Global regularity for the critical dispersive dissipative surface quasi-geostrophic equation

Alexander Kiselev and Fedor Nazarov

Department of Mathematics, University of Wisconsin, Madison, WI 53706, USA
E-mail: kiselev@math.wisc.edu and nazarov@math.wisc.edu

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Abstract

We consider the surface quasi-geostrophic equation with dispersive forcing and critical dissipation. We prove the global existence of smooth solutions given sufficiently smooth initial data. This is done using a maximum principle for the solutions involving conservation of a certain family of moduli of continuity.

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1. Introduction

In this paper, we study the following dispersive dissipative surface quasi-geostrophic (SQG) equation:

\[ \theta_t = u \cdot \nabla \theta - (-\Delta)^{1/2} \theta + Au, \quad \theta(x, 0) = \theta_0(x). \tag{1.1} \]

Here \( \theta \) is a scalar real-valued function, \( A \) is an amplitude parameter and the velocity \( u \) is given by \( u = (-R_2 \theta, R_1 \theta) \) with \( R_1, R_2 \) the usual Riesz transforms. We will consider (1.1) on a torus \( \mathbb{T}^2 \) (or, equivalently, on \( \mathbb{R}^2 \) with periodic initial data). Equation (1.1) appears in atmosphere and ocean modelling. It describes the evolution of potential temperature on the surface of strongly rotating half-space, where equations in the bulk are compressible Euler or Navier–Stokes equations coupled with the temperature equation, continuity equation and equation of state. Under certain assumptions, the system can be formally simplified (see Held et al [4]), leading to (1.1) on the surface of the half-space without the last term \( Au \). This additional term appears when the effects of spherical geometry are taken into account via so-called \( \beta \)-plane approximation (see, e.g., [7], section 6.5). The presence of the background gradient gives rise to dispersive waves and hence the system supports both wave-like and turbulent motions (see Held et al [4] for additional physical insight into (1.1) and Sukhatme and Smith [9] for its reinterpretation as part of a broader family).

In recent years, the SQG equation has been the focus of intense mathematical research, initiated by Constantin et al [2]. The equation is physically motivated, and it is perhaps the...
simplest equation of fluid dynamics for which the question of global existence of smooth solutions is still poorly understood. Global regularity for the SQG equation without dispersion is known in the subcritical and critical regime, when the dissipative term is \((-\Delta)^\alpha, \alpha \geq 1/2\). The subcritical case \(\alpha > 1/2\) goes back to Resnick [8] while the critical case \(\alpha = 1/2\) was recently settled in [1, 5]. The supercritical case \(\alpha < 1/2\) in general remains open.

Mathematically, the key ingredient in regularity proofs in the subcritical case is the maximum principle for \(\|\theta(x, t)\|_{L^\infty}\) (see, e.g., [8]). In the critical case, the crucial improvement comes from the stronger nonlocal maximum principle for a certain modulus of continuity ([5]) or DiGiorgi-type iterative estimates establishing Hölder regularity of \(\theta\) ([1]). In this work, we extend the nonlocal maximum principle technique to the case where dispersion is also present. The main result is the following theorem:

**Theorem 1.1.** The dispersive critical SQG equation (1.1) with smooth periodic initial data has a unique global smooth solution.

**Remark.**

(1) We note that in the case of stronger dissipation \(\alpha > 1/2\) the result also holds true and can be proved in a standard way (once the control of \(\|\theta(x, t)\|_{L^\infty}\) is established—which is a part of our proof and can be extended to the subcritical case in a straightforward manner).

(2) The smoothness assumption on the initial data can be relaxed to \(\theta_0 \in H^1\). Indeed, local existence of the solution smooth for \(t > 0\) starting from such initial data can be proved by standard methods. The linear dispersive part does not present any difficulty in this respect (see, e.g., [3] for the SQG case or [6] for an argument in the case of the Burgers equation, which can be easily adapted to our situation). Once \(t > 0\), one can apply the theorem above to get a global smooth solution.

(3) The proof of the uniqueness in the setting of theorem 1.1 is also standard.

(4) The key step in the proof is, as in [5], the derivation of a uniform estimate on \(\|\nabla \theta\|_{L^\infty}\) by using a family of moduli of continuity preserved by the evolution. Once one has this estimate, the proof of the global existence of the regular solution is achieved by the well-known approach of using the local existence theorem and differential inequalities for the Sobolev norms of the solution. Thus, in what follows we will focus on the essential issue of gaining control of \(\|\nabla \theta\|_{L^\infty}\).

2. The Proof

Our first observation is that the \(L^2\) norm of the solution over a single period cell is non-increasing.

**Lemma 2.1.** The \(L^2\) norm of a smooth solution of (1.1) is non-increasing.

**Proof.** Multiplying the equation by \(\theta(x, t)\) and integrating, we obtain

\[
\frac{1}{2} \partial_t \|\theta\|_{L^2} = -\int_{\mathbb{T}^2} \theta (-\Delta)^{1/2} \theta \, dx + A \int_{\mathbb{T}^2} \theta u_2 \, dx.
\]

The first term on the right-hand side is negative, while the second is, up to a constant factor, equal to

\[
\sum_{k \in \mathbb{Z}^2, k \neq 0} \frac{k^4}{|k|} \hat{\theta}(k_1, k_2)^2.
\]

The latter expression is zero since \(\theta\) is real-valued and so \(\hat{\theta}\) is even. \(\square\)
Our next step is gaining control of the $L^\infty$ norm of the solution of (1.1). One can no longer claim it is non-increasing as in the non-dispersive case (it is not), but it remains uniformly bounded.

**Lemma 2.2.** There exists a constant $D = D(A, \theta_0)$ such that the $L^\infty$ norm of a smooth solution $\theta(x, t)$ of (1.1) satisfies

$$\|\theta(x, t)\|_{L^\infty} \leq D$$

for all times while the solution remains smooth.

**Remark.** Observe that in contrast to the non-dispersive case, there is no $L^\infty$ norm maximal principle: the $L^\infty$ norm can grow, and numerical computations suggest it often does [9]. Instead, we just have an upper bound on the $L^\infty$ norm.

**Proof.** Consider a point $x$ where $\theta(x, t)$ reaches its maximum, $M$ (the case of a minimum is similar). At the point of maximum, we have

$$\partial_t \theta(x, t) = -\left(-\Delta\right)^{1/2} \theta(x, t) + AR_1 \theta(x, t).$$

We will use the representation

$$-\left(-\Delta\right)^{1/2} \theta(x, t) = \frac{d}{dh} \mathcal{P}_h * \theta|_{h=0} = \lim_{h \to 0} \frac{1}{h} \int_{\mathbb{R}^2} \frac{h}{(|y|^2 + h^2)^{3/2}} \theta(x - y) - M \ dy,$$  

(2.1)

where $\mathcal{P}_h$ is the usual Poisson kernel in $\mathbb{R}^2$. Observe that we can pass to the limit in (2.1) obtaining the kernel $|y|^{-3}$. On the other hand,

$$AR_1 \theta(x, t) = A \int_{\mathbb{R}^2} f(\hat{y}) \frac{1}{|y|^2} \theta(x - y) \ dy,$$  

(2.2)

where $f$ is a smooth mean zero function on the unit circle and $\hat{y} = y/|y|$. The integral converges in the principal value sense. Because of the mean zero property of $f$ we can replace $\theta(x - y)$ in (2.2) with $\theta(x - y) - M$.

Consider the ball $B_\rho$ of radius $\rho$ centred at zero, and the portion of the integrals in (2.1), (2.2) corresponding to that ball:

$$\int_{B_\rho} \left( \frac{1}{|y|^3} + \frac{Af(\hat{y})}{|y|^2} \right) \theta(x - y) - M \ dy.$$  

(2.3)

We can choose $\rho = \rho(A)$ sufficiently small independently of $M$ so that (2.3) does not exceed

$$\frac{1}{2} \int_{B_\rho} \frac{1}{|y|^3} \theta(x - y) - M \ dy.$$  

(2.4)

Let us denote by $m$ the Lebesgue measure on $\mathbb{T}^2$. Since by lemma 2.1, $\|\theta(x, t)\|_{L^2(\mathbb{T}^2)} \leq \|\theta_0\|_{L^2(\mathbb{T}^2)}$, we have that

$$m \left( y \in \mathbb{T}^2 : |\theta(y, t)| \geq \frac{M}{2} \right) \leq \frac{4\|\theta_0\|^2_{L^2(\mathbb{T}^2)}}{M^2}.$$  

(2.5)

Assume that $\rho$ is sufficiently small so that $B_\rho$ fits into a single period cell. Since $|y|^{-3}$ is monotone decreasing, the expression in (2.4) is maximal if points where $\theta(y, t)$ is large are concentrated near $y = 0$. In particular, assuming that $M$ is sufficiently large, we see from (2.5) that the expression in (2.4) is less than or equal to

$$-\frac{1}{2} \int_{B_\rho \setminus B_{\rho/2}} \frac{M}{2|y|^3} \ dy.$$
where \( r = 2\pi^{-1/2}\|\theta_0\|_{L^2(\mathbb{R}^2)}^2 M^{-1} \). Therefore, the expression in (2.4) does not exceed

\[
-\frac{\pi}{2} \int_{|y| < r} M \frac{M - |y|}{{|y|}\|y\|^2} \leq -\frac{\pi^{3/2}}{4\|\theta_0\|_{L^2(\mathbb{R}^2)}^2} M^2 + C(A)M.
\]  

(2.6)

The integral over the complement of \( B_\rho \) in (2.1) is negative, so it remains to control

\[
A \int_{R^2 \setminus B_\rho} f(\tilde{\gamma}) \frac{\omega(\tilde{\gamma})}{|\tilde{\gamma}|^2} (\theta(x) - \theta(y) - M) \, dy.
\]  

(2.7)

Note that due to the mean zero property of \( f \), we can replace \( M \) in (2.7) by \( \theta_0 \), the mean value of \( \theta \) over a period cell. Then for any period cell \( C \) lying entirely in \( \mathbb{R}^2 \setminus B_\rho \) with centre at distance \( L \) from the origin, we have

\[
\left| \int_C \frac{f(\tilde{\gamma})}{|\tilde{\gamma}|^2} (\theta(x) - \theta(y) - \theta_0) \, dy \right| \leq C M \max_C \left| \nabla \left( \frac{f(\tilde{\gamma})}{|\tilde{\gamma}|^2} \right) \right| \leq C M L^{-3}.
\]

Adding up contributions of the different cells, we get the total bound \( C A M \) for (2.7). Therefore, we have

\[
\partial_t \theta(x, t) \leq -C(\theta_0^2)M^2 + C(A)M,
\]

which is negative provided that \( M \) is large enough; define \( \tilde{D}(A, \theta_0) \) so that this is true if \( M \geq \tilde{D}(A, \theta_0) \). But then it is clear that \( \theta(x, t) \) can never reach such a value of \( M \) unless \( \|\theta_0\|_{L^\infty} \) was already larger—but in this case, \( \|\theta\|_{L^\infty} \) will decay until reaching \( \tilde{D}(A, \theta_0) \). Setting \( D(A, \theta_0) = \max\{\tilde{D}(A, \theta_0), \|\theta_0\|_{L^\infty}\} \), we obtain the result of the lemma.

Now we introduce a family of moduli of continuity. This is the same family that was considered in [5] in the case of the critical SQG. Namely, let \( \omega(\xi) \) be continuous and defined by

\[
\omega(\xi) = \xi - \xi^{3/2}, \quad 0 \leq \xi \leq \delta;
\]

\[
\omega(\xi) = \frac{\xi}{4 + \log(\xi/\delta)}, \quad \xi \geq \delta,
\]

and set \( \omega_B(\xi) = \omega(B\xi) \). Here \( 0 < \gamma < \delta \) are certain constants defined in [5]; the modulus of continuity \( \omega \) is increasing, concave and differentiable at every point except \( \xi = \delta \).

We will need the following lemma from [5]:

**Lemma 2.3.** If the function \( \theta \) has modulus of continuity \( \omega_B \), then \( u = (-R_2\theta, R_1\theta) \) has modulus of continuity

\[
\Omega_B(\xi) = C \left( \int_0^\xi \frac{\omega_B(\eta)}{\eta} \, d\eta + \xi \int_\xi^\infty \frac{\omega_B(\eta)}{\eta^2} \, d\eta \right)
\]

with some universal constant \( C > 0 \).

Observe that by a simple change of coordinates and definition of \( \omega_B \), \( \Omega_B(\xi) = \Omega(B\xi) \).

Our next lemma can be proven exactly as in [5], using that \( \omega''(0) = -\infty \) :

**Lemma 2.4.** Assume that a smooth solution of (1.1) \( \theta(x, t) \) has modulus of continuity \( \omega_B \) at some time \( t_0 \). The only way this modulus of continuity may be violated is if there exist \( t_1 \geq t_0 \) and \( y, z, y \neq z \) such that \( \theta(y, t_1) - \theta(z, t_1) = \omega_B(|y - z|) \), while for all \( t < t_1 \), the solution has modulus of continuity \( \omega_B \).

Next, consider two points \( y, z \) and time \( t_1 \) as in lemma 2.4. Observe that

\[
\partial_t (\theta(y, t) - \theta(z, t))|_{t = t_1} = u \cdot \nabla \theta(y, t_1) - u \cdot \nabla \theta(z, t_1) - (\Delta)^{1/2} \theta(y, t_1)
\]

\[
+ (-\Delta)^{1/2} \theta(z, t_1) + A\theta(y, t_1) - A\theta(z, t_1).
\]  

(2.9)
Let us denote $|y - z| = \xi$. We have the following lemma:

**Lemma 2.5.** For $y, z$ and $t_1$ as in lemma 2.4, we have

$$|u \cdot \nabla \theta(y, t_1) - u \cdot \nabla \theta(z, t_1)| \leqslant \omega'_B(\xi) \Omega_B(\xi)$$

(2.10)

and

$$|Au_2(y, t_1) - Au_2(z, t_1)| \leqslant A \Omega_B(\xi).$$

(2.11)

Moreover, $\delta > \gamma > 0$ can be chosen so that

$$- (\Delta)^{1/2} \theta(y, t_1) + (\Delta)^{1/2} \theta(z, t_1) \leqslant -2 \omega'_B(\xi) \Omega_B(\xi).$$

(2.12)

**Proof.** The inequality (2.11) follows immediately from lemma 2.3. The proof of the inequality (2.10) is identical to that provided in [5]. The proof of (2.12) is also the same as the treatment of the dissipative term given in [5]. Although the result is not stated in [5] in the same form, it follows immediately from the arguments provided there. In the estimates above at the point $x = \delta$ one should use the larger value of the two one-sided derivatives (which is the left derivative).

Now we are ready to prove our main technical result, from which theorem 1.1 follows as explained in the introduction.

**Theorem 2.6.** Assume that the initial data $\theta_0(x)$ is smooth and periodic. Then there exists a constant $B(A, \theta_0)$ such that while the solution of (1.1) $\theta(x, t)$ remains smooth, it satisfies

$$\|\nabla \theta(\cdot, t)\|_{L^\infty} \leqslant B.$$  

(2.13)

**Proof.** Consider $B_0$ large enough so that $\theta_0(x)$ has modulus of continuity $\omega_B$ for any $B > B_0$. Suppose the solution $\theta(x, t)$ loses this modulus of continuity $\omega_B$. Then by lemma 2.4 we can find $y, z$ and $t_1$ so that $\theta(y, t_1) - \theta(z, t_1) = \omega_B(|y - z|)$ and $\theta(x, t)$ has $\omega_B$ for all $t \leqslant t_1$. By lemma 2.5, we have

$$\partial_t (\theta(y, t) - \theta(z, t))|_{t = t_1} \leqslant -\omega'_B(\xi) \Omega_B(\xi) + A \Omega_B(\xi).$$

(2.14)

Moreover, by lemma 2.2, $\|\theta(\cdot, t)\|_{L^\infty} \leqslant D(A, \theta_0)$ and so

$$\omega_B(\xi) = \theta(y, t) - \theta(z, t) \leqslant 2D(A, \theta_0).$$

Since $\omega_B(\xi) = \omega(B \xi)$, it follows that $B \xi \leqslant \omega^{-1}(2D(A, \theta_0))$. But then since $\omega'$ is decreasing,

$$\omega'_B(\xi) = B \omega'(B \xi) \geqslant B \omega'(\omega^{-1}(2D(A, \theta_0))).$$

In particular, the right-hand side in (2.14) is strictly negative if $B \geqslant A/\omega'(\omega^{-1}(2D(A, \theta_0)))$. This gives a contradiction with the definition of $t_1$ since by smoothness the modulus of continuity should have been violated at an earlier time. Thus moduli of continuity corresponding to sufficiently large $B$ are preserved by evolution, as claimed by the theorem.

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