Reinforcement Learning for Multi-Objective and Constrained Markov Decision Processes

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Abstract
In this paper, we consider the problem of optimization and learning for constrained and multi-objective Markov decision processes, for both discounted rewards and expected average rewards. We formulate the problems as zero-sum games where one player (the agent) solves a Markov decision problem and its opponent solves a bandit optimization problem, which we here call Markov-Bandit games. We extend Q-learning to solve Markov-Bandit games and show that our new Q-learning algorithms converge to the optimal solutions of the zero-sum Markov-Bandit games, and hence converge to the optimal solutions of the constrained and multi-objective Markov decision problems. We provide a numerical example where we calculate the optimal policies and show by simulations that the algorithm converges to the calculated optimal policies. To the best of our knowledge, this is the first time learning algorithms guarantee convergence to optimal stationary policies for the constrained MDP problem with discounted and expected average rewards, respectively.

1 Introduction

1.1 Motivation
Reinforcement learning has made great advances in several applications, ranging from online learning and recommender engines, natural language understanding and generation, to mastering games such as Go [19] and Chess. The idea is to learn from extensive experience how to take actions that maximize a given reward by interacting with the surrounding environment. The interaction teaches the agent how to maximize its reward without knowing the underlying dynamics of the process. A classical example is swinging up a pendulum in an upright position. By making several attempts to swing up a pendulum and balancing it, one might be able to learn the necessary forces that need to be applied in order to balance the pendulum without knowing the physical
model behind it, which is the general approach of classical model based control theory \[4\].

However, many applications in practice require that we take actions that are subject to additional constraints that need to be fulfilled simultaneously \[3\]. One example is wireless communication where the total transmission power of the connected wireless devices is to be minimized subject to maximum delay constraints \[8\]. Also, safety requirements in decision making can be modeled as constrained Markov decision processes \[11\].

Informally, the most common problem description of constrained Markov Decision Processes (MDP:s) is as follows. Given a stochastic process with state \(s_k\) at time step \(k\), reward function \(r\), and a discount factor \(0 < \gamma < 1\), the constrained MDP problem is that for the optimizing agent to find a stationary policy \(\pi(s_k)\) that minimizes the discounted reward

\[
E \left( \sum_{k=0}^{\infty} \gamma^k r(s_k, \pi(s_k)) \right) 
\]

or the expected average reward

\[
\lim_{T \to \infty} E \left( \frac{1}{T} \sum_{k=0}^{T-1} r(s_k, \pi(s_k)) \right) 
\]

subject to constraints of the forms

\[
E \left( \sum_{k=0}^{\infty} \gamma^k r_j(s_k, \pi(s_k)) \right) \geq 0 
\]

and

\[
\lim_{T \to \infty} E \left( \frac{1}{T} \sum_{k=0}^{T-1} r_j(s_k, \pi(s_k)) \right) \geq 0 
\]

respectively, for \(j = 1, ..., J - 1\). The multi-objective or multi-criteria reinforcement learning problem of Markov decision processes is that of finding a policy that satisfies a number of of constraints in the form \[3\] or \[4\], which corresponds to the constrained Markov decision problem when the reward function \(r(s, a) = 0\) for all \((s, a)\).

**Example 1** (Altman, 1999). Consider a discrete time single-server queue with a buffer of finite size \(L\). For a given time slot, we assume that at most one customer may join the system. The state of the system at a given time slot is the number of customers in the queue. There is a delay cost \(c(s)\) given a state \(s \in \{S_1, ..., S_n\}\). The probability of a service to be completed is \(a_1\), where \(1/a_1\) is the Quality of Service (QoS). The probability of queue arrival at time \(t\) is \(a_2\). The actions are given by \(a_1\) and \(a_2\). Let \(c^1(a_1)\) be the cost to complete the service \((c^1\) is increasing in \(a_1\)). There is a cost corresponding to the throughput, \(c^2(a_2)\), \((c^2\) is decreasing in \(a_2\)). We assume that the number of actions is finite and actions sets are given by \(a_1 \in \{A_1, ..., A_n\}\) and \(a_2 \in \{B_1, ..., B_l\}\) where \(0 < A_1 \leq \cdots \leq A_n \leq 1\) and \(0 \leq B_1 \leq \cdots \leq B_l \leq 1\).
The transition probability \( P(s_{k+1}, s_k, a_1^k, a_2^k) \) from state \( s_k \) to \( s_{k+1} \) given actions \( a_1^k \) and \( a_2^k \) is given by

\[
P(s_+, s, a_1^k, a_2^k) = \begin{cases} 
B_{l_2} a_1^k & \text{if } L \geq s \geq 1, \\
a_2^k a_1^k + B_{l_2} A_{l_1} & \text{if } L \geq s \geq 1, \\
a_2^k A_{l_1} & \text{if } L \geq s \geq 0, \\
1 - a_2^k A_{l_1} & \text{if } L \geq s \geq 0, \\
1 & \text{if } L \geq s = 0 
\end{cases}
\]

For \( \gamma \in (0, 1) \), the constrained Markov decision process problem is given by

\[
\min_{\pi_1, \pi_2} \mathbb{E} \left( \sum_{k=0}^{\infty} \gamma^k c(s_k) \right) \\
\text{s. t. } \mathbb{E} \left( \sum_{k=0}^{\infty} \gamma^k c^1(\pi_1^k(s_k)) \right) \leq v^1 \tag{5} \\
\mathbb{E} \left( \sum_{k=0}^{\infty} \gamma^k c^2(\pi_2^k(s_k)) \right) \leq v^2
\]

which is equivalent to

\[
\max_{\pi_1, \pi_2} \mathbb{E} \left( \sum_{k=0}^{\infty} \gamma^k r(s_k, \pi(s_k)) \right) \\
\text{s. t. } \mathbb{E} \left( \sum_{k=0}^{\infty} \gamma^k r^1(s_k, \pi(s_k)) \right) \geq 0 \tag{6} \\
\mathbb{E} \left( \sum_{k=0}^{\infty} \gamma^k r^2(s_k, \pi(s_k)) \right) \geq 0
\]

where \( a_k = (a_1^k, a_2^k) \), \( \pi(s_k) = (\pi_1^k(s_k), \pi_2^k(s_k)) \), \( r(s_k, a_k) = -c(s_k) \), \( r^1(s_k, a_k) = -c^1(a_k^1) + \frac{v_1^1}{1-\gamma} \), \( r^2(s_k, a_k) = -c^2(a_k^2) + \frac{v_2^1}{1-\gamma} \).

Surprisingly, although constrained MDP problems are fundamental and have been studied extensively in the literature (see [3] and the references therein), the reinforcement learning counterpart seems to be still open. When an agent have to take actions based on the observed states, rewards outputs, and constraint-outputs solely (without any knowledge about the dynamics, reward functions, and/or constraint-functions), a general solution seems to be lacking to the best of the authors' knowledge for both the discounted and expected average rewards cases.
1.2 Previous Work

Constrained MDP problems are convex and hence one can convert the constrained MDP problem to an unconstrained zero-sum game where the objective is the Lagrangian of the optimization problem [3]. However, when the dynamics and rewards are not known, it doesn’t become apparent how to do it as the Lagrangian will itself become unknown to the optimizing agent. Previous work regarding constrained MDP:s, when the dynamics of the stochastic process are not known, considers scalarization through weighted sums of the rewards, see [18] and the references therein. Another approach is to consider Pareto optimality when multiple objectives are present [17]. However, none of the aforementioned approaches guarantee to satisfy lower bounds for a given set of reward functions simultaneously. In [10], a multi-criteria problem is considered where the search is over deterministic policies. In general, however, deterministic policies are not optimal [3]. Also, the multi-criteria approach in [10] may provide a deterministic solution to a multi-objective problem in the case of two objectives and it’s not clear how to generalize to a number of objectives larger than two. In [12] the author considers a single constraint and allowing for randomized policies. However, no proofs of convergence are provided for the proposed sub-optimal algorithms. Sub-optimal solutions with convergence guarantees were provided in [21] for the single constraint problem, allowing for randomized policies. In [5], an actor-critic sub-optimal algorithm is provided for one single constraints and it’s claimed that it can generalized to an arbitrary number of constraints. Sub-optimal solutions to constrained reinforcement learning problems with expected average rewards in a wireless communications context were considered in [8]. Sub-optimal reinforcement learning algorithms were presented in [15] for controlled trial analysis with multiple rewards, again by considering a scalarization approach. In [9], multi-objective bandit algorithms were studied by considering scalarization functions and Pareto partial orders, respectively, and present regret bounds. As previous results, the approach in [9] doesn’t guarantee to satisfy the constraints that correspond to the multiple objectives. In [2], constrained policy optimization is studied for the continuous MDP problem and some heuristics algorithms were suggested.

1.3 Contributions

We consider the problem of optimization and learning for constrained and multi-objective Markov decision processes, for both discounted rewards and expected average rewards. We formulate the problems as zero-sum games where one player (the agent) solves a Markov decision problem and its opponent solves a bandit optimization problem, which here we call Markov-Bandit games which are interesting on their own. The opponent acts on a finite set of actions (and not on a continuous space of actions). This transformation is essential in order to achieve a tractable optimal algorithm. The reason is that using Lagrange duality without model knowledge requires infinite dimensional optimization in the learning algorithm since the Lagrange multipliers are continuous (compare to the intractability of a partially observable MDP, where the beliefs are continuous variables). We extend Q-learning to solve Markov-Bandit games and show that our new Q-learning algorithms converge to the optimal solutions of the
zero-sum Markov-Bandit games, and hence converge to the optimal solutions of the constrained and multi-objective Markov decision problems. The proof techniques are different for solving the discounted and average rewards problems, respectively, where the latter becomes much more technically involved. We provide a numerical example where we calculate the optimal policies and show by simulations that the algorithm converges to the calculated optimal policies. To the best of our knowledge, this is the first time learning algorithms guarantee convergence to optimal stationary policies for the constrained and multi-objective MDP problem with discounted and expected average rewards, respectively.

1.4 Outline

In the problem formulation section 2, we present a precise mathematical definition of the constrained reinforcement learning problem for MDP:s. Then, we give a brief introduction to reinforcement learning with applications to zero-sum games and some useful results in the section on reinforcement learning for zero-sum Markov games 3. A solution to the constrained reinforcement learning problem is then presented in the section that follows 4. We demonstrate the algorithms by a small simulation 5 and we finally conclude the paper and discuss future work in section 6. Most of the proofs are relegated to the appendix.

1.5 Notation

\( \mathbb{N} \) The set of nonnegative integers.
\( \mathbb{Z} \) The set of integers \( \{0, 1, ..., J - 1\} \).
\( \mathbb{R} \) The set of real numbers.
\( \mathbb{E} \) The expectation operator.
\( \Pr \) \( \Pr(x \mid y) \) denotes the probability of the stochastic variable \( x \) given \( y \).
\( \arg \max \) \( \pi^* = \arg \max_{\pi \in \Pi} f_\pi \) denotes an element \( \pi^* \in \Pi \) that maximizes the function \( f_\pi \).
\( \geq \) For \( \lambda = (\lambda_0, ..., \lambda_{J-1}) \), \( \lambda \geq 0 \) denotes that \( \lambda_i \geq 0 \) for \( i = 0, ..., J - 1 \).
\( 1_{\{X\}}(x) \) \( 1_{\{X\}}(x) = 1 \) if \( x \in X \) and \( 1_{\{X\}}(x) = 0 \) if \( x \notin X \).
\( 1_n \) \( 1_n = (1, 1, ..., 1) \in \mathbb{R}^n \).
\( N(t, s, a, b) \) \( N(t, s, a, b) = \sum_{k=1}^n 1_{\{s, a, b\}}(s_k, a_k, b_k) \).
\( e \) \( e : (s, a, o) \mapsto 1 \).
\( |S| \) Denotes the number of elements in \( S \).
\( s_+ \) For a state \( s = s_k \), we have \( s_+ = s_{k+1} \).

2 Problem Formulation

Consider a Markov Decision Process (MDP) defined by the tuple \( (S, A, P) \), where \( S = \{S_1, S_2, ..., S_n\} \) is a finite set of states, \( A = \{A_1, A_2, ..., A_m\} \) is a finite set of
actions taken by the agent, and $P : S \times A \times S \to [0, 1]$ is a transition function mapping each triple $(s, a, s_+)$ to a probability given by

$$P(s, a, s_+) = \Pr(s_+ \mid s, a)$$

and hence,

$$\sum_{s_+ \in S} P(s, a, s_+) = 1, \quad \forall (s, a) \in S \times A$$

Let $\Pi$ be the set of policies that map a state $s \in S$ to a probability distribution of the actions with a probability assigned to each action $a \in A$, that is $\pi(s) = a$ with probability $\Pr(a \mid s)$. The agent’s objective is to find a stationary policy $\pi \in \Pi$ that maximizes the expected value of the total discounted reward (1) or the expected value of the average reward (2), for $s_0 = s \in S$, where $r : S \times A \to \mathbb{R}$ is some unknown reward function. The parameter $\gamma \in (0, 1)$ is a discount factor which models how much weight to put on future rewards. The expectation is taken with respect to the randomness introduced by the policy $\pi$ and the transition mapping $P$.

Constrained reinforcement learning is concerned with optimizing the total (discounted or average) reward subject to a set of constraints of the form (3) and (4), respectively, where $r^j : S \times A \to \mathbb{R}$ are bounded functions, for $j = 1, \ldots, J - 1$.

**Definition 1** (Unichain MDP). An MDP is called unichain, if for each policy $\pi$ the Markov chain induced by $\pi$ is ergodic, i.e. each state is reachable from any other state.

Unichain MDP:s are usually considered in reinforcement learning problems with discounted rewards, since they guarantee that we learn the process dynamics from the initial states. Thus, for the discounted reward case we will make the following assumption.

**Assumption 1** (Unichain MDP). The MDP $(S, A, P)$ is assumed to be unichain.

For the case of expected average reward, we will make a simpler assumption regarding the existence of a recurring state, a standard assumption in Markov decision process problems with expected average rewards to ensure that the expected reward is independent of the initial state.

**Assumption 2.** There exists a state $s^* \in S$ which is recurrent for every stationary policy $\pi$ played by the agent.

Assumption 2 implies that $E(r^j(s_k, a_k))$ is independent of the initial state at stationarity. Hence, the constraint (4) is at stationarity equivalent to $E(r^j(s_k, a_k)) \geq 0$, for all $k$. We will use this constraint in the sequel which turns out to be very useful in the game-theoretic approach to solve the problem.

**Assumption 3.** The absolute values of the reward functions $r$ and $\{r^j\}_{j=1}^{J-1}$ are strictly bounded by some constant $c$ known to the agent.
3 Reinforcement Learning for Zero-Sum Markov-Bandit Games

A Markov-Bandit zero-sum game is defined by the tuple \((S, A, O, P, R)\), where \(S\), \(A\) and \(P\) are defined as in section 2, \(O = \{o_1, o_2, ..., o_q\}\) is a finite set of actions made by the agent’s opponent. For the zero-sum Markov-Bandit game, we define the reward \(R: S \times A \times O \rightarrow \mathbb{R}\) which is assumed to be bounded. The agent’s objective is to maximize the minimum (average or discounted) reward obtained due to the opponent’s malicious action. The difference between a zero-sum Markov game and a Markov-Bandit game is that the opponent’s action doesn’t affect the state and it chooses a constant action \(o_k = o \in O\) for all time steps \(k\). This will be made more precise in the following sections.

3.1 Discounted Rewards

Consider a zero-sum Markov-Bandit game where the agent is maximizing the total discounted reward given by

\[
V(s) = \min_{o \in O} \mathbb{E}\left( \sum_{k=0}^{\infty} \gamma^k R(s_k, a_k, o) \right)
\]

for the initial state \(s_0 = s \in S\). Let \(Q(s, a, o)\) be the expected reward of the agent taking action \(a_0 = a \in A\) from state \(s_0 = s\), and continuing with an optimal policy thereafter when the opponent takes action \(o\). Note that this is different from zero-sum Markov games with discounted rewards \([14]\), where the opponent’s actions may vary over time, that is \(o_k\) is not a constant. Then for any stationary policy \(\pi\), we have that

\[
Q(s, a, o) = R(s, a, o) + \mathbb{E}\left( \sum_{k=1}^{\infty} \gamma^k R(s_k, \pi(s_k), o) \right)
\]

\[
= R(s, a, o) + \gamma \cdot \mathbb{E}(Q(s_{+}, \pi(s_{+}), o))
\]

Equation (8) is known as the Bellman equation, and the solution to (8) with respect to \(Q\) that corresponds to the optimal policy \(\pi^*\) is denoted \(Q^*\). If we have the function \(Q^*\), then we can obtain the optimal policy \(\pi^*\) according to the equation

\[
\pi^*(s) = \arg \max_{\pi \in \Pi} \min_{o \in O} \mathbb{E}(Q^*(s, \pi(s), o))
\]

which maximizes the total discounted reward

\[
\min_{o \in O} \mathbb{E}\left( \sum_{k=0}^{\infty} \gamma^k R(s_k, \pi^*(s), o) \right) = \min_{o \in O} \mathbb{E}(Q^*(s, \pi^*(s), o))
\]

for \(s = s_0\). Note that the optimal policy may not be deterministic, as opposed to reinforcement learning for unconstrained Markov Decision Processes, where there is always an optimal policy that is deterministic.

In the case we don’t know the process \(P\) and the reward function \(R\), we will not be able to take advantage of the Bellman equation directly. The following results show that we will be able to design an algorithm that always converges to \(Q^*\).
Theorem 1. Consider a Markov-Bandit zero-sum game given by the tuple \((S, A, O, P, R)\) where \((S, A, P)\) is unichain, and suppose that \(R\) is bounded. Let \(Q = Q^*\) be the solution to the Bellman equation

\[
Q(s, a, o) = R(s, a, o) + \gamma \cdot \mathbb{E}\left(Q\left(s^+, \pi^*(s^+), o\right)\right)
\]

\[
\pi^*(s) = \arg \max_{a \in A} \min_{o \in O} \mathbb{E}\left(Q(s, a, o)\right)
\]  \hspace{1cm} (9)

Let \(\alpha_k(s, a, o)\) satisfy

\[
0 \leq \alpha_k(s, a, o) < 1
\]

\[
\sum_{k=0}^{\infty} \alpha_k(s, a, o) = \infty
\]

\[
\sum_{k=0}^{\infty} \alpha_k^2(s, a, o) < \infty
\]

\[
\forall (s, a, o) \in S \times A \times O
\]

Then, the update rule

\[
a_+ = \arg \max_{a \in A} \min_{o \in O} Q_k(s^+, a, o)
\]

\[
Q_{k+1}(s, a, o) = (1 - \alpha(s, a, o))Q_k(s, a, o) + \\
+ \alpha(s, a, o)(R(s, a, o) + \gamma Q_k(s^+, a_+, o))
\]  \hspace{1cm} (11)

converges to \(Q^*\) with probability 1.

3.2 Expected Average Rewards

The agent’s objective is to maximize the minimal average reward obtained due to the opponent’s malicious actions, that is maximizing the total reward given by

\[
V(s_0) = \min_{o \in O} \lim_{T \to \infty} \mathbb{E}\left(\frac{1}{T} \sum_{k=0}^{T-1} R(s_k, a_k, o)\right)
\]  \hspace{1cm} (12)

for some initial state \(s_0 \in S\). Note that this problem is different from the zero-sum game considered in [16] where the opponent has to pick a fixed value for its action, \(o_k = o\), as opposed to the work in [16] where \(o_k\) is allowed to vary over time. Thus, from the opponent’s point of view, the opponent is performing bandit optimization.

Under Assumption 3 and for a given stationary policy \(\pi\), the value of

\[
V(o) = \lim_{T \to \infty} \mathbb{E}\left(\frac{1}{T} \sum_{k=0}^{T-1} R(s_k, \pi(s_k), o)\right)
\]  \hspace{1cm} (13)

is independent of the initial state \(s_0\) for any fixed value of the parameter \(o\). We will make this standard assumption in Markov decision process control problems.
Proposition 1. Consider an MDP \((S, A, P)\) with a total reward \((13)\) for a fixed number \(o\). Under Assumption 2 and for a fixed policy stationary policy \(\pi\), there exists a number \(v(o)\) and a vector \(H(s, o) = (H(S_1, o), ..., H(S_n, o)) \in \mathbb{R}^n\), such that for each \(s \in S\), we have that

\[
H(s, o) + v(o) = \left( R(s, \pi(s), o) + \sum_{s+ \in S} P(s+ | s, \pi(s)) H(s+, o) \right)
\]

Furthermore, the value of \((13)\) is \(V(o) = v(o)\).

Proof. Consult \([5]\). \(\square\)

Introduce

\[
Q(s, a, o) - v(o) = R(s, a, o) + \sum_{s+ \in S} P(s+ | s, a) H(s+, o)
\]

and let \(Q^*, v^*, \text{ and } H^*\) be solutions to Equation (14) corresponding to the optimal policy \(\pi^*\) that maximizes (12). Then we have that

\[
\pi^*(s) = \arg \max_{\pi \in \Pi} \min_{o \in O} \mathbb{E}(Q^*(s, \pi(s), o))
\]

\[
Q^*(s, \pi^*(s), o) - v^*(o) = R(s, \pi^*(s), o) + \sum_{s+ \in S} P(s+ | s, \pi^*(s)) H^*(s+, o)
\]

In the case we don’t know the process \(P\) and the reward function \(R\), we will not be able to take advantage of (15) directly. It’s worth to note here that the operator \(T\) in Equation (14) is not a contraction, so the standard Q-learning that is commonly used for reinforcement learning in Markov decision processes with discounted rewards can’t be applied here.

Assumption 4 (Learning rate). The sequence \(\gamma_k\) satisfies:

1. \(\gamma_{k+1} \leq \gamma_k\) eventually
2. For every \(0 < x < 1\), \(\sup_k \gamma_{\lfloor xk \rfloor} \gamma_k < \infty\)
3. \(\sum_{k=1}^{\infty} \gamma_k = \infty\) and \(\sum_{k=0}^{\infty} \gamma_k^2 < \infty\).
4. For every \(0 < x < 1\), the fraction \(\frac{\sum_{k=1}^{[xt]} \gamma_k}{\sum_{k=1}^{\infty} \gamma_k}\) converges to 1 uniformly in \(y \in [x, 1]\) as \(t \to \infty\).
For example, $\gamma_k = \frac{1}{k}$ and $\gamma_k = \frac{1}{k \log k}$ (for $k > 1$) satisfy Assumption 4.

Now define $N(k, s, a, o)$ as the number of times that state $s$ and actions $a$ and $o$ were played up to time $k$, that is

$$N(k, s, a, o) = \sum_{t=1}^{k} 1_{\{s, a', o\}}(s_t, a_t, o_t)$$

The following assumption is needed to guarantee that all combinations of the triple $(s, a, o)$ are visited often.

**Assumption 5 (Often updates).** There exists a deterministic number $d > 0$ such that for every $s \in S$, $a \in A$, and $o \in O$, we have that

$$\liminf_{k \to \infty} \frac{N(k, s, a, o)}{k} \geq d$$

with probability 1.

**Definition 2.** We define the set $\Phi$ as the set of all functions $f : \mathbb{R}^{n \times m \times q} \to \mathbb{R}$ such that

1. $f$ is Lipschitz
2. For any $c \in \mathbb{R}$, $f(cQ) = cf(Q)$
3. For any $r \in \mathbb{R}$ and $\hat{Q}(s, a, o) = Q(s, a, o) + r$ for all $(s, a, o) \in \mathbb{R}^{n \times m \times q}$, we have $f(\hat{Q}) = f(Q) + r$

The next result shows that we will be able to design an algorithm that always converges to $Q^*$.

**Theorem 2.** Consider a Markov-Bandit zero-sum game given by the tuple $(S, A, O, P, R)$ and suppose that $R$ is bounded. Suppose that Assumption 4 and 5 hold. Let $f \in \Phi$ be given, where the set $\Phi$ is defined as in Definition 2. Then, the asynchronous update algorithm given by

$$a_k = \arg \max_a \min_o Q_k(s_{k+1}, a, o)$$

$$Q_{k+1}(s, a, o) = Q_k(s, a, o) + 1_{\{s, a, o\}}(s_k, a_k, o) \times \gamma_{N(k, s, a, o)}(R(s_k, a_k, o) + Q_k(s_{k+1}, a_k, o) - f(Q_k(s_k, a_k, o)) - Q_k(s, a, o))$$

converges to $Q^*$ in (16) with probability 1.
4 Reinforcement Learning for Constrained Markov Decision Processes

4.1 Discounted Rewards

Consider the optimization problem of finding a stationary policy \( \pi \) that maximizes the reward (1) subject to the initial state \( s_0 = s \) and the constraints (3), that is

\[
\max_{\pi \in \Pi} \mathbb{E} \left( \sum_{k=0}^{\infty} \gamma^k r(s_k, \pi(s_k)) \right) \\
\text{s. t. } \mathbb{E} \left( \sum_{k=0}^{\infty} \gamma^k r_j(s_k, \pi(s_k)) \right) \geq 0 \\
\text{for all } k \in \mathbb{N}, \text{ and } j = 1, \ldots, J - 1 \tag{18}
\]

The next theorem states that the optimization problem (18) is equivalent to a zero-sum Markov game, in the sense that an optimal strategy of the agent in the zero-sum game is also optimal for (18).

**Theorem 3.** Consider optimization problem (18) and suppose that Assumption 3 holds. Introduce \( r_0(s, a) = r(s, a) - c \) and the set

\[
\Lambda = \{ \lambda : \lambda = (\lambda^0, \ldots, \lambda^{J-1}), \lambda^j \geq 0, \lambda^0 + \cdots + \lambda^{J-1} = 1 \} \tag{19}
\]

Let \( \pi^* \) be an optimal stationary policy in the zero-sum Markov-Bandit game

\[
\max_{\pi \in \Pi} \min_{\lambda \in \Lambda} \sum_{j=0}^{J-1} \mathbb{E} \left( \sum_{k=0}^{\infty} \gamma^k \lambda^j r_j(s_k, \pi(s_k)) \right) \tag{20}
\]

Then, \( \pi^* \) is an optimal policy to (18).

The interpretation of the game (20) is that the minimizer chooses index \( j \) with probability \( \lambda^j \), and hence, the minimizer applies mixed strategies given by \( \lambda \in \Lambda \).

Now that we are equipped with Theorem 1 and 3, we are ready to state and prove our next result.

**Theorem 4.** Consider the constrained MDP problem (18) and suppose that Assumption 1 and 3 hold. Introduce \( r^0(s, a) = r(s, a) - c \). Also, introduce \( O = \mathbb{Z}, o = j \) and

\[
R(s, a, o) = R(s, a, j) \triangleq r_j(s, a), \quad j = 0, \ldots, J - 1
\]

Let \( Q_k \) be given by the recursion according to (11). Then, \( Q_k \to Q^* \) as \( k \to \infty \) where \( Q^* \) is the solution to (9). Furthermore, the policy

\[
\pi^*(s) = \arg \max_{\pi \in \Pi} \min_{o \in O} \mathbb{E} (Q^*(s, \pi(s), o)) \tag{21}
\]

is an optimal solution to (18) for all \( s \in S \).
Proof. According to Theorem 3, (18) is equivalent to the zero-sum Markov-Bandit game (20), which is equivalent to
\[
\max_{\pi \in \Pi} \min_o \mathbb{E} \left( \sum_{k=0}^{\infty} \gamma^k R(s_k, \pi(s_k), o) \right)
\]
where \( o = j \) with probability \( \lambda_j \). Assumption 3 implies that \( |R(s, a, o)| \leq 2c \) for all \((s, a, o) \in S \times A \times O\). Now let \( Q^* \) be the solution to the maximin Bellman equation (9).

According to Theorem 1, \( Q_k \) in the recursion given by (10)-(11) converges to \( Q^* \) with probability 1. By definition, the optimal policy \( \pi^* \) achieves the value of the zero-sum Markov-Bandit game in (21), and thus achieves the value of (22). Hence,
\[
\pi^*(s) = \arg \max_{\pi \in \Pi} \min_o \mathbb{E} (Q^*(s, \pi(s), o))
\]
and the proof is complete. \( \square \)

4.2 Expected Average Rewards

Consider the optimization problem of finding a stationary policy \( \pi \) that maximizes the expected average reward (2) subject to the constraints (4), that is
\[
\max_{\pi \in \Pi} \lim_{T \to \infty} \mathbb{E} \left( \frac{1}{T} \sum_{k=0}^{T-1} r(s_k, \pi(s_k)) \right)
\]
subject to
\[
\lim_{T \to \infty} \mathbb{E} \left( \frac{1}{T} \sum_{k=0}^{T-1} r^j(s_k, \pi(s_k)) \right) \geq 0
\]
for all \( k \in \mathbb{N} \), and \( j = 1, \ldots, J - 1 \)

The next theorem states that the optimization problem (23) is equivalent to a zero-sum Markov-Bandit game, in the sense that an optimal strategy of the agent in the zero-sum game is also optimal for (23).

Theorem 5. Consider optimization problem (23) and suppose that Assumption 2 and 3 hold. Introduce \( r^0(s, a) = r(s, a) - c \) and the set
\[
\Lambda = \{ \lambda | \lambda = (\lambda^0, \ldots, \lambda^{J-1}), \lambda^0 \geq 0, \lambda^0 + \cdots + \lambda^{J-1} = 1 \}
\]
Let \( \pi^* \) be an optimal policy in the zero-sum game
\[
\max_{\pi \in \Pi} \min_{\lambda \in \Lambda} \lim_{T \to \infty} \sum_{j=0}^{J-1} \mathbb{E} \left( \frac{1}{T} \sum_{k=0}^{T-1} \lambda_k^j r^j(s_k, \pi(s_k)) \right)
\]
Then, \( \pi^* \) is an optimal policy to (23).

Now that we are equipped with Theorem 2 and 5, we are ready to state and proof the second main result.
Theorem 6. Consider the constrained Markov Decision Process problem \((23)\) and suppose that Assumption 2 and 3 hold. Introduce 
\[
\begin{align*}
    r_0(s,a) &= r(s,a) - c, \\
    O &= \mathbb{Z}_J, \\
    o = j
\end{align*}
\]
Let \(Q_k\) be given by the recursion according to \((17)\) and suppose that Assumptions 4 and 5 hold. Then, \(Q_k\) \(\to\) \(Q^\star\) as \(k \to \infty\) where \(Q^\star\) is the solution to \((16)\). Furthermore, the policy
\[
    \pi^\star(s) = \arg \max_{\pi \in \Pi} \min_{o \in O} \mathbb{E}(Q^\star(s,\pi(s),o))
\]
is an optimal solution to \((18)\) for all \(s \in S\).

Proof. According to Theorem 5, \((23)\) is equivalent to the zero-sum Markov-Bandit game \((25)\), which is equivalent to
\[
    \max_{\pi \in \Pi} \min_{o \in O} \lim_{T \to \infty} \mathbb{E}\left(\frac{1}{T} \sum_{k=0}^{T-1} R(s_k,\pi(s_k),o)\right)
\]
where \(o = j\) with probability \(\lambda^j_k\). Assumption 3 implies that \(|R(s,a,o)| \leq 2c\) for all \((s,a,o) \in S \times A \times O\). Now let \(Q^\star\) be the solution to the maximin optimality equation \((16)\). According to Theorem 2, \(Q_k\) in the recursion given by \((17)\) converges to \(Q^\star\) with probability 1 under Assumptions 2, 3, 4, and 5. By definition, the optimal policy \(\pi^\star\) maximizes the expected average reward of the zero-sum Markov-Bandit game \((27)\). Hence,
\[
    \pi^\star(s) = \arg \max_{\pi \in \Pi} \min_{o \in O} \mathbb{E}(Q^\star(s,\pi(s),o))
\]
and the proof is complete.

5 Simulations

In this section we will consider a simple example of a static process (that is, the state is constant) and an agent that takes action from the action set \(A = \{0, 1, 2\}\). There are three objectives given by the reward functions \(r_0, r_1, r_2\) defined as
\[
    r^j(a) = \begin{cases} 
    1 & \text{if } a = j \\
    0 & \text{otherwise}
    \end{cases}
\]
Note that we have dropped the dependence of the reward functions \(r_j\) on the state \(s\) as the state \(s\) is assumed to be constant. Let the discount factor be \(\gamma = \frac{1}{2}\) and let
\[
    \gamma_0 = \gamma_1 = \gamma_2 = \gamma = \frac{1}{3}
\]
The agent would then be looking for a probability distribution over the set \(A\), \(\Pr(a)\) for \(a \in A\), that simultaneously satisfies the objectives
\[
    \mathbb{E}\left(\sum_{k=0}^{\infty} \gamma^k r^j(a_k)\right) \geq \frac{1}{3}, \quad j = 0, 1, 2
\]
Figure 1: A plot of the maximum of \(|p_0 - ˆp_0| + |p_1 - ˆp_1| + |p_2 - ˆp_2|\) over 1000 iterations, as a function of the number of time steps.

Now suppose that the agent takes action \(a_k = 0\) with probability \(p_0\). Then we have that

\[
E \left( \sum_{k=0}^{\infty} \gamma^k r^0(a_k) \right) = p_0
\]

Similarly, we find that if the agent takes the action \(a_k = j\) with probability \(p_j, j = 1, 2\), then

\[
E \left( \sum_{k=0}^{\infty} \gamma^k r^j(a_k) \right) = p_j
\]

Without loss of generality, suppose that \(p_0 \leq p_1 \leq p_2\). Now the equality \(p_0 + p_1 + p_2 = 1\) together with the Arithmetic-Geometric Mean Inequality imply that

\[
\frac{1}{3} = \frac{p_0 + p_1 + p_2}{3} \geq \sqrt[3]{p_0 p_1 p_2} \geq p_0
\]

with equality if and only if \(p_0 = p_1 = p_2 = \frac{1}{3}\). Thus, in order to satisfy all of the three objectives, the agent’s mixed strategy is unique and given by \(p_0 = p_1 = p_2 = \frac{1}{3}\).

We have run 1000 iterations of a simulation of the learning algorithm as given by Theorem 4 over 5000 time steps (with respect to the time index \(k\)). As the above calculations showed, the probability distribution of the optimal policy is given by \(p_0 = p_1 = p_2 = \frac{1}{3}\). Let \(\hat{p}_0, \hat{p}_1, \hat{p}_2\) be the estimated probabilities based on the \(Q\)-learning algorithm given by Theorem 4. In Figure 1 we see a plot of the maximum of the total error

\[
|p_0 - \hat{p}_0| + |p_1 - \hat{p}_1| + |p_2 - \hat{p}_2|
\]

over all iterations, as a function of the number of time steps. We see that it converges after 1000 time steps and stays stable for the rest of the simulation.
6 Conclusions

We considered the problem of optimization and learning for constrained and multi-objective Markov decision processes, for both discounted rewards and expected average rewards. We formulated the problems as zero-sum games where one player (the agent) solves a Markov decision problem and its opponent solves a bandit optimization problem, which we call Markov-Bandit games. We extended Q-learning to solve Markov-Bandit games and proved that our new Q-learning algorithms converge to the optimal solutions of the zero-sum Markov-Bandit games, and hence converge to the optimal solutions of the constrained and multi-objective Markov decision problems. We provided a numerical example showed by simulations that the algorithm converges to the optimal policies.

It would be interesting to combine our algorithms with deep reinforcement learning and study the performance of the maximin Q-learning approach in this paper when the value function $Q$ is modeled as a deep neural network.

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Appendix

Proof of Theorem \[1\]

Lemma 1. Let the operator \( T \) be given by

\[
TQ(s, a, o) = \sum_{s'} P(s, a, s') \left( R(s, a, o) + \gamma Q(s', a, o) \right)
\]

Then,

\[
\|TQ_1 - TQ_2\|_\infty \leq \gamma \|Q_1 - Q_2\|_\infty
\]

Proof. Introduce

\[
a_+^1 = \arg \max_{a \in A} \min_{o \in O} Q_1(s+, a, o)
\]

and

\[
a_+^2 = \arg \max_{a \in A} \min_{o \in O} Q_2(s+, a, o)
\]

Then,

\[
\|TQ_1 - TQ_2\|_\infty = \max_{s, a, o} \left| \sum_{s'} P(s, a, s') \left( R(s, a, o) + \gamma Q_1(s+, a_+^1, o) - R(s, a, o) - \gamma Q_2(s+, a_+^2, o) \right) \right|
\]

\[
= \max_{s, a, o} \gamma \left| \sum_{s'} P(s, a, s+) \left( Q_1(s+, a_+^1, o) - Q_2(s+, a_+^2, o) \right) \right|
\]

\[
\leq \max_{s, a, o} \gamma \left| \sum_{s} P(s, a, s+) \left( Q_1(s+, a_+^1, o) - Q_2(s+, a_+^2, o) \right) \right|
\]

where the last inequality follows from the triangle inequality and the fact that \( P(s, a, s+) \geq \)
0. Now we have that
\[
|Q_1(s_+, a_+^1, o) - Q_2(s_+, a_+^2, o)| \leq
\]
\[
\leq |Q_1(s_+, a_+^1, o) - \min_o Q_2(s_+, a, o)|
\]
\[
= |Q_1(s_+, a_+^1, o) - \max_a \min_o Q_2(s_+, a, o)|
\]
\[
\leq |Q_1(s_+, a_+^1, o) + \min_o (-Q_2(s_+, a_+^1, o))|
\]
\[
\leq \max_o (Q_1(s_+, a_+^1, o) - Q_2(s_+, a_+^1, o))
\]
\[
\leq \max_o |Q_1(s_+, a_+^1, o) - Q_2(s_+, a_+^1, o)|
\]
\[
\leq \max_{s, a, o} |Q_1(s_+, a, o) - Q_2(s_+, a, o)|
\]
\[
= \|Q_1 - Q_2\|_\infty
\]

Combining (30) and (31-38) implies that
\[
\|TQ_1 - TQ_2\|_\infty \leq
\]
\[
\leq \max_{s, a, o} \sum_{s_+} P(s, a, s_+) \|Q_1 - Q_2\|_\infty
\]
\[
= \gamma \|Q_1 - Q_2\|_\infty
\]

and the proof is complete. ∎

Before proceeding, we need the following result.

**Proposition 2.** The random process \(\{\Delta_k\}\) taking values in \(\mathbb{R}\) and defined as
\[
\Delta_{k+1}(x) = (1 - \alpha_k(x))\Delta_k(x) + \alpha_k(x) F_k(x)
\]
converges to zero with probability 1 under the following assumptions:

i. For all \(x\), \(0 \leq \alpha_k(x) \leq 1\), \(\sum_k \alpha_k(x) = \infty\), and \(\sum_k \alpha_k^2(x) < \infty\)

ii. \(\|E(F_k(x) \mid \mathcal{F}_k)\|_\infty \leq \gamma \|\Delta_k\|_\infty\), with \(\gamma < 1\)

iii. \(E(F_k - E(F_k(x)) \mid \mathcal{F}_k))^2 \leq C(1 + \|\Delta_k\|^2_\infty)\), for some constant \(C > 0\)

where \(\mathcal{F}_k\) is the sigma algebra \(\sigma(\Delta_t, F_{t-1}, \alpha_{t-1}, t \leq k)\).

**Proof.** Consult [13]. ∎

**Proof of Theorem 1.** Define the operator \(T\) as in (28). According to Lemma [1] \(T\) is a contraction for \(\gamma < 1\). Now let
\[
\Delta_k(s, a, o) = Q_k(s, a, o) - Q^*(s, a, o)
\]
Subtracting $Q^*$ from the right and left hand sides of the second equality in (11) implies that
\[
\Delta_{k+1}(s,a,o) = (1 - \alpha(s,a,o))\Delta_k(s,a,o) + \\
\alpha(s,a,o)(R(s,a,o) + \gamma Q_k(s+,a+,o) - Q^*(s,a,o))
\]
Define
\[
F_k(s,a,o) = R(s,a,o) + \gamma Q_k(s+,a+,o) - Q^*(s,a,o)
\]
We will show that $\Delta_k$ satisfies the conditions of Proposition 2. Introduce the sigma algebra $\mathcal{F}_k = \sigma(\Delta_t, F_{t-1}, \alpha_{t-1}, t \leq k)$. Then we have that
\[
E(F_k(s,a,o) | \mathcal{F}_k) = \sum_{s+} P(s,a,s+) (R(s,a,o) + \\
\gamma Q_k(s+,a+,o) - Q^*(s,a,o))
\]
Using the fact that $TQ^* = Q^*$, we get
\[
\|E(F_k(s,a,o) | \mathcal{F}_k)\|_\infty = \|(TQ_k)(s,a,o) - (TQ^*)(s,a,o)\|_\infty \\
\leq \gamma \|Q_k - Q^*\|_\infty
\]
Also, we have that
\[
E(F_k - E(F_k(x)) | \mathcal{F}_k)^2 = \\
= E(R(s,a,o) + \gamma Q_k(s+,a+,o) - Q^*(s,a,o) - \\
- (TQ_k)(s,a,o) + Q^*(s,a,o))^2
\]
\[
= E(R(s,a,o) + \gamma Q_k(s+,a+,o) - Q^*(s,a,o) - \\
- (TQ_k)(s,a,o) - (TQ^*)(s,a,o))^2
\]
\[
= E(R(s,a,o) + \gamma Q_k(s+,a+,o) - Q^*(s,a,o) - \\
- (T\Delta_k)(s,a,o))^2
\]
\[
\leq C(1 + \|\Delta_k\|_\infty^2)
\]
where the last inequality holds since $R, Q_k, Q^*$, and $T$ are bounded. Thus, $\Delta_k = Q_k - Q^*$ satisfies the conditions of Proposition 2 and hence converges to zero with probability 1, i.e. $Q_k$ converges to $Q^*$ with probability 1.

\textbf{Proof of Theorem 2}

\textbf{Lemma 2.} The operator $T$ given by (28) with $\gamma = 1$ is a span semi-norm, that is
\[
\|TQ_1 - TQ_2\|_s \leq \|Q_1 - Q_2\|_s
\]
where
\[
\|Q\|_s \triangleq \max_{s,a,o} Q(s,a,o) - \min_{s,a,o} Q(s,a,o)
\]
Proof. We start off by noting the trivial inequalities
\[
\begin{align*}
\max_{s',a',o'} (Q_1(s', a', o') - Q_2(s', a', o')) \\
\geq Q_1(s_+, a_+, o) - Q_2(s_+, a_+, o) \\
\geq \min_{s',a',o'} (Q_1(s', a', o') - Q_2(s', a', o'))
\end{align*}
\]
(41)

The definition of the span semi-norm implies that
\[
\|TQ_1 - TQ_2\|_s = \\
= \max_{s,a,o} \sum_{s_+} P(s,a,s_+) (Q_1(s_+, a_+, o) - Q_2(s_+, a_+, o)) \\
= \max_{s,a,o} \sum_{s_+} P(s,a,s_+) (Q_1(s_+, a_+, o) - Q_2(s_+, a_+, o)) \\
- \min_{s,a,o} \sum_{s_+} P(s,a,s_+) (Q_1(s_+, a_+, o) - Q_2(s_+, a_+, o)) \\
\leq \max_{s,a,o} \sum_{s_+} P(s,a,s_+) \times \\
\times \max_{s',a',o'} (Q_1(s', a', o') - Q_2(s', a', o')) \\
- \min_{s',a',o'} \sum_{s_+} P(s,a,s_+) \times \\
\times \min_{s',a',o'} (Q_1(s', a', o') - Q_2(s', a', o')) \\
= \max_{s',a',o'} (Q_1(s', a', o') - Q_2(s', a', o')) \\
- \min_{s',a',o'} (Q_1(s', a', o') - Q_2(s', a', o')) \\
= \|Q_1 - Q_2\|_s
\]
(42)

For convenience, let \( e : (s,a,o) \mapsto 1 \) be a constant tensor with all elements equal to 1

Lemma 3. Let \( f \in \Phi \) be given, where the set \( \Phi \) is defined as in Definition \(^2\) and let
\[
T'(Q) = T(Q) - f(Q) \cdot e
\]
The ordinary differential equation (ODE)
\[
\dot{Q}(t) = T'(Q(t)) - Q(t)
\]
(43)
has a unique globally asymptotically stable equilibrium \( Q^* \), with \( f(Q^*) = v^* \), where \( Q^* \) and \( v^* \) satisfy \(^{16}\).
Proof. Introduce the operator

$$\hat{T}(Q) = T(Q) - v \cdot e$$

Applying lemma 1 with $\gamma = 1$, we have that

$$\|TQ_1 - TQ_2\|_\infty \leq \|Q_1 - Q_2\|_\infty$$

and hence, $T$ is Lipschitz. It’s easy to verify that

$$\hat{T}(Q_1) - \hat{T}(Q_2) = T(Q_1) - T(Q_2)$$

and therefore

$$\|\hat{T}(Q_1) - \hat{T}(Q_2)\|_\infty \leq \|Q_1 - Q_2\|_\infty$$

and

$$\|\hat{T}(Q_1) - \hat{T}(Q_2)\|_s \leq \|Q_1 - Q_2\|_s$$

Now consider the ODE:s

$$\dot{Q}(t) = \hat{T}(Q(t)) - Q(t) \quad (44)$$

and

$$\dot{Q}(t) = T'(Q(t)) - Q(t) = \hat{T}(Q(t)) + (v - f(Q)) \cdot e \quad (45)$$

Note that since $T$ and $f$ are Lipschitz, the ODE:s (44) and (45) are well posed.

Since $T$ is Lipschitz and span semi-norm, the rest of the proof becomes identical to Theorem 3.4 along with Lemma 3.1, 3.2, and 3.3 in [1] and hence omitted here. □

Proposition 3 (Borkar & Meyn, 2000: Theorem 2.5). Consider the asynchronous algorithm given by

$$Q_{k+1} = Q_k + \alpha_k (h(Q_k) + M_{k+1})$$

where $\alpha_k(s,a,o) = 1_{(s,a,o)}(s_k,a_k,o) \times \gamma_{N(k,s,a,o)}$. Suppose that

1. $M_k$ is a martingale sequence with respect to the sigma algebra $F_k = \sigma(Q_t, M_t, t \leq k)$, that is

$$\mathbb{E}(M_{k+1} | F_k) = 0$$

and that there exists a constant $C_1 > 0$ such that

$$\mathbb{E}(|M_{k+1}|^2 | F_k) \leq C_1 (1 + \|Q_k\|^2)$$

2. Assumptions 4 and 5 hold.

3. The limit

$$h_\infty(X) = \lim_{z \to \infty} \frac{h(zX)}{z}$$

exists.

4. $\dot{Q}(t) = h(Q(t))$ has a unique globally asymptotically stable equilibrium $Q^*$.

Then, $Q_k \to Q^*$ with probability 1 as $k \to \infty$ for any initial value $Q(0)$. 21
Proof of Theorem 2. Introduce the operator

\[ a_+ = \arg \max_{a \in A} \min_{o \in O} Q(s_+, a, o) \]

\[ (TQ)(s, a, o) = \sum_{s_+} P(s, a, s_+) (R(s, a, o) + Q(s_+, a_+, o)) \]

For convenience, let

\[ \alpha_k(s, a, o) = 1 \]

\[ M_{k+1}(s, a, o) = R(s, a, o) + Q_k(s_+, a_+, o) - (TQ_k)(s, a, o), \]

and

\[ h(Q) = TQ - f(Q) \cdot e - Q \]

Then,

\[ Q_{k+1} = Q_k + \alpha_k(h(Q_k) + M_{k+1}) \]

We will now show that conditions 1 - 4 in Proposition 3 hold, and therefore \( Q_k \to Q^* \) with probability 1, where \( Q^* \) is the solution to (16).

1. Let \( \mathcal{F}_k \) be the sigma algebra \( \sigma(Q_t, M_t, t \leq k) \). Clearly,

\[ E(M_{k+1} | \mathcal{F}_k) = 0 \]

and

\[ E(\|M_{k+1}\|^2 | \mathcal{F}_k) \leq C_1(1 + \|Q_k\|^2) \]

for some constant \( C_1 > 0 \).

2. We have supposed that assumptions 4 and 5 hold.

3. Let \( h(X) = T(X) - X - f(X) \cdot e \) and introduce

\[ a_+ = \arg \max_{a \in A} \min_{o \in O} Q(s_+, a, o) \]

\[ (TQ)(s, a, o) = \sum_{s_+} P(s, a, s_+) Q(s_+, a_+, o) \]

Then the limit

\[ h_\infty(X) = \lim_{z \to \infty} h(zX)/z \]

\[ = T(X) - X - f(X) \cdot e \]

exists.

4. By noting that

\[ h(x) = T(X) - X - f(X) \cdot e = T'(X) - X \]

we can apply Lemma 3 and conclude that \( \dot{Q}(t) = h(Q(t)) \) has a unique globally asymptotically stable equilibrium \( Q^* \).

Thus, according to Proposition 3 the iterators \( Q_k \) in (17) converge to \( Q^* \), where \( h(Q^*) = 0 \) and hence the unique solution to (16). \(\square\)
Proof of Theorem 3

Let

\[ C = \frac{1}{1 - \gamma} \cdot c = \left( \sum_{k=0}^{\infty} \gamma^k \right) c \]

Since \( C \) is just a constant, optimization problem (18) is equivalent to

\[
\max_{\pi \in \Pi} E \left( \sum_{k=0}^{\infty} \gamma^k r(s_k, \pi(s_k)) \right) - C
\]

s. t. \( E \left( \sum_{k=0}^{\infty} \gamma^k r_j(s_k, \pi(s_k)) \right) \geq 0 \)

for all \( k \in \mathbb{N} \), and \( j = 1, ..., J - 1 \)

(47)

Now let \( r^0(s, a) = r(s, a) - c < 0 \). Then we have that

\[
\sum_{k=0}^{\infty} \gamma^k r^0(s_k, \pi(s_k)) = \sum_{k=0}^{\infty} \gamma^k r(s_k, \pi(s_k)) - C
\]

and optimization problem (47) is equivalent to

\[
\max_{\pi \in \Pi} E \left( \sum_{k=0}^{\infty} \gamma^k r^0(s_k, \pi(s_k)) \right) - C
\]

s. t. \( E \left( \sum_{k=0}^{\infty} \gamma^k r_j(s_k, \pi(s_k)) \right) \geq 0 \)

for all \( k \in \mathbb{N} \), and \( j = 1, ..., J - 1 \)

(48)

Note that the value of (48) is negative since \( r^0(s, a) < 0 \) for all \( (s, a) \in S \times A \). The Lagrangian of the constrained optimization problem (48) is given by

\[
\mathcal{L}(\pi, \lambda) = \sum_{j=0}^{J-1} E \left( \sum_{k=0}^{\infty} \gamma^k \lambda^j r_j(s_k, \pi(s_k)) \right)
\]

where \( \lambda^0 \triangleq 1 \) and \( \lambda^j \geq 0 \) for all \( j \). Thus, optimization problem (48) is equivalent to

\[
\max_{\pi \in \Pi} \min_{\lambda \geq 0} \mathcal{L}(\pi, \lambda)
\]

(49)

Introduce the set \( \Lambda \) given by (19) and consider the zero-sum game

\[
\max_{\pi \in \Pi} \min_{\lambda \in \Lambda} \mathcal{L}(\pi, \lambda)
\]

(50)

First we note that \( 0 \notin \Lambda \), and so \( \lambda = 0 \) is not a feasible vector. We note that for \( \lambda = (1, 0, 0, ..., 0) \in \Lambda \), we have \( \mathcal{L}(\pi, \lambda) < 0 \) since \( r^0(s, a) < 0 \) for all \( (s, a) \in S \times A \). Hence, any optimal \( \lambda \) must be such that

\[
\mu = \max_{\pi \in \Pi} \min_{\lambda \in \Lambda} \mathcal{L}(\pi, \lambda) < 0
\]
This implies that for the minimizer in the game (50), \( \lambda^0 = 0 \) can’t be optimal since if \( \lambda^0 = 0 \), we would have \( \mathcal{L}(\pi, \lambda) \geq 0 \) for any policy \( \pi \in \Pi \) that satisfies the constraints (3). Hence, the minimizer must have \( \lambda^0 > 0 \). Since \( \mathcal{L}(\pi, \lambda) \) is concave in \( \pi \) and convex in \( \lambda \), we have that

\[
\max_{\pi \in \Pi} \min_{\lambda \in \Lambda} \mathcal{L}(\pi, \lambda) = \min_{\lambda \in \Lambda} \max_{\pi \in \Pi} \mathcal{L}(\pi, \lambda) \tag{51}
\]

Now for each \( \lambda \in \Lambda \) with \( \lambda^0 > 0 \), an optimal policy that maximizes \( \mathcal{L}(\pi, \lambda) \) also maximizes \( \mathcal{L}(\pi, \lambda)/\lambda^0 \). That is, it also maximizes \( \mathcal{L}(\pi, \lambda) \) for \( \lambda \geq 0 \) and \( \lambda^0 = 1 \). Thus, an optimal policy to the games (49) and (51) is also optimal for the game

\[
\min_{\lambda \geq 0} \lambda^0 \max_{\pi \in \Pi} \mathcal{L}(\pi, \lambda) = \max_{\pi \in \Pi} \min_{\lambda \geq 0} \lambda^0 \mathcal{L}(\pi, \lambda) \tag{52}
\]

which is the game in (50). Now let \( \pi^* \in \Pi \) be an optimal policy that maximizes \( \min_{\lambda \in \Lambda} \mathcal{L}(\pi, \lambda) \) and let \( \lambda^* = ((\lambda^0)^*, (\lambda^1)^*, \ldots, (\lambda^{J-1})^*) \) be the optimal vector in \( \Lambda \) that minimize \( \mathcal{L}(\pi^*, \lambda) \). Then, \( \pi^* \) is an optimal policy to (49) and \( \lambda = \lambda^*/(\lambda^0)^* \) is an optimal sequence of vectors for the minimizer and the value of (49) is \( \mu/(\lambda^0)^* \), and the proof is complete.

**Proof of Theorem 5**

First note that \( r^0(s, a) = r(s, a) - c < 0 \). Then we have that

\[
\lim_{T \to \infty} \mathbb{E} \left( \frac{1}{T} \sum_{k=0}^{T-1} r^0(s_k, \pi(s_k)) \right) = \lim_{T \to \infty} \mathbb{E} \left( \frac{1}{T} \sum_{k=0}^{T-1} r(s_k, \pi(s_k)) - c \right)
\]

Note that Assumption (2) implies that the value of (23) is independent of the initial state and that the following limit exists

\[
\mathbb{E}(r^j(s_k, \pi(s_k))) = \lim_{T \to \infty} \mathbb{E} \left( \frac{1}{T} \sum_{k=0}^{T-1} r(s_k, \pi(s_k)) \right)
\]

for all feasible stationary policies \( \pi \in \Pi, j = 0, ..., J - 1 \). Since \( c \) is just a constant, optimization problem (23) is equivalent to

\[
\max_{\pi \in \Pi} \mathbb{E} (r^0(s_k, \pi(s_k))) \quad \text{s. t. } \mathbb{E} (r^j(s_k, \pi(s_k))) \geq 0 \quad \text{for all } k \in \mathbb{N}, \text{ and } j = 1, ..., J - 1 \tag{52}
\]
Note that the value of (52) is negative since \( r^0(s,a) < 0 \) for all \((s,a) \in S \times A\). The Lagrangian of the constrained optimization problem (52) is given by

\[
\mathcal{L}(\pi, \lambda) = \sum_{j=0}^{J-1} \mathbb{E}\left( \lambda^j r^j(s_k, \pi(s_k)) \right)
\]

\[
= \lim_{T \to \infty} \sum_{j=0}^{J-1} \mathbb{E}\left( \frac{1}{T} \sum_{k=0}^{T-1} \lambda^j r^j(s_k, \pi(s_k)) \right)
\]

where the expectation is taken over \(s_k\) and \(\pi\), \(\lambda^0 \equiv 1\) and \(\lambda^j \geq 0\) for all \(j\). Thus, optimization problem (52) is equivalent to

\[
\max_{\pi \in \Pi} \min_{\lambda \geq 0 \atop \lambda^0 = 1} \mathcal{L}(\pi, \lambda) \tag{53}
\]

The rest of the proof is similar to the proof of Theorem 3.