Algebraic weighted colimits

by

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Abstract

In this thesis weighted colimits in 2-categories equipped with promorphisms are studied. Such colimits include most universal constructions with counits, like ordinary colimits in categories, weighted colimits in enriched categories, and left Kan extensions.

In the first chapter we recall the notion of 2-categories equipped with promorphisms (also simply called equipments), that provide a coherent way of adding bimodule-like morphisms to a 2-category. In the second chapter we recall two ways of defining weighted colimits in equipments. Most important to us is their original definition, introduced by Wood, in equipments that are endowed with a closed structure. The second notion, of what we call pointwise weighted colimits, was introduced by Grandis and Paré. It requires no extra structure and generalises Street’s notion of pointwise left Kan extensions in 2-categories. The main result of the second chapter gives a condition, on closed equipments, under which these two notions coincide.

In the third chapter we consider monads on equipments. The main idea of this thesis, given in the fourth chapter, generalises the notions of lax and colax morphisms, of algebras over a 2-monad, to notions of lax and colax promorphisms, of algebras over a monad on an equipment. One of these, that of right colax promorphisms, is well suited to the construction of weighted colimits. In particular, given a monad $T$ on an equipment $\mathcal{K}$, we will show that $T$-algebras, colax $T$-morphisms and right colax $T$-promorphisms form a double category $T Prom_{rc}$. Although weaker than equipments, double categories still allow definition of weighted colimits, and our main result states that the forgetful functor $T Prom_{rc} \to \mathcal{K}$ lifts all weighted colimits whenever $\mathcal{K}$ is closed, under some mild conditions on $T$. 
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Introduction

The main subject of this thesis is a result of Getzler, that was given in [Get09], which gives conditions ensuring that the left Kan extension of a symmetric monoidal functor, between symmetric monoidal categories, is again symmetric monoidal. Briefly speaking, the aim is to generalise this result to similar situations, like that of double functors between double categories, that of continuous order preserving maps between ordered compact Hausdorff spaces, or that of monoidal globular functors between monoidal globular categories.

Getzler’s result concerns the well known notion of left Kan extension, which we recall first. Given a pair of functors $d: A \to M$ and $j: A \to B$, between small categories and with common source, loosely speaking this notion describes the ‘best approximation’ to an extension $B \to M$ of $d$ along $j$. Formally, such approximations are considered to be pairs $(e, \zeta)$, where $e: B \to M$ is a functor and $\zeta: d \to e \circ j$ is a natural transformation, as on the left below. The left Kan extension of $d$ along $j$ is the approximation $(l, \eta)$ that is the ‘best’ in the sense that the transformation $\zeta$ of any other approximation $(e, \zeta)$ factors uniquely through $\eta$ as $\zeta': l \to e$:

$$
\begin{array}{c}
A \\
\downarrow^d \\
M
\end{array}
\xrightarrow{\zeta}
\begin{array}{c}
B \\
\downarrow^e
\end{array}
\quad = 
\begin{array}{c}
A \\
\downarrow^d \\
M
\end{array}
\xrightarrow{\eta}
\begin{array}{c}
B \\
\downarrow^e
\end{array}

(1)

The pair $(l, \eta)$ is called universal among the pairs $(e, \zeta)$, while the transformation $\eta$ is called the unit; one also says that $\eta$ exhibits $l$ as the left Kan extension of $d$ along $j$. Any two universal pairs are isomorphic.

If the target category $M$ is cocomplete then the left Kan extension $l$ always exists, and can be constructed using ‘coends’ in $M$:

$$
l(z) = \int_{y \in A} dy \otimes B(jy, z),
$$

(2)

where $dy \otimes B(jy, z)$ denotes the copower $\bigsqcup_{f \in B(jy, z)} dy$ in $M$. A coend is a certain colimit, see for example Section IX.6 of [ML98]. Choosing a left Kan extension $l = \text{lan}_j d: B \to M$ for every $d: A \to M$ gives the left adjoint in the adjunction

$$
\text{lan}_j: [A, M] \Rightarrow [B, M]: j^*,
$$

(3)

where $[A, M]$ denotes the category of functors $A \to M$ and natural transformations, and where $j^*$ is given by precomposition with $j$. The units $\eta$ in (1) form the components of the unit transformation of this adjunction.

More generally, the notion of left Kan extension above makes sense not only in the 2-category $\text{Cat}$ of categories, functors and natural transformations, but in any
2-category. In particular we can consider Kan extensions in the 2-category $\text{Cat}_{\otimes}$ of symmetric monoidal categories, symmetric monoidal functors and monoidal transformations. To be precise, here we mean ‘pseudomonoidal’ categories and functors, whose coherence maps are invertible. Examples of such categories include the category of vector spaces and their tensor products, or the cartesian monoidal structure on any category with finite products. We thus obtain the notion of ‘symmetric monoidal left Kan extension’ of a symmetric monoidal functor $d: A \to M$ along a symmetric monoidal functor $j: A \to B$: it is a universal pair $(l_{\otimes}, \eta_{\otimes})$ among all pairs $(e_{\otimes}, \zeta_{\otimes})$ that consist of a symmetric monoidal functor $e_{\otimes}: B \to M$ and a monoidal natural transformation $\zeta_{\otimes}: d \Rightarrow e_{\otimes} \circ j$.

Writing $U: \text{Cat}_{\otimes} \to \text{Cat}$ for the forgetful 2-functor, Getzler’s result gives conditions on $U_!j$ along $j$ for any symmetric monoidal functor $j: A \to B$: it is a universal pair $(l_{\otimes}, \eta_{\otimes})$ among all pairs $(e_{\otimes}, \zeta_{\otimes})$ that consist of a symmetric monoidal functor $e_{\otimes}: B \to M$ and a monoidal natural transformation $\zeta_{\otimes}: d \Rightarrow e_{\otimes} \circ j$.

Let $j: A \to B$ be a symmetric monoidal functor, and suppose that $M$ is cocomplete. If

(a) the binary tensor product $\otimes: M \times M \to M$ preserves colimits in both variables;

(b) for each $x$ in $A$ and $z_1, \ldots, z_n$ in $B$, the canonical map

$$\int (y_1, \ldots, y_n) \in A^x \quad A(x, y_1 \otimes \cdots \otimes y_n) \times \prod_{i=1}^n B(\j y_1, z_i) \to B(jx, z_1 \otimes \cdots \otimes z_n),$$

that is induced by applying $j$ to the map in $A$, tensoring the maps in $B$, and composing using the coherence map $j(y_1 \otimes \cdots \otimes y_n) \xrightarrow{\sim} jy_1 \otimes \cdots \otimes jy_n$, is an isomorphism,

then, for any symmetric monoidal functor $d: A \to M$, the ordinary left Kan extension of $Ud$ along $Uj$ can be lifted to a symmetric monoidal left Kan extension of $d$ along $j$. In particular, in this case the adjunction $(3)$ lifts to an adjunction

$$\text{lan}_j: [A, M]_{\text{ps}} \xrightarrow{\sim} [B, M]_{\text{ps}}: j^*,$$

where $[A, M]_{\text{ps}}$ denotes the category of symmetric monoidal functors $A \to M$ and monoidal transformations.

Condition (b) means that any map $f: jx \to z_1 \otimes \cdots \otimes z_n$ in $B$ can be decomposed as

$$f = [jx \xrightarrow{\eta_y} j(y_1 \otimes \cdots \otimes y_n) \xrightarrow{\sim} jy_1 \otimes \cdots \otimes jy_n \xrightarrow{h_1 \otimes \cdots \otimes h_n} z_1 \otimes \cdots \otimes z_n],$$

where $g: x \to y_1 \otimes \cdots \otimes y_n$ lies in $A$ and the maps $h: jy_i \to z_i$ lie in $B$. This decomposition should be unique in the sense that if a second tuple $(g', h'_1, \ldots, h'_n)$ also decomposes $f$ in the same way, then it is identified with $(g, h_1, \ldots, h_n)$ in the coend above. To get a feeling for this condition we shall consider a detailed example, in which we show how Getzler’s proposition can be used to prove that the forgetful functor from ‘bicommutative Hopf algebras’ to cocommutative coalgebras has a left adjoint.
Example: Hopf monoids

An important source of symmetric monoidal functors are ‘algebras over PROPs’, as follows. Informally, a ‘PROP’ is an algebraic structure that defines a type of algebra that lives in a symmetric monoidal category, and whose operations may have multiple inputs and multiple outputs. A typical example of such an algebra is a ‘Hopf monoid’:

**Definition.** Let $M = (M, \otimes, 1)$ be a symmetric monoidal category. A Hopf monoid $H$ is a bimonoid $(H, \mu, \eta, \Delta, \varepsilon)$ in $M$, with multiplication $\mu: H \otimes H \to H$ and unit $\eta: 1 \to H$, and comultiplication $\Delta: H \to H \otimes H$ and counit $\varepsilon: H \to 1$, that is equipped with a morphism $S: H \to H$ satisfying the identity

$$
\mu \circ (S \otimes \text{id}) \circ \Delta = \eta \circ \varepsilon = \mu \circ (\text{id} \otimes S) \circ \Delta,
$$

of maps $H \to H$. An Hopf monoid $H$ is called bicommutative if it is both commutative and cocommutative as a bimonoid.

A PROP is a symmetric strict monoidal category $P$ that has the set of natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$ as objects, on which its tensor product acts as addition: $n_1 \otimes \cdots \otimes n_m = n_1 + \cdots + n_m$. A $P$-algebra $A$, in a symmetric monoidal category $M$, is a symmetric pseudomonoidal functor $A: P \to M$, while morphisms of $P$-algebras are monoidal transformations.

As an example, Wadsley shows in [Wad08] that the PROP $H$, whose algebras are bicommutative Hopf monoids, has as hom-sets $H(m, n)$ the sets of $n \times m$-matrices with integer coefficients. Composition in $H$ is the usual matrix multiplication and, if $f: m_1 \to n_1$ and $g: m_2 \to n_2$ are two matrices, their tensor product $f \otimes g: m_1 + m_2 \to n_1 + n_2$ is the block matrix $(f \ 0 \ g)$. Note that $0$ is both the initial and terminal object of $H$, because there is precisely one matrix with $n$ rows and no columns, and one matrix with no rows and $n$ columns. Any $H$-algebra $A: H \to M$ induces the structure of a bicommutative Hopf monoid on the image $A = A(1)$ by taking the composites

$$
\mu = [A \otimes A \cong A(2) \xrightarrow{A(1)} A], \quad \eta = [1 \cong A(0) \xrightarrow{A(0)} A],
$$

$$
\Delta = [A \xrightarrow{A(1)} A(2) \cong A \otimes A], \quad \varepsilon = [A \xrightarrow{A(0)} A(0) \cong 1],
$$

$$
S = [A \xrightarrow{A(-1)} A],
$$

where the isomorphisms are the coherence maps of $A$. That these composites satisfy the bimonoid axioms and the Hopf monoid axiom follows directly from various matrix equations. For example, the identity $\mu \circ (S \otimes \text{id}) \circ \Delta = \eta \circ \varepsilon$ is the $A$-image of the matrix equation

$$
\begin{pmatrix}
1 & 1 \\
-1 & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 \\
0
\end{pmatrix}
= (0).
$$

Thus every strict $H$-algebra induces a bicommutative Hopf monoid in $M$; that this extends to an isomorphism between $[H, M]_{\text{ps}}$ and the category of bicommutative Hopf monoids in $M$ is shown in [Wad08].

Commutative monoids are modelled by a simpler PROP $F$, that is defined as follows. Writing $[n]$ for the set $\{1, \ldots, n\}$ if $n \geq 1$, and $[0] = \emptyset$, the morphisms $m \to n$ of $F$ are functions $f: [m] \to [n]$. Composition is the usual composition of functions, while the tensor product $f \otimes g: m_1 + m_2 \to n_1 + n_2$ is given by

$$
(f \otimes g)(i) = \begin{cases}
  f(i) & \text{if } i \leq m_1; \\
  g(i - m_1) + n_1 & \text{if } i > m_1.
\end{cases}
$$
It is readily seen that F-algebras in M are precisely the commutative monoids in M. Dually algebras for the opposite category F\textsuperscript{op} are the cocommutative comonoids in M.

Notice that the PROP F\textsuperscript{op} can be embedded into H by mapping each morphism m \rightarrow n, corresponding to a function f: [n] \rightarrow [m], to the n \times m-matrix (f_{ij}) that is given by f_{ij} = 1 if f = j and 0 otherwise. Thus the image of F\textsuperscript{op} in H is the subPROP consisting of all matrices that contain precisely one non-zero entry in each row, whose coefficient is 1. This gives a symmetric strict monoidal functor j: F\textsuperscript{op} \hookrightarrow H whose induced action

\[ j^*: [H, M]_{\text{ps}} \rightarrow [F\textsuperscript{op}, M]_{\text{ps}} \]

on the categories of algebras is simply the forgetful functor, that maps each bicommutative Hopf monoid to its underlying cocommutative comonoid.

We claim that the embedding j satisfies condition (b) of Getzler’s proposition. To see this we have to show that every (I \times I) matrix of identity matrices f in H decomposes as f = (∑ h \times l) \circ jg, where the h_i are n_i \times l_i-matrices in H and jg is a (I \times I) matrix in j(F\textsuperscript{op}), as in (5) above. It is easy to find matrices h_i, i = 1,\ldots,s, and jg that satisfy this equation: for example we can take l_i = m and let h_i be the blocks of f as shown below. As jg we can then simply take the block matrix of identity matrices I_m of dimension m, as on the right.

\[
\begin{pmatrix}
  h_1 \\
  h_2 \\
  \vdots \\
  h_s
\end{pmatrix} = \begin{pmatrix}
  h_1 & 0 & \ldots & 0 \\
  0 & h_2 & \ldots & \vdots \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & \ldots & 0 & h_s
\end{pmatrix}
\]

It is now not hard to check that the assignment f \mapsto (g, h_1,\ldots,h_s) does in fact give an inverse to the canonical map (4), so that condition (b) is satisfied.

We conclude that, for every cocomplete symmetric monoidal category M satisfying condition (a), there exists an adjunction of categories of PROP-algebras

\[
\text{lan}_j: [F\textsuperscript{op}, M]_{\text{ps}} \overset{\sim}{\longrightarrow} [H, M]_{\text{ps}}: j^*.
\]

Hence we obtain a formal proof for the following corollary.

**Corollary.** Let M be a cocomplete symmetric monoidal category, whose binary tensor product \(\otimes: M \times M \rightarrow M\) preserves colimits in both variables. The forgetful functor \(j^*: [H, M]_{\text{ps}} \rightarrow [F\textsuperscript{op}, M]_{\text{ps}}\), from bicommutative Hopf monoids in M to cocommutative comonoids in M, has a left adjoint.

Of course, for an explicit description of the free bicommutative Hopf monoid on a cocommutative comonoid \(C = (C, \Delta, \varepsilon)\) in M, we have to unpack the coend (2), which may not be easy. In the noncommutative case, with M the category of vector spaces, a description of free Hopf algebras generated by coalgebras has been given by Takeuchi [Tak71].

On the other hand we may consider the subPROP \(\Sigma \subset H\) that is generated by the symmetries of H, and write i: \(\Sigma \hookrightarrow H\) for the embedding. In other words \(\Sigma\) is the free symmetric strict monoidal category on the single object 1 \(\in H\), and the composite

\[ [H, M]_{\text{ps}} \xrightarrow{i^*} [\Sigma, M]_{\text{ps}} \cong M \]

is the forgetful functor from bicommutative Hopf monoids in M to M. That this forgetful functor does not admit a left adjoint is considered ‘folklore’, e.g. it is
not possible to construct a ‘free Hopf algebra’ on a vector space. In regard to Getzler’s proposition, this manifests as the fact that the embedding $i: \Sigma \hookrightarrow H$ does not satisfy condition (b). Indeed $\Sigma$ contains only isomorphisms, so that it is impossible to decompose the matrix $\begin{pmatrix} 1 \\ 1 \end{pmatrix}: 1 \to 2$, that models the comultiplication, as a composition $1 \xrightarrow{g} 2 \xrightarrow{h_1 \otimes h_2} 2$, with $g \in \Sigma$, as is required in (5).

**Similar settings**

The question that Getzler’s proposition answers can be asked in many similar settings, as follows. In abstract terms, we might consider any 2-monad $T$ on any 2-category $\mathcal{C}$, and let $T\text{-Alg}$ denote the 2-category of $T$-algebras, $T$-morphisms and $T$-cells (see Section 3.2). Like in any 2-category, the notion of left Kan extension can be defined in both $\mathcal{C}$ and $T\text{-Alg}$. Thus, given $T$-algebras $A$, $B$ and $M$, and $T$-morphisms $j: A \to B$ and $d: A \to M$, assuming their left Kan extension $l$ exists in $\mathcal{C}$ we may ask the question

“Under what conditions on $j$ and $M$ does $l$ lift to a left Kan extension in $T\text{-Alg}$?”

Getzler’s result, in the way stated above, answers this question in the specific case of the ‘free symmetric strict monoidal category’-monad on the 2-category $\mathcal{C} = \text{Cat}$ of categories, functors and transformations. To convince the reader that this situation is common three interesting examples are listed below.

As the first example we consider the ordered compact Hausdorff spaces as studied by Tholen [Tho09, Example 2]. Such spaces are preordered sets equipped with a compact Hausdorff topology, which is compatible with the ordering. Since preordered sets can be thought of as categories enriched over the category $2 = (\bot \to \top)$ of truth values, they can be considered as the objects of the 2-category $\mathcal{C} = 2\text{-Cat}$, where left Kan extensions of order preserving maps are defined. Precisely, the left Kan extension of a pair of order preserving maps $d: A \to M$ and $j: A \to B$ is given by

$$(\text{lan}_j) d(z) = \sup\{dx : jx \leq z\}$$

provided these suprema exist in $M$, see the discussion following Example 2.10. Moreover there is a 2-monad on $2\text{-Cat}$, called the ‘ultrafilter’-monad, whose algebras are precisely the ordered compact Hausdorff spaces. In this case the question becomes: “Given continuous order preserving maps $j$ and $d$, when is $\text{lan}_j d$ above a continuous left Kan extension?”

The second example is that of double categories, which we will use throughout the thesis. Informally, a double category $\mathcal{K}$ consists of

- objects $A, B, C, \ldots$,
- vertical morphisms denoted $f: A \to B$,
- horizontal morphisms denoted $J: A \rightrightarrows B$, with a barred arrow,
- cells $\phi$ that are shaped as squares, with vertical and horizontal morphisms as edges, as shown below:

$$\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow & \downarrow_{\phi} & \downarrow \\
C & \xrightarrow{g} & D
\end{array}$$
Pairs of vertical morphisms can be composed, as can pairs of horizontal morphisms, while pairs of cells can be composed both vertically, along a common horizontal edge, and horizontally, along a common vertical edge. In ‘strict’ double categories both compositions are strictly associative. Ones in which horizontal composition is only associative up to coherent invertible cells are called ‘pseudo’ double categories; these we will use often. Underlying any double category $K$ is a diagram

$$
\begin{array}{c}
\mathcal{K}_1 \\
\downarrow L \\
\mathcal{K}_0 \\
\uparrow R
\end{array}
$$

of categories and functors, as follows. The category $\mathcal{K}_0$ consists of the objects and vertical morphisms of $\mathcal{K}$, while $\mathcal{K}_1$ has horizontal morphisms as objects; its maps $J \to K$ are cells $\phi$ with horizontal source $J$ and target $K$. The functors $L$ and $R$ map $J: A \to B$ to its source $A$ and target $B$ respectively, while the cell $\phi$ above is mapped to $L\phi = f$ and $R\phi = g$. Thus the assignment $\mathcal{K} \mapsto (\mathcal{K}_1 \rightrightarrows \mathcal{K}_0)$ forgets the horizontal composition of $\mathcal{K}$. The diagram $\mathcal{K}_1 \rightrightarrows \mathcal{K}_0$ above can be considered as an internal category in the category $\mathcal{G}_1$ of presheaves on $\mathcal{G}_0 = (0 \rightrightarrows 1)$ and thus as an object of the 2-category $\text{Cat}(\mathcal{G}_1)$ of internal categories, internal functors and internal natural transformations in $\mathcal{G}_1$, where we have a notion of Kan extension. Moreover there is a 2-monad on $\text{Cat}(\mathcal{G}_1)$, that is induced by the ‘free category’-monad on graphs, whose algebras are the double categories as introduced above. Thus, taking double functors $S: \mathcal{K} \to \mathcal{L}$ and $F: \mathcal{K} \to \mathcal{M}$, we can forget the structure of horizontal compositions and regard them as functors $S'$ and $F'$ in $\text{Cat}(\mathcal{G}_1)$. Supposing that the left Kan extension $L'$ of $F'$ exists in $\text{Cat}(\mathcal{G}_1)$, this leads to the same question: ‘When can $L'$ be lifted to a left Kan extension of double functors?’

Generalising the previous example, the globular categories considered by Batanin in his paper [Bat98] are contravariant functors $\mathcal{G}^{\text{op}} \to \text{Cat}$, where $\mathcal{G}$ is the ‘globe category’. Batanin considers a monoidal structure on such categories and uses the resulting ‘monoidal globular categories’ to define weak $n$-categories. This situation too is like the ones described above: there is a 2-monad on $\text{Cat}(\mathcal{G})$, induced by the ‘free $\omega$-category’-monad on globular sets, whose algebras are the monoidal globular categories.

The aim of the thesis is to answer the question asked above in a general, conceptual way that, among others, can be applied to each of the three examples given above.

The idea

In each of the three examples given above, the 2-category $\mathcal{C}$ and the 2-monad $T$ that we considered are part of a bigger structure, in the following sense.

There exists a ‘closed’ pseudo double category $\mathcal{K}$ that ‘equips $\mathcal{C}$ with promorphisms’; moreover $T$ can be extended to form a monad $T'$ on $\mathcal{K}$.

To understand what this means, we first remark that each double category $\mathcal{K}$ induces a 2-category $V(\mathcal{K})$ consisting of its objects, vertical morphisms and vertical cells (cells whose horizontal source and target are identity morphisms). That $\mathcal{K}$ ‘equips $\mathcal{C}$ with promorphisms’ means that $V(\mathcal{K}) \cong \mathcal{C}$ and that each morphism $f: A \to B$ in $\mathcal{C}$ has both a ‘companion’ $B(f, \text{id}): A \to B$ and a ‘conjoint’ $B(\text{id}, f): B \to A$ in $\mathcal{K}$ (for the details see the comments following Definition 1.10). One can think of the companion $B(f, \text{id})$ as being the horizontal morphism ‘isomorphic’ to $f$, and the conjoint $B(\text{id}, f)$ being the horizontal morphism ‘adjoint’ to $f$. In this case the horizontal morphisms of $\mathcal{K}$ are called the ‘promorphisms of $\mathcal{C}$’, while $\mathcal{K}$ is called an equipment.
choice of weight $J : A \Rightarrow B$ resulting weighted colimit

$B = *$ (terminal category) ordinary weighted colimit

$J = B(j, \text{id})$ where $j : A \Rightarrow B$ ‘pointwise’ left Kan extension

$J = *(!, \text{id})$ where $!: A \Rightarrow *$ ordinary (conical) colimit

Table 1: Various types of weighted colimits in $\text{Cat}$.

The typical example of a 2-category that can be equipped with promorphisms is the 2-category $\text{Cat}$ of categories, functors and natural transformations. Here the promorphisms are the profunctors: a profunctor $J : A \Rightarrow B$ is simply a functor $J : A^{op} \times B \rightarrow \text{Set}$. That is, the pseudo double category $K = \text{Prof}$ consists of categories, functors, profunctors and natural transformations. The composition of profunctors is similar to the composition of relations (see Example 1.3), while the companion $B(j, \text{id}) : A \Rightarrow B$ and conjoint $B(\text{id}, f) : B \Rightarrow A$, of a functor $f : A \Rightarrow B$, are given by the representable profunctors $(x, y) \mapsto B(f x, y)$ and $(y, x) \mapsto B(y, f x)$.

Furthermore, a double category $K$ is ‘closed’ whenever, for each promorphism $J : A \Rightarrow B$, the functors given by pre- and postcomposition with $J$ both have right adjoints, which generalises the notion of a closed monoidal category. The equipment $\text{Prof}$ is closed: the right adjoint of precomposition with a profunctor $J : A \Rightarrow B$ maps any profunctor $K : A \Rightarrow C$ to the ‘left hom’ profunctor $J \circ K : B \Rightarrow C$ that is given by

$$(J \circ K)(y, z) = \{\text{natural transformations } J(-, y) \Rightarrow K(-, z)\}.$$ 

These left homs allow us to define many kinds of ‘well behaved’ weighted colimits in $\text{Cat}$, as follows. Given a profunctor $J : A \Rightarrow B$ (the weight) as well as a functor $d : A \rightarrow M$ (the diagram), the $J$-weighted colimit of $d$, if it exists, is a functor $\text{colim}_J d : B \rightarrow M$ that ‘represents’ the left hom $J \circ M(d, \text{id})$ in the following sense: there is a natural isomorphism of profunctors

$$M(\text{colim}_J d, \text{id}) \cong J \circ M(d, \text{id}).$$

One can verify that, taking various kinds of weights $J$, this generalises the usual notions of colimits in $\text{Cat}$, as described in Table 1. For us, the most important of these is that colimits weighted by companions are left Kan extensions, that is there is a natural isomorphism

$$\text{lan}_j d \cong \text{colim}_{B(j, \text{id})} d : B \rightarrow M,$$ 

for any pair of functors $d : A \rightarrow B$ and $j : A \rightarrow B$. In fact the functor $\text{colim}_{B(j, \text{id})} d$ satisfies a stronger property, that defines it as the ‘pointwise’ left Kan extension of $d$ along $j$, see [ML98, Corollary X.5.4]. We shall discuss the notion of pointwise left Kan extensions at the end of this introduction.

For well behaved $\mathcal{V}$ the 2-category $\mathcal{V}\text{-Cat}$ of $\mathcal{V}$-enriched categories can be equipped with $\mathcal{V}$-enriched profunctors, generalising the profunctors of $\text{Cat}$, and the resulting double category $\mathcal{V}\text{-Prof}$ will be closed. The same is true for the 2-category $\text{Cat}(\mathcal{E})$ of internal categories in a well behaved category $\mathcal{E}$ with finite limits (e.g. $\mathcal{E}$ is a presheaf category). In each of these cases the left homs can be used to define weighted colimits, just like we did in the case of $\text{Cat}$ above.

We remark that weighted colimits can also be defined in pseudo double categories, that are neither equipments nor closed: Grandis and Paré consider in [GP08] the notion of, what they call, ‘Kan extensions’ in ordinary double categories. We shall see in Proposition 2.14 that, in a closed pseudo double category, every weighted colimit, in our sense, is a Kan extension in their sense.
INTRODUCTION

Thus, given a 2-monad \( T \) on \( C \), we will assume there exists a closed double category \( K \) that equips \( C \) with promorphisms, together with a monad \( T' \) on \( K \) that restricts to \( T \) on \( C \). In each of the examples given in the previous section such \( K \) and \( T' \) exist. Remember that, in general, a ‘colax morphism’ \( A \to B \) between two \( T \)-algebras \( A \) and \( B \) is a morphism \( f: A \to B \) in \( C \) that is equipped with a ‘structure cell’ \( \bar{f}: f \circ a \Rightarrow b \circ Tf \), as shown on the left below, where \( a: TA \to A \) and \( b: TB \to B \) define the \( T \)-algebra structures on \( A \) and \( B \). This structure cell is required to satisfy an associativity and unit axiom. There is a dual notion of ‘lax morphism’ \( g: A \to B \), in which the direction of \( \bar{g} \) is reversed, as shown on the right below. Colax and lax morphisms whose structure cells are invertible are called ‘pseudomorphisms’.

\[
\begin{array}{c}
TA \xrightarrow{a} A \\
\xrightarrow{Tf} f \\
TB \xrightarrow{b} B
\end{array} \\
\begin{array}{c}
TA \xrightarrow{Tg} TB \\
\xrightarrow{a} \bar{g} \xrightarrow{\bar{f}} b \\
A \xrightarrow{g} B
\end{array}
\]

In the case of the ‘free symmetric strict monoidal category’-monad on \( \text{Cat} \), a colax morphism \( f: A \to B \) is a symmetric colax monoidal functor, that comes equipped with structure maps \( f(x_1 \otimes \cdots \otimes x_n) \to fx_1 \otimes \cdots \otimes fx_n \). Getzler’s proposition involves ‘pseudomonoidal functors’, for which these structure maps are invertible.

The main idea of this thesis is to generalise the above notions, of morphisms between \( T \)-algebras in \( C \), to notions of promorphisms between \( T' \)-algebras in \( K \). For example, we shall define a ‘lax promorphism’ \( A \Rightarrow B \) to be a promorphism \( J: A \Rightarrow B \) that comes equipped with a structure cell

\[
\begin{array}{c}
T'A \xrightarrow{T'J} T'B \\
\xrightarrow{a} \bar{J} \xrightarrow{\bar{\lambda}} b \\
A \xrightarrow{J} B
\end{array}
\]

compared to the structure cell \( \bar{g} \) of a lax morphism above. We shall see that colax structures on a morphism \( f: A \to B \) correspond precisely to lax structures on its companion \( B(f, \text{id}) \): \( A \Rightarrow B \).

Considering lax promorphisms \( J: A \Rightarrow B \) whose structure cells \( J \) are invertible in \( K_1 \) (the category of horizontal morphisms and cells) does not give the right notion of pseudopromorphism, because \( J \) being invertible in \( K_1 \) implies that the structure maps \( a: T'A \to A \) and \( b: T'B \to B \) are invertible. Instead, we shall consider the two horizontal cells (i.e. cells with identities as vertical source and target)

\[
\begin{array}{c}
T'A \xrightarrow{T'J} T'B \\
\xrightarrow{a} \bar{J} \xrightarrow{\bar{\lambda}} b \\
A \xrightarrow{J} B
\end{array} \\
\begin{array}{c}
A \xrightarrow{J} B \\
\xrightarrow{\rho} \bar{J} \\
B \xrightarrow{\text{id}} T'B
\end{array}
\]

that correspond to \( J \) under the ‘orthogonal flipping’ operation that was introduced by Grandis and Paré in [GP04, Section 1.6] (see also Proposition 1.24), and call the lax promorphism \( J \) a ‘left pseudopromorphism’ or ‘right pseudopromorphism’ when respectively \( \lambda J \) or \( \rho J \) is invertible. For example, condition (b) of Getzler’s proposition is equivalent to asking that the lax structure on the companion \( B(j, \text{id}) \) of the monoidal functor \( j: A \to B \) is right pseudo.

Under a mild condition on the monad \( T' \), the two notions of pseudopromorphism lead to corresponding notions of colax promorphism: a ‘left’ and a ‘right’ variant.
In Chapter 4 we will see that the second variant, that of ‘right colax promorphisms’, is well suited to constructing algebraic weighted colimits. More precisely, we will show in Section 4.1 that $T'$-algebras, colax morphisms, right colax promorphisms form a pseudo double category $T'$-Promrc, so that the main result of the thesis can be stated as follows.

**Theorem 4.32** Let $T'$ be a ‘right suitable normal’ monad on a closed equipment $K$. The forgetful functor $U_{T'} : T'$-Promrc $\to K$ ‘lifts’ all weighted colimits. Moreover its lift of a weighted colimit $\text{colim}_J d : B \to M$, where $d : A \to M$ is a pseudomorphism and $J : A \Rightarrow B$ is a right pseudopromorphism, is a pseudomorphism whenever the canonical vertical cell

$$\text{colim}_{T'} J (m \circ T'd) \Rightarrow m \circ T'(\text{colim}_J d) \tag{7}$$

is invertible, where $m : T'M \to M$ is the structure map of $M$.

As before, that $U_{T'}$ lifts all weighted colimits means that, if the weighted colimit $k = \text{colim}_{T'} J (U_{T'} d)$ exists in $K$ then the weighted colimit $l = \text{colim}_J d$ exists in $T'$-Promrc, and can be chosen such that $U_{T'} l = k$. The assumption that $T'$ is a ‘right suitable normal’ monad is the mild condition on $T'$ that was mentioned above, which is necessary to be able to define right colax promorphisms.

If we take for $T'$ the extension of the ‘free symmetric strict monoidal category’-monad on $\text{Cat}$, to the pseudo double category $\text{Prof}$ of profunctors, then the second assertion of the theorem above reduces to Getzler’s proposition. More precisely, for monoidal functors $d : A \to M$ and $j : A \Rightarrow B$, condition (b) of the latter means that the companion $B(j, \text{id})$ of $j$ is a right pseudopromorphism while condition (a) is equivalent to the invertibility of (7).

**Pointwise weighted colimits**

In his paper [Str74] Street gave the following refinement of the notion of left Kan extension, in any 2-category $\mathcal{C}$ that has ‘comma objects’ (a certain 2-limit; see Definition 2.20), like the 2-category $\text{Cat}(\mathcal{E})$ of categories, functors and transformations internal to a category $\mathcal{E}$ with finite limits. Street says that the cell $\eta : d \Rightarrow l \circ j$ in the diagram below exhibits $l$ as the pointwise left Kan extension of $d$ along $j$ if, for each $f : C \to B$, the composition of cells below exhibits $l \circ f$ as the left Kan extension of $d \circ \pi_A$ along $\pi_C$, where $\pi$ defines $j/f$ as the comma object of $j$ and $f$.

$$\begin{array}{ccc}
A & \xrightarrow{d} & M \\
\downarrow{\pi_A} & & \downarrow{\eta_j} \\
C & \xrightarrow{\pi} & B \\
\downarrow{j/f} & \downarrow{j} & \downarrow{\pi} \\
\pi_C & \xleftarrow{\pi} & j \\
\end{array}$$

For example the left Kan extensions in $\text{Cat}$, that are given as colimits weighted by companions, as in (6), are pointwise.

As we remarked before, Grandis and Paré have defined left Kan extensions in any pseudo double category in [GP08]; in the same paper they also define the stronger pointwise variants, by generalising Street’s notion above and using ‘double comma objects’. Modifying their definition somewhat, so that it fits in better with our notion of weighted colimits, we likewise consider pointwise weighted colimits in Section 2.3. The main result of Chapter 2 (Theorem 2.37) gives a condition on equipments under which pointwise weighted colimits coincide with the ordinary ones. For example, the pseudo double category $\text{Prof}(\mathcal{E})$ of internal profunctors in $\mathcal{E}$
category $\mathcal{E}$ with finite limits, that equips $\text{Cat}(\mathcal{E})$ with promorphisms, satisfies this condition.

Finally, in Theorem 4.30 we shall prove that, roughly, given a monad $T$ on an equipment $\mathcal{K}$, the equipment $T\text{-Prom}_{rc}$ of right colax $T$-profunctors satisfies the condition of Theorem 2.37 whenever $\mathcal{K}$ does, so that in that case weighted colimits in $T\text{-Prom}_{rc}$ are also pointwise.
Chapter 1

2-Categories equipped with promorphisms

This chapter contains the preliminary definitions that will be used in the coming chapters. We start by recalling the definition of pseudo double categories and, using this, we recall the definition of 2-categories equipped with promorphisms. These, also simply called ‘equipments’, will form the main setting of the theory presented here. Throughout, the main examples that we consider are the equipments $\mathcal{V} \text{-Prof}$ of $\mathcal{V}$-enriched categories and $\mathcal{V}$-profunctors, where $\mathcal{V}$ is some ‘well behaved’ symmetric monoidal category, and the equipments $\text{Prof}(\mathcal{E})$ of categories internal to $\mathcal{E}$ and internal profunctors, where $\mathcal{E}$ is some ‘well behaved’ category with finite limits.

For most parts of this chapter we follow the papers [Shu08] by Shulman and [CS10] by Cruttwell and Shulman, borrowing much of their notation as well. In particular we name double categories after their horizontal morphisms, like they do. However, we shall have no need to consider the more general ‘virtual double categories’ that are the main subject of [CS10]. The notion of 2-categories equipped with promorphisms was originally introduced by Wood [WooS2].

1.1 Pseudo double categories

In this section we recall the formal definition of the double categories, that we described in the introduction. We will consider the ‘pseudo’ variants, whose horizontal compositions satisfy the associativity and unit axioms only up to coherent invertible cells.

In the introduction double categories were described as generalisations of 2-categories where, in contrast to the ordinary case, there are two types of morphisms, one drawn vertically and the other drawn horizontally. Moreover the cells of such double categories are shaped like squares, with both a pair of vertical morphisms as source and target, as well as a pair of horizontal morphisms. In the introduction we briefly mentioned the example of the double category $\text{Prof}$, with categories as objects and functors and profunctors as vertical and horizontal morphisms. Before recalling the formal definition of double categories we shall describe, in more detail, another typical example.

Let us consider rings $A, B, \ldots$: besides the usual ring homomorphisms $f: A \to B$ between rings there are other objects that we may consider as morphisms $A \to B$, namely $(A, B)$-bimodules. Remember that an $(A, B)$-bimodule $J$ is an abelian group $J$ equipped with a left action $l: A \otimes J \to J$: $(a, x) \mapsto a \cdot x$ and a right action $r: J \otimes B \to J$ that are associative and unital, and that commute (the tensor products here are tensor products of abelian groups). To distinguish it from ring
homomorphisms we shall write \( J: A \to B \). Of course, bimodules only deserve to be considered as morphisms if they allow composition of ‘composable bimodules’, as well as ‘unit bimodules’ that act as units for this composition. But they do: given two bimodules \( J: A \to B \) and \( H: B \to C \) we can take the tensor product \( J \otimes_B H \) as the composite \( J \circ H: A \to C \) and, since we have \( J \otimes_B B \cong J \) and \( A \otimes A \cong J \), we can take the unit \((A,A)\)-bimodule \( A \to A \) to be the ring \( A \) itself, considered as an \((A,A)\)-bimodule. Thus, to define composition for bimodules we have to pick, for every composable pair \( J \) and \( H \), a tensor product \( J \otimes_B H \), that is, formally, a coequaliser of the parallel pair
\[
J \otimes B \otimes H \xrightarrow{r \otimes \text{id}} J \otimes H \xrightarrow{\text{id} \otimes I} J \otimes_B H
\]
(1.1)
in the category of abelian groups. Notice that, if we just pick any tensor product \( J \otimes_B H \) for each pair of composable bimodules \( J \) and \( H \) (and we do), then the isomorphism \( A \otimes_A J \cong J \) need not be an identity. Thus the composition of bimodules is not as strict as we are used to: instead of the composite of \( A \otimes_A J \) being equal to \( J \) it is isomorphic to \( J \). The same is true for associativity: the universal property of the coequalisers above implies that \( (J \otimes_B H) \otimes_C K \cong J \otimes_B (H \otimes_C K) \), but this will in general not be an identity. However, since all these isomorphisms are ‘canonical’, in the sense that they are obtained by universal properties, the family of all such isomorphisms is ‘coherent’: an example of this coherence is the commuting of the famous pentagon diagram of Mac Lane [ML98, Section VII.1], which states that the two ways of using the isomorphisms, that compare the tensor products of three factors, to obtain an isomorphism \( ((J \otimes_B H) \otimes_C K) \otimes_D L \cong J \otimes_B (H \otimes_C (K \otimes_D L)) \), that compares the tensor product of four, coincide.

Thus, to summarise, \((A,B)\)-bimodules can be considered as morphisms \( A \to B \) which can be composed, but the resulting composition will only be associative and unital ‘up to coherent isomorphisms’. Now, if we think about the mathematical object containing rings, ring homomorphisms and bimodules, it is natural to wonder how morphisms of bimodules fit in. The most general of these are morphisms \( J \to K \) from an \((A,B)\)-bimodule \( J \) to a \((C,D)\)-bimodule \( K \), as follows. Given two ring homomorphisms \( f: A \to C \) and \( g: B \to D \), an \((f,g)\)-bilinear map \( \phi: J \to K \) is a homomorphism \( J \to K \) of abelian groups such that \( \phi(a \cdot x) = f(a) \cdot \phi(x) \) and \( \phi(x \cdot b) = \phi(x) \cdot g(b) \). Drawing ring homomorphisms vertically and bimodules horizontally, we can depict the bilinear map \( \phi \) as a square shaped cell
\[
\begin{array}{ccc}
A & \xrightarrow{J} & B \\
\downarrow f & & \downarrow g \\
C & \xrightarrow{K} & D
\end{array}
\]
This is a useful representation of such maps, for it allows us to represent the composite \( \psi \circ \phi: J \to L \), of \( \phi \) and a second bilinear map \( \psi: K \to L \), as the two squares on the left below. We call \( \psi \circ \phi \) the vertical composite of \( \phi \) and \( \psi \); as the diagram below suggests it is an \((h \circ f,k \circ g)\)-bilinear map \( J \to L \).
Likewise given a third \((g, l)\)-bilinear map \(\chi: M \Rightarrow N\) as on the right above, we can construct the horizontal composite \(\phi \otimes_g \chi: J \otimes_B M \Rightarrow K \otimes_D N\), by taking the unique factorisation of the composite

\[
J \otimes M \xrightarrow{\phi \otimes \chi} K \otimes N \to K \otimes_D N
\]

through the coequaliser \(J \otimes M \to J \otimes_B M\). Again this composition is represented nicely by drawing the cells of \(\phi\) and \(\chi\) side-by-side, as we have done above.

We conclude that to capture rings, ring homomorphisms, bimodules and bilinear maps between bimodules in a ‘double category’ \(\text{Mod}\), such a double category should have rings as objects, and two types of morphisms: the vertical ones being ring homomorphisms and the horizontal ones being bimodules. Moreover it should have 2-dimensional structure that consists of square shaped cells representing the bilinear maps between bimodules. Any two morphisms of the same type can be composed in \(\text{Mod}\), while any pair of ‘adjacent’ cells can be composed, either horizontally or vertically. Finally we have seen that the composition of horizontal morphisms need not be strictly associative and unital, but only up to isomorphisms of bimodules (that is cells). This is exactly the description of a ‘pseudo double category’, whose formal definition we shall now recall.

Consider a diagram of an internal category

\[
\begin{array}{ccc}
K_1 \times_{K_0} K_1 & \xrightarrow{\circ} & K_1 \\
\downarrow{\pi_1} & & \downarrow{L} \\
K_1 & \xrightarrow{R} & K_0
\end{array}
\]  

(1.2)

denoted \(\mathcal{K}\), in the 2-category \(\textbf{Cat}\) of categories, functors and natural transformations, where \(K_1 \times_{K_0} K_1\) is the pullback

\[
\begin{array}{ccc}
K_1 \times_{K_0} K_1 & \xrightarrow{\pi_2} & K_1 \\
\downarrow{\pi_1} & & \downarrow{L} \\
K_1 & \xrightarrow{R} & K_0
\end{array}
\]

and such that the following equalities hold.

\[
\begin{align*}
L \circ \circ &= L \circ \pi_1: K_1 \times_{K_0} K_1 \to K_0 \\
R \circ \circ &= R \circ \pi_2: K_1 \times_{K_0} K_1 \to K_0 \\
R \circ \text{id} &= L \circ U: K_0 \to K_0
\end{align*}
\]  

(1.3)

The objects of \(K_0\) are called objects of \(\mathcal{K}\) while the morphisms \(f: A \to C\) of \(K_0\) are called vertical morphisms of \(\mathcal{K}\). An object \(J\) of \(K_1\) such that \(LJ = A\) and \(RJ = B\) is denoted by the barred arrow

\[
J: A \Rightarrow B
\]

and called a horizontal morphism of \(\mathcal{K}\). A morphism \(\phi: J \to K\) in \(K_1\), such that \(L\phi = f: A \to C\) and \(R\phi = g: B \to D\), is depicted

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{J} & & \downarrow{g} \\
C & \xrightarrow{\phi} & D
\end{array}
\]

and called a cell of \(\mathcal{K}\). We will call \(J\) and \(K\) the horizontal source and target of \(\phi\), while we call \(f\) and \(g\) its vertical source and target. A cell \(\phi: J \Rightarrow K\) whose vertical source and target are identities is called horizontal.
The fact that $K_0$ is a category means that we can compose two vertical morphisms $f: A \to C$ and $h: C \to E$ to form $h \circ f$, which we will draw as $A \xrightarrow{f} C \xrightarrow{h} E$, as is customary. Likewise two horizontal morphisms $J: A \Rightarrow B$ and $H: B \Rightarrow F$ can be composed using the functor $\circ$ of (1.2); we will denote the result $J \circ H$ and draw it as $A \xrightarrow{J} B \xrightarrow{H} F$. If the horizontal source of a cell $\psi$ coincides with the horizontal target of $\phi$ then their vertical composite $\psi \circ \phi$ can be formed in $K_1$; likewise for a cell $\chi$ whose vertical source is equal to the vertical target of $\phi$, the horizontal composite $\phi \circ \chi$ is given by the image of $(\phi, \chi)$ under the functor $\circ$.

Notice that the vertical composition of cells is strictly associative, and that each horizontal morphism $J: A \Rightarrow B$ comes with an identity $A \xrightarrow{id} A$, which is a unit for vertical composition. The edges of the vertical composite $\phi \circ \psi$ can be read off from the diagram obtained by drawing $\phi$ on top of $\psi$, sharing the common horizontal edge, while the diagram of $\phi$ and $\chi$ side-by-side, sharing their common vertical edge, represents the horizontal composite $\phi \circ \chi$. That the edges of these diagrams indeed coincide with the edges of $\psi \circ \phi$ and $\phi \circ \chi$ follows from the fact that $L, R$ and $\circ$ preserve composition, as well as the identities (1.3).

Thus compositions of cells can be naturally represented by joining them into a grid — we will do so many times. Grids that consist of one or two columns always define a unique composite because $\circ$ is a functor. For example, suppose we are given four composable cells as in

$$
\begin{array}{c}
A \\
\downarrow \phi \\
\hline
\downarrow \psi \\
C
\end{array}
\begin{array}{c}
A \\
\downarrow \chi \\
\hline
\downarrow \xi \\
C
\end{array},
$$

then $(\psi \circ \phi) \circ (\xi \circ \chi) = (\psi \circ \xi) \circ (\phi \circ \chi)$; this is called the interchange law.

The horizontal morphism $U_A = UA$ that is the image of $A$ under $U$ of (1.2), is called the horizontal identity on $A$; in diagrams we will depict it simply by $A \xrightarrow{id} A$. Likewise, given a vertical morphism $f: A \to C$, the horizontal identity cell $U_f = Uf$ will be depicted as

$$
\begin{array}{ccc}
A & & A \\
\downarrow f & & \downarrow U_f \\
C & & C
\end{array}
$$

Cells whose horizontal source and target are horizontal identities, like $U_f$ above, are called vertical.

**Definition 1.1.** A pseudo double category $K$ is a pseudo category internal to $\mathbf{Cat}$. That is, $K$ is given by a diagram of functors (1.2) satisfying the conditions (1.3), together with natural isomorphisms

- $a: \circ \circ (\circ \times \text{id}) \Rightarrow \circ \circ (\text{id} \times \circ): K_1 \times_{K_0} K_1 \times_{K_0} K_1 \to K_1$;
- $l: \circ \circ (U \times \text{id}) \Rightarrow \pi_2: K_0 \times_{K_0} K_1 \to K_1$;
- $r: \circ \circ (\text{id} \times U) \Rightarrow \pi_1: K_1 \times_{K_0} K_0 \to K_1$. 


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whose components are horizontal cells in $\mathcal{K}$, and such that the usual coherence axioms for a monoidal category or bicategory are satisfied (see e.g. [ML98, Section VII.1]).

Thus, in contrast to composition of vertical morphisms, composition of horizontal morphisms is only associative up to invertible horizontal cells

$$a: (J \circ H) \circ K \Rightarrow J \circ (H \circ K),$$

for any $A \xrightarrow{J} B \xrightarrow{H} C \xrightarrow{K} D$, called the associators of $\mathcal{K}$, and unital up to invertible horizontal cells

$$1: U_A \circ J \Rightarrow J \text{ and } \tau: J \circ U_B \Rightarrow J,$$

for any $J: A \Rightarrow B$, called the unitors of $\mathcal{K}$.

Given composable horizontal morphisms $J_i: A_{i-1} \rightarrow A_i$, for $i = 1, \ldots, n$, we will abbreviate by $J_{1} \circ \cdots \circ J_{n}$ their composition with right to left bracketing $J_{1} \circ \cdots \circ (J_{n-1} \circ J_n) \cdots$, and likewise for horizontal composites of cells. Moreover in writing down compositions of cells we will often leave out the associators and unitors. For example the composite

$$J \circ H \xrightarrow{\varphi} U_A \circ (J \circ H) \xrightarrow{\circ \circ \circ \circ} K \circ (L \circ M) \xrightarrow{\delta} (K \circ L) \circ M \xrightarrow{\chi} (P \circ Q) \circ U_X \xrightarrow{\varepsilon} P \circ Q,$$

will be abbreviated to simply

$$J \circ H \xrightarrow{\phi} K \circ L \circ M \xrightarrow{\phi} P \circ Q.$$

That this does not introduce ambiguity follows from the coherence axioms: like for monoidal categories and bicategories, they imply that any two composites of $(\circ)$-products of (inverses of) associators, (inverses of) unitors and identity cells, that have the same source and target, are equal. Thus, any two ways of completing an identity cell will give the same result. In fact, by a result of Grandis and Paré [GP99, Theorem 7.5] every pseudo double category is equivalent to a strict double category, whose associators and unitors are identities.

Likewise when drawing grids to represent compositions we shall always leave out associators and unitors. For example by the grid

\[
\begin{array}{c c c c c c}
A & \xrightarrow{J} & B & \xrightarrow{D(id)} & D \\
\downarrow{f} & & \downarrow{\phi} & & \\
C & \xrightarrow{K} & D & \xrightarrow{D(id)} & D \\
\end{array}
\]

that is taken from Proposition [1.23] below, we mean the composite

$$J \circ D(g, id) \xrightarrow{\varphi} U_A \circ (J \circ D(g, id)) \xrightarrow{\circ \circ \circ \circ \circ \circ} C(f, id) \circ (K \circ U_D) \xrightarrow{\varepsilon} C(f, id) \circ K$$

or, in its abbreviated version, $J \circ D(g, id) \xrightarrow{\phi} C(f, id) \circ K$.

The following are examples of pseudo double categories. In the next section we shall see that each of those are ‘promorphism equipments’.

**Example 1.2.** As will be clear rings, ring homomorphisms, bimodules and bilinear maps form a pseudo double category $\text{Mod}$. More precisely $\text{Mod}_0 = \text{Rng}$, the category of rings and their homomorphisms, while $\text{Mod}_1$ has bimodules as objects and bilinear maps as morphisms. The functors $L$ and $R: \text{Mod}_1 \rightarrow \text{Rng}$ map $(A, B)$-bimodules $J$ to $LJ = A$ and $RJ = B$, and each $(f, g)$-bilinear map $\phi: J \Rightarrow K$ to $L\phi = f$ and $R\phi = g$. As discussed the components of the associator $a$ and unitors $I$ and $\tau$ are obtained from the universality of tensor products.
Example 1.3. In the introduction we briefly mentioned the pseudo double category \( \text{Prof} \) of profunctors; in detail it is given as follows. Its objects and vertical morphisms are small categories and functors, i.e. \( \text{Prof}_0 = \text{Cat} \). Its horizontal morphisms \( J: A \nrightarrow B \) are profunctors, that is functors \( J: A^{\text{op}} \times B \to \text{Set} \), where \( A^{\text{op}} \) is the dual of \( A \), obtained by reversing the arrows in \( A \). We think of the elements of \( J(a,b) \) as morphisms, and denote them \( j: a \to b \). Likewise we think of the actions of \( A \) and \( B \) on \( J \) as compositions; hence, for morphisms \( s: a' \to a \) and \( t: b \to b' \), we will write \( t \circ j \circ s = J(s,t)(j) \). To describe the cells of \( \text{Prof} \), first consider a profunctor \( K: C \nrightarrow D \) and functors \( f: A \to C \) and \( g: B \to D \); we shall write \( K(f,g) \) for the composite \( K \circ (f^{\text{op}} \times g): A \nrightarrow B \). A cell

\[
A \xrightarrow{j} B \\
f \downarrow \phi \downarrow g \\
C \xrightarrow{k} D
\]

of \( \text{Prof} \) is simply a natural transformation \( \phi: J \to K(f,g) \). Such transformations clearly vertically compose so that they form a category \( \text{Prof}_1 \) with profunctors as objects.

The horizontal composition \( J \odot_B H \) of composable profunctors \( J: A \nrightarrow B \) and \( H: B \nrightarrow C \) is given by the reflexive coequalisers

\[
\prod_{b_1, b_2 \in B} J(a, b_1) \times B(b_1, b_2) \times H(b_2, c) \rightrightarrows \prod_{b \in B} J(a, b) \times H(b, c) \to (J \odot_B H)(a, c) \quad (1.4)
\]

of sets, for \( a \in A \) and \( c \in C \), where the pair of maps let \( B(b_1, b_2) \) act on \( J(a, b_1) \) and \( H(b_2, c) \) respectively. This can be thought of as a multi-object (and cartesian) variant of the tensor product of bimodules \( \mathbb{L} \). Often it is denoted shortly using the coend notation

\[
(J \odot_B H)(a, c) = \int^{b \in B} J(a, b) \times H(b, c).
\]

The associator \( a: (J \odot_B H) \odot_C K \cong J \odot_B (H \odot_C K) \) is induced by the fact that taking binary products preserves colimits in both factors. The unit profunctor \( U_A \) on the category \( A \) is given by its hom-objects \( U_A(a_1, a_2) = A(a_1, a_2) \), with actions given by composition, while the horizontal unit \( U_f \) for a functor \( f: A \to C \) is simply the restriction \( A(a_1, a_2) \to B(fa_1, fa_2) \) of \( f \) to the set of maps \( A(a_1, a_2) \). The unitor \( 1: U_A \odot A \cong J \) is induced by the actions \( A(a_1, a_2) \times J(a_2, b) \to J(a_1, b) \): it is not hard to check that its inverse has as components the composites

\[
J(a, b) \to A(a, a) \times J(a, b) \to (U_A \odot A) J(a, b)
\]

whose first maps are given by \( j \mapsto (\text{id}_a, j) \) while the second are insertions. The isomorphisms \( U_A \odot A \cong J \) and \( J \odot_B U_B \cong J \) are called the Yoneda isomorphisms.

Finally the horizontal composition \( \phi \odot_y \psi \) of the natural transformations

\[
A \xrightarrow{j} B \xrightarrow{H} E \\
f \downarrow \phi \downarrow \psi \downarrow g \downarrow h \\
C \xrightarrow{k} D \xrightarrow{L} F
\]

is given by the following composition of transformations

\[
(J \odot_B H)(a, c) = \int^{b \in B} J(a, b) \times H(b, c) \to \int^{b \in B} K(fa, gb) \times L(gb, hc) \\
\to \int^{d \in D} K(fa, d) \times L(d, hc) = (K \odot_D L)(f, h)(a, c), \quad (1.5)
\]
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where the first is induced by the products $\phi_{a,b} \times \psi_{b,c}$ and the second exists by the universality of the second coend.

Example 1.4. There is a $\mathcal{V}$-enriched variant of Prof as follows, for any symmetric pseudomonoidal category $\mathcal{V}$ (with invertible coherence maps) that is cocomplete, such that its binary tensor product $\odot$ preserves colimits on both sides. Examples include the category of vector spaces over a field and their tensor products, the cartesian monoidal category $\text{Cat}$ of categories and functors or the category $2 = (\bot \to \top)$ of truth values. The monoidal structure taken on the latter is cartesian, that is it is given by conjunction of truth values, while its coproducts are given by disjunction.

The pseudo double category $\mathcal{V}$-Prof of $\mathcal{V}$-profunctors has small $\mathcal{V}$-categories as objects and $\mathcal{V}$-functors as vertical morphisms. Generalising the unenriched case, the horizontal morphisms $J : A \to B$ of $\mathcal{V}$-Prof are $\mathcal{V}$-functors of the form $J : A^{\text{op}} \otimes B \to \mathcal{V}$. The dual $A^{\text{op}}$ of $A$ and tensor product $A^{\text{op}} \otimes B$ here are constructed using the symmetry on $\mathcal{V}$; see [KGS82, Section 1.4] for details. We shall again write $K(f, g) = K \circ (f^{\text{op}} \otimes g)$ where $K : C \to D, f : A \to C$ and $g : B \to D$; a cell

$$A \xrightarrow{J} B \xleftarrow{K} D$$

of $\mathcal{V}$-Prof is a $\mathcal{V}$-transformation $\phi : J \to K(f, g)$.

Finally horizontal composites and units are both enriched variants of those of $\mathcal{V}$-Prof: the horizontal composition of composable $\mathcal{V}$-profunctors $J : A \to B$ and $H : B \to C$ is given by the coends

$$(J \otimes_B H)(a, c) = \int^{b \in B} J(a, b) \otimes H(b, c),$$

which are again computed as the coequalisers [1.3], but with the cartesian products of $\text{Set}$ replaced by the tensor products of $\mathcal{V}$. The unit $\mathcal{V}$-profunctor $U_A$ on the $\mathcal{V}$-category $A$ is given by its hom-objects $U_A(a_1, a_2) = A(a_1, a_2)$, with actions given by composition; in particular there are enriched versions of the Yoneda isomorphisms $U_A \otimes_A J \cong J$ and $J \otimes_B U_B \cong J$, see for example [KGS82, Formula 3.71]. The horizontal composition $\phi \odot_g \psi$ of $\mathcal{V}$-natural transformations $\phi$ and $\psi$ is likewise given as the enriched variant of [1.3].

In the simple case $\mathcal{V} = 2$, a 2-enriched category $A$ is the same as a preordered set, that is a set $A'$ with an ordering $\leq$ that is reflexive and transitive, by taking $A' = \text{ob} A$ and $x \leq y$ if and only if $A(x, y) = \top$, for any pair of objects $x$ and $y$. Similarly, a 2-profunctor $J : A \to B$ can be thought of as a relation $\sim_J$ between the preordered sets $A$ and $B$, such that if $x_1 \leq x_2$ in $A$, $x_2 \sim_J y_1$ and $y_1 \leq y_2$ in $B$, then $x_1 \sim_J y_2$ as well. The horizontal composite of such relations $J : A \to B$ and $H : B \to C$ is given by the usual composite of relations: $x \sim_J \otimes_H z$ if and only if there exists a $y$ in $B$ such that $x \sim_J y$ and $y \sim_H z$. A 2-functor $f : A \to B$ is simply an order preserving map while a cell $\phi$, as above, exists (and is unique) if and only if $x \sim_J y$ implies $fx \sim_K gy$ for all $x$ in $A$ and $y$ in $B$.

Shulman shows in [Shu08, Section 11] that $\mathcal{V}$-categories and $\mathcal{V}$-profunctors can be considered as ‘monoids’ and ‘bimodules’ in the simpler double category $\mathcal{V}$-$\text{Mat}$ of $\mathcal{V}$-matrices. This will turn out useful for us, as many of our questions about $\mathcal{V}$-Prof are easier to answer by considering the ‘underlying questions’ in $\mathcal{V}$-$\text{Mat}$. This is why we shall recall this construction later, in Definition 1.16. In the same way, for every category $\mathcal{E}$ with finite limits, there is a relatively simple pseudo double category $\text{Span}(\mathcal{E})$ of ‘spans in $\mathcal{E}$’, and monoids and bimodules in $\text{Span}(\mathcal{E})$ are respectively
categories and profunctors ‘internal in $E$’. The pseudo double categories $\mathcal{V}$-$\text{Mat}$ and $\text{Span}(\mathcal{E})$ which, as we shall see in the next section, are promorphism equipments as well, are introduced below.

Example 1.5. Let $\mathcal{V}$ be a monoidal category that has all coproducts which are preserved by its tensor product. The pseudo double category $\mathcal{V}$-$\text{Mat}$ of $\mathcal{V}$-matrices is given as follows. Its objects are sets and its vertical morphisms are maps of sets, that is $\mathcal{V}$-$\text{Mat}_0 = \text{Set}$, while a horizontal morphism $J : A \rightarrow \rightarrow B$, that is an object of $\mathcal{V}$-$\text{Mat}_1$, consists of a ‘matrix’ $(J(a,b))_{a \in A, b \in B}$ of $\mathcal{V}$-objects. A cell is given by a family of maps $\phi_{a,b} : J(a,b) \rightarrow K(fa,gb)$ in $\mathcal{V}$. Such cells clearly compose vertically, thus forming a category $\mathcal{V}$-$\text{Mat}_1$. Horizontal composition of $\mathcal{V}$-matrices $J : A \rightarrow \rightarrow B$ and $H : B \rightarrow \rightarrow E$ is given by ‘matrix multiplication’:

$$(J \otimes H)(a,e) = \bigsqcup_{b \in B} J(a,b) \otimes H(b,e).$$

The existence of the associators follows from the assumption that the tensor product preserves coproducts, while the unit $U_A$ on a set $A$ is given by

$$U_A(a,a') = \begin{cases} 1 & \text{if } a' = a; \\ \emptyset & \text{otherwise}, \end{cases}$$

where 1 denotes the monoidal unit of $\mathcal{V}$, and $\emptyset$ its initial object. Finally, the composition of composable cells is given by the compositions

$$(J \circ H)(a,e) \rightarrow \bigsqcup_{b \in B} K(fa,gb) \otimes L(gb,he) \rightarrow (K \otimes L)(fa,he),$$

where the first map is the coproduct of the tensor products $\phi_{a,b} \otimes \psi_{b,e}$, and the second map exists by the universal property of coproducts.

In case $\mathcal{V} = 2$, a 2-matrix $J : A \rightarrow \rightarrow B$ is a relation on $A$ and $B$. Writing $a \sim_J b$ whenever $a$ and $b$ are related in $J$, a cell $\phi$ as above exists (and is unique) if and only if $a \sim_J b$ implies $fa \sim_K gb$ for all $a \in A$ and $b \in B$. Horizontal composition of relations is given by the usual composition of relations.

Example 1.6. For a category $\mathcal{E}$ with finite limits, the pseudo double category $\text{Span}(\mathcal{E})$ of spans in $\mathcal{E}$ is defined as follows. The objects and vertical morphisms of $\text{Span}(\mathcal{E})$ are those of $\mathcal{E}$, while a horizontal morphism $J : A \rightarrow \rightarrow B$ is a span $A \xleftarrow{j_A} J \xrightarrow{j_B} B$ in $\mathcal{E}$. A cell $\phi$ as on the left below is a map $\phi : J \rightarrow \rightarrow K$ in $\mathcal{E}$ such that the diagram on
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the right commutes.

\[ \begin{array}{ccc}
A & \xrightarrow{J} & B \\
\downarrow{f} & \phi & \downarrow{g} \\
C & \xrightarrow{K} & D
\end{array} \quad \begin{array}{ccc}
A & \xrightarrow{J} & B \\
\downarrow{f} & \phi & \downarrow{g} \\
C & \xrightarrow{K} & D
\end{array} \]

Given spans \( J: A \rightarrow B \) and \( H: B \rightarrow C \), their composition \( J \circ H \) is given by the usual composition of spans: after choosing a pullback \( J \times_B H \) of \( j_B \) and \( h_B \) below, it is taken to consist of the two sides of the diagram below.

\[ \begin{array}{ccc}
J \times_B H & \xrightarrow{\phi} & K \\
\downarrow{j_A} & & \downarrow{k_C} \\
J & \xrightarrow{J} & H \\
\downarrow{j_B} & & \downarrow{h_B} \\
A & \xrightarrow{j} & B & \xrightarrow{f} & C \\
\downarrow{g} & & \downarrow{k} \\
C & \xrightarrow{h_B} & D
\end{array} \]

That this composition is associative and unital up to coherent isomorphisms, with spans of the form \( A \xleftarrow{id} A \xrightarrow{id} A \) as units, follows from the universality of pullbacks; horizontal composition of cells is also given by using this universality.

Since we assume \( \mathcal{E} \) to have all finite limits we might equivalently regard a span \( J: A \rightarrow B \) in the slice category \( \mathcal{E}/A \times B \). In this case we will still write \( j = (j_A, j_B) \). A cell \( \phi \) as above thus corresponds to a commutative square in \( \mathcal{E} \) as follows.

\[ \begin{array}{ccc}
J \times_B H & \xrightarrow{\phi} & K \\
\downarrow{j_A} & & \downarrow{k_C} \\
J & \xrightarrow{j_B} & H \\
\downarrow{j_B} & & \downarrow{h_B} \\
A \times B & \xrightarrow{f} & C \times D
\end{array} \] \quad (1.6)

Regarding spans in \( \mathcal{E} \) as objects in slice categories, their horizontal composition can be given as follows. Given a morphism \( f: X \rightarrow Y \) notice that there is an adjunction

\[ f_*: \mathcal{E}/X \rightleftarrows \mathcal{E}/Y: f^* \]

whose left adjoint \( f_* \) is given by postcomposition with \( f \) and whose right adjoint \( f^* \) is given by pullback along \( f \).

**Lemma 1.7.** Let \( \mathcal{E} \) be a category with finite limits. Given a span \( j: J \rightarrow A \times B \) in \( \mathcal{E} \), horizontal composition \( J \circ -: \mathcal{E}/B \times C \rightarrow \mathcal{E}/A \times C \) can be given as

\[ J \circ - = (j_A \times \text{id}_C)_* \circ (j_B \times \text{id}_C)^* \]

where \( j = (j_A, j_B) \).

**Proof.** Let \( h: H \rightarrow B \times C \) be a second span in \( \mathcal{E} \). By definition \( J \circ H \) is given by \( (j_A \circ p, h_C \circ q): J \times_B H \rightarrow A \times C \), where \( p \) and \( q \) are the projections of the pullback \( W = J \times_B H \), onto \( J \) and \( H \) respectively. An easy calculation will show that \( (W, p, q) \) is the pullback of \( j_B \) and \( h_B \) if and only if

\[ \begin{array}{ccc}
W & \xrightarrow{q} & H \\
\downarrow{(p, h_C \circ q)} & & \downarrow{h} \\
J \times C & \xrightarrow{j_B \times \text{id}_C} & B \times C
\end{array} \]
is a pullback square. Here the map on the left is, by definition, \((j_B \times \text{id}_C)^* h\); composing it with \(j_A \times \text{id}_C\) does indeed give \(J \circ H: W \to A \times C\) as asserted. It is easily seen that this extends to the right action on morphisms of spans as well. \(\square\)

Having introduced the main examples, we close this section by recalling that every pseudo double category \(K\) contains an underlying vertical 2-category \(V(K)\), of vertical morphisms and cells, as well as a horizontal bicategory \(H(K)\), of horizontal morphisms and cells, as follows (a bicategory is a category ‘weakly enriched’ in \(\mathbf{Cat}\), see [Lei04, Definition 1.5.1]). The objects of both \(V(K)\) and \(H(K)\) are those of \(K\). Given objects \(A\) and \(B\), the category \(H(K)(A, B)\) is the subcategory of \(K_{1}\) consisting of all horizontal morphisms \(A \to B\) and horizontal cells, that is cells with identities as vertical maps, between them. The composition functors

\[
H(K)(A, B) \times H(K)(B, C) \to H(K)(A, C)
\]

are restrictions of \(\circ: K_{0} \times_{K_{0}} K_{1} \to K_{1}\), and we will denote composition in \(H(K)\) again by \(\circ\).

On the other hand, the category \(V(K)(A, B)\) consists of vertical maps \(A \to B\) in \(K_{0}\), while the morphisms \(\phi: f \Rightarrow g\) are the vertical cells of \(K\), that is the horizontal source and target of \(\phi\) are horizontal identities. The vertical (in terms of the 2-category \(V(K)\)) composition \(\psi \circ \phi\) of \(\phi\) with a second vertical cell \(\psi: g \Rightarrow h\) is given by the composition of the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{f} & A \\
\downarrow & & \downarrow \\
B & \xrightarrow{g} & B
\end{array}
\]

where the top and bottom cells are given by (inverses of) unitors of \(K\). That this composition is strictly associative follows from the fact that the unitors satisfy the identity coherence axiom (see [ML98, Diagram (7) of Section VII.1]). The horizontal composition functors

\[
\circ: V(K)(A, B) \times V(K)(B, C) \to V(K)(A, C)
\]

are given by vertical composition in \(K\), which is strictly associative.

**Example 1.8.** Remember that a vertical cell \(\phi: f \Rightarrow g\), of functors \(f\) and \(g\): \(A \to B\), in the pseudo double category \(\text{Prof}\) of (unenriched) profunctors, is a natural transformation of profunctors \(\phi: U_A \Rightarrow U_B(f, g)\), where \(U_B(f, g)(a_1, a_2) = B(fa_1, ga_2)\). By naturality \(\phi\) is completely determined by its images \(\phi_a = \phi(\text{id}_a): fa \Rightarrow ga\) in \(B\); indeed we have \(\phi(u) = \phi_{u_2} \circ f(u) = g(u) \circ \phi_{u_1}\) for any \(u: a_1 \to a_2\) in \(A\). From this identity it follows that the maps \(\phi_a\) form a natural transformation of functors \(\phi: f \Rightarrow g\), in the usual sense; in fact this gives a bijective correspondence between the vertical cells \(f \Rightarrow g\) of \(\text{Prof}\) and the natural transformations \(f \Rightarrow g\).

It follows from the descriptions of the unitors of \(\text{Prof}\), in Example 1.3, that the vertical composition of two vertical cells \(\phi: f \Rightarrow g\) and \(\psi: g \Rightarrow h\) corresponds to the usual vertical composition \(\psi \circ \phi\) of natural transformations of functors, given by \((\psi \circ \phi)_a = \psi_a \circ \phi_a\). Moreover, the horizontal composition of two vertical cells
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\( \phi: f \Rightarrow g \) and \( \psi: h \Rightarrow k \), where \( h \) and \( k \) are functors \( B \rightarrow C \), is determined by the images

\[(\phi \circ \psi)_{a} = \psi_{ga} \circ h(\phi_{a}) = k(\phi_{a}) \circ \psi_{fa}: (h \circ f)(a) \rightarrow (k \circ g)(a),\]

that is it corresponds to the usual horizontal composite \( \phi \circ \psi: h \circ f \rightarrow k \circ g \). We conclude that \( V(\text{Prof}) \cong \text{Cat} \), the 2-category of small categories, functors and natural transformations. On the other hand clearly the horizontal bicategory \( H(\text{Prof}) \) of \( \text{Prof} \) is the bicategory of categories and profunctors.

Likewise in the enriched case we get \( V(\mathcal{V}-\text{Prof}) \cong \mathcal{V}-\text{Cat} \), the 2-category of small \( \mathcal{V} \)-categories, \( \mathcal{V} \)-functors and \( \mathcal{V} \)-natural transformations.

1.2 2-Categories equipped with promorphisms

In the introduction we described 2-categories equipped with promorphisms, ‘promorphism equipments’ in short, as pseudo double categories \( \mathcal{K} \) in which every vertical map \( f: A \rightarrow C \) is accompanied by two horizontal morphisms \( C(f, \text{id}): A \Rightarrow C \) and \( C(\text{id}, f): C \Rightarrow A \), respectively its ‘companion’ and ‘conjoint’. This description is close to the original one given by Wood [Woo82]. However we will keep on following Shulman’s paper [Shu08] and define promorphism equipments as pseudo double categories in which all ‘cartesian fillers’ exist. Afterwards we will recall [Shu08, Theorem A.2] which shows that the two definitions coincide.

In a pseudo double category \( \mathcal{K} \), a diagram of the form

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow \phi & & \downarrow \psi \\
C & \xleftarrow{g} & D,
\end{array}
\]

also written as \( \begin{array}{c}
\begin{array}{c}
A \\
\downarrow f
\end{array} \xrightarrow{K} \begin{array}{c}
B \\
\downarrow g
\end{array}
\end{array} \), is called a niche.

**Definition 1.9.** A cartesian filler for the niche above is a cell

\[
\begin{array}{ccc}
A & \xrightarrow{K(f, g)} & B \\
\downarrow f & \downarrow \phi & \downarrow g \\
C & \xleftarrow{K} & D,
\end{array}
\]

that is universal in that every other filler \( \phi: J \Rightarrow K \) factorises uniquely as

\[
\begin{array}{ccc}
X & \xrightarrow{j} & Y \\
\downarrow b & & \downarrow k \\
A & \xrightarrow{\phi} & B \\
\downarrow f & & \downarrow \phi_a \\
C & \xrightarrow{K(f, g)} & D.
\end{array}
\]

The horizontal morphism \( K(f, g) \) is called the restriction of \( K \) along \( f \) and \( g \).

Cartesian fillers are unique up to precomposition with invertible horizontal cells.

**Definition 1.10.** A pseudo double category in which every niche has a cartesian filler is called a 2-category equipped with promorphisms, or simply a (promorphism) equipment.
We will call the vertical morphisms of a promorphism simply morphisms; the horizontal morphisms will be called promorphisms. In the example below we will see all the examples of the previous section are promorphism equipments, but first we recall the definition of the companion and conjoint of a morphism.

Given a morphism \( f : A \to C \), assume that the cartesian fillers

\[
\begin{array}{ccc}
A & \xrightarrow{U_C(f, id)} & C \\
\downarrow f & \Downarrow \varepsilon_f & \downarrow f \\
C & \xrightarrow{C(id, f)} & C
\end{array}
\text{ and }
\begin{array}{ccc}
A & \xrightarrow{U_C(id, f)} & C \\
\downarrow \eta_f & \Downarrow f & \downarrow \varepsilon_f \\
C & \xrightarrow{C(id, f)} & C
\end{array}
\]

exist; we will abbreviate \( C(f, id) = U_C(f, id) \) and \( C(id, f) = U_C(id, f) \). Factorising the identity cells \( U_f \) through these cartesian fillers gives cells

\[
\begin{array}{ccc}
A & \xrightarrow{\varepsilon_f} & C \\
\downarrow f & \Downarrow \eta_f & \downarrow f \\
C & \xrightarrow{C(f, id)} & C
\end{array}
\text{ and }
\begin{array}{ccc}
A & \xrightarrow{\eta_f} & C \\
\downarrow f & \Downarrow \varepsilon_f & \downarrow f \\
C & \xrightarrow{C(id, f)} & C
\end{array}
\]

that satisfy

\[
\begin{array}{ccc}
A & \xrightarrow{A} & A \\
\downarrow f & \Downarrow \varepsilon_f & \downarrow f \\
C & \xrightarrow{C} & C
\end{array} =
\begin{array}{ccc}
A & \xrightarrow{A} & A \\
\downarrow f & \Downarrow \eta_f & \downarrow f \\
C & \xrightarrow{C} & C
\end{array}
\text{ and }
\begin{array}{ccc}
A & \xrightarrow{A} & A \\
\downarrow f & \Downarrow \varepsilon_f & \downarrow f \\
C & \xrightarrow{C} & C
\end{array} =
\begin{array}{ccc}
A & \xrightarrow{A} & A \\
\downarrow f & \Downarrow \varepsilon_f & \downarrow f \\
C & \xrightarrow{C} & C
\end{array}.
\]

Moreover the uniqueness of factorisations through cartesian fillers implies that the composites

\[
\begin{array}{ccc}
A & \xrightarrow{A} & A \\
\downarrow \eta_f & \Downarrow \varepsilon_f & \downarrow \eta_f \\
C & \xrightarrow{C} & C
\end{array}
\text{ and }
\begin{array}{ccc}
A & \xrightarrow{A} & A \\
\downarrow \eta_f & \Downarrow \varepsilon_f & \downarrow \eta_f \\
C & \xrightarrow{C} & C
\end{array}
\]

are, after both pre- and postcomposition with (inverses of) unitors, equal to the identity cells of \( C(f, id) \) and \( C(id, f) \) respectively. The two equalities involving \( \varepsilon_f \) and \( \eta_f \) above define \( C(f, id) \) as the companion for \( f \), while those involving \( \varepsilon_f \) and \( \eta_f \) define \( C(id, f) \) as the conjoint for \( f \); the equalities themselves are called the companion and conjoint identities respectively. One can think of the companion \( C(f, id) \) as the promorphism ‘isomorphic’ to \( f \), while the conjoint \( C(id, f) \) can be thought of as being ‘adjoint’ to \( f \). Companions and conjoints were introduced by Grandis and Paré in [GP04, Sections 1.2 and 1.3] where they are called ‘orthogonal companions’ and ‘orthogonal adjoints’. As we remarked in the introduction, Wood’s original definition of equipments, given in [Woo82, is close to defining equipments as pseudo double categories in which every vertical morphism has a companion and conjoint.

**Example 1.11.** For a niche \( f : A \xrightarrow{\xi} C \xleftarrow{\eta} D \xrightarrow{\varphi} B \) in \( \text{Mod} \), where \( f \) and \( g \) are ring homomorphisms and \( K \) is a \( (C,D) \)-bimodule, the restriction \( K(f,g) : A \to B \) is \( K \) regarded as a \( (A,B) \)-bimodule via \( f \) and \( g \), with left action \( A \otimes K \xrightarrow{f \otimes id} C \otimes K \xrightarrow{\varphi} K \).
and similar right action. The cartesian filler \( \varepsilon \): \( K(f,g) \Rightarrow K \) is simply the identity for \( K \); it is clear that any \( (f \circ h, g \circ k) \)-bilinear map \( \phi \): \( J \Rightarrow K \), as in Definition 1.9, can be regarded as a \((h,k)\)-bilinear map \( J \Rightarrow K(f,g) \), which is the unique factorisation \( f \circ g \) of \( \phi \) through \( \varepsilon \). In particular the factorisation \( f \eta \) of the horizontal unit \( U_f \) through the companion cell \( f \varepsilon \), where \( U_f \) is the ring homomorphism \( f: A \to C \) regarded as a \((f,f)\)-bilinear map, is again simply \( f \), now regarded as an \((id,f)\)-bilinear map \( A \Rightarrow C(f,f, id) \).

For a niche \( A \xrightarrow{f} C \xrightarrow{K} D \xleftarrow{g} B \) in \( \text{Prof} \) we have, in Example 1.3 already used the notation \( K(f,g) \) to denote the composite \( K \circ (f^{op} \times g) : A \Rightarrow B \). It is readily seen that \( K \circ (f^{op} \times g) \), together with the cell \( \varepsilon \): \( K(f,g) \Rightarrow K \) that is the identity transformation on \( K \circ (f^{op} \times g) \), do indeed form the cartesian filler: any other filler

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
h & \downarrow & \downarrow k \\
A & \xleftarrow{\phi} & B \\
f & \downarrow \varepsilon & g \\
C & \xleftarrow{K} & D
\end{array}
\]

(1.7)

in \( \text{Prof} \), that is a transformation \( \phi : J \Rightarrow K(f \circ h, g \circ k) \), can be regarded as a cell \( \phi : J \Rightarrow K(f,g) \), because \( K \circ (f^{op} \times g) \circ (h^{op} \times k) = K \circ ((f \circ h)^{op} \times (g \circ k)) \). In particular the companion \( C(f, id) \) of a functor \( f: A \to C \) is given by the representable profunctor \( C(f, id)(a,c) = C(fa, c) \). Conjoints are given likewise.

The previous example generalises directly to the enriched case: in \( \mathcal{V}-\text{Prof} \) the restriction \( K(f,g) \) of a \( \mathcal{V} \)-profunctor \( K \) is given by \( K(f,g) = K \circ (f^{op} \otimes g) : A \Rightarrow B \).

In \( \mathcal{V}-\text{Mat} \) the restriction \( K(f,g) : A \Rightarrow B \) is simply given by \( K(f,g)(a,b) = K(fa, gb) \); the cartesian filler \( K(f,g) \Rightarrow K \) has identities as components. It follows that the companion \( C(f, id) \) of \( f \) is given by

\[
C(f, id)(a,c) = \begin{cases} 
1 & \text{if } fa = c; \\
\emptyset & \text{otherwise},
\end{cases}
\]

while both \( f \varepsilon \) and \( f \eta \) consist of identities. The conjoint \( C(id, f) \) is given likewise.

In \( \text{Span}(\mathcal{E}) \) the restriction \( K(f,g) \) is given by the pullback \( K(f,g) = (f \times g)^{op} k \), where \( k: K \to C \times D \); the cartesian filler is the projection \( K(f,g) \to K \). The universal property follows immediately from the fact that cells \( \phi \) in \( \text{Span}(\mathcal{E}) \) are commutative squares like (1.10), which factor through the pullback \( K(f,g) \). Hence the companion \( C(f, id) \) is the span \( A \xrightarrow{id} A \xleftarrow{f} C \), with \( f \varepsilon = f \) and \( f \eta = id A \), while the conjoint \( C(id, f) \) is the span \( C \xleftarrow{id} A \xrightarrow{f} A \).

A pseudo double category has all cartesian fillers if and only if it has both companions and conjoints for vertical morphisms, as the following shows.

**Proposition 1.12** (Shulman). Let \( A \xrightarrow{f} C \xleftarrow{K} D \xrightarrow{g} B \) be a niche in a pseudo double category \( K \). If \( f \) has a companion \( C(f, id) \) and \( g \) has a conjoint \( D(id, g) \) then the composition

\[
C(f, id) \odot K \odot D(id, g) \xrightarrow{f \varepsilon \odot id_k \odot \varepsilon_g} U_C \odot K \odot U_D \cong K,
\]

is a cartesian filler. In particular there is a canonical isomorphism \( C(f, id) \odot K \odot D(id, g) \cong K(f,g) \).
Sketch of the proof. This is shown in the proof of [Shu08 Theorem 4.1], and follows directly from the companion and conjoint identities. For example, any other filler \( \phi: J \Rightarrow K \) as in \([126] \) factorises uniquely through the cell above as the composition

\[
J \cong U_X \odot J \odot U_Y \xrightarrow{(\eta_U \circ \phi \circ (\eta_U \circ U)} C(f, id) \odot K \odot D(id, g).
\]

Factorising the cartesian fillers \( C(f, id) \odot K \odot D(id, g) \) and \( K(f, g) \) through each other we obtain the canonical isomorphism \( C(f, id) \odot K \odot D(id, g) \cong K(f, g) \). \( \square \)

Example 1.13. In \( \text{Prof} \) the isomorphism \( C(f, id) \odot C \odot D(id, g) \cong K(f, g) \) is simply an instance of the Yoneda isomorphisms, which were described in Example 1.3. The same holds for \( \mathcal{V} \text{-Prof} \).

Remember that an adjunction \( l \dashv r \) in a bicategory \( \mathcal{B} \) consists of morphisms \( l: X \to Y \) and \( r: Y \to X \), as well as cells \( \eta: id_X \Rightarrow r \circ l \) (the unit) and \( \epsilon: l \circ r \Rightarrow id_Y \) (the counit), satisfying the triangle identities

\[
[l \odot id_Y \circ r \circ l \circ id_Y] = id_Y \quad \text{and} \quad [r \odot id_Y \circ r \circ l \circ id_Y] = id_Y.
\]

These generalise the usual adjunctions between categories, in the 2-category \( \text{Cat} \) of categories, functors and natural transformations. The following, which is a direct consequence of the companion and conjoint identities, will be used throughout.

Proposition 1.14. For any morphism \( f: A \to C \) in an equipment \( K \), the companion \( C(f, id) \) is left adjoint to the conjoint \( C(id, f) \) in the horizontal bicategory \( H(K) \). The unit \( f \eta_f \) and counit \( f \epsilon_f \) of this adjunction are given by

\[
f \eta_f = [U_A \cong U_A \odot U_A \xrightarrow{\eta_U \odot id} C(f, id) \odot C(id, f)]
\]

and

\[
f \epsilon_f = [C(id, f) \odot C(f, id) \xrightarrow{id \odot \epsilon_f} U_C \odot U_C \cong U_C].
\]

Example 1.15. For a functor \( f: A \to C \) the unit \( f \eta_f \) of the adjunction \( C(f, id) \dashv C(id, f) \) is, under the Yoneda isomorphism \( C(f, id) \odot C(id, f) \cong C(f, f) \), simply given by the action \( f: A(a_1, a_2) \to C(fa_1, fa_2) \) of \( f \) on maps in \( A \), while the counit \( f \epsilon_f \) is induced by composition \( C(c_1, fb) \times C(fb, c_2) \to C(c_1, c_2) \) in \( C \).

The following definition recalls the notions of ‘monoid’ and ‘bimodule’ in pseudo double categories, from [Shu08 Section 11]; see also [Lei04 Section 5.3]. These are generalisations of the usual notions of monoids and bimodules in a monoidal category, including the famous example of rings being monoids in the monoidal category \( \text{Ab} \) of abelian groups and their tensor product.

Definition 1.16. Let \( K \) be a pseudo double category.

- A monoid \( A \) in \( K \) consists of \( A = (A_0, A, m, e) \) where \( A_0 \to A_0 \) is a horizontal morphism in \( K \) and \( m: A \odot A \Rightarrow A \) and \( e: U_{A_0} \Rightarrow A \) are horizontal cells satisfying the associativity and unit axioms below. The cells \( m \) and \( e \) are called the multiplication and unit of \( A \).

\[
m \circ (id_A \odot m) = m \circ (m \odot id_A) \quad \text{and} \quad m \circ (e \odot id_A) = id_A = m \circ (id_A \odot e)
\]

- Given monoids \( A \) and \( B \), a morphism of monoids \( f: A \to B \) consists of a vertical morphism \( f_0: A_0 \to B_0 \) and a cell

\[
A_0 \xrightarrow{A} A_0 \quad \xrightarrow{f_0} \quad B_0 \xrightarrow{B} B_0
\]

\[
f_0 \downarrow \quad f_0 \downarrow \quad f_0
\]
that is compatible with the multiplications and units of $A$ and $B$:

$$m_B \circ (f \circ f) = f \circ m_A \quad \text{and} \quad f \circ e_A = e_B \circ U_{f_0}.$$  

- Given monoids $A$ and $B$, an $(A,B)$-bimodule $J$ consists of $J = (J,l,r)$, where $J: A_0 \rightarrow B_0$ is a horizontal morphism in $K$ and $l: A \circ J \Rightarrow J$ and $r: J \circ B \Rightarrow J$ are horizontal cells defining the actions of $A$ and $B$ on $J$. These actions each satisfy the associativity and unit axioms

$$l \circ (id \circ l) = l \circ (m \circ id) \quad \text{and} \quad l \circ (e \circ id) = id,$$

likewise for $r$, and together satisfy the commutativity axiom

$$l \circ (id \circ r) = r \circ (l \circ id).$$

- Given monoids $A$, $B$, $C$ and $D$, morphisms $f: A \rightarrow B$ and $g: C \rightarrow D$ and bimodules $J: A \Rightarrow B$ and $K: C \Rightarrow D$, a cell $\phi$ as on the left below

$$
\begin{array}{ccc}
A & \overset{J}{\rightarrow} & B \\
\downarrow f & \phi & \downarrow g \\
C & \overset{K}{\rightarrow} & D
\end{array}
\quad
\begin{array}{ccc}
A_0 & \overset{J_0}{\rightarrow} & B_0 \\
\downarrow f_0 & \phi_0 & \downarrow g_0 \\
C_0 & \overset{K_0}{\rightarrow} & D_0
\end{array}
$$

is a cell in $K$ as on the right, satisfying the following compatibility axioms.

$$l_K \circ (f \circ \phi) = \phi \circ l_J \quad \text{and} \quad r_K \circ (\phi \circ g) = \phi \circ r_J.$$

Before giving examples we recall [Shulman, Proposition 11.10], which asserts that monoids and bimodules in an equipment $K$ form themselves an equipment $\text{Mod}(K)$, under a mild condition. Recall that a reflexive pair in a category $C$ is a pair of parallel maps $f$, $g: X \Rightarrow Y$ together with a common section $s: Y \rightarrow X$, that is $f \circ s = id_Y = g \circ s$. A reflexive coequaliser is a coequaliser of a reflexive pair.

**Proposition 1.17** (Shulman). Let $K$ be an equipment such that the categories $H(K)(A,B)$ have reflexive coequalisers, that are preserved by horizontal composition on both sides. Then the monoids and bimodules of $K$, together with their morphisms and cells, again form an equipment $\text{Mod}(K)$.

**Sketch of the proof.** We briefly recall the constructions that we will use here; for the details see Shulman’s article [Shulman, Proposition 11.10]. The vertical composition of morphisms of monoids and that of cells between bimodules is given by vertical composition in $K$. For the horizontal composition, consider two composable bimodules $J: A \Rightarrow B$ and $H: B \Rightarrow C$; their horizontal composite $J \circ_B H: A \Rightarrow C$ is constructed as the reflexive coequaliser

$$
\begin{array}{c}
J \circ_B H \\
\xrightarrow{r \circ \text{id}} \\
\xrightarrow{\text{id} \circ \phi} \\
J \circ H
\end{array}
\xrightarrow{\phi \circ g} \\
\xrightarrow{\text{id} \circ \phi} \\
J \circ_B H
$$

in $H(K)(A_0, C_0)$. By the unit axioms for the actions of $B$ on $J$ and $H$, a section for the parallel cells can be given by the composite

$$J \circ H \cong J \circ U_{B_0} \circ H \xrightarrow{id \circ \phi \circ id} J \circ_B H.$$

The unit for horizontal composition, on a monoid $A$, is itself considered as an $(A,A)$-bimodule.
Given horizontally composable cells
\[
\begin{array}{c}
A \xrightarrow{f} B \xleftarrow{g} E \\
\downarrow \phi \downarrow \psi \\
C \xleftarrow{h} D \xrightarrow{h}
\end{array}
\]
of bimodules, their horizontal composition \(\phi \circ g \psi\) is defined as follows. First consider the composition \(J \circ H \xrightarrow{\phi \circ g \psi} K \circ L \Rightarrow K \circ_D L\), which factors as a horizontal cell \(J \circ H \Rightarrow (K \circ_D L)(f, h)\) through the cartesian filler for the niche \(A \xrightarrow{J} C \xleftarrow{K \circ_D L} F \xleftarrow{h} E\). The fact that the cells \(\phi\) and \(\psi\) are compatible with the actions implies that this factorisation factors further as \(J \circ_B H \Rightarrow (K \circ_D L)(f, h)\) which, composed with the cartesian filler, corresponds to a cell \(J \circ_B H \Rightarrow K \circ_D L\). This is what \(\phi \circ g \psi\) is defined to be.

Finally consider a niche \(A \xrightarrow{J} C \xleftarrow{K} D \xleftarrow{\varepsilon} B\) consisting of morphisms of monoids \(f\) and \(g\) and a bimodule \(K\). The underlying niche \(A_0 \xrightarrow{f_0} C_0 \xleftarrow{g_0} D_0 \xleftarrow{\varepsilon_0} B_0\) has a cartesian filler \(\varepsilon : K(f_0, g_0) \Rightarrow K\) in \(K\), which can be made into a cell of bimodules as follows. The action \(A \circ K(f_0, g_0) \Rightarrow K(f_0, g_0)\) of \(A\) is given by the factorisation of \(A \circ K(f_0, g_0) \xrightarrow{\varepsilon} C \circ K \Rightarrow K\) through \(\varepsilon\); the action of \(B\) is given similarly. That \(\varepsilon\) is compatible with these actions follows immediately. Now the underlying cell of any other filler \(\phi\) will factor through \(\varepsilon\) as a cell \(\phi'\) in \(K\). It is easy to see that the universality of \(\varepsilon\) in \(K\) implies that \(\phi'\) is compatible with the actions on its source and target bimodules, so that it is a cell of bimodules: we conclude that \(\varepsilon\) is universal in \(\text{Mod}(K)\) as well.

**Example 1.18.** If \(V\) is a cocomplete symmetric pseudomonoidal category whose binary tensor product \(- \otimes -\) preserves colimits in both variables, then the equipment \(V\)-Mat of \(V\)-matrices satisfies the condition of the previous proposition, so that the monoids and bimodules in \(V\)-Mat form an equipment \(\text{Mod}(V\text{-Mat})\).

A monoid \(A\) in \(V\text{-Mat}\) consists of a set \(A_0\) and a \(V\)-object \(A(a, b)\) for every pair \(a, b \in A_0\), such that the maps \(m : A(a, b) \otimes A(b, c) \Rightarrow A(a, c)\) and \(e : 1 \Rightarrow A(a, a)\), induced by its multiplication and unit, make \(A\) into a \(V\)-category. Likewise a morphism \(f : A \Rightarrow B\) of monoids is a \(V\)-functor. Given \(V\)-categories \(A\) and \(B\), an \((A, B)\)-bimodule \(J : A \Rightarrow B\) consists of \(V\)-objects \(J(a, b)\), for \(a \in A, b \in B\), together with action maps
\[
l : A(a_1, a_2) \otimes J(a_2, b) \Rightarrow J(a_1, b) \quad \text{and} \quad r : J(a, b_1) \otimes B(b_1, b_2) \Rightarrow J(a, b_2),
\]
which can be reconsidered as \(V\)-functors \(J(-, b) : A^{\text{op}} \Rightarrow V\) and \(J(a, -) : B \Rightarrow V\) respectively, for each \(a \in A\) and \(b \in B\). The compatibility axiom for \(l\) and \(r\) implies that giving such families of \(V\)-functors is equivalent to giving a single \(V\)-profunctor \(J : A^{\text{op}} \otimes B \Rightarrow V\) (see [Kel82 Section 1.4]). Likewise, a cell of bimodules is precisely a natural transformation of \(V\)-profunctors.

One readily checks that the horizontal compositions, that make \(\text{Mod}(V\text{-Mat})\) into a pseudo double category and that are given in the proof above, coincide with the horizontal compositions of \(V\)-Prof as given in Example 1.4. We conclude that the equipments \(\text{Mod}(V\text{-Mat})\) and \(V\)-Prof are ‘isomorphic’, even though we do not yet know what that means; functors between pseudo double categories will be treated in Chapter 3.

Monoids in \(\text{Span}(\mathcal{E})\) are ‘internal categories in \(\mathcal{E}\)’, as follows.

**Example 1.19.** A monoid \(A\) in \(\text{Span}(\mathcal{E})\) is given by a span \(d : A \rightarrow A_0 \times A_0\) in \(\mathcal{E}\) equipped with a multiplication and unit as follows; we will write \(d = (d_0, d_1)\), and
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\[ d_2 = (d_0 \circ p, d_1 \circ q) : A \times A_0 A \to A_0 \times A_0 \] for the induced map on the composite of \( A \) with itself, where \( p \) and \( q \) are the projections onto the first and second factor of \( A \times A_0 A \). The multiplication and unit of \( A \) are given by morphisms

\[
\begin{align*}
A \times A_0 A & \xrightarrow{m} A \\
A_0 \times A_0 & \xrightarrow{d} A
\end{align*}
\]

in the slice category \( \mathcal{E}/A_0 \times A_0 \). Satisfying the associativity and unit axioms, these make \( A \) into a category internal in \( \mathcal{E} \) (see [ML98, Section XII.1]), with an ‘object of objects’ \( A_0 \) and an ‘object of morphisms’ \( A \). The maps \( d_0 : A \to A_0 \) and \( d_1 : A \to A_0 \) are the source and target maps, so that the pullback \( A \times A_0 A \) of \( d_0 \) and \( d_1 \) forms the ‘object of composable morphisms’. The map \( m \) then gives the composition of \( A \) while \( e \) supplies its identities. In the case that \( \mathcal{E} = \text{Set} \) this recovers ordinary categories as categories internal in \( \text{Set} \), with the difference that where an ordinary category \( A' \) has sets of maps \( A'(a, b) \) for any pair of objects \( a \) and \( b \), a category \( A \) internal in \( \text{Set} \) has a single set of maps \( A \), and comes with a map \( d : A \to A_0 \times A_0 \) that specifies sources and targets.

Likewise a map of monoids \( f : A \to B \) is an internal functor: it consists of a pair of maps \( f : A \to B \) and \( f_0 : A_0 \to B_0 \) making

\[
\begin{align*}
A & \xrightarrow{f} B \\
A_0 \times A_0 & \xrightarrow{d_0 \times d_0} B_0 \times B_0
\end{align*}
\]

commute, and which are compatible with the composition and units of \( A \) and \( B \).

An \((A, B)\)-bimodule \( J \) is given by a span \( d : J \to A_0 \times B_0 \), which can be thought of as an object of ‘\( \mathcal{E} \)-objects indexed by pairs of objects in \( A_0 \) and \( B_0 \)’, with action maps \( l : A \times A_0 J \to J \) and \( r : J \times B_0 B \to J \), over \( A_0 \times B_0 \), that are associative and unital. Bimodules in \( \text{Span}(\mathcal{E}) \) are called internal profunctors (‘internal hom-functors’ in [ML98]). Again profunctors \( J \) that are internal in \( \text{Set} \) are equivalent to the usual profunctors, but are given as a single set \( J \) of morphisms, as opposed to giving a set \( J(a, b) \) of morphisms for each pair of objects \( a \in A \) and \( c \in C \). A cell \( \phi \) as on the left below is given by a morphism \( \phi : J \to K \) in \( \mathcal{E} \) on the right, compatible with actions of the internal categories involved. Such a cell is called an internal transformation of internal profunctors.

\[
\begin{align*}
A & \xrightarrow{J} B \\
C & \xrightarrow{K}
\end{align*}
\]

Just like there is a notion of natural transformations between ordinary functors, there is a notion of ‘internal transformation’ between a pair of internal functors \( f \) and \( g : A \to B \) as well: see (1.10) below. Like in the ordinary case (Example [1.8]), such transformations correspond to vertical cells \( \phi \) between internal profunctors, as we shall see in Proposition [1.22].

If \( \mathcal{E} \) has reflexive coequalisers preserved by pullbacks then \( \text{Span}(\mathcal{E}) \) satisfies the condition of the proposition above, so that categories and profunctors internal to \( \mathcal{E} \) form an equipment, which we shall denote \( \text{Prof}(\mathcal{E}) \). Its horizontal composition
$J \circ_B H$ of internal profunctors $J: A \to B$ and $H: B \to C$ can be given by the reflexive coequaliser of pullbacks

$$J \times_{B_0} B \times_{B_0} H \rightrightarrows J \times_{B_0} H \to J \circ_B H,$$

where the parallel pair of maps is given by the actions of $B$ on $J$ and $H$. The internal unit profunctor $U_A$ is simply $d: A \to A_0 \times A_0$ itself, with actions given by composition. Moreover, the horizontal composite $\phi \circ_g \psi$ of two cells on the left below is given by the unique factorisation on the right.

\[
\begin{array}{ccc}
A & \xrightarrow{J} & B & \xrightarrow{H} & E \\
\downarrow^f & \phantom{\downarrow} \downarrow^\phi & \downarrow^g & \phantom{\downarrow} \downarrow^\psi & \downarrow^h \\
C & \xrightarrow{K} & D & \xrightarrow{L} & F
\end{array}
\quad
\begin{array}{ccc}
J \times_{B_0} H & \xrightarrow{J \circ_B H} & J \circ_B H \\
\downarrow^f \times_{B_0} \phi & \phantom{\downarrow} \downarrow\phi \times_{B_0} \psi & \downarrow^i \phi \circ_g \psi \\
K \times_{D_0} L & \xrightarrow{K \circ_D L} & K \circ_D L
\end{array}
\]

Finally, the proof of the proposition above shows that the cartesian filler $K(f, g)$ of a niche $A \xleftarrow{f} C \xrightarrow{\phi} D \xrightarrow{\psi} B$ in $\text{Prof}(\mathcal{E})$ can be constructed in $\text{Span}(\mathcal{E})$, so the span underlying the profunctor $K(f, g)$ is the pullback $(f_0 \times g_0)^* k$, where $k: K \to C_0 \times D_0$ is the span underlying $K$. The factorisation $f \circ \phi: J \to K(f, g)$ of any internal transformation $\phi$ of the form (1.8), is obtained by factoring its underlying map $\phi: J \to K$ through the pullback $K(f, g)$.

Although internal categories can be considered in any category $\mathcal{E}$ with finite limits, we will mostly be interested in the case of $\mathcal{E}$ being the category $\mathbb{S} = \left[\mathbb{S}^{\text{op}}, \text{Set}\right]$ of presheaves on a small category $\mathbb{S}$.

**Example 1.20.** Let $\mathbb{S}$ be a small category. Since pullbacks of presheaves are given pointwise, it follows that giving the multiplication $m$ and unit $e$ of a monoid $A: A_0 \to A_0$ in $\text{Span}(\mathbb{S})$ is the same as giving the sets $(A_0)_s$ and $A_s$, for each $s \in \text{ob} \mathbb{S}$, the structure of a category. Moreover, the fact that $m$ and $e$ of $A$ are maps of presheaves is equivalent to the two images of each $u: r \to s$ in $\mathbb{S}$, under $A$ and $A_0$, constituting a functor $A_u: A_s \to A_r$. Thus monoids in $\text{Span}(\mathbb{S})$ are categories indexed by $\mathbb{S}$. Similarly a map of monoids $f: A \to B$ is an $\mathbb{S}$-indexed functor, that is a natural transformation $f: A \to B$.

An $(A, B)$-bimodule $J$ in $\text{Span}(\mathbb{S})$ is, by definition, a natural transformation of presheaves $d: J \to A_0 \times B_0$ together with actions $l: A \times A_0 J \to J$ and $r: J \times_{B_0} B \to J$, which are transformations over $A_0 \times B_0$ as well, and satisfy the bimodule axioms. We will call such bimodules $\mathbb{S}$-indexed profunctors. An elementary calculation will show that the coequalisers of $\mathcal{E} = \text{Set}$ are preserved by pullbacks; that is the case for any category of presheaves $\mathcal{E} = \mathbb{S}$ follows since limits and colimits of presheaves are computed pointwise. Hence we can apply Proposition 1.17 so that $\mathbb{S}$-indexed categories and $\mathbb{S}$-indexed profunctors form an equipment $\text{Prof}(\mathbb{S}) = \text{Mod}(\text{Span}(\mathbb{S}))$.

As an example we consider the equipment $\text{Prof}(\mathbb{G}_1)$ of categories and profunctors indexed by $\mathbb{G}_1 = \{0 \Rightarrow 1\}$. Briefly speaking, categories indexed by $\mathbb{G}_1$ are ‘double categories without horizontal compositions’.

**Example 1.21.** A $\mathbb{G}_1$-indexed category $A: \mathbb{G}_1^{\text{op}} \to \text{Cat}$ consists of two categories $A_0$ and $A_1$ together with a pair of functors $L$ and $R: A_1 \to A_0$, that is it consists of the data underlying a category internal in $\text{Cat}$ as in (1.2). Hence, like a double category, we think of the objects $a, a', \ldots$ of $A_0$ as objects of $A$, the morphisms $p: a \to a'$ of $A_0$ as vertical morphisms of $A$, the objects $j$ of $A_1$, with $Lj = a_1$.

\(^{1}\)We should warn the reader that the same terminology is used in literature to mean pseudofunctors $\mathbb{S}^{\text{op}} \to \text{Cat}$, where $\text{Cat}$ is the 2-category of categories, functors and transformations.
and $Rj = a_2$, as horizontal morphisms $j: a_1 \Rightarrow a_2$ of $A$ and, lastly, the morphisms $x: j \Rightarrow k$ of $A_1$, with $Lx = p$ and $Rx = q$, as cells $x$ of $A$ of the form as shown below. The composition of $A_0$ and $A_1$ gives vertical composition of vertical morphisms and cells respectively. In other words, a $\mathcal{G}_1$-indexed category $A$ is a ‘double category without horizontal compositions’. In fact, in Section 3.2 we shall consider the ‘free strict double category’-monad $D$ on the double category of $\mathcal{G}_1$-indexed categories and their profunctors, and a $D$-algebra structure on a $\mathcal{G}_1$-indexed category $A$ will equip $A$ with horizontal compositions.

\[
\begin{array}{c}
a_1 \\ \downarrow p \\ a_1'
\end{array}
\begin{array}{c}
\downarrow x \\
q
\end{array}
\begin{array}{c}
a_2 \\ \downarrow j \\ a_2'
\end{array}
\]

A $\mathcal{G}_1$-indexed functor $f: A \Rightarrow B$ is simply given by two functors $f_0: A_0 \Rightarrow B_0$ and $f_1: A_1 \Rightarrow B_1$ such that $L \circ f_1 = f_0 \circ L$ and $R \circ f_1 = f_0 \circ R$. Thus $f$ maps objects, vertical morphisms and horizontal morphisms, as well as cells, of $A$ to those of $B$, in a way that preserves vertical and horizontal sources and targets, as well as vertical compositions.

A $\mathcal{G}_1$-indexed profunctor $J: A \Rightarrow B$ is given as follows. Its underlying span in $\text{Span}(\mathcal{G}_1)$ consists of sets $J_0$ and $J_1$, together with functions $L$, $R$, $d_0$ and $d_1$, that make both the ‘$L$-square’ and the ‘$R$-square’ in the diagram on the left below commute.

\[
\begin{array}{c}
J_1 \\
\downarrow \downarrow \\
J_0 
\end{array}
\begin{array}{c}
d_1 \\
R \\
d_0
\end{array}
\begin{array}{c}
ob A_0 \times \text{ob} B_0 \\
\times \\
ob A_1 \times \text{ob} B_1
\end{array}
\begin{array}{c}
a_1 \\
\downarrow s \\
b_1
\end{array}
\begin{array}{c}
h \\
\downarrow u \\
\downarrow t
\end{array}
\begin{array}{c}
a_2 \\
b_2
\end{array}
\]

Given a pair of objects $a \in A$ and $b \in B$ we write $J_0(a, b) = d_0^{-1} \{(a, b)\}$, and likewise $J_1(h, k) = d_1^{-1} \{(h, k)\}$ for horizontal morphisms $h: a_1 \Rightarrow a_2$ in $A$ and $k: b_1 \Rightarrow b_2$ in $B$. As with ordinary profunctors (Example 1.3) we shall think of elements $s \in J_0(a_1, b_1)$ as vertical morphisms $s: a_1 \Rightarrow b_1$, and of $u \in J_1(h, k)$, with $Lu = s$ and $Ru = t$, as a cell $u$ as shown on right above. That the sources and targets of $s$, $h$, $u$ and $t$ indeed coincide as drawn follows from the commuting of the two squares on the left above. The left action of $A$ on $J$ is given by functions $\circ: A_1 \times \text{ob} A_1 \rightarrow J_1$, for $i = 0, 1$, that make both squares in the diagram on the left below commute, and that satisfy associativity and unit axioms.

\[
\begin{array}{c}
A_1 \times \text{ob} A_1 \\
\downarrow \downarrow \\
A_0 \times \text{ob} A_0
\end{array}
\begin{array}{c}
\circ \\
L \times_L \downarrow R \times R \\
\downarrow \downarrow L
\end{array}
\begin{array}{c}
\circ \\
J_1 \\
J_0
\end{array}
\]

This lets us ‘precompose’ the morphisms and cells of $A$ with those of $J$: for example we can compose the cell $x$ in $A$ with $u$ above to obtain a cell $u \circ x: j \Rightarrow k$ in $J$. The commuting of the diagram on the left states that these compositions need to be compatible with $L$ and $R$ (e.g. $L(u \circ x) = s \circ p$); of course they also need to be associative and unital. Analogously $J$ comes equipped with an action of $B$, given by functions $\circ: B_1 \times \text{ob} B_1 \rightarrow J_1$ that satisfy similar axioms, while the actions of $A$ and $B$ commute.

Given a second $\mathcal{G}_1$-profunctor $H: B \Rightarrow C$, the horizontal composition $J \circ_B H$ is given by the coequalisers

\[
J_i \times \text{ob} B_i \leftarrow B_i \times \text{ob} B_i \\
\rightarrow H_i \Rightarrow (J \circ_B H)_i,
\]  

(1.9)
for \( i = 0,1 \), where the parallel pair of maps is given by the actions of \( B \) on \( J \) and \( H \). Thus, for an object \( a \) in \( A \) and \( c \) in \( C \), the set of vertical maps \((J \circ_B H)_0(a,c)\) is the set of equivalence classes of formal vertical composites

\[
\phi s \overset{a}{\rightarrow} b \overset{t}{\rightarrow} c, \quad s \in J \text{ and } t \in H,
\]

modulo the usual coend relation, which coequalises post- and precomposition with maps of \( B_0 \). The sets of cells \((J \circ_B H)_1\) are given similarly: they consist of equivalence classes of formal vertical composites of cells.

This leaves the \( \mathbb{G}_1 \)-indexed cells in \( \text{Prof}(\mathbb{G}_1) \), as on the left below, where \( J \) and \( K \) are \( \mathbb{G}_1 \)-indexed profunctors and \( f \) and \( g \) are \( \mathbb{G}_1 \)-indexed functors.

\[
\begin{array}{ccc}
A & \overset{J}{\rightarrow} & B \\
\downarrow{f} & \downarrow{g} & \downarrow{\phi} \\
C & \overset{K}{\rightarrow} & D \\
\end{array}
\]

Such a \( \phi \) is given by a pair of maps \( \phi_i : J_i \rightarrow K_i \), for \( i = 0,1 \), that make the two diagrams on the right above commute, and that are compatible with the actions of \( A, B, C \) and \( D \). This means that \( \phi_0 \) and \( \phi_1 \) map the vertical morphisms and vertical cells of \( J \) to those of \( K \), in a way that is compatible with horizontal and vertical sources and targets, as shown below.

\[
\begin{array}{c}
\phi : (a \overset{h}{\rightarrow} b) \mapsto (fa \overset{\phi s}{\rightarrow} gb) \\
\phi : a_1 \overset{h}{\rightarrow} a_2 \mapsto f_{a_1} \overset{\phi s}{\rightarrow} f_{a_2} \\
b_1 \overset{k}{\rightarrow} b_2 \mapsto \phi s_1 \overset{\phi s_2}{\rightarrow} gb_1 \overset{\phi s_2}{\rightarrow} gb_2
\end{array}
\]

These assignments are compatible with respect to the actions of \( A \) and \( B \) in the usual sense: for example \( \phi(u \circ x) = \phi u \circ \phi x \), for the composite of a cell \( u \) in \( J \) and \( x \) in \( A \).

We now briefly return to the comment in Example 1.13 about the vertical cells in the equipment \( \text{Prof}(\mathcal{E}) \) of profunctors internal to \( \mathcal{E} \). Just like natural transformations between ordinary functors, there is a notion of \textit{internal transformations} \( \phi : f \rightarrow g \) between internal functors \( f \) and \( g : A \rightarrow B \), as defined in e.g. [Str74, page 107]. Such transformations are given by a map \( \phi \) as on the left below, such that the naturality diagram on the right commutes. In the proposition below we shall prove that, like in \( \text{Prof} \) (see Example 1.8), the vertical cells of \( \text{Prof}(\mathcal{E}) \) are exactly such internal transformations.

\[
\begin{array}{ccc}
A_0 & \overset{\phi}{\rightarrow} & B \\
\downarrow{(f_0, g_0)} & \downarrow{d_B} & \\
B_0 \times B_0 & \overset{d_B}{\rightarrow} & B \\
\end{array}
\quad
\begin{array}{c}
A \overset{(f, \phi \circ d_1)}{\rightarrow} B \times B_0 \\
\downarrow{(\phi \circ d_0, g)} & \downarrow{m_B} & \\
B \times B_0 \overset{m_B}{\rightarrow} B
\end{array}
\]

(1.10)

Internal categories, functors and the transformations above form a 2-category \( \text{Cat}(\mathcal{E}) \), where the natural transformations compose as follows. The vertical composition of the transformations \( \phi : f \rightarrow g \) and \( \psi : g \rightarrow h \), where \( f, g, h : A \rightarrow B \), is given by

\[
A_0 \cong A_0 \times_A_0 A_0 \overset{\phi \times \psi}{\rightarrow} B \times B_0 \overset{m_B}{\rightarrow} B,
\]

while the horizontal composite of \( \phi : f \rightarrow g \) and \( \psi : h \rightarrow k \), now with \( f, g : A \rightarrow B \) and \( h, k : B \rightarrow C \), is given by

\[
A_0 \cong A_0 \times A_0 A_0 \overset{\phi \times g_0}{\rightarrow} B \times B_0 \overset{h \times \psi}{\rightarrow} C \times C_0 \overset{m_C}{\rightarrow} C.
\]
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**Proposition 1.22.** Let $\mathcal{E}$ be a category with finite limits. Transformations $\phi$ of internal profunctors (see Example 1.19), that are of the form as on the left below, correspond to maps $\phi_0$ in $\mathcal{E}$, as in the middle, that make the diagram on the right commute.

\[
\begin{array}{ccc}
A & \xrightarrow{f} & A \\
C & \xrightarrow{g} & D
\end{array}
\quad
\begin{array}{ccc}
A_0 & \xrightarrow{\phi_0} & K \\
C_0 \times D_0 & \xrightarrow{\phi} & C \times D
\end{array}
\quad
\begin{array}{ccc}
A & \xrightarrow{(f, \phi_0 \circ d_1)} & C \times D_0 \\
K & \xrightarrow{I} & K
\end{array}
\]  

This correspondence preserves the factorisations through the restriction $K(f, g)$ (again see Example 1.19) in the sense that $(f \phi_0) : A_0 \to K(f, g)$ coincides with the factorisation of $\phi_0$ through the pullback $K(f, g)$. Moreover it preserves vertical composition in the sense that $(\chi \circ \phi)_0 = \chi \circ \phi_0$, for any transformation $\chi$ with horizontal source $K$, and horizontal composition in the sense that

\[
\begin{array}{ccc}
A_0 \times A_0 & \xrightarrow{H \times \phi} & K \times D_0 \\
A \circ_A & \xrightarrow{\phi \circ \psi} & K \circ_D
\end{array}
\]

commutes, for any transformation $\psi : K \Rightarrow L$ with left vertical map $g$.

If $\mathcal{E}$ has coequalisers preserved by pullbacks, so that internal profunctors in $\mathcal{E}$ from an equipment $\text{Prof}(\mathcal{E})$ by Proposition 1.17, then this correspondence induces an isomorphism of 2-categories $V(\text{Prof}(\mathcal{E})) \cong \text{Cat}(\mathcal{E})$.

We will often abuse notation and denote $\phi_0$ by $\phi$ as well; it will always be clear which of the two is meant.

**Proof.** To a cell $\phi$ above, which is given by a map $\phi : A \to K$ in $\mathcal{E}$, we assign the composition $\phi_0 = [A_0 \xrightarrow{\phi} A \xrightarrow{\phi} K]$. That it makes the diagram on the right in (1.11) commute follows from the fact that

\[
\begin{array}{ccc}
id \times \epsilon_A & \xrightarrow{\text{id} \times \epsilon_A} & A \times A_0 \\
A \times A_0 & \xrightarrow{\phi} & A \\
A_0 \times A_0 & \xrightarrow{\phi} & A \\
\epsilon_A \times \text{id} & \xrightarrow{\text{id} \times \epsilon_A} & A \times A_0 \\
A \times A_0 & \xrightarrow{\phi} & A \\
A \times A_0 & \xrightarrow{\phi} & A
\end{array}
\]

commutes, where the commuting squares describe the naturality of $\phi$ and the commuting triangles form the unit axiom for $A$. Conversely a map $\phi_0 : A_0 \to K$ induces a cell $\phi : U_A \Rightarrow K$ given by the composite $A \to K$ of either leg of the naturality diagram (1.11) of $\phi_0$ above. That the resulting map is natural follows from the facts that $f$ and $g$ are compatible with composition and that the actions $l$ and $r$ are associative. That the two assignments $\phi \mapsto \phi_0$ and $\phi_0 \mapsto \phi$ thus given are inverse to each other follows from the commutativity diagram above, together with the fact that $f$ and $l$ (or $g$ and $r$) are compatible with the units of $A, B$ and $C$.

We saw in Example 1.19 that the restriction $K(f, g)$ is the pullback of $K \to C_0 \times D_0$ along $f_0 \times g_0$; that the correspondence above preserves the factorisations
through \(K(f, g)\) in the way as claimed follows directly from the universal property of the pullback \(K(f, g)\). That the diagram for the horizontal composition commutes follows from the commuting of

\[
\begin{array}{ccc}
A_0 \times A_0 & \xrightarrow{\phi \times \eta} & K \times D_0 \\
\downarrow & & \downarrow \\
H & \cong & K \circ D L
\end{array}
\]

where the square on the right commutes by definition of \(\phi \circ \eta \psi\), while the triangles making up the square on the left commute by the unit axiom for \(H\) and the definition of the Yoneda isomorphism \(H \cong A \circ_A H\).

Of course if \(\phi\) is vertical, i.e. \(K = U_B\) from some internal category \(B\), then \(\phi_0\) is an internal transformation as given in \([CS10]\), checking that this induces an isomorphism \(V(\text{Prof}(E)) \cong \text{Cat}(E)\) is straightforward.

Having introduced the main examples, the following propositions record some useful properties of equipments. Some of these can be found in \([Shul8]\); they all follow as specialisations of the Theorems and Corollaries 7.16 – 7.22 of \([CS10]\) for ‘virtual equipments’ to the ordinary equipments that we consider. Remember that the companion \(B(f, \text{id})\) of a map \(f : A \to B\) is defined by two cells \(f \eta\) and \(f \eta\) that satisfy the companion identities; see the discussion following Definition \([1.10]\).

**Proposition 1.23** (Shulman). If \(B(f, \text{id})\) and \(C(g, \text{id})\) are the companions of composable morphisms \(f : A \to B\) and \(g : B \to C\), in an equipment \(\mathcal{K}\), then the compositions of the diagrams below define the composition \(B(f, \text{id}) \circ C(g, \text{id})\) as the companion of \(g \circ f\).

Consequently, any other companion \(C(g \circ f, \text{id})\) of the composite \(g \circ f\) induces a canonical horizontal invertible cell \(B(f, \text{id}) \circ C(g, \text{id}) \Rightarrow C(g \circ f, \text{id})\) that is the factorisation of the cartesian filler above through \(g \circ f \varepsilon\) : \(C(g \circ f, \text{id}) \Rightarrow U_C\).

Analogously the composition \(C(\text{id}, g) \circ B(\text{id}, f)\) can be taken as the conjoint of \(g \circ f\), and any other conjoint \(C(\text{id}, g \circ f)\) for \(g \circ f\) induces a canonical horizontal invertible cell \(C(\text{id}, g) \circ B(\text{id}, f) \Rightarrow C(\text{id}, g \circ f)\).

Notice that, using the unit axiom for \(\mathcal{K}\), the cartesian filler on the left above can also be written as the composition

\[
B(f, \text{id}) \circ C(g, \text{id}) \overset{f \circ \eta}{\Rightarrow} U_B \circ C(g, \text{id}) \overset{\varepsilon}{\Rightarrow} C(g, \text{id}) \overset{\varepsilon}{\Rightarrow} U_C.
\]

**Proof.** It is clear that the companion identities imply that the compositions of the two diagrams above vertically compose to the identity of \(g \circ f\) and horizontally compose to the horizontal identity on \(B(f, \text{id}) \circ C(g, \text{id})\), that is they define \(B(f, \text{id}) \circ C(g, \text{id})\) as the companion of \(g \circ f\). The existence of the canonical isomorphisms follows from the fact that any two companions of \(g \circ f\) are isomorphic (via invertible horizontal cells). \(\square\)
Although the bijective correspondence itself, below, is given by Grandis and Paré in [GP04, Section 1.6] (see also [CS10, Corollary 7.21]), the author has not seen the explicit descriptions of its functoriality before.

**Proposition 1.24** (Grandis and Paré). In any equipment \( K \) there are bijective correspondences between the three sets consisting of all cells that are of the form

\[
\begin{array}{c}
A \xrightarrow{f} C \xrightarrow{g} B \xrightarrow{d} D \\
\end{array} \quad \begin{array}{c}
A \xrightarrow{f} J \\
\end{array} \quad \begin{array}{c}
C \xrightarrow{g} A \xrightarrow{d} B \\
\end{array}
\]

respectively. For a cell \( \phi \) that is of the second form above, we write \( \lambda \phi \) for the corresponding cell of the first form and \( \rho \phi \) for that of the third form; they are given by the compositions

\[
\lambda \phi = A \xrightarrow{f} B \xrightarrow{d} C \xrightarrow{g} D \xrightarrow{\eta} \quad \rho \phi = C \xrightarrow{g} A \xrightarrow{d} B \xrightarrow{\eta}.
\]

The assignments \( \phi \mapsto \lambda \phi \) and \( \phi \mapsto \rho \phi \) preserve horizontal identity cells as well as compositions, the latter in the following sense. For vertically composable cells as on the left below the diagram of horizontal cells on the right commutes, where the canonical cells \( \lambda \circ \) are given by the previous proposition.

\[
\begin{array}{c}
A \xrightarrow{f} B \\
\end{array} \quad \begin{array}{c}
C(f, id) \circ K \circ F(k, id) \\
\end{array} \quad \begin{array}{c}
J \circ D(g, id) \circ F(k, id) \\
\end{array} \quad \begin{array}{c}
C(f, id) \circ E(h, id) \circ L \\
\end{array}
\]

\[ (1.12) \]

If \( \phi \) is a horizontal cell, that is \( f = id_C \) and \( g = id_D \), this reduces to

\[ \lambda(\psi \circ \phi) = \lambda \psi \circ (\phi \circ id) : J \circ D(g, id) \Rightarrow C(f, id) \circ L; \]

likewise when \( \psi \) is a horizontal cell. Secondly, given horizontally composable cells as on the left below, the diagram on the right commutes.

\[
\begin{array}{c}
A \xrightarrow{f} B \\
\end{array} \quad \begin{array}{c}
J \circ M \circ H(l, id) \xrightarrow{\lambda(\phi \circ \chi)} C(f, id) \circ K \circ N \\
\end{array}
\]

\[ (1.13) \]

Analogously for \( \phi \mapsto \rho \phi \) the following hold, where the cells \( \rho \circ \) are given by previous proposition.

\[ \rho(\phi \circ \psi) \circ (\rho \circ id) = (id \circ \rho \circ \phi \circ id) \circ (id \circ \rho \circ id) \]

\[ \rho(\phi \circ \chi) = (id \circ \rho \circ \phi) \circ (\rho \circ id) \]
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Proof. Let \( \psi \) be a horizontal cell \( J \circ D(g, \text{id}) \Rightarrow C(f, \text{id}) \circ K \), i.e. \( \psi \) is of the first form above. It is clear that the companion identities for \( C(f, \text{id}) \) and \( D(g, \text{id}) \) imply that mapping \( \psi \) to the composition of the diagram below gives an inverse to \( \phi \mapsto \lambda \phi \).

Similarly composing a cell, that is of the third form above, with the conjoint cells \( \eta_f \) and \( \varepsilon_g \) gives an inverse to \( \phi \mapsto \rho \phi \).

That the assignments \( \lambda \) and \( \rho \) preserve horizontal identity cells follows from the companion and conjoint identities. To show that \( \lambda \) preserves vertical compositions, consider the cells \( \phi \) and \( \psi \) of (1.12); we will show that the diagram (1.12) commutes.

The composite of \( \lambda \phi \circ \text{id} \) and \( \text{id} \circ \lambda \psi \) in the top leg of the diagram can be written as the composition of the following grid of cells, where the empty cells denote identity cells.

\[
\begin{array}{cccc}
A & B & F \\
\downarrow^f & & \\
A & B & D \quad D & F \\
\downarrow^g & & \\
C & D & F \\
\end{array}
\]

It follows from the definition (see Proposition 1.23) of the canonical cells \( \lambda \circ \phi \) that postcomposing the square of companion cells on the left with \( \lambda \circ g \) gives the companion cell \( h \circ \eta \); in the same way precomposing the companion cell \( k \circ \varepsilon \) in \( \lambda(\psi \circ \phi) \) with \( \lambda \circ \phi \) gives the square of companion cells on the right above. This shows that both legs of the diagram (1.12) are equal.

Now suppose that \( \phi \) in (1.12) is a horizontal cell. In that case postcomposing the left subsquare in the diagram above with \( \lambda \circ \text{id} \) reduces it to the single companion cell \( g \circ \varepsilon \), since both are factorisations of \( U_h \) through \( h \circ \varepsilon \).

Likewise, precomposing the right subsquare with the composite

\[
F(k, \text{id}) \xrightarrow{\text{id}} U_D \otimes F(k, \text{id}) \xrightarrow{\eta_f \circ \text{id}} D(\text{id}, \text{id}) \otimes F(k, \text{id})
\]

reduces it to \( k \circ \varepsilon \). It follows that precomposing the top leg of (1.12) with the above composition gives \( \lambda \phi \circ (\phi \circ \text{id}) \). On the other hand, the universal property of the companion cell \( k \circ \varepsilon \) implies that the composite above, followed by \( \lambda \circ \phi \), is equal to the identity on \( F(k, \text{id}) \), so that the bottom leg of (1.12), precomposed with the composite above, reduces to \( \lambda(\psi \circ \phi) \). We conclude that \( \lambda(\psi \circ \phi) = \lambda \psi \circ (\phi \circ \text{id}) \).

Finally it is easily seen that the diagram (1.13) for the vertically composable cells \( \phi \) and \( \chi \) commutes because, in the vertical composition of \( \text{id} \circ \lambda \chi \) and \( \lambda \phi \circ \text{id} \), the companion cell \( g \circ \varepsilon \) in \( \lambda \phi \) cancels with the companion cell \( g \circ \eta \) in \( \lambda \chi \).

Recall that any equipment \( K \) induces a 2-category \( V(K) \) of morphisms and vertical cells, as well as a bicategory \( H(K) \) of promorphisms and horizontal cells. The
assignments $\phi \mapsto \lambda \phi$ and $\phi \mapsto \rho \phi$ restrict to ‘pseudofunctors’ between $V(K)$ and $H(K)$, as follows. Recall that a pseudofunctor $F: \mathcal{B} \to \mathcal{C}$ between bicategories is a ‘weakly’ Cat-enriched functor, preserving composition and identities only up to coherent invertible cells in $\mathcal{C}$, which are called compositors and unitors (see Definition 1.5.8). A pseudofunctor $F$ is called locally fully faithful whenever it restricts to fully faithful functors $\mathcal{B}(X, Y) \to \mathcal{C}(FX, FY)$ on the morphism-categories. Given a bicategory $\mathcal{B}$ we write $\mathcal{B}^{op}$ and $\mathcal{B}^{co}$ for the bicategories obtained by reversing respectively the morphisms or the cells of $\mathcal{B}$, that is $\mathcal{B}^{op}(X, Y) = \mathcal{B}(Y, X)$ and $\mathcal{B}^{co}(X, Y) = \mathcal{B}(X, Y)^{op}$.

**Proposition 1.25** (Shulman). Choosing a companion $B(f, \text{id})$ for every morphism $f: A \to B$, together with the assignment

$$\phi \mapsto [B(g, \text{id}) \xrightarrow{\text{co}} U_A \circ B(g, \text{id}) \xrightarrow{\lambda \phi} B(f, \text{id}) \circ U_B \xrightarrow{\delta} B(f, \text{id})],$$

for each vertical cell $\phi: f \Rightarrow g$ and where $\lambda \phi$ is given in the previous proposition, gives a pseudofunctor $\lambda: V(K)^{co} \to H(K)$ that restricts to the identity on objects and that is locally fully faithful. Its compositor components

$$\lambda_{\circ}: B(f, \text{id}) \circ C(h, \text{id}) \Rightarrow C(h \circ f, \text{id})$$

are the canonical cells given by Proposition 1.25, while its unitors $U_A \Rightarrow A(\text{id}, \text{id})$ are given by the companion cells $\text{id} \circ \eta$. 

Dually, choosing a conjoint $B(\text{id}, f)$ for each $f: A \to B$ induces a locally fully faithful pseudofunctor $\rho: V(K)^{op} \to H(K)$ that restricts to the identity on objects; its action on a vertical cell $\phi: f \Rightarrow g$ is given by

$$\phi \mapsto [B(\text{id}, f) \xrightarrow{\text{co}} B(\text{id}, f) \circ U_A \circ \rho \phi \xrightarrow{\delta} U_B \circ B(\text{id}, g) \xrightarrow{\text{co}} B(\text{id}, g)].$$

We are abusing notation by denoting the composition (1.14) again by $\lambda \phi$. This will not lead to confusion however, since we will never (except in the proof below) use the original cell $\lambda \phi: U_A \circ B(\text{id}, \text{id}) \Rightarrow B(f, \text{id}) \circ U_B$, as given by the previous proposition, when $\phi$ is vertical: it is far more natural to consider its composition with the unitors, as given above. Since in the proof below we will use both the original and the new definition of $\lambda \phi$, we will denote (1.14) temporarily by $\lambda' \phi$.

**Proof.** To see that the action of $\lambda'$ on vertical cells preserves vertical composition, consider composable vertical cells $\phi: f \Rightarrow g$ and $\chi: g \Rightarrow h$, where $f$, $g$, and $h$ are morphisms $A \to B$; we need

$$\lambda'(\tau \circ (\phi \circ \chi) \circ \Gamma^{-1}) = \lambda' \phi \circ \lambda' \chi: B(h, \text{id}) \Rightarrow B(f, \text{id})$$

(1.15)

where, by definition of $V(K)$, the left-hand side is the image of the vertical composite of $\phi$ and $\chi$. Under the isomorphisms $U_A \otimes B(h, \text{id}) \cong B(h, \text{id})$ and $B(f, \text{id}) \circ U_B \cong B(f, \text{id})$, given by the unitors of $K$, the left-hand side equals

$$U_A \circ U_A \circ U_A \circ B(h, \text{id}) \xrightarrow{\text{co} \circ \phi \circ \chi \circ \delta} B(f, \text{id}) \circ U_B \circ U_B \circ U_B,$$

while the right-hand side, under the isomorphisms $U_A \otimes B(h, \text{id}) \cong B(h, \text{id})$ and $B(f, \text{id}) \circ U_B \cong B(f, \text{id})$, equals the composite

$$U_A \circ U_A \circ U_A \circ B(h, \text{id}) \xrightarrow{\text{co} \circ \phi \circ \chi \circ \delta} U_A \circ U_A \circ B(g, \text{id}) \circ U_B \circ U_B \xrightarrow{\text{co} \circ \phi \circ \chi \circ \delta} B(f, \text{id}) \circ U_B \circ U_B \circ U_B \circ U_B.$$
Cancelling \( g \eta \) against \( g \varepsilon \) here we see that the two cells above coincide up to composition with unitors, so that the equality (1.15) follows.

Given vertically composable vertical cells \( \phi \) and \( \psi \), the fact that the compositors \( \lambda \odot \) are natural, that is \( \lambda \odot (X \phi \odot X \psi) = X (\psi \circ \phi) \circ \lambda \odot \), is equivalent to the commuting, for \( \phi \) and \( \psi \), of the diagram (1.12) of the previous proposition. That the compositors satisfy the associativity axiom follows from the naturality of the associator \( a \) and the uniqueness of factorisations through cartesian fillers. Finally it is easy to show that the cells \( \text{id} \eta: U_A \Rightarrow \text{id} \) can be taken as the unitors for \( \lambda \).

**Example 1.26.** Recall, from Example 1.8, that a vertical cell \( \phi: f \Rightarrow g \) between functors \( f: A \rightarrow B \) and \( g: A \rightarrow B \), in the equipment \( \text{Prof} \) of (unenriched) profunctors, is simply an ordinary natural transformation \( \phi: f \Rightarrow g \). In this case the corresponding horizontal cell \( \lambda \phi: B(g, \text{id}) \Rightarrow B(f, \text{id}) \) is given by the composite

\[
B(g, \text{id}) \cong U_A \odot_A U_A \odot_A B(g, \text{id}) \cong B(f, \text{id}) \odot_B U_B \odot_B U_B \cong B(f, \text{id}).
\]

Recalling the definition of the Yoneda isomorphisms here (Example 1.3) we find that under this composite a map \( s: ga \rightarrow b \) in \( B(ga, b) \) is mapped as \( s \mapsto \text{id}_a, \text{id}_a, s \mapsto (\text{id}_f, \phi_a, s) \mapsto s \circ \phi_a \), where \( s \circ \phi_a \) is contained in \( B(fa, b) \). Thus \( \lambda \phi \) is given by precomposition with the components of \( \phi \). In the case of enriched profunctors \( \mathcal{V} \text{-Prof} \) the action of \( \lambda \) on \( \mathcal{V} \)-natural transformations is given similarly.

We end this section with a definition that will be crucial later on, as explained by the remark following it.

**Definition 1.27.** In an equipment \( K \) consider a cell \( \phi \), together with its corresponding horizontal cells \( \lambda \phi \) and \( \rho \phi \) as given by Proposition 1.24. We say that \( \phi \) is *left invertible* if \( \lambda \phi \) has an inverse in \( H(K) \); if \( \rho \phi \) has an inverse then we say that \( \phi \) is *right invertible*.

**Remark 1.28.** Once we have defined what a monad \( T \) on an equipment \( K \) is, in Definition 3.12 a ‘strict’ \( T \)-algebra \( A \) will be defined as usual: it is an object \( A \) together with a structure morphism \( a: TA \rightarrow A \), satisfying associativity and unit axioms. Given a second \( T \)-algebra \( B \), a ‘lax \( T \)-promorphism’ \( J \) can then be defined (see Definition 4.1) as a promorphism \( J: A \Rightarrow B \) equipped with a ‘lax structure cell’

\[
\begin{array}{ccc}
TA & \xrightarrow{\lambda} & TB \\
\downarrow a & & \downarrow b \\
A & \xrightarrow{J} & B,
\end{array}
\]

satisfying associativity and unit axioms; compare the (well-known) definition of a colax \( T \)-morphism (Definition 3.20). Usually an object with some ‘lax structure cell’ is called ‘pseudo’ whenever the structure cell is invertible. In this case however, asking that \( J \) be invertible in \( K_1 \) is clearly too strong: it would mean that the structure maps \( a \) and \( b \) are isomorphisms. The solution to this, it will turn out, is to consider ‘left pseudo lax’ and ‘right pseudo lax’ \( T \)-promorphisms instead, for which \( J \) is respectively left or right invertible.

**Example 1.29.** A cell of \( \mathcal{V} \)-matrices

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow \phi & & \downarrow \eta \\
C & \xrightarrow{k} & D
\end{array}
\]
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Corresponds to the horizontal cell \( \rho \phi : C(\text{id}, f) \circ J \Rightarrow K \odot D(\text{id}, g) \) that is given by

\[
(C(\text{id}, f) \circ J)(c, b) = \prod_{fa = c} J(a, b)^{j} K(c, gb) = (K \odot D(\text{id}, g))(c, b);
\]

thus \( \phi \) is right invertible if and only if the maps \( \prod_{fa = c} \phi_{a,b} \) are isomorphisms, for all \( b \in B \) and \( c \in C \).

**Example 1.30.** In \( \text{Span}(\mathcal{E}) \) the horizontal cell \( \rho \phi : C(\text{id}, f) \times_J J \Rightarrow K \times_D D(\text{id}, g) \), corresponding to a cell \( \phi \) as above, is given by the morphism of spans on the left below. Here \( K \times_D D \) is the pullback of \( k_D \) and \( g \), with projections \( p \) and \( q \).

\[
\begin{array}{ccc}
A & \xrightarrow{J} & B \\
f \downarrow \phi & & \downarrow g \\
C & \xrightarrow{K \times_D D} & B \\
\end{array}
\]

We conclude that \( \phi \) is right invertible if and only if the square on the right above is a pullback square.

The final result of this section, which seems to be new, concerns the right invertibility of cells between bimodules. It is the first of several results that reduce statements about bimodules in \( \text{Mod}(\mathcal{K}) \) to statements about the underlying promorphisms in \( \mathcal{K} \), which are usually easier to check.

**Proposition 1.31.** Let \( \mathcal{K} \) be an equipment that satisfies the condition of Proposition 1.17 so that bimodules in \( \mathcal{K} \) form an equipment \( \text{Mod}(\mathcal{K}) \). A cell of bimodules \( \phi \), on the left below, is right invertible whenever the underlying cells \( f : A \Rightarrow C \) and \( \phi : J \Rightarrow K \), on the right, are right invertible in \( \mathcal{K} \).

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow \phi & & \downarrow j \circ A_0 \\
C & \xrightarrow{\mathcal{K}} & D \\
\end{array}
\]

\[
\begin{array}{ccc}
A_0 & \xrightarrow{A} & A_0 \\
f_0 & \downarrow f_0 & \downarrow \phi \\
C_0 & \xrightarrow{\mathcal{K}} & C_0 \\
\end{array}
\]

\[
\begin{array}{ccc}
A_0 & \xrightarrow{J} & B_0 \\
f_0 & \downarrow \phi \circ g_0 & \downarrow j_0 \\
C_0 & \xrightarrow{\mathcal{K}} & D_0 \\
\end{array}
\]

**Proof.** To compute \( \rho \phi = \varepsilon_f \circ f \phi \odot g \eta_0 \) we will use the fact that cartesian fillers (and thus companion and conjoint cells) in \( \text{Mod}(\mathcal{K}) \) can be computed in \( \mathcal{K} \), and likewise the factorisations through them, as was described in the proof of Proposition 1.17. Thus \( \phi' = \phi \odot g_0 \eta_0 \), which is the factorisation, in \( \text{Mod}(\mathcal{K}) \), of \( \phi \) through the restriction \( K(\text{id}, g) \), is computed in \( \mathcal{K} \) as the factorisation of \( \phi \) through \( K(\text{id}, g_0) \): we conclude that the cell \( \phi \circ g_0 \eta_0 \) of \( \mathcal{K} \) underlies the cell \( \phi' \) of \( \text{Mod}(\mathcal{K}) \). On the other hand the cell underlying \( \varepsilon_f \) is the cartesian filler of the niche \( C_0 \xrightarrow{\text{id}} C_0 \xrightarrow{g_0} C_0 \xrightarrow{f_0} A_0 \) which, by Proposition 1.17, we can take to be the composite \( \text{id}_C \odot \varepsilon_{f_0} \).

We recall from the proof of Proposition 1.17 that the composite \( \rho \phi = \varepsilon_f \odot f \phi' \) is defined by the following diagram in \( H(\mathcal{K}) \), where both rows are coequalisers.

\[
\begin{array}{ccc}
C(\text{id}, f) \odot A \circ J & \xrightarrow{\varepsilon_f \odot f \phi'} & C(\text{id}, f) \odot J \\
\downarrow r \odot \text{id} & & \downarrow r \odot f \phi' \\
C \odot C \odot K(\text{id}, g) & \xrightarrow{\varepsilon_f \odot f \phi'} & C \odot K(\text{id}, g) \\
\end{array}
\]

As we have seen, the cell underlying \( \varepsilon_f \odot f \phi' \) is the composite \( \text{id}_C \odot \varepsilon_{f_0} \odot \phi \odot \eta_0 = \text{id}_C \odot \rho \phi_0 \), which is invertible by assumption. Likewise the cell underlying \( \varepsilon_f \odot f \odot
$\phi'$ is the composite $\id_C \circ \varepsilon_{f_0} \circ f \circ \phi \circ \eta_{g_0} = \id_C \circ \rho(f \circ \phi)$. The latter is also invertible because $\rho(f \circ \phi)$ can be written as a vertical composite of $\rho f$ and $\rho \phi$ by Proposition 1.24, both of which are invertible by assumption. Hence in the diagram above the two cells, that are drawn both solidly and vertically, are invertible and it follows that the dashed cell, which equals $\rho \phi$, is invertible as well. This concludes the proof. \hfill \Box
Chapter 2

Weighted colimits

Having introduced equipments in the previous chapter we can now recall the notion of weighted colimits in such equipments. These are the main objects of our study and include, for example, ‘pointwise’ left Kan extensions. Weighted colimits can be introduced in two ways, both of which will be given here. The first way goes back to Wood’s original paper [Woo82], and defines weighted colimits in so-called ‘closed’ equipments—equipments equipped with extra structure generalising that of closed monoidal categories. For well-behaved $\mathcal{V}$ and $\mathcal{E}$, the equipments $\mathcal{V}$-$\text{Prof}$ and $\text{Prof}(\mathcal{E})$, of $\mathcal{V}$-enriched profunctors and internal profunctors in $\mathcal{E}$, are closed, and this first approach to defining weighted colimits is very close to how weighted colimits are defined in the 2-category $\mathcal{V}$-$\text{Cat}$ of $\mathcal{V}$-categories. In particular, colimits in $\mathcal{V}$-$\text{Prof}$ that are ‘weighted by companions’ are precisely the ‘enriched’ left Kan extensions in $\mathcal{V}$-$\text{Cat}$ that were introduced by Dubuc [Dub70, Section I.4]. Satisfying a stronger universal property than ordinary Kan extensions, this is the usual notion of Kan extensions of enriched functors, see for example Kelly’s book [Kel82, Formula 4.20].

The second approach uses the notion of ‘left Kan extensions in a pseudo double category’, as given by Grandis and Paré in [GP18, Section 2], which generalises the classical notion of Kan extension, in ordinary 2-categories. We will show that, in a closed equipment, a certain type of such left Kan extensions (which we will call ‘top absolute’ extensions) coincide with the weighted colimits as they were defined in the first approach. Using this alternative definition has several advantages. Firstly it allows us to define weighted colimits in pseudo double categories, which are much more common than equipments (even more so than closed equipments). This turns out to be very useful later on, because the pseudo double categories $T$-$\text{Prom}_{\text{rc}}$ of ‘algebraic promorphisms’, for a monad $T$ on an equipment (Definition 4.13), will, when $T$ is not a pseudomonad, in general not be an equipment (see Proposition 4.25).

Secondly we can follow Grandis and Paré and, using their ‘double comma objects’, introduce a notion of ‘pointwise’ weighted colimits, generalising Street’s notion of pointwise left Kan extensions in 2-categories, that was introduced in [Str74]. Just like Dubuc’s notion of ‘enriched’ extensions is the right notion when considering enriched functors, so is Street’s notion of ‘pointwise’ extensions the right one for internal functors. In fact Street’s notion can be used for enriched functors as well, but is too strong in general: it is easy to give a pair of 2-functors (i.e. $\text{Cat}$-enriched) whose enriched left Kan extension exists, but whose pointwise left Kan extension does not, see Example 2.24. Returning to weighted colimits in equipments, the main idea of this chapter is to, in the language of equipments, describe a property of double comma objects whose failure to hold can be thought of as the cause of the difference between pointwise and ordinary weighted colimits. Calling double comma objects with this property ‘strong’ the main result (Theorem 2.37) of this chapter states that if all double comma objects in an equipment $K$ are strong then
all weighted colimits in $\mathcal{K}$ are pointwise. Since double comma objects in $\text{Prof}(\mathcal{E})$ are strong (Proposition 2.30), we conclude that the notion of weighted colimits in equipments unifies the ‘right notions’ of weighted colimits in $\mathcal{V}$-$\text{Prof}$ and $\text{Prof}(\mathcal{E})$: in the first they generalise the notion of enriched left Kan extension while in the second they generalise the notion of pointwise left Kan extension. Later we will prove that, given a monad $T$ on an equipment $\mathcal{K}$, the equipment of ‘algebraic profunctors’ $T$-$\text{Prom}_{\text{rc}}$ has strong comma objects whenever $\mathcal{K}$ does, so that in that case the weighted colimits in $T$-$\text{Prom}_{\text{rc}}$ are also pointwise; see Theorem 4.30.

## 2.1 Weighted colimits in closed equipments

We start with the notion of closed equipment, which generalises that of a closed monoidal category. The main ideas here go back to Wood’s original paper [Woo82], while the author was introduced to them by the nLab article on equipments [Shu11], which is mostly written by Shulman. Nothing in this section is new.

Let $\mathcal{B}$ be a bicategory, with horizontal composition denoted $\circ$. Recall that $\mathcal{B}$ is called closed if for any morphism $H : B \to C$ the functors

$$H \circ - : \mathcal{B}(C,D) \to \mathcal{B}(B,D)$$

and

$$- \circ H : \mathcal{B}(A,B) \to \mathcal{B}(A,C)$$

both have right adjoints, in that case denoting them by $H \leftarrow -$ and $- \triangleright H$ respectively. This means that, for morphisms $J : A \to B$, $H : B \to C$ and $K : C \to C$, there are correspondences of cells

$$\mathcal{B}(B,C)(H, J \triangleleft K) \cong \mathcal{B}(A,C)(J \circ H, K) \cong \mathcal{B}(A,B)(J, K \triangleright H) \quad (2.1)$$

of which, on first sight, the first is natural in $H$ and $K$, while the second is natural in $J$ and $K$. However, the right adjoints can be made into functors

$$\triangleright : \mathcal{B}(A,B)^{op} \times \mathcal{B}(A,C) \to \mathcal{B}(B,C)$$

and

$$\leftarrow : \mathcal{B}(A,C) \times \mathcal{B}(B,C)^{op} \to \mathcal{B}(A,B)$$

in the usual way, and with respect to these both the correspondences above are natural in each of the variables $J$, $H$ and $K$. For example, given cells $\phi : H \Rightarrow J$ and $\psi : K \Rightarrow L$, their image $\phi \triangleright \psi : J \triangleright K \Rightarrow H \triangleright L$ is the adjoint of the composite

$$H \circ (J \triangleright K) \xrightarrow{\phi \circ id} J \circ (J \triangleright K) \xrightarrow{\text{ev}_{\phi}} K \xrightarrow{\psi} L.$$

Here $\text{ev}_{\phi} : J \circ (J \triangleright K) \Rightarrow K$ denotes the counit, called evaluation, of the adjunction $J \circ - \dashv J \triangleright -$; the unit, coevaluation, will be denoted $\text{coev}_{\phi} : H \Rightarrow J \triangleright (J \circ H)$. Likewise the counit and unit of $- \circ H \dashv - \triangleright H$ are given by cells $\text{ev}_{\psi} : (K \triangleright H) \circ H \Rightarrow K$ and $\text{coev}_{\psi} : J \Rightarrow (J \circ H) \triangleright H$. Often we will suppress the subscripts $\triangleright$ and $\leftarrow$ from these notations and simply write ev and coev, when no confusion can arise. Finally, we will call $J \triangleright K$ the left hom of $J$ and $K$, while $K \triangleright H$ will be called the right hom.

Recall that every pseudo double category $\mathcal{K}$ contains an underlying bicategory $H(\mathcal{K})$ of horizontal morphisms and horizontal cells.

**Definition 2.1.** A pseudo double category $\mathcal{K}$ is called closed whenever $H(\mathcal{K})$ is closed.

**Example 2.2.** The equipment $\mathcal{V}$-$\text{Mat}$ is closed whenever $\mathcal{V}$ is a closed symmetric monoidal category that has products. In that case the left hom $J \triangleright K$ of $\mathcal{V}$-matrices $J : A \Rightarrow B$ and $K : A \Rightarrow C$ can be given by

$$(J \triangleright K)(b,c) = \prod_{a \in A} [J(a,b), K(a,c)],$$
2.1. WEIGHTED COLIMITS IN CLOSED EQUIPMENTS

where $[-,-]$ denotes the inner hom functor of $\mathcal{V}$; the right homs are given similarly. To show that with these definitions $B = H(\mathcal{V}-\textbf{Mat})$ satisfies the correspondences (2.1) notice that, for any $H: B \rightarrow C$, a map of $\mathcal{V}$-matrices $\phi: J \circ H \Rightarrow K$ is exactly a family of maps $\phi_{a,b}: J(a,b) \otimes H(b,c) \rightarrow K(a,c)$, for $a \in A$, $b \in B$ and $c \in C$. Under the tensor-hom adjunction of $\mathcal{V}$, this corresponds to a family $\phi^\triangleright_{a,b,c}: H(b,c) \rightarrow [J(a,b), K(a,c)]$, forming a map $\phi^\triangleright: H \Rightarrow J \bullet K$.

To give a condition under which the equipment $\text{Span}(\mathcal{E})$ of spans in $\mathcal{E}$ is closed, where $\mathcal{E}$ has finite limits, notice that the slice categories $\mathcal{E}/X$, with $X \in \text{ob}\mathcal{E}$, are cartesian: the products of $\mathcal{E}/X$ are pullbacks in $\mathcal{E}$, over $X$, while the identity map on $X$ is the terminal object of $\mathcal{E}/X$. The category $\mathcal{E}$ is called locally cartesian closed whenever all its cartesian slice categories are cartesian closed. In that case each functor $f^*: \mathcal{E}/Y \rightarrow \mathcal{E}/X$, given by pullback along any $f: X \rightarrow Y$ in $\mathcal{E}$, has, besides a left adjoint $f_*$, given by postcomposition with $f$, a right adjoint $\prod_j: \mathcal{E}/X \rightarrow \mathcal{E}/Y$ as well (see [MLM92, Theorem I.9.4]).

**Example 2.3.** The horizontal bicategory underlying the equipment $\text{Span}(\mathcal{E})$ is the bicategory of spans and, as we saw in Lemma 1.7, horizontal composition with a span $j: J \rightarrow A \times B$ can be given as

$$J \circ - = (J_A \times \text{id}_C)_* \circ (J_B \times \text{id}_C)^*: \mathcal{E}/B \times C \rightarrow \mathcal{E}/A \times C.$$  

It follows from the discussion above that a right adjoint to this functor can be given by $J \triangleleft = \prod_{j_A \times \text{id}_C} \circ (J_A \times \text{id}_C)^*$, whenever $\mathcal{E}$ is locally cartesian closed. For $h: H \rightarrow B \times C$ the right homs $\triangleright H$ can be constructed in the same way, as the roles of $J$ and $H$ in $J \circ H$ are completely symmetric. Hence $\text{Span}(\mathcal{E})$ is closed if $\mathcal{E}$ is locally cartesian closed.

The equipment $\text{Mod}(\mathcal{K})$ of monoids and bimodules (see Definition 1.16) in a closed equipment $\mathcal{K}$ is often again closed by the following result, which is [Shulman] Proposition 11.16.

**Proposition 2.4** (Shulman). Let $\mathcal{K}$ be a closed equipment such that every category $H(\mathcal{K})(A,B)$ has reflexive coequalisers and equalisers. Then $\text{Mod}(\mathcal{K})$ is again closed.

**Sketch of the proof.** The fact that $\mathcal{K}$ is closed means that the reflexive coequalisers in $H(\mathcal{K})(A,B)$ are preserved by $\triangleright$ on both sides so that, by Proposition 1.17 the monoids and bimodules in $\mathcal{K}$ form an equipment $\text{Mod}(\mathcal{K})$. Given bimodules $J: A \Rightarrow B$ and $K: A \Rightarrow C$, their left hom $J \triangleleft K$ can be constructed as the equaliser

$$J \triangleleft K \rightarrow J \bullet K \rightrightarrows (A \circ J) \triangleleft K$$

in $H(\mathcal{K})(B_0, C_0)$, where the pair of parallel morphisms are induced by the action of $A$ on $J$ and $K$ respectively. \hfill $\square$

**Example 2.5.** If $\mathcal{V}$ is a closed symmetric monoidal category that is both complete and cocomplete then the equipment $\mathcal{V}$-$\textbf{Prof}$ of $\mathcal{V}$-profunctors is closed by the previous proposition: the left hom $J \triangleleft K$ of $\mathcal{V}$-profunctors $J: A \Rightarrow B$ and $K: A \Rightarrow C$ is given by the equalisers

$$(J \triangleleft K)(b,c) \rightarrow \prod_{a \in A} [J(a,b), K(a,c)] \rightleftharpoons \prod_{a_1, a_2 \in A} [A(a_1, a_2) \otimes J(a_2, b), K(a_1, c)],$$

where the pair of maps are induced by the action of $A(a_1, a_2)$ on $J(a_2, b)$ and $K(a_2, c)$ respectively; shortly this is written as the end

$$J \triangleleft K = \int_{a \in A} [J(a,-), K(a,-)].$$

\footnote{It is customary to denote the cell $H \Rightarrow J \circ K$, that corresponds to $\phi: J \circ H \Rightarrow K$ under (2.1), by $\phi^\triangleright$. Likewise, the cell $J \circ H \Rightarrow K$ corresponding to $\psi: H \Rightarrow J \circ K$ is denoted $\psi^\triangleright$.}
Notice that, by definition (see [Kel82, Formula 2.10]), this means that \((J \triangleleft_A K)(b, c)\) is the object of \(\mathcal{V}\)-natural transformations \(J(-, b) \to K(-, c)\):

\[
(J \triangleleft_A K)(b, c) = \{A, \mathcal{V}|J(-, b), K(-, c)\}.
\] (2.2)

The right homs are given similarly.

The symmetric monoidal category \(2 = (\bot \to \top)\) is closed with respect to implication, that is

\[
[x, y] = \begin{cases} 
\bot & \text{if } (x, y) = (\top, \bot); \\
\top & \text{otherwise.}
\end{cases}
\]

Notice that equalisers in \(2\) are always trivial, since \(2\) consists of a single non-identity map. Thus, the left hom \(J \triangleleft K\) of \(2\)-profunctors \(J : A \to B\) and \(K : A \to C\) (see Example [13]) is given by

\[
(J \triangleleft K)(y, z) = \bigwedge_{x \in A} [J(x, y), K(x, z)],
\]

in other words \(x \sim_{J \triangleleft K} z\) if and only if \(x \sim_J y\) implies \(x \sim_K z\) for all \(x \in A\).

Before defining weighted colimits in closed equipments, we record some useful properties of the adjoints \(\triangleleft\) and \(\triangleright\).

**Proposition 2.6.** Given composable horizontal morphisms \(A \xrightarrow{H} B \xleftarrow{K} C \xrightarrow{D} D\) in a closed pseudo double category \(\mathcal{K}\), as well as \(M : A \to D\), the following pairs of horizontal morphisms are naturally isomorphic in \(H(\mathcal{K})\).

- \(K \triangleleft (H \triangleleft M) \cong (H \circ K) \triangleleft M\)
- \(H \triangleleft (M \triangleright L) \cong (H \triangleleft M) \triangleright L\)
- \((M \triangleright L) \triangleright K \cong M \triangleright (K \circ L)\)

**Proof.** Using the Yoneda lemma, these follow directly from applying the isomorphisms \([2.13]\), once or twice, to the sides of the natural isomorphism below, that is given by precomposition with the associator for horizontal composition.

\[
H(\mathcal{K})(A, D)((H \circ K) \circ L, M) \cong H(\mathcal{K})(A, D)(H \circ (K \circ L), M)
\]

The following is [Shul08, Proposition 5.11].

**Proposition 2.7** (Shulman). For a niche \(A \xrightarrow{J} C \xleftarrow{K} D \xrightarrow{B} B\) in a closed equipment \(\mathcal{K}\), the following triples of promorphisms are naturally isomorphic in \(H(\mathcal{K})\).

- \(C(f, \text{id}) \circ K \cong K(f, \text{id}) \cong C(\text{id}, f) \triangleleft K\)
- \(K \circ D(\text{id}, g) \cong K(\text{id}, g) \cong K \triangleright D(g, \text{id})\)

From left to right, these isomorphisms are given by the horizontal cells that are adjoint to the counits of the companion-conjoint adjunction, see Proposition [1.14].

Combining them gives isomorphisms

\[
C(f, \text{id}) \circ K \circ D(\text{id}, g) \cong K(f, g) \cong C(\text{id}, f) \triangleleft K \triangleright D(g, \text{id}).
\]

**Proof.** We will only treat the first triple, the other being dual. The first isomorphism \(C(f, \text{id}) \circ K \cong K(f, \text{id})\) is given by Proposition [1.12]. That \(C(f, \text{id}) \circ K \cong C(\text{id}, f) \triangleleft K\) holds as well follows from the correspondences below, where \(J : A \to D\) is any promorphism. They are given by the companion-conjoint adjunction and the composition-left hom adjunction respectively.

\[
H(\mathcal{K})(A, D)(J, C(f, \text{id}) \circ K) \cong H(\mathcal{K})(C, D)(C(\text{id}, f) \circ J, K)
\]

\[
\cong H(\mathcal{K})(A, D)(J, C(\text{id}, f) \triangleleft K)
\]

Proof. We will only treat the first triple, the other being dual. The first isomorphism \(C(f, \text{id}) \circ K \cong K(f, \text{id})\) is given by Proposition [1.12]. That \(C(f, \text{id}) \circ K \cong C(\text{id}, f) \triangleleft K\) holds as well follows from the correspondences below, where \(J : A \to D\) is any promorphism. They are given by the companion-conjoint adjunction and the composition-left hom adjunction respectively.
Combining the previous propositions we obtain the following corollary. These isomorphisms will be used extensively.

**Corollary 2.8.** Given promorphisms $H : A \to B$, $K : A \to C$ and $L : C \to D$ in a closed equipment $K$, as well as morphisms $f : A \to C$ and $g : D \to B$, the following natural isomorphisms exist in $H(K)$.

- $H \circ (C(f, id) \circ L) \cong (C(id, f) \circ H) \circ L$
- $B(g, id) \circ (H \circ L) \cong (H \circ B(id, g)) \circ K$

The first is adjoint to evaluation followed by the counit, as follows from Proposition 1.7, while the second is adjoint to $g \circ g$ followed by evaluation.

In a closed equipment weighted colimits and weighted limits can be defined as follows.

**Definition 2.9.** Let $K$ be a closed equipment. Given a promorphism $J : A \to B$ and a morphism $d : A \to M$ in $K$, the $J$-weighted colimit of $d$, if it exists, is a morphism $\text{colim}_J d : B \to M$ together with an isomorphism

$$M(\text{colim}_J d, id) \cong J \circ M(d, id)$$

in $H(K)$, of promorphisms $B \to M$. Likewise, given a morphism $e : B \to M$, the $J$-weighted limit of $e$, if it exists, is a morphism $\text{lim}_J e : A \to M$ together with an isomorphism

$$M(id, \text{lim}_J e) \cong M(id, e) \circ J$$

in $H(K)$, of promorphisms $M \to A$.

We will call the pair of morphisms $M \leftarrow d A \to B$ above a colimit diagram in $K$; the pair $A \to B \to M$ is called a limit diagram.

**Example 2.10.** Given a $V$-profunctor $J : A \to B$ and a $V$-functor $d : A \to M$ in $V$-$\text{Prof}$, the $J$-weighted colimit of $d$ is the $V$-functor $\text{colim}_J d : B \to M$ satisfying

$$M(\text{colim}_J d(b), m) \cong [A^{op}, V](J(-, b), M(d-, m)), \quad (2.3)$$

natural in both $b$ and $m$, as follows from (2.2), where the $V$-object on the right is that of $V$-natural transformations $J(-, b) \to M(d-, m)$. This means that at each $b$ the image $\text{colim}_J d(b)$ is the usual $J(-, b)$-weighted colimit of $d$ in $M$, see [Kel82 Formula 3.5]. Conversely if all weighted colimits $\text{colim}_J J(-, b) d$ exist in $M$, then $\text{colim}_J d$ exists and can be given by $\text{colim}_J d(b) = \text{colim}_J J(-, b) d$. This follows from the fact that the assignment $(W, d) \mapsto \text{colim}_d W$ for pairs consisting of a $V$-weight $W : A^{op} \to V$ and $V$-diagram $d : A \to M$ whose weighted colimit exists, is functorial, see [Kel82 Formula 3.11]. Thus $\text{colim}_J d$ can be thought of as being a ‘parametrised weighted colimit’.

It follows that $\text{colim}_J d$ can be given as

$$\text{colim}_J d(b) = \int_{a \in A} da \otimes J(a, b), \quad (2.4)$$

in terms of coends of copowers in $M$, see [Kel82 Formula 3.70], whenever such colimits exist in $M$. Here $da \otimes J(a, b)$ is the copower of $da$ by $J(a, b)$, defined by an isomorphism $M(da \otimes J(a, b), m) \cong [J(a, b), M(da, m)]$ that is natural in $m$.

In the case that $B = 1$, the unit $V$-category consisting of a single object $*$ and morphism object $1(*, *) = 1$, the $V$-profunctor $J$ is simply a $V$-functor $J : A^{op} \to V$,
under the isomorphism $A^{op} \cong 1 \cong A^{op}$, and \( \text{colim}_J d : 1 \to M \) is the ordinary \( J \)-weighted colimit of \( d \).

If \( J \) is the companion \( B(j, \text{id}) \) of a \( \mathcal{V} \)-functor \( j : A \to B \) then the defining isomorphism becomes

\[
M(\text{colim}_{B(j, \text{id})} d(b), m) \cong [A^{op}, \mathcal{V}](B(j \cdot, b), M(d \cdot, m)),
\]

which is exactly the identity defining \( \text{colim}_{B(j, \text{id})} d \) as the \textit{enriched} left Kan extension of \( d \) along \( j \), in the sense of [Kel82 Formula 4.20], where they are simply called ‘left Kan extensions’. Enriched left Kan extensions are in particular (ordinary) left Kan extensions, see e.g. [Kel82 Theorem 4.43], and were first introduced (in the case of \( M \) having \( \mathcal{V} \)-copowers) by Dubuc in his lecture notes [Dun70 Section 1.4], where they are called ‘pointwise’ Kan extensions. We shall however reserve the prefix ‘pointwise’ for a different notion of left Kan extension, given by Street, which is generally stronger than Dubuc’s notion.

If \( J : A \to B \) is a 2-profunctor and \( d : A \to M \) an order preserving map then the defining isomorphism (2.3) is equivalent to

\[
\text{colim}_J d(y) = \sup \{dx : x \sim_J y \}.
\]

In the case that \( J = B(j, \text{id}) \) we recover the description of the left Kan extension of a pair of order preserving maps that was given in the introduction.

So far we have recovered both ordinary weighted colimits and pointwise left Kan extensions as weighted colimits in \( \mathcal{V} \text{-} \text{Cat} \); compare the unenriched situation as described in Table 1 of the introduction. To see that ordinary (conical) colimits can be recovered as well, recall that the assignment \( M \mapsto [M] \) mapping a \( \mathcal{V} \)-category \( M \) to its underlying category \( [M] \) has a left adjoint \( C \mapsto \mathcal{V}[C] \), as follows. Given an ordinary category \( C \), the free \( \mathcal{V} \)-category \( \mathcal{V}[C] \) has the same objects, while its hom-objects \( \mathcal{V}[C](x, y) \) are the copowers \( C(x, y) \otimes 1 \), see [Kel82 Section 2.5]. Notice that \( \mathcal{V}[*] = 1 \), where \( * \) is the terminal category. Given an ordinary category \( C \), consider the companion \( 1(\mathcal{V}[!], \text{id}) \) of the \( \mathcal{V} \)-functor \( \mathcal{V}[!] : \mathcal{V}[C] \to 1 \) induced by the terminal functor \( ! : C \to * \). Then, for any unenriched functor \( d : C \to [M] \), the \( 1(\mathcal{V}[!], \text{id}) \)-weighted colimit of the \( \mathcal{V} \)-functor \( \mathcal{V}[!] d^\mathcal{V} : \mathcal{V}[C] \to M \), that corresponds to \( d \) under the adjunction \( \mathcal{V}[\cdot] \dashv [-] \), is the conical colimit of \( d \) in \( M \). Indeed, the \( \mathcal{V} \)-profunctor \( 1(\mathcal{V}[!], \text{id}) \), considered as a \( \mathcal{V} \)-functor \( \mathcal{V}[C]^{op} \to \mathcal{V} \), coincides with the adjoint \( \Delta^*_1 \) of the constant functor \( \Delta^*_1 : C^{op} \to [\mathcal{V}] \) at \( 1 \in [\mathcal{V}] \). It follows that \( \text{colim}_1(\mathcal{V}[!], \text{id}) d^\mathcal{V} \) is the ordinary \( \Delta^*_1 \)-weighted colimit of \( d^\mathcal{V} \), which is by definition the conical colimit of \( d \), see [Kel82 Section 3.8].

### 2.2 Weighted colimits in pseudo double categories

In this section we compare the definition above, of weighted colimits in a closed equipment, to that of ‘left Kan extensions in a pseudo double category’, as introduced by Grandis and Paré in [GP08 Section 2]. The results in this section are new.

Remember that in a 2-category \( \mathcal{C} \) the (ordinary) left Kan extension of \( d : A \to M \) along \( j : A \to B \) is a pair \( (l : B \to M, \eta : d \Rightarrow l \circ j) \) such that, for any other pair \( (c : B \to M, \phi : d \Rightarrow c \circ j) \) there is a unique cell \( \phi : l \Rightarrow c \) such that

\[
[d \xrightarrow{j} l \circ j \xleftarrow{\eta} e \circ j] = \phi.
\]

In this case \( \eta \) is said to \textit{exhibit} \( l \) as the left Kan extension of \( d \) along \( j \); also \( \eta \) is called the unit of \( l \). Grandis and Paré generalise this notion to pseudo double categories as follows.
2.2. WEIGHTED COLIMITS IN PSEUDO DOUBLE CATEGORIES

Definition 2.11. In a pseudo double category $K$ consider vertical morphisms $d: A \to M$ and $l: B \to M$, as well as horizontal morphisms $J: A \rightrightarrows B$ and $I: M \rightrightarrows N$. The cell $\eta$ below exhibits $l$ as the left Kan extension of $d$ from $J$ to $I$ if every cell $\phi$ factors uniquely through $\eta$ as a vertical cell $\phi'$, as shown. The cell $\eta$ is called the unit of $l$.

\[
\begin{array}{c}
A & \xrightarrow{J} & B \\
\downarrow{d} & & \downarrow{\phi'} \\
M & \xrightarrow{\eta} & N
\end{array}
\]

A left Kan extension $l$ is called an absolute extension by Grandis and Paré when cells $\phi$ of the more general form below also factor uniquely through $\eta$, as shown. In the case that such factorisations only exist for cells $\phi$ with $K = U_N$ we will call $l$ a top absolute extension.

\[
\begin{array}{c}
A & \xrightarrow{J} & B \\
\downarrow{d} & \downarrow{\phi} & \downarrow{\phi'} \\
M & \xrightarrow{\eta} & N
\end{array}
\]

Remark 2.12. Notice that the weakest kind of factorisation above, that of cells $\phi: J \Rightarrow I$, implies that any two cells exhibiting the left Kan extension of $d$ from $J$ to $I$ coincide, up to horizontal composition on the right with a vertical isomorphism. In particular, if we know that the (top) absolute left Kan extension of $d$ from $J$ to $I$ exists then any exhibition of $l$ as the (ordinary) left Kan extension of $d$ from $J$ to $I$ also exhibits it as the (top) absolute extension.

The proposition below proves that $J$-weighted colimits in a closed equipment coincide with top absolute extensions from $J$ to a unit promorphism. To state it notice that every cell below corresponds to a horizontal cell $\lambda\eta: J \otimes M(i, id) \Rightarrow M(d, id)$ (see Proposition 1.24) that in turn is adjoint, under the adjunction (2.1), to a horizontal cell $(\lambda\eta)^\flat: M(i, id) \Rightarrow J \otimes M(d, id)$.

\[
\begin{array}{c}
A & \xrightarrow{J} & B \\
\downarrow{d} & \downarrow{\eta} & \downarrow{\phi'} \\
M & \xrightarrow{\lambda\eta} & N
\end{array}
\]

Example 2.13. Of course in the case of the equipment $V$-Prof of $V$-profunctors, the cell $\eta$ corresponding to $M(i, id) \cong J \otimes M(d, id)$ is the $V$-natural transformation $J \Rightarrow M(d, l)$ which, restricted to each $b$ in $B$, is the unit $\eta_{-b}: J(-, b) \Rightarrow M(d-, lb)$ that defines $lb$ as the $J(-, b)$-weighted colimit of $d$ (see Example 2.10).

Proposition 2.14. For a cell $\eta$ above, in a closed equipment $K$, the following are equivalent:

- $\eta$ defines $l$ as the top absolute left Kan extension of $d$ from $J$ to $U_M$ (Definition 2.11);

- the corresponding horizontal cell $(\lambda\eta)^\flat$, as given above, is an isomorphism, thus defining $l$ as the $J$-weighted colimit of $d$ (Definition 2.9).
Proof. The first statement means that we have unique factorisations

\[
\begin{array}{ccc}
A & \xrightarrow{J} & B \\
\downarrow \phi & & \downarrow e \\
M & \xrightarrow{M} & M
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
A & \xrightarrow{J} & B \\
\downarrow \eta & & \downarrow \psi' \\
M & \xrightarrow{M} & M
\end{array}
\]

which we claim is equivalent to the existence of the following unique factorisations of horizontal cells. This follows directly from the fact that under the correspondence \( \phi \mapsto \lambda \phi \) (which preserves composition, see Proposition 1.24) a factorisation of the first type corresponds to one of the second type, and vice versa.

\[
\begin{array}{ccc}
A & \xrightarrow{J} & B \\
\downarrow \psi & & \downarrow \psi' \\
M_{(d,\text{id})} & \xrightarrow{M} & M
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
A & \xrightarrow{J} & B \\
\downarrow \eta & & \downarrow \lambda \eta \\
M_{(d,\text{id})} & \xrightarrow{M} & M
\end{array}
\]

But the factorisation of horizontal cells above means precisely that \( \lambda \eta \) forms a ‘universal arrow from the functor \( J \circ - \) to the object \( M(d,\text{id}) \)', as defined on [ML98, Page 58]. Since evaluation \( \text{ev}: J \circ J \circ M(d,\text{id}) \Rightarrow M(d,\text{id}) \), which is the counit of the adjunction \( J \circ - \dashv J \circ - \), forms a universal arrow from \( J \circ - \) to \( M(d,\text{id}) \) as well (see [ML98, Theorem IV.1.2]), and because universal arrows are unique up to isomorphism, we conclude that the factorisation of horizontal cells above exists if and only if there is a unique invertible horizontal cell \( \chi: M(l,\text{id}) \Rightarrow J \circ M(d,\text{id}) \) such that \( \text{ev} \circ (\text{id} \circ J \circ \chi) = \lambda \eta \). This equation means that \( \chi \) is adjoint to \( \lambda \eta \), so that the latter statement is equivalent to \( (\lambda \eta)^{} \] \( = \chi \) being invertible, which is what we wanted to show.

Consequently we extend Definition 2.9 of weighted colimits in closed equipments, to one for pseudo double categories, as follows.

**Definition 2.15.** Given morphisms \( J: A \Rightarrow B \) and \( d: A \Rightarrow M \) in a pseudo double category \( K \), the \( J \)-weighted colimit of \( d \) is the top absolute left Kan extension of \( d \) from \( J \) to \( M_{\text{M}} \), as defined in Definition 2.11.

In Example 2.10 we saw that colimits weighted by companions in \( \mathcal{V}-\text{Prof} \) are left Kan extensions. Recall that in any double category \( K \), the companion of a vertical map \( j: A \rightarrow B \) is defined as the horizontal morphism \( B(j,\text{id}) : A \Rightarrow B \) that comes equipped with a cartesian filler \( j \varepsilon : B(j,\text{id}) \Rightarrow B \) of the niche \( B \Rightarrow B \) \( \Rightarrow A \), that defines \( B(j,\text{id}) \) as the restriction of \( U_{B} \) along \( j \) and \( \text{id} \); see the discussion following Definition 1.30. Factorising the horizontal unit \( U_{j} \) through \( j \varepsilon \) gives \( j \eta: U_{A} \Rightarrow B(j,\text{id}) \).

**Proposition 2.16.** Let \( j: A \rightarrow B \), \( d: A \rightarrow M \) and \( l: B \rightarrow M \) be vertical maps in a pseudo double category \( K \) and assume that \( j \) has a companion. If the cell

\[
\begin{array}{ccc}
A & \xrightarrow{B(j,\text{id})} & B \\
\downarrow d & & \downarrow \varepsilon \\
M & \xrightarrow{M} & M
\end{array}
\]

then

\[
\begin{array}{ccc}
A & \xrightarrow{B(j,\text{id})} & B \\
\downarrow d & & \downarrow \eta \\
M & \xrightarrow{M} & M
\end{array}
\]
exhibits $l$ as the $B(j, \text{id})$-weighted colimit of $d$ then the vertical cell $\zeta \circ \eta: d \Rightarrow l \circ j$ exhibits $l$ as the left Kan extension of $d$ along $j$ in the vertical 2-category $V(K)$. The converse holds whenever the $B(j, \text{id})$-weighted colimit of $d$ is known to exist.

Proof. The implication follows directly from the fact that there is a correspondence between cells of the form

\[
\begin{array}{ccc}
A & \xrightarrow{\phi} & B \\
\downarrow {e} & & \downarrow {j} \\
M & \xrightarrow{\chi} & M
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
A & \xrightarrow{\phi} & B \\
\downarrow {e} & & \downarrow {j} \\
M & \xrightarrow{\chi} & M
\end{array}
\]

given by $\phi \mapsto \phi \circ j \eta$ and $\chi \mapsto \chi \odot (U_e \circ j \varepsilon)$, that is natural with respect to horizontal composition on the right with vertical maps $e \Rightarrow f$, for any second map $f: B \Rightarrow M$. The converse follows from Remark 2.12.

Consequently we make the following definition.

**Definition 2.17.** Consider morphisms $j: A \rightarrow B$ and $d: A \rightarrow M$ in a pseudo double category $K$, such that $j$ has a companion $B(j, \text{id})$. We define the *weighted left Kan extension* $\text{lan}_j d: B \rightarrow M$ of $d$ along $j$ to be the weighted colimit $\text{lan}_j d = \text{colim}_{B(j, \text{id})} d$.

As we saw in Example 2.10, the weighted left Kan extensions in the equipment $\mathbf{V}$-$\mathbf{Prof}$ of $\mathbf{V}$-profunctors are precisely the enriched Kan extensions in $\mathbf{V}$-$\mathbf{Cat}$. Proving that the weighted left Kan extensions in the equipment $\mathbf{Prof}(\mathcal{E})$ of profunctors internal to $\mathcal{E}$ are precisely the ‘pointwise’ Kan extensions in $\mathbf{Cat}(\mathcal{E})$ will be the aim of the final section of this chapter.

We close this section by giving two useful properties of weighted colimits. The first is a ‘Fubini theorem’ for iterated weighted colimits, like the Fubini theorems for iterated limits in ordinary categories ([ML98, Section IX.8]) and for iterated ordinary weighted limits in enriched categories ([Kel82, Section 3.3]). Its proof is easy and omitted.

**Proposition 2.18.** Consider horizontally composable cells

\[
\begin{array}{ccc}
A & \xrightarrow{j} & B \\
\downarrow {d} & & \downarrow {h} \\
M & \xrightarrow{k} & M
\end{array}
\quad \begin{array}{ccc}
A & \xrightarrow{l} & C \\
\downarrow {\eta} & & \downarrow {\zeta} \\
M & \xrightarrow{\kappa} & M
\end{array}
\]

in a pseudo double category. If $\eta$ exhibits $l$ as the $J$-weighted colimit of $d$ and $\zeta$ exhibits $k$ as the $H$-weighted colimit of $l$ then $\eta \circ \zeta$ exhibits $k$ as the $J \odot H$-weighted colimit of $d$.

Remember that a cell $\phi$ in an equipment is called right invertible whenever its corresponding horizontal cell $\rho \phi$, that is obtained by horizontally composing $\phi$ with conjoint cells (see Proposition 1.24), is invertible.
Proposition 2.19. Consider vertically composable cells

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow & & \downarrow \phi \\
C & \xrightarrow{d} & M \\
\downarrow & & \downarrow \zeta \\
M & = & M \\
\end{array}
\quad \quad
\begin{array}{ccc}
J & \xrightarrow{g} & K \\
\downarrow & & \downarrow l \\
M & = & M \\
\end{array}
\]

in an equipment. If \( \phi \) is right invertible then \( \zeta \) exhibits \( l \) as a weighted colimit if and only if \( \zeta \circ \phi \) exhibits \( l \circ g \) as a weighted colimit.

Proof. Consider cells that are of one of the four forms shown below.

\[
\begin{array}{ccc}
A & \xrightarrow{J} & B & \xrightarrow{H} & E \\
\downarrow f & & \downarrow \psi & & \downarrow e \\
C & \xrightarrow{d} & M & = & M \\
\end{array}
\quad \quad
\begin{array}{ccc}
C & \xrightarrow{K} & D & \xrightarrow{D(id, g)} & B & \xrightarrow{H} & E \\
\downarrow d & & \downarrow \chi & & \downarrow \chi' \\
M & = & M & = & M \\
\end{array}
\]

Here the solid double headed arrows denote the following bijective correspondences. Cells that are of form of \( \psi' \) correspond to those of the form of \( \chi' \) under the assignments \( \psi' \mapsto (U_l \circ \varepsilon_g) \circ \psi' \) and \( \chi' \mapsto \chi' \circ (\eta_g \circ \text{id}_H) \); that these give a bijective correspondence follows from the conjoint identities. Likewise cells of the form of \( \psi \) correspond to those of the form of \( \chi \), by assigning \( \psi \mapsto ((U_d \circ \varepsilon_f) \circ \psi) \circ ((\rho \phi)^{-1} \circ \text{id}_H) \) and \( \chi \mapsto \chi \circ (\phi \circ \eta_g \circ \text{id}_H) \). That the latter give a bijective correspondence too follows from the conjoint identities and the fact that \( \rho \phi = \varepsilon_f \circ \phi \circ \eta_g \), see Proposition 1.24.

Moreover \( \zeta \circ \phi \) exhibits \( l \circ g \) as a weighted colimit if and only if every cell \( \psi \) above factors uniquely through \( \zeta \circ \phi \) as a cell \( \psi' \) as above, such that \( \zeta \circ \phi \circ \psi' = \psi \), as is denoted by the dashed double headed arrow on the left. Likewise if \( \zeta \) exhibits \( l \) as a weighted colimit of \( d \) then every cell \( \chi \) above factors uniquely through \( \zeta \) as a cell \( \chi' \) as above, such that \( \zeta \circ \chi' = \chi' \), as denoted on the right by the dashed double headed arrow. The latter are in fact equivalent: by taking \( g = \text{id}_B \) and vertically precomposing the cells \( \chi \) and \( \chi' \) with the isomorphisms \( H \cong B(id, id) \circ H \) we see that, in terms of Proposition 2.14, \( \zeta \) exhibits \( l \) as the top absolute left Kan extension of \( d \) along \( J \).

We claim that these two conditions are equivalent. Indeed consider four cells \( \psi, \chi, \psi' \) and \( \chi' \) above, such that the pairs \( (\psi, \chi) \) and \( (\psi', \chi') \) correspond under the solid correspondences. To see that the correspondence on the right, that is \( \zeta \circ \chi' = \chi' \), implies that on the left, i.e. \( (\zeta \circ \phi) \circ \psi' = \psi \), consider the simplified grids of cells below: still representing composites of cells these only describe the shape of
non-identity cells and how they fit together; to save space objects, morphisms and promorphisms are left out. The identities follow from the assumptions that \( \psi' \) and \( \chi' \) correspond, \( \zeta \circ \chi' = \chi \) and that \( \psi \) and \( \chi \) correspond.

\[
\begin{array}{ccc}
\phi & \eta_y & \\
\zeta & \chi' & \\
\end{array}
& \Rightarrow &
\begin{array}{ccc}
\phi & \eta_y & \\
\zeta & \chi' & \\
\end{array}
=
\begin{array}{ccc}
\phi & \eta_y & \\
\zeta & \chi' & \\
\end{array} = \begin{array}{ccc}
\psi
\end{array}
\]

For the converse assume that \( (\zeta \circ \phi) \circ \psi' = \psi \) holds. Then

\[
\begin{array}{ccc}
\zeta & \chi' & \\
\end{array} = \begin{array}{ccc}
(\rho \phi)^{-1} & \\
\zeta & \chi' & \\
\end{array} = \begin{array}{ccc}
(\rho \phi)^{-1} & \\
\zeta & \chi' & \\
\end{array} = \begin{array}{ccc}
(\rho \phi)^{-1}
\end{array}
\]

where the identities follow from \( \rho \phi = \varepsilon_f \circ \phi \circ \eta_g \) (Proposition 1.24) and the assumption that \( \psi' \) and \( \chi' \) correspond; \( (\zeta \circ \phi) \circ \psi' = \psi \); the assumption that \( \psi \) and \( \chi \) correspond. This completes the proof.

Remark. The economic representation of composites that was introduced in the proof above will be used regularly from now on. This will save some space as the composites of cells that we need to consider are getting bigger.

2.3 Pointwise weighted colimits

In his paper [Str74] Street uses ‘comma objects’ to give a notion of ‘pointwise’ left Kan extensions in 2-categories, which was generalised by Grandis and Paré in [GP08, Section 4.1] to a notion of ‘pointwise’ Kan extensions in pseudo double categories, using ‘double comma objects’. By modifying the latter slightly, we will introduce our notion of ‘pointwise’ weighted colimits, given in Definition 2.30. Extending the notion of pointwise left Kan extensions, these are in general stronger than the ordinary weighted colimits, as we will see in Example 2.24. To understand this difference we will consider ‘strong’ double comma objects in equipments, that satisfy a property isolating the cause of this difference. The main result of this section (Theorem 2.37) proves that if each double comma object in an equipment \( \mathcal{K} \) is strong, then the pointwise and ordinary notion of weighted colimit coincide. In particular this holds for the equipment \( \mathcal{K} = \text{Prof}(\mathcal{E}) \) of profunctors internal to \( \mathcal{E} \) so that, where weighted colimits in \( \mathcal{V}-\text{Prof} \) generalise Dubuc’s enriched left Kan extensions (see Example 2.10), in \( \text{Prof}(\mathcal{E}) \) they generalise Street’s pointwise left Kan extensions. The notion of strong double comma objects introduced here as well as all results in this section about them seem to be new.

We begin with the well-known notion of comma objects in 2-categories.

**Definition 2.20.** Given two morphisms \( f: A \to C \) and \( g: B \to C \) in a 2-category \( \mathcal{C} \), the **comma object** \( f/g \) of \( f \) and \( g \) consists of an object \( f/g \) equipped with projections \( \pi_A \) and \( \pi_B \), as in the diagram below, as well as a cell \( \pi: f \circ \pi_A \Rightarrow g \circ \pi_B \), satisfying
the following 1-dimensional and 2-dimensional universal properties.

\[
\begin{array}{ccc}
  f/g & \xrightarrow{\pi_A} & A \\
  \pi_B & \xleftarrow{\varphi} & \downarrow \pi \downarrow f \\
  B & \xrightarrow{g} & C
\end{array}
\]

Given any other cell \( \phi \) in \( \mathcal{C} \) as on the left below, the 1-dimensional property states that there exists a unique morphism \( \phi' : U \to f/g \) such that \( \pi_A \circ \phi' = \phi_A \), \( \pi_B \circ \phi' = \phi_B \) and \( \pi \circ \phi' = \phi \).

\[
\begin{array}{ccc}
  U & \xrightarrow{\phi_A} & A \\
  \phi_B & \xleftarrow{\varphi} & \downarrow f \\
  B & \xrightarrow{g} & C
\end{array}
\]

The 2-dimensional property is the following. Suppose we are given a further cell \( \psi \) as on the right above, which factors through \( \pi \) as \( \psi : U \to f/g \), like \( \phi \) factors as \( \phi' \). Then for any pair of cells \( \xi_A : \phi_A \Rightarrow \psi_A \) and \( \xi_B : \phi_B \Rightarrow \psi_B \), such that \( \psi \circ (f \circ \xi_A) = (g \circ \xi_B) \circ \phi \), there exists a unique cell \( \xi' : \phi' \Rightarrow \psi' \) such that \( \pi_A \circ \xi' = \xi_A \) and \( \pi_B \circ \xi' = \xi_B \).

We remark that formally the comma object \( f/g \) can also be defined as the \( \text{Cat} \)-enriched \( J \)-weighted limit of the diagram of \( f \) and \( g \), with the weight as follows, where \( * \) denotes the terminal category and \( 2 = (\perp \to \top) \).

\[
J = \begin{array}{ccc}
  * & \to & \top \\
  \downarrow & & \\
  * & \to & 2
\end{array}
\]

**Example 2.21.** In the case that \( f \) and \( g \) are ordinary functors in \( \mathcal{C} = \text{Cat} \) the comma object \( f/g \) is the usual comma category, whose objects are triples \( (a, x, b) \) with \( a \in A \), \( b \in B \) and \( x : fa \to gb \) in \( \mathcal{C} \), while a map \( (a, x, b) \to (a', x', b') \) is a pair \( (u, v) : (a, b) \to (a', b') \) in \( A \times B \) making the diagram

\[
\begin{array}{ccc}
  fa & \xrightarrow{x} & gb \\
  \downarrow f_u & & \downarrow g_v \\
  fa' & \xrightarrow{x'} & gb'
\end{array}
\]

commute. The natural transformation \( \pi : f \circ \pi_A \Rightarrow g \circ \pi_B \) is given by \( \pi_{(a, x, b)} = x \).

Comma categories can be generalised in several ways: firstly if \( f : A \to C \) and \( g : B \to C \) are 2-functors then the comma object \( f/g \) in \( 2\text{-Cat} \), the 2-category of 2-categories, 2-functors and 2-natural transformations, has as objects and morphisms the same ones as given above, while a cell \( (u, v) \Rightarrow (u', v') \), for morphisms \( (u, v) \) and \( (u', v') : (a, b) \to (a', b') \), consists of a pair of cells \( m : u \Rightarrow u' \) in \( A \) and \( n : v \Rightarrow v' \) in \( B \) that make the diagram below commute. The 2-natural transformation \( \pi : f \circ \pi_A \Rightarrow g \circ \pi_B \) is again given by \( \pi_{(a, x, b)} = x \).
Secondly if $C = \text{Cat}(S)$ then $f/g$, for two $S$-indexed functors $f: A \to B$ and 
\[ g: B \to C, \]
is given pointwise, that is $(f/g)_s = f_s/g_s$, where $f_s$ and $g_s$ is the comma
category of the functors $f_s$ and $g_s$ as given above. In general the comma object of
two internal functors $f: A \to C$ and $g: B \to C$ in any category $E$ with finite limits
can be similarly constructed. We shall not do so but instead treat, in detail, the
more general construction of internal double comma objects below.

We can now give Street’s definition of pointwise left Kan extensions that was
introduced in [Str74] Section 4.

**Definition 2.22.** Let $d: A \to M$ and $j: A \to B$ be morphisms in a 2-category
that has all comma objects. The cell $\eta: d \Rightarrow l \circ j$ in the diagram below is said to
exhibit $l$ as the pointwise left Kan extension of $d$ along $j$ if, for each $f: C \to B$, the
composition of cells below exhibits $l \circ f$ as the left Kan extension of $d \circ \pi_A$ along
$\pi_C$.

\[
\begin{array}{ccc}
A & \stackrel{\pi A}{\longrightarrow} & M \\
\downarrow f & & \downarrow M \\
B & \stackrel{\pi}{\longrightarrow} & C \\
\downarrow l & & \downarrow C \\
& B & \stackrel{\eta}{\longrightarrow} & \pi C \\
\end{array}
\]

(2.5)

**Remark 2.23.** Taking $f = \text{id}$ one can show that the cell $\eta$ above exhibits $l$ as the
ordinary left Kan extension of $d$ along $j$, see [Str74] Corollary 19. The following
example shows that, in the case of $C = \mathcal{V}$-$\text{Cat}$, Street’s notion of pointwise left
Kan extension is, in general, stronger than Dubuc’s notion of enriched left Kan
extension, that was given in Example 2.10. Thus enriched left Kan extensions in
$\mathcal{V}$-$\text{Prof}$, as defined in Definition 2.17 will in general not be pointwise. In fact, Street
remarks at the end of the introduction to [Str74] that “For $\mathcal{V} = \text{Set}$ and $\mathcal{V} = 2$ the
definitions do agree; for $\mathcal{V} = \text{AbGp}$ and $\mathcal{V} = \text{Cat}$, they do not.”

**Example 2.24.** To construct an enriched left Kan extension of 2-functors that is not
pointwise consider the 2-categories

\[
B = \left( x \cdot \begin{array}{c}
u \\ v \end{array} \cdot y \right) \quad \text{and} \quad M = \left( x' \cdot \begin{array}{c}u' \\ v' \end{array} \cdot y' \right).
\]

Writing $x: * \to B$ and $x': * \to M$ for the 2-functors that map the single object
of the terminal 2-category $*$ to the objects $x$ in $B$ and $x'$ in $M$ respectively, we
claim that the enriched left Kan extension $l: B \to M$ of $x'$ along $x$ is the constant
2-functor mapping all of $B$ to $x'$. Indeed since $*$ has no non-identity morphisms
the coends in (2.4) reduce to $l(x) = x'$ and $l(y) = 2 \otimes x'$, where $2 = (\bot \to \top)$. It
is easily checked that the latter copower exists and is equal to $x'$. Notice that
$l \circ x = x'$; the unit $\eta$ exhibiting $l$ is the identity transformation for $x'$.

To show that $l$ fails to be pointwise take $f = y: * \to B$ in (2.5): we claim that
the composite of $\pi$ and $\eta$ does not exhibit $l \circ y = x'$ as the left Kan
extension of $x' \circ \pi_A$ along $\pi_C$. To see this notice that the comma object $f/y$ (as given in
Example 2.21) is the discrete 2-category consisting of the triples $(*, u: x \to y, *)$ and
$(*, v: x \to y, *)$, so that the assignments $(*, u, *) \mapsto u'$ and $(*, v, *) \mapsto v'$
trivially form a 2-natural transformation $\phi: x' \circ \pi_A \to y' \circ \pi_C$. However both
possible candidates $\phi': x' = l \circ y \to y'$, for a factorisation of $\phi$ through (2.5), consist
of a single map in $M$ that is either $u'$ or $v'$ so that, precomposed with (2.5), both
$\phi'$ do not equal $\phi$. We conclude that $l$ is a left Kan extension that is not pointwise.

If a pointwise left Kan extension of $x$ and $x'$ exists then it coincides with $l$
(indeed, both are in particular ordinary left Kan extensions of $x$ and $x'$, which
are unique up to isomorphism), so we can conclude that the pointwise left Kan extension of $x$ and $x'$ does not exist.

The following definition introduces comma objects of a horizontal and vertical morphism, in a double category. It is one of several possible definitions, the strongest of which is used by Grandis and Paré in [GP08, Section 3.2], to define pointwise left Kan extensions in double categories; details are given in Remark 2.31.

**Definition 2.25.** Given a horizontal morphism $J: A \rightarrow B$ and a vertical morphism $f: C \rightarrow B$ in a pseudo double category $K$, the (vertical) double comma object $J/f$ of $J$ and $f$ consists of an object $J/f$ equipped with projections $\pi_A$ and $\pi_C$ as well as a cell $\pi$, as in the diagram below, satisfying the following 1-dimensional and 2-dimensional universal properties.

$$
\begin{array}{c}
J/f \\
\pi_A \downarrow \phi \downarrow \phi_C \downarrow f \\
\pi_C \downarrow \psi \downarrow f \\
A \rightarrow J \rightarrow B
\end{array}
$$

Given any other cell $\phi$ in $K$ as on the left below, the 1-dimensional property states that there exists a unique vertical morphism $\phi': X \rightarrow J/f$ such that $\pi_A \circ \phi' = \phi_A$, $\pi_C \circ \phi' = \phi_C$ and $\phi \circ U_{\phi'} = \phi$.

$$
\begin{array}{c}
X \downarrow \phi_C \downarrow f \\
A \rightarrow J \rightarrow B
\end{array}
$$

The 2-dimensional property is the following. Suppose we are given a further cell $\psi$ as in the middle above, which factors through $\pi$ as $\psi$ factors as $\phi'$. Then for any horizontal map $K: X \rightarrow Y$ and any pair of cells $\xi_A$ and $\xi_C$, such that the identity below holds, there exists a unique cell $\xi'$, as on the right above, such that $U_{\xi_A} \circ \xi' = \xi_A$ and $U_{\xi_C} \circ \xi' = \xi_C$.

$$
\begin{array}{c}
X \downarrow \phi_C \downarrow f \\
A \rightarrow J \rightarrow B
\end{array}
$$

Just like comma objects can be defined as weighted limits, the vertical double comma objects $J/f$ is the ‘double limit’ of the functor $D: I \rightarrow K$, where $I$ is the double category $0 \rightarrow 1 \leftarrow 2$, that maps onto $J$ and $f$. Analogous to the definition of classical limits, the double limit of $D$ is defined in [GP99, Section 4] as the ‘double terminal object’ of the ‘double comma category’ $\Delta/D$, where $\Delta: K \rightarrow K'$ is the diagonal ‘double functor’ and $D: s \rightarrow K'$ is the constant functor on $D \in K'$. Note that Grandis and Paré draw horizontal morphisms (i.e. promorphisms) vertically and vertical morphisms horizontally. We discuss the dual notion, that of ‘double colimit’, in more detail in Section 4.3.
Example 2.26. The construction of double comma objects is similar to that of comma objects, as given in Example 2.21. For example, the double comma object \( J/f \) of an (unenriched) profunctor \( J: A \to B \) and a functor \( f: C \to B \) is the category with triples \((a,j; a \to fc,c)\), where \( j \in J(a,fc) \), as objects and pairs \((x,y): (a,j,c) \to (a',j',c')\), that make the diagram below commute in \( J \), as morphisms.

\[
\begin{array}{ccc}
  a & \overset{j}{\to} & fc \\
  x & \downarrow & \downarrow fy \\
  a' & \overset{j'}{\to} & fc'
\end{array}
\]  

(2.6)

The transformation \( \pi: U_{J/f} \to J(\pi_A, f \circ \pi_C) \) maps each morphism \((x,y)\) to the diagonal \(j' \circ x = fy \circ j\) of the square above. To see that \( \pi \) satisfies the 1-dimensional universal property consider a second transformation \( \phi: U_X \to J(\phi_A, f \circ \phi_B) \); it factors uniquely through \( \pi \) as the functor \( \phi': X \to J/f \) that is given by \( \phi'(v) = \{\phi_A(v), \phi_B(v)(id_c), \phi_C(v)\} \) on objects, with \( \phi_B(v)(id_c) : \phi_Av \to f \circ \phi_C(v) \) in \( J \), and by \( \phi'(s) = (\phi_A s, \phi_B s) \) on morphisms \( s: u \to v \).

Moreover given a further natural transformation \( \psi: U_Y \to J(\psi_A, f \circ \psi_B) \) that factors as \( \psi': Y \to J/f \), as well as transformations \( \xi_A: K \to A(\phi_A, \psi_A) \) and \( \xi_C: K \to C(\phi_C, \psi_C) \), as in the definition above, the transformation \( \xi: K \to J/f(\phi', \psi') \), that is unique such that \( U_{\pi_A} \circ \xi' = \xi_A \) and \( U_{\pi_C} \circ \xi' = \xi_C \), can be given by mapping each morphism \( k: u \to v \) in \( K(u,v) \) to the morphism \( (\xi_A k, \xi_B k) \) in \( J/f(\phi u, \psi v) \). This shows that \( \pi \) satisfies the 2-dimensional universal property as well.

If \( J \) is a 2-profunctor (i.e. \( \text{Cat}-\text{enriched} \)) and \( f \) is a 2-functor, then the double comma object \( J/f \) has again the same objects and morphisms as the ones given above, while a cell \((x,y) \Rightarrow (x',y')\), consists of a pair of cells \( m: x \Rightarrow x' \) in \( A \) and \( n: y \Rightarrow y' \) in \( C \), so that the diagram below commutes in \( J \).

\[
\begin{array}{ccc}
  a & \overset{j}{\to} & fc \\
  x & \downarrow & \downarrow fy \\
  a' & \overset{j'}{\to} & fc'
\end{array}
\]  

(2.6)

We now turn to constructing the double comma object \( J/f \) of an internal profunctor \( J: A \to B \) and an internal functor \( f: C \to B \) in a category \( \mathcal{E} \) with finite limits, leaving the details to Appendix A. As its \( \mathcal{E} \)-object of objects take the wide pullback \((J/f)_0 = A_0 \times_{A_0} J \times_{B_0} C_0 \) of the diagram \( A_0 \overset{id}{\to} A_0 \overset{d_0}{\to} B_0 \overset{d_0}{\to} C_0 \) and define its object of morphisms \( J/f \) to be the pullback

\[
\begin{array}{ccc}
  J/f & \to & (J/f)_0 \times_{C_0} C \\
  \downarrow & & \downarrow r \\
  A \times_{A_0} (J/f)_0 & \overset{l}{\to} & (J/f)_0,
\end{array}
\]  

(2.7)

where the corners are pullbacks of \((J/f)_0 \to C_0 \xleftarrow{d_0} C \) and \( A \xrightarrow{d_1} A_0 \leftarrow (J/f)_0 \) respectively, and the maps \( r \) and \( l \) are the compositions

\[
\begin{align*}
(J/f)_0 \times_{C_0} C & \to A_0 \times_{A_0} J \times_{B_0} B \times_{C_0} C_0 \xrightarrow{id \times r \times id} (J/f)_0 \\
A \times_{A_0} (J/f)_0 & \cong A_0 \times_{A_0} A \times_{A_0} J \times_{B_0} C_0 \xrightarrow{id \times l \times id} (J/f)_0.
\end{align*}
\]  

(2.8)
Here the unlabelled map is given by applying \( f \) to \( C \) after using \( C_0 \times_{C_0} C \cong C \times_{C_0} C_0 \), while the isomorphism is induced by \( A \times A_0 \), \( A_0 \cong A_0 \times A_0 \) and the maps \( l \) and \( r \) denote the actions of \( A \) and \( B \) on \( J \). Notice that \( (J/f)_0 \) and \( J/f \) are the internal versions of the sets of objects and morphisms that define the comma category of ordinary functors, as given above. Indeed \( (J/f)_0 \) is the internal version of the set of triples \( (a, j: a \to f e, c) \), while \( J/f \) is the \( \mathcal{E} \)-object of commutative squares of the form (2.6). The source and target maps \( d = (d_0, d_1): J/f \to (J/f)_0 \times (J/f)_0 \) are given by the projections

\[
d_0 = [J/f \to (J/f)_0 \times_{C_0} C \to (J/f)_0]
\]

and

\[
d_1 = [J/f \to A \times_{A_0} (J/f)_0 \to (J/f)_0].
\]

So far we have defined an \( \mathcal{E} \)-span \( d: g \to (f/g)_0 \times (f/g)_0 \). That it can be made into an internal category that is the double comma object of \( J \) and \( f \) is asserted by the following proposition, whose proof is given in Appendix A.

**Proposition 2.27.** The \( \mathcal{E} \)-span \( d: J/f \to (J/f)_0 \times (J/f)_0 \) above can be given the structure of a category internal to \( \mathcal{E} \). Moreover, \( J/f \) is made into the double comma object of \( J \) and \( f \) in \( \text{Prof}(\mathcal{E}) \) by the internal functors \( \pi_A: J/f \to A \), given by the projections

\[
(\pi_A)_0: (J/f)_0 \to A_0 \quad \text{and} \quad \pi_A = [J/f \to A \times_{A_0} (J/f)_0 \to A],
\]

and \( \pi_C: J/f \to C \), given similarly, together with the cell

\[
\begin{array}{ccc}
J/f & \longrightarrow & J/f \\
\downarrow \pi_A & & \downarrow \pi_C \\
A & \underset{f}{\longrightarrow} & B \\
\end{array}
\]

of internal profunctors that, under the correspondence of Proposition 1.22, is given by the projection \( (J/f)_0 \to J \).

**Remark 2.28.** Remember from Example 1.19 that the restriction \( J(id, f) \) is given by the pushout of the maps \( d: J \to A_0 \times B_0 \) and \( id 	imes f_0: A_0 \times C_0 \to A_0 \times B_0 \). It will be useful to notice that we can take \( J(id, f) = (J/f)_0 \) as this pullback, such that the cartesian filler \( J(id, f) \to J \) is simply the projection \( (J/f)_0 \to J \). It is easily checked that with this choice the actions of \( A \) and \( C \) (as given in the proof of Proposition 1.19), that make \( J(id, f) \) into an internal profunctor, are the maps \( l \) and \( r \) given in (2.5), that were used to define \( J/f \).

Double comma objects of companions are closely related to comma objects, as follows.

**Proposition 2.29.** Let \( j: A \to B \) and \( f: C \to B \) be morphisms in an equipment \( \mathcal{K} \). If the cell

\[
\begin{array}{ccc}
X & \longrightarrow & X \\
\downarrow \pi_A & & \downarrow \pi_C \\
A & \underset{B(j, id)}{\longrightarrow} & B \\
\end{array}
\]

...
defines \( X \) as the double comma object of \( B(j, \text{id}) \) and \( f \) then the vertical cell \( j \varepsilon \circ \pi \) defines \( X \) as the comma object of \( j \) and \( f \) in the vertical 2-category \( V(K) \). The converse holds whenever the double comma object \( B(j, \text{id})/f \) is known to exist.

**Proof.** This follows from the correspondence between cells of the form 

\[
\begin{array}{ccc}
Y & \xrightarrow{\phi_A} & Y \\
\downarrow & \phi_C & \downarrow \\
C & \xrightarrow{i} & B
\end{array}
\quad \text{and} \quad 
\begin{array}{ccc}
Y & \xrightarrow{\phi_A} & Y \\
\downarrow & \phi_C & \downarrow \\
A & \xrightarrow{j} & B
\end{array}
\]

that is given by \( \phi \mapsto j \varepsilon \circ \phi \) and \( \psi \mapsto (j \eta \circ U_{\phi_A}) \circ \psi \), and that is natural with respect to precomposition with vertical cells and horizontal composition with vertical cells \( h \Rightarrow \phi_A \) or \( \phi_C \Rightarrow k \), where \( h: Y \to A \) and \( k: Y \to C \). It follows that the 1-dimensional property of \( \pi \) coincides with that of \( j \varepsilon \circ \pi \), while the 2-dimensional property of \( \pi \) coincides with that of \( j \varepsilon \circ \pi \) when restricted to cells \( \xi_A \) and \( \xi_C \) with \( K = U_X \) (see Definition 2.25). The implication follows, and its converse follows from the fact that double comma objects are uniquely determined, up to isomorphism, by their 1-dimensional property alone.

Following Grandis and Paré [GP08, Section 4.1] we generalise Street’s notion of pointwise left Kan extensions (Definition 2.22) to pointwise weighted colimits. The remark below explains the differences between our and Grandis and Paré’s definition. Suppose that the cell \( \pi \) defines the double comma object \( J/f \) of morphisms \( f: C \to B \) and \( J: A \rightharpoonup B \), in a pseudo double category \( K \). If the companion of \( \pi_C \) exists in \( K \), and is defined by the cartesian filler \( \pi_C \varepsilon \), then we will denote the composite \( \pi \circ (U_f \circ \pi_C \varepsilon) \) again by \( \pi \); it is of the form

\[
\begin{array}{ccc}
J/f & \xrightarrow{C(\pi_C, \text{id})} & C \\
\downarrow & \pi & \downarrow f \\
A & j & B
\end{array}
\]

**Definition 2.30.** Consider morphisms \( d: A \to M \) and \( J: A \rightharpoonup B \) in a pseudo double category that has companions and double comma objects. The cell \( \eta \) in the diagram below exhibits \( l \) as the pointwise \( J \)-weighted colimit if, for each \( f: C \to B \), the composition below exhibits \( l \circ f \) as the \( C(\pi_C, \text{id}) \)-weighted colimit of \( d \circ \pi_A \).

\[
\begin{array}{ccc}
J/f & \xrightarrow{C(\pi_C, \text{id})} & C \\
\downarrow & \pi & \downarrow f \\
A & j & B
\end{array}
\]

\[
\begin{array}{ccc}
M & \xrightarrow{d} & M \\
\downarrow & \eta & \downarrow \\
A & j & B
\end{array}
\]

In terms of Definition 2.17 this means that \( l \circ f \) is the weighted left Kan extension of \( d \circ \pi_A \) along \( \pi_C \). Also we will, when \( J = B(j, \text{id}) \) for \( j: A \rightharpoonup B \), say that \( \eta \) exhibits \( l \) as the pointwise weighted left Kan extension of \( d \) along \( j \).

**Remark 2.31.** Grandis and Paré use in their definition of pointwise left Kan extensions in strict double categories the following stronger notion of double comma
objects \( J/f \), that is given in [GP08, Section 3.2]. If the cell \( \pi \) defines \( J/f \) then, besides asking that \( \pi \) satisfies the 1-dimensional ‘vertical’ universal property of Definition 2.25, they ask the corresponding cell \( \pi' = \pi_A, \eta \circ \pi \circ (U_f \circ \pi_C) \) below to satisfy the following 1-dimensional ‘horizontal’ universal property: each cell \( \phi \) below factors uniquely through \( \pi' \) as a horizontal morphism \( \phi' : X \rightarrow J/f \), such that

\[
\begin{array}{ccc}
X & \xrightarrow{\phi_C} & C \\
\downarrow & & \downarrow \\
A & \xrightarrow{\phi_A} & B
\end{array}
\]

\[
\begin{array}{ccc}
X & \xrightarrow{\phi'} & J/f \\
\downarrow & & \downarrow \\
A & \xrightarrow{\phi'} & B
\end{array}
\]

Similarly there is a ‘horizontal’ counterpart to the ‘vertical’ 2-dimensional property of \( J/f \) given in Definition 2.25 and the double comma objects used by Grandis and Paré satisfy a ‘global’ 2-dimensional property, that combines both the horizontal and vertical directions. They remark that these are very strong conditions and write that one “…can only expect [their] comma objects to exist in very particular double categories, having some sort of symmetry in themselves …”. Their main example is the double category \( \text{Dbl} \), that has pseudo double categories as objects, lax functors (Definition 3.1) as vertical morphisms and colax functors (see Section 3.2) as horizontal morphisms, see [GP08, Section 1.4]. On the other hand, the double comma objects in the typical equipment \( \text{Prof} \) of profunctors seem not to satisfy the horizontal 1-dimensional property above.

The second difference is that Grandis and Paré ask the composite (2.9) to exhibit \( l \circ f \) just as an ordinary extension of \( d \circ \pi_A \), whereas we ask it to be a top absolute extension, which fits in better with our notion of weighted colimits (Proposition 2.14). We conclude that our notion of pointwise weighted colimits can be applied to more double categories than that of Grandis and Paré, because our requirements on double comma objects are weaker, and that it is stronger, because we ask the extension \( l \circ f \) to be top absolute. A further consequence of our weaker double comma objects is that it does not seem possible to prove that, in general, every pointwise weighted colimit is a weighted colimit. In the case of Grandis and Paré’s definition this does hold, as is shown in [GP08, Theorem 4.2], whose proof uses the horizontal 1-dimensional property of their comma objects. We will however be able to prove such a result in the case of the equipments \( \text{Prof}(\mathcal{E}) \). In fact the main result (Theorem 2.37) of this section implies that in \( \text{Prof}(\mathcal{E}) \) the notion of pointwise weighted colimits (Definition 2.30) coincides with that of weighted colimits (Definition 2.9).

The following is a consequence of Proposition 2.16 and Proposition 2.29.

Proposition 2.32. Let \( K \) be a pseudo double category that has companions and double comma objects. If the cell

\[
\begin{array}{ccc}
A & \xrightarrow{B(j, \text{id})} & B \\
\downarrow d & & \downarrow l \\
M & \xrightarrow{\zeta} & M
\end{array}
\]

exhibits \( l \) as the pointwise weighted left Kan extension of \( d \) along \( j \) then the vertical composite \( \zeta \circ j \) exhibits \( l \) as the pointwise left Kan extension of \( d \) along \( j \) in \( V(K) \).

Proof. Let \( f : C \rightarrow B \) be any morphism and let us write \( \zeta' = \zeta \circ j \eta \). If the cell \( \pi \) defines the double comma object \( B(j, \text{id})/f \) then the vertical composite \( \pi' = j \circ \pi \) defines \( B(j, \text{id})/f \) as the comma object of \( j \) and \( f \) by Proposition 2.29 and we
claim that the vertical cell \((\zeta' \circ U_{\pi_A}) \circ (U_I \circ \pi') = \zeta \circ \pi\) (i.e. (2.19)) with \(\eta = \zeta'\) and \(\pi = \pi'\) exhibits \(l \circ f\) as the left Kan extension of \(d \circ \pi_A\) along \(\pi_C\), which completes the proof. Indeed \((\zeta \circ \pi) \circ (U_{\pi_f} \circ \pi_C \circ \varepsilon)\) (i.e. (2.21)) with \(\eta = \zeta\) exhibits \(l \circ f\) as the weighted left Kan extension of \(d \circ \pi_A\) along \(\pi_C\) by assumption, so that the claim follows from Proposition 2.10.

The following definition is the main idea of this chapter. It describes a condition on double comma objects that, when satisfied by all double comma objects, implies that all weighted colimits in an equipment \(\mathcal{K}\) are pointwise. Remember that, if

\[
\begin{array}{ccc}
J/f & \xrightarrow{C(\pi_C,id)} & C \\
\pi_f & \downarrow \pi & \frown f \\
A & \downarrow j & B
\end{array}
\]

defines \(J/f\) as the double comma object of \(J\) and \(f\) (as in Definition 2.3), then we write \(\pi_f\) for its factorisation \(C(\pi_C,id) \Rightarrow J(id,f)\) through the restriction \(J(id,f)\), see Definition 1.3.

**Definition 2.33.** Assume that the cell \(\pi\) above defines \(J/f\) as the double comma object of \(J: A \rightarrow B\) and \(f: C \rightarrow B\). We will call \(J/f\) strong if every cell \(\phi\) below factors uniquely through \(\pi_f\) as follows.

\[
\begin{array}{ccc}
J/f & \xrightarrow{C(\pi_C,id)} & C \\
\pi_f & \downarrow \phi & \frown f \\
A & \downarrow j & B
\end{array}
\]

\[
\begin{array}{ccc}
J/f & \xrightarrow{C(\pi_C,id)} & C \\
\pi_f & \downarrow \phi & \frown f \\
A & \downarrow j & B
\end{array}
\]

**(2.10)**

**Example 2.34.** In \(\text{Prof}\) any double comma object \(J/f\) of a profunctor \(J: A \rightarrow B\) and a functor \(f: C \rightarrow B\) is strong. To see this recall that \(J/f\) has triples \((a,j: a \rightarrow fc, c)\) as objects and pairs \((x,y): (a,j, c) \rightarrow (a',j', c')\), with \(x: a \rightarrow a'\) and \(y: c \rightarrow c'\) such that \(fy \circ j = j' \circ x\), as morphisms, and notice that the natural transformation \(\pi_f: C(\pi_C,id) \Rightarrow J(\pi_A, f)\) is simply given by

\[
(\pi_f)^{(a,j,c), (a',j', c')} (x: c \rightarrow c') = [a \xrightarrow{j} fc \xrightarrow{f_x} fc'].
\]

For a natural transformation \(\phi: H(\pi_C, id) \Rightarrow M(h \circ \pi_A, k)\) as above, a factorisation \(\phi': J(id, f) \circ C H \Rightarrow M(h, k)\) consists of components

\[
\phi_{(a,d)}: \int_{c \in C} J(a, fc) \times H(c, d) \rightarrow M(ha, kd)
\]

which we take to be induced by the composites

\[
J(a, fc) \times H(c, d) \cong \coprod_{j \in J(a, fc)} H(c, d) \xrightarrow{\coprod_j \phi_{(a, j, c), d}} M(ha, kd).
\]

That these induce a well-defined map on the coend above follows from the naturality of \(\phi\) with respect to maps of the form \((\text{id}, x)\) in \(J/f\), where \(x: c \rightarrow c'\). Indeed this naturality means that the diagrams

\[
\begin{array}{ccc}
H(c', d) & \xrightarrow{\phi_{(a, f \circ j, c'), d}} & M(ha, kd) \\
H(x, d) & \xrightarrow{\phi_{(a, j, c), d}} & H(c, d)
\end{array}
\]

\[
\begin{array}{ccc}
H(c', d) & \xrightarrow{\phi_{(a, f \circ j, c'), d}} & M(ha, kd) \\
H(x, d) & \xrightarrow{\phi_{(a, j, c), d}} & H(c, d)
\end{array}
\]
commute, for every \( j: a \to c \) in \( J \). Finally notice that the \(((a, j, b), d)\)-component of the composite \( \pi_f \circ_C \id_H : C(\pi_C, \id) \circ_C H \to J(\id, f) \circ_C H \), under the isomorphism \( H(\pi_C, \id) \cong C(\pi_C, \id) \circ_C H \), is simply the composite of insertions

\[
H(b, d) \to J(a, fb) \times H(b, d) \to \int_{c \in C} J(a, fc) \times H(c, d),
\]

the first at \( j \in J(a, fb) \) and the second at \( c = b \). From this it easily follows that \( \phi' \) is the unique transformation such that \( \phi = \phi' \circ (\pi_f \circ_C \id_H) \).

**Example 2.35.** To give an example of a comma object in the equipment \( \text{Cat-Prof} \), of 2-categories, 2-functors, 2-profunctors and 2-natural transformations, that is not strong, we can again consider the 2-categories of Example 2.24:

\[
B = \left( \begin{array}{ccc} u & \downarrow & v' \\ x & \to & y' \end{array} \right) \quad \text{and} \quad M = \left( \begin{array}{ccc} u' & \downarrow & v' \\ x' & \to & y' \end{array} \right).
\]

Taking \( A = \ast = C \) to be the terminal 2-category, we choose \( f = y: C \to B \), the inclusion that maps the single object of \( A = \ast \) to \( y \), and likewise \( j = x: A \to B \), \( h = x': A \to M \) and \( k = y': C \to M \). Then the double comma object \( B(j, \id)/f \) (as in Example 2.24) is the discrete 2-category consisting of the triples \((\ast, u: x \to y, \ast)\) and \((\ast, v: x \to y, \ast)\), and the assignments \(((\ast, u, \ast), \ast) \mapsto u'\) and \(((\ast, v, \ast), \ast) \mapsto v'\) trivially form a 2-natural transformation \( \phi: C(\pi_C, \id) \to M(h \circ \pi_A, k) \). However, giving a cell

\[
\begin{array}{ccc}
A & \overset{f}{\longrightarrow} & B \\
\downarrow_{h} & & \downarrow_{k} \\
M & \overset{\phi'}{\longrightarrow} & M
\end{array}
\]

is equivalent to giving a functor \( \phi': B(x, y) \to M(x', y') \), and \( \phi' \circ \pi_f = \phi \) implies \( \phi' u = u' \) and \( \phi' v = v' \). We conclude that such a \( \phi' \) cannot exist, since there is no cell from \( u' \) to \( v' \) in \( M \).

**Proposition 2.36.** Let \( \mathcal{E} \) be a category with finite limits and reflexive coequalisers preserved by pullbacks, so that internal profunctors in \( \mathcal{E} \) form an equipment \( \text{Prof}(\mathcal{E}) \) by Proposition 1.17. The double comma objects of \( \text{Prof}(\mathcal{E}) \), that exist by Proposition 2.27, are all strong.

**Proof.** Let \( J: A \to B \) be an internal profunctor and \( f: C \to B \) be an internal functor, and consider an internal transformation \( \phi: H(\pi_C, \id) \to M \) as in (2.10), which we consider to be precomposed with the isomorphism \( H(\pi_C, \id) \cong C(\pi_C, \id) \circ_C H \) given by Proposition 1.12. We have to show that there exists a unique factorisation \( \phi': J(\id, f) \circ_C H \to M \) such that (2.11) holds.

We begin with computing the horizontal composite

\[
H(\pi_C, \id) \cong C(\pi_C, \id) \circ_C H \overset{\pi_f \circ_C \id_H}{\longrightarrow} J(\id, f) \circ_C H,
\]

where the cell \( \pi_f \), after being composed with unitors, coincides with the horizontal composite

\[
\begin{array}{ccc}
J/f & \overset{C(\pi_C, \id)}{\longrightarrow} & C \\
\downarrow_{\pi_a} & & \downarrow_{\pi_C \circ_C \id} \\
A & \overset{\id}{\longrightarrow} & C
\end{array}
\]

and

\[
\begin{array}{ccc}
J/f & \overset{\id}{\longrightarrow} & C \\
\downarrow_{\pi_f} & & \downarrow_{\pi_C \circ_C \id} \\
A & \overset{\id}{\longrightarrow} & C
\end{array}
\]
where \( \pi' \) denotes the factorisation of the cell \( \pi \), that defines \( J/f \), through the restriction \( J(id, f) \), and \( \pi \circ \varepsilon \) is the cartesian filler defining the companion \( C(\pi, id) \).

Recall from Remark 2.26 that we can take \( J(id, f) = (J/f)_0 \). With that choice the cartesian filler \( J(id, f) \to J \) is the projection \( (J/f)_0 \to J \) and the factorisation \( \pi' : J/f \to J(id, f) \), as a map \( (J/f)_0 \to (J/f)_0 \) (see Proposition 1.22), is simply the identity. Likewise the companion \( C(\pi, id) \) is the pullback of \( d : C \times C_0 \to C_0 \times C_0 \) along \( (\pi C)_0 \times id : (J/f)_0 \times C_0 \to C_0 \times C_0 \), so that we can take \( C(\pi, id) = (J/f)_0 \times C_0 \); it follows that the transformation \( \pi_j = \pi' \circ \pi \circ \varepsilon : C(\pi, id) \to J(id, f) \) is the right action \( r : (J/f)_0 \times C_0 \to (J/f)_0 \) given by \( (\pi, f) \). Finally we also take \( H(\pi, id) = (J/f)_0 \times C_0 H \). Recall from the proof of Proposition 1.12 that the isomorphism \( H(\pi, id) \cong C(\pi, id) \cong C H \) is the composite

\[
H(\pi, id) \cong U_{J/f} \otimes_{J/f} H(\pi, id) \circ \varepsilon_{\pi, \eta} \cong C(\pi, id) \circ C H,
\]

where \( \varepsilon_{\pi, \eta} \) corresponds to \( (id, e \circ (\pi C))_0 : (J/f)_0 \to (J/f)_0 \times C_0 \) under Proposition 1.22 while \( \varepsilon \) is the cartesian filler, i.e. the projection \( (J/f)_0 \times C_0 \to H \). Thus, using the remark on horizontal compositions in Proposition 1.22 we conclude that the composite \( 2.11 \) equals

\[
(J/f)_0 \times C_0 H \xrightarrow{(id, e \circ (\pi C)_0) \times C_0 id_H} ((J/f)_0 \times C_0 C) \times C_0 H \xrightarrow{r \times C_0 id_H} (J/f)_0 \times C_0 H \to (J/f)_0 \circ C H.
\]

Here the first two maps cancel by the unit axiom for actions, so that the coequaliser \( (J/f)_0 \times C_0 H \to (J/f)_0 \circ C H \) remains.

It follows that the identity \( 2.10 \) is equivalent to the commuting of the diagram

\[
\begin{array}{ccc}
(J/f)_0 \times C_0 C \times C_0 H & \xrightarrow{r \times id} & (J/f)_0 \times C_0 H \\
\xrightarrow{id \times t} & & \xrightarrow{\pi \circ C id_H} (J/f)_0 \circ C H \\
& \xrightarrow{\phi} & M \\
& \xrightarrow{(\pi C)_0 \times h_{D_0}} & A_0 \times D_0 \\
& \xrightarrow{h_{0} \times k_{0}} & M_0 \times M_0.
\end{array}
\]

That is, to prove that every internal transformation \( \phi \) induces a unique \( \phi' \) that satisfies \( 2.10 \) we have to show that \( \phi \) forks the actions \( r \times id \) and \( id \times t \), that is \( \phi \circ (r \times id) = \phi \circ (id \times t) \). But this is a consequence of the naturality of \( \phi \), as follows. Its naturality with respect to \( J/f \) (defined as the pullback \( 2.4 \)) means that the diagram

\[
\begin{array}{ccc}
J/f \times (J/f)_0 & \xrightarrow{\phi} & (J/f)_0 \times C_0 H \\
\xrightarrow{A \times A_0} & & \xrightarrow{h \times A_0} \phi M \times M_0 M \\
\xrightarrow{m} & & M
\end{array}
\]

commutes, where the unlabelled maps are induced by the projections \( J/f \to (J/f)_0 \times C_0 C \), \( (J/f)_0 \times C_0 H \to H \) and \( J/f \to A \times A_0 (J/f)_0 \to A \) and where \( m \) is the composition of \( M \). Next consider the map

\[
(J/f)_0 \times C_0 C \times C_0 H \to J/f \times (J/f)_0 ((J/f)_0 \times C_0 H),
\]
that is induced by \((J/f)_0 \times G_0 \rightarrow J/f\), which is in turn induced by the identity on \((J/f)_0 \times G_0 \times C_0 \rightarrow (J/f)_0\) and the composite \((J/f)_0 \times G_0 \rightarrow (J/f)_0\) and the identity on \(H\). It is easy to see that composing this map with the diagram above gives a commuting square whose top leg is \(\phi \circ (\text{id} \times G_0 l)\) and \((\text{using the unit axioms for } h \text{ and } m)\) whose bottom leg is \(\phi \circ (r \times C_0 \text{id})\), so that \(\phi\) forks \(\text{id} \times G_0 l\) and \(r \times C_0 \text{id}\). This completes the proof. 

We close this chapter with its main result.

**Theorem 2.37.** Consider a pseudo double category \(K\) that has companions as well as double comma objects, which are all strong. A cell

\[
\begin{array}{ccc}
A & \xrightarrow{J} & B \\
\downarrow{d} & & \downarrow{1} \\
M & \xrightarrow{l} & M
\end{array}
\]

exhibits \(l\) as the \(J\)-weighted colimit of \(d\) (Definition 2.15) if and only if it exhibits \(l\) as the pointwise \(J\)-weighted colimit of \(d\) (Definition 2.30).

**Proof.** Let \(H : C \rightarrow D\), \(f : C \rightarrow B\) and \(e : D \rightarrow M\) be any morphisms in \(K\), and consider cells that are of the form as in the diagram below:

\[
\begin{array}{ccc}
J/f & \overset{C(\pi_c, \text{id})}{\longrightarrow} & C \\
\downarrow{\pi_A} & & \downarrow{H} \\
A & \xrightarrow{\phi} & D \\
\downarrow{d} & & \downarrow{e} \\
M & \longrightarrow & M
\end{array}
\quad
\begin{array}{ccc}
A & \xrightarrow{J} & B \\
\downarrow{d} & & \downarrow{1} \\
M & \xrightarrow{\psi} & M
\end{array}
\]

\[
\begin{array}{ccc}
\begin{array}{ccc}
J/f & \overset{C(\pi_c, \text{id})}{\longrightarrow} & C \\
\downarrow{\pi_A} & & \downarrow{H} \\
A & \xrightarrow{\phi} & D \\
\downarrow{d} & & \downarrow{e} \\
M & \longrightarrow & M
\end{array}
\quad
\begin{array}{ccc}
\begin{array}{ccc}
A & \xrightarrow{J} & B \\
\downarrow{d} & & \downarrow{1} \\
M & \xrightarrow{\psi} & M
\end{array}
\end{array}
\]

Here the top double headed arrow denotes the correspondence between cells of the form \(\phi\) and \(\psi\) that follows from the fact that \(J/f\) is a strong double comma object: for each \(\phi\) there is a unique \(\psi\) such that \(\phi = \psi \circ (\pi \circ \eta_f \circ \text{id}_H)\) (here we have taken \(J(\text{id}, f) = J \circ B(\text{id}, f)\) in Definition 2.33 so that \(\pi_f = \pi \circ \eta_f\)). Also the bottom arrow denotes a bijective correspondence, that is given by the assignments \(\phi' \mapsto (U_f \circ \varepsilon_f) \circ \phi'\) and \(\psi' \mapsto \psi' \circ (\eta_f \circ \text{id}_H)\), where \(\varepsilon_f\) and \(\eta_f\) are the conjoint cells defining \(B(\text{id}, f)\).

Moreover \(\zeta\) exhibits \(l\) as the pointwise \(J\)-weighted colimit of \(d\) if and only if, for any \(f\), there is a correspondence between cells of the form \(\phi\) and \(\phi'\) as is denoted by the dashed arrow on the left: for each \(\phi\) there is a unique \(\phi'\) such that \(\phi = (\zeta \circ \pi) \circ \phi'\). Likewise if \(\zeta\) exhibits \(l\) as the ordinary \(J\)-weighted colimit of \(d\) then there exists a correspondence between cells of the form \(\psi\) and \(\psi'\) as on the right: for each \(\psi\) there is a unique \(\psi'\) such that \(\psi = \zeta \circ \psi'\). The latter are in fact equivalent: by taking
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Let \( f = \text{id}_H \) and vertically precomposing the cells \( \psi \) and \( \psi' \) with the isomorphisms \( H \cong B(\text{id}, \text{id}) \circ H \) we see that, in terms of Proposition 2.14, \( \zeta \) exhibits \( \iota \) as the top absolute left Kan extension of \( d \) along \( J \).

We claim that the two ‘dashed’ correspondences above are equivalent, thus proving the theorem. To see this, consider pairs of cells \((\phi, \psi)\) and \((\phi', \psi')\) that correspond under the top and bottom correspondence above. Then \( \psi = \zeta \circ \psi' \) implies

\[
\begin{array}{ccc}
\pi & \phi' \\
\zeta & \psi
\end{array}
= \begin{array}{ccc}
\pi & \eta_f \\
\zeta & \psi'
\end{array}
= \begin{array}{ccc}
\pi & \eta_f \\
\psi
\end{array}
= \phi
\]

For the converse assume \( \phi = (\zeta \circ \pi) \circ \phi' \). Then

\[
\begin{array}{ccc}
\pi_f & \\
\zeta & \psi'
\end{array}
= \begin{array}{ccc}
\pi & \eta_f \\
\zeta & \phi'
\end{array}
= \begin{array}{ccc}
\pi & \phi' \\
\zeta
\end{array}
\]

which, since factorisations through \( \pi_f \) are unique, implies \( \psi = \zeta \circ \psi' \). This concludes the proof.

Combined with Proposition 2.36 we obtain:

**Corollary 2.38.** Let \( \mathcal{E} \) be a category with finite limits and reflexive coequalisers preserved by pullbacks, so that internal profunctors in \( \mathcal{E} \) form an equipment \( \text{Prof}(\mathcal{E}) \) by Proposition 2.17. The pointwise weighted colimits in \( \text{Prof}(\mathcal{E}) \) coincide with the ordinary weighted colimits.
Chapter 3

Monads on equipments

Here we recall the notion of monads on equipments which, analogous to monads on
an ordinary category, allow us to describe algebraic structures in equipments. To do
so, we will first need to consider functors of equipments and their transformations:
these will be recalled in the first section below, and again mostly from [Shu08 Section
6]. Following this, Section 3.2 introduces the two main examples of monads
that we will consider: the ‘free symmetric strict monoidal V-category’-monad S on
V-Prof and the ‘free strict double category’-monad D on Prof(G_1), where G_1 is the
category (0 ⇒ 1). Both of them were briefly discussed in the introduction to this
thesis. In general the functor underlying a monad T, on an equipment K, does need
not preserve the horizontal units and horizontal composition of K strictly. It will be
important that the monads T we consider are ‘normal’, meaning that they preserve
horizontal units. Many monads, including S and D, are normal, and every nor-
mal monad T induces a ‘vertical 2-monad’ V(T) on the vertical 2-category V(K).

3.1 Functors, transformations and monads

We start with the definition of functors.

Definition 3.1. Let K and L be pseudo double categories. A lax functor F: K → L
consists of a pair of functors F_0: K_0 → L_0 and F_1: K_1 → L_1, such that LF_1 = F_0L
and RF_1 = F_0R, together with natural transformations

\[
\begin{array}{ccc}
\K_1 \times_{\K_0} \K_1 & \xrightarrow{\otimes_{\K}} & \K_1 \\
F_1 \times_{\K_0} F_1 & \xleftarrow{F_0} & F_1 \\
\L_1 \times_{\L_0} \L_1 & \xrightarrow{\otimes_{\L}} & \L_1 \\
F_1 & \xleftarrow{F_0} & F_0
\end{array}
\]

whose components are horizontal cells, and that satisfy the usual associativity and
unit axioms for lax monoidal functors, see e.g. [ML98 Section XI.2]. The transforma-
tions F_0 and F_U are called the compositor and unitor of F. A normal functor
$F: \mathcal{K} \to \mathcal{L}$ is a lax functor for which $F_U$ is the identity; a normal functor whose compositor $F_\circ$ is invertible is called a pseudofunctor.

Unpacking this, a lax functor $F: \mathcal{K} \to \mathcal{L}$ maps the objects, vertical and horizontal morphisms, as well as cells, of $\mathcal{K}$ to those of $\mathcal{L}$, in a way that preserves horizontal and vertical sources and targets. Moreover, composition of vertical maps is preserved while horizontal compositions and units are preserved only up to natural coherent cells $F_\circ: F_j \circ FH \Rightarrow F(J \circ H)$ and $F_U: U_{FA} \Rightarrow FU_A$.

The notions of normal functors and pseudofunctors above are slightly stronger than the ones in [Shul08 Section 6], where the unitors $F_U$ are only required to be invertible: such functors we will call weakly normal. The stronger notions are preferable as they are easier to work with, while Proposition 3.17 below shows that every weakly normal functor is equivalent to a normal one. Most of the functors that we encounter will be normal.

We will often use the unit axiom: for a normal functor $F$ it means that for any horizontal morphism $J: A \Rightarrow B$, the composition

$$U_{FA} \circ FJ = FU_A \circ FJ \xRightarrow{F \circ \epsilon} F(U_A \circ J) \xRightarrow{\epsilon} FJ$$

is equal to $\mathbf{1}$: $U_{FA} \circ FJ \Rightarrow FJ$, and similar for $\tau$. Also notice that, with $F$ still normal, the naturality of the unitor implies that $FU_f = U_{Ff} : U_{FA} \Rightarrow U_{FC}$ for any $f: A \Rightarrow C$.

**Example 3.2.** Any pullback preserving functor $F: \mathcal{D} \to \mathcal{E}$ between categories with finite limits induces a pseudofunctor $\text{Span}(F): \text{Span}(\mathcal{D}) \to \text{Span}(\mathcal{E})$, simply by applying $F$ to the spans in $\mathcal{D}$. Clearly this preserves unit spans strictly, while the existence of invertible compositors follows from the fact that $F$ preserves pullbacks. The functor underlying the ‘free strict category’-monad is an example.

Recall that the companion $C(f, \text{id})$ of a morphism $f: A \to C$ is defined by two cells $\eta: C(f, \text{id}) \Rightarrow UC$ and $\epsilon: U_A \Rightarrow C(f, \text{id})$ that satisfy the companion identities, see the discussion following Definition 1.9. That, between equipments, normal functors are the right morphisms to consider is because they preserve companions and conjoints, as follows.

The following is a direct consequence of [Shul08 Proposition 6.8]. Although the proof is relatively simple, it is important to our theory: without the inverses to the compositors below the main result (Theorem 4.32) could not have been proven for monads that are not pseudo.

**Proposition 3.3 (Shulman).** Let $F: \mathcal{K} \to \mathcal{L}$ be a normal functor between equipments. For any morphism $f: A \to C$ in $\mathcal{K}$ the images

$$Ff\varepsilon: FC(f, \text{id}) \Rightarrow U_{FC} \quad \text{and} \quad Ff\eta: U_{FA} \Rightarrow FC(f, \text{id})$$

define $FC(f, \text{id})$ as the companion of $Ff$ in $\mathcal{L}$. In particular the canonical map $FC(f, \text{id}) \Rightarrow FC(Ff, \text{id})$, that is induced by the cartesian filler of $FC(Ff, \text{id})$, is invertible. Moreover the compositor

$$F_\circ: FC(f, \text{id}) \circ FK \Rightarrow F(C(f, \text{id}) \circ K)$$

is invertible for any promorphism $K: C \Rightarrow D$. Analogous results hold for conjoints: for each morphism $g: B \to D$ the image $FD(\text{id}, g)$ is the conjoint of $Fg$ and the compositor

$$F_\circ: FK \circ FD(\text{id}, g) \Rightarrow F(K \circ D(\text{id}, g))$$

is invertible.
The images $F_f \varepsilon$ and $F_f \eta$ satisfy the vertical companion identity because $F$ preserves vertical composition; to see that they satisfy the horizontal one as well notice that

$$r \circ (F_f \eta \odot F_f \varepsilon) \circ \Gamma^{-1} = F_r \circ F_{\odot} \circ (F_f \eta \odot F_f \varepsilon) \circ \Gamma^{-1} = F_r \circ F_f \circ (F_f \eta \odot F_f \varepsilon) \circ \Gamma^{-1} = F_r \circ (F_f \eta \odot F_f \varepsilon) \circ \Gamma^{-1} = \text{id}_{F(C(f, \text{id}))},$$

which follows from the unit axiom for $F$, the naturality of $F_{\odot}$ and the horizontal companion identity for $C(f, \text{id})$. We claim that the inverse to $F_{\odot}$ above is given by the composite

$$F(C(f, \text{id}) \odot K) \xrightarrow{\Gamma^{-1}} U_{F_A} \odot F(C(f, \text{id}) \odot K) \xrightarrow{F_f \eta \odot F_f \varepsilon \odot \text{id}} FC(f, \text{id}) \odot F(U_C \odot K) \xrightarrow{\text{id} \odot \text{id}} FC(f, \text{id}) \odot FK.$$ 

Indeed, using the unit axiom for $F$ we find that pre- and postcomposing this with $F_{\odot}$ gives respectively

$$F(C(f, \text{id}) \odot K) \xrightarrow{F_f^{-1}} F(U_A \odot C(f, \text{id}) \odot K) \xrightarrow{F_f \eta \odot F_f \varepsilon \odot \text{id}} F(C(f, \text{id}) \odot U_C \odot K) \xrightarrow{\text{id} \odot \text{id}} F(C(f, \text{id}) \odot K)$$

and

$$FC(f, \text{id}) \odot FK \xrightarrow{\Gamma^{-1}} U_{F_A} \odot FC(f, \text{id}) \odot FK \xrightarrow{F_f \eta \odot F_f \varepsilon \odot \text{id}} FC(f, \text{id}) \odot U_{F_C} \odot K \xrightarrow{\text{id} \odot \text{id}} FC(f, \text{id}) \odot K$$

so that the claim follows from the horizontal companion identities for $FC(f, \text{id})$ and $C(f, \text{id})$ and the fact that $(\text{id} \odot \text{id}) \circ a = r \circ \text{id}$, which follows from the coherence axioms for pseudo double categories.

Normal functors between equipments are compatible with the correspondences of Proposition 1.24 as follows.

**Proposition 3.4.** Let $F: \mathcal{K} \to \mathcal{L}$ be a normal functor. The diagram on the left commutes for any cell $\phi$ on the right, where the isomorphisms are given by the previous proposition, and the cells $\lambda F \phi$ and $\lambda F \phi$ are given by Proposition 1.24.

$$\begin{array}{ccc}
FJ \circ FD(Fg, \text{id}) & \cong & FJ \circ FD(g, \text{id}) \\
\lambda F \phi \downarrow & & A \xrightarrow{J} B \\
FC(Ff, \text{id}) \odot FK & \xrightarrow{\varphi} & F(J \odot D(g, \text{id})) \\
\otimes & & f \downarrow \varphi \downarrow g \\
FC(f, \text{id}) \odot FK \xrightarrow{\otimes} F(C(f, \text{id}) \odot K)
\end{array}$$

A similar diagram commutes for the cells $p F \phi$ and $p F \phi$.

**Proof.** Writing $f \varepsilon$ and $F_f \varepsilon$ for the companion cells defining the companions $C(f, \text{id})$ and $FC(Ff, \text{id})$ respectively, the isomorphism on the left is the unique factorisation $FC(Ff, \text{id}) \Rightarrow FC(f, \text{id})$ of $F_f \varepsilon$ through $F_f \varepsilon$. To show that the diagram commutes, we will postcompose it with $F(f \varepsilon \odot \text{id}) : F(C(f, \text{id}) \odot K) \to FK$. Notice that, since the composite $F(f \varepsilon \odot \text{id}) \circ F_{\odot} = F_f \varepsilon \odot \text{id}$ is a cartesian filler, and $F_{\odot}$ is an isomorphism by the previous proposition, the cell $F(f \varepsilon \odot \text{id})$ is a cartesian.
fller as well. Hence, by uniqueness of factorisations through cartesian fillers, if the
diagram commutes after postcomposition with \(F(f \circ \xi \circ \text{id})\), then it commutes
itself as well. The bottom leg of the diagram above, postcomposed with \(F(f \circ \xi \circ \text{id})\),
is the composite
\[
\left[ FJ \circ FD(Fg, \text{id}) \xrightarrow{\lambda F \phi} FC(Ff, \text{id}) \circ FK \xrightarrow{\phi \circ \text{id}} FK \right]
\]
where the equality follows from \(\lambda F \phi = Ff \eta \circ \phi \circ Fg \xi\) and the vertical companion
identity \(Ff \xi \circ Ff \eta = UFf\). Likewise, precomposing the bottom leg with \(F(f \circ \xi \circ \text{id})\) gives
\[
FJ \circ FD(Fg, \text{id}) \Rightarrow FJ \circ FD(g, \text{id}) \xrightarrow{F \circ \xi} F(J \circ D(g, \text{id})) \xrightarrow{F(\phi \circ \eta \circ \xi)} FK
\]
as well, and the proof follows. \(\square\)

Lax functors between closed equipments are compatible with left homs, as follows.

**Proposition 3.5.** Let \(F \colon K \to \mathcal{L}\) be a lax functor between closed equipments. The
coherence cells \(F\circ\) induce canonical horizontal cells
\[
F(J \circ H) \circ K \Rightarrow FJ \circ (FH \circ K)
\]
in \(\mathcal{L}\), for any triple of promorphisms \(J \colon A \Rightarrow B\) and \(H \colon A \Rightarrow C\) in \(K\), and
\(K \colon FC \Rightarrow D\) in \(\mathcal{L}\).

**Proof.** Simply take the adjoint of
\[
FJ \circ F(J \circ H) \circ K \xrightarrow{F_0 \circ \text{id}} F(J \circ J \circ H) \circ K \xrightarrow{F \circ \text{id}} FH \circ K.\] \(\square\)

Having introduced functors we turn to transformations.

**Definition 3.6.** A transformation \(\xi \colon F \to G\) between lax functors \(F\) and \(G \colon K \to \mathcal{L}\) is
given by natural transformations \(\xi_0 \colon F_0 \to G_0\) and \(\xi_1 \colon F_1 \to G_1\), with \(L_1 = \xi_0 L\)
and \(R_1 = \xi_0 R\), and such that the following diagrams commute, which we will call
the composition and unit axiom respectively.

Unpacking this, a natural transformation \(\xi\) consists of a vertical morphism
\(\xi_A \colon FA \to GA\) for every object \(A\) of \(K\), as well as a cell \(\xi_J\) for every horizontal
morphism \(J \colon A \Rightarrow B\) in \(K\), as on the left below. The composition and unit axioms
state that the diagrams on the right commute. In the case of normal functors \(F\) and \(G\) the unit diagram below reduces to \((\xi)_{UA} = U\xi_A\), for every object \(A\) of \(K\).

\[\begin{array}{ccc}
FJ & F & FJ \circ FH \xrightarrow{\xi_J} F(J \circ H) & U_{FA} \circ FJ \xrightarrow{UJ} FU_A \\
\xi_A & \downarrow \xi_L & \downarrow \xi_{J \circ H} & \downarrow U_{J \circ H} \circ U_{\xi_A} \\
GA & \xrightarrow{FJ} GB & GJ \circ GH \xrightarrow{\xi_G} G(J \circ H) & U_{GA} \circ GU_A
\end{array}\]
Proposition 3.7. Let \( \xi: F \to G \) be a transformation of normal functors \( F, G: \mathcal{K} \to \mathcal{L} \) between equipments. For any morphism \( f: A \to C \) the components \( \xi_{C(f, \text{id})} \) and \( \xi_{G(f, \text{id})} \) coincide with the composites

\[
\begin{array}{cccccc}
Ff & Ff & Ff & FA & FA & FA \\
\xi_A & \xi_A & \xi_A & FA & FA & FA \\
Gf & Gf & Gf & FC & FC & FC \\
GA & GA & GA & FC & FC & FC \\
Gf & Gf & Gf & GC & GC & GC \\
\xi_C & \xi_C & \xi_C & GC & GC & GC \\
\end{array}
\]

Proof. That \( \xi_{C(f, \text{id})} \) equals the composite on the left follows directly by horizontally composing the following identity, which follows from the unit axiom of \( \xi \) and its naturality, on the right by \( Ff \varepsilon \).

\[
Gf \eta \circ U\xi_A = Gf \eta \circ \xi U_A = \xi_{C(f, \text{id})} \circ Ff \eta
\]

A similar argument shows that \( \xi_{G(f, \text{id})} \) coincides with the composite on the right. \( \square \)

We recall [Shul08, Proposition 6.17].

Proposition 3.8 (Shulman). Small equipments, lax functors and transformations form a strict 2-category \( \mathbf{Equip}_n \). Restricting to normal functors and pseudofunctors we obtain respectively sub-2-categories \( \mathbf{Equip}_n \) and \( \mathbf{Equip}_{ps} \).

Of course \( \mathbf{Equip}_n \) is contained in the larger 2-category \( \mathbf{Dbl}_l \) of small pseudo double categories, lax functors and transformations, which in turn contains sub-2-categories \( \mathbf{Dbl}_n \) and \( \mathbf{Dbl}_{ps} \) consisting of its normal functors and pseudofunctors respectively. The following is [Shul08, Example 6.20], where we have denoted by \( \mathbf{Cat}_n \) the 2-category of categories with finite limits, pullback preserving functors, and natural transformations.

Proposition 3.9 (Shulman). The assignment \( \mathcal{E} \mapsto \text{Span}(\mathcal{E}) \) given by Example 1.6 extends to a 2-functor \( \text{Span}: \mathbf{Cat}_n \to \mathbf{Equip}_{ps} \).

Following [Shul08], we denote by \( \mathbf{Equip}_0 \) and \( \mathbf{Equip}_n \) the sub-2-categories of \( \mathbf{ Equip}_0 \) and \( \mathbf{Equip}_n \) consisting of equipments \( \mathcal{K} \) for which every \( H(\mathcal{K})(A, B) \) has reflexive coequalizers that are preserved by \(- \circ -\) on both sides, that is those equipments that satisfy the conditions of Proposition 1.17. The following result is [Shul08, Proposition 11.12].

Proposition 3.10 (Shulman). The assignment \( \mathcal{K} \mapsto \text{Mod}(\mathcal{K}) \) given by Proposition 1.17 extends to a 2-functor \( \text{Mod}: \mathbf{Equip}_0 \to \mathbf{Equip}_n \). The image \( \text{Mod}(F) \) of a pseudofunctor \( F: \mathcal{K} \to \mathcal{L} \) is again a pseudofunctor whenever \( F \) preserves the reflexive coequalizers of \( H(\mathcal{K})(A, B) \).

Sketch of the proof. A lax functor \( F: \mathcal{K} \to \mathcal{L} \) between equipments \( \mathcal{K} \) and \( \mathcal{L} \) preserves monoids, morphisms of monoids, bimodules and cells of bimodules in the same way that lax monoidal functors do. Now suppose that \( \mathcal{K} \) and \( \mathcal{L} \) are contained in \( \mathbf{Equip}_0 \), so that their monoids and bimodules form equipments \( \text{Mod}(\mathcal{K}) \) and \( \text{Mod}(\mathcal{L}) \) by Proposition 1.17. Since the horizontal unit \( U_A \) for a monoid \( A: A_0 \to A_0 \) in \( \mathcal{K} \) is \( A \) itself, we have \( FU_A = U_{FA} \); so we can take the unitor of \( \text{Mod}(F) \) to be the identity. To construct the compositor \( \text{Mod}(F)_\circ: FJ \circ_{FB} FH \Rightarrow F(J \circ_{FB} H) \) for
composable bimodules $J: A \to B$ and $H: B \to C$, consider the following diagram in $H(\mathcal{L})(A_0, C_0)$.

\[
\begin{array}{ccc}
FJ \circ FB \circ FH & \cong & FJ \circ FH \\
\downarrow & & \downarrow \circ \ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \cir
\end{array}
\]

Here the top row is the coequaliser defining $FJ \circ FB \circ FH$ in $\text{Mod}(\mathcal{L})$, the bottom row is the image of the coequaliser defining $J \circ_B H$ and the solid vertical cells are given by the compositors of $F$. We can take the compositor $\text{Mod}(F)\circ$ to be the unique cell given by the universal property of the coequaliser $FJ \circ FB \circ FH$, making $\text{Mod}(F)$ into a normal functor $\text{Mod}(\mathcal{K}) \to \text{Mod}(\mathcal{L})$. Moreover, if $F$ is a pseudofunctor, so that the solid vertical cells are invertible, then $\text{Mod}(F)\circ$ is again invertible when coequaliser at the bottom row is preserved by $F$, thus proving the second assertion. Similarly the components of a transformation $\xi: F \to G$ induce a transformation $\text{Mod}(\xi): \text{Mod}(F) \to \text{Mod}(G)$. \qed

Remember that every pseudo double category $\mathcal{K}$ induces a 2-category $V(\mathcal{K})$ of vertical morphisms and vertical cells. In the proposition below $2\text{-Cat}$ denotes the 2-category of 2-categories, 2-functors and 2-transformations.

**Proposition 3.11.** The assignment $\mathcal{K} \mapsto V(\mathcal{K})$ extends to a strict 2-functor

$$V: \text{Dbl}_n \to \text{2-Cat}.$$ 

**Proof.** A normal functor $F: \mathcal{K} \to \mathcal{L}$ preserves vertical cells, so $V(F)$ can be defined simply by applying $F$ to the objects, vertical morphisms and vertical cells. That $V(F)$ preserves composition strictly follows from the unit axiom for $F$. Likewise the vertical part $\xi_0: F_0 \to G_0$, of any transformation $\xi: F \to G$, forms a 2-transformation $V(\xi): V(F) \to V(G)$: the naturality with respect to vertical cells follows from the unit axiom for $\xi$. \qed

Having introduced functors and transformations we are ready to define monads on pseudo double categories.

**Definition 3.12.** A lax monad $T$ on a pseudo double category $\mathcal{K}$ is a monad $T = (T, \mu, \eta)$ on $\mathcal{K}$ in $\text{Dbl}$, consisting of a lax functor $T: \mathcal{K} \to \mathcal{K}$, a multiplication $\mu: T^2 \to T$ and a unit $\eta: \text{id}_{\mathcal{K}} \to T$, satisfying the usual axioms (see [ML98 Section VI.1]). A lax monad $T$ on $\mathcal{K}$ is called a normal monad or pseudomonad whenever the underlying endofunctor $T: \mathcal{K} \to \mathcal{K}$ is a normal functor or pseudofunctor.

By Proposition 3.11 a normal monad $T$ on $\mathcal{K}$ induces a 2-monad $V(T)$ on $V(\mathcal{K})$. As mentioned in the introduction, this allows us in the next section to define $T$-algebras, $T$-morphisms and vertical $T$-cells as $V(T)$-algebras, $V(T)$-morphisms and $V(T)$-cells in $V(\mathcal{K})$. Notice that for $\mathcal{K}$ in $\text{Equip}^\delta$, any lax monad $S$ on $\mathcal{K}$ induces a normal monad $\text{Mod}(S)$ on $\text{Mod}(\mathcal{K})$, by Proposition 3.10.

**Example 3.13.** We saw in Example 1.19 that if $\mathcal{E}$ is a category with finite limits and reflexive coequalisers, the latter preserved by pullbacks, then categories internal to $\mathcal{E}$, together with internal profunctors, form an equipment $\text{Prof}(\mathcal{E}) = \text{Mod}(\text{Span}(\mathcal{E}))$. In that case any pullback preserving monad $T$ on $\mathcal{E}$ induces a pseudomonad $\text{Span}(T)$ on $\text{Span}(\mathcal{E})$ by Proposition 3.9 which, by Proposition 3.10, induces a normal monad $\text{Prof}(T)$ on $\text{Prof}(\mathcal{E})$. This is how, in the next section, the ‘free strict double category’-monad on $\text{Prof}(G_1)$ will be obtained from the ‘free category’-monad on the category

---

*Note: This text contains mathematical content and is formatted as a natural language representation.*
3.1. FUNCTORS, TRANSFORMATIONS AND MONADS

\[ G_1 \] of presheaves on \( G_1 = (0 \Rightarrow 1) \). Even though the latter is a pseudomonad the former is not, as we will see in Proposition 3.25.

Many interesting monads on bicategories of spans arise in this way, see for example [Lei04, Section 4.1] where Leinster discusses ‘cartesian monads’: a monad \((T, \mu, \eta)\) on a category \( \mathcal{E} \) with finite limits is cartesian whenever \( T \) preserves pullbacks and the naturality squares of \( \mu \) and \( \eta \) are pullbacks. In our terms the second condition means that for each span \( J \) the cells \( \mu_J \) and \( \eta_J \) are right invertible, see Example 1.30.

Remember that weighted colimits can be defined in any pseudo double category, see Definition 2.15. The following two propositions record some useful consequences of the universality of weighted colimits, with respect to normal functors of pseudo double categories and the transformations between them.

**Proposition 3.14.** Let \( F: \mathcal{K} \to \mathcal{L} \) be a normal functor of pseudo double categories. Given a colimit diagram \( M \leftarrow A \xrightarrow{d} B \) in \( \mathcal{K} \), any vertical morphism \( f: FM \to N \) in \( \mathcal{L} \) induces a canonical vertical cell

\[
\colim FJ (f \circ Fd) \Rightarrow f \circ F (\colim J d)
\]

of morphisms \( FB \to N \) in \( \mathcal{L} \), provided these weighted colimits exist. In the case that \( \mathcal{K} \) and \( \mathcal{L} \) are closed equipments this cell corresponds, after applying \( \lambda \) (Proposition 1.24) and under the canonical isomorphisms

\[
N(f \circ F (\colim J d), \text{id}) \cong F (J \otimes M (d, \text{id})) \circ N(f, \text{id})
\]

and

\[
N(\colim FJ (f \circ Fd), \text{id}) \cong FJ \circ (FM (d, \text{id}) \circ N(f, \text{id}))
\]

to the horizontal cell \( F (J \otimes M (d, \text{id})) \circ N(f, \text{id}) \Rightarrow FJ \circ (FM (d, \text{id}) \circ N(f, \text{id})) \) given by Proposition 3.5.

**Proof.** Supposing that the weighted colimits exist, denote by \( \eta \) and \( \zeta \) the cells that exhibit respectively \( l = \colim J d \) as the \( J \)-weighted colimit of \( d \) and \( k = \colim FJ (f \circ Fd) \) as the \( FJ \)-weighted colimit of \( f \circ Fd \), as in Proposition 2.14. It follows from the universality of \( \zeta \) (Definition 2.11) that there exists a unique vertical cell \( \phi: k \Rightarrow f \circ Fl \) such that

\[
\begin{array}{ccc}
FA & \xrightarrow{FJ} & FB \\
\downarrow Fd & & \downarrow Fd \\
FM & \xrightarrow{f} & FM \\
\downarrow f & & \downarrow f \\
N & \xrightarrow{f} & N
\end{array}
\]

and

\[
\begin{array}{ccc}
FA & \xrightarrow{FJ} & FB \\
\downarrow Fd & & \downarrow Fd \\
FM & \xrightarrow{f} & FM \\
\downarrow f & & \downarrow f \\
N & \xrightarrow{f} & N
\end{array}
\]

which is the canonical vertical cell that we seek. To prove the second assertion we apply \( \lambda \) to the latter identity. Using the functoriality of \( \lambda \) (Proposition 1.24), we obtain the top half of the following commuting diagram, while the bottom half is given by Proposition 3.4

\[
\begin{array}{ccc}
FJ \circ N(f \circ Fl, \text{id}) & \xrightarrow{\text{id} \circ \lambda \phi} & FJ \circ N(k, \text{id}) \\
\downarrow FJ \circ FM (l, \text{id}) \circ N(f, \text{id}) & \xrightarrow{\lambda Fg \circ \text{id}} & \downarrow FJ \circ N(f, \text{id}) \\
N(f \circ Fd, \text{id}) & \xrightarrow{\lambda Fg \circ \text{id}} & N(f, \text{id})
\end{array}
\]

or

\[
\begin{array}{ccc}
F(J \otimes M (l, \text{id})) \circ N(f, \text{id}) & \xrightarrow{\text{id} \circ \lambda \phi} & FJ \circ N(k, \text{id}) \\
\downarrow F(J \otimes M (l, \text{id})) \circ N(f, \text{id}) & \xrightarrow{\lambda Fg \circ \text{id}} & \downarrow FM (d, \text{id}) \circ N(f, \text{id}) \\
N(f \circ Fd, \text{id}) & \xrightarrow{\lambda Fg \circ \text{id}} & N(f, \text{id})
\end{array}
\]

or

\[
\begin{array}{ccc}
FJ \circ N(f \circ Fl, \text{id}) & \xrightarrow{\text{id} \circ \lambda \phi} & FJ \circ N(k, \text{id}) \\
\downarrow FJ \circ FM (l, \text{id}) \circ N(f, \text{id}) & \xrightarrow{\lambda Fg \circ \text{id}} & \downarrow FJ \circ N(f, \text{id}) \\
N(f \circ Fd, \text{id}) & \xrightarrow{\lambda Fg \circ \text{id}} & N(f, \text{id})
\end{array}
\]
We claim that under (2.1) this diagram is adjoint to
\[ \begin{array}{ccc}
N(f \circ Fl, \text{id}) & \xrightarrow{\lambda \phi} & N(k, \text{id}) \\
\downarrow & & \downarrow \nu^{(\lambda \phi)} \\
FM(l, \text{id}) \circ N(f, \text{id}) & \xrightarrow{FJ \circ N(f \circ Fd, \text{id})} & FJ \circ N(f, \text{id}) \\
\downarrow_{F(\lambda \phi)} & & \downarrow \\
F(M(d, \text{id}) \circ N(f, \text{id})) & \rightarrow & FJ \circ (FM(d, \text{id}) \circ N(f, \text{id})),
\end{array} \]
where the bottom cell is that given by Proposition 3.5. Indeed it is clear that the top legs are adjoint; to see that the bottom ones are too notice that, in general, the following (which is Lemma B.1(c)) holds: given any horizontal cell \( H \Rightarrow J \circ K \) that is adjoint to \( \psi : J \Rightarrow H \Rightarrow K \), the composite (where the second cell is given by Proposition 3.5)
\[ FH \circ L \Rightarrow F(J \circ K) \circ L \Rightarrow FJ \circ (FK \circ L) \]
is adjoint to \( FJ \circ FH \circ L \xrightarrow{F \circ \text{id}} F(J \circ H) \circ L \xrightarrow{F \circ \text{id}} FK \circ L \). Applying this to the bottom leg of the bottom diagram above we find it is adjoint to that of the top diagram. This completes the proof, since the sides of the diagram above are the isomorphisms of the assertion. □

Example 3.15. If \( K = \mathcal{V}-\text{Prof} \) and \( F \) is the identity, then the canonical transformation \( \text{colim}_J (f \circ d) \Rightarrow f \circ \text{colim}_J d \) is the usual unique transformation \( \phi \) whose components make the diagrams
\[ \begin{array}{ccc}
J(id, b) & \xrightarrow{\eta} & M(d, \text{colim}_J d(b)) \\
\downarrow_{\zeta} & & \downarrow f \\
M(f \circ d, \text{colim}_J (f \circ d)(b)) & \xrightarrow{M(id, \phi_b)} & M(f \circ d, (f \circ \text{colim}_J d)(b))
\end{array} \]
commute for each \( b \) in \( B \), where \( \eta \) and \( \zeta \) are the units of the weighted colimits \( \text{colim}_J (f \circ d)(b) \cong \text{colim}_{J(-, b)} d \) and \( \text{colim}_J (f \circ d)(b) \cong \text{colim}_{J(-, b)} (f \circ d) \), see Example 2.13. In particular, if the target \( N \) of \( f \) is cocomplete, then \( \phi_b \), as a map \( \int^y (f \circ d)(x) \otimes J(x, b) \Rightarrow f(\int^y dy \otimes J(y, b)) \) is induced by the composites
\[ (f \circ d)(x) \otimes J(x, b) \Rightarrow f(dx \otimes J(x, b)) \Rightarrow f(\int^y dy \otimes J(y, b)), \]
where the first map exists by universality of the copower \( (f \circ d)(x) \otimes J(x, b) \) and the second map is the \( f \)-image of insertion into the coend.

Proposition 3.16. Let \( \xi : F \Rightarrow G \) be a natural transformation between normal functors \( F \) and \( G : K \Rightarrow \mathcal{L} \) of pseudo double categories. Given a colimit diagram \( M \xleftarrow{A} A \xrightarrow{B} B \) in \( K \), any vertical morphism \( g : GM \Rightarrow N \) in \( \mathcal{L} \) induces a canonical vertical cell
\[ \text{colim}_{F,J}(g \circ Gd \circ \xi_A) \Rightarrow \text{colim}_{G,J}(g \circ Gd) \circ \xi_B, \]
provided that these colimits exist. This cell makes the following diagram commute, where the other unlabelled cells are given by the previous proposition. Moreover it
is invertible whenever $\xi_J$ is right invertible.

$$
\begin{align*}
\text{colim}_{GJ}(g \circ Gd) \circ \xi_B & \Rightarrow \text{colim}_{FJ}(g \circ Gd \circ \xi_A) \\
& \Rightarrow g \circ (\text{colim}_J d) \circ \xi_B \\
& \Rightarrow \text{colim}_{FJ}(g \circ \xi_M \circ Fd) \\
& \Rightarrow g \circ \xi_M \circ F(\text{colim}_J d)
\end{align*}
$$

**Proof.** Denoting by $\zeta$ and $\theta$ the cells that define respectively the weighted colimits $l = \text{colim}_{GJ}(g \circ Gd)$ and $k = \text{colim}_{FJ}(g \circ Gd \circ \xi_A)$, then the canonical vertical cell above is the unique factorisation $\phi$ of $\zeta \circ \xi_J$ through $\theta$: $\begin{align*}
FA & \xrightarrow{FJ} FB \\
GJ \downarrow \zeta & \quad \zeta_A \quad \zeta_B \quad \zeta_D \downarrow \\
GM \downarrow \zeta & \quad \zeta_M \quad \zeta_N \downarrow \\
\end{align*}$

That it makes the diagram above commute follows from the fact that, precomposed with $\theta$, the top and bottom leg equal respectively the left and right-hand side of

$$
U_g \circ G\kappa \circ \xi_J = U_g \circ U_{\xi_M} \circ F\kappa,
$$

where $\kappa$ is the cell defining $\text{colim}_J d$. The last assertion follows directly from Proposition 2.19 which shows that $\zeta \circ \xi_J$ exhibits $l \circ \xi_B$ as weighted colimit whenever $\xi_J$ is right invertible, so that in that case the canonical cell $\phi$ is an isomorphism.

To close this section we give the following simple coherence result, which shows that every weakly normal functor can be replaced by a normal one, so that our choice of considering just normal functors is not much of a restriction. Recall that a lax functor is called weakly normal if its unitor is invertible; we will denote by $\mathbf{Dbl}_{wn}$ the 2-category of pseudo double categories, weakly normal functors and transformations.

Recall that a 2-functor $F: \mathcal{C} \to \mathcal{D}$ is called a local equivalence if each of its hom-functors $F: \mathcal{C}(x, y) \to \mathcal{D}(Fx, Fy)$ is an equivalence, and that it is called essentially surjective on objects if for each object $y$ in $\mathcal{D}$ there is an object $x$ in $\mathcal{C}$ with $Fx \simeq y$. If $F$ is both a local equivalence and essentially surjective on objects then it is called a biequivalence. Analogous to the case of ordinary categories, one can show (see [Lei04, Proposition 1.5.13]) that the latter is equivalent to the existence of a pseudofunctor $G: \mathcal{D} \to \mathcal{C}$ together with equivalences $\text{id}_\mathcal{C} \simeq GF$ in $[\mathcal{C}, \mathcal{C}]$ and $FG \simeq \text{id}_\mathcal{D}$ in $[\mathcal{D}, \mathcal{D}]$, where $[\mathcal{C}, \mathcal{C}]$ is the 2-category of pseudofunctors $\mathcal{C} \to \mathcal{C}$, pseudonatural transformations and modifications; likewise for $[\mathcal{D}, \mathcal{D}]$.

**Proposition 3.17.** The inclusion $\mathbf{Dbl}_n \hookrightarrow \mathbf{Dbl}_{wn}$ is a biequivalence.

**Proof.** Denote the inclusion by $j$. Clearly $j$ is essentially surjective on objects so we have to prove that, for each pair of pseudo double categories $K$ and $L$, the inclusion $j: \mathbf{Dbl}_n(K, L) \hookrightarrow \mathbf{Dbl}_{wn}(K, L)$ is an equivalence. To do so consider a weakly normal functor $F: \mathcal{K} \to \mathcal{L}$ between pseudo double categories. We will construct a normal
functor $F': \mathcal{K} \to \mathcal{L}$ from $F$ simply by replacing the $F_1$-image of each horizontal unit $U_A$ by $U_{F,A}$. Simultaneously we will construct an invertible transformation 

$$\rho = \rho_F: F' \to F.$$ 

First we take $F'_0 = F_0$ and $\rho_0 = \mathrm{id}$; we define $F'_1$ and $\rho_1$ on horizontal morphisms by 

$$F'_1 J = \begin{cases} U_{F,A} & \text{if } J = U_A; \\
F_1 J & \text{otherwise} \end{cases} \quad \text{and} \quad (\rho_1)_J = \begin{cases} F_U: U_{F,A} \Rightarrow FU_A & \text{if } J = U_A; \\
\mathrm{id} & \text{otherwise}. \end{cases}$$

For a cell $\phi: J \Rightarrow K$ in $\mathcal{K}_1$, take $F'_1 \phi = \rho_K^{-1} \circ F_1 \phi \circ \rho_J$. It is clear that this makes $F'_1$ into a functor $\mathcal{K}_1 \to \mathcal{L}_1$ and that it makes $\rho_1$ natural. That $F'_1$ is compatible with $L, R: \mathcal{K}_1 \to \mathcal{K}_0$ and $L, R: \mathcal{L}_1 \to \mathcal{L}_0$ follows from the fact that the components of $\rho_1$ are horizontal cells, thus $L F'_1 = LF_1 = F_0 L$; likewise for $R$. Of course the unitor $F'_1$ we take to be the identity, making $\rho$ compatible with the unitors of $F'_1$ and $F_1$.

Finally let the compositor $F'_0$ be given by the composites

$$F' J \odot F' H \xrightarrow{\rho_J \odot \rho_H} F J \odot F H \xrightarrow{F \circ \rho_H} F (J \odot H) \xrightarrow{\rho_{J \odot H}} F' (J \odot H),$$

for any pair of horizontally composable cells $J$ and $H$ in $\mathcal{K}$. Checking that these satisfy the unit and associativity axiom is straightforward, while $\rho$ is compatible with the compositors by definition. This completes the definition of $F'$ and $\rho_F: F' \to F$. It is clear now that the assignment $F \mapsto F'$ extends to a functor 

$$(\cdot)': \mathrm{Dbl}_{\mathrm{pro}}(\mathcal{K}, \mathcal{L}) \to \mathrm{Dbl}_{\mathrm{bn}}(\mathcal{K}, \mathcal{L})$$

that maps a transformation $\xi: F \Rightarrow G$ to the composite $\xi' = \rho_G^{-1} \circ \xi \circ \rho_F: F' \Rightarrow G'$, so that the invertible transformations $\rho_F$ form a natural isomorphism $j \circ (-)': \mathrm{Dbl}_{\mathrm{pro}}(\mathcal{K}, \mathcal{L}) \cong \mathrm{id} \mathrm{Dbl}_{\mathrm{bn}}(\mathcal{K}, \mathcal{L})$. On the other hand clearly $(\cdot') \circ j = \mathrm{id} \mathrm{Dbl}_{\mathrm{bn}}(\mathcal{K}, \mathcal{L})$, and we conclude that the inclusion $j: \mathrm{Dbl}_{\mathrm{bn}}(\mathcal{K}, \mathcal{L}) \hookrightarrow \mathrm{Dbl}_{\mathrm{bn}}(\mathcal{K}, \mathcal{L})$ is an equivalence.

3.2 Examples

Here we recall the main examples: the ‘free strict monoidal $\mathcal{V}$-category’-monad $M$ and the ‘free symmetric strict monoidal $\mathcal{V}$-category’-monad $S$ on the equipment $\mathcal{V}$-$\mathrm{Prof}$ of $\mathcal{V}$-profunctors, as well as the ‘free strict double category’-monad $D$ on the equipment $\mathcal{V}$-$\mathrm{Prof}(\mathcal{G}_1)$ of $\mathcal{G}_1$-categories, where $\mathcal{G}_1 = (0 
arrow 1)$. Using Proposition 3.10, the monads $M$ and $S$ will be obtained from monads on $\mathcal{V}$-$\mathrm{Mat}$, while $D$ is obtained from the ‘free category’-monad on $\mathcal{V}$-$\mathrm{Prof}$ and $\mathcal{V}$-$\mathrm{Prof}(\mathcal{G}_1)$ respectively. All three monads are (of course) well-known, so not much in this section is original. However, our way of defining the ‘free symmetric strict monoidal category’-monad $S$, by letting it be induced by a monad on $\mathcal{V}$-$\mathrm{Mat}$, might be new. In fact, Cruttwell and Shulman discuss $S$ immediately after remarking that “Not every monad on $\mathcal{V}$-$\mathrm{Prof}$ or $\mathcal{V}$-$\mathrm{Prof}(\mathcal{E})$ is induced by one on $\mathcal{V}$-$\mathrm{Mat}$ or $\mathcal{V}$-$\mathrm{Span}(\mathcal{E})$ however. The following examples are also important.” [CS10] Example 3.14, suggesting that $S$ is not induced by a monad on $\mathcal{V}$-$\mathrm{Mat}$. That it is induced by such a monad is useful, for it allows a relatively easy proof of the fact that $S$ is a pseudomonad. Although stated in [CS10] Example 5.12 without a proof, the author has not yet seen a proof of this fact in the literature.

Monoidal $\mathcal{V}$-categories

We start with the monad for monoidal $\mathcal{V}$-categories. Throughout this subsection we assume $\mathcal{V}$ to be a cocomplete symmetric pseudomonoidal category whose colimits are preserved by the binary monoidal product $\otimes$ in both variables, so that $\mathcal{V}$-matrices and $\mathcal{V}$-profunctors form equipments $\mathcal{V}$-$\mathrm{Mat}$ and $\mathcal{V}$-$\mathrm{Prof}$; see Example 1.5.
and Example [1.3]. As mentioned above the ‘free strict monoidal \(\mathcal{V}\)-category’-monad on \(\mathcal{V}\)-\(\text{Prof}\) will be obtained from a monad on the pseudo double category \(\mathcal{V}\)-\(\text{Mat}\) of \(\mathcal{V}\)-matrices, namely the ‘free monoid’-monad, that we will also denote by \(M\) and that is given as follows.

Restricted to the objects of \(\mathcal{V}\)-\(\text{Mat}\), which are simply sets, \(M: \mathcal{V}\text{-Mat} \to \mathcal{V}\text{-Mat}\) is the usual ‘free monoid’-monad: each set \(A\) is mapped to the set
\[
MA = \prod_{n \geq 0} A^{\times n}
\]
of finite sequences of its elements. Such sequences will be denoted by \(\underline{x} = (x_1, \ldots, x_n)\) and we will write \(|\underline{x}| = n\) for the length of \(\underline{x}\); the empty sequence will be denoted \(\emptyset\).

The image \(Mf\) of a function \(f: A \to B\) applies \(f\) indexwise, so that \((Mf)(\underline{x}) = f(x)\).

The image of a \(\mathcal{V}\)-matrix \(J: A \to B\) is given by
\[
MJ(\underline{x}, \underline{y}) = \begin{cases} 
M_n J(\underline{x}, \underline{y}) & \text{if } |\underline{x}| = n = |\underline{y}|; \\
\emptyset & \text{otherwise},
\end{cases}
\]
where \(M_n J(\underline{x}, \underline{y}) = \otimes_{i=1}^{n} J(x_i, y_i)\) when \(n > 0\) and \(M_0((),()) = 1\) (the monoidal unit of \(\mathcal{V}\)), and where \(\emptyset\) is the initial object of \(\mathcal{V}\). The assignment \(J \mapsto MJ\) clearly extends to cells \(\phi: J \Rightarrow K\) of \(\mathcal{V}\)-matrices by tensoring the components of \(\phi\). One readily checks that the assignments above form two endofunctors \(M_0\) and \(M_1\) on the categories \(\mathcal{V}\text{-Mat}_0 = \text{Set}\) and \(\mathcal{V}\text{-Mat}_1\).

Notice that the unitors of \(\mathcal{V}\) induce natural isomorphisms \(MU: MUA \cong UMA\) for each set \(A\). Likewise, using the fact that \(\otimes\) preserves coproducts on both sides, the symmetric structure of \(\mathcal{V}\) induces natural isomorphisms \(M_\circ\); \(MJ \otimes MH \cong M(J \circ H)\). The coherence axioms for \(\mathcal{V}\) induce the associativity and unit axioms for \(M_U\) and \(M_\circ\), and we conclude that \(M_0\) and \(M_1\) combine to form a pseudo-endofunctor \(M\) on the equipment \(\mathcal{V}\text{-Mat}\). Finally the multiplication \(\mu: M^2 \to M\) and unit \(\eta: \text{id} \to M\), that make \(M\) into a pseudomonad on \(\mathcal{V}\text{-Mat}\), are given as follows. Given a set \(A\), the set \(M^2A\) consists of sequences of sequences of elements in \(A\), which we will call double sequences and denote by \(\underline{\underline{x}}\). The map \(\mu_A: M^2A \to MA\) is given by concatenation of double sequences, while \(\eta_A: A \to MA\) maps elements of \(A\) to the corresponding one-element sequences. Likewise the cell \(\mu_J: M^2J \Rightarrow MJ\), for a \(\mathcal{V}\)-matrix \(J: A \to B\), is induced by the isomorphisms
\[
M_{m_1} J(x_1, y_1) \otimes \cdots \otimes M_{m_n} J(x_n, y_n) \cong M_{m_1 + \cdots + m_n} J(\mu A, \mu B) (3.1)
\]
that are given by the associator of \(\mathcal{V}\); the cell \(\eta_J: J \Rightarrow MJ\) is simply given by the identities \(J(x, y) = J(\eta A, \eta B)\). That \(\mu\) satisfies the associativity axiom follows form the associativity axiom for \(\mathcal{V}\); that the unit axiom holds is clear. This completes the definition of the ‘free monoid’-monad \(M\) on \(\mathcal{V}\text{-Mat}\).

Having given \(M\) we turn to the normal monad \(\text{Mod}(M)\), on the equipment \(\text{Mod}(\mathcal{V}\text{-Mat}) = \mathcal{V}\text{-Prof}\) of \(\mathcal{V}\)-profunctors, that is induced by \(M\) by Proposition 3.10. The monad \(\text{Mod}(M)\) is the ‘free strict monoidal \(\mathcal{V}\)-category’-monad; we will again write \(M = \text{Mod}(M)\). It is shown below that \(\text{Mod}(M)\) is a pseudomonad as well.

Given a \(\mathcal{V}\)-category \(A\), the \(\mathcal{V}\)-category \(MA\) has finite sequences of \(A\)-objects as objects, i.e. \(\text{ob} MA = \prod_{n \geq 0} (\text{ob} A)^{\times n}\), while its hom-objects are given by
\[
MA(\underline{x}, \underline{y}) = \begin{cases} 
M_n A(\underline{x}, \underline{y}) & \text{if } |\underline{x}| = n = |\underline{y}|; \\
\emptyset & \text{otherwise.}
\end{cases}
\]
(3.2)
The composition of \(A\) is given indexwise after reordering:
\[
MA(\underline{x}, \underline{y}) \otimes MA(\underline{y}, \underline{z}) \cong \bigotimes_{i=1}^{n} A(x_i, y_i) \otimes A(y_i, z_i) \to \bigotimes_{i=1}^{n} A(x_i, z_i) = MA(\underline{x}, \underline{z}),
\]
where \( \underline{x}, \underline{y} \) and \( \underline{z} \) are of equal length \( n \), so that the identity on \( \underline{x} \) is given by the composition \( 1 \cong 1 \circ \frac{\text{id}_{1} \otimes \cdots \otimes \text{id}_{n}}{\text{id}_{1} \otimes \cdots \otimes \text{id}_{n}} : MA(\underline{x}, \underline{z}) \). The assignment \( A \mapsto MA(\underline{x}, \underline{z}) \) extends to \( \mathcal{V} \)-functors \( f : A \to B \) by letting \( Mf \) act indexwise: the functor \( Mf : MA \to MB \) is given on objects by \( (Mf)(\underline{x}, \underline{y}) = f(\underline{x}) \), while on hom-objects it is given by tensor products of the maps \( A(\underline{x}, \underline{y}) \to B(f(\underline{x}), f(\underline{y})) \) that define the action of \( f \) on the hom-objects of \( A \). Of course the action of \( M \) on a \( \mathcal{V} \)-profunctor \( J : A \Rightarrow B \) is given precisely like (3.2), when \( A \) is replaced by \( J \) (both \( A \) and \( J \) are \( \mathcal{V} \)-matrices). Likewise, its action on natural transformations between \( \mathcal{V} \)-profunctors is given by taking tensor products of their components, as in the definition of \( Mf \). The definitions of the multiplication \( \mu \) and unit \( \eta \) can be easily read off from those of the multiplication and unit of \('\text{free monoid}'\)-monad \( M \) on \( \mathcal{V} \text{-Mat} \) above.

Before describing the algebras for \('\text{free strict monoidal } \mathcal{V} \text{-category}'\)-monad \( M = \text{Mod}(M) \) on \( \mathcal{V} \text{-Prof} \), we remark that the \('\text{free monoid}'\)-monad \( M \) on \( \mathcal{V} \text{-Mat} \) preserves the reflexive coequalisers of \( H(\mathcal{V} \text{-Mat})(A, B) \), for any sets \( A \) and \( B \), so that by Proposition 3.10 \( \text{Mod}(M) \) is a pseudomonad. Since such coequalisers are computed pointwise this follows directly from the following proposition, which is [Rez06, Lemma 2.3.2(a)]. In general this result is very helpful when considering monads that involve tensor products.

**Proposition 3.18** (Rezk). Let \( \mathcal{V} \) be a symmetric monoidal category whose tensor product \( \otimes : \mathcal{V} \times \mathcal{V} \to \mathcal{V} \) preserves reflexive coequalisers in both variables. For each \( i = 1, \ldots, n \), let \( k_{i} : Y_{i} \to Z_{i} \) be the coequaliser of the reflexive pair \( f_{i}, g_{i} : X_{i} \rightrightarrows Y_{i} \) in \( \mathcal{V} \), with common section \( s_{i} : Y_{i} \to X_{i} \) in \( \mathcal{V} \). Then the induced diagram

\[
X_{1} \otimes \cdots \otimes X_{n} \rightrightarrows Y_{1} \otimes \cdots \otimes Y_{n} \to Z_{1} \otimes \cdots \otimes Z_{n}
\]

is again a reflexive coequaliser.

Writing \( \mathcal{D} \) for the category generated by three maps \( u : 0 \to 1, v : 0 \to 1 \) and \( s : 1 \to 0 \), under the relations \( u \circ s = \text{id}_{1} = v \circ s \), notice that a parallel pair \( f, g : X \rightrightarrows Y \) in any category \( \mathcal{C} \) is a reflexive pair if and only if there exists a functor \( R : \mathcal{D} \to \mathcal{C} \) with \( Ru = f \) and \( Rv = g \); in that case \( Rs \) is a common section for \( f \) and \( g \). The proof of the proposition above follows from the fact that the diagonal functor \( \mathcal{D} \to \mathcal{D} \times \cdots \times \mathcal{D} \) is ‘final’, see [ML98, Section IX.3].

Recall that \( M \) restricts to a 2-monad \( V(M) \) on the 2-category \( \mathcal{V} \text{-Cat} = \mathcal{V} \text{-Prof} \), of \( \mathcal{V} \)-categories, \( \mathcal{V} \)-functors and \( \mathcal{V} \)-natural transformations, by Proposition 3.1. We shall now recall the definition of colax algebras for 2-monads, as well as those of colax morphisms and cells between them, describing each of these notions in the case of the 2-monad \( V(M) \). Even though the algebras considered there are lax algebras we will follow [Lac02, Section 1], using largely the same notation.

Consider a 2-monad \( T = (T, \mu, \eta) \) on a 2-category \( \mathcal{C} \), consisting of a 2-functor \( T : \mathcal{C} \to \mathcal{C} \) and 2-natural transformations \( \mu : T^{2} \to T \) and \( \eta : \text{id}_{\mathcal{C}} \to T \) satisfying the usual axioms (see [ML98, Section VI.1]): \( \mu \circ T \mu = \mu \circ \mu T \) and \( \mu \circ T \eta = \mu \circ \eta T \).

**Definition 3.19.** Let \( T = (T, \mu, \eta) \) be a 2-monad on a 2-category \( \mathcal{C} \). A colax \( T \)-algebra \( A \) is a quadruple \( A = (A, a, \alpha, \alpha_{0}) \) consisting of an object \( A \) of \( \mathcal{C} \) equipped with a morphism \( a : TA \to A \), its structure map, and cells \( \alpha : a \circ \mu A \Rightarrow \mu \circ a T A \) and \( \alpha_{0} : a \circ \eta A \Rightarrow \text{id}_{A} \), respectively the associator and the unitor of \( A \). The latter satisfy...
the following coherence conditions:

\[
\begin{align*}
T^2 A & \xrightarrow{\mu_A} TA & T^2 A & \xrightarrow{\mu_A} TA \\
\mu_{TA} & \downarrow & \alpha & \downarrow & a & \downarrow & a \\
T^3 A & \xrightarrow{T\alpha} TA & = & T^3 A & \xrightarrow{T\alpha} TA \\
T^3 A & \xrightarrow{\eta_{TA}} TA & \xrightarrow{\alpha_0} A & \xrightarrow{id} A & \xrightarrow{id} A \\
\end{align*}
\]

We call \( A \) a pseudo \( T \)-algebra if the cells \( \alpha \) and \( \alpha_0 \) are isomorphisms; if they are identities then \( A \) is called strict. If just the unitor \( \alpha_0 \) is an identity then \( A \) is called normal.

Hence a normal colax \( V(M) \)-algebra \( A \), where \( V(M) \) is the vertical restriction of the ‘free strict monoidal \( V \)-category’-monad on \( V \)-Prof, as given above, comes equipped with a structure map that is a \( V \)-functor \( a: MA \to A \). The latter defines the tensor product \( a(x) \) of each sequence \( x \) of \( A \)-objects; we will write \( x_1 \otimes \cdots \otimes x_n = a(x) \). The fact that \( A \) is normal means that \( a(x) = x_1 \) for each sequence \( x \) of length 1. Moreover \( A \) comes with an associator \( \alpha: \alpha \circ \mu_{MA} \to \alpha \circ Ma \), that consists of \( V \)-natural maps

\[
a_x: x_{11} \otimes \cdots \otimes x_{1m_1} \otimes \cdots \otimes x_{m_1} \otimes \cdots \otimes x_{m_n} \to (x_{11} \otimes \cdots \otimes x_{1m_1}) \otimes \cdots \otimes (x_{m_1} \otimes \cdots \otimes x_{m_n}),
\]

for each double sequence \( x \). At \( x = ((x_1, x_2), (x_3, x_4, x_5), (l)) \) for example, the component \( a_x \) is a map

\[
x_1 \otimes x_2 \otimes x_3 \otimes x_4 \otimes x_5 \to (x_1 \otimes x_2) \otimes (x_3 \otimes x_4 \otimes x_5) \otimes l.
\]

Roughly speaking the associativity axiom above, for \( a \), says that the two ways of adding brackets to the tensor product \( \mu_A \circ \mu_{MA}(x) \), where \( x \) is a triple sequence, either starting with the ‘outer’ brackets or with the ‘inner’ ones, coincide. Formally it states that the following diagram commutes, for each triple sequence \( x \), where we have written \( |x| = n \) and \( |x_{ij}| = m_i \) and \( |x_{ijk}| = l_{ij} \) and where \( \otimes^{j=m_k,k=l_{ij}} x_{ijk} \) denotes the tensor product \( x_{i11} \otimes \cdots \otimes x_{im_1} \otimes \cdots \otimes x_{im_{m_1}} \otimes \cdots \otimes x_{im_{m_n}} \otimes l_{m_k} \). In the top leg
the inner brackets are added first while in the bottom one it is the outer brackets.

\[
\begin{align*}
\bigotimes_{i=1,j=1,k=1}^{i=n,j=m,l} x_{ijk} & \xrightarrow{a} \bigotimes_{i=1,j=1}^{i=n,j=m} \bigotimes_{k=1}^{k=l} x_{ijk} \\
\bigotimes_{i=1,j=1,k=1}^{i=n,j=m,l} x_{ijk} & \xrightarrow{a} \bigotimes_{i=1,j=1}^{i=n,j=m} \bigotimes_{k=1}^{k=l} x_{ijk}
\end{align*}
\]

The unit axioms for \(a\) mean that for double sequences \(x\) that are of the form \(x = (x_{11}, \ldots, x_{1n})\) or \(x = (x_{11}, \ldots, x_{nn})\) the component \(a_x\) is the identity.

In the case that \(V = \text{Set}\) the description of normal colax \(M\)-algebras above is closely related to that of ‘lax monoidal categories’ of e.g. [Lei04, Definition 3.1.1]. Indeed reversing the direction of the associator \(a\) in the discussion above we obtain the notion of a lax monoidal category whose unitors are identities. This is why, for general \(V\), normal colax \(M\)-algebras are called normal colax monoidal \(V\)-categories; we will often leave out the prefix ‘normal’. Returning to \(V = \text{Set}\), pseudo \(M\)-algebras, i.e. whose associators are invertible, are often called unbiased monoidal categories. The prefix ‘unbiased’ refers to the fact that the tensor products of all arities are part of the structure of \(A\), in contrast to the classical definition of a monoidal category, where only the nullary (unit) and binary tensor products are given. In the case of unenriched monoidal categories it is known that the biased and unbiased definitions are equivalent, see for example [Lei04, Section 3.2]. That this result can be extended to the monoidal enriched categories that are treated here seems plausible, but the author does not know of such an extension.

Next is the definition of morphisms between colax algebras.

**Definition 3.20.** Given colax \(T\)-algebras \(A = (A, a, \alpha, \alpha_0)\) and \(B = (B, b, \beta, \beta_0)\), a colax \(T\)-morphism \(f: A \to B\) is a morphism \(f: A \to B\) equipped with a cell \(\bar{f}: f \circ a \Rightarrow b \circ Tf\), its structure cell, satisfying the following associativity and unit coherence conditions.

If the structure cell \( \bar{f} \) is invertible then \(f\) is called a pseudo \(T\)-morphism; if it is an identity cell then \(f\) is called strict.

For \(T = V(M)\) a colax \(V(M)\)-morphism \(f: A \to B\) as above, between normal colax monoidal \(V\)-categories \(A\) and \(B\), is a \(V\)-functor \(f: A \to B\) equipped with a \(V\)-natural transformation \(f_\otimes: f \circ a \Rightarrow b \circ Mf\), the compositor, that consists of maps

\[f_\otimes: f(x_1 \otimes \cdots \otimes x_n) \to f(x_1) \otimes \cdots \otimes f(x_n)\]
for any sequence $x$; in particular this includes a map $f_{1_A} \to 1_B$. The unit axiom means that $f_\otimes$ restricts to the identity on sequences of length 1, while the associativity axiom states that the diagram below commutes, for each double sequence $x$.

Colax $V(M)$-morphisms are called **colax monoidal $V$-functors**.

Thus, like pseudo $V(M)$-algebras are unbiased variants of ordinary monoidal $V$-categories, colax monoidal $V$-functors between them are unbiased variants of ordinary colax monoidal $V$-functors which, instead of being equipped with comparison maps $f(x \otimes y) \to fx \otimes fy$ and $f_{1_A} \to 1_B$, come with comparison maps for each arity $n \geq 0$.

This leaves the definition of cells between $T$-morphisms.

**Definition 3.21.** Given colax $T$-morphisms $f$ and $g$: $A \to B$, a **$T$-cell** between $f$ and $g$ is a cell $\phi: f \Rightarrow g$ satisfying

\[
\begin{align*}
\begin{array}{c}
TA \xrightarrow{a} A \\
TB \xrightarrow{b} B
\end{array}
\end{align*}
\]

Thus in case $T = M$, the ‘free strict monoidal $V$-category’-monad, normal colax $M$-algebras and colax $M$-morphisms are respectively normal colax monoidal $V$-categories and colax monoidal $V$-functors, as we have seen.
Symmetric monoidal \( \mathcal{V} \)-categories

Here we discuss the ‘free symmetric strict monoidal \( \mathcal{V} \)-category’-monad \( S \) on the equipment \( \mathcal{V}\text{-Prof} \) of \( \mathcal{V} \)-profunctors, which is also briefly mentioned in \cite{CS10} Example 3.14 and whose restriction to \( \mathcal{V}\text{-Cat} \) is described in \cite{Get09} page 683. Its colax algebras are ‘colax symmetric monoidal \( \mathcal{V} \)-categories’, as we will see. Like the ‘free strict monoidal \( \mathcal{V} \)-category’-monad \( M \), the monad \( S \) is also induced by a monad on \( \mathcal{V}\text{-Mat} \), also denoted \( S \), that we will introduce first.

On \( \mathcal{V}\text{-Mat}_0 = \text{Set} \) we take \( S_0 = M_0 \) while the endofunctor \( S_1 \) on \( \mathcal{V}\text{-Mat}_1 \) is given as follows. Let \( \Sigma_n \) be the group of permutations of the set \( \{1,\ldots,n\} \); given \( \sigma \in \Sigma_n \) we will write \( \underline{x} \cdot \sigma \) for the permuted sequence given by \((\underline{x} \cdot \sigma)_i = x_{\sigma^{-1}(i)} \). This gives a right action on the sequences of length \( n \), i.e. \((\underline{x} \cdot \sigma) \cdot \tau = \underline{x} \cdot (\sigma \circ \tau) \). The image \( SJ \) of a \( \mathcal{V} \)-matrix \( J : A \rightarrow B \) is given by

\[
SJ(\underline{x}, \underline{y} ) = \begin{cases} 
\prod_{\sigma \in \Sigma_n} M_n J(\underline{x} \cdot \sigma, \underline{y}) & \text{if } |\underline{x}| = n = |\underline{y}|; \\
\emptyset & \text{otherwise}, 
\end{cases}
\]

where \( M_n J(\underline{x} \cdot \sigma, \underline{y}) \equiv \bigotimes_{i=1}^n J(\underline{x}_i, \underline{y}_i) \). As before the image \( S \phi \) of a cell \( \phi : J \Rightarrow K \) is given by taking tensor products of the components of \( \phi \); this extends \( J \Rightarrow SJ \) to an endofunctor \( S_1 \) on \( \mathcal{V}\text{-Mat}_1 \).

To combine the endofunctors \( S_0 \) and \( S_1 \) into a lax endofunctor \( S \) on \( \mathcal{V}\text{-Mat} \) we have to supply horizontal cells \( S_{\otimes} : SJ \circ SK \Rightarrow S(J \circ K) \), for \( \mathcal{V} \)-matrices \( J : A \rightarrow B \) and \( K : B \rightarrow C \), that form the compositor of \( S \) as well as horizontal cells \( S_U : U_{SA} \Rightarrow SU_A \), for each set \( A \), that form the unitor of \( S \). For the compositor notice that there is an induced action of \( \Sigma_n \) on the tensor products \( M_n J(\underline{x}, \underline{y}) \) given by the isomorphisms

\[
\sigma : M_n J(\underline{x}, \underline{y}) \xrightarrow{\sim} M_n J(\underline{x} \cdot \sigma, \underline{y} \cdot \sigma).
\]

that permute the factors, using the symmetric structure of \( \mathcal{V} \). Using this we define the components of \( S_{\otimes} \) to be the compositions below, for each pair of sequences \( \underline{x} \) and \( \underline{z} \) of length \( n \).

\[
(SJ \circ SK)(\underline{x}, \underline{z}) \cong \prod_{\rho, \sigma \in \Sigma_n, \underline{y} \in M_n B} M_n J(\underline{x} \cdot \rho, \underline{y}) \otimes M_n K(\underline{y} \cdot \sigma, \underline{z}) \\
\xrightarrow{\prod \sigma \otimes \text{id}} \prod_{\rho, \sigma \in \Sigma_n, \underline{y} \in M_n B} M_n J(\underline{x} \cdot (\rho \circ \sigma), \underline{y} \cdot \sigma) \otimes M_n K(\underline{y} \cdot \sigma, \underline{z}) \\
\cong \prod_{\tau \in \Sigma_n} M_n J(\underline{x} \cdot \tau, \underline{w}) \otimes M_n K(\underline{w}, \underline{z}) \cong S(J \circ K)(\underline{x}, \underline{z})
\]

Here the isomorphisms exists by the symmetric structure of \( \mathcal{V} \) and the fact that \( \otimes \) preserves coproducts in both variables, while the unlabelled map is induced by inserting each component, that is indexed by \( \rho, \sigma \) and \( \underline{y} \), at \( \tau = \rho \circ \sigma \) and \( \underline{w} = \underline{y} \cdot \sigma \). Checking that this makes \( S_{\otimes} \) a natural transformation, and that it satisfies the associativity axiom, is straightforward. The component of the unitor \( S_U \) for a set \( A \) is given by

\[
U_{SA}(\underline{x}, \underline{y}) = 1 \cong 1^n \rightarrow SU_A(\underline{x}, \underline{y})
\]

if \( \underline{x} = \underline{y} \), where the unlabelled map is insertion at \( \sigma = \text{id} \); if \( \underline{x} \neq \underline{y} \) then it is simply the initial map \( \emptyset \rightarrow SU_A(\underline{x}, \underline{y}) \). Checking that, with this definition, the unitors and associators satisfy the unit axioms is again straightforward. This concludes the definition of the lax endofunctor \( S \) on \( \mathcal{V}\text{-Mat} \).
The endofunctor $S$ is made into a lax monad as follows. Its multiplication
\[ \mu_A : S^2 A \to SA \]
for a set $A$ concatenates the double sequences of elements in $A$. To
define the cell $\mu_J : S^2 J \Rightarrow SJ$ for a $V$-matrix $J : A \to B$ notice that $S^2 J$ is given by
\[ S^2 J(x, y) = \prod_{\sigma \in \Sigma_n(x, y)} \prod_{i=1}^n m_i \, J((x_i, \sigma_{i,j_i}, (y_i))_{j=1}^i), \]
where $\Sigma_n(x, y) \subset \Sigma_n$ is the subgroup of permutations $\sigma$ for which $[x_{\sigma i}] = y_i$, for each index $i$. In here we can rewrite $(y_i)_j = (\mu_B y)_{k(i,j)}$ with
\[ k(i,j) = m_1 + \cdots + m_{i-1} + j_i; \]
similarly we have $(x_i)_{\sigma_{i,j_i}} = (\mu_A x)_{(\sigma_1, \ldots, \sigma_n) \circ (\sigma_{m_1}, \ldots, m_n) k(i,j_i)}$, where the permutations $\sigma_{(m_1, \ldots, m_n)}$ and $(\sigma_1, \ldots, \sigma_n)$ are given as follows. Writing $N = \Sigma m_i$, the block permutation $\sigma_{(m_1, \ldots, m_n)} \in \Sigma_N$ permutes the $n$ blocks of the $(m_1, \ldots, m_n)$-partition of $\{1, \ldots, N\}$ exactly like $\sigma$ permutes the elements of $\{1, \ldots, n\}$, while preserving the ordering of the elements in each block. The permutation $(\sigma_1, \ldots, \sigma_n)$ of $\Sigma_N$ is simply the disjoint union of the $\sigma_i$, letting each $\sigma_i$ act on the subset
\[ \{ l : \Sigma_{v < i} m_v + 1 \leq l \leq \Sigma_{v \leq i} m_v \} \subset \{1, \ldots, N\}. \]

It follows that, using the fact that $\otimes$ preserves coproducts in both variables, we can rewrite $S^2 J(x, y)$ as
\[ S^2 J(x, y) \cong \prod_{\sigma \in \Sigma_n(x, y)} M_N J(\mu_A x) \cdot \left( (\sigma_1, \ldots, \sigma_n) \circ (\sigma_{m_1}, \ldots, m_n), \mu_B y \right) \tag{3.3} \]
the multiplication $\mu_J : S^2 J \Rightarrow SJ$ is given by this isomorphism followed by inserting each component above, that is indexed by $(\sigma, \sigma_1, \ldots, \sigma_n)$, into $SJ(\mu_A x, \mu_B y)$ as the component indexed by $(\sigma_1, \ldots, \sigma_n) \circ (\sigma_{m_1}, \ldots, m_n)$. That the components $\mu_J$ are natural with respect to the cells of $V$-$\text{Mat}$, and that they satisfy the unit axiom (Definition $3.4$), is clear. To see that they satisfy the composition axiom as well, that is
\[ S^2 J \circ S^2 K \xrightarrow{\mu_J \otimes \mu_K} S^2 (J \circ K) \]
\[ SJ \circ SK \xrightarrow{\mu_J \otimes \mu_K} S(J \circ K) \]
commutes for every pair of $V$-matrices $J : A \Rightarrow B$ and $K : B \Rightarrow C$, notice that under the isomorphisms (3.3) the action of $S^2 \circ \sigma$ on $S^2 = S S \circ S (S \times S)$ is given as follows. Consider the component of $(S^2 J \circ S^2 K)(x, y)$ that is indexed by $y \in M_N B$ (for the horizontal composite), $(\sigma, \sigma_1, \ldots, \sigma_n)$ (for $S^2 J$) and $(\tau, \tau_1, \ldots, \tau_m)$ (for $S^2 K$). The first factor of $S^2 \circ \sigma$ permutes the tensor product of $J$-components in $S^2 J \circ S^2 K$ with $(\tau(k_1, \ldots, k_n))$, where $[x_i] = k_i$, while the second factor permutes the same product with $(\tau_1, \ldots, \tau_m)$. We conclude that the top leg of the diagram above is given by permuting the $J$-components with $(\tau_1, \ldots, \tau_m) \circ \tau(k_1, \ldots, k_n)$, followed by insertion into $S(J \circ K)$, which is exactly what the bottom leg does.

The components $\eta_A : A \to SA$ of the unit, for any set $A$, are simply given by
\[ \eta_A \]
by mapping the elements of $A$ to their corresponding 1-element sequences; the cells $\eta_J$ for $V$-matrices $J$ are given by inserting the components of $J$ into those of $SJ$. That this makes $\eta$ into a well-defined transformation $id \to S$ is clear. Finally, to
see that \( \mu \) and \( \eta \) satisfy the associativity and unit axioms of a monad, notice that their actions on the permutations, that are the indices of the coproducts in \( S^2 J \) and \( SJ \), coincide with the composition and unit of the ‘operad of symmetries’, see e.g. \cite{Lei04}: the operadic associativity and unit axioms for this operad imply the associativity and unit axioms for \( \mu \) and \( \eta \). This completes the definition of the lax monad \( S \) on the equipment \( \mathcal{V}\text{-Mat} \) of \( \mathcal{V} \)-matrices.

We will often use the fact that there exists a transformation \( \theta: M \to S \) embedding the ‘free monoid’-monad \( M \) on \( \mathcal{V}\text{-Mat} \) into \( S \), that restricts to the identity on \( \mathcal{V}\text{-Mat}_0 = \text{Set} \) and whose cells \( \theta_J: MJ \Rightarrow SJ \) insert at \( \sigma = \text{id} \). Notice that \( \theta \) is compatible with the multiplications and units, in the sense that \( \theta \circ \mu_M = \mu_S \circ \theta^2 \) (where \( \theta^2 = \theta S \circ M\theta \)) and \( \theta \circ \eta_M = \eta_S \), i.e. \( \theta \) is a morphism of monads.

As mentioned in the introduction, we are interested in the monad \( \text{Mod}(S) \) on the equipment \( \mathcal{V}\text{-Prof} \) of \( \mathcal{V} \)-profunctors that is induced by \( S \), see Proposition \( \ref{prop:mod-adj} \).

The monad \( \text{Mod}(S) \) is the ‘free symmetric strict monoidal \( \mathcal{V} \)-category’-monad; we will again write \( S = \text{Mod}(S) \). Its image \( SA \) of a \( \mathcal{V} \)-category \( A \) has finite sequences of elements of \( A \) as objects, while the hom-objects \( SA(x, y) \) are given by

\[
SA(x, y) = \begin{cases} 
\prod_{\sigma \in \Sigma_n} M_n A(x, \sigma, y) & \text{if } |x| = |y| = n; \\
\emptyset & \text{otherwise.}
\end{cases}
\]

Thus generalised morphisms \( x \to y \) in \( SA \) (that is \( \mathcal{V} \)-morphisms \( 1 \to SA(x, y) \)) can be thought of as a permutation of the sequence \( x \) followed by a sequence of maps from the permuted \( x \) to \( y \). Given \( x, y \) and \( z \) of equal length \( n \), the composition \( SA(x, y) \otimes SA(y, z) \to SA(x, z) \) is determined by

\[
M_n A(x, \sigma, y) \otimes M_n A(y, \tau, z) \xrightarrow{\tau \otimes \text{id}} M_n A(x, (\sigma \circ \tau), y) \otimes M_n A(y, \tau, z) \\
\quad \to M_n A(x, (\sigma \circ \tau), z),
\]

where the second map is given by composition in \( A \). Given a sequence \( x \) of length \( n \) and a permutation \( \sigma \) in \( \Sigma_n \) we denote by \( \sigma_x: x \to x \cdot (\sigma) \) the isomorphism in \( SA \) that is given by the composition

\[
1 \cong 1 \otimes \text{id}_x \xrightarrow{\otimes \sigma_x} M_n A(x, x) \xrightarrow{\sigma} M_n A(x, \sigma \cdot x, x) \to SA(x, x \cdot (\sigma)) \quad \text{(3.4)}
\]

where the last map is insertion at \( \sigma \). Note that the assignment \( (\sigma, x) \mapsto \sigma_x \) is functorial in the sense that \( \tau \cdot \sigma_x = (\sigma \circ \tau)_x \), while taking \( \sigma = \text{id} \) gives the identity for \( x \). In particular \( (\sigma^{-1})_x \sigma \) is the inverse of \( \sigma_x \).

Given a \( \mathcal{V} \)-functor \( f: A \to B \) the image \( SF: SA \to SB \) is given on objects by \( (SF)(x) = f(x) \), while on hom-objects it is determined by tensor products of the maps \( A(x, y) \to B(f(x), f(y)) \) that define the action of \( f \) on the hom-objects of \( A \). The formula for the action of \( S \) on \( \mathcal{V} \)-profunctors is identical to that on the hom-objects of \( \mathcal{V} \)-categories above. Likewise the actions of \( (SA)^{op} \) and \( SB \) on \( SJ \) are given in a similar fashion to the composition maps of \( SA \) above, using the actions of \( A^{op} \) and \( B \) on \( J \) instead of the composition maps of \( A \). Finally the images of \( \mathcal{V} \)-natural transformations of \( \mathcal{V} \)-profunctors are given by taking tensor products of their components.

Notice that, by Proposition \( \ref{prop:mod-adj} \), the monad \( S \) is in fact normal. More is true: the following was stated in \cite{CS10} Example 5.12 without proof.

**Proposition 3.23.** The ‘free symmetric strict monoidal \( \mathcal{V} \)-category’-monad \( S \) on \( \mathcal{V}\text{-Prof} \), as described above, is a pseudomonad.

Before giving a proof we will describe the algebras of \( S \). Notice that the embedding \( \theta: M \to S \) of monads on \( \mathcal{V}\text{-Mat} \) induces a morphism of monads \( \theta = \)
3.2. EXAMPLES

\[\text{Mod}(\theta) : \text{Mod}(M) \rightarrow \text{Mod}(S) \text{ on } \mathcal{V}-\text{Prof} \text{ and therefore a morphism of 2-monads } V(\theta) : V(M) \rightarrow V(S) \text{ on } \mathcal{V}-\text{Cat} = V(\mathcal{V}-\text{Prof})\]. In general, such a morphism of monads induces a functor \(V(S)\)-\text{Alg} \rightarrow V(M)\)-\text{Alg} between the categories of colax algebras over \(V(S)\) and \(V(M)\), see [Lei04, Section 6.1]. It follows that the algebra structure on a normal colax \(S\)-algebra \(A = (A, a', a')\) (see Definition 3.19) restricts to normal colax monoidal \(\mathcal{V}\)-category structure on \(A\), whose tensor products are given by the \(\mathcal{V}\)-functor \(a = [MA \xrightarrow{\theta_1} SA \xrightarrow{a'} A]\) and whose associator is the \(\mathcal{V}\)-natural transformation \(a = a' \circ \theta^2 : a \circ MA \rightarrow a \circ Ma\).

Furthermore, given a sequence \(x\) of length \(n\) and a permutation \(\sigma\) in \(\Sigma_n\), we write \(\langle x\rangle = a'(\langle \sigma \langle x \rangle \rangle) : x_1 \otimes \cdots \otimes x_n \rightarrow x_1 \otimes \cdots \otimes x_n\), where \(\sigma : x \rightarrow \pi : x \cdot \sigma\) is the isomorphism in \(SA\) that is given by the composition \((3.3)\). Writing \(a_n\) for the restriction of \(a\) to the full subcategory \(M_n\) of \(MA\) consisting of sequences of length \(n\), checking that the maps \(\langle s, \sigma \rangle\) combine to form a \(\mathcal{V}\)-natural isomorphism \(s_{\sigma} : a_n \rightarrow a_n \circ \sigma\), where the action of \(\sigma\) on \(a_n\) is given by permutation of tensor factors, is straightforward. The functoriality of \((3.3)\) implies that \(\sigma \mapsto s_{\sigma}\) is functorial in the sense that \((s_{\sigma} \circ \sigma) \circ s_{\sigma} = s_{\sigma \circ \sigma}\), and \(s_{id} = id\). Thus the original \(S\)-algebra structure \(\mathcal{V}\)-functor \(a' : SA \rightarrow A\) restricts to a \(\mathcal{V}\)-functor \(a : MA \rightarrow A\) that makes \(A\) into a colax monoidal \(\mathcal{V}\)-category, as well as coherent natural isomorphisms \(s_{\tau}\) that permute the factors of this monoidal structure. In fact giving \(a'\) is equivalent to giving both \(a\) and natural transformations \(s_{\sigma}\), provided that they satisfy the properties just described. Indeed this follows from the fact that they are related by the commuting of the following diagram.

\[
\begin{align*}
M_n A(\langle x, \sigma, y \rangle) &\xrightarrow{a_n} A(a_n(\langle x, \sigma \rangle), a_n(y)) \\
SA(\langle x, y \rangle) &\xrightarrow{a'} A(a_n(\langle x \rangle), a_n(y))
\end{align*}
\]

(3.5)

Since the embedding \(\theta : M \rightarrow S\) is the identity on objects, the original associator \(a'\) is completely determined by its restriction \(a = a' \circ \theta^2\). In particular the associativity axioms for \(a'\) and its restriction \(a\) are equivalent. Moreover the \(\mathcal{V}\)-naturality of \(a'\), as a transformation of \(\mathcal{V}\)-functors \(SA \rightarrow A\), splits into the \(\mathcal{V}\)-naturality of \(a\) and the naturality of \(a\) with respect to the symmetries \(s_{\tau}\). The latter means that for each double sequence \(x_i\) with \(|x_i| = n\) and \(|x_i| = n_i\), as well as permutations \(\sigma \in \Sigma_n\) and \(\tau \in \Sigma_{n_i}\), for each \(i = 1, \ldots, n\), the following diagram commutes. Here \(\otimes_{i=1}^{n_i} x_i\) denotes the tensor product \(x_{11} \otimes \cdots \otimes x_{m_1} \otimes \cdots \otimes x_{1} \otimes \cdots \otimes x_{m_2}\), and \((\tau_1, \ldots, \tau_n) \circ (\sigma_{m_1}, \ldots, \sigma_{m_2})\) is the permutation that was used in the definition of the multiplication \(\mu\) of the monad \(S\).

\[
\begin{align*}
\otimes_{i=1}^{n_i} x_{i_1} &\xrightarrow{a} \otimes_{i=1}^{n} x_{i_1} \\
\otimes_{i=1}^{n_i} x_{i_1} &\xrightarrow{a} \otimes_{i=1}^{n_i} x_{i_1}
\end{align*}
\]

Summarising, a normal colax \(S\)-algebra \(A\) is a normal colax monoidal \(\mathcal{V}\)-category \(A\) equipped with \(\mathcal{V}\)-natural symmetry isomorphisms \(s_{\tau} : a_n \rightarrow a_n \circ \tau\), for each permutation \(\sigma \in \Sigma_n\), where \(a_n\) denotes the restriction of the structure \(\mathcal{V}\)-functor \(a : MA \rightarrow A\) to the full subcategory \(M_n\) of sequences of length \(n\). The assignment \(\sigma \mapsto s_{\sigma}\) is required to be functorial in the sense that \((s_{\sigma} \circ \sigma) \circ s_{\sigma} = s_{\sigma \circ \sigma}\), and \(s_{id} = id\), making \(A\) into a monoidal \(\mathcal{V}\)-category. This completes the construction of a \(\mathcal{V}\)-monoidal \(\mathcal{V}\)-category from a \(\mathcal{V}\)-algebra.
while the associator \( a \colon a \circ \mu_A \to a \circ Ma \) is required to be natural with respect to the symmetries \( s_a \), in that the diagrams above commute. Normal colax \( S \)-algebras are called \textit{symmetric normal colax monoidal} \( \mathcal{V} \)-\textit{categories}.

Likewise a colax \( S \)-morphism \( f \colon A \to B \) between symmetric normal colax monoidal \( \mathcal{V} \)-\textit{categories} \( A \) and \( B \) is a colax monoidal \( \mathcal{V} \)-\textit{functor} \( f \colon A \to B \) whose \( \mathcal{V} \)-natural compositor \( f_B \colon f \circ a \to B \circ f \) is natural with respect to the symmetries of \( A \) and \( B \), in the sense that the diagrams below commute for any sequence \( x \) of length \( n \) and permutation \( \sigma \) in \( \Sigma_n \). Colax \( S \)-morphisms are called \textit{symmetric colax monoidal} \( \mathcal{V} \)-\textit{functors}. Finally \( S \)-\textit{cells} between such functors are simply monoidal \( \mathcal{V} \)-natural transformations.

\[
\begin{array}{ccc}
\prod_{x \in A} f(x) &=& \prod_{x \in A} f_0(x) \\
\downarrow s_x & & \downarrow s_x \\
\prod_{x \in A} f(x) &=& \prod_{x \in A} f_0(x)
\end{array}
\]

We now turn to the proof of Proposition 3.23 which, the author believes, has not appeared in the literature before. Where Cruttwell and Shulman remark in [CS10, Example 5.12] that such a proof can be given using "a more involved computation with coequalisers", we will give a simple proof that uses the fact that \( \mathcal{V} \)-\textit{profunctors} are given by the reflexive coequaliser of the two parallel maps below, that are given by letting \( SA \) act on \( S \) and \( M \) respectively.

\[
\prod_{x \in A} M_n(A(x, y) \otimes M_n A(y, z) \otimes M_n J(z, z))
\]

We now turn to the proof of Proposition 3.24 which, the author believes, has not appeared in the literature before. Where Cruttwell and Shulman remark in [CS10, Example 5.12] that such a proof can be given using "a more involved computation with coequalisers", we will give a simple proof that uses the fact that \( \mathcal{V} \)-\textit{profunctors} are given by the reflexive coequaliser of the two parallel maps below, that are given by letting \( MA \) act on \( SA \) and \( M \) respectively.

\[
\prod_{x \in A} M_n(A(x, y) \otimes M_n A(y, z) \otimes M_n J(z, z))
\]

We now turn to the proof of Proposition 3.23 which, the author believes, has not appeared in the literature before. Where Cruttwell and Shulman remark in [CS10, Example 5.12] that such a proof can be given using "a more involved computation with coequalisers", we will give a simple proof that uses the fact that \( \mathcal{V} \)-\textit{profunctors} are given by the reflexive coequaliser of the two parallel maps below, that are given by letting \( MA \) act on \( SA \) and \( M \) respectively.

\[
\prod_{x \in A} M_n(A(x, y) \otimes M_n A(y, z) \otimes M_n J(z, z))
\]
Pseudo double categories

As the third example we recall the ‘free strict double category’-monad \( D \) on the equipment \( \text{Prof}(\overset{\sim}{G}_1) = \text{Mod}(\text{Span}(\overset{\sim}{G}_1)) \) of \( G_1 \)-indexed profunctors (see Example 1.19), whose pseudoalgebras are ‘unbiased pseudo double categories’. Its definition will involve fewer ‘details’ than that of \( S \), because it is induced by the relatively simple monad \( F \) of free categories on the category \( \overset{\sim}{G}_1 \) of presheaves on \( G_1 = (0 \to 1) \). The latter induces \( D \) by the formula \( D = \text{Prof}(F) \), as described in Example 3.13.

A presheaf \( A : G_1^{op} \to \text{Set} \) consists of two sets \( A_0 \) and \( A_1 \) as well as a pair of functions \( A_1 \rightrightarrows A_0 \) that we will denote \( L \) and \( R \). Such a presheaf is a directed graph, with a set of vertices \( A_0 \) and a set of edges \( A_1 \): every \( j \in A_1 \) is a directed edge from the vertex \( Lj \) to the vertex \( Rj \). The free category monad \( F = (F, \mu, \eta) \) on \( \overset{\sim}{G}_1 \) is given as follows. The directed graph \( FA \) has the same vertices as \( A \), while its vertices are (possible empty) paths of the directed edges in \( A \). In formulae:

\[
(FA)_0 = A_0 \quad \text{and} \quad (FA)_1 = A_0 \amalg \prod_{n \geq 1} A_1 \times_{A_0} A_1 \times_{A_0} \cdots \times_{A_0} A_1, \quad (3.6)
\]

where the set under the brace denotes the limit, called a wide pullback, of the diagram \( A_1 \xrightarrow{R} A_0 \xleftarrow{L} A_1 \xrightarrow{R} \cdots \xleftarrow{L} A_1 \) consisting of \( n \) copies of \( A_1 \) and \( n - 1 \) copies of \( A_0 \). On these limits the functions \( L \) and \( R : (FA)_1 \to A_0 \) restrict to applying \( L \) to the leftmost copy of \( A_1 \) and \( R \) to the rightmost copy respectively. It is clear that this assignment extends to an endofunctor on \( \overset{\sim}{G}_1 \). The edges of the directed graph \( F^2A \) consist of paths of paths of edges in \( A \) and the multiplication \( \mu_A : F^2A \to FA \) is given by concatenation; the unit \( \eta_A : A \to FA \) maps each edge to its corresponding 1-edge path. Of course the strict algebras for \( F \) are categories.

An elementary calculation shows that \( F \) preserves pullbacks, so that we can apply the 2-functor \( \text{Span}(-) \) of Proposition 3.10 to obtain a monad \( \text{Span}(F) \) on the equipment \( \text{Span}(\overset{\sim}{G}_1) \) of spans in \( \overset{\sim}{G}_1 \). Following this we apply the 2-functor \( \text{Mod}(-) \) of Proposition 3.10 obtaining a monad \( D = \text{Prof}(F) \) on the equipment \( \text{Prof}(\overset{\sim}{G}_1) = \text{Mod}(\text{Span}(\overset{\sim}{G}_1)) \) of \( G_1 \)-indexed categories and \( G_1 \)-indexed profunctors. We have already described the equipment \( \text{Prof}(\overset{\sim}{G}_1) \) in detail in Example 1.21: its objects are ‘double categories without horizontal composition’, while a \( G_1 \)-indexed profunctor \( J : A \leftrightarrow B \) consists of vertical morphisms \( p : a \to b \), where \( a \in A \) and \( b \in B \), and cells \( u : j \Rightarrow k \), where \( j \) and \( k \) are horizontal morphisms in \( A \) and \( B \). The vertical morphisms and cells of \( A \) and \( B \) act on those of \( J \).
Applying the 2-functors \( \text{Span}(-) \) and \( \text{Mod}(-) \) we find that the monad \( D = \text{Mod}(\text{Span}(F)) \) is given as follows. The image \( DA \) of a \( \mathbb{G}_1 \)-indexed category \( A \) is again given by the formulae (3.6) where now the wide pullback is a limit of categories. The functors \( L \) and \( R: (DA)_1 \to A_0 \) are again given by applying \( L \) and \( R \) respectively to the leftmost and rightmost copy of \( A_1 \). Thus the objects and vertical maps of \( DA \) are those of \( A \), while its horizontal morphisms and cells are (possibly empty) ‘formal horizontal composites’

\[
\mathcal{J} = (a_0 \xrightarrow{j_0} a_1, a_1 \xrightarrow{j_2} a_2, \ldots, a_{n-1} \xrightarrow{j_n} a_n)
\]

and

\[
\mathcal{U} = \left( \begin{array}{cccc}
  a_0 & j_1 & a_1 & j_2 & a_2 & \ldots & a_{n-1} & j_n & a_n \\
  p_1 & k_1 & p_2 & k_2 & \ldots & p_{n-1} & k_n & p_n \\
  c_0 & k_1 & c_1 & k_2 & \ldots & c_{n-1} & k_n & c_n
\end{array} \right)
\]

(3.7)

of horizontal morphisms and cells in \( A \). Vertical composition of such sequences of cells is given coordinatewise. Again \( |j| = n \) denotes the length of \( j \), while the empty sequence at the object \( a \) will be denoted by \( (a) \), and that at the vertical morphism \( p \) by \( (p) \).

The action of \( D \) on a \( \mathbb{G}_1 \)-indexed profunctor \( J: A \Rightarrow B \) is given similarly: on vertical morphisms by \((DJ)_0(a,b) = J_0(a,b)\) and on cells by

\[
(DJ)_1(j,k) = \begin{cases} 
  J_0(a,b) & \text{if } j = (a), k = (b); \\
  J_1(j_1,k_1) \times_{J_0} J_1(j_2,k_2) \times_{J_0} \cdots \times_{J_0} J_1(j_n,k_n) & \text{otherwise}
\end{cases}
\]

where, in the second case, the wide pullback is of the maps \( L \) and \( R: J_1(j_i,k_i) \to J_0 \) into the set \( J_0 \) of all vertical morphisms of \( J \). Thus the cells \( \mathcal{U} \) of \( DJ \) look like the cells \( \mathcal{U} \) of \( DA \) above, except that now the horizontal morphisms \( j_i \) and \( k_i \) are horizontal morphisms of \( A \) and \( B \) respectively, while the vertical morphisms \( p_i \) are those of \( J_0 \). The actions of \( DA \) and \( DB \) are given coordinatewise.

The assignments \( A \Rightarrow DA \) and \( J \Rightarrow DJ \) above extend to a normal endofunctor on the equipment \( \text{Prof}(\mathbb{G}_1) \): the natural \( \mathbb{G}_1 \)-indexed transformations \( DJ \circ DB \Rightarrow DH \) making up the compositors of \( D \) are given by the universal property of coends. Exactly as for \( F \) the multiplication \( \mu: D^2 \to D \) is given by concatenation, while the unit \( \eta: \text{id} \Rightarrow D \) maps horizontal morphisms and cells to the corresponding sequences of length 1. Unlike \( \text{Span}(F) \), which is a pseudofunctor, the compositors of \( D = \text{Mod}(\text{Span}(F)) \) need not be invertible, as Proposition (3.25) below shows.

We shall now describe the colax algebras of \( D \). As with monoidal categories we simplify matters slightly by considering normal colax \( D \)-algebras, whose unitors \( \alpha_0: a \circ \eta A \Rightarrow \text{id} \) are identities.

A normal colax \( D \)-algebra \( A = (A, \circ, a) \) consists of a \( \mathbb{G}_1 \)-indexed category \( A \) that is equipped with a \( \mathbb{G}_1 \)-indexed functor \( \circ: DA \Rightarrow A \). The strict unitor \( \circ \circ \eta = \text{id} \) implies that \( \circ \) restricts to the identity on objects, vertical morphisms and sequences consisting of a single horizontal morphism or cell. It maps each sequence \( j = (j_1, \ldots, j_n) \) of composable horizontal morphisms of length \( n > 1 \), as in (3.7), to its composite \( j_1 \circ \cdots \circ j_n = \circ(j): a_0 \Rightarrow a_n \), while it picks a horizontal unit \( 1_a = \circ(a): a \Rightarrow a \) for each object \( a \). Likewise the sequence \( \mathcal{U} = (u_1, \ldots, u_n) \) of composable cells in (3.7) is mapped to a cell \( u_1 \circ \cdots \circ u_n = \circ(\mathcal{U}) \) that is of the form as on the left below, and it picks a horizontal unit cell \( 1_p = \circ(\mathcal{U}): \text{id}_a \Rightarrow \text{id}_c \)
for each vertical morphism \( p: a \to c \), as on the right.

\[
\begin{array}{ccc}
  a_1 & \overset{j_1 \ldots j_n}{\longrightarrow} & a_n \\
  \downarrow^p & & \downarrow^p \\
  c_1 & \overset{k_1 \ldots k_n}{\longrightarrow} & c_n
\end{array}
\]

The associator \( \alpha \) is a natural transformation \( \alpha: \circ \circ \mu \to \circ \circ D \circ \) consisting of cells

\[
\alpha: j_{11} \circ \ldots \circ j_{1m_1} \circ \ldots \circ j_{nm_n} \to (j_{11} \circ \ldots \circ j_{1m_1}) \circ \ldots \circ (j_{1m_1} \circ \ldots \circ j_{nm_n})
\]

which, by the unit axiom for \( \alpha \) (see Definition 3.19), are horizontal. Its associativity axiom says that the two ways of adding brackets in the composite of a triple sequence of horizontal morphisms coincide, exactly like the associativity axiom for normal colax monoidal categories. In [CS10, Example 9.3] normal colax \( D \)-algebras are called ‘normal oplax double categories’ by Cruttwell and Shulman; we shall call them normal colax double categories. In [Lei04, Section 5.2], Leinster calls non-normal pseudo \( D \)-algebras ‘weak double categories’. The latter can be considered as the ‘unbiased’ versions of the the familiar ‘biased’ pseudo double categories (Definition 1.1), that we have been using throughout, in the same way that unbiased monoidal categories are related to their biased counterparts.

Likewise colax morphisms \( f: A \to B \) between unbiased pseudo double categories \( A \) and \( B \) are colax double functors: they are \( \mathcal{G}_1 \)-indexed functors that come equipped with natural vertical cells \( f: \circ \circ j \to \circ \circ f j \) which satisfy an associativity axiom, exactly as for unbiased colax monoidal functors (also compare Definition 3.1 of biased lax functors).

**Proposition 3.25.** The ‘free strict double category’-monad \( D \) on \( \text{Prof}(\mathcal{G}_1) \), as given above, is not a pseudomonad.

**Proof.** Remember that the cells of \( (DJ)_1 \) are formal horizontal composites of cells in \( J \). Therefore, to find \( \mathcal{G}_1 \)-indexed profunctors \( J: A \Rightarrow B \) and \( H: B \Rightarrow C \) whose coequaliser computing \( J \circ B H \times H \circ B D \) is not invertible, we should try to find \( J \) and \( H \) that, each on its own, have no horizontally composable cells but so that, by taking the quotients that form the composite \( J \circ B H \), such composable cells are created. This can be easily achieved as follows. First we choose \( A \) and \( C \) to be the terminal \( \mathcal{G}_1 \)-indexed category, with single object \(*\) and single horizontal morphism \(* \Rightarrow *\). Let \( B \) be given by \( B_0 = (\bot \to \top) \) and with \( B_1 \) consisting of a single horizontal morphism \( \bot \Rightarrow \top \). Now choose \( J \) and \( H \) to consist of the single cells

\[
J = \left( \begin{array}{c} * \ \vdash \ * \\
\downarrow & \ \vee \\
\downarrow & \ \top \\
\end{array} \right) \quad \text{and} \quad H = \left( \begin{array}{c} \bot \ \vdash \ \top \\
\downarrow & \ \vee \\
\downarrow & \ * \ \Rightarrow \ * \\
\end{array} \right).
\]

Those cells are not horizontally composable with themselves, hence \( DJ = J \) and \( DH = H \). On the other hand the coregular computing \( J \circ B H \) (see 1.9) identifies the two formal composites, \((* \Rightarrow \bot, \bot \Rightarrow *)\) and \((* \Rightarrow \top, \top \Rightarrow *)\), of vertical morphisms in \( J \circ H \), so that \( J \circ B H \) only contains a single vertical morphism \(* \Rightarrow *\). Under this identification the formal vertical composite of the cells above has the form

\[
\left( \begin{array}{c} * \ \vdash \ * \\
\downarrow \\
\downarrow \\
\end{array} \right)
\]

\[
\left( \begin{array}{c} * \ \Rightarrow \ * \\
\downarrow \\
\end{array} \right)
\]

and we conclude that $J \otimes_B H$ is the terminal $\mathbb{G}_1$-indexed profunctor $* : A \Rightarrow C$. We conclude that the compositor of $J$ and $H$ is the natural $\mathbb{G}_1$-indexed transformation $* \to D*$, which is clearly not invertible because the single cell of $*$ above is horizontally composable with itself.

To close this chapter we now briefly look at the ‘free strict $\omega$-category’-monad and the ‘ultrafilter’-monad, whose algebras are respectively ‘monoidal globular categories’ and ‘ordered Hausdorff spaces’, leaving out most of the details.

**Monoidal globular categories**

As a generalisation of the category $\mathbb{G}_1$, that we used above to define double categories, we here consider the globe category $\mathbb{G}$ that has the natural numbers as objects and the maps

$$\begin{align*}
0 \xrightarrow{\sigma_1} 1 \xrightarrow{\sigma_2} 2 \xrightarrow{\sigma_3} \cdots
\end{align*}$$

as morphisms, that satisfy the relations

$$\sigma_n \circ \sigma_{n-1} = \tau_n \circ \sigma_{n-1} \quad \text{and} \quad \sigma_n \circ \tau_{n-1} = \tau_n \circ \tau_{n-1}.$$

A presheaf $A$ on $\mathbb{G}$ is called a globular set; it consists of sets $A_n$, $n \in \mathbb{N}$, whose elements are thought of as ‘$n$-dimensional globular cells’, and it comes equipped with ‘source and target functions’ $s$ and $t : A_n \to A_{n-1}$. For example an element of $A_2$ is thought of as a cell

$$\begin{array}{c}
\phi \\
\downarrow f \\
a \)/\ \phi \\
\downarrow g \\
\phi \\
\downarrow b
\end{array}$$

where $s\phi = f$, $t\phi = g$, $sf = a = sg$ and $tf = b = tg$ (notice that the images of relations above, on the $\sigma_i$ and $\tau_i$, hold).

More general, a $\mathbb{G}$-indexed category $A$, that is a presheaf $A : \mathbb{G}^{\text{op}} \to \text{Cat}$, is called a $\omega$-globular category by Batanin, who uses ‘monoidal’ variants of such globular categories in [Bat98] to introduce weak $\omega$-categories. There exists a ‘free strict $\omega$-category’-monad $T$ on the equipment $\text{Prof}(\mathbb{G})$ by Proposition 3.9 and Proposition 3.10, and pseudoalgebras for $\text{Prof}(T)$ are precisely the ‘unbiased’ versions of the monoidal globular categories of [Bat98]; see [CS10, Example 9.11]. Loosely speaking, the image $TA$ of a globular set $A$ consists of ‘formal pasting-diagrams’ of the cells of $A$: a typical element of $(TA)_2$ looks like

$$\begin{array}{c}
\phi_1 \\
\downarrow f_1 \\
a \\
\phi_2 \\
\downarrow f_2 \\
b \\
\phi_3 \\
\downarrow f_3 \\
c \\
\phi_4 \\
\downarrow f_4 \\
d
\end{array}$$

where the 0-cells, 1-cells and 2-cells those of $A$. The details can be found in [Lei04] Chapter 8.
3.2. EXAMPLES

Ordered compact Hausdorff spaces

The second example that we briefly mention are the ‘ordered compact Hausdorff spaces’ considered by Tholen in [Tho09]. Different from the usual way of defining a topological space, a topology on a set \( A \) can be given using ‘ultrafilters’ in \( A \), as follows. An ultrafilter \( \mathfrak{f} \) in \( A \) is defined to be a subset of the powerset \( P(A) \) on \( A \), that is upward-closed, closed under intersection and ‘satisfies dichotomy’, as follows:

- if \( S \in \mathfrak{f} \) and \( T \supseteq S \) then \( T \in \mathfrak{f} \);
- if \( S, T \in \mathfrak{f} \) then \( S \cap T \in \mathfrak{f} \);
- for any \( S \subseteq A \), either \( S \in \mathfrak{f} \) or \( A - S \in \mathfrak{f} \).

The set of ultrafilters in \( A \) is denoted \( \beta A \). One can think of ultrafilters in \( A \) as families of subsets that ‘move in a certain direction within \( A \)’: for each \( S \in \mathfrak{f} \) and any smaller subset \( T \subseteq S \), either \( T \in \mathfrak{f} \) or \( S - T \in \mathfrak{f} \), so that \( \mathfrak{f} \) either ‘moves into \( T \)’ or ‘moves into \( S - T \)’.

If \( A \) is a topological space then an ultrafilter \( \mathfrak{f} \) in \( A \) is said to converge to a point \( x \) of \( A \) if each open subset \( U \) containing \( x \) belongs to \( \mathfrak{f} \). If \( A \) is a compact Hausdorff space then every ultrafilter in \( A \) converges to precisely one point, giving a function \( a: \beta A \to A \), that is called the convergence of \( A \). Manes showed in his thesis [Man69] that the endofunctor \( \beta: \text{Set} \to \text{Set} \) can be extended to a monad \( (\beta, \mu, \eta) \), the ultrafilter-monad, such that strict \( \beta \)-algebra structures \( a: \beta A \to A \) on a set \( A \) correspond to compact Hausdorff topologies on \( A \).

Now consider the equipment \( 2\text{-Mat} \), where \( 2 = (\bot \to \top) \), that consists of sets and relations. The ultrafilter-monad \( \beta \) can be extended to a pseudomonad on \( 2\text{-Mat} \) by mapping the relation \( J: A \to B \) to the relation \( \beta J: \beta A \to \beta B \) that is given by

\[
(\beta J)(\mathfrak{f}, \mathfrak{g}) = \bigwedge_{S \in \mathfrak{f}, T \in \mathfrak{g}} \bigvee_{x \in S, y \in T} J(x, y).
\]

Hence, applying Proposition 3.10, we obtain a monad \( \text{Mod}(\beta) \) on the equipment \( 2\text{-Prof} \) of preordered sets and modular relations (see Example 1.4). A pseudoalgebra \( A \) for \( \text{Mod}(\beta) \) consists of a preordered set equipped with a compact Hausdorff topology, so that the convergence \( a: \beta A \to A \) is monotone. In [Tho09, Example 2] such spaces are called ordered compact Hausdorff spaces.
CHAPTER 3. MONADS ON EQUIPMENTS
Chapter 4

Algebraic promorphisms

In this final chapter we propose generalisations of the well-known notions of lax and colax morphisms, of algebras over a 2-monad, to notions of lax and colax promorphisms, of algebras over a monad on an equipment. The lax notion recovers natural notions of algebraic promorphisms, some of which have been considered before. For example a ‘lax monoidal profunctor’ \( J: A \to B \), of monoidal categories \( A \) and \( B \), comes equipped with a coherent tensor product

\[
(j_1: a_1 \to b_1, \ldots, j_n: a_n \to b_n) \mapsto j_1 \otimes \cdots \otimes j_n: a_1 \otimes \cdots \otimes a_n \to b_1 \otimes \cdots \otimes b_n
\]

of morphisms \( j: a \to b \) in \( J \). Another example, that of lax double profunctors between double categories, is discussed on the nLab page [Shu10].

When defining lax promorphisms we shall, following Remark 1.28, also consider two kinds of ‘pseudo lax promorphisms’: ‘left pseudo’ and ‘right pseudo’ ones. These correspond to the two ways of making a cell into a horizontal cell, either by using companions or by using conjoint cells. Right pseudopromorphisms generalise the situation of Getzler’s proposition, that was discussed in the introduction, and satisfy a decomposition condition. For example the lax monoidal profunctor \( J: A \to B \) above is a right pseudopromorphism if each morphism \( j: a \to c_1 \otimes \cdots \otimes c_n \) in \( J \) decomposes, in some sense uniquely, as a composite

\[
a \xrightarrow{f} b_1 \otimes \cdots \otimes b_n \xrightarrow{j_1 \otimes \cdots \otimes j_n} c_1 \otimes \cdots \otimes c_n,
\]

where \( f: a \to b_1 \otimes \cdots \otimes b_n \) belongs to \( A \) and each \( j_i: b_i \to c_i \) belongs to \( J \).

In Definition 4.13 the notion of right pseudopromorphisms leads to that of right colax promorphisms. Roughly speaking the latter are equipped with ‘splittings’ like the decomposition above, without being equipped with a lax structure. For example, a right colax monoidal profunctor \( H: A \to B \) does not define tensor products of its maps, like \( J \) does above, but only gives a coherent way of ‘splitting’ each morphism \( h: a \to c_1 \otimes \cdots \otimes c_n \) into a morphism \( f: a \to b_1 \otimes \cdots \otimes b_n \) in \( A \) and morphisms \( h_1: b_1 \to c_1, \ldots, h_n: b_n \to c_n \) in \( H \). Notice that, since we cannot take the tensor product of the maps \( h_1, \ldots, h_n \) in \( H \), there is no way of comparing \( h \) with its ‘splitting’ \( (f, h_1, \ldots, h_n) \).

Although the author does not yet know of any ‘true’ right colax promorphisms, that are not right pseudopromorphisms, they have the following formal advantage over lax promorphisms. It is, in general, not possible to coherently horizontally compose lax promorphisms, for a monad that is not pseudo. In particular it is not possible to compose lax double profunctors between double categories in a coherent way, as discussed in [Shu10]. Right colax promorphisms do not suffer from this: in fact the main result (Proposition 4.21) of Section 4.1 below asserts that colax
algebras, colax morphisms and right colax promorphisms over a normal monad $T$, that satisfies some mild conditions, form a pseudo double category $T$-$\text{Prom}_{rc}$.

Since $T$-$\text{Prom}_{rc}$ is a pseudo double category it allows a notion of ‘algebraic weighted colimits’, as given in Definition 2.15. Moreover, using the results of Chapter 2 and by generalising a notion of Grandis and Paré, we shall consider ‘pointwise weighted colimits’ in $T$-$\text{Prom}_{rc}$ as well. In fact Theorem 4.30 below enhances Theorem 2.37 by proving that all weighted colimits in $T$-$\text{Prom}_{rc}$ are pointwise whenever the underlying equipment $K$ has strong double comma objects (Definition 2.33).

Finally in Section 4.2 we consider the main result (Theorem 4.32), which states that if $T$ is a ‘suitable’ normal monad on a closed equipment $K$, then the forgetful functor $U : T$-$\text{Prom}_{rc} \to K$ lifts all weighted colimits. This means that an algebraic weighted colimit $\text{colim}_Jd$ in $T$-$\text{Prom}_{rc}$ can be computed as the ordinary weighted colimit $\text{colim}_UJ Ud$ in $K$. Following this, in Section 4.3 applications of the main result are considered, while it is also compared to results in [Get09], [GP07] and [MT08]. The author believes that all results in this chapter are original.

### 4.1 Algebraic promorphisms

We start with the notion of lax promorphisms which, even though being a straightforward generalisation of that of lax morphisms, appears to be new. That it is the right notion follows from Proposition 4.4 below, which shows that colax algebra structures on a morphism $f : A \to C$ correspond to lax algebra structures on its companion $C(f, \text{id})$. Given a normal monad $T$ on an equipment $K$, recall that by a colax $T$-algebra we mean a colax algebra for the 2-monad $\mathcal{V}(T)$ on $\mathcal{V}(K)$.

**Definition 4.1.** Let $T$ be a normal monad on an equipment $K$. Given colax $T$-algebras $A = (A, a, \alpha, \alpha_0)$ and $B = (B, b, \beta, \beta_0)$, a lax $T$-promorphism $A \Rightarrow B$ is a promorphism $J : A \Rightarrow B$ equipped with a *structure cell*

$$
\begin{array}{ccc}
T A & T J & T B \\
\downarrow^{\mu_A} & \downarrow^{\mu_B} & \downarrow^{Tb} \\
A & B & B
\end{array}
$$

satisfying the following coherence conditions, respectively the associativity axiom and the unit axiom.

$$
\begin{array}{ccc}
T^2 A & T^2 B & T^2 B \\
\downarrow^{\mu_A} & \downarrow^{\mu_B} & \downarrow^{Tb} \\
T A & T B & T B \\
\downarrow^{TJ} & \downarrow^{Tb} & \downarrow^{\beta} \\
A & B & B
\end{array} = 
\begin{array}{ccc}
T^2 A & T^2 J & T^2 B \\
\downarrow^{\mu_A} & \downarrow^{\mu_B} & \downarrow^{Tb} \\
T A & T J & T b \\
\downarrow^{\alpha} & \downarrow^{TJ} & \downarrow^{Tb} \\
A & A & B
\end{array}
$$

$$
\begin{array}{ccc}
A & J & B \\
\downarrow^{\eta_A} & \downarrow^{\eta_B} & \downarrow^{\beta_0} \\
T A & T J & T B \\
\downarrow^{TJ} & \downarrow^{\beta_0} & \downarrow^{b} \\
A & B & B
\end{array} = 
\begin{array}{ccc}
A & J & B \\
\downarrow^{\eta_A} & \downarrow^{\eta_B} & \downarrow^{b} \\
A & A & B
\end{array}
$$

$$
\begin{array}{ccc}
A & J & B \\
\downarrow^{\eta_A} & \downarrow^{\eta_B} & \downarrow^{\beta_0} \\
T A & T J & T B \\
\downarrow^{TJ} & \downarrow^{\beta_0} & \downarrow^{b} \\
A & B & B
\end{array} = 
\begin{array}{ccc}
A & J & B \\
\downarrow^{\eta_A} & \downarrow^{\eta_B} & \downarrow^{b} \\
A & A & B
\end{array}
$$
4.1. ALGEBRAIC PROMORPHISMS

Following Definition 4.27 we call $J$ a left pseudo $T$-promorphism when the horizontal cell $\lambda J: T J \circ B(b, id) \Rightarrow A(a, id) \circ J$ is invertible, and right pseudo when $\rho J: A(id, a) \circ T J \Rightarrow J \circ B(id, b)$ is invertible.

Taking for $T$ the ‘free strict monoidal $V$-category’-monad $M$ or the ‘free strict double category’-monad $D$, both introduced in Section 5.2, we obtain the notion of ‘lax monoidal $V$-profunctors’ between monoidal $V$-categories and that of ‘lax double profunctors’ between pseudo double categories, as follows.

Example 4.2. Choosing $T = M$, the ‘free strict monoidal $V$-category’-monad, let $A$ and $B$ be normal colax monoidal $V$-categories with structure $V$-functors $\alpha: MA \to A$ and $\beta: MB \to B$. A lax $M$-promorphism $J: A \Rightarrow B$, as defined above, will be called a lax monoidal $V$-profunctor. It consists of a $V$-functor $J: A^{op} \otimes B \to V$ equipped with a $V$-natural transformation $J_\otimes = J: M J \to a(b)$ of $V$-profunctors $MA \Rightarrow MB$, that satisfies the coherence axioms above. Unpacking this we find that $J_\otimes$ consists of morphisms

$$J_\otimes: J(x_1, y_1) \otimes \cdots \otimes J(x_n, y_n) \to J(x_1 \otimes \cdots \otimes x_n, y_1 \otimes \cdots \otimes y_n)$$

that are natural in $x_i \in M_n A$ and $y_i \in M_n B$, as well as $J_1: 1 \to J(1_A, 1_B)$, such that the following associativity and unit axioms are satisfied. The associativity axiom states that, for each pair of double sequences $x$ and $y$, of equal length and such that $|x_i| = m_i = |y_i|$, the following diagram commutes. Here $\otimes_{i=1}^{i=n,j=m_i} x_{ij}$ denotes the tensor product $x_{11} \otimes \cdots \otimes x_{m_1} \otimes \cdots \otimes x_{m_m}$.

$$\begin{array}{ccc}
\otimes_{i=1}^{i=n} J(x_{ij}, y_{ij}) & \overset{\otimes_{i=1}^{i=n} J_\otimes}{\longrightarrow} & \otimes_{i=1}^{i=n} J(J(x_{ij}, y_{ij}), \otimes_{i=1}^{i=n} x_{ij} \otimes_{i=1}^{i=n} y_{ij}) \\
\Downarrow & & \Downarrow
\end{array}$$

The unit axiom says that the map $J_\otimes: J(x, y) \to J(x, y)$, for any pair of objects $x$ and $y$, is the identity.

Formally the lax monoidal $V$-profunctor $J: A \Rightarrow B$ above is a right pseudo-profunctor whenever the horizontal composite $\rho J_\otimes = \varepsilon_a \circ J_\otimes \circ \eta_b$ is invertible. Computing $\rho J_\otimes$ we find that this means that the composites

$$\int^a A(x, y_1) \otimes \cdots \otimes y_n) \otimes J(y_1, z_1) \otimes \cdots \otimes J(y_n, z_n)$$

$$\overset{\int^a \otimes J_\otimes}{\longrightarrow} \int^a A(x, y_1) \otimes \cdots \otimes y_n) \otimes J(y_1 \otimes \cdots \otimes y_n, z_1 \otimes \cdots \otimes z_n)$$

$$\to J(x, z_1) \otimes \cdots \otimes z_n)$$

are isomorphisms, for all $x \in A$ and $z \in M_n B$, where the coends range over $y \in M_n A$ and where the second map is induced by the action of $A$ on $J$. When $n = 0$ this reduces to $A(x, 1_A) \otimes 1 \overset{\int^a \otimes J_\otimes}{\longrightarrow} A(x, 1_A) \otimes J(1_A, 1_B) \to J(x, 1_B)$.

In the case that $V = Set$ this means that every morphism $j: x \to x_1 \otimes \cdots \otimes x_n$ in $J$ can be decomposed as

$$j = (j_1 \otimes \cdots \otimes j_n) \circ f,$$

(4.2)
with \( j : y_1 \to y_n \) and \( f : x \to y_1 \otimes \cdots \otimes y_n \), and that this decomposition is unique in the sense that another pair \((f', j')\) also decomposes \( j \) if and only if \((f, j)\) and \((f', j')\) are identified in the first coend above.

Lax symmetric monoidal \( V \)-profunctors, that is lax \( S \)-profunctors for the ‘free symmetric strict monoidal \( V \)-category’-monad \( S \), can be described similarly.

**Example 4.3.** Consider normal colax double categories \( A \) and \( B \), that is normal colax \( D \)-algebras, with \( \mathbb{G}_1 \)-indexed structure functors \( \odot_A : DA \to A \) and \( \odot_B : DB \to B \). A lax \( D \)-promorphism \( J : A \Rightarrow B \) will be called a lax double profunctor: it consists of a \( \mathbb{G}_1 \)-indexed profunctor \( J : A \Rightarrow B \) (see Example 1.21) equipped with a structure transformation \( \odot_J = J : DJ \Rightarrow J(\odot_A, \odot_B) \), that satisfies the associativity and unit axioms. Giving \( \odot_J \) is equivalent to specifying horizontal composites

\[
\begin{array}{ccc}
\begin{array}{ccc}
\node(a1) at (0,0){a_0} ; \\
\node(a2) at (1,0){a_1} ; \\
\node(a3) at (2,0){a_2} ; \\
\node(a4) at (3,0){a_3} ; \\
\node(a5) at (4,0){a_4} ; \\
\node(a6) at (5,0){a_n} ; \\
\node(b1) at (0,-1){b_0} ; \\
\node(b2) at (1,-1){b_1} ; \\
\node(b3) at (2,-1){b_2} ; \\
\node(b4) at (3,-1){b_3} ; \\
\node(b5) at (4,-1){b_4} ; \\
\node(b6) at (5,-1){b_n} ; \\
\end{array}
& \xrightarrow{j_{a_0}} & \begin{array}{ccc}
\node(a1) at (0,0){a_0} ; \\
\node(a2) at (1,0){a_1} ; \\
\node(a3) at (2,0){a_2} ; \\
\node(a4) at (3,0){a_3} ; \\
\node(a5) at (4,0){a_4} ; \\
\node(a6) at (5,0){a_n} ; \\
\node(b1) at (0,-1){b_0} ; \\
\node(b2) at (1,-1){b_1} ; \\
\node(b3) at (2,-1){b_2} ; \\
\node(b4) at (3,-1){b_3} ; \\
\node(b5) at (4,-1){b_4} ; \\
\node(b6) at (5,-1){b_n} ; \\
\end{array}
\end{array}
\end{array}
\]

for the cells of \( J \), as well as horizontal units which are of the form as on the left below. The associativity axiom says that the diagram on the right commutes for every double sequence \( \mathbb{L} : k \Rightarrow l \).

\[
\begin{array}{ccc}
\begin{array}{ccc}
\node(a1) at (0,0){a} ; \\
\node(a2) at (1,0){a} ; \\
\node(b1) at (0,-1){b} ; \\
\node(b2) at (1,-1){b} ; \\
\end{array}
& \xrightarrow{\phi \uparrow \psi \uparrow} & \begin{array}{ccc}
\node(a1) at (0,0){a} ; \\
\node(a2) at (1,0){a} ; \\
\node(b1) at (0,-1){b} ; \\
\node(b2) at (1,-1){b} ; \\
\end{array}
\end{array}
\]

In the case that \( A \) and \( B \) are strict double categories this recovers the description that is given on the nLab page on double profunctors [Shu10].

That \( J \) is right pseudo means that the morphisms

\[
\int_{k \in DA} A(h, k_1 \odot_A \cdots \odot_A k_n) \times J(k_1, l_1) \times \cdots \times J(k_n, l_n) \to J(h, \odot_B \cdots \odot_B l_n),
\]

induced by \( \odot_J \) and the action of \( A \) on \( J \), are invertible for each horizontal morphism \( h \) in \( A \) and sequence \( l \) in \( DB \). This means that every cell \( w \) in \( J \), that is of the form below, can be decomposed as

\[
\begin{array}{ccc}
\begin{array}{ccc}
\node(c0) at (0,0){c_0} ; \\
\node(c1) at (1,0){c_1} ; \\
\node(c2) at (2,0){c_2} ; \\
\node(c3) at (3,0){c_3} ; \\
\node(c4) at (4,0){c_4} ; \\
\node(c5) at (5,0){c_n} ; \\
\end{array}
& \xrightarrow{\phi \uparrow \psi} & \begin{array}{ccc}
\node(c0) at (0,0){c_0} ; \\
\node(c1) at (1,0){c_1} ; \\
\node(c2) at (2,0){c_2} ; \\
\node(c3) at (3,0){c_3} ; \\
\node(c4) at (4,0){c_4} ; \\
\node(c5) at (5,0){c_n} ; \\
\end{array}
\end{array}
\]

where \( u \in A \) and \( w \in DJ \), and that this decomposition is unique up to the identification of the pairs \((u, w)\) in the coend above.

The following proposition shows that colax morphisms correspond to companions with lax promorphism structures. If \( f \) is a colax morphism for which the lax promorphism \( C(f, \text{id}) \) is right pseudo then we will say that \( f \) has a right pseudo companion. Such colax morphisms have appeared several times in literature. For example, they are crucial to the main results of [MT08] and [GP07], where such
morphisms respectively are called ‘$T$-operadic’ or said to satisfy the ‘left Conduché property’. Also the second condition of the result of Getzler, that is the motivation for this thesis and that was recalled in the introduction, simply means that the symmetric monoidal functor, along which the Kan extension is taken, has a right pseudo companion. For us they will be important because they are the vertical morphisms of $T$-Prom$_{MC}$ (Proposition 4.21) that admit companions, and thus can be used to define left Kan extensions.

**Proposition 4.4.** Let $T$ be a normal monad on an equipment $K$. For any morphism $f : A \to C$ in $K$, there is bijective correspondence between vertical cells $f : f \circ a \Rightarrow c \circ T f$, that make $f$ into a colax morphism, and cells $C(f, \text{id})$, of the form as on the left below, that make the companion $C(f, \text{id})$ into a lax promorphism.

$$
\begin{array}{ccc}
 TA & \xrightarrow{TC(f, \text{id})} & TC \\
 a & \downarrow & \circ \downarrow \eta \\
 A & \xrightarrow{C(f, \text{id})} & C
\end{array}
\quad
\begin{array}{ccc}
 TA & \xrightarrow{TA} & TA \xrightarrow{TC(f, \text{id})} & TC \\
 a & \downarrow & \circ \downarrow T f & \circ \downarrow T \eta \\
 A & \xrightarrow{f} & A & \xrightarrow{T f} & TC & \xrightarrow{TC} & TC
\end{array}
\quad
\begin{array}{ccc}
 TA & \xrightarrow{TC(f, \text{id})} & TC \\
 A & \xrightarrow{f} & C & \xrightarrow{T f} & C
\end{array}
$$

Under this assignment a structure cell $f$ is mapped to the composite on the right.

**Proof.** First notice that mapping $f$ to the composite above gives a bijection: it follows from the companion identities that an inverse is given by the assignment $C(f, \text{id}) \mapsto f \circ \varepsilon \circ C(f, \text{id}) \circ T f \eta$. We have to check that the associativity and unit axioms of $f$ are equivalent to those of $C(f, \text{id})$. To see this, notice that the unit axiom of $T$ implies that the image $T C(f, \text{id})$ is equal to the horizontal composite of $T f \eta$, $T f$ and $T^2 f \varepsilon$. Substituting this into the right-hand side of the associativity axiom for $C(f, \text{id})$, the cells $T f \eta$ (of $T C(f, \text{id})$) and $T f \varepsilon$ (of $C(f, \text{id})$) cancel and what remains is the right-hand side of the associativity axiom for $f$, horizontally composed on the left with $T f \eta$ and on the right with $T^2 f \varepsilon$.

The same is true for the left-hand side: recall from Proposition 3.4 that $\mu_{C(f, \text{id})}$ equals to the horizontal composition of $T f \eta$, the identity cell of $T f \circ \mu_A = \mu_C \circ T^2 f$ and $T^2 f \varepsilon$. Hence after replacing $\mu_{C(f, \text{id})}$ with this composite, the cells $T f \eta$ and $T f \varepsilon$ cancel in the left-hand side of the axiom for $C(f, \text{id})$ and we see that it too is equal to the left-hand side of the corresponding axiom for $f$, again after composing the latter on the left with $T f \eta$ and on the right with $T^2 f \varepsilon$. It thus follows from the companion identities that the associativity axioms for $f$ and $C(f, \text{id})$ are equivalent. A similar, easier argument shows that the unit axioms for $f$ and $C(f, \text{id})$ are equivalent as well, finishing the proof.

**Example 4.5.** Given a colax monoidal $\mathcal{V}$-functor $f : A \to C$ the structure cell $C(f, \text{id}) : MC(f, \text{id}) \Rightarrow C(f \circ \otimes, \otimes)$ consists of the composites

$$
C(f, \text{id}) \otimes \cdots \otimes C(f, \text{id}) \otimes C(\text{id}, \otimes \cdots \otimes \text{id}) \xrightarrow{\otimes \cdots \otimes \mu_n} C(f \circ \otimes, \otimes \cdots \otimes \otimes) \\
C(f, \text{id}) \xrightarrow{\text{id}} C(f \circ \otimes, \otimes \cdots \otimes \otimes). \quad (4.3)
$$

In this case, replacing $J_\otimes$ in (4.1) by the composite above, we find that $C(f, \text{id})$ being right pseudo coincides with the second condition of Getzler’s proposition $\text{(Get09, Proposition 2.3)}$, that was recalled in the introduction, and which has been the main motivation for developing the theory presented here. In particular the companion...
of the symmetric monoidal (unenriched) functor \(j: \mathbf{F}^{\text{op}} \to \mathbf{H}\) of PROPs, that was given as an example in the introduction, is right pseudomonoidal.

As in the previous example, the companion \(C(f, \text{id})\) of every colax double functor \(f: A \to C\) admits a lax double profunctor structure, which maps each sequence \(w = (w_1: f l_1 \Rightarrow l_1, \ldots, w_n: f l_n \Rightarrow l_n)\) in \(DC(f, \text{id})\) to the composite

\[
  J \circ f \colon A \to C
\]

in \(C(f \circ \circ, \circ)\), where \(\bar{J}\) is the compositor of \(J\). (Biased) colax double functors \(f\) for which \(C(f, \text{id})\) is right pseudo play an important role in the main theorem of [GP07], where the existence of a decomposition as in Example 4.3 for \(J = C(f, \text{id})\) and restricted to the biased case of \(n = 2\), is called the ‘left Conduché condition’. This result will be discussed in Section 4.3.

**Remark 4.6.** Notice that, in general, there is no obvious way to define the horizontal composition of two lax promorphisms \(J: A \Rightarrow B\) and \(H: B \Rightarrow C\): for that we need an inverse to \(T_\circ: TJ \circ TH \Rightarrow T(J \circ H)\) so that we can take

\[
  \frac{J \circ H = [T(J \circ H) \xrightarrow{T_\circ^{-1}} TJ \circ TH \xrightarrow{J \circ R} J \circ H]}{
}

as a structure cell for the horizontal composite. But, since \(T\) is not a pseudo functor in general, such inverses need not exist, so that colax algebras, colax morphisms and lax promorphisms will not in general form a pseudo double category. Instead one can prove that they admit a weaker structure, that of a ‘virtual’ double category. Virtual double categories are related to pseudo double categories in the same way that multicategories are related to monoidal categories: instead of cells \(i\) and \(\epsilon\), which suggests that structure cells for right colax promorphisms \(H: A \Rightarrow B\) should be horizontal cells of the form

\[
  \bar{H}: H \circ B(\text{id}, b) \Rightarrow A(\text{id}, a) \circ TH.
\]

The associativity and unit axioms for \(\bar{H}\) should then be chosen in such a way that when \(\bar{H} = (\rho J)^{-1}\), for a right pseudo lax promorphism \(J: A \Rightarrow B\), they are equivalent to the corresponding axioms for \(J\). To do this we first have to transform the associativity and unit axioms for lax promorphisms into identities in terms of \(\rho J\) and \(T \rho J\). First we introduce the following convention.

**Convention.** Let \(T = (T, \mu, \eta)\) be a normal monad on an equipment \(\mathbf{K}\). In the remainder of this chapter we will assume that a companion cell \(f \varepsilon: C(f, \text{id}) \Rightarrow \mathbf{U}_C\) and...
correspond, under Proposition 1.24, to the vertical structure cells of the form

\[ \rho \alpha : TA(a, id) \circ A(a, id) \Rightarrow TA(\mu_A, id) \circ A(a, id), \]
\[ \rho_0 : U_A \Rightarrow TA(\eta_A, id) \circ A(a, id), \]
\[ \rho \alpha : A(id, a) \circ TA(id, \mu_A) \Rightarrow A(id, a) \circ TA(id, a), \]
and
\[ \rho_0 : A(id, a) \circ TA(id, \eta_A) \Rightarrow U_A. \]

Likewise the horizontal cells \( \lambda \tilde{f} \) and \( \rho \tilde{f} \) corresponding to the structure cell \( \tilde{f} \) of a lax T-promorphism \( f : A \to C \) will be of the form

\[ \lambda \tilde{f} : TC(f, id) \circ C(c, id) \Rightarrow A(a, id) \circ C(f, id) \]
and
\[ \rho \tilde{f} : C(id, f) \circ A(id, a) \Rightarrow C(id, c) \circ TC(id, f), \]
while the T-image of the structure cell of a lax T-promorphism \( J : A \to B \) corresponds to the horizontal cells

\[ \rho T \tilde{J} : TA(id, a) \circ T^2 J \Rightarrow T^2 J \circ TB(id, b) \]
and
\[ \lambda T \tilde{J} : T^2 J \circ TB(b, id) \Rightarrow TA(a, id) \circ T J. \]

**Proposition 4.7.** Let \( T \) be a normal monad on an equipment \( K \), and let \( A \) and \( B \) be colax T-algebras. Given a promorphism \( J : A \Rightarrow B \) and a cell

\[ \begin{array}{ccc}
TA & \xrightarrow{TJ} & TB \\
\downarrow^a & & \downarrow^b \\
A & \xrightarrow{J} & B,
\end{array} \]

the following are equivalent.

- The cell \( J \) makes \( J \) into a lax \( T \)-promorphism, that is it satisfies the associativity and unit axioms.

- The horizontal cell \( \rho \tilde{J} : A(id, a) \circ T J \Rightarrow J \circ B(id, b) \) satisfies the following coherence conditions in \( H(K) \).
The horizontal cell $\lambda J: T J \circ B(b, id) \Rightarrow A(a, id) \circ J$ satisfies the following coherence conditions in $H(K)$.

Moreover in the associativity condition for $\rho T J$ the horizontal cell $\rho T J$ can be rewritten in terms of $T \rho J$ as below, where the inverse of the compositor exists by Proposition 3.3. Likewise $\lambda T J = T^{-1} \circ T \lambda J$.

\[ \rho T J = [T A(id, a) \circ T^2 J \xRightarrow{T \rho J} T(A(id, a) \circ T J)] \]

\[ \xRightarrow{T^{-1}} T(J \circ B(id, b)) \xRightarrow{T^{-1}} T J \circ T B(id, b) \]

Proof. For each lax $T$-promorphism $J: A \Rightarrow B$ it follows, from the naturality of $\rho$ (Proposition 1.24), that we obtain the two identities for $\rho J$ above when, under the convention above, we apply $\rho$ to the associativity and unit axioms for the structure cell $J$ (Definition 4.1). On the other hand, since the assignment $\rho$ is invertible, the two identities above imply the associativity and unit axioms for $J$. The second assertion is Proposition 3.4. For $\lambda J$ the arguments are symmetric.

To be able to rewrite the coherence conditions of Proposition 4.7 in terms of $(\rho J)^{-1}$ (provided this inverse exists) we need the inverses of $\rho \mu J$ and $\rho \eta J$ to exist. This leads to the following definition.

Definition 4.8. A normal monad $T = (T, \mu, \eta)$ on an equipment $K$ is called right suitable if for each promorphism $J: A \Rightarrow B$ the cells $\mu J$ and $\eta J$ are right invertible.

Cruttwell and Shulman consider pseudomonads with the property that all cells $\mu J$ and $\eta J$ are left invertible in [CS10, Appendix A]. They call such monads ‘horizontally strong’.
4.1. ALGEBRAIC PROMORPHISMS

Example 4.9. Remember the ‘free monoid’-monad $M$ on the equipment $\mathcal{V}$-$\text{Mat}$ of $\mathcal{V}$-matrices that was introduced in Section 3.2; it maps a set $A$ to the free monoid $MA$ generated by $A$ while the image $MJ$ of a $\mathcal{V}$-matrix $J: A \rightarrow B$ is given by $MJ(x, y) = M_{\eta}J(x, y) = \bigoplus_{\tau \in \Sigma} J(x, y)$. The monad induces the ‘free strict monoidal $\mathcal{V}$-category’-monad $M = \text{Mod}(M)$ on the equipment $\mathcal{V}$-$\text{Prof}$ of $\mathcal{V}$-profunctors. Also recall the modification $S$ of $M$, that is given by $SJ(x, y) = \left( \coprod_{v \in S} M_{\eta}J(v \cdot x, y) \right)$, which induces the ‘free symmetric strict monoidal $\mathcal{V}$-category’-monad $S = \text{Mod}(S)$ on $\mathcal{V}$-$\text{Prof}$. Both monads $M$ and $S$ on $\mathcal{V}$-$\text{Mat}$ are right suitable, as is shown below, while it follows from the proposition further below that the monads $M = \text{Mod}(M)$ and $S = \text{Mod}(S)$, that they induce, are right suitable as well.

Let $J: A \rightarrow B$ be a $\mathcal{V}$-matrix; we have to show that the $\mathcal{V}$-natural transformation $\rho_M: MA(id, \mu_A) \otimes M^2J = MJ(id, \mu_B)$ is invertible. Working out the components of its source we find

$$\left( MA(id, \mu_A) \otimes M^2J \right)(x, y) = \coprod_{v \in M^2A} MA(v, \mu_A) \otimes M^2J(v, y),$$

where $MA(id, \mu_A) \cong 1$ if $\mu_B = \emptyset$ and $0$ otherwise, while $M^2J(v, y) \neq 0$ only if $|x| + |y| = n = |y|$ and $|\mu| = m = |\mu|$, for all $i = 1, \ldots, n$. So the conditions, under which the $v$-component in the coproduct above is not the initial object, determine $v$ uniquely, if it exists. Notice that it does exist if and only if $|x| = \Sigma_{i=1}^{|x|} |y|$, since in that case we can take $v = (\Sigma_{i=1}^{|x|} |y|)$, where $\Sigma_{i=1}^{|x|} |y|$ is right suitable. As mentioned above this implies, by the proposition below, that $M = \text{Mod}(M)$ on $\mathcal{V}$-$\text{Prof}$ is right suitable as well. In particular the inverse of $\rho_M$, for a $\mathcal{V}$-profunctor $J: A \rightarrow B$, consists of the maps

$$MJ(x, y) \cong M^2J(v, y) \rightarrow \coprod_{u \in M^2A} MA(u, \mu_A) \otimes M^2J(u, v),$$

for $x$ and $y$ such that $|y| + \cdots + |y| = |x|$, where $v$ and the isomorphism are as given above and where the second map inserts at $w \rightarrow v$, using the identity $1 \rightarrow MA(v, \mu_A)$. That the monad $S$ on $\mathcal{V}$-$\text{Mat}$ is right suitable as well follows from applying $\rho$ to the identities $(\mu_S)_J \circ \theta_J^2 = \theta_J \circ (\mu_M)_J$ and $(\eta_S)_J = \theta_J \circ (\eta_M)_J$, that hold because $\theta: M \rightarrow S$ is a morphism of monads. Indeed, they give commuting diagrams in which we know of all but the cells $\rho(\mu_S)_J$ and $\rho(\eta_S)_J$ that they are invertible, by Proposition 3.21; the latter are therefore invertible as well.

Example 4.10. A monad $T = (T, \mu, \eta)$ on a category $\mathcal{E}$ with finite limits is called cartesian if $T$ preserves pullbacks and all the naturality squares of $\mu$ and $\eta$ are pullbacks. It follows from Example 6.5.3 that every such cartesian monad $T$ induces a right suitable monad $\text{Span}(T)$ on $\text{Span}(\mathcal{E})$. For example the free category monad $F$ on $\mathcal{G}_1$ is cartesian (see [Lei04, Example 6.5.3] or for an elementary proof [DPP96, Proposition 2.3]), so that the induced monad $\text{Span}(F)$ on $\text{Span}(\mathcal{G}_1)$ is right suitable. It follows from the proposition below that $D = \text{Mod}(\text{Span}(F))$, the free strict double category monad, is right suitable as well.
Example 4.11. The ‘free strict $\omega$-category’-monad $T$, that we briefly mentioned at
the end of Section 3.2, is cartesian as well. This is shown by Leinster in [Lei04,
Appendix F]. It follows that the induced monad $\text{Prof}(T)$, for monoidal globular
categories, is right suitable. The ultrafilter-monad $\beta$, that we also mentioned, is
not right suitable: one can verify that components of its multiplication are right
invertible, but those of its unit are not, see [CS10, Example A.7].

Recall that $\text{Equip}^{0}$ denotes the 2-category of equipments $\mathcal{K}$, for which every
$H(\mathcal{K})(A, B)$ has reflexive coequalisers preserved by $\circ$ on both sides, lax functors
and the transformations between them. Its sub-2-category $\text{Equip}^{n}$ consists of normal
functors.

Proposition 4.12. The 2-functor $\text{Mod}: \text{Equip}^{0} \to \text{Equip}^{n}$ of Proposition 3.10 pre-
serves right suitable monads.

Proof. The multiplication $\text{Mod}(\mu)$ of the image of a monad $T = (T, \mu, \eta)$ on $\mathcal{K}$ is
given by the cells $\mu_A$ that form morphisms of monoids $\mu_A: T^2A \to TA$ (remember
a monoid $A$ is a promorphism $A: A_0 \to A_0$ equipped with horizontal multiplication
and unit cells), and by the cells $\mu_J$ that form cells of bimodules $\mu_J: T^2J \Rightarrow TJ$
for every bimodule $J: A \Rightarrow B$. If $T$ is right suitable then the $\mathcal{K}$-cells underlying
$\mu_A$ and $\mu_J$ are right invertible, so that by Proposition 3.31 the cell $\mu_J$, as a cell of
bimodules, is right invertible in $\text{Mod}(\mathcal{K})$. In the same way we see that each $\eta_J$ is
right invertible in $\text{Mod}(\mathcal{K})$ as well, which completes the proof.

We are now ready to define right colax promorphisms.

Definition 4.13. Let $T$ be a right suitable normal monad on an equipment $\mathcal{K}$.
Given colax $T$-algebras $A$ and $B$, a right colax $T$-promorphism $J: A \Rightarrow B$ is a
promorphism $J: A \Rightarrow B$ equipped with a horizontal structure cell

\[ A \xrightarrow{J} B \xrightarrow{T(id, b)} TB \]

\[ \begin{array}{c}
    \downarrow
    \\
    A \xrightarrow{\rho_A} TA \xrightarrow{TJ} TB
\end{array} \]

satisfying the following associativity and unit axioms in $H(\mathcal{K})$. The diagrams in the
document represent these axioms.
Remark 4.14. It is readily seen that, for a right pseudopromorphism \( H: A \to B \), the axioms above for \( J = (pH)^{-1} \) are indeed equivalent to those of Proposition 4.7 for \( pH \), which was our goal. It follows that right pseudo lax promorphisms correspond to pseudo right colax promorphisms. We will not distinguish these two notions and call promorphisms of either type simply right pseudopromorphisms.

Also notice the necessity of the existence of the inverses of \( \rho \mu J \) and \( \rho \mu J' \), and the necessity of considering colax algebras, in the above definition. Indeed, in the associativity axiom both \( (\rho \mu J)^{-1} \) and \( \rho \alpha \) are 'kept in place' by \( J \) and \( T_0 \); when neither \( J \) nor \( T_0 \) is invertible it is not possible to rewrite the axiom in terms of \( \rho \mu J \) instead of \( (\rho \mu J)^{-1} \), or in terms of a lax algebra structure cell \( \alpha' : a \circ Ta \Rightarrow a \circ \mu A \) instead of \( \alpha \). In the same way \( (\rho \mu J')^{-1} \) is kept in place in the unit axiom.

We shall first describe the relatively simple notion of (unenriched) right colax monoidal profunctors. Before we start however, the author must admit that the only such profunctors he knows are all right pseudomonoidal profunctors, that is with invertible structure cells, as described in Example 4.2.

Consider an ordinary profunctor \( J: A \to B \) between normal colax unenriched monoidal categories \( A \) and \( B \). Loosely speaking, a right colax structure on \( J \) is a weakening of a right pseudo structure on \( J \); while the latter gives a decomposition (4.2) of each map \( j: x \to z_1 \otimes \cdots \otimes z_n \) in \( J \) as \( j = (j_1 \otimes \cdots \otimes j_n) \circ f \), where \( f: x \to y_1 \otimes \cdots \otimes y_n \) and \( j_i: y_i \to z_i \), a right colax structure only gives a coherent assignment \( j \mapsto (f, j) \), with no further relation between \( (f, j) \) and \( j \). In particular a right colax structure does not define tensor products \( J_n \) of maps in \( J \).

To make this formal, by a right splitting of a map \( j: x \to z_1 \otimes \cdots \otimes z_n \) in \( J \) we mean a pair \((f, j)\) consisting of a map \( f: x \to y_1 \otimes \cdots \otimes y_n \) in \( A \) and a sequence \( j: y \to z \) in \( M_n J \), of maps \( j_i: y_i \to z_i \) in \( J \). Such a right splitting \((f, j)\) of \( j \) is drawn as

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow \cong & & \downarrow \cong \\
\otimes & & \otimes \\
\vdots & & \vdots \\
z & \xrightarrow{j} & z
\end{array}
\]

Two splittings \((f, j)\) and \((f', j')\) of \( j \) are called equivalent if they are identified in the coend \( \int_{y \in M_n A} A(x, y_1 \otimes \cdots \otimes y_n) \times M_n J(y, z) \). Thus the equivalence relation on splittings \((f, j)\) is generated by the pairs \((f, j) \sim (f', j')\) for which there exists a sequence \( g: y \to y' \) such that \((y_1 \otimes \cdots \otimes y_n) \circ f = f' \) and \( j' \circ g = j \).

Likewise we consider \((right)\) double splittings of maps

\[
j: x \to (\otimes_1 \otimes \cdots \otimes \otimes_m) \otimes (\otimes_{n+1} \otimes \cdots \otimes \otimes_m)
\]

to be pairs \((f, j)\) where \( f: x \to (y_1 \otimes \cdots \otimes y_m) \otimes (y_{n+1} \otimes \cdots \otimes y_{n+m}) \) is a map in \( A \) and \( j: y \to z \) is a double sequence in \( M^2 J \). Again we call \((f, j)\) and \((f', j')\) equivalent if they are identified in the coend \( \int_{y \in M^2 A} A(x, (\otimes \otimes M \otimes)(y)) \times M^2 J(y, z) \).

Example 4.15. A right colax monoidal profunctor \( J: A \to B \), between normal colax monoidal categories \( A \) and \( B \), comes equipped with a right splitting \( J(j) = (f, j) \) of each map \( j: x \to z_1 \otimes \cdots \otimes z_n \), as above, such that the following naturality, associativity and unit axioms hold.

- For \( g: y \to x \) in \( A \), \( j: x \to z_1 \otimes \cdots \otimes z_n \) in \( J \) and \( h: z \to z' \) in \( M_n B \), if \( J(j) = (f, j) \) then the splittings \( J((h_1 \otimes \cdots \otimes h_n) \circ j \circ g) \) and \((f \circ g, h \circ j)\) are equivalent.

- For any \( j: x \to z_1 \otimes \cdots \otimes z_m \), \( \alpha_B \circ j \) the double splittings of \( \alpha_B \circ j \)

\[
(a_A \circ f, j) \quad \text{and} \quad ((h_1 \otimes \cdots \otimes h_n) \circ g, j)
\]
as in the diagrams below, are equivalent. Here the left-hand side is obtained by first splitting \( j \) as \( \bar{J}(j) = (f, J) \), where the target of \( f \) is \( y_1 \otimes \cdots \otimes y_N \), with 
\[ N = m_1 + \cdots + m_n \]
Then there is a unique way of decomposing \( j \) into a double sequence \( \bar{J}j \):

\[ y \rightarrow z \]

such that \( \mu_B j = \cdot \), making \((A \circ f, j)\) a double splitting of \( A_B \circ j \).

The right-hand side is given by first splitting \( A_B \circ j \) into \( \bar{J}(A_B \circ j) = (g, k) \) followed by splitting each of the \( k_i \) as \( \bar{J}(k_i) = (h, l) \). Together the latter form sequences \( h \) in \( M_nA \) and \( l \) in \( M^2J \), making \((h_1 \otimes \cdots \otimes h_n) \circ g, k \) a double splitting of \( A_B \circ j \) as well.

- For each \( j: x \rightarrow z \) the splitting \( \bar{J}(j) = (f, J) \) satisfies \( j \circ f = j \).

It is easily seen that, using the axiom of choice if necessary, choosing a right splitting of each map \( j: x \rightarrow z \) in \( J \), such that the first axiom above holds, is equivalent to giving a natural transformation \( J: J(id, \otimes) \rightarrow A(id, \otimes) \times_{MA} MJ \) as in Definition 4.13. It is then straightforward to check that the associativity axiom above corresponds to that of Definition 4.13 once we have postcomposed the latter with the inverse of the compositor \( M \circ \) (which exists because \( M \) is a pseudomonad), and that the unit axioms coincide as well.

The enriched version is as follows.

---

\[ x \rightarrow z \]

1 In fact \( j \) is the image of \( j \) under the cell \((\rho, \mu_J)^{-1}: TJ(id, \mu_B) \Rightarrow TA(id, \mu_A) \circ T^2A \) in the associativity axiom of Definition 4.13.
Example 4.16. Given normal colax monoidal \( V \)-categories \( A \) and \( B \), a \emph{right colax monoidal \( V \)-profunctor} \( J : A \Rightarrow B \) is a \( V \)-profunctor \( J : A \Rightarrow B \) equipped with \( V \)-natural maps

\[
J : \int^y A(x, \underbrace{y_1 \otimes \cdots \otimes y_n}_m) \to \int^y A(x, \underbrace{y_1 \otimes \cdots \otimes y_n}_m) \otimes J(\underbrace{y_1, \cdots, y_n}_m)
\]

for \( x \in A \) and \( z \in M_nB \), that satisfy the following associativity and unit axioms.

The associativity axiom states that, for every \( x \in A \) and \( z \in M^2B \), the composite

\[
\int^y A(x, \underbrace{y_1 \otimes \cdots \otimes y_n}_m) \Rightarrow \int^y A(x, \underbrace{y_1 \otimes \cdots \otimes y_n}_m) \otimes J(\underbrace{y_1, \cdots, y_n}_m)
\]

where the coends range over \( y \in MA \) or \( y \in M^2A \) and where the isomorphism is given by \( J_{\alpha, \beta} \), equals

\[
\int^y A(x, \underbrace{y_1 \otimes \cdots \otimes y_n}_m) \otimes \left( \bigotimes_{i=1}^n J(\underbrace{y_1, \cdots, y_n}_m) \right)
\]

Here the last map is induced by maps that, after permuting the factors, use the monoidal structure of \( A \) to map the tensor product of the factors \( A(\underbrace{y_1, \cdots, y_n}_m) \) into \( A(\underbrace{y_1, \cdots, y_n}_m) \otimes \cdots \otimes \cdots \) \( A(\underbrace{y_1, \cdots, y_n}_m) \), followed by composing the latter with \( A(x, \underbrace{y_1 \otimes \cdots \otimes y_n}_m) \). The unit axiom (remember that \( A \) and \( B \) are assumed to be normal) means that the composites

\[
\int^y A(x, y) \otimes J(y, z) \Rightarrow \int^y A(x, y) \otimes J(y, z) \cong J(x, z),
\]

where the isomorphism is the Yoneda isomorphism, equal the identity.

Right colax structures on double profunctors are similar to those on monoidal profunctors, as is shown in the following example. However, as a consequence of the fact that the ‘free strict double category’-monad \( D \) is not a pseudofunctor, the associativity axiom is slightly harder to state.

Example 4.17. Let \( A \) and \( B \) be normal colax double categories. A \emph{right colax double profunctor} \( J : A \Rightarrow B \) is a \( G_1 \)-indexed profunctor (Example 4.11), that comes equipped with \emph{right splittings} \( J(u) = (w, v) \); for every cell \( w \) : \( h \Rightarrow \underbrace{k_1 \otimes \cdots \otimes k_n}_m \) in \( J \), where \( u : h \Rightarrow \underbrace{k_1 \otimes \cdots \otimes k_n}_m \) is a cell in \( A \) and \( w : k \Rightarrow \underbrace{k_1 \otimes \cdots \otimes k_n}_m \) is a sequence of cells in...
$D_nJ$, as follows, that satisfies the axioms below.

\[
\begin{array}{ccc}
\text{a} & h & \text{b} \\
\downarrow \text{p} & \downarrow \text{w} & \downarrow \text{q} \\
\text{c}_0 & \cdots & \text{c}_{n'} & \text{c}_n
\end{array}
\]

The axioms are:

- for each splitting $\bar{J}(w) = (u, w)$ as above, $p_0 \circ f = p$ and $p_n \circ g = q$;
- for each $\bar{J}(w) = (u, w)$ above, $x: j \Rightarrow h$ in $A$ and $y: h \Rightarrow m$ in $D_nB$, the images of $\bar{J}((y_1 \circ \cdots \circ y_n) \circ w \circ x)$ and $(u \circ x, y \circ w)$ coincide in the coend $\int^D A(j, k_1 \circ \cdots \circ k_n) \times DJ(k_m, m)$;
- for each cell $w: h \Rightarrow l_1 \circ \cdots \circ l_{m_1} \circ \cdots \circ l_1 \circ \cdots \circ l_{m_n}$ in $J$, the images of the double splittings $\tilde{w}$
  \[
  \begin{array}{c}
  \left( h \xrightarrow{\tilde{w}} (\circ \circ \mu_A)(j) \xrightarrow{\Delta} (\circ \circ D \circ)(j), ((D \circ(j))_s \xrightarrow{id} \circ(j_s), j_s \xrightarrow{w_s} l_s)_{1 \leq s \leq n} \right)
  \end{array}
  \]
  and
  \[
  \begin{array}{c}
  \left( h \xrightarrow{\tilde{w}} (k), (k_s \xrightarrow{w_s} m_s, m_s \xrightarrow{w_s} l_s)_{1 \leq s \leq n} \right) \end{array}
  \]
  that are given as follows, coincide in $A(id, \circ) \times DA \times DA \times DA \times DA$.

Afterwards we will prove that, together with colax algebras and colax morphisms, these form a pseudo double category. In the definition we will use the following notation: for any colax $T$-morphism $f: A \to C$, with vertical structure cell $f: f \circ a \Rightarrow c \circ Tf$, we will denote the composite of the cells below by $\rho_1 f: A(id, a) \Rightarrow C(id, c)$. The subscript 1 in the notation represents the fact that in each vertical edge of $f$ we only ‘slid one map around the corner’, as opposed to $\rho f$, where both maps are moved. Notice that $f \mapsto \rho_1 f$ is functorial, in the sense that $\rho_1(g \circ f) = (\rho_1 g) \circ (\rho_1 f)$ for any other colax morphism $g: C \to E$. 

\[
\begin{array}{ccc}
A \xrightarrow{A(id, a)} TA & TA & TA \\
\downarrow \rho_1 & \downarrow \tau f & \downarrow \tau f \\
A & \sqrt{T} & TC & TC \\
\downarrow \rho_1 & \downarrow \psi & \downarrow \psi \\
C & C & C & C(id, c)
\end{array}
\]

Notice that while the image of the first double splitting here lies in the image of $A(id, \circ) \circ DA \times DA(id, \circ) \circ DDA \times DDA$, this is generally not true for the image of the second splitting. Indeed successive cells $y_s$ and $y_{s+1}$ need not have common vertical maps, so that they may not form a sequence $y \in DA$. In fact this failure is the reason for the monad $D$ not being pseudo.
Example 4.18. For a colax monoidal $V$-functor $f : A \to C$ the $V$-natural transformation $\rho_1 f_\otimes$ is given by the composites

$$A(x, y_1 \otimes \cdots \otimes y_n) \xrightarrow{J} C(f x, f(y_1 \otimes \cdots \otimes y_n)) \xrightarrow{C(id, f_\otimes)} C(f x, f y_1 \otimes \cdots \otimes f y_n).$$

Definition 4.19. Let $T$ be a right suitable normal monad on an equipment $\mathcal{K}$. Consider a cell

$$A \xrightarrow{f} B \quad \xrightarrow{\phi} \quad C \xrightarrow{g} D$$

where $f$ and $g$ are colax $T$-morphisms and where $J$ and $K$ are right colax $T$-promorphisms. We will call $\phi$ a $T$-cell whenever the following identity is satisfied, where $\rho_1 f_\otimes$ is defined above.

Notice that if $\phi$ is a vertical cell $\phi : f \Rightarrow g$ then the $T$-cell axiom for $\phi$ above is equivalent to the vertical $T$-cell axiom given in Definition 3.21: the first equals the horizontal composite of the latter with conjoint cells $\varepsilon_a$ on the left and $\eta_c$ on the right. Thus the definition above generalises that of vertical cells between colax morphisms.

As an example we describe cells between right colax monoidal functors; cells between right colax double functors are similar.

Example 4.20. When $T$ is the ‘free strict monoidal $V$-category’-monad $M$ we will call a $M$-cell $\phi : J \Rightarrow K(f, g)$ above, where $J$ and $K$ are right colax monoidal $V$-profunctors, a monoidal transformation. The $M$-cell axiom above means that the following diagram commutes, for objects $x \in A$ and $\underline{z} \in M_n B$, where the coends range over $y \in M_n A$ and $\underline{w} \in M_n C$.

Recall that every normal monad $T$ on an equipment $\mathcal{K}$ induces a 2-monad $V(T)$ on the vertical 2-category $V(\mathcal{K})$, by Proposition 3.11.

Proposition 4.21. Let $T$ be a right suitable normal monad on an equipment $\mathcal{K}$. Colax $T$-algebras, colax $T$-morphisms, right colax $T$-promorphisms and $T$-cells form
a pseudo double category \( T\text{-Prom}_{rc} \), in which the horizontal composite \( J \circ H \) of two right colax promorphisms \( J : A \to B \) and \( H : B \to C \) is equipped with the structure cell

\[
\overline{J \circ H} = [J \circ H \circ C(id, c) \xrightarrow{id \circ H} J \circ B(id, b) \circ TH]
\]

Moreover the vertical 2-category \( V(T)\text{-Alg}_{rc} \) is the 2-category \( V(T)\text{-Alg} \) of colax \( V(T)\)-algebras, colax \( V(T)\)-morphisms and vertical \( V(T)\)-cells.

**Proof.** We know that colax \( T\)-algebras (remember these are in fact colax \( V(T)\)-algebras, see Definition 3.22) and colax morphisms form a category \( (T\text{-Prom}_{rc})_0 \). On the other hand it follows readily from the functoriality of \( f \mapsto \rho_1 f \) that \( T\)-cells can be vertically composed by vertically composing their underlying cells in \( K \), so that they too form a category \( (T\text{-Prom}_{rc})_1 \), with right colax promorphisms as objects. The last assertion follows from the remark following Definition 4.19 above, once we have shown that \( (T\text{-Prom}_{rc})_0 \) and \( (T\text{-Prom}_{rc})_1 \) combine to a pseudo double category \( T\text{-Prom}_{rc} \).

We claim that the functor \( \circ : K_1 \times K_0 \to K_1 \), that equips \( K \) with horizontal composites, lifts to a functor

\[
\circ : (T\text{-Prom}_{rc})_1 \times (T\text{-Prom}_{rc})_0 \to (T\text{-Prom}_{rc})_1,
\]

by equipping each horizontal composite \( J \circ H \) with the structure cell \( \overline{J \circ H} \) above. First we have to check that \( \overline{J \circ H} \) satisfies the associativity and unit axioms. That it satisfies the associativity axiom is shown in Figure 4.1; proving the unit axiom is similar. Thus the structure cell \( \overline{J \circ H} \) above makes \( J \circ H \) into a well defined right colax promorphism.

To show that the horizontal composite of the cells underlying two \( T\)-cells

\[
A \xrightarrow{J} B \xrightarrow{H} C
\]

is again a \( T\)-cell as well consider the schematic diagrams below. Each diagram represents a composition \( J \circ H \circ C(id, c) \Rightarrow D(id, d) \circ (K \circ L) \) and, in particular, the first and the last represent the left and right-hand side of the \( T\)-cell axiom for \( \phi \circ \psi \). The equalities follow from the naturality of \( T\circ \), the \( T\)-cell axiom for \( \phi \) and the \( T\)-cell axiom for \( \psi \).

Finally notice that, for each colax algebra \( A \), the unit cell \( U_a \) trivially makes \( U_A \) into a lax promorphism, which is right pseudo because \( \rho U_a = \id_{A(id, a)} \). Thus each unit promorphism \( U_A \) admits a right colax promorphism structure; to finish the proof we remark that the associativity and unit axiom for \( T \) imply that the components of the associator \( \alpha \) and unitors \( I \) and \( \tau \) of \( K \) are \( T\)-cells, so that they can be taken as the associator and unitors for \( T\text{-Prom}_{rc} \). \( \square \)
4.1. Algebraic Promorphisms

Figure 4.1: A schematic proof of the associativity axiom for the composite \( J \circ H \) of right colax promorphisms \( J : A \to B \) and \( H : B \to C \), with structure cell as given by Proposition [L21]. Each of the diagrams above represents a vertical composition \( J \circ H \circ C(id, c) \circ TC(id, \mu_C) \Rightarrow A(id, a) \circ T(A(id, a) \circ T(J \circ H)) \) of horizontal cells: to save space only non-identity cells are denoted, while objects and promorphisms are left out. For example the first two blocks in the upper left diagram represent the cells (with \( \gamma \) the associator of \( C \))

\[
J \circ H \circ C(id, c) \circ TC(id, \mu_C) \xrightarrow{id \circ id \circ \rho_C} J \circ H \circ C(id, c) \circ TC(id, c) \xrightarrow{id \circ B \circ id} J \circ B(id, b) \circ TH \circ TC(id, c).
\]

In this way the top left diagram represents the right-hand side of the associativity axiom for \( J \circ H \), while the bottom right diagram represents its left-hand side. The equalities follow from (1) the associativity and naturality of \( T \); (2) the associativity axiom for \( H \); (3) the associativity axiom for \( J \); (4) the composition axiom for \( \mu \) (see Definition [L6]).
Example 4.22. Applying the previous proposition to the ‘free strict monoidal \(\mathcal{V}\)-category’-monad \(M\), on the equipment \(\mathcal{V}\)-\text{Prof} of \(\mathcal{V}\)-profunctors, we obtain the pseudo double category \(\mathcal{V}\)-\text{MonProf}_{rc}\) of colax monoidal \(\mathcal{V}\)-categories, colax monoidal \(\mathcal{V}\)-functors, right colax monoidal \(\mathcal{V}\)-profunctors (Example 4.16) and monoidal \(\mathcal{V}\)-natural transformations (Example 4.20) between them. Analogously, by applying the proposition to the ‘free symmetric strict monoidal \(\mathcal{V}\)-category’-monad \(S\), we obtain the pseudo double category \(\mathcal{V}\)-\text{sMonProf}_{rc}\) of symmetric right colax monoidal \(\mathcal{V}\)-profunctors.

Since \(M\) is a pseudomonad it is possible to replace the right colax monoidal \(\mathcal{V}\)-profunctors above by lax monoidal \(\mathcal{V}\)-profunctors (Example 4.12), obtaining a pseudo double category \(\mathcal{V}\)-\text{MonProf}_1\). However \(\mathcal{V}\)-\text{MonProf}_{rc}\) is best suited to constructing weighted colimits, as we shall see in Section 4.3.

Example 4.23. Applying the previous proposition to the ‘free strict double category’-monad \(D\), on the equipment \(\text{Prof}(\hat{\mathcal{G}}_1)\) of \(\mathcal{G}_1\)-indexed profunctors, we obtain the pseudo double category \(\text{DblProf}_{rc}\) of normal colax double categories, colax functors, right colax double profunctors (Example 4.17) and transformations between them. Unlike the monads \(M\) and \(S\), the monad \(D\) is not pseudo (Proposition 4.24), and it is not clear how to construct a pseudo double category of double categories and lax double profunctors (Example 4.2); see also the discussion in [Shul10].

Example 4.24. At the end of Section 3.2 we described how the monad for monoidal globular categories is induced by the ‘free strict \(\omega\)-category’-monad on \(\text{Span}(\mathcal{G})\). Since the latter is right suitable, see Example 4.11, we obtain a pseudo double category of ‘right colax monoidal globular profunctors’.

On the other hand, the ultrafilter-monad \(\beta\) on \(2\text{-Mat}\), that induces the monad for ordered compact Hausdorff spaces, is not right suitable, as the components of its unit are in general not right invertible. However, since \(\beta\) is a pseudo monad and since the components of its multiplication \(\mu\) are right invertible, it seems likely that it is still possible to arrange ordered compact Hausdorff spaces, their morphisms and right colax promorphisms, into a pseudo double category. Furthermore, it seems likely that a variant of our main theorem (Theorem 4.32) then holds, hence giving conditions under which the left Kan extension of a pair of morphisms between ordered compact Hausdorff spaces, as considered in the introduction, is again continuous.

Knowing that right colax promorphisms form a pseudo double category, we wonder whether it is an equipment. The proposition below shows that \(T\text{-Prom}_{rc}\) has all conjoints, but need not have all companions.

In general, consider a normal functor \(F\colon \mathcal{K} \to \mathcal{L}\) between pseudo double categories \(\mathcal{K}\) and \(\mathcal{L}\), and let \(f\) be a vertical morphism in \(\mathcal{K}\). We say that \(F\) lifts the companion of \(f\) if the existence of a companion for \(Ff\) in \(\mathcal{L}\) implies the existence of a companion for \(f\) in \(\mathcal{K}\) that is preserved by \(F\). Functors that lift conjoints are defined analogously.

Proposition 4.25. Let \(T\) be a right suitable normal monad on an equipment \(\mathcal{K}\). The forgetful functor \(U^T\colon T\text{-Prom}_{rc} \to \mathcal{K}\) lifts all conjoints, while it lifts companions of colax \(T\)-morphisms \(f\colon A \to C\) whose corresponding companion \(\hat{C}(f,\text{id})\) is right pseudo (see the comments preceding Proposition 4.4). In particular \(T\text{-Prom}_{rc}\) has restrictions \(K(f,g)\) for such \(f\).

Proof. First consider a colax morphism \(f\colon A \to C\). We saw in Proposition 4.12 that its companion \(\hat{C}(f,\text{id})\) is a lax promorphism, and we assume that it is right pseudo. This means that the structure cell \(\rho \hat{C}(f,\text{id})\) is invertible and hence its inverse \((\rho \hat{C}(f,\text{id}))^{-1}\) forms a right colax promorphism structure on \(\hat{C}(f,\text{id})\). Thus, if we prove that the companion cells \(\varepsilon_f\) and \(\eta_f\), that define \(\hat{C}(f,\text{id})\), are \(T\)-cells...
with respect this structure, then it follows that \( f \varepsilon \) is a companion cell in \( T{\text{-Prom}}_{ec} \) as well, because \( f \varepsilon \) and \( f \eta \) will still satisfy the companion identities in \( T{\text{-Prom}}_{ec} \).

But this is easy: by precomposing \((\rho, \text{resp. postcomposing})\) with \( \rho C(f, \text{id}) \), the \( T \)-cell axioms for \( f \varepsilon \) and \( f \eta \) are equivalent to \( \rho_1 f \circ T f \varepsilon = (f \varepsilon \circ \text{id}_{C(\varepsilon, \delta)}) \circ \rho C(f, \text{id}) \) and \( f \eta \circ \rho_1 f = \rho C(f, \text{id}) \circ (\text{id}_{A(\eta, \delta)} \circ T f \eta) \), which both follow from the fact that \( \rho C(f, \text{id}) = \rho f \). This shows that applying \( \rho \) to the associativity axiom for \( g \). This gives the following identity of cells
\[
D(id, g) \circ B(id, b) \Rightarrow D(id, d) \circ TD(id, g)
\]

We claim that the horizontal cell \( \rho g \) makes \( D(id, g) \) into a right colax promorphism. Indeed, to see that it satisfies the associativity axiom we apply \( \rho \) to the associativity axiom for \( g \). This gives the following identity of cells
\[
D(id, g) \circ B(id, b) \Rightarrow D(id, d) \circ T D(id, g)
\]

Secondly let \( g : B \Rightarrow D \) be a colax morphism with vertical structure cell \( g \).

We claim that the horizontal cell \( \rho g \) makes \( D(id, g) \) into a right colax promorphism. Indeed, to see that it satisfies the associativity axiom we apply \( \rho \) to the associativity axiom for \( g \). This gives the following identity of cells
\[
D(id, g) \circ B(id, b) \Rightarrow D(id, d) \circ T D(id, g)
\]

Of course \( U \) is its own inverse, but now considered as a cell \( \rho U \) on one hand, while it gives \( \rho T D(id, g) \) on the other, by Proposition 3.7. Therefore \( \rho U = (\rho U D(id, g))^{-1} \) in the left-hand side above. Finally our convention implies that \( \rho T g = T^{-1}_\varepsilon \circ T \rho g \circ T \eta \), and we conclude that postcomposing both sides with \( T \circ \varepsilon \) gives the associativity axiom for \( D(id, g) \). That the unit axiom holds for \( D(id, g) \) follows in the same way from that of \( g \).

To show that \( \varepsilon g \) is a conjoint cell in \( T{\text{-Prom}}_{ec} \), it remains to check that the cells \( \varepsilon g \) and \( \eta g \) form \( T \)-cells with respect to the algebra structure on \( D(id, g) \).

But this follows easily from the fact that we have taken \( D(id, g) = \rho g \). To complete the proof notice that the last assertion follows from Proposition 1.12.

In closing this section we consider weighted colimits in \( T{\text{-Prom}}_{ec} \) and, following [GP08, Section 4.1], define pointwise weighted colimits in \( T{\text{-Prom}}_{ec} \) (even though it may not be an equipment). Remember that, in Definition 2.15, the definition of weighted colimits in closed equipments was extended to pseudo double categories as follows. Given morphisms \( J : A \Rightarrow B \) and \( d : A \Rightarrow M \) in a pseudo double category, the \( J \)-weighted colimit of \( d \) is a vertical morphism \( l = \text{colim}_J d : B \Rightarrow M \) equipped with a cell \( \eta \), the unit, as below, such that any cell \( \phi \) factors uniquely through \( \eta \) as shown.

\[
\begin{array}{c}
A \xrightarrow{J} B \xrightarrow{H} C \\
\downarrow d \quad \downarrow e \quad \downarrow d \quad \downarrow \eta \quad \downarrow \eta' \\
M \xrightarrow{\phi} M \xrightarrow{M} M,
\end{array}
\]

Applied to \( T{\text{-Prom}}_{ec} \) we obtain the notion of a \textit{weighted colimit} \( \text{colim}_J d : B \Rightarrow M \) of a colax morphism \( d : A \Rightarrow M \), weighted by a right colax promorphism \( J : A \Rightarrow B \). In particular we can use the previous proposition in Definition 2.17 to obtain, for any colax morphism \( j : A \Rightarrow B \) that has a right pseudo companion \( B(j, \text{id}) \), the definition of a \textit{weighted left Kan extension} \( \text{lan}_j d : B \Rightarrow M \), of any colax morphism \( d : A \Rightarrow M \) along \( j \), as the weighted colimit \( \text{lan}_j d = \text{colim}_{B(j, \text{id})} d \) in \( T{\text{-Prom}}_{ec} \). We may then apply Proposition 2.10 to find that every weighted left Kan extension \( \text{lan}_j d \) is in particular a left Kan extension of \( d \) along \( l \) in the vertical 2-category.
$V(T\text{-Prom}_\text{rc}) = V(T\text{-Alg}_\text{c})$ of colax $V(T)$-algebras, in the usual sense. Our main result, given in the next section, states that the forgetful functor $U^T: T\text{-Prom}_\text{rc} \to K$ lifts weighted colimits. This means that if the weighted colimit $l' = \text{colim}_{K \to J} U^T d$ exists in $K$ then it can be lifted to a colax morphism $l$ (that is $U^T l = l'$) such that $l = \text{colim}_{J} d$.

We now turn to the notion of pointwise weighted colimits in $T\text{-Prom}_\text{rc}$. To generalise Definition 4.24 we first need double comma objects in $T\text{-Prom}_\text{rc}$. The following proposition shows that the forgetful functor $U^T: T\text{-Prom}_\text{rc} \to K$ lifts double comma objects $J//f$ of right pseudomorphisms $J: A 	o B$ and pseudomorphisms $f: C \to B$. This generalises [Lac05, Proposition 4.6] which shows that, for any 2-monad $T$ on a 2-category $C$, the forgetful 2-functor $U^T: T\text{-Alg}_\text{c} \to C$ lifts comma objects $f//g$ of colax $T$-morphisms $f$ and $g$ between strict $T$-algebras, when $g$ is a strict $T$-morphism. There it follows as an easy consequence of a more general theorem, and Lack remarks that giving a direct proof would strengthen his result to include the case of $g$ being only a pseudomorphism. We will give a direct proof of the proposition that is somewhat long (but straightforward) and therefore postponed to Appendix A.

Given a functor $F: K \to \mathcal{L}$ between pseudo double categories and morphisms $J: A \Rightarrow B$ and $f: C \Rightarrow B$ in $K$, we say that $F$ lifts the double comma object of $J$ and $f$ when, if a cell $\pi$ defines the double comma object of $FJ$ and $Ff$ in $\mathcal{L}$, then there exists a cell $\zeta$ in $K$ such that $F\zeta = \pi$, that defines the double comma object of $J$ and $f$.

**Proposition 4.26.** Consider a right suitable normal monad $T$ on an equipment $K$. The forgetful functor $U^T: T\text{-Prom}_\text{rc} \to K$ lifts double comma objects $J//f$ where $J: A \Rightarrow B$ is a right pseudomorphism and $f: C \Rightarrow B$ is a pseudomorphism. The projections of $J//f$ are strict $T$-morphisms, and $J//f$ is a pseudoalgebra whenever both $A$ and $C$ are.

As an example we consider the double comma object $J//f$ of a right pseudo double profunctor $J: A \Rightarrow B$ and a pseudo double functor $f: C \Rightarrow B$. In case $J = B(\text{id}, g)$ is the companion of a colax double profunctor $g: A \Rightarrow B$ this recovers the unbiased variants of the ‘comma double categories’ of [GP07, Section 1.3].

**Example 4.27.** Consider a right pseudo double profunctor $J: A \Rightarrow B$ and a pseudo double functor $f: C \Rightarrow B$ between normal colax double categories $A$, $B$, and $C$. Forgetting the horizontal composites for a moment, remember that the comma object $J//f$ in $K = \text{Cat}(\mathcal{G}_1)$ is constructed indexwise. Thus $J//f$ is the $\mathcal{G}_1$-indexed category that has objects $(a, p: a \Rightarrow fc, c)$, where $a \in A$ and $c \in C$ are objects and $p$ is a vertical morphism in $J$. A vertical morphism $(a, p: a \Rightarrow fc, c) \Rightarrow (a', p': a' \Rightarrow fc', c')$ is given by a pair $(s: a \Rightarrow a', t: c \Rightarrow c')$ of vertical morphisms $s$ in $A$ and $t$ in $C$ making the diagram on the left below commute, while a horizontal morphism $(a_0, p_0: a_0 \Rightarrow fc_0, c_0) \Rightarrow (a_1, p_1: a_1 \Rightarrow fc_1, c_1)$ consists of a triple $(j: a_0 \Rightarrow a_1, w: k: c_0 \Rightarrow c_1)$ of horizontal morphisms $j$ in $A$ and $k$ in $C$, and a cell $w$ in $J$ that is of the form as on the right below.

\[ \begin{array}{ccc} a & \overset{p}{\Rightarrow} & fc \\ s \downarrow & & t \downarrow \\ a' \overset{q}{\Rightarrow} fc' \end{array} \quad \begin{array}{ccc} a_0 & \overset{j}{\Rightarrow} & a_1 \\ p_0 \downarrow & \Rightarrow & w \downarrow \Rightarrow \ \downarrow \Rightarrow \ \\ f_{c_0} & \overset{k}{\Rightarrow} & f_{c_1} \end{array} \]

Finally, given a second horizontal morphism $(j', w', k')$: $(a_0', p_0', c_0') \Rightarrow (a_1', p_1', c_1')$, a
cell \( w \Rightarrow w' \) in \( J/f \) consists of a pair \((u, v)\) of cells

\[
\begin{array}{ccc}
  a_0 & \rightarrow & a_1 \\
  s_0 & \searrow & s_1 \\
  a_0' & \rightarrow & a_1'
\end{array}
\quad
\text{and}
\quad
\begin{array}{ccc}
  c_0 & \rightarrow & c_1 \\
  t_0 & \searrow & t_1 \\
  c_0' & \rightarrow & c_1'
\end{array}
\]

in \( A \) and \( C \) respectively, such that \( w' \circ u = f v \circ w \). With vertical composition given indexwise, this completes the description of \( J/f \) as a \( \mathbb{G}_1 \)-indexed category.

The \( \mathbb{G}_1 \)-indexed category \( J/f \) can be made into a normal colax double category as follows. On horizontal morphisms the structure functor \( \circ : D(J/f) \to J/f \) is given by the assignment

\[
\begin{array}{cccc}
  a_0 & j_0 & a_1 & a_n \\
  (p_0, j_0) & \downarrow & (p_1, j_1) & \cdots & (p_n, j_n) \\
  f c_0 & f k_1 & f c_1 & \cdots & f c_n
\end{array}
\]

where \( w_1 \circ \cdots \circ w_n \) is the horizontal composite given by the lax structure on \( J \), and where \( f^{-1} \) is the inverse of the colax structure cell of \( f \). By mapping a sequence \( ((u_1, v_1), \ldots, (u_n, v_n)) \) of composable cells in \( D(J/f) \) to the pair of composites \( (w_1 \circ \cdots \circ w_n, v_1 \circ \cdots \circ v_n) \) this extends to a \( \mathbb{G}_1 \)-indexed functor; that the latter is a well-defined cell follows from the naturality of \( f_\circ \) and of the horizontal composition of \( J \). Lastly the associator \( \alpha : \circ \circ f_{j,f} \to \circ f_{\circ} \) is given by \( \alpha_{\circ_{(w_1 \circ \cdots \circ w_n)} \circ_{(v_1 \circ \cdots \circ v_n)}} = (\langle (a_A)_j, (a_B)_k \rangle, \Xi) \), where \( a_A \) and \( a_B \) denote the associators of \( A \) and \( B \).

Recall that, in a pseudo double category \( K \) that has companions and double comma objects, a cell \( \eta \) below exhibits \( l \) as the pointwise \( J \)-weighted colimit of \( d \) if, for each morphism \( f : C \to B \), the composite \( \eta \circ \pi \) exhibits \( l \circ f \) as the weighted colimit, where \( \pi \) corresponds to the universal cell defining the double comma object \( J/f \), see Definition 2.35.

\[
\begin{array}{ccc}
  J/f & \overset{C(\pi_C, \text{id})}{\longrightarrow} & C \\
  \pi_A & \downarrow \pi & \downarrow f \\
  A & \overset{J}{\longrightarrow} & B \\
  d & \downarrow \eta & \downarrow l \\
  M & \overset{M}{\longrightarrow} & M
\end{array}
\]

In \( T\text{-Prom}_n \) we only know that the double comma object \( J/f \) exists when \( J \) is right pseudo and \( f \) is pseudo. Notice that in that case the cell \( \pi \) above can be constructed as well, since the companion of \( \pi_C : J/f \to C \) exists in \( T\text{-Prom}_n \) by Proposition 4.25 because \( \pi_C \) is a strict morphism. This leads to the following definition.

**Definition 4.28.** Let \( T \) be a right suitable normal monad on an equipment \( K \) that has double comma objects. Given a right pseudopromorphism \( J : A \to B \) and colax morphisms \( d : A \to M \) and \( l : B \to M \), we say that the cell \( \eta \) above exhibits \( l \) as the pointwise \( J \)-weighted colimit of \( d \) if, for each pseudomorphism \( f : C \to B \), the composition \( \eta \circ \pi \) above exhibits \( l \circ f \) as the \( C(\pi_C, \text{id}) \)-weighted colimit of \( d \circ \pi_A \).

As before, if \( J \) is the companion \( B(j, \text{id}) \) of a colax morphism \( j : A \to B \) then we call \( l \) the pointwise weighted left Kan extension of \( d \) along \( j \).
The idea of the previous definition is [GP08, Section 6.4] where ‘pointwise normal left Kan extensions’ of pairs of colax double functors are defined as follows. They consider a double category \( \mathsf{CxDbl}^u \) with normal colax double functors as vertical morphisms and colax double functors as horizontal ones and, after having shown that the ‘comma double category \( \mathsf{J}/f \)’ (see [GP08, Section 3.4]) can be constructed in \( \mathsf{CxDbl}^u \) whenever \( f \) is a pseudo double functor, define the ‘pointwise normal left Kan extension’ of a colax double functor \( D: A \to M \) along a colax double functor \( J: A \to B \) as a normal colax double functor \( l: B \to M \), equipped with a cell \( \eta \) as below, such that for every normal pseudo double functor \( f \) the composite \( \eta \circ \pi \) exhibits \( l \circ f \) as the left Kan extension of \( \pi_A \), from \( C(\pi_C, \text{id}) \) to \( D \) (see Definition 2.11).

\[
\begin{array}{ccc}
J/f & \overset{C(\pi_C, \text{id})}{\longrightarrow} & C \\
\pi_A \downarrow \pi & & \downarrow f \\
A & \overset{f}{\longrightarrow} & B \\
\eta \downarrow & & \downarrow \eta \\
A & \overset{id}{\longrightarrow} & M
\end{array}
\]

Assuming that the double comma objects of \( \mathcal{K} \) are all strong, our final aim is to apply Theorem 2.37 to \( T\text{-Prom}_{rc} \), thus proving that all colimits in \( T\text{-Prom}_{rc} \), that are weighted by right pseudopromorphisms, are pointwise. To do so we first have to show that \( T\text{-Prom}_{rc} \) has strong double comma objects. Notice that the cells \( \pi_f: C(\pi_C, \text{id}) \Rightarrow J(\text{id}, f) \), that are used in Definition 2.33 to define strong double comma objects, exist when \( J: A \to B \) is right pseudo and \( f: C \to B \) is pseudo, since both the companion \( C(\pi_C, \text{id}) \) and the restriction \( J(\text{id}, f) \) exist in \( T\text{-Prom}_{rc} \) by Proposition 4.25. The following enhances Proposition 4.26.

**Proposition 4.29.** Consider a right suitable normal monad \( T \) on an equipment \( \mathcal{K} \). The forgetful functor \( U^T: T\text{-Prom}_{rc} \to \mathcal{K} \) lifts strong double comma objects \( J/f \) where \( J: A \to B \) is a right pseudopromorphism and \( f: C \to B \) is a pseudomorphism.

**Proof.** In \( \mathcal{K} \) consider the double comma object \( J/f \) of a right pseudopromorphism \( J: A \to B \) and a pseudomorphism \( f: C \to B \), defined by a universal cell \( \pi \). Then \( J/f \) can be given the structure of a colax algebra so that the projections \( \pi_A: J/f \to A \) and \( \pi_C: J/f \to C \) become strict morphisms and the cell \( \pi \) becomes a \( T \)-cell, by Proposition 4.28. As discussed above, the corresponding cell \( \pi_f = \pi \circ (\eta_f \circ \pi_C \varepsilon) \), that is used in the factorisations (2.10) that define \( J/f \) as a strong double comma object, is a \( T \)-cell as well. Now consider such a factorisation \( \phi = \phi' \circ (\pi_f \circ \text{id}_B) \) in \( \mathcal{K} \); to prove the proposition, we have to show that if \( \phi \) is a \( T \)-cell then so is \( \phi' \). That the \( T \)-cell axiom for \( \phi' \), when precomposed with \( \pi_f \), holds follows from the following identities, that follow from the definition of \( \phi' \); the \( T \)-cell axiom for \( \phi \); the definition of \( \phi' \) again as well as the naturality of \( T \circ \).
4.2. ALGEBRAIC WEIGHTED COLIMITS

The $T$-cell axiom for $\pi_f$.

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\pi_f
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\phi'
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\rho_1 k
\end{array}
\end{array}
\end{array}
\end{array}
= \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\phi
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\rho_1 k
\end{array}
\end{array}
\end{array}
\end{array}
= \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
H
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
C(\pi_C, \text{id})
\end{array}
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\rho_1 \pi A
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
T\phi
\end{array}
\end{array}
\end{array}
\end{array}
= \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
H
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
C(\pi_C, \text{id})
\end{array}
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\rho_1 \pi A
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\rho_1 h
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
J(\text{id}, f)
\end{array}
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\rho_1 h
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
T\phi'
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}

The $T$-cell axiom for $\phi'$ follows from the uniqueness of factorisations through $\pi_f$. □

Finally let us consider a right suitable normal monad $T$ on an equipment $K$ that has strong double comma objects. Then, by restricting the proof of Theorem 2.37 to pseudomorphisms $f: C \to B$ and right pseudopromorphisms $J: A \Rightarrow B$, as in Definition 4.25 and so that the double comma object $J/f$ exists by Proposition 4.26 and is strong by Proposition 4.29, it can be applied to the pseudo double category $T$-$\text{Prom}_{\text{rc}}$ without change. We thus obtain the following variation of Theorem 2.37.

**Theorem 4.30.** Let $T$ be a right suitable normal monad on an equipment $K$, that has strong double comma objects. Given a colax morphism $d: A \to M$ as well as a right pseudopromorphism $J: A \Rightarrow B$, a $T$-cell

\[
A \xrightarrow{J} B
\]
\[
d \downarrow \downarrow \quad \downarrow \downarrow l
\]
\[
M \to \overline{M}
\]

exhibits $l$, in $T$-$\text{Prom}_{\text{rc}}$, as the $J$-weighted colimit of $d$ (Definition 2.15) if and only if it exhibits $l$ as the pointwise $J$-weighted colimit of $d$ (Definition 4.28).

### 4.2 Algebraic weighted colimits

We are now ready to state the main result. The following definition is motivated by [LS12, Definition 3.7], where the notion of ‘lifting limits along enriched functors’ is considered.

**Definition 4.31.** Consider a normal functor $F: K \to \mathcal{L}$ between pseudo double categories. Given a horizontal morphism $J: A \Rightarrow B$ in $K$, we say that $F$ lifts $J$-weighted colimits if, for any vertical morphism $d: A \to M$ in $K$, the following holds. If the cell $\zeta$ on the left below exhibits $k$ as the $FJ$-weighted colimit of $Fd$ in $\mathcal{L}$ then there exists a cell $\eta$, as on the right, that exhibits $l$ as the $J$-weighted colimit of $d$ in $K$, such that $F\eta = \zeta$ (and hence $Fl = k$).

\[
FA \xrightarrow{FJ} FB
\]
\[
Fd \downarrow \downarrow \quad \downarrow \downarrow k
\]
\[
FM \to \overline{FM}
\]

\[
A \xrightarrow{J} B
\]
\[
d \downarrow \downarrow \quad \downarrow \downarrow l
\]
\[
M \to \overline{M}
\]
The main result is as follows.

**Theorem 4.32.** Let $T$ be a right suitable normal monad on a closed equipment $\mathcal{K}$. The forgetful functor $U^T : T\text{-Prom}_{rc} \rightarrow \mathcal{K}$ lifts all weighted colimits. Moreover its lift of a weighted colimit $\text{colim}_T d : B \rightarrow M$, where $d : A \rightarrow M$ is a pseudomorphism and $J : A \rightsquigarrow B$ is a right pseudopromorphism, is a pseudomorphism whenever the canonical vertical cell (see Proposition 3.17)

$$\text{colim}_T J (m \circ T d) \Rightarrow m \circ T (\text{colim}_T d)$$

is invertible, where $m : TM \rightarrow M$ is the structure map of $M$.

By Theorem 2.37 and Theorem 4.30 in both $T\text{-Prom}_{rc}$ and $\mathcal{K}$ a cell $\eta$ exhibits a morphism $l$ as a pointwise weighted colimit if and only if it exhibits $l$ as an ordinary weighted colimit. Combining this with the main result we obtain the following corollary.

**Corollary 4.33.** Let $T$ be a right suitable normal monad on a closed equipment $\mathcal{K}$ that has strong double comma objects. The forgetful functor $U^T : T\text{-Prom}_{rc} \rightarrow \mathcal{K}$ lifts all pointwise weighted colimits $\text{colim}_T d$, where $J$ is a right pseudopromorphism.

In the proof of the main result the following lemma is crucial. Recall that a right colax $T$-promorphism $J : A \rightsquigarrow B$ comes equipped with a horizontal structure cell $J : J \odot B(\text{id}, b) \Rightarrow A(\text{id}, a) \odot T J$, while the structure cell $\bar{K} : T K \odot C(c, \text{id}) \Rightarrow (A(a, \text{id}) \odot K)$ that satisfies the $\lambda$-images of the associativity and unit axioms for $K$, see Proposition 1.24, to the horizontal cell $\lambda \bar{K} : TK \odot C(c, \text{id}) \Rightarrow A(a, \text{id}) \odot K$ that is given by Corollary 2.8.

**Lemma 4.34.** Under the hypothesis of Theorem 4.32, consider a right colax $T$-promorphism $J : A \rightsquigarrow B$ as well as a lax $T$-promorphism $K : A \Rightarrow C$. The left hom $J \triangleleft K : B \Rightarrow C$ admits a lax $T$-promorphism structure cell $\tilde{J} \triangleleft \bar{K}$ that corresponds, under Proposition 1.24, to the horizontal cell $\lambda \tilde{J} \triangleleft \bar{K}$ that is given by

$$T(J \triangleleft K) \odot C(c, \text{id}) \Rightarrow T J \triangleleft (TK \odot C(c, \text{id})) \Rightarrow \tilde{J} \triangleleft \bar{K} : \tilde{J} \odot (\text{id} \odot \bar{K}) \Rightarrow \tilde{J} \odot (A(a, \text{id}) \odot K)$$

$$\cong (A(a, \text{id}) \odot T J) \triangleleft K \Rightarrow \tilde{J} \odot (J \odot B(\text{id}, b)) \triangleleft K \cong B(b, \text{id}) \odot J \triangleleft K, \quad (4.5)$$

where the first cell is given by Proposition 3.9 and where the two isomorphisms are given by Corollary 2.8.

The proof of the above lemma consists of two (the first lengthy) calculations showing that the composite above satisfies the associativity and unit axioms, and will be postponed to Appendix B. We will denote the composition of the first four cells of (4.5) above by $\omega(J, \bar{K}) : T(J \triangleleft K) \odot C(c, \text{id}) \Rightarrow (J \odot B(\text{id}, b)) \triangleleft K$. It is useful to compute the adjoint $\omega(J, \bar{K})^2$: using item (b) of Lemma 3.14 we find that it is the composite

$$J \odot B(\text{id}, b) \odot T(J \triangleleft K) \odot C(c, \text{id}) \Rightarrow A(a, \text{id}) \odot T J \odot T(J \triangleleft K) \odot C(c, \text{id})$$

$$\Rightarrow A(a, \text{id}) \odot T(J \odot J \triangleleft K) \odot C(c, \text{id})$$

$$\Rightarrow A(a, \text{id}) \odot TK \odot C(c, \text{id})$$

$$\Rightarrow A(a, \text{id}) \odot A(a, \text{id}) \odot K \Rightarrow A(a, \text{id}) \odot K.$$  \(4.6\)

Here $\varepsilon_a$ is the counit of the companion-conjoint adjunction $A(a, \text{id}) \dashv A(\text{id}, a)$, see Proposition 1.14.

The proof of the main theorem takes up the remainder of this section. In the next section we consider its applications for the algebras for some of the monads that we described in Section 3.2.
4.2. ALGEBRAIC WEIGHTED COLIMITS

**Proof of Theorem 4.32.** Let \( J : A \to B \) be a right colax promorphism and \( d : A \to M \) be a colax morphism, and assume that the \( J \)-weighted colimit \( l = \text{colim}_J d : B \to M \) exists in \( K \). By Proposition 2.14 this means that it comes with a unit

\[
A \xrightarrow{J} B \xleftarrow{d} M \xrightarrow{l} M
\]

such that \((\lambda \eta)^\flat : M(l, id) \Rightarrow J \circ M(d, id)\) is invertible. We have to supply \( l \) with a colax structure such that \( \eta \) becomes a \( T \)-cell, and then show that \( \eta \) defines \( l \) as the \( J \)-weighted colimit of \( d \) in \( T \text{-Prom}_{\text{rec}} \). Recall from Proposition 4.4 that giving \( l \) a colax structure is the same as giving \( M(l, id) \) a lax structure. Likewise the colax structure on \( d \) corresponds to a lax structure on \( M(d, id) \) so that the lemma above the left hom \( J \circ M(d, id) \cong M(l, id) \) admits a lax structure. We thus obtain a colax structure on \( l \), whose vertical structure cell \( l \circ b \Rightarrow m \circ Tl \) corresponds to the composite

\[
\lambda l = [TM(l, id) \circ M(m, id) \xrightarrow{T(\lambda \eta)^\flat \circ id} T(J \circ M(d, id)) \circ M(m, id) \xrightarrow{id \circ (\lambda \eta)^\flat^{-1}} B(b, id) \circ J \circ M(d, id) \xrightarrow{id \circ \lambda \eta} B(b, id) \circ M(m, id)]. \tag{4.7}
\]

Notice that \( J \circ M(d, id) \) is invertible whenever \( \rho J \), \( \bar{l} \) and the canonical cell \( T(J \circ M(d, id)) \circ M(m, id) \Rightarrow TJ \circ (TM(d, id) \circ M(m, id)) \) are invertible. As we have seen in Proposition 3.14 the latter is, up to isomorphism, the \( \lambda \)-image of the canonical cell \( \text{colim}_T(J \circ m \circ Td) \Rightarrow m \circ T(\text{colim}_J d) \), so that the last assertion of the theorem follows.

Having obtained a colax structure on \( l : B \to M \), we show that \( \eta \) is a \( T \)-cell, which amounts to proving that the following equality holds.

\[
\begin{array}{c}
A \xrightarrow{J} B \xrightarrow{B(id, b)} TB \\
A \xrightarrow{A(id, a)} TA \xrightarrow{TJ} TB \\
M \xrightarrow{M(id, m)} TM \xrightarrow{Tl} TM
\end{array}
\]

\[
\begin{array}{c}
A \xrightarrow{J} B \xrightarrow{B(id, b)} TB \\
A \xrightarrow{\lambda \rho_1 l} \xrightarrow{\lambda \rho_1 I} \xrightarrow{\lambda \eta} \xrightarrow{\lambda \rho_1 l} \xrightarrow{Tl} TM \\
M \xrightarrow{M(id, m)} TM
\end{array}
\]

Equivalently we may prove that its image under \( \lambda \) holds, which is the equality of cells \( J \circ B(id, b) \circ TM(l, id) \Rightarrow M(d, id) \circ M(id, m) \) that is represented by the following diagram, in which all details but the non-identity cells are left out.

\[
\begin{array}{cc}
\xrightarrow{J} & \xrightarrow{\lambda \rho_1 l} \\
\xrightarrow{\lambda T \eta} & \xrightarrow{\lambda \eta}
\end{array}
\]

\[
\tag{4.8}
\]

It follows from the definitions of \( \rho_1 \) and \( \lambda \) (Definition 2.19 and Proposition 4.24), as well as the vertical companion identity, that the horizontal cell \( \lambda \rho_1 l \) here can be rewritten in terms of \( \lambda \bar{l} \) as

\[
\lambda \rho_1 l = [B(id, b) \circ TM(l, id) \xrightarrow{id \circ m \eta} B(id, b) \circ TM(l, id) \circ M(m, id) \circ M(id, m) \xrightarrow{id \circ \lambda \bar{l} \circ id} B(id, b) \circ B(b, id) \circ M(l, id) \circ M(id, m) \xrightarrow{\lambda \bar{l} \circ id} M(l, id) \circ M(id, m)] ,
\]
We have already seen that the composition of $T \circ \ev$ by evaluation. This implies that (4.9) equals id joint of (4.7) into the above we find that $\lambda \rho l$ equals the composite

$$B(id, b) \circ TM(l, id) \xrightarrow{id \circ TM(\eta)^{b} \circ \eta m} B(id, b) \circ T(J \triangleleft M(d, id)) \circ M(m, id) \circ M(id, m)$$

$$\xrightarrow{id \circ JM(id, id) \circ id} B(id, b) \circ B(b, id) \circ J \triangleleft M(d, id) \circ M(id, m)$$

$$\xrightarrow{id \circ \epsilon \circ \eta} J \triangleleft M(d, id) \circ M(id, m) \xrightarrow{(\eta^*)^{-1} \circ \eta} M(l, id) \circ M(id, m).$$

Now in the right-hand side of (4.8) we can rewrite $\lambda \eta \circ id = (\lambda \eta)^{b} \circ \eta$. Precomposed with the last cell of the composite above this gives $((\lambda \eta)^{b} \circ (id \circ (\lambda \eta)^{a} \circ \eta)) \circ id = ((\lambda \eta)^{b} \circ (\lambda \eta)^{a}) \circ id = ev \circ \eta$, where we have used the naturality of $-^{b}$ in its second argument. It follows that the right-hand side of (4.8) is given by

$$J \circ B(id, b) \circ TM(l, id) \xrightarrow{id \circ TM(\eta)^{b} \circ \eta m} J \circ B(id, b) \circ T(J \triangleleft M(d, id)) \circ M(m, id) \circ M(id, m)$$

$$\xrightarrow{id \circ JM(id, id) \circ id} J \circ B(id, b) \circ B(b, id) \circ J \triangleleft M(d, id) \circ M(m, id)$$

$$\xrightarrow{id \circ \epsilon \circ \eta} J \circ J \triangleleft M(d, id) \circ M(id, m) \xrightarrow{ev \circ \eta} M(d, id) \circ M(id, m).$$

In the above consider the composition of the last two cells together with the last cell of the composite $J \triangleleft M(d, id)$, as given by (4.8):

$$J \circ B(id, b) \circ (J \circ B(id, b)) \triangleleft M(d, id) \cong J \circ B(id, b) \circ B(b, id) \circ J \triangleleft M(d, id) \xrightarrow{id \circ \epsilon \circ \eta} J \circ J \triangleleft M(d, id) \xrightarrow{ev} M(d, id).$$

Writing $\phi = ev \circ (id \circ \epsilon \circ \eta \circ id)$ for the last two cells here, notice that the isomorphism, from right to left, equals $id \circ B(id, b) \circ \phi^{b}$, by Corollary 2.3. Of course $ev \circ (id \circ B(id, b) \circ \phi^{b}) = \phi$, and we conclude that the composite above is simply given by evaluation. This implies that (4.9) equals $id \circ TM(\eta)^{b} \circ \eta m$ followed by the adjoint of $\omega(J, \lambda \rho)$, which is the composite of the first four cells of (4.3). The latter we have already computed in (4.6), from which it follows that the right-hand side of (4.8) can be rewritten as

$$J \circ B(id, b) \circ TM(l, id) \xrightarrow{J \circ TM(\eta)^{b}} A(id, a) \circ T(J \triangleleft M(d, id))$$

$$\xrightarrow{id \circ T \circ \lambda \rho} A(id, a) \circ T(J \circ J \triangleleft M(d, id))$$

$$\xrightarrow{id \circ TM(\eta)^{a} \circ \eta m} A(id, a) \circ TM(d, id) \circ M(id, m)$$

$$\xrightarrow{id \circ \lambda \rho \circ \eta m} A(id, a) \circ A(a, id) \circ M(id, m)$$

$$\xrightarrow{\epsilon \circ \eta \circ id} M(id, m) \circ M(id, m).$$

We have already seen that the composition of $\lambda \rho \circ \eta$ and $\epsilon \circ \eta$ here equals $\lambda \rho \circ \eta$. Also notice that $T \circ ev \circ T \circ (id \circ TM(\eta)^{a}) = T \circ ev \circ T(id \circ (\lambda \eta)^{b}) \circ T \circ = T \lambda \eta \circ T \circ = \lambda \eta^* \eta$, where the last equality follows from the unit axioms for $T$. So, finally, the composite above equals

$$J \circ B(id, b) \circ TM(l, id) \xrightarrow{J \circ \eta \circ TM(l, id)} A(id, a) \circ TM(l, id) \xrightarrow{id \circ \lambda \eta} A(id, a) \circ TM(d, id) \xrightarrow{\lambda \rho \circ \eta \circ id} M(id, m).$$
which is the left-hand side of (4.8). This shows that, with respect to the chosen colax structure on $l$, the cell $\eta$ is a $T$-cell.

To complete the proof we have to show that $\eta$ defines $l$ as the $J$-weighted colimit of $d$ in $T$-$\text{Prom}_{rc}$. Thus we have to show that every $T$-cell $\phi$ as below factors uniquely through $\eta$, as shown.

\[
\begin{array}{ccc}
A & \xrightarrow{J} & B \\
\downarrow d & & \downarrow \phi \\
M & \cong & M
\end{array}
\quad = \quad
\begin{array}{ccc}
A & \xrightarrow{J} & B \\
\downarrow \eta & & \downarrow \phi'
\end{array}
\]

Forgetting about the $T$-algebra structures, such a factorisation does exist in $\mathcal{K}$, since $\eta$ defines $l$ as a weighted colimit. To show that this factorisation lifts to $T$-$\text{Prom}_{rc}$ we have to prove that $\phi'$ is a $T$-cell. That the $T$-cell axiom for $\phi'$ holds after it is horizontally precomposed with $\eta$ is shown below, where the identities are the $T$-cell axiom for $\eta$, the factorisation $T\phi = T(\eta \circ \phi')$, the $T$-cell axiom for $\phi$, and again the factorisation $\phi = \eta \circ \phi'$.

Since factorisations through $\eta$ are unique the $T$-cell axiom for $\phi'$ follows, completing the proof.

**4.3 Applications**

In this section we apply the main theorem to some of the monads that were described in Section 3.2. We shall also compare our main result to a result of Melliès and Tabareau, given in [MT08], which gives a different approach to obtaining algebra structures on left Kan extensions.

**Monoidal $\mathcal{V}$-categories**

We start with the ‘free (symmetric) strict monoidal $\mathcal{V}$-category’-monads $M$ and $S$ on the equipment $\mathcal{V}$-$\text{Prof}$ of $\mathcal{V}$-profunctors. Applying the main theorem we find that the forgetful functors of pseudo double categories

\[
U^M : \mathcal{V}$-$\text{MonProf}_{rc} \to \mathcal{V}$-$\text{Prof} \quad \text{and} \quad U^S : \mathcal{V}$-$\text{sMonProf}_{rc} \to \mathcal{V}$-$\text{Prof}
\]

lift weighted colimits, where $\mathcal{V}$-$\text{MonProf}_{rc}$ is the pseudo double category of right colax monoidal $\mathcal{V}$-profunctors and $\mathcal{V}$-$\text{sMonProf}_{rc}$ is that of symmetric right colax monoidal $\mathcal{V}$-profunctors, see Example 4.22.

Unpacking this result for $U^M$ we obtain the following. Recall that weighted colimits in a cocomplete $\mathcal{V}$-category can be computed using its coends and copowers, see Example 2.10.

**Proposition 4.35.** Given a colax monoidal $\mathcal{V}$-functor $d : A \to K$ into a cocomplete $\mathcal{V}$-category $K$ and a right colax monoidal $\mathcal{V}$-profunctor $J : A \to B$. Then the
weighted colimit \( l = \operatorname{colim}_J d \colon B \to K \) exists in \( \mathcal{V}\text{-Prof} \) and can be given by the coends 
\[
l(\underline{a}_1 \otimes \cdots \otimes \underline{a}_n) = \int^x dx \otimes J(x, \underline{a}_1 \otimes \cdots \otimes \underline{a}_n)
\]

\[
\xrightarrow{\int \text{id} \otimes J} \int^x dx \otimes \left( \int^y A(x, y_1 \otimes \cdots \otimes y_n) \otimes J(y_1, \underline{a}_1) \otimes \cdots \otimes J(y_n, \underline{a}_n) \right)
\]

\[
\cong \int^y \left( \int^x dx \otimes A(x, y_1 \otimes \cdots \otimes y_n) \otimes J(y_1, \underline{a}_1) \otimes \cdots \otimes J(y_n, \underline{a}_n) \right)
\]

\[
\cong \int^y d(y_1 \otimes \cdots \otimes y_n) \otimes (J(y_1, \underline{a}_1) \otimes \cdots \otimes J(y_n, \underline{a}_n))
\]

\[
\xrightarrow{\int d_0 \otimes \text{id}} \int^y (dy_1 \otimes \cdots \otimes dy_n) \otimes (J(y_1, \underline{a}_1) \otimes \cdots \otimes J(y_n, \underline{a}_n))
\]

\[
\to \int^y (dy_1 \otimes J(y_1, \underline{a}_1)) \otimes \cdots \otimes \int^y (dy_n \otimes J(y_n, \underline{a}_n)) = l_{\underline{a}_1} \otimes \cdots \otimes l_{\underline{a}_n} \quad (4.10)
\]

The second isomorphism here is the Yoneda isomorphism while the unlabelled map is a component of the canonical transformation

\[
\operatorname{colim}_{M_n J}(k_n \circ M_n d) \to k_n \circ M_n(\operatorname{colim}_J d)
\]

given by Proposition \[3.14\] where \( k_n : M_n K \to K \) is a restriction of the structure map of \( K \), that defines the \( n \)-fold tensor products of \( K \). The coends above range over single objects in \( A \) and sequences in \( M_n A \).

The monoidal structure for \( l \) above is invertible whenever \( d \) is a pseudomonoidal \( \mathcal{V} \)-functor, \( J \) is a right pseudomonoidal \( \mathcal{V} \)-profunctor and the canonical cells above are invertible for each \( n \geq 1 \). For the latter to hold it suffices that \( K \) is a pseudomonoidal \( \mathcal{V} \)-category such that \( k_2 : K \otimes K \to K \) preserves weighted colimits in both variables.

Before giving the proof we compare Getzler’s Proposition 2.3 of \[Get09\]. In stating it Getzler uses the fact that the \( \mathcal{V} \)-category of \( \mathcal{V} \)-presheaves \( A^{\text{op}} \to \mathcal{V} \) can be given a symmetric pseudomonoidal structure, as follows. The monoidal product of \( \mathcal{V} \)-presheaves \( d_1, \ldots, d_n : A^{\text{op}} \to \mathcal{V} \) is given by the formula

\[
(d_1 \otimes \cdots \otimes d_n)(x) = \int^{y \in M_n A} A(x, y_1 \otimes \cdots \otimes y_n) \otimes d_1(y_1) \otimes \cdots \otimes d_n(y_n).
\]

This is called the \textit{Day convolution} of \( d_1, \ldots, d_n \), which was introduced in \[Day70\] Section 4], and can be extended to a symmetric pseudomonoidal structure on \([A^{\text{op}}, \mathcal{V}]\). Moreover any \( \mathcal{V} \)-functor \( j^* : [B^{\text{op}}, \mathcal{V}] \to [A^{\text{op}}, \mathcal{V}] \), that is given by pre-composition with a symmetric colax monoidal \( \mathcal{V} \)-functor \( j : A \to B \), can be made into a symmetric lax monoidal \( \mathcal{V} \)-functor by the following structure maps, where \( d_1, \ldots, d_n : B^{\text{op}} \to \mathcal{V} \), where the coends range over \( y \in M_n A \) and \( z \in M_n B \) and
where the unlabelled map is induced by the universality of coends.

\[
(j^*d_1 \otimes \cdots \otimes j^*d_n)(x) = \int^y A(x, y_1 \otimes \cdots \otimes y_n) \otimes d_1 \circ j(y_1) \otimes \cdots \otimes d_n \circ j(y_n)
\]

\[
\frac{\int j \circ \id}{\int B(j(x), j(y_1) \otimes \cdots \otimes j(y_n)) \otimes d_1 \circ j(y_1) \otimes \cdots \otimes d_n \circ j(y_n)}
\]

\[
\to \int^z B(j(x), z_1 \otimes \cdots \otimes z_n) \otimes d_1(z_1) \otimes \cdots \otimes d_n(z_n)
\]

\[
= j^*(d_1 \otimes \cdots \otimes d_n)(x)
\]

(4.11)

Getzler calls a symmetric colax monoidal \(\mathcal{V}\)-functor \(j: A \to B\), for which the lax monoidal coherence map for \(j^*\) above is invertible, a \(\mathcal{V}\)-pattern. Using this notion, Proposition 2.3 of \cite{Get09} is stated as follows.

**Proposition 4.36** (Getzler). Let \(j: A \to B\) be a \(\mathcal{V}\)-pattern, and let \(M\) be a cocomplete symmetric pseudomonoidal \(\mathcal{V}\)-category whose binary monoidal product \(\otimes\) preserves colimits in both variables. The adjunction between the categories of \(\mathcal{V}\)-functors

\[
\lan_j: \left[A, M\right] \rightleftarrows \left[B, M\right]: j^*
\]

induces an adjunction between the categories of symmetric pseudomonoidal \(\mathcal{V}\)-functors

\[
\lan_j: \left[A, M\right]_{\text{ps}} \rightleftarrows \left[B, M\right]_{\text{ps}}: j^*.
\]

We claim that a symmetric colax monoidal \(\mathcal{V}\)-functor \(j: A \to B\) is a \(\mathcal{V}\)-pattern precisely if its companion \(B(j, \id): A \Rightarrow B\) is a right pseudopromorphism, so that Proposition 4.35 specialises to Getzler’s result in the case that the weight is taken to be the companion \(J = B(j, \id)\). To prove the claim we consider the components of the structure map (4.11) for \(j^*\) at representable \(\mathcal{V}\)-presheaves \(d_i = B(-, u_i)\), for \(u_i \in M_n B\). In that case the source of (4.11) is \((j^*d_1 \otimes \cdots \otimes j^*d_n)(x) = (A(id, a) \otimes M_A M B(j, id))(x, u)\), where \(a: M_n A \to A\) is the structure map of \(A\), while its target is

\[
(j^*d_1 \otimes \cdots \otimes d_n)(x) = \int^y B(j(x), z_1 \otimes \cdots \otimes z_n) \otimes M_n B(z, u) \cong B(j(x), u_1 \otimes \cdots \otimes u_n),
\]

using the enriched Yoneda isomorphism. It is readily checked that, composed with this isomorphism, (4.11) coincides with the horizontal cell \(\rho B(j, \id)\), corresponding to the structure cell \(B(j, \id)\) that makes \(B(j, \id)\) into a symmetric lax monoidal \(\mathcal{V}\)-functor; \(\rho B(j, \id)\) is obtained by substituting (1.13) into (4.11). This shows that \(j\) is a \(\mathcal{V}\)-pattern, that is \(j^*\) is pseudomonoidal, only if \(B(j, \id)\) is right invertible, i.e. \(B(j, \id)\) is a right pseudopromorphism. The converse follows from the fact that any \(\mathcal{V}\)-presheaf \(d: A^{\text{op}} \to \mathcal{V}\) can be written as a colimit of representable \(\mathcal{V}\)-presheaves \(d = \int^x A(-, x) \otimes d(x)\), see \cite{Kela} Formula 3.72.

**Proof of Proposition 4.35.** We can apply Theorem 4.32 to obtain a colax monoidal structure vertical cell \(l_\otimes: l \circ b \Rightarrow k \circ Ml\) for \(l\). To see that \(l_\otimes\) coincides with (1.10) recall that it corresponds to the horizontal cell \(\lambda \otimes\) given by the composite (1.5), with \(T = M\) and \(K = K(d, \id)\). Working out \(\lambda \otimes\) we find that its components are
The source and target of the composite above define the objects
\[ T J \triangleright \lambda \]
where we have used that the first isomorphism in \((4.5)\) is in fact the composite
\[ l \circ \mu \]
\[ \lambda \circ \mu \]
easily seen that, under these isomorphisms, the five maps in the composite
\[ K \]
where the first and third isomorphism are given by the defining isomorphisms of
\[ \lambda \circ \mu \]
unlabelled map this follows from Proposition \(3.14\). This proves the first assertion
\[(\lambda \circ \mu)\]
\[ \lambda \circ \mu \]
of the proposition, while the second one immediately follows from Theorem \(4.32\).

To prove the last assertion assume that \( K \) is a pseudomonoidal \( \mathcal{V} \)-category, with
invertible associator. We have to show that the canonical maps
\[
\int^n k_n(dy_1, \ldots, dy_n) \otimes (J(y_1, z_1) \otimes \cdots \otimes J(y_n, z_n))
\rightarrow k_n\left(\int^{y_1} dy_1 \otimes J(y_1, z_1), \ldots, \int^{y_n} dy_n \otimes J(y_n, z_n)\right) \tag{4.12}
\]
are isomorphisms for all \(n \geq 1\) where, for clarity, we have refrained from using the tensor product notations for the \(k_n\)-images. They are induced by the maps
\[
k_n(dy_1, \ldots, dy_n) \otimes (J(y_1, z_1) \otimes \cdots \otimes J(y_n, z_n))
\rightarrow k_n(dy_1 \otimes J(y_1, z_1), \ldots, dy_n \otimes J(y_n, z_n)),
\]
given by the universality of the copowers, followed by the \(k_n\)-image of the insertions into the coends \(\int^{y_i} dy_i \otimes J(y_i, z_i)\). A straightforward calculation shows that, when \(n \geq 2\), the canonical map above can be rewritten as (here \(n' = n - 1\))
\[
\int^n k_n(dy_1, \ldots, dy_n) \otimes (J(y_1, z_1) \otimes \cdots \otimes J(y_n, z_n))
\overset{(1)}{=} \int^n k_2(dy_1, k_1(dy_2, \ldots, dy_n) \otimes (J(y_2, z_2) \otimes \cdots \otimes J(y_n, z_n))) \otimes J(y_1, z_1)
\overset{(2)}{=} \int^n k_2(dy_1 \otimes J(y_1, z_1), k_1(dy_2, \ldots, dy_n) \otimes (J(y_2, z_2) \otimes \cdots \otimes J(y_n, z_n)))
\overset{(3)}{=} \int_{y_1, y_2, \ldots, y_n} k_2\left(dy_1 \otimes J(y_1, z_1), k_1(dy_2, \ldots, dy_n) \otimes (J(y_2, z_2) \otimes \cdots \otimes J(y_n, z_n))\right)
\overset{(2)}{=\overset{(1)}}{\rightarrow} \int_{y_1} k_2(dy_1 \otimes J(y_1, z_1), k_1(dy_2, \ldots, dy_n) \otimes (J(y_2, z_2) \otimes \cdots \otimes J(y_n, z_n)))
\rightarrow \int_{y_1}^{y_1} k_2(dy_1 \otimes J(y_1, z_1), k_1(dy_2, \ldots, dy_n) \otimes (J(y_2, z_2) \otimes \cdots \otimes J(y_n, z_n)))
\overset{(2)}{\overset{(1)}}{=} k_2\left(dy_1 \otimes J(y_1, z_1), k_1(dy_2, \ldots, dy_n) \otimes (J(y_2, z_2) \otimes \cdots \otimes J(y_n, z_n))\right)
\overset{(1)}{=\overset{(2)}}{\rightarrow} k_n(dy_1 \otimes J(y_1, z_1), \ldots, dy_n \otimes J(y_n, z_n)),
\]
where the isomorphisms are induced by (1) the restriction \(k_n \cong k_2 \circ (\text{id} \otimes k_1)\) of the invertible associator of \(K\), (2) the assumption that \(k_2\) preserves weighted colimits in both variables and (3) Fubini’s theorem for coends. The unlabelled map is the canonical map \(\mathbf{(4.12)}\) for \(n'\), and we conclude that \(\mathbf{(4.12)}\) is an isomorphism for \(n\) whenever it is one for \(n'\). Since all monoidal \(\mathcal{V}\)-categories are assumed to be normal, we have \(k_1 = \text{id}\), so that \(\mathbf{(4.12)}\) is the identity for \(n = 1\); that it is an isomorphism for all \(n \geq 1\) follows by induction. \(\square\)

**Double categories**

Applying the main theorem to the ‘free strict double category’-monad \(D\) on the equipment \(\text{Prof}(\hat{G}_1)\) of \(\hat{G}_1\)-indexed categories we find that the forgetful functor \(U^D : \text{DblProf}_{rc} \rightarrow \text{Prof}(\hat{G}_1)\) lifts all weighted colimits, where \(\text{DblProf}_{rc}\) is the pseudo double category of right colax double profunctors, see Example \(\mathbf{4.17}\). Furthermore, the equipment \(\text{Prof}(\hat{G}_1)\) has strong double comma objects by Proposition \(\mathbf{2.36}\) so that we can apply Corollary \(\mathbf{4.33}\) as well, obtaining the following.
Proposition 4.37. The forgetful functor $U^D: \text{DblProf}_{rc} \to \text{Prof}(\mathbb{S}_1)$ lifts all pointwise weighted colimits. In particular, it lifts pointwise weighted left Kan extensions $\text{lan}_d$ (Definition 4.22) where $d$ and $j$ are colax double functors such that $B(j, \text{id})$ is a right pseudo double profunctor.

The second statement here, about pointwise weighted left Kan extensions, can be compared (but seems unlikely to be equivalent) to the main result of [GP07]. To discuss it we need the notion of ‘double colimits’ in double categories, that were introduced in [GP99, Section 4]. First, a double cocone $x$ for a colax double functor $p: A \to M$ consists of an object $x$ in $M$ equipped with a vertical map $x_a: pa \to x$ for every object $a$ of $A$ and a cell

$$
\begin{array}{ccc}
  pa & \xrightarrow{pf} & pb \\
  x_a & \Downarrow{x_j} & x_b \\
  x & \xrightarrow{x} & x
\end{array}
$$

for every horizontal morphism $j: a \Rightarrow b$ in $A$, satisfying the following naturality conditions:
- $x_b \circ pf = x_a$ for every vertical cell $f: a \to b$ in $A$;
- $x_b \circ pu = x_j$ for every cell $u: j \Rightarrow k$ in $A$;
- $(x_j \circ x_b) \circ p_{c_1} = x_{j \circ k}$ for composable horizontal morphisms $j: a \Rightarrow b$ and $k: b \Rightarrow c$, where $p_{c_1}: p(j \circ k) \Rightarrow p j \circ pk$ is the compositor of $p$.

The double colimit of $p: A \to M$ is a double cocone $\text{colim} p = c$ for $p$ that is universal in the sense that, for any other double cocone $x$, there exists a unique vertical map $x': c \to x$ such that $x' \circ c_a = x_a$ and $1_{x'} \circ c_j = x_j$, for every object $a$ and horizontal morphism $j: a \Rightarrow b$, where $1_{x'}$ denotes the horizontal identity for $x'$. Dually one can define double limits; the double comma objects that we considered in Chapter 2 are examples of double limits.

The universality of double colimits implies that any transformation $\xi: p \Rightarrow q$ (Definition 5.10) of colax functors $p$ and $q: A \to M$ induces a canonical vertical morphism colim $\xi: \text{colim} p \to \text{colim} q$ in $M$, whenever these colimits exist. Remember that such transformations $\xi$ consist of natural vertical maps $\xi_a: pa \to qa$ for all $a$ in $A$, as well as natural cells $\xi_j: pq \Rightarrow qj$ for every $j: a \Rightarrow b$ in $A$. Symmetrically one can consider horizontal transformations $\chi: p \Rightarrow q$, which consist of horizontal maps $\chi_a: pa \Rightarrow qa$ for every object $a$ in $A$ as well as cells

$$
\begin{array}{ccc}
  pa & \xrightarrow{\chi_a} & qa \\
  pf & \Downarrow{\chi_f} & qf \\
  pb & \xrightarrow{\chi_b} & qb
\end{array}
$$

for every vertical morphism $f: a \to b$ in $A$, satisfying certain coherence conditions. Taking double colimits need not be functorial with respect to these horizontal transformations; consequently Grandis and Paré consider in [GP07, Section 4] cocomplete double categories $M$ equipped with a colax functorial choice of double colimits. Roughly speaking, such a choice picks out a horizontal morphism colim $\chi: \text{colim} p \Rightarrow \text{colim} q$ in $M$ for every horizontal transformation $\chi: p \Rightarrow q$, where $p$ and $q: A \to M$ are colax double functors, in a coherent way.

After recalling Grandis and Paré’s notion of ‘pointwise normal left Kan extension’ of a pair of colax double functors, that we discussed following Definition 4.28 the colax variant of the main theorem of [GP07] can now be stated, in our terms, as follows.
4.3. APPLICATIONS

**Theorem 4.38** (Grandis and Parè). Let $M$ be a normal pseudo double category in which every vertical isomorphism has a companion. The following are equivalent:

- for any pair of colax double functors $j: A \to B$ and $d: A \to M$ the pointwise normal left Kan extension $\text{lan}_j d: B \to M$ exists whenever the companion $B(j, \text{id})$ is a right pseudo double profunctor;

- $M$ is an equipment that admits a colax functorial choice of double colimits.

The author does not know how to compare this result with the second assertion of Proposition 4.37 above. In particular the respective pointwise left Kan extensions, that are considered in the two different results, belong to, what seem to be, very different double categories. However, the condition of the companion $B(j, \text{id})$ being a right pseudo double profunctor, in both results, is striking.

**Comparison to the work of Melliès and Tabareau**

While writing up this thesis the author discovered the unpublished paper [MT08] by Melliès and Tabareau, that is also devoted to producing algebraic left Kan extensions. Fortunately their approach, which we discuss below, is very different to the one presented here, leading to many differences in the results obtained.

Whereas we assume and use a closed structure on our equipments, their approach is based on the notion of ‘Yoneda situations’, as follows. In our terms, a morphism $y: M \to \widehat{M}$ in an equipment $\mathcal{K}$ is a Yoneda situation if

- $y$ is fully faithful, that is the unit $y_\eta y: U_M \Rightarrow \widehat{M}(y, y)$ of the companion-conjoint adjunction (Proposition 1.14) is invertible in $H(\mathcal{K})$;

- for each object $A$ the composition

$$V(\mathcal{K})(A, \widehat{M}) \xrightarrow{\rho} H(\mathcal{K})(\widehat{M}, A) \xrightarrow{\widehat{M}(y, \text{id}) \otimes} H(\mathcal{K})(A, M),$$

where $\rho$ is given by Proposition 1.25 is fully faithful.

In $\mathcal{K} = \text{Prof}$ the Yoneda embedding $y: M \to \widehat{M}$, with $\widehat{M} = [M^{op}, \text{Set}]$ the category of presheaves on $M$, is a Yoneda situation. Propositions 1 and 2 of [MT08] imply the following result.

**Proposition 4.39** (Melliès and Tabareau). In an equipment $\mathcal{K}$ consider morphisms $j$, $d$, $y$ and $k$ as below, such that $y$ is a Yoneda situation. If

- $y$ has a left adjoint colim: $\widehat{M} \to M$ and

- there exists an isomorphism $M(\text{id}, d) \circ B(j, \text{id}) \cong \widehat{M}(y, \text{id}) \circ \widehat{M}(\text{id}, k)$ in $H(\mathcal{K})$,

then the composite colim$\circ k: B \to M$ is the (ordinary) left Kan extension of $d$ along $j$.

$$
\begin{array}{ccc}
A & \xrightarrow{j} & B \\
\downarrow{d} & & \downarrow{k} \\
M & \xrightarrow{y} & \widehat{M}
\end{array}
$$

In our terms, their main theorem [MT08, Theorem 1] can now be stated as follows. Given a normal pseudomonad $T$ on an equipment $\mathcal{K}$ we denote by $V(T)$-$\text{Alg}_{ps}$ the 2-category of pseudo $V(T)$-algebras, pseudo $V(T)$-morphisms and $V(T)$-cells.
Theorem 4.40 (Melliès and Tabareau). Let $T$ be a normal pseudomonad on an equipment $K$ and consider the situation of the previous proposition in $K$, such that

- $A$, $B$, $M$ and $\hat{M}$ are pseudo $T$-algebras;
- $d$, $j$, $\text{colim}$ and $k$ are pseudo $T$-morphisms;
- the induced lax structures on $B(j,id)$ and $\hat{M}(y,id)$ are right pseudo.

Then $\text{colim}\circ k$, which is the left Kan extension of $d$ along $j$ by the previous proposition, can be given the structure of a pseudo $T$-morphism, which makes it the left Kan extension of $d$ along $j$ in $V(T)$-$\text{Alg}_{ps}$.

Thus the main difference between our main result and that of Melliès and Tabareau is that where we assume a closed structure on equipments, Melliès and Tabareau assume the existence of Yoneda situations. Moreover, where they work with ordinary left Kan extensions, we consider the more general, but also stronger, notion of weighted colimits. Finally, and perhaps most importantly, our main theorem works in the case of normal monads, while Melliès and Tabareau consider only pseudomonads.
Appendix A

Double comma objects

This appendix collects some proofs of results concerning comma objects that we have used. Each of them long, the author feared they would have been distracting when part of the main text.

Recall that the double comma object \( J/f \) of a pair of morphisms \( J: A \to B \) and \( f: C \to B \), in a pseudo double category \( K \), consists of a cell \( \pi \) as below that satisfies the following universal properties.

Firstly, any other cell \( \phi \) as above factors uniquely through \( \pi \) as a map \( \phi' : X \to J/f \), such that \( \pi_A \circ \phi' = \phi_A \), \( \pi_C \circ \phi' = \phi_C \) and \( \pi \circ U\phi' = \phi \), and, secondly, if the cell \( \psi \) below likewise factors as \( \psi' : Y \to J/f \), then any pair \( (\xi_A, \xi_C) \) of cells satisfying the identity below corresponds to a unique cell \( \xi' \), as on the right below, such that \( U\pi_A \circ \xi' = \xi_A \) and \( U\pi_C \circ \xi' = \xi_C \).

Informally, it is useful to think of the cells \( \phi = (\phi_A, \phi, \phi_C) \) and \( \psi = (\psi_A, \psi, \psi_C) \) above as objects, and of a pair \( \xi = (\xi_A, \xi_C) \) as a morphism \( \phi \to \psi \). Then the universality of \( \pi \) gives assignments \( \phi \mapsto \phi' \) and \( \xi \mapsto \xi' \), the latter with \( L\xi' = \phi' \) and \( R\xi' = \psi' \), that map \( \phi \) and \( \xi \) to their factorisations through the universal cell \( \pi \). The uniqueness of these factorisations implies that the assignments \( \phi \mapsto \phi' \) and \( \xi \mapsto \xi' \) are functorial in the following sense, where \( h: S \to X \) is a vertical map and \( \zeta \) and \( \theta \) are cells of the following form.

\[
\begin{align*}
X \xrightarrow{K} Y & \quad X \xrightarrow{K} Y \\
A \xrightarrow{\phi_A} A & \quad \phi_A \\
B \xrightarrow{f} B & \quad f
\end{align*}
\]

\[
\begin{align*}
X \xrightarrow{K} Y & \quad X \xrightarrow{K} Y \\
\phi & \quad \phi \\
\psi & \quad \psi
\end{align*}
\]

\[
\begin{align*}
X \xrightarrow{K} Y & \quad X \xrightarrow{K} Y \\
\phi \quad \psi & \quad \phi' \quad \psi'
\end{align*}
\]

\[
\begin{align*}
S \xrightarrow{L} T & \quad S \xrightarrow{L} T \\
h \quad h & \quad h \\
X \xrightarrow{K} Y & \quad X \xrightarrow{K} Y
\end{align*}
\]

\[
\begin{align*}
S \xrightarrow{L} T & \quad S \xrightarrow{L} T \\
h \quad h & \quad h \\
X \xrightarrow{K} Y & \quad X \xrightarrow{K} Y
\end{align*}
\]
- If \((\phi_A, \phi_C) \mapsto \phi'\) then \([\phi \circ h = (\phi_A \circ h, \phi \circ U_h, \phi_C \circ h)] \mapsto \phi' \circ h;\)
- if \((\xi_A, \xi_C) \mapsto \xi'\) then \([[(\xi_A \circ \zeta, \xi_C \circ \zeta) : \phi \circ h \to \psi \circ k] \mapsto \xi' \circ \zeta;\)
- if \((\phi_A, \phi_C) \mapsto \phi'\) then \([U_{\phi_A} \circ \theta, U_{\phi_C} \circ \theta]: \phi \circ h \to \phi \circ l] \mapsto U_{\phi'} \circ \theta;\)
- if \([(\rho_A, \rho_C): \phi \to \psi] \mapsto \rho'\) and \([(\sigma_A, \sigma_C): \psi \to \chi]\) \mapsto \sigma' then
  \[\rho_A \circ \sigma_A, \rho_C \circ \sigma_C): \phi \to \chi] \mapsto \rho' \circ \sigma'.\]

This functoriality will be useful, especially in the proof of the second proposition below.

We first turn to the proof of Proposition \ref{prop:double-comma-objects} which describes double comma objects in the equipment \(\text{Prof}(\mathcal{E})\) of internal profunctors in a category \(\mathcal{E}\) with finite limits. Given an internal profunctor \(J: A \Rightarrow B\) and an internal functor \(f: C \to B\) we have already constructed an \(\mathcal{E}\)-span \(J/f \to (J/f)_0 \times (J/f)_0\), in the discussion preceding Proposition \ref{prop:double-comma-objects} which asserts that this span underlies the internal category that is the double comma object of \(J\) and \(f\). We shall first recall the construction of \((J/f)_0\) and \(J/f\). The former we took to be the wide pullback \((J/f)_0 = A_0 \times_{A_0} J \times_{B_0} C_0\) of the diagram \(A_0 \xrightarrow{id} A_0 \xleftarrow{d_0} J \xrightarrow{d_1} B_0 \xleftarrow{l_0} C_0\), while for the latter we chose the pullback

\[
\begin{array}{ccc}
J/f & \rightarrow & (J/f)_0 \times_{C_0} C \\
\downarrow & & \downarrow r \\
A \times_{A_0} (J/f)_0 & \rightarrow & (J/f)_0,
\end{array}
\]

where the corners are pullbacks of \((J/f)_0 \to C_0 \xleftarrow{d_0} C\) and \(A \xrightarrow{d_1} A_0 \leftarrow J \xrightarrow{l_0} C_0\) respectively, and the maps \(r\) and \(l\) are the compositions

\[
(J/f)_0 \times_{C_0} C \to A_0 \times_{A_0} J \times_{B_0} B \times_{C_0} C_0 \xrightarrow{id \times r \times id} (J/f)_0
\]

and

\[
A \times_{A_0} (J/f)_0 \cong A_0 \times_{A_0} A \times_{A_0} J \times_{B_0} C_0 \xrightarrow{id \times l \times id} (J/f)_0.
\]

Here the unlabelled map is given by applying \(f\) to \(C\) after using \(C_0 \times_{C_0} C \cong C \times_{C_0} C_0\), while the isomorphism is induced by \(A \times_{A_0} A_0 \cong A_0 \times_{A_0} A\) and the maps \(l\) and \(r\) denote the actions of \(A\) and \(B\) on \(J\). The source and target maps \(d = (d_0, d_1): J/f \to (J/f)_0 \times (J/f)_0\) are given by the projections

\[
d_0 = [J/f \to (J/f)_0 \times_{C_0} C \to (J/f)_0]
\]

and

\[
d_1 = [J/f \to A \times_{A_0} (J/f)_0 \to (J/f)_0].
\]

**Proposition \ref{prop:double-comma-objects}** The \(\mathcal{E}\)-span \(d: J/f \to (J/f)_0 \times (J/f)_0\) above can be given the structure of a category internal to \(\mathcal{E}\). Moreover, \(J/f\) is made into the double comma object of \(J\) and \(f\) in \(\text{Prof}(\mathcal{E})\) by the internal functors \(\pi_A: J/f \to A\), given by the projections

\[
(\pi_A)_0: (J/f)_0 \to A_0 \quad \text{and} \quad \pi_A = [J/f \to A \times_{A_0} (J/f)_0 \to A],
\]

and \(\pi_C: J/f \to C\), given similarly, together with the cell

\[
\begin{array}{ccc}
J/f & \rightarrow & J/f \\
\downarrow & & \downarrow \pi_C \\
A & \xrightarrow{f} & B
\end{array}
\]

and the structure of a category internal to \(\mathcal{E}\). Moreover, \(J/f\) is made into the double comma object of \(J\) and \(f\) in \(\text{Prof}(\mathcal{E})\) by the internal functors \(\pi_A: J/f \to A\), given by the projections

\[
(\pi_A)_0: (J/f)_0 \to A_0 \quad \text{and} \quad \pi_A = [J/f \to A \times_{A_0} (J/f)_0 \to A],
\]

and \(\pi_C: J/f \to C\), given similarly, together with the cell

\[
\begin{array}{ccc}
J/f & \rightarrow & J/f \\
\downarrow & & \downarrow \pi_C \\
A & \xrightarrow{f} & B
\end{array}
\]
of internal profunctors that, under the correspondence of Proposition \ref{prop}, is given by the projection \((J/f)_0 \to J\).

Proof. To give the multiplication \(m : J/f \times (J/f)_0, J/f \to J/f\), notice that the source \(W = J/f \times (J/f)_0, J/f\) of \(m\) forms the wide pullback of the diagram

\[
\begin{array}{c}
\begin{array}{ccc}
(J/f)_0 \times C_0 & \to & A \times A_0 (J/f)_0 \\
\downarrow & & \downarrow \nu \times \id \\
(J/f)_0 & \to & (J/f)_0 \\
\end{array}
\end{array}
\]

where the first factor of \(W\) is the pullback of the objects labelled (1) and (2), the second factor is the pullback of those labelled (3) and (4), and where the unlabelled maps are projections. In the case of \(E = \Set\), \(W\) consists of quadruples of pairs of maps, one in each of the sets on the top row above, such that the pairs in (1) and (2), as well as those in (3) and (4), form commutative squares \((2.0)\), and such that the bottom map of the first square coincides with the top map of the second. Let \(m\) be induced by the compositions

\[
\begin{array}{c}
\begin{array}{ccc}
W & \to & A \times A_0 (J/f)_0 \\
\downarrow \nu \times \id & & \downarrow \nu \times \id \\
(J/f)_0 & \to & (J/f)_0 \\
\end{array}
\end{array}
\]

and

\[
\begin{array}{c}
\begin{array}{ccc}
W & \to & (J/f)_0 \times C_0 C \times C_0 C \\
\downarrow \nu \times \id & & \downarrow \nu \times \id \\
(J/f)_0 & \to & (J/f)_0 \\
\end{array}
\end{array}
\]

where the unlabelled maps are induced by projections from \(W\) onto (factors of) the pullbacks in the top row of \((A.1)\), as indicated by the labels. When \(E = \Set\), this simply composes the two pairs of maps making up the sides of two adjacent commutative squares \((2.0)\). That the maps above indeed induce a map into the pullback \(J/f\) follows from

\[
\begin{array}{c}
\begin{array}{ccc}
[W & \to & A \times A_0 (J/f)_0] \\
\downarrow \nu \times \id & & \downarrow \nu \times \id \\
(J/f)_0 & \to & (J/f)_0 \\
\end{array}
\end{array}
\]

Here the first equality follows from the fact that we can replace \(m_A \times \id\) by \(\id \times l\) (by associativity of the action of \(A\) on \(J\)) and that precomposing \(l\) with the projection onto (4) coincides with precomposing \(r\) with the projection onto (3), by definition of \(W\). Likewise the projection of \(W\) onto the factor \((J/f)_0\) of (3) equals the projection onto the same factor of (2) and, together with the fact that \(r\) and \(l\) commute, the second equality follows. Finally the third equality is symmetric to the first.

That the multiplication \(m : J/f \times (J/f)_0, J/f \to J/f\) thus defined is compatible with the source and target maps is clear; that it is associative follows from the associativity of \(m_A\) and \(m_C\). Moreover one readily verifies that the unit \(c : (J/f)_0 \to J/f\) can be taken to be induced by the compositions

\[
\begin{array}{c}
\begin{array}{ccc}
(J/f)_0 \cong (J/f)_0 \times C_0 C & \xrightarrow{\id \times C} & (J/f)_0 \times C_0 C \\
\end{array}
\end{array}
\]

and

\[
\begin{array}{c}
\begin{array}{ccc}
(J/f)_0 \cong A_0 \times A_0 (J/f)_0 & \xrightarrow{\nu \times \id} & A \times A_0 (J/f)_0 \\
\end{array}
\end{array}
\]
which completes the definition of the structure making $J/f$ into a category internal to $\mathcal{E}$. Checking that the projections $\pi_A: J/f \to A$ and $\pi_C: J/f \to C$, as well as the cell $\pi: U_{J/f} \to J$, are compatible with this structure is straightforward, which leaves us to prove that $\pi$ satisfies the universal properties that define $J/f$ as the double comma object.

So let us consider a second internal transformation $\phi$ as on the left below, given by a map $\phi: X_0 \to J$; we have to show that there exists a unique factorisation $\phi': X \to J/f$ such that $\pi_A \circ \phi' = \phi_A$, $\pi_C \circ \phi' = \phi_C$ and $\pi \circ U\phi' = \phi$.

\[
\begin{array}{ccc}
X & \xrightarrow{\phi_A} & Y \\
\downarrow & & \downarrow \\
A & \rightarrow & B
\end{array} \quad \begin{array}{ccc}
X & \xrightarrow{\phi_C} & Y \\
\downarrow & & \downarrow \\
C & \rightarrow & C
\end{array}
\]

Using the fact that, under Proposition 1.22, the composite $\pi \circ U\phi': X \to J$ corresponds to $X_0 \xrightarrow{\phi_0'} (J/f)_0 \xrightarrow{\pi} J$, it is readily checked that these identities are satisfied by taking

\[
\phi_0' = ((\phi_A)_0, \phi, (\phi_C)_0): X_0 \to (J/f)_0
\]
and
\[
\phi' = ((\phi_0' \circ d_0, \phi_C), (\phi_A, \phi_0' \circ d_1)): X \to J/f.
\]  

(A.2)

Here the factors $(\phi_A)_0$ in $\phi_0'$ and $\phi_A$ in $\phi'$ are forced by the identity $\pi_A \circ \phi' = \phi_A$, while the factors $(\phi_C)_0$ and $\phi_C$ are forced by $\pi_C \circ \phi' = \phi_C$. Similarly the factor $\phi$ in $\phi_0'$ is forced by $\pi \circ U\phi' = \phi$, while compatibility with the source and target maps forces the factors $\phi_0' \circ d_0$ in $\phi'$. We conclude that $\phi'$ is uniquely determined.

Finally, to prove that the 2-dimensional universal property holds as well, consider a second transformation $\psi$, as on the right above, and suppose it factors through $\pi$ as $\psi': Y \to J/f$. Given cells

\[
\begin{array}{ccc}
X & \xrightarrow{\xi_A \circ \psi} & Y \\
\downarrow & & \downarrow \\
A & \rightarrow & A
\end{array} \quad \begin{array}{ccc}
X & \xrightarrow{\xi_C \circ \psi} & Y \\
\downarrow & & \downarrow \\
C & \rightarrow & C
\end{array}
\]

with $\xi_A \circ \psi = \phi \circ (U_f \circ \xi_C)$, it is easy to see that the unique cell $\xi': K \to J/f$ such that $U_{\pi_A} \circ \xi' = \xi_A$ and $U_{\pi_C} \circ \xi' = \xi_C$, is given by $\xi' = ((\phi_0' \circ d_0, \xi_C), (\xi_A, \psi_0' \circ d_1))$. \qed

The second proposition whose proof we have postponed is Proposition 4.26. Given a right suitable normal monad $T$ on an equipment $\mathcal{K}$, it concerns the forgetful functor $U_T: T\text{-Prom}_{rc} \to \mathcal{K}$, where $T\text{-Prom}_{rc}$ is the pseudo double category of colax $T$-algebras, colax $T$-morphisms, right colax $T$-promorphisms and $T$-cells, as defined in Section 4.3.

**Proposition 4.26.** Consider a right suitable normal monad $T$ on an equipment $\mathcal{K}$. The forgetful functor $U_T: T\text{-Prom}_{rc} \to \mathcal{K}$ lifts double comma objects $J/f$ where $J: A \Rightarrow B$ is a right pseudopromorphism and $f: C \Rightarrow B$ is a pseudomorphism. The projections of $J/f$ are strict $T$-morphisms, and $J/f$ is a pseudoalgebra whenever both $A$ and $C$ are.

**Proof.** For a right pseudopromorphism $J: A \Rightarrow B$ and a pseudomorphism $f: C \Rightarrow B$
suppose that the double comma object $J/\mathcal{f}$, as on the left below, exists in $\mathcal{K}$.

\[
\begin{array}{ccc}
J/\mathcal{f} & \xrightarrow{\pi} & J/\mathcal{f} \\
\downarrow \pi_A & & \downarrow \pi_C \\
A & \xrightarrow{f} & B
\end{array}
\quad
\begin{array}{ccc}
TA & \xrightarrow{TJ} & TB \\
\downarrow a & & \downarrow b \\
A & \xrightarrow{j} & B
\end{array}
\]

Remember that the right colax algebra structure on $J$ is given by a horizontal cell $\bar{\rho}_1: \bar{J} \to \mathcal{J}$ (Definition 4.13), which, because $J$ is a right pseudomorphism, has an inverse $\bar{J}^{-1}$. Under Proposition 4.7, this inverse corresponds to a cell $\bar{\rho}_1$ above that makes $\mathcal{J}$ into a lax promorphism (Definition 4.1). Likewise the inverse of the vertical cell $\mathcal{f}: \mathcal{f} \circ \mathcal{C} \Rightarrow b \circ T\mathcal{f}$, that makes $\mathcal{f}$ into a colax morphism, gives $\mathcal{f}$ the structure of a lax morphism; we will write $\bar{\mathcal{f}} = \mathcal{f}^{-1}$ as well.

We will show that the algebra structures of $A$ and $C$ induce an algebra structure on $J/\mathcal{f}$, that this makes $\pi_A$ and $\pi_C$ into strict $T$-morphisms and $\pi$ into a $T$-cell, and that $\pi$ satisfies the universal properties making $J/\mathcal{f}$ into a double comma object in $T\text{-}\text{Prom}_{rc}$. As structure map $r: TJ/\mathcal{f} \to J/\mathcal{f}$ we take the factorisation of the composition

\[
\begin{array}{ccc}
TJ/\mathcal{f} & \xrightarrow{T\pi_A} & TJ/\mathcal{f} \\
\downarrow T\pi & & \downarrow T\pi_C \\
TA & \xrightarrow{TJ} & TB \\
\downarrow a & & \downarrow b \\
A & \xrightarrow{j} & B
\end{array}
\quad
\begin{array}{ccc}
T\pi & & T\pi_C \\
\downarrow j & & \downarrow C \\
\pi & & \pi_C \\
\downarrow f & & \downarrow \mathcal{f}
\end{array}
\]

through the cell $\pi$, so that $\pi_A \circ r = a \circ T\pi_A$ and $\pi_C \circ r = c \circ T\pi_C$, while $\pi \circ U_r$ equals the composite above. Note that the first two identities mean that $\pi_A$ and $\pi_C$ are strict morphisms. That $\pi$ is a $T$-cell (Definition 3.21) is shown by the following chain of equalities, where the identity cell right of $\pi$ is that of $\pi_C \circ r = c \circ T\pi_C$ and that left of $T\pi$ is that of $\pi_A \circ r = a \circ T\pi_A$. The equalities follow from the definition of $\rho_1$ (see the discussion preceding Definition 4.13); that $\pi \circ U_r$ equals the composite above; the definition of $\rho_1$ and $\bar{\mathcal{f}} = \mathcal{f}^{-1}$; that $(\bar{\mathcal{J}})^{-1} = \varepsilon_a \circ \bar{J} \circ \eta_b$.

Having chosen a structure map we have to give the associator and unitor of $J/\mathcal{f}$,
that is cells

\[ T^2J/f \xrightarrow{\mu_{J/f}} TJ/f \]
\[ T/J/f \xrightarrow{\nu} \xrightarrow{\rho} r \]
\[ T/J/f \xrightarrow{\text{id}} \xrightarrow{\rho_0} \]
\[ \xrightarrow{\text{r}} J/f, \]

such that \( \rho \) satisfies the associativity axiom, displayed below, and \( \rho_0 \) satisfies the unit axioms (see Definition 3.19).

\[ \begin{align*}
T^2J/f & \xrightarrow{\mu_{J/f}} TJ/f \\
T/J/f & \xrightarrow{\nu} \xrightarrow{\rho} r \\
T/J/f & \xrightarrow{\text{id}} \xrightarrow{\rho_0} \\
\xrightarrow{\text{r}} J/f, \\
\end{align*} \]

Both the associator and unitor are obtained by using the 2-dimensional universal property of \( J/f \), as follows. Notice that the source and target morphisms of the associator correspond to the cells \( \pi \circ U_r \circ U_{\mu_{J/f}} \) and \( \pi \circ U_r \circ U_{\nu} \), which equal the composites of the solid subdiagrams of the respective diagrams below. After adding the dashed cells \( \alpha \) and \( \gamma \) (the associators for \( A \) and \( C \)), we claim that the composites of the full diagrams coincide. Indeed, using the associativity axiom for \( \tilde{f} \) (which can be obtained from that of \( \bar{f} = (\tilde{f})^{-1} \), given in Definition 3.20), we can replace, in the diagram on the left-hand side, the subdiagram consisting of the identity cell for \( \mu_B \circ T^2f = Tf \circ \mu_C \) and the cells \( \tilde{f} \) and \( \gamma \) by the likewise shaped diagram consisting of the cells \( \beta, T\tilde{f} \) and \( \tilde{f} \). Similarly, in the diagram thus obtained we can replace the subdiagram consisting of \( \beta, \mu_J \) and \( \tilde{J} \) by subdiagram consisting of the cells \( \alpha, T\tilde{J} \) and \( \tilde{J} \), now by using the associativity axiom for \( \tilde{J} \) (Definition 4.1). The resulting diagram is that on the right-hand side.

We conclude, using the 2-dimensional universal property of \( J/f \), that there exists a unique cell \( r \circ \mu_{J/f} \Rightarrow r \circ Tr \), which we take \( \rho \) to be, such that \( U_{\pi_A} \circ \rho = \alpha \circ U_{T\pi_A} \) and \( U_{\pi_C} \circ \rho = \gamma \circ U_{T\pi_B} \). Notice that \( \rho \) is invertible if \( \alpha \) and \( \gamma \) are, because the assignment \( (\xi_A, \xi_B) \mapsto \xi' \) induced by the 2-dimensional universal property of \( \pi \) is functorial with respect to horizontal composition, as we have seen in the introduction to this appendix. Thus, once we have defined the algebra structure on \( J/f \), the final assertion of the proposition follows.
To show that \( \rho \) satisfies the associativity axiom (A.3) above, consider first the vertical cells making up its right-hand side: the functoriality of the factorisations through \( \pi \) and the naturality of \( \mu \) mean that

- The composite \( \rho \circ U_{T\mu J/f} \) corresponds to \( \alpha \circ U_{T^2\pi\alpha} \circ T\mu J/f = \alpha \circ U_{T^3\pi\alpha} \circ T\mu J/f \) and \( \gamma \circ U_{T\mu C} \circ T\pi C \);

- the composite \( U_r \circ T\rho \) corresponds to \( U_{o\circ T\pi A} \circ T\rho = U_a \circ T\alpha \circ U_{T^2\pi A} \) and \( U_c \circ T\gamma \circ U_{T^3\pi C} \);

- Denoting by id\( _{\mu J/f} \) the identity cell for \( \mu J/f \circ T\mu J/f = \mu J/f \circ T\mu J/f \), the composite \( U_r \circ \text{id}_{\mu J/f} \) corresponds to \( U_{o\circ T\pi A} \circ \text{id}_{\mu J/f} = U_a \circ \text{id}_{\mu A} \circ U_{T^2\pi A} \) and \( U_c \circ \text{id}_{\mu C} \circ U_{T^3\pi C} \).

That is each of the cells in the right-hand side of (A.3) are factorisations of the corresponding cells in the right-hand side of the associativity axiom for \( A \) and \( C \) and, since factorisation through \( \pi \) is functorial with respect to horizontal composition, we can conclude that the right-hand side of (A.3) corresponds to the right-hand sides of the associativity axioms of \( \alpha \) and \( \gamma \). In the same way the left-hand side of (A.3) corresponds to the left-hand sides of the associativity axioms for \( \alpha \) and \( \gamma \). Therefore, since for \( \alpha \) and \( \gamma \) both the left-hand and right-hand sides coincide, it follows that the left-hand and right-hand sides of (A.3) coincide too. This shows that \( \rho \) satisfies the associativity axiom.

The unitor \( \rho_0: r \circ \eta J/f \Rightarrow \text{id}_{J/f} \) is obtained similarly: its source and target morphisms correspond to the composites of the solid subdiagrams of the respective diagrams below. Adding the dashed unitors \( \gamma_0 \) and \( a_0 \), the full diagrams coincide because of the unit axioms for \( f \) (that can be obtained from that for \( f = (f)^{-1} \) and...
It follows that there is a unique cell $\rho_0: r \circ \eta_{J/f} \Rightarrow \text{id}_{J/f}$ such that $U_{\pi_A} \circ \rho_0 = \alpha_0 \circ U_{\pi_A}$ and $U_{\pi_C} \circ \rho_0 = \gamma_0 \circ U_{\pi_C}$. That $\rho_0$ satisfies the unit axioms (Definition 3.19) follows from the fact that $\alpha_0$ and $\gamma_0$ do, in the same way that the associativity axiom for $\rho$ followed from those for $\alpha$ and $\gamma$. This completes the definition of the colax algebra structure on $J/f$.

We have already seen that the induced algebra structure on $J/f$ makes $\pi_A$ and $\pi_C$ into strict morphisms, and $\pi$ into a $T$-cell: this leaves showing that $\pi$ satisfies the universal properties making $J/f$ into a double comma object of $T$-$\text{Prom}_{\text{rc}}$. Thus consider a second $T$-cell $\phi$ below. Since $J/f$ is a double comma object in $K$, $\phi$ factors through $\pi$ as a morphism $\phi': X \to J/f$, which we have to give a structure cell $\bar{\phi}'$ as on the right.

Again we can follow the same recipe: the source and target morphisms of $\bar{\phi}'$ correspond to the solid subdiagrams in

and it is shown below that the full diagrams coincide, where the identities follow from the definition of $\rho_1 \bar{\phi}_A$ (see Definition 4.19); the $T$-cell axiom for $\phi$ and the definition of $J_\bar{f}$; cancelling $J_\bar{f}$ against its inverse as well as the definitions of $\rho_1 \bar{\phi}_C$ and $\rho_1 \bar{f}$; the fact that $\bar{f} = \bar{f}^{-1}$. Using the 2-dimensional universal property of $J/f$, we obtain a unique cell $\bar{\phi}'$, as above, such that $U_{\pi_A} \circ \bar{\phi}' = \bar{\phi}_A$ and $U_{\pi_C} \circ \bar{\phi}' = \bar{\phi}_C$. That $\bar{\phi}'$ satisfies the associativity and unit axioms follows from the fact that $\bar{\phi}_A$ and $\bar{\phi}_C$ do. We conclude that there is a unique way of making $\phi'$ into a $T$-cell such that...
$U_{\pi_A} \circ \phi' = \phi_A$ and $U_{\pi_C} \circ \phi' = \phi_C$, as colax $T$-morphisms. This completes the proof showing that $\pi$ satisfies the 1-dimensional universal property in $T$-$\text{Prom}_{rc}$.

Finally let $\psi = (\psi_A, \psi, \psi_C)$ be a second $T$-cell, with 0-dimensional source $Y$, that factors through $\pi$ as the colax $T$-morphism $\psi': Y \to J/f$. Consider cells $(\xi_A, \xi_C): \phi \to \psi$, in $K$, as in the introduction to this appendix, so that there exists a unique cell $\xi'$ with $U_{\pi_A} \circ \xi' = \xi_A$ and $U_{\pi_C} \circ \xi' = \xi_C$, because $J/f$ is a double comma object in $K$. To prove that $J/f$ satisfies the 2-dimensional universal property in $T$-$\text{Prom}_{rc}$ as well, we have to show that if $\xi_A$ and $\xi_B$ are $T$-cells then so is $\xi'$. To see this consider the following identity, which is equivalent to the $T$-cell axiom (Definition 4.19) for $\xi$ and is obtained by vertically postcomposing the latter with the companion cell $\varepsilon_r$.

$$
\begin{array}{c}
\begin{array}{c}
X \xrightarrow{\phi'} \phi' \\
J/f \xrightarrow{\phi'} J/f
\end{array} \\
\begin{array}{c}
X \xrightarrow{\phi} \phi \\
J/f \xrightarrow{\phi} J/f
\end{array}
\end{array}
= 
\begin{array}{c}
\begin{array}{c}
X \xrightarrow{\varepsilon_r} X \\
J/f \xrightarrow{\varepsilon_r} J/f
\end{array} \\
\begin{array}{c}
X \xrightarrow{\varepsilon_r} X \\
J/f \xrightarrow{\varepsilon_r} J/f
\end{array}
\end{array}
= 
\begin{array}{c}
\begin{array}{c}
\phi \\
\phi
\end{array} \\
\phi \\
\phi
\end{array}
\]

To see that the identity above holds, we again use the functoriality of the 2-dimensional factorisations through $\pi$ and find that, by postcomposing the composites above with $U_{\pi_A}$, we obtain the two sides of the corresponding $T$-cell axiom for $\xi_A$, postcomposed with $\varepsilon_a$. Similarly, when postcomposed with $U_{\pi_C}$, we obtain the two sides of the corresponding $T$-cell axiom for $\xi_C$, postcomposed with $\varepsilon_c$. Therefore, since the $T$-cell axioms for $\xi_A$ and $\xi_C$ hold, and since factorisations through $\pi$ are unique, it follows that the identity above, and thus the $T$-cell axiom for $\xi'$, hold as well. This shows that $J/f$ satisfies the 2-dimensional property as well, in $T$-$\text{Prom}_{rc}$, which finishes the proof.
Appendix B

Proof of Lemma 4.34

In this appendix we prove Lemma 4.34 reproduced below, which was used in proving the main theorem.

Lemma 4.34. Let $T$ be a right suitable normal monad on a closed equipment $K$ (Definition 4.8) and consider a right colax $T$-promorphism $J: A \Rightarrow B$ (Definition 4.13) as well as a lax $T$-promorphism $K: A \Rightarrow C$ (Definition 4.1). The left hom $J \triangleleft K: B \Rightarrow C$ admits a lax $T$-promorphism structure cell $\lambda J \triangleleft K$ that corresponds, under Proposition 1.24, to the horizontal cell $\lambda J \triangleleft K$ that is given by

$$T(J \triangleleft K) \circ C(a, id) \Rightarrow T J \triangleleft (A(a, id) \circ K) \cong (A(id, a) \circ T J) \triangleleft K \frac{J \circ id}{\Rightarrow} (J \circ B(id, b)) \triangleleft K \cong B(b, id) \circ J \triangleleft K,$$  \hspace{1cm} (B.1)

where the first cell is given by Proposition 3.5 and where the two isomorphisms are given by Corollary 2.8.

In the proof we use the facts recorded in the following lemma. Each of them follows easily from the naturality of the isomorphisms defining $J \triangleleft -$ as the right adjoint of $J \circ -$ and the naturality of the associators $F \circ$ of lax functors.

Lemma B.1. In each of the statements below $J$, $H$, $K$, $L$ and $M$ are horizontal morphisms in a closed equipment $K$, while $\phi$ and $\psi$ are horizontal cells. In the last statement $F: K \rightarrow L$ is assumed to be a lax functor between closed equipments.

(a) If $L \Rightarrow K \triangleleft M$ is adjoint to $\phi: K \circ L \Rightarrow M$ and $H \circ K \triangleleft M \Rightarrow J \triangleleft M$ is adjoint to $J \circ H \circ K \triangleleft M \cong K \circ M \cong M$, then the composition $H \circ L \Rightarrow H \circ K \triangleleft M \Rightarrow J \triangleleft M$ is adjoint to $J \circ H \circ L \cong K \circ L \cong M$.

(b) If $K \Rightarrow H \triangleleft M$ is adjoint to $\phi: H \circ K \Rightarrow L$ and $H \triangleleft L \Rightarrow (J \circ H) \triangleleft M$ is adjoint to $J \circ H \circ H \triangleleft L \cong (J \circ L) \cong M$, then the composition $K \Rightarrow (J \circ H) \triangleleft M$ is adjoint to $J \circ H \circ K \cong J \circ L \cong M$.

(c) If $H \Rightarrow J \triangleleft K$ is adjoint to $\phi: J \circ H \Rightarrow K$ and $F(J \triangleleft K) \circ L \Rightarrow F(J \triangleleft (FK \circ L))$ is the horizontal cell given by Proposition 3.5, then the composition $FH \circ L \Rightarrow F(J \triangleleft K) \circ L \Rightarrow F(J \triangleleft (FK \circ L))$ is adjoint to $FJ \circ FH \circ L \Rightarrow F(J \circ H) \circ L \Rightarrow FK \circ L$. 

$\hspace{1cm}$

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Proof of Lemma 4.34. Recall that we denote the first four cells of (3.1) by $\omega(J, K)$, so that the lemma states that the left hom $J \triangleleft K: B \Rightarrow C$ can be given the structure of a lax promorphism whose structure cell $\overline{J \triangleleft K}$ corresponds, under Proposition 1.24, to the composite

$$\lambda J \triangleleft K = [T(J \triangleleft K) \circ C(c, id)] \overset{\omega(J, K)}{\Rightarrow} (J \circ B(id, b)) \triangleleft K \cong B(b, id) \circ J \triangleleft K,$$

where the isomorphism, from right to left, is adjoint to the counit $\varepsilon_{b, b}$ of $B(id, b)$, followed by evaluation (Corollary 2.3). Applying Lemma 3.11(b) to $\omega(J, K)$ we find that its adjoint is the composite

$$J \circ B(id, b) \circ T(J \triangleleft K) \circ C(c, id) \overset{id \circ T \circ C(c, id)}{\Rightarrow} A(id, a) \circ T(J \circ J \triangleleft K) \circ C(c, id) \overset{id \circ \varepsilon_c \circ id}{\Rightarrow} A(id, a) \circ TK \circ C(c, id) \overset{id \circ \lambda \ast}{\Rightarrow} A(id, a) \circ A(a, id) \circ K \triangleleft K,$$  \hspace{1cm} (B.2)

We will show that $J \triangleleft K$ satisfies the associativity and unit axioms of Definition 4.1 by proving that the corresponding axioms for $\lambda J \triangleleft K$ hold, as given in Proposition 1.7.

We start with the associativity axiom for $\lambda J \triangleleft K$, whose right-hand side is the following composite

$$T^2(J \triangleleft K) \circ TC(c, id) \circ C(c, id) \overset{T_{C \circ id}}{\Rightarrow} T(T(J \triangleleft K) \circ C(c, id)) \circ C(c, id) \overset{\omega(J, K) \circ id}{\Rightarrow} T \left( (J \circ B(id, b)) \triangleleft K \circ C(c, id) \cong T \left( B(b, id) \circ J \triangleleft K \circ C(c, id) \right) \right) \overset{id \circ \omega(J, K)}{\Rightarrow} TB(b, id) \circ (J \circ B(id, b)) \triangleleft K \overset{\lambda \circ \varepsilon_{id}}{\Rightarrow} TB(id, B, id) \circ B(b, id) \circ J \triangleleft K,$$  \hspace{1cm} (B.3)

where the inverse of the $T \circ -$-component exists by Proposition 3.3. In this chain of horizontal cells we consider the subchain $T \left( (J \circ B(id, b)) \triangleleft K \right) \circ C(c, id) \Rightarrow TB(id, B, id) \circ B(b, id) \circ J \triangleleft K$ of the last five cells, which also occurs as the top leg of the diagram below.

Here the single unlabelled cell is given by Proposition 3.5 and each of the isomorphisms is given by Corollary 2.3 using the counts for the several companion-conjoint adjunctions. For example, the top isomorphism of the subdiagram labelled
III is adjoint to the counit $b_⊙ Tb C_⊙ Tb$, for the adjunction $T B(b, id) \odot B(b, id) \cong B(id, b) \odot T B(id, b)$ of the companion and conjoint of $b_⊙ Tb$, followed by evaluation (remember our convention, preceeding Proposition 1.17 concerning the companions and conjoints of composites like $b_⊙ Tb$).

We will show that the diagram above commutes. First consider the bottom leg of the subdiagram labelled I, which consists of $id_⊙ J ⊳ K$ by first applying part (b) of Lemma B.1 to its second isomorphism and then part (c) to the unlabelled cell. Thus it is adjoint to

$$J \odot B(id, b) \odot T B(id, b) \odot T B(b, id) \odot T(J \circ K) \circ C(c, id)$$

That the two adjoints above coincide follows from the associativity axiom for $T$ and from $T(\varepsilon_b) \odot T_0 = T_ε_b \odot T_b = \varepsilon_0 \odot T_ε_b = T_b \varepsilon_b = T_ε_b$; we conclude that the subdiagram I commutes. Similarly we can use Lemma B.1(a) to compute the adjoint of the top leg of the subdiagram labelled II, obtaining the composite

$$J \odot B(id, b) \odot T B(id, b) \odot T B(b, id) \odot B(b, id) \odot J \circ K$$

The composite of the first two cells here coincides with $id_⊙ J ⊳ K \odot id$, so that the full composite above coincides with the adjoint of the bottom cell of II: the subdiagram II commutes too. Finally the top leg of the subdiagram labelled III is adjoint to

$$J \odot B(id, b) \odot T B(id, B(b, id) \odot B(b, id) \odot J \circ K$$

It follows from the definitions of $\rho β$ and $λ β$ (Proposition 1.24), as well as the vertical companion and conjoint identities, that $b_⊙ Tb C_⊙ Tb \circ (\rho β \odot id) = b_⊙ Tb C_⊙ Tb \circ (id \odot λ β)$,
so that the composite above equals the adjoint of the bottom leg of the subdiagram III, which therefore commutes as well.

Replacing in (B.3) the top leg of the diagram above with the bottom one, as well as expanding \( T(\omega) \), we obtain the composite

\[
T^2(J \triangleleft K) \odot TC(c, \text{id}) \odot C(c, \text{id}) \xRightarrow{T \odot \text{id}} T(T(J \triangleleft K) \odot C(c, \text{id})) \odot C(c, \text{id})
\]

\[\Rightarrow T\left(T(J \triangleleft (TK \odot C(c, \text{id}))) \odot C(c, \text{id})\right) \xRightarrow{T(id \odot \lambda K) \odot \text{id}} T\left(TJ \odot (A(a, \text{id}) \odot K)\right) \odot C(c, \text{id})\]

\[\cong T\left((A(id, a) \odot TJ) \triangleleft K\right) \odot C(c, \text{id}) \xRightarrow{TJ(id) \odot \text{id}} T\left((J \odot B(id, b)) \triangleleft K\right) \odot C(c, \text{id})\]

\[\Rightarrow T(J \odot B(id, b) \triangleleft (TK \odot C(c, \text{id}))) \xRightarrow{id \odot \lambda K} T(J \odot B(id, b) \triangleleft (A(id, a) \odot K)\right) \cong (A(id, a) \odot T(J \odot TB(id, b)) \triangleleft K\right) \]

\[\xRightarrow{(J \odot id) \odot \text{id}} (J \odot B(id, b) \odot TB(id, b)) \triangleleft K\]

\[\xRightarrow{(id \odot \rho) \odot \text{id}} (J \odot B(id, b) \odot TB(id, \mu B)) \triangleleft K\]

\[\cong TB(\mu B, \text{id}) \odot B(b, \text{id}) \odot J \triangleleft K.\]  \hspace{1cm} \text{(B.4)}

Our aim is to move \( T(J \triangleleft \text{id}) \odot \text{id} \) past \( \lambda K \), so that the rear of the composite above consists of the left hom of the right-hand side of the associativity axiom for \( J \) (Definition 4.13) and the identity on \( K \). To do this, consider the subchain \( T\left((A(id, a) \odot TJ) \triangleleft K\right) \odot C(c, \text{id}) \Rightarrow (A(id, a) \odot T(J \odot TB(id, b)) \triangleleft K \right) \) in the chain above, that starts with \( T(J \triangleleft \text{id}) \odot \text{id} \) and ends with the second isomorphism. Applying part (b) of Lemma B.1 to the latter, followed by applying part (c) to the unlabelled cell, we find that this subchain is adjoint to the composite

\[
A(id, a) \odot TJ \odot TB(id, b) \odot T\left((A(id, a) \odot TJ) \triangleleft K\right) \odot C(c, \text{id}) \]

\[\xRightarrow{id \odot T(id) \odot \text{id}} A(id, a) \odot T\left(J \odot B(id, b) \odot (A(id, a) \odot TJ) \triangleleft K\right) \odot C(c, \text{id})\]

\[\xRightarrow{id \odot T(id) \odot \text{id}} A(id, a) \odot T\left(A(id, a) \odot TJ \odot (A(id, a) \odot TJ) \triangleleft K\right) \odot C(c, \text{id})\]

\[\xRightarrow{id \odot T(id) \odot \text{id}} A(id, a) \odot TK \odot C(c, \text{id}) \xRightarrow{id \odot \lambda K} A(id, a) \odot A(id, a) \odot K \cong \text{id}\]

Using the associativity and naturality of \( T \odot \) it is straightforward to show that the composite above is also the adjoint of the composite below, which therefore coincides with the subchain of (B.4) that we are considering.

\[
T\left((A(id, a) \odot TJ) \triangleleft K\right) \odot C(c, \text{id}) \Rightarrow T(A(id, a) \odot TJ) \triangleleft (TK \odot C(c, \text{id}))) \]

\[
\xRightarrow{id \odot \lambda K} T(A(id, a) \odot TJ) \triangleleft (A(id, a) \odot K) \cong (A(id, a) \odot T(A(id, a) \odot TJ)) \triangleleft K\]

\[
\xRightarrow{(id \odot TJ) \odot \text{id}} \left(A(id, a) \odot T(J \odot B(id, b))\right) \triangleleft K\]

\[
\xRightarrow{(id \odot T(id)) \odot \text{id}} \left(A(id, a) \odot TJ \odot TB(id, b)\right) \triangleleft K\]

As was our aim, substituting the above into (B.4) we obtain a chain whose rear forms the left hom of the right-hand side of the associativity axiom for \( J \). Replacing the
latter with the left hom of the corresponding left-hand side we obtain

\[ T^2(J \triangleleft K) \circ TC(c, \text{id}) \circ C(c, \text{id}) \xrightarrow{T \circ \text{id}} T(T(J \triangleleft K) \circ C(c, \text{id})) \circ C(c, \text{id}) \]
\[ \Rightarrow T\left( T(J \triangleleft (TK \circ C(c, \text{id}))) \right) \circ C(c, \text{id}) \]
\[ \xrightarrow{T((\text{id} \circ \lambda K) \circ \text{id})} T\left( T(J \triangleleft (A(a, \text{id}) \circ K) \right) \circ C(c, \text{id}) \]
\[ \cong T\left( (A(id, a) \circ T(J) \triangleleft K) \circ C(c, \text{id}) \right) \Rightarrow T\left( A(id, a) \circ T(J) \triangleleft (TK \circ C(c, \text{id})) \right) \]
\[ \xrightarrow{\text{id} \circ \lambda K} T(A(id, a) \circ T(J)) \triangleleft (A(id, a) \circ K) \cong \left( A(id, a) \circ T\left( A(id, a) \circ T(J) \right) \right) \triangleleft K \]
\[ \xrightarrow{(\text{id} \circ \text{id}) \circ \text{id}} \left( A(id, a) \circ TA(id, a) \circ T^2(J) \right) \triangleleft K \]
\[ \xrightarrow{(\rho A \circ \text{id}) \circ \text{id}} \left( A(id, a) \circ TA(id, \mu A) \circ T^2(J) \right) \triangleleft K \]
\[ \xrightarrow{\text{id} \circ (\mu A)^{-1} \circ \text{id}} \left( A(id, a) \circ T(J) \circ TB(id, \mu B) \right) \triangleleft K \]
\[ \cong TB(id, id) \circ B(b, id) \circ J \triangleleft K. \]
\[ (B.5) \]

We claim that the subchain consisting of the first eight cells here, with target \( A(id, a) \circ TA(id, \mu A) \circ T^2(J) \triangleleft K \), equals the composite

\[ T^2(J \triangleleft K) \circ TC(c, \text{id}) \circ C(c, \text{id}) \Rightarrow T^2(J \triangleleft (T^2K \circ TC(c, \text{id}) \circ C(c, \text{id})) \]
\[ \xrightarrow{\text{id} \circ (\lambda TK \circ \text{id})} T^2(J \circ (TA(id, \text{id}) \circ TK \circ C(c, \text{id})) \]
\[ \xrightarrow{\text{id} \circ (\text{id} \circ \lambda K)} T^2(J \triangleleft (TA(id, \text{id}) \circ A(id, \text{id}) \circ K) \]
\[ \xrightarrow{\text{id} \circ (\lambda \circ \text{id})} T^2(J \triangleleft (TA(\mu A, id) \circ A(id, \text{id}) \circ K) \]
\[ \cong \left( A(id, a) \circ TA(id, \mu A) \circ T^2(J) \right) \triangleleft K, \]
\[ (B.6) \]

where the first cell is given by Proposition \[ \ref{prop:prop} \] now for \( F = T^2 \). Notice that the composite above contains the right-hand side of the associativity axiom for \( \lambda K \) (Proposition \[ \ref{prop:lambdaK} \]). To see that the claim holds we again compute adjoints: that of the composite above, using Lemma \[ \ref{lem:adj} \] (b), equals

\[ A(id, a) \circ TA(id, \mu A) \circ T^2J \circ T(J \triangleleft K) \circ TC(c, \text{id}) \circ C(c, \text{id}) \]
\[ \xrightarrow{\text{id} \circ T^2 \circ \text{id}} A(id, a) \circ TA(id, \mu A) \circ T^2(J \circ J \triangleleft K) \circ TC(c, \text{id}) \]
\[ \xrightarrow{\text{id} \circ T^2 \circ \text{ev} \circ \text{id}} A(id, a) \circ TA(id, \mu A) \circ T^2K \circ TC(c, \text{id}) \circ C(c, \text{id}) \]
\[ \xrightarrow{\text{id} \circ \lambda \circ \text{id}} A(id, a) \circ TA(id, \mu A) \circ TA(a, id) \circ TK \circ C(c, \text{id}) \]
\[ \xrightarrow{\text{id} \circ \lambda A \circ \text{id}} A(id, a) \circ TA(id, \mu A) \circ TA(id, a) \circ A(id, \text{id}) \circ K \]
\[ \xrightarrow{\text{id} \circ \lambda \circ A \circ \text{id}} A(id, a) \circ TA(id, \mu A) \circ TA(id, \mu A, id) \circ A(id, \text{id}) \circ K \xrightarrow{\text{id} \circ \lambda \circ A \circ \text{id}} K \]

As we have seen before, in showing that the subdiagram III commutes, we have \( a_{\mu A} \circ (\text{id} \circ \lambda a) = a_{\lambda A} \circ (\rho A \circ \text{id}) \). After substituting this into the composite above, we move \( \rho A \) all the way to the front, and rewrite \( a_{\lambda A} \circ (\text{id} \circ \lambda \circ \text{id}) \circ (\text{id} \circ \lambda \circ \text{id}) \). Following this we also rewrite

\[ (T_a \circ a_{\epsilon_a} \circ (T_a \circ \lambda K)) \circ (T_a \circ \lambda K) \]
\[ = T(a_{\epsilon_a} \circ (T_a \circ \lambda K)) \circ (T_a \circ \lambda K) \circ TA(id, a) \circ T^2K \circ TC(c, \text{id}) \Rightarrow TK, \]
where we have used that $\lambda T K = T^{-1}_\circ \circ T \lambda K \circ T_\circ$, see Proposition [BL.7] so that the 
adjoint above is equal to

$$A(id, a) \circ TA(id, \mu A) \circ T^2 J \circ T^2(J \triangleleft K) \circ TC(c, id) \circ C(c, id)$$

$$\rho J \circ T^2 \circ id \rightarrow A(id, a) \circ TA(id, a) \circ T^2 J \circ T(T(J \triangleleft K) \circ C(c, id)) \circ C(c, id)$$

$$id \circ T^2 \circ ev \circ id \rightarrow A(id, a) \circ TA(id, a) \circ T^2 K \circ TC(c, id) \circ C(c, id)$$

$$id \circ T \circ id \circ id \rightarrow A(id, a) \circ T(A(id, a) \circ TK \circ C(c, id)) \circ C(c, id)$$

$$id \circ T(a \cdot \lambda K) \circ id \rightarrow A(id, a) \circ T(A(id, a) \circ A(a, id) \circ K) \circ C(c, id)$$

$$id \circ \lambda K \rightarrow A(id, a) \circ A(a, id) \circ K \xrightarrow{id \circ \lambda} K.$$

On the other hand, computing the adjoint of the first eight cells of (B.5), using 
parts (b) and (c) of Lemma [BL.1] we find

$$A(id, a) \circ TA(id, \mu A) \circ T^2 J \circ T^2(J \triangleleft K) \circ TC(c, id) \circ C(c, id)$$

$$\rho J \circ T^2 \circ id \rightarrow A(id, a) \circ TA(id, a) \circ T^2 J \circ T(T(J \triangleleft K) \circ C(c, id)) \circ C(c, id)$$

$$id \circ T^2 \circ ev \circ id \rightarrow A(id, a) \circ TA(id, a) \circ T^2 K \circ TC(c, id) \circ C(c, id)$$

$$id \circ T \circ id \circ id \rightarrow A(id, a) \circ T(A(id, a) \circ TK \circ C(c, id)) \circ C(c, id)$$

$$id \circ T(a \cdot \lambda K) \circ id \rightarrow A(id, a) \circ T(A(id, a) \circ A(a, id) \circ K) \circ C(c, id)$$

$$id \circ \lambda K \rightarrow A(id, a) \circ A(a, id) \circ K \xrightarrow{id \circ \lambda} K.$$

Using the associativity and naturality of $T_\circ$, it is easily seen that the two composites
above coincide, which proves the claim.

Next we replace in (B.6) the right-hand side of the associativity axiom for $\lambda K$
by the corresponding left-hand side (see Proposition [BL.7]), and then substitute it for
the first eight cells in (B.5). This gives the composite

$$T^2(J \triangleleft K) \circ TC(c, id) \circ C(c, id) \Rightarrow T^2 J \triangleleft (T^2 K \circ TC(c, id) \circ C(c, id))$$

$$\xrightarrow{id \circ (id \circ \lambda J)} T^2 J \triangleleft (T^2 K \circ TC(\mu C, id) \circ C(c, id))$$

$$\xrightarrow{id \circ (\lambda J \circ id)} T^2 J \triangleleft (TA(\mu A, id) \circ TK \circ C(c, id))$$

$$\xrightarrow{id \circ (id \circ \lambda K)} T^2 J \triangleleft (TA(id, \mu A) \circ A(a, id) \circ K)$$

$$\cong (A(id, a) \circ TA(id, \mu A) \circ T^2 J) \triangleleft K$$

$$\xrightarrow{(id \circ (\rho J) \circ id)} (A(id, a) \circ T J \circ TB(id, \mu B)) \triangleleft K$$

$$\xrightarrow{(J \circ id) \circ id} (J \circ B(id, b) \circ TB(id, \mu B)) \triangleleft K$$

$$\cong TB(\mu B, id) \circ B(b, id) \circ J \triangleleft K. \quad (B.7)$$

Remember that the composite above equals the right-hand side of the associativity
axiom for $\lambda J \triangleleft K$ (Proposition [BL.7]). On the other hand, the left-hand side is the
composite

$$T^2(J \triangleleft K) \circ TC(c, id) \circ C(c, id) \xrightarrow{id \circ \lambda_{\mu \varepsilon_{\mu, J \triangleleft K}}} T^2(J \triangleleft K) \circ TC(\mu_c, id) \circ C(c, id)$$

$$\xrightarrow{\lambda_{\mu, J \triangleleft K} \circ id} TB(\mu_B, id) \circ T(J \triangleleft K) \circ C(c, id)$$

$$\xrightarrow{id \circ \omega(B, \mu_B)} TB(\mu_B, id) \circ (J \circ B(id, b)) \triangleleft K$$

$$\cong (J \circ B(id, b) \circ T B(id, \mu_B)) \triangleleft K$$

$$\cong TB(\mu_B, id) \circ B(b, id) \circ J \triangleleft K,$$  \hspace{1cm} (B.8)

where we have replaced the single isomorphism

$$TB(\mu_B, id) \circ (J \circ B(id, b)) \triangleleft K \cong TB(\mu_B, id) \circ B(b, id) \circ J \triangleleft K$$

by the pair of isomorphisms above (proving that these are equal is analogous to proving that the subdiagram II above commutes). Notice that both sides above have their final isomorphism in common; we will show that they coincide by proving that, after taking these isomorphisms away, the adjoints of the remaining subchains coincide. Computing the adjoint of the first four cells of the composite above, we obtain

$$J \circ B(id, b) \circ T B(id, \mu_B) \circ T^2(J \triangleleft K) \circ TC(c, id) \circ C(c, id)$$

$$\xrightarrow{J \circ id \circ \lambda_{\mu \varepsilon_{\mu, J \triangleleft K}}} A(id, a) \circ TJ \circ TB(id, \mu_B) \circ T^2(J \triangleleft K) \circ TC(\mu_c, id) \circ C(c, id)$$

$$\xrightarrow{id \circ \lambda_{\mu \varepsilon_{\mu, J \triangleleft K}} \circ id} A(id, a) \circ TJ \circ TB(id, \mu_B) \circ TB(\mu_B, id) \circ T(J \triangleleft K) \circ C(c, id)$$

$$\xrightarrow{id \circ \omega(B, \mu_B) \circ id} A(id, a) \circ TJ \circ T(J \triangleleft K) \circ C(c, id)$$

$$\xrightarrow{id \circ \epsilon_K \circ id} A(id, a) \circ TK \circ C(c, id)$$

$$\xrightarrow{id \circ \lambda_K} A(id, a) \circ A(id, a) \circ K \xrightarrow{id \circ \lambda_{\mu \varepsilon_{\mu, J \triangleleft K}}} K,$$  \hspace{1cm} (B.9)

where we have used Lemma [3.1(a)] and the fact that $\omega(J, K)$ is adjoint to $[B.2]$. As before we have $(\mu_{\mu \varepsilon_{\mu, J \triangleleft K}} \circ id) \circ (id \circ \lambda_{\mu \varepsilon_{\mu, J \triangleleft K}}) = (id \circ \lambda_{\mu \varepsilon_{\mu, J \triangleleft K}}) \circ (\mu_{\mu J \triangleleft K} \circ id)$, where $\mu_{\mu J \triangleleft K}$ can be rewritten as follows. The composition axiom (Definition [3.4]) and the naturality of $\mu$ imply that

$$[T^2 J \circ T^2(J \triangleleft K) \xrightarrow{\mu_J \circ \mu_{J \triangleleft K}} T J \circ T(J \triangleleft K) \xrightarrow{T ev} T(J \circ J \triangleleft K) \xrightarrow{T ev} TK]$$

$$= [T^2 J \circ T^2(J \triangleleft K) \xrightarrow{T ev} T^2(J \circ J \triangleleft K) \xrightarrow{\mu_J \circ \mu_{J \triangleleft K}} T(J \circ J \triangleleft K) \xrightarrow{T ev} TK]$$

$$= [T^2 J \circ T^2(J \triangleleft K) \xrightarrow{T ev} T^2(J \circ J \triangleleft K) \xrightarrow{T ev} T^2 K \xrightarrow{\lambda_K} TK]$$  \hspace{1cm} (B.10)

which, after applying $\rho$ and precomposing with $(\rho_{\mu_J})^{-1} \circ id$, gives the identity

$$(T ev \circ id) \circ (T ev \circ id) \circ (id \circ \rho_{\mu J \triangleleft K}) = \rho_{\mu_K} \circ (id \circ T ev) \circ (id \circ T^2 ev) \circ ((\rho_{\mu_J})^{-1} \circ id)$$

of horizontal cells $T J \circ T B(id, \mu_B) \circ T^2(J \triangleleft K) \Rightarrow TK \circ TC(\mu_c, id)$, where we have also used the functoriality of $\rho$, see Proposition [1.24]. Substituting this into
That the two adjoints above coincide follows from a and (associativity axiom for isomorphism, equals On the other hand, the adjoint of the right-hand side (B.7), after removing the final isomorphism, equals

\[
J \circ B(id, b) \circ TB(id, \mu_B) \circ T^2(J \triangleleft K) \circ TC(c, id) \circ C(c, id)
\]

\[
J \circ B(id, b) \circ TB(id, \mu_B) \circ T^2(J \triangleleft K) \circ TC(\mu_C, id) \circ C(c, id)
\]

\[
A(id, a) \circ TA(id, \mu_A) \circ T(J \circ J \triangleleft K) \circ TC(\mu_C, id) \circ C(c, id)
\]

\[
A(id, a) \circ TA(id, \mu_A) \circ T^2(J \circ J \triangleleft K) \circ TC(\mu_C, id) \circ C(c, id)
\]

\[
A(id, a) \circ TA(id, \mu_A) \circ T^2K \circ TC(\mu_C, id) \circ C(c, id)
\]

\[
A(id, a) \circ TK \circ TC(id, \mu_C) \circ TC(\mu_C, id) \circ C(c, id)
\]

\[
A(id, a) \circ TK \circ C(c, id)
\]

\[
A(id, a) \circ A(a, id) \circ K \xrightarrow{\alpha_a \circ id} K.
\]

On the other hand, the adjoint of the right-hand side (B.7), after removing the final isomorphism, equals

\[
J \circ B(id, b) \circ TB(id, \mu_B) \circ T^2(J \triangleleft K) \circ TC(c, id) \circ C(c, id)
\]

\[
A(id, a) \circ TA(id, \mu_A) \circ T^2J \circ T^2(J \triangleleft K) \circ TC(\mu_C, id) \circ C(c, id)
\]

\[
A(id, a) \circ TA(id, \mu_A) \circ T^2(J \circ J \triangleleft K) \circ TC(\mu_C, id) \circ C(c, id)
\]

\[
A(id, a) \circ TA(id, \mu_A) \circ T^2K \circ TC(\mu_C, id) \circ C(c, id)
\]

\[
A(id, a) \circ TA(id, \mu_A) \circ TA(\mu_A, id) \circ TK \circ C(c, id)
\]

\[
A(id, a) \circ TA(id, \mu_A) \circ A(a, id) \circ K \xrightarrow{\alpha_a \circ id} K.
\]

That the two adjoints above coincide follows from \(\alpha_{id} \circ \mu_A = \alpha_a \circ (id \circ \mu_A \circ \mu_A \circ id)\) and \((\mu_A \circ \mu_C \circ id) \circ (id \circ \mu_K) = (id \circ \mu_C \circ \mu_A) \circ (\mu_K \circ id)\). It follows that composites (B.7) and (B.8), which equal respectively the right-hand and left-hand side of the associativity axiom for \(\lambda J \triangleleft K\), coincide, and we conclude that the lax structure cell \(J \triangleleft K\), as it was defined in (B.1), satisfies the associativity axiom.

The unit axiom for \(\lambda J \triangleleft K\), given in Proposition (H.7) is proved similarly, as follows. Its left-hand side is the composite

\[
J \triangleleft K \circ TC(\eta_C, id) \circ C(c, id)
\]

\[
J \triangleleft K \circ TC(\eta_C, id) \circ C(c, id)
\]

\[
TB(\eta_B, id) \circ T(J \triangleleft K) \circ C(c, id) \Rightarrow TB(\eta_B, id) \circ TJ \circ (TK \circ C(c, id))
\]

\[
TB(\eta_B, id) \circ T(J \triangleleft K) \circ (A(a, id) \circ K) \cong TB(\eta_B, id) \circ (A(id, a) \circ TJ) \circ K
\]

\[
TB(\eta_B, id) \circ (J \circ B(id, b)) \triangleleft K \cong (J \circ B(id, b) \circ TB(id, \eta_B)) \triangleleft K
\]

\[
TB(\eta_B, id) \circ (J \circ B(id, b)) \triangleleft K \cong TB(\eta_B, id) \circ B(b, id) \circ J \triangleleft K
\]

where, like in (B.8), we have replaced the single isomorphism

\[
TB(\eta_B, id) \circ (J \circ B(id, b)) \triangleleft K \cong TB(\eta_B, id) \circ B(b, id) \circ J \triangleleft K
\]

by the pair of isomorphisms above. Using parts (a) and (b) of Lemma (B.4) we find
that the adjoint of the composite above, after removing the last isomorphism, equals

\[ J \circ B(id, b) \circ TB(id, \eta_B) \circ J \triangleleft K \]

\[
\xrightarrow{J \circ id \circ \lambda_\eta} A(id, a) \circ TJ \circ TB(id, \eta_B) \circ J \triangleleft K \circ TC(\eta_C, id) \circ C(c, id)
\]

\[
\xrightarrow{id \circ \lambda_\eta \circ \lambda_\eta} A(id, a) \circ TJ \circ TB(id, \eta_B) \circ TB(\eta_B, id) \circ T(J \triangleleft K) \circ C(c, id)
\]

\[
\xrightarrow{id \circ \lambda_\eta \circ \lambda_\eta} A(id, a) \circ TJ \circ T(J \triangleleft K) \circ C(c, id)
\]

\[
\xrightarrow{id \circ T \circ \lambda_\eta} A(id, a) \circ T(J \circ J \triangleleft K) \circ C(c, id)
\]

\[
\xrightarrow{id \circ \lambda_\eta} A(id, a) \circ T(J \triangleleft K) \circ C(c, id)
\]

\[
\xrightarrow{id \circ \lambda_\eta} A(id, a) \circ T(J \triangleleft K) \circ C(c, id)
\]

As before, here \((\eta_B \varepsilon_{\eta_B} \circ id) \circ (id \circ \lambda_\eta J \triangleleft K) = (id \circ \lambda_\eta \varepsilon_{\eta_C} \circ id) \circ (\rho J \triangleleft K) \circ id)\) and, like \([B.10]\), we have \(Tev \circ T \circ (\eta_J \circ \eta_J \triangleleft K) = \eta_K \circ ev: J \circ J \triangleleft K \Rightarrow TK\), so that the composite above can be rewritten as

\[ J \circ B(id, b) \circ TB(id, \eta_B) \circ J \triangleleft K \quad \xrightarrow{J \circ id} \quad A(id, a) \circ TJ \circ TB(id, \eta_B) \circ J \triangleleft K \]

\[
\xrightarrow{id \circ (\rho J)^{-1} \circ id} A(id, a) \circ TA(id, \eta_A) \circ J \circ J \triangleleft K
\]

\[
\xrightarrow{id \circ ev \circ \lambda_\eta} A(id, a) \circ TA(id, \eta_A) \circ K \circ TC(\eta_C, id) \circ C(c, id)
\]

\[
\xrightarrow{id \circ \rho K \circ \lambda_\eta} A(id, a) \circ TK \circ TC(id, \eta_C) \circ C(c, id)
\]

\[
\xrightarrow{id \circ \lambda_\eta \circ \lambda_\eta} A(id, a) \circ A(id, a) \circ K \quad \xrightarrow{\rho K \circ \lambda_\eta} K.
\]

Here we can first replace \((id \circ \lambda_\eta \varepsilon_{\eta_A} \circ id) \circ (id \circ \lambda_\eta \varepsilon_{\eta_A} \circ id)\) by \(\lambda_{\varepsilon_{\eta_A} \varepsilon_{\eta_A} \circ id}\), thus obtaining

\[ J \circ B(id, b) \circ TB(id, \eta_B) \circ J \triangleleft K \quad \xrightarrow{J \circ id} \quad A(id, a) \circ TJ \circ TB(id, \eta_B) \circ J \triangleleft K \]

\[
\xrightarrow{id \circ (\rho J)^{-1} \circ id} A(id, a) \circ TA(id, \eta_A) \circ J \circ J \triangleleft K
\]

\[
\xrightarrow{id \circ ev \circ \lambda_\eta} A(id, a) \circ TA(id, \eta_A) \circ K \circ TC(\eta_C, id) \circ C(c, id)
\]

\[
\xrightarrow{id \circ \lambda_\eta \circ \lambda_\eta} A(id, a) \circ TA(id, \eta_A) \circ TA(\eta_A, id) \circ TK \circ C(c, id)
\]

\[
\xrightarrow{id \circ \lambda_\eta} A(id, a) \circ TA(id, \eta_A) \circ TA(\eta_A, id) \circ A(id, a) \circ K \quad \xrightarrow{\lambda_{\varepsilon_{\eta_A} \varepsilon_{\eta_A} \circ id}} K.
\]

Now the unit axiom for \(\lambda K\) (Proposition [1.7]) means that we can substitute \(\lambda \alpha_0\) for the composite of \(\lambda K\), \(\lambda \eta K\) and \(\lambda \gamma_0\). Following this, we can rewrite \(\lambda_{\varepsilon_{\eta_A} \varepsilon_{\eta_A} \circ id} \circ (id \circ \lambda_\eta \varepsilon_{\eta_A} \circ id) = \rho \alpha_0 = \rho \varepsilon_{\eta_A} \circ \varepsilon_{\eta_A} \circ \lambda_\eta \varepsilon_{\eta_A} \circ id)\) and then use the unit axiom for \(J\) (Definition [1.13]) to substitute \(\rho \varepsilon_0\) for the composite of \(\rho \varepsilon_0\), \((\rho J)^{-1}\) and \(J\). We conclude that the composite above equals

\[ J \circ B(id, b) \circ TB(id, \eta_B) \circ J \triangleleft K \quad \xrightarrow{id \circ \rho \varepsilon_0 \circ \rho \varepsilon_0 \circ id} \quad J \circ J \triangleleft K \quad \varepsilon_0 K
\]

which coincides with the adjoint of

\[ J \circ K \quad \xrightarrow{\lambda \beta_0 \circ id} TB(\eta_B, id) \circ B(b, id) \circ J \triangleleft K \quad \equiv \quad (J \circ B(id, b) \circ TB(id, \eta_B) \circ J \triangleleft K) \circ K.
\]

From this we conclude that \(\lambda \beta_0 \circ id\) equals the composite \([B.11]\), thus proving that the unit axiom for \(\lambda J \triangleleft K\) holds. This completes the proof.
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