Line-Bundle-Valued Ternary Quadratic Bundles
Over Schemes

Dedicated to the Memory of Professor Martin Kneser

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July, 2005

Abstract

We study degenerations of rank 3 quadratic forms using those of rank 4
Azumaya algebras, and extend what is known for good forms and Azumaya
algebras. By considering line-bundle-valued forms, we extend the theorem
of Max-Albert Knus that the Witt-invariant—the even-Clifford algebra of a
form—suffices for classification. The general, usual and special orthogonal
groups of a form are determined in terms of automorphism groups of its Witt-
invariant. Martin Kneser’s characteristic-free notion of semiregular form is
used. Examples of non-existence of good forms and Azumaya structures are
given.

Keywords: Azumaya algebra, Clifford algebra, orthogonal group, quaternion
algebra, semiregular form, Witt-invariant.

MSC: 11E, 14A25, 14F05, 14L15, 14M, 14Q, 15A63, 15A66, 15A75, 15A78,
16H05, 16S60, 16W20, 20G05, 20G35

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To explain the results of this work, in this introduction we shall restrict ourselves to the case of affine schemes for the sake of simplicity, leaving the general formulation to §3. So let \( R \) be a commutative ring with 1. The objects of our study are ternary quadratic forms \( q : V \to R \) on projective \( R \)-modules \( V \) of rank 3. Zariski-locally these are quadratic forms in three variables. We shall denote such forms by pairs \((V, q)\). A similarity from \((V, q)\) to \((V', q')\) is a pair \((g, m)\), where \( g : V \cong V' \) and \( m : R \cong R \) are linear isomorphisms such that \( q' \circ g = m \circ q \). When \( m \) is the identity, we call the similarity an isometry.

Our broad aim is to study the sets of similarities between ternary forms (when they exist) in terms of their even-Clifford algebras and indicate applications. We are especially interested in bad quadratic forms, and would like to develop a theory
for such forms. We obtain results extending (though not deduced from, but rather motivated by) what is known for the good ones.

What are good forms? These are the same as the usual regular (or nonsingular) forms, provided none of the residue fields of \( R \) is of characteristic two. The correct technical notion is that of \textit{semiregular} form, due to Martin Kneser, on which we will elaborate in \S 2.3, page 10. At this point we only remark that this notion works universally, so includes the case of regular forms as well.

For a semiregular form \( q : V \rightarrow R \), its even-Clifford algebra \( A = C_0(V,q) \) is an Azumaya \( R \)-algebra with underlying module of rank 4. Azumaya means that the natural \( R \)-algebra homomorphism \( A \otimes_R A^{op} \rightarrow \text{End}_{R-\text{Mod}}(A) : a \otimes b^{op} \mapsto (x \mapsto axb^{op}) \)

is an isomorphism, where the superscript refers to the opposite algebra. Matrix algebras, for example algebras of \((2 \times 2)\) matrices in our case, are simple examples of Azumaya algebras, and they are all one gets if say \( R \) were an algebraically closed field. Any similarity \((g,m) : (V,q) \cong (V',q')\) induces an isomorphism of \( R \)-algebras \( C_0(g,m) : C_0(V,q) \cong C_0(V',q') \). So if we denote the set of isometry classes of semiregular ternary quadratic forms by \( Q_{sr}^3(R) \), and the set of \( R \)-algebra isomorphism classes of rank 4 Azumaya \( R \)-algebras by \( AZU_4(R) \), then we get a natural map, which is in fact functorial in \( R \):

\[
\begin{aligned}
Q_{sr}^3(R) &\rightarrow AZU_4(R) \text{ via } (V,q) \mapsto C_0(V,q).
\end{aligned}
\]

It is well-known that the above map is surjective but may not be injective. In fact, given an Azumaya algebra \( A \) of rank 4, there is a well-defined unique standard involution \( \sigma_A \) on \( A \), which defines a regular quadratic form: the norm \( n_A : A \rightarrow R \) and a linear form: the trace \( tr_A : A \rightarrow R \). Then the restriction of \( n_A \) to the kernel \( A' \) of \( tr_A \), which is a rank 3 \( R \)-module (due to the surjectivity of the trace), remains good i.e., is a semiregular quadratic form, and one has an isomorphism of \( R \)-algebras

\[
C_0(A',n_A|A') \cong A.
\]

Thus the map (1) is surjective. Let us explain why it is not injective. Let \( I \) be a rank 1 projective \( R \)-module (i.e., an invertible module) equipped with an \( R \)-linear isomorphism

\[
h : I^2 := I \otimes I \cong R.
\]

The pair \((I,h)\) is called a \textit{discriminant module} and can be thought of as a good symmetric bilinear form on \( I \) as well as a prescribed square root of \( R \). Now given a ternary quadratic module \((V,q)\), we may define a new quadratic module:

\[
(V,q) \otimes (I,h) := (V \otimes I, q \otimes h) : (q \otimes h)(v \otimes l) := q(v)h(l \otimes l).
\]

Then it can be verified that we have an isomorphism of \( R \)-algebras

\[
C_0(V,q) \cong C_0(V \otimes I, q \otimes h).
\]

Thus the map in (1) may not be injective. Now the set of isomorphism classes of \((I,h)\) forms a commutative group of exponent 2, the group operation induced by
the usual tensor product. It is denoted by $\text{Disc}(R)$ and acts on the left side of $(\dagger)$. It is a Theorem of Max-Albert Knus that we have a bijection

\begin{equation}
Q^3_3(R)/\text{Disc}(R) \cong A\zeta_4(R),
\end{equation}

which may also be thought of as a statement in cohomology (cf. discussion following Theorem 2.10, page 20). A natural question is to ask if there is a statement analogous to $(\dagger)$ if we consider non-semiregular forms on the left side as well.

Our central result, Theorem 3.1 (page 24), shows that the answer is yes, thus providing a limiting version of the cohomological statement $(\dagger)$. However, when considering non-semiregular forms, we would need to make some changes to our present formulation; these we set out to explain next.

Firstly, since we wanted to include degenerate forms in the left side of $(\dagger)$, and since such forms are limits, locally in the Zariski-topology, of semiregular forms, it is natural to consider similar limits on the right side. In other words, we consider rank 4 $R$-algebras which are schematic limits of rank 4 Azumaya algebras. They were introduced in [17], and are recalled in §2.9 (page 21). Instead of going now into the definition of such limiting algebras, we note that it follows from op. cit., that they are precisely those rank 4 $R$-algebras which are Zariski-locally isomorphic to even-Clifford algebras of ternary quadratic forms. Let us denote the set of isomorphism classes of such algebras by $\mathcal{S}\zeta_4(R)$. This will be the replacement for the right side of $(\dagger)$.

Given an algebra $A$ representing an element of $\mathcal{S}\zeta_4(R)$, all we know by definition is that there are elements $f_i$ that generate $R$, and ternary quadratic $R_{f_i}$-forms $(V_i, q_i)$ such that there are isomorphisms $A \otimes R_{f_i} \cong C_0(V_i, q_i)$. It is not clear a priori that there is a choice for which the $V_i$ would glue to give a rank 3 $R$-module $V$, and even if this were so, that the $q_i$ would glue to give a quadratic form on $V$. In fact, it may not. We show that there is a choice for which the $V_i$ glue to give a $V$, but that the $q_i$ glue to give a quadratic form $q$ with values in the invertible module $I := \det^{-1}(A)$, such that

\begin{equation}
C_0(V, q, I) \cong A,
\end{equation}

(cf. Theorem 3.6, part (a), page 25), and further that that we could get a quadratic form with values in $R$ iff $\det(A) \in 2\text{Pic}(R)$ (cf. Theorem 3.6, part (b)). At this point we note that the even Clifford algebra $C_0(V, q, I)$ is the one defined by Bichsel-Knus in [1], briefly recalled in §2.4, page 11, which reduces to the familiar even-Clifford algebra when the invertible module is $R$ itself. Thus, going back to our idea of replacing the left side of $(\dagger)$ with degenerate forms, we see that we should work with ternary quadratic forms with values in invertible $R$-modules. Let us denote the set of isometry classes of such quadratic modules by $Q_3(R)$.

Further, since we had to divide out by the action of discriminant modules to get the bijection $(\dagger)$, and since such discriminant modules are square roots of $R$, it is natural that we should consider square roots of invertible $R$-modules as well. Such objects could be called twisted discriminant modules, and they do form a group under the tensor product which we denote by $T\text{-Disc}(R)$. Any such object is given by a triple $(L, h, J)$ where $h : L \otimes L \cong J$ is an $R$-linear isomorphism with
L, J being invertible R-modules (h makes L a specified square-root of J). Given a quadratic form \( q : V \rightarrow I \) with values in the invertible module I, one defines in the obvious way:

\[
(V, q, I) \otimes (L, h, J) := (V \otimes L, q \otimes h, I \otimes J) : v \otimes l \mapsto q(v) \otimes h(l \otimes I),
\]

and there is a canonical isomorphism of R-algebras

\[
C_0(V \otimes I, q \otimes h, L \otimes J) \cong C_0(V, q, I).
\]

It happens that this generalisation to values in invertible modules does not disturb the bijection (††) i.e., if we considered semiregular quadratic modules with values in invertible modules on the left side modulo the action of T-Disc(R), we still obtain a bijection to the right side. Theorem \[3.1\] (page 24) shows that we do have a bijection

\[
(†) \quad \Omega_3(R)/T-Disc(R) \cong \text{SPA}_{224}(R)
\]

which is a limiting version of (††). We next indicate the ingredients that go into the proof of (†), which are of interest by themselves, since they involve the study of the sets of similarities between ternary quadratic modules.

Given ternary quadratic forms \( q, q' : V \rightarrow I \), we may consider the set \( \text{Sim}(q, q') \) of similarities from \( q \) to \( q' \); these are pairs \((g, l)\) where \( g \in \text{GL}(V) \) and \( l \in R^* \) that satisfy \( q'g = \lambda_lq \) where \( \lambda_l \) is the element of \( \text{Aut}(I) \) given by multiplication by \( l \). The second member \( l \) of \((g, l)\) is called the multiplier of the similarity. The set \( \text{Sim}(q, q') \) is a subset of \( \text{GL}(V) \times R^* \). The subset \( \text{Iso}(q, q') \) of orthogonal transformations or isometries are those similarities which have trivial multipliers. A smaller subset, \( \text{S-Iso}(q, q') \), consists of special orthogonal transformations, and corresponds to those for which \( \det(g) = 1 \). The latter two subsets may be considered as subsets of \( \text{GL}(V) \) and \( \text{SL}(V) \) respectively.

Since the even-Clifford algebra is functorial in \( q \), any similarity \( q \rightarrow q' \) gives rise to an isomorphism of \( R \)-algebras \( C_0(V, q, I) \cong C_0(V, q', I) \). Theorem \[3.4\] page 26 identifies the images of the subsets of similarities (defined in the previous paragraph) in the set of isomorphisms \( \text{Iso}(C_0(q), C_0(q')) \) from \( C_0(q) \) to \( C_0(q') \). It shows firstly, that each isomorphism of even-Clifford algebras lifts to a similarity; further, given such an isomorphism \( \phi \), Prop. \[3.2\] page 27 shows how to define an \( R \)-linear automorphism \( \phi_{\Lambda^2} \) of \( \Lambda^2(V) \otimes I^{-1} \). We may thus consider those isomorphisms \( \phi \) for which \( \det(\phi_{\Lambda^2}) \in R^* \) is a square, (respectively, is 1), and denote it by \( \text{Iso}'(C_0(q), C_0(q')) \) (respectively, by \( \text{S-Iso}'(C_0(q), C_0(q')) \)). Then Theorem \[3.3\] shows that the image of \( \text{Iso}(q, q') \) is precisely \( \text{Iso}'(C_0(q), C_0(q')) \) and that of \( \text{S-Iso}(q, q') \) is precisely \( \text{S-Iso}(C_0(q), C_0(q')) \).

These surjectivities are accomplished by explicitly constructing families of sections indexed by the integers. The proof of the existence of \( \phi_{\Lambda^2} \) involves the use of Bourbaki's Tensor Operations for forms with values in invertible modules (cf. \[2.5\]). In order to construct the sections, we make computations analysing the correspondence from \[17\] between locally-even-Clifford algebras and specialisations of Azumaya algebras with the same underlying module. The further uses of Theorem \[3.4\] are principally threefold: it is used to prove the injectivity part.
of (4) (Theorem 3.1) and later on in the proof of the surjectivity part (Theorem 3.6 (a)). It is also used in Theorem 3.5 page 27 in the description of the general, usual and special orthogonal groups \( \text{GO}(q) := \text{Sim}(q, q), \text{O}(q) := \text{Iso}(q, q) \) and \( \text{SO}(q) := \text{S-Iso}(q, q) \) respectively, as fitting into a neat commutative diagram.

Further, \( \text{GO}(q) \) is shown to be a semi-direct product, and when \( R \) is an integral domain, and \( q \) is semiregular even at one point of \( \text{Spec}(R) \), then all the subgroups of automorphisms of \( C_0(q) \) described above are seen to coincide, so that \( \text{O}(q) \) is the semi-direct product of the multiplicative subgroup \( \mu_2(R) \) consisting of the square roots of 1 and of \( \text{SO}(q) \).

The injectivity of Theorem 3.1 is the statement that two ternary quadratic modules with isomorphic even-Crifford algebras differ by an isometry after one of them is tensored with a suitable twisted discriminant module. The existence of such a module is reduced to the lifting of a given isomorphism of even-Criffords to a similarity in the free case (cf. § 4).

A suitable member of the family of sections constructed to show the surjectivities in the proof of Theorem 3.4 allows one to get a cocycle defining a ternary quadratic module from a suitably chosen cocycle for a given specialised algebra (cf. § 6).

Since Bourbaki’s Tensor Operations form a key tool in our computations, we explain how they are of use to us. Given two ternary quadratic forms \( q, q' : V \rightarrow I \), one may ask how one would compare, if at all, the underlying modules of their even-Crifford algebras. Now a bilinear form \( b : V \times V \rightarrow I \) defines a quadratic form \( q_b : V \rightarrow I \) by \( q_b(v) := b(v, v) \). If there exists a \( b \) so that \( q_b = q - q' \), then the Tensor Operations provide us with an explicitly computable \( R \)-linear isomorphism

\[
\psi_b : C_0(V, q, I) \cong C_0(V, q', I).
\]

Since we can always find a \( b \) such that \( q_b = q \), by taking \( q' = 0 \) we see that we could, via \( \psi_b \), transport the algebra structure on \( C_0(V, q, I) \) to one on the underlying module \( W \) of

\[
C_0(V, 0, I) = R \oplus \Lambda^2(V) \otimes I^{-1}
\]

with multiplicative identity \( w = 1_R \). In this way, we may map the space of \( J \)-valued bilinear forms on \( V \) into the space of \( R \)-algebra structures on the same underlying module \( W \) with identity \( w \). And we could do this functorially in \( R \)-algebras \( S \). Theorem 3.8 page 30 shows that in this way we obtain bijectively for any \( S \), all the locally-even-Crifford \( S \)-algebra structures on \( W \otimes S \) with identity \( w \otimes S \). In other words, bilinear forms correspond to such algebras. This generalizes a similar result due to S. Ramanan for the case \( R = k \) an algebraically closed field (cf. § 2.3 of [16]). When \( \text{char.}(k) \neq 2 \), it was given another interpretation by C. S. Seshadri in [14] (cf. Theorem 3.10 page 32), which will also be our main key to computations.

Not only in the proof of Theorem 3.8 but throughout this work, we use the description from [17] of the locally-even-Crifford algebras as schematic specialisations of Azumaya algebras since it is this description that allows us to compute without losing track of global information.
For the rest of this introduction, we depart to the language of schemes. Suppose \( W \) is a vector bundle of rank 4 over a scheme \( X \). If there exists an Azumaya algebra structure on \( W \), then it can be seen that \( W \) has to be self-dual. One may ask for a converse: if \( W \) is self dual, does there exist a global Azumaya algebra structure on \( W \)? We answer this question more generally, and show in Theorem 3.17, page 36 that this is indeed the case if on the one hand let \( X \) to be a proper scheme of finite type over a base scheme with connected fibres, while on the other hand we assume that \( W \) has square rank \( n^2 \) with \( n \geq 2 \), and moreover that there exists some associative unital algebra structure \( A \) on \( W \) which is Azumaya at some point of each fiber over the base. We then show in fact that \( A \) has to be Azumaya, giving a “punctual to global” result.

By an application of Theorem 3.1 to this result, we deduce that the hypothesis of self-duality on the underlying bundle of the even-Clifford algebra of a ternary quadratic bundle implies that the quadratic bundle is semiregular everywhere if it is semiregular even at a single point of each fiber over the base scheme.

Such results can be used to give examples of rank 4 vector bundles \( W \) over which there do not exist any Azumaya algebra structures but which nevertheless admit algebra structures that are Azumaya on a nonempty open subscheme. They can also be used to give examples of pairs \((V, I)\), consisting of a rank 3 vector bundle \( V \) and a line bundle \( I \), such that there exists no semiregular quadratic form on \( V \) with values in \( I \), though there do exist forms that are semiregular on a nonempty open subscheme. For any given irreducible smooth projective curve \( C \) of genus \( g \geq 2 \) over an algebraically closed field, such examples naturally occur on certain smooth projective varieties \( N_C(4,0) \) of dimension \( 4g - 3 \), where \( g \) is the genus of \( C \). In fact these \( N_C(4,0) \) appear as fine moduli spaces of certain parabolic stable vector bundles of rank 4 and degree zero over \( C \) and arise as Seshadri desingularisations of the moduli space of semistable vector bundles of rank 2 and degree 0 over \( C \). The whole situation can be generalised to the case of a curve relative to a base scheme and the results are given in Theorem 3.18, page 37. A detailed account with proofs will appear in [18].

Suppose \( X \) is a scheme and \( W \) a rank 4 vector bundle on \( X \) with a nowhere-vanishing global section \( w \). In Theorem 3.19, page 37, we delve a little into the topological and geometric properties of the scheme of specialised algebra structures on \( W \) with identity \( w \) (functorially in \( X \)-schemes). If \( X \) is locally-factorial, we show that the natural homomorphism from the Picard group of \( X \) to that of the scheme of specialised algebra structures on \( W \) with unit \( w \) is an isomorphism. We also study the specialised algebras that are nowhere Azumaya. When \( X \) is locally-factorial and \( W \) is self-dual, these algebras define a Weil divisor \( D_X \) such that some positive integer multiple \( n.D_X \) is principal. Therefore the natural surjective homomorphism from the Picard group of the scheme of specialisations to that of the open subscheme of Azumaya algebra structures is an isomorphism iff \( n = 1 \) and in this situation, every specialised algebra structure on \( W \otimes T \) arises from a quadratic form with values in the trivial line bundle, for any \( X \)-scheme \( T \). For example, this happens if \( X \) is locally-factorial and if there exists an Azumaya algebra structure on \( W \). The proof uses the stratification of the variety of specialisations of \((2 \times 2)\)-matrix algebras over an algebraically closed field (Theorem 3.20).
Plan of the Paper. Background material is to be found in §2, which also fixes some notation, explains definitions and recalls results for future use. The formulation of the statements of the main results follow in §3. The proofs will occupy §4 through §8.

Note on Numbering. Various items such as theorems, propositions, definitions etc are all consecutively numbered within a section, irrespective of the subsection that they may appear in.

2 Notations and Preliminaries

This section collects together definitions and results necessary for future use. We omit proofs for statements involving quadratic and bilinear forms, since these can be reduced to the corresponding results for the affine case which are treated in Knus’ book [9]. For the systematic treatment of the generalised Clifford algebra and its properties, we refer the reader to the paper of Bichsel-Knus [1]. We also collect some preliminaries on the notion of schematic image and recall some background material from Part A of [17].

2.1 Quadratic and Bilinear Forms with Values in a Line Bundle

Let $V$ be a vector bundle (of constant positive rank) and $I$ a line bundle on a scheme $X$. Let $\mathcal{V}$ and $\mathcal{I}$ respectively denote the coherent locally-free sheaves corresponding to $V$ and $I$. We define the coherent locally-free sheaves of bilinear and alternating forms on $V$ with values in $I$ respectively as follows:

$$\text{Bil}_{(V,I)} := (T_{\mathcal{O}_X}(V))^\vee \otimes \mathcal{I} \quad \text{and} \quad \text{Alt}^2_{(V,I)} := (\Lambda^2_{\mathcal{O}_X}(V))^\vee \otimes \mathcal{I}.$$ 

We let $\text{Bil}_{(V,I)}$ and $\text{Alt}^2_{(V,I)}$ denote the corresponding vector bundles. Now define the (coherent, locally-free) sheaf of $I$-valued quadratic forms on $V$ by the exactness of the following sequence:

$$0 \rightarrow \text{Alt}^2_{(V,I)} \rightarrow \text{Bil}_{(V,I)} \rightarrow \text{Quad}_{(V,I)} \rightarrow 0.$$ 

Let the corresponding bundle of $I$-valued quadratic forms on $V$ be denoted by $\text{Quad}_{(V,I)}$. Thus a bilinear form (resp. alternating form, resp. quadratic form) with values in $I$ on $V$ over an open set $U \rightarrow X$ is by definition a section over $U$ of the vector bundle

$$\text{Bil}_{(V,I)} \quad \text{(resp. of} \text{Alt}^2_{(V,I)}, \text{resp. of Quad}_{(V,I)}) \quad \text{or equivalently, an element of}$$

$$\Gamma(U, \text{Bil}_{(V,I)}) \quad \text{(resp. of} \Gamma(U, \text{Alt}^2_{(V,I)}), \text{resp. of} \Gamma(U, \text{Quad}_{(V,I)})).$$
In terms of the corresponding (geometric) vector bundles over \(X\), the exact sequence above translates into the following sequence of morphisms of vector bundles, with the first one a closed immersion and the second one a Zariski locally-trivial principal \(\text{Alt}^2_{(V,I)}\)-bundle:

\[
\text{Alt}^2_{(V,I)} \hookrightarrow \text{Bil}_{(V,I)} \rightarrow \text{Quad}_{(V,I)}.
\]

Given a quadratic form \(q \in \Gamma(U, \text{Quad}_{(V,I)})\), recall that the usual ‘associated’ symmetric bilinear form \(b_q \in \Gamma(U, \text{Bil}_{(V,I)})\) is given on sections (over open subsets of \(U\)) by

\[
v \otimes v' \mapsto q(v + v') - q(v) - q(v').
\]

Given a (not-necessarily symmetric!) bilinear form \(b\), we also have the induced quadratic form \(q_b\) given on sections by \(v \mapsto b(v \otimes v)\). A global quadratic form may not be induced from a global bilinear form, unless we assume something more, for e.g., that the scheme is affine, or more generally that the sheaf cohomology group \(H^1(X, \text{Alt}^2_{(V,I)}) = 0\).

### 2.2 Sets of Similarities of Quadratic Bundles

Let \((V, q, I)\) and \((V', q', I')\) be quadratic bundles on the scheme \(X\). We denote by

\[
\text{Sim}[(V, q, I), (V', q', I')]
\]

the set of generalised similarities from \((V, q, I)\) to \((V', q', I')\). These consist of pairs \((g, m)\) such that \(g : V \cong V'\) and \(m : I \cong I'\) are linear isomorphisms and the following diagram commutes (where \(q\) and \(q'\) are considered as morphisms of sheaves of sets):

\[
\begin{array}{ccc}
V & \xrightarrow{g} & V' \\
\downarrow{q} & & \downarrow{q'} \\
I & \xrightarrow{m} & I'
\end{array}
\]

When \(I = I'\), since an \(m \in \text{Aut}(I)\) may be thought of as multiplication by a scalar \(l \in \Gamma(X, \mathcal{O}_X) = \text{Aut}(I)\), we may call the isomorphism \((g, m)\) as an \(I\)-similarity with multiplier \(l\). In such a case we may as well denote \((g, m)\) by the pair \((g, l)\) and we often write

\[
g : (V, q, I) \cong_i (V', q', I).
\]

Let

\[
\text{Iso}[(V, q, I), (V', q', I)]
\]

be the subset of isometries (i.e., those pairs \((g, m)\) with \(m = \text{Identity}\) or \(I\)-similarities with trivial multipliers). When \(V = V'\), the subset of isometries with trivial determinant is denoted

\[
\text{S-Iso}[(V, q, I), (V, q', I)].
\]
On taking \( q = q' \) these sets naturally become subgroups of
\[
\text{Aut}(V) \times \Gamma(X, \mathcal{O}_X^*) = \text{GL}(V) \times \Gamma(X, \mathcal{O}_X^*)
\]
and we define
\[
\text{Sim}[(V, q, I), (V, q, I)] =: \text{GO}(V, q, I) \supset \text{Iso}[(V, q, I), (V, q, I)] =: \text{O}(V, q, I) \supset \text{S-Iso}[(V, q, I), (V, q, I)] =: \text{SO}(V, q, I).
\]
Of course, \( \text{O}(V, q, I) \) and \( \text{SO}(V, q, I) \) may be thought of as subgroups of \( \text{GL}(V) \equiv \text{GL}(V) \times \{1\} \) and \( \text{SL}(V) \equiv \text{SL}(V) \times \{1\} \) respectively.

### 2.3 Semiregular Bilinear forms

Fundamental problems in dealing with quadratic forms over arbitrary base schemes arise essentially from two abnormalities in characteristic two: firstly, the mapping that associates a quadratic form to its symmetric bilinear form is not bijective and secondly, there do not exist regular quadratic forms on any odd-rank bundle. The remedy for this is to consider semiregular quadratic forms, a concept due to M.Kneser [8] and elaborated upon by Knus in [9], which in fact works over an arbitrary base scheme (and hence in a characteristic-free way) and further reduces to the usual notion of regular form in characteristics \( \neq 2 \).

Let \( \text{Spec}(R) = U \hookrightarrow X \) be an open affine subscheme of \( X \) such that \( V|U \) is trivial. Let \( \text{Quad}_V := \text{Quad}_{(V, O_X)} \). Consider a quadratic form \( q \in \Gamma(U, \text{Quad}_V) \) on \( V|U \) and its associated symmetric bilinear form \( b_q \). The matrix of this bilinear form relative to any fixed basis is a symmetric matrix of odd rank and in particular, if \( U \) is of pure characteristic two (i.e., the residue field of any point of \( U \) is of characteristic two), then this matrix is also alternating and is hence singular, immediately implying that \( q \) cannot be regular. However, computing the determinant of such a matrix in *formal variables* \( \{\zeta_i, \zeta_{ij}\} \) shows that it is twice the following polynomial
\[
P_3(\zeta_i, \zeta_{ij}) = 4\zeta_1\zeta_2\zeta_3 + \zeta_{12}\zeta_{13}\zeta_{23} - (\zeta_1\zeta_{23}^2 + \zeta_2\zeta_{13}^2 + \zeta_3\zeta_{12}^2).
\]
The value \( P_3(q(e_i), b_q(e_i, e_j)) \) corresponding to a basis \( \{e_1, e_2, e_3\} \) is called the *half-discriminant* of \( q \) relative to that basis, and \( q \) is said to be semiregular if its half-discriminant is a unit. It turns out that this definition is independent of the basis chosen (§3, Chap.IV, [9]).

Even if \( V|U \) were only projective (i.e., locally-free but not free), the semiregularity of \( q \) may be defined as the semiregularity of \( q \otimes_R R_m \) for each maximal ideal \( m \subset R \), and it turns out that with this definition, the notion of a quadratic form being semiregular is local and is well-behaved under base-change (Prop.3.1.5, Chap.IV, [9]).

We may thus define the subfunctor of \( \text{Quad}_V := \text{Quad}_{(V, O_X)} \) of semiregular quadratic forms. This subfunctor is represented by a \( \text{GL}_V \)-invariant open subscheme
\[
i : \text{Quad}_V^{\text{sr}} \hookrightarrow \text{Quad}_V.
\]
because, over each affine open subscheme \( U \rightarrow X \) which trivialises \( V \), it corresponds to localisation by the non-zerodivisor \( P_3 \). Note that this canonical open immersion is affine and schematically dominant as well. We next turn to semiregular bilinear forms. Recall that we had defined a bilinear form \( b \) to be semiregular iff its induced quadratic form \( q_b \) is semiregular. Thus by definition, \( \text{Bil}^\text{sr}_V \) is the fiber product:

\[
\begin{array}{c}
\text{Bil}_V \xrightarrow{p} \quad \text{Quad}_V \\
\uparrow i' \quad \uparrow i \\
\text{Bil}^\text{sr}_V \xrightarrow{p'} \quad \text{Quad}^\text{sr}_V
\end{array}
\]

Since \( p \) is a Zariski-locally-trivial principal \( \text{Alt}_V^2 \)-bundle, it is smooth and surjective (in particular faithfully flat). It therefore follows that the affineness and schematic dominance of \( i \) imply those of \( i' \). We record these facts below.

**Proposition 2.1** The open immersion

\[
\text{Bil}^\text{sr}_V \hookrightarrow \text{Bil}_V
\]

is a \( \text{GL}_V \)-equivariant schematically dominant affine morphism. Further this open immersion behaves well under base-change (relative to \( X \)).

### 2.4 The Generalised Clifford Algebra of Bichsel-Knus

Let \( R \) be a commutative ring (with 1), \( I \) an invertible \( R \)-module and \( V \) a projective \( R \)-module. Consider the Laurent-Rees algebra of \( I \) defined by

\[
L[I] := R \oplus \left( \bigoplus_{n > 0} (T^n(I) \oplus T^n(I^{-1})) \right)
\]

and define the \( \mathbb{Z} \)-gradation on the tensor product of algebras \( TV \otimes L[I] \) by requiring elements of \( V \) (resp. of \( I \)) to be of degree one (resp. of degree two). Let \( q : V \rightarrow I \) be an \( I \)-valued quadratic form on \( V \). Following the definition of Bichsel-Knus \([1]\), let \( J(q, I) \) be the two-sided ideal of \( TV \otimes L[I] \) generated by the set

\[
\{(x \otimes TV x) \otimes 1_{L[I]} - 1_{TV} \otimes q(x) \mid x \in V\}
\]

and let the generalised Clifford algebra of \( q \) be defined by

\[
\tilde{C}(V, q, I) := TV \otimes L[I]/J(q, I).
\]

This is an \( \mathbb{Z} \)-graded algebra by definition. Let \( C_n \) be the submodule of elements of degree \( n \). Then \( C_0 \) is a subalgebra, playing the role of the even Clifford algebra in the classical situation (i.e., \( I = R \)) and \( C_1 \) is a \( C_0 \)-bimodule. Bichsel and Knus baptize \( C_0 \) and \( C_1 \) respectively as the even Clifford algebra and the Clifford module associated to the triple \((V, q, I)\). The generalised Clifford algebra satisfies
an appropriate universal property which ensures it behaves well functorially. Since $V$ is projective, the canonical maps

$$V \rightarrow \tilde{C}(V, q, I) \text{ and } L[I] \rightarrow \tilde{C}(V, q, I)$$

are injective. For proofs of these facts, see Sec.3 of [1]. If $(V, q, I)$ is an $I$-valued quadratic form on the vector bundle $V$ over a scheme $X$, with $I$ a line bundle, then the above construction may be carried out to define the generalised Clifford algebra bundle $\tilde{C}(V, q, I)$ which is an $\mathbb{Z}$-graded algebra bundle on $X$. Its degree zero subalgebra bundle is denoted $C_0(V, q, I)$ and is called the even Clifford algebra bundle of $(V, q, I)$.

### 2.5 Bourbaki’s Tensor Operations with Values in a Line Bundle

Let $R$ be a commutative ring and $L[I] := R \oplus \left( \bigoplus_{n>0}(T^n(I) \oplus T^n(I^{-1})) \right)$ as above. We denote by $\otimes_T$ (resp. by $\otimes_L$) the tensor product and by $1_T$ (resp. $1_L$) the unit element in the algebra $TV$ (resp. in $L[I]$).

**Theorem 2.2** (with the above notations)

1. Let $q : V \rightarrow I$ be an $I$-valued quadratic form on $V$ and $f \in \text{Hom}_R(V, I)$. Then there exists an $R$-linear endomorphism $t_f$ of the algebra $TV \otimes L[I]$ which is unique with respect to the first three of the following properties it satisfies:

   a. for each $\lambda \in L[I]$ we have $t_f(1_T \otimes \lambda) = 0$;
   b. for any $x \in V$, $y \in TV$, and $\lambda \in L[I]$ we have
      $$t_f((x \otimes_T y) \otimes \lambda) = y \otimes (f(x) \otimes_L \lambda) - (x \otimes_T 1_T)t_f(y \otimes \lambda);$$
   c. if $J(q, I)$ is the two-sided ideal of $TV \otimes L[I]$ as defined in §2.4 above, then we have
      $$t_f(J(q, I)) \subset J(q, I);$$
   d. $t_f$ is homogeneous of degree $+1$ (except for elements which it does not annihilate);
   e. if $\tilde{C}(V, q, I)$ is the generalised Clifford algebra of $(V, q, I)$ as defined in §2.4 above, then due to assertion (c) above, $t_f$ induces a $\mathbb{Z}$-graded endomorphism of degree $+1$ denoted by
      $$d_f^+ : \tilde{C}(V, q, I) \rightarrow \tilde{C}(V, q, I);$$
   f. $t_f \circ t_f = 0$;
   g. if $g \in \text{Hom}_R(V, I)$, then $t_f \circ t_g + t_g \circ t_f = 0$;
   h. if $\alpha \in \text{End}_R(V)$ and $\alpha^* f \in \text{Hom}_R(V, I)$ is defined by $x \mapsto f(\alpha(x))$ then
      $$t_f \circ (T(\alpha) \otimes \text{Id}_{L[I]}) = (T(\alpha) \otimes \text{Id}_{L[I]}) \circ t_{\alpha^* f};$$
(i) $t_f \equiv 0$ on the $R$-subalgebra of $TV \otimes L[I]$ generated by

$$\text{kernel}(f) \otimes L[I].$$

In fact, at least when $V$ is projective, the smallest $R$-subalgebra of $TV \otimes L[I]$ containing

$$\text{kernel}(f) \otimes R.1_L$$

is given by

$$\text{Image}(T(\text{kernel}(f)) \otimes R.1_L$$

and $t_f$ vanishes on this $R$-subalgebra.

(2) Let $q, q' : V \to I$ be two $I$-valued quadratic forms whose difference is the quadratic form $q_b$ induced by an $I$-valued bilinear form

$$b \in \text{Bil}_R(V, I) := \text{Hom}_R(V \otimes_R V, I).$$

This means that

$$q'(x) - q(x) = q_b(x) := b(x, x) \forall x \in V.$$

Further, for any $x \in V$ denote by $b_x$ the element of $\text{Hom}_R(V, I)$ given by $y \mapsto b(x, y)$. Then there exists an $R$-linear automorphism $\Psi_b$ of $TV \otimes L[I]$ which is unique with respect to the first three of the following properties it satisfies:

(a) for any $\lambda \in L[I]$ we have $\Psi_b(1_T \otimes \lambda) = (1_T \otimes \lambda)$;
(b) for any $x \in V$, $y \in TV$ and $\lambda \in L[I]$ we have

$$\Psi_b((x \otimes_T y) \otimes \lambda) = (x \otimes 1_L) \Psi_b(y \otimes \lambda) + t_{b_x}(\Psi_b(y \otimes \lambda));$$
(c) $\Psi_b(J(q', I)) \subset J(q, I)$;
(d) by the previous property, $\Psi_b$ induces an isomorphism of $\mathbb{Z}$-graded $R$-modules

$$\psi_b : \tilde{C}(V, q', I) \cong \tilde{C}(V, q, I);$$

in particular, given a quadratic form $q_1 : V \to I$, since there always exists an $I$-valued bilinear form $b_1$ that induces $q_1$ (i.e., such that $q_1 = q_{b_1}$), setting $q' = q_1$, $q = 0$ and $b = b_1$ in the above gives an $\mathbb{Z}$-graded linear isomorphism

$$\psi_{b_1} : \tilde{C}(V, q_1, I) \cong \tilde{C}(V, 0, I) = \Lambda(V) \otimes L[I];$$

(e1) $\Psi_b(T^{2n}V \otimes L[I]) \subset \oplus_{i \leq n}(T^{2i}V \otimes L[I])$;
(e2) $\Psi_b(T^{2n+1}V \otimes L[I]) \subset \oplus_{\text{odd } i \leq 2n+1}(T^iV \otimes L[I])$;
(e3) $\Psi_b(T^{2n}V \otimes I^{-n}) \subset \oplus_{i \leq n}(T^{2i}V \otimes I^{-i})$;
(e4) $\Psi_b(T^{2n+1}V \otimes I^{-n}) \subset \oplus_{\text{odd } i \leq 2n+1}(T^iV \otimes I^{i+1})$. 


(f) in particular, for \(x, x' \in V\),
\[
\Psi_b((x \otimes_T x') \otimes 1_L) = (x \otimes_T x') \otimes 1_L + 1_T \otimes b(x, x')
\]
so that for
\[
\psi_b : C_0(V, q_b, I) \cong C_0(V, 0, I) = \oplus_{n \geq 0} (A^{2n}(V) \otimes I^{-n})
\]
we have
\[
\psi_b((x \otimes_T x') \otimes \zeta) \mod J(q_b, I) = (x \wedge x') \otimes \zeta + \zeta(b(x, x')).1
\]
for any \(x, x' \in V\) and \(\zeta \in I^{-1} \equiv \text{Hom}_R(I, R)\);

(g) if \(f \in \text{Hom}_R(V, I)\), and \(t_f\) is given by (1) above, then \(\Psi_b \circ t_f = t_f \circ \Psi_b\);

(h) for \(I\)-valued bilinear forms \(b\) on \(V\),
\[
\Psi_{b_1 + b_2} = \Psi_{b_1} \circ \Psi_{b_2} \quad \text{and} \quad \Psi_0 = \text{Identity on } TV \otimes L[I];
\]

(i) for any \(\alpha \in \text{End}_R(V)\), we have
\[
\Psi_b \circ (T(\alpha) \otimes \text{Id}_{L[I]}) = (T(\alpha) \otimes \text{Id}_{L[I]}) \circ \Psi_{(b, \alpha)}
\]
where \((b, \alpha)(x, x') := b(\alpha(x), \alpha(x')) \forall x, x' \in V\);

(j) by property (h), one has a homomorphism of groups
\[
(\text{Bil}_R(V, I), +) \longrightarrow (\text{Aut}_R(TV \otimes L[I]))_{\circ} : b \mapsto \Psi_b;
\]
the associative unital monoid \((\text{End}_R(V), \circ)\) acts on \(\text{Bil}_R(V, I)\) on the right by \(b' \sim b'.\alpha\) and acts on the left (resp. on the right) of \(\text{End}_R(TV \otimes L[I])\) by
\[
\alpha.\Phi := (T(\alpha) \otimes \text{Id}_{L[I]}) \circ \Phi \quad \text{(resp. by } \Phi.\alpha := \Phi \circ (T(\alpha) \otimes \text{Id}_{L[I]}))
\]
and the homomorphism \(b \mapsto \Psi_b\) satisfies
\[
\alpha.\Psi_{(b, \alpha)} = \Psi_{b, \alpha};
\]
the group \(\text{Aut}_R(V) = \text{GL}_R(V)\) acts on the left of \(\text{Bil}_R(V, I)\) by
\[
g.b : (x, x') \mapsto b(g^{-1}(x), g^{-1}(x'))
\]
and on the left of \(\text{Aut}_R(TV \otimes L[I])\) by conjugation via the natural group homomorphism
\[
\text{GL}_R(V) \longrightarrow \text{Aut}_R(TV \otimes L[I]) : g \mapsto T(g) \otimes \text{Id}_{L[I]}
\]
namely we have
\[
g.\Phi := (T(g) \otimes \text{Id}_{L[I]}) \circ \Phi \circ (T(g^{-1}) \otimes \text{Id}_{L[I]}),
\]
and the homomorphism \(b \mapsto \Psi_b\) is \(\text{GL}_R(V)\)-equivariant: \(\Psi_{g.b} = g.\Psi_b\).
(3) For a commutative $R$-algebra $S$ (with 1), let

$$(q \otimes_R S), (q' \otimes_R S) : (V \otimes_R S =: V_S) \rightarrow (I \otimes_R S =: I_S)$$

be the $I_S$-valued quadratic forms induced from the quadratic forms $q, q'$ of (2) above and

$$(b \otimes_R S) \in \text{Bil}_S(V_S, I_S)$$

the $I_S$-valued $S$-bilinear form induced from the bilinear form $b$ of (2) above. Then as a result of the uniqueness properties (2a)–(2c) satisfied by $\Psi_b$ and $\Psi_{(b \otimes_R S)}$, the $S$-linear automorphisms $(\Psi_b \otimes_R S)$ and $\Psi_{(b \otimes_R S)}$ may be canonically identified. In particular, the $\mathbb{Z}$-graded $S$-linear isomorphism

$$(\psi_b \otimes_R S) : \tilde{C}(V_S, (q' \otimes_R S), I_S) \cong \tilde{C}(V_S, (q \otimes_R S), I_S)$$

induced from $\psi_b$ of (2d) above may be canonically identified with $\psi_{(b \otimes_R S)}$.

**Remark 2.3** As mentioned in [1], F. van Oystaeyen has observed that $L[I]$ is a faithfully-flat splitting for $I$, and the generalised Clifford algebra $\tilde{C}(V, q, I)$ is nothing but the “classical” Clifford algebra of the triple

$$(V \otimes_R L[I], q \otimes_R L[I], I \otimes_R L[I] \equiv L[I])$$

over $L[I]$. In the same vein, $I$-valued forms (both multilinear and quadratic) on an $R$-module $V$ can be treated as the usual ($L[I]$-valued) forms on $V \otimes_R L[I]$. With this in mind, the proof of Theorem 2.2 follows from the usual Bourbaki tensor operations with respect to $V \otimes_R L[I]$ on $L[I]$. (See §9, Chap.9, [2] or para.1.7, Chap.IV, [3] for the “classical” Bourbaki operations). However one needs to remember that the $\mathbb{Z}$-gradation on $TV \otimes_R L[I]$ as defined above is different from the usual $\mathbb{Z}_{\geq 0}$-gradation on $T(V \otimes_R L[I])$.

### 2.6 Tensoring by Bilinear- and Twisted Discriminant Bundles

Let $V, M$ be vector bundles on a scheme $X$ and let $I, J$ be line bundles on $X$. Let $q$ be a quadratic form on $V$ with values in $I$ and let $b$ be a symmetric bilinear form on $M$ with values in $J$. By abuse of notation, we also use $b$ to denote the corresponding $J$-valued linear form on $M \otimes M$.

**Proposition 2.4** (with the above notations)

(1) The tensor product of $(V, q, I)$ with $(M, b, J)$ gives a unique quadratic bundle

$$(V \otimes M, q \otimes b, I \otimes J).$$

The quadratic form $q \otimes b$ on $V \otimes M$ is given on sections by

$$v \otimes m \mapsto q(v) \otimes b(m \otimes m)$$

and has associated bilinear form $b_{q \otimes b} = b_q \otimes b$. 
(2) When $M$ is a line bundle, $(M, b, J)$ is regular (=nonsingular) iff $(M, q_b, J)$ is semiregular iff

$$b : M \otimes M \cong J$$

is an isomorphism.

(3) Let $V$ be of odd rank and $M$ a line bundle such that $(M, b, J)$ is regular. Then $(V, q, I)$ is semiregular iff

$$(V, q, I) \otimes (M, b, J) = (V \otimes M, q \otimes b, I \otimes J)$$

is semiregular.

Definition 2.5 A triple $(L, h, J)$, consisting of a linear isomorphism $h : L \otimes L \cong J$, with $L$ and $J$ both line bundles, is called a twisted discriminant bundle on $X$.

A twisted discriminant bundle $(L, h, J)$ specifies $L$ as a square root of the line bundle $J$ via $h$. The terminology is motivated by the following: when $J$ is the trivial line bundle, such a datum is referred to as a discriminant bundle in §3, Chap.III, of Knus’ book [9]. By part (2) of the preceding Proposition, $h$ is a regular bilinear form on $L$ (necessarily symmetric) with values in $J$, so that we may speak of an isometry between two twisted discriminant bundles $(L, h, J)$ and $(L', h', J')$: it is a pair $(\zeta, \eta)$ consisting of linear isomorphisms $\zeta : L \cong L'$ and $\eta : J \cong J'$ such that $\eta h = h'(\zeta \otimes \zeta)$.

Lemma 2.6 On the set $T$-Disc($X$) of isometry classes of twisted discriminant bundles on $X$, we have a natural group structure induced by the tensor product. $T$-Disc($X$) is functorial in $X$. If we consider only isometry classes of discriminant bundles, i.e., of triples $(L, h, \mathcal{O}_X)$, then we obtain a subgroup Disc($X$) $\subset T$-Disc($X$) which is of exponent 2.

Proposition 2.7 Let $V$ and $V'$ be vector bundles of the same rank on the scheme $X$, $(L, h, J)$ a twisted discriminant line bundle on $X$ and

$$\alpha : V' \cong V \otimes L$$

an isomorphism of bundles.

(1) Over any open subset $U \hookrightarrow X$, given a bilinear form

$$b' \in \Gamma (U, \text{Bil}_{(V', I)})$$

we can define a bilinear form

$$b \in \Gamma (U, \text{Bil}_{(V \otimes J^{-1})})$$

using $\alpha$ and $h$ as follows: we let

$$b := (b' \otimes J^{-1}) \circ (\zeta_{(\alpha,h)})^{-1}$$

where

$$\zeta_{(\alpha,h)} : V' \otimes V' \otimes J^{-1} \cong V \otimes V$$

is the linear isomorphism given by the composition of the following natural morphisms:
Then the association \( b' \mapsto b \) induces linear isomorphisms shown by vertical upward arrows in the following diagram of associated locally-free sheaves (with exact rows) making it commutative:

\[
\begin{array}{cccccc}
0 & \longrightarrow & \text{Alt}^{2}_{(V,J \otimes J^{-1})} & \longrightarrow & \text{Bil}_{(V,J \otimes J^{-1})} & \longrightarrow & \text{Quad}_{(V,J \otimes J^{-1})} & \longrightarrow & 0 \\
& & \cong & & \cong & & \cong & & \\
0 & \longrightarrow & \text{Alt}^{2}_{(V',J)} & \longrightarrow & \text{Bil}_{(V',J)} & \longrightarrow & \text{Quad}_{(V',J)} & \longrightarrow & 0.
\end{array}
\]

Therefore one also has the following commutative diagram of vector bundle morphisms with the vertical upward arrows being isomorphisms:

\[
\begin{array}{cccccc}
\text{Alt}^{2}_{(V,I \otimes J^{-1})} & \underset{\text{closed immersion}}{\longrightarrow} & \text{Bil}_{(V,I \otimes J^{-1})} & \underset{\text{locally trivial}}{\longrightarrow} & \text{Quad}_{(V,I \otimes J^{-1})} \\
\cong & & \cong & & \cong \\
\text{Alt}^{2}_{(V',I)} & \underset{\text{closed immersion}}{\longrightarrow} & \text{Bil}_{(V',I)} & \underset{\text{locally trivial}}{\longrightarrow} & \text{Quad}_{(V',I)}
\end{array}
\]

(2) Let \( b' \in \Gamma(X, \text{Bil}_{(V',J)}) \) be a global bilinear form and let it induce

\[
b \in \Gamma(X, \text{Bil}_{(V,I \otimes J^{-1})})
\]

via \( \alpha \) and \( h \) as defined in (1) above. Let

\[
\Psi_{b'} \in \text{Aut}_{\mathcal{O}_X}(TV' \otimes L[I]) \quad \text{(resp.} \quad \Psi_b \in \text{Aut}_{\mathcal{O}_X}(TV \otimes L[I \otimes J^{-1}])\text{)}
\]

be the \( \mathbb{Z} \)-graded linear isomorphism induced by \( b' \) (resp. by \( b \)) defined locally (and hence globally) as in (2), Theorem 2.2 above. Let

\[
Z_{(\alpha,h)} : \oplus_{n \geq 0}(T_{\mathcal{O}_X}^{2n}(V') \otimes I^{-n}) \cong \oplus_{n \geq 0}(T_{\mathcal{O}_X}^{2n}(V) \otimes I^{-n} \otimes J^n)
\]

be the \( \mathcal{O}_X \)-algebra isomorphism induced via the isomorphism \( \zeta_{(\alpha,h)} \) defined in (1) above. Then, taking into account (2e), Theorem 2.2 the following diagram commutes:

\[
\begin{array}{cccc}
\oplus_{n \geq 0}(T_{\mathcal{O}_X}^{2n}(V') \otimes I^{-n}) & \underset{\cong}{\longrightarrow} & \oplus_{n \geq 0}(T_{\mathcal{O}_X}^{2n}(V) \otimes I^{-n} \otimes J^n) \\
\Psi_{b'} & \cong & \Psi_b \\
\oplus_{n \geq 0}(T_{\mathcal{O}_X}^{2n}(V') \otimes I^{-n}) & \underset{\cong}{\longrightarrow} & \oplus_{n \geq 0}(T_{\mathcal{O}_X}^{2n}(V) \otimes I^{-n} \otimes J^n)
\end{array}
\]
thereby inducing by (2d), Theorem 2.2 the following commutative diagram of $O_X$-linear isomorphisms

$$
\begin{array}{ccc}
C_0(V', q_{b'}, I) & \xrightarrow{\text{via } Z_{(a,b)}} & C_0(V, q_b, I \otimes J^{-1}) \\
\psi_{b'} & \cong & \cong \\
\oplus_{n \geq 0} (\Lambda^2_{O_X}(V') \otimes I^{-n}) & \xrightarrow{\text{via } Z_{(a,b)}} & \oplus_{n \geq 0} (\Lambda^2_{O_X}(V) \otimes I^{-n} \otimes J^n)
\end{array}
$$

(3) Let $b$ and $b'$ be as in (2) above. Then $\alpha : V' \cong V \otimes L$ induces an isometry of bilinear form bundles

$$
\alpha : (V', b', I) \cong (V, b, I \otimes J^{-1}) \otimes (L, h, J)
$$

and also an isometry of the induced quadratic bundles

$$
\alpha : (V', q_{b'}, I) \cong (V, q_b, I \otimes J^{-1}) \otimes (L, h, J).
$$

Moreover, if we are just given a global $I \otimes J^{-1}$-valued quadratic form $q$ on $V$ (resp. an $I$-valued $q'$ on $V'$), then we may define the global $I$-valued quadratic form $q'$ on $V'$ (resp. $I \otimes J^{-1}$-valued $q$ on $V$) via

$$
q' := (q \otimes h) \circ \alpha \quad \text{(resp. via } q := (q' \circ \alpha^{-1} \otimes (h')^{-1}) \text{)}
$$

and again we have an isometry of quadratic bundles

$$
\alpha : (V', q', I) \cong (V, q, I \otimes J^{-1}) \otimes (L, h, J).
$$

**Proposition 2.8** Let $g : (V, q, I) \cong (V', q', I)$ be an $I$-similarity with multiplier $l \in \Gamma(X, O^*_X)$.

(1) There exists a unique isomorphism of $O_X$-algebra bundles

$$
C_0(g, l, I) : C_0(V, q, I) \cong C_0(V', q', I)
$$

such that for sections $v, v'$ of $V$ and $s$ of $I^{-1}$ we have

$$
C_0(g, l, I)(v.v'.s) = g(v).g(v').l^{-1}s.
$$

(2) There exists a unique vector bundle isomorphism

$$
C_1(g, l, I) : C_1(V, q, I) \cong C_1(V', q', I)
$$

such that the following hold for any section $v$ of $V$ and any section $c$ of $C_0(V, q)$:

(a) $C_1(g, l, I)(v.c) = g(v).C_0(g, l, I)(c)$ and
(b) $C_1(g, l, I)(c.v) = C_0(g, l, I)(c).g(v)$. 

Thus $C_1(g,l,I)$ is $C_0(g,l,I)$-semilinear.

(3) If $g_1 : (V',q',I) \cong (V'',q'',I)$ is another similarity with multiplier $l_1$, then the composition

$$g_1 \circ g : (V,q,I) \cong (V'',q'',I)$$

is also a similarity with multiplier given by the product of the multipliers. Further

$$C_i(g_1 \circ g,l_1,I) = C_i(g_1,l_1,I) \circ C_i(g,l,I)$$

for $i = 0, 1$.

A local computation shows that tensoring by a twisted discriminant bundle is locally the same as applying a similarity. In this case also one gets a global isomorphism of even Clifford algebras:

**Proposition 2.9** Let $(V,q,I)$ be a quadratic bundle on a scheme $X$ and $(L,h,J)$ be a twisted discriminant bundle. There exists a unique isomorphism of algebra bundles

$$\gamma_{(L,h,J)} : C_0((V,q,I) \otimes (L,h,J)) \cong C_0(V,q,I)$$

such that for any sections $v,v'$ of $V$, $\lambda,\lambda'$ of $L$, $s$ of $I^{-1} \equiv I'$ and $t$ of $J^{-1} \equiv J'$ we have

$$\gamma_{(L,h,J)}((v \otimes \lambda),(v' \otimes \lambda'),(s \otimes t)) = t(h(\lambda \otimes \lambda'))v.v'.s.$$

### 2.7 The Theorem of Max-Albert Knus

For any scheme $X$, denote by $Q_3^\text{sr}(X;O_X)$ the set of isomorphism classes of semiregular ternary quadratic bundles with values in the trivial line bundle on $X$. Here isomorphism stands for isometry as defined in §2.2. Recall the group $\text{Disc}(X)$ of discriminant bundles on $X$ (Lemma 2.6). By assertions (2) and (3) of Prop. 2.4, this group acts on $Q_3^\text{sr}(X;O_X)$.

For a ternary quadratic form $q : V \to O_X$, recall from §2.4 the Bichsel-Knus even-Clifford algebra $C_0(V,q,O_X)$, namely the degree zero subalgebra of the generalised Clifford algebra $\widetilde{C}(V,q,O_X)$. Since the values of the form are in the trivial line bundle, this even-Clifford algebra is the same as the classically-defined even-Clifford algebra.

Now the even-Clifford algebra of a semiregular form is Azumaya (see for instance, Prop.3.2.4, §3, Chap.IV, [9]). So if we let $\mathcal{A}_2\mathcal{U}_4(X)$ to denote the set of algebra-isomorphism classes of rank 4 Azumaya bundles over $X$, then by Prop. 2.7, the association $(V,q,O_X) \sim C_0(V,q,O_X)$ induces a map

$$\text{Witt-Invariant}_{(X;O_X)}^\text{sr} : Q_3^\text{sr}(X;O_X)/\text{Disc}(X) \to \mathcal{A}_2\mathcal{U}_4(X),$$

where the left side represents the set of orbits.

**Theorem 2.10** (Max-Albert Knus, § 3, Chap.V, [9]) For any scheme $X$, the map Witt-Invariant$_{(X;O_X)}^\text{sr}$ defined above is a bijection.
By Prop.3.2.2, §3, Chap.III, [9], the group Disc(\(X\)) is naturally isomorphic to the cohomology (abelian group) \(\check{H}^1_{\text{fppf}}(X, \mu_2)\). Further, by Lemma 3.2.1, §3, Chap.IV, [9], the cohomology \(\check{H}^1_{\text{fppf}}(X, \mathcal{O}_3)\) classifies the set of isomorphism classes of semiregular rank 3 quadratic bundles with values in the trivial line bundle, so it is the same as \(\mathcal{O}_3(X; \mathcal{O}_X)\). On the other hand, the set of isomorphism classes of rank 4 Azumaya algebras on \(X\) may be interpreted as the cohomology \(\check{H}^1_{\text{\acute{e}tale}}(X, \text{PGL}_2)\) (see page 145, §5, Chap.III, [9]). Thus the bijection of Theorem 2.10 can be thought of as a statement in cohomology:

\[
\check{H}^1_{\text{fppf}}(X, \mathcal{O}_3)/\check{H}^1_{\text{fppf}}(X, \mu_2) \cong \check{H}^1_{\text{\acute{e}tale}}(X, \text{PGL}_2).
\]

### 2.8 Results on the notion of Schematic Image

**Definition 2.11** (Defs.6.10.1-2, Chap.I, EGA I [6]) Let \(f : X \rightarrow Y\) be a morphism of schemes. If there exists a smallest closed subscheme \(Y' \hookrightarrow Y\) such that the inverse image scheme

\[
f^{-1}(Y') := Y' \times_Y (f_X)
\]

is equal to \(X\), one calls \(Y'\) the **schematic image** of \(f\) (or of \(X\) in \(Y\) under \(f\)). If \(X\) were a subscheme of \(Y\) and \(f\) the canonical immersion, and if \(f\) has a schematic image \(Y'\), then \(Y'\) is called the **schematic limit** or the **limiting scheme** of the subscheme \(X \hookrightarrow Y\).

**Proposition 2.12** (Prop.6.10.5, Chap.I, EGA I) The schematic image \(Y'\) of \(X\) by a morphism \(f : X \rightarrow Y\) exists in the following two cases:

1. \(f_*(\mathcal{O}_X)\) is a quasi-coherent \(\mathcal{O}_Y\)-module, which is for example the case when \(f\) is quasi-compact and quasi-separated;
2. \(X\) is reduced.

**Proposition 2.13** In each of the following statements whenever a schematic image is mentioned, we assume that one of the two hypotheses of the above Prop. 2.12 is true so that the schematic image does exist.

1. Let \(Y'\) be the schematic image of \(X\) under a morphism \(f : X \rightarrow Y\) and let \(f\) factor as

\[
X \xrightarrow{g} Y' \xrightarrow{j} Y.
\]

Then \(Y'\) is topologically the closure of \(f(X)\) in \(Y\), the morphism \(g\) is schematically dominant i.e.,

\[
g^\# : \mathcal{O}_{Y'} \rightarrow g_*(\mathcal{O}_X)
\]

is injective and the schematic image of \(X\) in \(Y'\) (under \(g\)) is \(Y'\) itself. If \(X\) is reduced (respectively integral) then the same is true of \(Y'\).

2. The schematic image of a closed subscheme under its canonical closed immersion is itself.
3. (Transitivity of Schematic Image.) Let there be given morphisms

\[ X \xrightarrow{f} Y \xrightarrow{g} Z, \]

such that the schematic image \( Y' \) of \( X \) under \( f \) exists, and further such that if \( g' \) is the restriction of \( g \) to \( Y' \), the schematic image \( Z' \) of \( Y' \) by \( g' \) exists. Then the schematic image of \( X \) under \( g \circ f \) exists and equals \( Z' \).

4. Let \( f : X \to Y \) be a morphism which factors through a closed subscheme \( Y_1 \) of \( Y \) by a morphism \( f_1 : X \to Y_1 \). Then the scheme-theoretic image \( Y' \) of \( X \) in \( Y \) is the same as the scheme-theoretic image \( Y'_1 \) of \( X \) in \( Y_1 \) considered canonically as closed subscheme of \( Y \).

5. If \( f : X \to Y \) has a schematic image \( Y' \) then \( f \) is schematically dominant iff \( Y' = Y \).

6. The formation of schematic image commutes with flat base change: if \( f : X \to Y \) is a morphism of \( S \)-schemes which has a schematic image \( Y' \) then for a flat morphism \( S' \to S \), one has that the induced \( S' \)-morphism

\[ f \times_S S' : X \times_S S' \to Y \times_S S' \]

has a schematic image and it may be canonically identified with \( Y' \times_S S' \). In particular this means that the formation of schematic image is local over the base.

Assertions (1) and (3) are respectively Prop.6.10.5 and Prop.6.10.3 in EGA I. The defining property of schematic image gives (2), while (4) can be deduced from the first three. As for (5), from (1) it follows that \( Y' = Y \) implies \( f = g \) is schematically dominant. For the other way around, one uses the following characterisation of a schematically dominant morphism in Prop.5.4.1 of EGA I: if \( f : X \to Y \) is a morphism of schemes, then \( f \) is schematically dominant iff for every open subscheme \( U \) of \( Y \) and every closed subscheme \( Y_1 \) of \( U \) such that there exists a factorisation

\[ f^{-1}(U) \xrightarrow{g_1} Y_1 \xrightarrow{j_1} U, \]

of the restriction \( f^{-1}(U) \to U \) of \( f \) (where \( j_1 \) is the canonical closed immersion), one has \( Y_1 = U \). Given \( f \) is schematically dominant, one just has to take \( U = Y \), \( Y_1 = Y' \) and \( g_1 = g \). Assertion (6) follows from statement (ii) (a) of Theorem 11.10.5 of EGA IV \[5\].

### 2.9 Specialisations of Rank 4 Azumaya Algebras: Recap

Until further notice we assume that \( W \) is a vector bundle of fixed positive rank on the scheme \( X \) with associated coherent locally-free sheaf \( W \). Given any \( X \)-scheme \( T \), by a \( T \)-algebra structure on \( W_T := W \times_X T \) (also referred to as \( T \)-algebra bundle), we mean a morphism

\[ W_T \times_T W_T \to W_T \]
of vector bundles on $T$ arising from a morphism of the associated locally-free sheaves. Given such a $T$-algebra structure and $T' \to T$ an $X$-morphism, it is clear that one gets by pullback (i.e., by base-change) a canonical $T'$-algebra structure on $W_{T'}$. Thus one has a contravariant “functor of algebra structures on $W$” from $\{X - \text{Schemes}\}$ to $\{\text{Sets}\}$ denoted $\text{Alg}_W$ with

\[ \text{Alg}_W(T) = \{T - \text{algebra structures on } W_T\} = \text{Hom}_{O_T}(W_T \otimes W_T, W_T). \]

It follows from Prop.9.6.1, Chap.I of EGA I [6] that the functor $\text{Alg}_W$ is represented by the $X$-scheme

\[ \text{Alg}_W := \text{Spec} \left( \text{Sym}_X \left( (W_X^\vee \otimes_X W_X^\vee) \otimes_X W_X \right) \otimes X \right). \]

Hence $\text{Alg}_W$ is affine (therefore separated), of finite presentation over $X$ and in fact smooth of relative dimension $\text{rank}_X(W)^3$. If $X' \to X$ is an extension of base, then the construction $\text{Alg}_W$ base-changes well i.e., one may canonically identify $\text{Alg}_W \times_X X'$ with $\text{Alg}_{W'}$ where $W' = W \times_X X'$ (cf. Prop.9.4.11, Chap.I, EGA I [6]).

The general linear groupscheme associated to $W$ viz $\text{GL}_W$ naturally acts on $\text{Alg}_W$ on the left, so that for each $X$-scheme $T$, $\text{Alg}_W(T)$ mod $\text{GL}_W(T)$ is the set of isomorphism classes of $T$-algebra structures on $W \times_X T$.

We remark that an algebra structure may fail to be associative and may fail to have a (two-sided) identity element for multiplication. However, a multiplicative identity for an associative algebra structure must be a nowhere vanishing section (Lemma 2.3, and (2) $\Rightarrow$ (4) of Lemma 2.4, Part A, [17]).

Let $w \in \Gamma(X, W)$ be a nowhere vanishing section. For any $X$-scheme $T$, let $\text{Assoc}_{W,w}(T)$ denote the subset of $\text{Alg}_W(T)$ consisting of associative algebra structures with multiplicative identity the nowhere vanishing section $w_T$ over $T$ induced from $w$. Thus we obtain a contravariant subfunctor $\text{Assoc}_{W,w}$ of $\text{Alg}_W$.

Let $\text{Stab}_{w}(T) \subset \text{GL}_W(T)$ denote the stabiliser subgroup of $w_T$, so that one gets a subfunctor in subgroups $\text{Stab}_{w} \subset \text{GL}_W$. It is in fact represented by a closed subgroupscheme (also denoted by) $\text{Stab}_{w}$ and further behaves well under base change relative to $X$ i.e., $\text{Stab}_{w} \times_X T$ can be canonically identified with $\text{Stab}_{w_T}$ for any $X$-scheme $T$. These follow from para 9.6.6 of Chap.I, EGA I [6].

It is clear that the natural action of $\text{GL}_W$ on $\text{Alg}_W$ induces one of $\text{Stab}_{w}$ on $\text{Assoc}_{W,w}$. It is easy to check (p.489, Part A, [17]) that the functor $\text{Assoc}_{W,w}$ is a sheaf in the big Zariski site over $X$ and further that this functor is represented by a natural closed subscheme

\[ \text{Assoc}_{W,w} \hookrightarrow \text{Alg}_W \]

which is $\text{Stab}_{w}$-invariant. Further the construction $\text{Assoc}_{W,w}$ behaves well with respect to base-change (relative to $X$). Consider the subfunctor

\[ \text{Azu}_{W,w} \hookrightarrow \text{Assoc}_{W,w} \]

corresponding to Azumaya algebras.
Theorem 2.14 (Theorem 3.4, Part A, [17])

(1) $\text{Azu}_{W,w}$ is represented by a $\text{Stab}_w$-stable open subscheme

$$\text{Azu}_{W,w} \hookrightarrow \text{Assoc}_{W,w}$$

and the canonical open immersion is an affine morphism.

(2) $\text{Azu}_{W,w}$ is affine (hence separated) and of finite presentation over $X$, and $\text{Azu}_{W,w}$ behaves well with respect to base-change (relative to $X$).

(3) Further, $\text{Azu}_{W,w}$ is smooth of relative dimension $(m^2 - 1)^2$ and geometrically irreducible over $X$, where $m^2 := \text{rank}_X(W)$.

Theorem 2.15 (Theorem 3.8, Part A, [17])

(1) The open immersion $\text{Azu}_{W,w} \hookrightarrow \text{Assoc}_{W,w}$ has a schematic image denoted

$$\text{SpAzu}_{W,w} \hookrightarrow \text{Assoc}_{W,w}$$

which is affine (hence separated) and of finite type over $X$ and is naturally a $\text{Stab}_w$-stable closed subscheme of $\text{Assoc}_{W,w}$, the action extending the natural one on the open subscheme $\text{Azu}_{W,w}$.

(2) When the rank of $W$ over $X$ is 4, $\text{SpAzu}_{W,w}$ is locally (over $X$) isomorphic to relative 9-dimensional affine space; in fact over every open affine subscheme $U$ of $X$ where $W$ becomes trivial and $w$ becomes part of a global basis we have

$$\text{SpAzu}_{W,w}|_U \cong \mathbb{A}^9_U.$$  

For the explicit isomorphism, see Theorem 3.10, page 32. Thus $\text{SpAzu}_{W,w}$ is smooth of relative dimension 9 and geometrically irreducible over $X$. In particular, it is of finite presentation over $X$.

(3) When $\text{rank}_X(W) = 4$, the construction $\text{SpAzu}_{W,w} \rightarrow X$ behaves well with respect to base change (relative to $X$).

3 Statements of the Main Results

The theory of semiregular/regular quadratic forms of low rank is well-known, as in Chap.V of Knus’ book [9]. This theory is satisfactory for such forms, since it classifies them in terms of various invariants and also gives information on the groups of generalised similarities, isometries and special isometries of such forms. It is natural to look for a corresponding theory for limiting or degenerate forms. We formulate in the following such a theory for the case of ternary quadratic forms.
3.1 A Limiting Version of a Theorem in Cohomology

For any scheme $X$, denote by $Q_3(X)$ (respectively, by $Q^{sr}_3(X)$) the set of isomorphism classes of line-bundle-valued ternary quadratic bundles (respectively line-bundle-valued semiregular ternary quadratic bundles) on $X$. Here isomorphism stands for isometry as defined in §2.2. Consider the group $T$-$\text{Disc}(X)$ of twisted discriminant bundles on $X$ (cf. Lemma 2.6). By Prop. 2.4 this group acts on $Q_3(X)$ and on the subset $Q^{sr}_3(X)$.

For a line-bundle-valued quadratic form $(V,q,I)$, recall from §2.4 the Bichsel-Knus even-Clifford algebra $C_0(V,q,I)$, which is the degree zero subalgebra of the generalised Clifford algebra $\tilde{C}(V,q,I)$ and reduces to the usual even Clifford algebra for a quadratic form with values in the structure sheaf.

Let $SPA4(X)$ (respectively, $AZ4(X)$) denote the set of isomorphism classes of associative unital algebra structures on vector bundles of rank 4 over $X$ that are Zariski-locally isomorphic to even-Clifford algebras of rank 3 quadratic bundles (respectively, that are Azumaya). Recall that the Theorem 2.10 (page 19) of Max-Albert Knus gives a bijection

$$\text{Witt-Invariant}^{sr}_{X} : Q^{sr}_3(X;/O_X)/\text{Disc}(X) \cong AZ4(X).$$

It follows that $AZ4(X) \subset SPA4(X)$. By Prop. 2.4 the association $(V,q,I) \sim C_0(V,q,I)$ induces a map

$$\text{Witt-Invariant}_X : Q_3(X)/T-\text{Disc}(X) \rightarrow SPA4(X),$$

where the left side represents the set of orbits. Since the even-Clifford algebra of a semiregular quadratic module is Azumaya, it follows that the above map restricts to a map:

$$\text{Witt-Invariant}^{sr}_X : Q^{sr}_3(X)/T-\text{Disc}(X) \rightarrow AZ4(X).$$

**Theorem 3.1** For any scheme $X$, both the map $\text{Witt-Invariant}_X$ as well as its restriction $\text{Witt-Invariant}^{sr}_X$ are bijections.

Thus the above Theorem 3.1 may be viewed as a “limiting version” of the Theorem 2.10 of Knus which as noted in page 20 may be interpreted as a statement in cohomology.

We shall see later ((b1), Theorem 3.6) that it is necessary to consider line-bundle-valued quadratic forms to obtain the surjectivity of Theorem 3.1 in those cases for which a given $A$ representing an element in $SPA4(X)$ is such that $\det(A) \not\in 2\cdot\text{Pic}(X)$.

It was shown in Part A, 17 that algebra bundles belonging to $SPA4(X)$ are precisely the scheme-theoretic specialisations (or limits) of rank 4 Azumaya algebra bundles on $X$ (Theorem 2.16).

Thus one may also restate the surjectivity as schematic specialisations of rank 4 Azumaya bundles arise as even-Clifford algebras of ternary quadratic bundles and the injectivity as follows: if the even-Clifford algebras of two ternary quadratic
The main result of [17] was the smoothness of the schematic closure of Azumaya algebra structures on a fixed vector bundle of rank 4 over any scheme. Part B of [17] had applied this result to obtain the generalised Seshadri desingularisation of the moduli space of semistable rank 2 degree zero vector bundles on a smooth proper curve relative to a locally universally-japanese (Nagata) base scheme and also to obtain the generalised Nori desingularisation of the Artin moduli space of invariants of several matrices in rank 2 over such a base scheme together with good specialisation properties over \(\mathbb{Z}\). The present work is concerned with applications to ternary quadratic forms and the results obtained follow from an analysis of the computations that lead to the smoothness.

The good algebraic properties of Azumaya algebras are reflected as good geometric properties of the scheme of Azumaya algebra structures on a fixed vector bundle: this scheme is separated, of finite type and smooth relative to the base scheme (over which the vector bundle is fixed) and also base-changes well relative to the base scheme (Theorem 2.14, page 23). When the vector bundle is of rank 4, the nice thing that happens is that all these good properties also pass over to the limit i.e., to the scheme of specialisations, defined to be the schematic image of the scheme of Azumaya algebra structures (Theorem 2.15, page 23). In the same vein, the present work shows that the theories of rank 3 semiregular quadratic forms and of rank 4 Azumaya algebras and their inter-relationships extend to the limit.

### 3.2 Study of Groups of Similitudes

A bilinear form \(b\), with values in a line bundle \(I\), defined on a vector bundle \(V\) over the scheme \(X\) induces an \(I\)-valued quadratic form \(q_b\) given on sections by \(x \mapsto b(x, x)\). Let

\[
L[I] := \mathcal{O}_X \oplus \left( \bigoplus_{n>0} (T^n(I) \oplus T^n(I^{-1})) \right)
\]

be the Laurent-Rees algebra of \(I\), where sections of \(V\) (resp. of \(I\)) are declared to be of degree one (resp. of degree two). Then, as we saw in (2d), Theorem 2.24 \(b\) naturally defines an \(\mathbb{Z}\)-graded linear isomorphism

\[
\psi_b : \tilde{C}(V, q_b, I) \cong \Lambda(V) \otimes L[I].
\]

In fact we have

\[
\psi_0 : \tilde{C}(V, 0, I) = \Lambda(V) \otimes L[I].
\]

Since in general a quadratic bundle \((V, q, I)\) on a non-affine scheme \(X\) may not be induced from a global \(I\)-valued bilinear form, one is unable to identify the \(\mathbb{Z}\)-graded vector bundle underlying its generalised Clifford algebra bundle with \(\Lambda(V) \otimes L[I]\). The following result overcomes this problem.

**Proposition 3.2** To every isomorphism of algebra-bundles

\[
\phi : C_0(V, q, I) \cong C_0(V', q', I'),
\]

bundles are isomorphic, then the quadratic bundles are isometric up to tensoring by a twisted discriminant bundle.
one may naturally associate an isomorphism of bundles
\[ \phi_{\Lambda^2} : \Lambda^2(V) \otimes I^{-1} \cong \Lambda^2(V') \otimes (I')^{-1} \]
inducing a map
\[ \zeta_{\Lambda^2} : \text{Iso}[C_0(V, q, I), C_0(V', q', I')] \to \text{Iso}[\Lambda^2(V) \otimes I^{-1}, \Lambda^2(V') \otimes (I')^{-1}] : \phi \mapsto \phi_{\Lambda^2} \]
where
\[ \text{Iso}[C_0(V, q, I), C_0(V', q', I')] \]
is the set of algebra bundle isomorphisms.

**Definition 3.3** When \( V = V' \) and \( I = I' \), we may thus denote the subset of those \( \phi \) for which
\[ \det(\phi_{\Lambda^2}) \in \text{Aut}[\Lambda^2(V) \otimes I^{-1}] \equiv \Gamma(X, \mathcal{O}_X^*) \]
is a square by
\[ \text{Iso}'[C_0(V, q, I), C_0(V, q', I)] \]
and those for which \( \det(\phi_{\Lambda^2}) = 1 \) by the smaller subset
\[ \text{S-Iso}[C_0(V, q, I), C_0(V, q', I)]. \]
Taking \( q = q' \) in these sets and replacing “Iso” by “Aut” in their notations respectively defines the groups
\[ \text{Aut}(C_0(V, q, I)) \supset \text{Aut}'(C_0(V, q, I)) \supset \text{S-Aut}(C_0(V, q, I)). \]

**Theorem 3.4** For \( I \)-valued quadratic forms \( q \) and \( q' \) on a rank 3 vector bundle \( V \) over a scheme \( X \), we have the following commuting diagram of natural maps of sets of with the downward arrows being the canonical inclusions, the horizontal arrows being surjective and the top horizontal arrow being bijective:

\[
\begin{array}{ccc}
\text{S-Iso}[(V, q, I), (V, q', I)] & \xrightarrow{\cong} & \text{S-Iso}[C_0(V, q, I), C_0(V, q', I)] \\
\text{inj} & & \text{inj} \\
\text{Iso}[(V, q, I), (V, q', I)] & \xrightarrow{\text{onto}} & \text{Iso}'[C_0(V, q, I), C_0(V, q', I)] \\
\text{inj} & & \text{inj} \\
\text{Sim}[(V, q, I), (V, q', I)] & \xrightarrow{\text{onto}} & \text{Iso}[C_0(V, q, I), C_0(V, q', I)]
\end{array}
\]

With respect to the surjections of the horizontal arrows in the diagram above, we further have the following (where \( l \) is the function that associates a similarity to its multiplier, \( \det(g, l) := \det(g) \) for an \( I \)-similarity \( g \) with multiplier \( l \) and \( \zeta_{\Lambda^2} \) is the map of Prop \ref{prop3.2} above):
(a) there is a family of sections
\[ s_{2k+1} : \text{Iso}[C_0(V, q, I), C_0(V, q', I)] \rightarrow \text{Sim}[(V, q, I), (V, q', I)] \]
indexed by the integers such that
\[ l \circ s_{2k+1} = \det^{2k+1} \circ \zeta_{\Lambda^2} \quad \text{and} \quad (\det^2 \circ s_{2k+1}) \times (l^{-3} \circ s_{2k+1}) = \det \circ \zeta_{\Lambda^2}; \]

(b) there is also a section
\[ s' : \text{Iso}'[C_0(V, q, I), C_0(V, q', I)] \rightarrow \text{Iso}[(V, q, I), (V, q', I)] \]
such that \( \det^2 \circ s' = \det \circ \zeta_{\Lambda^2}; \)

(c) there is a family of sections
\[ s_{2k+1}^+ : \text{Iso}[C_0(V, q, I), C_0(V, q', I)] \rightarrow \text{Sim}[(V, q, I), (V, q', I)] \]
indexed by the integers which is multiplicative when followed by the natural inclusions into \( \text{GL}(V) \times \Gamma(X, \mathcal{O}_X^*) \), i.e., if
\[ \phi_i \in \text{Iso}[C_0(V, q_i, I), C_0(V, q_{i+1}, I)] \]
then
\[ s_{2k+1}^+(\phi_2 \circ \phi_1) = s_{2k+1}^+(\phi_2) \circ s_{2k+1}^+(\phi_1) \in \text{GL}(V) \times \Gamma(X, \mathcal{O}_X^*). \]
Further,
\[ l \circ s_{2k+1}^+ = \det^{2k+1} \circ \zeta_{\Lambda^2} \quad \text{and} \quad (\det^2 \circ s_{2k+1}^+) \times (l^{-3} \circ s_{2k+1}^+) = \det \circ \zeta_{\Lambda^2}. \]

(d) The maps \( s_{2k+1} \) and \( s' \) above may not be multiplicative but are multiplicative up to \( \mu_2(\Gamma(X, \mathcal{O}_X)) \) i.e., these maps followed by the canonical quotient map, on taking the quotient of \( \text{GL}(V) \times \Gamma(X, \mathcal{O}_X^*) \) by \( \mu_2(\Gamma(X, \mathcal{O}_X)).\text{Id}_V \times \{1\} \), become multiplicative.

**Theorem 3.5** For a rank 3 quadratic bundle \( (V, q, I) \) on a scheme \( X \), one has the following natural commutative diagram of groups with exact rows, where the downward arrows are the canonical inclusions and where \( l \) is the function that associates to any \( I \)-(self)similarity its multiplier:

```
\[
\begin{array}{ccccccccc}
SO(V, q, I) & \xrightarrow{\cong} & S\text{-Aut}(C_0(V, q, I)) \\
\downarrow \text{inj} & & \downarrow \text{inj} \\
1 & \xrightarrow{\mu_2(\Gamma(X, \mathcal{O}_X))} & O(V, q, I) & \xrightarrow{\text{Aut}^+(C_0(V, q, I))} & 1 \\
\downarrow \text{inj} & & \downarrow \text{inj} & & \downarrow \text{inj} \\
1 & \xrightarrow{\Gamma(X, \mathcal{O}_X^*)} & \text{GO}(V, q, I) & \xrightarrow{\text{Aut}(C_0(V, q, I))} & 1 \\
\downarrow \text{det}^{2 \times l^{-3}} & & \downarrow \text{det} & & \downarrow \text{det} \\
\Gamma(X, \mathcal{O}_X^*) & \xrightarrow{\cong} & \Gamma(X, \mathcal{O}_X^*)
\end{array}
\]
```

Further, we have:
(a1) There are splitting homomorphisms

\[ s_{2k+1}^+: \text{Aut}(C_0(V,q,I)) \rightarrow \text{GO}(V,q,I) \]

such that

\[ l \circ s_{2k+1}^+ = \det^{2k+1} \quad \text{and} \quad (\det^2 \circ s_{2k+1}^+) \times (I^{-3} \circ s_{2k+1}^+) = \det. \]

In particular, \( \text{GO}(V,q,I) \) is a semidirect product.

(a2) The restriction of \( s_{2k+1}^+ \) to \( \text{Aut}^\prime(C_0(V,q,I)) \) does not necessarily take values in \( \text{O}(V,q,I) \), but the further restriction to \( \text{S-Aut}(C_0(V,q,I)) \) does take values in \( \text{SO}(V,q,I) \).

(a3) The maps \( s_{2k+1} \) and \( s' \) of Theorem 3.4 above (under the current hypotheses) may not be homomorphisms but are homomorphisms up to \( \mu_2(\Gamma(X,\mathcal{O}_X)) \).

(b) Suppose \( X \) is integral and \( q \otimes \kappa(x) \) is semiregular at some point \( x \) of \( X \) with residue field \( \kappa(x) \). Then any automorphism of \( C_0(V,q,I) \) has determinant 1. Hence

\[ \text{Aut}(C_0(V,q,I)) = \text{Aut}'(C_0(V,q,I)) = \text{S-Aut}(C_0(V,q,I)) \]

and \( \text{O}(V,q,I) \) is the semidirect product of \( \mu_2(\Gamma(X,\mathcal{O}_X)) \) and \( \text{SO}(V,q,I) \).

The proofs of the above results, and of the injectivity part of Theorem 3.1 will be given in §4 and §5.

### 3.3 Study of Bilinear Forms and Interpretation as Specialisations

As for the proof of the surjectivity part of Theorem 3.1 we have the following which will be proved in §6

**Theorem 3.6** Let \( X \) be a scheme and \( A \) a specialisation of rank 4 Azumaya algebra bundles on \( X \). Let \( \mathcal{O}_X.1_A \rightarrow A \) be the line sub-bundle generated by the nowhere-vanishing global section of \( A \) corresponding to the unit for algebra multiplication.

(a) There exist a rank 3 vector bundle \( V \) on \( X \), a quadratic form \( q \) on \( V \) with values in the line bundle \( I := \det^{-1}(A) \), and an isomorphism of algebra bundles \( A \cong C_0(V,q,I) \). This gives the surjectivity in the statement of Theorem 3.1 Further, the following linear isomorphisms may be deduced:

1. \( \det(A) \otimes \Lambda^2(V) \cong A/\mathcal{O}_X.1_A \), from which follows:
2. \( \det(\Lambda^2(V)) \cong (\det(A))^{\otimes-2} \);
3. \( V \cong (A/\mathcal{O}_X.1_A)^\vee \otimes \det(V) \otimes \det(A) \);
4. \( \det(A^\vee) \cong (\det(A))^{\otimes-3} \otimes (\det(V))^{\otimes-2} \) which implies that \( \det(A) \otimes \det(A^\vee) \in 2.\text{Pic}(X) \).
(b) There exists a quadratic bundle \((V', q', I')\) such that \(C_0(V', q', I') \cong A\) and with

1. \(I' = \mathcal{O}_X\) iff \(\det(A) \in 2\text{Pic}(X)\);
2. \(q'\) induced from a global \(I'\)-valued bilinear form iff \(\mathcal{O}_X.1_A\) is an \(\mathcal{O}_X\)-direct summand of \(A\);
3. with both \(I' = \mathcal{O}_X\) and with \(q'\) induced from a global bilinear form (with values in \(I'\)) iff \(\mathcal{O}_X.1_A\) is an \(\mathcal{O}_X\)-direct summand of \(A\) and \(\det(A) \in 2\text{Pic}(X)\).

If \(2 \in \Gamma(X, \mathcal{O}_X^*)\), then any quadratic form is induced from a symmetric bilinear form. Therefore from assertion (b2) of the above, we have the following:

**Corollary 3.7** Suppose \(2 \in \Gamma(X, \mathcal{O}_X^*)\). Then for any specialisation \(A\) of rank 4 Azumaya algebras on \(X\), \(\mathcal{O}_X.1_A\) is an \(\mathcal{O}_X\)-direct summand of \(A\).

There are two ingredients in the proof of part (a) of Theorem 3.6. The first is Theorem 3.4. The second is the following theorem which describes specialisations as bilinear forms under certain conditions. As a preparation towards its statement, we briefly remind the reader of a few results from from Part A, [17], (cf. §2.9).

For a rank \(n^2\) vector bundle \(W\) on a scheme \(X\) and \(w \in \Gamma(X, W)\) a nowhere-vanishing global section, recall that if \(\text{Azu}_{W,w}\) is the open \(X\)-subscheme of Azumaya algebra structures on \(W\) with identity \(w\) then its schematic image (or the scheme of specialisations or the limiting scheme) in the bigger \(X\)-scheme \(\text{Assoc}_{W,w}\) of associative \(w\)-unital algebra structures on \(W\) is the \(X\)-scheme \(\text{SpAzu}_{W,w}\). By definition, the set of distinct specialised \(w\)-unital algebra structures on \(W\) corresponds precisely to the set of global sections of this last scheme over \(X\).

If \(\text{Stab}_w \subset \text{GL}_W\) is the stabiliser subgroupscheme of \(w\), recall from Theorems 3.4 and 3.8, Part A, [17], that there exists a canonical action of \(\text{Stab}_w\) on \(\text{SpAzu}_{W,w}\) such that the natural inclusions

\[\text{Azu}_{W,w} \hookrightarrow \text{SpAzu}_{W,w} \hookrightarrow \text{Assoc}_{W,w}\]

are all \(\text{Stab}_w\)-equivariant. Now let \(V\) be a rank 3 vector bundle on the scheme \(X\) and \(\text{Bil}_{(V, I)}\) be the associated rank 9 vector bundle of bilinear forms on \(V\) with values in the line bundle \(I\). We say that a bilinear form \(b\) over an open subset \(U \hookrightarrow X\) is semiregular if there is a trivialisation \(\{U_i\}\) of \(I|U\), such that over each open subscheme \(U_i\), the quadratic form \(q_i\) with values in the trivial line bundle induced from \(q_b|U_i\) is semiregular (it may turn out that a semiregular bilinear form may be degenerate). This definition is independent of the choice of a trivialisation, since \(q_i\) is semiregular iff \(\lambda q_i\) is semiregular for every \(\lambda \in \Gamma(U_i, \mathcal{O}_X^*)\) (for further details see [24, 25]). In this way we obtain the open subscheme \(\text{Bil}_{(V, I)} \hookrightarrow \text{Bil}_{(V, I)}\) of semiregular bilinear forms on \(V\) with values in \(I\). We next take for \(W\) the following special choice:

\[W := \Lambda^{even}(V, I) := \bigoplus_{n \geq 0} \Lambda^{2n}(V) \otimes I^{-\otimes n}\]
and we let \( w \in \Gamma(X, W) \) be the nowhere-vanishing global section corresponding to the unit for the natural multiplication in the twisted even-exterior algebra bundle \( W \). There is an obvious natural action of \( GL_V \) on \( \text{Bil}_{(V, I)} \). There is also a natural morphism of groupschemes \( GL_V \to \text{Stab}_w \) given on valued points by

\[
g \mapsto \bigoplus_{n \geq 0} \Lambda^{2n}(g) \otimes \text{Id}
\]

and therefore the natural inclusions marked by (♠) above are \( GL_V \)-equivariant.

Finally, note that there is an obvious involution \( \Sigma \) on \( \text{Assoc}_{W, w} \) given by \( A \mapsto \text{opposite}(A) \) which leaves the open subscheme \( \text{Azu}_{W, w} \) invariant.

**Theorem 3.8**

(1) Let \( V \) be a rank 3 vector bundle on the scheme \( X \), \( W := \Lambda^{\text{even}}(V, I) \) and \( w \in \Gamma(X, W) \) correspond to \( I \) in the twisted even-exterior algebra bundle. There is a natural \( GL_V \)-equivariant morphism of \( X \)-schemes

\[
\Upsilon' = \Upsilon'_X : \text{Bil}_{(V, I)} \to \text{Assoc}_{W, w}
\]

whose schematic image is precisely the scheme of specialisations

\[
\text{SpAzu}_{W, w}.
\]

Further if \( \Upsilon' \) factors canonically through

\[
\Upsilon = \Upsilon_X : \text{Bil}_{(V, I)} \to \text{SpAzu}_{W, w},
\]

then \( \Upsilon \) is a \( GL_V \)-equivariant isomorphism and it maps the \( GL_V \)-stable open subscheme \( \text{Bil}_{(V, I)}^{W, r} \) isomorphically onto the \( GL_V \)-stable open subscheme \( \text{Azu}_{W, w} \).

(2) The involution \( \Sigma \) of \( \text{Assoc}_{W, w} \) defines a unique involution (also denoted by \( \Sigma \)) on the scheme of specialisations \( \text{SpAzu}_{W, w} \) leaving the open subscheme \( \text{Azu}_{W, w} \) invariant, and therefore via the isomorphism \( \Upsilon \), it defines an involution on \( \text{Bil}_{(V, I)} \). This involution is none other than the one on valued points given by \( B \mapsto \text{transpose}(-B) \).

(3) For an \( X \)-scheme \( T \), let \( V_T \) (resp. \( W_T \), resp. \( I_T \)) denote the pullback of \( V \) (resp. \( W \), resp. \( I \)) to \( T \), and let \( w_T \) be the global section of \( W_T \) induced by \( w \). Then the base-changes of \( \Upsilon'_X \) and \( \Upsilon_X \) to \( T \), namely

\[
\Upsilon'_X \times_X T : \text{Bil}_{(V, I)} \times_X T \to \text{Assoc}_{W, w} \times_X T
\]

and

\[
\Upsilon_X \times_X T : \text{Bil}_{(V, I)} \times_X T \cong \text{SpAzu}_{W, w} \times_X T
\]

may be canonically identified with the corresponding ones over \( T \) namely with

\[
\Upsilon'_T : \text{Bil}_{(V_T, I_T)} \to \text{Assoc}_{W_T, w_T}
\]

and

\[
\Upsilon_T : \text{Bil}_{(V_T, I_T)} \cong \text{SpAzu}_{W_T, w_T}
\]

respectively.
The explicit computation of the morphism \( \Upsilon \) locally over \( X \) is an important step in proving the above theorem. To describe this, suppose that \( I \) is trivial and \( V \) is free of rank 3 over \( X \), so that we may fix a basis \( \{ \epsilon_1, \epsilon_2, \epsilon_3 \} \) for \( V \), which naturally gives rise to a basis of \( \text{Bil}_V \).

For any \( X \)-scheme \( T \), a \( T \)-valued point \( B \) of \( \text{Bil}_V \) is just a global bilinear form with values in \( \mathcal{O}_T \) on the pull-back \( V \otimes_X T \) of \( V \) to \( T \). Such a \( B \) is given uniquely by a \((3 \times 3)\)-matrix \( (b_{ij}) \) with the \( b_{ij} \) being global sections of the trivial line bundle \( \mathbb{A}^1_T \) (or equivalently, elements of \( \Gamma(T, \mathcal{O}_T) \)). The chosen basis for \( V \) also gives rise to the basis

\[
\{ \epsilon_0 := w = 1 ; \epsilon_1 := e_1 \wedge e_2 , \epsilon_2 := e_2 \wedge e_3 , \epsilon_3 := e_3 \wedge e_1 \}
\]

of \( W = \Lambda^{even}(V) \). A \( T \)-valued point \( A \) of \( \text{Assoc}_{W,w} \) is just a \( w_T := (w \otimes_X T) \)-unital associative algebra structure on the bundle \( W_T := W \otimes_X T \). Let \( \cdot_A \) denote the multiplication in the algebra bundle \( A \), and for ease of notation, let \( s^o \) denote the section \( s \otimes_X T \) induced from a section \( s \) (for example, \( (w \otimes_X T) = w^o, \epsilon_i \otimes_X T = \epsilon_i^o \) etc).

**Theorem 3.9** In addition to the hypothesis of Theorem 3.8, assume that \( V \) is free of rank 3 and that \( I = \mathcal{O}_X \). Then fixing a basis for \( V \) and adopting the notations above, the map \( \Upsilon(T) \) takes \( B = (b_{ij}) \) to \( (A, 1_A, \cdot_A) = (W_T, w_T = w^o, \cdot_A) \) with multiplication given as follows, where \( M_{ij}(B) \) is the determinant of the minor of the element \( b_{ij} \) in \( B \):

- \( \epsilon_1^o \cdot_A \epsilon_2^o = -M_{33}(B)w^o + (b_{21} - b_{12})\epsilon_1^o \)
- \( \epsilon_2^o \cdot_A \epsilon_2^o = -M_{11}(B)w^o + (b_{32} - b_{23})\epsilon_2^o \)
- \( \epsilon_3^o \cdot_A \epsilon_3^o = -M_{22}(B)w^o + (b_{13} - b_{31})\epsilon_3^o \)
- \( \epsilon_1^o \cdot_A \epsilon_2^o = -M_{31}(B)w^o - b_{23}\epsilon_1^o - b_{12}\epsilon_2^o - b_{22}\epsilon_3^o \)
- \( \epsilon_2^o \cdot_A \epsilon_3^o = +M_{12}(B)w^o - b_{33}\epsilon_1^o - b_{31}\epsilon_2^o - b_{23}\epsilon_3^o \)
- \( \epsilon_3^o \cdot_A \epsilon_1^o = +M_{23}(B)w^o - b_{31}\epsilon_1^o - b_{11}\epsilon_2^o - b_{12}\epsilon_3^o \)
- \( \epsilon_1^o \cdot_A \epsilon_3^o = +M_{32}(B)w^o + b_{13}\epsilon_1^o + b_{11}\epsilon_2^o + b_{21}\epsilon_3^o \)
- \( \epsilon_2^o \cdot_A \epsilon_1^o = -M_{13}(B)w^o + b_{32}\epsilon_1^o + b_{21}\epsilon_2^o + b_{22}\epsilon_3^o \)
- \( \epsilon_3^o \cdot_A \epsilon_2^o = -M_{21}(B)w^o + b_{33}\epsilon_1^o + b_{13}\epsilon_2^o + b_{32}\epsilon_3^o \)

The key to the proofs of Theorems 3.1, 3.4 and 3.5 lies in an analysis of a different identification of the scheme of specialisations, namely one related to the scheme of \( \mathcal{O}_X \)-valued quadratic forms on a trivial rank 3 bundle in the special situation when \( W \) is free and \( w \) part of a global basis. Without loss of generality we may in this situation therefore take \( V \) to be a free rank 3 vector bundle on \( X \) and

\[
(W, w) = (\Lambda^{even}(V), 1),
\]
so that we are in the situation of Theorem 3.9 above. This relationship with quadratic forms was shown in Theorem 5.3, Part A, [17], which we briefly recall next. Let $Quad_V$ denote the bundle of quadratic forms on $V$ (with values in $O_X$) and $Quad^r_V$ the open subscheme of semiregular quadratic forms. Let $A_0$ denote the algebra bundle structure (with unit $w = 1$) on $W = \Lambda^{even}(V)$ given by $\Lambda^{even}(V)$ itself. Fix a basis for $V$ and adopt the notations preceding Theorem 3.9 above. Then $Stab_w$ is the semidirect product of a commutative 3-dimensional subgroupscheme

$$L_w \cong (A^3_X, +)$$

with the stabiliser subgroupscheme $Stab_{A_0}$ of $A_0$ in $Stab_w$ (Lemma 5.1, Part A, [17]).

**Theorem 3.10 (Definition 5.2 & Theorem 5.3, Part A, [17])** There is a natural isomorphism

$$\Theta : Quad_V \times_X L_w \cong SpAzu_{W,w}$$

which maps the open subscheme $Quad^r_V \times_X L_w$ isomorphically onto the open subscheme $Azu_{W,w}$.

The isomorphism $\Theta$ was first defined by C. S. Seshadri in [14] for the case $X = Spec(k)$, $k$ an algebraically closed field of characteristic $\neq 2$. Section 5 is essentially devoted to studying $\Theta$. There we compute $\Theta$ explicitly and in Theorem 5.1 we write out the multiplication table of every specialised algebra structure on any fixed free rank 4 vector bundle with fixed unit that is part of a global basis. It turns out that $\Theta$ is not equivariant with respect to $GL_V$, but nevertheless satisfies a ‘twisted’ form of equivariance (Theorem 5.4). A $T$-valued point $q$ of

$$Quad_V \cong A^6_X$$

may be identified with a 6-tuple $(\lambda_1, \lambda_2, \lambda_3, \lambda_{12}, \lambda_{13}, \lambda_{23})$ corresponding to the quadratic form

$$(x_1, x_2, x_3) \mapsto \Sigma_i \lambda_i x_i^2 + \Sigma_{i<j} \lambda_{ij} x_i x_j.$$ 

A $T$-valued point $t$ of $L_w \cong (A^3_X, +)$ may be identified with a 3-tuple $(t_1, t_2, t_3)$ which corresponds to the valued point of $Stab_w$ given by the $(4 \times 4)$-matrix

$$\begin{pmatrix} 1 & t_1 & t_2 & t_3 \\ 0 & I_3 \end{pmatrix}$$

where $I_3$ is the $(3 \times 3)$-identity matrix. With these notations, the identification of Theorems 3.3 and 3.10 may be compared with that of the above Theorem 3.10 as follows.

**Theorem 3.11** The isomorphism $\Upsilon^{-1} \circ \Theta : Quad_V \times_X L_w \cong Bil_V$ takes the valued point

$$\left( q, t \right) = \left( (\lambda_1, \lambda_2, \lambda_3, \lambda_{12}, \lambda_{13}, \lambda_{23}), (t_1, t_2, t_3) \right)$$

as follows.
to the valued point $B = (b_{ij})$ given by

$$B = \begin{pmatrix}
\lambda_1 & t_1 & \lambda_{13} - t_3 \\
\lambda_{12} - t_1 & \lambda_2 & t_2 \\
t_3 & \lambda_{23} - t_2 & \lambda_3
\end{pmatrix}.$$  

Moreover, under this identification, the involution $B \mapsto (-B)^t$ on $\text{Bil}_V$ (induced from the isomorphism $\Upsilon$ of Theorem 3.8) translates into the involution on

$$\text{Quad}_V \times_X \mathcal{L}_w$$

given by

$$(q, (t_1, t_2, t_3)) \mapsto (-q, (t_1 - \lambda_{12}, t_2 - \lambda_{23}, t_3 - \lambda_{13})).$$

### 3.4 Degenerations of Azumaya Bundles as Quaternion Algebra Bundles

We next make some comments on specialised algebras. Let $R$ be a unital commutative ring and $A$ a unital associative $R$-algebra. Given any involution $\sigma$ on $A$, we may define the trace and norm associated to this involution by

$$\text{tr}_\sigma : x \mapsto x + \sigma(x) \quad \text{and} \quad n_\sigma : x \mapsto x\sigma(x).$$

In para.1.3, Chap.I, [9], Knus calls $\sigma$ standard if $\sigma$ fixes $R.1_A$ and both $\text{tr}_\sigma$ and $n_\sigma$ take values in $R.1_A$. In Prop.1.3.4 of the same chapter, he proves that a standard involution is unique if it exists, provided the $R$-module underlying $A$ is finitely generated projective and faithful.

In para.1.3.7, op.cit., Knus defines $A$ to be a quaternion algebra if $A$ is a projective $R$-module of rank 4 and $A$ has a standard involution. Thus we may define a rank 4 algebra bundle on a scheme $X$ to be a quaternion algebra bundle if it is locally (in the Zariski topology) a quaternion algebra in Knus’ sense, and it would follow that the local standard involutions glue to define a unique global standard involution on the bundle.

**Proposition 3.12** Any specialised algebra bundle is a quaternion algebra bundle.

This result can be deduced from the following two facts:

1. Any specialised algebra is locally (in the Zariski topology) the even Clifford algebra of a rank 3 quadratic bundle (assertion (3), Theorem 2.15 and Theorem 3.10).

2. The even Clifford algebra of a quadratic module of rank 3 over a commutative ring has a standard involution which is none other than the restriction of the ‘standard’ involution on the full Clifford algebra (Prop.3.1.1, Chap.V, [9]).

Q.E.D.
We hasten to remark that even over an algebraically closed field there are quaternion algebras that are not even-Clifford algebras of quadratic forms.

The proof of the bijection stated in Theorem 2.10 follows from Prop.3.2.3 and Prop.3.2.4, Chap.V, [9] generalised to the scheme-theoretic setting. We recall how the surjectivity is established.

Let \( A \) be a specialised algebra bundle on the scheme \( X \). By the results just quoted if \( A \) is Azumaya, or more generally by Prop.3.12, we have the existence of a unique standard involution \( \sigma_A \) on \( A \), to which are associated the norm

\[ n_{\sigma_A} : A \to \mathbb{A}^1_X \text{ given on sections by } x \mapsto x.\sigma_A(x) \]

and the trace

\[ tr_{\sigma_A} : A \to \mathbb{A}^1_X \text{ given on sections by } x \mapsto x + \sigma_A(x). \]

Let \( A' := \ker(tr_{\sigma_A}) \hookrightarrow A \) be the subsheaf of trace zero elements. As the calculations in para.3.2, Chap.V, [9] show, the trace map is surjective if \( A \) is itself an Azumaya algebra; if this is the case, then it is further shown there that the rank 3 quadratic bundle

\[ (V, q, I) := (A', n_{\sigma_A}|_{A'}, \mathcal{O}_X.1_A) \]

is semiregular and its even Clifford algebra \( C_0(V, q, I) \cong A \). However the above method of retrieving a canonical rank 3 quadratic bundle fails badly for specialised non-Azumaya algebras. Consider even the case of \( X = \text{Spec}(k) \) where \( k \) is a field of characteristic two and the Clifford algebra \( A = C_0(V, q) \) of a quadratic form \( q \) on \( V = k^3 \) which is a perfect square (i.e., a square of a linear form or equivalently a sum of squares). In this case an easy computation shows that the subspace \( A' \) of trace zero elements is the full space \( A \).

**Proposition 3.13** Let \( S \) be a commutative semilocal ring that is 2-perfect i.e., such that the square map

\[ S \to S : s \mapsto s^2 \]

is surjective, and \( V \) a free rank 3 \( S \)-module. Then the set of semiregular quadratic \( S \)-forms on \( V \) forms a single \( \text{GL}(V) \)-orbit; in other words, upto isometry, \( \exists \) only one semiregular quadratic \( S \)-module structure on \( V \).

**Corollary 3.14** Let \( S \) be a commutative local ring that is 2-perfect. Then any two rank 4 Azumaya \( S \)-algebras are isomorphic. If \( S \) were only semilocal, the conclusion still holds provided the identity elements for multiplication for each of the two Azumaya \( S \)-algebras can be completed to an \( S \)-basis.

The proof of the above Proposition will be given in [S]. In view of Theorem 2.10 taking \( X = \text{Spec}(S) \) with \( S \) as in Prop.3.13 proves the first assertion of the above corollary. The second may be deduced by an application of Theorem 3.10 along with Prop.6.3.
3.5 Non-existence of Azumaya Structures and Semiregular Forms

In this and the next subsections, we study what happens when we impose the condition on self-duality on the bundle underlying an algebra (or on one on which a quadratic form is defined). In this subsection, our aim is to indicate examples of rank 4 vector bundles on which there do not exist any global Azumaya structures; for these examples it also turns out that there do not exist global regular quadratic forms with values in the trivial line bundle. However, in these examples there do exist algebra structures which are Azumaya on a nontrivial dense open subscheme and there do exist quadratic forms with values in the trivial bundle which are semiregular on a dense open subscheme. We use Theorem 3.1 to obtain examples of rank three bundles on which there do not exist any global semiregular quadratic forms with values in certain line bundles; however, there do exist forms which are semiregular on a dense open subscheme. We shall only outline the results and the proofs will be given in [18].

In the following we let $X$ be a scheme and $W$ a rank $n^2$ vector bundle over $X$. The following result describes the behaviour of the locus where an algebra structure is Azumaya.

**Proposition 3.15 (Prop.3.3, Part A, [17])**

1. Let $T$ be an $X$-scheme and $A$ an associative unital algebra structure on $W_T := W \otimes_X T$. Then the subset

   $$U(T, A) := \{ t \in T \mid A_t \text{ is an Azumaya } \mathcal{O}_{T,t} \text{ - algebra} \}$$

   is an open (possibly empty) subset. When $U(T, A)$ is nonempty, denote by the same symbol the canonical open subscheme structure. Then if $f : T' \to T$ is an $X$-morphism such that the topological image intersects $U(T, A)$, then

   $$U(T', f^*(A) = A \otimes_T T') \cong U(T, A) \times_T T'$$

   as open subschemes of $T'$. Further $U(T, A) \hookrightarrow T$ is an affine morphism.

2. $U(T, A)$ is the maximal open subset restricted to which $A$ is Azumaya.

3. Further let $f : T' \to T$ be a morphism of $X$-schemes such that $f^*(A)$ is Azumaya. Then $f$ factors through the open subscheme $U(T, A)$ defined above.

Next we let $V$ denote a vector bundle and $I$ a line-bundle over $X$. The following result describes the behaviour of the locus where a quadratic form is good.

**Proposition 3.16**

1. Let $T$ be an $X$-scheme and $q : V_T \to I_T$ a quadratic bundle with $V_T := V \otimes_X T$ and $I_T := I \otimes_X T$. Then the subset

   $$U(T, q) := \{ t \in T \mid q \otimes \kappa(t) : V_T \otimes \kappa(t) \to I_T \otimes \kappa(t) \text{ is “good”} \},$$

   ...
where “good” means semiregular if $V$ is of odd rank and regular if $V$ is of even rank, is an open (possibly empty) subset. When $U(T, q)$ is nonempty, denote by the same symbol the canonical open subscheme structure. Then if $f : T' \to T$ is an $X$-morphism such that the topological image intersects $U(T, q)$, and if $f^*(q) := q_{T'} : V_{T'} \to I_{T'}$ is the induced quadratic bundle on $T'$, then

$$U(T', q_{T'}) \cong U(T, q) \times_T T'$$

as open subschemes of $T'$. Further $U(T, q) \inj T$ is an affine morphism.

2. $U(T, q)$ is the maximal open subset restricted to which $q$ is good.

3. Further let $f : T' \to T$ be a morphism of $X$-schemes such that $q_{T'}$ is good. Then $f$ factors through the open subscheme $U(T, q)$ defined above.

We now let $S$ be a scheme, and let $X \to S$ be an $S$-scheme which is proper, of finite-type and has connected fibers relative to $S$. Again, $W$ denotes a vector bundle on $X$, $A$ an algebra structure on $W$, and $q : V \to I$ a quadratic form on the vector bundle $V$ with values in the line bundle $I$.

**Theorem 3.17** (With the above notations)

1. Suppose $W$ is self-dual and $U(X, A) \to S$ is surjective. Then $U(X, A) = X$ i.e., $A$ is Azumaya.

2. Suppose that the rank of $W$ is $4$, that there exists an algebra structure $A'$ on $W$ such that $U(X, A') \to S$ is surjective and that $U(X, A') \neq X$. Then there does not exist any global Azumaya algebra structure on $W$. Neither does there exist any global regular quadratic form on $W$ with values in the trivial line bundle.

3. Let $V$ be of rank 3 and suppose that the underlying bundle of $C_0(V, q, I)$ is self-dual. If $U(X, q) \to S$ is surjective, then $q$ is semiregular.

4. Let $V$ be of rank 3 such that $U(X, q) \to S$ is surjective but $U(X, q) \neq X$. Then there does not exist any global semiregular quadratic form on $V$ with values in $I$. If $(L, h, J)$ is a twisted discriminant bundle, then there does not exist any global semiregular quadratic form on $V \otimes L$ with values in $I \otimes J$.

Assertion (3) follows from (1) and Theorem 3.1. The proofs may be reduced to the case $B = \text{Spec}(k)$ where $k$ is a field. In this case, assertions (1) and (2) may be restated as follows. Let $X$ be a connected proper scheme of finite type over a field and let $W$ be a vector bundle on $X$ of rank $n^2$ for some $n \geq 2$.

(a) Let $W$ be self-dual and $A$ an associative unital algebra structure on $W$. If a section to $\text{Assoc}_{W, w}$ over $X$ meets $\text{Azu}_{W, w}$ topologically, then it factors as a morphism through the open subscheme $\text{Azu}_{W, w}$, where $w := 1_A$ and $A$ corresponds to the given section;
Let the rank of $W$ be 4. If there is a section over $X$ of $\text{Assoc}_{W,w}$ that topologically meets both the open subscheme $\text{Azu}_{W,w}$ and its complement (with $w = 1_A$ where $A$ corresponds to the given section), then the $X$-schemes $\text{Assoc}_{W,w'}$ (with $w'$ global nowhere-vanishing) cannot have sections that land topologically inside $\text{Azu}_{W,w'}$ and hence in particular the $X$-schemes $\text{Azu}_{W,w'}$ have no sections over $X$.

We leave it to the reader to formulate similar statements corresponding to the other assertions of Theorem 3.17. We next indicate situations where the above results apply to give examples of rank 4 bundles that do not admit any Azumaya algebra structures and of rank 3 vector bundles which do not admit any semiregular form with values in certain line bundles. The examples occur naturally on certain moduli spaces of vector bundles over a relative curve. As remarked earlier, we quickly mention the relevant objects, but we shall not get into definitions, proofs etc, which will appear in [18].

We first recall some facts about Nagata Rings. The standard reference is Chap.12 of Matsumura’s book [11]. An integral domain $A$ is said to satisfy condition N-1 if its integral closure $A_K$ in its quotient field $K$ is a finite $A$-module. It is said to satisfy condition N-2 if for every finite extension field $L/K$, the integral closure $A_L$ of $A$ in $L$ is a finite $A$-module. The properties N-1 and N-2 are preserved under localisation and N-2 $\Rightarrow$ N-1 whereas noetherianness with N-1 $\Rightarrow$ N-2 only in char.0; there exists an example of Y. Akizuki of a noetherian domain of positive char. which is not N-1. A commutative ring $B$ is called a Nagata ring (pseudo-geometric ring in Nagata’s own terminology and universally japanese ring in Grothendieck’s) if it is noetherian and $B/p$ is N-2 for each prime $p$ of $B$. Every localisation of $B$ and every finitely generated (commutative) $B$-algebra are then also Nagata, and complete noetherian local rings are Nagata as well. Dedekind domains of characteristic zero such as $\mathbb{Z}$ are Nagata.

Let $S$ be a normal integral scheme, which we shall assume is locally-universally Japanese (Nagata). This means that there is a covering of $S$ by affine open subschemes that are isomorphic to spectra of Nagata rings. Such an assumption on $S$ is necessary to be able to obtain finite-type quotients by reductive group schemes following Seshadri’s Geometric Invariant Theory over a General Base as in [15].

Let $C \to S$ be smooth projective of relative dimension 1 (i.e., a curve relative to $S$) with geometric fibres irreducible and of constant genus $g \geq 2$. Let $\mathcal{U}^{ss}_C(2,0)$ denote the Seshadri moduli space of semistable vector bundles on $C$ of rank 2 and degree zero, generalising the construction in [13] to the base $S$ using the methods of [15]. It is a coarse moduli scheme, obtained as a Geometric Invariant Theory quotient, and is a normal geometrically irreducible projective scheme of relative dimension $4g - 3$ over $S$. Let $\mathcal{U}^s_C(2,0) \subset \mathcal{U}^{ss}_C(2,0)$ denote the open subscheme corresponding to stable vector bundles. Then $\mathcal{U}^s_C(2,0)$ is precisely the locus where $\mathcal{U}^{ss}_C(2,0) \to S$ is smooth, unless $g = 2$ in which case $\mathcal{U}^{ss}_C(2,0) \to S$ is itself smooth. The Seshadri-desingularisation

$$\pi_2 : N_C^s(4,0) \to \mathcal{U}^{ss}_C(2,0)$$

may be constructed, extending the case $S = \text{Spec}(k)$, $k$ a field, carried out in [13].
for $\text{Char.}(k) \neq 2$ and in a characteristic-free way in § 6, Part B of [17]. $\pi_2$ is an isomorphism over the open subscheme $C_\mathbb{C}(2, 0)$. The scheme $N_C(4, 0)$ appears as a fine moduli space for certain parabolic stable vector bundles of rank 4 and degree 0 on $C$ and it turns out to be a geometrically irreducible smooth projective scheme relative to $S$. By the very construction of $N_C(4, 0)$, there is a naturally defined rank 4 vector bundle $W$ on $N_C(4, 0)$ which is equipped with an associative unital algebra structure $A$. Further, the locus where $A$ is Azumaya is precisely the dense open subscheme $N_C(4, 0) := \pi_2^{-1}(C_\mathbb{C}(2, 0))$. Let $(V, q, I)$ be any ternary quadratic bundle on $N_C(4, 0)$ such that $C_0(V, q, I) \cong A$ (such a choice exists by the surjectivity part of Theorem 3.1). Applying Theorem 3.17 to the present situation gives the following.

**Theorem 3.18** (With the above notations:)

1. $W$ does not admit any (global) Azumaya algebra structure.

2. There does not exist any global regular quadratic form on $W$ with values in the trivial line bundle.

3. There does not exist any global semiregular quadratic form on $V$ with values in $I$. If $(L, h, J)$ is a twisted discriminant bundle, there does not exist any global semiregular quadratic form on $V \otimes L$ with values in $I \otimes J$.

### 3.6 Algebra Structures on Self-Dual Bundles

In this subsection, we investigate again the effect of the condition of self-duality on the vector bundle underlying an algebra. Our aim is to study the Picard group of the scheme of specialisations of Azumaya structures on a fixed rank 4 vector bundle over suitable schemes. Recall that an integral separated noetherian scheme is said to be locally-factorial if each of its local rings is a unique factorisation domain ($=\text{UFD}=\text{factorial ring}$). The proofs of the following results will be given in [17].

**Theorem 3.19** Let $X$ be a scheme and $W$ a rank 4 vector bundle on $X$ with a global nowhere-vanishing section $w$. Let $D_X$ denote the closed subset

$$\text{SpAzu}_{W, w} \backslash \text{Azu}_{W, w}.$$

(a1) $X$ is irreducible iff $\text{SpAzu}_{W, w}$ is irreducible iff $\text{Azu}_{W, w}$ is irreducible iff $D_X$ is irreducible.

(a2) The set of irreducible components of $X$ is locally finite—for example this happens when $X$ is locally noetherian—iff the same is true of the corresponding set for $\text{SpAzu}_{W, w}$ or for $\text{Azu}_{W, w}$.

(a3) If $X$ is noetherian and finite-dimensional then the same are true for $\text{SpAzu}_{W, w}$ and $\text{Azu}_{W, w}$.
(b) If \( X' \rightarrow X \) is a morphism of schemes, and if \((W', w')\) denotes the pullback of \((W, w)\), then we have a canonical isomorphism

\[
\text{Azu}_{W', w'} \cong \text{Azu}_{W, w} \times_{\text{SpAzu}_{W, w}} \text{SpAzu}_{W', w'}
\]

In particular, the topological image of \( D_{X'} \) is \( D_X \). Moreover, when \( X' \rightarrow X \) is a homeomorphism onto its topological image—which is for example the case when it is a closed or an open immersion, then

\[
D_X \cap \text{SpAzu}_{W', w'}
\]

can be identified with \( D_{X'} \).

(c) If \( X \) is a scheme which is finite dimensional and whose set of irreducible components is locally finite, then the closed subset \( D_X \) is a divisor i.e., it has codimension 1 in \( \text{SpAzu}_{W, w} \).

(d) \( X \) is affine iff \( \text{SpAzu}_{W, w} \) is affine iff \( \text{Azu}_{W, w} \) is affine. If \( X \) is regular in codimension 1 (respectively locally-factorial) then so are \( \text{SpAzu}_{W, w} \) and \( \text{Azu}_{W, w} \).

(e) Assume that \( X \) is locally-factorial. Then the canonical homomorphism

\[
\text{Pic}(X) \rightarrow \text{Pic}(\text{SpAzu}_{W, w})
\]

is an isomorphism.

(f) Assume that \( X \) is locally-factorial and \( W \) is self-dual (i.e., \( W \cong W^\vee \)). Then the (Weil) divisor \( n.(D_X) \) is principal for some positive integer \( n \), so that the natural homomorphism given by restriction of line bundles

\[
\text{Pic}(\text{SpAzu}_{W, w}) \rightarrow \text{Pic}(\text{Azu}_{W, w})
\]

is an isomorphism iff \( n = 1 \). In this case, for any \( X \)-scheme \( T \), any specialised algebra structure on \( W \otimes_X T \) arises from a quadratic form with values in the trivial line bundle.

(g) Assume that \( X \) is locally-factorial and that there exists an Azumaya algebra structure on \( W \) with unit \( w \). Then the canonical homomorphisms

\[
\text{Pic}(X) \rightarrow \text{Pic}(\text{Azu}_{W, w}) \text{ and } \text{Pic}(\text{SpAzu}_{W, w}) \rightarrow \text{Pic}(\text{Azu}_{W, w})
\]

are isomorphisms and the divisor \( D_X \) is principal.

3.7 Stratification of the Variety of Specialisations

Let \( W \) be a rank 4 vector bundle on a scheme \( X, w \in \Gamma(X, W) \) a nowhere-vanishing global section and \( \text{Stab}_w \subset \text{GL}_W \) the stabiliser subgroupscheme of \( w \). Recall from page 29 that the natural inclusion

\[
\text{Azu}_{W, w} \hookrightarrow \text{SpAzu}_{W, w}
\]

is \( \text{Stab}_w \)-equivariant. When \( X = \text{Spec}(k) \) where \( k \) is an algebraically closed field, there is a canonical \( \text{Stab}_w \)-stratification of the \( k \)-variety underlying \( \text{SpAzu}_{W, w} \) as follows (the proof will be given in [3].
Theorem 3.20

(1) Let \( k \) be a quadratically closed field and \( X = \text{Spec}(k) \). Then the set of ternary quadratic modules up to similarity has 4 elements which correspond to

(a) semiregular quadratic modules;
(b) rank 2 quadratic modules i.e., those that are not semiregular but which are regular on a two-dimensional subspace;
(c) nonzero perfect squares (= squares of linear forms) and
(d) the zero form.

If \( V \) is a 3-dimensional vector space over \( k \) and \{\( e_1, e_2, e_3 \)\} a \( k \)-basis for \( V \), then representatives for these 4 \( \text{GL}_V \)-orbits in the space \( \text{Quad}_V \) of quadratic forms on \( V \) can respectively be taken to be:

(a) \( q^{(1)}(\sum_{i=1}^3 x_i e_i) = x_1 x_2 + x_3^2; \)
(b) \( q^{(2)}(\sum_{i=1}^3 x_i e_i) = x_1 x_2; \)
(c) \( q^{(3)}(\sum_{i=1}^3 x_i e_i) = x_3^2 \) and
(d) \( q^{(4)} = 0. \)

(2) In addition to the hypotheses and notations of (1) above, assume that \( k \) is an algebraically closed field. Then the four orbits

\[ \text{Quad}_V^{(i)} := \text{GL}_V \cdot q^{(i)} \text{ for } 1 \leq i \leq 4 \]

form a stratification of the \( k \)-variety \( \text{Quad}_V \) in the sense that we have

\[ \text{Quad}_V^{(i)} = \text{Quad}_V \text{ and } \text{Quad}_V^{(i+1)} = \text{Quad}_V^{(i)} \setminus \text{Quad}_V^{(i)}, \text{ 1 \leq i \leq 3} \]

and further we also have

\[ \text{Sing}(\text{Quad}_V^{(i+1)}) = \text{Quad}_V^{(i+1)} \setminus \text{Quad}_V^{(i+1)} \text{ for } 1 \leq i \leq 2 \]

unless the characteristic of \( k \) is 2 in which case \( \text{Quad}_V^{(3)} \) is itself smooth (the notation \( T \) denotes the orbit closure and \( \text{Sing}(T) \) denotes the subset of singular (non-smooth) points of \( T \), each given the canonical reduced induced closed subscheme structure).

(3) Continuing with the notations and hypotheses of (2) above, set

\[ (W, w) := (\Lambda^{\text{even}}(V), 1). \]

For ease of notation denote \( \text{SpAz}_W \) by \( \text{SpAz}_W \) and \( \text{Stab}_W \) by \( H. \) Then the four orbits

\[ \text{SpAz}_W^{(i)} := H \cdot \Theta(q^{(i)}, I_4) \text{ for } 1 \leq i \leq 4 \]
form a stratification of the $k$-variety $\text{SpA}_{zu}$ in the sense that we have

$$\text{SpA}_{zu}^{(1)} = \text{SpA}_{zu} \quad \text{and} \quad \text{SpA}_{zu}^{(i+1)} = \text{SpA}_{zu}^{(i)} \setminus \text{SpA}_{zu}^{(i)}, \quad 1 \leq i \leq 3$$

and further we also have

$$\text{Sing}(\text{SpA}_{zu}^{(i+1)}) = \text{SpA}_{zu}^{(i)} \setminus \text{SpA}_{zu}^{(i+1)} \quad \text{for} \quad 1 \leq i \leq 2$$

unless the characteristic of $k$ is 2 in which case $\text{SpA}_{zu}^{(3)}$ is itself smooth.

4 Injectivity: Reduction to Lifting to Similarities in the Free Case

Proof of Prop. 3.2. Start with an isomorphism of algebra-bundles

$$\phi : C_0(V, q, I) \cong C_0(V', q', I').$$

Let $\{U_i\}_{i \in I}$ be an affine open covering of $X$ (which may also be chosen so as to trivialise some or any of the involved bundles if needed). Choose bilinear forms

$$b_i \in \Gamma(U_i, \text{Bil}(V, I)) \quad \text{and} \quad b'_i \in \Gamma(U_i, \text{Bil}(V', I'))$$

such that

$$q|_{U_i} = q_{b_i} \quad \text{and} \quad q'|_{U_i} = q'_{b'_i} \quad \text{for each} \quad i \in I.$$

By (2d), Theorem 2.2, we have isomorphisms of vector bundles $\psi_{b_i}$ and $\psi_{b'_i}$, which preserve 1 by (2a) of the same Theorem, and we define the isomorphism of vector bundles $(\phi_{\Lambda^{+\circ}})_i$ so as to make the following diagram commute:

$$\begin{array}{ccc}
C_0(V, q, I)|_{U_i} & \xrightarrow{\phi|_{U_i}} & C_0(V', q', I')|_{U_i} \\
\psi_{b_i} & \cong & \psi_{b'_i} \\
\left(\mathcal{O}_X \oplus \Lambda^2(V) \otimes I^{-1}\right)|_{U_i} & \xrightarrow{\cong_{(\phi_{\Lambda^{+\circ}})_i}} & \left(\mathcal{O}_X \oplus \Lambda^2(V') \otimes (I')^{-1}\right)|_{U_i}
\end{array}$$

The linear isomorphism $(\phi_{\Lambda^{+\circ}})_i$ preserves 1 and therefore it induces a linear isomorphism

$$(\phi_{\Lambda^{+\circ}})_i : (\Lambda^2(V) \otimes I^{-1})|_{U_i} \xrightarrow{\cong} (\Lambda^2(V') \otimes (I')^{-1})|_{U_i}.$$}

Observe that $(\phi_{\Lambda^{+\circ}})_i$ is independent of the choice of the bilinear forms $b_i$ and $b'_i$. For, replacing these respectively by $\widehat{b}_i$ and $\widehat{b}'_i$, it follows from (2f), Theorem 2.2, that

$$\psi_{b_i} \circ (\psi_{\widehat{b}_i})^{-1} \quad (\text{resp.} \quad \psi_{b'_i} \circ (\psi_{\widehat{b}'_i})^{-1})$$

followed by the canonical projection onto

$$(\Lambda^2(V) \otimes I^{-1})|_{U_i} \quad (\text{resp. onto} \quad (\Lambda^2(V') \otimes (I')^{-1})|_{U_i})$$
is the same as the projection itself. By this observation, it is also clear that the
isomorphisms \( \{(\phi_\Lambda^2)_{i} \}_{i \in I} \) agree on (any open affine subscheme of, and hence on
all of) any intersection \( \tilde{U}_i \cap \tilde{U}_j \). Therefore they glue to give a global isomorphism
of vector bundles
\[
\phi_\Lambda^2 : \Lambda^2(V) \otimes \mathbb{I}^{-1} \cong \Lambda^2(V') \otimes (I')^{-1}
\]
as required. \textbf{Q.E.D, Prop.3.2}

\textbf{Reduction of Proof of Injectivity of Theorem 3.1 to Theorem 3.4}

We start with an isomorphism of algebra-bundles
\[
\phi : C_0(V, q, I) \cong C_0(V', q', I'),
\]
construct the isomorphism of vector bundles
\[
\phi_\Lambda^2 : \Lambda^2(V) \otimes \mathbb{I}^{-1} \cong \Lambda^2(V') \otimes (I')^{-1}
\]
and keep the notations introduced in the proof of Prop.3.2. Firstly we deduce a
linear isomorphism
\[
\det((\phi_\Lambda^2)')^{-1} : \det((\Lambda^2(V) \otimes \mathbb{I}^{-1})') \cong \det((\Lambda^2(V') \otimes (I')^{-1})').
\]
Since \( V \) and \( V' \) are of rank 3, there are canonical isomorphisms
\[
\eta : \Lambda^2(V) \equiv V^\vee \otimes \det(V) \quad \text{and} \quad \eta' : \Lambda^2(V') \equiv (V')^\vee \otimes \det(V').
\]
It follows therefore that if we set
\[
L := \det(V') \otimes (\det(V))^{-1} \quad \text{and} \quad J := I' \otimes \mathbb{I}^{-1}
\]
then we get a twisted discriminant line bundle \( (L \otimes \mathbb{I}^{-1}, h, J) \) and a vector bundle
isomorphism
\[
\alpha : V' \cong V \otimes (L \otimes \mathbb{I}^{-1}).
\]
Now for each \( i \in I \), the bilinear form \( b_i \in \Gamma(U_i, \text{Bil}(V, I)) \) induces, via \( \alpha|_U \) and
\( (L \otimes \mathbb{I}^{-1}, h, J)|_{U_i} \) and (1), Prop.2.7, a bilinear form
\[
b_i'' \in \Gamma(U_i, \text{Bil}(V', I)).
\]
By (3) of the same Proposition, over each \( U_i \) we get an isometry of bilinear form bundles
\[
\alpha|_{U_i} : (V'|U_i, b_i''', IJ|_{U_i}) \cong (V|U_i, b_i, I|_{U_i}) \otimes (L \otimes \mathbb{I}^{-1}, h, J)|_{U_i}
\]
and also an isometry of quadratic bundles
\[
\alpha|_{U_i} : (V'|U_i, q''', IJ|_{U_i}) \cong (V|U_i, q_i = q|_{U_i}, I|_{U_i}) \otimes (L \otimes \mathbb{I}^{-1}, h, J)|_{U_i}.
\]
On the other hand, by an assertion in (3), Prop.2.7 we could also define the global
quadratic bundle \( (V', q'', IJ) \) using \( (V, q, I), \alpha \) and \( (L \otimes \mathbb{I}^{-1}, h, J) \), so that we have
an isometry of quadratic bundles
\[
\alpha : (V', q'', IJ) \cong (V, q, I) \otimes (L \otimes \mathbb{I}^{-1}, h, J).
\]
It follows therefore that the \( q_{b''} \) glue to give \( q'' \). Notice that in general the \( b''_i \) (resp. the \( b_i \)) need not glue to give a global bilinear form \( b'' \) (resp. \( b \)) such that \( q_{b''} = q'' \) (resp. \( q_b = q \)). By (1), Prop. 2.8, there exists a unique isomorphism of algebra bundles

\[
C_0(\alpha, 1, IJ) : C_0(V', q'', IJ) \cong C_0((V, q, I) \otimes (LJ^{-1}, h, J))
\]

and by Prop. 2.9 we have a unique isomorphism of algebra bundles

\[
\gamma_{(LJ^{-1}, h, J)} : C_0((V, q, I) \otimes (LJ^{-1}, h, J)) \cong C_0(V, q, I).
\]

Therefore the composition of the following sequence of isomorphisms of algebra bundles on \( X \)

\[
C_0(V', q'', I') \xrightarrow{\text{similarity}} C_0(V', q'', IJ) \xrightarrow{\phi^{(\otimes)}} C_0(V', q, I') \xrightarrow{\gamma_{(LJ^{-1}, h, J)}} C_0(V, q, I)
\]

is an element of

\[
\text{Iso}[C_0(V', q'', I'), C_0(V', q', I')],
\]

which, granting Theorem 3.4, is induced by a similarity

\[
(g, l) \in \text{Sim}[(V', q'', I'), (V', q', I')].
\]

Hence we would have

\[
g : (V', q'', I') \cong (V', q', I') \otimes (\mathcal{O}_X, (s \otimes s' \mapsto s.s'.l^{-1}), \mathcal{O}_X)
\]

where \( l \in \Gamma(X, \mathcal{O}_X^*) \). This combined with the fact that \((V, q, I)\) and

\[
(V', q'', I') = (V', q'', IJ)
\]

are isomorphic up to the twisted discriminant bundle \((LJ^{-1}, h, J)\) by the construction above, would imply that \((V, q, I)\) and \((V', q', I')\) also differ by a twisted discriminant bundle. Therefore the proof of the injectivity asserted in Theorem 3.1 reduces to the proof of Theorem 3.4.

**Reduction of Theorem 3.4 to the Case when \( I \) is free.** For a similarity \( g \) with multiplier \( l \), we have \( C_0(g, l, I) \) given by (1), Prop. 2.8, so that we may define the map

\[
\text{Sim}[(V, q, I), (V, q', I)] \longrightarrow \text{Iso}[C_0(V, q, I), C_0(V, q', I)] : g \mapsto C_0(g, l, I).
\]

The equality

\[
\det(C_0(g, l, I)) = \det((C_0(g, l, I))_{\lambda^3}) = l^{-3}\det^2(g)
\]

will be shown to hold (locally, hence globally) in (1), Lemma 5.11 page 59. Thus

\[
\text{Iso}[(V, q, I), (V, q', I)] \text{ and } \text{S-Iso}[(V, q, I), (V, q', I)]
\]
are respectively mapped into 

\( \text{Iso}'[C_0(V, q, I), C_0(V, q', I)] \) and \( \text{S-Iso}[C_0(V, q, I), C_0(V, q', I)] \) as claimed. We start with an isomorphism of algebra-bundles

\[ \phi: C_0(V, q, I) \cong C_0(V, q', I), \]

which by Prop 3.2 leads to the automorphism of vector bundles

\[ \phi_{\Lambda^2} \in \text{Aut}(\Lambda^2(V) \otimes I^{-1}). \]

Firstly, define the global bundle automorphism

\[ g' \in \text{GL} \left( V \otimes (\text{det}(V))^{-1} \otimes I \right) \]

so that the following diagram commutes

\[
\begin{array}{ccc}
(\Lambda^2(V))^\vee \otimes I & \xrightarrow{(\eta')^{-1} \otimes I} & (\Lambda^2(V))^\vee \otimes I \\
\eta' \equiv & \equiv & \equiv \\
V \otimes (\text{det}(V))^{-1} \otimes I & \xrightarrow{g'} & V \otimes (\text{det}(V))^{-1} \otimes I
\end{array}
\]

where \( \eta: \Lambda^2(V) \equiv V^\vee \otimes \text{det}(V) \) is the canonical isomorphism (since \( V \) is of rank 3). Now let

\[ g \in \text{GL}(V) \leftarrow \text{GL}(V \otimes (\text{det}(V))^{-1} \otimes I) \]

be the image of \( g' \) i.e., the image of \( g' \otimes \text{det}(V) \otimes I^{-1} \) under the canonical identification

\[ \text{GL}(V \otimes (\text{det}(V))^{-1} \otimes I \otimes \text{det}(V) \otimes I^{-1}) \equiv \text{GL}(V). \]

Next, let \( l \in \Gamma(X, \mathcal{O}_X^*) \) be a global section such that

\[ \gamma(l) := (l^3).\text{det}(\phi_{\Lambda^2}) \]

has a square root in \( \Gamma(X, \mathcal{O}_X^*). \) For example, we have the following independent special cases when this is true:

**Case 1.** If \( \phi \in \text{S-Iso}[C_0(V, q, I), C_0(V, q', I)] \) i.e., if \( \text{det}(\phi_{\Lambda^2}) = 1 \), then set \( l = 1 \) and \( \sqrt{\gamma(l)} = 1. \)

**Case 2.** If \( \phi \in \text{Iso}'[C_0(V, q, I), C_0(V, q', I)] \) i.e., \( \text{det}(\phi_{\Lambda^2}) \) is a square, then set \( l = 1 \) and let \( \sqrt{\gamma(l)} \) denote any fixed square root of \( \text{det}(\phi_{\Lambda^2}). \)

**Case 3.** Given an integer \( k \), take \( l = (\text{det}(\phi_{\Lambda^2}))^{2k+1} \) and let \( \sqrt{\gamma(l)} \) denote any fixed square root of \( (\text{det}(\phi_{\Lambda^2}))^{6k+4}. \)
For each integer $k$, we now associate to $\phi$ the element

$$g_1^\phi := (l^{-1} \sqrt{\gamma(l)}) g$$

with $g$ as defined above. We shall show the following locally for the Zariski topology on $X$ (more precisely, for each open subscheme of $X$ over which $V$ and $I$ are free):

1. that $g_1^\phi$ is an $I$-similarity from $(V, q, I)$ to $(V, q', I)$ with multiplier $l$ (Lemma 5.9, page 57);

2. that $g_1^\phi$ induces $\phi$, i.e., with the notations of (1), Prop. 2.8, that $C_0(\phi^\Lambda_2) = \phi$ (Lemma 5.10, page 58);

3. that $\det(g_1^\phi) = \sqrt{\gamma(l)}$ so that $\det^2(g_1^\phi) = \det(\phi_2^\Lambda)$ in cases 1 and 2 (Lemma 5.8, page 57) and

4. that the map

$$S\text{-Iso}[V, q, I], (V, q, I) \to S\text{-Iso}[C_0(V, q, I), C_0(V, q', I)]$$

is injective (Lemma 5.12, page 60).

It would follow then that these statements are also true globally. The maps

$$s_{2k+1} : \phi \mapsto g_1^\phi$$

and

$$s' : \phi \mapsto g_1^\phi$$

will then give the sections to the maps (which would imply their surjectivities) as mentioned in Theorem 3.4. But these maps are not necessarily multiplicative since a computation reveals that if

$$\phi_i \in \text{Iso}(C_0(V, q_i, I), C_0(V, q_i+1, I))$$

is associated to

$$g_i^\phi \in \text{Sim}[(V, q_i, I), (V, q_i+1, I)]$$

and $\phi_2 \circ \phi_1$ to $g_{l_21}^\phi \circ g_{l_11}^\phi$, then

$$g_{l_21}^\phi \circ g_{l_11}^\phi = \delta g_{l_21}^\phi \circ g_{l_11}^\phi$$

for $\delta \in \mu_2(\Gamma(X, \mathcal{O}_X))$ because of the ambiguity in the initial global choices of square roots for $\gamma(l_i)$ and $\gamma(l_{21})$. However this can be remedied as follows. For any given $\phi \in \text{Iso}(C_0(V, q, I), C_0(V, q', I))$, irrespective of whether or not $\det(\phi_2^\Lambda)$ is a square, take

$$l = (\det(\phi_2^\Lambda))^{2k+1}, \gamma(l) = l^3 \det(\phi_2^\Lambda), \sqrt{\gamma(l)} := (\det(\phi_2^\Lambda))^{3k+2}$$

and

$$s_{2k+1}^+(\phi) := g_1^\phi = \left(l^{-1} \sqrt{\gamma(l)}\right) g.$$
5 The Free Case: Investigation of the Isomorphism Theta

Throughout this section, we work with \( I = \mathcal{O}_X \) and shorten our earlier notations \((V, q, I), C_0(V, q, I), C_0(g, l, I)\) etc respectively to \((V, q), C_0(V, q), C_0(g, l)\) etc. We conclude the proofs of the injectivity of Theorem 3.1 and Theorem 3.4 which were begun in §4 and also prove Theorem 3.5.

As means to these ends, we carry out two explicit computations. Firstly we compute the isomorphism \( \Theta \) of Theorem 3.10. This provides us with the multiplication table of every specialised algebra structure on any fixed free rank 4 vector bundle with fixed unit which is part of a global basis (Theorem 5.1 below). This result will also be used in §6 in the proof of Theorem 3.11. It turns out that \( \Theta \) is not equivariant with respect to \( \text{GL}_V \), but nevertheless satisfies a ‘twisted’ form of equivariance (Theorem 5.4). Secondly, we explicitly compute the algebra bundle isomorphism
\[
C_0(g, l) : C_0(V, q) \cong C_0(V, q')
\]
of (1). Prop. 2.8 induced by a similarity \( g : (V, q) \cong (V, q') \) with multiplier \( l \in \Gamma(X, \mathcal{O}_X \otimes \mathcal{O}_X) \) in the case when \( V \) is free of rank 3 (Theorem 5.5).

5.1 The Action of \( \text{GL} \) on Forms

Let \( V \) be a vector bundle over a scheme \( X \) with associated locally-free sheaf \( \mathcal{V} \). The \( X \)-smooth \( X \)-groupscheme \( \text{GL}_V \) acts naturally on the left on the sheaves \( \text{Alt}^2_V, \text{Bil}_V, \text{Quad}_V \) of alternating, bilinear and quadratic forms on \( V \) (with values in \( \mathcal{O}_X \)). Namely, for \( U \hookrightarrow X \) an open subscheme, and for
\[
b \in \Gamma(U, \text{Bil}_V) \quad (\text{resp. } a \in \Gamma(U, \text{Alt}^2_V), \text{ resp. } q \in \Gamma(U, \text{Quad}_V)),
\]
and for \( g \in \Gamma(U, \text{GL}_V) = \text{GL}(V|U) \), the corresponding form of the same type \( g.b \) (resp. \( g.a \), resp. \( g.q \)) is defined on sections (over open subsets of \( U \)) by
\[
(g.b)(v, v') := b(g^{-1}(v), g^{-1}(v')) \quad (\text{resp. } (g.a)(v, v') := a(g^{-1}(v), g^{-1}(v'))),
\]
\[
\quad \text{resp. } (g.q)(v) := q(g^{-1}(v))).
\]

It is immediate that the following short-exact-sequence of sheaves, indicated in §2.4 is equivariant with respect to this action:
\[
\begin{array}{c}
\text{(♠)} \quad 0 \longrightarrow \text{Alt}^2_V \longrightarrow \text{Bil}_V \longrightarrow \text{Quad}_V \longrightarrow 0.
\end{array}
\]

Equivalently, the \( X \)-groupscheme \( \text{GL}_V \) acts on the corresponding geometric vector bundles such that both of the \( X \)-morphisms of \( X \)-vector bundles in the following sequence are \( \text{GL}_V \)-equivariant:
\[
\text{Alt}^2_V \leftrightarrow \text{Bil}_V \leftrightarrow \text{Quad}_V.
\]

Notice that it is one and the same thing to require that
\[
\text{GL}(V|U) \ni g : (V|U, q) \cong (V|U, q')
\]
be a similitude with multiplier \( l \in \Gamma(U, \mathcal{O}_X) \), and to require that \( g.q = l^{-1}q' \).
5.2 Computation of the Isomorphism Theta

We briefly recall the definition of $\Theta$ from Part A of [17]. We keep the notations introduced just before Theorem 3.10; for ease of notation, the pullback of a section $s$ (of a vector bundle or its associated sheaf) is denoted by $s^\circ$. Since $V$ is free of rank 3 on $X$, we choose an identification
\[ V \equiv \mathcal{O}_X.e_1 \oplus \mathcal{O}_X.e_2 \oplus \mathcal{O}_X.e_3. \]

This gives the identification of the dual bundle as
\[ V^\vee \equiv \mathcal{O}_X.f_1 \oplus \mathcal{O}_X.f_2 \oplus \mathcal{O}_X.f_3 \]
(defined uniquely by $f_i(e_j) = \delta_{ij}$, the Kronecker delta). Therefore the dual of the sheaf of quadratic forms on $V$, which is
\[ (\text{Quad}_V)^\vee := (\text{Bil}_V/\text{Alt}_V^2)^\vee = ((T^2V)^\vee/(\Lambda^2V)^\vee), \]
has global $\mathcal{O}_X$-basis given by
\[ \{e_i \otimes e_i; (e_i \otimes e_j + e_j \otimes e_i)\}. \]

This leads to an identification of the associated sheaf of symmetric algebras
\[ \text{Sym}_{\mathcal{O}_X}(\text{Quad}_V^\vee) \equiv \mathcal{O}_X[Y_1, Y_2, Y_3, Y_{12}, Y_{13}, Y_{23}], \]
where $e_i \otimes e_i \equiv Y_i$ and $e_i \otimes e_j + e_j \otimes e_i \equiv Y_{ij}$, and therefore
\[ \text{Quad}_V := \text{Spec} \left( \text{Sym}_{\mathcal{O}_X}(\text{Quad}_V^\vee) \right) \equiv \text{Spec}(\mathcal{O}_X[Y_1, Y_2, Y_3, Y_{12}, Y_{13}, Y_{23}]) = \mathbb{A}^6_X. \]

Consider the universal quadratic bundle $(V, q)$ where $V$ is the pullback of $V$ by $\text{Quad}_V \to X$. The universal quadratic form $q$ is given by
\[ (x_1, x_2, x_3) \mapsto \Sigma_i Y_i(x_i)^2 + \Sigma_{i<j} Y_{ij}.x_i.x_j \]
and moreover the global bilinear form on $V$ given by
\[ b(q) : ((x_1, x_2, x_3), (x'_1, x'_2, x'_3)) \mapsto \Sigma_i Y_i.x_i.x'_i + Y_{12}.x_2.x'_1 + Y_{23}.x_3.x'_2 + Y_{13}.x_1.x'_3 \]
induces $q$ (the bilinear form ‘associated in the usual sense’ to $q$, viz. $b_q$ is not $b(q)$ but in fact its symmetrisation). Therefore, by (2d), Theorem 2.2, we get an isomorphism of vector bundles
\[ \psi_{b(q)} : C_0(V, q = b_q) \cong \Lambda^\text{even}(V) =: W \]
which, according to (2a) and (2f) of the same Theorem, carries the ordered Poincaré-Birkhoff-Witt basis
\[ \{1; e_1^o, e_2^o, e_3^o, e_4^o, e_5^o, e_6^o\} \]
The choices \( e_3^0, e_1^0 \) and \( e_3^0 \wedge e_1^0 \) instead of the usual \( e_1^0, e_3^0 \) and \( e_1^0 \wedge e_3^0 \) are deliberate—for example, \( \psi_{b(q)} \) would carry
\[
\{e_1^0, e_2^0, e_3^0, e_1^0, e_3^0\}
\]
on to
\[
\{w^0 = 1^o = 1; e_1^0 \wedge e_2^0, e_2^0 \wedge e_3^0, e_1^0 \wedge e_3^0 + Y_{13}.w^0\}
\]
which depends on \( Y_{13} \). Thus the even Clifford algebra bundle \( C_0(V, q = q_{b(q)}) \) induces via \( \psi_{b(q)} \) a \( w^0 \)-unital algebra structure on the pullback bundle \( W \) of
\[
W := \Lambda^{\text{even}}(V) \]
where \( w \) corresponds to 1 in \( \Lambda^{\text{even}}(V) \). But by definition, this algebra structure corresponds precisely to an \( X \)-morphism
\[
\theta: \text{Quad}_V \longrightarrow \text{Id-w-Sp-Azu}_W.
\]
The isomorphism \( \Theta \) is now given by the composition of the following \( X \)-morphisms (cf. Def.5.2, Part A, [17]):
\[
\begin{align*}
\text{Quad}_V \times_X L_w & \xrightarrow{\theta \times \text{Id}} \text{Id-w-Sp-Azu}_W \times_X L'_w \\
L'_w \times_X \text{Id-w-Sp-Azu}_W & \xrightarrow{\text{ACTION}} \text{Id-w-Sp-Azu}_W.
\end{align*}
\]
The association of \( q \) with \( b(q) \) also defines a splitting of the exact sequence (\( \star \)) of page 40 above, so that more generally, given a valued point \( q \in (\text{Quad}_V)(T) \), we may associate uniquely a valued point \( b(q) \in (\text{Bil}_V)(T) \) which induces it. That this association is not \( \text{GL}_V \)-equivariant is reflected in the lack of equivariance of the isomorphism \( \Theta \) (Theorem 5.4).

**Theorem 5.1** Let \( T \) be an \( X \)-scheme. Let \( q \) be a \( T \)-valued point of \( \text{Quad}_V \equiv A^6_X \) which is identified uniquely with a 6-tuple
\[
(\lambda_1, \lambda_2, \lambda_3, \lambda_{12}, \lambda_{13}, \lambda_{23})
\]
corresponding to the quadratic form
\[
(x_1, x_2, x_3) \mapsto \Sigma_i \lambda_i x_i^2 + \Sigma_{i<j} \lambda_{ij} x_i x_j.
\]
Let \( \mathfrak{L} \) be a \( T \)-valued point of \( L_w \equiv (A^3_X, +) \) which is identified uniquely with a 3-tuple \((t_1, t_2, t_3)\) that corresponds to the \( T \)-valued point of \( \text{Stab}_w \) given by the \((4 \times 4)\)-matrix
\[
\begin{pmatrix}
1 & t_1 & t_2 & t_3 \\
0 & 1 & t_2 & t_3 \\
0 & 0 & 1 & t_3 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]
where \( I_3 \) is the \((3 \times 3)\)-identity matrix. Then in terms of the global basis
\[
\{w^0 = 1^o = 1; e_1^0 := e_1^0 \wedge e_2^0, e_2^0 := e_2^0 \wedge e_3^0, e_3^0 := e_3^0 \wedge e_1^0\}
\]
on the corresponding ordered basis of the even exterior algebra (=even Clifford algebra of the zero quadratic form on \( V \)) given by
\[
\{w^0 = 1^o = 1; e_1^0 \wedge e_2^0, e_2^0 \wedge e_3^0, e_1^0 \wedge e_3^0\}.
\]
induced from that of $W = \Lambda^{even}(V)$, the multiplication table for the specialised algebra structure

$$\Theta(q, \mathbf{1}) = \mathbf{1}_\theta(q)$$
onumber

on the pullback bundle $W_T$ with unit $w^o = w_T$ is given as follows:

- $e_1^1.e_1^1 = (t_1\lambda_{12} - \lambda_{11}\lambda_2 - t_1^2).w^o + (\lambda_{12} - 2t_1).e_1^1$;
- $e_2^1.e_2^1 = (t_2\lambda_{23} - \lambda_{22}\lambda_3 - t_2^3).w^o + (\lambda_{23} - 2t_2).e_2^1$;
- $e_3^1.e_3^1 = (t_3\lambda_{13} - \lambda_{11}\lambda_3 - t_3^3).w^o + (\lambda_{13} - 2t_3).e_3^1$;
- $e_1^2.e_1^2 = (\lambda_{12}\lambda_3 - \lambda_{11}t_3 - t_1^2).w^o - t_2^2e_1^2 - t_1e_2^2 - \lambda_2e_3^2$;
- $e_2^2.e_2^2 = (\lambda_{31}\lambda_2 - \lambda_{32}t_1 - t_2^3).w^o - \lambda_3e_1^2 + t_3e_3^2 - t_2e_3^2$;
- $e_3^2.e_3^2 = (\lambda_{11}\lambda_3 - \lambda_{12}t_2 - t_3^3).w^o - \lambda_1e_3^2 - t_1e_3^3$;
- $e_1^3.e_1^3 = (\lambda_{12}t_3 - (\lambda_{12} - t_1)(\lambda_{23} - t_2)).w^o + (\lambda_{23} - t_3).e_1^2 + \lambda_2e_3^3$;
- $e_2^3.e_2^3 = (\lambda_3t_1 - (\lambda_{13} - t_3)(\lambda_{23} - t_2)).w^o + \lambda_3e_1^2 + \lambda_1t_3 - t_3e_3^2 + (\lambda_{23} - t_2).e_3^3$;
- $e_3^3.e_3^3 = (\lambda_{11}t_2 - (\lambda_{12} - t_1)(\lambda_{13} - t_3)).w^o + (\lambda_{13} - t_3).e_1^2 + \lambda_1t_2 + (\lambda_{12} - t_1).e_3^3$.

**Proof of Theorem 5.1** For clarity, let $*_q$ denote the multiplication in the algebra $C_0(V_T, q)$ and, for uniformity, let $e_0 := w$. Since $q = q_b(q)$, we have by (2d), Theorem 2.2 the isomorphism

$$\psi_b(q) : C_0(V_T, q) \cong \Lambda^{even}(V_T) = W_T.$$

Let $*_b(q)$ denote the product in the algebra structure $\theta(q)$ thus induced on $W_T$. Since the $e_i^q$ are a basis for $W_T$, it is enough to compute the products $e_i^q *_b(q) e_j^q$ for $1 \leq i, j \leq 3$. For example, consider the product $e_2^q *_b(q) e_1^q$. Using the properties of the multiplication in $C(V_T, q)$, and the properties of the isomorphism $\psi_b(q)$ from (2), Theorem 2.2 we get the following:

$$e_2^q *_b(q) e_1^q = \psi_b(q) \left( \{ \psi_b^{-1}(e_2^q \wedge e_3^q) \} \ast_q \{ \psi_b^{-1}(e_1^q \wedge e_3^q) \} \right)$$

$$= \psi_b(q) \left( (e_2^q \ast q e_3^q) \ast_q (e_1^q \ast q e_2^q) \right)$$

$$= \psi_b(q) \left( (\lambda_{23}(1^o) - e_3^q \ast q e_2^q) \ast_q (\lambda_{12}(1^o) - e_2^q \ast q e_1^q) \right)$$

$$= \psi_b(q) \left( \lambda_{23}\lambda_{12}(1^o) - \lambda_{23}e_3^q \ast q e_1^q - \lambda_{12}e_3^q \ast q e_2^q + (e_3^q \ast q e_2^q) \ast_q (e_2^q \ast q e_1^q) \right)$$

$$= \psi_b(q) \left( \lambda_{23}\lambda_{12}(1^o) - \lambda_{23}(1^o) e_3^q - e_1^q \ast q e_2^q \right)$$

$$- \lambda_{23}(1^o) e_2^q - e_3^q \ast q (e_2^q \ast q e_1^q)$$

$$= \lambda_{23}\lambda_{12}(1^o) - \lambda_{23}(1^o) e_3^q + e_3^q \ast q (e_2^q \ast q e_1^q)$$

In a similar fashion, the other products may be computed; this amounts to computing $\theta$ on $T$-valued points. The following result is needed to compute $\Theta$ from $\theta$. 


Lemma 5.2 Let \( *_{(b(q),t)} \) denote the multiplication in the algebra \( \Theta(q,t) = t \theta(q) \) and as before, \( *_{b(q)} \) denote the multiplication in \( \theta(q) \). Then we have

1. \( t(\epsilon_i^o) = t_i w^o + \epsilon_i^o \) for \( 1 \leq i \leq 3 \);
2. \( (t)^{-1}(\epsilon_i^o) = -t_i w^o + \epsilon_i^o \) for \( 1 \leq i \leq 3 \);
3. \( \epsilon_i^o \cdot_{(b(q),t)} \epsilon_j^o = t((t^{-1}(\epsilon_i^o)) \cdot_{b(q)} (t^{-1}(\epsilon_j^o))) \).

While the first two of the above formulae follow easily by direct computation, the third follows by using the first two alongwith the following:

\[ \epsilon_i^o \cdot_{(b(q),t)} \epsilon_j^o = t((t^{-1}(\epsilon_i^o)) \cdot_{b(q)} (t^{-1}(\epsilon_j^o))) \].

We may now compute the multiplication in the algebra \( \Theta(q,t) = t \theta(q) \) by making use of the formulas listed in the above lemma and the expressions for the products of the form \( \epsilon_i^o \cdot_{b(q)} \epsilon_j^o \) whose computation had already been illustrated before the lemma. Q. E. D., Theorem 5.1.

5.3 Computation of the Isomorphism arising from a Similarity

We continue with the notations introduced above. In the following we study the lack of equivariance of the isomorphism \( \Theta \) relative to \( \text{GL}_V \) and show that it satisfies a curious ‘twisted’ version of equivariance. Firstly we consider the morphism of \( X \)-groupeschemes

\[ \Lambda^{\text{even}} : \text{GL}_V \rightarrow \text{Stab}_w \] (given on valued points by) \( g \mapsto \Lambda^{\text{even}}(g) \).

Recall that

\[ \Lambda^{\text{even}}(V) =: A_0 \in \text{Id-w-Sp-Azu}_w(X) \]

is the even graded part of the Clifford algebra of the zero quadratic form on \( V \). A simple computation reveals the following result.

Lemma 5.3 For each \( X \)-scheme \( T \), define the map

\[ \text{GL}(V_T) \rightarrow \text{Stab}(A_0)_T : g \mapsto \left( \begin{array}{cc} 1 & 0 \\ 0 & B(g) \end{array} \right) \in \text{Stab}(w_T) \]

where \( B(g) := \text{det}(g) \left( E_{12} E_{23} (g^{-1})^t E_{23} E_{12} \right) \) with

\[ E_{12} = \left( \begin{array}{ccc} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{array} \right) \text{ and } E_{23} = \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{array} \right) \].

Then the above maps define a morphism of \( X \)-groupeschemes which in fact is none other than

\[ \Lambda^{\text{even}} : \text{GL}_V \rightarrow \text{Stab}_w; \]

in other words: \( B(g) = \Lambda^2(g) \).
Recall from Lemma 5.1, Part A, [17], that $\text{Stab}_w$ is the semidirect product of $\text{Stab}_{A_0}$ and $L_w$, so that $\text{Stab}_{A_0}$ naturally acts on $L_w$ by “outer conjugation”. Let $GL_V$ act on $L_w$ via the homomorphism $\Lambda^{even}$ i.e., for $g \in GL(V_T)$ and $l \in L_w(T)$,

$$g l := \Lambda^{even}(g) l := \Lambda^{even}(g) \Lambda^{even}(g^{-1}).$$

Any element $h \in \text{Stab}(w_T)$ can be uniquely written as

$$h = h_s h_l = h'_s h'_l$$

where $h_s \in \text{Stab}((A_0)_T)$ and $h_l, h'_l \in L_w(T)$.

Then the relation between $h_l$ and $h'_l$ can be written as

$$h'_l = h_s h_l$$

where “.” stands for the action of $\text{Stab}_{A_0}$ on $L_w$.

Since $\text{Id}-w\text{-Sp-Azu}_W$ comes with a canonical action of $\text{Stab}_w$ on it, we let $GL_V$ act on $\text{Id}-w\text{-Sp-Azu}_W$ via $\Lambda^{even}$. The following result describes the lack of $GL_V$-equivariance of the isomorphism $\Theta$.

**Theorem 5.4** Let $T$ be an $X$-scheme. For $T$-valued points $g, q, t$ respectively of $GL_V$, Quad$_V$, and $L_w$, there exists a unique $T$-valued point of $L_w$ given by an isomorphism $h'_l(g, q)$ of $O_T$-algebra bundles

$$h'_l(g, q) : g.\Theta(q, t) \longrightarrow \Theta(g.q, g.t).$$

Further, $h'_l(g, q)$ satisfies the formula

$$h'_l(gg', q) = h'_l(g, g'.q)(g, h'_l(g', q)).$$

Therefore $\Theta$ satisfies a ‘twisted’ version of $GL_V$-equivariance. The next theorem, which was originally motivated by the proof of this ‘twisted equivariance’, will be of central importance to us for the rest of this section.

**Theorem 5.5** Given a similarity

$$g : (V_T, q) \cong_l (V_T, q')$$

with multiplier $l \in \Gamma(T, O^*_T)$, let $h(g,l,q,q')$ be the automorphism of $(W_T, w_T)$ given by the composition of the following isomorphisms:

$$W_T \xrightarrow{\psi_{h(q)}} C_0(V_T, q) \xrightarrow{C_0(g.l)} C_0(V_T, q') \xrightarrow{\psi_{h(q')}} W_T$$

where the algebra bundle isomorphism $C_0(g,l)$ comes from (1), Prop [28] and the linear isomorphisms $\psi_{h(q)}$ and $\psi_{h(q')}$. In terms of actions, this means that

$$h(g,l,q,q'), \theta(q) = \theta(q').$$
Write \( h(g, l, q, q') \in \text{Stab}(w_T) \) uniquely as a product

\[
h(g, l, q, q') = h_s(g, l, q, q') h(u, l, q, q')
\]

with the first factor in \( \text{Stab}_{\mathcal{V}_l}(T) \) and the second in \( L_w(T) \) as explained earlier. Then \( h_s(g, l, q, q') \) depends only on \( g \) and \( l \) and not on \( q \) or \( q' \). In fact, one has

\[
h_s(g, l, q_1, q_2) = h_s(g, l) := \begin{pmatrix} 1 & 0 \\ 0 & l^{-1} A^2(g) \end{pmatrix} \forall q_1, q_2 \in \text{Quad}(V_T).
\]

**Proof:** We directly compute the \( O_T \)-linear automorphism \( h(g, l, q, q') \) of \( W_T \) as follows. Of course, this automorphism fixes \( w^o = w_T \). So we need to only compute the images of the three remaining basis elements

\[
e_1^o = e_1^o \wedge e_2^o, e_2^o = e_2^o \wedge e_3^o \text{ and } e_3^o = e_3^o \wedge e_1^o
\]

in terms of the basis elements \( w^o \) and \( e_i^o \). Let \( q \) and \( l(g.q) = q' \) respectively correspond to the 6-tuples

\[
(\mu_1, \mu_2, \mu_3, \mu_1, \mu_2, \mu_3) \text{ and } (\nu_1, \nu_2, \nu_3, \nu_{12}, \nu_{13}, \nu_{23}) \in \Gamma(T, O_T^{o6}).
\]

(We caution the reader that \( l(g.q) \neq (lg).q = l^{-2}(g.q) \)) Let \( g \in \text{GL}(V_T) \equiv \text{GL}_3(\Gamma(T, O_T)) \) be given by the matrix \((g_{ij})\). Observe that the \( \nu \) are polynomials in the \( \mu \) and \( g_{ij} \). In the following computation, for the sake of clarity, we denote the product in \( C(V_T, q) \) by \(*_q\). For example, we have

\[
h(g, l, q, q') e_1 = \psi_{b(q')} C_0(g, l) \left( (\psi_{b(q)})^{-1}(e_1^o \wedge e_2^o) \right)
\]

\[
= \psi_{b(q')} \left( C_0(g, l) (e_1^o *_q e_2^o) \right) \quad \text{(by (2f), Theorem 2.2)}
\]

\[
= \psi_{b(q')} \left( (l^{-1}(g(e_1^o) *_q g(e_2^o))) \right) \quad \text{(by (1), Prop 2.8)}
\]

\[
= l^{-1} \psi_{b(q')} \left( (g_{11} e_1^o + g_{21} e_2^o + g_{31} e_3^o) *_q (g_{12} e_1^o + g_{22} e_2^o + g_{32} e_3^o) \right)
\]

\[
= l^{-1} \psi_{b(q')} \left( ((g_{11} g_{12} \nu_1 + g_{21} g_{22} \nu_2 + g_{31} g_{32} \nu_3 + g_{21} g_{12} \nu_{12} + g_{31} g_{32} \nu_{13} + g_{11} g_{32} \nu_{23}) w^o + (g_{21} g_{22} - g_{21} g_{12}) e_1^o *_q e_2^o + (g_{21} g_{12} - g_{21} g_{32}) e_2^o *_q e_3^o + (g_{31} g_{12} - g_{31} g_{32}) e_3^o *_q e_1^o) \right)
\]

\[
= l^{-1} \left( P_1(g, l, q, q') w^o + C_{33}(g) e_1^o + C_{13}(g) e_2^o + C_{23}(g) e_3^o \right)
\]

(by (2f), Theorem 2.9)

where \( P_1(g, l, q, q') \) is the polynomial in the \( \nu \) and \( g_{ij} \) (as computed in the previous step) and where \( C_{ij}(g) \) represents the cofactor determinant of the element \( g_{ij} \) of the matrix \( g = (g_{ij}) \). Similarly one computes the values of \( h(g, l, q, q') e_2 \) and \( h(g, l, q, q') e_3 \). Then the matrix of \( h(g, l, q, q') \) is given by

\[
h(g, l, q, q') = \begin{bmatrix}
1 & l^{-1} P_1(g, l, q, q') & l^{-1} P_2(g, l, q, q') & l^{-1} P_3(g, l, q, q') \\
l^{-1} C_{33}(g) & l^{-1} C_{31}(g) & l^{-1} C_{32}(g) & \\
l^{-1} C_{11}(g) & l^{-1} C_{11}(g) & l^{-1} C_{12}(g) & \\
l^{-1} C_{21}(g) & l^{-1} C_{21}(g) & l^{-1} C_{22}(g) & \\
\end{bmatrix}
\]
which implies that

\[
h_s(g, l, q, q') = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & l^{-1}C_{33}(g) & l^{-1}C_{31}(g) & l^{-1}C_{32}(g) \\
0 & l^{-1}C_{13}(g) & l^{-1}C_{11}(g) & l^{-1}C_{12}(g) \\
0 & l^{-1}C_{23}(g) & l^{-1}C_{21}(g) & l^{-1}C_{22}(g)
\end{bmatrix}
\]

depends only on \( g \) and \( l \).

Next define the matrix

\[
\hat{g} = \begin{bmatrix}
g_{33} & g_{13} & g_{23} \\
g_{31} & g_{11} & g_{21} \\
g_{32} & g_{12} & g_{22}
\end{bmatrix}
\]

so that

\[
h_s(g, l, q, q') = \begin{bmatrix}
1 & 0 \\
0 & l^{-1}C(\hat{g})^t
\end{bmatrix}
\]

where \( C(\hat{g}) \) is the cofactor matrix of \( \hat{g} \). Now if \( E_{12} \) and \( E_{23} \) are the matrices defined in Lemma 5.3 above, premultiplying by \( E_{ij} \) has the effect of interchanging the \( i \)th and \( j \)th rows, while postmultiplying has a similar effect on the columns. Thus we get

\[
\hat{g} = E_{12}E_{23}(g^t)E_{23}E_{12}
\]

from which it follows that

\[
C(\hat{g})^t = \text{Adjoint } (\hat{g}) = \det (\hat{g}).(\hat{g})^{-1} = \det (g). \left( E_{12}E_{23}(g^{-1})^tE_{23}E_{12} \right),
\]

showing that \( C(\hat{g})^t = \Lambda^2(g) \) by Lemma 5.3 Q.E.D., Theorem 5.5

**Proof of Theorem 5.4** Note that \( g \) is an isometry from \( (V_T, q) \) to \( (V_T, g.q) \) and hence according to (1), Prop 2.8 induces the algebra isomorphism

\[
C_0(g, l = 1) : C_0(V_T, q) \cong C_0(V_T, g.q).
\]

Let \( h(g, q) := h(g, 1, q, g.q) \) where \( h(g, l, q, q') \) was defined in Theorem 5.5 above. As explained in page 51 there are two canonical decompositions of \( \text{Stab}(w_T) \), which lead to unique (ordered) decompositions of \( h(g, q) \) as

\[
h'_1(g, q)h_s(g, q) \text{ and } h_s(g, q)h_1(g, q).
\]

By Theorem 5.5 above,

\[
h_s(g, q_1) = h_s(g, q_2) = h_s(g, 1) = \Lambda_{\text{even}}(g) =: h_s(g) \forall q_1, q_2 \in \text{Quad}(V_T)
\]
and hence we get:

\[
\Theta(g.q, g.l) := (g.l).\theta(g.q)
\]

\[
= (\Lambda^{even}(g)\Lambda^{even}(g^{-1}))(h(g, 1, q, g.q)\theta(q))
\]

\[
= (h_s(g)h_s^{-1}(g))(h(g, q)\theta(q))
\]

\[
= ((h_s(g)h_s^{-1}(g))(h_s(g, q)h_l(g, q)))\theta(q)
\]

\[
= (h_s(g)h_l(g, q))\theta(q)
\]

\[
= (h_s(g)h_l(g, q)).(\Lambda \theta(q))
\]

\[
= (h_s(g, q)h_l(g, q)).(\Theta(q, l))
\]

\[
= h'_l(g.q).h_s(g, q).\Theta(q, l)
\]

\[
= h'_l(g.q).g.\Theta(q, l).
\]

Note that \(h_l(g, q)\) was explicitly computed in the proof of Theorem 5.3. above to be

\[
h_l(g, q) = \begin{bmatrix} 1 & P_1(g, 1, q, g.q) & P_2(g, 1, q, g.q) & P_3(g, 1, q, g.q) \\ 0 & I_3 \end{bmatrix} \in \mathcal{L}_w(T).
\]

The formula for \(h'_l(g_1g_2, q)\) stated in the theorem is gotten thus:

\[
h'_l(g_1g_2, q) = (h'_l(g_1g_2, q)h_s(g_1g_2))h_s^{-1}(g_1g_2)
\]

\[
= h(g_1g_2, q)h_s^{-1}(g_1g_2).
\]

Now by (3) of Prop.2.8 it follows that

\[
h'_l(g_1g_2, q) = (h(g_1, g_2.q)h(g_2, q))h_s^{-1}(g_1g_2)
\]

\[
= h'_l(g_1, g_2.q)h_s(g_1, g_2.q)h_s(g_2, q)h_l(g_2, q)h_s^{-1}(g_1g_2)
\]

\[
= h'_l(g_1, g_2.q)h_s(g_1, g_2)h_l(g_2, q)h_s^{-1}(g_1g_2)
\]

\[
= h'_l(g_1, g_2.q)h_s(g_1g_2)h_l(g_2, q)h_s^{-1}(g_1g_2)
\]

\[
= h'_l(g_1, g_2.q)(g_1g_2).h_l(g_2, q).
\]

On the other hand

\[
g_2^{-1}h'_l(g_2, q) = h_s^{-1}(g_2)(h'_l(g_2, q)h_s(g_2)) = h_s^{-1}(g_2)(h_s(g_2)h_l(g_2, q)) = h_l(g_2, q)
\]

and therefore

\[
h'_l(g_1g_2, q) = h'_l(g_1, g_2.q)(g_1g_2).h'_l(g_2, q)\)

\[
= h'_l(g_1, g_2.q)(g_1, h'_l(g_2, q)).
\]

Finally, one has to show the uniqueness of \(h'_l(g, q) \in \mathcal{L}_w(T)\). Suppose \(h_l \in \mathcal{L}_w(T)\) is also an algebra isomorphism

\[
h_l : g.\Theta(q, l) \rightarrow \Theta(g.q, g.l),
\]
which means

\[ h_l(g, \Theta(q, \mathbf{L})) = \Theta(g.q, g.\mathbf{L}). \]

Notice that while showing the ‘twisted’ equivariance of \( \Theta \) above, we have also proved that

\[ \Theta(g.q, g.\mathbf{L}) = h_l(g.q)\Theta(q, \mathbf{L}). \]

Therefore we get

\[ (h_lh_s(g)).(\mathbf{L}\Theta(q)) = h_l(g.q)\Theta(q, \mathbf{L}) \]

\[ \Rightarrow (h_lh_s(g)).(\mathbf{L}\Theta(q)) = (h_s(g)h_l(q, g)).\Theta(q, \mathbf{L}) \]

\[ \Rightarrow (h_l^{-1}(g))h_s(g)).(\mathbf{L}\Theta(q)) = (h_l(g.q)\mathbf{L}\Theta(q)) \]

\[ \Rightarrow \Theta(q, (g^{-1}, h_l)\mathbf{L}) = \Theta(q, h_l(g, q)\mathbf{L}). \]

But since \( \Theta \) is an isomorphism by Theorem 3.10, this implies that

\[ (g^{-1}, h_l)\mathbf{L} = h_l(g, q)\mathbf{L} \]

which gives

\[ h_l = h_s(g)h_l(g, q)h_l^{-1}(g) = h_l'(g, q). \]

Q.E.D., Theorem 5.4.

5.4 Conclusion of Proof of Injectivity

We remind the reader that towards the end of §4, we had reduced the proof of the injectivity of Theorem 3.1 to that of Theorem 3.4, and had indicated in page 45 that it would be enough to prove the latter in the case when \( V \) and \( I \) are both free—which has been the case in this section so far. Starting with an isomorphism of algebra bundles

\[ \phi : C_0(V, q) \cong C_0(V, q') \]

we arrive at the element \( g_l^0 \in \text{GL}(V) \) as defined in page 45 to briefly recall this, firstly \( g \in \text{GL}(V) \) was defined by the following commuting diagram:

\[
\begin{align*}
(C_0(V, q))^\vee \quad & \xrightarrow{((\psi_h(q))^{-1})} \quad (A_{even}(V))^\vee \quad \xrightarrow{\text{surjection}} \quad (A^2(V))^\vee \quad \xrightarrow{=} \\
\phi^\vee \quad & \xrightarrow{=} \quad (\phi_{A^{even}})^\vee \quad \xrightarrow{=} \quad (\phi_{A^2})^\vee \quad \xrightarrow{=} \\
(C_0(V, q'))^\vee \quad & \xrightarrow{((\psi_h(q'))^{-1})} \quad (A_{even}(V))^\vee \quad \xrightarrow{\text{inclusion}} \quad (A^2(V))^\vee \quad \xrightarrow{=} \\
(A^2(V))^\vee \quad & \xrightarrow{(\gamma')^{-1}} \quad V \otimes (\text{det}(V))^{-1} \quad \xrightarrow{\otimes \text{det}(V)} \quad V \\
(\phi_{A^2})^\vee \quad & \xrightarrow{=} \quad (\gamma')^{-1} \quad \xrightarrow{=} \quad g^{-1} \quad \xrightarrow{=} \\
(A^2(V))^\vee \quad & \xrightarrow{(\gamma')^{-1}} \quad V \otimes (\text{det}(V))^{-1} \quad \xrightarrow{\otimes \text{det}(V)} \quad V
\end{align*}
\]
Secondly, we had defined
\[ g_\phi^l := (l^{-1} \sqrt{\gamma(l)}) g. \]
Our current special choices of bilinear forms \( b(q) \) and \( b(q') \) that induce \( q \) and \( q' \) respectively do not affect the generality, as was observed in the proof of Prop. 3.2. We shall now show that \( g_\phi^l \) is a similitude from \((V, q)\) to \((V, q')\) with multiplier \( l \) and that this similitude induces \( \phi \) i.e., with the notations of (1), Prop. 2.8, that \( C_0(g_\phi^l, l) = \phi \). We proceed with the proof which will follow from several lemmas.

**Lemma 5.6** Consider the element
\[
h_s h_l = h'_l h_s = h := \phi_{\Lambda^{ev}} \in (\text{Stab}_w)(X)
\]
written uniquely as an ordered product in two different ways with \( h_l, h'_l \in (\text{L}_w)(X) \) and \( h_s \in (\text{Stab}_{\Lambda_0})(X) \) as explained in page 51; let \( B \) be the matrix corresponding to \( \phi_{\Lambda^2} \), and let the matrices \( E_{ij} \) be as defined in Lemma 5.3. Then we have matrix representations:
\[
h_s = \begin{pmatrix} 1 & 0 & 0 \\ 0 & B \end{pmatrix} \quad \text{and} \quad g_\phi^l = \begin{pmatrix} l^{-1} \sqrt{\gamma(l)} & 0 \\ 0 & l \times I_3 \end{pmatrix} E_{23} E_{12}((B^t)^{-1} E_{12} E_{23}
\]
The proof of the above lemma follows from the fact that the matrix of the canonical isomorphism \( \eta : \Lambda^2(V) \equiv V^\vee \otimes \det(V) \) is given by \( E_{23} E_{12} \), which can be verified by a simple computation.

**Lemma 5.7** We have the formulae
\[
h_s(\text{Identity}, l^{-1}, q', l^{-1}q') = \begin{pmatrix} 1 & 0 \\ 0 & l \times I_3 \end{pmatrix}
\]
and
\[
h_s(\text{Identity}, l^{-1}, q', l^{-1}q'), \theta(q') = \theta(l^{-1}q').
\]
The identity map on \( V \) is obviously a similarity with multiplier \( l^{-1} \) from \((V, q')\) to \((V, l^{-1}q')\). Hence the above lemma follows by taking \( T = X \), \( g = \text{Identity} \), and the \( l^{-1} \) and the \( q' \) at hand for the \( l \) and the \( q \) of Theorem 5.5 (caution: the \( q' \) there would have to be replaced by \( l^{-1}q' \)). This can also be seen directly from the multiplication tables for
\[
\theta(l^{-1}q') = \Theta(l^{-1}q', I_4) \quad \text{and} \quad \theta(q') = \Theta(q', I_4)
\]
written out in Theorem 5.3 where we must take \( T = X \) and \( I = I_4 \) i.e., \( t_i = 0 \forall i \). We observe from the multiplication table that each of the coefficients of \( \epsilon_i \) for \( 1 \leq i \leq 3 \) is a single \( \lambda \), whereas each coefficient of \( w = 1 = \epsilon_0 \) is a product of two \( \lambda \)'s, and this observation implies the lemma above.

As the reader might have noticed, there are two crucial facts about the identifications in this section; namely, firstly, for any \( X \)-scheme \( T \), each of the maps \( \psi_{b(q)} \) (for different \( q \)) identify \((C_0(V_T, \gamma), 1)\) with the same \((W_T, w_T)\) and secondly,
relative to the bases chosen, all these identifying maps have trivial determinant. The latter is also true of the identification \( \eta \), since it is given by the matrix \( E_{23}E_{12} \) (as was noted after Lemma 5.6). It therefore follows that

\[
\det(\phi) = \det(\phi_{\Lambda^2}) = \det(\phi_{\Lambda^2}) = \det(g') = \det(g) = \det(B^{-1}).
\]

But we had chosen \( l \in \Gamma(X, \mathcal{O}_X^*\) such that

\[
\gamma(l) := (l^3) \det(\phi_{\Lambda^2}) = l^3 \det(B).
\]

Using these facts along with Lemma 5.6 above, a straightforward computation gives the following.

**Lemma 5.8** We have the equality

\[
\det(g_l^\phi) = \sqrt{\gamma(l)}
\]

from which it follows that

\[
B(g_l^\phi) = l \times B
\]

where \( B(g_l^\phi) \) and \( B \) are as defined in Lemmas 5.8 and 5.6 respectively. In particular,

\[
\det^2(g_l^\phi) = \det(\phi_{\Lambda^2})
\]

when \( \det(\phi_{\Lambda^2}) \) is itself a square and for the cases 1 and 2 of page 44 where we had chosen \( l := 1 \).

**Lemma 5.9** \( g_l^\phi \) is a similitude from \((V, q)\) to \((V, q')\) with multiplier \( l \).

The hypothesis \( \phi : C_0(V, q) \cong C_0(V, q') \) is an algebra isomorphism translates in terms of actions into \( h.\theta(q) = \theta(q') \) where \( h = \phi_{\Lambda^2} \in (\text{Stab}_w)(X) \). Let

\[
h(g_l^\phi, q) := h(g_l^\phi, 1, q, g_l^\phi.q)
\]
where \( h(g, l, q, q') \) was defined in Theorem 5.5 above. Then we have

\[
\begin{align*}
\Theta(g_1^\phi, q_4) := \theta(g_1^\phi, q) \theta(q) = h(g_1^\phi, q)(h^{-1} \theta(q')) \\
= (h_1(g_1^\phi, q)h_s(g_1^\phi, q)h_s^{-1}(h_1')^{-1}) \theta(q') \\
= (h_1(g_1^\phi, q) \begin{pmatrix} 1 & 0 \\ 0 & I_3 \end{pmatrix}) (h_1')^{-1} \theta(q') \\
(\text{by Theorem 5.5, Lemmas 5.3 & 5.6})
\end{align*}
\]

\[
= (h_1'(g_1^\phi, q)h_1') \theta(l^{-1}q') \\
=: \Theta(l^{-1}q', (h_1'(g_1^\phi, q)h_1')).
\]

But since \( \Theta \) is an isomorphism (Theorem 5.10), this implies the claim of the above lemma namely, that

\[ g_1^\phi, q = l^{-1}q' \text{ and further that } h_1'(g_1^\phi, q) = (h_1')^{-1}. \]

**Lemma 5.10** The similarity

\[ g_1^\phi : (V, q) \cong (V, q') \]

induces \( \phi \) i.e., with the notations of (1), Prop 2.8 \( C_0(g_1^\phi, l) = \phi. \)

We have \( C_0(g_1^\phi, l) = \phi \) iff

\[ h(g_1^\phi, l, q, q') := \psi_{b(q')} \circ C_0(g_1^\phi, l) \circ \psi_{b(q)}^{-1} \]

\[ = \psi_{b(q')} \circ \phi \circ \psi_{b(q)}^{-1} =: \phi_{\Lambda_{\psi}} =: h. \]

Now using successively Theorem 5.5, Lemma 5.3, Lemma 5.8 and Lemma 5.6, we get the following sequence of equalities:

\[ h_s(g_1^\phi, l, q, q') = \begin{pmatrix} 1 & 0 \\ 0 & l^{-1} \times A^2(g_1^\phi) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & l^{-1} \times B(g_1^\phi) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix} = h_s. \]
Therefore the present hypotheses translated in terms of actions give

\[ h(g^\phi_l, l, q, q').\theta(q) = \theta(q') = h.\theta(q) \]

\[ \Rightarrow h_s(g^\phi_l, l, q, q').(h_l(g^\phi_l, l, q, q').\theta(q)) = h_s(h_l.\theta(q)) \]

\[ \Rightarrow \Theta(q, h_l(g^\phi_l, l, q, q')) = \Theta(q, h_l). \]

But \( \Theta \) being an isomorphism (Theorem 3.10), the last equality implies that

\[ h_l(g^\phi_l, l, q, q') = h_l \]

which gives \( h(g^\phi_l, l, q, q') = h. \)

**Lemma 5.11**

(1) For a similarity \( g \in \text{Sim}[(V, q), (V, q')] \) with multiplier \( l \) and the induced isomorphism

\[ C_0(g, l) \in \text{Iso}[C_0(V, q), C_0(V, q')] \]

given by (1), Prop [2.8] we have the equality

\[ \det((C_0(g, l))_{\Lambda^2}) = l^{-3}\det^2(g). \]

Therefore the map

\[ \text{Sim}[(V, q), (V, q')] \rightarrow \text{Iso}[C_0(V, q), C_0(V, q')] : g \mapsto C_0(g, l) \]

maps the subsets

\[ \text{Iso}[(V, q), (V, q')] \text{ and } \text{S-Iso}[(V, q), (V, q')] \]

respectively into the subsets

\[ \text{Iso}'[C_0(V, q), C_0(V, q')] \text{ and } \text{S-Iso}[C_0(V, q), C_0(V, q')]. \]

(2) In the case \( q' = q \), if \( C_0(g, l) \) is the identity on \( C_0(V, q) \), then

\[ g = l^{-1}\det(g) \times \text{Id}_V, \]

and further if \( g \in \text{O}(V, q) \) then

\[ g = \det(g) \times \text{Id}_V \text{ with } \det^2(g) = 1. \]

By definition,

\[ (C_0(g, l))_{\Lambda^2} = \psi_b(q') \circ C_0(g, l) \circ \psi_b^{-1}(q), \]

and the latter isomorphism is \( h(g, l, q, q') \) from Theorem 5.9 which further gives a formula for \( h_s(g, l, q, q') \). Now using the facts that \( \psi_b(q) \) and \( \psi_b(q') \) have trivial determinant (as noted before Lemma 5.8) we get assertion (1):

\[ \det ((C_0(g, l))_{\Lambda^2}) = \det ((C_0(g, l))_{\Lambda^2}) = \det(h(g, l, q, q')) \]

\[ = \det(h_s(g, l, q, q')) = l^{-3}\det^2(g). \]
If \( q = q' \) and \( C_0(g, l) \) is the identity, then the same argument in fact shows that
\[
I^{-1}A^2(g) = I_3
\]
and by using the formula in Lemma 5.3 for \( B(g) = A^2(g) \), we get
\[
g = l^{-1}(\det(g))I_3;
\]
taking determinants in the last equality gives \( \det^2(g) = I^3 \), so that when \( g \in O(V, q) \) i.e., \( l = 1 \),
\[
\det^2(g) \in \mu_2(\Gamma(X, 0_X))
\]
and assertion (2) follows.

**Lemma 5.12** The following map is a bijection:

\[
S-Iso[(V, q), (V, q')] \longrightarrow S-Iso[C_0(V, q), C_0(V, q')] : g \mapsto C_0(g, 1, q, q').
\]

Given \( \phi \in S-Iso[C_0(V, q), C_0(V, q')] \), by definition 3.3 we have \( \det(\phi_{A^2}) = 1 \), so by Lemma 5.8
\[
\det(g_\phi^0) = \sqrt{\gamma(l)} := 1
\]
for our choice under Case 1 on page 44. Therefore the corresponding element
\[
g_\phi^0 \in S-Iso[(V, q), (V, q')]
\]
and is, according to Lemma 5.10 such that
\[
C_0(g_\phi^0, l = 1, q, q') = \phi
\]
which gives the surjectivity. As for the injectivity, if
\[
g_1, g_2 : (V, q) \cong_1 (V, q')
\]
are isometries with determinant 1 such that
\[
C_0(g_1, 1, q, q') = C_0(g_2, 1, q, q'),
\]
then we have
\[
h(g_1, 1, q, q') = h(g_2, 1, q, q') \text{ so that } h_s(g_1, 1, q, q') = h_s(g_2, 1, q, q')
\]
whence by Theorem 5.5 and Lemma 5.3
\[
B(g_1) = B(g_2) \Rightarrow g_1 = g_2.
\]
Q.E.D., Theorem 3.4 and injectivity of Theorem 3.1
Proof of Theorem 3.5. Taking $q' = q$ in Theorem 3.4 gives the commutative diagram of groups and homomorphisms as asserted in the statement of the theorem. We continue with the notations above. For $g \in \text{GO}(V, q, I)$ with multiplier $l$, that the equality

$$\det(C_0(g, l, I)) = \det((C_0(g, l, I))_{\Lambda^2}) = l^{-3}\det^2(g)$$

holds (locally, hence globally) was shown in (1), Lemma 5.11. Assertion (2) of the same Lemma shows the following (locally, and hence globally): if $C_0(g, l, I)$ is the identity on $C_0(V, q, I)$, then

$$g = l^{-1}\det(g).\text{Id}_V,$$

and further if $g \in \text{O}(V, q, I)$ then

$$g = \det(g).\text{Id}_V \text{ with } \det^2(g) = 1.$$

The map

$$\Gamma(X, \mathcal{O}_X) \to \text{GO}(V, q, I)$$

is the natural one given by sending $\lambda$ to the similarity $\lambda.\text{Id}_V$ with multiplier $\lambda^2$. It follows from the formula in (1), Prop. 2.8 that

$$C_0(\lambda.\text{Id}_V, \lambda^2, I) = \text{Identity}.$$

This gives exactness at $\text{GO}(V, q, I)$ and at $\text{O}(V, q, I)$. We proceed to prove assertion (b). Let

$$\phi \in \text{Aut}(C_0(V, q, I)),$$

and consider the self-similarity

$$g_1^\phi = s_{2k+1}^+(\phi) \text{ with multiplier } l = \det(\phi)^{2k+1}.$$

For the moment, assume that $V$ and $I$ are trivial over $X$. Fix a global basis $\{e_1, e_2, e_3\}$ for $V$ and set $e'_i = g_i^\phi(e_i)$. It follows from Kneser’s definition of the half-discriminant $d_0$—see formula (3.1.4), Chap.IV, [9]—that

$$d_0(q, \{e_i\}) = d_0(q, \{e'_i\})\det^2(g_1^\phi).$$

Since we have

$$g_1^\phi.q = l^{-1}q,$$

a simple computation shows that

$$d_0(q, \{e'_i\}) = l^3d_0(q, \{e_i\}).$$

The hypothesis that $q \otimes \kappa(x)$ is semiregular means that the image of the element

$$d_0(q, \{e_i\}) \in \Gamma(X, \mathcal{O}_X)$$
in \( \kappa(x) \) is nonzero. Since \( X \) is integral, we therefore deduce that

\[
\det^2(g^\phi_l) = l^{-3}.
\]

On the other hand, we know that

\[
\det^2(g^\phi_l)l^{-3} = \det(\phi).
\]

It follows that

\[
\det^{12k+7}(\phi) = 1 \forall k \in \mathbb{Z},
\]

which forces \( \det(\phi) = 1 \). In general, even if \( V \) and \( I \) are not necessarily trivial, since this equality holds over a covering of \( X \) which trivialises both \( V \) and \( I \), it also holds over all of \( X \). \textbf{Q.E.D., Theorem 3.5}

### 6 Surjectivity of Theorem 3.1: Bilinear Forms as Specialisations

In this section we reduce the proof of Theorem 3.8 to Theorem 3.9. We prove the latter and using it alongwith Theorem 5.1, deduce Theorem 3.11. The surjectivity of Theorem 3.1 is also established.

**Reduction of Proof of Theorem 3.8 to the case \( I = \mathcal{O}_X \).** We adopt the notations introduced just before Theorem 3.8. Let \( T \) be an \( X \)-scheme. Given a bilinear form

\[
b \in \text{Bil}(V,I)(T) = \Gamma(T, \text{Bil}(V_T,I_T)),
\]

consider the linear isomorphism

\[
\psi_b : C_{0}(V_T,q_b,I_T) \cong \mathcal{O}_T \oplus \Lambda^2(V_T) \otimes (I_T)^{-1} = W_T
\]

of (2d), Theorem 2.2 Let \( A_b \) denote the algebra bundle structure on \( W_T \) with unit \( w_T = 1 \) induced via \( \psi_b \) from the even Clifford algebra \( C_{0}(V_T,q_b,I_T) \). By definition, \( A_b \in \text{Assoc}_{W,w}(T) \) and we get a map of \( T \)-valued points

\[
\Upsilon'(T) : \text{Bil}(V,I)(T) \rightarrow \text{Assoc}_{W,w}(T) : b \mapsto A_b.
\]

This is functorial in \( T \) because of (3), Theorem 2.2 and hence defines an \( X \)-morphism

\[
\Upsilon' : \text{Bil}(V,I) \rightarrow \text{Assoc}_{W,w}.
\]

The morphism \( \Upsilon' \) is \( \text{GL}_V \)-equivariant due to (2), Theorem 2.2. Notice that the schemes \( \text{Bil}(V,I), \text{Bil}_0(V,I) \) and \( \text{Assoc}_{W,w} \) are well-behaved relative to \( X \) with respect to base-change. In fact, so are \( \text{Azu}_{W,w} \) and \( \text{SpAzu}_{W,w} \), as may be recalled from Theorems 2.14 and 2.15 of page 24. In view of these observations, by taking a trivialisation for \( I \) over \( X \), we may reduce to the case when \( I \) is trivial. We treat this case next.
Reduction of Proof of Theorem 3.8 for $I = \mathcal{O}_X$ to Theorem 3.9

We first recall the following crucial fact (see (1), Prop.3.2.4, Chap.IV [9]): The even Clifford algebra of a semiregular quadratic form is an Azumaya algebra. Using this fact and the definition of $\Upsilon'$, we see that the morphism $\Upsilon'$ restricted to $\text{Bil}_V^{sr}$ factors through $\text{Azu}_{W,w}$ by a morphism $\Upsilon^{sr}$ such that the following diagram is commutative

\[
\begin{array}{ccc}
\text{Bil}_V & \overset{\Upsilon'}{\longrightarrow} & \text{Assoc}_{W,w} \\
\uparrow & & \uparrow \\
\text{Bil}_V^{sr} & \overset{\Upsilon^{sr}}{\longrightarrow} & \text{Azu}_{W,w}
\end{array}
\]

where the vertical arrows are the canonical open immersions. The above diagram base changes well in view of (2), Theorem 2.14, Prop.2.1 and (3), Theorem 2.2.

Notice that since the structure morphism $\text{Bil}_V \longrightarrow X$ is an affine morphism, and since the same is true of $\text{Assoc}_{W,w} \longrightarrow X$, it is also true of $\Upsilon'$. In particular, $\Upsilon'$ is quasi-compact and separated, and therefore has a schematic image by case (1) of Prop.2.12. The same is true of each of the two vertical arrows and of $\Upsilon^{sr}$ in view of Prop.2.1 and (1) of Theorem 2.14. Further, as noted in Prop.2.1, $\text{Bil}_V^{sr} \subseteq \text{Bil}_V$ is schematically dominant and therefore by (5), Prop.2.13, the limiting scheme of the former in the latter is the latter itself. So using the commutative diagram above, the transitivity of the schematic image (assertion (3), Prop.2.13), and the definition of $\text{SpAzu}_{W,w}$ (assertion (1), Theorem 2.15), we see that in order to prove (1), Theorem 3.8, it is enough to show that

\[\Upsilon^{sr} \text{ is schematically dominant and surjective, and } \Upsilon' \text{ is a closed immersion.}\]

We now claim that the above properties are equivalent to

\[\Upsilon^{sr} \text{ is proper and } \Upsilon' \text{ is a closed immersion.}\]

Suppose (**) holds. To show (*), we only need to show that $\Upsilon^{sr}$ is surjective and schematically dominant. From (**) it follows that

\[\Upsilon^{sr}_K := \Upsilon^{sr} \otimes_X K\]

is functorially injective and proper for each algebraically closed field $K$ with an $X$-morphism $\text{Spec}(K) \longrightarrow X$.

That both the $K$-schemes

\[\text{Bil}_V^{sr} \otimes_X K \text{ and } \text{Azu}_{W,w} \otimes_X K\]

are integral and smooth of the same dimension follows from the smoothness of relative dimension 9 and geometric irreducibility $/X$ of $\text{Bil}_V^{sr}$ (which is obvious),
and of Azu\textsubscript{W,w} from (3), Theorem 2.14. Since $\Upsilon_{K}^{sr}$ is differentially injective at each closed point, it has to be a smooth morphism by Theorem 17.11.1 of EGA IV and thus has to be an open map.

But by (**) it is also proper and hence a closed map. Thus $\Upsilon_{K}^{sr}$ is bijective etale, and hence an isomorphism. This also gives that $\Upsilon^{sr}$ is surjective. Now from Cor.11.3.11 of EGA IV and from the flatness of $\text{Bil}_{V}^{sr}$ over $X$, it follows that $\Upsilon^{sr}$ is itself flat, and hence schematically dominant since it is faithfully flat (being already surjective). Therefore (**) $\Rightarrow$ (*).

The property of a morphism being proper is local on the target (see for e.g., (f), Cor.4.8, Chap.IV, [7]) and the same is true of the property of being a closed immersion. Therefore, in verifying (**), we may assume that $V$ is free over $X$ so that

$$W = \Lambda^{even}(V)$$

is also free over $X$ and $w$ is part of a global basis. We are now in the situation of Theorem 3.9. Granting it, we see immediately from the multiplication table that (**) holds. For the table shows that the composition of the following $X$-morphisms

$$\text{Bil}_{V} \xrightarrow{\Upsilon'} \text{Assoc}_{W,w}^{\text{Closed}} \xrightarrow{\text{Alg}_{W}}$$

is a closed immersion, which implies that $\Upsilon'$ is also a closed immersion. Further, the multiplication table also shows that both $\Upsilon'$ and $\Upsilon^{sr}$ satisfy the valuative criterion for properness, and are therefore proper. Thus the conditions (**) are verified. So we have reduced the proof of (1), Theorem 3.8 to Theorem 3.9.

As for statement (2) of Theorem 3.8, firstly, the involution $\Sigma$ of $\text{Assoc}_{W,w}$ defines a unique involution (also denoted by $\Sigma$) on the scheme of specialisations $\text{SpAzu}_{W,w}$ (leaving the open subscheme $\text{Azu}_{W,w}$ invariant) because of the defining property of the schematic image involved; for we may verify that an automorphism of a scheme $T$ which leaves an open subscheme $U$ stable will also leave stable the limiting scheme of $U$ in $T$ (of course we assume here that the canonical open immersion $U \hookrightarrow T$ is a quasi-compact open immersion, which ensures the existence of the limiting scheme). Secondly, a glance at the multiplication table of Theorem 3.9 keeping in view the definition of opposite algebra shows that indeed the induced $\Sigma \in \text{Aut}_{X}(\text{Bil}_{V})$ takes the $T$-valued point $B = (b_{ij})$ to $\text{transpose}(-B) = (-b_{ji})$. Finally, assertion (3) of Theorem 3.8 is a consequence of (1) taking into account (3), Theorem 2.15.

**Proof of Theorem 3.9** Given $B = (b_{ij}) \in \text{Bil}_{V}(T)$, by our definition above, $(\Upsilon'(T))(B) = A_{B}$ is the algebra structure induced from the linear isomorphism

$$\psi_{B} : C_{0}(V_{T}, q_{B}) \cong \Lambda^{even}(V_{T})$$

of (2d), Theorem 2.2. The stated multiplication table for $A = A_{B}$ is a consequence of straightforward calculation, keeping in mind (2f), Theorem 2.2 and the standard properties of the multiplication in the even Clifford algebra $C_{0}(V_{T}, q_{B})$. Q.E.D., Theorems 3.9 & 3.8.
Proof of Theorem 3.11 The proof follows by comparing the multiplication table relative to $\Theta$ as computed in Theorem 5.1 with the multiplication table relative to $\Upsilon$ of Theorem 3.9 computed above. Q.E.D., Theorem 3.11

Proofs of assertions in (a), Theorem 3.10 and Surjectivity part of Theorem 3.11 Let $W$ be the rank 4 vector bundle underlying the specialised algebra $A$ and $w \in \Gamma(X,W)$ be the global section corresponding to $1_A$. We choose an affine open covering $\{U_i\}_{i \in \mathcal{J}}$ of $X$ such that $W|U_i$ is trivial and $w|U_i$ is part of a global basis $\forall i$. Therefore on the one hand, for each $i \in \mathcal{J}$, we can find a linear isomorphism

$$\zeta_i : \Lambda^\even (\mathcal{O}_X^{\oplus 3}) \cong W|U_i$$

taking $1_{\Lambda^\even}$ onto $w|U_i$. The $(w|U_i)$-unital algebra structure $A|U_i$ induces via $\zeta_i$ an algebra structure $A_i$ on $\Lambda^\even (\mathcal{O}_X^{\oplus 3})$ (so that $\zeta_i$ becomes an algebra isomorphism). Recall that $A_i$ is also a specialised algebra structure by (3), Theorem 2.15. Hence by Theorem 3.8 applied to $X = U_i$, $V = \mathcal{O}_U^{\oplus 3}$ and $I = \mathcal{O}_{U_i}$, we can also find an $\mathcal{O}_{U_i}$-valued quadratic form $q_i$ on $\mathcal{O}_X^{\oplus 3}|U_i$, induced from a bilinear form $b_i$ so that the algebra structure $A_i$ is precisely the one induced by the linear isomorphism

$$\psi_{b_i} : C_0 (\mathcal{O}_X^{\oplus 3}|U_i, q_i) \cong \Lambda^\even (\mathcal{O}_X^{\oplus 3}|U_i)$$

given by (2d) of Theorem 2.2. For each pair of indices $(i,j) \in \mathcal{J} \times \mathcal{J}$, let $\zeta_{ij}$ and $\phi_{ij}$ be defined so that the following diagram commutes:

$$\begin{align*}
C_0(\mathcal{O}_X^{\oplus 3}|U_{ij}, q_j|U_{ij}) &\xrightarrow{\psi_{b_i}|U_{ij}} \Lambda^\even (\mathcal{O}_X^{\oplus 3}|U_{ij}) \xrightarrow{\zeta_{ij}|U_{ij}} A|U_{ij} \\
\phi_{ij} \downarrow \cong &\quad \zeta_{ij} \downarrow \cong \quad = \downarrow \\
C_0(\mathcal{O}_X^{\oplus 3}|U_{ij}, q_j|U_{ij}) &\xrightarrow{\psi_{b_j}|U_{ij}} \Lambda^\even (\mathcal{O}_X^{\oplus 3}|U_{ij}) \xrightarrow{\zeta_{ij}|U_{ij}} A|U_{ij}
\end{align*}$$

The above diagram means that the algebras $A_i$ glue along $U_{ij} := U_i \cap U_j$ via $\zeta_{ij}$ to give (an algebra bundle isomorphic to) $A$, and in the same vein, the even Clifford algebras $C_0(\mathcal{O}_X^{\oplus 3}, q_i)$ glue along the $U_{ij}$ via $\phi_{ij}$ to give $A$ as well. Now consider the similarity

$$g^\phi_{ij} := s_{-1}^+(\phi_{ij}) : (\mathcal{O}_X^{\oplus 3}|U_{ij}, q_i|U_{ij}) \cong s_{i,j} (\mathcal{O}_X^{\oplus 3}|U_{ij}, q_j|U_{ij})$$

with multiplier

$$l_{ij} := \det(\phi_{ij})^{-1}$$

given by (c), Theorem 3.2. Since $s_{-1}^+$ is multiplicative, and since $\phi_{ij}$ satisfy the cocycle condition, it follows that $s_{-1}^+(\phi_{ij})$ also satisfy the cocycle condition and therefore glue the $\mathcal{O}_X^{\oplus 3}|U_i$ along the $U_{ij}$ to give a rank 3 vector bundle $V$ on $X$. While the $q_i$ do not glue to give an $\mathcal{O}_X$-valued quadratic form on $V$, the facts that the multipliers $\{l_{ij}\}$ form a cocycle for

$$I := \det^{-1}(A)$$
and that $s_{(-1)}^+$ is a section together imply, taking into account the uniqueness in (1), Prop. 2.8, that actually the $q_i$ glue to give an $I$-valued quadratic form $q$ on $V$ and that $C_0(V, q, I) \cong A$. We shall now revert to the notations of Section 3. By Theorem 3.6, we have

$$h_s(g_{ij}, l_{ij}, q_i | U_{ij}, q_j | U_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & l_{ij}^{-1} \Lambda^2(g_{ij}^{\phi_{ij}}) \end{pmatrix}$$

which means that

$$(\phi_{ij})_{\Lambda^2} = \det(\phi_{ij}) \times \Lambda^2(g_{ij}^{\phi_{ij}}).$$

This immediately implies part (1) of assertion (a) of Theorem 3.6, from which parts (2)—(4) can be deduced using the standard properties of the determinant and the perfect pairings between suitable exterior powers of a bundle.

Proofs of assertions in (b), Theorem 3.6. We first prove (b1). Let $A$ be a given specialisation, and let $A \cong C_0(V, q, I)$ as in part (a) of Theorem 3.6 with $I = \det^{-1}(A)$. By the injectivity part of Theorem 3.1, we have

$$C_0(V, q, I) \cong A \cong C_0(V', q', O_X)$$

iff there exists a twisted discriminant bundle $(L, h, J)$ and an isomorphism

$$(V, q, I) \cong (V', q', O_X) \otimes (L, h, J).$$

The latter implies that $I \cong J \cong L^2$ and hence $\det(A) \in 2 \text{Pic}(X)$. On the other hand, if this last condition holds, we could take for $L$ a square root of $J := I^{-1}$, along with an isomorphism $h : L^2 \cong J$ and we would have by Prop. 2.8 an algebra isomorphism

$$\gamma_{(L, h, J)} : C_0(V \otimes L, q \otimes h, O_X) \cong C_0((V, q, I) \otimes (L, h, J)) \cong C_0(V, q, I) \cong A.$$  

For the proof of (b2), suppose that the line subbundle $O_{X^{-1}}A \hookrightarrow A$ is a direct summand of $A$. We may choose a splitting

$$A \cong O_{X^{-1}}A \oplus (A/O_{X^{-1}}A).$$

Using assertion (1) of (a), Theorem 3.6, we see that there exists a rank 3 vector bundle $V$ on $X$ such that

$$A \cong O_{X^{-1}}A \oplus (A/O_{X^{-1}}A)$$

$$\cong O_{X^{-1}}A \oplus (\Lambda^2(V) \otimes I^{-1}) \cong O_{X^{-1}}A \oplus \Lambda^2(V) \otimes I^{-1} =: W$$

where $I := \det^{-1}(A)$ and the last isomorphism is chosen so as to map $O_{X^{-1}}A$ isomorphically onto $O_{X^{-1}}A$. Therefore if

$$(W, w) := (O_X^{-1} \oplus \Lambda^2(V) \otimes I^{-1}, 1),$$
then by the above identification \( A \) induces an element of \( \text{SpAz}_W,w(X) \), and since

\[
\Upsilon : \text{Bil}(V, I) \cong \text{SpAz}_W,w
\]

is an \( X \)-isomorphism by (1), Theorem 3.8, it follows that there exists an \( I \)-valued global quadratic form \( q = q_b \) induced from an \( I \)-valued global bilinear form \( b \) on \( V \) such that the algebra structure \( \Upsilon(b) \cong A \). (We recall that \( \Upsilon(b) \) is the algebra structure induced from the linear isomorphism

\[
\psi_b : C_0(V, q = q_b, I) \cong \mathcal{O}_X I \cong \Lambda^2(V) \otimes I^{-1} = W
\]

of (2d), Theorem 2.2, which preserves 1 by (2a) of the same Theorem). The proof of (b3) follows from a combination of those of (b1) and (b2). Q.E.D., Theorem 3.6 and surjectivity of Theorem 3.1.

7 Specialised Algebras on Self-Dual Bundles

In this section, we investigate the specialised algebras when the underlying bundle is self-dual and prove Theorem 3.19. We first have the following general result, which yields part of the assertions in (a), Theorem 3.19 that concern only rank 4 vector bundles.

**Proposition 7.1** Let \( X \) be any scheme, \( n \) an integer \( \geq 2 \) and \( W \) a rank \( n^2 \) vector bundle over \( X \) with nowhere-vanishing global section \( w \). Then \( X \) is irreducible (resp. reduced) iff \( \text{Az}_W,w \) is irreducible (resp. reduced) iff \( \text{SpAz}_W,w \) is irreducible (resp. reduced).

**Proof:** We first make certain observations when \( X \) is any reduced scheme and \( W \) is a rank \( n^2 \) vector bundle over \( X \) with nowhere-vanishing global section \( w \). It is not hard to see that the structure morphism

\[
\text{Az}_W,w \to X
\]

is in fact a morphism of finite presentation. Hence, in view of Prop.17.5.7, EGA IV [5, assertion (3) of Theorem 2.14] implies that \( \text{Az}_W,w \) is reduced. Since \( \text{SpAz}_W,w \) is the schematic image of \( \text{Az}_W,w \), it follows from (1), Prop.2.13 that \( \text{SpAz}_W,w \) is reduced as well. When \( X \) is integral, we next verify that \( \text{Az}_W,w \) is also integral. Consider any affine open subscheme

\[
U = \text{Spec}(R) \hookrightarrow X
\]

such that \( W|U \) is trivial and \( w|U \) is part of a global basis. There are \( (w|U) \)-unital Azumaya algebra structures on \( W|U \) which are isomorphic to the \( (n \times n) \)-matrix algebra over \( R \). Consider the orbit morphism

\[
\text{Stab}_{w|U} \to \text{Az}_{W|U,w|U}
\]
corresponding to one such algebra structure. Assertions (2) and (3) of Theorem 2.14 show that the topological image of this morphism is dense. Further, Stab_{w|U} is integral since it is $\cong A^1_{U^2}$ and since $U$ is integral. Thus

$$Az_{W|U, w|U}$$

is integral. Recall (e.g., Prop.2.1.6 & Cor.2.1.7, Chap.0.2, EGA I [6]) that a nonempty topological space, whose set of irreducible components is locally finite, is locally irreducible iff each irreducible component is open; further it is irreducible iff it is locally irreducible and connected. Now since $Az_{W, w}$ can be covered by irreducible open subschemes which pairwise intersect (as $X$ is irreducible), it follows that $Az_{W, w}$ is integral as well.

Putting the above facts together with (1), Prop.2.13 shows that $SpAz_{W, w}$ is also integral if $X$ is integral. Using again the facts that $Az_{W, w}$ behaves well under base change (Theorem 2.14) and that the schematic image of an irreducible scheme is irreducible (by (1), Prop.2.13), we may infer from the foregoing that $X$ is irreducible iff $Az_{W, w}$ is irreducible iff $SpAz_{W, w}$ is irreducible. Q.E.D., Prop.7.1

**Proof of assertions (a)—(d) of Theorem 3.19.** The proofs follow essentially from the properties of

$$\star \star \star$$

$$SpAz_{W, w} \rightarrow X \text{ and } Az_{W, w} \rightarrow X$$

as mentioned in Theorems 2.15 and 2.13. Part of the assertions in (a) are valid more generally and were proved in Prop.7.1. We indicate proofs for the not-so-obvious assertions, especially (c) and for the implication

$$X \text{ irreducible } \Rightarrow D_X \text{ irreducible}$$

in statement (a). The converse implication would follow from the fact that $D_X \rightarrow X$ is topologically surjective. Since the structure morphisms (\star \star \star) are smooth, they are faithfully flat (hence surjective). Therefore in view of (b), which is actually the result of good base-change properties of $SpAz_{W, w}$ and $Az_{W, w}$ relative to $X$, the irreducible components of $D_X$ and of $SpAz_{W, w}$ are induced from those of $X$ by pullback—provided we check the particular case when $X$ is irreducible and reduced.

So let $X$ be integral and first assume that $W$ is trivial and $w$ is part of a global basis. Without loss of generality, we may take

$$(W, w) = (\Lambda^{even}(V), 1)$$

for $V$ a rank 3 trivial vector bundle over $X$. After fixing a suitable basis for $V$, we may define the morphism $\Theta$, which by Theorem 3.10 is an isomorphism that maps the closed subset

$$(Quad_V \times_X L_w) \setminus (Quad_V^{x_r} \times_X L_w)$$

onto

$$D_0 := SpAz_{W, w} \setminus Az_{W, w}.$$
Therefore the irreducibility of $D_0$ is equivalent to that of the closed subset $\Quad_V \setminus \Quad_V^{sr}$.

Recall from the discussion on semiregular forms (page 10, Section 2) that the open subscheme $\Quad_V^{sr}$ corresponds to localisation by the polynomial $P_3$. This polynomial is irreducible as an element of $R[\zeta_i, \zeta_{ij}] \approx R[\Quad_V]$ when $X = \text{Spec}(R)$ is affine, though it is not clear if it is a prime element (unless we assume something more e.g., $R$ a UFD). The closed subset $\Quad_V \setminus \Quad_V^{sr}$ may be given the canonical closed subscheme structure $Z(P_3)$ corresponding to the vanishing of $P_3$. Let $q^{(2)}$ be the global quadratic form on $V$ given by $(x_1, x_2, x_3) \mapsto x_1 x_2$.

It can be checked that $q^{(2)}$ is not semiregular, but that its restriction to the rank two (direct summand) vector subbundle generated by $\{e_1, e_2\}$ is regular. Therefore the $X$-valued point corresponding to $q^{(2)}$ lands topologically inside the closed subset underlying $Z(P_3)$. Consider the orbit morphism $O(q^{(2)}): \GL_V \rightarrow \Quad_V$ corresponding to this $X$-valued point which also lands topologically inside $Z(P_3)$. It will follow from assertion (2), Theorem 3.20, that the topological image of $O(q^{(2)})$ is dense in $Z(P_3)$. On the other hand this topological image is irreducible, since $\GL_V \cong \mathbb{A}^9_X$. It follows that $\Quad_V \setminus \Quad_V^{sr}$, and hence $D_0$, is irreducible in the case when $W$ is trivial and $w$ is part of a global basis over $X$. Since the reduced closed subscheme structure on $Z(P_3)$ is given by the radical of the ideal $(P_3)$, it follows that $\text{rad}(P_3)$ is the minimal prime divisor of $(P_3)$ and by Krull’s ‘Hauptidealsatz’ this prime has height 1. Therefore, the codimension of $D_0$ is also 1 in the present case. Now consider the case of a general $(W, w)$, choose an affine open covering $\{U_i = \text{Spec}(R_i)\}_{i \in I}$ of $X$ such that $W_i := W|U_i$ is trivial and $w_i := w|U_i$ is part of a global basis $\forall i$. The subset $D_i := \text{SpAz}_W, w_i \setminus \text{Az}_W, w_i$ is irreducible for each $i$ by the preceding discussion. On the other hand, by Theorems 2.13 and 2.15 the subsets $D_i$ form an open cover of $D := \text{SpAz}_W, w \setminus \text{Az}_W, w$.

Since $X$ is irreducible, we have $D_i \cap D_j \neq \emptyset$ when $i \neq j$. Thus $D$ is locally irreducible and connected, and hence irreducible (for e.g., by Cor.2.1.7, Chap.0, EGA I [3]).
Since $X$ is integral and noetherian, assertion (2) of Theorem 2.15 implies that $\text{SpAzu}_{W,w}$ is integral and noetherian as well. Therefore the codimension of $D$ in $\text{SpAzu}_{W,w}$ is at least 1. On the other hand (for e.g., by Prop.14.2.3, Chap.0, EGA IV, [5]) this codimension is bounded above by

$$1 = \text{Codim}(D_i, \text{SpAzu}_{W_i,w_i})$$

for any $i$, since

$$\text{SpAzu}_{W_i,w_i} \hookrightarrow \text{SpAzu}_{W,w}$$

is an open subset whose intersection with $D$ is precisely $D_i$.

**Proof of assertion (e) of Theorem 3.19** Let $X$ be an integral separated quasi-compact scheme and let $\pi : \text{SpAzu}_{W,w} \rightarrow X$ be the structure morphism. We need the following result.

**Lemma 7.2** The canonical map $H^0(X, \mathcal{O}^*) \rightarrow H^0(\text{SpAzu}_{W,w}, \mathcal{O}^*)$ induced by $\pi$ is an isomorphism.

For the proof, we first consider the case when $X = \text{Spec}(R)$ is affine, $W$ is trivial and further $w$ is part of a global basis for $W$. In this case, by Theorem 3.10, $\text{SpAzu}_{W,w}$ may be identified with $A^n_X = \text{Spec}(R[X_1, \ldots, X_9])$ and the assertion of the Lemma is trivial since $R[X_1, \ldots, X_9]^* = R^*$, $R$ being an integral domain.

Since $X$ is quasi-compact, we can always find a finite affine open covering which trivializes $W$ and restricted to each member of which $w$ becomes part of a global basis. We now proceed by induction on the number $n$ of open sets in such an affine covering $\mathcal{V}$ of $X$. The case $n = 1$ was dealt with in the previous paragraph. Consider $n = 2$ i.e., let $\mathcal{V} = \{V_0, V_1\}$ be a covering by two members. We have the following natural commutative diagram with exact rows and with the vertical arrows given by the natural maps induced by $\pi$, where $\mathcal{U} := \pi^{-1}\mathcal{V}$ is the Zariski-open affine covering of $\text{SpAzu}_{W,w}$ induced from $\mathcal{V}$ via $\pi$. We write $H^0(\cdot)$ for $H^0(\cdot, \mathcal{O}^*)$ for typographical reasons.

$$
\begin{array}{cccc}
0 & \rightarrow & H^0(\text{SpAzu}_{W,w}) & \rightarrow & H^0(U_0) \times H^0(U_1) & \rightarrow & H^0(U_{01}) \\
& & \uparrow & & \uparrow & & \uparrow \\
0 & \rightarrow & H^0(X) & \rightarrow & H^0(V_0) \times H^0(V_1) & \rightarrow & H^0(V_{01})
\end{array}
$$

Here $V_{01} := V_0 \cap V_1$ and $U_{01} := \pi^{-1}(V_{01})$. Note that both $V_{01}$ and $U_{01}$ are affine since $X$ is separated and since $\pi$ is an affine morphism. Also note that $U_i$ (resp. $U_{ij}$) may be identified with $\text{SpAzu}_{W_i,w_i}$ (resp. with $\text{SpAzu}_{W_{ij},w_{ij}}$) due to the good base-change property (3) of Theorem 2.15, where $W_i := W|_{V_i}$ etc. By the previous paragraph, the middle and right vertical arrows are isomorphisms, hence so is the
first. Suppose therefore that Lemma \[7.2\] is true for any integral separated quasi-compact \(X\) with an affine open covering of cardinality \(\leq n\) which trivializes \(W\) and to each member of which \(w\) restricts to part of a basis. Let

\[
V = \{V_i^n | 0 \leq i \leq n\}
\]

be such a covering with \((n + 1)\) members and let \(U := \pi^{-1}V\) be the covering of \(\text{SpAzu}_{W,w}\) induced by \(\pi\) from \(V\). Let \(V_0\) denote the union of the first \(n\) open sets \(V_i^n, U_0 := \pi^{-1}(V_0), V_1 := V_i^n\) and \(U_1 := \pi^{-1}(V_1)\). Then writing out a commutative diagram as done above for the case of a covering with \(2\) members, and using the induction hypotheses (for \(n\) for \(V_0\), for \(n = 1\) for \(V_1\) and again for \(n\) for \(V_{01}\), we conclude the proof of Lemma \[7.2\].

We now proceed with the proof of assertion \((e)\) of Theorem \[3.19\]. So let \(X\) be locally-factorial. We want to show that \(\text{Pic}(\pi)\) is an isomorphism. Recall that we have canonical isomorphisms \(\text{Pic}(T) \equiv H^1(T, \mathcal{O}_T^*)\) (Ex.4.5, Ch.III, \[7\]) and \(\text{Pic}(T) \equiv \text{CL}(T)\) for \(T\) locally-factorial (Cor.6.16, Ch.II, \[7\]). Here CL denotes the Weil divisor class group. We shall use these identifications in what follows without particular mention.

Consider first the case when \(X = \text{Spec}(R)\) is affine, \(W\) is trivial and further \(w\) is part of a global basis for \(W\). In this case, by Theorem \[3.10\] \(\text{SpAzu}_{W,w}\) may be identified with \(\mathbb{A}^n_X\) and so the assertion \((e)\) is a consequence of Prop.6.6, Ch.II, \[7\].

Next, let \(V = \{V_0, V_1\}\) be a Zariski-open affine covering of \(X\) which trivializes \(W\) and makes \(w|V_i\) part of a global basis for \(W|V_i\) \((i = 0, 1)\). We have the following commutative diagram with exact rows given by Mayer-Vietoris sequences (Ex.2.24, Ch.III, \[12\]) and with the vertical arrows given by the natural maps induced by \(\pi\), where \(U := \pi^{-1}V\) is the Zariski-open affine covering of \(\text{SpAzu}_{W,w}\) induced from \(V\) via \(\pi\). We again write \(H^i(-)\) for \(H^i(-; \mathcal{O}^*)\) for typographical reasons.

\[
\begin{array}{cccccc}
H^0(U_0) \times H^0(U_1) & \longrightarrow & H^0(U_{01}) & \longrightarrow & H^1(\text{SpAzu}_{W,w}) & \overset{(\pi)}{\longrightarrow} \\
\uparrow & & \uparrow & & \uparrow & \\
H^0(V_0) \times H^0(V_1) & \longrightarrow & H^0(V_{01}) & \longrightarrow & H^1(X) & \overset{(\pi)}{\longrightarrow} \\
\downarrow^{(\pi)} & \downarrow & \downarrow & \downarrow & \downarrow & \\
H^1(\text{SpAzu}_{W,w}) & \longrightarrow & H^1(U_0) \times H^1(U_1) & \longrightarrow & H^1(U_{01}) & \\
\downarrow^{(\pi)} & \downarrow & \downarrow & \downarrow & \downarrow & \\
H^1(X) & \longrightarrow & H^1(V_0) \times H^1(V_1) & \longrightarrow & H^1(V_{01}) & \\
\end{array}
\]

Here \(V_{01} := V_0 \cap V_1\) and \(U_{01} := \pi^{-1}(V_{01})\). By Lemma \[7.2\], the first and second vertical arrows are isomorphisms. By the previous paragraph, the fourth and fifth vertical arrows are also isomorphisms. Therefore the 5-Lemma implies that the central vertical arrow is an isomorphism too. We now conclude the proof of assertion \((e)\) by induction on the number of members in an open covering of \(X\) (just as in the proof of Lemma \[7.2\]).
Proof of assertion (f) of Theorem 3.19. Let us remind the reader that the pullbacks

\[ W' := W \otimes_X \text{SpAzu}_{W,w} \quad \text{and} \quad W \otimes_X \text{Azu}_{W,w} \]

are naturally endowed with \((w \otimes_X \text{SpAzu}_{W,w})\)-unital associative algebra structures \(A'\) and \(A'|\text{Azu}_{W,w}\) with which they become respectively the universal specialisation and universal Azumaya algebra relative to the pair \((W, w)\)—for details see the proof of Theorem 3.4, Part I, [17]. By (c) and (d), the natural map

\[ \text{Pic}(\text{SpAzu}_{W,w}) \to \text{Pic}(\text{Azu}_{W,w}) \]

is surjective and its kernel is generated by the image of \(Z(D_X)\). Given an isomorphism \(\phi : W \to W\), we consider its pullback \(\phi'\) to \(\text{SpAzu}_{W,w}\), and let

\[ (X', A', \phi') := (\text{SpAzu}_{W,w}, W \otimes_X \text{SpAzu}_{W,w}, \phi \otimes_X \text{SpAzu}_{W,w}) \]

Note that \(D_X = X'\setminus U(X', A')\). Consider the composition of the following morphisms of vector bundles:

\[ W' \otimes_X W' \equiv A' \otimes_X A'^{\text{op}} \xrightarrow{(a \mapsto ba)} \text{End}(A') \equiv W' \otimes_X W' \xrightarrow{\phi' \otimes \text{Id}} W' \otimes_X W'. \]

The above composite gives an endomorphism of the vector bundle \(W' \otimes W'\) which is an isomorphism precisely at the local rings of the points of \(U(X', A')\). Therefore, the induced element

\[ s(\phi') \in \Gamma(X', O_{X'}) \equiv H^0(X', \text{End}(\det(W' \otimes W'))) \]

goes into the maximal ideal of the local ring at each point of \(D(X', A')\) and to a unit of the local ring at each point of \(U(X', A')\). It follows that \(n.(D_X)\) is principal for some \(n \geq 1\). Now \(A'|\text{Azu}_{W,w}\) is an Azumaya algebra bundle, and as seen in Theorem 2.10 (cf. also the discussion following Prop. 3.12) it is isomorphic to the even Clifford algebra of a canonically obtained rank 3 quadratic bundle on \(\text{Azu}_{W,w}\) with values in the structure sheaf. Therefore it follows from (b1), Theorem 3.6 that

\[ \det(W \otimes_X \text{SpAzu}_{W,w}) \]

maps to an element of \(2.\text{Pic}(\text{Azu}_{W,w})\). The last assertion in (f) is now again a consequence of (b1), Theorem 3.6.

Proof of assertion (g) of Theorem 3.19. Suppose that \(s\) is a section to \(\pi' : \text{Azu}_{W,w} \to X\) corresponding to an Azumaya algebra structure on \(W\) with identity \(w\). By assertions (e) and (f) (notice that \(W\) is self-dual!) of Theorem 3.19 the homomorphism

\[ \text{Pic}(\pi') : \text{Pic}(X) \to \text{Pic}(\text{Azu}_{W,w}) \]

is surjective. But \(\pi' \circ s = \text{Identity}\) implies that \(\text{Pic}(\pi')\) is injective as well. The rest of the assertions in (g) now follow from (f). Q.E.D., Theorem 3.19.
8 Stratification of the Variety of Specialisations

In this section, we prove Prop. 3.13 and Theorem 3.20.

Proof of Prop. 3.13. Fix an $S$-basis $\{e_1, e_2, e_3\}$ for $V$, and with respect to this basis, let $q^1$ denote the quadratic form given by

$$(x_1e_1 + x_2e_2 + x_3e_3) \mapsto x_1x_2 + x_3^2.$$ 

It is easy to see that this quadratic form is semiregular. We show that any semiregular quadratic form $q$ can be moved to $q^1$ i.e., that $\exists g \in \text{GL}(V)$ such that $g \cdot q = q^1$.

By Prop. 3.17, Chap. IV, [9], there exists a basis $\{e_1', e_2', e_3'\}$ for $V$ such that $q$ restricted to the submodule generated by $e_1'$ and $e_2'$ is regular and further such that

$$q(e_3') \in S^*, \ b_q(e_1', e_2') = 1 \text{ and } b_q(e_1', e_3') = 0 = b_q(e_2', e_3').$$

Let $g' \in \text{GL}(V)$ be the automorphism that maps $e_i'$ onto the $e_i$ for each $i$ and consider the quadratic form $q' := g' \cdot q$. Then by definition of the GL(V)-action on the set Quad(V) of quadratic $S$-forms on $V$ we have

$$q'(e_3') \in S^*, \ b_{q'}(e_1, e_2) = 1 \text{ and } b_{q'}(e_1, e_3) = 0 = b_{q'}(e_2, e_3).$$

So if we assume that

$$q'(e_i) = \lambda_i \ (\iff q'(e_i') = \lambda_i),$$

then we would have

$$q'(x_1e_1 + x_2e_2 + x_3e_3) = \lambda_1x_1^2 + \lambda_2x_2^2 + \lambda_3x_3^2 + x_1x_2 \ \forall \ x_i \in S.$$

Thus it is enough to show that $q'$ can be moved to $q^1$. We look for an invertible matrix

$$g'' = (u_{ij}) \in \text{GL}(V) \text{ such that } g'' \cdot q' = q^1.$$ 

Writing this condition equivalently as $q' = (g'')^{-1} \cdot q^1$ and comparing the polynomials in the $x_i$ gives 6 equations in terms of the $u_{ij}$ and the $\lambda_i$ which are to be satisfied. We choose the $u_{ij}$ as follows. First set

$$u_{11} = u_{22} = 0 \text{ and let } u_{12} \in S^*$$

be a free parameter. Since every element of $S$ has square roots in $S$, it makes sense to choose

$$u_{31} = \pm \sqrt{\lambda_1} \text{ and } u_{32} = \pm \sqrt{\lambda_2}.$$ 

We let

$$\alpha = 1 + 2u_{31}u_{32}, \ \beta = 1 - 2u_{31}u_{32} \text{ and } u_{21} = \beta/u_{12}.$$ 

Since $q' = g \cdot q$ and since $q$ is semiregular, $q'$ is also semiregular. Its half-discriminant relative to the present basis of $V$ is (remembering that $\lambda_3 \in S^*$)

$$d_q(e_1, e_2, e_3) = \lambda_3(4\lambda_1\lambda_2 - 1) \in S^*.$$
This implies that 
\[ \alpha \beta = 1 - 4\lambda_1 \lambda_2 \in S^* \implies \alpha, \beta \in S^*. \]

Therefore it makes sense to define 
\[ u_{33} = \pm \sqrt{(\beta \lambda_3)/\alpha}, u_{13} = -2u_{31}u_{33}/u_1 \] and 
\[ u_{23} = -2u_{32}u_{33}/u_1. \]

Note that \( u_{33} \in S^* \). A computation shows that the determinant of the matrix \( g'' = (u_{ij}) \) defined above is \(-u_{33} \alpha \in S^* \) and hence \( g'' \) is invertible. It is also easily checked that \( g'' \cdot q' = q' \). \textbf{Q.E.D., Prop 3.13.}

**Proof of assertion (2) of Theorem 3.20.** (We shall not prove assertion (1) since it is well-known). Recall from the discussion on semiregular forms (page 10, Section 5) that the open subscheme \( \text{Quad}_V^{sr} \) corresponds to localisation by the polynomial \( P_3 \) and that this polynomial is prime as an element of \( k[\zeta_i, \zeta_{ij}] \sim k[\text{Quad}_V] \).

Here a quadratic form \( q \) corresponding to the point \((\lambda_i, \lambda_{ij}) \in \mathbb{A}_k^6 \) is given by 
\[ (x_1, x_2, x_3) \mapsto \sum_i \lambda_i x_i^2 + \sum_{i<j} \lambda_{ij} x_i x_j. \]

For ease of readability (and typesetting !) let us denote the closure \( \overline{T} \) of a subset \( T \) (given the reduced closed subscheme structure) by \( \langle T \rangle \) in what follows. Since \( \text{Quad}_V^{(1)} \) is the same as the variety underlying the open subscheme \( \text{Quad}_V^{sr} \) of semiregular quadratic forms, that 
\[ \langle \text{Quad}_V^{(1)} \rangle = \text{Quad}_V \]
follows from the fact that \( \text{Quad}_V \) is irreducible. By assertion (1) of the present Theorem, \( \text{Quad}_V \) is the disjoint union of the \( \text{Quad}_V^{(i)} \), therefore 
\[ \langle \text{Quad}_V^{(1)} \rangle \setminus \text{Quad}_V^{(1)} \]
is the disjoint union of 
\[ \{ \text{Quad}_V^{(i)} | 2 \leq i \leq 4 \} \]
and also equals the closed subset \( Z(P_3) \) defined by the vanishing of \( P_3 \). An explicit computation shows that the dimension of the stabilizer of \( q^{(2)} \) in \( \text{GL}(V) \) is 4. Since \( \text{Quad}_V^{(2)} \) is an open dense subvariety of \[ \langle \text{Quad}_V^{(2)} \rangle \subset V(P_3), \]
its closure is thus 5-dimensional. But since \( P_3 \) is an irreducible polynomial, \( Z(P_3) \) is also an irreducible 5-dimensional subvariety. It follows that 
\[ \langle \text{Quad}_V^{(2)} \rangle = \langle \text{Quad}_V^{(1)} \rangle \setminus \text{Quad}_V^{(1)}. \]
Since $\text{Quad}^{(2)}_V$ is smooth in its closure ($= Z(P_3)$ as seen above), the singularities of its closure are contained in

$$\text{Quad}^{(3)}_V \cup \text{Quad}^{(4)}_V$$

which consists of quadratic forms that are perfect squares i.e., squares of linear forms. These singularities may be identified with points

$$(\lambda_i, \lambda_{ij}) \in \mathbb{A}^6_k \cong \text{Quad}_V$$

at which all the partial derivatives of $P_3$ vanish. A simple computation shows that this set is the symmetric determinantal variety given by the vanishing of the $(2 \times 2)$-minors of the matrix of the symmetric bilinear form associated to the generic quadratic form given by

$$(x_1, x_2, x_3) \mapsto \sum_i \zeta_i x_i^2 + \sum_{i<j} \zeta_{ij} x_i x_j;$$

but it can also be shown that this set precisely corresponds to the perfect squares. Therefore

$$\text{Sing } (\langle \text{Quad}^{(2)}_V \rangle) = \langle \text{Quad}^{(2)}_V \rangle \setminus \text{Quad}^{(2)}_V.$$

That

$$\langle \text{Quad}^{(i+1)}_V \rangle = \langle \text{Quad}^{(i)}_V \rangle \setminus \text{Quad}^{(i)}_V$$

for $i = 2$ follows from the above and the obvious fact that any quadratic form can be specialised to the zero quadratic form. The case $i = 3$ is trivial. To see that

$$\langle \text{Quad}^{(3)}_V \rangle$$

is smooth if $\text{char}(k) = 2$, we first note from the above and assertion (1) of the present Theorem that the closure of $\text{Quad}^{(3)}_V$ consists of perfect squares. Since $\text{char}(k) = 2$, under the identification

$$\text{Quad}_V \cong \mathbb{A}^6_k,$$

the perfect squares are seen to correspond to the copy of $\mathbb{A}^3_k$ in $\mathbb{A}^6_k$ given by the vanishing of the last three coordinates $\lambda_{ij}, \ 1 \leq i < j \leq 3$. To see that the zero quadratic form is a singularity of

$$\langle \text{Quad}^{(3)}_V \rangle$$

if $\text{char}(k) \neq 2$, we note from the above that $\langle \text{Quad}^{(3)}_V \rangle$ is defined by the same equations that define the singularities of $Z(P_3)$ and is a certain symmetric determinantal variety. It is a well-known fact—an application of Standard Monomial Theory (for e.g., see [10] or [3])—that the ideal defining these equations is itself reduced, i.e., it is the ideal of the variety. Checking the Jacobian criterion now shows that the zero quadratic form is indeed a singular point.
The proof will follow from a series of lemmas. Recall that on page 48 we had defined the $X$-morphism
\[ \theta : \text{Quad}_V \to \text{Id-w-Sp-Azu}_W \]
which was used to define the isomorphism $\Theta$.

**Lemma 8.1** There are exactly 4 $H$-orbits in $\text{SpAzu}$. 

By Theorem 3.10, any point of $\text{SpAzu}$ is of the form $t \cdot \theta(q)$. Its $H$-orbit is $H \cdot \theta(q)$. There exists a unique $i$, $(1 \leq i \leq 4)$, and some $g \in \text{GL}(V)$ such that $q = g \cdot q^{(i)}$. Consider the algebra isomorphism $C_0(g, 1)$ of (1), Prop. 2.8 and the induced isomorphism $h(g, 1, q^{(i)}, q)$ of Theorem 5.5. By definition,
\[ \theta(q) = \theta(g \cdot q^{(i)}) = h(g, 1, q^{(i)}, q) \cdot \theta(q^{(i)}) \]
so that there are at most 4 orbits. To see that there are at least 4, we use the following result.

**Lemma 8.2** For $q \in \text{Quad}_V$, we have
\[ L_w \cdot \Theta((\text{GL}(V) \cdot q) \times \{ I_4 \}) = \Theta((\text{GL}(V) \cdot q) \times L_w) = H \cdot \theta(q). \]

On the one hand, we have
\[ \Theta((\text{GL}(V) \cdot q) \times L_w) = \{ t \cdot \theta(g \cdot q) \mid t \in L_w \text{ and } g \in \text{GL}(V) \} \]
\[ \subset H \cdot \theta(q). \]

Conversely take any $h \cdot \theta(q) \in H \cdot \theta(q)$. By Theorem 3.10 there exists a unique $t' \in L_w$ and a unique $q' \in \text{Quad}_V$ such that
\[ h \cdot \theta(q) = t' \cdot \theta(q'). \]

Therefore
\[ ((t')^{-1} \cdot h) \cdot \theta(q) = \theta(q'). \]

Since $k$ is algebraically- and hence quadratically closed, by (b), Theorem 3.4 there exists $g \in \text{GL}(V)$ such that $q' = g \cdot q$ and
\[ ((t')^{-1} \cdot h) = h(g, 1, q, q'). \]

Hence
\[ h \cdot \theta(q) = t' \cdot \theta(g \cdot q) = \Theta(g \cdot q, t') \Rightarrow H \cdot \theta(q) \subset \Theta((\text{GL}(V) \cdot q) \times L_w). \]

This settles Lemma 8.2. As for Lemma 8.1 if $q, q' \in \text{Quad}_V$ are such that their $\text{GL}(V)$-orbits are distinct, then because $\Theta$ is an isomorphism, Lemma 8.2 shows that $H \cdot \theta(q)$ is distinct from $H \cdot \theta(q')$. 

---

**Proof of Assertion (3) of Theorem 3.20**
Lemma 8.3 For each \( i \), with \( 1 \leq i \leq 4 \),
\[
\langle \text{Quad}(i) \rangle = \langle \text{Quad}(i) \rangle \times L_w.
\]
If \( f : X \to Y \) is a smooth morphism and \( U \to Y \) is an open subset, then
\[
f^{-1}(U) = \langle f^{-1}(U) \rangle.
\]
Since \( L_w \to \text{Spec}(k) \) is smooth, so is the induced morphism
\[
\langle \text{Quad}(i) \rangle \times L_w = \langle \text{Quad}(i) \rangle.
\]
Taking \( f \) to be this morphism and \( U = \text{Quad}(i) \) gives Lemma 8.3.

Lemma 8.4 The \( GL(V) \)-stratification of \( \text{Quad} \) induces a \( GL(V) \)-stratification of
\[
\text{Quad} \times L_w
\]
(the \( GL(V) \)-action on \( L_w \) taken to be trivial) with strata given by
\[
\langle \text{Quad}(i) \rangle : = \langle \text{Quad}(i) \rangle \times L_w, \quad (1 \leq i \leq 4).
\]
To prove Lemma 8.4 the only thing that needs to be checked is that
\[
\langle \text{Quad}(i+1) \rangle = \langle \text{Quad}(i) \rangle \setminus \langle \text{Quad}(i) \rangle.
\]
This follows by applying Lemma 8.3 twice:
\[
\langle \text{Quad}(i+1) \rangle = \langle \text{Quad}(i) \rangle \times L_w
\]
\[
= \langle \text{Quad}(i) \rangle \times L_w
\]
\[
= ((\text{Quad}(i)) \setminus \text{Quad}(i)) \times L_w
\]
\[
= \langle \text{Quad}(i) \rangle \times L_w \setminus \langle \text{Quad}(i) \rangle.
\]
Now according to Lemma 8.4, we have
\[
\text{SpAzu}(i) = \Theta(GL(V) \cdot \langle i \rangle \times L_w) = \Theta((\text{Quad} \times L_w)(i)).
\]
This combined with Lemma 8.4 and the fact that \( \Theta \) is an isomorphism (Theorem 3.10) completes the proof of assertion (3) of Theorem 3.20 Q.E.D., Theorem 3.20

Acknowledgements The author is grateful for the Postdoctoral Fellowship of the Graduiertenkolleg Gruppen und Geometrie under support from the Deutschen Forschungsgemeinschaft and the State of Niedersachsen at the Mathematisches Institut Göttingen where this paper was written.

It is a pleasure to thank the National Board for Higher Mathematics, Department of Atomic Energy, Government of India, for its Postdoctoral Fellowship (August 2002–August 2003) at the Chennai Mathematical Institute, Chennai, India, during the latter half of which certain special cases of the results of this work were obtained.
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