Associated Conformable Fractional Legendre Polynomials

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Abstract. Along with the work of Abul-Ez et al. [37], we introduce the associated conformable fractional Legendre polynomials (ACFLPs), from which the fractional differential equation of ACFLPs is established. Subsequently, some of interesting properties are derived such as generating function, hypergeometric representation, analytical formula, besides various of recurrence relations. Also, orthogonal properties of ACFLPs are developed in conformable context. We append our study by presenting the shifted ACFLPs and driving some of important properties such as Rodrigues’ type representation formula of fractional order derivative and explicit formula. An interesting compact closed-form expression is derived from the definite integral using a convenient analytical formula for the shifted ACFLPs. This result is easily generalized for integrands involving products of an arbitrary number of shifted associated Legendre polynomials in conformable sense.

1. Introduction

Known among the Mathematicians that special functions have emerged from a wide variety of practical problems that interest not only mathematicians, it is of interest to other researchers in science to study their properties, characteristics, and applications. in general, the special functions induced as solutions of famous differential equations, such as Legendre, Laguerre, Hermite, Chebyshev, hypergeometric, etc.

Legendre's differential equation is a type of ordinary differential equation (ODE) that is commonly used in physics and engineering. It appears when equation of Laplace is solved in spherical coordinates particularly. In 1784, the significant of Legendre polynomials is sensed when the attraction of spheroids and ellipsoids was studying by A. Legendre. It is used in several areas in physics and mathematics. They may arise from solutions of Legendre ODE, such as the analog ODEs in spherical polar coordinates and the famous Helmholtz equation. Also, they appear as a result of requiring a complete, functions orthogonal sequence over the interval [-1,1]. (Gram-Schmidt orthogonalization). They are angular momentum eigen-functions in quantum mechanics.

Nowadays, the theory of fractional calculus is classified as generalized fractional integrals or derivatives in addition to its origin, as old classical calculus. Because fractional order derivatives are
important in describing many physical phenomena in real life, an increasing number of mathematicians are studying fractional calculus. [1]. Fractional derivatives, for example, provide a better description of the model of nonlinear earthquake oscillation [2], and fractional derivatives can be used to model fluid dynamics, eliminating the deficiency that arises from the occurrence of continuous traffic flow [3, 4]. As a consequence, fractional calculus becomes a more useful approach to understand mathematical models in a wide range of dynamical systems and physical [5-8]. The main approach used to achieve the mentioned results were fractional order derivatives in the sense of Riemann-Liouville or Caputo. Further advancements in fractional calculus have been successfully applied to a wide range of disciplines, including engineering and science [9-18].

An extension of the classical limit definitions of the derivatives of a function the Conformable derivative had been proposed in Khalil et al. [19]. There were rapid developments in the properties of the Leibniz rule and Chain rule, exponential functions, Gronwall’s inequality, integration by parts and Taylor power series for conformable fractional calculus [22, 20, 21]. The concept of conformable derivative usability has a wide range of theoretical and practical applications [23, 24]. Ma et al. [25] conformable derivative is applied to a gray system model and showed that the conformable derivative was suitable. Moreover, physical interpretation, Able’s formula, existence, stability, and application in many fields for conformable differential equations had been studied extensively in [26, 27, 28]. The authors of [29, 30, 31, 32] introduced some applications via partial differential equations in the conformable sense.

Saadatmandi and Aehghan [33] used spectral methods to solve some types of fractional differential equations (FAEs) with initial boundary conditions by deriving operational matrices of fractional derivatives for the shifted Legendre polynomials. The authors in [34] proposed orthogonal Legendre functions with fractional order based on shifted Legendre polynomials to obtain a solution for some FAEs. The conformable fractional Legendre differential equation was studied by Abu Hammad and Khalil [35] in 2014. Analog to the classical case, Analog discovered that some solutions were fractional polynomials in some cases. Mathematicians also looked into the fundamental properties of conformal fractional polynomials. Recently, the authors in [36] introduced a generalized study on shifted Legendre type polynomials of arbitrary fractional orders in the Caputo sense, based on some Rodrigues formulas in the matrices framework. Abul-Ez et al. [37] conducted a thorough investigation of conformable fractional Legendre polynomials and discovered several intriguing properties, which they published in early 2020.

Now, owing to the significance of earlier mentioned work regrading to Legendre type polynomials, the main contributions of this paper is to develop many properties of the associated Legendre polynomials in the conformable fractional sense. To this end, we investigate in detail some useful results related to the associated conformable fractional Legendre polynomials (ACFLPs).

The content and structure of this paper are arranged as follows. Some preliminaries of conformable fractional calculus are displayed in section 2. We introduced in section 3 the ACFLPs as a solution of associated conformable fractional Legendre differential equation. Some of interesting convergence properties of ACFLPs such as generating function, hypergeometric representation, and analytical formula are displayed in section 4. In section 5, various recurrence relations of ACFLPs are established. The orthogonality properties and expansion of analytic function are the subject of section 6. In section 7, we present shifted ACFLPs and drive some of important properties such as Rodrigues’ type representation formula of fractional order derivative and explicit formula. Further, an interesting compact closed-form expression is derived from the definite integral using a convenient analytical formula for the shifted ACFLPs.
2. Preliminaries

As it has been mentioned, Khalil et al. define the CFA in [19], as the following:

For a function \( f: (0, \infty) \to \mathbb{R} \), the CFA ordered \( x \), where \( 0 < x \leq 1 \), of \( f(a) \) at \( a > 0 \) was defined by\(^{1}\):

\[
A^x f(a) = \lim_{\varepsilon \to 0} \frac{f(a + \varepsilon a^{1-x}) - f(a)}{\varepsilon}
\]

(1)

and when \( a = 0 \) we have \( A^x f(0) = \lim_{a \to 0^+} A^x f(a) \).

2.1. Remark

We can observe the following remarks:

1) the authors in [19] found that the behaves of the CFA doing well in the rules of product and the chain unlike the case of the classic fractional calculus

2) Surprisingly enough, the CFA of the constant function is zero, however, this is not the case for the Riemann-Liouville fractional derivative.

3) when \( x = 1 \) in (1), we can easily obtain the analogous ordinary classical derivatives. Moreover, even if a function is not differentiable, it can be \( x \)-differentiable at a point; for example, if \( f(a) = 2\sqrt{a} \), then \( A^1 f(a) = 1 \). As a result, \( A^1 f(0) = 1 \). As well as, \( A^1 f(0) \) does not exist. This is in stark contrast to what is known about classical derivatives.

4) to solve the simple fractional differential equation \( A^1 b + b = 0 \), by applying the Riemann-Liouville or Caputo definitions, then it is have to use either fractional power series technique or the Laplace transform. Moreover, the corresponding definition and the fact \( A^1 (e^{\frac{a}{2}}) = e^{\frac{a}{2}} \), is used, then, we can observe that the general solution is \( b = ce^{-\frac{a}{2}} \) easily.

Moreover, Khalil et al. [19] introduced the \( x \)-fractional integral of a real function \( f \) as follows:

**Definition 2.1** Let \( f: (0, \infty) \to \mathbb{R} \), \( x \)-differentiable and \( x \in (0,1] \), then the \( x \)-fractional integral of \( f \) is defined by:

\[
I^x f(t) = \frac{t^x}{\Gamma(x)} \int_0^t f(a) \, da, \quad t \geq 0
\]

Neat, we’ll talk about the function of Gauss hypergeometric \( _2F_1(x,y;c;a) \), by the following formula:

\[
_2F_1(x,y;c;a) = \sum_{n=0}^{\infty} \frac{(x)_n(y)_n}{(c)_n} \frac{a^n}{n!} \quad (|a| < 1)
\]

where \((x)_n\) denotes the Pochhammer symbol, which is defined in terms of Gamma functions as:

\[
(x)_n = \frac{\Gamma(x + n)}{\Gamma(x)} = x(x+1)(x+2)\ldots(x+n-1), \quad n \in \mathbb{N} \text{ and } (x)_0 = 1
\]

In [35], the authors introduced the conformable fractional Legendre differential equation solution as:

\[
(1 - a^{2x})A^xA^b - 2xaA^b + x^2n(n+1)b = 0
\]

(2)

and presented the CFLPs, \( P_{xk}(a) \) as its solution. The authors in [37] gave an explicit formula of \( P_{xk}(a) \) as follows:

\[
b := P_{xk}(a) = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^{k(n-2k)}}{2^k k!(n-k)!(n-2k)!} a^{x(n-2k)}, \quad x \in (0,1]
\]

(3)

\(^{1}\) Unless otherwise mentioned in the article, the fractional number is taken to have the value \( 0<\alpha\leq1 \).
They established the recurrence relations of the conformable fractional Legendre polynomials as follows:

\[ P_{x(n+1)}(a) = \frac{(2n+1)}{(n+1)}xP_{x(n)}(a) - \frac{n}{n+1}P_{x(n-1)}(a) \]  \quad (4)

\[ A^xP_{x(n+1)}(a) - (n + 1)xP_{x(n)}(a) - a^xA^xP_{x(n)}(a) = 0 \]  \quad (5)

\[ a^xA^xP_{x(n)}(a) - nxP_{x(n)}(a) - A^xP_{x(n-1)}(a) = 0 \]  \quad (6)

\[ A^xP_{x(n+1)}(a) - A^xP_{x(n-1)}(a) = (2n + 1)xP_{x(n)}(a) \]  \quad (7)

In much of the work discussed here, rearranging terms in iterated sequence is a popular method. The fundamental lemma that follows is of the type that will be used to simplify the proofs in work. We have the following Lemma for the infinite double series (see [38]).

**Lemma 2.1**

Let we have:

\[ \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_{k,n} = \sum_{m=0}^{\infty} \sum_{j=0}^{m} a_{j,m-j} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_{k,n-k} \]  \quad (8)

\[ \sum_{n=0}^{\infty} \sum_{k=0}^{n} b_{k,n} = \sum_{m=0}^{\infty} \sum_{j=0}^{m} b_{j,m-j} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} b_{k,n+k} \]  \quad (9)

Based on the notation of conformable fractional derivative, we have \( A^x^n = A^xA^x\ldots A^x \) \( m \)-times, as well as the fact \( A^x a^p = pa^{p-x} \), Abul-Ez et al [37], presented the following Rodrigues formula of the conformable fractional Legendre polynomials \( P_{x,n}(a) \) in the form:

\[ P_{x,n}(a) = \frac{1}{x^{2n}}A^x(a^{2x} - 1)^n \]  \quad (10)

Now, we may assume that the function of associated conformable Legendre type in the form

\[ P_{x,n}^m(a) = \frac{1}{x^{2n}}(1 - a^{2x})^m A^x(a^{2x} - 1)^n = (1 - a^{2x})^m A^x A^x a^p P_{x,n}(a) \]  \quad (10)

For \( m = 0 \), these functions reduce to the conformable fractional Legendre polynomials defined in [37], and it is clear that if \( m > n \), then \( P_{x,n}^m(a) = 0 \).

3. **Associated Legendre differential equation via conformable calculus**

**Theorem 3.1** For \( x \in (0,1] \), the associated conformable fractional Legendre polynomials (10) is a solution of the fractional differential equation

\[ (1 - a^{2x})A^x A^x b - 2xa^x A^x b + x^2 \left[ n(n+1) - \frac{m^2}{1-a^{2x}} \right]b = 0 \]  \quad (11)

**Proof.** Since \( P_{x,n}(a) \) is a solution of conformable fractional Legendre equation (3), we have

\[ (1 - a^{2x})A^x P_{x,n}(a) - 2xa^x A^x P_{x,n}(a) + x^2 n(n+1) P_{x,n}(a) = 0 \]  \quad (12)

Using conformable derivative and by differentiating equation (12) \( m \)-times with respect to \( a \), we get

\[ A^x \left[ (1 - a^{2x})A^x P_{x,n}(a) - 2xa^x A^x P_{x,n}(a) + x^2 n(n+1) A^x P_{x,n}(a) \right] = 0 \]

Using Leibniz’s theorem, we obtain

\[ (1 - a^{2x})A^x(m+2) P_{x,n}(a) + m A^x(1 - a^{2x})A^x(m+1) P_{x,n}(a) + \frac{1}{2}m(m-1)A^x(1 - a^{2x})A^x P_{x,n}(a) \]

\[ - 2xa^x A^x(m+1) P_{x,n}(a) - 2xm A^x A^x A^x A^x A^x P_{x,n}(a) + x^2 n(n+1) A^x P_{x,n}(a) = 0 \]

Therefore,

\[ (1 - a^{2x})A^x(m+2) P_{x,n}(a) - 2xm A^x P_{x,n}(a) - m(m-1)A^x P_{x,n}(a) \]

\[ - 2xa^x A^x(m+1) P_{x,n}(a) - 2x^2 m A^x P_{x,n}(a) + x^2 n(n+1) A^x P_{x,n}(a) = 0 \]
Hence,

\[(1 - a^{2x}) A^{x(m+2)} P_{x}\{R\}(a) - 2x(m + 1)a^x A^{x(m+1)} P_{x}(a) + x^2(n(n + 1) - m(m + 1)) P_{x}(a) = 0\]  \hspace{1cm} (13)

Aenoting that \(A^{x} P_{x}(a) = Y(a)\), \(13\) becomes

\[(1 - a^{2x}) A^{x} Y(a) - 2x(m + 1)a^x A^{x} Y(a) + x^2\{n(n + 1) - m(m + 1)\} Y(a) = 0\]  \hspace{1cm} (14)

But

\[P_{x}^{x}(a) = \left(1 - a^{2x}\right)^{-\frac{m}{2}} A^{x} P_{x}(a)\]

Therefore, \(14\) may be written in the form

\[(1 - a^{2x}) A^{x} \left\{(1 - a^{2x})^{-m/2} P_{x}(a)\right\} - 2x(m + 1)a^x A^{x} \left\{(1 - a^{2x})^{-m/2} P_{x}(a)\right\} + x^2\{n(n + 1) - m(m + 1)\} P_{x}(a) = 0\]  \hspace{1cm} (15)

Substituting from \(16\) and \(17\), Eq.(15) becomes:

\[(1 - a^{2x})(1 - a^{2x})^{-m/2} A^{x} P_{x}(a) + 2xa^x(1 - a^{2x})^{-m/2} A^{x} P_{x}(a) + x^2\{m(m + 2)a^2x(1 - a^{2x})^{-m/2} - 2x(m + 1)a^x(1 - a^{2x})^{-m/2} A^{x} P_{x}(a) + x^2\{n(n + 1) - m(m + 1)\} A^{x} P_{x}(a) = 0\]

as required.

4. Generating function, Hypergeometric representation, and the analytical formula of the associated conformable fractional Legendre polynomials

In this section, we will investigate some of interesting properties of the associated conformable fractional Legendre polynomials such as generating function, hypergeometric representation, and the analytical formula.
4.1. Generating function

Generating functions are important way to transform formal power series into functions and to analyze asymptotic properties of sequences. The authors in [37] established the generating function of the conformable fractional Legendre polynomials as follow

\[
\frac{1}{\sqrt{1-2a^x t^x + t^2}} = \sum_{n=0}^{\infty} P_{nx}(a) t^n
\]  

(18)

In what follows we characterize the associated conformable fractional Legendre polynomials \( P_{anm}(a) \) by means of generating function as:

**Theorem 4.1** Let \( a \in (0,1] \), \( m \leq n \), then following generating function holds true:

\[
\frac{x^m(2m)!!(1-a^{2x})^m T}{2^{m!}(1-2a^{x}t^x + t^2x)^{m+\frac{1}{2}}} = \sum_{n=0}^{\infty} P_{x(n+m)}^{xm}(a) t^n
\]  

(19)

Proof. In view of conformable derivative (1), and by differentiating relation (18), \( m \)-times, we have

\[
\left(\frac{-3}{2}\right) \left(\frac{-5}{2}\right) \cdots \left(\frac{-1}{2} - m + 1\right) (-2xt^x)^m (1 - 2a^x t^x + t^2x)^{-\frac{1}{2} - m} = \sum_{n=0}^{\infty} A^n x^n P_{nx}(a) t^n
\]

Hence,

\[
\left(1.3.5 \cdots (2m - 1)\right) x^m t^x (1 - 2a^x t^x + t^2x)^{-\frac{1}{2} - m} = \sum_{n=0}^{\infty} A^n x^n P_{nx}(a) t^n
\]

Multiplying by \((1 - a^{2x})^m\), we get:

\[
\frac{(2m - 1)!! x^m}{(1 - 2a^x t^x + t^2x)^{m+\frac{1}{2}}} = \sum_{n=0}^{\infty} A^n x^n P_{nx}(a) t^n
\]

Replacing \( n \) by \( n + m \), we have:

\[
\frac{(2m - 1)!! x^m}{(1 - 2a^x t^x + t^2x)^{m+\frac{1}{2}}} = \sum_{n=0}^{\infty} A^n x^n P_{x(n+m)}(a) t^n
\]

as required. \( \square \)

4.2. Hypergeometric representation

In this subsection, we introduce the associated conformable fractional Legendre polynomials \( P_{anm}(a) \) in terms of Gauss hypergeometric function as follow:

**Theorem 4.2** For \( x \in (0,1] \), the associated Legendre polynomials \( P_{anm}(a) \) may be written as:

\[
P_{x^n}(a) = \frac{x^m(n+m)!(1-a^{2x})^m}{2^{m!(n-m)!}} 2F_1 \left( m-n, m+n+1; 1+m; \frac{1-a^x}{2} \right)
\]  

(20)

Proof. The authors in [37], deduce the relation

\[
P_{x^n}(a) = 2F_1 \left(-n, n+1; \frac{1-a^x}{2} \right)
\]  

(21)

By differentiating relation (21) \( m \)-times, and using the relation:
\[ A^m 2F_1(a, b; c; a^x) = \frac{x^m(a)m(b)}{c_m} F_1(a + m, b + m; c + m; a^x), \]

we get:
\[ A^m P_{x^n}(a) = \frac{(-n)_m(n + 1)_m}{(1)_m} \left( -\frac{x}{2} \right)^m \frac{2F_1}{2} \left( m - n, m + n + 1; 1 + m; \frac{1 - a^x}{2} \right) \]

Using the identity \((-n)_m(n + 1)_m = \frac{(-1)^{m(n+m)!}}{(n-m)!}\), we get
\[ A^m P_{x^n}(a) = \frac{x^m(n + m)!}{2^m(n - m)!} 2F_1 \left( m - n, m + n + 1; 1 + m; \frac{1 - a^x}{2} \right) \]

Multiplying by \((1 - a^{2x})^m\), the required result is established.

4.3 The analytical formula of \(P_{x^n}(a)\)

**Theorem 4.3** For \(x \in (0,1]\), the following analytical formula of associated conformable fractional Legendre polynomials \(P_{x^n}(a)\) is true
\[ P_{x^n}(a) = \sum_{k=0}^{\left\lfloor \frac{n-m}{2} \right\rfloor} \eta_{n,m}^{k,x} (1 - a^{2x})^m \frac{a^x(n-m-2k)}{2^n(n-k)! (n-m-2k)! k!}, \tag{22} \]

where
\[ \eta_{n,m}^{k,x} = \frac{(-1)^k(2n - 2k)! x^m}{2^n(n-k)! (n-m-2k)! k!}. \]

**Proof.** In view of (1), using the binomial expansion, we get:
\[ P_{x^n}(a) = \frac{1}{2^n n! x^n} (1 - a^{2x})^m \sum_{k=0}^{n} \frac{(-1)^k n!}{(n-k)! k!} a^{x(2n-2k)} \]

Using the linearity of conformable derivative, we have
\[ P_{x^n}(a) = \frac{1}{2^n n! x^n} (1 - a^{2x})^m \sum_{k=0}^{n} \frac{(-1)^k n!}{(n-k)! k!} A^{x(n+m)} a^{x(2n-2k)} \]
\[ = (1 - a^{2x})^m \sum_{k=0}^{n} \frac{(-1)^k x^m}{2^n(n-k)! k!} \frac{a^{x(n-m-2k)}}{(n-m-2k)!} \]

Therefore,
\[ P_{x^n}(a) = \sum_{k=0}^{n} \frac{(-1)^k x^m}{2^n(n-k)! k!} \frac{a^{x(n-m-2k)}}{(1 - a^{2x})^m} \]
as required.
5. Recurrence Relations of Associated conformable fractional Legendre polynomials

In view of (1), we establish some of recurrence relations of associated conformable fractional Legendre polynomials as follows

**Theorem 5.1** For \( a \in (0,1]\), the associated conformable fractional Legendre polynomials \( P_{an}^m(a) \), satisfy the following recurrence relations:

\[ i) \quad P_{x(n+1)}^m(a) - 2x(n+1)A^mP_{x(n+1)}^m(a) + x^2\{n(n+1) - m(m+1)\} A^m P_{x(n-1)}^m(a) = 0 \]

\[ ii) \quad (2n+1)a^x P_{x(n+1)}^m(a) = (n-m+1)A^m P_{x(n+1)}^m(a) + (n+m)A^m P_{x(n-1)}^m(a) \]

\[ iii) \quad P_{x(n+1)}^m(a) - P_{x(n-1)}^m(a) = x(2n+1)\sqrt{1-a^x} \]

**Proof.** Along with the scalar case, we have

\[ i) \quad This \ is \ the \ fundamental \ relationship \ linking \ three \ associated \ conformable \ fractional \ Legendre \ polynomials \ with \ the \ same \ \( na \) \ and \ consecutive \ \( ma \) \ values. \]

From equation (13), we have:

\[ (1-a^x)^{m+1} A^{(m+1)} P_{x(n+1)}^m(a) - 2x(m+1)a^x A^{(m+1)} P_{x(n+1)}^m(a) + x^2\{n(n+1) - m(m+1)\} A^m P_{x(n-1)}^m(a) = 0 \]  

(23)

Multiplying by \((1-a^x)^{m+1}/2\), we get:

\[ (1-a^x)^{m+1} A^{(m+1)} P_{x(n+1)}^m(a) - 2x(m+1)a^x (1-a^x)^{m+1} A^{(m+1)} P_{x(n+1)}^m(a) + x^2\{n(n+1) - m(m+1)\}(1-a^x)^{m+1} A^m P_{x(n-1)}^m(a) = 0 \]

Using (10), we obtain:

\[ P_{x(n+1)}^m(a) - P_{x(n-1)}^m(a) = (2n+1)x A^m P_{x(n-1)}^m(a) \]  

(25)

Now, differentiating (7) \((m-1)\) times via conformable derivative, we get:

\[ A^m P_{x(n+1)}^m(a) - A^m P_{x(n-1)}^m(a) = (2n+1)x A^{(m-1)} P_{x(n-1)}^m(a) \]  

(26)

Substituting from (25) in (26), we get:

\[ (n+1)A^m P_{x(n+1)}^m(a) - (2n+1)A^m P_{x(n+1)}^m(a) - mA^m P_{x(n+1)}^m(a) + nA^m P_{x(n-1)}^m(a) = 0 \]

Hence,

\[ (2n+1)a^x A^m P_{x(n+1)}^m(a) = (n-m+1)A^m P_{x(n+1)}^m(a) + (n+m)A^m P_{x(n-1)}^m(a) \]
Multiplying by \((1 - a^{2x})^m\), we get
\[
(2n + 1)a^x P_{x(n+1)}(a) = (n - m + 1)P_{x(n+1)}(a) + (n + m)P_{x(n-1)}(a)
\]
as required.

\[
\cdot \text{ Multiplying equation (25) by } (1 - a^{2x})^m, \text{ we obtain}
\]
\[
(1 - a^{2x})^m A^{x}P_{x(n+1)}(a) - (1 - a^{2x})^m A^{x}P_{x(n-1)}(a) = x(2n + 1)(1 - a^{2x})^m A^{x(m-1)}P_{x(n)}(a)
\]
Using (10), we get
\[
P_{x(n+1)}(a) - P_{x(n-1)}(a) = x(2n + 1)\sqrt{1 - a^{2x}} P_{x(n)}(a)
\]
which is required relation. \(\square\)

6. Orthogonality relations and expansion of functions in terms of associated CFLPs

The general theory of the expansion of analytic functions in terms of an arbitrary set of orthogonal polynomials is a basic subject in analysis, started by various authors to whom we may mention Boas [39], Faber [40], and Whittaker [41] and later on in higher dimension by Abul-Ez et al. [42, 43, 44, 45, 46]. In the case of the usual classical calculus, there are some functions that do not have Taylor power series representation about certain points but in the theory of conformable fractional calculus, they do. This fact had been shown by Abdeljawad [22], where he also stated the expansion of fractional power series for an infinity a-differentiable function within the fractional Taylor series. The expansion of a dispersed real function in a series of associated conformable fractional Legendre polynomials is useful in various applications, for example, in giving the solution of certain fractional differential equations.

In this section, we introduced the orthogonality relations of the associated conformable fractional Legendre polynomials and then use it to establish the expansion of a real valued functions in terms of associated CFLPs.

6.1. Orthogonality

It is well known that the orthogonality relations play an important role in many applications. The authors in [37] deduced the orthogonality relations of conformable fractional Legendre polynomials as follows:
\[
\int_{-1}^{1} P_{x}^{m}(a)P_{x}^{m}(a)\,da = \frac{2}{x(2n+1)}\delta_{nn'},
\]
where \(\delta_{nn'}\) is a familiar kroner delta and \(da = a^{x-1}\,da\).

In this section, we introduce the orthogonality relations of associated conformable fractional Legendre polynomials as follows:

**Theorem 6.1** Let \(x \in (0,1]\), then the associated conformable fractional Legendre polynomials are orthogonal over \([-1,1]\) with respect to the weight function \(w(a) = a^{x-1}\) as follows:
\[
\int_{-1}^{1} P_{x}^{m}(a)P_{x}^{m}(a)\,da = \int_{-1}^{1} P_{x}^{m}(a)P_{x}^{m}(a)a^{x-1}\,da = \frac{2^{x}2^{m}(n+m)}{(2n+1)(n-m)!}\delta_{nn'},
\]

**Proof.** In view of (1), we may write:
\[
\int_{-1}^{1} P_{x}^{m}(a)P_{x}^{m}(a)\,da = \int_{-1}^{1} (1 - a^{2x})^{m}A^{x}P_{x}(a)A^{x}P_{x}'(a)\,da
\]
Applying integration by parts [22], we get:
\[
\int_{-1}^{1} P_{x}^{m}(a)P_{x}^{m}(a)\,da = \left[(1 - a^{2x})^{m}A^{x}P_{x}(a)A^{x}(m-1)P_{x}'(a)\right]_{-1}^{1}
\]
\[- \int_{-1}^{1} A^{x(m-1)}P_{x}'(a)A^{x}(1 - a^{2x})^{m}P_{x}(a)\,da
\]

(28)
Now, from equation (13) with replacing $m$ by $(m - 1)$, we have:

$$(1 - a^{2x}) A^{x(m+1)} P_{x^n}(a) - 2xma^x A^{x(m)} P_{x^n}(a) + x^2(n(n+1) - m(m-1)) A^{x(m-1)} P_{x^n}(a) = 0$$

Multiplying by $(1 - a^{2x})^{m-1}$, we obtain:

$$(1 - a^{2x})^{m} A^{x(m+1)} P_{x^n}(a) - 2xma^x (1 - a^{2x})^{m-1} A^{x(m)} P_{x^n}(a) + x^2(n(n+1) - m(m-1)) (1 - a^{2x})^{m-1} A^{x(m-1)} P_{x^n}(a) = 0$$

Hence,

$$A^x[(1 - a^{2x})^m A^{x^n} P_{x^n}(a)] = -x^2[n(n+1) - m(m-1)] (1 - a^{2x})^{m-1} A^{x(m-1)} P_{x^n}(a)$$

(29)

Substituting from (29) in (28), we get:

$$\int_{-1}^{1} P_{x^n}^{x^n}(a) P_{x^n}'(a) d_x \, da = x^2[n(n+1) - m(m-1)] \times \int_{-1}^{1} P_{x^n}^{x(m-1)}(a) A^{x(m)}(a) A^{x(m-1)} P_{x^n}(a) d_x \, da$$

$$= x^2 (n+m)(n-m+1) \int_{-1}^{1} P_{x^n}^{x(m-1)}(a) P_{x^n,}'(a) d_x \, da$$

Applying this result again gives:

$$\int_{-1}^{1} P_{x^n}^{x^m}(a) P_{x^n}'(a) d_x \, da = x^4(n+m)(n-m+1)(n-m-1)(n-m+2)$$

$$\times \int_{-1}^{1} P_{x^n}^{x(m-2)}(a) P_{x^n,}'(a) d_x \, da$$

Repeating this process $m$-times, we obtain:

$$\int_{-1}^{1} P_{x^n}^{x^m}(a) P_{x^n}'(a) d_x \, da = x^{2m}(n+m)(n+m-1) ... (n+1)(n-m+1)(n-m+2) ... n \times \int_{-1}^{1} P_{x^n}^{x(m-2)}(a) P_{x^n,}'(a) d_x \, da$$

$$= \frac{x^{2m}(n+m)!}{(n-m)!} \int_{-1}^{1} P_{x^n}^{x(m-2)}(a) P_{x^n,}'(a) d_x \, da$$

Using (26), we have:

$$\int_{-1}^{1} P_{x^n}^{x^m}(a) P_{x^n}'(a) d_x \, da = \frac{2x^{2m-1}(n+m)!}{(2n+1)(n-m)!} \delta_{nn'}.$$ 

which is the required results.

6.2. Expansion of functions in terms of associated CFLPs

Having the orthogonality property (26), one can easily represent a given function $f(a)$ over the interval $[-1,1]$ in a series of associated conformable fractional Legendre polynomials such as:

$$f(a) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} A_{n,m} P_{x^n}^{x^m}(a),$$

where $A_{n,m}$ are determined by the relation

$$A_{n,m} = \frac{(2n+1)(n-m)!}{2x^{2m-1}(n+m)!} \int_{-1}^{1} P_{x^n}^{x^m}(a) f(a) d_x \, da,$$

where $d_x a = a^{x-1} da$. 

7. Shifted associated conformable fractional Legendre polynomials

From (11), one may write the associated conformable fractional Legendre equation as

\[ (1 - t^{2x}) A_t^x A_t^x P_{x/n}^x(t) - 2x t^{x} A_t^x P_{x/n}^x(t) + x^2 \left[ n(n + 1) - \frac{m^2}{1 - t^{2x}} \right] P_{x/n}^x(t) = 0, -1 \leq t \leq 1 \]

The transformation \( t^x = 2a^x - 1 \) gives the shifted associated conformable fractional Legendre equation:

\[ a^x (1 - a^x) A_a^x A_a^x \tilde{P}_{x/n}^x(a) - 2x (2a^x - 1) A_a^x \tilde{P}_{x/n}^x(a) + x^2 \left[ n(n + 1) - \frac{m^2}{4a^x(1-a^x)} \right] \tilde{P}_{x/n}^x(a) = 0, \quad a \in [0,1] \quad (30) \]

where \( P_{x/n}^x(a) \) and \( \tilde{P}_{x/n}^x(a) \) represent the ACFLPs and shifted ACFLPs respectively.

As shown in theorem 6.1, the shifted ACFLPs are orthogonal over the interval \([0,1]\), hence we may write:

\[ \int_0^1 \tilde{P}_{x/n}^x(a) \tilde{P}_{x/n}^x(a) da = \int_0^1 \tilde{P}_{x/n}^x(a) \tilde{P}_{x/n}^x(a) a^{x-1} da = \frac{x^{2m-1(n+m)}}{(2n+1)(n-m)!} \delta_{nn}'. \quad (31) \]

Now, for the shifted ACFLPs, we will use the following famous formula.

7.1. Rodrigues formula for shifted ACFLPs

Rajković and Kiryakova [47] discussed and defined the special functions based on a Rodrigues formula for shifted Legendre polynomials. Moreover, they tested the orthogonality property, which only applied in certain situations. The Rodrigues formula for Shifted Conformable Fractional Legendre Polynomials (SCFLP) was also introduced by authors in [37] as:

\[ \tilde{P}_{x/n}^x(a) = \frac{1}{x^n} A_x^m [a^x - 1]^n]. \quad (32) \]

Now, we will establish the Rodrigues formula for shifted associated conformable fractional Legendre polynomials.

The following theorem, in addition to formula (10), gives the Rodrigues formula for shifted ACFLPs.

**Theorem 7.1** The shifted associated conformable fractional Legendre polynomials \( \tilde{P}_{x/n}^{anm}(a) \) can be written in the sense of conformable derivative as

\[ \tilde{P}_{x/n}^{anm}(a) = 2^m a^{x/n} (1 - a^x)^m A_x^m \tilde{P}_{x/n}(a) \quad (33) \]

where \( \tilde{P}_{x/n}(a) \) is the (SCFLP).

**Proof.** Directly, by taking the transformation \( t^x = 2a^x - 1 \) in (10), we can establish the required result.

7.2. Analytical formula of associated of (SCFLP)

The authors in [37], deduced the analytical formula of (SCFLP):

\[ \tilde{P}_{x/n}(a) = \sum_{k=0}^{n} \frac{(-1)^{n+k}(n+k)!}{(n-k)!k!} a^k. \quad (34) \]
Using theorem 7.1, one can drive the analytical formula of the shifted ACFLPs as follow:

In view of (1), we have:

\[
A^{x^m} \tilde{p}_{x^n}(a) = \sum_{k=0}^{n} \frac{(-1)^{n+k}(n+k)!}{(n-k)!(k)!} A^{x^m} a^k
\]

\[
= \sum_{k=m}^{n} \frac{(-1)^{n+k}(n+k)!}{(n-k)!(k)!} \frac{x^m k!}{(k-m)!} a^{x^{k-m}}
\]

Taking \( r = k - m \), we get:

\[
A^{x^m} \tilde{p}_{x^n}(a) = \sum_{r=0}^{n-m} \frac{(-1)^{n+m+r} x^m (n + m + r)!}{(n - m - r)!(r + m)! r!} a^{x^r}
\]

Hence, using (33), we obtain:

\[
\tilde{p}_{x^n}(a) = \sum_{r=0}^{n-m} I_{n,m}^{r,x} (1 - a^x)^\frac{m}{2} a^{x^{r+m/2}}
\]

(35)

where

\[
I_{n,m}^{r,x} = \frac{(-1)^{n+m+r} x^m 2^m (n + m + r)!}{(n - m - r)!(r + m)! r!}
\]

Figs.1, 2 and 3 show graphs of the shifted associated conformable fractional Legendre polynomials (35) for different values of \( n, m \) and \( a \).

**Figure 1**: Illustrates the shifted ACFLPs when \( x = 0.5, m = 1 \) and different values of OR "for" \( n = 2,3,4,5 \).

**Figure 2**: Illustrates the shifted ACFLPs when \( m=2, n=5 \) and different values of \( a=0.2,0.4,0.6,0.8,1 \).
Figure 3: Illustrates shifted ACFLPs with $a=0.5$, $n=5$ and different values of $m=1,2,3,4$.

7.3. **Overlap integral of shifted ACFLPs**

There has been interest in deriving closed expression for the definite integral

$$ I_x(n_1, m_1; n_2, m_2) = \int_0^1 P_{x n_1}^m (a) P_{x n_2}^m (a) \, dx, $$

involving a product of two shifted ACFLPs.

Notice that $n_1, m_1, n_2$ and $m_2$ are integers with $0 \leq m_1 \leq n_1$ and $0 \leq m_2 \leq n_2$.

In this subsection, one can use the analytical form of $P_{x n}^m (a)$ to drive a simple closed expression of the integral (36).

Substituting (35) into (36), we obtain:

$$ I_x(n_1, m_1; n_2, m_2) = \sum_{r_1=0}^{n_1-m_1} \sum_{r_2=0}^{n_2-m_2} I_{x, r_1, r_2}^{n_1, m_1} I_{x, r_1, r_2}^{n_2, m_2} I_{x, r_1, r_2} \left( n_1, m_1; n_2, m_2 \right) $$

(37)

where

$$ I_{x, r_1, r_2} \left( n_1, m_1; n_2, m_2 \right) = \int_0^1 (1 - a^x)^{m_1+m_2} a^{x(r_1+r_2+m_1+m_2)} \, dx \, a = \int_0^1 \left( 1 - a^x \right)^{m_1+m_2} a^{x(r_1+r_2+m_1+m_2)} \, dx \, a $$

(38)

Integral (38) can be easily evaluated by noting that it possesses the form of the beta function, we have:

$$ I_{x, r_1, r_2} \left( n_1, m_1; n_2, m_2 \right) = \frac{1}{x} \beta \left( \frac{m_1+m_2}{2} + 1, \frac{m_1}{2} + \frac{1}{2} (2r_1 + 2r_2 + m_1 + m_2) + 1 \right) = \frac{1}{x} \beta \left( \frac{m_1+m_2+2}{2} + 1, \frac{1}{2} (2r_1 + 2r_2 + m_1 + m_2 + 2) \right) $$

(39)
Hence, Integral (36) becomes:

\[
I_x(n_1, m_1; n_2, m_2) = \frac{1}{x} \sum_{r_1=0}^{n_1-m_1} \sum_{r_2=0}^{n_2-m_2} I_{r_1}^{r_2} I_{x}^{r_2} \beta \left( \frac{m_1+m_2+2}{2}, 1, \frac{1}{2}(2r_1+2r_2+m_1+m_2+2) \right) \tag{40}
\]

By further expressing the beta function in terms of the gamma functions, we get:

\[
I_x(n_1, m_1; n_2, m_2) = \frac{1}{x} \sum_{r_1=0}^{n_1-m_1} \sum_{r_2=0}^{n_2-m_2} I_{r_1}^{r_2} I_{x}^{r_2} \times \frac{\Gamma \left( \frac{1}{2}(m_1+m_2+2) \right) \Gamma \left( \frac{1}{2}(2r_1+2r_2+m_1+m_2+2) \right)}{\Gamma \left( \frac{1}{2}(2r_1+2r_2+2m_1+2m_2+4) \right)} \tag{41}
\]

Equation (41) is thus the analytical solution of the integral (36) which involves summing terms of alternating sign. For example, we can evaluate the integral (36) with various (arbitrary) values of \(n_1, m_1, n_2, m_2\) and \(x\) as shown in table 1.

| \(n_1\) | \(m_1\) | \(n_2\) | \(m_2\) | \(x\) | \(I_x(n_1, m_1; n_2, m_2)\) |
|---|---|---|---|---|---|
| 3 | 2 | 2 | 1 | 0.5 | 70.685834705770347 |
| 5 | 1 | 4 | 2 | 0.4 | -29.157906816130264 |
| 7 | 4 | 6 | 4 | 0.8 | 0 |
| 9 | 3 | 8 | 3 | 0.5 | 0 |
| 10 | 6 | 9 | 5 | 0.2 | 1.184232571084228 \times 10^9 |

It is clear that this method can be easily generalized in a straightforward manner for integrals containing a product of any finite number of associated conformable fractional Legendre polynomials.

8. Conclusion

In view of many practical problems that interest not only for mathematicians and other researchers in sciences, it was found that special functions induced as solutions of famous differential equations, like as Legendre, Bessel, Laguerre, Hermite, hypergeometric, etc. Special functions of fractional calculus emerged and drew a lot of attention because of their wide range of applications in science and engineering. It is known that Legendre polynomials play an important role in solving many differential equations which are related to real life phenomena.

Motivated by the applications involved Legendre differential equation, the current paper introduced the associated Legendre polynomials in the conformable fractional sense. Some interesting properties such as hypergeometric representation, analytical formula, various recurrence relations are established. Orthogonal properties of such associated conformable fractional Legendre polynomials are developed. An interesting compact closed-form expression is derived from the definite integral using a convenient analytical formula for the shifted ACFLPs. All obtained results are newly presented and can be extended as well as they are useful for applications.

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