On the thermodynamic origin of the Hawking entropy and a measurement of the Hawking temperature

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Abstract

In the spherically symmetric case the Einstein field equations take on their simplest form for a matter-density $\rho = 1/(8\pi r^2)$, from which a radial metric coefficient $g_{rr} \propto r$ follows. The boundary of an object with such an interior matter-density is situated slightly outside of its gravitational radius. Its surface-redshift scales with $z \propto \sqrt{r}$, so that any such large object is practically indistinguishable from a black hole, as seen from exterior space-time.

The interior matter has a well defined temperature, $T \propto 1/\sqrt{r}$. Under the assumption, that the interior matter can be described as an ultra-relativistic gas, the object’s total entropy and its temperature at infinity can be calculated by microscopic statistical thermodynamics. They are equal to the Hawking result up to a possibly different constant factor.

The simplest solution of the field equations with $\rho = 1/(8\pi r^2)$ is the so called holographic solution, short "holostar". It has an interior string equation of state. The strings are densely packed, explaining why the solution does not collapse to a singularity. The holographic solution has been shown to be a very accurate model for the universe as we see it today in Ref\[7\].

The factor relating the holostar’s temperature at infinity to the Hawking temperature can be expressed in terms the holostar’s interior (local) radiation temperature and its (local) matter-density, allowing an experimental verification of the Hawking temperature law. Using the recent experimental data for the CMBR-temperature and the total matter-density in the universe measured by WMAP, the Hawking formula is verified to an accuracy better than 1%.

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1 Introduction:

In [6] several new exact solutions to the Einstein field equations were derived. These solutions are characterized by a spherical boundary membrane, consisting out of tangential pressure.

One of the new solution turned out to be of particular interest. The so-called holographic solution is characterized by the property that its boundary membrane carries a stress-energy-content equal to its gravitating mass. The membrane’s pressure is equal to the pressure of the - fictitious - membrane attributed to a black hole by the membrane paradigm. This guarantees, that the holostar’s action on the exterior space-time is by all practical purposes identical to that of a black hole.

The holostar’s geometric properties have been discussed extensively in [7]. The holographic solution has no free parameters, yet it turned out to be an astoundingly accurate description of the universe as we see it today.

The holostar’s interior matter-state can be interpreted as a collection of radially outlayed strings, attached to the holostar’s spherical boundary membrane [9]. The interior strings are densely packed, their mutual transverse separation is exactly one Planck area. This dense package explains, why the holostar doesn’t collapse to a singularity, although its boundary membrane lies just roughly two Planck coordinate distances outside of its gravitational radius.

Although the holostar’s total interior matter-density has a definite string character, at least part of the matter can be interpreted in terms of particles. In this paper a simple thermodynamic model for the interior matter state is explored, which allows us to derive the Hawking entropy and temperature relations for a spherically symmetric black hole by microscopic statistical thermodynamics in the ideal gas approximation.

2 A short introduction to the holographic solution

The holographic solution is an exact solution to the Einstein field equations with zero cosmological constant. The spherically symmetric metric of the holographic solution has been derived in [6]:

\[
\begin{align*}
 ds^2 &= g_{tt}(r)dt^2 - g_{rr}(r)dr^2 - r^2 d\Omega^2 \\
 g_{tt}(r) &= 1/g_{rr}(r) = \frac{r_0}{r} (1 - \theta(r - r_h)) + (1 - \frac{r_+}{r}) \theta(r - r_h) \\
 r_h &= r_+ + r_0 \\
 r_+ &= 2M
\end{align*}
\]
All quantities are expressed in geometric units \( c = G = 1 \). For clarity \( \hbar \) will be shown explicitly. \( \theta \) and \( \delta \) are the Heavyside-step functional and the Dirac-delta functional respectively. \( r_h \) denotes the radial coordinate position of the holostar’s surface, which divides the space-time manifold into an interior source region with a non-zero matter-distribution and an exterior vacuum space-time. \( r_+ \) is the radial coordinate position of the gravitational radius (Schwarzschild radius) of the holostar. \( r_+ \) is directly proportional to the gravitating mass \( M = r_+/2 \). \( r_0 \) is a fundamental length parameter.\(^1\)

The matter fields (mass density, principal pressures) of any spherically symmetric gravitationally bound object can be derived from the metric by simple differentiation (see for example [6]). For the discussion in this paper only the radial metric coefficient \( g_{rr}(r) \) is essential. In the spherically symmetric case the total mass-energy density \( \rho \) can be calculated solely from the radial metric coefficient. For any spherically symmetric self gravitating object the following general relation holds:

\[
\left( \frac{r}{g_{rr}} \right)' = 1 - 8\pi r^2 \rho \tag{3}
\]

It is obvious from the above equation, that a matter-density \( \rho = 1/(8\pi r^2) \) is special. It renders the differential equation for \( g_{rr} \) homogeneous and leads to a strictly linear dependence between \( g_{rr} \) and the radial distance coordinate \( r \).

With \( g_{rr} \) given by equation (2) the energy-density turns out to be:

\[
\rho(r) = \frac{1}{8\pi r^2} (1 - \theta(r - r_h)) \tag{4}
\]

Within the holostar’s interior the mass-energy density follows an inverse square law. Outside of the membrane, i.e. for \( r > r_h \), it is identical zero. Note, that \( r_h \) must not necessarily be finite.

In the following discussion the argument \((r - r_h)\) of the \( \theta \)- and \( \delta \)-distributions will be omitted.

The radial and tangential pressures also follow from the metric:

\[
P_r = -\rho = -\frac{1}{8\pi r^2} (1 - \theta) \tag{5}
\]

\[
P_\theta = P_\phi = \frac{1}{16\pi r_h} \delta \tag{6}
\]

\( P_r \) is the radial pressure. It is equal in magnitude but opposite in sign to the mass-density. \( P_\theta \) denotes the tangential pressure, which is zero everywhere, except for a \( \delta \)-functional at the holostar’s surface. The ”stress-energy-content” of the two principal tangential pressure components in the membrane is equal to the gravitating mass \( M \) of the holostar.

\(^1\)\( r_0 \) has been assumed to be roughly twice the Planck-length in [6, 7]. The analysis in [5] indicates \( r_0^2 \approx 4\sqrt{3}/4 \) at low energies. In this paper a more definite relationship in terms of the total number of particle degrees of freedom at high temperatures will be derived.
In order to determine the principal pressures from the metric, the time-coefficient of the metric $g_{tt}$ must be known. For the holostar equation of state with $P_r = -\rho$ we have $g_{tt} = 1/g_{rr}$. Other equations of state lead to different time-coefficients, and therefore different principal pressures.

Neither the particular form of the time-coefficient of the metric, nor the particular form of the principal pressures are important for the main results derived in this paper, which are based on equilibrium thermodynamics, where time evolution is irrelevant (as long as the relevant time scale is long enough, that thermal equilibrium can be attained). The essential assumptions are:

- spherical symmetry
- a radial metric coefficient $g_{rr} = r/r_0$
- a total energy density $\rho = 1/(8\pi r^2)$
- microscopic statistical thermodynamics of an ideal gas of ultra-relativistic fermions and bosons (in the context of the grand-canonical ensemble)

If the validity of Einstein’s field equations with zero cosmological constant is assumed, conditions two and three are interchangeable.

Throughout this paper I will frequently use the term holographic solution, or holostar, to refer to an object with the above stated properties. The reader should keep in mind, though, that the holographic solution is just a special case of a solution with a matter-density $\rho = 1/(8\pi r^2)$. The results derived in this paper refer to any solution with the above properties.

In the following sections I assume that $r_0^2$ is nearly constant, i.e. more or less independent of the size of the holostar and comparable to the Planck area $A_{Pl} = \hbar$:

$$r_0^2 = \beta r_{Pl}^2 = \beta \hbar$$  \hspace{1cm} (7)

This assumption will be justified later.

### 3 A simple derivation of the Hawking temperature and entropy

The interior metric of the holostar solution is well behaved and the interior matter-density is non-zero. The solution is static: The matter appears to exert a radial pressure preventing further collapse to a point singularity. This can be best seen in the string picture \[9\]. However, the solution gives no direct indication with respect to the state of the interior matter and the origin of the pressure.

In this section I will discuss a very simple model for the interior matter state of the holostar, which is able to explain many phenomena attributed to black holes. Let us assume that the interior matter distribution is dominated by ultra-relativistic weakly interacting fermions and the pressure is produced
by the exclusion principle. Due to spherical symmetry the mean momentum of the fermions \( p(r) \) and their number density per proper volume \( fn(r) \) will only depend on the radial distance coordinate \( r \). \( f \) denotes the effective number of degrees of freedom of the fermions. For ultra-relativistic fermions the local energy-density will be given by the product of the number density of the fermions and their mean momentum. This energy density must be equal to the interior mass-energy density of the holostar:

\[
\rho = p(r)fn(r) = \frac{1}{8\pi r^2} \quad (8)
\]

If the fermions interact only weakly, their mean momenta can be estimated by the exclusion principle:

\[
p(r)^3 \frac{1}{n(r)} = (2\pi\hbar)^3 \quad (9)
\]

These two equations can be solved for \( p(r) \) and \( n(r) \):

\[
p(r) = \frac{\hbar^2\pi^{\frac{3}{2}}}{f\frac{4}{3}} \frac{1}{r^\frac{3}{2}} \quad (10)
\]

\[
fn(r) = \frac{f\frac{4}{3}}{\hbar^28\pi^{\frac{3}{2}}} \frac{1}{r^\frac{3}{2}} \quad (11)
\]

The mean momenta of the fermions within the holostar fall off from the center as \( 1/r^{1/2} \) and the number density per proper volume with \( 1/r^{3/2} \). Similar dependencies, however without definite factors, have already been found in [7] by analyzing the geodesic motion of the interior massless particles in the holostarmetric. It is remarkable, that equilibrium thermodynamics combined with the uncertainty principle gives the same results as the geodesic equations of motion. This is not altogether unexpected. In [4] it has been shown, that the field equations of general relativity follow from thermodynamics and the Bekenstein entropy bound [I].

The momentum of the fermions at a Planck-distance \( r = r_{Pl} = \sqrt{\hbar} \) from the center of the holostar is of the order of the Planck-energy \( E_{Pl} = \sqrt{\hbar} \). It is also interesting to note, that for both quantities \( p(r) \) and \( n(r) \) the number of degrees of freedom \( f \) can be absorbed in the radial coordinate value \( r \rightarrow \sqrt{f} r \), so that \( p \) and \( n \) effectively depend on \( \sqrt{f} r \). We will see later that the square root of \( f \) plays an important role in the scaling of the fundamental length parameter \( r_0 \).

From (10) one can derive the following momentum-area law for holostars, which resembles the Stefan-Boltzmann law for radiation from a black body:

\[
p(r)^4r^2f = \hbar^3\pi^2 \quad (12)
\]

Note that this law not only refers to the holostar’s surface \( (r = r_h) \) but is valid for any concentric spherical surface of radius \( r \) within the holostar. Therefore it is reasonable to assume that the holostar has a well defined interior temperature \( T(r) \) proportional to the mean momentum \( p(r) \):
\[ p(r) = \sigma T(r) \]  

(13)

\( \sigma \) is a constant factor. We will see later, that it is related to the entropy per particle.

The local surface temperature of the holostar is given by:

\[ T(r_h) = \frac{p(r_h)}{\sigma} = \frac{\hbar^2}{\sigma f^2} \frac{1}{\sqrt{r_h}} \]  

(14)

The surface redshift \( z \) is given by:

\[ z = \frac{1}{\sqrt{g_{tt}(r_h)}} = \sqrt{g_{rr}(r_h)} = \sqrt{\frac{r_h}{r_0}} \]  

(15)

where \( g_{tt}(\infty) = 1 \) is assumed.

The local surface temperature can be compared to the Hawking temperature of a black hole. The Hawking temperature is measured at infinity. Therefore the red-shift of the radiation emitted from the holostar’s surface with respect to an observer at spatial infinity has to be taken into account, by dividing the local temperature at the surface by the gravitational red shift factor \( z \). With \( g_{rr}(r_h) = r_h^{1/2}(\beta \bar{h})^{-1/4} \) we find:

\[ T_\infty = \frac{T(r_h)}{\sqrt{g_{rr}(r_h)}} = \frac{\pi^{1/2}}{\sigma} \left( \frac{\beta}{\bar{f}} \right)^{1/4} \frac{\hbar}{r_h} \]  

(16)

The surface-temperature measured at infinity has the same dependence on the gravitational radius \( r_h \) as the Hawking temperature, which is given by:

\[ T_H = \frac{\hbar}{4\pi r_h} = \frac{\hbar}{8\pi M} \]  

(17)

Up to a possibly different constant factor the Hawking temperature of a spherically symmetric black hole and the respective temperature of the holostar at infinity are equal.

As the Hawking temperature of a black hole only depends on the properties of the exterior space-time, and the exterior space-times of a black hole and the holostar are equal (up to a small Planck-sized region outside the horizon), it is reasonable to assume that the Hawking temperature should be the true temperature of a holostar measured at spatial infinity. With this assumption, the constant \( \sigma \) can be determined by setting the temperatures of equations (16) and (17) equal:

\[ \sigma = \left( \frac{\beta}{\bar{f}} \right)^{1/4} \frac{4\pi^{3/2}}{\bar{f}^2} \]  

(18)

The total number of fermions within the holostar is given by the proper integral over the number-density:
\[ N = \int f n(r) dV \]  

(19)

\( dV \) is the proper volume element, which can be read off from the metric:

\[ dV = 4\pi r^2 \sqrt{g_{rr}} dr = 4\pi r^{\frac{1}{2}} (\beta h)^{\frac{1}{4}} dr \]  

(20)

Integration over the total interior volume of the holostar gives:

\[ N = \left( \frac{f}{\beta} \right)^{\frac{1}{4}} \frac{1}{4\pi} \frac{\pi r_0^2}{h} = \frac{1}{S} \frac{A}{4h} = \frac{S_{BH}}{\sigma} \]  

(21)

\( S_{BH} \) is the Bekenstein-Hawking entropy for a spherically symmetric black hole with horizon surface area \( A \).

Therefore the number of fermions within the holostar is proportional to its surface area and thus proportional to the Hawking entropy. This result is very much in agreement with the holographic principle [13, 12], giving it quite a new and radical interpretation: The degrees of freedom of a highly relativistic self-gravitating object don’t only “live on the surface”, the object contains a definite number of particles and their total number is proportional to the object’s surface area, measured in units of the Planck-area, \( A_{Pl} = \hbar \). This result is an immediate consequence of the interior metric \( g_{rr} \propto r \), the energy-momentum relation for relativistic particles \( E = p \) and the exclusion principle. It can be easily shown, that for any other spherically symmetric metric, for example \( g_{rr} \propto r^n \), the number of interior (fermionic) particles is not proportional to the boundary area.

From equation (21) we can see that \( \sigma \) is the entropy per particle. This allows a rough estimate of \( \beta \): The entropy of an ultra-relativistic particle should be of order unity (\( \sigma \approx 3 - 4 \)). The degrees of freedom in the Standard Model of particle physics - with the usual counting rule, weighting the fermionic degrees of freedom with \( 7/8 \) - amount to \( f \approx 100 \). Supersymmetry essentially doubles this number. It is expected, that a unified theory will not vastly exceed this number. For \( \sigma = 3 \) and \( f = 256 \) we find \( 4\pi\beta \approx 1.06 \). This justifies the assumption, that the fundamental length parameter \( r_0 \) should be of order Planck-length.

By help of equation (18) the local temperature can be expressed in terms of \( \beta \) alone:

\[ T(r) = \frac{\hbar^2}{4\pi\beta^{\frac{3}{2}} r^2} \frac{1}{\sqrt{4\pi} \frac{\hbar}{(r_0 r)^{\frac{3}{2}}}} \]  

(22)

Note that \( \beta \) depends explicitly on the (effective) number of degrees of freedom \( f \) of the ultra-relativistic particles within the holostar via equation (18). At the center of the holostar all the fermion momenta are comparable to the Planck energy, as can be seen from equation (22). All fermions of the Standard Model of particle physics will be ultra-relativistic. Quite likely there will be other fundamental particles of a grand unified theory (GUT), as well as other
entities such as strings and branes. Thus, close to the holostar’s center the number of ultra-relativistic degrees of freedom will be at its maximum and $\beta$ will be close to unity. The farther one is distanced from the center, the lower the local temperature gets. At $r \approx 10^6 km$ the electrons will become non-relativistic. The only particles of the Standard Model that remain relativistic at larger radial positions will be the neutrinos. If all neutrinos are massive, the mass of the lightest neutrino will define a characteristic radius of the holostar, beyond which there are no relativistic fermions contributing to the holostar’s internal pressure. If at least one of the neutrinos is massless, there will be no limit to the spatial extension of a holostar.

Note, that the radial coordinate position at which the holostar’s interior radiation temperature is equal to the temperature of the cosmic microwave background radiation, $T_{CMBR} = 2.725 K$, corresponds to roughly $r \approx 10^{28} m \approx 10^{12} ly$, i.e. quite close to the radius of the observable universe. This is just one of several coincidences, which point to the very real possibility, that the holostar or a variant thereof actually might serve as an alternative, beautifully simple model for the universe. For a more detailed discussion including some definite cosmological predictions, which are all experimentally verified within an error of maximally 15 % see [7].

Whenever the temperature within the holostar becomes comparable to the mass of a particular fermion species, a phase transition is expected to take place at the respective $r$-position. Such a transition will lower the effective value of $f$, as one of the particles ”freezes” out. Whenever $f$ changes, either $\sigma$ or $\beta$ must adjust due to equation (18). The question is, whether $\sigma$ or $\beta$ (or both) will change. Presumably $\sigma$ will at least approximately retain a constant value: The entropy per ultra-relativistic fermion, as well as the mean particle momentum per temperature, appears to be a local property which should not depend on the (effective) number of degrees of freedom of the particles at a particular $r$-position.

Under the assumption that $\sigma$ is nearly constant, the ratio of $\beta/f$ must be nearly constant as well, as can be seen from equation (18). Whenever $f$ changes, $\beta$ will adjust accordingly. Lowering the effective number of degrees of freedom leads to a flattening of the temperature-curve, as heat (and entropy) is transferred to the remaining ultra-relativistic particles. At any radial position of a phase transition, where a fermion becomes non-relativistic and annihilates with its anti-particle, the temperature is expected to deviate from the expression $T \propto 1/\sqrt{r}$. This is quite similar to what is believed to have happened in the very early universe, when the temperature fell below the electron-mass threshold and the subsequent annihilation of electron/positron pairs heated up the photon gas, keeping the temperature of the expanding universe nearly constant until all positrons were destroyed.

If the ”freeze-out” happens without significant heat and entropy transfer to the remaining gas of ultra-relativistic particles, such as when the particle that ”freezes” out has an appreciable non-zero chemical potential, the effective value of $f$ will remain nearly constant, which would imply that $\beta$ be nearly constant as well. In this case $\beta$ as well as $f$ would be nearly constant universal quantities.
There is evidence that this might actually be the case.\footnote{See \cite{5, 7, 8}.}

4 Thermodynamics of an ultra-relativistic fermion and boson gas

In this section I will discuss a somewhat more sophisticated model for the thermodynamic properties of the holostar.

As has been demonstrated in the previous section, if the holostar contains at least one fermionic species, its properties very much resemble the Schwarzschild vacuum black hole solution, when viewed from the outside: Due to Birkhoff’s theorem the external gravitational field cannot be distinguished from that of a Schwarzschild black hole. Its temperature measured at infinity is proportional to the Hawking temperature.

Due to its non-zero surface-temperature and entropy the holostar will gradually lose particles by emission from its surface. The (exterior) time scale of this process will be comparable to the Hawking evaporation time scale \( \propto r_h^3 \) (see for example \cite{7}). The (exterior) time for a photon to travel radially through the holostar is proportional to \( r_h^2 \). Therefore even comparatively small holostars are expected to have an evaporation time several orders of magnitude longer than their interior relaxation time.

This allows us, with the possible exception of near Planck-size holostars, to consider any spherical thin shell within the holostar’s interior to be in thermal equilibrium with its surroundings. Each shell can exchange particles, energy and entropy with adjacent shells on a time scale much shorter than the life-time of the holostar. Under these assumptions the thermodynamic parameters within each shell can be calculated via the grand canonical ensemble.

We mentally partition the holostar into a collection of thin spherical shells. The temperature scales as \( 1/\sqrt{r} \) and thus varies very slowly with \( r \). For the chemical potential(s) let us assume a slowly varying function with \( r \) as well. This assumption will be justified later. Under these circumstances the thickness of each shell \( \delta r \) can be chosen such, that it is large enough to be considered macroscopic, and at the same time small enough, so that the temperature, pressure and chemical potential(s) are effectively constant within the shell.

An accurate thermodynamic description has to take into account a possible potential energy of position. For the holostar a significant simplification arises from the fact, that the effective potential \( V_{eff}(r) \) for the motion of massless, i.e. ultra-relativistic, particles is nearly constant. The equations of motion for ultra-relativistic particles within the holostar’s interior were given by \cite{7}:

\[
V_{eff}(r) = \frac{r_i^3}{r^3} \quad (23)
\]

and

\footnote{See \cite{5, 7, 8}.}
\[
\beta^2_\perp (r) = \frac{r_i^3}{r^3}
\]  

(24)

\(\beta_\perp(r)\) is the radial velocity of a photon, expressed as a fraction to the local velocity of light in the (purely) radial direction. \(\beta_\perp(r)\) is the tangential velocity of the photon, expressed as a fraction to the local velocity of light in the (purely) tangential direction. \(r_i\) is the turning point of the motion. For pure radial motion \(r_i = 0\). For pure radial motion the effective potential is constant with \(V_{\text{eff}}(r) = 0\). In the case of angular motion \((r_i \neq 0)\) the effective potential approaches zero with \(1/r^3\), i.e. becomes nearly zero very rapidly, whenever \(r\) is greater than a few \(r_i\). Therefore, to a very good approximation we can regard the ultra-relativistic particles to move freely within each shell. Their total energy will only depend on the relativistic energy-momentum relation, not on the radial position.

With these preliminaries the grand canonical potential \(\delta J\) of a small spherical shell of thickness \(\delta r\) for a gas of relativistic fermions at radial position \(r\) will be given by:

\[
\delta J(r) = -T(r) \frac{f}{(2\pi \hbar)^3} \int \int \int d^3p \ln (1 + e^{-\frac{p - \mu(r)}{T(r)}})
\]

\[= -T^4 \delta V \frac{f}{2\pi^2 \hbar^3} \int_0^\infty z^2 \ln (1 + e^{-z + \frac{\mu(r)}{T(r)}}) dz
\]  

(25)

\(z = p/T(r)\) is a dimensionless integration variable. \(\mu(r)\) is the chemical potential at radial coordinate position \(r\). \(T(r)\) is the local temperature at this position.

Note that even when the radial coordinate extension \(\delta r\) of the shell is small, the proper radial extension \(\delta l = (r/r_0)^{1/2} \delta r\) of the shell will become quite large because of the large value of the radial metric coefficient in the holostar’s outer regions.

Knowing the results presented at the end of this section it is not difficult to show that - with the exception of the central region - it is possible to choose the radial extension of the shell such that the number of particles within the shell \(N\) is macroscopic and at the same time \(T(r)\) and \(\mu(r)\) are constant to a very good approximation within the shell. The proper volume of the shell \(\delta V\) is given by the volume element of equation (20).

The ratio of chemical potential \(\mu\) to local temperature \(T\) is assumed to be a very slowly varying function of \(r\). In fact, we will see later that this ratio is virtually independent of \(r\). The ratio \(\mu/T\) will be denoted by \(u\), keeping in mind that \(u\) might depend on \(r\):

\[
u = \frac{\mu(r)}{T(r)}
\]

The integral in equation (25) can be transformed to the following integral by a partial integration:
\[
\delta J(r) = -T^4 \delta V \frac{f}{2\pi^2 \hbar^3} \frac{1}{3} \int_0^\infty z^3 n_F(z, u) dz
\]  
(26)

where \( n_F \) is the mean occupancy number of the fermions:

\[
n_F(z, u) = \frac{1}{e^{z-u} + 1} = \frac{1}{e^{\frac{-\mu}{T}} + 1}
\]  
(27)

Knowing the grand canonical potential \( \delta J \) the entropy within the shell can be calculated:

\[
\delta S(r) = -\frac{\partial (\delta J)}{\partial T} = \frac{f}{2\pi^2 \hbar^3} T^3 \delta V \left( \frac{4}{3} Z_{F,3}(u) - u Z_{F,2}(u) \right)
\]  
(28)

By \( Z_{F,n} \) the following integrals are denoted:

\[
Z_{F,n}(u) = \int_0^\infty z^n n_F(z, u) dz
\]  
(29)

Such integrals commonly occur in the evaluation of Feynman-integrals in QFT and can be evaluated by the poly-logarithmic function \( Li_n(z) \):

\[
Z_{F,n}(u) = -\Gamma(n + 1) Li_{n+1}(-e^u)
\]  
(30)

For the derivation of the entropy the following identity has been used, which is easy to derive from the power-expansion of \( Li_n(z) \).

\[
\frac{\partial Z_{F,3}(u)}{\partial x} = 3 Z_{F,2}(u) \frac{\partial u}{\partial x}
\]  
(31)

The pressure in the shell is given by:

\[
P(r) = -\frac{\partial (\delta J)}{\partial (\delta V)} = \frac{f}{2\pi^2 h^3} T^4 \frac{Z_{F,3}(u)}{3}
\]  
(32)

The total energy in the shell can be calculated from the grand canonical potential via:

\[
\delta E(r) = \delta J - \left( T \frac{\partial}{\partial T} + \mu \frac{\partial}{\partial \mu} \right) \delta J = \frac{f}{2\pi^2 h^3} T^4 \delta V Z_{F,3}(u)
\]  
(33)

The total number of particles within the shell is given by:

\[
\delta N(r) = -\frac{\partial (\delta J)}{\partial \mu} = \frac{f}{2\pi^2 h^3} T^3 \delta V Z_{F,2}(u)
\]  
(34)

The total energy per fermion within the shell is proportional to \( T \), as can be seen by combining equations (33, 34):

\[
\epsilon = \frac{\delta E}{\delta N} = \frac{Z_{F,3}(u)}{Z_{F,2}(u)} T(r)
\]  
(35)
\( \epsilon \) only depends indirectly on \( r \) via \( u \). We will see later that \( u \) is essentially independent of \( r \), so that the mean energy per particle is proportional to the temperature with nearly the same constant of proportionality at any radial position \( r \).

The entropy per particle within the shell can be read off from equations (28, 34):

\[
\sigma = \frac{\delta S}{\delta N} = 4 \frac{Z_{F,1}(u)}{3 Z_{F,2}(u)} - u
\]

Again, \( \sigma \) only depends on \( r \) via \( u \).

The calculations so far have been carried through for fermions. It is likely, that the holostar will also contain bosons in thermal equilibrium with the fermions. The equations for an ultra-relativistic boson gas are quite similar to the above equations for a fermion gas. We have to replace:

\[
n_F(z, u) \rightarrow n_B(z, u) = \frac{1}{e^{z-u} - 1}
\]

\[
Z_{F,n} \rightarrow Z_{B,n} = \int_0^\infty z^n n_B(z, u) dz
\]

\[
Z_{B,n} = \Gamma(n + 1) Li_{n+1}(e^u)
\]

Let us assume that the fermion and boson gases are only weakly interacting. In such a case the extrinsic quantities, such as energy and entropy, can be simply summed up. The same applies for the partial pressures.

The number of degrees of freedom for fermions and bosons can differ. The fermionic degrees of freedom will be denoted by \( f_F \), the bosonic degrees of freedom by \( f_B \). In general, the different particle species will have different values for the chemical potentials. There are some restraints. Bosons cannot have a positive chemical potential, as \( Z_{B,n}(u) \) is a complex number for positive \( u \). Photons and gravitons, in fact all massless gauge-bosons, have a chemical potential of zero, as they can be created and destroyed without being restrained by a particle-number conservation law.

We are however talking of a gas of ultra-relativistic particles. In this case particle-antiparticle pair production will take place abundantly, so that we also have to consider the antiparticles. The chemical potentials of particle and anti-particle add up to zero: \( \mu + \overline{\mu} = 0 \). As bosons cannot have a positive chemical potential, the chemical potential of any ultra-relativistic bosonic species must be zero, i.e. \( \mu_B = \overline{\mu_B} = 0 \), whenever the energy is high enough to create boson/anti-boson pairs. This restriction does not apply to the fermions, which can have a non-zero chemical potential at ultra-relativistic energies, as both signs of the chemical potential are allowed. So for ultra-relativistic fermions we can fulfill the relation \( \mu_F + \overline{\mu_F} = 0 \) with non-zero \( \mu_F \).

It is convenient to use the ratio of the chemical potential to the temperature \( u = \mu/T \) as the relevant parameter instead of the chemical potential itself. If the number of degrees of freedom of fermions and bosons respectively, i.e. \( f_F \)
and $f_B$ is known, there are only two undetermined parameters in the model, $u_F$ and $\beta$. In order to determine $u_F$ and $\beta$ one needs two independent relations. These can be obtained by comparing the holostar temperature and entropy to the Hawking temperature and entropies respectively.

Alternatively $u_F$ can be determined without reference to the Hawking temperature law, solely by a thermodynamic argument. It is also possible to determine $\beta$ by a theoretical argument as proposed in [5].

The thermodynamic energy of a shell consisting of an ultra-relativistic ideal fermion and boson gas is given by:

$$\delta E_{th} = \frac{F_E}{2\pi^2\hbar^3} \delta V T^4$$  \hspace{1cm} (40)

with

$$F_E(u_F) = f_F(Z_{F,3}(u_F) + Z_{F,3}(-u_F)) + 2f_BZ_{B,3}(0)$$  \hspace{1cm} (41)

with the identities of the polylog-function and with $Z_{B,3}(0) = \pi^4/15$ one can express $F_E$ as a quadratic function of $u_F^2/\pi^2$ [10]:

$$F_E(u_F) = 2f_F \frac{\pi^4}{15} \left( \frac{15}{8} \left( 1 + \frac{\pi^2}{u_F^2} \right)^2 + \frac{f_B}{f_F} - 1 \right)$$  \hspace{1cm} (42)

We take the convention here, that $f_F$ and $f_B$ denote the degrees of freedom of one particle species, including particle and antiparticle. With this convention a photon gas ($g = 2$) is described by $f_B = 1$ (There are two photon degrees of freedom and the photon is its own anti-particle). All other particle characteristics, such as helicities, are counted extra. The total number of the degrees of freedom in the gas, i.e. counting particles and anti-particles separately, will be given by

$$f = 2(f_F + f_B)$$  \hspace{1cm} (43)

The total energy of the holostar solution is given by the proper integral over the mass density. The proper energy of the shell therefore is:

$$\delta E_{BH} = \rho \delta V = \frac{\delta V}{8\pi r^2} = \frac{1}{2}(\beta\hbar)^{-\frac{4}{3}} r^{\frac{2}{3}} \delta r$$  \hspace{1cm} (44)

Setting the two energies equal gives the local temperature within the holostar:

$$T^4 = \frac{\pi\hbar^3}{4 F_E r^2}$$  \hspace{1cm} (45)

Thus we recover the $1/\sqrt{r}$-dependence of the local temperature, at least if $F_E$ is constant.

$F_E$ is a function of $f_F$, $f_B$ and $u_F$. We will see later, that $u_F$ only depends on the ratio of $f_F$ and $f_B$. Therefore in any range of $r$-values where the number of degrees of freedom of the ultra-relativistic particles (or rather their ratio)
doesn’t change, the local temperature as determined by equation (45) will not deviate from an inverse square root law.

If the temperature of equation (45) is inserted into equation (32), the thermodynamic pressure is derived as follows:

\[ P(r) = \frac{1}{24\pi r^2} = \frac{\rho}{3} \]

This is the equation of state for an ultra-relativistic gas, as expected.

4.1 Comparing the holostar’s thermodynamic temperature and entropy to the Hawking result

By inserting the temperature derived in equation (45) into equation (28) we get the following expression for the thermodynamic entropy within the shell:

\[ \delta S(r) = \left( \frac{F_E}{4\pi\beta} \right)^{\frac{4}{3}} \frac{F_S r \delta r}{F_E \hbar} \]

(46)

with

\[ F_S(u_F) = f_F \left( \frac{4}{3} \{ Z_{F,3}(u_F) \} - u_F [Z_{F,2}(u_F)] \right) + 2f_B \frac{4}{3} (Z_{B,3}(0)) \]

(47)

We have used commutator [] and anti-commutator {} notation in order to render the above relation somewhat more compact.

Using the identities for the polylog function it is possible to express the above relation as a quadratic function of the variable \( u_F^2/\pi^2 \).

\[ F_S = \frac{4}{3} F_E(u_F) - f_F \frac{\pi^4 u_F^2}{3 \pi^2} \left( 1 + \frac{u_F^2}{\pi^2} \right) \]

(48)

with \( F_E \) is given by equation (42)

By comparing the temperature (45) and the entropy (46) of the holostar solution derived in the context of our simple model to the Hawking entropy and temperature, two important relations involving the two unknown parameters of the model \( u_F \) and \( \beta \) can be obtained.

We have already seen in section 3 that the holostar’s temperature at infinity is proportional to the Hawking temperature. As can be seen from equation (45) this general result remains unchanged in the more sophisticated thermodynamic analysis, as long as the quantity \( F_E(u_F, f_F, f_B) \) can be considered to be nearly constant. In order to determine \( F_E \) we can set the temperature at the holostar’s surface equal to the blue shifted Hawking temperature at the holostar’s surface, which can be obtained by multiplying the Hawking temperature (at infinity) with the red-shift factor \( z \) of the surface given in Eq. (15). We find:

\[ T^4 = T_{BH}^4 z^4 = \frac{\hbar^4}{2^{8} \pi^4 \beta \hbar^4} \cdot \frac{r_h^2}{\beta \hbar} = \frac{\hbar^3}{2^{8} \pi^4 \beta \hbar^2} \]

(49)
Comparing this to equation (45) we find:

\[
\frac{F_E}{4\pi \beta} = (2\pi)^4
\]  
(50)

This is an important result. It relates the fundamental area \(4\pi r_0^2 = 4\pi \beta \hbar\) to the thermodynamic parameters of the system, i.e. the number of degrees of freedom and the chemical potential of the fermions.

Another important relation is the ratio \(F_S/F_E\) in the interior holostar space-time, which can be obtained by comparing the Hawking entropy of a black hole with thermodynamic entropy of the holostar’s interior constituent matter.

The entropy of the holostar can be calculated by integrating equation (46). We will assume that \(F_E/\beta = \text{const}\), as follows from equation (50), and that \(F_S/F_E = \text{const}\), which will be justified shortly. If this is the case, the integral can be performed easily:

\[
S = \int_0^{r_h} \delta S(r)dV = \left(\frac{F_E}{4\pi \beta}\right)^\frac{1}{4} \frac{1}{2\pi} \frac{F_S A}{F_E 4\hbar}
\]  
(51)

with

\[
A = 4\pi r_h^2
\]

Setting this equal to the Hawking entropy, \(S_{BH} = A/(4\hbar)\), and using equation (50) we find the important result:

\[
\frac{F_S}{F_E} = 1
\]  
(52)

Writing out the above equation we get:

\[
\frac{f_F}{f_E} \left(\frac{1}{3} \{Z_{F,3}(u_F)\} - u_F [Z_{F,2}(u_F)]\right) + 2f_B \left(\frac{1}{3} Z_{B,3}(0)\right) = 1
\]  
(53)

Using the identities for the polylog function one can reduce the above equation to a very simple quadratic equation in the variable \(u_F^2/\pi^2\):

\[
\left(1 + \frac{u_F^2}{\pi^2}\right) \left(1 - 3\frac{u_F^2}{\pi^2}\right) + \frac{8}{15} \left(\frac{f_B}{f_F} - 1\right) = 0
\]  
(54)

The important message is, that whenever the bosonic and fermionic degrees of freedom - or rather their ratio \(f_B/f_F\) - is known, \(u_F\) can be calculated. Knowing \(u_F\), \(\beta\) can be determined via (50). Thus the two relations (50, 52) allow us to determine all free parameters of the model, whenever the number of particle degrees of freedom, \(f_F\) and \(f_B\) are known.
4.2 An alternative derivation of the relation $F_S/F_E = 1$

Before discussing the specifics of the thermodynamic model, I would like to point out another derivation of equation (52), which does not depend on the Hawking result. This alternative derivation only depends on the following fundamental thermodynamic relation

$$\frac{\delta S}{\delta E} T = 1 \quad (55)$$

and on the fact, that the holostar’s interior matter state is completely rigid, i.e. the interior matter state at any particular radial position depends only on $r$, but not on the overall size of the holostar.

Consider a process, where an infinitesimally small spherical shell of matter is added to the outer surface of the holostar. This process doesn’t affect the inner matter of the holostar, as the interior matter-state of the holostar at a given radial co-ordinated position $r$ does not depend in any way on the size of the holostar or on any other global quantity. Therefore, when adding a new layer of matter we don’t have to consider any interaction, such as heat-, energy- or entropy-transfer between the newly added matter layer and the interior matter. It is an adiabatic process, for which we can calculate the entropy-change of the whole system via equation (55). Let $r$ be the radial position of the holostar’s surface. The entropy of the newly added shell is given by equation (46), its energy by equation (44), and its temperature by equation (45). One finds that the thermodynamic relation (55) is only fulfilled, when $F_S = F_E$. We have derived equation (52) only from thermodynamics.

4.3 A closed formula for $u_F$ and some special cases

The chemical potential per temperature $u_F$ can be determined by finding the root of equation (54). The value of $u_F$ depends only on the ratio of fermionic to bosonic degrees of freedom.\(^3\). Let us denote the ratio of the degrees of freedom by

$$r_f = \frac{f_B}{f_F} \quad (56)$$

Then $u_F$ is given by:

$$\frac{u_F^2}{\pi^2} = \frac{2}{3} \sqrt{1 + \frac{2}{5} (r_f - 1) - \frac{1}{3}} \quad (57)$$

For $r_f = 0$ (only fermions) we find the following result:

$$u_F = \pi \sqrt{\frac{4}{15} - \frac{1}{3}} = 1.34416 \quad (58)$$

\(^3\)and on the constant ratio $F_S/F_E$, which has been shown to be unity for the interior holostar solution
For \( r_f = 1 \) (equal number of fermions and bosons) we get:

\[
    u_F = \frac{\pi}{\sqrt{3}} = 1.8138
\]  

(59)

From equation (57) one can see that \( u_F \) is a monotonically increasing function of \( r_f \). It attains its minimum value, when there are no bosonic degrees of freedom, i.e. \( f_B = r_f = 0 \). When the bosonic degrees of freedom vastly exceed the fermionic degrees of freedom \( u_F \) can - in principle - attain high values. For large \( r_f \) we have \( u_F \propto (r_f - 1)^{1/4} \). For all practical purposes one can assume that the number of bosonic degrees of freedom is not very much higher than the number of fermionic degrees of freedom. This places \( u_F \) in the range \( 1.34 < u_F < 3 \).

It is important to notice, that equation (57) only has a solution when the number of fermionic degrees of freedom, \( f_F \), is non-zero, whereas \( f_B \) can take arbitrary values for any non-zero \( f_F \). Therefore at least one fermionic (massless) particle species with a non-vanishing chemical potential proportional to the local radiation temperature is necessary.

### 4.4 Thermodynamic relations, which are independent from the Hawking formula

If \( u_F \) is known, all thermodynamic quantities of the model, such as \( F_E(u_F) \) and \( F_N(u_F) \) etc. can be evaluated. Note that in order to determine \( u_F \) we only needed the relation \( F_E = F_S \), whose derivation didn’t require the Hawking temperature/entropy relation. Yet in order to fix \( \beta \) via equation (50) we had to compare the holostar’s temperature (or entropy) to the Hawking-result. Therefore the particular relation between \( \beta \) and \( F_E \) derived in equation (50) is tied to the validity of the Hawking temperature formula.

Although there is no doubt that the Hawking temperature of a large black hole must be inverse proportional to its mass\(^4\), the exact numerical factor has not yet been determined experimentally and thus might be questioned. For a determination of this factor it is good know what thermodynamic relations in the interior holostar space-time are independent from the Hawking formula. The following derivations only make use of equation (52), i.e. \( F_E = F_S \).

Knowing \( u_F \) from equation (57) the entropy per particle \( \sigma \) can be easily calculated by equations (28, 34):

\[
    \sigma = \frac{\delta S}{\delta N} = \frac{F_S}{F_N} = \frac{F_E}{F_N}
\]

(60)

with

\[
    F_N(u_F) = f_F(Z_F \cdot 2(u_F) + Z_F \cdot (-u_F)) + 2f_B Z_B \cdot 2(0)
\]

(61)

\(^4\)This already follows from the Bekenstein-argument, that the entropy of a black hole should be proportional to the surface of its event horizon.
The energy per relativistic particle is given by equations (40, 67). We find, just as in the previous section, that the mean particle energy per temperature is constant and equal to the mean entropy per particle:

$$\epsilon = \delta E \delta N = \frac{F_E}{F_N}T = \sigma T$$  \hspace{1cm} (62)

$\sigma$ only depends on the number of degrees of freedom of the ultra-relativistic bosons and fermions in the model. In fact, $\sigma$ only depends on the ratio $r_f = f_B/f_F$ and is a very slowly varying function of this ratio. Figure 1 shows the dependence of $\sigma$ on $r_f$.

![Figure 1: mean entropy per particle of the ultra-relativistic fermions in the holographic solution as a function of the ratio of bosonic to fermionic degrees of freedom $r_f = f_B/f_F$.](image)

The relation $\epsilon = \sigma T$, which relates the mean energy per particle to the mean entropy times the local radiation temperature can be viewed as the fundamental thermodynamic characteristic of the holostar. Keep in mind that this relation is only valid for the mean energy per particle and the mean entropy per particle, evaluated with respect to all particles. It isn’t fulfilled for the bosonic and fermionic species individually. In general, except for the special case $f_B = 0$, we have $\epsilon_B \neq \sigma_B T$ and $\epsilon_F \neq \sigma_F T$. 

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The relation $\epsilon = \sigma T$, which is equivalent to $F_S = F_E$, has the remarkable side-effect, that the free energy is identical zero in the holostar solution:

$$F = E - ST = N(\epsilon - \sigma T) = 0 \quad (63)$$

Usually a closed system has the tendency to minimize it’s free energy, which is a compromise between minimizing it’s energy and maximizing it’s entropy. The holostar is the prototype of a closed system. It is a self-gravitating static solution to the Einstein field equations. It’s only form of energy-exchange with the exterior space-time is through Hawking-radiation, which is an utterly negligible mode of energy-exchange for a large holostar. In this respect it is remarkable that the holostar solution minimizes the free energy to zero, e.g. the smallest possible value that a sensible measure of energy in general relativity can have. This indicates, that the free energy in general relativity might be more than a mere book-keeping device.

With the help of equation (52), but not using equation (50), the entropy within the shell can be expressed as:

$$\delta S(r) = \left(\frac{F_E}{4\pi\beta}\right)^\frac{1}{4} \frac{r\delta r}{\hbar} \quad (64)$$

If the total entropy of the holostar, i.e. the integral over the entropy-contributions of the respective shells, is to be proportional to the Hawking entropy of a black hole with the same gravitational radius, $F_E/\beta$ must be constant. Integration of equation (64) gives the result:

$$S = \frac{1}{2\pi} \left(\frac{F_E}{4\pi\beta}\right)^\frac{1}{4} A \quad (65)$$

The Hawking result is reproduced, whenever:

$$\omega = \frac{1}{2\pi} \left(\frac{F_E}{4\pi\beta}\right)^\frac{1}{4} = 1 \quad (66)$$

$\omega$, which depends on the ratio $F_E/\beta$, is the constant of proportionality between the holostar entropy and the Hawking entropy. Setting $\omega = 1$ is equivalent to equation (50), which fixes $\beta$ with respect to the Hawking temperature. If the Hawking entropy/temperature formula have to be rescaled, $\omega$ is nothing else than the (nearly constant) scale factor. Therefore let us express all thermodynamic relations in terms of $\omega$.

The number of particles within the shell is given by equation (64), which is extended to encompass the bosonic degrees of freedom:

$$\delta N(r) = \frac{F_N}{F_E} \left(\frac{F_E}{4\pi\beta}\right)^\frac{1}{4} \frac{r\delta r}{\hbar} = \omega \frac{2\pi r\delta r}{\sigma} \quad (67)$$

The total number of particles is given by a simple integration, assuming that $\omega = \text{const}$.
\[ N = \left( \frac{F_E}{4\pi\beta} \right)^\dagger \frac{1}{2\pi} \frac{1}{\sigma} \frac{A}{4h} = \frac{\omega}{\sigma} \frac{A}{4\bar{h}} \]  

Therefore, as derived in the previous section, the total number of particles within the holostar is proportional to its surface area, whenever \( F_E/\beta = \text{const} \) and \( \sigma = \text{const} \).

The temperature of the holostar at infinity is given by

\[ T_\infty = T(r_h) \sqrt{g_{tt}(r_h)} = 2\pi \left( \frac{4\pi\beta}{F_E} \right)^\dagger \frac{\hbar}{4\pi r_h} = \frac{1}{\omega} \frac{\hbar}{4\pi r_h} \]  

Again, if we set \( \omega = 1 \) we get the Hawking temperature. The important result is, that \( \omega \) could in principle take on any arbitrary (nearly constant) value. This is possible, because the factor in the temperature is just the inverse as the factor in the entropy. As is well known from black hole physics, any constant rescaling of the Hawking entropy must necessarily rescale the temperature such, that the product of temperature and entropy is equal for the scaled and unscaled quantities, i.e. \( ST \) must be unaffected by the rescaling. This is necessary, because otherwise the thermodynamic identity

\[ \frac{\partial S}{\partial E} = 1 \]

would not be fulfilled in the exterior space-time. (In the exterior space-time the energy \( E \) is fixed and is taken to be the gravitating mass \( M = r_h/2 \) of the black hole.)

As can be seen from equations (65, 69), entropy and temperature at infinity of the holostar fulfill the rescaling condition. Furthermore, entropy and temperature at infinity are exactly proportional to the Hawking temperature and entropy. This result is not trivial. It depends on the holostar metric, which has just the right value at the position of the membrane, so that the temperature at infinity scales correctly with respect to the entropy.

### 4.5 Relating the local thermodynamic temperature to the Hawking temperature

Now we are ready to set \( \omega = 1 \), which gives us the desired relation between \( \beta \) and \( F_E \), as already expressed in equation (60).

With \( \omega = 1 \), the local thermodynamic temperature of any interior shell can be expressed solely in terms of \( \beta \). It turns out to be equal to the expression in equation (22) of the previous section:

\[ T^4 = \frac{\hbar^3}{(4\pi)^3 \beta v^2} \]  

or

\[ T^4 = \frac{1}{(4\pi)^3 \beta v^2} \]  

(70)
5 A measurement of the Hawking temperature

In the previous section the internal temperature of the holostar has been derived by "fixing" it with respect to the Hawking temperature. Although Hawking’s calculations are robust \(^3\) and there appears to be no reason, why the Hawking equation should be modified - at least for large black holes\(^5\) - it has been speculated whether the factor in the entropy-area law (or in the temperature formula) might take a different value. A single measurement of the Hawking temperature (or entropy) of a large black hole could settle the question. However, with no black hole available in our immediate vicinity and taking into account the extremely low temperatures of even comparatively small black holes, there appeared to be no feasible means to measure the Hawking entropy or temperature of a black hole directly or indirectly.

It would be of high theoretical value, if the Hawking temperature/entropy formula could be verified (or falsified) by an explicit measurement. The holostar provides such a means.

For this purpose let us assume, that the Hawking temperature formula were modified by a constant factor, i.e

\[
T = \frac{1}{\omega} \frac{\hbar}{4\pi r} \quad (71)
\]

where \(\omega\) is a dimensionless factor, whose value can be determined experimentally.

If we set the temperature of the holostar equal to the modified Hawking temperature we get the following result for \(F_E\):

\[
\frac{F_E}{4\pi \beta} = (2\pi \omega)^4 \quad (72)
\]

The local temperature within the holostar is then given by equation (45):

\[
T^4 = \frac{\hbar^3}{2^{8}\pi^4\beta^2 \omega^4} = \frac{1}{\omega^3} \frac{\hbar^3}{2^6\pi^3\beta^3 \rho} \quad (73)
\]

\(\rho = 1/(8\pi r^2)\) is the total (local) energy density of the matter within the holostar. The above equation can be solved for \(\omega\):

\[
\omega^4 = \frac{\hbar^3}{2^6\beta^3 \rho} \quad (74)
\]

\(^5\)The only ingredient in Hawking’s derivation is the propagation of a quantum field in the exterior vacuum space-time of a black hole. Both concepts (quantum field in vacuum; exterior space-time of a black hole) are very accurately understood.
The local radiation temperature $T$ and the total local energy density $\rho$ within a holostar are both accessible to measurement. Note that the local temperature within a holostar is much easier to measure than its (Hawking) temperature at infinity: The local interior temperature only scales with $1/\sqrt{M}$, whereas the temperature at infinity scales with $1/M$. Therefore even a very large holostar will have an appreciable interior local radiation temperature, although its Hawking temperature at infinity will be unmeasurable by all practical means.

In order to determine $\omega$ the value of $\beta$ need to be known. In [5] the following formula for $\beta$ has been suggested:

$$\frac{\beta}{4} = \frac{\alpha}{2} + \sqrt{\left(\frac{\alpha}{2}\right)^2 + \frac{3}{4}}$$

(75)

$\alpha$ is the running value of the fine-structure constant, which depends on the local energy scale. Note that the above relation for $\beta$ has not been derived rigorously in [5], but was suggested by analogy, i.e. by extrapolating the (exact) relation between mass, charge, boundary area and $r_0$ derived for an extremely charged holostar to the charged/rotating case. Angular momentum was introduced in straightforward way, giving the correct formula for a Kerr-Newman black hole in the macroscopic limit and the correct formula for the a charged, non-rotating holostar for $J = 0$. The formula with non-zero $J$ then was applied to a microscopic object, a spin-1/2 extremely charged holostar of minimal mass, in order to obtain equation (75). One must keep in mind though, that in principle there are several ways to extend the formula to the rotating case giving the correct macroscopic limit, but which might differ in their microscopic predictions.

We want to apply equation (75) in order to derive the value of $r_0$ which determines the interior radial metric coefficient of a large holostar, $g_{rr} = r_0/r$. The implicit assumption which lies at the heart of equation (75), is that $r_0^2 = \beta \hbar$ is a universal quantity, not dependent on the nature of the system in question and only - moderately - dependent on the energy-scale. It requires quite a leap of faith to do this. It is not possible to fully justify this assumption in the context of this paper. See [8, 5] for a more detailed discussion.

Due to the appearance of $\alpha$ in the formula for $r_0^2$ it is suggestive to interpret $r_0$ as a running length scale, which depends on the energy $E$ via $\alpha(E)$. This means that for high temperatures $r_0$ is expected to increase with energy as a function of $\alpha(E)$. This makes sense, because we have already seen, that $r_0^2$ is proportional to the effective degrees of freedom, which are also known to increase at high energies. Therefore, if we treat $r_0^2$ as a universal quantity, the only sensible way is to interpret $\alpha$ as the running value of the relevant coupling constants depending on the energy scale.

With this interpretation, whenever $\alpha$ is small, such as for the typical energies encountered today, it can be set to zero in the above equation to a very good approximation, so that $\beta \approx 4\sqrt{3/4}$.

Let us now make the assumption, that we live in a large holostar. In [7] several observational facts have been accumulated which suggest that such a
claim is not too far fetched. Then the local radiation temperature will be nothing else than the microwave-background temperature and the total (local) energy density will be the total matter density of the universe at the present time (= present radial position). Both quantities have been determined quite precisely in the recent past. With the following value for the temperature of the microwave background radiation

\[ T_{CMBR} = 2.725K \]

and with the total matter density determined from the recent WMAP-measurements [2]

\[ \rho = 0.26 \rho_c = 2.465 \cdot 10^{-27} \text{kg m}^{-3} \]

and with \( \beta \) determined from equation (75) using the present (low energy) value of the fine-structure constant, \( \alpha \),

\[ \beta = 3.479 \]

we find:

\[ \omega^4 = 1.0116 \] \quad (76)

or

\[ \omega = 1.003 \]

If we set the fine-structure constant to zero, i.e. \( \beta = 4 \sqrt{3}/4 \), the agreement is almost as good: \( \omega = 1.004 \). The very high accuracy suggested in the above results is somewhat deceptive. With \( T \) known to roughly 0.1% the error in \( \omega \) will be dominated by the uncertainty in \( \rho \). A conservative estimate for this uncertainty should be roughly 5%. Taking the fourth square root suppresses the relative error by roughly a factor of four, so that the error in \( \omega \) will be roughly 1%. Therefore, within the uncertainties of the determination of \( \rho \) and \( T \) the Hawking-entropy formula is reproduced to a remarkably high degree of accuracy of roughly 1%.

Not knowing \( \beta \), the experimental data only allow us to determine \( \omega^4 \beta \approx 3.519 \). Therefore, as long as equation (75) has not been verified independently, it is prudent to keep this caveat in mind.

6 Matter-dominated holostars

So far we have assumed, that the interior matter-state consists of an ultra-relativistic gas. At low temperatures, well below the rest-masses of the fundamental particles this is not the case. Is a matter-dominated holostar also compatible with the Hawking entropy and temperature?

The entropy of a massive particle (with zero chemical potential) is given by:
\[ \sigma_m = \frac{m}{T} \]  

(77)

As long as there is at least one relativistic particle left, we will have a radiation temperature given by equation (22). Let us assume, that the matter is dominated by one massive particle species, such as the nucleon. Then the number-density of the massive particles is simply given by:

\[ n_m = \frac{\rho}{m} = \frac{1}{8\pi r^2 m} \]  

(78)

The local entropy-density is given by the product of equations (77, 78) and is independent of particle mass.

\[ s = n_m \sigma_m = \frac{1}{8\pi r^2} \frac{1}{T} = \frac{1}{2\pi \hbar} \frac{r_0}{r} \]  

(79)

Therefore the above result equally applies to a mixture of particles with different masses. The total entropy follows from a proper integration over the interior entropy-density:

\[ S = \int_0^{r_h} s dV = \frac{\pi r_h^2}{\hbar} = S_{BH} \]  

(80)

7 Discussion and Outlook

A simple thermodynamic model for a compact self-gravitating object with an interior matter-density \( \rho = 1/(8\pi r^2) \) has been presented which fits well into the established theory of black holes. From the viewpoint of an exterior observer the object appears very similar to a classical black hole. The modifications are minor and only "visible" at close distance:

The event horizon is replaced by a two dimensional membrane with high tangential pressure, situated roughly two Planck coordinate lengths outside of the object’s gravitational radius. The surface redshift at the membrane scales with \( z = \sqrt{r/r_0} \), where \( r_0 \) is a fundamental length, roughly equal to two Planck lengths. A solar mass object has a surface redshift \( z \approx 10^{20} \).

Simply by assuming (i) spherical symmetry, (ii) Einstein’s field equations with zero cosmological constant and (iii) microscopic statistical thermodynamics in the ideal gas approximation it could be shown that any compact self-gravitating object with an interior matter-density \( \rho = 1/(8\pi r^2) \) has a thermodynamic entropy and a temperature at infinity exactly proportional to the Hawking entropy and -temperature. The number of interior ultra-relativistic particles is proportional to the proper area of the object’s boundary-membrane, measured in Planck units, indicating that the holographic principle is valid for compact self-gravitating objects of arbitrary size.

The object has a well-defined interior temperature with \( T \propto 1/\sqrt{r} \). The object’s surface temperature can be related to the Hawking temperature. By this correspondence one can set up a specific relation between the Hawking
temperature (measured at infinity), the interior radiation temperature and the interior matter density. This correspondence allows an experimental verification of the Hawking-temperature law, by measurements in the object’s interior.

At ultra-relativistic energies the fermions acquire a non-zero chemical potential. The chemical potential per temperature \( u = \mu / T \) can be calculated by a closed formula. Its value depends only on the ratio of bosonic to fermionic degrees of freedom and is a monotonically increasing function of this ratio. If there are only fermions, \( u \approx 1.34 \). In the supersymmetric case (equal fermionic and bosonic degrees of freedom) \( u = \pi / \sqrt{3} \approx 1.8 \). The non-zero chemical potential of the fermions naturally induces a profound matter-antimatter asymmetry at high temperatures. The implications of this finding are discussed in [10, 8].

One particularly interesting solution to the field equations with an interior matter-density \( 1/(8\pi r^2) \) is the so called holographic solution, short holostar. Its remarkable properties have been discussed extensively in [7, 5, 9, 8]. The holostar’s membrane has a pressure equal to the pressure derived from the so called “membrane paradigm” for black holes [14, 11]. This guarantees, that the holostar’s action on the exterior space-time is practically indistinguishable from that of a same-sized black hole. The membrane has zero energy-density, as expected from string theory. Its interior matter has an overall string equation of state. The strings are densely packed, each string occupying a transverse extension of exactly one Planck area. This dense package of strings is the fundamental reason why the holographic solution does not collapse to a singularity, although its membrane lies barely two Planck coordinate distances outside its gravitational radius.

The holostar solution has no singularity and no event horizon. Information is not lost: The total information content of the space-time is encoded in its constituent matter, which can consist out of strings or particles. Unitary evolution of particles is possible throughout the full space-time manifold. Every ultra-relativistic particle carries a definite entropy, which can be calculated when the ratio of bosonic to fermionic degrees of freedom is known.

The holostar solution has been shown in [7] to be an astoundingly accurate model for the universe, as we see it today. By comparing the CMBR-temperature to the total matter density of the universe as determined by WMAP the Hawking temperature law has been experimentally verified to an accuracy of roughly 1 %. However, the exact numerical verification depends on an equation which has been suggested by analogy in [5], but stills lacks a formal derivation.

Having two or more solutions for the field equations (black hole vs. holostar) makes the question of how these solutions can be distinguished from each other experimentally an imminently important question. Can we find out by experiment or observation, which of the known solutions, if any, is realized in nature? At the present time the best argument in favor of the holostar solution appears to be the accurate measurement of the Hawking temperature via the CMBR-temperature and the matter-density of the universe.

Yet it would be helpful if more direct experimental evidence were available. Due to Birkhoff’s theorem the holostar cannot be distinguished from a Schwarzschild black hole by measurements of its exterior gravitational field.
But whenever holostars come close to each other or collide, their characteristic interior structure should produce observable effects, which deviate from the collisions of black holes. Presumably a collision of two holostars will be accompanied by an intense exchange of particles, with the possible production of particle jets along the angular momentum axis.

In accretion processes the membrane might produce a noticeable effect. The rather stiff membrane with its high surface pressure might be a better "reflector" for the incoming particles, than the vacuum-region of the event horizon of a Schwarzschild-type black hole. There are observations of burst-like emissions from compact objects, which are assumed to be black holes because of their high mass \(M > 3 - 5 M_\odot\), but that have "hard" spectra rather characteristic for neutron stars. A more accurate observation of these objects might provide important experimental clues to decide the issue.

For holostars of sub-stellar size \(r_h \approx 1\, km\) the local temperature at the membrane becomes comparable to the nucleon rest mass energy. A rather hot particle gas at the position of the membrane could produce noticeable effects with respect to the relative abundances of the "reflected" particles, due to high energy interactions with the membrane or the holostar’s interior.

On the other hand, the extreme surface red-shifts on the order of \(z \approx 10^{20}\) for a solar mass holostar, and larger yet for higher mass objects \(z \propto \sqrt{M}\), might not allow a conclusive interpretation of the experimental data with regard to the true nature of any such black hole type object.

The most promising route therefore appears to be, to study the holostar from its interior. In \cite{7} it has been demonstrated, that the holostar has the potential to serve as an alternative model for the universe. The recent WMAP-measurements have determined the product of the Hubble constant \(H\) times the age of the universe \(\tau\) to be \(H\,\tau \simeq 1.02\) experimentally with \(H = 71\, (km/s)/Mpc\) and \(\tau = 13.7\, Gy\). The holostar solution predicts \(H\,\tau = 1\) exactly. There are other predictions which fit astoundingly well with the observational data. This in itself is remarkable, because the holostar-solution has practically no free parameters. It’s unique properties arise from a delicate cancelation of terms in the Einstein field equations, which only occurs for the "special" matter density \(\rho = 1/(8\pi r^2)\) in combination with a string equation of state, leading to the "special" radial metric coefficient \(g_{rr} = r/r_0\) and a time coefficient \(g_{tt} = r_0/r\). That the holostar solution with its completely "rigid" structure has so much in common with the universe as we see it today, either is the greatest coincidence imaginable, or not a coincidence at all.

With the holostar solution we have a beautifully simple model for a singularity free compact self gravitating object, which is easily falsifiable. Its metric and fields are simple, its properties are not. It is an elegant solution, as anyone studying its properties will soon come to realize. However, in science it is experiments and observations, not aesthetics, that will have to decide, which solution of the field equations has been chosen by nature. It is our task, to find out. The work has just begun.

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