Quantum Hall Effect and Dyson-Swinger Equations

P.A. Kurashvili

Department of Theoretical Physics, Tbilisi State University, 0128 Tbilisi, Georgia

Abstract

In this paper we make attempt to obtain a description of the Quantum Hall Effect (both integer and fractional) by means of electron’s Green functions of three-dimensional (planar) electrodynamics. We show that expression for the free electron propagator yields an integer number for the second Chern-Simons term, that corresponds to the quantized Hall conductivity in the approximation of non-interacting particles for integer filling factors, when there exists a gap for all excitations in the system. Then we try to check correspondence between fractional case and "interacting" Green functions, so it requires taking into consideration "full-fledged" propagators, including interactions. We are going to obtain them from Dyson-Swinger equations. We attempt to reach out from the perturbation theory regime using a specific method, called scale approximation. Our solutions are found in general gauge.

1 Introduction

The Quantum Hall Effect, found more then twenty years ago [1], [3], have been a subject of great interest since its discovery. As there was discovered, the planar system of electrons, placed in strong homogenous magnetic field perpendicular to the plane at low temperatures shows unusual behavior near the values of the field strengths corresponding to the integer fillings of Landau levels. The antisymmetric part of the conductivity tensor corresponding to the Hall conductance appears constant for the certain ranges of values of the magnetic field instead of equally increasing, as one could suppose starting from naive microscopic considerations, while the diagonal part is almost equal to zero at the same ranges. The quantization value of the Hall conductance have been shown to be equal to the ratio of two fundamental physical constants - the elementary electric charge and the Plank constant $\frac{e}{h}$.

The explanation for this behavior have been found in extremely "quantum" nature of electrons in two-dimensional systems due to the strong magnetic field and particularities of the single electron spectrum, when Landau levels, corresponding to the single-particle spectrum get widened due to local impurities and defects and there exist regions of localized states between them, interfering to the further increase of the Hall conductance while the Fermi level "travels" through these regions [2].

The precise quantization of the Hall conductance have been explained in works of Laughlin [4], Haldane [7], Thouless et al. [8] and others. The quantized Hall conductance expressed as a topological invariant, corresponds to an integer number in the case of non-degenerate ground (vacuum) state, or to a simple fraction with an odd denominator when any degeneracy is relevant. In works of Robert Laughlin there have been discovered some unusual properties of Quantum Hall state excitations, such as modes with fractional charges and statistics [5], [6]. After works of Zhang, Hu and others the field-theory language descriptions of the Quantum Hall Liquid was obtained that revealed the parallels to superconductivity and three-dimensional $2 + 1$ Chern-Simons Electrodynamics. According to these models the Quantum Hall conductance is interpreted as a Chern-Simons mass of the gauge field. In this paper we follow the relativistic approach based on relativistic Kubo formula for the quantized Hall conductance and the Dyson-Swinger equation for the full electronic propagator [12], [13].
2 Derivation of Kubo formula

The Hall conductivity is defined as the linear part of the correlation function [12]:

\[
\Pi_{\mu\nu}(q) = \frac{1}{(2\pi)^3} \int d^3 e^{iqx} \langle 0 | J_\mu(x) J_\nu(0) | 0 \rangle. \tag{1}
\]

Where \( J_\mu(x) \) is the conserved current:

\[
\partial^\mu J_\mu(x) = 0. \tag{2}
\]

Inserting this equation to the above expression we obtain:

\[
\Pi_{\mu\nu}(q) = -\frac{e^2}{(2\pi)^3} \int d^3 T r (\Lambda_\mu(p, p + q) S(p) \Lambda_\nu(p, q + p) S(p - q)).
\]

We used the following definitions:

\[
\int d^3 x d^3 y_1 d^3 y_2 e^{i(p x_1 - q y_1 - q x_2)} \langle 0 | T (J_\mu(x) \psi(x_1) \psi^+(x_2)) | 0 \rangle = S(p) \Lambda_\mu(p, p + q) S(p) (2\pi)^3 \delta^{(3)}(p' - p - q),
\tag{3}
\]

and

\[
\int d^3 y_1 d^3 y_1 e^{i(p x_1 - q y_1)} \langle 0 | T (\psi(x_1) \psi^+(y_1)) | 0 \rangle.
\tag{4}
\]

The linear part of current correlation function is given by

\[
\frac{\partial}{\partial q^\alpha} \Pi_{\mu\nu} |_{q=0} = -\frac{e^2}{(2\pi)^3} \int d^3 T r \{(\frac{\partial}{\partial q_\alpha} \Lambda_\nu(p, p + q)) |_{q=0} S(p) \Lambda_\nu(p, p) + \Lambda_\mu(p, p) S(p) \partial q_\alpha \Lambda_\nu(p, p + q) |_{q=0} S(p) - \Lambda_\mu(p, p) S(p) \Lambda_\nu(p, p) \frac{\partial}{\partial p_\alpha} S(p)\}. \tag{5}
\]

The first two terms give no contribution to the antisymmetric part of the conductivity tensor, which can be expressed as

\[
\sigma_{xy} = \lim_{\varepsilon \to 0} \frac{\varepsilon_{\mu\nu} q^2 \Pi_{\mu\nu}(q)}{q^2}. \tag{6}
\]

And the antisymmetric part of this is given by:

\[
\frac{1}{6} \varepsilon^{\mu\nu\rho} \int \frac{d^3 p}{(2\pi)^3} \left( \frac{\partial S^{-1}(p)}{\partial p_\mu} S(p) \frac{\partial S^{-1}(p)}{\partial p_\nu} S(p) \frac{\partial S^{-1}(p)}{\partial p_\rho} S(p) \right). \tag{7}
\]

Finally we arrive at

\[
N = \frac{1}{24\pi^2} \int d^3 p \varepsilon^{\mu\nu\rho} T r \{ (\partial_\mu S^{-1}) S(\partial_\nu S^{-1}) S(\partial_\rho S^{-1}) S(p) \}. \tag{8}
\]

Deriving this expression we made use of the Ward-Takahashi identity \( \Lambda_\mu(p, p) = \frac{\partial S^{-1}(p)}{\partial p_\mu} \). \( N \) denotes the topological invariant of mapping from 3-dimensional momentum \( p \)-space to matrices \( S(p) \) as if the integration area were 3-dimensional sphere \( S^3 \). When the value of momentum tends to infinity, the direction plays no role in our consideration and one can stereographically map \( R^3 \) to \( S^3 \). \( N \) topological invariant known as Pontriagin index and is always integer. Therefore the antisymmetric part of the conductivity tensor is given by the expression:

\[
\sigma_{xy} = \frac{e^2}{2\pi h} N = \frac{e^2}{h} N. \tag{9}
\]
where $N$ is some integer. So far, we have seen that Hall constant is quantized in the framework of the relativistic field theory. Below we attempt to get those numbers from the microscopical theory considerations.

First let us assume that fermions in the loop are non-interacting, so we can insert free propagator $S^{-1} = \hat{p} - m$ directly into our formula and obtain:

$$N = \frac{i}{2\pi^2} \int d^3p \frac{m(p^2 - m^2)}{(p^2 - m^2)^3} = -\frac{1}{2}.$$  \hspace{1cm} (8)

After the integration we must multiply this result by two, because we have two spin degrees of freedom for an electron and finally get an integer number: $N = 1$.

### 3 Dyson-Swinger equation

Dyson-Swinger equation is one of the most sensible ways of finding non-perturbational solutions in the quantum field theory. As we have seen already, the quantum Hall conductivity can be expressed by the integral that contains the electron propagator $S(p)$. The computation of this quantity is very difficult task and is the object of studies up to the present days. The main difficulty is that the Dyson-Swinger system is not closed and self-contained and one have to make additional assumptions to arrive at the closed and self-consistent system of equations.

For simplicity let us write our equations for the simplest case of the zero bare mass (in what follows we also neglect the Chern-Simons mass of photon):

$$S^{-1} = \hat{p} - i\frac{e^2}{(2\pi)^3} \int d^3k \gamma^\mu D_{\mu\nu}(p, k) S(k) \Gamma^\nu(p, k)$$ \hspace{1cm} (9)

On the right-hand side of this expression there stands the electron-photon vortex function, $\Gamma^\nu(p, k)$ and the full propagator of photon $D_{\mu\nu}(p - k)$. One can write down the Dyson-Swinger equation in the graphical form.

We can see, that right-hand side contains a new factor $K$, electron-electron scattering amplitude. In order to obtain a closed system, one has to make any assumption for vertex function an for $D_{\mu\nu}$, only then can this be considered as an equation.

One of the most convenient ways of studying the equation at hand is so-called ladder approximation, when one replaces the full vortex function by simple factor $\gamma^\mu$ and the full photon propagator is substituted by free one. This approximation was widely used in the middle of the last century by Landau, Berestetsky an Pitaevski during studies of the bound systems. It inherits many features of the perturbation theory but also includes summation over diagrams of certain type as a way of reaching out of the perturbation approximation, reliable only in the week coupling regime.

The second method mostly used at last times is based on approximation that conserves gauge invariance at every succeeding step of calculations. The method makes wide use of Ward-Takahashi identities conserving gauge invariance at every step of calculations. By this method one can explicitly find the longitudinal part of the vortex function but the transversal part remains totally undefined. This method is frequently used in studies of the infrared part of electron propagator. In our case...
this method is of no use, despite its non-perturbativity, because all solutions found in literature are constructed so that vortex factor is proportional to the transferred momentum, that has to be multiplied by photon propagator and gives zero because of gauge invariance. In other words, the longitudinal factor falls out from the Dyson-Swinger equation and the information is lost. The transversal part is completely undefined and this method is useless for our task. Therefore one can have more profit using the ladder approximation.

4 Landau gauge

We begin with the Landau gauge when the photon propagator is (we assume zero Chern-Simons mass)

\[ D_{\mu\nu} = -\frac{1}{q^2}(g_{\mu\nu} - \frac{q_{\mu}q_{\nu}}{q^2}) \] (10)

The electron propagator is written in form:

\[ S^{-1}(p) = \beta(p^2)\hat{p} - \alpha(p^2) \] (11)

Where \( \alpha \) and \( \beta \) are functions, we have to find. Now let us put this expression into DS equation and use the properties of three-dimensional Dirac matrices:

\[ Tr\gamma^\mu = 0, \]
\[ Tr\gamma^\mu\gamma^\nu = 2g_{\mu\nu}, \]
\[ Tr\gamma^\mu\gamma^\nu\gamma^\lambda = -2ie^{\mu\nu\lambda}, \]
\[ Tr\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma = 2[g^{\mu\nu}g^{\lambda\sigma} + g^{\mu\sigma}g^{\nu\lambda} - g^{\mu\lambda}g^{\nu\sigma}], \]
\[ Tr\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\rho = -2i[\epsilon^{\mu\nu\lambda\rho} + \epsilon^{\mu\lambda\nu\rho} + \epsilon^{\nu\lambda\mu\rho} - \epsilon^{\nu\lambda\rho\mu}] \]

We also employ following relations: \( \epsilon^{\mu\nu\sigma}g_{\sigma\lambda} = \epsilon^{\mu\nu}_{\lambda} \) Then we can derive the properties of the \( \gamma \) matrices of the form \( \gamma^\mu\gamma^\nu = g^{\mu\nu} - ie^{\mu\nu\lambda}\gamma^\lambda \)

After calculating traces from both sides

\[ \hat{p}(\beta(p^2) - 1) - \alpha(p^2) = \frac{ie^2}{(2\pi)^3} \int d^3k \frac{2\alpha(k^2) - 2\beta(k^2)\frac{k^2(q^2)q}{q^2}}{q^2(k^2)\beta^2(k^2) - \alpha^2(k^2)} \]

Then let us multiply both sides of the expression by \( \hat{p} \) and again take the trace:

\[ \alpha(k^2) = -2ie^2 \int d^3k \frac{\alpha(k^2)}{q^2[k^2\beta^2(k^2) - \alpha(k^2)]} \]

If we integrate by angles at right-hand side of this expression we get zero and therefore.

\[ \beta(p^2) - 1 = -2ie^2 \int d^3k \frac{\beta(k^2)}{k^2\beta^2(k^2) - \alpha(k^2)} \frac{(q,k)(p,q)}{q^2p^2} \]

Calculating traces of gamma-matrices we get the closed expression for \( \alpha \):

\[ \alpha(k^2) = -2ie^2 \int d^3k \frac{\alpha(k^2)}{q^2[k^2 - \alpha(k^2)]} \]

(12)

This equation is strongly nonlinear and one can not resolve it explicitly, therefore we need to make any reasonable approximation for it. The tool for performing this item have been provided by
Maris, Hertscovitz and Jacob [14], they showed that an approximation \( \alpha \equiv m \) is indeed very powerful non-perturbative supposition in many problems, when the gap function \( \alpha \) is replaced by the physical mass of electron: We can express the self-energy (in Euclidian momenta) so: 
\[
\alpha(p^2) = (p^2 + m^2)\chi(p^2)
\]
and perform the Furrier-transform to new variables:
\[
\chi(r) = \frac{1}{(2\pi)^3} \int d^3k \chi(k^2) e^{\text{i} k \cdot \vec{r}}
\]
Then we obtain the differential equation:
\[
\frac{d^2}{dr^2} \chi(r) + \frac{2}{r} \frac{d}{dr} \chi(r) - \left(m^2 - \frac{e^2}{2\pi r}\right)\chi(r) = 0 \tag{13}
\]
This expression is very similar to one for the wave function of electron in zero angular momentum state of hydrogen atom. The solution of this equation has the form of hypergeometrical function
\[
\chi(r) = e^{\nu r} F(a, c, \frac{x}{\lambda})
\]
Here we must take \( \nu = m \) and assume that \( a \) is equal to any negative integer \(-n\) or to zero in order to obtain a solution, falling exponent multiplied by a finite polinomial, that decreases to zero on spatial infinity: \( a = 1 - \frac{e^2}{4\pi m} \) hence, we conclude, that the "mass" of electron is \( m = \frac{e^2}{4\pi(n+1)} \), resembling the expression for the energy levels of the hydrogen atom. In the simplest case of \( n = 0 \) we have:
\[
\chi(r) = Ce^{-mr}.
\]
We convert this expression into the momentum representation again and choose the value of the arbitrary constant \( C \) in order to get \( \alpha(p^2 = 0) = m \). From this by Furrier-transformation we get for \( \alpha \) in momentum representation:
\[
\alpha(-p^2) = \frac{m^3}{p^2 + m^2}.
\]
If we insert again this expression to our formula, then
\[
N = -\frac{2}{3\pi} \int p^2 dp \frac{3\alpha(-p^2) - 2p^2 \alpha'(-p^2)}{(p^2 + \alpha^2(-p^2))^2} \tag{14}
\]
Thus
\[
N = -\frac{2}{3\pi} \int p^2 dp \frac{m^3(5p^2 + 3m^2)}{(p^2 + m^2)^4}
\]
This integral can be rewritten as
\[
N = -\frac{1}{3\pi} \int dx \sqrt{x(x + 5x)} \frac{1}{(1 + x)^4}
\]
and the integration yields the result in terms of the beta function:
\[
N = -\frac{8}{3} B\left(\frac{3}{2}, \frac{5}{2}\right) \tag{15}
\]
Finally we obtain:
\[
N = -\frac{1}{6} \tag{16}
\]
After taking into account the multiplicity due to the electron spin states, we obtain the result
\[
\sigma_{xy} = \frac{1}{3} \frac{e^2}{2\pi}
\]
5 Solving of DS equations

In the previous section we obtained the quantized Hall conductance by means of Kubo’s relativistic formula and expression for the free electron propagator. As we mentioned above, in the framework of non-relativistic theory one has to take into account the interactions between the charge carriers (electrons) in order to obtain fractional numbers. On relativistic language this imports that one have to insert the full propagator obtained from Dyson-Swinger equations.

In previous section we made it for Landau gauge and obtained a simple fraction, now we are going to solve the problem in the general gauge and see, if it is possible to obtain any other simple fractions and the explicit form for electron propagator. In general gauge, the propagator for the free photon is:

\[ D_{\mu\nu}(k) = \frac{1}{k^2}[(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}) + a \frac{k_\mu k_\nu}{k^2}] \]  

(17)

If we denote the vortex function by \( \Gamma_\nu(p, k) \), it is possible to decompose it into two, longitudinal and transverse parts:

\[ \Gamma_\nu(p, k) = \Gamma^T_\nu(p, k) + \Gamma^L_\nu(p, k) \]  

(18)

Here

\[ \Gamma^L_\nu(p, k) = \frac{q_\nu}{q^2}[S^{-1}(p) - S^{-1}(k)] \]

Here we again use the properties of three-dimensional Dirac matrices and compute the traces from both sides of the equation, then multiplying the equation by \( \hat{p} \) to obtain:

\[ \alpha(-p^2) = -\frac{e^2}{(2\pi)^3} \int d^3k \frac{1}{q^2[k^2\beta^2(-k^2) + \alpha^2(-k^2)]} \]

\[ [2\alpha(-k^2)\beta(-p^2_{\text{max}}) - \frac{2a}{q^2}\alpha(-k^2)\beta(-p^2)(p, q) - \frac{2a}{q^2}\alpha(-k^2)\beta(-p^2)(q, k)] \]  

(20)

\[ p^2(\beta(-p^2) - 1) = \frac{e^2}{(2\pi)^3} \int d^3k \frac{1}{q^2[k^2\beta^2(-k^2) + \alpha^2(-k^2)]} \]

\[ \frac{2a}{q^2}[\beta(-k^2)\beta(-p^2)p^2(q, k) + \alpha(-k^2)\alpha(-p^2)(p, q)] \]  

(21)

In Landau gauge we had \( \beta \equiv 1 \). This approximation would enable us to simplify the first expression. In order to clarify the reliability of this approximation, we have to evaluate the \( \alpha \). We work in linear approximation, where \( \alpha \) is so small, that one may neglect its value in the denominator under integral. Then we get the closed expression for \( \beta \):

\[ p^2(\beta(-p^2) - 1) = 2a \frac{e^2}{(2\pi)^3} \int d^3k \frac{p^2(p, k)}{q^4k^2\beta^2(-k^2)\beta(-p^2)} \]
or
\[ 1 - \frac{1}{\beta(-p^2)} = 2a - e^2 \frac{1}{(2\pi)^3} \int d^3k \frac{p^2(q, k)}{q^2k^2} \frac{1}{\beta^2(-k^2)} \]

The solution of this equation is possible by Førreier-transform method: if we put \( \frac{1}{\beta(-p^2)} = p^2\varphi(p^2) \) and transform it to new variables: 
\[ \varphi(p^2) = \int d^3qe^{ikq} \varphi(r), \]
we arrive at the following equation for the radial function:
\[ r^2\frac{d^2\varphi}{dr^2} + (2r - 2abr^2)\frac{d\varphi}{dr} - abr\varphi(r) = \frac{1}{4\pi} \delta(r) \]

Here \( b = \frac{e^2}{2\pi} \). If we denote \( abr = \rho \), we get:
\[ \rho^2\frac{d^2\varphi}{dr^2} + 2\rho(1 - \rho)\frac{d\varphi}{dr} - \rho\varphi = \frac{ab}{4\pi} \delta(\rho) \]

Because of the presence of the singular Dirac function in the right side, we have to perform one more Førreier transformation to variables \( k: \varphi(k) = \int d\rho\varphi(\rho)e^{ik\rho} \) Then after the denomination \( ik = t, \varphi'(t) = p(t) \)
\[ \varphi''(t)(t^2 + 2t) + (2t + 5)\varphi'(t) = \frac{\lambda}{4\pi} \]

From this non-homogenous equation we find:
\[ p(t) = \varphi'(t) = \frac{1}{\pi} \left( \frac{1}{2t} - \frac{3}{4t^2} - \frac{3}{2} \lambda \frac{1}{t^{5/2}(t + 2)} \right) \ln \frac{s - 1}{s + 1} \]

Where \( \lambda = ab, s = \left( \frac{t + 2}{t} \right)^{\frac{1}{2}} \) We can reconstruct \( \varphi(p^2) \) by reversing Førreier integrations:
\[ \varphi(p^2) = \int d^3qe^{ikq} \varphi(r) = \frac{\pi}{\nu} \int \varphi(r)(e^{ipr} - e^{-ipr})rdr = \frac{\pi}{\nu} \int \frac{dk}{2\pi} \int dr \varphi(k)e^{-ik\lambda r}(e^{ipr} - e^{-ipr})r = \]
\[ \frac{\pi}{\nu} \left( \frac{\partial}{\partial \nu}(\int dk\varphi(k)\delta(k\lambda + \nu) + \int dk\varphi(k)\delta(k\lambda - \nu)) = \right) \]
\[ \frac{\pi}{\nu}(\varphi'(k) - \varphi(-q)) = \frac{1}{p^2} + \frac{3}{8} \frac{\lambda^2}{p^2} \ln \frac{s - 1}{s + 1} - \frac{\lambda}{(-p)^{1/2}(-p + 2\lambda)^{1/2}} \ln \frac{s^* - 1}{s^* + 1} \]

\( \beta(p^2) \) is equal to unity when \( p = 0 \) and \( p = \infty \) and very close to this value at intermediate values of the argument and in the case of relevance we can put it equal exactly to one and solve the equations for \( \lambda \):
\[ \alpha(-p^2) = \frac{e^2}{(2\pi)^3} \int \frac{d^3k}{q^2(k^2 + m^2)} \{2\alpha(-k^2) - \frac{2a}{q^2}\alpha(-k^2)(p, q) - \alpha(-p^2)(q, k)\} \] (22)

Here again we use substitution \( \alpha(-p^2) = (p^2 + m^2)\chi(p^2) \) and Førreier-transformation to real variables.
First we get:
\[ (p^2 + m^2)[1 - \frac{e^2}{(2\pi)^3} \left( \frac{2\pi^2}{p} \arctan \frac{m}{p} - 2\pi^2 \frac{m}{p^2 + m^2} \right)\chi(p^2)] = \]
\[ \frac{2e^2}{(2\pi)^3} \int \frac{d^3k}{q^2} (1 - \frac{a}{q^2(p, k)})\chi(k^2) \]

The arctan on the left-hand side is suggestive of very strong non-linearity. We may substitute it by \( \frac{\pi}{4} \) if we put that space part of the momentum is small and the value of the total momentum is of the same order as mass.
After this approximations and transformation to radial variables, we have:

\[
r \frac{\partial^2}{\partial r^2} \chi(r) + \left(2 - \frac{e^2a}{4\pi B} r\right) \frac{\partial}{\partial r} \chi(r) - \left[(m^2 - \frac{e^2a}{4\pi B})r - \frac{e^2}{2\pi B}\right] \chi(r) = 0 \quad (23)
\]

Here \( B = 1 - \frac{e^2a}{16\pi} \).

This equation can be rewritten as one for the degenerate hypergeometrical function and its solution is:

\[
\chi(r) = e^{\nu x} F(a, 2, \frac{x}{\lambda})
\]

Where \( a = -2\nu - \frac{e^2}{2\pi B} \). It can be only zero or any negative integer value to fall in infinity. The quantities \( \nu, \lambda \) are defined from the following equations:

\[
\nu^2 - \frac{e^2a}{4\pi B} - (m^2 - \frac{e^2a}{4\pi B}),
\]

\[
0 = 1 + 2\lambda(2\nu - \frac{e^2a}{4\pi B})
\]

Going to the previous representation again, we can extract the propagators in momentum representation again. One also can see from this solutions, that in Landau gauge, this expression coincides with the one, obtained in the previous section.

6 Concluding remarks

In this paper we have outlined the scheme of calculating the topological Quantum Hall numbers using the relativistic microscopical theory.

Besides some successes, one can pose some questions, that are difficult to answer. First, how one can obtain all Quantum Hall fractures, basing on this theory? The calculations of the filling factor based on further hypergeometrical solutions of equation in Landau gauge yielded very poor results, compared with the brilliant \( \frac{1}{3} \) for the case \( n = 0 \) and the very sense of the “higher energy levels” is still unclear. Also we had very poor progress when trying to solve Dyson-Swinger equations in the general gauge, because removing the strong non-linearity gives only expression, trivially going to the one obtained in Landau gauge when gauge parameter \( a \) tends to zero.

One can see, that this nonlinearity needs more subtle treatment in order to maintain all the information about the full propagator, probably a slight modification of the ladder approximation itself, or considering strong-coupling limit (corresponding to the big \( n \)-s in our solutions).

It would be of interest to try this method to the new-found four and eight-dimensional generalizations of the quantum Hall effect [15]. We shall try to do it in the following papers.

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