ASYMPTOTIC BEHAVIOR OF THE SOLUTION OF A DIFFUSION EQUATION WITH NONLOCAL BOUNDARY CONDITIONS

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ABSTRACT. In this paper, we consider a particular type of nonlinear McKendrick–von Foerster equation with a diffusion term and Robin boundary condition. We prove the existence of a global solution to this equation. The steady state solutions to the equations that we consider have a very important role to play in the study of long time behavior of the solution. Therefore we address the issues pertaining to the existence of solution to the corresponding state equation. Furthermore, we establish that the solution of McKendrick–von Foerster equation with diffusion converges pointwise to the solution of its steady state equations as time tends to infinity.

1. Introduction. In the mathematical modeling of population dynamics and quasi static thermoelasticity nonlocal partial differential equations arise very naturally (see [2, 3]). Theoretical and numerical study of such nonlocal models have attracted attention of many mathematicians and engineers. In this paper, we would like to study the following McKendric–von Foerster type equation with diffusion. Let \( a^\dagger > 0 \) and \( D = (0, a^\dagger) \subset \mathbb{R}, \)

\[
\begin{align*}
  u_t(t, x) + u_x(t, x) + d(x) u(t, x) &= u_{xx}(t, x), \quad t > 0, \quad x \in D, \\
  u(t, 0) - u_x(t, 0) &= g \left( \int_D B(0, y) u(t, y) dy \right), \quad t > 0, \\
  u(t, a^\dagger) + u_x(t, a^\dagger) &= g \left( \int_D B(a^\dagger, y) u(t, y) dy \right), \quad t > 0, \\
  u(0, x) &= u_0(x), \quad x \in D, \quad u_0 \in C(\bar{D}).
\end{align*}
\]

Here \( d, B, g \) and \( u_0 \) are given nonnegative continuous functions. Moreover, \( d \) is a Hölder continuous function. Throughout the paper, we assume that:

\((H1)\) \( g \) is an increasing function;

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\((H2) \exists M_0 > 0\) such that \(\forall M > M_0,\)
\[
M \geq \max \left\{ g(M \int_D B(0, x)dx), g(M \int_D B(a^\dagger, x)dx) \right\};
\]

\((H3) \exists \alpha > 0, \beta > 0\) such that \(\alpha \leq d(,)\), \(B(,,) \leq \beta\).

In [5, 6, 7], Gladkov et al have studied equation (1) with a particular type of nonlinear nonlocal boundary condition. Furthermore, the authors of those papers have proved existence, uniqueness and blow-up phenomena of the solutions in finite time. In [3], Day has considered a linear thermoelasticity model similar to (1) and established its dynamic theories. Pao has developed the method of subsolution and supersolution to prove an existence and uniqueness result for a reaction diffusion equation with nonlocal boundary conditions. Moreover, the asymptotic behaviour and dynamical properties for some specific type of nonlinear differential equations are also studied using this method (see [12]–[15]). In [10], Iannelli et al have used the semigroup theory techniques to study the regularized parabolic problem. In [1] [8] [9], the authors have considered the renewal equation with diffusion with \(a^\dagger = +\infty\) and studied the existence, uniqueness of weak solution. Moreover, the long time behavior has been studied in some particular cases.

The paper is organized as follows. An existence result for equation (1) has been discussed in Section 2. We demonstrate that there exists a unique solution to (1) which is bounded above by a supersolution and bounded below by a subsolution. We define the steady state equations corresponding to (1) and show the existence of its solution in Section 3. In Section 4, we study the long time behavior of the solution of (1). In particular, we prove that the solution of (1) converges pointwise to the solution of the corresponding steady state equations. In Section 5, we present some numerical results to justify the convergence results that are discussed in the article.

2. Existence of solution. In this section, we prove an existence result to equation (1) using the method of subsolution and supersolution. Strong maximum principle for the parabolic equations plays a crucial role in order to prove the existence of solution of (1). To this end, we first define supersolution and subsolution.

Definition 2.1. A function \(m \in C^{1,2}((0, T) \times D) \cap C((0, T) \times \bar{D})\) is said to be a supersolution of (1) on \((0, T) \times D\) if it satisfies the following inequalities
\[
\begin{align*}
    m_t(t, x) + m_x(t, x) + d(x)m(t, x) & \geq m_{xx}(t, x), \quad t \in (0, T), \quad x \in D, \\
    m(t, 0) - m_x(t, 0) & \geq g \left( \int_D B(0, x)m(t, x)dx \right), \quad t \in (0, T), \\
    m(t, a^\dagger) + m_x(t, a^\dagger) & \geq g \left( \int_D B(a^\dagger, x)m(t, x)dx \right), \quad t \in (0, T), \\
    m(0, x) & \geq u_0(x), \quad x \in \bar{D}.
\end{align*}
\]
(2)

Similarly, one can define a subsolution of (1) as a function which satisfies all the reversed inequalities present in (2).

In what follows, unless specified otherwise, we always denote nonnegative supersolution and nonnegative subsolution to equation (1) by \(\tilde{u}\) and \(\hat{u}\), respectively.
For $M > \max\{M_0, \|u_0\|_{L^\infty}\}$, thanks to the nonnegativity of $g$, the constant function $\tilde{u} \equiv M$ is a supersolution and $\tilde{u} \equiv 0$ is a subsolution of \textbf{(1)}. This ensures the existence of subsolutions and supersolutions of \textbf{(1)}.

We consider the following sequence of problems for $n \geq 1$,

$$
\left\{
\begin{array}{ll}
\tilde{u}^{(n)}_t(t, x) - \tilde{u}^{(n)}_{xx}(t, x) + \tilde{u}^{(n)}_x(t, x) + d(x)\tilde{u}^{(n)}(t, x) = 0, & t > 0, \ x \in D, \\
\tilde{u}^{(n)}(t, 0) - \tilde{u}^{(n)}_x(t, 0) = g \left( \int_D B(0, x)\tilde{u}^{(n-1)}(t, x)dx \right), & t > 0, \\
\tilde{u}^{(n)}(t, a_t) + \tilde{u}^{(n)}(t, a_t) = g \left( \int_D B(a_t, x)\tilde{u}^{(n-1)}(t, x)dx \right), & t > 0, \\
\tilde{u}^{(n)}(0, x) = u_0(x), & x \in \bar{D}.
\end{array}
\right.
$$

If $u^{(0)} = \tilde{u}$ then we denote the sequence of solutions to the problems in \textbf{(3)} by $\{\tilde{u}^{(n)}\}$. Similarly, if $u^{(0)} = \hat{u}$ then denote corresponding sequence by $\{\hat{u}^{(n)}\}$.

**Theorem 2.1.** Let $\hat{u}$ and $\tilde{u}$ be a nonnegative supersolution and nonnegative subsolution to \textbf{(1)}, respectively, such that $\hat{u} \leq \tilde{u}$. Then the sequences $\{\tilde{u}^{(n)}\}$ and $\{\hat{u}^{(n)}\}$ satisfy the following inequality:

$$
\tilde{u} \leq \hat{u}^{(n)} \leq \hat{u}^{(n+1)} \leq \tilde{u} \leq \tilde{u}^{(n)} \leq \hat{u} \text{ in } [0, T] \times [0, a_t], \ \forall n \in \mathbb{N}, \tag{4}
$$

where $\hat{u}$ and $\tilde{u}$ are the pointwise limits of the sequences $\{\hat{u}^{(n)}\}$ and $\{\tilde{u}^{(n)}\}$, respectively.

**Proof.** First, we set $w = \tilde{u}^{(0)} - \tilde{u}^{(1)} = \hat{u} - \tilde{u}^{(1)}$, then $w$ satisfies

$$
w_t - w_{xx} + w_x + d(x)w = \tilde{u}_t - \tilde{u}_{xx} + \tilde{u}_x + d(x)\tilde{u} \geq 0,
$$

and

$$
w(t, 0) - w_x(t, 0) = \hat{u}(t, 0) - \tilde{u}_x(t, 0) - (\tilde{u}^{(1)}(t, 0) - \tilde{u}^{(1)}_x(t, 0)) = \hat{u}(t, 0) - \tilde{u}_x(t, 0) - g \left( \int_D B(0, x)\tilde{u}^{(0)}(t, x)dx \right) \geq 0.
$$

Similarly, one can prove that $w(t, a_t) + w_x(t, a_t) \geq 0$. Furthermore, we find that $w(0, x) = \hat{u}(0, x) - u_0(x) \geq 0$. Thanks to the maximum principle for the parabolic operators, we get

$$w \geq 0, \ \text{or} \ \hat{u}^{(1)} \leq \tilde{u}, \text{ in } [0, T] \times [0, a_t].$$

Using the similar argument it is straightforward to prove that $\hat{u}^{(1)} \geq \tilde{u}$.

Next we define $\tilde{w} = \tilde{u}^{(1)} - \hat{u}^{(1)}$. By using the definitions of nonnegative supersolution and nonnegative subsolution, we get

$$\tilde{w}_t - \tilde{w}_{xx} + \tilde{w}_x + d(x)\tilde{w} = 0.$$

Moreover, we have

$$
\tilde{w}(t, 0) - \tilde{w}_x(t, 0) = \tilde{u}^{(1)}(t, 0) - \tilde{u}^{(1)}_x(t, 0) - (\hat{u}^{(1)}(t, 0) - \hat{u}^{(1)}_x(t, 0)) = g \left( \int_D B(0, x)\hat{u}^{(0)}(t, x)dx - \int_D B(0, x)\tilde{u}^{(0)}(t, x)dx \right) \geq 0.
$$

and the last inequality follows from the monotonicity of $g$. Using the similar arguments we arrive at

$$\tilde{w}(t, a_t) + \tilde{w}_x(t, a_t) \geq 0.$$
Furthermore, we have
\[ \tilde{w}(0, x) = u_0(x) - u_0(x) = 0. \]
Again, from the maximum principle we get $\tilde{u}^{(1)} \geq \overline{u}^{(1)}$. So far, we have proved
\[ \hat{u} \leq \underline{u}^{(1)} \leq \overline{u}^{(1)} \leq \tilde{u} \text{ in } [0, T] \times [0, a].\]
Now, using the principle of induction and the arguments used in the proof so far, one can easily prove the required inequality
\[ \hat{u} \leq u^{(n)} \leq u^{(n+1)} \leq \overline{u}^{(n+1)} \leq \tilde{u} \text{ in } [0, T] \times [0, a^e], \forall n \in \mathbb{N}. \]
Therefore the pointwise limits of \( \{u^{(n)}\} \) and \( \{\overline{u}^{(n)}\} \), say $u$ and $\overline{u}$, respectively, exist and satisfy the relation (4). This completes a proof of the promised result.

In the next theorem we prove that the limits $u$ and $\overline{u}$ which are obtained in Theorem 2.1 are indeed solutions to (1). The main ingredient in the proof is the representation of classical solution to the linear parabolic equation given in the books of Friedman and Pao (see [4, 11]).

**Theorem 2.2. (Existence)** Consider the notation that is introduced in Theorem 2.1. Then both $u$ and $\overline{u}$ are solutions to equation (1).

**Proof.** First, observe that the solution $u^{(n)}$ to equation (3) is given by (see [4] for more details)
\[
\begin{aligned}
\underline{u}^{(n)}(t, x) &= \int_0^t \left[ \Gamma(t, x; \tau, a^e)\phi^{(n-1)}(\tau, a^e) - \Gamma(t, x; \tau, 0)\phi^{(n-1)}(\tau, 0) \right] d\tau \\
&+ \int_D \Gamma(t, x; 0, \xi)u_0(\xi)d\xi.
\end{aligned}
\]
(5)

Here $\Gamma(t, x; \tau, \xi)$ is the fundamental solution to (3) and the corresponding density function $\phi^{(n)}$, for $x = 0$ and $a^e$, is given by
\[
\phi^{(n)}(t, x) = 2F^{(n)}(t, x) + 2 \sum_{j=1}^{\infty} \int_0^t \{ M_j(t, x; \tau, a^e)F^n(\tau, a^e) \\ - M_j(t, x; \tau, 0)F^n(\tau, 0) \} d\tau,
\]
where
\[
F^{(n)}(t, x) = \int_D M(t, x; 0, \xi)u_0(\xi)d\xi + g \left( \int_D B(x, y)\underline{w}^{(n)}(t, y)dy \right),
\]
\[
M_1 = M = 2 \left( \frac{\partial \Gamma(t, x; \tau, \xi)}{\partial \nu(t, x)} + \Gamma(t, x; \tau, \xi) \right),
\]
and
\[
M_{n+1}(t, x; \tau, \xi) = \int_0^t \{ M(t, x; \sigma, a^e)M_n(\sigma, a^e; \tau, \xi) \\ - M(t, x; \sigma, 0)M_n(\sigma, 0; \tau, \xi) \} d\sigma.
\]
Define $\phi(t, x)$, the density function for $x = 0$ and $a^e$, by
\[
\phi(t, x) = 2F(t, x) + 2 \sum_{j=1}^{\infty} \int_0^t \{ M_j(t, x; \tau, a^e)F(\tau, a^e) \\ - M_j(t, x; \tau, 0)F(\tau, 0) \} d\tau,
\]
(6)
where
\[ F(t, x) = \int_D M(t, x; 0, \xi)u_0(\xi)d\xi + g\left(\int_D B(x, y)\bar{u}(t, y)dy\right). \] (7)

We show that \( \bar{u} \) has the integral representation given by
\[ \bar{u}(t, x) = \int_0^t \left[ \Gamma(t, x; \tau, a_t)\phi(\tau, a_\tau) - \Gamma(t, x; 0, 0)\phi(\tau, 0) \right]d\tau + \int_D \Gamma(t, x; 0, \xi)u_0(\xi)d\xi. \] (8)

Denote \( g^{(n)}(x) := g\left(\int_D B(x, y)\bar{u}^{(n)}(t, y)dy\right) \) and \( G(x) := g\left(\int_D B(x, y)\bar{u}(t, y)dy\right) \) for \( x = 0 \) and \( a_t \).

From Theorem 2.2, we get \( \hat{u} \) is a solution of (1). By repeating the same argument with \( \bar{u} \) replaced by \( \hat{u} \) we can deduce that \( \hat{u} \) is also a solution.

**Remark 2.1.** Since the constant functions \( \bar{u} \equiv M, \hat{u} = 0 \) are supersolution and subsolution of (1), respectively, thanks to Theorem 2.1 and Theorem 2.2 there exists a solution to (1) such that \( 0 \leq \bar{u}(t, x) \leq M \), for all \( t > 0, x \in \bar{D} \).

**Remark 2.2.** If \( u^\ast \) is any other solution to (1) such that \( \hat{u} \leq u^\ast \leq \bar{u} \) then \( \bar{u} \leq u^\ast \leq \hat{u} \).

For, consider \( \bar{u}, u^\ast \) as a pair of subsolution and supersolution. From Theorem 2.1 we get \( \bar{u} \leq \bar{u} \leq u^\ast \). On the other hand, considering \( (u^\ast, \bar{u}) \) as a pair of subsolution and supersolution we readily obtain \( u^\ast \leq \bar{u} \leq \bar{u} \). This completes the proof of Remark 2.2.

**Theorem 2.3.** Consider the notation that is introduced in Theorem 2.1. Assume that
\[ 0 \leq g^\prime(\cdot) < \min\left\{ \frac{1}{\int_D B(0, x)dx}, \frac{1}{\int_D B(a_1, x)dx} \right\}. \] (9)

If \( u^\ast \) is a solution to (1) such that \( \hat{u} \leq u^\ast \leq \bar{u} \), then \( \bar{u} = u^\ast = \hat{u} \).

**Proof.** From Remark 2.2 we immediately get that \( \bar{u} \leq u^\ast \leq \hat{u} \). In order to prove the required result it is enough to prove \( \bar{u} = \bar{u} \). To this end, we introduce \( w(t, x) := \bar{u}(t, x) - \bar{u}(t, x) \).

By Theorem 2.1 we get \( w(t, x) \geq 0 \), for all \( (x, t) \in [0, a_1] \times [0, T] \). Moreover, thanks to Theorem 2.2, \( w(t, x) \) satisfies the following equation, for \( t \in [0, T] \),
\[
\begin{cases}
w_t(t, x) - w_{xx}(t, x) + w_x(t, x) + d(t, x)w(t, x) = 0, & x \in D, \\
w(t, 0) - w_x(t, 0) = g\left(\int_D B(0, x)\bar{u}(t, x)dx\right) - g\left(\int_D B(0, x)\bar{u}(t, x)dx\right), \\
w(t, a_1) + w_x(t, a_1) = g\left(\int_D B(a_1, x)\bar{u}(t, x)dx\right) - g\left(\int_D B(a_1, x)\bar{u}(t, x)dx\right), \\
w(0, x) = 0, & x \in D.
\end{cases}
\]
If \( w \) attains its positive maximum, then from the maximum principle, it is attained on the boundary, say at \((t_0, 0)\). Since \( w(0, x) = 0 \), for all \( x \in D \), therefore we have \( t_0 > 0 \). From the mean value theorem there exists \( \xi \geq 0 \) such that
\[
 w(t_0, 0) - w_x(t_0, 0) = g'(\xi) \int_D B(0, x) w(t_0, x) dx
\leq w(t_0, 0) g'(\xi) \int_D B(0, x) dx.
\]
Thus, we have
\[
(1 - g'(\xi) \int_D B(0, x) dx) w(t_0, 0) - w_x(t_0, 0) \leq 0.
\]
On the other hand, we know that from the strong maximum principle \( w_x(t_0, 0) < 0 \). This contradicts assumption (9).

Similarly, we can get a contradiction if we assume that the maximum is attained at \((t_1, a_t)\) for some \( t_1 > 0 \). Therefore we have \( u \equiv 0 \). This shows that \( u \equiv \bar{u} \). \( \square \)

Theorems 2.2 and 2.3 ensure the existence and uniqueness of a bounded solution to (1).

3. Steady state equation. In this section, we define the steady state equations corresponding to (1). Furthermore, we prove an existence result for these steady state equations. In order to do that, we use the method of sub and supersolutions which has been discussed in detail in Section 2. We say that \( U \) is a steady state corresponding to (1) if it solves the following second order ordinary differential equation,
\[
\begin{cases}
 U_x(x) + d(x) U(x) = U_{xx}(x), \quad x \in D, \\
 U(0) - U_x(0) = g \left( \int_D B(0, x) U(x) dx \right), \\
 U(a_1) + U_x(a_1) = g \left( \int_D B(a_1, x) U(x) dx \right).
\end{cases}
\]
Furthermore, we call (10) as the equation of steady state corresponding to (1).

**Definition 3.1.** A function \( v \in C^2(D) \cap C(\bar{D}) \) is said to be a supersolution of (10) if it satisfies the following inequalities,
\[
\begin{cases}
 v_x(x) + d(x) v(x) \geq v_{xx}(x), \quad x \in D, \\
 v(0) - v_x(0) \geq g \left( \int_D B(0, x) v(x) dx \right), \\
 v(a_1) + v_x(a_1) \geq g \left( \int_D B(a_1, x) v(x) dx \right).
\end{cases}
\]

Similarly, we can define a subsolution of (10) as a function which satisfies the reversed inequalities in (11). As in Section 2, we construct the following sequence of ordinary differential equations,
\[
\begin{cases}
 -U_x^{(n)}(x) + U_{xx}^{(n)}(x) + d(x) U^{(n)}(x) = 0, \quad x \in D, \\
 U^{(n)}(0) - U_x^{(n)}(0) = g \left( \int_D B(0, x) U^{(n-1)}(x) dx \right), \\
 U^{(n)}(a_1) + U_x^{(n)}(a_1) = g \left( \int_D B(a_1, x) U^{(n-1)}(x) dx \right).
\end{cases}
\]
In what follows, unless specified otherwise, we always use \( \bar{U} \) and \( \hat{U} \) to denote a nonnegative supersolution and nonnegative subsolution to equation (10), respectively. If \( U(0) = \bar{U} \) then denote the sequence of solutions to the problems in (12) by \( \{ U^{(n)} \} \). If \( U(0) = \hat{U} \) then the sequence of solutions to the problems in (12) is denoted by \( \{ \bar{U}^{(n)} \} \). With this notation we have the following result similar to Theorems 2.1 and 2.3.

**Proposition 3.1.** (Existence) Let \( \bar{U} \) and \( \hat{U} \) be a nonnegative supersolution and a nonnegative subsolution to (10), respectively, such that \( \hat{U} \leq \bar{U} \). Then the sequences \( \{ \bar{U}^{(n)} \} \) and \( \{ \bar{U}^{(n)} \} \) possess the following monotonicity property:

\[
\hat{U} \leq U^{(n)} \leq U^{(n+1)} \leq U \leq \bar{U} \leq \bar{U}^{(n+1)} \leq \bar{U}^{(n)} \leq \bar{U} \quad \text{in} \quad D, \quad \forall n \in \mathbb{N},
\]

(13)

where \( U \) and \( \bar{U} \) are the pointwise limits of the sequences \( \{ U^{(n)} \} \) and \( \{ \bar{U}^{(n)} \} \), respectively. Moreover \( U \) and \( \bar{U} \) are solutions to (10).

The inequalities in (13) can be obtained from the same arguments given in the proof of Theorem 2.1. Using the arguments given in [11, 16] one can show that \( U \) and \( \bar{U} \) are solutions to (10). Therefore we omit the proof.

Now, we have the following remark similar to Remark 2.2.

**Remark 3.1.** If \( U^\ast \) is any other solution to (10) such that \( \hat{U} \leq U^\ast \leq \bar{U} \) then \( U \leq U^\ast \leq \bar{U} \).

We conclude this section by stating a uniqueness result for the steady state equation (10), which says that there is a unique solution for (10) between any pair of subsolution \( \bar{U} \) and subsolution \( \hat{U} \) with \( \hat{U} \leq \bar{U} \). Details are given in the following result.

**Proposition 3.2.** (Uniqueness) Assume that inequality (9) holds and consider the notation that is introduced in Proposition 3.1. Furthermore, if \( U^\ast \) is a solution to (10) with \( \hat{U} \leq U^\ast \leq \bar{U} \) then \( U^\ast = \bar{U} \).

One can prove this result by following similar lines of the proof of Theorem 2.3.

4. Long time behavior. In this section, we prove that the solution of (1) converges to the solution of the steady state problem (10) with time. First we consider the case when \( g(0) = 0 \). In this case, it is easy to observe that \( U \equiv 0 \) is the solution to (10). Furthermore, we prove that the solution of (1) converges to the trivial steady state with an exponential rate.

**Proposition 4.1.** Assume that inequality (9) holds and \( g(0) = 0 \). Then \( \exists C > 0 \) such that

\[
\| u(t, \cdot) \|_{L^\infty} \leq Ce^{-\alpha t}, \quad \forall t > 0,
\]

(14)

where \( \alpha \) is given in (H3).

**Proof.** Set \( C := \| u_0 \|_{L^\infty} \). It is easy to verify that the function \( \bar{u} := Ce^{-\alpha t} \) is a supersolution and the constant function \( \hat{u} = 0 \) is a subsolution to (1). From Theorems 2.1 and 2.3, there exists a unique solution to (1), say \( u \), that lies between \( \bar{u} \) and \( \hat{u} \). Therefore, \( 0 \leq u(t, x) \leq Ce^{-\alpha t} \), for all \( x \in D, \ t > 0 \). This completes the proof.

Now, we turn our attention to the case when \( g(0) \neq 0 \). In this case, the steady state cannot be trivial. We show that the solution to (1) converges pointwise to the nontrivial steady state.
Theorem 4.1. Assume \[ \tilde{U}, \tilde{u}(0) \neq 0 \] and \[ u_0(\cdot) \] satisfies
\[ \tilde{U}(x) \leq u_0(x) \leq \tilde{U}(x), \ \forall x \in \tilde{D}. \] (15)
Furthermore, assume that \[ \tilde{u}(t,x) \] and \[ \tilde{\rho}(t,x) \] are solutions of (1) with the initial data \[ \tilde{U} \] and \[ \tilde{U} \], respectively. If \[ u \] is a solution to (1), then we have
\[ \tilde{u}(t,x) \leq u(t,x) \leq \tilde{\rho}(t,x), \ \forall t > 0, \ x \in \tilde{D}, \] (16)
and
\[ \lim_{t \to \infty} u(t,x) = U(x). \] (17)

Proof. Step 1. We prove (16) in this step. Set \[ w(t,x) = u(t,x) - \tilde{\rho}(t,x) \], then \[ w \] satisfies the equations, for \[ t > 0 \],
\[
\begin{cases}
    w_t(t,x) - w_{xx}(t,x) + w_x(t,x) + d(x)w(t,x) = 0, & x \in D, \\
    w(t,0) - w_x(t,0) = g \left( \int_D B(0,x)u(t,x)dx \right) - g \left( \int_D B(0,x)\tilde{\rho}(t,x)dx \right), \\
    w(t,a_t) + w_x(t,a_t) = g \left( \int_D B(a_t,x)u(t,x)dx \right) - g \left( \int_D B(a_t,x)\tilde{\rho}(t,x)dx \right), \\
    w(0,x) = u_0(x) - \tilde{U}(x) \leq 0, & x \in D.
\end{cases}
\]

Claim. The function \[ w \] is nonpositive in \( (0,\infty) \times \tilde{D} \).
Suppose not, then \[ w(t,x) > 0 \] for some \( (t,x) \in (0,\infty) \times D \) and \[ w \] attains its positive maximum on the boundary, say at \( (t_0,0) \). Since \[ w(0,x) \leq 0 \], we must have \[ t_0 > 0 \]. Now, there exists \( \xi > 0 \) such that
\[
w(t_0,0) - w_x(t_0,0) = g'(\xi) \int_D B(0,x)w(t_0,x)dx \leq w(t_0,0)g'(\xi) \int_D B(0,x)dx.
\]
This readily implies
\[
1 - g'(\xi) \int_D B(0,x)dx \leq 0.
\]
By the same argument that is given in the proof of Theorem 2.3 we conclude that \[ w \leq 0 \]. Thus, we have
\[ u(t,x) \leq \tilde{\rho}(t,x), \ \forall t > 0, x \in D. \]

Similarly, we can get the other inequality
\[ \tilde{\rho}(t,x) \leq u(t,x), \ \forall t > 0, x \in D. \]

Step 2. We begin with the observation that \[ \tilde{U} \] and \[ \tilde{U} \] are supersolution and subsolution to (1), respectively. Hence, Theorems 2.1 and 2.2 ensure that \[ \tilde{\rho} \] and \[ \bar{\rho} \] satisfy the relation
\[ \tilde{U}(x) \leq \rho(t,x) \leq \tilde{\rho}(t,x) \leq \tilde{U}(x), \ \forall t > 0, x \in D. \] (18)
On the other hand, for \( \delta t > 0 \), set \( \tilde{\omega}(t,x) = \tilde{\rho}(t + \delta t,x) - \tilde{\rho}(t,x) \). Then we get
\[ \tilde{\omega}(0,x) = \tilde{\rho}(\delta t,x) - \tilde{\rho}(0,x) = \tilde{\rho}(\delta t,x) - \tilde{U}(x) \leq 0. \]
Using the arguments that are given in Step 1, we obtain \[ \tilde{\omega}(t,x) \leq 0 \]. Since \( \delta t \) is arbitrary, \( \tilde{\rho} \) decreases with time. Similarly, one can show that \( \bar{\rho} \) increases with time. Hence the pointwise limits
\[
\lim_{t \to \infty} \tilde{\rho}(t,x) = \bar{U}^*(x), \ \lim_{t \to \infty} \rho(t,x) = \bar{U}^*(x),
\]
exist. Moreover, from equation (18), we have
\[ \dot{U} \leq U^* \leq \bar{U} \leq \bar{\bar{U}}, \ x \in D. \]

By regularity arguments (see \cite{11}), one can show that \( U^* \), \( \bar{U} \) are the solutions to equation (10). From the uniqueness result in Proposition 3.2 we have

\[ \lim_{t \to \infty} \bar{\rho} = \lim_{t \to \infty} \rho = U \]

In view of (14), we have \( \lim_{t \to \infty} u(t, x) = U(x) \). Hence the theorem is proved. \( \square \)

5. **Numerical simulations.** In this section, we present some examples in which we have performed numerical experiments to determine the long time behavior of solution to (1). We broadly divide these examples into two categories. In the first category, we provide some examples that satisfy the main hypothesis of Section 4 i.e., inequality (9). In the next category, we give another set of examples which do not satisfy (9). We use the finite difference method to discretize the the problem. Let \( \Delta x \) and \( \Delta t \) denote the discretization step sizes for space and time, respectively. In all these examples we have taken \( a_1 = 20 \), \( B(a_1, x) = 0 \), \( \Delta x = 0.05 \) and \( \Delta t = 0.0012 \) in our numerical computations.

**Example 5.1.** Here we assume that \( d, B(0, x), g \) and \( u_0 \) are given by

\[
\begin{align*}
&d(x) = \frac{x}{1 + x}, \quad B(0, x) = \frac{e^{-x}}{2}, \quad g(x) = \sqrt{1 + x}, \quad u_0(x) = 2e^{-x}, \ \forall x \in (0, 20).
\end{align*}
\]

It is easy to verify that the functions \( B \) and \( g \) satisfy (9). In Figure 1 (left), the graphs of the solution \( u \) to equation (11) at \( t = 12 \) and solution \( U \) to (10) are presented. From this figure, we readily observe at \( t = 12 \), the solution to (11) is in well agreement with the corresponding steady state. The graph of the absolute difference between \( u \) at \( t = 12 \) and \( U \), i.e., \( |u(12, x) - U(x)| \) is plotted in Figure 1 (right). From both the graphs we find that \( u \) indeed converges to the nontrivial steady state \( U \) with time.

**Example 5.2.** In this example, we consider the case when

\[
\begin{align*}
&d(x) = 6 - 3e^{-x}, \quad B(0, x) = \frac{e^{-x}}{2}, \quad g(x) = \sqrt{1 + x}, \quad u_0(x) = 2, \ \forall x \in (0, 20).
\end{align*}
\]

We notice that these functions also satisfy (9). We have given the graphs of the solution of (11) at \( t = 12 \) and solution of (10) in Figure 2 (left). In Figure 2 (right), we have depicted the absolute value of the difference between \( u(12, .) \) and \( U(\cdot) \). We notice that \( u \) converges to \( U \) as time increases.

**Example 5.3.** In this example, we assume

\[
\begin{align*}
&d(x) = 4 - 2e^{-x}, \quad B(0, x) = 10 - 5e^{-x}, \quad g(x) = \sqrt{x}, \quad u_0(x) = 1, \ \forall x \in (0, 20).
\end{align*}
\]

The asymptotic behavior of the solution \( u \) to (11) when \( g(0) = 0 \) is discussed in Proposition 4.1, but the functions \( g \) and \( B \) in this example do not satisfy assumption (9) which is vital in it’s proof. In this example, we want to investigate whether \( u \) converges pointwise (if not uniformly) to a steady state. To this end, first we have found a nonzero numerical solution \( \bar{U} \) to (10). In Figure 3 (left), we have depicted the graphs of solution \( u \) to (11) at \( t = 12 \) and a nontrivial solution \( \bar{U} \) to (10). In Figure 3 (right), we have shown the corresponding absolute error. We have computed \( u(t, x) \) at \( x = 2, 3 \) and 4 to see the asymptotic behavior of \( u \) at these points. Graphs of \( u(t, x) \) at \( x = 2, 3 \) and 4 are presented in Figure 4 (left). From this figure, we readily observe that \( u(\cdot, x) \) eventually tends to a constant which...
depends only on $x$. In Figure 4 (right) we have plotted $|u(t, x) - U(x)|$ for $x = 2, 3$ and 4 to confirm that $u$ indeed converges to the nonzero steady state $U$ pointwise.

\textbf{Example 5.4.} Assume that

\[ d(x) = 6 - 3e^{-x}, \quad B(0, x) = e^{-x}, \quad g(x) = \sqrt{x}, \quad u_0(x) = 2, \quad \forall x \in (0, 20). \]

In this example also we notice that inequality (9) does not hold for the functions $B$ and $g$. As in the previous example, we have computed a nontrivial steady state $U$ (a numerical solution to (10)). In Figure 5 (left), we have presented the solution to (1) at $t = 12$ and a nontrivial solution to (10). In Figure 5 (right) we have shown the absolute difference between $u(12, x)$ and $U(x)$. In Figure 6 (left), graphs of $u(t, 1), u(t, 2)$ and $u(t, 4)$ are depicted. In this example also we observe that for every fixed value of $x$, the solution is eventually constant. Furthermore, we have computed $|u(12, x) - U(x)|$ for $x = 1, 2$ and 4 to establish that $u(t, x)$ converges to $U(x)$ at $x = 1, 2$ and 4, see Figure 6 (right). These numerical experiments suggest that the $u$ converges pointwise to a corresponding nonzero steady state with time.
6. Conclusions. We have proved the existence and uniqueness of a bounded classical solution to the nonlocal McKendric–von Foerster equation with diffusion. Moreover, an existence and uniqueness result has been established for the corresponding steady state equation also. If the nonlinear function \( g \) in the boundary condition is such that \( g(0) = 0 \), then the solution to \( (1) \) converges uniformly to the trivial steady state with an exponential rate in time. On the other hand, we have established that if the initial data is trapped in between a pair of sub and supersolutions of the steady state problem then the solution to \( (1) \) converges pointwise to the nontrivial solution to \( (10) \) for large time. We have also performed numerical simulations to show that the solution \( u \) converges to the nontrivial solution \( U \) to the steady state equation for large time. We have provided some examples where the crucial assumption in Section 4, i.e., inequality \( (9) \), does not hold but still the numerical experiments show that \( u \) converges to \( U \) as time increases. In view of these empirical results,
Figure 5. Solutions to (1) and (10) with $d, g, B$ and $u_0$ given in Example 5.4. Left: $u(12, x)$ (continuous line) and $U(x)$ (dotted line) for $0 \leq x \leq 4$, Right: The absolute difference between $u(12, x)$ and $U(x)$.

Figure 6. Solution to (1) with $d, g, B$ and $u_0$ given in Example 5.4. Left: $u(t, 1)$ (continuous line), $u(t, 2)$ (dashed line) and $u(t, 4)$ (dotted line) for $0 \leq t \leq 8$, Right: $|u(t, 2) - U(2)|$ (continuous line), $|u(t, 3) - U(3)|$ (dashed line) and $|u(t, 4) - U(4)|$ (dotted line).

we believe that the assumption (9) can be weakened in order to get the pointwise convergence of $u(t, .)$ to a nontrivial steady state $U(\cdot)$ as $t \to \infty$.

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