THE BOUNDEDNESS OF MULTI-LINEAR AND
MULTI-PARAMETER PSEUDO-DIFFERENTIAL OPERATORS

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Abstract. In this paper, we establish the boundedness on $L^r(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ of bilinear and bi-parameter pseudo-differential operators whose symbols $\sigma(x, \xi, \eta) \in S_{(1,1), (0,0)}^{(\delta_1, \delta_2)}$ for $x, \xi, \eta \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ and $0 \leq \delta_1, \delta_2 < 1$, which extends the result of Dai and Lu in [8].

1. Introduction. The theory of multi-linear and multi-parameter multipliers and pseudo-differential operators play important roles in many aspects of harmonic analysis. We denote by $M(\mathbb{R}^d)$ the set of all bounded functions $m$ on $\mathbb{R}^d \setminus \{0\}$, and satisfies the classical Hörmander-Mikhlin condition

$$|\partial^\alpha\xi m(\xi)| \leq C_\alpha \frac{1}{|\xi||\alpha|}$$  \hspace{1cm} (1.1)

for sufficiently many multi-indices $\alpha \in \mathbb{N}^d$ and each $\xi \in \mathbb{R}^d \setminus \{0\}$. The $N$-linear Fourier multiplier operator $T_m$ is defined by

$$T_m(f_1, \ldots, f_N)(x) = \int_{\mathbb{R}^{Nn}} m(\xi)e^{2\pi i \xi \cdot (x_1 + \cdots + x_N)} \widehat{f_1}(\xi_1) \cdots \widehat{f_N}(\xi_N) d\xi$$  \hspace{1cm} (1.2)

for $f_1, \ldots, f_N$ are Schwartz functions on $\mathbb{R}^n$, where $x \in \mathbb{R}^n$, $\xi = (\xi_1, \ldots, \xi_N) \in (\mathbb{R}^n)^N$ and $d\xi = d\xi_1 \cdots d\xi_N$. A classical result of Coifman and Meyer says that

Theorem 1.1. Let $m \in M(\mathbb{R}^n)$. Then $T_m$ is bounded from $L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_N}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for $1 < p_1, \ldots, p_N \leq \infty$ satisfying $0 < \frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_N}$.  

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The case $1 \leq p < \infty$ was proved by Coifman and Meyer [7] and the extension to the range $p \leq 1$ was established by Kenig and Stein [15] and Grafakos and Torres [12].

The bilinear and bi-parameter Fourier multiplier operator $J_m$ is defined by

$$J_m(f, g)(x) := \int_{\mathbb{R}^4} m(\xi, \eta) e^{2\pi i x \cdot (\xi + \eta)} \hat{f}(\xi) \hat{g}(\eta) d\xi d\eta,$$

(1.3)

where the symbol $m$ is smooth away from the planes $(\xi_1, \eta_1) = (0, 0)$ and $(\xi_2, \eta_2) = (0, 0)$ in $\mathbb{R}^2 \times \mathbb{R}^2$ and satisfies the less restrictive Marcinkiewicz conditions

$$\left| \partial_{\xi_1}^{\alpha_1} \partial_{\xi_2}^{\alpha_2} \partial_{\eta_1}^{\beta_1} \partial_{\eta_2}^{\beta_2} m(\xi, \eta) \right| \leq C \frac{1}{(|\xi_1| + |\eta_1|)^{\alpha_1} + |\beta_1|} \frac{1}{(|\xi_2| + |\eta_2|)^{\alpha_2} + |\beta_2|}$$

(1.4)

for sufficiently many multi-indices $\alpha = (\alpha_1, \alpha_2)$, $\beta = (\beta_1, \beta_2)$ and $\xi = (\xi_1, \xi_2)$, $\eta = (\eta_1, \eta_2)$. Muscalu, Pipher, Tao and Thiele [20] proved the following $L^r$ estimates for $J_m$ with $0 < r < \infty$.

**Theorem 1.2.** The bilinear and bi-parameter operator $J_m$ defined by (1.3) maps $L^p(\mathbb{R}^2 \times \mathbb{R}^2) \rightarrow L^q(\mathbb{R}^2)$ boundedly, as long as $1 < p, q \leq \infty$, $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ and $0 < r < \infty$.

In [21, 22], Muscalu, Pipher, Tao and Thiele generalized Theorem 1.2 to the $n$-linear and $K$-parameter setting for any $n \geq 1$, $K \geq 2$.

After that, J. Chen and G. Lu [4] provided an alternative proof of the $L^p$ estimates for the multilinear and multi-parameter Coifman-Meyer Fourier multipliers established in [20, 21] using the multi-parameter Littlewood-Paley theory instead of the time-frequency and para-product theory. More precisely, to describe their theorem, we need recall some notation. Let $\omega \in \mathcal{S}(\mathbb{R}^d)$ be a Schwartz function satisfying

$$\text{supp}\omega \subset \{ \xi \in \mathbb{R}^d : \frac{1}{2} \leq |\xi| \leq 2 \}, \sum_{j \in \mathbb{Z}} \omega\left(\frac{2^j \xi}{2}\right) = 1 \text{ for all } \xi \in \mathbb{R}^d \setminus \{0\}.$$  

(1.5)

For $s_1, s_2 \in \mathbb{R}$, the bi-parameter Sobolev space $W^{s_1, s_2}(\mathbb{R}^{2n})$ consists of all $f \in \mathcal{S}'(\mathbb{R}^{2n})$ such that

$$\|f\|_{W^{s_1, s_2}} = \|(I - \Delta)^{s_1/2, s_2/2} f\|_{L^2} < \infty,$$

where

$$(I - \Delta)^{s_1/2, s_2/2} f = \mathcal{F}^{-1}[(1 + |\xi_1|^2 + |\eta_1|^2)^{s_1/2}(1 + |\xi_2|^2 + |\eta_2|^2)^{s_2/2} \hat{f}(\xi_1, \xi_2, \eta_1, \eta_2)]$$

where $\xi_1, \xi_2, \eta_1, \eta_2 \in \mathbb{R}^n$. In [4], the following theorem was established by Chen and Lu.

**Theorem 1.3.** Let $m \in L^\infty(\mathbb{R}^{4n})$. Set

$$m_{j,k}(\xi_1, \xi_2, \eta_1, \eta_2) = m(2^j \xi_1, 2^k \xi_2, 2^j \eta_1, 2^k \eta_2) \omega_1(\xi_1, \eta_1) \omega_2(\xi_2, \eta_2),$$

where $\omega_1, \omega_2$ are the same as (1.5) with $d = 2n$. Let $s_1, s_2 > n$, $s = \min(s_1, s_2)$, $1 < p, q < \infty$, $p > \frac{2n}{s}$, $q > \frac{2n}{s}$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ with $0 < r < \infty$. If $m \in L^\infty(\mathbb{R}^{4n})$ satisfies

$$\sup_{j,k \in \mathbb{Z}} \|m_{j,k}\|_{W^{s_1, s_2}(\mathbb{R}^{4n})} < \infty,$$

then $\tilde{T}_m$ is bounded from $L^p(\mathbb{R}^{2n}) \times L^q(\mathbb{R}^{2n})$ to $L^r(\mathbb{R}^{2n})$, where $\tilde{T}_m$ is defined by

$$\tilde{T}_m(f, g)(x) = \int_{\mathbb{R}^{4n}} m(\xi, \eta) e^{2\pi i x \cdot (\xi + \eta)} \hat{f}(\xi) \hat{g}(\eta) d\xi d\eta.$$
On the other hand, let us recall some results about pseudo-differential operator. For the corresponding pseudo-differential operator of the classical Coifman-Meyer theorem, suppose that the symbol $\sigma(x, \xi, \eta)$ belongs to the Hörmander symbol class $BS_{1,0}^0(\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d)$, that is, $\sigma$ satisfies the following condition

\[ |\partial_{x}^\alpha \partial_{\xi}^\gamma \partial_{\eta}^\delta \sigma(x, \xi, \eta)| \leq C_{\kappa, \alpha, \beta} \frac{1}{(1 + |\xi| + |\eta|)^{\kappa + |\alpha| + |\beta|}} \]  

(1.6)

for sufficiently many multi-indices $\kappa, \alpha, \beta \in \mathbb{N}^d$. For these symbols, the following multi-linear, single parameter case has been studied (see [2], and see [3] for $d = 1$ case).

**Theorem 1.4.** Let $\sigma(x, \xi, \eta) \in BS_{1,0}^0(\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d)$. The operator

\[ \hat{T}_\sigma(f, g)(x) = \int_{\mathbb{R}^{2d}} \sigma(x, \xi, \eta)e^{2\pi \xi \cdot x}\hat{f}(\xi)\hat{g}(\eta)d\xi d\eta \]

is bounded from $L^{p_1}(\mathbb{R}^d) \times L^{p_2}(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$ for $1 < p_1, p_2 \leq \infty$ satisfying $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} > 0$, where $f, g \in S(\mathbb{R}^d)$.

For large amounts of literature involving estimates for multi-linear Calderón-Zygmund operators and multi-linear pseudo-differential operators, refer to [1, 6, 12, 15].

The corresponding bilinear and bi-parameter pseudo-differential operator was studied in [8].

**Theorem 1.5.** Define

\[ T_a(f, g)(x) = \int_{\mathbb{R}^4} a(x, \xi, \eta)e^{2\pi \xi \cdot x}\hat{f}(\xi, \eta_{2})\hat{g}(\eta_{1}, \eta_{2})d\xi d\eta \]

where

\[ |\partial_{x_{1}}^{\alpha_{1}} \partial_{x_{2}}^{\alpha_{2}} \partial_{\xi_{1}}^{\gamma_{1}} \partial_{\xi_{2}}^{\gamma_{2}} \partial_{\eta_{1}}^{\delta_{1}} \partial_{\eta_{2}}^{\delta_{2}} a(x, \xi, \eta)| \leq C \frac{1}{(1 + |\xi_{1}| + |\eta_{1}|)^{\kappa_{1}}} \frac{1}{(1 + |\xi_{2}| + |\eta_{2}|)^{\kappa_{2}}} \cdot \]

Then $T_a$ is bounded on $L^p(\mathbb{R}^2) \times L^q(\mathbb{R}^2) \to L^r(\mathbb{R}^2)$ provided that $1 < p, q \leq \infty$ and $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} > 0$.

The $L^p$ estimates for the trilinear pseudo-differential operators with flag symbols has also been established by G. Lu and L. Zhang in [17] while the $L^p$ estimates for the trilinear Fourier multipliers with flag singularity was proved by C. Muscalu [19].

The main purpose of this paper is to establish the $L^p$ estimates for the bilinear and bi-parameter pseudo-differential operators extend the result of Dai and Lu [8]. Now we state the result for the corresponding multi-linear and multi-parameter pseudo-differential operators. Let

\[ T_\sigma(f, g)(x) := \int_{\mathbb{R}^{n_1+n_2} \times \mathbb{R}^{n_1+n_2}} \sigma(x, \xi, \eta)\hat{f}(\xi)\hat{g}(\eta)e^{2\pi \xi \cdot x}\hat{\sigma}(\xi, \eta_{2})\hat{\sigma}(\eta_{1}, \eta_{2})d\xi d\eta, \]  

(1.7)

where $f, g \in S(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, and the bi-parameter smooth symbols $a \in BS_{(1,1),(\delta_1, \delta_2)}^{(0,0)}$ with $0 \leq \delta_1, \delta_2 < 1$ satisfies the following condition

\[ |\partial_{x_{1}}^{\alpha_{1}} \partial_{x_{2}}^{\alpha_{2}} \partial_{\xi_{1}}^{\gamma_{1}} \partial_{\xi_{2}}^{\gamma_{2}} \partial_{\eta_{1}}^{\delta_{1}} \partial_{\eta_{2}}^{\delta_{2}} \sigma(x, \xi, \eta)| \]

\[ \leq C \frac{1}{(1 + |\xi_{1}| + |\eta_{1}|)^{\delta_{1}}} \frac{1}{(1 + |\xi_{2}| + |\eta_{2}|)^{\delta_{2}}} \cdot \]

(1.8)
for every $x = (x_1, x_2), \xi = (\xi_1, \xi_2), \eta = (\eta_1, \eta_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ and all multi-indices $
abla = (\alpha_1, \alpha_2), \beta = (\beta_1, \beta_2)$ and $\gamma = (\gamma_1, \gamma_2) \in \mathbb{N}^{n_1} \times \mathbb{N}^{n_2}$. We will prove the following theorem.

**Theorem 1.6.** Let $0 \leq \delta_1, \delta_2 < 1$. The bilinear and bi-parameter pseudo-differential operator $T_{\sigma}$ with $\sigma \in B_{(1,1)}^{(0,0)}(\delta_1, \delta_2)$ defined by (1.7) maps $L^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) \times L^q(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) \to L^r(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ for $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ and $0 < r < \infty$.

The organization of this paper is as follows: In Section 2 we recall some preliminary facts and give some relevant definitions. In Section 3, we decompose the symbol and prove Theorem 1.6.

2. Preliminaries. First, we make some conventions. Throughout the paper, $C$ denotes a positive constant that is independent of the main parameters involved, but whose value may vary from line to line. For two nonnegative quantities $A$ and $B$, the notation $A \approx B$ means that $A \leq CB$ and $C^{-1}B \leq A$ for some unspecified constant $C > 0$. We use the symbol $N$ to denote the class of all natural numbers, that is, $N = \{0, 1, 2, 3, \ldots\}$.

Let $S(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ and $S'(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ be the Schwartz class of all rapidly decreasing smooth functions and tempered distributions, respectively. We define the Fourier transform $\mathcal{F}f$ and the inverse Fourier transform $\mathcal{F}^{-1}f$ of $f \in S(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ by

$$\mathcal{F}f(\xi, \eta) = \hat{f}(\xi, \eta) = \int_{\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}} f(x_1, x_2)e^{-2\pi i(x_1 \cdot \xi_1 + x_2 \cdot \xi_2)}dx_1dx_2$$

and

$$\mathcal{F}^{-1}f(x_1, x_2) = f^{\mathcal{F}}(x_1, x_2) = \int_{\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}} f(\xi, \eta)e^{2\pi i(x_1 \cdot \xi_1 + x_2 \cdot \xi_2)}d\xi_1d\xi_2.$$

For $m_1, m_2 \in \mathbb{R}$ and $0 \leq \rho_1, \rho_2, \delta_1, \delta_2 \leq 1$, the bilinear and bi-parameter H"{o}rmander symbol class $B_{(\rho_1, \rho_2), (\delta_1, \delta_2)}^{(m_1, m_2)}$ consists of all $\sigma(x, \xi, \eta) \in C^\infty(\mathbb{R}^{3n_1} \times \mathbb{R}^{3n_2})$ such that

$$|\partial_{x_1}^{\alpha_1}\partial_{\xi_1}^{\alpha_2}\partial_{\eta_1}^{\alpha_3}\partial_{x_2}^{\beta_1}\partial_{\xi_2}^{\beta_2}\partial_{\eta_2}^{\beta_3}\sigma(x_1, x_2, \xi_1, \xi_2, \eta_1, \eta_2)| \leq C(1 + |\xi_1| + |\eta_1|)^{m_1 + \delta_1|\alpha_1| - \rho_1(|\alpha_1| + |\gamma_1|)}(1 + |\xi_2| + |\eta_2|)^{m_2 + \delta_2|\alpha_2| - \rho_2(|\beta_2| + |\gamma_2|)}$$

for all multi-indices $\alpha_i, \beta_i \in \mathbb{N}^{n_i}$ with $i = 1, 2$ and $x = (x_1, x_2), \xi = (\xi_1, \xi_2), \eta = (\eta_1, \eta_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$.

Let $\Phi(\mathbb{R}^d)$ (with $d$ changing from time to time as needed) be the set of $\varphi \in S(\mathbb{R}^d)$ such that

$$\text{supp}\varphi \subset \{\zeta \in \mathbb{R}^d : 1/2 \leq |\zeta| \leq 2\},$$

$$\sum_{j \in \mathbb{Z}} \varphi(\zeta/2^j) = 1 \text{ for every } \zeta \in \mathbb{R}^d \setminus \{0\},$$

$$\varphi_0(\zeta) = 1 - \sum_{j=1}^{\infty} \varphi(\zeta/2^j), \varphi_j(\zeta) = \varphi(\zeta/2^j), j \geq 1. \hspace{1cm} (2.1)$$

Notice that $\varphi \in \Phi(\mathbb{R}^{n_i})$ with $d = n_i$, we have $\text{supp}\varphi_0 \subset \{\zeta \in \mathbb{R}^{n_i} : |\zeta| \leq 2\}, \text{supp}\varphi_j \subset \{\zeta \in \mathbb{R}^{n_i} : 2^{j-1} \leq |\zeta| \leq 2^j\}$ for $j \geq 1$, and $\sum_{j=0}^{\infty} \varphi_j(\zeta) = 1$ for all $i = 1, 2$.

Ding, Lu and Zhu [9] defined the bi-parameter local Hardy spaces $h^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ and obtained some properties of $h^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$. 
Definition 2.1. For $0 < p < \infty$ and $\varphi(x_i) \in \Phi(\mathbb{R}^{n_i})$ with $d = n_i$ for $i = 1, 2$. Then the bi-parameter local Hardy space $h^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ is defined by

$$h^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) = \{ f \in S'(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) : \| f \|_{h^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} = \| S(f) \|_{L^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} < \infty \},$$

where

$$S(f) = \left( \sum_{j_1, j_2 = 0}^{\infty} |\varphi_{j_1}(D_1)\varphi_{j_2}(D_2)f|^2 \right)^{1/2}.$$

Remark 1. They point out that the definition of bi-parameter local Hardy space is well defined. That is, the above definition is independent of the choice of the functions $\varphi(x_1)$ and $\varphi(x_2)$.

Remark 2. They also prove that the bi-parameter local Hardy spaces $h^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ are the same as the Lebesgue spaces $L^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ if $p > 1$ and the Schwartz function spaces $S(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ are dense in $h^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ for $0 < p < \infty$. For more details, we refer the reader to see the work of [9].

The strong maximal operator $M_s$ for a function $f$ on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ is defined by

$$M_s f(x_1, x_2) = \sup_{r_1, r_2 > 0} \frac{1}{r_1^{n_1} r_2^{n_2}} \int_R |f(y_1, y_2)| dy_1 dy_2,$$

where $R = \{(y_1, y_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} : |y_1 - x_1| < r_1, |y_2 - x_2| < r_2 \}$ and $f$ is a locally integrable function. We recall the following inequality, see [11, 4]

**Lemma 2.2.** Let $1 < p, q < \infty$. Then there exists a constant $C > 0$ such that

$$\| \{ \sum_{k \in \mathbb{N}} (M_s f_k)^q \}^{1/q} \|_{L^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} \leq C \| \{ \sum_{k \in \mathbb{N}} |f_k|^q \}^{1/q} \|_{L^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})}$$

for all sequences $\{ f_k \}_{k \in \mathbb{N}}$ of locally integrable functions on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$.

And the following lemma is well known and can be proved by the characterization of Hardy spaces using Littlewood-Paley square functions([18]).

**Lemma 2.3.** Let $A > 1$ and $0 < p < \infty$, Then

$$\| \sum_j f_j \|_{H^p(\mathbb{R}^n)} \leq C \| (\sum_j |f_j|^2)^{1/2} \|_{L^p(\mathbb{R}^n)}$$

for all sequences $\{ f_j \}_{j \in \mathbb{N}}$ satisfying $\text{supp } \hat{f}_j \subset \{ \xi \in \mathbb{R}^n : A^{-1} 2^j \leq |\xi| \leq A 2^j \}$.

For the characterization of product Hardy spaces using Littlewood-Paley square functions, it can be founded in [5] and [13].

**Lemma 2.4.** Let $A > 1, B > 1$ and $0 < p < \infty$, Then

$$\| \sum_{j_1, j_2} f_{j_1, j_2} \|_{H^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} \leq C \| (\sum_{j_1, j_2} |f_{j_1, j_2}|^2)^{1/2} \|_{L^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})}$$

for all sequences $\{ f_{j_1, j_2} \}_{j_1, j_2}$ satisfying $\text{supp } \hat{f}_{j_1, j_2} \subset \{ (\xi_1, \xi_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} : A^{-1} 2^{j_1} \leq |\xi_1| \leq A 2^{j_1}, B^{-1} 2^{j_2} \leq |\xi_2| \leq B 2^{j_2} \}$.
3. Proof of Theorem 1.6.

Proof. To prove this theorem, we start with the decomposition and reduction of the symbol $\sigma(x, \xi, \eta)$. First, we use the standard decomposition for $\sigma(x, \xi, \eta)$. Let $\varphi \in \Phi(\mathbb{R}^n)$ as in (2.1) for $i = 1, 2$, then we can write

$$1 = \sum_{k_1=0}^{\infty} \sum_{j_1=0}^{\infty} \sum_{k_2=0}^{\infty} \sum_{j_2=0}^{\infty} \varphi_{k_1}(\xi_1)\varphi_{j_1}(\eta_1)\varphi_{k_2}(\xi_2)\varphi_{j_2}(\eta_2)$$

$$= (\sum_{k_1 > j_1 + 2} + \sum_{k_1 + 2 < j_1} + \sum_{|k_1 - j_1| \leq 2})\varphi_{k_1}(\xi_1)\varphi_{j_1}(\eta_1)$$

$$\times (\sum_{k_2 > j_2 + 2} + \sum_{k_2 + 2 < j_2} + \sum_{|k_2 - j_2| \leq 2})\varphi_{k_2}(\xi_2)\varphi_{j_2}(\eta_2).$$

Applying the above decomposition to $\sigma(x, \xi, \eta)$, we obtain that

$$\sigma(x, \xi, \eta) = (\sum_{k_1=0}^{\infty} \sum_{j_1=0}^{\infty} \sum_{k_2=0}^{\infty} \sum_{j_2=0}^{\infty} \varphi_{k_1}(\xi_1)\varphi_{j_1}(\eta_1))$$

$$\times (\sum_{k_3=0}^{\infty} \sum_{j_3=0}^{\infty} \sum_{k_4=0}^{\infty} \sum_{j_4=0}^{\infty} \varphi_{k_3}(\xi_3)\varphi_{j_3}(\eta_3)) \varphi_{k_4}(\xi_4)\varphi_{j_4}(\eta_4).$$

By symmetry it suffices to consider the following symbols:

$$\sigma_1(x, \xi, \eta) = \sum_{k_1=3}^{\infty} \sum_{k_2=3}^{\infty} \sigma(x, \xi, \eta)\varphi_{k_1}(\xi_1)\varphi_0(2^{-k_1+3}\eta_1)\varphi_{k_2}(\xi_2)\varphi_0(2^{-k_2+3}\eta_2),$$

$$\sigma_2(x, \xi, \eta) = \sum_{k_1=3}^{\infty} \sum_{k_2=3}^{\infty} \sigma(x, \xi, \eta)\varphi_{k_1}(\xi_1)\varphi_0(2^{-k_1+3}\eta_1)\varphi_0(2^{-k_2+3}\xi_2)\varphi_{k_2}(\eta_2),$$

$$\sigma_3(x, \xi, \eta) = \sum_{k_1=3}^{\infty} \sum_{k_2=3}^{\infty} \sigma(x, \xi, \eta)\varphi_{k_1}(\xi_1)\varphi_0(2^{-k_1+3}\eta_1)\varphi_{k_2}(\xi_2)\varphi_{k_2}(\eta_2),$$

$$\sigma_4(x, \xi, \eta) = \sum_{k_1=3}^{\infty} \sum_{k_2=3}^{\infty} \sigma(x, \xi, \eta)\varphi_{k_1}(\xi_1)\varphi_{k_2}(\eta_1)\varphi_{k_2}(\xi_2)\varphi_{k_2}(\eta_2),$$

where

$$\varphi_j(\zeta) = \sum_{|j-l| \leq 2} \varphi_j(2^{-l}\zeta)$$

for fixed $j \in \mathbb{N}$ and $l \geq 0$.

We now rewrite these symbols using their Fourier series expansions. Hence, we can have

$$\sigma_1(x, \xi, \eta) = \sum_{k_1,k_2=0}^{\infty} \sum_{l_1,l_2 \in \mathbb{Z}} \sum_{l_1',l_2' \in \mathbb{Z}} c_{k_1,k_2,l_1,l_2,l_1',l_2'}^{(1)}(x) e^{il_1(2^{-k_1+3}\xi_1)} e^{il_1'(2^{-k_1+3}\eta_1)}$$

$$\times e^{il_2(2^{-k_2+3}\xi_2)} e^{il_2'(2^{-k_2+3}\eta_2)} \varphi_{k_1}(\xi_1)\varphi_0(2^{-k_1+3}\eta_1)\varphi_{k_2}(\xi_2)\varphi_0(2^{-k_2+3}\eta_2),$$

where

$$c_{k_1,k_2,l_1,l_2,l_1',l_2'}^{(1)}(x) = \int_{\mathbb{R}^n \times \mathbb{R}^n} \sigma(x, 2^{k_1}\xi_1, 2^{k_2}\xi_2, 2^{k_1}\eta_1, 2^{k_2}\eta_2) \varphi(\xi_1)\varphi_0(2^{k_1}\eta_1)$$

$$\times \varphi(\xi_2)\varphi_0(2^{k_2}\eta_2) e^{il_1(2^{-k_1+3}\xi_1)} e^{il_1'(2^{-k_1+3}\eta_1)} e^{il_2(2^{-k_2+3}\xi_2)} e^{il_2'(2^{-k_2+3}\eta_2)} d\xi d\eta.$$
It follows from integration by parts that
\[ \sup_{k_1, k_2 \in \mathbb{N}} 2^{-k_1 \delta_1 |\alpha_1|} 2^{-k_2 \delta_2 |\alpha_2|} |\partial_{x_1}^\alpha x_2^\beta e^{(1)}_{k_1, k_2, l_i, l'_i, d_i, d'_i}(x)| \leq C \frac{1}{(1 + |l_1| + |l'_1|)^{2M_1}} \frac{1}{(1 + |l_2| + |l'_2|)^{2M_2}} \]
for sufficiently large \( M_1, M_2 > 0 \) and all multi-indices \( \alpha_1, \alpha_2 \).
Moreover, for simplicity of presentation, we denote by
\[
\psi_{k_1}^l(\xi_t) = (1 + |l|)^{-M_1} e^{i l_t (2^{-k_1} \xi_t)} \varphi_{k_1}(\xi_t),
\phi_{k_1}^{l_1}(\xi_t) = (1 + |l|)^{-M_1} e^{i l'_1 (2^{-k_1} \xi_t)} \varphi_{0}(2^{-k_1} + 3 \xi_t),
\psi_{k_1}^{l_1}(\xi_t) = (1 + |l|)^{-M_1} e^{i l_2 (2^{-k_1} \xi_t)} \varphi_{k_1}(\xi_t),
\phi_{k_1}^{l_1, l_1, l'_1, l'_2}(x) = (1 + |l_1| + |l'_1|) 2^{M_1} (1 + |l_2| + |l'_2|)^{2M_2} e^{(1)}_{k_1, k_2, l_i, l'_i, d_i, d'_i}(x),
\]
where \( t = 1, 2, M_1 = M'_1 + M''_1 \) and \( M_2 = M'_2 + M''_2 \).
Then we have
\[
\sigma_1(x, \xi, \eta) = \sum_{l_1, l_2 \in \mathbb{Z}^n} \sum_{l'_1, l'_2 \in \mathbb{Z}^n} (1 + |l_1| + |l'_1|)^{-M_1} (1 + |l_2| + |l'_2|)^{-M_2} \sum_{k_1, k_2 = 0}^\infty a_{l_1, l_2, l'_1, l'_2}^{l_1, l_1, l'_1, l'_2} (x) \psi_{k_1}^{l_1}(\xi_1) \psi_{k_2}^{l_2}(\xi_2) \phi_{k_1}^{l_1}(\eta_1) \phi_{k_2}^{l_2}(\eta_2). \tag{3.1}
\]
Due to the decay in the coefficient, we can fix \( l_1, l_2, l'_1, l'_2 \) and only take the summation over \( k_1, k_2 \).
Hence, we can reduce the (3.1) to
\[
\sigma_1(x, \xi, \eta) = \sum_{k_1, k_2 = 0}^\infty a_{k_1, k_2}(x) \psi_{k_1}(\xi_1) \psi_{k_2}(\xi_2) \phi_{k_1}(\eta_1) \phi_{k_2}(\eta_2),
\]
where \( \{a_{k_1, k_2}\} \) satisfies
\[
\|\partial_{x_1}^\alpha \partial_{x_2}^\beta a_{k_1, k_2}\|_{L^\infty} \leq C 2^{k_1 \delta_1 |\alpha_1|} 2^{k_2 \delta_2 |\alpha_2|}, \tag{3.2}
\]
\( \{\psi_{k_1}\} \) satisfies
\[
\text{supp} \psi_{k_1} \subseteq \{ \xi_1 \in \mathbb{R}^n : |\xi_1| \leq C \}, \quad \|\partial_{x_1}^\alpha \psi_{k_1}\|_{L^\infty} \leq C 2^{-k_1 |\alpha_1|},
\]
\( \{\psi_{k_1}'\} \) satisfies
\[
\text{supp} \psi_{k_1}' \subseteq \{ \xi_1 \in \mathbb{R}^n : |\xi_1| \leq C \}, \quad \|\partial_{x_1}^\alpha \psi_{k_1}'\|_{L^\infty} \leq C, \quad k_1 = 0,
\]
\( \{\phi_{k_1}\} \) satisfies
\[
\text{supp} \phi_{k_1} \subseteq \{ \xi_1 \in \mathbb{R}^n : |\xi_1| \approx 2^{k_1} \}, \quad \|\partial_{x_1}^\alpha \phi_{k_1}\|_{L^\infty} \leq C 2^{-k_1 |\alpha_1|}, \quad k_1 \geq 1, \tag{3.3}
\]
and \( \{\phi_{k_1}\} \) satisfies
\[
\text{supp} \phi_{k_1} \subseteq \{ \xi_1 \in \mathbb{R}^n : |\xi_1| \leq C 2^{k_1} \}, \quad \|\partial_{x_1}^\alpha \phi_{k_1}\|_{L^\infty} \leq C 2^{-k_1 |\alpha_1|} \tag{3.4}
\]
for \( i = 1, 2 \).
Similarly, we can reduce \( \sigma_i(x, \xi, \eta) \) for \( i = 2, 3, 4 \) to
\[
\sigma_2(x, \xi, \eta) = \sum_{k_1, k_2 = 0}^\infty b_{k_1, k_2}(x) \psi_{k_1}(\xi_1) \phi_{k_2}(\xi_2) \phi_{k_1}(\eta_1) \phi_{k_2}(\eta_2),
\]
in the boundedness of pseudo-differential operators.
\[
\sigma_3(x, \xi, \eta) = \sum_{k_1, k_2=0}^{\infty} c_{k_1, k_2}(x) \psi_{k_1}(\xi_1) \psi_{k_2}(\xi_2) \psi'_{k_1}(\eta_1) \phi_{k_2}(\eta_2),
\]
\[
\sigma_4(x, \xi, \eta) = \sum_{k_1, k_2=0}^{\infty} d_{k_1, k_2}(x) \psi_{k_1}(\xi_1) \psi_{k_2}(\xi_2) \psi'_{k_1}(\eta_1) \phi_{k_2}(\eta_2),
\]
where the coefficients \(a_{k_1, k_2}(x), b_{k_1, k_2}(x)\) and \(c_{k_1, k_2}(x)\) satisfy the same estimate
\[
\|\partial^{\alpha_1} \partial^{\alpha_2} a_{k_1, k_2}\|_{L^\infty} \leq C 2^{k_1 \delta_1[|\alpha_1|] + k_2 \delta_2[|\alpha_2|]},
\]
and \(\psi_{k_1}, \psi'_{k_2}\) and \(\phi_{k_2}\) also satisfy (3.3), (3.4) and (3.5), respectively. Furthermore, to consider the symbol \(\sigma_i(x, \xi, \eta)\) for \(i = 1, 2, 3\), we need to decompose the coefficients \(a_{k_1, k_2}(x), b_{k_1, k_2}(x)\) and \(c_{k_1, k_2}(x)\). To decompose \(a_{k_1, k_2}(x)\), we use the partition of unity that defined in (2.1). Since
\[
1 = (\varphi_0(2^{-(k_1+3)\xi_1}) + \sum_{k'_1=1}^{\infty} \varphi(2^{-(k_1-3+k'_1)\xi_1})) (\varphi_0(2^{-(k_2+3)\xi_2}) + \sum_{k'_2=1}^{\infty} \varphi(2^{-(k_2-3+k'_2)\xi_2}))
\]
\[
= \varphi_0(2^{-(k_1+3)\xi_1}) \varphi_0(2^{-(k_2+3)\xi_2}) + \sum_{k'_1, k'_2=1}^{\infty} \varphi(2^{-(k_1-3+k'_1)\xi_1}) \varphi_0(2^{-(k_2-3+k'_2)\xi_2})
\]
\[
+ \sum_{k'_1=1}^{\infty} \varphi_0(2^{-(k_1+3)\xi_1}) \varphi(2^{-(k_2-3+k'_2)\xi_2})
\]
\[
+ \sum_{k'_2=1}^{\infty} \sum_{k'_1=1}^{\infty} \varphi(2^{-(k_1-3+k'_1)\xi_1}) \varphi(2^{-(k_2-3+k'_2)\xi_2}).
\]
Hence, we can decompose \(a_{k_1, k_2}(x)\) as
\[
a_{k_1, k_2}(x) = a^{(0,0)}_{k_1, k_2}(x) + \sum_{k'_1=1}^{\infty} a^{(k'_1,0)}_{k_1, k_2}(x) + \sum_{k'_2=1}^{\infty} a^{(0,k'_2)}_{k_1, k_2}(x) + \sum_{k'_1=1}^{\infty} \sum_{k'_2=1}^{\infty} a^{(k'_1,k'_2)}_{k_1, k_2}(x),
\]
with
\[
a^{(0,0)}_{k_1, k_2}(x) = \varphi_0(2^{-(k_1+3)}D_1) \varphi_0(2^{-(k_2+3)}D_2) a_{k_1, k_2}(x),
\]
\[
\sum_{k'_1=1}^{\infty} a^{(k'_1,0)}_{k_1, k_2}(x) = \sum_{k'_1=1}^{\infty} \varphi(2^{-(k_1-3+k'_1)}D_1) \varphi_0(2^{-(k_2+3)}D_2) a_{k_1, k_2}(x),
\]
\[
\sum_{k'_2=1}^{\infty} a^{(0,k'_2)}_{k_1, k_2}(x) = \sum_{k'_2=1}^{\infty} \varphi_0(2^{-(k_1+3)}D_1) \varphi(2^{-(k_2-3+k'_2)}D_2) a_{k_1, k_2}(x),
\]
\[
\sum_{k'_1=1}^{\infty} \sum_{k'_2=1}^{\infty} a^{(k'_1,k'_2)}_{k_1, k_2}(x) = \sum_{k'_1=1}^{\infty} \sum_{k'_2=1}^{\infty} \varphi(2^{-(k_1-3+k'_1)}D_1) \varphi(2^{-(k_2-3+k'_2)}D_2) a_{k_1, k_2}(x).
\]
By the moment condition of \(\Psi = \mathcal{F}^{-1}\varphi\) and Taylor’s formula,
\[
a^{(k'_1,k'_2)}_{k_1, k_2}(x)
\]
\[
= \delta^{(k_1-3+k'_1)n_1} \delta^{(k_2-3+k'_2)n_2} \int \Psi(2^{k_1-3+k'_1}(x_1 - y_1)) \Psi(2^{k_2-3+k'_2}(x_2 - y_2))
\]
\[
	imes (a_{k_1, k_2}(y_1, y_2) - \sum_{|\alpha_1|<N_1} \frac{\partial^{\alpha_1} a_{k_1, k_2}(x_1, x_2)}{\alpha_1!}(y_1 - x_1)^{\alpha_1}) dy.
\]
\[ = 2^{(k_1 - 3 + k_1')_N_1} 2^{(k_2 - 3 + k_2')_N_2} \int \Psi(2^{k_1 - 3 + k_1'} (x_1 - y_1)) \Psi(2^{k_2 - 3 + k_2'} (x_2 - y_2)) N_1 \]
\[ \times \left( \sum_{|\alpha_1| = N_1} \frac{(y_1 - x_1)^{\alpha_1}}{\alpha_1!} \int_0^1 (1 - t_1)^{N_1 - 1} (\partial^{\alpha_1} a_{k_1, k_2})(x_1 + t_1(y_1 - x_1), x_2) dt_1 dy \right) \]
\[ = 2^{(k_1 - 3 + k_1')_N_1} 2^{(k_2 - 3 + k_2')_N_2} \int \Psi(2^{k_1 - 3 + k_1'} (x_1 - y_1)) \Psi(2^{k_2 - 3 + k_2'} (x_2 - y_2)) \]
\[ \times \left( \sum_{|\alpha_1| = N_1} \sum_{|\alpha_2| = N_2} \frac{(y_1 - x_1)^{\alpha_1} (y_2 - x_2)^{\alpha_2}}{\alpha_1! \alpha_2!} \int_0^1 \int_0^1 (1 - t_1)^{N_1 - 1} (1 - t_2)^{N_2 - 1} (\partial^{\alpha_1} \partial^{\alpha_2} a_{k_1, k_2})(x_1 + t_1(y_1 - x_1), x_2 + t_2(y_2 - x_2)) dt_1 dt_2 \right) dy, \]
where \( k_1', k_2' \geq 1 \), \( dt = dt_1 dt_2 \) and \( dy = dy_1 dy_2 \). Then, by (3.2), we obtain the following estimate
\[ \| a_{k_1, k_2}(k_1', k_2') \|_{L^\infty} \leq C 2^{-k_1' N_1} 2^{-k_2' N_2} 2^{-k_1 N_1(1 - \delta_1)} 2^{-k_2 N_2(1 - \delta_2)}, \]
where \( N_1 \) and \( N_2 \) can be chosen arbitrary large and \( 0 \leq \delta_1, \delta_2 < 1 \). By the same argument as the above estimate, we also have
\[ \| a_{k_1, k_2}(0, 0) \|_{L^\infty} \leq C, \]
\[ \| a_{k_1, k_2}(k_1', 0) \|_{L^\infty} \leq C 2^{-k_1' N_1} 2^{-k_1 N_1(1 - \delta_1)}, \]
\[ \| a_{k_1, k_2}(0, k_2') \|_{L^\infty} \leq C 2^{-k_2' N_2} 2^{-k_2 N_2(1 - \delta_2)}. \]

On the other hand, recall the definition of \( a_{k_1, k_2}(k_1', k_2') \), we obtain the following support conditions
\[ \text{supp} \mathcal{F} [a_{k_1, k_2}(0, 0)] \subseteq \{ |\xi_1| \leq C 2^{k_1}, |\xi_2| \leq C 2^{k_2} \}, \]
\[ \text{supp} \mathcal{F} [a_{k_1, k_2}(k_1', 0)] \subseteq \{ |\xi_1| \approx 2^{k_1 + k_1'}, |\xi_2| \leq C 2^{k_2} \}, \]
\[ \text{supp} \mathcal{F} [a_{k_1, k_2}(0, k_2')] \subseteq \{ |\xi_1| \leq C 2^{k_1}, |\xi_2| \approx 2^{k_2 + k_2'} \}, \]
\[ \text{supp} \mathcal{F} [a_{k_1, k_2}(k_1', k_2')] \subseteq \{ |\xi_1| \approx 2^{k_1 + k_1'}, |\xi_2| \approx 2^{k_2 + k_2'} \}, k_1', k_2' \geq 1. \]

By this decomposition, we obtain that
\[ \sigma_1(x, \xi, \eta) = \sum_{k_1, k_2 = 0}^\infty \sum_{k_1', k_2' = 0}^\infty a_{k_1, k_2}(k_1', k_2') (x) \psi_{k_1} (\xi_1) \psi_{k_2} (\xi_2) \phi_{k_1} (\eta_1) \phi_{k_2} (\eta_2), \]
\[ \sigma_2(x, \xi, \eta) = \sum_{k_1, k_2 = 0}^\infty \sum_{k_1', k_2' = 0}^\infty b_{k_1, k_2}(k_1', k_2') (x) \psi_{k_1} (\xi_1) \phi_{k_2} (\xi_2) \phi_{k_1} (\eta_1) \psi_{k_2} (\eta_2), \]
\[ \sigma_3(x, \xi, \eta) = \sum_{k_1, k_2 = 0}^\infty \sum_{k_1', k_2' = 0}^\infty c_{k_1, k_2}(k_1', k_2') (x) \psi_{k_1} (\xi_1) \psi_{k_2} (\xi_2) \phi_{k_1} (\eta_1) \phi_{k_2} (\eta_2), \]
where \( b_{k_1, k_2}(k_1', k_2') (x) \) and \( c_{k_1, k_2}(k_1', k_2') (x) \) also satisfy the similar estimates as \( a_{k_1, k_2}(k_1', k_2') (x) \), we omit the details. For the sake of simplicity of notations, we use
\[ \tilde{\Delta}_j h(\xi) = \psi_j(\xi) \hat{h}(\xi), \quad \tilde{\Delta}_j^\prime h(\xi) = \psi_j^\prime(\xi) \hat{h}(\xi), \quad \tilde{S}_j h(\xi) = \phi_j(\xi) \hat{h}(\xi). \]
Then finally we reduce our original operator to the study of the following two cases

\[ T_{\sigma_1}(f, g)(x) = \sum_{k_1, k_2=0}^{\infty} \sum_{k_1', k_2'=0}^{\infty} a_{k_1, k_2}^{(k_1', k_2')} \Delta_{k_1} \Delta_{k_2} f S_{k_1} S_{k_2} g, \]

\[ T_{\sigma_2}(f, g)(x) = \sum_{k_1, k_2=0}^{\infty} \sum_{k_1', k_2'=0}^{\infty} b_{k_1, k_2}^{(k_1', k_2')} \Delta_{k_1} S_{k_2} f S_{k_1} \Delta_{k_2} g, \]

\[ T_{\sigma_3}(f, g)(x) = \sum_{k_1, k_2=0}^{\infty} \sum_{k_1', k_2'=0}^{\infty} c_{k_1, k_2}^{(k_1', k_2')} \Delta_{k_1} \Delta_{k_2} f \Delta_{k_1'} S_{k_2} g, \]

and

\[ T_{\sigma_4}(f, g)(x) = \sum_{k_1, k_2=0}^{\infty} d_{k_1, k_2} \Delta_{k_1} \Delta_{k_2} f \Delta_{k_1'} \Delta_{k_2'} g. \]

Notice the fact in \( T_{\sigma_1} \) and \( T_{\sigma_2} \), the support for each of the Fourier transform of \((\Delta_{k_1} \Delta_{k_2} f S_{k_1} S_{k_2} g)(x), (\Delta_{k_1} S_{k_2} f S_{k_1} \Delta_{k_2} g)(x)\) are contained in \( \{|\xi_1| \approx 2^{k_1}, |\xi_2| \approx 2^{k_2}, k_1, k_2 \geq 1\} \) with the usual modification when \( k_i = 0 \) for \( i = 1, 2 \).

### 3.1. Estimates for \( T_{\sigma_1} \)

We prove the \( L^p \times L^q \to L^r \) estimate for \( 1 < p, q < \infty \) for the operator \( T_{\sigma_1} \). We can rewrite \( T_{\sigma_1}(f, g)(x) \) that

\[
T_{\sigma_1}(f, g)(x) = \sum_{k_1, k_2=0}^{\infty} \sum_{k_1', k_2'=0}^{\infty} a_{k_1, k_2}^{(k_1', k_2')} \Delta_{k_1} \Delta_{k_2} f S_{k_1} S_{k_2} g
\]

\[
= \sum_{k_1, k_2=0}^{\infty} a_{k_1, k_2}^{(0,0)} \Delta_{k_1} \Delta_{k_2} f S_{k_1} S_{k_2} g + \sum_{k_1, k_2=0}^{\infty} \sum_{k_1', k_2'=0}^{\infty} a_{k_1, k_2}^{(k_1', k_2')} \Delta_{k_1} \Delta_{k_2} f S_{k_1} S_{k_2} g
\]

\[
+ \sum_{k_1, k_2=0}^{\infty} \sum_{k_1', k_2'=0}^{\infty} a_{k_1, k_2}^{(0, k_2')} \Delta_{k_1} \Delta_{k_2} f S_{k_1} S_{k_2} g
\]

\[
+ \sum_{k_1, k_2=0}^{\infty} \sum_{k_1', k_2'=0}^{\infty} a_{k_1, k_2}^{(k_1', 0)} \Delta_{k_1} \Delta_{k_2} f S_{k_1} S_{k_2} g
\]

\[
= T_{\sigma_1}^{(0,0)}(f, g)(x) + T_{\sigma_1}^{(1,0)}(f, g)(x) + T_{\sigma_1}^{(0,1)}(f, g)(x) + T_{\sigma_1}^{(1,1)}(f, g)(x).
\]

**Case 1**: Estimates for \( T_{\sigma_1}^{(0,0)} \).

Since the support of the Fourier transform of \( T_{\sigma_1}^{(0,0)} \) for fixed \( k_1, k_2 \) is included in \( \{|\xi_1| \approx 2^{k_1}, |\xi_2| \approx 2^{k_2}\} \) with the usual modification when \( k_i = 0 \) for \( i = 1, 2 \). Then, by Hölder’s inequality and Littlewood-Paley theory, the \( L^r \)-norm is estimated by

\[
\| T_{\sigma_1}^{(0,0)}(f, g) \|_{L^r} = \Big\| \sum_{k_1, k_2=0}^{\infty} a_{k_1, k_2}^{(0,0)} \Delta_{k_1} \Delta_{k_2} f S_{k_1} S_{k_2} g \Big\|_{L^r}
\]

\[
\leq C \| ( \sum_{k_1, k_2=0}^{\infty} | a_{k_1, k_2}^{(0,0)} \Delta_{k_1} \Delta_{k_2} f S_{k_1} S_{k_2} g |^2 )^{\frac{1}{2}} \|_{L^r}
\]

\[
\leq C \| ( \sum_{k_1, k_2=0}^{\infty} | \Delta_{k_1} \Delta_{k_2} f |^2 )^{\frac{1}{2}} \sup_{k_1, k_2 \in \mathbb{N}} | S_{k_1} S_{k_2} g | \|_{L^r}
\]
for fixed operator and the estimate of \( a_{k_1, k_2}(x) \) for \( 0 < \frac{1}{r} = \frac{1}{p} + \frac{1}{q} \) with \( 1 < p, q < \infty \).

**Case 2:** Estimates for \( T_{\sigma_1}^{(1, 0)} \) and \( T_{\sigma_1}^{(0, 1)} \)

Since the situations of \( T_{\sigma_1}^{(1, 0)} \) and \( T_{\sigma_1}^{(0, 1)} \) are symmetrical, we only consider the operator \( T_{\sigma_1}^{(1, 0)} \). By our estimates, the support of the Fourier transform of \( T_{\sigma_1}^{(1, 0)} \) for fixed \( k_2 \geq 1, k_1, k_1' \) is included in \( \{ |\xi_1| \leq C2^{k_1+k_1'}, |\xi_2| \approx 2^{k_2} \} \), and for fixed \( k_2 = 0, k_1', k_1 \) is included in \( \{ |\xi_1| \leq C2^{k_1+k_1'}, |\xi_2| \leq C \} \). Hence, by Hölder inequality and the estimate of \( a_{k_1, k_2}(x) \), we obtain that

\[
\|T_{\sigma_1}^{(1, 0)}(f, g)\|_{L^r} = \left\| \sum_{k_1, k_2 = 0}^{\infty} \sum_{k_1' = 1}^{\infty} a_{k_1, k_2}^{(k_1', 0)}(x) \Delta_{k_1} \Delta_{k_2} f S_{k_1} S_{k_2} g \right\|_{L^r}
\]

\[
\leq C \left\| \left( \sum_{k_2 = 0}^{\infty} \sum_{k_1 = 0}^{\infty} \sum_{k_1' = 1}^{\infty} a_{k_1, k_2}^{(k_1', 0)}(x) \Delta_{k_1} \Delta_{k_2} f S_{k_1} S_{k_2} g |^{2} \right)^{\frac{1}{2}} \right\|_{L^r}
\]

\[
\leq C \left\| \left( \sum_{k_2 = 0}^{\infty} \sum_{k_1 = 0}^{\infty} \sum_{k_1' = 1}^{\infty} 2^{-k_1' N_1} 2^{-k_1 N_1(1-\delta_1)} \Delta_{k_1} \Delta_{k_2} f S_{k_1} S_{k_2} g \right) \right\|_{L^r}
\]

\[
\leq C \left\| \left( \sum_{k_2 = 0}^{\infty} \sum_{k_1 = 0}^{\infty} \sum_{k_1' = 1}^{\infty} 2^{-k_1' N_1} 2^{-k_1 N_1(1-\delta_1)} \Delta_{k_1} \Delta_{k_2} f S_{k_1} S_{k_2} g \right)^{2} \right\|_{L^r}
\]

where

\[
\|a_{k_1, k_2}^{(k_1', 0)}\|_{L^\infty} \leq C 2^{-k_1' N_1} 2^{-k_1 N_1(1-\delta_1)}
\]

for \( k_1' \geq 1 \) and \( 0 < \frac{1}{r} = \frac{1}{p} + \frac{1}{q} \) with \( 1 < p, q < \infty \).

**Case 3:** Estimates for \( T_{\sigma_1}^{(1, 1)} \)

Since the support of the Fourier transform of \( T_{\sigma_1}^{(1, 1)} \) for fixed \( k_1, k_2, k_1', k_2' \) is included in \( \{ |\xi_1| \leq C2^{k_1+k_1'}, |\xi_2| \leq C2^{k_2+k_2'} \} \). Then, by Hölder inequality and (3.6), the \( L^r \)-norm is estimated by

\[
\|T_{\sigma_1}^{(1, 1)}(f, g)\|_{L^r} = \left\| \sum_{k_1, k_2 = 0}^{\infty} \sum_{k_1' = 1}^{\infty} a_{k_1, k_2}^{(k_1', k_2')}(x) \Delta_{k_1} \Delta_{k_2} f S_{k_1} S_{k_2} g \right\|_{L^r}
\]

\[
\leq C \left\| \left( \sum_{k_2 = 0}^{\infty} \sum_{k_1 = 0}^{\infty} \sum_{k_1' = 1}^{\infty} 2^{-k_1' N_1} 2^{-k_1 N_1(1-\delta_1)} 2^{-k_2' N_2} 2^{-k_2 N_2(1-\delta_2)} \Delta_{k_1} \Delta_{k_2} f S_{k_1} S_{k_2} g \right) \right\|_{L^r}
\]

\[
\leq C \left\| \left( \sum_{k_2 = 0}^{\infty} \sum_{k_1 = 0}^{\infty} \sum_{k_1' = 1}^{\infty} 2^{-k_1' N_1} 2^{-k_1 N_1(1-\delta_1)} 2^{-k_2' N_2} 2^{-k_2 N_2(1-\delta_2)} \Delta_{k_1} \Delta_{k_2} f S_{k_1} S_{k_2} g \right)^{2} \right\|_{L^r}
\]
3.2. Estimates for $T_{\sigma_2}$. In this subsection, We consider the boundedness of $T_{\sigma_2}$. As before, we decompose $T_{\sigma_2}(f, g)(x)$ as

$$T_{\sigma_2}(f, g)(x) = \sum_{k_1, k_2 = 0}^\infty \sum_{k_1', k_2' = 0}^\infty b_{k_1, k_2}^{(0, 0)}(x) \Delta_{k_1} S_{k_2} f S_{k_1} \Delta_{k_2} g$$

$$= \sum_{k_1, k_2 = 0}^\infty b_{k_1, k_2}^{(0, 0)}(x) \Delta_{k_1} S_{k_2} f S_{k_1} \Delta_{k_2} g + \sum_{k_1, k_2 = 0}^\infty \sum_{k_1' = 1}^\infty b_{k_1, k_2}^{(k_1', 0)}(x) \Delta_{k_1} S_{k_2} f S_{k_1} \Delta_{k_2} g$$

$$+ \sum_{k_1, k_2 = 0}^\infty \sum_{k_2' = 1}^\infty b_{k_1, k_2}^{(0, k_2')}(x) \Delta_{k_1} S_{k_2} f S_{k_1} \Delta_{k_2} g$$

$$+ \sum_{k_1, k_2 = 0}^\infty \sum_{k_1', k_2' = 1}^\infty b_{k_1, k_2}^{(k_1', k_2')}(x) \Delta_{k_1} S_{k_2} f S_{k_1} \Delta_{k_2} g$$

$$= T_{\sigma_2}^{(0, 0)}(f, g)(x) + T_{\sigma_2}^{(0, 1)}(f, g)(x) + T_{\sigma_2}^{(1, 0)}(f, g)(x) + T_{\sigma_2}^{(1, 1)}(f, g)(x).$$

**Case I:** Estimates for $T_{\sigma_2}^{(0, 0)}$

Since the support of the Fourier transform of $T_{\sigma_2}^{(0, 0)}$ for fixed $k_1, k_2$ is included in $\{||\xi|| \approx 2^{k_1}, ||\xi|| \approx 2^{k_2}\}$ with the usual modification when $k_i = 0$ for $i = 1, 2$. Then, by H"older inequality and Lemma 2.2, the $L^r$-norm is estimated by

$$\|T_{\sigma_2}^{(0, 0)}(f, g)\|_{L^r} = \| \sum_{k_1, k_2 = 0}^\infty b_{k_1, k_2}^{(0, 0)}(x) \Delta_{k_1} S_{k_2} f S_{k_1} \Delta_{k_2} g \|_{L^r}$$

$$\leq C \| \left( \sum_{k_1, k_2 = 0}^\infty |b_{k_1, k_2}^{(0, 0)}(x) \Delta_{k_1} S_{k_2} f S_{k_1} \Delta_{k_2} g|^2 \right)^{\frac{1}{2}} \|_{L^r}$$

$$\leq C \| \left( \sum_{k_1 = 0}^\infty \left( \sup_{k_2} |\Delta_{k_1} S_{k_2} f|^2 \right)^{\frac{1}{2}} \left( \sum_{k_1 = 0}^\infty \left( \sup_{k_2} |S_{k_1} \Delta_{k_2} g|^2 \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} \right) \|_{L^r}$$

$$\leq C \| \left( \sum_{k_1 = 0}^\infty \left( \sup_{k_2} |\Delta_{k_1} f|^2 \right)^{\frac{1}{2}} \left( \sum_{k_1 = 0}^\infty \left( \sup_{k_2} |S_{k_1} \Delta_{k_2} g|^2 \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} \right) \|_{L^r}$$

$$\leq C \| \left( \sum_{k_1 = 0}^\infty \left( \sup_{k_2} |\Delta_{k_1} f|^2 \right)^{\frac{1}{2}} \left( \sum_{k_2 = 0}^\infty |\Delta_{k_2} g|^2 \right)^{\frac{1}{2}} \right) \|_{L^r} \leq C \| f \|_{L^p} \| g \|_{L^q},$$

where $0 < \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ with $1 < p, q < \infty$. 

**Case II:** Estimates for $T_{\sigma_2}^{(1, 0)}$ and $T_{\sigma_2}^{(0, 1)}$
By symmetry, it is sufficient to consider $T_{\sigma_2}^{(1,0)}$, and the operator $T_{\sigma_2}^{(0,1)}$ can be treated in the same way. Since the support of the Fourier transform of $T_{\sigma_1}^{(1,0)}$ for fixed $k'_1, k_1, k_2$ is included in $\{ |\xi_1| \leq C2^{k_1 + k'_1}, |\xi_2| \approx 2^{k_2} \}$ with the usual modification when $k_2 = 0$. Hence, we obtain that

$$\|T_{\sigma_2}^{(1,0)}(f, g)\|_{L^r} = \left\| \sum_{k_1, k_2 = 0}^{\infty} \sum_{k'_1, k'_2 = 0}^{\infty} b(k'_1, k'_2) f S_{k_2} \Delta_{k_1} \Delta_{k_2} g \right\|_{L^r} \leq C \left\| \sum_{k_2 = 0}^{\infty} \sum_{k'_1, k'_2 = 0}^{\infty} b(k'_1, k'_2) f S_{k_2} \Delta_{k_1} \Delta_{k_2} g \right\|_{L^r} \leq C \left\| \sup_{k_1, k_2 \in \mathbb{N}} |\Delta_{k_1} S_{k_2} f| \right\|_{L^p} \left\| \sum_{k_2 = 0}^{\infty} (\sup_{k_1} |S_{k_1} \Delta_{k_2} g|)^2 \right\|_{L^q} \leq C \left\| f \right\|_{L^p} \|g\|_{L^q},$$

where $0 < \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ with $1 < p, q < \infty$.

**Case III:** Estimates for $T_{\sigma_2}^{(1,1)}$

As before, the $L^r$-norm of $T_{\sigma_2}^{(1,1)}$ is estimated by

$$\|T_{\sigma_2}^{(1,1)}(f, g)\|_{L^r} \leq C \left\| \sum_{k_1, k_2 \in \mathbb{N}} |\Delta_{k_1} S_{k_2} f| \right\|_{L^p} \left\| \sum_{k_1, k_2 \in \mathbb{N}} |S_{k_1} \Delta_{k_2} g| \right\|_{L^q} \leq C \left\| f \right\|_{L^p} \|g\|_{L^q},$$

where $0 < \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ with $1 < p, q < \infty$.

Hence, we obtain

$$\|T_{\sigma_2}(f, g)\|_{L^r} \leq C \left\| f \right\|_{L^p} \|g\|_{L^q}$$

for $0 < \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ with $1 < p, q < \infty$.

### 3.3. Estimates for $T_{\sigma_3}$

In this subsection, We consider the boundedness of $T_{\sigma_3}$. As before, we decompose $T_{\sigma_3}(f, g)(x)$ as

$$T_{\sigma_3}(f, g)(x) = \sum_{k_1, k_2 = 0}^{\infty} c_{k_1, k_2}^{(0)}(x) \Delta_{k_1} \Delta_{k_2} S_{k_2} g$$

$$+ \sum_{k_1, k_2 = 0}^{\infty} c_{k_1, k_2}^{(1)}(x) \Delta_{k_1} \Delta_{k_2} \Delta_{k_1} S_{k_2} g$$

$$= T_{\sigma_3}^{(0)}(f, g)(x) + T_{\sigma_3}^{(1)}(f, g)(x).$$

**Case A:** Estimates for $T_{\sigma_3}^{(0)}$

Similarly, by Lemma 2.3 for the second variable, the $L^r$-norm is estimated by

$$\|T_{\sigma_3}^{(0)}(f, g)\|_{L^r} = \left\| \sum_{k_1, k_2 = 0}^{\infty} c_{k_1, k_2}^{(0)}(x) \Delta_{k_1} \Delta_{k_2} S_{k_2} g \right\|_{L^r}$$
\[ \leq C \| \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} c_{k_1,k_2}^{(0)}(x) \Delta_{k_1} \Delta_{k_2} f \Delta_{k_1}' \Delta_{k_2}' S_{k_2} g \|^2_{L^r} \]

\[ \leq C \| \sum_{k_1,k_2=0}^{\infty} \Delta_{k_1} \Delta_{k_2} f \|^2_{L^r} \cdot \| \{ \sum_{k_2=0}^{\infty} (\sup_{k_1} |S_{k_1} \Delta_{k_2} g|)^2 \}^{\frac{1}{2}} \|_{L^q} \]

\[ \leq C \| f \|_{L^r} \| g \|_{L^q}, \]

where \( 0 < \frac{1}{r} = \frac{1}{p} + \frac{1}{q} \) with \( 1 < p, q < \infty \).

**Case B:** Estimates for \( T_{\sigma_3}^{(1)} \).

As before, the \( L^r \)-norm of \( T_{\sigma_3}^{(1)} \) is estimated by

\[ \| T_{\sigma_3}^{(0)}(f,g) \|_{L^r} = \| \sum_{k_1,k_2=0}^{\infty} \sum_{k_2'=1}^{\infty} c_{k_1,k_2}^{(1)}(x) \Delta_{k_1} \Delta_{k_2} f \Delta_{k_1}' \Delta_{k_2}' S_{k_2} g \|_{L^r} \]

\[ \leq C \| \sum_{k_1,k_2=0}^{\infty} \sum_{k_2'=1}^{\infty} c_{k_1,k_2}^{(1)}(x) \Delta_{k_1} \Delta_{k_2} f \Delta_{k_1}' \Delta_{k_2}' S_{k_2} g \|_{L^r} \]

\[ \leq C \| \sum_{k_1=0}^{\infty} (\sup_{k_2} |\Delta_{k_1} \Delta_{k_2} f|) (\sup_{k_2} |\Delta_{k_1}' \Delta_{k_2}' S_{k_2} g|) \|_{L^r} \]

\[ \leq C \| \left\{ \sum_{k_1=0}^{\infty} (\sup_{k_2} |\Delta_{k_1} \Delta_{k_2} f|)^2 \right\}^{\frac{1}{2}} \|_{L^p} \cdot \| \left\{ \sum_{k_2=0}^{\infty} (\sup_{k_1} |\Delta_{k_1}' \Delta_{k_2}' S_{k_2} g|)^2 \right\}^{\frac{1}{2}} \|_{L^q} \]

\[ \leq C \| f \|_{L^r} \| g \|_{L^q}, \]

where \( 0 < \frac{1}{r} = \frac{1}{p} + \frac{1}{q} \) with \( 1 < p, q < \infty \).

Hence, we obtain

\[ \| T_{\sigma_3}(f,g) \|_{L^r} \leq C \| f \|_{L^r} \| g \|_{L^q} \]

for \( 0 < \frac{1}{r} = \frac{1}{p} + \frac{1}{q} \) with \( 1 < p, q < \infty \).

### 3.4. Estimates for \( T_{\sigma_4} \).

In this subsection, We consider the boundedness of \( T_{\sigma_4} \).

It is easy to see that

\[ \| T_{\sigma_4}(f,g) \|_{L^r} = \| \sum_{k_1,k_2=0}^{\infty} d_{k_1,k_2}(x) \Delta_{k_1} \Delta_{k_2} f \Delta_{k_1}' \Delta_{k_2}' g \|_{L^r} \]

\[ \leq C \| \sum_{k_1,k_2=0}^{\infty} |\Delta_{k_1} \Delta_{k_2} f \Delta_{k_1}' \Delta_{k_2}' g| \|_{L^r} \]

\[ \leq C \| \left\{ \sum_{k_1,k_2=0}^{\infty} |\Delta_{k_1} \Delta_{k_2} f|^2 \right\}^{\frac{1}{2}} \|_{L^p} \cdot \| \left\{ \sum_{k_1,k_2=0}^{\infty} |\Delta_{k_1}' \Delta_{k_2}' g|^2 \right\}^{\frac{1}{2}} \|_{L^q} \]

\[ \leq C \| f \|_{L^r} \| g \|_{L^q}, \]

This completes the proof of Theorem 1.6. \( \square \)

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