Minimal Schemes for Large Neutrino Mixings with Inverted Hierarchy

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Abstract

Existing oscillation data point to nonzero neutrino masses with large mixings. We analyze the generic features of the neutrino Majorana mass matrix with inverted hierarchy and construct realistic minimal schemes for the neutrino mass matrix that can explain the large (but not maximal) $\nu_e - \nu_\mu$ mixing of MSW-LAM as well as the nearly maximal $\nu_\mu - \nu_\tau$ mixing and the small (or negligible) $\nu_e \to \nu_\tau$ transition. These minimal schemes are quite unique and turn out to be extremely predictive. Implications for neutrinoless double beta decay, tritium beta decay and cosmology are analyzed.

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1. Introduction

The large, rather than small, neutrino mixings confirmed by atmospheric and solar oscillation experiments \([1, 2]\) over the recent years have brought neutrino physics to an exciting new era. It indicates that lepton flavor mixing is very different from quark flavor mixing, and neutrino mass generation may have a distinct origin from the traditional Dirac-type Yukawa interactions for the charged quarks and leptons in the standard model (SM). In fact, the neutrino masses can be naturally of Majorana nature, generated from either a seesaw mechanism \([3]\) at high scales or a radiative mechanism around the weak scale \([4, 5]\).

The current global fit strongly favors Mikheyev-Smirnov-Wolfenstein Large Angle Mixing (MSW-LAM) in which the solar mixing angle \(\theta_{\odot}\) (giving the \(\nu_e \leftrightarrow \nu_\mu\) transition) is large but significantly deviates from the maximal value 45°, i.e., \(25^\circ \leq \theta_{\odot} \leq 39^\circ\) at 95% C.L. \([4]\), with a central value at \(\theta_{\odot} \simeq 32^\circ\) \([5, 6, 7]\). On the other hand, the atmospheric data indicate a maximal mixing angle \(\theta_{\text{atm}}\) (representing the \(\nu_\mu \rightarrow \nu_\tau\) transition), with the 95% C.L. limit \(33^\circ \leq \theta_{\text{atm}} \leq 57^\circ\) and the central value \(\theta_{\text{atm}} \simeq 45^\circ\) \([5]\). This is also supported by the K2K long baseline experiment \([7]\).

The CHOOZ \([10]\) and Palo Verde \([11]\) long baseline reactor experiments (in combination with the mass range of atmospheric data \([4]\)) bound \(\sin^2 \theta_{\text{CHOOZ}} \lesssim 0.04\) at 95% C.L., where the angle \(\theta_{\text{CHOOZ}}\) measures the \(\nu_e \rightarrow \nu_\tau\) transition. Furthermore, the solar oscillations constrain the mass-square difference \(\Delta_{\odot} = \lvert m_1^2 - m_2^2 \rvert\) to be, \(1.8 \times 10^{-5} \text{eV}^2 \leq \Delta_{\odot} \leq 4.1 \times 10^{-4} \text{eV}^2\), for MSW-LAM at 99% C.L., while the atmospheric oscillations confine the mass-square difference \(\Delta_{\text{atm}} = \lvert m_{12}^2 - m_3^2 \rvert\) as, \(1.3 \times 10^{-3} \text{eV}^2 \leq \Delta_{\text{atm}} \leq 5.5 \times 10^{-3} \text{eV}^2\), at 99% C.L. This suggests two generic patterns for the light neutrino mass-eigenvalues, \((m_1, m_2, m_3) \geq 0\), namely, the “Normal Hierarchy” (called Type-A) and “Inverted Hierarchy” (called Type-B),

\[
A: \quad m_1 < m_2 \ll m_3; \quad B: \quad m_1 \sim m_2 \gg m_3. \quad (1)
\]

Hereafter, we will focus on the Type-B scenario with inverted mass hierarchy. Our present goal is to construct a realistic scheme for the neutrino Majorana mass matrix, containing only a minimal set of parameters to describe the neutrino data, especially the non-maximal solar neutrino mixing à la MSW-LAM (which is hard \([4, 5]\) to realize in models with an approximate \(L_\nu - L_\mu - L_\tau\) symmetry \([12]\)). We show that such a Minimal Scheme can be quite uniquely derived and is highly predictive. Implications for neutrinoless double \(\beta\) decay, tritium \(\beta\) decay, and cosmology are analyzed.

2. Minimal Schemes for Neutrino Majorana Masses with Inverted Hierarchy

Consider the generic \(3 \times 3\) symmetric Majorana mass matrix \(M_\nu\) for 3 light flavor-neutrinos \((\nu_e, \nu_\mu, \nu_\tau)\), at the weak scale and with leptons in the mass-eigenbasis,

\[
M_\nu = \begin{pmatrix} m_{ee} & m_{e\mu} & m_{e\tau} \\ m_{e\mu} & m_{\mu\mu} & m_{\mu\tau} \\ m_{e\tau} & m_{\mu\tau} & m_{\tau\tau} \end{pmatrix}. \quad (2)
\]

With extra new heavy fields integrated out, Eq. (2) is the most general description of the Majorana masses of three active neutrinos based upon Weinberg’s unique dimension-5 effective operator \([13]\),

\[
\frac{C_{ij}}{\Lambda} \sum_i \sum_j L_i^\alpha H^\alpha H^\beta \epsilon^{\alpha\beta} \epsilon_{\alpha\beta},
\]

which gives a mass term, \(\frac{1}{2} \nu^T M_\nu \nu\), with \(M_i^j = C_{ij} \epsilon^{ij}/\Lambda\), where \(\langle H \rangle = 1^\text{The maximal value 45° is also excluded by the 99% C.L. limit of the MSW-LAM solution, }24^\circ \leq \theta_{\odot} \leq 43^\circ\) \([6, 7, 8]\).
\( v/\sqrt{2} \) is the vacuum expectation value of the SM Higgs doublet. The mass matrix \(^2\) contains nine independent real parameters, which can be equivalently chosen as three mass eigenvalues \((m_1, m_2, m_3) \geq 0\), three mixing angles \((\theta_{12}, \theta_{23}, \theta_{13})\), and three CP-violation phases \((\phi, \phi', \phi'')\) with \(\phi\) the usual Dirac phase and \((\phi', \phi'')\) the Majorana phases (which do not affect the neutrino oscillation). The neutrino mixing matrix \(V\) of diagonalizing \(M_\nu\), via \(V^T M_\nu V = M_\nu^{\text{diag}}\), contains six parameters (three rotation angles and three phases) and can be decomposed into a matrix \(U\) (à la Cabibbo-Kobayashi-Maskawa) and a diagonal matrix \(U'\) with only two Majorana phases,

\[
U = \begin{pmatrix}
c_{13} & -s_{13} & -s_3 e^{-i\phi} \\
s_1 c_2 - c_1 s_2 s_3 e^{i\phi} & c_1 c_2 + s_1 s_2 s_3 e^{i\phi} & -s_2 c_3 \\
s_1 s_2 + c_1 c_2 s_3 e^{i\phi} & c_1 s_2 - s_1 c_2 s_3 e^{i\phi} & c_2 c_3
\end{pmatrix}
\]

and \(U' = \text{diag}(1, e^{i\phi'}, e^{i\phi''})\). Here we use the notations \((\theta_1, \theta_2, \theta_3) \equiv (\theta_{12}, \theta_{23}, \theta_{13})\), for convenience. From the mass diagonalization, we can reconstruct the neutrino mass matrix \(M_\nu\) via the relation,

\[
M_\nu = V^* M_\nu^{\text{diag}} V^\dagger,
\]

which gives,

\[
\begin{align*}
m_{ee} &= c_3^2 \left[ c_1^2 m_1 + s_1^2 m_2' \right] + p^2 s_2^2 m_3', \\
m_{\mu\mu} &= s_1 c_2 - \bar{p} c_1 s_2 s_3)^2 m_1 + (c_1 c_2 + \bar{p} s_1 s_2 s_3)^2 m_2' + s_2^2 c_3^2 m_3', \\
m_{\tau\tau} &= (s_1 c_2 + \bar{p} c_1 s_2 s_3)^2 m_1 + (c_1 c_2 - \bar{p} s_1 s_2 s_3)^2 m_2' + c_2^2 c_3^2 m_3', \\
m_{e\mu} &= c_3 \left[ s_1 c_1 c_2 (m_1 - m_2') - \bar{p} s_2 s_3 (c_1^2 m_1 + s_1^2 m_2') + ps_2 s_3 m_3' \right], \\
m_{e\tau} &= c_3 \left[ s_1 c_1 c_2 (m_1 - m_2') + \bar{p} c_2 s_3 (c_1^2 m_1 + s_1^2 m_2') - pc_2 s_3 m_3' \right], \\
m_{\mu\tau} &= (s_1 c_2 + \bar{p} c_1 s_2 s_3)(s_1 c_2 - \bar{p} c_1 s_2 s_3)m_1 + (c_1 c_2 - \bar{p} s_1 s_2 s_3)(c_1 c_2 + \bar{p} s_1 s_2 s_3)m_2' - s_2 c_2 c_3^2 m_3',
\end{align*}
\]

where \((m_2', m_3') \equiv (m_2 e^{-i2\phi'}, m_3 e^{-i2\phi''})\) and \(p = \bar{p}^* \equiv e^{i\phi}\).

The precise form of the neutrino mass matrix \(M_\nu\) in Eq. \((2)\) should be predicted by an appropriate full theory where the mass-mechanism is known. On the other hand, Eq. \((1)\) shows how \(M_\nu\) can be fully reconstructed in terms of nine directly measurable quantities, the mass-eigenvalues, the mixing angles and the CP-phases. Before knowing the underlying full theory, this suggests an important and reliable bottom-up approach, namely, we ask: given the existing neutrino experiments, can we construct a simple, realistic \(M_\nu\) with only a minimal set of input parameters which describes all the oscillation data? To be concrete, we will focus on the Type-B scenario with inverted mass hierarchy in Eq. \((3)\).\(^3\) We will show that such a minimal scheme can be quite uniquely derived and is highly predictive. We can, of course, further extend or elaborate the Minimal Scheme with more fine structure and more input parameters if that is needed to match with an underlying theory (once specified). However, the essential structure of the Minimal Scheme and its capability for describing the existing oscillation data\(^4\) will remain in any realistic extension.

### 2.1. Minimal Scheme of Type-B1

The neutrino mass matrices of inverted hierarchy (Type-B) can be classified into Type-B1 and -B2 \(^7\), which we will analyze in turn. We start from the simplest, naive mass matrix \(M_{\nu,0}\) of

\(^7\)For a very recent analysis of the normal hierarchy (Type-A) via a bottom-up approach, see Ref. \(^8\).

\(^8\)The result from Liquid Scintillation Neutrino Detector (LSND) \(^9\) awaits confirmation by the Fermilab mini-BooNE experiment \(^10\) and will not be considered in the present study.
Type-B1 \[7\],

\[
M_{\nu}[B1] = \frac{\overline{m}_0}{\sqrt{2}} \begin{pmatrix}
0 & 1 & 1 \\
1 & 0 & 0 \\
1 & 0 & 0
\end{pmatrix},
\]

(6)

which generates a mass spectrum \((m_1, m_2, m_3) = (m_1, -m_2', m_3') = \overline{m}_0(1, 1, 0)\) and exact bi-maximal mixing, \(\theta_1 = \theta_2 = 45^\circ\). This simple structure (3) is motivated by \(L_e - L_\mu - L_\tau\) symmetry (It was also shown to be generic for the minimal radiative Zee-model and its various extensions \[4, 5\].) However, (3) is not realistic and is excluded by the solar oscillation data since it predicts \(\Delta_\odot = |m_1^2 - m_2^2| = 0\), and, more seriously, a maximal solar angle \(\theta_\odot = \theta_1 = 45^\circ\) which is difficult to reconcile with the MSW-LAM \[12\]. We observe that such a failure is due to the small but nonzero ratio of the two measured mass-square differences \(\Delta_\odot/\Delta_{\text{atm}} = \mathcal{O}(10^{-1} - 10^{-2})\) and a moderate angular deviation \((45^\circ - \theta_1)/\theta_1 \sim (45^\circ - 32^\circ)/32^\circ \sim 0.4\), for MSW-LAM. Therefore, it is justified to take \(\overline{M}_0\) as our zeroth order mass matrix and build in the necessary Minimal Perturbations to make a realistic Type-B1 neutrino mass-matrix \(M_\nu = \overline{M}_0 + M_\nu\). Such minimal perturbations represent proper \(L_e - L_\mu - L_\tau\) violation effects. What is the minimal set of extra parameters which we need for a realistic perturbation \(\Delta M_\nu\)? First, we need a sizable parameter \(\kappa = \mathcal{O}(0.5)\) to accommodate the solar angular deviation of \((45^\circ - \theta_1)/\theta_1 \sim 0.4\); second, we need a small parameter \(\delta' \sim |m_1 - m_2|/\overline{m}_0 \lesssim \mathcal{O}(0.1)\) to account for the minor mass ratio \(\Delta_\odot/\Delta_{\text{atm}} = \mathcal{O}(10^{-1} - 10^{-2})\); finally, to ensure the Type-B mass spectrum (3) we should impose a condition \(|m_1 - m_2|/\overline{m}_0 \sim m_3/\overline{m}_0 = \mathcal{O}(\delta')\), which can be naturally realized only if we introduce an “interplay” parameter \(\delta\) lying between \(\kappa\) and \(\delta'\). In summary, to construct a realistic perturbation to \(\overline{M}_0\), we have to start with three dimensionless parameters \((\kappa, \delta, \delta')\) satisfying the proper hierarchies,

\[
1 > |\kappa| > |\delta| > |\delta'|, \quad |\kappa| \gg |\delta'|,
\]

\[
m_3/\overline{m}_0 \sim |m_1 - m_2|/\overline{m}_0 = \mathcal{O}(\delta').
\]

With these, we can almost uniquely determine the pattern of the perturbation \(\Delta M_\nu\), and derive the following Minimal Scheme-B1:

\[
M_\nu[B1] = \frac{\overline{m}_0}{\sqrt{2}} \begin{pmatrix}
\kappa & 1 & 1 \\
1 & -\kappa & -\delta \\
1 & -\delta & -\delta'
\end{pmatrix}.
\]

(8)

The relative sign between 11- and 22-entry is uniquely fixed by the requirement \(|m_1 - m_2|/\overline{m}_0 = \mathcal{O}(\delta')\). Note that to affect \(\theta_1\), \(\kappa\) cannot be put in 12- and 21-entry as \(M_\nu\) is symmetric. Another reason to arrange the 11-entry to be of \(\mathcal{O}(\kappa)\) rather than \(\mathcal{O}(\delta, \delta')\) comes from the generic observation about the nature of \(m_{ee}\) by using the Type-B mass spectrum (3) and the general Eq. (3),

\[
m_1 c_3^2 \gtrsim |m_{ee}| \gtrsim m_1 c_3^2 |\cos 2\theta_1| \simeq m_1 |\cos 2\theta_1|,
\]

where the upper [lower] bound corresponds to the CP-conserving values of the Majorana phase \(\phi' = 0\), or, \(\pi \ [\pi/2, \text{ or, } 3\pi/2]\). For solar mixing within the 95\% C.L. range, \(25^\circ \leq \theta_1 \leq 39^\circ\) (MSW-LAM), we deduce the lower limit,

\[
|m_{ee}|/m_1 \approx |m_{ee}|/\overline{m}_0 \gtrsim (0.21 - 0.64),
\]

(10)

\footnote{For some recent non-minimal approaches with certain \(U(1)\) flavor symmetry and additional fields, see Ref. [7].}
so that we can identify \( \sqrt{2}|m_{ee}|/m_0 = \mathcal{O}(\kappa) \). It is important to note that, for general Type-B scenarios, the significant deviation of \( 0.15 \leq (45^\circ - \theta_1)/\theta_1 \leq 0.8 \) for \( 25^\circ \leq \theta_1 \leq 39^\circ \) à la MSW-LAM already requires a sizable \( m_{ee} \) which is potentially observable via 0\(\nu\beta\beta\)-decay experiments \( [22] \), depending on the overall scale \( m_0 \). As will be shown in Sec. 3.1, due to the condition \( [7] \) and the smallness of \( s_2^3 \) \( [10] \), we can reduce the dimensionless inputs \((\kappa, \delta, \delta')\) down to a single parameter \( \kappa \) (or, equivalently; the solar angle \( \theta_\odot \simeq \theta_1 \)). This makes our minimal scheme-B1 highly predictive.

After a scan for all possible variations of the minimal Scheme-B1 under the condition \( [7] \), we find a few other acceptable minimal schemes with \( M_\nu \equiv (m_0/\sqrt{2}) \mathcal{M} \) and \( \mathcal{M} \) given by

\[
\begin{pmatrix}
\kappa & 1 & 1 \\
1 & -\delta' & -\delta \\
1 & -\delta & -\kappa
\end{pmatrix}, \quad
\begin{pmatrix}
\kappa & 1 - \delta' & 1 \\
1 - \delta' & -\kappa & -\delta \\
1 & -\delta & 0
\end{pmatrix}, \quad
\begin{pmatrix}
\kappa & 1 - \delta' & 1 \\
1 - \delta' & 0 & -\delta \\
1 & -\delta & -\kappa
\end{pmatrix}.
\]

(11)

Here the first matrix has the same mass eigenvalues as Eq. (8); its rotation angles \( (\theta_3, \theta_2) \) contain a sign flip for the small \( \mathcal{O}(s_3) \) terms. The third matrix in Eq. (11) is a variation of the first matrix by relocating \( \delta' \); similarly, the second matrix above is constructed by relocating \( \delta' \) from Eq. (8). Hence, Eq. (11) differs from Eq. (8) only by small terms of \( \mathcal{O}(s_3, \delta') \), with no conceptual difference. We will focus on the minimal scheme [8] hereafter. We also note that all the realistic minimal schemes we find for the Type-B can have at most one independent texture zero [cf. the above Eq. (11) and the following Eq. (13) with \( \xi' = 0 \) or \( \xi = 0 \)]. A recent interesting analysis \([19]\) classified viable schemes with two independent texture-zeros, which, as expected, do not contain Type-B schemes.

### 2.2. Minimal Scheme of Type-B2

The naive form of Type-B2 is defined as \([17]\),

\[
M_{\nu^0[B2]} = m_0 \begin{pmatrix}
1 & 0 & 0 \\
0 & 1/2 & 1/2 \\
0 & 1/2 & 1/2
\end{pmatrix},
\]

(12)

which has a Type-B mass-spectrum \((m_1, m_2, m_3) = (m_1, m'_2, m'_3) = m_0(1, 1, 0)\), a maximal mixing angle \( \theta_2 = 45^\circ \), and vanishing \((\theta_1, \theta_3)\). To be realistic, the zeroth order matrix (12) has to be properly perturbed for generating the observed small but nonzero \( \Delta_\odot = |m_1^2 - m_2^2| \) and the large \((\text{rather} \text{ than maximal}) \) mixing \( \theta_1 \) for MSW-LAM. Using the fact of \( m_1 \simeq m_2 \gg m_3 \) and the general relation \([3]\), we find all the perturbations in \( \Delta M_\nu = M_\nu - M_{\nu^0} \) to be of \( \mathcal{O}(m_1 - m_2, m_3, s_2^3m_0) \) or smaller. This suggests a completely different perturbation structure from Type-B1, namely, we use only small perturbation parameters of \( \mathcal{O}(|m_1 - m_2|/m_0, m_3/m_0) \) and the large solar mixing angle \( 25^\circ \leq \theta_1 \leq 39^\circ \) (95\% C.L.) can be naturally generated from an \( \mathcal{O}(1) \) ratio of two small perturbation parameters. Inspecting the structure of \( \Delta M_\nu = M_\nu - M_{\nu^0} \) for Type-B2 and using Eq. (3), we are quite uniquely led to the following Scheme-B2,

\[
M_{\nu[B2]} = m_0 \begin{pmatrix}
1 + \delta & \xi & \xi' \\
\xi & 1/2 & 1/2 \\
\xi' & 1/2 & 1/2
\end{pmatrix},
\]

(13)

where we impose

\[
\frac{m_3}{m_0} \lesssim \frac{|m_1 - m_2|}{m_0} = \mathcal{O}(\delta, \xi, \xi') \ll 1.
\]

(14)
We can define a truly Minimal Scheme-B2 by setting $\xi' = 0$. Choosing $\xi' = \xi$ will give, $\theta_3 = \mathcal{O}(\delta^2, \xi^2, s_3^2, \delta s_3, \xi s_3) \simeq 0$, and is consistent with the Chooz bound \([10]\). But Type-B2 generally has negligible $s_3$ even for nonzero $\xi' \neq \xi$. In Eq. (13), there is no need to perturb the $2 \times 2$ block of $\nu_\mu - \nu_\tau$ as the maximal mixing is favored by the atmospheric data \([1]\); also the $2 \times 2$ block of $\nu_e - \nu_\mu$ invokes two small perturbations so that an $\mathcal{O}(1)$ ratio is generated to explain the non-maximal solar mixing (MSW-LAM). Unlike the Type-B1, the scheme-B2 has larger $m_{ee} = m_0(1 + \delta) \simeq m_0$ at the zeroth order and is more sensitive to the $0\nu\beta\beta$ experiments.

### 3. Analysis of the Minimal Schemes and Predictions for Neutrino Oscillation

In this section, we systematically solve the diagonalization equations in (5) for the Minimal Scheme-B1 (8) and -B2 (13) [$\xi' = 0$] with CP-conservation. We then study their predictions for the neutrino oscillations.

#### 3.1. Analyzing the Minimal Scheme of Type-B1

The parameters ($\kappa, \delta$) will be retained up to all orders without approximation. But, from the solar and Chooz oscillation data, it is justified to treat the small parameters ($\delta', s_3$) as perturbations to first power and ignore terms of $\mathcal{O}(\delta^2, s_3^2) \lesssim \mathcal{O}(10^{-2})$ or smaller. As will be shown below, the expansion of $s_3$ also plays a key role for eliminating $\delta'$ from inputs.

From Eq. (8), we deduce the mass-eigenvalues of $M_\nu$, up to $\mathcal{O}(\delta')$,

$$m_{1,2} = m_0 \left[ 1 \pm \frac{1}{2} \left( 1 - \frac{x}{2 + \omega} \right) \delta' \right], \quad m_3 = m_0 \frac{x}{2 + \omega} \delta',$$

where we expand $\delta'$ to first order and define,

$$\kappa \equiv \frac{\kappa}{\omega}, \quad \delta' \equiv \frac{\delta}{\omega}, \quad \delta \equiv \sqrt{2 + \omega} = \sqrt{\frac{2}{1 - \kappa^2 - \delta'^2}},$$

$$\omega \equiv \kappa^2 + \delta^2 = \frac{2(\kappa^2 + \delta^2)}{1 - (\kappa^2 + \delta^2)}, \quad m_0 \equiv \frac{m_{0}}{\omega} \sqrt{1 + \frac{\omega}{2}}.$$ (16)

The parameter $x = \mathcal{O}(1)$ in Eq. (15) will be determined by the consistency condition,

$$(1 - \delta^2) \kappa - 2\delta + (1 + \kappa^2) \delta' = x \delta',$$

due to the requirement $m_3/m_0 = \mathcal{O}(\delta')$ in Eq. (6). With the definition of Eq. (16), we can rewrite the neutrino mass matrix (8) scaled by $m_0$,

$$M_\nu[B1] = m_0 \begin{pmatrix} \kappa & \omega^{-1} & \omega^{-1} \\ \omega^{-1} & -\kappa & -\delta \\ \omega^{-1} & -\delta & -\delta' \end{pmatrix}.$$ (18)

From Eq. (15), we deduce

$$\frac{\Delta_\odot}{\Delta_{\text{atm}}} = \frac{|m_1^2 - m_2^2|}{|m_{1,2}^2 - m_3^2|} \simeq 2 \left[ 1 - \frac{x}{2 + \omega} \right] \delta'.$$

(19)

With the mass-eigenvalues given, we can then solve for the mixing angles by substituting $M_\nu$ [cf. Eq. (8) or Eq. (18)] into the six diagonalization equations in (5) and expanding ($\delta'$, $s_3$) systematically.
to first order. At this order, we find that only five of the six equations are independent. Note that we have three dimensionless parameters ($\kappa$, $\delta$, $\delta'$) in $M_{\nu}$ (in which the overall scale $m_{0}$ is irrelevant to the diagonalization) and three mixing angles ($\theta_1$, $\theta_2$, $\theta_3$). Hence, from the five equations, we can solve five out of the six parameters as functions of a single dimensionless input parameter which will be chosen as the angle $\theta_1$ (measured in the solar oscillation). To explicitly understand this nontrivial reduction of input parameters, we first note that even though we have three dimensionless inputs ($\kappa$, $\delta$, $\delta'$) in (8), the condition $m_3/m_0 = O(\delta')$ in [2] [or, (17)] relates $\kappa$ and $\delta$ at zeroth order of $\delta'$ so that only two inputs among ($\kappa$, $\delta$, $\delta'$) are independent under the expansion of $\delta'$. Then, we summarize two relevant relations derived from Eq. (5) [and Eq. (18)],

\[ \delta = \frac{1}{2} \kappa, \quad \delta' = \frac{\kappa}{4(1-r)}, \]

\[ \theta_2 = \frac{\pi}{4} - \frac{3-4r}{8(1-r)} \kappa^2, \quad \theta_3 \simeq s_3 = \frac{3-4r}{8(1-r)} \kappa \sqrt{1-\kappa^2}, \]

with $r \equiv \frac{x}{2 + \omega}$. Now we see that the absence of $O(s_3)$ term in Eq. (20) and the smallness of $s_3^2 (\ll 0.04 [14])$ lead us to have three constraints [two in Eq. (20) and one in Eq. (17)] among the four parameters ($\kappa$, $\delta$, $\delta'$, $x$). This feature remains if we include higher order terms via iteration. This makes our scheme end up with a single input for all mixings and thus extremely predictive. After a lengthy and careful derivation, we arrive at the following complete set of solutions of our Minimal Scheme-B1, up to $O(\delta', s_3)$,

\[ \kappa = \frac{8}{9} \cos 2\theta_1, \quad \delta = \frac{1}{2} \kappa, \quad \delta' = \frac{\kappa}{4(1-r)}, \]

\[ \theta_2 = \frac{\pi}{4} - \frac{3-4r}{8(1-r)} \kappa^2, \quad \theta_3 \simeq s_3 = \frac{3-4r}{8(1-r)} \kappa \sqrt{1-\kappa^2}, \]

where we have,

\[ 2 + \omega = \frac{2}{1-\frac{4}{3} \kappa^2}, \quad r = \frac{1-\frac{5}{4} \kappa^2}{2(1-\kappa^2)}, \quad x = \frac{1}{1-\kappa^2}. \]

Finally, inputting the solar angle $\theta_C (\simeq \theta_1)$, we deduce the following numerical predictions,

\[ 25^\circ \leq \theta_1 \leq 39^\circ, \quad [\text{Input of MSW–LAM, 95\%C.L.}]; \]

\[ 39.8^\circ \leq \theta_2 \leq 44.5^\circ, \quad 0.13 \leq s_3 \simeq \theta_3 \leq 0.046, \]

\[ 0.57 \geq \kappa \geq 0.19, \quad 0.29 \geq \delta \geq 0.092, \quad 0.25 \geq \delta' \geq 0.091; \]

\[ 0.28 \geq \frac{\Delta_{\odot}}{\Delta_{\text{atm}}} \geq 0.095, \]

and also $0.44 \leq r \leq 0.50, \quad 1.48 \geq x \geq 1.01$. The results in Eq. (23) agree well with the oscillation data [12, 13, 14, 15], i.e., $33^\circ \leq \theta_2 \simeq \theta_{\text{atm}} \leq 57^\circ$ and $s_3 = \sin \theta_{\text{cha}} \lesssim 0.2$ at 95\% C.L., and $4.2 [3.3] \times 10^{-3} \leq \Delta_{\odot}/\Delta_{\text{atm}} \leq 0.17 [0.32]$ at 95\%[99\%] C.L. The on-going KamLAND experiment [23] will more precisely test the MSW-LAM parameter space (though it will not be sensitive to $s_3$ [23]). We have checked the numerical accuracy of the above solutions by substituting Eq. (23) back into Eqs. (8) and (18) and evaluating the difference of the two sides in each equation. We find that the difference (uncertainty) is always less than $0.015 (0.0009)$ for $\theta_1 = 25^\circ (39^\circ)$. Thus, our systematical expansion works well up to $O(\delta', s_3)$, as expected, since the ignored terms are of $O(\delta'^2, s_3^2)$ and
become smaller for larger $\theta_1$ as shown in Eq. (23). For Type-B schemes, the mass scale $m_0$ is generally bounded by

$$0.036 \text{ eV} \leq m_0 \simeq m_{1,2} \simeq \Delta^{1/2}_{\text{atm}} \leq 0.074 \text{ eV}, \quad [99\% \text{ C.L.}].$$ \hspace{1cm} (24)

### 3.2. Analyzing the Minimal Scheme of Type-B2

We now turn to the minimal scheme-B2 in (13) with $\xi' = 0$, which has the mass-eigenvalues, up to $O(\delta, \xi)$,

$$m_{1,2} = m_0 \left[ 1 + \frac{\delta}{2} \pm \frac{1}{2} \sqrt{\delta^2 + 2\xi^2} \right], \quad m_3 = 0. \hspace{1cm} (25)$$

Substituting (13) into (5), expanding up to $O(\delta, \xi, s_3)$ and using (25), we derive the solutions,

$$\theta_1 = \frac{\pi}{4} - \frac{1}{2} \arcsin \frac{\delta}{\sqrt{\delta^2 + 2\xi^2}}, \quad \theta_2 = \frac{\pi}{4}, \quad \theta_3 = -\frac{\xi}{\sqrt{2}}, \quad (\delta, \xi) > 0;$$

and

$$\frac{\Delta_\odot}{\Delta_{\text{atm}}} = \frac{|m_1^2 - m_2^2|}{|m_{1,2}^2 - m_3^2|} = 2\sqrt{\delta^2 + 2\xi^2}. \hspace{1cm} (26)$$

We see that $\theta_2$ is maximal at this order. The sizable deviation of $\theta_1 - \frac{\pi}{4}$ is indeed naturally generated by an $O(1)$ ratio of two small parameters $(\delta, \xi) \ll 1$. [Allowing $\xi' \neq 0$, the corresponding formulas for Eqs. (23)-(24) can be directly obtained by the simple replacements, $\xi \to \xi - \xi'$ for $\theta_3$ and $\xi \to \xi + \xi'$ for all other quantities.] Using the inputs for LAM [6, 7, 8], $25^\circ \leq \theta_\odot \simeq \theta_1 \leq 39^\circ$ and $4.2 \times 10^{-3} \leq \Delta_\odot / \Delta_{\text{atm}} \leq 0.17$ at 95% C.L., we deduce,

$$1.2 \geq \delta/\xi \geq 0.3, \quad 1.1 \times 10^{-3} \leq \xi \leq 0.06. \hspace{1cm} (27)$$

Thus we have

$$8 \times 10^{-4} \leq -\theta_3 \leq 0.04. \hspace{1cm} (28)$$

The mass scale $m_0 \simeq \Delta^{1/2}_{\text{atm}}$ is bounded as in Eq. (24). As mentioned above, allowing nonzero $\xi' = O(\xi)$, we can derive,

$$s_3 \simeq \theta_3 = \frac{\xi' - \xi}{\sqrt{2}}, \hspace{1cm} (29)$$

which remains of the same order. Hence, $|\theta_3| \lesssim O(10^{-2})$ generally holds for Type-B2, implying negligible CP-violation from the Dirac phase $\phi$. It has been shown [24] that combining the data from two near-future long baseline accelerator experiments [23], the Main Injector Neutrino Oscillation Search (MINOS) and the Imaging Cosmic And Rare Underground Signals (ICARUS), may place a 95% C.L. lower bound, $\theta_3 \geq O(0.05)$ (when $\theta_3$ lies within their combined sensitivity), which could possibly discriminate Type-B2 from Type-B1.

### 4. Implications for Neutrinoless Double $\beta$-Decay, Tritium $\beta$-Decay and Cosmology

The oscillation data may already give a strong hint on the neutrino mass scale [cf. Eq. (24)] so long as the neutrino masses exhibit the hierarchy structure [cf. Eq. (1)], but the possibility of three nearly degenerate neutrinos ($m_1 \sim m_2 \sim m_3$) could allow a higher scale. Hence, the laboratory experiment on neutrinoless double $\beta$-decay ($0\nu\beta\beta$) [23, 26, 28] is indispensable to pin
down the absolute mass scale, as well as the Majorana nature of active neutrinos. For the Minimal Scheme-B1 and -B2, we have,

\[ m_{ee}[B1] = \kappa m_0, \quad m_{ee}[B2] \simeq m_0. \]  

(30)

Thus, using Eqs. (30) and (24), we derive, at 99% C.L.,

\[ 0.014 \text{ eV} \leq m_{ee}[B1] \leq 0.029 \text{ eV}, \quad 0.036 \text{ eV} \leq m_{ee}[B2] \leq 0.074 \text{ eV}, \]  

(31)

where we input the central value of solar fit (LAM), \( \theta_1 \simeq 32^\circ \), for Type-B1. The \( m_{ee}[B2] \) is already sensitive to the current \( 0\nu\beta\beta \) measurement \[27\]. The experiments of \( 0\nu\beta\beta \) decay \[27\], such as the on-going NEMO3 and the upcoming CUORE, can probe \( |m_{ee}| \sim 0.1 \text{ eV} \), while the near-future measurements at GENIUS, EXO, MAJORANA and MOON aim at a sensitivity of \( |m_{ee}| \sim 0.01 \text{ eV} \), which is decisive for testing the whole mass range \([21]\) of Type-B1 and -B2 schemes.

Tritium \( \beta \)-decay requires \[29\], \( m_{\nu_e} < 2.2 \text{ eV} \), at 95% C.L., where \( m_{\nu_e} \equiv (M_1 M_2)^{1/2} \simeq m_{1,2} \simeq m_0 \) for Type-B. This is well above the range given in Eq. (24). The sensitivity of \( H^3 \beta \)-decay could eventually reach \( m_{\nu_e} \sim 0.5 \text{ eV} \) \[30\].

The latest cosmology measurements of the power spectrum for the Cosmic Microwave Background (CMB), Galaxy Clustering and Lyman Alpha Forest \[31\] put a 95% C.L. upper bound on the neutrino masses, \( \sum_j m_j \leq 4.2 \text{ eV} \). This gives, for our Type-B schemes, \( m_1 \simeq m_2 \simeq m_0 \leq 2.1 \text{ eV} \), which is about the same as the tritium \( \beta \)-decay bound. The newest analysis \[32\] from the 2dF Galaxy Redshift Survey arrives at an upper bound, \( \sum_j m_j \leq 2.2 \text{ eV} \), which results in, \( m_1 \simeq m_2 \simeq m_0 \leq 1.1 \text{ eV} \), for Type-B schemes. Stronger constraints of \( \sum_j m_j \lesssim 0.4 \text{ eV} \) are expected from the forthcoming Microwave Anisotropy Probe (MAP) and PLANCK satellite experiments \[33, 34\].

It is interesting to note that the neutrino mass scale may also be determined from the so-called \( Z \)-bursts \[35\] due to the resonant annihilation of ultra high energy neutrinos with cosmological relic (anti-)neutrinos into \( Z \) bosons (whose decay produces protons and photons). In the most plausible case where the ordinary cosmic rays are protons of extragalactic origin, the required neutrino mass range is \[36\],

\[ 0.01 [0.02] \text{ eV} \leq m_{\nu}(\text{heaviest}) \leq 3.0 [2.1] \text{ eV}, \quad \text{at 99% C.L. [95% C.L]}, \]  

(32)

which is compatible with the neutrino oscillation bound \[24\] for the inverted mass hierarchy.

5. Conclusions

In this study, we have considered two essential and distinct scenarios for the neutrino Majorana mass matrix with inverted hierarchy, called Type-B1 [cf. Eq. (1)] and -B2 [cf. Eq. (12)]. For Type-B1, we start with the form of Eq. (6) at zeroth order and perturb it into the realistic form of Eq. (8) with three parameters \( (\kappa, \delta, \delta') \) under the hierarchy (1) that is necessary to correctly predict the oscillation data, especially, the non-maximal solar neutrino mixing of MSW-LAM. The sizes of \( \delta' \sim |m_1 - m_2| \geq m_0 \Delta_\odot / \Delta_{\text{atm}} \ll 1 \) \[2\] and \( s_3 = \sin \theta_{\text{cha}} \ll 1 \) \[10\] justify the expansion of \( (\delta', s_3) \), which enables us to reduce the number of inputs down to a single parameter \( \kappa \), or, equivalently, \( \theta_\odot (\simeq \theta_1) \). Thus, using only the measured solar angle \( \theta_\odot \) as input, we predict the atmospheric mixing angle, \( \theta_{\text{atm}} (\simeq \theta_2) \), the value of \( \theta_{\text{cha}} (\simeq \theta_3) \), and the mass ratio \( \Delta_\odot / \Delta_{\text{atm}} \), in complete agreement with the existing data. We also note that the Minimal Scheme-B1 in Eq. (8) points to a generic way for
naturally extending the minimal Zee-model [4] in which $M_{\nu}$ exhibits the following structure,

$$M^{\text{Zee}}_{\nu} = \begin{pmatrix} 0 & m_{e\mu} & m_{e\tau} \\ m_{e\mu} & 0 & m_{\mu\tau} \\ m_{e\tau} & m_{\mu\tau} & 0 \end{pmatrix},$$

(33)

where the pattern $m_{e\mu} \simeq m_{e\tau} \gg m_{\mu\tau}$ can be realized [20], which ensures an approximate $L_e - L_\mu - L_\tau$ symmetry. The necessity of modifying Eq. (33) in the minimal Zee-model for accommodating the MSW-LAM was noted recently [12]. Our minimal construction of Scheme-B1 in Eq. (8) demonstrates a generic way to extend $M^{\text{Zee}}_{\nu}$ under an appropriate perturbation [cf. Eq. (7)]. The choice of $m_{ee} = (m_0/\sqrt{2})\kappa = m_0 \kappa$ is due to the general observation in Eq. (9) for Type-B1 and a sizable $\cos 2\theta_1 \in (0.21 - 0.64)$ for $25^\circ \leq \theta_1 \leq 39^\circ$ (95% C.L., MSW-LAM); while $m_{\mu\mu} \simeq -m_{ee}$ is enforced by the Type-B mass-spectrum, $m_{1,2} \gg |m_1 - m_2| \sim m_3$. The $\kappa$ terms in $M_{\nu}[B1]$ represent the generic leading modification to the minimal Zee-model (33).

For Type-B2, we start with the leading order mass matrix (12) and find the perturbation structure in Eq. (13) based on the general relations in Eq. (5) and the smallness of $\theta_{\text{cha}}(= \theta_3)$. The realistic Minimal Scheme-B2 contains only two small parameters ($\delta, \xi \ll 1$, defined as in Eq. (13), when $\xi' = 0$. In contrast to Typy-B1, the non-maximal solar mixing angle $\theta_3$ is naturally accommodated by a ratio of two small parameters, $\delta/\xi = O(1)$, while the atmospheric mixing angle remains maximal. Using the measured values of solar angle $\theta_3$ and mass ratio $\Delta_{\text{atm}}$, we derive the ranges for $\theta_3(= \theta_{\text{cha}})$ and the perturbation parameters ($\delta, \xi$). The angle $\theta_3$ is found to be of $O(10^{-2})$ or smaller. Combining the data from both MINOS and ICARUS experiments [23] may reach the sensitivity [24] to discriminate between the minimal Type-B1 and Type-B2 schemes.

The overall neutrino mass scale $m_0$ for the inverted mass hierarchy is quite uniquely fixed by the atmospheric neutrino data on the mass-squared difference $\Delta_{\text{atm}} = |m_{1,2}^2 - m_3^2| \simeq m_{1,2}^2 \simeq m_0^2$ [cf. Eq. (24)]. Thus, the mass matrix of our minimal scheme-B1 or -B2 is known and highly predictive. Some implications of the Type-B1 and -B2 minimal schemes for the neutrinoless double $\beta$-decay, tritium $\beta$-decay and cosmology are given above.

Note Added: After the submission of this work, a new announcement [37] from the Sudbury Neutrino Observatory (SNO) collaboration appeared on April 20, 2002, which further confirms the MSW-LAM as the best solution to the solar neutrino oscillations.

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