On a subclass of bi-univalent functions related to shell-like curves connected with Fibonacci number

Jenifer Arulmani¹*

Abstract
In this paper we defined a new subclass of bi-univalent functions related to shell-like curves connected with Fibonacci number using the Frasin differential operator. We find some coefficient bounds and solve the linear functional $|a_3^+ - \mu a_3^-|$. Also we obtained various results proved by several authors as particular cases.

Keywords
Bi-Univalent, Shell-like, Fibonacci Number, Differential operator.

AMS Subject Classification
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¹Department of Mathematics, Presidency College(Autonomous), Chennai-600005, Tamil Nadu, India.
*Corresponding author: jeniferarulmani06@gmail.com;
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1. Introduction
We denote by $A$ the class of regular functions defined in the open unit disk $\Delta = \{z/|z| < 1\}$ with the normalization conditions $f(0) = f'(0) - 1 = 0$ and the Taylor series expansion,

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k.$$ (1.1)

Consider $S$ to be the class of univalent functions in $A$. For any two analytic functions $f(z)$ and $g(z)$ in $\Delta$, we say that $f(z)$ is subordinate to $g(z)$ [9], (symbolically, $f \prec g$) if there exists a function $\phi(z)$ analytic in $\Delta$ satisfying $\phi(0) = 0$ and $|\phi(z)| < 1$ such that

$$f(z) = g(\phi(z)), (|z| < 1).$$

By the Koebe-one quarter theorem[4] (Theorem 2.3 pg.31) , we know that "The range of every function of the class $S$ contains a disk $\{w : |w| < 1/4\}". Hence there exists inverse $f^{-1}$ for every function $f \in S$, defined by

$$f^{-1}(f(z)) = z, (z \in \Delta); \quad \text{and}$$

$$f(f^{-1}(w)) = w, (|w| < r_0(f) : r_0(f) \geq 1/4).$$

Where the inverse of $f$ is given by,

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 w^2 - a_3) w^3 - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + ... = g(w).$$

A function $f \in A$ is said to be bi-univalent if both $f$ and $f^{-1}$ (its inverse) are univalent in $\Delta$. We denote by $\Sigma$ the class of bi-univalent and analytic functions in $\Delta$ of the form (1.1).
Using the binomial series,

$$(1 - \lambda)^m = \sum_{j=0}^{m} \binom{m}{j} (-1)^j \lambda^j,$$

$m \in \mathbb{N} = 1, 2, ... \quad \text{and} \quad j \in \mathbb{N}_0 = 0, 1, 2, ...$

Frasin [5] defined the following differential operator for function $f \in A$,

$$D^\phi f(z) = f(z)$$

$$D_{m,\lambda}^n f(z) = (1 - \lambda)^m f(z) + (1 - (1 - \lambda)^m) z f'(z) = D_{m,\lambda} f(z), \ (\lambda > 0; m \in \mathbb{N}).$$

In general,

$$D_{m,\lambda}^n = D_{m,\lambda}(D_{m,\lambda}^{n-1} f(z)), n \in \mathbb{N}_0$$

$$= z + \sum_{k=2}^{\infty} |1 + (k - 1)c_j^m(\lambda)|^n a_k z^k$$
where, \( c_j^n(\lambda) = \sum_{j=1}^{m} \binom{m}{j} (-1)^{j+1} \lambda^j \).

**Remarks:**

1. For \( m = 1 \), we get the Al-oboudi differential operator,
   \( D^n_{1,\lambda} \) [1].

2. For \( m = \lambda = 1 \), we get the Salagean differential operator,
   \( D^n \) [11].

For \( f \in A \) the class \( SL \) of shell-like functions which is the sub-class of the class \( S^\ast \) of starlike functions was first introduced by Sokol [12] in 1995 as below,

**Definition 1.1.** [12] A function \( f \in A \) having the series expansion (1.1) is said to be in the class \( SL \) of starlike shell-like functions if it satisfies the following conditions:

\[
\frac{zf'(z)}{f(z)} < \tilde{p}(z) = \frac{1 + \tau^2 z^2}{1 - \tau z - \tau^2 z^2}
\]

where \( \tau = \frac{(1-\sqrt{5})}{2} \approx -0.618 \).

In the year 2011, Dziok et al. [2], introduced the class \( KSL \) of convex functions related to a shell-like curves as follows:

**Definition 1.2.** [2] A function \( f \in A \) of the form (1.1) belongs to the class \( KSL \) of convex shell-like functions if it satisfies the following condition:

\[
1 + \frac{zf'(z)}{f(z)} < \tilde{p}(z) = \frac{1 + \tau^2 z^2}{1 - \tau z - \tau^2 z^2}
\]

where \( \tau = \frac{(1-\sqrt{5})}{2} \approx -0.618 \).

Again Dziok et al. [3] in the year 2011, defined the following class \( SLM_\alpha \) of \( \alpha \)-convex shell-like functions.

**Definition 1.3.** [3] A function \( f \in A \) of the form (1.1) belongs to the class \( SLM_\alpha \) of \( \alpha \)-convex shell-like functions if it satisfies the following condition:

\[
(1-\alpha)\left(1 + \frac{zf'(z)}{f(z)}\right) + \alpha \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} < \tilde{p}(z) = \frac{1 + \tau^2 z^2}{1 - \tau z - \tau^2 z^2}
\]

where \( \tau = \frac{(1-\sqrt{5})}{2} \approx -0.618 \).

We note that \( SLM_0 \equiv SL \) and \( SLM_1 \equiv KSL \). We consider \( \tau = \frac{1-\sqrt{5}}{2} \approx -0.618 \) and \( \tilde{p}(z) = \frac{1+\tau^2 z^2}{1-\tau z - \tau^2 z^2} \) throughout this paper.

The function \( \tilde{p}(z) \) does not belongs to the class \( S \). Since \( \tilde{p}(z) \) is univalent in the disc \( |z| < \tau^2 \approx 0.38 \). We can observe the following from \( \tilde{p}(z) \) [6]; \( \tilde{p}(0) = \tilde{p}(-1) = 1 \); \( \tilde{p} \) takes the unit circle to a curve described by \((10x - \sqrt{5})^2 = (\sqrt{5}x - 1)^2\), which is translated and revolved trisecrix of Maclaurin. The curve \( \tilde{p}(re^{i\theta}) \) is a closed curve without any loop for \( 0 < r \leq r_0 = \tau^4 \approx 0.38 \).

For \( r_0 < r < 1 \), it has a loop, and for \( r = 1 \) it has a vertical asymptote. In the year 2016, Raina and Sokol [10] proved the following,

\[
\tilde{p}(z) = \frac{1 + \tau^2 z^2}{1 - \tau z - \tau^2 z^2} = 1 + \sum_{n=1}^{\infty} \left( u_{n-1} + u_{n+1} \right) \tau^n z^n
\]

where \( u_n = \frac{(1-\tau)^n - \tau^n}{\sqrt{5}} \), such that

\[
u_n = u_{n-2} + u_{n-1} \text{ for } n = 2, 3, ...
\]

By simple calculation we can decompose all the higher powers \( \tau^n \) as a linear combination of \( \tau \) and 1. The resulting recurrence relationships yield Fibonacci number \( u_n \),

\[
\tau^n = u_n \tau + u_{n-1}.
\]

Thus \( \tilde{p}(z) \) is related to Fibonacci number. So we can rewrite \( \tilde{p}(z) \) as,

\[
\tilde{p}(z) = 1 + \sum_{n=1}^{\infty} \tilde{p}_n \tau^n z^n
\]

where \( \tilde{p}_n = (u_{n-1} + u_{n+1}) \). Now using (1.2) in (1.3) we have,

\[
\tilde{p}(z) = 1 + \tau z + 3 \tau^2 z^2 + 4 \tau^3 z^3 + ...
\]

(1.4)

Motivated by the works of earlier authors we define a new subclass of bi-univalent functions related to shell-like curves connected to Fibonacci number using Frasin differential operator.

**Definition 1.4.** A function \( f(z) \in \sum \) given by (1.1) is said to be in the class \( \alpha - SLM_\Sigma(n,m,\lambda,\tilde{p}(z)) \) if the following conditions are satisfied,

\[
(1-\alpha) \left( \frac{D^{n+1}_{m,\lambda} f(z)}{D^n_{m,\lambda} f(z)} \right) + \alpha \left( \frac{D^{n+1}_{m,\lambda} f(z)}{D^n_{m,\lambda} f(z)} \right) < \tilde{p}(z)
\]

(1.5)

and

\[
(1-\alpha) \left( \frac{D^{n+1}_{m,\lambda} g(w)}{D^n_{m,\lambda} g(w)} \right) + \alpha \left( \frac{D^{n+1}_{m,\lambda} g(w)}{D^n_{m,\lambda} g(w)} \right) < \tilde{p}(w)
\]

(1.6)

**Remarks**

1. \( \alpha - SLM_\Sigma(1,n,\lambda,\tilde{p}(z)) = SLM_{\alpha} \Sigma(n,\tilde{p}(z)) \), the class of bi-univalent functions defined by Gurmeet singh et al. [7].

2. \( \alpha - SLM_\Sigma(1,0,1,\tilde{p}(z)) = SLM_{\alpha} \Sigma(\tilde{p}(z)) \), the class of bi-univalent functions defined by Guney et al. [6].

We consider \( \mathbb{P} \) to be the class of Caratheodary functions. i.e., for \( p(z) \in \mathbb{P} \), \( \Re\{p(z)\} > 0 \), \( p(z) \) is analytic in \( \Delta \) and have the series expansion

\[
p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n, \quad z \in \Delta.
\]

**Lemma 1.5.** If \( p(z) \in \mathbb{P} \), then \( |p_n| \leq 2 \) for each \( n = 1, 2, ...
\]
Theorem 2.1. If \( f(z) \) is in the class \( \alpha - SLM_{\Sigma}(n,m,\lambda, \tilde{p}(z)) \) then,

\[
|a_2| \leq \frac{|\tau|}{\sqrt{(c_m^\alpha(\lambda)(|\tau\varsigma + \psi|)}}
\]

and

\[
|a_3| \leq \frac{|\tau|(|\psi - \tau(1 + 3\alpha)c_m^\alpha(\lambda)(1 + c_m^\alpha(\lambda))^{2n}|}{2\Psi[\tau\varsigma + \psi]}
\]

where

\[
\zeta = 2(1 + 2\alpha)(1 + 2c_m^\alpha(\lambda))^{n} - (1 + 3\alpha)(1 + c_m^\alpha(\lambda))^{2n},
\]

\[
\psi = (1 + 3\alpha)^2c_m^\alpha(\lambda)(1 + c_m^\alpha(\lambda))^{2n}(1 - 3\tau)\text{ and}
\]

\[
\Psi = (1 + 2\alpha)(c_m^\alpha(\lambda))^2(1 + 2c_m^\alpha(\lambda))^n.
\]

Proof. Since \( f \in \alpha - SLM_{\Sigma}(n,m,\lambda, \tilde{p}(z)) \), from the definition 1.4, we have

\[
(1 - \alpha)\frac{D_{m,\lambda}^{n+1}f(z)}{D_{m,\lambda}^{n}f(z)} + \alpha \frac{(D_{m,\lambda}^{n+1}f(z))'}{(D_{m,\lambda}^{n}f(z))'} = \tilde{p}(r(z)) \tag{2.1}
\]

and

\[
(1 - \alpha)\frac{D_{m,\lambda}^{n+1}g(w)}{D_{m,\lambda}^{n}g(w)} + \alpha \frac{(D_{m,\lambda}^{n+1}g(w))'}{(D_{m,\lambda}^{n}g(w))'} = \tilde{p}(s(w)) \tag{2.2}
\]

where \( r(z) \) and \( s(w) \) are analytic functions in \( \Delta \) with \( r(0) = s(0) = 0 \) and \( |r(z)| < 1 \) and \( |s(w)| < 1 \).

Now define the function,

\[
h(z) = \frac{1 + r(z)}{1 - r(z)} = 1 + r_1z + rz^2 + ...
\]

Then,

\[
\tilde{p}(r(z)) = 1 + \frac{r_1}{2}\tau z + \frac{1}{2}(r_2 - \frac{r_1^2}{2} + \frac{3r_2^2}{4}\tau)\tau z^2 + ...
\]  \tag{2.3}

Similarly we define the function,

\[
k(w) = \frac{1 + s(z)}{1 - s(z)} = 1 + s_1z + sz^2 + ...
\]

Then,

\[
\tilde{p}(s(w)) = 1 + \frac{s_1}{2}\tau w + \frac{1}{2}(s_2 - \frac{s_1^2}{2} + \frac{3s_2^2}{4}\tau)\tau w^2 + ...
\]  \tag{2.4}

and by considering the LHS of (2.1), we have

\[
(1 - \alpha)\frac{D_{m,\lambda}^{n+1}f(z)}{D_{m,\lambda}^{n}f(z)} + \alpha \frac{(D_{m,\lambda}^{n+1}f(z))'}{(D_{m,\lambda}^{n}f(z))'} = 1 + (1 + \alpha)c_m^\alpha(\lambda)(1 + c_m^\alpha(\lambda))^n a_2z + [2(1 + 2\alpha)c_m^\alpha(\lambda)(1 + 2c_m^\alpha(\lambda))^n a_2z^2 + \frac{2(1 + 2\alpha)c_m^\alpha(\lambda)(1 + 3\alpha)c_m^\alpha(\lambda)(1 + c_m^\alpha(\lambda))^{2n} a_2^2 z^2}{4(c_m^\alpha(\lambda)(|\tau\varsigma + \psi|)}} + ...
\]

and

\[
(1 - \alpha)\frac{D_{m,\lambda}^{n+1}g(w)}{D_{m,\lambda}^{n}g(w)} + \alpha \frac{(D_{m,\lambda}^{n+1}g(w))'}{(D_{m,\lambda}^{n}g(w))'} = 1 - (1 + \alpha)c_m^\alpha(\lambda)(1 + c_m^\alpha(\lambda))^n a_2w + [2(1 + 2\alpha)c_m^\alpha(\lambda)(1 + 2c_m^\alpha(\lambda))^n (2a_2^2 - a_3) - (1 + 3\alpha)c_m^\alpha(\lambda)(1 + c_m^\alpha(\lambda))^{2n} a_2^2 w^2 + ...
\]

Using (2.3), (2.4), and the above two equations in (2.1) and (2.2) and equating the coefficients of \( z, z^2, w \) and \( w^2 \) we have the following equations,

\[
c_m^\alpha(\lambda)(1 + c_m^\alpha(\lambda))^n(1 + \alpha) a_2 = \frac{r_1}{2}\tau \tag{2.5}
\]

\[
2(1 + 2\alpha)c_m^\alpha(\lambda)(1 + 2c_m^\alpha(\lambda))^n a_3 - (1 + 3\alpha)c_m^\alpha(\lambda)(1 + c_m^\alpha(\lambda))^{2n} a_2^2 = \left(\frac{r_2}{2}\right)\tau + \frac{3r_2^2}{4}\tau^2 \tag{2.6}
\]

\[
-c_m^\alpha(\lambda)(1 + c_m^\alpha(\lambda))^n(1 + \alpha) a_2 = \frac{s_1}{2}\tau \tag{2.7}
\]

\[
2(1 + 2\alpha)c_m^\alpha(\lambda)(1 + 2c_m^\alpha(\lambda))^n(2a_2^2 - a_3) - (1 + 3\alpha)c_m^\alpha(\lambda)(1 + c_m^\alpha(\lambda))^{2n} a_2^2 = \left(\frac{s_2}{2}\right)\tau + \frac{3s_2^2}{4}\tau^2 \tag{2.8}
\]

from (2.5) and (2.6),

\[
r_1 = -s_1, \tag{2.9}
\]

also

\[
2[c_m^\alpha(\lambda)]^2[1 + c_m^\alpha(\lambda)]^{2n}(1 + \alpha)^2 a_2^2 = \frac{1}{4}(r_1^2 + s_1^2)\tau^2
\]

\[
r_1^2 + s_1^2 = \frac{8[c_m^\alpha(\lambda)]^2[1 + c_m^\alpha(\lambda)]^{2n}(1 + \alpha)^2}{\tau^2}. \tag{2.10}
\]

Adding (2.6) and (2.8), we get

\[
a_2^2 \left[4c_m^\alpha(\lambda)(1 + 2\alpha)(1 + 2c_m^\alpha(\lambda))^n - 2c_m^\alpha(\lambda)(1 + c_m^\alpha(\lambda))^{2n}(1 + 3\alpha)\right] = \left(r_2 + s_2\right)\tau - \frac{1}{4}(r_1^2 + s_1^2)\tau + \frac{3}{4}(r_1^2 + s_1^2)\tau^2. \tag{2.11}
\]

Using (2.10) in the above equation we get

\[
a_2^2 = \frac{(r_2 + s_2)\tau^2}{4[c_m^\alpha(\lambda)(|\tau\varsigma + \psi|)}} \tag{2.12}
\]

where \( \zeta = 2(1 + 2\alpha)(1 + 2c_m^\alpha(\lambda))^n - (1 + 3\alpha)(1 + c_m^\alpha(\lambda))^{2n} \)

and \( \psi = (1 + 2\alpha)^2c_m^\alpha(\lambda)(1 + c_m^\alpha(\lambda))^{2n}(1 - 3\tau) \).
By using Lemma 1.5 and triangular inequality we get the required inequality for $\vert a_3 \vert$.

To estimate $\vert a_3 \vert$ first we subtract (2.8) from (2.6) and then by using (2.9), we get

$$4e_j^n(\lambda)(1+2\alpha)(1+2e_j^n(\lambda))^n(a_3-a_3^2) = (r_2-s_2)\frac{\tau}{2}. \quad (2.13)$$

Now by using (2.12) in the above equation we get the coefficient bound for $\vert a_3 \vert$.

For $m = 1$ in theorem 2.1 we get the following corollary,

**Corollary 2.2.** If $f(z) \in SLM_{\alpha, \lambda}(n, \tilde{p}(z))$, then

$$\vert a_2 \vert \leq \frac{\vert \tau \vert}{\sqrt{\xi}}$$

and

$$\vert a_3 \vert \leq \frac{\vert \tau \vert (1+\lambda)\alpha^2(1+\alpha)^2(1-3\lambda)^2 - \tau(1+3\lambda)\alpha}{2\xi\lambda(1+2\alpha)(1+2\lambda)^n}$$

where $\xi = \tau[2(1+2\alpha)(1+\lambda)^n - (1+3\alpha)(1+\lambda)^{2n}] + (1+\alpha)^2\lambda(1+\lambda)^{2n}(1-3\tau)$ which agrees with the results of Gurmeet Singh et al. [7] Theorem 6.

For $m = \lambda = 1$ in theorem 2.1 gives the following corollary,

**Corollary 2.3.** If $f(z) \in SLM_{\alpha, \lambda}(n, \tilde{p}(z))$, then

$$\vert a_2 \vert \leq \frac{\vert \tau \vert}{\sqrt{4^n(1+\alpha)^2 + [2(1+2\alpha)3^n - \eta 4^n]\tau}}$$

and

$$\vert a_3 \vert \leq \frac{\vert \tau \vert 4^n(1+\alpha)^2 - \eta \tau}{2(1+2\alpha)3^n[4^n(1+\alpha)^2 + [2(1+2\alpha)3^n - \eta 4^n]\tau]}$$

where $\eta = 3\alpha^2 + 9\alpha + 4$ which agrees with the results of Gurmeet Singh et al. [7] Corollary 7.

On substituting $m = \lambda = 1$ and $n = 0$ in theorem 2.1 gives the following corollary,

**Corollary 2.4.** If $f(z) \in SL_{\alpha, \lambda}(\tilde{p}(z))$, then

$$\vert a_2 \vert \leq \frac{\vert \tau \vert}{\sqrt{(1+\alpha)^2 - (1+\alpha)(2+3\alpha)\tau}}$$

and

$$\vert a_3 \vert \leq \frac{\vert \tau \vert [(1+\alpha)^2 - (3\alpha^2 + 9\alpha + 4)\tau]}{2(1+2\alpha)(1+\alpha)^2 - (1+\alpha)(2+3\alpha)\tau]}$$

which agrees with the results of Guney et al. [6] Corollary 1.

On substituting $m = \lambda = 1$ and $n = \alpha = 0$ in theorem 2.1 gives the following corollary,

**Corollary 2.5.** If $f(z) \in SLM_{\alpha, \lambda}(\tilde{p}(z))$, then

$$\vert a_2 \vert \leq \frac{\vert \tau \vert}{\sqrt{1-2\tau}}$$

and

$$\vert a_3 \vert \leq \frac{\vert \tau \vert (1-4\tau)}{2(1-2\tau)}$$

which agrees with the results of Guney et al. [6] Corollary 2.

### 3. Fekete-Szego Inequality for the function class $\alpha - SLM_{\alpha, \lambda}(n, m, \lambda, \tilde{p}(z))$

**Theorem 3.1.** If $f(z)$ is in the class $\alpha - SLM_{\alpha, \lambda}(n, m, \lambda, \tilde{p}(z))$ then,

$$\vert a_3 - \mu a_2^2 \vert\leq \left\{ \begin{array}{ll} \frac{\vert \tau \vert}{1-\mu \tau^2}, & \vert \mu - 1 \vert \leq \frac{\tau}{X} \\ \frac{\tau}{1-\mu \tau^2}, & \vert \mu - 1 \vert \geq \frac{\tau}{X} \end{array} \right. \quad (3.1)$$

where

$$Y = e_j^n(\lambda)\tau[2(1+2\alpha)(1+2e_j^n(\lambda))^n - (1+3\alpha)(1+e_j^n(\lambda))2^n] + e_j^n(\lambda)(1+e_j^n(\lambda))2^n(1+\alpha)^2(1-3\tau) + X = 2\tau [e_j^n(\lambda)(1+2\alpha)(1+2e_j^n(\lambda))^n]$$

**Proof.** From (2.12) and (2.13), we have

$$a_3 - \mu a_2^2 = \frac{\tau(c_2-d_2)}{8e_j^n(\lambda)(1+2\alpha)(1+2e_j^n(\lambda))^n} + (c_2+d_2)\chi(\mu) \quad (3.2)$$

where

$$\chi(\mu) = \frac{(1-\mu)\tau^2}{4e_j^n(\lambda)(\tau\bar{\zeta} + \phi)}$$

with $\bar{\zeta} = 2(1+2\alpha)(1+2e_j^n(\lambda))^n - (1+3\alpha)(1+e_j^n(\lambda))2^n$ and $\phi = e_j^n(\lambda)(1+e_j^n(\lambda))^2(1+\alpha)^2(1-3\tau)$. 

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The above equation can be expressed as,

\[ a_3 - \mu a_2^2 = |\chi(\mu)| + \frac{\tau}{8e_\mu(\lambda)(1 + 2\alpha)(1 + 2e_\mu(\lambda))^n}c_2 + |\chi(\mu)| - \frac{\tau}{8e_\mu(\lambda)(1 + 2\alpha)(1 + 2e_\mu(\lambda))^n}d_2. \]  

Taking modulus on the above equation, we obtain,

\[ |a_3 - \mu a_2^2| \leq \begin{cases} \left| \frac{\tau}{8e_\mu(\lambda)(1 + 2\alpha)(1 + 2e_\mu(\lambda))^n} \right|, & 0 \leq |\chi(\mu)| \leq \frac{\tau}{8\delta} \\ \left| \frac{\chi(\mu)}{4} \right|, & |\chi(\mu)| \geq \frac{\tau}{8\delta} \end{cases} \]  

where \( \delta = e_\mu(\lambda)(1 + 2\alpha)(1 + 2e_\mu(\lambda))^n. \) Using the above equation we can get the desired bound for the Fekete-Szego problem.

By varying the parameters in Theorem 3.1 we get the following corollaries. When we consider \( m = 1 \) in Theorem 3.1 we get the following corollary, which is proved by Gurmeet Singh et al.[8] in Theorem 11.

**Corollary 3.2.** If \( f(z) \in \text{SLM}^\lambda_{a, \Sigma}(n, \bar{p}(z)), \) then

\[ |a_3 - \mu a_2^2| \leq \begin{cases} \frac{\tau |\zeta|}{8(1 + 2\alpha)(1 + 2\alpha)^n}, & |\mu - 1| \leq \frac{1 - 2\zeta}{2\tilde{\tau}} \\ \frac{X}{\lambda(1 + 2\alpha)(1 + 2\alpha)^n}, & |\mu - 1| \geq \frac{1 - 2\zeta}{2\tilde{\tau}} \end{cases} \]

where

\[ X = \lambda \left[ 2(1 + 2\alpha)(1 + 2\lambda)^n - (1 + 3\alpha)(1 + \lambda)^n \right] + \lambda (1 + \lambda)^3(1 + 3\tau). \]

If we consider \( m = \lambda = 1 \) in Theorem 3.1 we get the following corollary, which is proved by Gurmeet Singh et al.[8] in Corollary 12.

**Corollary 3.3.** If \( f(z) \in \text{SLM}^\lambda_{a, \Sigma}(n, \bar{p}(z)), \) then

\[ |a_3 - \mu a_2^2| \leq \begin{cases} \frac{\tau |\zeta|}{8(1 + 2\alpha)(1 + 2\alpha)^n}, & |\mu - 1| \leq \frac{1 - 2\zeta}{2\tilde{\tau}} \\ \frac{Z}{\lambda(1 + 2\alpha)(1 + 2\alpha)^n}, & |\mu - 1| \geq \frac{1 - 2\zeta}{2\tilde{\tau}} \end{cases} \]

where

\[ Z = (2(1 + 2\alpha)3^n - (3\alpha^2 + 9\alpha + 4)^n)\tau + (1 + \alpha)2^4n. \]

If we consider \( m = \lambda = 1 \) and \( n = 0 \) in Theorem 3.1 we get the following corollary, which is proved by Guney et al.[6] in Theorem 11.

**Corollary 3.4.** If \( f(z) \in \text{SLM}^\lambda_{a, \Sigma}(\bar{p}(z)), \) then

\[ |a_3 - \mu a_2^2| \leq \begin{cases} \frac{\tau |\zeta|}{8(1 + 2\alpha)(1 + 2\alpha)^n}, & |\mu - 1| \leq \frac{1 - 2\zeta}{2\tilde{\tau}} \\ \frac{P}{\lambda(1 + 2\alpha)(1 + 2\alpha)^n}, & |\mu - 1| \geq \frac{1 - 2\zeta}{2\tilde{\tau}} \end{cases} \]

where \( P = (1 + \alpha)(1 + \alpha) - (2 + 3\alpha)\tau \)

If we consider \( m = \lambda = 1 \) and \( n = 0 \) in Theorem 3.1 we get the following corollary, which is proved by Guney et al.[6] in corollary 4.

**Corollary 3.5.** If \( f(z) \in \text{SLM}^\lambda_{a, \Sigma}(\bar{p}(z)), \) then

\[ |a_3 - \mu a_2^2| \leq \begin{cases} \frac{\tau |\zeta|}{8(1 + 2\alpha)(1 + 2\alpha)^n}, & |\mu - 1| \leq \frac{1 - 2\zeta}{2\tilde{\tau}} \\ \frac{1 - \mu^2}{2(1 - 2\zeta)}, & |\mu - 1| \geq \frac{1 - 2\zeta}{2\tilde{\tau}} \end{cases} \]

If we consider \( m = \lambda = \alpha = 1 \) and \( n = 0 \) in Theorem 3.1 we get the following corollary, which is proved by Guney et al.[6] in corollary 5.

**Corollary 3.6.** If \( f(z) \in \text{KSL}^\lambda_{a, \Sigma}(\bar{p}(z)), \) then

\[ |a_3 - \mu a_2^2| \leq \begin{cases} \frac{\tau |\zeta|}{8(1 + 2\alpha)(1 + 2\alpha)^n}, & |\mu - 1| \leq \frac{1 - 2\zeta}{2\tilde{\tau}} \\ \frac{1 - \mu^2}{2(1 - 2\zeta)}, & |\mu - 1| \geq \frac{1 - 2\zeta}{2\tilde{\tau}} \end{cases} \]

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