (and Eqs. (5.18), (5.19)). It is easy to guess that we should correct the formal result for $\det(\bar{D}_n D_n)$ by taking

$$\det(\bar{D}_n^\dagger \bar{D}_n) = \text{const.} \det(K_0)^{-1} \det(\partial_{L_2}^\dagger \partial_{L_2}^\dagger) \left( \frac{\det'(-\Delta)}{\text{area}} \right) \det'(\partial_{L_2}^\dagger \partial_{L_2}^\dagger)$$

$$\cdot e^{-\frac{2i}{\pi} \int_{\Sigma} (b, \wedge b) \left( \int_\Sigma \delta(BV) e^{-\|CV\|^2} DV \right)^{-1}}. \quad (7.23)$$

Indeed, Eq. (7.21) is replaced then by the relation

$$\det(\bar{D}_n^\dagger \bar{D}_n) \det(\Omega(1, n))^{-1} = \text{const.} |z|^2 \left| \det(f \kappa^j \omega^j \wedge b)_{<g} \right|^2 e^{-\frac{2i}{\pi} \int_{\Sigma} (b, \wedge b) \cdot \det(H_0)^{-1} \det(\Omega(1, n))^{-1} \det(\partial_{L_2}^\dagger \partial_{L_2}^\dagger) \left( \frac{\det'(-\Delta)}{\text{area}} \right) \det'(\partial_{L_2}^\dagger \partial_{L_2}^\dagger), \quad (7.24)$$

and (7.22) follows. We shall show in Appendix D that formula (7.23) is, indeed, the right expression for the zeta-function regularized determinant. Putting it together with the Eq. (6.39) from the end of Section 4.3, we obtain the following explicit expression

$$\det(\bar{D}_n^\dagger \bar{D}_n) \det(\Omega(1, n))^{-1} \left| \det(\int_{\Sigma} \tr \omega^j(n) \wedge \frac{\partial A_0^j(n)}{\partial \alpha_0}) \right|^2 \prod_{\alpha} d^2 n_\alpha = \text{const.} i^{-N}$$

$$\cdot \det(H_0)^{-1} \det(\Omega(1, n))^{-1} \det(\partial_{L_2}^\dagger \partial_{L_2}^\dagger) \left( \frac{\det'(-\Delta)}{\text{area}} \right) \det'(\partial_{L_2}^\dagger \partial_{L_2}^\dagger)$$

$$\cdot e^{-\frac{2i}{\pi} \int_{\Sigma} (b, \wedge b) \epsilon_{\alpha_1, \ldots, \alpha_N} z^{\alpha_1} dz^{\alpha_2} \wedge \ldots \wedge dz^{\alpha_N}}$$

$$\cdot \left( \frac{1}{i} \sum_{j=1}^g (-1)^j \det(\int_{\Sigma} \kappa^j \omega^j \wedge b)_{<\alpha_j} \omega^j(x) \right)^2. \quad (7.25)$$

8. Functional integral over $hh^\dagger$

We shall attempt now a direct calculation of the functional integral (6.12). Only the $e^{(k+4)S(hh^\dagger, A_x, b)}$ term under it depends on $hh^\dagger$ and, as noticed before, its $hh^\dagger$-integral
should give the partition function of the $SL(2, \mathbb{C})/SU(2)$-valued WZW model. Below, we shall find its form somewhat surprising. The action $S(hh^\dagger, A_x, b)$ is explicitly given by Eq. (6.19) in the parametrization (6.18) of $hh^\dagger$ by real functions $\varphi$ and $v \in \Gamma(L_0^{-2})$. It will be more convenient to use $w \equiv e^{-\varphi}v$ instead of $v$. The formal measure $D(hh^\dagger)$ becomes then the product of the formal Lebesgue measures $Dw$ and $D\varphi$ determined by the $L^2$ scalar products $\int_\Sigma \langle w, w \rangle \text{vol}$ and $\int_\Sigma |\varphi|^2 \text{vol}$. Thus we have:

$$
\int e^{(k+4)S(hh^\dagger, A_x, b)} D(hh^\dagger) = e^{\frac{k+4}{2\pi i} \int_\Sigma \langle b, \wedge b \rangle} \int e^{\frac{k+4}{2\pi i} \int_\Sigma \left[ -\varphi(\partial \bar{\varphi} - 2F_0) + \langle e^{-\varphi}b + (\partial + (\bar{\partial} \varphi))w, \wedge (e^{-\varphi}b + (\bar{\partial} + (\partial \varphi))w) \rangle \right]} DwD\varphi, \quad (8.1)
$$

where $\bar{\partial} \equiv \bar{\partial}_{L^{-2}}$ when acting on $w$ and $F_0$ is the curvature form of the holomorphic connection of $L_0$ preserving the fixed hermitian metric. Note that the field $w$ enters quadratically in the action so that the $w$-integral is Gaussian and may be easily performed:

$$
\int e^{\frac{k+4}{2\pi i} \int_\Sigma \left( e^{-\varphi}b + (\partial + (\bar{\partial} \varphi))w, \wedge (e^{-\varphi}b + (\bar{\partial} + (\partial \varphi))w) \right)} Dw = e^{\frac{k+4}{2\pi i} \int_\Sigma \left( P_\varphi(e^{-\varphi}b), \wedge P_\varphi(e^{-\varphi}b) \right) \det (\langle \partial + (\bar{\partial} \varphi) \rangle^\dagger (\bar{\partial} + (\partial \varphi))^\dagger)^{-1}, \quad (8.2)
$$

where $P_\varphi$ denotes the orthogonal projector on the kernel of $(\partial + (\partial \varphi))^\dagger$. Explicitly,

$$
P_\varphi(e^{-\varphi}b) = i e^{\varphi} \left( \langle \eta^\alpha, (H_\varphi)^{-1}\beta \rangle \right) \int \sum_{\beta} \eta^\beta \wedge b, \quad
i \int_{\Sigma} \left( P_\varphi(e^{-\varphi}b), \wedge P_\varphi(e^{-\varphi}b) \right) = \left( \int \sum_{\beta} \eta^\beta \wedge b \right) \left( (H_\varphi)^{-1}\beta \right) \eta^\beta \equiv z^\alpha \left( (H_\varphi)^{-1}\beta \right) \eta^\beta, \quad (8.3)
$$

with the matrix of the modified scalar products of the vectors of the basis $(\eta^\alpha)$ of $H^0(L_2^2 K)$

$$
(H_\varphi)^{\alpha\beta} \equiv \frac{1}{i} \int \sum_{\beta} \eta^\alpha \wedge \eta^\beta, \quad (8.4)
$$

compare Eq. (8.38). For convenience, we shall rewrite the exponential on the right hand side of Eq. (8.2) as a finite-dimensional Gaussian integral:

$$
e^{\frac{k+4}{2\pi i} \int_\Sigma \left( P_\varphi(e^{-\varphi}b), \wedge P_\varphi(e^{-\varphi}b) \right)} = e^{-\frac{k+4}{2\pi i} \int_\Sigma \left( (H_\varphi)^{-1}\beta \right) z^\beta} = \left( \frac{2}{k+4} \right)^N \det (H_\varphi) \int e^{-\frac{k+4}{2\pi i} \int_\Sigma \left( (H_\varphi)^{-1}\beta \right) \eta^\beta + i \bar{c}_{\alpha} z^\alpha + i c_{\alpha} \bar{z}^\alpha \prod_{\alpha} d^2 c_{\alpha}. \quad (8.5)
$$

The $\varphi$-dependence of the product of determinants $\det (H_\varphi) \det \left( \langle \partial + (\partial \varphi) \rangle^\dagger (\partial + (\bar{\partial} \varphi)) \right)^{-1}$ with the second one regularized by the zeta-function prescription (or any other gauge invariant procedure) is given by the chiral anomaly:

$$
\delta \ln \left( \det (H_\varphi) \det \left( \langle \partial + (\partial \varphi) \rangle^\dagger (\partial + (\bar{\partial} \varphi)) \right)^{-1} \right) = \frac{2}{\pi i} \int_{\Sigma} \langle \bar{\partial} \varphi, \partial \varphi - F_0 \rangle + \frac{1}{2\pi i} \int_{\Sigma} \langle \delta \varphi \rangle R, \quad (8.6)
$$

where $R$ is the metric curvature form of (the holomorphic tangent bundle of) $\Sigma$ normalized so that $\int_{\Sigma} R = 4\pi i(g - 1)$. The global form of the formula (8.6) is

$$
\det (H_\varphi) \det \left( \langle \partial + (\partial \varphi) \rangle^\dagger (\partial + (\bar{\partial} \varphi)) \right)^{-1}
$$
compared to the genus zero and one cases studied in [13, 14] and [27], respectively. There, a new difficulty in the calculation of the scalar product of CS states, as calculation of the Gaussian $c$-integral is easy to perform:

$$\int e^{\frac{k}{2\pi i} \int_{\Sigma} \varphi (\partial \overline{\partial} \varphi - 2F_0) + \frac{1}{2\pi i} \int_{\Sigma} \varphi R} \det (H_0) \det (\overline{\partial}_{L_1^\alpha} \overline{\partial}_{L_2\alpha})^{-1} .$$

(8.7)

Gathering Eq. (8.2), (8.5) and (8.7), we obtain

$$\int e^{\frac{k}{2\pi i} \int_{\Sigma} \varphi (\partial \overline{\partial} \varphi - 2F_0) + \frac{1}{2\pi i} \int_{\Sigma} \varphi R} \det (H_0) \det (\overline{\partial}_{L_1^\alpha} \overline{\partial}_{L_2\alpha})^{-1} \cdot \int e^{-\frac{2\pi}{k+4} \overline{\epsilon}_a (H_0)_{\alpha \beta} c_{\beta} + i \overline{c}_a z^a + i \overline{c}_a z^a} \prod_\alpha d^2 c_\alpha .$$

(8.8)

Here appears a new difficulty in the calculation of the scalar product of CS states, as compared to the genus zero and one cases studied in [13, 14] and [27], respectively. There, the $hh^I$ integral for the partition function of the $SL(2, \mathbb{C})/SU(2)$-valued WZW theory [28, 29] led, after parametrization of $hh^I$ by $\varphi$ and $w$, to an iterative Gaussian integral: after calculation of the Gaussian $w$ integral, the remaining $\varphi$ integral was, miraculously, also becoming Gaussian. This does not seem to be the case here. The right hand side of Eq. (8.8) includes the term

$$e^{-\frac{2\pi}{k+4} \overline{\epsilon}_a (H_0)_{\alpha \beta} c_{\beta} = e^{\frac{2\pi}{k+4} \int_{\Sigma} e^{2\varphi} (\overline{\epsilon}_a \eta^a, \wedge c_{\beta} \eta^\beta)}$$

(8.9)

with the Liouville type terms containing $e^{2\varphi}$ in the exponential. So the $\varphi$-integral obtained after integrating out $w$ seems to be of a non-Gaussian type, in contrast to the low genera situation.

We shall show however, that this difficulty may be solved by a trick used in the Liouville theory [28, 29]. The functional integral over $\varphi$ which we are left with has the form:

$$I_\varphi \equiv \int e^{\frac{k}{2\pi i} \int_{\Sigma} \varphi (\partial \overline{\partial} \varphi - 2F_0) + \frac{1}{2\pi i} \int_{\Sigma} \varphi R - \frac{2\pi}{k+4} \overline{\epsilon}_a (H_0)_{\alpha \beta} c_{\beta} \} D\varphi .$$

(8.10)

We shall integrate first over the zero mode of $\varphi_0 \equiv (\int_{\Sigma} \varphi \text{vol})/(\text{area})^{1/2}$. For this purpose, let us multiply $I_\varphi$ by $1 = (\text{area})^{1/2} \cdot \int \delta(\varphi_0 - a (\text{area})^{1/2}) da$. Changing the order of integration, shifting $\varphi$ to $\varphi + a$ and setting $M \equiv (k + 1)(g - 1)$, we obtain

$$(\text{area})^{-1/2} I_\varphi = \int e^{-2aM} \cdot e^{\frac{k}{2\pi i} \int_{\Sigma} \varphi (\partial \overline{\partial} \varphi - 2F_0) + \frac{1}{2\pi i} \int_{\Sigma} \varphi R - \frac{2\pi}{k+4} \overline{\epsilon}_a (H_0)_{\alpha \beta} c_{\beta} \} \int \delta(\varphi_0 - a (\text{area})^{1/2}) da \delta(\varphi_0) D\varphi = \frac{1}{2} \Gamma(-M) (\frac{2\pi}{k+4})^M \int e^{\frac{k}{2\pi i} \int_{\Sigma} \varphi (\partial \overline{\partial} \varphi - 2F_0) + \frac{1}{2\pi i} \int_{\Sigma} \varphi R (\overline{\epsilon}_a (H_0)_{\alpha \beta} c_{\beta})^M \delta(\varphi_0) D\varphi , (8.11)$$

where we have used the relations $\frac{1}{2\pi i} \int_{\Sigma} F_0 = \text{deg}(L_0) = g - 1$ and $\frac{1}{2\pi i} \int_{\Sigma} R = \text{deg}(K^{-1}) = 2(1 - g)$. As we see, the integration over the zero mode of $\varphi$ diverges but may be easily (multiplicatively) regularized by removing the overall divergent factor $(-1)^M \Gamma(-M)$.

Now, the $c$-integral is easy to perform:

$$\int (\overline{\epsilon}_a (H_0)_{\alpha \beta} c_{\beta})^M e^{i c_{\alpha} z^a + i \overline{c}_a z^a} \prod_\alpha d^2 c_\alpha = (2\pi)^{2N} (\overline{(H_0)_{\alpha \beta} \partial_{z^{a}} \partial_{\overline{z}^{\beta}}})^M \prod_\alpha \delta(z^a) . (8.12)$$
Gathering the above results, we obtain the following “Coulomb gas representation” for the higher genus partition function of the $SL(2, \mathbb{C})/SU(2)$-valued WZW model:

$$
\int e^{(k+4)S(hh^\dagger,A_{x,b})} D(hh^\dagger) = \text{const.} \cdot \left( \int e^{-\frac{k+4}{2\pi} \int_{\Sigma} \varphi^2 (\partial \varphi - 2F_0) + \frac{\pi}{2\pi} \int_{\Sigma} \varphi R} \right) \cdot \left( \prod_m \delta(z^\alpha) \right)
$$

where $m$ runs from 1 to $M$. Observe, that the integrand in the functional integral over $\varphi$ is now invariant under constant shifts of $\varphi$, except for the term $\delta(\varphi_0)$ (the neutrality of the Coulomb gas). The $\varphi$-integral is of the Gaussian form

$$
\int e^{-\frac{k+4}{2\pi} \int_{\Sigma} \varphi^2 (\partial \varphi - 2F_0) + \frac{\pi}{2\pi} \int_{\Sigma} \varphi R} \cdot \left( \prod_m \delta(z^\alpha) \right).
$$

The substitution of this result into Eq. (8.13) results in the relation

$$
\int e^{(k+4)S(hh^\dagger,A_{x,b})} D(hh^\dagger) = \text{const.} \cdot \left( \prod_m \delta(z^\alpha) \right)
$$

which is the final formula for the higher genus partition function of the $SL(2, \mathbb{C})/SU(2)$-valued WZW model.

Equation (8.13) reduces the functional integral over $hh^\dagger$ to a finite dimensional integral over $M$ copies of $\Sigma$. The integrand is a smooth function except for $O(d(x_1, x_2))$ singularities at coinciding points. Power counting shows, that the integral converges for $g = 2$ but for higher genera it diverge unless special combinations

$$
\gamma_{(\alpha_m), (\beta_m)} \prod_m \langle \eta^\alpha_m, \eta^\beta_m \rangle(x_m)
$$

(8.17)
of forms are integrated. We shall return to this issue below. Another feature of the right hand side of Eq. (8.16) may look even more surprising in a candidate for the partition function: its dependence on the external field $A_{x,b}$ is not functional but distributional! The entire dependence on $b \in \Lambda^0 (L_0)$ resides in the term $\prod_m \partial_{z^{\alpha_m}} \partial_{\bar{z}^{\beta_m}} \delta (z^\alpha)$ (recall that $z^\alpha \equiv \int_\Sigma \eta^\alpha \wedge b$). This is not so astonishing in view of the fact that the partition function of the $SL(2, \mathbb{C})/SU(2)$ WZW may be expected, by formal arguments similar to the ones used in [30], to be the hermitian square of a holomorphic section of a negative power of the determinant bundle. But there are no such sections. There exist, however distributional solutions of the corresponding Ward identities and the right hand side of (8.16) is one of them.

9. Assembling the final formula

The main results (7.25) and (8.16) of the calculations of the last two Sections permit to reduce the formal scalar product formula (6.12) to the following finite-dimensional integral:

$$
\|\Psi\|^2 = \text{const. } i^{-N} \det (\text{Im } \tau)^{-1} \left( \frac{\det (-\Delta)}{\text{area}} \right)^{1/2} e^{\frac{\text{tr } A_0 b \wedge A_0^*}{2\pi}} \cdot \int_\Sigma \det (f (\kappa^r, \kappa^s) \text{vol})^{-1} \det \left( \partial_{\bar{z}}^{i} \partial_{z}^{j} \right) e^{-2\pi k (\int_{x_0} \omega) (\text{Im } \tau)^{-1} (\int_{x_0} \omega)}
$$

$$
\cdot \left[ \prod_{j=1}^{g} (-1)^j \det (f (\kappa^r, \kappa^s) \text{vol})^{-1} \det \left( \partial_{\bar{z}}^{i} \partial_{z}^{j} \right) e^{-\frac{4\pi}{k+2} G(x_m, x_m)} \right] \cdot \left( \prod_{m \neq m_2} e^{-\frac{4\pi}{k+2} G(x_m, x_m)} \right),
$$

(9.1)

The integral is, for fixed $x$, over the $(N-1) \equiv (3g-4)$-dimensional projective space with homogeneous coordinates $(z^\alpha)$, over the Cartesian product of $M \equiv (k+1)(g-1)$ copies of $\Sigma$ (variables $x_m$) and, finally, over the projection $x$ of $x \in \Sigma$ to $\Sigma$. Let us discuss first the $z^\alpha$ integral. It has the form

$$
I_z \equiv \int_{\mathbb{P} H^1 (L_x^2)} |P(z)|^2 \left( \prod_m \partial_{z^{\alpha_m}} \partial_{\bar{z}^{\beta_m}} \delta (z^\alpha) \right) |\epsilon_{\alpha_1, \ldots, \alpha_N} z^{\alpha_1} dz^{\alpha_2} \wedge \ldots \wedge dz^{\alpha_N}|^2,
$$

(9.2)

where

$$
P(z) = \sum_{j=1}^{g} (-1)^j \det (f (\kappa^r, \kappa^s) \text{vol})^{-1} \det \left( \partial_{\bar{z}}^{i} \partial_{z}^{j} \right) e^{-\frac{4\pi}{k+2} G(x_m, x_m)} \cdot \left( \prod_{m \neq m_2} e^{-\frac{4\pi}{k+2} G(x_m, x_m)} \right),
$$

(9.3)

is a homogeneous polynomial in $(z^\alpha)$ of degree $M$ (with values in $K|_x$). The integrand is a distributional $2(N-1)$-form on $H^1 (L_x^2)$ invariant under complex rescalings of $z$. Note that formally

$$
\int_{H^1 (L_x^2)} |P(z)|^2 \left( \prod_m \partial_{z^{\alpha_m}} \partial_{\bar{z}^{\beta_m}} \delta (z^\alpha) \right) |dz^{1} \wedge \ldots \wedge dz^{N}|^2 = \frac{1}{(N-1)!} \left( \int_{\mathbb{C}} |\lambda^{-1} d\lambda|^2 \right) I_z,
$$

(9.4)
where the divergent integral on the right hand side is over the fibers of the projection of \( H^1(L^2_\mathbb{R}) \) onto \( \mathbb{P}H^1(L^2_\mathbb{R}) \). Of course, the left hand side is perfectly well defined and we shall take it as a definition of the right hand side. One may expect to reabsorb this way the infinite constant \((-1)^M \Gamma(-M)\) produced by the integration of the zero mode of the field \( \varphi \), see Eq. (5.11). This is more then formal gymnastics. In Appendix E, we show that changing the order of integration in the above arguments by computing the integral over \( \mathbb{P}H^1(L^2_\mathbb{R}) \) modular degrees of freedom just after the \( w \) functional integration and the one over the scalar field \( \varphi \) only afterwards, one obtains the same final result but no infinite constants, apart of those of the Wick ordering, appear in the intermediate steps. This way, it is rather the convergent integration over the (part of) the modular degrees of freedom then the divergent Gupta-Trivedi-Wise-Goulian-Li trick which removes the cumbersome Liouville-type terms from the effective action for \( \varphi \) and renders the \( \varphi \) integral calculable. It is an interesting question whether similar arguments may be used to substantiate the Goulian-Li trick in the gravity case.

With the above interpretation of \( I_z \), we obtain the following expression for the scalar product of genus \( g \) CS states:

\[
||\Psi||^2 = \text{const.} \text{ det (Im } \tau)^{-1} \left( \frac{\text{det}'(-\Delta)}{\text{area}} \right)^{1/2} e^{-\frac{i k}{2} \int \tau \frac{\text{tr}}{2} A_0^a A_0^b} \cdot \int \text{det} \left( f \left( \kappa^x, \kappa^x \right) \text{ vol} \right)^{-1} \text{ det} \left( \frac{\partial L_2^x}{\partial L_2^x} \right) e^{-2\pi k (f^x_{x_0} \tilde{\omega}) (\text{Im } \tau)^{-1} (f^x_{x_0} \omega)} \cdot \prod_{m \neq n} \frac{\partial^2 \kappa^x \partial^2 \kappa^x}{|_{b=0}} \left( \frac{1}{g} \sum_{j=1}^g (-1)^j \text{ det} \left( f \kappa^x \omega^j \wedge \kappa^x \right) \omega_j^j(x) \psi(x, b) \right)^2 \cdot \left( \prod_{m_1 \neq m_2} e^{-\frac{i}{24 \pi} G(x_{m_1}, x_{m_2})} \right) \left( \prod_{m} e^{-\frac{i}{24 \pi} G(x_m, x_m)} \right) \left( \frac{1}{g} \left( \eta^\alpha_{x_0} \wedge \eta^\beta_{x} \right)(x_m) \right),
\]

(9.5)

with the \((M + 1)\)-fold integration over \( \Sigma \) (over the projection \( x \) of \( \mathbf{x} \) to \( \Sigma \) and over the positions \( x_m \) of \( M \) “screening charges”). We have restored the \( \mathbf{x} \) subscript to stress the \( \mathbf{x} \) dependence of various entries in the integrated function. In fact, it is easy to see that the latter depends only on \( x \). In Appendix F, we submit formula (9.3) to few consistency checks showing that \( ||\Psi||^2 \) does not depend on the choices of the bases \( (\kappa^x) \) of \( H^0(L^2_\mathbb{R}) \) and \( (\eta_x^a) \) of \( H^0(L^2_K) \), and of the choice of a hermitian structure of \( L_0 \). We also show that upon multiplication of the Riemannian metric of \( \Sigma \) by a function \( e^\sigma \), \( ||\Psi||^2 \) picks up the factor

\[
\exp \left[ -\frac{i}{24 \pi} \frac{3k}{k+2} \int \left( \frac{1}{2} \partial \sigma \wedge \bar{\partial} \sigma + \sigma R \right) \right]
\]

(9.6)

which guarantees the right value \( c = \frac{3k}{k+2} \) of the Virasoro central charge of the theory with partition function given by Eq. (1.3).

We shall rewrite the scalar product formula putting it into a form both more geometric and closer to the spirit of discussion of Sect. 4. To this end consider

\[
\chi(\mathbf{x}; x_1, \ldots, x_{k(g-1)}) = \sum_{(\alpha_m)} \prod_{m=1}^{k(g-1)} \left( \eta^\alpha_{x_m} (x_m) \partial_{x_m}^\beta \right) \psi(\mathbf{x}, b).
\]

(9.7)
Clearly, the relation (3.18) holds so that $\chi$ is the holomorphic $k(g-1), 0$-form on $\Sigma \times \Sigma^k(g-1)$ with values in the h.l.b. $B_k$ of Eq. (3.19) discussed at the end of Sect. 3. It is essentially the same object as $\chi$ introduced by Eq. (1.7) in Sect. 4, directly related to Bertram’s picture $\Pi$ of CS states. The precise relation between the two $\chi$’s is given by Eq. (C.18) of Appendix C. With its use, one obtains from Eq. (9.7) a fully normalized formula for the scalar product which uses the description of CS states by polynomials $\psi'$ discussed in Sect. 4.

The expression
\[ e^{-2\pi k(f_{x_0}\omega)(\text{Im}\tau)^{-1}(f_{x_0}\omega)} |\Phi(x)|^2 \]  
(9.8)
with $\Phi$ as in Eq. (3.16) defines an admissible hermitian structure on $L^2K((2-2g)x_0)$, see the beginning of Sect. 5: it induces a connection with the curvature $2\pi \omega(\text{Im}\tau)^{-1}\wedge \bar{\omega} = -4\pi i g\alpha$. In order to find the geometric interpretation, of the other terms in the scalar product formula (9.3), let us return to the linear map (6.28)
\[ H^0(K) \ni \nu \xrightarrow{l(x,b)} \nu[b] \in H^1(L^{-2}K) \cong H^0(L^2)^* . \]  
(9.9)
Recall from Sect. 6.3 that surjectivity of $l(x,b)$ assured the local regularity of the projection from $A^{01}$ into the orbit space $A^{01}/G^\mathbb{C}$. We may view $\wedge^{-1}l(x,b)$ as a holomorphic $1,0$-form on $\Sigma$ with values in the bundle $\det^{-1}H^0(K) \otimes \det^{-1}R^0pr_1\mathcal{L}_0^2$ with the representation
\[ \wedge^{-1}l(x,b) = \sum_{j=1}^g (-1)^j \det(f \kappa_\omega^i \wedge b) \nu_j \omega^{i}(x) \otimes (\wedge \omega^{i})^{-1}(\wedge \kappa_\omega^i)^{-1} , \]  
(9.10)
compare the discussion after (5.3) in Sect. 5. Setting
\[ \phi(x; x_1, \ldots, x_{g-1}) \equiv \sum_{(\alpha_r)} \prod_{r=1}^{g-1} \eta_{\alpha_r}^r(x_r) \partial_{\zeta_{\alpha_r}} \wedge^{-1}l(x,b) \]
\[ = (g-1)! \mathcal{S}_{g-1} \left( \sum_{j=1}^g (-1)^j \det(\kappa_\omega^i(x_r) \omega^{i}(x_r)) \nu_j \omega^{i}(x) \right) \otimes (\wedge \omega^{i})^{-1}(\wedge \kappa_\omega^i)^{-1} , \]  
(9.11)
with $\mathcal{S}_{g-1}$ symmetrizing the variables $x_r$, we obtain a holomorphic $g, 0$-form on $\Sigma \times \Sigma^{g-1}$ with values in
\[ C_k \equiv \det^{-1}H^0(K) \otimes pr_1(\det^{-1}R^0pr_1\mathcal{L}_0^2) S^{g-1}(\mathcal{L}_0^2) \cong \det^{-1}H^0(K) \otimes pr_1(\det^{-1}R^0pr_1\mathcal{L}_0^2) S^{g-1}(\mathcal{L}_0^2) \equiv C'_k \]  
(9.12)
where the isomorphism of the h.l.b.’s on $\Sigma \times \Sigma^{g-1}$ $C'_k$ and $C_k$ is induced by the isomorphism (4.9). In the right hand realization, $\phi$ coincides with the one introduced by Eq. (5.5) in Sect. 5. Now, notice that
\[ \sum_{(\alpha_m)} \prod_{m=1}^{M} \left( \eta_{\alpha_m}^m(x_m) \partial_{\zeta_{\alpha_m}} \wedge^{-1}l(x,b) \psi(x,b) \right) \]
\[ = (g-1)! \mathcal{S}_M (\phi(x; x_1, \ldots, x_{g-1}) \chi(x; x_g, \ldots, x_M)) . \]  
(9.13)
Clearly, the right hand side is a holomorphic \((1 + M), 0\)-form on \(\Sigma \times \Sigma^M\) with values in the \(h.i.b.
\begin{align*}
det^{-1}H^0(K) \otimes pr_1^*((\det^{-1}R^0 pr_1^*L_0^2)(L^2K((2-2g)x_0))^kS^M(L_0^2)) \\
\equiv \det^{-1}H^0(K) \otimes pr_1^*((\det^{-1}R^0 pr_1^*L_0^2)(L^2K))^kS^M(L_0^2). \tag{9.14}
\end{align*}

It is easy to see that
\begin{align*}
det(\text{Im} \tau)^{-1}e^{-2\pi k(\int_{x_0}^{x} \omega)} \frac{1}{\text{Im} \tau} (\int_{x_0}^{x} \omega) \sum_{(\alpha_m)} \prod_{m=1}^{M} \frac{1}{(\eta_k^{\alpha_m}(x_m) \partial_{x_k}^{\alpha_m})} \sum_{j=1}^{g} (-1)^j \det(f \kappa_x^r \omega^j \wedge b)_{\eta \neq j} \\
\cdot \omega^j(x) \psi(x, b) \bigg|_{S^M(\phi(x; x_1, \ldots, x_{g-1}) \chi(x; x_g, \ldots, x_M))} \bigg|_{S^M(\phi(x; x_1, \ldots, x_{g-1}) \chi(x; x_g, \ldots, x_M))}^{\wedge 2} = (M_{g-1})^2 |S_M(\phi(x; x_1, \ldots, x_{g-1}) \chi(x; x_g, \ldots, x_M))|^{\wedge 2}, \tag{9.15}
\end{align*}

where, if we interpret \(S_M(\phi \chi)\) as a \((1 + M), 0\)-form on \(\Sigma \times \Sigma^M\) with values in the bundle on the left hand side of \((9.14)\), we should use on the latter the hermitian metric induced by the Riemannian metric of \(\Sigma\), the hermitian structure of \(L_0\) and an admissible hermitian structure of \(L^2K((2-2g)x_0)\). It will be then simpler to work only with the admissible hermitian metrics on all occurring line bundles, including the holomorphic tangent bundle whose hermitian structure is given by the Riemannian metric. With such choices, we may rewrite
\begin{align*}
\|\Psi\|^2 = \text{const.} \cdot \int \det'((\bar{\partial}_{L(-x)^2} \bar{\partial}_{L(-x)^2}) |S_M(\phi(x; x_1, \ldots, x_{g-1}) \chi(x; x_g, \ldots, x_M))|^{\wedge 2} \prod_{m_1 \neq m_2} e^{-\frac{4\pi}{x+\pi}G(x_{m_1}, x_{m_2})} \prod_{m} e^{-\frac{4\pi}{x+\pi}G(x_{m})}, \tag{9.16}
\end{align*}

including the prefactors into the normalization of the hermitian metric and replacing \(x\) by \(x\) in accordance with the interpretation of the \(x\)-dependence as giving rise to geometric objects on \(\Sigma\). The Green function \(G(x, y)\) in \((9.16)\) should be orthogonal to the 2-form \(\alpha \equiv \frac{1}{2g}(\text{Im} \tau)^{-1} \wedge \bar{\omega}\). We have rewritten \(\det'((\bar{\partial}_{L} \bar{\partial}_{L}) \equiv \det'((\bar{\partial}_{L(-x)^2} \bar{\partial}_{L(-x)^2}) \equiv \det((\bar{\partial}_{L} \bar{\partial}_{L})\) using the fact that the latter determinant is independent of the normalization of the hermitian structure on \(L(-x)\) so that it takes the same value for any admissible metric on \(L(-x)\).

When specified to the case of Arakelov metric on \(\Sigma\), this is exactly the expression \((9.7)\) of Sect. 5, if we reinterpret \(S_M(\phi \chi)\) according to the right hand side of \((9.14)\) and use the relation between the hermitian structures induced by isomorphism \((9.3)\), see the discussion in Appendix B.

Similarly as for the lower genus case, see \[13\], the natural conjecture is that the integral on the right hand side of Eq. \((9.16)\) converges if and only if the function \(\psi\) defines a globally non-singular CS state \(\Psi\). The singularities under the integral in Eq. \((9.16)\) come from the product
\begin{align*}
\prod_{m_1 \neq m_2} e^{-\frac{4\pi}{x+\pi}G(x_{m_1}, x_{m_2})} \sim \prod_{m_1 \neq m_2} d(x_{m_1}, x_{m_2})^{-\frac{2}{x+\pi}}. \tag{9.17}
\end{align*}

The power counting when \(Q + 1 \) of \(x_m\) converge shows that
\begin{align*}
S_M(\phi \chi)(x; y, y + y_1, \ldots, y + y_Q, x_{Q+2}, \ldots, x_M) \tag{9.18}
\end{align*}
has to have the vanishing Taylor expansion at zero in $y_1, \ldots, y_Q$ up to order $\leq Q + k$. This gives also a set of sufficient conditions for the convergence of the integral in (9.10). Notice that for $g = 2$ when $M = k + 1$ these conditions are always satisfied. For $g > 2$, taking $Q = k + 1$, we infer that if the integral converges then $S_M(\phi \chi)(x; x_1, \ldots, x_M)$ has to vanish whenever $k + 2$ of $x_m$'s coincide. Let us see that this condition is, indeed, satisfied for $\phi$ corresponding to CS states. As we have explained at the end of Sect. 4, such states give sections $\chi(x; x_g, \ldots, x_M)$ which vanish whenever $k + 1$ of $x_m$'s coincide. On the other hand,

$$
\sum_{j=1}^{g} (-1)^j \det(\kappa^r_x(x_r) \omega^j(x_r))_{i \neq j} \omega^j(x) = - \det \begin{pmatrix}
\kappa^1_x(x_1) \omega^j(x_1) \\
\vdots \\
\kappa^{g-1}_x(x_{g-1}) \omega^j(x_{g-1}) \\
\omega^j(x)
\end{pmatrix}
$$

(9.19)

vanishes whenever two of $x_r$'s coincide. Hence $S_M(\phi \chi)(x; x_1, \ldots, x_M)$ vanishes whenever $k + 2$ of $x_m$'s are equal. It is clear that a complete analysis of the convergence of the integral in Eq. (9.16) and of the related “fusion rule conditions” should be based on the geometry studied in [1][22] and we shall postpone it to a future work.

Appendix A

Let us show that, for $c$ a non-vanishing function on $\Sigma$, for $g_c = \begin{pmatrix} c^{-1} & 0 \\ 0 & c \end{pmatrix}$ and for $h_c = U g_c U^{-1}$,

$$
\exp[S(h_c, A_{0}^{10} + A_{0}^{01})] = e^{\frac{i}{2\pi} \int_{\Sigma} c^{-1} \partial c \wedge (c^{-1} \partial c + 2a_x) \nu(c) },
$$

(A.1)

where $\nu$ is given by Eq. (B.11). Recall, that $U$ is a smooth isomorphism of rank 2 vector bundles with trivial determinant, $U : L_0 \oplus L_0^{-1} \to \Sigma \times \mathbb{C}^2$. The gauge field $A_{0}^{10} + A_{0}^{01}$ represents the image under $U$ of the diagonal connection $\nabla + \bar{\partial} - \sigma^3 a_x$. First note that, by the gauge invariance of the WZW action, $S(h_c, A_{0}^{10} + A_{0}^{01})$ is independent of the choice of $U$. Moreover, since under an infinitesimal change of the field $h$,

$$
\delta S(h, A) = \frac{i}{2\pi} \int_{\Sigma} \text{tr} h^{-1} \delta h \ F(A_{10}^{40} + h^{-1} A_{0}^{01})
$$

(A.2)

which transforms covariantly, one infers that under small changes of the function $c$,

$$
\delta S(h_c, A_{0}^{10} + A_{0}^{01}) = \frac{i}{\pi} \int_{\Sigma} c^{-1} \delta c \ (\partial(c^{-1} \partial c) + \partial a_x + F_0),
$$

(A.3)

which is the variation of the right hand side of Eq. (A.1). We may then assume that $c = 1$ on a small disc $D$. Using a trivialization of the bundle $L_0$ over $D$ and over $\Sigma \setminus D$ with the
transition function $f$ defined around the boundary of $D$, we may take the isomorphism $U$ equal to the identity on $\Sigma \setminus D$ and interpolating smoothly \( \left( f^{-1}_0 \ 0 \right) \) inside $D$. In any case, $h_c = \left( c^{-1} \ 0 \ 0 \right) = g_c$ everywhere. Let $a_0$ be the 1-form representing on $\Sigma \setminus D$ the metric connection of $L_0$. It follows easily, that

\[
S(h_c, A_0^{10} + A_0^{01}) = S(g_c) - \frac{i}{\pi} \int_{\Sigma \setminus D} c^{-1} dc \wedge (a_0 + a_x) \tag{A.4}
\]

from which Eq. (A.3) follows by integration by parts on the cut surface.

Let us identify the flat bundle corresponding to the character $\Pi_1 \ni p \mapsto \nu(c_p) \in S^1$ of the fundamental group $\Pi_1$ of $\Sigma$. First note that $\nu(c_p)$ is independent of the choice of the metric on the h.l.b. $L_0$. Suppose that $L_0 = L(-x_0)$ has divisor $D \equiv \sum_{m=1}^{Q+g} y_m - \sum_{n=0}^{Q} x_n$ so that $L_0 \equiv O(D)$. Let us choose a hermitian metric on $O(D)$ so that $1(1(x)) = \exp[4\pi(\sum G(x, y_m) - \sum G(x, x_n))]$ where 1 is the canonical section of $O(D)$ with zeros at $y_m$ and poles at $x_n$ and $G(x, y)$ is a Green function of the Laplacian on $\Sigma$. 1 trivializes $O(D)$ on $\Sigma \setminus \{ y_m, x_n \}$ and the 1,0-form

\[
a = 4\pi \partial (\sum_{m=1}^{Q+g} G(\cdot, y_m) - \sum_{n=0}^{Q} G(\cdot, x_n)) \tag{A.5}
\]

represents there the metric connection of $O(D) \equiv L_0$. In particular, its curvature $F_0$ is equal to $\bar{\partial}\partial a$. Cutting $\epsilon$-balls around the points $y_m$ and $x_n$ (region $B_\epsilon$) and integrating by parts, we obtain

\[
\nu(c_p) \equiv e^{\frac{\pi}{\nu} \int_{\Sigma \setminus B_\epsilon} \ln c_p} \prod_{j=1}^{g} \left( W_{a_j}^{-1} \int_{B_\epsilon} c_p^{-1} dc_p W_{b_j} \right)
\]

\[
= \exp[\lim_{\epsilon \to 0} \int_{\Sigma \setminus B_\epsilon} (\partial a) \ln c_p] \prod_{j=1}^{g} \left( W_{a_j}^{-1} \int_{B_\epsilon} c_p^{-1} dc_p W_{b_j} \right)
= \exp\left[\lim_{\epsilon \to 0} \int_{\Sigma \setminus B_\epsilon} (a d \ln c_p - \int_{B_\epsilon} a d \ln c_p)\right]. \tag{A.6}
\]

Since $\bar{\partial}G(x, y) = \frac{1}{4\pi(x-y)} +$ a smooth function, the boundary term contributes

\[
\prod_{m=1}^{Q+g} c_p(y_m)^2 \prod_{n=0}^{Q} c_p(x_n)^{-2} \tag{A.7}
\]

whereas the volume term $\lim_{\epsilon \to 0} \int_{\Sigma \setminus B_\epsilon} a d \ln c_p$ may be shown to vanish by using Eq. (A.3) and integrating once more by parts ($c_p$ is harmonic). It is easy to see that the flat bundle corresponding to the character (A.7) of $\Pi_1$ is equivalent to the trivial bundle with the $\bar{\partial}$-operator

\[
\bar{\partial} = 2\pi \left( \sum_{m} \frac{y_m}{x_0} - \sum_{n} \frac{x_n}{x_0} \right) (\text{Im} \tau)^{-1} \omega
\]

and is isomorphic to $O(D - (g - 1)x_0)^2 \equiv L(-gx_0)^2$. 

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Appendix B

Consider a function \( f_{x_0} \) on \( \bar{\Sigma} \times \Sigma \) given by

\[
f_{x_0}(x, y) \equiv e^{2\pi i(\int_{x_0}^x \omega(\text{Im} \tau)^{-1}(\int_{x_0}^y \omega) \, \vartheta(-a + \int_{x_0}^x \omega | \tau) \, \vartheta(a + \int_{x_0}^y \omega | \tau)) \, \vartheta(a + \int_{x_0}^y \omega | \tau)}, \tag{B.1}
\]

where \( a \) is an odd characteristic. One has

\[
f_{x_0}(px, y) = c_p(y)^{-1} f_{x_0}(x, y), \quad (\bar{\vartheta} + \pi(\text{Im} \tau)^{-1} \bar{\omega}(y)) f_{x_0}(x, y) = 0.
\]

Besides, \( f_{x_0} \) has first order zeros at \( x = x_0 \) and \( y = x_0 \) and a first order pole at \( x = y \). It follows that multiplication by \( f_{x_0} \) establishes an isomorphism between the h.l.b.'s \( \mathcal{L}_0' \) and \( pr_1^*(\mathcal{O}(-x_0)) \mathcal{L}_0 \) over \( \Sigma \times \Sigma \).

Note that the hermitian structure on the h.l.b. \( \mathcal{L}_0 \) coming from an admissible hermitian metric on the bundle \( L_0 \) (see the beginning of Sect. 5 for the definition of admissibility) induces the connection with curvature

\[
\pi pr_1^*(\omega)(\text{Im} \tau)^{-1} pr_2^*(\bar{\omega}) - \pi pr_1^*(\bar{\omega})(\text{Im} \tau)^{-1} pr_2^*(\omega) + (g - 1) pr_2^* \alpha.
\]

Taking also an admissible hermitian structure on the bundle \( \mathcal{O}(-x_0) \) we obtain a hermitian metric on the h.l.b. \( pr_1^*(\mathcal{O}(-x_0))\mathcal{L}_0 \) corresponding to the curvature form

\[
- pr_1^* \alpha + \pi pr_1^*(\omega)(\text{Im} \tau)^{-1} pr_2^*(\bar{\omega}) - \pi pr_1^*(\bar{\omega})(\text{Im} \tau)^{-1} pr_2^*(\omega) + (g - 1) pr_2^* \alpha, \tag{B.2}
\]

the same as the curvature induced by the hermitian structure of \( \mathcal{L}_0' \) described around Eq. \((5.2)\) in Sect. 5. Hence multiplication by \( f_{x_0} \) must carry one hermitian structure into the other one, up to a constant factor.

Appendix C

Let us discuss in more detail the relation between the two descriptions of the CS states: the one discussed in Sect. 3 using functions \( \psi(x, b) \) with \( x \in \bar{\Sigma}, \ b \in \wedge^{01}(L_0^{-2}) \) and the one of \( \bar{\Sigma} \), discussed in Sect. 4, employing polynomials \( \psi'(b') \), \( b' \in \wedge^{01}(L^{-2}) \). Let us fix \( x \) with \( x \neq x_0 \) and \( b \). Viewing \( b \) as an element of \( \wedge^{01}(L_x^{-2}) \), we may define a form \( b'' \in \wedge^{01}(L(-x)^{-2}) \) by setting

\[
b'' \equiv f_{x_0}(x, \cdot)^2 b \tag{C.1}
\]

since the multiplication by \( f_{x_0}(x, \cdot) \) given by Eq. \((B.1)\) establishes an isomorphism between \( L(-x) \) and \( L_x \). We shall choose \( \kappa \in H^0(L^2(-x)) \setminus H^0(L(-x)^2) \) s.t.

\[
\int_{\Sigma} \kappa \nu \wedge b'' = 0 \quad \text{for} \quad \nu \in H^0(K(-x)) \tag{C.2}
\]
Except on a subset of \( b'' \)'s of codimension at least 2 in \( H^1(L(-x)^{-2}) \), such \( \kappa \) exists and is unique up to normalization (compare the discussion around \( (5.28) \) in Sect. 6.3). Eq. \((C.2)\) guarantees that there exists a function \( f_1 \in \Gamma(\mathcal{O}(x)) \) s. t.

\[
\bar{\partial} f_1 = b \kappa .
\]

Let \( f_t = 1 + tf_1 \in \Gamma(\mathcal{O}(x)) \) for \( t \in \mathbb{C} \). Since, by our assumptions, \( \kappa(x) \neq 0 \) as an element of \( L^2(-x) \) and, for small \( t \), \( f_t \) may have zeros only close to \( x \), it follows that, for such \( t \), the map

\[
L^{-1} \ni l \mapsto (f_t, l\kappa) \in L(-x)^{-1} \oplus L(-x)
\]

is an embedding which, moreover, is holomorphic if we modify the \( \bar{\partial} \) operator of \( L(-x)^{-1} \oplus L(-x) \) by replacing it by \( \bar{\partial} + (0 \quad iv') \). Choose now \( \zeta \in \Gamma(\mathcal{O}(-x)) \) and \( s \in \Gamma(L^{-2}(x)) \) s. t.

\[
\zeta - s\kappa = 1 .
\]

Note that \( s(x) \) has to be a non-vanishing element of \( L^{-2}(x) \) because otherwise Eq. \((C.5)\) could not be satisfied as \( \zeta \), viewed as a function on \( \Sigma \), vanishes at \( x \). Now, We shall perturb \( \zeta \) and \( s \) by taking \( \zeta_t \in \Gamma(\mathcal{O}(x)) \) and \( s_t \in \Gamma(L^{-2}(x)) \) s. t.

\[
\zeta_t f_t - s_t \kappa = 1
\]

and \( \zeta_t = \zeta + t\zeta_1 + o(t), \ s_t = s + ts_1 + o(t) \) are analytic in (small) \( t \). This may be easily achieved by solving Eq. \((C.6)\) for \( \zeta_t \) with \( s_t = s \) outside a small ball \( B_t(x) \) around \( x \) and for \( s_t \) with \( \zeta_t = \zeta \) on \( B_2(x) \) and by interpolating between the two solutions in \( B_2(x) \setminus B_t(x) \). Consider now the smooth isomorphism

\[
V_t : L^{-1} \oplus L \longrightarrow L(-x)^{-1} \oplus L(-x) , \quad V_t = \begin{pmatrix} f_t & s_t \\ \kappa & \zeta_t \end{pmatrix}
\]

depending analytically on (small) \( t \). A straightforward computation shows that

\[
V_t^{-1} \begin{pmatrix} 0 & tv' \\ 0 & 0 \end{pmatrix} V_t + V_t^{-1} \bar{\partial} V_t = \begin{pmatrix} 0 & t\zeta_1^2 b'' + \zeta_1 \bar{\partial} s_t - s_t \bar{\partial} \zeta_t \\ 0 & 0 \end{pmatrix} .
\]

Notice that \( t\zeta_1^2 b'' + \zeta_1 \bar{\partial} s_t - s_t \bar{\partial} \zeta_t \equiv b'_0 + tb'_1 + o(t) \in \Lambda^0(L^{-2}) \). In particular,

\[
b'_0 = \zeta \bar{\partial} s - s \bar{\partial} \zeta = (1 + s\kappa) \bar{\partial} s - s \bar{\partial}(s\kappa) = \bar{\partial} s ,
\]

\[
b'_1 = \zeta_1^2 b'' + \zeta_1 \bar{\partial} s_1 + \zeta_1 \bar{\partial} s - s \bar{\partial} \zeta_1 - s_1 \bar{\partial} \zeta = b'' - \bar{\partial}(f_1s) + \bar{\partial}s_1 ,
\]

where we have used the relations \( \zeta = 1 + s\kappa \) and \( \zeta_1 = -\zeta f_1 + s_1 \kappa \) following from Eqs. \((C.5)\) and \((C.6)\). Let us define two gauge fields

\[
A_{x, tb}^{01} = U \begin{pmatrix} 0 & tb \\ 0 & 0 \end{pmatrix} U^{-1} + U \bar{\partial} U^{-1} ,
\]

\[
A_{b'_t}^{01} = U' \begin{pmatrix} 0 & b'_1 \\ 0 & 0 \end{pmatrix} U'^{-1} + U' \bar{\partial} U'^{-1} ,
\]
where \( U : L_{0}^{-1} \oplus L_{0} \rightarrow \Sigma \times \mathbb{C}^{2} \) and \( U' : L_{0}^{-1} \oplus L \rightarrow \Sigma \times \mathbb{C}^{2} \) are smooth isometric isomorphisms, see Sects. 3 and 4. \( A_{x,\theta}^{01} \) and \( A_{b_{t}}^{01} \) are gauge related:

\[
A_{b_{t}}^{01} = h_{t}^{-1} A_{x,\theta}^{01} h_{t} + h_{t}^{-1} \partial h_{t},
\]

where

\[
h_{t} = U \begin{pmatrix} f_{x_{0}}(x_{\cdot})^{-1} & 0 \\ 0 & f_{x_{0}}(x_{\cdot}) \end{pmatrix} V_{t} U'^{-1}.
\]

Expressing the same CS state \( \Psi \) in two descriptions corresponding to Eq. (4.3) and Eq. (B.4) and comparing them using the gauge invariance of \( \Psi \), we obtain the relation

\[
\psi'(b_{t}') = \exp[k S(h_{t}, A_{x,\theta}^{01}) + \frac{ik}{2\pi} \int_{\Sigma} \text{tr} (A_{0}^{10} \wedge A_{b_{t}}^{01} - A_{0}^{10} \wedge A_{x,\theta}^{01})] \psi(x, \theta b).
\]

Now, due to the homogeneity of \( \psi(x, \cdot) \),

\[
\psi(x, \theta b) = t^{k(g-1)} \psi(x, b).
\]

On the other side, it is easy to see with the use of Eq. (C.9) that, for \( \eta' \in H^{0}(L^{2}K) \),

\[
\int_{\Sigma} \eta' \wedge b_{0}' = \int_{\Sigma} \eta' \wedge \partial s = \lim_{t \rightarrow 0} \int_{\Sigma_{\eta, b_{0}(x)}} \eta' \wedge \partial s = \lim_{t \rightarrow 0} \int_{\partial B_{\eta}(x)} \eta' = 2\pi i \frac{\eta'(x)}{d(s^{-1})(x)/dx},
\]

where \( s^{-1} \) is differentiated as a section of \( L^{2} \) vanishing at \( x \). Hence the class of \( b_{0}' \) in \( \mathbb{P} H^{1}(L^{2}) \) coincides with the image of \( x \in \Sigma \) under the embedding \( (1.12) \) of \( \Sigma \) into \( \mathbb{P} H^{1}(L^{2}) \). Specifying Eq. (C.8) to \( t = 0 \), we infer that the corresponding rank 2 holomorphic bundle is isomorphic by \( V_{0} \) with the split bundle \( L(-x)^{-1} \oplus L(-x) \). Using the integral presentation (4.3) of \( \psi' \) and Theorem 2a of [1], see Sect. 4, we obtain the relation

\[
\psi'(b_{t}') = \frac{(2\pi i)^{k}}{k!} t^{k(g-1)} \left( \frac{d(s^{-1})(x)}{dx} \right)^{k} \int_{\Sigma^{k}(g-1)} \chi'(x_{1}, \ldots, x_{k}, x_{k+1}, \ldots, x_{k})
\]

\[
\cdot b_{1}'(x_{k+1}) \cdots b_{1}'(x_{k}) + o(t^{k(g-1)}).
\]

Besides, using Eq. (C.10), we may replace \( b_{t}' \) by \( b'' = f_{x_{0}}(x, \cdot)^{2} b \) on the right hand side. Hence the relations (C.14), (C.15) and (C.17) imply that

\[
\chi'(x, \ldots, x; x_{k+1}, \ldots, x_{k}) = \mathcal{U}_{x_{0}}(x) \chi(x; x_{k+1}, \ldots, x_{k})
\]

\[
\cdot f_{x_{0}}(x, x_{k+1})^{-2} \cdots f_{x_{0}}(x, x_{k})^{-2},
\]

where

\[
\mathcal{U}_{x_{0}}(x) \equiv \frac{k!}{(2\pi i)^{k}} \left( \frac{d(s^{-1})(x)}{dx} \right)^{k} \exp[k S(h_{0}, A_{x,\theta}^{01}) + \frac{ik}{2\pi} \int_{\Sigma} \text{tr} (A_{0}^{10} \wedge A_{b_{0}}^{01} - A_{0}^{10} \wedge A_{x,\theta}^{01})].
\]

\( \mathcal{U}_{x_{0}}(x) \) takes values in \( (L^{2}K)^{k} \). It must be independent of the choice of \( \kappa, \zeta \) and \( s \) since the other terms in Eq. (C.18) are. It gives an explicit realization of the isomorphism between functions \( \Phi(x) \) on \( \Sigma \) transforming by Eq. (3.19) and behaving like \( (x-x_{0})^{-2k(g-1)} \) around \( x_{0} \) and sections of \( (L^{2}K)^{k} \), see Appendix A. Such isomorphism is unique up to
normalization and Eq. (C.19) fixes this normalization completely (in a way dependent
on the isometric isomorphisms \( U \) and \( U' \)). Eq. (C.18) establishes the precise relation
between functions \( \psi \) used to represent the CS states in this paper and polynomials \( \psi' \)
introduced in Sect. 4 and corresponding to the description of [22].

Appendix D

We shall prove here the formula (7.23) for the zeta-function regularized determinant
of the operator \( \bar{D}_n^\dagger \bar{D}_n \). Let us consider the determinant line bundle \( \mathcal{F} \) of the \( \bar{\partial} \)-family
\( (\bar{\partial} + [A^{01}_{x,b}, \cdot] \equiv \bar{D}_n) \) of operators acting in the trivial bundle \( \Sigma \times sl(2, \mathbb{C}) \). It is a
holomorphic line bundle over the space of pairs \((x, b)\) with the fibers
\[
\det(\ker(\bar{D}_n))^{-1} \det(\coker(\bar{D}_n)).
\]
Generically, \( \ker(\bar{D}_n) = 0 \) and the dual space to the \( \coker(\bar{D}_n) \) is spanned by the \( sl(2, \mathbb{C}) \)-valued 1,0-forms \( \omega^\alpha(x, b) \equiv U (\frac{\mu^\alpha}{\eta^\alpha} \lambda^\alpha) U^{-1} \) constructed in Sect. 5.3., see formulae (6.30) and (6.31). The complex gauge transformations \( h_{c,v} \equiv Ug_{c,v}U^{-1} \), with \( g_{c,v} \) as in
Eq. (2.13), \( c \) a non-zero constant or \( c = c_p \), act on \( \mathcal{F} \). Division by their action gives
the 4th power (\( 4 = 2 \times \) the dual Coxeter number of \( SU(2) \)) of the h.l.b. DET over the
compact space \( \mathbb{P}W_0 \) discussed in Sect. 3. The formula
\[
\|1 \otimes \omega^\alpha(x, b)\|^2 = \det(\bar{D}_n^\dagger \bar{D}_n) \det(\Omega(1, n))^{-1}
\]
defines Quillen’s hermitian metric [33] on \( \mathcal{F} \). Its curvature is easily calculable (from the
Riemann-Roch-Grothendick Theorem, see e.g. [14]) to be
\[
\frac{2}{\pi i} \int_{\Sigma} \text{tr} (\delta A^{01}_{x,b})^\dagger \wedge \delta A^{01}_{x,b} = \frac{2}{\pi} \int_{\Sigma} (2\bar{\delta a}_x \wedge \delta a_x + \langle \delta b, \wedge \delta b \rangle).
\]
The change of the Quillen metric on \( \mathcal{F} \) under the complex gauge transformations \( h_{c,v} \)
may be inferred from the chiral anomaly formula (6.10) with \( S(h_{c,v}h_{c,v}^\dagger, A(n)) \) given by
Eqs. (6.18) and (6.19). It is then easy to see that the modified metric
\[
\| \cdot \|^2 = \| \|^2 e^{\frac{2i}{\pi} \int_{\Sigma} \langle b, \wedge b \rangle}
\]
is invariant under the \( h_{c,v} \) transformations and descends to \( \text{DET}^4 \). Its curvature is
\[
\frac{1}{\pi i} \int_{\Sigma} \bar{\delta a}_x \wedge \delta a_x.\] On the other hand, the right hand side of Eq. (7.24) multiplied by
\( \exp\frac{2}{\pi} \int_{\Sigma} \langle b, \wedge b \rangle \) also defines a hermitian structure on the h.l.b. \( \text{DET}^4 \) and the Riemann-
Roch-Grothendick Theorem shows that the curvatures agree. Hence, the two metrics are
proportional with the proportionality constant which might a priori depend on the metric
(i.e. also on the complex structure) of \( \Sigma \).

In order to see that Eq. (7.23) for \( \det(\bar{D}_n^\dagger \bar{D}_n) \) represents properly also the dependence
on the metric of \( \Sigma \), it is enough to show that it produces the right behavior of the
On the other hand, that, when (in the notation of Eq. (6.30). Using also the relation (6.34), we obtain
\[
\det(\bar{D}_n^t \bar{D}_n) = R t^{2g} \det'((\bar{D}_n^t \bar{D}_n)|_{t=0} + o(t^{2g}).
\] (D.5)

in the leading nontrivial order (we have assumed that \(z^\alpha(b) = 0\) for \(\alpha > 1\)). It follows that, when \(t \to 0\) (and with the zeta-functions regularized determinants),
\[
\lim_{t \to 0} \det(\Omega(1, x, tb)) = \frac{2}{\pi} (f \mu_1 \wedge \mu_1) \det_{\alpha, \beta \neq 1} (H_0^{\alpha\beta})
\] (D.6)
in the notation of Eq. (7.30). Using also the relation (6.34), we obtain
\[
\lim_{t \to 0} \det(\Omega(1, x, tb)) = 4 |\det(f \kappa^\omega \wedge b\{h<g\})|^2
\]
\[
\cdot \sum_{\iota', \iota''} (-1)^{\iota' + \iota''} \det(f \kappa^\omega \wedge b\{h<\iota', \iota''\}) \det(f \kappa^{\omega_2} \wedge b\{h<\iota, \iota''\}) \det(H_0) (H_0^{-1})_{11}
\]
\[
= 4 \det(\Im \tau) \det(H_0) (H_0^{-1})_{11} \left|\det(f \kappa^\omega \wedge b\{h<\iota\})\right|^{-2}
\]
\[
\cdot \det((f \kappa^\omega \wedge b\{h<\iota\}) \Im \tau)^{-1}_ij \det(f \kappa^\omega \wedge b\{h<\iota\})
\] (D.7)

where the last equality is a consequence of the identity \(\det(\sum_j A^\iota_j A^\j) = \sum_j |\det(A^\iota_j)|^2\) for \((A^\iota_j)\) a \((g-1) \times g\) matrix which may be easily verified by taking \((A^\iota_j)\) with first \((g-1)\) columns forming a unit matrix. It follows now from Eq. (5.37) that
\[
\lim_{t \to 0} t^{-2g} \left(\int \delta(BV) e^{-\|CV\|^2} D V\right)^{-1} = 4 |z^1|^2 (H_0^{-1})_{11}
\]
\[
\cdot \det((f \kappa^\omega \wedge b\{h<\iota\}) \Im \tau)^{-1}_ij \det(f \kappa^{\omega_2} \wedge b\{h<\iota\})
\] (D.8)

and, consequently, the right hand side of Eq. (7.23) behaves when \(t \to 0\) in accordance with (D.3). This ends the proof of formula (7.23).

Appendix E

Let us denote
\[
\mathcal{P}(\varphi) \equiv \det(H_\varphi)^{-1} |z^1|^2 \int_{\mathbb{C}^N-1} |P(z)|^2 e^{-\frac{4\pi}{\varphi} z^\alpha(H_\varphi^{-1})_{\beta\alpha} z^\beta} \prod_{\alpha=2}^N d^2 z^\alpha
\]
where \( P(z) \) is a homogeneous polynomial in variables \((z^\alpha)\) of degree \(M \equiv (k+1)(g-1)\):

\[
P(z) = \sum_{(\alpha_m)} \frac{1}{M!} \left( \prod_{m=1}^{M} \partial_{z^{\alpha_m}} P(z) \right) \prod_{m=1}^{M} z^{\alpha_m} = \sum_{(\alpha_m)} P_{(\alpha_m)} \prod_{m=1}^{M} z^{\alpha_m}. \tag{E.2}
\]

\( \mathcal{P}(\phi) \) with \( P(z) \) given by Eq. (E.3) is the \( z \)-integral to be computed after the \( w \)-integration in Sect. 7 if we postpone the \( \phi \) integrals in (E.1) clearly converge. We shall show that the zero-mode integral

\[
\int_{\mathbb{R}} e^{-2aM} \mathcal{P}(\varphi + a) \ da = \frac{1}{2\pi} \frac{M-1}{\frac{k+4}{k+1}} \sum_{(\alpha_m), (\beta_m)} \mathcal{P}_{(\alpha_m)} P_{(\beta_m)} \prod_{m=1}^{M} H^\alpha_{\beta_m}. \tag{E.3}
\]

Consequently, integrating in our calculation of the right hand side of (9.12) first over \( w \) then over \( z^\alpha, \alpha > 1 \) and at the end over \( \varphi \) one obtains the expression (9.5) without encountering other infinities then the standard ones removed by the zeta-function regularization of the determinants and the Wick ordering of the \( \varphi \)-field exponentials. In order to prove formula (E.3), let us rewrite

\[
\int_{\mathbb{R}} e^{-2aM} \mathcal{P}(\varphi + a) \ da = \frac{1}{2\pi} \int_{\mathbb{C}} |t|^{2(M-1)} \mathcal{P}(\varphi - \ln |t|) \ dv t
\]

\[
= \frac{1}{2\pi} \det (H_\varphi)^{-1} |z|^2 \prod_{\alpha=2}^N d^2 z^\alpha. \tag{E.4}
\]

The integral clearly converges. By the change of variables

\[
\zeta^1 = tz^1, \quad \zeta^2 = tz^2, \quad \ldots, \quad \zeta^N = tz^N, \tag{E.5}
\]

one obtains

\[
\int_{\mathbb{R}} e^{-2aM} \mathcal{P}(\varphi + a) \ da = \frac{1}{2\pi} \det (H_\varphi)^{-1} \prod_{\alpha=1}^N d^2 \zeta^\alpha. \tag{E.6}
\]

Appendix F

We shall check the consistency of the formula (9.3) for the scalar product of the CS states. First of all, the integrand on the right hand side is independent of the choice of the bases \((\kappa_\chi^L)\) of \( H^0(L_\chi K) \) and \((\eta_\chi^L)\) of \( H^0(L_\chi^L K) \). Indeed, a change of \((\kappa_\chi^L)\) in
\[ \sum_{j=1}^{g} (-1)^j \det(f^j, \kappa^j \omega^j \land b)_{\psi, j} \omega^j(x) \wedge^2 \] is compensated by that in \( \det(f^j, \kappa^j \omega^j \land b)_{\psi, j} \omega^j(x) \) and \( \eta^2_\Sigma \partial_{\Sigma^2} \) is independent of the choice of \( (\eta^2_\Sigma) \). Another consistency check is the independence of the integrand under change of the hermitian structure of \( L_0 \). Recall, that the the Green function \( G(x, y) \) of the Laplacian was chosen so that \( \int_{L_0} G(\cdot, y) ((k + 2)F_0(y) + \frac{1}{k+2}R(y)) = 0 \), where \( F_0 \) is the curvature form of \( L_0 \). The multiplication of the hermitian metric of \( L_0 \) by a positive function \( e^{\varphi/(k+2)} \) leads to the replacement \( F_0 \mapsto F_0 + \frac{\partial \varphi}{k+2} \) and

\[
G(x, y) \mapsto G(x, y) + \frac{1}{4\pi M}(\varphi(x) + \varphi(y))
- \frac{i}{8\pi^2 M^2} \int_{\Sigma} (\partial \varphi \land \bar{\partial} \varphi + \varphi(2(k + 2)F_0 + \frac{1}{2}R)) \ .
\]

(F.1)

Since \( \langle \eta^\alpha_m, \eta^\beta_m \rangle(x_m) \) picks up the factor \( e^{2\varphi(x_m)/(k+2)} \), the last line of Eq. (9.5) is multiplied by

\[
\exp\left[ \frac{i}{2\pi(k + 2)} \int_{\Sigma} (\partial \varphi \land \bar{\partial} \varphi + \varphi(2(k + 2)F_0 + \frac{1}{2}R)) \right].
\]

By the chiral anomaly formula, \( \det(f^j, \kappa^j \omega^j \land b)_{\psi, j} \omega^j(x) \) changes by the factor

\[
\exp\left[ -\frac{i}{\pi(k + 2)^2} \int_{\Sigma} (\partial \varphi \land \bar{\partial} \varphi + (k + 2)\varphi(2F_0 + \frac{1}{2}R)) \right].
\]

Finally, we shall show that \( \exp\left[ -\frac{ik}{2\pi} \int_{\Sigma} \text{tr} A_{00} \wedge A_{00} \right] |\psi(x, b)|^2 \), which depends on the metric of \( L_0 \) through the smooth isometric isomorphism \( U \) \( : \) \( L_0^{-1} \oplus L_0 \rightarrow \Sigma \times \mathbb{C}^2 \), yields upon the multiplication of the metric by \( e^{\varphi/(k+2)} \) the factor

\[
\exp\left[ -\frac{ik}{2\pi} \int_{\Sigma} \text{tr} A_{00} \wedge A_{00} \right] |\psi(x, b)|^2
\]
cancelling the previous ones. First note that, by Eq. (6.17),

\[
e^{-\frac{i}{2\pi} \int_{\Sigma} \text{tr} A_{00} \wedge A_{00}} |\psi(x, b)|^2 = c(x) \left| \Psi(A_{00}) \right|^2 e^{-\frac{ik}{2\pi} \int_{\Sigma} \text{tr} (A_{00}^k \wedge A_{00}^k + \frac{ik}{2\pi} \int_{\Sigma} (b \wedge b))}, \]

(F.2)

with \( c(x) \) independent of the choice of the metric on \( L_0 \). As follows from the transformation properties of \( \Psi \) and of the WZW action \( S \),

\[
\left| \Psi(h^{-1}A_{01}) \right|^2 e^{-\frac{i}{2\pi} \int_{\Sigma} \text{tr} (h^{-1}A_{01}^t \wedge h^{-1}A_{01})}
= e^{kS(hh_1, -(A_{01}^t) + A_{01})} \left| \Psi(A_{01}) \right|^2 e^{-\frac{ik}{2\pi} \int_{\Sigma} \text{tr} (A_{01}^t \wedge A_{01})},
\]

(F.3)

In particular, the expression does not change if \( h \) takes values in the compact group. It follows that, for a fixed metric on \( L_0 \), \( \exp\left[ -\frac{ik}{2\pi} \int_{\Sigma} \text{tr} A_{00} \wedge A_{00} \right] |\psi(x, b)|^2 \) is independent of the choice of the isometric isomorphism \( U \) \( : \) \( L_0^{-1} \oplus L_0 \rightarrow \Sigma \times \mathbb{C}^2 \) since \( A_{00}^b \) for two different choices are related by an \( SU(2) \)-valued gauge transformation.

After the multiplication of the hermitian structure on \( L_0 \) by \( e^{\varphi} \), we may take \( U e^{-\varphi \sigma^3 / 2} \) as the new isometric isomorphism from \( L_0^{-1} \oplus L_0 \) to \( \Sigma \oplus \mathbb{C}^2 \). As a result of the change
of \( U, A^{01}_{x,b} \) changes to \( h^{-1}A^{01}_{x,b} \) with \( h = Ue^{\varphi/2}U^{-1} \). By virtue of Eqs. (F.2) and (F.3),
e\int_\Sigma \tr A^0_{\lambda} \ A^{01}_{x,b} |\psi(x, b)|^2 \) is then multiplied by
\[
e^{kS(h h^\dagger, -(A^{01}_{x,b})^\dagger + A^{01}_{x,b}) - \frac{i}{2\pi} \int_\Sigma (e^{-2\varphi} - 1)(b, \wedge b). \tag{F.4}
\]
We are left with the calculation of \( S(h h^\dagger, -(A^{01}_{x,b})^\dagger + A^{01}_{x,b}) \). By Eq. (A.2), under the infinitesimal change of \( \varphi \)
\[
\delta S(h h^\dagger, -(A^{01}_{x,b})^\dagger + A^{01}_{x,b}) = \frac{i}{2\pi} \int_\Sigma tr U \varphi \sigma^3 U^{-1} F(-(A^{01}_{x,b})^\dagger + (h h^\dagger)^{-1}A^{01}_{x,b})
\]
\[
= \frac{i}{2\pi} \int_\Sigma tr \varphi \sigma^3 (\nabla + \partial + (a_x - a_x + \partial \varphi) e^{-2\varphi} b)
\]
\[
= \frac{i}{2\pi} \int_\Sigma tr \varphi \sigma^3 \left( -F_0 + \partial \varphi - e^{-2\varphi} (b, \wedge b) \nabla (e^{-2\varphi} b) + 2e^{-2\varphi} a_x \wedge b \right) F_0 - \partial \varphi + e^{-2\varphi} (b, \wedge b)
\]
\[
= \delta \left( -\frac{i}{2\pi} \int_\Sigma (\partial \varphi \wedge \partial \varphi + 2\varphi F_0 - (e^{-2\varphi} - 1)(b, \wedge b)) \right) \tag{F.5}
\]
so that the factor (F.4) becomes \( e^{-\frac{i}{2\pi} \int_\Sigma (\partial \varphi \wedge \partial \varphi + 2\varphi F_0)} \), as required.

As for the change of \( \psi(x, b) \) itself, a straightforward calculation shows that if \( h : \Sigma \to SL(2, \mathbb{C}) \) then
\[
e^{\frac{i}{2\pi} \int_\Sigma \tr h A^0_{\lambda} A^{01}_{x,b} \psi(h^{-1}A^{01}_{x,b})} = \left( e^{\frac{i}{2\pi} \int_\Sigma \tr A^0_{\lambda} (h h^\dagger)^{-1} \delta h^\dagger} \right) \left( e^{kS(h h^\dagger, A^{01}_{x,b}) - kS(h h^\dagger) + kS(h) - \frac{i}{2\pi} \int_\Sigma \tr A^0_{\lambda} (h h^\dagger)^{-1} \delta h^\dagger} \right) \psi(A^{01}_{x,b}) \tag{F.6}
\]
It follows that if \( U \mapsto U' = h^{-1}U \) with \( SU(2) \)-valued \( h \) then \( \psi(x, b) \) changes only by a constant factor \( e^{-kS(h) - \frac{i}{2\pi} \int_\Sigma \tr A^0_{\lambda} (h h^\dagger)^{-1} \delta h^\dagger} \). On the other hand, if the metric of \( L_0 \) is multiplied by \( e^\varphi \) and \( U \mapsto U' = U e^{-\varphi/2} \equiv h^{-1}U \), then \( \psi(x, b) \) picks up the factor
\[
e^{-kS(h h^\dagger, A^{01}_{x,b}) - kS(h h^\dagger) + kS(h) - \frac{i}{2\pi} \int_\Sigma \tr A^0_{\lambda} (h h^\dagger)^{-1} \delta h^\dagger} \tag{F.7}
\]
where the last inequality may be checked by differentiating \( S(h h^\dagger, A^{01}_{x,b}) \) with respect to \( \varphi \) with the use of Eq. (A.2). Again \( \psi(x, b) \) is multiplied by a constant independent of \( x \) and \( b \).

In the next check, let us find the dependence of the right hand side of Eq. (15) on the conformal factor of the Riemannian metric of \( \Sigma \). If the metric is multiplied by a positive function \( e^\sigma \), then
\[
G(x, y) \mapsto G(x, y) + \frac{1}{8\pi M} (\sigma(x) + \sigma(y))
\]
\[
- \frac{i}{32\pi^2 M^2} \int_\Sigma \tr (\partial \sigma \wedge \partial \sigma + 2\sigma(2(k + 2)F_0 + R)), \tag{F.8}
\]
\[
:G(x, x) : \mapsto :G(x, x) : - \frac{M-1}{4\pi M} (\sigma(x) + \sigma(y))
\]
\[
- \frac{i}{32\pi^2 M^2} \int_\Sigma \tr (\partial \sigma \wedge \partial \sigma + 2\sigma(2(k + 2)F_0 + R)),
\]

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and the last line of Eq. (9.5) is multiplied by
\[ \exp\left[ \frac{i}{4\pi(k+2)} \int_{\Sigma} \left( \frac{1}{2} \partial \sigma \wedge \bar{\partial} \sigma + \sigma (2(k+2)F_0 + R) \right) \right]. \]

In virtue of the conformal anomaly formula, \( \det(f_\Sigma(\kappa^r, \kappa^s) \text{vol})^{-1} \) \( \det'(\bar{\partial}_{\Lambda_2}^\dagger \partial_{\Lambda_2}) \) and \( \left( \frac{\text{det}'(-\Delta)}{\text{area}} \right)^{1/2} \) change, respectively, by the factors
\[ \exp[- \frac{i}{12\pi} \int_{\Sigma} \left( \frac{1}{2} \partial \sigma \wedge \bar{\partial} \sigma + \sigma (6F_0 + R) \right)] \quad \text{and} \quad \exp[- \frac{i}{24\pi} \int_{\Sigma} \left( \frac{1}{2} \partial \sigma \wedge \bar{\partial} \sigma + \sigma R \right)]. \]

Altogether, Eq. (9.5) picks the factor
\[ \exp[- \frac{i}{24\pi} \frac{3k}{k+2} \int_{\Sigma} \left( \frac{1}{2} \partial \sigma \wedge \bar{\partial} \sigma + \sigma R \right)] \quad (F.9) \]
when the Riemannian metric is multiplied by \( e^\sigma \). This guarantees the right value \( c = \frac{3k}{k+2} \) of the Virasoro central charge of the theory with partition function given by the formula (1.5).

Another easy check of formula (9.5) shows that its right hand side does not depend on the choice of \( x_0 \) used to fix the bundle \( L_0 = L(-x) \). We leave it to the reader. A more involved problem which we have not addressed is the independence of the scalar product expression of the choice of the h.l.b. \( L \) of degree \( g \).

References

[1] A. Bertram: Moduli of Rank-2 Vector Bundles, Theta Divisors and the Geometry of Curves in Projective Space. J. Diff. Geom. 35, 429-469 (1992)

[2] E. Witten: Quantum Field Theory and Jones Polynomial, Commun. Math. Phys. 121, 351-399 (1989)

[3] E. Witten: Non-Abelian Bosonization in Two Dimensions. Commun. Math. Phys. 92, 455-472 (1984)

[4] C. S. Seshadri: Space of Unitary Vector Bundles on a Compact Riemann Surface. Ann. of Math. 85, 303-336 (1967)

[5] M. S. Narasimhan, S. Ramanan: Moduli of Vector Bundles on a Compact Riemann Surface. Ann. Math. 89, 19-51 (1969)

[6] E. Verlinde: Fusion rules and modular transformations in 2-d conformal field theory. Nucl. Phys. B 300, 360-376 (1988)
[7] D. Bernard: *On the Wess-Zumino-Witten Models on Riemann Surfaces*. Nucl. Phys. **B 309**, 145-174 (1988)

[8] N. Hitchin: *Flat Connections and Geometric Quantization*. Commun. Math. Phys. **131**, 347-380 (1990)

[9] K. Gawędzki: *Constructive Conformal Field Theory*. In: Functional integration, geometry and strings, eds. Z. Haba, J. Sobczyk, Birkhäuser, Basel, Boston, Berlin 1989, pp. 277-302

[10] S. Axelrod, S. Della Pietra, E. Witten: *Geometric Quantization of Chern-Simons Gauge Theory*. J. Differential Geometry **33**, 787-902 (1991)

[11] V. Knizhnik, A. B. Zamolodchikov: *Current Algebra and Wess-Zumino Model in Two Dimensions*. Nucl. Phys. **B 247**, 83-103 (1984)

[12] K. Gawędzki, A. Kupiainen: *SU(2) Chern-Simons Theory at Genus Zero*. Commun. Math. Phys. **135**, 531-546 (1991)

[13] K. Gawędzki: *Quadrature of Conformal Field Theories*. Nucl. Phys. **B 328**, 733-752 (1989)

[14] F. Falceto, K. Gawędzki, A. Kupiainen: *Scalar Product of Current Blocks in WZW Theory*. Phys. Lett. **B260**, 101-108 (1991)

[15] P. Christe, R. Flume: *The Four-Point Correlations of All Primary Operators of the d=2 Conformally Invariant SU(2) σ-Model with Wess-Zumino Term*. Nucl. Phys. **B 282**, 466-494 (1987)

[16] A. B. Zamolodchikov, V. A. Fateev: *Operator Algebra and Correlation Functions of the Two-Dimensional Wess-Zumino SU(2) × SU(2) Chiral Model*. Yad. Fiz. **43**, 1031-1044 (1986)

[17] V. V. Schechtman, A. N. Varchenko: *Hypergeometric Solutions of Knizhnik-Zamolodchikov Equations*. Lett. Math. Phys. **20**, 279-283 (1990)

[18] N. Reshetikhin: seminar in Strasbourg, May 93

[19] A. Varchenko: *Critical Points of the Product of Powers of Linear Functions and Families of Bases of Singular Vectors*. Chapel Hill preprint 1993

[20] H. M. Babujian, R. Flume: *Off-Shell Bethe Ansatz Equation for Gaudin Magnets and Solutions of Knizhnik-Zamolodchikov Equations*. Preprint [hep-th/9310110]

[21] B. Feigin, E. Frenkel, N. Reshetikhin: *Gaudin Model, Bethe Ansatz and Correlation Functions at the Critical Level*. Preprint [hep-th/9402022]

[22] M. Thaddeus: *Stable Pairs, Linear Systems and the Verlinde Formula*, Berkeley preprint 1993

[23] A. Jaffe, F. Quinn: *“Theoretical Mathematics”: towards a Cultural Synthesis of Mathematics and Theoretical Physics*. B.A.M.S. **29**, 1-13 (1993)
[24] K. Gawędzki: \textit{SU}(2) WZNW Model at Higher Genera from Gauge Field Functional Integral. Preprint IHES/P/93/66, hep-th 9312051

[25] B. Feigin: lectures at the Conference “Vector Bundles on Curves”. Bombay Dec. 13-24, 1993

[26] Z. Haba: \textit{Correlation Functions of \( \sigma \) Fields with Values in a Hyperbolic Space.} Int. J. Mod. Phys. \textbf{A 4}, 267-286 (1989)

[27] K. Gawędzki, A. Kupiainen, \textit{Coset Construction from Functional Integrals:} Nucl. Phys. \textbf{B 320}, 625-668 (1989)

[28] A. Gupta, S. P. Trivedi, M. B. Wise: \textit{Random Surfaces in Conformal Gauge.} Nucl. Phys. \textbf{B 340}, 475-490 (1990)

[29] M. Goulian, M. Li: \textit{Correlation Functions in Liouville Theory.} Phys. Rev. Lett. \textbf{66}, 2051-2055 (1991)

[30] E. Witten: \textit{On Holomorphic Factorization of WZW and Coset Models.} Commun. Math. Phys. \textbf{144}, 189-212 (1992)

[31] L. Alvarez-Gaumé, J.-B. Bost, G. Moore, P. Nelson, C. Vafa: \textit{Bosonization on Higher Genus Riemann Surfaces.} Commun. Math. Phys. \textbf{112}, 503-552 (1987)

[32] S. J. Arakelov: \textit{Intersection Theory of Divisors on an Arithmetic Surface.} Math. USSR Izv. \textbf{8}, 1167-1180 (1974)

[33] D. Quillen: \textit{Determinants of Cauchy-Riemann Operators over a Riemann Surface.} Funct. Anal. Appl. \textbf{19}, 31-34 (1985)

[34] J.-B. Bost: \textit{Fibrés Déterminants, Déterminants Régularisés et Mesures sur les Espaces de Modules des Courbes Complexes.} Astérisque \textbf{152-153}, 113-149 (1987)

[35] T. Kato: Perturbation theory for linear operators. Berlin, Heidelberg, New York: Springer 1966