Different faces of harmonic oscillator

Alexander Turbiner†

Instituto de Ciencias Nucleares, UNAM, Apartado Postal 70–543, 04510 Mexico D.F., Mexico

Abstract
Harmonic oscillator in Fock space is defined. Isospectral as well as polynomiality-of-eigenfunctions preserving, translation-invariant discretization of the harmonic oscillator is presented. Dilatation-invariant and polynomiality-of-eigenfunctions preserving discretization is also given.

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†On leave of absence from the Institute for Theoretical and Experimental Physics, Moscow 117259, Russia
E-mail: turbiner@xochitl.nuclecu.unam.mx
Undoubtedly the harmonic oscillator plays a fundamental role in science. Goal of present note is to give a review of different representations of quantum harmonic oscillator and its deformations in terms of the elements of Heisenberg algebra, differential, finite-difference, discrete operators.

The Hamiltonian of harmonic oscillator is defined by
\[ H = -\frac{\partial^2}{\partial x^2} + \omega^2 x^2, \]  
where \( \omega \) is the oscillator frequency. The eigenfunctions and eigenvalues are given by
\[ \Psi_k(x) = H_k(\sqrt{\omega}x)e^{-\omega x^2/2}, \quad E_k = \omega(2k + 1), \quad k = 0, 1, \ldots \]  
where \( H_k \) is the \( k \)th Hermite polynomial in standard notation. Without a loss of generality all normalization constants we put equal to 1. The Hamiltonian (1) is \( \mathbb{Z}_2 \)-invariant, \( x \to -x \), which leads to two families of eigenstates: even and odd, symmetric and anti-symmetric with respect of reflection, correspondingly. This property is coded in parity of the Hermite polynomials:
\[ H_{2n+p}(\sqrt{\omega}x) = x^p L_n^{(p-\frac{1}{2})}(\omega x^2), \quad n = 0, 1, \ldots \]  
where \( L_n^{(\alpha)}(y) \) is the \( n \)th associated Laguerre polynomial in standard notation, and \( p = 0, 1 \) has a meaning of parity. Hereafter we can call a ground state the lowest energy state of parity \( p \):
\[ \Psi_0^{(p)}(x) = x^p e^{-\omega x^2/2}, \]  
Thus the formula (4) makes an unification of both possible values of parity and for the sake of simplicity we will call (4) the ground state eigenfunctions without specifying parity.

Make a gauge rotation of the Hamiltonian (1) taking a gauge factor the ground state eigenfunction (4) and change variable \( x \to y = \omega x^2 \), which incorporate the reflection symmetry. Finally, after dropping off the constant terms we get an operator
\[ h(y, \partial_y) = -\frac{1}{\omega} (\Psi_0^{(p)}(x))^{-1} H \Psi_0^{(p)}(x) \bigg|_{y = \omega x^2} = 4y\partial_y^2 - 4(y - p - \frac{1}{2})\partial_y \]  
with the spectrum \((-4n)\), where \( n = 0, 1, 2, \ldots \). The operator (5) simultaneously describes a family of eigenstates of positive parity if \( p = 0 \) and a family of eigenstates of negative parity if \( p = 1 \). We will call (5) the algebraic form of the Hamiltonian of the harmonic oscillator. The word ‘algebraic’ reflects the fact that the operator (5) has a form of linear differential operator with polynomial coefficients and furthermore possesses infinitely-many polynomial eigenfunctions. The latter implies that any eigenfunction can be found by algebraic means by solving a system of linear algebraic equations.

The algebraic form (5) admits a generalization of the original Hamiltonian (1) we started with. If we assume that the parameter \( p \) can take any
real value, \( p > -1/2 \), one can make an inverse gauge transformation of the operator (5) back to the Hamiltonian form and we arrive at

\[
\omega \frac{y^{p/2}e^{-y/2}}{2} \left[ 4y\partial_y^2 - 4(y - p - \frac{1}{2})\partial_y \right] y^{-p/2}e^{y/2} \bigg|_{y=\sqrt{x}} = \left[ \partial_x^2 - \omega^2x^2 - \frac{p(p-1)}{x^2} \right] \equiv -\mathcal{H}_k ,
\]

(6)

which is known in literature as Kratzer Hamiltonian. It is worth to mention that this Hamiltonian coincides also with 2-body Calogero Hamiltonian. Hereafter we will call the system characterized by the Hamiltonian (6) the harmonic oscillator.

The resulting Hamiltonian (6) is characterized by the eigenfunctions

\[
\Psi_k(x) = x^pL_n^{1+p}(\omega x^2)e^{-\omega x^2/2} ,
\]

(7)

which coincides with (2) at \( p = 0, 1 \). The spectrum (6) is still equidistant with energy gap \( \omega \) and after appropriate shift of the reference point it coincides with the spectrum of the original harmonic oscillator (1). Thus, the deformation of (1) to (6) is isospectral, which is, of course, well-known.

In order to move ahead let us introduce a notion of the Fock space. Take two operators \( a \) and \( b \) obeying the commutation relation

\[
[a, b] \equiv ab - ba = I ,
\]

(8)

with the identity operator \( I \) on the r.h.s. – they span a three-dimensional Lie algebra which is called the Heisenberg algebra \( h_3 \). By definition the universal enveloping algebra of \( h_3 \) is the algebra of all normal-ordered polynomials in \( a, b \): any monomial is taken to be of the form \( b^k a^m \). If, besides the polynomials, all entire functions in \( a, b \) are considered, then the extended universal enveloping algebra of the Heisenberg algebra appears or in other words, the extended Heisenberg-Weyl algebra. In the (extended) Heisenberg-Weyl algebra one can find the non-trivial embedding of the Heisenberg algebra: non-trivial elements obeying the commutation relations (8), whose can be treated as a certain type of quantum canonical transformations. We say that the (extended) Fock space, \( \mathcal{F} \) is determined if we take the (extended) universal enveloping algebra of the Heisenberg algebra and attach to it the vacuum state \( |0> \) such that

\[
a|0> = 0 .
\]

(9)

It is easy to check that the following statement holds if the operators \( a, b \) obey (8), then the operators

\[
J_n^+ = b^2a - nb ,
\]

\[
J_n^0 = ba - \frac{n}{2} ,
\]

(10)

\(^1\)Sometimes this is called the Heisenberg-Weyl algebra
$J_n^- = a$, span the $sl_2$-algebra with the commutation relations:

$$[J^0, J^\pm] = \pm J^\pm, \ [J^+, J^-] = -2J^0, \tag{10}$$

where $n \in \mathbb{C}$. For the realization (10) the quadratic Casimir operator is equal to

$$C_2 \equiv \frac{1}{2}\{J^+_n, J^-_n\} - J^0_nJ^0_n = -\frac{n}{2}\left(\frac{n}{2} + 1\right), \tag{11}$$

where $\{,\}$ denotes the anticommutator and is $c$-number. If $n \in \mathbb{Z}_+$, then (10) possesses a finite-dimensional, irreducible representation in Fock space leaving invariant the space of polynomials in $b$:

$$\mathcal{P}_n(b) = \langle 1, b, b^2, \ldots, b^n | 0 \rangle, \tag{12}$$

of dimension $\dim \mathcal{P}_n = (n + 1)$. The spaces $\mathcal{P}_n$ possess a property that $\mathcal{P}_n \subset \mathcal{P}_{n+1}$ for each $n \in \mathbb{Z}_+$ and form an infinite flag and

$$\bigcup_{n \in \mathbb{Z}_+} \mathcal{P}_n = \mathcal{P}.$$

It is evident that any polynomial in generators $J^0_n$– operator preserves the flag of $\mathcal{P}$. Such an operator we will call $sl_2$-exactly-solvable operator.

Take as an example the $sl_2$-exactly-solvable operator of the form

$$h_f(b, a) = 4J^0J^- - 4J^0 + 4(p + \frac{1}{2})J^- = 4ba^2 - 4(b - p - \frac{1}{2})a, \tag{13}$$

where $p$ is a parameter and $J^\pm, 0 \equiv J^\pm, 0$ (see (10)). One can demonstrate that the eigenfunctions of $h_f$ are the associated Laguerre polynomials of the argument $b$, $L_n^{(p-\frac{1}{2})}(b)$ and their eigenvalues, $E_n = -4n$.

As the next step we consider two different realization of the Heisenberg algebra (8) in terms of differential and finite-difference operators. A traditional realization of (8) appearing in all text-books is the coordinate-momentum representation:

$$a = \frac{d}{dy} \equiv \partial_y, \ b = y, \tag{14}$$

where the operator $b = y$ stands for the multiplication operator on $y$ in a space of functions $f(y)$. In this case the vacuum is a constant and without a loss of generality we put $|0> = 1$. Recently, a finite-difference analogue of (14) has been found (8):

$$a = \mathcal{D}_+, \ b = y(1 - \delta \mathcal{D}_-), \tag{15}$$

where

$$\mathcal{D}_\pm f(y) = \frac{f(y \pm \delta) - f(y)}{\pm \delta},$$

2For details and discussion see, for example, [3]
is the finite-difference operator, $\delta$ is real number and $D_{\pm}(-\delta) = D_{\mp}(\delta)$. A remarkable property of this realization is that the vacuum remains the same for both cases (14)-(15) and it can be written as $|0\rangle = 1$.

Substitution of (14) into (13) leads to the operator (5) – the algebraic form of the Hamiltonian of the harmonic oscillator. Thus, the operator (13) can be called the algebraic form of the Hamiltonian of the harmonic oscillator in the Fock space. It is evident that the procedure of realization of the Heisenberg generators $a, b$ by concrete operators (differential, finite-difference, discrete) provided that the vacuum remains unchanged leaves any polynomial operator in $a, b$ isospectral. Now let us study another ‘face’ of harmonic oscillator by substituting the realization (15) in (13). Finally, we obtain

$$h_d(y, D_{\pm}) = \frac{4}{\delta}[y + \delta(p + \frac{1}{2})]D_+ - 4(1 + \frac{1}{\delta})yD_-.$$  

(16)

Thus, in the realization by finite-difference operators the corresponding spectral problem can be defined as

$$\frac{4}{\delta^2}[y + \delta(p + \frac{1}{2})]\phi(y + \delta) - \frac{4}{\delta}[(1 + \frac{2}{\delta})y + p + \frac{1}{2}]\phi(y) + \frac{4}{\delta}(1 + \frac{1}{\delta})y\phi(y - \delta) = E\phi(y).$$  

(17)

The operator $h_d(y, D_{\pm})$ is a non-local, three-point, finite-difference operator. It is illustrated by Fig.1.

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**Fig. 1.** Graphical representation of the operator $h_d(y, D_{\pm})$

In general, the function $\phi(y)$ in the rhs of (17) can be replaced by $\phi(y + \delta)$ or $\phi(y - \delta)$, or by a linear combination of $\phi(y \pm \delta), \phi(y)$. It does not change the statement that (17) has infinitely-many polynomial eigenfunctions.

It is easy to check that

$$[ye^{-\delta \partial_y}]^n I = y^{(n)} I.$$  

where $y^{(n+1)} = y(y - \delta) \ldots (y - n\delta)$ is a so-called *quasi-monomial* and $I$ is the identity operator. Using this relation one can show that the eigenfunctions of (15) remain polynomials and furthermore the solutions of (17) are given by

$$\hat{L}_n^{(p+\frac{1}{2})}(y, \delta) = \sum_{\ell=0}^{n} a_{\ell}^{(p+\frac{1}{2})} y^{(\ell)},$$  

(18)

where $a_{\ell}^{(p+\frac{1}{2})}$ are the coefficients in the expansion of the Laguerre polynomials, $L_n^{(p+\frac{1}{2})}(y) = \sum_{\ell=0}^{n} a_{\ell}^{(p+\frac{1}{2})} y^{\ell}$. We call these polynomials the *modified associated Laguerre polynomials*. Simultaneously, the eigenvalues of the
equation (16) remain equal to \((-4n), n = 0, 1, 2 \ldots\) and they are the same as the eigenvalues of the harmonic oscillator problem (1), (6) and (13). Thus, one can say that the operator (16) defines a finite-difference form of harmonic oscillator Hamiltonian.

A natural question can be posed about the most general second-order linear differential operator, which (i) has infinitely-many polynomial eigenfunctions and (ii) is isospectral to the harmonic oscillator (1). Following the Theorem [1] one can show that this operator has a form

\[
h_g(y, \partial_y) = 4(AJ^0 + BJ^-)J^- - 4J^0 + 4(p + \frac{1}{2})CJ^- =
\]

\[
4(Ay + B)\partial_y^2 - 4[y - (p - \frac{1}{2})C]\partial_y
\]

(19)

where \(A, B, C\) are arbitrary constants and the generators (11) are realized by differential operators (14). However, by a linear change of variable, \(y \rightarrow \alpha y + \beta\) the operator (19) is transformed to (5). Thus, without loss of generality we can put \(A = \alpha = 1\) and also \(C = 1\). The eigenfunctions of (19) remain the Laguerre polynomials but of a shifted argument, \(L_n^{(p-\frac{1}{2})}(y + \beta)\). It leads to a statement that among the second-order differential operators there exist no non-trivial isospectral deformation of the harmonic oscillator potential preserving polynomiality of the eigenfunctions.

The operator (19) can be rewritten in the Fock space formalism by using (14)

\[
h_g(b, a) = 4(b + B)a^2 - 4[b - (p - \frac{1}{2})]a .
\]

(20)

It is evident that the operator (20) is the most general second order polynomial in \(a\), which is isospectral to (13) and also preserves the space of polynomials (12). By substitution (13) into the operator (20) it becomes transformed into a finite-difference operator

\[
h_g(y, D_{\pm}) = 4BD_+^2 + \frac{4}{\delta^2}[y + \delta(p + \frac{1}{2})]D_+ - 4(1 + \frac{1}{\delta})yD_- - 4\delta[y - 2B + \delta(p + \frac{1}{2})]\phi(y + \delta)
\]

\[
- \frac{4}{\delta^2}[(1 + \frac{2}{\delta})y - B + (p + \frac{1}{2})]\phi(y) + \frac{4}{\delta}(1 + \frac{1}{\delta})y\phi(y - \delta) = E\phi(y)
\]

(22)

and it has infinitely-many polynomial eigenfunctions.

The operator \(h_d(y, D_{\pm})\) now becomes the four-point finite-difference operator, see Fig.2.
\[ \phi(y - \delta) \quad \phi(y) \quad \phi(y + \delta) \quad \phi(y + 2\delta) \]

Fig. 2. Graphical representation of the operator (21)

It is quite surprising that a linear shift of variable \( y \) in differential operator (16) (which gives nothing non-trivial, see discussion above) leads to occurrence of the extra point in their isospectral finite-difference counterpart.

Now it is time to ask what would happen if in the expressions (13),(20) the operators \( a, b \) are not the generators of the Heisenberg algebra (8) but the generators of the \( q \)-deformed Heisenberg algebra

\[ [a, b]_q \equiv ab - qba = 1, \]

where \( q \) is a parameter. Following the Theorem proved in [1], one can demonstrate that within the \( q \)-deformed Fock space built on using \( q \)-deformed Heisenberg algebra (23) there exists the flag of linear spaces of polynomials in \( b, P \) (see (12)), which is preserved by the operators (13),(20). By a simple calculation one can find the eigenvalues of (13),(20)

\[ E_n^{(q)} = -4\{n\}, \quad n = 0, 1, \ldots, \]

where

\[ \{n\} = \frac{1 - q^n}{1 - q}, \]

is a so-called \( q \)-number and \( \{n\} \to n \), if \( q \to 1 \). If the parameter \( q \) the spectra of (13),(20) are real.

The algebra (23) has a realization in terms of discrete operators (see, for example, [2])

\[ a = D_q, \quad b = y, \]

where

\[ D_qf(y) = \frac{f(qy) - f(y)}{y(1 - q)}. \]

This realization has a property that the vacuum remains the same as well as for the cases (14)-(15) and it can be written as \( |0> = 1 \). Now we can substitute (25) in (13) the following operator emerges

\[ h_q(y, \mathcal{D}_q) = 4\tilde{J}_0^+\tilde{J}_- - 4\tilde{J}_0 - 4(p + \frac{1}{2})\tilde{J}_- \]

\[ = 4yD_q^2 - 4(y - p - \frac{1}{2})D_q, \]

where the generators \( \tilde{J}_0 = ba, \quad \tilde{J}_- = a \) have the same functional form as (10) but obey the \( q \)-deformed commutation relation

\[ [\tilde{J}_0, \tilde{J}_-]_{1/q} \equiv \tilde{J}_0\tilde{J}_- - \frac{1}{q}\tilde{J}_0\tilde{J}_- = -\tilde{J}_- , \]

forming the \( q \)-deformed Borel subalgebra \( b(2)_q \) of the \( q \)-deformed algebra \( sl(2)_q \).
The operator $h_q(y, D_q)$ is a non-local, three-point, discrete, dilatation-invariant operator illustrated by Fig.3.

The spectral problem for the operator (26) has a form

$$4 \frac{\phi(q^2 y)}{y(q-1)^2} - 4 \frac{1 + q + (y - p - \frac{1}{2}) q(1 - q)}{y(q-1)^2} \frac{\phi(qy)}{y(q-1)^2} + 4 \frac{1 + (y - p - \frac{1}{2})(1 - q)}{y(q-1)^2} \frac{\phi(y)}{y(q-1)^2} = E(q) \phi(y) .$$

or, the rhs can be taken as

$$= E(q) \phi(qy) .$$

or as

$$= E(q) \phi(q^2 y) .$$

If for the case (27) the eigenvalues are given by (24) while for (28), (29) the eigenvalues are equal to

$$E^{(q)}_n = -4q^n \{ n \} , \ n = 0, 1, \ldots$$

and

$$E^{(q)}_n = -4q^{2n} \{ n \} , \ n = 0, 1, \ldots$$

correspondingly, while in the limit $q \to 1$ all three expressions coincide. The spectral problems (27)–(29) can be considered as a possible definition of a $q$–deformed harmonic oscillator. In the literature it is known many other definitions of the $q$–deformed harmonic oscillator. Such a situation reflects an existence of an ambiguity appearing when a $q$– deformation is performed and absence of clear criteria, which can remove or reduce this ambiguity. For instance, in the literature it is exploited three types of the $q$–Laguerre polynomials (see, for example, an excellent review [4]), but it is not clear why other possible $q$–deformations of Laguerre polynomials are not studied.

Substitution of (25) in (20) gives a slight modification of the expressions (26)–(27). Unlike translation-invariant case it does not lead to a change of the number of points in the operator (26) as it happened for the operators (16) and (21).

\footnote{For example, any term in non-deformed expression can be modified by multipliers of the type $q^n$ and it can be added extra terms with vanishing coefficients in the limit $q \to 1$ like $(1 - q)^b$.}
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