New emergency facility construction problem under the given facility and block-wisely different construction cost

Hiroaki ISHII

Department of Mathematical Sciences, School of Science and Technology
Kwansei Gakuin University, 2-1 Gakuen Sanda Hyogo 669-1337 Japan

Abstract

This paper considers a model of emergency facility construction problem with given emergency facility under the block-wisely difference accident occurrence probabilities and difference construction costs. That is, we consider the following problem. There exists a polygonal area $X$ where new ambulance service station should be located beside a given one (hereafter we call this facility as old one) and there exist $m$ hospitals. We assume that possible candidates are in a finite number of blocks that construction costs are different block-wisely. If an accident occurs, the ambulance cars in the nearer station among either given or new one rushes to the scene of accident and bring the injured persons to the nearest hospitals as soon as possible. Demand points (possible accident occurrence points) are distributed with block-wisely uniform probabilities in $X$. Weighted sum of $A$-distances of the route from the nearer station among old and new to the hospital via accident point is considered. For the weighted distance from the nearer station to each block with uniform accident occurrence probability, maximum expected distance is considered with respect to this distance. We also consider the construction cost where in possible site of each block it is constant. The expected distance should be minimized and construction cost be minimized. Above setting, we formulate the model with bi-objectives. After definition of non-dominated site, we propose some solution procedure to find some non-dominated sites. Finally we conclude the results and discuss further research problems.

Keywords: Ambulance facility, $A$-distance, Accident occurrence probability, Facility construction cost, Given ambulance station, Expected weighted distance

1. Introduction

There are huge amount of papers on facility location problems since Weber published his paper [1] and Hamacher et. al. [2] tried to classify them by using the similar codes to queuing and scheduling problems. We considered many models on emergency facility location problem ([3,4,5,6]) and propose an extended model of them under the old facility and block-wise fixed construction costs. This paper is organized as follows. Section 2 formulates our model and defines non-dominated solutions. Section 3 proposes solution procedures to seek some non-dominated solutions. Section 4 summarizes results of our paper and discusses further research problems.

2. Problem formulation

We consider the following model:

(1) There exists a polygonal area $X$ where new ambulance service station should be located beside a given one and there exist $m$ hospitals $H_1, H_2, \ldots, H_m$. We assume that possible candidates are in a finite number of disjoint rectangular blocks $D_1, D_2, \ldots, D_t$ that construction cost is constant $c_j$ in each block $D_j$, $j=1,2,\ldots,t$ and construction cost of the site in $X - D_1 - D_2 - \cdots - D_t$ is very high. This means we should consider the construction site should be limited to $D_1, D_2, \ldots, D_t$. If an accident occurs, the ambulance cars in the nearer station among either given or new one rushes to the scene of accident and bring the injured persons to the nearest hospitals as soon as possible.

(2) Demand points (possible accident occurrence points) are distributed within disjoint rectangular blocks $B_1, B_2, \ldots, B_n$ in $X$ and in each block $B_j$ accident occurrence probability is constant $p_i, i=1,2,\ldots,n$. The probability of point in $X - B_1 - B_2 - \cdots - B_n$ is 0. Therefore we consider only $B_1, B_2, \ldots, B_n$.

(3) $S(Q)$ denotes the nearest hospital to the point $Q \in X$. Weighted sum of $A$-distances of the route from the nearer station to the hospital via accident point is considered.

(4) For the maximum weighted distance from the nearer station among the old facility location site $p_o$ and new
facility location site \( p \) to each block with uniform accident occurrence probability, the satisfaction degree is considered with respect to A distance introduced in [7] and shown here.

(A distance)

There exists a set of directions \( A = \{ \alpha_1, \alpha_2, \ldots, \alpha_a \} \), where \( \alpha_i \) is an angle from \( x \) axis in an orthogonal coordinates and let \( 0^\circ \leq \alpha_i < \alpha_i < \cdots < \alpha_a < 180^\circ \). Hereafter if no confusion occurs, directions \( \alpha_i, i = 1, 2, \ldots, a \) are used as the same meaning. Directions \( \alpha_i, \alpha_{i+1} \) are called neighboring where \( \alpha_{i+1} \) is also called neighboring, that is \( \alpha_{i+1} \) is interpreted as \( \alpha_i \). Further A line, a half line and a line segment are called A-directional (or A-oriented) if their directions are ones of \( \{ \alpha_1, \ldots, \alpha_a \} \).

A distance is a so called angular distance where allowable direction are limited to a set of \( A = \{ \alpha_1, \alpha_2, \ldots, \alpha_a \} \). Especially, rectilinear distance is a special case of A distance with directions \( \{ 0^\circ, 90^\circ \} \). That is A distance is a general version of rectilinear distance.

Then A distance \( d_A \) between two points \( p_1, p_2 \in \mathbb{R}^2 \) is defined as follows.

\[
d_A(p_1, p_2) = \begin{cases} 
  d_z(p_1, p_2) & \text{if direction } \overline{p_1p_2} \text{ is A oriented} \\
  \min_{n=1} d_z(p_1, p_1) + d_z(p_3, p_2) & \text{Otherwise}
\end{cases}
\]

where \( d_z(p_1, p_2) \) is the Euclidean distance between \( p_1, p_2 \).

According to the result in [3], when \( \alpha_j < \) an angle of the line connecting demand point \( i \) with the facility site \( (x, y) \) \( < \alpha_{j+1} \),

\[
d_i = M_1 |m_2(p_i - x) - (q_j - y)| + M_2 |m_1(p_i - x) - (q_j - y)|
\]

where \( m_1 = \max(\tan \alpha_i, \tan \alpha_{i+1}) \),

\[
m_2 = \min(\tan \alpha_j, \tan \alpha_{j+1})
\]

(If either \( \alpha_j \) or \( \alpha_{j+1} \) is \( 90^\circ \), then we interpret

\[
M_1 = \lim_{n \to \infty} \sqrt{1 + m_1^2} = 1, \quad M_2 = \lim_{n \to \infty} \sqrt{1 + m_2^2}
\]

Further A-circle from center \( O \), that is, the points equidistance points from \( O \) is a following \( 2a \) polygon with vertices are on the usual circle (that is, equidistance points in a euclidian . distance)

![Figure 2. A-circle with respect to directions \( \alpha_1, \alpha_2, \alpha_3 \)](image)

Then the maximum weighted distance from the nearer station among the given facility location site \( p_0 \) and new facility location site \( p \) to the accident site \( Q \) is

\[
\min \{ R(p_0, Q), R(p, Q) \} \quad \text{where}
\]

\[
R(p_0, Q) = w_1 d_A(p_0, Q) + w_2 d_A(Q, S(Q))
\]

\[
R(p, Q) = w_1 d_A(p, Q) + w_2 d_A(Q, S(Q)) \quad \text{and} \quad w_1, w_2
\]

are positive weights corresponding to the importance (emergency) of A-distance from the demand point to the station and that from the demand point to the nearest hospital.

Then we should consider

\[
R(p) = \max_{0 \leq i \leq n} \min \{ R(p_0, Q), R(p, Q) \} \quad \text{should be minimized.}
\]

Note that \( d_A(Q, S(Q)) \) is calculated as below using Voronoi diagram with respect to hospitals \( H_1, H_2, \ldots, H_n \).
New emergency facility construction problem under the given facility and block-wise different construction cost

(Voronoi diagram)

For a set of s points \( V = \{V_1, V_2, \ldots, V_s\} \), Voronoi polygon \( V_d(V_i) \) on point \( V_i \) with respect to \( V \) with A-distance on \( X \) is defined as follows:

\[
V_d(V_i) = \bigcap_{j \neq i} \{Q | d_a(Q, V_i) \leq d_a(Q, V_j), Q \in X\}.
\]

The set of all Voronoi polygons for the points in \( V \) is a partition of \( X \). Edge of Voronoi polygon is called Voronoi edge. Then we construct Voronoi diagram \( VD_\alpha(H) \) with respect to the set of hospital points \( H = \{H_1, H_2, \ldots, H_m\} \) and A-distance on the area \( X \). It is done in \( O(m \log m) \) computational time (see [7]). Figure 3 illustrates Voronoi diagram in this case.

Figure 3. Voronoi diagram with respect to hospitals \( H_1, H_2, H_3, H_4, H_5, H_6 \).

(5) We also consider the construction cost of construction site of the new facility site should be minimized.

(6) \( R(p) = \max_{t \in b} \min_{Q \in k} \{R(p, Q), R(p, Q)\} \) should be minimized and construction cost minimized.

Above setting, we formulate the model with bi-objective. Since usually the optimal site about the new emergency facility station, we usually seek non-dominated site after defining non-domination site in the next section 3.

3. Solution Procedure

(Non-dominated site)

For two candidate construction sites \( p_1 \in D, p_2 \in D \), if \( R(p_1) \leq R(p_2) \) for all \( k \leq \ell \) and at least inequality holds as strictly, then site \( p_1 \) dominate \( p_2 \). If there does not exist site dominating \( p \), \( p \) is called as non-dominated site.

First we show the useful theorems to useful to find non-dominated sites.

Theorem 1. For the line segment \( DE \) with endpoints \( D, E \) and points \( B, C \) not on \( DE \), suppose line segments \( BD \) and \( BE \) are included between two A-oriented half-lines with adjacent orientations \( \alpha_j, \alpha_{j+1} \) from \( B \) respectively, then the weighted sum of A-distance among paths between \( B \) and \( C \) via point \( T \) on the line segment \( DE \),

\[
w_d(B, T) + w_d(T, C) \leq w_d(B, T) + w_d(T, C)
\]

is maximized when \( T = D \) or \( E \).

(Proof) Content of Theorem 1 is only slightly different from that of Theorem 1 in [3]. That is, Theorem 1 in this paper extends the content of that of [3] in two points. One point is that the former assumes “line segments \( BD \) and \( BE \) are included between two A-oriented half-lines with adjacent orientations \( \alpha_j, \alpha_{j+1} \) from \( B \) respectively” and the latter “line segments \( BD \) and \( BE \) are on two A-oriented half-lines with adjacent orientations \( \alpha_j, \alpha_{j+1} \) from \( B \) respectively”. The other point is that the former is sum of weighted A-distances and the latter simply the sum of two A-distances. But ideas of proof in this Theorem 1 is very same as Theorem of [3].

Q.E.D.

Further we relax the constraints that \( BD \) and \( BE \) have \( \alpha_j \) and \( \alpha_{j+1} \) orientations respectively from Theorem 1

Theorem 2. For the line segment \( DE \) with endpoints \( D, E \) and points \( B, C \) not on \( DE \), \( w_d(B, T) + w_d(T, C), T \notin DE \) is maximized when \( T = D \) or \( E \).

(Proof) We draw all A-oriented half lines from \( B \) and \( C \), and let all intersections of these lines and \( DE \) be \( T_1, T_2, \ldots, T_{i-1} \) by ordering from \( D \). Further let \( T_0 = D \) and \( T_i = E \). Then the situation may be interpreted as Figure 3. By Theorem 1, when consider the subinterval \( T \in [T_{i-1}, T_{i+1}] \), \( w_d(B, T) + w_d(T, C) \) is maximized at \( T_{i-1} \) or \( T_{i+1} \). So \( T_i \) is dropped from candidates of maximizer. In turn, when considering \( T \in [T_{i-2}, T_{i-3}] \), \( T_{i-1} \) is dropped by Theorem 1. Continuing this way, only remaining candidates are \( D, E \) and points as \( T_{i+7} \) which are intersections of \( DE \) and certain A-lines from both \( B \) and \( C \). Let all points on \( DE \) with same property as \( T_{i+7} \) be \( T_1', \ldots, T_l' \). Then

\[
w_d(B, T_i') + w_d(T_i', C) = w_d(B, T_i') + w_d(T_i', C)
\]

, \( i = 1, \ldots, l \), since both \( BT_i' \) and \( CT_i' \) are A-oriented.

Since Euclidean distance is a convex function, then \( w_d(B, T) + w_d(T, C) \) is maximized at \( T = D \) or \( E \). Thus

\[
w_d(B, T) + w_d(D, C) \geq w_d(B, D) + w_d(D, C)
\]

and these inequalities imply \( w_d(B, T) + w_d(T, C) \), \( T \in DE \) is maximized at \( D \) or \( E \) since.
Based on Theorem 1 and 2, we have the following Theorem 3.

**Theorem 3.** For fixed $p \in D_1 + D_2 + \cdots + D_t$ and $B_j$, candidate of maximizer of $\min \{ R(p_u, Q), R(p, Q) \}$ in $Q \in B_j$, we have the following result.

(a) the intersection points of Voronoi edges and $B_j$,
(b) the intersection points of boundary of $X$ and $B_j$,
(c) vertices of $B_j$.

**Proof** Direct result from Theorem 1 and Theorem 2.

Q. E. D.

Let these points (a), (b), and (c) in Theorem 3 be $Q_1, Q_2, \ldots, Q_n, i = 1, 2, \ldots, n$.

Based on these theorems, now we consider the solution procedure.

First we assume that $c_1 < c_2 < c_3 < \cdots < c_i$ without any loss of generality for $D_1, D_2, \ldots, D_t$. Let

$$D_j = \{ (x, y) | R_j' \leq x \leq R_j, R_j \leq y \leq R_j' \}, j = 1, 2, \ldots, t.$$

Then we consider the following sub-problem $P'$, $j = 1, 2, \ldots, t$.

$P'$: Minimize

$$\{ \min \{ R(p_u, Q), R(p, Q) \} | \ell = 1, 2, \ldots, n \},$$

subject to $p \in D_j$

Since $R(p_u, Q) = w_1 d_a(1, T_u') + w_2 d_a(T_u', C) = w_2 d_a(1, T_u') + w_2 d_a(T_u', C) \leq \max \left\{ \frac{w_1}{w_2} \left( d_a(T_1, C) + w_2 d_a(T, D) \right), \frac{w_1}{w_2} \left( d_a(T_2, C) + w_2 d_a(T, E) \right) \right\}$

$$= \max \left\{ \frac{w_1}{w_2} d_a(B, D) + w_2 d_a(T, D), \frac{w_1}{w_2} d_a(B, E) + w_2 d_a(T, E) \right\} \leq \max \left\{ \frac{w_1}{w_2} d_a(B, D) + w_2 d_a(T, D), \frac{w_1}{w_2} d_a(B, E) + w_2 d_a(T, E) \right\} \leq \max \left\{ \frac{w_1}{w_2} d_a(B, D) + w_2 d_a(T, D), \frac{w_1}{w_2} d_a(B, E) + w_2 d_a(T, E) \right\}$$

Q. E. D.

Figure 4: Intersections and line segment DE

Based on Theorem 1 and 2, we have the following Theorem 3.

$$P': \text{Min } \gamma$$

subject to $w_i p, \ell \leq \gamma, \lambda_i \geq \beta_i, \ell' \geq \beta_i$.

Now we consider a solution method of $P'$. First we divide $D_j$ into the following subsidiary problems $P'(s_1, s_2, \ldots, s_j)$ where $s_i = 1$ or $0$, $i = 1, 2, \ldots, n$.

$$P'(s_1, s_2, \ldots, s_j) :$$

Min $\gamma$

subject to $w_i P_i, \ell \leq \gamma, i = 1, 2, \ldots, n$

$\lambda_i \geq \beta_i, \ell' \geq \beta_i$.

for $i = \{ u | s_u = 1, u = 1, 2, \ldots, n \}$

$\lambda_i \geq \xi_i + k_i, \ell = 1, 2, \ldots, n$ for $i = \{ u | s_u = 0, u = 1, 2, \ldots, n \}$

$p = (x, y) \in D_j(s_1, \ldots, s_j)$

where

$$D_j(s_1, \ldots, s_j) = \{ p = (x, y) | d_a(p, Q) \leq k_i, \ell = 1, 2, \ldots, n \} \cap D_j$$

Note that each sub-region $D_j(s_1, \ldots, s_j)$ is a polygon since each A-circle $\{ p = (x, y) | d_a(p, Q) \leq \xi_i, \ell = 1, 2, \ldots, n \}$ is a polygon and $D_j$ is a rectangular region. Each condition $d_a(p, Q) \leq k_i$ is equivalent to the following problem $P'$ where we set

$k_i = \frac{w_2}{w_1} d_a(Q, S(Q)), \ell = 1, 2, \ldots, n\), and$k_i = d_a(p_u, Q)$.

$P$: Min $\gamma$

subject to $w_i p, \ell \leq \gamma, \lambda_i \geq \beta_i, \ell' \geq \beta_i$.

for $i = \{ u | s_u = 1, u = 1, 2, \ldots, n \}$

$\lambda_i \geq \xi_i + k_i, \ell = 1, 2, \ldots, n$ for $i = \{ u | s_u = 0, u = 1, 2, \ldots, n \}$,

$p = (x, y) \in D_j(s_1, \ldots, s_j)$

where

$D_j(s_1, \ldots, s_j) = \{ p = (x, y) | d_a(p, Q) \leq \xi_i, \ell = 1, 2, \ldots, n \} \cap D_j$.
New emergency facility construction problem under the given facility and block-wisely different construction cost

\[ Q_i = (p_i, q_i), \quad m_i = \max(\tan \alpha_i, \tan \alpha_{i+1}), \]
\[ m_i = \min(\tan \alpha_i, \tan \alpha_{i+1}), \]
\[ M_i = \sqrt{1 + m_i^2}, \quad M_i = \sqrt{1 + m_i^2}. \]

That is, we further consider the following auxiliary problems

\[ \mathbf{P}^i(s_1, s_2, \ldots, s_n)(v_i^1, v_i^2, \ldots, v_i^n, v_i^{n+1}, \ldots, v_i^n, v_i^1, \ldots, v_i^n, v_i^{n+1}; v_i^n, v_i^{n+1}, \ldots, v_i^n, v_i^1, \ldots, v_i^n, v_i^{n+1}) \]

of subsidiary problem \( \mathbf{P}^i(s_1, s_2, \ldots, s_n) \);  

Min \( \gamma \)

subject to \( w_i\beta_i \leq \gamma, i = 1, 2, \ldots, n \)
\[ \beta_i \geq \beta_i + k_i, \quad M_i | m_i | (p_i - x) - (q_i - y) | \]
\[ + M_i \frac{|m_i | (p_i - x) - (q_i - y)}{\beta_i}, \]
\[ \ell = 1, 2, \ldots, n, \quad i \in \{u | s_i = 1, u = 1, 2, \ldots, n\} \]
\[ \lambda_i = \bar{R} + k_i, \quad \ell = 1, 2, \ldots, n; \quad \lambda_i \leq \lambda_i \]

These auxiliary problems are easily transformed to linear programming problem. We solve them by changing \( \alpha_i \in \alpha \), \( i = 1, 2, \ldots, n \), \( i \in \{u | s_i = 1, u = 1, 2, \ldots, n\} \) and obtain their optimal values. An optimal solution of \( \mathbf{P}^i(s_1, s_2, \ldots, s_n) \) is an optimal solution giving minimal one among optimal values of

\[ \mathbf{P}^i(s_1, s_2, \ldots, s_n)(v_i^1, v_i^2, \ldots, v_i^n, v_i^{n+1}, \ldots, v_i^n, v_i^1, \ldots, v_i^n, v_i^{n+1}) \]

of subsidiary problem \( \mathbf{P}^i(s_1, s_2, \ldots, s_n) \).

Let the optimal solution be \( p_i(s_1, \ldots, s_n) \) and denote corresponding \( R(p_i(s_1, \ldots, s_n)) \) with \( R_i(s_1, \ldots, s_n) \).

Then an optimal site \( p(j) = (x(j), y(j)) \) of \( \mathbf{P}^i \) is the optimal solution \( p_j(s_1, \ldots, s_n) \) of \( \mathbf{P}^i(s_1, s_2, \ldots, s_n) \) with the smallest value among all \( R_i(s_1, \ldots, s_n) \). Then with respect to \( p(j) = (x(j), y(j)) \), construction cost \( c_j \) is and \( R(p(j)) \). We solve all \( \mathbf{P}^i, j = 1, \ldots, t \) as above and check them about non-dominated site. Then we have some non-dominated sites. Note that \( p(1) = (x(1), y(1)) \) is a non-dominated site.

4. Conclusion

We have proposed a new emergency facility construction model with a given old one, accident occurrence probabilities and block-wise construction cost. But our solution procedure to this model is straightforward and so more refinement method should be constructed such as a minimal covering A-circle including all small a-circles in [3]. Further auxiliary problem is a linear programming problem and so we may use sensibility analysis. Moreover actual construction cost is not fixed but random since it is fluctuate and suitability of construction site depends on not only construction cost but many factors. Therefore more realistic location models should be considered though they become difficult to solve.

References

[1] Weber A. Über Den stand Der Industrien, 1, Teil, Rein Theorie Des Stabdores (1909)
[2] Hamacher W. H. and Stefan N. Classification of location models, Location Science, 6 (1998) 229-242.
[3] Matutomi T. and Ishii, H. (1998). Minimax location problem with A-distance, Journal of the Operations Research Society of Japan, 41(1998) 181-195.
[4] Hsia, H. C, Ishii H. and Yeh K. Y. Ambulance Service facility location problem, Journal of the Operations Research Society of Japan, 52(2009) 339-354.
[5] Ishii H, Lee Y. K and Yeh K. Y. Mathematical ranking method for facility location Problem, International Journal of Association for Management Systems, 4 (2012) 73-76.
[6] Ishii H and Lee Y. K. Mathematical ranking method for emergency facility location problem with block wisely different accident occurrence probabilities. Procedia Computer Science, 22 (2013) 1065-1072.
[7] Widmayer P. et al. On some distance problems in fixed orientations, SIAM J. on Computing, 16 (1987) 728-746.
