A Connection Approach to Numerical Relativity

D. C. Salisbury and L.C. Shepley
Center for Relativity, The University of Texas at Austin,
Austin, TX 78712-1081 USA

Allan Adams
LBJ High School, Austin, Texas USA

Darren Mann and Larry Turvan
Austin College, Sherman, Texas 75091 USA

Brian Turner
Department of Physics, University of Texas at Dallas,
Richardson, Texas USA

Abstract

We discuss a general formalism for numerically evolving initial data in general relativity in which the (complex) Ashtekar connection and the Newman-Penrose scalars are taken as the dynamical variables. In the generic case three gauge constraints and twelve reality conditions must be solved. The analysis is applied to a Petrov type $\{1111\}$ planar spacetime where we find a spatially constant volume element to be an appropriate coordinate gauge choice.
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1Permanent address: Department of Physics, Austin College, Sherman, Texas 75091
e-mail: dsalis@austinc.edu
1 Introduction

Since their introduction by Abhay Ashtekar in 1986 [1] [2], Ashtekar variables have been the basis of remarkable conceptual and technical advances toward a quantum theory of gravity [3] [4] [5]. (Brügmann maintains an updated e-mail bibliography on publications related to Ashtekar variables [6].) Applications have also been found in classical general relativity. The most fruitful of these have probably been in the study of Bianchi cosmologies [7] [8] [9]. It was suggested soon after the formalism was discovered that it might lend itself to classical numerical work, mainly because of the simplicity of the constraints in terms of the new variables. Meanwhile Capovilla, Dell, and Jacobson (CDJ) discovered a procedure for solving algebraically all of the diffeomorphism constraints [10]. This work has opened up new connection-based approaches to general relativity. (For further references and an informative schematic tracing the roots of this current industry to the work of Plebanski [11], see the review by Peldán [12].) Our intention here is to introduce the CDJ formalism in a general form suitable for undertaking a finite differencing approach to planar cosmology. Recently, Miller and Smolin have proposed a discretization scheme which is based on the same (CDJ) action [13]. The variables and discretization they choose are likely most suitable for quantum lattice calculations.

Our plan is first to introduce Ashtekar variables, displaying the full constraint, evolution, and reality conditions in the 3+1 formalism. In the next section we derive the general dynamical, constraint, and reality relations using a variation of the (CDJ) action in which a complex SO(3) connection and a symmetric complex second rank SO(3) tensor are our variables. These latter objects turn out to be linear combinations of the five complex Newman-Penrose scalars, and we present arguments why we might wish to specify them freely and to monitor their evolution.

In Section 3 we apply the formalism to a spacetime which possesses two commuting spacelike Killing fields. The simplest such model requires only two real Newman-Penrose scalars, and is of type \{1111\}. It is not possible to treat either flat, plane sandwich wave, or colliding plane waves (prior to collision) in this formalism. Our variable and coordinate choices are preserved in a spatially constant (but time dependent) volume gauge.

We conclude with a prognosis of ongoing work in solving numerically the planar model presented in Section 3 and a discussion of potential future applications. Appendix A presents a translation between planar Ashtekar and York variables employed in work by Anninos, Centrella, and Matzner. In Appendix B we derive the relation between our scalar variables and the Newman-Penrose scalars.

2 Ashtekar Variables

The constraints and evolution equations in the Ashtekar formalism can be most easily obtained through a variation of the Hilbert action amended with a term which vanishes automatically by virtue of the cyclic Bianchi identity [14] [20]. We first rewrite the Hilbert action in terms of tetrad fields \(E^\alpha_I\), using the associated Ricci rotation coefficients \(\Omega^I_{\mu J}\) and curvature \(\mathcal{R}^I_{\mu \nu J J}\)

\[
\mathcal{R}^I_{\mu \nu J} = 2\partial_{[\mu}\Omega^I_{\nu] J} + 2\Omega^I_{[\mu M}\Omega^M_{\nu] J}.
\]

The Ricci scalar \(\mathcal{R}\) is then just

\[
\mathcal{R} = E^\alpha_I E^\beta_J \mathcal{R}_{\alpha \beta}^{I J} = E^\alpha_I E^\beta_J \mathcal{R}_{\alpha \beta}^{I} K^J\eta^{JK}.
\]

The vanishing object which we add is simply \(-i\) times the scalar formed from the dual \(^*\mathcal{R}^I_{\mu \nu J J}\) of \(\mathcal{R}^I_{\mu \nu J J}\)

\[
^*\mathcal{R}^I_{\mu \nu J} := \frac{1}{2} \epsilon^{IJKLM} \mathcal{R}_{\mu \nu KL}.
\]
Our conventions are that greek letters ($\mu, \nu, \rho, \ldots$) range over the four spacetime coordinate indices, while upper case latin letters from the middle of the alphabet ($I, J, K, \ldots$) range over the four internal SO(1,3) indices. The SO(1,3) indices are raised and lowered with the Minkowski metric, which we take to be $\eta_{IJ} = diag(-1, 1, 1, 1)$. Lower case latin indices from the beginning of the alphabet ($a, b, c, \ldots$) range over the three spatial indices, while lower case latin indices from the middle of the alphabet ($i, j, k, \ldots$) range over the three internal SO(3) indices. Since we shall frequently deal with concrete indices, we find it useful to represent contravariant objects with a capital letters, and corresponding covariant objects with lower case letters. Thus, we represent the covariant tetrad fields as $e^I_\mu$, with inverse contravariant field $E^\mu_I$. Where possible we shall distinguish between four-dimensional objects and three-dimensional objects by representing the former in either latin script or capital greek letters.

We achieve a suitable 3+1 decomposition by taking $E^\mu_0$ to be normal to our spacelike foliation

$$ E^\mu_0 = (N^{-1}, -N^{-1}N^a), \tag{4} $$

where $N$ and $N^a$ are the lapse and shift, respectively. Then

$$ E^a_i = \delta^a_\mu T^\mu_i, \tag{5} $$

where $T^a_i$ is the triad field. The canonical fields in the Ashtekar initial value formulation of general relativity are the densitized triads $\tilde{T}^a_i$ and the connections $A^i_a$. Let

$$ t := \det t^i_a. \tag{6} $$

The densitized triad of weight one, $\tilde{T}^a_i$, and the densitized lapse of weight minus one, $\tilde{N}$, are thus

$$ \tilde{T}^a_i := t T^a_i, \tag{7} $$

$$ \tilde{N} := t^{-1} N. \tag{8} $$

We can of course express the metric in terms of $\tilde{T}^a_i$, but Rovelli has shown that the densitized triad itself has a simple geometrical interpretation: $\tilde{T}^a_i$ corresponds to an SO(3)-Lie-algebra-valued area two-form, and the SO(3) norm of this two-form is the area [21].

The Ashtekar connection is constructed from the 3-dimensional Ricci rotation coefficients $\omega^i_a$ and the extrinsic curvature $K_{ab}$ in the following manner,

$$ A^i_a := \omega^i_a - i K^i_a, \tag{9} $$

where

$$ K^i_a := T^a i K_{ab}, \tag{10} $$

and $\omega^i_a$ is the dual of $\omega^i_a$,

$$ \omega^i_a := \frac{1}{2} \varepsilon^{ijk} \omega^j_k $$

$$ = \frac{1}{2} \varepsilon^{ijk} (T^b[i \mid t^j_k]_a - T^b[i \mid t^j_k]_b + T^c[i \mid T^b[j]_a t^j_k b^k c). \tag{11} $$

(Note that since the SO(3) indices are raised and lowered with the Kronecker delta, we can raise and lower them at our convenience.)

The curvature $F^i_a$ associated with the the Ashtekar connection is

$$ F^i_a = 2 \partial^i_a A^j_a + 2 A^i_a A^k_a \tag{12} $$

$$ = 2 \partial^i_a A^j_a - 2 A^i_a A^j_b, $$
and the amended Hilbert action $S$ takes the form

$$S = \int d^4x \left( i\tilde{T}^a_i(\dot{A}^i_a - D_a A^i) - iN^a_\flat \tilde{T}^b_j F_{ab}^i + \frac{1}{2} \tilde{N}^a_\flat \tilde{T}^b_j \tilde{T}^c_k F_{ab}^{ij} \right).$$  \hfill (13)

In (13) we have introduced the covariant derivative operator defined through

$$D_a A^i := \partial_a A^i + A^i a A^a_{ji},$$  \hfill (14)

where

$$A^i a := \Omega^i a_0 - i\epsilon^{ijk}\Omega^j a_0 k.$$  \hfill (15)

Following the Palatini method, we now vary $\tilde{T}^a_i$, $A^i a$, $N$, and $A^i$ independently in the action (13). The resulting scalar diffeomorphism constraint $C$ takes the form

$$C := \tilde{T}^a_i \tilde{T}^b_j F_{ab}^{ij} = 0,$$  \hfill (16)

while the vector diffeomorphism constraint $C_a$ is

$$C_a := F_{ab} \tilde{T}^b_i = 0.$$  \hfill (17)

We have in addition a gauge constraint $C^i$ which reflects or freedom to perform local triad rotations

$$C^i := D_a \tilde{T}^a_i = 0.$$  \hfill (18)

Our equations of motion are

$$\dot{A}^i_a = D_a A^i + N^b F_{ba}^i + iN^a_\flat \tilde{T}^b_j F_{ab}^{ij},$$  \hfill (19)

$$\dot{\tilde{T}}^a_i = \epsilon^{ijk} \tilde{T}^a_k A^j + 2D_a N^{[b} \tilde{T}^a_c] + i\epsilon^{ijk} D_b (N \tilde{T}^b_j \tilde{T}^c_k).$$  \hfill (20)

Finally, we must restrict ourselves to a real metric. A polynomial form of these reality conditions is achieved by requiring first that the doubly densitized contravariant metric $\tilde{g}^{ab} := \ell^2 g^{ab}$ be real

$$\Im(\tilde{g}^{ab}) = 0.$$  \hfill (21)

The second reality condition stems ultimately from our definition (9) of the Ashtekar connection. If our triads were real (as we shall assume below in our planar example), we would require that the real part of the Ashtekar connection be precisely the Ricci rotation coefficient derived from the triads. It is clear that this condition cannot involve explicitly a gauge choice. (We employ a generalized notion of gauge which encompasses not only the traditional internal gauge, but also coordinate choices. So we are referring to the freedom in $N$ and $N^a$ as well as $A^i$. ) The required conditions can be obtained most efficiently in polynomial form by requiring that the time derivative of $\tilde{g}^{ab}$ be real \[22\]. We find that with the choices $A^i = 0$ and $N^a = 0$ in the equation of motion (20),

$$\dot{\tilde{g}}^{ab} = 2iN^c \epsilon_{ijk} D_c(\tilde{T}^c_j k \tilde{T}^a_i).$$  \hfill (22)

So our second reality condition is

$$\Re(\epsilon_{ijk} D_c(\tilde{T}^c_j k \tilde{T}^a_i)) = 0.$$  \hfill (23)

The approach we have described seems at this stage to offer no practical advantage from a numerical perspective. Indeed, we may wonder whether any gain at all may be achieved given that we have increased the number of constraints by three, and we have twelve non-trivial reality conditions in the generic case. Nor do the evolution equations undergo any significant simplifications. Ultimately, of course, it must be possible to choose gauge conditions such that
the conventional and the Ashtekar numerical approach are exactly equivalent. One merely constructs the real part of the Ashtekar connection from the assumed functional form of the real triad fields, as prescribed in the definition (9). Thus after fulfilling all reality conditions in this manner, the formalism simply amounts to a change of variables, which may or may not be useful from the point of view of generating numerical solutions. This point of view is illustrated in Appendix A in which we show the equivalence with a numerical construction of planar cosmologies due to Anninos, Centrella, and Matzner.

Thus a truly new method demands the use of new gauge conditions. It is to this strategy we turn next.

3 A Connection Approach

The Ashtekar approach was itself stimulated by work in complex general relativity, and it was from this perspective that a procedure was discovered by Capovilla, Dell, and Jacobson for solving the four diffeomorphism constraints (16) and (17). They discovered that if $\Psi_{ij}$ is an arbitrary traceless, symmetric SO(3) tensor, then the following densitized triad satisfies the diffeomorphism constraints

$$\tilde{T}^a_i = \tilde{B}_j^a \Psi^{-1}_{ji},$$

(24)

where $\tilde{B}^a_j$ is the double dual of the Ashtekar curvature

$$\tilde{B}^a_j := \frac{1}{4} \epsilon_{abc} \epsilon_{ijk} F_{bc}^{jk}.$$  

(25)

In hindsight this result is not a surprise since the diffeomorphism constraints are merely algebraic and polynomial in $\tilde{T}^a_i$ and $\tilde{B}^a_i$. Indeed, in substituting our Ansatz (24) into the constraints (16) and (17) we find

$$C = \epsilon_{abc} \epsilon^{ijk} \tilde{T}^a_i \tilde{T}^b_j \tilde{B}^c_k = \tilde{B} \Psi^{-1} \Psi_{ii},$$

(26)

$$C_a = \epsilon_{abc} \tilde{B}^{ci} \tilde{T}^b_i = \epsilon_{abc} \tilde{B}^{ci} \tilde{B}^{bj} \Psi_{ji},$$

(27)

where $\Psi := det(\Psi_{ij})$ and $\tilde{B} := det(\tilde{B}^a_i)$. We note in passing that the traceless condition could be relaxed if we wished to contemplate degenerate metrics, in which case we would permit $\tilde{B} = 0$. However, henceforth we shall assume that $\tilde{B} \neq 0$.

The components of $\Psi_{ij}$ are simply linear combinations of the complex Newman-Penrose scalars $\Psi_i$, where $i = 0, \ldots, 4$. In Appendix B we show that

$$\Psi_{11} = \frac{1}{2} (-\Psi_0 + 2\Psi_2 - \Psi_4),$$

(28)

$$\Psi_{12} = \frac{i}{2} (\Psi_0 - \Psi_4),$$

(29)

$$\Psi_{13} = \Psi_1 - \Psi_3,$$

(30)

$$\Psi_{22} = \frac{1}{2} (\Psi_0 + 2\Psi_2 + \Psi_4),$$

(31)

$$\Psi_{23} = -i\Psi_1 - i\Psi_3,$$

(32)

$$\Psi_{33} = -2\Psi_2,$$

(33)

with the inverses,

$$\Psi_0 = \frac{1}{2} (-\Psi_{11} + \Psi_{22} - 2i\Psi_{12}),$$

(34)

$$\Psi_1 = \frac{1}{2} (\Psi_{13} + i\Psi_{23}),$$

(35)

$$\Psi_2 = \frac{1}{2} (\Psi_{11} + \Psi_{22}),$$

(36)
\( \Psi_3 = \frac{1}{2}(-\Psi_{13} + i\Psi_{23}) \),

\( \Psi_4 = \frac{1}{2}(-\Psi_{11} + \Psi_{22} + 2i\Psi_{12}) \),

(37)

(38)

These scalars possess a readily measurable physical significance pointed out by Szekeres. \( \Psi_0 \) and \( \Psi_4 \) measure transverse contributions to pure gravitational waves, while \( \Psi_1 \) and \( \Psi_3 \) correspond to longitudinal contributions. \( \Psi_2 \) is a Coulombic component which we do not permit to vanish since we require that \( \Psi_{ij} \) exist. Another quantity of physical interest is the Bel-Robinson superenergy tensor whose positive-definite properties can often be useful in monitoring gravitational wave disturbances. Its tensorial expression in terms of the Newman-Penrose scalars is given in Appendix B. We view these physical interpretations as motivation for directly monitoring the dynamics of the Newman-Penrose scalars. We must note, however, that in assuming that \( \Psi_{ij} \) is invertible, we are excluding Petrov types \{31\}, \{4\} and \{-\} from consideration, as we discuss in the Appendix [24] [25].

We shall now derive the equations of motion for \( \Psi_{ij} \). They can fact be obtained from the following action [24]

\[
S = i \int d^4x \left( (\dot{A}^i_a - D_a \dot{A}^i) \tilde{E}^a_{ij} \Psi^{-1}_{jk} + \mu_i \epsilon^{ijk} \Psi_{jk} + \rho \Psi_{ii} \right).
\]

(39)

(We have added the Lagrange multipliers \( \mu_i \) and \( \rho \) to enforce the symmetry and tracelessness of \( \Psi \).) However, we shall simply substitute the Ansatz [24] for \( \tilde{T}^a_i \) directly into the equations of motion (19) and (20).

The characteristic equation for an arbitrary 3 \times 3 matrix \( \Psi_{ij} \) is

\[
\Psi \delta_{ij} = \Psi_{ik} \Psi_{kl} \Psi_{lj} - \Psi_{kk} \Psi_{ii} \Psi_{ij} + \frac{1}{2}(\Psi_{kk})^2 \Psi_{ij}.
\]

(40)

Thus in our case

\[
\Psi^{-1}_{ij} = \Psi^{-1}_{ij} (\Psi_{ik} \Psi_{kj} - \frac{1}{2}(\Psi_{kk})^2 \delta_{ij}).
\]

(41)

Our Ansatz [24] for \( \tilde{T}^a_i \) takes the form

\[
\tilde{T}^a_i = \tilde{B}^a_j \Psi^{-1} (\Psi_{jk} \Psi_{kl} - \frac{1}{2} \delta_{lj} \Psi_{kk} \Psi_{ik}).
\]

(42)

Substituting into the equation of motion (13) we find

\[
\dot{A}^i_a = D_a A^i + \epsilon_{abc} \tilde{B}^b k^c \Psi^{-1}_{ij} \tilde{B}^i_{jk} = D_a A^i + \epsilon_{abc} \tilde{B}^b k^c \Psi^{-1}_{ij} \tilde{B}^i_{jk} \Psi_{jk} \Psi_{kl}.
\]

(43)

(44)

The equation of motion for \( \Psi_{ij} \) requires some more work and we will not include all of the details. For this purpose we require the time derivative of \( \tilde{B}^a_i \)

\[
\dot{\tilde{B}}^a_i = \frac{1}{2} \epsilon^{abc} \tilde{B}^b_{ic}.
\]

(45)

(46)

So according to our Ansatz [24] we have

\[
\dot{\tilde{T}}^a_i = \dot{\tilde{B}}^a_j \Psi^{-1}_{ji} - \tilde{B}^a j \Psi^{-1}_{jm} \tilde{B}^m_{ni} \Psi^{-1}_{ni}.
\]

(47)

\[
= \epsilon^{ijkl} \tilde{B}^a k \tilde{B}^i_{jk} \Psi^{-1}_{ji} + 2 D_b (N^{[b}] \tilde{T}^a_i) - 2 N^{[b]} \tilde{B}^{aj} \Psi^{-1}_{ji} \\
- i \epsilon^{abc} D_b (\tilde{B}^{aj} \tilde{B}^{i}_{jk} \Psi^{-1}_{ji}) + i \epsilon^{abc} \tilde{B}^{aj} \tilde{B}^{i}_{jk} \Psi^{-1}_{ji} \tilde{B}^{k}_{jl} \Psi^{-1}_{ji}.
\]

(48)
In (48) we used
\[ F_{bc}^{\ jk} A^k = 2\mathcal{D}[D_c]A^j \] (49)
and the evolution equation (47). Comparing the two expressions for \( \dot{T}^i_{ij} \), (20) and (48), we find ultimately that
\[
\dot{\Psi}_{ij} = 2\epsilon_{kl}\psi_{kl}\Psi^{-1}_{ij} + N^a\mathcal{D}_a\Psi_{ij} - iN^{il}D_a\psi_{kj}\tilde{B}^a_{ml}\Psi^{-1}_{mi}.
\] (50)

In deriving (50) we made liberal use of the gauge constraint (18). We note here that since
\[ D_a\tilde{B}^a_{ij} \equiv 0, \] (51)
an equivalent form of the gauge constraint is
\[
C_i = \tilde{B}_j\mathcal{D}_a\Psi^{-1}_{ji} = 0.
\] (52)
Notice that \( \dot{\Psi}_{ij} \) is manifestly traceless. It is also symmetric, since
\[
\epsilon^{ijk}\dot{\Psi}_{ij} = -iN^{il}\psi_{kj}\tilde{B}^a_{li}\Psi^{-1}_{ij} + iN^a\mathcal{D}_a\Psi_{jl}\tilde{B}^a_{ji}\Psi^{-1}_{lk} = iN^{ij}\psi_{kl}\tilde{B}^a_{jl}\Psi^{-1}_{ij} + 2iN\epsilon_{mli}\Psi_{im}\tilde{B}^a_{ji}\Psi^{-1}_{jk} = 0.
\] (53)
The first term in the last line vanishes due to the gauge constraint (52), while the second term is zero because of the symmetry of \( \Psi_{ij} \).

4 Application to Planar Cosmology

We shall now apply the preceding formalism to a spacetime which is presumed to possess two spacelike commuting Killing vectors. In this case it is possible to introduce a spatial coordinate \( x^3 = z \) and time \( t \) such that the only nonvanishing densitized triad and connection variables are \( A^1_1, A^1_2, A^2_1, A^2_2, A^3_3, K^1_1, K^1_2, K^2_1, K^2_2, \) and \( K^3_3 \) [27]. We attempt a further simplification motivated by our experience with conventional variables described in Appendix A. We take the diagonal components of \( A^i_a \) to be pure imaginary, and the off-diagonal components to be real, so that our non-vanishing connection components are, in addition to the real \( A^1_2 \) and \( A^2_1 \) (see the definition (8)),
\[
A^1_1 := -iK^1_1, \tag{54}
\]
\[
A^2_2 := -iK^2_2, \tag{55}
\]
\[
A^3_3 := -iK^3_3, \tag{56}
\]
where the \( K \)'s are real. The double dual \( \tilde{B}^a_i \) defined in (23) is therefore ( \( t \) is \( \partial/\partial z \))
\[
\tilde{B}^a_i = \begin{pmatrix}
-A^1_1' + K^2_2 K^3_3 & iK^2_2' - iA^1_2 K^3_3 \\
-iK^1_1' - iA^2_1 K^3_3 & A^2_1' + K^1_1 K^3_3
\end{pmatrix}
\] (57)

Let us take \( \Psi_{ij} \) to be diagonal, so that it has the form
\[
\Psi = \begin{pmatrix}
-a & -b \\
-b & a + b
\end{pmatrix},
\] (58)
with \(a\) and \(b\) real. Thus we have selected the following nonvanishing Newman-Penrose scalars

\[
\begin{align*}
\Psi_0 &= \frac{1}{2}(a - b), \\
\Psi_2 &= \frac{1}{2}(a + b), \\
\Psi_4 &= \frac{1}{2}(a - b) = \Psi_0.
\end{align*}
\]

This corresponds to a spacetime of Petrov type \(\{1111\}\) \cite{Petrov}. We note that we can treat neither plane sandwich waves nor colliding waves within this formalism since in both cases there exist open regions which are of Petrov type \(\{4\}\) and/or \(\{-\}\). Our spacetime possesses equal left and right moving transverse waves, described by \(\Psi_0\) and \(\Psi_4\), respectively, and an everywhere present Coulombic part described by \(\Psi_2\). As mentioned in the introduction, we can now easily monitor the Bel-Robinson super-energy tensor. In Appendix B, we show for example, that the super-energy density for observers in the orthonormal basis \(E_\mu^\nu\) is

\[
T_{0000} = \frac{1}{2}(a^2 + ab + b^2).
\]

Isenberg, Jackson, and Moncrief have investigated the behavior of this object in a Gowdy \(T^3 \times \mathbb{R}\) planar spacetime \cite{IsenbergJacksonMoncrief}. Then our densitized triads \cite{DensitizedTriads} are

\[
\tilde{T}_i^a = \begin{pmatrix}
-a^{-1}\tilde{B}_1^1 & -b^{-1}\tilde{B}_1^2 \\
-a^{-1}\tilde{B}_2^1 & -b^{-1}\tilde{B}_2^2 \\
(a + b)^{-1}\tilde{B}_3^3
\end{pmatrix}.
\]

We shall require that \(\tilde{T}_i^a\) be real. Referring to \(\tilde{B}_i^a\) in (67), we are led to our first reality conditions

\[
K_2^2 - A_1^2K_3^3 = 0,
\]

\[
K_1^1 + A_2^2K_3^3 = 0.
\]

Thus our densitized triads are diagonal

\[
\tilde{T}_i^a = \begin{pmatrix}
-a^{-1}\tilde{B}_1^1 \\
-b^{-1}\tilde{B}_2^2 \\
(a + b)^{-1}\tilde{B}_3^3
\end{pmatrix},
\]

yielding the metric

\[
g_{ab} = \begin{pmatrix}
(\tilde{B}_1^1)^{-1}\tilde{B}_2^2\tilde{B}_3^3b(a + b) \\
(\tilde{B}_2^2)^{-1}\tilde{B}_1^1\tilde{B}_3^3a(a + b) \\
(\tilde{B}_3^3)^{-1}\tilde{B}_1^1\tilde{B}_2^2ab
\end{pmatrix}.
\]

We shall also assume that the only nonvanishing gauge functions are \(A^3 := iA^3\), \(N^3\), and \(N\). Substituting \(\tilde{T}_i^a\) from (66) into (19) we find the equations of motion

\[
\dot{K}_1^1 = -A^3A_1^2 + Nab^{-1}(a + b)^{-1}\tilde{B}_2^2\tilde{B}_3^3,
\]

\[
\dot{K}_2^2 = A^3A_2^1 + Nba^{-1}(a + b)^{-1}\tilde{B}_1^1\tilde{B}_3^3,
\]

\[
\dot{K}_3^3 = A^3 + N(ab)^{-1}(a + b)\tilde{B}_1^1\tilde{B}_2^2,
\]
\[ \dot{A}_2^1 = A^3K_2^3 - N^3\tilde{B}_1^1, \quad (71) \]
\[ \dot{A}_1^2 = -A^4K_1^1 + N^3\tilde{B}_2^2. \quad (72) \]

The remaining equations of motion are (from (58) and (54))
\[ \dot{a} = N^3a'' + N_K^3K_3^3b^3b^{-1}(b + 2a) + N_K^3K_3^3\tilde{B}_3^3(a - b)(a - b)^{-1}, \quad (73) \]
\[ \dot{b} = N^3b'' + N_K^3K_1^1\tilde{B}_1^1a^{-1}(a + 2b) + N_K^3K_3^3(a - b)(a - b)^{-1}. \quad (74) \]

Let us next impose the reality condition (23). In our case this is simply the condition that \( A_2^1 \) and \( A_1^2 \) are the only non-vanishing Ricci rotation coefficients formed from our diagonal densitized triads (24):
\[ A_2^1 = \omega_2^1 = \frac{1}{2T_1^3} \left( \frac{T_3^3\tilde{T}_1^3}{T_2^3} \right)' = -\frac{a}{B_1^1} \left( \frac{a\tilde{B}_1^3\tilde{B}_3^3}{(a + b)aB_2^2} \right)', \quad (75) \]
\[ A_1^2 = \omega_1^2 = -\frac{1}{2T_2^3} \left( \frac{T_3^3\tilde{T}_2^3}{T_1^3} \right)' = \frac{b}{B_2^2} \left( \frac{b\tilde{B}_2^3\tilde{B}_3^3}{(a + b)bB_1^1} \right)', \quad (76) \]

It turns out that the gauge constraints \( C_1 = 0 \) are now automatically satisfied. This fact is easiest to see referring to (13) and the connection definition (18):
\[ C_i = \partial_a\tilde{T}_a^i + \omega_a^j\tilde{T}_a^j - i\epsilon^{ijk}K^k_a\tilde{T}_a^j = 0. \quad (77) \]

The sum of the first two terms vanishes (the densitized triad is covariantly constant). The second term is the gauge constraint in a real triad formalism in which the canonical variables are \( \tilde{T}_a^i \) and \( K^i_a \) (28). In the present case this constraint vanishes identically since both \( \tilde{T}_a^i \) and \( K^i_a \) are diagonal.

Next we must insure that our vanishing variable components remain zero under time evolution. With our choices of gauge functions, the only non-trivial condition arising from the evolution of \( \Psi_{ij} \) in (75) and \( A^i_a \) in (14) is
\[ -i\Psi_{12} = (b - a)(A^3 - N^3K_3^3) - Nb'\tilde{B}_3^3(a + b)^{-1} + NA_1^2\tilde{B}_2^2b^{-1}(a + 2b) = 0. \quad (78) \]

Now we must require that the gauge choices (64) and (53) are preserved under time evolution. We have
\[ -i\dot{B}_1^2 = (A^3 - N^3K_3^3)\tilde{B}_1^1 - NA_2^2a^{-1}b^{-1}(a + b)\tilde{B}_1^1 + \left( Nba^{-1}(a + b)^{-1}\tilde{B}_1^1\tilde{B}_3^3 \right)' = 0, \quad (79) \]
\[ -i\dot{B}_2^1 = -(A^3 - N^3K_3^3)\tilde{B}_2^2 - NA_1^2a^{-1}b^{-1}(a + b)\tilde{B}_2^2 - \left( Nab^{-1}(a + b)^{-1}\tilde{B}_2^2\tilde{B}_3^3 \right)' = 0. \quad (80) \]

Two of the preceding three relations (23), (24), and (25) fix the gauge functions \( A^3 - N^3K_3^3 \) and \( N \), so the third must be a new constraint. We present the results without proof as the calculation is straightforward but tedious. The constraint turns out to be the requirement that the determinant of the densitized triads be spatially constant:
\[ 0 = \tilde{T}' = (\tilde{T}_1^1\tilde{T}_2^2\tilde{T}_3^3)' = \left( \frac{\tilde{B}_1^1\tilde{B}_2^2\tilde{B}_3^3}{ab(a + b)} \right)'. \quad (81) \]

We find that \( \tilde{N} \) satisfies the first order differential equation
\[ \tilde{N}'\tilde{N}^{-1} - A_1^2\tilde{B}_2^2b^{-1}(a + 2b)(a + b)(b - a)^{-1} + b'\tilde{B}_3^3(b - a)^{-1} = 0, \quad (82) \]
so that \( A^3 - N^3 K_3^3 \) may be expressed most simply as
\[
A^3 - N^3 K_3^3 = -N' (a + b)^{-1} \tilde{B}_3.
\] (83)

The constant volume gauge has been examined by Rovelli in the context of canonical quantization, although he specializes to volume elements which are time-independent [14].

Finally, the shift \( N^3 \) is fixed through the requirement that the volume element remain spatially constant under time evolution:
\[
0 = \frac{\dot{T}^3}{2} = 2 \left( N^{3'} \dot{T} + N \dot{T}(K_1^1 \dot{T}_1^1 + K_2^2 \dot{T}_2^2 + K_3^3 \dot{T}_3^3) \right) ',
\] (84)
so that
\[
N^{3''} = - \left( N((K_1^1 \dot{T}_1^1 + K_2^2 \dot{T}_2^2 + K_3^3 \dot{T}_3^3)) ' \right),
\] (85)
and this equation is readily integrable.

5 Discussion

We have developed a general formalism for fixing and evolving vacuum initial data consisting of a complex SO(3) connection and the set of Newman-Penrose scalars. Although it is certainly true that the constraints, reality conditions, and evolution equations are more complicated than those obtained using conventional variables, we believe that spacetimes may well exist for which this complication is compensated by our ability to fix arbitrarily and then to monitor the time evolution of the physically significant Newman-Penrose scalars.

We list here the full set of relations obtained in Section 3 for the planar case. The dynamical variables are \( a, b, K_1^1, K_2^2, \) and \( K_3^3 \), with gauge functions \( N^3, N, \) and \( A \). All are real, and they depend only only the coordinates \( t \) and \( z \). All constraints are satisfied identically by our dynamical variables, but we have the following set of reality conditions:
\[
K_2^2 - A_2^2 K_3^3 = 0,
\] (86)
\[
K_1^1 + A_1^1 K_3^3 = 0,
\] (87)
\[
A_2^2 = \omega_2 = \frac{1}{2T^1} \left( \frac{\tilde{T}_2^3}{\tilde{T}_1^1} \right) ' = -\frac{a}{B_2^3} \left( \frac{a \tilde{B}_1^3}{(a + b) a B_2^3} \right) ',
\] (88)
\[
A_1^1 = \omega_1 = \frac{1}{2T^2} \left( \frac{\tilde{T}_1^3}{\tilde{T}_2^2} \right) ' = \frac{b}{B_1^3} \left( \frac{b \tilde{B}_2^3}{(a + b) b B_1^3} \right) '.
\] (89)

The equations of motion are
\[
\dot{a} = N^3 a' + N K_2^2 \tilde{B}_2 b^{-1}(b + 2a) + N K_3^3 \tilde{B}_3 (a - b)(a - b)^{-1},
\] (90)
\[
\dot{b} = N^3 b' + N K_1^1 \tilde{B}_1 a^{-1}(a + 2b) + N K_3^3 \tilde{B}_3 (a - b)(a - b)^{-1},
\] (91)
\[
\dot{K}_1^1 = -A^3 A_1^1 + \frac{N ab^{-1} (a + b)^{-1}}{B_2^3} \tilde{B}_2 \tilde{B}_3,
\] (92)
\[
\dot{K}_2^2 = A^3 A_2^2 + \frac{N b a^{-1} (a + b)^{-1}}{\tilde{B}_1^3} \tilde{B}_1 \tilde{B}_3,
\] (93)
\[
\dot{K}_3^3 = A^3 + N (a b^{-1} (a + b)^{-1} \tilde{B}_1 \tilde{B}_2^2).
\] (94)
The gauge conditions are

\[(b - a)(A^i - N^3 K^i_3) - N b' \tilde{B}^3_3(a + b)^{-1} + \hat{N} A_{2}^i \tilde{B}^2_2 b^{-1}(a + 2b) = 0, \quad (95)\]

\[(A^i - N^3 K^i_3) \tilde{B}^3_1 - \hat{N} A_{2}^i a^{-1} b^{-1}(a + b) \tilde{B}^1_1 + \left( \hat{N} a b^{-1}(a + b)^{-1} \tilde{B}^1_1 \tilde{B}^3_3 \right)' = 0, \quad (96)\]

\[(A^i - N^3 K^i_3) \tilde{B}^2_2 + \hat{N} A_{2}^i a^{-1} b^{-1}(a + b) \tilde{B}^2_2 + \left( \hat{N} a b^{-1}(a + b)^{-1} \tilde{B}^2_2 \tilde{B}^3_3 \right)' = 0, \quad (97)\]

\[N^{3r} = - \left( \hat{N} ((K^1_1 \tilde{T}^2_1 + K^2_2 \tilde{T}^2_2 + K^3_3 \tilde{T}^3_3))' \right). \quad (98)\]

We are currently finite-differencing these relations. Our strategy is to solve the reality conditions for the variables \(K^1_1, K^2_2,\) and \(K^3_3\) so that the Newman-Penrose scalars \(a\) and \(b\) may be specified freely. We maintain that the complexity of these relations is in itself not compelling reason to reject this approach. The issue that counts is whether it is possible to construct a stable and accurate numerical code, and the jury is still out.

We anticipate that it might turn out to be useful to combine the connection approach with the conventional one, possibly in choosing one technique for evolution and the other for constraints, or applying both to monitor accuracy. We are also investigating the possibility of applying the connection approach to the characteristic initial value problem, where the spinorial formalism has proved especially useful in analyzing gravitational waves [25].

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Appendix A

We shall give here the change of variables between our variables \(T^i_a\) and \(K^i_a\) and the York variables employed by Anninos, Centrella, and Matzner (ACM) in a series of papers in which they develop a numerical code for solving Einstein’s equations in a plane-symmetric spacetime [3, 18, 17, 19]. The ACM line element is

\[ds^2 = -\alpha^2 dt^2 + \phi^4 dx^2 + \phi^4 h^2 dy^2 + \phi^4 (\beta dt + dz)^2, \quad (99)\]

corresponding to the covariant tetrad field (the components are functions only of \(z\) and \(t\))

\[e^\rho_\mu = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \phi^2 & 0 & 0 \\ 0 & 0 & \phi^2 h & 0 \\ \beta \phi^2 & 0 & 0 & \phi^2 \end{pmatrix}. \quad (100)\]

The Ashtekar connection is

\[A^i_\mu = \begin{pmatrix} i \alpha^{-1}(2\beta \phi \phi' - 2\phi \phi) & \phi^{-1}(2\phi' h + \phi h') & 0 \\ -2\phi^{-1} \phi' & i \alpha^{-1}(2\phi \phi' h + \phi^2 h' + \beta - 2\phi \phi h - \phi^2 h) & 0 \\ 0 & 0 & i \alpha^{-1}(2\phi \phi' \beta + \phi^2 \beta' - 2\phi \phi) \end{pmatrix}. \quad (101)\]
Thus
\[
K_{i\alpha} = \begin{pmatrix}
-\alpha^{-1}(2\beta\phi\phi' - 2\phi\dot{\phi}) & 0 & 0 \\
0 & -\alpha^{-1}(2\phi\phi'\beta h + \phi^2 h'\beta - 2\phi\dot{\phi}h - \phi^2 \dot{h}) & 0 \\
0 & 0 & -\alpha^{-1}(2\phi\phi'\beta + \phi^2 \beta' - 2\phi\dot{\phi})
\end{pmatrix},
\]
with corresponding equations of motion
\[
\dot{\phi} = \beta\phi' + \frac{1}{2}\alpha\phi^{-1} K_{1}^1, \quad (103)
\]
\[
\dot{h} = \alpha\phi^{-1}(K_2^2 - K_1^1) + h'\beta. \quad (104)
\]
The remaining equations of motion are
\[
\dot{K}_1^1 = A_1^2 A_3^3 + \beta(K_1^1 + A_2^2 K_3^3) - \alpha\phi^{-2}h^{-1}(h A_2^1 + h K_1^1 K_3^3 + A_2^2 A_1^1 + K_1^1 K_2^2), \quad (105)
\]
\[
\dot{K}_2^2 = -A_2^1 A_3^3 + \beta(K_2^2 - A_2^2 K_3^3) - \alpha\phi^{-2}(-A_2^1 + K_2^2 K_3^3 + A_2^2 A_1^1 + K_1^1 K_2^2), \quad (106)
\]
\[
\dot{K}_3^3 = -A_3^1 - \alpha\phi^{-2}h^{-1}(h A_2^1 + h K_1^1 K_3^3 - A_2^1 + K_2^2 K_3^3). \quad (107)
\]
The non-trivial constraints are the scalar constraint [17]:
\[
A_1^2 A_3^3 + K_1^1 K_2^2 + h(A_2^1 + K_1^1 K_3^3) - A_1^2 + K_2^2 K_3^3 = 0, \quad (108)
\]
and the vector constraint \(C_3\):
\[
h(K_1^1 + A_2^2 K_3^3) + K_2^2 = A_1^2 K_3^3 = 0. \quad (109)
\]
The gauge functions \(A_3 := i A_3^3\), \(\alpha\), and \(\beta\) are partially determined through the conditions that \(\tilde{T}_2^1 = \tilde{T}_2^2 = 0\) and \(\tilde{T}_1^1 = \tilde{T}_3^3\). We find
\[
A_3^3 = -\beta K_3^3 + \alpha' \phi^{-2}, \quad (110)
\]
and
\[
\beta' = \alpha\phi^{-2}(K_3^3 - K_1^1). \quad (111)
\]
As noted in Section 2, there are no further conditions, since all of our variables are real, and we have constructed the only nonvanishing \(\omega_{i\alpha}\)'s from the triads:
\[
\omega_1^1 = A_1^1 = 2\phi^{-1} \phi' h + h', \quad (112)
\]
\[
\omega_2^2 = A_2^2 = -2\phi^{-1} \phi'. \quad (113)
\]
We now give the change of variables to the ACM formalism, where a York conformal decomposition is employed. ACM define a traceless extrinsic curvature
\[
A_{ab} := K_{ab} - \frac{1}{3}\theta_{ab}K, \quad (114)
\]
where \(K := K_{a}^a\). The conformally transformed \(A_{ab}\) is
\[
\hat{A}_{ab} := \phi^2 A_{ab}, \quad (115)
\]
The independent variables are taken to be
\[
\hat{\eta} := A_1^1 - A_2^2, \quad (116)
\]
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in addition to $\hat{A}_3 := \hat{A}$ and $K$. They are obtained from our variables as follows:

$$\hat{\eta} = \phi^4(K_1^1 - h^{-1}K_2^2),$$

$$\hat{A} = \frac{1}{3}\phi^4(2K_3^3 - K_1^1 - h^{-1}K_2^2),$$

$$K = \phi^{-2}(K_1^1 + h^{-1}K_2^2 + K_3^3).$$

Substituting into the constraints (10) and (11) we find for the scalar constraint

$$(\phi'h')' = \frac{h\phi}{8}
\left(-2h^{-1}h'' + \frac{2}{3}K^2\phi^4 - \frac{1}{2}(\eta^2 + 3\hat{A}^2)\phi^{-8}\right),$$

while the vector constraint is

$$\hat{A}' - \frac{3}{2}h^{-1}h'\hat{A} + \frac{1}{2}h^{-1}h'\hat{\eta} - \frac{2}{3}\phi^{-2}K' = 0.$$  

(121)

As we pointed out in the text, it is by no means obvious that the system of evolution equations, constraints, and gauge conditions (103) - (111) are in any way superior from a numerical perspective to those obtained through the York procedure. We have written a fully constrained code in which we solve the constraint (108) for $K_1^1$ and substitute into the vector constraint to obtain a first order ordinary differential equation for $K_2^2$. The metric functions $h$ and $\phi$ are then chosen to represent incoming colliding waves. Data is evolved using a second-order accurate leapfrog procedure with downstream updating. This is, however, a procedure which is also available using conventional variables. Furthermore, it could be argued that the York procedure possesses greater physical motivation, since one is free to specify a conformal class of metrics. Finally, it is much more difficult to implement periodic boundary conditions with our choice of independent variables. On the other hand, the connection approach we describe in the text is well motivated physically, and there is no difficulty in implementing periodic boundary conditions.

Appendix B

We shall show that the components of the symmetric SO(3) tensor $\Psi_{ij}$ used in the construction of the densitized triad $\tilde{T}^a_i$ in (24) are the linear combinations of the Newman-Penrose scalars given in (28 - 33). For this purpose it is most convenient to work with both SU(2) and SL(2,C) spinors. Although all of the results we shall give here are known, they are not to be found summarized in one location, and we think our procedures may also be more accessible to non-experts in spinorial calculus. In the course of this derivation the relation between the tetrad and triad 3+1 decomposition will also be made apparent.

We let $\Sigma^i_{AA'}$ represent $\frac{1}{\sqrt{2}}$ times the 2x2 identity and Pauli matrices $\Lambda^i$

$$\Sigma^0_{AA'} = \frac{i}{\sqrt{2}}\begin{pmatrix}1 & 1 \\ 0 & 0 \end{pmatrix},$$

$$\Sigma^1_{AA'} = \frac{i}{\sqrt{2}}\Lambda^1 = \frac{i}{\sqrt{2}}\begin{pmatrix}0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\Sigma^2_{AA'} = \frac{i}{\sqrt{2}}\Lambda^2 = \frac{i}{\sqrt{2}}\begin{pmatrix}0 & -i \\ i & 0 \end{pmatrix},$$

$$\Sigma^3_{AA'} = \frac{i}{\sqrt{2}}\Lambda^3 = \frac{i}{\sqrt{2}}\begin{pmatrix}1 & 0 \\ 0 & -1 \end{pmatrix}. $$
The indices $A$ and $A'$ range over 0 and 1. For visualization purposes we shall conceive of the first index as representing the row and the second as representing the column. These $\text{SL}(2,\mathbb{C})$ indices may be raised and lowered with the Levi-Civita symbols

$$\epsilon_{AB} = \epsilon_{A'B'} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$  \hspace{1cm} (126)

Note that we are letting a capital latin letter from the middle of the alphabet represent a Minkowski index ranging from 0 to 3.

We also introduce $\text{SU}(2)$ matrices $\sigma^i_{A'B'} = \sigma^i_{B'A'}$ in terms of the Pauli matrices

\begin{align*}
\sigma_1^A &= \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
\sigma_2^A &= \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \\
\sigma_3^A &= \frac{i}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\end{align*}

We have the following useful identities:

\begin{align*}
\Sigma^I_{AA'} \Sigma^J_{BB'} &\equiv \frac{1}{2} \eta^{I^J} \delta^B_A + \left( \delta^I_0 \epsilon^{jkl} + \frac{i}{\sqrt{2}} \delta^I_j \delta^l_k \epsilon^{ijk} \right) \sigma^k_B A, \\
\sigma_i &\equiv \delta_{ij} \delta^A_B + \frac{1}{\sqrt{2}} \epsilon_{ijk} \sigma^k_A B.
\end{align*}

With the aid of these matrices and the tetrads given in (4) and (5), we can introduce the soldering forms $\Sigma^\mu_{AA'}$:

$$\Sigma^\mu_{AA'} := E^\mu_I \Sigma^I_{AA'}.$$  \hspace{1cm} (132)

As the name suggests, these objects ‘solder’ the tangent space to spinor space, and we shall now use them to effect our translation from tensorial to spinor expressions. For this purpose we construct a complex connection $A^{IJ}_\mu$:

$$A^{IJ}_\mu := \frac{1}{2} \left( \Omega^{IJ}_\mu - i \ast \Omega^{IJ}_\mu \right),$$  \hspace{1cm} (133)

where

$$\ast \Omega^{IJ}_\mu := \frac{1}{2} \epsilon^{IJKL} \Omega^{KL}_\mu,$$  \hspace{1cm} (134)

and $\epsilon^{IJKL}$ is the Levi-Civita symbol with $\epsilon^{0123} = -1$. Since

$$\ast A^{IJ}_\mu = i A^{IJ}_\mu,$$  \hspace{1cm} (135)

$A^{IJ}_\mu$ is said to be self-dual. Self-dual objects have natural spinorial analogues. In particular, we assert without proof that the spinorial equivalent of the tensorial curvature $F_{\mu \nu}^{IJ}$ is

$$F_{\mu \nu}^{IJ} = \Sigma^I_{AA'} \Sigma^J_{BB'} R_{\mu \nu}^{AB} \epsilon^{A'B'} = \Sigma^I_{AA'} \Sigma^J_{BB'} A' R_{\mu \nu}^{AB},$$  \hspace{1cm} (136)

where

$$F_{\mu \nu}^{IJ} := 2 \partial_{[\mu} A^{IJ}_{\nu]} + 2 A^{IJ}_{\mu [M} A^{LM}_{\nu]}^{J},$$  \hspace{1cm} (137)

$R_{\mu \nu}^{AB}$ is the spinorial curvature constructed from the spinorial connection $\chi^{AB}_\mu$ which leaves the metric $\epsilon_{AB}$ covariantly constant

$$\chi^{AB}_\mu = \Sigma^I_{AA'} \Sigma^J_{BB'} \Omega^{IJ}_\mu.$$  \hspace{1cm} (138)
Comparing $A^i_a$ in (133) with the definition of the Ashtekar connection $A^i_a$ in (9), we discover that

$$A^i_a = \frac{1}{2} A^i_a.$$  

(139)
Thus

$$F_{ab}^{ij} = 2 \partial_{[a} A_{b]}^{ij} + 2 A_{[a}^{ik} A_{b]k}^{j} + 2 A_{[a}^{i0} A_{b0}^{j} = \partial_{[a} A_{b]}^{ij} - A_{[a}^{i} A_{b]}^{j}$$  

(140)
$$= \frac{1}{2} F_{ab}^{ij},$$  

(141)
where in (140) we used the self-duality relation (133).

We are finally prepared to effect the spinorial translation of (24), which we rewrite as

$$\tilde{B}^a_i = \Psi_{ij} \tilde{T}^a_j.$$  

(142)

We have

$$\tilde{B}^a_i = \frac{1}{4} e^{abc} \epsilon^{ikl} F_{bc}^{kl} = \frac{1}{\sqrt{2}} e^{abc} \sigma^i_{AB} \Sigma_{AB}^{ab},$$  

(143)

where we made use of the identity (130). The 2-form $\Sigma_{ab}^{AB}$ will be useful in our manipulations

$$\Sigma_{ab}^{AB} := \Sigma_{a}^{AA'} \Sigma_{b}^{B} A' = \frac{1}{\sqrt{2}} \epsilon^{ijk} \sigma^i_{AB} \Sigma_{AB}^{ab}. $$  

(144)

(145)

(146)

In (144) we used the covariant tetrad $e^I_\mu$ which is the inverse of $E^I_\mu$ given in (1) and (2):

$$e^I_\mu = \begin{pmatrix} N & 0 & t^a_i \\ t^a_i N^a & 0 \\ 0 & t^a_i \end{pmatrix},$$  

(147)

while in (145) we used the identity (130) and the definition (2). We may therefore reexpress $\tilde{T}^a_i$ using the identity (131):

$$\tilde{T}^a_i = \frac{1}{\sqrt{2}} e^{abc} \Sigma_{ab}^{AB} \sigma_i^{AB}.$$  

(148)

Finally, substituting both (143) and (148) into (142), we conclude that

$$R_{ab}^{CD} = \sigma_i^{AB} \sigma_j^{CD} \Psi_{ij} \Sigma_{ab}^{AB} = \chi_{CDAB} \Sigma_{ab}^{AB},$$  

(149)

where

$$\chi_{CDAB} := \sigma_i^{AB} \sigma_j^{CD} \Psi_{ij}.$$  

(150)

Note that because of the symmetry and tracelessness of $\Psi_{ij}$, $\chi_{ABCD}$ is completely symmetric

$$\chi_{ABCD} = \chi_{(ABCD)}.$$  

(151)
All that remains is to show that $\chi_{ABCD}$ is simply the Weyl conformal spinor $\Psi_{ABCD}$. We refer to the spinorial form of the curvature

\[ \mathcal{R}_{AA'B'B'} = \sum_{\mu} A' \sum_{BB'} \mathcal{R}_{\mu}^{CD} = \Psi_{ABCD} \epsilon_{A'B'} + \Phi_{A'B'CD} \epsilon_{AB} \quad (152) \]

Thus

\[ \mathcal{R}_{ab} AB = \Psi_{ABCD} \sum_{ab}^{CD} + \sum_{a}^{C} \sum_{b}^{C} A'B'AB. \quad (153) \]

Comparing (149) and (153), we deduce that

\[ \chi_{ABCD} \epsilon_{A'B'CD} = \Psi_{ABCD} \epsilon_{A'B'} + \Phi_{A'B'CD} \epsilon_{AB}. \quad (154) \]

It follows that $\Phi_{A'B'CD} = 0$; in other words the Ricci tensor vanishes, and

\[ \chi_{ABCD} = \Psi_{ABCD} = \sigma_{i} AB \sigma_{j} CD \Psi_{ij}. \quad (155) \]

The definitions of the Newman Penrose scalars are

\[ \Psi_{0} := \Psi_{0000}, \quad (156) \]
\[ \Psi_{1} := \Psi_{0001}, \quad (157) \]
\[ \Psi_{2} := \Psi_{0011}, \quad (158) \]
\[ \Psi_{3} := \Psi_{0111}, \quad (159) \]
\[ \Psi_{4} := \Psi_{1111}. \quad (160) \]

so our claim (149) - (153) has been verified.

We conclude by giving the general tensorial form of the Bel-Robinson tensor $T_{\mu\nu\rho\sigma}$

\[ T_{\mu\nu\rho\sigma} := \sum_{\mu} A' A' \sum_{\nu} BB' \sum_{\rho} CC' \sum_{\sigma} DD' \Psi_{ABCD} \Psi_{A'B'C'D'}. \quad (161) \]

Thus in our tetrad basis the components are

\[ T_{IJKL} = \sum_{I} I I' \sum_{J} BB' \sum_{K} CC' \sum_{L} DD' \Psi_{ABCD} \Psi_{A'B'C'D'}. \quad (162) \]

We compute, for example, the Bel-Robinson super-energy density in this orthonormal basis for the planar cosmology considered in Section 4.

\[ T_{0000} = \frac{1}{4} \delta^{AA'} \delta^{BB'} \delta^{CC'} \delta^{DD'} \Psi_{ABCD} \Psi_{A'B'C'D'} \]
\[ = \frac{1}{4} \left( \Psi_{0}^{2} + 4 \Psi_{1}^{2} + 6 \Psi_{2}^{2} + 4 \Psi_{3}^{2} + \Psi_{4}^{2} \right) \]
\[ = \frac{1}{2} (a^{2} + ab + b^{2}) \quad (163) \]

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