General linear dynamics – quantum, classical or hybrid

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Abstract. We describe our recent proposal of a path integral formulation of classical Hamiltonian dynamics. Which leads us here to a new attempt at hybrid dynamics, which concerns the direct coupling of classical and quantum mechanical degrees of freedom. This is of practical as well as of foundational interest and no fully satisfactory solution of this problem has been established to date. Related aspects will be observed in a general linear ensemble theory, which comprises classical and quantum dynamics in the form of Liouville and von Neumann equations, respectively, as special cases. Considering the simplest object characterized by a two-dimensional state-space, we illustrate how quantum mechanics is special in several respects among possible linear generalizations.

1. The notorious linearity of the evolution equations
This article is motivated by discussions of the origin of quantum mechanics, as presented, for example, in Refs. [1, 2]. These suggest that quantum mechanics could possibly emerge by coarse-graining and sufficiently far from the Planck scale. Thus, the quantum phenomena would arise from dynamics beneath.

Numerous indications exist which support this view. Nevertheless, the general principles governing such hypothetical dynamics are under debate [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14] and, perhaps, have not even been touched upon, so far.

In this situation, it may be worth while to focus on aspects of quantum and classical mechanics which bring them as closely together as possible, yet sharpen remaining differences. The linearity of the respective evolution equations must certainly be counted among such aspects.

Indeed, it is remarkable that dynamical evolution in physics, to a large extent, can be encoded in one or the other of three linear equations:

\[ \partial_t \rho = \{ H, \rho \} , \quad \text{(1)} \]

\[ i \partial_t \Psi = \hat{H} \Psi , \quad \text{(2)} \]

\[ i \partial_t \hat{\rho} = [\hat{H}, \hat{\rho}] , \quad \text{(3)} \]

which, respectively, are the Liouville equation (1), governing the evolution of a classical phase space density \( \rho \) in terms of the Poisson bracket \( \{ , \} \) and the Hamilton function \( H \) of the object under study, the Schrödinger equation (2) for the quantum mechanical wave function(\( \Psi \)) in terms of the appropriate Hamilton operator \( \hat{H} \), and the von Neumann equation (3) for the density operator \( \hat{\rho} \) in terms of the commutator \( [ , ] \) with \( \hat{H} \).

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We emphasize that all three equations pertain to (or can be interpreted as) ensemble theories. They have the generic structure:

\[ i\partial_t \text{"state"} = \mathcal{L} \text{"state"}, \]

where the "state" appears only linearly in all cases. This first-order in time differential equation, of course, assumes widely different forms depending on the forces or dynamics under study. That is, the "Liouville superoperator" \( \mathcal{L} \) on the right-hand side allows to incorporate physics ranging from classical non-relativistic single particles all the way to relativistic quantum fields, employing a phase space ensemble of initial conditions in classical statistical mechanics and the functional Schrödinger picture for interacting quantum fields.

Since the Schrödinger equation describes special cases (pure states) of situations (pure and mixed states) covered by the von Neumann equation, we consider Eqs. (1) and (3), eventually in the form of Eq. (4), in the following.

In passing, we recall that nonlinear modifications of quantum mechanics have been proposed and severely constrained in various ways, see, for example, Refs. [15, 16, 17, 18].

In distinction, modifications of the linear Eq. (3), in particular, have been derived for "open systems" or quantum mechanical objects interacting with an environment (such as a heat bath of oscillators, etc.). - Here, one is clearly not interested in modifying quantum mechanics but in applying it to complex situations. – The generic form of the resulting consistent dynamics, replacing Eq. (3) while preserving the defining properties of the density operator, is known as the Lindblad equation [19, 20]. Its generator provides an important example of a completely positive map. Some aspects of this will be discussed in more detail in Section 4, in order to contrast it with modifications of the linear dynamics introduced there.

A hypothetical fundamental coupling of quantum mechanical and classical degrees of freedom – "hybrid dynamics" – instead, presents a departure from quantum mechanics. While this does not touch the linearity, in the sense of Eq. (4), consistency of the resulting theory has to be carefully examined.

We remark that there is also a practical interest in certain forms of hybrid dynamics. The Born-Oppenheimer approximation, for example, is based on a separation of interacting slow and fast degrees of freedom of a compound object. The former are treated as approximately classical while the latter as of quantum mechanical nature. \(^1\) It must be emphasized, however, that in these cases, hybrid dynamics is considered as relevant approximate description of an intrinsically quantum mechanical object. This has recently been employed, for example, in a re-derivation of geometric forces and Berry's phase [22].

Such considerations are and will become increasingly important for precise manipulations of quantum mechanical objects by apparently, for all practical purposes classical means, especially in (sub-)nanotechnological devices.

However, this may be also relevant for studies of the measurement problem: Namely, quantum theory, endowed with the Copenhagen interpretation, assumes a coupling of the quantum mechanical object to the classical measuring (or manipulating) apparatus. However, it remains profoundly silent about how this is realized and how the ensuing state reduction ("collapse of the wave function") proceeds. – Truly hybrid dynamics, which does not arise from an approximation, augmented by effects of environment induced decoherence and anharmonic forces, must have something to say about measurement situations, if it can be formulated consistently.

Presently, we are motivated by interest in the back-reaction effect of quantum fluctuations on classical degrees of freedom, in particular if they are physically distinct. We recall here

\(^1\) Mean field theory, based on the expansion of quantum mechanical variables into a classical part plus quantum fluctuations, leads to another approximation scheme and another form of hybrid dynamics. This is discussed more generally for macroscopic quantum phenomena in Ref. [21].
especially discussions of the “semiclassical” Einstein equation coupling the classical metric of spacetime to the expectation value of the energy-momentum tensor of quantized matter fields. Can this be made into a consistent hybrid theory leaving gravity unquantized? This has recently been re-examined, for example, in Refs. [23] and [24, 25, 26]; earlier related work includes Refs. [27, 28, 29, 30, 31, 32, 33, 34].

Concerning the origin of quantum mechanics from a coarse-grained deterministic dynamics [1, 2], to which we alluded above, the back-reaction problem can be more provocatively stated as the problem of the interplay of fluctuations among underlying deterministic and emergent quantum mechanical degrees of freedom. Or, in short: “Can quantum mechanics be seeded?”

In this paper, we present a few ingredients which may be helpful in further studies of these questions. – In the following Section 2, we begin with a brief recapitulation of our path integral formulation of classical Hamiltonian dynamics. Some simple observations following from this lead us to take up the subject of hybrid dynamics in Section 3. In Section 4, we continue the discussion from a different angle, namely with some remarks on problems encountered when trying to embed classical or quantum mechanics in a more general linear dynamics.

2. A path integral for classical Hamiltonian dynamics

Our aim here is to solve the Liouville equation (1) with the help of a suitable propagator which, in turn, is represented as a path integral. We summarize essential steps, while more details of the derivation can be found in Ref. [35].

The following considerations are rather independent of the number of degrees of freedom and apply to matrix or Grassmann valued variables as well; field theories require a classical functional formalism, which has been considered elsewhere [8, 36]. We consider a one-dimensional system, for simplicity.

2.1. The quantum-like version of the Liouville equation

We assume conservative forces acting on the classical object under study and that Hamilton’s equations are determined by the generic Hamiltonian function:

$$ H(x, p) := \frac{1}{2}p^2 + V(x) \ , \eqno(5) $$

where $x$ and $p$ denote generalized coordinate and momentum, respectively, and where $V(x)$ stands for an external potential (a mass parameter will be inserted later).

An ensemble of such objects can be described by a probability distribution function $\rho$ depending on the $x, p$-coordinates of phase space and time. Such a distribution evolves according to the Liouville equation:

$$ -\partial_t \rho = \frac{\partial H}{\partial p} \cdot \frac{\partial \rho}{\partial x} - \frac{\partial H}{\partial x} \cdot \frac{\partial \rho}{\partial p} = \{p\partial_x - V'(x)\partial_p\} \rho \ , \eqno(6) $$

with $V'(x) := dV(x)/dx$. – The relative minus sign in the Poisson bracket, or between terms here, reflects the symplectic phase space symmetry. It will give rise to a commutator structure, which reminds one of quantum mechanics, as we shall see momentarily.

Following a Fourier transformation, $\rho(x, p; t) = \int d y e^{-i p y} \rho(x, y; t)$, and a transformation of the effectively doubled number of spacelike coordinates,

$$ Q := x + y/2 \ , \qquad q := x - y/2 \ , \eqno(7) $$

the Liouville equation becomes:

$$ i\partial_t \rho = \{\hat{H}_Q - \hat{H}_q + \mathcal{E}(Q, q)\} \rho \ , \eqno(8) $$
\begin{align}
\hat{H}_\chi & := \frac{1}{2}\partial^2_{\chi} + V(\chi) , \quad \text{for } \chi = Q, q , \\
\mathcal{E}(Q, q) & := (Q - q)V'(\frac{Q + q}{2}) - V(Q) + V(q) = -\mathcal{E}(q, Q) .
\end{align}

Thus, it bears strong resemblance to the von Neumann equation, cf. Eq. (3), considering \( \rho(Q, q; t) \) as matrix elements of a density operator \( \hat{\rho}(t) \).

We automatically recover the Hamiltonian operator \( \hat{H} \) related to the Hamiltonian function, Eq. (5), as in quantum theory. Furthermore, reality and normalization of the phase space probability distribution \( \rho(x, p; t) \) translate into hermiticity and trace normalization of the density operator \( \hat{\rho}(t) \) [35].

However, an essential dynamical feature here consists in the interaction \( \mathcal{E} \) between bra- and ket-states. Thus, generally, the Hilbert space and its dual are coupled by a genuine superoperator, a concept to be defined in the following subsection. \(^2\)

It is remarkable that the interaction between bra- and ket-states vanishes under certain circumstances:

\( \mathcal{E} \equiv 0 \iff \text{potential } V(x) \text{ is constant, linear, or harmonic} , \)

rendering a Liouville superoperator of quantum mechanical form, i.e., as in Eq. (3). \(^3\) This has been discussed in Ref. [2] under the perspective of having quantum phenomena emerge due to discrete spacetime structure.

In the following, we will study in more detail the classical dynamics described by Eq. (8), or by appropriate generalizations, and pay particular attention to the presence of the superoperator \( \mathcal{E} \), when comparing with the von Neumann equation and its solution by a propagator.

### 2.2. The superspace

The dynamics of density operators in the general form of Eq. (4) can be conveniently rewritten by introducing the concept of superspace, also called Liouville space [35, 37]. \(^4\)

Let \( \hat{H} \) denote a Hamiltonian operator, as in quantum theory, and let \( \{|j\} \), \( j = 1, \ldots, N \), present a complete orthonormal set of basis states, assuming here that the relevant Hilbert space is \( N \)-dimensional. Then, the matrix elements of the von Neumann equation (3) read:

\[ i\partial_t \rho_{jk} = [(\hat{H} \rho)]_{jk} - (\rho \hat{H})_{jk} \ , \quad j, k = 1, 2, \ldots, N \ , \]

with a density matrix \( \rho \) of \( N^2 \) elements. Or, written as in Eq. (4):

\[ i\partial_t \rho_{jk} = \sum_{lm} \mathcal{L}_{jk, lm} \rho_{ln} \equiv (\hat{\mathcal{L}} \rho)_{jk} \ , \]

where the Liouville superoperator \( \hat{\mathcal{L}} \) is now defined by its matrix elements:

\[ \mathcal{L}_{jk, lm} := H_{jl} \delta_{mk} - \delta_{jl} H_{mk} \ . \]

\(^2\) The interaction \( \mathcal{E} \) is antisymmetric under \( Q \leftrightarrow q \). It follows that the complete Liouville superoperator on the right-hand side of Eq. (8), to be compared with Eq. (4), has a symmetric spectrum with respect to zero and, in general, will not be bounded below. Related observations were discussed, for example, in Refs. [6, 8, 9, 10].

\(^3\) Analogously, the vanishing of \( \mathcal{E} \) in a field theory amounts to having massive or massless free fields, with or without external sources, and with or without bilinear couplings. In these cases, anharmonic forces or interactions are absent.

\(^4\) No relation with the case of supersymmetry is implied.
This simple rewriting may suggest to introduce a space in which the density operator is a vector. This is the role of the Liouville space (or superspace).

The dynamics of density operators can then be described in parallel for classical and quantum mechanics. In general, they will differ, of course, by the precise form of the superoperator, examples of which we have seen here and in the previous subsection.

Given the Hilbert space, as above, the density operator can be expanded as:

$$\hat{\rho} = \sum_{j,k} \rho_{jk} |j\rangle \langle k| .$$

(15)

The family of $N^2$ operators $|j\rangle \langle k|$, with $j, k = 1, \cdots, N$, can be interpreted as a complete set of matrices, or vectors, such that the density operator becomes:

$$|\rho\rangle = \sum_{j,k} \rho_{jk} |j,k\rangle ,$$

(16)

where the “ket” $|j,k\rangle$ denotes the Liouville space vector representing the Hilbert space operator $|j\rangle \langle k|$. Similarly, we introduce a “bra”-vector $\langle j,k|$ as the Hermitian conjugate of $|j,k\rangle$.

Consequently, any operator $\hat{A}$ is represented by a vector, denoted by $|A\rangle$, and can be expanded as:

$$|A\rangle = \sum_{j,k} A_{jk} |j,k\rangle ,$$

(17)

where $A_{jk}$ are the usual matrix elements $\langle j|\hat{A}|k\rangle$. Furthermore, with a bra-vector $\langle j,k| \hat{B} |j,k\rangle$, the scalar product of two operators is defined by:

$$\langle (j,k|A\rangle \langle (B| : = \text{Tr}(\hat{B}^\dagger \hat{A}) .$$

(18)

This implies the orthonormality condition:

$$\langle (j,k|mn\rangle = \text{Tr}(|j,k\rangle \langle m|n\rangle ) = \delta_{kn} \delta_{jm} ,$$

(19)

in analogy to and based on $\langle j,k| = \delta_{j,k}$. Finally, consider the scalar product:

$$\langle (j,k|A\rangle = \text{Tr}(|k\rangle \langle j,\hat{A}|l\rangle = \sum_{l} \langle l|k\rangle \langle j,\hat{A}|l\rangle = \langle j|\hat{A}|k\rangle \equiv A_{jk} .$$

(20)

Upon substitution in Eq. (17), this yields:

$$|A\rangle = \sum_{j,k} |j,k\rangle \langle j,k|A\rangle .$$

(21)

This is consistent with the following completeness relation in Liouville space:

$$\sum_{j,k} |j,k\rangle \langle j,k| = 1 .$$

(22)

It is easy to see now that the Liouville space is a linear space and that the density operator $\hat{\rho}$, in particular, is a vector in this space. To conclude these formal considerations, we define a linear operator in terms of its matrix elements:

$$\hat{F} := \sum_{j,k,m,n} |j,k\rangle \langle (j,k|\hat{F}|mn\rangle \langle mn| = \sum_{j,k,m,n} \mathcal{F}_{jk,mn} |j,k\rangle \langle mn| .$$

(23)
an example of which is given in Eq. (14).

The importance of Liouville space for classical and quantum dynamics resides in that Liouville and von Neumann equations, both, assume the form of Eq. (4), incorporating an appropriate superoperator \( \hat{L} \), cf. Eqs. (8)–(10) and (12)–(14), respectively. Thus, we find a close formal similarity between the structure of these equations and the Schrödinger equation (2).

Therefore, it is plausible that formal derivations or techniques concerning the solution of the Schrödinger equation can be transferred to the cases of Liouville or von Neumann equations with the help of Liouville space notions. This regards perturbation theory as much as nonperturbative methods, the path integral approach in particular, to which we turn next.

### 2.3. The Liouville propagator and path integral

The technical ingredients of the Feynman path integral approach, for the derivation of quantum mechanical propagators in particular, are very well known [38, 39]. We will make use of these ideas, in order to derive a path integral for the propagator of density matrices based on the Liouville space formulation of the preceding subsection.

Our derivation relies on the close formal similarity between the classical Liouville equation and the von Neumann equation on one hand side and the Schrödinger equation on the other, in the appropriate representation that we discussed.\(^5\)

In particular, the formal solutions of the classical Liouville equation and of the quantum mechanical von Neumann equation, both, can be written in the form:

\[
|\rho(t)\rangle = e^{-i\hat{L}(t-t_0)/\hbar} |\rho(t_0)\rangle \ ,
\]

where \( \hat{L} \) is the relevant Liouville superoperator. Here, we have:

\[
\langle \langle Q, q | \hat{L} | Q', q' \rangle \rangle = \delta(Q - Q') \delta(q - q') (\hat{H}(Q) - \hat{H}(q) + \mathcal{E}(Q, q)) \ ,
\]

where \( \hat{H} \) denotes the appropriate Hamilton operator in coordinate representation, as indicated, which alone is relevant for the von Neumann equation, while \( \mathcal{E} \) represents the additionally present superoperator term for classical dynamics, cf. Subsection 2.1.

In order to solve the problem of time evolution in the present case, we need to know the (super)matrix elements entering the propagation equation:

\[
\langle \langle Q, q | \rho(t) \rangle \rangle = \int dQ' dq' \langle \langle Q, q | e^{-i\hat{L}(t-t_0)/\hbar} | Q', q' \rangle \rangle \langle \langle Q', q' | \rho(t_0) \rangle \rangle \ ,
\]

in analogy to the case of a state vector evolving according the Schrödinger equation. Not surprisingly, we can now follow the usual steps [38, 39], in order to construct the path integral representation of our propagator [35].

In this derivation, one has to pay attention to a suitable generalization of the Trotter product formula. This works out in a straightforward way; the relevant definitions and details of the proof are given in Ref. [40].

Rewriting the Eq. (26) as:

\[
\rho(Q, q; t) = \int dQ' dq' \mathcal{G}(Q, q; t | Q', q' ; t_0) \rho(Q', q' ; t_0) \ ,
\]

our interest is to know the superpropagator \( \mathcal{G} \).

\(^5\) In this subsection, we reinstate \( \hbar \) explicitly.
Following the recipe to arrive at a path integral representation, we make use here of suitably inserted complete sets of superspace vectors, such as:

\[ \int dQ dq \left| Q, q \right\rangle \langle \left| Q, q \right\rangle = 1, \]

(28)

and, correspondingly, for momentum space, cf. Eq. (22). Using the plane wave relation between coordinate and momentum eigenfunctions, we also employ:

\[ \langle \langle P, p \rangle \rangle = \frac{1}{2\pi\hbar} \int dQ dq \exp \left( -i\frac{\hbar}{\hbar}(PQ - pq) \right) \langle \langle Q, q \rangle \rangle. \]

(29)

Furthermore, the orthogonality relation \( \langle \langle Q, q \mid Q', q' \rangle \rangle = \delta(Q - Q')\delta(q - q') \), cf. Eq. (19), implies:

\[ \langle \langle P, p \mid Q, q \rangle \rangle = \frac{1}{2\pi\hbar} \exp \left( -i\frac{\hbar}{\hbar}(PQ - pq) \right) \]

(30)

Then, with all following steps of the derivation in parallel with the usual ones, it is straightforward to obtain the Liouville path integral [35, 40]:

\[ \mathcal{G}(Q_f, q_f; t|Q_i, q_i; t_0) = \int \mathcal{D}Q \mathcal{D}q \exp \left( i\hbar \mathcal{S}[\dot{Q}, \dot{q} ; q, q] \right). \]

(31)

with the boundary conditions \( Q(t_i) = Q_i, q(t_i) = q_i \), \( Q(t_f) = Q_f, q(t_f) = q_f \), otherwise unrestricted paths, and with the superaction \( \mathcal{S} \) defined by:

\[ \mathcal{S} := \int_{t_0}^{t} dt \left( \frac{m}{2} \dot{Q}^2 - V(Q) - \left( \frac{m}{2} \dot{q}^2 - V(q) \right) - \mathcal{E}(Q, q) \right). \]

(32)

for a particle of mass \( m \). We recall that \( \mathcal{E} \equiv 0 \) corresponds to evolution according to the von Neumann equation, whereas \( \mathcal{E} \neq 0 \) represents the only modification due to classical dynamics, in accordance with the Liouville equation, cf. Eqs. (8)–(10).

However simple this result may seem, the Eqs. (31)–(32) describe time evolution of the full density matrix. The particular new feature is that formally classical dynamics is treated on the same footing as quantum mechanics, differing only in the action entering the phase in the integrand of the path integral.

These considerations are limited neither by one dimension nor by single-particle physics, but can be extended all the way to relativistic field theories. This offers new calculational tools, new approximation methods in particular, and may be of interest for applications in classical statistical mechanics. In the following, however, we turn to the quantum-classical divide.

2.4. Simple properties of the Liouville path integral

First of all, it seems worth while to record a few simple properties of the Liouville path integral. We observe that in the absence of forces, \( V \equiv 0 \), the classical and quantum mechanical propagators coincide, according to Eqs. (31)–(32). Which implies that classical and quantum mechanical behaviour can differ at most in the initial states that are being propagated, in this case. – This holds true even for \( V \neq 0 \), if \( \mathcal{E} \equiv 0 \), i.e., for harmonic forces, cf. (11). – Thus, the cherished “textbook effect” of the spreading of wave packets is not a peculiar (kinematical) quantum effect, but rather an effect of the particular states considered!

A simple calculation, taking into account our above transformations from \( x, p \) to \( Q, q \) coordinates (and back), shows that a classical point particle with initial phase space distribution \( \propto \delta(x - x_0)\delta(p - p_0) \) is propagated to the distribution \( \propto \delta(x - x_0 - p_0t/m))\delta(p - p_0) \), i.e. along
a straight line path, as expected. – Instead, a free Gaussian wave packet in \(Q,q\)-space spreads in the way described in textbooks on quantum mechanics.

Finally, we can convince ourselves that the classical dynamics is properly represented by the Liouville propagator, in general, by undoing the coordinate transformations (7) directly in the path integral representation of the superpropagator:

\[
y := Q - q, \quad x := \frac{Q + q}{2},
\]

a transformation with unit Jacobian. Following a partial integration of the superaction, the path integral over the \(y\)-coordinate simply yields a functional \(\delta\)“function” and results in:

\[
\mathcal{G}(x_f, y_f; t|x_i, y_i; t_0) = \int Dx \exp(i m \dot{x} y_f) \delta[m \dot{x} + V'(x)] \exp(-i m \dot{x}_i y_i),
\]

with the boundary conditions \(x(t_{i,f}) = x_{i,f}\), similarly for the velocities, and where \(y_{i,f}\) denote the initial and final values of the \(y\)-variable, respectively. The phase factors stem from the partial integration of the superaction.

Thus, we find the expected result that only paths following solutions of the classical equations of motion contribute to the propagator. The role of the phase factors is easily understood by recalling Eq. (27): After the inverse coordinate transformations (33), one of the two ordinary integrations there becomes here, including the relevant (initial) phase factor:

\[
\int dy_1 e^{-i m \dot{x}_i y_1} \rho(x_i, y_i; t_0) = \rho(x_i, m \dot{x}_i \equiv p_i; t_0),
\]

i.e., this incorporates the inverse Fourier transformation, back to the momentum variable of phase space. The second (final) phase factor, then, is necessary for the propagator to fulfill the important semi-group property [38].

3. Hybrid dynamics for two interacting objects

We have seen that the path integral for the propagator of the classical Liouville equation shows significant similarity with the propagator of the quantum mechanical von Neumann equation. This suggests a new perspective on hybrid dynamics – the hypothetical direct coupling of quantum and classical degrees of freedom that we discussed in Section 1.

In particular, we are interested here in the propagator for the density matrix of a bi-partite system, composed of a classical and a quantum mechanical particle of masses \(m, m'\) in external potentials \(V, v\), respectively. Correspondingly, the relevant coordinates will be denoted by \(Q,q\) and \(Q',q'\), respectively. Our previous considerations lead us to propose the following hybrid superpropagator (cf. Eq. (31)):

\[
\mathcal{G}_h(Q_f, q_f; Q'_f, q'_f; t|Q_i, q_i; Q'_i, q'_i; t_0) = \int DQ'Dq'DQ''Dq'' \exp\left(i \frac{1}{R} \mathcal{S}[\hat{Q}, \hat{q}; \hat{Q'}, \hat{q'}; \hat{Q''}, \hat{q''}]\right),
\]

where the superaction naturally consists of three contributions, \(\mathcal{S} \equiv \mathcal{S}_{cl} + \mathcal{S}_{qm} + \mathcal{S}_h:\)

\[
\mathcal{S}_{cl} := \int_{t_0}^{t} d\tau \left( \frac{m}{2} (\dot{Q'}^2 - \dot{q'}^2) - (Q - q)V'(\frac{Q + q}{2}) \right),
\]

\[
\mathcal{S}_{qm} := \int_{t_0}^{t} d\tau \left( \frac{m}{2} (\dot{\hat{Q}}^2 - \dot{\hat{q}}^2) - (v(Q') - v(q')) \right),
\]

\[
\mathcal{S}_h := - \int_{t_0}^{t} d\tau \mathcal{V}(Q, q; Q', q') .
\]
Several remarks are in order here. The contribution $\mathcal{S}_{cl}$ is the classical superaction arrived at in Eqs. (31)–(32); we just inserted $\mathcal{L}$, Eq. (10), explicitly and collected terms. Correspondingly, the contribution $\mathcal{S}_{qm}$ is the action for the propagator of the von Neumann equation, i.e., it describes the quantum mechanical particle in the usual way. Thus, in the absence of the interaction term $\mathcal{S}_h$, the hybrid superpropagator $\tilde{\mathcal{S}}_h$ factorizes into the corresponding ones for Liouville and von Neumann equations. This would consistently describe a composite system of two independent, classical and quantum mechanical particles.

The interaction term $\mathcal{S}_h$ introduces the coupling responsible for hybrid dynamics. – For our present purposes, we restrict this coupling by two consistency requirements: 6
(A) $\mathcal{S}_{cl} + \mathcal{S}_h$ describes a classical particle interacting with another particle (coordinates $Q', q'$), as if the latter were classical;
(B) $\mathcal{S}_{qm} + \mathcal{S}_h$ describes a quantum mechanical particle interacting with another particle (coordinates $Q, q$), as if the latter were quantum mechanical.

In view of the general form of the external potential terms in Eqs. (37) and (38), these conditions can only be fulfilled, if $V(Q, q; Q', q')$ is harmonic, i.e., is a polynomial of degree less than or equal to two in all variables. Thus, the hybrid coupling is of a form that admits a classical or quantum mechanical interpretation, depending on whether it is viewed from the classical or quantum mechanical subsystem.

An example is provided by a distance dependent oscillator potential, $\lambda^{-1}V := (x - x')^2$. Following the derivation of the classical superpropagator, this becomes, in the above coordinates:

$$\lambda^{-1}V(Q, q; Q', q') = \left((Q - q)\partial_{(Q+q)/2} + (Q' - q')\partial_{(Q'+q')/2}\right)\left((Q + q)/2 - (Q' + q')/2\right)^2 = (Q - q)(Q + q) + (Q' - q')(Q' + q')^2 - 2(QQ' - qq'),$$

i.e., it can be seen as coupling between two classical particles, fulfilling condition (A). On the other hand, the separable oscillator terms here also equal $Q^2 - q^2 + Q'^2 - q'^2$, i.e., are of quantum mechanical form, in accordance with (11), and the bilinear coupling can as well be seen as quantum mechanical, fulfilling condition (B).

Interesting consequences of these conditions, determining a harmonic hybrid interaction, are summarized in the following Table:

| $\mathcal{V}$ | $\mathcal{v}$ | $\mathcal{V}$ | resulting dynamics |
|---|---|---|---|
| h | h | h | CL or QM |
| anh | h | h | CL |
| h | anh | h | QM |
| anh | anh | h | ??? |

Indicated are the nature of the potentials acting on the classical and quantum mechanical particles and their interaction, $\mathcal{V}$, $\mathcal{v}$, and $\mathcal{V}$, respectively - h: harmonic; anh: anharmonic – and the character of the resulting hybrid dynamics – CL: classical; QM: quantum mechanical.

For example, if the classical and quantum mechanical particles are governed by a harmonic potential and an anharmonic potential, respectively, then the resulting dynamics of the composite system can be considered as that of two bilinearly coupled quantum mechanical particles, one of which moves in a harmonic potential. – The case of both potentials being anharmonic is most interesting. However, no general statements can be made in this case without further study. The quantum mechanical or classical behaviour of the composite system might very well depend on where (concerning its variables) one is looking at this object! 7

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6 The question of consistency of the map induced by the superpropagator which evolves the quantum-classical hybrid density will be discussed in more detail elsewhere.

7 These findings might help with the problem posed in Section 1: “Can quantum mechanics be seeded?”
3.1. Intra- and interspace entanglement

There is a qualitative difference between CL and QM, contained in the full path integral for the superpropagator, Eqs. (36)–(39). Following standard arguments which lead back from a path integral to an equivalent equation of motion for the (hybrid) density operator $\hat{\rho}$, several related observations may be interesting.

The QM evolution is generated by a commutator of the Hamiltonian with the density operator. Generally, this superposes and, for multi-partite systems, in particular, entangles underlying bra- and ket-states separately, $\propto H_{ij} \rho_{jk} - \rho_{ij} H_{jk}$ (using a convenient notation with discrete indices). For a bi-partite system, it is revealing to write the relevant interaction terms explicitly:

$$[\hat{H}_{\text{int}}, \hat{\rho}] = \hat{H}_1 \rho_2 \otimes \hat{H}_2 \rho_2 - \hat{\rho}_1 \hat{H}_1 \otimes \rho_2 \hat{H}_2 ,$$

(41)

for an interaction $\hat{H}_{\text{int}} \coloneqq \hat{H}_1 \otimes \hat{H}_2$, with the factors acting on subsystems “1” and “2”, respectively, and where $\hat{\rho} = \rho_1 \otimes \rho_2$, for a separable initial state. This has been called dynamically assisted entanglement generation, see, for example, Refs. [41, 42, 43].

It may come as a surprise that the CL evolution does this just as well, due to the structure of the superoperator. For polynomial interactions, in particular, the superoperator always contains a contribution proportional to the usual QM terms.

However, the CL evolution, generally, produces additional correlations in $\hat{\rho}$, due to terms contained in $L_{ij}$, which entangle bra- and ket-states. – In comparison with Eq. (41), such terms can have the unfamiliar structure:

$$\hat{H}_i \rho_1 \otimes \rho_2 \hat{H}_2 - \hat{\rho}_1 \hat{H}_1 \otimes \rho_2 \hat{H}_2 ,$$

(42)

which differs decidedly from a commutator. – This leads us to distinguish intra- (i.e., within given tensor product Hilbert space of subsystems “1” and “2”) and inter-space entanglement (i.e., between said Hilbert space and its dual).

Consider, for example, the anharmonic potential $V(x_1 - x_2) = \lambda(x_1 - x_2)^4$ for a bi-partite system consisting of particles “1” and “2”. Similarly as before, this leads here to the interaction:

$$V(Q_1, q_1; Q_2, q_2) = \frac{1}{2} \lambda(Q_1 - q_1 - (Q_2 - q_2))(Q_1 + q_1 - (Q_2 + q_2))^3 ,$$

(43)

in terms of, by now, familiar variables, taking into account subsystems “1” and “2” with $Q$’s and $q$’s referring to bra- and ket-states, respectively. Besides separable terms, $\propto (Q_a - q_a)(Q_a + q_a)^3$, $a = 1, 2$, there are terms which mix (and entangle) variables of both subsystems, as usual in QM. However, there are also additional terms that refer to Hilbert space and its dual simultaneously (and entangle corresponding states), for example, $\propto Q_a Q_b \rho_b^a$, $b \neq a$.

In retrospect, somehow, such difference between CL and QM evolution had to be expected: instead with superstates $|Q, q\rangle$, we could have worked with superstates $|x, p\rangle$, relating to coordinates and momenta of the classical theory. There, coordinates and momenta end up tightly correlated, due to Hamilton’s equations, and produce inter-space entanglement in an interacting bi-partite system.

Thus, the confrontation of CL with QM, as in our side-by-side study, is quite revealing. In particular, we speculate that this opens new views on generating entanglement in multipartite systems, perhaps, by evolving through quasiclassical stages or by making use of decohered intermediary states. ⁸

⁸ Previous considerations of the semiclassical regime, such as in Refs. [41, 42], were motivated as suitable approximations of the quantum mechanical evolution, in particular, for studies of the different decoherence properties between classically regular and chaotic systems. Our results seem to show that crossing the quasiclassical divide may offer an additional resource for entanglement generation and related “truly quantum” phenomena. This might be related to a common “underlying reality” of CL and QM physics, assumed to consist, for example, only in statistical correlations in Refs. [44, 45].
Concerning the quantum-classical divide, the present analysis shows that there is an appealing, if not puzzling formal similarity between CL and QM. However, this demonstrates one more time that what has been discussed in various ways as CL limit of QM – and which is similarly relevant for “emergent QM” – deserves further study.

While our work has been concerned mainly with the evolution of CL or QM objects, we recall that V.J. Man’ko and collaborators have pointed out that classical states may differ widely from what could be obtained as the “\( \hbar \to 0 \)” limit of quantum mechanical ones. They show that all states can be classified by their ‘tomograms’ as either CL or QM, CL and QM, and neither CL nor QM [46].

The classical limit can be considered a limit “For All Practical Purposes”, gradually approached with decoherence as an essential but insufficient ingredient or, formally, following the mnemonic “\( \hbar \to 0 \)” rule. However, in order to truly bridge the qualitative difference between intra- and inter-space entanglement that we find, and explain the “Man’ko classes of states”, some unknown dynamics seems missing.\(^9\)

The problems discussed here lead us to the question: “Does the \( \hbar \to 0 \) deformation of QM provide the only interesting linear dynamics besides QM itself?”

4. How special is quantum mechanics?
We are interested here once more in the structure of the linear evolution equations that we have discussed side by side, namely the classical Liouville equation and the von Neumann equation of quantum mechanics. However, we consider this with respect to a hypothetical most general linear dynamics that can be represented in the generic form of Eq. (4):

\[
\partial_t \rho_{ij} = \frac{1}{i} \sum_{k,l} L_{ij,kl} \rho_{kl},
\]

where the indices \((i, j, k, l = 1, \ldots, N)\) refer to a discrete, finite dimensional Hilbert space, which we assume for simplicity. Thus, generally, there are \( N^2 \times N^2 \) complex coefficients to be specified. Imposing the constraint that the ‘density matrix’ is and remains Hermitian, which requires that \( L_{ij,kl} = -L_{ji,k\ell}^* \), the set of coefficients can be specified by \( N^4 \) real parameters.

This should be compared with the von Neumann equation – which we discussed, in order to introduce the concept of a superoperator, cf. Eqs. (12)–(14) in Section 2.2. In terms of a Hermitian Hamiltonian, \( \hat{H} \), the matrix elements of which are specified by \( N^2 \) real parameters, the corresponding superoperator has been obtained as: \( L_{ij,kl} \equiv \hat{H}_{ik} \delta_{lj} - \delta_{ik} \hat{H}_{lj} \).

Therefore, we can write a symbolic relation, concerning the number of real parameters which determine the respective dynamical equation:

\[
\text{“QM } \sim \sqrt{\text{GL} \text{”}},
\]

i.e., the number of real parameters entering the quantum mechanical evolution law (QM) scales with the Hilbert space dimension like the square root of the corresponding number for the most general linear dynamics (GL) which preserves hermiticity. By choosing the eigenstates of the Hamiltonian as basis of the Hilbert space, the number of relevant parameters in QM could be further reduced to \( N \), the number of real eigenvalues of \( \hat{H} \).

We may wonder whether the commutator in Eq. (12), \([\hat{H}, \_\] \), presents the minimal structure preserving hermiticity, normalization, and positivity of the density matrix. – Conversely, can the different numbers of real parameters between a more general linear evolution law and QM be attributed to an attractor mechanism or information loss if and when QM is emergent [1, 2]?\(^9\)

\(^9\) A simple attractor model, motivated by assumptions about effects of fundamental spacetime discreteness [2], has been discussed in Ref. [3].
As a first step to investigate these issues, we consider here whether QM admits a more general linear evolution. - We recall the following result for the QM of open systems, see Ref. [47] and references there:

- If and only if the Liouville superoperator \( \hat{\mathcal{L}} \) entering the right-hand side of Eq. (44) generates a Hermitian, trace preserving, completely positive map, \( \hat{\mathcal{M}}(t)\hat{\rho}(0) := \exp(-i\hat{\mathcal{L}}t)\hat{\rho}(0) \), then it can be written in canonical Kraus form

\[
\hat{\mathcal{M}}(t)\hat{\rho}(0) = \sum_k \hat{M}_k(t)\hat{\rho}(0)\hat{M}_k^\dagger(t),
\]

with \( \sum_k \hat{M}_k^\dagger \hat{M}_k = 1_{N\times N} \).

- Such a map \( \hat{\mathcal{M}} \) applied to a Hermitian, positive semidefinite, normalized density operator yields another one and linear maps between density operators can be written in Kraus form. An evolution equation corresponding to such a map, in general, contains the von Neumann equation as a special case. - In fact, the solution of the von Neumann equation (12) is provided by a unitary transformation, \( \hat{\rho}(t) = \hat{U}(t)\hat{\rho}(0)\hat{U}(t)^\dagger \), which presents the simplest case with only one unitary Kraus operator, \( \hat{\mathcal{M}}(t) \equiv \hat{U}(t) := \exp(-i\hat{H}t) \).

Therefore, any generalization which maintains the defining properties of density operators, generally, will need additional Kraus operators specified by additional parameters. - In this sense, the commutator defining the von Neumann equation presents the minimal structure indeed.

Furthermore, it has been shown that the dynamics generated by \( \hat{\mathcal{M}} \) of the Kraus form, in general, is equivalent to a non-unitary reduced dynamics of a unitary dynamics on a bigger (tensor product) Hilbert space “System \( \otimes \) Environment” [47]:

\[
\hat{\mathcal{M}}\hat{\rho}_S = \text{Tr}_E \left( \hat{U}_\text{big}^\dagger\hat{\rho}_S \otimes \hat{\rho}_E\hat{U}_\text{big} \right),
\]

with \( \hat{U}_\text{big}^\dagger \hat{U}_\text{big} = \hat{U}_\text{big}^\dagger \hat{U}_\text{big} = 1 \). Hence, we learn:

- Hermitian, trace preserving, completely positive maps, while generalizing the von Neumann dynamics, are not general enough to “leave QM”. They describe the QM of open systems. However, this kind of maps does not exhaust the larger set of maps generated by all possible Liouville superoperators.

We conclude that one cannot “leave QM” by invoking more general linear dynamics than that generated by trace preserving, completely positive maps without affecting properties and interpretation of the states represented by density matrices.  

It is useful to recall here (the motivation behind) the assumptions made concerning the properties and interpretation of density operators. - A density operator \( \hat{\rho} \) is required to be i) Hermitian, i.e., to have real eigenvalues, in order to qualify as an observable. This is needed, in turn, if one requires \( \hat{\rho} \) to be ii) positive-semidefinite and iii) normalized. All three properties, together, are necessary assumptions for the standard probability interpretation of the eigenvalues of a density matrix, according to the Born rule.

A recurrent theme, when comparing quantum with classical states, for example, with the help of the Wigner (function) transform of the density matrix, is the appearance of negative eigenvalues or “negative probabilities”. Numerous attempts have been made to give a satisfactory physical interpretation and mathematically consistent definition to these, see Refs. [45, 48, 49] and further references there.

Instead of entering this discussion, we look at the arguably simplest example of general linear dynamics in the following. This is obtained by giving up the requirement ii) above, i.e., by abandoning positivity, and by considering a two-dimensional Hilbert space.

\[\text{It will be interesting to see how (any form of) hybrid dynamics, cf. Section 3., fares in this respect.}\]
4.1. General linear $2 \times 2$ dynamics: a model

A convenient parametrization of the most general Liouville superoperator $\hat{L}$ for a two-
dimensional state space can be written with the help of the Pauli matrices $\sigma$. Thus, the
(super)matrix elements $\mathcal{L}_{ij,kl}$, $i,j,k,l = 1,2$, are defined by:

$$\mathcal{L}_{ij,kl} := iG_{\mu\nu}(|i\rangle a_{\mu} + \bar{b}_{\mu} \cdot \sigma|k\rangle \langle l|a_{\nu} + \bar{b}_{\nu} \cdot \sigma|j\rangle) ,$$

(48)

where a summation over repeated indices is understood; in this notation we have $(\hat{L}\rho)_{ij} =
\mathcal{L}_{ij,kl}\rho_{kl}$. Hermiticity requires $G_{\mu\nu} = G_{\nu\mu}^*$. Generally, three vectors $\bar{b}_\mu = 0,1,2 \in \mathbb{R}^3$ are needed. Furthermore, we define:

$$G := \begin{pmatrix} 1 & 0 & 0 \\ 0 & g_{11} & g_{12} \\ 0 & g_{12} & g_{22} \end{pmatrix}$$

(49)

i.e., where $g$ is a Hermitian $2 \times 2$ matrix. This Ansatz saturates the expected number of
$N^4 = 2^4 = 16$ real parameters, since $a_{\mu}, b_{\mu}$, and $g$ contribute 3, 9, and 4 parameters,
respectively.

In order that the map generated by $\hat{L}$ be trace preserving, we must have $0 = i\partial_t \text{Tr} \hat{\rho} = \text{Tr} \hat{L}\hat{\rho}$,
for all $\hat{\rho}$ which leads to $\mathcal{L}_{ii,kl} = 0$. This yields four real constraints,

$$a \cdot G \cdot a + \bar{b} \cdot G \cdot \bar{b} = 0 ,$$

(50)

$$2a \cdot \text{Re} G \cdot \bar{b} + ig_{ij} \bar{b}_i \times \bar{b}_j = 0 ,$$

(51)

which reduce the number of available real parameters to twelve.

Next, we conveniently choose the vectors $\bar{b}_{0,1,2}$ to form a right-handed orthogonal system;
with respect to suitable coordinates, this sets six vector components to zero. Thus, we are left
with six real parameters, three of which could characterize the QM evolution of a two-state
object, while three pertain to the generalization we are concerned with.

Finally, we assume for simplicity that the ‘QM part’ of $\hat{L}$ is diagonal, corresponding to a spin-
1/2 particle in a constant external magnetic field parallel to the quantization axis, for example.
Setting one of the remaining parameters to zero, we obtain a simplified model with altogether
three real parameters, $\alpha, \beta, \gamma \in \mathbb{R}$:

$$\mathcal{L}_{ij,kl} := \alpha \left\{ (|i\rangle \sigma_z |k\rangle \delta_{ij} - \delta_{ik}(|l\rangle \sigma_z |j\rangle) \right\}$$

$$+ \beta \gamma \left\{ (|i\rangle \sigma_z |k\rangle \langle l| \sigma_z |j\rangle) - (|i\rangle \sigma_z |k\rangle \langle l| \sigma_z |j\rangle) \right\}$$

$$+ i\beta \gamma \left\{ (|i\rangle \sigma_y |k\rangle \delta_{ij} + \delta_{ik}(|l\rangle \sigma_y |j\rangle) \right\}$$

$$+ i\beta \gamma \delta_{ik} \delta_{ij}$$

$$+ i\beta \gamma \delta_{ik} + i\beta \gamma \delta_{ik}$$

$$+ i\beta \gamma \delta_{ik} + i\beta \gamma \delta_{ik}$$

$$+ i\beta \gamma \delta_{ik}$$

(52)

where QM terms $\propto \alpha \sigma_z$ feature in the first line. All other contributions on the right-hand side
have no counterpart in QM; this will become obvious by explicitly solving the model. The first
two terms in the second line and the two last ones in the third couple bra- and ket-states, which
we discussed as a consequence of classical evolution in Sections 2.1 and 3.1. It is easy to verify
that the generator $-i\hat{L}$ is Hermitian and trace preserving, as it should.

Taking hermiticity and trace normalization of the density matrix into account by:

$$\hat{\rho} \equiv \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{12}^* & 1 - \rho_{11} \end{pmatrix} ,$$

(53)

we write the resulting evolution equation explicitly:

$$i\partial_t \hat{\rho} = \hat{L}\hat{\rho} =$$

$$\begin{pmatrix} i\gamma^2 (1 - 2\rho_{11}) & 2\beta \gamma + (2\alpha + i[2\beta^2 + \gamma^2])\rho_{12} - i\gamma^2 \rho_{12} \\ -2\beta \gamma - (2\alpha - i[2\beta^2 + \gamma^2])\rho_{12}^* - i\gamma^2 \rho_{12}^* \end{pmatrix} .$$

(54)
Note that for $\beta, \gamma \to 0$, we recover the von Neumann equation, e.g., for a spin-1/2 particle in a constant magnetic field. In this case, besides the Hamiltonian, $\hat{H} := \alpha \sigma_z$, also the total spin-squared $S^2 := \sigma^2 / 4 \propto 1$ is a constant of motion.

4.2. Constants of motion
We remark that if a constant operator, $\hat{C}$, obeys $\text{Tr}(\hat{C} \hat{\rho}(0)) = \text{Tr}(\hat{C} \hat{\rho}(t))$, for all solutions $\hat{\rho}$ of the general linear evolution equation (44), then $C_{ij} L_{ij,k} = 0$, which generalizes the vanishing of the commutator $[\hat{C}, \hat{H}]$ in QM, for constants of motion. This is fulfilled for any $\hat{C} \equiv c \mathbb{1}$, with constant $c$, due to the preservation of the trace normalization of the density matrix, incorporated by $\hat{L}_{ij,k} \equiv 0$. However, this raises also the interesting question, whether there exist conserved superoperators, $\hat{Q}$, which can be defined by the vanishing supercommutator:

$$[\hat{Q}, \hat{L}]_{ij,mn} := Q_{ij,km} L_{kl,mn} - L_{ij,km} Q_{kl,mn} \equiv 0 .$$

In the present case, this amounts to a $4 \times 4$ matrix equation, to be studied.

4.3. Solution of the 2 $\times$ 2 model
We now turn to the explicit solution of Eq. (54), which can be represented as follows. The matrix elements are:

$$\rho_{11}(t) = 1 - \rho_{22}(t) = \frac{1}{2} + \left( \rho_{11}(0) - \frac{1}{2} \right) e^{-\gamma^2 t} ,$$

$$\rho_{12}(t) = \rho_{21}^*(t) = r_+ e^{\Omega_+ t} + r_- e^{\Omega_- t} + r_c ,$$

where $r_\pm$ are determined by initial conditions. The remaining constants are defined by:

$$r_c := -\gamma \frac{\alpha - i \beta^2}{\alpha^2 + \beta^2 (\beta^2 + \gamma^2)} ,$$

$$\Omega_\pm := 2 \beta^2 + \gamma^2 \pm 2i\alpha(1 - \gamma^4/4\alpha^2)^{1/2} ,$$

and we may choose $\alpha \geq 0$. Diagonalizing the resulting normalized Hermitian density matrix, we obtain the eigenvalues:

$$\rho_\pm(t) = \frac{1}{2} \pm \left( (\rho_{11}(0) - 1/2) e^{-\gamma^2 t} + |\rho_{12}(t)|^2 \right)^{1/2} ,$$

with $|\rho_{12}|^2$ given by Eqs. (57)-(59).

Several features of the solutions of our model seem unexpected. To begin with, for $\beta, \gamma \to 0$, we obtain:

$$\rho_{11}(t) = 1 - \rho_{22}(t) = \rho_{11}(0) ,$$

$$\rho_{12}(t) = \rho_{21}^*(t) = r_+ e^{2i\alpha t} + r_- e^{-2i\alpha t} ,$$

which amounts to the usual QM solution only, if we choose $r_+ \equiv 0$. From a QM perspective, the larger set of solutions, parametrized by a larger set of initial conditions, is quite surprising.

11 Interestingly, the underlying equation for the off-diagonal matrix elements, for $\gamma \neq 0$ in particular, is of second order; it decouples into two first order equations in the limit $\gamma \to 0$, independently of the value of $\beta$.
It leads to time dependent ‘probabilities’, if we try to interpret the eigenvalues of $\hat{\rho}$ in the usual way:

$$
\rho_{\pm}(t) = \frac{1}{2} \pm \left( \left(\rho_{11}(0) - 1/2\right)^2 + |r_+|^2 + |r_-|^2 + 2\text{Re}(r_+ r_-^* e^{i2\pi t}) \right)^{1/2}.
$$

(63)

While these ‘probabilities’ are real and normalized, by construction of our model, they might temporarily fall outside the interval $[0,1]$, depending on the parameters $r_{\pm}$.

This remark applies also for the general case, with $\beta, \gamma \neq 0$, where amplitudes grow $\propto \exp(2\beta^2 + \gamma^2)t$, see Eqs. (57)-(59).

The effect of such anomalous ‘probabilities’ is clearly seen in the following mean square deviations sensitive to fluctuations. With $\langle \hat{O} \rangle := \text{Tr}(\hat{O}\hat{\rho})$, $\sigma_{x,y}^2 = 1$, and $\langle \sigma_x \rangle = 2\text{Re}(\rho_{12})$, $\langle \sigma_y \rangle = -2\text{Im}(\rho_{12})$, $\langle \sigma_z \rangle = 2\rho_{11} - 1$, we obtain:

$$
\Delta_x^2 := \langle \sigma_x^2 \rangle - \langle \sigma_x \rangle^2 = 1 - \left(2\text{Re}(\rho_{12})\right)^2,
$$

(64)

$$
\Delta_y^2 := \langle \sigma_y^2 \rangle - \langle \sigma_y \rangle^2 = 1 - \left(2\text{Im}(\rho_{12})\right)^2,
$$

(65)

$$
\Delta_z^2 := \langle \sigma_z^2 \rangle - \langle \sigma_z \rangle^2 = 4\rho_{11}(1 - \rho_{11}).
$$

(66)

While $\Delta_x^{2 \rightarrow \infty}$ 1, for $\gamma \neq 0$, which would correspond to a completely mixed state in QM, $\Delta_{x,y}^2$ can have oscillatory contributions with growing amplitude $\propto \exp \Omega_{\pm} t$, see Eqs. (56)-(59). - Also note that the would-be QM energy expectation varies in time, $\alpha(\sigma_x) = \alpha(2\rho_{11}(0) - 1) \exp(-\gamma^2 t)$, approaching the stationary QM mixed-state value zero.

To summarize: Modifying a QM two-state model by applying a general linear perturbation, however small – which preserves hermiticity and trace normalization of the density matrix – opens the possibility of a larger state space, reflected in a doubling of the number of degrees of freedom.\(^{12}\)

We emphasize that the QM evolution becomes exponentially unstable when such perturbation is introduced. Generically, this spoils the standard interpretation of the eigenvalues of the density matrix as probabilities and can result in interesting oscillatory effects, as we have seen.\(^{13}\)

It is an interesting question, whether there is a “classical” formulation of general linear dynamics, suggested by a doubled number of certain degrees freedom and “anomalous probabilities” that seem invariably to appear – and which remind of similar phenomena encountered when relating quantum and classical mechanics, e.g., via the Wigner function.

5. Conclusions

In this article we have summarized our earlier derivation of a path integral for classical Hamiltonian systems based on an ensemble description [35, 40].

This leads us to point out the correlation properties of classical dynamics in parallel to the quantum mechanical ones and to identify characteristic similarities and differences. In particular, it seems useful to distinguish intra- and interspace entanglement.

The former has been held characteristic of quantum mechanics and a feature of superpositions of tensor product states. The latter concerns classical mechanics only; it correlates Hilbert space states and their duals. However, surprisingly, for anharmonic potentials or interactions, classical mechanics additionally shows intraspace entanglement, as in quantum mechanics.

As a first application, we propose a new formulation of hybrid dynamics, i.e., based on a hypothetical direct coupling between quantum and classical objects. This may be of practical as well as foundational interest.

\(^{12}\) I.e., a doubling of the initial conditions for the off-diagonal matrix elements here; it will be interesting to see, whether this extends to all degrees of freedom in a more general model.

\(^{13}\) One might speculate on the relevance for flavour oscillations.
Finally, we study a generalization of quantum evolution, general linear dynamics, where the evolution is generated by a superoperator that preserves hermiticity and trace normalization of density matrices. We argue that one cannot “leave QM” without giving up one of the three defining properties of density matrices, to be Hermitian, normalized, and positive-semidefinite.

In the most simple case of a two-state system, we solve such dynamics explicitly. We show that the corresponding von Neumann dynamics becomes exponentially unstable under the influence of a general linear perturbation. Most interestingly, it leads to the appearance of “anomalous probabilities” and an enlargement of the state space, possibly pointing towards a sort of prequantum dynamics. – We intend to study more complex objects consisting of such two-state systems as building blocks which interact. This may be useful in trying to understand quantum mechanics as an emergent phenomenon [1].

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