Action of derived automorphisms on infinity-morphisms

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Abstract

In this paper we investigate how to simultaneously change homotopy algebras of a certain type and a corresponding infinity morphism between them, and show that this can be done in a homotopically unique way. More precisely, for a reduced cooperad $C$, given $\Omega(C)$-algebras $V$ and $W$ and an $\infty$-morphism $U : V \rightsquigarrow W$, for any $\varphi \in \text{Der}(\Omega(C))$ we produce new $\Omega(C)$-algebras and a new $\infty$-morphism $U^\varphi : V^\varphi \rightsquigarrow W^\varphi$, that are unique up to homotopy. Operads play the central role in answering this question, in particular a 2-colored operad $Cyl(C)$ that governs pairs of $\infty$-algebras and $\infty$-morphisms between them.

1 Introduction

Homotopy algebras and morphisms appear in many areas throughout mathematics; in homological algebra in the form of algebraic transfer theorems, in geometry in the study of iterated loop spaces, in deformation quantization in Kontsevich’s formality theorem, and so on. Much work has been done to find the correct framework in which to study homotopy algebras, and the theory of operads is one such attempt. The complicated coherence relations that define homotopy algebras are encoded in the language of operads, which are easily manipulated with homological or combinatorial techniques; see [11] for an excellent overview of these techniques. This is the approach we take in this paper.

While there is a notion of morphisms between homotopy algebras of a specific type, in practice and theory one is interested in the looser notions of $\infty$-morphisms, which themselves satisfy some complicated system of coherence relations. Since homotopy algebras can be defined as algebras over a specific operad, it seems natural to ask if $\infty$-morphisms can be defined in the language of operads. The answer is provided in [9] via a 2-colored “cylinder construction” operad, which we restate and study further in this paper; similar ideas were also considered in [12], [1] and [8].

Our main goal is to answer the following question; given a pair of homotopy algebras and an $\infty$-morphism between them, can we change the homotopy algebras and the $\infty$-morphism simultaneously to get new homotopy algebras and a new $\infty$-morphism (all of the same type)? More specifically, given a derivation of the operad $\Omega(C)$ governing the homotopy algebras $V$ and $W$, we can exponentiate that derivation to an automorphism of $\Omega(C)$ and use that automorphism to define new $\Omega(C)$-algebra structures on $V$ and $W$; can we do the same to a $\infty$-morphism between $V$ and $W$, to create a new $\infty$-morphism that respects the new $\Omega(C)$-algebra structures? We show that this is possible, and moreover that the answer is unique up to homotopy, using the previously mentioned techniques of operadic homological algebra, and in particular the aforementioned cylinder construction.

We briefly indicate how our results may be applied to justify a claim made in Section 10.2 of Thomas Willwacher’s paper [14], concerning the action of the Grothendieck-Teichmüller group $\text{GRT}_1$ on formality morphisms. The Grothendieck-Teichmüller group and Lie algebra are connected to Drinfeld associators, the absolute Galois group of $\mathbb{Q}$, Kontsevich’s graph complex, the
2 Preliminaries

First, we set notation and recall some basic definitions and facts. For a general introduction to the theory of operads, see [6] or [11]. Throughout, we work in the category of cochain complexes of graded vector spaces over a field $k$ of characteristic zero, $\text{Ch}_k$. We will freely use the abbreviation “dg” to stand for the phrase “differential graded.” Given an operad $O$,

$$\mu : O(n) \otimes O(k_1) \otimes \ldots \otimes O(k_n) \to O(k_1 + \ldots + k_n)$$

will denote operadic multiplication, while

$$\circ_i : O(n) \otimes O(k) \to O(n + k - 1)$$

denotes the $i$-th elementary insertion, as usual. Dually, given a cooperad $C$,

$$\Delta : C(k_1 + \ldots + k_n) \to C(n) \otimes C(k_1) \otimes \ldots \otimes C(k_n)$$

denotes the cooperadic comultiplication, while

$$\Delta_i : C(n + k - 1) \to C(n) \otimes C(k)$$

denotes the $i$-th elementary coinsertion. Occasionally, the more general (co)multiplications

$$\mu_t : \underline{O}_n(t) \to O(n) \quad \quad \Delta_t : C(n) \to \underline{C}_n(t)$$

from [6] will be needed, where $\underline{O}_n$ is the functor associated to the collection $O$ from the category $\text{Tree}(n)$ of $n$-labeled rooted planar trees to $\text{Ch}_k$, and $t \in \text{Tree}(n)$ (likewise for $\underline{C}_n$ and $C$). Later, we will need two subcategories of $\text{Tree}(n)$. The first is $\text{Tree}_2(n)$, the full subcategory of $\text{Tree}(n)$ consisting of trees with exactly 2 internal vertices. The second is $\text{PF}_k(n)$ (“pitchforks”), the full subcategory of $\text{Tree}(n)$ consisting of trees with exactly $k + 1$ internal vertices, one of which has height 1, and the other $k$ have height exactly 2. Some examples of such trees can be found in figures 1 and 2.

If an operad is augmented, $O_\circ$ will denote the kernel of the augmentation map, while $C_\circ$ will denote the cokernel of the coaugmentation map of a coaugmented cooperad. Throughout, $\partial_O$ will denote the differential of an operad $O$, and likewise $\partial_C$ for a cooperad $C$. 

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The endomorphism operad of a dg vector space $V$, $\text{End}_V$, is of particular importance, since it allows us to define algebras over an operad. We have $\text{End}_V(n) = \text{Hom}(V^\otimes n, V)$, with operadic multiplication defined by function composition, and the symmetric action defined by rearranging tensor factors. Then we say that $V$ is an algebra over the operad $\mathcal{O}$ (or that $V$ is an $\mathcal{O}$-algebra) if we have a map of operads

$$\mathcal{O} \rightarrow \text{End}_V$$

or, equivalently, we have multiplication maps

$$\mu_n : \mathcal{O}(n) \otimes V^\otimes n \rightarrow V$$

for all $n \geq 0$, satisfying appropriate associativity, equivariance, and unit axioms [13]. This second formulation leads to the definition of $\mathcal{C}$-coalgebras: we say that $V$ is a coalgebra over the cooperad $\mathcal{C}$ if we have comultiplication maps

$$\Delta_n : V \rightarrow \mathcal{C}(n) \otimes V^\otimes n$$

satisfying appropriate dual axioms. This then leads to the definitions of the free $\mathcal{O}$-algebra

$$\mathcal{O}(V) = \bigoplus_{n \geq 0} (\mathcal{O}(n) \otimes V^\otimes n)_{S_n}$$

and the cofree $\mathcal{C}$-coalgebra

$$\mathcal{C}(V) = \bigoplus_{n \geq 0} (\mathcal{C}(n) \otimes V^\otimes n)_{S_n}.$$

Given a collection $\mathcal{Q}$, we can form the free operad $\mathcal{OP}(\mathcal{Q})$. It is generally most convenient to think of elements of $\mathcal{OP}(\mathcal{Q})$ as rooted trees with internal vertices decorated by elements of $\mathcal{Q}$, subject to an appropriate symmetry relation; with this in mind, $(t; x_1, \ldots, x_k)$ is the element of $\mathcal{OP}(\mathcal{Q})(n)$ where the $n$-labeled tree $t$ has $k$ internal vertices decorated by the elements $x_1, \ldots, x_k$ of $\mathcal{M}$, according to the total ordering on internal vertices. We will frequently identify an element
\( x \in Q(n) \) with the standard \( n \)-corolla decorated by \( x \) in \( \mathcal{OP}(Q)(n) \). It is occasionally useful to remember that at each level \( \mathcal{OP}(Q)(n) = \operatorname{colim} \mathcal{Q}_n \) (strictly, speaking, this only defines the free pseudo-operad, with the free operad then obtained by formally adjoining a unit). Since the \( S_n \) action on \( \mathcal{OP}(Q)(n) \) permutes the labels, in this paper we will omit these labels when drawing elements of \( \mathcal{OP}(Q) \).

Then, given a coaugmented cooperad \( C \), we can define the cobar construction \( \Omega C \) to be \( \mathcal{OP}(s C) \) as an operad of graded vector spaces (where \( s \) is the suspension operator of degree 1), with the differential defined on generators \( s x \in s C \) by

\[
\partial \Omega(s x) = -s \partial C(x) - \sum_{z \in \operatorname{Isom}(\text{Tree}_2(n))} (-1)^{|x_1|} (t_z; s x_1, s x_2)
\]

where the sum is taken over all isomorphism classes of \( \text{Tree}_2(n) \), \( t_z \) is a representative of the isomorphism class \( z \in \operatorname{Isom}(\text{Tree}_2(n)) \), and \( \Delta t_z(x) = \sum x_1 \otimes x_2 \). Note that we use Sweedler type notation in the above equation, and will continue to do so throughout the paper.

Much of the above notation extends immediately to the colored setting, so we will only focus on certain ideas and notation; our primary reference is [4]. We will focus on the 2-colored setting for now, and the 3-colored versions should be clear (and will only be minimally needed in this paper). We will refer to our 2 colors as \( \alpha \) and \( \beta \). Given a 2-colored collection \( Q \), \( Q(a, b; \alpha) \) denotes the level of \( Q \) with \( a \) inputs of color \( \alpha \), \( b \) inputs of color \( \beta \), and output of color \( \alpha \). Similarly, \( Q(a, b; \beta) \) indicates that the output is of color \( \beta \). As in the single-color case, we have a free colored operad construction, governed by colored trees. The category of colored \( n \)-labeled trees is defined similarly to \( \text{Tree}(n) \), except that the edges carry colors, and morphisms must respect the coloring. In figures, edges with color \( \alpha \) will be represented by solid lines, while edges of color \( \beta \) will be represented by dashed lines.

As before, we will need two subcategories of colored \( n \)-labeled trees, but slightly more specialized than above. The first is \( \text{Tree}_2'(n) \), the full subcategory of \( \text{Tree}(n) \) consisting of trees with exactly 2 internal vertices, such that the root edge carries color \( \beta \), and all other edges carry color \( \alpha \). The second is \( \mathcal{PF}_k'(n) \), the full subcategory of colored \( \text{Tree}(n) \) consisting of trees with exactly \( k + 1 \) internal vertices, one of which has height 1, and the other \( k \) have height exactly 2, such that all leaf edges carry color \( \alpha \) and all other edges carry color \( \beta \). Some examples of such trees can be found in figures 3 and 4.

Finally, recall that a coaugmented cooperad \( C \) is called reduced if

\[
C(0) = \{0\} \quad C(1) = k
\]

and hence \( C_0(0) = C_0(1) = \{0\} \). We will assume that all cooperads are reduced for the remainder of the paper.

### 3 The 2-colored operad \( C_{yl}(C) \)

As mentioned in the introduction, our ultimate goal is to study homotopy algebras and \( \infty \)-morphisms using the techniques of operads. We will focus on the operad theory and results first, and later show
First, given a coaugmented cooperad $C$, define the 2-colored collection $\tilde{C}$ as follows:

$$\tilde{C}(n, 0; \alpha) = \tilde{C}(0, n; \beta) = sC_o(n)$$

$$\tilde{C}(n, 0; \beta) = C(n)$$

$$\tilde{C} = 0 \quad \text{otherwise.}$$

Then form the free 2-colored operad $OP(\tilde{C})$. We think of elements of $OP(\tilde{C})$ as 2-colored trees with vertices decorated by elements of $sC_o$ and $C$, such that vertices have incoming edges only of a single color, and vertices have input color $\alpha$ and output color $\beta$ exactly when the vertex is decorated by an element of $C$.

Note that unlike $\Omega(C)$, elements of $OP(\tilde{C})$ may have vertices decorated by unsuspended elements of $C$, and since $C = k \oplus C_o$, in particular may have vertices decorated by elements of $k$. We will reserve the notation $1_{\alpha\beta} \in OP(\tilde{C})(1, 0; \beta)$ for the 1-corolla decorated by $1 \in k$. This leads us to an alternate notion of degree: given $X \in OP(\tilde{C})$ or $X \in \Omega(C)$, we will say that the weight of $X$, or $\text{wt}(X)$, is the number of internal vertices of $X$ not decorated by elements of $k$ (in $\Omega(C)$, weight is just the number of internal vertices). We will say that a map $F$ has weight $m$ if it raises weight by exactly $m$. Note finally that we have $\Omega(C) \subseteq OP(\tilde{C})$ by declaring that an element $X \in \Omega(C)$ has edges only of color $\alpha$ or only of color $\beta$; we will denote these assignments by $X^\alpha$ and $X^\beta$, respectively. Similarly, given $x \in C(n)$, we will write $x^{\alpha\beta} \in OP(\tilde{C})(n, 0; \beta)$ to indicate the corolla with $n$ incoming edges of color $\alpha$ and outgoing edge of color $\beta$; this is consistent with our earlier notation for $1^{\alpha\beta} \in OP(\tilde{C})(1, 0; \beta)$. We will often mark vectors with superscripts in this way to clearly indicate their input and output colors.

On $OP(\tilde{C})$, define a derivation $\partial$ on generators as follows:

$$\partial(s \ x^\alpha) = \partial_{\Omega}(s \ x^\alpha) \quad \text{for} \quad s \ x \in \tilde{C}(n, 0; \alpha) = sC_o(n)$$

$$\partial(s \ x^\beta) = \partial_{\Omega}(s \ x^\beta) \quad \text{for} \quad s \ x \in \tilde{C}(0, n; \beta) = sC_o(n)$$

$$\partial(1^{\alpha\beta}) = 0 \quad \text{for} \quad 1 \in \tilde{C}(n, 0; \beta) = C(n)$$

$$\partial(x^{\alpha\beta}) = \partial_c(x)^{\alpha\beta} + \partial'(x^{\alpha\beta}) + \partial''(x^{\alpha\beta}) \quad \text{for} \quad x \neq 1 \in \tilde{C}(n, 0; \beta) = C(n)$$

how they relate to homotopy algebras, after giving a more thorough introduction to that subject.
where $\partial'$ is defined by

$$\partial'(x^{\alpha\beta}) = \sum_{z \in \text{Isom}(\text{Tree}'_2(n,0;\beta))} (-1)^{|x_1|} (t_z; x_1, s x_2)$$

with $t_z$ a representative of the isomorphism class $z \in \text{Isom}(\text{Tree}'_2(n))$ and $\Delta_{t_z}(x) = \sum x_1 \otimes x_2$, and where $\partial''$ is defined by

$$\partial''(x^{\alpha\beta}) = -\sum_k \sum_{z \in \text{Isom}(\text{PF}'_k(n,0;\beta))} (t_z; s x_0, x_1, \ldots, x_k)$$

with $t_z$ a representative of the isomorphism class $z \in \text{Isom}(\text{PF}'_k(n,0;\beta))$ and $\Delta_{t_z}(x) = \sum x_0 \otimes x_1 \otimes \ldots \otimes x_k$, and where both comultiplications are taken by forgetting the coloring on $t_z$. A visual interpretation of the last definition is found in figure 5.

$$\partial(x^{\alpha\beta}) = \partial_C(x) + \sum_{z \in \text{Isom}(\text{Tree}'_2(n,0;\beta))} (-1)^{|x_1|} (t_z; x_1, s x_2)$$

$$-\sum_k \sum_{z \in \text{Isom}(\text{PF}'_k(n,0;\beta))} (t_z; s x_0, x_1, \ldots, x_k)$$

Figure 5: The differential on $\text{Cyl}(C)$.

$\partial$ visibly has degree 1, and so with the following proposition, we see that $\partial$ gives $\mathcal{OP}(\tilde{C})$ the structure of a dg operad. Following [9], we will call this operad $\text{Cyl}(C)$.

**Proposition 3.1.** $\partial^2 = 0$.

**Proof.** The proof is a technical computation in the same spirit as showing $\partial^2 = 0$. It suffices to show that $\partial^2 = 0$ on corollas, and since it is clear for single-color corollas, we will give the general ideas behind the computation for mixed-color corollas. Since $\partial = \partial_C + \partial' + \partial''$, we have that $\partial^2 = 0$ from the following observations:
1. \( \partial^2 C = 0 \) because \( \partial C \) is a differential on \( C \);

2. \( \partial C \circ \partial' + \partial' \circ \partial C = \partial C \circ \partial' + \partial' \circ \partial C = 0 \) because \( \partial C \) is a coderivation;

3. \( \partial' \circ \partial' = 0 \) because of coassociativity;

4. \( \partial' \circ \partial'' + \partial'' \circ \partial' + \partial'' \circ \partial'' = 0 \) because of coassociativity and elementary combinatorial identities.

\( \square \)

The importance of \( \text{Cyl}(C) \) is that it governs pairs of homotopy algebras and \( \infty \)-morphisms between them, as we will later show. For now, we proceed to study \( \text{Cyl}(C) \) in more depth. Given that \( \text{Cyl}(C) \) is essentially a 2-colored modification of \( \Omega(C) \), one would expect their cohomology to be related somehow. This is indeed the case, at least if we restrict our attention to the weight 0 components of their respective differentials. Explicitly, the weight 0 component of \( \partial \Omega \) is just \( \partial C \); explicitly,

\[ \partial_C(s x) = -s \partial_C(x) \]

for \( s x \in s C_0 \). On \( \text{Cyl}(C) \), the weight 0 part of \( \partial \), to be denoted \( \partial_0 \), is given explicitly by

\[
\begin{align*}
\partial_0(s x^\alpha) &= \partial_C(s x)^\alpha & s x &\in \tilde{C}(n, 0; \alpha) = s C_0(n) \\
\partial_0(s x^\beta) &= \partial_C(s x)^\beta & s x &\in \tilde{C}(0, n; \beta) = s C_0(n) \\
\partial(1^\alpha) &= 0 & 1 &\in \tilde{C}(n, 0; \beta) = C(n) \\
\partial_0(x^\alpha^\beta) &= \partial_C(x)^\alpha^\beta + \partial_0^\prime(x^\alpha^\beta) + \partial_0^\prime(x^\alpha^\beta) & 1 \neq x &\in \tilde{C}(n, 0; \beta) = C(n)
\end{align*}
\]

where

\[ \partial_0^\prime(x^\alpha^\beta) = 1^\alpha \circ_1 s x^\alpha \]

and where

\[ \partial_0^\prime(x^\alpha^\beta) = -\mu(s x^\beta; 1^\alpha^\beta, \ldots, 1^\alpha^\beta). \]

It may seem as though we have overused the notation \( \partial_C \) by now, but all such uses are really just the original \( \partial_C \) acting as a derivation on a free operad, respecting suspensions and/or coloring. A visual representation of the action of \( \partial_0 \) on mixed-color generators is found in figure 6.

From weight considerations (or directly checking), both \( \partial^2 C = 0 \) and \( \partial^2_0 = 0 \), so we may consider \( \Omega(C) \) and \( \text{Cyl}(C) \) with respect to these simpler differentials. Then we have:

**Theorem 3.2.** The inclusion maps

\[
\begin{align*}
\iota_\alpha, \iota_\beta : \ (\Omega(C)(n), \partial_C) &\longrightarrow (\text{Cyl}(C)(n, 0; \beta), \partial_0) \\
\iota_\alpha : \ X &\mapsto 1^\alpha \circ_1 X^\alpha \\
\iota_\beta : \ X &\mapsto \mu(X^\beta; 1^\alpha^\beta, \ldots, 1^\alpha^\beta)
\end{align*}
\]
\[ \partial_0(x^{\alpha\beta}) = \partial_C(x) + s x - 1 \]

Figure 6: The weight 0 component of the differential on \( Cyl(C) \).

are quasi-isomorphisms for all \( n \geq 0 \), and furthermore, are homotopic.

**Proof.** Given that we will show that \( \iota_\alpha \) is homotopic to \( \iota_\beta \), it suffices to show that \( \iota_\beta \) is a quasi-isomorphism; we will begin with this. Introduce the following filtrations on \( \Omega(C)(n) \) and \( Cyl(C)(n; 0; \beta) \):

\[
\mathcal{F}_m^\Omega(C)(n) = \left\{ X \in \Omega(C)(n) \mid \text{(the number of edges in } X) - |X| \leq m \right\}
\]

\[
\mathcal{F}_m^Cyl(C)(n; 0; \beta) = \left\{ X \in Cyl(C)(n; 0; \beta) \mid \text{(the number of edges of color } \alpha \text{ in } X) - |X| \leq m \right\}
\]

These filtrations are ascending, cocomplete, and compatible with \( \iota_\beta \) (since they are essentially the same filtration). They also respect \( \partial_C \) and \( \partial_0 \); in particular, note that \( \partial_C \) and \( \partial_0 \) raise internal degree without changing the number of (straight) edges, so they lower the filtration index, while \( \partial_0 \) raises internal degree and the number of straight edges, so it preserves the filtration index. Consequently, when we consider the associated graded complexes, we have

\[
\text{Gr}_\mathcal{F} \Omega(C)(n) = (\Omega(C)(n), 0)
\]

\[
\text{Gr}_\mathcal{F} Cyl(C)(n; 0; \beta) = (Cyl(C)(n; 0; \beta), \partial_0)
\]

By Appendix A of [6], it suffices to show that \( \iota_\beta : (\Omega(C)(n), 0) \to (Cyl(C)(n; 0; \beta), \partial_0) \) is a quasi-isomorphism. For the remainder of this first section of the proof, when we refer to those complexes, they will carry those differentials.

For this, we need an auxiliary construction. Define the 3-colored collection \( Q \), with colors \( \alpha, \beta, \gamma \), by

\[
Q(a, 0, 0; \alpha) = s C_\alpha(a) \quad \text{with } \partial_Q = 0
\]

\[
Q(0, b, c; \beta) = s C_\beta(b + c) \quad \text{with } \partial_Q = 0
\]

\[
Q(a, 0, 0; \beta) = C_\alpha(a) \oplus s C_\alpha(a) \quad \text{with } \partial_Q : x \to s x
\]

\[
Q = 0 \quad \text{otherwise.}
\]

Note that \( H^\bullet(Q(0, b, c; \beta)) = H^\bullet(Q(b + c, 0, 0; \alpha)) = s C_\alpha(b + c) \), while \( H^\bullet(Q(a, 0, 0; \beta)) = 0 \).

When we form \( \circ \mathcal{P}(Q) \), we have that

\[
Cyl(C)(n; 0; \beta) \cong \bigoplus_{m=0}^n \circ \mathcal{P}(Q)(m, n - m; \beta)
\]
via the (backwards) identification $\mathcal{O}\mathbb{P}(\mathcal{Q})(m, 0, n - m; \beta) \to \text{Cyl}(\mathcal{C})(n, 0; \beta)$ determined by the following rules. First, send edges of color $\gamma$ to the element $1^{\alpha\beta} \in \text{Cyl}(\mathcal{C})(1, 0; \beta)$. Then perform the following identifications:

\[
\begin{align*}
\mathsf{s} \ x^\alpha & \in \mathcal{Q}(a, 0, 0; \alpha) & \mapsto & \mathsf{s} \ x^\alpha \in \tilde{\mathcal{C}}(a, 0; \alpha) \\
\mathsf{s} \ x^\beta & \in \mathcal{Q}(0, b, 0; \beta) & \mapsto & \mathsf{s} \ x^\beta \in \tilde{\mathcal{C}}(0, b; \beta) \\
x^{\alpha\beta} & \in \mathcal{C}_o \subseteq \mathcal{Q}(a, 0, 0; \beta) & \mapsto & x^{\alpha\beta} \in \tilde{\mathcal{C}}(a, 0; \beta) \\
\mathsf{s} \ x^{\alpha\beta} & \in \mathcal{S}\mathcal{C}_o \subseteq \mathcal{Q}(a, 0, 0; \beta) & \mapsto & 1^{\alpha\beta} \circ_1 \mathsf{s} \ x^\alpha \in \text{Cyl}(\mathcal{C})(a, 0; \beta)
\end{align*}
\]

An example of this identification is shown in figure 7.

![Diagram](image)

Figure 7: An element of $\text{Cyl}(\mathcal{C})$ (on the left) identified with an element of $\mathcal{O}\mathbb{P}(\mathcal{Q})$ (on the right). The dotted lines indicate edges of color $\gamma$.

It is not hard to check that this identification is an isomorphism of cochain complexes, and consequently

\[
H^*(\text{Cyl}(\mathcal{C})(n, 0; \beta)) \cong \bigoplus_{m=0}^{n} H^*(\mathcal{O}\mathbb{P}(\mathcal{Q})(m, 0, n - m; \beta))
\]

Since $\mathcal{O}\mathbb{P}(\mathcal{Q})(m, 0, n - m; \beta)$ is colim from a finite, disjoint union of connected groupoids (specifically, the groupoids consisting of members of isomorphism classes of 3-colored $n$-labeled planar trees), and carries only the differential structure coming from $\mathcal{Q}$, Lemma A.1 applies. In particular, since taking coinvariants is exact when working over a field of characteristic 0, we have from lemma A.1 that

\[
H^*(\mathcal{O}\mathbb{P}(\mathcal{Q})(m, 0, n - m; \beta)) = \mathcal{O}\mathbb{P}(H^*(\mathcal{Q}))(m, 0, n - m; \beta).
\]

But if $m > 0$, any element of $\mathcal{O}\mathbb{P}(\mathcal{Q})(m, 0, n - m; \beta)$ must contain at least one vertex decorated by an element of $\mathcal{Q}(a, 0, 0; \beta)$. Since $H^*(\mathcal{Q}(a, 0, 0; \beta)) = 0$, we have in this case that $\mathcal{O}\mathbb{P}(H^*(\mathcal{Q}))(m, 0, n - m; \beta) = 0$ also. On the other hand, if $m = 0$, all vertices are decorated
by elements of $Q(0, b, c; \beta)$, and in this case we have $H^\bullet(Q(0, b, c; \beta)) = Q(0, b, c; \beta)$. Consequently,

$$H^\bullet(Cyl(C)(n, 0; \beta)) \cong H^\bullet(\mathcal{O}(Q)(0, 0, n; \beta)) = \mathcal{O}(H^\bullet(Q))(0, 0, n; \beta) = \mathcal{O}(Q)(0, 0, n; \beta).$$

Passing back to $Cyl(C)(n, 0; \beta)$ via the earlier isomorphism, we see that

$$\mathcal{O}(Q)(0, 0, n; \beta) \cong \mathcal{O}(\Omega(C)(n)) \subseteq Cyl(C)(n, 0; \beta)$$

which shows that $\iota_\beta$ is a quasi-isomorphism; therefore $\iota_\beta$ is a quasi-isomorphism for the original complexes, as desired.

It remains to show that $\iota_\alpha$ is homotopic to $\iota_\beta$ in $Cyl(C)$ with the original differential $\partial_0$. Observe that in $Cyl(C)$, the presence of vertices decorated by $1^{\alpha \beta}$ is determined completely by the coloring of adjacent vertices, and whether they are decorated by suspended vectors or not. Therefore, given $X \in \Omega(C)$, we may define $X_i \in Cyl(C)$ by declaring that $X_i$ has the same underlying tree as $X$, it has the same internal vectors as $X$ but that the $i$th (nontrivial) vertex is no longer suspended (using the total order on vertices), that the edges before (nontrivial) vertex $i$ are of color $\beta$ and the edges after are of color $\alpha$ (using the total order on edges), and then finally adding trivial vertices $1^{\alpha \beta}$ and edges of color $\beta$ as necessary to make $X_i$ a legal element of $Cyl(C)$. Figure 8 provides an example of this construction.

![Figure 8](image-url)

**Figure 8:** The construction of $X_2 \in Cyl(C)$ given $X \in \Omega(C)$.

We may now construct the homotopy between $\iota_\alpha$ and $\iota_\beta$. Given $X = (t; s x_1, ..., s x_k) \in \Omega(C)$, define $h : \Omega(C) \to Cyl(C)$ by

$$h(X) = \sum_{i=1}^k (-1)^{|s x_1| + ... + |s x_{i-1}|} X_i.$$
We may think of the sign in the above term coming from the suspension decorating the $i$th nodal vertex $x_i$ “jumping over” the vertices $s x_1, ..., s x_i-1$ to leave the tree. Since $\partial_C(s x) = - s \partial_C(x)$ for $x \in C$, we have that $\partial_C \circ h + h \circ \partial_C = 0$. It is also easy to check that $\iota_\alpha - \iota_\beta = (\partial'_0 + \partial''_0) \circ h$; $\partial'_0$ applied to the the first term of $h(X)$ yields $\iota_\alpha(X)$, $\partial''_0$ applied to the last term of $h(X)$ yields $- \iota_\beta(X)$ (the sign from $h$ will cancel with the sign coming from $\partial''_0$ “jumping over” the nontrivial vertices before the final vertex $x_k$), and all middle terms cancel from similar sign considerations. We therefore have in general that $\iota_\alpha - \iota_\beta = \partial_0 \circ h + h \circ \partial_C$, which shows that $\iota_\alpha$ and $\iota_\beta$ are homotopic, which completes the proof.

In fact, Theorem 3.2 is true with respect to the full differentials on $\Omega(C)$ and $Cyl(C)$, not simply the weight 0 parts.

**Corollary 3.3.** The inclusion maps

$$
\begin{align*}
\iota_\alpha, \iota_\beta : \Omega(C)(n), \partial_\Omega &\longrightarrow \text{(Cyl(C)(n, 0; \beta), \partial)} \\
\iota_\alpha : X &\mapsto 1^{\alpha_1} \circ X^\alpha \\
\iota_\beta : X &\mapsto \mu(X^{\beta}, 1^{\alpha_1}, ..., 1^{\alpha_\beta})
\end{align*}
$$

are quasi-isomorphisms for all $n \geq 0$, and furthermore, are homotopic.

**Proof.** The argument that $\iota_\beta$ is a quasi-isomorphism is very similar, but requires an initial modification. First filter $\text{Cyl(C)(n, 0; \beta)}$ and $\Omega(C)(n)$ by weight and form the associated graded complexes; this then gives us the exact situation of Theorem 3.2, and the result holds.

A different argument is needed to show that $\iota_\alpha$ and $\iota_\beta$ are homotopic. For this we introduce the map $\Pi : \text{Cyl(C)(n, 0; \beta)} \longrightarrow \Omega(C)(n)$ defined as follows. If $X \in \text{Cyl(C)(n, 0; \beta)}$ contains any nontrivial mixed-color vertices (that is, vertices decorated by elements of $C_\circ$, $\Pi(X) = 0$. Otherwise, define $\Pi(X)$ by changing all edges to color $\alpha$ and delete all trivial mixed vertices $1^{\alpha_\beta}$, merging the adjacent edges; the result is an element of $\Omega(C)$ because $X$ contained no nontrivial mixed vertices. Figure 9 gives an example of this. It is easy to check that $\Pi$ is a map of cochain complexes and that $\Pi$ is a one-sided inverse to both $\iota_\alpha$ and $\iota_\beta$:

$$
\Pi \circ \iota_\alpha = 1_{\Omega(C)} = \Pi \circ \iota_\beta.
$$

Since we already know that $\iota_\beta$ is a quasi-isomorphism, it follows that $\iota_\alpha$ and $\iota_\beta$ induce the same map on cohomology, and therefore are homotopic.

Recall that a derivation of an operad $O$ is a $k$-linear map

$$D : O(n) \longrightarrow O(n)$$

for all $n \geq 0$ that satisfies an appropriate Leibniz rule:

$$D(X_1 \circ_i X_2) = D(X_1) \circ_i X_2 + (-1)^{|D||X_1|} X_1 \circ_i D(X_2)$$
Theorem 3.4. The maps

$$\text{res}_\alpha, \text{res}_\beta : \text{Der}(\text{Cyl}(\mathcal{C})) \rightarrow \text{Der}(\Omega(\mathcal{C}))$$

given by restricting to a single color $\alpha$ or $\beta$ are quasi-isomorphisms of dg Lie algebras.

Proof. It is clear that the above restriction maps are morphisms of dg Lie algebras. Since derivations are uniquely determined by their action on generators, we may equivalently consider the maps

$$\text{res}_\alpha, \text{res}_\beta : \text{Hom}(\tilde{\mathcal{C}}, \text{Cyl}(\mathcal{C})) \rightarrow \text{Hom}(s\mathcal{C}, \Omega(\mathcal{C}))$$

still determined by restricting to a single color $\alpha$ or $\beta$. Here, the differentials on $\text{Hom}(\tilde{\mathcal{C}}, \text{Cyl}(\mathcal{C}))$ and $\text{Hom}(s\mathcal{C}, \Omega(\mathcal{C}))$ take the following form:

$$\partial(F) = \partial \circ F - (-1)^{|F|} \hat{F} \circ \partial$$

where the map $F$ defined on generators extends uniquely to the derivation $\hat{F}$. This way, the identification from derivations to morphisms respects the differential structure.

Introduce the following descending filtration on $\text{Hom}(\tilde{\mathcal{C}}, \text{Cyl}(\mathcal{C}))$:

$$\mathcal{F}_m \text{Hom}(\tilde{\mathcal{C}}, \text{Cyl}(\mathcal{C})) = \{ F \in \text{Hom}(\tilde{\mathcal{C}}, \text{Cyl}(\mathcal{C})) \mid \text{wt}(F) \geq m \}.$$
Introduce the same filtration on \( \text{Hom}(sC_\circ, \Omega(C)) \). Note that we could have equivalently defined these filtrations on \( \text{Der}(Cyl(C)) \) and \( \text{Der}(\Omega(C)) \) (and will do so later in the paper). These filtrations are complete and compatible with the appropriate differentials and the restriction maps \( \text{res}_\alpha, \text{res}_\beta \).

As before, we will move to the associated graded complexes which carry simpler differentials; from Lemma E.1 of [3], it suffices to show that the restriction maps are quasi-isomorphisms in this simpler setting. When we move to the associated graded complexes for this filtration, only the part of the differentials coming from the weight 0 part of the internal differentials survives. Explicitly, \( \text{Hom}(\tilde{C}, Cyl(C)) \) carries the reduced differential

\[
\partial(F) = \partial_0 \circ F - (-1)^{|F|} \hat{F} \circ \partial_0
\]

for \( F \in \text{Hom}(\tilde{C}, Cyl(C)) \), and \( \text{Hom}(sC_\circ, \Omega(C)) \) carries the differential

\[
\partial(F) = \partial_c \circ F - (-1)^{|F|} \hat{F} \circ \partial_c
\]

for \( F \in \text{Hom}(sC_\circ, \Omega(C)) \).

Restricting our attention to level \( n \), we now decompose \( \text{Hom}(\tilde{C}, Cyl(C)) \) into subspaces (not subcomplexes) based on how a derivation acts on different color generators:

\[
\text{Hom}(sC_\circ(n), \Omega(C)(n))^\alpha + \text{Hom}(\Omega(C), Cyl(C)(n, 0; \beta))^\alpha \beta + \text{Hom}(sC_\circ(n), \Omega(C)(n))^\beta.
\]

Here, the first summand gives the action of a derivation on corollas purely of color \( \alpha \), the second summand on mixed-color corollas, and the third summand on corollas of color \( \beta \); the superscripts make this explicit. Before we can state how the differential structure respects this decomposition, we need to recall the earlier maps

\[
\iota_\alpha, \iota_\beta : \Omega(C)(n) \longrightarrow Cyl(C)(n, 0; \beta)
\]

\[
\iota_\alpha : X \mapsto 1^{\alpha \beta} \circ_1 x^{\alpha}
\]

\[
\iota_\beta : X \mapsto \mu(x^{\beta}, 1^{\alpha \beta}, \ldots, 1^{\alpha \beta})
\]

and use them to define new, degree 1 maps:

\[
\text{incl}_\alpha, \text{incl}_\beta : \text{Hom}(sC_\circ(n), \Omega(C)(n)) \longrightarrow \text{Hom}(\bar{C}(n), Cyl(C)(n, 0; \beta))
\]

\[
\text{incl}_\alpha(F)(x) = (-1)^{|F|} \iota_\alpha(F(sx))
\]

\[
\text{incl}_\beta(F)(x) = (-1)^{|F|} \iota_\beta(F(sx))
\]

for \( F \in \text{Hom}(sC_\circ(n), \Omega(C)(n)) \) and \( x \in \mathcal{C}(n) \). It is then straightforward to check that with respect to the above decomposition of \( \text{Hom}(\tilde{C}, Cyl(C)) \), \( \partial \) acts as follows:

\[
\partial(F + F' + F'') = \partial(F) - \text{incl}_\alpha(F) + \partial(F') + \text{incl}_\beta(F'') + \partial(F'')
\]

where \( F \in \text{Hom}(sC_\circ(n), \Omega(C)(n))^\alpha, F' \in \text{Hom}(\bar{C}(n), Cyl(C)(n, 0; \beta))^\alpha \beta, F'' \in \text{Hom}(sC_\circ(n), \Omega(C)(n))^\beta, \) and where

\[
\partial(F) = \partial_c \circ F - (-1)^{|F|} \hat{F} \circ \partial_c
\]

\[
\partial(F') = \partial_0 \circ F' - (-1)^{|F'|} \hat{F'} \circ \partial_0
\]

\[
\partial(F'') = \partial_c \circ F'' - (-1)^{|F''|} \hat{F''} \circ \partial_c.
\]
As a cochain complex, \( \text{Hom}(\tilde{C}, Cyl(C)) \) is therefore a “cylinder-type construction” as described in [7, Appendix A], and the maps \( \text{res}_\alpha \) and \( \text{res}_\beta \) are the natural projections onto the first and third summands. By the same reference, it is enough to show that the maps

\[
\text{s}^{-1} \text{incl}_\alpha, \text{s}^{-1} \text{incl}_\beta : \text{Hom}(\text{s}C_\circ(n), \Omega(C)(n)) \longrightarrow \text{s}^{-1} \text{Hom}(C(n), Cyl(C)(n, 0; \beta))
\]

are quasi-isomorphisms. But this is precisely the situation obtained by applying the functor \( \text{Hom}(\text{s}C_\circ, -) \) to the maps

\[
\iota_\alpha, \iota_\beta : \Omega(C)(n) \longrightarrow Cyl(C)(n, 0; \beta)
\]

and we know those maps are quasi-isomorphisms from Theorem 3.2. Since \( \text{Hom}_{S_n} \) is exact when working over a field of characteristic 0, \( \text{incl}_\alpha \) and \( \text{incl}_\beta \) are also quasi-isomorphisms. From the results of [7, Appendix A], we conclude that the maps \( \text{res}_\alpha, \text{res}_\beta \) are quasi-isomorphisms in the associated graded setting, and therefore in general as well.

To show that \( \text{res}_\alpha \) and \( \text{res}_\beta \) are homotopic, we will show that they induce the same map on cohomology. Recall the map \( \Pi : Cyl(C)(n, 0; \beta) \rightarrow \Omega(C)(n) \) from the proof of Corollary 3.3. Given closed \( D \in \text{Der}(\text{Cyl}(C)) \), define \( T \in \text{Der}(\Omega(C)) \) on generators by

\[
T(sx) = (-1)^{|D|}(\Pi \circ D)(x^{\alpha_\beta}).
\]

\( D \) is closed, so in particular

\[
0 = [\partial, D](x^{\alpha_\beta}) = (\partial \circ D)(x^{\alpha_\beta}) - (-1)^{|D|}(D \circ \partial)(x^{\alpha_\beta}).
\]

If we rearrange the above terms and apply \( \Pi \) we obtain the equation

\[
(\Pi \circ D \circ \partial)(x^{\alpha_\beta}) = (-1)^{|D|}(\Pi \circ \partial \circ D)(x^{\alpha_\beta}) = (-1)^{|D|}(\partial \circ \Pi \circ D)(x^{\alpha_\beta}) = (\partial \circ T)(sx)
\]

recalling that \( \Pi \) is a cochain map. It is straightforward to check that

\[
(\Pi \circ D \circ \partial)(x^{\alpha_\beta}) = (\text{res}_\alpha D - \text{res}_\beta D - (-1)^{|D|}(T \circ \partial))(sx)
\]

and so we substitute this into the previous equation and rearrange terms to see that

\[
(\text{res}_\alpha D - \text{res}_\beta D)(sx) = (\partial \circ T + (-1)^{|D|}T \circ \partial)(sx) = \partial(T)(sx).
\]

Thus \( \text{res}_\alpha \) and \( \text{res}_\beta \) induce the same map on cohomology, and hence are homotopic.

\[\square\]

A final remark: since the proof of Theorem 3.4 relies on the weight filtration \( F \), it would be true if we replaced \( \text{Der}(-) \) by \( F_m \text{Der}(-) \) for any \( m \geq 0 \). In particular, we will need the inverse quasi-isomorphism \( F_1 \text{Der}(\Omega(C)) \rightarrow F_1 \text{Der}(\text{Cyl}(C)) \).
4 Acting on homotopy structures by derivations

We will now turn our attention to homotopy algebras, and show how the operadic techniques and results developed earlier may be applied to their study. All of [6], [10], and [11] offer thorough introductions to the subject. Given a coaugmented, reduced cooperad $C$, we will use the following “pedestrian” definition of homotopy algebras: a homotopy algebra of type $C$ to be an algebra $V$ over $\text{Cobar}(C)$. That is, we have a map of operads

$$F : \text{Cobar}(C) \longrightarrow \text{End}_V.$$ 

The complicated systems of coherence relations needed in explicit definitions of homotopy algebras are built into the compatibility of the above map with the differentials. This is actually equivalent [6] to a coderivation $Q_V$ on $C(V)$, the cofree coalgebra generated by $V$ over the cooperad $C$, that satisfied the Maurer-Cartan equation.

While there is a natural notion of morphisms of $O$-algebras for any operad $O$, in this setting we have the richer notion of $\infty$-morphisms. More specifically, an $\infty$-morphism between two homotopy algebras $V, W$ of type $C$ is a map of dg coalgebras

$$U : (C(V), \partial_V + Q_V) \longrightarrow (C(W), \partial_W + Q_W).$$

We denote an $\infty$-morphism by $U : V \rightsquigarrow W$. We will use $\text{Cyl}(C)$ to study such morphisms, as indicated in the following proposition.

**Proposition 4.1.** A $\text{Cyl}(C)$-algebra structure on a pair of dg vector spaces $(V, W)$ is equivalent to the following triple of data:

1. a map $\Omega(C) \to \text{End}_V$;
2. a map $\Omega(C) \to \text{End}_W$;
3. an $\infty$-morphism $V \rightsquigarrow W$.

**Proof.** Following [4], given cochain complexes $V$ and $W$, let $\text{End}_{V,W}$ be the 2-colored endomorphism operad. The pair $V, W$, being algebras over $\text{Cyl}(C)$ means that there is a map of colored operads

$$F : \text{Cyl}(C) \longrightarrow \text{End}_{V,W}.$$ 

By construction, the single-color portions of the above map correspond exactly to maps

$$F_\alpha : \Omega(C) \longrightarrow \text{End}_V,$$

$$F_\beta : \Omega(C) \longrightarrow \text{End}_W.$$ 

Observe next that the mixed-color portion of $F$ can be expressed in terms of its components

$$F_{\alpha\beta}(n) : \text{Cyl}(C)(n, 0; \beta) \longrightarrow \text{End}_{V,W}(n, 0; \beta).$$
Equivalently,

\[ F_{\alpha\beta}(n) : \mathcal{C}(n) \longrightarrow \operatorname{Hom}_k(V^\otimes n, W). \]

This is, in turn, equivalent to a map

\[ U_n : (\mathcal{C}(n) \otimes V^\otimes n)^{S_n} \longrightarrow W. \]

which extends uniquely to (and is uniquely determined by) a coalgebra map

\[ U : \mathcal{C}(V) \longrightarrow \mathcal{C}(W). \]

Finally, it is a straightforward check that the compatibility of \( F \) with the differentials on \( \text{Cyl} \mathcal{C} \) and \( \text{End}_{V,W} \) is equivalent to \( U \) being a dg coalgebra map, respecting the coderivations \( Q_V \) and \( Q_W \) (which correspond to \( F_{\alpha} \) and \( F_{\beta} \)). \( \square \)

It is easy to see that, given an operad \( \mathcal{O} \), an \( \mathcal{O} \)-algebra \( V \) via the map \( F : \mathcal{O} \rightarrow \text{End}_V \), and an endomorphism \( \varphi \) of \( \mathcal{O} \), the composite \( F \circ \varphi \) defines a new \( \mathcal{O} \)-algebra structure on \( V \). This also holds true for colored operads and algebras over them. Thus, given a pair \( (V, W) \) that is an algebra over \( \text{Cyl} \mathcal{C} \), via the operad morphism \( F : \text{Cyl} \mathcal{C} \rightarrow \text{End}_{V,W} \), and given an endomorphism \( \varphi \) of \( \text{Cyl} \mathcal{C} \), the morphism \( F^\varphi = F \circ \varphi \) defines a new \( \text{Cyl} \mathcal{C} \)-algebra structure on \( (V, W) \). This result is very general, but requires that we start with an endomorphism of \( \text{Cyl} \mathcal{C} \), which may be difficult to construct. However, we can use the results of the preceding section to construct such an endomorphism out of some initial, more naturally-occurring data. We have the following result:

**Proposition 4.2.** Given a degree 0 derivation \( D \in F_1 \text{Der}(\Omega \mathcal{C}) \) for a reduced cooperad \( \mathcal{C} \), \( D \) is locally nilpotent: for all \( X \in \Omega \mathcal{C} \), \( D^m(X) = 0 \) for some \( m \geq 0 \). Consequently, assuming \( D \) is a cocycle, we may exponentiate \( D \) to an automorphism of \( \Omega \mathcal{C} \),

\[
\exp(D) = \sum_{m=0}^{\infty} \frac{1}{m!} D^m.
\]

Here, we use the same filtration as in the proof of Theorem 3.4.

**Proof.** It is a standard result that a locally nilpotent derivation exponentiates to an automorphism, so it is enough to show that, under our assumptions, all derivations \( D \in F_1 \text{Der}(\Omega \mathcal{C}) \) are locally nilpotent. This follows from straightforward weight considerations, since \( D \) raises weight by at least 1, and since all nodal vertices of \( \Omega \mathcal{C} \) have at least 2 incoming edges for reduced \( \mathcal{C} \). \( \square \)

We have an identical result for derivations of \( \text{Cyl} \mathcal{C} \).

**Proposition 4.3.** Given a degree 0 cocycle \( D \in F_1 \text{Der}(\text{Cyl} \mathcal{C}) \) for a reduced cooperad \( \mathcal{C} \), \( D \) is locally nilpotent, and therefore may be exponentiated to an automorphism of \( \text{Cyl} \mathcal{C} \).

**Proof.** The same weight argument works here as for \( \Omega \mathcal{C} \), with the observation that for a vertex to have only a single incoming edge, it must be decorated by \( 1^{\alpha\beta} \in \text{Cyl} \mathcal{C}(1, 0; \beta) \). \( \square \)
At this point, we can state and prove the main, and desired, theorem of the paper. Indeed, the proof is essentially just referencing the previous techniques and results and interpreting them correctly.

**Theorem 4.4.** Let $V$ and $W$ be $\Omega(C)$-algebras for a reduced cooperad $C$, and let $U : V \rightsquigarrow W$ be an $\infty$-morphism between them. Given a degree 0 cocycle $D \in \mathcal{F}_1 \text{Der}(\Omega(C))$, we may define new $\Omega(C)$-algebra structures on $V$ and $W$; call them $V^D$ and $W^D$. Furthermore, we may modify the $\infty$-morphism $U$ to a new $\infty$-morphism $U^D : V^D \rightsquigarrow W^D$ that is unique up to homotopy.

**Proof.** Let

$$F_V : \Omega(C) \longrightarrow \text{End}_V \quad F_W : \Omega(C) \longrightarrow \text{End}_W$$

be the $\Omega(C)$-algebra structure maps on $V$ and $W$, with the $\infty$-morphism $U : V \rightsquigarrow W$. Given $D \in \mathcal{F}_1 \text{Der}(\Omega(C))$ use a homotopy inverse $\Phi : \text{Der}(\Omega(C)) \rightarrow \text{Der}(\text{Cyl}(C))$ to the quasi-isomorphisms of Theorem 3.4 to obtain the derivation $\Phi(D) \in \mathcal{F}_1 \text{Der}(\text{Cyl}(C))$. Proposition 4.3 then gives the automorphism $\exp(\Phi(D))$, which acts on $\text{Cyl}(C)$ to determine a new $\text{Cyl}(C)$-algebra structure on the pair $(V, W)$; this gives us our new $\Omega(C)$-algebras $V^D$ and $W^D$, and a new $\infty$-morphism $U^D : V^D \rightsquigarrow W^D$. This construction is unique up to homotopy because $\Phi$ is a quasi-isomorphism. □

As an example of a situation in which $H^0(\text{Der}(\Omega(C)))$ is known to be nonzero, [14] gives that $H^0(\text{Der}(\Omega(Ger^V))) \cong \text{grt}$, the Grothendieck-Teichmüller Lie algebra (see Section 5.2 of [6] for information on $\text{Ger}^V$). This leads to an application of Theorem 4.4 to justify a statement made in Section 10.2 of [14], concerning $\text{GRT}_1$-equivariance of Tamarkin’s construction of formality morphisms. While the full explanation is outside the scope of this paper, the idea is that while [14] details how $\text{GRT}_1$ acts on homotopy algebras, it does not explicitly explain how it acts on $\infty$-morphisms, a gap filled by our Theorem 4.4. A subsequent paper [2] will deal with this situation fully; alternately, in the stable setting, this question will be addressed in [5].

## A Colim from connected groupoids

**Lemma A.1.** Let $F : g \rightarrow \text{Ch}_k$ be a functor from a connected groupoid $g$. Then $\text{colim} \ F = F(a)_{\text{Aut}(a)}$, for any object $a \in g$.

**Proof.** Choose $a \in g$; we need to show that $F(a)_{\text{Aut}(a)}$ is a co-cone for $F : g \rightarrow \text{Ch}_k$, and that it is universal. That is, for any other co-cone $X$ for $F$, there is a unique map $\tau : F(a)_{\text{Aut}(a)} \rightarrow X$, and for any $a, b \in g$ there are maps $\pi_b : F(b) \rightarrow F(a)_{\text{Aut}(a)}$ and $\pi_c : F(c) \rightarrow F(a)_{\text{Aut}(a)}$, such that the following diagram commutes:
Note that we trivially have this for \( b = c = a \), where \( g \) is any automorphism of \( a \). Then \( \pi_b = \pi_c = \pi \), the canonical projection \( F(a) \rightarrow F(a)_{\text{Aut}(a)} \), and \( \tau \) exists and is unique because of the universal property of quotients. Since \( g \) is a connected groupoid we have maps \( h_{ba} : b \rightarrow a \) and \( h_{ca} : c \rightarrow a \) (for simplicity, let the inverses of these maps be denoted \( h_{ab} \) and \( h_{ac} \), respectively). Then we have the commuting diagram

where the left and right triangles commute because \( X \) is a co-cone for \( F \), and the top rectangle commutes by construction. This then gives us the first diagram with \( \pi_b = \pi \circ F(h_{ba}) \) and \( \pi_c = \pi \circ F(h_{ca}) \), and therefore \( \text{colim} \ F = F(a)_{\text{Aut}(a)} \). \qed

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