SMALL FEEDBACK VERTEX SETS
IN PLANAR DIGRAPHS

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Abstract. Let $G$ be a directed planar graph on $n$ vertices, with no directed cycle of length less than $g \geq 4$. We prove that $G$ contains a set $X$ of vertices such that $G - X$ has no directed cycle, and $|X| \leq \frac{2n-5}{g}$ if $g = 4$, $|X| \leq \frac{2n-5}{5}$ if $g = 5$, and $|X| \leq \frac{2n-6}{g}$ if $g \geq 6$. This improves recent results of Golowich and Rolnick.

A directed graph $G$ (or digraph, in short) is said to be acyclic if it does not contain any directed cycle. The digirth of a digraph $G$ is the minimum length of a directed cycle in $G$ (if $G$ is acyclic, we set its digirth to $+\infty$). A feedback vertex set in a digraph $G$ is a set $X$ of vertices such that $G - X$ is acyclic, and the minimum size of such a set is denoted by $\tau(G)$. In this short note, we study the maximum $f_g(n)$ of $\tau(G)$ over all planar digraphs $G$ on $n$ vertices with digirth $g$. Harutyunyan [1, 4] conjectured that $f_3(n) \leq \frac{2n}{5}$ for all $n$. This conjecture was recently refuted by Knauer, Valicov and Wenger [5] who showed that $f_g(n) \geq \frac{n-1}{g-1}$ for all $g \geq 3$ and infinitely many values of $n$. On the other hand, Golowich and Rolnick [3] recently proved that $f_4(n) \leq \frac{7n}{17}$, $f_5(n) \leq \frac{8n}{15}$, and $f_g(n) \leq \frac{3n-6}{g}$ for all $g \geq 6$ and $n$. Harutyunyan and Mohar [4] proved that the vertex set of every planar digraph of digirth at least 5 can be partitioned into two acyclic subgraphs. This result was very recently extended to planar digraphs of digirth 4 by Li and Mohar [6], and therefore $f_4(n) \leq \frac{n}{2}$.

This short note is devoted to the following result, which improves all the previous upper bounds for $g \geq 5$ (although the improvement for $g = 5$ is rather minor). Due to the very recent result of Li and Mohar [6], our result for $g = 4$ is not best possible (however its proof is of independent interest and might lead to further improvements).

Theorem 1. For all $n \geq 3$ we have $f_4(n) \leq \frac{5n-5}{9}$, $f_5(n) \leq \frac{2n-5}{4}$ and for all $g \geq 6$, $f_g(n) \leq \frac{2n-6}{g}$.

In a planar graph, the degree of a face $F$, denoted by $d(F)$, is the sum of the lengths (number of edges) of the boundary walks of $F$. In the proof of Theorem 1 we will need the following two simple lemmas.

Lemma 2. Let $H$ be a planar bipartite graph, with bipartition $(U, V)$, such that all faces of $H$ have degree at least 4, and all vertices of $V$ have degree at least 2. Then $H$ contains at most $2|U| - 4$ faces of degree at least 6.

Proof. Assume that $H$ has $n$ vertices, $m$ edges, $f$ faces, and $f_6$ faces of degree at least 6. Let $N$ be the sum of the degrees of the faces of $H$, plus...
twice the sum of the degrees of the vertices of $V$. Observe that $N = 4m$, so, by Euler’s formula, $N \leq 4n + 4f - 8$. The sum of degrees of the faces of $H$ is at least $4(f - f_6) + 6f_6 = 4f + 2f_6$, and since each vertex of $V$ has degree at least 2, the sum of the degrees of the vertices of $V$ is at least $2|V|$. Therefore, $4f + 2f_6 + 4|V| \leq 4n + 4f - 8$. It follows that $f_6 \leq 2|U| - 4$, as desired.

\[ \square \]

**Lemma 3.** Let $G$ be a connected planar graph, and let $S = \{F_1, \ldots, F_k\}$ be a set of $k$ faces of $G$, such that each $F_i$ is bounded by a cycle, and these cycles are pairwise vertex-disjoint. Then $\sum_{F \in S}(3d(F) - 6) \geq \sum_{i=1}^{k}(3d(F_i)+6)-12$, where the first sum varies over faces $F$ of $G$ not contained in $S$.

**Proof.** Let $n$, $m$, and $f$ denote the number of vertices, edges, and faces of $G$, respectively. It follows from Euler’s formula that the sum of $3d(F) - 6$ over all faces of $G$ is equal to $6m - 6f = 6n - 12 \geq 6\sum_{i=1}^{k}d(F_i) - 12$. Therefore, $\sum_{F \in S}(3d(F) - 6) \geq 6\sum_{i=1}^{k}d(F_i) - 12 - \sum_{i=1}^{k}(3d(F_i) - 6) = \sum_{i=1}^{k}(3d(F_i)+6)-12$, as desired. \[ \square \]

We are now able to prove Theorem 1.

**Proof of Theorem 1.** We prove the result by induction on $n \geq 3$. Let $G$ be a planar digraph with $n$ vertices and digirth $g \geq 4$. We can assume without loss of generality that $G$ has no multiple arcs, since $g \geq 4$ and removing one arc from a collection of multiple arcs with the same orientation does not change the value of $\tau(G)$. We can also assume that $G$ is connected, since otherwise we can consider each connected component of $G$ separately and the result clearly follows from the induction (since $g \geq 4$, connected components of at most 2 vertices are acyclic and can thus be left aside). Finally, we can assume that $G$ contains a directed cycle, since otherwise $\tau(G) = 0 \leq \min\{\frac{2n-5}{3}, \frac{2n-5}{4}, \frac{2n-6}{5}\}$ (since $n \geq 3$).

Let $C$ be a maximum collection of arc-disjoint directed cycles in $G$. Note that $C$ is non-empty. Fix a planar embedding of $G$. For a given directed cycle $C$ of $C$, we denote by $\overline{C}$ the closed region bounded by $C$, and by $\mathring{C}$ the interior of $\overline{C}$. It follows from classical uncrossing techniques (see 2 for instance), that we can assume without loss of generality that the directed cycles of $C$ are pairwise non-crossing, i.e. for any two elements $C_1, C_2 \in C$, either $\overline{C_1}$ and $\overline{C_2}$ are disjoint, or one is contained in the other. We define the partial order $\preceq$ on $C$ as follows: $C_1 \preceq C_2$ if and only if $\overline{C_1} \subseteq \overline{C_2}$. Note that $\preceq$ naturally defines a rooted forest $\mathcal{F}$ with vertex set $C$: the roots of each of the components of $\mathcal{F}$ are the maximal elements of $\preceq$, and the children of any given node $C \in \mathcal{F}$ are the maximal elements $C' \preceq C$ distinct from $C$ (the fact that $\mathcal{F}$ is indeed a forest follows from the non-crossing property of the elements of $C$).

Consider a node $C$ of $\mathcal{F}$, and the children $C_1, \ldots, C_k$ of $C$ in $\mathcal{F}$. We define the closed region $R_C = \overline{C} - \bigcup_{1 \leq i \leq k} C_i$. Let $\phi_C$ be the sum of $3d(F) - 6$, over all faces $F$ of $G$ lying in $R_C$.

**Claim 4.** Let $C_0$ be a node of $\mathcal{F}$ with children $C_1, \ldots, C_k$. Then $\phi_{C_0} \geq \frac{3}{2}(g - 2)k + \frac{3}{2}g$. Moreover, if $g \geq 6$, then $\phi_{C_0} \geq \frac{3}{2}(g - 2)k + \frac{3}{2}g + 3$. 


Assume first that the cycles $C_0, \ldots, C_k$ are pairwise vertex-disjoint. Then, it follows from Lemma 2 that $\phi_{C_0} \geq (k + 1)(3g + 6) - 12$. Note that since $g \geq 4$, we have $(k + 1)(3g + 6) - 12 \geq \frac{3}{2}(g - 2)k + \frac{3}{2}g$. Moreover, if $g \geq 6$, $(k + 1)(3g + 6) - 12 \geq \frac{3}{2}(g - 2)k + \frac{3}{2}g + 3$, as desired. As a consequence, we can assume that two of the cycles $C_0, \ldots, C_k$ intersect, and in particular, $k \geq 1$.

Consider the following planar bipartite graph $H$: the vertices of the first partite set of $H$ are the directed cycles $C_0, C_1, \ldots, C_k$, the vertices of the second partite set of $H$ are the vertices of $G$ lying in at least two cycles among $C_0, C_1, \ldots, C_k$, and there is an edge in $H$ between some cycle $C_i$ and some vertex $v$ if and only if $v \in C_i$ in $G$ (see Figure 1). Observe that $H$ has a natural planar embedding in which all internal faces have degree at least 4. Since $k \geq 1$ and at least two of the cycles $C_0, \ldots, C_k$ intersect, the outerface also has degree at least 4. Note that the faces $F_1, \ldots, F_t$ of $H$ are in one-to-one correspondence with the maximal subsets $D_1, \ldots, D_t$ of $R_{C_0}$ whose interior is connected. Also note that each face of $G \cap R_{C_0}$ is in precisely one region $D_i$ and each arc of $\bigcup_{i=0}^k C_i$ (i.e. each arc on the boundary of $R_{C_0}$) is on the boundary of precisely one region $D_i$. For each region $D_i$, let $\ell_i$ be the number of arcs on the boundary of $D_i$, and observe that $\sum_{i=1}^t \ell_i = \sum_{j=0}^k \vert C_j \vert$. Let $\phi_{D_i}$ be the sum of $3d(F) - 6$, over all faces $F$ of $G$ lying in $D_i$. It follows from Lemma 3 (applied with $k = 1$) that $\phi_{D_i} \geq 3\ell_i - 6$, and therefore $\phi_{C_0} = \sum_{i=1}^t \phi_{D_i} \geq \sum_{i=1}^t (3\ell_i - 6)$.

![Figure 1. The region $R_{C_0}$ (in gray) and the planar bipartite graph $H$.](image)

A region $D_i$ with $\ell_i \geq 4$ is said to be of type 1, and we set $T_1 = \{1 \leq i \leq t \mid D_i \text{ is of type 1}\}$. Since for any $\ell \geq 4$ we have $3\ell - 6 \geq \frac{3\ell}{2}$, it follows from the paragraph above that the regions $D_i$ of type 1 satisfy $\phi_{D_i} \geq \frac{3\ell}{2}$. Let $D_i$ be a region that is not of type 1. Since $G$ is simple, $\ell_i = 3$. Assume first that $D_i$ is bounded by (parts of) two directed cycles of $C$ (in other words, $D_i$ corresponds to a face of degree four in the graph $H$). In this case we say that $D_i$ is of type 2 and we set $T_2 = \{1 \leq i \leq t \mid D_i \text{ is of type 2}\}$. Then the boundary of $D_i$ consists in two consecutive arcs $e_1, e_2$ of some directed cycle $C^+$ of $C$, and one arc $e_3$ of some directed cycle $C^-$ of $C$. Since $g \geq 4$, these three arcs do not form a directed cycle, and therefore their orientation is transitive. It follows that $\vert C^+ \vert \geq g + 1$, since otherwise
the closed region obtained from $C^+$ by replacing $e_1, e_2$ with $e_3$ would have length $g - 1$, contradicting that $G$ has digirth at least $g$. Consequently, $\sum_{i=0}^k |C_i| \geq (k + 1)g + |T_2|$. If a region $D_i$ is not of type 1 or 2, then $\ell_i = 3$ and each of the 3 arcs on the boundary of $D_i$ belongs to a different directed cycle of $C$. In other words, $D_i$ corresponds to some face of degree 6 in the graph $H$. Such a region $D_i$ is said to be of type 3, and we set $T_3 = \{1 \leq i \leq t \mid D_i$ is of type 3\}. It follows from Lemma 2 that the number of faces of degree at least 6 in $H$ is at most $2(k + 1) - 4$. Hence, we have $|T_3| \leq 2k - 2$.

Using these bounds on $|T_2|$ and $|T_3|$, together with the fact that for any $i \in T_2 \cup T_3$ we have $\phi_{D_i} \geq 3\ell_i - 6 = \frac{3k}{2} - \frac{3}{2}$, we obtain:

$$\phi_{C_0} = \sum_{i \in T_1} \phi_{D_i} + \sum_{i \in T_2} \phi_{D_i} + \sum_{i \in T_3} \phi_{D_i} \geq \sum_{i=1}^t \frac{3k}{2} - \frac{3}{2}|T_2| - \frac{3}{2}|T_3| \geq \frac{3}{2}\sum_{i=0}^k |C_i| - \frac{3}{2}|T_2| - \frac{3}{2}(2k - 2) \geq \frac{3}{2}(k + 1)g - 3k + 3 = \frac{3}{2}(g - 2)k + \frac{3}{2}g + 3,$$

as desired. This concludes the proof of Claim 4. \qed

Let $C_1, \ldots, C_{k_\infty}$ be the $k_\infty$ maximal elements of $\preceq$. We denote by $R_{k\infty}$ the closed region obtained from the plane by removing $\bigcup_{i=1}^{k\infty} \hat{C}_i$. Note that each face of $G$ lies in precisely one of the regions $R_C$ ($C \in \mathcal{C}$) or $R_{k\infty}$. Let $\phi_{k\infty}$ be the sum of $3d(F) - 6$, over all faces $F$ of $G$ lying in $R_{k\infty}$. A proof similar to that of Claim 4 shows that $\phi_{k\infty} \geq \frac{3}{2}k\infty(g - 2) + 3$, and if $g \geq 6$, then $\phi_{k\infty} \geq \frac{3}{2}k\infty(g - 2) + 6$.

We now compute the sum $\phi$ of $3d(F) - 6$ over all faces $F$ of $G$. By Claim 4,

$$\phi = \phi_{k\infty} + \sum_{C \in \mathcal{F}} \phi_C \geq \frac{3}{2}k\infty(g - 2) + 3 + (|C| - k\infty)\frac{3}{2}(g - 2) + |C| \cdot \frac{3}{2}g \geq (3g - 3)|C| + 3.$$

If $g \geq 6$, a similar computation gives $\phi \geq 3g|C| + 6$. On the other hand, it easily follows from Euler’s formula that $\phi = 6n - 12$. Therefore, $|C| \leq \frac{2n - 5}{g - 1}$, and if $g \geq 6$, then $|C| \leq \frac{2n - 6}{g}$.

Let $A$ be a set of arcs of $G$ of minimum size such that $G - A$ is acyclic. It follows from the Lucchesi-Younger theorem \cite{LucchesiYounger} (see also \cite{Golumbic}) that $|A| = |C|$. Let $X$ be a set of vertices covering the arcs of $A$, such that $X$ has minimum size. Then $G - X$ is acyclic. If $g = 5$ we have $|X| \leq |A| = |C| \leq \frac{2n - 5}{g}$ and if $g \geq 6$, we have $|X| \leq |A| = |C| \leq \frac{2n - 6}{g}$, as desired. Assume now that $g = 4$.

In this case $|A| = |C| \leq \frac{2n - 5}{3}$. It was observed by Golowich and Rolnick \cite{GolowichRolnick} that $|X| \leq \frac{1}{3}(n + |A|)$ (which easily follows from the fact that any graph on
n vertices and m edges contains an independent set of size at least $\frac{2n - m}{3}$, and thus, $|X| \leq \frac{5n - 3}{9}$. This concludes the proof of Theorem 1. □

**Final remark**

A natural problem is to determine the precise value of $f_g(n)$, or at least its asymptotical value as $g$ tends to infinity. We believe that $f_g(n)$ should be closer to the lower bound of $n - \frac{1}{g}$, than to our upper bound of $\frac{2n - 6}{g}$.

For a digraph $G$, let $\tau^*(G)$ denote the the infimum real number $x$ for which there are weights in $[0,1]$ on each vertex of $G$, summing up to $x$, such that for each directed cycle $C$, the sum of the weights of the vertices lying on $C$ is at least 1. Goemans and Williamson [2] conjectured that for any planar digraph $G$, $\tau(G) \leq \frac{3}{2} \tau^*(G)$. If a planar digraph $G$ on $n$ vertices has digirth at least $g$, then clearly $\tau^*(G) \leq \frac{n}{g}$ (this can be seen by assigning weight $1/g$ to each vertex). Therefore, a direct consequence of the conjecture of Goemans and Williamson would be that $f_g(n) \leq \frac{3n}{2g}$.

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