On light surfaces in black hole thermodynamics

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Abstract We discuss the fundamentals of classical black hole (BH) thermodynamics in a new framework determined by light surfaces and their frequencies. This new approach allows us to study BH transitions inside the Kerr geometry. In the case of BHs, we introduce a new parametrization of the metric in terms of the maximum extractable rotational energy or, correspondingly, the irreducible mass, which is an alternative to the spin parametrization. It turns out that BH spacetimes with spins \( a/M = \sqrt{5/9} \) and \( a/M = 1/\sqrt{2} \) show anomalies in the rotational energy extraction and surface gravity whereas the case \( a/M = \sqrt{3}/2 \) is of particular relevance to study the variations of the horizon area. We find the general conditions under which BH transitions can occur and express them in terms of the masses of the initial and final states. This shows that BH transitions in the Kerr geometry are not arbitrary but depend on the relationship between the mass and spin of the initial and final states. From an observational point of view, we argue that near the BH poles it is possible to detect photon orbits with frequencies that characterize the light surfaces analyzed in this work.

1 Introduction

A black hole (BH) singularity is mainly an “active” gravitational object in the sense that it can interact with its environment constituted usually by matter and fields. A BH transition from a state to another, accompanied by a change of the characteristic parameters, is regulated by the laws of BH thermodynamics. These changes are also crucial in the analysis of BH energy extraction detectable, for example, by observing jet emissions. The progenitor collapse into a BH appears as an irreversible process of information degeneration, where the information becomes inaccessible to the distant observers and the BH has no observable hair. A (stationary) BH macro-state is defined and determined only by the mass parameter \( M \) mass, spin \( J \), and electric charge \( Q \). However, the BH micro-states number can be enormously large, providing eventually a very high BH entropy. The number of BH micro-states increases with the BH size and the BH size, function of outer horizon area, becomes a measure of its entropy. The horizon area cannot decrease in any process with suitable conditions on energy and no repulsive gravity. Hawking’s area theorem, states that the total horizon area of classical BHs cannot decrease over time [1–3].

In the processes of BH energy and rotational energy extraction, underlying for instance jet emission processes, this result is maintained. The Penrose process is, for example, a classical method for extracting energy from a Kerr BH using its frame-dragging effects in the ergoregion. A small body may have in this region negative energy as measured by an observer at infinity. If the body is fragmented into two parts, and the negative-energy part is swallowed by the BH, decreasing its (total) mass, the other part can be ejected. In this process \( J \) decreases, consequently, its area increases (as determined by the outer horizon) [4]. The balance of the mass increase (which increases the BH area) and spin variations in this process is maintained within the constraint that the event horizon can increase or remains constant. For example, the recently observational confirmation of Hawking’s black-hole area theorem has been presented based on data from GW150914 [5].

Black holes are classical solutions of the Einstein field equations, representing spacetime regions bounded by the event horizon and causally disconnected from observers at infinity. Bekenstein first suggested the idea that the event horizon\(^1\) defines the BH entropy [8,9]. However, BH thermodynamic properties seem to be purely intrinsically geometric

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\( ^1 \) It should be noted that the inner horizon has been shown to be unstable and it has been conjectured it could end into a singularity [6,7].
characteristics, defined on the basis of the outer BH (event) horizon alone. Probably, an unknown quantum mechanism would explain and complete BH classical thermodynamics, leaving however open complex problems (for instance, the well-known information paradox as a consequence of Hawking thermal radiation), and touching the limit of applicability of quantum gravity.

However, if BHs have an entropy, we can expect them to have also a temperature. This implies that a BH must also radiate. This radiation was explained by effects of quantum fluctuation of matter fields in the vacuum close to the singularity as Hawking radiation. Fluctuations of the (matter) field vacuum produce couples of particles and antiparticles in the close vicinity of a black hole. A particle is captured by the singularity with negative energy, the other particle can escape to infinity with positive energy, constituting observable radiation with thermal profile.

In this semi-classical scenario, the BH would evaporate emitting only thermal Hawking radiation with non-vanishing (non constant) entropy; however, such an evolution appears not to be fully described by quantum mechanics (raising the so called information paradox). The derivation of Hawking radiation is indeed semi-classical and yet does not establish the BH situation at the Planck size, where quantum gravity is expected to play a major role. Specifically, it does not establish if, for example, the BH evaporates completely or viceversa some kind of remnants is left. A further question is how the emitted radiation, reaching infinity, may be entangled with such remnant, and what is its entropy. The BH may also continue the evaporation process completely, leaving no remnants. One could expect that (Hawking) radiation from the collapsing matter into a BH would carry out the information of the matter, the BH would evaporate away as a result of thermal emission and the process preserves the unitarity foreseen in quantum mechanics. This expectation, however, fails with the difficulties existing in obtaining a unitary description of BH evaporation (and BH formation). A concrete possibility is that the final radiation of the complete evaporation does not “bring” information on the initial matter state (contradicting quantum mechanics unitarity realized through the action of an unitary evolution operator). In this case, the quantum fluctuation would produce a mixed state of the outgoing emitted particles with the ingoing particles; consequently, the outgoing radiation would be entangled with the final hole state (the final state is a mixed state if the fate of the BH is to evaporate to nothing) – see, for example, [10] for a review on the BH information problem. All these are aspects of still open debated issues, at the heart of the quantization of the geometric theories of gravity, of the treatment of unified theories for matter field-quantized gravity, and very often addressed in quantum theories such as loop quantum gravity or in the context of string theories.

From the classical point of view the effects of a change of state in BHs are constrained by the laws of thermodynamics. One might ask if there are privileged or not allowed state transitions or if transitions are singled out by some anomalous characteristics. Constrains on these processes are generally determined by the initial state of the black hole, its spin J or by the specific mass ratios of the initial and final states. There are strong indications that the creation of a naked singularity (NS) through a conversion process from an extreme BH is forbidden and that no accretion process can bring a BH into a NS state, while the reverse process, assuming the NS existence, is still debated in different contexts. However, it is necessary to consider possible processes of rotational energy extraction. For example, geometrically thin accretion disks have been studied for possible mechanisms to convert asymptotically NSs into extreme BH states [11].

In this work, we study aspects of classic black hole thermodynamics reformulated in terms of light surfaces. We focus on Kerr singularities. Using some characteristics of specific null surfaces of the Kerr geometries, including the case of the horizons, we study the relation between the initial and final states of a BH transformation regulated by the laws of BH thermodynamics. BHs an NSs geometries are therefore represented in a plane called the extended plane. The surfaces are defined by a characteristic null frequency \( \omega \), which connect all the geometries described by the Kerr solution and all the points of these geometries where a null orbit with frequency \( \omega \) is defined. These geometries always include BHs and in some cases naked singularities. Characteristic frequencies are frequencies of the Kerr horizons. A special metric line parametrization, describing only BH spacetimes (no naked singularities), is given in terms of the maximum extractable rotational energy \( \xi \) or, correspondingly, the inertial mass parameter \( M_{\text{irr}} \), replacing the metric spin parameter. We focus on some particular transformations between one state and another. We identify some particular transition states, representing the initial spin of the black hole or a specific ratio of the masses. These emerge as properties of the classical thermodynamic variables, when viewed in the extended plane. Using the representation provided by the light surfaces, we obtain an overview of the possible states. Thus, we first introduce the new framework, writing the principal quantities in this new way, and the we study the laws of BH transformations in the extended plane in terms of the light surfaces.

The plan of the article is as follows. In Sect. 2, we introduce the Kerr geometries while, in Sect. 2.1, we discuss BH Killing horizons and the characteristic frequencies, building up the framework for the formulation of the laws of BH thermodynamics that are introduced in Sect. 3. In Sect. 3.1, masses and thermodynamic variables are explored in terms of the light surfaces. The analysis continues in Sect. 4, where we focus on the laws of BH thermodynamics. Rotational energy
and BH “rest” mass are the focus of Sect. 4.1 while the BH irreducible mass is addressed in Sect. 4.2. BH transformations in the new frame are investigated in Sect. 5. An in-depth discussion on the inner and outer horizons in BH transitions is presented in Sect. 5.1. The case of constant irreducible mass is described in Sect. 5.2. In Sect. 5.3, we discuss the the inner horizon relations in BH thermodynamics. In Sect. 6, we present the metric tensor re-parameterizations in the new framework in terms of inertial mass, extractable rotational energy, and surface gravity. Discussion and final remarks follow in Sect. 7. In Appendix A, there are further details of black holes in the extended plane. Some remarkable areas in the extended plane are described in Appendix B.

2 The Kerr geometry

The metric tensor of the Kerr geometry can be written in Boyer-Lindquist (BL) coordinates \{t, r, \theta, \phi\} as follows

\[
\begin{align*}
ds^2 &= -dt^2 + \frac{b^2}{\Delta} dr^2 + \rho^2 d\theta^2 + (r^2 + a^2) \sin^2 \theta d\phi^2 \\
&\quad + \frac{2M}{\rho^2} r(dt - a \sin^2 \theta d\phi)^2 \\
&\quad + \frac{\rho^2}{\Delta} (dr - a \sin^2 \theta d\phi)^2 \\
&\quad + \frac{\rho^2}{\Delta} r^2 \sin \theta \cos \theta, \\
&\Delta \equiv r^2 - 2Mr + a^2, \\
r_{\pm} \equiv M \pm \sqrt{M^2 - a^2}; \\
r_{e}^\pm \equiv M \pm \sqrt{M^2 - a^2} \cos \theta,
\end{align*}
\]

where \( r \in [0, +\infty], t \in [0, +\infty], \theta \in [0, \pi] \) and \( \phi \in [0, 2\pi] \). The radii \( r_+ \) and \( r_- \) are the outer and inner Killing horizons, respectively; \( r_{e}^+ \) and \( r_{e}^- \) represent the outer and inner ergospheres, respectively. Here \( M \) is the (ADM and Komar) mass parameter and the specific angular momentum is given as \( J = J/M \), where \( J \) is the total angular momentum of the gravitational source. For simplicity, here and in the following we consider dimensionless parameters defined as \( r \rightarrow r/M \) and \( a \rightarrow a/M \). Moreover, we will use the notation \( \sigma \equiv \sin^2 \theta \).

It is \( r_+ < r_{e}^+ \) on the planes \( \theta \neq 0 \) and \( r_{e}^+ = 2M \) on the equatorial plane \( \theta = \pi/2 \), where \( r_- = 0 \). Moreover, \( 0 < r_{e}^- < r_+ < r_{e}^+ \) for \( \sigma \neq 0 \) and \( a \in [0, M] \). For \( \sigma = 0 \), the functions \( r_{e}^\pm \) satisfy the properties \( 0 < r_{e}^- = r_- < r_+ < r_{e}^+ \). For \( a = \pm M \) (extreme Kerr BH), we have that \( r_{e}^\pm = M \). The case \( a = 0 \) corresponds to the static spherically symmetric Schwarzschild solution, where \( r_+ = 2M \) and there is no ergosphere. Naked singularities are defined for \( a^2 > M^2 \). The surfaces \( r_{e}^\pm \) in NSs are well-defined for \( \cos \theta = [-M/a, M/a] \) or, equivalently, \( a/M \in [1, 1/\text{sec}^2 \theta] \). In the region \( r \in [r_+, r_{e}^+] \) [(outer ergoregion or simply ergoregion), it is \( g_{tt} > 0 \) and the t-Boyer-Lindquist coordinate becomes spacelike. This fact implies that a static observer cannot exist inside the ergoregion. In the following analysis, we will refer to some properties of circular motion in these regions.

Since the metric is independent of \( \phi \) and \( t \), the covariant components \( p_\phi \) and \( p_t \) of a particle four-momentum are conserved along its geodesic. Consequently, \( E \equiv -g_{ab} \xi^a \xi^b \) and \( L \equiv g_{ab} \xi^a \xi^b \) are constants of motion for test particle orbits, where \( \xi = \partial_t \) is the Killing field representing the stationarity of the Kerr geometry and \( \xi_\phi = \partial_\phi \) is the rotational Killing field (the vector \( \xi \) becomes spacelike in the ergoregion). The constant \( L \) may be interpreted as the axial component of the angular momentum of a test particle following timelike geodesics and \( E \) as representing the total energy of the test particle coming from radial infinity, as measured by a static observer at infinity. The motion on the fixed plane \( \sigma = 1 \) is restricted to that plane \( (u^0 = 0) \) because the Kerr metric is symmetric under reflections with respect to the equatorial hyperplane \( \theta = \pi/2 \).

2.1 Killing horizons and characteristic frequencies

In this section, we build the framework for the BH thermodynamics laws by using the properties of special light surfaces and the BH horizons.

2.1.1 BH horizons

Let us introduce the Killing vector \( \mathcal{L} = \partial_\theta + \omega \partial_\phi \). We will consider mainly \( \omega = \text{constant} \). The quantity \( \mathcal{L} \mathcal{N} \equiv \mathcal{L} \cdot \mathcal{L} \) becomes null for photon-like particles with orbital frequencies \( \omega_{\pm} \).

The Killing vector \( \mathcal{L}_\pm \equiv \xi_\pm + \omega_{\pm} \xi_\phi \), satisfy the null condition on the norm of \( \mathcal{L} \), can be interpreted as generator of null curves \( (g_{ab} \mathcal{L}_\pm^a \mathcal{L}_\pm^b = 0) \) as the Killing vectors \( \mathcal{L}_\pm \) are also generators of Killing event horizons.

The Kerr horizons are null (lightlike) hypersurfaces generated by the flow of a Killing vector, whose null generators coincide with the orbits of a one-parameter group of isometries, i.e., in general, there exists a Killing field \( \mathcal{L} \), which is normal to the null surface. More precisely, the BH horizon \( r_+ \) is a non-degenerate (bifurcate) Killing horizon generated by the vector field \( \mathcal{L} \). In the case \( a = 0 \) (where \( \omega = 0 \)), \( \xi_\pm \) (now generator of \( r_+ \)) is hypersurface-orthogonal.

Thus, the null frequencies \( \omega_{\pm} \) evaluated on the BH horizons, are the BH inner and outer horizons frequencies \( \omega_{\pm}^H \).

2.1.2 Light surfaces and characteristic frequencies

The results discussed in this work follow from the investigation of the properties of the null vector \( \mathcal{L} \). The framework of BH thermodynamics is constructed upon the solutions \( \mathcal{M}_B : \mathcal{L}_N = 0, \) with \( \omega = \text{constant} \), where \( \mathcal{M}_B \) is a spin parameter of the set \( \mathcal{M}_B \in [a, a\sqrt{\sigma}] \). Solutions \( \mathcal{M}_B \) are
functions of \((\omega, r, \sigma)\), and can be represented as a curve called metric bundle \(\mathcal{MB}\) on a plane \(a - r\) or \(A - r\) (extended plane) where \(A \equiv a\sqrt{\sigma}\) \([12–18]\).

The quantity \(\omega : \mathcal{L}_N = 0\), for the null vector \(\mathcal{L}\) will be called characteristic \(\mathcal{MB}\) frequency.

A particular null vector \(\mathcal{L}\) is defined by \(\omega = \omega(r_{\pm}) \equiv \omega_{\pm}^H\), defining the Killing horizons of the metric. In fact, the event horizons of a spinning \(\text{BH}\) are Killing horizons with respect to the Killing field \(\mathcal{L}_H = \partial_t + \omega_{\pm}^H \partial_{\phi}\), where \(\omega_{\pm}^H\) is the angular velocity (frequency) of the horizons representing the \(\text{BH}\) rigid rotation. The event horizon of a stationary asymptotically flat solution with matter, satisfying suitable hyperbolic equations, is a Killing horizon. The strong rigidity theorem connects the event horizon with a Killing horizon. In the limiting case of spherically symmetric, static spacetimes, \(\xi\) is a Killing horizon. The strong rigidity theorem defines the Killing horizons of the metric. In fact, the event horizons of stationary observers frequencies. Specifically, the causal structure defined by timelike stationary observers is characteristic \(\mathcal{MB}\) frequency. In the extended plane, \(\mathcal{MB}\) are made by points \((a, r)\) or \((a\sqrt{\sigma}, r)\) distinguishing a geometry \(a\) (which can be also a naked singularity) and an orbit (and plane \(\sigma\)) where there is light-like frequency \(\omega\) corresponding to the \(\mathcal{MB}\) characteristic frequency.

Two important properties follow: all metric bundles are tangent to the horizon curves and the horizon curves are the envelope surfaces of the metric bundles. Thus, all the \(\mathcal{MB}\) characteristic frequencies, all the light surfaces frequencies (also in the naked singularity solutions) are frequencies of a \(\text{BH}\) (inner or outer) horizons, determined by the corresponding \(\mathcal{MB}\) tangent point. The point \(a_0 (A_0)\) at the origin \(r = 0\) (central singularity) is called the bundle origin.

Note that in this case we are assuming \(\omega > 0\) as \(a > 0\); however, the study of counter-rotating photon circular orbits with frequency \(\omega < 0\) is possible with \(\mathcal{MB}\)s in the extended plane \(a \in R\) with \(\omega > 0\), although in this analysis this extension is not necessary – see [13]. The \(\text{NS}\) light surfaces properties are defined by the \(\text{BH}\) light surfaces (through the condition of tangency of the bundles with the horizon curve). Vice versa it has been shown that most bundles \(\mathcal{MB}\) with \(\text{NS}\) origin are tangent to the inner horizons curve. The horizon curves point \(a = 0, r = 0\) is the central singularity, \((a = M, r = M)\) is the horizon of the extreme Kerr \(\text{BH}\), while \(a = 0\) with point \(r = 2M\) is the horizon of the Schwarzschild solution.

Each line \(a_{\pm} =\)constant in this extended plane is a \(\text{BH}\) geometry, where the inner horizons are for \(r \in [0, M]\) on the line \(a_{\pm}\), while \(r \in [M, 2M]\) contains the outer horizons. The maximum of the curve \(a_{\pm}\) is the point \(a = M\) and \(r = M\) which is the horizon in the extreme Kerr \(\text{BH}\). The Schwarzschild \(\text{BH}\) is for \(a = 0\); therefore, it corresponds to the zeros of the metric bundles curves, where \(r = 2M\) is its horizon-and in the extended plane it coincides also with the outer ergosurface on the equatorial plane of all the Kerr \(\text{BHs}\) and \(\text{NSs}\).

The characteristic frequency \(\omega\), the bundles origin \(a_0\), the tangent spin \(a_0\) and radius \(r_0\) to the horizon curve in the extended plane are related by the following conditions

\[
\begin{align*}
\omega &= \frac{1}{A_0} = \frac{1}{a_0\sqrt{\sigma}} = \frac{\sqrt{2 - r_{\pm}}}{2r_0} = \omega_{\pm}^H(r), \\
r_0 (\omega) &= \frac{2}{4\omega^2 + 1} = \frac{2A_0^2}{A_0^2 + 4}, \\
a_0 (a_0) &= \frac{4A_0}{A_0^2 + 4} = 4 \frac{\omega^2}{(4\omega^2 + 1)^2}, \\
a_0 &= \frac{2}{\sqrt{\sigma}} \sqrt{\frac{r_0}{2 - r_0}}.
\end{align*}
\]

where \(r_{\pm} \in [0, 2M], a_0 \in [0, +\infty], a_0 \in [0, M]\) (with the extension to negative values for the counter-rotating case), \(\omega \in [0, +\infty]\) and \(\sigma \in [0, 1]\) (see Fig. 1).

The bundle is tangent to the horizon curve at one point only; therefore, this defines uniquely the \(\text{BH}\) with the characteristic frequency \(\omega\) and, consequently, bundles can contain either only \(\text{BHs}\) or \(\text{BHs}\) and \(\text{NSs}\). Bundles are not defined in the region \([r_-, r_+]\) of \(\text{BH}\) spacetimes, but they are defined in the region \([0, 2M]\) of \(\text{NSs}\).

\(\mathcal{MB}\) curves in the extended plane connect points of different (\(\text{BH}\) or \(\text{BH}\) and \(\text{NS}\)) geometries having all the same characteristic null frequency \(\omega\). They are related to the light surface \(r_\omega (\omega) : \mathcal{L}_N = 0\) defined in a fixed spacetime as the light surfaces \(r_\omega\) are the collections of all points, crossing of the \(\mathcal{MB}\) with a line \(a_{\pm}\) constant on the extended plane.

2.1.3 Constrains on matter distribution

\(\mathcal{MBs}\), defined by light-surfaces, determine also the timelike (matter circular) motion providing the limiting orbital frequencies of stationary observers.

As the solutions \(\omega : \mathcal{L}_N = 0\), for each point \((a, r, \theta)\), are generally two frequencies \(\omega_{\pm}\), this implies that there are two bundles crossing at any point of the extended plane. The frequencies \(\omega_{\pm}\), at the crossing point \(r\), bound the possible stationary observers frequencies. Specifically, the causal structure defined by timelike stationary observers is character-
Fig. 1 Left and center panels: light surfaces $r^\pm_s$ as functions of the bundles frequency $\omega$ for different values (denoted on each curve) of the dimensionless spin $a/M$, representing $\text{NS}$ and $\text{BH}$ geometries. The quantities $r^\pm_s$ are solutions of $L \cdot L = 0$, where $\sigma = \sin^2 \theta$. Left panel: equatorial plane $\sigma = 1$. Right panel: Plane $\sigma = 0.5$. $r_\pm$ are the inner and outer horizons in the extended plane, where $r_+ \in [M, 2M]$, and $r_- \in [0, M]$, while $r_s$ represents the points of the metric bundles that are tangent to the horizon curve in the extended plane, i.e., it is the horizon curve as function of the bundles characteristic (and horizons) frequency. Gray region is $r < r_\pm$. There $a_0$ is the bundle origin spin as function of the frequency $a_0 = 1/\omega \sqrt{\sigma}$. Note that in slowly rotating $\text{NS}$, an increase of the spin $a_0$ leads to the appearance of the Killing bottleneck (a restriction of the light surfaces). Right panel: The central sphere represents $\text{BH}$s in the extended plane; each horizontal plane corresponds to a $a =$ constant surface, i.e., a particular Kerr geometry, an the points on the black surface represent the inner and outer $\text{BH}$ horizons. Planes not crossing the sphere are $\text{NS}$s. The sphere considers also the two rotation directions of the $\text{BH}$ spin and the central equatorial plane is the Schwarzschild spacetime. The points $p_s$ and $q_s$ correspond to replicas, where $p_s$ denote geometries related by the same frequency $\omega$ and $q_s$ include a point of the horizon.

On the other hand, static observers are defined by the limiting curves characteristic frequencies. The limiting frequencies $\omega_s$, which are photon orbital frequencies, solutions of the condition $L_N = 0$, determine the frequencies $\omega_s^0$ of the Killing horizons as well as the bundles characteristic frequencies.

The vector $L$ appears in the description of certain $\text{BH}$ evolution processes because it enters the definitions of thermodynamic variables and stationary observers. The vector $L$, the condition $L_N = 0$ and $\mathcal{MB}$s are closely related to the definition of stationary observers, i.e., observers with a tangent vector which is a Killing vector, that is, whose four-velocity $u^a$ is a linear combination of the two Killing vectors $\xi_\phi$ and $\xi_t$; therefore, $u^a = \gamma L^a = \gamma (\xi_t^a + \omega \xi_\phi^a)$, where $\gamma$ is a normalization factor and $d\phi/dt = u^\phi/u^t \equiv \omega$. The dimensionless quantity $\omega$ is the orbital frequency of the stationary observer.

It is clear that $\mathcal{MB}$s relate different geometries through their light surfaces defined by the characteristic frequencies which are $\text{BH}$s frequency. Thus, they are particularly adapted to describe $\text{BH}$ state transitions, constrained by the properties of the null vector $L$, which builds up many of the thermodynamic properties of $\text{BHs}$. In the next section, we shall detail the $\text{BH}$ thermodynamic properties in terms of the null vector $L$ and, therefore, of the metric bundles.

$\mathcal{MB}$s are structures that are related to the points of the light surfaces of different geometries and, therefore, provide relations between geometries. This different way of rewriting the properties of the null vector $L$, which defines also the $\text{BH}$ horizons, emphasizes some geometric characteristics and properties, repeating in the same spacetime or different geometries of a $\mathcal{MB}$. One of these properties, from the light surfaces and $\mathcal{MB}$ definitions is the frequency $\omega$, all the points of a $\mathcal{MB}$ curve are the replica of the $\text{BH}$ horizon frequency $\omega^0_H$ or $\omega^1_H$ of the $\text{BH}$ individuated by the tangent point $(a_s, r_s)$. In some cases, these horizons (frequencies) replicas are in the same spacetime. In [17, 18], the $\text{BH}$ poles $\sigma \approx 0$ have been studied for the observation of the photon orbits with $\mathcal{MB}$s characteristic frequency. These are light-like (circular) orbits having the same frequency as the black hole $\omega^\pm_H$.

In the context of replicas, we also introduce the concept of horizon confinement. Indeed, we say that there is a replica when in the spacetime it is possible to find at least a couple of points having the same value for the property $Q$. Then, we say that there is a confinement, when that value is not replicated. In the Kerr spacetime, part of the inner horizon frequencies are “confined”. The confinement analysis, which is the study of the topology of the curves $\omega =$ constant in the extended plane, provides information about the local properties of the spacetime replicated in regions more accessible to observers; for example, in the case of properties defined in the proximity of the $\text{BH}$ poles or of the inner horizons. An observer can register the presence of a replica at the point $p$ of the $\text{BH}$ spacetime with spin $a_p$, belonging to a Killing bundle. The observer will find the replica of the $\text{BH}$ horizon frequency $\omega^0_H(a_p)$ at the point $p$; therefore, her/his orbital stationary frequency is $\omega_p \in [\omega_s, \omega_s]$, where one of $(\omega_s, \omega_s)$ is the horizon’s frequency $\omega^0_H$, replicated on a pair of orbits $(r_s, r_p)$. The second light-like frequency $\omega_s$ is the frequency of a horizon in a $\text{BH}$ spacetime. The relation between the two
frequencies ($\omega_+$, $\omega_-$) is determined by a characteristic ratio, which we also study [17,18]. In general, we use in this work replicas to connect BH spacetimes related in a transition and governed by the thermodynamic laws in the extended plane. We reformulate BH thermodynamics on the replicas in terms of the light surfaces, exploring the thermodynamic properties of the geometries defined by the metric bundles.

3 Black hole thermodynamics

The norm $L_N \equiv L \cdot L$ is constant on the horizon. We start here our analysis of BH thermodynamics by introducing the BH surface gravity as the constant (acceleration)$^3$ $\ell : \nabla^a L_N = -2L^a \nabla a$, evaluated on the outer horizon $r_+$, or equivalently, $L^\beta \nabla_\beta L_a = -\ell L_a$ and $L_L \ell = 0$, where $L_L$ is the Lie derivative, a non affine geodesic equation, i.e., $\ell$ = constant on the orbits of $L$.

The BH surface gravity $\ell$, which is also a conformal invariant of the metric [21], may be defined as the rate at which the norm of the Killing vector $L$ vanishes from outside (i.e. $r > r_+$). For the Kerr spacetime it becomes $\ell_{Kerr} = (r_r^2 - a^2)/(r_r^2 + a^2)^2$.

The surface gravity re-scales with the conformal Killing vector, i.e., it is not the same on all generators but, because of the symmetries, it is constant along one specific generator.

The BH event horizon of stationary solutions has constant surface gravity, i.e., the surface gravity is constant on the horizon of stationary black holes, which is postulated as the zeroth BH law-area theorem (see for example [19,20]).

More generally, the BH horizon area is non-decreasing, a property which is considered as the second law of BH thermodynamics, establishing the impossibility to achieve with any physical process a BH state with zero surface gravity.

Clearly, in the extreme Kerr spacetime ($a = M$), where $r_\pm = M$, the surface gravity is zero. This implies that the temperature is also null ($T_H = 0$), with a non-vanishing entropy [19,20,22]. A non-extremal BH cannot reach the extremal limit in a finite number of steps, which is implied by the third law. (This fact has consequences also regarding the stability against Hawking radiation.).

On the other hand, the condition (constance of) $\nabla^a L = 0$ when $\ell = 0$ substantially constitutes the definition of the degenerate Killing horizon – degenerate BH –, in the case of Kerr geometries only the extreme BH case is degenerate; therefore, in the extended plane it corresponds to the point $a = M$, $r = M$. A fundamental theorem of Boyer shows that degenerate horizons are closed. This fact also establishes a topological difference between black holes and extreme black holes.

Now, the first law of BH thermodynamics, $\delta M = (1/8\pi)\ell_H^+ \delta A^+_{area} + \omega_H^+ \delta J$, relates the variation of the mass $\delta M$, the (outer) horizon area $\delta A^+_{area}$, and angular momentum $\delta J$ with the surface gravity $\ell_H^+$ and angular velocity $\omega_H^+$ on the outer horizon. Here all the quantities, including the surface gravity $\ell$, are evaluated on the outer Killing horizon so that the notation (+) can be omitted. The (Hawking) temperature term is related to the surface gravity by $T_H = h c/2\pi k_B$ ($k_B$ is the Boltzmann constant) and the horizon area $A^+_{area}$, to the entropy, $S = k_B A^+_{area}/l_P^2$ ($l_P$ is the Planck length, $c$ the speed of light).$^4$

We focus our analysis on two initial and final states for BH transition. We can interpret the terms $(\omega_H^+ \delta J)$ in the BH transition from one state $(0)$ to a new state $(1)$ considering the frequency as characteristic bundle frequency, therefore, with $\omega_H^+(0) = \omega(0)$ which is the MB frequency tangent to the outer horizon of the BH at the initial state $(0)$, the entropy term $(\ell(0)\delta A^+_{area})$ is also considered on the bundle, writing the relation in the extended plane.

All the quantities are expressed in terms of bundles at $\omega(0) =$ constant and $(\delta A^+_{area}, \delta J, \delta M)$, describing the transition from the initial to final state. Furthermore, as in the extended plane, bundles tangent to the inner horizons are also relevant. We write a corresponding relation between the quantities $(\delta A^-_{area}, \delta J, \delta M)$ and $(\omega_H^-(0), \ell^-(0))$ evaluated on the inner horizon $r_−$. Alternately, we express $(\delta A^\pm_{area}, \delta J, \delta M)$ and $(\omega_H^\mp(0), \ell^\mp(0))$ and their relation in a unique form as function of a generic point $r$ of the extended plane.

In the next sections we will consider quantities $(\ell, \omega)$ and $(M, A_{area}, J)$ and their variations in the extended plane, on the horizon curve. We express the relation between these in terms of the properties of the light surfaces.

3.1 Masses and BH thermodynamics

Here we explore BH thermodynamics in the extended plane in terms of the light surfaces using MB curves and discuss variations of the total mass $M$, area $A_{area}$, and momentum $J$ of the BH defined by the tangency property with the horizons curve in the extended plane, considering the bundle frequency $\omega$ and the acceleration $\ell$. We emphasize some (“conformal”) properties, as defined below, in these relations.

$^3$ We note that in this work we have used the notation $\ell$ for the surface gravity instead of the more usual $\kappa$, to emphasize the role of acceleration relating to the null vector $L$.

$^4$ If the BH temperature is $T = \ell/(2\pi)$, its entropy is $S = A_{area}/(4\hbar G)$, the pressure-term is $p = -\omega_H$, where the internal energy is $U \equiv GM (M = c^2m/G\approx mass$, where $m$ is a mass term). The term $\omega_H \delta J$ is interpreted as the “work”. The BH (horizontal) area $A_{area}$ is related to the outer horizon definition, $A_{area} = 8\pi |mr|$, while the volume term is $V = GJ/c^2$ (where $J = amc^3/G$). The BH horizon area will be considered here alternatively in the extended plane.
We will connect the BHs states before and after a transition by expressing their characteristics and parameters through the MBs (curves defined by the properties of the light surfaces for different geometries) and their representation in the extended plane. Particularly the condition of tangency with the curve of the horizons provides the constraint for BHs transformations. Although the extended plane represents also NSs solutions as the origin of the bundles and the MBs contain BHs and BHs and NSs (related to the tangency to the curve portion of the internal horizons) or depending on the σ angle (near the poles there are interesting properties of the ω frequencies explored in [17,18]), these transformations seem to confirm cut off of any BH-NS transition.

The problem of writing the relation between main BH quantities before and after a transition (on the horizon curve) can be rephrased as the problem of finding replicas in the extended plane. More precisely, let us consider two points, r and rP, along the horizon curve ar > 0, which can also be in the entire range r ∈ [0, 2M]. We relate the parameters \( P_± = (\ell_±, \omega_±) \), regulating the BH transition, as evaluated on the point r and rP.

We search for a quantity \( \kappa \) such that \( P_±(r) = \kappa P_±(rP) \) (conformal property). Note that it can also be \( \ell_± \) or \( \pm \ell_± \), connecting, therefore, properties defined on an outer horizon with properties defined on an inner horizon; for this reason the notation \( (\ast) \) represents a change in sign in one or more components of the couple. While the frequency \( \omega \) is always positive (according to the sign of \( J \)), the surface gravity \( \ell \) can be negative, when evaluated on a point of the inner horizon; therefore, it can be \( \ell_±(rP) = \ell_± < 0 \).

We also investigate the case \( \delta A_± = -\delta A_\text{area} \) occurring for \( \delta M = 0 \) (as \( \delta M^+ = \delta M^- \)). In here we use notation \( (\pm) \) to indicate quantities evaluated on the horizons \( r_± \). Moreover, \( \delta Q \equiv Q(1) - Q(0) \) denotes the change of the quantity \( Q \) from the initial (0) to the final state (1) of the transition. The point \( a = M \) and \( r = rP \) corresponds to the extreme Kerr BH, where \( \omega(rP) \neq \omega(r) \) for points \( (r, rP) \) of the horizon curve. For the surface gravity the situation is different. There can be replicas on the curve of the horizons, which depend on a and r and connect inner and outer horizons; this is clearly illustrated in Fig. 2.

For convenience, we report here some relations for the frequencies:

\[
\omega_± = \frac{\sqrt{2r - r^2}}{2\sqrt{r}}, \quad \omega_± = \pm \frac{\sqrt{(r-1)^2 - 1}}{2\sqrt{(2 - r)r}}.
\]  

In the first equation of (3), the frequency is defined as a function of a general point \( r \in a_± \) of the inner horizon for \( r \in [0, M] \) or the outer horizon for \( r \in [M, 2M] \). In the second expression, we explicit the inner and outer horizon frequencies.

When calculated on the inner horizon curve in the extended plane, the surface gravity reads

\[
\ell_±^H(r) = \frac{r - 1}{2r}.
\]

In first expression of Eq. (4), we denote with \( \ell_±^H \) the acceleration for the inner horizon \( \ell_±^H \) and outer horizon \( \ell_±^H \) as functions of the spin \( a \) or as functions of the radius \( r \) on the horizon curve. The second function, instead, is the acceleration defined as function of a general point \( r \in a_± \). We introduce the quantities \( \ell_±^H(r) \equiv \ell^H_\ast(r)/\ell^H_\ast(r) \) and \( \omega_±^H(r) \equiv \omega^H_\ast(r)/\omega^H_\ast(r) \). Using the relation \( \ell_\ast^H(r) = -\omega_\ast^H(r) \), we obtain

\[
\{\ell_±^H, \omega_±^H\} = \omega_±^H(r)-\ell_±^H(r),
\]

\[
\{\ell_±^H, H_\ast\} = \omega_±^H(r)-\ell_±^H(r),
\]

where \( \omega_±^H(r) = \frac{1}{\omega_±^H(r)} \).

see Fig. 3.

There is \( \ell_±^H = \pm \omega_±^H \) for \( a = aP \equiv 1/\sqrt{2} \).

An analogue relation holds for the quantities evaluated as functions of the spin \( a \). Considering the dependence from the spin \( a \) we find,

\[
\ell_±^H(a) = \pm \sqrt{1 - a^2},
\]

alternatively to Eq. (4). Defining \( \ell_±^H(a) = \ell_±^H(a)/\ell_±^H(a) \) and \( \omega_±^H(a) = \omega_±^H(a)/\omega_±^H(a) \), we obtain \( \omega_±^H(a) = -\ell_±^H(a) \) and

\[
\{\ell_±^H(a), \omega_±^H(a)\} = \omega_±^H(a)\ell_±^H(a)-\ell_±^H(a),
\]

\[
\{\ell_±^H(a), \omega_±^H(a)\} = \omega_±^H(a)-\ell_±^H(a),
\]

see also Eq. (5) and Fig. 2.

Here, we search for the relation \( \delta M(aP) = s \delta M(a)^* \), where \( a \in [0, M] \) and \( aP \in [0, M] \) are two BH geometries. For the particular case \( a = aP \), the couple \( (\ast, \ast) \) denotes that \( \ast \) or \( \ast \ast \) are ±, relating to the inner-inner horizons of the two geometries or outer-outer or inner-outer horizons of the two geometries \( (a, aP) \). The special case \( a = aP \) contains \( (\ast, \ast) = (\ast, s = 0) \). The case considered for the quantities as functions of \( r \), where \( \ast \ast = -\ast = \pm \) and \( s \), satisfies conditions (5). The apparent contradiction in this case is that for a fixed spacetime \( a \in BH \), we have that \( \delta M^+ = \delta M^- \), but the relation regulating this variation with the other characteristic quantities depends on the point as \( \delta M^+ = s \delta M^- \) (where

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functions $\ell$ evaluated on the horizons $r_{\pm}$ we have used the notation $\pm$ to stress quantities in relations evaluated on the horizons $r_{\pm}$, respectively).

Then, we obtain

$$\omega_H^\pm, \ell_H^\pm,$$ and the ratios are plotted as functions of the BH dimensionless spin. Bottom right panel: The quantities $r(s), r_p(s)$ and the spin $a(r) = a(r_p)$ from Eqs. (12) and (15) as functions of the parameter $s$. There $(\omega(r_p), \ell(r_p)) = (s(\omega(r), -\ell(r)))$. Bottom right panel: The frequency $\omega$ and the acceleration $\ell$ evaluated on $r$ and $r_p$ are plotted as functions of $s$ – see Eq. (14)

\[
\delta M^+ = \ell_H^+ \delta A^+_{\text{area}} + \omega_H^+ \delta J^+ = r \delta M^- \equiv \ell_H^- \delta A^-_{\text{area}} + \omega_H^- \delta J^-.
\]

We note also the coincidence $\omega_H^\pm(r) = \ell_H^\pm(r) = Q_{\pm} \equiv 1/\sqrt{2} - 1/2$ in the form of Eqs. (3) and (4) for $r = r_+ = (\sqrt{2} + 2)/2$. This is the outer horizon for the BH geometry with spin $a/M = a_r = 1/\sqrt{2}$, specifically, $\ell_H^+ = \pm \omega_H^+$. Therefore, for this spacetime the mass variation has a special spin for when written in the extended plane $\delta M^\pm = Q_{\pm} \delta M^\pm$ (where $\delta M^\pm \equiv \pm \delta A^\pm_{\text{area}} + \delta J^\pm$, where $Q_{\pm} = \omega_H^\pm = 1/2 + 1/\sqrt{2} = -\ell_H^\pm$).

We analyze the more general problem addressing firstly the special case $s = 1$ and secondly the general case with $s \neq 1$.

**The case** $s = 1$ This problem can be reduced to finding replicas for $\ell (r_+ (a_p)) = - \ell (r_- (a))$, solved for

$$\frac{2\sqrt{2}}{3} < a \leq 1,$$

\[
a_p = a_p^\pm = \frac{\sqrt{4 (1 - a^2)^{3/2} + 12 a^4 - 15 a^2 + 4}}{4 a^2 - 3} \quad (10)
\]
The frequencies and accelerations are
\[
\omega_H(r_p) = s \omega_H(r) \quad \text{and} \quad \ell_H(r_p) = -s \ell_H(r) \quad (11)
\]
for \( s \geq 0 \) (the situation for \( s < 0 \), implying a change of the BH spin rotation, provides a result analogue to the case \( s > 0 \) that is a conformal transformation from an outer horizon \( r_+ \) to an inner horizon \( r_- = r \). We obtain:
\[
r = \frac{2s}{s + 1} \quad \text{and} \quad r_p = \frac{2}{s + 1}, \quad (12)
\]
then
\[
(\omega(r_p), \ell(r_p)) = (s (\omega(r), -\ell(r)), \quad \text{where} \quad \frac{r}{r_p} = s
\]
with
\[
a(r) = a(r_p) = 2 \sqrt{\frac{s}{(s + 1)^2}} = \sqrt{sr_p}
\]
(15)
The radii \( r \) and \( r_p \) correspond to the horizons of the BH spacetime with \( a(r) = a(r_p) \). It is clear that \( (r, r_p) \) represent a horizon parametrization and \( a(r_p) \) can be seen as analogue to the tangent curve of the horizon \( a_{\text{ext}}(\omega) \), function of the frequency \( \omega \). It is clear from Eqs. (12) and (14) that there are two ranges, \( s < 1 \) and \( s > 1 \), describing the same spacetime (horizontal line in the plot of \( a(r_p) = a(r) \) as function of \( s \)).  

The remarkable aspect of this relation is that the conformal factor is different according with the spin \( \hat{s} \equiv [(2 - a^2) \mp 2 \sqrt{(1 - a^2)})/a^2, \) see Fig. 4.

In Sect. 5.1, we will study these relations in detail.

4 Exploring the laws of BH thermodynamics

Israel’s theorem establishes that the event horizon must be spherically symmetric, implying that the sphericity of an isolated and static BH cannot be broken. Information on the BH past, in the sense of BH “hair”, is to be considered as eliminated or inaccessible to observers. (Counterexamples might be considered in some multi-dimensional spacetimes with non-abelian fields).

In this section, we reformulate several aspects of BH thermodynamics in the extended plane and on the bundles.

To explore the laws of BH thermodynamics in the extended plane, we consider the Smarr’s formula connecting \( \{M, J, A_{\text{area}}\} \), where \( A_{\text{area}} \) is the horizon area.

In this section, we expand on the analysis of Sect. 3.1, where we considered masses and BH thermodynamics in the extended plane. In Sect. 4.1, we introduce the BH rotational energy and “rest” mass in the extended plane, which is studied using the light surfaces in Sect. 4.2, where we investigate also horizon areas in the extended plane and BH thermodynamics. Some of the concepts discussed here are the subject of Sect. 5.2. Moreover, we analyze the constant irreducible mass and the quantities evaluated on the inner horizons in Sect. 5.3.
4.1 Rotational energy and BH “rest” mass

The horizon surface area (on the event horizon \( r_+ \)) is \( A_{\text{area}} = 16\pi M^2_{BH} \), and determines the BH irreducible (or rest) mass

\[
M_{irr} = \sqrt{a^2 + r_+^2}/2.
\]

The total BH mass \( M \) of the Kerr BH can be decomposed into the mass \( M_{irr} \) and the rotational energy.

From the first law of BH thermodynamics, we obtain

\[
M^2 = M^2_{irr} + J^2/4M^2_{irr},
\]

where \( J \) is the BH angular momentum in units of mass \( M \), as measured in the asymptotical flat region.

The maximum rotational energy which can be extracted from the black hole is \( \xi \propto (M - M_{irr}) \). A result of Christodolou and Ruffini sets an upper limit on the amount of energy that can be extracted from a Kerr BH as the total rotational energy, assuming a Schwarzschild BH as the final state, and the the bottom limit\(^5\) of \( M \) at the end of stationary process of the energy extraction, has to be \( M_{irr} \).

Within these assumptions, the maximum rotational energy which can be extracted is the limit of \( \xi = \xi_M = (2 - \sqrt{2})/2 \), hence \( \xi \in [0, \xi_M] \), where at the state (0) (prior to the extraction) there is an extreme Kerr spacetime (with spin \( a = M \)). Considering (0) as the state prior the extraction, from the law \( M^2_{irr} = r_+(J, M)/2 \), we obtain

\[
M(0)^2 - M_{irr}(0)^2 = \left( \frac{J(0)M(0)}{M(0)(2M_{irr}(0))} \right)^2, \tag{16}
\]

from the variation we find

\[
\frac{\delta M_{irr}}{M_{irr}} = \frac{\delta M - \delta J \omega_H(0)}{\sqrt{M(0)^2 - J(0)^2/4}},
\]

where \( \delta M_{irr} \geq 0 \), thus \( \delta M - \delta J \omega_H(0) \geq 0 \) and \( \omega_H \equiv \omega^+_{H} \) is the frequency of the outer Killing horizon for the initial BH, imposing limits on the BH spin-shift\(^6\). We obtain \( M^2 = J(0)^2/(4M^2_{irr}) + M^2_{irr} \), and the (extracted rotational) energy is essentially

\[
\xi = M(0) - \sqrt{r_+(0)/2}, \quad \text{or} \quad \frac{\xi}{M(0)} = 1 - \sqrt{1 + \sqrt{1 - \frac{J(0)^2}{M(0)^2}}}.
\]

Note that here \( \xi \) has units of mass, the dimensionless spin \( a_{\xi}/M = J(0)/M(0)^2 \) refers to an initial state before the transition, defined by the mass \( M(0) \) and spin \( J(0) \), which is a function of the extracted rotational energy \( \xi/M(0) \), that is, \( 1-M_{irr}(0)/M(0) \), and, therefore, of the ratio \( M_{irr}(0)/M(0) \).

On the other hand, the radius \( r/M \) on the extended plane can be expressed in terms of the total mass \( M(0) \). We can write the dimensionless spin as \( a_{\xi} = a_{\xi} \) as

\[
a_{\xi}(\xi) \equiv 2\sqrt{(2 - \xi)(\xi - 1)^2}\xi \tag{17}.
\]

The function \( a_{\xi}(\xi) \) links the former state spin \( a_0 \) to the rotational extraction in the subsequent phase, where the BH is settled in a Schwarzschild spacetime. The function \( a_{\xi} \) is, therefore, an expression for the tangent spin in the extended plane as function of the rotational energy parameter \( \xi \). Here and in the following, we shall use dimensionless quantities.

Equivalently, we can express the rotational energy parameter as

\[
\xi_{\pm} = 1 \pm \sqrt{\frac{r_+}{2}} \tag{18}.
\]

Considering \( a_{\xi}(\xi) \equiv a_{\xi}^{(\pm)} \), solving for \( \xi \) there is

\[
\xi_{s}^{(\pm)} = 1\sqrt{1 - (a_{\xi}^{(\pm)})^2} = b, \quad \text{where} \quad b = \pm; \quad s = \pm, \tag{19}
\]

relating directly the energy parameter \( \xi \) to the horizons. As \( \xi \propto M - M_{irr} \) (here \( \xi \) has units of mass \( M \)), only the solution \( \xi_{\pm} \) has to be considered. The general solution \( a_{\xi}(\xi) = a_{\xi}^{(\pm)} \), for a couple of spins \( a_{\xi}^{(\pm)} \), provides the eight functions \( \xi_{s}^{(\pm)} \).

For the point \( \xi = 1 \), it is \( a_{\xi}(\xi) = 0 \) with \( \xi = 0 \). There is a maximum that depends on the spin, \( \xi_{\ell} \equiv (2 - \sqrt{2})/2 \) (and \( \xi_m \equiv (2 + \sqrt{2})/2 \), where \( a_{\xi}(\xi) = M \) and \( r_+ = M \) (the extreme Kerr BH)). In general, it is \( a_{\xi}(\xi) \in [0, M] \) (dimensionless) and the definition of the rotational energy as \( 1-M_{irr}/M \), with restricted range \( \xi \in [0, \xi_{\ell}] \), where \( \xi_{\ell} \equiv (2 - \sqrt{2})/2 \), limiting, therefore, the energy extracted to a maximum of \( \approx 29\% \) of the mass \( M \). We can express the extracted energy in terms of characteristic frequency of the bundle and through the tangency condition of the bundle in the extended plane. We introduce the curves \( \xi_{\ell}^{(\pm)} \) and \( \xi_{m}^{(\pm)} \) in the extension of the plane for extended values of \( \xi \), considering the BH horizon frequencies \( \omega^{\pm}_{H} \), the horizon curves in the plane \( \xi - r/M \), and the functions \( \xi_{s}^{(\pm)} \), \( \xi_{\ell}^{(\pm)} \) from the other two solutions of \( a = a_{\xi}(\xi) \):

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where \( r \) is a point of the horizon curve in the extended plane. The maximum rotational energy extractable is therefore expressed as function of the light-surface frequencies (or equivalently the \( MB \) origin \( A_0 \)). For the extreme case \( a = M \), where the frequency is \( \omega = 1/2 \), we obtain \( \xi = 1 \pm 1/\sqrt{2} \).

The horizon frequencies, which are also the characteristic frequencies of the bundles, in terms of the dimensionless parameter \( \xi \) are

\[
\xi^\pm \equiv \frac{1 \pm \sqrt{\frac{2}{4\omega^2 + 1}}}{\sqrt{\frac{2}{4\omega^2 + 1}}}, \quad \xi^\pm_\ell \equiv \frac{1 \pm \sqrt{\frac{2\omega^2}{\xi^\pm 4\omega^2 + 1}}}{1 \pm \sqrt{\frac{2\omega^2}{\xi^\pm 4\omega^2 + 1}}},
\]

\[ \xi^\pm_\ell = \frac{1 \pm \sqrt{\frac{2\omega^2}{\xi^\pm 4\omega^2 + 1}}}{\sqrt{\frac{2\omega^2}{\xi^\pm 4\omega^2 + 1}}}, \]

with \( \xi \) having a saddle point for energy \( \xi = \left(1 \pm \sqrt{2}/3\right) \), corresponding to the spin \( a/M = 2\sqrt{2}/3 \) and frequency\(^7 \) \( \omega = (1/2\sqrt{2}, 1/\sqrt{3}) \). Frequency \( \omega_H^\pm = \pm 1/2\sqrt{2} \) for the outer horizon \( r_g/M = 4/3 \) is also the saddle point of \( \xi^\pm_\ell \) as function of \( \omega \), the maximum extractable rotational energy decreases/increases faster with the horizon \( MB \) frequencies constrained by the limiting values \( \omega_H^\pm = \pm 1/2\sqrt{2} \).

4.2 BH irreducible mass

We express the irreducible mass of a BH in terms of the horizon curve in the extended plane. Thus, using also the results given in Sect. 3.1, we rewrite the mass function in terms of the inner and outer horizon radii in the extended plane. We then study the areas and the relation (9) in this representation, wiring first law of BH thermodynamics in this new frame.

We can express the BH irreducible mass \( M_{irr} \), a quantity defined on the outer BH horizon, in terms of its area \( M_{irr} = \sqrt{a^2 + r^2_+} = \sqrt{A_{area}/16\pi} \) as

\[ M_{irr}(all) = \sqrt{\frac{\mathcal{A}_0^2}{\mathcal{A}_0^2 + 4}} = \frac{r_g}{2} = \sqrt{\frac{1}{4\omega^2 + 1}}, \]

\[ M_{irr}^\pm = \sqrt{\frac{1}{2} \pm \frac{1}{2} \sqrt{\frac{(\mathcal{A}_0^2 - 4)^2}{(\mathcal{A}_0^2 + 4)^2} + 1}}, \]

\[ = M_{irr}(r_\pm) = \frac{r_\pm(a_g)}{2}. \]

in terms of the radius \( r_g \), the frequency \( \omega \), the bundle origin \( A_0 = a_g\sqrt{\sigma} \) or the horizon curve \( a_g \). (Note the dependence on \( \theta \) through the bundle origin, does not contradict the rigidity of the black hole horizon, being a representation of the light-surfaces frequencies on different planes \( \sigma \) s). Therefore, the mass function is defined also in the inner horizon. The function \( M_{irr}(all) \) denotes the irreducible mass evaluated on the horizon curve in the extended plane, as function of the bundle origin \( A_0 \), the tangent radius \( r_\pm \), or the frequencies \( \omega \).

The notation \( \pm \) refers to the radii \( r_\pm \).

The irreducible mass is, therefore, determined by the tangent radius in the extended plane. Note that

\[ \mathcal{M}_{irr}^\pm = \frac{M_{irr}}{\sqrt{2}}, \quad \mathcal{M}_{irr}^\pm = \frac{1}{2} \pm \frac{1}{2} \sqrt{(r_\pm - 1)^2}, \]

relating the irreducible masses, the frequencies and the surfaces gravities. Considering the expression for the irreducible mass in terms of the radius \( r \), we obtain the expressions

\[ \mathcal{M}_{irr}^\pm = \sqrt{\frac{\mathcal{F}_\pm}{\sqrt{2}}}, \quad \mathcal{M}_{irr}^\pm = \frac{1}{2} \pm \frac{1}{2} \sqrt{(r_\pm - 1)^2}, \]

which are evaluated for \( a = a_\pm \), respectively, see Fig. 5.

We use the concept of maximum extractable energy \( \xi \) introduced in Eq. (26), where \( M_{irr} = M - \xi, (\xi \) has units of mass \( M \) and \( (M_{irr}/M)_{\text{min}} = 1 - \xi_{\text{max}} = 1 - \xi_\ell = 1/\sqrt{2} \) for the extreme spacetime. We obtain

\[ \frac{M_{irr}}{M} = 1 - \xi_{\pm} \equiv \mp \sqrt{-\frac{r_\pm}{2}}, \quad \frac{M_{irr}(\nu)}{M} \equiv 1 - \xi_{\nu} = \pm \sqrt{-\frac{r_\pm}{2}} \]

\[ = \pm \frac{r_g}{2} = \pm \frac{a_g}{4\omega}. \]

On the other hand, from the functions \( \xi_{\ell}^\pm \) and \( \xi_{\ell}^\pm \) of Eq. (26), which are extensions of the rotational energy definition in the extended plane, we obtain

\[ \xi_{\ell}^\pm = \frac{1 \pm \sqrt{\frac{2\omega^2}{\xi^\pm 4\omega^2 + 1}}}{\sqrt{\frac{2\omega^2}{\xi^\pm 4\omega^2 + 1}}}, \]

\[ \xi_{\ell}^\pm = \frac{1 \pm \sqrt{\frac{2\omega^2}{\xi^\pm 4\omega^2 + 1}}}{\sqrt{\frac{2\omega^2}{\xi^\pm 4\omega^2 + 1}}}, \]

\[ = \frac{1 \pm \sqrt{\frac{2\omega^2}{\xi^\pm 4\omega^2 + 1}}}{\sqrt{\frac{2\omega^2}{\xi^\pm 4\omega^2 + 1}}}, \]

\[ = \frac{1 \pm \sqrt{\frac{2\omega^2}{\xi^\pm 4\omega^2 + 1}}}{\sqrt{\frac{2\omega^2}{\xi^\pm 4\omega^2 + 1}}}, \]

\[ = \frac{1 \pm \sqrt{\frac{2\omega^2}{\xi^\pm 4\omega^2 + 1}}}{\sqrt{\frac{2\omega^2}{\xi^\pm 4\omega^2 + 1}}}, \]

\[ = \frac{1 \pm \sqrt{\frac{2\omega^2}{\xi^\pm 4\omega^2 + 1}}}{\sqrt{\frac{2\omega^2}{\xi^\pm 4\omega^2 + 1}}}, \]

\[ = \frac{1 \pm \sqrt{\frac{2\omega^2}{\xi^\pm 4\omega^2 + 1}}}{\sqrt{\frac{2\omega^2}{\xi^\pm 4\omega^2 + 1}}}, \]

\[ = \frac{1 \pm \sqrt{\frac{2\omega^2}{\xi^\pm 4\omega^2 + 1}}}{\sqrt{\frac{2\omega^2}{\xi^\pm 4\omega^2 + 1}}}, \]
Fig. 5 Irreducible mass $M_{irr}$ as function of the spin $a/M$, the radius on the extended plane $r/M$, the origin spin of the bundle $A_0 = a_0 \sqrt{\sigma}$, where $\sigma \equiv \sin^2 \theta$, and the bundle characteristic frequency $\omega$. Here, $EBH$ denotes the extreme BH spacetime. The regions of the inner and outer horizons in the extended plane are denoted, respectively, on the left and right of the $EBH$ limit.

$$\frac{M_{irr}(\mu)}{M} = 1 - \xi_\mu^\pm \equiv \pm \sqrt{1 - \frac{r}{2}},$$

and

$$\frac{M_{irr}}{M} = 1 - \xi_\tau^\pm = \pm \frac{2\omega}{\sqrt{4\omega^2 + 1}} = \pm 2\omega \sqrt{\frac{r_g}{2}} = \pm 2\omega a_g.$$  \hspace{1cm} (26)

Note that for $r = 0$, according to Eq. (26), we find $M_{irr}/M = (1, 0)$, with $M_{irr}/M = 1/\sqrt{2}$ for $r = M$, and $M_{irr}/M = 0$ for $r = 2M$. In Sect. 5, we analyze further aspects of the irreducible mass.

To enlighten the properties of the mass $M_{irr}$ in the extended plane, we can consider some limiting values according to the MBs structures. From Eq. (25), we obtain $M_{irr}/M = 1/\sqrt{2}$ for the extreme Kerr spacetime. On the ergosurface and on the Schwarzschild horizon, we find $M_{irr}/M = 1$. For $\omega = +\infty$ (equivalently $r = 0$), it is $M_{irr}/M = 0$. From Eq. (22), we find

$$\lim_{A_0 \to X} M_{irr}(all) = Y \quad \text{and} \quad Y = \left\{ 1, 0, \frac{1}{\sqrt{5}}, \frac{1}{\sqrt{2}}, \frac{2}{\sqrt{5}} \right\},$$

$$\lim_{A_0 \to X} M_{irr}(r+) = Y \quad \text{and} \quad Y = \left\{ 1, 1, \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{2}}, \frac{2}{\sqrt{5}} \right\},$$

where respectively $X = \{ +\infty, 0, 1, 2, 4 \}$.  \hspace{1cm} (27)

Limits $M_{irr} = 2/\sqrt{5}$ or $M_{irr} = 1/\sqrt{5}$ refer to the definitions for inner and outer horizons for the tangent spin $a_g/M = 4/5$.

To clarify the relation between bundle NS origin and the BH irreducible mass through the tangency properties of the bundles, we invert some of the previous relations as functions of $M_{irr}$, obtaining the bundle origin and the characteristic frequency as the expressions

$$A_0(all) = \frac{2M_{irr}}{\sqrt{1 - M_{irr}^2}},$$

therefore $\omega = \frac{1}{A_0(all)} = 4A_0(r_+)$, \hspace{1cm} (28)

which are illustrated in Fig. 6. These expressions relate the origin spin $A_0$ to the irreducible mass $M_{irr}$.

Below we discuss this limiting value in relation of the BH transitions, revisiting the considerations of Sect. 4.1.

From the relation $M_{irr}^+ = r_+/2$, it follows that

$$\frac{M_{irr}^+}{M} = \sqrt{1 + \sqrt{1 - \frac{J(0)^2}{M(0)^2}}}.$$  \hspace{1cm} (29)
and using Eq. (19)

\[ M_{irr}^r = b \frac{\sqrt{1 - (a_\ell^{(\pm)})^2}}{\sqrt{2}}, \quad \text{where} \quad b = \mp; \quad \mp = \pm, \]

(31)

where we use dimensionless units and \((\delta M - \delta J(0)\omega_H^+ \geq 0)\). In particular, we find \(M_{irr}/M = [0, 1]\) for \(a = 0\) and \(M_{irr}/M = \pm 1/\sqrt{2}\) for \(a = M\) —see discussion in Sect. 5.

4.2.1 BH horizon areas in the extended plane

The BH inertial mass can be found from the BH area:

\[ A_{\text{area}} = \frac{2\pi}{\sqrt{\omega_H}} \int_0^{\pi} \int_0^{2\pi} \sqrt{g_{\phi\phi}} \sin \theta \, d\phi \, d\theta = 4\pi (a^2 + r^2). \]

(32)

Here we use the extended definition \(A_{\text{area}}(a_\pm) = 8\pi r\). Note that this expression depends explicitly on \(r = r_g\). Considering also the relations analyzed in Sect. 3.1, we obtain

\[ \delta M^\mp = \frac{\ell^\mp}{\delta A_{\text{area}}^\mp} + \omega_H^\mp \delta J^\mp, \quad da_\pm = \pm \frac{\ell(a_\pm)}{\omega(a_\pm)} dr, \]

(33)

\( \delta M^\mp = 0 \) and \( \delta M_{irr} > 0 \).

The last relation can be written also as \(\delta a = [\delta M(r) + \delta r(M-r)]/\sqrt{r(2M-r)}\). In Eq. (33), we considered the variation of the horizon area (and the irreducible masses) for the ADM mass \(M\), the area \(A_{\text{area}}\) and the momentum \(J\). The horizon frequencies \(\omega_H^\pm\) are the characteristic bundle frequencies. We consider other regions of the extended plane. In Sect. 5, there are also further considerations on the transitions with \(\delta M_{irr} = 0\) and \(\delta M_{irr} > 0\).

From the expressions for \(A_{\text{area}}(0)\) and \(A_{\text{area}}(\omega)\), we obtain the saddle points

for \(A_{\text{area}}(0) = 8\pi r_g(0)\), it is

\[ A_0 = \frac{2}{\sqrt{3}} \quad \text{where} \quad A_{\text{area}}(A_0) = 4\pi, \]

(34)

for \(A_{\text{area}}(\omega) = 8\pi r_g(\omega)\) it is

\[ \omega = \frac{1}{2\sqrt{3}} \quad \text{where} \quad A_{\text{area}} = 12\pi \]

(35)

Frequency \(\omega = \omega_H^+ = 1/2\sqrt{3}\) is the outer horizon frequency for the BH with spin \(a/M = \sqrt{3}/2\), where \(\omega_H^- = \sqrt{3}/2 = 1/A_0 = 2/\sqrt{3}\). For \(a/M = \sqrt{3}/2\) there is \(r+/M = 3/2\), \(r-/M = 1/2\), in this geometry there is \(a_\ell^+ = 0\) on \(r_+\) and \(a_\ell^- = 0\) on \(r_-\) where \(\ell = (r^2 - a^2)/(a^2 + r^2)\) is the acceleration, defined for \(r > 0\) equal the the BHs surfaces gravities when evaluated on the horizons.

The saddle point \(A_0 = 2/\sqrt{3}\) (NS) for the area is also the saddle point for the angular momentum, corresponding to the frequency \(\omega = 1/(L_{\ell})\) of the inner horizon \(r_- = 1/2\) for the spacetime \(a = \omega\), whose frequency of the outer horizon is a saddle point for the frequency function). It is easy to see that

\[ A_{\text{area}}^\pm (r_\pm) = \pm 8\pi r_\pm \quad \text{and} \quad \delta A_{\text{area}}^\pm = -\delta A_{\text{area}}^\mp \]

i.e. \(\partial a A_{\text{area}}^\mp \delta a = -\partial a A_{\text{area}}^\pm \delta a \)

(36)

which follows from relation \(\delta r_+ = -\delta r_-\), where \(r_+ r_- = a^2\) and \(r_+ + r_- = 2M\) and, therefore, it holds if \(\delta M = 0\).

Analogously, we obtain

\[ A_{\text{area}}(a_\pm) = 8\pi r, \quad \text{and} \quad \ell(r_\pm, a_\pm) = \frac{1}{2 + b/(\ell(r_\pm))^2}, \]

\( \ell = \sqrt{2(2 - r)/2 + 2(r - 1)^2} \).

(37)
The total frequencies in terms of the radius $r/M$ is
\[
\omega = \frac{\sqrt{2 - r}}{2\sqrt{r}}, \quad \ell(r, a_\pm) = \frac{r - 1}{2r}, \quad \text{and there is}
\]
\[
\ell = \omega_{\text{EBH}}^2 - \omega^2,
\]
(38)
or, equivalently, $\omega = \frac{1}{2}\sqrt{1 - 4\ell}$, that is, $\omega_{\text{EBH}}$ is the frequency of the extreme BH – see Eq. (B.3).

4.2.2 Notes on BH thermodynamics

We can now analyze BH thermodynamics in the extended plane, considering the mass variations and the interpretation of Eq. (33) for the inner horizon curve (–). However, in the extended plane, quantities relevant to BH transitions may be different, when referring to a point $r$ on the horizon curve for a BH state. (For simplicity, in some of the expressions we do not consider the factor $8\pi$. Moreover, we will consider in some expressions spin, radius, characteristic frequency and origin spin as dimensionless parameters). In these relations, we express the area (or the irreducible mass) in terms of the radius in the extended plane, considering dimensionless parameters.

Then,
\[
\delta M = \frac{\delta A_{\text{area}}(r - 1)}{2\ell} + \frac{\delta J\sqrt{2 - r}}{2\sqrt{r}}, \quad \text{and}
\]
\[
\delta M = \delta A_{\text{area}}\left(\frac{1}{4} - \omega^2\right) + \delta J\omega,
\]
\[
\delta M = \frac{1}{4} - \frac{1}{A_0^2} \delta A_{\text{area}} + \frac{\delta J}{A_0}.
\]
(39)

This the the mass variation is constrained by the $M$ tangent point $(r = r_2)$ or characteristic null frequency $\omega$ (origin $A_0$) regulating therefore the BHs transitions. Where $r = r_s \in [M, 2M]$, when evaluated on the outer curve. In the extended plane the relations are satisfied also for the inner horizons terms and we use again this property in Sect. 6 where we will study the doubled metric for the inner and outer horizon by writing two different metrics.

In Sect. 3.1, we discussed the relation between quantities defined on inner and outer horizon points in the extended plane. Here, we consider this relation in terms of surface gravity and frequencies. From Eq. (23), it follows that $(\omega_\pm^-)^{-1} = \omega_\pm^+ = \omega_H^- = -\ell_H^- = \omega_H^+ = \omega^2(a)$. Also, $A_{\text{area}}^- = -A_{\text{area}}^+ + 4M^2$; thus, we can write $\delta A_{\text{area}}^- = -\delta A_{\text{area}}^+ + 8M\delta M$, and $\delta A_{\text{area}}^+ = -\delta A_{\text{area}}^-$ for $\delta M = 0$, where i.f $\delta J = 0$ then $\delta M = \delta A_{\text{area}}^+\ell_H^-/2M$

A more precise analysis is provided in Sect. 5.3. Here we consider the condition $\delta A_{\text{area}}^+ = -\delta A_{\text{area}}^-$ (i.e. $\delta M = 0$, invariant masses). Then,

$\delta J/\delta A_{\text{area}}$ of Eq. (41), as functions of the tangent spin $a = a_\pm \in [0, M]$, the origin spin $A_0 \in [0, +\infty]$, the characteristic BH horizons frequencies $\omega$, the tangent radius $r = r_s \in [0, 2M]$. The BH area is $A_{\text{area}}$ $(A_{\text{area}}^\pm$ for $r = r_\pm$ BH horizons), $J$ is the BH spin

\[
0 = \delta M = \frac{1}{2} \delta A_{\text{area}}^+\left(\ell^+ - \ell^-\right) + \frac{1}{2} \delta J(\omega^-_H + \omega_H^+) = \frac{\delta J}{\delta A_{\text{area}}^-} = \frac{\ell^-_H - \ell^+_H}{\omega^-_H - \omega_H^+}.
\]
(40)

From the definition of horizon in the extended plane, we find $\delta M = (\omega\delta J + \ell\delta r)/ (3M - r)$, with the distinguished value $r = 3M$, which is related to the Schwarzschild case [13]. Considering $\delta M = 0$, we obtain

\[
\delta J = \frac{\delta A_{\text{area}}^-}{a} = \frac{\ell^-_H - \ell^+_H}{\omega^-_H - \omega_H^+} = \frac{1}{4\omega - 1} - \frac{1}{\sqrt{r - 2r}}.
\]
(41)

see Fig. 7.

In this way, we can consider the variation of masses, areas and angular momentum of the BHs in terms of the quantities defined on the horizon curves, for instance, the first law of BH thermodynamics in terms of the tangent point $r$ on the horizons, the origin $A_0$ and the frequency $\omega$, describing the outer horizon $r \in [M, 2M]$, $A_0 \in [2M, +\infty]$, $\omega \in [0, 1/2]$ or the inner horizon for $r \in [0, M]$, $A \in [0, 2M]$, $\omega \in [1/2, +\infty]$, respectively. Such relations express the variations in terms of the angular momentum of the horizons, corresponding in some cases to naked singularities. We can write Eq. (39) in terms of quantities relative to the extreme ($ext$) Kerr BH, considered as reference with $r_{\text{ext}} = M$, $\omega = 1/2$ and $A_0 = 2M$, as

\[
\delta M = \frac{1}{2r} \delta M \quad \text{where} \quad \delta M = \delta A_{\text{area}}(r - r_{\text{ext}}) + \delta J a_{\pm}(r)
\]
\[
\delta M = \frac{1}{(L_f)^2_{\text{extr}}} \delta M = \frac{1}{(L_f)^2_{\text{extr}}} - \frac{1}{(L_f)^2} \delta A_{\text{area}} + \frac{\delta J}{(L_f)}.
\]
(42)
where \((L_f) = 1/\omega\), BH angular momentum, coincident with the bundle origin. The case of extreme Kerr BHs in this sense is a limit where \(\delta M = \delta J/2\).

The Figs. 5 and 8 outline the four zones identified through the inertial mass as functions of the spin, \(A_0\), of the characteristic frequency of the bundles.

5 On BH transitions

5.1 Relating inner and outer horizons in BH transitions

Considering the discussion of Sect. 3.1, we use the expression \(\ell^\pm_H(\omega)\) in the extended plane and solve the equation \(\ell^\pm_H(\omega) = s\omega\) for a given frequency \(\omega\). Analyzing this special BH transition we look for a solution \(\ell\) in a fixed spacetime and obtain \(s = (1 - 4\omega^2)/4\omega\), which is represented in Fig. 9 for any \(\omega\). The condition \(s < 0\) implies that \(\ell\) is evaluated on the inner horizon (we assume \(\omega > 0\)) and, therefore, \(\omega^2 > 1/4\), bounded by the limiting value of the extreme Kerr spacetime. Then, \(\ell = 1/4 - \omega^2\). However, we are interested in the cases \(s = \text{constant}\) (\(s = \omega\left(1 - 4\omega^2\right)/4\omega\)). For \(s = \pm 1\) it is \(\omega = \pm \left(1 \pm \sqrt{2}\right)/2\) corresponding to \(a_p/M = 1/\sqrt{2}\).

Here we solve the more general problem where the frequencies are not equal, that is, where quantities \((\ell, \omega)\) and \(\omega_l\) are related by \(\ell(\omega) = s\omega_l\), with \(\omega \neq \omega_l\), and obtain \(\omega_l \neq 0\) and \(\omega = \pm 1/2\) (extreme Kerr BH) or \(\omega_l \neq 0\) and \(s = (1 - 4\omega^2)/4\omega_l\); for \(s = \pm 1\) we obtain \(\omega_l = \pm (1 - 4\omega^2)/4\), which is represented in Fig. 9. In this way we connect BHs, as initial and final state of these particular transformations.

We now consider the relation between frequencies and surface gravity as a function of a radius \(r\) of the horizon curve in the extended plane (that is, in this way we express the conditions for the inner and outer horizons relations), where \(\omega(r) = s\ell(r)\) at the same point \(r = r_p\), tangent point on the horizon curve). We obtain \(r = 1/\sqrt{1/(s^2 + 1)}\) for \(s < 0\) and \(s \geq 0\), respectively. This relates a BH inner and outer horizon (at fixed \(a\)). For \(s = \pm 1\) we obtain \(r = r_p = r_p = 1 \pm \sqrt{2}\), outer and inner horizons, respectively, for the BH with spin \(a_p/M = 1/\sqrt{2}\).

Consider the more general case where \(r \neq r_p\).

There is: \(\omega(r) = s\ell(r_p)\) for

\[
\left(\frac{r = 2M, s = 0, r_p > 0, \quad r = \frac{2p^2}{r_p + (r_p - M)^2s^2}, \quad [(r \leq 0, M), s < 0, (s > 0, r_p \geq M)]\right)
\]

\[\text{Eq} (43)\]
alternatively, defining $\tilde{s}_g \equiv \sqrt{\frac{2M-r}{r}} (r_p/(r_p-M))$ we can write the results as follows

$$s > 0 : \quad (r = 2M, r_p = M, s > 0),$$

and \((r_p > M, r \in [0, 2M], s = \tilde{s}_g)\) \hspace{1cm} (44)

$$s \leq 0 : \quad (r = 2M, r_p = M, s \leq 0),$$

$$[ (r_p \in [0, 2M], r \in [0, 2M]), (r_p > M, r = 2M)], s = \tilde{s}_g].$$ \hspace{1cm} (45)

In particular, for the static case there is \((r = 2M, s = 0, r_p > 0)\). For the special cases \(s = \pm 1\) we obtain

for \(s = 1\) it is \(r \in [M, 2M]\) and \(r_p = \frac{M}{M - \sqrt{2M-r}}\) \hspace{1cm} (46)

for \(s = -1\) it is \(r \in [0, 2M]\) and \(r_p = \frac{M}{M + \sqrt{2M-r}}\) \hspace{1cm} (47)

represented in Fig. 9, where the relation between the BH horizons (where the range \(r = r_s\) has been extended) connect couple of geometries as final and initial states of these special transitions.

5.2 Constant irreducible mass

A more complicated transition occurs when the final state is not a Schwarzschild static BH, implying \(J(1) = 0\), where (1) is for final state of transition and (0) for the initial state. We start by noting that the condition \(M_{\text{iff}} = \text{constant}\) and \(M(1) = M(0) = M\) implies \(J(1) = J(0) = J\), which can also be interpreted as \(J(1)^2 = J(0)^2 = J^2\) and consider the case of change in rotation orientation, implying a state to be considered a static BH, that is, with a state \(J = 0\) with a consequent increase of mass, (for \(\delta M_{\text{iff}} \geq 0\)).

Introducing the ratios \((k_m, k_j)\), with \(J_1 = k_j J_0\) and \(M_1 = k_m M_0\) (we use notation \(Q(0) = Q_0\) or \(Q(1) = Q_1\)), the condition \(\delta M_{\text{iff}} = 0\) is verified for

$$k_j^\pm = (\pm) \sqrt{\frac{2(k_m^2 - 1) M_0 r_+ (0) + a(0)^2}{a(0)}},$$ \hspace{1cm} (48)

(we consider with the sign (±) the possibility of \(J_1 J_0 < 0\)). Eventually, it can also be written in compact form as

$$k_j = \pm \sqrt{\frac{2(k_m^2 - 1) M_0 r_+ (0) + a(0)^2}{a(0)}} \hspace{1cm} (49)$$

where \(r_+(0)\) is the outer horizon radius at state (0). Therefore, \(k_j = \pm 1\) for \(k_m = 1\) (unaltered masses). Assuming now \(k_j = k_m k_j m\) (with \(k_m > 0\)), the condition \(\delta M_{\text{iff}} = 0\) implies

$$k_j m = \pm \sqrt{\frac{2(k_m^2 - 1) M_0 r_+ (0) + a(0)^2}{a(0)}} \hspace{1cm} \text{where}$$

$$k_j m \in k_m \left[- \sqrt{\frac{M(0)^2}{a(0)^2}}, \sqrt{\frac{M(0)^2}{a(0)^2}}\right].$$ \hspace{1cm} (50)

Then, there are

$$\left(k_m = \frac{1}{\sqrt{2}} : k_j m = 0\right),$$

$$\left(k_m \in \left[\frac{1}{\sqrt{2}}, 1\right], k_j m \pm \sqrt{2 - \frac{1}{k_m^2}}\right),$$ \hspace{1cm} (50)

where it is described a total or partial energy extraction respectively.

Let us introduce the quantities

$$\tilde{k}_\pm \equiv \sqrt{-\frac{2M_0^2}{a(0)^2 k_m^2}} \pm 2\sqrt{\frac{(k_m^2 - 1)^2 M_0^2 (M_0^2 - a(0)^2)}{a(0)^2 k_m^4}} + \frac{2M_0^2}{a(0)^2} + \frac{1}{k_m^2},$$

and \(\epsilon_a \equiv \frac{r_+(0)}{M_0}\) \hspace{1cm} (51)
Note that $c_a/\sqrt{2}$ is related to the definition of irreducible mass – Sect. 4.2. For the Schwarzschild case we obtain $(a(0) = 0, k_m = 1)$. For $a(0) \in ]0, M_0]$ we find

$$ (k_m = c_a/\sqrt{2}, k_{jm} = 0), \text{ or } (k_m = c_a/1/\sqrt{2}, j_m = -\tilde{k}_\pm). $$

for $a(0) \geq M_0$, it is $(k_m = c_a, j_m = 0)$, and $(k_m \in ]c_a, 1], j_m = -\tilde{k}_\pm)$. In particular, the condition $k_m = 1$ implies $a(0) = 0$ or $a(0) \in ]0, M_0]$ and $k_{jm} = \pm 1$. For $k_{jm} = 0$, that is, the final state of a Schwarzschild BH, or for $a(0) \in ]0, M_0]$ it is $k_m = c_a/\sqrt{2}$, which for an extreme BH, $a(0) = M_0$, is $k_m = 1/\sqrt{2}$. For $k_{jm} = 1$, it is $k_m = 1$, implying $k_j = 1$; in other words, the condition $k_j = k_m$ implies that the BH state is immutable. Analogously, $k_m = 1$ for $k_{jm} = -1$.

Let us now introduce the spins

$$ J_a = M_1 \sqrt{2M_0^2 - M_1^2}, \quad J_i = 2M_1 \sqrt{M_0^2 - M_1^2}, $$

$$ \tilde{J}_\pm = \sqrt{J_0^2 \pm 2 \left( (M_0^4 - J_0^2) (M_0^2 - M_1^2)^2 + M_0^2 (M_1^2 - M_2^2) \right)}. $$

(52)

We obtain the following conditions for a transition $\delta M_{irr} = 0$:

for $M_0 = M_1/\sqrt{2}$ : $(J_0 = 0, J_1 = \pm M_1^2)$;

$$ M_0 \in ]M_1/\sqrt{2}, M_1[ : (J_0 \in ]0, J_a[), J_1 = \pm \tilde{J}_\mp); $$

$$ (J_0 = J_a, J_1 = \pm M_1^2); $$

for $M_0 = M_1$ : $(J_0 = 0, J_1 = 0)$;

$(J_0 \in ]0, J_a[), J_1 = \pm \tilde{J}_\mp);$

$$ (J_0 = J_a, J_1 = \pm M_1^2); $$

for $M_0 = \sqrt{2}M_1$ : $(J_0 = M_0^2, J_1 = 0)$.

(53)

(54)

Finally, we see also the special case:

for $M_0 = \sqrt{3}/2$ : $(J_0 = J_a, J_1 = 0)$;

$$ (J_0 \in ]J_a, M_0^2[), J_1 = \pm \tilde{J}_\mp), $$

(55)

(note the symmetries in the state $0 \leftrightarrow \pm J_a$). In this case $J_a = J_1 = \sqrt{87/9}M_0^2$ while $\tilde{J}_\pm = J_0^2 \pm \frac{3}{2} \left( M_0^4 - J_0^2M_1^4 - M_0^4 \right)$, and in this transition, $a = \sqrt{87/9}M_0$ is the initial BH state.

If the transition ends with a static BH, $J_1 = 0$, then $J_0 = 0$ and $M_1 = M_0$, that is, there is no transition with $M_1 = M_0/\sqrt{2}$ or with $J_0 = M_0^2$, i.e., an extreme Kerr BH as initial state. Viceversa, in the case $J_0 \in ]0, M_0^2[ \text{ we have } M_1 = \sqrt{r_+(0)/\sqrt{2}}$. If $J_0 = 0$, that is, the starting state is a Schwarzschild BH, then $J_1 = 0$ with $M_0 = M_1$. Therefore, there is no transition for $M_1 \in ]M_0, \sqrt{2}M_0[ \text{ with } J_1 = 2M_0 \sqrt{M_1^2 - M_0^2} \text{ (note the symmetry with solution } J_1 \text{ in Eq. (52)). In the case, } M_1 = \sqrt{2}M_0, \text{ we get } J_1 = 2M_0 \text{ implying an extreme Kerr spacetime } (J_1 = M_1^2).$

In the extreme case, where $M_1^2 = J_1 = 1$, that is, a transition ending in a Kerr extreme BH, for $J_0 \in ]0, M_0[ \text{, we obtain } M_1 = \sqrt{r_+(0)}$. Then, for $J_0 = M_0^2$, that is, starting from a Kerr extreme BH and ending in Kerr extreme BH, we get $M_0 = M_1$. Note that the condition $J_1/M_1^2 = J_0/M_0^2$, that is, the BH dimensionless spin is invariant, implies that $J_0 = J_1$ and $M_1 = M_0$, that is, there is no transition from $(J_0, M_0)$ with $M_{irr} = \text{constant and } a/M = \text{constant}. \text{ This is particularly relevant for the case of Kerr extreme BH. If the initial state is an extreme Kerr BH, i.e. } J_0 = M_0^2, \text{ the transformation can lead to a Schwarzschild BH, i.e. } J_1 = 0, \text{ if } M_1 = M_0/\sqrt{2}; \text{ or can lead to a Kerr BH with}\n
$$ M_1 \in \left[ \frac{1}{\sqrt{2}}, 1 \right) \text{ and } J_1 = \pm M_0^2 \sqrt{\frac{2M_0^2 - 1}{M_0^2}}, $$

(56)

(note the symmetry with respect to the case $J_0 = \{0, 1\}$ or $J_1 = \{0, 1\}$ considered above). In the particular case $M_1/M_0 = \sqrt{2}/3$, we obtain

$$ J_0 = \frac{2\sqrt{2}}{3}, \text{ and } J_1 = 0, \text{ or } J_0 = \left[ \frac{2\sqrt{2}}{3}, 1 \right), $$

and $J_1 = \pm \frac{J_0^2 - 2M_0^2r_+(0)}{3}$. (57)

5.3 On the inner horizon relations in BH thermodynamics

Here and in the following analysis, we use the notation $M = M_\pm$ and $J = J_\pm$ to stress the quantities defined on a point of the outer or inner horizon, respectively. From the definition $M_{irr}^2 \equiv A_{irr}\text{area}/2$ per inner and outer horizon, we obtain the two relations
constituting the two branches of $a(\xi)$ functions and $M_{\text{irr}}$ in Figs. 5 and 10. The second relation is in terms of the dimensionless spin of the BH and dimensionless energy $\xi$. These are solutions of $a(\xi) = 2\sqrt{-(\xi - 2)(\xi - 1)^2} \xi$ for $\xi_+ \in \left[\frac{1}{2} \left(2 - \sqrt{2}\right), 1\right]$ and for $\xi_- \in [0, \frac{1}{2} \left(2 - \sqrt{2}\right)]$, respectively. Solving $a(\xi) = a$, we obtain the solutions:

$$\xi^\pm_+ \equiv M_+ - \frac{1}{2} \sqrt{\left(\frac{J_+}{M_+}\right)^2 + r_+^2}, \quad \text{equivalently} \quad \xi^\pm_+(a) = 1 - \frac{\sqrt{1 \mp \sqrt{1 - a^2_+}}}{\sqrt{2}} \quad (58)$$

$$\xi^\pm_- \equiv M_- - \frac{1}{2} \sqrt{\left(\frac{J_-}{M_-}\right)^2 + r_-^2}, \quad \text{equivalently} \quad \xi^\pm_-(a) = 1 + \frac{\sqrt{1 \mp \sqrt{1 - a^2_-}}}{\sqrt{2}} \quad (59)$$

defining the new functions $M^\pm_{\text{irr}} = -\sqrt{r_\pm^2}/2$ and constituting the other two branches of Figs. 5 and 10 (at $\xi > 1$). For the outer and inner horizons. Bottom-left panel: The spin $a_{\pm}(M^\pm_{\text{irr}})$ as function of irreducible mass $M_{\text{irr}}$. Bottom-right panel: The irreducible masses, $(M^u_{\text{irr}}, M^d_{\text{irr}})$ as functions of the surface gravity $\ell$, see Eq. (69). The static spacetime and the limit of extreme Kerr BH, EBH, are represented by the following cases: By using the ratio $\delta J/\delta M_+$, we find that condition $(\prec)_M$ holds for $J_+ \in [0, M^2]$ and $\delta M < 0$ (mass decreasing) for $\delta J/\delta M_+ < 2 M^2_+ r_+ / J_+$, and $\delta M > 0$ (mass increasing) for $\delta J/\delta M_+ > 2 M^2_+ r_+ / J_+$ = 4$\omega^2_+ L^2_+ / J_+$, or explicitly

$$J_+ \in [0, M^2_+], \quad [\delta M_+ \leq 0, \delta J_+ > \delta J^+_+] \quad \text{and} \quad [\delta M_+ > 0, \delta J_+ > \delta J^+_+],$$

where

$$\delta J^+_+ = \frac{2 M_+ \left(\delta M_+ M^2_+ \pm J_+ \sqrt{M^2_+ \left(M^4_+ - 1\right) / J^2_+ - 1}\right)}{J_+}. \quad (60)$$

In these relations we assumed $\delta M_{\text{irr}}$, for the variations $(\delta M, \delta J)$, to be null. However, during the BH evolution the irreducible mass can also increase; in this case, we get the relation $\delta M_{\text{irr}}^+ \geq 0$ for $\delta M^+ - \omega^2_0 \delta J^+ \geq 0$ and then $\delta J^+/\delta M^+ \leq 1/\omega^2_0 = \mathcal{A}(0) = (L_+)$, that is, the origin of its bundle (where $\mathcal{A}(0) = a_0 \sqrt{\sigma}$) constrains the transition.
This relation also holds for the angular momentum of the horizon. 

However,

$$ (M_{\pm}^\text{i}r)^2 = M^2 - (M_{i r}^\text{r})^2, \quad \text{then} \quad \xi = M - \sqrt{M^2 - (M_{i r}^\text{r})^2} = M - M_{i r}^\text{r}. \quad (61) $$

The condition $\delta M_{i r}^r > 0$ with $\delta M_{i r}^r > 0$ holds for

$$ M > M_{i r}^\text{r}, \quad \delta M_{i r}^r \in \left[ 0, \frac{\delta M M}{M_{i r}^\text{r}} \right] \quad (62) $$

viceversa, the condition $\delta M_{i r}^r < 0$ with $\delta M_{i r}^r > 0$ holds for

$$ M > M_{i r}^\text{r}, \quad \left[ \delta M \leq 0, \delta M_{i r}^r \geq 0 \right), \quad \left( \delta M > 0, \delta M_{i r}^r \geq \frac{\delta M M}{M_{i r}^\text{r}} \right). $$

While the condition $\delta M_{i r}^r \geq 0$, from Eq. (61), holds for

$$ (M = M_{i r}^+, \quad \delta M \geq 0); \quad \left( M > M_{i r}^+, \quad \left( \delta M > 0; \quad \delta M_{i r}^r \leq -\sqrt{\frac{\delta M^2 M^2}{M^2 - (M_{i r}^r)^2}} \right); \right) \quad \left( \delta M \leq 0; \quad \delta M_{i r}^r \leq \sqrt{\frac{\delta M^2 M^2}{M^2 - (M_{i r}^r)^2}} \right). \quad (63) $$

Interestingly, $\delta M_{i r}^r > 0$ for $J_+ \in [0, M_+^2], \delta M_+ = 0, \delta J_+ < 0$. In other words,

$$ \delta M_{i r}^r > 0, \quad \text{for} \quad J_+ = 0, \quad \delta M_+ > 0; \quad J_+ \in [0, M_+^2], \quad \left( \delta M_+ < 0, \quad \frac{\delta J_+}{\delta M_+} > \frac{1}{\omega_H^+} \right); \quad \left( \delta M_+ > 0, \quad \frac{\delta J_+}{\delta M_+} < \frac{1}{\omega_H^+} \right)$$

and

$$ \delta M_{i r}^r > 0, \quad \text{for} \quad J_- \in [0, M_-^2], \quad \left( \delta M_- < 0, \quad \frac{\delta J_-}{\delta M_-} < \frac{1}{\omega_H^-} \right); \quad \left( \delta M_- > 0, \quad \frac{\delta J_-}{\delta M_-} > \frac{1}{\omega_H^-} \right) \quad (64) $$

where $\omega_H^\pm = J_\pm/2M_{i r}^\pm$ and we consider the cases $\delta M_{i r}^\pm \geq 0$. In these relation we note how the bundle angular momentum and pitch constrains the transition, distinguishing NS origin with $A_0 > 2$, for the relations in term of the BH inner horizon (in the extended plane inner and outer horizons relations have to be considered).

Below we summarize some properties of the BH spacetimes highlighted in this analysis relation to the MBS characteristic. In the BH geometry with spin $a_\gamma \equiv \sqrt{1/2}$, it is $\ell_{H}^\pm = \pm \omega_H^-$. In this spacetime the transitions are essentially regulated by the characteristic bundle frequency. In this case, $r_{H}^\pm = r_{H}^\pm$, that is, the outer ergosurface on the equatorial plane of the BH is a geodesic orbit of the (corotating) photon. In this geometry, $\omega(r) = s \ell(r)$ at the same point $r MB$ tangent point with the horizon curve. We obtain $r = \frac{1}{2}\sqrt{1/(s^2 + 1)}$ for $s < 0$ and $s \geq 0$, respectively. For $s = \pm 1$, we obtain $r = r_\pm = \frac{1}{2}\sqrt{1/2}$, outer and inner horizons, respectively, for BHs with spin $a/M = 1/\sqrt{2}$ – Eq. (43). In other words, if we consider $\ell_\pm = s \omega$, for $s = \pm 1$, we obtain $\omega = \pm (1 \pm \sqrt{2})/2$, corresponding to $a/M = 1/\sqrt{2}$.

Furthermore, $\delta_\ell = \ln s = 0$ for $a = a_\gamma \equiv M/\sqrt{2}$, where $s \equiv \omega_H^+/\omega_H^-$. The analysis of the extremes of the accelerations $\ell$, solutions $\delta \ell = 0$, enlightens the role of the BH spin $a/M = \sqrt{3}/2$. The radius solution of $\delta \ell = 0$, is on the horizon curve in the extended plane, where $r = 3/2M$, which is the outer horizon of the BH spacetime with spin $a/M = \sqrt{3}/2$. Restricting to the horizon curve, we find the solution for $r = M/2$, which is the inner horizon of the BH spacetime with $a/M = \sqrt{3}/2$. Therefore, the extremes of the acceleration $\ell$ for $a$ and $r$ constitute, respectively, the inner and outer horizon for the BH with spin $a/M = \sqrt{3}/2$. (That is $\delta \ell = 0$ for $r_a = a/\sqrt{3}$, while $\delta \ell = 0$ for $r_e = \sqrt{3}a$; however, $r_0 = r_e$ and $r_e^+ = r_a$ for $a/M = \sqrt{3}/2$). The saddle point of $(L_f)(ao^\pm)$ (BH angular momentum or bundle origin) as function of $r$ is $r_{\pm}/M = 1/2$, which is the inner horizon of the BH spacetime, where the momentum and frequency are $(L_f)(ao^\pm) = 2/\sqrt{3} = 1/\omega$. (Note that in this case $a = \omega_{\pm}$). The saddle point of the function $\omega(r)$ is for the same BH spacetime on the outer horizon $r_{+}/M = 3/2$ where $(L_f) = 2/\sqrt{3} = 1/\omega$. The horizon (metric bundles and light surfaces) frequency $\omega_{\pm}$ of Eq. (21) has a saddle point for $\xi = (1 \pm \sqrt{2}/3)$ corresponding to the spin $a = 2\sqrt{2}/3$.
and frequency $\omega = (1/2\sqrt{2}, 1/\sqrt{2})$. For $a/M = 2\sqrt{2}/3$, we obtain also $r_{\text{mso}} = r_{+}^{0}$, that is, the marginally stable orbit of the corotating particle is the outer ergosurface on the equatorial plane. Spin $a/M = 2\sqrt{2}/3$ solves the problem of replicas for $\ell(r_{\text{+}} (a)) = - \ell(r_{-} (a))$ — see Eq. (10). On the other hand, Eq. (57) shows the relevance of this spin $J_{0}/M^{2}_{0} = 2\sqrt{2}/3$ in relation to BH transition with masses ratio $M_{1}/M_{0} = \sqrt{2}/3$. The origin spin $A_{0} = 2/\sqrt{3}$ is a saddle point for the area, Eq. (34), and it is also the saddle point for the angular momentum corresponding to the frequency $\omega$, whose frequency of the outer horizon is a saddle point for the frequency.

### 6 Inertial mass, extractable rotational energy, and surface gravity

We can connect two BH states, before and after a transition, representing them in the same extended plane. We analyze the tensor $g(\cdot, \cdot)$ in the extended plane by evaluating the metric along the horizon curve in terms of thermodynamic quantities. In the first case, we use the function $a = a_{\xi} (\xi)$ of Eq. (17) for the BH spin in terms of the maximum extractable rotational energy $\xi$. The tensor describes a set of BH geometries. Considering Fig. 10, where the function $a_{\xi} (\xi)$ is represented, there are multiple copies of the geometry, up to four, for the parameter $\xi$ in the range $\xi \in [0, 2]$ and $\xi \in [0, 1]$. The maximum extractable energy is for an initial extreme Kerr BH with parameter $\xi_{0}$, we consider however the range $\xi \in [0, 2]$. In the second case, we use the concept of irreducible mass with the relations Eqs. (22), (25), and (30).

In the third case, we use the acceleration $\ell$ and the surface gravity. This re-parametrization provide a way to connect different states often with multiple solutions, as in the extended plane, for different values of the parameters, creating a sort of BH cluster metric. Notably there are no NS solutions.

#### Rotational energy $\xi$

Considering the spin-function $a = a_{\xi} (\xi) = 2/\sqrt{2} (\xi - 2)(\xi - 1)^{2} \in [0, 1]$ of Eq. (17), we obtain up to four copies of the geometry for each $\xi$ value. In this way we use the symmetries of the function to create copies. The metric components are

\[
\begin{align*}
  g_{tt} &= -\frac{4(\xi - 2)(\xi - 1)^{2}(\sigma - 1) + (r - 2)r}{4(\xi - 2)(\xi - 1)^{2}(\sigma - 1) + r^{2}}, \\
  g_{\phi\phi} &= \frac{1}{4}\left[\frac{(r^{2} - 4(\xi - 2)(\xi - 1)^{2}\xi) + 4(\xi - 2)(\xi - 1)^{2}\sigma (r - 2)r}{4(\xi - 2)(\xi - 1)^{2}(\sigma - 1) + r^{2}}\right].
\end{align*}
\]

The metric describes a BH geometry for $\xi \in [0, \xi_{(\xi)}]$, where $\xi_{(\xi)}$ is the extreme BH spacetime. In the extended range $\xi \in [0, 2M]$, the metric tensor (65) describes multiple copies of the BHs with $a \in [0, M]$. The extreme Kerr BH corresponds to $\xi = \xi_{(\xi)}$ and $\xi \equiv \left(2 + \sqrt{2}\right)/2 > \xi_{(\xi)}$.

Then, the Kerr extreme BH and the Schwarzschild static metric are the limits

\[
\begin{align*}
  \lim_{\xi \to \xi_{(\xi)}} g &= \lim_{\xi \to \xi_{(\xi)}} g_{\text{Kerr}}; \\
  \lim_{\xi \to 0} g &= \lim_{\xi \to 0} g_{\text{Schw}}.
\end{align*}
\]

#### Irreducible mass $M_{irr}$

We use the function $a = a_{irr} \equiv 2\sqrt{M_{irr}^{2} - (1 - M_{irr}^{2})}$ for the spin; then, the metric tensor describes BH geometries. Note that the condition $a_{irr} = 0$ is a limiting value in the extended plane, corresponding (as for $M_{irr}$) to the static limit. In fact, the function $a_{irr}$ has been evaluated from the horizon curve in the extended plane, writing the irreducible mass for any point of the horizon curve. Therefore, it describes BH spacetimes ($a_{irr} \in [0, 1]$). From Fig. 10 it is also clear that there are two copies of one BH geometry $a$ for $M_{irr} = M_{irr}(a)$. Explicitly,

---

8 The irreducible mass is defined in Eq. (22), where $M_{irr} = \sqrt{r^{2} + a^{2}/2}$. Therefore, $M_{irr}^{2} = \sqrt{r^{2} + a^{2}/2}$. All the quantities are dimensionless.
The extreme Kerr spacetime are
\[
g_{tt} = \frac{2r}{4 \left( M^2_{irr} - 1 \right) M^2_{irr} \left( \sigma - 1 \right) + r^2} - 1, \quad g_{\phi\phi} = -\frac{4\sqrt{M^2_{irr} \left( 1 - M^2_{irr} \right) r \sigma}}{4 \left( M^2_{irr} - 1 \right) M^2_{irr} \left( \sigma - 1 \right) + r^2}
\]
\[
g_{rr} = \frac{4 \left( M^2_{irr} - 1 \right) M^2_{irr} \left( \sigma - 1 \right) + r^2}{4M^2_{irr} \left( 1 - M^2_{irr} \right) + (r - 2)r}, \quad g_{\theta\theta} = 4 \left( M^2_{irr} - 1 \right) M^2_{irr} \left( \sigma - 1 \right) + r^2
\]
\[
g_{\phi\theta} = 4 \left[ 4M^2_{irr} (1 - M^2_{irr}) + r^2 \right] - 4M^2_{irr} \left( M^2_{irr} - 1 \right) \sigma \left( 2M^2_{irr} - r \right) \left( 2M^2_{irr} + r - 2 \right)
\]
\[
4 \left( M^2_{irr} - 1 \right) M^2_{irr} \left( \sigma - 1 \right) + r^2
\]
\[
(66)
\]

The limits for the Schwarzschild spacetime and for the extreme Kerr spacetime are
\[
\lim_{M_{irr} \to 0} g = \lim_{M_{irr} \to 1^+} g = g_{\text{Schw}}, \quad \lim_{M_{irr} \to 1/\sqrt{2}} g = g_{\text{Kerr}}
\]
\[
(67)
\]

The singular points of the line elements are
\[
g_{tt} = 0: \quad \left[ r = 2: \left( \left( M_{irr} = 0, \sigma \in [0, 1) \right); \left( \sigma = 1, M_{irr} \in [0, 1) \right) \right); \left[ M_{irr} = \frac{1}{\sqrt{2}} : \left( 1 = 0, \sigma = 0; \left( \sigma \in [0, 1], \left( r = 1 + \sqrt{\sigma} \right) \right) \right) \right] \right]
\]
\[
\sigma \geq 0: \quad \left[ r = r^+_{M_{irr}} : \left( \sigma < 1, \left( M_{irr} \in [0, 1], M_{irr} \neq \frac{1}{\sqrt{2}} \right) \right) \right]
\]

\[
\left( M_{irr} \geq 1, \sigma \leq 1 \right), \quad \left[ r = r^+_{M_{irr}} : \sigma < 1, \left( M_{irr} \in [0, 1], M_{irr} \neq \frac{1}{\sqrt{2}} \right) \right],
\]
\[
g_{rr} = 0: \quad \left[ r = 0, \sigma = 1, \left( M_{irr} \in [0, 1], M_{irr} > 1 \right); \left[ r = r^+_{M_{irr}}, \sigma \in [0, 1], M_{irr} > 1 \right) \right]
\]
\[
g_{\phi\theta} = 0: \quad \left[ \sigma \in [0, 1], M_{irr} > 1, r = r^+_{M_{irr}} \right], \quad g_{\theta\theta} = 0: \quad \left[ \sigma \in [0, 1], \left( M_{irr} = 0, r = 2 \right) \right]
\]
\[
\left( M_{irr} = \frac{1}{\sqrt{2}} : r = 1 \right), \quad \left( r = 2M^2_{irr} \left( M_{irr} \neq \frac{1}{\sqrt{2}} \right) \right) \}
\]
\[
\left( r = 2(1 - M^2_{irr}), M_{irr} \in [0, 1], M_{irr} \neq \frac{1}{\sqrt{2}} \right) \}
\]

The extreme Kerr and the Schwarzschild BH limit are
\[
\lim_{\ell \to 0} g = g_{\text{Kerr}}, \quad \lim_{\ell \to 1/4} g = \lim_{\ell \to \pm \infty} g = g_{\text{Schw}}.
\]

(70)

There are the following singular points for the metric
\[
g_{tt} = 0:
\]
\[
\sigma = 0; \quad \left( \ell = 0, r = 1 \right), \quad \left( r = r^+_{a}, \ell \neq \frac{1}{2}, 0 \right) ;
\]
\[
\left( r = r^+_{a}, \ell < \frac{1}{4}, \ell \neq 0 \right)
\]

\[
(71)
\]
\[ \ell \neq \frac{1}{2} : (\sigma = 1, r = 2), \]
\[ \Big( \sigma \in [0, 1], \left[ r = r_H^\pm \left( r = r_H^-, \ell < \frac{1}{4} \right) \right] \Big), \]
\[ g_{rr} = 0 : \left( \sigma = 1, \left( \ell = \frac{1}{2}, r \geq 0 \right) \right), \]
\[ g_{tt} = 0 : \left( \sigma = 1, \left[ \ell = 0, \ell \neq \left\{ \frac{1}{2}, \frac{1}{4} \right\} \right] \right); \]
\[ \left( \sigma \in [0, 1], r = r_z, \ell > \frac{1}{4}, \ell \neq \frac{1}{2} \right), \]
\[ g_{tt}^{-1} = 0 : \left( \sigma \in [0, 1], [\ell > \frac{1}{4}, \ell \neq \frac{1}{2}], r = r_z \right). \]
\[ g_{rr}^{-1} = 0 : \left( \sigma \in [0, 1], (\ell = 0, r = 1), \left[ r = r_{\pm}(\ell), \left( \ell < \frac{1}{4}, \ell \neq 0, \ell > \frac{1}{2} \right) \right] \right) \]

(72)

with

\[ r_H^\pm \equiv 1 \pm 2 \sqrt{\frac{\ell^2}{(1 - 2\ell)^2}}, \quad r_B \equiv 1 \pm \sqrt{\frac{4\ell(\ell - \sigma) + \sigma}{(1 - 2\ell)^2}}, \]
\[ r_z \equiv \sqrt{\frac{(4\ell - 1)(1 - \sigma)}{(2\ell - 1)^2}}, \quad r_{\pm}(\ell) \equiv \frac{1}{1 - 2\ell}, \]

(73)

where \( r_{\pm}(\ell) \) are the horizon curves as functions of the acceleration \( \ell_H^2 \) – see Fig. 11.

7 Discussion and final remarks

The laws of BH classical thermodynamics have a geometric nature determined by the BH horizon, which has a prominent role in BH physics. In this work, we studied BH thermodynamics by using certain light surfaces, grouped in structures defined in metric bundles (MBs), tightly connected to the definitions of Killing horizons. The introduction of the extended plane allows an overview of all the possible BH (and possibly NS) states and their properties. It can be used to describe BH transitions from the MBs viewpoint, for instance, a transition from the tangent spin \( a_z(0) \) for the BH(0) to \( a_z(1) \) for the BH(1), corresponding to a transition between the respective tangent bundles. Consequently, we can also describe transitions in terms of the origin frequency and the tangent radius of the bundle.

In this new frame we reformulated, BH thermodynamics on the replicas in terms of the light surfaces, exploring the thermodynamic properties of the BHs geometries. Fundamentals of classical BH thermodynamics are discussed interpreting BH transformations according to the characteristic frequency which is also the BH horizon frequency. In particular, we considered the transformations with constant irreducible mass.

In the analysis of the thermodynamical properties, we apply the concepts of rotational energy parameter \( \xi \), surface gravity, frequency \( \omega \), irreducible mass \( M_{Ir} \), which we use to re-express the first law as well as the metric line element. A special metric line parametrization, describing only BH spacetimes, is given in terms of the maximum extractable rotational energy \( \xi \) or, correspondingly, the inertial mass parameter \( M_{Ir} \), replacing the metric spin parameter. In this work, we change the perspective in the analysis of some aspects of the Kerr solutions as seen in the extended plane, that is, we study properties unfolding along all the solutions; therefore, we re-consider the line element adapting it to this new frame. In Sect. 6, we rewrite the metric tensor on parts of the extended plane, providing alternative representations of the BH geometries, creating clusters of BH copies with respect to a new metric parameter, recovered from the bundles representations as curves in the extended plane. We used the irreducible mass \( M_{Ir} \) of the BH – Fig. 5, the BH rotational energy \( \xi \) (Fig. 10), the BH surface gravity (Fig. 8). The extended plane and MBs concepts include NSs; however, no transformation allows a BH-NS transition. NSs, as MBs components, can be used to represent some properties of the BH geometry. With these re-parameterizations with functions defined in the extended plane, we obtain structures constituted by replicas of the metric tensor describing the
same geometries or portions of the same geometry in the extended plane. We plan in a future work to study the cluster metric with extended range of parameters.

Replicas of light-surfaces with equal photon frequencies, are useful to connect different points of the spacetime, and different geometries following the BH transition with energy release. Here, we use the function $\xi(r)$, maximum extractable rotational energy – see [23–28]. In this context, we discuss how $\omega$, the bundle characteristic frequency, is tied to the extracted rotational energy $\xi$. The frequency, in fact, also appears as a component of the “work term” in the laws of BH thermodynamics. We use extensively this correspondence to formulate BH thermodynamics in terms of metric bundles. More precisely, the energy extraction process can take place regulated by the light surfaces which are, in general, two surfaces, one surface located outside the horizon $r_+$ and one within the ergoregion $[r_+, r_g^+]$. (In the case of the inner horizon, that is, of points characterized by the frequency $\omega^+_{\mathcal{H}} \equiv \omega_{\mathcal{H}}(r_-)$, where $r_-$ is the inner Killing horizon, the situation is more complex and it is not always possible to find a replica of the horizon [12]. Interestingly, the dependence of the orbit from the poloidal angle $\theta$, allows a more accurate study of the regions close to the BH rotational axis, revealing interesting observational implications [12].

In Sect. 4, we reformulated aspects of BH thermodynamics on the bundles, exploring the properties of the rotational energy and “rest” mass in the extended plane. The transition from a state $(0)$ to a new state $(1)$ is essentially regulated by the characteristics of the initial state, such as the BH surface gravity $\ell_{\mathcal{H}}(0)$ evaluated on the outer horizon, entering the entropy/temperature terms, and by the work term, which is expressed through the bundle frequency $\omega^+_{\mathcal{H}}$. We have expressed the relation between these quantities in terms of the tangency conditions of the bundles with the horizon curve and in terms of the bundle curves in the extended plane, leading to the alternative viewpoint in terms of replicas, i.e., orbits of the bundles located at $r > r_+$ (and some being in the $r < r_-$ region). In this sense, we describe the state transitions in terms of metric bundles transitions. In this regard, as bundles connect different points and geometries in the extended plane, we evaluated the line element along the bundles and in other portions of the extended plane, thus, connecting metrics particularly from the state with ($\ell(0)$, $\omega(0)$) to the state with parameters ($\ell(1)$, $\omega(1)$).

We highlighted classes of BHs related to the fact that a transition between initial and final state is subject to particular laws. The results point out BH spacetimes with spins $a/M = \sqrt{8/7}$ and $a/M = 1/\sqrt{2}$ showing anomalies in the rotational energy extraction, and surface gravity variation after BH state transitions and BH spin $a/M = \sqrt{3}/2$ relevant in the BH area variations. We also enlightened BHs masses ratios $M_1 = M_0/\sqrt{2}$, and $M_1/M_0 = \sqrt{2}/3$ (distinguishing spin $a/M = \sqrt{8/7}$), regulating the BH transformations studied here, and representing limiting values in the BH transformations–see for example Eqs. (55) and (56).

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Appendix A: Black holes in the extended plane

In this section, we discuss some aspects of the extended plane [12,13]. Here, we present the set of functions

\begin{align}
0_1 : \quad A_0(r) &\equiv A_0 \left(1 - \frac{r}{2M}\right), \\
0_2 : \quad \mathbf{A}_0(r) &\equiv \frac{2M}{A_0} r, \\
0_3 : \quad \mathbf{A}_0^{\text{tan}} &\equiv \frac{A_0^2 - 2}{2a_g(A_0)} + \left(\frac{4 - A_0^4}{A_0}\right) r, \quad \text{and} \\
0_4 : \quad \mathbf{A}_{\text{tangent}}(r) &\equiv \left(\frac{A_0^4 - 16}{8(A_0^2 + 4)} a_g(A_0) + \frac{32A_0}{16A_0^2}\right) - \frac{a_g(A_0)}{16A_0^2} \left(A_0^4 - 16\right), \quad (A.1)
\end{align}

see Fig. 12.

The four regions of Fig. 5, outlined in this analysis also through the inertial mass, the surface gravity $\ell$ and the BH area in the extended plane, can be obtained as functions of the origin spin $A_0$, the characteristic frequency of $\omega$ of the bundles, and the tangent radius (the first law of BH thermodynamics in the extended plane is defined along the boundary of these regions). Considering the curves of Eq. (A.1) and using $r_g = r_g(A_0)$ and $a_g = a_g(A_0)$, with the origin spin $A_0 = 2r_{\pm}/a_g$ and $A_0 = 2\sqrt{r_g}/\sqrt{2} - r_g$, we can express the curves as

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Using the irreducible mass $M_{\text{irr}}^{\pm}$ to express the bundle origin $\mathcal{A}_0$, we find $\mathcal{A}_0 = 2 \sqrt{1 - (M_{\text{irr}}^{\pm})^2/M_{\text{irr}}^{\pm}}$ and

$$\{0\} = \left\{ \begin{array}{l} \mathcal{A}_0(r), \quad 0_2 : \mathcal{A}_4^\pm(r), \quad 0_3 : \mathcal{A}_{\text{tangent}}^0, \quad 0_4 : \mathcal{A}_{\text{tangent}}(r) \end{array} \right\}$$

and

$$\{0\} = \left\{ \begin{array}{l} (2 - r) \sqrt{r_g}, \quad \frac{r}{\sqrt{2 - r_g}}, \quad \frac{r_g(r_g - 2) - 4r(r_g - 1)}{\sqrt{(2 - r_g)r_g}}, \quad \frac{r_g(1 - r) + r}{\sqrt{(2 - r_g)r_g}} \end{array} \right\},$$

in terms of the tangent radius $r_g \in [0, 2M]$ on the horizons. By inverting Eq. (22), we obtain $\mathcal{A}_0 = 2M_{\text{irr}}(\text{all})/(1 - M_{\text{irr}}(\text{all})^2)$ and

$$\{0\} = \left\{ \begin{array}{l} -M_{\text{irr}}(\text{all})(r - 2), \quad \frac{\sqrt{1 - M_{\text{irr}}(\text{all})^2} - 1}{M_{\text{irr}}(\text{all})}, \quad 2 \left[ 3M_{\text{irr}}(\text{all})^4 - M_{\text{irr}}(\text{all})^2(2r + 1) + r \right], \quad \frac{4M_{\text{irr}}(\text{all})^4 - 2M_{\text{irr}}(\text{all})^2(2r + 1) - 4\sqrt{M_{\text{irr}}(\text{all})^2} \left( M_{\text{irr}}(\text{all})^2 - 1 \right) M_{\text{irr}}(\text{all}) + r}{2\sqrt{M_{\text{irr}}(\text{all})^2(1 - M_{\text{irr}}(\text{all})^2)}} \end{array} \right\}.$$
Table 1 \( A_{area} \) represent the BH areas in the extended plane. \( a_g \) is the tangent spin to the horizon curve, \( r_g \) is the tangent radius of the bundle curve to the horizon in the extended plane, \( \omega \) is the horizon frequency and bundle characteristic frequency, \( a_{\pm} \) are the horizon curves in the extended plane, \( r_{\pm} \) are the outer and inner BH horizon, respectively. \( A_0 = a_0 \sqrt{\sigma} \) is the origin spin of the bundle, where \( \sigma = \sin^2 \theta \), \( M_{\text{irr}} \) is the irreducible mass of the BH, \( M_{\text{irr}(g)} \) is defined in Eq. (26).

| Tangent radius: | \[ \int r_g(\omega)d\omega = \arctan(2\omega) \] |
| Horizons curves: | \[ \int a_\pm \delta r = \frac{(r_\pm - a_\pm)}{2} - \arcsin \left[ \frac{M_{\text{irr}}}{2M} \right] \] |
| Tangent spin \( a_g \): | \[ \int a_g(\omega)d\omega = \frac{\log(w^2 + 1)}{2}; \int a_g(r_g)d\omega = \frac{\log(2)}{2}; \int a_g(r_g)d\sigma = \frac{2\pi}{\sqrt{\pi}} \] |
| Areas: | \[ \int A_{area}(a_{\pm})d\sigma = 4\pi r^2; \int_0^M A_{area}(a_{\pm})d\sigma = 4\pi; \int_0^M A_{area}(a_{\pm})d\sigma = 12\pi \] |

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