Notes on adding D6 branes wrapping $\mathbb{RP}^3$ in $\text{AdS}_4 \times \mathbb{CP}^3$

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Abstract

We deform the $\mathcal{N} = 6$ Chern Simons theory by adding extra matter hypermultiplets in a fundamental representation of one or both gauge groups. We compute the quantum corrected moduli space. We verify that the holographic dual of the modified theory consists of the usual $\text{AdS}_4 \times \mathbb{CP}^3$ background in presence of $\text{AdS}_4$ filling D6 branes which wrap $\mathbb{RP}^3 \subset \mathbb{CP}^3$. We extend the correspondence to a similar modification of more general known $\mathcal{N} = 3$ dual pairs

1 Introduction

$\mathcal{N} = 3$ CSM theories in three dimensions have a powerful property: they have a classically and conformally invariant action for any choice of matter content, with no marginal deformations. In this note we consider a modification of the $\mathcal{N} = 6$ ABJM theory $^{[1]}$, where extra matter is added, in the fundamental representations $(N,1)$ or $(1,N)$ of the $U(N) \times U(N)$ gauge group. This breaks the $\mathcal{N} = 6$ supersymmetry to $\mathcal{N} = 3$, but preserves conformal invariance. We relate the resulting theory to Type IIA string theory on $\text{AdS}_4 \times \mathbb{CP}^3$ in presence of $\text{AdS}_4$ filling D6 branes which wrap $\mathbb{RP}^3 \subset \mathbb{CP}^3$. In general there can be $m_1$ fundamentals on the first node, and $m_2$ on the second. This choice corresponds
in the IIA picture to the choice of the $\mathbb{Z}_2$ Wilson line on the $m_1 + m_2$ D6, living in $\pi_1(\mathbb{RP}^3)$. The IIA setup lifts to a purely geometric background of M-theory, $AdS_4 \times M_7$ for certain 3-Sasakian manifolds $M_7$. In M-theory, the choice of Wilson line lifts to a flat topologically nontrivial C-field.

There is a branch of the moduli space of the Chern-Simons-matter theory that corresponds to M2 branes probing this geometry. The classical moduli space in that branch receives a quantum correction. The 1-loop exact quantum correction to the metric on the hyperkähler moduli space was found by direct calculation in [2]. Here we construct the chiral ring. It looks very similar to the chiral ring of the uncorrected theory, except that the conformal dimension of the monopole operators that appear in certain chiral primaries becomes non-zero due to the presence of the fundamental matter. This beautifully matches the chiral ring of the expected moduli space. In the special case where the Chern Simons level is 1, and a single D6 brane is added, we can see the $SU(3)$ isometry of the resulting $N^{0,1,0}$ 3-Sasakian near horizon geometry.

There is another branch to the moduli space, in which the $N$ D2 branes become dissolved in the $m$ D6 branes. This is expected to give the moduli space of $N$ instantons of rank $m$ on the cone over $\mathbb{RP}^3$, $\mathbb{C}^2/\mathbb{Z}_2$. The Chern-Simons levels do not enter the analysis of this branch of the moduli space, and the result is identical to the Higgs branch of a Yang-Mills theory with the same matter content. In fact, this branch is characterized by the fact that the fundamentals have nonzero VEVs. This implies that the moment maps must all be vanishing due to the F-term and D-term equations. Then this branch of the moduli space of this 2+1 CSM quiver theory is exactly the moduli space of the same quiver interpreted as a 3+1 gauge theory. In fact the quiver is precisely the ADHM quiver for charge $N$ instantons of rank $m_1 + m_2$ on $\mathbb{C}^2/\mathbb{Z}_2$. This agrees beautifully with the expected result.

One can study the same question for the more general $\mathcal{N} = 3$ quiver CSM theories with $n$ nodes of [3, 4]. The branch in which the D2 branes are dissolved into the D6 branes again leads exactly to the ADHM quiver, this time for $\mathbb{C}^2/\mathbb{Z}_n$. Indeed in this case the D6 branes are wrapping an $S^3/\mathbb{Z}_n$ homologically trivial 3-cycle.

After this draft was completed, we received [5], which has significant overlap with this work, but reaches different conclusions regarding the precise match between the number of D6 branes and the number of fundamental fields $m_1, m_2$.

## 2 Adding fundamentals to Chern-Simons-matter theories

Consider the introduction of fundamental hypermultiplets in the $\mathcal{N} = 6$ theory of [1]. This cannot be done in a manner preserving more than $\mathcal{N} = 3$ supersym-
metry. The Lagrangian for a $\mathcal{N} = 3$ theory, with unbroken $SO(3)$ R-symmetry, with two bifundamental hypermultiplets and any number of fundamental hypermultiplets is uniquely determined. We will denote the bifundamental chiral fields $A_i^i, B_i$ for $i = 1, 2$, $m_1$ fundamentals of the first gauge group, $a^i, b_i$, and $m_2$ fundamentals of the second group, $c^i, d_i$.

The $\mathcal{N} = 2$ superpotential for the $\mathcal{N} = 3$ theory is

$$W = \frac{4\pi}{k_1} (B_i A_i^i + b_i a^i)^2 - \frac{4\pi}{k_2} (-A_i^i B_i - c^i d_i)^2$$ (2.1)

We see that even if $k_1 + k_2 = 0$, the coupling to the fundamental matter reduces the flavor symmetry to an $SO(3)_F$. This illustrates clearly the fact that the $\mathcal{N} = 3$ supersymmetry is not enhanced, as the original $SO(6)$ R-symmetry is reduced to $SO(3)_R \times SO(3)_F$. Although in the following we will keep $k_1 + k_2 = 0$, it would be very natural to relax this constraint, and introduce a Roman mass in the holographic dual, as in [6]. Along the same lines, we can add fundamental matter to a more general family $\mathcal{N} = 3$ CSM theories with known holographic duals. These theories were introduced in [4]. They are the unique $\mathcal{N} = 3$ quiver theories with unitary gauge groups organized in a necklace, with a single bifundamental hypermultiplet between adjacent nodes. If we add fundamental matter while preserving $\mathcal{N} = 3$ SUSY we get a superpotential

$$W = \sum_i \frac{4\pi}{k_i} (B^{(i+1)} A^{(i+1)} - A^{(i)} B^{(i)} + b_i^{(i)} a^{(i),t})^2$$ (2.2)

The resulting $\mathcal{N} = 3$ Chern-Simons-matter theory with fundamentals will still be conformally invariant, since there are no marginal or relevant operators which preserve the $\mathcal{N} = 3$ supersymmetry and $SO(3)$ R-symmetry, just as in the case without fundamentals [7].

3 Introducing D6 branes in AdS$_4 \times M_6$

3.1 D5 branes in the IIB configuration

We begin with IIB theory on a circle, with axio-dilaton $\tau = \frac{1}{g_s} + \chi$, and consider $N$ D3 branes along directions 0123 (where $x^3$ is the circle direction), and various D5 and $(1, p_i)$ fivebranes. The $(1, p_i)$ fivebrane, $i = 1, ..., n$, is extended along 0123 [7, 8, 9, 10, 11], where $\theta_i = \arg(\tau) - \arg(p_i + \tau)$. The $m$ D5 branes are extended along the 3-plane with $\theta = \arg(\tau)$. This configuration preserves $\mathcal{N} = 3$ supersymmetry [8, 9]. In the simplest example, D5 branes are added to the configuration that engineers the $\mathcal{N} = 6$ Chern-Simons-matter theory [10], with one NS5 brane and one $(1, k)$ fivebrane.

The T-dual and lift to M-theory of such a configuration of $(p, q)$ fivebranes was determined by [10]. In general, a Lagrangian description of the effective 2+1
field theory on \( N \) D3 branes stretched between a \((p, q)\) fivebrane and a \((p', q')\) fivebrane is not known. When all the fivebranes have a single unit of NS5 charge, the effective field on the D3 branes flows to a 2+1 conformal field given by an \( \mathcal{N} = 3 \) quiver Chern-Simons-matter theory [3, 4]. The Chern-Simons level on the D3 branes stretched between the successive \((1, p_i)\) and \((1, p_{i+1})\) fivebranes is given by \( k_i = p_{i+1} - p_i \). We now consider introducing some D5 branes as well.

The D5 branes may be distributed along the the \( x^3 \) circle between any pair of \((1, p_i)\) fivebranes, so we partition \( m = \sum_{i=1}^{n} m_i \). There will be \( m_i \) fundamental hypermultiplets attached to that node in the quiver, arising from the D3-D5 bifundamental strings. Note that in this configuration there are 4 Neumann-Dirichlet directions, thus one obtains precisely two chiral multiplets that make up the hypermultiplet.

### 3.2 M-theory lift

This IIB configuration can be T-dualized and lifted to M-theory, following [10]. Applying T-duality to the \( x^3 \) circle takes the D3 branes to D2 branes, now living in a seven dimensional transverse geometry. The \((1, p_i)\) fivebranes become Taub-NUT with D6 charge dissolved into \( F_2 \) flux. The D5 branes naturally become D6 branes in this geometry. The metric is much simpler after lifting to M-theory, so we will postpone the details of the IIA description until later.

This lifts to pure geometry in M-theory, with M2 branes probing an eight dimensional hyperkahler transverse space. As shown in [10], the metric on this \( T^2 \) fibration over a base \( \mathbb{R}^6 \) is given in terms of a two by two matrix of harmonic functions as

\[
\begin{align*}
\text{ds}^2 &= U_{ij} d\tilde{x}^i \cdot d\tilde{x}^j + U^{ij} \left( d\varphi_i + A_i \right) \left( d\varphi_j + A_j \right), \\
A_i &= d\tilde{x}^j \cdot \tilde{\omega}_{ji} = dx_a^i \omega^n_{ji}, \quad \partial_{x_{ki}} \omega^n_{ji} - \partial_{x_{jk}} \omega^n_{ji} = \epsilon^{abc} \partial_{x^c} U_{ki}, \\
\end{align*}
\]

where \( U^{ij} \) is the inverse of the matrix \( U_{ij} \). The matrix \( U \) obeys linear equations that follow from this ansatz. The metric of a single Kaluza-Klein monopole times \( \mathbb{R}^3 \times S^1 \) that arises from a single NS5 brane can be written in this form as a configuration with

\[
U = U_\infty + \begin{pmatrix} h_1 & 0 \\ 0 & 0 \end{pmatrix}, \quad h_1 = \frac{1}{2|x_1|}. \tag{3.2}
\]

The asymptotic value \( U_\infty \) encodes the complex and Kahler parameters of the torus fiber at infinity, which are determined in terms of the original IIB coupling \( \tau \), and the radius of the \( x^3 \) circle. The matrix of harmonic functions associated to a \((p, q)\) fivebrane can be obtained from the above \( U \) by application of the appropriate \( SL(2, \mathbb{Z}) \) transformation via

\[
U \mapsto g^\dagger U g, \quad \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} \mapsto g \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix}. \tag{3.3}
\]
The linearity of the harmonic equation that \( U \) satisfies implies that the metric obtained by lifting \((1,q_i)\) fivebranes, and \(m\) D5 branes is given by

\[
U = U_\infty + \sum_{i=1}^{n} \frac{1}{2|x_1 + q_i x_2|} \left( \begin{array}{cc} 1 & q_i \\ q_i & q_i^2 \end{array} \right) + \frac{m}{2|x_2|} \left( \begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right). \tag{3.4}
\]

The low energy effective theory on the stack of M2 branes will be determined by the local singularity at the origin in this geometry. This conical hyperkahler 8-manifold was shown by \([11]\) to be a particular abelian hyperkahler quotient. In particular, the pair of \(U(1)\) isometries of the \(T^2\) fiber are compatible with the hyperkahler structure, and one obtains the hypertoric manifold \(\mathbb{H}^{n+1}/\!\!/N\), where \(N\) is the kernel of the map

\[
\beta : U(1)^{n+1} \to U(1)^2, \quad \beta = \left( \begin{array}{ccc} 1 & 1 & \ldots & 1 & 0 \\ p_1 & p_2 & \ldots & p_n & m \end{array} \right). \tag{3.5}
\]

The hyperkahler quotient just described sets to zero the moment maps for all \(U(1)\)'s in the kernel of the \(\beta\), thus we see that the coordinates on the base \(\mathbb{R}^6\) are exactly

\[
\mu^\alpha_I = \sum_{i=1}^{n} \mu_i^\alpha, \\
\mu^\alpha_{II} = \sum_{i=1}^{n} p_i \mu_i^\alpha + m \mu^\alpha_{n+1}. \tag{3.6}
\]

The way the lifted geometry is modified from that found in \([4]\) by the inclusion of the D5 branes can be explained as follows. Consider the geometry obtained without D5 branes. If we erase the D5 brane, we have the hypertoric manifold \(\mathbb{H}^n/\!\!\!/\bar{N}\), where \(\bar{N}\) is the kernel of the map

\[
\beta : U(1)^{n} \to U(1)^2, \quad \beta = \left( \begin{array}{ccc} 1 & 1 & \ldots & 1 \\ p_1 & p_2 & \ldots & p_n \end{array} \right). \tag{3.7}
\]

The coordinates on the base \(\mathbb{R}^6\) are exactly

\[
\mu^\alpha_I = \sum_{i=1}^{n} \mu_i^\alpha, \\
\mu^\alpha_{II} = \sum_{i=1}^{n} p_i \mu_i^\alpha. \tag{3.8}
\]

In the IIA picture, before going to the near horizon limit, the \(m\) D6 branes wrap the cycle defined by \(\mu^\alpha_{II} = 0\). We will better characterize this cycle in the next subsection. For now, let’s keep looking at the backreacted M-theory geometry.

We want to compare the “chiral rings” on the two manifolds, i.e. the set of homogeneous, holomorphic functions on the hyperkahler cones in a given complex
structure. In a hypertoric manifold $\mathbb{H}^{n+1}/\mathbb{N}$, such functions are built out of monomials of the elementary quaternionic coordinates on $\mathbb{H}^{n+1}$, invariant under $\mathbb{N}$, modulo F-term relations.

With respect to $\mathbb{H}^{n}/\tilde{\mathbb{N}}$ we have an extra quaternionic coordinate $(u, v)$. The set of F-term relations

$$
\sum c_i u_i v_i + c uv = 0, \quad \sum c_i = 0 \quad \sum c_i p_i + mc = 0 \quad (3.9)
$$

is the union of the F-term relations $c = 0$ for $\mathbb{H}^{n}/\tilde{\mathbb{N}}$ and an extra one, which can be used to eliminate $uv$ from the monomials. We are led to look for functions of the general form $u^d f$ (for $d \geq 0$) or $v^{-d} f$ (for $d \leq 0$). Here $f$ is a function on $\mathbb{H}^{n}/\tilde{\mathbb{N}}$, which transforms with definite weights, $a_1, a_2$, under the pair of hypertoric $U(1)$ isometries, $(\lambda_1, \lambda_2) : f \mapsto \lambda_1^{a_1} \lambda_2^{a_2} f$. Now we should require $u^d f$ ($v^{-d} f$) to be invariant under all the $U(1)$’s except those in the kernel of the new $\beta$.

Of course, $f$ is already invariant under the quotients of the original geometry, so the only new requirement is that under a transformation with weights $(\lambda_1, \ldots, \lambda_n, \lambda_0)$, such that $\prod_{i=1}^n \lambda_i = 1$ and $\lambda_0^n \prod_{i=1}^n \lambda_i^{p_i} = 1$, then $u^d f \mapsto \lambda_0^d (\prod_{i=1}^n \lambda_i^{p_i})^{a_2} u^d f$ must be invariant. Therefore we must have

$$
d = ma_2.
$$

This says that the functions on the new hyperkahler manifold are the same as before, but dressed with an appropriate power of $u$. Thinking of these as elements in the chiral ring, we see that the spectrum of chiral operators is unchanged, but the conformal dimensions are shifted, by precisely $\frac{1}{2} m$ times the baryon number of the operator. The chiral ring relations are also modified, because the product $u^{d_1} f_1 u^{d_2} f_2$ will have to be rewritten to eliminate the factors of $uv$ with the F-term relation.

### 3.3 D6 branes on $\mathbb{RP}^3$ in $\mathbb{CP}^3$

Reducing to IIA on the $S^1$ associated to the baryonic current in the field theory, we obtain $N$ D2 branes in a seven dimensional transverse cone. The near horizon geometry is given by $\text{AdS}_4 \times \mathbb{CP}^3$ in the $\mathcal{N} = 6$ case, and for more general quivers the dilaton is varying in the internal six manifold.

Note that the D5 branes introduced above become D6 branes in the IIA reduction of the near horizon geometry. That is, the same circle is identified as the M-theory circle in the GGPT geometry and the 3-Sasakian near horizon. Intuitively, this is because we are taking an ’t Hooft limit for which the D5 charge goes to infinity while the NS5 charge remains fixed in the IIB configuration, hence the smallest cycle will be the M-theory circle that shrinks due to the D5 charge.

More concretely, recall that the 3-Sasakian internal seven manifold is the unit sphere in the singular hyperkahler quotient $\mathbb{H}^{n+1}/\mathbb{N}$. The ’t Hooft limit
involves scaling all of the fivebrane charges by $\left(1, q_i\right) \mapsto \left(1, kq_i\right)$, hence the kernel of $\beta$ contains a discrete subgroup $\mathbb{Z}_k$ acting by phase rotation on all of the $u_i$. This $\mathbb{Z}_k$ sits inside of a $U(1)$ isometry of the hyperkahler eight manifold, and results in a parametrically small cycle in the near horizon geometry. But this $U(1)$ is precisely the phase associated with the coordinate $\mu^\alpha_I$ on the base of the GGPT torus fibration. Thus we see that the cycle we reduce on to IIA is the same as the M-theory circle in the lift of the original brane configuration.

As we said above, the D6 branes are wrapping the locus $\mu^\alpha_I = 0$ in the seven dimensional cone. Notice that if we set $\mu^\alpha_I = 0$ and remove the corresponding $U(1)$ circle reduce to IIA theory, we are really doing an hyperkahler quotient. More precisely, the D6 branes are wrapped on the hyperkahler four manifold given by the quotient $\mathbb{H}^n///\mathbb{N}'$, where $\mathbb{N}'$ is the kernel of the map

$$\beta' : U(1)^n \to U(1), \quad \beta' = (1 \ 1 \ldots 1).$$

This is a standard realization of the hyperkahler geometry $\mathbb{C}^2/\mathbb{Z}_n$. In the near horizon limit, the cycle wrapped by the D6 branes is the unit sphere in this space, namely $S^3/\mathbb{Z}_n$. For $n = 2$, this is $\mathbb{R}^3$ in $\mathbb{C}^3$. It is preserved by a $SO(4) \subset SU(4)$ subgroup of the isometries of $\mathbb{C}^3$, which coincides with the symmetry group of our proposed SCFT.

Notice that $\pi_1(S^3/\mathbb{Z}_n) = \mathbb{Z}_n$, hence the D6 branes can carry a discrete $\mathbb{Z}_n$ Wilson line. This discrete parameter in the IIA description is the remnant of the position of the D5 branes in the IIB circle. They could sit in any of the $n$ intervals between $(1, p_i)$ and $(1, p_{i+1})$ fivebranes. They would correspondingly contribute a single fundamental hypermultiplet at the node of the necklace quiver with Chern-Simons coupling $p_{i+1} - p_i$.

### 3.4 Volumes and Free Energy

Consider the M-theory lift of $\text{AdS}_4 \times \mathbb{P}^3$ with $m$ D6 branes wrapping the $\mathbb{R}^3$ cycle. The internal tri-Sasakian seven manifold is the unit sphere in the hypertoric eight manifold $\mathbb{H}^3///U(1)$, where the $U(1)$ acts with charges $m, m, k$. The volume of the seven manifold is given by \cite{[12]}

$$Vol(M_7) = Vol(S^7) \frac{m + 2k}{2(m+k)^2}, \quad (3.10)$$

in terms of the volume of a sphere with the same radius of curvature. In the $\text{AdS}_4 \times M_7$ near horizon limit, the total integral of $*G_4$ on $M_7$ is by definition the total M2 brane charge of $N$. The effective four dimensional supergravity solution corresponding to a black M2 brane only depends on the Planck scale and the local value of the four form field strength in the $\text{AdS}_4$.

The number of degrees of freedom at high temperatures is determined from the AdS black hole, which is modified from the calculation of M2 branes in flat
space only by the change in the four dimensional Plank scale, i.e. in the volume of the internal space. We find that in area $V$ and at temperature $T$:

$$
\beta F = -2^{7/2} 3^{-2} \pi^2 N^{3/2} \frac{(m + k) \sqrt{2}}{\sqrt{m + 2k}} V^2 T^2 \sim \frac{N^2}{\sqrt{\lambda}} + \frac{3mN}{4} \sqrt{\lambda} + ...
$$

Intriguingly, there is actually an enhancement of the number of degrees of freedom in the fundamentals by a factor of $\sqrt{\lambda}$, relative to $m$ weakly coupled $U(N)$ fundamentals! Of course, the total number of degrees of freedom is still less than the counting of fields in a free theory, $N^2 + mN$ in this case, since we are assuming $m \ll k$ in the above expansion. It would be interesting to explore this phenomenon further.

### 4 Higgs branch moduli space

The superpotential is given by

$$
W = \frac{1}{k} \text{Tr} \left( B_i A^i + b_t a^t \right)^2 - \frac{1}{k} \text{Tr} \left( -A^i B_i - c^s d_s \right)^2.
$$

(4.1)

Define the hyperkähler moment maps

$$
\mu_1^\alpha = \left\{ B_i A^i + b_t a^t, A_i A_i^\dagger - B_i B_i^\dagger + a_t^\dagger a_t - b_t b_t^\dagger \right\}
$$

$$
\mu_2^\alpha = \left\{ -A^i B_i - c^s d_s, B_i B_i^\dagger - A_i A_i^\dagger + d_t^\dagger d_t - c_t^\dagger c_t \right\}.
$$

(4.2)

Then the bosonic potential vanishes if and only if

$$
A^i \mu_1^\alpha = B_i \mu_2^\alpha
$$

$$
a_t^\dagger \mu_1^\alpha = 0
$$

$$
b_t \mu_1^\alpha = 0
$$

$$
c_t \mu_2^\alpha = 0
$$

$$
d_t \mu_2^\alpha = 0.
$$

(4.3)

The branch in which the D2 branes dissolve into the D6 branes is $\mu_1^\alpha = 0$. Only in that case can one turn on the fundamentals. Now there is no Mukhi effect, since there are matter fields turned on which are charged under the entire gauge group. That is, the gauge symmetry is completely Higgsed. Therefore solving the F-term equations, together with $\mu_2^\alpha = 0$, gives exactly the moduli space of the same quiver interpreted as a 3+1 Yang-Mills quiver. But it is just the ADHM quiver describing $N$ instantons in $\mathbb{C}^2/\mathbb{Z}_2$ with rank $m_1 + m_2$.

More generally, the hyperkahler moduli space of a general $\mathcal{N} = 3$ theory is determined as follows. The bosonic potential is given by

$$
V = \sum_{\alpha=1}^3 \sum_{a} \left| (k^{-1})^{ij} \mu_i^\alpha T_{ij} q_{a} A^a \right|^2
$$

where where $q_{a}^A$ are the matter fields indexed by $a, b$ in a pseudoreal representation of the gauge group determined by $T$, $A$ is an $SU(2)_R$ doublet index. $\mu_i^\alpha$ are
the hyperkahler moment maps, $\alpha = 1, 2, 3$, $i, j$ are gauge group indices, and $k$ is the matrix defined by the Chern-Simons form. Therefore we have the equations
\[(k^{-1})^{ij} \mu_i^\alpha T^{ab}_j q_b^A = 0, \tag{4.4}\]

There may be branches where $\mu^\alpha \neq 0$. On such branches, the gauge group is not entirely Higgsed, since the moduli space equation implies the matter fields are invariant under some gauge transformations, determined by $k^{-1}\mu^\alpha$. Let’s assume for simplicity that the unbroken gauge group is Abelian. Hence we may have a contribution to the moduli space from the dualized gauge fields. The Chern-Simons coupling is a possible obstruction to the dualization. Note that (4.4) together with the definition of the moment map,
\[\mu_i^\alpha = T^{ab}_i q_a^A q_b^B \Gamma_{AB}^\alpha,\]
where $\Gamma^\alpha = \sigma^\alpha \sigma^2$ in terms of Pauli matrices are the tensor product coefficients, implies that
\[(k^{-1})^{ij} \mu_i^\alpha \mu_j^\beta = 0, \]
thus $k^{-1}\mu$ lies in a subgroup, $H$, of the unbroken gauge group which is null in the Chern-Simons form. This allows the corresponding gauge fields to be dualized, and contribute to the moduli space. At low energies, the “extra” directions in moduli space allowed by $\mu^\alpha \neq 0$ should combine with the dualized gauge bosons to give an hyperkahler manifold.

We can parameterize the extra directions in moduli space by the expectation value of a monopole operator with magnetic flux in $H$. When such a monopole is turned on, the Chern-Simons term is not invariant under all constant gauge transformations, $\Lambda$, due to the term $k \frac{1}{4\pi} \Lambda \int_{S^2} F$. For a given monopole background, the phase appearing in the partition function, $e^{iS_{CS}}$, defines a map from the gauge group to $U(1)$. One should only quotient by constant gauge transformations that are in the kernel of this map for all allowed monopole configurations (obeying the appropriate flux quantization) on a given branch of the moduli space. Much like the Coulomb branch of $\mathcal{N} = 4$ theories, the metric on these branches may receive quantum corrections [2]. Much information on the Coulomb branch of $\mathcal{N} = 4$ theories can be extracted by a careful analysis of monopole operators [13, 14, 15]. The same is true for $\mathcal{N} = 3$ CSM theories. We will present the analysis in the next section.

5 Quantum corrected chiral ring of $\mathcal{N} = 3$ CSM theories

The moduli spaces and chiral ring of $\mathcal{N} = 3$ conformal field theories display the rigidity of hyperkähler manifolds. Indeed chiral primary operators are the
highest weight components of $SU(2)_R$ multiplets, and their conformal dimension is determined by the spin of the representation. The $\mathcal{N} = 2$ superpotential, which determines the chiral ring, is fixed by $\mathcal{N} = 3$ supersymmetry, hence it is impossible for the F-term equations to receive quantum corrections. Simple chiral primary operators made out of the elementary scalar fields of the theory sit in $SU(2)_R$ multiplets determined immediately from the form of the operator, hence their conformal dimension is unaffected by quantum corrections.

If the expectation values of such simple chiral primary operators were sufficient to parameterize the vacua of the theory, this would be the end of the story. On the other hand, the $\mathcal{N} = 3$ CSM theories we consider in this paper have a larger set of chiral primary operators, and a more interesting moduli space. Indeed, as reviewed in the previous section the moduli space parameterized by the $A, B$ bifundamental fields is unusually large due to the $\sum k_i = 0$ constraint, which allows the moment maps of the gauge action to be non-zero, and furthermore allows a certain combination of the gauge fields to be dualized into an extra scalar field. Overall, the moduli space has one extra hyperkahler dimension for each unbroken $U(1)$, over which the shift of the dual photon acts tri-holomorphically.

The simple chiral primary operators are insufficient to parameterize the full moduli space, as they have charge zero under shifts of the dual photon. Operators charged under the shift of the dual photons have to carry magnetic charge, and can be realized as disorder operators in the three dimensional field theory, as in [16], [13] As we are dealing with a CFT, the definition is truly straightforward: we can use the state-operator map, and simply look at BPS states in the theory on a two-sphere, with magnetic flux on the sphere. In a gauge theory with Yang-Mills coupling such states are simply realized by turning on a constant gauge field on the sphere in a specific $U(1)$ subgroup of the gauge group, and a constant expectation value in the same $U(1)$ subgroup of one adjoint scalar of the gauge multiplet. There are three scalars in the gauge multiplet, and the choice of one of them corresponds to the choice of an $\mathcal{N} = 2$ subalgebra. There are fermion zeromodes from any fermion charged under the magnetic field, which need to be properly quantized. The final result is that the BPS vacuum for the fermion zeromodes carry an R-charge in the $\mathcal{N} = 2$ subalgebra equal to

$$ Q = \frac{1}{2} \left( \sum_{i \in \text{hyper}} - \sum_{i \in \text{vector}} \right) |q_i| $$

Here $q_i$ are the $U(1)$ gauge charges of fermions in either hypermultiplets or vectormultiplets. If this R-charge is positive, the states defined by different $\mathcal{N} = 2$ subalgebras can be organized into a finite-dimensional $SU(2)_R$ multiplet of conformal dimension $Q$.\footnote{If the R-charge is nonpositive, it signals a mistake in the assumption that the theory flows to an}

To study monopoles in a CSM theory we can add
a small Yang-Mills coupling as a regulator of the theory. The main effect of the CS coupling is that the Chern-Simons equations of motion are not satisfied by a constant magnetic flux on the sphere in the absence of an appropriate charge density, as $k \ast F = J$. Such a charge density can be generated by acting on the naive vacuum with creation operators of matter scalar fields in the s-wave. There is a certain tension here: if a scalar field is charged under the $U(1)$ magnetic field of the monopole, it has Landau levels on the sphere which do not include an s-wave state, and have an energy greater than the R-charge of the field. Hence only scalar fields with no charge under the $U(1)$ subgroup used to define the monopole can be used to dress the naive vacuum of the monopole. On the other hand, for generic choices of the CS levels and monopole charges, the required $k \ast F$ charge has a component along this $U(1)$, and the CS equations of motion can never be satisfied. This is why it is important that the $U(1)$ charge should be null in the Chern-Simons form.

In the specific CSM theories we consider, with $\sum k_i = 0$, a monopole generated by a $U(1)$ embedded the same way (say by an element $t$ of the Lie algebra) in all gauge groups requires a charge proportional to $(k_1t, k_2t, \cdots k_n, t)$, which is orthogonal to the $U(1)$ embedding $(t, t, \cdots t)$. Moreover, such charge can be generated by acting with creation operators from the bifundamental scalar fields, which are not charged under the monopole $U(1)$. From now on we will always consider such monopole operators. Overall, the dimension of the monopole operator will be the sum of the two contributions

$$Q_0 = \frac{1}{2} \left( \sum_{i \in \text{hyper}} - \sum_{i \in \text{vector}} \right) |q_i|$$

and the dimension of the scalar fields used in the dressing.

Hence if we change the matter content of the theory, for example by adding fundamental matter at some node, the extra charged hypermultiplets will contribute to the dimension of the monopole operator, and correct the dimensions of the chiral ring operators charged under shifts of the dual photon, by an amount $\frac{1}{2}qm$ proportional to the charge $q$ and to the number of fundamental fields $m = m_1 + m_2$. This exactly what we found in section 3.2.

5.1 Examples of quantum corrected “geometric” moduli space

We begin with the theory that arises at low energies from $N$ D3 branes intersecting an NS5, $(1, 1)$ fivebrane, and D5 brane. The classical moduli space simply consists of $\mathbb{C}^4$, since the vanishing of the bosonic potential implies that the fundamentals $a = 0$, $b = 0$. In \cite{1}, the ring of chiral operators was found to be IR fixed point with the same $SU(2)_R$ R-symmetry as in the UV. \cite{15}
\( T A_i, \tilde{T} B_i \), where \( T \) is the \('t Hooft operator\) that creates one unit of magnetic flux for the diagonal combination of gauge fields (and \( \tilde{T} \) creates \(-1\) units of that magnetic flux). That is, \( \int_{S^2} F_+ = 1 \). Note that this operator is mutually local (ie. has a non-singular OPE) with the bifundamental matter fields, since they are neutral under this combination of the gauge groups.

In the \( \mathcal{N} = 6 \) theory without the additional of the fundamentals, these operators are dimension zero. We now argue that the addition of the fundamental makes this operator have dimension \( 1/2 \), so the gauge invariant operators \( T A_i, T' B_i \) have dimension \( 1 \). Together with the dimension 1 mesonic operators, \( A' B_j \), these form the 8 dimensional representation of the flavor \( SU(3) \), and exactly give the ring of chiral operators on \( T^* \mathbb{P}^2 \). There have been various proposals in the literature for the conformal field theory dual to this \( \text{AdS}_4 \times \mathbb{N}^{0,1,0} \) \([17, 18]\) that are significantly different from ours, but we see here that the quantum correction to the moduli space is crucial for finding the correct answer.

We can construct the dimension 1 currents of the quantum \( SU(3) \) flavor symmetry, by acting with supersymmetry generators on the dimension 1 operators, which are nothing else but the moment maps of the flavor symmetry group.

### 5.2 OPEs

Consider a monopole operator, \( T \), with magnetic flux determined by a map \( \rho : U(1) \to G \), up to gauge equivalence. As we explained above, it may pick up an anomalous dimension, \( q/2 \), for \( q \) a positive integer. This operator lives in a dimension \( q + 1 \) representation of the \( SU(2)_R \). There is also a distinct conjugate operator \( \tilde{T} \), the \('t Hooft operator\) with magnetic flux associated to \( \tilde{\rho}(e^{i\phi}) = 1/\rho(e^{i\phi}) \), which has the same anomalous dimension. Suppose we want to compute the OPE of \( T \) and \( \tilde{T} \) in the chiral ring. Due to the Chern-Simons terms, the Gauss’ law in the monopole background is modified, so some of the zero modes of the matter field must be excited on the \( S^2 \). We want to focus on the contribution to the OPE from the magnetic flux configuration itself; the results should be dressed with matter operators appropriately in gauge invariant combinations. As discussed in \([16, 13]\), one should compute the partition function on a cylinder, \( I \times S^1 \), with the monopole configuration on the \( S^2 \).

Suppose one considers a particular embedding of the \( U(1) \) into the gauge group. One should think of the monopole operator that would be gauge invariant in the absence of Chern-Simons terms as being the average over the group of the associated operator. In such a particular configuration, one may apply the method of \([16, 13]\) to determine the OPE. It is not difficult to convince oneself that only when the configurations at the two ends of the cylinder are identical do any contributions to the chiral ring appear. The result of this calculation is that

\[
TT \sim \int dg_G (\mu \cdot \text{ad}_g(h))^q,
\]
where $h \in \mathfrak{g}$ is the generator of the $U(1)$ embedding.

In the special case $G = U(1)$ we reproduce the result of section 3.2. A gauge invariant operator containing $T$ corresponds to a function on moduli space which has a factor $u^q$. A gauge invariant operator containing $\tilde{T}$ corresponds to a function on moduli space which has a factor $v^q$. The OPE is expected to reproduce the fact that $u^qv^q = (uv)^q$ can be rewritten as the $q$-th power of the moment map of the baryonic symmetry $\sum p_i \mu_i$ by the F-term relations.

6 Branes and chiral operators

Let us first identify the operators dual to gravitons in the M-theory description. These will correspond to gravitons, D0 branes and their bound states in the IIA near horizon geometry. Note that the dilaton is not constant in the internal manifold, so the most natural definition of the “pure” D0 brane is the operator of smallest dimension charged under the $U(1)_{B}$.

The simplest operators are the mesons $\text{Tr} (A_i B_j)$ which are neutral under both $U(1)$ isometries, and transform in the adjoint of the $SU(2)$. These are dual to gravitons in M-theory with no momentum along the $T^2$ isometry directions. Note that there are fewer protected operators of this form than in the $\mathcal{N} = 6$ theory, since operators of the form $\text{Tr} (C_i C_j^i)$ include the likes of $\text{Tr} (A_i A_j^i)$ which is unprotected in the $\mathcal{N} = 3$ theory. It would be interesting to study these almost protected operators that would fill out the $SU(4)_R$ multiplet in the ABJM theory with a small number of fundamentals. On the field theory side, the anomalous dimensions will arise from loops of fundamental fields. On the gravity side, they are expected to arise from the interaction with the D6 branes.

To construct operators charged under the $U(1)_F$, recall that the bifundamental hypermultiplets are all charged equally. Thus the operator with smallest dimension, $n/2$, is $\text{Tr} (A_n A_{n-1} \ldots A_1)$. Chiral primaries constructed out of the matter fields alone cannot receive quantum corrections in these $\mathcal{N} = 3$ theories.

The minimal monopole operator from which a chiral primary may be constructed is $T$ with magnetic charge in a $U(1)$ subgroup of the diagonal $U(N)$ in $U(N)^n$. It carries $k_i$ fundamental indices under the $i^{th}$ gauge group, due to the Chern-Simons coupling. Therefore one can form gauge invariant chiral operators of the form $T \prod_i C_i^{d_i}$, for $d_i - d_{i+1} = k_i$, and where by $C_i^{d_i}$ we mean $A_i^{d_i}$ if $d_i > 0$ and $B_{-d_i}$ if $d_i < 0$. The obvious solution is to take $d_i = q_i + d$, the D5 charge of the $i^{th}$ fivebrane. This operator will have dimension $\frac{1}{2}(\sum d_i + m)$, and charge $\sum q_i k_i$ under the $U(1)_B$, so they are dual to D0 branes.

If there is more than one fundamental, the M-theory supergravity description is never strictly valid, since the 3-Sasakian internal manifold will have orbifold singularities. In the IIA description, the near horizon geometry is a warped compactification, so AdS$_4$ curvature and string coupling depend on the position
in the internal six manifold. This limit will be valid when the curvature and string coupling are small at the maximum value of the size of the M-theory circle. As an estimate of that radius, we will use the inverse of momentum of the lightest D0 brane, as determined from the field theory analysis.

In particular, we find that

$$R_{str}^2 \gtrsim \frac{R_{3M}^3}{m+k} = \frac{8\pi N^{1/2}}{\sqrt{m+2k}}.$$  

The radius of the 3-Sasakian manifold in eleven dimensional Planck units is $R_M/\ell_P \sim N^{1/6}(m+k)^{1/3}(m+2k)^{-1/6}$. Thus the size of the M-theory circle at its largest will be of order $N^{1/6}(m+k)^{-2/3}(m+2k)^{-1/6}$. Therefore, IIA supergravity will be valid when

$$m + 2k \ll N \ll (m + k)^4(m + 2k).$$

Note that the field theory becomes weakly coupled even for fixed Chern-Simons level if the number of fundamentals is much greater than $N$. In the regime where there is a gravity dual, but $m \gg k$, these reproduce the results of [19] for the D2-D6 system in flat space. This is natural, since the geometry is dominated by the $\mathbb{C}^2 \times \mathbb{C}^2/\mathbb{Z}_m$ singularity near the lift of the D6 branes.

7 $\mathcal{N} = 3$ Mass deformation

In completely Higgsed branch of the moduli space, the effect of turning on FI masses, which breaks conformal invariance, is to modify the equations to

$$\mu_i^\alpha = \zeta_i^\alpha,$$  

(7.1)

where $i$ indices the nodes in the quiver. This is the usual FI deformation of the D6-D2 system. For generic $\zeta_i^\alpha$, the geometric branch (with $A_i$, $B_i$ diagonal, and fundamentals set to zero) will be lifted.

In the IIB picture, there are clearly $\mathcal{N} = 3$ mass deformations corresponding to the relative positions of all of the fivebranes. If there are no D5 branes, these precisely correspond to the FI parameters of the CSM theory - the overall one does not change the potential, one linear combination is non-geometric as seen in [20], and the rest correspond to (partial) hyperkähler resolutions of the singularity. Once we have at least one fundamental, the overall FI parameter gives it a mass,

$$W = \sum_i \frac{1}{k_i} (A_i B_i - B_{i+1} A_{i+1} + p_i^s q_i^s - \zeta_i)^2,$$  

(7.2)

where $p_i^s$ are the fundamentals on the $i^{th}$ node. The FI deformation $\zeta_i = k_i \zeta$ completely cancels except for giving masses to the fundamentals. In the case of
a single fundamental added to the ABJM theory, this corresponds to a complete resolution of the hyperkahler singularity, $T^*\mathbb{C}P^2$. The OPE of the monopole operators gets modified to $TT^* \sim \mu - \zeta$. This is precisely what one expects for the hyperkahler resolution of the geometric branch, when it is not lifted for $\sum k_i = 0$.

It is also possible to give different masses to each fundamental, while preserving $\mathcal{N} = 3$ supersymmetry. This corresponds in the IIB picture to separating the D5 branes. The completely Higgsed branch of the moduli space will be lifted, generically, while the geometric branch will have the $\mathbb{Z}_m$ singularity resolved. The latter fact can be seen by noting that the quantum correction to the moduli space occurs along the locus where the fundamentals become massless. Giving them different explicit mass terms, $m_i^2 p_i^2 q_i^2$, means that this location will be different for each fundamental. In particular, the fundamental, $p_i^2$, becomes massless when

$$A_iB_i - B_{i+1}A_{i+1} = k_im_i^2.$$ 

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