REMARKS ON 5-DIMENSIONAL COMPLETE INTERSECTIONS

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Abstract. This paper will give some examples of diffeomorphic complex 5-dimensional complete intersections and remarks on these examples. Then a result on the existence of diffeomorphic complete intersections that belong to components of the moduli space of different dimensions will be given as a supplement to the results of P. Brückmann (J. reine angew. Math. 476 (1996), 209–215; 525 (2000), 213–217).

1. Introduction

Let $X_n(d) \subset \mathbb{C}P^{n+r}$ be a smooth complete intersection of multidegree $d := (d_1, \cdots, d_r)$, i.e., the transversal intersections of hypersurfaces of degrees $d_1, \cdots, d_r$ respectively. We call the product $d_1d_2\cdots d_r$ the total degree, denoted by $d$. It is well known that all complete intersections of fixed multidegree are diffeomorphic. On the other hand, there exist diffeomorphic complete intersections with different multidegrees. For lower dimensions, such as complex dimensions 2, 3, 4, the diffeomorphic examples can be found in [1, 2, 8]. W. Ebeling ([3]) and A.S. Libgober-J. Wood ([11]) independently found examples of homeomorphic complex 2-dimensional complete intersections but not diffeomorphic. In [6], F.Q. Fang and the author proved that, in dimensions $n = 5, 6, 7$, two complete intersections $X_n(d)$ and $X_n(d')$ are homeomorphic if and only if they have the same total degree, Pontrjagin classes and Euler characteristics. Particularly, by Traving’s result ([7, Theorem A] or [12]), to the prime factorization of total degree $d = \prod_{p \text{ primes}} p^{\nu_p(d)}$, if $\nu_p(d) \geq \frac{2n+1}{2(p-1)} + 1$ for all primes $p$ with $p(p-1) \leq n+1$, two homeomorphic complex $n$-dimensional complete intersections are diffeomorphic.

The first purpose of this paper is to give examples of diffeomorphic complex 5-dimensional complete intersections with different multidegrees. These examples, which are easy to check but hard to happen upon, were found by computer search. From these examples, we can deduce some interesting remarks about complete intersections.

Libgober and Wood ([10]) showed the existence of homeomorphic complete intersections of dimension 2 and diffeomorphic ones of dimension 3 which belong to components of the moduli space having different dimensions. In fact it was shown that there is a procedure which allows one to produce from a pair of homeomorphic complete intersections an arbitrarily long family, all members of which are homeomorphic. P. Brückmann ([1]) shows that the construction mentioned yields families of arbitrary length $t$ of complete intersections in $\mathbb{C}P^{4t-2}$ (resp. $\mathbb{C}P^{5t-2}$) consisting of homeomorphic complete intersections of dimension 2 (resp. diffeomorphic ones of dimension 3) but that belong to components of the moduli space of different dimensions. Furthermore, under Theorem 1 of [5],

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Brückmann also proves the similar result for the complete intersections of dimension 4 in $\mathbb{C}P^{d+2}$ ([2]).

Another purpose of this paper is to give the following theorem, which is a supplement to the results of Brückmann [1, 2].

**Theorem 1.1.** For each integer $t > 1$, there exist $t$ diffeomorphic complex 5-dimensional complete intersections in $\mathbb{C}P^{7t-2}$ isomorphism class of which lie in different dimensional components of the moduli space.

This paper is organized as follows: After presenting the basic formulas of characteristic classes of complete intersections in Section 2, we will give examples of diffeomorphic complex 5-dimensional complete intersections in Section 3. Section 4 proves Theorem 1.1. The last section will be devoted to the code of computer program to evaluate an inequality, which is a key to prove Theorem 1.1.

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### 2. Characteristic classes of complete intersections

For a complete intersection $X_n(d)$, let $H$ be the restriction of the dual bundle of the canonical line bundle over $\mathbb{C}P^{n+r}$ to $X_n(d)$, and $x = c_1(H) \in H^2(X_n(d); \mathbb{Z})$. Associate the multidegree $d = (d_1, d_2, \ldots, d_r)$, define the power sums $s_i = \sum_{j=1}^r d_j^i$ for $1 \leq i \leq n$. Then the Chern classes and Pontrjagin classes are presented as follows ([8]):

\[
\begin{align*}
    c_k &= \frac{1}{k!} g_k(n + r + 1 - s_1, \ldots, n + r + 1 - s_k)x^k, 1 \leq k \leq n, \\
    p_k &= \frac{1}{k!} g_k(n + r + 1 - s_2, \ldots, n + r + 1 - s_{2k})x^{2k}, 1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor.
\end{align*}
\]

The Euler characteristic is $(x^n \cap [X_n(d)]) = d = d_1 \cdots d_r)$

\[
e(X_n(d)) = c_n(X_n(d)) \cap [X_n(d)] = \frac{1}{n!} g_n(n + r + 1 - s_1, \ldots, n + r + 1 - s_n).
\]

Where the $g_k$’s are polynomials that can be iteratively computed from the Newton formula:

\[
s_k - g_1(s_1)s_{k-1} + \frac{1}{2} g_2(s_1, s_2)s_{k-2} + \cdots + (-1)^k \frac{1}{k!} g_k(s_1, s_2, \ldots, s_k)k = 0, k \geq 1.
\]
For example, the first six are
\[ g_1(s_1) = s_1, \]
\[ g_2(s_1, s_2) = s_1^2 - s_2, \]
\[ g_3(s_1, s_2, s_3) = s_1^3 - 3s_1s_2 + 2s_3, \]
\[ g_4(s_1, \ldots, s_4) = s_1^4 - 6s_1^2s_2 + 8s_1s_3 + 3s_2^2 - 6s_4, \]
\[ g_5(s_1, \ldots, s_5) = s_1^5 - 10s_1^2s_2 + 20s_1s_3 - 30s_1s_4 + 15s_2s_3 - 20s_2s_3 + 24s_5, \]
\[ g_6(s_1, \ldots, s_6) = s_1^6 - 15s_1^3s_2 + 40s_1^3s_3 - 90s_1^2s_4 + 45s_1^2s_2^2 - 120s_1s_2s_3 + 144s_1s_5 \]
\[ - 15s_2^3 + 90s_2s_4 + 40s_3^2 - 120s_6. \]

Note that the \( k \)-th Pontrjagin class \( p_k \) is a integral multiple of \( x^{2k} \), where \( x \) generates the second cohomology of the complete intersection. Thus we can compare this invariant for different complete intersections. For convenience, throughout the rest of the paper, we view the Pontrjagin class \( p_k \) of \( X_n(d) \) as the multiple of \( x^{2k} \).

3. **Examples of diffeomorphic complex 5-dimensional complete intersections**

For complex 5-dimensional complete intersections \( X_5(d_1, \ldots, d_r) \), its total degree, Pontrjagin classes and Euler characteristic are as follows:

\[ d = d_1 \times \cdots \times d_r, \quad (3.1) \]
\[ p_1 = 6 + r - s_2, \quad (3.2) \]
\[ p_2 = \frac{1}{2} \left[ (6 + r - s_2)^2 - (6 + r - s_4) \right], \quad (3.3) \]
\[ e = \frac{1}{5!} d \left[ (6 + r - s_1)^5 - 10(6 + r - s_1)^3(6 + r - s_2) + 20(6 + r - s_1)^2(6 + r - s_3) \right. \]
\[ - 30(6 + r - s_1)(6 + r - s_4) + 15(6 + r - s_1)(6 + r - s_2)^2 \]
\[ - 20(6 + r - s_2)(6 + r - s_3) + 24(6 + r - s_5) \]. \quad (3.4) \]

Here, \( p_1 \) and \( p_2 \) denote the Pontrjagin classes as appointed in the end of Section 2.

By Theorem 1.1 of [6], to find homeomorphic complex 5-dimensional complete intersections, we only need to find different multidegrees, such that (3.1)-(3.4) all agree respectively. Additionally, by [7, Theorem A], for the total degree \( d = \prod_{\text{primes}} p_\rho^{\nu_\rho(d)} \), if \( \nu_2(d) \geq 7 \) and \( \nu_3(d) \geq 4 \), the homeomorphic 5-dimensional complete intersections are diffeomorphic. This searching can completely be done by computer. According to [8, Proposition 7.3], let \( X_n(d) \subset \mathbb{C}P^{n+r} \) be a complete intersection of given codimension \( r \) with \( n > 2 \) and \( 2r \leq n + 2 \), then the total degree and Pontrjagin classes of \( X_n(d) \) determine the multidegree. Thus, it is impossible to find out such a homeomorphic or diffeomorphic example with different multidegrees in which one of the complete intersections has codimension 2 or 3 for complex dimension 5. Theoretically, there should exist a lot of homeomorphic complete intersections with codimension \( \geq 4 \). However, with the codimension becoming smaller, it will become more difficult to find out such examples. In fact, we can offer such examples with codimension 7 (See Section 4).
Example 3.1. Take two complete intersections $X_5(46, 36, 34, 21, 14, 13, 12, 11, 3, 2, 2)$, and $X_5(44, 42, 26, 23, 18, 17, 7, 6, 6, 4)$, we calculated the power sums of two multidegrees as follows:

| Multidegree | $s_1$ | $s_2$ | $s_3$ | $s_4$ | $s_5$ |
|-------------|-------|-------|-------|-------|-------|
| $(46, 36, 34, 21, 14, 13, 12, 11, 3, 2, 2)$ | 194   | 5655  | 20000 | 7790356 | 317267984 |
| $(44, 42, 26, 23, 18, 17, 7, 6, 6, 4)$   | 193   | 5655  | 20000 | 7790355 | 317267983 |

Although, the above two complete intersections have different power sums and codimensions, they have the same total degree and symmetric functions $r - s_1, \ldots, r - s_5$. By formulas (3.2),(3.3),(3.4), it is evident that they have the same Pontrjagin classes and Euler characteristic.

| $X_5(d)$ (codim=11, 10) | $d$   | $p_1$  | $p_2$  | $e/d$   |
|-------------------------|-------|--------|--------|---------|
| $X_5(46, 36, 34, 21, 14, 13, 12, 11, 3, 2, 2)$ | 340867118592 | -5639 | 19794330 | -6401091783 |
| $X_5(44, 42, 26, 23, 18, 17, 7, 6, 6, 4)$   | 340867118592 | -5639 | 19794330 | -6401091783 |

Since total degree satisfies $d = 2^9 \times 3^5 \times 7^2 \times 11 \times 13 \times 17 \times 23$, they are diffeomorphic complete intersections.

Example 3.2. (1), Take $X_5(66, 56, 45, 39, 16, 15, 8, 3)$, $X_5(64, 60, 42, 39, 20, 11, 9, 3)$, it is easy to get the following table:

| Multidegree | $s_1$ | $s_2$ | $s_3$ | $s_4$ | $s_5$ |
|-------------|-------|-------|-------|-------|-------|
| $(66, 56, 45, 39, 16, 15, 8, 3)$ | 248   | 11592 | 621368 | 35343636 | 2075677598 |
| $(64, 60, 42, 39, 20, 11, 9, 3)$ | 248   | 11592 | 621368 | 35343636 | 2075677598 |

The above two multidegrees have different power sums $s_3, s_5$, but they have the same total degree, Pontrjagin classes and Euler characteristic. Since $d = 37362124800 = 2^{11} \times 3^6 \times 5^2 \times 7 \times 11 \times 13$, so $X_5(66, 56, 45, 39, 16, 15, 8, 3)$ and $X_5(64, 60, 42, 39, 20, 11, 9, 3)$ are diffeomorphic.

(2), By deleting the last degree 3 from the multidegrees in (1), we take complete intersections $X_5(66, 56, 45, 39, 16, 15, 8)$ and $X_5(64, 60, 42, 39, 20, 11, 9)$.

| $X_5(d)$ (codim=8) | $d$   | $p_1$  | $p_2$  | $e/d$   |
|---------------------|-------|--------|--------|---------|
| $X_5(66, 56, 45, 39, 16, 15, 8)$ | 37362124800 | -11578 | 84696853 | -31485015068 |
| $X_5(64, 60, 42, 39, 20, 11, 9)$ | 37362124800 | -11578 | 84696853 | -31485015068 |

The different Euler characteristics imply that $X_5(66, 56, 45, 39, 16, 15, 8)$ is not homotopy equivalent to $X_5(64, 60, 42, 39, 20, 11, 9)$.

(3), By appending a degree 7 into the multidegrees in (1), we find that complete intersections $X_5(66, 56, 45, 39, 16, 15, 8, 7, 3)$ and $X_5(64, 60, 42, 39, 20, 11, 9, 7, 3)$ have different Euler characteristics.

| $X_5(d)$ (codim=9) | $d$   | $e/d$   |
|---------------------|-------|---------|
| $X_5(66, 56, 45, 39, 16, 15, 8, 7, 3)$ | 261534873600 | -33795490160 |
| $X_5(64, 60, 42, 39, 20, 11, 9, 7, 3)$ | 261534873600 | -33795524864 |
So $X_5(66, 56, 45, 39, 16, 15, 8, 7, 3)$ and $X_5(64, 60, 42, 39, 20, 11, 9, 7, 3)$ are not homotopy equivalent.

**Example 3.3.** For complex 4-dimensional complete intersections $X_4(d_1, \ldots, d_r)$, its Euler characteristic is as follows:

$$e = \frac{d}{4!} \left[(5 + r - s_1)^4 - 6(5 + r - s_1)(5 + r - s_2) + 8(5 + r - s_1)(5 + r - s_3) + 3(5 + r - s_2)^2 - 6(5 + r - s_4)\right].$$

Let $X_5(66, 56, 45, 39, 16, 15, 8, 3)$ and $X_5(64, 60, 42, 39, 20, 11, 9, 3)$, which are diffeomorphic by Example 3.2 (1), simultaneously make transversal intersection with hypersurface of homogeneous degree 2, we can construct two complex 4-dimensional complete intersections $X_4(66, 56, 45, 39, 16, 15, 8, 3, 2)$ and $X_4(64, 60, 42, 39, 20, 11, 9, 3, 2)$. They have different Euler characteristics,

| $X_4(d)$ (codim=9)            | $d$       | $e/d$    |
|------------------------------|-----------|----------|
| $X_4(66, 56, 45, 39, 16, 15, 8, 3, 2)$ | 74724249600 | 365019422 |
| $X_4(66, 56, 45, 39, 16, 15, 8, 3, 2)$ | 74724249600 | 365025086 |

So $X_4(66, 56, 45, 39, 16, 15, 8, 3, 2)$ and $X_4(64, 60, 42, 39, 20, 11, 9, 3, 2)$ are not homotopy equivalent. That is, although $X_5(66, 56, 45, 39, 16, 15, 8, 3)$ and $X_5(64, 60, 42, 39, 20, 11, 9, 3)$ are diffeomorphic, $X_4(66, 56, 45, 39, 16, 15, 8, 3, 2)$ and $X_4(64, 60, 42, 39, 20, 11, 9, 3, 2)$, which are the transversal intersection of diffeomorphic complex 5-dimensional complete intersections with the same hypersurface of homogeneous degree 2, do not have the same homotopy type.

**Example 3.4.** For complex 6-dim complete intersections $X_6(d_1, \ldots, d_r)$, its Euler characteristic is as follows:

$$e = \frac{d}{6!} \left[(7 + r - s_1)^6 - 15(7 + r - s_1)^4(7 + r - s_2) + 40(7 + r - s_1)^3(7 + r - s_3) - 90(7 + r - s_1)(7 + r - s_2)^2 + 45(7 + r - s_1)^2(7 + r - s_3) - 120(7 + r - s_1)(7 + r - s_2)(7 + r - s_3) + 144(7 + r - s_1)(7 + r - s_2) - 120(7 + r - s_2)^3 + 90(7 + r - s_2)(7 + r - s_2) + 40(7 + r - s_3)^2 - 120(7 + r - s_6)\right].$$

Take $X_6(66, 56, 45, 16, 15, 8, 3), X_6(64, 60, 42, 20, 11, 9, 3)$, it is easy to check that

| $X_6(d)$ (codim=7)            | $d$       | $e/d$    |
|------------------------------|-----------|----------|
| $X_6(66, 56, 45, 16, 15, 8, 3)$ | 958003200 | 1370218430570 |
| $X_6(64, 60, 42, 20, 11, 9, 3)$ | 958003200 | 1369971514442 |

The different Euler characteristics imply that $X_6(66, 56, 45, 16, 15, 8, 3)$ is not homotopy equivalent to $X_6(64, 60, 42, 20, 11, 9, 3)$. However, $X_5(66, 56, 45, 39, 16, 15, 8, 3)$ and $X_5(64, 60, 42, 39, 20, 11, 9, 3)$, which are the transversal intersection of the above two non-homotopy equivalent complex 6-dimensional complete intersections with the same hypersurface of homogeneous degree 39, are diffeomorphic by Example 3.2 (1).

Compare the above Examples 3.2, 3.3 and 3.4, we can obtain the following interesting remarks.
Remark 3.5. $X_n(d_1, \ldots, d_{r-1}, c)$ is homeomorphic (diffeomorphic, homotopy equivalent) to $X_n(d'_1, \ldots, d'_{r-1}, c)$, however, it may not be true not only for $X_n(d_1, \ldots, d_{r-1})$ and $X_n(d'_1, \ldots, d'_{r-1})$, but also for $X_n(d_1, \ldots, d_{r-1}, c, c')$ and $X_n(d'_1, \ldots, d'_{r-1}, c, c')$ (See Example 3.2 (1),(2),(3)).

Note that, in [4], Fang asked the following question: If $X_n(d)$ and $X_n(d')$ are diffeomorphic or homeomorphic or homotopy equivalent, is $X_n(d;a)$ diffeomorphic to $X_n(d';a)$ for a natural number $a$? Here $X_n(d;a)$ is the complete intersection with multidegree $(d_1, d_2, \ldots, d_r, a)$. Now, Remark 3.5 partially gives a negative answer to Fang’s question.

Remark 3.6. $X_{n+1}(d_1, \ldots, d_{r-1})$ is diffeomorphic to $X_{n+1}(d'_1, \ldots, d'_{r-1})$, but it may not be true for $X_n(d_1, \ldots, d_{r-1}, c)$ and $X_n(d'_1, \ldots, d'_{r-1}, c)$ (See Example 3.3), even if $c \leq \min\{d, d'\}$.

Remark 3.7. Even if $X_{n+1}(d_1, \ldots, d_{r-1})$ is not diffeomorphic to $X_{n+1}(d'_1, \ldots, d'_{r-1})$, $X_n(d_1, \ldots, d_{r-1}, c)$ can be diffeomorphic to $X_n(d'_1, \ldots, d'_{r-1}, c)$ (See Example 3.4).

4. Moduli spaces of complete intersections

In this section, we will prove Theorem 1.1.

Let $X_n(d) \subset \mathbb{C}P^N$, where $n \geq 2$, $d = (d_1, \ldots, d_r), d_i \geq 2$ and $r = N - n$. Then from [1, Lemma 3], the explicit formula for moduli space dimension is

$$m(d) \equiv m(X_n(d)) = 1 - (N + 1)^2 + \sum_{i=1}^r \binom{N + d_i}{N} + \sum_{i=1}^r \sum_{j=1}^r (-1)^j \sum_{1 \leq k_1 < \cdots < k_j \leq r} \binom{N + d_i - d_{k_1} - \cdots - d_{k_j}}{N}. \quad (4.1)$$

Where $\binom{m}{N} = 0$ for $m < N (m \in \mathbb{Z})$.

Theorem 4.1. For each integer $t > 1$, there exist $t$ diffeomorphic complex 5-dimensional complete intersections in $\mathbb{C}P^{t^2 - 2}$ isomorphism class of which lie in different dimensional components of the moduli space.

Proof. Consider the following two multidegrees

$$d = (88, 77, 72, 54, 48, 31, 29), \quad d' = (87, 81, 64, 62, 44, 33, 28).$$

We list the corresponding power sums, total degree, Pontrjagin classes and Euler characteristic in Table 1.

From Table 1, the total degree is $1136843237376 = 2^{11} \times 3^6 \times 7 \times 11^2 \times 29 \times 31$, so the two complete intersections $X_5(d)$ and $X_5(d')$ are diffeomorphic but have different moduli space dimensions:

$$m(d) = 1 382 270 197 857 128,$$
$$m(d') = 1 370 693 416 581 393.$$

There is a way to generate larger sets of diffeomorphic complete intersections from the above pairs $d$ and $d'$, which arose from [10] and had an application in [1, 2].
Denote the composed multidegree
\[ d_{\lambda,\mu} = (d_1, \ldots, d_\lambda, d'_1, \ldots, d'_\mu), \lambda + \mu = s \geq 1. \]

Then the composed multidegrees \( d_{0,s} \), \( d_{1,s-1} \), \ldots, \( d_{s,0} \) have the same power sums \( s_1, s_2, \ldots, s_5 \) respectively, so the corresponding complete intersections are diffeomorphic to each other. Let \( X_5(d_{\lambda,\mu}) \subset \mathbb{C}P^{7s+5} \) be 5-dimensional complete intersections with multidegree \( d_{\lambda,\mu} \). It is reasonable to expect that the corresponding \( m(d_{\lambda,\mu})'s \) will all be different (See [10]). There is no general way to prove this. However, for the dimension formula (4.1) and the above special pairs \( d \) and \( d' \), there are finite binomial coefficients \((N + d_i - d_{k_1} - \cdots - d_{k_j})/N \) different from zero \((N = 7s + 5)\). We can prove the following inequality:

\[ m(d_{\lambda+1,\mu-1}) - m(d_{\lambda,\mu}) > 0, \; 0 \leq \lambda < s. \]

This inequality will be proved in the coming Proposition.

Now, the sequence \( m(d_{\lambda,s-\lambda})|_{\lambda=0,1,\ldots,s-1} \) is strictly monotonously increasing. Let \( t = s+1 \), there exist \( t \) five-dimensional complete intersections \( X_5(d_{0,s}), X_5(d_{1,s-1}), \ldots, X_5(d_{s,0}) \) in \( \mathbb{C}P^{7s+5} = \mathbb{C}P^{7t-2} \) with the desired properties. The proof is finished. \( \square \)

**Proposition 4.2.**

\[ m(d_{\lambda+1,s-\lambda-1}) - m(d_{\lambda,s-\lambda}) > 0, \; 0 \leq \lambda < s. \]

**Proof.** For the chosen multidegrees \( d \) and \( d' \),

\[
m(d_{\lambda,s-\lambda}) = 1 - (N + 1)^2 + \left[ \frac{\lambda}{d_i \in d} \right] + \left( s - \lambda \right) \left[ \frac{d_i \in d}{d_i} \right] \left( \frac{N + d_i}{N} \right) + \left[ \frac{\lambda}{d_i \in d} \right] + \left( s - \lambda \right) \left[ \frac{d_i \in d}{d_i} \right] \sum_{j=1}^{3} (-1)^j \sum_{1 \leq k_1 < \cdots < k_j \leq 7s} \left[ \frac{d_i \in d, s-\lambda}{d_{k_1}, \ldots, d_{k_j}} \right] \left( \frac{N + d_i - d_{k_1} - \cdots - d_{k_j}}{N} \right),
\]

|      | \( s_1 \) | \( s_2 \) | \( s_3 \) | \( s_4 \) | \( s_5 \) |
|------|----------|----------|----------|----------|----------|
| \( d \) | 399      | 25879    | 1833489  | 137438707| 10682130249|
| \( d' \) | 399      | 25879    | 1833489  | 137438707| 10682130249|

**Table 1.** Power sum, total degree, Pontrjagin class, Euler characteristic
Where, the index $j$ is maximally 3 that is determined by $\max\{d, d'\} = 88$ and $\min\{d, d'\} = 28$. So,

$$m(d_{\lambda+1,s-\lambda-1}) - m(d_{\lambda,s-\lambda})$$

$$= \left[ \sum_{d_i \in \mathcal{d}} - \sum_{d_i \in \mathcal{d}'} \right] \left( \frac{N + d_i}{N} \right)$$

$$+ \left[ \lambda \sum_{d_i \in \mathcal{d}} + (s - \lambda) \sum_{d_i \in \mathcal{d}'} \right] \sum_{j=1}^{3} (-1)^j \sum_{1 \leq k_1 < \cdots < k_j \leq 7s} \sum_{d_{k_1}, \ldots, d_{k_j} \in d_{\lambda+1,s-\lambda-1}} \left( \frac{N + d_i - d_{k_1} - \cdots - d_{k_j}}{N} \right)$$

$$- \left[ \lambda \sum_{d_i \in \mathcal{d}} + (s - \lambda) \sum_{d_i \in \mathcal{d}'} \right] \sum_{j=1}^{3} (-1)^j \sum_{1 \leq k_1 < \cdots < k_j \leq 7s} \sum_{d_{k_1}, \ldots, d_{k_j} \in d_{\lambda,s-\lambda}} \left( \frac{N + d_i - d_{k_1} - \cdots - d_{k_j}}{N} \right)$$

(4.2)

$$\triangleq 3 \sum_{j=0}^{3} M_j(\lambda, s),$$

To prove (4.2) $> 0$, let us decompose (4.2) into the sum of $M_j(\lambda, s)$, $j = 0, 1, 2, 3$. In the following, we will describe $M_j(\lambda, s)$ as polynomials of invariants $s$ and $\lambda$ ($N = 7s + 5$). Firstly,

$$M_0(\lambda, s) \triangleq \left[ \sum_{d_i \in \mathcal{d}} - \sum_{d_i \in \mathcal{d}'} \right] \left( \frac{N + d_i}{N} \right),$$

(4.3)

$$M_1(\lambda, s) \triangleq \left[ - (\lambda + 1) \sum_{d_i \in \mathcal{d}} \sum_{d_k \in d_{\lambda+1,s-\lambda-1}} - (s - \lambda - 1) \sum_{d_i \in \mathcal{d}'} \sum_{d_k \in d_{\lambda,s-\lambda}} \right] \left( \frac{N + d_i - d_k}{N} \right)$$

$$= \left[ - (\lambda + 1)^2 \sum_{d_i \in \mathcal{d}} \sum_{d_k \in \mathcal{d}} - (s - \lambda - 1) \sum_{d_i \in \mathcal{d}'} \sum_{d_k \in \mathcal{d}'} \right] \left( \frac{N + d_i - d_k}{N} \right)$$

$$- (s - \lambda - 1)(\lambda + 1) \sum_{d_i \in \mathcal{d}'} \sum_{d_k \in d_{\lambda,s-\lambda}} - (s - \lambda - 1)^2 \sum_{d_i \in \mathcal{d}'} \sum_{d_k \in d_{\lambda,s-\lambda}}$$

$$+ \lambda^2 \sum_{d_i \in \mathcal{d}} \sum_{d_k \in \mathcal{d}} + \lambda(s - \lambda) \sum_{d_i \in \mathcal{d}'} \sum_{d_k \in \mathcal{d}}$$

$$+ (s - \lambda) \sum_{d_i \in \mathcal{d}'} \sum_{d_k \in \mathcal{d}} + (s - \lambda)^2 \sum_{d_i \in \mathcal{d}'} \sum_{d_k \in \mathcal{d}'} \right] \left( \frac{N + d_i - d_k}{N} \right)$$

$$= \left[ (-2\lambda - 1) \sum_{d_i \in \mathcal{d}} \sum_{d_k \in \mathcal{d}} + (1 + 2\lambda - s) \sum_{d_i \in \mathcal{d}'} \sum_{d_k \in \mathcal{d}'} \right] \left( \frac{N + d_i - d_k}{N} \right).$$

(4.4)
There are four summations in the third part \( M_2(\lambda, s) \),

\[
M_2(\lambda, s) \triangleq \left[ (\lambda + 1) \sum_{d_i \in d} \sum_{d_{k1}, d_{k2} \in \mathcal{L}_{\lambda+1,s-\lambda}} + (s - \lambda - 1) \sum_{d_i \in d'} \sum_{d_{k1}, d_{k2} \in \mathcal{L}_{\lambda+1,s-\lambda}} \right] \left( N + d_i - d_{k1} - d_{k2} \right)
\]

For the simplification of summations, let’s define

\[
\Gamma_{dd'd} = \sum_{d_i \in d} \sum_{d_j \in d'} \sum_{d_k \in d} \left( N + d_i - d_j - d_k \right) = \Gamma_{dd'd'}.
\]

\[
\Gamma_{dd'd} = \sum_{d_i \in d} \sum_{d_j \in d'} \sum_{d_k \in d} \sum_{d_k \in d} \left( N + d_i - d_j - d_k \right) = \Gamma_{dd'd}.
\]

\[
\Gamma_{dd'd} = \sum_{d_i \in d} \sum_{1 \leq k_1 < k_2 \leq 2s} \left( N + d_i - d_{k1} - d_{k2} \right),
\]

\[
\Gamma_{dd'd} = \sum_{d_i \in d} \sum_{1 \leq k_1 < k_2 \leq 2s} \left( N + d_i - d_{k1} - d_{k2} \right).
\]

Similarly, \( \Gamma_{dd'd}, \Gamma_{dd'd'}, \Gamma_{dd'd'}, \Gamma_{dd'd}, \Gamma_{dd'd'} \) can also be imitated and defined. By induction, it is easy to see that:

\[
\sum_{d_i \in d} \sum_{1 \leq k_1 < k_2 \leq 2s} \left( N + d_i - d_{k1} - d_{k2} \right) = \lambda \Gamma_{dd'd} + \frac{\lambda(\lambda - 1)}{2} \Gamma_{dd'd'} + (s - \lambda) \Gamma_{dd'd} + \frac{(s - \lambda)(s - \lambda - 1)}{2} \Gamma_{dd'd}.
\]

Similarly,

\[
\sum_{d_i \in d'} \sum_{1 \leq k_1 < k_2 \leq 2s} \left( N + d_i - d_{k1} - d_{k2} \right) = \lambda \Gamma_{dd'd} + \frac{\lambda(\lambda - 1)}{2} \Gamma_{dd'd'} + (s - \lambda) \Gamma_{dd'd} + \frac{(s - \lambda)(s - \lambda - 1)}{2} \Gamma_{dd'd}.
\]
Then,

\[ M_2(\lambda, s) = (\lambda + 1) \left[ (\lambda + 1) \Gamma_{d_1 <} + \frac{(\lambda + 1) \lambda}{2} \Gamma_{d_2 <} + (\lambda + 1)(s - \lambda - 1) \Gamma_{d_3 <} \right. \]
\[ + (s - \lambda - 1) \Gamma_{d_4 <} + \frac{(s - \lambda - 1)(s - \lambda - 2)}{2} \Gamma_{d_5 <} \left] \sum_{d_1 \in d} \sum \sum_{1 \leq k_1 < k_2 < k_3 \leq 7s} - (s - \lambda - 1) \sum_{d_i \in d'} \sum \sum_{1 \leq k_1 < k_2 < k_3 \leq 7s} \lambda \sum_{d_i \in d} \sum \sum_{1 \leq k_1 < k_2 < k_3 \leq 7s} d_1 - d_k - d_k' - d_k'' \right. \]
\[ + \left. \lambda \sum_{d_i \in d} \sum \sum_{1 \leq k_1 < k_2 < k_3 \leq 7s} d_1 - d_k - d_k' - d_k'' + (s - \lambda) \sum_{d_i \in d'} \sum \sum_{1 \leq k_1 < k_2 < k_3 \leq 7s} \right] \left( N + d_i - d_k - d_k' - d_k'' \right). \]

For the last part \( M_3(\lambda, s) \),

\[ M_3(\lambda, s) \triangleq \left\{ - (\lambda + 1) \sum_{d_1 \in d} \sum \sum_{1 \leq k_1 < k_2 < k_3 \leq 7s} \right. \]
\[ + \left. \lambda \sum_{d_i \in d} \sum \sum_{1 \leq k_1 < k_2 < k_3 \leq 7s} \right. \]
\[ + \left. \left( N + d_i - d_k - d_k' - d_k'' \right) \right\} \left( N + 88 - d_k - d_k' - d_k'' \right) \]
\[ + \left[ - (s - \lambda - 1) \sum_{d_1 \in d} \sum \sum_{1 \leq k_1 < k_2 < k_3 \leq 7s} \right. \]
\[ + \left. \lambda \sum_{d_i \in d} \sum \sum_{1 \leq k_1 < k_2 < k_3 \leq 7s} \right. \]
\[ + \left. \left( N + 87 - d_k - d_k' - d_k'' \right) \right\} \left( N + 87 - d_k - d_k' - d_k'' \right). \]
By induction, it is easy to see that

\[
\sum_{1 \leq k_1 < k_2 < k_3 \leq 7s, d_{k_1, d_{k_2, d_{k_3}}} \in d_{\lambda, s-\lambda}} \left( \frac{N + 88 - d_{k_1} - d_{k_2} - d_{k_3}}{N} \right)
\]

\[
= \lambda^2 (s - \lambda) + \left[ \lambda \left( \frac{s - \lambda}{2} \right) + \frac{1}{6} (\lambda - 2)(\lambda - 1)\lambda \right] \left( \frac{N + 1}{N} \right) + \frac{\lambda(\lambda - 1)(s - \lambda)}{2} \left( \frac{N + 2}{N} \right)
\]

\[+ \lambda \left( \frac{s - \lambda}{2} \right) \left( \frac{N + 3}{N} \right) + \frac{1}{6} (s - \lambda - 2)(s - \lambda - 1)(s - \lambda) \left( \frac{N + 4}{N} \right),
\]

\[\sum_{1 \leq k_1 < k_2 < k_3 \leq 7s, d_{k_1, d_{k_2, d_{k_3}}} \in d_{\lambda, s-\lambda}} \left( \frac{N + 87 - d_{k_1} - d_{k_2} - d_{k_3}}{N} \right)
\]

\[
= \lambda \left( \frac{s - \lambda}{2} \right) + \frac{1}{6} (\lambda - 2)(\lambda - 1)\lambda + \frac{\lambda(\lambda - 1)(s - \lambda)}{2} \left( \frac{N + 1}{N} \right)
\]

\[+ \lambda \left( \frac{s - \lambda}{2} \right) \left( \frac{N + 2}{N} \right) + \frac{1}{6} (s - \lambda - 2)(s - \lambda - 1)(s - \lambda) \left( \frac{N + 3}{N} \right).
\]

Note that \(M_3(\lambda, s)\) will non-trivially appear only when \(s \geq 2\). Thus

\[
M_3(\lambda, s) = \frac{1}{6} (12 - 21s + 12s^2 - 3s^3 + 44\lambda - 54s\lambda + 18s^2\lambda + 60\lambda^2 - 48s\lambda^2 + 40\lambda^3)
\]

\[+ \frac{1}{6} (-6 + 9s - 3s^2 - 23\lambda + 30s\lambda - 12s^2\lambda - 33\lambda^2 + 36s\lambda^2 - 28\lambda^3) \left( \frac{N + 1}{N} \right)
\]

\[- \frac{1}{2} (-1 + s - 2\lambda)(2 - 3s + s^2 + 4\lambda - 4s\lambda + 4\lambda^2) \left( \frac{N + 2}{N} \right)
\]

\[+ \frac{1}{3} (-1 + s - \lambda)(6 - 7s + 2s^2 + 13\lambda - 7s\lambda + 8\lambda^2) \left( \frac{N + 3}{N} \right)
\]

\[+ \frac{1}{6} (1 - s + \lambda)(2 - s + \lambda)(3 - s + 4\lambda) \left( \frac{N + 4}{N} \right).
\]

(4.6)

Summarize (4.3)-(4.6), we see that (4.2) is exactly a polynomial of \(s, \lambda\) with complicated coefficients and higher degree. Fortunately, using the technical computational software Mathematica, (4.3)-(4.6) can all be computed by executable program. Finally, we calculate the following results:

\[
m(d_{1,0}) - m(d_{0,1}) = 11 576 781 275 735,
\]

\[
m(d_{2,0}) - m(d_{1,1}) = 34 356 628 415 559 239 284,
\]

\[
m(d_{1,1}) - m(d_{0,2}) = 34 347 842 980 758 828 832.
\]

More generally,

\[
m(d_{\lambda+1,s-\lambda-1}) - m(d_{\lambda,s-\lambda}) > \begin{cases} 3148, & 0 \leq \lambda < s, \\ 4 \times 10^{24}, & 0 \leq \lambda < s, s \geq 3. \end{cases}
\]
Furthermore, the outputs of the following two cases in Mathematica program are false,

\[ m(d_{\lambda+1,s-\lambda-1}) - m(d_{\lambda,s-\lambda}) < \begin{cases} 
3148, & 0 \leq \lambda < s, \\
4 \times 10^{24}, & 0 \leq \lambda < s, s \geq 3. 
\end{cases} \]

Thus, it is clear that, with any fixed \( s \geq 1, s > \lambda \geq 0 \), \( m(d_{\lambda,s-\lambda}) \) form a strictly monotonously increasing sequence for \( \lambda \). Hence, the Proposition follows. \( \square \)

5. Mathematica Code and outputs

In this section, Mathematica code and outputs that are designed to evaluate the inequality in Proposition 4.2 are attached in a notebook(.nb format).
Evaluation of the inequality in Proposition 4.2

Input of Multidegrees

\[ A_1 = \{88, 77, 72, 54, 48, 31, 29\}; \]
\[ A_2 = \{87, 81, 64, 62, 44, 33, 28\}; \]

Sums of Binomial Coefficients

\[ \alpha[x, a_] := \text{Sum}[\text{Binomial}[7 \times 5 + a[k], a[k]], \{k, \text{Length[a]}\}]; \]
\[ \beta[x, a_, b_] := \text{Sum[} \]
\[ \quad \text{If}[a[k] - b[l] \geq 0, \text{Binomial}[7 \times 5 + a[k] - b[l], a[k] - b[l]], 0], \{k, \text{Length[a]}\}, \{l, \text{Length[b]}\}]; \]
\[ \gamma[x, a_, b_, c_] := \text{Sum[} \]
\[ \quad \text{If}[a[k] - b[l] - c[t] \geq 0, \text{Binomial}[7 \times 5 + a[k] - b[l] - c[t], a[k] - b[l] - c[t]], 0], \{k, \text{Length[a]}\}, \{l, \text{Length[b]}\}, \{t, \text{Length[c]}\}]}; \]
\[ \delta[x, a_, b_] := \text{Sum[} \]
\[ \quad \text{If}[a[k] - b[l] - b[l] \geq 0, \text{Binomial}[7 \times 5 + a[k] - b[l] - b[l], a[k] - b[l] - b[l]], 0], \{k, \text{Length[a]}\}, \{l, \text{Length[b]} - 1\}, \{t, 1 + 1, \text{Length[b]}\}]; \]

Polynomials \( M_j(y, x), j = 0, 1, 2, 3, y = \lambda, x = s \)

\[ M_0[y, x] := \alpha[x, A1] - \alpha[x, A2]; \]
\[ M_1[y, x] := (2y - 1) \beta[x, A1, A1] + (1 + 2y - x) \beta[x, A1, A2] + \]
\[ (1 + 2y - x) \beta[x, A2, A1] + (2x + 2y - 1) \beta[x, A2, A2]; \]
\[ M_2[y, x] := (2y + 1) \delta[x, A1, A1] + \frac{y(3y + 1)}{2} \gamma[x, A1, A1, A1] + \]
\[ (y + 1)^2 (x - y - 1) - y^2 (x - y) \gamma[x, A1, A2, A1] + \]
\[ (x - 2y - 1) \delta[x, A1, A2] + \delta[x, A2, A1] \]
\[ + \frac{(x - y - 1)(x - 3y - 2)}{2} \gamma[x, A1, A2, A2] + \frac{y(2x - 3y - 1)}{2} \gamma[x, A2, A1, A1] + \]
\[ (y + 1) (x - y - 1)^2 - y (x - y)^2 \gamma[x, A2, A2, A1] + \]
\[ (1 + 2x + 2y) \delta[x, A2, A2] + \frac{(x - y - 1)(2 - 3x + 3y)}{2} \gamma[x, A2, A2, A2]; \]
\[ M_3[y, x] := \frac{1}{6} \left(12 - 21x + 12x^2 - 3x^3 + 44y - 54xy + 18x^2y + 60y^2 - 48x^2y^2 + 40y^3\right) + \]
\[ \frac{1}{6} \left(-6 + 9x - 3x^2 - 23y + 30xy - 12x^2y - 33y^2 + 36x^2y^2 - 28y^3\right) \]
\[ \text{Binomial}[7x + 6, 1] - \]
\[ \frac{1}{2} \left(-1 + x - 2y\right) \left(2 - 3x + x^2 + 4y - 4xy + 4y^2\right) \text{Binomial}[7x + 7, 2] + \]
\[ \frac{1}{3} \left(-1 + x - y\right) \left(6 - 7x + 2x^2 + 13y - 7xy + 8y^2\right) \text{Binomial}[7x + 8, 3] + \]
\[ \frac{1}{6} \left(1 - x + y\right) (2 - x + y) (3 - x + 4y) \text{Binomial}[7x + 9, 4]; \]
The case \( s=x=1,2 \)

\[
m(1) = 1.382270197857128
\]

\[
m(2) = 1.370693416581393
\]

\[
m(1) - m(2) = 0.01557681275735
\]

\[
m(1, A1) = 3.4356628415559239284
\]

\[
m(1, A2) - m(2, A1) = 0.0000000000000000000000
\]

**Solution of**

\[
M_0(y, x) + M_1(y, x) + M_2(y, x) + M_3(y, x) > 4 \times 10^{24} 
\]

Reduce [FunctionExpand[M0[y, x] + M1[y, x] + M2[y, x] + M3[y, x]] >

\[
4 \times 10^{24} \land x > 0 \land y > 0, (x, y)
\]

**Solution of**

\[
M_0(y, x) + M_1(y, x) + M_2(y, x) + M_3(y, x) < 4 \times 10^{24}
\]

Reduce[FunctionExpand[M0[y, x] + M1[y, x] + M2[y, x] + M3[y, x]] <

\[
4 \times 10^{24} \land x > 0 \land y > 0, (x, y)
\]

**Solution of**

\[
M_0(y, x) + M_1(y, x) + M_2(y, x) + M_3(y, x) > 3148 \land x > 0
\]

Reduce[FunctionExpand[M0[y, x] + M1[y, x] + M2[y, x] + M3[y, x]] >

\[
3148 \land x > 0 \land y > 0, (x, y)
\]

**Solution of**

\[
M_0(y, x) + M_1(y, x) + M_2(y, x) + M_3(y, x) \leq 3148 \land x > 0
\]

Reduce[FunctionExpand[M0[y, x] + M1[y, x] + M2[y, x] + M3[y, x]] \leq

\[
3148 \land x > 0 \land y > 0, (x, y)
\]
REFERENCES

[1] P. Brückmann, A remark on moduli spaces of complete intersections, J. reine angew. Math. 476 (1996), 209–215.
[2] P. Brückmann, A remark on moduli spaces of 4-dimensional complete intersections, J. reine angew. Math. 525 (2000), 213–217.
[3] W. Ebeling, An example of two homeomorphic, nondiffeomorphic complete intersection surfaces, Invent. Math. 99(3) (1990), 651–654.
[4] F. Q. Fang, Topology of complete intersections, Comment. Math. Helv. 72 (1997), 466–480.
[5] F. Q. Fang and S. Klaus, Topological classification of 4-dimensional complete intersections, Manuscript Math. 90 (1996), 139–147.
[6] F. Q. Fang and J. B. Wang, Homeomorphism classification of complex projective complete intersections of dimensions 5, 6 and 7, Math. Z. 266 (2010), 7919–746.
[7] M. Kreck, Surgery and duality. Ann. of Math. 149(3) (1999), 707–754.
[8] A. S. Libgober and J. W. Wood, Differentiable structures on complete intersections I, Topology. 21 (1982), 469–482.
[9] A. S. Libgober and J. W. Wood, Differentiable structures on complete intersections II, Singularities, Proc. Symp. Math. 40, Part 2, Amer. Math. Soc., Providence, RI (1983), 123–133.
[10] A. S. Libgober and J. W. Wood, Remarks on moduli spaces of complete intersections. Contemp. Math. 58 (1986), 183–194.
[11] A. S. Libgober and J. W. Wood, Uniqueness of the complex structure on Kähler manifolds of certain homotopy types, J. Differential Geom. 32(1) (1990), 139–154.
[12] C. Traving, Klassifikation vollständiger Durchschnitte, Diplomarbeit, University of Mainz, 1985.

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