LOCAL ORTHOGONAL RECTIFICATION: DERIVING NATURAL COORDINATES TO STUDY FLOWS RELATIVE TO MANIFOLDS

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Abstract. We recently derived a method, local orthogonal rectification (LOR), that provides a natural and useful geometric frame for analyzing dynamics relative to a base curve in the phase plane for two-dimensional systems of ODEs (Letson and Rubin, SIAM J. Appl. Dyn. Syst., 2018). This work extends LOR to apply to any embedded base manifold in a system of ODEs of arbitrary dimension and establishes a corresponding system of LOR equations for analyzing dynamics within the LOR frame, which maps naturally back to the original phase space. The LOR equations encode geometric properties of the underlying flow and remain valid, in general, beyond a local neighborhood of the embedded manifold. In addition to developing a general theory for LOR that makes use of a given normal frame, we show how to construct a normal frame that conveniently simplifies the computations involved in LOR. Finally, we illustrate the utility of LOR by showing that a blow-up transformation on the LOR equations provides a useful decomposition for studying trajectories’ behavior relative to the embedded base manifold and by using LOR to identify canard behavior near a fold of a critical manifold in a two-timescale system.

1. Introduction and motivation. Often the dynamics of an ordinary differential equation (ODE) defies rectangular coordinate schemes; that is, the geometry induced by a flow may be difficult to represent in Cartesian coordinates. In fact, a common early step in analysis is to exchange Cartesian coordinates for a geometry better suited for the problem [11, 2, 10, 15, 16, 12, 17]. In this work, we present a technique that allows us to use any embedded manifold (not necessarily invariant) to generate a natural coordinate frame for a dynamical system. We call this technique Local Orthogonal Rectification (LOR). We previously derived the planar version of LOR and illustrated its utility in a variety of scenarios; for example, we used the LOR framework to precisely define transiently attracting or repelling rivers in the two-dimensional phase plane, to develop a convenient method to locate rivers, and to implement this method to study transient dynamics in some example systems [9]. The focus of the present work is on providing a rigorous, much more..."
general derivation for the LOR approach and its properties that applies in arbitrary dimensions and on illustrating its use in the non-planar setting.

Consider an ODE and initial condition

$$\dot{x} = f(x), \quad x(0) = x_0 \in \Omega,$$

where $f \in C^r(\Omega, \mathbb{R}^n)$ for $n, r \geq 1$ and $\Omega$ is an open subset of $\mathbb{R}^n$, which induces a flow $\Phi: \Omega \times \mathbb{R} \to \Omega$. For simplicity, we introduce the notation $\partial_t = \partial/\partial t$. Suppose that $M$ is a codimension-$k$ $C^r$-regular manifold embedded in $\Omega$. Specifically, suppose there exist an indexing set $A$ and an atlas of charts $\{(U_\alpha, \sigma_\alpha)\}_{\alpha \in A}$ where $M = \bigcup_{\alpha \in A} \sigma_\alpha(U_\alpha)$, such that for all $\alpha, \beta \in A$:

1. each $U_\alpha \subseteq \mathbb{R}^{n-k}$ is open with corresponding $\sigma_\alpha \in C^r(U_\alpha, \Omega)$ a homeomorphism on its image;
2. if $\sigma_\alpha(U_\alpha) \cap \sigma_\beta(U_\beta) \neq \emptyset$ then the map $\kappa_{\alpha, \beta} : U_\alpha \cap \sigma_\alpha^{-1} \circ \sigma_\beta(U_\beta) \to U_\beta \cap \sigma_\beta^{-1} \circ \sigma_\alpha(U_\alpha)$ defined by $\kappa_{\alpha, \beta} = \sigma_\beta^{-1} \circ \sigma_\alpha$ is a diffeomorphism;
3. for all $\eta \in U_\alpha$,
   \[ \text{dimgspan} \{ \partial_1 \sigma_\alpha, \ldots, \partial_{n-k} \sigma_\alpha \} = n-k; \]
4. $M$ can be equipped with a local normal frame: that is, there are mappings $N_j \sigma_\alpha \in C^3(U_\alpha, \mathbb{R}^n)$ such that
   \[ \langle N_{j_1} \sigma_\alpha(\eta), v \rangle = 0, \quad \langle N_{j_1} \sigma_\alpha(\eta), N_{j_2} \sigma_\alpha(\eta) \rangle = \delta_{j_1,j_2} \quad \forall \eta \in U_\alpha, \forall v \in T_{\sigma_\alpha(\eta)}M, \]
   where $\langle \cdot, \cdot \rangle$ denotes the standard Euclidean inner product, $j_1, j_2 \in \{1, \ldots, k\}$, and $\delta_{j_1,j_2}$ is the Kronecker delta.

We call these four conditions the LOR assumptions. These assumptions guarantee that the tangent space to $M$ at any point $p \in M$, denoted by $T_pM$, is an $n-k$ dimensional space.

To simplify indexing, we will use the convention that any index related to tangential objects will be denoted by $i$, or when necessary by $i_1, i_2, \ldots$, and any index related to normal objects will be denoted by $j$ or by $j_1, j_2, \ldots$. Stated simply, $\{N_j \sigma_\alpha(\eta)\}_{j=1}^k$ forms an orthonormal basis of $T_{\sigma_\alpha(\eta)}M^\perp$ on $U_\alpha$. The existence of such a basis is straightforward to establish locally, and by refining our domains $U_\alpha$ we can guarantee that such mappings exist.

With our notation and assumptions in hand, we now motivate the underlying geometric idea for constructing the LOR frame. Suppose that we are interested in studying the dynamics near a point $x_0 \in \Omega$ that lies close to our embedded manifold. Furthermore, suppose that $x_0$ can be written in the form

$$x_0 = \sigma_\alpha(\eta_0) + \sum_{j=1}^k \xi_{0,j} N_j \sigma_\alpha(\eta_0), \quad \eta_0 \in U_\alpha, (\xi_{0,1}, \xi_{0,2}, \ldots, \xi_{0,k}) \in \mathbb{R}^k. \quad (2)$$

To write equation (2), we have assumed that $x_0$ can be decomposed into a point on $M$, namely $\sigma_\alpha(\eta_0)$, and a vector in the orthogonal complement of $T_{\sigma_\alpha(\eta_0)}M$, namely $\sum_{j=1}^k \xi_{0,j} N_j \sigma_\alpha(\eta_0)$. We will establish that such a decomposition is generic, sufficiently close to $M$. We define $\Psi_\alpha : U_\alpha \times \mathbb{R}^k \to \mathbb{R}^n$ by

$$\Psi(\eta, (\xi_1, \ldots, \xi_k)) = \sigma_\alpha(\eta) + \sum_{j=1}^k \xi_j N_j \sigma_\alpha(\eta)$$

and denote $\xi = (\xi_1, \ldots, \xi_k)$, so that (2) can be more succinctly expressed as $x_0 = \Psi(\eta_0, \xi_0)$. 

Now, denote by $\phi(t)$ the trajectory of (1) such that $\phi(0) = x_0$. We want to continue tracking $\phi(t)$ in our decomposition. To do so, we seek smooth $\eta : (-\delta, \delta) \to \mathcal{U}_\alpha$, $\xi : (-\delta, \delta) \to \mathbb{R}^k$ such that $\phi(t) = \Psi(\eta(t), \xi(t))$ for $t \in (-\delta, \delta)$. We will establish that this continuation can be achieved, providing a convenient new set of coordinates $(\eta, \xi)$, which we will call the LOR frame. We will also derive a system of ODEs that govern the evolution of $\eta(t), \xi(t)$, the LOR equations, by using the ODE satisfied by $\phi(t)$.

We call $(\eta, \xi)$ the LOR coordinates for the base manifold $\mathcal{M}$. By choosing $\mathcal{M}$ to be dynamically relevant (i.e., a structure that can be observed to play a role in organizing the flow), we will be able to study the dynamics near $\mathcal{M}$ using the corresponding LOR frame. The geometric nature of the LOR frame can offer striking insights into the local behavior of the flow, representing a powerful approach for the theoretical study of dynamical systems.

The remainder of the paper is organized as follows. In Section 2, we state our main result, which stipulates the existence of the LOR frame and the form of the LOR equations. Next, in Section 3, we set up the notation and definitions needed to prove this result, while the proof itself appears in Section 4. In Section 5, we consider LOR in a variety of settings. We treat the special cases where $\mathcal{M}$ is codimension-1 or codimension-$(n-1)$, present an algorithm for constructing a particularly useful normal frame assuming that $\mathcal{M}$ is equipped with a Frenet-type atlas, and introduce a blow-up approach to studying how trajectories in $\Omega$ evolve relative to $\mathcal{M}$. We conclude in Section 6 with a computational example to explicitly illustrate the implementation of LOR, in this case to easily identify trajectories organizing canard behavior near a fold of a critical manifold in a two-timescale system.

2. Statement of the main result. In what follows, we will pick a specific chart of $\mathcal{M}$ and drop our indexing subscript; that is, we will study the LOR frame on the chart $(\mathcal{U}, \sigma)$ of $\mathcal{M}$. Once our theory is established locally, we will prove that transfer across charts is well-defined, a result that is critical to the utility of LOR.
To state our main result, we define $Tf : \mathcal{U} \times \mathbb{R}^k \to \mathbb{R}^{n-k}$, $Nf : \mathcal{U} \times \mathbb{R}^k \to \mathbb{R}^k$ by

$$
Tf(\eta, \xi) = \begin{pmatrix}
\langle f \circ \Psi(\eta, \xi), \partial_1 \sigma(\eta) \rangle \\
\vdots \\
\langle f \circ \Psi(\eta, \xi), \partial_{n-k} \sigma(\eta) \rangle
\end{pmatrix},
Nf(\eta, \xi) = \begin{pmatrix}
\langle f \circ \Psi(\eta, \xi), N_1 \sigma(\eta) \rangle \\
\vdots \\
\langle f \circ \Psi(\eta, \xi), N_k \sigma(\eta) \rangle
\end{pmatrix}.
$$

Note that $Tf(\eta, \xi)$ projects the vector field at $\Psi(\eta, \xi)$ into the normal space of $\mathcal{M}$ and $Nf(\eta, \xi)$ projects the vector field at $\Psi(\eta, \xi)$ into the normal space of $\mathcal{M}$. With this additional notation, we can state our main result.

**Main Result (LOR Equations Lemma).** Suppose that $x_0 \in \Omega$ can be expressed as $x_0 = \Psi(\eta_0, \xi_0)$ for $\eta_0 \in \mathcal{U}$ and $\xi_0 \in \mathbb{R}^k$. If $\|\xi_0\|$ is sufficiently small, then there exist $\delta > 0$, $\eta \in \mathcal{C}^1((-\delta, \delta), \mathcal{U}), \xi \in \mathcal{C}^1((-\delta, \delta), \mathbb{R}^k)$ such that $\Phi(x_0, t) = \Psi(\eta(t), \xi(t))$. Furthermore, $\eta(t), \xi(t)$ satisfy the initial value problem

$$
\dot{\eta} = S_\sigma(\eta, \xi)^{-1}Tf(\eta, \xi), \quad \eta(0) = \eta_0,
$$

$$
\dot{\xi} = Nf(\eta, \xi) - K_\sigma(\eta, \xi)S_\sigma(\eta, \xi)^{-1}Tf(\eta, \xi), \quad \xi(0) = \xi_0,
$$

where $S_\sigma(\eta, \xi) \in \mathbb{R}^{n-k \times n-k}, K_\sigma(\eta, \xi) \in \mathbb{R}^{k \times n-k}$ are defined in subsection 3.2.

The following section will supply the explicit forms of $S_\sigma(\eta, \xi)$ and $K_\sigma(\eta, \xi)$ and will establish some key results that underlie the LOR Equations Lemma. We call $S_\sigma(\eta, \xi)$ and $K_\sigma(\eta, \xi)$ the tangent and normal exchange operator, respectively, for reasons that will become apparent in subsection 3.2 and in Section 4, where we give the proof of the LOR Equations Lemma.

3. Constructing the LOR frame. To start the construction, we will present a set of tools for the local analysis of embedded manifolds and some of their properties, establish our notation and review relevant concepts for readers. Subsequently, in subsection 3.3, we establish an important tracking result.

3.1. First and second fundamental forms. When studying surfaces embedded in $\mathbb{R}^3$, the first and second fundamental forms, often denoted $I, II$ respectively, play a crucial role in representing local properties or shape. We will review the generalization of these concepts to an $(n-k)$-manifold embedded in $\mathbb{R}^n$.

**Remark 1.** There is a substantial body of work regarding the differential geometry of embedded manifolds including generalization of the concept of a second fundamental form [7]. While there may be a more elegant presentation of the following material, we set up just what we need for the problem at hand.

**Definition 3.1.** Given a chart $(\mathcal{U}, \sigma)$ of the $\mathcal{C}^r$ codimension-$k$ manifold $\mathcal{M}$ with $r \geq 2$ and a $\mathcal{C}^s$ normal frame $\{N_j \sigma(\eta)\}_{j=1}^s$ with $s \geq 1$, define the mappings $I, II_j : \mathcal{U} \to \mathbb{R}^{n-k \times n-k}$ entrywise by

$$
(I(\eta))_{i_1, i_2} = \langle \partial_{i_1} \sigma(\eta), \partial_{i_2} \sigma(\eta) \rangle,
$$

$$
(II_j(\eta))_{i_1, i_2} = \langle \partial_{i_1} \sigma(\eta), \partial_{i_2} N_j \sigma(\eta) \rangle
$$

where $i_1, i_2 \in \{1, \ldots, n-k\}, j \in \{1, \ldots, k\}$. We call $I(\eta), \{II_j(\eta)\}_{j=1}^k$ the first and second fundamental forms of $\sigma$, respectively.

In the instance where $n = 3$ and $k = 1$, such that $\mathcal{M}$ is a surface embedded in $\mathbb{R}^3$, our representations of the first and second fundamental forms reduce to the standard definition. For any manifold embedded in $\mathbb{R}^n$, the arclength of a curve along the manifold is defined using the first fundamental form; given a $\mathcal{C}^1$ curve...
\( \gamma : [0, 1] \to \mathcal{M} \), there is a parametrized curve \( \bar{\gamma} : [0, 1] \to \mathcal{U} \) such that \( \gamma(t) = \sigma \circ \bar{\gamma}(t) \), with \( \langle \gamma'(t), \gamma'(t) \rangle = \langle [D\sigma \circ \bar{\gamma}(t)]\gamma'(t), [D\sigma \circ \bar{\gamma}(t)]\gamma'(t) \rangle = \langle [I \circ \bar{\gamma}(t)]\gamma(t), \gamma(t) \rangle \), where the final equality follows from transposition. Therefore, the arclength functional may be expressed as

\[
\mathcal{L}(\gamma) = \int_0^1 \sqrt{\langle [I \circ \bar{\gamma}(t)]\gamma'(t), \gamma'(t) \rangle} dt.
\]

Note that the second fundamental forms depend on the choice of normal frame and hence are not (generically) intrinsic features of \( \mathcal{M} \). However, \( I(\eta) \) is preserved under diffeomorphic reparameterization. The following result presents the properties of these matrices that are relevant for our analysis.

**Lemma 3.2.** The maps \( I(\eta), \{I_j(\eta)\}_{j=1}^k \) are self-adjoint, \( I(\eta) \) is positive definite,

\[
(I_j(\eta))_{i_1,i_2} = -\langle \partial_{i_1} \partial_{i_2} \sigma(\eta), N_j \sigma(\eta) \rangle,
\]

and

\[
I(\eta) = D_\sigma \sigma(\eta)^T D_\sigma \sigma(\eta), \quad I_j(\eta) = D_\sigma \sigma(\eta)^T D_\sigma N_j \sigma(\eta).
\]  

**Proof.** For notational convenience we will suppress \( \eta \)-dependence. From the symmetry of the inner product, we note that \( I \) is self-adjoint. Furthermore, the regularity of \( \mathcal{M} \) (LOR assumption 3) implies that \( \{\partial_\eta \sigma\}_{i=1}^m \) forms a linearly independent set. Thus, \( I \) is a Gram matrix and hence is positive definite.

Now consider

\[
0 = \partial_{i_2} 0 = \partial_{i_2} \langle \partial_{i_1} \sigma, N_j \sigma \rangle = \langle \partial_{i_1} \partial_{i_2} \sigma, N_j \rangle + \langle \Pi_j \rangle_{i_1,i_2}.
\]

Equation (6) implies that \( \langle \Pi_j \rangle_{i_1,i_2} = -\langle \partial_{i_1} \partial_{i_2} \sigma, N_j \rangle \) and, as second derivatives commute, \( \Pi_j \) is self-adjoint. The equalities in (5) follow immediately from matrix multiplication. \( \square \)

### 3.2. The tangent and normal exchange operators

We will use the first and second forms to define a linear operator that is fundamental to the LOR frame. We call this operator the **tangent exchange operator** because it takes tangent vectors in \( T_{\sigma(\eta)}\mathcal{U} \) and exchanges them for other tangent vectors in \( T_{\sigma(\eta)}\mathcal{M} \), as will become clear later, in Section 4.

**Definition 3.3.** Given a chart \( (\mathcal{U}, \sigma) \) of the \( \mathcal{C}^r \) codimension-\( k \) manifold \( \mathcal{M} \) with \( r \geq 2 \) and a \( \mathcal{C}^s \) normal frame \( \{N_j \sigma(\eta)\}_{j=1}^k \) with \( s \geq 1 \), define the mapping \( S_\sigma : \mathcal{U} \times \mathbb{R}^k \to \mathbb{R}^{n-k \times n-k} \) by

\[
S_\sigma(\eta, \xi) = I(\eta) + \sum_{j=1}^k \xi_j \Pi_j(\eta).
\]

As noted in Section 2, we call \( S_\sigma(\eta, \xi) \) the tangent exchange operator of \( \sigma \).

For our analysis, we will require the following result, which shows that given an embedding into \( \mathbb{R}^n \), \( S_\sigma \) is invariant under change of normal frame (i.e., \( S_\sigma \) is extrinsically invariant).

**Lemma 3.4.** For any \( \eta \in \mathcal{U} \) and \( \xi \in \mathbb{R}^k \) sufficiently small in norm, \( S_\sigma(\eta, \xi) \) is invertible. Suppose that \( \{N_j \sigma(\eta)\}_{j=1}^k \), \( \{\tilde{N}_j \sigma(\eta)\}_{j=1}^k \) are distinct normal frames to \( \sigma(\eta) \). If

\[
\sigma(\eta) + \sum_{j=1}^k \xi_j N_j \sigma(\eta) = \sigma(\eta) + \sum_{j=1}^k \tilde{\xi}_j \tilde{N}_j \sigma(\eta),
\]

then \( \xi = \tilde{\xi} \).
then $S_{\sigma}(\eta, \xi) = \hat{S}_{\sigma}(\eta, \hat{\xi})$ where $S_{\sigma}$ is the tangent exchange operator calculated from normal basis $\{N_j \sigma(\eta)\}$ and $\hat{S}_{\sigma}$ is the tangent exchange operator calculated from $\{\hat{N}_j \sigma\}$.

Proof. Note that $S_{\sigma}(\eta, 0) = I(\eta)$, which is a positive matrix, and hence there is a $\delta > 0$ such that $\det S_{\sigma}(\eta, \xi) > 0$ for $\xi \in B_\delta(0)$, from the continuity of $S_{\sigma}$. Suppose that $\{N_j \sigma(\eta)\}_{j=1}^k$, $\{\hat{N}_j \sigma(\eta)\}_{j=1}^k$ are distinct normal frames to $\sigma(\eta)$, with corresponding second fundamental forms $(\mathbb{II}_j(\eta))_{i_1,i_2}, (\hat{\mathbb{II}}_j(\eta))_{i_1,i_2}$. As each $\{N_j \sigma(\eta)\}, \{\hat{N}_j \sigma(\eta)\}$ forms an orthonormal basis,

$$N_j, \sigma(\eta) = \sum_{j=1}^k \left(N_j \sigma(\eta), \hat{N}_{j_2} \sigma(\eta)\right) \hat{N}_{j_2} \sigma(\eta).$$

Let $B_{j_1,j_2}(\eta) = \left(N_j \sigma(\eta), \hat{N}_{j_2} \sigma(\eta)\right)$. Subtracting $\sigma(\eta)$ from both sides of (7) and projecting onto $N_j, \sigma(\eta)$ yields

$$\xi_{j_1} = \sum_{j=1}^k \hat{\xi}_{j_2} \left(N_j \sigma(\eta), \hat{N}_{j_2} \sigma(\eta)\right) = \sum_{j=1}^k B_{j_1,j_2}(\eta) \hat{\xi}_{j_2}.$$ 

Therefore, for fixed $i_1, i_2 \in \{1, \cdots, n - k\}$,

$$(S_{\sigma}(\eta, \xi))_{i_1,i_2} = \left(I(\eta)\right)_{i_1,i_2} + \sum_{j=1}^k \xi_{j_1} (\mathbb{II}_j(\eta))_{i_1,i_2}$$

$$= \left(I(\eta)\right)_{i_1,i_2} - \sum_{j=1}^k \left(\sum_{j_2=1}^k B_{j_1,j_2}(\eta) \hat{\xi}_{j_2}\right) \left(\mathbb{II}_{i_1,i_2}(\sigma(\eta)), N_{j_2} \sigma(\eta)\right)$$

$$= \left(I(\eta)\right)_{i_1,i_2} - \sum_{j=1}^k \sum_{j_2=1}^k \hat{\xi}_{j_2} B_{j_1,j_2}(\eta) \left(\mathbb{II}_{i_1,i_2}(\sigma(\eta)), B_{j_1,j_3}(\eta) \hat{N}_{j_2} \sigma(\eta)\right)$$

$$= \left(I(\eta)\right)_{i_1,i_2} + \sum_{j=1}^k \sum_{j_2=1}^k \sum_{j_3=1}^k \hat{\xi}_{j_2} B_{j_1,j_2}(\eta) B_{j_1,j_3}(\eta) \left(\mathbb{II}_{i_1,i_2}(\sigma(\eta)), \hat{N}_{j_2} \sigma(\eta)\right). \tag{8}$$

Next we derive a simple identity to simplify this flurry of indices. Note that

$$\hat{N}_j \sigma(\eta) = \sum_{j=1}^k B_{j_2,j_1}(\eta) N_{j_2} \sigma(\eta)$$

$$= \sum_{j_2=1}^k B_{j_2,j_1}(\eta) \left(\sum_{j_3=1}^k B_{j_2,j_3}(\eta) \hat{N}_{j_3} \sigma(\eta)\right)$$

$$= \sum_{j_2=1}^k \sum_{j_3=1}^k B_{j_2,j_1}(\eta) B_{j_2,j_3}(\eta) \hat{N}_{j_3} \sigma(\eta). \tag{9}$$
Projecting both sides of (9) onto $\tilde{N}_{j_1}\sigma(\eta)$ yields
\[
\delta_{j_1,j_4} = \sum_{j_2=1}^{k} \sum_{j_3=1}^{k} B_{j_2,j_1}(\eta)B_{j_3,j_4}(\eta)\delta_{j_3,j_4}
\]
\[
= \sum_{j_2=1}^{k} B_{j_2,j_1}(\eta)B_{j_2,j_4}(\eta).
\]
Applying (10) to (8) yields
\[
(S_{\sigma}(\eta,\xi))_{i_1,i_2} = (I(\eta))_{i_1,i_2} + \sum_{j_2=1}^{k} \sum_{j_3=1}^{k} \xi_{j_2}(\hat{\mathbb{U}}_{j_2}(\eta))_{i_1,i_2} (\delta_{j_2,j_3})
\]
\[
= (I(\eta))_{i_1,i_2} + \sum_{j_2=1}^{k} \xi_{j_2}(\hat{\mathbb{U}}_{j_2}(\eta))_{i_1,i_2}
\]
\[
= (\hat{S}_{\sigma}(\eta,\xi))_{i_1,i_2}.
\]
Since $i_1, i_2$ were arbitrary, the result follows.

\textbf{Corollary 1.} If \(\{N_j(\sigma(\eta))\}_{j=1}^{k}, \{\tilde{N}_j(\sigma(\eta))\}_{j=1}^{k}\) are normal frames and \(\Psi(\eta,\xi) = \hat{\Psi}(\eta,\xi)\)

then \((B(\eta))_{j_1,j_2} = \left\langle \tilde{N}_{j_1}\sigma(\eta), \tilde{N}_{j_2}\sigma(\eta) \right\rangle\) is a unitary matrix, and \(\xi = B(\eta)\xi\).

To conclude this subsection, we present a definition for another linear map, which has no obvious analogue in the theory of surfaces.

\textbf{Definition 3.5.} Given a chart \((\mathcal{U}, \sigma)\) of the $\mathcal{C}^r$ codimension-$k$ manifold $\mathcal{M}$ with

$r \geq 2$ and a $\mathcal{C}^s$ normal frame $\{N_j(\sigma(\eta))\}_{j=1}^{k}$ with $s \geq 1$, define the mapping $K_{\sigma}$ : $\mathcal{U} \times \mathbb{R}^k \rightarrow \mathbb{R}^{k \times n-k}$ by

\[
K_{\sigma}(\eta,\xi) = \sum_{j=1}^{k} \xi_j [N\sigma(\eta)][D_\eta N_j\sigma(\eta)],
\]

where

\[
N\sigma(\eta) := (N_1\sigma(\eta), \ldots, N_k\sigma(\eta))^T.
\]

As noted in Section 4, we call $K_{\sigma}$ the normal exchange operator.

\textbf{3.3. LOR tracking lemma.} The following lemma provides the backbone of our main result.

\textbf{Lemma (LOR Tracking Lemma).} Suppose that $x_0 \in \Omega$ can be expressed as $\Psi(\eta_0,\xi_0)$ for $\eta_0 \in \mathcal{U}$, $\xi_0 \in \mathbb{R}^k$ and that $\gamma : I \rightarrow \mathbb{R}^n$ is a $\mathcal{C}^1$ curve defined on an interval $I$ containing zero with $\gamma(0) = x_0$. If $\|\xi_0\|$ is sufficiently small, then there exist $\delta > 0$ such that $(-\delta,\delta) \subseteq I$ and functions $\eta \in \mathcal{C}^1((-\delta,\delta),\mathcal{U}), \xi \in \mathcal{C}^1((-\delta,\delta),\mathbb{R}^k)$ with $\eta(0) = \eta_0, \xi(0) = \xi_0$ such that $\gamma(t) = \Psi(\eta(t),\xi(t))$ for $t \in (-\delta,\delta)$.

\textbf{Proof.} We will apply the Implicit Function Theorem (IFT). Define $F : \mathcal{U} \times \mathbb{R}^k \times I \rightarrow \mathbb{R}^n$ by

\[
F(\eta,\xi,t) = \Psi(\eta,\xi) - \gamma(t).
\]

Note that $F$ is $\mathcal{C}^1$ and $F(\eta_0,\xi_0,0) = 0$. Hence we need only show that the matrix $D_{(\eta,\xi)}F(\eta_0,\xi_0,0)$ is invertible. We compute
\[
\frac{\partial \Psi}{\partial \xi_j}(\eta, \xi) = N_j \sigma(\eta) \\
\partial_i \Psi(\eta, \xi) = \partial_i \sigma(\eta) + \sum_{j=1}^{k} \xi_j \partial_i N_j \sigma(\eta)
\]

for \(i \in \{1, \ldots, n-k\}, j \in \{1, \ldots, k\}\). It suffices to prove that the \(n\) vectors described
in (13) are linearly independent. By way of contradiction suppose that there are 
\(n\) nonzero scalars \(\{\alpha_i\}_{i=1}^{n-k}, \{\beta_j\}_{j=1}^{k} \subseteq \mathbb{R}\) such that
\[
\sum_{i=1}^{n-k} \alpha_i \partial_i \Psi(\eta, \xi) + \sum_{j=1}^{k} \beta_j \frac{\partial \Psi}{\partial \xi_j}(\eta, \xi) = 0.
\]
Projecting both sides of this relation into \(\partial_i \sigma(\eta)\) and simplifying yields
\[
\sum_{i=1}^{n-k} \left( \langle \partial_i \sigma(\eta), \partial_i \sigma(\eta) \rangle + \sum_{j=1}^{k} \xi_j \langle \partial_i N_j \sigma(\eta), \partial_i \sigma(\eta) \rangle \right) \alpha_i = 0
\]
or, when vectorized, \(S_{\sigma}(\eta, \xi) \alpha = 0\). We can guarantee that there is a \(\delta > 0\) such that 
\(S_{\sigma}(\eta, \xi)\) is invertible for all \(\xi_0 \in B_\delta(0)\), by Lemma 3.4, hence \(\alpha = 0\). Furthermore,
\(\{N_j \sigma(\eta)\}_{j=1}^{k}\) is an orthonormal set and thus \(\beta_j = 0\) follows for all \(j\). Thus, we obtain the desired contradiction and conclude that 
\(D_{(\eta, \xi)}F(\eta_0, \xi_0, 0)\) is invertible, such that our claim follows from the IFT. \(\square\)

The non-constructive wording “for \(\xi_0\) sufficiently small in norm” in the LOR Tracking Lemma can actually be made exact.

**Definition 3.6.** Given a chart \((\mathcal{U}, \sigma)\) of the \(\mathcal{C}^r\) codimension-\(k\) manifold \(M\) with \(r \geq 2\) and a \(\mathcal{C}^s\) normal frame \(\{N_j \sigma(\eta)\}_{j=1}^{k}\) with \(s \geq 1\), for each \(\eta \in \mathcal{U}\), let \(P_\eta\) be the maximal, by inclusion, path-connected subset of \(\mathbb{R}^k\) containing the origin such that
\[\det S_{\sigma}(\eta, \xi) > 0 \quad \forall \xi \in P_\eta.\]
Define the maximal parameter domain \(P\) by
\[P = \bigcup_{\eta \in \mathcal{U}} \{\eta\} \times P_\eta.\]

The definition of the maximal parameter domain provides the following rewording of the LOR Tracking Lemma.

**Corollary 2.** Suppose that \(x_0 \in \Omega\) can be expressed \(\Psi(\eta_0, \xi_0)\) for \(\eta_0 \in \mathcal{U}, \xi_0 \in \mathbb{R}^k\), and that \(\gamma : I \to \mathbb{R}^n\) is a \(\mathcal{C}^1\) curve defined on an interval \(I\) containing zero with 
\(\gamma(0) = x_0\). If \((\eta_0, \xi_0) \in P\), then there exist \(\delta > 0\) such that 
\((-\delta, \delta) \subseteq I\) and functions \(\eta \in \mathcal{C}^1((-\delta, \delta), \mathcal{U}), \xi \in \mathcal{C}^1((-\delta, \delta), \mathbb{R}^k)\) such that \(\gamma(t) = \Psi(\eta(t), \xi(t))\) for \(t \in (-\delta, \delta)\).

With this result, the continuation of the LOR frame for curve tracking is guaranteed up to but not across \(P_0 := \text{cl}(P) \cap \{\det S_{\sigma}(\eta, \xi) = 0\}\). The reader may be worried that \(P\) depends on our choice of normal frame and thus \(\Psi(P), \Psi(P_0) \subseteq \mathbb{R}^n\) may be dependent on the choice of normal frame. Thankfully, we find that Lemma 3.4 can be used to eliminate any such uneasiness.

**Corollary 3.** \(\Psi(P)\) and \(\Psi(P_0)\) are invariant to choice of normal frame.
Proof. Pick any two normal frames \( \{ N_j \sigma(\eta) \}_{j=1}^k \) and \( \{ \hat{N}_j \sigma(\eta) \}_{j=1}^k \) and define corresponding maximal parameter domains \( \mathcal{P}, \hat{\mathcal{P}} \). For any \( x \in \Psi(\mathcal{P}) \), there exist \( (\eta, \xi) \in \mathcal{P} \) such that \( \Psi(\eta, \xi) = x \). Now, fix \( \eta \in \mathcal{U} \). Denote by \( \gamma : [0,1] \to \mathcal{P}_\eta \) any path connecting 0 to \( \xi \), let \( \Gamma(s) = (\eta, \gamma(s)) \) for all \( s \in [0,1] \), and note that \( \det S_\sigma \circ \Gamma(s) > 0 \).

From Lemma 3.4 we know that there is a unitary change of basis matrix, \( U \), between \( \{ N_j \sigma(\eta) \}_{j=1}^k \) and \( \{ \hat{N}_j \sigma(\eta) \}_{j=1}^k \). Therefore the curve \( \Gamma \) will lift to \( \hat{\Gamma}(s) = (\eta, U \gamma(s)) \). By Lemma 3.4, \( \det \hat{S}_\sigma \circ \Gamma(s) = \det S_\sigma \circ \Gamma(s) > 0 \), and hence \( (\eta, U \xi) \in \hat{\mathcal{P}} \) by maximality. Thus \( x \in \Psi(\hat{\mathcal{P}}) \) and \( \Psi(\mathcal{P}) \subseteq \hat{\Psi}(\hat{\mathcal{P}}) \).

Reversing the proof gives the opposite inclusion. Note that \( \Psi \) is defined on an appropriate domain of \( (\eta, \xi) \) independent of LOR tracking, so \( \Psi(\hat{\mathcal{P}}) \) is uniquely defined as desired, and the claim regarding \( \Psi(\mathcal{P}_0) \) follows analogously.

4. Proof of the main result. With our definitions and preliminary results laid out, we can prove our main result.

Proof of the LOR Equations Lemma. Suppose that \( x_0 \in \Omega \) can be expressed as \( \Psi(\eta_0, \xi_0) = x_0 \) for \( \eta_0 \in \mathcal{U}, \xi_0 \in \mathbb{R}^k \). Denote \( \phi(t) = \Phi(x_0, t) \) and note that \( \phi \in \mathcal{C}^1 \). If \( \| \xi_0 \| \) is sufficiently small, then there exist \( \delta > 0, \eta \in \mathcal{C}^1((-\delta, \delta), \mathcal{U}), \xi \in \mathcal{C}^1((-\delta, \delta), \mathbb{R}^k) \) such that \( \phi(t) = \Psi(\eta(t), \xi(t)) \) for \( t \in (-\delta, \delta) \), by the LOR Tracking Lemma.

Suppressing time dependence, we compute

\[
\dot{\phi} = \frac{d}{dt} \Psi(\eta, \xi) = \left( D_\eta \sigma(\eta) + \sum_{j=1}^k \xi_j D_\eta N_j \sigma(\eta) \right) \eta + \sum_{j=1}^k \xi_j N_j \sigma(\eta). 
\]

Note that \( \dot{\phi} = f \circ \phi = f \circ \Psi(\eta, \xi) \). Acting on the left by \( D_\eta \sigma(\eta)^T \) yields

\[
\left( D_\eta \sigma(\eta)^T D_\eta \sigma(\eta) + \sum_{j=1}^k \xi_j D_\eta \sigma(\eta)^T D_\eta N_j \sigma(\eta) \right) \dot{\eta} = D_\eta \sigma(\eta)^T f \circ \Psi(\eta, \xi),
\]

which reduces to

\[
S_\sigma(\eta, \xi) \dot{\eta} = T f(\eta, \xi). 
\]

Thus \( \dot{\xi} = S_\sigma(\eta, \xi)^{-1} T f(\eta, \xi) \), as \( S_\sigma \) is invertible near \( (\eta_0, \xi_0) \). The equation governing \( \xi \) follows analogously by left acting on equation (14) by the expression given in (12).

\[\Box\]

Remark 2. Equation (15) highlights the exchange of tangent vectors via the action of \( S_\sigma \) that motivates our choice to name this the exchange operator. Note that the standard shape operator from differential geometry also exchanges tangent vectors, but these operators are not identical.

We have once again used non-constructive language to constrain \( \xi_0 \). We can strengthen this language by using \( \mathcal{P} \) and “reverse” the statement of the main result.

Corollary 4. Suppose that \( \eta(t), \xi(t) \) solve system (3) for \( (\eta_0, \xi_0) \in \mathcal{P} \) for \( t \in (-\delta, \delta) \). Then \( \Psi(\eta(t), \xi(t)) = \Phi(\Psi(\eta_0, \xi_0), t) \) for \( t \in (-\delta, \delta) \).

Given a vector field \( f \) and a chart \( (\mathcal{U}, \sigma) \) of a smooth, regular, codimension- \( k \) manifold, we denote the LOR vector field by

\[
\mathcal{L}_\sigma f(\eta, \xi) = \left( S_\sigma(\eta, \xi)^{-1} T f(\eta, \xi), N f(\eta, \xi) - K_\sigma(\eta, \xi) S_\sigma(\eta, \xi)^{-1} T f(\eta, \xi) \right),
\]
Interestingly, the operator $\mathcal{L}$ is linear; that is,

$$\mathcal{L}(\alpha g + \beta f)(\eta, \xi) = \alpha \mathcal{L}g(\eta, \xi) + \beta \mathcal{L}f(\eta, \xi)$$

as $f, g$ only appear in $Tf, Tg, Nf, Ng$, which are vectors of inner products. Therefore the LOR dynamics respects additive perturbations. We hope to take advantage of this fact in future work.

It is natural to wonder whether the LOR flow is topologically conjugate to the flow induced by (1). It can be computationally difficult to find a codomain on which $\Psi$ is a homeomorphism. However, the flow induced by (1) is a submersion of the LOR flow via $\Psi$.

**Definition 4.1.** Suppose that $\Phi_X, \Phi_Y$ are both continuous flows on $X, Y$ respectively, with $\Phi_X$ defined on $\{(x, t) : x \in X, t \in I_x\}$ for time intervals $I_x \subset \mathbb{R}$. The flow $\Phi_Y$ is a submersion of $\Phi_X$ via $H$ if there is a continuous, surjective map $H : X \rightarrow Y$ such that

$$H \circ \Phi_X(x, t) = \Phi_Y(H(x), t) \quad \forall x \in X, t \in I_x.$$  

This is simply the definition of topological conjugacy of flows with the injectivity of the homeomorphism relaxed.

**Corollary 5.** The flow induced by $f$ is a submersion of the flow induced by $\mathcal{L}f$ via $\Psi$.

5. Extensions and applications. In this section, we highlight several scenarios where the computations involving LOR simplify in useful ways. We also discuss certain auxiliary ideas, namely a measure of near-invariance for a surface in a flow that can be defined based on LOR and a blow-up transformation that can be applied to the LOR equations, that enhance the utility of LOR. The near-invariance measure plays a useful role in Section 6.

5.1. Hypersurfaces and Frenet curves. In the cases where $k = 1$ and $k = n-1$, which correspond to $\mathcal{M}$ being a curve and a hypersurface, respectively, there is a canonical choice of normal frame that allows us to simplify the LOR dynamics. First, suppose that $k = n-1$ and $\mathcal{M}$ is a codimension-1 manifold. In this case, $T_p\mathcal{M}$ is a $n-1$ dimensional space, with a one-dimensional orthogonal complement in $\mathbb{R}^n$. Therefore there are exactly 2 unit vectors that could serve as a normal frame. We choose the normal vector in accordance with the right hand rule; specifically, we define

$$N(\eta) = \frac{\partial_1\sigma(\eta) \wedge \cdots \wedge \partial_{n-1}\sigma(\eta)}{\|\partial_1\sigma(\eta) \wedge \cdots \wedge \partial_{n-1}\sigma(\eta)\|}$$  

where $x_1 \wedge \cdots \wedge x_{n-1}$ is the outer product of $n-1$ vectors, which is a generalization of the cross product. We easily attain the following result.

**Proposition 1.** For $N(\eta)$ given in (16), $K(\eta, \xi) = 0$.

**Proof.** Note that $\langle N(\eta), N(\eta) \rangle = 1$ and hence

$$0 = \partial_i \langle N(\eta), N(\eta) \rangle = 2 \langle \partial_i N(\eta), N(\eta) \rangle.$$  

Therefore, $K(\eta, \xi) = 0$. 

Thus, with this choice of $N(\eta)$, the $\xi$ equation in (3) simplifies significantly.

The study of curves is more interesting. Suppose that $\sigma$ parameterizes a curve in $\mathbb{R}^n$. By imposing additional structure on $\sigma$ we can generate a natural normal
frame. Assume that \( \sigma \in C^{n-1}(\mathcal{U}) \) and that \( \{ \sigma'(\eta), \sigma''(\eta), \ldots, \sigma^{(n-1)}(\eta) \} \) are linearly independent for \( \eta \in \mathcal{U} \), where \( ' = d/d\eta \); if \( \sigma \) satisfies this condition, then it is called a Frenet curve. Note that, generically, \( \{ \sigma'(\eta), \sigma''(\eta), \ldots, \sigma^{(n-3)}(\eta) \} \) will be linearly independent almost everywhere in \( \mathcal{U} \), hence the Frenet condition is fairly mild.

When \( \sigma \) is a Frenet curve, we can construct the Frenet frame by performing Gram-Schmidt orthonormalization of the first \( n-1 \) derivatives of \( \sigma \); this will yield the vectors \( T\sigma(\eta), N_1\sigma(\eta), \ldots, N_{n-2}\sigma(\eta) \). The final normal vector, \( N_{n-1}\sigma(\eta) \), is chosen to be the unique vector such that \( \det F(\eta) = 1 \)

\[
F(\eta) = (T\sigma(\eta), N_1\sigma(\eta), \ldots, N_{n-1}\sigma(\eta))^T.
\]

The interesting feature of this frame is the so-called Frenet-Serret equation, which is given by

\[
\frac{d}{d\eta} F(\eta) = ||\sigma'(\eta)|| C(\eta) F(\eta) \tag{17}
\]

where

\[
C(\eta) = \begin{pmatrix} 0 & \kappa_1(\eta) & 0 & \cdots \\ -\kappa_1(\eta) & 0 & \kappa_2(\eta) & \cdots \\ 0 & -\kappa_2(\eta) & 0 & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix}
\]

is an \( n \times n \), antisymmetric, tridiagonal matrix whose nonzero entries \( \kappa_1(\eta), \ldots, \kappa_{n-1}(\eta) \) are the generalized curvatures of \( \sigma \). We note that \( \kappa_i(\eta) > 0 \) for \( i \in \{1, \ldots, n-2\} \) and

\[
\kappa_{n-1}(\eta) = \frac{\det(\sigma'(\eta), \ldots, \sigma^{(n)}(\eta))}{\alpha(\eta)}
\]

where \( \alpha(\eta) \) is some product of the norms of \( \sigma'(\eta), \ldots, \sigma^{(n)}(\eta) \). The quantities \( \kappa_i(\eta) \) are geometric invariants of the curve \( \sigma \).

Using this particular normal frame allows us to greatly simplify the LOR equations; we find that in this frame,

\[
I(\eta) = ||\sigma'(\eta)||^2 \\
\Pi_j(\eta) = -||\sigma'(\eta)||^2 \delta_{1,j} \kappa_1(\eta)
\]

hence \( S_\sigma(\eta) = ||\sigma'(\eta)||^2 (1 - \xi_1 \kappa_1(\eta)) \). With significantly more index wrangling, we find that

\[
K_\sigma(\eta, \xi) = ||\sigma'(\eta)|| \hat{C}(\eta) \xi
\]

where \( \hat{C}(\eta) \) is the (1,1)-principal minor of \( C(\eta) \). Therefore (3) simplifies to

\[
\dot{\eta} = \frac{\langle f \circ \Psi(\eta, \xi), T\sigma(\eta) \rangle}{||\sigma'(\eta)|| (1 - \xi_1 \kappa_1(\eta))} \\
\dot{\xi} = N f(\eta, \xi) - \frac{\langle f \circ \Psi(\eta, \xi), T\sigma(\eta) \rangle}{1 - \xi_1 \kappa_1(\eta)} \hat{C}(\eta) \xi. \tag{18}
\]

We call system (18) the Frenet LOR equations. This example illustrates that when \( \sigma \) is a Frenet curve, \( K_\sigma \) reduces to the matrix of curvatures. If we take \( \sigma(\eta) \) to be a trajectory of (1), then \( N f(\eta, \xi) = 0 \) and \( \dot{\eta}|_{\xi=0} = 1 \).
By dividing through by \( \dot{\eta} \) to eliminate time from system (18), we find

\[
\frac{d\xi}{d\eta} = \|\sigma'(\eta)\| (1 - \xi_1 \kappa_1(\eta)) \frac{N_f(\eta, \xi)}{\langle f \circ \Psi(\eta, \xi), T(\sigma(\eta)) \rangle} - \|\sigma'(\eta)\| \dot{C}(\eta) \xi.
\]  

(19)

This set of equations (19) is well-suited to study transition maps: suppose we choose \( \sigma \) to be a dynamically relevant trajectory, and we want to study how the normal space of \( \sigma \) at \( \eta = 0 \) is mapped to the normal space at \( \eta = 1 \). Usually, one has to deal with approximating the time it takes for trajectories to travel from one section \( \Sigma_1 \) to another section \( \Sigma_2 \), defined to represent this mapping. In (19), however, we have eliminated time dependence; thus, we can simply integrate trajectories from \( \eta = 0 \) to \( \eta = 1 \). Therefore it is very natural to represent Poincaré maps in the LOR frame.

5.2. A measure for near-invariance. In this section, we present a natural generalization of a technique used for curves in planar systems in [9]; namely, we construct a quantitative measure that quantifies how close a surface is to being invariant under a flow. We will focus on the simplest case \( k = 1 \), when \( M \) is a hypersurface embedded in \( \Omega \). Recall from the preceding section that the \( \xi \) dynamics in the LOR frame for a codimension-one chart is given by \( \dot{\xi} = Nf(\eta, \xi) \). Note that \( \sigma(U) \) is locally invariant if and only if \( \dot{\xi}|_{\xi=0} = 0 \); that is, \( Nf(\eta, 0) = 0 \) for all \( \eta \in U \). Geometrically, \( Nf(\eta, 0) = 0 \) if and only if \( f \circ \sigma(\eta) \in T\sigma(\eta)M \); thus, the normal vector \( Nf(\eta) \) is orthogonal to our vector field along \( \sigma \). Heuristically, we note that \( \sigma(U) \) is closer to being invariant if \( |Nf(\eta, 0)| \) is small, as LOR trajectories cannot rapidly escape \( U \times \{0\} \).

There are several natural candidates for constructing a measure of this “near-invariance” property. We choose one that we find particularly informative.

**Definition 5.1.** Given a chart \((U, \sigma)\) of the \( C^{r} \) codimension-1 manifold \( M \) with \( r \geq 2 \) and a \( C^{s} \) normal vector \( Nf(\eta) \) with \( s \geq 1 \), we define \( \mu_{\sigma} : U \to \mathbb{R}^{+} \) by

\[
\mu_{\sigma}(\eta) = \inf_{\xi \in P_{\eta}} \{ |\xi| : Nf(\eta, \xi) = 0 \}
\]

with the convention that \( \inf \emptyset = -\infty \). We call \( \mu_{\sigma} \) the near-invariance measure of \( \sigma \). Furthermore, define \( \Xi_{\sigma} : U \to \mathbb{R} \) by

\[
\Xi_{\sigma}(\eta) = \arg \inf_{\xi \in P_{\eta}} \{ |\xi| : Nf(\eta, \xi) = 0 \}.
\]

We call \( \Xi_{\sigma} \) the correction to \( \sigma \).

Intuitively, \( \mu_{\sigma}(\eta) \) represents the closest point \( (\eta, \xi) \in \{\eta\} \times P_{\eta} \) such that \( Nf(\eta, \xi) = 0 \); it measures how far off the manifold we must travel to reach the \( \xi \)-nullsurface. Note that \( \Xi_{\sigma}(\eta) \), when defined, is the smallest \( \xi \), in norm, such that \( Nf(\eta, \xi) = 0 \).

Initially, these definitions seem cumbersome; however, \( \mu_{\sigma}, \Xi_{\sigma} \) have interesting properties. For example, \( \mu_{\sigma}(\eta) = 0 \) if and only if \( \sigma(U) \) is locally invariant. The correction \( \Xi_{\sigma} \) allows us to study how non-invariant manifolds can create trapping regions. If \( Nf(\eta, 0) > 0 \) and \( \Xi_{\sigma}(\eta) > 0 \) for \( \eta \in U \), then the set \( W = \{(\eta, \xi) : \eta \in U, 0 \leq \xi \leq \Xi_{\sigma}(\eta)\} \) is positively locally invariant if \( \langle \nabla \Xi_{\sigma}(\eta), f \circ \Xi_{\sigma}(\eta) \rangle > 0 \) for \( \eta \in U \). This type of trapping region can help to explain how non-invariant manifolds can play an organizing role in dynamics. Indeed, the use of the near-invariance measure plays a key role in the precise definition and identification of rivers in planar systems [9].
Furthermore, we can utilize the function $\Xi_\sigma$ to obtain useful nearly invariant manifolds. If $\Xi_\sigma$ is well-defined and $C^1$ on $\mathcal{U} \subseteq \mathcal{U}$, then consider the function $\hat{\sigma} : \mathcal{U} \to \Omega$ defined by

$$\hat{\sigma}(\eta) = \sigma(\eta) + \Xi_\sigma(\eta)N\sigma(\eta).$$

Generically, $\hat{\sigma}$ is a regular chart itself; that is, it parametrizes a codimension-1 manifold $\hat{\mathcal{M}}$, in which case we call $\hat{\sigma}$ the corrected chart of $\sigma$. Interestingly, the corrected chart is often more nearly-invariant than the original chart, with $\mu_{\hat{\sigma}}(\eta) < \mu_{\sigma}(\eta)$ for $\eta \in \mathcal{U}$. We make use of this property in our computational example in Section 6 to identify canard behavior in the normal form system for a folded-saddle node bifurcation [6, 14].

5.3. Constructing a normal frame to a manifold à la Frenet. In the theory presented thus far, we have simply assumed that the user can provide a normal frame. In this section, we present a rigorous algorithm for constructing these frames.

In the construction of the Frenet frame for the $k = 1$ case, we used higher derivatives of the curve to “fill-up” $T_pM$ and then applied the Gram-Schmidt process to orthonormalize these derivatives and thus construct our moving frame. Here, we apply the same ideas, albeit with more notation, to generalize this construction to higher codimensions.

Suppose that $f \in C^r(\mathbb{R}^m, \mathbb{R}^n)$. Given an index $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m$ such that $\sum_{i=1}^{m} \alpha_i := |\alpha| \leq r$, we use $\partial_\alpha$ to denote the indicial derivative:

$$\partial_\alpha f(x) = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_m^{\alpha_m}}(x).$$

Note that, when represented indicially, $\partial_i := \partial/\partial x_i = \partial e_i$, where $e_i$ is the $i^{th}$ canonical basis vector.

**Definition 5.2.** A chart $(\mathcal{U}, \sigma)$ of a regular, $C^{r+2}$, $m$-manifold $M$ embedded in $\Omega \subseteq \mathbb{R}^n$ for $n > m$ is a Frenet chart if there are indices $\{\alpha_j\}_{j=m+1}^{n}$ such that $\alpha_j \in \mathbb{N}^m$, $2 \leq |\alpha_j| \leq r - 1$ for $m + 1 \leq j \leq n$, and

$$\text{dim span} \left\{ \partial_{\alpha_1}\sigma(\eta), \ldots, \partial_{\alpha_m}\sigma(\eta), \partial_{\alpha_{m+1}}\sigma(\eta), \ldots, \partial_{\alpha_n}\sigma(\eta) \right\} = n \quad \forall \eta \in \mathcal{U}. \quad (20)$$

In this case, $\{\alpha_j\}_{j=m+1}^{n}$ is a Frenet index set.

When considering curves, the $m = 1$ case, taking $\alpha_j = j$ for $m + 1 = 2 \leq j \leq n$, the above condition is equivalent to the curve in question being Frenet. Definition 5.2 allows for the use of more complicated configurations of derivatives to build a basis for $\mathbb{R}^n$.

To simplify notation, given a Frenet index set $\{\alpha_j\}_{j=m+1}^{n}$, we define $\alpha_j = e_j$ for $1 \leq j \leq m$, after which condition (20) simplifies to

$$\text{dim span} \left\{ \partial_{\alpha_j}\sigma(\eta) \right\}_{j=1}^{n} = n \quad \forall \eta \in \mathcal{U}.$$

Clearly we can perform the Gram-Schmidt process on $\{\partial_{\alpha_j}\sigma\}$ to construct an orthonormal basis. The first $m$ vectors will be linear combinations of $\{\partial_{e_j}\sigma\}$ and hence form an orthonormal basis for $T_{\sigma(\eta)}M$; thus, we denote them $T_{\sigma(\eta)}M$, $i = 1, \ldots, m$. The final $n - m$ orthonormal vectors produced will be orthogonal to $T_{\sigma(\eta)}M$ and can be denoted $N_{\sigma(\eta)}M$ without any abuse of notation; they form a normal frame per LOR assumption 4. Therefore, we can produce normal frames to manifolds in an algorithmic way, provided that we make a fairly generic assumption about the manifolds’ higher derivatives.
One advantage of the Frenet frame in the preceding section is that it allows us to easily compute \( \mathbf{II}, K_{\sigma} \) using inherent properties of the LOR base curve. We shall next establish that we can compute \( \mathbf{II}, K_{\sigma} \) from the properties of our generalized Frenet frame as well.

The Gram-Schmidt process, being linear, can be represented as a matrix multiplication. That is, there is an \( n \times n \) matrix \( G : \mathcal{U} \rightarrow \mathbb{R}^{n \times n} \) such that

\[
G(\eta) \begin{pmatrix}
\partial_{\alpha_1}\sigma(\eta) \\
\vdots \\
\partial_{\alpha_n}\sigma(\eta)
\end{pmatrix} = \begin{pmatrix}
T_1\sigma(\eta) \\
\vdots \\
N_k\sigma(\eta)
\end{pmatrix},
\]

and \( G \) is \( \mathcal{C}^1 \), everywhere invertible, and lower triangular. Note that \( G \) is of the form

\[
G(\eta) = \begin{pmatrix}
G_1(\eta) & 0 \\
G_2(\eta) & G_3(\eta)
\end{pmatrix}
\]

(21)

where \( G_1(\eta), G_3(\eta) \) are \( \mathcal{C}^1 \), invertible, \( m \times m, k \times k \) (respectively), lower triangular matrices, and \( G_2(\eta) \) is a \( \mathcal{C}^1 k \times m \) matrix. Define

\[
\hat{G}(\eta) = \begin{pmatrix}
I & 0 \\
G_2(\eta) & G_3(\eta)
\end{pmatrix}
\]

(22)

where \( I \) is the \( m \times m \) identity matrix, then by definition

\[
\hat{G}(\eta) \begin{pmatrix}
\partial_{e_1}\sigma(\eta) \\
\vdots \\
\partial_{e_m}\sigma(\eta) \\
\partial_{\alpha_{m+1}}\sigma(\eta) \\
\vdots \\
\partial_{\alpha_n}\sigma(\eta)
\end{pmatrix} = \begin{pmatrix}
\partial_{e_1}\sigma(\eta) \\
\vdots \\
\partial_{e_m}\sigma(\eta) \\
N_{1}\sigma(\eta) \\
\vdots \\
N_{k}\sigma(\eta)
\end{pmatrix}.
\]

(23)

To finish our generalization of the Frenet frame, we need to understand how the derivatives of \( T_i\sigma, N_j\sigma \) can be expressed as linear combinations of \( T_i\sigma, N_j\sigma \), as in (17). To this end, we define the following.

**Definition 5.3.** Suppose that \( (\mathcal{U}, \sigma) \) is a Frenet chart with index set \( \{ \alpha_j \}_{j=1}^n \). Then \( \{ \partial_{\alpha_j}\sigma(\eta) \}_{j=1}^n \) forms a basis for \( \mathbb{R}^n \) and therefore there are matrices \( A_i : \mathcal{U} \rightarrow \mathbb{R}^{n \times n} \) such that

\[
\partial_{e_i} \begin{pmatrix}
\partial_{\alpha_1}\sigma(\eta) \\
\vdots \\
\partial_{\alpha_n}\sigma(\eta)
\end{pmatrix} = \begin{pmatrix}
\partial_{\alpha_1+e_i}\sigma(\eta) \\
\vdots \\
\partial_{\alpha_n+e_i}\sigma(\eta)
\end{pmatrix} = A_i(\eta) \begin{pmatrix}
\partial_{\alpha_1}\sigma(\eta) \\
\vdots \\
\partial_{\alpha_n}\sigma(\eta)
\end{pmatrix}.
\]

We call these matrices the algebraic closures of the index set.

**Remark 3.** At first glance, the entries of \( A_i(\eta) \) appear to be arbitrary and thus could be difficult to compute. However, by filling out our normal frame with higher derivatives of \( \sigma \), we can greatly simplify \( A_i \). Note that \( \partial_{e_i}\partial_{\alpha_j}\sigma(\eta) = \partial_{\alpha_j+e_i}\sigma(\eta) \), and therefore if \( \alpha_s + e_i \in \{ \alpha_j \}_{j=1}^n \), then the \( s^{th} \) row of \( A_i(\eta) \) is a canonical basis vector; that is, if \( \alpha_s + e_i = \alpha_j \) then \( A_i(\eta)_s = e_j \), hence \( A_i \) can be fairly sparse. In fact, the remarkable simplicity of the Frenet-Serrat equation is caused by the sparseness of the relevant algebraic closure.
Proposition 2. If \((U, \sigma)\) is a Frenet chart with index set \(\{\alpha_j\}_{j=1}^n\) and algebraic closures \(\{A_j(\eta)\}_{j=1}^n\), then for \(i = 1, \ldots, m,\):
\[
\partial_{e_i} v(\eta) = \left[ (\partial_{e_i} \hat{G}(\eta) + \hat{G}(\eta) A_i(\eta)) \hat{G}^{-1}(\eta) \right] v(\eta),
\]
where \(\hat{G}\) is given in equation (22), and
\[
v(\eta) = (\partial_{e_i} \sigma(\eta), \ldots, \partial_{e_m} \sigma(\eta), N_1 \sigma(\eta), \ldots, N_s \sigma(\eta))^T.
\]

Proof. Taking derivatives of both sides of (23), using the definition of the algebraic closures, and applying (23) yields the result.

Proposition 2 allows us to compute \(\Pi_i, K_\sigma\) from first principles, after we define the following operators.

Definition 5.4. Given a chart \((U, \sigma)\) of the \(\mathcal{C}^r\) codimension-\(k\) manifold \(M\) with \(r \geq 2\) and a \(\mathcal{C}^s\) normal frame \(\{N_j \sigma(\eta)\}_{j=1}^k\) with \(s \geq 1\), we define mappings \(\Pi_{j_1} : U \to \mathbb{R}^{k \times n-k}\) entrywise by
\[
(\Pi_{j_1}(\eta))_{j_2,j} = (\partial_{N_j} \sigma(\eta), N_{j_2} \sigma(\eta)),
\]
where \(j_1, j_2 \in \{1, \ldots, k\}, i \in \{1, \ldots, n-k\}\). We call \(\{\Pi_i(\eta)\}_{i=1}^{n-k}\) the third fundamental forms of \(\sigma\).

These operators have no clear analogue in the theory of surfaces; however, they bear some resemblance to the Christoffel symbols. The third fundamental forms are useful insofar as they determine \(K_\sigma\).

Lemma 5.5. Given a chart \((U, \sigma)\) of the \(\mathcal{C}^r\) codimension-\(k\) manifold \(M\) with \(r \geq 2\) and a \(\mathcal{C}^s\) normal frame \(\{N_j \sigma(\eta)\}_{j=1}^k\) with \(s \geq 1\),
\[
K_\sigma(\eta) = \sum_{j=1}^k \xi_j \Pi_j(\eta).
\]

Now we can derive formulae for \(\Pi_j, \Pi_{j_1}\) using properties of our Frenet-type frame. Let
\[
A_i(\eta) = \begin{pmatrix}
A_{i,1}(\eta) & A_{i,2}(\eta) \\
A_{i,3}(\eta) & A_{i,4}(\eta)
\end{pmatrix}
\]
where \(A_{i,1}(\eta)\) is \(m \times m\), \(A_{i,2}(\eta), A_{i,3}(\eta)^T\) are \(m \times k\) and \(A_{i,4}(\eta)\) is \(k \times k\). Let
\[
B_1(\eta) = \begin{pmatrix}
\Pi_1(\eta)_{i,1} & \cdots & \Pi_k(\eta)_{i,1} \\
\vdots & \ddots & \vdots \\
\Pi_1(\eta)_{i,m} & \cdots & \Pi_k(\eta)_{i,m}
\end{pmatrix},
\]
\[
B_2(\eta) = \begin{pmatrix}
\Pi_1(\eta)_{i,1} & \cdots & \Pi_1(\eta)_{i,k} \\
\vdots & \ddots & \vdots \\
\Pi_k(\eta)_{i,1} & \cdots & \Pi_k(\eta)_{i,k}
\end{pmatrix}
\]

Corollary 6. If \((U, \sigma)\) is a Frenet chart with index set \(\{\alpha_j\}_{j=1}^n\), then
\[
B_1(\eta) = -A_{i,2}(\eta) G_3(\eta)^{-1}
\]
\[
B_2(\eta) = \partial_{e_i} G_3(\eta) G_3(\eta)^{-1} + G_3(\eta) A_{i,4}(\eta) G_3(\eta)^{-1}.
\]

Proof. From the definition of \(\Pi_i, \Pi_{j_1}\), and Lemma 3.2 we can compute
\[
\partial_{e_i} v(\eta) = \begin{pmatrix}
-B_1(\eta) \\
B_2(\eta)
\end{pmatrix} v(\eta)
\]
where the blank \(m \times m\) matrix block is irrelevant, and \(v(\eta)\) is defined in (25). Comparing this expression to (24) and using the definitions (21), (22) yields the result, after some simple block matrix computation.
5.4. Blowing up an invariant manifold to study local transient dynamics.

Suppose that \( (\mathcal{U}, \sigma) \) is a chart of a locally invariant codimension-\( k \) manifold \( \mathcal{M} \) that satisfies our LOR assumptions. We will study the LOR flow generated by (3) to determine the dynamics near \( \sigma(\mathcal{U}) \) and will introduce what we term LOR blow-up coordinates to serve in this process.

First we let \( \xi = r \xi \) where \( \|\xi\| = 1 \) such that \( r^2 = \langle \xi, \xi \rangle \). Intuitively, \( r \) represents the Hausdorff distance of \( \Psi(\eta, \xi) \) from the manifold \( \mathcal{M} \) and \( \xi \) represents the “angle” of \( \xi \) relative to the manifold.

It is straightforward, using the fact that \( \langle \xi, \xi \rangle = 1 \) and hence \( \langle \dot{\xi}, \xi \rangle = 0 \), to compute that for \( r > 0 \),

\[
\begin{align*}
\dot{r} &= \langle \dot{\xi}, \xi \rangle, \\
\dot{\xi} &= -\langle \dot{\xi}, \xi \rangle \dot{\xi}.
\end{align*}
\]

System (27) becomes potentially problematic in the \( r \to 0^+ \) limit, which is precisely what we are interested in, in order to study dynamics local to \( \mathcal{M} \). Note that if \( \lim_{r \to 0^+} \dot{\xi}/r \) is well-defined then we can continuously extend the dynamics of (27) to \( r = 0 \). In our new coordinate scheme, the LOR equations (3) give us

\[
\dot{\xi} = Nf(\eta, r\xi) - K_\sigma(\eta, r\xi)S_\sigma(\eta, r\xi)^{-1}Tf(\eta, r\xi).
\]

By definition, \( K_\sigma(\eta, r\xi) = rK_\sigma(\eta, \xi) \); hence, as \( r \to 0^+ \), the right hand side of (28) limits to 0. Therefore, we can compute the L'Hôpital type limit

\[
\lim_{r \to 0^+} \frac{\dot{\xi}}{r} = \frac{d}{dr}|_{r=0} = D\xi Nf(\eta, 0)\xi - K_\sigma(\eta, \xi)S_\sigma(\eta, 0)^{-1}Tf(\eta, 0) =: g(\eta, \xi)
\]

so that from (28), (29), and \( \xi = r\xi \), we have

\[
\begin{align*}
\dot{\eta} &= S_\sigma(\eta, 0)^{-1}Tf(\eta, 0) + O(r) \\
\dot{\xi} &= g(\eta, \xi) - \langle g(\eta, \xi), \xi \rangle \xi + O(r) \\
\dot{r} &= \langle g(\eta, \xi), \xi \rangle r + O(r^2).
\end{align*}
\]

Note that in the above representation, at leading order, the \( \dot{\eta} \) equation depends only upon \( \eta \) and the \( \dot{\xi} \) equation depends only on \( \eta, \xi \). Therefore, we have decoupled the dynamics along the manifold (the \( \eta \) dynamics) from the angular dynamics relative to the manifold (the \( \xi \) dynamics) from the contraction/expansion dynamics (the \( r \) dynamics) for \( r \) sufficiently small. This non-trivial decoupling is possible because we have chosen to represent our dynamics in their natural frame.

We call the \( (\eta, \xi, r) \) coordinates the LOR blow-up coordinates, as the corresponding phase space has the form \( \mathcal{U} \times S^{k-1} \times [0, \varepsilon) \), where \( S^{k-1} = \{ x \in \mathbb{R}^k \mid \|x\| = 1 \} \) is the \( k \)-sphere and \( 0 < \varepsilon \ll 1 \). This coordinate representation is effectively equivalent to performing geometric desingularization on the entirety of \( \mathcal{M} \).

The stability of \( \mathcal{M} \) depends only on the sign of \( \langle g(\eta, \xi), \xi \rangle \); \( \mathcal{M} \) is stable in regions where \( \langle g(\eta, \xi), \xi \rangle < 0 \) and unstable where \( \langle g(\eta, \xi), \xi \rangle > 0 \).

To study the dynamics near the manifold, we first consider the \( \eta \) dynamics on \( r = 0 \), denoting by \( \Phi_\eta \) the flow induced by \( \dot{\eta} = S_\sigma(\eta, 0)^{-1}Tf(\eta, 0) \). Next, we consider the \( \dot{\xi} \) dynamics on \( r = 0 \), treating \( \eta \) as a non-autonomous forcing term. Denote the corresponding flow by \( \Phi_\xi \). Finally, we can approximate the \( r \) solution
by
\[ r(t) \approx r_0 \exp \left( \int_0^t \langle g(\eta(s), \tilde{\xi}(s)), \tilde{\xi}(s) \rangle \, ds \right). \]

5.5. Fast-slow analysis. Consider now the $\mathcal{G}^r$ fast-slow system
\[
\begin{align*}
\dot{x}_1 &= f_1(x_1, x_2; \varepsilon) \\
\dot{x}_2 &= \varepsilon f_2(x_1, x_2; \varepsilon)
\end{align*}
\]
with $x_1 \in \mathbb{R}^k, x_2 \in \mathbb{R}^{n-k}$ and critical manifold $\mathcal{M}_0 := \{ f_1(x_1, x_2; 0) = 0 \}$. When $\varepsilon = 0$, $\mathcal{M}_0$ is a set of equilibria and hence is trivially invariant. Using the rescaled slow time variable $\tau = \varepsilon t$, equation (31) becomes
\[
\begin{align*}
\varepsilon \dot{x}'_1 &= f_1(x_1, x_2; \varepsilon) \\
\dot{x}'_2 &= f_2(x_1, x_2; \varepsilon)
\end{align*}
\]
where $' = d/d\tau$. The $\varepsilon \to 0^+$ limit of system (32) specifies the slow timescale dynamics on $\mathcal{M}_0$. Suppose that $(\mathcal{U}, \sigma)$ parameterizes a chart of $\mathcal{M}_0$. If $\sigma(\mathcal{U})$ is normally hyperbolic, then Fenichel’s Theorem [3] guarantees the existence of $\varepsilon^+ > 0$ for which there exists a chart $\tilde{\sigma} : \mathcal{U} \times [0, \varepsilon^+) \to \mathbb{R}^n$ such that: $\tilde{\sigma}$ parameterizes a locally-invariant, regular, codimension-$k$ manifold $\mathcal{M}_\varepsilon$ for each $\varepsilon \in [0, \varepsilon^+)$. If $\sigma(\eta; 0) = \sigma(\eta)$, and $\tilde{\sigma}(\eta; \varepsilon)$ is $\mathcal{G}^r \mathcal{O}(\varepsilon)$-close to $\sigma$.

As long as our chosen normal frame is continuous in $\varepsilon$, we can perform the LOR transformation, treating $\varepsilon$ as a parameter, which will yield dynamics of the form
\[
\begin{align*}
\dot{\eta} &= \varepsilon S_\sigma(\eta, \xi; \varepsilon)^{-1} \tilde{T} f(\eta, \xi; \varepsilon), \\
\dot{\xi} &= N f(\eta, \xi; \varepsilon) - \varepsilon K_\sigma(\eta, \xi; \varepsilon) S_\sigma(\eta, \xi; \varepsilon)^{-1} \tilde{T} f(\eta, \xi; \varepsilon)
\end{align*}
\]
where $\varepsilon \tilde{T} f(\eta, \xi; \varepsilon) = T f(\eta, \xi; \varepsilon)$, which remains well-defined as $T f(\eta, \xi; \varepsilon) = \mathcal{O}(\varepsilon)$. Hence the LOR equations are also fast-slow. Transforming to LOR blow-up coordinates and expanding in $\varepsilon$ yields, from equations (30) and (29),
\[
\begin{align*}
\dot{\eta} &= \varepsilon S_\sigma(\eta, 0; 0)^{-1} \tilde{T} f(\eta, 0; 0) + \mathcal{O}(r, \varepsilon^2) \\
\dot{\xi} &= B(\eta) \dot{\xi} - \langle B(\eta) \dot{\xi}, \dot{\xi} \rangle \dot{\xi} + \mathcal{O}(r, \varepsilon) \\
\dot{r} &= \langle B(\eta) \dot{\xi}, \dot{\xi} \rangle r + \mathcal{O}(r^2, \varepsilon)
\end{align*}
\]
where $B(\eta) = D_\xi N f(\eta, 0; 0)$. Note that since $S_\sigma, T f, N f$ are being evaluated on $\varepsilon = 0$, we use the existence of $\tilde{\sigma}$ only to justify our derivation; the equations in (34) that we have obtained depend only on the critical manifold $\mathcal{M}_0$. Thus, system (34) gives a natural framework to study dynamics relative to a slow manifold $\mathcal{M}_\varepsilon$ that relates the dynamics relative to $\mathcal{M}_\varepsilon$ to the dynamics relative to $\mathcal{M}_0$.

6. A computational example. In this final section, we present a long-form computational example of the LOR frame in action. We choose a well-known, fast-slow system for our analysis, the normal form for a folded saddle-node canard point [13], which is given by
\[
\begin{pmatrix}
\dot{x} \\
\dot{y} \\
\dot{z}
\end{pmatrix} = \begin{pmatrix}
\varepsilon \frac{\mu}{2} y - (\mu + 1) z \\
\varepsilon \\
x + z^2
\end{pmatrix} =: f(x, y, z; \varepsilon)
\]
for parameter $\mu > 0$ as well as $0 < \varepsilon \ll 1$.

In the $\varepsilon \to 0^+$ limit, the critical manifold $\mathcal{M}_0$, parameterized by $\sigma(\eta_1, \eta_2) = (-\eta_2^2, \eta_1, \eta_2)$, becomes a surface of critical points. Let $\mathcal{U}_L = \{(\eta_1, \eta_2) | \eta_2 < 0\}, \mathcal{U}_R =$
{\eta_1, \eta_2 > 0}, \mathcal{U}_F = \{(\eta_1, \eta_2)|\eta_2 = 0\}, then \sigma(\mathcal{U}_L) is a sheet of normally hyperbolic, stable fixed points in the \(\varepsilon = 0\) system, while \sigma(\mathcal{U}_R) is a sheet of normally hyperbolic, unstable fixed points. The set \sigma(\mathcal{U}_F), called the fold, is a line of nilpotent fixed points.

Fenichel theory guarantees, for \(\varepsilon > 0\) but sufficiently small, the existence of charts \(\sigma_{L,\varepsilon}, \sigma_{R,\varepsilon}\) such that \(\sigma_{i,\varepsilon}(\mathcal{U}_i)\) is a locally-invariant manifold, \(\|\sigma(\eta) - \sigma_{i,\varepsilon}(\eta)\| = \mathcal{O}(\varepsilon)\) for \(\eta \in \mathcal{U}_i\), and \(\sigma_{i,\varepsilon}(\mathcal{U}_i)\) has the same stability properties as \(\sigma(U_i)\) for \(i \in \{L, R\}\).

The most intriguing feature of (35) is the existence of trajectories beginning in \(\sigma_{L,\varepsilon}(\mathcal{U}_L)\) that cross into \(\sigma_{R,\varepsilon}(\mathcal{U}_R)\); these trajectories are referred to as canards and play an organizing role in various forms of fast-slow dynamics including mixed-mode oscillations.

The standard technique for identifying canards is to perform geometric desingularization in a neighborhood of the origin of system (35) and track carefully chosen trajectories across the multiple “charts” of the desingularized variables. Here, we present an approach based on LOR techniques.

To begin, we note that

\[ N\sigma(\eta) = \left( \frac{2\eta_1, 0, 1}{\sqrt{4\eta_1^2 + 1}} \right) \]

forms a normal frame to \(\sigma\). We compute

\[ I(\eta) = \left( \begin{array}{cc} 4\eta_1^2 + 1 & 0 \\ 1 & 1 \end{array} \right), \quad \Pi(\eta) = \left( \begin{array}{cc} \frac{2}{\sqrt{4\eta_1^2 + 1}} & 0 \\ 0 & 0 \end{array} \right), \]

and therefore

\[ S_\sigma(\eta, \xi) = \left( \begin{array}{cc} \alpha(\eta)^2 + \frac{2\varepsilon}{\alpha(\eta)} & 0 \\ 0 & 1 \end{array} \right) \]

where \(\alpha(\eta) = \sqrt{4\eta_1^2 + 1}\). Note that \(S_\sigma(\eta, \xi)\) is invertible as long as \(\xi \neq -\alpha(\eta)^3/2\).

Thus,

\[ P_\eta = \left\{ \xi \in \mathbb{R} : \xi > -\frac{\alpha(\eta)^3}{2} \right\}. \]

Note that the normal exchange operator is zero, as demonstrated in Proposition 1.

We find the LOR dynamics have the form

\[ \dot{\eta}_1 = \sum_{i=0}^{2} p_i(\eta)\xi^i \quad \text{(36)} \]

\[ \dot{\eta}_2 = \varepsilon \quad \text{(37)} \]

\[ \dot{\xi} = \sum_{i=0}^{2} q_i(\eta)\xi^i \quad \text{(38)} \]

where \(p_i, q_i\) are rational functions of \(\eta_1, \eta_2\). Interestingly, it turns out that \(q_0(\eta) = \mathcal{O}(\varepsilon)\), so that \(\xi = \mathcal{O}(\varepsilon, \|\eta\|)\).

We can solve \(\dot{\xi} = 0\) explicitly to obtain two solution manifolds \(\xi = \Xi_{\pm}(\eta)\). We find that \(\Xi_{+}(\eta) = \mathcal{O}(\eta_2^2)\) near \(\eta_2 = 0\), which corresponds to the fold. Furthermore, we find that \(\Xi_{-}(\eta) = \mathcal{O}(\eta_2, \varepsilon)\) and hence will lie closer to \(\xi = 0\) near the fold. Thus, for \(\eta_2\) sufficiently small, the correction \(\Xi(\eta)\) is given by \(\Xi(\eta) = \Xi_{-}(\eta)\).

In this system, the correction plays a central role in organizing the dynamics near \(\{\xi = 0\}\) and we can use it to easily build a simple trapping region and demonstrate the existence of a perturbed slow manifold. The trajectories of interest in this

\[ \frac{\partial}{\partial t}\left( \begin{array}{c} \eta_1 \\ \eta_2 \\ \xi \end{array} \right) = \left( \begin{array}{c} \sum_{i=0}^{2} p_i(\eta)\xi^i \\ \varepsilon \\ \sum_{i=0}^{2} q_i(\eta)\xi^i \end{array} \right) \]
system begin in \( \{ \eta_1 < 0, \eta_2 < 0 \} \) and evolve towards the fold \( \{ \eta_2 = 0 \} \). We find that
\[
\dot{\eta}|_{\xi=0} = \varepsilon \frac{\mu \eta_1 - 2(\mu + 1) \eta_2}{2 \alpha(\eta)}
\]
and note that if
\[
\eta_2 < 0 \text{ and } \frac{\eta_1}{\eta_2} < \frac{2(\mu + 1)}{\mu}
\]
then \( \dot{\eta}|_{\xi=0} > 0 \); for this reason, we define a first “funnel”
\[
\mathcal{F}_1 = \left\{ \eta \in \mathbb{R}^2 \left| \eta_2 < 0, \frac{\eta_1}{\eta_2} < \frac{2(\mu + 1)}{\mu} \right. \right\}.
\]
If \((\eta_0, \xi_0) \in \mathcal{F}_1 \times \mathbb{R}^+\) then \(\xi(t) > 0\) for as long as \(\eta(t) \in \mathcal{F}_1\). By refining \(\mathcal{F}_1\) we can also bound \(\xi(t)\) from above.

**Proposition 3.** Let
\[
\mathcal{F} = \{ (\eta_1, \eta_2) \in \mathcal{F}_1 | s(\eta) < 0 \}
\]
where \(s(\eta)\) is defined by
\[
s(\eta) = \frac{(-16\eta_2^3 - \mu^2 (2\eta_2 - \eta_1) (\eta_2^2 (-\eta_2 + \eta_1) + \eta_1) - 2 (12\alpha^4 + 1) \eta_2 \mu (-2\eta_2 + \eta_1))}{16\eta_2^3 (4\eta_2^2 + 1)^{3/2}} + \mathcal{O}(\varepsilon).
\]
If \(\eta(t) \in \mathcal{F}\) then \(0 < \xi(t) < \Xi_\varepsilon \circ \eta(t)\) for \(\varepsilon > 0\) sufficiently small. Furthermore, if \(\eta_0 \in \mathcal{F}\) and there exists a \(T > 0\) such that \(\eta(T) \notin \mathcal{F}\) then there exists \(0 < t_F < T\) such that \(\eta_2(t_F) = 0\).

**Proof.** If \(\xi_0 < \Xi_\varepsilon(\eta_0)\) and \(\eta_0 \in \mathcal{F}\) then we need only satisfy \(\dot{\xi}(t) < \left\langle \nabla \Xi_\varepsilon \circ \eta(t), \eta(t) \right\rangle\) for \(\eta(t) \in \mathcal{F}\) and \(\varepsilon > 0\) sufficiently small. An order expansion of \(\dot{\xi}|_{\xi=\Xi_\varepsilon^-}(\eta) - \left\langle \nabla \Xi_\varepsilon^- (\eta), \dot{\eta} \right\rangle|_{\xi=\Xi_\varepsilon^-(\eta)}\) at \(\varepsilon = 0\) yields the expression on the right hand side of equation (39), which we require to be negative, yielding the region \(\mathcal{F}\). We find that \(\Xi^- \circ \eta(t)\) is a supersolution of \(\dot{\xi}\) exactly when \(\eta(t) \in \mathcal{F}\). Hence the first part of the result follows.

The second claim follows from similar subsolution arguments. For \(\varepsilon > 0\) sufficiently small, the boundary of \(\mathcal{F}_1\), given by \(\eta_1 + 8\eta_1 \eta_2^2 - 8\eta_2^3 = 0\), is impassible when \(\eta_2 < 0\). Thus any trajectories that escape \(\mathcal{F} \times \mathbb{R}^+\) must cross the fold curve \(\{ \eta_2 = 0 \}\).

Two features of this result bear mentioning. First, we have demonstrated that any trajectories that escape from a neighborhood of the stable critical manifold must do so through the fold. Second, recalling that \(\Xi^- = \mathcal{O}(\varepsilon)\), we have verified a weak version Fenichel’s Theorem for system (35). The organizational role of the correction is demonstrated in the left half of Fig. 2.

By performing an additional transformation of our system, we can extract canard-type solutions, which pass across the fold \(\{ \eta_2 = 0 \}\), in a very natural way. We have demonstrated that the correction manifold plays a strong role in organizing the flow near the critical manifold, so it is natural to use the correction as a base manifold for another LOR transformation. That is, we define
\[
\dot{\sigma}(\dot{\eta}) = \sigma(\dot{\eta}) + \Xi_\varepsilon^- (\dot{\eta}) N \sigma(\dot{\eta}),
\]
which is a manifold embedded in our original space that satisfies all of our LOR assumptions, such that we can compute the LOR dynamics relative to \( \hat{\sigma} \). We suppress the computations for brevity; however, they follow exactly along the lines of the previous computations, and can be executed extremely quickly by symbolic computation software (e.g., Mathematica, Maple, Julia; see https://github.com/LORlab2020/Mathematica for Mathematica code that can be used for LOR). Note that in this transformation, \( \hat{\eta} \) is geometrically equivalent to \( \eta \); that is, \( \hat{\eta}_1 \) still corresponds to \( x \) in the original system, just as \( \hat{\eta}_2 \) corresponds to \( y \), but we need the \( \hat{\eta} \) notation to represent the change from \( \xi \) to \( \hat{\xi} \).

The lowest order terms of the resultant system are given by

\[
\begin{align*}
\dot{\hat{\eta}}_1 &= \varepsilon + \mathcal{O}(\varepsilon^2, \xi) \\
\dot{\hat{\eta}}_2 &= \frac{\varepsilon}{4} \left( 2 + \left( 2 - \frac{\hat{\eta}_1}{\hat{\eta}_2} \right) \mu \right) + \mathcal{O}(\varepsilon^2, \xi) \\
\dot{\hat{\xi}} &= 2\hat{\eta}_2 \hat{\xi} + \mathcal{O}(\varepsilon^2, \varepsilon^2).
\end{align*}
\]  

Note that \( \xi = 0 \) is invariant to \( \mathcal{O}(\varepsilon^2) \), which is an improvement to the previous system; trajectories starting on \( \xi = 0 \) must spend an \( \mathcal{O}(\varepsilon^{-2}) \) amount of time near \( \xi = 0 \) before escaping.

We remain interested in trajectories beginning in \( \{ \hat{\eta}_2 < 0 \} \) that cross the fold curve. If we numerically plot the \( \hat{\eta} \) dynamics on \( \{ \hat{\xi} = 0 \} \), we notice that there is a single trajectory that plays an outsized role in organizing the \( \hat{\eta} \) phase plane.

It is easily verified that \( \gamma(t) = (\varepsilon t, \varepsilon \mu t / 2, 0) \) solves (40) to \( \mathcal{O}(\varepsilon^2) \), and this is, in fact, the “central” trajectory of the approximate dynamics; see Fig. 3. We also note that \( \gamma \) is the only trajectory that remains well-defined as it crosses the fold, through a L’Hopital type limit. Finally, we note that \( \gamma(t) \) is constrained to lie in a
plane, as its trace in $\hat{\eta}$ is a line; stated equivalently, $\gamma(t)$ has everywhere vanishing curvature.

In earlier work [9], we suggest that rivers in planar systems, which are centrally organizing orbits with no apparent source, are trajectories for which curvature and torsion vanish simultaneously. Previous work has established the zero curvature set plays an organizational role in planar systems [1, 4, 5], we use the LOR frame to study where the zero curvature set is invariant. One advantage of this definition, is that it generalizes easily to arbitrary dimension. In $\mathbb{R}^n$ we define a river to be a Frenet trajectory $\Gamma(t)$ whose Frenet curvature $\kappa_{n-2}(t)$ vanishes to second order; that is, if there exists $t^*$ such that

$$\kappa_{n-2}(t^*) = \kappa_{n-2}(t^*) = 0$$

then $\Gamma(t)$ is a river, and we call $\Gamma(t^*)$ a confluence [9].

Our analysis suggests that $\gamma(t)$ is an organizing trajectory that lies in a plane and thus has identically zero Frenet curvature. This result suggests a strong connection between canards and rivers. In fact, if we study the zero curvature locus of the original system (35), we find that there are seven trajectories that have identically zero curvature and hence are rivers by our definition. The most interesting of these is given by

$$\Gamma(z) = \left( -z^2 + \frac{\varepsilon \mu}{2}, \frac{2z}{\mu}, z \right)$$

and is invariant, such that it can be reparameterized as a solution. Furthermore, $\Gamma(z)$ lies $O(\varepsilon)$ close to our approximate canard $\Psi_2(\gamma(t), 0)$. For $z < 0$, $\Gamma$ lies near the stable branch of the critical manifold, at $z = 0$ it passes near the fold, and for $z > 0$ it remains $O(\varepsilon)$ close to the unstable critical manifold; see Fig. 3.

We intend to formalize the connection between rivers and canards in future work. Additionally, it would be interesting to investigate how the LOR frame, specialized to the context of fast-slow systems, relates to other geometrically-motivated
manifold approximation schemes that apply in that setting, such as computational singular perturbation (CSP) methods [8, 18] or the zero derivative principle [1], and whether LOR provides a way to extend ideas like CSP beyond the fast-slow context. For the time being, we hope that the computations and sketch of ideas in this section serve as an amuse-bouche to demonstrate how the LOR frame can highlight nontrivial features of phase space.

7. Conclusions. LOR provides a general approach to deriving a natural coordinate frame, based on a geometric representation, that is well suited to study dynamics relative to any manifold embedded in the flow of an ODE of arbitrary dimensions. A variety of methods have been previously developed to describe the flow in the vicinity of a periodic orbit or other attractor [11, 2, 10, 15, 16, 12, 17]. LOR is advantageous because, to our knowledge, it is the unique approach with a full range of desirable properties: it is not based on linearization, it is not limited to scenarios involving periodic orbits, it generally extends well beyond the local neighborhood of the base manifold used to define it, and it applies naturally in arbitrary dimensions. LOR leads to LOR equations describing the evolution of trajectories in the LOR frame, which encode the geometry of the flow. These equations do not depend on the choice of normal frame used in their derivation, which allows a specific frame to be chosen to simplify associated calculations. An additional blow-up transformation of the LOR equations provides an especially convenient decomposition for studying transient aspects of the behavior of trajectories near structures in a phase space. Alternatively, we have seen that LOR can be used to identify canard behavior that organizes flow along a slow manifold. Given the derivation and illustration of these ideas and their properties in this work, we hope that LOR will serve as a useful tool for future studies of dynamical systems.

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