Nonlinear BSDEs with Two Optional Doob’s Class Barriers Satisfying Weak Mokobodzki’s Condition and Extended Dynkin Games

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Abstract
We study reflected backward stochastic differential equations (RBSDEs) on the probability space equipped with a Brownian motion. The main novelty of the paper lies in the fact that we consider the following weak assumptions on the data: barriers are optional of class (D) satisfying weak Mokobodzki’s condition, generator is continuous and non-increasing with respect to the value-variable (no restrictions on the growth) and Lipschitz continuous with respect to the control-variable, and the terminal condition and the generator at zero are supposed to be merely integrable. We prove that under these conditions on the data there exists a solution to corresponding RBSDE. In the second part of the paper, we apply the theory of RBSDEs to solve basic problems in Dynkin games driven by nonlinear expectation based on the generator mentioned above. We prove that the main component of a solution to RBSDE represents the value process in corresponding extended nonlinear Dynkin game. Moreover, we provide sufficient conditions on the barriers guaranteeing the existence of the value for nonlinear Dynkin games and the existence of a saddle point.

Keywords Reflected backward stochastic differential equations · Optional barriers · Dynkin games · Nonlinear expectation

Mathematics Subject Classification 60H20 · 60H25

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1 Introduction

Let $B$ be a standard $d$-dimensional Brownian motion on a given probability space $(\Omega, \mathcal{F}, P)$, $T$ be a strictly positive real number (horizon time) and let $\mathbb{F} := (\mathcal{F}_t)_{0 \leq t \leq T}$ be the standard augmentation of the filtration generated by $B$. In the present paper, we study Reflected Backward Stochastic Differential Equations (RBSDEs for short) of the following form

$$
\begin{aligned}
Y_t &= \xi + \int_t^T f(r, Y_r, Z_r) \, dr + R_T - R_t - \int_t^T Z_r \, dB_r, \quad t \in [0, T], \\
L_t &\leq Y_t \leq U_t, \quad t \in [0, T],
\end{aligned}
$$

where $\xi$ (terminal value) is an $\mathcal{F}_T$-measurable random variable, the mapping $f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ (generator) is an $\mathbb{F}$-progressively measurable process with respect to the first two variables and $L, U$ (barriers) are $\mathbb{F}$-optional processes of class (D). We look for a triple $(Y, Z, R)$ of finite variation and $\mathbb{F}$-progressively measurable processes, with $R$ of finite variation and $R_0 = 0$, that satisfies (1). Given a solution $(Y, Z, R)$ to (1), we call $Y$ the main part of the solution. The role of $R$ is to keep $Y$ between barriers $L, U$, and the role of $Z$ is to keep $Y$ adapted to $\mathbb{F}$. In order to get the uniqueness for problem (1) one requires $R$ to satisfy the so called minimality condition which states that

$$
\begin{aligned}
\int_0^T (Y_r - \limsup_{s \uparrow r} L_s) \, dR_r^{*,+} + \int_0^T (\liminf_{s \uparrow r} U_s - Y_r) \, dR_r^{*,-} &= 0, \\
\sum_{0 \leq r < T} (Y_r - L_r) \max\{R_r^{*,+} - R_r, 0\} + \sum_{0 \leq r < T} (U_r - Y_r) \max\{R_r - R_r^{*,+}, 0\} &= 0,
\end{aligned}
$$

where $R^*$ is the càdlàg part of $R$, i.e.

$$
R_t^* = R_t - \sum_{0 \leq s < t} (R_{s+}^* - R_s),
$$

and $R^{*,+}, R^{*,-}$ its Jordan decomposition.

Formulation of the problems In the paper, we merely assume that

(A1) $\mathbb{E}[\xi] + \mathbb{E} \int_0^T |f(r, 0, 0)| \, dr < \infty$,

(A2) There is $\lambda \geq 0$ such that $|f(t, y, z) - f(t, y, z')| \leq \lambda |z - z'|$ for $t \in [0, T], \ y \in \mathbb{R}, \ z, z' \in \mathbb{R}^d$,

(A3) $y \mapsto f(t, y, z)$ is non-increasing and continuous for fixed $t \in [0, T], \ z \in \mathbb{R}^d$,

(A4) $\int_0^T |f(r, y, 0)| \, dr < \infty$ for every $y \in \mathbb{R}$,

(Z) There exist $\gamma \geq 0, \ k \in [0, 1)$ and a non-negative $\mathbb{F}$-progressively measurable process $g$, satisfying $\mathbb{E} \int_0^T g_r \, dr < \infty$, such that

$$
|f(t, y, z) - f(t, y, 0)| \leq \gamma (g_t + |y| + |z|)^k, \quad t \in [0, T], \ y \in \mathbb{R}, \ z \in \mathbb{R}^d.
$$

Note that we do not impose any restrictions on the growth of $f$ with respect to $y$-variable. By [4], under (A1)–(A4), (Z), there exists a solution $(Y, Z)$ to (1) without
barriers (BSDE), i.e.

\[ Y_t = \xi + \int_t^T f(r, Y_r, Z_r) \, dr - \int_t^T Z_r \, dB_r, \quad t \in [0, T]. \tag{3} \]

Moreover, it is unique provided that \( Y \) is of class (D) and for some \( s > \kappa \), \( Z \in H^s(0, T) \)—the class of \( \mathbb{F} \)-progressively measurable processes that satisfy \( \mathbb{E}[\int_0^T |Z_r|^s \, dr]^{s/2} < \infty \). In the present paper, we focus on the existence and uniqueness problem for (1)–(2) under conditions (A1)–(A4), (Z). We shall also study the representation of the process \( Y \) as the value process in nonlinear Dynkin games.

The existence problem. First, observe that by the very definition of a solution to (1) its main part is an optional semimartingale (the sum of a càdlàg local martingale and an optional finite variation process). Consequently, we deduce at once, that the existence of an optional semimartingale between the barriers \( L, U \) is a necessary condition for the existence of a solution to (1) (intrinsic condition). The said condition is known in the literature (for càdlàg semimartingales) as weak Mokobodzki’s condition (see [21]), and we shall continue to use this nomenclature for optional semimartingales:

\( (WM) \) There exists an optional semimartingale \( X \) such that \( L_t \leq X_t \leq U_t, \ t \in [0, T] \).

Therefore, the natural question arises whether under (A1)–(A4), (Z) the above condition (WM) is also sufficient for the existence of a solution to (1)–(2). It is known that the stronger condition, so called Mokobodzki’s condition:

\( (M) \) Condition (WM) holds and \( \mathbb{E}([M]_T)^{p/2} + \mathbb{E}[|V|^p_T] < \infty \) for some \( p > 1 \), where \( X_t = X_0 + V_t + M_t \) is the Mertens decomposition of the optional semimartingale \( X \),

is sufficient for the existence of a solution to (1)–(2) under (A1)–(A4), (Z). The class of barriers that meet condition (WM) is significantly wider than its subclass determined by condition (M). In particular any pair of barriers satisfying so called complete separation condition:

\[ L, U \text{ are càdlàg, } L_t < U_t, \ t \in [0, T], \ L_{t-} < U_{t-}, \ t \in (0, T], \tag{4} \]

fulfills (WM) (see [38, Lemma 3.1]). Reflected BSDEs with complete separation condition (4) imposed on the barriers have been studied quite extensively in the literature (see e.g. [3, 5, 13, 20, 22–24, 38]). Our paper may also be seen as a continuation and extension of this research.

In the main result of the paper, we give a positive answer to the posed question, and even prove more, that condition (Z) can be dropped.

**Theorem 1** Assume that (A1)–(A4) hold and weak Mokobodzki’s condition (WM) is in force. Then there exists a solution \((Y, Z, R)\) to (1)–(2).

It appears, and it may seem surprising at first, that the above result does not hold for BSDEs (3) (see Remark 1). The explanation of this phenomenon is that in the case of reflected BSDEs barriers keep the main part of a solution in the class (D).
In Sect. 9 we give an easy criterion (not optimal) for barriers $L, U$ to satisfy (WM). Namely, we show that whenever

$$L, U \text{ are left-limited, } L_t^- < U_t^-, \quad t \in (0, T],$$

$$\limsup_{s \downarrow t} L_s < \liminf_{s \downarrow t} U_s, \quad t \in [0, T),$$

then (WM) holds.

The uniqueness problem. An interesting issue is also the problem of the uniqueness for solutions to (1)–(2). In the proof of the uniqueness for BSDEs (3) (see [4] and Theorem 2) the crucial roles were played by condition (Z) and the fact that for any solution $(Y, Z)$ to (3) we have, under conditions (A1)–(A4), (Z), that $Z \in H^s_T(0, T)$ for some $s > \kappa$ provided $Y$ is of class (D). For reflected BSDEs this property does not hold even if $f \equiv 0$ (see [26, Example 5.6]). Nevertheless, we are able to prove the following result.

**Theorem 2** Assume that (A1)–(A4), (Z) are in force. Then there exists at most one solution to RBSDE (1)–(2).

Solutions to RBSDEs as value processes in Dynkin games. The above theorem is a consequence of a much deeper result, which is our third main result of the paper. In order to formulate it, we use the notion of the nonlinear expectation introduced by Peng in [37]. For given stopping times $\nu \leq \zeta \leq T$ consider mapping

$$E^f_{\nu, \zeta} : L^1(F^\zeta) \to L^1(F^\nu),$$

by letting $E^f_{\nu, \zeta} \xi := Y^{\zeta}_\nu$, where $(Y^{\zeta}, Z^{\zeta})$ is a (unique) solution to (3), with $T$ replaced by $\zeta$, such that $Y^{\zeta}$ is of class (D). For given stopping times $\tau, \sigma \leq T$ and sets $H \in F_\tau, G \in F_\sigma$, we let

$$J(\tau, H; \sigma, G) := (L_\tau 1_H + \limsup_{h \searrow 0} L_{\tau + h} 1_{H^c}) 1_{[\tau \leq \sigma, \tau < T]} + (U_{\sigma} 1_G + \liminf_{h \searrow 0} U_{\sigma + h} 1_{G^c}) 1_{[\sigma < \tau]} + \xi 1_{[\tau = \sigma = T]},$$

with the convention that $L_t = L_{t \wedge T}, U_t := U_{t \wedge T}, t \geq 0$. We prove the following representation theorem.

**Theorem 3** Assume that (A1)–(A4), (Z) are in force. If $(Y, Z, R)$ is a solution to RBSDE (1)–(2), then

$$Y_\theta = \essinf_{\sigma \geq \theta, G \in F_\sigma} \esssup_{\tau \geq \theta, H \in F_\tau} E^f_{\theta, \tau \wedge \sigma} J(\tau, H; \sigma, G)$$

$$= \esssup_{\tau \geq \theta, H \in F_\tau} \essinf_{\sigma \geq \theta, G \in F_\sigma} E^f_{\theta, \tau \wedge \sigma} J(\tau, H; \sigma, G)$$

for any stopping time $\theta \leq T$. 

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In other words, we show that $Y$ is a value process in an extended nonlinear Dynkin game; “nonlinear” since we consider the nonlinear expectation, and “extended” since players may change payoffs $L$, $U$ on sets $H^c$, $G^c$, respectively, which extends the set of their strategies (in the classical Dynkin games the players are not allowed to choose sets $G$, $H$). The above theorem is a generalization of [18, Theorem 4.18], where the authors assumed that $(Y, Z, R)$ is a solution to RBSDE with $L^2$-data and Lipschitz continuous driver, and $L$, $U$ satisfy Mokobodzki’s condition (M) with $p = 2$ (in [18], however, the authors considered more general filtration). Observe that the above extended nonlinear Dynkin game reduces to the nonlinear Dynkin game provided that $L$ and $U$ are right-continuous. We prove, however, a stronger result (cf [18, Theorem 4.7]).

**Theorem 4** Assume that (A1)–(A4), (Z) are in force. Moreover, suppose that $L$ is right upper semicontinuous and $U$ is right lower semicontinuous. If $(Y, Z, R)$ is a solution to RBSDE (1)–(2), then

$$Y_\theta = \text{ess inf }_{\sigma \geq \theta} \text{ess sup }_{\tau \geq \theta} E_{\theta, \tau \land \sigma}^f J(\tau, \Omega; \sigma, \Omega)$$

for any stopping time $\theta \leq T$.

Thus, $Y$ represents the value process in a nonlinear Dynkin game provided $L$, $U$ are sufficiently regular as mentioned above. The above result was achieved by Bayraktar and Yao in [3] for continuous barriers $L$, $U$ satisfying (4) and under the following additional conditions: $\mathbb{E} \sup_{t \leq T} |L_t| + \mathbb{E} \sup_{t \leq T} |\hat{U}_t| < \infty$, generator $f$ admits the linear growth with respect to $Y$-variable, i.e. $|f(t, y, 0)| \leq g_t + \psi |y|$ for some $\psi \geq 0$. Note that in the present paper growth of $f$ with respect to $Y$-variable is subject to no restriction.

Finally, we show that further regularity assumptions on barriers $L$, $U$ allow one to indicate saddle points for nonlinear Dynkin games. For any stopping time $\theta \leq T$ set:

$$\tau^*_\theta := \inf \{t \geq \theta, Y_t = L_t\} \land T; \quad \sigma^*_\theta := \inf \{t \geq \theta, Y_t = U_t\} \land T$$

and

$$\bar{\tau}_\theta := \inf \{t \geq \theta, R_t^+ > R^+_\theta\} \land T; \quad \bar{\sigma}_\theta := \inf \{t \geq \theta, R_t^- > R^-_\theta\} \land T.$$

**Theorem 5** Assume that (A1)–(A4), (Z) are in force. Moreover, suppose that $L$ is upper semicontinuous and $U$ is lower semicontinuous. Then

$$E_{\theta, \tau^*_\theta \land \sigma^*_\theta}^f J(\tau^*_\theta, \Omega; \sigma^*_\theta, \Omega) = \text{ess inf }_{\sigma \geq \theta} \text{ess sup }_{\tau \geq \theta} E_{\theta, \tau \land \sigma}^f J(\tau, \Omega; \sigma, \Omega)$$

for any stopping time $\theta \leq T$.  

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Proof techniques and relations of main results to the existing literature First, note that the fact that the complete separation condition (4) implies weak Mokobodzki’s condition is an easy calculation and may be found e.g. in [21, 38] (in the case of continuous barriers). Reflected BSDEs with condition (4) imposed on the barriers have been considered in many papers (see [3, 5, 13, 20, 22–24, 38]). In the papers [5, 20, 22], \(L^2\)-data and sublinear growth of the generator with respect to \(Y\)-variable are required. In [23] authors considered bounded data and continuous generator with quadratic-growth with respect to \(Z\)-variable. Some results for RBSDEs with generators subject to sublinear growth with respect to \(Y\)-variable are described also in [3, 24] (\(L^1\)-data) and [13] (\(L^p\)-data, \(p \in (1, 2)\)). \(L^1\)-data and generator being merely monotone and continuous with respect to \(Y\)-variable were considered in [38].

In all the mentioned papers (besides [23]), the method of local solutions and pasting local solutions, introduced by Hamadène and Hassani in [20], has been applied to achieve the existence for underlying RBSDEs. This method is rather complicated, and this is perhaps the reason why the development of theory of RBSDEs with barriers satisfying complete separation condition is far from being satisfactory. The second drawback of the method is that it is based on the penalization scheme which is not available for RBSDEs with optional barriers. In [25], the author proposed a different method which applies to RBSDEs with barriers satisfying even more general than (4) weak Mokobodzki’s condition (WM). We call this method localization procedure. The advantage of the method is its simplicity and wide applicability. The method is based on the following simple observation: for any chain \((\tau_k)\), i.e. non-decreasing sequence of stopping times satisfying

\[ P(\tau_k < T, \; k \geq 1) = 0, \]

we have

\[
(Y, Z, R) \text{ solves } RBSDE^T(\xi, f, L, U) \iff (Y, Z, R) \text{ solves } RBSDE^{\tau_k}(Y_{\tau_k}, f, L, U), \; k \geq 1. 
\]

The method consists of finding a proper regular approximation \((Y^n)\), on the whole interval \([0, T]\), of a potential solution \(Y\) of a given problem (by “proper” we mean an approximation which does not blow up when passing to the limit). The terms of approximating sequence may solve BSDEs or RBSDEs of the generic form

\[
Y^n_t = \xi_n + \int_t^T f_n(r, Y^n_r, Z^n_r) \, dr + R^n_T - R^n_t - \int_t^T Z^n_r \, dB_r \quad t \in [0, T],
\]

with suitable chosen \(\xi_n, f_n, R^n\). In the first step one shows that \((Y^n)\) converges to a process \(Y\). After that, we show that \(Y\) is the main part of a solution to RBSDE\(^{\tau_k}(Y_{\tau_k}, f, L, U)\) for each \(k \geq 1\). Since \((\tau_k)\) is a chain, we conclude that \(Y\) is the main part of a solution to RBSDE\(^T(\xi, f, L, U)\).

By using localization procedure in [25], the first author of the present paper was able to provide an existence result for RBSDEs with merely càdlàg barriers of class \(\mathfrak{C} \) Springer
(D) satisfying (WM), $L^1$-data, and generator being continuous and non-increasing with respect to $Y$-variable (with no restrictions on the growth of the generator with respect to $Y$-variable).

As far as we know the only papers in the literature concerned with RBSDEs of the form (1) with non-càdlàg barriers satisfying (WM) are [28, 34]. In [28] RBSDEs on a general filtered space are studied under (WM) but with $f$ independent of $Z$-variable. A generalization of (4) to the case of làdlàg barriers was presented in [34], where the authors considered stochastic Lipschitz generator $f$ (on the Brownian–Poisson filtration).

As to the nonlinear Dynkin games, to the best of our knowledge, there are only few papers in the literature: [10–12, 17, 19]—all with $L^2$-data and Lipschitz generator—and [27, 28]—with $L^1$-data and continuous and monotone generator with respect to $Y$-variable and independent of $Z$-variable.

Comments on the related literature. Reflected backward stochastic differential equations with two barriers have been introduced by Cvitanić and Karatzas in [7] as a generalization of backward stochastic differential equations introduced by Pardoux and Peng in [35] (analogous results for one reflecting barrier, i.e. in case $U \equiv \infty$, have been presented for the first time by El Karoui et al. in [14]). In [7] the authors considered (1) with barriers being continuous processes satisfying Mokobodzki’s condition (M) with $p = 2$. Moreover, they assumed that data are $L^2$-integrable (i.e. $\sup_{t \leq T} |L_t|$, $\sup_{t \leq T} |U_t|, |\xi|$, $\int_0^T |f(r, 0, 0)| \, dr$ have second moments) and $f$ is Lipschitz continuous with respect to $(Y, Z)$-variable (uniformly in $(\omega, t)$). Under these assumptions a solution to (1) has been defined in [7] as a triple $(Y, Z, R)$ of $\mathcal{F}$-progressively measurable processes such that $Y$ is continuous, and $R$ is a continuous finite variation process, with $R_0 = 0$, satisfying the minimality condition of the form

$$\int_0^T (Y_r - L_r) \, dR^+_r = \int_0^T (U_r - Y_r) \, dR^-_r = 0,$$

where $R = R^+ - R^-$ is the Jordan decomposition of $R$. Observe that with continuous $Y, L, U, R$, condition (2) reduces to the above condition.

BSDEs and Reflected BSDEs are of great interest to scientists because of their numerous applications in various fields of mathematics and problems (e.g. partial differential equations, integro-differential equations, variational inequalities, optimization theory, control theory, mathematical finance etc., see [6, 36, 39] and the references therein). Over the past two decades, many interesting results have been obtained regarding RSBDEs. In particular, numerous existence results for RBSDEs, which strengthen the result of [7] by weakening assumptions on generator $f$, filtration $\mathbb{F}$, barriers $L, U$ and horizon time $T$, have been provided.

Despite of intensive research, until 2016, only RBSDEs with càdlàg barriers were considered in the literature. With the work by Grigorova et al. in [16] there was a change in this regard and papers on less regular barriers began to appear. Equations of that type with $L^2$-data and Lipschitz generator were studied in [33] (Brownian filtration), in [16, 17] (Brownian–Poisson filtration) and [1, 2, 18] (general filtration). RBSDEs with optional barriers and $L^1$-data were considered only in [29, 30], in the
case of Brownian filtration. Results on optional barriers, $L^1$-data and possibly infinite horizon time were presented in [28] but with $f$ independent of $Z$-variable. The case of $L^2$-data and $f$ being stochastic Lipschitz driver was presented in [32] (Brownian–Poisson filtration) and in [31, 34] (general filtration).

2 Basic Notation

We say that a function $y: [0, T] \to \mathbb{R}^d$ is regulated on $[0, T]$ if for any $t \in [0, T)$, there exists the limit $y_{t+} := \lim_{u \downarrow t} y_u$ and for any $s \in (0, T]$ there exists the limit $y_{s-} := \lim_{u \uparrow s} y_u$. For any regulated function $y$ on $[0, T]$ we define $\Delta^+ y_t := y_{t+} - y_t$, $t \in [0, T)$ and $\Delta^- y_s := y_s - y_{s-}$, $s \in (0, T]$.

For $x \in \mathbb{R}^d$ by $|x|$ we denote the euclidean norm. As mentioned in Sect. 1, $T$ stands for the set of all stopping times taking values in $[0, T]$. What is more, for $\nu, \zeta \in T$, $T_{\nu, \zeta} := \{ \tau \in T, \nu \leq \tau \leq \zeta \}$, $T_{\nu, T} := T_{\nu, T}$, $T^\xi := T_{0, \xi}$.

Let $\nu, \zeta \in T$, $\nu \leq \zeta$, and $p \geq 1$. By $\mathcal{S}_p^F(\nu, \zeta)$ we denote all $\mathcal{F}$-progressively measurable, $\mathbb{R}$-valued processes $Y = (Y_t)_{t \in [0, T]}$ such that (see [8, Theorem 33, p. 103])

$$||Y||_{\mathcal{S}_p^F(\nu, \zeta)} := \left( \mathbb{E} \left( \sup_{\nu \leq t \leq \zeta} |Y_t|^p \right) \right)^{\frac{1}{p}} < \infty.$$ 

$\mathcal{M}_{loc}(\nu, \zeta)$ is the space of all $\mathcal{F}$-local martingales on $[[\nu, \zeta]]$. Let $q \geq 1$. By $L_{p, q}^F(\nu, \zeta)$ we denote the set of all $\mathcal{F}$-progressively measurable, $\mathbb{R}$-valued processes $X = (X_t)_{t \in [0, T]}$ such that

$$||X||_{L_{p, q}^F(\nu, \zeta)} := \left( \mathbb{E} \left( \int_\nu^\zeta |X_r|^p \, dr \right)^{\frac{q}{p}} \right)^{\frac{1}{p}} < \infty.$$ 

$L_{p, q}^F(\nu, \zeta)$ is a shorthand for $L_{p, p}^F(\nu, \zeta)$.

Let $G \subset F$. $L^p(G)$ is the set of all $G$-measurable random variables $X$ such that

$$||X||_{L^p(G)} := \left( \mathbb{E}[|X|^p] \right)^{\frac{1}{p}} < \infty.$$ 

By $\mathcal{H}_F(\nu, \zeta)$, we denote the space of all $\mathcal{F}$-progressively measurable, $\mathbb{R}^d$-valued processes $Z = (Z_t)_{t \in [0, T]}$ such that

$$\int_\nu^\zeta |Z_r|^2 \, dr < \infty \quad P\text{-a.s.}$$
\( \mathcal{H}_F^2(\nu, \zeta), s > 0, \) is a subspace of \( \mathcal{H}_F(\nu, \zeta) \) consisting of \( Z \) satisfying

\[
\mathbb{E} \left( \int_0^\zeta |Z_r|^2 \, dr \right)^{\frac{1}{2}} < \infty.
\]

We say that \( \mathbb{F} \)-progressively measurable process \( X = (X_t)_{t \in [0,T]} \) is of class (D) on \( [[\nu, \zeta]] \) if the family \( \{X_\tau, \tau \in \mathcal{T}_{\nu,\zeta}\} \) is uniformly integrable. By \( D_\mathbb{F}^2(\nu, \zeta) \) we denote the set of all \( \mathbb{F} \)-progressively measurable, \( \mathbb{R} \)-valued processes \( Y = (Y_t)_{t \in [0,T]} \) such that \( |Y|^2 \) is of class (D) on \( [[\nu, \zeta]] \). We equip \( D_\mathbb{F}^2(\nu, \zeta) \) with the norm

\[
||Y||_{D_\mathbb{F}^2(\nu, \zeta)} := \left( \sup_{\sigma \in \mathcal{T}_{\nu,\zeta}} \mathbb{E}[|Y_\sigma|^2] \right)^{\frac{1}{2}}.
\]

A sequence \( (\tau_k)_{k \geq 1} \subset \mathcal{T}_{\nu,\zeta} \) is called a chain on \( [[\nu, \zeta]] \) if

\[
\forall \omega \in \Omega \exists n \in \mathbb{N} \forall k \geq n \tau_k(\omega) = \zeta(\omega).
\]

By \( \mathcal{V}_F(\nu, \zeta) \) (resp. \( \mathcal{V}_F^+(\nu, \zeta) \)) we denote a space of \( \mathbb{F} \)-progressively measurable, \( \mathbb{R} \)-valued processes \( V = (V_t)_{t \in [0,T]} \) with finite variation (resp. nondecreasing) on \( [[\nu, \zeta]] \) and \( \mathcal{V}_{0,\mathbb{F}}(\nu, \zeta) \) (resp. \( \mathcal{V}_{0,\mathbb{F}}^+(\nu, \zeta) \)) is a subspace of \( \mathcal{V}_F(\nu, \zeta) \) (resp. \( \mathcal{V}_F^+(\nu, \zeta) \)) consisting of processes \( V \) such that \( V_0 = 0. \mathcal{V}_{\mathbb{F}}^p(\nu, \zeta) \) (resp. \( \mathcal{V}_{\mathbb{F}}^{\infty,p}(\nu, \zeta) \)) is the set of all \( V \in \mathcal{V}_F(\nu, \zeta) \) (resp. \( V \in \mathcal{V}_F^+(\nu, \zeta) \)) such that \( E|V|_p^p < \infty \), where \( |V|_{\nu,\zeta} \) denotes the total variation of \( V \) on \( [[\nu, \zeta]] \). Let \( V \in \mathcal{V}_F(0, T) \). By \( V^* \) we denote the càdlàg part of the process \( V \), i.e.

\[
V_t^* := V_t - \sum_{0 \leq r < t} \Delta^+ V_r.
\]

Any stochastic process \( X \in \mathcal{M}_{loc}(0, T) + \mathcal{V}_{\mathbb{F}}(0, T) \) is called an \textit{optional semimartingale} (see e.g. [15]).

Throughout the paper all relations between random variables are supposed to hold \( \mathbb{P} \)-a.s. For processes \( X^1 = (X^1_t)_{t \in [0,T]} \) and \( X^2 = (X^2_t)_{t \in [0,T]} \) we write \( X^1 \leq X^2 \) if \( X^1_t \leq X^2_t, t \in [0, T], \mathbb{P} \)-a.s.

Let \( V^1, V^2 \in \mathcal{V}_{0,\mathbb{F}}(0, \tau) \). We write \( dV^1 \leq dV^2 \), if \( dV^{1,*} \leq dV^{2,*} \) and \( \Delta^+ V^1 \leq \Delta^+ V^2 \) on \( [0, \tau] \).

For an \( \mathbb{F} \)-optional process \( X = (X_t)_{t \in [0,T]} \) we set \( \widetilde{X}_s := \lim \sup_{r \uparrow s} X_r, \) \( \underline{X}_s := \lim \inf_{r \downarrow s} X_r, s \in (0, T) \) and \( \overline{X}_s := \lim \sup_{r \downarrow s} X_r, \underline{X}_s := \lim \inf_{r \uparrow s} X_r, s \in [0, T], s \in [0, T] \).

### 3 Backward SDEs

Let \( p \geq 1 \). We shall need the following hypotheses:
There exists an $\lambda \geq 0$ such that $|f(t, y, z) - f(t, y, z')| \leq \lambda |z - z'|$ for $t \in [0, T]$, $y \in \mathbb{R}$, $z, z' \in \mathbb{R}^d$.

There exists at most one solution $Y$.

For every $(t, z) \in [0, T] \times \mathbb{R}^d$ the mapping $\mathbb{R} \ni y \to f(t, y, z)$ is continuous, depending only on $y$.

We say that a pair $(Y, Z)$ is a solution to BSDE $\nu, \zeta (\hat{\xi}, f)$ for short if

\begin{align*}
\text{(a)} \quad & Y \text{ is a continuous process and } Z \in \mathcal{H}_\mathbb{F}(\nu, \zeta), \\
\text{(b)} \quad & \int_0^T |f(r, Y_r, Z_r)| dr < \infty, \\
\text{(c)} \quad & Y_t = \hat{\xi} + \int_t^T f(r, Y_r, Z_r) dr - \int_t^T Z_r dB_r, t \in [\nu, \zeta].
\end{align*}

Let $V \in \mathcal{V}_{0, \mathbb{F}}(\nu, \zeta)$.

We say that a pair $(Y, Z)$ is a solution to BSDE $\nu, \zeta (\hat{\xi}, f)$ for short if

\begin{align*}
\text{(a)} \quad & Y \text{ is a continuous process and } Z \in \mathcal{H}_\mathbb{F}(\nu, \zeta), \\
\text{(b)} \quad & \int_0^T |f(r, Y_r, Z_r)| dr < \infty, \\
\text{(c)} \quad & Y_t = \hat{\xi} + \int_t^T f(r, Y_r, Z_r) dr - \int_t^T Z_r dB_r, t \in [\nu, \zeta].
\end{align*}

Let us adopt the shorthand $\text{BSDE}_\nu^\xi := \text{BSDE}_0^0(\xi, f)$. The following results follow from [4, Proposition 3.2, Theorem 4.2, Remark 4.3].

**Theorem 1** Assume that (H1)–(H4) are in force. Suppose that (H5) holds with $p > 1$. Then the following assertions hold.

\begin{enumerate}[(i)]
\item There exists a solution $(Y, Z) \in \mathcal{S}_\mathbb{F}^p(0, T) \times \mathcal{H}_\mathbb{F}^p(0, T)$ to BSDE $\nu, \zeta (\xi, f)$.
\item There exists at most one solution $(Y, Z)$ to BSDE $\nu, \zeta (\hat{\xi}, f)$ such that $Y \in \mathcal{S}_\mathbb{F}^p(0, T)$.
\end{enumerate}

**Proposition 1** Assume that (H5), with $p > 1$, and (H1),(H2) are satisfied. Let $(Y, Z)$ be a solution to BSDE $\nu, \zeta (\hat{\xi}, f)$ such that $Y \in \mathcal{S}_\mathbb{F}^p(0, T)$. Then there exists $c > 0$, depending only on $\mu, \lambda, T, p$, such that

$$
\mathbb{E} \left[ \sup_{0 \leq t \leq T} |Y_t|^p + \left( \int_0^T |Z_r|^2 \, dr \right)^{\frac{p}{2}} + \left( \int_0^T |f(r, Y_r, Z_r)| \, dr \right)^p \right] 
\leq c \mathbb{E} \left[ |\xi|^p + \left( \int_0^T |f(r, 0, 0)| \, dr \right)^p \right].
$$

(8)

In case (H5) is satisfied with $p = 1$, we shall need for the existence and uniqueness of solutions to BSDEs additional hypothesis.

\begin{enumerate}[(Z)]
\item There exists an $\mathbb{F}$-progressively measurable process $g \in L_\mathbb{F}^1(0, T)$ and $\gamma \geq 0$, $\kappa \in [0, 1)$ such that

$$
|f(t, y, z) - f(t, y, 0)| \leq \gamma (g_t + |y| + |z|)^\kappa, \quad t \in [0, T], \quad y \in \mathbb{R}, \quad z \in \mathbb{R}^d.
$$
\end{enumerate}
Remark 1 Condition (Z) says that driver $f$ is allowed to have at most sublinear growth with respect to $z$-variable. A typical example of a driver satisfying (H1)–(H5), (Z) is of the following form

$$f(t, y, z) := f_0(t, y) + b(y)(1 + |z|)^\kappa,$$

where $f_0$ satisfies (H2)–(H5), $b$ is continuous, non-increasing and bounded, and $\kappa \in (0, 1)$.

Observe that, in general, under merely (H1)–(H4) and (H5) with $p = 1$, we cannot expect the existence of a solution $(Y, Z)$ to BSDE$^T(\xi, f)$ with positive $\xi$ such that $Y$ is positive and of class (D). Indeed, assume that $(Y, Z)$ is a solution to the following BSDE

$$Y_t = \xi + \int_t^T Z_r \, dr - \int_t^T Z_r \, dB_r, \quad t \in [0, T],$$

with positive $\xi \in L^1(\mathcal{F}_T)$, and $Y$ is positive of class (D). Let $(\tau_k)$ be chain such that $Y \in S^2(0, \tau_k), Z \in \mathcal{H}^2_F(0, \tau_k), k \geq 1$. Then, by Itô’s formula

$$Y_0 = \mathbb{E}\left[Y_{\tau_k} \exp\left(-\frac{\tau_k}{2} + B_{\tau_k}\right)\right]$$

Therefore, by applying Fatou’s lemma, we find that

$$e^TY_0 \geq \mathbb{E}[\xi \exp(B_T)].$$

If the above inequality was true for any positive $\xi \in L^1(\mathcal{F}_T)$, then $\exp(B_T)$ would be bounded, a contradiction.

Theorem 2 Assume that (H1)–(H4), (Z) are in force. Moreover, assume that (H5) is satisfied with $p = 1$. Then the following assertions hold.

(i) There exists a solution $(Y, Z)$ of BSDE$^T(\xi, f)$ such that $Y$ is of class (D) and $Z \in \mathcal{H}^s_F(0, T), s \in (0, 1)$.

(ii) There exists at most one solution $(Y, Z)$ to BSDE$^T(\xi, f)$ such that $Y$ is of class (D).

Proof The assertion (i) follows from [4, Theorem 6.3]. As to (ii), by [4, Theorem 6.2], there exists at most one solution $(Y, Z)$ to BSDE$^T(\xi, f)$ such that $Y$ is of class (D) and $Z \in \mathcal{H}^s_F(0, T), s \in (0, 1)$. So, it is enough to show that if $(Y, Z)$ is a solution to BSDE$^T(\xi, f)$ such that $Y$ is of class (D), then $Z \in \mathcal{H}^s_F(0, T), s \in (0, 1)$. This follows at once from [30, Remark 2.1] and [4, Lemma 3.1].
4 Reflected BSDEs with Two Optional Barriers Under Mokobodzki’s Condition

In this section we assume that processes $L$ and $U$ are merely $\mathbb{F}$-optional. Let $v, \zeta \in \mathcal{T}$, $v \leq \zeta$, and $\xi \in \mathcal{F}_\zeta$ such that $L_\zeta \leq \xi \leq U_\zeta$.

**Definition 3** We say that a triple $(Y, Z, R)$ of $\mathbb{F}$-adapted processes is a solution to reflected backward stochastic differential equation on the interval $[[v, \zeta]]$ with generator $f$, terminal value $\xi$, lower barrier $L$ and upper barrier $U$ (RBSDE$^{v,\zeta}$ $(\xi, f, L, U)$ for short) if

(a) $Y$ is a regulated process and $Z \in \mathcal{H}_p(v, \zeta)$,

(b) $R \in \mathcal{V}_{0,1}(v, \zeta)$, $L_t \leq Y_t \leq U_t$, $t \in [v, \zeta]$, and

\[
\int_v^\zeta \left( Y_r - \overline{L}_r \right) dR_r^{*,+} + \sum_{v \leq r < \zeta} (Y_r - L_r)(\Delta^+ R_r)^+ \\
= \int_v^\zeta \left( U_r - Y_r^- \right) dR_r^{*,-} + \sum_{v \leq r < \zeta} (U_r - Y_r^-)(\Delta^+ R_r)^- = 0,
\]

where $R^* = R^{*,+} - R^{*,-}$ is the Jordan decomposition of $R^*$,

(c) $\int_v^\zeta |f(r, Y_r, Z_r)| dr < \infty$,

(d) $Y_t = \hat{\xi} + \int_t^\zeta f(r, Y_r, Z_r) dr + R_t - R_t - \int_t^\zeta Z_r dB_r$, $t \in [v, \zeta]$.

In what follows we refer to condition (b) as the minimal condition.

Let us adopt the shorthand RBSDE$^\xi :=$RBSDE$^{0,\xi}$.

We consider the following condition, which we call strong Mokobodzki’s condition:

(H6) there exists a process $X \in \mathcal{M}_{loc}(0, T) + \mathcal{V}^p_{0,\mathbb{F}}(0, T)$ such that $L \leq X \leq U$, $X \in S^p_{\mathbb{F}}(0, T)$ and $f(\cdot, X, 0) \in L^1_{\mathbb{F}}(0, T)$.

Assume that $L_T \leq \xi \leq U_T$. The following result has been proven in [30, Proposition 3.2, Theorem 3.9].

**Theorem 3** Let $p > 1$. Assume (H1)–(H5).

(i) There exists at most one solution $(Y, Z, R)$ to RBSDE$^T(\xi, f, L, U)$ such that $Y \in S^p_{\mathbb{F}}(0, T)$.

(ii) There exists a solution $(Y, Z, R)$ to RBSDE$^T(\xi, f, L, U)$ such that $Y \in S^p_{\mathbb{F}}(0, T)$, $Z \in \mathcal{H}^p_{\mathbb{F}}(0, T)$ and $R \in \mathcal{V}^p_{0,\mathbb{F}}(0, T)$ if and only if (H6) holds.

In the case of $p = 1$, we consider the following version of strong Mokobodzki’s condition:

(H6*) there exists a process $X \in \mathcal{M}_{loc}(0, T) + \mathcal{V}^1_{\mathbb{F}}(0, T)$ such that $X$ is of class (D), $L \leq X \leq U$ and $f(\cdot, X, 0) \in L^1_{\mathbb{F}}(0, T)$.

The following result has been proven in [30, Theorem 3.8].

**Theorem 4** Let $p = 1$. Assume (H1)–(H5), (H6*), (Z). Then there exists a unique solution $(Y, Z, R)$ of RBSDE$^T(\xi, f, L, U)$ such that $Y$ is of class (D), $Z \in \mathcal{H}^p_{\mathbb{F}}(0, T)$, $q \in (0, 1)$ and $R \in \mathcal{V}^1_{0,\mathbb{F}}(0, T)$.
5 Nonlinear Expectation

Let \( p \geq 1 \). Throughout this section, we assume that either \( p = 1 \) and (H1)–(H5), (Z) are in force or \( p > 1 \) and (H1)–(H5) are in force. Let \( \nu, \zeta \in \mathcal{T}, \nu \leq \zeta \). We define the operator

\[
\mathbb{E}^{(1)}_{\nu, \zeta} : L^1(\mathcal{F}_\nu) \to L^1(\mathcal{F}_\nu),
\]

by letting \( \mathbb{E}^{(1)}_{\nu, \zeta} (\xi) := Y_\nu \), where \( (Y, Z) \) is a solution to BSDE\( ^{\zeta} (\xi, f) \) such that \( Y \) is of class (D). By Theorem 2, the operator \( \mathbb{E}^{(1)}_{\nu, \zeta} \) is well defined under conditions (H1)–(H5), (Z). By Theorem 1, under (H1)–(H5) (with \( p > 1 \)), we may define the operator

\[
\mathbb{E}^{(p)}_{\nu, \zeta} : L^p(\mathcal{F}_\mu) \to L^p(\mathcal{F}_\nu),
\]

with \( \mathbb{E}^{(p)}_{\nu, \zeta} (\xi) := Y_\nu \), where \( (Y, Z) \) is a solution to BSDE\( ^{\zeta} (\xi, f) \) such that \( Y \in S^p_\mathcal{F}(0, \zeta) \). Finally, we define operator

\[
\mathbb{E}^f_{\nu, \zeta} : L^1(\mathcal{F}_\nu) \to L^1(\mathcal{F}_\nu),
\]

by letting

\[
\mathbb{E}^f_{\nu, \zeta} (\xi) := \begin{cases} 
\mathbb{E}^{(1)}_{\nu, \zeta} (\xi) & \text{for } \xi \in L^1(\mathcal{F}_\nu) \setminus \bigcup_{p > 1} L^p(\mathcal{F}_\nu) \\
\mathbb{E}^{(p)}_{\nu, \zeta} (\xi) & \text{for } \xi \in \bigcup_{p > 1} L^p(\mathcal{F}_\nu). 
\end{cases}
\]

We say that a process \( X \) of class (D) is an \( \mathbb{E}^f \)-supermartingale (resp. \( \mathbb{E}^f \)-submartingale) on \([\nu, \zeta]\), if \( \mathbb{E}^f_{\sigma, \tau} (X_\tau) \leq X_\sigma \) (resp. \( \mathbb{E}^f_{\sigma, \tau} (X_\tau) \geq X_\sigma \)) for every \( \sigma, \tau \in \mathcal{T}_{\nu, \zeta}, \sigma \leq \tau \). \( X \) is an \( \mathbb{E}^f \)-martingale on \([\nu, \zeta]\), if \( X \) is at the same time an \( \mathbb{E}^f \)-supermartingale and an \( \mathbb{E}^f \)-submartingale on \([\nu, \zeta]\).

**Remark 2** Note that the process \( Y \) of class (D) is an \( \mathbb{E}^f \)-martingale on \([\nu, \zeta]\) if and only if \( Y \) is indistinguishable from the first component of the solution to BSDE\( ^{\nu, \zeta} (Y, f) \) on \([\nu, \zeta]\). Thus, in order to prove that \( Y \) is an \( \mathbb{E}^f \)-martingale on \([\nu, \zeta]\), it suffices to show that \( Y_\sigma = \mathbb{E}^f_{\sigma, \zeta} (Y_\zeta) \), for any \( \sigma \in \mathcal{T}_{\nu, \zeta} \).

**Proposition 2** Let \( \nu, \zeta \in \mathcal{T}, \nu \leq \zeta \).

(i) Let \( \xi \in L^p(\mathcal{F}_\zeta) \) and let \( V \) be an \( \mathbb{F} \)-adapted, finite variation process such that \( V_\nu = 0 \). Let \( (X, H) \) be a solution to BSDE\( ^{\nu, \zeta} (\xi, f + dV) \) such that \( X \) is of class (D), in case \( p = 1 \), and \( X \in S^p_\mathcal{F}(\nu, \zeta) \), in case \( p > 1 \). If \( V \) (resp. \( -V \)) is increasing, then \( X \) is \( \mathbb{E}^f \)-supermartingale (resp. \( \mathbb{E}^f \)-submartingale) on \([\nu, \zeta]\).

(ii) If \( \xi_1, \xi_2 \in L^p(\mathcal{F}_\zeta) \) and \( \xi_1 \leq \xi_2 \), then \( \mathbb{E}^f_{\nu, \zeta} (\xi_1) \leq \mathbb{E}^f_{\nu, \zeta} (\xi_2) \).
(iii) Let $\xi \in L^p(\mathcal{F}_\tau)$. For every $A \in \mathcal{F}_\nu$,

$$1_A \mathbb{E}_{\nu,\xi}^f(\xi) = \mathbb{E}_{\nu,\xi}^f(1_A \xi),$$

where $f_A(t, y, z) = f(t, y, z)1_{\{t \geq \nu\}}$.

(iv) Let $\xi \in L^p(\mathcal{F}_\tau)$. For every $\gamma \in T$ such that $\gamma \geq \zeta$,

$$\mathbb{E}_{\nu,\xi}^f(\xi) = \mathbb{E}_{\nu,\gamma}^f(\xi),$$

where $f^\xi(t, y, z) = f(t, y, z)1_{\{t \leq \gamma\}}$.

(v) Let $p = 1$. Assume that $f_1$, $f_2$ satisfies (H1)–(H5), (Z) and let $\nu$, $\xi_1$, $\xi_2 \in T$, $\nu \leq \xi_1 \leq \xi_2$. Assume that $\xi_1 \in L^1(\mathcal{F}_{\xi_1})$ and $\xi_2 \in L^1(\mathcal{F}_{\xi_2})$. Moreover, let $(Y^1, Z^1)$ be a solution to BSDE$^{\nu,\xi_2}(\xi_2, f_1^{\xi_1})$, where $f_1^{\xi_1}(t, y, z) = f_1(t, y, z)1_{\{t \leq \xi_1\}}$, and $(Y^2, Z^2)$ be a solution to BSDE$^{\nu,\xi_2}(\xi_2, f_2)$, such that $Y^1, Y^2$ are of class (D). If $(Y^1 - Y^2) \in S^2_p(\nu, \xi_2)$, then

$$\left[\mathbb{E}_{\nu,\xi_1}^{f_1}(\xi_1) - \mathbb{E}_{\nu,\xi_2}^{f_2}(\xi_2)\right]^2 \leq CE\left(\int_{\nu}^{\xi_1} |Y_r^1 - Y_r^2| |f_1 - f_2|(r, Y_r^2, Z_r^2) \, dr + |\xi_1 - \xi_2|^2 + \int_{\xi_1}^{\xi_2} |Y_r^1 - Y_r^2| |f_2(r, Y_r^2, Z_r^2)| \, dr\right),$$

for some $C$ depending only on $\lambda$, $\mu$, $T$.

(vi) Let $p > 1$. Assume that $f_1$, $f_2$ satisfies (H1)–(H5) and let $\nu$, $\xi_1$, $\xi_2 \in T$, $\nu \leq \xi_1 \leq \xi_2$. Assume that $\xi_1 \in L^p(\mathcal{F}_{\xi_1})$ and $\xi_2 \in L^p(\mathcal{F}_{\xi_2})$. Moreover, let $(Y^1, Z^1)$ be a solution to BSDE$^{\xi_2}(\xi_1, f_1^{\xi_1})$, with $Y^1 \in S^p_p(0, \xi_2)$, where $f_1^{\xi_1}(t, y, z) = f_1(t, y, z)1_{\{t \leq \xi_1\}}$, and $(Y^2, Z^2)$ be a solution to BSDE$^{\xi_2}(\xi_2, f_2)$, with $Y^2 \in S^p_p(0, \xi_2)$. Then there exists $c > 0$, depending only on $T, \mu, \lambda, p$, such that such that

$$\left\|\mathbb{E}_{\nu,\xi_1}^{f_1}(\xi_1) - \mathbb{E}_{\nu,\xi_2}^{f_2}(\xi_2)\right\|_{S^p_p(0, \xi_1)} \leq c \left[\mathbb{E}\left(\int_{0}^{\xi_1} |f_1 - f_2|(r, Y_r^2, Z_r^2) \, dr\right)^p + \mathbb{E}|\xi_1 - \xi_2|^p + \mathbb{E}\left(\int_{\xi_1}^{\xi_2} |f_2(r, Y_r^2, Z_r^2)| \, dr\right)^p\right]^{1/p}.$$

Proof See Appendix. $\Box$

6 Extended Nonlinear Dynkin Games

In the whole section, we assume that (H1)–(H5), (Z) are in force and that $L$ and $U$ are $\mathbb{F}$-optional processes of class (D).

Definition 4 Let $\tau \in T$ and $H \in \mathcal{F}_\tau$. A pair $\rho = (\tau, H)$ is called a stopping system if $\{\tau = T\} \subset H$. For brevity, we write $\tau | H$. $\blacksquare$ Springer
By \( \mathcal{U} \) we denote the set of all stopping systems. We then have \( T \subset \mathcal{U} \), by using embedding \( T \ni \tau \mapsto \tau|_{\Omega} \in \mathcal{U} \). For given \( \theta \in T \), we denote by \( \mathcal{U}_{\theta} \) the set of all stopping systems \( \tau|_{H} \) such that \( \tau \geq \theta \). For an optional, right-limited process \( \phi \) and \( \tau|_{H} \in \mathcal{U} \) we put

\[
\phi_{\tau|_{H}} := \phi_{\tau}1_{H} + \phi_{\tau +}1_{H^c}.
\]

In particular, we have \( \phi_{\tau|_{\Omega}} = \phi_{\tau} \). For an optional process \( \phi \) let

\[
\phi_{\tau|_{H}}^u := \phi_{\tau}1_{H} + \phi_{\tau|_{H}}\setminus \tau \quad \text{and} \quad \phi_{\tau|_{H}}^l := \phi_{\tau}1_{H} + \phi_{\tau|_{H}}\setminus \tau.
\]

Note that, when \( \phi \) is right-limited, then \( \phi_{\tau|_{H}}^u = \phi_{\tau|_{H}}^l = \phi_{\tau|_{H}} \).

For two stopping systems \( \tau|_{H}, \sigma|_{G} \in \mathcal{U} \) we define the pay-off

\[
J(\tau|_{H}, \sigma|_{G}) := L^{\mu}_{\tau|_{H}}1_{\{\tau \leq \sigma, \tau < T\}} + U^{\ell}_{\sigma|_{G}}1_{\{\sigma < \tau\}} + \xi1_{\{\tau = \sigma = T\}}.
\]

Note that \( J(\tau|_{H}, \sigma|_{G}) \) is \( \mathcal{F}_{\tau \wedge \sigma} \)-measurable random variable. Now, we shall proceed to the so called extended Dynkin games.

**Definition 5** Let \( \theta \in T \).

(i) Upper and lower value of the game are defined respectively as

\[
\overline{V}(\theta) := \text{ess inf } \text{ess sup }_{\sigma \in \mathcal{U}_{\theta}, \tau \subset \mathcal{U}_{\theta}} E^{\mu}_{\theta, \tau \wedge \sigma} J(\tau|_{H}, \sigma|_{G});
\]

\[
\underline{V}(\theta) := \text{ess sup } \text{ess inf }_{\tau \subset \mathcal{U}_{\theta}, \sigma \in \mathcal{U}_{\theta}} E^{\ell}_{\theta, \tau \wedge \sigma} J(\tau|_{H}, \sigma|_{G}).
\]

(ii) We say that an extended \( \mathbb{E}^{f} \)-Dynkin game with pay-off function \( J \) has a value if \( \overline{V}(\theta) = \underline{V}(\theta) \) for any \( \theta \in T \).

Let \( (Y, Z, R) \) be a solution to RBSDE\( ^T(\xi, f, L, U) \). For every \( \theta \in T \) and \( \varepsilon > 0 \) we define the following sets

\[
A^{\varepsilon} := \{(\omega, t) \in \Omega \times [0, T]: Y_{t}(\omega) \leq L^{\varepsilon}_{\theta}(\omega) + \varepsilon\};
\]

\[
B^{\varepsilon} := \{(\omega, t) \in \Omega \times [0, T]: Y_{t}(\omega) \geq U^{\varepsilon}_{\theta}(\omega) - \varepsilon\},
\]

where \( L^{\varepsilon}_{\theta} := L_{\theta}1_{[\theta < T]} + \xi1_{[\theta = T]}, U^{\varepsilon}_{\theta} := U_{\theta}1_{[\theta < T]} + \xi1_{[\theta = T]} \) for \( \theta \in T \). Let us also define the following stopping times

\[
\tau^{\varepsilon}_{\theta} := \inf \{t \geq \theta, Y_{t} \leq L^{\varepsilon}_{\theta} + \varepsilon\} \wedge T, \quad \sigma^{\varepsilon}_{\theta} := \inf \{t \geq \theta, Y_{t} \geq U^{\varepsilon}_{\theta} - \varepsilon\} \wedge T.
\]

We let

\[
H^{\varepsilon} := \{(\omega, \tau^{\varepsilon}_{\theta}(\omega)) \in A^{\varepsilon}\}; \quad G^{\varepsilon} := \{(\omega, \sigma^{\varepsilon}_{\theta}(\omega)) \in B^{\varepsilon}\}.
\]
Consider the following stopping systems

\[ \tau^\varepsilon_\theta | H^\varepsilon \quad \text{and} \quad \sigma^\varepsilon_\theta | G^\varepsilon. \]  

(12)

**Lemma 1** Let \((Y, Z, R)\) be a solution to RBSDE\(^T\) \((\xi, f, L, U)\). Then \(Y\) is an \(\mathbb{E}^f\)-submartingale on \([[\theta, \tau^\varepsilon_\theta]])\) and an \(\mathbb{E}^f\)-supermartingale on \([[\theta, \sigma^\varepsilon_\theta]])\).

**Proof** We show that \(Y\) is \(\mathbb{E}^f\)-submartingale on \([[\theta, \tau^\varepsilon_\theta]])\). The proof of the second assertion runs analogously. By the definition of \(\tau^\varepsilon_\theta\), we have \(Y_t \geq L_t + \varepsilon\) on \([[\theta, \tau^\varepsilon_\theta]])\. This implies, by the minimality condition, that \(R^+\) is constant on \([[\theta, \tau^\varepsilon_\theta]])\. By the last inequality, we also have \(Y_{\tau^\varepsilon_\theta} = \varepsilon\), so as a result, by the minimality condition again, we get \(\Delta^n R^+_t = 0\). Therefore, \(R^+\) is constant on \([[\theta, \tau^\varepsilon_\theta]])\. This implies that

\[ Y_t = Y_{\tau^\varepsilon_\theta} + \int_t^{\tau^\varepsilon_\theta} f(r, Y_r, Z_r) dr - R^+_{\tau^\varepsilon_\theta} + R^-_t - \int_t^{\tau^\varepsilon_\theta} Z_r dB_r, \quad t \in [\theta, \tau^\varepsilon_\theta]. \]

Thus, by Proposition 2 (i), \(Y\) is an \(\mathbb{E}^f\)-submartingale on \([[\theta, \tau^\varepsilon_\theta]])\). \(\square\)

**Lemma 2** Let \((Y, Z, R)\) be a solution to RBSDE\(^T\) \((\xi, f, L, U)\).

(i) For any \(\theta \in T\),

\[ Y_{\tau^\varepsilon_\theta} | H^\varepsilon \leq L^{\xi, \mu}_{\tau^\varepsilon_\theta} | H^\varepsilon + \varepsilon \quad \text{and} \quad Y_{\sigma^\varepsilon_\theta} | G^\varepsilon \geq U^{\xi, \ell}_{\sigma^\varepsilon_\theta} | G^\varepsilon - \varepsilon. \]  

(13)

(ii) For each \(\theta \in T\) and any \(\tau [A, \sigma] [B] \in \mathcal{U}_\theta\),

\[ \mathbb{E}^f_{\theta, \tau^\varepsilon_\theta \wedge \sigma} \left( Y_{\tau^\varepsilon_\theta} | H^\varepsilon 1_{\tau^\varepsilon_\theta \leq \sigma} + Y_{\sigma} | B 1_{\sigma < \tau^\varepsilon_\theta} \right) \geq Y_{\theta} \quad \text{and} \quad \mathbb{E}^f_{\theta, \tau \wedge \sigma^\varepsilon_\theta} \left( Y_{\tau} | A 1_{\tau \leq \sigma} + Y_{\sigma} | G^\varepsilon 1_{\sigma < \tau} \right) \leq Y_{\theta}. \]  

(14)

**Proof** (i) We shall prove the first inequality in (13), the proof of the second one runs analogously. Due to the definitions of \(\tau^\varepsilon_\theta | H^\varepsilon\), \(Y_{\tau^\varepsilon_\theta} | H^\varepsilon\), \(L^{\xi, \mu}_{\tau^\varepsilon_\theta} | H^\varepsilon\) and \(H^\varepsilon\), we have, on the set \(H^\varepsilon\), that \(Y_{\tau^\varepsilon_\theta} | H^\varepsilon = Y_{\tau^\varepsilon_\theta} \leq L^{\xi, \mu}_{\tau^\varepsilon_\theta} | H^\varepsilon + \varepsilon = L^{\xi, \mu}_{\tau^\varepsilon_\theta} \wedge H^\varepsilon + \varepsilon\), while on the set \(H^\varepsilon^{-c}\), we have

\[ Y_{\tau^\varepsilon_\theta} | H^\varepsilon = Y_{\tau^\varepsilon_\theta}^+ \quad \text{and} \quad L^{\xi, \mu}_{\tau^\varepsilon_\theta} \wedge H^\varepsilon = \overline{L}^{\xi, \varepsilon}_{\tau^\varepsilon_\theta}. \]  

(15)

On the other hand, by the definition of \(\tau^\varepsilon_\theta\), for P-a.e. \(\omega \in \Omega\) there exists nonincreasing sequence \((\tau_n)\) (depending on \(\omega \in \Omega\)) such that \(\tau_n \searrow \tau^\varepsilon_\theta\) and \(Y_{\tau_n} \leq L^{\xi, \varepsilon}_{\tau_n} + \varepsilon\) for all \(n \in \mathbb{N}\). Therefore

\[ \limsup_{n \to \infty} Y_{\tau_n} \leq \limsup_{n \to \infty} L^{\xi, \varepsilon}_{\tau_n} + \varepsilon. \]
Due to the definiton of $\bar{L}^\xi$, we have $\limsup_{n \to \infty} L^\xi_{t_n} \leq \bar{L}^\xi_{t_\theta}$. Since $Y$ is regulated, $\limsup_{n \to \infty} Y_{t_n} = Y_{t_\theta^+}$. Thus, $Y_{t_\theta^+} \leq \bar{L}^\xi_{t_\theta} + \epsilon$. This inequality combined with (15) implies that $Y_{t_\theta^+} \leq L^\xi_{t_\theta} + \epsilon$ on $H^{e,c}$.

(ii) First, we shall prove the first inequality in (14). We have

$$Y_{t_\theta^+}1_{\{\tau_\theta^+ \leq \sigma\}} + Y_{\sigma}1_{\{\sigma < \tau_\theta^\epsilon\}} = Y_{t_\theta^+}1_{\{\tau_\theta^+ \leq \sigma\}} + Y_{t_\theta^+}1_{\{\tau_\theta^+ \leq \sigma\}} + Y_{\sigma}1_{\{\sigma < \tau_\theta^\epsilon\}} + Y_{\sigma}1_{\{\sigma < \tau_\theta^\epsilon\}}$$

(16)

By Lemma 1 $Y$ is an $\mathbb{E}^f$-submartingale on $[\theta, \tau_\theta^\epsilon]$, which implies that

$$1_{\{\sigma < \tau_\theta^\epsilon\}} Y_{\sigma^+} \geq 1_{\{\sigma < \tau_\theta^\epsilon\}} Y_{\sigma \wedge \tau_\theta^\epsilon}.$$  

(17)

By the form of $H^e$ and the minimality condition, $1_{H^{e,c}} \Delta^+ R_{t_\theta} = 0$. Thus,

$$1_{H^{e,c}} Y_{t_\theta^+} \geq 1_{H^{e,c}} Y_{t_\theta^+}.$$  

(18)

By virtue of (16)--(18), we conclude that $Y_{t_\theta^+}1_{\{\tau_\theta^+ \leq \sigma\}} + Y_{\sigma}1_{\{\sigma < \tau_\theta^\epsilon\}} \geq Y_{\tau_\theta^\epsilon \wedge \sigma}$. Since the operator $\mathbb{E}^f$ is nondecreasing (see Proposition 2 (ii))

$$\mathbb{E}^f_{\theta, \tau_\theta^\epsilon \wedge \sigma}(Y_{t_\theta^+}1_{\{\tau_\theta^+ \leq \sigma\}} + Y_{\sigma}1_{\{\sigma < \tau_\theta^\epsilon\}}) \geq \mathbb{E}^f_{\theta, \tau_\theta^\epsilon \wedge \sigma}(Y_{t_\theta^+}1_{\{\tau_\theta^+ \leq \sigma\}}).$$

Using again the fact that $Y$ is $\mathbb{E}^f$-submartingale on $[\theta, \tau_\theta^\epsilon]$, we conclude that $\mathbb{E}^f_{\theta, \tau_\theta^\epsilon \wedge \sigma}(Y_{t_\theta^+}1_{\{\tau_\theta^+ \leq \sigma\}} + Y_{\sigma}1_{\{\sigma < \tau_\theta^\epsilon\}}) \geq Y_{\theta}$. The proof of the second inequality in (14) requires slightly different arguments. We have

$$Y_{\tau}1_{\{\tau \leq \sigma_\theta^\epsilon\}} + Y_{\sigma_\theta^\epsilon}1_{\{\sigma_\theta^\epsilon < \tau\}} = Y_{\tau}1_{\{\tau \leq \sigma_\theta^\epsilon\}} + Y_{\tau}1_{\{\tau \leq \sigma_\theta^\epsilon\}} + Y_{\sigma}1_{\{\sigma < \tau\}} + Y_{\sigma_\theta^\epsilon}1_{\{\sigma_\theta^\epsilon < \tau\}}.$$  

(19)

By the analogous argument as in the proof of (16)—the form of $G^e$ combined with the definition of a solution to RBSDE give $1_{G^{e,c}} \Delta^+ R_{t_\theta} = 0$—we find that

$$1_{G^{e,c}} Y_{\sigma_\theta^\epsilon^+} \leq 1_{G^{e,c}} Y_{\sigma_\theta^\epsilon}.$$  

(20)

Using the fact that $Y$ is an $\mathbb{E}^f$-supermartingale on $[\theta, \sigma_\theta^\epsilon]$ (see Lemma 1) we obtain that

$$1_{\{\tau < \sigma_\theta^\epsilon\}} Y_{\tau^+} \leq 1_{\{\tau < \sigma_\theta^\epsilon\}} Y_{\tau}.$$  

(21)

But we also need

$$1_{\{\tau = \sigma_\theta^\epsilon\}} Y_{\tau^+} \leq 1_{\{\tau = \sigma_\theta^\epsilon\}} Y_{\tau}.$$  

(22)
The above inequality may not hold only in case $\Delta^+ R_{\sigma_0}^- > 0$. The last relation implies, by using the minimality condition, that $Y_{\sigma_0} = U_{\sigma_0}$. On the other hand, by the definition of $\sigma_0^\varepsilon$, $U_t - Y_t \geq \varepsilon$, $t \in [\theta, \sigma_0^\varepsilon)$. This and the previous equation force left positive jump, so $\Delta R_{\sigma_0}^- > 0$. Consequently, by the minimality condition, $Y_{\sigma_0}^- = \inf_{\theta \leq \sigma_0} Y_\theta$. This contradicts the relation $U_t - Y_t \geq \varepsilon$, $t \in [\theta, \sigma_0^\varepsilon)$. Therefore, (22) must hold. Combining (19)–(22) gives

$$Y_\tau |A 1_{[\tau \leq \sigma_0^\varepsilon]} + Y_{\sigma_0^\varepsilon} 1_{[\sigma_0^\varepsilon < \tau]} \leq Y_{\tau \wedge \sigma_0^\varepsilon}.$$

With the aid of monotonicity of the operator $\mathbb{E}_{\theta, \varepsilon}^\tau$ and the fact that $Y$ is $\mathbb{E}^f$-supermartingale on $[\theta, \sigma_0^\varepsilon]$, we easily deduce from the above inequality the result.

**Lemma 3** Let $(Y, Z, R)$ be a solution to RBSDE$^T(\xi, f, L, U)$. We have the following inequalities:

$$\mathbb{E}_{\theta, \varepsilon}^\tau, \mathbb{E}_{\theta, \tau \wedge \sigma_0^\varepsilon} [J(\mathbb{E}_{\tau}^\sigma | G^\varepsilon) - C \varepsilon] \leq Y_\theta \leq \mathbb{E}_{\theta, \tau \wedge \sigma_0^\varepsilon} [J(\mathbb{E}_{\tau}^\sigma | H^\varepsilon, \sigma | B) + C \varepsilon], \quad (23)$$

where $C$ is a constant depending only on $\lambda, \mu, \kappa, T, \|g\|_{L^1}, \gamma$.

**Proof** Let $\theta \in T$ and $\varepsilon > 0$. We shall show the first inequality in (23) (the proof of the other one is analogous). By Lemma 2

$$Y_\theta \leq \mathbb{E}_{\theta, \tau \wedge \sigma_0^\varepsilon} \left( Y_{\tau_0^\varepsilon} | H^\varepsilon 1_{[\tau_0^\varepsilon \leq \sigma]} + Y_{\sigma} | B 1_{[\sigma < \tau_0^\varepsilon]} \right). \quad (24)$$

We have

$$Y_{\tau_0^\varepsilon} | H^\varepsilon 1_{[\tau_0^\varepsilon \leq \sigma]} + Y_{\sigma} | B 1_{[\sigma < \tau_0^\varepsilon]} = Y_{\tau_0^\varepsilon} | H^\varepsilon 1_{[\tau_0^\varepsilon \leq \sigma, \tau_0^\varepsilon < T]} + \xi 1_{[\tau_0^\varepsilon = \sigma = T]}.$$

By Lemma 2, $Y_{\tau_0^\varepsilon} | H^\varepsilon \leq L_{\tau_0^\varepsilon}^u | H^\varepsilon + \varepsilon$. Moreover, since $Y \leq U$ and $Y$ is right-limited, we have $Y_{\sigma} | B = Y_{\sigma} | B \leq U_{\sigma} | B$. Consequently,

$$Y_{\tau_0^\varepsilon} | H^\varepsilon 1_{[\tau_0^\varepsilon \leq \sigma]} + Y_{\sigma} | B 1_{[\sigma < \tau_0^\varepsilon]} \leq (L_{\tau_0^\varepsilon}^u | H^\varepsilon + \varepsilon) 1_{[\tau_0^\varepsilon \leq \sigma, \tau_0^\varepsilon < T]} + U_{\sigma} | B 1_{[\sigma < \tau_0^\varepsilon]} + \xi 1_{[\tau_0^\varepsilon = \sigma = T]} \leq J(\tau_0^\varepsilon | H^\varepsilon, \sigma | B) + \varepsilon.$$

By (24) and by properties of the operator $\mathbb{E}^f$ (see Proposition 2 (ii) and (v)), we get

$$Y_\theta \leq \mathbb{E}_{\theta, \tau_0^\varepsilon} [J(\tau_0^\varepsilon | H^\varepsilon, \sigma | B) + \varepsilon] \leq \mathbb{E}_{\theta, \tau_0^\varepsilon} [J(\tau_0^\varepsilon | H^\varepsilon, \sigma | B)] + C \varepsilon. \quad (25)$$

**Theorem 5** Let $(Y, Z, R)$ be a solution to RBSDE$^T(\xi, f, L, U)$. The extended $\mathbb{E}^f$-Dynkin game has a value. What is more, for any stopping time $\theta \in T$,

$$\mathcal{V}(\theta) = Y_\theta = \mathcal{V}(\theta).$$
Moreover, for every $\theta \in T$ and $\varepsilon > 0$ the pair of stopping systems $(\tau^\varepsilon_\theta \restriction H^\varepsilon, \delta^\varepsilon_\theta \restriction G^\varepsilon)$ defined in (12) is $\varepsilon$-saddle point in time $\theta$ for extended $\mathbb{F}$-Dynkin game, i.e. satisfies inequalities (13).

**Proof** Since right-hand side inequality in (24) is satisfied for all $\sigma \in \mathcal{U}_\theta$ we have that

$$Y_\theta \leq \text{ess inf}_{\sigma \in \mathcal{U}_\theta} \mathbb{E}^f_{\theta, \tau^\varepsilon_\theta \land \sigma} \left( J(\tau^\varepsilon_\theta \restriction H^\varepsilon, \sigma \restriction B) \right) + C\varepsilon$$

Thus, by the definition of $V_\theta$ (see (11)) we have that $Y_\theta \leq V(\theta) + C\varepsilon$. Similarly, we show that $V(\theta) - C\varepsilon \leq Y_\theta$ for all $\varepsilon > 0$. In consequence, $V(\theta) \leq Y_\theta$ gives us $V(\theta) = Y_\theta = V(\theta)$. $\Box$

The representation given in Theorem 5 yields, as a simple corollary, the uniqueness for solutions to RBSDEs with two optional barriers of class (D).

**Theorem 6** There exists at most one solution $(Y, Z, R)$ to RBSDE$^T(\xi, f, L, U)$.

### 7 Nonlinear Dynkin Games

Throughout the section, we assume that (H1)–(H5), (Z) are in force and that $L$ and $U$ are $\mathbb{F}$-optional processes of class (D).

For $\tau, \sigma \in T$ we define the pay-off

$$J_0(\tau, \sigma) := L_\tau 1_{[\tau \leq \sigma, \tau < T]} + U_\sigma 1_{[\tau < \sigma]} + \xi 1_{[\tau = \sigma = T]}.$$  

**Definition 6** Let $\theta \in T$. Upper and lower value of the game are defined respectively as

$$V_0(\theta) := \text{ess inf}_{\sigma \geq \theta} \text{ess sup}_{\tau \geq \theta} \mathbb{E}^f_{\theta, \tau \land \sigma} J_0(\tau, \sigma);$$

$$V_0(\theta) := \text{ess sup}_{\tau \geq \theta} \text{ess inf}_{\sigma \geq \theta} \mathbb{E}^f_{\theta, \tau \land \sigma} J_0(\tau, \sigma).$$  

**Lemma 4** Let $(Y, Z, R)$ be a solution to RBSDE$^T(\xi, f, L, U)$. Assume that $L$ is right upper semicontinuous and $U$ is right lower semicontinuous. Then

$$Y_{\tau^\varepsilon_\theta} \leq L_{\tau^\varepsilon_\theta} + \varepsilon, \quad Y_{\sigma^\varepsilon_\theta} \geq U_{\sigma^\varepsilon_\theta} - \varepsilon.$$  

**Proof** In case $\tau^\varepsilon_\theta = T$, (28) is obvious. Suppose that $\tau^\varepsilon_\theta < T$. Suppose by contradiction that $P(Y_{\tau^\varepsilon_\theta} > L_{\tau^\varepsilon_\theta} + \varepsilon) > 0$. Without loss of generality we may assume that properties attributed to $Y$, $M$, $R$, $L$, $U$ and holding $P$-a.s. hold for any $\omega \in \Omega$. 

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Fix \( \omega \in \{ Y_{\tau_{\theta}^e} > L_{\tau_{\theta}^e}^\xi + \varepsilon \} \). By the minimality condition \( \Delta^+ R_{\tau_{\theta}^e}(\omega) = 0 \), and so \( \Delta^+ Y_{\tau_{\theta}^e}(\omega) = -(\Delta^+ R_{\tau_{\theta}^e}^+ - \Delta^+ R_{\tau_{\theta}^e}^-)(\omega) = \Delta^+ R_{\tau_{\theta}^e}^- (\omega) > 0 \). Therefore

\[
Y_{(\tau_{\theta}^e)+}(\omega) > L_{\tau_{\theta}^e}^\xi (\omega) + \varepsilon. \tag{29}
\]

Take \( \omega \in \Omega \). By the definition of \( \tau_{\theta}^e \) there exists a non-increasing sequence \(( t_n(\omega) ) \) such that \( Y_{t_n}(\omega) \leq L_{t_n}^\xi (\omega) + \varepsilon \) for any \( n \in \mathbb{N} \). Hence \( \limsup_{n \to \infty} Y_{t_n}(\omega) \leq \limsup_{n \to \infty} L_{t_n}^\xi (\omega) + \varepsilon \). By the assumptions made \( L \) is right upper semicontinuous, thus \( \limsup_{n \to \infty} L_{t_n}^\xi (\omega) \leq L_{\tau_{\theta}^e}^\xi (\omega) \). On the other hand \( t_n(\omega) \searrow \tau_{\theta}^e (\omega) \) implies \( \limsup_{n \to \infty} Y_{t_n}(\omega) = Y_{(\tau_{\theta}^e)+}(\omega) \). Consequently, \( Y_{(\tau_{\theta}^e)+}(\omega) \leq L_{\tau_{\theta}^e}^\xi (\omega) + \varepsilon \), which contradicts (29). From this we deduce that \( Y_{\tau_{\theta}^e} \leq L_{\tau_{\theta}^e}^\xi + \varepsilon \) for \( P \)-a.e. \( \omega \in \Omega \). \( \Box \)

**Theorem 7** Assume that \( L \) is right upper semicontinuous and \( U \) is right lower semicontinuous. Let \(( Y, Z, R )\) be a solution to RBSDE\(^T\)(\( \xi, f, L, U \)). Then for any \( \theta \in \mathcal{T} \)

\[
Y_{\theta} = \overline{V}_0(\theta) = V_0(\theta). \tag{30}
\]

Moreover, for any \(( \tau, \sigma ) \in \mathcal{T}_{\theta} \times \mathcal{T}_{\theta} \)

\[
\mathbb{E}^f_{\theta, \tau \wedge \sigma} J_0(\tau, \sigma) - C \varepsilon \leq Y_{\theta} \leq \mathbb{E}^f_{\theta, \tau_{\theta}^e \wedge \sigma} J_0(\tau_{\theta}^e, \sigma) + C \varepsilon, \tag{31}
\]

where \( C \) is a constant depending only on \( \lambda, \mu, \nu, T, \| g \|_{L^1}, \gamma \).

**Proof** Let \( \theta \in \mathcal{T} \) and \( \varepsilon > 0 \). We shall prove that \(( \tau_{\theta}^e, \sigma_{\theta}^e )\) satisfies (31). By Lemma 1 \( Y \) is an \( \mathcal{E}^f \)-submartingale on \([ [ \theta, \tau_{\theta}^e ] ] \). We thus have

\[
Y_{\theta} \leq \mathbb{E}^f_{\theta, \tau_{\theta}^e \wedge \sigma} [ Y_{\tau_{\theta}^e \wedge \sigma} ]. \tag{32}
\]

By the assumptions made on \( L \) and Lemma 4, \( Y_{\tau_{\theta}^e} \leq L_{\tau_{\theta}^e}^\xi + \varepsilon \). From this and the fact that \( Y \leq U \) we have

\[
Y_{\tau_{\theta}^e \wedge \sigma} \leq ( L_{\tau_{\theta}^e}^\xi + \varepsilon ) [ 1_{ [ \tau_{\theta}^e \leq \sigma, \tau_{\theta}^e < T ] } + U_{\sigma} 1_{ [ \sigma < \tau_{\theta}^e ] } + \xi 1_{ [ \tau_{\theta}^e = \sigma = T ] } ] \leq J_0(\tau_{\theta}^e, \sigma) + \varepsilon.
\]

Applying (32) and properties of the operator \( \mathbb{E}^f \) (see Proposition 2 (ii) and (v)) yields

\[
Y_{\theta} \leq \mathbb{E}^f_{\theta, \tau_{\theta}^e \wedge \sigma} ( J_0(\tau_{\theta}^e, \sigma) + \varepsilon ) \leq \mathbb{E}^f_{\theta, \tau_{\theta}^e \wedge \sigma} J_0(\tau_{\theta}^e, \sigma) + C \varepsilon. \tag{33}
\]

By Lemma 1 \( Y \) is an \( \mathcal{E}^f \)-supermartingale on \([ [ \theta, \sigma_{\theta}^e ] ] \). As a result

\[
Y_{\theta} \geq \mathbb{E}^f_{\theta, \tau \wedge \sigma_{\theta}^e} ( Y_{\tau \wedge \sigma_{\theta}^e} ). \tag{34}
\]
By the assumptions made on $U$ and Lemma 4 we have $Y_{\sigma_0^e} \geq U_{\sigma_0^e} - \varepsilon$. Applying analogous arguments as in case of $L$ yields $Y_\theta \geq \mathbb{E}_{\theta, \tau \wedge \sigma_0^e}^f J_0(\tau, \sigma_0^e) - C\varepsilon$, which combined with (33) gives (31). Consequently,

$$Y_\theta \leq \text{ess inf}_{\sigma \geq \theta} \mathbb{E}_{\theta, \tau \wedge \sigma}^f J_0(\tau, \sigma) + \varepsilon \leq \text{ess sup}_{\tau \geq \theta} \mathbb{E}_{\theta, \tau \wedge \sigma}^f J_0(\tau, \sigma) + \varepsilon,$$

which combined with the definition of $V_0(\theta)$ yields $Y_\theta \leq V(\theta) + \varepsilon$. Letting $\varepsilon \to 0$ we find that $V(\theta) = Y_\theta \leq V(\theta)$.

### 8 Existence of Saddle Points

In the whole section, we assume that (H1)–(H5), (Z) are in force and that $L$ and $U$ are $\mathbb{F}$-optional processes of class (D).

Let $(Y, Z, R)$ be a solution to RBSDE$^T(\xi, f, L, U)$. We shall prove that there exists a saddle point for a nonlinear Dynkin game with sufficiently regular payoffs. For $\theta \in T$ we define:

$$(Y, Z, R) \text{ is a solution to RBSDE } T(\xi, f, L, U).$$

For $\theta \in T$ we define:

$$\tau^*_\theta := \inf \left\{ t \geq \theta, Y_t = L_t^\xi \right\} \wedge T; \quad \sigma^*_\theta := \inf \left\{ t \geq \theta, Y_t = U_t^\xi \right\} \wedge T$$

(35)

and

$$\bar{\tau}_\theta := \inf \left\{ t \geq \theta, R_t^+ > R_\theta^+ \right\} \wedge T; \quad \bar{\sigma}_\theta := \inf \left\{ t \geq \theta, R_t^- > R_\theta^- \right\} \wedge T.$$  

(36)

**Proposition 3** Assume that $L$ is right upper semicontinuous and $U$ is right lower semicontinuous. Let $(Y, Z, R)$ be a solution to RBSDE$^T(\xi, f, L, U)$ and let $\theta \in T$.

1. If $R^-,*$ is continuous, then $Y$ is an $\mathbb{E}^f$-supermartingale on $[\theta, \bar{\sigma}_\theta]$. Moreover,

$$Y_{\sigma^*_\theta} = U_{\sigma^*_\theta}^\xi \quad \text{and} \quad Y_{\bar{\sigma}_\theta} = U_{\bar{\sigma}_\theta}^\xi.$$  

(37)

2. If $R^+,*$ is continuous, then $Y$ is an $\mathbb{E}^f$-submartingale on $[\theta, \bar{\tau}_\theta]$. Moreover,

$$Y_{\tau^*_\theta} = L_{\tau^*_\theta}^\xi \quad \text{and} \quad Y_{\bar{\tau}_\theta} = L_{\bar{\tau}_\theta}^\xi.$$  

(38)

**Proof** Ad 1). Assume that $R^-,*$ is continuous. By the definition of $\bar{\sigma}_\theta$ we have that $R_{\bar{\sigma}_\theta}^- = R_\theta^-$. Thus,

$$Y_t = Y_{\bar{\sigma}_\theta} + \int_t^{\bar{\sigma}_\theta} f(r, Y_r, Z_r) dr + R_{\bar{\sigma}_\theta}^+ - R_t^+ - \int_t^{\bar{\sigma}_\theta} Z_r dB_r, \quad t \in [\theta, \bar{\sigma}_\theta].$$

By Proposition 2, $Y$ is an $\mathbb{E}^f$-supermartingale on $[\theta, \bar{\sigma}_\theta]$. We shall prove that $Y_{\bar{\sigma}_\theta} = U_{\bar{\sigma}_\theta}^\xi$. Assume that $\bar{\sigma}_\theta < T$ (in case $\bar{\sigma}_\theta = T$ the desired equality is obvious). Suppose,
by contradiction, that $P(Y_{\bar{\sigma}_0} < U_{\bar{\sigma}_0}^{\xi}) > 0$. By the minimality condition, $\Delta^+ R_{\bar{\sigma}_0}^- = 0$ on $\{Y_{\bar{\sigma}_0} < U_{\bar{\sigma}_0}^{\xi}\}$. Observe that $\Delta^+ Y_{\bar{\sigma}_0} = -(\Delta^+ R_{\bar{\sigma}_0}^+ - \Delta^+ R_{\bar{\sigma}_0}^-) = -\Delta^+ R_{\bar{\sigma}_0}^+ \leq 0$, which implies that $Y$ is right upper semicontinuous on $\{Y_{\bar{\sigma}_0} < U_{\bar{\sigma}_0}^{\xi}\}$. Let $a \in \mathbb{R}$ and $\varepsilon > 0$ (depending on $\omega \in \Omega$) be such that $U_{\bar{\sigma}_0}^{\xi} > a + \varepsilon$ and $Y_{\bar{\sigma}_0} < a - \varepsilon$. Since $Y$ is right upper semicontinuous on $\{Y_{\bar{\sigma}_0} < U_{\bar{\sigma}_0}^{\xi}\}$, and $U$ is right lower semicontinuous, there exists $\delta > 0$ (depending on $\omega \in \Omega$) such that $U_{\bar{\sigma}_0 + \delta}^{\xi} > a + \varepsilon$ and $Y_{\bar{\sigma}_0 + \delta} < a - \varepsilon$, $s \in [0, \delta]$. Furthermore, from the definition of $\bar{\sigma}_0$ we have $R_{\bar{\sigma}_0 + \delta}^{-,*} > R_{\bar{\sigma}_0}^{-,*}$. Consequently, on the set $\{Y_{\bar{\sigma}_0} < U_{\bar{\sigma}_0}^{\xi}\}$ the following holds

$$\int_{\bar{\sigma}_0}^{\bar{\sigma}_0 + \delta} (U_\tau^\xi - Y_{\tau -}) \, dR_{\tau}^{-,*} > 2\varepsilon \left( R_{\bar{\sigma}_0 + \delta}^{-,*} - R_{\bar{\sigma}_0}^{-,*} \right) > 0.$$ 

This contradicts the minimality condition.

What is left is to show that $Y_{\sigma_0^*} = U_{\sigma_0^*}^{\xi}$. In case $\sigma_0^* = T$ the equation follows at once. Suppose that $\sigma_0^* < T$. If $\Delta^+ R_{\sigma_0^*}^{-,*} (\omega) > 0$, then by the very definition of a solution to RBSDE, we have $Y_{\sigma_0^*} (\omega) = U_{\sigma_0^*}^{\xi} (\omega)$. Suppose that $\Delta^+ R_{\sigma_0^*}^{-,*} (\omega) = 0$. Observe that

$$\Delta^+ Y_{\sigma_0^*} (\omega) = -\left( \Delta^+ R_{\sigma_0^*}^+ (\omega) - \Delta^+ R_{\sigma_0^*}^- (\omega) \right) = -\Delta^+ R_{\sigma_0^*}^+ (\omega) \leq 0.$$ 

Thus, $Y_{\sigma_0^*} (\omega) \leq Y_{\sigma_0^*} (\omega)$. By the definition of $\sigma_0^*$ there exists a non-increasing sequence $(t_n (\omega))_{n \geq 1}$ such that $t_n (\omega) \searrow \sigma_0^* (\omega)$ and $Y_{t_n} (\omega) = U_{t_n}^{\xi} (\omega)$. Letting $n \to \infty$ and using right lower semicontinuity of $U$ we find that $Y_{\sigma_0^*} (\omega) \geq U_{\sigma_0^*}^{\xi} (\omega)$, which combined with $Y_{\sigma_0^*} (\omega) \leq Y_{\sigma_0^*} (\omega)$ gives the result.

Ad A2). The case when $R_{\tau}^{-,*}$ is supposed to be continuous runs analogously. □

**Corollary 1** Under assumptions of Proposition 3 we have that continuity of $R_{\tau}^{-,*}$ (resp. $R_{\tau}^{+,*}$) implies $\sigma_0^* \leq \bar{\sigma}_0$ (resp. $\tau_0^* \leq \bar{\tau}_0$).

**Proposition 4** Let $(Y, Z, R)$ be a solution to RBSDE$^T(\xi, f, L, U)$. If $L$ (resp. $U$) is left upper semicontinuous (resp. left lower semicontinuous), then $R_{\tau}^{+,*}$ (resp. $R_{\tau}^{-,*}$) is continuous.

**Proof** Let $\tau \in T$ be predictable. We shall prove that $\Delta^+ R_{\tau}^{+,*} = 0$. We have

$$\Delta Y_{\tau} = -\Delta R_{\tau}^{+,*} + \Delta R_{\tau}^{-,*} = -\Delta R_{\tau}^{+,*} 1_{\{Y_{\tau} = \overline{L}_{\tau} \cap D \}}$$

$$+ \Delta R_{\tau}^{-,*} 1_{\{Y_{\tau} = \underline{U}_{\tau} \cap D' \}}, \quad (39)$$

where $D := \{\Delta R_{\tau}^{+,*} > 0\}$ and $D' := \{\Delta R_{\tau}^{-,*} > 0\}$. Since $dR_{\tau}^+ \perp dR_{\tau}^-$, $D \cap D' = \emptyset$. Thus, on the set $D$, $\Delta Y_{\tau} \leq 0$. From this and the regularity assumption on $L$, $\overline{L}_{\tau} \leq L_{\tau} \leq Y_{\tau} \leq \underline{Y}_{\tau} \leq Y_{\tau -}$ on $D$. Consequently, $\Delta Y_{\tau} = 0$ on $\{Y_{\tau} = \overline{L}_{\tau} \cap D \}$. This combined with (39) implies $\Delta^+ R_{\tau}^{+,*} = 0$. Since the last inequality holds for any predictable
\( \tau \in \mathcal{T} \), we deduce that \( R^{+, \ast} \) is continuous. The similar reasoning may be applied to \( U \).

\[ \square \]

**Theorem 8** Suppose that \( L \) is upper semicontinuous and \( U \) is lower semicontinuous. Let \((Y, Z, R)\) be a solution to \( \text{RBSDE}^T(\xi, f, L, U) \). Then for any \( \theta \in \mathcal{T} \) couples (35) and (36) are saddle points at \( \theta \) for the nonlinear Dynkin game with the payoff function (26).

**Proof** Let \( \theta \in \mathcal{T} \). By Theorem 7 \( Y_\theta = \overline{V}_0(\theta) = V_0(\theta) \). By Proposition 4 \( R^{+, \ast}, R^{-, \ast} \) are continuous. Let \( \tau \in \mathcal{T}_\theta \). Since \( \sigma^\ast_\theta \leq \bar{\sigma}_\theta \) (see Corollary 1), by Proposition 3 process \( Y \) is an \( \mathcal{E}^f \)-supermartingale on \( [\theta, \tau \wedge \sigma^\ast_\theta] \). Therefore,

\[
Y_\theta \geq \mathbb{E}^{\mathcal{F}}_{\theta, \tau \wedge \sigma^\ast_\theta}[Y_{\tau \wedge \sigma^\ast_\theta}].
\]

Since \( Y \geq L \) and \( Y_{\sigma^\ast_\theta} = U_{\sigma^\ast_\theta} \) (see Proposition 3), we also have

\[
Y_{\tau \wedge \sigma^\ast_\theta} = Y_\tau 1_{[\tau \leq \sigma^\ast_\theta]} + Y_{\sigma^\ast_\theta} 1_{[\sigma^\ast_\theta < \tau]} \geq L_\tau 1_{[\tau \leq \sigma^\ast_\theta]} + U_{\sigma^\ast_\theta} 1_{[\sigma^\ast_\theta < \tau]} = J_0(\tau, \sigma^\ast_\theta).
\]

Using (40) and the fact that \( \mathbb{E}^f \) is a non-decreasing operator, we deduce that \( Y_\theta \geq \mathbb{E}^{\mathcal{F}}_{\theta, \tau \wedge \sigma^\ast_\theta}J_0(\tau, \sigma^\ast_\theta) \) for any \( \tau \in \mathcal{T}_\theta \), in particular \( \mathbb{E}^{\mathcal{F}}_{\theta, \tau \wedge \sigma^\ast_\theta}J_0(\tau_\theta^\ast, \sigma^\ast_\theta) \leq Y_\theta \). In the similar way we arrive at \( Y_\theta \leq \mathbb{E}^{\mathcal{F}}_{\theta, \tau_\theta^\ast \wedge \sigma^\ast_\theta}J_0(\tau_\theta^\ast, \sigma^\ast_\theta) \) for any \( \sigma \in \mathcal{T}_\theta \), and so \( Y_\theta \leq \mathbb{E}^{\mathcal{F}}_{\theta, \tau_\theta^\ast \wedge \sigma^\ast_\theta}J_0(\tau_\theta^\ast, \sigma^\ast_\theta) \). Consequently, \( Y_\theta = \mathbb{E}^{\mathcal{F}}_{\theta, \tau_\theta^\ast \wedge \sigma^\ast_\theta}J(\tau_\theta^\ast, \sigma^\ast_\theta) \) and \( (\tau_\theta^\ast, \sigma^\ast_\theta) \) is a saddle point at \( \theta \). Analogously, one shows, by using Proposition 3, that \( (\bar{\tau}_\theta, \bar{\sigma}_\theta) \) is a saddle point at \( \theta \).

\[ \square \]

**9 Existence Result**

In the whole section, we assume that \( L, U \) are \( \mathcal{F} \)-optional processes of class (D).

Let us consider the following assumption, which is called in the literature **weak Mokobodzki’s condition**.

(WM) There exists a process \( X \in \mathcal{M}_{loc}(0, T) + \mathcal{V}_{\mathbb{F}}(0, T) \) such that \( L \leq X \leq U \).

**Proposition 5** Assume that \( L, U \) are left-limited, and

\[
\widehat{L}_t < U_t, \quad L_{t-} < U_{t-}, \quad t \in [0, T].
\]

Then weak Mokobodzki’s condition (WM) holds for \( L, U \).

**Proof** We let \( \tau_0 := 0 \), and for \( n \geq 1 \),

\[
\tau_n := \inf \left\{ t > \tau_{n-1} : \left( \widehat{L}_{\tau_{n-1}} + U_{\tau_{n-1}} \right) < 2L_t \text{ or } \left( \widehat{L}_{\tau_{n-1}} + U_{\tau_{n-1}} \right) > 2U_t \right\} \wedge T.
\]
Obviously, \((\tau_n)\) is nondecreasing. Observe that by the definition of \(\tau_n\) for each \(\omega \in \Omega\) there exists a sequence \(\{t_m^\omega\}\) such that \(t_m^\omega \downarrow \tau_n(\omega)\) and for all \(m \in \mathbb{N}\)

\[
\begin{align*}
\hat{L}_{\tau_{n-1}(\omega)}(\omega) + U_{\tau_{n-1}(\omega)}(\omega) &< 2L_{t_m^\omega}(\omega) \quad \text{or} \\
\hat{L}_{\tau_{n-1}(\omega)}(\omega) + U_{\tau_{n-1}(\omega)}(\omega) &> 2U_{t_m^\omega}(\omega).
\end{align*}
\] (42)

Letting \(m \to \infty\) yields

\[
\begin{align*}
\hat{L}_{\tau_{n-1}(\omega)}(\omega) + U_{\tau_{n-1}(\omega)}(\omega) &\leq 2\hat{L}_{\tau_n(\omega)}(\omega) \quad \text{or} \\
\hat{L}_{\tau_{n-1}(\omega)}(\omega) + U_{\tau_{n-1}(\omega)}(\omega) &\geq 2U_{\tau_n(\omega)}(\omega), \quad n \geq 1.
\end{align*}
\] (43)

**Step 1** We shall prove that \((\tau_n)\) is a chain. First, note that

\[
P(\tau_{n-1} = \tau_n < T) = 0, \quad n \geq 1.
\] (44)

Indeed, suppose that for some \(n \geq 1\), \(P(\tau_{n-1} = \tau_n < T) > 0\). Let \(\omega \in \{\tau_{n-1} = \tau_n < T\}\). Then, by (43)

\[
U_{\tau_n(\omega)}(\omega) \leq \hat{L}_{\tau_n(\omega)}(\omega).
\]

Therefore, \(P(U_{\tau_n} \leq \hat{L}_{\tau_n}) > 0\), which contradicts (41). Suppose that \((\tau_n)\) is not a chain. Then, according to (44), there must exist \(\tau \in T\) such that \(\tau_n \not\to \tau\) and \(P(\cap_{n=1}^\infty \{\tau_n < \tau\}) > 0\). Let \(\omega \in \cap_{n=1}^\infty \{\tau_n < \tau\}\). By the second inequality in (41) for any \(\delta > 0\) there exists \(n_\delta \geq 1\) such that

\[
\begin{align*}
\hat{L}_{\tau_{n-1}(\omega)}(\omega) &\leq L_{\tau(\omega)-}(\omega) + \delta, \quad U_{\tau(\omega)-}(\omega) - \delta \leq U_{\tau_{n-1}(\omega)}(\omega), \quad n \geq n_\delta.
\end{align*}
\]

Suppose that the first inequality in (43) holds for infinitely many \(n \geq 1\) (the proof in the second case is analogous). Then, by the above inequalities, we conclude from (43) that

\[
L_{\tau(\omega)-}(\omega) + U_{\tau(\omega)-}(\omega) - 2\delta \leq L_{\tau(\omega)-}(\omega) + \delta.
\]

Letting \(\delta \searrow 0\), we obtain that \(U_{\tau(\omega)-}(\omega) \leq L_{\tau(\omega)-}(\omega)\). Therefore, \(P(U_{\tau-} \leq L_{\tau-}) > 0\), which contradicts (41). Thus, \((\tau_n)\) is a chain.

**Step 2** We shall construct an optional semimartingale lying between barriers \(L, U\).

Define

\[
X_t := \frac{1}{2} \sum_{n=1}^\infty \left(\left(\hat{L}_{\tau_{n-1}} + U_{\tau_{n-1}}\right)1_{(\tau_{n-1}, \tau_n)}(t) + \left(L_{\tau_{n-1}} + U_{\tau_{n-1}}\right)1_{[\tau_{n-1}]}(t)\right),
\]

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for \( t \in [0, T] \). Clearly, \( L_t \leq X_t \leq U_t, t \in [0, T] \) and \( X \) is \( \mathbb{F} \)-adapted. Since \((\tau_n)\) is a chain, we get that \( X \) is of finite variation, thus an optional semimartingale. This completes the proof. \( \Box \)

**Proposition 6** Assume that \( f_1, f_2 \) satisfy (H1)–(H4), \( \xi_1, \xi_2, f_1, f_2 \) satisfy (H5) with \( p = 2 \), and \( |f^1 - f^2|, (\cdot, Y^2, Z^2) \in L^{1,2}_{\mathbb{F}}(0, T) \). Let \((Y^i, Z^i, R^i)\) be a solution to RBSDE\(^T\)\((\xi, f^i, L, U)\) such that \( Y^i \in S^0_{\mathbb{F}}(0, T), i = 1, 2 \). Then there exists \( c > 0 \), depending only on \( T, \mu, \lambda \), such that

\[
\|Y^1 - Y^2\|_{D^2(0, T)} \leq c \left( \|\xi_1 - \xi_2\|_{L^2} + \|f_1 - f_2\|_{L^{1,2}_{\mathbb{F}}(0, T)} \right).
\]

**Proof** We let \( J^i \) denote the right-hand side of (10) but with \( \xi \) replaced by \( \xi_i, i = 1, 2 \). Let

\[
\tilde{f}(t, y, z) := f_1(t, Y^1_t, Z^1_t) - f_2(t, Y^1_t, Z^1_t) + f_2(t, y, z), \quad t \in [0, T], y \in \mathbb{R}, z \in \mathbb{R}^d.
\]

Observe that \((Y^1, Z^1, R^1)\) is a solution to RBSDE\(^T\)\((\tilde{\xi}, \tilde{f}, L, U)\). By Theorem 5

\[
Y^1_0 = \text{ess sup} \left[ \text{ess inf}_{\tau | A \in \mathcal{U}_0, \sigma | B \in \mathcal{U}_0} \mathbb{E}^{f}_{\theta, \tau \wedge \sigma} J^1(\tau | A, \sigma | B) \right]
\]

and

\[
Y^2_0 = \text{ess sup} \left[ \text{ess inf}_{\tau | A \in \mathcal{U}_0, \sigma | B \in \mathcal{U}_0} \mathbb{E}^{f_2}_{\theta, \tau \wedge \sigma} J^2(\tau | A, \sigma | B) \right].
\]

Hence

\[
|Y^1_0 - Y^2_0| \leq \text{ess sup} \left[ \text{ess sup}_{\tau | A \in \mathcal{U}_0, \sigma | B \in \mathcal{U}_0} \left| \mathbb{E}^{f}_{\theta, \tau \wedge \sigma} J^1(\tau | A, \sigma | B) - \mathbb{E}^{f_2}_{\theta, \tau \wedge \sigma} J^2(\tau | A, \sigma | B) \right| \right].
\]

(46)

Applying Proposition 2(vi) yields

\[
\mathbb{E} \left| \mathbb{E}^{f}_{\theta, \tau \wedge \sigma} J^1(\tau | A, \sigma | B) - \mathbb{E}^{f_2}_{\theta, \tau \wedge \sigma} J^2(\tau | A, \sigma | B) \right|^2 \leq c \mathbb{E} \left[ \|\xi_1 - \xi_2\|^2 + \left( \int_0^T |f_1 - f_2|(r, Y^2_r, Z^2_r) dr \right)^2 \right].
\]

Combining the last two inequalities gives at once the result. \( \Box \)

**Theorem 9** Assume that (H1)–(H4), (WM) are in force. Suppose that (H5) is satisfied with \( p = 1 \). Then there exists a solution \((Y, Z, R)\) to RBSDE\(^T\)\((\xi, f, L, U)\).

**Proof** Let \( X \) be the process appearing in (WM). Since \( X \) is a special semimartingale, there exists a chain \((\hat{y}_k)\) and processes \( H \in \mathcal{H}_{\mathbb{F}}(0, T) \) and \( C \in \mathcal{V}_{\mathbb{F}}(0, T) \) such that
$H \in \mathcal{H}_F^2(0, \hat{\gamma}_k), C \in \mathcal{V}_F^2(0, \hat{\gamma}_k), k \geq 1,$ and

$$X_t = X_0 + C_t + \int_0^t H_r dB_r, \quad t \in [0, T].$$

Let

$$f_{n,m}(t, y, z) = \{ f(t, y, z) \land n \} \lor (-m).$$

Note that $f_{n,m}$ is nondecreasing with respect to $n$ and nonincreasing with respect to $m$. Moreover, $f_{n,m}(t, y, z) \nearrow f(t, y, z) \lor (-m)$, when $n \to \infty$, and $f_{n,m}(t, y, z) \searrow f(t, y, z)$, when $m \to \infty$. By [9, p. 417] there exist regulated processes $\hat{L}, \hat{U}$ satisfying

$$\hat{L}_v = \text{ess inf}_{\tau \in T_v} \mathbb{E}(L_{\tau} | \mathcal{F}_v), \quad \hat{U}_v = \text{ess sup}_{\tau \in T_v} \mathbb{E}(U_{\tau} | \mathcal{F}_v), \quad v \in T.$$ 

Furthermore, by [29, Proposition 3.8], $-\hat{L}, \hat{U}$ are supermartingales of class (D) on $[0, T]$. As a result, there exist processes $F, G \in \mathcal{H}_F(0, T)$ and $A, D \in \mathcal{V}_F^{1,1}(0, T)$ such that

$$\hat{L}_t = \hat{L}_T - \int_t^T dA_r - \int_t^T F_r dB_r, \quad t \in [0, T].$$

and

$$\hat{U}_t = \hat{U}_T - \int_t^T dD_r - \int_t^T G_r dB_r, \quad t \in [0, T].$$

Obviously, $\hat{L} \leq L \leq U \leq \hat{U}$, since $\hat{L}$ and $\hat{U}$ are of class (D), by (H4) there exists a chain $(\tau^L_k)$ on $[0, T]$ such that

$$\mathbb{E}\left( \int_0^{\tau^L_k} \left| f(r, \hat{L}_r, 0) \right| dr \right)^2 + \mathbb{E}\left( \int_0^{\tau^L_k} \left| f(r, \hat{U}_r, 0) \right| dr \right)^2 \leq k. \quad (47)$$

Moreover, let us consider chain $(\tau^U_k)$ on $[0, T]$ such that $\hat{L}, \hat{U} \in \mathcal{S}_F^2(0, \tau^U_k), f(\cdot, 0, 0) \in L^{1,2}_F(0, \tau^U_k)$, and $A, D \in \mathcal{V}_F^2(0, \tau^U_k)$, $k \geq 1$. We let $\gamma_k := \hat{\gamma}_k \land \tau^L_k \land \tau^U_k$. Define

$$L^n_t = L_t 1_{\{t \leq \gamma_n\}} + \hat{L} 1_{\{t > \gamma_n\}}, \quad U^n_t = U_t 1_{\{t \leq \gamma_n\}} + \hat{U} 1_{\{t > \gamma_n\}}.$$ 

Note that

$$\hat{L} \leq L^n \leq L^{n+1} \leq L \leq U \leq U^{n+1} \leq U^n \leq \hat{U}, \quad n \geq 1. \quad (48)$$
Moreover, $L^n \not\subseteq L$ and $U^n \not\subseteq U$. Finally, we define

$$X_{i}^{n,m} = X_{i}^{1} 1_{\{t \leq \gamma_n \wedge \gamma_m\}} + \hat{L}_{i}^{1} 1_{\{t > \gamma_n \geq \gamma_m\}} + \hat{U}_{i}^{1} 1_{\{t > \gamma_m > \gamma_n\}}.$$ 

Note that $L^n \leq X_{i}^{n,m} \leq U^m$ and $X_{i}^{n,m}$ is a difference of two supermartingales of class (D). Therefore, by the definition of $f_{n,m}$, strong Mokobodzki’s condition (H6*) is satisfied with $L^n, U^m, X_{i}^{n,m}$ and $f_{n,m}$. By Theorem 4 there exists a unique solution $(Y^n, Z^n, R^n, m)$ to RBSDE$^T(\xi, f_{n,m}, L^n, U^m)$ such that $Y^n, Z^n, R^n, m \in \mathcal{H}^2_{\mathcal{F}}(0, T), q \in (0, 1)$ and $R^n, m \in \mathcal{Y}_{0,\mathcal{F}}^1(0, T)$. By [30, Proposition 3.2, Lemma 3.3], $Y^n, m$ is nondecreasing with respect to $n$ and nonincreasing with respect to $m$. Let us define

$$Y^m = \sup_{n \geq 1} Y^n, m, \quad Y = \inf_{m \geq 1} Y^m.$$ 

Obviously, $Y^m$ and $Y$ are of class (D). The remainder of the proof, we divide into two steps.

Step 1 We shall prove that or any $k \leq m$, process $Y^m$ is the first component of a solution to RBSDE$^{\gamma_k}(Y^n, \gamma_k, f_{m}, L, U^m)$. Let $k \leq m \leq n$. Since $\hat{L} \leq Y^n, m \leq \hat{U}$, we have $Y^n, m, Y^m \in S^2_{\mathcal{F}}(0, \gamma_k)$. According to Theorem 3—observe that (H5), (H6) are satisfied with $L, U^m, f_{m}, Y^m, X$ and $p = 2$ on $[[0, \gamma_k]]$—there exists a solution $(\tilde{Y}^k, m, \tilde{Z}^k, m, \tilde{R}^k, m)$ to RBSDE$^{\gamma_k}(Y^n, \gamma_k, f_{m}, L, U^m)$ such that $\tilde{Y}^k, m \in S^2_{\mathcal{F}}(0, \gamma_k), \tilde{Z}^k, m \in \mathcal{H}^2_{\mathcal{F}}(0, \gamma_k)$ and $\tilde{R}^k, m \in \mathcal{Y}_{0,\mathcal{F}}^2(0, \gamma_k)$. We shall show that $Y^m = \tilde{Y}^k, m$ on $[[0, \gamma_k]]$. By Proposition 6

$$\begin{align*}
\|\tilde{Y}^k, m - Y^n, m\|_{D^2(0, \gamma_k)}^2 & \leq c_{\mathbb{E}} \left[ \left( \int_{0}^{\gamma_k} |f_{m} - f_{n,m}|(r, \tilde{Y}^k, m, \tilde{Z}^k, m) \, dr \right)^2 \right] \\
+ & \|\tilde{Y}^k, m - Y^n, m\|_{D^2(0, \gamma_k)}^2 = c_{\mathbb{E}} \left[ \|Y^m - Y^n, m\|_{\gamma_k}^2 \\
+ & \left( \int_{0}^{\gamma_k} |f(r, \tilde{Y}^k, m, \tilde{Z}^k, m)| 1_{\{f(r, \tilde{Y}^k, m, \tilde{Z}^k, m) > n\}} \, dr \right)^2 \right].
\end{align*}$$

(49)

Observe that $0 \leq Y^m - Y^n, m \leq 2|Y^m| + |Y^n, m| \in L^2(\Omega, \mathcal{F}_{\gamma_k})$ (the last assertion is a consequence of the fact that $Y^m, Y^n, m \in S^2_{\mathcal{F}}(0, \gamma_k)$). Therefore, by the Lebesgue dominated convergence theorem, the first term on the right-hand side of (49) tends to
zero as $n \to \infty$. Note that, by the definition of $\gamma_k$, (H1), (47) and Jensen’s inequality

$$
\mathbb{E}\left( \int_0^{\gamma_k} \left| f(r, \tilde{Y}^{k,m}_r, \tilde{Z}^{k,m}_r) \right| dr \right) \leq \mathbb{E}\left( \lambda \int_0^{\gamma_k} \left| \tilde{Z}^{k,m}_r \right| dr \right)^2 + \mathbb{E}\left( \int_0^{\gamma_k} \left| f(r, \tilde{Y}_{r}^{k,m}, 0) \right| dr \right)^2 \leq T \lambda^2 \mathbb{E}\int_0^{\gamma_k} \left| \tilde{Z}^{k,m}_r \right|^2 dr
$$

Consequently, by the Lebesgue dominated convergence theorem, the most right term in (49) tends to zero as $n \to \infty$. As a result, letting $n \to \infty$ in (49), we obtain that $Y^{n,m}_r \to \tilde{Y}^{k,m}$ in $D^2_{\mathbb{F}}(0, \gamma_k)$. This completes the proof of step 1.

**Step 2** We shall prove that process $Y$ is the first component of a solution to RBSDE($\xi, f, L, U$). Let $k \leq m$. Since $\hat{L} \leq Y^m \leq \hat{U}$, we have that $Y^m, Y \in S^2_{\mathbb{F}}(0, \gamma_k)$. Observe that conditions (H5) and (H6) are met by $L, U, f, X$ on $[0, \gamma_k]$ with $p = 2$. Therefore, by Theorem 3, there exists a solution $(\tilde{Y}^k, \tilde{Z}^k, \tilde{R}^k)$ to RBSDE$^{\gamma_k}(Y^k, f, L, U)$ such that $\tilde{Y} ^k \in S^2_{\mathbb{F}}(0, \gamma_k)$, $\tilde{Z} ^k \in H^2_{\mathbb{F}}(0, \gamma_k)$, and $\tilde{R} ^k \in \mathcal{V}^2_{0,\mathbb{F}}(0, \gamma_k)$. We shall show that $Y = \tilde{Y} ^k$ on $[0, \gamma_k]$. By Proposition 6

$$
\left\| \tilde{Y} ^k - Y^n \right\|_{D^2_{\mathbb{F}}(0, \gamma_k)} \leq C \mathbb{E}\left[ \left( \int_0^{\gamma_k} \left| f(r, \tilde{Y} ^k_r, \tilde{Z} ^k_r) - f_m(r, \tilde{Y} ^k_r, \tilde{Z} ^k_r) \right| dr \right)^2 + \left| Y_{\gamma_k} - Y^m_{\gamma_k} \right|^2 \right]
$$

$$
= C \mathbb{E}\left[ \left( \int_0^{\gamma_k} \left| f(r, \tilde{Y} ^k_r, \tilde{Z} ^k_r) \right| 1_{\{f(r, \tilde{Y} ^k_r, \tilde{Z} ^k_r) < -m\}} dr \right)^2 + \left| Y_{\gamma_k} - Y^m_{\gamma_k} \right|^2 \right].
$$

By the Lebesgue dominated convergence theorem the first term on the right-hand side of (50) tends to zero when $m \to \infty$. By combining (47), the definition of $\gamma_k$, condition (H1), and Jensen’s inequality, we conclude that

$$
\mathbb{E}\left( \int_0^{\gamma_k} \left| f(r, \tilde{Y} ^k_r, \tilde{Z} ^k_r) \right| dr \right)^2 \leq \mathbb{E}\left( \lambda \int_0^{\gamma_k} \left| \tilde{Z} ^k_r \right| dr \right)^2 + \mathbb{E}\left( \int_0^{\gamma_k} \left| f(r, \tilde{Y} ^k_r, 0) \right| dr \right)^2 \leq T \lambda^2 \mathbb{E}\int_0^{\gamma_k} \left| \tilde{Z} ^k_r \right|^2 dr + \mathbb{E}\left( \int_0^{\gamma_k} \left| f(r, \hat{L}_r, 0) \right| dr \right)^2 \leq T \lambda^2 \mathbb{E}\int_0^{\gamma_k} \left| \tilde{Z} ^k_r \right|^2 dr + T^2 \cdot k^2 < \infty.
$$
Therefore, by the Lebesgue dominated convergence theorem, the most right term in (50) tends to zero as \( m \to \infty \). Consequently, letting \( m \to \infty \) in (50), we deduce that \( Y^m \to Y^k \) in \( D^4_{\mathbb{F}}(0, \gamma_k) \). Hence, \( Y^k = Y \) on \( [0, \gamma_k] \), \( k \geq 1 \). In other words, for any \( k \geq 1 \), process \( Y \) is the first component of a solution to RBSDE\(^Y_k\)(\( Y_{\gamma_k}, f, L, U \)). This in turn implies, by using the uniqueness argument (see Theorem 3), that \( \tilde{Z}^k = \tilde{Z}^{k+1} \), and \( \tilde{R}^k = \tilde{R}^{k+1} \) on \( [0, \gamma_k] \), \( k \geq 1 \). With the aid of these properties, one easily checks that the triple \( (Y, Z, R) \) is a solution to RBSDE\(^Y_k\)(\( Y_{\gamma_k}, f, L, U \)) for each \( k \geq 1 \), where

\[
Z_t := \sum_{k=0}^{\infty} \tilde{Z}^k_1_{(\gamma_k, \gamma_{k+1})}(t), \quad R_t := \sum_{k=0}^{\infty} \tilde{R}^k_1_{(\gamma_k, \gamma_{k+1})}(t). 
\]

This combined with the fact that \( (\gamma_k) \) is a chain implies that \( (Y, Z, R) \) is a solution to RBSDE\(^T\)(\( \xi, f, L, U \)). \( \square \)

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**Declarations**

**Conflict of interest**  The authors declare that they have no conflict of interest.

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**Appendix**

**Proof of Proposition 2** (i) Assume that \( V \) is increasing. Let \( \sigma, \tau \in \mathcal{T} \) be such that \( \nu \leq \sigma \leq \tau \leq \zeta \). Obviously, \((X, H)\) is a solution to BSDE\(^{\nu, \tau}\)(\( X_{\tau}, f + dV \)). Let \((\tilde{X}, \tilde{H})\) be a solution to BSDE\(^{\nu, \tau}\)(\( X_{\tau}, f \)) such that \( \tilde{X} \) is of class (D), in case \( p = 1 \), and \( \tilde{X} \in \mathcal{S}_F^p(\nu, \tau) \), in case \( p > 1 \). By [30, Proposition 3.2, Lemma 3.3] and Theorem 2, \( X \geq \tilde{X} \) on \([\nu, \tau]\), in particular, \( X_{\sigma} \geq \tilde{X}_{\sigma} \). Therefore, we have \( \mathbb{E}^f_{\nu, \tau}(X_{\tau}) = \tilde{X}_{\sigma} \leq X_{\sigma} \), hence \( X \) is \( \mathbb{E}^f \)-supermartingale. Analogously, we show that if \(-V\) is increasing, then \( X \) is \( \mathbb{E}^f \)-submartingale. This completes the proof of (i). The assertion (ii) follows directly from [30, Proposition 3.2, Lemma 3.3] and Theorem 2. As to (iii), let \( A \in \mathcal{F}_\nu \). Let \((Y, Z), (\tilde{Y}, \tilde{Z})\) be solutions to BSDE\(^\zeta\)(\( \xi, f \)) and BSDE\(^\zeta\)(\( 1_A \xi, f_A \)), respectively, such that \( Y_{\tau} = \mathbb{E}^f_{\nu, \zeta}(\xi), \tau \in \mathcal{T}^\zeta \) and \( \tilde{Y}_{\tau} = \mathbb{E}^{f_A}_{\nu, \zeta}(1_A \xi), \tau \in \mathcal{T}^\zeta \). It is easy to see that
$(1_A Y, 1_A Z)$ is a solution to BSDE$^{\nu, \zeta}(1_A \xi, f_A)$. Indeed, for $\sigma \in T_{v, \zeta}$,

$$1_A Y_t = 1_A \xi + \int_\sigma^\zeta 1_A f(r, Y_r, Z_r) \, dr - \int_\sigma^\zeta 1_A Z_r \, dB_r$$

$$= 1_A \xi + \int_\sigma^\zeta 1_A f(r, 1_A Y_r, 1_A Z_r) \, dr - \int_\sigma^\zeta 1_A Z_r \, dB_r.$$

By the uniqueness for BSDEs (see Theorems 1, 2) $1_A Y = \bar{Y}$ on $[[v, \zeta]]$, which implies (iii). For (iv), let $(Y, Z)$ be a solution to BSDE$^\xi(\xi, f)$ and let $(\bar{Y}, \bar{Z})$ be a solution to BSDE$^\nu(\bar{Y}, \bar{Z})$ such that $Y_\tau = \mathbb{E}_{\tau, \zeta}^{f, \xi}(\xi), \tau \in T^\zeta$ and $\bar{Y}_\tau = \mathbb{E}_{\tau, \nu}^{f, \xi}(\xi), \tau \in T^\nu$. Note that, $(\bar{Y}, \bar{Z})$ is a solution to BSDE$^{\nu, \zeta}(\bar{Y}, f)$. What is more, for $\sigma \in T_{v, \zeta}$,

$$\bar{Y}_{\sigma} = \mathbb{E}(\xi + \int_{\sigma}^{\nu} f(r, \bar{Y}_r, \bar{Z}_r) \, dr | \mathcal{F}_{\sigma}) = \mathbb{E}(\xi + \int_{\sigma}^{\zeta} f(r, \bar{Y}_r, \bar{Z}_r) \, dr | \mathcal{F}_{\sigma}),$$

therefore $\bar{Y}_t = \xi$. Hence, $(\bar{Y}, \bar{Z})$ is a solution to BSDE$^{\nu, \zeta}(\xi, f)$, which results, by the uniqueness argument, that $\bar{Y} = Y$ on $[[v, \zeta]]$ and $\mathbb{E}_{v, \zeta}^{f, \xi}(\xi) = \bar{Y}_v = \mathbb{E}_{v, \nu}^{f, \xi}(\xi), v \leq \zeta$. This concludes the proof of (iv). Now, we shall proceed to the proof of (v). Let $(Y^1, Z^1), (Y^2, Z^2)$ be defined as in the assertion (v). By (iv), we know that $\mathbb{E}_{v, \zeta_1}^{f_1}(\xi_{11}) = Y^1_v$, for $v \in T^{\zeta_1}$. Let us define

$$\tau_k = \inf \left\{ t \geq 0 : \int_0^t |Z^1_r - Z^2_r|^2 \, dr \geq k \right\} \wedge \zeta_2, \quad k \in \mathbb{N}.$$

From the definition of a solution to BSDE, we have that $\{\tau_k\}_{k \geq 1}$ is a chain. By Ito’s formula, for $v \in T^{\zeta_1}$,

$$e^{av} |Y^1_v - Y^2_v|^2 + \int_v^{\tau_k} e^{ar} |Z^1_r - Z^2_r|^2 \, dr + a \int_v^{\tau_k} e^{ar} |Y^1_r - Y^2_r|^2 \, dr$$

$$\leq e^{a\tau_k} |Y^1_{\tau_k} - Y^2_{\tau_k}|^2 + 2 \int_v^{\tau_k} e^{ar} (Y^1_r - Y^2_r) (f_1(r, Y^1_r, Z^1_r) - f_2(r, Y^2_r, Z^2_r)) \, dr$$

$$- 2 \int_v^{\tau_k} e^{ar} (Y^1_r - Y^2_r) (Z^1_r - Z^2_r) \, dB_r, \quad a \geq 0. \quad (51)$$

By (H1), (H2) (without loss of generality we may assume that $\mu = 0$), we have

$$(Y^1_r - Y^2_r) (f(r, Y^1_r, Z^1_r) - f(r, Y^2_r, Z^2_r)) \leq \lambda |Y^1_r - Y^2_r| |Z^1_r - Z^2_r|$$

$$+ |Y^1_r - Y^2_r| |f_1(r, Y^1_r, Z^1_r) - f_2(r, Y^2_r, Z^2_r)| \leq 4 \lambda^2 |Y^1_r - Y^2_r|^2 + \frac{1}{4} |Z^1_r - Z^2_r|^2$$

$$+ |Y^1_r - Y^2_r| |f_1(r, Y^1_r, Z^1_r) - f_2(r, Y^2_r, Z^2_r)|.$$
Therefore, by (51), we have
\[
e^{av} |Y_v^1 - Y_v^2|^2 + \int_v^\tau e^{ar} |Z_r^1 - Z_r^2|^2 \, dr + a \int_v^\tau e^{ar} |Y_r^1 - Y_r^2|^2 \, dr
\]
\[
\leq e^{a\tau_k} |Y_{\tau_k}^1 - Y_{\tau_k}^2|^2 + 8\lambda^2 \int_v^{\xi_2} e^{ar} |Y_r^1 - Y_r^2|^2 \, dr + \frac{1}{2} \int_v^{\xi_2} e^{ar} |Z_r^1 - Z_r^2|^2 \, dr
\]
\[
+ 2 \int_v^{\xi_2} e^{ar} |Y_r^1 - Y_r^2| |f_1^{\xi_1} - f_2|(r, Y_r^2, Z_r^2) \, dr
\]
\[
- 2 \int_v^{\tau_k} e^{ar} (Y_r^1 - Y_r^2)(Z_r^1 - Z_r^2) \, dB_r.
\]

Consequently, by the fact that \( \int_0^\tau e^{vr} (Y_r^1 - Y_r^2)(Z_r^1 - Z_r^2) \, dB_r \) is a martingale on \([v, \tau_k]\), we find that for \( a \geq 8\lambda^2 \),
\[
e^{av} |Y_v^1 - Y_v^2|^2 \leq \mathbb{E}\left(e^{a\tau_k} |Y_{\tau_k}^1 - Y_{\tau_k}^2|^2 \right.
\]
\[
+ 2 \int_v^{\xi_2} e^{ar} |Y_r^1 - Y_r^2| |f_1^{\xi_1} - f_2|(r, Y_r^2, Z_r^2) \, dr |\mathcal{F}_v\). \tag{52}
\]

Since \( Y^1 - Y^2 \in \mathcal{S}_\mathcal{F}_\mathcal{F}^2(\nu, \xi_2) \), we may conclude, by letting \( k \to \infty \) in the right-hand side of (52) and applying the Lebesgue dominated convergence theorem, that
\[
e^{av} |Y_v^1 - Y_v^2|^2 \leq \mathbb{E}\left(e^{a\xi_2} |\xi_1 - \xi_2|^2 \right.
\]
\[
+ 2 \int_v^{\xi_2} e^{ar} |Y_r^1 - Y_r^2| |f_1^{\xi_1} - f_2|(r, Y_r^2, Z_r^2) \, dr |\mathcal{F}_v\),
\]

which implies that
\[
|Y_v^1 - Y_v^2|^2 \leq C \mathbb{E}\left(|\xi_1 - \xi_2|^2 + \int_v^{\xi_2} |Y_r^1 - Y_r^2| |f_1^{\xi_1} - f_2|(r, Y_r^2, Z_r^2) \, dr |\mathcal{F}_v\right) \tag{53}
\]

for some \( C > 0 \) depending only on \( \lambda \) and \( \xi_2 \). Finally, note that
\[
\int_v^{\xi_2} |Y_r^1 - Y_r^2| |f_1^{\xi_1} - f_2|(r, Y_r^2, Z_r^2) \, dr = \int_v^{\xi_1} |Y_r^1 - Y_r^2| |f_1^{\xi_1} - f_2|(r, Y_r^2, Z_r^2) \, dr
\]
\[
+ \int_{\xi_1}^{\xi_2} |Y_r^1 - Y_r^2| |f_2|(r, Y_r^2, Z_r^2) \, dr,
\]

which combined with (53) completes the proof of (v). The inequality asserted in (vi) follows directly from Proposition 1. \( \square \)
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