Third Version of Weak Orlicz–Morrey Spaces and Its Inclusion Properties

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\textbf{Abstract}

Orlicz–Morrey spaces are generalizations of Orlicz spaces and Morrey spaces which were first introduced by Nakai. There are three versions of Orlicz–Morrey spaces, i.e: Nakai’s (2004), Sawano–Sugano–Tanaka’s (2012), and Deringoz–Guliyev–Samko’s (2014) versions. On this article we will discuss the third version of weak Orlicz–Morrey space which is seen as an enlargement of third version of (strong) Orlicz–Morrey space. Similar to its first version and second version, the third version of weak Orlicz-Morrey space is considered as a generalization of weak Orlicz spaces, weak Morrey spaces, and generalized weak Morrey spaces. In this study, we will investigate some properties of the third version of weak Orlicz–Morrey spaces, especially the sufficient and necessary conditions for inclusion relations between two these spaces. One of the keys to get our result is to estimate the quasi-norm of characteristics function of open balls in $\mathbb{R}^n$.

\textbf{Keywords:} Weak Orlicz spaces, Weak Morrey spaces, Weak Orlicz-Morrey space of third version, Inclusion property.

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\section{Introduction}

Orlicz-Morrey spaces are generalization of Orlicz spaces and Morrey spaces and it is firstly introduced by E. Nakai in 2004 [2,12,13]. These spaces are one of the important topics in mathematical analysis, particularly in harmonic analysis. There are three versions of Orlicz–Morrey spaces, i.e: Nakai’s (2004), Sawano–Sugano–Tanaka’s (2012) [2], and Deringoz–Guliyev–Samko’s (2014) \cite{4} versions.

For a Young function $\Theta : [0, \infty) \to [0, \infty)$ (i.e. $\Theta$ is convex, $\lim_{t \to 0} \Theta(t) = 0 = \Theta(0)$, continuous and $\lim_{t \to \infty} \Theta(t) = \infty$), we define $\Theta^{-1}(s) := \inf \{r \geq 0 : \Theta(r) > s\}$. Given two Young functions $\Theta_1, \Theta_2$, we write $\Theta_1 \prec \Theta_2$ if there exists a constant $C > 0$ such that $\Theta_1(t) \leq \Theta_2(Ct)$ for all $t > 0$.

Now, let $G_\theta$ be the set of all functions $\theta : (0, \infty) \to (0, \infty)$ such that $\theta(r)$ is decreasing but $\Theta^{-1}(t^{-\theta})\theta(t)^{-1}$ is almost decreasing for all $t > 0$. Let $\theta_1 \in G_{\theta_1}$ and $\theta_2 \in G_{\theta_2}$, we denote $\theta_1 \preceq \theta_2$ if there exists a constant $C > 0$ such that $\theta_1(t) \leq C\theta_2(t)$ for all $t > 0$. 

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First we recall definition of (strong) Orlicz–Morrey spaces of Deringoz–Guliyev–Samko’s (2014) version. Let \( \Theta \) be a Young function and \( \theta \in G_\Theta \), the Orlicz–Morrey space \( \mathcal{M}_{\theta,\Theta}(\mathbb{R}^n) \) is the set of measurable functions \( f \) on \( \mathbb{R}^n \) such that

\[
\|f\|_{\mathcal{M}_{\theta,\Theta}(\mathbb{R}^n)} := \sup_{a \in \mathbb{R}^n, r > 0} \frac{1}{\theta(|B(a, r)|^{\frac{1}{n}})} \Theta^{-1}\left(\frac{1}{|B(a, r)|}\right) \|f\|_{L_{\theta}(B(a, r))} < \infty,
\]

where \( \|f\|_{L_{\theta}(B(a, r))} := \inf\{b > 0 : \int_{B(a, r)} \Theta\left(\frac{|f(x)|}{b}\right) dx \leq 1\} \). Here, \( B(a, r) \) denotes the open ball in \( \mathbb{R}^n \) centered at \( a \in \mathbb{R}^n \) with radius \( r > 0 \), and \( |B(a, r)| \) for its Lebesgue measure.

Meanwhile, for \( \Theta \) is Young function and \( \theta \in G_\Theta \), the weak Orlicz–Morrey space \( w\mathcal{M}_{\theta,\Theta}(\mathbb{R}^n) \) is the set of all measurable functions \( f \) on \( \mathbb{R}^n \) such that

\[
\|f\|_{w\mathcal{M}_{\theta,\Theta}(\mathbb{R}^n)} := \sup_{a \in \mathbb{R}^n, r > 0} \frac{1}{\theta(|B(a, r)|^{\frac{1}{n}})} \Theta^{-1}\left(\frac{1}{|B(a, r)|}\right) \|f\|_{wL_{\theta}(B(a, r))} < \infty,
\]

where \( \|f\|_{wL_{\theta}(B(a, r))} := \inf\left\{ b > 0 : \sup_{t > 0} \Phi(t)\left\{|x \in B(a, r) : \frac{|f(x)|}{t} > t\right\} \leq 1 \right\} \).

The space \( w\mathcal{M}_{\theta,\Theta}(\mathbb{R}^n) \) is quasi-Banach spaces equipped with the quasi-norm \( \| \cdot \|_{w\mathcal{M}_{\theta,\Theta}(\mathbb{R}^n)} \). Note that, analog with \( \mathcal{M}_{\theta,\Theta}(\mathbb{R}^n) \) space, \( w\mathcal{M}_{\theta,\Theta}(\mathbb{R}^n) \) also covers many classical spaces, which shown in the following example.

**Example 1.1.** Let \( 1 \leq p \leq q < \infty \), \( \Phi \) be a Young function, and \( \theta \in G_\Theta \) then we obtain:

1. If \( \Theta(t) = t^p \) and \( \theta(t) = t^{\frac{p}{n}} \), then \( w\mathcal{M}_{\theta,\Theta}(\mathbb{R}^n) = wL^p(\mathbb{R}^n) \) is weak Lebesgue space.
2. If \( \Theta(t) = t^q \) and \( \theta(t) = t^{\frac{q}{n}} \), then \( w\mathcal{M}_{\theta,\Theta}(\mathbb{R}^n) = w\mathcal{M}_q(\mathbb{R}^n) \) is classical weak Morrey space.
3. If \( \Theta(t) = t^p \), then \( w\mathcal{M}_{\theta,\Theta}(\mathbb{R}^n) = w\mathcal{M}_p(\mathbb{R}^n) \) is generalized weak Morrey space.
4. If \( \theta(t) = \Theta^{-1}(t^{-n}) \), then \( w\mathcal{M}_{\theta,\Theta}(\mathbb{R}^n) = wL_\Theta(\mathbb{R}^n) \) is weak Orlicz space.

Moreover, the relationship between \( \mathcal{M}_{\theta,\Theta}(\mathbb{R}^n) \) space and \( w\mathcal{M}_{\theta,\Theta}(\mathbb{R}^n) \) space can be stated as the following lemma.

**Lemma 1.2.** Let \( \Theta \) be a Young function and \( \theta \in G_\Theta \). Then \( \mathcal{M}_{\theta,\Theta}(\mathbb{R}^n) \subseteq w\mathcal{M}_{\theta,\Theta}(\mathbb{R}^n) \) with \( \|f\|_{w\mathcal{M}_{\theta,\Theta}(\mathbb{R}^n)} \leq \|f\|_{\mathcal{M}_{\theta,\Theta}(\mathbb{R}^n)} \) for every \( f \in \mathcal{M}_{\theta,\Theta}(\mathbb{R}^n) \).

Many authors have been culminating important observations about inclusion properties of function spaces, see [7, 8, 9, 10, 11, 15], etc. Recently, Masta et al. [11] obtained sufficient and necessary conditions for inclusion of (strong) Orlicz–Morrey spaces of all versions. In the same paper, Masta et al. also proved the sufficient and necessary conditions for inclusion properties of weak Orlicz–Morrey spaces of Nakai’s and Sawano–Sugano–Tanaka’s versions.

In this paper, we would like to obtain the inclusion properties of weak Orlicz–Morrey space \( w\mathcal{M}_{\theta,\Theta}(\mathbb{R}^n) \) of Deringoz–Guliyev–Samko’s version, and compare it with the result for Nakai’s and Sawano–Sugano–Tanaka’s versions.

To achieve our purpose, we will use the similar methods in [5, 9, 11, 14] which pay attention to the characteristic functions of open balls in \( \mathbb{R}^n \), in the following lemma.
Lemma 1.3. \cite{4} Let $\Theta$ be a Young function, $\theta \in G_\theta$, and $r_0 > 0$, then there exists a constant $C > 0$ such that
\[
\frac{1}{\theta(r_0)} \leq \|\chi_{B(0,r_0)}\|_{M_{\theta_0,\Theta}(\mathbb{R}^n)} \leq \frac{C}{\theta(r_0)}.
\]

For weak Orlicz–Morrey spaces, we have the following lemma.

Lemma 1.4. Let $\Theta$ be a Young function, $\theta \in G_\theta$, and $r_0 > 0$, then there exists a constant $C > 0$ such that
\[
\frac{1}{\theta(r_0)} \leq \|\chi_{B(0,r_0)}\|_{wM_{\theta_0,\Theta}(\mathbb{R}^n)} \leq \frac{C}{\theta(r_0)}.
\]

Proof. Since $\Theta$ is a Young function and $\theta \in G_\theta$, by Lemmas 1.2 and 1.3, we have
\[
\|\chi_{B(0,r_0)}\|_{wM_{\theta_0,\Theta}(\mathbb{R}^n)} \leq \|\chi_{B(0,r_0)}\|_{M_{\theta_0,\Theta}(\mathbb{R}^n)} \leq \frac{C}{\theta(r_0)}.
\]

On the other hand,
\[
\|\chi_{B(0,r_0)}\|_{wM_{\theta_0,\Theta}(\mathbb{R}^n)} = \sup_{a \in \mathbb{R}^n, r > 0} \frac{1}{\Theta(\|B(a,r)\|/\|B(0,r_0)\|)} \theta^{-1} \left( \frac{1}{\|B(a,r)\|} \right) \|\chi_{B(0,r_0)}\|_{wL_B(a,r)}
\]
\[
= \sup_{a \in \mathbb{R}^n, r > 0} \frac{1}{\Theta(\|B(a,r)\|/\|B(0,r_0)\|)} \theta^{-1} \left( \frac{1}{\|B(a,r)\|} \right) \Theta^{-1} \left( \frac{1}{\|B(0,r_0)\|/\|B(a,r)\|} \right)
\]
\[
\geq \frac{1}{\theta(r_0)}.
\]

Consequently, we have $\frac{1}{\theta(r_0)} \leq \|\chi_{B(0,r_0)}\|_{wM_{\theta_0,\Theta}(\mathbb{R}^n)} \leq \frac{C}{\theta(r_0)}$. \qed

In this paper, the letter $C$ will be used for constants that may change from line to line, while constants with subscripts, such as $C_1, C_2$, do not change in different lines.

2 Results

First, we reprove sufficient and necessary conditions for inclusion properties of Orlicz–Morrey space $M_{\theta_0,\Theta}(\mathbb{R}^n)$ in the following theorem.

Theorem 2.1. \cite{11} Let $\Theta_1, \Theta_2$ be Young functions such that $\Theta_1 < \Theta_2$, $\Theta_1^{-1} \leq \Theta_2^{-1}$, $\theta_1 \in G_{\theta_1}$ and $\theta_2 \in G_{\theta_2}$. Then the following statements are equivalent:

1. $\theta_2 \lesssim \theta_1$.
2. $M_{\theta_2,\Theta_2}(\mathbb{R}^n) \subseteq M_{\theta_1,\Theta_1}(\mathbb{R}^n)$.
3. There exists a constant $C > 0$ such that
\[
\|f\|_{M_{\theta_1,\Theta_1}(\mathbb{R}^n)} \leq C \|f\|_{M_{\theta_2,\Theta_2}(\mathbb{R}^n)}
\]
for every $f \in M_{\theta_2,\Theta_2}(\mathbb{R}^n)$.
Proof. Assume that (1) holds and let \( f \in M_{\theta_1,\Theta_1}(\mathbb{R}^n) \). Since \( \Theta_1 < \Theta_2 \), by using similar arguments in the proof of Corollary 2.3 in [9], we have

\[
\|f\|_{L_{\theta_1}(B(a,r))} \leq C \|f\|_{L_{\theta_2}(B(a,r))}
\]

for every \( B(a,r) \subseteq \mathbb{R}^n \).

Knowing that, \( \Theta_1^{-1} \lesssim \Theta_2^{-1} \) and \( \theta_1 \lesssim \theta_2 \) (i.e. there exists constant \( C_1, C_2 > 0 \) such that \( \Theta_1^{-1}(t) \leq C_1 \Theta_2^{-1}(t) \) and \( \theta_2(t) \leq C_2 \theta_1(t) \) for every \( t > 0 \), we obtain

\[
\|f\|_{M_{\theta_1,\Theta_1}(\mathbb{R}^n)} := \sup_{a \in \mathbb{R}^n, \ r > 0} \frac{1}{\theta_1(|B(a,r)|^{\frac{1}{n}})} \Theta_1^{-1}\left(\frac{1}{|B(a,r)|}\right) \|f\|_{L_{\theta_1}(B(a,r))}
\]

\[
\leq \sup_{a \in \mathbb{R}^n, \ r > 0} \frac{1}{\theta_1(|B(a,r)|^{\frac{1}{n}})} \Theta_1^{-1}\left(\frac{1}{|B(a,r)|}\right) \|f\|_{L_{\theta_2}(B(a,r))}
\]

\[
\leq \sup_{a \in \mathbb{R}^n, \ r > 0} \frac{1}{\theta_2(|B(a,r)|^{\frac{1}{n}})} \Theta_2^{-1}\left(\frac{1}{|B(a,r)|}\right) \|f\|_{L_{\theta_2}(B(a,r))}
\]

\[
:= C C_1 C_2 \|f\|_{M_{\theta_2,\Theta_2}(\mathbb{R}^n)}
\]

This proves that \( M_{\theta_2,\Theta_2}(\mathbb{R}^n) \subseteq M_{\theta_1,\Theta_1}(\mathbb{R}^n) \).

Next, since \( (M_{\theta_2,\Theta_2}(\mathbb{R}^n), M_{\theta_1,\Theta_1}(\mathbb{R}^n)) \) is a Banach pair, it follows from [9] Lemma 3.3 that (2) and (3) are equivalent. It thus remains to show that (3) implies (1).

Assume that (3) holds. Let \( r_0 > 0 \). By Lemma 1.3, we have

\[
\frac{1}{\theta_1(r_0)} \leq \|\chi B(0,r_0)\|_{M_{\theta_1,\Theta_1}(\mathbb{R}^n)} \leq C \|\chi B(0,r_0)\|_{M_{\theta_2,\Theta_2}(\mathbb{R}^n)} \leq \frac{C}{\theta_2(r_0)}.
\]

Since \( r_0 > 0 \) is arbitrary, we conclude that \( \theta_2(t) \leq C \theta_1(t) \) for every \( t > 0 \).

Now we come to the inclusion property of weak Orlicz–Morrey spaces \( wM_{\theta_1,\Theta_1}(\mathbb{R}^n) \) and \( wM_{\theta_2,\Theta_2}(\mathbb{R}^n) \) with respect to Young functions \( \Theta_1, \Theta_2 \) and parameters \( \theta_1, \theta_2 \).

**Theorem 2.2.** Let \( \Theta_1, \Theta_2 \) be Young functions such that \( \Theta_1 < \Theta_2, \ \Theta_1^{-1} \lesssim \Theta_2^{-1}, \ \theta_1 \in G_{\theta_1} \) and \( \theta_2 \in G_{\theta_2} \). Then the following statements are equivalent:

1. \( \theta_2 \lesssim \theta_1 \).
2. \( wM_{\theta_2,\Theta_2}(\mathbb{R}^n) \subseteq wM_{\theta_1,\Theta_1}(\mathbb{R}^n) \).
3. There exists a constant \( C > 0 \) such that

\[
\|f\|_{wM_{\theta_1,\Theta_1}(\mathbb{R}^n)} \leq C \|f\|_{wM_{\theta_2,\Theta_2}(\mathbb{R}^n)}
\]

for every \( f \in wM_{\theta_2,\Theta_2}(\mathbb{R}^n) \).

**Proof.** Assume that (1) holds and let \( f \in wM_{\theta_2,\Theta_2}(\mathbb{R}^n) \). Since \( \Theta_1 < \Theta_2 \), by using similar arguments in the proof of Theorem 3.3 in [9], we have
Knowing that, \( \Theta_1^{-1} \leq \Theta_2^{-1} \) and \( \theta_1 \leq \theta_2 \) (i.e. there exists constant \( C_1, C_2 > 0 \) such that \( \Theta_1^{-1}(t) \leq C_1 \Theta_2^{-1}(t) \) and \( \theta_2(t) \leq C_2 \theta_1(t) \) for every \( t > 0 \)), we obtain

\[
\| f \|_{wM_{\theta_1,\Theta_1}(\mathbb{R}^n)} := \sup_{a \in \mathbb{R}^n, r > 0} \frac{1}{\theta_1(\| B(a, r) \|^{1/\gamma})} \Theta_1^{-1}\left( \frac{1}{\| B(a, r) \|} \right) \| f \|_{wL_{\Theta_1}(B(a, r))}
\]

\[
\leq \sup_{a \in \mathbb{R}^n, r > 0} \frac{C}{\theta_1(\| B(a, r) \|^{1/\gamma})} \Theta_1^{-1}\left( \frac{1}{\| B(a, r) \|} \right) \| f \|_{wL_{\Theta_2}(B(a, r))}
\]

\[
\leq \sup_{a \in \mathbb{R}^n, r > 0} \frac{CC_1}{\theta_1(\| B(a, r) \|^{1/\gamma})} \Theta_2^{-1}\left( \frac{1}{\| B(a, r) \|} \right) \| f \|_{wL_{\Theta_2}(B(a, r))}
\]

\[
\leq \sup_{a \in \mathbb{R}^n, r > 0} \frac{CC_1C_2}{\theta_2(\| B(a, r) \|^{1/\gamma})} \Theta_2^{-1}\left( \frac{1}{\| B(a, r) \|} \right) \| f \|_{wL_{\Theta_2}(B(a, r))}
\]

\[
:= CC_1C_2 \| f \|_{wM_{\theta_2,\Theta_2}(\mathbb{R}^n)}
\]

This proves that \( wM_{\theta_2,\Theta_2}(\mathbb{R}^n) \subseteq wM_{\theta_1,\Theta_1}(\mathbb{R}^n) \).

As a corollary of the Open Mapping Theorem [3, Appendix G], we are aware that [6, Chapter I, Lemma 3.3] still holds for quasi-Banach spaces, and so (2) and (3) are equivalent.

Assume that (3) holds. Let \( r_0 > 0 \). By Lemma 1.3, we have

\[
\frac{1}{\Theta_1(r_0)} \leq \| \chi_{B(0,r_0)} \|_{wM_{\theta_1,\Theta_1}(\mathbb{R}^n)} \leq C \| \chi_{B(0,r_0)} \|_{wM_{\theta_2,\Theta_2}(\mathbb{R}^n)} \leq \frac{C}{\Theta_2(r_0)}.
\]

Since \( r_0 > 0 \) is arbitrary, we conclude that \( \theta_2(t) \leq C \theta_1(t) \) for every \( t > 0 \).

For generalized weak Morrey spaces, we have the following corollary.

**Corollary 2.3.** Let \( 1 \leq p < \infty \), \( \theta_1 \in G_{\theta_1} \) and \( \theta_2 \in G_{\theta_2} \). Then the following statements are equivalent:

1. \( \theta_2 \leq \theta_1 \).
2. \( wM_{\theta_2}^p(\mathbb{R}^n) \subseteq wM_{\theta_1}^p(\mathbb{R}^n) \).
3. There exists a constant \( C > 0 \) such that

\[
\| f \|_{wM_{\theta_1}^p(\mathbb{R}^n)} \leq C \| f \|_{wM_{\theta_2}^p(\mathbb{R}^n)}
\]

for every \( f \in wM_{\theta_2}^p(\mathbb{R}^n) \).

## 3 Concluding Remarks

We have shown the sufficient and necessary conditions for the inclusion relation between weak Orlicz–Morrey space \( wM_{\theta,\Theta}(\mathbb{R}^n) \). In the proof of the inclusion property we used the norm of characteristic function on \( \mathbb{R}^n \). The inclusion properties of weak Orlicz-Morrey space \( wM_{\theta,\Theta}(\mathbb{R}^n) \)
(Theorem 2.2) and weak Orlicz–Morrey space \( w\mathcal{M}_{\psi,\Psi}(\mathbb{R}^n) \) of Sawano–Sugano–Tanaka’s version (Theorem 3.9 in [11]) generalize the inclusion properties of weak Morrey spaces and generalized weak Morrey spaces in [5]. Meanwhile, the inclusion properties of weak Orlicz-Morrey space \( wL_{\phi,\Theta}(\mathbb{R}^n) \) of Nakai’s version (Theorem 3.4 in [11]) generalize the inclusion properties of weak Orlicz spaces in [9].

Comparing Theorem 2.2 and Theorem 3.9 in [11], we can say that the condition on the Young function for the inclusion of the weak Orlicz–Morrey space \( w\mathcal{M}_{\psi,\Psi}(\mathbb{R}^n) \) is simpler than that for the weak Orlicz–Morrey space \( w\mathcal{M}_{\phi,\Theta}(\mathbb{R}^n) \).

As a corollary of Lemma 1.2, Theorem 2.1 and Theorem 2.2, we also have the following inclusion relations

\[
\mathcal{M}_{\theta_2,\Theta_2} \hookrightarrow \mathcal{M}_{\theta_1,\Theta_1} \\
\downarrow \\
w\mathcal{M}_{\theta_2,\Theta_2} \hookrightarrow w\mathcal{M}_{\theta_1,\Theta_1}
\]

for \( \Theta_1 \prec \Theta_2, \Theta_1^{-1} \subseteq \Theta_2^{-1} \) and \( \theta_2 \preceq \theta_1 \), where the arrows mean ‘contained in’ or ‘embedded into’.

References

[1] F. Deringoz, V. S. Guliyev, and S. Samko, “Boundedness of the maximal and singular operators on generalized Orlicz–Morrey spaces”, In: Operator Theory, Operator Algebra and Applications, Operator Theory: Advances and Applications 242 (2014), 139–158.

[2] S. Gala, Y. Sawano, and H. Tanaka, “A remark on two generalized Orlicz-Morrey spaces”, J. Approx. Theory 198 (2015), 1–9.

[3] L. Grafakos, Modern Fourier analysis, Graduate Texts in Mathematics, 250, Springer, New York, 2009.

[4] V. S. Guliyev, S. G. Hasanov, Y. Sawano, and T. Noi, “Non-smooth atomic decompositions for generalized Orlicz–Morrey spaces of the third kind”, Acta Appl. Math. 145–1 (2017), 133–174 [DOI: 10.1007/s10440-016-0052-7].

[5] H. Gunawan, D.I. Hakim, K.M. Limanta, A.A. Masta, “Inclusion properties of generalized Morrey spaces”, Math. Nachr. 290 (2017), 332-340 [DOI:10.1002/mana.201500425].

[6] S.G Kreĭn, Yu.I Petunîn, and E.M. Semënîov, Interpolation of Linear Operators, Translation of Mathematical Monograph vol. 54, American Mathematical Society, Providence, R.I., 1982.

[7] A. Kufner, O. John, and S. Fučík, Function Spaces, Noordhoff International Publishing, Czechoslovakia, 1977.

[8] L. Maligranda, Orlicz Spaces and Interpolation, Departamento de Matemática, Universidade Estadual de Campinas, 1989.

[9] A.A. Masta, H. Gunawan, and W. Setya-Budhi, “Inclusion property of Orlicz and weak Orlicz spaces”, J. Math. Fund. Sci. 48-3 (2016), 193–203 [DOI: http://dx.doi.org/10.56142Fj.math.fund.sci.2016.48.3.1].
[10] A.A. Masta, H. Gunawan, W.S. Budhi, “An inclusion property of Orlicz-Morrey spaces”, *J. Phys.: Conf. Ser.* **893** 012015 (2017), 1–7, [DOI: https://doi.org/10.1088/1742-6596/893/1/012015].

[11] A.A. Masta, H. Gunawan, and W. Setya-Budhi, “On inclusion properties of two versions of Orlicz–Morrey spaces”, *Mediterr. J. Math.*, **14-6** (2017), 1–12 [DOI: https://doi.org/10.1007/s00009-017-1030-7].

[12] E. Nakai, “On Orlicz-Morrey spaces”, research report [http://repository.kulib.kyoto-u.ac.jp/dspace/bitstream/2433/58769/1/1520-10.pdf, accessed on August 17, 2015.]

[13] E. Nakai, “Orlicz-Morrey spaces and some integral operators”, research report [http://repository.kulib.kyoto-u.ac.jp/dspace/bitstream/2433/26035/1/1399-13.pdf, accessed on August 17, 2015.]

[14] A. Osançiol, “Inclusion between weighted Orlicz spaces”, *J. Inequal. Appl.* **2014**-390 (2014), 1–8, [DOI: https://doi.org/10.1186/1029-242X-2014-390].

[15] M. Taqiyuddin M and A. A. Masta, “Inclusion properties of Orlicz spaces and weak Orlicz spaces generated by concave function”, *IOP Conf. Ser.: Mater. Sci. Eng.*, **288**-012103, 1–5 [DOI: https://doi.org/10.1088/1757-899X/288/1/012103].