A Fast Proximal Point Method for Computing Wasserstein Distance

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Abstract

Wasserstein distance plays increasingly important roles in machine learning, stochastic programming and image processing. Major efforts have been under way to address its high computational complexity, some leading to approximate or regularized variations such as Sinkhorn distance. However, as we will demonstrate, regularized variations with large regularization parameter will degrade the performance in several important machine learning applications, and small regularization parameter will fail due to numerical stability issues with existing algorithms. We address this challenge by developing an Inexact Proximal point method for Optimal Transport (IPOT) with the proximal operator approximately evaluated at each iteration using projections to the probability simplex. We prove the algorithm has linear convergence rate. We also apply IPOT to learning generative models, and generalize the idea of IPOT to a new method for computing Wasserstein barycenter.

1 Introduction

Many practical tasks in machine learning rely on computing a Wasserstein distance between probability measures or between their sample points [3, 36, 38, 33]. However, the high computational cost of Wasserstein distance has been a thorny issue and has limited its application to challenging machine learning problems.

In this paper we focus on Wasserstein distance for discrete distributions the computation of which amounts to solving the following discrete optimal transport (OT) problem,

$$W(\mu, \nu) = \min_{\Gamma \in \Sigma(\mu, \nu)} \langle C, \Gamma \rangle.$$  \hspace{1cm} (1)

Here $\mu, \nu$ are two probability vectors, $W(\mu, \nu)$ is the Wasserstein distance between $\mu$ and $\nu$. Matrix $C = [c_{ij}] \in \mathbb{R}^{n \times n}_+$ is the cost matrix, whose element $c_{ij}$ represents the distance between the $i$-th support point of $\mu$ and the $j$-th one of $\nu$. Notation $\langle \cdot, \cdot \rangle$ represents the Frobenius dot-product and $\Sigma(\mu, \nu) = \{ \Gamma \in \mathbb{R}^{n \times n}_+ : \Gamma 1_n = \mu, \Gamma^T 1_n = \nu \}$, where $1_n$ represents $n$-dimensional vector of ones. This is a linear programming problem with typical super $O(n^3)$ complexity.

An effort by Cuturi to reduce the complexity leads to a regularized variation of (1) giving rise the so-called Sinkhorn distance [7]. It aims to solve an entropy regularized optimal transport problem

$$W_\epsilon(\mu, \nu) = \min_{\Gamma \in \Sigma(\mu, \nu)} \langle C, \Gamma \rangle + \epsilon h(\Gamma).$$  \hspace{1cm} (2)

The entropic regularizer $h(\Gamma) = \sum_{i,j} \Gamma_{ij} \ln \Gamma_{ij}$ results in an optimization problem (2) that can be solved efficiently by iterative Bregman projections [5],

$$a^{(i+1)} = \frac{\mu}{G b^{(i)}}, \quad b^{(i+1)} = \frac{\nu}{G^T a^{(i+1)}}$$

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starting from $b^{(0)} = \frac{1}{n} \mathbf{1}_n$, where $G = [G_{ij}]$ and $G_{ij} = e^{-C_{ij}/\epsilon}$. The optimal solution $\Gamma^*$ then takes the form $\Gamma^* = a_i G_{ij} b_j$. The iteration is also referred as Sinkhorn iteration, and the method is referred as Sinkhorn algorithm which, recently, is proven to achieve a near-$O(n^2)$ complexity [2].

The choice of $\epsilon$ has two distinct numerical implications. On one hand, $G_{ij} = e^{-C_{ij}/\epsilon}$ and if $\epsilon$ is very small, $G$ tends to underflow. The methods in [5, 6, 17] try to address this numerical instability by performing the computation in log-space, but they require a significant amount of extra exponential and logarithmic operations, and thus, reducing the efficiency. More significantly, even with the benefits of log-space computation, the linear convergence rate of the Sinkhorn algorithm is determined by the contraction ratio $\kappa(G)$, which approaches 1 as $\epsilon \to 0$ [13]. Consequently, we observe drastically increased number of iterations for Sinkhorn method when using small $\epsilon$.

Can we just employ Sinkhorn distance with a moderately sized $\epsilon$ for machine learning problems so that we can get the benefits of the reduced complexity? Some applications show Sinkhorn distance can generate good results with a moderately sized $\epsilon$ [14, 18]. However, we show that in several important problems such as generative model learning and Wasserstein barycenter computation, a moderately sized $\epsilon$ will significantly degrade the performance while the Sinkhorn algorithm with a very small $\epsilon$ becomes prohibitively expensive (also shown in [32]).

In this paper, we propose a new framework to compute the Wasserstein distance using generalized proximal point iterations. The proximal operator is based on Bregman divergence, and its numerical evaluation can be accomplished by a Sinkhorn-like iterative process. To enhance efficiency, we can terminate the inner iteration only after a few iterations (in fact, one inner iteration is used in our numerical experiments), leading to an inexact proximal point method which is referred as Inexact Proximal point method for Optimal Transport (IPOT). Different from existing Sinkhorn algorithms, the result of IPOT method converges to the original optimal transport problem, while its computational complexity is comparable to the Sinkhorn algorithm with a moderately sized $\epsilon$. Furthermore, IPOT can achieve the result with very small $\epsilon$ with properly chosen early stopping.

We first apply IPOT to generative model learning (see section 5), and contrast our method with the one based on Sinkhorn algorithm [14]. We find that the smoothness introduced by moderate entropic regularization used in Sinkhorn distance tends to shrink the learned distribution towards the mean of target distribution, and hence cannot cover the support of the target distribution adequately,\(^1\) while IPOT result with smaller regularization can avoid the problem.

We then focus on the computation of Wasserstein barycenter, again we show better performance is obtained by using very small regularization (with much sharper images) rather than the larger ones. We develop a new method to compute Wasserstein barycenter by extending the idea of IPOT (see Section 6). It turns out that the inexact evaluation of the proximal operator blends well with Sinkhorn-like barycenter iteration.

Regarding IPOT, we carefully analyze its convergence behavior both in terms of theoretical analysis and empirical experiments. We also provide conditions on the number of inner iterations that will guarantee the linear convergence of IPOT. In fact, empirically, IPOT behaves better than our theoretical analysis: the algorithm seems to be linearly convergent with just one inner iteration, demonstrating its efficiency.

## 2 Preliminaries

### 2.1 Wasserstein Distance and Optimal Transport

Wasserstein distance is a metric for two probability measures. Given two distributions $\mu$ and $\nu$, the $p$-Wasserstein distance between them is defined as

$$W_p(\mu, \nu) := \left\{ \inf_{\gamma \in \Sigma(\mu, \nu)} \int_{\mathcal{X} \times \mathcal{Y}} d^p(x, y) d\gamma(x, y) \right\}^{\frac{1}{p}},$$

where $\Sigma(\mu, \nu)$ is the set of joint distributions whose marginals are $\mu$ and $\nu$, respectively. The above optimization problem is also called the Monge-Kantorovitch problem or optimal transport problem [16]. In the following, we focus on the 2-Wasserstein distance, and for convenience we write $W = W_2^2$.

\(^1\)This is a problem related to the mode dropping or mode collapsing issues in generative model learning.
Wasserstein distance is computed by (1). In particular, given samples \( \mu, \nu \in \mathbb{R}_+^n \), where \( \| \mu \|_1 = \| \nu \|_1 = 1 \). Then the Wasserstein distance is computed by (1). In particular, given samples \( \{x_i\} \) and \( \{y_i\} \), we have empirical distributions \( \hat{\mu} = \frac{1}{|x_i|} \sum x_i \delta_{x_i} \) and \( \hat{\nu} = \frac{1}{|y_i|} \sum y_i \delta_{y_i} \), respectively, where \( | \cdot | \) is the cardinality of a set. The supports of \( \hat{\mu} \) and \( \hat{\nu} \) are finite, so the samples are readily used as the support points, and we have \( \mu = \frac{1}{|x_i|} 1\{x_i\} \), \( \nu = \frac{1}{|y_i|} 1\{y_i\} \), and \( C = [c(x_i, y_j)] \in \mathbb{R}_{+}^{[|x_i|] \times [|y_i|]} \) in (1).

The optimization problem (1) is a linear programming (LP) problem. LP tends to provide a sparse solution, which is preferable in applications like histogram calibration or color transferring [22]. However, the cost of LP scales at least \( O(n^3 \log n) \) for general metric, where \( n \) is the number of support points [20]. As aforementioned, an alternative optimization method is the Sinkhorn algorithm in [7]. Following the same strategy, many variants of the Sinkhorn algorithm have been proposed [2, 9, 35]. Unfortunately, all these methods only approximate original optimal transport by its regularized version and their performance both in terms of numerical stability and computational complexity is very sensitive to the choice of \( \epsilon \).

### 2.2 Proximal Point Method

Proximal point methods are widely used in optimization [1, 19, 25, 26]. Since we are dealing with Problem (1), we focus on its convex programming formulation. Given a convex objective function \( f \) defined on \( \mathcal{X} \) with optimal solution set \( \mathcal{X}^* \subset \mathcal{X} \), we can generate a sequence \( \{x(t)\}_{t=1,2,...} \) by the following generalized proximal point iterations:

\[
x^{(t+1)} = \text{arg min}_{x \in \mathcal{X}} f(x) + \beta^{(t)} d(x, x^{(t)}),
\]

where \( d \) is a regularization term used to define the proximal operator, usually defined to be a closed proper convex function. Commonly, \( d \) is defined as the square of Euclidean distance, i.e., \( d(x, y) = \|x - y\|_2^2 \).

### 3 Bregman Divergence Based Proximal Point Method

#### 3.1 Proposed Method

Our key idea is to use Bregman divergence \( D_h \) as the regularization in evaluating the proximal operator in (4), i.e.

\[
\Gamma^{(t+1)} = \text{arg min}_{\Gamma \in \Sigma(\mu, \nu)} \langle C, \Gamma \rangle + \beta^{(t)} D_h(\Gamma, \Gamma^{(t)}),
\]

where Bregman divergence \( D_h \) based on entropy function \( h(x) = \sum x_i \ln x_i \) takes the form (see supplementary material for more) \( D_h(x, y) = \sum_{i=1}^n x_i \log \frac{x_i}{y_i} - \sum_{i=1}^n x_i + \sum_{i=1}^n y_i \). Substituting Bregman divergence into proximal point iteration (5), with simplex constraints, we obtain

\[
\Gamma^{(t+1)} = \text{arg min}_{\Gamma \in \Sigma(\mu, \nu)} \langle C - \beta^{(t)} \log \Gamma, \Gamma \rangle + \beta^{(t)} h(\Gamma).
\]

Denote \( C' = C - \beta^{(t)} \ln \Gamma^{(t)} \). The optimization (6) can be solved by Sinkhorn iteration by replacing \( G_{i,j} \) by \( G_{i,j}' = e^{-C'_{i,j}/\beta^{(t)}} = \Gamma^{(t)}_{i,j} e^{-C_{i,j}/\beta^{(t)}} \).

The algorithm is shown in Algorithm 1. For simplicity we use \( \beta = \beta^{(t)} \). Denote \( \text{diag}(a) \) the diagonal matrix with \( a \) as its ith diagonal elements. Denote \( \odot \) as element-wise matrix multiplication and \( \frac{1}{|x|} \) as element-wise division. We use warm start to improve the efficiency, i.e. in each proximal point iteration, we use the final value of \( a \) and \( b \) from last proximal point iteration as initialization instead of \( b^{(0)} = 1_m \). Later we will show empirically IPOT will converge under a large range of \( \beta \) with \( L = 1 \), a single inner iteration will suffice.

#### 3.2 Theoretical Analysis

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Theorem 3.1. Let \( \{x^{(t)}\} \) be a sequence generated by the proximal point algorithm
\[
x^{(t+1)} = \arg \min_{x \in \mathcal{X}} f(x) + \beta^{(t)} D_h(x, x^{(t)}),
\]
where \( f \) is continuous and convex. Assume \( f^* = \min f(x) > -\infty \). Then, with \( \sum_{t=0}^{\infty} \beta^{(t)} = \infty \), we have
\[
f(x^{(t)}) \downarrow f^*.
\]
If we further assume \( f \) is linear and \( \mathcal{X} \) is bounded, the algorithm has linear convergence rate.

More importantly, the following theorem gives us a guarantee of convergence when (6) is solved inexactly.

Theorem 3.2. Let \( \{x^{(t)}\} \) be the sequence generated by the Bregman distance based proximal point algorithm with inexact scheme (i.e., finite number of inner iterations are employed). Define an error sequence \( \{e^{(t)}\} \) where
\[
e^{(t+1)} \in \beta^{(t)} \left[ \nabla f(x^{(t+1)}) + \partial \iota_{\mathcal{X}}(x^{(t+1)}) \right] + \left[ \nabla h(x^{(t+1)}) - \nabla h(x^{(t)}) \right],
\]
where \( \iota_{\mathcal{X}} \) is the indicator function of set \( \mathcal{X} \). If the sequence \( \{e^{(k)}\} \) satisfies \( \sum_{k=1}^{\infty} \|e^{(k)}\| < \infty \) and \( \sum_{k=1}^{\infty} (e^{(k)}, x^{(k)}) \) exists and is finite, then \( \{x^{(k)}\} \) converges to \( x^\infty \) with \( f(x^\infty) = f^* \). If the sequence \( \{e^{(t)}\} \) satisfies that exist \( \rho \in (0, 1) \) such that \( \|e^{(t)}\| \leq \rho^t \), \( (e^{(t)}, x^{(t)}) \leq \rho^t \) and with assumptions that \( f \) is linear and \( \mathcal{X} \) is bounded, then \( \{x^{(t)}\} \) converges linearly.

The proof of both theorems is given in the supplementary material. Theorem 3.2 guarantees the convergence of inexact proximal point method — as long as \( L \) satisfies the given conditions, the IPOT algorithm would converge linearly.

Now we know IPOT can converge to the exact Wasserstein distance. What if an entropic regularization is wanted? The following remark shows IPOT can achieve regularizations with early stopping.

Remark. Heuristically, if Sinkhorn does not underflow, with enough iteration, the result of IPOT is approximately the same as Sinkhorn with \( \epsilon^{(t)} = \beta/t \). The difference lies in IPOT is a principled way to avoid underflow and can converge to arbitrarily small regularization, while Sinkhorn always causes numerical difficulty when \( \epsilon \to 0 \), even with scheduled decreasing \( \epsilon \) like [6]. More specifically, in IPOT, we can factor \( \Gamma = \text{diag}(\mu_1) G^t \text{diag}(\mu_2) \), where \( (\cdot)^t \) is element-wise exponent operation, and \( \mu_1 \) and \( \mu_2 \) are two scaling vectors. So we have \( \epsilon^{(t)} = \beta/t \). As \( t \) goes infinity, all entries of \( G^t \) would underflow if we use Sinkhorn with \( \epsilon^{(t)} = \beta/t \). But we know \( \Gamma^* \) is neither all zeros nor contains infinity. So instead of computing \( G^t \), \( u_1 \) and \( u_2 \) directly, we use \( \Gamma^t \) to record the multiplication of \( G^t \) with part of \( u_1 \) and \( u_2 \) in each step, so the entries of \( \Gamma^t \) will not over/underflow. The explicit computation of \( G^t \) is not needed.

Therefore, by tuning \( \beta \) and iteration number, we can achieve the result of arbitrary amount of regularization with IPOT. In the experimental section, we will verify our theoretical analysis and show the superiority of IPOT.

4 Empirical Analysis

In this section we will illustrate the convergence behavior w.r.t. \( L \), the transportation plan, and the scalability of IPOT. We leverage the implementation of Sinkhorn iteration and LP solver based on Python package POT [12], and use Pytorch to parallel some of the implementation.

4.1 Convergence Rate
Figure 1: The plot of differences in computed Wasserstein distances w.r.t. number of iterations. Here, $W$ are the Wasserstein distance computed at current iteration. $W_{LP}$ is computed by simplex method, and is used as ground truth. The right lower figure is the two input margins for the test.

Figure 2: (a) The transportation plan generated by Sinkhorn and IPOT methods at different iteration number. The red colormap is the result from Sinkhorn or IPOT method, while the black wire is the result of simplex method for comparison. In the right lower plans, the red and the black is almost identical. (b) Log-log plot of average time used to achieve 1e-4 relative precision with error bar. Each point is obtained by the average of 6 tests on different datasets.

4.2 Sparsity of the Transportation Plan

Figure 2 describes the transportation plans of optimal transport. The plans are generated with the same input as Figure 1.

A larger $\epsilon$ leads to more blur in the Sinkhorn plans. In applications such as histogram calibration and color transferring, the non-sparse structure of transportation plan make it difficult to extract the plan from source distribution to target distribution. While in IPOT iteration, the result always converge to ground truth with enough iteration. On the other hand, a smaller $\epsilon$ needs significantly more iterations to converge. The Sinkhorn $\epsilon = 0.0001$ case still cannot converge after 2000 iterations. This figure also demonstrated that IPOT can approximate the Sinkhorn result with early stopping.

4.3 Scalability

The scalability test of IPOT is shown in Figure 2 (b). The two probability measures tested are the empirical distributions (described in Section 2.1) of 16D uniformly distributed data. The Sinkhorn algorithm follows [7] and the stabilized Sinkhorn algorithm follows [6]. We also notice there is a method [29] for $\epsilon$ scaling, to help the convergence when $\epsilon \to 0$. However, although it is faster than Sinkhorn method when data size is smaller than 1024, the time used at 1024 is already around 2000s. Therefore we do not spend time on it. The LP solver has a good performance under the current experiment settings. But LP solver is not guaranteed to have approximately $O(n^2)$ convergence as shown here. Readers who are interested please refer to experiments in [7].

Sinkhorn and IPOT can be paralleled conveniently, so we provide both CPU and GPU tests here. Under this setting, IPOT takes approximately the same resources as Sinkhorn at $\epsilon = 0.01$. For smaller $\epsilon$, original Sinkhorn will underflow, and we need to use stabilized Sinkhorn. StabilizedSinkhorn is more expensive than IPOT, especially for large datasets and small $\epsilon$. 
5 Extension to Learning Generative Models

The WGAN method [3] uses Wasserstein distance in learning generative models. With Wasserstein distance as the loss function, the generative networks are more stable and easier to train. The IPOT algorithm provides an alternative way to compute the Wasserstein distance, instead of the dual formulation used in [3].

For simplicity, we assume $\{x_i\} = \{z_j\} = n$. Given a dataset $\{x_i\}$ and some noise $\{z_j\}$ [4, 15], our goal is to find a parameterized function $g_{\theta}(\cdot)$ that minimize $W(\{x_i\}, \{g_{\theta}(z_j)\})$, 

$$W(\{x_i\}, \{g_{\theta}(z_j)\}) = \min_{\Gamma} \langle C(\theta), \Gamma \rangle \quad \text{s.t.} \quad \Gamma 1_n = \frac{1}{n} 1_n, \quad \Gamma^T 1_n = \frac{1}{n} 1_n, \quad (7)$$

where $C(\theta) = [c(x_i, g_{\theta}(z_j))]$. Usually, $g_{\theta}$ is parameterized by a neural network with parameter $\theta$, and the minimization over $\theta$ is done by stochastic gradient descent.

In particular, given current estimation $\theta$, we can obtain optimum $\Gamma^*$ by IPOT, and compute the Wasserstein distance by $\langle C(\theta), \Gamma^* \rangle$ accordingly. Then, we can further update $\theta$ by the gradient of current Wasserstein distance. There are two ways to solve the gradient: One is auto-diff based method such as [14], the other is based on the envelope theorem [1]. Different from the auto-diff based methods, the back-propagation based on envelope theorem does not go into proximal point iterations because the derivative over $\Gamma^*$ is not needed, which accelerates the learning process greatly. This also has significant implications numerically because the derivative of a computed quantity tends to amplify the error. Therefore, we adopt envelope based method.

Omitting some derivation, we can compute the derivative of Wasserstein distance efficiently by 

$$\frac{\partial W(\{x_i\}, \{g_{\theta}(z_j)\})}{\partial \theta} = \langle \Gamma^*, 2(g_{\theta}(z_j) - x_i) \frac{\partial g_{\theta}(z_j)}{\partial \theta} \rangle,$$

where we assume $C_{ij}(\theta) = \| x_i - g_{\theta}(z_j) \|_2^2$, but the algorithm can also adopt other metrics. The derivation is in supplement materials. The flowchart is shown in Figure 3, and the algorithm is shown in Algorithm 2.

Note Sinkhorn distance is defined as $S(\{x_i\}, \{g_{\theta}(z_j)\}) = \langle C(\theta), \Gamma^* \rangle$, where $\Gamma^* = \arg \min_{\Gamma} \mathbb{E} \Sigma_{1/n, 1/n}(C(\theta), \Gamma) + \epsilon h(\Gamma)$. If Sinkhorn distance is used in learning generative models, envelope theorem cannot be used because the loss function for optimizing $\theta$ and $\Gamma$ is not the same.

In the tests, we observe the method in [14] suffers from shrinkage problem, i.e. the generated distribution tends to shrink towards the target mean. The recovery of target distribution is sensitive to the weight of regularization term $\epsilon$. Only relatively small $\epsilon$ can lead to a reasonable generated distribution.

5.1 Experiments on 2D Synthetic Data

First, we do a 2D toy example to demonstrate the shrinkage problem and the necessity to use small $\epsilon$. We use a 2D-2D NN as generator to learn a mapping from uniformly distributed noise to mixture of Gaussian distributed real data. IPOT uses 200 iterations and Sinkhorn uses 500 iterations. The implementation of dual method (i.e. WGAN method) follows [3].

Figure 4 shows the results. The result for Sinkhorn at $\epsilon = 0.01$ is acceptable, although the sample density at upper right corner is smaller than expected. For $\epsilon = 0.1$, the generated samples totally shrink to the mean. IPOT does not suffer from this problem.
5.2 Experiments on Real Data

Figure 5 shows the generated result for MNIST dataset. The generator \(g_\theta : \mathbb{R}^2 \mapsto \mathbb{R}^{784}\) use 2D noise data \(\{z_j\}_{i=1}^n \sim \text{Unif}(0,1)^2\) as input, and one fully connected hidden layer with 500 nodes. The images shown in Figure 5 is generated by uniform grid points on \([0, 1]^2\), to demonstrate how the latent space is mapped to images.

Algorithm 2 Learning generative networks

```
Input: real data \(\{x_i\}\), initialized generator \(g_\theta\)

while not converged do
    Sample a batch of real data \(\{x_i\}_{i=1}^n\)
    Sample a batch of noise data \(\{z_j\}_{i=1}^n \sim q\)
    \(C_{ij} := c(x_i, g_\theta(z_j)) := ||x_i - g_\theta(z_j)||^2\)
    \(\Gamma = \text{IPOT}(\frac{1}{n^2}1_n, \frac{1}{n^2}1_n, C)\)
    Update \(\theta\) with \(\langle \Gamma, [2(x_i - g_\theta(z_j)) \frac{\partial g_\theta(z_j)}{\partial \theta} \rangle\)
end while
```

IPOT generator maps the latent space to all ten digits, while Sinkhorn generator suffers shrinkage and cannot cover all ten digits. If a larger \(\epsilon\) is used (see supplement), Sinkhorn generator would shrink to one point, and hence cannot learn anything.

This test is for demonstration purpose to argue for the potential of the method to prevent shrinkage, so we do not pursue a performance of state-of-art. For future improvement, a more sophisticated network should be adopted. Also, Euclidean distance as we used here might not be able to describe the structure of high dimension data, especially RGB images. A proper distance such as the one in [14] or [28] would largely improve the capability of the learning model.

5.3 Reason of Shrinkage

The result of Sinkhorn method with moderate size \(\epsilon\) tends to shrink towards the mean, so the learned distribution cannot cover all the support of target distribution. To demonstrate the reason of this trend, consider the extreme condition when \(\epsilon \to \infty\), the loss function becomes

\[
\Gamma^* = \arg\min_{\Gamma} h(\Gamma) = \arg\min_{\Gamma} D_h (\Gamma, 11^T / n^2).
\]

So \(\Gamma^* = 11^T / n^2\). If we view \(\{x_i\}\) and \(\{y_j : y_j = g_\theta(z_j)\}\) as the realizations of random variables \(X\) and \(Y\), the optimal Sinkhorn distance \(W_\epsilon\) is expected to be

\[
\mathbb{E}_{X,Y}[W_\epsilon] = \mathbb{E}_{X,Y}[\langle \Gamma^*, C \rangle] = \mathbb{E}_{X,Y}\sum_{i,j}||x_i - y_j||_2^2 = n^2(\text{Var}(X) + (\bar{X} - \bar{Y})^2 + \text{Var}(Y)),
\]

where \(n\) is the data size, \((\bar{\cdot})\) is the mean of random variable, and \(\text{Var}(\cdot)\) is the variance. At the minimum of Wasserstein distance, the mean of generated data \(\{y_j\}\) is the same as \(\{x_i\}\), but the variance is zero. Therefore, a large \(\epsilon\) would cause the learned distribution shrink toward the data mean due to smoothing the effect of regularization.
6 Extension to Wasserstein Barycenter

Wasserstein barycenter is widely used in machine learning and computer vision due to its nice property [5, 23]. Given a set of distributions \( P = \{p_1, p_2, ..., p_K\} \), their Wasserstein barycenter is defined as

\[
q^*(P, \lambda) = \arg \min_{q \in Q} \sum_{k=1}^{K} \lambda_k W(q, p_k)
\]

where \( W \) is the Wasserstein distance, and \( Q \) is in the space of probability distributions, and \( \sum_{k=1}^{K} \lambda_k = 1 \).

The idea of our IPOT method can be generalized to learn Wasserstein barycenter. In particular, plugging the definition of Wasserstein distance in (1) into (8) with some derivation (see supplementary material for more), we get the proximal point iteration for barycenter analogous to (5) is

\[
\{r^{(t+1)}_k\} = \arg \min_{\{r_k\}} \sum_{k=1}^{K} \lambda_k \{\langle r_k, C' \rangle + \beta^{(t)} D_h(r_k, r^{(t)}_k)\}
\]

s.t. \( r_k 1 = p_k, \forall k, \exists q, r^T_k 1 = q \)

The minimization in each proximal step is solved by Sinkhorn barycenter iteration [5]. We provide the detailed algorithm in the supplement material. The same as Algorithms 1, this algorithm can also converge with \( L = 1 \) and a large range of \( \beta \).

6.1 Learning Barycenter

We test our proximal point barycenter algorithm on MNIST dataset, borrowing the idea from [8]. Here, the images in MNIST dataset is randomly uniformly reshape to half to double of its original size, and the reshaped images have random bias towards corner. After that, the images are mapped into 50 \( \times \) 50 grid. For each digit we use 50 of the reshaped images with the same weights as the dataset to compute the barycenter. All results are computed using 50 iterations and under \( \epsilon, \beta = 0.001 \). So for proximal point method, the regularization is approximately the same as \( \epsilon = 2 \times 10^{-5} \), which is pretty small. We compare our method with state-of-art Sinkhorn based methods [8], [31] and [5]. Among the four methods, the convolutional method [31] is different in terms of that it only handles structural input tested here and does not require \( O(n^2) \) storage, unlike other three general purpose methods. The results (Figure 6) from proximal point algorithm are clear, while the results of Sinkhorn based algorithms suffer blurry effect due to entropic regularization.

While the time complexity of our method is in the same order of magnitude with Sinkhorn algorithm [5], the space complexity is \( K \) times of it, because \( K \) different transport maps need to be stored. This might cause pressure to memory for large \( K \). Therefore, a sequential method is needed. We left this to future work.

7 Conclusion

We proposed a proximal point method - IPOT - based on Bregman distance to solve optimal transport problem. The algorithm can converge to ground truth even if the inner optimization iteration only performs a few times. We also demonstrate its potential in generative models and Wasserstein barycenter. For generative models, we show it is a good alternative to avoid shrinkage problem, and the back-propagation of learning process can be largely simplified by envelope theorem. For barycenter, our new method can generate much sharper results than state-of-art.

Figure 6: The result of barycenter. For each digit, we randomly choose 8 of 50 scaled and shifted images to demonstrate the input data. From the top to the bottom, we show (top row) the demo of input data; (second row) the results based on [8]; (third row) the result based on [31]; (fourth row) the result based on [5]; (bottom row) the results based on inexact proximal point algorithm.
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Appendix

A More Analysis on IPOT

A.1 Intuitive Illustration

Figure 7 illustrates how the accuracy improves w.r.t. proximal point iterations in the solution space. First, let’s consider Sinkhorn algorithm. The loss function of Sinkhorn has regularization term $\epsilon h(\Gamma)$, which can be rewritten as constraint $D_h(\Gamma, 1^n) \leq \eta$ for some $\eta > 0$. So in the left figure Sinkhorn solution is feasible within the $D_h$ constraints and the closest to optimal solution set. Proximal point algorithm, on the other hand, solves optimization with $D_h$ constraints iteratively, until an optimal solution is reached. Exact proximal point method is when (6) solved exactly. It provides a feasible solution that is closest to the optimal solution set in each step, and finally reach an optimal solution. Inexact proximal point method is when (6) is solved inexacty. Since we use Sinkhorn iteration to solve (6), this is when the iteration number of the inner optimization is not enough to converge. In each step, the solution might not be feasible or closest to the optimal set, but eventually it converges to an optimal solution.

A.2 Convergence w.r.t. $L$

As mentioned in Section 4.1, we provide the test result of 64D Gaussian distributed data here. We choose the computed Wasserstein distance $\langle \Gamma, C \rangle$ as the indicator of convergence, because while the optimal transportation plan might not be unique, the computed Wasserstein distance at convergence must be unique and minimized to ground truth. We use the empirical distribution as input distributions, i.e.,

$$W(\{x_i\}, \{g_\theta(z_j)\}) = \min_{\Gamma} \langle C(\theta), \Gamma \rangle$$

s.t. $\Gamma 1_n = 1_n, \Gamma^T 1_n = 1_n$. \hspace{1cm} (9)

As shown in Figure 8, the convergence rate is also linear. For comparison, we also provide the convergence path of Sinkhorn iteration. The result cannot converge to ground truth because the method is essentially regularized.

A.3 Convergence w.r.t. $\beta$

Figure 9 shows how the computed Wasserstein distances change w.r.t. the number of iterations under different $\beta$. Here, a smaller $\beta$ usually lead to quicker convergence of proximal point iterations. On the other hand, the convergence of inner Sinkhorn iteration, is quicker when $\beta$ is large. Therefore, the choice of $\beta$ is a trade-off between inner and outer convergence rates. The convergence rate increases w.r.t. $\beta$ when $\beta$ is small, and decreases when $\beta$ is large.

Figure 7: Schematic of the convergence path of (a) Sinkhorn algorithm, (b) exact proximal point algorithm and (c) inexact proximal point algorithm. The distance shown is in Bregman sense. Sinkhorn solution is feasible and the closest to optimal solution set within the $D_h$ constraints. Proximal point algorithm, however, solves optimization with $D_h$ constraints iteratively, until an optimal solution is reached.

Figure 8: The plot of differences in computed Wasserstein distances w.r.t. number of iterations for 64D Gaussian distributed data. Here, $W$ are the Wasserstein distance computed at current iteration. $W_{LP}$ is computed by simplex method, and is used as ground truth. The test adopts $c(x, y) = ||x - y||^2$. Due to random data is used, the number of iteration that the algorithm reaches $10^{-17}$ varies from 1000 to around 5000 according to our tests.
The choice of $L$ also appears to be a trade-off. While a larger $L$ takes more resources in each step, it also achieves a better accuracy, so less proximal point iterations are needed to converge. So the choice of best $L$ is relevant to the choice of $\beta$. For large $\beta$, the inner Sinkhorn iteration can converge faster, so smaller $L$ should be used. For small $\beta$, larger $L$ should be used, which is not efficient, and also improve the risk of underflow for the inner Sinkhorn algorithm. So unless there are specific need for accuracy, we do not recommend using very small $\beta$ and large $L$.

But in practice, all the tests adopt $L = 1$ for simplicity.

Remark. When we are talking about amount of regularization, usually we are referring to the magnitude of $\epsilon$ for Sinkhorn, or the equivalent magnitude of $\epsilon$ computed from remark in Section 3 for IPOT method. However, the amount of regularization in a loss function should be quantified by $\epsilon / ||C||$, instead of $\epsilon$ alone. That is why in this paper, different magnitude of $\epsilon$ is used for different application.

**B Learning Generative Models**

In this section, we show the derivation for the learning algorithm, and more tests result.

**Theorem B.1. Envelope theorem.** Let $f(x, \theta)$ and $l(x)$ be real-valued continuously differentiable functions, where $x \in \mathbb{R}^n$ are choice variables and $\theta \in \mathbb{R}^m$ are parameters. Denote $x^*$ to be the optimal solution of $f$ with constraint $l = 0$ and fixed $\theta$, i.e.

$$x^* = \arg \min_x f(x, \theta) \quad s.t. \quad l(x) = 0.$$  

Then, assume that $V$ is continuously differentiable function defined as $V(\theta) \equiv f(x^*(\theta), \theta)$, the derivative of $V$ over parameters is

$$\frac{\partial V(\theta)}{\partial \theta} = \frac{\partial f}{\partial \theta}.$$  

In our case, because $\Gamma^*$ is the minimization of $(\Gamma, C(\theta))$ with constraints, we have

$$\frac{\partial W(x_i, \{g_\theta(z_j)\})}{\partial \theta} = \frac{\partial (\Gamma^*, C(\theta))}{\partial \theta}$$  

$$= (\Gamma^*, \frac{\partial C(\theta)}{\partial \theta}) = (\Gamma^*, 2(g_\theta(z_j) - x_i) \frac{\partial g_\theta(z_j)}{\partial \theta}),$$

A detailed flowchart for the forward and backward propagation is shown in Figure 10.

**B.1 Synthetic Test**

In section 5.1, we show the learning result of Sinkhorn with $\epsilon = 0.1, 0.01$ and IPOT with $\beta = 0.1, 0.01$. Two results of each method is not enough to demonstrate how the learning result changes w.r.t. parameters. Therefore, in Figure 11 and 12 we show sequences of results for a 1D-1D and 2D-2D generator, respectively. The upper sequence is IPOT with $\beta = 0.01, 0.025, 0.05, 0.075, 0.1$. The results barely change w.r.t. $\beta$. The lower sequence is the corresponding Sinkhorn results. The results shrink to the mean of target data, as expected. Also, we observe the learned distribution tends to have a tail that is not in the range of target data (also in 2D result, we do not include that part for a better view). It might be because the range of support that has a small probability has very small gradient when updated. Once the distribution is initialized to have a tail with small probability, it can hardly be updated. But this theory cannot explain why larger $\epsilon$ corresponds to longer tails. The tails can be on the left or right. We pick the ones on the left for easier comparison.

**B.2 MNIST Test**

The same shrinkage can be observed in MNIST data as well. See figure 13. While $\epsilon = 0.1$ covers most shapes of the numbers, $\epsilon = 1$ only covers a fraction, and $\epsilon = 10$ seems to cover only the mean of images.

**C Wasserstein Barycenter**

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**Figure 9:** The plot of differences in computed Wasserstein distances with different $\beta$ for the same input as figure 1.
Figure 11: The sequences of learning result of IPOT, Sinkhorn. In each figure, the orange histogram is the histogram of generated data, while the red line represents the PDF of the ground truth of target distribution.

Figure 12: The sequences of learning result of IPOT, Sinkhorn. In each figure, the orange dots are samples of generated data, while the contour represents the ground truth distribution.

Here is the full derivation of Wasserstein barycenter. Given a set of distributions \( P = \{p_1, p_2, \ldots, p_K\} \), Wasserstein barycenter is defined as

\[
q^\ast(P, \lambda) = \arg \min_q \sum_{k=1}^K \lambda_k W(q, p_k)
\]  

(10)

where \( W \) is the Wasserstein distance, and \( q \) is in the space of probability distributions, and \( \sum_{k=1}^K \lambda_k = 1 \).

Denote

\[
B(P, \lambda) = \min_q \sum_{k=1}^K \lambda_k W(q, p_k).
\]  

(11)

Recall the Wasserstein distance takes the form

\[
W(q, p_k) = \min_{\Gamma} \langle \Gamma, C \rangle, \quad \text{s.t.} \quad \Gamma 1 = p_k, \ \Gamma^T 1 = q.
\]  

(12)

Substitute (12) into (11), we have

\[
B(P, \lambda) = \min_q \sum_{k=1}^K \lambda_k \min_{\Gamma_k} \langle \Gamma_k, C \rangle, \quad \text{s.t.} \quad \Gamma_k 1 = p_k, \ \Gamma_k^T 1 = q
\]

\[
= \min_{\Gamma_k} \sum_{k=1}^K \lambda_k \langle \Gamma_k, C \rangle, \quad \text{s.t.} \quad \Gamma_k 1 = p_k, \ \text{and} \ \exists q, \ \Gamma_k^T 1 = q.
\]
Analogous to IPOT, the proximal point iteration for barycenter is

\[
\Gamma_{k}^{(t+1)} = \arg \min_{\Gamma_k} \sum_{k=1}^{K} \lambda_k \{ \langle \Gamma_k, C \rangle + \beta(t) D_h(\Gamma_k, \Gamma_k(t)) \} \\
\text{s.t.} \quad \Gamma_k 1 = p_k, \text{ and } \exists q, \Gamma_k^T 1 = q
\]

The algorithm is shown in 3. In programming, the steps can be modified to have better efficiency. Please see our code for detail.

## D Color Transferring

Optimal transport is directly applicable to many applications, such as color transferring and histogram calibration. We will show the result of color transferring and why accurate transportation map is superior to entropically regularized ones.

The goal of color transferring is to transfer the tonality of a target image into a source image. This is usually done by imposing the histogram of the color palette of one image to another image. Since Reinhard et al. [24], many methods [22, 37] are developed to do so by learning the transformation between the two histograms. Experiments in [27] have shown that transformation based on optimal transport map outperforms state-of-the-art techniques for challenging images.

Same as other prime-form Wasserstein distance solvers [20, 7], the proximal point method provide a transportation map. By definition, the map is a transportation from the source distribution to a target one with minimum cost. Therefore it provides a way to transform a histogram to another.

One example is shown in figure 14. We use three different maps to transform the RGB channels, respectively. For each channel, there are at most 256 bins. Therefore, using three channels separately is more efficient than treating the colors as 3D data. Figure 14 shows proximal point method can produce identical result as linear programming at convergence, while the results produced by Sinkhorn method differ w.r.t. $\epsilon$. 
In the main body of the paper, we discussed the proximal point algorithm with specific Bregman distance, which is generated through the traditional entropy function. In this section, we generalize our results by proving the effectiveness of proximal point algorithm with general Bregman distance. Bregman distance is applied to measure the discrepancy between different matrices which turns out to be one of the key ideas in regularized optimal transport problems. Its special structure also give rise to proximal-type algorithms and projectors in solving optimization problems.

E.1 Basic Algorithm

The fundamental iterative scheme of general Bregman proximal point algorithm can be denoted as

$$x^{(t+1)} = \arg \min_{x \in X} \left\{ f(x) + \beta^t D_h(x, x^{(t)}) \right\},$$

(13)

where $t \in \mathbb{N}$ is the index of iteration, and $D_h(x, x^{(t)})$ denotes a general Bregman distance between $x$ and $x^{(t)}$ based on a Legendre function $h$ (The definition is presented in the following). In the main body of the paper, $h$ is specialized as the classical entropy function and as follows the related Bregman distance reduces to the generalized KL divergence. Furthermore, the Sinkhorn-Knopp projection can be introduced to compute each iterative subproblem. In the following, we present some fundamental definitions and lemmas.

**Definition E.1.** Legendre function: Let $h : X \to (-\infty, \infty]$ be a lsc proper convex function. It is called...
1. Essentially smooth: if \( h \) is differentiable on int \( \text{dom} \, h \), with moreover \( \| \nabla h(x(t)) \| \to \infty \) for every sequence \( \{x(t)\} \subset \text{int} \, \text{dom} \, h \) converging to a boundary point of \( \text{dom} \, h \) as \( t \to +\infty \);

2. Legendre type: if \( h \) is essentially smooth and strictly convex on \( \text{int} \, \text{dom} \, h \).

**Definition E.2. Bregman distance:** any given Legendre function \( h \),
\[
D_h(x, y) = h(x) - h(y) - \langle \nabla h(y), x - y \rangle, \quad \forall x \in \text{dom} \, h, \forall y \in \text{int} \, \text{dom} \, h, \tag{14}
\]
where \( D_h \) is strictly convex with respect to its first argument. Moreover, \( D_h(x, y) \geq 0 \) for all \( (x, y) \in \text{dom} \, h \times \text{int} \, \text{dom} \, h \), and it is equal to zero if and only if \( x = y \). However, \( D_h \) is in general asymmetric, i.e., \( D_h(x, y) \neq D_h(y, x) \).

**Definition E.3. Symmetry Coefficient:** Given a Legendre function \( h : X \to (-\infty, \infty] \), its symmetry coefficient is defined by
\[
\alpha(h) = \inf \left\{ \frac{D_h(x, y)}{D_h(y, x)} \mid (x, y) \in \text{int} \, \text{dom} \, h \times \text{int} \, \text{dom} \, h, \ x \neq y \right\} \in [0, 1]. \tag{15}
\]

**Lemma E.4.** Given \( h : X \to (-\infty, +\infty), D_h \) is general Bregman distance, and \( x, y, z \in X \) such that \( h(x), h(y), h(z) \) are finite and \( h \) is differentiable at \( y \) and \( z \),
\[
D_h(x, z) - D_h(x, y) - D_h(y, z) = \langle \nabla h(y) - \nabla h(z), x - y \rangle \tag{16}
\]

**Proof.** The proof is straightforward as one can easily verify it by simply subtracting \( D_h(y, z) \) and \( D_h(x, y) \) from \( D_h(x, z) \).

### E.2 Theorem 3.1 and Theorem 3.2

In this section, we first establish the convergence of Bregman proximal point algorithm, i.e., **Theorem 3.1**, while our analysis is based on \([10, 11, 34]\). Further, we establish the convergence of inexact version Bregman proximal point algorithm, i.e., **Theorem 3.2**, in which the subproblem in each iteration is computed inexactly within finite number of sub-iterations. Before proving both theorems, we propose several fundamental lemmas. The first Lemma is the fundamental descent lemma, which is popularly used to analysis the convergence result of first-order methods.

**Lemma E.5.** (Descent Lemma) Consider a closed proper convex function \( f : X \to (-\infty, \infty] \) and for any \( x \in X \) and \( \beta^{(t)} > 0 \), we have:
\[
f(x^{(t+1)}) \leq f(x) + \beta^{(t)} \left[ D_h(x, x^{(t)}) - D_h(x, x^{(t+1)}) - D_h(x^{(t+1)}, x^{(t)}) \right], \quad \forall x \in X. \tag{17}
\]

**Proof.** The optimality condition of (13) can be written as
\[
(x - x^{(t+1)})^T \left[ \nabla f(x^{(t+1)}) + \beta^{(t)} \left( \nabla h(x^{(t+1)}) - \nabla h(x^{(t)}) \right) \right] \geq 0, \quad \forall x \in X.
\]

Then with the convexity of \( f \), we obtain
\[
f(x) - f(x^{(t+1)}) + \beta^{(t)} \left( x - x^{(t+1)} \right)^T \left( \nabla h(x^{(t+1)}) - \nabla h(x^{(t)}) \right) \geq 0. \tag{18}
\]

With (16) it follows that
\[
(x - x^{(t+1)})^T \left( \nabla h(x^{(t+1)}) - \nabla h(x^{(t)}) \right) = D_h(x, x^{(t)}) - D_h(x, x^{(t+1)}) - D_h(x^{(t+1)}, x^{(t)}).
\]

Substitute the above equation into (18), we have
\[
f(x^{(t+1)}) \leq f(x) + \beta^{(t)} \left[ D_h(x, x^{(t)}) - D_h(x, x^{(t+1)}) - D_h(x^{(t+1)}, x^{(t)}) \right], \quad \forall x \in X. \tag{19}
\]

Next, we prove the convergence result in **Theorem 3.1**.

**Theorem 3.1** Let \( \{x^{(t)}\} \) be the sequence generated by the general Bregman proximal point algorithm with iteration (13) where \( f \) is assumed to be continuous and convex. Further assume that \( f^* = \min f(x) > -\infty \). Then we have that \( \{f(x^{(t)})\} \) is non-increasing, and \( f(x^{(t)}) \to f^* \). Further assume there exists \( \eta \), s.t.
\[
f^* + \eta d(x) \leq f(x), \quad \forall x \in X, \tag{19}
\]

The algorithm has linear convergence.
Proof. 1. First, we prove the sufficient decrease property:
\[ f(x^{(t+1)}) \leq f(x^{(t)}) - \beta(t)(1 + \alpha(h))D_h(x^{(t+1)}, x^{(t)}). \quad (20) \]
Let \( x = x^{(t)} \) in (17), we obtain
\[
\begin{align*}
  f(x^{(t+1)}) & \leq f(x^{(t)}) - \beta(t) \left[D_h(x^{(t)}, x^{(t+1)}) + D_h(x^{(t+1)}, x^{(t)})\right] \\
  & \leq f(x^{(t)}) - \beta(t)(1 + \alpha(h))D_h(x^{(t+1)}, x^{(t)}).
\end{align*}
\]
With the sufficient decrease property, it is obvious that \{f(x^{(t)})\} is non-decreasing.

2. Summing (20) from \( i = 0 \) to \( i = t - 1 \) and for simplicity assuming \( \beta(t) = \beta \), we have
\[
\sum_{i=0}^{k-1} \frac{1}{\beta(t)} \left(f(x^{(i+1)}) - f(x^{(i)})\right) \leq \left[-1 + \alpha(h)\right] \sum_{i=0}^{k-1} D_h(x^{(i+1)}, x^{(i)})
\]
\[ \Rightarrow \sum_{i=0}^{\infty} D_h(x^{(i+1)}, x^{(i)}) < \frac{1}{\beta(1 + \alpha(h))} f(x^{(0)}) < \infty, \]
which indicates that \( D_h(x^{(i+1)}, x^{(i)}) \to 0 \). Then summing (17) from \( i = 0 \) to \( i = t - 1 \), we have
\[
 k \left(f(x^{(t)}) - f(x)\right) \leq \sum_{i=0}^{t-1} \left(f(x^{(i+1)}) - f(x)\right) \leq \beta D_h(x, x^{(0)}) < \infty, \quad \forall x \in X.
\]
Let \( t \to \infty \), we have \( \lim_{t \to \infty} f(x^{(t)}) \leq f(x) \) for every \( x \), as a result we have \( \lim_{t \to \infty} f(x^{(t)}) = f^* \).

3. Finally, we prove the convergence rate is linear. Assume \( x^* = \arg \min f(x) \) is the unique optimal solution. Denote \( d(x) = D_h(x^*, x) \). Let also \( \beta(t) = \beta \), we will prove
\[
\frac{d(x^{(t+1)})}{d(x^{(t)})} \leq \frac{1}{1 + \frac{\eta}{\beta}} \quad (21)
\]
Replace \( x \) with \( x^* \) in inequality (17), we have
\[
f(x^{(t+1)}) \leq f^* + \beta \left[d(x^{(t)}) - d(x^{(t+1)}) - D_h(x^{(t+1)}, x^{(t)})\right]. \quad (22)
\]
Using assumption (19), we have
\[
f^* + \eta d(x^{(t+1)}) \leq f(x^{(t+1)}) \quad (23)
\]
Sum 22 and 23 up, we have
\[
\frac{\eta}{\beta} d(x^{(t+1)}) \leq d(x^{(t)}) - d(x^{(t+1)}) - D_h(x^{(t+1)}, x^{(t)}) \leq d(x^{(t)}) - d(x^{(t+1)})
\]
Therefore,
\[
\frac{d(x^{(t+1)})}{d(x^{(t)})} \leq \frac{1}{1 + \frac{\eta}{\beta}}
\]
Therefore, we have a linear convergence in Bregman distance sense.

Unfortunately, assumption (19) does not always hold when \( f \) is linear. However, in our specific case, \( \Gamma \) is bounded in \([0, 1]^n \times n\). In this case, for \( \eta \) to exist, we only need
\[
\lim_{r \to \Gamma^*} \frac{D_h(\Gamma^*, \Gamma)}{\langle C, \Gamma^* - \Gamma \rangle} < \infty
\]
Easy to prove the above holds.

Inequality (21) shows how the convergence rate is linked to \( \beta \). This is the reason we claim in Section 4.1 that a smaller \( \beta \) would lead to quicker convergence in exact case.

From above, we showed that the general Bregman proximal point algorithm with constant stepsize can guarantee convergence to the optimal solution \( f^* \), and has linear convergence rate with some assumptions. Further, we prove the convergence result for the general Bregman proximal point algorithm with inexact scheme in Theorem 3.2.
Theorem 3.2 Let \( \{x^{(t)}\} \) be the sequence generated by the general Bregman proximal point algorithm with inexact scheme (i.e., finite number of inner iterations are employed). Define an error sequence \( \{e^{(t)}\} \) where

\[
e^{(t+1)} \in \beta^{(t)} \left[ \nabla f(x^{(t+1)}) + \partial_{X} (\{x^{(t+1)}\}) \right] + \left[ \nabla h(x^{(t+1)}) - \nabla h(x^{(t)}) \right],
\]

where \( \partial_{X} \) is the indicator function of set \( X \). If the sequence \( \{e^{(t)}\} \) satisfies \( \sum_{t=1}^{\infty} ||e^{(t)}|| < \infty \) and \( \sum_{t=1}^{\infty} (e^{(t)}, x^{(t)}) \) exists and is finite, then \( \{x^{(t)}\} \) converges to \( x^{\infty} \) with \( f(x^{\infty}) = f^{*} \). If the sequence \( \{e^{(t)}\} \) satisfies that exist \( \rho \in (0, 1) \) such that \( ||e^{(t)}|| \leq \rho^{t} \), \( \langle e^{(t)}, x^{(t)} \rangle \leq \rho^{t} \) and with assumption \( (19) \), then \( \{x^{(t)}\} \) converges linearly.

Remark: If exact minimization is guaranteed in each iteration, the sequence \( \{x^{(t)}\} \) will satisfy that

\[
0 \in \beta^{(t)} \left[ \nabla f(x^{(t+1)}) + \partial_{X} (x^{(t+1)}) \right] + \frac{1}{\beta^{(t)}} \left[ \nabla h(x^{(t+1)}) - \nabla h(x^{(t)}) \right].
\]

As a result, with enough inner iteration, the guaranteed \( e^{(t)} \) will go to zero.

Proof. This theorem is extended from [11, Theorem 1], and we propose a brief proof here. The proof contains the following four steps:

1. We have for all \( k \geq 0 \), through the three point lemma

\[
D_{h}(x, x^{(t+1)}) = D_{h}(x, x^{(t)}) - D_{h}(x^{(t+1)}, x^{(t)}) - \langle \nabla h(x^{(t)}) - \nabla h(x^{(t+1)}), x^{(t+1)} - x \rangle,
\]

which indicates

\[
D_{h}(x, x^{(t+1)}) = D_{h}(x, x^{(t)}) - D_{h}(x^{(t+1)}, x^{(t)}) - \langle \nabla h(x^{(t)}) - \nabla h(x^{(t+1)}), e^{(t+1)} + x^{(t)} - x \rangle + \langle e^{(t+1)}, x^{(t+1)} - x \rangle.
\]

Since \( \frac{1}{\beta^{(t)}} \left( \nabla h(x^{(t)}) - \nabla h(x^{(t+1)}) + e^{(t+1)} + x^{(t+1)} - x \right) \) \( \in \nabla f(x^{(t+1)}) + \partial_{X}(x^{(t+1)}) \) and \( 0 \in \nabla f(x^{*}) + \partial_{X}(x^{*}) \) if \( x^{*} \) be the optimal solution, we have

\[
\langle \nabla h(x^{(t)}) - \nabla h(x^{(t+1)}) + e^{(t+1)} + x^{(t+1)} - x \rangle = \beta^{(t)} \left( \frac{1}{\beta^{(t)}} \left( \nabla h(x^{(t)}) - \nabla h(x^{(t+1)}) + e^{(t+1)} \right) \right) - 0, x^{(t+1)} - x^{*} \geq 0,
\]

because \( \nabla f + \partial_{X} \) is monotone (\( f + \partial_{X} \) is convex). Further we have

\[
D_{h}(x^{*}, x^{(t+1)}) \leq D_{h}(x^{*}, x^{(t)}) - D_{h}(x^{(t+1)}, x^{(t)}) + \langle e^{(t+1)}, x^{(t+1)} - x^{*} \rangle.
\]

2. Summing the above inequality from \( i = 0 \) to \( i = t - 1 \), we have

\[
D_{h}(x^{*}, x^{(t)}) \leq D_{h}(x^{*}, x^{(0)}) - \sum_{i=0}^{t-1} D_{h}(x^{(i+1)}, x^{(i)}) + \sum_{i=0}^{t-1} \langle e^{(i+1)}, x^{(i+1)} - x^{*} \rangle.
\]

Since \( \sum_{t=1}^{\infty} ||e^{(t)}|| < \infty \) and \( \sum_{t=1}^{\infty} \langle e^{(t)}, x^{(t)} \rangle \) exists and is finite, we guarantee that

\[
\bar{E}(x^{*}) = \sup_{t \geq 0} \left\{ \sum_{i=0}^{t-1} \langle e^{(i+1)}, x^{(i+1)} - x^{*} \rangle \right\} < \infty,
\]

Together with \( D_{h}(x^{(i+1)}, x^{(i)}) > 0 \), we have

\[
D_{h}(x^{*}, x^{(t)}) \leq D_{h}(x^{*}, x^{(0)}) + \bar{E}(x^{*}) < \infty,
\]

which indicates

\[
0 \leq \sum_{i=0}^{\infty} D_{h}(x^{(i+1)}, x^{(i)}) \leq D_{h}(x^{*}, x^{(0)}) + \bar{E}(x^{*}) < \infty,
\]

and hence \( D_{h}(x^{(i+1)}, x^{(i)}) \to 0 \).

3. Based on the above two items, we know that the sequence \( \{x^{(t)}\} \) must be bounded and has at least one limit point \( x^{\infty} \). The most delicate part of the proof is to establish that \( 0 \in \nabla f(x^{\infty}) + \partial_{X}(x^{\infty}) \). Let \( T = \nabla f + \partial_{X} \), then \( T \) denotes the subdifferential mapping of a closed proper convex function \( f + \partial_{X} \) (\( f \) is a linear function and \( X \) is a closed convex set). Let \( \{x^{(i)}\} \) be the sub-sequence such that \( x^{(i)} \to x^{\infty} \). Because \( x^{(i)} \in X \) and \( X \) is a closed convex set, we know \( x^{\infty} \in X \). We know that
Based on the above four items, we guarantee the convergence results in this theorem. Let

\[ D_h(x^*, x^{(t+1)}) \leq D_h(x^*, x^{(t)}) + (e^{(t+1)}, x^{(t+1)} - x^*) \] and \( \sum_{k=0}^{\infty} (e^{(t+1)}, x^{(t+1)} - x^*) \) exists and is finite. From [21, Section 2.2], we guarantee that \( \{D_h(x^*, x^{(t)})\} \) converges to \( 0 \leq d(x^*) < \infty \).

Define \( y^{(t+1)} := \lambda_k \left( \nabla h(x^{(t)}) - \nabla h(x^{(t+1)}) + e^{(t+1)} \right) \), we have

\[ \lambda_k(y^{(t+1)}, x^{(t+1)} - x^*) = D_h(x^*, x^{(t)}) - D_h(x^*, x^{(t+1)}) - D_h(x^{(t+1)}, x^{(t)}) + (e^{(t+1)}, x^{(t+1)} - x^*). \]

By taking the limit of both sides and \( \lambda_k = \lambda > 0 \), we obtain that

\[ \langle y^{(t+1)}, x^{(t+1)} - x^* \rangle \to 0. \]

For the reason that \( y^{(t+1)} \) is a subgradient of \( f + \lambda x \) at \( x^{(t+1)} \), we have

\[ f(x^*) \geq f(x^{(t+1)}) + \langle y^{(t+1)}, x^* - x^{(t+1)} \rangle, \quad x^* \in X, \quad \lambda \in \mathbb{R}^+ \]

Further let \( \lambda \to \infty \) and using \( f \) is lower semicontinuous, \( \langle y^{(t+1)}, x^{(t+1)} - x^* \rangle \to 0 \), we obtain

\[ f(x^*) \geq f(x^{\infty}), \quad x^{\infty} \in X \]

which implies that \( 0 \in \nabla f(x^{\infty}) + \lambda x(x^{\infty}) \).

4. Recall the inexact scheme (24), we can equivalently guarantee that

\[ \langle x - x^{(t+1)} \rangle^T \left\{ \beta^{(t)} \nabla f(x^{(t+1)}) + \left[ \nabla h(x^{(t+1)}) - \nabla h(x^{(t)}) \right] - e^{(t+1)} \right\} \geq 0, \forall x \in X. \]

Together the convexity of \( f \) and the three point lemma, we obtain

\[ f(x^{(t+1)}) \leq f(x) + \frac{1}{\beta^{(t)}} \left[ D_h(x, x^{(t)}) - D_h(x, x^{(t+1)}) - D_h(x^{(t+1)}, x^{(t)}) - (x - x^{(t+1)})^T e^{(t+1)} \right]. \]

Let \( x = x^* \) in the above inequality and recall the assumption (19), i.e.,

\[ f(x) - f(x^*) \geq \eta d(x), \]

we have with \( \beta^{(t)} = \beta \)

\[ \eta d(x^{(t+1)}) \leq \frac{1}{\beta} \left[ d(x^{(t)}) - d(x^{(t+1)}) \right] + \frac{1}{\beta} \left( (x^{(t+1)} - x^*)^T e^{(t+1)} \right) \]

\[ \leq \frac{1}{\beta} \left[ d(x^{(t)}) - d(x^{(t+1)}) \right] + \frac{1}{\beta} \left( C||e^{(t+1)}|| + (x^{(t+1)}, e^{(t+1)}) \right), \]

where \( C := \sup_{x \in X} \{||x||\} \). The second inequality is obtained through triangle inequality. Then

\[ d^{(t+1)} \leq \mu d^{(t)} + \mu \left( C||e^{(t+1)}|| + (x^{(t+1)}, e^{(t+1)}) \right), \]

where \( \mu = \frac{1}{\beta^{(t)}} < 1 \). With our assumptions and according to Theorem 2 and Corollary 2 in [30], we guarantee the generated sequence converges linearly in the order of \( O\left(e^t\right) \), where \( e = \sqrt{\frac{1+\max_{\nu,\rho}}{2}} \in (0, 1) \).

Based on the above four items, we guarantee the convergence results in this theorem. ☐