Lagrangian–Hamiltonian unified formalism for autonomous higher order dynamical systems

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Abstract

The Lagrangian–Hamiltonian unified formalism of Skinner and Rusk was originally stated for autonomous dynamical systems in classical mechanics. It has been generalized for non-autonomous first-order mechanical systems, as well as for first-order and higher order field theories. However, a complete generalization to higher order mechanical systems is yet to be described. In this work, after reviewing the natural geometrical setting and the Lagrangian and Hamiltonian formalisms for higher order autonomous mechanical systems, we develop a complete generalization of the Lagrangian–Hamiltonian unified formalism for these kinds of systems, and we use it to analyze some physical models from this new point of view.

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1. Introduction

In recent decades, a strong development in the intrinsic study of a wide variety of topics in theoretical physics, control theory and applied mathematics has been done, using methods of differential geometry. Thus, the intrinsic formulation of Lagrangian and Hamiltonian formalisms has been developed both for autonomous and non-autonomous systems. This study has been carried out mainly for first-order dynamical systems, that is, those whose Lagrangian or Hamiltonian functions depend on the generalized coordinates of position and velocity (or momentum). From the geometric point of view, this means that the phase space of the system is in most cases the tangent or cotangent bundle of the smooth manifold representing the configuration space.

However, there are a significant number of relevant systems in which the dynamics have explicit dependence on accelerations or higher order derivatives of the generalized coordinates of position. These systems, usually called higher order dynamical systems, can be modeled geometrically using higher order tangent bundles [17]. These models are typical
of theoretical physics, for example those describing the interaction of relativistic particles with spin, string theories from Polyakov and others, Hilbert’s Lagrangian for gravitation or Podolsky’s generalization of electromagnetism (see [8] and references cited there). They also appear in a natural way in numerical models arising from the discretization of first-order dynamical systems that preserve their inherent geometric structures [16]. There are a lot of works devoted to the development of the formalism of these kinds of theories and their application to many models in mechanics and field theory (see, for instance, [2–4, 7, 9, 12, 20, 25, 26, 35, 37, 38]).

Furthermore, a generalization of the Lagrangian and Hamiltonian formalisms exists that compresses them into a single formalism. This is the so-called Lagrangian–Hamiltonian unified formalism, or Skinner–Rusk formalism due to the authors’ names of the original paper. It was originally developed for first-order autonomous mechanical systems [39], and later generalized to non-autonomous dynamical systems [6, 14], control systems [5], first-order classical field theories [15, 18] and, more recently, to higher order classical field theories [10, 41]. Nevertheless, although the geometrization of both higher order Lagrangian and Hamiltonian formalisms was already developed for autonomous mechanical systems [11, 17, 23, 29], a complete generalization of the Skinner–Rusk formalism for higher order mechanical systems is yet to be developed. A first attempt was outlined in [13], with the aim of providing a geometric model for studying optimal control of underactuated systems, although a deep analysis of the model and its relation with the standard Lagrangian and Hamiltonian formalisms was not performed.

Thus, the aim of this work is to provide a detailed and complete description of the Lagrangian–Hamiltonian unified formalism for higher order autonomous mechanical systems. Our approach is different from that given in [13] (these differences are commented in section 5).

The paper is organized as follows.

Section 2 consists of a review of the basic definitions and the geometric structures of higher order tangent bundles, some of which are generalizations of the geometric structures of tangent bundles, namely the canonical vector fields, the almost-tangent structures and semisprays, whereas others such as the Tulczyjew derivation are needed for developing the Lagrangian and Hamiltonian formalisms of higher order mechanical systems, which are also described in this section. In particular, higher order regular and singular systems are distinguished.

The main contribution of the work is found in section 3, where the geometric formulation of the Lagrangian–Hamiltonian unified formalism for higher order autonomous mechanical systems is described in detail, including the study of how the Lagrangian and Hamiltonian formalisms are recovered from that formalism.

Finally, in section 4, two examples are analyzed in order to show the application of the formalism: the first is a regular system, the so-called Pais–Uhlenbeck oscillator, while the second is a singular one, the second-order relativistic particle.

The paper concludes in section 5 with a summary of results, discussion and future research.

All the manifolds, the maps and the structures are smooth. In addition, all the dynamical systems considered are autonomous. Summation over crossed repeated indices is understood, although on some occasions the symbol of summation is written explicitly in order to avoid confusion.

2. Higher order dynamical systems

2.1. Geometric structures of higher order tangent bundles

(See [17, 36, 23, 24, 27] for details).
2.1.1 Higher order tangent bundles. Let $Q$ be a $n$-dimensional differentiable manifold, and $k \in \mathbb{N}$. The $k$th-order tangent bundle of $Q$, denoted by $T^kQ$, is the $(k+1)$-dimensional manifold made of the $k$-jets with source at $0 \in \mathbb{R}$ and target $Q$, that is, $T^kQ = J^k_0(\mathbb{R}, Q)$. It is a submanifold of $J^k(\mathbb{R}, Q)$.

We have the following canonical projections: if $r \leq k$,
\[
\rho^k_r : T^kQ \to T^rQ, \quad \beta^k : T^kQ \to Q
\]
where $\hat{\sigma}^k(0)$ denotes a point in $T^kQ$, that is, the equivalence class of a curve $\sigma : I \subset \mathbb{R} \to Q$ by the $k$-jet equivalence relation. Note that $\rho^k_0 = \beta^k$, where $T^0Q$ is canonically identified with $Q$, and $\rho^k_k = 1_{T^kQ}$.

If $(U, \varphi)$ is a local chart in $Q$, with $\varphi = (\varphi^A)$, $1 \leq A \leq n$, and $\sigma : \mathbb{R} \to Q$ is a curve in $Q$ such that $\sigma(0) \in U$, by writing $\sigma^A = \varphi^A \circ \sigma$, the $k$-jet $\hat{\sigma}^k(0)$ is given in $(\beta^k)^{-1}(U) = T^kU$ by $(q^1, \ldots, q^k)$, where $q^i = \sigma^A(0)$ and $q^i = \frac{\partial \sigma^A}{\partial t^i}(0)$ $(1 \leq i \leq k)$. Usually, we write $q^0_i$ instead of $q^i$, and so we have the local chart $(\beta^k)^{-1}(U)$ in $T^kQ$ with local coordinates $(q^0_0, q^1_1, \ldots, q^k_k)$.

Local coordinates in $T(T^kQ)$ are denoted by $(q^0_0, q^1_1, \ldots, q^k_k, v^1_1, \ldots, v^k_k)$.

Using these coordinates, the local expression of the canonical projections is $\rho^k_r (q^0_0, q^1_1, \ldots, q^k_k) = (q^0_0, q^1_1, \ldots, q^r_r)$, and then for the tangent maps $T\rho^k_r : T(T^kQ) \to T(T^rQ)$ we have the local expression $T\rho^k_r (q^0_0, q^1_1, \ldots, q^k_k, v^1_1, \ldots, v^k_k) = (q^0_0, q^1_1, \ldots, q^r_r, v^1_1, \ldots, v^k_k)$.

If $\sigma : \mathbb{R} \to Q$ is a curve in $Q$, the canonical lifting of $\sigma$ to $T^kQ$ is the curve $\hat{\sigma}^k : \mathbb{R} \to T^kQ$ defined as $\hat{\sigma}^k(t) = \hat{\sigma}^k(0)$, where $\sigma^A(t) = \sigma^A(s + t)$, (that is, the $k$-jet lifting of $\sigma$). If $k = 1$, we will write $\hat{\sigma}^1 \equiv \hat{\sigma}$.

Let $V(\rho^k_{k-1})$ be the vertical sub-bundle of $T^kQ$ in $T^{r-1}Q$. In the above coordinates, for every $p \in T^kQ$ and $u \in V_p(\rho^k_{k-1})$, we have that its components are $u = (0, \ldots, 0, v^1_1, \ldots, v^k_k)$. Furthermore, if $i_{k-r+1} : V(\rho^k_{k-1}) \to T(T^kQ)$ is the canonical embedding, then
\[
i_{k-r+1} (q^0_0, q^1_1, \ldots, q^k_k, v^1_1, \ldots, v^k_k) = (q^0_0, q^1_1, \ldots, q^k_k, 0, \ldots, 0, v^1_1, \ldots, v^k_k).
\]

Now consider the induced bundle of $\tau_{T^rQ} : T(T^{r-1}Q) \to T^{r-1}Q$ by the canonical projection $\rho^k_{k-1}$, denoted by $T^kQ \times_{T^rQ} T(T^{r-1}Q)$, which is a vector bundle over $T^kQ$. We have the following commutative diagrams:

\[
\begin{array}{ccc}
T^kQ & \xrightarrow{T^kQ} & T(T^{r-1}Q) \\
\downarrow{\rho^k_{k-1}} & & \downarrow{\tau_{T^{r-1}Q}} \\
T^kQ & \xrightarrow{\rho^k_{k-1}} & T^{r-1}Q \\
\end{array}
\quad
\begin{array}{ccc}
T^kQ & \xrightarrow{\rho^k_{k-1}} & T^{r-1}Q \\
\downarrow{\rho^k_{k-1}} & & \downarrow{\tau_{T^{r-1}Q}} \\
T^kQ & \xrightarrow{\rho^k_{k-1}} & T^{r-1}Q \\
\end{array}
\]

Then, there exists a unique bundle morphism $s_{k-r+1} : T(T^kQ) \to T^kQ \times_{T^rQ} T(T^{r-1}Q)$ such that the following diagram is commutative:
It is defined by $s_{k-r+1}(u) = (\tau_{T^kQ}(u), T\rho^r_{r-1}(u))$, for every $u \in T(T^kQ)$. Its local expression is

$$s_{k-r+1}(q^0_0, q^1_k, v^0_0, \ldots, v^d_k) = (q^0_0, q^1_k, v^0_0, \ldots, v^d_k, 0, \ldots, 0).$$

As $s_{k-r+1}$ is a surjective map and $\text{Im} (i_{k-r+1}) = \ker (s_{k-r+1})$, we have the exact sequence

$$0 \rightarrow V(\rho^k_{k-1}) \rightarrow T(T^kQ) \rightarrow T^kQ \times_{T^rQ} T(T^{r-1}Q) \rightarrow 0,$$

which is called the $(k - r + 1)$-fundamental exact sequence. In local coordinates, it is given by

$$0 \rightarrow (q^0_0, q^1_k, v^0_0, \ldots, v^d_k) \mapsto (q^0_0, q^1_k, 0, \ldots, 0, v^0_0, \ldots, v^d_k)$$

$$\mapsto (q^0_0, q^1_k, v^0_0, \ldots, v^d_k) \mapsto (q^0_0, q^1_k, q^r_0, q^1_k, v^0_0, \ldots, v^d_k) \mapsto 0.$$

Thus, we have $k$ exact sequences

$$\text{1st: } 0 \rightarrow V(\rho^k_{k-1}) \rightarrow T(T^kQ) \rightarrow T^kQ \times_{T^rQ} T(T^{r-1}Q) \rightarrow 0$$

$$\vdots$$

$$\text{rth: } 0 \rightarrow V(\rho^k_{k-r}) \rightarrow T(T^kQ) \rightarrow T^kQ \times_{T^rQ} T(T^{r-1}Q) \rightarrow 0$$

$$\vdots$$

$$\text{kth: } 0 \rightarrow V(\beta^k) \rightarrow T(T^kQ) \rightarrow T^kQ \times_{T^rQ} T^rQ \rightarrow 0,$$

where $V(\beta^k) \equiv V(\rho^k_0)$ denotes the vertical subbundle of $T^kQ$ on $Q$. These sequences can be connected by means of the connecting maps

$$h_{k-r+1} : T^kQ \times_{T^rQ} T(T^{k-r}Q) \rightarrow V(\rho^k_{r-1})$$

locally defined as

$$h_{k-r+1}(q^0_0, q^1_k, v^0_0, \ldots, v^d_k)$$

$$= (q^0_0, q^1_k, 0, \ldots, 0, \frac{r!}{0!} v^0_0, \frac{(r + 1)!}{1!} v^1_0, \ldots, \frac{k!}{(k-r)!} v^{r-1}_k).$$

These maps are globally well defined and are vector bundle isomorphisms over $T^kQ$. Then, we have the following connection between exact sequences:

$$0 \rightarrow V(\rho^k_{k-1}) \rightarrow T(T^kQ) \rightarrow T^kQ \times_{T^rQ} T(T^{r-1}Q) \rightarrow 0$$

$$\rightarrow h_{k-r+1}$$

$$0 \rightarrow V(\rho^k_{k-r}) \rightarrow T(T^kQ) \rightarrow T^kQ \times_{T^rQ} T(T^{r-1}Q) \rightarrow 0.$$
2.1.2. Higher order canonical vector fields: vertical endomorphisms and almost-tangent structures. The canonical injection is the map

\[ j_r : \mathbb{T}^k Q \longrightarrow \mathbb{T}(\mathbb{T}^{-1} Q) \]

where

\[ \gamma : \mathbb{R} \longrightarrow \mathbb{T}^{-1} Q \] \[ t \longmapsto \gamma(t) = \tilde{\sigma}_t^{-1}(0). \]

In local coordinates

\[ j_r (q_0^i, \ldots, q_k^i) = (q_0^i, \ldots, q_{r-1}^i; q_r^i, q_1^i, \ldots, q_k^i). \]

Then, the following composition allows us to define a vector field \( \Delta_r \in \mathfrak{X}(\mathbb{T}^k Q) \),

\[
\begin{array}{c}
\mathbb{T}^k Q \xrightarrow{\text{Id} \times j_{k-r+1}} \mathbb{T}^k Q \times_{\mathbb{T}^{k-r} Q} \mathbb{T}(\mathbb{T}^{k-r} Q) \xrightarrow{h_{k-r+1}} V(\rho_{k-r+1}^r) \xrightarrow{i_{k-r+1}} \mathbb{T}(\mathbb{T}^k Q) \quad \Delta_r
\end{array}
\]

that is, \( \Delta_r = i_{k-r+1} \circ h_{k-r+1} \circ (\text{Id} \times j_{k-r+1}) \). From the local expressions of \( i_{k-r+1}, h_{k-r+1} \) and \( j_{k-r+1} \) we obtain that \( \Delta_r (q_0^i, \ldots, q_k^i) = (q_0^i, \ldots, q_{r-1}^i; q_r^i, 0, \ldots, 0, r! q_1^i, (r + 1)! q_2^i, \ldots, (k-r)! q_{k-r+1}^i) \), or what is equivalent,

\[
\Delta_r = \sum_{i=0}^{k-r} \frac{(r+i)!}{i!} q_{i+1}^i \frac{\partial}{\partial q_{r+i}^i} = r! q_1^i \frac{\partial}{\partial q_r^i} + (r + 1)! q_2^i \frac{\partial}{\partial q_{r+1}^i} + \cdots + \frac{k!}{(k-r)!} q_{k-r+1}^i \frac{\partial}{\partial q_k^i}.
\]

In particular,

\[
\Delta_1 = \sum_{i=1}^{k} q_i^i \frac{\partial}{\partial q_i^i} = \sum_{i=0}^{k-1} (i+1) q_{i+1}^i \frac{\partial}{\partial q_{r+i}^i} = q_1^i \frac{\partial}{\partial q_1^i} + 2q_2^i \frac{\partial}{\partial q_2^i} + \cdots + k q_k^i \frac{\partial}{\partial q_k^i}.
\]

**Definition 1.** The vector field \( \Delta_r \) is the \( r \)th-canonical vector field in \( \mathbb{T}^k Q \). In particular, \( \Delta_1 \) is called the Liouville vector field in \( \mathbb{T}^k Q \).

Remember that, if \( N \) is a \((k+1)n\)-dimensional manifold, an almost-tangent structure of order \( k \) in \( N \) is an endomorphism \( J \) in \( TN \) such that \( J^{k+1} = 0 \) and \( \text{rank } J = kn \). Then, \( \mathbb{T}^k Q \) is endowed with a canonical almost-tangent structure. In fact:

**Definition 2.** For \( 1 \leq r \leq k \), let \( i_{k-r+1}, h_{k-r+1}, s_r \) be the morphisms of the fundamental exact sequences introduced above. The map

\[ J_r : i_{k-r+1} \circ h_{k-r+1} \circ s_r : \mathbb{T}(\mathbb{T}^k Q) \longrightarrow \mathbb{T}(\mathbb{T}^k Q) \]

defined by the composition

\[
\begin{array}{c}
\mathbb{T}(\mathbb{T}^k Q) \xrightarrow{s_r} \mathbb{T}^k Q \times_{\mathbb{T}^{k-r} Q} \mathbb{T}(\mathbb{T}^{k-r} Q) \xrightarrow{h_{k-r+1}} V(\rho_{k-r+1}^r) \xrightarrow{i_{k-r+1}} \mathbb{T}(\mathbb{T}^k Q) \quad J_r
\end{array}
\]

is called the \( r \)th-vertical endomorphism of \( \mathbb{T}(\mathbb{T}^k Q) \).
From the local expressions of $s_r$, $h_{k-r+1}$, $i_{k-r+1}$ we obtain that
\[ J_r(q^A_0, \ldots, q^A_k, v^A_0, \ldots, v^A_k) = \left( q^A_0, \ldots, q^A_k, 0, \ldots, 0, r!v^A_0, (r+1)!v^A_1, \ldots, \frac{k!}{(k-r)!}v^A_{k-r} \right), \]
that is, $J_r = \sum_{i=0}^{k-r} \frac{r+i}{i!} dq^A_i \otimes \frac{\partial}{\partial v^A_i}$. In particular, $J_1 = \sum_{i=0}^{k-1} (i+1) dq^A_i \otimes \frac{\partial}{\partial v^A_i}$.

The $r$th-vertical endomorphism $J_r$ has constant rank $(k-r+1)n$ and satisfies that
\[
(J_r)^i = \begin{cases} 0 & \text{if } rs \geq k + 1 \\ J_{rs} & \text{if } rs < k + 1 \end{cases}.
\]
As a consequence, the 1st-vertical endomorphism $J_1$ defines an almost-tangent structure of order $k$ in $T^k Q$, which is called the canonical almost-tangent structure of $T^k Q$. Then, any other vertical endomorphism $J_r$ is obtained by composing $J_1$ with itself $r$ times. Furthermore, we have the following relation:
\[
J_r \circ \Delta_r = \begin{cases} 0 & \text{if } r + s \geq k + 1 \\ \Delta_{r+s} & \text{if } r + s < k + 1 \end{cases}.
\]
As a consequence, starting from the Liouville vector field and the vertical endomorphisms, we can recover all the canonical vector fields. However, as all the vertical endomorphisms are obtained from $J_1$, we conclude that all the canonical structures in $T^k Q$ are obtained from the Liouville vector field and the canonical almost-tangent structure.

Consider now the dual maps $J_r^\ast$ of $J_r$, $1 \leq r \leq k$, that is, the endomorphisms in $T^r(T^k Q)$, and their natural extensions to the exterior algebra $\wedge(T^k Q)$ (also denoted by $J_r^\ast$). Their action on the set of differential forms is given by
\[
J_r^\ast \omega(X_1, \ldots, X_p) = \omega(J_r(X_1), \ldots, J_r(X_p)),
\]
for $\omega \in \Omega^p(T^k Q)$ and $X_1, \ldots, X_p \in \mathfrak{T}(T^k Q)$, and for every $f \in C^\infty(T^k Q)$ we write $J_r^\ast(f) = f$.

The endomorphism $J_r^\ast : \Omega(T^k Q) \to \Omega(T^k Q), 1 \leq r \leq k$, is called the vertical operator, and it is locally given by
\[
J_r^\ast(f) = f, \quad \text{for every } f \in C^\infty(T^k Q)
\]
\[
J_r^\ast(dq^A_i) = \begin{cases} 0, & \text{if } i < r \\ \frac{i!}{(i-r)!} dq^A_{i-r}, & \text{if } i \geq r \end{cases}.
\]

2.1.3. Vertical derivations and differentials: Tulczyjew’s derivation. The inner contraction of the vertical endomorphisms $J_r$ with any differential $p$-form $\omega \in \Omega^p(T^k Q)$ is the $p$-form $i(J_r)\omega$ defined as follows: for every $X_1, \ldots, X_p \in \mathfrak{T}(T^k Q)$
\[
i(J_r)\omega(X_1, \ldots, X_p) = \sum_{i=1}^{p} \omega(X_1, \ldots, J_r(X_i), \ldots, X_p),
\]
and taking $i(J_r)f = 0$, for every $f \in C^\infty(T^k Q)$, we can state

**Definition 3.** The map
\[
\Omega(T^k Q) \to \Omega(T^k Q)
\]
\[
i(J_r)\omega
\]
is a derivation of degree 0 in $\Omega(T^k Q)$, which is called the $r$th-vertical derivation in $\Omega(T^k Q)$.

Its local expression is
\[
i(J_r)(dq^A_i) = \begin{cases} 0, & \text{if } i < r \\ \frac{i!}{(i-r)!} dq^A_{i-r}, & \text{if } i \geq r \end{cases}.
\]
**Definition 4.** The operator \( d_{J_r} = [i(J_r), d] \) is a skew-derivation of degree 1, which is called the \( r \)th-vertical differential.

Its local expression is given by

\[
d_{J_r}(f) = \frac{\partial f}{\partial q_i} dq_i - d_{J_r}(dq_i), \quad \text{for every } f \in C^\infty(T^kQ).
\]

\( d_{J_r}(dq_i) = 0 \)

For \( 1 \leq r, s \leq k \), we have that \( d_{J_r} d = -dd_{J_s} \).

In the set \( \bigoplus_{r \geq 0} \Omega(T^kQ) \), we can define the following equivalence relation: for \( \omega \in \Omega(T^kQ) \) and \( \lambda \in \Omega(T^kQ) \),

\[
\omega \sim \lambda \iff \begin{cases} 
(\rho^k_{r+1})^*(\lambda), & \text{if } k' < k \\
(\rho^k_r)^*(\omega), & \text{if } k' \geq k.
\end{cases}
\]

Then, we consider the quotient set \( \Omega = \bigoplus_{k \geq 0} \Omega(T^kQ)/\sim \), which is a commutative graded algebra. In this set we can define the so-called Tulczyjew’s derivation [40, 17], denoted by \( d_T \), as follows: for every \( f \in \mathcal{C}^\infty(T^kQ) \) we construct the function \( d_T f \in C^\infty(T^{k+1}Q) \) given by

\[
(d_T f)(\tilde{\sigma}^{k+1}(0)) = (d_{J_r})((\rho^k_{r+1})(\tilde{\sigma}^{k+1}(0))).
\]

where \( \rho^k_{r+1} : T^{k+1}Q \to T(T^kQ) \) is the canonical injection introduced in section 2.1.2. From the coordinate expression for \( \rho^k_{r+1} \), we obtain that

\[
d_T f(q_0^1, \ldots, q_{k+1}^1) = \sum_{i=0}^{k} q_{i+1}^1 \frac{\partial f}{\partial q_i}(q_0^1, \ldots, q_{k}^1).
\]

The map \( d_T \) extends to a derivation of degree 0 in \( \Omega \) and, as \( d_T d = dd_T \), it is determined by its action on functions and by the property \( d_T(dq_i^r) = dq_i^{r+1} \).

Furthermore, the maps \( i(J_r), d_{J_r}, i(\Delta_r) \) and \( L(\Delta_r) \) extend to \( \Omega \) in a natural way.

### 2.1.4. Higher order semisprays.

**Definition 5.** A vector field \( X \in \mathfrak{X}(T^kQ) \) is a semispray of type \( r \), \( 1 \leq r \leq k \), if for every integral curve \( \sigma \) of \( X \), we have that, if \( \gamma = \beta^k \circ \sigma \), then \( \tilde{\gamma}^{k-r+1} = \rho^k_{k-r+1} \circ \sigma \) (where \( \tilde{\gamma}^{k-r+1} \) is the canonical lifting of \( \gamma \) to \( T^{k-r+1}Q \)).

![Diagram of semispray](image)

**In particular,** \( X \in \mathfrak{X}(T^kQ) \) is a semispray of type 1 if for every integral curve \( \sigma \) of \( X \), we have that \( \gamma = \beta^k \circ \sigma \), then \( \tilde{\gamma}^k = \sigma \).
The local expression of a semispray of type \( r \) is
\[
X = q_1 \frac{\partial}{\partial q_0^1} + q_2 \frac{\partial}{\partial q_0^2} + \cdots + q_{k + 1} \frac{\partial}{\partial q_0^{k + 1}} + X_{k + 1} \frac{\partial}{\partial q_0^{k + 1}} + \cdots + X_k \frac{\partial}{\partial q_0^k},
\]

Proposition 1. The following assertions are equivalent.

1. A vector field \( X \in \mathcal{X}(T^kQ) \) is a semispray of type \( r \).
2. \( T\rho_{k-r} \circ X = j_{k-r+1} \), that is, the following diagram commutes:

\[
\begin{array}{ccc}
T(T^kQ) & \xrightarrow{T\rho_{k-r}} & T(T^{k-r}Q) \\
\downarrow X & & \downarrow j_{k-r+1} \\
T^kQ & \xrightarrow{j_{k-r+1}} & T(T^{k-r}Q)
\end{array}
\]

3. \( J_r(X) = \Delta_r \).

Obviously, every semispray of type \( r \) is a semispray of type \( s \), for \( s \geq r \).

If \( X \in \mathcal{X}(T^rQ) \) is a semispray of type \( r \), a curve \( \sigma \) in \( Q \) is said to be a path or solution of \( X \) if \( \tilde{\sigma}^k \) is an integral curve of \( X \), that is, \( \tilde{\sigma}^k = X \circ \tilde{\sigma}^k \), where \( \tilde{\sigma}^k \) denotes the canonical lifting of \( \sigma \) from \( T^kQ \) to \( T(T^kQ) \). Then, in coordinates, \( \sigma \) verifies the following system of differential equations of order \( k + 1 \):

\[
\frac{d^{k+r+2} \sigma^A}{dr^{k+r+2}} = X_{k+r+1}^A \left( \sigma, \frac{d\sigma}{dr}, \ldots, \frac{d^k\sigma}{dr^k} \right)
\]

\[
\vdots
\]

\[
\frac{d^{k+1} \sigma^A}{dr^{k+1}} = X_{k}^A \left( \sigma, \frac{d\sigma}{dr}, \ldots, \frac{d^k\sigma}{dr^k} \right).
\]

Observe that, taking \( k = 1 \), then \( r = 1 \) and \( \rho_{1-1+1} = \text{Id}_{T^1Q} \), we recover the definition of the holonomic vector field (SODE in \( TQ \)). So, semisprays of type 1 in \( T^kQ \) are the analogue to the holonomic vector fields in \( TQ \); that is, they are the vector fields whose integral curves are the canonical liftings to \( T^kQ \) of curves on the basis \( Q \). Their local expressions are

\[
X = q_1 \frac{\partial}{\partial q_0^1} + q_2 \frac{\partial}{\partial q_0^2} + \cdots + q_{k} \frac{\partial}{\partial q_0^{k}} + X_{k+1} \frac{\partial}{\partial q_0^{k+1}} + \cdots + X_{k} \frac{\partial}{\partial q_0^{k}}.
\]

2.2. Lagrangian formalism

Let \( Q \) be a \( n \)-dimensional differentiable manifold and \( \mathcal{L} \in C^\infty(T^kQ) \). We say that \( \mathcal{L} \) is a Lagrangian function of order \( k \).

Definition 6. The Lagrangian 1-form \( \theta_{\mathcal{L}} \in \Omega^1(T^{2k-1}Q) \) associated with \( \mathcal{L} \) is defined as

\[
\theta_{\mathcal{L}} = \sum_{r=1}^{k} (-1)^{r+1} \frac{1}{r!} d^{r-1}_{T} d_{j_r} \mathcal{L}.
\]

Then, the Lagrangian 2-form, \( \omega_{\mathcal{L}} \in \Omega^2(T^{2k-1}Q) \), associated with \( \mathcal{L} \) is

\[
\omega_{\mathcal{L}} = -d \theta_{\mathcal{L}} = \sum_{r=1}^{k} (-1)^{r+1} \frac{1}{r!} d^{r-1}_{T} d_{j_r} \mathcal{L}.
\]
Observe that the Lagrangian 1-form is a semibasic form of type $k$ in $T^{2k-1}Q$.

We assume that $\omega_L$ has constant rank (we refer to this fact by saying that $L$ is a geometrically admissible Lagrangian).

**Definition 7.** The Lagrangian energy, $E_L \in C^\infty(T^{2k-1}Q)$, associated with $L$ is defined as

$$E_L = \left( \sum_{r=1}^{k} (-1)^{r-1} \frac{1}{r!} \Delta_r(L) \right) - (\rho^{2k-1})^* L.$$

It is usual to write $L$ instead of $(\rho^{2k-1})^* L$, and we will do this in the following.

The coordinate expressions of these elements are

$$\theta_L = \sum_{r=1}^{k} \sum_{i=0}^{k-r} (-1)^i d_i \left( \frac{\partial L}{\partial q^{A}_{r+i}} \right) dq^{A}_{r-i},$$

$$\omega_L = \sum_{r=1}^{k} \sum_{i=0}^{k-r} (-1)^i d_i \left( \frac{\partial L}{\partial q^{A}_{r+i}} \right) \wedge dq^{A}_{r-i},$$

$$E_L = \sum_{r=1}^{k} \sum_{i=0}^{k-r} (-1)^i d_i \left( \frac{\partial L}{\partial dq^{A}_{r+i}} \right) - L.$$

**Definition 8.** A Lagrangian function $L \in C^\infty(T^kQ)$ is said to be regular if $\omega_L$ is a symplectic form. Otherwise $L$ is a singular Lagrangian.

To say that $L$ is a regular Lagrangian is locally equivalent to saying that the Hessian matrix $\left( \frac{\partial^2 L}{\partial q^B \partial q^A} \right)$ is regular at every point of $T^kQ$.

**Definition 9.** A Lagrangian system of order $k$ is a couple $(T^{2k-1}Q, L)$, where $Q$ represents the configuration space and $L \in C^\infty(T^kQ)$ is the Lagrangian function. It is said to be a regular (respectively singular) Lagrangian system if the Lagrangian function $L$ is regular (respectively singular).

Thus, in the Lagrangian formalism, $T^{2k-1}Q$ represents the phase space of the system. The dynamical trajectories of the system are the integral curves of any vector field $X_L \in \mathfrak{X}(T^{2k-1}Q)$ satisfying that:

1. it is a solution to the equation
   $$i(X_L)\omega_L = dE_L;$$
   (5)
2. it is a semispray of type 1 in $T^{2k-1}Q$.

Equation (5) is the higher order Lagrangian equation, and a vector field $X_L$ solution to (5) (if it exists) is called a Lagrangian vector field of order $k$. If, in addition, $X_L$ satisfies condition 2, then it is called an Euler–Lagrange vector field of order $k$, and its integral curves on the base are solutions to the higher order Euler–Lagrange equations.

In natural coordinates of $T^{2k-1}Q$, if

$$X_L = \sum_{i=0}^{2k-1} f_i^{A} \frac{\partial}{\partial q^{A}} = f_0^{A} \frac{\partial}{\partial q^{A}_0} + f_1^{A} \frac{\partial}{\partial q^{A}_1} + \cdots + f_{2k-1}^{A} \frac{\partial}{\partial q^{A}_{2k-1}},$$
as
\[
\frac{dE_L}{dt} = \sum_{r=1}^{k-r} \sum_{i=0}^{k-r} (-1)^i d_i^r \left( \frac{\partial L}{\partial q_i^r} \right) dq_i^r + \sum_{r=1}^{k-r} \sum_{i=0}^{k-r} (-1)^i d_i^r \left( \frac{\partial^2 L}{\partial q_i^r \partial q_i^{r+1}} \right)
\]
\[
- \sum_{r=0}^{k} \frac{\partial L}{\partial q_i^r} dq_i^r,
\]
from (5) we obtain
\[
(f_0^a - q_1^a) \frac{\partial^2 L}{\partial q_i^r \partial q_i^r} = 0
\]
\[
(f_1^b - q_2^b) \frac{\partial^2 L}{\partial q_i^r \partial q_i^r} - (f_0^b - q_1^b) (\cdots) = 0
\]
\[
\vdots
\]
\[
(-1)^k (f_{2k-1}^a - d_T (q_{2k-1}^b)) \frac{\partial^2 L}{\partial q_i^r \partial q_i^r} + \sum_{i=0}^{k-r} (-1)^i d_i^r \left( \frac{\partial L}{\partial q_i^r} \right) - \sum_{i=0}^{k-r} (f_i^b - q_i^b) (\cdots) = 0,
\]
where the terms in brackets (\cdots) contain relations involving partial derivatives of the Lagrangian and applications of \(d_T\), which for simplicity are not written. These are the local expressions of the Lagrangian equations for \(X_L\).

Now, if \(\sigma : \mathbb{R} \to T^{2k-1}Q\) is an integral curve of \(X_L\), from (5) we obtain that \(\sigma\) must satisfy the Euler–Lagrange equation
\[
i(\tilde{\sigma})(\omega_L \circ \sigma) = dE_L \circ \sigma,
\]
where \(\tilde{\sigma}\) denotes the canonical lifting of \(\sigma\) to \(T(T^{2k-1}Q)\); and as \(X_L\) is a semispray of type 1, we have that \(\sigma\) is the canonical lifting of a curve \(\gamma : \mathbb{R} \to Q\) to \(T^{2k-1}Q\); that is, \(\sigma = \tilde{\gamma}\).

Now, if \(L \in C^\infty(T^4Q)\) is a regular Lagrangian, then \(\omega_L\) is a symplectic form in \(T^{2k-1}Q\), and as a consequence we have that:

**Theorem 1.** Let \((T^{2k-1}Q, L)\) be a regular Lagrangian system of order \(k\).

1. There exists a unique \(X_L \in \mathfrak{X}(T^{2k-1}Q)\) which is a solution to the Lagrangian equation (5) and is a semispray of type 1 in \(T^{2k-1}Q\).
2. If \(\gamma : \mathbb{R} \to Q\) is an integral curve of \(X_L\), then \(\sigma = \tilde{\gamma}\) is a solution to the Euler–Lagrange equations:
\[
\frac{\partial L}{\partial q_i^r} \circ \tilde{\gamma}^{2k-1} - \frac{d}{dt} \frac{\partial L}{\partial q_i^r} \circ \tilde{\gamma}^{2k-1} + \frac{d^2}{dt^2} \frac{\partial L}{\partial q_i^r} \circ \tilde{\gamma}^{2k-1} + \cdots + (-1)^k \frac{d^k}{dt^k} \frac{\partial L}{\partial q_i^r} \circ \tilde{\gamma}^{2k-1} = 0.
\]

If \(L \in C^\infty(T^4Q)\) is a singular Lagrangian, then \(\omega_L\) is a presymplectic form, so the existence and uniqueness of solutions to the Lagrangian equation (5) is not assured, except in special cases (for instance, when \(\omega_L\) is a presymplectic horizontal structure [17]). In general, in the most favourable cases, equation (5) has solutions \(X_L \in \mathfrak{X}(T^{2k-1}Q)\) in some submanifold \(S_f \hookrightarrow T^{2k-1}Q\), for which these vector field solutions are tangent. This submanifold is obtained by applying the well-known constraint algorithms (see, for instance, [22, 21, 30]). Nevertheless, these vector field solutions are not necessarily semisprays of type 1 on \(S_f\), but only on the
points of another submanifold \( M_f \hookrightarrow S_f \hookrightarrow T^{2k-1}Q \) (see [21, 30]). On the points of this last submanifold, the integral curves of \( X_L \in \mathfrak{X}(T^{2k-1}Q) \) are solutions to the Euler–Lagrange equations (8).

A detailed study of higher order singular Lagrangian systems can be found in [23, 24].

### 2.3. Hamiltonian formalism

**Definition 10.** Let \((T^{2k-1}Q, \mathcal{L})\) be a Lagrangian system. The Legendre–Ostrogradsky map (or generalized Legendre map) associated with \(\mathcal{L}\) is the map \(\mathcal{F}\mathcal{L} : T^{2k-1}Q \rightarrow T^*(T^{2k-1}Q)\) defined as follows: for every \(u \in T(T^{2k-1}Q)\),

\[
\theta_{\mathcal{L}}(u) = (\hat{\rho}^{2k-1}_{\mathcal{L}}(u), \mathcal{F}\mathcal{L}(\tau_{T^{2k-1}}Q(u))).
\]

This map verifies that \(\pi_{T^{2k-1}} \circ \mathcal{F}\mathcal{L} = \rho^{2k-1}_{\mathcal{L}}\), where \(\pi_{T^{2k-1}} : T^*(T^{2k-1}Q) \rightarrow T^{2k-1}Q\) is the natural projection. Furthermore, if \(\theta_{k-1} \in \Omega^1(T^*(T^{2k-1}Q))\) and \(\omega_{k-1} = -d\theta_{k-1} \in \Omega^2(T^*(T^{2k-1}Q))\) are the canonical 1 and 2 forms of the cotangent bundle \(T^*(T^{2k-1}Q)\), we have that

\[
\mathcal{F}\mathcal{L}^*\omega_{k-1} = \theta_{\mathcal{L}}, \quad \mathcal{F}\mathcal{L}^*\omega_k = \omega_{\mathcal{L}}.
\]

Given a local natural chart in \(T^{2k-1}Q\), we can define the following local functions:

\[
\hat{p}^{-1}_{rA} = \sum_{i=0}^{k-r} (-1)^i d^r_q \left( \frac{\partial \mathcal{L}}{\partial q_{r+i}} \right).
\]

Observe that

\[
\hat{p}^{-1}_{rA} = -d_q (\hat{p}^{rA}_r), \quad 1 \leq r \leq k - 1.
\]

Thus, bearing in mind the local expression (3) of the form \(\theta_{\mathcal{L}}\), we can write \(\theta_{\mathcal{L}} = \sum_{r=1}^{k} r^{-1} d_q \hat{p}^{-1}_{rA},\) and we obtain that the expression in natural coordinates of the map \(\mathcal{F}\mathcal{L}\) is

\[
\mathcal{F}\mathcal{L}(q_0, q^A_1, \ldots, q^A_{2k-1}) = (q_0, q^A_1, \ldots, q^A_{2k-1}, \hat{p}^0_A, \hat{p}^1_A, \ldots, \hat{p}^{k-1}_A), \quad \text{with} \quad \mathcal{F}\mathcal{L} = \hat{p}^1_A.
\]

\(\mathcal{L}\) is a regular Lagrangian if, and only if, \(\mathcal{F}\mathcal{L} : T^{2k-1}Q \rightarrow T^*(T^{2k-1}Q)\) is a local diffeomorphism.

As a consequence of this, we have that, if \(\mathcal{L}\) is a regular Lagrangian, then the set \((q^0_i, \hat{p}^0_i)\), \(0 \leq i \leq k - 1\), is a set of local coordinates in \(T^{2k-1}Q\), and \((\hat{p}^1_A)\) are called the Jacobi–Ostrogradsky momentum coordinates.

Observe that relation (9) means that we can recover all the Jacobi–Ostrogradsky momentum coordinates from the set \((\hat{p}^{2k-1}_A)\).

**Definition 11.** \(\mathcal{L} \in C^\infty(T^2Q)\) is said to be a hyperregular Lagrangian of order \(k\) if \(\mathcal{F}\mathcal{L}\) is a global diffeomorphism. Then, \((T^{2k-1}Q, \mathcal{L})\) is a hyperregular Lagrangian system of order \(k\).
As \( \pi_{T^k} \circ \mathcal{F} \mathcal{L} = \rho_{k-1}^{2k-1} \), this condition is equivalent to demanding that the restriction of \( \rho_{k-1}^{2k-1} : T^{2k-1}Q \to T^1Q \) to every fiber be one-to-one.

In order to explain the construction of the canonical Hamiltonian formalism of a Lagrangian higher order system, we first consider the case of hyperregular systems (the regular case is the same, but restricting on the suitable open submanifolds where \( \mathcal{F} \mathcal{L} \) is a local diffeomorphism).

So, \((T^{2k-1}Q, \mathcal{L})\) being a hyperregular Lagrangian system, since \( \mathcal{F} \mathcal{L} \) is a diffeomorphism, there exists a unique function \( h \in C^\infty(T^*(T^{k-1}Q)) \) such that \( \mathcal{F} \mathcal{L}^*h = E_L \), which is called the Hamiltonian function associated with this system. Then, the triad \((T^*(T^{k-1}Q), \omega_{k-1}, h)\) is called the canonical Hamiltonian system associated with the hyperregular Lagrangian system \((T^{2k-1}Q, \mathcal{L})\). Thus, in the Hamiltonian formalism, \( T^*(T^{k-1}Q) \) represents the phase space of the system.

The dynamical trajectories of the system are the integral curves of a vector field \( X_h \in \mathfrak{X}(T^*(T^{k-1}Q)) \) which is a solution to the Hamilton equation

\[
i(X_h)\omega_{k-1} = dh.
\]

As \( \omega_{k-1} \) is symplectic, there is a unique vector field \( X_h \) solution to this equation, and it is called the Hamiltonian vector field.

In natural coordinates of \( T^*(T^{k-1}Q) \), \((q_i^A, p_A^j) \) (with \( 0 \leq i \leq k-1; 1 \leq A \leq n \)), taking \( X_h = f_i^A \frac{\partial}{\partial q_i^A} + g_A^j \frac{\partial}{\partial p_A^j} \), as \( dh = \frac{\partial h}{\partial q_i^A} dq_i^A + \frac{\partial h}{\partial p_A^j} dp_A^j \), and \( \omega_{k-1} = dq_i^A \wedge dp_A^j \), from (10) we obtain that

\[
f_i^A = -\frac{\partial h}{\partial p_A^j}, \quad g_A^j = -\frac{\partial h}{\partial q_i^A}.
\]

Now, if \( \sigma : \mathbb{R} \to T^*(T^{k-1}Q) \) is an integral curve of \( X_h \), we have that \( \sigma \) must satisfy the Hamiltonian equation

\[
i(\tilde{\sigma})(\omega_{k-1} \cdot \sigma) = dh \cdot \sigma,
\]

and, if \( \sigma(t) = (q_i^A(t), p_A^j(t)) \) in coordinates, it gives the classical expression of the Hamilton equations:

\[
\frac{dq_i^A}{dt} = \frac{\partial h}{\partial p_A^j} \circ \sigma, \quad \frac{dp_A^j}{dt} = -\frac{\partial h}{\partial q_i^A} \circ \sigma.
\]

For the case of singular higher order Lagrangian systems, in general there is no way to associate a canonical Hamiltonian formalism, unless some minimal regularity condition is imposed [23]. In particular:

**Definition 12.** A Lagrangian \( \mathcal{L} \in C^\infty(T^nQ) \) is said to be an almost-regular Lagrangian function of order \( k \) if:

1. \( \mathcal{F} \mathcal{L}(T^{2k-1}Q) = P_0 \) is a closed submanifold of \( T^*(T^{k-1}Q) \).
   (We denote the natural embedding by \( j_{Po} : P_0 \hookrightarrow T^*(T^{k-1}Q) \)).
2. \( \mathcal{F} \mathcal{L} \) is a surjective submersion on its image.
3. For every \( p \in T^{2k-1}Q \), the fibers \( \mathcal{F} \mathcal{L}^{-1}(\mathcal{F} \mathcal{L}(p)) \) are connected submanifolds of \( T^{2k-1}Q \).

Then \((T^{2k-1}Q, \mathcal{L})\) is an almost-regular Lagrangian system of order \( k \).

Denoting the map defined by \( \mathcal{F} \mathcal{L} = j_{Po} \circ \mathcal{F} \mathcal{L}_h \) by \( \mathcal{F} \mathcal{L}_h : T^{2k-1}Q \to P_0 \), we have that the Lagrangian energy \( E_L \) is a \( \mathcal{F} \mathcal{L}_h \)-projectable function, and then there is a unique function \( h_0 \in C^\infty(P_0) \) such that \( \mathcal{F} \mathcal{L}_h h_0 = E_L \) (see [23]).
This \( h_0 \) is the canonical Hamiltonian function of the almost-regular Lagrangian system and, taking \( \omega_o = j^*_k \omega_{k-1} \), the triad (\( P_o, \omega_o, h_o \)) is the canonical Hamiltonian system associated with the almost regular Lagrangian system \((T^{2k-1}Q, L)\). For this system we have the Hamilton equation

\[
i(X_{h_o}) \omega_o = dh_o, \quad X_{h_o} \in \mathfrak{X}(P_o).
\] (11)

As \( \omega_o \) is, in general, a presymplectic form, in the best cases, this equation has some vector field \( X_{h_o} \) solution only on the points of some submanifold \( P_f \leftarrow P_o \rightarrow T^*(T^{k-1}Q) \), for which \( X_{h_o} \) is tangent to \( P_f \). This vector field is not unique, in general. It can be proved that \( P_f = \mathcal{F}_L(S_f) \), where \( S_f \leftarrow T^{2k-1}Q \) is the submanifold where there are vector field solutions to the Lagrangian equation (5) which are tangent to \( S_f \) (see the above section). Furthermore, as \( \mathcal{F}_L \) is a submersion, for every vector field \( X_{h_o} \in \mathfrak{X}(T^*(T^{k-1}Q)) \) which is a solution to the Hamilton equation (11) on \( P_f \), and tangent to \( P_f \), there exists some semispray of type 1, \( X_L \in \mathfrak{X}(T^{2k-1}Q) \), which is a solution of the Euler–Lagrange equation on \( S_f \), and tangent to \( S_f \), such that \( \mathcal{F}_L \circ X_L = X_{h_o} \). This \( \mathcal{F}_L \)-projectable semispray of type 1 could be defined only on the points of another submanifold \( M_f \leftarrow S_f \). (See [23, 24] for a detailed exposition of all these topics.)

3. Skinner–Rusk unified formalism

3.1. Unified phase space. Geometric and dynamical structures

Let \( L \in C^\infty(T^kQ) \) be the Lagrangian function of order \( k \) of the system. First we construct the unified phase space

\[
\mathcal{W} = T^{2k-1}Q \times_{T^1Q} T^*(T^{k-1}Q)
\]

(the fiber product of the above bundles), which is endowed with the canonical projections \( \text{pr}_1 : T^{2k-1}Q \times_{T^1Q} T^*(T^{k-1}Q) \to T^{2k-1}Q \); \( \text{pr}_2 : T^{2k-1}Q \times_{T^1Q} T^*(T^{k-1}Q) \to T^*(T^{k-1}Q) \), and also with the canonical projections onto \( T^{2k-1}Q \). So we have the diagram

\[
\begin{array}{ccc}
W & \xrightarrow{pr_1} & T^{2k-1}Q \\
\downarrow & & \downarrow \\
T^*(T^{k-1}Q) & \xrightarrow{pr_2} & T^{k-1}Q \\
\beta^{2k-1} & & \beta^{k-1}
\end{array}
\]

If \( (U, q_0^A) \) is a local chart of coordinates in \( Q \), denoting by \((\beta^{2k-1})^{-1}(U); q_0^A, q_1^A, \ldots, q_{2k-1}^A)\) and \((\pi_{T^1Q} \circ \beta^{k-1})^{-1}(U); q_0^A, q_1^A, \ldots, q_{k-1}^A, p_0^A, p_1^A, \ldots, p_{k-1}^A)\) the induced charts in \( T^{2k-1}Q \) and in \( T^*(T^{k-1}Q) \), respectively, we have that \( (q_0^A, \ldots, q_{k-1}^A; q_0^A, \ldots, q_{2k-1}^A; p_0^A, \ldots, p_{k-1}^A) \) are the natural coordinates in the suitable open domain \( W \subset \mathcal{W} \). Note that \( \text{dim}(\mathcal{W}) = 3kn \).
Definition 13. Let \( p \in T^{2k-1}Q \), its projection \( q = \rho_{2k-1}^{2k-1}(p) \) to \( T^{k-1}Q \) and a covector \( \alpha_q \in T_q^{\ast}(T^{k-1}Q) \). The coupling function \( C \in \mathbb{C}^{\infty}(W) \) is defined as follows:

\[
C : T^{2k-1}Q \times_{T^{k-1}Q} T^{k-1}Q \longrightarrow \mathbb{R} \\
(p, \alpha_q) \longmapsto \langle \alpha_q \mid j_k(p)q \rangle,
\]

where \( j_k : T^{2k-1}Q \to T(T^{k-1}Q) \) is the canonical injection introduced in (1), \( j_k(p)q \) is the corresponding tangent vector to \( T^{k-1}Q \) in \( q \) and \( \langle \alpha_q \mid j_k(p)q \rangle = \alpha_q(j_k(p)q) \) denotes the canonical pairing between vectors of \( T_q(T^{k-1}Q) \) and covectors of \( T_q^{\ast}(T^{k-1}Q) \).

Observe that, if \( k = 1 \), the map \( j_1 : TQ \to TQ \) is the identity on \( TQ \), and we recover the standard canonical coupling between vectors in \( T_pQ \) and covectors in \( T_p^{\ast}Q \).

Using the coupling function, given a Lagrangian function \( L \in \mathbb{C}^{\infty}(T^\ast Q) \), we can define the Hamiltonian function \( H \in \mathbb{C}^{\infty}(W) \) as

\[
H = C - (\rho_{2k-1}^{2k-1} \circ pr_1)^{\ast}L,
\]

whose coordinate expression is

\[
H = p_A^i q_{i+1}^A - L(q_0^A, \ldots , q_k^A).
\]

Now, \((W, \Omega, H)\) is a presymplectic Hamiltonian system.

Finally, in order to give a complete description of the dynamics of higher order Lagrangian systems, we need to introduce the following concept.

Definition 14. A vector field \( X \in \mathfrak{X}(W) \) is said to be a semispray of type \( r \) in \( W \) if, for every integral curve \( \sigma : I \subset \mathbb{R} \to W \) of \( X \), the curve \( \sigma_1 = pr_1 \circ \sigma : I \to T^{2k-1}Q \) satisfies that, if \( \gamma = \beta^{2k-1} \circ \sigma_1 \), \( \tilde{\gamma}^{2k-r} = \rho_{2k-r}^{2k-1} \circ \sigma_1 \).

In particular, \( X \in \mathfrak{X}(W) \) is a semispray of type \( 1 \) if \( \tilde{\gamma}^{2k-1} = \sigma_1 \).
The local expression of a semispray of type \( r \) in \( \mathcal{W} \) is
\[
X = \sum_{i=0}^{2k-r} q^A_i \frac{\partial}{\partial q^A_i} + \sum_{i=2k-r}^{2k-1} X^A_i \frac{\partial}{\partial q^A_i} + \sum_{i=0}^{k-1} G^A_i \frac{\partial}{\partial p^A_i},
\]
and, in particular, for a semispray of type 1 in \( \mathcal{W} \) we have
\[
X = \sum_{i=0}^{2k-2} q^A_i \frac{\partial}{\partial q^A_i} + \sum_{i=2k-2}^{2k-1} X^A_i \frac{\partial}{\partial q^A_i} + \sum_{i=0}^{k-1} G^A_i \frac{\partial}{\partial p^A_i}.
\]

3.2. Dynamical vector fields

3.2.1. Dynamics in \( \mathcal{W} = T^{2k-1}Q \times_{T^{k-1}Q} T^*(T^{k-1}Q) \). As we know, the dynamical equation of the presymplectic Hamiltonian system \((\mathcal{W}, \Omega, H)\) is geometrically written as
\[
i(X)\Omega = dH; \quad X \in \mathcal{X}(\mathcal{W}). \tag{18}
\]

Then, according to [22] we have

**Proposition 2.** Given the presymplectic Hamiltonian system \((\mathcal{W}, \Omega, H)\), a solution \( X \in \mathcal{X}(\mathcal{W}) \) to equation (18) exists only on the points of the submanifold \( \mathcal{W}_o \hookrightarrow \mathcal{W} \) defined by
\[
\mathcal{W}_o = \{ p \in \mathcal{W} : \xi(p) \equiv (i(Y)dH)(p) = 0, \forall Y \in \ker \Omega \}. \tag{19}
\]

We denote by \( j_o : \mathcal{W}_o \hookrightarrow \mathcal{W} \) the natural embedding and by \( \mathcal{X}_{\mathcal{W}_o}(\mathcal{W}) \) the set of vector fields in \( \mathcal{W} \) at support on \( \mathcal{W}_o \). We have the following characterization of \( \mathcal{W}_o \):

**Proposition 3.** The submanifold \( \mathcal{W}_o \hookrightarrow \mathcal{W} \) is the graph of the Legendre–Ostrogradsky map defined by \( \mathcal{L} \), that is, \( \mathcal{W}_o = \text{graph} \, \mathcal{L} \).

**Proof.** As \( \mathcal{W}_o \) is defined by (19), it suffices to prove that the constraints defining \( \mathcal{W}_o \) are those defining the graph of the Legendre–Ostrogradsky map associated with \( \mathcal{L} \). We make this calculation in coordinates. Taking the local expression (17) of the Hamiltonian function \( H \in \mathcal{C}^\infty(\mathcal{W}) \), we have
\[
dH = \sum_{i=0}^{k-1} (q^A_i dp^A_i + p^A_i dq^A_i) = - \sum_{i=0}^{k} \frac{\partial \mathcal{L}}{\partial q^A_i} dq^A_i,
\]
and using the local basis of \( \ker \Omega \) given in (13), we obtain that the equations defining the submanifold \( \mathcal{W}_o \) are
\[
i(Y)dH = 0 \iff p^{k-1}_A - \frac{\partial \mathcal{L}}{\partial q^A_i} = 0.
\]

Observe that these expressions relate the momentum coordinates \( p^{k-1}_A \) with the Jacobi–Ostrogradsky functions \( p^{k-1}_A = \frac{\partial \mathcal{L}}{\partial q^A_i} \), and so we obtain the last group of equations of the Legendre–Ostrogradsky map. Furthermore, in section 2.3 we have seen that the other Jacobi–Ostrogradsky functions \( p^{r-1}_A (1 \leq r \leq k-1) \) satisfy relations (9). Thus, we can consider that \( \mathcal{W}_o \) is the graph of a map
\[
F : T^{2k-1}Q \quad \rightarrow \quad T^*(T^{k-1}Q) \quad \rightarrow \quad (q^A_i) \quad \mapsto \quad (q^A_i, \ldots, q^A_{k-1}, p^A_k, \ldots, p^{k-1}_A)
\]
which we identify with the Legendre–Ostrogradsky map by making the identification \( p^{i-1}_A = \tilde{p}^{i-1}_A \).

Hence, we look for vector fields \( X \in \mathcal{X}_{\mathcal{W}}(\mathcal{W}) \) which are solutions to equation (18) at support on \( \mathcal{W}_{\nu} \), that is

\[
[i(X)\Omega - dH]|_{\mathcal{W}_{\nu}} = 0.
\]  

(20)

In natural coordinates a generic vector field in \( \mathcal{X}(\mathcal{W}) \) is

\[
X = \sum_{i=0}^{k-1} f_i^A \frac{\partial}{\partial q_i^A} + \sum_{i=k}^{2k-1} F_i^A \frac{\partial}{\partial q_i^A} + \sum_{i=0}^{k-1} G_A \frac{\partial}{\partial p^{i-1}_A},
\]

bearing in mind the local expressions of \( \Omega \) and \( dH \), from (18), we obtain the following system of \((2k+1)n\) equations:

\[
f_i^A = q_{i+1}^A,
\]

(21)

\[
G_A^0 = \frac{\partial L}{\partial q_0^A}, \quad G_A = \frac{\partial L}{\partial q_i^A} - p^{i-1}_A = d_T(p^{i}_A),
\]

(22)

\[
p^{i-1}_A - \frac{\partial L}{\partial q_k^A} = 0,
\]

(23)

where \( 0 \leq i \leq k-1 \) in (21) and \( 1 \leq i \leq k-1 \) in (22). Therefore,

\[
X = \sum_{i=0}^{k-1} q_{i+1}^A \frac{\partial}{\partial q_i^A} + \sum_{i=k}^{2k-1} F_i^A \frac{\partial}{\partial q_i^A} + \sum_{i=0}^{k-1} G_A \frac{\partial}{\partial p^{i-1}_A} + \sum_{i=1}^{k-1} d_T(p^{i}_A) \frac{\partial}{\partial p^{i}_A}.
\]

(24)

We can observe that equations (23) are just a compatibility condition that, together with the other conditions for the momenta, say that the vector fields \( X \) exist only with support on the submanifold defined by the graph of the Legendre–Ostrogradsky map. So we recover, in coordinates, the result stated in propositions 2 and 3. Furthermore, this local expression shows that \( X \) is a semispray of type \( k \) in \( \mathcal{W} \).

The component functions \( F_i^A, k \leq i \leq 2k-1 \), are undetermined. Nevertheless, we must study the tangency of \( X \) to the submanifold \( \mathcal{W}_{\nu} \); that is, we have to impose that \( L(X)\xi|_{\mathcal{W}_{\nu}} \equiv X(\xi)|_{\mathcal{W}_{\nu}} = 0 \), for every constraint function \( \xi \) defining \( \mathcal{W}_{\nu} \). So, taking into account proposition 3, these conditions lead to

\[
\left( \sum_{i=0}^{k-1} q_{i+1}^A \frac{\partial}{\partial q_i^A} + \sum_{i=k}^{2k-1} F_i^A \frac{\partial}{\partial q_i^A} + \sum_{i=0}^{k-1} G_A \frac{\partial}{\partial p^{i-1}_A} \right) \left( p^{k-1}_A - \frac{\partial L}{\partial q_k^A} \right) = 0
\]

\[
\left( \sum_{i=0}^{k-1} q_{i+1}^A \frac{\partial}{\partial q_i^A} + \sum_{i=k}^{2k-1} F_i^A \frac{\partial}{\partial q_i^A} + \sum_{i=0}^{k-1} G_A \frac{\partial}{\partial p^{i-1}_A} \right) \times \left( p^{k-2}_A - \sum_{i=0}^{k-2} (-1)^i d_T \left( \frac{\partial L}{\partial q^{k-1-i}_A} \right) \right) = 0
\]

\[
\vdots
\]

\[
\left( \sum_{i=0}^{k-1} q_{i+1}^A \frac{\partial}{\partial q_i^A} + \sum_{i=k}^{2k-1} F_i^A \frac{\partial}{\partial q_i^A} + \sum_{i=0}^{k-1} G_A \frac{\partial}{\partial p^{i-1}_A} \right) \times \left( p^{2-2}_A - \sum_{i=0}^{2-2} (-1)^i d_T \left( \frac{\partial L}{\partial q^{2-1-i}_A} \right) \right) = 0
\]

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\[
\left(\sum_{i=0}^{k-1} q_{i+1}^A \frac{\partial}{\partial q_i^A} + \sum_{i=k}^{2k-1} F_i^A \frac{\partial}{\partial q_i^A} + \sum_{i=k}^{k-1} \frac{\partial L}{\partial q_i^A} \frac{\partial}{\partial p_i^A} + \sum_{i=1}^{k-1} d_T(p_i^A) \frac{\partial}{\partial p_i^A}\right) \\
\times \left(p_i^A - \sum_{i=0}^{k-1} (-1)^i d_T \left(\frac{\partial L}{\partial q_i^A}\right)\right) = 0,
\]

and, from here, we obtain the following \(k^n\) equations:

\[
(F_{k^2-2}^A - q_{k^2-1}^A) \frac{\partial^2 L}{\partial q_i^A \partial q_i^A} - \sum_{l=0}^{k-1} (F_{k^2+l}^A - q_{k^2+l}^A) (\cdots) = 0
\]

\[
(-1)^k (F_{k^2-1}^A - d_T(q_{k^2-1})) \frac{\partial^2 L}{\partial q_i^A \partial q_j^A} + \sum_{l=0}^{k-1} (-1)^l d_T \left(\frac{\partial L}{\partial q_i^A}\right) - \sum_{l=0}^{k-2} (F_{k^2+l}^A - q_{k^2+l}^A) (\cdots) = 0,
\]

where the terms in brackets (\(\cdots\)) contain relations involving partial derivatives of the Lagrangian and applications of \(d_T\) which for simplicity are not written. These are just the Lagrangian equations for the components of \(X\), as we have seen in (6). These equations can be compatible or not, and a sufficient condition to ensure compatibility is the regularity of the Lagrangian function. In particular, we have:

**Proposition 4.** If \(L \in C^\infty(T^4Q)\) is a regular Lagrangian function, then there exists a unique vector field \(X \in \mathcal{X}_{\mathcal{W}_\omega}(\mathcal{W})\) which is a solution to equation (20); it is tangent to \(\mathcal{W}_\omega\), and is a semispray of type 1 in \(\mathcal{W}\).

**Proof.** As the Lagrangian function \(L\) is regular, the Hessian matrix \(\left(\frac{\partial^2 L}{\partial q_i^A \partial q_j^A}\right)\) is regular at every point, and this allows us to solve the above \(k\) systems of \(n\) equations (25) determining all the functions \(F_i^A\) uniquely as follows:

\[
F_i^A = q_{i+1}^A, \quad (k \leq i \leq 2k - 2)
\]

\[
(-1)^k (F_{k^2-1}^A - d_T(q_{k^2-1})) \frac{\partial^2 L}{\partial q_i^A \partial q_j^A} + \sum_{l=0}^{k-1} (-1)^l d_T \left(\frac{\partial L}{\partial q_i^A}\right) = 0.
\]

In this way, the tangency condition holds for \(X\) at every point on \(\mathcal{W}_\omega\). Furthermore, equalities (26) show that \(X\) is a semispray of type 1 in \(\mathcal{W}\).

However, if \(L\) is not regular, equations (25) can be compatible or not. In the most favourable cases, there is a submanifold \(\mathcal{W}_f \hookrightarrow \mathcal{W}_\omega\) (it could be \(\mathcal{W}_f = \mathcal{W}_\omega\)) such that there exist vector fields \(X \in \mathcal{X}_{\mathcal{W}_f}(\mathcal{W})\), tangent to \(\mathcal{W}_f\), which are solutions to the equation

\[
[i(X)\Omega - dH]|_{\mathcal{W}_f} = 0.
\]

In this case, equations (25) are not compatible, and the compatibility condition gives rise to new constraints.
3.2.2. Dynamics in $T^{2k-1}Q$. Now we study how to recover the Lagrangian dynamics from the dynamics in the unified formalism, using the dynamical vector fields.

First we have the following results.

**Proposition 5.** The map $\widetilde{pr}_1 = pr_1 \circ jo : \mathcal{W}_o \to T^{2k-1}Q$ is a diffeomorphism.

**Proof.** As $\mathcal{W}_o = \text{graph } \mathcal{F} \mathcal{L}$, we have that $T^{2k-1}Q \simeq \mathcal{W}_o$. Furthermore, $\widetilde{pr}_1$ is a surjective submersion and, by the equality between dimensions, it is also an injective immersion and hence it is a diffeomorphism. \qed

**Lemma 1.** If $\omega_{k-1} \in \Omega^2(T^*(T^{k-1}Q))$ is the canonical symplectic 2-form in $T^*(T^{k-1}Q)$, and $\omega_\mathcal{L} = \mathcal{F} \mathcal{L}^* \omega_{k-1}$ is the Lagrangian 2-form, then $\Omega = \pr_1^* \omega_\mathcal{L}$.

**Proof.** In fact,

$$\pr_1^* \omega_\mathcal{L} = \pr_1^* (\mathcal{F} \mathcal{L}^* \omega_{k-1}) = (\mathcal{F} \mathcal{L} \circ \pr_1)^* \omega_{k-1} = \pr_1^* \omega_{k-1} = \Omega.$$ \qed

**Lemma 2.** There exists a unique function $E_\mathcal{L} \in C^\infty(T^{2k-1}Q)$ such that $\pr_1^* E_\mathcal{L} = H$. This function $E_\mathcal{L}$ is the Lagrangian energy.

**Proof.** As $\widetilde{pr}_1$ is a diffeomorphism, we can define the function $E_\mathcal{L} = (\widetilde{pr}_1^{-1} \circ jo)^* H \in C^\infty(T^{2k-1}Q)$, which obviously verifies that $\pr_1^* E_\mathcal{L} = H$.

In order to prove that $E_\mathcal{L}$ is the Lagrangian energy defined previously, we calculate its local expression in coordinates. Thus, from (17) we obtain that

$$\pr_1^* E_\mathcal{L} = H = \sum_{i=0}^{k-1} p_i^* q_i^k = L(q_0^k, \ldots, q_k^k),$$

but $\mathcal{W}_o \hookrightarrow \mathcal{W}$ is the graph of the Legendre–Ostrogradsky map, and by proposition 3 we have

$$p_i^* = \sum_{j=0}^{k-i-1} (-1)^i d_i^l \left( \frac{\partial L}{\partial q_{i+1}^l} \right),$$

and then

$$\pr_1^* E_\mathcal{L} = \sum_{i=0}^{k-1} \sum_{j=0}^{k-i-1} q_i^k (-1)^i d_i^l \left( \frac{\partial L}{\partial q_{i+1}^l} \right) - L(q_0^k, \ldots, q_k^k)$$

$$= \sum_{i=1}^{k} \sum_{j=0}^{k-i} q_i^k (-1)^i d_i^l \left( \frac{\partial L}{\partial q_{i+1}^l} \right) - L(q_0^k, \ldots, q_k^k).$$

Now, as $\widetilde{pr}_1 = pr_1 \circ jo$ and $pr_1^* q_i^j = q_i^j$, we finally obtain

$$E_\mathcal{L} = \sum_{i=1}^{k} \sum_{j=0}^{k-i} q_i^k (-1)^i d_i^l \left( \frac{\partial L}{\partial q_{i+1}^l} \right) - L(q_0^k, \ldots, q_k^k)$$

which is the local expression (4) of the Lagrangian energy. \qed

Using these results, we can recover an Euler–Lagrange vector field in $T^{2k-1}Q$ starting from a vector field $X \in \mathfrak{X}_{\mathcal{W}_o}(\mathcal{W})$ tangent to $\mathcal{W}_o$, a solution to (20). First we have:
Lemma 3. Let $X \in \mathfrak{X}(W)$ be a vector field tangent to $W_o$. Then, there exists a unique vector field $X_L \in \mathfrak{X}(T^{2k-1}Q)$ such that $X_L \circ \text{pr}_1 \circ j_o = T\text{pr}_1 \circ X \circ j_o$.

Proof. As $X \in \mathfrak{X}(W)$ is tangent to $W_o$, there exists a vector field $X_o \in \mathfrak{X}(W_o)$ such that $Tj_o \circ X_o = X \circ j_o$. Furthermore, as $\text{pr}_1$ is a diffeomorphism, there is a unique vector field $X_L \in \mathfrak{X}(T^{2k-1}Q)$ which is $\text{pr}_1$-related with $X_o$; that is, $X_L \circ \text{pr}_1 = T\text{pr}_1 \circ X_o$. Then

$$X_L \circ \text{pr}_1 \circ j_o = X_L \circ \text{pr}_1 = T\text{pr}_1 \circ X_o = T\text{pr}_1 \circ Tj_o \circ X_o = T\text{pr}_1 \circ X \circ j_o.$$ 

And as a consequence we obtain

Theorem 2. Let $X \in \mathfrak{X}_{W_o}(W)$ be a vector field solution to equation (20) and tangent to $W_o$ (at least on the points of a submanifold $W_f \hookrightarrow W_o$). Then, there exists a unique semispray of type $k$, $X_L \in \mathfrak{X}(T^{2k-1}Q)$, which is a solution to the equation

$$i(X_L)\omega_L - dE_L = 0$$

(at least on the points of $S_f = \text{pr}_1(\mathcal{W}_f)$). In addition, if $L \in \mathcal{C}^\infty(T^3Q)$ is a regular Lagrangian, then $X_L$ is a semispray of type 1, and hence it is the Euler–Lagrange vector field.

Conversely, if $X_L \in \mathfrak{X}(T^{2k-1}Q)$ is a semispray of type $k$ (respectively of type 1), which is a solution to equation (28) (at least on the points of a submanifold $S_f \hookrightarrow T^{2k-1}Q$), then there exists a unique vector field $X \in \mathfrak{X}_{W_0}(W)$ which is a solution to equation (20) (at least on $W_f = \overline{T^{2k-1}-1}(S_f) \hookrightarrow W_o \hookrightarrow W$), and it is a semispray of type $k$ in $W$ (respectively of type 1).

Proof. Applying lemmas 1, 2, and 3, we have

$$0 = [i(X)\Omega - dH]|_{W_o} = [i(X)\text{pr}_1^*\omega_L - d\text{pr}_1^*E_L]|_{W_o} = \text{pr}_1^*[i(X_L)\omega_L - dE_L]|_{W_o},$$

but as $\text{pr}_1$ is a surjective submersion, this is equivalent to

$$0 = [i(X_L)\omega_L - dE_L]|_{\text{pr}_1(W_o)} = [i(X_L)\omega_L - dE_L]|_{T^{2k-1}Q} = 0,$$

since $\text{pr}_1(W_o) = T^{2k-1}Q$. The converse is immediate, reversing this reasoning.

In order to prove that $X_L$ is a semispray of type $k$, we proceed in coordinates. From the local expression (24) for the vector field $X$ (where the functions $F_i^A$ are the solutions of the equations (25)), and using lemma 3, we obtain that the local expression of $X_L \in \mathfrak{X}(T^{2k-1}Q)$ is

$$X_L = \sum_{i=0}^{k-1} q_{i+1}^A \frac{\partial}{\partial q_i^A} + \sum_{i=k}^{2k-1} F_i^A \frac{\partial}{\partial q_i^A},$$

and then

$$J_k(X_L) = \sum_{i=0}^{k-1} \frac{(k+i)!}{i!} q_{i+1}^A \frac{\partial}{\partial q_i^A} = \Delta_k,$$

so $X_L$ is a semispray of type $k$ in $T^{2k-1}Q$.

Finally, if $L \in \mathcal{C}^\infty(T^3Q)$ is a regular Lagrangian, equations (25) become (26), and hence the local expression of $X$ is

$$X = \sum_{i=0}^{2k-2} q_{i+1}^A \frac{\partial}{\partial q_i^A} + F_{2k-1}^A \frac{\partial}{\partial q_{2k-1}^A} + \frac{\partial L}{\partial q_0^A} \frac{\partial}{\partial p_A} + \sum_{i=1}^{k-1} dp_i^A \frac{\partial}{\partial p_i^A}.$$
Therefore
\[ X_L = \sum_{i=0}^{2k-2} q_{i+1}^A \frac{\partial}{\partial q_i^A} + F_{2k-1}^A \frac{\partial}{\partial q_{2k-1}^A}, \]
and then \[ J_1(X_L) = \sum_{i=0}^{2k-2} (i+1)q_{i+1}^A \frac{\partial}{\partial q_i^A}, \]
which shows that \( X_L \) is a semispray of type 1 in \( T^{2k-1}Q \).

**Remarks**

- It is important to point out that, if \( L \) is not a regular Lagrangian, then \( X_L \) is a semispray of type \( k \) in \( W \), but not necessarily a semispray of type 1. This means that \( X_L \) is a Lagrangian vector field, but it is not necessarily an Euler–Lagrange vector field (it is not a semispray of type 1, but just a semispray of type \( k \)). Thus, for singular Lagrangians, this must be imposed as an additional condition in order that the integral curves of \( X_L \) verify the Euler–Lagrange equations. This is different from the case of first-order dynamical systems (\( k = 1 \)), where this condition (\( X_L \) is a semispray of type 1, that is, a holonomic vector field) is obtained straightforwardly in the unified formalism.

In general, only in the most interesting cases have we assured the existence of a submanifold \( W_f \hookrightarrow W_o \) and vector fields \( X \in X_{W_o}(W) \) tangent to \( W_f \) which are solutions to equation (27). Then, considering the submanifold \( S_f = pr_1(W_f) \hookrightarrow T^{2k-1}Q \), in the best cases (see [8, 23, 24]), we have that those Euler–Lagrange vector fields \( X_L \) exist, perhaps on another submanifold \( M_f \hookrightarrow S_f \) where they are tangent, and are solutions to the equation

\[ [i_{X_L} \omega_L - dE_L]_{|M_f} = 0. \quad (29) \]

- Observe also that theorem 2 states that there is a one-to-one correspondence between vector fields \( X \in X_{W_o}(W) \) which are solutions to equation (20) and \( X_L \in X(T^{2k-1}Q) \) solutions to (28), but not uniqueness, unless \( L \) is regular. In fact:

**Corollary 1.** If \( L \in C^\infty(T^4Q) \) is a regular Lagrangian, then there is a unique \( X \in X_{W_o}(W) \) tangent to \( W_o \) which is a solution to equation (20), and it is a semispray of type 1.

**Proof.** As \( L \) is regular, by proposition 1 there is a unique semispray of type 1, \( X_L \in X(T^{2k-1}Q) \), which is a solution to equation (28) on \( T^{2k-1}Q \). Then, by theorem 2, there is a unique \( X \in X_{W_o}(W) \), tangent to \( W_o \), which is a solution to (20) on \( W_o \).

**3.2.3. Dynamics in** \( T^*(T^{2k-1}Q) \).

**Hyperregular and regular Lagrangians.** In order to recover the Hamiltonian formalism, we distinguish between the regular and non-regular cases. We start with the regular case, although by simplicity we analyze the hyperregular case (the regular case is recovered from this by restriction on the corresponding open sets where the Legendre–Ostrogadsky map is a local diffeomorphism). For this case we have the following commutative diagram:
Theorem 3. Let \( L \in C^\infty(T^kQ) \) be a hyperregular Lagrangian, \( h \in C^\infty(T^*(T^{k-1}Q)) \) the Hamiltonian function such that \( \mathcal{F}L^*h = E_L \) and \( X \in \mathfrak{X}_{W_o}(V^*) \) the vector field solution to equation (20), tangent to \( W_o \). Then, there exists a unique vector field \( X_h = \mathcal{F}L^*X_L \in \mathfrak{X}(T^*(T^{k-1}Q)) \) which is a solution to the equation
\[
i(X_h)\omega_{k-1} - dh = 0.
\]
(30)

Conversely, if \( X_h \in \mathfrak{X}(T^*(T^{k-1}Q)) \) is a solution to equation (30), then there exists a unique vector field \( X \in \mathfrak{X}_{W_o}(V^*) \), tangent to \( W_o \), which is a solution to equation (20).

Proof. If \( L \) is hyperregular, then \( \mathcal{F}L \circ pr_1 = \mathcal{F}L \circ pr_2 \) is a diffeomorphism, since it is a composition of diffeomorphisms; then, there exists a unique vector field \( X_h \in \mathfrak{X}(V^*) \) such that \( \mathcal{F}L \circ pr_1 = X_h \), and there is a unique \( X \in \mathfrak{X}_{W_o}(V^*) \) such that \( j_o X_h = X|_{W_o} \).

Now, as \( \mathcal{F}L^*h = E_L \), by applying lemma 2 we have that \( pr_1^* (\mathcal{F}L^*(h)) = pr_2^*E_L = H \); but \( \mathcal{F}L \circ pr_1 = pr_2 \), and then \( pr_2^*h = H \). Therefore, by the definition of \( \Omega \), we have
\[
0 = [i(X)\Omega - dh]|_{W_o} = [i(X)pr_2^*\omega_{k-1} - d(pr_2^*h)]|_{W_o} = pr_2^* [i(X_h)\omega_{k-1} - dh]|_{W_o}.
\]
However, as \( pr_2 \) is a surjective submersion and \( pr_2(V_{W_o}) = T^*(T^{k-1}Q) \), we finally obtain that
\[
0 = [i(X_h)\omega_{k-1} - dh]|_{pr_2(V_{W_o})} = [i(X_h)\omega_{k-1} - dh]|_{T^*(T^{k-1}Q)}.
\]
Singular (almost-regular) Lagrangians. Remember that, for almost-regular Lagrangians, only in the most interesting cases have we assured the existence of a submanifold \( W_f \hookrightarrow W_0 \) and vector fields \( X \in X_{W_0}(W) \) tangent to \( W_f \) which are solutions to equation (27). In this case, the dynamical vector fields in the Hamiltonian formalism cannot be obtained straightforwardly from the solutions in the unified formalism, but rather by passing through the Lagrangian formalism and using the Legendre–Ostrogradsky map.

Thus, we can consider the submanifolds \( S_f = pr_1(W_f) \) and \( P_f = pr_2(W_f) = \mathcal{F}\mathcal{L}(S_f) \hookrightarrow T^*(T^{k-1}Q) \). Then, using theorem 2, from the vector fields \( X \in X_{W_0}(W) \) we obtain the corresponding \( X_L \in X(T^{2k-1}Q) \), and from these the semisprays of type 1 (if they exist) which are perhaps defined on a submanifold \( M_f \hookrightarrow S_f \) are tangent to \( M_f \) and are solutions to equation (29). So we have the following commutative diagram:

\[
\begin{array}{ccc}
W & \xrightarrow{pr_1} & W_P = T^{2k-1}Q \times T^{k-1}Q \hookrightarrow P_o \\
\downarrow j_{W_f} & & \downarrow j_P \\
S_f & \xrightarrow{pr_1} & T^{2k-1}Q \\
\downarrow j_{S_f} & & \downarrow j_o \\
M_f & \xrightarrow{pr_1} & T^{k-1}Q \\
\end{array}
\]

Now, it is proved ([24]) that there are Euler–Lagrange vector fields (perhaps only on the points of another submanifold \( M_f \hookrightarrow M_f \)), which are \( \mathcal{F}\mathcal{L} \)-projectable on \( P_f = \mathcal{F}\mathcal{L}(S_f) \hookrightarrow P_o \hookrightarrow T^*(T^{k-1}Q) \). These vector fields \( X_{ho} = \mathcal{F}\mathcal{L}_o X_L \in \mathfrak{X}(T^*(T^{k-1}Q)) \) are tangent to \( P_f \) and are solutions to the equation

\[
[i_{X_{ho}}\omega_o - \partial h_o]_{P_f} = 0.
\]

Conversely, as \( \mathcal{F}\mathcal{L}_o \) is a submersion, for every solution \( X_{ho} \in \mathfrak{X}(T^*(T^{k-1}Q)) \) to the last equation, there is a semispray of type 1, \( X_L \in \mathfrak{X}(T^{2k-1}Q) \), such that \( \mathcal{F}\mathcal{L}_o X_L = X_{ho} \), and we can recover solutions to equation (27) using theorem 2.
3.3. Integral curves

After studying the vector fields which are solutions to the dynamical equations, we analyze their integral curves, showing how to recover the Lagrangian and Hamiltonian dynamical trajectories from the dynamical trajectories in the unified formalism.

Let \( X \in \mathcal{X}(\mathcal{W}) \) be a vector field tangent to \( \mathcal{W} \) which is a solution to equation (20), and let \( \sigma : I \subset \mathbb{R} \to \mathcal{W} \) be an integral curve of \( X \), on \( \mathcal{W}_o \). As \( \tilde{\sigma} = X \circ \sigma \), this means that the following equation holds:

\[
i(\tilde{\sigma})(\Omega \circ \sigma) = dH \circ \sigma.
\]  

(31)

Furthermore, if \( \sigma_o : I \to \mathcal{W}_o \) is a curve on \( \mathcal{W}_o \) such that \( j_o \circ \sigma_o = \sigma \), we have that \( \sigma_o \) is an integral curve of the vector field \( X_o \in \mathcal{X}(\mathcal{W}_o) \) associated with \( X \), and \( \tilde{\sigma}_o = X_o \circ \sigma_o \).

In local coordinates, if \( \sigma(t) = (q^A(t), p^A(t)) \), we have that

\[
q^A_i(t) = q^A_{i+1} \circ \sigma \quad (0 \leq i \leq k - 1);
\]

\[
p^A_i(t) = \frac{\partial L}{\partial q^A_i} \circ \sigma; \quad p^A_i(t) = d_T(\tilde{p}^A_i) \circ \sigma \quad (1 \leq i \leq k - 1),
\]

where \( \tilde{p}^A_i \) are solutions to equations (25).

Now, for the Lagrangian dynamical trajectories we have the following result.

**Proposition 6.** Let \( \sigma : I \subset \mathbb{R} \to \mathcal{W} \) be an integral curve of a vector field \( X \) solution to (20), on \( \mathcal{W}_o \). Then, the curve \( \sigma_L = \text{pr}_1 \circ \sigma : I \to T^{2k-1}Q \) is an integral curve of \( X_L \).

**Proof.** As \( \sigma = \tilde{j}_o \circ \sigma_o \), using that \( T\tilde{j}_o \circ X_o = X \circ j_o \), and that \( T\text{pr}_1 \circ X = X_L \circ \text{pr}_1 \), we have

\[
\tilde{\sigma}_L = \text{pr}_1 \circ \tilde{j}_o \circ \sigma_o = \text{pr}_1 \circ \tilde{j}_o \circ \sigma_0 = T\text{pr}_1 \circ T\tilde{j}_o \circ \tilde{\sigma}_o = T\text{pr}_1 \circ T\tilde{j}_o \circ X_o \circ \sigma_o
\]

\[
= T\text{pr}_1 \circ X \circ \tilde{j}_o \circ \sigma_0 = X_L \circ \text{pr}_1 \circ \tilde{j}_o \circ \sigma_0 = X_L \circ \sigma_L.
\]

\[\square\]

**Corollary 2.** If \( \mathcal{L} \in C^\infty(T^2Q) \) is a regular Lagrangian, then the curve \( \sigma_L = \text{pr}_1 \circ \sigma : I \to T^{2k-1}Q \) is the canonical lifting of a curve on \( Q \); that is, there exists \( \gamma : I \subset \mathbb{R} \to Q \) such that \( \sigma_L = \tilde{\gamma}^{2k-1} \).

**Proof.** It is a straightforward consequence of proposition 6 and theorem 2. \[\square\]

And for the Hamiltonian trajectories, we have

**Proposition 7.** Let \( \sigma : I \subset \mathbb{R} \to \mathcal{W} \) be an integral curve of a vector field \( X \) solution to (20), on \( \mathcal{W}_o \). Then, the curve \( \sigma_h = \mathcal{F}\mathcal{L} \circ \sigma_L : I \to T^*(T^{2k-1}Q) \) is an integral curve of \( X_h = \mathcal{F}\mathcal{L}_x (X_L) \).

**Proof.** Given that \( \sigma_L \) is an integral curve of \( X_L \), proposition 6, and the definitions of \( X_h \) and \( \sigma_h \), we obtain

\[
\tilde{\sigma}_h = \mathcal{F}\mathcal{L} \circ \tilde{\sigma}_L = T\mathcal{F}\mathcal{L} \circ \tilde{\sigma}_L = T\mathcal{F}\mathcal{L} \circ X_L \circ \sigma_L = X_h \circ \mathcal{F}\mathcal{L} \circ \sigma_L = X_h \circ \sigma_h.
\]

Thus, \( \sigma_h \) is an integral curve of \( X_h \). \[\square\]

The relation among all these integral curves is summarized in the following diagram:
Remark. Observe that in propositions 6 and 7 we make no assumption on the regularity of the system. The only considerations in the almost-regular case are that, in general, the curves are defined in some submanifolds which are determined by the constraint algorithm, and that \( \sigma_L \) is not necessarily the lifting of any curve in \( Q \) and this condition must be imposed. In particular,

- if the Lagrangian is regular (or hiperregular), then \( \text{Im}(\sigma) \subset W_o, \text{Im}(\sigma_L) \subset T^{2k-1}Q \) and \( \text{Im}(\sigma_h) \subset T^*(T^{k-1}Q) \);
- if the Lagrangian is almost-regular, then \( \text{Im}(\sigma) \subset W_f \hookrightarrow W_o, \text{Im}(\sigma_L) \subset S_f \hookrightarrow T^{2k-1}Q \) and \( \text{Im}(\sigma_h) \subset P_f \hookrightarrow P_o \hookrightarrow T^*(T^{k-1}Q) \).

4. Examples

4.1. The Pais–Uhlenbeck oscillator

The Pais–Uhlenbeck oscillator is one of the simplest (regular) systems that can be used to explore the features of higher order dynamical systems, and has been analyzed in detail in many publications [32, 28]. Here, we study it using the unified formalism.

The configuration space for this system is a one-dimensional smooth manifold \( Q \) with local coordinate \( (q_0) \). Taking natural coordinates in the higher order tangent bundles over \( Q \), the second-order Lagrangian function \( \mathcal{L} \in C^\infty(T^2Q) \) for this system is locally given by

\[
\mathcal{L}(q_0, q_1, q_2) = \frac{1}{2} \left( q_1^2 - \omega^2 q_0^2 - \gamma q_2^2 \right),
\]

where \( \gamma \) is some nonzero real constant, and \( \omega \) is a real constant. \( \mathcal{L} \) is a regular Lagrangian function, since the Hessian matrix of \( \mathcal{L} \) with respect to \( q_2 \) is

\[
\left( \frac{\partial^2 \mathcal{L}}{\partial q_2 \partial q_2} \right) = -\gamma
\]

which has maximum rank, since we assume that \( \gamma \) is nonzero. Note that, if we take \( \gamma = 0 \), then \( \mathcal{L} \) becomes a first-order regular Lagrangian function, and thus it is a nonsense to study this system using the higher order unified formalism.
As this is a second-order dynamical system, the phase space that we consider is

\[ \mathcal{W} = T^3 \mathcal{Q} \times_{\text{pr}_1} T^\ast(\mathcal{Q}) \times_{\text{pr}_2} T^\ast(\mathcal{Q}). \]

Denoting the canonical symplectic form by \( \omega_1 \in \Omega^2(T^\ast(\mathcal{Q})) \), we define the presymplectic form \( \Omega \in \operatorname{pr}_2 \omega_1 \in \Omega^2(\mathcal{W}) \) with the local expression

\[ \Omega = dq_0 \wedge dp^0 + dq_1 \wedge dp^1. \]

The Hamiltonian function \( H \in C^\infty(\mathcal{W}) \) in the unified formalism is

\[ H(q_0, q_1, q_2, q_3, p^0, p^1) = p^0 q_1 + p^1 q_2, \]

and then the Hamiltonian function can be written locally

\[ H(q_0, q_1, q_2, q_3, p^0, p^1) = p^0 q_1 + p^1 q_2 - \frac{1}{2}(q_1^2 - \omega^2 q_0^2 - \gamma q_2^2). \]

As stated in the above sections, we can describe the dynamics for this system in terms of the integral curves of vector fields \( X \in \mathfrak{X}(\mathcal{W}) \) which are solutions to equation (18). If we take a generic vector field \( X \) in \( \mathcal{W} \), given locally by

\[ X = f_0 \frac{\partial}{\partial q_0} + f_1 \frac{\partial}{\partial q_1} + F_2 \frac{\partial}{\partial q_2} + F_3 \frac{\partial}{\partial q_3} + G^0 \frac{\partial}{\partial p^0} + G^1 \frac{\partial}{\partial p^1}, \]

taking into account that

\[ dH = \omega^2 q_0 dq_0 + (p^0 - q_1) dq_1 + (p^1 + \gamma q_2) dq_2 + q_1 dp^0 + q_2 dp^1. \]

from the dynamical equation \( i(X)\Omega = dH \), we obtain the following system of linear equations for the coefficients of \( X \):

\[ f_0 = q_1 \] (32)

\[ f_1 = q_2 \] (33)

\[ G^0 = -\omega^2 q_0 \] (34)

\[ G^1 = q_1 - p^0 \] (35)

\[ p^1 + \gamma q_2 = 0k. \] (36)

Equations (32) and (33) give us the condition of semispray of type 2 for the vector field \( X \). Furthermore, equation (36) is an algebraic relation stating that the vector field \( X \) is defined along a submanifold \( \mathcal{W}_o \) that can be identified with the graph of the Legendre–Ostrogradsky.
map, as we have seen in propositions 2 and 3. Thus, using (32), (33), (34) and (35), the vector field $X$ is given locally by

$$X = q_1 \frac{\partial}{\partial q_0} + q_2 \frac{\partial}{\partial q_1} + q_3 \frac{\partial}{\partial q_2} - \omega^2 q_0 \frac{\partial}{\partial p^0} + (q_1 - p^0) \frac{\partial}{\partial p_1}. \quad (37)$$

As our goal is to recover the Lagrangian and Hamiltonian solutions from the vector field $X$, we must require $X$ to be a semispray of type 1. Nevertheless, as $L$ is a regular Lagrangian function, this condition is naturally deduced from the formalism, as we have seen in (25).

Notice that the functions $F_2$ and $F_3$ in (37) are not determined until the tangency of the vector field $X$ on $W_o$ is required. The Legendre–Ostrogradsky transformation is the map $FL : T^3Q \rightarrow T^*(TQ)$ given in local coordinates by

$$FL^* p^0 = \frac{\partial L}{\partial q_1} - d_T \left( \frac{\partial L}{\partial q_2} \right) \equiv \frac{\partial L}{\partial q_1} - d_T (p^1) = q_1 + \gamma q_3$$

$$FL^* p^1 = \frac{\partial L}{\partial q_2} = -\gamma q_2$$

and, as $\gamma \neq 0$, we see that $L$ is a regular Lagrangian since $FL$ is a (local) diffeomorphism. Then, the submanifold $W_o = \text{graph} FL$ is defined by

$$W_o = \{ p \in W : \xi_0(p) = \xi_1(p) = 0 \},$$

where $\xi_r = p^r - FL^* p^r$, $r = 1, 2$. The diagram for this situation is

Next we compute the tangency condition for $X \in \mathcal{X}(W)$ given locally by (37) on the submanifold $W_o \hookrightarrow W$, by checking if the following identities hold:

$$L(X)\xi_0 \mid_{W_o} = 0, \quad L(X)\xi_1 \mid_{W_o} = 0.$$  

As we have seen in section 3.2.1, these equations give us the Lagrangian equations for the vector field $X$; that is, on the points of $W_o$ we obtain

$$L(X)\xi_0 = -\omega^2 q_0 - q_2 - \gamma F_3 = 0 \quad (38)$$

$$L(X)\xi_1 = \gamma (F_2 - q_3) = 0. \quad (39)$$

Equation (39) gives us the condition of semispray of type 1 for the vector field $X$ (recall that $\gamma \neq 0$), and equation (38) is the Euler–Lagrange equation for the vector field $X$. Note that, as $\gamma$ is nonzero, these equations give us a unique solution for $F_2$ and $F_3$. Thus, there is a unique vector field $X \in \mathcal{X}(W)$ solution to the equation $[i(X)\Omega - dH] \mid_{W_o} = 0$ which is tangent to the submanifold $W_o \hookrightarrow W$, and it is given locally by

$$X = q_1 \frac{\partial}{\partial q_0} + q_2 \frac{\partial}{\partial q_1} + q_3 \frac{\partial}{\partial q_2} - \frac{1}{\gamma} (\omega^2 q_0 + q_2) \frac{\partial}{\partial q_3} - \omega^2 q_0 \frac{\partial}{\partial p^0} + (q_1 - p^0) \frac{\partial}{\partial p_1}.$$
Then, if \( \sigma : \mathbb{R} \to \mathcal{W} \) is an integral curve of \( X \) locally given by
\[
\sigma(t) = (q_0(t), q_1(t), q_2(t), q_3(t), p^0(t), p^1(t)),
\]
(40)
and its component functions are solutions to the system
\[
\begin{align*}
\dot{q}_0(t) &= q_1(t); \\
\dot{q}_1(t) &= q_2(t); \\
\dot{q}_2(t) &= q_3(t); \\
\dot{q}_3(t) &= -\frac{1}{\gamma} (\omega^2 q_0(t) + q_2(t)); \\
\dot{p}^0(t) &= -\omega^2 q_0(t); \\
\dot{p}^1(t) &= q_1(t) - p^0(t).
\end{align*}
\]
(41)-(46)
Finally, we recover the Lagrangian and Hamiltonian solutions for this system. For the Lagrangian solutions, as we have shown in lemma 3 and theorem 2, the Euler–Lagrange vector field is the unique semispray of type 1, \( X_L \in \mathfrak{X}(T^3Q) \), such that \( X_L \circ pr_1 \circ j_o = Tpr_1 \circ X \circ j_o \). Thus, the vector field \( X_L \) is locally given by
\[
X_L = q_1 \frac{\partial}{\partial q_0} + q_2 \frac{\partial}{\partial q_1} + q_3 \frac{\partial}{\partial q_2} - \frac{1}{\gamma} (\omega^2 q_0 + q_2) \frac{\partial}{\partial q_3}.
\]
For the integral curves of \( X_L \) we know from proposition 6 that if \( \sigma : \mathbb{R} \to \mathcal{W} \) is an integral curve of \( X \), then \( \sigma_L = pr_1 \circ \sigma \) is an integral curve of \( X_L \). Thus, if \( \sigma \) is given locally by (40), then \( \sigma_L \) has the following local expression:
\[
\sigma_L(t) = (q_0(t), q_1(t), q_2(t), q_3(t)),
\]
(47)
and its components satisfy equations (41), (42), (43) and (44). Note that equations (41), (42) and (43) state that \( \sigma_L \) is the canonical lifting of a curve in the basis; that is, there exists a curve \( \gamma : \mathbb{R} \to Q \) such that \( \tilde{\gamma}^\gamma = \sigma_L \). Furthermore, equation (44) is the Euler–Lagrange equation for this system.

Now, for the Hamiltonian solutions, as \( L \) is a regular Lagrangian, theorem 3 states that there exists a unique vector field \( X_h = F L \circ X_L \in \mathfrak{X}(T^3Q) \) which is a solution to the Hamilton equation. Hence, it is given locally by
\[
X_h = q_1 \frac{\partial}{\partial q_0} + q_2 \frac{\partial}{\partial q_1} - \omega^2 q_0 \frac{\partial}{\partial p^0} + (q_1 - p^0) \frac{\partial}{\partial p_1}.
\]
For the integral curves of \( X_h \), proposition 7 states that if \( \sigma_L : \mathbb{R} \to T^3Q \) is an integral curve of \( X_L \) coming from an integral curve \( \sigma \) of \( X \), then \( \sigma_h = F L \circ \sigma_L \) is an integral curve of the vector field \( X_h \). Therefore, if \( \sigma \) is given locally by (40), then \( \sigma_L \) is given by (47) and so \( \sigma_h \) can be locally written
\[
\sigma_h(t) = (q_0(t), q_1(t), p^0(t), p^1(t)),
\]
and its components must satisfy equations (41), (42), (45) and (46). Note that these equations are the standard Hamilton equations for this system.
4.2. The second-order relativistic particle

Let us consider a relativistic particle whose action is proportional to its extrinsic curvature. This system was analyzed in [34, 33, 8, 31], and here we study it using the Lagrangian–Hamiltonian formalism.

The configuration space is an \( n \)-dimensional smooth manifold \( Q \) with local coordinates \( (q^0_A) \), \( 1 \leq A \leq n \). Then, if we take the natural set of coordinates on the higher order tangent bundles over \( Q \), the second-order Lagrangian function for this system, \( \mathcal{L} \in C^\infty(T^2Q) \), can be written locally as

\[
\mathcal{L}(q^0_A, q^1_A, q^2_A) = \frac{\alpha}{(q^1_A)^2} \left[ (q^1_A)^2 (q^2_A)^2 - (q^1_A q^2_A)^2 \right]^{1/2} = \frac{\alpha}{(q^1_A)^2} \sqrt{g}.
\]

where \( \alpha \) is some nonzero constant. It is a singular Lagrangian, as we can see by computing the Hessian matrix of \( \mathcal{L} \) with respect to \( q^2_A \), which is

\[
\frac{\partial^2 \mathcal{L}}{\partial q^2_A \partial q^2_A} = \begin{cases} 
\frac{\alpha}{2(q^1_A)^2 \sqrt{g}} \left[ (q^1_A q^2_A)^2 - 2(q^1_A)^2 (q^2_A)^2 \right] q^1_A q^1_A & \text{if } B \neq A \\
\frac{\alpha}{\sqrt{g}} \left[ g - (q^2_A)^2 (q^2_A)^2 + 2(q^1_A q^2_A) q^1_A q^2_A - (q^1_A)^2 (q^2_A)^2 \right] & \text{if } B = A;
\end{cases}
\]

then after a long calculation we obtain that \( \det \left( \frac{\partial^2 \mathcal{L}}{\partial q^2_A \partial q^2_A} \right) = 0 \). In particular, \( \mathcal{L} \) is an almost-regular Lagrangian.

As this is a second-order dynamical system, the phase space that we consider is

\[
\mathcal{W} = T^3Q \times_{TQ} T^*(TQ)
\]

As \( \mathcal{L} \) is almost-regular, the ‘natural’ phase space for this system would be \( T^3Q \times_{TQ} P_T \), where \( P_T \rightarrow T^*(TQ) \) denotes the image of the Legendre–Ostrogradsky map. However, as we have a set of natural coordinates defined in \( \mathcal{W} \), it is easier to work in \( \mathcal{W} \) and then to obtain the constraints as a consequence of the formalism.

If \( \omega_0 \in \Omega^2(T^*(TQ)) \) is the canonical symplectic form, we define the presymplectic form \( \Omega = pr_2^* \omega_0 \in \Omega^2(\mathcal{W}) \), whose local expression is

\[
\Omega = dq^0_A \wedge dp^0_A + dq^1_A \wedge dp^1_A.
\]

The Hamiltonian function \( H \in C^\infty(\mathcal{W}) \) is \( H = \mathcal{C} - (\rho^3_A \circ pr_1)^* \mathcal{L} \), where \( \mathcal{C} \) is the coupling function, whose local expression is \( \mathcal{C} (q^0_A, q^1_A, q^2_A, p^0_A, p^1_A) = p^0_A q^1_A + p^1_A q^2_A \), and then the Hamiltonian function can be written locally as

\[
H (q^0_A, q^1_A, q^2_A, p^0_A, p^1_A) = p^0_A q^1_A + p^1_A q^2_A - \frac{\alpha}{(q^1_A)^2} \left[ (q^1_A)^2 (q^2_A)^2 - (q^1_A q^2_A)^2 \right]^{1/2}.
\]

The dynamics for this system are described as the integral curves of vector fields \( X \in \mathfrak{X}(\mathcal{W}) \) which are solutions to equation (18). If we take a generic vector field \( X \in \mathfrak{X}(\mathcal{W}) \), given locally by

\[
X = f^A_0 \frac{\partial}{\partial q^0_A} + f^A_1 \frac{\partial}{\partial q^1_A} + F^A_2 \frac{\partial}{\partial q^2_A} + F^A_3 \frac{\partial}{\partial q^3_A} + G_A^0 \frac{\partial}{\partial p^0_A} + G_A^1 \frac{\partial}{\partial p^1_A},
\]

As this is a second-order relativistic particle, the phase space that we consider is

\[
\mathcal{W} = T^3Q \times_{TQ} T^*(TQ)
\]

As \( \mathcal{L} \) is almost-regular, the ‘natural’ phase space for this system would be \( T^3Q \times_{TQ} P_T \), where \( P_T \rightarrow T^*(TQ) \) denotes the image of the Legendre–Ostrogradsky map. However, as we have a set of natural coordinates defined in \( \mathcal{W} \), it is easier to work in \( \mathcal{W} \) and then to obtain the constraints as a consequence of the formalism.

If \( \omega_0 \in \Omega^2(T^*(TQ)) \) is the canonical symplectic form, we define the presymplectic form \( \Omega = pr_2^* \omega_0 \in \Omega^2(\mathcal{W}) \), whose local expression is

\[
\Omega = dq^0_A \wedge dp^0_A + dq^1_A \wedge dp^1_A.
\]

The Hamiltonian function \( H \in C^\infty(\mathcal{W}) \) is \( H = \mathcal{C} - (\rho^3_A \circ pr_1)^* \mathcal{L} \), where \( \mathcal{C} \) is the coupling function, whose local expression is \( \mathcal{C} (q^0_A, q^1_A, q^2_A, p^0_A, p^1_A) = p^0_A q^1_A + p^1_A q^2_A \), and then the Hamiltonian function can be written locally as

\[
H (q^0_A, q^1_A, q^2_A, p^0_A, p^1_A) = p^0_A q^1_A + p^1_A q^2_A - \frac{\alpha}{(q^1_A)^2} \left[ (q^1_A)^2 (q^2_A)^2 - (q^1_A q^2_A)^2 \right]^{1/2}.
\]

The dynamics for this system are described as the integral curves of vector fields \( X \in \mathfrak{X}(\mathcal{W}) \) which are solutions to equation (18). If we take a generic vector field \( X \in \mathfrak{X}(\mathcal{W}) \), given locally by

\[
X = f^A_0 \frac{\partial}{\partial q^0_A} + f^A_1 \frac{\partial}{\partial q^1_A} + F^A_2 \frac{\partial}{\partial q^2_A} + F^A_3 \frac{\partial}{\partial q^3_A} + G_A^0 \frac{\partial}{\partial p^0_A} + G_A^1 \frac{\partial}{\partial p^1_A},
\]

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taking into account that

\[ \text{d}H = q_1^A \text{d}p_A^0 + q_2^A \text{d}p_A^1 + \left[ p_A^0 + \frac{\alpha}{(q_1^A)^2 \sqrt{g}} \left[ (q_1^A)^2 (q_2^A) \frac{\partial}{\partial q_1^A} \right] dq_1^A + \left[ p_A^1 - \frac{\alpha}{(q_1^A)^2 \sqrt{g}} \left( (q_1^A)^2 q_2^A - (q_1^A q_2^A) q_1^A \right) \right] dq_2^A, \]

from the dynamical equation we obtain the following linear systems for the coefficients of \( X \):

\[ f_0^A = q_1^A \]  
\[ f_1^A = q_2^A \]  
\[ G_0^A = 0 \]  
\[ G_1^A = -p_A^1 - \frac{\alpha}{(q_1^A)^2 \sqrt{g}} \left[ \left( (q_1^A)^2 (q_2^A)^2 - 2 (q_1^A q_2^A)^2 \right) q_1^A + (q_1^A q_2^A) (q_1^A)^2 q_2^A \right] \]

\[ p_A^1 = \frac{\alpha}{(q_1^A)^2 \sqrt{g}} (q_1^A)^2 q_2^A - (q_1^A q_2^A) q_1^A = 0. \]  

Note that from equations (49) and (50) we obtain the condition of semispray of type 2 for \( X \). Furthermore, equations (53) are algebraic relations between the coordinates in \( W \) stating that the vector field \( X \) is defined along a submanifold \( W \), that is identified with the graph of the Legendre–Ostrogradsky map, as we stated in propositions 2 and 3. Thus, the vector field \( X \) is given locally by

\[ X = q_1^A \frac{\partial}{\partial q_0^A} + q_2^A \frac{\partial}{\partial q_1^A} + F^2_A \frac{\partial}{\partial q_2^A} + F^3_A \frac{\partial}{\partial q_3^A} + G_1^A \frac{\partial}{\partial p_A^1}, \]  

where the functions \( G_1^A \) are determined by (52). As we want to recover the Lagrangian solutions from the vector field \( X \), we must require \( X \) to be a semispray of type 1. This condition reduces the set of vector fields \( X \in X(W) \) given by (54) to the following ones:

\[ X = q_1^A \frac{\partial}{\partial q_0^A} + q_2^A \frac{\partial}{\partial q_1^A} + q_3^A \frac{\partial}{\partial q_2^A} + F^3_A \frac{\partial}{\partial q_3^A} + G_1^A \frac{\partial}{\partial p_A^1}. \]

Note that the functions \( F_2^A \) are not determined until the tangency of the vector field \( X \) on \( W \) is required. Now, the Legendre–Ostrogradsky transformation is the map \( \mathcal{F}L : T^*Q \to T^*(TQ) \) locally given by

\[ \mathcal{F}L(p_0^A) = \frac{\partial L}{\partial q_0^A} - d_T \left( \frac{\partial L}{\partial q_1^A} \right) = \frac{\partial L}{\partial q_0^A} - d_T (p_1^A) \]

\[ = \frac{\alpha}{(q_1^A)^2 \sqrt{g}} \left[ \left( (q_2^A)^2 (q_1^A)^2 + (q_1^A)^2 (q_2^A) (q_1^A q_2^A) - (q_1^A)^2 (q_1^A q_2^A) (q_2^A q_1^A) \right) q_1^A \right] \]

\[ + \frac{\alpha}{(q_1^A)^2 \sqrt{g}} \left[ \left( ((q_1^A)^2)^2 (q_2^A q_1^A) - (q_1^A)^2 (q_1^A q_2^A)(q_1^A q_2^A) - (q_1^A q_2^A) (q_1^A)^2 q_2^A \right) \right] \]

\[ \mathcal{F}L(p_1^A) = \frac{\partial L}{\partial q_1^A} = \frac{\alpha}{(q_1^A)^2 \sqrt{g}} \left[ (q_1^A)^2 q_2^A - (q_1^A q_2^A) q_1^A \right]. \]
and, in fact, \( L \) is an almost-regular Lagrangian. Thus, from the expression in local coordinates of the map \( \mathcal{FL} \), we obtain the (primary) constraints that define the closed submanifold \( \mathcal{P}_o = \text{Im} \mathcal{FL} \), which are

\[
\phi_1^{(0)} = p_i^1 q_i^1 = 0; \quad \phi_2^{(0)} = (p_1^1)^2 - \frac{\alpha^2}{(q_1^1)^2} = 0.
\]

Let \( \mathcal{FL}_o : T^3Q \to \mathcal{P}_o \). Then, the submanifold \( \mathcal{W}_o = \text{graph} \mathcal{FL}_o \) is defined by

\[
\mathcal{W}_o = \{ p \in \mathcal{W} : \xi_0^A(p) = \xi_1^A(p) = \phi_1^{(0)}(p) = \phi_2^{(0)}(p) = 0, \; 1 \leq A \leq \dim \mathcal{Q} \},
\]

where \( \xi_t^A \equiv p_A^t - \mathcal{FL}_* p_A^t \). The diagram for this situation is

![Diagram](image)

Note that \( \mathcal{W}_o \) is a submanifold of \( T^3Q \times T_Q \mathcal{P}_o \), and that \( \mathcal{W}_o \) is the real phase space of the system, where the dynamics take place.

Next we compute the tangency condition for \( X \in \mathfrak{X}(\mathcal{W}) \) given locally by (55) on the submanifold \( \mathcal{W}_o \hookrightarrow \mathcal{W}_{P_o} \hookrightarrow \mathcal{W} \), by checking if the following identities hold:

\[
L(X)\xi_0^A|_{\mathcal{W}_o} = 0, \quad L(X)\xi_1^A|_{\mathcal{W}_o} = 0
\]

\[
L(X)\phi_1^{(0)}|_{\mathcal{W}_o} = 0, \quad L(X)\phi_2^{(0)}|_{\mathcal{W}_o} = 0.
\]

As we have seen in section 3.2.1, equations (57) give us the Lagrangian equations for the vector field \( X \). However, equations (58) do not hold since

\[
L(X)\phi_1^{(0)} = L(X)(p_1^0 q_1^1) = -p_1^0 q_1^1, \quad L(X)\phi_2^{(0)} = L(X)((p_i^1)^2 - \alpha^2/(q_i^1)^2) = -2p_1^0 q_1^1,
\]

and hence we obtain two first-generation secondary constraints

\[
\phi_1^{(1)} = p_1^0 q_1^1 = 0, \quad \phi_2^{(1)} = p_1^0 p_1^1 = 0
\]

that define a new submanifold \( \mathcal{W}_1 \hookrightarrow \mathcal{W}_o \). Now, checking the tangency of the vector field \( X \) to this new submanifold, we obtain

\[
L(X)\phi_1^{(1)} = L(X)(p_1^0 q_1^1) = 0, \quad L(X)\phi_2^{(1)} = L(X)(p_1^0 p_1^1) = -(p_1^0)^2,
\]

and a second-generation secondary constraint appears

\[
\phi_2^{(2)} = (p_1^0)^2 = 0,
\]

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which defines a new submanifold $\mathcal{W}_2 \hookrightarrow \mathcal{W}_1$. Finally, the tangency of the vector field $X$ on this submanifold gives no new constraints, since

$$L(X)\phi^{(2)} = L(X)(p^0_2) = 0.$$ 

So we have two primary constraints (56), two first-generation secondary constraints (59) and a single second-generation secondary constraint (60). Note that these five constraints only depend on $q^1_A$, $p^0_A$ and $p^1_A$, and so they are $pr_2$-projectable. Thus, we have the following diagram:

\[
\begin{array}{c}
\mathcal{W} \\
pr_1 \\
pr_2 \\
W_{P_0} \\
pr_1, W_{P_0} \\
pr_2, W_{P_0} \\
T^*Q \\
S_1 \\
S_2 \\
\mathcal{W}_0 \\
\mathcal{W}_1 \\
\mathcal{W}_2 \\
T^*(TQ) \\
P_0 \\
P_1 \\
P_2 \\
\end{array}
\]

where

$$P_1 = \{ p \in P_0 : \phi^{(1)}_1(p) = \phi^{(1)}_2(p) = 0 \} = pr_2(\mathcal{W}_1)$$

$$P_2 = \{ p \in P_0 : \phi^{(2)}_2(p) = 0 \} = pr_2(\mathcal{W}_2)$$

$$S_1 = FL^{-1}_o(P_1) = pr_1(\mathcal{W}_1)$$

$$S_2 = FL^{-1}_o(P_2) = pr_1(\mathcal{W}_2).$$

Focusing only on the Legendre–Ostrogradsky map, and ignoring the unified part of the diagram, we have

\[
\begin{array}{c}
T^3Q \\
\mathcal{F}L \\
T*(TQ) \\
S_1 \\
S_2 \\
P_0 \\
P_1 \\
P_2 \\
\end{array}
\]
Note that we still have to check (57). As we have seen in section 3.2.1, we will obtain the following equations:

\[
(F^B_2 - d_T (q^B_2)) \frac{\partial^2 \mathcal{L}}{\partial q^a_0 \partial q^2_2} + \frac{\partial \mathcal{L}}{\partial q^0_2} - d_T \left( \frac{\partial \mathcal{L}}{\partial q^A_2} \right) + \frac{q^B_2}{2} \frac{\partial^2 \mathcal{L}}{\partial q^2_2 \partial q^2_2} = 0
\]

(61)

\[
(F^B_2 - q^B_2) \frac{\partial^2 \mathcal{L}}{\partial q^a_0 \partial q^2_2} = 0.
\]

(62)

As we have already required the vector field \( X \) to be a semispray of type 1, equations (62) are satisfied identically and equations (61) become

\[
(F^B_2 - d_T (q^B_2)) \frac{\partial^2 \mathcal{L}}{\partial q^a_0 \partial q^2_2} + \frac{\partial \mathcal{L}}{\partial q^0_2} - d_T \left( \frac{\partial \mathcal{L}}{\partial q^A_2} \right) + \frac{q^B_2}{2} \frac{\partial^2 \mathcal{L}}{\partial q^2_2 \partial q^2_2} = 0.
\]

(63)

A long calculation shows that this equation is compatible and so no new constraints arise. Thus, we have no Lagrangian constraint appearing from the semispray condition. If some constraint had appeared, it would not be \( \mathcal{F} \mathcal{L}_c \)-projectable (see [24]).

Thus, the vector fields \( X \in \mathfrak{X}(\mathcal{W}) \) given locally by (55) which are solutions to the equation

\[
[i(X)\Omega - dH]|_{\mathcal{W}_2} = 0
\]

are tangent to the submanifold \( \mathcal{W}_2 \hookrightarrow \mathcal{W}_c \). Therefore, taking the vector fields \( X_\sigma \in \mathfrak{X}(\mathcal{W}_2) \) such that \( T_{\mathcal{W}_2}X_\sigma = X \circ j_\sigma \), the form \( \Omega_\sigma = (j_{\mathcal{W}_2} \circ j_\sigma \circ j_1 \circ j_2)^* \Omega \) and the canonical Hamiltonian function \( H_\sigma = (j_{\mathcal{W}_2} \circ j_\sigma \circ j_1 \circ j_2)^* H \), the above equation leads to

\[
i(X_\sigma)\Omega_\sigma - dH_\sigma = 0,
\]

(4.2.)

but a simple calculation in local coordinates shows that \( H_\sigma = 0 \), and thus the last equation becomes

\[
i(X_\sigma)\Omega_\sigma = 0.
\]

One can easily check that, if the semispray condition is not required at the beginning and we perform all this procedure with the vector field given by (54), the final result is the same. This means that, in this case, the semispray condition does not give any additional constraint.

As final results, we recover the Lagrangian and Hamiltonian vector fields from the vector field \( X \in \mathfrak{X}(\mathcal{W}) \). For the Lagrangian vector field, by using lemma 3 and theorem 2 we obtain a semispray of type 2, \( X_\mathcal{L} \in \mathfrak{X}(T^3 Q) \), tangent to \( S_2 \). Thus, requiring the condition of semispray of type 1 to be satisfied (perhaps on another submanifold \( M_2 \hookrightarrow S_2 \)), the local expression for the vector field \( X_\mathcal{L} \) is

\[
X_\mathcal{L} = q_1^A \frac{\partial}{\partial q^0_0} + q_2^A \frac{\partial}{\partial q^1_1} + q_3^A \frac{\partial}{\partial q^2_2} + F^3_3 \frac{\partial}{\partial q^3_3},
\]

where the functions \( F^3_3 \) are determined by (63). For the Hamiltonian vector fields, recall that \( \mathcal{L} \) is an almost-regular Lagrangian function. Thus, we know that there are Euler–Lagrange vector fields which are \( \mathcal{F} \mathcal{L}_c \)-projectable on \( P_2 \), tangent to \( P_2 \) and solutions to the Hamilton equation.

5. Conclusions and outlook

After introducing the natural geometric structures needed for describing higher order autonomous dynamical systems, we review their Lagrangian and Hamiltonian formalisms, following the exposition made in [17].

The main contribution of this work is that we develop the Lagrangian–Hamiltonian unified formalism for higher order dynamical systems, following the ideas of the original article [39].
We pay special attention to showing how the Lagrangian and Hamiltonian dynamics are recovered from this, both for regular and singular systems.

A first consideration is to discuss the fundamental differences between the first-order and the higher order unified Lagrangian–Hamiltonian formalisms. In particular:

- as there is no canonical pairing between the elements of $T^{2k-1}_qQ$ and of $T^*_q(T^{k-1}_qQ)$, in order to define the higher order coupling function $C$ in an intrinsic way, we use the canonical injection that transforms a point in $T^{2k-1}_qQ$ into a tangent vector along $T^{k-1}_qQ$;

- when the equations that define the Legendre–Ostrogradsky map are recovered from the unified formalism (both in the characterization of the compatibility submanifold $W_o$ as the graph of $FL$, and in the equations in local coordinates of the vector field $X \in \mathcal{X}(W)$ solution to the dynamical equations), the only equations that are recovered are those that define the highest order momentum coordinates, and the remaining equations that define the map must be recovered using the relations between the momentum coordinates;

- the regularity of the Lagrangian function is more relevant in the higher order case, because the condition of semispray of type 1 (the holonomy condition) of the Lagrangian vector field cannot be deduced from the dynamical equations if the Lagrangian is singular, unlike the first-order case, where this holonomy condition is deduced straightforwardly from the equations independently of the regularity of the Lagrangian function. When the Lagrangian is singular, we can only ensure that the Lagrangian vector field is a semispray of type $k$. It is therefore necessary, in general, to require the condition of semispray of type 1 as an additional condition.

Then, for regular Lagrangian systems, when the tangency condition of the vector field $X \in \mathcal{X}(W)$ solution in the unified formalism along the submanifold $W_o$ is required, we obtain not only the Euler–Lagrange equations for the vector field, but also the remaining $k-1$ systems of equations that the vector field must satisfy to be a semispray of type 1.

As we point out in the introduction, a previous and quick presentation of a unified formalism for higher order systems was outlined in [13]. Our formalism differs from this one, since in that article the authors take $T^kQ \oplus_{T^{k-1}Q} T^*(T^{k-1}Q)$ as the phase space in the unified formalism, instead of ours, which is $T^{2k-1}_qQ \oplus_{T^{2k-1}Q} T^*(T^{k-1}_qQ)$. This is a significant difference, since when we want to recover the dynamical solutions of the Lagrangian formalism from the unified formalism, the Lagrangian phase space is $T^{2k-1}_qQ$, instead of $T^kQ$, which is the bundle where the Lagrangian function is defined. This fact makes it more natural to obtain the Lagrangian dynamics as well as the Hamiltonian dynamics, which in turn is obtained from the Lagrangian one using the Legendre–Ostrogradsky map.

By using any suitable generalization of some of the several formalisms for first-order non-autonomous dynamical systems [1, 6, 19], a future avenue of research consists in generalizing this unified formalism for higher order non-autonomous dynamical systems. This generalization should also be recovered as a particular case of the corresponding unified formalism for higher order classical field theories. As regards this topic, a proposal for a unified formalism for higher order classical field theories has recently been made [10, 41], which is based on the model presented in [13]. This formulation allows us to improve some previous models for describing the Lagrangian and Hamiltonian formalisms. Nevertheless, some ambiguities arise when considering the solutions of the field equations. We hope that a suitable extension of our formalism to field theories will enable these difficulties to overcome and complete the model given in [10, 41].
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