Estimating time-changes in noisy Lévy models

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Abstract
In quantitative finance, we often model asset prices as a noisy Itô semimartingale. As this model is not identifiable, approximating by a time-changed Lévy process can be useful for generative modelling. We give a new estimate of the normalised volatility or time change in this model, which obtains minimax convergence rates, and is unaffected by infinite-variation jumps. In the semimartingale model, our estimate remains accurate for the normalised volatility, obtaining convergence rates as good as any previously implied in the literature.

1 Introduction
In quantitative finance, we often wish to predict the distribution of future asset prices using historical data; this problem is of interest when pricing options or evaluating investment strategies. From economic considerations, we know that log-prices must be given by a noisy semimartingale; however, this model cannot in general be identified from price data.

We will therefore consider modelling log-prices as a noisy time-changed Lévy process. We note that this model is general enough to describe the salient features of price data – stochastic volatility, jumps and noise – while still being simple enough to identify its parameters from data. It thus serves as a useful approximation to the semimartingale model for generative modelling.

Our goal will be to estimate the normalised volatility or time-change process in this model. Previous estimates have failed to achieve minimax convergence rates when the jumps are of infinite variation, as is suggested by empirical evidence. We will therefore describe a new estimate, which obtains minimax rates, and is unaffected by arbitrary jump activity.

We will further show that in the semimartingale model, our estimate remains accurate for the normalised volatility, obtaining convergence rates

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as good as any previously implied in the literature. Our estimate thus achieves the best of both worlds: good convergence when the time-changed approximation is accurate, and no penalty when it is not.

We begin by describing the statistical models we will consider. We will suppose we have a single asset whose efficient log-price is given by an Itô semimartingale,

$$X_t = X_0 + \int_0^t b_s \, ds + \int_0^t \sqrt{c_s} \, dB_s + \int_0^t \int_{\mathbb{R}} x (\mu(dx,ds) - 1_{|x| < 1} \nu_s(dx) \, ds), \quad (1)$$

where $b_t \in \mathbb{R}$ is a drift process, $c_t > 0$ a volatility process, $\nu_t$ a jump measure process, $\mu(dx,dt)$ a Poisson random measure with intensity $\nu_t(dx) \, dt$, and the above decomposition holds with respect to a filtration $\mathcal{F}_t$. (We refer to Jacod and Shiryaev, 2003, for definitions.)

We note that the assumption (1) is extremely common in quantitative finance, and is motivated by economic no-arbitrage arguments, as in Delbaen and Schachermayer (1994). The model (1) reproduces common features of price data, such as stochastic volatility – given by the dependence of the characteristics $(b_t, c_t, \nu_t)$ on time – and jumps – given by the presence of the jump measure process $\nu_t$.

To fit this model to price data, however, it is widely accepted that we must also account for a third feature, known as microstructure noise. The quoted price of assets in general can diverge from the efficient market price, due to economic artefacts such as the bid-ask spread, tick sizes, transaction costs and communication delays. Indeed, empirical studies confirm that high-frequency price data is too volatile to be explained solely by an efficient price process (Andersen et al., 2000; Mykland and Zhang, 2005; Hansen and Lunde, 2006).

A popular model for microstructure noise is to assume that the log-prices are observed under zero-mean errors. We thus consider observations

$$Y_j = X_{Tj/n} + \varepsilon_j, \quad j = 0, \ldots, n - 1, \quad (2)$$

over a time interval $[0, T]$, and with errors $\varepsilon_j$ satisfying $\mathbb{E}[\varepsilon_j \mid \mathcal{F}_{Tj/n}] = 0$. (We refer to Jacod et al., 2009, for a discussion of this model.)

Unfortunately, the observations $Y_j$ are insufficient to identify the parameters of the model (1). Even given noiseless observations, letting the time horizon $T \to \infty$, and the step size $T/n \to 0$, we cannot in general identify the drift process $b_t$, or jump measure process $\nu_t$.

In the following, we will therefore also consider a time-changed Lévy process model. Here, we instead suppose the log-price

$$X_t = L_{R_t}, \quad (3)$$

for a Lévy process

$$L_t = L_0 + bt + \sqrt{c} B_t + \int_0^t \int_{\mathbb{R}} x (\mu(dx,ds) - 1_{|x| < 1} \nu(dx) \, ds),$$
with drift $b \in \mathbb{R}$, volatility $c > 0$, jump measure $\nu$, and Poisson random measure $\mu(dx, dt)$ with intensity $\nu(dx) dt$, and a time-change process

$$R_t = \int_0^t r_s ds,$$

given by a rate process $r_t > 0$.

The model (3) was popularised by Carr and Wu (2004), and its applications also discussed by Cont and Tankov (2004). Intuitively, this model describes prices which move faster or slower according to an activity rate $r_t$; this rate can be thought of as the cumulative effect of factors such as trading activity and volume, investor liquidity, and general economic uncertainty.

Formally, the time-changed model (3) is the subset of the semimartingale model (1) which satisfies the separability condition

$$b_t = br_t, \quad c_t = cr_t, \quad \nu_t = \nu r_t. \tag{4}$$

This condition requires, for example, that the jump measure $\nu_t$ be governed by the rate process $r_t$, and contain no idiosyncratic jump component.

Since these parameters are defined only up to a multiplicative constant, we must also choose a normalisation for $r_t$. In the following, for simplicity we will set $r_t$ to integrate to one (although we will also discuss alternative normalisations). Equivalently, using (4) we may define

$$r_t = \frac{c_t}{\int_0^t c_s ds}; \tag{5}$$

we note that this definition is then also meaningful for the semimartingale model (1).

The separability condition (4) can be thought of as similar to the additivity condition in an additive model. We take a fully nonparametric model, which is difficult to fit, and restrict it to a lower-dimensional one, which is less so. As our smaller model (3) reproduces the salient features of price data – stochastic volatility, jumps and noise – it can potentially offer a good approximation to the full model (1).

This approximation can be useful in a variety of settings. If we wish to predict the distribution of future asset prices, for example to price options or evaluate investment strategies, we must fit a generative model to the data. We already know we cannot fit the full model (1), as we cannot identify its parameters $b_t$ and $\nu_t$. As we will see below, the parameters of the model (3) can all be identified from price data; it may thus be used either directly as a generative model, or as a starting point to identify suitable parametric alternatives (Carr and Wu, 2004; Cont and Tankov, 2004).

To fit the model (3) to data, we must estimate the parameters $b, c, \nu$, and $r_t$. If the time horizon $T \to \infty$, and the step size $T/n \to 0$, the drift $b$ and volatility $c$ can be estimated using standard techniques. Estimation
of the Lévy measure $\nu$, while more involved, has also been considered by several authors (Figueroa-López, 2009, 2011; Belomestny, 2011; Belomestny and Panov, 2013), and extensions of Figueroa-López’s approach to include noise are possible as in Vetter (2014).

In the following, we will focus specifically on estimation of the rate process $r_t$. We first note that some of the factors contributing to this process, in particular trading activity and volume, can be observed directly. While such side information may be useful in practice, we can expect that not all such factors are observable, and the relationship between observable factors and efficient prices may be unclear, especially after accounting for microstructure noise.

In the following, we will therefore restrict ourselves to estimating $r_t$ directly from price data. While previous work has provided such estimates in a variety of settings (Winkel, 2001; Woerner, 2007; Rosenbaum and Tankov, 2011; Figueroa-López, 2012), these authors have not considered our setting (2) and (3). Even accounting for microstructure noise, we cannot apply their methods here to obtain minimax rates of convergence.

An alternative route to estimating $r_t$ is to first use the identification (4), and then estimate the volatility $c_t$ in the semimartingale model (1). Many authors have described approaches for this problem, under various assumptions on the jump measure process $\nu_t$.

If there are no jumps present, the integrated volatility $\int_0^1 c_s \, ds$ can be recovered using multiscale estimators (Zhang et al., 2005; Zhang, 2006), realised kernels (Barndorff-Nielsen et al., 2008), or pre-averaging (Jacod et al., 2009; Podolskij and Vetter, 2009b). The spot volatility $c_t$ can likewise be recovered using kernel estimators (Kristensen, 2010; Mancini et al., 2014), Fourier series (Munk and Schmidt-Hieber, 2010b; Reiß, 2011), or wavelets (Hoffmann et al., 2012).

In each case, these methods can achieve minimax convergence rates, equivalent for fixed $T$ to observing $c_t$ under Gaussian white noise of size $n^{-1/4}$. In fact, it can be shown this link is a formal statistical equivalence (Reiß, 2011).

When jumps are present, however, we must account for them before estimating $c_t$. Methods for doing so include jump thresholding (Mancini, 2001, 2009; Fan and Wang, 2007; Jing et al., 2011), bipower variation (Barndorff-Nielsen and Shephard, 2004; Podolskij and Vetter, 2009a,b; Hautsch and Podolskij, 2013), and characteristic functions (Todorov and Tauchen, 2012; Jacod and Reiß, 2014; Jacod and Todorov, 2014).

Unfortunately, if the jumps are of infinite variation, in general these methods can no longer achieve the same convergence rates. Even given noiseless observations of the efficient prices, it is known that the minimax convergence rate for $c_t$ suffers, unless we assume the infinite-variation part is a scaled $\beta$-stable process (Jacod and Reiß, 2014; Jacod and Todorov, 2014).

Nonetheless, empirical evidence suggests that price data does indeed
contain infinite-variation jumps (Aït-Sahalia and Jacod, 2009; Jing et al., 2011). In the following, we will therefore construct a novel estimate of the rate process $r_t$. We will show that our estimate achieves good rates of convergence in both models, and in the time-changed model is unaffected by arbitrary jump activity.

Our estimate will be constructed in three stages. We will first obtain pre-averaged estimates of price increments, and estimates of the microstructure noise, as in Jacod et al. (2009) or Podolskij and Vetter (2009b).

We will then construct local estimates of the spot volatility, derived by estimating the characteristic function of the price process. While our approach will be similar to ones considered by previous authors (Todorov and Tauchen, 2012; Jacod and Reiß, 2014; Jacod and Todorov, 2014), the precise construction necessary to obtain minimax rates will be new.

Finally, we will smooth our local estimates of the volatility, using standard tools from nonparametric regression. While many such approaches are possible, we will use local polynomials, as described for example by Tsybakov (2009). We will also discuss how the various parameters required can be chosen automatically from the data.

We will then prove results on the convergence rates of our estimates. We note that our results will apply in two settings: a standard nonparametric setting, where the characteristics of $X_t$ are assumed fixed and smooth; and a setting more natural in quantitative finance, where these characteristics are themselves described by Itō semimartingales with locally bounded characteristics.

For simplicity, our results will focus on the high-frequency case, where the fixed time horizon $T = 1$. We note, however, that similar results can also be proved when $T \to \infty$, provided that the step size $T/n \to 0$.

In the time-changed Lévy model (3), we will then show that our procedure estimates $r_t$ with minimax convergence rates, equal to those in the Gaussian white noise model with noise level $n^{-1/4}$. Our results will hold under arbitrary jump activity, and without knowledge of the Lévy parameters $b, c$ and $\nu$.

In the general semimartingale model (1), we will show that our procedure continues to estimate $r_t$. While lower bounds for this problem are still unknown, the convergence rates we will obtain are as good as any implied by previous work. Our estimate thus achieves the best of both worlds: good convergence when the time-changed approximation is accurate, and no penalty when it is not.

In Section 2, we will give the construction of our estimates, and in Section 3, describe the specific assumptions we consider. In Section 4, we will then state our results on convergence rates, and in Section 5, give proofs.
2 Local characteristic-function estimates

In this section, we will define our estimates of the volatility and rate processes. As described in the introduction, our estimates are constructed in three stages: pre-averaging, spot volatility estimation, and smoothing.

We begin with the pre-averaging step, and proceed using the construction of Reiß (2011). We must first subdivide the time interval $[0, 1]$ into a number $n_0$ of equal bins. To define $n_0$, we choose $n_1, n_2 \in \mathbb{N}$ in terms of $n$, so that

$$n_m \sim h_m^{-1} n^{(2m-1)/8}, \quad m = 1, 2,$$

for bandwidths $h_1, h_2 > 0$, and set $n_0 = n_1 n_2$.

We then divide $[0, 1]$ into $n_0$ bins, and compute on each one a pre-averaged estimate $\hat{X}_k$ of the increments of $X_t$. We will compute $\hat{X}_k$ by integrating the observed increments against a scaling function $\Phi_n(t)$; we define

$$\Phi_n(t) = \sqrt{n_0} \Phi(n_0 t), \quad \Phi(t) = 2 \sin(2\pi t).$$

The specific choice of scaling function $\Phi_n$ is motivated by Reiß (2011), who shows that in a Gaussian setting, functions of this form are most efficient at extracting information from noisy data. We note that our choice of $\Phi$ includes a full period of the sine wave in each bin, rather than a half period; this choice allows us to ensure that the pre-averaged increments are approximately symmetrically distributed, a property we will require when modelling the behaviour of infinite-variation jumps.

We may now define the pre-averaged increments $\hat{X}_k$. For $k = 0, \ldots, n_0 - 1$, define index sets $J_k = (n/n_0)[k, k + 1) \cap \mathbb{Z}$, and let

$$\hat{X}_k = \sum_{j, j+1 \in J_k} p_j(Y_{j+1} - Y_j), \quad p_j = \Phi_n(j/n).$$

The estimate $\hat{X}_k$ thus averages the observed increments of $X_t$ over the time interval $[k, k+1)/n_0$. We can also define an estimate $\hat{\sigma}_k^2$ of the microstructure noise over the interval. We set

$$\hat{\sigma}_k^2 = \frac{n_0}{2n} \sum_{j, j+1 \in J_k} (Y_{j+1} - Y_j)^2,$$

proportional to the realised quadratic variation of the observations.

We next describe our spot volatility estimation step. We will subdivide $[0, 1]$ into $n_2$ larger bins, and on each one, construct an estimate $\hat{c}_l(u)$ of the volatility $c_t$. While our approach will be based on local characteristic function estimates, similar to those considered by previous authors (Todorov and Tauchen, 2012; Jacod and Reiß, 2014; Jacod and Todorov, 2014), the precise construction necessary to obtain minimax rates will be new.
For \( l = 0, \ldots, n_2 - 1 \), we define index sets \( K_l = n_1[l, l + 1) \cap \mathbb{Z} \), and local estimates \( \hat{\varphi}_l(u) \) of the characteristic function of increments of \( X_t \), given by

\[
\hat{\varphi}_l(u) = \frac{1}{n_1} \sum_{k \in K_l} \cos(u \hat{X}_k).
\]

We note that \( \hat{\varphi}_l(u) \) thus averages the cosines of the \( \hat{X}_k \) over the time interval \([l, l + 1)/n_2\). We can also define an estimate \( \hat{\psi}_l(u) \) of the corresponding contribution of the microstructure noise; we set

\[
\hat{\psi}_l(u) = \frac{1}{n_1} \sum_{k \in K_l} \exp(-\kappa u \hat{\sigma}_k^2), \quad \kappa = \frac{4\pi^2 n_0^2}{n}.
\]

If the log-prices \( X_t \) and noises \( \varepsilon_j \) were Gaussian, then by considering their characteristic functions, we would expect

\[
\hat{\varphi}_l(u) \approx \exp(-\frac{c_l}{n^2} + \kappa \sigma_{l/n_2}^2 u^2), \quad \hat{\psi}_l(u) \approx \exp(-\kappa \sigma_{l/n_2}^2 u^2).
\]

We could then rearrange these quantities to obtain an estimate

\[
-\frac{1}{u^2} \log \left| \frac{\hat{\varphi}_l(u)}{\hat{\psi}_l(u)} \right|
\]

of the volatility \( c_{l/n_2} \).

In fact, such an estimate would be biased. We can, however, provide bias-corrected estimates \( \tilde{\varphi}_l(u) \) of \( c_{l/n_2} \); we define

\[
\tilde{\varphi}_l(u) = -\frac{1}{u^2} \left( \log \left| \frac{\hat{\varphi}_l(u)}{\hat{\psi}_l(u)} \right| + \frac{\tilde{\sigma}_l^2(u)}{2} \right),
\]

where the bias-correction term

\[
\tilde{\sigma}_l^2(u) = \frac{1}{n_1} \left( \frac{1 + \hat{\varphi}_l(2u)}{2 \hat{\varphi}_l(u)^2} - 1 \right).
\]

We have thus defined an estimate \( \tilde{\varphi}_l(u) \) of the spot volatility. The advantage of this procedure over other such estimates is that it naturally accounts for the presence of jump activity: we will show that, for general semimartingales, \( \tilde{\varphi}_l(u) \) is an asymptotically-unbiased estimate of the quantity \( c_{l/n_2}(u) \), given by the adjusted volatility process

\[
c_l(u) = c_t + \frac{1}{n_0 u^2} \int_0^1 \int_R (1 - \cos(\sqrt{n_0} \Phi(w)ux)) \nu_t(dx) \, dw.
\]

The process \( c_t(u) \) thus includes both the volatility \( c_t \), and a term depending on the jump measure \( \nu_t \). As \( n \to \infty \), the term involving \( \nu_t \) vanishes;
however, when the jump activity $\beta$ is large, this term will not vanish fast enough to be negligible, and so we must consider it explicitly. Crucially in the time-changed model (3), we have that both terms enter $c_t(u)$ linearly, and so $c_t(u)$ is proportional to the rate process $r_t$.

In either model, since $r_t$ integrates to one, we may estimate it by normalising our estimates of $c_t(u)$. However, to estimate $r_t$ optimally we will not be able to use the preliminary estimates $\hat{c}_t(u)$ directly, as their variance is too large. First, we must smooth them, using standard tools from nonparametric regression.

While many such approaches are possible, in the following we will use a local polynomial estimate of $c_t(u)$, as described for example by Tsybakov (2009). To define our estimate, fix a non-negative kernel function $K : \mathbb{R} \to \mathbb{R}$, supported on $[-1, 1]$, and satisfying $\int_{\mathbb{R}} K(t) \, dt = 1$. Also fix an order $N \in \mathbb{N}$, and bandwidth $h > 0$. Then let $\tilde{c}_t(u)$ denote a local polynomial estimate of $c_t(u)$ of degree $N - 1$, using the observations $\hat{c}_t(u)$, kernel $K$, and bandwidth $h$.

In other words, let

$$\tilde{c}_t(u) = \sum_{l=0}^{n_2-1} W_{n,l}(t) \hat{c}_t(u),$$

where the weight functions $W_{n,l}(t)$ are given by

$$W_{n,l}(t) = \frac{1}{n_2h} K(\lambda_{n,l}(t))U(0)^T V_n(t)^{-1} U(\lambda_{n,l}(t)),$$

for the terms

$$\lambda_{n,l}(t) = \frac{1}{h} \left(t - \frac{l}{n_2}\right),$$

$$U(\lambda) = \begin{pmatrix} 1, \lambda, \ldots, \lambda^{N-1} \end{pmatrix}^T,$$

$$V_n(t) = \frac{1}{n_2h} \sum_{l=0}^{n_2-1} K(\lambda_{n,l}(t)) U(\lambda_{n,l}(t)) U(\lambda_{n,l}(t))^T.$$

The constant $N \in \mathbb{N}$ serves as an upper bound for the smoothness we expect of the volatility process $c_t$, and other processes related to $X_t$. We include here the case where $N$ is large, so that our estimate can match known nonparametric lower bounds for a wide range of smoothness.

In practice, however, we may believe that these processes are Itô semimartingales, as in most common financial models. We will see later that in this case it suffices to take $N = 1$; the above estimate then reduces to the Nadaraya-Watson kernel estimate, with weights $W_{n,l}(t)$ given by

$$W_{n,l}(t) = \frac{K(\lambda_{n,l}(t))}{\sum_{l=0}^{n_2-1} K(\lambda_{n,l}(t))}.$$
In either case, we then have an estimate $\tilde{c}_t(u)$ of the volatility $c_t$. To estimate the normalised volatility or rate process $r_t$, we likewise define the normalised estimate

$$\tilde{r}_t(u) = \frac{\tilde{c}_t(u)}{\sum_{m=0}^{n_2-1} \hat{c}_m(u)} = \sum_{l=0}^{n_2-1} W_{n,l}(t) \hat{r}_l(u),$$

where

$$\hat{r}_l(u) = \frac{\hat{c}_l(u)}{\sum_{m=0}^{n_2-1} \hat{c}_m(u)}.$$

In the following sections, we will prove results on the theoretical performance of our estimates $\tilde{c}_t(u)$ and $\tilde{r}_t(u)$. We first, however, briefly discuss their implementation. In particular, we note that the above estimates require the choice of a number of parameters: the kernel $K$, order $N$, frequency $u$, and bandwidths $h_1$, $h_2$ and $h$.

In general, good performance in nonparametric regression can be obtained with a range of kernels $K$; popular choices include the uniform, Epanechnikov and biweight kernels, given by Beta($k,k$) densities for $k = 1,2$ and 3 respectively. If we believe the volatility $c_t$, and other characteristic processes, are given by Itô semimartingales, then as noted above, we may also take $N = 1$.

The remaining parameters $u$, $h_1$, $h_2$, and $h$ are more important. We will show in the following that the variances of our estimates $\hat{c}_l(u)$ depend crucially on the choice of the frequency $u$, and bandwidths $h_1, h_2$. The correct choice of the bandwidth $h$ is likewise known to be crucial generally in nonparametric regression.

To select these parameters, we can borrow methods from nonparametric statistics. While many such approaches are available, we will briefly mention the heuristic of generalised cross-validation, a popular method for choosing the bandwidth $h$ in nonparametric regression (Golub et al., 1979).

The GCV criterion

$$GCV(u, h_1, h_2, h) = \frac{1}{n_2} \sum_{l=0}^{n_2-1} \frac{(\hat{r}_{l/n_2}(u) - \hat{r}_l(u))^2}{\left(\frac{1}{n_2} \sum_{l=0}^{n_2-1} W_{n,l}(l/n_2)\right)^2}$$

provides an estimate of the $L^2$ error in $\hat{r}_l(u)$. We can then choose $u$, $h_1$, $h_2$ and $h$ to minimise this criterion, using any standard global optimisation algorithm.

Simple tests on simulated data show that minimising this criterion provides sensible choices of the parameters, for both estimates $\tilde{c}_l(u)$ and $\tilde{r}_l(u)$. (We note that it is inadvisable to apply GCV to $\tilde{c}_l(u)$ directly, as the criterion then favours parameters which shrink the estimate to zero.)
We have thus described new estimates of the volatility $c_t$, and normalised volatility or rate process $r_t$; however, we have yet to consider their performance. In the following sections, we will show that, for suitable choices of the parameters, these estimates can obtain good rates of convergence over both the semimartingale model (1), and time-changed Lévy model (3).

3 Semimartingale and Lévy models

In this section, we will describe the assumptions we make on our data. Our assumptions will be satisfied by common models in both nonparametric statistics and quantitative finance. Under these assumptions, we may then proceed to show that our estimates $\tilde{c}_t(u)$ and $\tilde{r}_t(u)$ achieve good rates of convergence.

We first assume that the log-prices $X_t$ are generated under the general Itô semimartingale model (1), and our observations $Y_j$ come from the microstructure noise model (2), with fixed time horizon $T = 1$. For simplicity, we do not consider further other choices of $T$, but we note that similar results can also be proved when $T \to \infty$, $T/n \to 0$.

Our assumptions will then be stated in terms of a filtration $\mathcal{F}_t$, $t \in [0,1]$, with respect to which the semimartingale decomposition (1) and zero-mean condition of (2) hold. As in Jacod et al. (2009), to allow for the modelling of microstructure noise, we will not assume that the filtration $\mathcal{F}_t$ is right-continuous. Instead, we will require that the semimartingale decomposition (1) is also valid with respect to the filtration $\mathcal{F}^+_t = \bigcap_{s>t} \mathcal{F}_s$, and that the noises $\varepsilon_j$ are $\mathcal{F}^+_{j/n}$-measurable.

We then let $\mathcal{S}$ denote the class of probability measures $\mathbb{P}$ satisfying the above conditions, on some filtered measurable space $(\Omega, \mathcal{F}, \mathcal{F}_t)$. We will also make some further assumptions on the characteristics $b_t$, $c_t$ and $\nu_t$, and errors $\varepsilon_j$.

We begin by defining a smoothness assumption on $\mathcal{F}_t$-adapted processes. We will require in the following that the volatility $c_t$, and other characteristic processes of the log-prices $X_t$ and noises $\varepsilon_j$, satisfy this assumption with high probability.

**Definition 1.** Let $S \in [0,1]$ be an $\mathcal{F}_t$-stopping time, $\alpha \geq \frac{1}{2}$, $D > 0$, and set $\alpha_0 = 1 \land \alpha$. We define $\mathcal{T}^\alpha(D, S)$ to be the class of $\mathcal{F}_t$-adapted complex-valued processes $Z_t$, for which the stopped process $Z_{t\wedge S}$ satisfies:

(i) $|Z_{t\wedge S}| \leq D$, $t \in [0,1]$;

(ii) $\mathbb{E} \left[ |Z_{s\wedge S} - Z_{t\wedge S}|^2 \mid \mathcal{F}_t^+ \right] \leq D^2(s-t)^{2\alpha_0}$, $0 \leq t \leq s \leq 1$; and

(iii) if $\alpha > 1$, then letting $m$ denote the largest integer smaller than $\alpha$, $Z_{t\wedge S}$
has \(m\)-th real derivative \(Z_{t+}^{(m)}\) satisfying

\[
\mathbb{E}\left[\left|Z_{s+}^{(m)} - Z_{t+}^{(m)}\right|^2 \mid \mathcal{F}_t^+\right] \leq D^2(s-t)^{2(\alpha-m)}, \quad 0 \leq t \leq s \leq 1.
\]

The classes \(\mathcal{I}^\alpha(D,S)\) thus contain all processes \(Z_t\) which, when stopped by \(S\), are bounded and smooth in quadratic mean. We note that these classes describe a variety of processes. Firstly, the class \(\mathcal{I}^\alpha(D,1)\) contain all processes which are almost-surely \(\alpha\)-Hölder, with constant \(D\). More generally, the following lemma shows that the classes \(\mathcal{I}^{1/2}(D,S)\) can describe all càdlàg Itô semimartingales with locally bounded characteristics.

**Lemma 1.** Let \(Z_t\) be a càdlàg Itô semimartingale, having decomposition

\[
Z_t = Z_0 + \int_0^t b_{Z,s} \, ds + \int_0^t \sqrt{c_{Z,s}} \, dB_{Z,s} \\
+ \int_0^t \int_\mathbb{R} x (\mu_{Z}(dx,ds) - 1_{|x|<1} \nu_{Z,s}(dx)) \, ds
\]

with respect to both filtrations \(\mathcal{F}_t\) and \(\mathcal{F}_t^+\). Suppose the processes \(b_{Z,t}\) and \(c_{Z,t}\) are locally bounded, as are the processes \(\int_{|x|<R} x^2 \nu_{Z,t}(dx)\) for all \(R > 0\). Then for each \(D > 0\), there exists an event \(\Omega_0 \in \mathcal{F}_0\), and \(\mathcal{F}_t\)-stopping time \(S\), such that on \(\Omega_0\), \(Z_t \in \mathcal{I}^{1/2}(D,S)\), with \(\mathbb{P}(\Omega_0 \cap \{S = 1\}) \to 1\) as \(D \to \infty\).

We now define our assumptions on the observations \(Y_j\). Essentially, our results will be proved in models where with high probability, the drift \(b_t\) is bounded, and the stochastic volatility \(c_t\), jump process \(\nu_t\), and noise variance \(\sigma_t^2\) are bounded and smooth.

**Definition 2.** Let \(\alpha \geq \frac{1}{2}\), \(\beta \in [0,2]\), \(\gamma \in [0,1]\), and \(0 < C \leq D\). We define \(S_{\alpha,\beta}(C,D)\) to be the class of probability measures \(\mathbb{P} \in \mathcal{S}\) which satisfy the following conditions, on an event \(\Omega_0 \in \mathcal{F}_0\), and for a stopping time \(S \in [0,1]\), with \(\mathbb{P}(\Omega_0 \cap \{S = 1\}) \geq 1 - \gamma\).

(i) The noises \(\varepsilon_j\) have variance

\[
\mathbb{E}[\varepsilon_j^2 \mid \mathcal{F}_{j/n}] = \sigma_j^2/n,
\]

for a latent process \(\sigma_j^2 \in \mathcal{I}^\alpha(D,S)\); and have bounded fourth moment,

\[
\mathbb{E}[\varepsilon_j^4 \mid \mathcal{F}_{j/n}] \leq D^2, \quad j/n \leq 1.
\]

(ii) The drift process \(b_t\) is bounded,

\[
|b_t| \leq D, \quad t \in [0,S].
\]
(iii) The volatility process $c_t \in I^\alpha(D,S)$, and is also bounded below,
$$c_t \geq C, \quad t \in [0,S].$$

(iv) The jump activity is of index at most $\beta$,
$$\int_{\mathbb{R}} (1 \wedge |x|^\beta) \nu_t(dx) \leq D, \quad t \in [0,S];$$
and if $\beta > 1$, for any measurable function $f : \mathbb{R} \to \mathbb{C}$ with
$$|f(x)| \leq (1 \wedge x^2),$$
we have
$$\int_{\mathbb{R}} f(x) \nu_t(dx) \in I^\alpha(D,S).$$

Also define $S^{\alpha,\beta}(C,D) = S^{\alpha,\beta}_0(C,D)$, and let $S^{\alpha,\beta}$ denote the class of probability measures $\mathbb{P}$ which lie in some $S^{\alpha,\beta}_\gamma(C,D)$ for each $0 < C \leq D$, with $\gamma \to 0$ as $C \to 0$, $D \to \infty$.

In the following, our theoretical results will be first proved for the classes $S^{\alpha,\beta}(C,D)$, where the characteristics of the log-price $X_t$ and noises $\varepsilon_j$ are almost-surely bounded and smooth. These classes will be the most convenient for our analysis, and will allow us to draw comparisons with previous nonparametric results in the literature.

We note that these classes impose quite strong conditions on our process $X_t$; in particular, they require the volatility $c_t$ to be bounded away from zero. However, we will also generalise our results to the larger classes $S^{\alpha,\beta}$, which only require these conditions to hold locally; in particular, they only impose the weaker bound that $c_t > 0$ almost-surely.

The parameters $\alpha$ and $\beta$ govern two different smoothness properties of $X_t$. The parameter $\alpha$ measures the smoothness of the characteristics of $X_t$ and the $\varepsilon_j$ over time: if $X_t$ is a Lévy process, and the $\varepsilon_j$ have constant variance, the above conditions can hold for any value of $\alpha$. In contrast, the parameter $\beta$ governs the jump activity of $X_t$, and thus also the smoothness of its sample paths.

We note that in typical semimartingale models, the parameter $\alpha = \frac{1}{2}$; we include here the case $\alpha > \frac{1}{2}$ to allow for comparison with previous results in nonparametrics. We also note that, since the smoothness $\alpha$ is measured in square mean, the case $\alpha = \frac{1}{2}$ allows for jumps in the volatility, or in other characteristics; it thus allows the characteristics to depend smoothly on $X_t$, for any level $\beta$ of jump activity. More generally, Lemma 1 shows that the classes $S^{1/2,\beta}$ contain most common models for financial processes.

As some of our results will be specific to the time-changed model (3), we additionally define submodels describing this case. We note that our definition includes a choice of normalisation; as in (5), we assume the rate process $r_t$ integrates to one.
Definition 3. Let $\mathcal{T}$ denote the class of probability measures $\mathbb{P} \in \mathcal{S}$ satisfying (4), for a drift $b \in \mathbb{R}$, volatility $c > 0$, Lévy measure $\nu$, and rate process $r_t > 0$ given by (5). Also define the models $\mathcal{T}^\alpha(C, D) = \mathcal{S}^{\alpha, 2}(C, D) \cap \mathcal{T}$, and $\mathcal{T}^\alpha = \mathcal{S}^{\alpha, 2} \cap \mathcal{T}$.

This choice of normalisation is most convenient for our results, but we note that others are also possible; for example, we might prefer the deterministic normalisation $E[\int_0^T r_s ds] = T$. If we made an ergodicity assumption on the process $r_t$, as in Figueroa-López (2009), then for suitable $T \rightarrow \infty$, $T/n \rightarrow 0$, we would have that $\int_0^T r_s ds$ is close to $E[\int_0^T r_s ds]$. The two normalisations would then also be close, and our arguments would apply equally in either setting.

In the following, for simplicity, we will concentrate on Definition 3. We then have that in particular, the class $\mathcal{T}^{1/2}$ covers most common financial models for time-changed Lévy processes. With these definitions, we are now ready to give our results on the performance of our estimates.

4 Convergence results

In this section, we will show that our estimates $\hat{c}_l(u)$ and $\hat{r}_l(u)$ have good rates of convergence, in both the general semimartingale models $\mathcal{S}^{\alpha, \beta}$, and time-changed Lévy models $\mathcal{T}^\alpha$. In particular, we will establish that in $\mathcal{T}^\alpha$, the time-change $r_t$ can be recovered at minimax rates under arbitrary jump activity.

We first define some additional processes which will be relevant to our results. We set

$$\varphi_l(u) = \exp(-c_l(u)u^2)\psi_l(u), \quad \psi_l(u) = \exp(-\kappa\sigma^2_l u^2),$$

processes we will show describe the means of the estimates $\hat{\varphi}_l(u)$ and $\hat{\psi}_l(u)$.

We also set

$$\rho^2_l(u) = \frac{1}{2}(1 + \varphi_l(2u)) - \varphi^2_l(u), \quad \tau^2_l(u) = \rho^2_l(u)/n_1 \varphi^2_l(u),$$

processes we will show describe the variances of the estimates $\hat{\varphi}_l(u)$ and $\hat{c}_l(u)$.

We now begin with a result on the accuracy of the preliminary estimates $\widehat{c}_l(u)$. At this stage, our results will be proved solely in the bounded semimartingale model $\mathcal{S}^{\alpha, \beta}(C, D)$; we will return later to the consequences for our other models.

We can establish that, on events with high probability, our preliminary estimates $\widehat{c}_l(u)$ have asymptotic mean $c_{l/n_2}(u)$, and variance $\tau^2_{l/n_2}(u)$. We can further show that the errors in these statements are of order $n^{-\alpha_1}$ and $n^{-\alpha_2}$ respectively, where the rates

$$\alpha_1 = \frac{1}{4} \land \frac{3\alpha}{8}, \quad \alpha_2 = \frac{\alpha_1}{2} + \frac{1}{16}.$$
Theorem 1. Fix $u, h_1, h_2 > 0$, $\alpha \geq \frac{1}{2}$, $\beta \in [0, 2]$, $0 < C \leq D$, and suppose $\mathbb{P} \in S^{\alpha,\beta}(C, D)$. Then the local volatility estimates $\hat{c}_l(u)$ are $\mathcal{F}_{(l+1)/n_2}$-measurable, and we have events $E_l \in \mathcal{F}_{(l+1)/n_2}$, satisfying
\[
\mathbb{P}(E_l^c \mid \mathcal{F}_{l/n_2}) \leq \exp(-A n^{1/8})
\]
for a constant $A > 0$, on which
\[
\begin{align*}
\mathbb{E}[(\hat{c}_l(u) - c_{l/n_2}(u))1(E_l) \mid \mathcal{F}_{l/n_2}] &= O(n^{-\alpha_1}), \\
\mathbb{E}[(\hat{c}_l(u) - c_{l/n_2}(u))^2 1(E_l) \mid \mathcal{F}_{l/n_2}] &= \tau_{l/n_2}^2(u) + O(n^{-\alpha_2}).
\end{align*}
\]
Furthermore, these results are uniform over $l = 0, \ldots, n_2 - 1$, and $\mathbb{P} \in S^{\alpha,\beta}(C, D)$.

We thus have that the estimates $\hat{c}_l(u)$ behave roughly like $n^{3/8}$ observations of the adjusted volatility process $c_l(u)$, under errors with variance $n^{-1/8}$. In other words, we obtain an accuracy like observing the process $c_l(u)$ under $n^{-1/4}$ white noise. While our estimates $\hat{c}_l(u)$ also include an additional bias term, and are accurate only on a set of high probability, we will nonetheless see that they are good enough to accurately recover the volatility $c_l$ or time-change $r_t$.

We now establish that our regression estimate $\hat{c}_l(u)$ is a good estimate of the adjusted volatility $c_l(u)$. We will measure the accuracy of our estimates both pointwise, and in the $L^2$-norm,
\[
\|f\|_2^2 = \int_0^1 f(t)^2 \, dt.
\]
In these metrics, we will show that $c_l(u)$ can be recovered at the rate $n^{-\alpha_3}$, where
\[
\alpha_3 = \frac{\alpha}{2(2\alpha + 1)}
\]
is the standard minimax rate for recovering a function of smoothness $\alpha$ under $n^{-1/4}$ white noise.

Theorem 2. Fix a kernel $K$ as in Section 2, $N \in \mathbb{N}$, $u \in \mathbb{R}$, $h_1, h_2 > 0$, $\alpha \in [\frac{1}{2}, N]$, $\beta \in [0, 2]$, $0 < C \leq D$, let $h \sim n^{-1/(2\alpha+1)}$, and suppose $\mathbb{P} \in S^{\alpha,\beta}(C, D)$. We then have an event $E$, satisfying
\[
\mathbb{P}(E^c \mid \mathcal{F}_0) \leq \exp(-A n^{1/8})
\]
for a constant $A > 0$, on which
\[
\begin{align*}
\mathbb{E}[\|\hat{c}_l(u) - c_l(u)\|_2^2 1(E) \mid \mathcal{F}_0]^{1/2} &= O(n^{-\alpha_3}),
\end{align*}
\]
uniformly in $t \in [0, 1]$, and
\[
\begin{align*}
\mathbb{E}[\|\hat{c}(u) - c(u)\|_2^2 1(E) \mid \mathcal{F}_0]^{1/2} &= O(n^{-\alpha_3}).
\end{align*}
\]
Furthermore, these results are uniform over $\mathbb{P} \in S^{\alpha,\beta}(C, D)$. 
We thus have that the regression estimates $\hat{c}_t(u)$ accurately recover $c_t(u)$, in the model $S^{\alpha,\beta}(C, D)$. It remains to deduce consequences for the volatility $c_t$ and time-change $r_t$, in the more general models $S^{\alpha,\beta}$ and $T^{\alpha}$. In $S^{\alpha,\beta}$, we will obtain the rate $n^{-\alpha_4}$, where
\[ \alpha_4 = \alpha_3 \wedge \frac{2 - \beta}{4} \]
depends also on the jump activity $\beta$ of the log-price $X_t$. When estimating $r_t$ in $T^{\alpha}$, however, we will retain the convergence rate $n^{-\alpha_3}$, even under arbitrary jump activity.

**Corollary 1.** Let the parameters $K, N, u, h_1, h_2, \alpha, \beta, C, D$ and $h$ be as in Theorem 2.

(i) If $P \in S^{\alpha,\beta}$, the estimates $\hat{c}_t(u)$ and $\hat{r}_t(u)$ satisfy
\[ |\hat{c}_t(u) - c_t|, |\hat{r}_t(u) - r_t| = O_p(n^{-\alpha_4}), \]
uniformly in $t \in [0, 1]$, and
\[ \|\hat{c}(u) - c\|_2, \|\hat{r}(u) - r\|_2 = O_p(n^{-\alpha_4}). \]
Furthermore, these results are uniform over $P \in S^{\alpha,\beta}(C, D)$.

(ii) If also $P \in T$, the results for $\hat{r}_t(u)$ hold with improved convergence rate $n^{-\alpha_3}$.

We note that convergence does depend on the choice of parameters $K, N, u, h_1, h_2$ and $h$, and in particular requires the bandwidth $h$ to be chosen as in Theorem 2. Adaptive results in this setting are possible, for example applying Lepski’s method to choose $h$, and using Azuma’s inequality to control the deviations in $\hat{c}_l(u)$ and $\hat{\psi}_l(u)$ (Lepski et al., 1997). For simplicity, however, we will treat these parameters as fixed, noting that they can be chosen heuristically as in Section 2.

For the time-changed Lévy model $T^{\alpha}$, as a simple consequence of results in Munk and Schmidt-Hieber (2010a), we can further show that our rates are optimal. We can likewise provide a partially matching lower bound for the general semimartingale model $S^{\alpha,\beta}$.

**Theorem 3.** Let $\alpha > \frac{1}{2}, \beta \in [0, 2], \text{ and } 0 < C < D$.

(i) No estimate $c^*_t$ of $c_t$ can satisfy
\[ \|c^*_t - c\|_2 = o_p(n^{-\alpha_3}), \]
uniformly over $P \in S^{\alpha,\beta}(C, D) \cap T$, or
\[ |c^*_t - c_t| = o_p(n^{-\alpha_3}), \]
uniformly over $t \in [0, 1]$ and $P \in S^{\alpha,\beta}(C, D) \cap T$.  

15
The same results hold for any estimate $r_t^*$ of $r_t$.

In the general semimartingale model $S^{\alpha,\beta}$, if $\beta$ is large, we have $\alpha_3 > \alpha_4$, and matching lower bounds are more difficult to establish. We note, however, that our estimates $\tilde{c}_t(u)$ and $\tilde{r}_t(u)$ already obtain rates as good as those implied by previous work under noise. Furthermore, the recent paper of Jacod and Reiß (2014) on the noiseless problem suggests that the rate $n^{-\alpha_4}$ is indeed optimal, up to log factors.

It may at first be surprising that the results for $r_t$ in the time-changed model $T^\alpha$ are better than in the general semimartingale model $S^{\alpha,\beta}$, when the jump activity is large. However, we know that the difficulty in estimating the volatility $c_t$ in $S^{\alpha,\beta}$ comes primarily from distinguishing $c_t$ and $\nu_t$. We obtain improved convergence rates in $T^\alpha$ because in this model, we can estimate the rate process $r_t$ without having to separate $c_t$ and $\nu_t$.

We have thus shown that our estimate $\tilde{r}_t(u)$ can recover the time change in a noisy Lévy model at the minimax rate, equivalent to observing $r_t$ under $n^{-1/4}$ white noise. It can do so without knowledge of the distribution of the Lévy process, and under arbitrary jump activity.

Furthermore, in the general semimartingale setting, where the Lévy assumption may be violated, $\tilde{r}_t(u)$ remains a valid estimate of the normalised volatility. In this setting, we again achieve good rates, governed either by the noise level of $n^{-1/4}$, or by a bias due to jump activity, common to all volatility estimates.

5 Proofs

We now give proofs of our results. We prove results on the preliminary estimates $\hat{c}_t(u)$ in Section 5.1, and results on convergence rates in Section 5.2. Technical proofs are given in Section 5.3.

5.1 Proofs on preliminary estimates

We first prove Theorem 1, our result bounding the error in our preliminary estimates $\tilde{c}_t(u)$. Our proof will use a series of lemmas, controlling the behaviour of the various components of $\tilde{c}_t(u)$. We begin by stating some technical lemmas; proofs are given in Section 5.3.

Lemma 2. In the setting of Theorem 1, fix $u \in \mathbb{R}$, and let $\xi_t$ denote (i) $c_t(u)$, (ii) $\varphi_t(u)$, or (iii) $\psi_t(u)$. In each case, for $n \in \mathbb{N}$, $0 \leq t \leq s \leq 1$, we have

$$\mathbb{E}[(\xi_s(u) - \xi_t(u))^2 | \mathcal{F}_t] = O((s-t)^{2\alpha_0} + n^{-1/2}).$$

Furthermore, we have (iv) $c_t(u) \leq 3D$, almost surely.
Lemma 3. In the setting of Theorem 1, for \( k = 0, \ldots, n_0 - 1 \), and \( u \in \mathbb{R} \), we have

\[
\mathbb{E}[\cos(u \hat{X}_k) \mid \mathcal{F}_{k/n_0}] = \varphi_{k/n_0}(u) + O(n^{-1/4}),
\]

\[
\mathbb{V}[\cos(u \hat{X}_k) \mid \mathcal{F}_{k/n_0}] = \rho_{k/n_0}^2(u) + O(n^{-1/4}).
\]

Lemma 4. In the setting of Theorem 1, for \( k = 0, \ldots, n_0 - 1 \), and \( u \in \mathbb{R} \), we have

\[
\mathbb{E}[\exp(-\kappa \hat{\sigma}_k^2 u^2) \mid \mathcal{F}_{k/n_0}] = \psi_{k/n_0}(u) + O(n^{-1/4}),
\]

\[
\mathbb{V}[\exp(-\kappa \hat{\sigma}_k^2 u^2) \mid \mathcal{F}_{k/n_0}] = O(n^{-1/4}).
\]

We are now in a position to describe the behaviour of the estimates \( \hat{\varphi}_l(u) \) and \( \hat{\psi}_l(u) \). First, we will define the event \( E_l \) mentioned in the statement of Theorem 1. We set

\[
E_l = \{ \hat{\varphi}_l(u) \geq \zeta(u) \} \cap \{ \hat{\psi}_l(u) \geq \zeta(u) \},
\]

where the constant \( \zeta(u) = \frac{1}{2} \exp(- (\kappa + 3) D u^2) \). We then have the following result.

Lemma 5. In the setting of Theorem 1, for \( l = 0, \ldots, n_2 - 1 \), we have:

(i) \( \mathbb{E}[\hat{\varphi}_l(u) - \varphi_{l/n_2}(u) \mid \mathcal{F}_{l/n_2}] = O(n^{-\alpha_1}) \);

(ii) \( \mathbb{E}[\hat{\psi}_l(u) - \psi_{l/n_2}(u) \mid \mathcal{F}_{l/n_2}] = O(n^{-\alpha_1}) \);

(iii) \( \mathbb{E}[(\hat{\varphi}_l(u) - \varphi_{l/n_2}(u))^2 \mid \mathcal{F}_{l/n_2}] = \rho_{l/n_2}^2(u)/n_1 + O(n^{-\alpha_1}) \);

(iv) \( \mathbb{E}[(\hat{\psi}_l(u) - \psi_{l/n_2}(u))^2 \mid \mathcal{F}_{l/n_2}] = O(n^{-1/4}) \);

(v) for \( p = 3, 4 \), \( \mathbb{E}[(\hat{\varphi}_l(u) - \varphi_{l/n_2}(u))^p \mid \mathcal{F}_{l/n_2}] = O(n^{-\alpha_1}) \); and

(vi) \( \mathbb{P}(E_l^c \mid \mathcal{F}_{l/n_2}) \leq \exp(-A n^{1/8}) \), for a constant \( A > 0 \).

Proof. We first note that

\[
\hat{\varphi}_l(u) - \varphi_{l/n_2}(u) = \frac{1}{n_1} \sum_{k \in K_l} Z_{\delta,k},
\]

where the random variables

\[
Z_{\delta,k} = \cos(u \hat{X}_k) - \varphi_{l/n_2}(u), \quad k \in K_l;
\]

we will begin by proving some facts about the \( Z_{\delta,k} \). We have \( |Z_{\delta,k}| \leq 2 \), and

\[
\mathbb{E}[\mathbb{E}[Z_{\delta,k} \mid \mathcal{F}_{k/n_0}]^2 \mid \mathcal{F}_{l/n_2}] = \mathbb{E}[(\varphi_{k/n_0}(u) - \varphi_{l/n_2}(u) + O(n^{-1/4}))^2 \mid \mathcal{F}_{l/n_2}],
\]

17
using **Lemma 3**,  
\[
= O(1) \mathbb{E}[(\varphi_{k/n_0}(u) - \varphi_{l/n_2}(u))^2 \mid \mathcal{F}_{l/n_2}] + O(n^{-1/2}) \\
= O(n^{-2\alpha_1}), \tag{6}
\]
using **Lemma 2**(ii).

We also have  
\[
\mathbb{E}[Z_{\delta,k}^2 \mid \mathcal{F}_k/n_0] = \mathbb{V}ar[Z_{\delta,k}^2 \mid \mathcal{F}_k/n_0] + \mathbb{E}[Z_{\delta,k}^2 \mid \mathcal{F}_k/n_0]^2 \\
= \rho_{k/n_0}^2(u) + (\varphi_{k/n_0}(u) - \varphi_{l/n_2}(u))^2 + O(n^{-1/4}),
\]
using **Lemma 3**, so  
\[
\mathbb{E}[(\mathbb{E}[Z_{\delta,k}^2 \mid \mathcal{F}_k/n_0] - \rho_{l/n_2}^2(u))^2 \mid \mathcal{F}_{l/n_2}] \\
= O(1) \mathbb{E}[(\rho_{k/n_0}^2(u) - \rho_{l/n_2}^2(u))^2 + (\varphi_{k/n_0}(u) - \varphi_{l/n_2}(u))^2 \mid \mathcal{F}_{l/n_2}] \\
+ O(n^{-1/2}) \\
= O(n^{-2\alpha_1}). \tag{7}
\]
using **Lemma 2**(ii). We may now prove the claims of the theorem.

(i) We have  
\[
\mathbb{E}[[\hat{\varphi}_l(u) - \varphi_{l/n_2}(u) \mid \mathcal{F}_{l/n_2}] = \frac{1}{n_1} \sum_{k \in K_l} \mathbb{E}[Z_{\delta,k} \mid \mathcal{F}_{l/n_2}] \\
= O(1) \sum_{k \in K_l} \mathbb{E}[[\mathbb{E}[Z_{\delta,k} \mid \mathcal{F}_k/n_0]] | \mathcal{F}_{l/n_2}] \\
= O(n^{-\alpha_1}),
\]
using (6).

(ii) The result follows similarly to (i), using **Lemma 2**(iii) and **Lemma 4**.

(iii) We have  
\[
\mathbb{E}[Z_{\delta,k}^2 \mid \mathcal{F}_{l/n_2}] = \rho_{l/n_2}^2(u) + \mathbb{E}[(\mathbb{E}[Z_{\delta,k}^2 \mid \mathcal{F}_k/n_0] - \rho_{l/n_2}^2(u) \mid \mathcal{F}_{l/n_2}) \\
= \rho_{l/n_2}^2(u) + O(n^{-\alpha_1}),
\]
using (7). Likewise, for \(k, k_1 \in K_l, k > k_1\), we have  
\[
\mathbb{E}[Z_{\delta,k}Z_{\delta,k_1} \mid \mathcal{F}_{l/n_2}] = \mathbb{E}[\mathbb{E}[Z_{\delta,k} \mid \mathcal{F}_k/n_0]Z_{\delta,k_1} \mid \mathcal{F}_{l/n_2}] \\
= O(1) \mathbb{E}[(\mathbb{E}[Z_{\delta,k} \mid \mathcal{F}_k/n_0] \mid \mathcal{F}_{l/n_2}) \\
= O(n^{-\alpha_1}),
\]

using (6). We deduce that
\[
\mathbb{E}[(\hat{\varphi}_l(u) - \varphi_{l/n_2}(u))^2 \mid \mathcal{F}_{l/n_2}]
\]
\[
= \mathbb{E}\left[ \frac{1}{n_1} \sum_{k \in K_l} Z_{\delta,k}^2 + \frac{2}{n_1} \sum_{k,k_1 \in K_l, k > k_1} Z_{\delta,k} Z_{\delta,k_1} \mid \mathcal{F}_{l/n_2} \right]
\]
\[
= \rho_{l/n_2}^2 (u)/n_1 + O(n^{-\alpha_1}).
\]

(iv) For \( k \in K_l \), by a similar argument, we have
\[
\mathbb{E}[(\exp(-\kappa \delta^2 k u^2) - \psi_{l/n_2}(u))^2 \mid \mathcal{F}_{l/n_2}] = O(n^{-1/4}),
\]
using Lemma 4. The result follows.

(v) We first consider the case \( p = 3 \). For \( k \in K_l \), we have
\[
\mathbb{E}[Z_{\delta,k}^3 \mid \mathcal{F}_{l/n_2}] = O(1),
\]
and for \( k, k_1 \in K_l, k > k_1 \),
\[
\mathbb{E}[Z_{\delta,k}^2 Z_{\delta,k_1} \mid \mathcal{F}_{l/n_2}]
\]
\[
= \mathbb{E}[\mathbb{E}[Z_{\delta,k}^2 \mid \mathcal{F}_{k/n_0}] Z_{\delta,k_1} \mid \mathcal{F}_{l/n_2}]
\]
\[
= \rho_{l/n_2}^2 (u) \mathbb{E}[Z_{\delta,k_1} \mid \mathcal{F}_{l/n_2}]
\]
\[
+ O(1) \mathbb{E}[\mathbb{E}[Z_{\delta,k}^2 \mid \mathcal{F}_{k/n_0}] \mid \mathcal{F}_{l/n_2}]
\]
\[
= O(n^{-\alpha_1}),
\]
using (6) and (7).

Similarly, for \( k, k_1, k_2 \in K_l, k > k_1, k_2 \), we have
\[
\mathbb{E}[Z_{\delta,k} Z_{\delta,k_1} Z_{\delta,k_2} \mid \mathcal{F}_{l/n_2}]
\]
\[
= \mathbb{E}[\mathbb{E}[Z_{\delta,k} \mid \mathcal{F}_{k/n_0}] Z_{\delta,k_1} Z_{\delta,k_2} \mid \mathcal{F}_{l/n_2}]
\]
\[
= O(1) \mathbb{E}[\mathbb{E}[Z_{\delta,k} \mid \mathcal{F}_{k/n_0}] \mid \mathcal{F}_{l/n_2}]
\]
\[
= O(n^{-\alpha_1}),
\]
using (6). We deduce that
\[
E[(\hat{\phi}_l(u) - \phi_{l/n_2}(u))^3 \mid \mathcal{F}_{l/n_2}] \\
= E \left[ \left( \frac{1}{n_1} \sum_{k \in K_l} Z_{\delta,k} \right)^3 \mid \mathcal{F}_{l/n_2} \right] \\
= \frac{O(1)}{n_1^3} E \left[ \sum_{k \in K_l} Z_{\delta,k}^3 + \sum_{k,k_1 \in K_l, k > k_1} Z_{\delta,k}^2 Z_{\delta,k_1} \right. \\
\left. + \sum_{k,k_1,k_2 \in K_l, k > k_1, k_2} Z_{\delta,k} Z_{\delta,k_1} Z_{\delta,k_2} \mid \mathcal{F}_{l/n_2} \right] \\
= O(n^{-\alpha_1}).
\]

For \( p = 4 \), by a similar argument, we have that for \( k,k_1,k_2,k_3 \in K_l, k > k_1,k_2,k_3 \),
\[
E[Z_{\delta,k}^4 \mid \mathcal{F}_{l/n_2}], E[Z_{\delta,k}^3 Z_{\delta,k_1} \mid \mathcal{F}_{l/n_2}], E[Z_{\delta,k}^2 Z_{\delta,k_1}^2 \mid \mathcal{F}_{l/n_2}] = O(1),
\]
\[
E[Z_{\delta,k} Z_{\delta,k_1} Z_{\delta,k_2} Z_{\delta,k_3} \mid \mathcal{F}_{l/n_2}] = O(n^{-\alpha_1}),
\]
and if \( k_1 > k_2 \),
\[
E[Z_{\delta,k}^2 Z_{\delta,k_1} Z_{\delta,k_2} \mid \mathcal{F}_{l/n_2}] = O(n^{-\alpha_1}).
\]

We thus obtain that
\[
E[(\hat{\phi}_l(u) - \phi_{l/n_2}(u))^4 \mid \mathcal{F}_{l/n_2}] \\
= E \left[ \left( \frac{1}{n_1} \sum_{k \in K_l} Z_{\delta,k} \right)^4 \mid \mathcal{F}_{l/n_2} \right] \\
= \frac{O(1)}{n_1^4} E \left[ \sum_{k \in K_l} Z_{\delta,k}^4 + \sum_{k,k_1 \in K_l, k > k_1} Z_{\delta,k}^3 Z_{\delta,k_1} \right. \\
\left. + \sum_{k,k_1,k_2 \in K_l, k > k_1, k_2} Z_{\delta,k}^2 Z_{\delta,k_1} Z_{\delta,k_2} + \sum_{k,k_1,k_2,k_3 \in K_l, k > k_1, k_2 > k_2} Z_{\delta,k} Z_{\delta,k_1} Z_{\delta,k_2} Z_{\delta,k_3} \mid \mathcal{F}_{l/n_2} \right] \\
= O(n^{-\alpha_1}).
\]
(vi) We first note that the quantity 

\[ \varphi_l(u) = \frac{1}{n_l} \sum_{k \in K_l} E[\cos(u \hat{X}_k) \mid \mathcal{F}_{k/n_0}] \]

\[ = \frac{1}{n_l} \sum_{k \in K_l} \varphi_{k/n_0}(u) + O(n^{-1/4}) \]

using Lemma 3,

\[ \geq 2 \zeta(u) + O(n^{-1/4}), \]

using Lemma 2(iv). Then using Azuma’s inequality, we have

\[ \Pr(\hat{\varphi}_l(u) \leq \zeta(u) \mid \mathcal{F}_{l/n_2}) \]

\[ \leq \Pr(\hat{\varphi}_l(u) - \varphi_l(u) \leq -\zeta(u) + O(n^{-1/4}) \mid \mathcal{F}_{l/n_2}) \]

\[ \leq \exp(-A'n^{1/8}), \]

for a constant \( A' > 0 \). By a similar argument, we also have

\[ \Pr(\hat{\psi}_l(u) \leq \zeta(u) \mid \mathcal{F}_{l/n_2}) \leq \exp(-A''n^{1/8}), \]

for a constant \( A'' > 0 \). The result follows.

Finally, we may prove Theorem 1.

**Proof of Theorem 1.** From the definitions, we have that the estimates \( \hat{\varphi}_l(u) \) are \( \mathcal{F}_{(l+1)/n_2} \)-measurable, and the events \( E_l \in \mathcal{F}_{(l+1)/n_2} \). The bound on the probability of \( E_l^c \) likewise follows directly from Lemma 5(vi).

It thus remains to prove the bounds on the mean and variance of \( \hat{\varphi}_l(u) \). We will decompose the error in \( \hat{\varphi}_l(u) \) into three terms, controlling the error in each of \( \log(\hat{\varphi}_l(u)) \), \( \log(\hat{\psi}_l(u)) \), and \( \hat{\tau}^2_l(u) \).

We first consider \( \log(\hat{\varphi}_l(u)) \), and define the random variable

\[ Z_{\varphi,l} = \frac{\hat{\varphi}_l(u)}{\varphi_{l/n_2}(u)} - 1. \]

We then have that

\[ \log(\hat{\varphi}_l(u)) - \log(\varphi_{l/n_2}(u)))1(E_l) \]

\[ = \log(1 + Z_{\varphi,l})1(E_l) \]

\[ = (Z_{\varphi,l} - \frac{1}{2}Z_{\varphi,l}^2 + \frac{1}{3}Z_{\varphi,l}^3 + O(Z_{\varphi,l}^4))1(E_l), \]

using Taylor’s theorem, since on \( E_l \),

\[ 1 + Z_{\varphi,l} \geq \frac{\zeta(u)}{\varphi_{l/n_2}(u)} \geq \zeta(u) > 0. \]  

(8)
To bound the error in $\log(\hat{\phi}_l(u))$, we will now take expectations of the $Z_{\phi,l}$ terms. We have that

$$E[Z_{\phi,l}1(E_l) | F_{l/n_2}] = E[\hat{\phi}_l(u) - 1 | F_{l/n_2}] + O(E_{l}^{c} | F_{l/n_2})$$

since $\hat{\phi}_l(u)$ is bounded, and $\varphi_{l/n_2}(u) \geq 2\zeta(u) > 0$, 

$$= O(n^{-\alpha_1}),$$

using Lemma 5(i) and (vi). Similarly, we also have

$$E[Z_{\phi,l}^21(E_l) | F_{l/n_2}] = \tau_{l/n_2}^2(u) + O(n^{-\alpha_1}),$$

$$E[Z_{\phi,l}^31(E_l) | F_{l/n_2}] = O(n^{-\alpha_1}),$$

$$E[Z_{\phi,l}^41(E_l) | F_{l/n_2}] = O(n^{-\alpha_1}),$$

using Lemma 5(i), (iii), (v) and (vi); as a consequence, we deduce

$$E[|Z_{\phi,l}|^31(E_l) | F_{l/n_2}] \leq E[Z_{\phi,l}^21(E_l) | F_{l/n_2}]^{1/2}E[Z_{\phi,l}^31(E_l) | F_{l/n_2}]^{1/2} = O(n^{-\alpha_2}),$$

using Cauchy-Schwarz.

We can now bound the error in $\log(\hat{\phi}_l(u))$. We conclude that

$$E[(\log(\hat{\phi}_l(u)) - \log(\varphi_{l/n_2}(u)))1(E_l) | F_{l/n_2}]$$

$$= E[(Z_{\phi,l} - \frac{1}{4}Z_{\phi,l}^2 + \frac{1}{3}Z_{\phi,l}^3 + O(Z_{\phi,l}^4))1(E_l) | F_{l/n_2}]$$

$$= -\frac{1}{2}\tau_{l/n_2}^2(u) + O(n^{-\alpha_1}),$$

and similarly,

$$E[(\log(\hat{\psi}_l(u)) - \log(\psi_{l/n_2}(u)))21(E_l) | F_{l/n_2}]$$

$$= E[(Z_{\psi,l} + O(Z_{\psi,l}^2))^21(E_l) | F_{l/n_2}]$$

$$= E[(Z_{\psi,l} + O(Z_{\psi,l}^3 + Z_{\psi,l}^4))1(E_l) | F_{l/n_2}]$$

$$= \tau_{l/n_2}^2(u) + O(n^{-\alpha_2}).$$

We next consider the error in $\log(\hat{\psi}_l(u))$. By a similar argument, we can obtain that

$$E[(\log(\hat{\psi}_l(u)) - \log(\psi_{l/n_2}(u)))1(E_l) | F_{l/n_2}] = O(n^{-\alpha_1}),$$

$$E[(\log(\hat{\psi}_l(u)) - \log(\psi_{l/n_2}(u)))21(E_l) | F_{l/n_2}] = O(n^{-1/4}),$$

using Lemma 5(ii), (iv) and (vi).
Finally, we prove bounds on \( \hat{\tau}_l^2(u) \), defining the random variable
\[
Z_{\tau,l} = \hat{\varphi}_l(2u) - \varphi_{l/n_2}(2u).
\]
We then have
\[
(\hat{\tau}_l^2(u) - \tau_{l/n_2}^2(u))1(E_l) = \frac{1}{n_1} \left( \frac{1 + \varphi_{l/n_2}(2u) + Z_{\tau,l} - 1/2}{\varphi_{l/n_2}^2(u)(1 + \varphi_{l/n_2}(2u))} \right)1(E_l)
\]
\[
= \frac{1}{n_1} \left( \frac{-2(1 + \varphi_{l/n_2}(2u)Z_{\varphi,l} + Z_{\tau,l} + \varphi_{l/n_2}(2u))}{\varphi_{l/n_2}^2(u)} \right)1(E_l),
\]
using (8), and that \( \varphi_l(u) \) is bounded below.
Using Lemma 5(i), (iii) and (vi) as above, we also have that
\[
\mathbb{E}[Z_{\tau,l}1(E_l) | \mathcal{F}_{l/n_2}] = O(n^{-\alpha_1}),
\]
\[
\mathbb{E}[Z_{\tau,l}^21(E_l) | \mathcal{F}_{l/n_2}] = O(n^{-1/8}).
\]
We therefore conclude that
\[
\mathbb{E}[(\hat{\tau}_l^2(u) - \tau_{l/n_2}^2(u))1(E_l) | \mathcal{F}_{l/n_2}]
\]
\[
= O(n^{-1/8})\mathbb{E}[Z_{\varphi,l} + Z_{\tau,l} + Z_{\varphi,l}^2 + |Z_{\varphi,l}||Z_{\tau,l}| | \mathcal{F}_{l/n_2}],
\]
since \( \varphi_l(u) \) is bounded below,
\[
= O(n^{-1/4}),
\]
using Cauchy-Schwarz. Likewise,
\[
\mathbb{E}[(\hat{\tau}_l^2(u) - \tau_{l/n_2}^2(u))^21(E_l) | \mathcal{F}_{l/n_2}]
\]
\[
= O(n^{-1/4})\mathbb{E}[Z_{\varphi,l}^2 + Z_{\tau,l}^2 | \mathcal{F}_{l/n_2}]
\]
\[
= O(n^{-3/8}).
\]
We have thus bounded the error in each of \( \log(\hat{\varphi}_l(u)) \), \( \log(\hat{\psi}_l(u)) \), and \( \hat{\tau}_l^2(u) \). Combining these results, we deduce that
\[
\mathbb{E}[(\hat{\tau}_l(u) - \tau_{l/n_2}(u))1(E_l) | \mathcal{F}_{l/n_2}]
\]
\[
= O(1)\mathbb{E}[(\log(\hat{\varphi}_l(u)) - \log(\varphi_{l/n_2}(u))) + \frac{1}{2}\tau_{l/n_2}^2(u)
\]
\[
- \log(\hat{\psi}_l(u)) - \log(\psi_{l/n_2}(u))) + \frac{1}{2}((\hat{\tau}_l^2(u) - \tau_{l/n_2}^2(u)))1(E_l) | \mathcal{F}_{l/n_2}]
\]
\[
= O(n^{-\alpha_1}),
\]
and
\[
\mathbb{E}[(\hat{\tau}_l(u) - \tau_{l/n_2}(u))^21(E_l) | \mathcal{F}_{l/n_2}]
\]
\[
= O(1)\mathbb{E}[(\log(\hat{\varphi}_l(u)) - \log(\varphi_{l/n_2}(u)))^2 + (\log(\hat{\psi}_l(u)) - \log(\psi_{l/n_2}(u)))^2
\]
\[
+ ((\hat{\tau}_l^2 - \tau_{l/n_2}^2(u))^2 + O(n^{-1/4}))1(E_l) | \mathcal{F}_{l/n_2}]
\]
\[
= \tau_{l/n_2}^2(u) + O(n^{-\alpha_2}).
\]
Finally, it can be checked that these results are uniform over \( l = 0, \ldots, n_2 - 1 \), and \( P \in S^{\alpha,\beta}(C, D) \).

### 5.2 Proofs of convergence rates

We next prove Theorem 2, our result on the performance of our regression estimate \( \tilde{c}_t(u) \). Our argument follows from Tsybakov (2009), taking care to account for the extra error terms in the statement of Theorem 1, and the stochastic nature of the target \( c_t(u) \).

**Proof of Theorem 2.** To begin, we will state some facts about local polynomial regression, as given in the proof of Theorem 1.7 in Tsybakov (2009).

Since the design points \( l/n_2 \) are uniform, we have that for large \( n \), the matrices \( V_n(t) \) are invertible, and the weight functions \( W_{n,l}(t) \) well-defined. Furthermore, the weights \( W_{n,l}(t) \) satisfy:

\[
|W_{n,l}(t)| = O\left(\frac{1}{n_2 h}\right) 1\left(\left|\frac{t - l}{n_2}\right| \leq h\right),
\]

uniformly in \( l = 0, \ldots, n_2 - 1; \) \( n_2 - 1 \sum_{l=0}^{n_2-1} |W_{n,l}(t)| = O(1); \)

and

\[
\sum_{l=0}^{n_2-1} \left(\frac{t - l}{n_2}\right)^p W_{n,l}(t) = \begin{cases} 1, & p = 0, \\ 0, & p = 1, \ldots, N - 1. \end{cases}
\]

We now prove the results on our estimate \( \tilde{c}_t(u) \). We must first define the high-probability event \( E \) given in the statement of the theorem. We let \( E_{a,b} = \bigcap_{l=a}^{b-1} E_l \), and set \( E = E_{0,n_2} \). We then note that from Theorem 1, we have

\[
P(E^c \mid \mathcal{F}_0) \leq \sum_{l=0}^{n_2-1} \mathbb{E}[P(E^c_l \mid \mathcal{F}_{l/n_2}) \mid \mathcal{F}_0]
\]

\[
= O(n^{3/8}) \exp(-A n^{1/8})
\]

\[
\leq \exp(-A' n^{1/8}),
\]

for constants \( A, A' > 0 \). Similarly, for \( l = 0, \ldots, n_2 - 1, k \geq l \), we have

\[
P(E^c_{k,n_2} \mid \mathcal{F}_{l/n_2}) \leq \exp(-A' n^{1/8}).
\]

We next split the estimates \( \tilde{c}_t(u) \) into bias and variance parts. Let

\[
\tilde{c}_t(u) = c_{l/n_2}(u) + \tilde{c}_{1,l}(u) + \tilde{c}_{2,l}(u),
\]
where the bias term
\[
\hat{c}_{1,t}(u) = \frac{\mathbb{E}[(\hat{c}_t(u) - c_{l/n^2}(u))1(E_t) \mid \mathcal{F}_{l/n^2}]}{\mathbb{P}(E_t \mid \mathcal{F}_{l/n^2})},
\]
setting \(\hat{c}_{1,t}(u) = 0\) when \(\mathbb{P}(E_t \mid \mathcal{F}_{l/n^2}) = 0\), and the variance term \(\hat{c}_{2,t}(u)\) is then defined by (13).

We can similarly split the regression estimates \(\tilde{c}_t(u)\) into bias and variance parts. Let
\[
\tilde{c}_t(u) = c_t(u) + \hat{c}_{1,t}(u) + \hat{c}_{2,t}(u) + \hat{c}_{3,t}(u),
\]
where the estimator bias and variance, \(\tilde{c}_{1,t}(u)\) and \(\tilde{c}_{2,t}(u)\), are given by
\[
\tilde{c}_{k,t}(u) = \sum_{l=0}^{n_2-1} W_{n,l}(t) \hat{c}_{k,l}(u), \quad k = 1, 2,
\]
and the regression bias
\[
\tilde{c}_{3,t}(u) = \sum_{l=0}^{n_2-1} W_{n,l}(t)c_{l/n^2}(u) - c_t(u).
\]

To bound the error in \(\tilde{c}_t(u)\), we must show that all three terms \(\tilde{c}_{k,t}(u)\) are small. We begin with the estimator bias \(\tilde{c}_{1,t}(u)\), and note that for large \(n\),
\[
|\tilde{c}_{1,t}(u)| = 1(E_{0,t}) \frac{\mathbb{E}[(\hat{c}_t(u) - c_{l/n^2}(u))1(E_t) \mid \mathcal{F}_{l/n^2}]}{\mathbb{P}(E_t \mid \mathcal{F}_{l/n^2})}
\]
\[
= O(1)\mathbb{E}[(\hat{c}_t(u) - c_{l/n^2}(u))1(E_t) \mid \mathcal{F}_{l/n^2}],
\]
using (12),
\[
= O(1)(\mathbb{E}[(\hat{c}_t(u) - c_{l/n^2}(u))1(E_t) \mid \mathcal{F}_{l/n^2}] + \mathbb{P}(E_{l+1,n^2} \mid \mathcal{F}_{l/n^2})),
\]
since \((\hat{c}_t(u) - c_{l/n^2}(u))1(E_t)\) is bounded,
\[
= O(n^{-\alpha_1}),
\]
using (12) and Theorem 1. We thus have
\[
|\tilde{c}_{1,t}(u)| \leq \sum_{l=0}^{n_2-1} |W_{n,l}(t)||\hat{c}_{1,l}(u)| = O(n^{-\alpha_1}),
\]
using (10) and (15).
We next consider the estimator variance $\tilde{c}_{2,t}(u)$. We first note that
\[
\mathbb{E}[\tilde{c}_{2,t}(u)1(E) | \mathcal{F}_{l/n^2}] = \mathbb{E}[(\hat{c}_t(u) - c_{l/n^2}(u) - \tilde{c}_{1,t}(u))1(E) | \mathcal{F}_{l/n^2}]
= 0,
\]
and
\[
\mathbb{E}[\tilde{c}_{2,t}^2(u)1(E) | \mathcal{F}_{l/n^2}] = O(1)(\mathbb{E}[(\hat{c}_t(u) - c_{l/n^2}(u))^21(E) | \mathcal{F}_{l/n^2}] + \tilde{c}_{1,t}^2(u))
= O(n^{-1/8}),
\]
using (15) and Theorem 1. We thus have
\[
\mathbb{E}[\tilde{c}_{2,t}^2(u)1(E) | \mathcal{F}_0] = \mathbb{E} \left[ \left( \sum_{l=0}^{n^2-1} W_{n,l}(t)\tilde{c}_{2,t}(u)1(E) \right)^2 | \mathcal{F}_0 \right]
= \sum_{l=0}^{n^2-1} W_{n,l}^2(t)\mathbb{E}[\tilde{c}_{2,t}^2(u)1(E) | \mathcal{F}_0]
= O(n^{-1/8}) \left( \sum_{l=0}^{n^2-1} W_{n,l}(t) \right) \left( \sum_{l=0}^{n^2-1} |W_{n,l}(t)| \right)
= O(n^{-2\alpha_3}),
\]
using (9) and (10).

Finally, we bound the regression bias $\tilde{c}_{3,t}(u)$. Let $m$ denote the largest integer smaller than $\alpha$. Using Taylor’s theorem, for $t \in [0,1]$, and $l = 0, \ldots, n^2 - 1$, we then have that
\[
c_{l/n^2}(u) = c_t(u) + \sum_{r=1}^{m-1} \frac{(t - l/n^2)^r}{r!} c_t^{(r)}(u) + \frac{(t - l/n^2)^m}{m!} c_t^{(m)}(u),
\]
for some $t_l \in [0,1]$ lying between $t$ and $l/n^2$. We deduce that
\[
\mathbb{E}[\tilde{c}_{3,t}^2(u) | \mathcal{F}_0]
= \mathbb{E} \left[ \left( \sum_{l=0}^{n^2-1} W_{n,l}(t)(c_{l/n^2}(u) - c_t(u)) \right)^2 | \mathcal{F}_0 \right],
\]
using (11),
\[
= \mathbb{E} \left[ \left( \sum_{l=0}^{n^2-1} W_{n,l}(t) \frac{(t - l/n^2)^m}{m!} (c_t^{(m)}(u) - c_t^{(m)}(u)) \right)^2 | \mathcal{F}_0 \right],
\]
26
again using (11),

\[
O(h^{2m}) \sum_{k,l=0}^{n-1} |W_{n,k}(t)||W_{n,l}(t)|1 \left( \left| t - \frac{k}{n^2} \right|, \left| t - \frac{l}{n^2} \right| \leq h \right) \]

\[ \times \mathbb{E}[|c_k^{(m)}(u) - c_l^{(m)}(u)||c_t^{(m)}(u) - c_t^{(m)}(u)| | \mathcal{F}_0], \]

using (9),

\[
O(h^{2\alpha}) \left( \sum_{i=0}^{n-1} |W_{n,i}(t)| \right)^2,
\]

using Cauchy-Schwarz,

\[
O(n^{-2\alpha_3}),
\]

using (10).

Combining these results, we obtain that

\[
\mathbb{E}[(\tilde{c}_t(u) - c_t(u))^2 1(E) \mid \mathcal{F}_0] \\
= O(1)\mathbb{E}[(\tilde{c}_{t,1}^2(u) + \tilde{c}_{t,2}^2(u) + \tilde{c}_{t,3}^2(u)) 1(E) \mid \mathcal{F}_0] \\
= O(n^{-2\alpha_3}),
\]

as required. For the $L^2$ risk, we likewise obtain

\[
\mathbb{E}[(\tilde{c}_t(u) - c_t(u))^2 1(E) \mid \mathcal{F}_0] = \mathbb{E} \left[ \int_0^1 (\tilde{c}_t(u) - c_t(u))^2 1(E) \, dt \mid \mathcal{F}_0 \right] \\
= \int_0^1 \mathbb{E}[(\tilde{c}_t(u) - c_t(u))^2 1(E) \mid \mathcal{F}_0] \, dt \\
= O(n^{-2\alpha_3}).
\]

Finally, we can check that these rates are uniform over $t \in [0,1]$, and $\mathbb{P} \in S^{\alpha,\beta}(C,D), \square$

We may now deduce our corollary describing the performance of $\tilde{c}_t(u)$ and $\tilde{c}_t(u)$.

**Proof of Corollary 1.** We first fix $0 < C \leq D$, and prove bounds on the error of $\tilde{c}_t(u)$ under the assumption that $\mathbb{P} \in S^{\alpha,\beta}(C,D)$. For $t \in [0,1]$, we have

\[
|\tilde{c}_t(u) - c_t| \leq |\tilde{c}_t(u) - c_t(u)| + |c_t(u) - c_t|,
\]

and from Theorem 2,

\[
|\tilde{c}_t(u) - c_t(u)| = O_p(n^{-\alpha_3}).
\]
It thus remains to bound \(|c_t(u) - c_t|\). We have that
\[
|c_t(u) - c_t| = \frac{1}{n_0 a^2} \int_0^1 \int_0^1 (1 - \cos(\sqrt{n_0}\Phi(w)ux)) \, dw \, \nu_t(dx)
\]
\[
= O(n_0^{-1}) \int_0^1 (1 \wedge n_0 x^2) \nu_t(dx)
\]
\[
= O(n_0^{-1}) \int_0^1 (1 \wedge (n_0 x^2)^{\beta/2}) \nu_t(dx)
\]
\[
= O(n_0^{-(2-\beta)/2}) \int_0^1 (1 \wedge |x|^\beta) \nu_t(dx)
\]
\[
= O(n_0^{-(2-\beta)/2}) = O(n^{-(2-\beta)/4}).
\]

We thus conclude that
\[
|\tilde{c}_t(u) - c_t| = O_p(n^{-\alpha_4});
\]
by a similar argument, the same holds also for the \(L^2\) error, \(\|\tilde{c}(u) - c\|_2\). We can further check that these limits hold conditionally on \(F_0\), and uniformly over all \(t \in [0,1]\), \(\mathbb{P} \in S^{\alpha,\beta}(C, D)\).

We next consider the case that \(\mathbb{P} \in S^{\alpha,\beta}(C, D)\), for some \(\gamma \in [0,1]\). Create, on an extended probability space, a process \(X_t^S, \ t \in [0,1]\), which almost-surely agrees with \(X_t\) at times \(t \in [0,S]\). For times \(t \in [S,1]\), we require that \(X_t\) is a Lévy process with respect to both \(F_t\) and \(\mathcal{F}^i_t\), with characteristic triplet \((b_S, c_S, \nu_S)\).

Also create observations
\[
Y_j^S = X_{j/n}^S + \xi_j^S, \quad j = 0, \ldots, n - 1,
\]
where for \(j/n \leq S\), the errors \(\xi_j^S = \varepsilon_j\). When \(j/n > S\), we require that the errors \(\xi_j^S\) are \(\mathcal{F}_{j/n}^+\)-measurable, and equal to \(\pm \sigma_S\) each with probability \(\frac{1}{2}\) given \(\mathcal{F}_{j/n}\).

Then let \(\tilde{c}_t^S(u)\) denote the estimate of \(c_t\) defined similarly to \(\tilde{c}_t(u)\), but using the observations \(Y_j^S\). Conditionally on \(\Omega_0\), the law of the \(X_j^S\) and \(Y_j^S\) lies in \(S^{\alpha,\beta}(C, D)\), so we can apply (16) to \(\tilde{c}_t^S(u)\). We obtain that
\[
|\tilde{c}_t^S(u) - c_{t\wedge S}|1(\Omega_0) = O_p(f(C, D)n^{-\alpha_4}),
\]
unformly in \(\gamma, C, D\), and for a function \(f(C, D) > 0\).

We now consider the case \(\mathbb{P} \in S^{\alpha,\beta}\), and suppose we are given an arbitrary sequence \(\delta_n > 0, \delta_n \to \infty\). If we choose \(C_n \to 0, D_n \to \infty\) slowly enough as \(n \to \infty\), we will obtain that \(f(C_n, D_n) = O(\delta_n)\). Since \(\mathbb{P} \in S^{\alpha,\beta}\), we also have that \(\mathbb{P} \in S^{\alpha,\beta}(C_n, D_n)\) for some \(\gamma_n \to 0\); let \(\Omega_{0,n} \in \mathcal{F}_0\) and \(S_n \in [0,1]\) denote the associated events and stopping times.

Applying (17), we deduce that
\[
|\tilde{c}_t^{S_n}(u) - c_{t\wedge S_n}|1(\Omega_{0,n}) = O_p(\delta_n n^{-\alpha_4}).
\]

28
Since
\[ P(\Omega_{0,n} \cap \{ S_n = 1 \}) \geq 1 - \gamma_n \to 1, \]
this implies that
\[ |\tilde{c}_t(u) - c_t| = O_p(\delta_n n^{-\alpha_4}). \]
Since this result holds for any diverging sequence $\delta_n$, we conclude that
\[ |\tilde{c}_t(u) - c_t| = O_p(n^{-\alpha_4}). \]
Again, the result for the $L^2$ error follows similarly.

Next, we suppose that $P \in T^\alpha(C,D)$, and bound the accuracy of the estimate $\tilde{r}_t(u)$. We begin by bounding its normalising constant,
\[ \frac{1}{n^2} \sum_{l=0}^{n^2-1} \tilde{c}_l(u) = \sum_{l=0}^{n^2-1} \tilde{W}_{n,l}(t) \tilde{c}_l(u), \]
where the weights $\tilde{W}_{n,l}(t) = 1/n^2$. Since these weights satisfy (9) and (10) for the bandwidth $h = 1$, we have that
\[ \left| \frac{1}{n^2} \sum_{l=0}^{n^2-1} (\tilde{c}_l(u) - c_{l/n^2}(u)) \right| = \left| \sum_{l=0}^{n^2-1} \tilde{W}_{n,l}(t) (\tilde{c}_l(u) - c_{l/n^2}(u)) \right| = O_p(n^{-\alpha_1}), \]
arguing as in Theorem 2.

We also have
\[
\mathbb{E} \left[ \left( \frac{1}{n^2} \sum_{l=0}^{n^2-1} c_{l/n^2}(u) - \int_0^1 c_t(u) \, dt \right)^2 \right] = \mathbb{E} \left[ \left( \sum_{l=0}^{n^2-1} \int_{l/n^2}^{(l+1)/n^2} (c_{l/n^2}(u) - c_t(u)) \, dt \right)^2 \right] \leq \mathbb{E} \left[ \sum_{l=0}^{n^2-1} \int_{l/n^2}^{(l+1)/n^2} (c_{l/n^2}(u) - c_t(u))^2 \, dt \right],
\]
by Jensen’s inequality,
\[
= \sum_{l=0}^{n^2-1} \int_{l/n^2}^{(l+1)/n^2} \mathbb{E}[(c_{l/n^2}(u) - c_t(u))^2] \, dt = O(n^{-2\alpha_1}).
\]
We thus deduce that
\[ \left| \frac{1}{n^2} \sum_{l=0}^{n^2-1} \tilde{c}_l(u) - \int_0^1 c_t(u) \, dt \right| = O_p(n^{-\alpha_1}). \] (18)
From Theorem 2, we also have that

$$|\tilde{c}_t(u) - c_t(u)| = O_p(n^{-\alpha_3}). \tag{19}$$

Combining these results, we obtain that

$$|	ilde{r}_t(u) - r_t| = \left| \frac{1}{n_2} \sum_{l=0}^{n_2-1} \tilde{c}_l(u) - \frac{\int_0^1 c_t(u) dt}{\int_0^1 c_t(u) dt} \right| = O_p(n^{-\alpha_3}),$$

since $\int_0^1 c_t(u) dt \geq C > 0$.

We can again check that this limit holds conditionally on $F_0$, and uniformly over all $t \in [0, 1]$, $\mathbb{P} \in \mathcal{T}^\alpha(C, D)$. Arguing as above, we then conclude that for $\mathbb{P} \in \mathcal{T}^\alpha$, 

$$|	ilde{r}_t(u) - r_t| = O_p(n^{-\alpha_3}).$$

As above, we can also conclude that these results likewise hold for the $L^2$ error $\|\tilde{r}(u) - r\|_2$.

Finally, we bound the performance of $\tilde{r}_t(u)$ for $\mathbb{P} \in S^{\alpha, \beta}(C, D)$. Combining (16), (18) and (19), we have

$$\left| \frac{1}{n_2} \sum_{l=0}^{n_2-1} \tilde{c}_l(u) - \int_0^1 c_t dt \right|, |\tilde{c}_l(u) - c_l| = O_p(n^{-\alpha_4}).$$

Arguing as above, we obtain that

$$|	ilde{r}_t(u) - r_t| = O_p(n^{-\alpha_4}),$$

and that this can be extended to $\mathbb{P} \in S^{\alpha, \beta}$, and the $L^2$ error $\|\tilde{r}(u) - r\|_2$. \qed

Finally, we can also prove our lower bound on the rate of estimation, which is a simple corollary of results in Munk and Schmidt-Hieber (2010a).

**Proof of Theorem 3.** We begin with part (i), and appeal to the proof of Theorem 2.1 in Munk and Schmidt-Hieber (2010a). The authors give a lower bound on the $L^2$ estimation rate of $c_t$, in a setting similar to our $S^{\alpha, \beta}(C, D)$.

Munk and Schmidt-Hieber consider a setting where $\sigma_t^2 = \sigma^2 > 0$ is a deterministic constant, $c_t$ is deterministic, and $b_t = \nu_t = 0$. They then construct a large number of choices $c_{\omega,t}$ for the volatility, separated from each other in $L^2$ norm at a rate at least $n^{-\alpha_1}$. They further establish that, given observations $Y_j$ under one such volatility function $c_{\omega,t}$, we cannot consistently estimate $\omega$. They thus show that no estimate $\hat{c}_t$ of $c_t$ can satisfy $\|c^* - c\|_2 = o_p(n^{-\alpha_3})$.

It can be checked that, when $C < 1 < D$, their models lie in $S^{\alpha, \beta}(C, D) \cap \mathcal{T}$ for large $n$, so their lower bound holds also in that setting. By rescaling
their volatility functions $c_{\omega,t}$, we can obtain the same results also for general $0 < C < D$.

A pointwise lower bound can be proved by a similar argument; we sketch a proof below. Define two choices for the volatility,

$$c_{0,t} = 1, \quad c_{1,t} = 1 + h^a K((1 - t)/h),$$

where $h = n^{-1/2(2a+1)}$, and $K : \mathbb{R} \to \mathbb{R}$ is a smooth non-increasing non-negative function, satisfying $K(0) = 1, K(1) = 0$.

We note that when $C < 1 < D$, these models lie within $\mathcal{S}^{\alpha,\beta}(C,D)$ for large $n$; as above, by rescaling we can work with general $0 < C < D$. We also have that $c_{0,1}$ and $c_{1,1}$ are separated at a rate $n^{-\alpha}$. It thus suffices to show that we cannot consistently distinguish $c_0$ from $c_1$ given the $Y_j$.

We therefore need consider only the observations $X_t$ and $Y_j$, $j > nt$. Arguing similarly to Munk and Schmidt-Hieber, it can be shown that these observations are insufficient to distinguish $c_0$ and $c_1$, thereby establishing our lower bound.

For part (ii), it can be checked that the rate functions $r_{\omega,t} = c_{\omega,t}/\int_0^1 c_{\omega,s} \, ds$ are again separated, in $L^2$ norm or pointwise, at a rate at least $n^{-\alpha}$. We thus conclude that our lower bounds hold also for $r_t$. \hfill $\square$

## 5.3 Technical proofs

We now give proofs of our technical lemmas. We begin with a simple proof of Lemma 1, our result establishing that càglàd Itô semimartingales with locally bounded characteristics satisfy our assumptions.

**Proof of Lemma 1.** For $D > 0$, we define events $\Omega_0$ and stopping times $S$ with the desired properties. When $D < 2$, we may set $\Omega_0 = \emptyset$. When $D \geq 2$, we set $\Omega_0 = \{|Z_0| \leq D\}$, and

$$S = \sup \left\{ s \in [0,1] : |Z_s| \leq R, \quad |b_{Z,s}| + \int_{1\leq|x|<3R} |x| \nu_{Z,s}(dx), c_{Z,s} + \int_{|x|<3R} x^2 \nu_{Z,s}(dx) \leq D/2 \right\},$$

where $R \in [0,D]$ is to be determined. We must then show that on the event $\Omega_0$, $Z_t \in \mathcal{I}_{1/2}(D,S)$, with $\mathbb{P}(\Omega_0 \cap \{S = 1\}) \to 1$ as $D \to \infty$.

On $\Omega_0$, we first note that since $Z_t$ is càglàd, the condition $|Z_t| \leq D$ follows from the definitions of $R$ and $S$. To establish $Z_t \in \mathcal{I}_{1/2}(D,S)$, we
then split $Z_t$ into a predictable term,

$$Z_{P,t} = \int_0^{t-} b_{Z,u} \, du + \int_0^{t-} \int_{1 \leq |x| < 3R} x \nu_{Z,u}(dx) \, du,$$

martingale term

$$Z_{M,t} = \int_0^{t-} \sqrt{c_{Z,u}} dB_{Z,u} + \int_0^{t-} \int_{|x| < 3R} x (\mu_{Z}(dx, ds) - \nu_{Z,u}(dx) \, du),$$

and large-jump term

$$Z_{J,t} = \int_0^{t-} \int_{|x| \geq 3R} x \mu_{Z}(dx, ds).$$

We thus have

$$Z_t = Z_{P,t} + Z_{M,t} + Z_{J,t}.$$

Furthermore, the stopped process

$$Z_{t∧S} = Z_{P,t∧S} + Z_{M,t∧S},$$

since the stopped large-jump term $Z_{J,t∧S} = 0$. We thus deduce that, for $0 \leq t \leq s \leq 1$,

$$\mathbb{E}[(Z_{s∧S} - Z_{t∧S})^2 | \mathcal{F}_t^+] \leq 2\mathbb{E}[(Z_{P,s∧S} - Z_{P,t∧S})^2 + (Z_{M,s∧S} - Z_{M,t∧S})^2 | \mathcal{F}_t^+]$$

$$\leq 2\mathbb{E} \left[ \left( \int_{s∧S}^{t∧S} \left( |b_{Z,u}| + \int_{1 \leq |x| < 3R} |x| \nu_{Z,u}(dx) \right) du \right)^2 \right]$$

$$+ \int_{s∧S}^{t∧S} \left( c_{Z,u} + \int_{|x| < 3R} \nu_{Z,u}(dx) \right) du | \mathcal{F}_t^+]$$

$$\leq D^2 (s - t)^2 / 2 + D(s - t)$$

$$\leq D^2 (s - t),$$

using the definition of $S$, and that $D \geq 2$.

We conclude that $Z_t \in \mathcal{I}^{1/2}(D, S)$, so it remains to show that $\mathbb{P}(\Omega_0 \cap \{ S = 1 \}) \to 1$ as $D \to \infty$. We first consider the event $\Omega_0$, and note that since $Z_0$ is finite, we have $\mathbb{P}(\Omega_0) \to 1$ as $D \to \infty$.

We next consider the stopping time $S$. As the integrals $\int_{|x| < R} x^2 \nu_{Z,t}(dx)$ are locally bounded, we have that if $R \to \infty$ slowly enough with $D$, then

$$\mathbb{P} \left( \sup_{t \in [0,1]} \int_{|x| < R} x^2 \nu_{Z,t}(dx) > \frac{D}{4} \right) \to 0$$
as \( D \to \infty \). Since \( b_t \) and \( c_t \) are also locally bounded, we likewise have
\[
\mathbb{P} \left( \sup_{t \in [0,1]} |b_t| > \frac{D}{4} \right), \mathbb{P} \left( \sup_{t \in [0,1]} c_t > \frac{D}{4} \right) \to 0.
\]
Finally, as \( X_t \) is càglàd, it is again locally bounded, and
\[
\mathbb{P} \left( \sup_{t \in [0,1]} |X_t| > R \right) \to 0
\]
as \( D \to \infty \). Combining these results, we obtain that \( \mathbb{P}(S = 1) \to 1 \) as \( D \to \infty \), as required.

We next establish our technical lemmas Lemmas 2–4, used in the proof of Theorem 1; we begin with some new definitions. We will write the characteristic functions of the log-prices \( X_t \) in terms of the spot characteristic exponent,
\[
\theta_t(u) = ib_t u - \frac{1}{2} c_t u^2 + \int \left( e^{iux} - 1 - iux \mathbb{1}_{|x| < 1} \right) \nu_t(dx).
\]

We will also describe the pre-averaged increments \( \hat{X}_k \) using constants \( p_j, q_j \). For \( j \in J_k, k = 0, \ldots, n_0 - 1 \), we define
\[
p_j = \begin{cases} 
\Phi_n(j/n), & j + 1 \in J_k, \\
0, & \text{otherwise},
\end{cases}
\]
and set \( p_{-1} = 0 \). (Note that this does not conflict with our earlier definition of \( p_j \), which held only for \( j, j + 1 \in J_k \).) Then for \( j = 0, \ldots, n - 1 \), we define
\[
q_j = p_{j-1} - p_j.
\]
We may now proceed with the lemmas.

**Lemma 6.** In the setting of Theorem 1, let \( 0 \leq t \leq s \leq 1 \), \( u, v \in \mathbb{R} \), and \( k = 0, \ldots, n_0 - 1 \). We then have:

(i) \( |\theta_t(u)| = O(1 + u^2) \);
(ii) \( |\theta_t(u) - \theta_t(v)| = O(1 + (|u| + |v|)|u - v|) \);
(iii) \( \mathbb{E}[|\theta_s(u) - \theta_t(u)|^2 \mid \mathcal{F}_t] = O(1 + u^2 + u^4(s-t)^{2\alpha_0}) \);
(iv) \( \sum_{j \in J_k} q_j^2 = 2\kappa + O(n^{-1/2}) \); and
(v) \( \int_{k/n_0}^{(k+1)/n_0} \theta_{k/n_0}(\Phi_n(w)u) \, dw = -c_{k/n_0}(u)u^2. \)

**Proof.** We prove each statement in turn.
(i) For \( t \in [0, 1] \), \( u, x \in \mathbb{R} \), we have
\[
|e^{ix} - 1 - iux1_{|x|<1}| \leq \frac{1}{2} |ux|^2 1_{|x|<1} + 2 \cdot 1_{|x|\geq 1} \\
\leq 2(1 + u^2)(1 \wedge x^2),
\]
so
\[
|\theta_t(u)| = O(1 + u^2) \left( |b_t| + |c_t| + \int_{\mathbb{R}} (1 \wedge x^2) \nu_t(dx) \right)
= O(1 + u^2).
\] (20)

(ii) For \( t \in [0, 1] \), \( w \in \mathbb{R} \), \(|x| < 1\), we likewise have
\[
|ix(e^{iwx} - 1)| \leq |wx|^2,
\]
so for \( u, v \in \mathbb{R} \),
\[
|\theta_t(u) - \theta_t(v)| \leq \int_u^v \left| \frac{d}{dw} \left( \theta_t(w) - \int_{|x|\geq 1} (e^{iwx} - 1) \nu_t(dx) \right) \right| dw \\
+ \int_{|x|\geq 1} |(e^{iwx} - 1) - (e^{iwx} - 1)| \nu_t(dx) \\
= \int_u^v \left| ib_t - wc_t + \int_{|x|<1} ix(e^{iwx} - 1) \nu_t(dx) \right| dw \\
+ \int_{|x|\geq 1} |e^{iwx} - e^{iwx}| \nu_t(dx) \\
= \int_u^v O(1 + |w|) \left( |b_t| + |c_t| + \int_{|x|<1} x^2 \nu_t(dx) \right) dw \\
+ O(1) \int_{|x|\geq 1} \nu_t(dx) \\
= O(1 + (|u| + |v|)|u - v|).
\]

(iii) For \( u, x \in \mathbb{R} \), we first set
\[
f(x) = \frac{e^{iux} - 1 - iux1_{|x|<1}}{2(1 + u^2)},
\]
and for \( t \in [0, 1] \),
\[
Z_{\nu,t} = \int_{\mathbb{R}} f(x) \nu_t(dx).
\]
From (20), we have that \(|f(x)| \leq (1 \wedge x^2)\), so if \( \beta > 1 \), \( Z_{\nu,t} \in \mathcal{L}^\alpha(D, S) \), and for \( 0 \leq t \leq s \leq 1 \),
\[
\mathbb{E}[(Z_{\nu,s} - Z_{\nu,t})^2 \mid \mathcal{F}_t^+] = O((s - t)^{2\alpha_0}).
\]
If $\beta \leq 1$, we likewise have

$$|f(x)| \leq \frac{|ux|1_{|x|<1} + 1_{|x|\geq 1}}{1 + u^2} \leq \frac{2(1 \wedge |x|)}{1 + |u|},$$

so $|Z_{\nu,t}| = O(1/(1 + |u|))$, and

$$(Z_{\nu,s} - Z_{\nu,t})^2 \leq 2(Z_{\nu,s}^2 + Z_{\nu,t}^2) = O(1/(1 + u^2)).$$

In either case, since

$$\theta_t(u) = ib_t u - \frac{1}{2} c_t u^2 + 2(1 + u^2)Z_{\nu,t},$$

we deduce that

$$E[|\theta_s(u) - \theta_t(u)|^2 | F_t^+] = O\left(\frac{1}{1 + u^2} + u^4(s - t)^2\right).$$

(iv) For $k = 0, \ldots, n_0 - 1$, we have

$$\sum_{j \in J_k} q_j^2 = \sum_{j \in J_k} \left(|\Phi_n(t)\right)_{(j-1)/n}^2 + O(n^{-1/2}),$$

since for $j \in J_k$, $q_j = -|\Phi_n(t)\right)_{(j-1)/n}$ unless $j - 1 \not\in J_k$ or $j + 1 \not\in J_k$,

in which case both $q_j$ and $|\Phi_n(t)\right)_{(j-1)/n}$ are $O(n^{-1/4})$,

$$= \sum_{j \in J_k} \left(\int_{(j-1)/n}^{j/n} \Phi_n'(t) \, dt\right)^2 + O(n^{-1/2})$$

$$= n^{-2} \sum_{j \in J_k} \Phi_n'(j/n)^2 + O(n^{-1/2}),$$

since for $|s - t| \leq 1/n$, $|\Phi_n'(s) - \Phi_n'(t)| = O(n^{1/4})$,

$$= n^{-1} \int_{k/n_0}^{(k+1)/n_0} \Phi_n'(t)^2 \, dt + O(n^{-1/2})$$

$$= 2\kappa + O(n^{-1/2}).$$

(v) For $k = 0, \ldots, n_0 - 1$, we have

$$\int_{k/n_0}^{(k+1)/n_0} \theta_{k/n_0}(\Phi_n(w)u) \, dw = \frac{1}{n_0} \int_0^1 \theta_{k/n_0}(\sqrt{n_0}\Phi(w)u) \, dw$$

$$= \text{Re} \left(\frac{1}{n_0} \int_0^1 \theta_{k/n_0}(\sqrt{n_0}\Phi(w)u) \, dw\right),$$

35
since $\theta_t$ is Hermitian, and $\Phi$ is antisymmetric about $\frac{1}{2}$,

$$= -c_{k/n_0}(u)u^2. \quad \square$$

We may now prove Lemma 2.

**Proof of Lemma 2.** We consider each process $\xi_t$ in turn, proving (iv) as a corollary to (i).

(i) For $u, x \in \mathbb{R}$, we set

$$f(x) = \frac{1}{2n_0 u^2} \int_0^1 (1 - \cos(\sqrt{n_0} \Phi(w)ux)) dw,$$

and note that

$$0 \leq f(x) \leq \frac{1 \wedge \sqrt{n_0} |ux|}{n_0 u^2} \leq 1 \wedge \frac{1 \wedge n_0 u^2 x^2}{n_0 u^2} \leq 1 \wedge x^2.$$

If $\beta > 1$, the process

$$Z_{c,t} = \int_{\mathbb{R}} f(x) \nu_t(dx)$$

is thus in $\mathcal{I}^\alpha(D,S)$, so for $0 \leq t \leq s \leq 1$,

$$\mathbb{E}[(Z_{c,s} - Z_{c,t})^2 | \mathcal{F}_t^+] = O((s-t)^{2\alpha_0}).$$

If instead $\beta \leq 1$, we likewise have

$$0 \leq f(x) \leq \frac{1 \wedge \sqrt{n_0} |ux|}{n_0 u^2} \leq \frac{1 \wedge |x|}{\sqrt{n_0} |u|},$$

so $|Z_{c,s}| = O(n^{-1/4})$, and

$$(Z_{c,s} - Z_{c,t})^2 \leq 2(Z_{c,s}^2 + Z_{c,t}^2) = O(n^{-1/2}).$$

In either case, since

$$c_t(u) = c_t + 2Z_{c,t}(u),$$

we then have that

$$\mathbb{E}[(c_s(u) - c_t(u))^2 | \mathcal{F}_t^+] = O(1)\mathbb{E}[(c_s - c_t)^2 + (Z_{c,s} - Z_{c,t})^2 | \mathcal{F}_t^+]$$

$$= O((s-t)^{2\alpha_0} + n^{-1/2}).$$

Since

$$Z_{c,t} \leq \int_{\mathbb{R}} 1 \wedge |x|^2 \nu_t(dx) \leq D,$$

we also have $c_t(u) \leq 3D$, almost surely.
(ii) For $0 \leq t \leq s \leq 1$, we have
\[
\mathbb{E}[(\varphi_s(u) - \varphi_t(u))^2 \mid \mathcal{F}_t^+] \\
\leq 2\mathbb{E}[(c_s(u) - c_t(u))^2 + (\sigma_s^2(u) - \sigma_t^2(u))^2 \mid \mathcal{F}_t^+] \\
= O((s-t)^{2\alpha_0} + n^{-1/2}),
\]
using (i).

(iii) The result follows similarly to (ii).

We next describe the characteristic functions of the increments of $X_t$ and noises $\varepsilon_j$. For $j = 0, \ldots, n-2$, define the increments
\[
\Delta X_j = X_{(j+1)/n} - X_{j/n},
\]
and set $\Delta X_{n-1} = 0$. We then have the following result.

**Lemma 7.** In the setting of Theorem 1, let $j = 0, \ldots, n-1$, $u \in \mathbb{R}$. Then

(i) if $u = o(1)$,
\[
\mathbb{E}[\exp(iu\varepsilon_j) \mid \mathcal{F}_{j/n}] = \exp(-\frac{1}{2}\sigma_{j/n}^2 u^2) + O(|u|^3); \text{ or}
\]

(ii) if $j \neq n-1$, and $u = O(n^{1/2})$,
\[
\mathbb{E}[\exp(iu\Delta X_j) \mid \mathcal{F}_{j/n}] = \exp(n^{-1}\theta_{j/n}(u)) + O(n^{-1}(1 + |u| + u^2 n^{-\alpha_0})).
\]

**Proof.** We prove each result in turn.

(i) We note that $\varepsilon_j \mid \mathcal{F}_{j/n}$ has bounded fourth moment, so we can expand its characteristic function to third order using Taylor’s theorem. We obtain that
\[
\mathbb{E}[\exp(iu\varepsilon_j) \mid \mathcal{F}_{j/n}] = 1 + iu\mathbb{E}[\varepsilon_j \mid \mathcal{F}_{j/n}] - \frac{1}{2}u^2\mathbb{E}[\varepsilon_j^2 \mid \mathcal{F}_{j/n}] + O(|u|^3)\mathbb{E}[|\varepsilon_j|^3 \mid \mathcal{F}_{j/n}] \\
= 1 - \frac{1}{2}\sigma_{j/n}^2 u^2 + O(|u|^3)
\]
for small enough $u$, since $\sigma_{j/n}^2$ is bounded.

(ii) We define
\[
\Theta_t = \int_0^t \theta_s(u) \, ds, \quad t \in [0,1],
\]
and note from Lemma 6(i) that $\theta_s(u)$ is bounded. The process $\Theta_t$ is thus bounded, continuous and of finite variation. We deduce that its stochastic exponential,
\[
\mathcal{E}(\Theta)_t = \exp(\Theta_t) \neq 0.
\]
From Corollary II.2.48 in Jacod and Shiryaev (2003), we then have that the process
\[ M_{X,t} = \exp(iuX_t) \]
is a local \( F^+_t \)-martingale; since \( M_{X,t} \) is bounded, it is also a true martingale.

We can thus use \( M_{X,t} \) to compute the characteristic functions of the increments \( \Delta X_j \). From Lemma 6(i), we have that for \( u = O(n^{1/2}) \),
\[ \frac{M_{X,(j+1)/n}}{M_{X,j/n}} = \exp \left( iu \Delta X_j - \int_{j/n}^{(j+1)/n} \theta_s(u) \, ds \right) = O(1), \]
and similarly, the random variable
\[ Z_{X,j} = \int_{j/n}^{(j+1)/n} (\theta_s(u) - \theta_{j/n}(u)) \, ds = O(1). \]

We therefore obtain that
\[
\exp(-n^{-1}\theta_{j/n}(u))\mathbb{E}[\exp(iu\Delta X_j) \mid \mathcal{F}^+_j] \\
= \mathbb{E} \left[ \frac{M_{X,(j+1)/n}}{M_{X,j/n}} \exp(Z_{X,j}) \mid \mathcal{F}^+_j \right] \\
= 1 + \mathbb{E} \left[ \frac{M_{X,(j+1)/n}}{M_{X,j/n}}(\exp(Z_{X,j}) - 1) \mid \mathcal{F}^+_j \right],
\]
since \( M_{X,t} \) is a martingale,
\[ = 1 + O(1)\mathbb{E}[|Z_{X,j}| \mid \mathcal{F}^+_j], \]
since \( M_{X,(j+1)/n}/M_{X,j/n} \) and \( Z_{X,j} \) are bounded,
\[ = 1 + O(1)\mathbb{E} \left[ \int_{j/n}^{(j+1)/n} |\theta_s(u) - \theta_{j/n}(u)| \, ds \mid \mathcal{F}^+_j \right] \\
= 1 + O(1) \int_{j/n}^{(j+1)/n} \mathbb{E}[|\theta_s(u) - \theta_{j/n}(u)|^2 \mid \mathcal{F}^+_j]^{1/2} \, ds \\
= 1 + O(n^{-1}(1 + |u| + u^2 n^{-\alpha_0})),
\]
using Lemma 6(iii). The result follows since \( n^{-1}\theta_{j/n}(u) \) is bounded, using Lemma 6(i).

We may now prove Lemma 3.
Proof of Lemma 3. We first note we may assume \( n \) is large enough that \( J_k \) is non-empty. We then express the distribution of the pre-averaged increments \( \hat{X}_k \) in terms of the increments \( \Delta X_j \), and noises \( \varepsilon_j \). From the definitions, we obtain

\[
\hat{X}_k = \sum_{j \in J_k} p_j (\Delta X_j - \varepsilon_j + \varepsilon_{j+1}) = \sum_{j \in J_k} (p_j \Delta X_j + q_j \varepsilon_j).
\]

We now compute the characteristic functions of this sum, conditional on \( \mathcal{F}_{k/n_0} \); we begin by writing down the characteristic functions of the terms \( p_j \Delta X_j \). For \( j \in J_k \), set

\[
Z_{\theta,j} = |\theta_{j/n} (p_j u) - \theta_{k/n_0} (p_j u)|.
\]

We then have that for \( j, j + 1 \in J_k \),

\[
\mathbb{E}[\exp(iu p_j \Delta X_j) \mid \mathcal{F}_{j/n}^+] = \exp(n^{-1}\theta_{j/n} (p_j u)) + O(n^{-3/4}),
\]

since \( |p_j| = O(n^{1/4}) \), using Lemma 7(ii),

\[
\mathbb{E}[\exp(iu p_j \Delta X_j) \mid \mathcal{F}_{j/n}^+] = \exp(n^{-1}\theta_{k/n_0} (p_j u)) + O(n^{-3/4} + n^{-1}Z_{\theta,j}), \tag{21}
\]

since by Lemma 6(i), \( n^{-1}\theta_{j/n} (p_j u) \) is bounded. The result (21) also holds when \( j \in J_k \), \( j + 1 \notin J_k \), since then \( p_j = 0 \).

We can also write down the characteristic functions of the terms \( q_j \varepsilon_j \). For \( j \in J_k \), set

\[
Z_{\sigma,j} = |\sigma_{j/n}^2 - \sigma_{k/n_0}^2|.
\]

We then have that

\[
\mathbb{E}[\exp(iu q_j \varepsilon_j) \mid \mathcal{F}_{j/n}^+] = \exp(-\frac{1}{2}q_j^2 \sigma_{j/n}^2 u^2) + O(n^{-3/4}),
\]

since \( |q_j| = O(n^{-1/4}) \), using Lemma 7(i),

\[
\mathbb{E}[\exp(iu q_j \varepsilon_j) \mid \mathcal{F}_{j/n}^+] = \exp(-\frac{1}{2}q_j^2 \sigma_{k/n_0}^2 u^2) + O(n^{-3/4} + n^{-1/2}Z_{\sigma,j}), \tag{22}
\]

since \( \frac{1}{2}\sigma^2 u^2 \) is bounded.

We may thus compute inductively the characteristic functions of the sums

\[
\hat{X}_{k,m} = \sum_{j \in J_k : j \geq m} (p_j \Delta X_j + q_j \varepsilon_j).
\]
By induction on $m$, we will show that
\[
\mathbb{E}[\exp(iu\hat{X}_{k,m}) | \mathcal{F}_{m/n}] = \exp\left(\sum_{j \in J_k : j \geq m} (n^{-1}\theta_{k/n_0}(p_ju) - \frac{1}{2}q_j^2\sigma_{k/n_0}^2u^2)\right) + O(1) \sum_{j \in J_k : j \geq m} \mathbb{E}[n^{-3/4} + n^{-1}Z_{\theta,j} + n^{-1/2}Z_{\sigma,j} | \mathcal{F}_{m/n}] .
\]
(23)

In the base case, when $m = \max(J_k) + 1$, the result is trivial. In the inductive case, we assume that $m \in J_k$, and (23) holds for $m + 1$. Since
\[
\mathbb{E}[\exp(iu\hat{X}_{k,m}) | \mathcal{F}_{m/n}] = \mathbb{E}[\exp(iuq_m\varepsilon_m)\mathbb{E}[\exp(iup_m\Delta X_m)] \\
\times \mathbb{E}[\exp(iu\hat{X}_{k,m+1}) | \mathcal{F}_{(m+1)/n}] | \mathcal{F}_{m/n}],
\]
and using (21) and (22), we have that (23) holds also for $m$.

We therefore have that (23) holds when $m = \min(J_k)$, in which case $\hat{X}_{k,m} = \hat{X}_k$. We conclude that
\[
\mathbb{E}[\cos(u\hat{X}_k) | \mathcal{F}_{k/n_0}] = \Re\left(\mathbb{E}[\exp(iu\hat{X}_k) | \mathcal{F}_{k/n_0}]\right) = \Re\left(\Re[\mathbb{E}[\exp(iu\hat{X}_{k,m}) | \mathcal{F}_{m/n}] | \mathcal{F}_{k/n_0}]\right)
\]
\[
= \Re\left(\exp\left(\sum_{j \in J_k} (n^{-1}\theta_{k/n_0}(p_ju) - \frac{1}{2}q_j^2\sigma_{k/n_0}^2u^2)\right)\right) + O(n^{-1/4}),
\]

using (23) and Lemma 6(iii),
\[
= \Re\left(\exp\left(\int_{\theta_{k/n_0}}^{(k+1)/n_0} \theta_{k/n_0}(\Phi_n(w)u) dw - \kappa\sigma_{k/n_0}^2u^2\right)\right) + O(n^{-1/4}),
\]
since for $|s-t| \leq 1/n$, $|\Phi_n(s) - \Phi_n(t)| = O(n^{-1/4})$, and using Lemma 6(i), (ii) and (iv),
\[
= \varphi_{k/n_0}(u) + O(n^{-1/4}),
\]
using Lemma 6(v). As a consequence, we also obtain
\[
\text{Var}[\cos(u\hat{X}_k) | \mathcal{F}_{k/n_0}] = \mathbb{E}[\cos(u\hat{X}_k)^2 | \mathcal{F}_{k/n_0}] - \mathbb{E}[\cos(u\hat{X}_k) | \mathcal{F}_{k/n_0}]^2
\]
\[
= \frac{1}{2}\mathbb{E}[1 + \cos(2u\hat{X}_k) | \mathcal{F}_{k/n_0}] - \mathbb{E}[\cos(u\hat{X}_k) | \mathcal{F}_{k/n_0}]^2
\]
\[
= \frac{1}{2}(1 + \varphi_{k/n_0}(2u) + O(n^{-1/4})) - (\varphi_{k/n_0}(u) + O(n^{-1/4}))^2,
\]
\[
= p_{k/n_0}^2(u) + O(n^{-1/4}),
\]
since $\varphi_t(u)$ is bounded. \qed
We next move on to our bounds on the noise estimates $\hat{\sigma}_{k}^{2}$. We will first need to decompose the log-price process $X_{t}$ into two parts: we set

$$X_{t} = X_{I,t} + X_{J,t},$$

where the square-integrable component

$$X_{I,t} = \int_{0}^{t} b_{s} \, ds + \int_{0}^{t} \sqrt{c_{s}} \, dB_{s} + \int_{0}^{t} \int_{|x| < 1} x (\mu(dx,ds) - \nu_{s}(dx)) \, ds,$$

and the large-jump component

$$X_{J,t} = \int_{0}^{t} \int_{|x| \geq 1} x \mu(dx,ds).$$

We begin by proving a technical result on the variation of the process $X_{I,t}$.

**Lemma 8.** In the setting of Theorem 1, for $j = 0, \ldots, n-1$, $p = 2, 4$, we have

$$\mathbb{E}\left[ (X_{I,(j+1)/n} - X_{I,j/n})^{p} \mid \mathcal{F}_{j/n}^{+} \right] = O(n^{-1}).$$

**Proof.** Our argument follows Luschgy and Pagès (2008). We define the $\mathcal{F}_{t}^{+}$-martingale

$$M_{I,t} = X_{I,t} - \int_{0}^{t} b_{s} \, ds,$$

and note that

$$\mathbb{E}\left[ (X_{I,(j+1)/n} - X_{I,j/n})^{p} \mid \mathcal{F}_{j/n}^{+} \right] = O(1)\mathbb{E}\left[ (M_{I,(j+1)/n} - M_{I,j/n})^{p} \mid \mathcal{F}_{j/n}^{+} \right] + O(n^{-p}),$$

so it suffices to prove an equivalent bound for $M_{I,t}$.

If $p = 2$, we note that

$$\mathbb{E}[(M_{I,(j+1)/n} - M_{I,j/n})^{2} \mid \mathcal{F}_{j/n}^{+}] = \mathbb{E}[\|M_{I}\|_{(j+1)/n} - \|M_{I}\|_{j/n} \mid \mathcal{F}_{j/n}^{+}] = \mathbb{E}\left[ \int_{j/n}^{(j+1)/n} c_{s} \, ds + \int_{j/n}^{(j+1)/n} \int_{|x| < 1} x^{2} \mu(dx,ds) \mid \mathcal{F}_{j/n}^{+} \right]$$

$$= \int_{j/n}^{(j+1)/n} \mathbb{E}\left[ c_{s} + \int_{|x| < 1} x^{2} \nu_{s}(dx) \mid \mathcal{F}_{j/n}^{+} \right] \, ds = O(n^{-1}),$$

(24)
as required.

If instead \( p = 4 \), then since the quadratic variation \([M_I]_t\) is integrable, we may define the martingale

\[
M_{V,t} = [M_I]_t - \mathbb{E}[M_I]_t.
\]  

(25)

We then note that

\[
\mathbb{E}[(M_{V,(j+1)/n} - M_{V,j/n})^2 | \mathcal{F}_{j/n}^+] = \mathbb{E}[(M_{V,(j+1)/n} - [M_I]_{j/n})^2 | \mathcal{F}_{j/n}^+] = \mathbb{E} \left[ \int_{j/n}^{(j+1)/n} \int_{|x|<1} x^4 \mu(dx,ds) | \mathcal{F}_{j/n}^+ \right],
\]

since \([M_V]_t\) depends only on the jumps in \( X_{I,t} \),

\[
\leq \int_{j/n}^{(j+1)/n} \mathbb{E} \left[ \int_{|x|<1} x^2 \nu_s(dx) | \mathcal{F}_{j/n}^+ \right] ds = O(n^{-1}).
\]  

(26)

We thus have that

\[
\mathbb{E}[(M_{I,(j+1)/n} - M_{I,j/n})^4 | \mathcal{F}_{j/n}^+] = O(1) \mathbb{E}[(M_{I,(j+1)/n} - [M_I]_{j/n})^2 | \mathcal{F}_{j/n}^+],
\]

using the Burkholder-Davis-Grundy inequality,

\[
= O(1) \mathbb{E}[(M_{V,(j+1)/n} - M_{V,j/n} + O(n^{-1}))^2 | \mathcal{F}_{j/n}^+] = O(n^{-1}),
\]

using (24) and (25),

\[
= O(1) \mathbb{E}[(M_{V,(j+1)/n} - M_{V,j/n})^2 | \mathcal{F}_{j/n}^+] + O(n^{-2}) = O(n^{-1}),
\]

using (26).

We may now prove Lemma 4.

Proof of Lemma 4. We first denote by \( E_{\sigma,k} \) the event that \( X_{I,t} \), the large-jump component of the log-price, is constant over the interval \([k, k+1]/n_0\).

Since the expected number of jumps in \( X_{I,t} \) over that interval is

\[
\int_{k/n_0}^{(k+1)/n_0} \int_{|x| \geq 1} \nu_s(dx) ds = O(n^{-1/2}),
\]

42
we have that $E_{\sigma,k}$ holds with high probability,

$$\mathbb{P}(E_{\sigma,k}^c) = O(n^{-1/2}).$$

On the event $E_{\sigma,k}$, we have that $X_{J,t}$ makes no contribution to our estimate $\tilde{\sigma}_k^2$. In this case, $\tilde{\sigma}_k^2$ will be equal to

$$\tilde{\sigma}_k^2 = \frac{n_0}{2n} \sum_{j,j+1 \in J_k} (Z_{\varepsilon,j} + Z_{I,j})^2,$$

where the random variables

$$Z_{\varepsilon,j} = \varepsilon_{j+1} - \varepsilon_j, \quad Z_{I,j} = X_{I,(j+1)/n} - X_{I,j/n}.$$

Since $E_{\sigma,k}$ holds with high probability, we may therefore proceed by bounding $\tilde{\sigma}_k^2$. We set

$$S_k = \tilde{\sigma}_k^2 - \sigma_k^2/n_0,$$

and then have

$$S_k = S_{k,0} + S_{k,1} + S_{k,2} + S_{k,3} + O(n^{-1/2}),$$

for the sums

$$S_{k,0} = \frac{n_0}{2n} \sum_{j,j+1 \in J_k} (\sigma_{j/n}^2 + \sigma_{(j+1)/n}^2 - 2\sigma_k^2/n_0),$$

$$S_{k,1} = \frac{n_0}{2n} \sum_{j,j+1 \in J_k} (Z_{\varepsilon,j}^2 - \sigma_{j/n}^2 - \sigma_{(j+1)/n}^2),$$

$$S_{k,2} = \frac{n_0}{n} \sum_{j,j+1 \in J_k} Z_{\varepsilon,j}Z_{I,j},$$

$$S_{k,3} = \frac{n_0}{2n} \sum_{j,j+1 \in J_k} Z_{I,j}^2,$$

which we will bound in turn.

To bound $S_{k,0}$, we note that if $j, j + 1 \in J_k$, we have

$$\mathbb{E}[(\sigma_{j/n}^2 + \sigma_{(j+1)/n}^2 - 2\sigma_k^2/n_0)^2 | \mathcal{F}_{k/n_0}]$$

$$= O(1)\mathbb{E}[(\sigma_{j/n}^2 - \sigma_k^2/n_0)^2 + (\sigma_{(j+1)/n}^2 - \sigma_k^2/n_0)^2 | \mathcal{F}_{k/n_0}]$$

$$= O(n^{-1/2}),$$

so $\mathbb{E}[S_{k,0}^2 | \mathcal{F}_{k/n_0}] = O(n^{-1/2})$. Similarly, to bound $S_{k,1}$, we note that if also $j_1, j_1 + 1 \in J_k$, we have

$$\mathbb{E}[(Z_{\varepsilon,j}^2 - \sigma_{j/n}^2 - \sigma_{(j+1)/n}^2)(Z_{\varepsilon,j_1}^2 - \sigma_{j_1/n}^2 - \sigma_{(j_1+1)/n}^2) | \mathcal{F}_{k/n_0}]$$

$$= \begin{cases} O(1), & |j - j_1| \leq 1, \\ 0, & \text{otherwise}. \end{cases}$$

43
so $\mathbb{E}[S_{k,1}^2 \mid \mathcal{F}_{k/n}] = O(n^{-1/2})$.

To bound $S_{k,2}$, we note that

$$
\mathbb{E}[Z_{k,j}^2 Z_{I,j}^2 \mid \mathcal{F}_{k/n}] = \mathbb{E}[\epsilon_j^2 \mathbb{E}[Z_{I,j}^2 \mid \mathcal{F}_{j/n}] + Z_{I,j} \mathbb{E}[\epsilon_j^2 \mid \mathcal{F}_{(j+1)/n}] \mid \mathcal{F}_{k/n}]
= O(n^{-1}),
$$

using Lemma 8, so $\mathbb{E}[S_{k,2}^2 \mid \mathcal{F}_{k/n}] = O(n^{-1})$. For $S_{k,3}$, we likewise have

$$
\mathbb{E}[S_{k,3}^2 \mid \mathcal{F}_{k/n}] = O(n^{-1/2}),
$$

and thus $\mathbb{E}[S_k \mid \mathcal{F}_{k/n}] = O(n^{-1/4})$.

We have thus controlled the deviation of $S_k$; it remains to compute its moment generating function,

$$
f(v) = \mathbb{E}[\exp(-vS_k) \mid \mathcal{F}_{k/n}].
$$

Since $S_k \geq -\sigma_{k/n}^2$, for $v \geq 0$ we may take derivatives under the expectation, obtaining that

$$
|f'(v)| = |\mathbb{E}[S_k \exp(-vS_k) \mid \mathcal{F}_{k/n}]|
\leq \exp(v\sigma_{k/n}^2)\mathbb{E}[|S_k| \mid \mathcal{F}_{k/n}]
= O(n^{-1/4}),
$$

uniformly over $v \in [0,u]$, for fixed $u \geq 0$.

From Taylor’s theorem, we then have that for some $v \in [0,u]$,

$$
f(u) = f(0) + uf'(v) = 1 + O(n^{-1/4}).
$$

We thus deduce that

$$
\mathbb{E}[\exp(-u(\tilde{\sigma}_k^2 - \sigma_{k/n}^2)) \mid \mathcal{F}_{k/n}]
= \mathbb{E}[\exp(-u(\tilde{\sigma}_k^2 - \sigma_{k/n}^2))1(E_{\sigma,k}) \mid \mathcal{F}_{k/n}] + O(n^{-1/2}),
$$

since $\tilde{\sigma}_k^2 \geq 0$, and $\sigma_k^2$ is bounded,

$$
\mathbb{E}[\exp(-uS_k)1(E_{\sigma,k}) \mid \mathcal{F}_{k/n}] + O(n^{-1/2})
= f(u) + O(n^{-1/2}),
$$

44
since $S_k$ is bounded below,

$$= 1 + O(n^{-1/4}).$$

We have thus computed the moment generating function of $\hat{\sigma}_k^2 - \sigma^2_{k/n_0}$; we may now deduce our results. Since $\psi_t(u)$ is bounded, we conclude that

$$E[\exp(-\kappa \hat{\sigma}_k^2 u^2) \mid \mathcal{F}_{k/n_0}] = \psi_{k/n_0}(u) + O(n^{-1/4}),$$

and

$$\text{Var}[\exp(-\kappa \hat{\sigma}_k^2 u^2) \mid \mathcal{F}_{k/n_0}] = E[\exp(-2\kappa \hat{\sigma}_k^2 u^2) \mid \mathcal{F}_{k/n_0}] - E[\exp(-\kappa \hat{\sigma}_k^2 u^2) \mid \mathcal{F}_{k/n_0}]^2 = \psi^2_{k/n_0}(u)(1 + O(n^{-1/4}) - 1 + O(n^{-1/4})) = O(n^{-1/4}),$$

as required. ⊓⊔

**Notations**

- $X_t$: efficient price process
- $b_t, c_t, \nu_t$: semimartingale characteristics of $X_t$
- $B_t$: Brownian motion in $X_t$
- $\mu(dx, dt)$: Poisson random measure in $X_t$
- $\varepsilon_j$: microstructure noises
- $Y_j$: observed prices
- $n$: number of observations
- $L_t$: Lévy process in time-changed model
- $b, c, \nu$: Lévy characteristics of $L_t$
- $R_t$: time-change process
- $r_t$: normalised volatility
- $n_0, n_1, n_2$: numbers of bins used in estimates
- $h_1, h_2$: bandwidths in $n_1, n_2$
- $\Phi, \Phi_n$: scaling functions for pre-averaging
- $\hat{\sigma}_k$: pre-averaged increments
- $J_k$: index sets for pre-averaging
- $\hat{\sigma}_k^2$: estimated noise variance
- $\hat{\psi}_l(u)$: estimated characteristic functions
- $K_l$: index sets for volatility estimation
- $\hat{\psi}_l(u)$: estimated noise characteristic function
- $\kappa$: constant in noise characteristic function
- $\hat{c}_l(u)$: estimated volatilities
- $\hat{\tau}_l^2(u)$: bias-correction term in $\hat{c}_l(u)$
\begin{itemize}
  \item $c_t(u)$: adjusted volatility process
  \item \(\tilde{c}_t(u)\): local-polynomial estimate of \(c_t\)
  \item \(K, N, h\): parameters in \(\tilde{c}_t(u)\)
  \item \(W_{n,t}(t)\): weights in \(\tilde{c}_t(u)\)
  \item \(\tilde{r}_t(u)\): local-polynomial estimate of \(r_t\)
  \item \(T\): time horizon
  \item \(F_t, F_t^+\): filtrations of \(X_t\) before and after noise
  \item \(\mathcal{S}\): base class of probability measures
  \item \(I_{\alpha}(D,S)\): class of \(\alpha\)-smooth processes
  \item \(\alpha_0\): Lipschitz smoothness rate
  \item \(S_{\alpha,\beta}^\rho(C,D)\): class of semimartingale models
  \item \(\Omega_0, S\): probable event and stopping time in \(S_{\alpha,\beta}^\rho(C,D)\)
  \item \(T_{\alpha}(C,D), T^\alpha\): classes of time-changed Lévy models
  \item \(\varphi_t(u), \psi_t(u)\): asymptotic means of \(\hat{\varphi}_t(u), \hat{\psi}_t(u)\)
  \item \(\rho_t^2(u), \tau_t^2(u)\): asymptotic variances of \(\hat{\varphi}_t(u), \hat{\psi}_t(u)\)
  \item \(\alpha_1, \alpha_2\): convergence rates in mean and variance of \(\hat{\varphi}_t(u)\)
  \item \(\alpha_3\): convergence rate for \(\tilde{c}_t(u)\) estimating \(c_t(u)\)
  \item \(\alpha_4\): convergence rate for \(\tilde{c}_t(u)\) estimating \(c_t(u)\)
  \item \(E_t, \zeta(u)\): probable events and constant in Theorem 1
  \item \(\theta_t\): spot characteristic function of \(X_t\)
  \item \(p_j, q_j\): weights for pre-averaging, and their differences
  \item \(\Delta X_t\): increments of \(X_t\)
  \item \(X_{i,t}, X_{J,t}\): integrable and large-jump components of \(X_t\)
\end{itemize}

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47
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