WEIGHTED ESTIMATES FOR THE BERGMAN PROJECTION ON THE HARTOGS TRIANGLE

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Abstract. We apply modern techniques of dyadic harmonic analysis to obtain sharp estimates for the Bergman projection in weighted Bergman spaces. Our main theorem focuses on the Bergman projection on the Hartogs triangle. The estimates of the operator norm are in terms of a Bekollé-Bonami type constant. As an application of the results obtained, we give, for example, an upper bound for the $L^p$ norm of the Bergman projection on the generalized Hartogs triangle $\mathbb{H}_{m/n}$ in $\mathbb{C}^2$.

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1. Introduction

Let $\Omega \subseteq \mathbb{C}^n$ be a bounded domain. Let $L^2(\Omega)$ denote the space of square-integrable functions with respect to the Lebesgue measure $dV$ on $\Omega$. Let $A^2(\Omega)$ denote the subspace of square-integrable holomorphic functions. The Bergman projection $P$ is the orthogonal projection from $L^2(\Omega)$ onto $A^2(\Omega)$. Associated with $P$, there is a unique function $K_{\Omega}$ on $\Omega \times \Omega$ such that for any $f \in L^2(\Omega)$:

$$P(f)(z) = \int_{\Omega} K_{\Omega}(z; \bar{w}) f(w) dV(w). \quad (1.1)$$

Let $P^+$ denote the positive Bergman projection defined by:

$$P^+(f)(z) := \int_{\Omega} |K_{\Omega}(z; \bar{w})| f(w) dV(w). \quad (1.2)$$

A question of importance in analytic function theory and harmonic analysis is to understand the boundedness of $P$ or $P^+$ on the space $L^p(\Omega, \mu dV)$, where $\mu$ is some non-negative locally integrable function on $\Omega$.

For the unweighted case ($\mu \equiv 1$), the $L^p$ boundedness for the Bergman projection have been studied in various settings. On a wide class of domains, the Bergman projection is $L^p$ regularity for all $1 < p < \infty$. See for instance [Pec74, PS77, McN89, McN94a, NRSW88, McN94b, MS94, CD06, EL08, BS12]. In all these results, the domain needs to satisfy certain boundary conditions. On some other domains, the projection has only a finite range of mapping regularity. See for example [Zey13, CZ16, EM16, EM17, Che17a]. One important example is the Hartogs triangle $\mathbb{H}$. In [CZ16], Chakrabarti and Zeytuncu showed that the Bergman projection on the Hartogs triangle is $L^p$-regular if and only if $\frac{4}{3} < p < 4$.

Less is known about the situation when the weight $\mu \not\equiv 1$, and results and progress depend upon the domains being studied. For the case of the unit ball in $\mathbb{C}^n$, the boundedness of $P$
and $P^+$ in the weighted $L^p$ space was studied by Bekollé and Bonami in [BB78] and [Bek82].

Let $T_z$ denote the Carleson tent over $z$ in the unit ball $B_n$ defined as below:

- $T_z := \{ w \in B_n : |1 - \frac{\bar{w}}{|z|}| < 1 - |z| \}$ for $z \neq 0$, and
- $T_z := B_n$ for $z = 0$.

Then the result of Bekollé and Bonami can be stated as follows:

**Theorem 1.1.** (Bekollé-Bonami) Let the weight $\mu(w)$ be a positive, locally integrable function on the unit ball $B_n$. Let $1 < p < \infty$. Then the following conditions are equivalent:

1. $P : L^p(B_n, \mu) \mapsto L^p(B_n, \mu)$ is bounded.
2. $P^+ : L^p(B_n, \mu) \mapsto L^p(B_n, \mu)$ is bounded.
3. The Bekollé-Bonami constant

$$B_p(\mu) := \sup_{z \in B_n} \frac{f_{T_z}(\mu \| \cdot \|_p(w) dV(w))}{f_{T_z}(\mu dV(w))^{p-1}}$$

is finite.

Motivated by recent developments on the $A_2$-Conjecture [Hyt12] for singular integrals in the setting of Muckenhoupt weighted $L^p$ spaces, people have made progress on the dependence of the operator norm $\|P\|_{L^p(B_n, \mu)}$ on $B_p(\mu)$. In [PR13], Pott and Reguera gave a weighted $L^p$ estimate for the Bergman projection on the upper half plane. Their estimates are in terms of the Bekollé-Bonami constant and the upper bound estimate is sharp. Later, Rahm, Tchoundja, and Wick [RTW17] generalized the results of Pott and Reguera to the unit ball case, and also obtained sharp estimates for the Berezin transform.

The purpose of this paper is to establish sharp weighted inequalities for the Bergman projection on the Hartogs triangle. We give a Bekollé-Bonami type constant and obtain weighted $L^p$-norm estimates for $P$ and $P^+$. Recall that the Hartogs triangle $\mathbb{H}$ is defined by

$$\mathbb{H} = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < |z_2| < 1\}.$$ 

It is known that the kernel $K(z_1, z_2; \bar{w}_1, \bar{w}_2)$ has the following form:

$$K_{\mathbb{H}}(z_1, z_2; \bar{w}_1, \bar{w}_2) = \frac{1}{\pi^2 z_1^{2} \bar{w}_2^{1}(1 - \frac{z_1}{z_2})^{2}(1 - z_2 \bar{w}_2)^{2}}.$$ 

Given functions of several variables $f$ and $g$, we use $f \lesssim g$ to denote that $f \leq C g$ for a constant $C$. If $f \lesssim g$ and $g \lesssim f$, then we say $f$ is comparable to $g$ and write $f \approx g$.

The main result obtained in this paper is:

**Theorem 1.2.** Let $1 < p < \infty$, and $p'$ denote the Hölder conjugate to $p$. Let $\mu$ be a positive, locally integrable weight on $\mathbb{H}$ of the form

$$\mu(z_1, z_2) = \mu_1(z_1/z_2) \mu_2(z_2).$$

Set $\nu = |z_2|^{-p'} \mu \frac{1}{|z_2|}$. Then the Bergman projection $P$ is bounded on the weighted function space $L^p(\mathbb{H}, \nu dV)$ if and only if $[\mu, \nu]_p < \infty$. Moreover,

$$[\mu, \nu]_p^{\frac{1}{p}} \lesssim \|P\|_{L^p(\mathbb{H}, \nu dV)} \lesssim \|P^+\|_{L^p(\mathbb{H}, \nu dV)} \lesssim (pp')^{\frac{1}{2}} [\mu, \nu]_p^{\frac{1}{p}}$$

(1.3)

For the definitions of the Bekollé-Bonami constant $[\mu, \nu]_p$ see Section 2.
There has been some recent interest in analyzing the $L^p$ regularity properties of the projection via characteristics of the weight. In [Che17b], Chen considered an $A^+_p$ condition, which is equivalent to the Bekollé-Bonami condition in the upper half plane setting, and obtained the $L^p$ regularity of the weighted Bergman projection with some special weights on the Hartogs triangle. Using the $A^+_p$ condition, Chen, Krantz, and Yuan [CKY19] obtained the $L^p$ regularity results for the Bergman projections on domains covered by the polydisc through a rational proper holomorphic map. The result of Chakrabarti and Zeytuncu in [CZ16] can be recovered from [Che17b] by showing that the $A^+_p$ constant of the weight $\mu \equiv 1$ blows up for $p \not\in (\frac{4}{3}, 4)$. Similarly, Theorem 1.2 provide another proof for this result.

The approach we employ in this paper is similar to the ones in [PR13] and [RTW17]. The lower bound estimate follows from a weak-type inequality argument. To obtain the upper bound estimate, we show that $P$ and $P^+$ are controlled by a positive dyadic operator. Then an analysis on the weighted $L^p$ norm of the dyadic operator yields the desired estimate. Here we use harmonic analysis strategy from [Moe12] and [Lac17]. Our upper bound is sharp. In Section 4.1, we provide an example of weights and functions where the sharp bound is attained. As applications of our results, we recover the $L^p$-regularity results in [CZ16] and [ENL17] and give upper bound estimates for the $L^p$-norm of the Bergman projections on the Hartogs triangle $\mathbb{H}$ and the generalized Hartogs triangle $\mathbb{H}_{m/n}$. See Sections 4.2 and 4.4. It is worth noting that the construction of the positive dyadic operator relies on a dyadic structure on the unit disc where the measure of the set in the structure can be used to estimate the Bergman kernel function. Since the dyadic structures on the disc $\mathbb{D}$ and the ball $\mathbb{B}_n$ are well understood, the approach we use in this paper can also be applied to the setting where the domain is related to the unit disc or ball, such as the polydisc, the product of unit balls, and domains that are biholomorphically equivalent to them.

The paper is organized as follows: In Section 2, we introduce a dyadic structure on the unit disc and a corresponding structure on the Hartogs triangle and provide the results that will be used throughout the paper. In Section 3, we present the dyadic operator $Q^+_{m,n,v}$ and prove Theorem 1.2. In Section 4, we give a sharp example for our upper bound estimate. We also provide some examples where the upper bound estimates can be explicitly computed. In Section 5, we make several remarks and possible directions for generalization.

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2. Preliminaries

Let $\mathbb{D}$ denote the unit disc in $\mathbb{C}$. Let $\mathbb{D}^*$ denote the punctured disc $\mathbb{D}\setminus\{0\}$. The Hartogs triangle $\mathbb{H}$ is defined by

$$\mathbb{H} = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < |z_2| < 1\}.$$ (2.1)
Note that the mapping \((z_1, z_2) \mapsto (\overline{z}_1, z_2)\) is a biholomorphism from \(\mathbb{H}\) onto \(\mathbb{D} \times \mathbb{D}^*\). The biholomorphic transformation formula (see [Kra01]) then implies that
\[
K_\mathbb{H}(z_1, z_2; \overline{w}_1, \overline{w}_2) = \frac{1}{z_2 \overline{w}_2} K_{\mathbb{D} \times \mathbb{D}^*}(\frac{z_1}{z_2}, \frac{\overline{w}_1}{\overline{w}_2}, \frac{\overline{w}_2}{\overline{w}_2}) = \frac{1}{z_2 \overline{w}_2} K_{\mathbb{D} \times \mathbb{D}^*}(\frac{z_1}{z_2}, \frac{\overline{w}_1}{\overline{w}_2}, \frac{\overline{w}_2}{\overline{w}_2}) = \pi^2 z_2 \overline{w}_2 (1 - \frac{\overline{w}_1}{z_2 \overline{w}_2})^2 (1 - z_2 \overline{w}_2)^2.
\]
Hence, the Bergman projection \(P\) and the absolute Bergman projection \(P^+\) on the Hartogs triangle can be expressed as follows
\[
P(f)(z) = \int_\mathbb{H} \frac{f(w)}{\pi^2 z_2 \overline{w}_2 (1 - \frac{\overline{w}_1}{z_2 \overline{w}_2})^2 (1 - z_2 \overline{w}_2)^2} dV(w); \tag{2.3}
\]
\[
P^+(f)(z) = \int_\mathbb{H} \frac{f(w)}{\pi^2 |z_2 \overline{w}_2| (1 - \frac{\overline{w}_1}{z_2 \overline{w}_2}) (1 - z_2 \overline{w}_2)^2} dV(w). \tag{2.4}
\]
We next introduce a dyadic structure on the unit disk. A related construction appears in [ARS06]. Let \(D = \{D^k\}\) be a dyadic system on the unit circle with
\[
D^k_j = \{e^{2\pi i \theta} : (j - 1)2^{-k} \leq \theta < j2^{-k}\}, \quad \text{for } j = 1, \ldots, 2^k.
\]
Let \(d(\cdot, \cdot)\) denote the Bergman metric on the unit disc \(\mathbb{D}\). For \(z \in \mathbb{D}\), let \(B(z, r)\) denote the ball centered at point \(z\) with radius \(r\) under this metric. Set \(r = 2^{-1}\ln 2\). For \(k \in \mathbb{N}\), let \(S_{kr}\) denote the circle centered at the origin with radius \(kr\) in the Bergman metric. Let \(P_{kr}z\) be the radial projection of \(z\) onto the sphere \(S_{Nr}\). By the proof of [RTW17, Lemma 9], \(\{P_{kr}D^k_j\}\) satisfy the following three properties:

1. \(S_{kr} = \bigcup_{j=1}^{2^k} P_{kr}D^k_j\);
2. \(P_{kr}D^k_j \cap P_{kr}D^k_i = \emptyset\) for \(i \neq j\);
3. For \(w^k_j = P_{kr}e^{2\pi i (j - \frac{1}{2})2^{-k}}\), \(S_{kr} \cap B(w^k_j, \lambda) \subseteq P_{kr}D^k_j \subseteq S_{kr} \cap B(w^k_j, C\lambda)\).

Define subsets, \(K^0_j\) of \(\mathbb{D}\) by:
\[
K^0_j := \{z \in \mathbb{D} : d(0, z) < r\},
\]
\[
K^k_j := \{z \in \mathbb{D} : kr \leq d(0, z) < (k + 1)r \text{ and } P_{kr}z \in P_{kr}D^k_j\}, k \geq 1, j \geq 1.
\]
Now, let \(c^k_j \in K^k_j\) be defined by \(P_{(k+\frac{1}{2})r}w^k_j\). For \(\alpha = c^k_j\), the set \(K_\alpha := K^k_j\) is referred to as a kube and the point \(\alpha = c^k_j\) is the center of the kube. We define a Bergman tree structure \(\mathcal{T} := \{c^k_j\}\) on centers of the kubes. We say that \(c^{k+1}_j\) is a child of \(c^k_j\) if \(P_{kr}D^k_j \subseteq P_{kr}D^k_{k+1}\). We say \(c^m_i \geq c^k_j\) if \(m \geq n\) and \(P_{kr}c^m_i \in P_{kr}D^k_j\). We define \(\hat{K}_\alpha\) to be the dyadic tent under \(K_\alpha\):
\[
\hat{K}_\alpha := \bigcup_{\beta \in \mathcal{T} : \beta \geq \alpha} K_\beta. \tag{2.5}
\]
For \(z \in \mathbb{D}\), we say the generation \(\text{gen}(z) = N\) if \(z \in K^N_j\) for some \(j\).

Using shifted dyadic systems \(D_l = \{D^k_j(l)\}\) on the unit circle with
\[
D^k_j(l) = \{e^{2\pi i \theta} : (j - 1)2^{-k} + l \leq \theta < j2^{-k} + l\}, \quad \text{for } j = 1, \ldots, 2^k \text{ and } l \in \mathbb{R},
\]
For a subset $U$, we use the notation $|U|$ to denote the Lebesgue measure of $U$. The following three lemmas relate the Carleson tent $T_z$ to the dyadic tent $\hat{K}_\alpha$ and the Bergman kernel function on $\mathbb{D}$.

**Lemma 2.1.** Let $\mathcal{T}$ be a Bergman tree constructed as above. For $\alpha \in \mathcal{T}$,

$$|T_\alpha| \approx |\hat{K}_\alpha| \approx |K_\alpha| \approx (1 - |\alpha|)^2.$$  

**Proof.** Suppose $\text{gen}(\alpha) = k$. Let $R_{kr}$ denote the Euclidean distance between $S_{kr}$ and the origin. Then $|\hat{K}_\alpha| = \pi 2^{-k}(1 - R_{kr}^2)$ and $|K_\alpha| = \pi 2^{-k}(R_{(k+1)r}^2 - R_{kr}^2)$. Recall that $r = 2^{-1}\ln 2$. By the definition of the Bergman distance, $1 - R_{kr} \approx e^{-2kr} = 2^{-k}$. Thus $|\hat{K}_\alpha| \approx |K_\alpha| \approx 2^{-2k}$. Since $\alpha$ is the center of the kube $K_\alpha$, the Bergman distance $d(0, \alpha) = (k + \frac{1}{2})r$. Hence we obtain

$$(1 - |\alpha|)^2 = (1 - R_{(k+\frac{1}{2})r})^2 \approx 2^{-2(k+\frac{1}{2})} \approx |\hat{K}_\alpha| \approx |K_\alpha|.$$  

Notice that the Carleson tent $T_\alpha$ is the intersection set of the unit disc $\mathbb{D}$ and the disc centered at the point $\frac{z}{|z|}$ with Euclidean radius $1 - |\alpha|$. A geometric consideration then yields

$$|T_\alpha| \approx (1 - |\alpha|)^2.$$  

□

**Lemma 2.2** ([RTW17] Lemma 9). There is a finite collection of Bergman trees $\{\mathcal{T}_l\}_{l=1}^N$ such that for all $\alpha \in \mathbb{D}$, there is a tree $\mathcal{T}$ from the finite collection and an $\beta \in \mathcal{T}$ such that the dyadic tent $\hat{K}_\beta$ contains the tent $T_\alpha$ and $\sigma(\hat{K}_\beta) \approx |T_\alpha|$.

**Lemma 2.3** ([RTW17], Lemma 15). For $z, w \in \mathbb{D}$, there is a Carleson tent, $T_\alpha$, containing $z$ and $w$ such that

$$|T_\alpha| \approx |1 - zw|^2 = \pi^{-1}|K_\mathbb{D}(z, w)|^{-1}.$$  

(2.6)

**Lemma 2.4.** For any dyadic tent $\hat{K}_\beta$ with $\beta \in \mathcal{T}_l$ for some $l$, there exists a Carleson tent $T_z$ such that $\hat{K}_\beta \subseteq T_z$ and $|\hat{K}_\beta| \approx |T_z|$.

**Proof.** Given a dyadic tent $\hat{K}_\beta$, we can find a Carleson tent $T_z$ such that $\hat{K}_\beta$ is a largest dyadic tent in $T_z$. Without loss of generality, we may assume that $z$ is a positive real number. By Lemma 2.1 $|\hat{K}_\beta| \approx |K_\alpha|$. It suffices to show that the top kube $K_\beta$ of the tent $\hat{K}_\beta$ satisfies the inequality $|K_\beta| \approx |T_z|$. Since $K_\beta$ is a largest kube contained in $T_z$, all of its ancestors are not contained in $T_z$. Let $k$ be the generation $\text{gen}(\beta)$ of $\beta$. Then $T_z$ intersects with at most two of the Borel subsets $\{Q_{j1}^{(k-1)}\}_{j=1}^{2^{k-1}}$ of $S_{(k-1)r}$. Let $R_{(k-1)r}$ denote the Euclidean distance between $S_{(k-1)r}$ and the origin. The arc length of the set $P_{(k-1)r}D_{j1}^{(k-1)}$ equals $R_{(k-1)r}2\pi 2^{1-k}$. Thus the arc length of the intersection set $S_{(k-1)r} \cap T_z$ is less than $2R_{(k-1)r}2\pi 2^{1-k}$. Note that the point $z$ is a positive real number. $T_z$ is symmetric about the real number axis. Therefore the point $R_{(k-1)r}e^{2\pi i 2^{1-k}}$ is not in $T_z$, i.e.

$$|1 - R_{(k-1)r}e^{2\pi i 2^{1-k}}| \geq 1 - z.$$  

one can obtain different dyadic structures on $\mathbb{D}$ with their corresponding Bergman trees $T_i$. Recall the Carleson tent $T_z$ over $z \in \mathbb{D}$:

- $T_z := \{w \in \mathbb{D} : |1 - w\bar{z}| < 1 - |z|\}$ for $z \neq 0$, and
- $T_z := \mathbb{D}$ for $z = 0$. 

For a subset $U$, we use the notation $|U|$ to denote the Lebesgue measure of $U$. The following three lemmas relate the Carleson tent $T_z$ to the dyadic tent $\hat{K}_\alpha$ and the Bergman kernel function on $\mathbb{D}$.
Since $1 - R_{Nt} \approx e^{-2Nt}$ and $|1 - e^{2\pi it}| \approx t$ for $t \in \mathbb{R}$, we have
\[
|1 - R_{(k-1)r}e^{2\pi i2^{1-k}}| \leq |1 - R_{(k-1)r}| + |R_{(k-1)r} - R_{(k-1)r}e^{2\pi i2^{1-k}}| \\
\approx e^{-2(k-1)r} - 1 + 2^{1-k} - e^{-(k-1)\ln 2} \approx 2^{-2(k-1)}.
\]
Hence $2^{-(k-1)} \gtrsim 1 - z = 1 - |z|$. Lemma 2.1 then implies that $|T_z| \lesssim 2^{-2(k-1)}$. Since $\text{gen}(\beta) = k$, the Bergman distance $d(\beta, 0)$ equals $(k + \frac{1}{2})r$. Recall that $r = 2^{-1} \ln 2$. We have
\[
1 - |\beta| \approx e^{-2(k+\frac{1}{2})} = 2^{-(k+\frac{1}{2})}.
\]
Applying Lemma 2.1 again yields $|K_\beta| \approx 2^{-2(k+\frac{1}{2})} \gtrsim |T_z|$. By the containment $K_\beta \subseteq T_z$, there holds $|K_\beta| \leq |T_z|$. Combining these inequalities, we conclude that $|K_\beta| \approx |T_z|$ and the proof is complete.

Combining Lemmas 2.2 and 2.3 we obtain the following estimate for arbitrary $z, w \in \mathbb{D}$:
\[
|1 - z\bar{w}|^{-2} \approx |T_\alpha|^{-1} \approx |\hat{K}_\beta|^{-1} \leq \sum_{m=1}^{M} \sum_{\gamma \in \mathcal{T}_m} 1_{\hat{K}_\gamma}(z) 1_{\hat{K}_\gamma}(w) |\hat{K}_\gamma|.
\]
(2.7)

Here $\{\mathcal{T}_m\}_{m=1}^{M}$ is the finite collection in Lemma 2.2.

Similarly, on the bidisk, $\mathbb{D}^2$, we have:
\[
|1 - z_1\bar{w}_1|^{-2} |1 - z_2\bar{w}_2|^{-2} \\
\approx |T_{\alpha_1}|^{-1} |T_{\alpha_2}|^{-1} \\
\approx |\hat{K}_{\beta_1}|^{-1} |\hat{K}_{\beta_2}|^{-1} \\
\leq \sum_{m,n=1}^{M} \sum_{\gamma \in \mathcal{T}_m, \eta \in \mathcal{T}_n} 1_{\hat{K}_\gamma \times \hat{K}_\eta}(z_1, z_2) 1_{\hat{K}_\gamma \times \hat{K}_\eta}(w_1, w_2) |\hat{K}_\gamma \times \hat{K}_\eta|.
\]
(2.8)

Given a tree structure $\mathcal{T}_m \times \mathcal{T}_n$ on $\mathbb{D}^2$ and a dyadic tent $\hat{K}_{\beta_1} \times \hat{K}_{\beta_2}$ we define the induced tree structure $\mathcal{T}_{\beta_1, \beta_2}$ and dyadic tent $\hat{K}_{\beta_1, \beta_2}$ on $\mathbb{H}$ to be:
\[
\mathcal{T}_{\beta_1, \beta_2} := \left\{(c_1, c_2) \in \mathbb{H} : \left(\frac{c_1}{c_2}\right) \in \mathcal{T}_m \times \mathcal{T}_n\right\},
\]
(2.9)
\[
\hat{K}_{\beta_1, \beta_2} := \left\{(z_1, z_2) \in \mathbb{H} : \left(\frac{z_1}{z_2}\right) \in \hat{K}_{\beta_1} \times \hat{K}_{\beta_2}\right\}.
\]
(2.10)

Similarly the induced Carleson tent $T'_{z_1, z_2}$ on $\mathbb{H}$ can be defined by
\[
T'_{z_1, z_2} := \{(w_1, w_2) \in \mathbb{H} : \left(\frac{w_1}{w_2}\right) \in T_{z_1} \times T_{z_2}\}.
\]
(2.11)

Set $du = |w_1|^{-2}dV$. For a weight $\mu$ and a subset $U \subseteq \mathbb{H}$, we set $\mu(U) := \int_U \mu dV$ and let $\langle f \rangle_{U}^{\mu dV}$ denote the average of the function $|f|$ with respect to the measure $\mu dV$ on the set $U$:
\[
\langle f \rangle_{U}^{\mu dV} = \frac{\int_U |f(w_1, w_2)| \mu dV}{\mu(U)}.
\]
(2.12)

Given weights $\mu$ on $\mathbb{H}$ and $\nu = |z_2|^{-p'} \mu^{-p'/p}$, we define the characteristic of two weights $\mu, \nu$ by
\[
[\mu, \nu]_p := \sup_{z_1, z_2 \in \mathbb{D}} \langle \mu |w_2|^{2-p} dV \rangle_{T_{z_1, z_2}}^{\mathbb{D}} \left(\langle |w_2|^{2} \nu dV \rangle_{T_{z_1, z_2}}^{\mathbb{D}}\right)^{p-1}.
\]
(2.13)
By Lemmas 2.2 and 2.4 we can replace $T'_{z_1, z_2}$ by $\tilde{K}'_{\gamma, \eta}$ to obtain a quantity of comparable size:

$$\langle \mu, \nu \rangle_p \approx \sup_{1 \leq m, n \leq M} \sup_{(\gamma, \eta) \in T_{m, n}} (|\mu|_{2^{-p}})^{\frac{1}{p}} \left( \frac{\langle \mu \rangle_{L^2}}{K'_{\gamma, \eta}} \right)^{\frac{1}{p-1}}.$$  \hspace{1cm} (2.14)

From now on, we will abuse the notation $\langle \mu, \nu \rangle_p$ to represent both the supremum in $T'_{z_1, z_2}$ and the supremum in $\tilde{K}'_{\gamma, \eta}$.

The proof of Theorem 1.2 will use the weighted strong maximal function on $\mathbb{H}$.

**Definition 2.5.** For a weight $\mu$, and a Bergman tree $T'_{m, n}$, we define the following maximal function:

$$M_{T'_{m, n}, \mu} f(w_1, w_2) := \sup_{(\beta_1, \beta_2) \in T'_{m, n}} \frac{1_{K'_{\beta_1, \beta_2}}(w_1, w_2)}{\mu(K'_{\beta_1, \beta_2})} \int_{K'_{\beta_1, \beta_2}} |f(z_1, z_2)| \mu(z_1, z_2) dV(z_1, z_2).$$  \hspace{1cm} (2.15)

We set $\langle f \rangle_{Q, \mu} := \frac{\int_Q |f| d\mu}{\mu(Q)}$, then we also have:

$$M_{T'_{m, n}, \mu} f(w_1, w_2) = \sup_{(\beta_1, \beta_2) \in T'_{m, n}} \frac{1_{K'_{\beta_1, \beta_2}}(w_1, w_2)}{\mu(K'_{\beta_1, \beta_2})} \langle f \rangle_{K'_{\beta_1, \beta_2}, \mu}.$$  \hspace{1cm} (2.16)

We have the following $L^p$ regularity result for $M_{T'_{m, n}, \mu}$.

**Lemma 2.6.** Let $\mu(z_1, z_2)$ the same as in Theorem 1.2, then $M_{T'_{m, n}, \mu}$ is bounded on $L^p(\mathbb{H}, \mu)$ for $1 < p \leq \infty$. Moreover, $\|M_{T'_{m, n}, \mu}\|_{L^p(\mathbb{H}, \mu)} \lesssim (p/(p - 1))^2$ for $1 < p < \infty$.

**Proof.** When $p = \infty$, the boundedness of $M_{T'_{m, n}, \mu}$ is obvious. We turn to the case $1 < p < \infty$. Set $\mu_2(w_2) := |w_2|^2 \mu_2(w_2)$. Using the biholomorphism $h : (w_1, w_2) \mapsto (w_1 w_2, w_2)$ from $D \times D^*$ onto $\mathbb{H}$, we transform $M_{T'_{m, n}, \mu}$ into the following maximal function on $D \times D^*$:

$$M_{T_{m, n}, \mu} f(w_1, w_2) := \sup_{(\beta_1, \beta_2) \in T_{m, n}} \frac{1_{K_{\beta_1}}(w_1) 1_{K_{\beta_2}}(w_2)}{\mu_1(K_{\beta_1}) \mu_2(K_{\beta_2})} \int_{K_{\beta_1} \times K_{\beta_2}} |f(z_1, z_2)| \mu_1(z_1) \mu_2(z_2) dV(z_1, z_2),$$  \hspace{1cm} (2.17)

and it suffices to show that $M_{T_{m, n}, \mu}$ is $L^p$ bounded on $L^p(D \times D^*, \mu_2 w_2^2 \circ h)$ for $1 < p \leq \infty$.

Defining the following two 1-parameter maximal functions:

$$M_{T_{m, \mu_1}} f(w_1, w_2) := \sup_{\beta_1 \in T_{m}} \frac{1_{K_{\beta_1}}(w_1)}{\mu_1(K_{\beta_1})} \int_{K_{\beta_1}} |f(z_1, w_2)| \mu_1(z_1) dV(z_1);$$  \hspace{1cm} (2.18)

$$M_{T_{\mu_2}} f(w_1, w_2) := \sup_{\beta_2 \in T_{m}} \frac{1_{K_{\beta_2}}(w_2)}{\mu_2(K_{\beta_2})} \int_{K_{\beta_2}} |f(w_1, z_2)| \mu_2(z_2) dV(z_2),$$  \hspace{1cm} (2.19)

we obtain that $M_{T_{m, \mu}} f \leq M_{T_{m, \mu_1}} \circ M_{T_{\mu_2}} f$. By Fubini’s Theorem, it is enough to show that $M_{T_{m, \mu_1}}$ is bounded on $L^p(D, \mu_1 dV)$ and $M_{T_{\mu_2}}$ is bounded on $L^p(D, \mu_2 dV)$. Here we show the $L^p$ boundedness of $M_{T_{m, \mu_1}}$. The boundedness of $M_{T_{\mu_2}}$ follows from an analogous argument.

Note that $M_{T_{m, \mu_1}}$ is bounded on $L^\infty(D, \mu_1)$. By interpolation, the weak-type $(1,1)$ estimate

$$\mu_1(\{ z \in D : M_{T_{m, \mu_1}} f(z) > \lambda \}) \lesssim \frac{\|f\|_{L^1(D, \mu_1)}}{\lambda}$$  \hspace{1cm} (2.20)
is sufficient to finish the proof. For a point \( w \in \{ z \in \mathbb{D} : \mathcal{M}_{\mathcal{T}; \mathcal{M}_1} f(z) > \lambda \} \), there exists a unique maximal tent \( \hat{K}_\alpha \) that contains \( w \) and satisfies:

\[
\frac{1}{\mu_1(\hat{K}_\alpha)} \int_{\hat{K}_\alpha} |f(z)| \mu_1(z) dV(z) > \frac{\lambda}{2}.
\]  
(2.21)

Let \( \mathcal{A}_\lambda \) be the set of indices of all such maximal tents \( \hat{K}_\alpha \). The union of these maximal tents covers the set \( \{ z \in \mathbb{D} : \mathcal{M}_{\mathcal{T}; \mathcal{M}_1} f(z) > \lambda \} \). Since the tents \( \hat{K}_\alpha \) are maximal, they are also pairwise disjoint and hence

\[
\mu_1(\{ z \in \mathbb{D} : \mathcal{M}_{\mathcal{T}; \mathcal{M}_1} f(z) > \lambda \}) \leq \sum_{\alpha \in \mathcal{A}_\lambda} \mu_1(\hat{K}_\alpha) \leq \sum_{\alpha \in \mathcal{A}_\lambda} \frac{2}{\lambda} \int_{\hat{K}_\alpha} f(z) \mu_1(z) dV(z) \leq \frac{2\|f\|_{L^p(\mathbb{D}, \mu_1)}}{\lambda}.
\]

Thus inequality (2.20) holds and \( \mathcal{M}_{\mathcal{T}; \mathcal{M}_1} \) is weak-type (1,1). Using a standard argument for the Hardy-Littlewood maximal function, we further have

\[
\|\mathcal{M}_{\mathcal{T}; \mathcal{M}_1}\|_{L^p(\mathbb{D} \times \mathbb{D}^*, |w_2|^2 \mu_0 h)} \lesssim \frac{p}{p - 1}.
\]

Since the same inequality holds for \( \mathcal{M}_{\mathcal{T}; \mathcal{M}_2} \),

\[
\|\mathcal{M}_{\mathcal{T}; \mathcal{M}_1}\|_{L^p(\mathbb{D} \times \mathbb{D}^*, |w_2|^2 \mu_0 h)} \leq \|\mathcal{M}_{\mathcal{T}; \mathcal{M}_1} \circ \mathcal{M}_{\mathcal{T}; \mathcal{M}_2}\|_{L^p(\mathbb{D} \times \mathbb{D}^*, |w_2|^2 \mu_0 h)} \lesssim \left( \frac{p}{p - 1} \right)^2.
\]

Finally, we define two operators \( Q \) and \( Q^+ \). Let \( p' \) be the conjugate index of \( p \). We set

\[
Q(f)(z_1, z_2) = \int_{\mathbb{H}} \frac{1}{\pi^2 z_2(1 - \frac{z_1 w_1}{z_2 w_2})^2(1 - \frac{z_2 w_2}{z_1 w_1})^2} f(w_1, w_2) dV(w_1, w_2),
\]  
(2.22)

\[
Q^+(f)(z_1, z_2) = \int_{\mathbb{H}} \frac{1}{\pi^2 z_2[1 - \frac{z_1 w_1}{z_2 w_2}]^2[1 - \frac{z_2 w_2}{z_1 w_1}]^2} f(w_1, w_2) dV(w_1, w_2).
\]  
(2.23)

It is clear that \( P = QM_1|w_2| \) and \( P^+ = Q^+M_1|w_2| \). Moreover, the weighted \( L^p \) norm of the projection, \( \|P^+ : L^p(\mathbb{H}, \mu dV) \rightarrow L^p(\mathbb{H}, \mu dV)\| \), is equal to the weighted norm of \( Q^+M_\nu \) acting between two different weighted \( L^p \) spaces.

**Lemma 2.7.** Let \( \mu \) be a weight on the Hartogs triangle. Set \( \nu := \mu^{\frac{p'}{p}} |w_2|^{-p'} \). Then

\[
\|P : L^p(\mathbb{H}, \mu dV) \rightarrow L^p(\mathbb{H}, \mu dV)\| = \|QM_\nu : L^p(\mathbb{H}, \nu dV) \rightarrow L^p(\mathbb{H}, \nu dV)\|;
\]  
(2.24)

\[
\|P^+ : L^p(\mathbb{H}, \mu dV) \rightarrow L^p(\mathbb{H}, \mu dV)\| = \|Q^+M_\nu : L^p(\mathbb{H}, \nu dV) \rightarrow L^p(\mathbb{H}, \nu dV)\|.
\]  
(2.25)

**Proof.** We show (2.25) here as the proof for (2.24) is similar. Given \( f \in L^p(\mathbb{H}, \mu) \), we have

\[
\int_{\mathbb{H}} |f|^p \mu dV(w_1, w_2) = \int_{\mathbb{H}} \left| \frac{f}{w_2} \right|^p |w_2|^p \mu dV(w_1, w_2) = \int_{\mathbb{H}} \left| M_{\frac{1}{|w_2|}} f \right|^p |w_2|^p \mu dV(w_1, w_2). \quad (2.26)
\]

Thus \( \|f\|_{L^p(\mu dV)} = \|M_{1/|w_2|} f\|_{L^p(\mu |w_2|^p dV)} \) and

\[
\|P^+ : L^p(\mathbb{H}, \mu dV) \rightarrow L^p(\mathbb{H}, \mu dV)\| = \|Q^+ : L^p(\mathbb{H}, |w_2|^p \mu dV) \rightarrow L^p(\mathbb{H}, \mu dV)\|.
\]
We claim further that for \( f \in L^p(\mathbb{H}, |w_2|^p \mu dV) \), \( \|f\|_{L^p(|w_2|^p \mu dV)} = \|M_{1/\nu} f\|_{L^p(\nu dV)} \). Then (2.25) holds. Recall that \( \nu := \frac{\mu}{\pi^2} |w_2|^{-p} \). We have

\[
\int_{\mathbb{H}} \left| \frac{f}{\nu} \right|^p \nu dV = \int_{\mathbb{H}} |f|^p \nu^{1-p} dV = \int_{\mathbb{H}} |f|^p (\frac{\mu}{\pi^2} |w_2|^{-p})^{1-p} dV = \int_{\mathbb{H}} |f|^p |w_2|^p \mu dV.
\]

Hence the claim is shown and the proof is complete.

\[\square\]

3. Proof of Theorem 1.2

It is sufficient to prove that inequality (1.3) holds.

3.1. Proof for the upper bound. For the upper bound inequality

\[
\|P^+\|_{L^p(\mathbb{H}, \mu dV)} \lesssim (pp')^2 [\mu, \nu]_p^{\max \{1, \frac{1}{p-1}\}},
\]

we first consider the case \( p \geq 2 \). The case \( 1 < p < 2 \) will follow from a duality argument.

Recall the tree structure \( \{T'_{m,n}\}_{m=1}^M \) and the dyadic tent \( \{\hat{K}'_{\beta_1,\beta_2}\} \) from (2.9) and (2.10). Set the measure \( du := |w_2|^{-2} dV \). By Lemma 2.2 and the inequality (2.8), there is a finite collection \( M \) such that for \((z_1, z_2)\) and \((w_1, w_2)\) in \( \mathbb{H} \), there exists \( \hat{K}_{\beta_1} \) and \( \hat{K}_{\beta_2} \) such that

\[
1 - \frac{z_1 \bar{w}_1}{z_2 \bar{w}_2} = \left| 1 - \frac{z_1 \bar{w}_1}{z_2 \bar{w}_2} \right|^{-2} \approx |\hat{K}_{\beta_1}|^{-1} |\hat{K}_{\beta_2}|^{-1}
\]

\[
\leq \sum_{m,n=1}^M \sum_{\gamma \in T_{m,n}} \frac{1_{\hat{K}_\gamma \times \hat{K}_\eta} (z_1/z_2, z_2) 1_{\hat{K}_\gamma \times \hat{K}_\eta} (w_1/w_2)}{|\hat{K}_\gamma \times \hat{K}_\eta|}
\]

\[= \sum_{m,n=1}^M \sum_{(\gamma, \eta) \in T_{m,n}} \frac{1_{\hat{K}_{\gamma,n}'} (z_1, z_2) 1_{\hat{K}_{\gamma,n}'} (w_1, w_2)}{u(\hat{K}_{\gamma,n}')} (3.1)
\]

Applying this inequality to the operator \( Q^+ M_{p'} \) yields

\[
\left| Q^+ M_{p'} f(z_1, z_2) \right| = \left| \int_{\mathbb{H}} \frac{|z_2|^{-1} M_{p'} f(w_1, w_2)}{\pi^2 |1 - \frac{z_1 \bar{w}_1}{z_2 \bar{w}_2}|^2} dV(w_1, w_2) \right|
\]

\[
\lesssim \int_{\mathbb{H}} \sum_{m,n=1}^M \sum_{(\gamma, \eta) \in T_{m,n}} \frac{1_{\hat{K}_{\gamma,n}'} (z_1, z_2) 1_{\hat{K}_{\gamma,n}'} (w_1, w_2) |M_{p'} f(w_1, w_2)|}{|z_2| u(\hat{K}_{\gamma,n}')} dV(w_1, w_2)
\]

\[= \sum_{m,n=1}^M \sum_{(\gamma, \eta) \in T_{m,n}} \frac{1_{\hat{K}_{\gamma,n}'} (z_1, z_2)}{|z_2|} \left( p' |w_2|^2 du \right)_{K_{\gamma,n}'} (3.2)
\]

Set \( Q_{m,n,p'} f(z_1, z_2) := \sum_{(\gamma, \eta) \in T_{m,n}} 1_{\hat{K}_{\gamma,n}'} (z_1, z_2) |z_2|^{-1} (p' |w_2|^2 du)_{K_{\gamma,n}'} \). Then it suffices to estimate the \( L^p \) norm for each \( Q_{m,n,p'}^+ \). The proof given below uses the idea of how to prove the linear bound for sparse operators in weighted theory of harmonic analysis, see for example [Moe12].
and [Lac17]. For arbitrary $g \in L^p(\mathbb{H}, \mu)$,

$$
\langle Q_{m,n,\nu}^+(z_1, z_2), g(z_1, z_2) \rangle
= \int_{\mathbb{H}} Q_{m,n,\nu}^+(z_1, z_2) g(z_1, z_2) \mu dV(z_1, z_2)
$$

$$
= \int_{\mathbb{H}} \sum_{(\gamma, \eta) \in T_{m,n}} 1_{K_{\gamma,\eta}'}(z_1, z_2) \left| z_2 \right|^{-1} \langle f |w_2|^2 \rangle_{K_{\gamma,\eta}'}^{du} g(z_1, z_2) \mu dV(z_1, z_2)
$$

$$
= \sum_{(\gamma, \eta) \in T_{m,n}} \langle f |w_2|^2 \rangle_{K_{\gamma,\eta}'}^{du} \int_{K_{\gamma,\eta}'} g(z_1, z_2) |z_2|^{-1} \mu dV(z_1, z_2)
$$

$$
= \sum_{(\gamma, \eta) \in T_{m,n}} \langle f |w_2|^2 \rangle_{K_{\gamma,\eta}'}^{du} \langle g |w_2|^{p-1} \rangle_{K_{\gamma,\eta}'}^{du} \langle |w_2|^{2-p} \mu \rangle_{K_{\gamma,\eta}'}^{du} u(K_{\gamma,\eta})
$$

$$
\leq |\mu|_p \sum_{(\gamma, \eta) \in T_{m,n}} \langle f |w_2|^2 \rangle_{K_{\gamma,\eta}'}^{du} \langle g |w_2|^{p-1} \rangle_{K_{\gamma,\eta}'}^{du} \langle |w_2|^{2-p} \mu \rangle_{K_{\gamma,\eta}'}^{du} u(K_{\gamma,\eta}) \langle |w_2|^{2-p} \rangle_{K_{\gamma,\eta}'}^{du} (\nu(K_{\gamma,\eta}))^{2-p}.
$$

(3.3)

Recall from Lemma 2.1 that $|\hat{K}_n| \approx |K_n|$ for the tree structure $T$ with Lebesgue measure $\sigma$ on the unit disc. Hence for the induced tree structure $T_{m,n}$ with the induced weighted measure $u$ on the Hartogs triangle, we also have $u(\hat{K}_{\gamma,\eta}') \approx u(K_{\gamma,\eta}')$. The facts that $p \geq 2$ and $K_{\gamma,\eta}' \subseteq \hat{K}_{\gamma,\eta}'$ gives the inequality $(\nu(\hat{K}_{\gamma,\eta}'))^{2-p} \leq (\nu(K_{\gamma,\eta}'))^{2-p}$. Combining these facts, we have

$$
(\nu(K_{\gamma,\eta}'))^{p-1} (\nu(K_{\gamma,\eta}'))^{2-p} \approx (\nu(K_{\gamma,\eta}'))^{p-1} (\nu(K_{\gamma,\eta}'))^{2-p}.
$$

(3.4)

By Hölder’s inequality,

$$
u(K_{\gamma,\eta}') \leq \left( \int_{K_{\gamma,\eta}'} |w_2|^{-p} \mu dV \right)^{1/p}.
$$

Therefore,

$$
\left( \int_{K_{\gamma,\eta}'} |w_2|^{-p} \mu dV \right)^{1/p}.
$$

(3.5)

Applying these inequalities to the last line of (3.3), we have

$$
|\mu|_p \sum_{(\gamma, \eta) \in T_{m,n}} \langle f |w_2|^2 \rangle_{K_{\gamma,\eta}'}^{du} \langle g |w_2|^{p-1} \rangle_{K_{\gamma,\eta}'}^{du} \langle |w_2|^{2-p} \mu \rangle_{K_{\gamma,\eta}'}^{du} u(K_{\gamma,\eta}) \langle |w_2|^{2-p} \rangle_{K_{\gamma,\eta}'}^{du} (\nu(K_{\gamma,\eta}'))^{2-p}.
$$

(3.6)
Applying Hölder’s inequality again the sum above yields:

\[
\sum_{(\gamma,\eta) \in T_{m,n}} \langle f \rangle_{K'_{\gamma,\eta}} \nu(K'_{\gamma,\eta}) \left( g \left| w_2 \right|^{p-1} \left| w_2 \right|^{-p} \mu \right) \left( \frac{1}{\left| w_2 \right|} \right)^{\frac{1}{p}} \left( \int_{K'_{\gamma,\eta}} \left| w_2 \right|^{-p} \mu dV \right)^{\frac{1}{p}} \leq \left( \sum_{(\gamma,\eta) \in T_{m,n}} \left( \left| \langle f \rangle_{K'_{\gamma,\eta}} \right| \right)^{p} \nu(K'_{\gamma,\eta}) \int_{K'_{\gamma,\eta}} \left| w_2 \right|^{-p} \mu dV \right)^{\frac{1}{p}} \left( \sum_{(\gamma,\eta) \in T_{m,n}} \left( \langle g \left| w_2 \right|^{p-1} \left| w_2 \right|^{-p} \mu \right) \left( \int_{K'_{\gamma,\eta}} \left| w_2 \right|^{-p} \mu dV \right)^{\frac{1}{p}} \right)^{\frac{1}{p}}. \tag{3.7}
\]

By the disjointness of $K'_{\gamma,\eta}$ and Lemma 2.6, we have

\[
\sum_{(\gamma,\eta) \in T_{m,n}} \left( \left| \langle f \rangle_{K'_{\gamma,\eta}} \right| \right)^{p} \nu(K'_{\gamma,\eta}) \leq \int_{E} (M_{T_{m,n},p} f)^{p} \nu dV \leq \left( p' \right)^{2p} \| f \|_{L^{p}(\mathbb{H})}^{2p}. \tag{3.8}
\]

Note that \( \| g \left| w_2 \right|^{p-1} \|_{L^{p'}(\mathbb{H})} = \| g \|_{L^{p}(\mathbb{H})}. \) A similar argument using the maximal function $M_{T_{m,n},p} w_2^{-p}\mu$ will also give the inequality

\[
\sum_{(\gamma,\eta) \in T_{m,n}} \left( \int_{K'_{\gamma,\eta}} \left| w_2 \right|^{-p} \mu dV \right)^{p} \left( \int_{K'_{\gamma,\eta}} \left| w_2 \right|^{-p} \mu dV \right)^{\frac{1}{p}} \leq \left( p' \right)^{2p} \| g \|_{L^{p'}(\mathbb{H})}^{2p}. \tag{3.9}
\]

Substituting (3.8) and (3.9) into (3.7) and (3.3) finally yields

\[
\left\langle Q_{m,n,\nu}^{+}, f, g \mu \right\rangle \lesssim \left[ \mu, \nu \right]_{p} \left( \left| f \right| \right)_{L^{p}(\mathbb{H})} \| g \|_{L^{p}(\mathbb{H})}. \tag{3.10}
\]

Therefore \( \| P^{+} \|_{L^{p}(\mathbb{H})} \lesssim \| Q_{m,n,\nu}^{+} f \|_{L^{p}(\mathbb{H})} \lesssim \left( \left| f \right| \right)_{L^{p}(\mathbb{H})} \left. \left[ \mu, \nu \right]_{p} \right. \| g \|_{L^{p}(\mathbb{H})}. \)

Now we turn to the case \( 1 < p < 2 \) and show that

\[
\left\langle Q_{m,n,\nu}^{+}, f, g \mu \right\rangle \lesssim \left( \left| \mu, \nu \right|_{p} \right)^{\frac{1}{p'}} \| f \|_{L^{p}(\mathbb{H})} \| g \|_{L^{p}(\mathbb{H})}. \tag{3.11}
\]

for all $f \in L^{p}(\mathbb{H})$ and $g \in L^{p'}(\mathbb{H}, \mu dV)$. By the definition of $Q_{m,n,\nu}^{+}$,

\[
\left\langle Q_{m,n,\nu}^{+}, f, g \mu \right\rangle = \left\langle \sum_{(\gamma,\eta) \in T_{m,n}} 1_{K'_{\gamma,\eta}}(w_1, w_2) \left| w_2 \right|^{-1} \langle f \nu \left| w_2 \right|^{2} \rangle_{K'_{\gamma,\eta}}, g \mu \right\rangle = \sum_{(\gamma,\eta) \in T_{m,n}} 1_{K'_{\gamma,\eta}}(w_1, w_2) \langle f \nu \left| w_2 \right|^{2} \rangle_{K'_{\gamma,\eta}} \left| w_2 \right|^{-1} \mu \left( w_2 \right) = \sum_{(\gamma,\eta) \in T_{m,n}} \langle f \nu \left| w_2 \right|^{2} \rangle_{K'_{\gamma,\eta}} \left( \langle g \left| w_2 \right|^{p-1} \left| w_2 \right|^{-p} \mu \left| w_2 \right|^{2} \rangle_{K'_{\gamma,\eta}} \langle w_2, f \nu \rangle \right) = \sum_{(\gamma,\eta) \in T_{m,n}} \langle f \nu \left| w_2 \right|^{2} \rangle_{K'_{\gamma,\eta}} \left( \langle g \left| w_2 \right|^{p-1} \left| w_2 \right|^{-p} \mu \left| w_2 \right|^{2} \rangle_{K'_{\gamma,\eta}} \langle w_2, f \nu \rangle \right). \tag{3.12}
\]

Set $h = g \left| w_2 \right|^{p-1}$ and $\psi = \left| w_2 \right|^{-p} \mu$. Then $\| h \|_{L^{p'}(\mathbb{H}, \mu dV)} = \| g \|_{L^{p'}(\mathbb{H}, \mu dV)}$. Setting the weight $\omega$ to satisfies $\left| w_2 \right|^{-p} \omega \mu^{\frac{1}{p'}} = \psi$, we have $\omega = \mu^{\frac{1}{p'}} = \nu \omega^{\frac{1}{p'}}$. Replacing $p$ by $p'$, $\mu$ by $\omega$, and $\nu$ by
ψ and going through same argument for the case \( p \geq 2 \) yields that

\[
\| M_{|z_2|Q_{m,n,w_2}^+}P \|_{L^{p'}(H,\nu dV)} = \| Q_{m,n,w_2}^+P \|_{L^{p'}(H,|z_2|\nu dV)} \\
\lesssim (pp')^2 \sup_{(\gamma,\eta)\in T_{m,n}} \left( \langle \mu|w_2|2-p\rangle_{K_{\gamma,\eta}} \right)^{p'-1} \left( \langle |w_2|^2-p\nu|z_2|^2 \rangle_{K_{\gamma,\eta}} \right)^{\frac{1}{p'}} \\
= (pp')^2 \left( \sup_{(\gamma,\eta)\in T_{m,n}} \langle \mu|w_2|2-p\rangle_{K_{\gamma,\eta}} \left( \langle |w_2|^2\nu \rangle_{K_{\gamma,\eta}} \right)^{p-1} \right)^{\frac{1}{p'}} \\
= (pp')^2([\mu,\nu]_p)_{p-1}. 
\]

Thus we have

\[
\langle Q_{m,n,p}, g \mu \rangle \lesssim (pp')^2([\mu,\nu]_p)_{p-1} \| g \|_{L^{p'}(H,\nu dV)} \| f \|_{L^p(H,\nu dV)},
\]

and

\[
\| P^+ \|_{L^p(H,\nu dV)} \lesssim (pp')^2([\mu,\nu]_p)_{p-1}.
\]

Combining the results for \( 1 < p < 2 \) and \( p \geq 2 \) gives the upper bound in Theorem 1.2

\[
\| P^+ \|_{L^p(H,\nu dV)} \lesssim (pp')^2([\mu,\nu]_p)_{\max\{1,\frac{1}{p-1}\}}.
\]

3.2. Proof for the lower bound. Now we turn to show the lower bound

\[
[\mu,\nu]_{p-1} \lesssim \| P \|_{L^p(H,\nu dV)}
\]

in Theorem 1.2. By the proof of Lemma 2.7

\[
\| P \|_{L^p(H,\nu dV)} = \| M_{z_2}QM_{\nu} : L^p(H,\nu dV) \to L^p(H,\mu dV) \|. 
\]

It suffices to show that \( [\mu,\nu]_p \lesssim \| M_{z_2}QM_{\nu} : L^p(H,\nu dV) \to L^p(H,|z_2|^{-p}\mu dV) \|^{2p} \). For simplicity, we set \( A := \| M_{z_2}QM_{\nu} : L^p(H,\nu dV) \to L^p(H,|z_2|^{-p}\mu dV) \|. \) Set \( \mu_p \{ (w_1, w_2) \in H : |M_{z_2}QM_{\nu}f(w_1, w_2)| > \lambda \} \lesssim \frac{A^p}{\lambda p} \| f \|_{L^p(H,\nu dV)}^p. \) We choose \( f(w_1, w_2) = 1_{K_{\gamma,\eta}}(w_1, w_2) \) with \( \gamma \) and \( \eta \) to be determined. Then

\[
|M_{z_2}QM_{\nu}1_{K_{\gamma,\eta}}(z_1, z_2)| \\
= \left| \int_{K_{\gamma,\eta}} \frac{1}{\pi^2(1 - \frac{z_1w_1}{z_2w_2})^2(1 - \frac{z_2w_2}{z_2w_2})^2} \nu(w_1, w_2) dV(w_1, w_2) \right| \\
= \left| \int_{K_{\gamma,\eta}} \frac{1}{\pi^2(1 - \frac{z_1t_1}{z_2t_2})^2(1 - \frac{z_2t_2}{z_2t_2})^2} \nu(t_1 w_2, w_2) |w_2|^2 dV(t_1, w_2) \right| \\
= \left| P_{D^2}(|w_2|^2 \nu(t_1 w_2, w_2) 1_{K_{\gamma \times K_{\eta}}}(t_1, w_2))(z_1/z_2, z_2) \right|.
\]

Here \( P_{D^2} \) is the Bergman projection on the polydisc \( D^2 \).

Recall that for a point \( z \in D \) and a tree structure \( T \), the generation \( \text{gen}(z) \) equals \( N \) if \( z \in K_j^N \) for some \( j \). By [Bek82], Lemma 5], there exists an integer \( N \) so that for \( \gamma \in T_m \) with \( \text{gen}(\gamma) > N \), there is a \( \gamma' \in T_{m'} \) with \( \text{gen}(\gamma') = \text{gen}(\gamma) \) such that for any fixed \( z \in K_{\gamma'} \) and all \( w \in K_{\gamma} \) there holds,

\[
(1 - z\bar{w})^{-2} = (1 - z\bar{\gamma})^{-2} + ((1 - z\bar{w})^{-2} - (1 - z\bar{\gamma})^{-2})
\]
where \(|(1-z\bar{w})^{-2}-(1-z\bar{\gamma})^{-2}| \leq 2^{-1}|1-z\bar{\gamma}|^{-2}\) and \(|1-z\bar{\gamma}|^2 \approx |\hat{K}_{\gamma}|\). Moreover, an elementary geometric argument yields that \(\text{arg}((1-z\bar{w})^{-2}, (1-z\bar{\gamma})^{-2}) \leq \pi/6\) for all \(w \in \hat{K}_{\gamma}\). Thus for \((\gamma, \eta) \in T_m \times T_n\) with \(\text{gen}(\gamma), \text{gen}(\eta) > N\), there is a \((\gamma', \eta') \in T_{m'} \times T_{n'}\) with \(\text{gen}(\gamma) = \text{gen}(\gamma')\) and \(\text{gen}(\eta) = \text{gen}(\eta')\) such that for any fixed \((z_1/z_2, \bar{z}_2) \in \hat{K}_{\gamma} \times \hat{K}_{\eta}\) there holds:

\[
\text{arg} \left( \left(1 - \frac{z_1}{z_2} \right)^{-2} \left(1 - z_2 \bar{w}_2 \right)^{-2}, \left(1 - \frac{z_1}{z_2} \right)^{-2} \left(1 - z_2 \bar{\eta} \right)^{-2} \right) \leq \pi/3,
\]

for all \((t_1, w_2) \in \hat{K}_{\gamma} \times \hat{K}_{\eta}\). Hence

\[
|P_{D^2}(w_2^2 \nu(t_1 w_2, w_2)1_{K_{\gamma} \times K_{\eta}}(t_1, w_2))(z_1/z_2, z_2)|
= \int_{K_{\gamma} \times K_{\eta}} \frac{1}{\pi^2 \left(1 - \frac{z_1}{z_2} \right)^2 \left(1 - z_2 \bar{w}_2 \right)^2} \nu(t_1 w_2, w_2) |w_2|^2 dV(t_1, w_2)
\geq 16^{-1} \int_{K_{\gamma} \times K_{\eta}} \frac{1}{\pi^2 \left(1 - \frac{z_1}{z_2} \right)^2 \left(1 - z_2 \bar{\eta} \right)^2} \nu(t_1 w_2, w_2) |w_2|^2 dV(t_1, w_2)
> c_1 \langle |w_2|^2 \nu(t_1 w_2, w_2) \rangle_{K_{\gamma} \times K_{\eta}}^d,
\]

for some constant \(c_1\). Thus via the biholomorphism between \(D \times D^*\) and \(\mathbb{H}\), the following containment holds:

\[
\hat{K}'_{\gamma', \eta'} \subseteq \{ (w_1, w_2) \in \mathbb{H} : |M_{z_1}QM_{\bar{w}_1}f(w_1, w_2)| > c_1 \langle |w_2|^2 \nu(t_1 w_2, w_2) \rangle_{K_{\gamma} \times K_{\eta}}^d \}.
\tag{3.17}
\]

Note also that \(\langle |w_2|^2 \nu(t_1 w_2, w_2) \rangle_{K_{\gamma} \times K_{\eta}}^d = \langle |w_2|^2 \nu \rangle_{K_{\gamma} \times K_{\eta}}^d\). Inequality (3.15) then implies

\[
\mu_p(\hat{K}'_{\gamma', \eta'}) \leq \mathcal{A}^p \left( \langle |w_2|^2 \nu \rangle_{K_{\gamma} \times K_{\eta}}^d \right)^{-p} \nu(\hat{K}'_{\gamma', \eta'}),
\tag{3.18}
\]

which is equivalent to \(\langle |w_2|^{2-p} \mu \rangle_{K_{\gamma} \times K_{\eta}}^d \left( \langle |w_2|^2 \nu \rangle_{K_{\gamma} \times K_{\eta}}^d \right)^{p-1} \lesssim \mathcal{A}^p\). Since one can interchange the roles of \(\gamma, \eta\) and \(\gamma', \eta'\) in the proof of \([\text{Bek82}, \text{Lemma 5}]\), there holds

\[
\langle |w_2|^{2-p} \mu \rangle_{K_{\gamma} \times K_{\eta}}^d \left( \langle |w_2|^2 \nu \rangle_{K_{\gamma} \times K_{\eta}}^d \right)^{p-1} \lesssim \mathcal{A}^p.
\tag{3.19}
\]

Combining these two inequalities, we have

\[
\left( \langle |w_2|^{2-p} \mu \rangle_{K_{\gamma} \times K_{\eta}}^d \left( \langle |w_2|^2 \nu \rangle_{K_{\gamma} \times K_{\eta}}^d \right)^{p-1} \right) \left( \langle |w_2|^{2-p} \mu \rangle_{K_{\gamma} \times K_{\eta}}^d \left( \langle |w_2|^2 \nu \rangle_{K_{\gamma} \times K_{\eta}}^d \right)^{p-1} \right) \lesssim \mathcal{A}^{2p}.
\tag{3.19}
\]

By Hölder’s inequality,

\[
u(\hat{K}'_{\gamma, \eta})^p \leq \int_{K_{\gamma} \times K_{\eta}} |w_2|^{2-p} \mu d\nu \left( \int_{K_{\gamma} \times K_{\eta}} |w_2|^2 \nu d\nu \right)^{p-1}
\tag{3.20}
\]

for any \((\gamma, \eta) \in T_{m,n}\). Therefore \(\langle |w_2|^{2-p} \mu \rangle_{K_{\gamma} \times K_{\eta}}^d \left( \langle |w_2|^2 \nu \rangle_{K_{\gamma} \times K_{\eta}}^d \right)^{p-1} \gtrsim 1\) for all \((\gamma, \eta) \in T_{m,n}\).

Applying this to (3.19) and taking the supremum of the left side of (3.19) for \(\text{gen}(\gamma) > N\) and \(\text{gen}(\eta) > N\), there holds

\[
\sup_{(\gamma, \eta) \in T_{m,n}, \\text{gen}(\gamma), \text{gen}(\eta) > N} \langle |w_2|^{2-p} \mu \rangle_{K_{\gamma} \times K_{\eta}}^d \left( \langle |w_2|^2 \nu \rangle_{K_{\gamma} \times K_{\eta}}^d \right)^{p-1} \lesssim \mathcal{A}^{2p}.
\tag{3.21}
\]
We turn to show that (3.21) also holds when the supremum is taken over tents where either $\text{gen}(\gamma) \leq N$ or $\text{gen}(\eta) \leq N$.

Suppose that both $\text{gen}(\gamma) \leq N$ and $\text{gen}(\eta) \leq N$. Then $\hat{K}_\gamma$ and $\hat{K}_\eta$ are big tents on the unit disk $\mathbb{D}$ and $|\hat{K}_\gamma| = |\hat{K}_\eta| \approx 1$. Set $B_{1/4} = \{z \in \mathbb{C} : |z| < 1/4\}$. Then for any given $z \in \mathbb{D}$, $|z\bar{w}| < 1/4$ for $w \in B_{1/4}$. Therefore $\text{Arg}((1 - z\bar{w})^2) \subseteq [-\frac{\pi}{6}, \frac{\pi}{6}]$. Applying this fact, we obtain

$$\left| P_{\mathbb{D}^2}(|w|^2 B_{1/4} \times B_{1/4}(t_1, w_2)) \left( \frac{z_1}{z_2}, z_2 \right) \right|$$

$$= \left| \int_{B_{1/4} \times B_{1/4}} \pi^2 (1 - \frac{z_1}{z_2} t_1)^2 (1 - z_2 \bar{w}_2)^2 dV(t_1, w_2) \right|$$

$$\geq 16^{-1} \left| \int_{B_{1/4} \times B_{1/4}} \pi^{-2} |w|^2 dV(t_1, w_2) \right| > c_2$$

for some constant $c_2$. Therefore,

$$\mathbb{D}^2 = \{(z_1, z_2) \in \mathbb{D}^2 : |P_{\mathbb{D}}(1 B_{1/4} \times B_{1/4})(z_1, z_2)| > c_2 \} . \quad (3.22)$$

Let $B'_{1/4}$ denote the set $\{(w_1, w_2) \in \mathbb{H} : (\frac{w_1}{w_2}, w_2) \in B_{1/4} \times B_{1/4}\}$. Via the biholomorphism between $\mathbb{D} \times \mathbb{D}^*$ and $\mathbb{H}$, we obtain

$$\mu_{p}(\mathbb{H}) = \mu_{p} \left\{(w_1, w_2) \in \mathbb{H} : |P_{\mathbb{D}}(1 B'_{1/4})(w_1, w_2)| > \frac{c}{2} \right\} \leq \frac{A^p}{c^2} \|1 B'_{1/4}\|_{L^p(\mathbb{H}, \nu dV)} . \quad (3.23)$$

By [Bek82] Lemma 4, there holds that $\nu(\mathbb{H}) < \infty$. Therefore

$$\langle |w|^2 \rangle_{K_{\gamma, \eta}}^{du} (\langle |w|^2 \rangle_{K_{\gamma, \eta}}^{du})^{p-1} = \frac{\int_{K_{\gamma, \eta}} |w|^2 \nu dV}{\mu(K_{\gamma, \eta})} \left( \frac{\int_{K_{\gamma, \eta}} \nu dV}{\mu(K_{\gamma, \eta})} \right)^{p-1}$$

$$\lesssim \int_{\mathbb{H}} |w|^2 \nu dV \left( \int_{\mathbb{H}} \nu dV \right)^{p-1} \lesssim A^p . \quad (3.24)$$

For the case $\text{gen}(\gamma) \leq N$ and $\text{gen}(\eta) > N$, we combine the arguments for both the big tents and the small tents. There exists an $\eta'$ with $\text{gen}(\eta') = \text{gen}(\eta)$ such that for all $\frac{w_1}{z_2} \in \mathbb{D}^*$ and $z_2 \in \hat{K}_{\eta'}$, there holds:

$$|P_{\mathbb{D}^2}(|w|^2 \nu(t_1, w_2) 1_{B_{1/4} \times K_\eta}(t_1, w_2))(z_1/z_2, z_2)|$$

$$= \left| \int_{B_{1/4} \times K_\eta} \pi^2 (1 - \frac{z_1}{z_2} t_1)^2 (1 - z_2 \bar{w}_2)^2 dV(t_1, w_2) \right|$$

$$\geq 16^{-1} \int_{B_{1/4} \times K_\eta} \pi^{-2} |w|^2 \nu(t_1, w_2)^2 dV(t_1, w_2) > c_3 \langle |w|^2 \nu(t_1, w_2)^2 \rangle_{K_\eta \times \hat{K}_\eta}$$

for some constant $c_3$. Set $B'_{1/4, \eta} = \{(w_1, w_2) \in \mathbb{H} : (\frac{w_1}{w_2}, w_2) \in B_{1/4} \times \hat{K}_\eta\}$. Via the biholomorphism between $\mathbb{D} \times \mathbb{D}^*$ and $\mathbb{H}$, and $\hat{K}_\eta$, the following containment holds:

$$\hat{K}_{0, \eta'} \subseteq \{(w_1, w_2) \in \mathbb{H} : |M_{z_2} Q_{M_{\nu}} 1_{B'_{1/4, \eta}}(w_1, w_2)| > \frac{c_3}{32} \langle |w|^2 \nu(t_1, w_2)^2 \rangle_{K_\eta \times \hat{K}_\eta} \} . \quad (3.25)$$

Applying the proof for inequalities (3.21) and (3.24) to (3.25) gives

$$\langle |w|^2 \rangle_{K_{\gamma, \eta}}^{du} (\langle |w|^2 \rangle_{K_{\gamma, \eta}}^{du})^{p-1} \lesssim \langle |w|^2 \rangle_{K_{0, \eta'}}^{du} (\langle |w|^2 \rangle_{K_{0, \eta'}}^{du})^{p-1} \lesssim A^p . \quad (3.26)$$
Similarly, for \(\eta\) we use a similar argument with the role of \(\gamma\) interchanged. Combining all these estimates, we obtain the desired lower bound:

\[
[p, \nu]_p = \sup_{(\gamma, \eta) \in \mathcal{T}_{m,n}} \left( |w_2|^{2-p}\mu_{\mathcal{T}_{m,n}} \left( \|w_2\|^2\nu_{\mathcal{T}_{m,n}} \right) \right)^{p-1} \lesssim A^{2p}, \tag{3.27}
\]

which completes the proof of Theorem 1.2.

4. Examples

We begin by providing a sharp example for the upper bound estimate in Theorem 1.2.

4.1. A sharp example for the upper bound. We give an example for the case \(1 < p \leq 2\). The case \(p > 2\) follows from a duality argument. The idea is based on the construction of the sharp examples in [PR13] and [RTW17]. Given a number \(1 > s > 0\), we set

\[
\mu(w_1, w_2) = |w_2|^{p-2} \left( \frac{(1 - w_1/w_2)(1 - w_2)}{(1 + w_1/w_2)(1 + w_2)} \right)^{2(p-1)(1-s)}.
\]

Then for \((t_1, t_2) \in \mathbb{D}^2\),

\[
\int_{T_{t_1} \times T_{t_2}} \left| w_2 \right|^{2-p} \mu(w_1, w_2) d\nu(w_1, w_2) = \int_{T_{t_1} \times T_{t_2}} \left| \frac{(1 - w_1)(1 - w_2)}{(1 + w_1)(1 + w_2)} \right|^{(p-1)(2-2s)} d\nu(w_1, w_2)
\]

\[
= \prod_{j=1}^2 \int_{\{ w_j \in \mathbb{D} : |1 - w_j| < 1 - |t_j| \}} \left| \frac{1 - w_j}{1 + w_j} \right|^{(p-1)(2-2s)} d\nu(w_j).
\]

Using the changes of variables \(z_j = i \frac{1-w_j}{1+w_j}\), we have

\[
\int_{\{ w_j \in \mathbb{D} : |1 - w_j| < 1 - |t_j| \}} \left| \frac{1 - w_j}{1 + w_j} \right|^{(p-1)(2-2s)} d\nu(w_j) \approx \int_{\{ z_j \in \mathbb{C} : |z_j| < 1 - |t_j|, \text{Im} z_j > 0 \}} |z_j|^{p-1}(2-2s) |i + z_j|^{-4} d\nu(z_j) \approx \frac{(1 - |t_j|)^{(p-1)(2-2s)+2}}{(p-1)(2-2s) + 2}.
\]

Thus

\[
\int_{T_{t_1} \times T_{t_2}} \left| w_2 \right|^{2-p} \mu(w_1, w_2) d\nu(w_1, w_2) \approx \prod_{j=1}^2 \frac{(1 - |t_j|)^{(p-1)(2-2s)+2}}{(p-1)(2-2s) + 2}.
\]

Similarly, for \(\nu = |w_2|^{-p'} \mu^{\frac{s}{p}}\),

\[
\int_{T_{t_1} \times T_{t_2}} \left| w_2 \right|^{2-p} \nu(w_1, w_2) d\nu(w_1, w_2) \approx \prod_{j=1}^2 \frac{(1 - |t_j|)^{2s}}{2s}.
\]

Since \(1 < p \leq 2\) and \(0 < s < 1\), we have

\[
[p, \nu]_p = \sup_{t_1, t_2 \in \mathbb{D}} \left( |w_2|^{2-p}\mu_{T_{t_1} \times t_2} \left( \|w_2\|^2\nu_{T_{t_1} \times t_2} \right) \right)^{p-1} \lesssim s^{-2(p-1)}.
\]
When \( w_1/w_2, w_2 \in T_{1/2} \), there holds \(|1 + w_1/w_2|^{-1} |1 + w_2|^{-1} |w_2|^{-1} \approx 1 \). Thus

\[
\|f\|_{L^p(\mathbb{H}, \mu)}^p \approx \int_{T_{1/2,1/2}} \mu^{\frac{p}{2}}(w_1, w_2) \mu(w_1, w_2) dV(w_1, w_2) \approx \langle |w_2|^2 \nu \rangle^2_{T_{1/2,1/2}} \approx s^{-2}. \tag{4.7}
\]

For \( z_1/z_2, z_2 \in T_{-1/2} \), we claim that

\[
|P(f)(z_1, z_2)| \gtrsim \left| \frac{(1 - z_1/z_2)(1 - z_2)}{(1 + z_1/z_2)(1 + z_2)} \right|^{-2} \langle f \rangle_{T_{1/2,1/2}}. \tag{4.8}
\]

Using formula (2.2) for the Bergman projection, we have

\[
|P(f)(z_1, z_2)| = \int_{T_{1/2,1/2}} \frac{f(w_1, w_2)}{\pi^2 z_2 w_2 (1 - \frac{w_1}{z_2})^2 (1 - w_2 z_2)^2} dV(w_1, w_2) \]

\[
= |z_2|^{-1} \int_{T_{1/2,1/2}} \frac{f(w_1 w_2, w_2)}{\pi^2 (1 - \frac{w_1}{z_2})^2 (1 - w_2 z_2)^2} dV(w_1, w_2). \tag{4.9}
\]

Set \( \tilde{h}(t_1, t_2) = (h(t_1), h(t_2)) \) to be the biholomorphic mapping from the product of upper-half plane \( \mathcal{H}^2 \) to the bidisk \( \mathbb{D}^2 \) with

\[
h(t_j) = \frac{i - t_j}{i + t_j} \quad \text{and} \quad h^{-1}(t_j) = \frac{1 - t_j}{1 + t_j}. \tag{4.10}
\]

Applying the changes of variables \( h(t_j) = w_j \), the biholomorphism transformation formula, and using the fact \(|z_2| > 1/2\) for \( z_2 \in T_{-1/2} \) yield

\[
|z_2|^{-1} \left| \int_{T_{1/2,1/2}} \frac{f(w_1 w_2, w_2)}{\pi^2 (1 - \frac{w_1}{z_2})^2 (1 - w_2 z_2)^2} dV(w_1, w_2) \right|
\]

\[
\approx |J_{\mathcal{H}} \tilde{h}^{-1}(\frac{z_1}{z_2}, z_2)| \int_{T_{1/2,1/2}} w_2 f(w_1 w_2, w_2) K_{\mathcal{H}^2}(\tilde{h}^{-1}(\frac{z_1}{z_2}, z_2); h^{-1}(w_1, w_2)) J_{\mathcal{H}} h^{-1}(w_1, w_2) dV(w_1, w_2)
\]

\[
\approx |J_{\mathcal{H}} h^{-1}(\frac{z_1}{z_2}, z_2)| \int_{\tilde{h}^{-1}(T_{1/2,1/2})} \frac{|h(t_2)|^{2s-2}}{(i + t_1)^{-4}(i + t_2)^{-4}} K_{\mathcal{H}^2}(\tilde{h}^{-1}(\frac{z_1}{z_2}, z_2); \tilde{t}_1, \tilde{t}_2) J_{\mathcal{H}} \tilde{h}(t_1, t_2) dV(t_1, t_2)
\]

\[
= |J_{\mathcal{H}} h^{-1}(\frac{z_1}{z_2}, z_2)| \int_{h^{-1}(T_{1/2})} K_{\mathcal{H}}(h^{-1}(\frac{z_1}{z_2}); \tilde{t}_1) |t_1|^{2(s-1)} dV(t_1)
\]

\[
\times \int_{h^{-1}(T_{1/2})} K_{\mathcal{H}}(h^{-1}(z_2); \tilde{t}_2) |t_2|^{2(s-1)} |h(t_2)|^{\frac{2s-2}{s}} dV(t_2). \tag{4.11}
\]

Since \( z_1/z_2, z_2 \in T_{-1/2} \), we have \(|J_{\mathcal{H}} h^{-1}(\frac{z_1}{z_2}, z_2)| = 4|1 + z_1/z_2|^{-2} |1 + z_2|^{-2} \geq 1 \). Furthermore, a hyperbolic geometry argument implies that \( h^{-1}(T_{-1/2}) \subseteq \{ w \in \mathbb{D} : \text{Im}w > 0, |w| > 3 \} \) and \( h^{-1}(T_{1/2}) \subseteq \{ w \in \mathbb{D} : \text{Im}w > 0, |w| < 1/3 \} \). Hence for \( z \in h^{-1}(T_{-1/2}) \) and any points \( \zeta_1, \zeta_2 \in h^{-1}(T_{1/2}) \), we have \( \arg \{(z - \zeta_1)^2, (z - \zeta_2)^2\} \leq \frac{\pi}{2} \) and \( |z - \zeta_j| \lesssim |z| \). Applying these
Thus the claim (4.8) holds. Using a change of variables

\[ \int_{h^{-1}(T_{1/2})} \left| t_1 \right|^{2(s-1)} |h(t_2)|^{\frac{p-2}{p-1}} dV(t_2) \]

\[ \gtrsim \left( \int_{h^{-1}(T_{1/2})} \left| t_1 \right|^{2(s-1)} |h(t_2)|^{\frac{p-2}{p-1}} dV(t_2) \right)^2 \]

\[ \gtrsim \left( \int_{h^{-1}(T_{1/2})} \left| t_1 \right|^{2(s-1)} dV(t) \right)^2 \]

\[ \gtrsim \left( \frac{(1 - z_1/z_2)(1 - z_2)}{(1 + z_1/z_2)(1 + z_2)} \right)^{-2} s^{-2}. \]  

(4.12)

Note that

\[ \langle f \rangle_{T_{1/2,1/2}}^{T_{1/2,1/2}} \approx \int_{T_{1/2,1/2}} \left| w_2 \right|^{\frac{p-2}{p}} \left( \frac{(1 - w_1/w_2)(1 - w_2)}{(1 + w_1/w_2)(1 + w_2)} \right)^{2(s-1)} dV(w_1, w_2) \]

\[ \approx \langle |w_2|^{2} \rangle_{T_{1/2,1/2}}^{T_{1/2,1/2}} \approx s^{-2}. \]  

(4.13)

Thus the claim (4.8) holds. Using a change of variables \( w_1' = w_1 \) and \( w_2' = -w_2 \) and inequality (4.8), there holds

\[ \| Pf \|_{L^p(\mathbb{H}, \mu dV)}^p = \int_{\mathbb{H}} \left| Pf(w_1, w_2) \right|^p \mu(w_1, w_2) dV(w_1, w_2) \]

\[ \geq s^{-2p} \int_{T_{1/2,1/2}} \left| \frac{(1 - w_1/w_2)(1 - w_2)}{(1 + w_1/w_2)(1 + w_2)} \right|^{-2p} \mu(w_1, w_2) dV(w_1, w_2) \]

\[ = s^{-2p} \int_{T_{1/2,1/2}} \left| w_2 \right|^{2p-4} \left( \frac{(1 - w_1'/w_2')(1 - w_2')}{(1 + w_1'/w_2')(1 + w_2')} \right)^{2p} \mu^{-1}(w_1', w_2') dV(w_1', w_2') \]

\[ \approx s^{-2p} \int_{T_{1/2,1/2}} \left( \frac{(1 - w_1')(1 - w_2')}{(1 + w_1')(1 + w_2')} \right)^{(2p-1)(s-1)+2p} dV(w_1', w_2'). \]  

(4.14)

Using the changes of variables \( t_j = \frac{i + w_j'}{1 - w_j} \), we have for a sufficiently large constant \( R \)

\[ s^{-2p} \int_{T_{1/2,1/2}} \left( \frac{(1 - w_1')(1 - w_2')}{(1 + w_1')(1 + w_2')} \right)^{(2p-1)(s-1)+2p} dV(w_1', w_2') \]

\[ \gtrsim s^{-2p} \prod_{j=1}^{2} \int_{\{t_j \in \mathbb{C} : 2R > |t_j| > R, \Im t_j > 0\}} \left| t_j \right|^{2p-1(1-s)-2p} |i + t_j|^{-4} dV(t_j) \]

\[ \gtrsim s^{-2p} \prod_{j=1}^{2} \int_{\{t_j \in \mathbb{C} : 2R > |t_j| > R, \Im t_j > 0\}} \left| t_j \right|^{-2p-2} dV(t_j) \]

\[ \gtrsim s^{-2p-2}(p-1)^{-2}. \]  

(4.15)

Thus \( \| Pf \|_{L^p(\mathbb{H}, \mu dV)} \gtrsim (p-1)^{-2} s^{-2p-2} \gtrsim \left( \frac{p^4}{(p-1)^2} \right)^{\frac{1}{p}} \left( \mu, \nu \right)^{\max \left(1, \frac{p-1}{p-2} \right)} \| f \|_{L^p(\mathbb{H}, \mu dV)}. \)
4.2. \( L^p \) regularity of the Bergman projection on the Hartogs triangle. If weight \( \mu \) is identically 1, then \( \mu dV \) is the Lebesgue measure on the Hartogs triangle, and \( \|P^+\|_{L^p(D, \mu)} \) is the unweighted \( L^p \) norm of the Bergman projection. Chakrabarti and Zeytuncu showed in [CZ16] that the Bergman projection on the Hartogs triangle is \( L^p \) regular if and only if \( \frac{4}{3} < p < 4 \). Using Theorem 1.2 we give an alternative proof of this \( L^p \) regularity result.

Set \( \mu \equiv 1 \). Then \( \nu = |w_2|^{-p'} \) and

\[
[\mu, \nu]_p = \sup_{(\gamma, \eta) \in T_{m,n}} \left( \frac{\|w_2\|_{K_{\gamma, \eta}}^{2-p} \|w_2\|_{K_{\gamma, \eta}}^{2-p'}}{1 \leq m, n \leq M} \right)^{p-1}.
\]

When \( p \geq 4 \) or \( p \leq 4/3 \), we have \( \int_{\mathbb{D}} |w_2|^{-p} du \int_{\mathbb{D}} |w_2|^{-p'} du = \infty \). Thus \( [\mu, \nu]_p = \infty \) for \( p \notin \left( \frac{4}{3}, 4 \right) \). By Theorem 1.2 the Bergman projection \( P \) is not bounded on \( L^p(\mathbb{D}) \).

When \( p \in \left( \frac{4}{3}, 4 \right) \), we have

\[
[\mu, \nu]_p = \sup_{(\gamma, \eta) \in T_{m,n}} \left( \frac{\|w_2\|_{K_{\gamma, \eta}}^{2-p} \|w_2\|_{K_{\gamma, \eta}}^{2-p'}}{1 \leq m, n \leq M} \right)^{p-1} = \sup_{\eta \in T_{m,n}} \left( \frac{\|w_2\|_{K_{\eta}}^{2-p} \|w_2\|_{K_{\eta}}^{2-p'}}{1 \leq n \leq M} \right)^{p-1}.
\]

The supremum above is taken over finitely many Bergman trees on the unit disk \( \mathbb{D} \). Therefore for all \( w_2 \in K_{\eta} \) with \( \eta \neq 0 \), we have \( |w_2| \geq 1 \) and

\[
\left( \frac{\|w_2\|_{K_{\eta}}^{2-p} \|w_2\|_{K_{\eta}}^{2-p'}}{1 \leq n \leq M} \right)^{p-1} \approx 1.
\]

When \( \eta = 0 \), \( \hat{K}_{\eta} = \mathbb{D} \) and

\[
\left( \frac{\|w_2\|_{\mathbb{D}}^{2-p} \|w_2\|_{\mathbb{D}}^{2-p'}}{1 \leq n \leq M} \right)^{p-1} = \frac{2}{4-p} \left( \frac{2(p-1)}{3p-4} \right)^{p-1}.
\]

Thus \( [\mu, \nu]_p \approx \frac{2}{4-p} \left( \frac{2(p-1)}{3p-4} \right)^{p-1} < \infty \). Theorem 1.2 gives the \( L^p \) boundedness of the Bergman projection \( P \) for \( \frac{4}{3} < p < 4 \), and implies the blowing up of \( \|P\|_{L^p(\mathbb{D})} \) as \( p \to \frac{4}{3}^+ \) or \( p \to 4^- \). This fact can also be checked by computing \( \|P(\bar{z}_2)\|_{L^p(\mathbb{D})} \) for the case \( p \to 4^- \) and computing the quotient \( \|P(\bar{z}_2^{-p} \bar{z}_2)\|_{L^p(\mathbb{D})}/\|\bar{z}_2^{-p} \bar{z}_2\|_{L^p(\mathbb{D})} \) for the case \( p \to \frac{4}{3}^+ \).

4.3. The case \( \mu(w_1, w_2) = |w_1|^a |w_2|^b \). When the weight \( \mu(w_1, w_2) = |w_1|^a |w_2|^b \), the weight \( \nu = |w_1|^{-ap'/p} |w_2|^{-p'(1+b/p)} \). By a change of variables we obtain

\[
[\mu, \nu]_p = \sup_{(\gamma, \eta) \in T_{m,n}} \left( \frac{|w_1|^a |w_2|^{2-p+b} \|w_2\|_{K_{\gamma, \eta}}^{2-p} \|w_2\|_{K_{\gamma, \eta}}^{2-p'}}{1 \leq m, n \leq M} \right)^{p-1} = \sup_{(\gamma, \eta) \in T_{m,n}} \left( \frac{|w_1|^a \|w_2\|_{K_{\eta}}^{2-p} \|w_2\|_{K_{\eta}}^{2-p'}}{1 \leq n \leq M} \right)^{p-1}.
\]

By a similar argument as in the previous example, we have the following estimate for \( [\mu, \nu]_p \):

- \( [\mu, \nu]_p \approx (a+2)^{-1} (2 - \frac{ap'}{p})_1^{p-1} (4 - p + a + b)^{-1} (4 - p'(1 + \frac{a+b}{p})^{1-p} \), for \(-2 < a < 2(p-1)\) and \(p - 4 < a + b < 3p - 4\);
Here operator is not self-adjoint on the generalized Hartogs triangle and the measure $u$. Using the proper map $\omega$ under a rescaling of $2$ if and only if $\max\{1, \frac{a+1}{2}, \frac{a+b+4}{3}\} < p < a + b + 4$. The $p$ range we obtain here is not of form $(\alpha, \frac{a}{a-1})$. This is because, for the case $a$ or $b$ is not zero, the Bergman projection operator is not self-adjoint on $L^p(\mathbb{H}, \mu dV)$, and is not necessarily $L^2$ bounded.

### 4.4. $L^p$ regularity of the Bergman projection on the generalized Hartogs triangle.

In [EM17], Edholm and McNeal studied the $L^p$ boundedness of the Bergman projection on the generalized Hartogs triangle

$$\mathbb{H}_{m/n} = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^m < |z_2|^n < 1\},$$

where $m, n \in \mathbb{Z}^+$ with $\gcd(m, n) = 1$. A crucial step in their paper (see [EM17, Proposition 3.4]) is to analyze the $L^p$ regularity of the integral operator $K_A$ defined by

$$K_A(f)(z_1, z_2) := \int_{\mathbb{H}_{m/n}} \frac{|z_2\bar{w}_2|^A}{|1 - z_2\bar{w}_2|^2 |z_1^m\bar{w}_1^m - z_1^m\bar{w}_1^m|^2} f(w_1, w_2) dV(w_1, w_2).$$

Using the proper map $h : (w_1, w_2) \mapsto (w_1^m w_2^{m-1}, w_2)$ from $\mathbb{H}_{m/n}$ to $\mathbb{H}$, we can relate the $L^p$ norm of $K_A$ on $\mathbb{H}_{m/n}$ to the weighted $L^p$ norm of the absolute Bergman projection on $\mathbb{H}$:

$$\|K_A\|_{L^p(\mathbb{H}_{m/n})} = m^{-1} \|M_h P^+ M_h : L^p(\mathbb{H}, \omega_1 dV) \rightarrow L^p(\mathbb{H}, \omega_2 dV)\| = m^{-1} \|P^+ M_h : L^p(\mathbb{H}, \omega_1 dV) \rightarrow L^p(\mathbb{H}, \omega_2 h^p dV)\|,$$

where the weights $\omega_1(w_1, w_2) = |w_1|^{2n-2} |w_2|^{2m(n-1)}$, $\omega_2(w_1, w_2) = |w_1|^{-2+2/n} |w_2|^{2(n-1)}$, and $h(w_1, w_2) = |w_2|^{A-2n+1}$. Setting $\mu := \omega_2 h^p$ and $\nu := \omega_1^\frac{1}{p} |w_2|^{(A-2n)p'}$, we obtain

$$\|P^+ M_h : L^p(\mathbb{H}, \omega_1 dV) \rightarrow L^p(\mathbb{H}, \omega_2 h^p dV)\| = \|Q^+ M_\nu : L^p(\mathbb{H}, \nu dV) \rightarrow L^p(\mathbb{H}, \mu dV)\|. \quad (4.19)$$

Here $\nu$ is no longer equal to $|w_2|^{-p} \mu^\frac{1}{p}$ which is the dual weight of $|w_2|^{-p} \mu$ with respect to the measure $u$. Still, by Hölder’s inequality,

$$\int_{K'_{\gamma, \eta}} |w_2|^{-1} \mu^\frac{1}{p} \nu^\frac{1}{p'} dV \leq \left( \nu(K'_{\gamma, \eta}) \right)^\frac{1}{p'} \left( \int_{K'_{\gamma, \eta}} |w_2|^{-p} \mu dV \right)^\frac{1}{p}.$$

Set $\lambda := |w_2|^{-1} \mu^\frac{1}{p} \nu^\frac{1}{p'}$. When $Am + 2n + 2m - 2nm > 2nm - Am > 0$, we further have

$$u(K'_{\gamma, \eta}) = \left( \langle |z_2|^2 \lambda \rangle_{K'_{\gamma, \eta}} \right)^{-1} \int_{K'_{\gamma, \eta}} |w_2|^{-1} \mu^\frac{1}{p} \nu^\frac{1}{p'} dV \leq \left( \nu(K'_{\gamma, \eta}) \right)^\frac{1}{p'} \left( \int_{K'_{\gamma, \eta}} |w_2|^{-p} \mu dV \right)^\frac{1}{p} \left( \langle |z_2|^2 \lambda \rangle_{K'_{\gamma, \eta}} \right)^{-1} < \infty.$$

Set

$$[\mu, \nu, \lambda]_{p,a} := \sup_{(\gamma, \eta) \in \Gamma, \nu, \lambda} \langle |w_2|^{2-p} \mu^\frac{1}{p} \nu^\frac{1}{p'} \rangle_{K'_{\gamma, \eta}} \left( \langle |w_2|^2 \nu \rangle_{K'_{\gamma, \eta}} \right)^{p-1} \left( \langle |z_2|^2 \lambda \rangle_{K'_{\gamma, \eta}} \right)^{-a}.$$

We remark here that, by the definition of $\mu$, $\nu$, and $\lambda$, the constant $[\mu, \nu, \lambda]_{p,a}$ is invariant under a rescaling of $\omega_1$ and $\omega_2$ by the same scaler $c$. By an similar argument as in Example
4.1, we obtain for \( p \in \left( \frac{2n+2m}{Am+2n+2m-2nm}, \frac{2n+2m}{2nm-Am} \right) \),

\[
[\mu, \nu, \lambda]_{p, p-1} \approx m \left( (A - 2n)p + \frac{2n + 2m}{m} + 2 \right)^{-1} \left( \frac{2A + \frac{2n}{m} + 2 - 4n}{m(n-1) + (A - 2n)\frac{p}{p-1} + 2 + \frac{2}{m}} \right)^{p-1}
\]

\[
= \frac{m(p-1)^{p-1}(2Am + 2n + 2m - 4nm)^{p-1}}{(2n - A)(Am + 2n + 2m - 2nm)^{p-1}} \left( \frac{2n + 2m}{2nm - Am - p} \right)^{-1}
\]

\[
\times \left( p - \frac{2n + 2m}{Am + 2n + 2m - 2nm} \right)^{1-p},
\]

(4.20)

and

\[
[\mu, \nu, \lambda]_{p, 1} \approx m^{p-1} \frac{2A + \frac{2n}{m} + 2 - 4n}{(A - 2n)p + \frac{2n + 2m}{m} + 2} \left( \frac{2}{m(n-1) + (A - 2n)\frac{p}{p-1} + 2 + \frac{2}{m}} \right)^{1-p}
\]

\[
= \frac{m^{2p-3}(p-1)^{p-1}(2Am + 2n + 2m - 4nm)^{p-1}}{(2n - A)(Am + 2n + 2m - 2nm)^{p-1}} \left( \frac{2n + 2m}{2nm - Am - p} \right)^{-1}
\]

\[
\times \left( p - \frac{2n + 2m}{Am + 2n + 2m - 2nm} \right)^{1-p}.
\]

(4.21)

Replacing \([\mu, \nu]_p\) in (3.3) by \([\mu, \nu, \lambda]_{p, p-1}\) for the case \( p \geq 2 \) and using a duality argument as in (3.12) for the case \( 1 < p < 2 \), we recover [EM17, Proposition 3.4]:

\[
\mathcal{K}_A \text{ is bounded on } L^p(\mathbb{H}_{m/n}) \text{ if } p \in \left( \frac{2n + 2m}{Am + 2n + 2m - 2nm}, \frac{2n + 2m}{2nm - Am} \right)
\]

whenever \( Am + 2n + 2m - 2nm > 2nm - Am > 0 \).

and obtain an \( L^p \) norm estimate for such a bounded \( \mathcal{K}_A \):

\[
\|\mathcal{K}_A\|_{L^p(\mathbb{H}_{m/n})} \lesssim \frac{p^4}{(p-1)^2m} [\mu, \nu, \lambda]_{p, 1-p} \text{ for } p \geq 2,
\]

(4.22)

\[
\|\mathcal{K}_A\|_{L^p(\mathbb{H}_{m/n})} \lesssim \frac{p^4}{(p-1)^2m} ([\mu, \nu, \lambda]_{p, 1})^{\frac{1}{p-1}} \text{ for } 1 < p < 2.
\]

(4.23)

By [EM17, Theorem 3.4], the Bergman projection \( |P_{\mathbb{H}_{m/n}}(f)(z)| \lesssim m^2 \mathcal{K}_A(|f|)(z) \) with \( A = 2n - 1 + \frac{1}{n} \). Applying (4.22) and (4.20) to this inequality of \( P_{\mathbb{H}_{m/n}} \), we recover the \( L^p \) regularity result of the Bergman projection on \( \mathbb{H}_{m/n} \), obtained in [EM17, Corollary 4.7], and obtain an estimate for the \( L^p \) norm of \( P_{\mathbb{H}_{m/n}} \):

**Theorem 4.1.** For \( p \in \left( \frac{2m+2n}{m+n+1}, \frac{2m+2n}{m+n-1} \right) \),

\[
\|P_{\mathbb{H}_{m/n}}\|_{L^p(\mathbb{H}_{m/n})} \lesssim m^3(p-1)^{p-1}2^{\max\{p-1, \frac{1}{p-1}\}}(2n + 2m - p(m + n - 1))^{-\max\{1, \frac{1}{p-1}\}}
\]

\[
\times ((2m + 2n + 1)p - 2m - 2m)^{-\max\{p-1, 1\}}.
\]

5. **Remarks and Generalizations**

1. The assumption \( \mu(z_1, z_2) = \mu_1(z_1/z_2)\mu_2(z_2) \) in Theorem 1.2 is used only in the proof of Lemma 2.6. Because of this fact, our lower bound in Theorem 1.2 holds without this assumption:
Corollary 5.1. Let $\mu$ be a weight on $\mathbb{H}$ and set $\nu = |w_2|^{-p'} \mu^{-\frac{1}{p'}}$. If the Bergman projection $P$ is bounded on the corresponding weighted space $L^p(\mathbb{H}, \mu dV)$, then $[\mu, \nu]_p < \infty$. Moreover, there holds
\[ \|P\|_{L^p(\mathbb{H}, \mu dV)} \gtrsim ([\mu, \nu]_p)^{\frac{1}{p'}}. \]

We can also generalize our upper bound estimate for $P$ and $P^+$ as follows:

Corollary 5.2. Let $\mu$ be a weight on $\mathbb{H}$ and set $\nu = |w_2|^{-p'} \mu^{-\frac{1}{p'}}$. Suppose the quantities $\|M_{T_{m,n},\nu}\|_{L^p(\mathbb{H}, \mu dV)}$, $\|M_{T_{m,n},|w_2|^{-p}\mu}\|_{L^p(\mathbb{E}, |z_2|^{-p} \mu dV)}$, and $[\mu, \nu]_p$ are all finite. Then the operators $P$ and $P^+$ are bounded on $L^p(\mathbb{H}, \mu dV)$. Moreover,
\[ \|P\|_{L^p(\mathbb{H}, \mu dV)} \leq \|P^+\|_{L^p(\mathbb{H}, \mu dV)} \lesssim \|M_{T_{m,n},\nu}\|_{L^p(\mathbb{H}, \mu dV)} \|M_{T_{m,n},|w_2|^{-p}\mu}\|_{L^p(\mathbb{E}, |z_2|^{-p} \mu dV)} \times ([\mu, \nu]_p)^{\max\{1, \frac{1}{p} - 1\}}. \]

In [Fer81], Fefferman gave a sufficient condition for the boundedness of the maximal operator $M^{(n)}_{\mu}$ on $\mathbb{R}^n$ defined by
\[ M^{(n)}_{\mu}(f)(x) = \sup_{x \in R} \int_R |f(t)| \mu(t) dV(t), \]
where $R$ is any rectangle in $\mathbb{R}^n$ with sides parallel to the coordinate axes. He showed that, if the weight $\mu$ on $\mathbb{R}^n$ is uniformly in the class $A_\infty$ in each variable separately, then $M^{(n)}_{\mu}$ is $L^p$ bounded on $L^p(\mathbb{R}^n, \mu)$ for all $1 < p < \infty$. In [APR17], Aleman, Pott, and Reguera studied the $B_\infty$ weights on the unit disc which is the analogue of the $A_\infty$ weights in the Bergman setting. Using their results and Fefferman’s proof, it is possible to give a sufficient condition for the boundedness of $\|M_{T_{m,n},\nu}\|_{L^p(\mathbb{H}, \mu dV)}$ and $\|M_{T_{m,n},|w_2|^{-p}\mu}\|_{L^p(\mathbb{E}, |z_2|^{-p} \mu dV)}$ in the corollary above. To obtain an upper bound estimate, one also needs to understand the dependence of the quantities $\|M_{T_{m,n},\nu}\|_{L^p(\mathbb{H}, \mu dV)}$ and $\|M_{T_{m,n},|w_2|^{-p}\mu}\|_{L^p(\mathbb{E}, |z_2|^{-p} \mu dV)}$ on the sufficient condition for the weight $\nu$ and $|z_2|^{-p}\mu$.

2. The example in Section 4.1 showed the upper bound estimate in Theorem 1.2 is sharp. It is not clear if the lower bound estimates given in Theorem 1.2 or in [PR13] and [RTW17] are sharp. It would be interesting to see what a sharp lower bound is in terms of the Bekollé-Bonami constant.

3. We focus on the weighted estimates for the Bergman projection on the Hartogs triangle for the simplicity of the computation. In [RTW17], Rahm, Tchoudja, and Wick obtained the weighted estimates for operators $S_{a,b}$ and $S^+_{a,b}$ defined by
\[
S_{a,b}f(z) := (1 - |z|^2)^a \int_{\mathbb{H}} \frac{f(w)(1 - |w|^2)^b}{(1 - zw)^{n+1+a+b}} dV(w); \]
\[
S^+_{a,b}f(z) := (1 - |z|^2)^a \int_{\mathbb{H}} \frac{f(w)(1 - |w|^2)^b}{|1 - zw|^{n+1+a+b}} dV(w), \]
on the weighted space $L^p(\mathbb{E}, (1 - |w|^2)^b \mu dV)$. Using the methods in this paper, it is possible to obtain weighted estimates for analogues of $S_{a,b}$ and $S^+_{a,b}$ in the Hartogs triangle setting. On the other hand, the dyadic structure we construct for the Hartogs triangle is induced by the dyadic structure on the unit disc via the biholomorphism between $\mathbb{H}$ and $\mathbb{D} \times \mathbb{D}^*$. When the domain $\Omega$ is covered by the polydisc through a rational proper holomorphic map as in [CKY19], an induced dyadic structure on $\Omega$ can be obtained via the proper map. One
direction for generalization is to obtain the Bekollé-Bonami type estimates for the Bergman projection, and analogues of $S_{a,b}$ and $S^+_{a,b}$ on such a domain $\Omega$.

4. In the proof of Theorem 1.2, the positive dyadic operator $Q^+_{m,n,\nu}$ is used to relate the Bergman projection to the maximal operator. The constant $\frac{p^4}{(p-1)^2}$ appeared in Theorem 1.2 dominates the $L^p$ and $L^{p'}$ norms of the maximal operator on the $D^2$. In [ˇC17], Čučković showed that the $L^p$-norm of the Bergman projection on a smooth bounded strongly pseudoconvex domain is dominated by $\frac{p^2}{p-1}$. This fact suggests the possibility to relate the Bergman projection to the maximal function via a dyadic harmonic analysis argument. It would be interesting to see what is the appropriate dyadic structure and the dyadic operator for the Bergman projection on the strongly pseudoconvex domain, and establish Bekollé-Bonami estimates for weighted $L^p$ norm of the projection.

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