Uniformity of Coarse Spaces

Elisa Hartmann

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Abstract

This paper is the first in a series presenting a new version of boundary on coarse spaces. The space of ends functor maps coarse metric spaces to uniform metric spaces and coarse maps to uniformly continuous maps.

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0 What is this paper about

Coarse Geometry of metric spaces studies the large scale properties of a metric space. Meanwhile uniformity of metric spaces is about small scale properties.

Our purpose is to pursue a new version of duality between the coarse geometry of metric spaces and the uniformity of metric spaces. We present a notion of boundary on coarse metric spaces which is a uniform metric space. The methods are very basic and do not require any deep theory.

Note that the topology of metric spaces is well understood and there are a number of topological tools that can be applied on coarse metric spaces which have not been used before. The new discovery may lead to new insight on the topic of coarse geometry.

Before introducing the space of ends of a coarse space we are going to present other notions of boundary which have been studied in coarse geometry:
If $X$ is a proper metric space the Higson corona $\nu X$

- is the boundary of the Higson compactification $hX$ of $X$ which is a compact topological space that contains the underlying topological space of $X$ as a dense subset.

- The compactification $hX$ is determined by a subset of $C(X)$, the bounded continuous functions on $X$, which are called the Higson functions.

- By a comment on [13, p. 31] the Higson corona can be defined for any coarse space. The same does not work for the Higson compactification $hX$.

- The [13, Proposition 2.41] implies that the Higson corona is a covariant functor that sends coarse maps modulo closeness to continuous maps.

- The topology of $\nu X$ has been studied in [9]. It was shown in [9, Theorem 1] that for every $\sigma$-compact subset $A \subseteq \nu X$ the closure $\overline{A}$ of $A$ in $\nu X$ is equivalent to the Stone-Čech compactification of $A$. It has been noted in [13, Exercise 2.49] that the topology of $\nu X$ for $X$ an unbounded proper metric space is never second countable.

- In [4, Theorem 1.1] and [3, Theorem 7.2] it was shown that if the asymptotic dimension $\text{asdim}(X)$ of $X$ is finite then

$$\text{asdim}(X) = \dim(\nu X)$$

where the right side denotes the topological dimension of $\nu X$. Note that one direction of the proof uses the notion of coarse cover.

The space of ends $\Omega Y$ of a locally connected, connected and locally compact Hausdorff space $Y$ is the boundary of the Freudenthal compactification $\varepsilon Y$.

- it is totally disconnected and every other compactification of $Y$ that is totally disconnected factors uniquely through $\varepsilon Y$ by [12, Theorem 1]. The points of $\Omega Y$ are called endpoints or ends.

- Now [12, Theorem 5] shows that if $Y$ is a connected locally finite countable CW-complex every endpoint of $Y$ can be represented by a proper map $a : \mathbb{R}_+ \to Y$. Two proper maps $a_1, a_2 : \mathbb{R}_+ \to Y$ represent the same endpoint if they are connected by a proper homotopy.

- $\Omega$ is functorial for proper continuous maps.

- If $Y$ is a locally compact Hausdorff space then $\Omega Y$ can be constructed using a proximity relation which is a relation on the subsets of $Y$. See [10] for that one.

If $X$ is a proper Gromov hyperbolic metric space then the Gromov boundary $\partial X$

- consists of equivalence classes of sequences that converge to infinity in $X$.

- The topology on $\partial X$ is generated by a basis of open neighborhoods. Loosely speaking two points on the boundary are close if the sequences that represent them stay close for a long time.

- By [3, Proposition 2.14] the topological spaces $\partial X$ and $\partial X \cup X$ are compact and by [3, Theorem 2.1] the topology on $\partial X$ is metrizable.

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1 for which the topology of $X$ needs to be locally compact which is given if the metric is proper.

2 but under a different name.
If \( f : X \to Y \) is a quasi-isometry between proper Gromov hyperbolic groups then it extends to a homeomorphism 
\[
\partial f : \partial X \to \partial Y
\]
by \([8, \text{Proposition 2.20}]\).

In \([5]\) is studied a notion of morphisms for which the Gromov boundary is a functor: If 
\( f : X \to Y \) is a visual function between proper Gromov hyperbolic metric spaces then 
there is an induced map 
\[
\partial f : \partial X \to \partial Y
\]
which is continuous by \([5, \text{Theorem 2.8}]\).

If \( X \) is a metric space then the space of ends \( E(X) \) of \( X \)

- consists of the endpoints of \( X \) which are equivalence classes of coarse maps \( \varphi : \mathbb{Z}_+ \to X \).
- The uniformity on \( E(X) \) is generated by a basis \((D_U)_U\) of entourages over coarse covers \( U \) of \( X \).
- If \( f : X \to Y \) is a coarse map then it induces a uniformly continuous map \( E(f) : E(X) \to E(Y) \) between spaces of ends. That way the space of ends \( E(\cdot) \) is a functor from the category of metrizable coarse spaces to the category of uniform spaces.
- If \( Z \subseteq Y \) is a subspace then the inclusion \( i : Z \to Y \) induces a uniform embedding \( E(i) : E(Z) \to E(Y) \) by Proposition \(49\). The functor \( E(\cdot) \) preserves coproducts by Lemma \(53\).
- The uniformity on \( E(X) \) is totally bounded by Lemma \(55\) and metrizable by Corollary \(61\).

The outline of this paper is as follows:

- Chapter 1 introduces the basic notions. While the space of ends functor is defined for all metric spaces the most reasonable presentation is for coarsely proper coarsely geodesic metric spaces. Subchapter 1.1 presents the coarse structure and cocoarse structure associated to a metric space. In Subchapters 1.2, 1.3 we introduce the notions coarsely proper and coarsely geodesic.
- The Chapter 2 creates the basic tool sets which are going to be used in this and the following study.
- Finally Chapter 3 is about the space of ends functor. Subchapter 3.1 gives the definition and Subchapter 3.2 studies a few basic properties. The Chapter closes with Subchapter 3.3 which sets the space of ends functor in context with other notions.

1 Introduction

In this chapter we present the objects of this study. We will study coarsely geodesic coarsely proper metric spaces and coarse maps between them.
1.1 Metric Spaces

Definition 1. Let \((X, d)\) be a metric space.

- Then the **bounded coarse structure associated to** \(d\) on \(X\) consists of those subsets \(E \subseteq X^2\) for which \[\sup_{(x, y) \in E} d(x, y) < \infty\]

We call an element of the coarse structure **entourage**.

- The **bounded cocoarse structure associated to** \(d\) on \(X\) consists of those subsets \(C \subseteq X^2\) such that every sequence \((x_i, y_i)\) in \(C\) is either bounded (which means both of the sequences \((x_i)\) and \((y_i)\) are bounded) or \[d(x_i, y_i) \to \infty\]

for \(i \to \infty\). We call an element of the cocoarse structure **coentourage**.

- In what follows we assume the metric \(d\) to be finite for every \((x, y) \in X^2\).

Remark 2. Note that there is a more general notion of coarse spaces. By [13, Theorem 2.55] a coarse structure on a coarse space \(X\) is the bounded coarse structure associated to some metric \(d\) on \(X\) if and only if the coarse structure has a countable base.

Definition 3. If \(X\) is a metric space a subset \(B \subseteq X\) is **bounded** if the set \(B^2\) is an entourage in \(X\).

Remark 4. Note the following duality:

- A subset \(F \subseteq X^2\) is an entourage if and only if for every coentourage \(C \subseteq X^2\) there is a bounded set \(A\) such that \[F \cap C \subseteq A^2\]

- A subset \(D \subseteq X^2\) is a coentourage if and only if for every entourage \(E \subseteq X^2\) there is a bounded set \(B\) such that \[E \cap D \subseteq B^2\]

Definition 5. A map \(f : X \to Y\) between metric spaces is called **coarse** if

- \(E \subseteq X^2\) being an entourage implies that \(f^2(E)\) is an entourage (**coarsely uniform**);

- and if \(A \subseteq Y\) is bounded then \(f^{-1}(A)\) is bounded (**coarsely proper**).

Or equivalently

- \(B \subseteq X\) being bounded implies that \(f(B)\) is bounded;

- and if \(D \subseteq Y^2\) is a coentourage then \(f^{-2}(D)\) is a coentourage.

Two maps \(f, g : X \to Y\) between metric spaces are called **close** if \[f \times g(\Delta_X)\]

is an entourage in \(Y\). Here \(\Delta_X\) denotes the diagonal in \(X\).

Notation 6. A map \(f : X \to Y\) between metric spaces
is called $K$–coarsely surjective if
\[ E(Y, K)[\text{im } f] = Y \]

- $f$ is called coarsely surjective if $f$ is $K$–coarsely surjective for some $K > 0$.
- $f$ is called coarsely injective if
  1. for every entourage $F \subseteq Y^2$ the set $f^{-2}(F)$ is an entourage in $X$.
  2. or equivalently if for ever coentourage $C \subseteq X^2$ the set $f^2(C)$ is a coentourage in $Y$.
- two subsets $A, B \subseteq X$ are called coarsely disjoint if $A \times B$ is a coentourage.

Remark 7. We study metric spaces up to coarse equivalence. A coarse map $f : X \to Y$ is a coarse equivalence if
- There is a coarse map $g : Y \to X$ such that $f \circ g$ is close to $\text{id}_Y$ and $g \circ f$ is close to $\text{id}_X$.
- or equivalently if $f$ is both coarsely injective and coarsely surjective.

Notation 8. If $X$ is a metric space we write
\[ B(p, r) = \{ x \in X : d(x, p) \leq r \} \]
for a point $p \in X$ and $r \geq 0$. If we did not specify a coarse space we write
\[ E(Y, r) = \{ (x, y) \in Y^2 : d(x, y) \leq r \} \]
for $Y$ a metric space and $r \geq 0$.

1.2 Coarsely Proper:
This is [2, Definition 3.D.10]:

Definition 9. (coarsely proper) A metric space $X$ is called coarsely proper if there is some $R_0 > 0$ such that for every bounded subset $B \subseteq X$ the cover
\[ \bigcup_{x \in B} B(x, R_0) \]
of $B$ has a finite subcover.

Remark 10. (proper)
- A metric space $X$ is proper if the map
\[ r_p : X \to \mathbb{R}_+ \]
\[ x \mapsto d(x, p) \]
is a proper continuous map for every $p \in X$.
- Every proper metric space is coarsely proper. A coarsely proper metric space is proper if it is complete.

\[^3\text{as in the reverse image of compact sets is compact}\]
1 INTRODUCTION

• If $X$ has a proper metric then the topology of $X$ is locally compact.

Lemma 11. • If $f : X \to Y$ is a coarse map between metric spaces and $X' \subseteq X$ a coarsely proper subspace then

$$f(X') \subseteq Y$$

is coarsely proper.

• being coarsely proper is a coarse invariant.

Proof. • 1. Suppose $R_0 > 0$ is such that every bounded subset of $X'$ can be covered by finitely many $R_0$-balls. Because $f$ is a coarsely uniform map there is some $S_0 > 0$ such that $d(x, y) \leq R_0$ implies $d(f(x), f(y)) \leq S_0$. We show that $f(X')$ is coarsely proper with regard to $S_0$.

2. Let $B \subseteq f(X')$ be a bounded subset. Then $f^{-1}(B)$ is bounded in $X$ thus there is a finite subcover of $\bigcup_{x \in B} B(x, R_0)$ which is

$$f^{-1}(B) = B(x_1, R_0) \cup \cdots \cup B(x_n, R_0)$$

But then

$$B = f \circ f^{-1}(B)$$

$$= f(B(x_1, R_0) \cup \cdots \cup B(x_n, R_0))$$

$$= f(B(x_1, R_0)) \cup \cdots \cup f(B(x_n, R_0))$$

$$\subseteq B(f(x_1), S_0) \cup \cdots \cup B(f(x_n), S_0)$$

is a finite cover of $B$ with $S_0$-balls.

• 1. Suppose $f : X \to Y$ is a coarsely surjective coarse map between metric spaces and $X$ is coarsely proper. We show that $Y$ is coarsely proper:

2. By point 1 the subset $\text{im } f \subseteq Y$ is coarsely proper. Suppose $\text{im } f$ is coarsely proper with regard to $R_0 \geq 0$ and suppose $K \geq 0$ is such that $E(Y, K)[\text{im } f] = Y$, we show that $Y$ is coarsely proper with regard to $R_0 + K$.

3. Let $B \subseteq Y$ be a bounded set. Then there are $x_1, \ldots, x_n$ such that

$$B \cap \text{im } f \subseteq B(x_1, R_0) \cup \cdots \cup B(x_n, R_0)$$

and then

$$B \subseteq B(x_1, R_0 + K) \cup \cdots \cup B(x_n, R_0 + K).$$

Example 12. Note that every countable group is a proper metric space.

1.3 Coarsely Geodesic:

The following definition can also be found on [2] p. 10:

Definition 13. (coarsely connected) Let $X$ be a metric space.

• Let $x, y \in X$ be two points. A finite sequence of points $a_0, \ldots, a_n$ in $X$ is called a $c$–path joining $x$ to $y$ if $x = a_0, y = a_n$ and $d(a_i, a_{i+1}) \leq c$ for every $i$. 
• then $X$ is called \textit{c-coarsely connected} if for every two points $x, y \in X$ there is a $c$–path between them

• the space $X$ is called \textit{coarsely connected} if there is some $c \geq 0$ such that $X$ is $c$–coarsely connected.

\textbf{Example 14.} Not an example:

$$\{2^n : n \in \mathbb{N}\} \subseteq \mathbb{Z}^+$$

\textbf{Lemma 15.} \textit{Being coarsely connected is invariant by coarse equivalence.}

\textit{Proof.} Note that this is [2, Proposition 3.B.7]. The argument for the proof can be found in [2, Proposition 3.B.4]. For the convenience of the reader we recall it:

If $f : X \rightarrow Y$ is a coarsely surjective coarse map and $X$ is coarsely connected we will show that $Y$ is coarsely connected. Suppose $X$ is $c$-coarsely connected. Let $y, y'$ be two points in $Y$. Note that by coarse surjectivity of $f$ there is some $K \geq 0$ such that $E(Y, K)[\text{im } f] = Y$. And by coarseness of $f$ there is some $d \geq 0$ such that $f^2(E(X, c)) \subseteq E(Y, d)$. Now denote by

$$e = \max(K, d)$$

Choose points $x, x' \in X$ such that $d(y, f(x)) \leq K$ and $d(y', f(x')) \leq K$ and a $c$-path $x = a_0, a_1, \ldots, a_n = x'$. Then

$$y, f(x), f(a_1), \ldots, f(x'), y'$$

is an $e$-path in $Y$ joining $y$ to $y'$. Thus $Y$ is $e$-coarsely connected which implies that $Y$ is coarsely connected. \hfill \square

\textbf{Example 16.} By [2, Proposition 4.B.8] a countable group is coarsely connected if and only if it is finitely generated.

This one is [2, Definition 3.B.1(b)]:

\textbf{Definition 17.} \textbf{(coarsely geodesic)} A metric space $X$

• is called \textit{c–coarsely geodesic} if it is $c$-coarsely connected and there is a function

$$\Phi(X, c) : \mathbb{R}_+ \rightarrow \mathbb{N}$$

(called the upper control) such that for every $x, y \in X$ there is a $c$–path $x = a_0, \ldots, a_n = y$ such that

$$n + 1 \leq \Phi(X, c)(d(x, y))$$

• the space $X$ is called \textit{coarsely geodesic} if there is some $c \geq 0$ such that $X$ is $c$–coarsely geodesic.

\textbf{Lemma 18.} \textit{Being coarsely geodesic is a coarse invariant.}

\textit{Proof.} Suppose that $f : X \rightarrow Y$ is a coarse equivalence between metric spaces and that $X$ is $c$–coarsely geodesic. We proceed as in the proof of Lemma[15] using the same notation:

1. There is a constant $K \geq 0$ such that $E(Y, K)[\text{im } f] = Y$,

2. There is a constant $d \geq 0$ such that $f^2(E(X, c)) \subseteq E(Y, d)$. By the proof of Lemma[15] the space $Y$ is $e = \max(K, c)$–coarsely connected.
3. For every \( r \geq 0 \) there is some \( s \geq 0 \) such that
\[
\ker f^{-2}(E(Y,r)) \subseteq E(X,s)
\]
we store the association \( r \mapsto s \) in the map \( \varphi : \mathbb{R}^+ \to \mathbb{R}^+ \).

Define
\[
\Phi(Y,e) : \mathbb{R}^+ \to \mathbb{N} \\
r \mapsto \Phi(X,c) \circ \varphi(r + 2K) + 2
\]

Then \( \Phi(Y,e) \) is an upper bound for \( Y \): Let \( y, y' \in Y \) be two points. Consider the same \( e \)-path \( y, f(x), a_1, \ldots, f(x'), y' \) as in the proof of Lemma 15 with the additional condition that \( n + 1 \leq \Phi(X,e) \). Then \( d(y, y') \leq r \) implies that \( d(f(x), f(x')) \leq r + 2K \) which implies \( d(x, x') \leq \varphi(r + 2K) \).

Thus \( Y \) is \( e \)-coarsely geodesic which implies that \( Y \) is coarsely geodesic.

Example 19. Note that by [2, Proposition 1.A.1] every finitely generated group is coarsely geodesic.

2. **Groundwork**

This Chapter is denoted to the boring part. We develop the technical preliminaries needed for this and the following studies.

2.1 **Coarse Rays**

In [11] every metric space that is coarsely equivalent to \( \mathbb{Z}^+ \) is called a coarse ray. We keep with that notation:

**Definition 20. (coarse ray)** If \( X \) is a metric space a sequence \( (x_i)_i \subseteq X \) is called a coarse ray in \( X \) if there is a coarsely injective coarse map \( \rho : \mathbb{Z}^+ \to X \) such that \( x_i = \rho(i) \) for every \( i \).

**Lemma 21.** If \( X \) is a \( c \)-coarsely geodesic metric space, \( (x_i)_i \) a sequence in \( X \) and if for every \( i < j \) the sequence
\[
x_i, \ldots, x_j
\]
is a \( c \)-path such that \( \Phi(X,c)(d(x_i, x_j)) \geq |i - j| + 1 \) then the association
\[
i \mapsto r_i
\]
defines a coarsely injective coarse map \( \rho : \mathbb{Z}^+ \to X \).

**Proof.** We show that \( \rho \) is coarsely injective coarse:

1. \( \rho \) is coarsely uniform: Let \( n \in \mathbb{N} \) be a number. Then for every \( i, j \in \mathbb{Z}^+ \) if \( |i - j| \leq n \) then \( d(x_i, x_j) \leq cn \).
2. \( \rho \) is coarsely injective: Let \( k \geq 0 \) be a number. Then \( d(x_i, x_j) \leq k \) implies \( |i - j| \leq \Phi(X,c)(k) - 1 \) for every \( ij \).

**Proposition 22.** If \( X \) is a coarsely geodesic coarsely proper metric space and
• if $X$ is not bounded then there is at least one coarse ray in $X$.

• in fact if $(x_i)_i$ is a sequence in $X$ that is not bounded then there is a subsequence $(x_{i_k})_k$ that is not bounded, a coarse ray $(r_i)_i$ and an entourage $E \subseteq X^2$ such that

$$(x_{i_k})_k \subseteq E[(r_i)_i]$$

Remark 23. Point 1 is the same as [11, Lemma 4]. The proof is quite different though.

Proof. Suppose $X$ is coarsely proper with regard to $R_0$ and $c$–coarsely geodesic. We will determine a sequence $(V_i)_i$ of subsets of $X$ and a sequence $(r_i)_i$ of points in $X$.

• Define $r_0 = x_0$ and $V_0 = X$.

• Then define for every $y \in X$ the number $d_0(y)$ to be the minimal length of a $c$–path joining $x_0$ to $y$.

• We define a relation on the points of $X$: $y \leq z$ if $d_0(y) \leq d_0(z)$ and $y$ lies on a $c$-path of minimal length joining $x_0$ to $z$. This makes $(X, \leq)$ a partially ordered set.

• for every $i \in \mathbb{N}$ do:

1. denote by

$$C_i = \{x \in X : d_0(x) = i\}$$

2. there are $y_1, \ldots, y_m \in X$ such that

$$C_i = B(y_1, R_0) \cup \cdots \cup B(y_m, R_0)$$

3. Now $(x_i)_i \cap V_{i-1}$ is not bounded and $V_{i-1}$ is coarsely geodesic. Thus for every $n \in \mathbb{N}$ there is some $x_{n_k} \in V_{i-1}$ with $d_0(x_{n_k}) \geq n + i$.

4. Then there is one of $j = 1, \ldots, m$ such that for infinitely many $n \in \mathbb{N}$: there is $y \in B(y_j, R_0)$ such that $y \leq x_{n_k}$. Then define

$$V_i := \{x \in V_{i-1} : \exists y \in B(y_j, R_0) : y \leq x\}$$

Note that $V_i$ is coarsely geodesic and that $V_i \cap (x_i)_i$ is not bounded.

5. Define $r_i := y_j$.

• We show that $(r_i)_i$ and $E = E(X, R_0)$ have the desired properties:

1. $(r_i)_i$ is a coarse ray: for every $i$ the sequence $r_0, \ldots, r_i$

is $R_0$–close to a $c$–path of minimal length which implies that every subsequence is $R_0$–close to a $c$–path of minimal length.

2. The set

$$P := \bigcup_i V_i \cap C_i$$

contains infinitely many of the $(x_i)_i$ and $(r_i)_i$ is $R_0$–coarsely dense in $P$. Thus the result.
2.2 Totally Bounded Uniformity

Definition 24. (close relation) Let $X$ be a coarse space. Two subsets $A, B \subseteq X$ are called close if they are not coarsely disjoint. We write

$$A \bowtie B$$

Then $\bowtie$ is a relation on the subsets of $X$.

Lemma 25. In every metric space $X$:

1. if $B$ is bounded, $B \not\bowtie A$ for every $A \subseteq X$
2. $U \bowtie V$ implies $V \bowtie U$
3. $U \bowtie (V \cup W)$ if and only if $U \bowtie V$ or $U \bowtie W$

Proof. 1. easy.

2. easy.

3. easy. 

Proposition 26. Let $X$ be a metric space. Then for every subspaces $A, B \subseteq X$ with $A \not\bowtie B$ there are subsets $C, D \subseteq X$ such that $C \cap D = \emptyset$ and $A \not\bowtie (X \setminus C), B \not\bowtie (X \setminus D)$.

Proof. Note this is the same as [7, Proposition 4.5] where the same statement was proven in a similar fashion. Let $E_1 \subseteq E_2 \subseteq \cdots$ be a symmetric basis for the coarse structure of $X$. Then for every $x \in A^c \cap B^c$ there is a least number $n_1(x)$ such that $x \in E_{n_1(x)}[A]$ and a least number $n_2(x)$ such that $x \in E_{n_2(x)}[B]$. Define:

$$V_1 = \{x \in A^c \cap B^c : n_1(x) \leq n_2(x)\}$$

and

$$V_2 = A^c \cap B^c \setminus V_1$$

Now for every $n$

$$E_n[V_1] \cap B \subseteq E_{2n}[A] \cap B$$

because for every $x \in V_1$, if $x \in E_n[B]$ then $x \in E_n[A]$. Now define

$$C = A \cup V_1$$

and

$$D = B \cup V_2$$

Remark 27. Compare $\bowtie$ with the notion of proximity relation [14, chapter 40, pp. 266]. By Lemma 25 and Proposition 26 the close relation satisfies [14, P-1),P-3)-P-5) of Definition 40.1 but not P-2).

Remark 28. If $f : X \rightarrow Y$ is a coarse map then whenever $A \bowtie B$ in $X$ then $f(A) \bowtie f(B)$ in $Y$.

We recall [6, Definition 45]:
**Definition 29.** (coarse cover) If $X$ is a metric space and $U \subseteq X$ a subset a finite family of subsets $U_1, \ldots, U_n \subseteq U$ is said to coarsely cover $U$ if

$$U^2 \cap (\bigcup_i U_i^c)^c$$

is a coentourage in $X$.

**Remark 30.** Note that coarse covers determine a Grothendieck topology on $X$. If $f : X \to Y$ is a coarse map between metric spaces and $(V_i)_i$ a coarse cover of $V \subseteq Y$ then $(f^{-1}(V_i))_i$ is a coarse cover of $f^{-1}(V) \subseteq X$.

**Lemma 31.** Let $X$ be a metric space. A finite family $U = \{U_\alpha : \alpha \in A\}$ is a coarse cover if and only if there is a finite cover $V = \{V_\alpha : \alpha \in A\}$ of $X$ as a set such that $V_\alpha \not\in U_\alpha^c$ for every $\alpha$.

**Proof.**

- $n = 1$:
  1. a subset $V$ covers $X$ as a set if and only if $V = X$.
  2. a subset $U$ coarsely covers $X$ if and only if $U_c^c$ is bounded if and only if $U_c^c \not\in X$.

- $n + 1 \to n + 2$: subsets $U, V, U_1, \ldots, U_n$ coarsely cover $X$ if and only if $U, V$ coarsely cover $U \cup V$ and $U \cup V, U_1, \ldots, U_n$ coarsely cover $X$ at the same time.

  1. Suppose $U, V, U_1, \ldots, U_n$ coarsely cover $X$. By induction hypothesis there is a cover of sets $V_1', V_2'$ of $U \cup V$ such that $V_1' \not\in U_c^c$ and $V_2' \not\in V_c^c$ and there is a cover of sets $W, V_1, \ldots, V_n$ such that $W \not\in (U \cup V)_c^c$ and $V_i \not\in U_i^c$ for every $i$. Then

$$B := W \cap (U \cup V)_c^c$$

is bounded. Then $V_1' \cup B, V_2', V_1, \ldots, V_n$ is a finite cover of $X$ with the desired properties.

- Suppose $(V_\alpha)_\alpha$ cover $X$ as sets and $V_\alpha \not\in U_\alpha^c$ for every $\alpha$. Let $E \subseteq X^2$ be an entourage. Then $E[U_\alpha^c] \cap V_\alpha$ is bounded for every $\alpha$. Then

$$\bigcap_{\alpha} E[U_\alpha^c] = \bigcap_{\alpha} E[U_\alpha^c] \cap (\bigcup_{\alpha} V_\alpha)$$

$$= \bigcup_{\alpha} (V_\alpha \cap \bigcap_{\alpha} E[U_\alpha])$$

is bounded.

**Remark 32.** The [14, Theorem 40.15] states that every proximity relation on a set is induced by some totally bounded uniformity on it. Note that a coarse cover on a metric space $X$ does not precisely need to cover $X$ as a set. Except for that the collection of all coarse covers of a metric space satisfies [14, a),b) of Theorem 36.2]. We can compare coarse covers of $X$ with a base for a totally bounded uniformity on $X$: the collection of all sets $\bigcup_i U_i^c$ for $(U_i)_i$ a coarse cover satisfies [14, b)-e) of Definition 35.2] but not a). Note that by [14, Definition 39.7] a diagonal uniformity is totally bounded if it has a base consisting of finite covers.
Lemma 33. \textbf{(separation cover)} If $U_1, U_2$ coarsely cover a metric space $X$ (or equivalently if $U_1^c, U_2^c$ are coarsely disjoint) then there exists a coarse cover $V_1, V_2$ of $X$ such that $V_1 \not\subset U_1^c$ and $V_2 \not\subset U_2^c$.

Proof. 1. By Proposition 26 there are subsets $C, D \subseteq X$ such that $C \cap D = \emptyset$, $U_1^c \not\subset C^c$ and $U_2^c \not\subset D^c$. Thus $U_1, C$ is a coarse cover of $X$ such that $C \not\subset U_2^c$.

2. By Proposition 26 there are subsets $A, B \subseteq X$ such that $A \cap B = \emptyset$, $A^c \not\subset U_1^c$ and $B^c \not\subset C^c$.

Then $B, C$ are a coarse cover of $X$ such that $B \not\subset U_1^c$.

3. Then $V_1 = B$ and $V_2 = C$ have the desired properties. \qed

Notation 34. \textbf{(coarse star refinement)} Let $\mathcal{U} = (U_i)_{i \in I}$ be a coarse cover of a metric space $X$.

1. If $S \subseteq X$ is a subset then
   \[ \text{cst}(S, \mathcal{U}) = \bigcup \{ U_i : S \subsetneq U_i \} \]
   is called the coarse star of $S$.

2. A coarse cover $\mathcal{V} = (V_j)_{j \in J}$ of $X$ is called a coarse barycentric refinement of $\mathcal{U}$ if for every $j_1, \ldots, j_k \in J$ such that there is an entourage $E \subseteq X^2$ such that
   \[ \bigcap_k E[V_{j_k}] \]
   is not bounded then there is some $i \in I$ and entourage $F \subseteq X^2$ such that
   \[ \bigcup_k V_{j_k} \subseteq F[V_i] \]

3. A coarse cover $\mathcal{V} = (V_j)_{j \in J}$ of $X$ is called a coarse star refinement of $\mathcal{U}$ if for every $j \in J$ there is some $i \in I$ and entourage $E \subseteq X^2$ such that
   \[ \text{cst}(V_j, \mathcal{V}) \subseteq E[U_i] \]

Lemma 35. If $\mathcal{V} = (V_i)_i$ is a coarse star refinement of a coarse cover $\mathcal{U} = (U_i)_i$ of a metric space $X$ then

- if $S \subseteq X$ is a subset then there is an entourage $E \subseteq X^2$ such that
  \[ \text{cst}(\text{cst}(S, \mathcal{V}), \mathcal{V}) \subseteq E[\text{cst}(S, U)] \]

- if $f : X \to Y$ is a coarse map between metric spaces, $(U_i)_i$ a coarse cover of $Y$ and $S \subseteq X$ a subset then
  \[ f(\text{cst}(S, f^{-1}(U))) \subseteq \text{cst}(f(S), \mathcal{U}) \]

Proof. • Suppose $E \subseteq X^2$ is an entourage such that for every $V_j$ there is an $U_i$ such that $\text{cst}(V_j, \mathcal{V}) \subseteq U_i$. Note that $S \subsetneq V_j$ implies $S \not\subset U_i$ in that case. Then

\[ \text{cst}(\text{cst}(S, \mathcal{V}), \mathcal{V}) = \text{cst}(\bigcup \{ V_i : V_i \subsetneq S \}, \mathcal{V}) \]
\[ = \bigcup_{V_i \subsetneq S} \text{cst}(V_i, \mathcal{V}) \]
\[ \subseteq \bigcup_{S \subsetneq U_j} E[U_j] \]
\[ = E[\text{cst}(S, U)] \]

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\[ f(\text{cst}(S, f^{-1}(U_i))) = \bigcup \{ f \circ f^{-1}(U_i) : S \setminus f^{-1}(U_i) \} \]
\[ \subseteq \bigcup \{ f \circ f^{-1}(U_i) : f(S) \setminus f \circ f^{-1}(U_i) \} \]
\[ \subseteq \bigcup \{ U_i : f(S) \setminus U_i \} \]
\[ = \text{cst}(f(S), U) \]

**Lemma 36.** If \( U \) is a coarse cover of a metric space \( X \) then there exists a coarse cover \( V \) of \( X \) that coarsely star refines \( U \).

**Proof.** There are three steps:

- If \( V = (V_j)_j \) is a coarse barycentric refinement of \( U \) and \( W = (W_k)_k \) is a coarse barycentric refinement of \( V \) then \( W \) is a coarse star refinement of \( U \):
  1. fix \( W_k \) and denote \( J = \{ j : W_k \setminus W_j \} \).
  2. Then for every \( j \in J \) there is some \( V_j \) and entourage \( E_j \subseteq X^2 \) such that \( W_k \cup W_j \subseteq E_j[V_j] \).
  3. Define \( E = \bigcup_j E_j \). Then \( \bigcap_j E[V_j] \supseteq W_k \). Thus there is some \( U_i \) and entourage \( F \subseteq X^2 \) such that \( \bigcup_j V_j \subseteq F[U_i] \).
  4. For every \( j \in J \):

    \[
    W_j \subseteq E[V_j] \subseteq E \circ F[U_i]
    \]

    thus \( \text{cst}(W_k, W) \subseteq E \circ F[U_i] \).

- We show there is a coarse barycentric refinement \( V = (V_i)_i \) of \( U \): First we show if \( U_1, U_2 \) is a coarse cover of \( X \) then there is a coarse barycentric refinement \( V_1, V_2, V_3 \) of \( U_1, U_2 \):
  1. By Lemma 33 there is a coarse cover \( W_1, W_2 \) of \( X \) such that \( W_1 \not\approx U_1^c \) and \( W_2 \not\approx U_2^c \).
  2. Then \( W_1^c, U_1 \) and \( W_2^c, U_2 \) are coarse covers of \( X \).
  3. By Proposition 26 there are \( C, D \subseteq X \) such that \( C \cap D = \emptyset \), \( D^c \not\approx U_2^c \), \( C^c \not\approx W_2 \).
  4. Also there are \( A, B \subseteq X \) such that \( A \cap B = \emptyset \), \( A^c \not\approx U_1^c \), \( B^c \not\approx W_1 \).

- Then \( V_1 = W_1, V_2 = C \cup B, V_3 = W_2 \) has the desired properties:
  1. \( (V_i)_i \) is a coarse cover:
    (a) Note that by \( B^c \not\approx W_1 \) and \( W_1^c \not\approx W_2^c \) the sets \( W_2, B \) are a coarse cover of \( X \).
    (b) Note that by \( C^c \not\approx W_2 \) and \( W_2^c \not\approx W_1^c \) the sets \( W_1, C \) coarsely cover \( X \).
    (c) Note that \( (W_1 \cap W_2) \not\approx (C^c \cup B^c) \). Then, combining items i,ii, we get that \( W_1, W_2, B \cap C \) is a coarse cover as required.
2. $(V_i)_i$ is a coarse barycentric refinement of $U_1, U_2$:
   (a) There is some entourage $E \subseteq X^2$ such that $V_1 \cup V_2 \subseteq E[U_1^c]$: For $W_1$ we use that $W_1 \not\in U_1^c$. For $C \cap B$ we use that $A^c \not\in U_1^c$ and $B \subseteq A^c$.
   (b) There is an entourage $E \subseteq X^2$ such that $V_2 \cup V_3 \subseteq E[U_2^c]$; For $W_2$ we use that $W_2 \not\in U_2^c$. For $C \cap B$ we use that $D^c \not\in U_2^c$ and $C \subseteq D^c$.
   (c) $V_1 \not\in V_3$: We use $W_1^c \not\in W_2^c$.

- Now we show the general case: Suppose $U_i \subseteq X$ are subsets such that $\mathcal{U} = (U_i)_i$. We show there is a coarse barycentric refinement $V$ of $\mathcal{U}$.

1. For every $i$ the sets $U_i, \bigcup_{j \neq i} U_j$ coarsely cover $X$. By Lemma 33 there are subsets $W_i^1, W_i^2$ that coarsely cover $X$ such that $W_i^1 \not\in U_i^c$ and $W_i^2 \not\in (\bigcup_{j \neq i} U_j)^c$.
2. Then there is a coarse barycentric refinement $V_i^1, V_i^2, V_i^3$ of $W_i^1, W_i^2$ for every $i$.
3. Then we define
   \[ V := \left( \bigcap_i V_i^{\sigma(i)} \right)_\sigma, \]
   here $\sigma(i) \in \{1, 2, 3\}$ is all possible permutations.
4. We show $V$ is a coarse cover of $X$ that is a coarse barycentric refinement of $\mathcal{U}$:
   (a) $V$ is a coarse cover: by design.
   (b) $V$ is a coarse barycentric refinement of $\mathcal{U}$: Suppose there is an entourage $E \subseteq X^2$ and a subindex $(\sigma_k)_k$ such that $\bigcap_{\sigma_k} E[\bigcap_{\sigma_k} V_{\sigma_k(i)}^{\sigma_k}]$ is not bounded. Then
   \[ \bigcap_{i, \sigma_k} E[V_{\sigma_k(i)}^{\sigma_k}] \]
   is not bounded. Then there is an entourage $F \subseteq X^2$ such that for every $i$:
   \[ \bigcup_{\sigma_k} V_{\sigma_k(i)}^{\sigma_k} \subseteq F[W_i^{\sigma_k}] \]
   where $l_i$ is one of 1, 2. Then
   \[ \bigcup_{\sigma_k} V_{\sigma_k(i)}^{\sigma_k} \subseteq \bigcap_{\sigma_k} V_{\sigma_k(i)}^{\sigma_k} \subseteq \bigcap_{i} F[W_i^{l_i}] \]
   if $l_i = 1$ for one $i$ then we are done. Otherwise
   \[ \bigcup_{\sigma_k} V_{\sigma_k(i)}^{\sigma_k} \subseteq \bigcap_{i} F[W_i^{2}] \]
   and $F[W_i^{2}] \not\in (\bigcup_{j \neq i} U_j)^c$ implies
   \[ \bigcap_{i} F[W_i^{2}] \not\in (\bigcup_{j \neq i} U_j)^c \]
   which implies that $\bigcap_{i} F[W_i^{2}]$ is bounded, a contradiction. □
3 Space of Ends

We introduce the space of ends of a coarse space which is a functor $E$ from the category of coarse metric spaces to the category of uniform metric spaces. In the course of this chapter we show that $E$ preserves coproducts.

3.1 Definition

Definition 37. (endpoint) Let $X$ be a metric space,

- two coarse maps $\phi, \psi : \mathbb{Z}_+ \to X$ are said to represent the same endpoint in $X$ if there is an entourage $E \subseteq X^2$ such that $E[\psi(\mathbb{Z}_+)] = \phi(\mathbb{Z}_+)$

- if $U = (U_i)_i$ is a coarse cover of $X$ and $p, q$ are two endpoints in $X$ which are represented by $\phi, \psi : \mathbb{Z}_+ \to X$. Then $q$ is said to be in a $U$-neighborhood of $p$, denoted $q \in U[p]$, if there is an entourage $E \subseteq X^2$ such that $E[\text{cst}(\phi(\mathbb{Z}_+), U)] \supseteq \psi(\mathbb{Z}_+)$ and $E[\text{cst}(\psi(\mathbb{Z}_+), U)] \supseteq \phi(\mathbb{Z}_+)$

Lemma 38. If $V \subseteq U$ is a refinement of a coarse cover of a metric space $X$ then for every two endpoints $p, q$ of $X$ the relation $q \in V[p]$ implies the relation $q \in U[p]$.

Proof. Suppose $V = (V_i)_i$ and $U = (U_i)_i$. If $p, q$ are represented by $\phi, \psi : \mathbb{Z}_+ \to X$ then

$$\text{cst}(\phi(\mathbb{Z}_+), V) = \bigcup \{V_i : \phi(\mathbb{Z}_+) \times V_i\} \subseteq \bigcup \{U_i : \phi(\mathbb{Z}_+) \times U_i\} = \text{cst}(\phi(\mathbb{Z}_+), U)$$

in the same way $\text{cst}(\psi(\mathbb{Z}_+), V) \subseteq \text{cst}(\psi(\mathbb{Z}_+), U)$. Then if $q \in V[p]$ there is some entourage $E \subseteq X^2$ such that

$$\psi(\mathbb{Z}_+) \subseteq E[\text{cst}(\phi(\mathbb{Z}_+), V)] \subseteq E[\text{cst}(\phi(\mathbb{Z}_+), U)]$$

and $\phi(\mathbb{Z}_+) \subseteq E[\text{cst}(\psi(\mathbb{Z}_+), U)]$. Thus $q \in U[p]$. □

Definition 39. (space of ends) Let $X$ be a metric space. As a set the space of ends $E(X)$ of $X$ consists of the endpoints in $X$. A subset $U \subseteq E(X)$ is open if for every $p \in U$ there is a coarse cover $U$ of $X$ such that $U[p] \subseteq U$.

This defines a topology on $E(X)$.

Remark 40. The topology on the set of endpoints $E(X)$ is generated by a uniformity: If $U$ is a coarse cover of $X$ then $D_U = \{(p, q) : q \in U[p]\}$ is the entourage associated to $U$. Then $(D_U)_U$ over coarse covers $U$ of $X$ are a base for a diagonal uniformity on $E(X)$. 15
Lemma 41. If \( X \) is a metric space then \( E(X) \) is indeed a uniform space. Coarse covers of \( X \) give rise to a base for the uniform structure.

Proof. We check that \((D_U)\) over coarse covers are a base for a uniformity on \( E(X) \):

1. If \( \mathcal{U} \) is a coarse cover of \( X \) then \( \Delta \subseteq D_U \), where \( \Delta = \{(p,p) : p \in E(X)\} : p \in U[p] \).
2. If \( \mathcal{U}, \mathcal{V} \) are coarse covers of \( X \) then \( D_U \cap D_V \) is an entourage: Suppose \( U = (U_i)_i, V = (V_i)_i \) then define
   \[ D_U \cap V := (U_i \cap V_j)_{ij} \]
   Suppose \( p, q \) are represented by \( \phi, \psi : Z_+ \to X \). Then \( q \in (U \cap V)[p] \) implies
   \[
   \psi(Z_+) \subseteq E[\text{cst}(\phi(Z_+),U \cap V)]
   = E[\bigcup\{U_i \cap V_i : \phi(Z_+) \wedge U_i \cap V_i\}]
   \subseteq \bigcup_{U_i \cap V_i \wedge \phi(Z_+)} E[U_i] \cap E[V_i]
   \subseteq (\bigcup_{U_i \wedge \phi(Z_+)} E[U_i]) \cap (\bigcup_{V_i \wedge \phi(Z_+)} E[V_i])
   = E[\text{cst}(\phi(Z_+),U)] \cap E[\text{cst}(\psi(Z_+),V)]
   \]
   In the same way \( \phi(Z_+) \subseteq E[\text{cst}(\psi(Z_+),U)] \cap E[\text{cst}(\phi(Z_+),V)] \). Thus \( q \in U[p] \cap V[p] \). This way we have proven:
   \[ D_U \cap V \subseteq D_U \cap D_V \]
3. If \( \mathcal{U} \) is a coarse cover of \( X \) then there is a coarse cover \( \mathcal{V} \) of \( X \) such that \( D_V \circ D_V \subseteq D_U \): By Lemma 36 there is a coarse star refinement \( \mathcal{V} \) of \( \mathcal{U} \). And by Lemma 48 item 2 the uniform cover \( (\mathcal{V}[p])_p \) star refines the uniform cover \( (U[p])_p \) thus the result.
4. If \( \mathcal{U} \) is a coarse cover then \( D_U = D_U^{-1} \).

A subset \( D \subseteq E(X)^2 \) is an entourage of the uniform structure of \( E(X) \) if there is a coarse cover \( \mathcal{U} \) of \( X \) such that

\[ D_U \subseteq D. \]

\[ \square \]

Theorem 42. If \( f : X \to Y \) is a coarse map between metric spaces then the induced map

\[
E(f) : E(X) \to E(Y) \\
[\varphi] \mapsto [f \circ \varphi]
\]

is a continuous map between topological spaces.

Proof. \( \bullet \) well defined: if \( \phi, \psi : Z_+ \to X \) represent the same endpoint in \( X \) then there is some entourage \( E \subseteq X^2 \) such that \( E[\psi(Z_+)] = \phi(Z_+) \). But then

\[
f^2(E)[f \circ \psi(Z_+)] \supseteq f(E[\psi(Z_+)])
= f \circ \phi(Z_+)
\]

Thus \( f \circ \phi, f \circ \psi \) represent the same endpoint in \( Y \).
• $E(f)$ continuous:

1. We show that the reverse image of an open set is an open set.
2. Let $U \subseteq E(Y)$ be open and $p \in E(f)^{-1}(U)$ be a point. Suppose that $p$ is represented by a coarse map $\phi : Z_+ \to X$. Then $f \circ \phi$ represents $E(f)(p) \in U$. Now there is a coarse cover $U = (U_i)_i$ of $Y$ such that $U[E(f)(p)] \subseteq U$. Then $f^{-1}(U) = (f^{-1}(U_i))_i$ is a coarse cover of $X$.
3. If $q \in f^{-1}(U)[p]$ we show that $E(f)(q) \in U[E(f)(p)]$; Suppose that $q$ is represented by a coarse map $\psi : Z_+ \to X$. Then there is some entourage $F \subseteq X^2$ such that

\[
F[cst(\phi(Z_+), f^{-1}(U))] \supseteq \psi(Z_+)
\]

and

\[
F[cst(\psi(Z_+), f^{-1}(U))] \supseteq \phi(Z_+)
\]

By Lemma 35

\[
f^2(F)[cst(f \circ \phi(Z_+), U)] \supseteq f^2(F)[f[cst(\phi(Z_+), f^{-1}(U)))]
\]

\[
\supseteq f(F[cst(\phi(Z_+), f^{-1}(U))])
\]

\[
\supseteq f \circ \psi(Z_+)
\]

and

\[
f^2(F)[cst(f \circ \psi(Z_+), U)] \supseteq f^2(F)[f[cst(\psi(Z_+), f^{-1}(U)))]
\]

\[
\supseteq f(F[cst(\psi(Z_+), f^{-1}(U))])
\]

\[
\supseteq f \circ \phi(Z_+)
\]

Now $f \circ \psi$ represents $E(f)(q)$ which by the above is in $U[E(f)(p)]$.

\[\square\]

Remark 43. The proof of Theorem 12 uses the following: if $f : X \to Y$ is a coarse map and $D_U$ the entourage of $E(Y)$ associated to a coarse cover $U$ of $Y$ then there is an entourage $D_{f^{-1}(U)}$ of $E(X)$ associated to the coarse cover $f^{-1}(U)$ of $X$ such that $(p,q) \in D_{f^{-1}(U)}$ implies $(E(f)(p), E(f)(q)) \in D_U$. Thus $E(f)$ is a uniformly continuous map between uniform spaces $E(X)$ and $E(Y)$.

Lemma 44. If two coarse maps $f, g : X \to Y$ are close then $E(f) = E(g)$.

Proof. Let $p \in E(X)$ be a point that is represented by $\varphi$. Now $f, g$ are close thus $H := f \times g(\Delta_X)$ is an entourage. But then

\[
H[g \circ \varphi(Z_+)] \supseteq f \circ \varphi(Z_+)
\]

thus $E(f)(p) = E(g)(p)$.

\[\square\]

Corollary 45. If $f$ is a coarse equivalence then $E(f)$ is a homeomorphism between topological spaces $E(X)$ and $E(Y)$. In fact $E(f)$ is a uniform isomorphism between uniform spaces $E(X)$ and $E(Y)$.

Corollary 46. If $\mathcal{mCoarse}$ denotes the category of metric spaces and coarse maps modulo closeness and $\Top$ the category of topological spaces and continuous maps then $E$ is a functor

\[
E : \mathcal{mCoarse} \to \Top
\]

If $\mathcal{Uniform}$ denotes the category of uniform spaces and uniformly continuous maps then $E$ is a functor

\[
E : \mathcal{mCoarse} \to \mathcal{Uniform}
\]
Example 47. \( E(\mathbb{Z}_+) \) is a point.

3.2 Properties

Lemma 48. If \( X \) is a metric space

- \( \mathcal{V} \) is a coarse star refinement of a coarse cover \( \mathcal{U} \) of \( X \) then \( q \in \mathcal{V}[p] \) and \( r \in \mathcal{V}[q] \) implies \( r \in \mathcal{U}[p] \).
- if \( \mathcal{V} \) coarsely star refines \( \mathcal{U} \) then \( (\mathcal{V}[p])_p \) star refines \( (\mathcal{U}[p])_p \)

Proof. Suppose \( p \) is represented by \( \phi : \mathbb{Z}_+ \to X \), \( q \) is represented by \( \psi : \mathbb{Z}_+ \to X \) and \( r \) is represented by \( \rho : \mathbb{Z}_+ \to X \). Then \( E[\text{cst}(\phi(\mathbb{Z}_+)), \mathcal{V}] \supseteq \psi(\mathbb{Z}_+) \) and \( E[\text{cst}(\psi(\mathbb{Z}_+)), \mathcal{V}] \supseteq \phi(\mathbb{Z}_+) \), \( E[\text{cst}(\rho(\mathbb{Z}_+)), \mathcal{V}] \supseteq \psi(\mathbb{Z}_+) \) and \( E[\text{cst}(\psi(\mathbb{Z}_+)), \mathcal{V}] \supseteq \rho(\mathbb{Z}_+) \). By Lemma 35 there is an entourage \( F \subseteq \mathbb{R}^2 \) such that \( \text{cst}(\text{cst}(\phi(\mathbb{Z}_+)), \mathcal{V}), \mathcal{V}) \subseteq F[\text{cst}(\phi(\mathbb{Z}_+), \mathcal{U})] \). Then

\[
E^{o2} \circ F[\text{cst}(\phi(\mathbb{Z}_+), \mathcal{U})] \supseteq E^{o2}[\text{cst}(\text{cst}(\phi(\mathbb{Z}_+), \mathcal{V}), \mathcal{V})] \\
\supseteq E[\text{cst}(\psi(\mathbb{Z}_+), \mathcal{V})] \\
\supseteq \rho(\mathbb{Z}_+)
\]

the other direction works the same way.

- Fix \( p \in E(X) \). Then

\[
st(\mathcal{V}[p], (\mathcal{V}[p])_p) \subseteq \mathcal{U}[p]
\]

because if \( q \in \mathcal{V}[p] \) and \( q \in \mathcal{V}[r] \) then \( r \in \mathcal{U}[p] \) by Item 1 \( \square \)

Proposition 49. If \( i : Z \to Y \) is an inclusion of metric spaces then \( E(i) : E(Z) \to E(Y) \) is a uniform embedding.

Proof. \( E(i) \) injective: easy.

- Define a map

\[
\Phi : E(i)(E(Z)) \to E(Z) \\
E(i)(p) \mapsto p
\]

We show \( \Phi \) is a uniformly continuous map:

- If \( \mathcal{U} = (U_i)_i \) is a coarse cover of \( Z \) we show there is a coarse cover \( \mathcal{V} \) of \( Y \) such that for every \( p, q \in E(Z) \) the relation \( E(i)(q) \in \mathcal{V}[E(i)(p)] \) implies \( q \in \mathcal{U}[p] \).

- Note that for every \( i \) the sets \( U_i^c \not\subset (\bigcup_{j \neq i} U_j)^c \) are coarsely disjoint in \( Y \). By Lemma 35 there are subsets \( W_1, W_2 \subseteq Y \) that coarsely cover \( Y \) and \( W_1^c \not\subset U_i^c \) and \( W_2^c \not\subset (\bigcup_{j \neq i} U_j)^c \). Now define

\[
\mathcal{V} := (\bigcap_i W_{\sigma(i)}^i)_{\sigma}
\]

with \( \sigma(i) \in \{1, 2\} \) all possible permutations. Note that there is some entourage \( E \subseteq \mathbb{R}^2 \) such that for every \( \sigma \) there is some \( U_j \) such that

\[
\bigcap_i W_{\sigma(i)}^i \cap Z \subseteq E[U_j]
\]
Let \( p, q \in E(Z) \) such that \( E(i)(q) \in V(E(i)(p)) \). Suppose \( p, q \) are represented by \( \phi, \psi : Z_+ \to Z \). Then there is some entourage \( F \subseteq Y^2 \) such that

\[
F[\text{cst}(\phi(Z_+), V)] \supseteq \psi(Z_+)
\]

and

\[
F[\text{cst}(\psi(Z_+), V)] \supseteq \phi(Z_+)
\]

then

\[
F \circ E[\text{cst}(\phi(Z_+), U)] \supseteq F[\text{cst}(\phi(Z_+), V) \cap Z] \supseteq \psi(Z_+)
\]

The other direction works the same way.

Remark 50. By Proposition 49 and Corollary 45 every coarsely injective coarse map \( f : X \to Y \) induces a uniform embedding. We identify \( E(X) \) with its image \( E(f)(E(X)) \) in \( E(Y) \).

Example 51. There is a coarsely surjective coarse map \( \omega : Z_+ \to Z^2 \). Now \( E(\omega) : E(Z_+) \to E(Z^2) \) is not a surjective map obviously.

Lemma 52. If two subsets \( U, V \) coarsely cover a metric space \( X \) then

\[
E(U \cap V) = E(U) \cap E(V).
\]

Proof. The inclusion \( E(U \cap V) \subseteq E(U) \cap E(V) \) is obvious.

• we show the reverse inclusion: if \( p \in E(U) \cap E(V) \) then it is represented by \( \phi : Z_+ \to U \) in \( E(U) \) and \( \psi : Z_+ \to V \) in \( E(V) \). Then there is an entourage \( E \subseteq X^2 \) such that

\[
E[\psi(Z_+)] = \phi(Z_+)
\]

Note that \( E \cap V^c \times U^c \) is bounded. Denote by \( F \) the set of indices \( i, j \) for which

\[
E \cap (\phi(i), \psi(j)) \subseteq V^c \times U^c
\]

Now we construct a coarse map \( \varphi : Z_+ \to U \cap V \): for every \( i \in \mathbb{N} \setminus F \) do:

1. if \( \phi(i) \in V \) then define \( \varphi(i) := \phi(i) \).
2. if \( \phi(i) \in V^c \) then define \( \varphi(i) := \psi(i) \)

Fix a point \( x_0 \in U \cap V \) then for every \( i \in F \) define: \( \varphi(i) = x_0 \). Then \( \varphi \) represents \( p \) in \( E(U \cap V) \).

Lemma 53. The functor \( E(\cdot) \) preserves coproducts.
Proof. Let \( X = A \sqcup B \) be a coarse disjoint union of metric spaces. Without loss of generality we assume that \( A, B \) cover \( X \) as sets. Fix a point \( x_0 \in X \). Then there is a coarse map
\[
r : X \to \mathbb{Z}
\]
\[
x \mapsto \begin{cases} d(x, x_0) & x \in A \\ -d(x, x_0) & x \in B \end{cases}
\]
Note that \( E(\mathbb{Z}) = \{-1, 1\} \) is a space which consists of two points with the discrete uniformity. Then \( E(r)(E(A)) = 1 \) and \( E(r)(E(B)) = -1 \). Thus \( E(X) \) is the uniform disjoint union of \( E(A), E(B) \).

Proposition 54. Let \( X \) be a metric space. The uniformity \( E(X) \) is separating.

Proof. If \( p \neq q \) are two points in \( E(X) \) we show there is a coarse cover \( U \) such that
\[ q \notin U[p] \]
1. Suppose \( p \) is represented by \( \phi : \mathbb{Z}_+ \to X \) and \( q \) is represented by \( \psi : \mathbb{Z}_+ \to X \). Now there is one of two cases:
   (a) there is a subsequence \( (i_k)_k \subseteq \mathbb{N} \) such that \( \phi(i_k)_k \not\in \psi(\mathbb{Z}_+) \).
   (b) there is a subsequence \( (j_k)_k \subseteq \mathbb{N} \) such that \( \psi(j_k)_k \not\in \phi(\mathbb{Z}_+) \).
Without loss of generality we can assume the first case holds. By Lemma 33 there is a coarse cover \( U = \{ U_1, U_2 \} \) of \( X \) such that \( U_1 \not\in \phi(i_k)_k \) and \( U_2 \not\in \psi(\mathbb{Z}_+) \). Then \( q \notin U[p] \).
2. Now
\[ q \notin U[p] = st(p, [U[r]]_r) \]
thus the result.

Lemma 55. If \( X \) is a metric space,
\begin{itemize}
  \item \( U = (U_i)_{i \in I} \) is a coarse cover of \( X \) and \( p \in E(X) \) is represented by \( \phi : \mathbb{Z}_+ \to X \) then define \( I(p) := \{ i \in I : \phi(\mathbb{Z}_+) \not\subseteq U_i \} \)
  \[ U(S) = \{ p \in E(X) : \phi(\mathbb{Z}_+) \not\subseteq E[\bigcup_{i \in S} U_i] \} \]
  here \( \phi : \mathbb{Z}_+ \to X \) represents \( p \) and \( E \subseteq X^2 \) is an entourage. If \( q \in E(X) \) then \( q \in U[p] \) if and only if \( q \in U(I(p)) \) and \( p \in U(I(q)) \).
  \item Define \( U(S) = \{ p \in E(X) : p \in U(S), I(p) \supseteq S \} \)
  Then \( q \in U[p] \) if and only if there is some \( S \subseteq I \) such that \( p, q \in U(S) \). The uniform cover
  \[ (U(S))_{S \subseteq I} \]
  associated to \( D_U \) is a finite cover.
\end{itemize}
The uniform space $E(X)$ is totally bounded.

**Proof.** • easy.

- We just need to show: if $q \in U(p)$ then $p \in U(I(p) \cap I(q))$. For that it is sufficient to show if $\phi : Z_+ \to X$ represents $p$ then there is an entourage $E \subseteq X^2$ such that $\phi(Z_+) \subseteq E[\bigcup_{i \in I(q)} U_i]$. Assume the opposite: there is some subsequence $(i_k)_k \subseteq Z_+$ such that

$$\phi(i_k) \not\in \bigcup_{i \in I(p) \cap I(q)} U_i$$

Now $\phi(Z_+) \not\in \bigcup_{i \not\in I(p)} U_i$ thus

$$\phi(i_k) \not\in (\bigcup_{i \in I(p) \cap I(q)} U_i) \cup (\bigcup_{i \not\in I(p)} U_i)$$

And thus $\phi(i_k) \not\in \bigcup_{i \in I(q)} U_i$ a contradiction to the assumption.

- easy

**Notation 56.** If $A, B \subseteq X$ are two subsets of a metric space

- and $x_0 \in X$ a point then define

$$\chi_{A,B} : \mathbb{N} \to \mathbb{R}_+$$

$$i \mapsto d(A \setminus B(x_0, i), B \setminus B(x_0, i))$$

if $A \not\subseteq B$ then $\chi_{A,B}$ is a coarse map there is a bound

$$\chi_{A,B}(i) \leq 2i$$

- Now $A \subseteq B$ if and only if $\chi_{A,B}$ is bounded.

- If $A_1, A_2 \subseteq X$ are subsets with $A_1 = E[A_2]$ then

$$i \mapsto |\chi_{A_1,B}(i) - \chi_{A_2,B}(i)|$$

is bounded.

- If $\chi \in \mathbb{R}_+^\mathbb{N}$ is a coarse map then the class $m(\chi)$ of $\chi$ is at least $\varepsilon \geq 0$ if

$$\chi(i) \geq \varepsilon i + c$$

where $c \leq 0$ is a constant. If two coarse maps $\chi_1, \chi_2 \in \mathbb{R}_+^\mathbb{N}$ are close then they have the same class.

- The following properties hold:

1. If $A_1 \subseteq A_2$ then $\chi_{A_1,B} \geq \chi_{A_2,B}$.
2. If $A, B, C \subseteq X$ then $\chi_{A,B} + \chi_{B,C} \geq \chi_{A,C}$.
3. If $A, B, C \subseteq X$ then $m(\chi_{A,B}) + m(\chi_{B,C}) \geq m(\chi_{A,B} + \chi_{B,C})$.

**Proof.** • easy.
are represented by \( \phi, \psi, \rho \).

We define \( d \) a subsequence (\( \phi, \psi \) maps \( Z \) to \( X \)) such that one of the following holds:

1. \( \phi(Z_+) \not\sim \psi(i_k)_k \) and \( m(\chi_{\phi(Z_+)}(\psi(i_k)_k)) \geq \varepsilon \)
2. \( \psi(Z_+) \not\sim \phi(i_k)_k \) and \( m(\chi_{\psi(Z_+)}(\phi(i_k)_k)) \geq \varepsilon \)

We define \( d(p, q) = 0 \) if and only if \( p = q \).

**Lemma 58.** The map \( d : E(X)^2 \to \mathbb{R}_+ \) indeed defines a metric on endpoints.

**Proof.** The properties 1,2 are easy to prove. We show the triangle inequality: If \( p, q, r \in E(X) \) are represented by \( \phi, \psi, \rho : Z_+ \to X \) and \( d(p, r) \geq \varepsilon \), we show \( d(p, q) + d(q, r) \geq \varepsilon \).

- If \( d(p, r) \geq \varepsilon \) then there is a subsequence \( (i_k)_k \subseteq Z_+ \) such one of two cases holds:
  1. \( \phi(Z_+) \not\sim \rho(i_k)_k \) and \( m(\chi_{\phi(Z_+)}(\rho(i_k)_k)) \geq \varepsilon \)
  2. \( \rho(Z_+) \not\sim \phi(i_k)_k \) and \( m(\chi_{\rho(Z_+)}(\phi(i_k)_k)) \geq \varepsilon \)

Without loss of generality we assume the first case holds.

- If we add \( q \) then for every subsequences \( (j_k)_k, (l_k)_k \subseteq Z_+ \) there are 4 cases that can arise:
  1. \( \phi(Z_+) \not\sim \psi(j_k)_k \) and \( m(\chi_{\phi(Z_+)}(\psi(j_k)_k)) \geq \varepsilon_1 \)
  2. \( \psi(Z_+) \not\sim \phi(j_k)_k \) and \( m(\chi_{\psi(Z_+)}(\phi(j_k)_k)) \geq \varepsilon_1 \)
  3. \( \psi(Z_+) \not\sim \rho(l_k)_k \) and \( m(\chi_{\psi(Z_+)}(\rho(l_k)_k)) \geq \varepsilon_2 \)
  4. \( \rho(Z_+) \not\sim \psi(l_k)_k \) and \( m(\chi_{\rho(Z_+)}(\psi(l_k)_k)) \geq \varepsilon_2 \)

Which means exactly: Either (1 and 3) or (1 and 4) or (2 and 3) or (2 and 4) are minimal choices. We check each case:

- 1,3: Now

\[
m(\chi_{\phi(Z_+)}(\psi(j_k)_k)) + m(\chi_{\psi(Z_+)}(\rho(i_k)_k)) \geq m(\chi_{\phi(Z_+)}(\psi(j_k)_k) + \chi_{\psi(Z_+)}(\rho(i_k)_k))
\geq m(\chi_{\phi(Z_+)}(\psi(j_k)_k) + \chi_{\psi(Z_+)}(\rho(i_k)_k))
\geq m(\chi_{\phi(Z_+)}(\rho(i_k)_k))
\]

- 1,4: similar.

- 2,3: similar.

- 2,4: similar.

\[\square\]
Lemma 59. If $X$ is a metric space and $\mathcal{U}$ a coarse cover of $X$ then there is an $\varepsilon > 0$ such that for every two endpoints $p, q \in E(X)$ the relation $q \notin \mathcal{U}[p]$ implies $d(p, q) \geq \varepsilon$.

Proof. • By Lemma 59 the uniform space $E(X)$ is totally bounded. Without loss of generality we can fix an endpoint $p \in E(X)$ and study the endpoints $q \in E(X)$ for which $q \notin \mathcal{U}[p]$.

• We will define an $\varepsilon \geq 0$ as the minimum of a finite collection $\varepsilon_0, \ldots, \varepsilon_n \geq 0$ of numbers.

• If $q \notin \mathcal{U}[p]$ there are 2 cases:
  1. $p \notin \mathcal{U}(I(q))$: There is a subset $(i_k)_{k \in Z^+}$ such that
     \[ \phi(i_k) \notin \bigcup_{i \in I(q)} U_i \]
     Now for every $S \subseteq I$ if $p \notin \mathcal{U}(S)$: then there is some subset $(i_k)_{k \in Z^+}$ such that
     \[ \phi(i_k) \notin \bigcup_{i \in S} U_i. \]
     Thus if $I(q) = S$ then $d(p, q) > m(\chi_{\phi(i_k)} \cup_{i \in S} U_i)$. Define
     \[ \varepsilon_S := m(\chi_{\phi(i_k)} \cup_{i \in S} U_i) \]
     in this case.
  2. $q \notin \mathcal{U}(I(p))$: There is some $(i_k)_{k \in Z^+}$ such that
     \[ \psi(i_k) \notin \bigcup_{i \in I(p)} U_i \]
     Now $\psi(i_k) \subseteq \bigcup_{i \notin I(p)} U_i$ and $\phi(Z^+) \notin \bigcup_{i \notin I(p)}$. Then define
     \[ \varepsilon_b := m(\chi_{\phi(Z^+)} \cup_{i \notin I(p)} U_i) \]

• Then
  \[ \varepsilon := \min_S(\varepsilon_S, \varepsilon_b) \]
  has the desired properties.
\[ \square \]

Proposition 60. If $X$ is a metric space then the uniformity on $E(X)$ has a countable base.

Proof. 1. Note that the metric $d$ on $E(X)$ as defined in Definition 57 induces a uniformity, say $\tau_d$.

2. By Lemma 59 every entourage in $E(X)$ is a neighborhood of an entourage of $\tau_d$ on $E(X)$.

3. Now $\tau_d$ has a countable base, thus the result.
\[ \square \]

Corollary 61. If $X$ is a metric space then $E(X)$ is a metrizable uniformity.

Proof. We use [14, Corollary 38.4] by which a uniformity is metrizable if it is separating and has a countable base. By Proposition 54 the space $E(X)$ is separating and by Proposition 60 it admits a countable base.
\[ \square \]
3.3 Side Notes

Remark 62. (large-scale category) Large-scale geometry\(^4\) (LargeScale) studies metric spaces and large-scale maps modulo closeness. Note the following facts:

1. Every large-scale map is already coarsely uniform.
2. Isomorphisms in LargeScale are called quasi-isometries.
3. A metric space is coarsely geodesic if and only if it is coarsely equivalent to a geodesic metric space.
4. A metric space is large-scale geodesic if and only if it is quasi-isometric to a geodesic metric space.
5. A coarse map \(f : X \to Y\) between large-scale geodesic metric spaces is already large-scale.
6. A coarse equivalence \(f : X \to Y\) between large-scale geodesic metric spaces is already a quasi-isometry.

Proof. 1. easy.

2. definition.

3. see [2] Lemma 3.B.6,(5)]

4. see [2] Lemma 3.B.6,(6)]

5. see [2] Proposition 3.B.9,(1)]

6. see [2] Proposition 3.B.9,(2)]

Lemma 63. (Higson corona) If \(X\) is a metric space then the \(C^*\)-algebra that determines the Higson corona is a sheaf. That means exactly that the association

\[ U \mapsto C(\nu X) = B_h(X)/B_0(X) \]

for every subset \(U \subseteq X\) is a sheaf with values in \(\text{CStar}\).

Proof. We recall a few definitions which can be found in [2], p.29,30].

- The algebra of bounded functions that satisfy the Higson condition is denoted by \(B_h\).
- A bounded function \(f : X \to \mathbb{C}\) satisfies the Higson condition if for every entourage \(E \subseteq X^2\) the function

\[ df|_E : E \to \mathbb{C} \]

\( (x,y) \mapsto f(y) - f(x) \)

wends to 0 at infinity.
- the ideal of bounded functions that tend to 0 at infinity is called \(B_0\).
- A function \(f : X \to \mathbb{C}\) tends to 0 at infinity if for every \(\epsilon > 0\) there is a bounded subset \(B \subseteq X\) such that \(|f(x)| \geq \epsilon\) implies \(x \in B\).

\(^4\)The notation is from [2]
We check the sheaf axioms:

1. global axiom: if $U_1, U_2$ coarsely cover a subset $U \subseteq X$ and $f \in B_h(X)$ such that $f|_{U_1} \in B_0(U_1)$ and $f|_{U_2} \in B_0(U_2)$ we show that $f \in B_0(U)$ already. Let $\varepsilon > 0$ be a number. Then there are bounded subsets $B_1 \subseteq U_1$ and $B_2 \subseteq U_2$ such that $|f(x)| \geq \varepsilon$ implies $x \in B_i$ for $i = 1, 2$. Now

$$B := B_1 \cup B_2 \cup (U_1 \cup U_2)^c$$

is a bounded subset of $U$. Then $|f(x)| \geq \varepsilon$ implies $x \in B$. Thus $f \in B_0(U)$.

2. gluing axiom: if $U_1, U_2$ coarsely cover a subset $U \subseteq X$ and $f_1 \in B_h(U_1), f_2 \in B_h(U_2)$ are functions such that

$$f_1|_{U_2} = f_2|_{U_1} + g$$

where $g \in B_0(U_1 \cap U_2)$. We show there is a function $f \in B_h(U)$ which restricts to $f_1$ on $U_1$ and $f_2 + g$ on $U_2$. Define:

$$f : U \rightarrow \mathbb{C}$$

$$x \mapsto \begin{cases} f_1(x) & x \in U_1 \\ f_2(x) + g & x \in U_2 \\ 0 & \text{otherwise} \end{cases}$$

then $f$ is a bounded function. We show $f$ satisfies the Higson condition: Let $E \subseteq U^2$ be an entourage and $\varepsilon > 0$ be a number. Then there are bounded subsets $B_1 \subseteq U_1$ and $B_2 \subseteq U_2$ such that $|df|^E_{E \cap U^2}(x, y) \geq \varepsilon$ implies $x \in B_i$ for $i = 1, 2$. There is a bounded subset $B_3 \subseteq U$ such that

$$E \cap (U_1^2 \cup U_2^2)^c \cap U^2 \subseteq B_3^2$$

Define

$$B := B_1 \cup B_2 \cup B_3$$

then $|df|_E(x) \geq \varepsilon$ implies $x \in B$. Thus $f$ has the desired properties.

\[\square\]

**Lemma 64.** If $X$ is a proper geodesic metric space denote by $\sim$ the relation on $E(X)$ of belonging to the same uniform connection component in $E(X)$ then there is a continuous bijection

$$E(X)/\sim \rightarrow \Omega(X)$$

where the right side denotes the space of ends of $X$ as a topological space.

**Proof.** There are several different definitions for the space of ends of a topological space. We use [1] Definition 8.27.

- An end in $X$ is represented by a proper continuous map $r : [0, \infty) \rightarrow X$. Two such maps $r_1, r_2$ represent the same end if for every compact subset $C \subseteq X$ there is some $N \in \mathbb{N}$ such that $r_1[N, \infty), r_2[N, \infty)$ are contained in the same path component of $X \setminus C$.

- If $r : [0, \infty) \rightarrow X$ is an end then there is a coarse map $\varphi : \mathbb{Z}_+ \rightarrow X$ and an entourage $E \subseteq X^2$ such that

$$E[r[0, \infty]] = \varphi(\mathbb{Z}_+)$$

we construct $\varphi$ inductively:
4 Remarks

1. \( \varphi(0) := r(0) \)
2. If \( \varphi(i - 1) = r(t_{i-1}) \) is already defined then \( t_i := \min \{ t > t_{i-1} : d(\varphi(t_{i-1}), \varphi(t)) = 1 \} \). Set \( \varphi(i) := r(t_i) \).

By the above construction \( \varphi \) is coarsely uniform. The map \( \varphi \) is coarsely proper because \( r \) is proper and \( X \) is proper.

- Note that every geodesic space is also a length space. If for some compact subset \( C \subseteq X \) the space \( X \setminus C \) has two path components \( X_1, X_2 \) then for every \( x_1 \in X_1, x_2 \in X_2 \) a path (in particular the shortest) joining \( x_1 \) to \( x_2 \) contains a point \( c \in C \). Thus

\[
d(x_1, x_2) = \inf_{c \in C} (d(x_1, c) + d(x_2, c))
\]

Then \( X \) is the coarse disjoint union of \( X_1, X_2 \). On the other hand if \( X \) is the coarse disjoint union of subspaces \( X_1, X_2 \) then there is a bounded and in particular because \( X \) is proper compact subset \( C \subseteq X \) such that

\[
X \setminus C = X_1' \sqcup X_2'
\]

is a path disjoint union and \( X_1' \subseteq X_1, X_2' \subseteq X_2 \) differ only by bounded sets.

- Now we show the association is continuous:

1. We use [1] Lemma 8.28 in which \( G_{x_0}(X) \) denotes the set of geodesic rays issuing from \( x_0 \in X \). Then [1] Lemma 8.28 states that the canonical map

\[
G_{x_0} \to \Omega(X)
\]

is surjective. Fix \( r \in G_{x_0} \). Then \( \tilde{V}_n \subseteq G_{x_0} \) denotes the set of proper rays \( r' : \mathbb{R}_+ \to X \) such that \( r'(n, \infty), r(n, \infty) \) lie in the same path component of \( X \setminus B(x_0, n) \). Now [1] Lemma 8.28 states the sets \( \{ V_n = \{ r' : r' \in \tilde{V}_n \} \} \) form a neighborhood base for \( [r] \in \Omega(X) \).

2. Now to every \( n \) we denote by \( U_1^n \) the path component of \( X \setminus B(x_0, n) \) that contains \( r(\mathbb{R}_+) \) and we define \( U_2^n := X \setminus U_1^n \). For every \( n \in \mathbb{N} \) the sets \( U_1^n, U_2^n \) are a coarse cover of \( X \).

3. Suppose \( \rho : \mathbb{Z}_+ \to X \) is a coarse map associated to \( r \) and represents \( \tilde{r} \in E(X) \). If \( s \in G_0 \) suppose \( \sigma : \mathbb{Z}_+ \to X \) is the coarse map associated to \( s \) and represents \( \tilde{s} \in E(X) \). If \( [s] \not\in V_n \) then \( \sigma(\mathbb{Z}_+) \not\subseteq U_1^n \). This implies \( \tilde{s} \not\in \{ U_1^n, U_2^n \}[\tilde{r}] \).

4. Thus for every \( n \in \mathbb{N} \) there is an inclusion \( \{ U_1^n, U_2^n \}[\tilde{r}] / \sim \subseteq V_n \) by the association.

\( \square \)

4 Remarks

The starting point of this research was an observation in the studies of [3]: coarse cohomology with twisted coefficients looked like singular cohomology on some kind of boundary. We tried to find a functor from the coarse category to the category of topological spaces that would reflect that observation.

And then we noticed that two concepts play an important role: One is the choice of topology on the space of ends and one is the choice of points. The points were designed such that
• coarse maps are mapped by the functor to maps of sets
• and the space $\mathbb{Z}_+$ is mapped to a point

If the metric space is Gromov hyperbolic then coarse rays represent the points of the Gromov boundary, thus the Gromov boundary is a subset of the space of ends. The topology was trickier to find. We looked for the following properties:

• coarse maps are mapped to continuous maps
• coarse embeddings are mapped to topological embeddings

Now a proximity relation on subsets of a topological space helps constructing the topology on the space of ends of Freudenthal. We discovered that coarse covers on metric spaces give rise to a totally bounded uniformity and thus used that a uniformity on a space gives rise to a topology.

Finally, after a lucky guess, we came up with the uniformity on the set of endpoints. In which way does the space of ends functor reflect isomorphism classes will be studied in a paper that follows.

It would be possible, conversely, after a more thorough examination to find more applications. Coarse properties on metric spaces may give rise to topological properties on metrizable uniform spaces.

We wonder if this result will be of any help with classifying coarse spaces up to coarse equivalence. However, as of yet, the duality has not been studied in that much detail.

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