On embeddings of locally finite metric spaces into $\ell_p$

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Abstract

It is known that if finite subsets of a locally finite metric space $M$ admit $C$-bilipschitz embeddings into $\ell_p$ ($1 \leq p \leq \infty$), then for every $\varepsilon > 0$, the space $M$ admits a $(C + \varepsilon)$-bilipschitz embedding into $\ell_p$. The goal of this paper is to show that for $p \neq 2, \infty$ this result is sharp in the sense that $\varepsilon$ cannot be dropped out of its statement.

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1 Introduction and Statement of Results

During the last decades, the study of bilipschitz embeddings of metric spaces into Banach spaces has become a field of intensive research with a great number of applications. The latter are not restricted to the area of Functional Analysis, but also include Graph Theory, Group Theory, and Computer Science. We refer to

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This work is focused on the study of relations between the embeddability into $\ell_p$ of an infinite metric space and its finite pieces. Let us recollect some needed notions.

**Definition 1.1.** A metric space is called *locally finite* if each ball of finite radius in it has finite cardinality.

**Definition 1.2.** Let $(A, d_A)$ and $(Y, d_Y)$ be metric spaces. Given, $1 \leq C < \infty$, a map $f : A \rightarrow Y$, is called a *$C$-bilipschitz embedding* if there exists $r > 0$ such that
\[
\forall u, v \in A \quad rd_A(u, v) \leq d_Y(f(u), f(v)) \leq rCd_A(u, v).
\]

A map $f$ is a *bilipschitz embedding* if it is $C$-bilipschitz for some $1 \leq C < \infty$. The smallest constant $C$ for which there exists $r > 0$ such that (1) is satisfied, is called the *distortion* of $f$.

Unexplained terminology can be found in [14, 23].

It has been known that the bilipschitz embeddability of locally finite metric spaces into Banach spaces is finitely determined in the sense described by the following theorem.

**Theorem 1.3** ([22]). Let $A$ be a locally finite metric space whose finite subsets admit bilipschitz embeddings with uniformly bounded distortions into a Banach space $X$. Then, $A$ also admits a bilipschitz embedding into $X$.

Theorem 1.3 has many predecessors, see [2, 3, 4, 20, 21]. Applications of this theorem to the coarse embeddings important for Geometric Group Theory and Topology are discussed in [22]. To expand on the theme, the argument of [22] yields a stronger result, namely the one stated as Theorem 1.4. In order to formulate Theorem 1.4 it is handy to employ the parameter $D(X)$ of a Banach space $X$ introduced in [19]. Let us recollect its definition. Given a Banach space $X$ and a real number $\alpha \geq 1$, we write:

- $D(X) \leq \alpha$ if, for any locally finite metric space $A$, all finite subsets of which admit bilipschitz embeddings into $X$ with distortions $\leq C$, the space $A$ itself admits a bilipschitz embedding into $X$ with distortion $\leq \alpha \cdot C$;
- $D(X) = \alpha$ if $\alpha$ is the least number for which $D(X) \leq \alpha$;
- $D(X) = \alpha^+$ if, for every $\varepsilon > 0$, the condition $D(X) \leq \alpha + \varepsilon$ holds, while $D(X) \leq \alpha$ does not;
- $D(X) = \infty$ if $D(X) \leq \alpha$ does not hold for any $\alpha < \infty$.

In addition, we use inequalities like $D(X) < \alpha^+$ and $D(X) < \alpha$ with the natural meanings, for example $D(X) < \alpha^+$ indicates that either $D(X) = \beta$ for some $\beta \leq \alpha$ or $D(X) = \beta^+$ for some $\beta < \alpha$. 
Theorem 1.4. There exists an absolute constant $D \in [1, \infty)$, such that for an arbitrary Banach space $X$ the inequality $D(X) \leq D$ holds.

Recently, new estimates of the parameter $D(X)$ for some classes of Banach spaces have been obtained in [19]. Recall that a family of finite-dimensional Banach spaces $\{X_n\}_{n=1}^{\infty}$ is said to be nested if $X_n$ is a proper subspace of $X_{n+1}$ for every $n \in \mathbb{N}$. For such families, an estimate for $D(X)$ from above is expressed by:

**Theorem 1.5 ([19, Theorem 1.9]).** Let $1 \leq p < \infty$. If $\{X_n\}_{n=1}^{\infty}$ is a nested family of finite-dimensional Banach spaces, then $D\left(\bigoplus_{n=1}^{\infty}X_n\right)_p \leq 1^+.$

The next assertion is an immediate consequence of Theorem 1.5:

**Corollary 1.6 ([19, Corollary 1.10]).** If $1 \leq p < \infty$, then $D(\ell_p) \leq 1^+.$

It should be mentioned that the case where $p = \infty$ was discarded because the classical result of Fréchet [8] implies that $D(\ell_\infty) = 1$. Observe also that it is a well-known fact that $D(\ell_2) = 1$. Although the paper [19] contains some estimates for $D(X)$ from below, the following question was left open: whether $D(\ell_p) = 1^+$ or $D(\ell_p) = 1$ for $1 \leq p < \infty$, $p \neq 2$?

The main goal of this paper is to complete the picture by proving that $D(\ell_p) \geq 1^+$ if $p \in [1, \infty)$, $p \neq 2$. See Theorem 1.10 and Corollary 1.8. It is worth pointing out that our proofs for the cases $p = 1$ and $p > 1$ are different from each other.

Recall that a Banach space is called strictly convex if its unit sphere does not contain line segments. In the present work, it is shown that $D(X) > 1$ for a large class of strictly convex Banach spaces $X$ implying that $D(X) = 1^+$ for all strictly convex Banach spaces satisfying the assumption of Theorem 1.5. To be more specific, the following statement will be proved (see Section 2):

**Theorem 1.7.** Let $X$ be a strictly convex Banach space such that all finite subsets of $\ell_2$ admit isometric embeddings into $X$, but $\ell_2$ itself does not admit an isomorphic embedding into $X$. Then $D(X) > 1$.

With the help of Theorem 1.7, one derives:

**Corollary 1.8.** Let $p \in (1, \infty)$, $p \neq 2$. Then every strictly convex Banach space of the form $X = \left(\bigoplus_{n=1}^{\infty}X_n\right)_p$, where $\{X_n\}_{n=1}^{\infty}$ is a nested sequence of finite-dimensional Banach spaces satisfies $D(X) > 1$.

Combining Theorem 1.5 and Corollary 1.8 one obtains:

**Corollary 1.9.** Let $p \in (1, \infty)$, $p \neq 2$, and let $\{X_n\}_{n=1}^{\infty}$ be a nested family of finite dimensional strictly convex Banach spaces. Then, the space $X = \left(\bigoplus_{n=1}^{\infty}X_n\right)_p$ satisfies $D(X) = 1^+$. The equality $D(\ell_p) = 1^+$ for $p \in (1, \infty)$, $p \neq 2$, follows as a special case of this result.

The case $p = 1$ is quite different because $\ell_1$ is not strictly convex. This case is examined in Section 3 where we prove:
Theorem 1.10. $D(\ell_1) > 1$.

Juxtaposing this outcome with Theorem 1.5 we reach:

Corollary 1.11. $D(\ell_1) = 1^+$. 

Remark 1.12. It should be mentioned that the above results are not the first known ones claiming $D(X) > 1$. Before now, results of this kind were obtained in [13, Theorem 2.9] and [19, Theorem 1.12] for some other Banach spaces and their classes.

2 Proof of Theorem 1.7

Prior to presenting the proof of Theorem 1.7 let us provide some auxiliary information. By developing the notion of a linear triple [6, p. 56], we introduce the following:

Definition 2.1. A collection $r = \{r_i\}_{i=1}^n$ of points in a metric space $(A,d_A)$ is called a linear tuple if the sequence $\{d_A(r_i,r_1)\}_{i=1}^n$ is strictly increasing and, for $1 \leq i < j < k \leq n$, the equality below holds:

$$d_A(r_i,r_k) = d_A(r_i,r_j) + d_A(r_j,r_k).$$

(2)

It is not difficult to see that, for strictly convex Banach spaces, the following statement is valid.

Observation 2.2. An isometric image of a linear tuple $r = \{r_i\}_{i=1}^n$ in a strictly convex Banach space is contained in the line segment joining the images of $r_1$ and $r_n$.

For the sequel, the next fact is needed (by $B_Z$ we denote the unit ball of a Banach space $Z$):

Lemma 2.3. Let $Z$ be a finite-dimensional Banach space and $F$ be a Banach space. Then, for each $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon,Z,F) > 0$ such that if a $\delta$-net in $B_Z$ admits an isometric embedding into $F$, then $F$ contains a subspace whose Banach-Mazur distance to $Z$ does not exceed $(1 + \varepsilon)$.

Lemma 2.3 is an immediate consequence of Bourgain’s discretization theorem [7]. It should be emphasized that this theorem provides a much stronger claim because Bourgain found an explicit estimate for $\delta$ as a function of $\varepsilon$ and the dimension of $Z$; besides in Bourgain’s theorem, the distortion of embedding of $Z$ is estimated in terms of distortion of embedding of a $\delta$-net of $B_Z$. See [8, 9] for simplifications of Bourgain’s proof, see also its presentation in [23, Section 9.2]. Meanwhile, the existence of $\delta(\varepsilon,Z,F)$ can be derived from earlier results of Ribe [24] and Heinrich and Mankiewicz [10], see [9, p. 818].

Proof of Theorem 1.7 Denote the unit vector basis of $\ell_2$ by $\{e_i\}_{i=1}^\infty$. Our intention is to find a locally finite subset $M$ of $\ell_2$ in such a way that:
(A) $M$ contains a $\delta(\frac{1}{n}, \ell_2^n, X)$-net $M_n$ of a shifted unit ball $y_n + B_{\ell_2^n}$, $n \in \mathbb{N}$.

(B) There exists a sequence $\{\alpha_i\}_{i=1}^{\infty}$ of positive numbers, such that if $T : M \rightarrow X$ is an isometry satisfying $T(0) = 0$, then the image of $T(M_n)$ is contained in the linear span of $\{T(\alpha_1 e_1), \ldots, T(\alpha_n e_n)\}$.

The existence of such a set $M$ will prove Theorem 17 because, by the assumption of the theorem, finite subsets of $M$ admit isometric embeddings into $X$. On the other hand, $M$ itself does not admit an isometric embedding into $X$. In fact, such an embedding $T$ could be assumed to satisfy $T(0) = 0$. Therefore conditions (A) and (B), combined with Lemma 2.3, would imply that the Banach-Mazur distance between the linear span of $\{T(\alpha_1 e_1), \ldots, T(\alpha_n e_n)\}$ and $\ell_2^n$ does not exceed $(1 + \frac{1}{n})$.

It is well known (see [12]) that, in this case, $X$ contains a subspace isomorphic to $\ell_2$, which is a contradiction.

We let

$$M = \left( \bigcup_{n=1}^{\infty} M_n \right) \cup \{0\},$$

where $M_n$ are finite sets constructed in the way described hereinafter.

Denote by $R_i, i \in \mathbb{N}$, the positive rays generated by $e_i$, that is, $R_i = \{\alpha e_i : \alpha \geq 0\}$. Let $M_1$ be the $\delta(1, \ell_1^n, X)$-net in the line segment $[0, 2e_1]$, where we assume that $M_1$ includes $e_1$. It is clear that $M_1$ satisfies (A).

For $n > 1$ sets $\{M_n\}_{n=1}^{\infty}$ will be constructed inductively. Suppose that we have already created $M_1, \ldots, M_{n-1}$. To construct $M_n$, we pick points $s^n_i \in R_i, 1 \leq i \leq n$, and one more point, $s^n_{n+1} \in R_n$ - so that $R_n$ contains both $s^n_n$ and $s^n_{n+1}$ - in such a way that $\text{conv}(\{s^n_i\}_{i=1}^{n+1})$ is at distance at least $n$ from the origin, and $\text{conv}(\{s^n_i\}_{i=1}^{n+1})$ contains a shift $y_n + B_{\ell_2^n}$ of the unit ball (for some $y_n$). This is clearly possible.

Next, we select a $\delta(\frac{1}{n}, \ell_2^n, X)$-net $N_n$ in this shifted unit ball $y_n + B_{\ell_2^n}$ and include it in $M_n$ together with $\{s^n_i\}_{i=1}^{n+1}$. At this point, it is evident that condition (A) is satisfied.

To ensure that condition (B) is also satisfied - as it will be seen later - we add, for each element $z \in N_n$, finitely many additional elements of $\text{conv}(\{s^n_i\}_{i=1}^{n+1})$ to $M_n$ according to the procedure suggested below:

- If $z \in \{s^n_i\}_{i=1}^{n+1}$, there is nothing to include. If $z \notin \{s^n_i\}_{i=1}^{n+1}$, we find and include in $M_n$ an element $w_1(z)$ in a convex hull of an $n$-element subset $W_1(z)$ of $\{s^n_i\}_{i=1}^{n+1}$ with $z$ being on the line segment joining $w_1(z)$ and $s^n_i \in (\{s^n_i\}_{i=1}^{n+1} \setminus W_1(z))$.

- If $w_1(z) \in \{s^n_i\}_{i=1}^{n+1}$, there is nothing else to include. If $w_1(z) \notin \{s^n_i\}_{i=1}^{n+1}$, we find and include in $M_n$ an element $w_2(z)$ in a convex hull of an $(n - 1)$-element subset $W_2(z)$ of $\{s^n_i\}_{i=1}^{n+1}$ such that $w_1(z)$ is on the line segment joining $w_2(z)$ and $s^n_i \in (\{s^n_i\}_{i=1}^{n+1} \setminus W_2(z))$.

- We continue in an obvious way.
If we do not terminate the process in one of the previous steps, we arrive at the situation when \( w_n(z) \) is in a convex hull of a 2-element subset of \( \{s^n_i\}_{i=1}^{n+1} \), and hence it is on some line segment of the form \([s^n_i, s^n_j]\). At this point we stop.

It has already been stated that condition (A) is satisfied for \( M \). Now, let us verify condition (B). To do this, it suffices to prove that, for each isometry \( T : (M_n \cup \{0\}) \to X \) satisfying \( T(0) = 0 \), the image \( T(M_n) \) is contained in the linear span of \( \{Ts^n_1, \ldots, Ts^n_n\} \). This condition looks slightly different from the one in (B). However, defining \( \{\alpha_i\}_{i=1}^\infty \) by \( \alpha_1 = 1 \) and \( \alpha_i e_i = s^n_i \) one can see that in essence the conditions are equivalent because, by Observation 2.2, the images \( \{Ts^n_i\}_{n=i}^\infty \) are multiples of each other.

To show that \( T(M_n) \) is contained in the linear span \( L \) of \( \{Ts^n_1, \ldots, Ts^n_n\} \), the procedure outlined underneath is applied, where in each step Observation 2.2 is used.

- Since \( 0, s^n_n, \) and \( s^n_{n+1} \) form a linear tuple, and \( T(0) = 0 \), we have \( T(s^n_{n+1}) \in L \).
- Whenever \( w_n(z) \) is defined, one has \( Tw_n(z) \in L \) because \( w_n(z) \in [s^n_i, s^n_j] \).
- Likewise, for each \( z \) such that \( w_{n-1}(z) \) is defined, one obtains \( Tw_{n-1}(z) \in L \) since \( w_{n-1}(z) \) is in the line segment joining \( w_n(z) \) and one of \( s^n_i \).
- We proceed in a straightforward way till we get \( Tz \in L \).

In addition, it is easy to see that the assumption that \( \text{conv}(\{s^n_i\}_{i=1}^{n+1}) \) is at distance at least \( n \) from the origin together with the fact that each set \( M_n \) is finite and is contained in \( \text{conv}(\{s^n_i\}_{i=1}^{n+1}) \), implies that the set \( \bigcup_{n=1}^\infty M_n \) is locally finite.

**Proof of Corollary 1.8.** To check that this \( X \) satisfies the conditions of Theorem 1.7 the two well-known facts come in handy:

1. Each finite subset of \( L_p[0,1] \) admits an isometric embedding into \( \ell_p \), see [1].
2. The space \( L_p[0,1] \) contains a subspace isometric to \( \ell_2 \), see [11, p. 16].

Combining (1) and (2) we conclude that all finite subsets of \( \ell_2 \) are isometric to subsets of \( L_p \), and, thence, to subsets of \( X \). On the other hand, it is known that each infinite-dimensional subspace of \( X \) contains a subspace isomorphic to \( \ell_p \) (this can be done using a slight variation of the argument used to prove [14, Proposition 2.a.2]), and as such it is not isomorphic to \( \ell_2 \).

**Remark 2.4.** The first part of the proof of Theorem 1.7 demonstrates that its statement can be strengthened by replacing the condition “\( \ell_2 \) does not admit an isomorphic embedding into \( X \)” by “there is \( \alpha > 1 \) such that \( X \) does not contain a subspace whose Banach-Mazur distance to \( \ell_2 \) does not exceed \( \alpha \)”. It is known [18] that the latter condition is strictly weaker. In addition, it is not difficult to see that although Joichi did not formally state the pertinent modification of the main result of [12], it arises from the proof.
3 The case of $\ell_1$

Proof of Theorem 1.10. Recall [1] that each finite subset of $L_1(-\infty, \infty)$ admits an isometric embedding into $\ell_1$. To prove Theorem 1.10 we construct in $L_1(-\infty, \infty)$ a locally finite metric space $M$ such that its isometric embeddability into $\ell_1$ would imply that $\ell_1$ contains a unit vector $x$, for every $n \in \mathbb{N}$, can be represented as a sum of $2^n$ vectors with pairwise disjoint supports and of norm $2^{-n}$ each. This leads to a contradiction: consider the maximal in absolute value coordinate of the vector $x$, let it be $\alpha$. If, for some $n \in \mathbb{N}$, $|\alpha| > 2^{-n}$, it is clearly impossible to partition vector into $2^n$ vectors of norm $2^{-n}$ each with pairwise disjoint supports.

The starting point of the construction is the fact that the indicator function $1_{(0,1]}$ has, for each $n \in \mathbb{N}$, a representation as a sum of $2^n$ pairwise disjoint vectors of norm $2^{-n}$. To be specific, we adopt the writing:

- $1_{(0,1]} = d_0 + d_1$, where $d_0 = 1_{(0,\frac{1}{2}], \ell_1} = 1_{(\frac{1}{2},1]}$
- $1_{(0,1]} = d_{00} + d_{01} + d_{10} + d_{11}$, where $d_{00} = 1_{(0,\frac{1}{4]}, d_{01} = 1_{(\frac{1}{4},\frac{1}{2}], d_{10} = 1_{(\frac{1}{2},\frac{3}{4}], d_{11} = 1_{(\frac{3}{4},1]}}$
- We carry on in an obvious way.

In the sequel, the following notation will be employed: let $d = 1_{(0,1]}$ and denote the functions introduced above by $d_\sigma$, where $\sigma$ is a finite string of 0’s and 1’s. Denote by $\ell(\sigma)$ the length of the string $\sigma$. For each $\sigma = \{\sigma_i\}_{i=1}^{\ell(\sigma)}$, the subinterval $I(\sigma)$ of $(0,1]$ is defined by:

$$I(\sigma) = \left( \sum_{i=1}^{\ell(\sigma)} \sigma_i 2^{-i}, \sum_{i=1}^{\ell(\sigma)} \sigma_i 2^{-i} + 2^{-\ell(\sigma)} \right).$$

With this notation $d_\sigma = 1_{I(\sigma)}$ and the mentioned above representation of $1_{(0,1]}$ as a sum of $2^n$ terms can be written as:

$$d = \sum_{\sigma, \ell(\sigma) = n} d_\sigma,$$

where the summands are disjointly supported. Now, denote by $\mathcal{T}$ the set of all finite strings of 0’s and 1’s. It is obvious that $\{d_\sigma\}_{\sigma \in \mathcal{T}}$ is not a locally finite set. Nonetheless, we can add to $\{d_\sigma\}$ pairwise disjoint functions in such a way that a locally finite subset of $L_1(-\infty, \infty)$ will be obtained, and the existence of an isometric embedding of this set into $\ell_1$ would imply the existence in $\ell_1$ of a vector $x$ with the properties described at the beginning of the proof.

First, opt for an injective map $\Psi$ from the collection of all finite strings of 0’s and 1’s into $\mathbb{Z}\backslash\{0\}$.

Now, we consider the locally finite set $M$ satisfying the conditions: It contains both functions $d$ and 0, and, in addition, it includes all sums $f_\sigma := d_\sigma + \ell(\sigma) \cdot 1_{(\Psi(\sigma),\Psi(\sigma)+1]}$, where $\sigma \in \mathcal{T}$. 

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Since $\ell(\sigma)$ is less than any fixed constant only for finitely many strings $\sigma$, this set is a locally finite subset of $L_1(-\infty, \infty)$. It has to be shown that isometric embeddability of this set into $\ell_1$ implies the existence in $\ell_1$ of a vector $x$ with the properties stated in the first paragraph of the proof, thus resulting in a contradiction.

Indeed, if there is an isometric embedding of $M$ into $\ell_1$, then there is an isometric embedding $F$ which maps $0$ to $0$ and - as it will be proved - in such a case $x = Fd$ is the desired vector. More elaborately put, the existence of such an isometric embedding implies that there exist vectors $\{x_\sigma\}_{\sigma \in T}$ so that, for each $n \in \mathbb{N}$, the vectors $\{x_\sigma\}_{\ell(\sigma)=n}$ are disjointly supported, have norm $2^{-n}$, and

$$x = Fd = \sum_{\sigma, \ell(\sigma)=n} x_\sigma.$$ 

Each element $a = \sum_{i=1}^{\infty} a_i c_i$ of $\ell_1$ can be considered as a possibly infinite union of intervals in the coordinate plane which join $(i,0)$ and $(i,a_i)$. The total length of all intervals is equal to $||a||$.

The proposed construction guarantees that if $\ell(\sigma) = n$, then $||f_\sigma - d|| = ||f_\sigma|| + ||d|| - 2 \cdot 2^{-n}$. Since $F$ is an isometry, $F(0) = 0$ and $Fd = x$, this implies $||Ff_\sigma - x|| = ||Ff_\sigma|| + ||x|| - 2 \cdot 2^{-n}$. Consequently, the total length of intersections of the intervals corresponding to $x$ and to $Ff_\sigma$ is $2^{-n}$ for $\sigma, n \in \{0,1\}$.

On the other hand, if $\sigma \neq \tau$ and $\ell(\sigma) = \ell(\tau) = n$, the functions $f_\sigma$ and $f_\tau$ are disjointly supported and, therefore, $||f_\sigma - f_\tau|| = ||f_\sigma - 0|| + ||f_\tau - 0||$. As a result, $||Ff_\sigma - Ff_\tau|| = ||Ff_\sigma|| + ||Ff_\tau||$. This means that the intersections of the intervals corresponding to $Ff_\sigma$ and $Ff_\tau$ have total length $0$. It does not immediately imply that vectors $Ff_\sigma$ and $Ff_\tau$ are disjointly supported: one can imagine, for example, that $Ff_\sigma$ contains the interval joining $(i,0)$ and $(i, \frac{1}{4})$ and $Ff_\tau$ contains the interval joining $(i,0)$ and $(i, -\frac{1}{4})$.

Let us define the vector $x_\sigma$ for $\sigma$ satisfying $\ell(\sigma) = n$ as a vector for which the corresponding intervals are intersections of the intervals corresponding to $x$ and to $Ff_\sigma$. The previous paragraphs imply that $x_\sigma$ and $x_\tau$ satisfy $||x_\sigma|| = ||x_\tau|| = 2^{-n}$ and have disjoint supports when $\ell(\sigma) = \ell(\tau) = n$ and $\sigma \neq \tau$ (for the latter we use the fact that the interval corresponding to $x$ at $i$ can have ‘positive’ or ‘negative’ part, but not both).

Finally, let $s = \sum_{\sigma, \ell(\sigma)=n} x_\sigma$. With the preceding arguments, we conclude that $||s|| = 1$ and $|s_i| \leq |x_i|$ for each $i \in \mathbb{N}$. Thus, $s = x$, and the desired decomposition of $x$ is completed. \qed

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