Quantum Phases in Partially Filled Landau Levels

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We compare the energies of different electron solids, such as bubble crystals with triangular and square symmetry and stripe phases, to those of correlated quantum liquids in partially filled intermediate Landau levels. Multiple transitions between these phases when varying the filling of the top-most partially filled Landau level explain the observed reentrance of the integer quantum Hall effect. The phase transitions are identified as first-order. This leads to a variety of measurable phenomena such as the phase coexistence between a Wigner crystal and a two-electron bubble phase in a Landau-level filling-factor range $4.15 \lesssim \nu \lesssim 4.26$, which has recently been observed in transport measurements under micro-wave irradiation.

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1. Introduction

The most prominent phenomena in two-dimensional electron systems in a perpendicular magnetic field $B$ are the integer and fractional quantum Hall effects (IQHE and FQHE, respectively). In spite of the similarity between the two effects, their origin is different: on the one hand, the IQHE is a manifestation of the energy quantization of electrons (mass $m$ and charge $-e$) in highly degenerate Landau levels (LLs), with a level separation of $\hbar eB/m$. The ratio $\nu = n_{el}/n_B$, between the electronic density $n_{el}$ and the density of states per level, $n_B = B/(\hbar/e)$, determines the filling of the LLs, and the IQHE occurs if $\nu = N$, with integral $N$. The signature of this effect is a plateau in the Hall resistance, accompanied by a vanishing longitudinal resistance. On the other hand, the FQHE is due to strongly correlated quantum liquids formed by the electrons in a partially filled LL and occurs at some of the “magical” filling factors $\nu = p/(2ps + 1)$ [and at their particle-hole symmetric fillings $\nu = 1 - p/(2ps + 1)$], with integral $s$ and $p$. Also in the first excited LL, fractional quantum Hall states have been observed at $\bar{\nu} = 1/3, 2/3, 1/5$, and $4/5$, where $\bar{\nu} = \nu - N$ denotes the filling of the topmost level.
In higher LLs, the strong Coulomb repulsion between electrons in the partially filled level may lead to phases different from quantum liquids: calculations in the Hartree-Fock approximation have revealed the existence of electron-solid phases, such as stripes around $\tilde{\nu} = 1/2$, 1,2 A stripe phase has indeed been observed in transport measurements, which show a large anisotropy in the longitudinal magneto-resistance around $\nu = 9/2, 11/2, 13/2, \ldots$.3 Eisenstein et al. have furthermore measured a non-monotonic behavior of the Hall resistance in the first excited LL $n = 1$:4 the FQHE at $\tilde{\nu} = 1/3$ and $1/5$ is surrounded by pinned electron-solid phases, which are insulating and thus cause an integer quantization of the Hall resistance, as for the neighboring IQHE. This reentrant IQHE is reminiscent of an effect observed before in the second excited LL.5 We have shown that the effect may be understood in terms of an alternation between quantum-liquid and electron-solid phases when varying the filling of the topmost LL.6 Here, we furthermore investigate bubble crystals with different symmetry. Whereas the quantum-liquid phases are favored at $\bar{\nu} = 1/(2s + 1)$, at $\bar{\nu} \neq 1/(2s + 1)$ quasi-particles are excited and raise the energy of the quantum liquids above that of the competing electron solids.

2. Energy Calculation for the Different Phases

In order to describe the low-energy degrees of freedom, which, at non-zero values of the partial filling factor $\tilde{\nu}$, consist of intra-LL excitations, we adopt a model of spin-polarized electrons,

$$\hat{H} = \frac{1}{2} \sum_q v_n(q) \hat{\rho}(-q) \hat{\rho}(q), \quad \text{with} \quad v_n(q) = \frac{2\pi e^2}{eq} \left| F_n(q) \right|^2,$$

where only the components of the density operator in the $n$-th LL are taken into account, $\rho_n(q) = F_n(q) \bar{\rho}(q)$.a The LL form factor $F_n(q) = L_n(q^2/2) \exp(-q^2/4)$ is given in terms of Laguerre polynomials $L_n(x)$, and $\epsilon$ is the dielectric constant. The quantum-mechanical properties of the model are revealed by the unusual commutation relations for the projected density operators, $[\hat{\rho}(q), \hat{\rho}(k)] = 2i \sin [(q \times k)_+]/2 \hat{\rho}(q + k)$. This model allows for a common description of all LLs.

The electron-solid phases are characterized by an order parameter $\Delta(q)$, which determines the density profile of the phase, given by the local filling factor $\nu(r)$ and the area $A$, $\Delta(q) \equiv \langle \hat{\rho}(q) \rangle / n_B A = \int d^2 r \nu(r) \exp(iq \cdot r)/A$. The cohesive energy of the electron-solid phases becomes in the Hartree-Fock approximation1,2,6

$$E_{\text{coh}}^{\text{sol}}(n; \tilde{\nu}) = \frac{n_B}{2\tilde{\nu}} \sum_q u_n^{HF}(q) |\Delta(q)|^2,$$

where the Hartree-Fock potential $u_n^{HF}(q)$ takes into account quantum-mechanical exchange effects.

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aWe use a system of units, in which the magnetic length $l_B = \sqrt{\hbar/eB} \equiv 1$
The bubble crystal with an arbitrary lattice symmetry is characterized by the local filling factor \( \bar{\nu}(r) = \Theta(r_B - |r - R_j|) \), where \( \Theta(x) \) is the step function, and \( R_j \) are the lattice vectors. The area of the primitive cell \( A_{pc} = 2\pi M/\bar{\nu} \) is determined by the partial filling factor and the bubble radius \( r_B = \sqrt{2M} \) containing \( M \) electrons.

The order parameter of the bubble crystal
\[
\Delta_B^M(q) = \frac{2\pi}{Aq} J_1(q\sqrt{2M}) \sum_j e^{iq\cdot R_j}
\]
yields the cohesive energy
\[
E_{coh}^B(n; M, \bar{\nu}) = \frac{n_B\bar{\nu}}{M} \sum_{G_l \neq 0} u_n^{HF}(G_l) J_1^2(\sqrt{2M}|G_l|)/|G_l|^2,
\]
where the lattice symmetry is specified only by the reciprocal lattice vectors \( G_l \).

In the case of stripes with width \( a \) oriented parallel to the \( y \)-direction, the ansatz \( \bar{\nu}(x) = \Theta(a/2 - |x - x_j|) \) leads to the order parameter,
\[
\Delta_S^S(q) = \frac{2}{L_x} \delta_{q_y,0} \sin \left( \frac{q_x A S \bar{\nu}}{2} \right) \sum_j e^{iq_x J_S},
\]
where \( A_S = a/\bar{\nu} \) is the stripe periodicity. This yields the cohesive energy
\[
E_{coh}^S(n; A_S, \bar{\nu}) = \frac{n_B}{2\pi^2 a} \sum_{l \neq 0} u_n^{HF} \left( q = \frac{2\pi}{A_S} l \right) \frac{\sin^2(\pi l \bar{\nu}/l^2)}{l^2},
\]
which is to be minimized with respect to the variational parameter \( A_S \).

The quantum-liquid phases, which we investigate here, may not be characterized by an order parameter, but they are described by Laughlin’s wavefunctions. Their cohesive energy is given in terms of Haldane’s pseudopotentials, \( V_{2m+1}^n = (2\pi/A) \sum_q v_n(q)L_{2m+1}(q^2)\exp(-q^2/2) \),
\[
E_{coh}^{q-l}(n; s, \bar{\nu}) = \bar{\nu} \sum_{m=0}^{\infty} c_{2m+1}^s V_{2m+1}^n + [\bar{\nu}(2s + 1) - 1]|\Delta^n(s)|,
\]
where the expansion coefficients \( c_{2m+1}^s \) specify the Laughlin wavefunction. The second term in Eq. (5) takes into account the energies \( \Delta^n(s) \) of the excited quasiparticles of charge \( 1/(2s + 1) \) [at \( \bar{\nu} > 1/(2s + 1) \)] and quasi-holes of charge \( -1/(2s + 1) \) [at \( \bar{\nu} < 1/(2s + 1) \)], in units of the electronic charge. They may be calculated analytically in the Hamiltonian theory of the FQHE, established by Murthy and Shankar.

3. Results

Here, we concentrate on some aspects of the phases in the first and second excited LLs, \( n = 1 \) and \( n = 2 \), respectively. A more detailed discussion, including a quantitative study of the role of impurities, may be found in Ref. 6.
Fig. 1(a) shows the energies for different electronic phases in \( n = 1 \). The quantum-liquid phases are favored around \( \bar{\nu} = 1/3 \) and \( 1/5 \), whereas in an intermediate range, \( 0.23 < \bar{\nu} < 0.3 \), a Wigner crystal (WC, \( M = 1 \)) has a lower energy. Because the Wigner is pinned by impurities, one observes an integer quantization of the Hall resistance in this range, whereas one finds the FQHE around \( \bar{\nu} = 1/3 \) and \( 1/5 \).\(^4\) Above \( \bar{\nu} \sim 0.38 \), the FQHE disappears because the quantum liquid has a higher energy than a two-electron bubble crystal, which competes with a stripe phase. The latter has a lower energy as one approaches half-filling. Experimentally, however, an anisotropic longitudinal resistance, which is the signature of stripe phases,\(^3\) has only been observed in a tilted magnetic field.\(^10\) Notice that non-Laughlin-type quantum liquids, which are not considered in our energy investigations, also compete in this filling-factor range. At \( \nu = 5/2 \), e.g., a Pfaffian state, which is a special case \(( k = 2 \) of the parafermionic ones at \( \bar{\nu} = k/(2 + k) \) with integral \( k \), gives rise to a FQHE.\(^11\) A recently observed FQHE\(^12\) at \( \bar{\nu} = 2/5 \) in \( n = 1 \) is likely to be a parafermionic hole state with \( k = 3 \). Also the nature of the FQHE state at \( \bar{\nu} = 1/3 \) remains controversial because numerical studies on a few number of electrons indicate a rather small overlap with a Laughlin-type state.\(^13\) Although we consider only such Laughlin-type states here, it cannot be ruled out that other quantum-liquid phases have a lower energy and are responsible for the FQHE at these fillings.

Our energy calculations suggest that quantum-liquid phases may also be found below \( \bar{\nu} = 1/5 \) in the absence of impurities. However, the energy of the WC is lowered by impurities, due to the deformation of its lattice structure. This effect is most relevant at small \( \bar{\nu} \), and the FQHE is therefore unstable in this limit,\(^6\) where one observes the IQHE.\(^4\) The energies for the bubble crystals are shown both for the case of a triangular (continuous lines) and a square lattice symmetry (broken lines). The energy difference between these two cases is extremely tiny (on the order of 1\%). From classical considerations, one would expect that a triangular lattice has a lower energy than a square lattice.\(^14\) Our energy calculations indicate that this is correct in the low-\( \bar{\nu} \) limit, whereas at larger densities a WC with square-lattice symmetry has a lower energy than the triangular one. A similar behavior is found for the two-electron bubble crystal. However, this change of symmetry occurs at filling-factor values, where other phases have a lower energy; the square-lattice symmetry of the WC, e.g., is favored only above \( \bar{\nu} \sim 0.3 \), where quantum-liquid, two-electron bubble, and stripe phases are the ground state.

The energy results for \( n = 2 \) are shown in Fig. 1(b). In contrast to \( n = 1 \), a quantum liquid is unstable around \( \bar{\nu} = 1/3 \), where a two-electron bubble crystal has the lowest energy. Our energy calculations suggest that a FQHE might be found around \( \bar{\nu} = 1/5 \) or \( 1/7 \). Note, however, that the energies of the quantum-liquid phases are very close to that of the WC and, in the case of \( \bar{\nu} = 1/5 \), to a mixed phase of a WC and a two-electron bubble crystal, which is represented by the tangent. It is therefore not clear whether the quantum liquid remains stable in the presence of
impurities, which lower the energy of the crystal phases, as shown by the gray curves. They have been calculated for an impurity strength \( V_0 / \xi = 0.005 e^2 / \epsilon l_B \), where \( V_0 \) is the characteristic energy of a short-range Gaussian potential with correlation length \( \xi \).\(^6\) Experimentally, a small maximum in the longitudinal resistance around \( \bar{\nu} = 1/5 \) indicates an incipient melting of a crystal phase.\(^5\) This feature has recently been studied in more detail by Gervais et al.,\(^15\) who found that the maximum, which decreases when lowering the temperature \( T \), splits into two peaks separated by a small local minimum precisely at \( \bar{\nu} = 1/5 \) with increasing \( T \). A reminiscent \( T \)-dependent effect has been observed in the WC regime in the lowest LL.\(^16\) Even if this effect may indicate a quantum-liquid ground state in extremely pure samples, it may also be understood in different terms: whereas the crystal, which in this scenario remains the \( T = 0 \) ground state, melts at rather low \( T \) (on the order of the energy difference between the WC and the quantum-liquid phase), the quantum coherence of the liquid displaying FQHE features is only destroyed at higher \( T \).\(^17\)

4. Phase Transitions

Our energy calculations suggest that the transitions between the different phases are first-order. The first-order phase transitions between the quantum-liquid and the insulating bubble crystals may cause a hysteretical behavior in the Hall resistance around the transition points, which, to the knowledge of the authors, has not been reported yet. Also the phase transitions between bubble crystals with different \( M \) per site are first-order, in agreement with time-dependent Hartree-Fock calculations by Côté et al.\(^18\) This leads to a phase coexistence, or a mixed phase, around the transition points in a filling-factor range, which is described by a tangent on the energy curves, \( e.g. \) at \( 0.15 \lesssim \bar{\nu} \lesssim 0.26 \) in \( n = 2 \) [c.f. Fig 1(b)]. Experimentally, there is evidence for such a mixed phase, which is revealed by a double-peak structure in transport measurements under micro-wave irradiation, recently performed by Lewis.
5. Conclusions

In conclusion, we have performed energy calculations for competing quantum phases in intermediate LLs. An alternation between insulating electron-solid and quantum-liquid phases, which display the FQHE, is at the origin of the observed reentrance of the IQHE in $n = 1$ and $n = 2$. The transitions between the different phases are found to be first-order and may lead to a variety of observable phenomena. In the case of transitions between bubble crystals with different electron number per site, a phase coexistence is expected. This scenario is supported by recent micro-wave experiments, in which a double-peak structure has been observed in the longitudinal conductivity in a filling-factor range $0.16 \lesssim \nu \lesssim 0.28$, in good agreement our theoretical investigations.

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