Research Article

Risk strategy analysis for an online rental problem of durable equipment with a transaction cost

Chunlin Xin¹, Jianwen Zhang¹ and Ziping Wang²

Abstract
This study introduces the second-hand market into the famous ski-rental model, presents an online rental problem of durable equipment with a transaction cost, and designs an optimal deterministic competitive strategy. The traditional competitive analysis is based on the worst-case scenario; hence, its results are too conservative. Even though investors want to manage and control their risks in reality, in some cases, they are willing to undertake higher risk to obtain greater benefits. Considering this situation, this study designs a risk strategy combining the decision makers’ risk tolerance with certain and probabilistic forecasts. Numerical analysis shows that the proposed risk strategy can improve the competitive ratio. This study introduces the idea of risk compensation into traditional competitive analysis and designs strategies for online rental of durable equipment based on forecast. The decision maker selects a strategy according to risk tolerance and forecast. If the forecast is correct, then a reward is obtained; otherwise, the risk is guaranteed to be within the decision maker’s risk tolerance. The optimal restricted ratio, that is, the competitive ratio of a risk strategy, is less than the optimal competitive ratio of a deterministic strategy. Therefore, the performance of the proposed risk strategy is better than a deterministic strategy. At the same time, the risk strategy based on the probabilistic forecast represents an extension of the strategy based on a certain forecast. In other words, the risk strategy based on a certain forecast is a special case of the risk strategy based on the probabilistic forecast.

Keywords
Durable equipment rental, online problems, competitive analysis, risk strategy, transaction cost

Received 17 December 2020; revised 17 December 2020; accepted 19 March 2021

Introduction
In recent decades, the total financial lease business in China increased from 370 billion RMB to 7000 billion RMB. Also, the financial lease market penetration reached 12% in 2019, while the world average financial lease market penetration was 21%. This indicates broad prospects and immense potentials of the financial lease market in China. Some experts even predicted that the financial lease business could be “sunrise industry” of China in the 21st century. Additionally, financial lease plays an important role in expanding domestic demand and guiding the rational allocation of capital. By adopting a financial lease, enterprises can obtain advanced technologies and equipment in a short time, optimize the capital structure, and avoid the risk of equipment obsolete caused by technology updates. Therefore, it is necessary to investigate leasing strategies from the perspective of investors.

In most cases, the objects of a financial lease are expensive and durable equipment. Therefore, the decision maker must decide whether to rent or buy equipment without knowing how long the equipment will be needed. Obviously, a financial lease is a typical online problem with dynamic characteristics. Therefore, the

¹School of Economics and Management, Beijing University of Chemical Technology, Beijing, China
²Department of Information Science & Systems, Morgan State University, Baltimore, MD, USA

Corresponding author:
Chunlin Xin, School of Economics and Management, Beijing University of Chemical Technology, Beijing 100029, China.
Email: xinchl@mail.buct.edu.cn

Creative Commons Non Commercial CC BY-NC. This article is distributed under the terms of the Creative Commons Attribution-NonCommercial 4.0 License (https://creativecommons.org/licenses/by-nc/4.0/) which permits non-commercial use, reproduction and distribution of the work without further permission provided the original work is attributed as specified on the SAGE and Open Access pages (https://us.sagepub.com/en-us/nam/open-access-at-sage).
competitive analysis\(^1\) has been recognized as a complementary approach in decision making under uncertainty. Using this approach, an online algorithm processes an input sequence generated by the adversary a piece at a time. As a response to each input piece, the online algorithm produces the output without knowing future input pieces. The competitive ratio gauges the performance of an online algorithm. Assume that for an input sequence \(\theta\), the cost of an online algorithm \(A\) is denoted as \(\text{COST}_A(\theta)\), and the cost of the offline algorithm \(\text{OPT}\) by \(\text{COST}_{\text{OPT}}(\theta)\). As to a cost minimization problem, if there exist \(\lambda_A\) and \(\varepsilon\) such that the inequality \(\text{COST}_A(\theta) \leq \lambda_A \text{COST}_{\text{OPT}}(\theta) + \varepsilon\) holds, then the online algorithm \(A\) is called \(\lambda_A\)-competitive, and \(\lambda^* = \inf_A \lambda_A\) represents the optimal competitive ratio.

The online rental problem has been a hot topic since the famous ski-rental model\(^2\) was proposed. A review of the literature reveals that recent research on online rental mainly focuses on two research directions:

1. Considering various realistic factors of leasing activities.

   Albers et al.\(^3\) considered delayed time in an online investment decision problem and set up the delayed action models, which represented the generalizations of the classical ski-rental model. Liu et al.\(^4\) designed an optimal deterministic strategy for the online rental problem of durable equipment with a transaction cost based on the traditional competitive analysis. Xin et al.\(^5\) studied an online device replacement problem and proposed two time-independent strategies, which represent variants of the ski-rental model. Liu et al.\(^6\) considered that the rental company promoted the rental demand by discount cards and presented a promotion strategy. Zhang et al.\(^7\) have found that an online player often has multiple rental options in realistic rental markets and developed a ski-rental model with multiple options. Furthermore, Hu et al.\(^8\) researched a discrete version of multiple online rental problems and presented an approximation algorithm for a risk control strategy. Dai et al.\(^9\) considered an online version of the financial lease decision problem, where the lessee had two options: financial lease or lease.

2. Providing more decision-making information and conditions, and thereby improving the competitive ratio.

   Fujiwara and Iwama\(^10\) reconsidered the conventional online ski-rental problem through an average-case competitive analysis and obtained the optimal competitive strategies for an exponential distribution \(f(x) = ke^{-\lambda x}\). Xu et al.\(^11\) introduced the interest rate and tax rate into Fujiwara and Iwama’s model and researched the discrete version both with and without the interest rate in the probabilistic environments. Chen and Xu\(^12\) considered two payment options to lease a piece of equipment and proposed a non-additive two-option rental problem, and in 2018, introduced the compound interest rate into the continuous version.

The traditional competitive ratio analysis assumes that an online decision maker knows nothing about the adversary, while the adversary knows everything about the online decision maker. Obviously, this represents the worst-case analysis, and the result generated by the traditional competitive analysis is too conservative. In addition, the above-mentioned literature assumes that the equipment becomes worthless at the end of the usage period. However, expensive and durable objects of a financial lease with some surplus values can be sold in the second-hand market.

This study specifies the characteristic of durable equipment and establishes an online durable equipment rental model, which makes the rental model more realistic. Moreover, this study considers that investors want to manage and control the risk but are also willing to undertake higher risk in exchange for larger benefits. That is, the decision maker gets a reward when the forecast is correct, while risk can be limited within the decision maker’s risk tolerance even though the forecast is incorrect. In addition, certain and probabilistic forecast models are presented. The numerical result shows that the proposed risk-reward model can improve the competitive ratio performance significantly.

**Mathematical model and deterministic strategy**

Suppose that equipment will be used for \(n\) periods, which is unknown to the decision maker. Denote the lease cost per period as \(h\), the price of the new equipment as \(B\), the depreciation per period by \(d\), the price of the used equipment in period \(t = 1, 2, \ldots, n\) by \(B_t\), the transaction expenses of buying the equipment by \(i\), and the transaction expenses of selling the equipment by \(j\).

The two following requirements are made for the parameters: (1) \(h > d\) so that it is profitable to the leasing company, and (2) \(i + d + j > h\) so that a rental is a valid option to the decision maker at least in period 1. For the convenience of discussion, in the following, it is assumed that the equipment price is stable, that is, the price of the used durable equipment equals the price of the new equipment minus the depreciation.

Based on the assumption that the equipment will be used for \(n\) periods, the offline algorithm can be presented as follows

\[
\text{COST}_{\text{OPT}}(n) = \min\{hn, i + B - (B_j - j)\},\text{ where } B_j = B - dn
\]
Then, the cost of the offline algorithm can be expressed as

\[
\text{COST}_{\text{OPT}}(n) = \min\{hn, i + dn + j\} = \begin{cases} 
  hn, & n < t^* \\
  i + dn + j, & n \geq t^*
\end{cases}
\]

(1)

where \( t^* = (i + j)/(h - d) \).

The decision maker may decide to buy the equipment after renting it for \((t - 1)\) periods, and then use it continuously during the remaining \((n - t + 1)\) periods, and finally sell it on the second-hand market. Hence, the cost of an online strategy can be expressed as

\[
\text{COST}_{\text{ON}}(n) = \begin{cases} 
  hn, & n < t \\
  h(t - 1) + i + d(n - t + 1) + j, & n \geq t
\end{cases}
\]

(2)

**Lemma 1:** An optimal deterministic strategy is to buy durable equipment after renting it for \((t^* - 1)\) periods, and then use it continuously during the remaining \((n - t^* + 1)\) periods, and finally, sell it on the second-hand market. The optimal competitive ratio of this strategy is expressed as \( \lambda^* = 1 + (h - d)(i + d + j - h)/(i + j)h \). The proof is presented in Liu et al.\(^4\).

**Risk strategy based on certain forecast**

Lemma 1 is based on the traditional competitive analysis, which assumes that an online decision maker knows nothing about an adversary, whereas the adversary knows everything about the online decision maker. Thus, as mentioned above, the traditional competitive analysis represents the worst-case analysis, which results in too conservative solutions. However, in reality, the decision maker rarely has no information about the past and future. Instead, the decision maker can utilize historical data and market information to make a forecast and design a strategy accordingly. The decision maker will get a reward when the forecast is correct, and risk will be within the risk tolerance range even though the forecast is incorrect. These types of competitive strategies are referred to as risk strategies.

Al-Binah\(^13\) proposed the concept of risk and reward and constructed the first risk-reward model. Denote the competitive ratio of strategy \( A \) by \( \lambda_A \), and the optimal competitive ratio by \( \lambda^* \). Then, the risk of algorithm \( A \) is computed as \( \lambda_A/\lambda^* \). Assume the risk tolerance of the investor is denoted as \( r \), where \( r \geq 1 \), and a higher value of \( r \) implies higher risk tolerance. Denote the set of all strategies that respect the investor’s risk tolerance by \( S_r = \{ A : \lambda_A \leq r \lambda^* \} \) and the set of possible forecasts by \( F \). Define \( \overline{\lambda}_A = \sup_{\theta \in F} \{ \text{COST}_A(\theta)/\text{COST}_{\text{OPT}}(\theta) \} \) as the restricted ratio of algorithm \( A \) when the forecast is correct and \( f_A = \lambda^*/\overline{\lambda}_A \) as the reward of algorithm \( A \). Note that the reward of \( A \) is measured as an improvement over the optimal online algorithm.

**Theorem 1:** When the risk tolerance is \( r \), if the forecast is \( r < t^* - 1 \), then the deterministic strategy of Lemma 1 is still an optimal risk strategy; if the forecast is \( r \geq t^* - 1 \), then an optimal risk strategy is to buy the equipment after renting it for \( s = (i + d + j - h r \lambda^*)/(h r \lambda^* - h) \) periods, then use it continuously during the remaining \((n - s)\) periods, and finally, sell it on the second-hand market, and the restricted ratio is expressed as \( \overline{\lambda}_A = 1 + (h - d)(i + d + j - h r \lambda^*)/(d(i + d + j) + h(i + j)(r \lambda^* - 1)) \).

**Proof:** According to the risk-reward framework, there are two possible forecasts: one is that the durable equipment will be used less than \((t^* - 1)\) periods, and another is that the equipment will be used longer than \((t^* - 1)\) periods.

**Forecast 1:** \( F_1 = \{ n : n < t^* - 1 \} \), if this forecast is true, the decision maker will lease the durable equipment from beginning to the end, which is also what the optimal offline algorithm renders, so the restricted ratio of this strategy is \( 1 \).

**Forecast 2:** \( F_2 = \{ n : n \geq t^* - 1 \} \), assume strategy \( A(s) \) is that the online decision maker buys the durable equipment after renting it for \( s \) periods. When the input sequence is \((s + 1)\), the ratio of the online algorithm’s cost to the offline algorithm’s cost achieves its maximum, which means the adversary should continue the input sequence until the online decision maker buys the equipment. Then, the competitive ratio of \( A(s) \) is expressed as \( \lambda_A(s) = (hs + i + d + j)/\min\{i + d(s + 1) + j, h(s + 1)\} \). Given a risk tolerance \( r \), the strategy should also belong to \( S_r = \{ A : \lambda_A(s) \leq r \lambda^* \} \). The corresponding analysis steps are as follows:

1. if \( i + d(s + 1) + j > h(s + 1) \), that is, \( s < t^* - 1 \), then \( \frac{hs + i + d + j}{i + d(s + 1) + j} \leq \lambda_A \), we have \( s \geq \frac{i + d(s + 1) + j}{i + d(s + 1) + j} \).

2. if \( i + d(s + 1) + j \leq h(s + 1) \), that is, \( s \geq t^* - 1 \), then \( \frac{hs + i + d + j}{i + d(s + 1) + j} \leq \lambda_A \), we have \( s \leq \frac{i + d(s + 1) + j}{i + d(s + 1) + j} \). Then, the value range of \( s \) is obtained as

\[
s_1 \equiv \frac{i + d + j - h r \lambda^*}{h r \lambda^* - h} \leq s \leq \frac{(i + d + j)(r \lambda^* - 1)}{h - d r \lambda^*} \equiv s_2
\]

(3)

If Forecast 2 is correct, the optimal offline strategy is to buy the equipment at the beginning, so the restricted ratio is expressed as \( \overline{\lambda}_A = (hs + i + d + j)/(i + d(s + 1) + j) \). It should be noted that the smaller the value of \( \overline{\lambda}_A \) is, the greater the obtained reward will be when the forecast is correct. Moreover, it can be shown that \( \partial \overline{\lambda}_A/\partial s > 0 \). According to (3), \( \overline{\lambda}_A \) achieves its minimum
when \( S = S_1 = (i + d + j - h r^*)/(h r^* - h) \), so \( \lambda_{A,i} = 1 + (h - d)(i + d + j - h r^*)/(d(i + d + j) - h) + h(i + j) (r^* - 1) \) denotes the minimum restricted ratio.

**Risk strategy based on probabilistic forecast**

The risk strategy of Al-Binali is based on a certain forecast that is described to be a subset of \( I \). Dong et al.\(^{14} \) extended the certain forecast to the probabilistic forecast, based on which the strategy is more flexible. In this study, this strategy is defined as a risk strategy based on the probabilistic forecast. Let \( F_1, F_2, \ldots, F_m \) be a group of subsets of \( I \), where \( \cup F_i = I \) and \( F_i \cap F_j = \emptyset \) for \( i \neq j \). Denote the probability that the online decision maker expects as \( \Pi_i \) on the probabilistic forecast the restricted ratio of the risk strategy based on the first \( m \) periods, and finally, sell it on the second-hand market, the durable equipment after renting it for \( s \) periods. According to the definition of restricted ratio based on the probabilistic forecast, it holds that \( \lambda_A = \sum_{i=1}^{m} P_i \lambda_{A,i} \), where

\[
\lambda_{A,i} = \sup_{\beta \in P_i} \frac{\text{COST}(\beta)}{\text{COST}_{\text{opt}}(\beta)}
\]

Therefore, the following results can be obtained.

Based on the forecast \( F_1 \), we have:

\[
\lambda_{A,1} = \left\{ \begin{array}{ll}
\frac{hs + i + d + j}{h(s + 1)}, & s < t^* - 1 \\
1, & s \geq t^* - 1
\end{array} \right.
\]

Based on the forecast \( F_2 \), we have:

\[
\lambda_{A,2} = \left\{ \begin{array}{ll}
\frac{hs + i + d + j}{i + d(s + 1) + j}, & s < t^* - 1 \\
\frac{hs + i + d + j}{i + d(s + 1) + j}, & s \geq t^* - 1
\end{array} \right.
\]

Hence

\[
\lambda_A = \left\{ \begin{array}{ll}
\frac{P_1 hs + i + d + j}{h(s + 1)} + (1 - P_1) \frac{hs + i + d + j}{i + d(s + 1) + j}, & s < t^* - 1 \\
P_1 + (1 - P_1) \frac{hs + i + d + j}{i + d(s + 1) + j}, & s \geq t^* - 1
\end{array} \right.
\]

Taking the partial derivative of \( \lambda_A \) in terms of \( s \), we get

\[
\delta \lambda_A/\delta s = \left\{ \begin{array}{ll}
-P_1 \frac{i + d + j - h}{h(s + 1)} + (1 - P_1) \frac{d(i + d + j)}{[i + d(s + 1) + j]} & s < t^* - 1 \\
(1 - P_1) \frac{d(i + d + j)}{[i + d(s + 1) + j]}, & s \geq t^* - 1
\end{array} \right.
\]

Let \( F(s) = -\frac{i + d + j - h}{h(s + 1)} + \frac{d(i + d + j)}{[i + d(s + 1) + j]} \).

In the following, three scenarios are considered.

1. When \( \frac{1}{2} \leq P_1 \leq 1 \), \( \frac{\delta \lambda_A}{\delta s} \leq \frac{1}{2} F(s) \) holds. If \( s < t^* - 1 \), then \( \delta F(s)/\delta s > 0 \), and \( \delta \lambda_A/\delta s \leq \frac{1}{2} F(s) < \frac{1}{2} F(t^* - 1) = 0 \).

   Also, if \( s \geq t^* - 1 \), it can be verified that \( \delta \lambda_A/\delta s > 0 \).

   Therefore, \( \lambda_A \) is monotone decreasing at \( s < t^* - 1 \), and monotone increasing at \( s \geq t^* - 1 \).

   Consequently, the optimal risk strategy based on the probabilistic forecast is to buy the durable equipment after renting it for \( t^* - 1 \) periods.

2. When \( \theta < P_1 < \frac{1}{2} \), \( \lambda_A \) is monotone decreasing at \( s < N - 1 \), and monotone increasing at \( s \geq N - 1 \), where \( N = \frac{1}{\theta} \sqrt{\frac{i + j}{h}} \). Inequality (3) leads to \( N - 1 \geq s \), so

   \[ P_1 \geq \frac{\sqrt{\frac{i + j}{h}}}{1 + \sqrt{\frac{i + j}{h}}} = \theta \]. As a result, \( \lambda_A \) achieves its minimum at \( s = N - 1 \), i.e., the optimal risk strategy based on the probabilistic forecast is to buy the durable equipment after renting it for \( (N - 1) \) periods.

3. When \( 0 \leq P_1 < \theta \), \( P_2 \) becomes so large that the probabilistic forecast \( \{F_1(P_1), F_2(P_2)\} \) is equivalent to the certain forecast \( F_2 \). Therefore, the optimal
risk strategy based on the probabilistic forecast is to buy the durable equipment after renting if for \( s = s_1 \) periods.

In summary, the above analysis suggests that the optimal risk strategy based on the probabilistic forecast is to buy the durable equipment after renting it for \( s^* \) periods, as given by equation (4).

**Corollary 1:** Based on equation (4), \( s^* = t^* - 1 \) when \( P_1 = 1 \), while \( s^* = s_1 = \frac{d+s-k}{hr} \) when \( P_1 = 0 \). It reduces to the optimal risk strategy based on a certain forecast given by Theorem 1. Thus, it can be concluded that the risk strategy based on the probabilistic forecast is an extension of the risk strategy based on a certain forecast. That is, the risk strategy based on a certain forecast is a special case of the risk strategy based on the probabilistic forecast.

### Numerical analysis

Assume that a company needs to use durable equipment, whose rental cost is \( h = $300 \) per month, the depreciation rate is \( d = $1400 \) per month, the transaction fee of buying this equipment is \( i = $9000 \), and the transaction fee of selling it is \( j = $7000 \). According to Lemma 1, the optimal deterministic strategy is to buy the durable equipment after renting it for \( t^* - 1 = 9 \) months with the competitive ratio of \( \lambda^* = 1.48 \). If the risk tolerance is \( r = 1.2 \), then according to Theorem 1, the optimal risk strategy based on certain forecast is to buy the durable equipment after renting it for \( s = 5 \) months with the restricted ratio of \( \lambda_A = 1.34 \). Theorem 2 leads to the optimal risk strategy based on the probabilistic forecast, which is to buy the durable equipment after renting it for \( s^* \) months with a restricted ratio \( \lambda_A \). The calculation results are presented in Table 1.

Based on the results presented in Table 1, the following conclusions can be drawn: (1) optimal restricted ratio is less than the optimal competitive ratio in all the cases; (2) based on the probabilistic forecast, the decision maker will buy the equipment earlier if he predicts that it can be used for a longer time, i.e., \( P_1 \) decreases while \( P_2 \) increases, which coincides well with practice; (3) the risk strategy based on a certain forecast is a special case of the risk strategy based on the probabilistic forecast when \( P_1 = 0 \), which confirms Corollary 1.

### Conclusion

This study addresses the management problem of enterprise managers of whether to buy or rent the equipment to meet the production requirements. Considering the characteristic of wanted durable equipment, from the perspective of risk-reward, the risk strategy based on a certain forecast, which overcomes the drawback of the traditional competitive analysis of providing too conservative results, is designed. Furthermore, a more flexible risk strategy based on the probabilistic forecast is also designed, which provides a better theoretical basis.

However, there are still some problems that need further consideration. This study assumes that the durable equipment price is stable, but in practice, equipment price commonly fluctuates due to the intensive market competition. In addition, the depreciation rate is assumed to be uniform. However, it would be interesting to study the online rental problem with an accelerated depreciation rate. Finally, a more thorough study should investigate the impact of other realistic factors, such as interest and tax on the model.

### Acknowledgements

We thank LetPub (www.letpub.com) for its linguistic assistance during the preparation of this manuscript.

### Declaration of Conflicting Interests

The author(s) declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

### Funding

The author(s) disclosed receipt of the following financial support for the research, authorship, and/or publication of this article: The work was partly supported by National Key Research and Development Program of China (No. 2019YFC1906100), National Key Technology Research and Development Program of the Ministry of Science and Technology of China (No. 2015BAK39B00).

**ORCID iD**

Chunlin Xin https://orcid.org/0000-0003-0812-4975
References

1. Albers S. Online algorithms: a survey. *Math Program, Ser B* 2003; 97: 3–26.
2. Karp RK. Online algorithms versus offline algorithms: how much is it worth to know the future. In: *Processing of the IFIP 12th World Computer on Algorithms, Software, and Architecture Information Processing* 1992; 416–429.
3. Albers S, Charikar M and Mitzenmacher M. Delayed information and action in on-line algorithms. *Informat Comput* 2001; 170: 135–152.
4. Liu B, Xin CL and Shen FW. Online leasing for durable equipment with transaction cost. In: *4th International Conference on cooperation and Promotion of Information Resources in Science and Technology* 2009; 252–254.
5. Xin CL, Yi FL and Xu YF. Competitive analysis of the on-line replacement problems with continuous time. *Information* 2009; 12: 21–31.
6. Liu B, Cui WT, Xin CL, et al. Online algorithm for ski rental with promotion strategy. *Information* 2010; 13: 5–14.
7. Zhang GQ, Xu YF and Wang Y. Competitive analysis for the online rental problem with multiple options. *Operat Res Manag Sci* 2012; 21: 11–18.
8. Hu ML, Xu WJ, Li HY, et al. Competitive analysis for discrete multiple online rental problems. *J Manag Sci Eng* 2018; 3: 125–140.
9. Dai WQ, Dong YC and Zhang XT. Competitive analysis of the online financial lease problem. *Eur J Operat Res* 2016; 250: 865–873.
10. Fujiwara H and Iwama K. Average-case competitive analyses for ski-rental problems. *Algorithmica* 2005; 42: 95–107.
11. Xu YF, Xu WJ and Li HY. On the on-line rent-or-buy problem in probabilistic environments. *J Glob Optim* 2007; 38: 1–20.
12. Chen XL and Xu WJ. Risk-reward strategies for the non-additive two-option online leasing problem. *Int J Innovat Comput Inform Contr* 2017; 13: 2051–2065.
13. Al-Binali S. A risk-reward framework for the competitive analysis of financial games. *Algorithmica* 1999; 25: 99–115.
14. Dong YC, Xu YF and Xu WJ. The on-line rental problem under risk-reward model with probabilistic forecast. *Information* 2011; 14: 89–96.