Empirical model of magnetic field line spreading in isotropic turbulence with varying mean field

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Abstract. In many astrophysical phenomena, understanding the diffusion of the magnetic field line random walk (FLRW) is central to understand cosmic ray transport. In 3D fluctuations, the behavior of the FLRW can be characterized by the Kubo number $R = (b/B_0)(l_{\parallel}/l_{\perp})$ [1], where the parameters $b$ and $B_0$ are the rms magnetic fluctuation and the large-scale mean field, respectively. The parameters $l_{\parallel}$ and $l_{\perp}$ are coherence scales parallel and perpendicular to $B_0$, respectively. For isotropic turbulence, in which $l_{\parallel} = l_{\perp}$, Sonsrettee et al. [2] found that Corrsin-based theories can generally describe the FLRW’s behavior for a whole range of $R$ by varying $b/B_0$. Sonsrettee et al. [2] used Corrsin-based theory with three models of field line spreading to examine the $R$-scaling of the asymptotic diffusion coefficients for the FLRW. The models are the diffusive decorrelation (DD) model [3-5], the random ballistic decorrelation (RBD) model [6], and the ordinary differential equation (ODE) model [7]. To improve the theory of the FLRW in isotropic turbulence with $B_0 = 0$, Sonsrettee [8] proposed the empirical (EMP) model of magnetic field line spreading to determine the asymptotic diffusion coefficients. Benchmarked against the previous models, the EMP model is the best model to predict computer simulation results (with $\leq 0.9\%$ error). In this work, we extend the previous works [2, 8] by formulating the EMP model to explore the $R$-scaling FLRW behavior in isotropic magnetic turbulence by varying $B_0$. In the limit of very low $R$, we obtain the the closed-form solution of the FLRW for the EMP model. In order to develop the closed-form solution at any $R$, we employ the Padé approximants to the EMP model. The EMP model predicts that, with increasing $R$, the FLRW behavior transits from quasilinear diffusion to Bohm diffusion. This work shows that the theoretical results of the EMP model match the computer simulation results for the FLRW in Kolmogorov turbulence better than the other models significantly.

1. Introduction
In many astrophysical phenomena, magnetic field lines become a turbulent random walk. The diffusion of the magnetic field line random walk (FLRW) is important to understand the transport of energetic charged particles. In 3D magnetic turbulence $\mathbf{B}$, the behavior of the FLRW can be classified by the Kubo number $R = (b/B_0)(l_{\parallel}/l_{\perp})$ [1], where the parameters $b$ and $B_0$ are the rms magnetic fluctuation and the large-scale mean field, respectively. The parameters $l_{\parallel}$ and $l_{\perp}$ are coherence scales parallel and perpendicular to $B_0$, respectively. For isotropic turbulence, in which $l_{\parallel} = l_{\perp}$, Sonsrettee et al. [2] found that Corrsin-based theories can generally describe the FLRW’s behavior for a whole range of $R$ by varying $b/B_0$. Sonsrettee et al. [2] used the Corrsin-based theory with three models of field line spreading to examine the $R$-scaling of the asymptotic diffusion coefficients for the FLRW. The models are the diffusive decorrelation (DD)
model [3–5], the random ballistic decorrelation (RBD) model [6] and the ordinary differential equation (ODE) model [7]. All models qualitatively agree with direct computer simulations. To improve the theory of the FLRW in isotropic turbulence with \( B_0 = 0 \), Sonsrettee [8] proposed the empirical (EMP) model of magnetic field line spreading to determine the asymptotic diffusion coefficients. Motivated by variance of the DD model \( (\sigma^2 \propto b^2 |\Delta \tau|) \) and of the RBD model \( (\sigma^2 \propto b^2 |\Delta \tau|^2) \), the variance of the EMP model is assumed as \( \sigma^2 = \beta b^2 |\Delta \tau|^\alpha \), where the proportional constant \( \beta \) is extrapolated with the proportional constants of the DD and RBD models. The exponent \( \alpha \) is empirically determined by comparing the solution of EMP model with a computer simulation result of turbulence at \( B_0 = 0 \) [8]. The parameter \( \tau \) is the field line displacement parameter defined as \( d\tau = ds/B \), where the parameter \( s \) is the arc length along the field line. Benchmarked against the other models, the EMP model is the best model to predict computer simulation results for the FLRW in isotropic turbulence with \( B_0 = 0 \) [8].

In this work, we use the EMP model in a nonperturbative analytic framework to explore the \( R \)-scaling FLRW behavior in isotropic magnetic turbulence by varying \( B_0 \). For very low \( R \), we obtain a closed-form solution. In order to develop the closed-form solution of the FLRW for the EMP model at any \( R \), we formulate the Padé approximants to interpolate the asymptotic diffusion coefficient to any \( B_0 \). In order to verify the predictions of the EMP model, we compare them with the simulation results of Kolmogorov turbulence for varying \( B_0 \) with a finite \( b \) obtained by [2].

2. Empirical model for the FLRW with a Uniform Mean Field

2.1. Theoretical frame work

Without loss of generality, the 3D magnetic field turbulence with a uniform mean field can be described as \( \mathbf{B}(x, y, z) = B_0 \hat{z} + \mathbf{b}(x, y, z) \), where \( B_0 \hat{z} \) and \( \mathbf{b} \) are the mean field and the fluctuation field, respectively. Here, we assume that \( \mathbf{b} \) is homogeneous and isotropic turbulence. By magnetic field line definition, the relation between the magnetic field \( \mathbf{B} \) and arc length along the field line \( ds \) can be described as

\[
\frac{dx}{ds} = \frac{b_x}{B}, \quad \frac{dy}{ds} = \frac{b_y}{B}, \quad \frac{dz}{ds} = \frac{B_0 + b_z}{B}. \tag{1}
\]

With equation (1), the field line trajectory can be traced. Unfortunately, the diffusion coefficients determined by using equation (1) are nonlinear. To avoid the nonlinear problems, we describe the field line trajectory in terms of the parameter \( \tau \), defined as \( d\tau = ds/B \) [2,9], as

\[
\frac{dx}{d\tau} = b_x, \quad \frac{dy}{d\tau} = b_y, \quad \frac{dz}{d\tau} = B_0 + b_z. \tag{2}
\]

Then the diffusion coefficients are defined as

\[
D_x(\tau) = \frac{1}{2} \frac{d(x^2)}{d\tau}, \quad D_y(\tau) = \frac{1}{2} \frac{d(y^2)}{d\tau}, \quad D_z(\tau) = \frac{1}{2} \frac{d((z - \langle z \rangle)^2)}{d\tau}. \tag{3}
\]

Since the asymptotic diffusion coefficients \( D_i \equiv D_i(\tau \to \infty) \), where \( i = x, y, z \), is a constant, we obtain, by integrating equation (3),

\[
\langle x^2 \rangle \cong 2D_x \tau, \quad \langle y^2 \rangle \cong 2D_y \tau, \quad \langle z^2 \rangle \cong 2D_z \tau \tag{4}
\]

for very large \( \tau \).

Note that, in order to focus on the FLRW process, the previous works [2,9] and this work consider the case of axisymmetric turbulence with \( \langle b_x^2 \rangle = \langle b_y^2 \rangle \) and thus \( D_x = D_y \). Here, we
apply the nonperturbative analytic framework of [2, 8, 9] in order to determine the asymptotic field line diffusion coefficients for the FLRW in isotropic turbulence with variance

\[ \sigma^2_{x_i}(|\Delta \tau|) = \beta_{i,\alpha} |\Delta \tau|^\alpha, \]  

(5)

where \( \beta_{i,\alpha} \) is a proportional constant and \( \alpha \) is an exponent of \( |\Delta \tau| \). Note that the variance definition in equation (5) is more general than that in [8]. The variance definition in equation (5) can be applied to the variance in [8] and end up with the same results. Let \((x; y; z)\) be the field line trajectory relative to its location at \( \tau = 0 \). By integrating equation (2), the field line deviations from the mean field trajectory are

\[ x_i - \langle x_i \rangle = x_i - B_{0,i}\tau = \int_0^{\tau} b_i[x(\tau'), y(\tau'), z(\tau')]d\tau', \]  

(6)

where \((B_{0,x}, B_{0,y}, B_{0,z}) = B_0 = (0, 0, B_0)\). Since the fluctuation \( b \) is homogeneous, an ensemble average variance of \( x_i \) is

\[ \langle x^2_i \rangle = \int_0^{\tau} \int_{-\tau}^{\tau} \mathcal{L}_{ii}(\Delta \tau) d\Delta \tau d\tau', \]  

(7)

where \( \Delta \tau = \tau'' - \tau' \) and \( \mathcal{L}_{ii}(\Delta \tau) \) is the Lagrangian correlation of the magnetic turbulence at two positions along the magnetic field line separated by \( \Delta \tau \). Calculating Lagrangian correlation is difficult to theoretically understand since the positions themselves depend on the representation of the turbulence. In order to avoid this complexity, by using Corrsins independence hypothesis [10], we approximate the Lagrangian correlation in terms of the standard (Eulerian) correlation \( R_{ii} \) as

\[ \mathcal{L}_{ii}(\Delta \tau) = \int R_{ii}(\Delta x)P(\Delta x|\Delta \tau)d\Delta x, \]  

(8)

where \( R_{ii} \equiv \langle b_i(0, 0) b_i(x_{\perp}, z) \rangle \) and \( P(\Delta x|\Delta \tau) = P(\Delta x|\Delta \tau)P(\Delta y|\Delta \tau)P(\Delta z|\Delta \tau) \) is the probability to find the displacement \( \Delta x \) of field lines after \( \Delta \tau \). By definition, \( R_{ii} \) can be expressed in terms of the inverse Fourier transform of the power spectrum \( S_{ii}(k) \):

\[ R_{ii}(\Delta x) = \int e^{ik \cdot \Delta x} S_{ii}(k)dk. \]  

(9)

Note that, for isotropic turbulence, \( S_{ii}(k) \) is a function only of the wavevector magnitude \( k \) as \( S_{ii}(k) = S_{ii}(k) \). According to the central limit theorem, the probability distribution of the magnetic field line displacement \( P(\Delta x|\Delta \tau) \), which is a result of the summation of random walks, can be approximated as a Gaussian distribution:

\[ P(\Delta x_i|\Delta \tau) = \frac{1}{\sqrt{2\pi \sigma_i^2}} \exp \left( -\frac{(\Delta x_i - B_{0,i}\Delta \tau)^2}{2\sigma_i^2} \right). \]  

(10)

Substituting equations (8), (9) and (10) into equation (7) and integrating over \( \Delta x \), we obtain

\[ \langle x^2_i \rangle = \int S_{ii}(k) \int_0^\tau \int_{-\tau}^{\tau} \exp \left( ikzB_0D\tau - \sum_i k_i^2 \sigma_i^2 \langle \Delta \tau \rangle |/2 \right)d\Delta \tau d\tau'dk. \]  

(11)

In order to determine asymptotic diffusion, we focus on \( \langle x^2_i \rangle \) in the limit of large \( \tau \). Thus, in equation (11), we can extend the limits of the \( \Delta \tau \) integration to \( \pm \infty \) [2].

\[ \langle x^2_i \rangle = \int S_{ii}(k) \int_0^\tau \int_{-\infty}^{\infty} \exp \left( ikzB_0\Delta \tau - \sum_i k_i^2 \sigma_i^2 \langle \Delta \tau \rangle |/2 \right)d\Delta \tau d\tau'dk \]  

(12)

\[ = 2\tau \int_{-\infty}^{\infty} S_{ii}(k) \exp \left( ikzB_0\Delta \tau - \sum_i k_i^2 \sigma_i^2 \langle \Delta \tau \rangle |/2 \right)dk d\Delta \tau. \]  

(13)
With equations (4), (5) and (13), the asymptotic diffusion coefficients for any $\alpha$ become

$$D_{i,\alpha} = \int S_{ii}(k) \int_0^\infty \exp \left( ik_z B_0 \Delta \tau - \sum_i k_i^2 \sigma_{\tau,\alpha}^2 (|\Delta \tau|)/2 \right) d\Delta \tau d^3k$$

$$= \int S_{ii}(k) \int_0^\infty \exp \left( ik_z B_0 \Delta \tau - \sum_i k_i^2 \beta_i \Delta \tau^\alpha / 2 \right) d\Delta \tau d^3k$$

In order to extend the works of [8], we assume that the variances of the EMP model are $\sigma_{\tau,\alpha}^2 (|\Delta \tau|) = \beta_i \Delta \tau^\alpha$ with $\alpha = 0.8694$. To determine $\beta_i \Delta \tau^\alpha$, we start with taking log of $\sigma_{\tau,\alpha}^2$ as $\ln \sigma_{\tau,\alpha}^2 = \ln \beta_i + \alpha \ln |\Delta \tau|$. With the proportional constants of the DD model ($\beta_{i,1}$) and the RBD model ($\beta_{i,2}$), the $\ln \sigma_{\tau,\alpha}^2$ is linearly extrapolated as

$$\frac{(\alpha - 1)}{(2 - 1)} = \frac{(\ln \beta_{i,\alpha} + \alpha \ln |\Delta \tau|)}{(\ln \beta_{i,1} + \alpha \ln |\Delta \tau|)} - \frac{(\ln \beta_{i,1} + \alpha \ln |\Delta \tau|)}{(\ln \beta_{i,2} + 2 \alpha \ln |\Delta \tau|)}. \quad (16)$$

Then

$$\beta_{i,\alpha} = \frac{\beta_{i,1}}{\beta_{i,2}^\frac{1-\alpha}{\alpha}}. \quad (17)$$

According to equations (15) and (17), we can determine the diffusion coefficients for any $\alpha$ and $B_0$ with $\beta_{i,1} = 2D_{i,1}$ and $\beta_{i,2} = (b_i^2)$ as

$$D_{i,\alpha} = \int S_{ii}(k) \int_0^\infty e^{ik_z B_0 \Delta \tau - \sum_i k_i^2 (2D_{i,1})^\alpha (b_i^2)^\alpha (|\Delta \tau|^\alpha / 2)} d\Delta \tau d^3k.$$ 

Here, with equation (18), we can determine $D_{i,\alpha}$ for the FLRW in isotropic turbulence at any $R = b/B_0$. 

2.2. Diffusion coefficient at very high $B_0$

In general, to determine $D_{i,\alpha}$, we have to solve equation (18) numerically. Fortunately, for very high $B_0$ corresponding to very low $R$, we can estimate the closed-form solution for equation (18). In order to analyze the diffusion coefficients at $B_0 \to \infty$, we start by rewriting equation (15) as

$$D_{i,\alpha} = \int S_{ii}(k) \left\{ \int_0^\infty \exp (ik_z B_0 \Delta \tau) \times \exp \left( - \sum_i k_i^2 \beta_i \Delta \tau^\alpha / 2 \right) d\Delta \tau \right\} d^3k$$

$$= \int S_{ii}(k) \mathcal{F}(k_x, k_y, B_0 k_z) d^3k. \quad (20)$$

With the term $\exp (ik_z B_0 \Delta \tau)$ in equation (19), the function $\mathcal{F}$ in equation (20) is the Fourier transform in $\Delta \tau$ and the quantity $k_z B_0$ acts as the frequency of the Fourier transform. Basically, if the oscillation term, $\exp (ik_z B_0 \Delta \tau)$, changes faster than the term $\exp (- \sum_i k_i^2 \beta_i \Delta \tau^\alpha / 2)$, the Fourier transform $\mathcal{F}$ becomes zero. Thus, in the limit $B_0 \to \infty$, the frequency $k_z B_0$ goes to infinity and $\mathcal{F}$ goes to zero for any $k_z$, except at $k_z \to 0$ [2]. This leads to $\mathcal{F} \propto \delta(B_0 k_z) = \delta(B_0 \cos \theta) = \delta(\cos \theta) / B_0 k$. Then, for isotropic turbulence in which $S_{ii}(k) \propto S(k)$,

$$D_{i,\alpha} \propto \int_0^\infty \int_{-1}^1 \int_0^{2\pi} k^2 S(k) \frac{\delta(\cos \theta)}{B_0 k} d\varphi d(\cos \theta) dk$$

$$= \frac{2\pi}{B_0} \int_0^\infty k S(k) dk \quad (21)$$

$$\propto \frac{b^2}{B_0} \lambda, \quad (22)$$

$$\propto \lambda,$$
where $\lambda_c = (2\pi^2/b^2) \int k S(k) dk$ is the correlation length [11, 12]. Sonsrettee et al. [2] showed that, for the DD ($\alpha = 1$) and RBD ($\alpha = 2$) models, their diffusion coefficients agree that

$$D_x (B_0 \to \infty) = D_y (B_0 \to \infty) = \frac{\lambda_c b^2}{4 B_0}, \quad (24)$$

$$D_z (B_0 \to \infty) = \frac{\lambda_c b^2}{2 B_0}, \quad (25)$$

which are called quasilinear behavior. Thus, in the limit $B_0 \to \infty$, the diffusion coefficients $D_{i,\alpha}$ for any reasonable $\sigma_{i,\alpha}^2$ tend to the same limiting forms as the DD and RBD models (equations (24) and (25)) [2]. Figure 1 shows that the $D_{i,\alpha}$ for various $\alpha$ at high $B_0$, calculated by numerical evaluating equation (18)\(^1\), agree with equations (24) and (25). In figure 1, we use $D_{\perp}$ as $D_x$ and $D_y$, and $D_0$ as $D_z$. Since, the Kubo number $R = b/B_0$ for the FLRW in isotropic turbulence, equation (23) also point out that the FLRW behavior at very low $R$ is quasilinear.

![Figure 1](image-url)  
**Figure 1.** Magnetic field line diffusion coefficients in directions (a) perpendicular and (b) parallel to the mean field as a function of $B_0$ at high $B_0$ from selected $\sigma_{i,\alpha}^2$ models of the Corrsin-based theory calculated by numerical evaluation of equation (18). The dashed lines are provided for comparison to the quasilinear behavior as in equation (24) for $D_{\perp}$, and as in equation (25) for $D_{||}$.

### 2.3. Estimated solutions of the EMP model at any $B_0$

Now, for the EMP model, at very high $B_0$, we have a closed-form solutions of $D_i \propto b^2/B_0$ as in equations (24) and (25). At $B = 0$, according to the $\sigma$ definition in equation (5), the closed-form solution obtained by [8] is

$$D_{i,\alpha} (B_0 \to 0) \equiv D_{0,\alpha} = \frac{b}{\sqrt{3}} \Gamma \left(1 + \frac{1}{\alpha}\right) \left(2^{1-\alpha} \tilde{\lambda}^{2-\alpha}\right)^{-1/\alpha} \frac{\int k^2 Sk^{-2/\alpha}dk}{\int k^2 Sdk} \quad (26)$$

$$= \Lambda_\alpha \frac{b}{\sqrt{3}}, \quad (27)$$

where the constant $\Lambda_\alpha = \Gamma(1 + 1/\alpha)(2^{1-\alpha} \tilde{\lambda}^{2-\alpha})^{-1/\alpha}(\int k^2 Sk^{-2/\alpha}dk)/(\int k^2 Sdk)$ and the asymptotic diffusion coefficient $D_{0,\alpha}$ is the asymptotic diffusion coefficient at $B_0 = 0$. Even

\(^1\) we integrate over a sufficient range of $\Delta \tau$ to obtain asymptotic diffusion coefficients.
we cannot determine the closed-form solution of $D_{i,0}$ for intermediate $B_0$ with equation (18), as suggested by [2], we can formulate the estimated formula of the EMP model at any $B_0$ with a Padé approximant as

$$D_{i,0}^2 = \frac{D_{0,0}^2}{1 + A (B_0/b)^2} \quad (28)$$

in order to allow $D_{i,0} \propto B_0^{-1}$ at very high $B_0$. Note that, in this Padé approximant, we treat $B_0$ as the only independent variable with $b$ and $A$ constants. With the $D_{i,0}$ at $B_0 \to \infty$ in equations (24) and (25), we can obtain $A$ for each $D_{i,0}$ and

$$D_{x,0} = D_{y,0} = \frac{\Lambda_0 b}{\sqrt{3 \left(1 + \frac{16 \Lambda_0^2}{3 \Lambda_0^2} \left(\frac{B_0}{b}\right)^2\right)}} \quad (29)$$

$$D_{z,0} = \frac{\Lambda_0 b}{\sqrt{3 \left(1 + \frac{4 \Lambda_0^2}{3 \Lambda_0^2} \left(\frac{B_0}{b}\right)^2\right)}} \quad (30)$$

With the Padé approximants in equations (29) and (30), we have the simple formulas to evaluate $D_{i,0}$ at any $B_0$ (or $R$) and can understand the FLRW behavior scaled by $B_0$ (or $R$).

### 3. Results and discussion

In order to benchmark the EMP model for the $R$–scaling of FLRW against the other models, we compare the asymptotic diffusion coefficients of the EMP model with those of the DD, RBD, ODE models and of the simulations for varying $B_0$ with $b = 1$ obtained by [2]. For the DD and RBD models, Sonsrettee et al. [2] assumed that the FLRW spreads diffusively with $\sigma_i^2 = 2D_i |\Delta \tau|$ and ballistically in random directions with $\sigma_i^2 = b^2 |\Delta \tau|^2$ over the decorrelation scale of random walk, respectively. The variances of the DD and RBD models are equivalent to $\sigma_i^2 = 1$ and $\sigma_i^2 = 2$, respectively. The variance of the ODE model is the ensemble average of the solution of equation (2). For the direct computer simulations, Sonsrettee et al. [2] used a zero padding technique to generate the isotropic magnetic turbulence on a regular 3D Fourier grid in $k$-space with a power spectrum consistent with Kolmogorov turbulence [11, 13]. The 500,000 field lines were traced by numerically solving equation (2) using a fifth-order Runge-Kutta method with adaptive timestepping regulated by a fourth-order error estimate step. The equation (2) was solved over a sufficient range $\tau$ to obtain asymptotic diffusion coefficients. The mean squared displacement as a function of $\tau$ for each field line was collected to evaluate the ensemble average running diffusion coefficient $D_i(\tau)$. With the axisymmetry of turbulence, the perpendicular diffusion coefficient $D_\perp$ is calculated from $D_\perp = (D_x + D_y)/2$. The parallel diffusion coefficient is $D_\parallel = D_z$. To obtain asymptotic diffusion coefficients for the EMP model, we integrate equation (18) over a sufficient range of $|\Delta \tau|$.

In order to compare the EMP results with the simulation results, we evaluate the integral in equations (18), (29) and (30) over the range of the $k$-vector magnitude used in the simulations.

In figure 2, we compare the asymptotic diffusion coefficients predicted by the EMP model, calculated by evaluating the integral in equation (18), with those of the DD, RBD, ODE models, and computer simulations for $0.0625 \leq R \leq 400$. Moreover, in figure 3, the Padé approximants (equations (29) and (30)) for the EMP model are compared with the computer simulations and the Padé approximants for the other three models for $0.0625 \leq R \leq 400$. 


Figure 2. Magnetic field line diffusion coefficients in directions (a) perpendicular and (b) parallel to the mean field as a function of the mean field $B_0$ and the Kubo number $R$ from direct computer simulations and from four models of the Corrsin-based theory calculated by numerical evaluation of the analytic theory. A dotted line is provided for comparison to the slope of quasilinear behavior. A dashed line is provided for comparison to the slope of Bohm behavior. The stared data are obtained by [2].

Figure 3. Magnetic field line diffusion coefficients in directions (a) perpendicular and (b) parallel to the mean field as a function of the mean field $B_0$ and the Kubo number $R$ from direct computer simulations and from four models of the Corrsin-based theory calculated by Padé approximation. A dotted line is provided for comparison to the slope of quasilinear behavior. A dashed line is provided for comparison to the slope of Bohm behavior. The stared data are obtained by [2].

At $0.0625 \leq R \leq 0.25$ corresponding to the very high $B_0$ limit, figures 2 and 3 show that the results of the analytic theory and the Padé approximants to the theory of all models are closely agree with the simulation results. For analytic diffusions, the EMP, DD, RBD and ODE models predict the computer simulation results with %errors of $\leq 3.74\%$, $\leq 4.02\%$, $\leq 4.50\%$ and $\leq 3.47\%$, respectively. For the Padé approximants to the theory, the EMP, DD, RBD and ODE
models predict the computer simulation results with %errors of \( \leq 1.88\% \), \( \leq 2.17\% \), \( \leq 2.54\% \) and \( \leq 1.43\% \), respectively.

At \( 0.25 < R \leq 5 \) corresponding to the range of intermediate \( B_0 \), the results of the analytic theory and the Padé approximants to the theory for the EMP model qualitatively agree with those of simulations with the same accuracy as those of the other three models.

At \( 5 < R \leq 400 \) corresponding to the very low \( B_0 \) limit, the EMP model predicts the simulation results better than the other three models significantly. For the analytic theory (figure 2), the EMP, DD, RBD and ODE models predict the simulation results with %errors of \( \leq 8.37\% \), \( \leq 15.51\% \), \( \leq 20.33\% \) and \( \leq 22.53\% \), respectively. For the Padé approximants to the theory (figure 3), the EMP, DD, RBD and ODE models predict the computer simulation results with %errors of \( \leq 1.97\% \), \( \leq 15.60\% \), \( \leq 20.43\% \) and \( \leq 21.92\% \), respectively.

Even the EMP model generally well matches the simulation results at \( 0.25 < R \leq 400 \), there is a slight difference between the results of the simulation and the EMP model at \( 1 < R \leq 5 \) (or at \( 0.2 \leq B_0 < 1 \)) for \( D_z \). This difference comes from the field line trapping near the locations with \( B = (b_x, b_y, b_z + B_0) = 0 \) [2]. When the \( B_0 \) is in the order of \( b \approx 1 \), the \( P(\Delta z|\Delta \tau) \) in simulation is slightly non-Gaussian. However, the locations with \( b_x = 0 \), \( b_y = 0 \) and \( b_z = -B_0 \) is exceedingly rare at high \( B_0 (B_0 \gg 1) \). This is the reason why the Corrsin-based theory assuming the Gaussian distribution of field line displacement gives results slightly different from the simulation results at \( 1 < R \leq 5 \) (or at \( 0.2 \leq B_0 < 1 \)). Note that even the \( P(\Delta z|\Delta \tau) \) are non-Gaussian at \( 5 < R \leq 400 \) (or at \( 0.0025 \leq B_0 < 0.2 \)), the EMP can predict the simulation results very well. The reason is that the EMP model is empirically formulated from the simulation result at \( R \to \infty \) (or at \( B_0 = 0 \)).

Finally, between the analytic theory and Padé approximation for the EMP model, the Padé approximants match the results of the simulation for \( D_j \) as a function of \( B_0 \) better than the analytic theory. The reason is that the analytic theory for the EMP model absorbs the errors of the DD model through the \( \beta \) varying as \( B_0 \) (see equations (17) and (18)) while the Padé approximant absorbs the errors of the DD model through the \( \beta \) only at \( B_0 = 0 \) and \( B_0 \to \infty \) of which errors are very small.

4. Conclusions
In the present work, we apply the EMP model to improve the \( R \)-scaling of the FLRW diffusion coefficient in isotropic turbulence. In order to examine the diffusion coefficient, we use a nonperturbative analytic framework based on Corrsin’s hypothesis and the assumption of \( \sigma_{2,n}^2(\Delta \tau) = \beta_{n,\alpha}|\Delta \tau|^\alpha \). The proportional constant \( \beta \) as a function of \( B_0 \) is extrapolated with the proportional constants of the DD and RBD models. We obtain the closed-form solutions of the EMP model (\( \alpha = 0.8694 \)) for very small \( R \). For any \( R \), we obtain a numerical solution of the diffusion coefficients of the EMP model by solving equation (18) numerically.

In order to develop the closed-form solution of the FLRW for the EMP model at any \( R \), we employ the Padé approximants to the EMP model with boundary conditions at \( B_0 = 0 \) and \( B_0 \to \infty \).

Generally, the EMP model predicts quasilinear behavior at very low \( R \) (or very high \( B_0 \)) and the Bohm behavior at very high \( R \) (or very low \( B_0 \)) of simulation results as well as or better than the DD, RBD and ODE models. At very low \( R \) (at \( 0.0625 \leq R \leq 0.25 \)), the results of all models and the simulations trend to be quasilinear diffusion. Thus the results from all models of the Corrsin-based theory and the simulations are in close agreement at \( 0.0625 \leq R \leq 0.25 \).

At high \( R \) (at \( 5 < R \leq 400 \)), the EMP model predicts the simulation results better than the other three models significantly. Moreover, for the EMP model, the Padé approximants give more accurate results than the analytic theory. This difference comes from the different ways to parameterize the diffusion coefficient in isotropic turbulence.
\( \beta \) in the analytic theory and Padé approximation.

In the grand conclusion, the EMP model is the best model to generally match the asymptotic diffusion coefficients obtained by simulations for the FLRW in isotropic turbulence. The EMP model predicts that, with increasing \( R \), the FLRW behavior transits from quasilinear diffusion to Bohm diffusion. The Padé approximants for the EMP model better match the simulation results than the analytic solutions for the EMP model.

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