Crossover between universality classes in a magnetically disordered metallic wire

Guillaume Paulin\textsuperscript{1,3} and David Carpentier\textsuperscript{2}

\textsuperscript{1} Institut für Theoretische Physik, Universität zu Köln, Zülpicher Str. 77, 50937 Köln, Deutschland
\textsuperscript{2} CNRS—Laboratoire de Physique de l’Ecole Normale Supérieure de Lyon, 46, Allée d’Italie, 69007 Lyon, France

E-mail: gpaulin@thp.uni-koeln.de

New Journal of Physics \textbf{14} (2012) 023026 (25pp)
Received 14 October 2011
Published 13 February 2012
Online at http://www.njp.org/
doi:10.1088/1367-2630/14/2/023026

Abstract. In this paper, we present the numerical results of conduction in a disordered quasi-one-dimensional wire in the possible presence of magnetic impurities. Our analysis leads us to the study of universal properties in different conduction regimes, such as the localized and metallic ones. In particular, we analyze the crossover between universality classes occurring when the strength of magnetic disorder is increased. For this purpose, we use a numerical Landauer approach, and derive the scattering matrix of the wire from the electron’s Green’s function.
1. Introduction

The interplay between disorder and quantum interferences leads to one of the most remarkable phenomena in condensed matter: the Anderson localization of waves. The possibility of directly probing the properties of this localization with cold atoms [1, 2] has greatly renewed the interest in this fascinating physics. In this paper, we focus on the particular situation where electrons encounter two kinds of disorder: a usual scalar potential at the origin of diffusion and a magnetic potential, arising from a collection of frozen random magnetic moments. This situation is naturally realized experimentally in the study of transport properties of metallic spin glass wires [3–6]. In these wires, the spins freeze at low temperatures when entering the spin glass phase due to the frustrating magnetic couplings. In this glassy phase, and neglecting any residual Kondo effect in this regime, the impurities act effectively as a (weak) magnetic potential. We study numerically the effect of both types of disorder on the statistical properties of the wire conductance. In particular, we focus on the experimentally relevant crossover of (weak) localization properties of the wire as a function of the magnetic disorder strength.

One-dimensional (1D) disordered electronic systems are always localized. Following the scaling theory [7] this implies that by increasing the length $L_x$ of the wire for a fixed amplitude of disorder, its typical conductance ultimately reaches vanishingly small values. The localization length $\xi$ separates the metallic regime for small length $L_x \ll \xi$ from the asymptotic insulating regime. In the present paper, we focus on several universal properties of both metallic and insulating regimes of these wires in the simultaneous presence of two kinds of disorder. The first type corresponds to scalar potentials induced by the impurities, for which the system has time reversal symmetry (TRS) and spin rotation degeneracy. In this class the Hamiltonian...
belongs to the so-called Gaussian orthogonal ensemble (GOE) of the random matrix theory (RMT) classification \(^8\), corresponding to the class AI in the modern classification of Anderson universality classes (see e.g. \(^9\)). If impurities do have a spin, the TRS is broken as well as spin rotation invariance. The Hamiltonian is then a unitary matrix, which corresponds in RMT to the Gaussian unitary ensemble (GUE) with the breaking of Kramers degeneracy \(^10\) and to the Anderson class A \(^9\). However, for the experimentally relevant case of a magnetic potential weaker than the scalar potential, the system is described neither by the GUE class nor by the GOE class, but extrapolates in between. This intermediate regime, of particular relevance experimentally, is the main object of study of the present paper. Moreover, the present work paved the way for a numerical study of the correlation of conductances in the crossover regime \(^11\).

This paper is organized as follows: in section 2, the model and the numerical method used will be described in detail. In section 3, we identify the localized and metallic regimes of transport of the system. The localization length will be determined by two different methods and the crossover between both universality classes (GOE, GUE) will be highlighted. In section 4, the insulating regime is studied with a particular focus on the statistical distribution of the conductance, which allows us to highlight universal behavior. In section 5, we turn to the study of the metallic regime and perform a careful analysis of each of the first three cumulants of the statistical distribution of the conductance. We focus on the universal properties of conductance fluctuations, and the non-analyticity of the complete distribution is discussed. Finally, section 6 is devoted to the conclusion.

2. The model and the method

2.1. The model

In this paper, we study numerically the scaling of transport properties of wires in the presence of magnetic and scalar disorders. We will focus on the regime of phase coherent transport, reached experimentally at low temperature (see, in particular, \(^6\)). In this regime, the phase coherence length \(L_\phi\), which phenomenologically accounts for inelastic scattering of electrons on impurities \(^12\), is larger than (or comparable with) the wire’s length \(L_x\), so that phase coherence for the propagating electrons is preserved in the whole sample. Note that this phase coherence length \(L_\phi\) includes, in particular, a contribution from inelastic scattering on the non-frozen magnetic impurities through a Kondo dephasing, which is strongly reduced when entering the magnetic glass phase \(^6\).

We describe the behavior of electrons inside the disordered wire using a tight-binding Anderson lattice model with two kinds of disorder potentials:

\[
\mathcal{H} = \sum_{(i,j),s} t_{ij} c_{j,s}^\dagger c_{i,s} + \sum_{i,s} v_i c_{i,s}^\dagger c_{i,s} + J \sum_{i,s,s'} \mathbf{\hat{S}}_i \cdot \mathbf{\hat{S}}_{s,s'} c_{i,s}^\dagger c_{i,s'}.
\]  

(1)

\(t_{ij}\) is the hopping term from site \(i\) to \(j\). In the following, \(t_{ij}\) will take two different values: \(t_{ij} = t_{\parallel}\) in the longitudinal \(x\)-direction and \(t_{ij} = t_{\perp}\) in the transverse \(y\)-direction. The scalar disorder potential \(V = \{v_i\}_i\) is diagonal in electron-spin space. We choose the \(v_i\) to be random scalars uniformly distributed in the interval \([-W/2, W/2]\). In this work, we have chosen without loss of generality to fix \(t_{\perp} = 1\) so that all energy scales are relative to the bandwidth \(t_{\parallel} = 1\), and the amplitude of disorder \(W = 0.6\). In equation (1), \(s, s'\) label the SU(2) spin of electrons and the \(\mathbf{\hat{S}}_i\) account for spins of the frozen magnetic impurities. A realistic choice for these frozen
spins in a spin glass phase consists in considering classical spins with random orientations, thereby neglecting any small spatial correlation in a spin glass configuration [13]. The coupling $J$ between the electron spins and the magnetic impurities fixes the amplitude of the magnetic disorder. In this work, we will monitor the behavior of the transport properties of the sample as a function of this amplitude $J$, from $J = 0$ to $J = 0.4$. Indeed, variation of the amplitude of magnetic disorder $J$ allows us to extrapolate from GOE/class Al for $J = 0$ to GUE/class A for $J \neq 0$. We will also show in section 5 that, as a bonus, the presence of the magnetic disorder allows also a numerically easier settlement of the universal metallic regime of the wire.

For a given realization of both scalar $\{v_i\}_i$ and magnetic disorder $\{\vec{S}_i\}_i$, the Landauer conductance of this model on a 2D square lattice of size $L_x \times L_y$ is evaluated numerically using a recursive Landauer method described in detail in the next paragraph.

### 2.2. The method

We have chosen to use a numerical method based on the lattice model (1), as opposed to, e.g., the random matrix or Dorokhov–Mello–Pereyra–Kumar [14, 15] method, so as to provide a numerical study allowing for an easy comparison with the experimental situation [6]. Moreover, this method allows for further natural developments such as the study of the conductance change upon magnetic impurities spin flipping, which would be difficult to reach by an alternative method. Starting from a lattice model such as (1), the natural method providing the conductance of a finite size sample is based on the Landauer formalism [16].

We consider a two-terminal setup, with electrodes connected to the wire at $x = 0$ and at $x = L_x$. These electrodes are described as semi-infinite ribbons with the same transverse geometry as the sample, and described with (1) but without randomness. Electrons are then confined in the transverse $y$-direction via a potential that has the form

$$V(y) = 0 \quad \text{if } 0 \leq y \leq L_y; \quad V(y) = \infty \quad \text{otherwise.} \quad (2)$$

This confining potential in the $y$-direction leads to the appearance in the electrodes of $N$ modes propagating in the $x$-direction. The complete wave function of an electron in this tight binding lattice model reads then

$$\psi(x, y) = \phi_n(y) e^{ik_x x}, \quad (3)$$

where $k_x$ is the momentum of electrons in the longitudinal direction and

$$\phi_n(y) = \sqrt{\frac{2}{N_y + 1}} \sin \left( \frac{n \pi y}{N_y + 1} \right). \quad (4)$$

We used $N_y = L_y$ in units of lattice spacing. The group velocity of this mode reads

$$v_n = 2 \frac{t_{//}}{\hbar} \sin(k_x), \quad (5)$$

where $t_{//}$ is the longitudinal hopping amplitude. This velocity depends on the momentum $k_x$, which is determined for a constant energy by the dispersion relation

$$E_n = \mu - 2t_\perp \cos \left( \frac{n \pi}{N_y + 1} \right) - 2t_{//} \cos(k_x). \quad (6)$$
At given energy $E - \mu$, we end up with the following relation for the longitudinal part of the momentum of the electron:

$$k_x = \arccos \left( \frac{\mu - E}{2t_{//}} - \frac{t_{\perp}}{t_{//}} \cos \left( \frac{n\pi}{N_y + 1} \right) \right).$$

(7)

To optimize the efficiency of the numerical study, we fix $t_{//} = 2t_{\perp}$ and stay near the band center, avoiding in particular the presence of fluctuating states studied in [17].

To compute the conductance of such a wire, the Landauer–Büttiker formalism of coherent transport is used [18]. It allows us to relate the dimensionless conductance $g$ of a diffusive wire with the scattering matrix $T$:

$$g = \sum_{\text{modes } m,n} T_{mn},$$

(8)

where $T_{mn}$ is the transmission coefficient between modes $m$ and $n$. The dimensionless conductance $g$ is defined from the conductance $G$ of the system as $g = G/(2e^2/h)$ when spin degeneracy is present ($J = 0$), and $g = G/(e^2/h)$ otherwise ($J \neq 0$).

Following Fisher and Lee [19], we relate the scattering matrix to the electronic retarded Green functions of the system through

$$t_{mn} = i\hbar \sqrt{v_n v_m} \sum_{y,y'} \phi_n(y) G^R(y, x = 0|y', x = L_x) \phi_m(y')$$

(9)

with $T = \text{Tr}(t^\dagger t)$, and $v_n$ and $\phi_n$ (resp. $v_m$ and $\phi_m$) are the group velocity and the eigen wave function of propagating mode $n$ (resp. $m$). Mode $n$ belongs to the left lead, whereas mode $m$ belongs to the right one. In (9), $G^R(y, x = 0|y', x = L_x)$ represents the retarded Green’s function of an electron between the point $(x = 0, y)$ in the left contact between the system and the electrode and the point $(x = L_x, y')$ in the right contact. Consequently, by calculating the retarded Green’s functions of the system only between both sides of it, we are able to determine the dimensionless conductance of the wire. An efficient method of calculating $G^R(y, x = 0|y', x = L_x)$, which takes advantage of the quasi-1D nature of the ribbon, consists in obtaining it recursively [20]. In figure 1, the principle of the method is sketched: using a Dyson equation we deduce the boundary Green’s function of a system of size $n + 1$ from the corresponding Green’s function of a subsystem of size $n$, and the exact Green’s function of the $n + 1$ row. This allows us to perform matrix inversion only of the simple row system. At both initial and final steps, we reconnect the system to the semi-infinite electrodes (see figure 2) described by a standard self-energy:

$$\Sigma^R_{\text{bound}}(y_1, y_2) = -\frac{1}{t_{//}} \sum_{n=1}^{N_y} \phi_n(y_1) e^{i k_n} \phi_n(y_2).$$

(10)

The last step consists of combining the Green’s functions of the sample of desired longitudinal size with the one of the right lead, which is also given by equation (10).

The unitarity of the corresponding scattering matrix $[T_{mn}]_{m,n}$ is used to monitor the accuracy of the numerical method. Such a test yielded typical relative error of the order of $10^{-4}$ for a system size $L_x = 1600$ and $L_y = 40$. This method allows one to compute the conductance of a wire of length $L_x$ and width $L_y$ for any given configuration of scalar disorder $V$ and for any configuration of frozen classical spins $\{\vec{S}_i\}$.
**Figure 1.** Principle of recursive calculation of retarded Green’s functions of the wire. The use of a Dyson: $G^{R}_{n+1} = G^{R}_n + G^{R}_n V G^{R}_{n+1}$. At each step the longitudinal length $L_x$ is increased by one lattice spacing.

**Figure 2.** Boundary conditions: the wire is connected to two leads represented by two semi-infinite metallic wires.

In the next sections, we study the properties of this conductance for one given configuration of magnetic disorder and for many different configurations of scalar disorder. Universal properties are identified by varying the transverse length $L_y$ from 10 to 80, with the aspect ratio $L_x/L_y$ taken from 1 to 6000. The typical number of configurations of scalar disorder $V$ we used was $N_d = 5000$, with exceptions for the study of the localization properties where for $L_y = 10$ we sampled the conductance distribution for 50 000 different configurations of disorder.

### 3. Localization length

#### 3.1. Determination of $\xi$

We start our analysis with a determination of the parameters corresponding to the metallic and localized regimes, through a careful determination of the localization length of the system. While experimentally the only accessible regime of a phase coherent wire is the metallic regime, numerically this regime is difficult to reach and describe quantitatively, as opposed to the localized regime. This is related to the extreme reduction in the number of propagating modes in the numerical system which is associated with a corresponding reduction of the localization length. Hence, in order to clearly identify the conditions to access the universal properties of transport in the metallic regime, we start with a detailed determination of this localization length in the experimentally relevant crossover situation. Afterwards, we will take the opportunity of the present study to describe other characteristics of the localized regimes in the crossover situation, before turning to our main interest: the universal metallic regime.
The localization length separates short wires of metallic behavior from an insulating long wires. A first method to access the localization length $\xi$ from the conductance consists in considering the scaling behavior of the typical conductance $g_{\text{typ}}$ defined as

$$g_{\text{typ}} = e^{\langle \log g \rangle},$$

where $\langle \cdot \rangle$ represents the average over the different configurations of scalar disorder $V$. This typical conductance decays exponentially with the longitudinal length of the wire [21, 22]:

$$g_{\text{typ}} \sim e^{-2L_x/\xi}$$

in the regime of long wires $L_x \gg \xi$.

Figures 3–6 show the behavior of $\langle \log g \rangle$ as a function of longitudinal length $L_x$ for different widths $L_y$. Different curves correspond to different values of magnetic disorder. The linear fit of the large length part of the curves allows for a precise determination of the corresponding localization length for each value of $L_y$ and $J$.

A second method to determine this localization length consists in considering the Lyapunov exponent $\gamma$ of the transfer matrix of the system, following a standard RMT approach [21, 23, 24]. This exponent can be deduced from the conductance as

$$\gamma(L_x) = \frac{1}{2L_x} \log \left( 1 + \frac{1}{g(L_x)} \right),$$

and the localization length follows from its asymptotic behavior:

$$\xi^{-1} = \lim_{L_x \to \infty} \gamma(L_x).$$

In figures 7 and 8, we have plotted the Lyapunov exponent versus the inverse of the longitudinal length for different values of magnetic disorder. Different curves correspond to different widths of the wire.

With this method, a simple extrapolation of the curve is necessary to obtain $\xi$, without any fit, since $\gamma$ tends to the inverse of the localization length for an infinite wire. Nevertheless, we
found that this method shows less accuracy than the preceding one: as shown in figure 7 or 8, the Lyapunov exponent is still varying for the longest longitudinal length. We find that both methods give fully compatible results, while the Lyapunov exponent method requires much larger system sizes than the typical conductance method for a given required accuracy.

A first manifestation of the universality of the Anderson localization classes appears through the dependence of $\xi$ on the transverse length $L_y$ (or the number of propagating modes). It is expected to follow [21, 25]:

$$\xi = (\beta L_y + 2 - \beta)l_e,$$

where $l_e$ is the elastic mean free path and $\beta$ encodes the universal class of the model: $\beta = 1$ corresponds to the orthogonal universality class GOE, while $\beta = 2$ for GUE. Note that this
Figure 6. Evolution of $\langle \log g \rangle$ as a function of longitudinal size for different transverse lengths ($L_y = 10, 20, 30, 40$) and $J = 0.4$. The linear part of the curve allows one to get the localization length $\xi(J = 0.4)$ from the scaling form in the insulating regime $\langle \log g \rangle = -\frac{2L_x}{\xi}$.

Figure 7. Evolution of the Lyapunov exponent with the inverse of longitudinal length in a semi-log plot. Circles correspond to $J = 0$, squares to $J = 0.05$, diamonds to $J = 0.2$ and triangles to $J = 0.4$. The value of the transverse length is $L_y = 10$. The localization length can be extrapolated from the value of $\gamma$ for $L_x \to \infty$.

change in $\beta$ is accompanied by an artificial doubling of the number of transverse modes $N_y \equiv L_y \to 2N_y$ due to the breaking of Kramers degeneracy [21]. This effective factor 4 when breaking the spin rotation symmetry has been discussed in [26, 27] in detail, when discussing the magnetic field dependance of this localization length, in comparison with the random matrix and nonlinear sigma models. Comparison of numerical localization lengths for different $J$ with (15) is shown in figure 9. Excellent agreement is found for $J = 0$ (GOE class, $\beta = 1$) and with the GUE class for $J \geq 0.2$. For intermediate values of $J \neq 0$ we observe a crossover between the two extreme GOE and GUE laws, which cannot be described by equation (15), and for which no analytical work exists to our knowledge.
Figure 8. Evolution of the Lyapunov exponent with the inverse of longitudinal length in a semi-log plot. Circles correspond to $J = 0$, squares to $J = 0.05$, diamonds to $J = 0.2$ and triangles to $J = 0.4$. The value of the transverse length is $L_y = 20$. The localization length can be extrapolated from the value of $\gamma$ for $L_x \to \infty$.

Figure 9. Evolution of localization length as a function of transverse length. $l_e$ is the mean free path of the diffusive sample. A different behavior of the localization length is seen if $J = 0$ or $J \neq 0$. Inset: scaling of the typical conductance $\langle \log g \rangle = -\frac{2L_x}{\xi}$.

From these results, we also note that the localization regime is reached for much longer wires in the presence of magnetic impurities (GUE case) than without (GOE): localization is hampered by the presence of these magnetic impurities.

As we will discuss below, this property helps in observing numerically the universal weak localization regime and the associated universal conductance fluctuations (UCF).

3.2. The insulating and metallic regimes

The localization length helps discriminate between both insulating and metallic regimes: the ribbon behaves indeed as a metal ($g \gg 1$) for lengths $L_x \ll \xi$ and as an insulator ($g \ll 1$) if...
$L_x \gg \xi$. In both asymptotic regimes, the shape of the probability density function (PDF) of the conductance is known: it is log-normal for insulating wires [21] and Gaussian for metallic ones. By varying the longitudinal length we can study the evolution of this PDF from a Gaussian to a log-normal distribution, as shown in figure 10. This plot is created for a given value of $L_y$ and $J$. One can note that in the metallic regime, the PDF is very well approximated by a Gaussian for relatively small wires: the Gaussian regime is easily reached, whereas it takes a length much larger than the localization length for the distribution to become log-normal in the insulating regime. This point will be discussed more precisely below on the cumulants of this PDF.

The insulating regime is then characterized by $\langle g \rangle < 1$ and the metallic one by $\langle g \rangle > 1$.

In order to characterize samples by the average $\langle g \rangle$ and, in particular, plot higher cumulants as a function of $\langle g \rangle$, we now turn to a short study of the behavior of this first cumulant as a function of system size. In figure 11, we have plotted $\langle g \rangle(L_x)$ for different values of magnetic disorder $J$ (hence different localization lengths). These curves approximately collapse when plotted against the scaling variable $L_x/\xi$, as shown in figure 12. We remind the reader that for $J = 0$, the average conductance is defined by $g = G/2G_0$, which explains why $J = 0$ and $J = 0.4$ curves coincide in figure 11: according to figure 9, only for these values of magnetic disorder, universality classes are reached.

This study of $\langle g \rangle(L_x)$ allows us to proceed with the study of higher cumulants of the PDF of $g$ and test the prediction of the single-parameter scaling of distributions [28].

4. Universal insulating regime

4.1. Probability density functions

In the insulating regime $L_x \geqslant \xi$, we expect a log-normal conductance statistical distribution, as seen previously. However, in the weakly insulating regime $\langle g \rangle \lesssim 1$ this log-normal asymptotic...
Figure 11. Evolution of average conductance versus longitudinal length for different values of magnetic disorder for $L_y = 10$ and $J = 0, 0.05, 0.1, 0.2$ and $0.4$. The line $\langle g \rangle = 1$ is plotted as a frontier between insulating and metallic regimes.

Figure 12. Evolution of average conductance versus the scaling variable $L_x/\xi$, for $L_y = 10$ and $J = 0, 0.05, 0.1, 0.2$ and $0.4$. All curves nearly collapse in a single curve. The frontier between insulating and metallic is drawn again.

form is not reached. Instead, as shown in figure 10 we find a non-analytical behavior of $P(g)$ in agreement with [29–32].

In order to study the dependance of this non-analyticity on the universality class, we have plotted in figure 13 the distribution $P(g)$ for similar values of $\langle g \rangle$ but different magnetic strengths $J = 0$ (GOE) and $J = 0.2$ (GUE). The shapes of these distributions are highly similar if $\langle g \rangle \ll 1$, showing that distributions for $J = 0$ and $J \neq 0$ reach the same log-normal distribution at large system sizes, in agreement with the super-universality scenario [33]. In the intermediate regime ($\langle g \rangle \approx 1$), shapes are symmetry dependent. Moreover, we find that the non-analyticity appears for different values of conductance (close to 1) and the rate of
Figure 13. Comparison of PDF of conductance for $J = 0$ (plain curves) and $J = 0.2$ (dashed curves). Plots are performed for different values of average conductance. (a) $\langle g \rangle (J = 0) = 0.84$ and $\langle g \rangle (J = 0.2) = 0.79$. (b) $\langle g \rangle (J = 0) = 0.67$ and $\langle g \rangle (J = 0.2) = 0.62$. (c) $\langle g \rangle (J = 0) = 0.45$ and $\langle g \rangle (J = 0.2) = 0.42$. (d) $\langle g \rangle (J = 0) = 0.21$ and $\langle g \rangle (J = 0.2) = 0.18$.

the exponential decay [29] in the metallic regime seems to differ from one class to the other (see, for instance, curves (a) or (b)). Finally, the plot in figure 14 represents the distribution of conductance of mean value just above and below the threshold $\langle g \rangle = 1$. Plain lines represent Gaussian interpolations with a mean and a variance given by the first and the second cumulant of each numerical conductance distribution. For $\langle g \rangle > 1$ the Gaussian interpolation approximates the full distribution very well, while as soon as $\langle g \rangle < 1$ the Gaussian approximation only applies in the tail $g \geq 1$ of the distribution of conductance [32]. The shape of the distribution for $g < 1$ (figure 13) agrees qualitatively with the numerical results of [34] (see in particular their figure 4). Unfortunately, a more accurate comparison proves to be difficult due to the lack of an analytical description of the distribution. To quantize further these results on the whole distribution of $\log g$, we now turn to a quantitative study of the second and the third cumulant of this distribution.

4.2. Study of cumulants

This conductance distribution converges to the log-normal only deeply in the insulating regime, the convergence being very slow (much slower than in the metallic regime). This qualitative result is confirmed by the study of moments: in the insulating regime the second cumulant is expected to follow [25, 35]:

$$
\langle (\log g - \langle \log g \rangle)^2 \rangle = \langle (\log g)^2 \rangle_c = -2\langle \log g \rangle.
$$

(16)
Figure 14. PDF of conductance for $\langle g \rangle < 1$ ($J = 0.2$) and $\langle g \rangle > 1$ ($J = 0$) and Gaussian interpolations. $L_y = 10$.

Figure 15. Plot of the variance of $\log g$ as a function of the mean for the orthogonal ($J = 0$) and the unitary ($J = 0.2$) case, for $L_y = 10$ and 20. The slope of the linear fit is equal to $-1.88$. This plot also shows super-universality as the behavior of the second cumulant does not depend either on geometry of the wire or on the universality class.

Our numerical results are in agreement with this scaling with, however, very slow convergence towards this law: corrections are measurable even for the largest system size where the system is deeply in the localized state, as shown in figure 15. More precisely, we find that for the deep insulating regime $\langle (\log g)^2 \rangle = -1.88 \langle \log g \rangle$, with a slight discrepancy with (16).

This plot also shows that in the deep insulating regime the behavior of the second cumulant as a function of the first one does not depend on the value of magnetic disorder: both curves for $J = 0$ and $J = 0.2$ follow the same law. This is in agreement with our previous result on statistical distributions: there is a super-universal behavior in the deep insulating regime.
Finally, we have studied the third cumulant of $\log g$ scaled in figure 16 as a function of the first cumulant. The linear behavior for each value of magnetic disorder in the deep insulating regime ($\langle \log g \rangle < -4$) is in agreement with the single-parameter scaling. We find that contrary to the second cumulant the coefficient of proportionality between the skewness and the average depends on the symmetry of disorder, which denotes a lack of super-universality concerning this cumulant. For instance, dots and diamonds (which correspond to the case $J = 0$) have the same behavior, as opposed to the case $J = 0.2$ (squares or triangles). A systematic study of this point with even larger system sizes and other numerical methods would be of great interest but is definitely beyond the scope of the present paper.

The study of cumulants of the distribution of $\log g$ confirms the single-parameter scaling of the distribution, with a slight discrepancy concerning the value of the coefficient of proportionality between the second and the first cumulant. Moreover, super-universality has been highlighted concerning the second cumulant, but is lacking concerning the third cumulant.

4.3. Comparison with exact results in the crossover regime

For localization in wires connected to ideal contacts, the exact formulae for the average conductance [36] and conductance fluctuations [37] have been derived for the two universal orthogonal and unitary classes. These formulae are of particular interest in the intermediate regime between the metallic ($L_x \ll \xi$) and the deeply localized ($L_x \gg \xi$) regime. We have numerically evaluated the formulae (3.105) and (3.106) of [37] and we compare them with our numerical data in figures 17 and 18. We find excellent agreement between the $J = 0$ data and the orthogonal exact formulae, on the one hand, and the $J \neq 0$ data and the unitary formulae, on the other. This comparison naturally breaks down for small sizes where the quasi-1D assumption for diffusion breaks down and a non-universal regime takes place. As shown in figure 18, the exact unitary behavior is recovered for $J = 0.05$ beyond the crossover length $L_m$ (see the next section). Whether the scale dependence of $\langle g \rangle$ and $\langle \delta g^2 \rangle$ in the crossover regime ($J$ small such that $L_m \simeq \xi$) is amenable to an exact formula along the lines of [37] is an open question of great interest.

Figure 16. Plot of the skewness of $\log g$ as a function of the mean for the orthogonal ($J = 0$) and the unitary ($J = 0.2$) case, for $L_y = 10$ and $20$. 

New Journal of Physics 14 (2012) 023026 (http://www.njp.org/)
Figure 17. Comparison of the numerical average conductance $\langle g \rangle$ as a function of the reduced length $L_x/\xi$ with the exact expressions of [37] for the orthogonal ($J = 0$) and the unitary ($J = 0.2$) case, for various transverse sizes $L_y$.

Figure 18. Comparison of the numerical average conductance fluctuations $\langle \delta g^2 \rangle$ as a function of the reduced length $L_x/\xi$ with the exact expressions of [37] for the orthogonal ($J = 0$) and the unitary ($J = 0.4$) case, for various transverse sizes $L_y$.

5. Universal metallic regime

We now focus on the universal metallic regime described by weak localization. By definition weak localization corresponds to metallic diffusion, expected for lengths of wire $l_x \ll L_x \ll \xi$. For this regime to be reached, we thus need to increase $\xi$ through an increase of the number of
transverse modes $L_y$ with all other parameters fixed (see equation (15)). Moreover, for a fixed geometry, this regime will be easier to reach with magnetic impurities than without. As we saw in figure 13, the shape of the PDF of conductance is a truncated Gaussian in this regime. In the following, we study its three first cumulants quantitatively, starting with the variance.

5.1. Conductance fluctuations and universal crossover

In the weak localization regime, the conductance fluctuations $\langle g^2 \rangle_c$ are expected to be independent of the size of the system and only depend on the universal localization class of the model. Figure 19 shows that for a suitable value of transverse length, the system reaches the expected plateau in conductance fluctuations. The value of the plateau identifies with the expected values ($1/15$ and $4/15$ for GUE and GOE, respectively) with a high accuracy. However, the presence of this plateau depends strongly on the value of transverse length $L_y$ and on the magnetic disorder. Figures 20 and 21 illustrate this point with further details: these plots show the conductance fluctuations as a function of longitudinal size for $J = 0$ and $J = 0.2$ and for two values of transverse length $L_y$. In the first plot, for both values of magnetic disorder the universal plateau arises, whereas it appears only for $J = 0$ if $L_y = 40$.

The evolution of this variance of the PDF of $g$ depends on the longitudinal length through

$$\langle \delta g^2 \rangle = \langle g^2 \rangle_c = \frac{1}{4} F(0) + \frac{3}{4} F \left( x \sqrt{\frac{2}{3}} \right) + \frac{1}{4} F \left( x \sqrt{\frac{3}{2}} \right),$$

where $x = L_x/L_m$ and the scaling function $F(x)$ depends only on dimension [11, 12]. The universality occurs in this equation in the two limit $x = 0$ corresponding to $J = 0$ or $x \gg 1$ where $F(x) \to 0$. In both cases the variance becomes geometry independent. Moreover, this expression shows that the whole crossover between the two classes is described by a universal crossover function parameterized solely by the length $L_m$, called the magnetic dephasing length.
Figure 20. The second cumulant of $g$ as a function of longitudinal length, for $L_y = 80$. Dots represent data for $J = 0$ and squares for $J = 0.2$. The value of UCF is shown in each symmetry class (with or without magnetic disorder). The UCF regime is reached in both cases.

Figure 21. The second cumulant of $g$ as a function of longitudinal length, for $L_y = 40$. Dots represent data for $J = 0$ and squares for $J = 0.2$. The value of UCF is shown in each symmetry class (with or without magnetic disorder). The UCF regime is reached in the case of magnetic impurities but not for scalar impurities.

scale [12]. In figure 22, these conductance fluctuations are plotted as a function of longitudinal length $L_x$ for different values of $J$. A single-parameter fit by (17) provides the determination of the magnetic dephasing length $L_m$ as a function of the magnetic disorder $J$. The Universal behavior is also highlighted for strong enough magnetic disorder. The determination of this scattering length allows a precise study of average conductance, and in particular the study of the classical part as described in the next subsection. Figure 23 shows the scaling form of these fluctuations (as a function of $L_x/L_m(J)$) in excellent agreement with the theory (17). Moreover, for long wires (and large values of $J$) conductance fluctuations are no longer $L_x$ dependent and are equal to $1/15$, in units of $G_0$. This is the so-called UCF regime, which is precisely identified numerically in the present work.
Figure 22. Variance of $g$ as a function of longitudinal size. UCF are shown. Different curves correspond to different values of magnetic disorder $J$ ($J = 0.025 \rightarrow 0.4$). Transverse length $L_y = 40$. The only free parameter in analytical fits is the magnetic length $L_m$.

Figure 23. Variance of $g$ as a function of $L_x/L_m$. Transverse length $L_y = 40$.

The plot of figure 19 confirms the analytical results of [31] both qualitatively in the shape of the curves and quantitatively in the values of fluctuations in both universality classes. In our study, values of UCF are reached with a maximal error of 1% for GOE and 3% for GUE with respect to the analytical value of the UCF in the regime independent of $\langle g \rangle$ (i.e. with much higher precision than e.g [38] and [30]).

Let us finally comment on the work of Qiao et al [33] who have performed a similar numerical Landuer study of 1D transport in various universality classes, focusing mostly on the metallic regime. While both studies agree on the UCF (although we have a higher accuracy for $\beta = 1$), we did not find evidence for a second universal conductance plateau. This result definitely deserves further study.
5.2. The average conductance

The main contribution to the average conductance is of classical origin. However, a weak localization correction must be added when the quantum behavior of electrons is taken into account [12]. This quantum part manifests itself in the magneto-conductance behavior of long wires (larger than the phase coherence length) where a weak magnetic field is sufficient to destroy this quantum correction by dephasing the various diffusing paths with respect to each other [12]. We can write this average conductance as (without any magnetic field)

\[
\langle g \rangle(J, L_x, L_m) = g_{\text{class}}(J, L_x) + \delta g_{\text{WL}}(L_x, L_m),
\]

where \( g_{\text{class}} \) is the classical part of the conductance and \( \delta g_{\text{WL}} \) is the weak localization correction.

For a quasi-1D system the quantum correction has the simple form [12]

\[
\delta g_{\text{WL}} = \sum_{n=1}^{\infty} \left( \frac{-1/\pi^2}{n^2 + 2 (L_x/L_m)^2} - \frac{3/\pi^2}{n^2 + \frac{3}{2} (L_x/L_m)^2} \right).
\]

The knowledge of the magnetic length \( L_m \) we gained in the previous study on conductance fluctuations can now be used to completely characterize this weak localization contribution to the conductance. By subtracting the corresponding contribution to the average conductance, we obtain the classical conductance, plotted in figure 24 as a function of longitudinal length.

Taking into account the contact resistance [39] in the two-terminal setup, the expected expression for this classical conductivity reads

\[
g_{\text{class}}(J, L_x) = \frac{1}{\frac{1}{L_y} + \frac{L_x}{\sigma_0(J)}}.
\]

Figure 24 shows this expression plotted for different values of magnetic disorder. The corresponding fitting parameter \( \sigma_0 \) is plotted as a function of \( J \) in figure 25.
To compare these results with theory, consider the Einstein relation which links the (Einstein) conductivity to the diffusion constant:

$$\sigma_0 = s e^2 \rho_0(\varepsilon_F) D,$$

where $s$ is the spin degeneracy and $\rho_0(\varepsilon_F)$ the electronic density of states at the Fermi level. By definition, the diffusion coefficient reads, for non-magnetic impurities,

$$D = v_F^2 \tau_e,$$

with $v_F$ the Fermi velocity and $\tau_e$ the elastic scattering time. It can be related to the scalar disorder by

$$\tau_e = \frac{1}{2\pi \rho_0 n_i v_0^2},$$

where $n_i$ is the impurity density and $v_0^2 = W^2/12$. For more than one diffusive process, it is compulsory to use the Matthiesen rule that modifies scattering time $\tau_e$ in the following way:

$$\frac{1}{\tau_e} \rightarrow \frac{1}{\tau_e} + \frac{1}{\tau_m},$$

where $\tau_m = L_m^2/D$ and is related to the magnetic disorder:

$$\tau_m = \frac{1}{2\pi \rho_0 n_i J^2 \langle S^2 \rangle}.$$  

Using this allows one to get the $J$ dependance of the Einstein conductivity:

$$\sigma_0(J) = \frac{\sigma_0(J = 0)}{1 + \frac{3}{W^2} J^2}.$$  

In figure 25, we compare this expression with a numerical evaluation of the conductivity. The good agreement between both curves provides an additional check of the correct determination.
of the magnetic dephasing length $L_m$. Including the magnetic disorder dependance of the diffusion coefficient (through the Matthiesen rule), we obtain a perturbative expression to second order in $J$ for this magnetic dephasing length:

$$L_m(J) = \sqrt{D(J) \tau_m(J)} \propto \frac{1}{J \sqrt{\frac{W^2}{12} + \frac{J^2}{4}}}.$$  (27)

In figure 26, we have plotted the numerical evaluation of the magnetic length as a function of magnetic disorder and the corresponding fit with equation (27). We note that it is also possible to obtain the magnetic length via the study of correlations of conductance, i.e. via the study of $\langle g(V, \{S_i^{(1)}\}_i) g(V, \{S_i^{(2)}\}_i) \rangle_c$, which is beyond the scope of the present paper [11].

5.3. The third cumulant

Finally, we consider the third cumulant of the distribution of conductance. According to the analytical study of [31], this cumulant decays to zero in a universal way as $\langle g \rangle$ increases. Here in figure 28 we find the dependance of this decrease on the symmetry class: for GOE $\langle g^3 \rangle_c$ goes to zero in a monotonic way, whereas it decreases, changes its sign and then goes to zero in the GUE case. For $\langle g \rangle > 4$ numerical errors are dominant; then this part of the curve is irrelevant. Moreover, for GUE this decrease seems to be universal, whereas it depends on the transverse length for GOE. In figure 27 the convergence of the skewness is represented when increasing the number of configurations used to perform averages $N_{des}$ for both GOE and GUE. Plots show a good enough convergence of averages to conclude that the third cumulant of conductance is not zero for all values of $\langle g \rangle$. Note that the maximal number of averages is 50,000. Moreover, this fast vanishing of the third cumulant confirms the faster convergence of the whole distribution towards the Gaussian, compared to what happens in the insulating regime.
Based on our numerical results, we cannot confirm nor refute the expected law $\langle g^3 \rangle_c \propto 1/\langle g \rangle^n$, with $n = 2$ in GOE and $n = 3$ in GUE [40, 41].

6. Conclusion

To conclude, we have conducted extensive numerical studies of electronic transport in the presence of random frozen magnetic moments. Comparing and extending previous analytical and numerical studies, we have identified the insulating and metallic regimes described by the universality classes GOE and GUE. We have paid special attention to the dependence on this symmetry of cumulants of the distribution of conductance in both metallic and insulating universal regimes. In particular, we have identified with high accuracy the domain of UCF, and determined its extension in the present model. We have also determined precisely the so-called magnetic length $L_m$ which represents the elastic scattering length of the spin on magnetic impurities. This length is of primary importance in experiments as it controls the crossover...
between universality classes. This work paves the way for further studies of transport in metals with frozen magnetic impurities, since we have clearly identified the range of parameters to access the experimentally relevant metallic diffusive regime. One possible extension consists of considering the evolution of the statistics of conductance as the magnetic disorder is varied, e.g. by rotating or flipping the spins of impurities. Comparing the conductance obtained in both spin configurations mimics the experimental measurement of the conductance of a low-temperature canonical spin glass after two successive quenches [11, 42], without the necessary restrictions of analytical approaches [11]. Experimentally, this approach could give access to the fundamental properties of a spin glass, which have never been measured.

Acknowledgments

We thank X Waintal for useful discussions. This work was supported by the ANR grants QuSpins and Mesoglass. All numerical calculations were performed on the computing facilities of the ENS-Lyon calculation center (PSMN).

References

[1] Billy J, Josse V, Zuo Z, Bernard A, Hambrecht B, Lugen P, Clément D, Sanchez-Palencia L, Bouyer P and Aspect A 2008 Direct observation of anderson localization of matter waves in a controlled disorder Nature 453 891–4
[2] Roati G, D’Errico C, Fallani L, Fattori M, Fort C, Zaccanti M, Modugno G, Modugno M and Inguscio M 2008 Anderson localization of a non-interacting Bose–Einstein condensate Nature 453 895–8
[3] de Végvar P G N, Lévy L P and Fulton T A 1991 Conductance fluctuations of mesoscopic spin glasses Phys. Rev. Lett. 66 2380
[4] Jaroszynski J, Wrobel J, Karczewski G, Wojtowicz T and Dietl T 1998 Magnetococonductance noise and irreversibilities in submicron wires of spin-glass n+-cdmnte Phys. Rev. Lett. 80 5635
[5] Neuttiens G, Eom J, Strunk C, Pattyn H, Van Haesenendonck C, Bruynseraede Y and Chandrasekhar V 1998 Thermoelectric effects in mesoscopic AuFe spin-glass wire Europhys. Lett. 42 185
[6] Capron T, Perrat-Mabilon A, Peaucelle C, Meunier T, Carpentier D, Levy L P, Baueerle C and Saminadayar L 2011 Remanence effects in the electrical resistivity of spin glasses Europhys. Lett. 93 27001
[7] Abrahams E, Anderson P W, Liciardiello D C and Ramakrishnan T V 1979 Scaling theory of localization: absence of quantum diffusion in two dimensions Phys. Rev. Lett. 42 673
[8] Evers F and Mirlin A D 2008 Anderson transitions Rev. Mod. Phys. 80 1355
[9] Ryu S, Schnyder A P, Furusaki A and Ludwig A W W 2010 Topological insulators and superconductors: tenfold way and dimensional hierarchy New J. Phys. 12 065010
[10] Mirlin A D 2000 Statistics of energy levels and eigenfunctions in disordered systems Phys. Rep. 326 259
[11] Paulin G, Capron T, Carpentier D, Meunier T, Bäuerle C, Lévy L P and Saminadayar L 2011 Spin flip induced conductance fluctuations Preprint
[12] Akkermans E and Montambaux G 2007 Mesoscopic Physics of Electrons and Photons (Cambridge: Cambridge University Press)
[13] Mézard M, Parisi G and Virasoro M 1987 Spin Glass Theory and Beyond (Singapore: World Scientific)
[14] Dorokhov O N 1983 Zh. Eksp. Teor. Fiz. 85 1040
[15] Mello P A, Pereyra P and Kumar N 1988 Macroscopic approach to multichannel disordered conductors Ann. Phys. 181 290–317
[16] Landauer R 1957 Spatial variation of currents and fields due to localized scatterers in metallic conduction IBM J. Res. Dev. 1 223

New Journal of Physics 14 (2012) 023026 (http://www.njp.org/)
[17] Deych L I, Erementchouk M V and Lis yansky A A 2003 Scaling in the one-dimensional anderson localization problem in the region of fluctuation states Phys. Rev. Lett. 90 126601
[18] Buttiker M, Imry Y, Landauer R and Pinhas S 1985 Generalized many-channel conductance formula with application to small rings Phys. Rev. B 31 6207–15
[19] Fisher D S and Lee P A 1981 Relation between conductivity and transmission matrix Phys. Rev. B 23 6851–4
[20] Croy A, Roemer R A and Schreiber M 2006 Localization of electronic states in amorphous materials: recursive Green’s function method and the metal–insulator transition at E ≠ 0 Lecture Notes in Computational Science and Engineering ed K Hoffman and A Meyer (Berlin: Springer) pp 203–26
[21] Beenakker C W J 1997 Random-matrix theory of quantum transport Rev. Mod. Phys. 69 731
[22] Slevin K, Markos P and Ohtsuki T 2001 Reconciling conductance fluctuations and the scaling theory of localization Phys. Rev. Lett. 86 3594
[23] Markos P 1993 Weak disorder expansion of Lyapunov exponents of products of random matrices: a degenerate theory J. Stat. Phys. 70 899
[24] Pichard J-L 2001 Random matrix theory of scattering in chaotic and disordered media Waves and Imaging through Complex Media ed P Sebah (Dordrecht: Kluwer)
[25] Pichard J-L 1991 Quantum Coherence in Mesoscopic Systems (NATO ASI Series B254) ed B Kramer (New York: Plenum) p 369
[26] Larkin I 1983 Zh. Eksp. Teor. Fiz. 85 764
[27] Lerner I V and Imry Y 1995 Magnetic-field dependence of the localization length in anderson insulators Europhys. Lett. 29 49–54
[28] MacKinnon A and Kramer B 1981 One-parameter scaling of localization length and conductance in disordered systems Phys. Rev. Lett. 47 1546
[29] Muttalib K A and Wölfle P 1999 ‘One-sided’ log-normal distribution of conductances for a disordered quantum wire Phys. Rev. Lett. 83 3013
[30] Markos P 2002 Dimension dependence of the conductance distribution in the nonmetallic regime Phys. Rev. B 65 104207
[31] Froufe-Pérez L S, Garcia-Mochales P, Serena P A, Mello P A and Saenz J J 2002 Conductance distributions in quasi-one-dimensional disordered wires Phys. Rev. Lett. 89 246403
[32] Muttalib K A, Wölfle P, Garcia-Martín A and Gopar V A 2003 Nonanalyticity in the distribution of conductances in quasi-one-dimensional wires Europhys. Lett. 61 95
[33] Qiao Z, Xing Y and Wang J 2010 Universal conductance fluctuation of mesoscopic systems in the metal–insulator crossover regime Phys. Rev. B 81 085114
[34] Gopar V A, Muttalib K A and Wölfle P 2002 Conductance distribution in disordered quantum wires: crossover between the metallic and insulating regimes Phys. Rev. B 66 174204
[35] Beenakker C W J and van Houten H 1991 Quantum transport in semiconductor nanostructures Solid State Phys. 44 1
[36] Zirnbauer M R 1992 Super Fourier analysis and localization in disordered wires Phys. Rev. Lett. 69 1584
[37] Mirlin A D, Mueller-Groeling A and Zirnbauer M R 1994 Conductance fluctuations of disordered wires: Fourier analysis on supersymmetric spaces Ann. Phys. 236 325
[38] Cieplak M, Bulka B R and Dietl T 1991 Universal conductance fluctuations in spin glasses Phys. Rev. B 44 12337
[39] Engquist H L and Anderson P W 1981 Definition and measurement of the electrical and thermal resistances Phys. Rev. B 24 1151
[40] Macêdo A M S 1994 Random-matrix approach to the quantum-transport theory of disordered conductors Phys. Rev. B 49 1858
[41] van Rossum M C W, Igor Lerner V, Boris Altshuler L and Nieuwenhuizen Th M 1997 Deviations from the Gaussian distribution of mesoscopic conductance fluctuations Phys. Rev. B 55 4710
[42] Carpentier D and Orignac E 2008 Measuring overlaps in mesoscopic spin glasses via conductance fluctuations Phys. Rev. Lett. 100 057207