Hot Big Bangs from Quotients of AdS

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Abstract

We perform a double quotient of global AdS$_4$. Single quotients of global AdS have proven invaluable tools for studying the AdS/CFT correspondence due to the fact that they provide interesting structure while retaining trivial local properties of global AdS. We find that a double quotient yields a spacetime similar to a big bang, however instead of the entire spacetime originating from a point, the “big bang” transitions the spacetime from $(2 + 1)$ to $(3 + 1)$ dimensions. We also study the entanglement of the dual CFT to find that, at early times, the spacetime has thermal properties which disappear after a critical time.

1 Introduction

Over two decades ago, a new solution to Einstein’s equations in $2 + 1$ dimensions with a negative cosmological constant was discovered by Bañados et al. This solution, known as the BTZ black hole, is an eternal black hole spacetime which is locally isometric to Anti-de Sitter (AdS) space. It was also shown that this spacetime could be obtained from a quotient, or identification of points by a discrete group, of global AdS [5] [11].

The BTZ black hole has proven to be an invaluable tool for studying quantum gravity, especially in regards to the AdS/CFT correspondence due its exhibiting interesting global properties, while remaining locally trivial [2] [3] [4]. While the BTZ black hole has been generalized to higher dimensions, the quantum gravity results are much more sparse. This could be attributed to the fact that the higher-dimensional generalizations usually include time-dependences, making calculations more complicated.

Of interest to this paper is the case of a double quotient of global AdS. In $2 + 1$ dimensions, it has been shown that this either produces a third exterior region and creates a multi-boundary wormhole or identifies the two asymptotic boundaries of the BTZ black hole to create a type (3 + 1)-dimensional global AdS (AdS$_4$). In 2 + 1 dimensions, it has been shown that this either produces a third exterior region and creates a multi-boundary wormhole or identifies the two asymptotic boundaries of the BTZ black hole to create a type (3 + 1)-dimensional global AdS (AdS$_4$). In 2 + 1 dimensions, it has been shown that this either produces a third exterior region and creates a multi-boundary wormhole or identifies the two asymptotic boundaries of the BTZ black hole to create a type (3 + 1)-dimensional global AdS (AdS$_4$).

2 Single Quotient of Global AdS$_4$

To perform the first quotient, we begin with global AdS$_4$, defined as the surface

$$- T^2_1 - T^2_2 + X^2_1 + X^2_2 + X^2_3 = - l^2$$

(1)

embedded in $\mathbb{R}^{2,3}$ with metric

$$ds^2 = -dT^2_1 - dT^2_2 + dX^2_1 + dX^2_2 + dX^2_3$$

(2)

where $l$ is the AdS radius. We will consider the boost-like isometry given by

$$\xi = -X_1 \partial_{T_1} - T_1 \partial_{X_1}, \quad \xi^2 = -X^2_1 + T^2_1$$

(3)

to generate the quotient. To avoid closed timelike curves, we remove regions of the spacetime where $\xi^2 < 0$ after the quotient [5] [9]. This creates a singularity in the causal structure where timelike geodesics can end at

$$-X^2_1 + T^2_1 = 0$$

(4)

or equivalently, by using [1],

$$- T^2_2 + X^2_2 + X^2_3 = - l^2.$$  \hspace{1cm} \text{(5)}

This singularity asymptotes to the null cone given by

$$T^2_2 = X^2_2 + X^2_3 \Rightarrow X^2_2 - T^2_1 = - l^2$$

(6)

which we identify as the event horizon of the black hole. These surfaces are plotted in Figure [4].
coordinates the singularity is given by the surface created from a quotient by $\xi$. Here, the $T_1$ and $X_1$ coordinates are suppressed.

We can define coordinates on the surface $[1]$, which are related to the coordinates of the embedding space by

$$T_1 = \frac{l}{1 + t^2 - y_1^2 - y_2^2} \cosh \left( \frac{r + \phi}{l} \right)$$

$$T_2 = \frac{2l}{1 + t^2 - y_1^2 - y_2^2}$$

$$X_1 = l \frac{1 - t^2 + y_1^2 + y_2^2}{1 + t^2 - y_1^2 - y_2^2} \cosh \left( \frac{r + \phi}{l} \right)$$

$$X_2 = \frac{2ly_1}{1 + t^2 - y_1^2 - y_2^2}$$

$$X_3 = \frac{2ly_2}{1 + t^2 - y_1^2 - y_2^2}$$  \(7\)

where $r_+$ is a constant. In terms of these coordinates the induced metric on the surface becomes

$$ds^2 = \frac{4l^2}{(1 + t^2 - y_1^2 - y_2^2)^2} (-dt^2 + dy_1^2 + dy_2^2)$$

$$+ r_+^2 \left( \frac{1 - t^2 + y_1^2 + y_2^2}{1 + t^2 - y_1^2 - y_2^2} \right)^2 d\phi^2$$  \(8\)

and the event horizon is given by

$$T_1^2 - X_1^2 = l^2 \left( \frac{1 - t^2 + y_1^2 + y_2^2}{1 + t^2 - y_1^2 - y_2^2} \right)^2 = l^2$$

$$\rightarrow - t^2 + y_1^2 + y_2^2 = 0.$$

We can also see that the embedding coordinates diverge at the surface

$$1 + t^2 - y_1^2 - y_2^2 = 0$$  \(11\)

which we associate with the boundary of the spacetime. Here, we note that, aside from the singularity, the spacetime in these coordinates is geodesically complete, and therefore no further maximal extension is necessary. In addition we see that this spacetime has only a single, connected asymptotic boundary, as opposed to the $(2 + 1)$-dimensional case, which has two asymptotically distinct boundaries $[6]$. The first quotient of AdS$_4$ is an interesting case to study due to its single asymptotic boundary. Once we perform the second quotient of the spacetime, this single boundary will be split into four distinct asymptotic boundaries, and will also exhibit features which are similar to those of a big bang spacetime.

3 Second Quotient of AdS$_4$

To perform a second quotient of AdS$_4$, we first recognize that the metric (8) has a boost-like isometry with Killing vector

$$\xi' = y_1 \partial_t + t \partial_{y_1}.$$  \(12\)

As we will be performing a quotient defined by this Killing vector, we will again need to excise regions of the spacetime where $\xi'^2 < 0$ to avoid closed timelike curves. These regions correspond to

$$- y_1^2 + t^2 < 0.$$  \(13\)

Since the boundary of these regions are null lines $t = \pm y_1$, they are not bounded by event horizons and therefore are naked. Figure 2 shows these naked singularities. It is also worthy to note that the introduction of these new singularities separates the spacetime into four asymptotically distinct regions: two future regions and two past regions.
In these coordinates, the naked singularity is located at $\xi^2 = 0$ or $\tau = 0$ which is reminiscent of a big bang spacetime since the $\tau$ coordinate plays the role of time. However, by examining the metric (15), we see that, as $\tau \to 0$, the spacetime does not collapse to a single point, but instead, transitions from $3 + 1$ to $2 + 1$ dimensions.

Since the $\theta$ coordinate is scaled by $\tau$, for very small $\tau$, we may neglect the $d\theta$ term and we are left with the metric of the BTZ black hole [2]

$$ds^2 = \frac{4l^2}{1 - y_2^2} (-d\tau^2 + dy_2^2) + r_+^2 \left(\frac{1 + y_2^2}{1 - y_2^2}\right) d\phi^2. \quad (20)$$

Therefore, for times near $\tau = 0$, we should expect the black hole to radiate at inverse temperature $[11]$ $\beta = \frac{r_+}{2\pi}.$ \quad (21)

However, away from $\tau = 0$, the black hole cools to zero temperature. This can be seen by defining the coordinates

$$\tau = \rho \sinh(\alpha \zeta) \quad y_2 = \rho \cosh(\alpha \zeta) \quad (22)$$

which transform the metric (15) to

$$ds^2 = \frac{4l^2}{(1 - \rho^2)^2} \left[-\alpha^2 \rho^2 d\zeta^2 + d\rho^2 + \alpha^2 \rho^2 \sinh^2(\alpha \zeta) d\theta^2\right]$$

$$+ r_+^2 \left(\frac{1 + \rho^2}{1 - \rho^2}\right)^2 d\phi^2. \quad (23)$$

Here, $\zeta \in (-\infty, \infty)$ and $\rho \in [0, 1)$. These coordinates correspond to an observer living in one exterior of the black hole, whose event horizon is located at $\rho = 0$. It is worth noting that, in these coordinates, the event horizon is 1-dimensional. We can perform a Wick rotation by taking $\zeta \to -i\zeta_E$ and the metric becomes

$$ds^2 = \frac{4l^2}{(1 - \rho^2)^2} \left[\alpha^2 \rho^2 d\zeta_E^2 + d\rho^2 + \alpha^2 \rho^2 \sin^2(\alpha \zeta_E) d\theta^2\right]$$

$$+ r_+^2 \left(\frac{1 + \rho^2}{1 - \rho^2}\right)^2 d\phi^2. \quad (24)$$

The term in square brackets is simply the metric on $\mathbb{R}^3$ in spherical coordinates where $\zeta_E$ is the polar coordinate and is therefore non-periodic. Since imaginary time is non-periodic after a Wick rotation, this implies that the black hole is not radiating far from $\tau = 0$. We will see more evidence for the freezing out of radiation in the next section.

### 4 Entanglement of Boundary CFTs

In the AdS/CFT correspondence, it is common for thermal properties to be shared by AdS spacetimes and their CFT duals $[11, 2, 3, 4]$. Therefore, AdS/CFT is an important tool for understanding the thermodynamics of a spacetime, especially in the quantum gravity regime.

Since the metric (15) exhibits a symmetry under $\tau \to -\tau$, we should be able to assign a Hartle-Hawking initial condition to the spacetime $[10]$. However, we cannot obtain a Euclidean-signature metric from a single Wick rotation, $\tau \to -i\tau$. Instead, we will utilize
the fact that as $\tau \to 0$, the $\theta$ coordinate collapses and
the metric becomes that of BTZ. We can therefore
smoothly glue the 3-dimensional Euclidean metric of
the BTZ spacetime, given by

$$ds_E^2 = \frac{4l^2}{(1 - r_E^2 - y_E^2)} (d\tau_E^2 + dy_E^2)$$
$$+ r_+^2 \left( \frac{1 + r_E^2 + y_E^2}{1 - r_E^2 - y_E^2} \right) d\phi^2,$$

(25)
to the metric [15] along $\tau = 0$ as shown in Figure 3.

![Figure 3: Hartle-Hawking initial state of the big bang
spacetime. Region I is a 3-dimensional Euclidean space,
where Region II is a (3 + 1)-dimensional Lorentzian
spacetime. Region I is a 3-dimensional Euclidean space,
and in Region II, the $\phi$ and $\theta$ dimensions are suppressed.
Despite the spaces having different dimensionality, we
can smoothly glue them along $\tau = 0$ since the $\theta$ dimen-
sion collapses as $\tau \to 0$.

Since the initial condition of the spacetime is identical
to that of BTZ, the state of the CFTs will also be identical very near $\tau = 0$. Namely they will be in a
thermofield double state, given by

$$|\Psi\rangle = \frac{1}{\sqrt{Z(\beta)}} \sum_n e^{-\beta E_n/2} |E_n\rangle_L \otimes |E_n\rangle_R$$

(26)

where $L$ and $R$ refer to the two CFTs on the separate
asymptotic boundaries [2]. We see that at early times,
there is entanglement between the two future asymp-
totic boundaries and so each independent CFT will be
in a mixed state. More importantly, this mixed state
can be interpreted as thermal, reflecting the fact that
the spacetime is thermal at early times.

Since the boundary of the spacetime is time-dependent,
we should expect the entanglement between the two fu-
ture boundary CFTs to also be time-dependent. To
examine this time dependence, we will appeal to the
AdS/CFT correspondence and calculate entanglement
entropy holographically.

The holographic entanglement entropy conjecture of
Ryu and Takayanagi states that the entanglement en-
tropy, $S_A$ of a spatial region $A$ of a CFT can be calcu-
lated from the area of the minimal area surface, $\gamma$, in
the dual spacetime which terminates on the boundary
of $A$. This relationship is given by

$$S_A = \frac{\text{Area}(\gamma)}{4G_N^{(d)}}$$

(27)

where $G_N^{(d)}$ is the $d$-dimensional gravitational con-
stant [3, 13].

Since this proposal is not covariant, it typically re-
quires a spacetime which exhibits a global timelike
Killing vector. However, since the big bang spacetime
we have introduced is formed from quotients of AdS$_4$
we can find the entanglement entropy by simply relating
coordinates [15] to Poincaré coordinates [3, 4, 13]. The
metric of global AdS$_4$ in Poincaré coordinates is
given by

$$ds^2 = \frac{l^2}{Z^2} (dW_+ dW_- + dY^2 + dZ^2)$$

(28)

where $W_+, Y \in (-\infty, \infty)$ and $Z \in (0, \infty)$ [4]. Note that
this is the null coordinate form of the Poincaré metric
where $W_+ = X \pm T$ with $X$ and $T$ being timelike and
spacelike coordinates, respectively. The boundary of
the spacetime is at $Z = 0$. If we consider a strip on the
boundary at fixed time given by

$$W_+ - W_- = \text{const} \Rightarrow \Delta W_+ - \Delta W_- = 0,$$

(29)
the length of the strip will be given by $\Delta Y = L$ and the
width will be given by

$$R^2 = \Delta W_+ \Delta W_-.$$  (30)

![Figure 4: Minimal-area surface, $\gamma_P$, (green) corresponding
to the strip, $A$, (red) at a constant-time slice of
global AdS$_4$.]

If we choose a minimal-area surface, $\gamma_P$, which termi-
nates only on the boundary in the $X = 1/2(W_+ + W_-)$
direction as shown in Figure 4 the area of $\gamma_P$ will be given by

$$\text{Area}(\gamma_P) = 2l^2 \left( \frac{L}{\epsilon} \right) - \kappa l^2 \left( \frac{L}{R} \right)$$

(31)
where $\epsilon$ is a cutoff introduced to prevent the expression from diverging and
\[
\kappa = 4\pi \left( \frac{\Gamma \left( \frac{3}{4} \right)}{\Gamma \left( \frac{1}{4} \right)} \right)^2 \tag{32}
\]
is a positive constant \[1, 13\]. For the expression \[31\] to be consistent, we can see that the area of the surface $\gamma_P$ must vanish when the region, $A$, on the boundary also vanishes, i.e., $\text{Area}(\gamma_P) \to 0$ as $\epsilon, R \to 0$.

The coordinates in \[15\] are related to Poincaré coordinates by
\[
W_\pm = \frac{\pm 2|\tau|}{1 - \tau^2 + y_2^2} e^{\frac{\tau}{2} \phi \pm \alpha \theta} \\
Y = \frac{2y_2}{1 - \tau^2 + y_2^2} e^{\frac{\tau}{2} \phi} \\
Z = l \left( \frac{1 + \tau^2 - y_2^2}{1 - \tau^2 + y_2^2} \right) e^{\frac{\tau}{2} \phi}. \tag{33}
\]
The boundary of the spacetime will be located at $Z = 0$ or $y_2 = \sqrt{1 + \tau^2}$. If we anchor the surface, $\gamma$, to the boundary at fixed time $\tau = \tau_0$, such that $\theta \in [\theta_1, \theta_2]$ and $\phi \in [\phi_1, \phi_2]$, the cutoff $Z = \epsilon \ll 1$ will be given by
\[
\epsilon_{1,2} = l a e^{\frac{\tau_0}{2} \phi_{1,2}} \tag{34}
\]
where $0 < la \ll 1$ and $\epsilon_{1,2} \to 0$ corresponds to $a \to 0$. For simplicity, we will center the strip such that $\theta_1 = -\Theta$ and $\theta_2 = \Theta$. Then, the width of the strip is given by
\[
R^2 = l \left( e^{\frac{\tau_0}{2} \phi_2} - e^{\frac{\tau_0}{2} \phi_1} \right)^2 \left[ 1 - \tau_0^2 \sinh^2 (\alpha \Theta) \right] \tag{35}
\]
and the length is given by
\[
L = l |\tau_0| \left( e^{\frac{\tau_0}{2} \phi_2} + e^{\frac{\tau_0}{2} \phi_1} \right) \sinh (\alpha \Theta). \tag{36}
\]
The width, $R$, vanishes either when $\phi_1 = \phi_2$ or $\tau_0 \sinh (\alpha \Theta) = 1$.

Using \[34\], \[35\], and \[36\] along with $\epsilon^2 = \epsilon_1 \epsilon_2$ in \[31\], we obtain an expression for the area of the minimal area surface in coordinates \[15\]
\[
\text{Area}(\gamma) = 4l^2 \left| \tau_0 \right| \frac{\sinh (\alpha \Theta)}{a} \cosh \Delta \\
- \kappa l^2 \left( \frac{\left| \tau_0 \right| \sinh (\alpha \Theta)}{\sqrt{1 - \tau_0^2 \sinh^2 (\alpha \Theta)}} \right) \coth \Delta. \tag{37}
\]
We can see that, as $a \to 0$ and $\tau_0 \sinh (\alpha \Theta) \to 1$, the area goes to zero and the region on the boundary is pure. This implies that there is a minimum time for a region of the boundary to reach a pure state
\[
\tau_{\text{min}} = \frac{1}{\sinh (\alpha \Theta)}. \tag{38}
\]
When $\Theta = \pi$, the region, $A$, will be one full future asymptotic boundary. Therefore, the minimum time, $\tau_{\text{crit}}$, for the CFT on this copy of the future boundary of the spacetime to purify will be
\[
\tau_{\text{crit}} = \frac{1}{\sinh (\alpha \pi)}. \tag{39}
\]
For $\tau < \tau_{\text{crit}}$, the boundary of either future exterior is in a mixed state, which we know from the Hartle-Hawking state to be a thermofield double. For $\tau > \tau_{\text{crit}}$, the boundary of both future exteriors are in pure states. Again, this is a reflection of the fact that, for early times, the spacetime is thermal but cools after a certain time.

5 Conclusions

We have arrived at a spacetime from a double quotient of global AdS$_4$ which behaves similar to a big bang spacetime with interesting thermal properties. However, instead of the spacetime originating from a point, as is typical for a big bang, it transitions from a $(2+1)$-dimensional to a $(3+1)$-dimensional spacetime. At early times, when one spatial dimension is small, the black hole emits thermal radiation. During this period, we also see entanglement between two of the boundary CFTs. After a critical time, the radiation stops and the entanglement is broken between the CFTs.

This work further exemplifies the power of the AdS/CFT correspondence. Using relativity or quantum mechanics alone, it is only possible to study the early and late time limits of the spacetime, but not the transition period. However, this is the epoch in which the spacetime exhibits particularly interesting characteristics. By applying AdS/CFT through the holographic entanglement entropy calculation, we were able uncover specific behaviors of this transition period. Moreover, this work further confirms the holographic entanglement entropy conjecture, as the holographic calculation agreed with both the early and late time limits in the relativistic and quantum calculations.

\footnote{Since the asymptotic boundaries are distinct, the region $A$ can only exist on one of these boundaries, so we choose the one corresponding to $y_1, \tau > 0$.}
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