REPRESENTATIONS OF THE KAUFFMAN SKEIN ALGEBRA OF SMALL SURFACES

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Abstract. We prove a uniqueness result for finite-dimensional representations of the Kauffman skein algebra $S_A(S)$ of a surface $S$, when $A$ is a root of unity and when the surface $S$ is a sphere with at most four punctures or a torus with at most one puncture. We show that, if two irreducible representations of $S_A(S)$ have the same classical shadow and the same puncture invariants, and if this classical shadow is sufficiently generic in the character variety $\mathcal{X}_{SL_2(\mathbb{C})}(S)$, then the two representations are isomorphic.

Let $S$ be an oriented surface (without boundary) with finite topological type. The Kauffman skein algebra $S_A(S)$ of $S$ is a certain quantization of the character variety $\mathcal{X}_{SL_2(\mathbb{C})}(S)$ = \{group homomorphisms $r: \pi_1(S) \to SL_2(\mathbb{C})$\} // $SL_2(\mathbb{C})$ depending on a parameter $A = e^{\pi i h} \in \mathbb{C} - \{0\}$. The elements of $S_A(S)$ are represented by linear combinations of framed links in the thickened surface $S \times [0, 1]$, considered modulo certain relations; see [10, 6, 9] and §1.

In [2, 3, 4], Bonahon and Wong consider finite-dimensional representations of the skein algebra $S_A(S)$ when $A$ is a root of unit, construct invariants of irreducible representations, and show that any set of invariant is realized. The purpose of the current paper is to provide a uniqueness component to their results when the surface $S$ is small.

The precise result of [2] is the following. The setup requires $A^2$ to be an $N$-root of unity with $N$ odd, and we will restrict attention to the case where $A^N = -1$ as results are easier to state; because $N$ is odd, this case is equivalent to requiring that $A$ be an $N$-root of $-1$. The case $A^N = +1$ can be deduced from the case $A^N = -1$ by using spin structures on the surface; see [1, 2].

By definition of the geometric invariant theory quotient involved in the definition of $\mathcal{X}_{SL_2(\mathbb{C})}(S)$, two homomorphisms $r, r': \pi_1(S) \to SL_2(\mathbb{C})$ define the same element of $\mathcal{X}_{SL_2(\mathbb{C})}(S)$ if and only if they associate the same trace $\text{Tr} r(\gamma) = \text{Tr} r'(\gamma)$ to each element $\gamma \in \pi_1(S)$.

Theorem 1 ([2]). Suppose that $A$ is a primitive $N$-root of $-1$ with $N$ odd, and let $\rho: S_A(S) \to \text{End}(V)$ be an irreducible finite-dimensional representation of the Kauffman skein algebra. Let $T_N(x)$ be the $N$-th normalized Chebyshev polynomial of the first kind, defined by the trigonometric equality that $\cos N\theta = \frac{1}{2}T_N(2\cos \theta)$.

(1) There exists a unique character $r \in \mathcal{X}_{SL_2(\mathbb{C})}(S)$ such that

$$T_N(\rho([K])) = -\left(\text{Tr} r(K)\right) \text{Id}_V$$

in $\text{End}(V)$ for every framed knot $K \subset S \times [0, 1]$ whose projection to $S$ has no crossing and whose framing is vertical.

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(2) Let $P_k$ be a small simple loop going around the $k$-th puncture of $S$, and consider it as a knot in $S \times [0, 1]$ with vertical framing. Then there exists a number $p_k \in \mathbb{C}$ such that $\rho(P_k) = p_k \text{Id}_V$.

(3) The number $p_k$ of (2) is related to the character $r \in \chi_{\text{SL}_2(\mathbb{C})}(S)$ of (1) by the property that $T_N(p_k) = -\text{Tr} r(P_k)$.

The character $r \in \chi_{\text{SL}_2(\mathbb{C})}(S)$ is the classical shadow of the representation $\rho$, while the numbers $p_k$ are its puncture invariants. Bonahon and Wong also show in [3, 4] that every character $r \in \chi_{\text{SL}_2(\mathbb{C})}(S)$ and set of numbers $p_k \in \mathbb{C}$ associated to the punctures of $S$ that satisfy the relation (3) of Theorem 1 can be realized as the set of invariants of an irreducible representation of $\mathcal{S}_A(S)$.

We address the following conjecture which asserts that, generically, an irreducible representation of $\mathcal{S}_A(S)$ should be determined by its classical shadow and its puncture invariants.

**Conjecture 2** (Generic Uniqueness Conjecture). There exists a Zarisky dense open subset $\mathcal{U} \subset \chi_{\text{SL}_2(\mathbb{C})}(S)$ such that, if two irreducible representations $\rho, \rho' : \mathcal{S}_A(S) \to \text{End}(V)$ have the same classical shadow and the same puncture invariants, and if the classical shadow $r$ belongs to this dense subset $\mathcal{U} \subset \chi_{\text{SL}_2(\mathbb{C})}(S)$, then the representations $\rho$ and $\rho'$ are isomorphic.

Our main result is the following.

**Theorem 3.** The Generic Uniqueness Conjecture holds when the surface $S$ is a sphere with at most 4 punctures, or a torus with 0 or 1 puncture.

The cases of the sphere with at most 3 punctures are essentially trivial; see [3, 4]. When $S$ is the one-puncture torus, we rely in [4] on a presentation of $\mathcal{S}_A(S)$ given by Bullock-Przytycki [7] and on the systematic classification of all irreducible representations of the corresponding algebra by Havlíček-Pošta [5]; it is however non-trivial to relate the invariants of [5] to those of Theorem 1. The case of the unpunctured torus follows from the one-puncture case; see [5]. For the four-puncture sphere, we again use in [6] a presentation of $\mathcal{S}_A(S)$ exhibited in [7], and extend to this context the arguments that we had developed for the one-puncture torus; however, the situation is very significantly more complicated than for the one-puncture torus.

The analysis of [8] provides many different representations of the skein algebra of the one-puncture torus whose classical shadow is the trivial character. As a consequence, Conjecture 2 cannot hold without the genericity hypothesis.

1. The Kauffman skein algebra

Let $S$ be an oriented surface of finite topological type, without boundary. This means that $S = S_{g,p}$ is obtained by removing $p$ points from a closed oriented surface $\hat{S}$ of genus $g$. We consider framed links in the thickened surface $S \times [0, 1]$, namely unoriented 1-dimensional submanifolds $K \subset S \times [0, 1]$ endowed with a continuous choice of a vector transverse to $K$ at each point of $K$. A framed knot is a framed link that is connected.

The following definition provides a convenient way to describe a framing, in particular when representing a link by a picture. If the projection of $K \subset S \times [0, 1]$ to $S$ is an immersion, the vertical framing for $K$ is the framing that is everywhere parallel to the $[0, 1]$ factor and points towards 1.

The framed link algebra $\mathcal{K}(S)$ is the vector space over $\mathbb{C}$ freely generated by the isotopy classes of all framed links $K \subset S \times [0, 1]$. This vector space $\mathcal{K}(S)$ can be endowed with a multiplication, where the product of $K_1$ and $K_2$ is defined as the framed link $K_1 \cdot K_2 \subset$
$S \times [0, 1]$ that is the union of $K_1$ rescaled in $S \times [0, \frac{1}{2}]$ and $K_2$ rescaled in $S \times [\frac{1}{2}, 1]$. In other words, the product $K_1 \cdot K_2$ is defined by superposition of the framed links $K_1$ and $K_2$. This superposition operation is compatible with isotopies, and therefore provides a well-defined algebra structure on $\mathcal{K}(S)$.

Three framed links $K_1$, $K_0$ and $K_\infty$ in $S \times [0, 1]$ form a Kauffman triple if the only place where they differ is above a small disk in $S$, where they are as represented in Figure 1 (as seen from above) and where the framing is vertical and pointing upwards (namely the framing is parallel to the $[0, 1]$ factor and points towards 1).

![Figure 1. A Kauffman triple](image)

For $A \in \mathbb{C} \setminus \{0\}$, the Kauffman skein algebra $S_A(S)$ is the quotient of the framed link algebra $\mathcal{K}(S)$ by the linear subspace generated by:

1. all elements $K_1 - A^{-1}K_0 - AK_\infty \in \mathcal{K}(S)$ as $(K_1, K_0, K_\infty)$ ranges over all Kauffman triples;
2. the element $\bigcirc + (A^2 + A^{-2})\emptyset$ where $\bigcirc \subset S \times [0, 1]$ is an unknot projecting to a simple loop in $S$ and endowed with the vertical framing, and where $\emptyset$ is the empty link.

The superposition operation descends to a multiplication in $S_A(S)$, and endows $S_A(S)$ with the structure of an algebra. The class $[\emptyset]$ of the empty link is an identity element in $S_A(S)$, and we usually identify it to the scalar 1.

An element $[K] \in S_A(S)$ represented by a framed link $K \subset S \times [0, 1]$ is a skein in $S$.

Throughout the article, we will assume that the parameter $A$ is a primitive $N$–root of $-1$ with $N$ odd; for instance, $A = e^{\pi i/N}$. Because $N$ is odd, this implies that $A^2$ and $A^4$ are primitive $N$–roots of unity, a property that is frequently used in the article.

2. Chebyshev polynomials

The normalized $n$-th Chebyshev polynomial of the first kind is the polynomial $T_n(x)$ such that $\text{Tr} M^n = T_n(\text{Tr} M)$ for every $M \in \text{SL}_2(\mathbb{C})$. It can be inductively computed by the recurrence relation that $T_n(x) = xT_{n-1}(x) - T_{n-2}(x)$, combined with the initial conditions $T_0(x) = 2$ and $T_1(x) = x$.

The polynomial $T_n(x)$ contains only even powers of $x$ when $n$ is even, and only odd powers of $x$ when $n$ is odd. Also, applying the relation $\text{Tr} M^n = T_n(\text{Tr} M)$ to a rotation matrix gives the trigonometric identity that $2 \cos n\theta = T_n(2 \cos \theta)$.

We will use the following computations.

**Lemma 4.**

1. $T_n(a + a^{-1}) = a^n + a^{-n}$;
2. $T_N(x) - a^N - a^{-N} = \prod_{k=1}^{N} (x - aA^{2k} - a^{-1}A^{-2k})$.

**Proof.** Property (a) is a simple application of the relation $\text{Tr} M^n = T_n(\text{Tr} M)$ to a matrix $M \in \text{SL}_2(\mathbb{C})$ with eigenvalues $a, a^{-1}$. 
For (b), note that (a) implies that the solutions of the equation $T_N(x) = a^N + a^{-N}$ are the numbers $x = b + b^{-1}$ with $b^N = a^N$. Since $A^2$ is a primitive $N$-root of unity, these are the numbers of the form $x = aA^{2k} + a^{-1}A^{-2k}$. The equality then follows from the fact that the highest degree term of $T_N(x)$ is $x^N$. \hfill \Box

3. The sphere with at most three punctures

This case is essentially trivial. Indeed, in the sphere with at most 3 punctures, every simple closed curve is isotopic, either to a trivial knot, or to a simple loop $P_k$ going around one of the punctures. It follows that the algebra $S_A(S)$ is generated by the skeins $[P_k]$, so that every irreducible representation $\rho: S_A(S) \to \text{End}(V)$ is completely determined by its puncture invariants $p_k$ (which also determine the classical shadow).

4. The one-puncture torus $S_{1,1}$

This section is devoted to the case of the one-puncture torus $S_{1,1}$.

4.1. A presentation for the skein algebra $S_A(S_{1,1})$. We will use the presentation of $S_A(S_{1,1})$ given by Bullock and Przytycki in [7].

Identify the punctured torus $S_{1,1}$ to the quotient space obtained from a square $[0, 1] \times [0, 1]$ by removing the four corners and gluing together opposite sides in the usual fashion. Let $X_1$ be the closed curve that is the image in $S_{1,1}$ of the vertical line segment $\{\frac{1}{2}\} \times [0, 1]$, let $X_2$ be the image of the horizontal line segment $[0, 1] \times \{\frac{1}{2}\}$, and let $X_3$ be the closed curve that is the image of the two slope 1 line segments respectively parametrized by $t \mapsto (t + \frac{1}{2}, t)$ and $t \mapsto (t, t + \frac{1}{2})$ with $0 \leq t \leq \frac{1}{2}$. See Figure 2

Also, let $P \subset S_{1,1}$ be a simple loop going around the puncture.

![Figure 2. Curves on the one-puncture torus](image-url)
Proposition 5. The Kauffman skein algebra $\mathcal{S}_A(S_{1,1})$ admits a presentation by generators $X_1$, $X_2$, and $X_3$ and relations

$$AX_1X_2 - A^{-1}X_2X_1 = (A^2 - A^{-2})X_3$$
$$AX_2X_3 - A^{-1}X_3X_2 = (A^2 - A^{-2})X_1$$
$$AX_3X_1 - A^{-1}X_1X_3 = (A^2 - A^{-2})X_2$$

where the $X_i$ are represented by the closed curves indicated above, endowed with vertical framing.

If the loop $P$ going around the puncture is also endowed with the vertical framing, the corresponding element of $\mathcal{S}_A(S_{1,1})$ is equal to

$$P = AX_1X_2X_3 - A^2X_1^2 - A^{-2}X_2^2 - A^2X_3^2 + A^2 + A^{-2}.$$ 

This element is central in $\mathcal{S}_A(S_{1,1})$. 

The above presentation is also a presentation of another algebra, the algebra $U_q(\mathfrak{so}_3)$ whose irreducible representations were classified by Havlíček and Pošta. We will heavily rely on the arguments of [8], while adapting them to our goals.

4.2. Reconstructing an irreducible representation of $\mathcal{S}_A(S_{1,1})$.

Theorem 6. Let $\rho: \mathcal{S}_A(S_{1,1}) \rightarrow \text{End}(V)$ be an irreducible representation with classical shadow $r \in \mathcal{X}_{\text{SL}_2(\mathbb{C})}(S_{1,1})$ and puncture invariant $p \in \mathbb{C}$. Suppose that

$$\text{Tr}r(X_3) \neq \pm 2$$

and

$$\text{Tr}r(X_1)\text{Tr}r(X_2)\text{Tr}r(X_3) + \text{Tr}r(X_1)^2 + \text{Tr}r(X_2)^2 \neq 0$$

for the curves $X_1$, $X_2$, $X_3$ of [4.4]. Then, up to isomorphism, the representation $\rho: \mathcal{S}_A(S_{1,1}) \rightarrow \text{End}(V)$ is completely determined by $r$ and $p$. In addition, $V$ has dimension $N$.

Theorem 3 proves our Generic Uniqueness Theorem 1 in the case of the one-puncture torus $S_{1,1}$, since its hypotheses describe a Zariski dense open subset of the character variety $\mathcal{X}_{\text{SL}_2(\mathbb{C})}(S)$. We will later provide a slightly more general result, Theorem 15.

The proof of Theorem 6 will take a while, and we will split it into several lemmas. We assume that the hypotheses of Theorem 6 hold, for the remainder of this section.

For notational convenience, set $t_i = -\text{Tr}r(X_i)$ for $i = 1, 2, 3$, so that $T_N(\rho(X_i)) = t_i\text{Id}_V$. The hypotheses of Theorem 6 are then that $t_3 \neq \pm 2$ and $t_1t_2t_3 + t_1^2 + t_2^2 \neq 0$.

The numbers $t_1$, $t_2$, $t_3$ and the puncture invariant $p$ are related by the following equation.

Lemma 7.

$$T_N(p) = -t_1t_2t_3 - t_1^2 - t_2^2 - t_3^2 + 2$$

Proof. This is a consequence of the special case where $N = 1$ and $A = -1$. Indeed, an observation of Bullock [5] (see also [9]) shows that the character $r \in \mathcal{X}_{\text{SL}_2(\mathbb{C})}(S_{1,1})$ uniquely determines a homomorphism $\Theta_r: \mathcal{S}_{-1}(S_{1,1}) \rightarrow \mathbb{C}$ by the property that $\Theta_r([K]) = -\text{Tr}r(K)$ for every framed knot $K \subset S_{1,1} \times [0, 1]$.

In the special case $A = -1$ considered, the second half of Proposition 5 states that the elements $X_1$, $X_2$, $X_3$, $P \in \mathcal{S}_{-1}(S_{1,1})$ satisfy the relation

$$P = -X_1X_2X_3 - X_1^2 - X_2^2 - X_3^2 + 2$$
Choose a number \( x_3 \in \mathbb{C} \) such that \( t_3 = x_3^N + x_3^{-N} \).

Because \( T_N(\rho(X_3)) = t_3 \text{Id}_V \), all possible eigenvalues \( \lambda \) of \( \rho(X_3) \) are such that \( T_N(\lambda) = t_3 = x_3^N + x_3^{-N} \), and therefore equal to one of the numbers \( \lambda_k = x_3^{2k} + x_3^{-1}A^{-2k} \) by Lemma 8(a). Since \( t_3 \neq \pm 2 \) by hypothesis, the numbers \( \lambda_k \) with \( k = 1, 2, \ldots, N \) are distinct.

It is convenient to consider all integer indices \( k \), so that \( \lambda_{k+N} = \lambda_k \).

Define \( V_k = \{ v \in V; \rho(X_3)v = \lambda_kv \} \). Namely, \( V_k \) is the eigenspace corresponding to \( \lambda_k \) if \( \lambda_k \) is really an eigenvalue of \( \rho(X_3) \), and is 0 otherwise.

Our key tool is provided by the operators \( U_k \) and \( D_k \in \text{End}(V) \) defined by

\[
U_k = A\rho(X_1) - x_3A^{2k}\rho(X_2) \\
D_k = A\rho(X_1) - x_3^{-1}A^{-2k}\rho(X_2).
\]

These “up” and “down” operators \( U_k \) and \( D_k \) are borrowed from [8]. What makes them so useful is the following property.

**Lemma 8.** The operator \( U_k \) sends the subspace \( V_k \) to \( V_{k+1} \), and \( D_k \) sends \( V_k \) to \( V_{k-1} \).

**Proof.** We want to show that \( \rho(X_3)U_kv_k = \lambda_{k+1}U_kv_k \) for every \( v_k \in V_k \). For this, expand

\[
\rho(X_3)U_kv_k = \rho(X_3)\left(A\rho(X_1) - x_3A^{2k}\rho(X_2)\right)v_k = \left(A\rho(X_3X_1) - x_3A^{2k}\rho(X_3X_2)\right)v_k.
\]

Using the relations of Proposition 5,

\[
\rho(X_3)U_kv_k = A\rho(A^{-2}X_1X_3 + A^{-1}(A^2 - A^{-2})X_2)v_k \\
= A\rho(A^{-2}X_1X_3 + A^{-1}(A^2 - A^{-2})X_2)v_k \\
- x_3A^{2k}\rho(A^2X_2X_3 - A(A^2 - A^{-2})X_1)v_k \\
= A^{-1}\rho(X_1)\rho(X_3)v_k + (A^2 - A^{-2})\rho(X_2)v_k \\
- x_3A^{2k+2}\rho(X_3)v_k + x_3A^{2k+1}(A^2 - A^{-2})\rho(X_1)v_k.
\]

Using the facts that \( \rho(X_3)v_k = \lambda_kv_k \) and \( \lambda_k = x_3A^{2k} + x_3^{-1}A^{-2k} \), we obtain

\[
\rho(X_3)U_kv_k = A^{-1}\rho(X_1)\lambda_kv_k + (A^2 - A^{-2})\rho(X_2)v_k \\
- x_3A^{2k+2}\rho(X_2)v_k + x_3A^{2k+1}(A^2 - A^{-2})\rho(X_1)v_k \\
= (x_3A^{2k+3} + x_3^{-1}A^{-2k-1})\rho(X_1)v_k - (x_3A^{2k+2} + A^{-2})\rho(X_2)v_k \\
= \lambda_{k+1}(A\rho(X_1) - x_3A^{2k}\rho(X_2))v_k = \lambda_{k+1}U_kv_k.
\]

which shows that \( U_kv_k \) belongs to \( V_{k+1} \).

The proof that \( D_k \) sends \( V_k \) to \( V_{k-1} \) is similar. \( \square \)

The indexing of the eigenspaces \( V_k \) of \( \rho(X_3) \) and of the operators \( U_k, D_k \) depends on our choice of \( x_3 \) such that \( t_3 = x_3^N + x_3^{-N} \). In particular, replacing \( x_3 \) by \( x_3^{2i} \) replaces \( V_k, U_k, D_k \) by \( V_{k-1}, U_{k-1} \) and \( D_{k-1} \), respectively. Similarly, replacing \( x_3 \) by \( x_3^{-1} \) flips the order and replaces \( V_k, U_k, D_k \) by \( V_{N-k}, D_{N-k}, U_{N-k} \).
Lemma 9. For every vector \( v_k \in V_k \),

\[
\begin{align*}
\rho(X_1)v_k &= -\frac{x_3^{-1}A^{-2k-1}}{x_3A^{2k} - x_3^{-1}A^{-2k}}U_kv_k + \frac{x_3A^{2k-1}}{x_3A^{2k} - x_3^{-1}A^{-2k}}D_kv_k \\
\rho(X_2)v_k &= -\frac{1}{x_3A^{2k} - x_3^{-1}A^{-2k}}U_kv_k + \frac{1}{x_3A^{2k} - x_3^{-1}A^{-2k}}D_kv_k \\
\rho(X_3)v_k &= (x_3A^{2k} + x_3^{-1}A^{-2k})v_k.
\end{align*}
\]

Proof. The first two lines are obtained by solving for \( \rho(X_1)v_k \) and \( \rho(X_3)v_k \) in the definition of \( U_k \) and \( D_k \). We just need to check that the denominators are non-zero. But this immediately follows from the hypothesis that \( t_3 = x_3^N + x_3^{-N} \) is different from \( \pm 2 \).

The last equation is just a rephrasing of the definition of the subspace \( V_k \).

Our next computation shows how the operators \( U_k \) and \( D_k \) interact with each other. Note that \( U_k \) sends \( V_k \) to \( V_{k+1} \), and that \( D_{k+1} \) sends \( V_{k+1} \) back to \( V_k \).

Lemma 10. For every \( v_k \in V_k \),

\[ D_{k+1}U_kv_k = -(p + x_3^2A^{4k+2} + x_3^{-2}A^{-4k-2})v_k. \]

Proof. We begin by expanding

\[
D_{k+1}U_kv_k = (\rho(X_1) - x_3^{-1}A^{-2k-2}\rho(X_2))(\rho(X_1) - x_3A^{2k}\rho(X_2))v_k
\]

\[ = (A^2\rho(X_1^2) - x_3^{-1}A^{-2k-1}\rho(X_2X_1) - x_3A^{2k+1}\rho(X_1X_2) + A^{-2}\rho(X_3^2))v_k. \]

The puncture invariant \( p \) is defined by the property that \( \rho(P) = p \text{Id}_V \) for the puncture element

\[ P = AX_1X_2X_3 - A^2X_1^2 - A^{-2}X_2^2 - A^2X_3^2 + A^2 + A^{-2}. \]

Therefore, using the fact that \( \rho(X_3)v_k = \lambda_kv_k = (x_3A^{2k} + x_3^{-1}A^{-2k})v_k \) and the relation that \( AX_1X_2 - A^{-1}X_2X_1 = (A^2 - A^{-2})X_3 \),

\[
D_{k+1}U_kv_k + pv_k = A^2\rho(X_1^2)v_k - x_3^{-1}A^{-2k-1}\rho(X_2X_1)v_k - x_3A^{2k+1}\rho(X_1X_2)v_k
\]

\[ + A^{-2}\rho(X_2^2)v_k + A\rho(X_1X_2X_3)v_k - A^2\rho(X_3^2)v_k
\]

\[ - A^{-2}\rho(X_3^2)v_k - A^2\rho(X_3^2)v_k + (A^2 + A^{-2})v_k
\]

\[ = -(x_3^2A^{4k+2} + x_3^{-2}A^{-4k-2})v_k \]

after simplifications. This concludes the proof.

Lemma 11. Consider the map

\[
\prod_{j=1}^N D_{k+j} \prod_{j=1}^N U_{k+N-j} = D_{k+1}D_{k+2} \cdots D_{k+N-1}D_{k+N}U_{k+N-1}U_{k+N-2} \cdots U_{k+1}U_k.
\]

For every \( v_k \in V_k \),

\[
\prod_{j=1}^N D_{k+j} \prod_{j=1}^N U_{k+N-j}v_k = (t_1t_2t_3 + t_1^2 + t_2^2)v_k
\]
Proof. Note that $\mathcal{U}_{k+j-1}\mathcal{U}_{k+j-2} \ldots \mathcal{U}_{k+1}\mathcal{U}_k v_k$ is an element of $V_j$ for every $j$. By successive applications of Lemma 10, we conclude that

$$\prod_{j=1}^N D_{k+j} \prod_{j=1}^N \mathcal{U}_{k+N-j} v_k = -\prod_{j=1}^N (p + x_3^2 A^{4k+4j+2} + x_3^{-2} A^{-4k-4j-2}) v_k.$$ 

Because $A^2$ and $A^4$ are primitive $N$-roots of unity, the set of powers $A^{4k+4j}$ with $j = 1, 2, \ldots, N$ is the same as the set of powers $A^{2l}$ with $l = 1, 2, \ldots, N$. Therefore

$$\prod_{j=1}^N (p + x_3^2 A^{4k+4j+2} + x_3^{-2} A^{-4k-4j-2}) = \prod_{l=1}^N (p + x_3^2 A^{2l} + x_3^{-2} A^{-2l}) = T_N(p) + x_3^{2N} + x_3^{-2N} = -t_1 t_2 t_3 - t_1^2 - t_2^2$$

by Lemmas 3(b) and 14 remembering that $t_3 = x_3^N + x_3^{-N}$. This completes the proof. \qed

We now use the fact that the representation $\rho: S_3(S_1,1) \to \text{End}(V)$ is irreducible.

**Lemma 12.** The space $V$ has dimension $N$, and all eigenspaces $V_k$ of $\rho(X_3)$ are 1-dimensional.

More precisely, $V$ admits a basis $\{v_1, v_2, \ldots, v_N\}$ where each $v_k$ generates the eigenspace $V_k$, and where for some $u \neq 0$

$$\mathcal{U}_k v_k = \begin{cases} v_{k+1} & \text{if } 1 \leq k \leq N - 1 \\ uv_1 & \text{if } k = N \end{cases}$$

and

$$D_k v_k = \begin{cases} -\frac{1}{u} (p + x_3^2 A^2 + x_3^{-2} A^{-2}) v_N & \text{if } k = 1 \\ -(p + x_3^2 A^{4k-2} + x_3^{-2} A^{-4k+2}) v_{k-1} & \text{if } 2 \leq k \leq N. \end{cases}$$

**Proof.** The operator $\rho(X_3) \in \text{End}(V)$ admits at least one non-zero eigenvector. Therefore, some $V_{k_0}$ is different from 0.

The map $\prod_{j=1}^N \mathcal{U}_{k_0+N-j} = \mathcal{U}_{k_0+N-1}\mathcal{U}_{k_0+N-2} \ldots \mathcal{U}_{k_0+1}\mathcal{U}_{k_0}$ of Lemma 14 sends $V_{k_0}$ to $V_{k_0}$. Because $t_1 t_2 t_3 + (t_1)^2 + (t_2)^2 \neq 0$ by hypothesis in Theorem 10 Lemma 11 shows that this map is not the 0 map on $V_{k_0}$. Therefore, there is an eigenvalue $u \neq 0$ and a non-zero eigenvector $v_{k_0} \in V_{k_0}$ such that

$$\mathcal{U}_{k_0+N-1}\mathcal{U}_{k_0+N-2} \ldots \mathcal{U}_{k_0+1}\mathcal{U}_{k_0} v_{k_0} = uv_{k_0}.$$ 

We can now arrange that $k_0 = 0$. Indeed, if we apply $\mathcal{U}_N \mathcal{U}_{N-1} \ldots \mathcal{U}_{k_0+1}\mathcal{U}_{k_0}$ to both sides of the above equality and if we set $v_1 = \mathcal{U}_N\mathcal{U}_{N-1} \ldots \mathcal{U}_{k_0+1}\mathcal{U}_{k_0} v_{k_0} \in V_{N+1} = V_1$, we obtain that

$$\mathcal{U}_N \mathcal{U}_{N-1} \ldots \mathcal{U}_2 \mathcal{U}_1 v_1 = uv_1.$$ 

Then, set $v_k = \mathcal{U}_{k+1} \mathcal{U}_{k+2} \ldots \mathcal{U}_2 \mathcal{U}_1 v_1 \in V_k$ for every $k = 1, 2, \ldots, N$. Let $W$ be the $N$-dimensional linear subspace of $V$ spanned by the vectors $v_1, v_2, \ldots, v_N$.

By construction,

$$\mathcal{U}_k v_k = \begin{cases} v_{k+1} & \text{if } 1 \leq k \leq N - 1 \\ uv_1 & \text{if } k = N. \end{cases}$$
Also, by Lemma 10,

\[
D_k v_k = \begin{cases} 
-\frac{1}{3}(p + x_3^2 A^2 + x_3^{-2} A^{-2})v_N & \text{if } k = 1 \\
-(p + x_3^2 A^{2k-2} + x_3^{-2} A^{-4k+2})v_{k-1} & \text{if } 2 \leq k \leq N.
\end{cases}
\]

An immediate consequence of these formulas is that, for every \( k \), \( U_k v_k \) and \( D_k v_k \) both belong to \( W \). Lemma 9 then shows that \( \rho(X_1) v_k, \rho(X_2) v_k \) and \( \rho(X_3) v_k \) belong to \( W \). This proves that \( W \) is invariant under the image of the representation \( \rho : S_k(S_{1,1}) \to \text{End}(V) \). By irreducibility of \( \rho \), it follows that \( W = V \). Therefore, \( V \) has dimension \( N \), and each eigenspace \( V_k \) of \( \rho(X_3) \) is the line generated by the vector \( v_k \).

Combining the above formulas for \( U_k v_k \) and \( D_k v_k \) with Lemma 9 completely determines \( \rho(X_1) \) and \( \rho(X_2) \) in terms of the parameters \( p, x_3 \) and \( u \). Here, \( p \) and \( x_3 \) are given by the data of Theorem 6. We now determine \( u \) in terms of the rest of this data.

**Lemma 13.**

\[ u = -t_1 - x_3^N t_2. \]

**Proof.** We need to relate the number \( u \) to the numbers \( t_i \) such that \( T_N(\rho(X_i)) = t_i \text{Id}_V \).

By Lemma 9, \( \rho(X_1) \) can be decomposed as \( \rho(X_1) = U + D \) where, in the basis \( \{v_1, v_2, \ldots, v_N\} \) provided by Lemma 12,

\[
U v_k = -\frac{x_3^{-1} A^{-2k-1}}{x_3 A^{2k} - x_3^{-1} A^{-2k}} U_k v_k
\]
\[
D v_k = \frac{x_3 A^{2k-1}}{x_3 A^{2k} - x_3^{-1} A^{-2k}} D_k v_k.
\]

The crucial property is that \( U \) and \( D \) cyclically permute the eigenspaces \( V_k \) in opposite directions, namely \( U \) sends each \( V_k \) to \( V_{k+1} \) while \( D \) sends \( V_k \) to \( V_{k-1} \).

Expanding \( T_N(\rho(X_1)) = T_N(U + D) \) gives a linear combination of terms of the form \( A_1 A_2 \cdots A_m \), where each \( A_i \) is either \( U \) or \( D \), and where \( m \leq N \) is odd. If such a monomial contains \( n \) terms that are \( U \) and \( m - n \) terms that are \( D \), it sends \( V_k \) to \( V_{k+2n-m} \). Therefore, since \( T_N(\rho(X_1)) = t_1 \text{Id}_V \), this expansion of \( T_N(U + D) \) contains only terms such that \( 2n - m = 0 \mod N \). Since \( m \) is odd and \( 0 \leq n \leq m \leq N \), we only have two possibilities: \((m, n) = (N, 0)\) or \((m, n) = (N, N)\). Since the highest degree term of \( T_N(x) \) is \( x_N \), this proves that

\[
T_N(\rho(X_1)) = T_N(U + D) = U^N + D^N.
\]

As \( A^2 \) is a primitive \( N \)-root of unity, \( \prod_{k=1}^N (x_3 A^{2k} - x_3^{-1} A^{-2k}) = x_3^N - x_3^{-N} \). It follows that

\[
U^N v_k = \frac{x_3^{-N} v_k}{x_3^N - x_3^{-N}} U_{k+N-1} U_{k+N-2} \cdots U_{k+1} U_k v_k
\]
\[
= \frac{u x_3^{-N} v_k}{x_3^N - x_3^{-N}} v_k
\]

using the expression for \( U_l v_l \) given by Lemma 12.
Similarly,
\[ \mathcal{D}^N v_k = -\frac{x_3^N}{x_3^N - x_3^{-N}} \mathcal{D}_{k-N+1} \mathcal{D}_{k-N+2} \cdots \mathcal{D}_{k-1} \mathcal{D}_k v_k \]
\[ = +\frac{x_3^N}{x_3^N - x_3^{-N}} \frac{1}{u} \prod_{j=1}^N (p + x_3^2 A^{4k-4j+2} + x_3^{-2} A^{-4k+4j-2}) v_k \]
\[ = -\frac{u^{-1} x_3^N}{x_3^N - x_3^{-N}} (t_1 t_2 t_3 + t_1^2 + t_2^2) v_k \]
where the last equality is proved by the same computation as in the proof of Lemma 11.

Comparing the equalities \( T_N(\rho(X_1)) = t_1 \text{ Id}_V \) and \( T_N(\rho(X_1)) = U^N + \mathcal{D}^N \), it follows that
\[ t_1 = -\frac{u x_3^{-N}}{x_3^N - x_3^{-N}} - \frac{u^{-1} x_3^N}{x_3^N - x_3^{-N}} (t_1 t_2 t_3 + t_1^2 + t_2^2). \]

This almost determines \( u \), up to two possibilities. To resolve the ambiguity, we perform similar computations for \( T_N(\rho(X_2)) = t_2 \text{ Id}_V \), which give
\[ t_2 = \frac{u}{x_3^N - x_3^{-N}} + \frac{u^{-1}}{x_3^N - x_3^{-N}} (t_1 t_2 t_3 + t_1^2 + t_2^2). \]
Combining these expressions for \( t_1 \) and \( t_2 \) shows that \( u = -t_1 - t_2 x_3^N \).

We now just need to combine Lemmas 9, 12 and 13 to obtain:

**Lemma 14.**
\[ \rho(X_1)^v_k = \begin{cases} -\frac{x_3^{-1} A^{-3}}{x_3 A^2 - x_3^{-1} A^{-2}} v_2 - \frac{x_3 A(p + x_3 A^2 + x_3^{-2} A^{-2})}{x_3 A^2 - x_3^{-1} A^{-2}} v_{k+1} & \text{if } k = 1 \\
\frac{x_3 A^{2k-1}}{x_3 A^2 - x_3^{-1} A^{-2}} v_{k+1} & \text{if } 2 \leq k \leq N - 1 \\
\frac{1}{x_3 A^2 - x_3^{-1} A^{-2}} v_1 \end{cases} \]
\[ \rho(X_2)^v_k = \begin{cases} -\frac{x_3 A^{2k-1}}{x_3 A^2 - x_3^{-1} A^{-2}} v_{k+1} & \text{if } k = 1 \\
\frac{1}{x_3 A^2 - x_3^{-1} A^{-2}} v_1 + \frac{p + x_3 A^{2k-1}}{x_3 A^2 - x_3^{-1} A^{-2}} v_{k-1} & \text{if } 2 \leq k \leq N - 1 \\
\frac{t_1}{x_3 A^2 - x_3^{-1} A^{-2}} v_1 + \frac{p + x_3 A^{2k-1}}{x_3 A^2 - x_3^{-1} A^{-2}} v_{k-1} & \text{if } k = N \end{cases} \]
\[ \rho(X_3)^v_k = (x_3 A^{2k} + x_3^{-1} A^{-2k}) v_k. \]

This proves that, up to isomorphism, the representation \( \rho \) is completely determined by the numbers \( t_1, t_2, x_3 \) and \( p \). Since \( x_3 \) was chosen as an arbitrary number such that \( x_3^N = x_3^{-N} = t_3 \), it follows that \( \rho \) is completely determined by \( t_1, t_2, t_3, p \), namely by its classical shadow \( r \in \mathcal{X}_{\text{SL}_2(\mathbb{C})}(S_{1,1}) \) and its puncture invariant \( \rho \).

This concludes the proof of Theorem 6 and therefore of the Uniqueness Theorem 3 in the case of the one-puncture torus \( S_{1,1} \).

**4.3. A more general statement.** The following statement slightly improves Theorem 6 by providing fewer exceptions.

**Theorem 15.** Consider a character \( r \in \mathcal{X}_{\text{SL}_2(\mathbb{C})}(S_{1,1}) \). Setting \( t_i = -\text{Tr} r(X_i) \) for the generators \( X_i \) of Proposition 5, suppose that at least one of the following conditions fails:

1. \( t_i = \pm 2 \) for each \( i = 1, 2, 3 \);
(2) \( t_i = 0 \) for each \( i = 1, 2, 3 \);
(3) one trace \( t_i \) is equal to \( \pm 2 \), the other two \( t_j \) are equal to \( \pm \frac{2}{\sqrt{3}} i \), and the signs are such that \( t_1 t_2 t_3 = -\frac{8}{3} \).

Then, up to isomorphism, there exists a unique representation \( \rho: S_A(S_{1,1}) \to \text{End}(V) \) with classical shadow \( r \) and puncture invariant \( p \) for every \( p \in \mathbb{C} \) with
\[
T_N(p) = -t_1 t_2 t_3 - t_1^2 - t_2^2 - t_0^2 + 2.
\]

Proof. An easy case-by-case analysis show that, if the hypotheses of Theorem 6 hold, it is possible to cyclically reindex the curves \( X_1, X_2, X_3 \) (which does not change their relations) so that \( t_3 \neq \pm 2 \) and \( t_1 t_2 t_3 + t_1^2 + t_2^2 \neq 0 \), namely so that the hypotheses of Theorem 6 are satisfied. Theorem 6 then proves the uniqueness of the representation \( \rho \).

To prove the existence, one can rely on [3, 4] or check by brute force computation that the operators explicitly given in Lemma 14 really satisfy the relations of Proposition 5. \( \square \)

5. The unpunctured torus \( S_{1,0} \)

For the unpunctured torus \( S_{1,0} \), Bullock and Przytycki show that the skein algebra \( S_A(S_{1,0}) \) admits a presentation with the same generators and relations as \( S_A(S_{1,1}) \) in Proposition 5 but with the additional relation that \( P + A^2 + A^{-2} = 0 \). As a consequence, an irreducible representation \( \rho: S_A(S_{1,0}) \to \text{End}(V) \) is equivalent to the data of an irreducible representation \( \rho: S_A(S_{1,1}) \to \text{End}(V) \) with puncture invariant \( p = -A^2 - A^{-2} \). The Generic Uniqueness Theorem 3 then follows from the case of the one-puncture torus, as proved by Theorem 6.

6. The four-puncture sphere \( S_{0,4} \)

6.1. A presentation for the skein algebra \( S_A(S_{0,4}) \). A presentation of the skein algebra \( S_A(S_{0,4}) \) of the four-puncture sphere \( S_{0,4} \) can again be found in Bullock-Przytycki [7].

![Figure 3. Curves on the four-puncture sphere](image)

Consider the sphere as a “pillowcase” obtained from a rectangle \([0, 2] \times [0, 1]\) by identifying each point \((0, y)\) to \((1, y)\), each \((x, 0)\) to \((2 - x, 0)\), and each \((x, 1)\) to \((2 - x, 1)\). Identify the
four-puncture sphere $S_{0,4}$ to the surface obtained from this pillowcase by removing the four points that are the images of the six points of $[0, 2] \times [0, 1]$ with integer coordinates.

This enables us to single out several simple closed curves in $S_{0,4}$. The first curve $X_1$ is in the image of the two vertical arcs $\left\{ \frac{1}{2}, \frac{3}{2} \right\} \times [0, 1]$. The second curve $X_3$ is the image of the horizontal line segment $[0, 1] \times \left\{ \frac{1}{2} \right\}$. The third curve is the image of three slope 1 segments, respectively parametrized by $t \mapsto \left( \frac{1}{2}t, \frac{1}{2}t + \frac{1}{2} \right)$, $t \mapsto (t + \frac{1}{2}, t)$ and $t \mapsto \left( \frac{1}{2}t + \frac{3}{2}, \frac{1}{2}t \right)$ for $0 \leq t \leq 1$.

We also consider small loops $P_0, P_1, P_2, P_3$ going around the punctures, indexed in such a way that for $i = 1, 2, 3$, the closed curve $X_i$ separates $P_0$ and $P_i$ from the other two puncture loops. See Figure 3.

We consider the elements $X_1, X_2, X_3, P_0, P_1, P_2, P_3 \in S_A(S_{0,4})$ represented by these simple closed curves, endowed with the vertical framing.

**Proposition 16** ([7]). The skein algebra $S_A(S_{0,4})$ of the four-puncture sphere admits a presentation with generators $X_1, X_2, X_3, P_0, P_1, P_2, P_3$ as above, and with the following relations:

1. the $P_i$ are central;
2. $A^2X_1X_2 - A^{-2}X_2X_1 = (A^4 - A^{-4})X_3 + (A^2 - A^{-2})(P_0P_3 + P_1P_2)$
3. $A^2X_2X_3 - A^{-2}X_3X_2 = (A^4 - A^{-4})X_1 + (A^2 - A^{-2})(P_0P_1 + P_2P_3)$
4. $A^2X_3X_1 - A^{-2}X_1X_3 = (A^4 - A^{-4})X_2 + (A^2 - A^{-2})(P_0P_2 + P_1P_3)$
5. $A^2X_1X_2X_3 - A^{-4}X_1^3 - A^4X_2^3 - A^4X_3^3 = A^2(P_0P_1 + P_2P_3)X_1 - A^{-2}(P_0P_2 + P_1P_3)X_2 - A^2(P_0P_3 + P_1P_2)X_3 + (A^2 + A^{-2})^2 = P_0P_1P_2P_3 + P_0^2 + P_1^2 + P_2^2 + P_3^2$.

We will take advantage of the fact that the relations (2–4) are very similar to the relations what we already encountered for the one-puncture torus.

### 6.2. Reconstructing an irreducible representations of $S_A(S_{0,4})$ from its invariants.

Let $\rho: S_A(S_{0,4}) \to \text{End}(V)$ be an irreducible representation with classical shadow $r \in \mathcal{X}_{\text{SL}_2(\mathbb{C})}(S_{0,4})$ and puncture invariants $p_0, p_1, p_2, p_3$. As in the proof of Theorem 6 we want to show that $\rho$ can be reconstructed from these invariants, provided that they are generic enough.

**Theorem 17.** Let $\rho: S_A(S_{0,4}) \to \text{End}(V)$ be an irreducible representation with classical shadow $r \in \mathcal{X}_{\text{SL}_2(\mathbb{C})}(S_{0,4})$ and puncture invariants $p_0, p_1, p_2, p_3 \in \mathbb{C}$. Suppose that,

$$\text{Tr} r(X_3) \neq \pm 2$$

and $\text{Tr} r(X_3) \neq T_N(r)$ for every solution $r$ of the equation

$$(r^2 + p_0p_3r + p_0^2 + p_3^2 - 4)(r^2 + p_1p_2r + p_1^2 + p_2^2 - 4) = 0.$$

Then, up to isomorphism, the representation $\rho: S_A(S_{1,1}) \to \text{End}(V)$ is completely determined by $r$ and $p$. In addition, $V$ has dimension $N$.

Theorem 17 proves the Generic Uniqueness Theorem 3 in the case of the four-puncture sphere.

The rest of this section is devoted to the proof of Theorem 17. In particular, we henceforth assume that its hypotheses hold. The proof follows the general lines of the argument used for Theorem 6 but most steps will be more complicated. As a consequence, our reconstruction of the representation $\rho$ from its classical shadow $r \in \mathcal{X}_{\text{SL}_2(\mathbb{C})}(S_{0,4})$ and its puncture invariants $p_i$ will not be as explicit as in that earlier case.
As in the case of the one-puncture torus, set \( t_i = -\text{Tr} r(X_i) \), so that \( T_N(\rho(X_i)) = t_i \text{Id}_V \).

Also, in view of the relations of Proposition 16, it is convenient to introduce

\[
q_1 = p_0p_1 + p_2p_3 \\
q_2 = p_0p_2 + p_1p_3 \\
q_3 = p_0p_3 + p_1p_2 \\
\Delta = p_0p_1p_2p_3 + p_6^2 + p_7^2 + p_8^2,
\]

Choose \( x_3 \in \mathbb{C} \), such that \( t_3 = x_3^N + x_3^{-N} \). Since \( T_N(\rho(X_3)) = t_3 \text{Id}_V \), all possible eigenvalues of \( \rho(X_3) \) are of the form \( \lambda_k = x_3A^{4k} + x_3^{-1}A^{-4k} \) for \( k = 1, 2, \ldots, N-1 \). These \( N \) numbers \( \lambda_k \) are distinct by our hypothesis that \( t_3 \neq \pm 2 \). As before, set \( V_k = \{ v \in V; \rho(X_3)v = \lambda_kv \} \).

The “up” and “down” operators \( \mathcal{U}_k \) and \( \mathcal{D}_k \) are now given by more complicated formulas.

\[
\mathcal{U}_k = A^2\rho(X_1) - x_3A^{4k}\rho(X_2) + \frac{q_2 + x_3A^{4k+2}q_1}{x_3A^{4k+2} - x_3^{-1}A^{-4k-2}} \text{Id}_V \\
\mathcal{D}_k = A^2\rho(X_1) - x_3^{-1}A^{-4k}\rho(X_2) + \frac{-q_2 - x_3^{-1}A^{-4k+2}q_1}{x_3A^{4k-2} - x_3^{-1}A^{-4k+2}} \text{Id}_V
\]

if we set \( \beta_k^+ = \frac{-q_2 + x_3A^{4k+2}q_1}{x_3A^{4k+2} - x_3^{-1}A^{-4k+2}} \) and \( \beta_k^- = \frac{-q_2 - x_3^{-1}A^{-4k+2}q_1}{x_3A^{4k-2} - x_3^{-1}A^{-4k+2}} \) to simplify further computations.

Lemma 18. For every \( v_k \in V_k \),

\[
\rho(X_1)v_k = \frac{-x_3^{-1}A^{-4k-2}}{x_3A^{4k} - x_3^{-1}A^{-4k}} \mathcal{U}_k v_k + \frac{x_3A^{4k-2}}{x_3A^{4k} - x_3^{-1}A^{-4k}} \mathcal{D}_k v_k \\
+ \beta_k^+ - \beta_k^- \frac{1}{x_3A^{4k} - x_3^{-1}A^{-4k}} v_k
\]

\[
\rho(X_2)v_k = \frac{-1}{x_3A^{4k} - x_3^{-1}A^{-4k}} \mathcal{U}_k v_k + \frac{1}{x_3A^{4k} - x_3^{-1}A^{-4k}} \mathcal{D}_k v_k \\
+ \beta_k^+ - \beta_k^- \frac{1}{x_3A^{4k} - x_3^{-1}A^{-4k}} v_k
\]

\[
\rho(X_3)v_k = (x_3A^{4k} + x_3^{-1}A^{-4k})v_k. \quad \Box
\]

Lemma 19. The operator \( \mathcal{U}_k \) sends each subspace \( V_k \) to \( V_{k+1} \), and \( \mathcal{D}_k \) sends \( V_k \) to \( V_{k-1} \).

Proof. Given a vector \( v_k \in V_k \), we want to show that \( \rho(X_3)\mathcal{U}_k v_k = \lambda_{k+1}\mathcal{U}_k v_k \). As in the proof of Lemma 18, we expand

\[
\rho(X_3)\mathcal{U}_k v_k = A^2\rho(X_3X_1)v_k - x_3A^{4k}\rho(X_3X_2)v_k + \beta_k^+ \rho(X_3)v_k
\]

\[
= A^{-2}\rho(X_1)\rho(X_3)v_k + (A^4 - A^{-4})\rho(X_2)v_k + (A^2 - A^{-2})q_2v_k \\
- x_3A^{4k+4}\rho(X_2)\rho(X_3)v_k + x_3A^{4k+2}(A^4 - A^{-4})\rho(X_1)v_k \\
+ x_3A^{4k+2}(A^2 - A^{-2})q_1v_k + \beta_k^+ \rho(X_3)v_k
\]

using the relations of Proposition 16.
Since $\rho(X_3) v_k = \lambda_k v_k$ and $\lambda_k = (x_3 A^{4k} + x_3^{-1} A^{-4k}) v_k$, it follows that

$$
\rho(X_3) U_k v_k = A^{-2} \rho(X_1) \lambda_k v_k + (A^4 - A^{-4}) \rho(X_2) v_k + (A^2 - A^{-2}) q_2 v_k
$$

$$- x_3 A^{4k+2} \rho(X_1) v_k + x_3 A^{4k+2} (A^4 - A^{-4}) \rho(X_1) v_k
$$

$$+ x_3 A^{4k+2} (A^2 - A^{-2}) q_1 v_k + \beta_k^+ \lambda_k v_k
$$

$$= (x_3 A^{4k+6} + x_3 A^{-4k-2}) \rho(X_1) v_k - (x_3 A^{8k+4} + A^{-4}) \rho(X_2) v_k
$$

$$+ \beta_k^+ (x_3 A^{4k+4} + x_3 A^{-4k-4}) v_k
$$

$$= \lambda_{k+1} (A^2 \rho(X_1) - x_3 A^{4k} \rho(X_2) + \beta_k^+) v_k = \lambda_{k+1} U_k v_k
$$

which shows that $U_k v_k$ belongs to $V_{k+1}$.

The proof that $D_k$ sends $V_k$ to $V_{k-1}$ is very similar. \qed

We now have the equivalent of Lemma 10.

**Lemma 20.** For every $v_k \in V_k$,

$$
D_{k+1} U_k v_k = R_k v_k
$$

where

$$
R_k = - (\Delta - 2 + x_3^2 A^{8k+4} + x_3^{-2} A^{-8k-4} + (x_3 A^{4k+2} + x_3^{-1} A^{-4k-2}) q_3 - \beta_{k+1}^- \beta_k^+)
$$

**Proof.** We begin by expanding

$$
D_{k+1} U_k v_k = (A^2 \rho(X_1) - x_3 A^{-4k-4} \rho(X_2) + \beta_{k+1}^- \text{Id}_V)(A^2 \rho(X_1) - x_3 A^{4k} \rho(X_2) + \beta_k^+ \text{Id}_V) v_k
$$

$$= A^4 \rho(X_1) v_k - x_3^{-1} A^{-4k-2} \rho(X_2 X_1) v_k + A^2 \beta_{k+1}^- \rho(X_1) v_k
$$

$$- x_3 A^{4k+2} \rho(X_1 X_2) v_k + A^{-4} \rho(X_2^2) v_k - x_3 A^{4k} \beta_{k+1}^- \rho(X_2) v_k + k
$$

$$+ A^2 \beta_k^+ \rho(X_1) v_k - x_3 A^{-4k-4} \beta_k^+ \rho(X_2) v_k + \beta_{k+1}^- \beta_k^+ v_k
$$

Using Relation (5) of Proposition 10,

$$
D_{k+1} U_k v_k + \Delta v_k = A^4 \rho(X_1^2) v_k - x_3 A^{-4k-2} \rho(X_2 X_1) v_k + A^2 \beta_{k+1}^- \rho(X_1) v_k
$$

$$- x_3 A^{4k+2} \rho(X_1 X_2) v_k + A^{-4} \rho(X_2^2) v_k - x_3 A^{4k} \beta_{k+1}^- \rho(X_2) v_k
$$

$$+ A^2 \beta_k^+ \rho(X_1) v_k - x_3^{-1} A^{-4k-4} \beta_k^+ \rho(X_2) v_k + \beta_{k+1}^- \beta_k^+ v_k
$$

$$+ A^2 \rho(X_1 X_2 X_3) v_k - A^4 \rho(X_2^2) v_k - A^{-4} \rho(X_3^2) v_k - A^4 \rho(X_3^2) v_k
$$

$$- A^2 q_1 \rho(X_1) v_k - A^{-2} q_2 \rho(X_2) - A^2 q_3 \rho(X_3) v_k + (A^2 + A^{-2}) q_2 v_k
$$

$$= - x_3^{-1} A^{-4k-2} \rho(X_2 X_1) v_k + A^2 \beta_{k+1}^- \rho(X_1) v_k - x_3 A^{4k+2} \rho(X_1 X_2) v_k
$$

$$- x_3 A^{4k} \beta_{k+1}^- \rho(X_2) v_k + A^2 \beta_k^+ \rho(X_1) v_k - x_3^{-1} A^{-4k-4} \beta_k^+ \rho(X_2) v_k
$$

$$+ \beta_{k+1}^- \beta_k^+ v_k + A^2 \rho(X_1 X_2 X_3) v_k - A^4 \rho(X_2^2) v_k - A^4 \rho(X_3^2) v_k - A^2 q_1 \rho(X_1) v_k
$$

$$- A^2 q_2 \rho(X_2) v_k - A^2 q_3 \rho(X_3) v_k + (A^2 + A^{-2}) q_2 v_k.
$$

We simplify this complicated expression in a few steps. We first observe that all terms with $\rho(X_1)$ and $\rho(X_2)$ cancel each other out. Indeed,

$$
A^2 \beta_{k+1}^- \rho(X_1) v_k + A^2 \beta_k^+ \rho(X_1) v_k - A^2 q_1 \rho(X_1) v_k = A^2 (\beta_{k+1}^- + \beta_k^+ - q_1) \rho(X_1) v_k = 0
$$
Lemma 22. As in the proof of Lemma 11, this follows from Lemma 20.

Proof. Therefore

\[ -x_3 A^{4k} \beta_{k+1}^{-1} \rho(X_2) v_k - x_3^{-1} A^{-4k-4} \beta_k^+ \rho(X_2) v_k - A^{-2} q_2 \rho(X_2) v_k = -A^{-2} (x_3 A^{4k+2} \beta_{k+1}^{-1} + x_3^{-1} A^{-4k-2} \beta_k^+ + q_2) \rho(X_2) v_k = 0, \]

so that we are left with

\[ \mathcal{D}_{k+1} \mathcal{U}_k v_k + \Delta v_k = -x_3^{-1} A^{-4k-2} \rho(X_2 X_1) v_k - x_3 A^{4k+2} \rho(X_1 X_2) v_k + \beta_{k+1}^- \beta_k^+ v_k + A^2 \rho(X_1 X_2 X_3) v_k - A^4 \rho(X_2^3) v_k = A^2 \rho(X_1 X_2) \lambda_k v_k = A^2 (x_3 A^{4k} + x_3^{-1} A^{-4k}) \rho(X_1 X_2) v_k. \]

We now use the fact that \( v_k \) is an eigenvalue of \( \rho(X_3) \):

\[ A^2 \rho(X_1 X_2 X_3) v_k = A^2 \rho(X_1 X_2) \lambda_k v_k = A^2 (x_3 A^{4k} + x_3^{-1} A^{-4k}) \rho(X_1 X_2) v_k. \]

Therefore

\[ \mathcal{D}_{k+1} \mathcal{U}_k v_k + \Delta v_k = -x_3^{-1} A^{-4k-2} \rho(X_2 X_1) v_k + x_3 A^{4k+2} \rho(X_1 X_2) v_k + \beta_{k+1}^- \beta_k^+ v_k - A^4 \rho(X_2^3) v_k = A^2 \rho(X_1 X_2) \lambda_k v_k = A^2 (x_3 A^{4k} + x_3^{-1} A^{-4k}) \rho(X_1 X_2) v_k. \]

after using Relation (1) of Proposition [16]. The formula of Lemma 20 then follows from a last application of the property that \( \rho(X_3) v_k = (x_3 A^{4k} + x_3^{-1} A^{-4k}) v_k. \)

\[ \square \]

We now turn to the analogue of Lemma [11] and consider the product

\[ \prod_{j=1}^{N} \mathcal{D}_{k+j} \prod_{j=1}^{N} \mathcal{U}_{k+N-j} = \mathcal{D}_{k+1} \mathcal{D}_{k+2} \cdots \mathcal{D}_{k+N-1} \mathcal{D}_{k+N} \mathcal{U}_{k+N-1} \mathcal{U}_{k+N-2} \cdots \mathcal{U}_{k+1} \mathcal{U}_k. \]

Since the output of Lemma 20 is more complicated than that of Lemma [10] the computation will be more elaborate. We begin with a simple step.

Lemma 21. For every \( v_k \in V_k \),

\[ \prod_{j=1}^{N} \mathcal{D}_{k+j} \prod_{j=1}^{N} \mathcal{U}_{k+l-j} v_k = \prod_{j=1}^{N} R_j v_k \]

where \( R_j \) is as in Lemma 20.

Proof. As in the proof of Lemma [11], this follows from Lemma 20 \( \square \)

We next tackle the product of Lemma 21

Lemma 22.

\[ \prod_{j=1}^{N} R_j = -\frac{(t_3 - T_N(r_0))(t_3 - T_N(r_1))(t_3 - T_N(r_2))(t_3 - T_N(r_3))}{t_3^2 - 4} \]

where \( r_0, r_1, r_2 \) and \( r_3 \) are the solutions of the equation

\[ (r^2 + p_0 p_3 r + p_0^2 + p_3^2 - 4)(r^2 + p_1 p_2 r + p_1^2 + p_2^2 - 4) = 0. \]
Proof. After substituting back the values of $\beta_j^+ = \frac{q_2 + x_3 A^{4j+2} q_1}{x_3 A^{4j-2} - x_3 - A^{4j-2}}$, $\beta_j^- = \frac{-q_2 - x_3 A^{4j+2} q_1}{x_3 A^{4j-2} - x_3 - A^{4j-2}}$
and $\Delta = p_0 p_1 p_2 p_3 + p_0^2 + p_1^2 + p_2^2 + p_3^2$ and expanding,

$$R_j = -(\Delta - 2 + x_3 A^{8j+4} - x_3^{-2} A^{-8j-4} + (x_3 A^{4j+2} + x_3^{-1} A^{-4j-2}) q_3 - \beta_j^- \beta_j^+)$$
can be factored as $R_j = -S_j S_j' S_j''$ where

$S_j = (x_3 A^{-8j-4} + p_0 p_3 x_3^{-1} A^{-4j-2} + (p_0^2 + p_3^2 - 2) + p_0 p_3 x_3 A^{4j+2} + x_3^2 A^{8j+4})$

$S_j' = (x_3 A^{-8j-4} + p_1 p_3 x_3^{-1} A^{-4j-2} + (p_1^2 + p_3^2 - 2) + p_1 p_3 x_3 A^{4j+2} + x_3^2 A^{8j+4})$

and $S_j'' = (x_3 A^{4j+2} - x_3^{-1} A^{-4j-2})^2$.

The term $S_j$ looks nicer in terms of $u = x_3 A^{4j+2}$. Then,

$$S_j = u^{-2} + u^2 + p_0 p_3 u + p_0 p_3 u^{-1} + p_0^2 + p_3^2 - 2$$

$$= (u + u^{-1})^2 + p_0 p_3(u + u^{-1}) + p_0^2 + p_3^2 - 4$$

$$= (u + u^{-1} - r_0)(u + u^{-1} - r_3)$$

$$= (r_0 - x_3 A^{4j+2} - x_3^{-1} A^{-4j-2})(r_3 - x_3 A^{4j+2} - x_3^{-1} A^{-4j-2})$$

where $r_0$ and $r_3$ are the solutions of the equation $r^2 + p_0 p_3 r + p_0^2 + p_3^2 - 4 = 0$. Note that $r_0$ and $r_3$ do not depend on $j$. Lemma 21(b) then shows that

$$\prod_{j=1}^N S_j = (T_N(r_0) - x_3^N - x_3^{-N})(T_N(r_3) - x_3^N - x_3^{-N}) = (T_N(r_0) - t_3)(T_N(r_3) - t_3).$$

Similarly,

$$\prod_{j=1}^N S_j' = (T_N(r_1) - t_3)(T_N(r_2) - t_3).$$

where $r_1$ and $r_2$ are the solutions of the equation $r^2 + p_1 p_2 r + p_1^2 + p_2^2 - 4 = 0$.

Finally

$$\prod_{j=1}^N S_j'' = \prod_{j=1}^N (x_3 A^{4j+2} - x_3^{-1} A^{-4j-2})^2 = (x_3^N - x_3^{-N})^2 = t_3^2 - 4,$$

which concludes the computation. \qed

Lemma 23. The space $V$ has dimension $N$, and all eigenspaces $V_k$ of $\rho(X_3)$ are 1-dimensional.

More precisely, $V$ admits a basis $\{v_1, v_2, \ldots, v_N\}$ where each $v_k$ generates the eigenspace $V_k$, and where for some $u \neq 0$

$$U_k v_k = \begin{cases} v_{k+1} & \text{if } 1 \leq k \leq N - 1 \\ uv_1 & \text{if } k = N \end{cases}$$

and

$$D_k v_k = \begin{cases} \frac{1}{u} R_N v_N & \text{if } k = 1 \\ R_{k-1} v_{k-1} & \text{if } 2 \leq k \leq N. \end{cases}$$

where $R_k$ is defined as in Lemma 20.

Proof. By the combination of Lemmas 21 and 22, and by hypothesis on $t_3$, the product $\prod_{j=1}^N D_{k+j} \prod_{j=1}^N U_{k+l-j}$ is different from 0. The proof is then identical to that of Lemma 12. \qed
Lemma 24. The number \( u \) occurring in Lemma 23 is completely determined by \( x_3, t_1, t_2 \) and the puncture invariants \( p_i \).

Proof. Although we cannot be as specific as in the proof of Lemma 12, we will follow a very similar argument.

By Lemma 18 we can write \( \rho(X_i) \) as a sum \( \rho(X_i) = U + D + I \) where, in the basis \( \{v_1, v_2, \ldots, v_N\} \) of Lemma 23

\[
\begin{align*}
Uv_k &= -\frac{x_3^{-1}A^{-4k-2}}{x_3A^{4k} - x_3^{-1}A^{-4k}}U_kv_k \\
Dv_k &= \frac{x_3A^{4k-2}}{x_3A^{4k} - x_3^{-1}A^{-4k}}D_kv_k \\
Iv_k &= \frac{x_3^{-1}A^{-4k-2}\beta^+ - x_3A^{4k-2}\beta^-}{x_3A^{4k} - x_3^{-1}A^{-4k}}v_k.
\end{align*}
\]

The key observation is that, by the combination of Lemma 23 with the above definition, \( Uv_k = a_kv_{k+1} \) where \( a_k \) depends only on \( x_3 \) if \( 1 \leq k \leq N - 1 \), and where \( a_N \) is \( u \) times a quantity depending only on \( x_3 \). Similarly, \( Dv_k = b_kv_{k-1} \) where \( b_k \) depends only on \( x_3 \) and the puncture invariants \( p_i \) if \( 2 \leq k \leq N \), and where \( b_1 \) is \( \frac{1}{u} \) times a quantity depending only on \( x_3 \) and the \( p_i \). And \( Iv_k = c_kv_k \) where \( c_k \) depends only on \( x_3 \) and the \( p_i \).

If we expand \( T_N(\rho(X_i)) = T_N(U + D + I) \), we obtain a linear combination of terms \( A_1A_2\ldots A_m \) with \( m \leq N \), where each \( A_i \) is equal to \( U, D \) or \( I \). Since \( T_N(\rho(X_i)) = t_1\text{Id}_V \) the only monomials that do not cancel out in this linear combination are those for which the line \( A_1A_2\ldots A_{m-1}A_mV_i \) is equal to \( V_i \).

For most such monomials \( A_1A_2\ldots A_m \) with a nontrivial contribution to \( T_N(\rho(X_i)) \), the sequence of lines \( V_i, A_mV_i, A_{m-1}A_mV_i, \ldots, A_2\ldots A_{m-1}A_mV_i, A_1A_2\ldots A_{m-1}A_mV_i = V_i \) switches as many times from \( V_i \) to \( V_N \) as it does from \( V_N \) to \( V_i \). This implies that, when we compute \( A_1A_2\ldots A_mv_i \), any term \( u \) is balanced by a term \( \frac{1}{u} \) and conversely, so that \( A_1A_2\ldots A_mv_i \) is equal to \( v_i \) times a scalar depending only on \( x_3 \) and the \( p_i \).

Because \( m \leq N \), there are exactly two exceptions to this property, namely \( A_1A_2\ldots A_m = U^N \) and \( D^N \). These two exceptions occur with coefficient 1 in the expression of \( T_N(\rho(X_i)) \) since the highest degree of \( T_N(x) \) is \( x^N \). Also,

\[
\begin{align*}
U^Nv_1 &= -u \prod_{k=1}^N \frac{x_3^{-1}A^{-4k-2}}{x_3A^{4k} - x_3^{-1}A^{-4k}}v_1 = -u \frac{x_3^{-N}}{x_3^N - x_3^{-N}}v_1 \\
D^Nv_1 &= u^{-1} \prod_{k=1}^N \frac{x_3A^{4k-2}}{x_3A^{4k} - x_3^{-1}A^{-4k}}R_{k-1}v_1 = u^{-1} \frac{x_3^N}{x_3^N - x_3^{-N}} \prod_{k=1}^N R_kv_1
\end{align*}
\]

by Lemma 23.

Since \( T_N(\rho(X_i)) = t_1\text{Id}_V \), the conclusion of this discussion is that

\[
t_1 = -u \frac{x_3^{-N}}{x_3^N - x_3^{-N}} + u^{-1} \frac{x_3^N}{x_3^N - x_3^{-N}} \prod_{k=1}^N R_k + f(x_3, p_0, p_1, p_2, p_3)
\]

for an explicit function \( f(x_3, p_0, p_1, p_2, p_3) \) of \( x_3 \) and of the puncture invariants \( p_i \).
The same argument applied to $T_N(\rho(X_2)) = t_2 \text{Id}_V$ gives that

$$t_2 = -u\frac{1}{x_3^N - x_3^{-N}} + u^{-1}\frac{1}{x_3^N - x_3^{-N}} \prod_{k=1}^{N} R_k + g(x_3, p_0, p_1, p_2, p_3)$$

for another function $g(x_3, p_0, p_1, p_2, p_3)$ of $x_3$ and of the puncture invariants $p_i$.

Then

$$u = t_1 - x_3^N t_2 - f(x_3, p_0, p_1, p_2, p_3) + x_3^N g(x_3, p_0, p_1, p_2, p_3).$$

The combination of Lemmas 18, 23 and 24 then shows that, after isomorphism of $\rho$, the operators $\rho(X_1)$, $\rho(X_2)$ and $\rho(X_3)$ are completely determined by the $t_i$ and $p_j$. We are not able to give expressions as explicit as in Lemma 14 but this is enough to prove Theorem 17.

References

[1] John W. Barrett. Skein spaces and spin structures. Math. Proc. Cambridge Philos. Soc., 126(2):267–275, 1999.
[2] Francis Bonahon and Helen Wong. Representations of the Kauffman skein algebra I: invariants and miraculous cancellations. preprint, arXiv:1206.1638, 2012.
[3] Francis Bonahon and Helen Wong. Representations of the Kauffman skein algebra II: punctured surfaces. preprint, arXiv:1206.1639, 2012.
[4] Francis Bonahon and Helen Wong. Representations of the Kauffman skein algebra III: closed surfaces and canonicity. preprint, 2015.
[5] Doug Bullock. Estimating a skein module with $\text{SL}_2(\mathbb{C})$ characters. Proc. Amer. Math. Soc., 125(6):1835–1839, 1997.
[6] Doug Bullock, Charles Frohman, and Joanna Kania-Bartoszyńska. The Kauffman bracket skein as an algebra of observables. Proc. Amer. Math. Soc., 130(8):2479–2485 (electronic), 2002.
[7] Doug Bullock and Józef H. Przytycki. Multiplicative structure of Kauffman bracket skein module quantizations. Proc. Amer. Math. Soc., 128(3):923–931, 2000.
[8] Miroslav Havlíček and Severin Pošta. On the classification of irreducible finite-dimensional representations of $U'_q(\text{so}_3)$ algebra. J. Math. Phys., 42(1):472–500, 2001.
[9] Józef H. Przytycki and Adam S. Sikora. On skein algebras and $\text{Sl}_2(\mathbb{C})$-character varieties. Topology, 39(1):115–148, 2000.
[10] Vladimir G. Turaev. Skein quantization of Poisson algebras of loops on surfaces. Ann. Sci. École Norm. Sup. (4), 24(6):635–704, 1991.

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