VANISHING RESULTS FOR CHROMATIC LOCALIZATIONS OF ALGEBRAIC $K$-THEORY

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Abstract. We show that algebraic $K$-theory preserves $n$-connective $L_f^n$-equivalences between connective ring spectra, generalizing a result of Waldhausen for rational algebraic $K$-theory to higher chromatic heights. We deduce various vanishing results for telescopic localizations of algebraic $K$-theory and use them to discuss a purity property of telescopically localized algebraic $K$-theory.

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1. INTRODUCTION AND RESULTS

We are interested in the algebraic $K$-theory of rings and ring spectra. One reason algebraic $K$-theory is hard to compute is that it lacks features like nilinvariance, excision, homotopy invariance, or étale descent. Using the arithmetic fracture square, one may reduce the calculation of the $K$-theory of a ring spectrum to the calculation of its $p$-adic $K$-theory for all primes $p$ and its rational $K$-theory. These localizations of $K$-theory tend to have better properties. For example, using module structures on Nil-groups Weibel [Wei81] proved that rational $K$-theory satisfies excision and nilinvariance on rings that are $N$-torsion for some integer $N > 0$, i.e. that are $H\mathbb{Q}$-acyclic, and that $p$-adic $K$-theory satisfies excision and nilinvariance on rings

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in which $p$ is invertible, i.e. which are $HF_p$-acyclic. However, $p$-adic $K$-theory of rings in which $p$ is not invertible does neither satisfy excision nor nilinvariance.

Whereas $\mathbb{Q}$ and the finite fields $\mathbb{F}_p$ are all the prime fields in algebra, in higher algebra there is an additional family of ‘prime fields’ for each prime number $p$, ‘interpolating’ between $\mathbb{Q}$ and $\mathbb{F}_p$: the Morava $K$-theories $K(n)$ at chromatic height $n \in \mathbb{N}$ and an implicit prime number $p$. Mitchell [Mit90] has shown that the $K$-theory of the integers, $K(\mathbb{Z})$, vanishes $K(n)$-locally for every $n \geq 2$ and every prime $p$. It follows that the same vanishing happens for all $HZ$-algebras.

On the other hand, $K(1)$-local $K$-theory is an interesting invariant already for rings: By work of Thomason, Clausen, Mathew, Naumann, and Noël [Tho85, TT 90, CMNN16, CM19] $K(1)$-local $K$-theory satisfies étale descent, and is in fact closely related to étale $K$-theory, i.e. the étale sheafification of algebraic $K$-theory.

For the study of $K$-theory of ring spectra it becomes important not only to consider $K(n)$-localization, but also the (a priori finer) telescopic localizations at the spectra $T(n)$. Here $T(n)$ is the telescope of a $v_n$-self map on a finite type $n$-complex. Any $T(n)$-local equivalence is a $K(n)$-local equivalence, and the converse holds in many cases, e.g. if $n = 1$ or for maps of MU-modules. It is also often convenient to consider the localizations at $T(i)$ for all $i \leq n$, respectively at $K(i)$ for all $i \leq n$, at once. The associated localization functors are denoted by $L_i$ and $L_n$. We refer to Section 2.1 for more details about Morava $K$-theories and telescopic localizations.

Contrary to the case of discrete rings, the $T(n)$-local $K$-theory of a ring spectrum does not necessarily vanish if $n \geq 2$, see e.g. [Aus10, AR02]. Moreover, $T(n)$-local $K$-theory has several pleasant properties. In particular, Clausen and Mathew [CM19] prove étale descent in general for $K$-theory of $\mathbb{E}_2$-ring spectra after $T(n)$-localization for any $n$, or equivalently after applying the functor $L_i^n$.

One goal of the present paper is to add to the list of pleasant properties of $T(n)$-local $K$-theory by generalizing Weibel’s excision and nilinvariance results to this context. In order to motivate our main theorem we recall an alternative argument for Weibel’s results on excision. By the main theorem of [LT19], the birelative $K$-theory of a given Milnor square of rings is equivalent to the relative $K$-theory of an associated 1-connective map between connective ring spectra. Weibel’s excision result is then an immediate consequence of the following result of Waldhausen [Wal78]: If $A \to B$ is a 1-connective map between connective ring spectra which is a rational, respectively an $p$-adic equivalence, then also the induced map $K(A) \to K(B)$ is a rational, respectively an $p$-adic equivalence. Using unstable chromatic homotopy theory, we prove the following analog at higher chromatic heights, which is our main result.

**Theorem A.** Let $n \geq 1$ and let $f : A \to B$ be an $n$-connective $L_i^n$-equivalence between connective ring spectra. Then the induced map $K(A) \to K(B)$ is again an $L_i^n$-equivalence.

**Remark.** Using the theorem of Dundas–Goodwillie–McCarthy [DGM13, Theorem VII.0.0.2], it follows that Theorem A holds verbatim with $K$-theory replaced by topological cyclic homology. The same remark applies to several of its consequences as listed below. We give more details in Subsection 2.6.

The following is an immediate consequence of Theorem A and known localization sequences in algebraic $K$-theory, see Corollary 2.10 for a detailed proof and a slightly more general statement. Further immediate applications are discussed in Section 2.3.

**Corollary.** For integers $0 < n < m$, the algebraic $K$-theory $K(K(m))$ of Morava $K$-theory at height $m$ vanishes $T(n)$-locally.
We recall from [LT19] that a localizing invariant $E$ is called truncating if the canonical map $E(A) \to E(\pi_0(A))$ is an equivalence for every connective ring spectrum $A$. Combining a variant of Theorem A which omits the $T(0)$-information with results from [LT19], we prove the following theorem.

**Theorem B.** $K(1)$-local algebraic $K$-theory is truncating on $K(1)$-acyclic ring spectra. In particular, $K(1)$-local $K$-theory of discrete rings and schemes satisfies excision, cdh-descent, and is invariant under quotients by a nilpotent two-sided ideal.

It is a direct consequence of Quillen’s computation [Qui72] of the $K$-theory of finite fields that $K(\mathbb{F}_p)$ vanishes $K(1)$-locally. Using Theorem B we thus find the following corollary, see Corollary 2.20 for a more general statement.

**Corollary.** The $K(1)$-localization of $K(\mathbb{Z}/p^n)$ vanishes for all $n \geq 1$.

This result was previously obtained by Bhatt–Clausen–Mathew [BCM] by different techniques: Using finite flat descent for $K$-theory and trace methods they reduce it to the $K(1)$-local vanishing of $TC(O_{\mathbb{F}_p}/p^n)$, which they deduce from a calculation in prismatic cohomology.

Combining methods of the current paper and of [LT19] with a result of Hahn [Hah16] we extend Theorem B to higher chromatic heights in Section 2.5:

**Theorem C.** $T(n)$-local $K$-theory is truncating on $T(1) \oplus \cdots \oplus T(n)$-acyclic ring spectra.

As any bounded above spectrum is $T(i)$-acyclic for every integer $i \geq 1$, Theorem C together with Mitchell’s theorem imply that the $K$-theory of every bounded, connective ring spectrum vanishes $T(n)$-locally for all $n \geq 2$. For example, $L_{T(n)}K(\tau_{\leq m}\mathbb{S}) = 0$ for all $n \geq 2$ and the same vanishing holds for every $E_1$-algebra over a truncation of the sphere. We also obtain the following redshift result; see Corollary 2.29.

**Corollary.** Let $A$ be an $E_\infty$-ring spectrum. If $A$ is $T(1)$-acyclic, then $K(A)$ is $T(n)$-acyclic for every integer $n \geq 2$. If $A$ is $S[1/p]$-acyclic, then $K(A)$ is $T(n)$-acyclic for every integer $n \geq 1$.

Using the corollary to Theorem B, Bhatt–Clausen–Mathew deduce that for any $HZ$-algebra $A$, the canonical map $K(A) \to K(A[1/p])$ is a $K(1)$-local equivalence, thereby exhibiting many properties of $K(1)$-local $K$-theory (including that it is truncating on $HZ$-algebras and homotopy invariant). We interpret this as a purity property of telescopeically localized algebraic $K$-theory and discuss a possible extension to ring spectra and to higher heights in Section 5.2.

To formulate this purity property, we consider a variant of the $L_n^f$-localization, which we denote by $L_n^{p,f}$, namely the Bousfield localization functor at the spectrum $S[\frac{1}{p}] \oplus T(1) \oplus \cdots \oplus T(n)$. For an $HZ$-algebra $A$, one has $L_n^{p,f}A \approx A[\frac{1}{p}]$. We ask:

**Question.** Let $A$ be a ring spectrum, and let $n \geq 1$ be an integer. Is the canonical map

$$K(A) \to K(L_n^{p,f}A)$$

a $T(i)$-local equivalence for $1 \leq i \leq n$?

This question trivially has an affirmative answer for $L_n^{p,f}$-local ring spectra as e.g. $E(m)$ and $T(m)$ for $m \leq n$. It also has an affirmative answer for Morava $K$-theories $K(m)$ and their connective covers $k(m)$; see Proposition 3.8. To go further, we explain in Section 3.1...
the argument of Bhatt–Clausen–Mathew that allows one to prove the $T(i)$-local vanishing of relative $K$-theory spectra. The main results of the final subsections can be summarized as follows.

**Theorem D.** The above question has an affirmative answer for $ko$- and hence in particular for $H\mathbb{Z}$-algebras. For $n \geq 2$, it also has an affirmative answer for $tmf$-algebras.

Of course, the case of $HZ$-algebras is just the result of Bhatt–Clausen–Mathew mentioned above.

**Conventions.** We fix a prime number $p$ which will be the implicit prime in all Morava $K$-theories $K(i)$ below. We adopt the convention that $K(0) = H\mathbb{Q}$ and $K(\infty) = H\mathbb{F}_p$. Whenever we speak of a ring spectrum, we mean an $E_1$-ring spectrum, i.e. an algebra in the symmetric monoidal $\infty$-category of spectra. By a module over a ring spectrum we mean a right module. Given an $E_k$-ring spectrum $R$ for $k \geq 2$, an $R$-algebra is an algebra in the monoidal $\infty$-category $R\text{Mod}_R$ of $R$-modules. For a spectrum $E$, we denote by $L_E$ the Bousfield localization functor at $E$. For a spectrum $X$ and a pointed space $Y$, we write $X \otimes Y$ for the smash product $X \otimes \Sigma^\infty Y$.

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## 2. The Chromatic Behaviour of Algebraic K-theory

### 2.1. Recollections from Chromatic Homotopy Theory

For an integer $n \geq 1$, we denote by $V_n$ a type $n$-complex, i.e. a pointed finite CW-complex with $K(i) \otimes V_n = 0$ for $i < n$ and $K(n) \otimes V_n \neq 0$. We denote by $v_n$ a $v_n$-self map of $V_n$, i.e. a map $\Sigma^d V_n \to V_n$ for some positive integer $d$ inducing an isomorphism on $K(n)$-homology and nilpotent maps on $K(i)$-homology for $i \neq n$. Such maps exist by [HS98].

If $X$ is a pointed space or a spectrum, we define its $v_n$-periodic homotopy groups $v_n^{-1}\pi_\ast(X; V_n)$ by the formula

$$v_n^{-1}\pi_\ast(X; V_n) = \mathbb{Z}[v_n^{\pm 1}] \otimes_{\mathbb{Z}[v_n]} \pi_\ast \text{Map}_\ast(V_n, X).$$

**Definition 2.1.** We call a map of pointed spaces or spectra a $v_n$-periodic equivalence (with $n \geq 1$) if it induces an isomorphism on $v_n$-periodic homotopy groups. A $v_0$-periodic equivalence is by definition a rational equivalence. For a spectrum $E$, we say that another spectrum $X$ is $E$-acyclic if $E \otimes X = 0$ and say that a map is an $E$-equivalence if its fibre is $E$-acyclic.

For a fixed pair $(V_n, v_n)$ we denote by $T(n) = \Sigma^\infty V_n[v_n^{-1}]$ the telescope of $v_n$. We follow the convention that $T(0) = H\mathbb{Q}$. We recall that the Bousfield class of a spectrum $E$ is the full subcategory of $\text{Sp}$ consisting of the $E$-acyclic spectra. For the convenience of the reader not familiar with chromatic homotopy theory, we note the following well-known properties.

**Lemma 2.2.** Let $X$ be a spectrum and $Y$ be a pointed space.

(i) We have $v_n^{-1}\pi_\ast(X; V_n) \cong v_n^{-1}\pi_\ast(\Omega^\infty X; V_n)$.

(ii) The maps $\tau_{\geq k}X \to X$ and $\tau_{\geq k}Y \to Y$ are $v_n$-periodic equivalences for all $k$ and all $n \geq 1$.

(iii) The spectra $K(m)$ are $T(n)$-acyclic for $n \neq m$. 

Any $T(n)$-acyclic spectrum is $K(n)$-acyclic.

A spectrum which is $S/p$-acyclic is also $T(n)$-acyclic for all $n \geq 1$.

The map $X \to X_p^\infty$ is a $T(n)$-equivalence for all $n \geq 1$.

The Bousfield class of $T(n)$ does not depend on the choice of $(V_n, v_n)$.

A spectrum is $T(n)$-acyclic if and only if its $v_n$-periodic homotopy groups vanish.

Proof. Part (i) follows immediately from the definitions and the equivalence $\text{Map}_n(V_n, X) \simeq \text{Map}_n(V_n, \Omega^\infty X)$. Assertion (ii) follows from the observation that the $v_n$-periodic homotopy groups of a bounded above spectrum or space vanish. This in turn follows from the fact that the degree $d$ of the self map $v_n$ is positive. Claim (iii) follows from the fact that $K(m) \otimes T(n) \simeq (K(m) \otimes V_n)[v_n^{-1}]$ which vanishes as $v_n$ is nilpotent on Morava $K$-homology if $n$ is different from $m$. To see (iv), assume that $X$ is $T(n)$-acyclic. Then we have $0 = K(n) \otimes T(n) \otimes X$. But $K(n) \otimes T(n) \neq 0$, so, since $K(n)$ is a field spectrum (any module is a direct sum of shifted copies of $K(n)$), we must have $K(n) \otimes X = 0$.

For part (v) observe that some power of $p$, say $p^k$, is zero on $V_n$ and hence on $T(n)$. Given an $S/p$-acyclic spectrum $X$, we have $X/p^k = 0$. Thus we see that

$$0 = X/p^k \otimes T(n) \simeq X \otimes T(n)/p^k \simeq X \otimes (T(n) \oplus \Sigma T(n)),$$

and the latter term has $X \otimes T(n)$ as a retract. Thus $X$ is $T(n)$-acyclic. Statement (vi) follows from (v), since the fibre of $X \to X_p^\infty$ is $S/p$-acyclic.

For (vii), as in [MS95, Lemma 2.1], we fix a pair $(V_n, v_n)$ and consider the full subcategory of finite $p$-local spectra consisting of those $Y$ which admit a $v_n$-self map $y$ and such that $T(n) \otimes Z = 0$ implies that $Y[y^{-1}] \otimes Z = 0$ as well. This is a thick subcategory, as follows from [HS93, Corollary 3.8]. Since it contains $V_n$, this thick subcategory is given by $C_{\geq n}$: see [HS98, Theorem 7]. Hence if $T(n) \otimes Z = 0$ and $(V_n', v_n')$ is another choice, also $T(n)' \otimes Z = 0$. Running the same argument also with $V_n'$ instead of $V_n$ gives the claim.

To see (viii), consider a spectrum $X$ and observe that we may calculate its $v_n$-periodic homotopy groups using the mapping spectrum map($V_n, X$) instead of the mapping space map($V_n, X$) due to the positivity of the degree of the self-map $v_n$. Thus, the $v_n$-periodic homotopy groups of $X$ are isomorphic to the homotopy groups of the spectrum $(DV_n \otimes X) [DV_n^{-1}]$, where $DV_n$ denotes the dual of the finite spectrum $\Sigma^\infty V_n$ (which is again of type $n$). This spectrum is equivalent to $T(n) \otimes X$ where $T(n)$ is the telescope of $DV_n$. The claim then follows from (vii). \qed

The converse of (iv) for all $n \geq 1$ is equivalent to the telescope conjecture. It is known to be true in height $n = 1$ and is open in general, see e.g. [Bar19] for a survey. The following lemma was indicated to us by Dustin Clausen.

**Lemma 2.3.** Let $R$ be a ring spectrum. Then $R$ is $K(n)$-acyclic if and only if it is $T(n)$-acyclic.

In fact, coherent associativity of $R$ is not needed; it suffices that $R$ is a spectrum with a homotopy unital multiplication.

Proof. The “if”-part follows from Lemma 2.2 (iv). To see the “only if” statement, we argue first that one can assume that $T(n)$ is a ring spectrum. Indeed, by replacing $V_n$ by $W_n = V_n \otimes DV_n$, we can assume that the suspension spectrum of our type $n$-complex is an $E_1$-ring spectrum. Moreover, the $v_n$-self-map of $V_n$ defines an element $w \in \pi_*W_n$, multiplication with which is a $v_n$-self map again. Applying [HS98, Theorem 11], we see that a power of
w lies in the center of $\pi_n W_n$. Thus the localization $W_n[w^{-1}]$ admits the structure of an $E_1$-ring spectrum. As the Bousfield class of $T(n)$ does not depend on the choice of the type $n$ complex, we can thus indeed assume that $T(n)$ is a ring spectrum. It is then a consequence of the nilpotence theorem [HS98, Theorem 3] that the ring spectrum $T(n) \otimes R$ is zero if and only if $K(m) \otimes (T(n) \otimes R) = 0$ for all $0 \leq m \leq \infty$. If $m \neq n$ then $K(m) \otimes T(n) \otimes R = 0$ as $K(m) \otimes T(n) = 0$ by Lemma 2.2(iii). If $R$ is $K(n)$-acyclic, then also $K(n) \otimes T(n) \otimes R = 0$, and hence $T(n) \otimes R$ vanishes. □

Finally, the following is a variant of a result of Bousfield and gives crucial information about the relationship between $v_i$-periodic equivalences and $T(i)$-equivalences in the unstable context.

**Proposition 2.4.** Fix an integer $n \geq 1$ and let $F$ be an $n$-connected pointed space whose $v_i$-periodic homotopy groups vanish for $0 \leq i \leq n$. Then $F$ is $T(i)$-acyclic for $0 \leq i \leq n$.

**Proof.** Lemma 2.2(ii) implies that the $v_i$-periodic homotopy groups of $\tau_{\geq N} F$ vanish for any integer $N \geq 0$ and $0 \leq i \leq n$. For $N \geq n + 1$ large enough, a result of Bousfield ([BHM18, Theorem 3.1], [Bon01, Corollary 4.8]) together with [BHM18, Lemma 3.3] implies that $\tau_{\geq N} F$ is $T(i)$-acyclic for all $0 \leq i \leq n$. It hence suffices to prove that $\tau_{\leq N} F$ is $T(i)$-acyclic for $0 \leq i \leq n$. This is an $n$-connected space with only finitely many non-trivial homotopy groups of which are torsion groups. By an induction over the Postnikov tower, using that $T(i)$-homology commutes with filtered colimits, it hence suffices to show that $T(i) \otimes K(\pi, m) = 0$ if $m \geq i$ is a finite group. As in the proof of Lemma 2.3 we may assume that $T(i)$ is a ring spectrum. As $T(i) \otimes K(m) = 0$ for all $m > i$ by Lemma 2.2(iii), [CSY18, Theorem E] implies the desired vanishing. □

### 2.2. Proof of the main theorem

For the reader’s convenience, we repeat the statement of Theorem A below. We wish to thank Akhil Mathew for sharing with us a simplification of our original proof. We call a map $n$-connective if its fibre is $n$-connective, i.e. if it induces an isomorphism on $\pi_k$ for $k < n$ and a surjection on $\pi_n$.

**Definition 2.5.** For $n \geq 0$, we denote by $L^i_n$ the Bousfield localization of spectra at the spectrum $T(0) \oplus T(1) \oplus \cdots \oplus T(n)$. Thus, a map of spectra is an $L^i_n$-equivalence if and only if it is a $T(i)$-equivalence for all $0 \leq i \leq n$.

**Theorem 2.6.** Let $n \geq 1$ and let $f : A \to B$ be an $n$-connective $L^i_n$-equivalence between connective ring spectra. Then the induced map $K(A) \to K(B)$ is again an $L^i_n$-equivalence.

**Proof.** Let $F$ be the fibre of the map $BGL(A) \to BGL(B)$. First, we observe that $F$ is $n$-connected. Furthermore, since the map $A \to B$ is a $v_i$-periodic equivalence, the same holds for the map $M(A) \to M(B)$ between their matrix ring spectra. Since $\tau_{\geq 1} GL(A) \simeq \tau_{\geq 1} M(A)$, we deduce from Lemma 2.2(ii) that the map $GL(A) \to GL(B)$ is also a $v_i$-periodic equivalence, hence the same holds for the map $BGL(A) \to BGL(B)$. We thus find that $F$ is $n$-connected and its $v_i$-periodic homotopy groups vanish for $0 \leq i \leq n$. Thus $T(i) \otimes F = 0$ for $0 \leq i \leq n$ by Proposition 2.4.

The map $K(A) \to K(B)$ is $(n + 1)$-connective and a rational equivalence, see [Wal78, Propositions 1.1, 2.2] or also [LT19, Lemma 2.4]. It hence suffices to show that $\tau_{\geq 1} K(A) \to \tau_{\geq 1} K(B)$ is a $T(i)$-local equivalence for $1 \leq i \leq n$. For this recall the plus-construction of algebraic $K$-theory which satisfies that $\Omega^\infty \tau_{\geq 1} K(R) \simeq BGL(R)^+$ and that $\Sigma^\infty BGL(R) \to$...
$\Sigma^\infty \text{BGL}(R)^+$ is an equivalence for every connective ring spectrum $R$. From the above we thus deduce that

$$\Sigma^\infty \text{BGL}(A) \cong \Sigma^\infty \Omega^\infty \tau_{\geq 1} K(A) \rightarrow \Sigma^\infty \Omega^\infty \tau_{\geq 1} K(B) \cong \Sigma^\infty \text{BGL}(B)$$

is a $T(i)$-local equivalence for $1 \leq i \leq n$. Now we observe that the canonical composite

$$\Omega^\infty (-) \rightarrow \Omega^\infty \Sigma^\infty \Omega^\infty (-) \rightarrow \Omega^\infty (-)$$

is a natural equivalence. Applying the Bousfield–Kuhn functor $\Phi_i$, and the property that $\Phi_i \circ \Omega^\infty \cong L_{T(i)}$, see [Kuh08, Theorem 1.1], we find that every map $f$ of spectra is $T(i)$-locally a retract of the map $\Sigma^\infty \Omega^\infty f$. Applying this for $f$ being the map $\tau_{\geq 1} K(A) \rightarrow \tau_{\geq 1} K(B)$, the proposition follows.

\begin{remark}
One can sometimes use Theorem A to get consequences also for iterated algebraic $K$-theory: Assume, for instance, that $A$ and $B$ are connective $E^\infty$-ring spectra so that $K(A), K(B)$ are again $E^\infty$-ring spectra. If $A \rightarrow B$ is an $n$-connective $L^f_n$-equivalence, then so is the map $K(A) \rightarrow K(B)$. If we further assume that $K(A)$ and $K(B)$ are connective, then we can apply Theorem A to the latter map and deduce that $K(K(A)) \rightarrow K(K(B))$ is an $L^f_n$-equivalence.
\end{remark}

The following proposition shows that one can omit the knowledge about $T(0)$-localizations, i.e. the rationalizations, if we adjust the connectivity of the map accordingly.

\begin{proposition}
Let $f: A \rightarrow B$ be an $(n+1)$-connective map between connective ring spectra. Assume that $f$ is a $T(i)$-local equivalence for $1 \leq i \leq n$. Then the map $K(A) \rightarrow K(B)$ is a $T(i)$-local equivalence for $1 \leq i \leq n$.
\end{proposition}

\begin{proof}
We consider the diagram

$$
\begin{array}{ccc}
A & \rightarrow & B \\
\downarrow & & \downarrow \\
A_Q & \rightarrow & B_Q \\
\end{array}
$$

where $P$ is defined to be the pullback. First we observe that the diagram

$$
\begin{array}{ccc}
K(P) & \rightarrow & K(B) \\
\downarrow & & \downarrow \\
K(A_Q) & \rightarrow & K(B_Q) \\
\end{array}
$$

is a pullback by [LT19, Main Theorem] since $P \rightarrow A_Q$ is a rationalization. Since $A \rightarrow B$ is a $\pi_0$-isomorphism we find that the relative $K$-theory $K(A_Q, B_Q)$ is itself rational: The map $A_Q \rightarrow B_Q$ is a $\pi_0$-isomorphism and an $S/p$ equivalence for all primes $p$. By [LT19, Lemma 2.4] the same holds true for the map $K(A_Q) \rightarrow K(B_Q)$, which precisely means that any prime acts invertibly on the fibre. In particular we deduce that the fibre vanishes $T(i)$-locally for all $i \geq 1$. From the above pullback, we conclude that that map $K(P) \rightarrow K(B)$ is also a $T(i)$-local equivalence for all $i \geq 1$.

We now aim to show that the map $K(A) \rightarrow K(P)$ is a $T(i)$-local equivalence for all $0 \leq i \leq n$, i.e. an $L^f_n$-equivalence. For this we observe that the map $A \rightarrow P$ is itself an
$L^f_n$-equivalence and $n$-connective by the assumption that $f$ is $(n+1)$-connective. Thus we conclude the lemma from Theorem [A] □

Remark 2.9. The above proof shows that for the conclusion of Proposition [2.8] to hold, it suffices to assume that $f$ is a $T(i)$-local equivalence for $1 \leq i \leq n$, is a $\pi_0$-isomorphism, and that the canonical map from the fibre of $f$ to its rationalisation is $n$-connective.

2.3. Examples. We find the following immediate corollary of Theorem [A]

Corollary 2.10. The spectrum $K(K(m))$ is $L^f_{m-1}$-equivalent to $K(F_p) \oplus \Sigma K(F_p)$. In particular $K(K(m))$ vanishes $T(n)$-locally for $0 < n < m$.

Proof. Using the theorem of the heart [AGH19 Proposition 4.4] (in fact, if $k$), the spectrum $K(K(m))$ vanishes $T(n)$-locally for $0 < n < m$. Upon applying any localizing invariant, this gives the zero map. From (1) we thus obtain a fibre sequence

\[ K(F_p) \to K(k(m)) \to K(K(m)) \]

where $k(m)$ is the connective cover of $K(m)$ and the first map is induced by the functor $\text{Perf}(F_p) \to \text{Perf}(k(m))$ given by restriction of scalars along the canonical map $k(m) \to F_p$. This map is $(2p^m - 2)$-connective and an $L^f_{m-1}$-equivalence by Lemma [2.2]iii). As $2p^m - 2 \geq m > n$, Theorem [A] gives that the map $K(k(m)) \to K(F_p)$ induced by extension of scalars $\text{Perf}(k(m)) \to \text{Perf}(F_p)$ is an $L^f_{m-1}$-equivalence. The composite

\[ K(F_p) \to K(k(m)) \to K(F_p) \]

is induced by the functor $\text{Perf}(F_p) \to \text{Perf}(k(m)) \to \text{Perf}(F_p)$ sending $X$ to $X \otimes k(m) F_p$, which is equivalent to $\text{id} \oplus \Sigma 2p^m-1$ as there is a fibre sequence

\[ \Sigma 2p^m-2 k(m) \to k(m) \to H F_p. \]

Upon applying any localizing invariant, this gives the zero map. From (1) we thus obtain a fibre sequence

\[ L^f_{m-1} K(F_p) \to L^f_{m-1} K(F_p) \to L^f_{m-1} K(K(m)) \]

which shows the first claim. From Quillen’s calculation [Qui72] we know that $K(F_p)_{p} \simeq HZ_p$, so that $K(F_p)$ vanishes $T(n)$-locally for all $n > 0$ by Lemma [2.2]vi),(ii). This implies the second claim. □

The following is an example that arose from a discussion with George Raptis. Recall that for a connected space $X$ its Waldhausen $A$-theory is given by $A(X) = K(S[\Omega X]) = K(\Sigma^\infty \Omega X)$. In particular, $A(*) = K(S)$. In the following proposition we assume that $n \geq 1$; the case $n = 0$ is due to Waldhausen.

Corollary 2.11. Let $W$ be an $(n-1)$-connected space. If $\Sigma^\infty W$ is $L^f_n$-acyclic, then the canonical maps $A(*) \to A(\Sigma W)$ are mutually inverse $L^f_n$-equivalences.

For instance, $W$ could be an $(n-1)$-connected type $m$-complex for $m > n$.

Proof. The James splitting (see [Ada72 Chapter 10, Theorem 5]) gives an equivalence

\[ \Sigma^\infty \Omega \Sigma W \simeq \bigcup_{k \geq 1} W^\wedge k, \]

which implies that $\Sigma^\infty \Omega \Sigma W$ is $L^f_n$-acyclic. Hence, since $\Omega \Sigma W$ is $(n-1)$-connected, the canonical map $S[\Omega \Sigma W] \to S$ is an $n$-connective $L^f_n$-equivalence. The claim thus follows from Theorem [A] □
Likewise, we can reprove and extend (parts of) a recent theorem of Angelini-Knoll and Quigley about the chromatic localization of $K$-theory of certain Thom spectra $y(m)$ considered in [MRS01, Section 3]. To explain the setup, we recall that for a fixed prime $p$, there is an essentially unique map of $E_2$-spaces

$$\Omega^2\Sigma^2 S^1 \to BGL_1(S^\wedge_p)$$

sending a generator of $\pi_1$ to the element $1 - p \in \pi_1(BGL_1(S^\wedge_p)) \cong \mathbb{Z}_p^*$. It is a theorem of Mahowald (for $p = 2$) and Hopkins (for odd primes) that its Thom spectrum is $H\mathbb{F}_p$ [Mah79]. We note that $\Omega^2\Sigma^2 S^1 \simeq \Omega(\Omega S^3)$ and that $\Omega S^3$ has a canonical cell structure with one cell in every even dimension; see [Hil63, Corollary 17.4]. Let us denote by $F_m(\Omega S^3)$ the $2m$-skeleton of this cell structure. One then obtains maps of $E_1$-spaces

$$\Omega F_{p^m-1}(\Omega S^3) \to \Omega^2 S^3 \to BGL_1(S^\wedge_p)$$

whose Thom spectra are denoted by $y(m)$, leaving the prime $p$ implicit as always. One has $y(0) = S^\wedge_p$ and $y(\infty) = H\mathbb{F}_p$. The above filtration of $\Omega S^3$ can also be described as the James filtration on $\Omega \Sigma^2 S^2$, compare [MRS01, Section 3.1].

**Lemma 2.12.** For $m \geq 0$, the map $y(m) \to y(\infty) = H\mathbb{F}_p$ is $(2p^m - 2)$-connective.

**Proof.** For $m = 0$ this is clear. For $m > 0$ we distinguish cases: For odd primes $p$, this is stated in the paragraph preceding [MRS01, Equation 3.7]. For $p = 2$, the map is even $(2p^m - 1)$-connective by [AKQ19, Lemma 2.5]. \hfill \Box

**Lemma 2.13.** The spectrum $y(m)$ is $L^j_{m-1}$-acyclic.

**Proof.** We need to show that $y(m)$ is $T(n)$-acyclic for $n < m$. For $n = 0$ this follows easily from the previous lemma, which shows that $y(m)$ is $p$-torsion, and thus rationally trivial. We now discuss the case where $n > 0$. Again, we distinguish the cases of even and odd primes. For $p = 2$ this follows from [AKQ19, Proposition 2.20] and Lemma 2.12. For odd primes, it is explained in [MRS01] that for a finite type $n$ spectrum $V_n$, the Adams spectral sequence for the spectrum $V_n \otimes y(m)$ has a vanishing line of slope $\frac{1}{2p^{m-1}-2}$, because this is true for $y(m)$. On the other hand, the element $v_n$ acting on $V_n$ gives an element of slope $\frac{1}{|v_n|}$ for the Adams spectral sequence. Hence, if $n < m$, it follows that the element $v_n$ is nilpotent on $V_n \otimes y(m)$, so that $T(n) \otimes y(m)$ vanishes as claimed. \hfill \Box

**Corollary 2.14.** The map $K(y(m)) \to K(\mathbb{F}_p)$ is an $L^j_{m-1}$-equivalence. In particular, $K(y(m))$ vanishes $T(n)$-locally for $0 < n < m$.

**Proof.** By Lemmas 2.12 and 2.13 the map $y(m) \to \mathbb{F}_2$ is $(2^{m+1} - 2)$-connective and an $L^j_{m-1}$-equivalence. The corollary then follows from Theorem A as $n < m \leq 2^{m+1} - 2$ for all $m \geq 1$.

**Remark 2.15.** In [AKQ19] it is shown that in the case where the implicit prime $p$ is 2 the $K(n)$-localization of $K(y(m))$ vanishes for $n \leq m$, assuming a conjecture about a particular compatibility of inverse limits with Morava $K(n)$-homology holds. Our methods do not need such inputs, but we cannot treat the case of $T(m)$-localized $K(y(m))$ as $y(m)$ is not $T(m)$-acyclic.

We obtain a similar result for the integral versions $z(m)$ of $y(m)$ which appear in [AKQ19] when $p = 2$. Again, there are versions for odd primes, but we refrain from spelling them out here.
Corollary 2.16. The map \( K(z(m)) \rightarrow K(Z_{(2)}) \) is a \( T(n) \)-equivalence for \( 0 < n < m \).

Proof. The proof of [AKQ19, Lemma 2.5] shows that the map \( z(m) \rightarrow Z_{(2)} \) is \((2^{m+1} - 2)\)-connective. Furthermore, by [AKQ19, Proposition 2.20], \( z(m) \) is \( K(n) \)-acyclic for \( 1 \leq n < m \), and hence also \( T(n) \)-acyclic for \( 1 \leq n < m \), again by Lemma 2.3. We can thus apply Proposition 2.8 since \( 2^{m+1} - 2 \geq m \geq n + 1 \).

2.4. Applications in height 1. In this subsection we study implications of the main theorem for \( K(1) \)-local \( K \)-theory, in particular proving Theorem B. We start with the following lemma.

Lemma 2.17. Let
\[
\begin{array}{ccc}
A & \rightarrow & B \\
\downarrow & & \downarrow \\
A' & \rightarrow & B'
\end{array}
\]
be a pullback square of ring spectra in which \( A \) is connective. If \( A \rightarrow A' \) is \( n \)-connective and \( A \rightarrow B \) is \( m \)-connective, then the induced map \( A' \odot_A^B B \rightarrow B' \) is \((m + n + 2)\)-connective.

Here \( A' \odot_A^B B \) denotes the ring spectrum associated to the displayed pullback square by [LT19, Main Theorem].

Proof. Denote by \( I \) the common fibre of the vertical maps, by \( J \) that of the horizontal maps. From [LT19, Remark 1.16] we have an equivalence
\[
\text{fib}(A' \odot_A^B B \rightarrow B') \simeq \Sigma \text{fib}(I \otimes_A B \xrightarrow{\mu} I)
\]
where the map \( \mu \) is induced by the right \( B \)-module structure on \( I \). Now \( \mu \) has a section \( \sigma : I \rightarrow I \otimes_A B \) induced by the map \( A \rightarrow B \). The fibre of \( \sigma \) identifies with \( I \otimes_A J \) which is \((n + m)\)-connective as \( A \) is connective. In other words, \( \sigma \) is an isomorphism in degrees \( \leq m + n - 1 \) and surjective in degree \( m + n \). Since \( \mu \circ \sigma \simeq \text{id}_I \) and so \( \mu \) is surjective in every degree, it then follows that \( \mu \) is an isomorphism in degrees \( \leq m + n \) and surjective in degree \( m + n + 1 \), i.e. \( \mu \) is \((m + n + 1)\)-connective. By the above equivalence, \( A' \odot_A^B B \rightarrow B' \) is then \((m + n + 2)\)-connective, as desired.

We are now able to prove the first part of Theorem B from the introduction. While our proof gives actually a \( T(1) \)-local result, we will phrase the statement in terms of the better known \( K(1) \). By the proven telescope conjecture at height 1 this is no loss of information.

Theorem 2.18. \( K(1) \)-local \( K \)-theory is truncating on \( K(1) \)-acyclic ring spectra. In other words: The canonical map \( K(A) \rightarrow K(\pi_0(A)) \) is a \( K(1) \)-local equivalence for every \( K(1) \)-acyclic, connective ring spectrum \( A \).

Proof. Let \( A \) be a connective ring spectrum which is \( K(1) \)-acyclic and first consider the map \( A \rightarrow \tau_{\leq 1} A \). This map is \( 2 \)-connective and a \( K(1) \)-local equivalence, hence the map \( K(A) \rightarrow K(\tau_{\leq 1} A) \) is again a \( K(1) \)-local equivalence by Proposition 2.8. We may thus assume that \( A \) is itself \( 1 \)-truncated. In this case, \( A \rightarrow \pi_0(A) \) is a square-zero extension with fibre \( \pi_1(A)[1] \) and thus there is a pullback square of \( \mathbb{E}_1 \)-ring spectra
\[
\begin{array}{ccc}
A & \rightarrow & \pi_0(A) \\
\downarrow & & \downarrow \\
\pi_0(A) & \rightarrow & \pi_0(A) \oplus \pi_1(A)[2];
\end{array}
\]
see [Lur17, 7.4.1.29]. For ease of notation we denote the ring in the lower right corner by $A'$.

It follows from Lemma 2.17 that the map

$$
\pi_0(A) \odot_A^{A'} \pi_0(A) \rightarrow \pi_0(A)
$$

is 2-connective. It is also a $K(1)$-local equivalence as both spectra are $K(1)$-acyclic: The ring $\odot$ is a module over the $K(1)$-acyclic ring $\pi_0(A)$. From Proposition 2.8 we then deduce that the map

$$
K(\pi_0(A) \odot_A^{A'} \pi_0(A)) \rightarrow K(\pi_0(A))
$$

is a $K(1)$-local equivalence. By [LT19, Main Theorem] the following is a pullback square:

$$
\begin{array}{ccc}
K(A) & \rightarrow & K(\pi_0(A)) \\
\downarrow & & \downarrow \\
K(\pi_0(A)) & \rightarrow & K(\pi_0(A) \odot_A^{A'} \pi_0(A))
\end{array}
$$

Since the composite

$$
K(\pi_0(A)) \rightarrow K(\pi_0(A) \odot_A^{A'} \pi_0(A)) \rightarrow K(\pi_0(A))
$$

is an equivalence, we deduce from the above that the lower horizontal map in the above square is a $K(1)$-local equivalence. Thus so is the upper horizontal map as claimed.

**Corollary 2.19.** $K(1)$-local $K$-theory is nilinvariant: Let $A$ be a discrete ring and $I \subseteq A$ a nilpotent two-sided ideal. Then the map $K(A) \rightarrow K(A/I)$ is a $K(1)$-local equivalence.

**Proof.** Since $K(1)$-local $K$-theory is truncating on $K(1)$-acyclic ring spectra, it is in particular truncating on $HZ$-algebras. By [LT19, Corollary 3.5], it is then also nilinvariant. □

In particular, we obtain the following result, which was first obtained by Bhatt–Clausen–Mathew using different methods. We will discuss consequences of this result in Section 3.

**Corollary 2.20.** The $K(1)$-local $K$-theory of $\mathbb{Z}/p^n$ vanishes for every $n \geq 1$. More generally, $L_{K(1)}K(A) = 0$ if $A$ is a $K(1)$-acyclic and $p^\infty$-torsion ring spectrum which is connective or $E_2$.

**Proof.** From Quillen’s calculation [Qui72] we know that $K(F_p)^\wedge \simeq H\mathbb{Z}_p$. Hence

$$
L_{K(1)}K(\mathbb{Z}/p^n) \simeq L_{K(1)}K(F_p) \simeq L_{K(1)}H\mathbb{Z}_p = 0
$$

by Lemma 2.2(vi), Corollary 2.19 and since bounded above spectra are $K(1)$-acyclic. For connective $A$, the second assertion follows since $L_{K(1)}K(A) \simeq L_{K(1)}K(\pi_0(A))$ by Theorem 2.18 and $\pi_0(A)$ is a $\mathbb{Z}/p^n$-algebra for a suitable $n$. If $A$ is $E_2$, it follows that $A$ is an $E_1$-algebra over its connective cover $\tau_{\geq 0}A$ so that $L_{K(1)}K(A)$ is a module over $L_{K(1)}(\tau_{\geq 0}A)$. However, $L_{K(1)}(\tau_{\geq 0}A) = 0$ by the above, as $\tau_{\geq 0}A$ is again $K(1)$-acyclic by Lemma 2.2(ii). □

From [LT19, Theorem 3.3 & Theorem A.2], we also find the following consequence. We refer to Remark 3.2 and Appendix A of [LT19] for the precise meaning of excision and cdh-descent.

**Corollary 2.21.** $K(1)$-local $K$-theory satisfies excision for Milnor squares of discrete rings and cdh-descent.
2.5. Applications in higher height. We now investigate the behaviour of $T(n)$-local $K$-theory for higher heights $n$. First, we prove the following theorem at height 2.

**Theorem 2.22.** $T(2)$-local $K$-theory is truncating on $T(1) \oplus T(2)$-acyclic ring spectra. In other words: $T(2)$-local $K$-theory vanishes on connective ring spectra which are $T(1) \oplus T(2)$-acyclic.

**Proof.** Let $A$ be a connective and $T(1) \oplus T(2)$-acyclic ring spectrum. By Proposition 2.8 the map $A \to \tau_{\leq 2}A$ induces an equivalence in $T(2)$-local $K$-theory. We now show that also the square zero extension $\tau_{\leq 2}A \to \tau_{\leq 1}A$ induces an equivalence in $T(2)$-local $K$-theory. We consider the pullback diagram

$$
\begin{array}{ccc}
\tau_{\leq 2}A & \longrightarrow & \tau_{\leq 1}A \\
\downarrow & & \downarrow \\
\tau_{\leq 1}A & \longrightarrow & \tau_{\leq 1}A \oplus \pi_2(A)[3]
\end{array}
$$

and let $\odot$ be the ring provided by [LT19, Main Theorem]. We observe that $\odot$ is $T(1) \oplus T(2)$-acyclic as it is a module over $\tau_{\leq 1}A$. Furthermore, it follows from Lemma 2.17 that the map $\odot \to \tau_{\leq 1}A$ is 3-connective, and thus that the map $K(\odot) \to K(\tau_{\leq 1}A)$ is a $T(2)$-local equivalence by Proposition 2.8. Hence, in the pullback diagram

$$
\begin{array}{ccc}
K(\tau_{\leq 2}A) & \longrightarrow & K(\tau_{\leq 1}A) \\
\downarrow & & \downarrow \\
K(\tau_{\leq 1}A) & \longrightarrow & K(\odot)
\end{array}
$$

the lower horizontal map is a $T(2)$-local equivalence, as it has a section which is a $T(2)$-local equivalence. Thus $K(\tau_{\leq 2}A) \to K(\tau_{\leq 1}A)$ is also a $T(2)$-local equivalence.

Finally, we claim that the map $K(\tau_{\leq 1}A) \to K(\pi_0(A))$ is a $T(2)$-local equivalence. We first argue that this is the case if $\pi_1(A) \otimes \mathbb{Q} = 0$. To this end, we consider the pullback diagram

$$
\begin{array}{ccc}
\tau_{\leq 1}A & \longrightarrow & \pi_0(A) \\
\downarrow & & \downarrow \\
\pi_0(A) & \longrightarrow & \pi_0(A) \oplus \pi_1(A)[2]
\end{array}
$$

and again consider the associated ring $\odot$. As before, the map $\odot \to \pi_0(A)$ is 2-connective. It is also a rational equivalence since by assumption $\tau_{\leq 1}A \to \pi_0(A)$ is a rational equivalence. Applying Theorem A we find that the map $K(\odot) \to K(\pi_0(A))$ is a $T(2)$-local equivalence. Using Mitchell’s result [Mit90] we deduce that $L_{T(2)}K(\tau_{\leq 1}A) = 0$ provided $\pi_1(A) \otimes \mathbb{Q} = 0$. In particular, this vanishing holds for the 1-truncated sphere spectrum $\tau_{\leq 1}S$. As any 1-truncated connective ring spectrum is a $\tau_{\leq 1}S$-algebra, it follows that $L_{T(2)}K(\tau_{\leq 1}A) = 0$ without any additional hypothesis. Together with the first part of the proof we have finally shown that $L_{T(2)}K(A)$ vanishes. \qed

**Remark 2.23.** It is not true that $T(n)$-local $K$-theory is truncating on $T(n)$-acyclic ring spectra: For example, $ku$ is $T(2)$-acyclic, but $L_{T(2)}K(ku)$ does not vanish, at least if $p \geq 5$ by [Aus10, Theorem 1.1], whereas $L_{T(2)}K(\mathbb{Z}) = 0$ by [Mit90].

Using a result of Hahn we can now prove the following result.

**Theorem 2.24.** The $K$-theory of the truncated sphere $\tau_{\leq m}S$ vanishes $T(n)$-locally for all $n \geq 2$. 

Proof. Theorem 2.22 implies the case \( n = 2 \). In particular, we have \( L_{K(2)}(K_{\leq m}S) = 0 \) for any integer \( m \geq 0 \). Since \( \tau_{\leq m}S \) is an \( E_\infty \)-ring spectrum, so is \( K(\tau_{\leq m}S) \). From [Hah16] Theorem 1.1 it then follows that \( L_{K(n)}(K_{\leq m}S) = 0 \) for all \( n \geq 2 \). Again using that \( K(\tau_{\leq m}S) \) is a ring spectrum, we find that it is also \( T(n) \)-acyclic for all \( n \geq 2 \) by Lemma 2.3.

Remark 2.25. Using arguments similar to the ones in the proof of Theorem 2.22 one can show that \( L_{T(n)}(K_{\leq m}S) = 0 \) for all \( n \) such that \( 4p - 4 \geq n \), where \( p \) is the implicit prime in \( T(n) \), without alluding to [Hah16]. This is due to Ben Antieau [Ant19]. He also conjectured that Theorem 2.21 should be true.

Corollary 2.26. Let \( A \) be a bounded above ring spectrum which is connective or \( E_2 \), and let \( n \geq 2 \). Then \( L_{T(n)}(K(A)) = 0 \).

Proof. Any connective, bounded above ring spectrum is an algebra over \( \tau_{\leq m}S \) for some \( m \).

We thus conclude from Theorem 2.21. If \( A \) is \( E_2 \), then it is an algebra over its connective cover, so the result also follows.

Corollary 2.27. For all \( n \geq 1 \), \( T(n) \)-local \( K \)-theory is truncating on \( T(1) \oplus \cdots \oplus T(n) \)-acyclic ring spectra.

For \( n \geq 2 \), an equivalent statement is that \( T(n) \)-local \( K \)-theory vanishes on connective, \( T(1) \oplus \cdots \oplus T(n) \)-acyclic ring spectra. Consequently, for \( n \geq 2 \), \( T(n) \)-local \( K \)-theory vanishes on \( T(1) \oplus \cdots \oplus T(n) \)-acyclic \( E_2 \)-ring spectra.

Proof. The equivalent \( T(1) \)-local version of Theorem 2.18 provides the case \( n = 1 \). For \( n \geq 2 \), let \( A \) be connective and \( T(i) \)-acyclic for \( 1 \leq i \leq n \). Then the map \( K(A) \to K(\tau_{\leq n}A) \) is a \( T(n) \)-local equivalence by Proposition 2.8. Hence we conclude by Corollary 2.26.

Remark 2.28. The conclusion of the above corollary also holds for \( n = \infty \), where we interpret \( T(\infty) = HF_p \). A connective ring spectrum \( A \) is \( HF_p \)-acyclic if and only if it is \( S/p \)-acyclic, in other words if it is an algebra over \( S[p^{-1}] \).

By Waldhausen’s result \( p \)-adic \( K \)-theory is truncating on \( S[p^{-1}] \)-algebras. Since \( H_{HF_p}K(A) \simeq H_{HF_p}K(A)_p \), also \( HF_p \)-local \( K \)-theory is truncating on \( S[p^{-1}] \)-algebras. This fits well with the formulation of Corollary 2.27 once we observe that a connective ring which is \( HF_p = T(\infty) \)-acyclic is also \( T(n) \)-acyclic for all \( n \geq 1 \) by Lemma 2.22(v).

For \( E_\infty \)-rings we can say more and find the following redshift phenomenon:

Corollary 2.29. Let \( A \) be an \( E_\infty \)-ring spectrum. If \( A \) is \( T(1) \)-acyclic, then \( K(A) \) vanishes \( T(n) \)-locally for every integer \( n \geq 2 \). If \( A \) is \( S\langle 1/p \rangle \)-acyclic, then \( K(A) \) vanishes \( T(n) \)-locally for every integer \( n \geq 1 \).

The same vanishing results thus also hold for an \( E_1 \)-algebra over a \( T(1) \)-acyclic, respectively \( S\langle 1/p \rangle \)-acyclic \( E_\infty \)-ring, as its \( T(n) \)-local \( K \)-theory is a module over 0.

Proof. Since every \( E_\infty \)-ring spectrum is an algebra over its connective cover, it suffices to prove the claim in the case where \( A \) is connective. We use here that if a spectrum is \( T(1) \)-respectively \( S\langle 1/p \rangle \)-acyclic, then so is its connective cover; see Lemma 2.2(ii). If \( A \) is \( T(1) \)-acyclic, then it is also \( T(i) \)-acyclic for all \( i \geq 1 \) by [Hah16] and Lemma 2.3. So we deduce the corollary in this case from Corollary 2.27 and Mitchell's theorem [Mit90]. If \( A \) is \( S\langle 1/p \rangle \)-acyclic, then it is \( T(i) \)-acyclic for all \( i \geq 0 \) by [MNN15, Proposition 4.2]. We conclude by Corollary 2.29 and the previous case.
Remark 2.30. Similarly as in Remark 2.7, one can also say something for $i$-fold iterated algebraic $K$-theory $K^{(i)}$. For example, the canonical maps $K^{(i)}(\mathbb{Z}/p^k) \to K^{(i-1)}(\mathbb{Z})$, where the latter is induced by the truncation map $K(\mathbb{Z}/p^k) \to \mathbb{Z}$, are $L_f^2$-equivalences for all $i \geq 1$. Indeed, the case $i = 1$ follows directly from Corollary 2.29 and the fact that $K_j(\mathbb{Z}/p^k) \otimes \mathbb{Q} = 0$ for $j > 0$ and the non-positive $K$-groups of a ring are invariant under quotients by a nilpotent ideal, as non-positive $K$-theory is truncating by [BGT13, Theorem 9.53].

As in the proof of Theorem A, the map $K(K(\mathbb{Z}/p^k)) \to K(\mathbb{Z})$ is 2-connective and in particular $K(K(\mathbb{Z}/p^k))$ is again connective. Inductively, the spectra $K^{(i)}(\mathbb{Z}/p^k)$ and $K^{(i)}(\mathbb{Z})$ are connective for every $i$. Now Theorem A and Corollary 2.29 imply that the map $K(K(\mathbb{Z}/p^k)) \to K(\mathbb{Z})$ is a $L_f^n$-equivalence for all $n$. Using Remark 2.7, the claim follows for all $i \geq 2$.

We finish this subsection with the observation that the assumption on the connectivity of the map $f$ in Theorem 2.6 is not in general necessary.

Lemma 2.31. Let $f: A \to B$ be a map between connective $\mathbb{H}Z$-algebras which induces an isomorphism on $\pi_0$ and is a rational equivalence. Then $K(A) \to K(B)$ is a $T(i)$-local equivalence for all $i \geq 0$.

Proof. For $i = 0$, this is classical and due to Waldhausen. For $i = 1$ we consider the commutative diagram

$$
\begin{array}{ccc}
L_{T(1)}K(A) & \longrightarrow & L_{T(1)}K(B) \\
\downarrow & & \downarrow \\
L_{T(1)}K(\pi_0(A)) & \longrightarrow & L_{T(1)}K(\pi_0(B))
\end{array}
$$

where the vertical maps are equivalences by Theorem 2.18 and the lower horizontal map is an equivalence by assumption. Finally, if $i \geq 2$, then both $K(A)$ and $K(B)$ vanish $T(i)$-locally by Mitchell’s result [Mit90].

2.6. Applications to TC. Most of the results of Section 2 hold true if we replace algebraic $K$-theory by topological cyclic homology.

Theorem 2.32. Let $n \geq 1$ and let $f: A \to B$ be an $n$-connective $L_f^n$-equivalence between connective ring spectra. Then the induced map $TC(A) \to TC(B)$ is again an $L_f^n$-equivalence.

Proof. By the Dundas–Goodwillie–McCarthy theorem [DGM13, Theorem VII.0.0.2], there is a cartesian square

$$
\begin{array}{ccc}
K(A) & \longrightarrow & TC(A) \\
\downarrow & & \downarrow \\
K(B) & \longrightarrow & TC(B).
\end{array}
$$

is $L_f^n$ preserves cartesian squares, Theorem A implies the result.

Likewise, the Dundas–Goodwillie–McCarthy theorem also implies that the analogues of Proposition 2.8 and Remark 2.9 still hold true for topological cyclic homology.

With the single exception of Corollary 2.10 (which additionally relies on the theorem of the heart), all other results about $K$-theory in this section are formal consequences of our main theorem, its variant Proposition 2.8 and the following three facts:
(1) $K$-theory of ring spectra extends to a localizing invariant $\text{Cat}^\text{perf}_\infty \to \text{Sp}$ from the $\infty$-category of small, stable, idempotent complete $\infty$-categories in the sense of \cite{LT19} Definition 1.2. This is necessary to apply the results of \cite{LT19}.

(2) The spectrum $K(\mathbb{F}_p)$ is $T(i)$-acyclic for $i \geq 1$. (Quillen)

(3) The spectrum $K(\mathbb{Z})$ is $T(i)$-acyclic for $i \geq 2$. (Mitchell)

The corresponding facts are also true for topological cyclic homology.

(1) Blumberg, Gepner, and Tabuada establish in \cite[Section 10]{BGT13} that the $\text{TC}^n$ are localizing invariants and hence also their limit $\text{TC}^\infty$.

(2) Blumberg, Gepner, and Tabuada define in \cite[Theorems 6.14 and 7.4]{BGT13} for every $E_\infty$-ring spectrum $R$ a map $K(R) \to \text{TC}(R)$ of $E_\infty$-ring spectra. In particular, we obtain for every $i \geq 1$ an $E_\infty$-map

$$0 = L_{T(i)}K(\mathbb{F}_p) \to L_{T(i)}\text{TC}(\mathbb{F}_p),$$

implying also the vanishing of the target.

(3) The argument for the vanishing of $L_{T(i)}\text{TC}(\mathbb{Z})$ for $i \geq 2$ is analogous.

Thus, we can replace algebraic $K$-theory by topological cyclic homology in all results of this section, with the exception of Corollary \ref{3.20}.

3. A PURITY PROPERTY FOR TELEScopICALLY LOCALIZED ALGEBRAIC $K$-THEORY

In this section we will discuss a purity property of algebraic $K$-theory of ring spectra, viewed as sheaves on the Balmer spectrum of finite spectra. To put this into context, we first recall the result of Bhatt–Clausen–Mathew that $L_{K(1)}K(\mathbb{Z}/p^n) = 0$ for every $n \geq 0$ from Corollary \ref{3.20}. From this vanishing, Bhatt–Clausen–Mathew deduce that the canonical map

$$L_{K(1)}K(A) \to L_{K(1)}K(A[\frac{1}{p}])$$

is an equivalence for every $HZ$-algebra $A$ and we will give their argument below; see Corollary \ref{3.25}. We aim to find a similar statement that applies to more general ring spectra. Geometrically, we may think of $A$ as a sheaf on $\text{Spec}(\mathbb{Z})$ and the operation $A \to A[\frac{1}{p}]$ corresponds to restricting this sheaf to the basic open subset $D(p) = \text{Spec}(\mathbb{Z}) \setminus \{(p)\}$. To obtain a more general version for ring spectra, we consider the Balmer spectrum $\text{Sp}(\text{Sp}^\omega)$ of the category $\text{Sp}^\omega$ of finite spectra; see Figure \ref{1}.

There is one generic point and for every prime number $p$, there is a family of points $x_{p,n}$ indexed by $n \in \mathbb{N} \cup \{\infty\}$ with $x_{p,n}$ specializing to $x_{p,n+1}$. One can think of the $n$-th Morava $K$-theory $K(n)$ at the prime $p$ as residue field of $x_{p,n}$. Here $K(\infty, p) = HE_p$. The basic open sets are given by the complement $D(p,n)$ of a column of the form $\{x_{p,m} | m \geq n\}$ for some $n \geq 1$. We may again think of a spectrum as a sheaf on $\text{Sp}(\text{Sp}^\omega)$. In this picture, restriction to the open subset $D(p,n+1)$ is given by the localization functor $L_{\omega}^{p,f}$ defined as follows.

**Definition 3.1.** We denote by $L_{\omega}^{p,f}$ the Bousfield localization at $\mathbb{S}[\frac{1}{p}] \oplus T(1) \oplus \cdots \oplus T(n)$.

Write $\mathcal{C}_{>n}$ for the $\infty$-category of $p$-local, finite spectra which are of type greater than $n$, i.e. which are $K(0) \oplus \cdots \oplus K(n)$-acyclic.

**Lemma 3.2.** The category of $L_{\omega}^{p,f}$-acyclic spectra coincides with $\text{Ind}(\mathcal{C}_{>n})$. Hence, $L_{\omega}^{p,f}$ is a finite and thus smashing localization.\footnote{Note that in contrast to \cite{BGT13}, we do not require that our localizing invariants commute with filtered colimits.}
In particular, \( L^p_f \) is the identity functor and \( L^p_0 = L_{\mathbb{Z}[1/p]} \).

**Proof.** The Bousfield class \( \langle S_{\mathbb{Z}[1/p]} \oplus T(1) \oplus \cdots \oplus T(n) \rangle \) has as complement the Bousfield class \( \langle \Sigma V_{n+1} \rangle \) of a type \((n+1)\)-spectrum: every spectrum is acyclic for \( S_{\mathbb{Z}[1/p]} \oplus T(1) \oplus \cdots \oplus T(n) \oplus \Sigma V_{n+1} \)-acyclic. Indeed, this follows easily from the inductive construction of a type \((k+1)\)-complex as \( V_k/v \), where \( V_k \) is a type \( k \)-complex with \( v_k \)-self map \( v \). It follows from [MS95, Proposition 3.3] that every \( L^p_f \)-acyclic spectrum is a colimit of finite \( L^p_f \)-acyclic spectra. The thick subcategory theorem implies that the category of finite \( L^p_f \)-acyclic spectra is precisely \( C_{>n} \). \( \square \)

**Lemma 3.3.** For integers \( 0 \leq m < n \) and a spectrum \( X \) there is a pullback diagram

\[
\begin{array}{ccc}
L^p_f X & \longrightarrow & L_{T(m+1) \oplus \cdots \oplus T(n)} X \\
\downarrow & & \downarrow \\
L^p_m X & \longrightarrow & L^p_m L_{T(m+1) \oplus \cdots \oplus T(n)} X
\end{array}
\]

natural in \( X \).

**Proof.** This is a special case of the following well known lemma. \( \square \)

**Lemma 3.4.** Let \( E \) and \( F \) be spectra. Assume that \( L_E \) preserves \( F \)-acyclic spectra. Then there is a pullback diagram

\[
\begin{array}{ccc}
L_{E \oplus F} X & \longrightarrow & L_F X \\
\downarrow & & \downarrow \\
L_E X & \longrightarrow & L_E L_F X
\end{array}
\]

natural in \( X \).

We note that the assumptions of the lemma are for instance satisfied if \( L_E \) is smashing, or if \( L_F \) annihilates \( E \)-local objects.

**Proof.** Denote the pullback of the diagram \( L_E X \to L_E L_F X \leftarrow L_F X \) by \( P(X) \). There is a canonical map \( X \to P(X) \); it is easy to show that this map is an \((E \oplus F)\)-local equivalence, and that \( P(X) \) is \((E \oplus F)\)-local. \( \square \)
Proof. We consider the following commutative diagram.

(i) for $p$-local spectra $X$, the canonical map $L_n^{p,f}X \to L_n^i X$ is an equivalence, and
(ii) for $T(1)$-acyclic spectra $X$, the canonical map $L_n^{p,f}X \to X[1/p]$ is an equivalence.

As $HZ$-algebras are $T(1)$-acyclic, we can rewrite the Bhatt–Clausen–Mathew equivalence as $L_{T(1)}K(A) \simeq L_{T(1)}K(L_n^{p,f}A)$. We ask:

**Question 3.5.** Given a ring spectrum $A$ and an integer $n \geq 1$, is the canonical map

$$K(A) \to K(L_n^{p,f}A)$$

a $T(i)$-local equivalence for $1 \leq i \leq n$?

**Remark 3.6.** If $A$ is $T(i)$-acyclic for $i > n$, we have $L_m^{p,f}A \simeq L_n^{p,f}A$ for every $m \geq n$. Thus, in this case the question is asking whether $K(A) \to K(L_n^{p,f}A)$ a $T(i)$-equivalence for all $i \geq 1$.

The remainder of this paper is devoted to a discussion of Question 3.5. We will give an affirmative answer for Morava $K$-theories (Proposition 3.8), $HZ$-algebras and $K(1) \oplus \cdots \oplus K(n)$-acyclic connective ring spectra (Corollary 3.29), and $ko$-algebras (Corollary 3.33). Finally, we give an affirmative answer for $tmf$-algebras for $n \geq 2$ (Corollary 3.40). We start by listing some immediate examples for Question 3.5.

**Example 3.7.** Question 3.5 has an affirmative answer for

(i) $L_n^{p,f}$-local ring spectra. Examples include the Lubin–Tate, Johnson–Wilson, or Morava $E$-theory spectra at heights $m$, the Morava $K$-theory spectra $K(m)$, and the telescopes $T(m)$ for $0 \leq m \leq n$.

(ii) $L_n^{p,f}$-acyclic, connective ring spectra $A$.

Part (i) is obvious. To see (ii), note that for $1 \leq i \leq n$, we have $L_{T(i)}K(A) \simeq L_{T(i)}K(\pi_0(A))$ by Corollary 2.27. As $A$ is also $S[\frac{1}{p}]$-acyclic, $\pi_0(A)$ is a $\mathbb{Z}/p^n$-algebra for some $n$. We thus have

$$L_{T(i)}K(A) \simeq L_{T(i)}K(\pi_0(A)) \simeq 0$$

where the second equivalence holds by Corollary 2.20 for $i = 1$ and by Mitchell’s result for $i \geq 2$. Hence both, source and target of the map in Question 3.5 vanish.

In fact, Question 3.5 has an affirmative answer for the Morava $K$-theories $K(m)$ and their connective covers $k(m)$ for all $m$.

**Proposition 3.8.** For every pair of integers $m, n \geq 1$, the map

$$L_{T(i)}K(k(m)) \to L_{T(i)}K(L_n^{p,f}k(m))$$

is an equivalence for $1 \leq i \leq n$. The same is true if $k(m)$ is replaced by $K(m)$.

**Proof.** We consider the following commutative diagram.

$$
\begin{array}{c}
L_{T(i)}K(k(m)) \\
\simeq \\
L_{T(i)}K(k(m))
\end{array} 
\begin{array}{c}
\to \\
\to
\end{array} 
\begin{array}{c}
L_{T(i)}K(L_n^{p,f}k(m)) \\
\simeq \\
L_{T(i)}K(L_n^{p,f}k(m))
\end{array}
$$

The left vertical map is an equivalence because its fibre is given by $L_{T(i)}K(\mathbb{F}_p)$ (and $i$ is at least 1) and the right vertical map is an equivalence because the map $k(m) \to K(m)$ is an
$L_n^{p,f}$-equivalence for all $n$ as its fibre is bounded above and $p$-torsion. It hence suffices to argue that the lower horizontal map is an equivalence. If $m \leq n$ this is Example 3.7(i) and if $m > n$ it follows from Corollary 2.10.

We now note some equivalent formulations of Question 3.5.

**Lemma 3.9.** For a fixed integer $n \geq 1$ and $\mathbb{E}_1$-ring $A$, Question 3.5 has an affirmative answer for $A$ if and only if it does for the $p$-localization $A_{(p)}$ if and only if it does for the $p$-completion $A_p$.  

**Proof.** As $A$ and $A_{(p)}$ have the same $p$-completion, it suffices to prove the claim for the $p$-completion. Applying Lemma 3.4 to $S/p$ and $S[p] \bigoplus T(1) \oplus \cdots \oplus T(n)$ and using that $L_{S/p\otimes S[p]}$ is the identity functor, we obtain the following pullback diagram of $\mathbb{E}_1$-ring spectra.

\[
\begin{array}{ccc}
A & \longrightarrow & A_p \\
\downarrow & & \downarrow \\
L_n^{p,f}A & \longrightarrow & L_n^{p,f}A_p
\end{array}
\]

Since $L_n^{p,f}$ is smashing, this pullback diagram is Tor-independent and hence induces a pullback diagram upon applying $K$-theory [LT19, Corollary 1.4]. The claim follows from this.

**Lemma 3.10.** Question 3.5 has an affirmative answer if and only if the canonical map $L_n^{p,f}K(A) \rightarrow L_n^{p,f}K(L_n^{p,f}A)$ is a $p$-adic equivalence.

**Proof.** This follows immediately from the observation that for any spectrum $X$, the map $L_n^{p,f}X \rightarrow L_{T(1)\oplus \cdots \oplus T(n)}X$ is a $p$-completion, which in turn follows from Lemma 3.9 and the fact that the target is $p$-complete by Lemma 2.2(v).

**Proposition 3.11.** Question 3.5 has an affirmative answer for every $n \geq 1$ and every ring spectrum $A$ if and only if, for every $n \geq 1$ and every $L_n^{p,f}$-acyclic ring spectrum $A$, the $K$-theory $K(A)$ vanishes $T(n)$-locally.

**Proof.** The “only if”-statement is clear. For the converse, we consider a ring spectrum $A$. We note that Lemma 2.2 and the Thomason–Neeman localization theorem [Nee92, Theorem 2.1] imply that the sequence of small stable $\infty$-categories

\[C_{>n} \longrightarrow \text{Perf}(S) \longrightarrow \text{Perf}(L_n^{p,f}S)\]

is exact. Tensoring the above exact sequence with $\text{Perf}(A)$, using the fact that $L_n^{p,f}$ is smashing, we obtain the exact sequence

\[C_{>n} \otimes \text{Perf}(A) \longrightarrow \text{Perf}(A) \longrightarrow \text{Perf}(L_n^{p,f}A),\]

and $C_{>n} \otimes \text{Perf}(A)$ can be identified with the $\infty$-category of $L_n^{p,f}$-acyclic perfect $A$-modules. Since $C_{>n} = (\Sigma^\infty V_{n+1})$ is generated by $\Sigma^\infty V_{n+1}$, we deduce that $C_{>n} \otimes \text{Perf}(A)$ is generated by the perfect $A$-module $V_{n+1} \otimes A$. The Schwede–Shipley theorem [Lau17, Theorem 7.1.2.1] then asserts that $C_{>n} \otimes \text{Perf}(A) \simeq \text{Perf}(\text{End}_A(V_{n+1} \otimes A))$. We now observe that $\text{End}_A(V_{n+1} \otimes A) \simeq \text{Hom}_S(\Sigma^\infty V_{n+1}, V_{n+1} \otimes A) \simeq DV_{n+1} \otimes V_{n+1} \otimes A$ is an $L_n^{p,f}$-acyclic ring spectrum and notice that it is in particular $L_n^{p,f}$-acyclic for $1 \leq i \leq n$. So its $T(i)$-local $K$-theory vanishes by assumption. Since $T(i)$-local $K$-theory is localizing, the result follows from the above exact sequence.
3.1. $K$-theory and localizations away from an ideal. To approach Question 3.11 we would like to apply the vanishing results of the previous section to the fibre of the map $K(A) \to K(L_n f A)$, at least in specific examples. To do so we will use an approximation argument that we learned from Dustin Clausen and Akhil Mathew. In the simplest case, namely that of an $H\mathbb{Z}$-algebra $A$, one has to show that $K(A) \to K(A[1/p])$ is a $K(1)$-equivalence. Let us briefly sketch a direct proof of this fact. One can identify the fibre of the canonical map $\text{Perf}(\mathbb{Z}) \to \text{Perf}((\mathbb{Z})[1/p])$ with the colimit of the $\infty$-categories $\text{Mod}_{\mathbb{Z}/p^k}((\text{Perf}(\mathbb{Z}))$, i.e. of perfect $\mathbb{Z}$-modules equipped with the structure of a $\mathbb{Z}/p^k$-module. Tensoring over $\text{Perf}(\mathbb{Z})$ with $\text{Perf}(A)$, we obtain an exact sequence

$$\text{colim}_k \text{Perf}(A) \otimes_{\mathbb{Z}} \text{Mod}_{\mathbb{Z}/p^k}((\text{Perf}(\mathbb{Z})) \to \text{Perf}(A) \to \text{Perf}(A[1/p]).$$

The $K$-theory of $\text{Perf}(A) \otimes_{\mathbb{Z}} \text{Mod}_{\mathbb{Z}/p^k}((\text{Perf}(\mathbb{Z}))$ is a module over $K(\mathbb{Z}/p^k)$ and thus vanishes $K(1)$-locally by Corollary 2.20. As $K$-theory commutes with filtered colimits, this implies that $K(A) \to K(A[1/p])$ is a $K(1)$-equivalence.

There are technical problems with this approach if we replace $H\mathbb{Z}$ by other $\mathbb{E}_\infty$-ring spectra $R$. The most severe of these is that in general a quotient of $S/R$ does not need to possess any ring structure and even under favourable circumstances has usually at most an $\mathbb{E}_1$-ring structure. As the $K$-theory of an $\mathbb{E}_1$-ring has no natural ring structure, we cannot directly repeat the argument above. As a substitute, we use an argument based on a cobar construction indicated to us by Dustin Clausen and Akhil Mathew and that we explain next.

Let $R \to S$ be a map of $\mathbb{E}_\infty$-ring spectra. Then we can form the corresponding cobar construction $CB\Sigma_{S/R}^\bullet$ i.e. the cosimplicial $\mathbb{E}_\infty$-ring spectrum given by $S^\otimes_R(\bullet+1)$. See [MNN17, Construction 2.7] for a detailed construction. We write $\text{Tot}_m(CB\Sigma_{S/R}^\bullet)$ for the limit of the restriction $CB\Sigma_{S/R}|_{\Delta^m}$. For varying $m$, we obtain the tower $\{\text{Tot}_m(CB\Sigma_{S/R}^\bullet)\}_{m \geq 0}$ of $\mathbb{E}_\infty$-rings.

We will always assume that $S$ is perfect as an $R$-module. Then each finite totalization $\text{Tot}_m(CB\Sigma_{S/R})$ is an $\mathbb{E}_\infty$-$R$-algebra, perfect as an $R$-module, i.e. a commutative algebra in the symmetric monoidal $\infty$-category $\text{Perm}(R)$. We can view $\text{Perm}(R)$ as a commutative algebra object in the symmetric monoidal $\infty$-category $\text{Cat}_{\text{perf}}^{\infty}$ of small, stable, idempotent complete $\infty$-categories. By an $R$-linear category we mean a right module over $\text{Perm}(R)$ in $\text{Cat}_{\text{perf}}^{\infty}$. It follows from [Lur17, Remark 4.2.4.26] that any $R$-linear category $C$ is canonically right-tensored over $\text{Perm}(R)$ in the sense of [Lur17, Definition 4.2.1.19]. In particular, for an algebra $B$ in $\text{Perm}(R)$ we can form the $\infty$-category $\text{Mod}_B(C)$ of right $B$-modules in $C$.

We will use the following notation. If $C$ is an $R$-linear category, and $M$ is an object of $\text{Perm}(R)$, then $(M)_C \subseteq C$ denotes the smallest thick subcategory of $C$ containing the objects $X \otimes_R M$ for every $X \in C$. In the case $C = \text{Perm}(A)$ for an $R$-algebra $A$, $(S)_{\text{Perm}(A)}$ coincides with the thick subcategory of $\text{Perm}(A)$ generated by $A \otimes_R S$.

**Lemma 3.12.** Let $R \to S$ be a map of $\mathbb{E}_\infty$-ring spectra such that $S$ is perfect as an $R$-module. Let further $C$ be an $R$-linear category. Then the functor

$$\text{colim}_m \text{Mod}_{\text{Tot}_m(CB\Sigma_{S/R}^\bullet)}(C) \longrightarrow C$$

induced by restriction of scalars is fully faithful. Its essential image is $(S)_C \subseteq C$.

We remark that the forgetful functor $\text{Cat}_{\text{perf}}^{\infty} \to \text{Cat}_{\infty}$ preserves filtered colimits; see [Lur17, Proposition 1.1.4.6, Lemma 1.2.4.6]. In the proof of Lemma 3.12 we use the notion of the pro-category of an accessible $\infty$-category $C$ admitting finite limits as in [Lur18, §A.8.1] and the following fact, which we prove first.
Lemma 3.13. Let $x$ be an object of an accessible $\infty$-category $C$ which admits finite limits. Then the canonical functor $\text{Pro}(C_{x/}) \to \text{Pro}(C)_{x/}$ is an equivalence.

Proof. From the explicit description of the mapping spaces in slice categories in [Lur09, Lemma 5.5.5.12] and pro-categories in [Lur18, Remark A.8.1.5] and the fact that filtered colimits in spaces are stable under pullback, it follows that the functor under consideration is fully faithful. To see that it is also essentially surjective, we note that an arbitrary object of $\text{Pro}(C)_{x/}$ is of the form $x \to \{y_i\}_{i \in I}$ where $\{y_i\}_{i \in I}$ is the limit in $\text{Pro}(C)$ of a cofiltered diagram $I \to C \rightrightarrows \text{Pro}(C)$. The map $x \to \{y_i\}_{i \in I}$ gives rise to a diagram $I \to C_{x/}$. Composing with $C_{x/} \to \text{Pro}(C_{x/})$ and taking the limit, we get an object $\{x \to y_i\}_{i \in I}$ in $\text{Pro}(C_{x/})$. As the canonical functor between pro-categories preserves cofiltered limits, it sends the latter object to $x \to \{y_i\}_{i \in I}$, up to equivalence. □

Proof of Lemma 3.12. We write $R_n = \text{Tot}_m(\text{CB}^\bullet_{S/R})$ to ease notation. We start by proving the full faithfulness. For this, it suffices to show that for $X, Y \in \text{Mod}_{R_n}(C)$ the canonical map

$$\text{colim}_m \text{Hom}_{R_m}(X, Y) \to \text{Hom}_R(X, Y) = \text{map}_C(X, Y)$$

is an equivalence. This map is equivalent to the map

$$\text{colim}_m \text{Hom}_{R_m}(X \otimes_{R_m} R_n, Y) \to \text{Hom}_{R_n}(X \otimes_R R_n, Y).$$

So it is enough to show that for every $X \in \text{Mod}_{R_n}(C)$ the canonical map

$$(3) \quad X \otimes_R R_n \to \{X \otimes_{R_m} R_n\}_m$$

of pro-systems of $R_n$-modules is an equivalence. To see this, we first establish two auxiliary claims. We denote by $\text{CAlg}$ the $\infty$-category of $E_\infty$-ring spectra.

Claim 1. The map $R_n \to \{R_m \otimes_R R_n\}_m, x \mapsto 1 \otimes x$, is an equivalence in $\text{Pro}(\text{CAlg})$.

Proof. Fix some positive integer $q$. We have $R_m \otimes_R S^{\otimes nq} \simeq \text{Tot}_m(\text{CB}^\bullet_{S/R} \otimes_R S^{\otimes nq})$. The cosimplicial $E_\infty$-ring spectrum $\text{CB}^\bullet_{S/R} \otimes_R S^{\otimes nq}$ extends to a split augmented cosimplicial $E_\infty$-ring spectrum, where the augmentation is given by $S^{\otimes nq} \to \text{CB}^0_{S/R} \otimes_R S^{\otimes nq}, x \mapsto 1 \otimes x$. This implies that the associated Tot-tower is equivalent to the constant pro-object with value $S^{\otimes nq}$, see [Mat16a, Example 3.11]. Now the claim follows, because $R_n$ is the limit of a finite diagram consisting of $S^{\otimes nq}$s and the inclusion $\text{CAlg} \to \text{Pro}(\text{CAlg})$ preserves finite limits. The latter holds since filtered colimits preserve finite limits in spaces. □

Claim 2. The canonical map $R_n \otimes_R R_n \to \{R_m \otimes_{R_m} R_n\}_m$ is an equivalence in $\text{Pro}(\text{CAlg})$.

Proof. Using that the relative tensor product is the pushout in $\text{CAlg}$, we see that $R_n \otimes_{R_m} R_n \simeq (R_n \otimes_{R_n} R_n) \otimes (R_m \otimes_{R_m} R_n)$:

$$R_n \longrightarrow R_n$$
$$\downarrow \quad \downarrow$$
$$R_m \to R_m \otimes_R R_n \longrightarrow R_n$$
$$\downarrow \quad \downarrow \quad \downarrow$$
$$R_n \to R_n \otimes_R R_n \to R_n \otimes_{R_m} R_n$$

Every square in this diagram is a pushout. Informally, the equivalence $R_n \otimes_{R_m} R_n \to (R_n \otimes_{R_n} R_n) \otimes (R_m \otimes_{R_m} R_n)$ is given by $x \otimes y \mapsto x \otimes 1 \otimes y$ with inverse $(x \otimes y) \otimes z \mapsto x \otimes yz$. Similarly,
we can identify \( R_n \otimes R R_n \simeq (R_n \otimes R R_n) \otimes_{R_n} R_n \). Under these identifications, the map in Claim 2 is equivalent to the canonical map
\[
(R_n \otimes R R_n) \otimes_{R_n} R_n \to \{(R_n \otimes R R_n) \otimes R_n\}_{m} = (R_n \otimes_{R R_n} R_n)_{m}
\]
induced by the equivalence \( R_n \to \{R_m \otimes R R_n\}_{m} \) in \( \text{Pro}(\text{CAlg}) \) from Claim 1. So Claim 2 is now proven, as pushouts in \( \text{Pro}(\text{CAlg}) \) are computed levelwise. \( \square \)

Given Claim 2, Lemma 3.13 implies that the canonical map
\[
(R_n \otimes R R_n \xrightarrow{id} R_n \otimes R R_n) \to \{(R_n \otimes R R_n \to R_n \otimes_{R_m} R_n)\}_{m}
\]
is an equivalence in \( \text{Pro}(\text{CAlg}_{R_n \otimes R R_n/}) \). Applying the forgetful functor from \( \text{CAlg}_{R_n \otimes R R_n/} \) to the \( \infty \)-category of \((R_n, R_n)\)-bimodules \( \text{BiMod}(R_n, R_n) \) we obtain that the map \( R_n \otimes R R_n \to \{R_n \otimes_{R_m} R_n\}_{m} \) is in fact an equivalence in \( \text{Pro}(\text{BiMod}(R_n, R_n)) \). Tensoring this equivalence on the left with \( X \) over \( R_n \) then implies that the map (3) is an equivalence in \( \text{Pro}(\text{Mod}_{R_n}(C)) \) as desired. This finishes the proof of the full faithfulness.

It remains to prove the assertion about the essential image. It clearly contains \( X \otimes_R S \) for every \( X \in C \) and thus it contains \( \langle S \rangle_C \), as the essential image is idempotent complete. On the other hand, \( \langle S \rangle_{\text{Perf}(R)} \) for every \( q \geq 1 \), and so \( R_m \in \langle S \rangle_{\text{Perf}(R)} \) for each \( m \) as a finite limit of objects of \( \langle S \rangle_{\text{Perf}(R)} \). It follows that \( X \otimes_R R_m \in \langle S \rangle_C \) for every \( X \in C \). If \( X \) is an object of \( \text{Mod}_{R_m}(C) \), then \( X \) is a retract of \( X \otimes_R R_m \) in \( C \), so \( X \) belongs to \( \langle S \rangle_C \). \( \square \)

Several times we will use the following consequence.

**Proposition 3.14.** Let \( R \to S \) be a map of \( \mathbb{E}_\infty \)-ring spectra such that \( S \) is perfect as an \( R \)-module. Let \( D \) be a symmetric monoidal \( \infty \)-category which admits filtered colimits, and let \( E: \text{Cat}_{\infty}^{\text{perf}} \to D \) be a finitary, lax symmetric monoidal functor. Assume that \( E(\text{Tot}_m(\text{CB}^\bullet_{S/R})) \) vanishes for every \( m \geq 0 \). Then \( E(\langle S \rangle_C) = 0 \) for every \( R \)-linear category \( C \).

**Proof.** By Lemma 3.12 we have an equivalence
\[
\langle S \rangle_C \simeq \colim_m \text{Mod}_{\text{Tot}_m(\text{CB}^\bullet_{S/R})}(C).
\]
The \( \infty \)-category \( \text{Mod}_{\text{Tot}_m(\text{CB}^\bullet_{S/R})}(C) \) is \( \text{Tot}_m(\text{CB}^\bullet_{S/R}) \)-linear. So, as \( E \) is lax symmetric monoidal, applying \( E \) to it yields a module over \( E(\text{Tot}_m(\text{CB}^\bullet_{S/R})) \), which vanishes by assumption. As \( E \) preserves filtered colimits, the claim follows. \( \square \)

**Example 3.15.** We will apply the previous proposition with \( E \) being \( T(i) \)-local \( K \)-theory for some \( i \geq 1 \). This is a finitary, lax symmetric monoidal, localizing invariant when viewed as a functor from \( \text{Cat}_{\infty}^{\text{perf}} \) to \( T(i) \)-local spectra. The vanishing-assumption in the proposition is satisfied for example in the case where \( R \to S \) is a map of \( \mathbb{E}_\infty \)-rings, \( S \) is perfect as an \( R \)-module, and
\begin{itemize}
  \item[(i)] \( S \) is \( T(1) \)-acyclic (for \( i \geq 2 \)), or
  \item[(ii)] \( S \) is \( \mathbb{S}_{[1]}^{[1]} \)-acyclic (for \( i \geq 1 \)).
\end{itemize}

\textsuperscript{2}A functor from small, idempotent complete, stable \( \infty \)-categories is called \textit{finitary} if it preserves filtered colimits.
Indeed, if \( S \) is \( T(1) \)-acyclic or \( \mathbb{S}^{[\frac{1}{p}]} \)-acyclic, then so is \( S^{\otimes q} \) for every \( q \geq 1 \). Hence so is each \( \text{Tot}_n(CB^\bullet_{S/R}) \) as a finite limit of \( T(1) \)-acyclic respectively \( \mathbb{S}^{[\frac{1}{p}]} \)-acyclic spectra. Now Corollary 2.29 gives the desired \( T(i) \)-local vanishing of their \( K \)-theory.

Below we will apply the above vanishing result to kernels of localizations away from an ideal, which we recall next. Later, we will identify the \( L_{\eta}^f \)-localizations in specific cases as localizations away from an ideal. We fix an \( \mathbb{E}_\infty \)-ring spectrum \( R \) and an ideal \( a = (a_1, \ldots, a_k) \subseteq \pi_n(R) \) generated by homogenous elements \( a_1, \ldots, a_k \).

**Definition 3.16.** The functor \( \text{Mod}(R) \to \text{Mod}(R) \) given by Bousfield localization at the \( R \)-module spectrum \( R(\frac{1}{a_1}) \oplus \cdots \oplus R(\frac{1}{a_k}) \) is denoted by \( M \mapsto M[a^{-1}] \) and called localization away from \( a \).

This notion is taken from [GM95, Section 3 and 5], where it is also observed that \( M[a^{-1}] \) is independent of the choice of generators \( a_1, \ldots, a_k \) and in fact only depends only the radical of \( a \).

**Lemma 3.17.** The localization \( M[a^{-1}] \) can be identified with the limit over the pointed \( k \)-dimensional cube \( C_M \), which assigns to a non-empty subset \( S \subseteq \{1, \ldots, k\} \) the module \( C_M(S) = M[[\prod_{i \in S} a_i]^{-1}] \).

**Proof.** Denote the limit of the cube in the statement by \( M' \) and observe that

\[
M'[\frac{1}{a_i}] \simeq \lim_{\emptyset \neq S \subseteq \{1, \ldots, k\}} C_M(S)[\frac{1}{a_i}] \simeq M[\frac{1}{a_i}].
\]

Thus \( M \to M' \) is an \( R(\frac{1}{a_i}) \)-local equivalence for all \( 1 \leq i \leq k \). Moreover, as every \( C_M(S) \) is \( R(\frac{1}{a_1}) \oplus \cdots \oplus R(\frac{1}{a_k}) \)-local, so is the limit \( M' \). It follows that \( M' \simeq M[a^{-1}] \) as asserted. \( \square \)

**Lemma 3.18.** Let \( E \) be a localizing invariant, and let \( A \) be an \( R \)-algebra. Then \( E(A[a^{-1}]) \) is equivalent to the limit of the pointed cube sending a non-empty subset \( S \subseteq \{1, \ldots, k\} \) to \( A[[\prod_{i \in S} a_i]^{-1}] \).

**Proof.** We argue by induction over the number \( k \) of generators of the ideal. The case \( k = 1 \) is clear. For the inductive step, let \( b = (a_1, \ldots, a_{k-1}) \). Lemma 3.17 implies that there is a cartesian square

\[
\begin{array}{ccc}
A[a^{-1}] & \longrightarrow & A[a_k^{-1}] \\
\downarrow & & \downarrow \\
A[b^{-1}] & \longrightarrow & A[b^{-1}][a_k^{-1}].
\end{array}
\]

As \( R[a^{-1}][a_k^{-1}] \simeq R[a_k^{-1}] \), it follows that

\[
R[b^{-1}] \otimes_{R[a^{-1}]} R[a_k^{-1}] \to R[b^{-1}][a_k^{-1}]
\]

is an equivalence. Thus the main theorem of [Tam18] (or [LT19]) applies to give the result. \( \square \)

**Lemma 3.19.** Let \( A \) be an \( R \)-algebra. Then the kernel of the localization functor \( \text{Mod}(A) \to \text{Mod}(A[a^{-1}]) \) is compactly generated. The subcategory of compact objects is spanned by those \( M \in \text{Perf}(A) \) satisfying \( M[a^{-1}] = 0 \). It coincides with \( \langle R/(a_1, \ldots, a_k) \rangle_{\text{Perf}(A)} = \langle A \otimes_R R/(a_1, \ldots, a_k) \rangle \).
Proof. This is a standard result. For a detailed proof of the first two assertions in a slightly different setting see [Lur18 Proposition 7.1.1.12]. For the last one, note that \( R/(a_1, \ldots, a_k) \) vanishes under the localization functor \( \text{Perf}(R) \to \text{Perf}(R[a^{-1}]) \). It follows that the subcategory \( (R/(a_1, \ldots, a_k)_{\text{Perf}(A)}) \) is contained in the kernel of \( \text{Perf}(A) \to \text{Perf}(A[a^{-1}]) \).

Conversely, let \( M \in \text{Perf}(A) \) with \( M[a^{-1}] = 0 \). By definition, we know that \( M \otimes_R R/(a_1, \ldots, a_k) \) is contained in \( (R/(a_1, \ldots, a_k)_{\text{Perf}(A)}) \). Hence the same is true for \( M \otimes_R R/(a_1^\ell_1, \ldots, a_k^\ell_k) \) for any sequence of numbers \( (\ell_1, \ldots, \ell_k) \). Since \( M \) is compact and \( M[a^{-1}] = 0 \)-torsion, there are numbers \( \ell_i \) such that \( a_i^{\ell_i} : M \to M \) is null-homotopic for all \( i \). It follows that \( M \) is a retract of \( M \otimes_R R/(a_1^\ell_1, \ldots, a_k^\ell_k) \) and is thus also contained in \( (R/(a_1, \ldots, a_k)_{\text{Perf}(A)}) \).

We will use the previous lemma to give an explicit description of the \( L_n^{p,f} \)-localization of \( MU \)-modules, which we will use later. Similar results have appeared in work of Ravenel [Rav92 8.1.1] and [GM95 Theorem 6.1] with \( L_n \)-localizations instead of \( L_n^{p,f} \). Combining these results, one finds that \( L_n \)-localization and \( L_n^{p,f} \)-localization agree on \( MU \)-modules. For the precise statement, we fix an implicit prime \( p \) and denote by \( v_n \) the coefficient of \( xp^n \) in the \( p \)-series for the universal formal group law over \( \pi_*(MU) \) and by \( v_n \) the ideal \( (p, v_1, \ldots, v_n) \).

We recall that a graded formal group law over a graded ring \( A \). Its associated formal group has height at least \( n+1 \) for every \( v_n \) of type \( \mathbb{E}_{\infty} \)-acyclics is generated by \( \mathbb{E}_{\infty} \)-ring from [BL14, Definition 1.4].

Proposition 3.20. For every \( MU \)-module \( M \), the localizations \( L_n^{p,f} \) and \( M[v_n^{-1}] \) coincide.

Proof. We show first that \( M \to M[v_n^{-1}] \) is an \( L_n^{p,f} \)-equivalence. By the previous lemma, the fibre of the map \( M \to M[v_n^{-1}] \) is contained in the localizing subcategory of \( MU \)-modules generated by \( MU/v_n \), so it suffices to show that \( MU/v_n \) is \( L_n^{p,f} \)-acyclic. Using [Ang08], one can show that \( MU/v_n \) is \( L_n^{p,f} \)-acyclic. By Lemma [2.3] it now suffices to show that for \( 1 \leq i \leq n \) the spectrum \( K(i) \otimes MU/v_n \) is zero. But this is a complex orientable ring spectrum whose formal group has height both exactly \( i \) and at least \( n+1 \); it thus must be trivial.

It thus suffices to show that \( M[v_n^{-1}] \) is \( L_n^{p,f} \)-local. As this spectrum is a finite limit of the spectra \( M([\prod_i v_i]^{-1}) \) with \( \emptyset \neq I \subseteq \{0, \ldots, n\} \), it suffices to show that each such \( M([\prod_i v_i]^{-1}) \) is \( L_n^{p,f} \)-local. As observed in Lemma [3.2] the category of \( L_n^{p,f} \)-acyclics is generated by a type-(\( n+1 \)) complex \( \Sigma^\infty V_{n+1} \). Thus, we need to show that \( DV_{n+1} \otimes M([\prod_i v_i]^{-1}) = \text{map}(\Sigma^\infty V_{n+1}, M([\prod_i v_i]^{-1})) = 0 \). As this is a module over the ring spectrum \( DV_{n+1} \otimes MU[v_1^{-1}] \) for \( i \) in \( I \), it suffices to show that this ring spectrum vanishes. Since \( DV_{n+1} \) is again of type \( n+1 \), the nilpotence theorem implies that it suffices to show that \( K(m) \otimes MU[v_1^{-1}] = 0 \) for \( m > n \) (including \( m = \infty \)). Again, this follows since \( K(m) \otimes MU[v_1^{-1}] \) has two isomorphic formal group laws, one of which (the one through \( K(m) \)) has height at least \( m \), so that the same is true for the one through \( MU[v_1^{-1}] \). This implies that the unit \( v_i \) is sent to zero under the map \( MU[v_1^{-1}] \to K(m) \otimes MU[v_1^{-1}] \) so that the latter must be zero. The same argument works for \( m = \infty \).

We next recall the definition of a regular \( \mathbb{E}_{\infty} \)-ring from [BL14 Definition 1.4].
Definition 3.21. A connective $\mathbb{E}_\infty$-ring $R$ is called regular if $\pi_0(R)$ is a regular coherent ring, perfect as an $R$-module, and $\pi_n(R)$ is finitely presented as a $\pi_0(R)$-module for every $n \geq 0$.

Lemma 3.22. Let $R$ be a regular connective $\mathbb{E}_\infty$-ring. Then the subcategory $\langle \pi_0(R) \rangle \subseteq \text{Perf}(R)$ coincides with the subcategory $\text{Perf}^R(R)$ of bounded perfect $R$-modules.

Proof. Clearly $\langle \pi_0(R) \rangle \subseteq \text{Perf}^R(R)$. Conversely, let $P$ be a bounded perfect $R$-module, and let $\pi_N(P)$ be the lowest non-vanishing homotopy group of $P$. Then $\pi_N(P)$ is finitely presented as a $\pi_0(R)$-module [Lur17, Corollary 7.2.4.5]. As $\pi_0(R)$ is regular, it follows that $\pi_N(P)$ is a perfect $\pi_0(R)$-module. This implies that $\pi_N(P) \in \langle \pi_0(R) \rangle$ and that $\tau_{>N}P \in \text{Perf}^R(R)$. We can thus repeat the argument with $\tau_{>N}P$ to see that $\pi_{N+1}(P) \in \langle \pi_0(R) \rangle$. Inductively we get $P \in \langle \pi_0(R) \rangle$.

For the next two results, we fix a regular connective $\mathbb{E}_\infty$-ring, equipped with homogenous elements $a_1, \ldots, a_k \in \pi_*(R)$ of positive degree. We denote by $a = (a_1, \ldots, a_k)$ the ideal generated by the $a_i$ and by $(p, a) = (p, a_1, \ldots, a_k)$ the ideal generated by $a$ and $p$.

Proposition 3.23. Suppose that $R/(a_1, \ldots, a_k)$ is bounded above. Then for any $R$-algebra $A$, the canonical map $K(A) \to K(A[a^{-1}])$ is a $T(i)$-local equivalence for all $i \geq 2$.

Proof. We first claim that the kernel of the localization functor $\text{Perf}(A) \to \text{Perf}(A[a^{-1}])$ agrees with $\langle \pi_0(R) \rangle_{\text{Perf}(A)}$. By Lemma 3.19 we have to show that $\langle R/(a_1, \ldots, a_k) \rangle_{\text{Perf}(A)} = \langle \pi_0(R) \rangle_{\text{Perf}(A)}$. It suffices to show this when $A = R$. As the $a_i$ have positive degree, $\pi_0(R)[\frac{1}{a_i}]$ vanishes for every $i$, and hence $\pi_0(R)$ is in the kernel of the localization functor, so that we find $\langle \pi_0(R) \rangle_{\text{Perf}(A)} \subseteq \langle R/(a_1, \ldots, a_k) \rangle_{\text{Perf}(A)}$. Conversely, by assumption $R/(a_1, \ldots, a_k)$ is bounded, so we get the $\subseteq$-containment from Lemma 3.22.

From Lemma 3.19 and the Thomason–Neeman localization theorem again we thus get an exact sequence

$$\langle \pi_0(R) \rangle_{\text{Perf}(A)} \longrightarrow \text{Perf}(A) \longrightarrow \text{Perf}(A[a^{-1}])$$

of $R$-linear stable $\infty$-categories. Since $T(i)$-local $K$-theory is a localizing invariant, it suffices to argue that $L_{T(i)}K(\langle \pi_0(R) \rangle_{\text{Perf}(A)})$ vanishes. This vanishing follows from Proposition 3.14 and Example 3.15(i) applied to the map $R \to \pi_0(R)$.

Similarly we get:

Proposition 3.24. Suppose that $R/(a_1, \ldots, a_k)$ is bounded above. Then for any $R$-algebra $A$, the canonical map $K(A) \to K(A[(p, a)^{-1}])$ is a $T(i)$-local equivalence for $i \geq 1$.

Proof. The proof proceeds as the previous one. The kernel of the localization functor $\text{Perf}(A) \to \text{Perf}(A[(p, a)^{-1}])$ coincides with $\langle \pi_0(R)/p \rangle_{\text{Perf}(A)}$. As $\pi_0(R)/p$ is $T(1)$-acyclic and $p$-torsion, we conclude by Proposition 3.14 and Example 3.15(ii). Notice here that $\pi_0(R)/p$ denotes the cofibre of the multiplication by $p$ on the spectrum $\pi_0(R)$, not the discrete ring $\pi_0(R)$.

3.2. Applications to $HZ$-algebras and $K(1)$-acyclic ring spectra. As a direct consequence we get the following.

Corollary 3.25. Let $A$ be an $HZ$-algebra. Then the canonical map $K(A) \to K(A[\frac{1}{p}])$ is a $T(i)$-local equivalence for $i \geq 1$.

Of course, by Mitchell’s result [Mit90] source and target vanish $T(i)$-locally for $i \geq 2$.

Proof. Apply Proposition 3.24 to $R = HZ$ and $a = 0$. □
Remark 3.26. If $A$ is a regular ring, then the fibre of the map $K(A) \to K(A_{1/p}^1)$ is equivalent to the $G$-theory of $A/p$ and hence a $K(F_p)$-module. In this case, the assertion of Corollary 3.25 follows directly from Quillen’s computation of $K_*(F_p)$ without using the result of Bhatt–Clausen–Mathew (Corollary 2.20).

Remark 3.27. The analog of Corollary 3.25 for topological cyclic homology does not hold: As $\text{THH}(\mathbb{Z}[-1/p])$ is a $\mathbb{Z}_{1/p}^1$-algebra, it vanishes $p$-adically. So $\text{TC}(\mathbb{Z}[-1/p])$ vanishes $p$-adically, and a fortiori after $T(1)$-localization. However, $L_{T(1)}(\text{TC}(\mathbb{Z}))$ does not vanish: For odd primes $p$, Bökstedt and Madsen [BM94] computed the connective cover of $\text{TC}(\mathbb{Z})_{1/p}^1 \simeq \text{TC}(\mathbb{Z}_{1/p}^1)$ to be equivalent to $j \oplus \Sigma j \oplus \Sigma^3 k_{1/p} u$ where $j$ is the connective cover of the $K(1)$-local sphere. In particular, the $T(1)$- or equivalently $K(1)$-localization of $\text{TC}(\mathbb{Z})$ is given by

$$L_{K(1)}(\text{TC}(\mathbb{Z})) \simeq L_{K(1)}(\mathbb{Z}) \oplus \Sigma L_{K(1)}(\mathbb{Z}) \oplus \Sigma^3 KU^\wedge_1 \neq 0.$$  

(4)

For $p = 2$, [Rog99b, Theorem 0.5] and [Rog99a, Formula (0.2)] give a filtration of $L_{K(1)}(\text{TC}(\mathbb{Z}))$, whose associated graded essentially looks like the summands in (4). Using that $KU^\wedge_1$ is rationally non-trivial in infinitely many degrees, while the other two terms are rationally non-trivial only in finitely many degrees, one obtains that $L_{K(1)}(\text{TC}(\mathbb{Z}))$ is non-trivial at $p = 2$ as well.

We point out that, although $K(1)$-local TC commutes with filtered colimits of rings [CMM18, Theorem G], it does not commute with filtered colimits of categories: As explained in the beginning of Section 3.1, there is a fibre sequence

$$\text{colim}_k \text{Mod}_{\mathbb{Z}/(p^k)}(\text{Perf}(\mathbb{Z})) \to \text{Perf}(\mathbb{Z}) \to \text{Perf}(\mathbb{Z}[-1/p]).$$

Assuming that $L_{K(1)}$ TC commutes with filtered colimits we find that $L_{K(1)}$ TC of the fibre vanishes, as $L_{K(1)}(\text{TC}(\text{Mod}_{\mathbb{Z}/(p^k)}(\text{Perf}(\mathbb{Z}))))$ is a module over $L_{K(1)}(\text{TC}(\mathbb{Z}/(p^k)))$ which vanishes by the theorem of Bhatt–Clausen–Mathew (or Corollary 2.20 and McCarthy’s theorem [McC97]).

Corollary 3.28. Let $A$ be a connective, $K(1) \oplus \cdots \oplus K(n)$-acyclic ring spectrum. Then the canonical map $K(A) \to K(A_{1/p}^1)$ is a $T(i)$-local equivalence for all $1 \leq i \leq n$.

Proof. Using that $T(i)$-local $K$-theory is truncating on $K(1) \oplus \cdots \oplus K(n)$-acyclic ring spectra by Corollary 3.27, we reduce to the case of a discrete ring, where the statement follows from Corollary 3.25. \qed

We summarize this as follows.

Corollary 3.29. Question 3.3 at height $n$ has an affirmative answer for connective $K(1) \oplus \cdots \oplus K(n)$-acyclic ring spectra. It also has an affirmative answer for $HZ$-algebras.

Proof. We need to show that the map $K(A) \to K(L_{n,i}^p A)$ is a $T(i)$-local equivalence for $1 \leq i \leq n$ and $A$ a connective $K(1) \oplus \cdots \oplus K(n)$-acyclic ring spectrum. This follows from Corollary 3.28 since $L_{n,i}^p A \to A_{1/p}^1$ is an equivalence by Lemma 3.3. For $HZ$-algebras $A$, we again have $L_{n,i}^p A \simeq A_{1/p}^1$, so the claim follows from Corollary 3.25. \qed

We also get the following consequence, which (for $HZ$-algebras) was equally observed by Bhatt–Clausen–Mathew.
Corollary 3.30. Let $A$ be a connective and $K(1)$-acyclic ring spectrum. Then the canonical map $L_{K(1)}K(A) \to L_{K(1)}K(A[x])$ is an equivalence. In other words, $K(1)$-local $K$-theory is homotopy invariant on connective, $K(1)$-acyclic ring spectra.

Here, for any ring spectrum $A$, the symbol $A[x]$ denotes the ring spectrum $A \otimes \Sigma^\infty_+ \mathbb{Z}_{\geq 0}$.

Proof. We observe that $A[x] = A \otimes \mathbb{S}[x]$ is also $K(1)$-acyclic and connective. Hence, by Theorem 2.13 we may assume that $A$ is discrete so that $A[x]$ is the usual discrete polynomial ring $A \otimes_{\mathbb{Z}} \mathbb{Z}[x]$. By Corollary 3.28 we may furthermore assume that $p$ is invertible in $A$. In this case, Weibel [Wei81] has shown that $p$ is also invertible on $NK(A) = \text{fib}(K(A[x]) \to K(A))$. So the $p$-completion of $NK(A)$ vanishes, and hence $L_{K(1)}NK(A) = 0$ as well. □

Remark 3.31. For ring spectra, there are two canonical “affine lines”: The flat affine line $A[x]$ as above, and the smooth affine line $A \otimes \mathbb{S}\{x\}$, where $\mathbb{S}\{x\}$ is the free $\mathbb{E}_\infty$-ring on a degree 0 generator. Since the canonical map $\mathbb{S}\{x\} \to \mathbb{S}[x]$ is a $\pi_0$ isomorphism, we also obtain homotopy invariance with respect to the smooth affine line on connective, $K(1)$-acyclic ring spectra $A$: Both maps $K(A) \to K(A[x])$ and $K(A[x]) \to K(A[x])$ are $K(1)$-local equivalences.

3.3. Applications to ko-algebras. Let $KO$ be the real $K$-theory spectrum, $ko$ its connective cover, and let $\beta \in \pi_8(ko)$ be the Bott element, so that $KO \simeq ko[\frac{1}{\beta}]$. The Bott element for connective complex $K$-theory $ku$ will be denoted by $\beta_U \in \pi_2(ku)$.

In this subsection we apply the results about regular $\mathbb{E}_\infty$-ring spectra to $ko$-algebras. To connect these with Question 3.5, we first explain the relation of the $L_{1/F}^{p,f}$-localization with localization away from an ideal for $ko$-algebras.

Lemma 3.32. For a $ko$-module $M$ there is a pullback diagram

$$
\begin{array}{ccc}
L_{1/F}^{p,f}M & \longrightarrow & M[\frac{1}{\beta}] \\
\downarrow & & \downarrow \\
M[\frac{1}{p}] & \longrightarrow & M[\frac{1}{p\beta}].
\end{array}
$$

In other words, the $L_{1/F}^{p,f}$-localization coincides with localization away from $(p, \beta)$. Moreover, for $n \geq 1$, the natural maps $L_{n/F}^{p,f}M \to L_{1/F}^{p,f}M$ are equivalences.

Proof. We will first construct the corresponding cartesian square for a $ku$-module $M$. As there is a ring map $MU \to ku$, every $ku$-module obtains the structure of an $MU$-module. Thus, Proposition 3.20 identifies $L_{1/F}^{p,f}M$ with $M[(p, v_1)^{-1}]$. Concretely, the image of $v_1$ in $\pi_* ku$ is in our convention the coefficient of $x^p$ of the $p$-series $[p](x) = \frac{1}{p^0}(\beta_U x + 1)\beta_U^{p-1}-1$ of the multiplicative formal group law $x + y + \beta_U xy$, i.e. it equals $\beta_U^{-1}$. We obtain $M[(p, v_1)^{-1}] \simeq M[(p, \beta_U)^{-1}]$ and thus also the corresponding cartesian square from Lemma 3.17.

Let $C\eta$ denote the cone of the Hopf map $\eta : \Sigma S \to S$ and recall Wood’s theorem that $ko \otimes C\eta \simeq ku$. By the thick subcategory theorem, the thick subcategory generated by $C\eta$ contains the sphere spectrum as $C\eta$ has non-trivial rational homology. We see that a $ko$-module $M$ is $L_{1/F}^{p,f}$-local if and only if $M \otimes_{ko} ku \simeq M \otimes C\eta$ is $L_{1/F}^{p,f}$-local. As the image of $\beta$...
in $\pi_*ku$ equals $\beta^4_4$, we deduce that $M[1/\beta]$ is $L_{1}^{p,f}$-local. Thus the canonical map $M \to M[1/\beta]$ factors over $L_{1}^{p,f}$ and we get indeed a square of the form

$$\begin{array}{ccc}
L_{1}^{p,f}M & \longrightarrow & M[1/\beta] \\
\downarrow & & \downarrow \\
M[1/p] & \longrightarrow & M[1/p]\beta).
\end{array}$$

This is cartesian as it is cartesian after tensoring with $C\eta$, where it coincides with the corresponding square for $M \otimes_{ko} ku$.

By a similar argument, it suffices to show for the second claim that $L_{n}^{p,f}M \to L_{1}^{p,f}M$ is an equivalence for all $ku$-modules $M$. This follows from Proposition 3.20 and the fact that the image of $v_i$ in $\pi_*ku$ is zero for $i \geq 2$.

□

**Corollary 3.33.** Let $A$ be a $ko$-algebra. Then the natural map $K(A) \to K(A[1/\beta])$ is a $T(n)$-local equivalence for all $n \geq 2$, and the map $K(A) \to K(L_{n}^{p,f}A)$ is a $T(n)$-local equivalence for all $n \geq 1$.

**Proof.** The $E_\infty$-ring spectrum $ko$ is regular, see [BL14, Proposition 3.1], and $ko/\beta$ is bounded above. So Proposition 3.23 implies the first claim. By Lemma 3.32, $L_{n}^{p,f}A$ coincides with the localization of $A$ away from $(p, \beta)$, and thus the second claim follows from Proposition 3.24. □

**Remark 3.34.** By work of Blumberg–Mandell [BM08], there is a fibre sequence of connective $K$-theory spectra

$$K^c_*(Z) \longrightarrow K^c_*(ku) \longrightarrow K^c_*(KU)$$

and likewise for $ko$ and $KO$ in place of $ku$ and $KU$. Together with Mitchell’s result this implies Corollary 3.33 in the case where $A$ is $ko$ or $ku$.

Furthermore, we remark that from [AR02, CMNN] and [CMNN16] it is known that $K(ko)$ vanishes $T(n)$-locally for $n \geq 3$. For these cases, one can thus also deduce the results of Corollary 3.33 as all terms vanish.

### 3.4. Applications to $tmf$-algebras

Our last examples are algebras over the connective spectrum of topological modular forms $tmf$; for a general introduction we refer to [DFHH14] and to [Beh19]. By [Mat16b, Section 5.1], the $E_2$-term of the Adams–Novikov spectral sequence for $tmf$ coincides with the cohomology of a certain Hopf algebroid, whose zeroth cohomology coincides with the ring of modular forms. The latter has been computed in [Del75] (and also in [Bau08]) as $\mathbb{Z}[c_4, c_6, \Delta]/(c_4^3 - c_6^2 - 1728\Delta)$ with $|c_4| = 8$, $|c_6| = 12$, and $|\Delta| = 24$. The edge homomorphism of the Adams–Novikov spectral sequence for $tmf$ thus takes the form of a map

$$\pi_*(tmf) \longrightarrow \mathbb{Z}[c_4, c_6, \Delta]/(c_4^3 - c_6^2 - 1728\Delta).$$

After inverting 6, the spectral sequence collapses [Bau08, Proposition 4.4] and the edge homomorphism becomes an isomorphism.

Furthermore, the elements $c_4$ and $\Delta^{24}$ are permanent cycles: This can be checked locally after inverting 6 and at the primes 2 and 3. After inverting 6, it follows from the previous isomorphism. At the prime 3, $\Delta^{4}$ is a permanent cycle [Bau08, Section 6], and at the prime

\[3^{24} \text{Note that Bauer calls the Adams–Novikov spectral sequence for } tmf \text{ the elliptic spectral sequence.}\]
2, $\Delta^8$ is a permanent cycle [Bau08 Section 8]. Hence $\Delta^{24}$ is integrally a permanent cycle. For $c_4$ one again has to argue 2- and 3-locally; it is again contained in the respective sections of [Bau08]. One can thus choose lifts $\gamma$ of $c_4$ and $\delta$ of $\Delta^{24}$ to $\pi_*(\tmf)$.

**Lemma 3.35.** The quotient $\tmf/[\gamma, \delta]$ is bounded above.

**Proof.** As a first step, we observe that $\tmf[\frac{1}{6}]\rightarrow \tmf[\frac{1}{6}]$ is bounded above. Namely, its homotopy ring is given by

$$\mathbb{Z}[\frac{1}{6}, c_4, c_6]/(c_4, (\frac{c_4^8}{1728})^{24}) \cong \mathbb{Z}[\frac{1}{6}, c_6]/(c_6^{28})$$

which is clearly bounded above. To obtain the result, it thus suffices to show that $\tmf/[\gamma, \delta]$ is bounded above at the primes 2 and 3. By [Mat16b, Theorem 4.13], we have $\tmf_3 \otimes W \cong \tmf_1(2)(3)$ for a particular finite spectrum with non-trivial rational homology. We have $\pi_*(\tmf_1(2)(3)) = \mathbb{Z}[b_2, b_4]$, which agrees with the ring of modular forms of level 2. The image of $c_4$ in those is $16b_2^3 - 48b_4$ and the image of $\Delta$ is $16b_2^3b_4 - 64b_4^3$, which coincides modulo $c_4$ with $-16b_2^3$.

Thus, $\pi_*(\tmf_1(2)(3))/[\gamma, \delta] \cong \mathbb{Z}[b_2, b_4]/(b_2^3 - 3b_4, b_4^2)$ and $\tmf_1(2)(3)/[\gamma, \delta] \cong \tmf_3/[\gamma, \delta] \otimes W$ is bounded above. By the thick subcategory theorem, the thick subcategory of finite spectra generated by $W$ contains the sphere. Thus, $\tmf_3/[\gamma, \delta]$ is bounded above as well.

Likewise, by [Mat16b, Theorem 4.10] we have $\tmf_2 \otimes Z \cong \tmf_1(3)(3)$ for another finite spectrum $Z$ with non-trivial rational homology. We have $\pi_*(\tmf_1(3)(3)) = \mathbb{Z}[a_1, a_3]$, which agrees with the ring of modular forms of level 3. The image of $c_4$ in those is $a_1^3 - 24a_1a_3$ and the image of $\Delta$ is $a_1^3a_3^3 - 27a_3^4$. As above one concludes that $\tmf_2/[\gamma, \delta]$ is bounded above. The result follows. $\square$

We will now describe the localizations of $\tmf$ away from the ideals $(\gamma, \delta)$ and $(p, \gamma, \delta)$ in terms of more familiar objects. Recall that $\tmf$ is by definition the connective cover of an $E_\infty$-ring spectrum $Tmf$ that arises as the global sections of a sheaf $\mathcal{O}^{top}$ of $E_\infty$-ring spectra on the étale site of the compactified moduli stack of elliptic curves $\mathcal{M}_{ell}$. This stack carries a line bundle $\omega$, and $c_4$ and $\Delta$ can be viewed as sections of $\omega^4$ and $\omega^{{\otimes}12}$, respectively. Their non-vanishing loci $D(c_4)$ and $D(\Delta)$ cover $\mathcal{M}_{ell}$.

**Lemma 3.36.** There is an equivalence $Tmf \cong \tmf/[\gamma, \delta]^{-1}$ of $E_\infty$-algebras over $\tmf$.

**Proof.** As $\mathcal{O}^{top}$ is a sheaf, we obtain a cartesian square

$$\begin{array}{ccc}
\tmf & \longrightarrow & \mathcal{O}^{top}(D(\Delta)) \\
\downarrow & & \downarrow \\
\mathcal{O}^{top}(D(c_4)) & \rightarrow & \mathcal{O}^{top}(D(\Delta) \cap D(c_4)).
\end{array}$$

By the main theorem and Lemma 3.21 of [MML14], we can identify $\mathcal{O}^{top}(D(c_4))$ with $\tmf[\gamma^{-1}]$ and similarly for the other corners. Thus $\tmf$ obtains a map from $\tmf/[\gamma, \delta]^{-1}$ and it suffices to show that $\tmf/[\gamma^{-1}] \rightarrow Tmf/[\gamma^{-1}]$ and $\tmf/[\delta^{-1}] \rightarrow Tmf/[\delta^{-1}]$ are equivalences. It suffices to check this after inverting 6 and after localizing at 2 and 3. We will only detail the case of localizing at 2. Localizing at 3 instead involves $\tmf_1(2)$, while after inverting 6 we can work directly with $\tmf[\frac{1}{6}]$.

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5Here we have chosen the convention based on the Weierstraß curve $y^2 = x^3 + b_2x^2 + b_4x$ over $\mathbb{Z}[b_2, b_4]$, which becomes over $\mathbb{Z}[\frac{1}{6}][b_2, b_4, \Delta^{-1}]$ the universal elliptic curve with a non-trivial 2-torsion point over a $\mathbb{Z}[\frac{1}{6}]$-algebra.
By [Mat16b] again, $\text{tmf}(2) \otimes Z$ (with $Z$ as in the proof of Lemma 3.33) is equivalent to $\text{tmf}(3)(2)$. One computes via the descent spectral sequence

$$H^i(M_1(3); \omega^j) \to \pi_{2j-i}(\text{tmf}(3))$$

that all negative homotopy groups of $\text{tmf}(3)$ are concentrated in odd degrees, while $\text{tmf}(3) = \tau_{\geq 0} \text{tmf}(3)$ has homotopy concentrated in even degrees (see [HM17, Section 4.1] for details). Thus all homotopy groups in negative degrees are killed by powers of $\gamma$ and $\delta$ and we see that the maps $\text{tmf}(3)[\gamma^{-1}] \to \text{tmf}(3)[\gamma^{-1}]$ and $\text{tmf}(3)[\delta^{-1}] \to \text{tmf}(3)[\delta^{-1}]$ are equivalences. As the thick subcategory of $\text{Sp}^2$ generated by $Z$ contains the sphere, this implies the corresponding statement for $\text{tmf}(2)$ and $\text{tmf}(3)$. \hfill \Box

Putting everything together we get a variant of the first statement of Corollary [3.33] for $\text{tmf}$-algebras.

**Corollary 3.37.** Let $A$ be a $\text{tmf}$-algebra. Then $K(A) \to K(A \otimes_{\text{tmf}} \text{tmf})$ is a $T(i)$-equivalence for all $i \geq 2$. Equivalently, the square

$$\begin{array}{ccc}
K(A) & \longrightarrow & K(A[\delta^{-1}]) \\
\downarrow & & \downarrow \\
K(A[\gamma^{-1}]) & \longrightarrow & K(A[(\gamma \delta)^{-1}])
\end{array}$$

becomes cartesian after $T(i)$-localization for $i \geq 2$.

**Proof.** By Lemma 3.36 the localization of $A$ away from $(\gamma, \delta)$ is given by $A \otimes_{\text{tmf}} \text{tmf}$. According to [BL14, Proposition 4.1] the $E_\infty$-ring spectrum $\text{tmf}$ is regular, and $\text{tmf}/(\gamma, \delta)$ is bounded above by Lemma 3.35. We can thus apply Proposition 3.23 to deduce the first assertion. Applying Lemma 3.18 gives the equivalent formulation. \hfill \Box

**Remark 3.38.** In [BL14] Barwick and Lawson provide an analog of the Blumberg–Mandell localization sequence (see Remark 3.34) for certain regular ring spectra. In particular, there is a localization sequence of connective $K$-theory spectra

$$K^\text{cn}(\mathbb{Z}) \to K^\text{cn}(\text{tmf}) \to K^\text{cn}(\text{tmf})$$

which implies the result of the previous corollary for $A = \text{tmf}$ and certain other regular $\text{tmf}$-algebras. We also remark that in [AGH19] these localization sequences are extended to non-connective $K$-theory spectra. However, for the present application this is irrelevant as the difference vanishes after telescopic localization.

**Lemma 3.39.** There is a canonical equivalence $L^{p,f}_2 \text{tmf} \simeq \text{tmf}[(p, \gamma, \delta)^{-1}]$.

**Proof.** We claim first that the spectrum $\text{tmf}[(\gamma, \delta)^{-1}] = \text{tmf}(p)$ is $L^{p}_2$-local. Indeed, for every étale map $\text{Spec} A \to \mathcal{M}_{\mathbb{Z}(p)}$ the spectrum $\mathcal{O}^\text{top}(\text{Spec} A)$ is by construction an even, Landweber exact spectrum of height at most 2. As such it has the structure of an $\mathcal{M}U(p)$-module [HS99, Proposition 2.19]. As $\mathcal{O}^\text{top}(\text{Spec} A)/(p, v_1, v_2) = 0$, the map $\mathcal{O}^\text{top}(\text{Spec} A) \to \mathcal{O}^\text{top}(\text{Spec} A)/(p, v_1, v_2)^{-1}$ is an equivalence and thus Proposition 3.20 implies that $\mathcal{O}^\text{top}(\text{Spec} A)$ is $L^{p}_2$-local. As $L^{p}_2$-local spectra are closed under limits, $\text{tmf}$ is $L^{p}_2$-local as well. This implies that $\text{tmf}[(p, \gamma, \delta)^{-1}]$ is $L^{p,f}_2$-local and we obtain a canonical map

$$L^{p,f}_2 \text{tmf} \to \text{tmf}[(p, \gamma, \delta)^{-1}]$$
Let us next show that this map is an equivalence after inverting 6. The spectrum $tmf\frac{1}{6}$ is even and thus has a complex orientation and associated elements $v_1, v_2 \in \pi_*(tmf\frac{1}{6})$. The complex orientation gives $tmf\frac{1}{6}$ the structure of a (naive) $MU$-module and thus we can compute $L^0_2 tmf\frac{1}{6}$ as $tmf\frac{1}{6}((p, v_1, v_2)^{-1})$. As localization away from an ideal just depends on the radical, it remains to show in this case that the radicals of $(p, v_1, v_2)$ and of $(p, \gamma, \delta)$ agree.

We claim that $p, v_1, v_2$ is a regular sequence in $\pi_*(tmf\frac{1}{6}) \cong \mathbb{Z}\frac{1}{6}[c_4, c_6]$. Indeed: The ring $\pi_*(tmf\frac{1}{6})/p$ is an integral domain. The element $v_1$ cannot vanish in $\pi_*(tmf\frac{1}{6})/p$ as all elliptic curves over $\mathbb{F}_p$ had formal groups of height at least 2. Moreover, $v_2$ must be invertible in $\pi_*(tmf\frac{1}{6})/(p, v_1)$ as the heights of formal group laws of elliptic curves are bounded by 2.

Thus, we see that $\pi_*(tmf\frac{1}{6})/(p, v_1, v_2)$ has Krull dimension 0. By the theory of Artinian rings, this quotient must be a finite-dimensional $\mathbb{F}_p$-algebra and hence every element of positive degree must be nilpotent. We obtain that the radical of $(p, v_1, v_2)$ agrees with the ideal generated by $p$ and all elements of positive degree. As $tmf\frac{1}{6}(\gamma, \delta)$ is bounded, the radical of $(p, \gamma, \delta)$ is also the ideal generated by $p$ and the elements of positive degree.

It remains to show that the map $\left[ \text{3.35} \right]$ is an equivalence after localizing at 2 and 3. In the former case we tensor with the complex $Z$ (with $Z$ as in the proof of Lemma $\left[ \text{3.35} \right]$) and thus just have to show that the map

$$L^0_2 tmf_1(3)(2) \to tmf_1(3)(2)((p, \gamma, \delta)^{-1})$$

is an equivalence. As $\pi_*(tmf_1(3)(2)) \cong \mathbb{Z}(2)[a_1, a_3]$ is a polynomial ring in two variables, we can argue exactly as in the case of $tmf\frac{1}{6}$. If localizing at 3, we argue analogously with $tmf_1(2)$ instead. \qed

The same proof as for the previous corollary, but using Proposition $\left[ \text{3.24} \right]$ and Lemma $\left[ \text{3.39} \right]$ gives an affirmative answer to Question $\left[ \text{3.35} \right]$ at heights $n \geq 2$ for $tmf$-algebras:

**Corollary 3.40.** Let $A$ be a $tmf$-algebra. Then $K(A) \to K(L^0_{n,f} A)$ is a $T(i)$-equivalence for all $i \geq 1$ and $n \geq 2$.

**Proof.** The only thing to notice is that the canonical map $L^0_{n,f} A \to L^0_{p,f} A$ is an equivalence. This follows from the fact that $tmf$ is $T(i)$-acyclic for $i > 2$ as $tmf \to Tmf$ is a $T(i)$-equivalence and $Tmf_{(p)}$ is $L^0_2$-local as shown in the proof of Lemma $\left[ \text{3.39} \right]$. \qed

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