A Detailed investigation of the Blade Element Momentum (BEM) model based on analytical and numerical results and proposal for modifications of the BEM model

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Abstract. The paper presents the results of a comprehensive investigation of the BEM model based on detailed results from actuator disc simulations as well as analytical derivations. The objectives of this work has been to investigate the deficiencies in the BEM model, which is the most common engineering model for computation of the aerodynamic loads on wind turbine rotors and used widely within the industry. An additional objective has been to derive modifications to the BEM model in order to improve the accuracy of the model. Our comparisons of numerical results from the actuator disc simulations with BEM results have shown two areas of deviations. On the inner part of the rotor the BEM model overestimates the induction due to neglecting the pressure term from wake rotation. On the outer part of the rotor the tendency is opposite with an underestimation of the induction by the BEM model which seems to be an effect from expansion of the stream tubes. Two simple correction models to the BEM model were derived to account for these deviations and the results of the modified BEM model correlate very well with actuator disc results for different load distributions.

1. Introduction

It is well-known that the blade element momentum (BEM) model, being the most common engineering model for computation of aerodynamic forces in aerodynamic and aeroelastic design models, is derived on basis of a number of assumptions for the flow properties and by disregarding different terms in the describing set of equations. The BEM model has originally been developed by Glauert [1] who also discusses the assumptions that must be introduced in order to derive the simple relations for axial and tangential induction. One of these assumptions is to disregard the pressure term from rotation of the wake. This assumption has later been considered by e.g. Wilson and Lissaman [2] and de Vries [3] made a more detailed analytical study of the consequences of this assumption. From his study the influence of making the different assumptions about wake expansion and about the pressure term from wake rotation is shown in figure 1 in the form of the rotor power coefficient as function of local tip speed ratio. Including the pressure term from wake rotation and at the same time including the wake expansion de Vries shows that the power loss from wake rotation at low tip speed ratio is almost cancelled by the increased mass flow through the rotor due to the low pressure in the center of the wake from wake rotation. This is thus different from the theory of Glauert where only the
loss from the wake rotation is included and this leads to the decrease in rotor power coefficient at low tip speed ratio as shown by the dashed line in figure 1. However, the analysis was carried out for a constant bound circulation along the blade span and cannot just be expanded for use on arbitrary loadings on the rotor. Later Sharp [4] made a similar study and he also came to the conclusion that the rotor power coefficient does not decrease at low tip speed ratio but can even exceed the Lanchester-Betz limit.

Besides disregarding the pressure term from wake rotation there are other assumptions behind the simple set of equations used for computation of the induction and of the aerodynamic forces in the BEM model. For example the axial induction at the rotor disc is assumed to be half of its value in the far wake but the most uncertain point might be that the equations are derived using the momentum equation on integral form but at the end the induction equations are used on differential stream tubes assumed to be independent of each other. This has big advantages for the solutions of the equations and it also gives a robust model which is important when using the BEM model together with numerical optimization for rotor design. However, it is uncertain how good this assumption is when the loading on the rotor is varying considerably.

In order to validate the BEM model, comparison will be made with a numerical actuator disc (AD) model. The difference between the BEM model and the AD model is that the full set of equations are solved in the AD model with a CFD code which in the present case is the commercial code FIDAP. A comparison of the results from the two codes will thus reveal the importance of the assumptions behind the simplified set of equations in the BEM model.

In the present paper an analytical investigation of the influence of the pressure from wake rotation using a vortex model will be presented first. Then follows a short description of the AD model and how the conversion of energy is computed in this model. A comparison of results from the BEM and the AD model is then presented with identification of the main deviations. Finally follows the presentation of two correction models for the BEM model which improves the correlation with the AD model considerably.

2. Influence of pressure from wake rotation analyzed with a distributed vortex method

The analysis of the rotor flow is based on the use of a simple vortex model described by Øye [5]. The idealized wind turbine is modeled with an infinite number of blades, each represented by a lifting line with constant vortex strength from the root to the tip. The total bound vortex strength of all the blade is \( \Gamma \). Figure 2 depicts the system of trailing vortices from the root and the tip. The infinite amount of lifting lines forms a sheet of vorticity assuming a ball-bearing analogy with constant vortex strength. The vortex sheet is
divided into an axial and tangential component, $\gamma_a$ and $\gamma_t$, respectively, where

$$
\gamma_i = \frac{\Gamma}{2\pi R} \frac{V_i}{V_o} = 2w_a, \quad \gamma_a = \frac{\Gamma}{2\pi R} = 2V_i = 2w_i, \quad V_a = V_o - w_a
$$

(1)

and where $w_a$ and $w_i$ is the axial and tangential induced velocity at the disc. Note that the combined assumptions of a cylindrical wake and a constant pitch of the trailing vortex sheet, results in a model which produces a constant axial induced velocity $w_a$ over the disc. The relation between the total bound vortex strength $\Gamma$ and $w_a$ can be found by solving the above equations for $\Gamma$. The total vortex strength and corresponding axial and tangential loadings are then given by

$$
\Gamma = 2\pi R \left( - R \Omega + \sqrt{R^2 \Omega^2 + 4 w_a (V_o - w_a)} \right),
$$

(2)

$$
F_a(r) = \rho \cdot R \cdot (\Omega r + w_i), \quad F_t = \rho \cdot R \cdot (V_o - w_a)
$$

Introducing the usual non-dimensional local thrust and power coefficient $C_T$ and $C_{\text{mec}}$, explicit expressions may be derived as

$$
C_T(r) = \frac{F_a}{\frac{1}{2} \rho V_o^2} \frac{r}{2 \pi r} = 4a(1-a) \frac{\Omega + \Gamma}{4 \pi R^2}
$$

(3)

$$
C_{\text{mec}} = \frac{F_i \cdot \Omega \cdot r}{\frac{1}{2} \rho V_o^2} = 4a(1-a)^2 \frac{R \Omega}{R \Omega + w_i}
$$

showing that the added term $\Delta C_T$, compared to classic axial momentum theory, increases towards the root section. Figure 3 displays computed distribution from the above expressions as function of $\lambda=2$, 5, and 8, with constant induced velocity $w_a=1/3$. The increased loading captured by the model, becomes singular at the root section, thus predicting infinite axial loading for $r \to 0$. Closer inspection of the radial pressure gradient due to the centrifugal acceleration yields

$$
\frac{dp}{dr} = \rho \frac{w_i}{r} = \rho \frac{\Gamma^2}{4 \pi^2} \cdot \frac{1}{r^3} \Rightarrow \Delta p_r = \rho \frac{\Gamma^2}{8 \pi^2} \left( \frac{1}{r^2} - \hat{1} \right)
$$

(4)

$$
\Delta C_T = \frac{\Delta p_r}{\frac{1}{2} \rho V_o^2} = \left( \frac{\Gamma}{2 \pi V_o} \right)^2 \left( \frac{1}{r^2} - \hat{1} \right)
$$

revealing that a low pressure is present in the wake, corresponding exactly to the expression derived previously. In other words, the low pressure due the centrifugal acceleration should be included into the axial momentum equation. The mechanical power does not vary with radii and is seen to decrease slightly with increasing tip speed ratio. Regarding the limitations of the model, the wake is prescribed...
thereby assumed not to expand. The model appears promising in terms of serving as an improvement to the classical BEM model. Presently, however, the model still needs to mature further before it can be incorporated into the BEM method, mainly because the circulation is constant in the above derivation. Real blades have varying circulation and the number of blades is finite, meaning that tip correction is needed as well as some kind of root correction since finite trailed vortex lines are released at the root section too.

![Figure 3](image)

**Figure 3** Distribution of local CT and CP as function of the velocity ratio.

3. Actuator disc model computations

The loading on the actuator disc is derived from the lift and drag projections $C_y, C_x$, normal and tangential to the rotor plane:

$$
C_y = C_L \cos(\phi) + C_D \sin(\phi)
$$

$$
C_x = C_L \sin(\phi) - C_D \cos(\phi)
$$

where $\phi$ is the angle from the rotor plane to the relative velocity vector.

The infinitesimal thrust $dT$ and torque $dQ$ on an infinitesimal radius element $dr$ can be written as

$$
dT = \frac{1}{2} \rho V^2 c N_B \frac{dC_T}{d\phi} dr
$$

$$
dQ = \frac{1}{2} \rho V^2 c N_B \frac{dC_Q}{d\phi} r dr
$$

where $V_r$ is the velocity to the blade section, $\rho$ is density of mass, $c$ is chord length and $N_B$ is number of blades.

We then derive the local thrust coefficient $CT$ and the local torque coefficient $CQ$:

$$
CT = \frac{dT}{\frac{1}{2} \rho V_0^2 2\pi r dr} = \frac{V_r^2 C_T c N_B}{V_0^2 2\pi r}
$$

$$
CQ = \frac{dQ}{\frac{1}{2} \rho V_0^2 2\pi r dr} = \frac{V_r^2 C_Q c N_B}{V_0^2 2\pi r}
$$

where $V_0$ is the free stream velocity.

We now restrict the computations to an ideal energy conversion ($C_D=0$) and derive the ratio between $CT$ and $CQ$:
This equation shows that if we specify a rotor loading by $CT$, then we shall apply a corresponding loading $CQ$ as specified by eq. (8). However, as $CQ$ depends on the actual flow angle $\phi$ which is a part of the flow solution an iteration loop is necessary in order to reach a final solution for a specified axial loading $CT$. We can now study the energy conversion in a rotor simply by specifying the loading expressed by $CT$ and $CQ$.

It should be noted, that in the following all variables are non-dimensionalized as follow:

$$v^* = \frac{v}{V_0}, r^* = \frac{r}{R}, p^* = p \left( \frac{1}{2} \rho V_0^2 \right)$$

In order to describe the power conversion in the actuator disc we define the shaft power coefficient $CP_s$:

$$CP_s = CQ r \Omega$$

where $\Omega$ is the angular rotational speed of the rotor.

However, we can also define the power coefficients for the power conversion in the fluid:

$$CP_{ax} = CT v_a$$

$$CP_{tan} = v_a v_t^2$$

where $v_a$ and $v_t$ is the axial and tangential velocity through the rotor disc, respectively.

$CP_{ax}$ is the power coefficient for extraction of power from the flow whereas $CP_{tan}$ is the power coefficient for power added to the fluid in the form of rotation (swirl) of the wake. The difference between $CP_{ax}$ and $CP_{tan}$ is the total power coefficient $CP_{tot}$ and must therefore be equal to the above defined shaft power coefficient:

$$CP_{tot} = CP_{ax} - CP_{tan} = CP_s$$

4. The BEM model computations

The present implementation of the BEM model is chosen because the input to the model then are the same two load coefficients $CT, CQ$ as used for the AD model.

Momentum theory relates the induction $a$ to the axial thrust coefficient $CT$ by:

$$CT = 4a(1-a)$$

For angular momentum we have:

$$dQ = \rho (2\pi r dr) r (1-a) V_0 (2a' \Omega)$$

and the torque $dQ$ is derived as:

$$dQ = \frac{1}{2} \rho V_t^2 r C_s c N_y dr$$

Combining eq. (14) and (15) and introducing the tangential load coefficient $CQ$ defined in eq. (7) we get:

$$a' = CQ \frac{V_0}{4r(1-a)\Omega}$$

and introducing the non-dimensionalization we get

$$a' = CQ \frac{1}{4r(1-a)\Omega}$$

The flow through the disc as function of the load coefficients $CT, CQ$ can now be found by eq. (13) and (17). For solution of eq. (13) we express the induction as function of $CT$ using a third order polynomium:

$$a = k_3 CT^3 + k_2 CT^2 + k_1 CT$$
The constants have been found so that eq. (13) is fulfilled at loadings up to 0.7-0.8 but ensuring that we have a smooth transition to an empirical relation at higher loadings where eq. (13) is not valid. The following values for the constants have been used: $k_3 = 0.08921$, $k_2 = 0.05450$, $k_1 = 0.25116$.

5. Comparison of BEM and AD results

A number of different load distributions have been used in the numerical study [6] on which the present results are based. Here will first be shown the results for a tip speed ratio of 6 and at a constant loading of $CT = 0.95$ which decreases linearly to zero at the rotor centre in order to avoid a tangential loading increasing to infinity, figure 4.

![Figure 4](image)

**Figure 4** A constant thrust coefficient $CT = 0.95$ (except close to the rotor center) as function of radius and the corresponding tangential loading for an ideal rotor (airfoil $C_D = 0$) at a tip speed ratio of 6.

![Figure 5](image)

**Figure 5** The axial and tangential velocities at the rotor disc, computed by the AD and the BEM model, respectively.

![Figure 6](image)

**Figure 6** Comparison of the local power coefficients computed by the AD model and the BEM model.

![Figure 7](image)

**Figure 7** The local, average CP coefficient derived from the distributions shown in figure 6 (the average CP from the rotor centre and to the actual radial position).

The comparison of the axial and tangential velocities at the rotor disc, figure 5 shows that the induction computed by the AD model is less than the induction by the BEM model on the inner part of the rotor whereas the opposite tendency is seen on the outer part of the rotor. It should be noted that exactly the same loading has been used in the two models as the tangential loading only was found by the AD model using a few iterations.

The difference in flow velocities leads to considerable deviations in the local CP coefficients as seen in figure 6. On the inboard part of the rotor the CP coefficient computed by the AD model is considerably...
bigger than $CP$ from the BEM model and is seen to exceed the Betz limit of 0.59. It should also be noted that the shaft power $CP$ and the total $CP_{tot}$ derived from the power conversion in the fluid as shown by equation (12) coincide completely as they should do.

In order to see the integrated effect of the deviations the local, average $CP_{av}$ (shows the average power coefficient over the rotor area from rotor centre to the actual radial position) for the two models is shown in figure 7. At the rotor edge the two curves almost coincide showing that the deviations observed almost cancels each other.

The example shown here is representative for a big number of comparisons performed as background for the paper [6] although the deviations decrease for decreasing loading. In the following sections the causes of the deviations will be discussed as well as derivation of two correction models to the BEM model to account for the deviations.

6. Correction model for the influence of the pressure variation from wake rotation

To illustrate the cause of the deviations on the inboard part of the rotor the AD model was run with and without the tangential loading but with the same axial loading in the two cases. The axial loading was in this case also linear but increasing slightly from the tip to the root region resulting in an average CT equal to 0.95.

Excluding the tangential loading in the AD model results in an axial profile at the rotor disc correlating very well with the BEM results except on the outboard part of the rotor, figure 8. However, for the case with tangential loading and thus with wake rotation the induction in the AD results are much less on the inboard part of the rotor. The difference is due to the pressure component caused by the wake rotation as shown in figure 9. The wake rotation is discontinuous over the rotor disc and there is thus also a discontinuity in the pressure.

The mechanism behind the increased axial velocity due to wake rotation is thus simple and the same is the proposed correction model to the BEM model. The correction model contains two steps:

- compute the radial pressure variation at the rotor disc from the induced tangential velocity distribution
- derive the axial velocity component corresponding to this pressure variation and add it to the local velocity at the disc (free stream velocity minus induction)

The pressure term from the wake rotation is computed as:

$$p_w = \int r \frac{V^2}{r} dr$$
where \( v_i \) is the tangential velocity computed in the BEM code \( v_i = 2 \alpha' r \Omega \).

From the linearized equations for the actuator disc flow, Glauert [1] we have:

\[
\frac{\partial \nu_x}{\partial x} = -\frac{\partial p}{\partial x} + X 
\]

where \( X \) is the axial volume force. Integrating this equation, the velocity correction term \( \Delta \nu_x \) should be proportional to \( p_{w} \). However, comparisons with the numerical results from the AD model indicates that it is not the full pressure deficit that is converted to an increased velocity. The following relation gives the best correlation with numerical AD results:

\[
\Delta \nu_x = 0.7 \, p_{w} 
\]  \hspace{1cm} (21)

As mentioned above the pressure term from wake rotation is discontinuous over the rotor disc and in principle this also holds for \( \Delta \nu_x \). This is different from the axial loading on the disc which also causes a jump in pressure but the influence on the velocity from this jump is cancelled by an opposite jump in the axial volume forces. The steep gradient or discontinuity of \( \Delta \nu_x \) over the rotor disc could explain that it is not the full pressure gradient that is converted to velocity but instead to turbulence. The corrected axial velocity \( \nu_x \) through the rotor disc can now be computed as:

\[
\nu_x = (1 - \alpha) + \Delta \nu_x 
\]  \hspace{1cm} (22)

When comparing the velocity profiles at the rotor disc including this additional velocity term from wake rotation an excellent correlation with the AD model is observed on the inboard part of the rotor. On the outboard part there is still a deviation which will be discussed in the next section.

Figure 10: The effect of the correction of the BEM model to include an additional velocity term from the wake pressure due to wake rotation.

7. Correction model for the decreased inflow at the tip region due to wake expansion

The other characteristic deviation when comparing the BEM model with AD results is the reduced inflow at the tip region which is not modeled by the BEM model. This is illustrated in figure 11 where the axial velocity profiles computed by the AD model for a constant loading is shown. The BEM model gives a constant induction whereas the AD model shows a decreasing velocity (bigger induction) at the tip region but with the opposite tendency towards the rotor centre. However just 1D downstream the velocity profile is almost constant and far downstream it has become completely constant. The corresponding radial velocity profiles shown to the right in figure 12 are a good measure of the rate of expansion and show clearly the big difference in rate of expansion at the rotor disc to the
conditions further downstream. The reduced inflow at the tip is a general characteristic of the results from non-linear flow models as e.g. the results from actuator disc simulations presented by Sørensen [7].

The above results indicate that the reduced inflow at the tip region are due to the rate of wake expansion. A simple vortex model for a constant loaded actuator disc and with a cylindrical wake gives a constant induction over the disc as shown by Øye [5]. However, for an expanding vortex sheet the induction is no longer constant. This explains why the velocity distribution just 1D downstream is almost constant as shown above in figure 11 because the wake from this point and further downstream is almost cylindrical.

The proposed correction model to be included in the BEM model to account for the reduced inflow in the tip region is therefore based on the assumption of a close relation between the reduced inflow $\Delta v_a$ and the radial velocity distribution $v_r$. These two parameters $\Delta v_a$ and $v_r$ are shown in figure 13 and figure 14 for different constant axial loadings. $\Delta v_a$ was derived as the difference between the BEM and the AD velocity profiles for radial positions greater than 0.3 and all the $\Delta v_a$ were adjusted to 0 at the starting point, figure 13.

**Figure 11** Axial velocity profiles at several downstream positions from the rotor, computed with the AD model for constant loading and no tangential loading.

**Figure 12** Radial velocity profiles at the same positions as the axial velocity profiles in figure 11.

**Figure 13** Difference in axial velocity at the rotor disc computed with the BEM and the AD model, respectively.

**Figure 14** Radial velocity as function of radial position computed with the AD model for different, constant loadings.
The correlation between ΔνΦ and νr at different radial stations is now determined and approximated with third order polinomium trend lines. The variation of the factors in each these polynomia was then derived as function of radius with the following result:

\[ k3 = -15.83r + 20.73 \]  
\[ k2 = -0.52r - 0.90 \]  
\[ k1 = 1.58r^2 - 1.63r + 0.51 \]

The correction velocity \( \Delta νΦ \) can now be found as:

\[ \Delta νΦ = k3 νr^3 + k2 νr^2 + k1 νr \]  

However, the radial velocity cannot be computed with the BEM method. Therefore, an equation was derived from the analytical solution for \( νr \) for a two-dimensional actuator disc [8]. A modification of the analytical equation was made in order to fit the present AD results:

\[ νr = \frac{1}{2.24} \frac{CTav \cdot 0.04^2 + (r+1)^2}{\pi \cdot 4\pi \cdot 0.04^2 + (r-1)^2} \]

The average \( CTav \) as function of radius is derived as:

\[ CTav = \int_0^r \frac{CT \cdot 2\pi r \cdot dr}{\pi \cdot r^2} \]

The CT used in the above equation was found from equation (18) but using an induction factor corresponding to the axial flow velocity after the pressure correction as given by equation (22).

![Figure 15](results.png)

Figure 15 Results from the BEM model with the two correction models compared with the results of the standard BEM method and the AD model.
The final corrected axial velocity in the BEM model is now given by:

\[ v_a = k_{\text{massflow}} \left( (1 - a) + \Delta v_w - \Delta v_a \right) \]  \hspace{1cm} (29)

The factor \( k_{\text{massflow}} \) was a factor introduced to adjust the total mass flow through the rotor to be equal to the mass flow of the uncorrected BEM method as a general result of the comparisons of the uncorrected BEM model with the AD model results was that the power coefficient for the rotor correlates very well as e.g. shown in figure 7. However, the mass flow factor is in general close to one as the two corrections terms for the axial velocity has different signs and integrated over the rotor with respect to mass flow almost cancels each other.

An example of results with the corrected BEM model is shown in figure 15 in comparison with the AD results and results from the standard BEM model. The introduction of the correction models is seen to improve the correlation with the AD results substantially and this improvement has also been found for a number of other loadings investigated.

8. Summary
A detailed investigation of the BEM model has been carried on basis of numerical results from an AD model. The comparisons show that the BEM model overestimates the induction on the inboard part of the rotor due to neglecting the pressure term in the wake from wake rotation. At the tip the tendency is opposite and the increased induction on the outboard part of the rotor computed with the AD model seems to be due to wake expansion. Two correction models to be incorporated in the BEM model have been developed and the results of the corrected BEM model correlates very well with AD simulations for a number of different load forms. The corrected BEM model has also the advantage that the stream tubes are no longer independent as both correction terms are based on integral quantities. Finally, the study has shown that the integrated CP computed with the standard BEM model correlates very well with AD results and this also holds for the corrected BEM model. However, the distribution of local CP is quite different when comparing the standard BEM model with the corrected BEM model.

References
[1] Glauert, H. 1963 Airplane Propellers Aerodynamic Theory Volume IV edited by William Frederick Durand. The Dover edition 1963.
[2] Wilson R E and Lissaman P B S 1974 Applied aerodynamics of wind power machines Oregon State University
[3] M de Vries, O. 1979 Fluid Dynamic Aspects of Wind Energy Conversion AGARD Advisory Group for Aeroepace Research & Development AGARD-AG-243 July 1979 pp C4-C9
[4] Sharp D J 2004 A general momentum theory applied to an energy-extracting actuator disc Wind Energy 7 pp 177-188
[5] Øye, S. 1990 A Simple Vortex Model Proceedings of the third IEA Symposium on the Aerodynamics of Wind Turbines, ETSU, Harwell, 1990, pp. 4.1-4.15.
[6] Madsen H Aa, Mikkelsen R, Johansen J, Bak C, Øye S and Sørensen N N 2006 Inboard rotor.blade aerodynamics and its influence on blade design. Report Risø-R-1559(EN) Research in Aeroelasticity EFP-2005 edited by C Bak.
[7] Sørensen J N, Shen W Z and Mundate X 1998 Analysis of wake states by a full-field actuator disc model. Wind Ener. 1 pp 73-88
[8] Madsen H A 1996 A CFD Analysis of the Actuator Disc Flow compared with Momentum Theory Results In proceedings of IEA Joint Action of 10th Symposium onAerodynamics of Wind Turbines, Edinburg, December 16-17, 1996 edited by B.M. Pedersen, pp. 109-124.