Anti-self-dual gravity and supergravity from a pure connection formulation.

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(Dated: September 2, 2015)

We introduce a complex pure connection action with constraints which is diffeomorphism and gauge invariant. Taking as an internal group $SU(2)$, we obtain, from the equations of motion, anti-self-dual Einstein spaces together with the zero torsion condition thanks to Bianchi identity. By applying the same procedure, we take as internal symmetry the super group $OSp(1|2)$ and by means of the Bianchi identity and integrability conditions, the equations of motion are those that come from anti-self-dual supergravity $N = 1$ with cosmological constant sector.

PACS numbers: 00

I. INTRODUCTION

It was Einstein’s great achieve that gravity is a manifestation of spacetime curvature. In Riemannian geometry, a spacetime with curvature is describe by the metric and Einstein therefore used it to describe gravity. Being the metric the fundamental variable which determines spacetime intervals, and provides the causal structure to which all interactions including gravity itself must conform, it is astonishing that gravity can be reformulated in such a way that the metric doesn’t play the central role, instead it becomes a derived object, and then, as we do not need a pre-existing metric, the theory becomes background free.

In middle of 1970, Plebański adopted the viewpoint that the two forms are basic variables, and he exhibited a first order Palatini type action for complex General Relativity (GR) in

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which the field variables are certain triple of self-dual (SD) two-forms and triple of connections one-form. Later Ashtekar by considered a canonical transformation on the phase space of GR, obtained the SD formulation of GR given by Plebański.

Thus, it was realized by Capovilla, Dell and Jacobson[2], that the two-form fields of Plebański formulation can be integrated out to obtain a pure connection formulation of GR, where the only dynamical field is the SD connection one-form, becoming the main structure used to describe the gravitational field. The result is that gravity can be reformulated as a pure connection diffeomorphism invariant gauge theory.

Recently Krasnov[3] has shown that there is not a single diffeomorphism invariant gauge theory that shares the same key properties with GR, as they have the same number of propagating degrees of freedom (DoF), but an infinite parameter class of them. Even more, the fundamental scale is set not by the Newton’s constant, which doesn’t appear in the original formulation of the theory at all, but rather by the radius of curvature of the background that is used to expand the theory around. Among these diffeomorphism invariant gauge theories, González and Montesinos[4] had introduced pure gauge connection family of actions that contains not only as the gauge group the Lorentz group but those that contains a product of the local Lorentz group and an internal gauge group, considering vanishing and non-vanishing cosmological constant term.

One of the most interesting features given by gauge formulations is that they are suitable for the incorporation of supersymmetry, giving rise to the supersymmetric gauge theories. Self-dual and anti-self-dual formulations of supergravity where considered by Jacobson[5], and further developed by Capovilla, Dell and Jacobson[2]. Later, at the beginning of the 90’s, where considered supergravity self-dual models in the Atiyah-Ward space-time[7]. Also, MacDowell-Mansouri self-dual supergravity models in 3+1 dimensions with $OSp(4|1)$ gauge fields where considered by H. García-Compeán et. al. [8].

In this paper, we introduced a constrained gauge connection action for gravity in the Lorentzian signature case that can be easily generalized to obtain supergravity $N = 1$ without the needed to introduce new dynamical fields. The constrains that will be used imply algebraic relations, of the Plebański’s type, over the dynamical fields and provides to the equations of motion the correct shape needed in each case, for gravity and supergravity respectively. The organization of the paper is as follows. In Section 2 We present a brief introduction to BF theory that will be needed to introduce our proposal for a constrained
pure connection action. In Section 3 We show that SD Einstein spaces are obtained by considering anti-self-dual (ASD) connections. Section 4 We show that supergravity $N = 1$ with cosmological constant sector is obtained by considering ASD $OSp(1|2)$-valued connection. We conclude with a brief discussion.

II. FROM BF ACTION TO PURE CONNECTION ACTION

In this section, we briefly describe some features about Plebański theory as a constrained $BF$ theory that will be needed for the description of our work, for more details the reader is refereed to [1][9][10][11] and the references therein. We consider, in general, a principal fiber bundle $P$ over the four dimensional spacetime manifold $\mathcal{M}$ with a Lie group $G$ as an internal group, whose Lie algebra $\mathfrak{g}$ is equipped with a non degenerate bilinear invariant form, the Cartan-Killing form $\kappa$, and a $\mathfrak{g}$-valued connection $A$ which defines a curvature $F(A)$. It is important to note that $\mathcal{M}$ is a manifold without a metric structure on it and it will be constructed later by means of pure algebraic relations over the basic objects of the theory.

In order to construct an action for gravity for the $BF$ theory it is needed to introduce the two-form Lie algebra valued fields $B$, Lagrange multiplier $\Phi$ which is a Lie valued two-form automorphism whose effect is to constraint the DoF of the $B$ fields and in general without any restriction over it. So that to obtain the right DoF from the $\Phi$ field we need to put some constraints on it, to do so, we have two options, one is to impose the required restrictions by hand, or by means of Lagrange multipliers, we will use the second option and then introduce the four-form fields $\rho_1$ and Lie algebra valued fields $\rho_2$ (the restriction associated to $\rho_2$ is considered in [12] where it was used to construct a SUSY $N = 1$ extension of BF theory).

We consider the action

$$S_{BF}[A, B, \Phi, \rho_1, \rho_2] = \int_{\mathcal{M}} Tr \left( B \wedge F - \frac{1}{2} \Phi(B) \wedge B + \rho_1 (\Phi - \lambda) + \rho_2 \Phi \right)$$

(1)

as a constraint $BF$ action for gravity, where $\lambda$ is a constant, proportional to the cosmological constant (the case without cosmological constant could be derived if it is considered $\lambda = 0$). We have to note that we are considering as internal groups only $SU(2)$ and the supersymmetric extension $OSp(1|2)$.

Let us consider as the generators for Lie algebra $\mathfrak{g}$ the set $\{t_\alpha, t_\beta, t_\gamma, \ldots\}$ where $[t_\alpha, t_\beta] =$
\[ f_{\alpha\beta\gamma} t_\gamma, \text{ the Cartan-Killing form } \kappa_{\alpha\beta} = Tr(t_\alpha t_\beta), \text{ the structure constants } f_{\alpha\beta\gamma} = Tr(t_\alpha t_\beta t_\gamma) \text{ and } \Phi(B) = \Phi^{\alpha\beta} B_\alpha t_\beta. \]

We have to note that if we consider superalgebras, the brackets \[ , \] has to be changed by the graded brackets \[ , \} \] and the trace, \( Tr \), is changed by supertrace \( StTr \) in (1).

The trace for three generators is valid at last for the internal group and supergroup considered in this work \[13][14]. Then the action takes the form

\[
S_{BF}[A, B, \Phi, \rho_1, \rho_2] = \int_M B^\alpha \wedge F_\alpha - \frac{1}{2} \Phi^{\alpha\beta} B_\alpha \wedge B_\beta + \rho_1 (\Phi^\alpha - \lambda) + \rho_2^\alpha \Phi^\beta\gamma f_{\alpha\beta\gamma}. \tag{2}
\]

We have to note that, without loss of generality, we have not taken into account constant factors that can be singlet out the action and/or can be absorbed into the Lagrange multipliers. From (2) we observe that the equations of motion relative to \( \rho_2 \) and \( \rho_1 \) imply that the \( \Phi \) tensor is symmetric and its trace is proportional to the cosmological constant, respectively. So the action reads

\[
S_{BF}[A, B, \Phi] = \int_M B^\alpha \wedge F_\alpha - \frac{1}{2} \Phi^{T\alpha\beta} B_\alpha \wedge B_\beta - \frac{\lambda}{2} B^\alpha \wedge B_\alpha. \tag{3}
\]

on which the superindex on \( \Phi^T \) is related to the traceless part of the tensor field. The last action is how usually it is presented the BF action for gravity, for internal symmetries as \( SU(2) \) for complex or real fields or in the \( SO(3,1) \) with real fields and Immirzi parameter \[15][16]. Finally, The equations of motion coming from the last action are

\[
\delta_A S_{BF} = 0 \Rightarrow DB = 0 \tag{4}
\]

\[
\delta_B S_{BF} = 0 \Rightarrow F = \Phi^TB + \lambda B \tag{5}
\]

\[
\delta_\Phi S_{BF} = 0 \Rightarrow (B \wedge B)^T = 0 \tag{6}
\]

where \( D \) is the covariant derivative and it is defined as usual \( D = d + [A, \ ] \). The third equation is known in the literature as the simplicial or Plebański constraint \[1][11], which implies that the B-field is a simple two-form of the tetrad field, \( B \propto e \wedge e \), and it was introduced by Plebański at the middles of the seventies where he considered as the internal group \( G = SL(2,C) \otimes \overline{SL(2,C)} \), later it was considered only the one quiral part, the complexification of the gauge group \( G = SU(2) \), the self-dual(SD) or anti-self-dual(ASD) description. Once the explicit form of the B-field is given, the first and the second equations of motion imply the zero torsion condition, \( A = A(e) \), and the Einstein field equations,
respectively.

The action is invariant under local gauge transformation

\[ \delta^G_\alpha B = -[\alpha, B] \quad \delta^G_\alpha A = -D\alpha \quad \delta^G_\alpha \Phi = -[\alpha, \Phi]^{\mathfrak{g}\otimes\mathfrak{g}} \quad \delta^G_\alpha \rho_2 = -[\alpha, \rho_2] \] (7)

where \( \alpha \) are Lie algebra-valued scalars (gauge parameters) and the symbol \([\ , \ , ]^{\mathfrak{g}\otimes\mathfrak{g}}\) is due that \( \Phi \) is a Lie algebra valued bivector so the gauge transformation must be taken on each vector index. But even more, the action is invariant under the Kalb-Ramond(shift) symmetry \( \delta^C \)

\[ \delta C B = -DC \quad \delta C A = \Phi \cdot C \] (8)

where \( C \) are Lie algebra one-form transformation parameters. If we defined \( C = i_{vB} \), then the diffeomorphisms plus field dependent gauge transformations are found on-shell, but not only the transformation of the basic fields \( B \) and \( A \), but we can find the rules of transformation over the rest fields \( \Phi, \rho_1 \) and \( \rho_2 \) implying that the total symmetry group is the semi-direct product of these two groups.

Finally, let us construct a pure connection action that we will consider in this work. Inspired by the equation of motion for the B-field \( F = \Phi \cdot B \), consider \( \det \Phi \neq 0 \) so \( B = \Phi^{-1}F \), then the action (11) is rewritten as

\[
S_{BF}[A, \Phi, \Psi, \rho_1, \rho_2] = \int_M Tr \left( \frac{1}{2} \Psi(F) \wedge F + \rho_1(\Phi - \lambda) + \rho_2\Phi \right)
\] (9)

where we have defined \( \Psi = \Phi^{-1} \). But the relations on \( \Phi \) are algebraic so we have to rewrite the restrictions over \( \Psi \) in an algebraic manner. Then our proposal is to take the action (10) as topological pure gauge action for gravity independent of the \( BF \) functional. We consider for simplicity \( \Psi \) as a traceless object because opposite to the \( BF \) theory, the cosmological constant does not enter into the theory by this constraint and even more, we can observe that if we consider the trace term a constant different from zero, it only contributes as a boundary term in the form of the Chern-Simmons functional with no effect at the classical level which is the level presented in this work.

We have to note that it inherites the same gauge and diffeomorphism symmetry as in the \( BF \) theory that could be seen through the shift symmetry shown in equation (8), and has the same topological spirit, in the sense, that the metric doesn’t appear explicitly and it
can be seen as a perturbation of a topological field theory, for if we consider \( \Psi^{\alpha\beta} \approx \delta^{\alpha\beta} \), the Lagrangian becomes a total derivative, the second Chern Class.

In the next two sections we will show that the equation of motion coming from this action, considering complex field as well as the gauge group \( SU(2) \) and \( OSp(2|1) \), are ASD solutions of the Einstein equations, for gravity and supergravity respectively, where the zero torsion and supertorsion condition, comes by means of the Bianchi identity. Through the work we have labelled \( su(2) \) Lie algebra indices by the middle of the Latin alphabet lowercase letters \( \{i, j, k, \ldots\} \), \( so(1, 3) \) Lie algebra indices by the beginning of the Latin alphabet lowercase letters \( \{a, b, c, \ldots\} \), capital Latin letters for ASD spinorial indices \( \{A, B, C, \ldots\} \), overdotted capital Latin letters for SD spinorial indices \( \{\dot{A}, \dot{B}, \dot{C}, \ldots\} \), Lie superalgebra \( OSp(1|2) \) indices by the end of the Latin alphabet lowercase letters \( \{p, q, r \ldots\} \) and Greek alphabet letters for space-time indices \( \{\mu, \nu, \rho, \ldots\} \). Finally we define \( G^{(\alpha\beta)} = G^{\alpha\beta} + G^{\beta\alpha} \) and \( G^{[\alpha\beta]} = G^{\alpha\beta} - G^{\beta\alpha} \).

**III. ASD FORMULATION FOR GRAVITY**

Let us take the Lie group \( SU(2) \) as our gauge group, this group is semisimple so Cartan-Killing form is non-degenerated. We take as our spacetime a 4-dimensional globally hyperbolic, oriented smooth manifold \( \mathcal{M} \). Now choose a principal complex \( SU(2) \)-bundle \( P \) over \( M \) which is related to the ASD complex bundle. Let us take as our fundamental dynamical field a \( SU(2) \) complex connection \( A = A_i t^i \) \( (i = 1, 2, 3) \) where \( A^i \) is the ASD part of the real \( SO(3, 1) \) connection \( A_{ab} \) \( (a, b = 0, 1, 2, 3) \)

\[
A^i = \Pi^{(-)0i}_{ab} A^a_{\bar{b}} \quad \text{where} \quad \Pi^{(-)0i}_{ab} = \frac{1}{4} \left( \eta^{0i}_{ab} + i\epsilon^{0i}_{ab} \right),
\]

(11)

from which we have defined \( \eta_{ab,cd} = \eta_{ac}\eta_{bd} - \eta_{ad}\eta_{bc} \) and \( \eta = diag(-1, 1, 1, 1) \), the Minkowski metric. The Levi-Civita tensor is taken as \( \epsilon^{0123} = 1, \epsilon^{0ij} = \epsilon^{ijk} \) and \( \epsilon^{123} = 1 \). The generators of the \( su(2) \) Lie algebra in the adjoint representation satisfy

\[
[t_i, t_j] = f_{ij}^k t_k = 2i\epsilon_{ij}^k t_k.
\]

(12)

The Cartan-Killing form is calculated directly by the trace over the Lie algebra generators and we observe that is proportional to the Kronecker delta, then as the internal metric is defined upon an overall constant we consider \( \kappa_{ij} = \delta_{ij} \). The field strength is then an ASD two form given by

\[
F_i = dA_i + i\epsilon_i^{jk} A^j \wedge A^k.
\]

(13)
Our next step is to define the zero-form field $\Psi$ as a bivector Lie algebra valued field, $\Psi = \Psi^{ij}t_it_j$, and else, define the action over the two-from field strength as $\Psi(F) = \Psi^{ij}F_it_j$. At this point we have all the necessary ingredients needed in (10) so in this case the action is written as

$$S_{FF}[A, \Psi, \rho_1, \rho_2] = \int_{\mathcal{M}} \Psi^{ij}F_i \wedge F_j + \rho_1\Psi^i + \rho_2^i\Psi^{jk}\epsilon_{ijk}. \tag{14}$$

Let us now calculate the equations of motion, we consider first those that put constraints in the shape of $\Psi$ field. From $\rho_2$ we obtain that $\Psi$ is completely symmetric object and from $\rho_1$ we obtain that $\Psi$ is traceless. These equations of motion are algebraic so the action is equivalent to

$$S_{FF}[A, \Psi] = \int_{\mathcal{M}} \Psi^T_{ij}F^i \wedge F^j. \tag{15}$$

From the variation of $\Psi^T$ field, we obtain

$$(F^i \wedge F^j)^T = F^i \wedge F^j - \frac{1}{3} \delta^{ij}F^k \wedge F_k = 0. \tag{16}$$

The last equation is known as the instanton equation [20][21], it is a simplicial constraint similar to the Plebański constraint and it implies is that $F$ is an ASD simple two form

$$F_i = \lambda\Pi^{(-)0i}_{\,\,ab}\Sigma^{ab} = \lambda\Sigma^i \quad \text{where} \quad \Sigma^{ab} = e^a \wedge e^b \tag{17}$$

in which $\lambda$ is a constant proportional to the cosmological constant, by dimensional consistency, and it appears due to the fact that the algebraic equation is defined upon a constant term. Besides if we apply the covariant derivative in both sides of the equation (17) and by means of the Bianchi identitiy, we obtain

$$DF^i = D\left(\lambda\Pi^{(-)0i}_{\,\,ab}\Sigma^{ab}\right) = \lambda D\Sigma^i \Rightarrow D\Sigma^i = 0 \tag{18}$$

which is the zero torsion condition and it implies that the ASD connection can be written as a function of the tetrad field, $A = A(e)$. So if we demand that the tetrad field is nondegenerate then (17) leads to conformally ASD Einstein spaces where the field strength has SD Weyl tensor [15][19][20].

The equation of motion for the connection $A$, gives

$$D(\Psi^T_{ij}F^j) = 0 \tag{19}$$

which implies that $\Psi^T_{ij}F^j$ is covariantly constant. The equation (19) is a differential equation that gives a dynamical behavior to the $\Psi$ field, so it is interesting to note that it gives
us an additional algebraic equation by applying a covariant derivative over itself, called integrability condition. For simplicity instead of taking only the traceless part of $\Psi$, we consider the complete tensor field $\Psi$, and then put the traceless and antisymmetric restriction over it

$$D(D(\Psi_{ij} F^{ij})) = 0 \Rightarrow \epsilon_{jk}^i \Psi_{il} F^l \wedge F^j = \epsilon_{jk}^i \Psi_{il}^T F^l \wedge F^j = 0$$

and by the equation of motion imposed by $\rho_2$, as well as the not degeneracy of the tetrad field, it implies that the equation (20) is an identity, i.e. it does not put any new constraint into the theory. In general, the solution for the integrability condition is not trivial, as we shall see in the supersymmetric case, and put constraints over $\Psi$ that has to be taken into account in the theory, fortunately for us, the solution is very trivial in the bosonic case. Finally, we observe that we can obtain ASD Einstein gravity from the action (10) by the use of algebraic equations and the Bianchi identity.

The $FF$ action inherites the symmetry coming from the BF action, i.e., the action is invariant under the local gauge transformation

$$\delta_\alpha A^i = -D\alpha^i \quad \delta_\alpha \Psi^{ij} = -2i\epsilon_{kl}^i \alpha^k \Psi^{lj} - 2i\epsilon_{kl}^j \alpha^k \Psi^{il} \quad \delta_\alpha \rho_2^i = -2i\epsilon_{kl}^i \alpha^k \rho_2^j$$

where $\alpha$ are $su(2)$-valued scalars, and the shift symmetry

$$\delta_C A = -2C$$

for if $C$ are $su(2)$ one-form transformation parameters defined as $C = i_v F = v^\mu F_{\mu\nu} dx^\nu$, then diffeomorphisms plus field dependent gauge transformations of all the fields are given

$$\delta_v A^i = v^\mu \partial_\mu A^i + \partial_\nu v^\mu A^i_\mu - D_\nu A^i \quad \delta_v \Psi^{ij} = v^\mu \partial_\mu \Psi^{ij} - \delta_\alpha \Psi^{ij} \quad \delta_v \rho_1 = D_\mu (v^\mu \rho_1) \quad \delta_v \rho_2^i = \partial_\nu (v^\mu \rho_2^i) - \delta_\alpha \rho_2^i$$

and $\delta_\alpha$ are $SU(2)$ transformations with parameters $\alpha^i = v^\mu A^i_\mu$ and the four-forms $\rho_1$, $\rho_2^i$ transform as invariant densities. At this point we have to note that the rules of transformation can be found without the use of the equation of motion, i.e., they are obtained off-shell opposite to the tetrad field case where transformations can be found by means of the equation of motion (17).
Finally we end this section by pointing some remarks that will be useful in the next section, we have defined ASD fields as $su(2)$ Lie valued complex fields $v^i$ which can be related to $so(3, 1)$ Lie valued real fields $v^{ab}$ by the ASD projector $\Pi^{(-)}$, but originally, the action of Plebański was formulated in spinorial notation and we can recover the original equations by considering $v^i = \frac{1}{2} (\sigma^i)_A^B v_B^A$, where $(\sigma^i)$ are the Pauli matrices. On the other hand, for the Lorentz group the corresponding embedding is into $SL(2, \mathbb{C}) \otimes SL(2, \mathbb{C})$, given for the fundamental representation by $v^A^B = \frac{1}{2} (\sigma^a)^B_A v^a$, where $(\sigma^a)$ are the Pauli matrices. With these conventions, the adjoint representation of $SO(3, 1)$ decomposes as,

$$v^{ab} = (\sigma^{ab}) B^A v^i = \Pi^{(-)0i}_{ab} (\sigma^i)_A^B v^{ab} = -\frac{1}{2} (\sigma_{ab})_A^B v^{ab}. \tag{24}$$

From the last result we can obtain

$$u^i v_i = -\frac{1}{4} u^{(-)ab} v^{(-)}_{ab} = -\frac{1}{2} u^{AB} v_{AB}. \tag{25}$$

Now we can translate the equations of motion (17) and its covariant derivative, the zero torsion condition, into the spinor language

$$F^{AB} = \lambda_1 \Sigma^{AB} \quad , \quad D\Sigma^{AB} = 0 \tag{26}$$

where $\Sigma^{AB} = e^{A\dot{A}} \wedge e_{\dot{A}}^B [2]$. 

**IV. SUPERSYMMETRIC EXTENSION**

The generalization of action (10) for supergravity $N = 1$ is done by making its fundamental fields transforming under the adjoint representation of $OSp(2|1)$. The fermionic (odd) part is labelled by the spinorial indices and the bosonic (even) part by $su(2)$ indices, then the connection one-form is written as $A = A^p t_p = A^i t_i + A^B t_B$, $\Psi = \Psi^{pq} t_p t_q$ and $\Psi(F) = \Psi^{pq} F_q^p t_p$. 

The generators of the Lie superalgebra are given by \([t_p, t_q] = f_{pq}^r t_r\) where \(f_{pq}^r\) are the structure constants and are calculated as follows
\[
[t_i, t_j] = 2i \epsilon_{ij}^k \quad [t_i, t_A] = -(\sigma_i)_A^B t_B \quad \{t_A, t_B\} = (\sigma^i)_{AB} t_i.
\] (27)

The Cartan-Killing form is calculated straightforward
\[
k_{pq} = \begin{pmatrix}
\delta_{ij} & 0 \\
0 & \epsilon_{AB}
\end{pmatrix}.
\] (28)

The bosonic and fermionic sectors of the field strength are
\[
F^i = R^i - \frac{1}{2} A(\sigma^i) A, \quad F^B = dA^B - (\sigma_i)_C^B A_i \wedge A^C
\] (29)
where \(R^i\) is the bosonic ASD field strength \([13]\). Let us defined the covariant derivative for Lie superalgebra valued fields as \(\nabla \xi = \nabla \xi^i t_i + \nabla \xi^B t_B\), where
\[
\nabla \xi^i = D\xi^i - (\sigma^i)_C^B A^C \wedge \xi_B, \quad \nabla \xi^B = D\xi^B + (\sigma_j)_C^B A^C \wedge \xi^j
\]
and \(D\) is referred to the usual non supersymmetric covariant derivative which it is defined in such a way that it “knows” how to act over different non supersymmetric Lie algebra valued fields
\[
D\zeta^i = d\zeta^i + 2i \epsilon_{jk}^i A^j \wedge \zeta^k \quad D\zeta^{ab} = d\zeta^{ab} + A^{ac} \wedge \zeta^b + A^{bc} \wedge \zeta^a.
\]
\[
D\zeta^B = d\zeta^B + A^{BC} \wedge \zeta_C
\]
Now we have the basic ingredients needed in the action \([10]\) which in the supersymmetric case is written as follows
\[
S_{FF}[A, \Psi, \rho_1, \rho_2] = \int_M \Psi^{pq} F_p \wedge F_q + \rho_1 \Psi^p + \rho_2^P \Psi^{qr} f_{pqr}.
\] (30)

Now, from the variation of \(\rho_1\), we have that \(\Psi\) is supertraceless, \(\Psi^p = \Psi^i + \Psi^A_A = 0\). From the variation of \(\rho_2\) we obtain \(\Psi^{qr} f_{pqr} = 0\) which implies two independent relations
\[
2i \Psi^{ij} \epsilon_{ijk} - \Psi^{AB}(\sigma_k)_{AB} = 0 \quad (\Psi^{iA} + \Psi^{Ai})(\sigma_i)_{AB} = -\Psi^{D}_B \sigma^D_B + \Psi^{D}_B \sigma^D_B = 0.
\] (31) (32)

We note that in the supersymmetric formulation, the antisymmetric part of the bosonic sector does not vanish, instead it implies that
\[
\Psi^{[ij]} = -\frac{1}{4} [\sigma^i, \sigma^j]_{AB} \Psi^{AB}, \quad \Psi^{(AB)} = [\sigma^i, \sigma^j]^{AB} \Psi_{ij},
\]
fortunately, even when they both give contributions to the antisymmetric bosonic sector of \( \Psi^{ij} \) and to the symmetric part of the fermionic sector \( \Psi^{AB} \) they are not necessary because of the symmetry of the wedge product of the field strength with itself, i.e. \( F \wedge F \).

Now the action reads

\[
S_{FF}[A, \Psi] = \int_{\mathcal{M}} \Psi^{ij} F_i \wedge F_j + \Psi^{AB} F_A \wedge F_B + (\Psi^{iA} + \Psi^{Ai}) F_i \wedge F_A
\]

(33)

but in order to go further, we have to decompose \( \Psi \) into its irreducible components \(^{22}\)

\[
\Psi^{ij} = \Psi^{ij}_T + \frac{1}{3} \delta^{ij} \Psi^k_k
\]

(34)

\[
\Psi^{AB} = -\frac{1}{2} \epsilon^{AB} \Psi^C_C + \frac{1}{2} \Psi^{(AB)}
\]

(35)

\[
\Psi^{iA} + \Psi^{Ai} = - (\sigma^i)_{BC} \Psi^{(ABC)} + \frac{1}{3} (\sigma_i)^A \left[ - \Psi^{CD} + \Psi^{DC} \right]
\]

(36)

where \( \Psi^{(ABC)} \) and \( \Psi^{(AB)} \) are completely symmetric tensor fields. As \(^{32}\) is an algebraic equation and the product \( F_A \wedge F_B \) is antisymmetric, the action is equivalent to

\[
S_{FF}[A, \Psi] = \int_{\mathcal{M}} \Psi^T ij F_i \wedge F_j + \Psi^k_k \left( \frac{1}{3} F^i \wedge F_i - \frac{1}{2} F^B \wedge F_B \right) - \Psi^{(ABC)} F_{AB} \wedge F_C.
\]

(37)

From the last action, let us calculate the equations of motion. For the bosonic traceless part of \( \Psi \), we have the Plebański simplicial constraint

\[
(F_i \wedge F_j)^T = 0 \Rightarrow F_i = R_i - \frac{1}{2} A(\sigma_i) A = \lambda_1 \Pi^{(-)0i} \Sigma^{ab} = \lambda_1 \Sigma_i.
\]

(38)

From \( \Psi^{(ABC)} \) we obtain the well known CDJ constraint \(^{2}\), where the solution is defined, upon a constant factor, as

\[
F_{(AB} \wedge F_{C)} = 0 \Rightarrow DA_C = F_C = \lambda_2 (\sigma_a)_{CC} e^a \wedge \varphi^C = \lambda_2 \Pi^{ab} \Sigma_{ab} = \lambda_2 \Sigma_C
\]

(39)

where \( \varphi \) is a spin 3/2 one-form field which at the quantum level, together with reality conditions \(^{2,12}\), is known as the gravitino field, \( \varphi^C \) is its complex conjugate. Thus we have defined the spinorial projector \( \Pi^{ab}_C = (\sigma^a)_{CC} \varphi^b \).

For the equation of motion for the connection \( A \), we proceed as in the last section and, at first, we calculate the most general form of the equation of motion coming from the action \(^{30}\) and then put the constraints over it. Then we have for the connection

\[
\nabla \left[ (\Psi^{pq} + (-1)^{|q|} \Psi^{qp}) F_q \right] = 0
\]

(40)
where the symbol $| \cdot |$ is zero if the object is bosonic and one if it is a fermionic index. First let us apply a supercovariant derivative to (40) in order to obtain the integrability condition
\[
\nabla \left[ \nabla \left[ (\Psi_{pq} + (-1)^{|p||q|} \Psi_{qp}) F_q \right] \right] = (\Psi^{sq} + (-1)^{|s||q|} \Psi^{qs}) F^r \wedge F_q f_{rs}^p = 0, \tag{41}
\]
then by the use of the simplicial constraints, symmetry properties of $\Psi$ (and a little bit of algebra), we obtain
\[
\rho_2^D \Psi^A_A = 0, \tag{42}
\]
but we observe, by consistency with the equation of motion (32), that $\rho_2^D \neq 0$ (otherwise (32) couldn’t exists), and so $\Psi^A_A = 0$, implying by supertraceless condition, that $\Psi^i_A$ is zero too. As we can observe, in the supersymmetric case we obtain, from the integrability condition, a new constraint over the $\Psi$ field, which implies that each trace term in $\Psi$ vanish independently.

Finally, by the use of the Bianchi’s identity $\nabla F = \nabla F^it_i + \nabla F^A t_A = 0$ we obtain
\[
D \Sigma_i = \frac{\lambda_2}{\lambda_1} A^B \wedge \Sigma_C (\sigma_i)_C^B, \quad D \Sigma_A = -\frac{\lambda_1}{\lambda_2} A_B \wedge \Sigma^B_A. \tag{43}
\]
We observe that the solution of the CDJ simplicial constraint (39) force us to introduce into the game, the complex conjugate quiral part of the ASD sector, i.e., the objects with dotted indices, then for a correct description of the theory we have to translate the equations from the $SU(2)$ complex valued fields to the $SL(2, C) \otimes SL(2, C)$ language, given by the relations at the end of the last section (24). But even more it is also necessary if we wish to compare our result with those that we can find in the literature. Then let us identify $A_C = \sqrt{\Lambda} \varphi_C$, $\lambda_1 = -4\Lambda^2$ and $\lambda_2 = -i\Lambda^{3/2}$ and by (24), the equations of motion read
\[
(\sigma_{ab})_A^B \left[ \bar{R}^{ab} + 4\Lambda^2 \Sigma^{ab} \right] = -2\Lambda \varphi_A \wedge \varphi^B,
\]
\[
De^a = -\frac{i}{2} \bar{\varphi} (\sigma^a) \varphi
\]
\[
e^a (\bar{\sigma}_a)^A_B \wedge D \varphi_A = -2i\Lambda (\bar{\sigma}_ab)_B^A \Sigma^{ab} \wedge \varphi^B
\]
\[
e^a (\sigma_a)^A_B \wedge D \bar{\varphi}_A = +2i\Lambda (\sigma_{ab})_B^A \Sigma^{ab} \wedge \varphi^B.
\]
These equations can be compared to the usual equations of motion for supergravity N=1 in ASD formulation with cosmological constant sector [23][24], we observe that the first is a solution for supergravity which is the supersymmetric extension of (17), the second is the zero supertorsion condition and the last two gives the dynamical behavior of the gravitino.
field related by complex conjugation.

The rules of transformation for the fields are straightforward calculated based on the previous section results, the shift symmetry $\delta_C A^p = \nu^\mu F_{\nu\mu}^p dx^\nu$ implies off-shell transformations

$$\delta_v A^p_{\nu} = \nu^\mu \partial_\mu A^p_{\nu} + \partial_\nu \nu^\mu A^p_{\mu} - \nabla_\nu \alpha^p$$

$$\delta_v \Psi^{pq} = \nu^\mu \partial_\mu \Psi^{pq} - \delta_\alpha \Psi^{pq}$$

$$\delta_v \rho_1 = D_\mu (\nu^\mu \rho_1)$$

$$\delta_v \rho^p_2 = \partial_\mu (\nu^\mu \rho^p_2) - \delta_\alpha \rho^p_2$$

where $\delta_\alpha$ are $Osp(1|2)$ transformations with parameters $\alpha^p = \nu^\mu A^p_{\mu}$ and the four-forms $\rho_1$, $\rho^p_2$ transform as invariant densities and the rules for the tetrad field transformations can be found off-shell.

\section{V. Conclusions and Outlooks}

As was shown by Krasnov and Montesinos, there are a family of diffeomorphism invariant pure connection gauge theories that share the same key properties with General Relativity, then among those elements of the family, we introduce a simple one \[10\]. As we shown by the Kalb-Ramond symmetry, is gauge and diffeomorphism invariant, as expected. But even more, we showed how to obtain SD Einstein spaces by consider ASD complex $su(2)$ valued fields, and supergravity $N = 1$ with cosmological sector by considering ASD complex $OSp(1|2)$ fields. We observed, in both cases, that half of the equation of motions needed in the theory came by the variation of the action and the left half came by the use of the Bianchi identity. We also had that we haven’t put constraints over the $\Psi$ field, as it is usual done, instead this field has no intrinsic properties and it is constrained, in its shape, by the constrains given in the action, and by integrability conditions. As we showed the trace of the $\Psi$ field is not longer related to the cosmological constant and in both cases, pure bosonic and supersymmetric cases, vanishes. Finally, the appearance of the cosmological constant is due to algebraic relations and dimensional consistency, and it has no relation with the trace part, as in the $BF$ case.

It would be interesting to consider the canonical analysis of the class to verify that the theory has two degrees of freedom per spacetime point and to compare to those that could
be found in the literature, as special case, how different is from the Ashtekar formulation. All of these, in order to obtain the quantization of the theory proposed.

We can also consider \( so(3, 1) \) real valued fields in order to avoid reality conditions, and compare in the Lagrangian level with the Holst action and in the Hamiltonian level with the Barbero formulation in the pure bosonic case.

Also the introduction of matter into the action or the generalization to more general internal gauge groups and supergroups could be considered. The consequences of the presence of a cosmological constant regarding the deformation of the symmetry group of discretized models, such as consistent discretization approach, as well as the consequences of degenerated metrics.

**Acknowledgements**

We thank O. Obregón, C. Ramirez and M. Sabido for useful discussions. The author acknowledges support from a CONACyT scholarship (México) and PROMEP postdoctoral grant.

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