Quasi-invariants of dihedral systems

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A basis of quasi-invariant module over invariants is explicitly constructed for the two-dimensional Coxeter systems with arbitrary multiplicities. It is proved that this basis consists of $m$-harmonic polynomials, thus the earlier results of Veselov and the author for the case of constant multiplicity are generalized.

1 Introduction

We start with the introducing the main object of our considerations, the algebra of quasi-invariants. Consider a Coxeter system $\mathcal{R}$ in $\mathbb{R}^n$ which consists of the non-collinear pairs of vectors $\{\pm \alpha\}$ with prescribed multiplicities $m_\alpha \in \mathbb{Z}_+$. The corresponding Coxeter group $G$ is generated by the reflections $s_\alpha$, $\alpha \in \mathcal{R}$:

$$s_\alpha u = u - \frac{2(\alpha, u)}{(\alpha, \alpha)} \alpha, \quad u \in \mathbb{R}^n.$$ 

The group $G$ should be finite and the set of reflections $\{s_\alpha\}$ must be the set of all the reflections in $G$. The multiplicity function $m_\alpha = m(\alpha)$ is supposed to be invariant:

$$m_\alpha = m_{g(\alpha)}$$

for any $g \in G$, $\alpha \in \mathcal{R}$.

A polynomial $p(x)$ is called quasi-invariant related to the system $\mathcal{R}$ if it satisfies the condition

$$s_\alpha p(x) - p(x) = O((\alpha, x)^{2m_\alpha+1})$$

near the hyperplanes $(\alpha, x) = 0$ for any $\alpha \in \mathcal{R}$. Equivalently,

$$\partial_\alpha^{2s-1} p = 0, \quad \text{if } (\alpha, x) = 0, \quad s = 1, \ldots, m_\alpha.$$
For instance, if a polynomial $p(x)$ belongs to the ring $S^G$ of polynomials invariant under the geometric action of the Coxeter group, then $s_\alpha p(x) = p(x)$ for any $\alpha \in \mathcal{R}$, and therefore property (1) is satisfied. Thus the quasi-invariants $Q^R$ form a ring containing the ring of invariants $S^G$.

The rings of quasi-invariants were introduced by Chalykh and Veselov in 1990 in the context of quantum integrable systems [1]. The authors showed that to each quasi-invariant $q(x)$ it corresponds a differential operator $L_q = \chi(q)$ of the form

$$L_q(x, \partial_x) = q(\partial_x) + \text{lower order terms},$$

and all such operators $L_q$ commute. Under the homomorphism $\chi$ the quasi-invariant $q(x) = x^2_1 + \ldots + x^2_n$ corresponds to the generalized Calogero–Moser operator [2, 3, 4]

$$L_{x^2} = \Delta - \sum_{\alpha \in \mathcal{R}_+} \frac{2m_\alpha}{(\alpha, x)} \partial_\alpha. \quad (2)$$

Here $\mathcal{R}_+$ is a subset of vectors from $\mathcal{R}$ belonging to a halfspace, and $\Delta$ is the Laplace operator in $\mathbb{R}^n$. Therefore each quasi-invariant $q(x)$ leads to the quantum integral $L_q$ of the Calogero–Moser problem [2]. As it is explained in [2] the commutative ring of all quantum integrals for [2] is isomorphic to the ring of quasi-invariants $Q^R$.

We note that the homomorphism $\chi$ on the invariant polynomials was considered by Heckman and Opdam for a general multiplicity function [6]. It is crucial that for the integer multiplicity function the ring $Q^R$ of quantum integrals becomes larger than the ring of invariants $S^G$, and the ring depends on these integer parameters, as it all was discovered by Chalykh and Veselov [1]. The necessity to consider integer multiplicities to get larger commutative rings have been justified recently by Taniguchi [7].

The first investigation of the ring $Q^R$ was done by Volchenko, Kozachko, and Mishachev [8] in the case of dihedral system with multiplicity equal 1. The authors found the multiplicative generators of the ring. The more complete description of the ring $Q^R$ for the dihedral systems with constant multiplicity function $m_\alpha = m \in \mathbb{Z}_+$ was carried by Veselov and the author [5]. It turned out that for the dihedral system $I_2(N)$ consisting of $N$ lines on the plane $\mathbb{R}^2 \cong \mathbb{C}$ the ring $Q^{I_2(N)}$ is a free module over the subring of invariants $S^{I_2(N)} = \mathbb{C}[z\bar{z}, z^N + \bar{z}^N]$. These module has $2N$ generators $h_1, \ldots, h_{2N}$ such that the linear space $H_m = \langle h_1, \ldots, h_{2N} \rangle$ spanned by these generators is the
solution space of the following system of two equations
\[
\mathcal{L}_z \bar{z} h = 0,
\]
\[
\mathcal{L}_{\bar{z}N + \bar{z}N} h = 0,
\]
and the determinant formulas for \( h_i \) were found. It was conjectured in [9] that
the ring of quasi-invariants \( Q^R \) is free as a module over invariants \( S^G \) for any
Coxeter system \( R \). It was also conjectured that one can take \( m \)-harmonic polynomials as a basis in this free module. The space \( H_m \) of \( m \)-harmonic polynomials consists of the solutions to the following system of equations
\[
\chi(\sigma) h = 0,
\]
where \( \sigma \) is an arbitrary element from the ring \( S^G \) of invariant polynomials
with free term equal zero (see [9]).

Etingof and Ginzburg remarkably proved in [10] that the ring \( Q^R \) is free
over invariants \( S^G \) for any Coxeter system \( R \). But the choice of a basis in
\( H_m \) as a free basis for the quasi-invariant module is impossible in general (see
counterexample in [10]). Nevertheless the degrees of homogeneous generators
coincide with the degrees of homogeneous basis in \( H_m \), as it is proved in [10].
The free generators should form a basis in the complement to ideal generated
by homogeneous invariants of positive degree in \( Q^R \). The problem to produce
some particular choice of such a basis remains open.

Besides the Cohen–Macaulay property for the quasi-invariant algebra,
the Gorenstein property was also established by Etingof and Ginzburg [10].
One of the two proofs given in [10] is based on the results of Felder–Veselov
paper [11]. In that paper the authors in particular compute the Poincare
polynomial for the space \( H_m \) of \( m \)-harmonic polynomials. The Gorenstein
property corresponds to the palindromicity of the Poincare polynomial for
\( H_m \) which was conjectured in [9].

The differential operators on the singular algebraic variety corresponding
to \( Q^R \) were studied by Berest, Etingof, and Ginzburg in [12]. The authors
proved that the algebra of differential operators is simple and Morita equivalent to the Weyl algebra of polynomial differential operators in \( n \)-dimensional
space.

In the present paper following [13] we describe the quasi-invariants of
dihedral systems \( I_2(2N) \) with arbitrary (invariant) multiplicities \((m,n)\)\(^1\). We

\(^1\)In this case there exists two orbits under the action of the group \( G \) on the set of roots
\( \mathcal{R} \). The multiplicity of one of them equals \( m \in \mathbb{Z}_+ \), and it is equal to \( n \in \mathbb{Z}_+ \) on another
orbit.
give explicit formulas for the generators of the ring $Q^{I_2(2N)}$ as a module over invariants $S^{I_2(2N)}$ generalizing the formulas from [5]. It occurs that in this case as in the case of dihedral systems with constant multiplicity [5], one can take as the generators a homogeneous basis in the space $H_m^2$ of $m$-harmonic polynomials. Therefore we obtain explicit determinant formulas for the basic $m$-harmonic polynomials for arbitrary dihedral systems.

The technique used in this paper is different from the approach in [5] where dihedral systems with constant multiplicity were considered. Now we heavily use the results of Etingof–Ginzburg [10] and Felder–Veselov [11]. This approach allows to reduce the explicit calculations essentially. The scheme of our considerations is the following.

In Section 2 we review the result of Felder and Veselov on the Poincare polynomial for $m$-harmonics, and we write down explicitly the Poincare polynomial $P_{H_m}$ for dihedral $m$-harmonics. By Etingof–Ginzburg theorem the degrees of homogeneous generators for quasi-invariants have degrees given by $P_{H_m}$. In Section 3 we prove that these generators can be chosen to have a particular form. To do this we argue as follows. We take a general polynomial of degree under consideration and impose quasi-invariant conditions. From Etingof–Ginzburg theorem it follows that this system of equations must have a solution. From the form of equations it follows that it must also exist a solution of the desired form. Then we check that the linear span of the quasi-invariants of the form constructed does not belong to the ideal generated by homogeneous invariants of positive degree. This can be easily done using again the Etingof–Ginzburg theorem. Thus we conclude (by Etingof–Ginzburg theorem) that we have constructed a free basis with elements of particular form.

In Section 4 we check firstly that the constructed basis is $m$-harmonic. This is due to the special form of the chosen basis, here we again use the Etingof–Ginzburg theorem. Then we come back to the system of quasi-invariant conditions which defined our basis. We note that for a polynomial of our form the set of quasi-invariant conditions leads to $m$-harmonicity of this polynomial. In particular, the polynomial of our form is uniquely determined by the set of quasi-invariance equations. This fact allows us to get explicit determinant formulas for $m$-harmonic polynomials and therefore for the basis of quasi-invariant module.

\footnote{We use notation $H_m$ meaning by $m$ the function of multiplicity $m = m(\alpha)$. For the system $I_2(2N)$ the function of multiplicity takes two values $m$ and $n$.}
2 Poincaré polynomial for dihedral groups

Let \( P_{H_m}(t) \) be Poincaré polynomial for the space \( H_m \) of \( m \)-harmonic polynomials related to a Coxeter system \( \mathcal{R} \),

\[
P_{H_m}(t) = \sum_{i \geq 0} \dim (H_m^{(i)}) t^i,
\]

where \( H_m^{(i)} \) is the space of homogeneous \( m \)-harmonic polynomials of degree \( i \).

The following formula for \( P_{H_m}(t) \) was found by Felder and Veselov in [11],

\[
P_{H_m}(t) = \sum_{V_j} t^{\sum_a m_a d^-_a(V_j)} P_j(t),
\] (3)

In this formula the external sum is carried over all non-isomorphic irreducible representations of a Coxeter group \( G \). The internal sum is carried over the classes \( C_a \) of conjugate reflections in \( G \). Polynomial \( P_j \) is the Poincaré polynomial for representation \( V_j \) in the space \( H_0 \cong \mathbb{C}[x_1, \ldots, x_n]/I_0 \), where \( I_0 \) is the ideal in the ring of polynomials generated by homogeneous invariants of positive degree. That is, the coefficient at \( t^k \) equals the dimension of isotypical component of the representation \( V_j \) in the \( k \)th grade of \( \mathbb{C}[x_1, \ldots, x_n]/I_0 \).

Then \( d^-_a(V_j) = \frac{2N_a \dim V_{j,a}^-}{\dim V_j} \),

where \( N_a \) is a number of elements in the class \( C_a \), and

\[
V_{j,a}^- = \{ v \in V_j | s_a v = -v \text{ for any } s_a \in C_a \},
\]

\( m_a \) is a multiplicity for reflections \( s_a \in C_a \).

For dihedral group \( I_2(N) \) with odd \( N \) there are two one-dimensional non-isomorphic irreducible representations and \( \frac{N-1}{2} \) two-dimensional ones. All the reflections are conjugate and \( N_a = N \). Since in the complex coordinates the ring of invariants is isomorphic to \( \mathbb{C}[z \bar{z}, z^N + \bar{z}^N] \), one can choose generators in factor-space as follows

\[
\mathbb{C}[z, \bar{z}]/I_0 \simeq \langle 1, z, \bar{z}, \ldots, z^{N-1}, \bar{z}^{N-1}, z^N - \bar{z}^N \rangle.
\]

For one-dimensional representations the piece of sum (3) has the form

\[
1 + t^{(2m+1)N}
\]
where the first term corresponds to the trivial representation $V_{\text{triv}}$ which is realized on the constants. The second term corresponds to the sign representation $V_{\text{sign}}$ which is realized on the vector $z^N - \bar{z}^N$. We’ve also used that
\[
d^{-a}(V_{\text{triv}}) = \frac{2N \cdot 0}{1} = 0,
\]
\[
d^{-a}(V_{\text{sign}}) = \frac{2N \cdot 1}{1} = 2N.
\]
Consider now two-dimensional representations. Representation $V_i$ is realized as $\langle z_i, \bar{z}_i \rangle$ and as $\langle z_{N-i}, \bar{z}_{N-i} \rangle$. Then
\[
d^{-a}(V_i) = \frac{2N \cdot 1}{2} = N.
\]
Totally we get
\[
P_{H_m} = 1 + t^{(2m+1)N} + \sum_{i=1}^{N-1} 2t^m(t^i + t^{N-i}) = 1 + 2 \sum_{i=1}^{N-1} t^{mN+i} + t^{(2m+1)N}.
\]
Now let us consider even dihedral system $I_2(2N)$. We have two classes $C_a$ and $C_b$ of conjugate reflections each of which consists of $N$ elements. Denote the multiplicities of reflections as $m_a = m$ and $m_b = n$. As in the odd case
\[
\mathbb{C}[z, \bar{z}]/I_0 \cong \langle 1, z, \bar{z}, \ldots, z^{2N-1}, \bar{z}^{2N-1}, z^{2N} - \bar{z}^{2N} \rangle.
\]
At first we analyze one-dimensional representations and calculate the corresponding terms in the Poincaré polynomial. There are four one-dimensional representations for $I_2(2N)$ depending on whether each class of reflections $C_a, C_b$ acts as 1 or -1.

- $V_{\text{triv}} = \langle 1 \rangle$. The corresponding term in (3) is equal to 1.
- $V_{\text{sign}} = \langle z^{2N} - \bar{z}^{2N} \rangle$, $d^{-a}(V_{\text{sign}}) = d^{-b}(V_{\text{sign}}) = \frac{2N-1}{1} = 2N$. The contribution to the Poincaré polynomial is $t^{(m+n+1)2N}$.
- $V_{\text{sign}_1} = \langle z^{N} + \bar{z}^{N} \rangle$. One gets
\[
m_a d^{-a}(V_{\text{sign}_1}) + m_b d^{-b}(V_{\text{sign}_1}) = m \cdot \frac{2N \cdot 0}{1} + n \cdot \frac{2N \cdot 1}{1} = 2nN
\]
which leads to $t^{(2n+1)N}$ term in (3).
\(V_{\text{sign}}^2 = \langle z^N - \bar{z}^N \rangle\). One gets
\[
m_a d_a^-(V_{\text{sign}}^2) + m_b d_b^-(V_{\text{sign}}^2) = m \cdot \frac{2N \cdot 1}{1} + n \cdot \frac{2N \cdot 0}{1} = 2mN
\]
which leads to \(t^{(2m+1)N}\) term in (3).

Indeed, representations \(V_{\text{sign}}^1, V_{\text{sign}}^2\) are defined by the property that for any \(s \in C_a, \tau \in C_b\)
\[
s|_{V_{\text{sign}}^1} = \text{Id}, \quad s|_{V_{\text{sign}}^2} = -\text{Id},
\]
\[
\tau|_{V_{\text{sign}}^1} = -\text{Id}, \quad \tau|_{V_{\text{sign}}^2} = \text{Id}.
\]
The dihedral group is generated by two reflections \(s \in C_a, \tau \in C_b\) such that \((s\tau)^{2N} = 1\). Also it is generated by \(s\) and \(s\tau\). In our notations we can think of \(s\) as a reflection with respect to line \(z = \bar{z}\), and \(s\tau\) would be a rotation by the angle \(\phi = \frac{\pi}{N}\). Thus the following formulas hold
\[
s : z \rightarrow \bar{z}, \quad \bar{z} \rightarrow z,
\]
\[
s\tau : z \rightarrow \epsilon z, \quad \bar{z} \rightarrow \epsilon \bar{z},
\]
where \(\epsilon = e^{i\frac{\pi}{N}}\). Then representation \(V_{\text{sign}}^1\) is characterized by the property
\[
s|_{V_{\text{sign}}^1} = \text{Id}, \quad \tau|_{V_{\text{sign}}^1} = -\text{Id},
\]
or equivalently
\[
s|_{V_{\text{sign}}^1} = \text{Id}, \quad (s\tau)|_{V_{\text{sign}}^1} = -\text{Id}.
\]
Obviously the following action of group elements holds
\[
s(z^N + \bar{z}^N) = \bar{z}^N + z^N,
\]
and
\[
(s\tau)(z^N + \bar{z}^N) = \epsilon^N z^N + \epsilon^N \bar{z}^N = -(z^N + \bar{z}^N).
\]
Thus \(V_{\text{sign}}^1 = \langle z^N + \bar{z}^N \rangle\) and if \(m_s = m, m_\tau = n\) then the contribution of this irreducible representation to the Poincaré polynomial would be \(t^{(2n+1)N}\). Analogously \(V_{\text{sign}}^2 = \langle z^N - \bar{z}^N \rangle\) since
\[
s(z^N - \bar{z}^N) = \bar{z}^N - z^N = -(z^N - \bar{z}^N),
\]
and
\[
(s\tau)(z^N - \bar{z}^N) = \epsilon^N z^N - \epsilon^N \bar{z}^N = -(z^N - \bar{z}^N),
\]
the corresponding term in the Poincaré polynomial is \( t^{(2m+1)N} \).

Now let’s turn to two-dimensional irreducible representations. Representation \( V_i \) is realized twice in \( \mathbb{C}[z, \bar{z}] / I_0 \) as the space \( \langle z^i, \bar{z}^i \rangle \) and as \( \langle z^{2N-i}, \bar{z}^{2N-i} \rangle \), \( i = 1, \ldots, N - 1 \). Thus \( P_i(t) = 2t^i + 2t^{2N-i} \) in (4). For all the reflections \( s \in C_a, \tau \in C_b \) one has \( \dim V_i^- = 1 \), hence \( d_a^- (V_i) = d_b^- (V_i) = \frac{2N-1}{2} = N \) and we get the term in the Poincaré polynomial equal to \( 2t^{(m+n)N} (t^i + t^{2N-i}) \).

Summing up

\[
P_{H_m} = 1 + t^{(m+n+1)2N} + t^{(2m+1)N} + t^{(2n+1)N} + 2 \sum_{i=1}^{N-1} t^{(m+n)N} (t^i + t^{2N-i}). \tag{5}
\]

### 3 Generators of quasi-invariants

We recall the following key theorem proven by Etingof and Ginzburg [10] (see also [9]).

**Theorem 1** [10] For any Coxeter system \( \mathcal{R} \) the ring of quasi-invariants \( Q^\mathcal{R} \) is free as a module over invariants \( S^G \) of the corresponding Coxeter group \( G \). As the generators one can take any homogeneous representatives of the factor space \( Q^\mathcal{R} / I \), where \( I \) is the ideal in \( Q^\mathcal{R} \) generated by the homogeneous invariant polynomials of positive degrees. The degrees of the homogeneous generators coincide with the degrees of a homogeneous basis in the space \( H_m \) of \( m \)-harmonic polynomials.

We are going to determine what are the polynomials that can be taken as the generators for dihedral groups. As the case of constant multiplicity function is already considered in [9] we shall suppose that our group is even dihedral group \( I_2(2N) \) with different multiplicities, for instance \( m > n \). By Theorem[10] the degrees of generators for \( Q^{I_2(2N)} \) over invariants are same as the degrees of basic \( m \)-harmonic polynomials. According to (5) they satisfy

\[(2n+1)N < (m+n)N + i < (2m+1)N < (m+n+1)2N,\]

where \( 1 \leq i \leq 2N - 1, i \neq N \). And we get the following table for number of generators of corresponding degree:

| deg    | 0   | (2n + 1)N | (m + n)N + i | (2m + 1)N | (m + n + 1)2N |
|--------|-----|------------|--------------|------------|----------------|
| number of generators | 1   | 1          | 2            | 1          | 1              |
Let us introduce the normal vectors \( \alpha_i, i = 0, \ldots, 2N - 1 \) to the lines of reflections. Namely, \( \alpha_i = (-\sin \frac{\pi i}{2N}, \cos \frac{\pi i}{2N}) \), the multiplicities \( m_i = m \) for even \( i \) and \( m_i = n \) for odd \( i \). Let’s define the following four quasi-invariants

\[
q^0 = 1, \quad q^1 = (z^N + \bar{z}^N)^{2n+1} = \mu_1 \prod_{i=1; \ i=2j+1}^{2N-1} (\alpha_i, x)^{2n+1},
\]

\[
q^2 = (z^N - \bar{z}^N)^{2m+1} = \mu_2 \prod_{i=0; \ i=2j}^{2N-2} (\alpha_i, x)^{2m+1},
\]

\[
q^3 = (z^N + \bar{z}^N)^{2n+1}(z^N - \bar{z}^N)^{2m+1} = \mu_3 \prod_{i=0}^{2N-1} (\alpha_i, x)^{2m_i+1},
\]

where \( \mu_1, \mu_2, \mu_3 \) are some constants. It’s obvious that \( q^0 \) and \( q^3 \) are quasi-invariants. Polynomial \( q^1 \) is also quasi-invariant. Indeed, conditions on the lines \( (\alpha_i, x) = 0 \) with odd \( i \) are obviously satisfied. If \( i \) is even then the conditions follow from the invariance \( s_\alpha q^1 = q^1 \). Analogously, \( q^2 \in Q^{I_2(2N)} \).

These quasi-invariants will be a part of the basis for \( Q^{I_2(2N)} \) over invariants that we are constructing.

**Proposition 1**  Quasi-invariants \( q^0, q^1, q^2, q^3 \) defined by (7) do not belong to the ideal \( I \) generated by homogeneous invariants of positive degree \( S^{I_2(2N)}_+ \) in the ring \( Q^{I_2(2N)} \).

**Proof**  According to Theorem 1 the degrees of free generators for \( Q^{I_2(2N)} \) over invariants are given by the table on page 8. Therefore there are no quasi-invariants of degree less than \( \deg q^1 \) which are not invariants. Since \( q^1 \) is not invariant we conclude that \( q^1 \notin I \). Let us show that \( q^2 \notin I \). Let \( r_i^{1,2} \) be independent elements in the complement to \( I \) of degrees \( (m + n)N + i \), \( 1 \leq i \leq 2N - 1, i \neq N \). Suppose that \( q_2 \in I \), that is

\[
q^2 = s_0 q^0 + s_1 q^1 + \sum_i s_i^1 r_i^1 + \sum_i s_i^2 r_i^2,
\]

where \( s_0, s_1, s_i^1, s_i^2 \) are invariants, i.e. they belong to \( \mathbb{C}[z\bar{z}, z^{2N} + \bar{z}^{2N}] \). Since \( \deg r_i^{1,2} \neq 0 \mod N \) polynomials \( s_i^1, s_i^2 \) are divisible by \( z\bar{z} \). Let us consider relation (7) modulo terms divisible by \( z\bar{z} \). We get

\[
z^{(2m+1)N} - z^{(2m+1)N} = \lambda_1 (z^{2N} + \bar{z}^{2N})^{a} (z^{(2n+1)N} + \bar{z}^{(2n+1)N}) + \lambda_2 (z^{2N} + \bar{z}^{2N})^b + O(z\bar{z}).
\]
Now we get a contradiction as it follows both \( \lambda_1 + \lambda_2 = 1 \), and \( \lambda_1 + \lambda_2 = -1 \). Therefore relation (7) is impossible and \( q^3 \notin I \).

Let us finally show that \( q^3 \notin I \). As above we obtain that the following relation must hold

\[
z^{2(m+n+1)N} - \bar{z}^{2(m+n+1)N} = \lambda_1 (z^{2N} + \bar{z}^{2N})^a (z^{(2m+1)N} - \bar{z}^{(2m+1)N}) + \\
\lambda_2 (z^{2N} + \bar{z}^{2N})^b (z^{(2n+1)N} + \bar{z}^{(2n+1)N}) + \lambda_3 (z^{2N} + \bar{z}^{2N})^c + O(z\bar{z}). \tag{8}
\]

We obtain \( \lambda_1 = \lambda_2 = 0 \) as for any \( a, b \) the degrees of monomials in \( z \) and in \( \bar{z} \) which are obtained from the first two terms of the right hand-side in (8) have the form \( N \cdot (\text{odd number}) \). And in the left-hand side the result of dividing degree by \( N \) is \( 2(m+n+1) \) which is even. Further we conclude that the right-hand side of (8) takes the form \( \lambda_3 (z^{2Nc} + \bar{z}^{2Nc}) + O(z\bar{z}) \), and the equality (8) is impossible. Hence \( q^3 \notin I \) and the proposition is proved.

The proved proposition shows that polynomials \( q^0, q^1, q^2, q^3 \) may be chosen as a part of free basis for \( Q_{I}^{2(2N)} \) over invariants. The rest \( 2N - 4 \) polynomials in a basis consist of pairs of polynomials of degrees \( (m+n)N + i \) with \( 1 \leq i \leq 2N - 1, i \neq N \). The next Proposition and Theorem show that these quasi-invariants may be chosen to have more or less simple form.

**Proposition 2** There exist quasi-invariants \( q^{1,2}_i \) of the form

\[
q^1_i = \sum_{s=0}^{m+n} a_s z^{(m+n-s)N+i} \bar{z}^{Ns}, \quad a_0 = 1, \\
q^2_i = \sum_{s=0}^{m+n} b_s \bar{z}^{(m+n-s)N+i} z^{Ns}, \quad b_s = \bar{a}_s
\]

for \( 1 \leq i \leq 2N - 1, i \neq N \).

**Proof** Let \( r^{1,2}_i \) be some basis in the homogeneous complement to \( I \) of degree \( (m+n)N + i \), \( 1 \leq i \leq 2N - 1, i \neq N \). Let us consider two-dimensional space \( \langle r^1_i, r^2_i \rangle \). As it does not intersect \( I \) we may think that

\[
r^1_i = z^{(m+n)N+i} + O(z\bar{z}), \\
r^2_i = \bar{z}^{(m+n)N+i} + O(z\bar{z}).
\]
Let us consider the quasi-invariance conditions for the first polynomial

\[ r_i^1 = \sum_{s=0}^{(m+n)N+i-1} a_s z^{(m+n)N+i-s} \tilde{z}^{s}, \quad a_0 = 1. \]

The lines \((\alpha_j, x) = 0, j = 0, \ldots, 2N - 1\) are given by the equations \(z = \epsilon^j \tilde{z}\) where \(\epsilon^{2N} = 1\), and \(\epsilon\) is a primitive root. The quasi-invariance condition \(\partial_{\alpha_j}^2 r_i^1 = 0\) at \((\alpha_j, x) = 0 \Leftrightarrow z = \epsilon^j \tilde{z}\) takes the form

\[ \sum_{s=0}^{(m+n)N+i-1} a_s ((m+n)N+i-2s)^{2t-1} \epsilon^{((m+n)N+i-s)j} = 0, \]

or, equivalently,

\[ \sum_{s=0}^{(m+n)N+i-1} a_s ((m+n)N+i-2s)^{2t-1} \left( \frac{1}{\epsilon^s} \right)^j = 0. \]

Let \(t\) satisfy \(1 \leq t \leq n < m\). Then quasi-invariance conditions are nontrivial for all \(2N\) lines \(j = 0, \ldots, 2N - 1\). The conditions may be rewritten in the form

\[ \sum_{p=0}^{2N-1} A_p \left( \frac{1}{\epsilon^p} \right)^j = 0, \]

where

\[ A_p = \sum_{s \equiv p(2N)} a_s ((m+n)N+i-2s)^{2t-1} \]

This system of equations is of Vandermonde type. As \(\epsilon^{p_1} \neq \epsilon^{p_2}\) if \(p_1 \neq p_2\) we conclude that

\[ A_p = 0, \quad p = 0, \ldots, 2N - 1. \]

Now let us consider the left quasi-invariance conditions corresponding to \(n < t \leq m\). These conditions correspond to the lines with even \(j\). We have

\[ \sum_{s=0}^{(m+n)N+i-1} a_s ((m+n)N+i-2s)^{2t-1} \left( \frac{1}{\epsilon^s} \right)^j = 0, \]

where \(j = 0, 2, \ldots, 2N - 2\). Equivalently,

\[ \sum_{p=0}^{N-1} B_p \left( \frac{1}{\epsilon^{2p}} \right)^j = 0, \]
where
\[ B_p = \sum_{s \equiv p(N)} a_s ((m + n)N + i - 2s)2t-1, \]
and \( j = 0, 1, \ldots, N - 1 \). This system of equations is again of Vandermonde type. As \( \epsilon^{2p_1} \neq \epsilon^{2p_2} \) if \( p_1 \neq p_2 \) and \( 0 \leq p_i \leq N - 1 \) we conclude that
\[ B_p = 0, \quad p = 0, \ldots, N - 1. \]

Thus quasi-invariance of the polynomial
\[ r^1_i = \sum_{s=0}^{(m+n)N+i-1} a_s z^{(m+n)N+i-s}z^s \]
is equivalent to the following conditions on its coefficients
\[
\sum_{s \equiv p(2N)} a_s ((m + n)N + i - 2s)2t-1 = 0, \quad p = 0, 1, \ldots, 2N - 1, \quad 1 \leq t \leq n,
\]
\[
\sum_{s \equiv p(N)} a_s ((m + n)N + i - 2s)2t-1 = 0, \quad p = 0, 1, \ldots, N - 1, \quad n < t \leq m.
\]
(10)

These equations are split to the systems of equations for the sets of coefficients \( \{a_s\} \) with indexes \( s \) having same residue after division by \( N \). As we know that there exists solution to this system (10) with \( a_0 = 1 \), we conclude that there exists solution where the only nonzero coefficients are
\[ a_0, a_N, a_{2N}, \ldots, a_{(m+n+\delta)N}, \]
where \( \delta = 0 \) if \( 1 \leq i \leq N - 1 \), and \( \delta = 1 \) if \( N + 1 \leq i \leq 2N - 1 \). The corresponding quasi-invariant has the form
\[ R^1_i = \sum_{s=0}^{m+n+\delta} a_{sN} z^{(m+n-s)N+i}z^s \]
(11)
with the coefficients satisfying the following system of equations
\[
\sum_{0 \leq s \leq m+n+\delta} a_{sN} ((m + n - 2s)N + i)2t-1 = 0, \quad 1 \leq t \leq n,
\]
\[
\sum_{0 \leq s \leq m+n+\delta \atop s = 2k+1} a_{sN}(m + n - 2s)N + i)^{2t-1} = 0, \quad 1 \leq t \leq n, \quad (12)
\]
\[
\sum_{s=0}^{m+n+\delta} a_{sN}(m + n - 2s)N + i)^{2t-1} = 0, \quad n < t \leq m.
\]

Since the quasi-invariance equations (10 (or 12)) are real we conclude that complex conjugate to a quasi-invariant would also be a quasi-invariant. Therefore there exist quasi-invariants
\[
R^2_i = \bar{R}^1_i = \sum_{s=0}^{m+n+\delta} b_{sN} z^{(m+n-s)N+i} z^{Ns}
\]
with \(b_{sN} = \bar{a}_{sN}\). For \(i \leq N - 1\) we have \(\delta = 0\) and we may define \(q^1_i = R^1_i\). Then \(q^1_i\) have a form required by the statement of the Proposition. For \(i \geq N + 1\) the last nonzero coefficient in \(R^1_i\) is \(a_{(m+n+1)N}\). Let us show that one can make \(a_{(m+n+1)N} = 0\). For that consider quasi-invariant
\[
q^1_i = R^1_i - a_{(m+n+1)N}(z \bar{z})^{i-N} R^2_{2N-i}.
\]
We have
\[
(z \bar{z})^{i-N} R^2_{2N-i} = (z \bar{z})^{i-N} \sum_{s=0}^{m+n} b_{sN} z^{(m+n-s)N+2N-i} \bar{z}^{Ns} =
\]
\[
\sum_{s=0}^{m+n} b_{sN} z^{(m+n-s+1)N} \bar{z}^{Ns+i-N} = \sum_{s=0}^{m+n} b_{(m+n-s)N} z^{(m-n)N+i-N} \bar{z}^{Ns+N}.
\]
Therefore
\[
q^1_i = z^{(m+n)N+i} + \sum_{s=1}^{m+n+1} z^{(m+n-s)N+i} \bar{z}^{Ns} (a_{sN} - a_{(m+n+1)N} b_{m+n-s+1})
\]
and the last term in the sum above is equal to zero since \(b_0 = 1\). Thus \(q^1_i\) for \(i \geq N + 1\) also have a form required in the Proposition. Taking complex conjugates we get the quasi-invariants of the form
\[
q^2_i = \sum_{s=0}^{m+n} b_s z^{(m+n-s)N+i} \bar{z}^{Ns}, \quad b_s = \bar{a}_s.
\]
The Proposition is proved.
Theorem 2 The quasi-invariants $q_1^{1,2}$ together with $q^0, q^1, q^2, q^3$ form a free basis for $Q^{I_2(2N)}$ over $S^{I_2(2N)}$.

By Theorem 1 a set of vectors forms a basis for $Q^{I_2(2N)}$ over $S^{I_2(2N)}$ if it forms a basis in the complement to the ideal $I$. By Proposition 2 the polynomials $q^0, q^1, q^2, q^3$ form a part of a basis for $Q^{I_2(2N)}$ over invariants. We are left to prove that constructed in Proposition 2 quasi-invariants do not belong to the ideal $I$ generated by homogeneous invariants of positive degree. More exactly one should prove that the linear space $\langle q_1^{1}, q_2^{1} \rangle \cap I = 0$. If the intersection is not trivial then we would have that

$$\lambda_1 q_1^{1} + \lambda_2 q_2^{1} \in I,$$

and

$$\lambda_1 z^{(m+n)N+i} + \lambda_2 z^{(m+n)N+i} + O(z\bar{z}) = p(z, \bar{z}) \in I,$$

where $O(z\bar{z})$ is a polynomial divisible by $z\bar{z}$, and $\lambda_1, \lambda_2$ are some constants. Now, in the right-hand side we should have $p(z, \bar{z})$ as a linear combination of basis vectors for the complement to $I$ multiplied by polynomial coefficients in $z\bar{z}$ and $z^{2N} + \bar{z}^{2N}$. The degrees of basis vectors are given in the table on page 8. We see that there are no basis vectors whose degree would be less than $(m+n)N+i$ but equal to $(m+n)N+i$ modulo $2N$. Therefore $p(z, \bar{z})$ must be divisible by $z\bar{z}$, hence

$$\lambda_1 z^{(m+n)N+i} + \lambda_2 z^{(m+n)N+i} \text{ is divisible by } z\bar{z}$$

which is possible only if $\lambda_1 = \lambda_2 = 0$. Thus the theorem is proven.

4 Explicit formulas for $m$-harmonic basis

Let us show that the basis for $Q^{I_2(2N)}$ just considered in the theorem is actually $m$-harmonic. Let us denote the quantum integrals corresponding to the invariants $z\bar{z}$ and $z^{2N} + \bar{z}^{2N}$ by $L_1$ and $L_2$ respectively. The corresponding operators have the forms (c.f. [9])

$$L_1 = 4\partial_z \partial_{\bar{z}} - 4m \sum_{k=1}^{2N-1} e^{i\varphi_k} \partial_z - e^{-i\varphi_k} \partial_{\bar{z}} - 4n \sum_{k=1}^{2N-1} e^{i\varphi_k} \partial_z - e^{-i\varphi_k} \partial_{\bar{z}} \quad (13)$$

where $\varphi_k = \frac{\pi k}{2N}$. As to operator $L_2$ it has the following form (up to a constant)

$$L_2 = \partial_z^{2N} + \partial_{\bar{z}}^{2N} + \text{lower order terms}.$$
Theorem 3  Polynomials \(q^s\) and \(q^i_s\) are \(m\)-harmonic, that is, the following identities hold

1. \(\mathcal{L}_1(q^s) = \mathcal{L}_2(q^s) = 0, \quad 0 \leq s \leq 3\).
2. \(\mathcal{L}_1(q^s_i) = \mathcal{L}_2(q^s_i) = 0, \quad 1 \leq i \leq 2N - 1, i \neq N, \quad s = 1, 2\).

Proof Let us first establish that polynomials \(q^s\) are \(m\)-harmonic. We have \(\mathcal{L}_1(1) = \mathcal{L}_2(1) = 0\) as \(\deg \mathcal{L}_i(1) < 0\) but on the other hand \(\mathcal{L}_i(Q_{I_2(2N)}) \subset Q_{I_2(2N)}\) (see [9] where this is explained in general case), that is the image is polynomial. Further let us prove that quasi-invariant \(q^3\) is \(m\)-harmonic. Notice that

\[q^3 = \mu_3 \prod_{i=0}^{2N-1} (\alpha_i, x)^{2m_i+1}\]

is characterized by the property that it is anti-invariant quasi-invariant for the group \(I_2(2N)\) of minimal possible degree. Since operators \(\mathcal{L}_1, \mathcal{L}_2\) are \(I_2(2N)\)-invariant and since their degrees are negative we get that

\[\mathcal{L}_1(q^3) = \mathcal{L}_2(q^3) = 0.\]

Quasi-invariants \(q^1, q^2\) are anti-invariant quasi-invariants of minimal possible degree with respect to subgroups \(I_2(N) \subset I_2(2N)\) corresponding to two different orbits of mirrors of the system \(I_2(2N)\). Since \(\mathcal{L}_1, \mathcal{L}_2\) are invariants with respect to these subgroups we get the rest equalities of the first statement of the theorem, namely

\[\mathcal{L}_1(q^{1,2}) = \mathcal{L}_2(q^{1,2}) = 0.\]

Let us now prove the second part. Consider two-dimensional space \(V = \langle q^1_i, q^2_i \rangle\) generated by polynomials \(q^1_i, q^2_i\). Let us introduce two-dimensional spaces \(U, W\) generated by the image of \(V\) under the action of operators \(\mathcal{L}_1, \mathcal{L}_2\), namely

\[U = \langle L_1 q^1_i, L_1 q^2_i \rangle,\]
\[W = \langle L_2 q^1_i, L_2 q^2_i \rangle.\]

As operators \(\mathcal{L}_1, \mathcal{L}_2\) are invariant with respect to \(I_2(2N)\) action, they are intertwining operators between representations \(V\) and \(U\), and between \(V\) and \(W\) correspondingly. According to Schur lemma either these operators are zero operators or one or two of them is scalar operator. The first case corresponds
to the conditions 2 in the statement of the theorem. In the second case corresponding representation $U$ (or $W$) is isomorphic to representation $V$.

Representation $V$ is two-dimensional irreducible representation. In the basis $q_i^1, q_i^2$ generating matrices $s$ and $s\tau$ (see (11)) take the form

$$s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad s\tau = (-1)^{m+n} \begin{pmatrix} e^i & 0 \\ 0 & e^{-i} \end{pmatrix}. \tag{14}$$

For any $q \in W$ one has $\deg q = (m+n)N + i - 2N < (m+n)N$. Since $Q_{I_2(2N)}$ is freely generated by $q_i^1, q_i^2$ over $S_{I_2(2N)}$, the two-dimensional irreducible representations in $Q_{I_2(2N)}$ are met in the degrees more or equal to

$$\min_i (\deg q_i^{1,2}) = (m + n)N + 1.$$ 

Hence $W = 0$. Let us show that $U = 0$ as well. Notice that as it follows from (14) among representations $V_i = \langle q_i^1, q_i^2 \rangle$ each irreducible two-dimensional representation of $I_2(2N)$ is met twice. Namely, $V_i \cong V_{2N-i}$, and under this isomorphism

$$q_i^1 \rightarrow \lambda q_{2N-i}^2,$$

$$q_i^2 \rightarrow \lambda q_{2N-i}^1.$$ 

If $1 \leq i \leq N - 1$ then representation $V_i$ does not occur in the space $Q_{I_2(2N)}$ in the degrees less than $i$. Therefore

$$\mathcal{L}_1(q_i^1) = \mathcal{L}_1(q_i^2) = 0.$$ 

We are left to consider the case $N + 1 \leq i \leq 2N - 1$. Generally speaking we have

$$\mathcal{L}_1(q_i^1) = p_1 q_{2N-i}^2,$$

$$\mathcal{L}_1(q_i^2) = p_2 q_{2N-i}^1,$$

where $p_1, p_2 \in S_{I_2(2N)}$. Let us show that it must be $p_1 = p_2 = 0$. Recall that

$$q_i^1 = \sum_{s=0}^{m+n} a_s z^{(m+n-s)N+i} z^{Ns},$$

$$q_{2N-i}^2 = \sum_{s=0}^{m+n} b_s z^{(m+n-s)N+2N-i} z^{Ns}.$$
Notice that the degree of $q_1^i$ with respect to variable $\bar{z}$ is equal to $(m + n)N$ at most. Because of the form (13) of operator $L_1$ we get

$$\deg_{\bar{z}} L_1(q_1^i) \leq (m + n)N.$$ 

On the other hand if $p_2 \neq 0$ we obtain

$$\deg_{\bar{z}} (p_2 q_2^{2N-i}) \geq \deg_{\bar{z}} q_2^{2N-i} = (m + n)N + 2N - i > (m + n)N.$$ 

The contradiction shows that $p_2 = 0$. Similarly $p_1 = 0$, and therefore polynomials $q_1^i, q_2^i$ are $m$-harmonic. The theorem is proven.

**Corollary 1** Quasi-invariants $q_1^i, q_2^i$ of the form (9) are uniquely defined.

Indeed, the linear space of $m$-harmonics is defined uniquely (see [9]). The space $H_i$ of homogeneous $m$-harmonics of degree $(m + n)N + i$ is two-dimensional. Therefore quasi-invariants $q_1^i, q_2^i$ are uniquely characterized as homogeneous $m$-harmonics of the form

$$z^{(m+n)N+i} + O(z\bar{z}), \quad \bar{z}^{(m+n)N+i} + O(z\bar{z})$$

correspondingly, where $O(z\bar{z})$ denotes polynomials divisible by $z\bar{z}$.

Finally let us present explicit determinant formulas for quasi-invariants $q_1^i, q_2^i$. Let us introduce (see p. 18) the $(m + n + 1) \times (m + n + 1)$ matrix $A = A(i)$. 

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$A =$

$$
\begin{bmatrix}
z^{(m+n)N+i} & \Sigma z^{(m+n-1)N+i} & \Sigma^2 z^{(m+n-2)N+i} & \cdots & \Sigma^{(m+n)N+i} \\
(m+n)N+i & 0 & (m+n-4)N+i & 0 & (m+n-8)N+i & 0 \\
((m+n)N+i)^3 & 0 & ((m+n-4)N+i)^3 & 0 & ((m+n-8)N+i)^3 & 0 \\
\cdots \\
((m+n)N+i)^{2n-1} & 0 & ((m+n-4)N+i)^{2n-1} & 0 & ((m+n-8)N+i)^{2n-1} & 0 \\
0 & (m+n-2)N+i & 0 & (m+n-6)N+i & 0 \\
0 & ((m+n-2)N+i)^3 & 0 & ((m+n-6)N+i)^3 & 0 \\
0 & ((m+n-2)N+i)^{2n-1} & 0 & ((m+n-6)N+i)^{2n-1} & 0 \\
((m+n)N+i)^{2n+1} & ((m+n-2)N+i)^{2n+1} & ((m+n-4)N+i)^{2n+1} & ((m+n-6)N+i)^{2n+1} & ((m+n-8)N+i)^{2n+1} \\
((m+n)N+i)^{2m-1} & ((m+n-2)N+i)^{2m-1} & ((m+n-4)N+i)^{2m-1} & ((m+n-6)N+i)^{2m-1} & ((m+n-8)N+i)^{2m-1}
\end{bmatrix}
$$
Theorem 4 For generators $q_i^1, q_i^2 (1 \leq i \leq 2N - 1, i \neq N)$ the following formulas hold

$$q_i^1 = c \det A, \quad q_i^2 = c \det \bar{A},$$

where $c = (\det A_1)^{-1}$, and the matrix $A_1$ is obtained from $A$ by deleting the first column and the first row.

Proof Firstly let us show that $\det A_1 \neq 0$. During the proof of Theorem 2 we showed (see formulas (11), (12)) that quasi-invariance equations on the coefficients of polynomial

$$q_i^1 = \sum_{s=0}^{m+n} a_s z^{(m+n-s)N+i} z^N, \quad a_0 = 1$$

have the form

$$\sum_{0 \leq s \leq m+n \atop s=2k} a_s ((m+n-2s)N+i)^{2t-1} = 0, \quad 1 \leq t \leq n,$$

$$\sum_{0 \leq s \leq m+n \atop s=2k+1} a_s ((m+n-2s)N+i)^{2t-1} = 0, \quad 1 \leq t \leq n,$$

$$\sum_{s=0}^{m+n} a_s ((m+n-2s)N+i)^{2t-1} = 0, \quad n < t \leq m.$$

We know by theorem 2 that this system has a solution. Condition $\det A_1 \neq 0$ is equivalent to the statement that system (16) has a unique solution since the matrix of this linear system is exactly $A_1$. But the solution is unique according to Corollary 1.

To prove the theorem we need to show that

$$(-1)^{k+1} \frac{\det A_k}{\det A_1} = a_{k-1},$$

$k = 1, \ldots, m+n+1$, where the matrix $A_k$ is obtained from $A$ by deleting the $k$th column and the first row. According to Kramer's rule the solution $a_1, \ldots, a_{m+n}$ to linear system (16) is given by the formulas

$$a_{k-1} = \frac{\det \bar{A}_k}{\det A_1},$$

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where \( \hat{A}_k \) is matrix \( A \) where the first column and the first row are omitted, and instead of \( k \)th column it is written the first column with a minus sign. Changing the sign for this column and interchanging it to the first place we get

\[
\det \hat{A}_k = (-1)(-1)^{k-2} \det A_k,
\]

thus (17) is satisfied. Theorem is proven.

**Remark.** We have deduced formulas for \( q_{i,2} \) under assumption that \( n < m \). Nevertheless Theorem 4 remains true and gives formulas for the basis of quasi-invariant module consisting of \( m \)-harmonic polynomials also in the case \( m = n \). One can see that in this case formulas (15) reduce to the formulas for \( m \)-harmonic basis obtained in [5].

## 5 Concluding remarks

### 5.1 Generators for general Coxeter systems

In this paper it is completed the description of quasi-invariant modules for two-dimensional Coxeter systems. The description of generators of \( Q^R \) over \( S^G \) in the dimensions greater than two is an open question. In the general case the choice of \( m \)-harmonic polynomials is impossible as it is shown by Etingof and Ginzburg [10]. But the counterexample found by the authors looks exceptional as it consists of the Coxeter system \( C_6 \) with multiplicities on the long roots equal 0. The ring of corresponding quasi-invariants may be described as a module over invariants of the Coxeter group \( A^6_6 \) with the generators which are \( m \)-harmonic polynomials related to the group \( A^6_1 \). The question when \( m \)-harmonic polynomials can be chosen as the generators of \( Q^R \) over \( S^G \) is an open question when the dimension is greater than two.

### 5.2 Deformed systems

In the papers [14] [15] the quantum systems of Calogero–Moser type related to non-Coxeter configurations \( \mathcal{A} = \mathcal{A}_n(m), \mathcal{C}_{n+1}(m, l) \) were investigated. The corresponding second order operator has the form

\[
\mathcal{L}^\mathcal{A} = \Delta - \sum_{\alpha \in \mathcal{A}} \frac{2m_{\alpha}}{(\alpha, x)} \partial_\alpha.
\]
As in the Coxeter case the commutative ring of quantum integrals for $L^A$ is isomorphic to the ring $Q^A$ of quasi-invariants related to the system $A$. It is shown in [14] that the rings $Q^A$ are Cohen–Macaulay, that is they are freely generated over polynomial subrings $S^A$, which are analogs of the invariant rings in the case of Coxeter systems. The question of description of the rings $Q^A$, and in particular the question on the generators of $Q^A$ as a module over $S^A$ is an open problem. The natural generalization of $m$-harmonic polynomials to the non-Coxeter case does not lead to a basis already in the simplest case of the system $A_2(2)$ (see [14]).

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