Electronic Maxwell’s equations

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Abstract
To date, the wave nature of electron has been widely researched, together with its similarity to optics. To unify electronic waves and electromagnetic waves, we establish four equations analogous to Maxwell’s equations by expressing the Dirac equation in terms of the quaternions. We develop some fundamental theories from the electronic version of Maxwell’s equations. In practice, solving electron wave problem is convenient without appearance of quantum operators such as Pauli matrices. The benefit is its potential in the analysis and applications of diverse electron beams, for example, phase-shift control. Moreover, physical quantities represented by electronic vector fields are notably similar to those in optics, making it possible to apply some ideas developed in optics in the research of electron.

1. Introduction

The particle property of light was proposed by Einstein in 1905 [1], which, credit to de Broglie, gave rise to the discovery of wave-particle duality. This duality reveals the similar nature of photons and electrons. Inspired by Hamilton’s discovery on the analogy between geometric optics and classical mechanics [2], Schrodinger founded an undulatory theory of the mechanics [3], asserting that electrons are indeed described by waves. In relativity, the wave equation of electron became Klein–Gordon equation [4, 5], which was corrected into Dirac equation by Dirac in 1928 [6]. The hidden similarity between Dirac equation and Maxwell’s equations was then uncovered, when photon wave function was discovered [7]. Thereafter, Maxwell’s equations were written in a form of Dirac equation [8–10].

Dirac equation is good at describing the particle nature of electrons, such as spin and orbital angular momentum [11]. So is the optical Dirac equation at that of photons [9]. With the optical Dirac equation, the particle nature of light is well described. Applying the ideas of electrons to photons via optical Dirac equation, we can get a good picture of the photon properties, such as momentum and angular momentum, as well as their spin–orbit interaction and topological properties [9, 12–15]. Discussing the particle nature of photons via optical Dirac equation has proved helpful.

Analogous to photons, electrons have a wave nature as evident from electron vortices [16–20]. Many researches on electrons have shown its similarity to photons [21–27]. Just as we understand a photon’s particle nature as that of electrons via the optical Dirac equation, it is natural to ask whether we can describe the wave nature of electrons as that of photons via Maxwell’s equations. Therefore, we derive and develop ‘Maxwell’s equations for electrons’, which we refer to as the electronic Maxwell’s equations (EME). The electronic fields are therefore polarized and consist of two parts, the longitudinal and the transversal. A similar argument holds for a ‘massive photon’. If the photon has nonzero rest mass, the electromagnetic field contains a longitudinally polarized field [28, 29]. Moreover, electronic waves are polarized in space just like electromagnetic waves and thereby present a high similar situation to optics when representing the physical quantities by fields. In practice, the problem can be solved conveniently without dealing with quantum operators, thereby providing a computational benefit in the analysis and applications of diverse electron beams.
2. Electronic Maxwell’s equations

To begin, we introduce the quaternionic optical Dirac equation, which is Maxwell’s equations given in the form of the Dirac equation. It encourages us to find the electronic field vectors, which are key to obtaining the EME.

We convert the electromagnetic field vectors expressed in Gaussian units into quaternions writing \( \vec{E} = \tau \cdot \vec{E} \) and \( \vec{B} = \tau \cdot \vec{B} \), where \( \tau = (\tau_x, \tau_y, \tau_z) = (i, j, k) \) with \( i, j, k \) denoting the imaginary units of the quaternion algebra. And we write charge density as \( \rho \). The imaginary unit of the complex fields \( E \) and \( B \) is denoted as \( \epsilon \), which satisfies \( s \epsilon = s \tau = s^2 = -1 \). Note that \( \vec{E} \) and \( \vec{B} \) are complex quaternions. We thus arrive with the quaternionic optical Dirac equation,

\[
\begin{align*}
\hat{\mathbf{s}} \frac{\partial}{\partial t} \Psi &= c \left[ \sigma \cdot \mathbf{s} \tau \right] \cdot \hat{\mathbf{p}} \Psi + c \hbar \left[ \hat{\mathbf{s}} \frac{\mathbf{f}}{\mathbf{t}} \right],
\end{align*}
\]

where \( \hat{\mathbf{p}} = -\hbar \nabla \) and \( \Psi = [\hat{\mathbf{E}}, \hat{\mathbf{s}} \hat{\mathbf{B}}]^{\top} \). Comparing (1) with the electronic Dirac equation, we then derive the EME

\[
\begin{align*}
\left( i \hbar \frac{\partial}{\partial t} - \beta mc^2 \right) \left[ \begin{array}{c} \varphi \\ \chi \end{array} \right] &= \left( \left( c I - i \sigma \cdot \varphi \right) \cdot \hat{\mathbf{p}} + \left[ -c \phi \sigma \cdot \chi, \chi \right] \cdot \sigma \cdot \epsilon \right) \left[ \begin{array}{c} \varphi \\ \chi \end{array} \right],
\end{align*}
\]

we find that Dirac equation (2) is represented in quaternion as (1). The Pauli matrices and the unit matrix together form a basis for quaternion. Mapping \( i \) to \( s \) and \( -i \sigma \) to \( \tau \), we find that \( \varphi \) and \( \chi \) play the role as \( \vec{E} \) and \( s \vec{B} \). We therefore introduce electronic field vectors \( \varphi \) and \( \chi \) as

\[
\left[ \begin{array}{c} \varphi \\ \chi \end{array} \right] = \left[ \begin{array}{c} f I - i \sigma \cdot \varphi \\ g I - i \sigma \cdot i \chi \end{array} \right],
\]

where \( f \) and \( g \) are scalar functions and the field vectors \( \varphi \) and \( \chi \) are defined as \( \varphi = (\varphi_x, \varphi_y, \varphi_z) \), \( \chi = (\chi_x, \chi_y, \chi_z) \), respectively. Just like \( \vec{E} \) and \( \vec{B} \), the components \( f I - i \sigma \cdot \varphi \) and \( g I - i \sigma \cdot i \chi \) are understood to be quaternions. The state function (3) is a \( 4 \times 2 \) matrix and its two columns are Dirac states describing the same electron; indeed, it is a dyadic Dirac state. The two columns of state (3) must be linear dependent, and thus they must satisfy conditions

\[
\begin{align*}
f^2 + \varphi \cdot \varphi &= 0, \\
g^2 + \chi \cdot \chi &= 0, \\
f g + \varphi \cdot \chi &= 0, \\
g \varphi - f \chi &= \varphi \times \chi
\end{align*}
\]

in which (6) is deducible from (4), (5) and (7). Functions \( f \) and \( g \) determined by \( \varphi \) and \( \chi \) are not manifestly new. We shall see later that \( f \) and \( g \) are related to the longitudinal waves.

Substituting (3) into the Dirac equation (2), we then derive the EME

\[
\begin{align*}
\nabla \cdot \varphi + \left( -\frac{1}{c} \frac{\partial}{\partial t} - k \Phi + i \mu \right) g &= 0, \\
\nabla \cdot \chi + \left( \frac{1}{c} \frac{\partial}{\partial t} + k \Phi + i \mu \right) f &= 0, \\
\nabla \times \varphi + \nabla f &= \left( -\frac{1}{c} \frac{\partial}{\partial t} - k \Phi + i \mu \right) \chi, \\
\nabla \times \varphi + \nabla g &= \left( \frac{1}{c} \frac{\partial}{\partial t} + k \Phi + i \mu \right) \varphi
\end{align*}
\]

where \( k = -ie/(\hbar c) \), \( \nabla = (\nabla - k \mathbf{A}) \) and \( \mu^{-1} = \hbar/(mc) \) the reduced Compton wavelength of electron. Except for the rest mass terms and \( f \) and \( g \), EME (8) is similar to Maxwell’s equations if \( \Phi \) and \( \mathbf{A} \) are
understood as the ‘source’ like $\rho$ and $J$. If $\Phi$ and $A$ are both zero, we obtain

\[
\begin{cases}
\nabla \cdot \varphi + \left( -\frac{1}{c} \frac{\partial}{\partial t} + i\mu \right) g = 0 \\
\nabla \cdot \chi + \left( \frac{1}{c} \frac{\partial}{\partial t} + i\mu \right) f = 0 \\
\nabla \times \varphi + \nabla f = \left( -\frac{1}{c} \frac{\partial}{\partial t} + i\mu \right) \chi \\
\nabla \times \chi + \nabla g = \left( \frac{1}{c} \frac{\partial}{\partial t} + i\mu \right) \varphi
\end{cases}
\]

(9)

which can be reduced to Maxwell’s equations in form if we set $\mu = 0$ and $f = g = 0$.

Unlike massless photons, there is a third state of polarization, the longitudinal polarization, which arise from a nonzero rest mass, as we shall see later. To demonstrate that $f$ and $g$ describe longitudinal wave, we consider a momentum-energy eigenstate with momentum $p$ and energy $\varepsilon$, giving

\[
\begin{align*}
\varphi_\perp &= -\frac{p}{p} \left( \frac{\varepsilon + mc^2}{cp} \right) g \\
\chi_\perp &= \frac{p}{p} \left( \frac{\varepsilon - mc^2}{cp} \right) f \\
\varphi_\parallel &= - \left( \frac{\varepsilon + mc^2}{cp} \right) \frac{p}{p} \times \chi_\parallel \\
\chi_\parallel &= \left( \frac{\varepsilon - mc^2}{cp} \right) \frac{p}{p} \times \varphi_\parallel
\end{align*}
\]

(10)

with longitudinal fields $\varphi_\perp = p^{-1} (p \cdot \varphi) p$ and $\chi_\perp = p^{-1} (p \cdot \chi) p$ and transversal fields $\varphi_\parallel = p^{-1} p \times \varphi \times p$ and $\chi_\parallel = p^{-1} p \times \chi \times p$. The first two equations of (10) show that $f$ and $g$ are indeed the longitudinal waves that are polarized along the line of motion. If $f$ and $g$ are both zero, there is no longitudinal wave. For example, a circularly polarized wave $\varphi = 2^{-1/2} [1, i, 0]^T \exp(ih^{-1}(pz - ct))$ contains no longitudinal wave since $f$ and $g$ are both zero. Thus, there exist electronic waves that are transversely polarized.

### 3. Physical quantities represented by electronic fields

The physical quantities of the electron are described by Dirac states, and hence we can represent them with electronic fields in accordance with (3). We know that the dyadic Dirac state (3) has two columns and each column is a Dirac state. Therefore, the inner product is an average of the inner products of each column,

\[
\langle \psi_1 | \psi_2 \rangle = \int_V d^3r \frac{1}{2} \text{tr}(\psi_1^\dagger \psi_2) = \int_V d^3r \text{Re}(\psi_1^\dagger \psi_2)
\]

\[
= \int_V d^3r (f_1 f_2 + g_1 g_2 + \varphi_1^\dagger \cdot \varphi_2 + \chi_1^\dagger \cdot \chi_2),
\]

(11)

where

\[
\psi_1 = \begin{bmatrix} f_1 I - i\sigma \cdot \varphi_1 \\ ig_1 I - i\sigma \cdot i\chi_1 \end{bmatrix}
\]

and

\[
\psi_2 = \begin{bmatrix} f_2 I - i\sigma \cdot \varphi_2 \\ ig_2 I - i\sigma \cdot i\chi_2 \end{bmatrix}.
\]

The function $\text{Re}$ in (11) extracts the real part of the quaternion on which it acts. With the help of (11), we may evaluate the expectation of any mechanical operator $\hat{M}$ as

\[
\langle \hat{M} \rangle = \int_V d^3r \text{Re}(\psi_1^\dagger \hat{M} \psi).
\]

(12)

We are now able to represent all physical quantities of electron with $f, g, \varphi$ and $\chi$ via (12).

Some interesting results arise that reveal the similarity between electrons and electromagnetic waves. For example, evaluating the probability density of the electronic field,

\[
\rho = |f|^2 + |g|^2 + |\varphi|^2 + |\chi|^2
\]

(13)
reveals a similarity to the energy density of electromagnetic field [28]. The spin angular momentum (SAM) density,

\[ \Sigma = -i\hbar f^{\mu} \varphi^\mu - f^{\mu} \varphi^\mu + g^{\mu} x^\mu - g x^\mu + \varphi^\mu \times \varphi + \chi^\mu \times \chi \] (14)

is analogous to the SAM density of electromagnetic field [30]. Moreover, we find that the probability and SAM are not independent of each other. With conditions (4) and (5), we establish

\[ |\varphi|^4 = |f|^4 + (-i\varphi^\mu \times \varphi)^2 \] (15)

and

\[ |\chi|^4 = |g|^4 + (-i\chi^\mu \times \chi)^2. \] (16)

The second term of (15) or (16) contributes to the spin of the electron in accordance with (14). We also see that (15) and (16) are analogous to Einstein’s energy–momentum relation. With (13), (15) and (16), we find that

\[ \rho = \frac{(-i\varphi^\mu \times \varphi)^2}{|\varphi|^4 - |f|^4} + \frac{(-i\chi^\mu \times \chi)^2}{|\chi|^4 - |g|^4}. \] (17)

We see that the probability density and SAM density relate each other in accordance with (14) and (17). There is also an analogous case in optics. Relations (15) and (16) hold in optics if we substitute \( \varphi \) with \( E \) and \( \chi \) with \( B \), noting that \( f^2 = -E \cdot E \) and \( g^2 = -B \cdot B \). This correspondence implies that the energy density of electromagnetic field and its SAM density are also not independent of each other.

While there are similarities between the electronic and electromagnetic waves, there are differences as well. From the example above, the major difference between the physical quantities exhibited by electronic waves and by electromagnetic waves is that there are contributions from the longitudinal waves of an electron. As evident in the previous section, the dyadic Dirac state (3) is distinct from the quaternionic optical Dirac state \( [\hat{E}, \hat{S}\hat{B}] \), because of its nonzero real components. We will see in the next section that \( f \) and \( g \) are indispensable because the rest mass is nonzero. Therefore, we consider the rest mass of electron as the main reason for the occurrence of the difference.

4. Lorentz transformation of the electronic field

Maxwell’s equations are Lorentz invariant as are the EME. However, the electronic field transforms in a manner distinct from the electromagnetic field. To retain Lorentz invariance of the EME, both \( f \) and \( g \) are indispensable because of the nonzero rest mass of the electron.

Now, for simplicity, let us suppose that we have a coordinate system \( K' \) moving at a velocity \( v \) in the direction of increasing \( x \) of system \( K \) in which EME (9) is established. Applying Lorentz transformation to the partial derivative with respect to time \( t \), gives

\[ \frac{1}{c} \frac{\partial}{\partial t} = \gamma \frac{1}{c} \frac{\partial}{\partial t'} - \frac{v}{c} \frac{\partial}{\partial x}, \] (18)

with \( \gamma = 1/\sqrt{1 - (v/c)^2} \). If, for example, we apply (18) to one equation of EME (9)

\[ \frac{1}{c} \frac{\partial}{\partial t} \varphi_x - \frac{\partial}{\partial x} g + i\mu \varphi_x = \frac{\partial}{\partial y} \chi_z - \frac{\partial}{\partial z} \chi_y, \] (19)

it becomes

\[ \frac{1}{c} \frac{\partial}{\partial t'} \gamma \left( \varphi_x + \frac{v}{c} g \right) - \frac{\partial}{\partial x'} \gamma \left( g + \frac{v}{c} \varphi_x \right) + i\mu \varphi_x = \frac{\partial}{\partial y'} \chi_z - \frac{\partial}{\partial z'} \chi_y. \] (20)

Obviously, the term with the partial time derivative and the mass term are not consistent. It should be in a form of

\[ \frac{1}{c} \frac{\partial}{\partial t'} \varphi_x - \frac{\partial}{\partial x'} g + i\mu \varphi_x = \frac{\partial}{\partial y'} \chi_z - \frac{\partial}{\partial z'} \chi_y, \] (21)

which is the corresponding equation of (19) in system \( K' \). On the other hand, we obtain another form for (20) if we write (18) in another way,

\[ \frac{1}{c} \frac{\partial}{\partial t} = \frac{1}{\gamma c} \frac{\partial}{\partial t'} - \frac{v}{c} \frac{\partial}{\partial x}, \]

which leads to

\[ \frac{1}{c} \frac{\partial}{\partial t'} \varphi_x - \frac{\partial}{\partial x'} g + i\gamma \left( \varphi_x + \frac{v}{c} g \right) = \left| \frac{\partial}{\partial y'} \chi_z + \frac{v}{c} \varphi_x \right| \gamma \left( \chi_z - \frac{v}{c} \varphi_x \right). \] (22)
Again, this equation is not in accordance with (21), because of the annoying nonzero mass. Nevertheless, if we combine (20) and (22) together, we obtain
\[
\left( \frac{1}{c} \frac{\partial}{\partial t'} + i\mu \right) \left( (1 + \gamma) \psi_x + \gamma \frac{v}{c} g \right) - \frac{\partial}{\partial x} \left( (1 + \gamma) g + \gamma \frac{v}{c} \psi_x \right)
\]
\[
= \left| \begin{array}{ll}
\frac{\partial}{\partial y} & \frac{\partial}{\partial z'} \\
(1 + \gamma) \chi_y + \gamma \frac{v}{c} \varphi_y & (1 + \gamma) \chi_z - \gamma \frac{v}{c} \varphi_y 
\end{array} \right|
\]
which is consistent with (21). We are not able to retain Lorentz invariance if we eliminate both \( f \) and \( g \) in (3). We hence suppose that, \( f \) and \( g \), i.e., the longitudinal fields, are attributes of the rest mass of the electron.

There should be only a difference of a factor \( u(v) \) between (23) and (21), which is in common for all functions of the one system of equations, and is independent of \( t, x, y \) and \( z \) but depends upon \( v \). We apply the same trick to the remaining equations of EME (9) yielding relations
\[
\begin{aligned}
f' &= u(v) \left( (1 + \gamma) f - \gamma \frac{v}{c} \chi_x \right) \\
\varphi_x' &= u(v) \left( (1 + \gamma) \varphi_x + \gamma \frac{v}{c} g \right) \\
\varphi_y' &= u(v) \left( (1 + \gamma) \varphi_y - \gamma \frac{v}{c} \chi_z \right) \\
\varphi_z' &= u(v) \left( (1 + \gamma) \varphi_z + \gamma \frac{v}{c} \chi_y \right)
\end{aligned}
\]
and
\[
\begin{aligned}
g' &= u(v) \left( (1 + \gamma) g + \gamma \frac{v}{c} \varphi_x \right) \\
\chi_y' &= u(v) \left( (1 + \gamma) \chi_y - \gamma \frac{v}{c} f \right) \\
\chi_z' &= u(v) \left( (1 + \gamma) \chi_z + \gamma \frac{v}{c} \varphi_y \right) \\
\chi_y' &= u(v) \left( (1 + \gamma) \chi_y - \gamma \frac{v}{c} \varphi_y \right)
\end{aligned}
\]
If we transform EME from \( K' \) to \( K \), we have
\[
2(1 + \gamma) u(v) u(-v) = 1.
\]
The transformation from \( K' \) to \( K \) should be symmetric with respect to that from \( K \) to \( K' \). That is to say, if we rotate these two frames along their own \( y \) axes through angle \( \pi \), turning \( x \) to \(-x\) and \( z \) to \(-z\), the transformation from \( K \) to \( K' \) becomes that from \( K' \) to \( K \). Thus, we have
\[
u(v) = u(-v).
\]
with (27) and \( u(0) = 1/2 \), (26) gives
\[
u(v) = \frac{1}{\sqrt{2(1 + \gamma)}}.
\]
Generally, with transformation (24) and (25) and the factor (28), we establish the Lorentz transformation of electronic fields as
\[
\begin{aligned}
f' &= \sqrt{\frac{1 + \gamma}{2}} f - \frac{\gamma}{\sqrt{2(1 + \gamma)}} \beta \cdot \chi \\
g' &= \sqrt{\frac{1 + \gamma}{2}} g + \frac{\gamma}{\sqrt{2(1 + \gamma)}} \beta \cdot \varphi \\
\varphi' &= \sqrt{\frac{1 + \gamma}{2}} \varphi + \frac{\gamma}{\sqrt{2(1 + \gamma)}} (\beta g + \beta \times \chi) \\
\chi' &= \sqrt{\frac{1 + \gamma}{2}} \chi - \frac{\gamma}{\sqrt{2(1 + \gamma)}} (\beta f + \beta \times \varphi)
\end{aligned}
\]
with \( \beta = v/c \). The Lorentz transformation of electronic fields (29) is identical to the Lorentz transformation of Dirac state [31].

With Lorentz transformation (29), we are able to build condition (7) from conditions (4) and (5), manifesting that only conditions (4) and (5) are the independent conditions that apply to \( f \) and \( g \). According to the principle of relativity, electronic fields in frame \( K' \) should also satisfy conditions (4) and (5); indeed they gives
\[
\frac{\varphi}{f} - \frac{\chi}{g} = \frac{\varphi}{f} \times \frac{\chi}{g}.
\]
That is to say, in special relativity, conditions (4) and (5) imply the condition (7). Furthermore, for a plane wave in free space, we have
\[ \chi = \frac{e}{\varepsilon + mc^2} (p \times \varphi + pf), \quad g = -\frac{e\mathbf{p} \cdot \varphi}{\varepsilon + mc^2}. \] (30)

Substituting (30) for \( \chi \) and \( g \) in (6) and (7), we find that conditions (6) and (7) hold for all plane waves. This implies that conditions (6) and (7) are not strong constraints.

5. Similarity between electronic waves and electromagnetic waves

In this section, we give an example that illustrate the similarity in the wave nature between electronic waves and electromagnetic waves. As Klein famously did [32], we discuss the tunneling problem in which an electronic beam impinges on a potential barrier perpendicularly. We then discuss a similar instance in optics. We treat both problem in a ‘classical’ way without appearance of any quantum operator.

Suppose that there is a potential distribution
\[ V = \begin{cases} 0 & \text{for } z < 0, \\ V_0 > 0 & \text{for } z \geq 0. \end{cases} \] (31)
in which we have an electronic beam propagating perpendicularly to the \( z = 0 \) plane. According to EME (8), we have
\[ \begin{cases} \nabla \cdot \varphi + \left( \frac{1}{c} \frac{\partial}{\partial t} - \frac{eV}{c\hbar} + i\mu \right) g = 0 \\ \nabla \cdot \chi + \left( \frac{1}{c} \frac{\partial}{\partial t} + \frac{eV}{c\hbar} + i\mu \right) f = 0 \\ \nabla \times \varphi + \nabla f = \left( -\frac{1}{c} \frac{\partial}{\partial t} - \frac{eV}{c\hbar} + i\mu \right) \chi \\ \nabla \times \chi + \nabla g = \left( -\frac{1}{c} \frac{\partial}{\partial t} + \frac{eV}{c\hbar} + i\mu \right) \varphi \end{cases} \] (32)
We consider the incident wave to have momentum \( \mathbf{p}_i \) directed along increasing \( z \). The reflected wave has a momentum \( -\mathbf{p}_i \) and the transmitted wave has momentum \( \mathbf{p} \). From (32), we have two energy equations,
\[ \varepsilon^2 = c^2 p_i^2 + m^2 c^4, \quad (\varepsilon - V_0)^2 = c^2 p^2 + m^2 c^4. \] (33)
With (33), the momentum of transmitted wave is
\[ p = \frac{1}{c} \sqrt{(\varepsilon - V_0 + mc^2)(\varepsilon - V_0 - mc^2)}. \] (34)
Continuity of the wave function requires the boundary condition to simply reads
\[ \Psi_i + \Psi_r = \Psi_t, \] (35)
where \( \Psi_i \) denotes the incident wave, \( \Psi_r \) the reflected wave and \( \Psi_t \) the transmitted wave. All four fields \( f, g, \varphi \) and \( \chi \) must satisfy this boundary condition (35). With (32) and boundary condition (35), we establish
\[ \Psi_t = \left( \frac{p_0(p_0 - p)}{mV_0} - \frac{\varepsilon}{mc^2} \right) \Psi_i \] (36)
and
\[ \Psi_i = \left( 1 + \frac{p_0(p_0 - p)}{mV_0} - \frac{\varepsilon}{mc^2} \right) \Psi_t. \] (37)
To simplify the reflection coefficient in (36), we write
\[ V_0 = \varepsilon + mc^2 + a, \] (38)
where \( a \) is not restricted. Then (34) reduces to
\[ p = \frac{1}{c} \sqrt{a(a + 2mc^2)}. \] (39)
Putting (38) into (36) and (37), gives the reflection coefficient
\[ R = \frac{a(m - \varepsilon c^2) - p_0 p}{mV_0} - 1 \] (40)
and the transmission coefficient

\[ T = \frac{a(m - \varepsilon c^2) - p_0 p}{m V_0} \]

If \( a \) is smaller than \(-2mc^2\), the reflection coefficient is positive, reaching 1 when \( a = -2mc^2 \). As \( a \) increases in value in the interval \((-2mc^2, 0)\), reflection coefficient is an imaginary number that is manifested as a phase shift in the reflected wave. The phase of reflection coefficient (40) is

\[ \theta = \arctan \frac{cp_0 \sqrt{|a|(2mc^2 - |a|)}}{m^2c^4 + \varepsilon (mc^2 - |a|)} \]

When \( a = 0 \), the reflection coefficient is \(-1\), the magnitude of the phase shift being \( \pi \). If \( a \) is increased further, reflection coefficient falls below \(-1\), producing electron-positron pairs.

For optics, analogous to (31), we have a distribution of refractive index of

\[ n = \begin{cases} 1 & \text{for } z < 0 \\ n_0 > 1 & \text{for } z \geq 0 \end{cases} \]

Analogously, we define the effective potential for photon as

\[ V' = \varepsilon' - cp' = \varepsilon' - cp'_0/n_0, \]

where \( p'_0 \) denotes the momentum of the incident photon, \( \varepsilon' = cp'_0 \) the energy of the photon and \( p' \) the momentum of transmitted photon. Applying the Fresnel equation, reflection coefficient is given by

\[ R' = \frac{1 - n_0}{1 + n_0} = \frac{-V'}{2cp'_0 + V'} \]

and transmission coefficient is

\[ T' = \frac{2cp'_0}{2cp'_0 + V'}. \]

We see that reflection coefficient (40) and (44) are both determined by the potential. The optical reflection coefficient (44) is always a real number because the rest mass of photon is zero. Moreover, reflection coefficient (40) is an imaginary number when the potential energy of electron lies in the gap between \( \varepsilon - mc^2 \) and \( \varepsilon + mc^2 \) because of the nonzero rest mass. When the potential energy of the electron is \( \varepsilon + mc^2 \), there exist a half-wavelength loss in the reflection of electron, analogous to light. When the potential energy of electron is lower than \( \varepsilon - mc^2 \), the behavior of electronic wave is similar to that of light. As we can see, EME works as effective as Dirac equation.

6. Conclusions

We modify the Dirac equation and rearrange the equation into four EME. Because electronic waves are polarized in space, as electromagnetic waves are, physical quantities given by vectorial electronic waves are similar to those in optics. In addition, we find that the electronic field consist of two parts, namely, longitudinal and transversal. Without appearance of quantum operators, we are able to deal with many problems of electrons conveniently, just like the example given above. Except for the quantization of physical quantities, EME provides all that we want to know about the electron such as SAM, probability density and so forth. In practice, when we are not interested in the quantization of physical quantities, EME provides a shortcut to solving the problems because we do not need to deal with quantum operators. This approach offers advantages in many applications of electron beams, especially when the wave nature of electron is of main concern. For example, we can control the phase shift of an electron beam in accordance with (41) by adjusting the strength the potential barrier.

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